THE MODIFIED DECOMPOSITION METHOD FOR SOLVING LINEAR SECOND-ORDER FREDHOLM INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract

This paper applies Modified Decomposition Method (MDM) as numerical analysis linear second-order Fredholm Integro-differential Equations. The calculation of the approximate solutions are computed by mathematical package. The main aim of this paper is to demonstrate how effective this method minimizes the size of calculations and reaching the final solution in the shortest time and best result. When comparing the results with the (ADM) and with the exact solution, we will note how effective this method minimizes the size of calculations of the solution and reaches the exact solution. Accordingly, the (MDM) is the best method to be used to solve linear second-order Fredholm Integro-Differential equation. The conversion to the exact solution is notably fast and also a time saver, as it requires less computational work in solving equations. This is why the (MDM) is more efficient in solving this kind of equations.

Keywords: MDM, Integro-differential Equations, Fredholm integral Equation, approximate solutions.

I. Introduction

An integro-differential equation is a type of equations where two kinds of operators, differential and integral appear in the same equation. The major role of the integro differential equations significant in many field of science, engineering and physical phenomena. As examplified, but not limited by: nano-hydrodynamics, glass-forming process, dropwise condensation, wind ripple in the desert, and modeling the competition between tumor cells and the immune system. Many research works are motivated by this system of integro-differential equations in the last few years. And, thus, attention was focused on the study of these equations. For example, the
equations containing shifts of the unknown function and its derivatives, and also integro-differential-difference equations. For more illustrative examples, see [III] and [VII]. An integro-differential equation can be classified into two types: Fredholm integro-differential equation and Volterra integro-differential equation. The difference between these two types is that in the former equations, the variables have fixed bound limits whereas in the latter equations, the variables have an upper bound limit. In this study, we will use the (MDM) to solve linear Fredholm Integro-differential equations in second-order and we compare the result with previous updated work by MOHD F. KARIM and others [IX], where (ADM) is used to solve Fredholm Integro-Differential equations of second-Order. However, many researchers use some other method to solve Fredholm Integro-Differential equation, for instance, Conjugate Gradient method [IV], Legendre-Galerkin method [VIII], Tau method [XI], Generalize Minimal Residual [V], Trigonometric Scaling Function [VI], Taylor Polynomial [X] and Sinc-Collocation method [I].

Without loss of generality, a definition can be given to the system of second-order linear Fredholm integro-differential equations, as follows:

\[ A(r)u''(r) + B(r)u'(r) + C(r)u(r) = g(r) + \int_a^b \psi(r,s)u(s)ds; \]

and the initial condition is \( u(r) = p, \ u'(r) = q, \) where \( A(R), B(R), C(R) \) is a constant matrices, \( g(r) \) a function, \( \psi(r,s) \) separable kernel, and \( u(r) \) solution to be determined.

This paper is using (MDM) in solving linear second-order Fredholm Integro-differential equation. The paper is organized in the following way. Sec.II is a review of linear second-order Fredholm Integro-Differential equation. (MDM) is shown as a method to solve equations in Sec.III. Sec.IV, constructs examples of the equations and show how the result of using (MDM) converges to the exact solution and show that this method minimizes the size of calculations of the solution when comparing the results with the (ADM).

II. Preliminaries

This section is a sum up of the general form of linear second-order Fredholm Integro-Differential equation. The general form of that equation is in the following form:

\[ u''(r) = g(r) + \lambda \int_a^b \psi(r,s)u(s)ds; \]

in which the kernel \( \psi(r,s) \) is separable kernels and to be shown as a finite sum of the form:

\[ \psi(r,s) = \sum_{k=1}^{n} f_k(r)h_k(s); \]

now, This analysis will be performed on one term kernel of the form:

\[ \psi(r,s) = f(r)h(s). \]
An assumed standard form to Fredholm integro-differential equation is given by:

\[ u^m(r) = g(r) + \int_0^1 \psi(r, s)u(s)ds, \quad u^k(0) = a_k, \quad 0 \leq k \leq (m - 1). \tag{5} \]

in which \( u^m(r) \) indicates the \( m \)th derivative of \( u(r) \) with respect to \( r \) and \( a_k \) are constants that give the initial conditions. Now, by substituting (4) in (5), we get

\[ u^m(r) = g(r) + f(r) \int_0^1 h(s)u(s)ds, \quad u^k(0) = a_k, \quad 0 \leq k \leq (m - 1). \tag{6} \]

In our work, we will depend the \( m \)th derivative in second-order, \( m = 2 \). The definite integral in Integro-Differential (6) will easily be observed. It involves an integ which depends on the variable \( s \). In operator form, equation (6) can be written as

\[ Iu(r) = g(r) + f(r) \int_0^1 h(s)u(s)ds, \tag{7} \]

where \( I \) is the differential operator given by

\[ I = \frac{d^m}{dr^m}. \]

It is clear that \( I \) is an invertible operator, and when applying \( I^{-1} \) on the both sides of (7) we get

\[ u(r) = a_0 + a_1 r + \frac{1}{2!} a_2 r^2 + \cdots + \frac{1}{(m-1)!} a_{m-1} r^{m-1} + I^{-1}(g(r)) + \left( \int_0^1 h(s)u(s)ds \right) I^{-1}(f(r)). \tag{8} \]

We note that the equation obtained in (8) is a standard form of Fredholm Integro-Differential equation.

### III. The Modified Decomposition Method

In this section, we show how to apply the (MDM) to the equation. The modified decomposition method is a powerful technique that minimizes the size of calculations. The method requires that the data function \((g(r))\) in (7) to have more than one term, in order for it to be used. Hence, the data function \((g(r))\) can be segregated into two components, as introduced in [II] of the form

\[ g(r) = g_0(r) + g_1(r). \tag{9} \]

Now we must determine the \( u(r) \) for solving (8). The series solution of \( u(r) \) is find by

\[ u(r) = \sum_{k=0}^{\infty} u_m(r). \]

\[ u_0(r) + u_1(r) + u_2(r) + \cdots \]
\[ u(r) = \sum_{k=0}^{m-1} \frac{1}{k!} a_k r^k + L^{-1}(g(r)) + \left( \int_0^1 h(s) u_0(s) ds \right) L^{-1}(f(r)) \]
\[ + \left( \int_0^1 h(s) u_1(s) ds \right) L^{-1}(f(r)) + \left( \int_0^1 h(s) u_2(s) ds \right) L^{-1}(f(r)) \]
\[ + \cdots \quad (10) \]

When substituting (9) in (10) we can determine the unknown function \( u(r) \) by the above mentioned relation, where
\[ u_0(r) = \sum_{k=0}^{m-1} \frac{1}{k!} a_k r^k + L^{-1}(g(r)) \]
\[ u_1(r) = L^{-1}(g_1(r)) + \left( \int_0^1 h(s) u_0(s) ds \right) L^{-1}(f(r)) \]
\[ u_{m+1}(r) = \left( \int_0^1 h(s) u_m(s) ds \right) L^{-1}(f(r)), \quad m \geq 1. \quad (11) \]

The equations (11) above represent a formula (MDM), and through it, we can find a variety of solutions \( u(r) \). The advantage of this method, as we have mentioned earlier, is the fast result in reaching solution, as in illustrated exemplified by the clarified method which is compared with the (ADM) in both the speed and accuracy of access to the solution.

### IV. Illustrative Examples

In this section we present four examples and apply the method presented in the previous section for solving them. The expected results are to coverage approximately to the exact solution.

**Example 1.** Consider linear Fredholm Integro-Differential equation of second-order [IX]:
\[ u''(r) = r - \sin(r) - \int_0^{\pi/2} r s u(s) ds, \]
subject to the initial conditions:
\[ u(0) = 0, \quad u'(0) = 1, \]
with exact solution:
\[ u(r) = \sin(r). \]

Now, to exclude the second derivative, we integrating the two sides twice from 0 to \( r \); and by applying the initial conditions we get:
\[ u(r) = \sin(r) + \frac{1}{3!} r^3 - \frac{1}{3!} r^3 \int_0^{\pi/2} su(s)ds, \]

By choosing \( g_0(r) = \sin(r) \), \( g_1(r) = \frac{1}{3!} r^3 \) and applying the (MDM), we obtain
\[
\begin{align*}
  u_0(r) &= \sin(r) \\
  u_1(r) &= 0.
\end{align*}
\]

This means that since the value of \( u_1(r) \) equals zero, we do not need to calculate the rest of the values because the result will be zero in all cases and the final answer will be
\[
\begin{align*}
  u(r) &= u_0(r) + u_1(r) + \cdots \\
  u(r) &= \sin(r) + 0 \\
  u(r) &= \sin(r).
\end{align*}
\]

**Fig.1:** Plot of the exact solution and (MDM) solution, of Example 1.

Figure (1), shows a plot in a graph to compare between the exact solution and the (MDM) solution. It shows how the obtained answer by (MDM) is fully converged with the exact solution. And that the absolute error between the two solutions is Zero. And the two lead to the same answer.

Furthermore, when comparing between (MDM) and (ADM) in terms of speed and accuracy to reach the exact solution. As we have already shown, the (ADM) was used to solve the equation in [IX] the results were as shown in Table (1) and by comparing the results in Table (1) we conclude that (MDM) has minimized the steps of the calculation of the solution and given a solution identical to that of the exact solution.
Table 1: Results for Example 1 using (ADM) [IX].

| $u_m(r)$ | Approximate solution |
|----------|----------------------|
| $u_0(r)$ | $\sin(r) + \frac{r^3}{3!}$ |
| $u_1(r)$ | $\frac{r^3}{3!}(1.31877)$ |
| $u_2(r)$ | $\frac{r^3}{3!}(-0.42038)$ |
| $u_3(r)$ | $\frac{r^3}{3!}(0.134)$ |
| $u_4(r)$ | $\frac{r^3}{3!}(-0.04272)$ |

Example 2. When considering linear Fredholm Integro-Differential equation of second-order [IX]:

$$u''(r) = -\sin(r) + \cos(r) + \left(2 - \frac{\pi}{2}\right)(r) - \int_0^{\pi/2} rsu(s)ds,$$

subject to the initial conditions:

$$u(0) = -1, \quad u'(0) = 1,$$

with exact solution:

$$u(r) = \sin(r) - \cos(r).$$

Now to exclude the second derivative, we integrating both sides twice from 0 to $r$ and by using the initial conditions we get:

$$u(r) = \sin(r) - \cos(r) + \left(2 - \frac{\pi}{2}\right)\left(\frac{r^3}{3!}\right) - \int_0^{\pi/2} rsu(s)ds,$$

By choosing $g_0(r) = \sin(r) - \cos(r), \ g_1(r) = \left(2 - \frac{\pi}{2}\right)\left(\frac{r^3}{3!}\right)$ and applying the (MDM), we get

$$u_0(r) = \sin(r) - \cos(r)$$

$$u_1(r) = 0.$$

This means that since the value of $u_1(r)$ equals zero, we do not need to calculate the rest of the values because the result will be zero in all cases and the final answer will be

$$u(r) = u_0(r) + u_1(r) + \cdots$$

$$u(r) = \sin(r) - \cos(r) + 0$$

$$u(r) = \sin(r) - \cos(r).$$
Figure (2), plots a graph to compare the exact and (MDM) solutions. It clearly shows that the answer obtained by (MDM) is fully converged with the exact solution. And the absolute error between the two solutions is Zero. And, thus the answer (obtained from the use of MDM is the same of the exact solution) is the same. Also, we will compare between (MDM) and (ADM) in terms of speed and accuracy to reach the exact solution. As we have already shown that a (ADM) was used to solve this equation in [IX], the results were as shown in Table (2) and by comparing the results in Table (2), we conclude that (MDM) has minimize the steps of the calculation of the solution and given a solution identical to the exact solution.

**Table 2: Results for Example 2 using (ADM) [IX].**

| \(u_m(r)\) | Approximate solution |
|---|---|
| \(u_0(r)\) | \(\sin(r) - \cos(r) + (2 - \frac{\pi}{2}) \frac{r^3}{3!}\) |
| \(u_1(r)\) | \(-\frac{r^3}{3!}(0.56602)\) |
| \(u_2(r)\) | \(-\frac{r^3}{3!}(-0.18043)\) |
| \(u_3(r)\) | \(-\frac{r^3}{3!}(0.05752)\) |
| \(u_4(r)\) | \(-\frac{r^3}{3!}(-0.01843)\) |

**Example 3.** Consider linear Fredholm Integro-Differential equation of second-order [IX]:
\[ u''(r) = -e^r + \frac{1}{2} r + \int_0^1 rsu(s)ds, \]

subject to the initial conditions:

\[ u(0) = 0, \quad u'(0) = -1, \]

with exact solution:

\[ u(r) = 1 - e^r. \]

Now for exclude the second derivative, we integrating both sides twice from 0 to \( r \) and by using the initial conditions we get:

\[ u(r) = 1 - e^r + \frac{r^3}{12} + \frac{r^3}{6} \int_0^1 su(s)ds, \]

by choosing \( g_0(r) = 1 - e^r \), \( g_1(r) = \frac{r^3}{12} \) and applying the (MDM), we get

\[ u_0(r) = 1 - e^r \]

\[ u_1(r) = 0. \]

This means that since the value of \( u_1(r) \) equals zero, we do not need to calculate the rest of the values because the result will be zero in all cases and the final answer will be

\[ u(r) = u_0(r) + u_1(r) + \cdots \]

\[ u(r) = 1 - e^r + 0 \]

\[ u(r) = 1 - e^r. \]

Figure (3), shows a graph plot to compare exact and (MDM) solutions. It shows a full converge between the (MDM) resulted answer and the exact solution; with an absolute error between them to be Zero, i.e. we get the same answer with the (MDM) method as with the exact solution.

Also, we will compare between (MDM) and (ADM) in terms of speed and accuracy to reach the exact solution. As we have already shown that a (ADM) was used to solve this equation in [IX], the results were as shown in Table (3) and by comparing
the results in Table (3), we conclude that (MDM) has minimize the steps of the calculation of the solution and given a solution identical to the exact solution.

Table 3: Results for Example 3 using (ADM) [IX].

| $u_m(r)$ | Approximate solution |
|----------|----------------------|
| $u_0(r)$ | $1 - e^r + \frac{r^3}{12}$ |
| $u_1(r)$ | $\frac{r^3}{6}(-\frac{29}{60})$ |
| $u_2(r)$ | $\frac{r^3}{6}(-\frac{29}{1800})$ |
| $u_3(r)$ | $\frac{r^3}{6}(-\frac{29}{54000})$ |
| $u_4(r)$ | $\frac{r^3}{6}(-\frac{29}{162000})$ |

Example 4. Consider linear Fredholm Integro-Differential equation of second-order [IX]:

$$u''(r) + ru'(r) - ru(r) = e^r - 2 \sin(r) + \sin(r) \int_{-1}^{1} e^{-s}u(s)ds,$$

subject to the initial conditions:

$$u(-1) = e^{-1}, u(1) = e,$$

with exact solution:

$$u(r) = e^r.$$
Now, as we did in the previous examples, we excluded the second derivative by double integrating of both sides from 0 to r, with the use of initial condition, we get:

\[ u(r) = e^r + 2 \sin(r) - \sin(r) \int_{-1}^{1} e^{-s}u(s)ds, \]

by choosing \( g_0(r) = e^r, \ g_1(r) = 2 \sin(r) \) and applying the (MDM), the outcome is

\[ u_0(r) = e^r \]
\[ u_4(r) = 0. \]

This means that since the value of \( u_4(r) \) equals zero, we do not need to calculate the rest of the values because the result will be zero in all cases and the final answer will be

\[ u(r) = u_0(r) + u_4(r) + \ldots \]
\[ u(r) = e^r + 0 \]
\[ u(r) = e^r. \]

Figure (4), plots a graph comparing the exact solution and the (MDM). It clearly shows a full converge between results obtained by the two solutions (MDM & the exact), with an absolute error to be Zero.

Also, we will compare between (MDM) and (ADM) in terms of speed and accuracy to reach the exact solution. As we have already shown that a (ADM) was used to solve this equation in [III], the results were as shown in Table (4) and by comparing the results in Table (4), we conclude that (MDM) has minimize the steps of the calculation of the solution and given a solution identical to the exact solution.

**Fig. 4:** Plot of the exact solution and (MDM) solution, of Example 4.
Table 4: Results for Example 4 using (ADM) [IX].

| $u_m(r)$ | Approximate solution |
|----------|----------------------|
| $u_0(r)$ | $e^r + 2\sin(r)$    |
| $u_1(r)$ | $-\sin(r)(0.67301)$ |
| $u_2(r)$ | $-\sin(r)(0.44654)$ |
| $u_3(r)$ | $-\sin(r)(0.29628)$ |
| $u_4(r)$ | $-\sin(r)(0.19658)$ |
| $u_5(r)$ | $-\sin(r)(0.13043)$ |
| $u_6(r)$ | $-\sin(r)(0.08654)$ |
| $u_7(r)$ | $-\sin(r)(0.05742)$ |
| $u_8(r)$ | $-\sin(r)(0.03810)$ |
| $u_9(r)$ | $-\sin(r)(0.02528)$ |
| $u_{10}(r)$ | $-\sin(r)(0.01677)$ |

V. Concluding Remark

In this paper, the linear second-order Fredholm integro-differential equations are solved using (MDM). Some examples are solved using this method where the results show a perfect match with the exact solution. By using (MDM) and comparing the result with the other method (ADM), we showed that (MDM) significantly reduces the number of steps to reach the solution. As a result, the tiring method can be considered to be simple and the results to be more accurate and fast to obtain.
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