THE WEITZENBÖCK DERIVATIONS AND CLASSICAL INVARIANT THEORY. II. THE SYMBOLIC METHOD

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Abstract. A method based on the symbolic methods of the classical invariant theory is developed for a representation of elements of the kernel of Weitzenbök derivations.

1. Introduction

Let \( K \) be a field of characteristic 0 and let \( K[X] \) be a polynomial algebra in a set of variables \( X \). A linear locally nilpotent derivation \( D \) of the polynomial algebra \( K[X] \) is called a Weitzenbök derivation. Denote by \( D_d := (d_1, d_2, \ldots, d_s) \) the Weitzenbök derivation of the algebra \( K[X] \) with the Jordan normal form consisting of \( s \) Jordan blocks of size \( d_1 + 1, d_2 + 1, \ldots, d_s + 1 \), respectively. The only derivation which corresponds to a single Jordan block of size \( d + 1 \) is called the basic Weitzenbök derivation and denoted by \( D_d \).

The algebra \( \ker D_d = \{ f \in K[X] \mid D_d(f) = 0 \} \), is called the kernel of the derivation \( D_d \). It is well known that the kernel \( \ker D_d \) is isomorphic to the algebra joint covariants of \( s \) binary forms of orders \( d_1, d_2, \ldots, d_s \). Algebras of joint covariants of binary forms were an object of research in the 19th century. To describe the kernel of linear locally nilpotent derivations we should involve computational tools of classical invariant theory, including the famous symbolic method. The symbolic method was developed by Aronhold, Clebsch, and Gordan. It is the most powerful tool of the classical invariant theory, including the famous symbolic method. The symbolic method was developed by Aronhold, Clebsch, and Gordan. It is the most powerful tool of the classical invariant theory. A classic presentation of the symbolic method can be found in [5]–[8]. Recently, a rigorous foundation of the symbolic method has been given by Kung and Rota [9] and by Kraft and Weyman [10].

In this paper we develop an analogue of the classical symbolic method for the kernel of Weitzenbök derivations. In order to explain the essence of the method, we give some examples. Let \( D_4 \) be the basic Weitzenbök derivation of the polynomial algebra \( K[X_4] := K[x_0, x_1, x_2, x_3, x_4] \) i.e., \( D_4(x_i) = x_{i-1}, D_4(x_0) = 0, i = 0 \ldots 4 \). Consider the Weitzenbök derivation \( D_{(4,4)} \) of the algebra \( K[X_4, Y_4] : D_{(4,4)}(x_i) = D_4(x_i), D_{(4,4)}(y_i) = y_{i-1}, D_{(4,4)}(y_0) = 0 \). The following differential operator \( P^{(n)}_{y,x} : K[X_4, Y_4] \to K[X_4, Y_4] \) defined by

\[
P^{(n)}_{y,x} = y_0 \frac{\partial}{\partial x_0} + y_1 \frac{\partial}{\partial x_1} + \cdots + y_n \frac{\partial}{\partial x_n},
\]

is called the polarization operator. The operator \( R^{(n)}_{y,x} : K[X_4, Y_4] \to K[X_4] \), defined by

\[
R^{(n)}_{y,x}(F) = F \bigg|_{y_i=x_i}
\]
is called the restitution operator. If \( F \) is a homogeneous polynomial then Euler’s homogeneous function theorem implies that \( R_{y,x}^{(4)}(\mathcal{P}_2'(F)) = \deg(F) F \). The polarization operator commutes with the Weitzenböck derivations:

\[
P_{y,x}^{(4)}(\mathcal{D}_4(F)) = \mathcal{D}_{(4,4)}(\mathcal{P}^{(4)}_{y,x}(F)).
\]

It is easy to verify that the polynomial \( F = x_2^2 + 2x_0x_4 - 2x_1x_3 \) belongs to \( \ker \mathcal{D}_4 \) and its polarization

\[
P_{y,x}^{(4)}(F) = 2y_0x_4 - 2y_1x_3 + 2y_2x_2 - 2y_3x_1 + 2y_4x_0,
\]

belongs to \( \ker \mathcal{D}_{(4,4)} \). Let us change the variables by

\[
(1)
\]

\[
x_i = \frac{1}{i!}i^4 - i \alpha^i, \quad y_i = \frac{1}{i!}i^4 - i \beta^i, \quad i = 0, \ldots, 4.
\]

Then we get that

\[
P_{y,x}^{(4)}(F) = \frac{1}{12} (\alpha_0 \beta_1 - \beta_0 \alpha_1) := \frac{1}{12} [\alpha, \beta]^4,
\]

where \([\alpha, \beta] := \alpha_0 \beta_1 - \beta_0 \alpha_1\). The polynomial \( \Psi = \frac{1}{12} [\alpha, \beta]^4 \) is called the symbolic representation of the polynomial \( F \). The letters \( \alpha, \beta \) are called the symbol letters. Observe, that \( \Psi \) belongs to kernel of the derivation \( \mathcal{D}_{(1,1)} \) which acts on \( \mathbb{K}[\alpha_0, \alpha_1, \beta_0, \beta_1] \). Moreover, the polynomial \( \Psi \) has much simpler form than the polynomial \( F \).

On the other hand, let us consider the polynomial \( \Phi = \alpha_0^2 \beta_0^2 [\alpha, \beta]^2 \in \ker \mathcal{D}_{(1,1)} \). Then by (1) we get

\[
\Phi = \alpha_0^4 \beta_0^2 \beta_1^2 - 2 \alpha_0^3 \beta_0^3 \beta_1 \alpha_1 + \alpha_0^2 \beta_0^4 \alpha_1^2 = 2(x_0y_2 - x_1y_1 + x_2y_0),
\]

and \( R_{y,x}^{(4)}(\Phi) = 2(2x_0x_2 - x_1^2) \in \ker \mathcal{D}_4 \). Thus \( \frac{1}{2} \alpha_0^2 \beta_0^2 [\alpha, \beta]^2 \) is a symbolic representation for \( 2x_0x_2 - x_1^2 \). To get elements of degree 3 we should involve one more symbolic letter \( \gamma \). Similarly, one may show that

\[
\frac{1}{4!} [\alpha, \beta]^2 [\alpha, \gamma]^2 [\beta, \gamma]^2 \in \ker \mathcal{D}_{(1,1,1)},
\]

is a symbolic representation for the following element of the kernel of derivation \( \mathcal{D}_4 \):

\[
12x_0x_4x_2 + 6x_1x_3x_2 - 2x_2^3 - 9x_0x_3^2 - 6x_1^2x_4,
\]

and for all its polarizations.

For the general case consider the polynomial algebra \( \mathbb{K}[\alpha_i \mid \alpha \in \mathcal{J}, i = 0, 1] \) where \( \alpha \) runs a set \( \mathcal{J} \) of symbol letters. Elements of kernel of the Weitzenböck derivation

\[
\mathcal{D}_{\mathcal{J}} := \mathcal{D}_{(1,1, \ldots, 1)}_{|\mathcal{J}| \text{times}}
\]

defined by \( \mathcal{D}_{\mathcal{J}}(\alpha_0) = 0, \mathcal{D}_{\mathcal{J}}(\alpha_1) = \alpha_0, \) for all \( \alpha \in \mathcal{J} \) are called the symbolic expressions.

The following statement is a main point of the symbolic method:

**The Symbolic Method.**

- Any element of \( \ker \mathcal{D}_d \) allows a symbolic representation;
- Any symbolic expression is a symbolic representation for an element of \( \ker \mathcal{D}_d \) for some \( d \).

Thus we get a remarkable fact – the kernel of an arbitrary Weitzenböck derivation \( \mathcal{D}_d \) is completely defined by the kernel of the special Weitzenböck derivation \( \mathcal{D}_{\mathcal{J}} \). The kernel \( \ker \mathcal{D}_{\mathcal{J}} \) is well-known and generated by \( \alpha_0 \) and the brackets \([\alpha, \beta] \) where \( \alpha, \beta \) run over \( \mathcal{J} \).

The paper organized as follows. In section 2 we review some of the standard facts on the representation theory of the Lie algebra \( \mathfrak{sl}_2 \) and its maximal nilpotent subalgebra \( \mathfrak{u}_2 \).
In section 3 we develop an analogue of the classical symbolic method for Weitzenböck derivations. In section 4 we introduce the notions of the convolution and the semi-transvectant which used for calculation a generating set of kernel of derivations.

2. Basic facts

A representation of the Lie algebra $\mathfrak{g}$ on a finite-dimensional complex vector space $V$ is a homomorphism $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$, where $\mathfrak{gl}(V)$ is Lie algebra of endomorphisms of $V$. We say that such a map gives $V$ the structure of $\mathfrak{g}$-module. The algebra $\mathfrak{g}$ acts on $V$ by linear operators $\rho(g), g \in \mathfrak{g}$. When there is little ambiguity about the map $\rho$ we sometimes call $V$ itself a representation of $\mathfrak{g}$; in this vein we will suppress the symbol $\rho$ and write $gv$ for $\rho(g)v$.

If $U, V$ are representations, the tensor product $U \otimes V$ is also representation, the latter via

$$g(u \otimes v) = gu \otimes v + u \otimes gv.$$  

For a representation $V$, the tensor algebra $T(V)$ is again a representation of $\mathfrak{g}$ by this rule, and symmetric algebra $\text{Sym}(V)$ are subrepresentations of it. Thus, the algebra $\mathfrak{g}$ acts on $\text{Sym}(V)$ by derivations. An element $v \in V$ is called an invariant of $\mathfrak{g}$-module $V$ if $gv = 0$. Denote by $V^0$ the set of all invariants of the $\mathfrak{g}$-module $V$.

Let $\mathfrak{sl}_2$ is the Lie algebra of $2 \times 2$ traceless matrices and $\mathfrak{u}_2$ is its maximal nilpotent subalgebra. The canonical basis of $\mathfrak{sl}_2$ is the basis $(e, f, h)$, where

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$  

We have

$$[h, e] = 2e, [h, f] = -2f, [e, f] = h.$$  

For each nonnegative integer $n$, the algebra $\mathfrak{sl}_2$ has an irreducible representation $V_n$ of dimension $n + 1$, which is unique up to an isomorphism. The endomorphisms $\rho(e), \rho(f), \rho(h)$ act on $V_n := \{v_0, v_1, \ldots, v_n\}$ by the formulas

$$\rho(e)v_i = v_{i-1}, \rho(f)v_i = (n - i)(i + 1)v_{i+1}, \rho(h)v_i = (n - 2i)v_i.$$  

The action of $\mathfrak{sl}_2$ extends by derivations to the symmetric algebra $\text{Sym}(V_n)$ and to the algebra $\text{Sym}(V_d) := \text{Sym}(V_{d_1} \oplus V_{d_2} \oplus \ldots \oplus V_{d_s}), d := (d_1, d_2, \ldots, d_s)$. For convenience, the derivations of $\text{Sym}(V_d)$ which correspond to the operators $\rho(e), \rho(f), \rho(h)$ denote by $D, D_s, E$ respectively. Let us identify the algebra $\text{Sym}(V_n)$ with the polynomial algebra $\mathbb{K}[V_n] := \mathbb{K}[v_0, v_1, \ldots, v_n]$ and identify the algebra $\text{Sym}(V_d)$ with the polynomial algebra $\mathbb{K}[V_{d_1}, V_{d_2}, \ldots, V_{d_s}]$. Under this identification, the kernel of the derivation $D$ coincides with the algebra of invariants $\text{Sym}(V_d)^{\mathfrak{u}_2}$, and the algebra $\ker D \cap \ker D_s$ coincides with the algebra of invariants $\text{Sym}(V_d)^{\mathfrak{sl}_2}$. Any $\mathfrak{u}_2$-invariant is called the semi-invariant. It is clear that the derivation $D$ is exactly the Weitzenböck derivation $D_d$.

To begin, let us describe the algebras of invariants and semi-invariants of the symmetric algebra $\text{Sym}(V_1 \oplus V_1 \oplus \ldots \oplus V_1)$.

Let $\mathcal{G} = \{\alpha, \beta, \ldots\}$ be an alphabet consisting of an infinite supply of Greek letters. The letter in $\mathcal{G}$ are called the symbol letter. To each symbol letter $\alpha$ we associate two variables $\alpha_0, \alpha_1$ and the two-dimensional vector space $V_\alpha := \mathbb{K}\alpha_0 \oplus \mathbb{K}\alpha_1$. For a finite subset $\mathcal{J} \subset \mathcal{G}$ put $V_{\mathcal{J}} := \oplus_{\alpha \in \mathcal{J}} V_\alpha$. The algebra $\text{Sym}(V_\mathcal{J})$ turns into $\mathfrak{sl}_2$-module by the actions:

$$D(\alpha_0) = 0, D(\alpha_1) = \alpha_0, D_s(\alpha_0) = \alpha_1, D_s(\alpha_2) = 0, E(\alpha_i) = (1 - 2i)\alpha_i, i = 0, 1, \alpha \in \mathcal{J}.$$  

A direct check shows that the conditions (2) hold. We note that the symmetric group $S_{|\mathcal{J}|}$ naturally acts on $\text{Sym}(V_\mathcal{J})$, here $|\mathcal{J}|$ is the cardinality of the set $\mathcal{J}$. 
The algebra of invariants \( \text{Sym} (V_J)^{u_2} \) is generated by \( \alpha_0 \) and by the brackets

\[
[\alpha, \beta] := \begin{vmatrix} \alpha_0 & \alpha_1 \\ \beta_0 & \beta_1 \end{vmatrix}, \alpha, \beta \in J.
\]

The algebra of invariants \( \text{Sym} (V_J)^{u_2} \) is generated by the brackets \([\alpha, \beta]\). It is well-known results of classical invariant theory, see [7]. In the theory of locally nilpotent derivations the first part of this statement is known as the Nowicki conjecture, see, for instance, [3].

**Corollary.** As a vector space the algebra \( \text{Sym} (V_J)^{u_2} \) is generated by the polynomials

\[
P_{l,m} := \prod_{\alpha \neq \beta} [\alpha, \beta]^{l_{\alpha, \beta}} \prod_{\gamma} \gamma_0^{m_{\gamma}}, \alpha, \beta, \gamma \in J.
\]

The order \( \text{ord} P \) and the weight \( \text{wt} P := (\text{wt}_\alpha P)_{\alpha \in J} \) of the symbolic expression \( P \) are defined by

\[
\text{ord} P := \sum_{\gamma} m_{\gamma}, \text{wt}_\alpha P := \sum_{\beta} (l_{\alpha, \beta} + l_{\beta, \alpha}) + m_{\alpha}.
\]

In particular, \( \text{wt}_\alpha P \) is equal to the number of times the symbol \( \alpha \) occurs in the symbolic expression \( P \). Observe, that

\[
\text{ord} P = \min \{ k \in \mathbb{N} \mid D^{k+1}_x (P) = 0 \}, \ E(P) = \text{wt} P \cdot P
\]

The symbolic expression \( P \) is called decomposable if it can be written as a product \( P = P_1 P_2 \) in a non-trivial way where \( P_1 \) and \( P_2 \) are disjoint, i.e., no symbol occurs in both. We denote by \( \text{supp} P \) the support of \( P \), i.e., the set of symbols \( \alpha \in J \) occurring in \( P \). Put \( \alpha \sim \beta \) if \( \text{wt}_\alpha = \text{wt}_\beta \). Then the relation \( \sim \) is an equivalence relation defined on the set \( \text{supp} P \subseteq J \).

Denote by \( J_1, J_2, \ldots, J_t \) the equivalence classes and by \( m_1, m_2, \ldots, m_t \) denote their cardinality, \( \sum_{\gamma} \gamma = | \text{supp} P/ \sim | \). Denote by \( n_1, n_2, \ldots, n_t \) the corresponding weights of elements of the classes \( J_i \).

Let \( x_{\alpha, \beta}, x_{\gamma} \) for \( \alpha, \beta, \gamma \in J, \alpha \neq \beta \) denote independent variables and define the free polynomial algebra

\[
\text{Sym}_J := \mathbb{K}[x_{\alpha, \beta}, x_{\gamma} \mid \alpha, \beta, \gamma \in J, \alpha \neq \beta].
\]

Define the map \( \chi : \text{Sym}_J \to \text{Sym} (V_J)^{u_2} \) by

\[
\chi(x_{\alpha, \beta}) = [\alpha, \beta], \chi(x_\gamma) = \gamma_0,
\]

and extend it in the natural way to monomials and all of \( \text{Sym}_J \).

**The First Fundamental Theorem for \( \mathfrak{sl}_2 \).** There is a canonical isomorphism

\[
\text{Sym}_J / \ker \chi \cong \text{Sym} (V_J)^{u_2}
\]

where the ideal \( \ker \chi \) is generated by the elements

\[
x_{\alpha, \beta} + x_{\beta, \alpha} = 0, x_{\gamma} x_{\alpha, \beta} + x_{\beta} x_{\gamma, \alpha} + x_{\alpha} x_{\beta, \gamma} = 0,
\]

\[
x_{\alpha, \beta} x_{\gamma, \delta} + x_{\gamma, \alpha} x_{\beta, \delta} + x_{\beta, \gamma} x_{\alpha, \delta} = 0.
\]

The theorem implies that the following three relations(syzygies)

\[
[\alpha, \beta] + [\beta, \alpha] = 0,
\]

\[
\gamma_0 [\alpha, \beta] + \beta_0 [\gamma, \alpha] + \alpha_0 [\beta, \gamma] = 0,
\]

\[
[\alpha, \beta] [\gamma, \delta] + [\gamma, \alpha] [\beta, \delta] + [\beta, \gamma] [\alpha, \delta] = 0,
\]

for distinct \( \alpha, \beta, \gamma, \delta \in J \) generate all the relationship among the semi-invariants \( \text{Sym} (V_J)^{u_2} \).
The algebra $\text{Sym}(V_J)^{u_2}$ is graded by weight. Let $\text{Sym}(V_J)^{u_2}_w$ be those elements with weight $w$. Then $\text{Sym}(V_J)^{u_2} = \bigoplus_w \text{Sym}(V_J)^{u_2}_w$. If $w = (n, n, \ldots, n)$ we will write $w = (n)|_J|$. If $w = (n_1, n_2, \ldots, n_t)$, we write $w = (n_1)^{m_1}(n_2)^{m_2} \cdots (n_t)^{m_t}$, or more compact $w = n^m$. Here $n := (n_1, n_2, \ldots, n_t)$, $m := (m_1, m_2, \ldots, m_t)$ and $m_1 + m_2 + \cdots + m_t = |J|$. 

3. Symbolic method

We will show that symbolic expressions can be used in a very efficient way to describe and manipulate semi-invariants of the $u_2$-module $\text{Sym}(V_d)$, $d := (d_1, d_2, \ldots, d_s)$. Recall that $\text{Sym}(V_d)^{u_2} = \ker D_d$.

Let $\mathcal{R} = \{x, y, z, \ldots \}$ be an alphabet consisting of an infinite supply of ordered roman letters. Denote by $n_x$ the ordinal number of the letter $x$ in the set $\mathcal{R}$. To each letter $x$ and to each integer number $n$ associate the $n + 1$-dimension vector space

$$V_{x,n} := \mathbb{K}x_0 \oplus \mathbb{K}x_1 \oplus \cdots \oplus \mathbb{K}x_n \cong V_n.$$  

For a finite subset $I \subset \mathcal{R}$ and for $d := (d_1, d_2, \ldots, d_{|I|})$ put $V_{I,d} := \bigoplus_{x \in I} V_{x,d}$. The action

$$(4) \quad D(x_i) = x_{i-1}, D_\ast(x_i) = (d_{n_x} - 2i)x_i, i = 0, 1, \ldots, d_{n_x}, x \in I,$$

gives $\text{Sym}(V_{I,d})$ the structure of $\mathfrak{sl}_2$-module. The order $\text{ord} S$ of a homogeneous semi-invariant $S \in \text{Sym}(V_{I,d})^{u_2}$ is defined by

$$\text{ord} S := \min_k \{k \mid D^{k+1}_\ast(S) = 0\}.$$  

The algebra $\text{Sym}(V_{I,d})^{u_2}$ is graded by multidegree. Let $(\text{Sym}(V_{I,d})^{u_2})_m$ be those elements with multidegree $m$. Then

$$\text{Sym}(V_{I,d})^{u_2} = \bigoplus_m (\text{Sym}(V_{I,d})^{u_2})_m.$$  

The following result summarizes what is classically called "symbolic method":

**Theorem 3.1.** There is a surjective $u_2$-homomorphism of vector spaces

$$\Lambda : (\text{Sym}(V_J)^{u_2})_m \rightarrow (\text{Sym}(V_{I,d})^{u_2})_m, \quad m := (m_1, m_2, \ldots, m_t), |I| = t,$n

such that the composition $\Lambda \circ \chi : (\text{Sym}_J)^{d_m} \rightarrow (\text{Sym}(V_{I,d})^{u_2})_m$ is surjective with kernel

$$\ker \Lambda \circ \chi = (\ker \chi)_m + \{P - \sigma P \mid \sigma \in S_{|J|}, J \subseteq \text{Supp } P/\sim, i = 1, \ldots, t\}.$$

**Proof.** Let $V_n = \langle \alpha_0, \alpha_1 \rangle$ and $V_{x,n} = \langle x_0, x_1, \ldots, x_n \rangle$ be two $\mathfrak{sl}_2$-modules as above. It is well-known that the linear map $\alpha_0^{n-1} \alpha_1^i \rightarrow i!x_i$ is $\mathfrak{sl}_2$-isomorphism of the vector spaces $\text{Sym}^n(V_n)$ and $V_{x,n}$. In fact, we have

$$D(\alpha_0^{n-1}\alpha_1^i) = i\alpha_0^{n-1} \alpha_1^{i-1} \rightarrow i(i - 1)!x_{i-1} = i!D(x_i),$$

$$D_\ast(\alpha_0^{n-1}\alpha_1^i) = (n - i)\alpha_0^{n-1} \alpha_1^{i+1} \rightarrow (n - i)(i + 1)!x_{i+1} = i!D_\ast(x_i).$$

Let us consider a set $J \subset \mathcal{R}$, $|J| = n$. The linear multiplicative map $\alpha_0^d \alpha_1^i \rightarrow i!x_i$ for all $\alpha \in J$ determines the $\mathfrak{sl}_2$-homomorphism of the component $(\text{Sym}(V_J))_w$ into $\text{Sym}^n(V_d)$.

Let us now consider the component $(\text{Sym}(V_J)^{u_2})_w$, where $w = (n_1)^{m_1}(n_2)^{m_2} \cdots (n_t)^{m_t}$ and $m_1 + \cdots + m_t = |J|$. Let $P$ be a symbol expression of $(\text{Sym}(V_J)^{u_2})_w$. Let $\varphi$ be a surjective map of the coset $\text{Supp } P/\sim$ into a finite set $I \subset \mathcal{R}$. Define the map $\Lambda$ by

$$\alpha_0^{w_{t,n-1}} \alpha_1^i \rightarrow i!\varphi(\alpha)_i.$$
Let us prove the surjectivity of the map $\Lambda$. Rewrite the $u_2$-module $V_{\mathcal{I},d}$ as $V_{\mathcal{I},d} = \bigoplus_\alpha m_\alpha V_{\varphi(\alpha),n_\alpha}$, where $\alpha$ runs over all members of cosets. Every semi-invariant is a sum of multihomogeneous semi-invariants. Moreover, a multihomogeneous semi-invariant $S \in (\text{Sym}(V_{\mathcal{I},d})^{u_2})_m$ of multidegree $(m_1, m_2, \ldots, m_t)$ can be polarized to produce a multilinear semi-invariant $P(S)$ of multidegree $(1,1,\ldots,1)$. Note that $P(S) \in \left(\text{Sym}\left(\tilde{V}_{\mathcal{I},d}\right)^{u_2}\right)_{(1,1,\ldots,1)}$, where $\tilde{V}_{\mathcal{I},d} = \bigoplus_{x \in \mathcal{I}} V_{x,n_x}$, $\tilde{\mathcal{I}}$ is union $\mathcal{I}$ with the set of new polarizing variables, $|\tilde{\mathcal{I}}| = |\mathcal{J}|$. Clearly, $P(S)$ can be reconstructed from $S$ by restitution. Since $|\tilde{\mathcal{I}}| = |\mathcal{J}|$ we can associate to each symbol letter $\alpha \in \mathcal{J}$ a vector space $V_{x,n^*}$, for some natural $n^*$. Extend the map $x_k \mapsto 1/k! \alpha_0^{n^*-k}\alpha_1^k$ multiplicatively to all monomial and denote it by $\tilde{\Lambda}$. We have the commutative diagram

$$
\begin{array}{ccc}
(Sym(V_\mathcal{J})^{u_2})_{d^\mathcal{I}} & \xrightarrow{\Lambda} & (Sym(V_{\mathcal{I},d})^{u_2})_m \\
\tilde{\Lambda} \downarrow & & \downarrow \rho \\
\left(\text{Sym}\left(\tilde{V}_{\mathcal{I},d}\right)^{u_2}\right)_{(1,1,\ldots,1)} & & \\
\text{[\mathcal{I}] times} & & \\
\end{array}
$$

where all arrows are $\mathfrak{sl}_2$-homomorphisms, and $\Lambda \left(\tilde{\Lambda}(P(S))\right) = S$. Thus $\Lambda$ is a surjective map.

The part of theorem concerning $\ker \Lambda \circ \chi$ can be proved in the same manner as the proof in the preprint [10], see also [12].

Observe, that $(\text{Sym}(V_{\mathcal{I},d})^{u_2})_m = (\ker D_d)_m$, where $D_d$ is a derivation of the polynomial algebra $\mathbb{K}[\mathcal{I}]$. Therefore we got a handy tool for writing elements of the kernel of arbitrary linear locally nilpotent derivations.

It is best to look now at some examples.

**Example 3.1.** Let us consider the symbolic expression $P := [\alpha, \beta]^2[\beta, \gamma]\gamma_0^2$. Then $\text{supp } P = \{\alpha, \beta, \gamma\}$, $wt P = (2,3,3) = (2)_1(3)^2$, $\text{supp } P/_{\sim} = \{\{\alpha\}, \{\beta, \gamma\}\}$, $|\text{supp } P/_{\sim}| = 2$. Put $\mathcal{I} = \{x, y\}$ and associate $\alpha$ to $x$ and both $\beta, \gamma$ associate to $y$. Since $d = (2,3)$ then the map $\Lambda$ acts by

$$
\begin{align*}
\alpha_0^2 & \mapsto x_0, \alpha_0\alpha_1 \mapsto x_1, \alpha_1^2 \mapsto 2! x_2, \\
\beta_0^3 & \mapsto y_0, \beta_0^2\beta_1 \mapsto y_1, \beta_0\beta_1^2 \mapsto 2! y_2, \beta_1^3 \mapsto 3! y_3, \\
\gamma_0^3 & \mapsto y_0, \gamma_0^2\gamma_1 \mapsto y_1, \gamma_0\gamma_1^2 \mapsto 2! y_2, \gamma_1^3 \mapsto 3! y_3.
\end{align*}
$$
We have
\[ P := [\alpha, \beta]^2[\beta, \gamma] = \alpha_0^2 \beta_1 \gamma_0 \gamma_1 - \alpha_0^2 \beta_1 \gamma_0^3 - 2 \alpha_0 \alpha_1 \beta_0 \gamma_0 \gamma_1 + 2 \alpha_0 \alpha_1 \beta_0 \gamma_0^3 + \alpha_1 \beta_0 \gamma_0 \gamma_1 - \alpha_1 \beta_0^2 \beta_1 \gamma_0^3. \]

Thus
\[ \Lambda(P) = 2x_0y_2y_1 - 6x_0y_3y_0 - 2x_1y_2y_0 + 4x_1y_2y_1 + 2x_2y_0y_1 - 2x_2y_1y_0 = 2x_0y_2y_1 - 6x_0y_3y_0 - 2x_1y_2y_0 + 4x_1y_2y_1. \]

Consider the polynomial algebra \( \mathbb{K}[X_2, Y] := \mathbb{K}[x_0, x_1, x_2, y_1, y_2, y_3] \). The polynomial \( \Lambda(P) \) belongs to the kernel of the derivation \( D_{(2, 3)} \) of the algebra \( \mathbb{K}[X_2, Y] \) defined by
\[ D_{(2, 3)}(x_i) = x_{i-1}, i = 0, 1, 2, D_{(2, 3)}(y_i) = y_{i-1}, i = 0, 1, 2, 3, D_{(2, 3)}(x_0) = D_{(2, 3)}(y_0) = 0. \]

**Example 3.2.** Let \( P = [\alpha, \beta]^n \). Then supp \( P = \{\alpha, \beta\} \), wt \( P = (n, n) = (n)^2 \), supp \( P/ \sim \) = \( \{\alpha, \beta\} \), |supp \( P/ \sim \) | = 1. Put \( \mathcal{I} = \{x\} \) and associate \( \alpha, \beta \) to \( x \). The map \( \Lambda \) acts by
\[ \alpha_0^{n-i} \alpha_1^i \mapsto i!x_i, \beta_0^{n-i} \beta_1^i \mapsto i!x_i, i = 0, 1, \ldots, n. \]

We have
\[ \Lambda([\alpha, \beta]^n) = \Lambda \left( \sum_{i=0}^{n} (1)^{i} \alpha_0^{n-i} \alpha_1^i \beta_0^{n-i} \right) = \sum_{i=0}^{n} (1)^{i} i! (n - i)! x_{i} x_{n-i}. \]

Observe that \( [\alpha, \beta]^n = (-1)^n[\beta, \alpha]^n \) but, obviously, \( \Lambda([\alpha, \beta]^n) = \Lambda([\beta, \alpha]^n). \) It follows that \( \Lambda([\alpha, \beta]^n) = 0 \) for odd \( n \). Note, that the polynomial belongs to the kernel of the basic Weitzenböck derivation \( D_n \) of \( \mathbb{K}[X_{n, 2}] \) defined by \( D_n(x_i) = x_{i-1}. \)

**Example 3.3.** Consider the polynomial \( A = 3x_1x_2x_0 - 3x_3x_2^2 - x_3^3 \in \ker D_3 \). Find a symbolic expression of the semi-invariant \( A \). To get multilinear polynomial polarize \( A \) two times with respect to letters \( y \) and \( z \) :
\[ P_y(A) = 3y_0y_1x_2 - 6y_0x_3x_0 + 3y_1x_0x_2 - 3y_1x_2^2 + 3y_2x_0x_1 - 3y_3x_0^2, \]
\[ \mathcal{P}(A) = P_y(P_y(A)) = 3z_0y_1x_2 - 6z_0y_3x_0 + 3z_0x_2x_1 - 6z_0y_3x_0 + 3z_1y_0x_2 - 6z_1y_1x_1 + 3z_1y_2x_0 + 3z_2y_0x_1 + 3z_2y_1x_0 - 6z_3y_0x_0. \]

The polynomial \( \mathcal{P}(A) \) has the multidegree \( (1, 1, 1) \). The map \( \tilde{\Lambda} \) acts by
\[ x_i \mapsto 1/i! \alpha_0^{3-i} \alpha_1^i, y_i \mapsto 1/i! \beta_0^{3-i} \beta_1^i, z_i \mapsto 1/i! \gamma_0^{3-i} \gamma_1^i. \]

We have
\[ \tilde{\Lambda}(\mathcal{P}(A)) = -\gamma_0^3 \beta_0^3 \alpha_1^3 + 3/2 \gamma_0^3 \beta_0^2 \beta_1 \alpha_0 \alpha_1^2 + 3/2 \gamma_0^3 \beta_0^2 \beta_0 \alpha_0^3 \alpha_1 - \gamma_0^3 \beta_1^2 \alpha_0^3 \]
\[ + 3/2 \gamma_0^2 \gamma_1 \beta_0 \alpha_0 \alpha_1^2 - 6 \gamma_0^2 \gamma_1 \beta_0^2 \beta_1 \alpha_0^2 \alpha_1 + 3/2 \gamma_0^2 \gamma_1 \beta_0 \beta_1 \alpha_3 \alpha_0^2 + 3/2 \gamma_0 \gamma_1^2 \beta_0^2 \beta_1 \alpha_0^3 \alpha_1 + 3/2 \gamma_0 \gamma_1 \beta_0 \beta_1 \gamma_0^3 \alpha_0^2 \alpha_1. \]

After simplification we obtain
\[ 2 \tilde{\Lambda}(\mathcal{P}(A)) = 3 \beta_0 \gamma_0^2 [\alpha, \beta]^2[\alpha, \gamma] + 3 \beta_0^2 \gamma_0[\alpha, \beta][\alpha, \gamma]^2 - 2 \beta_0^3 [\alpha, \gamma]^3 - 2 \gamma_0^3 [\alpha, \beta]^3. \]

Taking into account \( \Lambda(\gamma_0^3 [\alpha, \beta]^3) = \Lambda(\beta_0^3 [\alpha, \gamma]^3) = 0 \), and
\[ \Lambda(\beta_0 \gamma_0^2 [\alpha, \beta]^2[\alpha, \gamma]) = \Lambda(\beta_0^2 \gamma_0 [\alpha, \beta][\alpha, \gamma]^2), \]
we get that \( \tilde{\Lambda}(\mathcal{P}(A)) = 3 \beta_0 \gamma_0^2 [\alpha, \beta]^2[\alpha, \gamma] \). Thus \( \beta_0 \gamma_0^2 [\alpha, \beta]^2[\alpha, \gamma] \) is a symbolic expression of \( A \) and
\[ A = \Lambda \left( \beta_0 \gamma_0^2 [\alpha, \beta]^2[\alpha, \gamma] \right). \]
Example 3.4. Put $P = \prod_{\alpha < \beta} [\alpha, \beta]^2$, $\text{supp} P = 2k$. We have, see proof in [2], that

$$\Lambda \left( \prod_{\alpha < \beta} [\alpha, \beta]^2 \right) = (k + 1)! \begin{vmatrix} x_0 & x_1 & 2!x_2 & \cdots & k!x_k \\ x_1 & 2x_2 & 3!x_3 & \cdots & (k+1)!x_{k+1} \\ 2x_k & (k+1)!x_{k+1} & \cdots & (2k-1)!x_{2k-1} \\ k!x_k & (k+1)!x_{k+1} & \cdots & (2k)!x_{2k} \end{vmatrix}.$$

This semi-invariant belongs to $\ker D_k$, has degree $k + 1$ and called the catalecticant.

4. CONVOLUTION AND SEMI-TRANVARIANT

There are simple and effective way to find semi-invariants of given multidegree $m$. The following differential operator on $\text{Sym}(\mathcal{J})$

$$\text{Conv}_{\alpha, \beta} := [\alpha, \beta] \frac{\partial^2}{\partial \alpha \partial \beta}, \alpha, \beta \in \mathcal{J}$$

is called the convolution with respect to the symbol letters $\alpha$ and $\beta$. Obviously, the convolution operator does not change the weight of a symbolic expression, so $\text{Conv}_{\alpha, \beta}$ is an endomorphism of the vector space $\text{Sym}(V^w)_{\alpha \beta}$.

Example 4.1 Let us find an elements of kernel of the derivation $D_{1,2,3}$ of the multidegree $(1, 2, 1)$ and the weight $(1, 2, 2, 3) = (1)^2(2)^2(3)^1$. Put $\mathcal{J} = \{\alpha, \beta, \gamma, \delta\}$, $\mathcal{I} = \{x, y, z\}$. To the symbol letter $\alpha$ we associate the roman letter $x$, to both letters $\beta, \gamma$ we associate the letter $y$ and to symbol letter $\delta$ we associate the letter $z$. The map $\Lambda$ acts by $\alpha_i \mapsto x_i$, $\beta_0^2 \beta_1^2 \gamma_0^2 \gamma_1^2 \delta_0 \delta_1 \delta_2 \delta_3 \mapsto i! y_i$, $\gamma_0^2 \gamma_1^2 \delta_0 \delta_1 \delta_2 \delta_3 \mapsto i! z_i$, $z = 0, 1, 2, 3$. It is clear that the following symbolic expression $\Phi = \alpha_0 \beta_0^2 \gamma_0^2 \delta_0 \delta_1 \delta_2 \delta_3$ has the weight $(1)^2(2)^2(3)^1$. The polynomial $\Lambda(\Phi)$ is equal to $x_0 y_0 z_0$ and has the multidegree $(1, 2, 1)$. By direct calculations we get

$$\text{Conv}_{\alpha, \beta}(\Phi) = 2 [\alpha, \beta] \beta_0 \gamma_0^2 \delta_0 \delta_1 \delta_2 \delta_3 \mapsto 2 y_0 z_0 \begin{vmatrix} x_0 & x_1 \\ y_0 & y_1 \end{vmatrix},$$

$$\text{Conv}_{\gamma, \delta}(\text{Conv}_{\alpha, \beta}(\Phi)) = 12 [\alpha, \beta] [\gamma, \delta] \beta_0 \gamma_0^2 \delta_0 \delta_1 \delta_2 \delta_3 \mapsto 12 \begin{vmatrix} x_0 & x_1 \\ y_0 & y_1 \end{vmatrix},$$

$$\text{Conv}_{\gamma, \delta}^2(\text{Conv}_{\alpha, \beta}(\Phi)) = 24 [\alpha, \beta] [\gamma, \delta]^2 \beta_0 \delta_0 \delta_1 \delta_2 \delta_3 \mapsto 48 (z_0 y_2 - y_1 z_1 + y_0 z_2) \begin{vmatrix} x_0 & x_1 \\ y_0 & y_1 \end{vmatrix},$$

$$\text{Conv}_{\beta, \delta}^2(\text{Conv}_{\gamma, \delta}(\text{Conv}_{\alpha, \beta}(\Phi))) = 24 [\alpha, \beta] [\gamma, \delta]^2 [\beta, \delta]^2 \mapsto 48 (3 x_0 y_0 z_3 y_1 - 2 x_0 y_2 y_0 y_2 - 2 x_0 y_1^2 z_2 + 3 x_0 y_2 y_1 z_1 - 2 x_0 y_2^2 z_0 - 3 y_0^2 x_1 z_3 + 3 y_1 x_1 y_0 y_2 - y_1^2 x_1 z_1 - y_0 x_1 y_1 y_2 + y_1 x_1 y_0 y_2).$$

For a symbolic expression $P$ denote by $\text{Conv}(P)$ the set of its all possible convolutions. For a subalgebra $\Delta \in \text{Sym}(V_{\mathcal{J}}^w)$ denote by $\text{Conv}(\Delta)$ the subalgebra generated by all possible convolutions $\text{Conv}(P)$, $P \in \Delta$. The following statement holds:

Lema 4.1.

$$\text{Sym}(V_{\mathcal{J}}^w) = \text{Conv}(\text{Sym}(\oplus_{\alpha \in \mathcal{J}} \mathbb{K} \alpha_0)).$$

Proof. Let $P \in (\text{Sym}(V_{\mathcal{J}}^w))^w$. Then $P$ is obtained by convolutions of the semi-invariants $\prod_{\alpha \in \mathcal{J}} \alpha_0^{w_0} \in \text{Sym}(\oplus_{\alpha \in \mathcal{J}} \mathbb{K} \alpha_0).$ \hfill $\Box$

Example 4.1 Let $\mathcal{J} = \{\alpha, \beta, \gamma\}$. The component $(\text{Sym}(V_{\mathcal{J}}^w))_{(2,1,1)}$ is generated by the following 5 semi-invariants

$$\alpha_0^2 \beta_0 \gamma_0, \alpha_0 \gamma_0 [\alpha, \beta], \alpha_0 \beta_0 [\alpha, \gamma], \alpha_0^2 [\beta, \gamma], [\alpha, \gamma] [\alpha, \beta].$$

All of them are the convolutions of the semi-invariant $\alpha_0^2 \beta_0 \gamma_0$. 

Let \( F \in \text{Sym}(V_T,d)^{u_2} \) and \( \Phi \in \text{Sym}(V_T)^{u_2} \) is its symbolic representation. Denote by \( \text{Conv}_\Lambda(F) \) the set of elements \( \Lambda(\text{Conv}(\Phi)) \). The elements of \( \text{Conv}_\Lambda(F) \) are called the \( \Lambda \)-convolutions. For a subalgebra \( T \subset \text{Sym}(V_T,d)^{u_2} \) denote by \( \text{Conv}_\Lambda(T) \) the algebra generated by all \( \Lambda \)-convolutions of all elements of the algebra \( T \).

**Theorem 4.1.** Let \( N_T := \mathbb{K}[x_0 | x \in I] \). Then \( \text{Sym}(V_T,d)^{u_2} = \text{Conv}_\Lambda(N_T) \).

**Proof.** Consider an arbitrary homogeneous semi-invariant \( F \in \text{Sym}(V_T,d)^{u_2} \). Let \( \Phi \) be its symbolic expression. By Lemma 4.1, \( \Phi \) belongs to \( \text{Conv}(\text{Sym}(\oplus_{\alpha \in J} \mathbb{K} \alpha_0)) \) and \( |J| = \deg F \). It is easy to see, that

\[
\Lambda(\text{Sym}(\oplus_{\alpha \in J} \mathbb{K} \alpha_0)) = N_T \text{ and } \Lambda(\text{Conv}(\text{Sym}(\oplus_{\alpha \in J} \mathbb{K} \alpha_0))) = \text{Conv}_\Lambda(N_T),
\]

thus \( \Lambda(\Phi) = F \in \text{Conv}(N_T) \). We get \( \text{Sym}(V_T,d)^{u_2} \subseteq \text{Conv}_\Lambda(N_T) \). The inclusion \( \text{Conv}_\Lambda(N_T) \subseteq \text{Sym}(V_T,d)^{u_2} \) is obvious. \( \square \)

We have got a tool to find a generating system of kernel for Weitzenböck derivations.

**Example 4.2** Let \( |I| = n \) and \( d = (1,1,\ldots,1) \). Prove that

\[
\ker D_d = \mathbb{K}[x_0,x_0y_1 - x_1y_0 | x \neq y, x, y \in I].
\]

In fact, let \( F \) be a homogeneous polynomial of \( N_T \) of degree \( m \). Then its symbolic representation \( \Phi \) has the form \( \Phi = \prod_{\alpha \in J} \alpha_0 \). Since all factors of \( \Phi \) have the degrees 1, we obtain that all possible convolutions of the polynomial \( \Phi \) have form \( \prod_{\alpha \in J} \alpha_0 \prod_{\beta,\gamma \in J}[\beta,\gamma] \), for \( \alpha \neq \beta, \gamma \). It follows

\[
\Lambda \left( \prod_{\alpha \in J} \alpha_0 \prod_{\beta,\gamma \in J}[\beta,\gamma] \right) = \prod_{x \in I} x_0 \prod_{y,z \in I} (y_0z_1 - y_1z_0), x \neq y, z.
\]

Thus \( \text{Conv}_\Lambda(F) \in \mathbb{K}[x_0,x_0y_1 - x_1y_0 | x \neq y, x, y \in I] \) and

\[
\text{Conv}_\Lambda(N_T) = \mathbb{K}[x_0,x_0y_1 - x_1y_0 | x \neq y, x, y \in I].
\]

**Example 4.3** Let \( |I| = n \geq 3, d = (2,2,\ldots,2) \) and let \( F \) be an homogeneous polynomial of \( N_T \) of degree \( m \). Then its symbolic representation \( \Phi \) has the form \( \Phi = \prod_{\alpha \in J} \alpha_0^2 \). For \( m = 1 \) we have \( \Phi = \alpha_0^2, \alpha \in J \). It is obvious that \( \text{Conv}(\Phi) = \{\alpha_0^2, \alpha \in J\} \).

For \( m = 2 \) we have \( \Phi = \alpha_0^2\beta_0^2, \alpha \neq \beta, \alpha, \beta \in J \). There are only two convolutions of \( \Phi : \alpha_0\beta_0[\alpha,\beta] \) and \([\alpha,\beta]^2\).

For \( m = 3 \) we have \( \Phi = \alpha_0^2\beta_0^2\gamma_0^2, \alpha, \beta, \gamma \in J \). There exists a unique not decomposable convolution: \([\alpha,\beta][\alpha,\gamma][\beta,\gamma]\).

If \( m > 3 \) then any symbol expression is either decomposable or belongs to \( \ker \Lambda \), see [7], page 162. We have

\[
\Lambda(\alpha_0^2) = x_0, \Lambda(\alpha_0\beta_0[\alpha,\beta]) = \begin{vmatrix} x_0 & x_1 \\ y_0 & y_1 \end{vmatrix}, \Lambda([\alpha,\beta]^2) = 2(x_0y_2 - x_1y_1 + x_2y_2),
\]

\[
\Lambda([\alpha,\beta][\alpha,\gamma][\beta,\gamma]) = \Lambda(\alpha_0^2\beta_1\gamma_1^2\beta_0 - \alpha_0^2\beta_1^2\gamma_1\gamma_0 + \alpha_0\beta_1^2\gamma_0^2\alpha_1 - \beta_0^2\alpha_0\gamma_1^2 + \beta_0^2\alpha_1^2\gamma_0\gamma_1 - \beta_0\alpha_1^2\gamma_0^2\beta_1) = 2x_0y_2y_1 - 2x_0y_2z_1 + 2x_1y_2z_0 - 2x_0y_1z_2 + 2y_0x_2z_1 - 2y_1x_2z_0 = 2\begin{vmatrix} x_0 & y_0 & z_0 \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}.
\]

Thus, the kernel \( \ker D_{(2,2,\ldots,2)} \) is generated by the semi-invariants of these four types.
With increasing of \( d \) it becomes difficult to apply the Theorem \([13]\). We will offer other similar but more effective approach to find the kernel of a Weitzenb"{o}ck derivations. In the paper \([13]\) we introduced the conception of semi-transvectant, an analogue of the classical transvectant. 

Recall that the algebra \( u_2 \) acts on \( \text{Sym} (V_{\mathcal{I}, d}) \) by the locally nilpotent derivation \( D = D_d \). Let \( F, G \in \text{Sym} (V_{\mathcal{I}, d})^{u_2} \) be two semi-invariants of degrees \( p \) and \( q \), respectively. The semi-invariant of the form

\[
[F, G]^r := \sum_{i=0}^{r} (-1)^i \binom{r}{i} D_i^d(F) D_{r-i}^d(G),
\]

\( 0 \leq r \leq \min(p, q) \), \( [m]_i := m(m-1) \ldots (m-(i-1)) \), \( m \in \mathbb{Z} \), is called the \( r \)-th semi-transvectant of the semi-invariants \( F \) and \( G \).

**Example 4.4.** The semi-transvectant \( [F, G]^1 := [F, G] \) is called the semi-Jacobian. If \( F, G, H \) are three semi-invariants of orders greater than unity, then the iterated semi-Jacobian \( [[F, G], H] \) is reducible \([8]\) and

\[
[[F, G], H] = \text{ord}(F) - \text{ord}(G) \cdot \frac{2 \text{ord}(F) + \text{ord}(G) - 2}{2} + \frac{1}{2} [F, G]^2 H + \frac{1}{2} [F, H]^2 G - \frac{1}{2} [G, H]^2 F.
\]

**Example 4.5.** The semi-invariant \( [F, G]^2 := \text{Hes}(F) \) is called the semi-Hessian. The square of a semi-Jacobian \([F, G]\) is given by the formula

\[
[F, G][F, G] = [F, G]^2 FG - \frac{1}{2} \text{Hes}(F)G^2 - \frac{1}{2} \text{Hes}(G)F^2.
\]

**Example 4.6.** We have

\[
(\alpha_0^2, \beta_0^2) = \frac{1}{3} \alpha_0 \alpha_1 \gamma_1 \gamma_0 + \frac{1}{6} \alpha_0^2 \beta_0 \gamma_1^2 - \frac{1}{3} \alpha_1 \alpha_0 \beta_1 \gamma_0^2 - \frac{2}{3} \alpha_1 \alpha_0 \gamma_1 \beta_0 \gamma_0 + \frac{1}{2} \alpha_1^2 \beta_0 \gamma_0^2 = \frac{1}{2} [\alpha, \gamma]^2 \beta_0 - \frac{1}{3} \alpha_0 [\alpha, \gamma][\beta, \gamma].
\]

The following statement holds:

**Lema 4.2 \([13]\).**

(i) for \( 0 \leq r \leq \min(\text{ord}(x_0), \max(\text{ord}(F), \text{ord}(G))) \) the semi-transvectant \([x_0, FG]^r\) is reducible ;

(ii) if \( \text{ord}(F) = 0 \), then \([x_0, FG]^r = [x_0, G]^r\);

(iii) \( \text{ord}([F, G]^r) = \text{ord}(F) + \text{ord}(G) - 2 r \).

There is a close relationship between the convolutions and the semi-transvectants. A \( k \)-fold contraction of two disjoint symbolic expressions \( \Phi \) and \( \Psi \) is a symbolic expression

\[
\left( \prod_{\alpha, \beta} \text{Conv}_{\alpha, \beta} \right) (\Phi \cdot \Psi)
\]

where the product runs over \( k \) pairs \( (\alpha, \beta) \in \text{Supp} \Phi \times \text{Supp} \Psi \).

**Lema 4.3 \([10]\).** Let \( \Phi \) and \( \Psi \) be two disjoint symbolic expressions. The semi-transvectant \([\Phi, \Psi]^k\) is a linear combination of semi-invariants \( T \) where \( T \) runs through the \( k \)-fold contractions of \( \Phi \) and \( \Psi \), and each such \( T \) occurs with a positive rational coefficient \( q_T \) where \( \sum_T q_T = 1 \).

Using the semi-transvectants we offer an algorithm for computation of the kernel of Weitzenb"{o}ck derivations. For \( x \in \mathcal{I} \) put \( \tau_{x,i}(F) := (x_0, F)^i \), \( i \leq \min(\text{ord}(x_0), \text{ord}(F)) \). For a subalgebra \( T \subseteq \text{Sym} (V_{\mathcal{I}, d})^{u_2} \) denote by \( \tau(T) \) the algebra generated by the elements \( \tau_{x,i}(F), F \in T, i \leq \min(\text{ord}(x_0), \text{ord}(F)) \). The following theorem is a weak form of Gordan’s theorem:
Theorem 4.2 \((\text{[14]}).
\) Let \(\mathcal{T}\) be a subalgebra of \(\mathcal{D}_d\) containing \(\mathcal{N}_i\) and \(\tau(\mathcal{T}) \subseteq \mathcal{T}\). Then \(\ker \mathcal{D}_d = \mathcal{T}\).

The theorem implies the following algorithm for \(\ker \mathcal{D}_d\). Define the series of subalgebras

\[ \mathcal{T}_1 \subseteq \mathcal{T}_2 \subseteq \mathcal{T}_3 \subseteq \cdots \subseteq \mathcal{T}_k \subseteq \cdots, \]

where \(\mathcal{T}_1 = \mathcal{N}_i\) if \(\mathcal{T}_1 := \tau(\mathcal{T}_{i-1})\). If for some \(i\) we have that \(\mathcal{T}_i = \mathcal{T}_{i+1}\), then \(\mathcal{T}_i = \ker \mathcal{D}_d\).

Example 4.7 Consider the basic Weitzenböck derivation \(\mathcal{D}_3\), \(\mathcal{I} = \{x\}\). We have \(\mathcal{T}_1 = \mathbb{K}[x_0]\). The subalgebra \(\mathcal{T}_2\) generated by the elements \(\tau_i(x_i^2)\). By Lemma 4.2 \((i)\) for \(j > 1\) all of them are reducible one. The only irreducible semi-invariant is \(dv = \tau_2(x_0)\). Therefore, \(\mathcal{T}_2 = \mathbb{K}[x_0, dv]\).

The subalgebra \(\mathcal{T}_3\) generated by \(\tau_i(x_i^0dv^i)\), \(i, k, l \in \mathbb{N}\). Since \(\text{ord}(dv) = 2\), then Lemma 4.2 \((i)\) implies that the algebra \(\mathcal{T}_3\) consists only the following elements \(\tau_1(dv), \tau_2(dv)\) and \(\tau_3(dv^2)\). By (5) we obtain that \(\tau_3(dv^2) = 0\), \(\tau_2(dv) = 0\) and \(tr := \tau_1(dv) \neq 0\). The direct calculation shows that \(tr\) does not belong to \(\mathcal{T}_2\), thus \(\mathcal{T}_3 = \mathbb{K}[t, dv, tr]\). The algebra \(\mathcal{T}_4\) generated by \(\tau_i(x_i^0dv^i tr^m)\), \(i, k, l, m \in \mathbb{N}\). As above we find the only new element \(ch = \tau_3(tr)\) for \(\mathcal{T}_4\). We have \(\text{ord}(ch) = 0\) but the algebra \(\mathcal{T}_3\) does not consist any invariants. It implies that \(\mathcal{T}_4 = \mathbb{K}[t, dv, tr, ch]\). By lemma 4.7 \((ii)\) we have that the algebra \(\mathcal{T}_5\) does not consist of any new semi-invariants. Thus \(\mathcal{T}_5 = \mathcal{T}_4\) and \(\ker \mathcal{D} = \mathbb{K}[t, dv, tr, ch]\), where

\[
\begin{align*}
dv &= x_1^2 - 2x_0x_2, \\
tr &= 3x_3x_0^2 + x_1^3 - 3x_0x_1x_2, \\
ch &= 8x_0x_2^3 + 9x_3^2x_0^2 + 6x_1^3x_3 - 3x_1^2x_2^2 - 18x_0x_1x_2x_3.
\end{align*}
\]

Up to a constant factor the symbolic representation of these semi-invariants have the form \(\alpha_0\beta_0[\alpha, \beta]^2, \beta_0\gamma_0^2[\alpha, \beta]^2[\gamma, \delta]^2\) and \([\alpha, \beta]^2[\gamma, \delta]^2\), respectively.

For \(d = 4, 5, 6\) the kernel of the basic Weitzenböck derivation was calculated in \([14]\). The cases \(d = 7, 8\) considered in \([15], [16]\). For \(d > 8\) the problem is still open however the corresponding algebras of invariants was calculated for \(d = 9, 10\) in \([17], [18]\).

Example 4.8 Consider the derivation \(\mathcal{D}_{(2,3)}, \mathcal{I} = \{x, y\}\), \(\text{ord}(x_0) = 2\), \(\text{ord}(y_0) = 3\). To the set \(\mathcal{J}_1 := \{\alpha, \beta, \gamma, \delta\}\) we associate the letter \(x\) and to set \(\mathcal{J}_2 := \{\varepsilon, \zeta, y, \mu\}\) we associate the letter \(y\). The maps \(\Lambda\) acts by \(v_0^{i_1-i_2}v_1^i \mapsto v_1^ix_i\), \(i = 0, 1, 2\) for \(v \in \mathcal{J}_1\) and by \(v_0^{i_1}v_1^i \mapsto v_1^iy_i\), \(i = 0, 1, 2, 3\) for \(v \in \mathcal{J}_2\). Put \(D := \Lambda ([\alpha, \beta]^2), \Delta := \Lambda ([\varepsilon, \zeta]^2 x_0x_0), Q := \Lambda ([\varepsilon, \zeta]^2[\eta, \varepsilon]x_0y_0^3), R := \Lambda ([\varepsilon, \zeta]^2[\eta, \varepsilon][\zeta, \mu][\eta, \mu]^2).\)

The minimal generating set for \(\ker \mathcal{D}_{(2,3)}\) consists of the following 15 elements:

\[
\begin{align*}
x_0, y_0, \\
D, [x_0, y_0]^2, \Delta, [x_0, y_0], \\
[x_0, \Delta]^2, [x_0^2, y_0]^3, [x_0, \Delta], Q, \\
R, [x_0, Q]^2, \\
[x_0^3, y_0]^6, [x_0^2, Q]^3, \\
[x_0^3, y_0Q]^6.
\end{align*}
\]

The proof for the corresponding algebras of covariants see in \([6]\) or in \([7], [8]\).

For instance, let us calculate the explicit form of the semi-invariant \((x_0, Q)^2\). We have

\[
(a_0^2[\varepsilon, \zeta]^2[\eta, \varepsilon]x_0y_0^3)^2 = \varepsilon[\zeta]^2[\eta, \varepsilon](a_0^2x_0y_0^3)^2 =
\]

\[
= \varepsilon[\zeta][\eta, \varepsilon]\left(\frac{1}{6}\alpha_0^2x_0^2\eta_1^2 + \frac{1}{3}\alpha_0^2\alpha_1\eta_1\eta_0 = \frac{2}{3}\alpha_1\alpha_0\eta_1\eta_0 - \frac{1}{3}\alpha_1\alpha_0\eta_1\eta_0 + \frac{1}{2}\alpha_0^2\alpha_1^2\right) =
\]
\[
\begin{align*}
&= \frac{5}{6} \alpha_0^2 \varepsilon_0 \eta_1 \varepsilon_0 \varepsilon_1^2 \eta_0 \varepsilon_1 - \frac{2}{3} \alpha_0^2 \varepsilon_0^2 \eta_1 \varepsilon_0 \varepsilon_1^2 \eta_0 \varepsilon_1 - \frac{2}{3} \alpha_0^2 \varepsilon_1^2 \eta_1 \varepsilon_0 \varepsilon_1^2 \eta_0 + \\
&+ \frac{2}{3} \alpha_1 \alpha_0 \eta_1^2 \varepsilon_0 \varepsilon_0 \varepsilon_1^3 \eta_1^2 + \frac{2}{3} \alpha_1 \alpha_0 \eta_1^2 \varepsilon_0 \varepsilon_0 \varepsilon_1^2 \eta_0 + \frac{2}{3} \alpha_1 \alpha_0 \varepsilon_1^2 \eta_0 \varepsilon_0 \varepsilon_1^2 + \\
&+ \varepsilon_0 \eta_0 \varepsilon_0^2 \varepsilon_1 \eta_1 + \frac{1}{2} \varepsilon_1 \eta_0 \varepsilon_0 \varepsilon_1^2 \eta_1 - \frac{1}{6} \alpha_0^2 \varepsilon_0 \eta_1 \varepsilon_0^3 \eta_1 \varepsilon_1 + \\
&+ \frac{1}{6} \alpha_0^2 \varepsilon_0 \eta_1 \varepsilon_0 \varepsilon_1 \eta_0 + \frac{1}{6} \alpha_0^2 \varepsilon_1 \eta_0 \varepsilon_0 \varepsilon_1^2 \eta_0 + \frac{1}{3} \alpha_0 \varepsilon_1 \eta_0 \varepsilon_0 \varepsilon_1^2 \eta_0 + \\
&+ \varepsilon_0 \eta_1^2 \eta_0 \varepsilon_0 \varepsilon_0 \varepsilon_1^3 \eta_1 - \frac{1}{3} \alpha_0 \varepsilon_0 \varepsilon_1 \eta_0 \varepsilon_0 \varepsilon_1^2 \eta_1 - \\
&- \frac{1}{2} \varepsilon_0 \eta_0 \varepsilon_0 \varepsilon_1 \varepsilon_1 \eta_0^2 - \frac{1}{2} \varepsilon_0 \eta_1^2 \varepsilon_0 \varepsilon_1 \varepsilon_1 \eta_1 + \frac{5}{3} \alpha_0 \alpha_0 \eta_1 \varepsilon_0 \varepsilon_1 \eta_1 \varepsilon_0 \varepsilon_1^2 + \\
&+ \frac{1}{3} \alpha_0^2 \varepsilon_0 \eta_0 \varepsilon_0 \varepsilon_1 \varepsilon_1 - \frac{1}{3} \alpha_0 \alpha_0 \varepsilon_0 \varepsilon_0 \varepsilon_1^3 \eta_0^3 \varepsilon_1.
\end{align*}
\]

Then
\[
(x_0, Q)^2 = \Lambda \left( \left( \alpha_0^2, [\varepsilon, \varepsilon]^{2}[\eta, \varepsilon] \varepsilon_0 \eta_0^3 \right)^2 \right) = \\
= 6 x_0 y_1 y_2 y_3 - 6 y_2 y_0 x_2 y_1 - 6 x_1 y_0 y_1 y_3 - 2 x_0 y_1 y_2^2 + 6 y_0^2 x_2 y_3 - 6 x_0 y_2 y_3 y_0 + 8 x_1 y_0 y_2^2 - \\
- 2 x_1 y_2 y_1^2 + 2 y_1^3 x_2.
\]

In [6, 7, 19] were calculated the algebras joint covariants which are isomorphic to the kernels of the following derivations \( D_{(2,4)}, D_{(3,3)} \) and \( D_{(3,3,3)} \).

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