COHOMOLOGICAL ASPECTS OF HOPF ALGEBRA LIFTINGS

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Abstract. A recent result of ours [GM] shows that all Hopf algebra liftings of a given diagram in the sense of Andruskiewitsch and Schneider are cocycle deformations of each other. Here we develop a ‘non-abelian’ cohomology theory, which gives a method for an explicit description of cocycles relevant to the lifting process.

0. Introduction

The Nichols algebra $B(V)$ of a crossed $kG$-module $V$ is a connected braided Hopf algebra. In terms of generators and relations it can be described via a certain pushout diagram

$$
\begin{array}{c}
K(V) \xrightarrow{\kappa} R(V) \\
\varepsilon \downarrow \quad \pi \downarrow \\
k \xrightarrow{\iota} B(V)
\end{array}
$$

of connected braided Hopf algebras. The Radford biproduct or bosonization $H(V) = B(V)\# kG$ has a similar presentation in the category of ordinary Hopf algebras. A lifting of $H(V)$ is a pointed Hopf algebra $H$ for which $gr^c H \cong H(V)$, where $gr^c H$ is the graded Hopf algebra associated with the coradical filtration of $H$. Such liftings are obtained by deforming the multiplication of $H(V)$. In this context the lifting problem for $V$ is asking for the characterization and classification of all liftings of $H(V)$. This problem has been solved by Andruskiewitsch and Schneider in [AS] for a large class of crossed $kG$-modules of finite Cartan type, which will carry the attribute ‘special’ in this paper. This allows, in particular, for a classification of all finite dimensional pointed Hopf algebras $A$ for which the order of the abelian group of points is not divisible by any prime $< 11$. In recent work [GM] we have shown that for any given $V$ in this class all liftings of $H(V)$ are cocycle deformations of each other (see also [Ma], Appendix). This is done via a description of the lifted Hopf algebras suitable for the application of results by Masuoka about Morita-Takeuchi equivalence [Ma] and by Schauenburg about Hopf-Galois extensions [Sch]. For some special cases such results had been obtained in [Ma, Di, BDR, Gr]. In addition, our results in [GM] show that
every lifting of $H(V)$, and therefore the corresponding cocycle, is completely determined by a $G$-invariant algebra map $f \in \text{Alg}_G(K(V), k)$, but without an explicit description of the corresponding cocycle in terms of $f$.

In the present paper we aim at making this connection between the $G$-invariant algebra map $f \in \text{Alg}_G(K, k)$ and the corresponding deforming cocycle $\sigma : B \otimes B \to k$ more explicit. For that purpose we first describe a non-abelian equivariant cohomology theory for braided Hopf algebras $X$ in the category of crossed $H$-modules and for their bosonizations $X \# H$, where $H$ is an ordinary Hopf algebra. The Radford biproduct $X \# H$ is an ordinary Hopf algebra and carries the obvious $H$-bimodule structure. A pushout diagram of (braided) Hopf algebras as above, in which $\kappa$ has a $H$-module coalgebra retraction gives rise to a Meier-Vietoris type 5-term exact sequence

$$ 1 \to \text{Alg}_H(B, k) \overset{\pi}{\longrightarrow} \text{Alg}_H(R, k) \overset{\kappa}{\longrightarrow} \text{Alg}_H(K, k) \overset{\delta}{\longrightarrow} H^2_H(B, k) \overset{\pi}{\longrightarrow} H^2_H(R, k) $$

of pointed sets. In the situation of the lifting problem, when $B$ is the Nichols algebra of a crossed $kG$-module of special finite Cartan type, then $\text{Alg}_G(R, k)$ is trivial. If, in addition, $K$ is a $K$-bimodule coalgebra retract in $R$, then the connecting map $\delta : \text{Alg}_G(K(V), k) \to H^2_H(B(V), k)$ exists and is injective. Then, in view of the characterization of liftings in [AS, GM], the cocycles obtained via the connecting map account for all liftings of $B(V) \# kG$.

The 5-term sequence for equivariant Hochschild cohomology

$$ 0 \to \text{Der}_H(B, k) \overset{\pi}{\longrightarrow} \text{Der}_H(R, k) \overset{\kappa}{\longrightarrow} \text{Der}_H(K, k) \overset{\delta}{\longrightarrow} H^2_H(B, k) \overset{\pi}{\longrightarrow} H^2_H(R, k) $$

has been established in [GM] and is an exact sequence of vector spaces. Here it suffices that $K$ is a $K$-bimodule retract in $R$, which in the liftings situation is always the case. The question about the relationship between Hochschild cohomology and non-abelian cohomology naturally arises in this context. In the cocommutative case there are Sweedler’s results. For quantum linear spaces, i.e.: for diagrams of type $A_1 \times A_1 \times \ldots \times A_1$, there is an exponential relationship between Hochschild cocycles and those ‘multiplicative’ cocycles which depend on the root vector parameters alone [GM]. Here we present some more general results on this topic involving linking as well. This includes an approach to quantum planes quite different from that of [ABM], Section 5. In the last section we also develop a program for the connected case, and apply it to diagrams of type $A_2$. Results for type $A_n$, $n > 2$, and for type $B_2$ will be part of a forthcoming paper.

The notation in the paper as in [GM] is pretty much standard: $m : A \otimes A \to A$ denotes multiplication, $\Delta : C \otimes C \to C$ comultiplication, $s : H \to H$ the antipode, and $* : \text{Hom}(C, A) \otimes \text{Hom}(C, A) \to \text{Hom}(C, A)$ the convolution multiplication $f * f' = m(f \otimes f')\Delta$. We use Sweedler’s notation in the form $\Delta(c) = c_1 \otimes c_2$ etc., and also $\Delta^{(n)} = (1 \otimes \Delta^{(n-1)})\Delta$ for $n \geq 1$ with $\Delta^{(0)} = 1$. The notation used for coactions of a Hopf algebra $\delta : X \to H \otimes X$ is $\delta(x) = x_{-1} \otimes x_0$. 
1. A non-abelian cohomology

Every lifting of the bosonisation \( A = B \# kG \) of the Nichols algebra \( B \) of a finite dimensional special crossed \( G \)-module \( V \) is determined by a \( G \)-invariant algebra map \( f \in \text{Alg}_G(K \# kG, k) \), and it is also a cocycle deformation \( A_\sigma \) of \( A \). The \( G \)-invariant ‘multiplicative’ cocycle \( \sigma : A \otimes A \to k \) must therefore be completely determined by the \( G \)-invariant algebra map \( f : K \to k \). In the examples presented in [GM] Section 3 the relation between the two entities is given explicitly. In this paper non-abelian cohomology will serve to clarify this relationship for some special diagrams of finite Cartan type.

1.1. The ‘multiplicative’ cohomology. The non-abelian equivariant cohomology of a braided Hopf algebra in the category of crossed \( H \)-modules \( X \) or its bosonization, which is an ordinary Hopf algebra, is defined via the cosimplicial group complex of regular elements

\[
\begin{array}{cccc}
\text{Reg}_H(k, k) & \overset{\partial^0}{\longrightarrow} & \text{Reg}_H(X, k) & \overset{\partial^1}{\longrightarrow} & \text{Reg}_H(X^2, k) & \overset{\partial^2}{\longrightarrow} & \text{Reg}_H(X^3, k) \\
\text{Hom}_H(k, k) & \overset{\partial^0}{\longrightarrow} & \text{Hom}_H(X, k) & \overset{\partial^1}{\longrightarrow} & \text{Hom}_H(X^2, k) & \overset{\partial^2}{\longrightarrow} & \text{Hom}_H(X^3, k),
\end{array}
\]

in the standard cosimplicial algebra complex

where \( X^i \) denotes the \( i \)-th tensor power of \( X \), and where

\[
\partial^i f = \begin{cases} 
\varepsilon \otimes f & \text{if } i = 0 \\
(1^i \otimes m \otimes 1^{n-i-1}) f & \text{if } 0 < i < n \\
f \otimes \varepsilon & \text{if } i = n
\end{cases}
\]

are the standard cofaces.

The first equivariant ‘non-abelian’ cohomology of \( X \) with coefficients in \( k \) is given by

\[
H^1_H(X, k) = Z^1_H(X, k) = \{ f \in \text{Reg}_H(X, k) | \partial^1 f = \partial^0 f \ast \partial^0 f \} = \text{Alg}_H(X, k)
\]

which is a group under the convolution multiplication. A 1-cocycle is therefore an element \( f \in \text{Reg}_H(X, k) \) such that \( fm = (f \otimes \varepsilon) \ast (\varepsilon \otimes f) = m_k(f \otimes f) \), that is an algebra map. For the second cohomology define the set of ‘non-abelian’ 2-cocycles by

\[
Z^2_H(X, k) = \{ \sigma \in \text{Reg}_H(X^2, k) | \partial^0 \sigma \ast \partial^2 \sigma = \partial^3 \sigma \ast \partial^1 \sigma, \sigma(\varepsilon \otimes 1) = \varepsilon = \sigma(1 \otimes \varepsilon) \}
\]

which means that \( \sigma \in \text{Reg}_H(X^2, k) \) is a cocycle if and only if the ‘multiplicative’ 2-cocycle conditions

\[
(\varepsilon \otimes \sigma) \ast \sigma(1 \otimes m) = (\sigma \otimes \varepsilon) \ast \sigma(m \otimes 1), \sigma(\varepsilon \otimes 1) = \varepsilon = \sigma(1 \otimes \varepsilon)
\]
are satisfied, in particular \(\sigma(y_1 \otimes z_1)\sigma(x \otimes y_2 z_2) = \sigma(x_1 \otimes y_1)\sigma(x_2 y_2 \otimes z)\) in the ordinary case and \(\sigma(y_1 \otimes (y_2)_0 z_1)\sigma(x \otimes (y_2)_0 0 z_2) = \sigma(x_1 \otimes (x_2)_0 y_1)\sigma((x_2)\_0 y_2 \otimes z)\) in the braided case. Define a relation on \(\text{Reg}_H(X^2, k)\) by declaring \(\sigma \sim \sigma'\) if and only if \(\sigma' = \partial^0 \chi * \partial^2 \chi * \sigma * \partial^1 \chi^{-1}\) for some \(\chi \in \text{Reg}_H(X, k)\).

**Lemma 1.1.** The relation \(\sim\) just defined on \(\text{Reg}_H(X^2, k)\) is an equivalence relation, which restricts to \(\mathbb{Z}^2_H(X, k)\). The second "non-abelian" cohomology \(\mathcal{H}_2(X, k) = Z^2_H(X, k) / \sim\) is a pointed set with distinguished element class \((\varepsilon \otimes \varepsilon) = \text{im}(\partial : \text{Reg}_H(X, k) \to \text{Reg}_H(X \otimes X, k))\), where \(\partial f = \partial^0 f * \partial^2 f * \partial^1 f^{-1}\). Moreover, there is a natural isomorphism \(\mathcal{H}_2(X, k) \cong \mathcal{H}_1(X \# H, k)\) and a natural injection \(\mathcal{H}_2(X, k) \to \mathcal{H}_2(X \# H, k)\) for braided Hopf algebras \(X\) in the category of crossed \(H\)-modules and their bosonisations \(Y = X \# H\).

**Proof.** First we will show that it is sufficient to prove the assertions for ordinary Hopf algebras. If \(X\) is a Hopf algebra in the category of crossed \(H\)-modules then \(Y = X \# H\) is an ordinary Hopf algebra. The linear map

\[\psi_n : Y^n \to X^n\]

defined inductively by \(\psi_1(xh) = x\varepsilon(h)\) and \(\psi_n(xh \otimes y) = x \otimes h\psi_{n-1}y\) is a \(H\)-bimodule map (diagonal left and trivial right \(H\)-action on \(X^n\)), which has linear right inverse \(\phi_n : X^n \to Y^n\) given by \(\phi_n(x) = x1\) and \(\phi_n(x \otimes y) = x1 \otimes \phi_{n-1}y\). It factors through \(Y^{(n)} = Y \otimes_H Y \otimes_H \cdots \otimes_H Y\) to give a left \(H\)-module isomorphism \(Y^{(n)} \otimes_H k \cong X^n\). Induction on \(n\) shows that it is also compatible with the "coalgebra structures" in that \(\Delta X \ast \psi_n = (\psi_n \otimes \psi_n) \Delta_{Y^n}\) and \(\varepsilon \psi_n = \varepsilon\). The induced injective algebra map

\[\psi^n : \text{Hom}_H(X^n, k) \to \text{Hom}_H(Y^n, k)\]

is then given by \(\psi^n(f) = f\psi_n\), that is \(\psi^n(f)(xh \otimes y) = f(x \otimes h\psi_{n-1}y)\) or \(\psi^n f(x^1 \otimes h^1 x^2 \otimes \cdots \otimes h^n \ast a^n) = f(x^1 \otimes h^1 x^2 \otimes \cdots \otimes h^n \ast a^n)\). It is an algebra map, since it preserves the convolution multiplication,

\[\psi^n(f * f') = (f * f')\psi_n = (f \otimes f')\Delta_{Y^n} \psi_n = (f \otimes f') (\psi_n \otimes \psi_n) \Delta_{Y^n} = (f \psi_n \otimes f' \psi_n) \Delta_{Y^n} = \psi^n(f) * \psi^n(f')\]

and the convolution identity, \(\psi^n(\varepsilon) = \varepsilon \psi_n = \varepsilon\). It therefore automatically restricts to an injective group homomorphism

\[\psi^n : \text{Reg}_H(X^n, k) \to \text{Reg}_H(Y^n, k)\]

between the groups of regular elements. This leads to a injective homomorphism of the standard cosimplicial groups.
compatible with the standard cofaces

\[
\sigma \quad \frac{\partial^0}{\partial^0} \quad \frac{\partial^1}{\partial^1} \quad \frac{\partial^2}{\partial^2} \quad \frac{\partial^3}{\partial^3}
\]

\[
\begin{array}{c|c|c|c|c}
\text{Reg}_H(k, k) & \text{Reg}_H(X, k) & \text{Reg}_H(X^2, k) & \text{Reg}_H(X^3, k) \\
\hline
\| & \downarrow \psi^1 & \downarrow \psi^2 & \downarrow \psi^3 \\
\text{Reg}_H(k, k) & \text{Reg}_H(Y, k) & \text{Reg}_H(Y^2, k) & \text{Reg}_H(Y^3, k)
\end{array}
\]

in which \( \psi^1 \) is an isomorphism. It then suffices to prove the first assertion for the ordinary Hopf algebra \( Y = X\#H \).

First observe that if \( f \in \text{Reg}_H(Y, k) \) then \( \partial f = \partial^0 f * \partial^2 f * \partial^1 f^{-1} \) is a 2-cocycle:

\[
\begin{align*}
(\varepsilon \otimes \partial f) * \partial f(1 \otimes m) & (x \otimes y \otimes z) \\
& = \partial f(y_1 \otimes z_1) \partial f(x \otimes y_2 z_2) \\
& = f(z_1) f(y_1) f^{-1}(y_2 z_2) f(y_3 z_3) f(x_1) f^{-1}(x_2 y_4 z_4) \\
& = f(x_1) f(y_1) f(z_1) f^{-1}(x_2 y_2 z_2) \\
& = f(y_1) f(x_1) f^{-1}(x_2 y_2) f(z_1) f(x_3 y_3) f^{-1}(x_4 y_4 z_2) \\
& = \partial f(x_1 \otimes y_1) \partial f(x_2 y_2 \otimes z) \\
& = (\partial f \otimes \varepsilon) * \partial f(m \otimes 1) (x \otimes y \otimes z)
\end{align*}
\]

Now we show that \( \sim \) is an equivalence relation even on \( \text{Reg}_H(Y, k) \), and that it restricts to \( Z^2_H(Y, k) \).

Reflexivity, \( \sigma \sim \sigma \) of the relation \( \sim \) obviously holds with \( \chi = \varepsilon \).

To check symmetry, observe that \( \sigma' = \partial^0 \chi * \partial^2 \chi * \sigma * \partial^1 \chi^{-1} \) for some \( \chi \in \text{Reg}_H(Y, k) \) implies that \( \sigma = \partial^0 \chi^{-1} * \partial^2 \chi^{-1} * \sigma' * \partial^1 \chi \) since \( (\partial^0 \chi)^{-1} = \partial^0 \chi^{-1} \) and \( \partial^2 \chi * \partial^0 \chi = \partial^0 \chi * \partial^2 \chi \).

For transitivity suppose that in addition \( \sigma'' = \partial^0 \psi * \partial^2 \psi * \sigma' * \partial^1 \psi^{-1} \) for some \( \psi \in \text{Reg}_H(Y, k) \). Then

\[
\begin{align*}
\sigma'' & = \partial^0 \psi * \partial^2 \psi * \partial^0 \chi * \partial^2 \chi * \sigma * \partial^1 \chi^{-1} * \partial^1 \psi^{-1} \\
& = \partial^0 (\psi * \chi) * \partial^2 (\psi * \chi) * \sigma * \partial^1 (\psi * \chi)^{-1}
\end{align*}
\]

since \( \partial^2 \psi * \partial^0 \chi = \partial^0 \chi * \partial^2 \psi \) and the \( \partial^i \) are group homomorphisms.

To show that the equivalence relation \( \sim \) restricts to \( Z^2_H(Y, k) \) it suffices to show that if \( \sigma \in Z^2_H(Y, k) \) and \( \chi \in \text{Reg}_H(Y, k) \) then \( \sigma' = \partial^0 \chi * \partial^2 \chi * \sigma * \partial^1 \chi^{-1} \) is a
cyclo as well:

\[
(\partial^0 \sigma' + \partial^2 \sigma')(x \otimes y \otimes z) = \sigma'(y_1 \otimes z_1)\sigma'(x \otimes y_2 z_2)
\]

\[
= \chi(z_1)\chi(y_1)\sigma(y_2 \otimes z_2)\chi^{-1}(y_3 z_3)\chi(y_4 z_4)\chi(x_1)\sigma(x_2 \otimes y_5 z_5)\chi^{-1}(x_3 y_6 z_6)
\]

\[
= \chi(x_1)\chi(y_1)\chi(z_1)\sigma(y_2 \otimes z_2)\sigma(x_2 \otimes y_3 z_3)\chi^{-1}(x_3 y_4 z_4)
\]

\[
= \chi(x_1)\chi(y_1)\chi(z_1)\sigma(x_2 \otimes y_2)\sigma(x_3 y_3 \otimes z_2)\chi^{-1}(x_4 y_4 z_3)
\]

\[
= \chi(y_1)\chi(x_1)\sigma(x_2 \otimes y_2)\chi^{-1}(x_3 y_3)\chi(z_1)\chi(x_4 y_4)\sigma(x_5 y_5 \otimes z_2)\chi^{-1}(x_6 y_6 z_3)
\]

\[
= \sigma'(x_1 \otimes y_1)\sigma'(x_2 y_2 \otimes z) = (\partial^3 \sigma' + \partial^1 \sigma')(x \otimes y \otimes z)
\]

and

\[
\sigma'(x \otimes 1) = \chi(x_1)\sigma(x_2 \otimes 1)\chi^{-1}(x_3) = \epsilon(x) = \chi(x_1)\sigma(1 \otimes x_2)\chi^{-1}(x_3) = \sigma'(1 \otimes x).
\]

This proves the assertions for the bosonisation \( Y = X \# H \). For the braided Hopf algebra \( X \) they are now a consequence of the properties of the diagram above. It follows that

\[
\text{Alg}_H(X, k) = \mathcal{H}_H(X, k) \cong \mathcal{H}_H(Y, k) = \text{Alg}_H(Y, k)
\]

since \( \psi^1 \) is an isomorphism and \( \psi^2 \) is injective. Since, in addition, \( \psi^3 \) is injective as well it follows that \( \sigma \in \text{Reg}_H(X^2, k) \) is a 2-cocycle if and only if \( \psi^2 \sigma \in \text{Reg}_H(Y, k) \) is a cocycle. In particular, if \( f \in \text{Reg}_H(X, k) \) then \( \partial f = \partial^0 f \circ \partial^2 f \circ \partial^1 f^{-1} \in Z^2(X, k) \). Moreover, the following argument shows that the induced map \( \mathcal{H}_H^2(X, k) \to \mathcal{H}_H^2(Y, k) \) is injective. Suppose that \( \sigma, \sigma' \in Z^2(X, k) \) are such that \( \psi^2 \sigma \sim \psi^2 \sigma' \) in \( Z^2(Y, k) \). This means that

\[
\psi^2 \sigma' = \partial^0 \psi^1 \phi \circ \partial^2 \psi^1 \phi \circ \psi^2 \sigma \circ \partial^1 \psi^1 \phi^{-1} = \psi^2 (\partial^0 \phi \circ \partial^2 \phi \circ \sigma \circ \partial^1 \phi^{-1})
\]

for some \( \phi \in \text{Reg}_H(X, k) \), and hence \( \sigma' = \partial^0 \phi \circ \partial^2 \phi \circ \sigma \circ \partial^1 \phi^{-1} \), where we used the fact that \( \psi^1 \) is an isomorphism and \( \psi^2 \) is an injective algebra map.

Our aim here is to describe cocycle deformations of the bosonizations \( Y = X \# H \) of braided Hopf algebras \( X \) in the category of crossed \( H \)-modules. The following calculation shows that equivalent cocycles lead to isomorphic deformations.

**Proposition 1.2.** Let \( Y = X \# H \) be the bosonization of a braided Hopf algebra \( X \) in the category of crossed \( H \)-modules. If \( \sigma, \sigma' \in Z^2_H(Y, k) \) are in the same cohomology class then the cocycle deformations \( Y_\sigma \) and \( Y_{\sigma'} \) are isomorphic.

**Proof.** Suppose that \( \sigma' = \partial^0 \chi \circ \partial^2 \chi \circ \partial^1 \chi^{-1} \) for some \( \chi \in \text{Reg}_H(X, k) \). It suffices to show that the equivariant coalgebra automorphism \( \psi = \chi^{-1} \circ 1 \circ \chi : Y \to Y \) is actually also an algebra map \( \psi : Y_\sigma \to Y_{\sigma'} \). And it is, since

\[
m_{\sigma'}(\psi x \otimes y) = \chi^{-1}(x_1)\chi^{-1}(y_1)m_{\sigma'}(x_2 \otimes y_2)\chi(x_3)\chi(y_3)
\]

\[
= \chi^{-1}(x_1)\chi^{-1}(y_1)\sigma'(x_2 \otimes y_2)x_3 y_3 \sigma'^{-1}(x_4 \otimes y_4)\chi(x_5)\chi(y_5)
\]

\[
= \sigma(x_1 \otimes y_1)\chi^{-1}(x_2 y_2)x_3 y_3 \chi(x_4 y_4)\sigma^{-1}(x_5 \otimes y_5)
\]

\[
= \sigma(x_1 \otimes y_1)\psi(x_2 y_2)\sigma^{-1}(x_3 \otimes y_3)
\]

\[
= \psi m_{\sigma}(x \otimes y)
\]
implies that $\psi m_\sigma = m_\sigma (\psi \otimes \psi)$.

2. A 5-term sequence in ‘non-abelian’ cohomology

A commutative ‘pushout’ square of (braided) Hopf algebras in the introduction and its bosonisation can help to get an explicit description of the deforming cocycles $\sigma$ on $B$ and of the corresponding cocycles on the bosonization $A = B#H$ in terms of the $H$-invariant algebra maps $f \in \text{Alg}_H(K,k)$,. Such squares of (braided) Hopf algebras

\[
\begin{array}{c}
K \xrightarrow{\kappa} R \\
\downarrow \varepsilon \quad \downarrow \pi \\
k \xrightarrow{\iota} B
\end{array}
\qquad
\begin{array}{c}
K#H \xrightarrow{\kappa#1} R#H \\
\downarrow \varepsilon#1 \quad \downarrow \pi#1 \\
B # H \xrightarrow{\iota#1} B#H
\end{array}
\]

induce a square of cosimplicial groups

\[
\begin{array}{cccc}
\text{Reg}_H(k,k) & \xrightarrow{\delta^0} & \text{Reg}_H(B,k) & \xrightarrow{\delta^0} & \text{Reg}_H(B^2,k) & \xrightarrow{\delta^0} & \text{Reg}_H(B^3,k) \\
\downarrow & & \downarrow \pi^* & & \downarrow (\pi^2)^* & & \downarrow (\pi^3)^* \\
\text{Reg}_H(k,k) & \xrightarrow{\delta^1} & \text{Reg}_H(R,k) & \xrightarrow{\delta^1} & \text{Reg}_H(R^2,k) & \xrightarrow{\delta^1} & \text{Reg}_H(R^3,k) \\
\downarrow & & \downarrow \kappa^* & & \downarrow (\kappa^2)^* & & \downarrow (\kappa^3)^* \\
\text{Reg}_H(k,k) & \xrightarrow{\delta^2} & \text{Reg}_H(K,k) & \xrightarrow{\delta^2} & \text{Reg}_H(K^2,k) & \xrightarrow{\delta^2} & \text{Reg}_H(K^3,k)
\end{array}
\]

where the trivial part has been omitted, and a similar square for the bosonisation.

The natural injective group homomorphism $\psi : \text{Reg}_H(X,k) \rightarrow \text{Reg}_H(X#H,k)$ induces a natural map between these squares. Here is a 5-term sequence for non-abelian cohomology in case $\kappa : K \rightarrow R$ has a $K$-bimodule coalgebra retraction $u : R \rightarrow K$.

**Theorem 2.1.** If $\kappa K \rightarrow R$ has a $K$-bimodule coalgebra retraction then there is an exact sequences of pointed sets

\[
1 \rightarrow \text{Alg}_H(B,k) \xrightarrow{\pi^*} \text{Alg}_H(R,k) \xrightarrow{\kappa^*} \text{Alg}_H(K,k) \xrightarrow{\delta} \mathcal{H}_H^2(B,k) \xrightarrow{\pi^*} \mathcal{H}_H^2(R,k)
\]

and an injective map induced by the cosimplicial group homomorphism $\psi^*$ into a similar exact sequence involving the bosonisations. The connecting map $\delta : \text{Alg}_H(K,k) \rightarrow \mathcal{H}_H^2(B,k)$ does not depend on the particular choice of the $K$-bimodule coalgebra retraction $u : R \rightarrow K$. 

Proof. It is clear that \( \pi^* : \text{Alg}_H(B, k) \rightarrow \text{Alg}_H(R, k) \) is injective and that \( \kappa^*\pi^* = (\pi\kappa)^* = (\varepsilon)^* = \varepsilon^*\tau^* \) is the trivial map. Moreover, if \( \kappa^*(f) = \varepsilon \) for \( f \in \text{Alg}_H(R, k) \) then, by the pushout property, there is a unique \( f' \in \text{Alg}_H(B, k) \) such that \( \pi^*(f') = f \). To construct \( \delta : \text{Alg}_H(K, k) \rightarrow \mathcal{H}_H^2(B, k) \) observe first that

\[
\text{Alg}_H(K, k) = Z_H^1(K, k) = \{ f \in \text{Reg}_H(K, k) | \partial^1 f = \partial^2 f * \partial^0 f \} = \{ f \in \text{Reg}_H(K, k) | \partial^0 f * \partial^2 f * \partial^1 f s = \varepsilon \otimes \varepsilon \}.
\]

The existence of a \( H \)-invariant \( K \)-module coalgebra retraction \( u : R \rightarrow K \) for the injection \( \kappa : K \rightarrow R \) implies that, for every \( f \in \text{Alg}_H(K, k) \), the map \( f_u \in \text{Hom}_H(R, k) \) is convolution invertible with inverse \( f_{su} \). Then by Lemma 14 the map

\[
\sigma_R = \partial u^* f = \partial^0 f_u * \partial^2 f_u * \partial^1 f_{su} : R \otimes R \rightarrow k
\]

is a convolution invertible 2-cocycle with inverse \( \sigma_R^{-1} = \partial^1 f_u * \partial^2 f_{su} * \partial^0 f_u \), in particular \( \sigma_R(x \otimes y) = f_u(x_1) f_u(y_1) f_{su}(x_2 y_2) \). It satisfies the 2-cocycle conditions

\[
\sigma_R(1 \otimes 1) = \varepsilon = \sigma_R(1 \otimes 1), \quad (\varepsilon \otimes \sigma_R) * \sigma_R(1 \otimes m) = (\sigma_R \otimes \varepsilon) * \sigma_R(m \otimes 1).
\]

Now \( (\kappa \otimes \kappa)^* \partial^i u^* = \partial^i \kappa^* u^* = \partial^i f \) for \( i = 0, 1, 2 \), so that \( (\kappa \otimes \kappa)^* \partial^i f_u = \partial^i f = \varepsilon \otimes \varepsilon \), since \( f : K \rightarrow k \) is an algebra map. Moreover, because \( u \) is a \( H \)-invariant \( K \)-bimodule coalgebra map and \( f : K \rightarrow k \) is a \( H \)-invariant algebra map it follows that \( (f u \otimes 1)c = (f u \otimes 1)\tau \) and \( fm_K = f \otimes f \), so that

\[
\partial u^* f = (\varepsilon \otimes f_u \otimes \varepsilon \otimes f_{sum})(\Delta_{R \otimes R} \otimes 1 \otimes 1) \Delta_{R \otimes R} = ((f_u \otimes f_u) c \otimes f_{sum} h_{R \otimes R} = (f_u \otimes f_u \otimes f_{sum}) \Delta_{R \otimes R} = (f_u \otimes f_u \otimes f_{sum})(1 \otimes \tau \otimes 1)(\Delta_R \otimes \Delta_R)
\]

and \( \partial f_u(x r \otimes r') = \varepsilon(x) \partial f_u(r \otimes r') = \partial f_u(r \otimes r' x) \) for all \( x \in K \) and \( r, r' \in R \), which says that \( \partial f_u : R \otimes R \rightarrow k \) is a \( K \)-bimodule map. This means in particular that

\[
\partial u^* f(K^+ R \otimes R + R \otimes RK^+) = 0
\]

and hence that the cocycle \( \sigma_R = \partial u^* f : R \otimes R \rightarrow k \) factors uniquely through \( \pi \otimes \pi : R \otimes R \rightarrow B \otimes B \), i.e. there exists a unique \( \sigma : B \otimes B \rightarrow k \) such that \( (\pi \otimes \pi)^* \sigma = \partial u^* f \). Since \( \pi : R \rightarrow B \) is a surjective Hopf algebra map, this \( \sigma : B \otimes B \rightarrow k \) is a 2-cocycle as well. So define

\[
\delta : \text{Alg}_H(K, k) \rightarrow Z_H^2(B, k)
\]

by \( \delta(f) = \sigma \).

Exactness at \( \text{Alg}_H(K, k) \): If \( f \in \text{Alg}_H(K, k) \) and \( \delta f = \partial \chi \) for some \( \chi \in \text{Reg}_H(B, k) \) then \( \partial f_u = (\pi \otimes \pi)^* \partial \chi = \partial \pi^* \chi \) and \( g = \pi^* \chi^{-1} * f_u \in \text{Reg}_H(R, k) \) and \( \kappa^* g = \kappa^*(\chi^{-1} \pi * f_u) = \chi^{-1} \pi \kappa * f_{ku} = \chi^{-1} \varepsilon * f = \varepsilon * f = f \). It remains to
show that \( g \in \text{Alg}_H(R, k) \). But \( \partial g = \varepsilon \otimes \varepsilon \), since
\[
\partial g = \partial^0 g \ast \partial^2 g \ast \partial^1 g^{-1} \\
= \partial^0 (\chi^{-1} \pi \ast f_u) \ast \partial^2 (\chi^{-1} \pi \ast f_u) \ast \partial^1 (f_su \ast \chi \pi) \\
= \partial^0 \chi^{-1} \pi \ast \partial^0 f_u \ast \partial^2 \chi^{-1} \pi \ast \partial^2 f_u \ast \partial^1 f_su \ast \partial^1 \chi \pi \\
= \partial^0 \chi^{-1} \pi \ast \partial^2 \chi^{-1} \pi \ast \partial^0 f_u \ast \partial^2 f_u \ast \partial^2 f_su \ast \partial^1 \chi \pi \\
= \partial^0 \chi^{-1} \pi \ast \partial^2 \chi^{-1} \pi \ast \partial^0 f_u \ast \partial^1 \chi \pi \\
= \partial^0 \chi^{-1} \pi \ast \partial^2 \chi^{-1} \pi \ast \partial^0 \chi \pi \ast \partial^1 \chi \pi \\
= \varepsilon \otimes \varepsilon,
\]
as \( \partial^0 f' \ast \partial^2 f'' = (f' \otimes f'')c = (f' \otimes f'')\tau = f'' \otimes f' = \partial^2 f'' \ast \partial^0 f' \) for \( f' \in \text{Reg}_H(R, k) \), so that \( g \) is an algebra map.

Conversely, if \( f \in \text{Alg}_H(R, k) \) then \( \kappa^* f \in \text{Alg}_H(K, k) \), \( \partial f_{ku} \in Z^2_H(R, k) \), \( \delta f_{\kappa} \in Z^2_B(B, k) \) and \( (\pi \otimes \pi) \ast \delta \kappa^* (f) = \partial f_{ku} \). Moreover,
\[
(f_{ku} \ast f_{s})(r\kappa(x)) = f_{ku}(r_{1}(r_{2} - 1)\kappa(x_{1}))f_{s}(r_{2})0\kappa(x_{2}) \\
= f_{ku}(r_{1})f_{\kappa}(r_{2} - 1)xf_{s}(r_{2})0\kappa(x_{2}) \\
= f_{ku}(r_{1})f_{\kappa}(x_{1})f_{s}(x_{2})f_{s}(r_{2}) \\
= \varepsilon(x)(f_{ku} \ast f_{s})(r)
\]
for \( r \in R \) and \( x \in K \), in particular \( (f_{ku} \ast f_{s})(RK^+) = 0 \). Hence, there is a unique \( \chi \in \text{Reg}_H(B, k) \) such that \( f_{ku} \ast f_{s} = \chi \pi \), and observe that \( \chi \) is convolution invertible since \( (f_{ku} \ast f_{s})^{-1}(RK^+) = (f \ast f_{sku})(RK^+) = 0 \) as well. This implies that \( f_{ku} = \chi \pi \ast f \) and
\[
\partial f_{ku} = \partial (\chi \pi \ast f) = \partial^0 (\chi \pi \ast f) \ast \partial^2 (\chi \pi \ast f) \ast \partial^1 (\chi \pi \ast f)^{-1} \\
= \partial^0 \chi \pi \ast \partial^0 f \ast \partial^2 \chi \pi \ast \partial^2 f \ast \partial^1 f_s \ast \partial^1 \chi^{-1} \pi \\
= \partial^0 \chi \pi \ast \partial^2 \chi \pi \ast \partial^0 f \ast \partial^2 f \ast \partial^1 f_s \ast \partial^1 \chi^{-1} \pi \\
= \partial^0 \chi \pi \ast \partial^2 \chi \pi \ast \partial^0 f \ast \partial^1 \chi^{-1} \pi \\
= \partial \chi \pi,
\]
so that \( (\pi \otimes \pi) \ast \delta f_{\kappa} = \partial f_{ku} = \partial \pi \ast \chi = (\pi \otimes \pi) \ast \partial \chi \) and \( \delta \kappa^* f = \delta f_{\kappa} = \partial \chi \), which is equivalent to \( \varepsilon \otimes \varepsilon \) under the equivalence relation on \( Z^2_H(B, k) \).

Exactness at \( H^1_{\text{H}}(B, k) \): If \( f \in \text{Alg}_H(K, k) \) then \( (\pi \otimes \pi) \ast \delta f = \partial f_u \), which is equivalent to \( \varepsilon \otimes \varepsilon \) in \( Z^2_H(B, k) \).

Conversely, if \( \sigma \in Z^2_H(B, k) \) and \( (\pi \otimes \pi) \ast \sigma = \partial f \) for some \( f \in \text{Reg}_H(R, k) \) then \( \partial \kappa^* f = (\kappa \otimes \kappa)^* \partial f = (\pi \kappa \otimes \pi \kappa)^* \sigma = \varepsilon \otimes \varepsilon \), so that \( \kappa^* f = f_{\kappa} \in \text{Alg}_H(K, k) \), \( \partial f_{ku} \in Z^2_H(R, k) \) and \( \delta (f_{\kappa}) \in Z^2_H(B, k) \). It suffices to prove that \( \partial f_{\kappa} \) is equivalent to \( \sigma \) in \( Z^2_B(B, k) \). Now, since \( \partial f(RK^+ \otimes R + R \otimes RK^+) = 0 \) it follows that \( \partial f(r \otimes \kappa(x)) = \varepsilon(x) \partial f(r \otimes 1) + \partial f(r \otimes (\kappa(x) - \varepsilon(x))) = \varepsilon(x) \partial f(r \otimes 1) = (\varepsilon \otimes \varepsilon)(r \otimes \kappa(x)) \), which implies that
\[
f(r\kappa(x)) = \partial^1 f(r \otimes \kappa(x)) = \partial^2 f \ast \partial^0 f(r \otimes \kappa(x)) = f(r)f_{\kappa}(x)
\]
for all \( r \in R \) and \( x \in K \). Then \((f \kappa_0u \ast f)(RK^+)=0\), since

\[
(f \ast f \kappa_0u)(\kappa(x)) = f((r_1(r_2)_1\kappa_0(x_1))f(\kappa_0u)(r_2)_0) = f((r_1)f((r_2)_1\kappa_0(x_1))f_\kappa_0u(r_2)_0) = f(r_1)f_\kappa_0u(x_1)f_\kappa_0u(r_2) = \epsilon(x)(f \ast f \kappa_0u)(r),
\]

and hence there is a unique \( \chi \in \text{Reg}_H(B, k) \) such that \( f \ast f \kappa_0u = \pi^* \chi \), that is \( f = \chi \pi \ast f \kappa_0u \). Then

\[
(\pi \otimes \pi)^* \sigma = \partial f = \partial^0(\chi \pi \ast f \kappa_0u) \ast \partial^2(\chi \pi \ast f \kappa_0u) \partial^1(\chi \pi \ast f \kappa_0u)^{-1}
\]

\[
= \partial^0 \chi \pi \ast \partial^0 f \kappa_0u \ast \partial^2 \chi \pi \ast \partial^2 f \kappa_0u \ast \partial^1 f \kappa_0u \ast \partial^1 \chi^{-1} \pi
\]

\[
= \partial^0 \chi \pi \ast \partial^2 \chi \pi \ast \partial^0 f \kappa_0u \ast \partial^2 f \kappa_0u \ast \partial^1 f \kappa_0u \ast \partial^1 \chi^{-1} \pi
\]

\[
= \partial^0 \chi \pi \ast \partial^2 \chi \pi \ast \partial^0 f \kappa_0u \ast \partial^2 f \kappa_0u \ast \partial^1 f \kappa_0u \ast \partial^1 \chi^{-1} \pi
\]

\[
= (\pi \otimes \pi)^*(\partial^0 \chi \ast \partial^2 \chi \ast \partial^0 f \kappa_0u \ast \partial^2 f \kappa_0u \ast \partial^1 f \kappa_0u \ast \partial^1 \chi^{-1}),
\]

since \( \partial^0 f \kappa_0u \ast \partial^2 \chi \pi = \partial^2 \chi \pi \ast \partial^0 f \kappa_0u \), and thus

\[
\sigma = \partial^0 \chi \ast \partial^2 \chi \ast \partial^0 f \kappa_0u \ast \partial^2 f \kappa_0u \ast \partial^1 \chi^{-1},
\]

so that \( \sigma \) is equivalent to \( \delta f \kappa \) in \( Z^2_H(B, k) \). The remaining assertions are now obvious.

Similar and somewhat simpler arguments lead to an exact sequence of pointed sets

\[
\mathcal{H}^1_H(B \# H, k) \xrightarrow{\pi^*} \mathcal{H}^1_H(R \# H, k) \xrightarrow{\pi^*} \mathcal{H}^1_H(K \# H, k) \xrightarrow{\delta} \mathcal{H}^2_H(B \# H, k) \xrightarrow{\pi^*} \mathcal{H}^2_H(R \# H, k)
\]

for the bosonisations, and the map \( \psi^* \) of cosimplicial groups induces an injective map between the two sequences. As an alternative, given the exact sequence for the bosonisations and the map \( \psi^* \) the sequence for the braided square also follows directly.

It remains to show that any two \( K \)-bimodule coalgebra retractions \( u, u' : K \to R \) lead to the same connecting map \( \delta : \text{Alg}_H(K, k) \to \mathcal{H}^2_H(B, k) \). Observe that \( \ker \pi = K^+R + RK^+ \). For \( f \in \text{Alg}_H(K, k) \) let \( \sigma, \sigma' \in Z^2_H(B, k) \) be such that \((\pi \otimes \pi)^* \sigma = \delta f u \) and \((\pi \otimes \pi)^* \sigma' = \delta f u' \). If \( x \in K^+ \) and \( r \in R \) then \( \Delta(xr) = x_1(x_2)_1r_1 \otimes (x_2)_2r_2 \) and

\[
u' \ast f su(xr) = f u'(x_1(x_2)_1r_1)f su((x_2)_2r_2) = f((x_2)_1u(r_1))f su((x_2)_2r_2) = \epsilon(x)f u'(r_1)f su(r_2) = 0
\]

and a similar argument shows that \( f u' \ast f su(xr) = 0 \), so that \( \chi = f u' \ast f su \in \text{Reg}_H(B, k) \). Moreover, since the faces \( \partial^i : \text{Reg}_H(R, k) \to \text{Reg}_H(R \otimes R, k) \) are
group homomorphisms and since $\partial^0 f' \ast \partial^2 f'' = \partial^2 f'' \ast \partial^0 f'$ it follows that

\[ l(\pi \circ \pi)^* (\partial^0 \chi \ast \partial^2 \chi \ast \sigma \ast \partial^1 \chi^{-1}) \]
\[ = \partial^0 (f u' \ast f su) \ast \partial^2 (f u' \ast f su) \ast \partial f u \ast \partial^1 (f u' \ast f su) \]
\[ = \partial^0 f u' \ast \partial^2 f u' \ast \partial^1 f su' = (\pi \circ \pi)^* \sigma'. \]

But $(\pi \circ \pi)^* : \text{Reg}_H(B \otimes B, k) \to \text{Reg}_H(R \otimes R, k)$ is injective, so that $\partial^0 \chi \ast \partial^2 \chi \ast \sigma \ast \partial^1 \chi^{-1} = \sigma'$, which means that $\sigma$ and $\sigma'$ are in the same cohomology class. \hfill \Box

**Remark.** For Hochschild cohomology, which in some cases can be viewed as the infinitesimal part of the ‘multiplicative’ cohomology, such a sequence (now of vector spaces) also exists [GM]. The proofs are similar but somewhat simpler in that case, and the requirement that the retraction $u : R \to K$ be a coalgebra map is not needed.

### 3. Applications to the lifting process

Every lifting of a given diagram of special finite Cartan type is by [GM] a cocycle deformation of the bosonisation $B(V) \# kG$ of the Nichols algebra $B(V)$ and is completely determined by a $G$-invariant algebra map $f : K(V) \to k$. In the presence of a $K(V)$-module algebra map $f : R(V) \to K(V)$ for the injection $\kappa : K(V) \to R(V)$ the deforming cocycle can be determined via the connecting map $\delta : \text{Alg}_G(K, k) \to \mathcal{H}_G^2(R, k)$ described in the last section. Observe that in our case, $\text{Alg}_G(B, k) = \text{Alg}_G(R, k) = \{ \varepsilon \}$, and that $\delta$ is injective. The simple root vectors $x_\alpha$, where $\alpha \in \Phi^+$ is a simple root, generate $R$ as an algebra. Moreover, $f(x_\alpha) = f(g x_\alpha) = \chi_\alpha(g) f(x_\alpha)$ for every $g \in G$ and $f \in \text{Alg}_G(R, k)$. It follows that $\text{Alg}_G(R, k) = \{ \varepsilon \}$, since $q_\alpha = \chi_\alpha(g_\alpha)$ is a non-trivial root of unity for every simple root $\alpha$.

By [GM] Theorem 2.2 ([AS], Theorem 2.6) it follows that the map $\vartheta : R \to B \otimes K$, given by $\vartheta(x^a z^\alpha) = x^a \otimes z^\alpha$, is a $K$-module isomorphism. The $K$-bimodule retraction

\[ u = (\varepsilon \otimes 1) \vartheta : R \to K \]

for the injection $\kappa : K \to R$ has kernel $B^+ R$ and is a $K$-bimodule map.

#### 3.1. Type $A_1$.

In this case the retraction $u : R \to K$ is a $K$-module coalgebra map, since the obvious injection $\nu : B \to R$ is a coalgebra map, so that $B^+ R$ is a coideal in $R$. The injective map

\[ \delta : \text{Alg}_G(K, k) \to \mathcal{H}_G^2(B, k) \]

is given by $\sigma = \delta f = (fu \otimes fu) \ast fsum(v \otimes v) = fsum(v \otimes v)$, that is $\sigma(x^i \otimes x^j) = f su(x^{i+j})$ and $\sigma^{-1}(x^i \otimes x^j) = f u(x^{i+j})$ for $0 \leq i, j < N$. Using

\[ \Delta(x^m \otimes x^n) = \sum_{0 \leq i \leq m; 0 \leq j \leq n} \begin{pmatrix} m \\ i \end{pmatrix} \begin{pmatrix} n \\ j \end{pmatrix} q^i q^j g^{m-i} \otimes x^i g^{n-j} \otimes x^m \otimes x^n \]


and the identity
\[
\sum_{i+j=r} \binom{m}{i}_q \binom{n}{j}_q q^{l(m-i)} = \binom{m+n}{r}_q = 1
\]
of \([Kn]\) it follows that
\[
m_\sigma(x^m \otimes x^n) = \begin{cases} 
  x^{m+n}, & \text{if } m + n < N \\
  f_s(z)x^{m+n-N}(1 - g^N), & \text{if } m + n \geq N
\end{cases}
\]

3.2. Quantum planes. The general quantum plane \(V = k x_1 \oplus k x_2\) has \(G\)-coaction \(\delta(x_i) = g_i \otimes x_i\) and \(G\)-action \(gx_i = \chi_1(g)x_i\), where \(\chi_1(g_2)\chi_2(g_1) = 1\) and \(q = \chi_1(g_1)\) is a primitive root of unity of order \(N\). Moreover, \(\chi_i^N = \varepsilon = \chi_1\chi_2\), so that \(\chi_1(g_i) = q\) and \(\chi_2(g_i) = q^{-1}\).

In the free Hopf algebra \(k < x_1, x_2 >\) the relation \(x_2x_1 = qx_1x_2 + z_{21}\), where \(z_{21} = [x_2, x_1] = x_2x_1 - qx_1x_2\), can be used to construct a PBW-basis. The following Lemma, which will also be used later, is helpful in this connection.

**Lemma 3.1.** For a quantum plane with linkable vertices, i.e. with \(\chi_1\chi_2 = \varepsilon\), the relations
\[
x_2^m x_1^n = \sum_{r=0}^{l} q^{(m-r)(n-r)r_1q} \binom{m}{r}_q \binom{n}{r}_q x_1^{n-r} x_2^{m-r} z_{21}^r + p_{mn}
\]
hold in \(k < x_1, x_2 >\), where \(l = \min\{m, n\}\) and \(p_{mn}\) is an element in the ideal generated by \([x_1, z_{21}]\) and \([x_2, z_{21}]\).

**Proof.** Since the vertices are linkable, we have \(z_{21}x_i = x_i z_{21} - [x_i, z_{21}]\). It follows by induction on \(m\) that
\[
x_2^m x_1 = q^m x_1 x_2^m + m q x_2^{m-1} z_{21} - \sum_{i=1}^{m-1} q^i x_2^{m-1-i} x_2 [x_1, z_{21}]
\]
where \([x_1, z_{21}] = \sum_{k=1}^{i-1} x_1^{i-k} [x_2, z_{21}] x_2^{k-1}\), and then, if \(m \geq n\), by induction on \(n\)
\[
x_2^m x_1^n = \sum_{r=0}^{l} q^{(m-r)(n-r)r_1q} \binom{m}{r}_q \binom{n}{r}_q x_1^{n-r} x_2^{m-r} z_{21}^r + p_{mn},
\]
where \(p_{m(n+1)} = p_{mn}x_1 + \sum_{r=0}^{n} q^{(m-r)(n-r)r_1q} \binom{m}{r}_q \binom{n}{r}_q x_2^{n-r} x_2^{m-r} [x_1, z_{21}] + p(m-r) z_{21}\) and \(p_{m1} = \sum_{i=1}^{m-1} q^i x_2^{m-i} [x_1, z_{21}]\). Here we used the identities
\[
\binom{m}{r}_q = \binom{m}{r}_q \quad \text{and} \quad \binom{n+1}{r}_q = \binom{n}{r}_q + q^{n+1-r} \binom{n}{r}_q
\]
on the other hand, by induction on \(n\) we get
\[
x_2 x_1^n = q^n x_1^n x_2 + n q x_1^{n-1} z_{22} - \sum_{i=1}^{n-1} (n-i) q x_1^{n-i-1} [x_1, z_{21}] x_1^{i-1}
\]
and then, if $m \leq n$, by induction on $m$

$$x_2^m x_1^n = \sum_{r=0}^{m} q^{(m-r)(n-r)} \binom{m}{r} q^{(n-r)} \binom{n}{r} q^{m-r} x_2^{m-r} x_2^r - p'_{mn}$$

with $p'_{(m+1)n} = x_2 p'_{mn} + \sum_{r=0}^{m} q^{(m-r)(n-r)} \binom{m}{r} q^{(n-r)} \binom{n}{r} q^{m-r} - q^{m+1-r} \binom{m}{r} q^{m+1-r} x_2^{m+1-r} - p'_{1(n-r)x_2^m} x_2^r + p'_{1n} = \sum_{i=1}^{n-1} (n-i) q^{n-i-1} \binom{2}{1} [x_1, z_2] x_2^i$. Here the identities

$$\binom{n}{r} q = q^{n-r} \binom{n}{r} q^{m+1-r} \binom{m}{r} q$$

were used. □

The elements $x_1^i = z_i$ and $[x_2, x_1] = z_2$ are primitive in $k < x_1, x_2 >$, and so are $[x_1, z_2], [x_2, z_1]$ and $[z_i, z_2]$ for $i = 1, 2$. The ideal generated by the elements $[x_1, z_2], [x_2, z_1], [z_1, z_2]$, and $[z_2, z_2]$ in the braided Hopf algebra $k < x_1, x_2 >$ is therefore a Hopf ideal, so that

$$R = k < x_1, x_2 > / ([x_1, z_2], [x_2, z_1], [z_1, z_2], [z_2, z_2])$$

is a Hopf algebra in the category of crossed $kG$-modules. It follows from the Lemma above that $[z_2, z_1] = z_2 z_1 - z_1 z_2 = 0$. Thus, if $K$ is the Hopf subalgebra of $R$ generated by $z_1, z_2$ and $z_2$, then $K = k[z_1, z_2, z_2]$ as an algebra, and

$$\begin{array}{ccc}
K & \xrightarrow{\kappa} & R \\
\varepsilon \downarrow & & \pi \downarrow \\
k & \xrightarrow{\iota} & B
\end{array}$$

is a pushout square of braided Hopf algebras, where $B = k < x_1, x_2 > / (z_1, z_2, z_2)$ is the Nichols algebra of the quantum plane. By the Lemma above $R \cong (B \otimes K) \otimes J$ as a vector space, where $J$ is the ideal in $R$ generated by $[x_1, z_2]$ and $[x_2, z_2]$, which is not a Hopf ideal.

**Proposition 3.2.** For the quantum plane the injection $\kappa : K \to R$ has a $K$-bimodule coalgebra retraction $u : R \to K$ defined by $u(x^a z^b + J) = \varepsilon(x^a) z^b$.

**Proof.** It is clear that the linear map $u : R \to K$ defined by $u(x^a z^b + J) = \varepsilon(x^a) z^b$ satisfies $u \kappa = 1_K$. It is a $K$-bimodule map, since in $R$ we have $[x_i, z_j] = 0$ and $[x_i, z_2] \in J$. It is also a coalgebra map, since its kernel $\ker u = (B^+ \otimes K) \oplus J$ is a coideal. □

Theorem 3.1 is therefore applicable, and since $\text{Alg}_G(R, k) = \{ \varepsilon \}$, it follows that the connecting map $\delta : \text{Alg}_G(K, k) \to \mathcal{H}_q(B, k)$ is injective (see proof of Proposition 5.7). It is determined by $(\pi \otimes \pi)^* \delta f = \partial(fu) = (fu \otimes fu) \ast f \text{sum}_R$ and, since the obvious injection $v : B \to R$ is a coalgebra map, we see that

$$\sigma(x^a \otimes x^b) = \partial(fu)(x^a \otimes x^b) = f su(x^a x^b)$$
for $0 \leq a_i, b_i < N$, where the cocycle $\sigma = \partial(fu)(u \otimes v)$ represents the cohomology class $\delta f \in H^2_G(B, k)$. In particular, in view of the definition of $u$ and Lemma 3.1

$$\sigma(x_i^m \otimes x_j^n) = \left\{ \begin{array}{ll} \delta_{ij}^{m+n} f(s(z_i)) & , \text{if } i \leq j \\ \delta_{ij}^{m+n}! f(s(z_i^n)) & , \text{if } i = 2 > j = 1 \end{array} \right.$$ 

for $0 \leq m, n < N$. Here is the connection to Hochschild cohomology, a result which is also applicable in a more general context. Recall first that by [GM] there is a Kunneth type isomorphism in equivariant Hochschild cohomology

$$H^2_G(B, k) \cong H^2_G(B_1, k) \oplus H^2_G(B_2, k) \oplus (H^1(B_1, k) \otimes H^1(B_2, k))_G,$$

where $B_i = k[x_i]/(x_i^N)$ are Nichols algebras of quantum lines and $H^1(B_i, k) = \text{Der}(B_i, k) \cong \text{Hom}(B_i^+/(B_i^+)^2, k)$. This means that every $\zeta \in H^2_G(B, k)$ has a unique decomposition of the form $\zeta = \zeta_1 + \zeta_2 + \zeta_3$. The following result has also been obtained recently with somewhat different methods in [ABM], section 5.

**Theorem 3.3.** For any quantum plane the diagram

$$\begin{array}{ccc}
\text{Der}_G(K, k) & \xrightarrow{\delta_{\text{Hoch}}} & H^2_G(B, k) \\
\exp & & \exp \\
\Alg_G(K, k) & \xrightarrow{\delta} & H^2_G(B, k)
\end{array}$$

commutes if $\exp(d) = e^d$ and $\exp_q(\zeta) = e^{\zeta_1} * e^{\zeta_2} * e^{\zeta_3}$, where $e^d = \sum_{n \geq 0} d^n n!$ and $\exp_q(\zeta) = e^\zeta = \sum_{n \geq 0} \frac{\zeta^n}{n!}$ are the convolution exponential and $q$-exponential, respectively.

**Proof.** It is clear that $\exp : \text{Der}_G(K, k) \rightarrow \Alg_G(K, k)$, given by the convolution power series $\exp(d) = e^d = \sum_{n \geq 0} d^n n!$, is an isomorphism of abelian groups, since the Hopf algebra $K = k[z_1, z_2, z_3]$ is a polynomial algebra. By [GM] there is a Kunneth type isomorphism in equivariant Hochschild cohomology

$$H^2_G(B, k) \cong H^2_G(B_1, k) \oplus H^2_G(B_2, k) \oplus (H^1(B_1, k) \otimes H^1(B_2, k))_G,$$

where $B_i = k[x_i]/(x_i^N)$ are Nichols algebras of quantum lines and $H^1(B_i, k) = \text{Der}(B_i, k) \cong \text{Hom}(B_i^+/(B_i^+)^2, k)$. The connecting map $\delta_{\text{Hoch}} : \text{Der}_G(K, k) \rightarrow H^2_G(B, k)$ is an isomorphism, since $\text{Der}_G(R, k) = 0$ and since $\dim \text{Der}_G(K, k) = 3 = \dim H^2_G(B, k)$. The connecting map $\delta : \Alg_G(K, k) \rightarrow H^2_G(B, k)$ is injective, as mentioned above, since $\Alg_G(R, k) = \{ \varepsilon \}$. Moreover, every element $d \in \text{Der}_G(K, k)$ has a unique expression of the form $d = d_1 + d_2 + d_3$, every $f \in \Alg_G(K, k)$ is uniquely of the form $f_1 * f_2 * f_3$, where the notation is self explanatory, and $e^d = e^{d_1} * e^{d_2} * e^{d_3}$. For a general $f = f_1 * f_2 * f_3 \in \Alg_G(K, k)$
one obtains the formula
\[
\delta f(x_1^{k} x_2^{m} \otimes x_1^{n} x_2^{l}) = \delta f_1 \delta f_2 \cdot \delta f_3 \delta f_4 \cdot \delta f_5 \delta f_6 (z_21)\]

where \(\delta f = \delta f_1 \delta f_2 \delta f_3 \delta f_4 \delta f_5 \delta f_6 (z_21)\),

Remark: \(\text{Reg}_G(R, k) \to \text{Reg}_G(R \otimes R, k)\) are algebra maps and that \(\partial(fu)(v \otimes v) = \partial^i (fu)(v \otimes v)\). This formula shows in particular that

\[
\delta e^d_1 (x_1^k x_2^m \otimes x_1^n x_2^l) = \delta e^d_0 \delta e^d_1 \cdot \delta e^d_2 \cdot \delta e^d_3 \delta e^d_4 \delta e^d_5 \delta e^d_6 (z_21)\]

On the other hand, drawing on Lemma 3.1 again, for \(d = d_1, d_2, d_21\) compute

\[
e^d_q \delta \text{coh} e^d = \sum_{t \geq 0} \frac{(dsum)^t}{t!} e^d_q = \sum_{t \geq 0} \frac{(dsum)^t}{t!} \delta \text{coh} e^d_q \]

by evaluating the convolution powers \((dsum)^t(x_1^k x_2^m \otimes x_1^n x_2^l)\) for \(t > 0\) to get

\[
(d_1 \text{sum})^t(x_1^k x_2^m \otimes x_1^n x_2^l) = \delta e^d_0 \delta e^d_1 \delta e^d_2 \delta e^d_3 \delta e^d_4 \delta e^d_5 \delta e^d_6 (z_21)\]

and therefore \(e^d_q \delta \text{coh} e^d = \delta e^d_q\) for the specified derivations. This means that the map \(\text{Exp}_q : \mathcal{H}_G^2(B, k) \to \mathcal{H}_G^2(B, k)\) is given by \(\text{Exp}_q (\zeta) = e^\zeta_q \cdot e^\zeta_q \cdot e^\zeta_q \cdot e^\zeta_q \cdot e^\zeta_q \cdot e^\zeta_q\).

Remark: Observe that in general \(\delta f_1\) and \(\delta f_2\) do not commute with \(\delta f_21\), since for example \(\delta f_2 = \delta f_21 (x_1^k x_2^m \otimes x_1^n x_2^l) = q^{-1} (1 + q) (x_2 x_2) = 21 (z_21)\), and \(\delta f_21 = \delta f_2 (x_1^k x_2^m \otimes x_1^n x_2^l) = (1 + q) (z_21) (z_21)\), so that in general \(\text{Exp}_q (\zeta) \neq e^\zeta_q\) (see also [ABM]). But, if \(\zeta_21 = 0\), that is \(f_2 = \zeta\), then \(\text{Exp}_q (\zeta) = e^\zeta_q\), since \(\delta f_2 \cdot \delta f_1 = \delta f_1 \cdot \delta f_2\), a result already obtained in [GM].

3.3. Linking. Let \(V = kx_1 \oplus \ldots \oplus kx_9\) be any special diagram of finite Cartan type, and suppose that \(i < j\) is a linkable pair, i.e. \(\chi_i \chi_j = \varepsilon\). Then \(i\) and \(j\) are in different components of the Dynkin diagram, and they are not linkable to any other vertices. Let \(B = TV/I\) be the Nichols algebra of \(V\), where \(I\) is the ideal generated by the usual set \(S\). If \(S_{ij} = S \setminus \{z_{ij}\}\), where \(z_{ij} = [x_i, x_j]\), then the ideal \(I_{ij}\) in \(TV\) generated by \(S_{ij}\) is still a Hopf ideal and \(R_{ij} = TV/I_{ij}\) is a braided Hopf
algebra. The kernel of the canonical projection $\pi : R_{ij} \to B$ is the ideal generated by $z_{ji}$, which is a Hopf ideal, since $z_{ji}$ is primitive. If $K_{ij}$ is the Hopf subalgebra of $R_{ij}$ generated by $z_{ji}$ then

$$
\begin{array}{ccc}
K_{ij} & \xrightarrow{\kappa} & R_{ij} \\
\varepsilon & \downarrow & \pi \\
k & \xrightarrow{i} & B
\end{array}
$$

is a pushout square. Moreover, as a vector space $\pi$ is a pushout square. Moreover, as a vector space $R_{ij} \cong (B \otimes K_{ij}) \oplus J_{ij}$, where $J_{ij}$ is the Hopf ideal generated by the set $\{[x_k, z_{ji}] | 1 \leq k \leq \theta\}$, which is not a Hopf ideal.

**Proposition 3.4.** For any special diagram of finite Cartan type and any linkable pair of vertices $i < j$ in its Dynkin diagram, the linear map $u : R_{ij} \to K_{ij}$, given by $u(x^a \otimes z_{ji}^b + J_{ij}) = \varepsilon(x^a) e_{ji}^b$, is a $K_{ij}$-bimodule coalgebra retraction for the inclusion $\kappa : K_{ij} \to R_{ij}$.

**Proof.** It is clear that the $u : R_{ij} \to K_{ij}$ just defined is a linear map satisfying $uK = 1_{K_{ij}}$. It is a $K_{ij}$-bimodule map, since in $R_{ij}$ the element $[x^a, z_{ji}]$ is in $J_{ij}$ for every $x^a \in B$. It is a coalgebra map, since $(B^+ \otimes K_{ij}) \oplus J_{ij}$ is a coideal in $R_{ij}$. \hfill $\Box$

Our Theorem 3.1 and the corresponding result for Hochschild cohomology are therefore applicable. Since, as an algebra, $R_{ij}$ is generated by the set $\{x_l | 1 \leq l \leq \theta\}$, and since the $\chi_l(x_l)$ are non-trivial roots of unity, we conclude that $\text{Der}_G(R_{ij}, k) = 0$ and $\text{Alg}_G(R_{ij}, k) = \{\varepsilon\}$. Moreover, for the polynomial Hopf algebra $K_{ij} = k[z_{ji}]$, the convolutian exponential map $\exp : \text{Der}_G(K_{ij}, k) \to \text{Alg}_G(K_{ij}, k)$ is an isomorphism of groups, and the diagram

$$
\begin{array}{ccc}
\text{Der}_G(k_{ij}, k) & \xrightarrow{\delta_{\text{Hoch}}} & H^2_G(B, k) \\
\exp & \downarrow & \ \\
\text{Alg}_G(K_{ij}, k) & \xrightarrow{\delta} & H^2_G(B, k)
\end{array}
$$

carries some information. In this generality there is no obvious map relating $H^2_G(B, k)$ to $H^2_G(k_{ij}, B, k)$, but the diagram relates the image of $\delta_{\text{Hoch}}$ to $H^2_G(B, k)$. More precisely, by the Kunneth formula for the equivariant Hochschild cohomology of Nichols algebras, $\text{im} \delta_{\text{Hoch}} \subseteq (\text{Der}(B_i, k) \otimes \text{Der}(B_j, k))_G$, where $B_i$ and $B_j$ are the Nichols algebras of the components of the Dynkin diagram containing the vertices $i$ and $j$, respectively.

**Corollary 3.5.** Let $i < j$ be a linkable pair of vertices in a special diagram of finite Cartan type. For the derivation $d \in \text{Der}_G(K_{ij}, k)$ the Hochschild cocycle representing $\delta_{\text{Hoch}}d \in H^2_G(B, k)$ is given by $\xi(x^a \otimes x^b) = \delta_{ij}d_{ij} e_{ij}^a(x_{ji})$ and $e_{ij}^i = e_{ij}d_{ij} \in H^2_G(B, k)$.

**Proof.** Replacing the pair $(1, 2)$ by $(i, j)$, Lemma 3.1 holds for any linkable pair $i < j$ in any special diagram of finite Cartan type. It shows that the Hochschild
cocycle $\zeta = \delta_{\text{Hoch}}$ is of the form specified. Together with arguments, similar to those used in Theorem 3.3, it also shows that $\delta e^d = e^\zeta_q$. □

3.4. Type $A_1 \times \ldots \times A_1$. The general quantum linear space $V = kx_1 \oplus \ldots \oplus kx_\theta$ of dimension $\theta$ has $G$-coaction $\delta(x_i) = g_i \otimes x_i$ and $G$-action $gx_i = \chi_i(g)x_i$, where $\chi_i = \varepsilon$ and $\chi_i(g_j)\chi_j(g_i) = 1$ for $i \neq j$.

A vertex $i$ is linkable to at most one other vertex, since the order $N_i$ of $q_{ii} = \chi_i(g_i)$ is supposed to be greater than 2. The vertex set $\{1, 2, \ldots, \theta\}$ can therefore be decomposed into a set $L$ of linkable pairs of the form $i < j$ and a set of non-linkable singletons $L^\perp$, and it can be ordered accordingly. A quantum linear space is therefore a collection of quantum planes together with a bunch of quantum lines with pushout squares

$$
\begin{array}{c@{\quad}\cdots@{\quad}c}
K_{ij} & \xrightarrow{\kappa_{ij}} & R_{ij} \\
\varepsilon & \downarrow \pi_{ij} & \varepsilon \\
k & \xrightarrow{\varepsilon} & B_{ij}
\end{array}
\quad
\begin{array}{c@{\quad}\cdots@{\quad}c}
K_l & \xrightarrow{\kappa_l} & R_l \\
\varepsilon & \downarrow \pi_l & \varepsilon \\
k & \xrightarrow{\varepsilon} & B_l
\end{array}
$$

for $(i, j) \in L$ and $l \in L^\perp$, respectively. The braided tensor product of all these squares represents the Nichols algebra of the quantum linear space. The following considerations about such braided tensor products together with the results for $\theta \leq 2$ will describe the deforming cocycles for all quantum linear spaces.

If a subset $S$ of $\{1, 2, \ldots, \theta\}$ is such that none of its vertices is linkable to any vertex not in $S$ then the complement $T$ has the same property and $\{1, 2, \ldots, \theta\} = S \cup T$. The elements of $K_S$ commute with the elements of $R_T$ and the elements of $K_T$ commute with those of $R_S$. It follows that there is a commutative diagram of coalgebras

$$
\begin{array}{c@{\quad}\cdots@{\quad}c}
K & \xrightarrow{\kappa} & R & \xrightarrow{\pi} & B \\
\rho_K & \downarrow & \rho_R & \downarrow & \rho_B \\
K_S \otimes K_T & \xrightarrow{\kappa_S \otimes \kappa_T} & R_S \otimes R_T & \xrightarrow{\pi_S \otimes \pi_T} & B_S \otimes B_T
\end{array}
$$

with $\rho = (p_S \otimes p_T)\Delta$ is an isomorphism with inverse $\rho^{-1} = m(i_S \otimes i_T)$. The projections $e_S = i_S p_s$ and $e_T = i_T p_T$ on $K$, $R$ and $B$ have the property that

$$
e_S * e_T = \rho^{-1} \rho = 1, \quad u e_S = e_S u, \quad u e_T = e_T u, \quad u = e_S u * e_T u = u e_S \ast u e_T
$$

and, moreover, since the elements of $K_S$ commute with those of $K_T$, also $e_T * e_S = e_S * e_T = 1_K$ on $K$. The latter is of course not true on $R$ and $B$, because $e_T * e_S(x^a x^b_T) = \chi^a(g^b)e_S * e_T(x^a x^b_T)$. With $u_S = p_S u i_S$ and $u_T = p_T u i_T$ the
The projections $e_S = i_S \pi_S$ and $e_T = i_T \pi_T$ on $R$ and $K$ satisfy

$$e_S u = u e_S, \quad e_T u = u e_T, \quad e_S * e_T = \rho_S^{-1} \rho_S = 1,$$

and the diagram

$$
\begin{array}{ccc}
R & \xrightarrow{p_S} & R_S \\
\downarrow u & & \downarrow u_S \\
K & \xrightarrow{p_S} & K_S
\end{array}
\begin{array}{ccc}
R_T & \leftarrow & R \\
\downarrow u_T & & \downarrow u_T \\
K_T & \leftarrow & K
\end{array}
\begin{array}{ccc}
R_S \otimes R_T & \xrightarrow{u_S \otimes u_T} & K_S \otimes K_T
\end{array}
$$

commutes. Moreover, since the elements of $K_S$ and $K_T$ commute, we have $e_T * e_S = e_S * e_T = 1_K$ on $K$. This is of course not the case on $R$ or on $B$, because $e_T * e_S (x^a x' a' z b') = \chi'(g'^a) e_S * e_T (x^a x' a' z b')$.

**Proposition 3.6.** Suppose that $\{1, 2, \ldots, \theta\} = S \cup T$ is such that none of the vertices of $S$ is linkable to any vertex of $T$, then $u = u_S * u_T : R \to K$, where $u_S = u e_S$ and $u_T = u e_T = e_S u * e_T u$ and the diagram

$$
\begin{array}{ccc}
\Alg_G(K_S, k) \times \Alg_G(K_T, k) & \xrightarrow{\delta} & \mathcal{H}_G^2(B_S, k) \times \mathcal{H}_G^2(B_T, k) \\
\rho^1 \downarrow & & \rho^2 \downarrow \\
\Alg_G(K, k) & \xrightarrow{\delta} & \mathcal{H}_G^2(B, k)
\end{array}
$$

commutes, where $\rho^1(f, f') = (f \otimes f') \rho$ and $\rho^2(\sigma, \sigma') = (\sigma \otimes \sigma')(1 \otimes c \otimes 1)(\rho \otimes \rho)$. Moreover, $\rho^1$ is an isomorphism, while $\rho^2$ is injective.

**Proof.** With our assumptions and $f \in \Alg_G(K, k)$ we have $f = f(e_S * e_T) = f e_S * f e_T$ and $fu = f e_S u * f e_T u$. Observe that the inverse of $\rho^1$ is given by

$$(\rho^1)^{-1}(f) = (f i_S \otimes f i_T) \rho = f(e_S * e_T) = f,$$

while

$$(\rho^1)^{-1} \rho^1(f, f_T) = ((f_S \otimes f_T) \rho i_S, (f_S \otimes f_T) \rho i_T) = (f_S \Delta, \varepsilon \otimes f_T) \Delta = (f_S, f_T),$$

since $\rho i_S = (1 \otimes \varepsilon) \Delta$ and $\rho i_T = (\varepsilon \otimes 1) \Delta$. A similar argument shows that $\rho^2$ has a left inverse $\psi$ given by $\psi(\sigma) = (\sigma(i_S \otimes i_S), \sigma(i_T \otimes i_T))$.

The diagram commutes, because

$$\partial^2(\rho^1(f_S, f_T) u) = \partial^2((f_S \otimes f_T) u) = \partial^2(\rho i_S u_S \rho i_T u_T)$$

$$= \partial^2(f_S u_S \rho i_T) \rho i_T (f u_T) = (\partial^2(f_S u_S \rho i_T) \rho i_T) = \partial^2(\rho i_S u_S \rho i_T) \rho i_T = \partial^2(\rho^2 f_S u_S \rho i_T).$$
where we used $d'(p_S \otimes p_S) = p_S d'$, $d'(p_T \otimes p_T) = p_T d'$ and $\rho_{R \otimes R} = (1 \otimes c \otimes 1)(\rho \otimes \rho) = (1 \otimes c \otimes 1)(p_S \otimes p_T \otimes p_S \otimes p_T)(\Delta_R \otimes \Delta_R) = (p_S \otimes p_S \otimes p_T \otimes p_T)\Delta_{R \otimes R}$.

Moreover,

$$\partial^i f u = \partial^i (f e_s * f e_T) u = \partial^i (f e_s u * f e_T u) = \partial^i f e_s u * \partial^i f e_T u = \partial^i f u_s p_S * \partial^i f u_T p_T = (\partial^i f u_s S \otimes \partial^i f u_T p_T)\rho_{R \otimes R}$$

as well.

Since $e_T * e_S = e_S * e_T = 1_K$ on $K$ and hence $\partial^i f u = \partial^i f e_s u * \partial^i f e_T u = \partial^i f e_T u * \partial^i f e_S u$, and since $\partial f u(v \otimes v) = \partial^i f u_S(v \otimes v)$, it follows that

$$\partial f u = \partial^i f e_S u * \partial^i f e_T u$$

as required.

A comparison with Hochschild cohomology can be obtained inductively via a generalized ‘exponential’ map, making use of the isomorphism

$$\text{Der}_G(K, k) \xrightarrow{\delta_{\text{hoch}}} H^2_G(B, k)$$

and Proposition 3.6 to get a commutative square

$$\text{Der}_G(K, k) \xrightarrow{\delta_{\text{hoch}}} H^2_G(B, k)$$

$$\text{Alg}_G(K, k) \xrightarrow{\delta} H^2_G(B, k)$$

which says that

$$\delta e^{d_S + d_T} = \delta e^{d_S} * \delta e^{d_T} = \text{Exp}_S(\delta_{\text{hoch}} d_S) * \text{Exp}_T(\delta_{\text{hoch}} d_T) = \text{Exp}(\delta_{\text{hoch}}(d_S + d_T))$$

by extending the notation naturally. In particular, if $f \in \text{Alg}_G(K, k)$ and $\sigma = \delta f$ then

$$\sigma(x_i^m \otimes x_j^n) = f s u(x_i^m x_j^n) = \begin{cases} f s u(z_i) & \text{if } i = j \text{ and } m + n = N_i \\ n_{i,j}^m f s (z_j)^n & \text{if } i > j \text{ linkable and } m = n \\ 0 & \text{otherwise.} \end{cases}$$

Remark. Observe that Proposition 3.6 holds for any special diagram of finite Cartan type, provided that a $K$-bimodule coalgebra retraction $u : R \to K$ exists. This is because $\partial^i (f e_S u) * \partial^i (f e_T u) = \partial^i (f e_T u) * \partial^i (f e_S u)$ for $f \in \text{Alg}_G(K, k)$ and $i \leq j$ if $S$ and $T$ are not linkable.
3.5. The connected case. Let $\mathcal{D}$ be a special connected datum of finite Cartan type with Cartan matrix $(a_{ij})$. The vector space $V = V(\mathcal{D})$ can also be viewed as a crossed module in $\mathbb{Z}[I]^\mathcal{D}$, where $\mathbb{Z}[I]$ is the free abelian group on the set of simple roots $I = \{\alpha_1, \ldots, \alpha_\theta\}$. The $\mathbb{Z}[I]$-degree of a word $x = x_{i_1}x_{i_2}\ldots x_{i_n}$ in the tensor algebra is defined by $\deg(x) = \sum_{i=1}^\theta n_i \alpha_i$, where $n_i$ is the number of occurrences of $x_i$ in $x$. The Weyl group $W \subset \text{Aut}(\mathbb{Z}[I])$ is generated by the automorphisms $s_i$ defined by $s_i(\alpha_j) = \alpha_j - a_{ij}\alpha_i$. The root system $\Phi = \cup_{i=1}^\theta W(\alpha_i)$ is the union of the orbits of simple roots in $[I]$, and

$$\Phi^+ = \{\alpha = \sum_{i=1}^\theta n_i \alpha_i \in \Phi|n_i \geq 0\}$$

is the set of positive roots. The Hopf algebra $\mathcal{A}(V)$, the quotient Hopf algebra $R(\mathcal{D}) = \mathcal{A}(V)/(ad^{1-a_{ij}} x_i(x_j)|1 \leq i \neq j \leq \theta)$ and its Hopf subalgebra $K(\mathcal{D})$ generated by $\{x_\alpha|\alpha \in \Phi^+\}$, as well as the Nichols algebra $B(\mathcal{D}) = R(\mathcal{D})/(x_\alpha)$, are all Hopf algebras in $\mathbb{Z}[I]$. In particular, their comultiplications are $\mathbb{Z}[I]$-graded. By construction, for $\alpha \in \Phi^+$, the root vector $x_\alpha$ is $\mathbb{Z}[I]$-homogeneous of $\mathbb{Z}[I]$-degree $\alpha$, so that $\delta(x_\alpha) = g_\alpha \otimes x_\alpha$ and $gx_\alpha = \chi_\alpha(g)x_\alpha$. For $1 \leq l \leq p$ and for $a = (a_1, a_2, \ldots, a_p) \in \mathbb{N}^p$ write $\underline{a} = \sum_{i=1}^\theta a_i \beta_i$ and

$$g^a = g_1^{a_1}g_2^{a_2}\ldots g_p^{a_p} \in G, \ x^a = \chi_1^{a_1}\chi_2^{a_2}\ldots \chi_p^{a_p} \in \tilde{G}, \ x^a = x^{a_1}_{\beta_1}x^{a_2}_{\beta_2}\ldots x^{a_p}_{\beta_p} \in R(\mathcal{D}).$$

In particular, for $e_l = (\delta_{kl})_{1 \leq k \leq p}$, where $\delta_{kl}$ is the Kronecker symbol, $e_l = \beta_i$ and $x^{e_l} = x^\beta_l$ and $x^{N_{e_l}} = x^{N_{\beta_l}} = z_l$ for $1 \leq l \leq p$. In this notation

$$\{x^a|0 \leq a_i\} \cup \{z^b|0 \leq b_i\} \cup \{x^{a}\} \cup \{x^{a}|0 \leq a_i < N\}$$

form a PBW-basis for $R(\mathcal{D})$, $K(\mathcal{D})$ and $B(\mathcal{D})$, respectively. The height of $\alpha = \sum_{i=1}^\theta n_i \alpha_i \in \mathbb{Z}[I]$ is defined to be the integer $ht(\alpha) = \sum_{i=1}^\theta n_i$. Observe that if $a, b, c \in \mathbb{N}^p$ and $\underline{a} = \underline{b} + \underline{c}$ then

$$g^a = g^b g^c, \ x^a = x^b x^c \text{ and } ht(b) < ht(a) \text{ if } \underline{c} \neq 0.$$

By AS (Theorem 2.6), the sets

$$\{z^b|0 \leq b_i\} \cup \{x^{a}z^b|0 \leq a_i < N, 0 \leq b_j\} \cup \{x^{a}|0 \leq a_i < N\}$$

form a basis for $K(\mathcal{D})$, $R(\mathcal{D})$ and $B(\mathcal{D})$, respectively. The squares

$$\begin{array}{ccc}
K(\mathcal{D}) & \xrightarrow{\kappa} & R(\mathcal{D}) \\
\varepsilon & \downarrow & \pi \\
k & \xrightarrow{\varepsilon^1} & B(\mathcal{D})
\end{array}$$

$$\begin{array}{ccc}
K\#kG & \xrightarrow{\kappa\#1} & R\#kG \\
\varepsilon^1 & \downarrow & \pi^1 \\
kG & \xrightarrow{\varepsilon^1} & B\#kG
\end{array}$$

are pushout squares of braided Hopf algebras and their bosonizations, respectively.

Moreover, the $K$-module isomorphism $\vartheta : R \to B \otimes K$ given by $\vartheta(x^a z^b) = x^a \otimes z^b$, can be used to get a $K$-module retraction $u = (\varepsilon \otimes 1)\vartheta : R(\mathcal{D}) \to K(\mathcal{D})$. 

$u(x^a z^b) = \varepsilon(x^a) z^b$, for the inclusion of $\kappa : K(V) \to R(V)$. Thus, the conditions for the 5-term sequence in Hochschild cohomology are satisfied. The connecting map

$$\delta_{hoch} : \text{Der}_G(K, k) \to \mathcal{H}_G^2(B, k)$$

which is injective since $\text{Der}_G(R, k) = 0$, is such that $\delta_{hoch} d(\pi \otimes \pi) = \partial_{hoch}(du) = -dum_R$, where $\pi : R(V) \to B(V)$ is the canonical projection. The $K$-module map $u : R \to K$ just defined is not a coalgebra map in general.

Observe that $K = k[z]_{\alpha \in \Phi^+}$ is a polynomial algebra, since by our assumption $\chi_i^N = \varepsilon$ for $1 \leq i \leq \theta$. The algebra isomorphism $\rho : \oplus_{\alpha \in \Phi^+} K_{\alpha} \to K$, given by $\rho(z_1^{n_1} \otimes z_2^{n_2} \otimes \ldots \otimes z_p^{n_p}) = z_1^{\alpha_1} z_2^{\alpha_2} \ldots z_p^{\alpha_p}$, induces a commutative diagram

$$\begin{array}{ccc}
\text{Der}_G(K, k) & \xrightarrow{\rho_{\text{Der}}} & \oplus_{\alpha \in \Phi^+} \text{Der}_G(K_{\alpha}, k) \\
\text{Exp} & & \text{Exp} \\
\text{Alg}_G(K, k) & \xrightarrow{\rho_{\text{Alg}}} & \oplus_{\alpha \in \Phi^+} \text{Alg}_G(K_{\alpha}, k)
\end{array}$$

of sets, with $\rho_{\text{Der}}(d) = (d_{\alpha})$, $\rho_{\text{Alg}}(f) = (f_{\alpha})$, $\text{Exp}(d_{\alpha}) = (e^{d_{\alpha}})$ and $\text{Exp}(d) = \rho_{\text{Alg}}^{-1} \text{Exp} \rho_{\text{Der}}$, where $i_{\alpha} : K_{\alpha} \to K$ and $p_{\alpha} : K \to K_{\alpha}$ are the obvious canonical injections and projections. This means more explicitly that

$$\text{Exp}(d)(z_1^{n_1} z_2^{n_2} \ldots z_p^{n_p}) = e^{d_1}(z_1^{n_1}) e^{d_2}(z_2^{n_2}) \ldots e^{d_p}(z_p^{n_p})$$

for $d \in \text{Der}_G(K, k)$.

If $\kappa : K \to R$ has a $K$-module coalgebra retraction $u_{\infty} : R \to K$ then Theorem 2.1 is applicable, and the diagram

$$\begin{array}{ccc}
\text{Der}_G(K, k) & \xrightarrow{\delta_{hoch}} & \mathcal{H}_G^2(B, k) \\
\text{Exp} & & \\
\text{Alg}_G(K, k) & \xrightarrow{\delta} & \mathcal{H}_G^2(B, k)
\end{array}$$

connects the relevant part of the Hochschild cohomology $\mathcal{H}_G^2(B, k)$ to the multiplicative cohomology $\mathcal{H}_G^2(B, k)$.

**Proposition 3.7.** Let $V$ be a special (connected) diagram of finite Cartan type. If $K(V)$ is a $K$-module coalgebra retract in $R(V)$ then the connecting map

$$\delta : \text{Alg}_G(K, k) \to \mathcal{H}_G^2(B, k)$$

is injective.

**Proof.** The simple root vectors $x_\alpha$, where $\alpha \in \Phi^+$ is a simple root, generate $R$ as an algebra. Moreover, $f(x_\alpha) = f(gx_\alpha) = \chi_\alpha(g)f(x_\alpha)$ for every $g \in G$ and every $f \in \text{Alg}_G(R, k)$. It follows that $\text{Alg}_G(R, k) = \{ \varepsilon \}$, since $q_\alpha = \chi_\alpha(q_\alpha)$ is a non-trivial root of unity for every simple root $\alpha \in \Phi^+$. 

Now suppose that $\delta f = \delta f'$ in $\mathcal{H}_G^2(B, k)$ for some $f, f' \in \text{Alg}_G(K, k)$. The representing cocycles $\sigma$ and $\sigma'$ are equivalent, so that $\sigma' = \partial^0 \chi \ast \partial^2 \chi \ast \sigma \ast \partial^1 \chi^{-1}$ for some $\chi \in \text{Reg}_G(B, k)$. It follows that

$$\partial^0 f' u \ast \partial^2 f' u \ast \partial^3 f' su = \partial f' u = (\pi \otimes \pi)^* \sigma' = (\pi \otimes \pi)^* (\partial^0 \chi \ast \partial^2 \chi \ast \sigma \ast \partial^1 \chi^{-1})$$

$$= \partial^0 (\pi^* \chi) \ast \partial^2 (\pi^* \chi) \ast \partial f u \ast \partial^1 (\pi^* \chi^{-1})$$

$$= \partial^0 (\pi^* \chi) \ast \partial^2 (\pi^* \chi) \ast \partial^0 f u \ast \partial^2 f u \ast \partial^3 f su \ast \partial^1 (\pi^* \chi^{-1})$$

$$= \partial^0 (\pi^* \chi) \ast \partial^2 (\pi^* \chi) \ast \partial f u \ast \partial^1 f su \ast \partial^1 (\pi^* \chi^{-1})$$

$$= \partial^0 (\pi^* \chi) \ast \partial^2 (\pi^* \chi \ast f u) \ast \partial^1 (f su \ast \pi^* \chi^{-1})$$

since the $\text{im} \partial^0$ and $\text{im} \partial^2$ commute elementwise, so that $\partial^2 (\pi^* \chi) \ast \partial^0 f u = (\pi^* \chi \otimes \varepsilon \otimes \varepsilon \otimes f u) \Delta_{R \otimes R} = (\varepsilon \otimes f u \otimes \pi^* \chi \otimes \varepsilon) \Delta_{R \otimes R} = \partial^0 f u \ast \partial^2 (\pi^* \chi)$. This means, again using the elementwise commutativity of $\text{im} \partial^0$ and $\text{im} \partial^2$, that

$$\partial^1 (f su \ast \pi^* \chi^{-1} \ast f' u) = \partial^1 (f su \ast \pi^* \chi^{-1}) \ast \partial^1 f' u$$

$$= \partial^2 (\pi^* \chi \ast f u)^{-1} \ast \partial^0 (\pi^* \chi \ast f u)^{-1} \ast \partial f' u \ast \partial^1 (f su \ast \pi^* \chi^{-1})$$

$$= \partial^0 (f su \ast \pi^* \chi^{-1}) \ast \partial^0 f' u \ast \partial^2 (f su \ast \pi^* \chi^{-1}) \ast \partial f' u$$

so that $f su \ast \pi^* \chi^{-1} \ast f' u \in \text{Alg}_G(R, k) = \{ \varepsilon \}$ and then $f' u = \pi^* \chi \ast f u$. But then

$$f' = f' u \kappa = (\pi^* \chi \ast f u) \kappa = f u \kappa \ast \pi \kappa = \chi \varepsilon \ast f = \varepsilon \ast f = f$$

as required. \qed

The multiplicative cocycle $\sigma$ representing the cohomology class $\delta f$ is given by

$$(\pi \otimes \pi)^* \sigma = \partial f u_{\infty} = \partial^0 f u_{\infty} \ast \partial^2 f u_{\infty} \ast \partial^1 f su_{\infty} = (f u_{\infty} \otimes f u_{\infty}) \ast f su_{\infty} \ast m_R$$

or, equivalently $\sigma = \partial f u_{\infty}(v \otimes v)$, where $v : B \to R$ is the obvious linear section of the canonical projection $\pi : R \to B$.

**Conjecture 1.** For every special connected diagram of finite Cartan type $V$ the braided Hopf subalgebra $K$ is a $K$-module coalgebra retract in $R$.

Here is a recursive procedure to verify the conjecture. Let $B_i$ be the linear span in $B$ of all ordered words involving root vectors of height $\leq i$ only. For $i > 1$ let $B_i \subset B_i$ be the linear span of all ordered monomials in $B_i$ containing at most $j$ distinct root vectors of height $i$. Then $B_i$ is a subcoalgebra of $B$. The inclusion $v_i : B_i \to R$ is not a coalgebra map, but $B_i \otimes K \subset R$ is a subcoalgebra under the coalgebra structure inherited from $R$ (not the tensor product coalgebra structure). This gives a finite filtration $B_i \subseteq B_{i+1} \subseteq B_{i+1}$ of $B$ and $\cup_{i \geq 0} B_i = B$. Observe that $B_i^0 = B_i$ for $i \geq 1$ and $B_i^0 = B_i$ for some $j$.

- For $B_1$ let $u_1 = \varepsilon \otimes 1 : B_1 \otimes K \to K$, which is a coalgebra map.
Suppose a coalgebra retraction \( u^j_{i+1} = m_K(\varphi^j_{i+1} \otimes 1) : B^j_{i+1} \otimes K \to K \) has been constructed. Extend \( \varphi^j_{i+1} \) linearly to \( B^{j+1}_{i+1} \) by sending to zero all PBW-monomials involving more than \( j \) distinct root vectors of height \( i + 1 \). For such a PBW-monomial \( x \in B^{j+1}_{i+1} \setminus B^j_{i+1} \) find a \( z \in K \) such that \( \Delta_K z - z \otimes 1 - 1 \otimes z = (u^j_{i+1} \otimes u^j_{i+1}) \Delta_R v^{j+1}_{i+1} x \). Now define \( \varphi^j_{i+1} : B^{j+1}_{i+1} \to K \) by \( \varphi^j_{i+1}(x) = z \) and \( \varphi^j_{i+1} |_{B^j_{i+1}} = \varphi^j_{i+1} \). Then \( u^j_{i+1} = m_K(\varphi^j_{i+1} \otimes 1) : B^{j+1}_{i+1} \otimes K \to K \) is a \( K \)-module coalgebra map.

Since \( B \) is finite dimensional \( B = B^1_i \) for some pair \((i,j)\). Then \( u_\infty = u^i_j = m_K(\varphi^i_j \otimes 1) \vartheta : R \otimes B \otimes K \to K \) is a retraction for the inclusion \( \kappa : K \to R \).

### 3.6. Type \( A_2 \)

Here we have a crossed \( kG \)-module \( V = kx_1 \otimes kx_2 \) with coaction \( \delta(x_1) = g_i \otimes x_1 \) and action \( g x_1 = \chi_i(g)x_1 \), where \( \chi_i(g_i) = q \) and \( \chi_i(g_j) \chi_j(g_i) = q_{ij} q_{ji} = q^{-1} \). If \( e_{12} = x_1 \), \( e_{23} = x_2 \) and \( e_{13} = [e_{12}, e_{23}] = [x_1, x_2] \) then \( \{e_{12}^m e_{13}^n e_{23}^l \mid 0 \leq m, n, l < N \}, \{e_{12}^m e_{13}^n e_{23}^l \mid 0 \leq m, n, l \} \) and \( \{z_{ij}^m z_{kl}^l \mid 0 \leq m, n, l \} \), where \( z_{ij} = e_{ij}^N \), form a basis for \( B(V), R(V) \) and \( K(V) \), respectively. In this notation, taken from [ASI], the comultiplications in the bosonisations are determined by

\[
\Delta(e_{ij}) = \sum_{i \leq p \leq j} \lambda_{ipj} e_{ip} g_p g_j \otimes e_{pj},
\]

where \( e_{ii} = 1 \) and

\[
\lambda_{ipj} = \begin{cases} 
1 & \text{if } i = p \text{ or } p = j \\
1 - q^{-1} & \text{if } i \neq p \neq j
\end{cases}
\]

#### Proposition 3.8

For diagrams of type \( A_2 \) the Hopf subalgebra \( K \subset R \) is a \( K \)-bimodule coalgebra retract, with retraction \( u_\infty = u_2 : R \to K \).

**Proof.** It will be necessary to deform the \( K \)-bimodule retraction \( u = (\varepsilon \otimes 1) \vartheta : R \otimes K \) somewhat to make it a coalgebra map in this case. Observe that \( u_1 = \varepsilon \otimes 1 : B_1 \otimes K \to K \) is a \( K \)-module coalgebra map. The following arguments show that its extension \( u = \varepsilon \otimes 1 : B \otimes K \to K \) is not a coalgebra map. In \( R \# kG \) we get

\[
\Delta(e_{12}^m e_{13}^n e_{23}^l) = \sum_{1 \leq p_1 \leq 2, 1 \leq p_2 \leq 3, 2 \leq r_x \leq 3} \lambda_{1p_12} \ldots \lambda_{1p_2m} \lambda_{1q_13} \ldots \lambda_{1q_n3} \lambda_{2r_13} \ldots \lambda_{2r_33}
\]

which contains the term \( \lambda_{123}^{n} \chi_{12}^{(2)} (g_{23}) e_{12}^m e_{13}^{n+l} e_{23}^l \), the only term that may make \((u \otimes u)\Delta(e_{12}^m e_{13}^n e_{23}^l) \neq 0 \). In particular, if \( m + n = N + n + l = m = l \) and this term

\[
\lambda_{123}^{n} \chi_{12}^{(2)} (g_{23}) e_{12}^N e_{13}^N e_{23}^N
\]
is a non-zero element in $(K \# kG) \otimes (K \# kG)$. It follows directly that
\[
(u \otimes u)\Delta(e_{12}^m e_{13}^n e_{23}^l) = u(e_{12}^m e_{13}^n e_{23}^l) \otimes 1 + g_{12}^m g_{13}^n g_{23}^l \otimes u(e_{12}^m e_{13}^n e_{23}^l)
+ \lambda_{123} \chi_{(12)}^N (g_{23}^l) u(e_{12}^m g_{23}^n) \otimes u(e_{23}^l)
\]
for $0 \leq m, n, l < N$. In particular, $(u \otimes u)\Delta(e_{12}^m e_{13}^n e_{23}^l) \neq 0$ if and only if $m + n = N = n + l$, and then
\[
(u \otimes u)\Delta(e_{12}^N e_{13}^N e_{23}^N) = \lambda_{123} \chi_{12}^N (g_{23}^l) z_{12} h_{23} \otimes z_{23}
\]
while $u(e_{12}^m e_{13}^n e_{23}^l) = 0$. The $K$-bimodule retraction $u : R \to K$ defined by $u(x^n z^b) = \varepsilon(x^n)z^b$ is therefore not a coalgebra map. To remedy this situation, observe that
\[
\Delta(z_{13}) = z_{13} \otimes 1 + h_{13} \otimes z_{13} + (1 - q^{-1}) N \chi_{12}^N (g_{23}) z_{12} h_{23} \otimes z_{23}
\]
and define $u_2 : R \to K$ by
\[
u_2(e_{12}^m e_{13}^n e_{23}^l z) = \delta_{1}^{m+n} \chi_{12}^N (1 - q^{-1})_{N} \chi_{12}^N (g_{23}) z_{12} h_{23} \otimes z_{23}
\]
for $z \in K$. Observe that $\nu_2 = m_K(\varphi_2 \otimes 1) \partial : R \to B \otimes K \to K \otimes K \to K$, where $\varphi_2 : B \to K$ is given by
\[
\varphi_2(e_{12}^m e_{13}^n e_{23}^l) = (1 - q^{-1})_{m} \chi_{12}^N (g_{23}) z_{12} h_{23} \otimes z_{23}
\]
with $t = \min(m, l)$. It then follows by construction that $u_\infty = u_2 : R \to K$ is a $K$-bimodule coalgebra retraction for $\kappa : K \to R$.

The connecting map $\delta : \text{Alg}_G(K, k) \to \mathcal{H}_G^2(B, k)$ guaranteed by Theorem 2.1 is injective by Proposition 3.7 since $\text{Alg}_G(R, k) = \{\varepsilon\}$ and since all elements of $\text{im} \partial^0$ commute with those of $\text{im} \partial^2$. The resulting cocycle deformations account for all liftings of $B \# kG$.

Results for type $A_n$, $n > 2$, and type $B_2$ are in the pipeline. They will be a subject of a forthcoming paper.

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