I. Introduction

Precursor shock waves are physically observable, causal discontinuities in classical and quantum fields propagating through regions of relativistic spacetime. By this is meant a phenomenon such that, before a certain time $t_0$ in some laboratory or equivalent, an array of signal detectors is in its no-signal state, but after that time, geometrically recognizable patterns of signals will be found to have been triggered in that array.

This subject has become topical on account of recent reports of the observation of gravitational shock waves by the LIGO and Virgo science collaborations [1], but it has a long and important history. The advent of special relativity raised a fundamental question: are Maxwell’s equations consistent with the lightcone veto on superluminal signal propagation? The issue was tackled in 1914 by Sommerfeld and subsequently by Brillouin [8]. This approach stands in contrast with conventional explicit methods, such as that of Sommerfeld and Brillouin and to the numerical simulation of relativistic shock waves [12].

It is generally assumed that all physical effects propagate at speeds limited by the light cone structure of relativistic spacetime, but important questions remain to be answered. For instance, two important speeds are generally discussed in quantum wave mechanics. One is the phase velocity $w$ conventionally associated with de Broglie waves and the other is the particle speed $v$, frequently referred to as group velocity. These speeds satisfy the de Broglie relation $w = c^2$, where $c$ is the speed of light. But for massive particles, neither of these speeds is equal to $c$ and since $v$ has to be less than $c$ for reasons of causality, we deduce that $w$ is superluminal. The conventional interpretation of this is that $w$ is associated with correlations. Two correlated events can be observed on a hyperplane of simultaneity (even in Newtonian mechanics), thereby giving the impression of superluminal speeds [16]. However, that does not prove any causal connection, even in classical physics. Correlations have everything to do with the context of observation, including how the correlated events were set up in the first place. Other questions concern the causality structure of the Feynman propagator in relativistic quantum field theory, and the propagation of higher spin fields, such as the Rarita-Schwinger field [13], as these appear to involve superluminal speeds [23]. We shall comment on all of these issues.

In this article we focus on a third speed involved in field theory, that is, the speed of propagation of shock waves.

---

1 Actually a speed, but the term phase velocity sounds better.
Any analysis of shock waves requires a careful interplay between reductionist and emergent concepts. On the one hand, field equations are generally derived from the reductionist principles of Lagrangian mechanics. On the other hand, shock waves are large scale, emergent processes highly sensitive to the details of those field equations, the non-local initial conditions setting off those shock waves, the laws of causality, and the protocols of observation.

In the standard approach to field theory, the mathematics is usually discussed from the perspective of an exophysical observer standing outside of some region of spacetime, monitoring the behavior of a system under observation in that region. The observer usually enters the picture in only two places: the first is where they initialize the equations of motion describing the system and the second is then when they observe the final state of that system. Certainly that is the way quantum theory is normally discussed when it is applied to scattering processes. In any discussion of shock waves, however, the situation becomes more complicated. Now the role of the observer becomes more intermingled with the dynamical evolution of the system under observation, requiring more care and detail in the analysis.

Throughout this paper, the term suitably arbitrary means arbitrary provided certain conditions such as differentiability are met. We set \( c = \hbar = 1 \) and work in a standard Minkowski spacetime inertial frame with metric tensor components \((1, -1, -1, -1)\) down the main diagonal and zero everywhere else.

In the next section we discuss a simplified model that serves as a template for all discussions in this paper.

II. First order linear PDE

In \(1 + 3\) spacetime and relative to an inertial frame with coordinates \((x^0 \equiv t, \mathbf{x})\), consider the field equation

\[
i \dot{\phi} + i \mathbf{a} \cdot \nabla \phi - m \phi = 0,
\]

where \( \phi \) is a real or complex scalar field, \( \mathbf{a} \) is a non-zero, constant real 3-vector, \( m \) is a real constant, \( \phi \equiv \partial \phi(t, \mathbf{x})/\partial t, \) and \( = \) denotes an equality holding only for solutions to (1). We investigate the possibility of finding shock wave solutions to equation (1) in five nominally different approaches. The first three approaches require us to solve the equation in one way or another. The merit of the fourth approach, which is based on the work of Fock [8] and on Kemmer’s notation [11], is that it is easier in this respect: we do not need to solve the differential equation but draw our conclusions based on the structure of the differential equations themselves and on the logic of observation and causality as it applies to shock waves. This approach is similar in spirit to standard discussions of characteristics given in [2] and applied by [22]. Towards the end of this paper we shall introduce a fifth approach, based on distribution theory, that justifies the heuristic approach of Fock.

A. The Fourier transform approach

Fourier transforming equation (1) with respect to the spatial coordinates, solving the transformed equation in transform space, and then inverting back gives the general solution

\[
\phi(t, \mathbf{x}) = e^{-imt}/c \Phi(x - at),
\]

where \( \Phi \) is a suitably arbitrary function of the variable \( z \equiv x - at \). Discontinuities can be embedded in the shape function \( \Phi \). For example, a typical plane wave tsunami solution is

\[
\phi(t, \mathbf{x}) = \delta(t - \mathbf{n} \cdot \mathbf{x}),
\]

where \( \mathbf{n} \equiv b / (a \cdot b) \) for any vector \( b \) such that \( a \cdot b \neq 0 \), \( \delta \) is suitably arbitrary, and \( \theta \) is the Heaviside step function.

B. The method of characteristics

In this approach we first rewrite (1) in matrix form:

\[
i [\{1, a^T\} \left( \frac{\partial}{\partial \mathbf{x'}} \right) \phi(t, \mathbf{x}) - m \phi(t, \mathbf{x}) = 0,
\]

where superscript \( T \) denotes transpose. Next, we make the passive linear-inhomogeneous coordinate transformation

\[
\begin{bmatrix} t' \\ \mathbf{x}' \end{bmatrix} = \begin{bmatrix} \alpha & \beta^T \\ \gamma & \delta I_3 \end{bmatrix} \begin{bmatrix} t \\ \mathbf{x} \end{bmatrix} + \begin{bmatrix} s \\ \mathbf{r} \end{bmatrix},
\]

where \( \alpha, \delta \) and \( s \) are real constants, \( \beta, \gamma \) and \( \mathbf{r} \) are real column three-vectors, and \( I_3 \) is the \( 3 \times 3 \) identity matrix. This transformation is invertible provided \((\alpha \delta - \beta \gamma) \delta^2 \neq 0\), which we assume. Given that \( \phi \) is a scalar field and defining \( \phi'(t', \mathbf{x}') \equiv \phi(t, \mathbf{x}) \), (3) becomes

\[
i \{(\alpha + \mathbf{a} \cdot \beta) \partial_t + (\gamma + \delta \mathbf{a}) \cdot \nabla'\} \phi'(t', \mathbf{x}') - m \phi'(t', \mathbf{x}') = 0.
\]

We now take advantage of the fact that the various constants in transformation (1) are suitably arbitrary. We choose to set

\[
\gamma + \delta \mathbf{a} = 0,
\]

and then (5) becomes

\[
i (\alpha + \mathbf{a} \cdot \beta) \partial_t \phi'(t', \mathbf{x}') - m \phi'(t', \mathbf{x}') = 0.
\]

Assuming \( \alpha + \mathbf{a} \cdot \beta \neq 0 \), the general solution to (8) is

\[
\phi'(t', \mathbf{x}') = U(\mathbf{x}') \exp \left\{ -\frac{imt'}{\alpha + \mathbf{a} \cdot \beta} \right\},
\]

where \( U \) is suitably arbitrary. Transforming back to the original coordinates and using (11) we get

\[
\phi(t, \mathbf{x}) = U(-\delta \mathbf{a}t + \delta \mathbf{x} + \mathbf{r}) \exp \left\{ -\frac{im(\alpha + \beta \cdot \mathbf{x} + s)}{\alpha + \mathbf{a} \cdot \beta} \right\},
\]

which is equivalent to (2), the solution found using the Fourier transform method.
C. The Schwinger-Pauli-Jordan function method

This is perhaps the most powerful method in standard free field theory, as it explicitly solves the initial value problem in Lorentz-signature spacetimes (critical to a satisfactory physical interpretation of what is going on) as well as explicitly revealing the causal singularity structure that ultimately underpins the propagation of shock waves.

Solutions to (11) are assumed to have the form

\[
\phi(t, x) = \int \frac{d^3y}{c} G^{(+)}(t, x - y) \eta(y)
\]  

(10)

for \( t > 0 \). Here \( \{\eta(y) : y \in \mathbb{R}^3\} \) represents the initial data, that is, the field values distributed over the spacelike hypersurface at initial laboratory time \( t = 0 \). Taking into account the fact that the observer necessarily exists before the shock wave is initiated, the Schwinger-Pauli-Jordan (SPJ) function \( G^{(+)} \) is taken here to be a distribution over the spacetime \( (-\infty, \infty) \times \mathbb{R}^3 \) with the following properties:

1. \( G^{(+)}(t, x) = 0, \quad t < 0; \)
2. \( (i\partial_t + a \cdot \nabla_x - m) G^{(+)}(t, x) = 0, \quad t > 0, \)
3. \( \lim_{t \to t_0^+} G^{(+)}(t, x) = \delta^3(x). \)

Given these conditions, the SPJ function is readily found to be

\[
G^{(+)}(t, x) = \theta(t)e^{-imt} \delta^3(at - x),
\]  

(11)

ignoring any inessential \( \delta(t) \) contribution. The interpretation of this solution is that it encodes the shock wave that would be propagated throughout future spacetime from a point event disturbance at the origin of space and time coordinates. The Heaviside function has been inserted here by hand to reinforce the classical causality condition that the field \( \phi \) cannot exist before it is created at initial time \( t_0 \). As a distribution over all spacetime, \( G^{(+)} \) does not satisfy the original homogeneous equation of motion but does satisfy the inhomogeneous equation

\[
(i\partial_t + a \cdot \nabla_x - m) G^{(+)}(t, x) = i\delta(t) \delta^3(at - x),
\]  

(12)

reflecting the creation of a point source at time zero.

Since the original wave equation (11) is linear, shock waves from different point sources would not interact with each other, but would superpose. Therefore, the combined effect of a collection of such events distributed over some spacelike hypersurface is given by integrals such as (10).

This also applies if for instance the initial shock wave is generated in some finite four-dimensional region \( V \) of spacetime. Assuming no dynamical interaction between fields created at different times, then the general solution outside of this region will be given by

\[
\phi(t, x) = \int_V dt_0 d^3y G^{(+)}(t - t_0, x - y) \phi(t_0, y),
\]  

(13)

where \( \phi(t_0, y) \) represents a spacetime density of source events and the Heaviside function in \( G^{(+)} \) ensures classical causality is obeyed at all times. By this we mean that in this scenario, every point source event can influence events only in its own relative future. A similar, implicit assumption is made in Schwinger’s source theory [18].

Inside the region \( V \), the field \( \phi(t, x) \) satisfies the inhomogeneous equation

\[
(i\partial_t + a \cdot \nabla_x - m) \phi(t, x) = i\phi(t, x),
\]  

(14)

which could be used to model the creation of a shock wave.

D. The Fock-Kemmer approach

The Fock-Kemmer approach to shock wave analysis is useful and economical because it does not require any solution per se of the differential equations involved for conclusions about shock waves to be reached. Before we can discuss the method, however, we need to introduce the concepts of Fock subsurface, Fock flow, subsurface normal velocity, and Kemmer bracket.

In the following, we assume we are an exophysical observer looking in over a region \( \mathcal{R} \) of 1 + 3 dimensional spacetime, using a coordinate patch \( P(t, x) \) covering \( \mathcal{R} \), such that the coordinate \( t \in [0, T] \) represents observer time indexing a spacelike foliation of \( \mathcal{R} \).

1. Fock subsurfaces

A Fock subsurface \( F_t \) at time \( t \) is the set of points in \( \mathcal{R} \) satisfying the condition

\[
F_t \equiv \{ x : F(x) = t, (t, x) \in \mathcal{R} \},
\]  

(15)

where \( F \) is some differentiable function of spatial coordinates only. Fock subsurface functions are in general defined contextually by observers, such as when torches and particle beams are switched on, or by natural causes such as underwater avalanches or the collision of two black holes as recently reported [1]. For example, in Newtonian space-time, a spherical pulse of light generated at the origin of space-time coordinates is subsequently distributed over a Fock surface defined by \( r = t \).

2. Fock flows

A Fock flow \( \mathcal{F}[F] \) is a family of Fock subsurfaces in \( \mathcal{R} \) indexed by the observer’s time \( t \), that is, a family of two-dimensional surfaces defined by the set of equations

\[
\mathcal{F}[F] \equiv \{ F_t : t \in [t_i, t_f] \},
\]  

(16)

where \( F \) is a Fock subsurface function and \( t_i < t_f \).

3. Subsurface normal velocity

The gradient \( \nabla F_P \) at a point \( P \) on a Fock surface \( F_t \) denotes the usual set of Cartesian spatial coordinate partial derivatives of \( F \) evaluated at \( P \). Given a Fock flow \( \mathcal{F}[F] \), by considering a point \( P \) on the Fock subsurface \( F_t \), projecting that point normally to that subsurface so
as to intersect the Fock subsurface \( F_{t+\delta t} \), and then taking the appropriate limit \( \delta t \to 0 \), it is straightforward to establish that the “velocity” \( w_P \) at a point \( P \) on a given Fock subsurface is given by

\[
\mathbf{w}_P = (\nabla F_P)^{-2} \nabla F_P.
\]  

This requires the gradient \( \nabla F \) not to vanish at \( P \). This velocity will be referred to as the subsurface normal velocity at \( P \). Its magnitude is the subsurface normal speed \( w_P \) and is given by

\[
w_P = |\nabla F|^{-1}.
\]  

4. Kemmer brackets

Kemmer brackets were introduced \[11\] as a powerful notational way to discuss Fock’s shock wave analysis \[8\]. Given a propagating field \( \phi \) and a Fock flow \( \mathcal{F}[F] \), the Kemmer bracket \( [\phi]_F \) of the field \( \phi \) relative to \( F \) is defined by

\[
[\phi]_F(x) \equiv \phi(F(x), x).
\]  

A Kemmer bracket is a function of spatial coordinates only. Here and elsewhere we shall make extensive use of the Fock-Kemmer identity

\[
\nabla [\phi]_F = \nabla F [\phi]_F + |\nabla \phi|^F,
\]  

where \( [\phi]_F(x) \equiv \partial_t \phi(t, x)|_{t=F(x)} \) and \( \nabla [\phi]_F(x) \equiv \nabla \phi(t, x)|_{t=F(x)} \). We may apply the Fock-Kemmer identity to derivatives of the field \( \phi \), giving for example

\[
\nabla [\phi]_F = [\phi]_F \nabla F + |\nabla \phi|^F,
\]  

and so on.

5. Application to equation (1)

Considering the vector \( \mathbf{a} \) in the original equation of motion (1), the Fock-Kemmer identity gives

\[
\mathbf{a} \cdot \nabla [\phi]_F = \mathbf{a} \cdot \nabla F [\phi]_F + [\mathbf{a} \cdot \nabla \phi]^F.
\]  

On the other hand, applying the Kemmer bracket to the equation of motion (1) directly gives

\[
i[\dot{\phi}]_F + i[\mathbf{a} \cdot \nabla \phi]^F - m[\phi]^F = 0.
\]  

Using (22) in (21) then gives

\[
(\mathbf{a} \cdot \nabla F - 1)[\dot{\phi}]_F = \mathbf{a} \cdot \nabla [\phi]_F + im[\phi]^F.
\]  

It is straightforward to verify that on the Fock subsurface \( F(x) = t \), the solution (23) satisfies (24), where now

\[
[\phi]^F(x) \equiv e^{-i \mathcal{F}[F(x)]} \Phi(x - \mathbf{a} F(x)).
\]  

6. The Fock shock wave condition

In the above, the Fock subsurface \( F \) function is suitably arbitrary. Now consider a specific choice, written \( F = W \), representing a shock wave of discontinuity. Fock’s heuristic argument \[8\] is that on such a shock wave, it should not be possible to work out the Kemmer bracket of \( [\phi]^W \) from a knowledge of \( [\phi]^W \) or its derivatives such as \( \mathbf{a} \cdot \nabla [\phi]^W \). The constructs \( [\phi]^W \) and \( \mathbf{a} \cdot \nabla [\phi]^W \) depend on initial data available in principle to the observer whilst \( [\phi]^W \) represents data that is causally unavailable. We shall call this chain of reasoning Fock’s argument. Our distribution theory approach in VIII fully justifies Fock’s heuristic argument.

Given the Fock argument, then the conclusion from (23) is that the coefficient of \( [\mathbf{a} \cdot \nabla \phi]^W \) on the left-hand side of (23) must vanish when \( F = W \), that is, on a surface of discontinuity. We deduce that a shock wave must satisfy the equation

\[
\mathbf{a} \cdot \nabla W = 1.
\]  

This also means that the right-hand side of (23) must vanish on such a surface also, giving the condition

\[
\mathbf{a} \cdot \nabla [\phi]^W + im[\phi]^W = 0.
\]  

It is readily confirmed that (24) does indeed satisfy (25) when \( W \) satisfies the shock wave condition (25).

7. Interpretation

To get some understanding of these results, we can without loss of generality take \( \mathbf{a} = (a, 0, 0) \) where \( a > 0 \). Then (25) reduces to

\[
a \partial_x W(x, y, z) = 1.
\]  

This equation has general solution

\[
W(x, y, z) = \frac{x}{a} + U(y, z), \quad a \neq 0,
\]  

where \( U \) is suitably arbitrary. Assuming the solution is of the form (24) we have

\[
[\phi]^W(x) = e^{-imx/a} \Phi(U(y, z), -y, -z).
\]  

Then we readily find that condition (26) is indeed satisfied.

The subsurface normal speed \( w(x) \) of a shock wave \( W(x) = t \) is given by \( w(x) = |\nabla W|^{-1} \). From (28) the subsurface normal speed is found to be

\[
w(x) = \frac{a}{\sqrt{1 + a^2 U_0^2 + a^2 U_z^2}}.
\]  

The following clarifies the shock wave geometry and kinematics relevant to equation (1). First, using (28), we write the shock wave in the form \( x = at - aU(y, z) \). At initial time \( t = 0 \), the shock wave surface is given by \( x = -aU(y, z) \). Subsequently, this surface moves uniformly in the positive \( x \)-direction with speed \( a \) in that direction. This motion is not generally perpendicular to the shock wave surface at all points, and (30) shows that the speed of the shock wave in the \( x \) direction is generally greater than the surface normal speed.
Because of their contextuality, shock waves require some care in their specification. For instance, it is not enough to define a two-dimensional Fock subsurface in three-dimensional space and think of it as a shock wave of discontinuity. We need to specify the direction of motion of this surface as well, because the Fock flow $\mathcal{F}^{(+)}[F]$ defined by $F(x) = t$ models Fock subsurfaces moving in the opposite direction to those belonging to the Fock flow $\mathcal{F}^{(-)}[F]$ defined by $F(x) = -t$.

In the real world, irreversibility is ubiquitous: a given Fock shock flow $\mathcal{F}^{(+)}[W]$ may be physically observable, such as an incoming photon or neutrino shock wave sent out from some approximate point source such as an exploding star, whilst its theoretical counterpart $\mathcal{F}^{(-)}[W]$ represents an incoming sphere of radiation that would never be seen naturally. This reinforces our earlier comments that shock waves are essentially emergent phenomena.

### III. Application to Maxwell’s equations in vacuo

The Fock-Kemmer analysis can be extended naturally to electromagnetic wave theory. In this section we consider the situation of free charges in vacuo. We shall treat the critical case of electromagnetic shock wave propagation in a polarizable and magnetizable medium in a later section.

In vacuo, Maxwell’s equations can be written in the form

$$
\begin{align*}
\nabla \times B - \dot{E} &= j_f, & \nabla \cdot E &= \rho_f, \\
\nabla \times E + \dot{B} &= 0, & \nabla \cdot B &= 0,
\end{align*}
$$

where $\dot{E} \equiv \partial E/\partial t$, etc., and $j_f$ and $\rho_f$ are the free charge current and charge densities respectively. To deal with the fact that these equations lead to second order wave equations, we are led to define the six component Maxwell bi-field $\Phi$, the bi-current density $J$, and the bi-charge density $\Omega$ by

$$
\Phi \equiv \begin{bmatrix} E \\ B \end{bmatrix}, \quad J \equiv \begin{bmatrix} j_f \\ 0 \end{bmatrix}, \quad \Omega \equiv \begin{bmatrix} \rho_f \\ 0 \end{bmatrix}, \tag{32}
$$

noting that the components of $\Phi$ and $J$ are vectorial while those of $\Omega$ are scalar in nature. Six-component fields such as $\Phi$ and $J$ are denoted with (square) brackets whilst two-component fields such as $\Omega$ are denoted with (round) parentheses.

We define derivatives as

$$
\nabla \times \Phi \equiv \begin{bmatrix} \nabla \times E \\ \nabla \times B \end{bmatrix}, \quad \nabla \cdot \Phi \equiv \begin{bmatrix} \nabla \cdot E \\ \nabla \cdot B \end{bmatrix}, \quad \dot{\Phi} \equiv \begin{bmatrix} \dot{E} \\ \dot{B} \end{bmatrix}, \tag{33}
$$

noting that the ‘divergence operator’ acting on a six-component field returns a two-component field. Then Maxwell’s equations (31) can be written in the form

$$
\Phi + S \nabla \times \Phi = -J, \quad \nabla \cdot \Phi = \Omega, \tag{34}
$$

where

$$
S \equiv \begin{bmatrix} 0 \ 3 & -I_3 \\
I_3 & 0 \ 3 \end{bmatrix}, \tag{35}
$$

$I_3$ being the $3 \times 3$ identity matrix and $O_3$ the $3 \times 3$ zero matrix. Equations (34) give

$$
\nabla \cdot \left( \Phi + S \nabla \times \Phi \right) = -\nabla \cdot \dot{\Phi} = -\nabla \cdot J, \quad \nabla \cdot \dot{\Phi} = \dot{\Omega}, \tag{36}
$$

and so we must have $\nabla \cdot J + \dot{\Omega} = 0$, which is equivalent to the charge continuity equation $\partial_t \rho_f + \nabla \cdot j_f = 0$.

**A. The Kemmer bracket of the Maxwell bi-field**

Taking the Kemmer bracket across (33), we have

$$
\left[ \Phi \right]^F + S \left( \nabla \times \Phi \right)^F = -\left[ J \right]^F, \quad \left[ \nabla \cdot \Phi \right]^F = \left[ \Omega \right]^F. \tag{37}
$$

Applying the Fock-Kemmer identity $\left[ \nabla \times \Phi \right]^F = \nabla \times \left[ \Phi \right]^F$ to the first equation in (37) then gives

$$
\{ I_6 - S \nabla F \times \} \left[ \Phi \right]^F = -S \nabla \times \left[ \Phi \right]^F - \left[ J \right]^F. \tag{38}
$$

where $I_6$ is the $6 \times 6$ identity matrix and $S \nabla F \times$ is a $6 \times 6$ antisymmetric matrix with components linearly dependent on the components of $\nabla F$.

We now apply the Fock argument. In (38), we should be able to know everything on the right-hand side, even in the case of a shock wave, $F = W$. This would then allow us to determine $\left[ \Phi \right] W$, which is forbidden by Fock’s argument, unless the shock wave function $W$ satisfies the condition

$$
\det \{ I_6 - S \nabla W \times \} = 0. \tag{39}
$$

We readily find

$$
(\nabla W)^2 = 1, \tag{40}
$$

which is precisely what we expect from special relativity ($c = 1$ in this section). We note that this condition is independent of any electric charges in the system.

### IV. The charged Dirac equation

The charged Dirac equation in external electromagnetic fields is given by

$$
i\gamma^\mu D_\mu \psi - m\psi = 0 \tag{41}
$$

in standard notation, where the $\gamma^\mu$ are the Dirac matrices (3) and $D_\mu \equiv \partial_\mu + ieA_\mu$ are the gauge covariant derivatives, with $A_\mu$ the components of the electromagnetic potential one-form. Separating spatial and temporal components and taking Kemmer brackets with respect to a Fock flow $\mathcal{F}[F]$ gives

$$
\gamma^0 \left[ D_0 \psi \right]^F = -\gamma^i \left[ D_i \psi \right]^F - im \left[ \psi \right]^F. \tag{42}
$$
Using the Fock-Kemmer identity \(^{20}\), we can show that
\[ [D_j \psi]^F = \partial_j [\psi]^F - \partial_j F [D_0 \psi]^F + i e (\partial_j FA_0 + A_j) [\psi]^F. \]
(43)

Using this in \(^{12}\) then gives
\[ (\gamma^0 - \gamma^i \partial_i F) [D_0 \psi]^F = \frac{-i e c \gamma^i \partial_i [\psi]^F}{c} = \frac{-i e c \gamma^1 ([\partial_1 FA_0 + A_1) [\psi]^F}{c} - \frac{im c [\psi]^F}{c}. \]
(44)

On a shock wave, Fock’s argument then gives the wavefront condition
\[ \det \{ \gamma^0 - \gamma^i \partial_i W \} = 0. \]
(45)

Using the standard representation of the Dirac matrices \(^3\), \(^{13}\), then gives \(^{10}\), that is, exactly the same condition as found for the electromagnetic field. Note that a veto on knowing \([D_0 \psi]^W\) is equivalent to a veto on knowing \([\psi]^W\), since we assume we can always determine \([A_0 \psi]^W\).

We comment here on the conventional observation that Dirac quantum fields do not commute at spacelike separations, leading to concern regarding causality. We have four points to make about this concern.

1) The conventional view is that Dirac fields are not observables but certain bilinear combinations of them are, such as the charge four-current operator, and these observables do commute at spacelike separations.

2) The Jordan-Wigner construction of fermion fields \(^3\) is manifestly non-local, supporting the view of Schwinger that ‘The mathematical machinery of quantum mechanics is a symbolic expression of the laws of atomic measurement, abstracted from the specific properties of individual techniques of measurement.’ \(^{17}\). This means that with fermions, there is implicit contextuality that may induce superluminal correlations. That does not imply superluminal signalling.

3) The SPJ function for the free Dirac field does indeed have a lightcone cutoff \(^3\), which guarantees no superluminal transmission of free field shock waves.

4) Somewhat surprisingly and perhaps disturbing, the conventional Feynman propagator does not have a lightcone cutoff. However, that propagator is used in conventional LSZ formalism scattering calculations \(^3\) based on remote past (limit of time tending to \(-\infty\)) \(m\) states propagating to remote future (limit of time tending to \(+\infty\)) \(m\) states. In between state preparation and outcome detection is the regime we call the information void, where no signal detection takes place. In this regime and in the absence of any signal detection, standard causality rules do not apply. The rules of quantum mechanical path integrals allow (indeed require) all dynamically possible intermediate processes to be taken into consideration, including acausal ones. The only thing that matters empirically is what the signal detectors register, not the imagined behaviour of the fields in the information void. We return to this point in \(^{\text{VIII}}\).

V. The free Klein-Gordon equation

As a second-order differential equation, the free particle Klein-Gordon equation (KGE)
\[ \ddot{\psi} - \nabla^2 \psi + m^2 \psi = 0 \]
(46)
presents an addition layer of structure that can be circumvented by suitable redefinition of variables. We take our cues from three places: i) Petiau \(^{14}\), Duffin \(^3\) and Kemmer \(^{10}\) discussed linearized approaches to the KGE along the lines of the Dirac equation; ii) the Dirac equation can be readily discussed in Kemmer bracket terms, and iii) the electromagnetic fields obey second order differential equations, but our linearization approach above gave us the required shock wave condition straightforwardly.

Our approach in this section is to introduce four extra auxiliary variables: \(\sigma \equiv \dot{\phi} \) and \(\eta_i \equiv \partial_i \phi, \ i = 1, 2, 3\), and define the five-component field \(\Phi\) by
\[ \Phi^T \equiv (\phi, \sigma, \eta_1, \eta_2, \eta_3), \]
(47)
where superscript \(T\) denotes transpose. Then \(^{46}\) can be written in the form
\[ \dot{\Phi}_A = \Gamma^i_{AB} \partial_i \Phi_B + K_{AB} \Phi_B, \]
(48)
where capital Latin indices run from 1 to 5, small Latin indices run from 1 to 3, and the constant matrices \(\Gamma_{AB}, K_{AB}\) can be readily determined from \(^{46}\) and \(^{48}\).

We find for example
\[ \Gamma^1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \text{ etc., and } K = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -m^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \]

We note that in order to recover the original KGE \(^{46}\) from \(^{48}\) we need the auxiliary equation \(\eta = \nabla \phi\).

The Fock shock wave condition for the KGE is found as before. Taking Kemmer brackets on both sides of \(^{48}\) with respect to an arbitrary Fock flow \(F[F]\) and applying the Fock-Kemmer identity, we readily deduce that
\[ (I_5 + \partial_i F \Gamma^i_5) [\Phi]^F = \Gamma^i_5 \partial_i [\Phi]^F + K [\Phi]^F, \]
(50)
where \(I_5\) is the \(5 \times 5\) identity matrix. On a shock wave \(F = W\), Fock’s argument then leads to the condition
\[ \det (\partial_i W T^i_5 + I_5) = 0, \]
(51)
or else we would be able to determine \([\Phi]^W\) from a knowledge of \([\Phi]^W\). Condition \(^{51}\) gives \((\nabla W)^2 = 1\), which is exactly the same as that found for the Dirac and Maxwell fields.
VI. The charged Klein-Gordon equation

The wave equation for a charged scalar particle in external electromagnetic potentials is given by

\[ D_\mu D^\mu \varphi + m^2 \varphi = 0, \]  

(52)

where \( D_\mu \equiv \partial_\mu + ieA_\mu \) is the same gauge covariant operator used in the charged Dirac equation discussed above.

Our approach is a synthesis of the methods used for the charged Dirac equation and the free Klein-Gordon equation above. First we define the fields

\[ \sigma \equiv D_0 \varphi = (\partial_t + ieA_0) \varphi, \quad \eta_i \equiv D_i \varphi, \quad i = 1, 2, 3. \]  

(53)

Then we find

\[ D_0 \eta_i = ieE^i \varphi + D_i \sigma, \]  

(54)

where \( E^i \equiv \partial_i A_1 - \partial_1 A_i \) is the electric field.

Next, we define the five component object \( \Phi \) as before, that is

\[ \Phi^F \equiv (\varphi, \sigma, \eta_1, \eta_2, \eta_3). \]  

(55)

Then \( \Phi \) satisfies the equation

\[ D_0 \Phi_A = \Gamma_{AB} D_i \Phi_B + \bar{K}_{AB} \Phi_B, \]  

(56)

where the \( \Gamma^i \) are as before but

\[ \bar{K} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -m^2 & 0 & 0 & 0 & 0 \\ ieE^1 & 0 & 0 & 0 & 0 \\ ieE^2 & 0 & 0 & 0 & 0 \\ ieE^3 & 0 & 0 & 0 & 0 \end{pmatrix}. \]  

(57)

Taking Kemmer brackets with respect to a Fock flow \( F \) we arrive at the relation

\[ (I_5 + \Gamma^i \partial_i F)|D_0 \Phi|^F = \frac{c}{c} \Gamma^i \partial_i [\Phi]^F + [\bar{K}] \Phi^F + i\epsilon \Gamma^i ([\partial_i F, A_0 + A_j] \Phi)^F. \]  

(58)

Applying Fock’s argument leads to the same condition

\[ \det(I_5 + \Gamma^i \partial_i W) = 0 \]  

as for the free Klein-Gordon field, consistent with the expected lightcone condition \((\nabla W)^2 = 1\).

VII. The Rarita-Schwinger equation

The success of the Standard Model is based on spin zero, spin half, and spin one fields. Particles associated with each such spin have been observed. The Rarita-Schwinger equation (RSE) was proposed as a model for spin three-halves particle fields. Such a field will have the vectorial characteristics of a spin one field and the spinorial characteristics of a spin half field. We shall denote such a field by \( \psi_\mu \), the spinorial index being understood.

In line with our comments on the Dirac equation above, we should expect an RS field not to be an observable per se. However, we did find that Dirac field and electromagnetic field shock waves obey special relativistic causality rules, so it is natural to see what happens in the case of RS fields. We shall look at the free RSE, on the grounds that if that gives superluminal shock waves, then we should not be surprised to find no stable RS particles in nature.

With a lack of empirical evidence to guide us in choice of equation for the RSE, we choose to work with the following RSE equation:

\[ (i\gamma^i \partial_i - m)\psi_\mu = 0, \quad \mu = 0, 1, 2, 3, \]  

(59)

supplemented by the constraint equations

\[ \gamma^\mu \psi_\mu = 0, \quad \partial^\mu \psi_\mu = 0. \]  

(60)

If we did not have these constraint equations, then we could use the same approach as we applied to the Dirac equation above to prove immediately that the RSE field does indeed satisfy lightcone causality. Taking the Kemmer bracket across equation (60) gives

\[ \gamma^0 [\psi_\mu]^F + \gamma^1 [\partial_1 \psi_\mu]^F = -im[\psi_\mu]^F. \]  

(61)

The Fock-Kemmer identity applied to \( \psi_\mu \) and then used in (61) gives

\[ \{\gamma^0 - \gamma^1 \partial_1 F\} [\psi_\mu]^F = -\gamma^1 [\partial_1 \psi_\mu]^F - im[\psi_\mu]^F. \]  

(62)

The Fock argument then gives us condition (45) for a shock wave, exactly as for the Dirac equation.

However, this does not prove that such shock waves can be constructed: the constraints (60) may make this impossible. Our resolution of the causality issues with the RSE equation is therefore the statement that if shock waves occurred with such fields, lightcone causality would necessarily be maintained. That does not prove that such shock waves could be constructed consistent with the constraints. Whatever the possibility of such construction, superluminal propagation of spin 3/2 particles is ruled out.

VIII. Distributional field approach

Anticipating our discussion below on electromagnetically polarizable and magnetizable media and motivated by a desire to see the Fock analysis in more than heuristic terms, we introduce an approach based on Fock’s ideas, but now explicitly incorporating the theory of distributions (generalized functions) and test functions. We give a brief review of relevant distribution concepts and our notation in the Appendix.

Because of its discontinuity and singularity structure, a shock wave is best not regarded as a smooth function but as a distributional-valued field. Doing this gives some mathematical justification for Fock’s argument. We shall apply distribution methods to several situations, the first being to revisit the first order equation discussed in (51).
A. First order equation revisited

Given equation (1), we make the shock wave ansatz
\[ \phi \equiv f D \theta_W + g D \delta_W, \] (63)
where \( \equiv \) denotes distributional equality, discussed in (A).

More explicitly, we take \( \phi \) to be a distribution-valued field of the form
\[ \phi(t, x) \equiv f(t, x) \theta(t - W(x)) + g(t, x) \delta(t - W(x)), \] (64)
where \( f \) and \( g \) are test functions, \( W \) is a Fock shock wave function, and \([g]^W \neq 0\). Then we find the derivatives
\[ \phi = f \delta_W + (f + \dot{g}) \delta_W + g \delta^{[1]}_W, \]
\[ a \cdot \nabla \phi = a \cdot \nabla f \theta_W + (a \cdot \nabla g - f a \cdot \nabla W) \delta_W - g a \cdot \nabla W \delta^{[1]}_W. \] (65)

Using these expressions in the distribution field equation
\[ i \phi + i a \cdot \nabla \phi - m \phi = 0, \] (66)
we find a distributional equation of the form
\[ A \theta_W + B \delta_W + C \delta^{[1]}_W = 0, \] (67)
where the coefficients \( A, B, \) and \( C \) are given by
\[ A \equiv i f + i a \cdot \nabla f - m f, \]
\[ B \equiv i f + i g - i a \cdot \nabla W + i a \cdot \nabla g - m g, \]
\[ C \equiv i g - i a \cdot \nabla W. \] (68)

Now applying the distributional independence theorem (A8 to (67)) we must have
\[ A(t, x) = 0, \quad t > W(x), \]
\[ [B]^W = [C^{[1]}]^W, \]
\[ [C]^W = 0. \] (69)

It is straightforward now to show that these conditions are equivalent to \( a \cdot \nabla W = 1 \) and \( a \cdot \nabla [g]^W + im [g]^W = 0 \), precisely agreeing with the results derived above using Fock’s argument. We note that inside the region \( t > W(x) \), which contains all events after the shock wave has passed, the field \( \phi \) is essentially given by the test function \( f \), which satisfies the original wave equation (1) and has no singularities or discontinuities.

B. Lorentzian signature propagation

Suppose \( \varphi \) is any field satisfying the distributional equivalence equation of motion
\[ \Box \varphi \equiv v^{-2} \partial^2_t \varphi - \nabla^2 \varphi = V(\partial_t \varphi, \varphi, \ldots), \] (70)
where \( v \) is a constant and the highest derivative on the right-hand side is first order in time. Consider the shock wave ansatz
\[ \varphi = f \theta_W + g_0 \delta_W + g_1 \delta^{[1]}_W + \ldots g_n \delta^{[n]}_W, \] (71)
for some finite integer \( n \geq 0 \), with \( \theta_W \equiv \theta(t - W(x)) \), \( \delta_W \equiv \delta(t - W(x)) \), and \( f \) and \( \{g_k : k = 0, 1, \ldots, n\} \) are a set of test functions with \([g^{[1]}]^W \neq 0\). Here \( W(x) \) is some Fock precursor shock wave function whose properties are to be determined from (71). Then applying the distributional equivalence theorem quoted in the Appendix, we readily conclude that
\[ [g^{[1]}]^W (1 - v^2 \nabla W \cdot \nabla W) = 0, \] (72)
plus other conditions not relevant to the conclusions. Since we have assumed \([g^{[1]}]^W \neq 0\), we deduce that the shock wave precursor function \( W \) must satisfy the condition \( v^2 \nabla W \cdot \nabla W = 1 \). From (18), the shock wave normal speed is therefore \( v \).

A particular issue arises with non-linear theories, because products of distributions are not defined here. Therefore, any terms on the right-hand side of (20) such as \( \varphi^2 \) would in principle create a problem with the distributional approach. Our resolution is to look at the physics of observation. It is well-known that conventional quantum field theory encounters renormalization divergences that are removed by an appeal to the finiteness of observed quantities. Indeed, products of quantum field operators are generally ill-defined. In our case, we would argue that non-linear interaction terms, such as \( \varphi^2 \) on the right-hand side of (20) should be re-interpreted, because shock waves are the results of field interactions. For example, we would propose an ansatz for \( \varphi^2 \) of the form
\[ \varphi^2 \sim f^2 \theta_W + G_0 \delta_W + G_1 \delta^{[1]}_W + \ldots G_n \delta^{[n]}_W, \] (73)
where \( f \) is the same test function as in equation (71) and the \( G_i \) are test functions. Such an ansatz does not then alter our conclusions. This argument is analogous to Wilson’s expansion of products of quantum fields [24]. We note that in our discussion of polarizable and magnetizable media in the next section, we take a similar approach in our modelling of the polarization and magnetization response to an incoming electromagnetic shock wave.

The same methodology allows us to deduce the same result for any higher order equation such as
\[ \alpha (\Box_v)^2 \varphi + \beta \Box_v \varphi = V(\partial_t \varphi, \varphi, \ldots), \] (74)
where \( \alpha \) and \( \beta \) are test functions.

C. Shock waves in polarizable and magnetizable media

The possibility that \( v \) is not the speed of light in vacuo (\( c = 1 \) throughout this paper) is of critical importance in the theory of propagation of electromagnetic waves through real media. It is possible, in certain cases of anomalous dispersion, to encounter situations where \( v > 1 \). We discuss what must happen in such cases in this section.

The modified d’Alembertian operator \( \Box_v \equiv v^{-2} \partial^2_t - \nabla^2 \) is the critical factor in any discussion of wave processes.
In electromagnetic wave theory the constant $v$ is referred to as the **phase velocity**. It is usually asserted that light propagates in a polarizable and/or magnetizable medium with this speed. When $v$ is less than $c$, the speed of light in vacuo, it is possible for particle speeds in such a medium to exceed $v$ (but still be less than $c$), and then Cerenkov or Askaryan radiation may be observed. These are all important observed phenomena. The problem however is that it is possible to encounter media for which $v$ exceeds $c$, as well as media in which the group velocity is greater than $c$. In such cases, the obvious question is whether precursor signals could ever propagate faster than $c$.

It is not enough to simply assert the traditional relativistic veto $v \leq c$: the dynamics of light propagation in media should predict that veto in a natural, accountable way. We discuss here how the distributional field method deals with this issue.

In any such discussion, it is important to understand that we are dealing with complex, emergent processes using reductionist equations of motion. Therefore, approximate, relatively simple models have to made, generally regarded as statistical in nature. Maxwell’s equations for electromagnetic fields in polarizable and magnetizable media are exactly of this type. We shall apply our distributional field method to electromagnetic waves in a nominally linearly polarizable and magnetizable homogeneous, isotropic medium, with charge-free field equations of motion

$$
\begin{align*}
\nabla \cdot \mathbf{D} &= 0, \quad \nabla \times \mathbf{E} + \frac{\partial}{\partial t} \mathbf{B} = 0, \\
\nabla \cdot \mathbf{B} &= 0, \quad \nabla \times \mathbf{H} - \frac{\partial}{\partial t} \mathbf{D} = 0.
\end{align*}
$$

(75)

Here $\mathbf{D} \equiv \varepsilon_0 \mathbf{E} + \mathbf{P}$ is the displacement field, where $\varepsilon_0$ is the permittivity of free space and $\mathbf{P}$ is the polarization field, and $\mathbf{H} \equiv \mathbf{B}/\mu_0 - \mathbf{M}$ is the magnetic intensity field, where $\mu_0$ is the permeability of free space and $\mathbf{M}$ is the magnetization field. Equations (75) are generally taken as exact equations, within the given context.

For linear, isotropic media, the polarization and magnetization fields are generally assumed to be given by

$$
\mathbf{P} = \chi_e \varepsilon_0 \mathbf{E}, \quad \mathbf{M} = \chi_m \mathbf{H},
$$

(76)

where $\chi_e$ is the electric susceptibility and $\chi_m$ is the magnetic susceptibility. Assuming these susceptibilities are scalar constants, then equations (75) give

$$
(\varepsilon \mu_0 \partial_t^2 - \nabla^2) \mathbf{E} = 0, \quad (\varepsilon \mu_0 \partial_t^2 - \nabla^2) \mathbf{B} = 0,
$$

(77)

where $\varepsilon \equiv (1 + \chi_e) \varepsilon_0$, $\mu \equiv (1 + \chi_m) \mu_0$, and so the phase velocity $v$ is given by $v \equiv 1/\sqrt{\varepsilon \mu}$. The significance to us here is that in the case of certain novel media, it is possible to encounter **negative** susceptibilities, leading to the result $v > c$. In such cases, our above discussion of Lorentzian signature propagation leads us to conclude that equations (77) must be incorrect equations for precursor wavefront propagation. We resolve this problem in two steps.

First, making no nonlinearity assumption about polarization or magnetization, equations (77) give the exact wave equations

$$
(\varepsilon_0 \mu_0 \partial_t^2 - \nabla^2) \mathbf{E} = -c_0^{-1} \nabla (\nabla \cdot \mathbf{P}) - \mu_0 \partial_t^2 \mathbf{P},
$$

$$
(\varepsilon_0 \mu_0 \partial_t^2 - \nabla^2) \mathbf{B} = \mu_0 \nabla \times \partial_t \mathbf{M},
$$

(78)

Second, we invoke causality. Suppose we have a nominally linear, isotropic medium at rest in the laboratory before any shock wave has passed through. Then clearly, all fields are zero then. Now suppose a precursor shock wave passes through the medium. The medium will consist of atoms and molecules that cannot react instantly. There must be some delay before the polarisation $\mathbf{P}$ and magnetization $\mathbf{M}$ can adjust to the sudden changes in the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$. Therefore, we make the following shock wave ansatz for the fields concerned, all of which are regarded now as distributional fields:

$$
\mathbf{E} = D \delta(t \delta_N^W) + \sum_{n=0}^{N-1} E_0 \delta_n^W, \quad \mathbf{P} = D \chi_e \varepsilon_0 \delta(t \delta_N^W) + \sum_{n=0}^{N-1} P_n \delta_n^W, \quad \mathbf{M} = D \chi_m \mu_0 \delta(t \delta_N^W) + \sum_{n=0}^{N-1} M_n \delta_n^W, \quad [E_N^W, [B_N^W] = 0,
$$

(79)

for some integer $N \geq 1$, noting the different upper limits on the summations in $\mathbf{E}$ and $\mathbf{B}$ compared to those in $\mathbf{P}$ and $\mathbf{M}$. This difference is our method of encoding causality. In these expressions, the coefficients of $\theta_W$ and the $\delta_n^W$ are assumed to be test function fields.

Applying the distributional equivalence theorem to equations (78) now considered as distributional field equations when ansatz (79) is used, there are two important conclusions. On the one hand, the coefficients of $\theta_W$ give the wave equations

$$
(\varepsilon \mu_0 \partial_t^2 - \nabla^2) \mathbf{E} = 0, \quad (\varepsilon \mu_0 \partial_t^2 - \nabla^2) \mathbf{B} = 0
$$

(80)

in the region of the medium where $t > W(x)$, that is, after the precursor shock wave has passed. These wave equations have phase velocity $v = 1/\sqrt{\varepsilon \mu}$, with no restriction on $v$ being greater than $c$. On the other hand, matching the effects of the $\delta_n^W$ terms in the ansatz leads to the conditions

$$
[E_N^W]^W (\varepsilon_0 \mu_0 - \nabla W \cdot \nabla W) = [B_N^W] (\varepsilon_0 \mu_0 - \nabla W \cdot \nabla W) = 0,
$$

(81)

from which we deduce the expected shock wave condition $c^2 \nabla W \cdot \nabla W = 1$. The distributional field approach therefore allows phase speeds greater than $c$ in media after signals have passed, but retains the relativistic lightcone limit on precursor signal propagation itself.

**D. Shock waves in quantum field theory**

Up to this point, we have been discussing classical fields. Shock waves in quantum field theory present new
challenges, principally on account of the uncertainty principle. If we prepare a localized-in-space signal state, then we can expect a spread in momentum associated with that state. Indeed, the concept of particle state in quantum field theory remains problematical [4]. We make two comments here, reserving this topic for future work.

First, the SPJ function $\Delta(x)$ for the scalar field demonstrates precisely the sort of structure that our distributional field approach has taken. Specifically, the SPJ function for the free Klein-Gordon equation (10) is given by

$$\Delta(t, r) = \frac{m}{4\pi \sqrt{t^2 - r^2}} J_1(m \sqrt{t^2 - r^2}) \theta(t-r) - \frac{1}{4\pi t} \delta(t-r),$$

for $t > 0$, which means that precursor shock waves are limited by the speed of light ($c = 1$ here).

The second point concerns the Feynman propagator. It is well-known that the scalar field propagator $\Delta_F(x)$ given by the famous $+i\epsilon$ prescription,

$$\Delta_F(x) \equiv \int \frac{d^4 p}{(2\pi)^4} \frac{e^{-i p x}}{p^2 - m^2 + i\epsilon},$$

has the merit of transmitting positive energy signals forwards in time, and ‘negative energy waves backwards in time’, according to the Feynman-Stueckelberg interpretation [2]. However, $\Delta_F(x)$ does not vanish outside the lightcone, raising the question of precursor shock wave speeds once again. The conventional resolution is to assert that real signals cannot be sent faster than $c$, and that whatever is transmitted outside the lightcone via the Feynman propagator concerns correlations, which are not signals. Correlations are emergent phenomena, underling the point that quantum field theory is really a theory of observation processes, rather than “things” such as fields or particles.

On the same point, it is remarkable that Julian Schwinger developed a novel approach to quantum field theory called source theory, in which the emphasis is on signal preparation and signal detection. In his approach, he postulated that the vacuum-to-vacuum amplitude $Z[J] = \langle 0^+ | 0^- \rangle^J$ in the presence of sources $J$ is of the form

$$Z[J] = \exp[(i/2) \int d^4 x d^4 y J(x) \Delta_+(x-y) J(y)].$$

Then Schwinger’s amplitude [51] can be written as

$$Z[J] \sim \exp \left[ i \frac{1}{2} \int_{R_2} d^4 x \int_{R_1} d^4 y J_1(x) \left\{ \Delta_+(x-y) \right\} J_1(y),$$

ignoring the pieces where apparatus in a given region interacts with itself.

We define $L(R_1)$ to be the ‘lightcone’ associated with $R_1$, by which we mean the set of all those events in spacetime that are each in or on the lightcone of at least one event in $R_1$. Now suppose $R_2$ has zero intersection with $L(R_1)$, which means that all events in $R_2$ are spacelike relative to all points in $R_1$. The point is, the Feynman propagator does not vanish between such points. Therefore, according to [55] a shock wave initiated at $R_1$ would have a non-zero effect on $Z[J]$, contrary to intuition.

We note that it is an inadequate argument to dismiss this result on the grounds that only correlations are involved, or that the effects are ‘small’. There is a problem here of principle touching on the relationship between classical relativity and quantum mechanics, and on the generally under-developed status of the theory of localized observation in quantum field theory.

The Feynman propagator is used conventionally because of the input that positive energies propagate forwards in time, but this input comes at the cost of violating the lightcone veto. It works conventionally because of the temporal limits to infinity being taken. Problems arise when this cannot be done, as in the case of shock waves. Our thoughts here are that it is possible to make an alternative choice in Schwinger’s formalism that uses the lightcone veto as an input, at the expense of the positive energy input. Specifically, we could make the replacement $\Delta_F(x) \rightarrow \Delta_C(x)$, where $\Delta_C(x) \equiv -\frac{1}{2} \left( \Delta_R(x) + \Delta_A(x) \right)$. The retarded and advanced propagators $\Delta_R$ and $\Delta_A$ satisfy the same inhomogeneous equation as $\Delta_F$ (up to a sign) but most significantly, vanish outside the lightcone. If we did this, then shock waves initiated in region $R_1$ would never affect detectors in $R_2$ if $R_2$ and $L(R_1)$ were disjoint. We note that $\Delta_C(x)$ differs from $\Delta_F$ only by a complementary function $\Delta_H(x)$.

There are two points about this suggestion. First, Schwinger aimed to avoid fields per se in his formalism. A replacement such as the one suggested here would need some interpretation in terms of standard field operators, particularly the creation and annihilation operators. We note that what observers see in their detectors are signals, not necessarily positive energy particles. Second, Schwinger did actually consider the possibility of adding complementary functions into his formalism [19].

IX. Concluding remarks

Quantum field theory shock wave analysis appears not to have been significantly explored yet. It is our belief that any development of it will require considerable attention to the observer concept and more explicit modelling of the processes of observation. This will require
taking emergent concepts such as irreversibility and finite time processes in quantum field theory into account much more than they are at present.

Our distributional approach uses the most singular term in the shockwave ansatz, such as in equations (11) and (10). However, much interesting detail can be found in the less singular terms, such as information concerning the flow of energy and momentum via shock wave fronts, particularly in the case of electromagnetic waves. We hope to report further on those details in subsequent articles.

Acknowledgements
R. H. A. is grateful for financial support from the Kurdistani regional government (KRG). He thanks his family for their deep patience in undertaking their son’s responsibilities during his absences. G. J. is indebted to Nicholas Kemmer for inspiring this work.

A. Distribution theory

There are two spaces of objects in our approach, referred to as distributions and test functions respectively.

If $D$ is a distribution and $f$ a real or complex-valued test function over $\mathbb{R}$, the action $\langle D, f \rangle$ of $D$ on $f$ is defined by

$$\langle D, f \rangle \equiv \int_{-\infty}^{\infty} D(x)f(x)dx, \quad \text{(A1)}$$

and is assumed to exist for all distributions and test functions.

The value $f(a)$ at $x = a$ of a test function $f$ is denoted by $[f]^a$. The $n^{th}$ derivative of a test function $f$ is also a test function and denoted by $f^{[n]}$, $n = 0, 1, 2, \cdots$, with $f^{[0]} \equiv f$.

test functions

A test function $\theta$ is an infinitely differentiable real or complex-valued function that falls off sufficiently rapidly as $|x| \to \infty$, such that

1. $(1, f^{[n]})$ exists for $n = 0, 1, 2, \ldots$

2. If $f$ and $g$ are test functions, and $\alpha$ and $\beta$ are real or complex constants, then $\alpha f g$ and $\alpha f + \beta g$ are test functions.

Distributions

A distribution $D$ is a process that maps a test function into $\mathbb{R}$ or $\mathbb{C}$ via the processes of integration, subject to the following conditions for any test function $f$:

3. For any constant $\alpha$ and any test function $f$, we have $\langle \alpha D, f \rangle = \alpha \langle D, f \rangle$.

4. For any distributions $D_1$, $D_2$ we define their sum $D_1 + D_2$ as $\langle (D_1 + D_2), f \rangle \equiv \langle D_1, f \rangle + \langle D_2, f \rangle$.

5. For any distribution $D$, we define its $n^{th}$ derivative $D^{[n]}$ in terms of its action on any test function $f$ by $\langle D^{[n]}, f \rangle \equiv (-1)^n \langle D, f^{[n]} \rangle$, $n = 0, 1, 2, \ldots$.

6. For any distribution $D$ and test functions $f$, $g$, we define the generalized function $fD$ by $\langle fD, g \rangle \equiv \langle D, fg \rangle$.

Important examples of distributions are

The Heaviside step $\theta_a$

The conventional notation for this distribution is $\theta_a(x) \equiv \theta(x - a)$, where $a$ is real. For any test function $f$, $\theta_a$ is defined by

$$\langle \theta_a, f \rangle \equiv \int_{-\infty}^{\infty} f(x)dx. \quad \text{(A2)}$$

The reverse Heaviside step $\bar{\theta}_a$

The conventional notation for this distribution is $\bar{\theta}_a(x) \equiv \theta(a - x)$. For any test function $f$, $\bar{\theta}_a$ is defined by

$$\langle \bar{\theta}_a, f \rangle \equiv \int_{-\infty}^{a} f(x)dx. \quad \text{(A3)}$$

The Dirac delta $\delta_a$

The conventional notation for this distribution is $\delta_a(x) \equiv \delta(a - x)$ or $\delta(x - a)$. For any test function, $\delta_a$ is defined by

$$\langle \delta_a, f \rangle = [f]^a. \quad \text{(A4)}$$

Distributional equivalence

Two distributions $D_1$, $D_2$ are distributionally equivalent, written $D_1 \equiv D_2$, if $\langle D_1, f \rangle = \langle D_2, f \rangle$ for any test function $f$.

Using the rules given above, a number of distributions involving Heaviside steps and Dirac deltas can be shown to be distributionally equivalent, such as

1. $\theta_a + \bar{\theta}_a = 1$,

2. $\theta_a^{[1]} \equiv \delta_a$,

3. $\bar{\theta}_a^{[1]} = -\delta_a$.

4. For the product of any test function $f$ and the Dirac delta, we can choose to evaluate $f$ at $x = a$ or not, that is,

$$f\delta_a \equiv [f]^a \delta_a. \quad \text{(A5)}$$

Differentiating this last distributional equivalence on both sides with respect to $x$ gives the rule

$$f^{[1]}\delta_a \equiv ([f]^a - f)\delta_a^{[1]}, \quad \text{(A6)}$$

and so on for higher derivatives.

Using the above rules, we can prove the following:
Theorem: The distributions $\theta_a, \, \theta_a^0, \delta_a, \delta_a^{[1]}$, etc., are distributionally independent, which means the following. Suppose $e, f, g_0, g_1, g_2, \ldots$, are test functions and we are given that
\[ e\theta_a + f\theta_a^0 + \sum_{n=0}^{\infty} g_n \delta_a^{[n]} = 0, \quad (A7) \]
where $\delta_a^{[n]} \equiv \delta_a$. Then assuming we can interchange orders of summation, we must have
\[ e(x) = 0, \quad x > a, \]
\[ f(x) = 0, \quad x < a, \]
\[ 0 = \sum_{p=0}^{\infty} (-1)^{m+p} \binom{m+p}{p} g_{m+p}^{[p]} a^m, \quad m = 0, 1, 2, 3, \ldots \quad (A8) \]