Shadowless rapidly rotating yet not ultraspinning Kerr-AdS$_4$ and Kerr-Newman-AdS$_4$ black holes

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We find that the Kerr-(Newman)-AdS$_4$ black hole will be shadowless if its rotation parameter is larger than a critical value and the shadowless-ness may be related to the appearance of the null hypersurface caustics (NHC) outside the event horizon. This is because the NHC exists outside the event horizon for the high dimensional Kerr-(Newman)-AdS superentropic black hole [37, 38]. The existence of the NHC is thought previously to mean that the causal structure of spacetime has some pathologies. However, it was recently found that the spacetime with the NHC is free of closed timelike curve [39].

For the ultraspinning Kerr-Sen-AdS$_4$ black hole, whether it is superentropic or not, the NHC always appears both inside and outside of the event horizon [39], which seems to indicate that the existence of the NHC outside the event horizon is related to ultraspinning rather than superentropincness. The ultraspinning means that the rotational angular velocity of the black hole is boosted to the speed of light. Then a further issue, which we are going to address, is whether the rotational angular velocity reaching the speed of light is a necessary requirement for existence of the NHC outside the event horizon.

Let us start with the four-dimensional Kerr-Newman-AdS black hole solution [40], whose metric has the form [41]

$$ds^2 = -\Delta_r \left( dt - \frac{a}{\Xi} \sin^2 \theta d\phi \right)^2 + \frac{\Delta_r}{\Delta_\theta} dr^2 + \frac{\Delta_r}{\Delta_\theta} d\theta^2 + 2 \frac{\Delta_r \sin^2 \theta}{\Sigma} \left( adt - \frac{r^2 + a^2}{\Xi} d\phi \right)^2,$$

in the Boyer-Lindquist coordinates, where

$$\Delta_r = (r^2 + a^2) (1 + r^2/l^2) - 2mr + q^2, \quad \Xi = 1 - a^2/l^2, \quad \Delta_\theta = 1 - a^2 \cos^2 \theta / l^2, \quad \Sigma = r^2 + a^2 \cos^2 \theta, \quad a$$ is the rotation parameter, $m$ the mass parameter, $q$ the electric charge parameter and $l$ the AdS radius. The horizon is determined by equation $\Delta_r = 0$. When $q = 0$, the metric in Eq. (2) reduces to that of the Kerr-AdS$_4$ black hole. The RN-AdS$_4$ black hole is obtained if $a = 0$ in Eq. (2). In Fig. 1, we plot the rescaled horizon radius $\bar{r}$ as a function of the

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scaled mass $\bar{m}$. It is easy to see that if the black hole mass is larger than a critical value both the RN-AdS black hole and the Kerr-(Newman)-AdS black hole have the Cauchy and event horizons. When the black hole mass equals to the critical value, the Cauchy and event horizons coincide. While the Schwarzschild-AdS black hole, which corresponds to the case with both $a = 0$ and $q = 0$, only has the event horizon.

To derive the condition for existence of the NHC for the Kerr-Newman-AdS black hole, we use the method given in Refs. [37–39, 42, 43]. After introducing the outgoing and ingoing Eddington-Finkelstein coordinates defined in terms of the "generalized tortoise coordinate" $r_\ast(r, \theta)$: $u = t - r_\ast(r, \theta)$, $v = t + r_\ast(r, \theta)$, the null hypersurfaces can be described by $u = \text{const}$, $v = \text{const}$, which are dubbed the outgoing and ingoing null congruences of the hypersurfaces, respectively. It is easy to obtain that the null hypersurfaces defined by $u = \text{const}$ or $v = \text{const}$ satisfy the equation

$$g^{uv} \partial_u (t + r_\ast) \partial_v (t + r_\ast) = g^{rr}(\partial_r r_\ast)^2 + g^{\theta \theta}(\partial_\theta r_\ast)^2 = 0.$$  \hspace{1cm} (3)

By using the contravariant components $g^{rr}$, $g^{\theta \theta}$, and $g^{\theta \theta}$ of the metric (2), Eq. (3) can be re-expressed as

$$\Delta_r (\partial_r r_\ast)^2 - \frac{\Sigma^2 (r^2 + a^2)^2}{\Lambda_r} = -\Delta_\theta (\partial_\theta r_\ast)^2 - \frac{\Sigma^2 a^2 \sin^2 \theta}{\Lambda_\theta}.$$  \hspace{1cm} (4)

This equation can be solved by the method of separation of variables. Introducing the so-called "constant of separation" $a^2 \lambda$ (hereinafter, $\lambda$ is referred to as the separation constant) for Eq. (4), we have

$$(\partial_r r_\ast)^2 = \frac{Q^2(r)}{\Lambda_r}, \quad (\partial_\theta r_\ast)^2 = \frac{P^2(\theta)}{\Lambda_\theta}. \hspace{1cm} (5)$$

where

$$Q^2(r) = \Sigma^2 \left( (r^2 + a^2)^2 - a^2 \lambda \Delta_r \right),$$  \hspace{1cm} (6)

$$P^2(\theta) = \Sigma^2 a^2 \left( \lambda \Delta_\theta - \sin^2 \theta \right).$$

In [37, 43], $\lambda$ is chosen as $\lambda < 1$ and it was then found that the NHC cannot exist outside the event horizon of the Kerr-Newman-AdS black hole. However, this choice is incorrect, since when $\theta = \pi/2$, $P^2(\theta) = \Sigma^2 a^2 (\lambda - 1) \geq 0$ must be satisfied. Thus, the allowed region of $\lambda$ should be $\lambda \geq 1$.

After some complicated computations, which are presented in detail in the Appendix, we find that $Q(r) = 0$ is a sufficient condition for the existence of the NHC, which means that

$$(r^2 + a^2)^2 - a^2 \lambda \Delta_r = 0. \hspace{1cm} (7)$$

Figure 2 shows the numerical results of the NHC condition (7). In addition, the photon sphere condition, which is governed by the equation $(r^2 + a^2)\Delta_r - \partial_r \Delta_r = 0$ [44, 45], and the horizon condition ($\Delta_r = 0$), are plotted, where a prime indicates the derivative with respect to $r$. In this figure, two intersection points between the dashed black line and the blue/purple line represent that the black hole has the Cauchy horizon and the event horizon. One can see that the NHC exists only inside the Cauchy horizon when $\bar{a} = 0.25$, and it appears both inside the Cauchy horizon and outside the event horizon of the black hole when $\bar{a} = 0.75$. This suggests that the NHC can appear outside the event horizon of the black hole for a large enough $a$ which is not necessarily equal to $l$. The photon sphere line indicates that the NHC, which appears outside of the event horizon, locates between the event horizon and the photon sphere of the black hole.

How large is the value of the rotational angular velocity required for existence of the NHC outside the event horizon?
After a detailed calculation, we find that when the rotation parameter $a$ satisfies
\[ a \geq \frac{l}{\sqrt{\lambda}}, \tag{8} \]
the NHC exists both outside the event horizon and inside the Cauchy horizon of the Kerr-(Newman)-AdS$_4$ black hole. Otherwise, there is the NHC only inside the Cauchy horizon. Thus, if the rotation parameter is larger than a critical value $a_c = l/\sqrt{\lambda}$, the NHC appears outside the event horizon. Apparently, this critical value is determined by $l$ and $\lambda$. When $\lambda$ equals unity, the NHC exists outside the event horizon only in the case of the ultraspinning Kerr-(Newman)-AdS$_4$ black hole ($a \to l$). If $\lambda > 1$, a large enough $a$, which is not necessarily equal to $l$, can lead to the existence of the NHC outside the event horizon of the black hole. Therefore, the value of $\lambda$ determines how large $a$ has to be to ensure the existence of the NHC outside the event horizon.

Now we demonstrate that when the rotation parameter is larger than a critical value, in addition to the existence of the NHC outside the event horizon, the Kerr-(Newman)-AdS black hole’s shadow cannot be obtained and thus it should be shadowless. Here, we take the Kerr-AdS$_4$ black hole as an example and investigate its shadows. The results are expected to be valid for the Kerr-Newman-AdS$_4$ case. Figure 3 gives the numerical results in the equatorial plane $\theta = \pi/2$. With the increase of the rotational angular velocity, the black hole shadows change from an ellipse to a “D” shape. Once $a > 0.741l$, the shadow of the Kerr-AdS$_4$ black hole disappears. Clearly, both the shadowless-ness and the existence of the NHC outside the event horizon of the Kerr-(Newman)-AdS black hole require that the black hole rotates rapidly enough with the rotation parameter larger than a critical value. This seems to suggest that the existence of the HNC outside the event horizon might be the cause of the shadowless-ness of a rapidly rotating Kerr-(Newman)-AdS black hole. Therefore, we can obtain the value of $\lambda$ from analyzing the shadow of the Kerr-(Newman)-AdS$_4$ black hole. From the inequality (8), we find that the value of $\lambda$ should be chosen to be about $\lambda \sim 1.821$ when $\bar{m} = 1$.

It is easy to obtain that when the rotation parameter of the black hole is larger than the critical value $a_c$, i.e. $a_c \simeq 0.741l$ for the Kerr-AdS$_4$ black hole with $\bar{m} = 1$, the NHC will appear outside the event horizon of the black hole. Apparently, the ultraspinning limit ($a \to l$) of the black hole is not a necessary but sufficient condition for the appearance of the NHC outside the event horizon of black hole. Thus, we further confirm that the superentropic-ness of ultraspinning black holes is unrelated to the presence of the NHC outside the event horizon. Finally, we believe that our work can be generalized to other rotating AdS black holes, and for these black holes there is also a critical value for the rotation parameter.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig3.png}
\caption{The shadows for the Kerr-AdS$_4$ black hole for different rescaled rotation parameter $\bar{a}$ with $\bar{m} = 1$, where $\bar{m} = m/l$, $\bar{a} = a/l$.}
\end{figure}

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Appendix A

Using the relations given in (5), one can express the total differential \( dr_s = \partial_t r_s dr + \partial_\theta r_s d\theta \)
as
\[
dr_s = \frac{Q(r, \lambda)}{\Delta_r} dr + \frac{P(\theta, \lambda)}{\Delta_\theta} d\theta .
\]

(A1)

By treating \( \lambda \) as a variable, the exact differential Eq. (A1) can be generalized to be
\[
dr_s = \frac{Q(r, \lambda)}{\Delta_r} dr + \frac{P(\theta, \lambda)}{\Delta_\theta} d\theta + c_1 F(r, \theta, \lambda) d\lambda ,
\]

(A2)

where \(c_1\) is an arbitrary constant and \( F(r, \theta, \lambda) \) is an arbitrary function. Since Eqs. (A1) and (A2) are functionally equivalent, the condition
\[
F(r, \theta, \lambda) = 0
\]

should be satisfied, which indicates \( dF(r, \theta, \lambda) = 0 \). Thus, we arrive at
\[
[\partial_t F(r, \theta, \lambda)] d\lambda + [\partial_\theta F(r, \theta, \lambda)] dr + [\partial_\lambda F(r, \theta, \lambda)] d\theta = 0 .
\]

(A4)

From the Poincaré lemma in the external differentiation theory, namely \( d(dr_s) = 0 \), one can obtain the following integrable conditions
\[
\frac{\partial_r Q(r, \lambda)}{\Delta_r} = c_1 \partial_t F(r, \theta, \lambda) , \quad \frac{\partial_\theta P(\theta, \lambda)}{\Delta_\theta} = c_1 \partial_\theta F(r, \theta, \lambda) .
\]

(A5)

From the definitions of \( Q(r, \lambda) \) and \( P(\theta, \lambda) \) given in (6), one finds straightforwardly
\[
\frac{\partial_r Q(r, \lambda)}{2Q(r, \lambda)} = -\frac{\Delta_r a^2}{2Q(r, \lambda)} , \quad \frac{\partial_\theta P(\theta, \lambda)}{2P(\theta, \lambda)} = -\frac{\Delta_\theta a^2}{2P(\theta, \lambda)} ,
\]

(A6)

and one can rewrite Eq. (A4) as
\[
vd\lambda = -\frac{dr}{Q(r, \lambda)} + \frac{d\theta}{P(\theta, \lambda)} ,
\]

(A7)

after choosing the constant \( c_1 \) to be \( \Xi^2 a^2 / 2 \) and defining \( v = -\partial_\lambda F(r, \theta, \lambda) \). By making use of Eq. (A1) and Eq. (A7), the metric of the Kerr-Newman-AdS\(_4\) metric given in (2) can be re-expressed in terms of the coordinates \((r, \theta, \varphi)\)
\[
ds^2 = \frac{\Delta_\theta \Delta^2 (dr^2 - dt^2) + R^2 \sin^2 \theta (d\phi - \Omega dt)^2}{\Xi^2 R^2} + \frac{\nu^2 P^2(\theta, \lambda)Q^2(r, \lambda)}{\Xi^2 R^2} d\lambda^2 ,
\]

(A8)

where
\[
R^2 = \frac{\sin^2 \varphi}{\sin^2 \lambda} = \frac{\Delta_\varphi (r^2 + a^2) - \Delta_\varphi a^2 \sin^2 \varphi}{\Xi^2 \Sigma},
\]

(A9)

\[
\Omega = -\frac{\sin \varphi}{\sin \lambda} = \frac{a \left[ \Delta_\varphi (r^2 + a^2) - \Delta_\varphi \right]}{\Xi^2 \Sigma R^2} .
\]

(A10)

Then, using Eqs. (A1) and (A7), one has
\[
dr = \frac{Q(r, \lambda)\Delta_\theta}{\Xi^2 R^2 \Sigma} \left[ \Delta_\theta dr_s - P^2(\theta, \lambda)vd\lambda \right] ,
\]

(A11)

\[
d\theta = \frac{P(\theta, \lambda)\Delta_\theta}{\Xi^2 R^2 \Sigma} \left[ \Delta_\theta dr_s + Q^2(r, \lambda)vd\lambda \right] .
\]

(A12)

Since the outgoing and ingoing null congruences are defined to be \( du = dv = 0 \), which means \( dr_s^2 = dt^2 \), the metric of the Kerr-Newman-AdS\(_4\) black hole on the null hypersurface is reduced to
\[
dh^2 = \frac{\nu^2 P^2(\theta, \lambda)Q^2(r, \lambda)}{\Xi^2 R^2} d\lambda^2 .
\]

(A13)

Because the volume element of a null hypersurface is the square root of the determinant of the induced metric, the points where the induced metric determinant goes to zero correspond to the NHC. Thus, the condition of the NHC is
\[
vP(\theta, \lambda)Q(r, \lambda) = 0 ,
\]

(A14)

which is the determinant of the induced metric (A13) of the Kerr-Newman-AdS\(_4\) black hole. Let us consider the ingoing null hypersurface case (same for the outgoing null hypersurface case), namely, \( \lambda = \text{const} \) and a decreasing \( r \), as an example to study the NHC. We need to analyze each factor in Eq. (A14). Since Eq. (A7) gives that the decreasing of \( \theta \) will lead to the decreasing of \( r \) for a fixed \( \lambda \), we have \( P(\theta, \lambda) > 0 \). Therefore, \( Q(r, \lambda) = 0 \) is a sufficient condition for the existence of the NHC.