Perturbation theory for propagating magnetic droplet solitons

L. D. Bookman and M. A. Hoefer

1Department of Mathematics, North Carolina State University, Raleigh, NC 27695, USA
2Department of Applied Mathematics, University of Colorado, Boulder, CO 80309, USA

Droplet solitons are strongly nonlinear, inherently dynamic structures in the magnetization of ferromagnets, balancing dispersion (exchange energy) with focusing nonlinearity (strong perpendicular anisotropy). Large droplet solitons have the approximate form of a circular domain wall sustained by precession and, in contrast to single magnetic vortices, are predicted to propagate in an extended, homogeneous magnetic medium. In this work, multiscale perturbation theory is used to develop an analytical framework for investigating the impact of additional physical effects on the behaviour of a propagating droplet. After first developing soliton perturbation theory in the general context of Hamiltonian systems, a number of physical phenomena of current interest are investigated. These include droplet–droplet and droplet–boundary interactions, spatial magnetic field inhomogeneities, spin transfer torque induced forcing in a nanocontact device and damping. Their combined effects demonstrate the fundamental mechanisms for a previously observed droplet drift instability and under what conditions a slowly propagating droplet can be supported by the nanocontact, important considerations for applications. This framework emphasizes the particle-like dynamics of the droplet, providing analytically tractable and practical predictions for modern experimental configurations.

1. Introduction

The ability to excite, probe and control magnetic media at the nanometre scale has enabled new applications in spintronics and magnonics as well as
the exploration of new physics. Of particular, recent interest is the generation of coherent structures within the magnetization, a vector field describing the magnetic dipole moments of a magnetic material. Coherent structures observed experimentally include those with non-trivial topology such as magnetic vortices [1,2] and skyrmions [3–5] as well as non-topological droplet solitons [6–9]. The magnetic droplet soliton (droplet hereafter) is a coherently precessing, nanometre-scale localized wave structure exhibiting strongly nonlinear effects [10]. Its first observation [6] was enabled by a nanocontact spin-torque oscillator (NC-STO) device, which provides the necessary forcing to oppose magnetic damping, hence has been termed a dissipative droplet soliton [11].

The droplet is theoretically understood as a solution of a conservative Landau–Lifshitz partial differential equation modelling spatio-temporal dynamics in an ultra-thin, two-dimensional magnetic film with uniaxial, perpendicular anisotropy. When the stationary droplet’s precessional frequency is close to the local, magnetic field-induced (Zeeman) frequency, it resembles a circular domain wall of large radius, almost reversed at its core. These large droplets can propagate, coinciding with a superimposed wave structure to the otherwise spatially homogeneous phase. An example is shown in figure 1.

There are a rich variety of physical mechanisms that can change the orientation of the local dipole moment in a ferromagnet and hence influence the particle-like behaviour of a droplet. An example already mentioned is that of the NC-STO and damping. This motivated us to develop a droplet soliton perturbation theory that allows for the analysis of droplet dynamics in the presence of a large number of physical perturbations [12]. This theory describes soliton dynamics via a finite dimensional dynamical system representing the adiabatic evolution of the droplet’s four parameters (centre, precessional frequency and phase) resulting from perturbations.
While the theory provided fundamental explanations of droplet physics, it was limited to almost stationary droplets, i.e. droplets of negligible momentum, which manifested as a constraint equation on the dynamics. Furthermore, important physical effects such as droplet acceleration due to a magnetic field gradient [13–15] and droplet interactions [16,17] were excluded from the theory.

Soliton perturbation theory has been successfully used to describe dynamics in many physical systems [18,19], notably solitons in nonlinear Schrödinger (NLS)-type equations modelling, for example optical fibres [20] and Bose–Einstein condensates [21]. The central idea is to project the perturbed PDE dynamics onto the unperturbed soliton solution manifold, allowing for adiabatic evolution of the soliton’s parameters. The resulting modulation equations can be obtained in different ways, of which multiple scales perturbation theory [22] or perturbed conservation laws [18] are perhaps the most common. Both approaches have been shown to be equivalent in some specific applications (e.g. [23]) but the conservation law approach is powerful in its simplicity. However, it can be unclear which balance laws to use, especially when higher order information is sought. The governing Landau–Lifshitz equation for magnetization dynamics is a strongly nonlinear, vectorial generalization of the NLS equation that lacks Galilean invariance. There exist two-dimensional moving droplet solutions [16,24] described by six independent parameters. As such, droplet perturbation theory is significantly more complex than its NLS soliton counterpart, so a structured approach is desirable.

The earliest work we have found in the literature is [25], which outlines a general soliton perturbation methodology based on the Lagrangian of the underlying system. More explicit work in general nonlinear systems is provided in [22], where multiple scales perturbation theory and Green’s function for the linearized operator are used. An alternative, rigorous approach was developed for the modulational stability of solitons in NLS equations by Weinstein in [26] based on multiple scales and the generalized nullspace of the linearized operator. Motivated by the challenges associated with magnetic droplet soliton perturbation theory, we revisit the general approach for perturbed Hamiltonian systems, using multiple scales and the generalized nullspace formulation. In §2, we provide necessary conditions and expressions for the resulting modulation equations. Our approach preserves much of the generality of the Green’s function approach with the comparative accessibility of Weinstein’s approach to NLS. Additionally, we demonstrate that these methods do generalize to soliton perturbation in higher dimension, beyond (1+1)D, and how they can be used to determine the evolution of large numbers of parameters for a single soliton.

Two-dimensional moving droplet solutions have been computed [16,24] and studied asymptotically in the weakly nonlinear regime [27]. They can be accelerated and controlled by an inhomogeneous, external magnetic field [15] and exhibit novel interaction properties [17]. Droplets can also experience a drift instability in NC-STOs, whose origin is not well understood [11]. Because droplets are relatively new physical features of nanomagnetic systems, they hold potential for applications such as spintronic information storage and transfer or probing material properties. Moreover, their fundamental physics are not well understood. A more general theory to describe the motion of droplets in realistic physical systems is desirable.

Towards this end, we here present an analytical framework for investigating the impact of a large class of physical effects on a six parameter propagating droplet soliton. In §3, we derive an approximate solution for the propagating droplet in the close to Zeeman frequency (large mass) and small velocity (large momentum) regime, which greatly reduces the complexity of the asymptotic theory. This allows for explicit analytical results, a powerful feature of soliton perturbation theory considering the droplet’s strongly nonlinear qualities. The main result of this paper, the modulation equations, is presented in §4, using the general soliton perturbation theory formulation from §2. The versatility of this framework is subsequently discussed in §§5 and 6 through an investigation of a series of perturbations of current physical interest. In particular, we analytically demonstrate a mechanism leading to the NC-STO droplet drift instability and droplet attraction or repulsion to a ferromagnetic boundary.
The model under study here is the perturbed Landau–Lifshitz equation for the magnetization, $\mathbf{m} = [m_x, m_y, m_z]$, of a two-dimensional ferromagnetic film in non-dimensional form

$$\frac{\partial \mathbf{m}}{\partial t} = -\mathbf{m} \times \mathbf{h}_{\text{eff}} + \varepsilon \mathbf{p}, \quad \mathbf{m} : \mathbb{R}^2 \times (0, \infty) \rightarrow \mathbb{S}^2$$

and

$$\mathbf{h}_{\text{eff}} = \nabla \mathbf{m} + (h_0 + m_z)\mathbf{z}, \quad \lim_{|\mathbf{x}| \rightarrow \infty} \mathbf{m} = \mathbf{z},$$

where $\mathbf{p}$ is a perturbation that preserves the magnetization’s length ($\mathbf{p} \cdot \mathbf{m} \equiv 0$) and $0 < \varepsilon \ll 1$ is a small parameter encoding the strength of the perturbation. The perturbation can depend explicitly on time and space so long as the variation is slow. That is, the perturbation depends only on the slow temporal $T = \varepsilon t$ and spatial $\mathbf{x} = \varepsilon \mathbf{x}$ coordinates. The perpendicular, external magnetic field $h_0\mathbf{z} = h_0(\mathbf{X}, T)\mathbf{z}$ can be large and slowly varying in space and time. Further generalizations to rapidly varying perturbations are possible. A necessary requirement for the existence of droplet solitons is strong perpendicular anisotropy, encoded in the orientation of the $m_z\mathbf{z}$ term in the effective field $\mathbf{h}_{\text{eff}}$. See [12] for the derivation and non-dimensionalization of this model.

2. Adiabatic dynamics for Hamiltonian systems

The main aim of this paper is to determine the evolution of parameters of the droplet soliton in response to a fairly general class of perturbations. The equations determining the time-evolution of parameters will be referred to as modulation equations. Before performing that analysis directly, we first consider the more general context of perturbed Hamiltonian systems. Note that the Landau–Lifshitz equation is a Hamiltonian system, with canonically conjugate variables $\cos(\Theta) = m_z, \Phi = \arg(m_x + im_y)$. That is, the Landau–Lifshitz system may be written as $\partial \cos(\Theta)/\partial t = \delta \mathcal{E}/\delta \Phi$ and $\partial \Phi/\partial t = -\delta \mathcal{E}/\delta \cos(\Theta)$, where the right-hand sides are expressed in terms of variational derivatives of the energy $\mathcal{E}$, defined later in equation (3.5). In such systems, it is convenient to execute perturbation theory in the Hamiltonian variables.

The basic procedure is to allow the parameters to vary on a time scale proportional to the strength of the perturbation, $\varepsilon$. By allowing the parameters to vary in this way, additional degrees of freedom are introduced which can be used to resolve the difficulties arising from singular perturbations. Expanding about the soliton solution in an asymptotic series, one obtains a linear problem at order $\varepsilon$. In general, this linear equation will not admit solutions bounded in time. However, as used in [26], a solvability condition exists whereby bounded solutions are assured, guaranteeing that the linear problem at order $\varepsilon$ does not break the asymptotic ordering. Imposing these conditions leads to the modulation equations. This procedure is equivalent to projecting the solution of the perturbed model onto the family of solitons. While one might wish to then solve the linear equation at order $\varepsilon$ to obtain a further correction, we will not do that in this work. As will be demonstrated by the examples in §5, quite satisfactory predictions can be made considering only the leading order parametric dynamics.

A Hamiltonian system requires a real inner product space, $\mathcal{X}$; a nonlinear functional, $H : \mathcal{X} \rightarrow \mathbb{R}$ and a skew adjoint operator $J : \mathcal{X} \rightarrow \mathcal{X}$. We will use the notation $\langle \cdot, \cdot \rangle$ for the inner product on $\mathcal{X}$. The standard form for a Hamiltonian system is

$$\frac{\partial z}{\partial t} = J\nabla H(z),$$

where $z \in \mathcal{X}$ is referred to as the state variable. We refer to $H$ as the Hamiltonian, which is often assigned the physical meaning of energy since it is automatically a conserved quantity of such a system. In this context, by $\nabla H$ we mean the first variation of this nonlinear functional and by $\Delta H$ we mean the second variation (both taken with respect to the state variable, $z$). We consider here Hamiltonians which depend explicitly upon additional parameters, $q \in \mathbb{R}^m$ ($m$ is the number of such parameters). These parameters may arise due to a change of coordinates, such as to a comoving reference frame. For the examples which arise in this work, the
parameters $q$ arise from just such a transformation, so we will typically refer to these parameters as ‘frequencies’.

We assume here that (2.1) admits a solitary wave solution, $u$, with sufficiently rapid decay as $|x| \to \infty$ such that

$$0 = \nabla H(u, q).$$

(2.2)

If $H$ depends on $q$, naturally $u$ will depend on $q$ as well. Typically, the parameters $q$ do not provide a full parametrization of the solitary wave manifold due to underlying symmetries in the equation such as translation invariance. Accordingly, we will allow $u$ to depend on a separate set of parameters $r \in \mathbb{R}^s$ ($s$ is the number of such parameters). For reasons that will become clear in later examples, we refer to these parameters as ‘phases’. Therefore, the solitary wave can be written $u = u(x, q, r)$. Often times, the Hamiltonian system (2.1) admitting solitary wave solutions (2.2) is idealized, neglecting important physical effects. While some perturbations may give rise to a different Hamiltonian system, in general such effects do not need to preserve the Hamiltonian structure. We will treat both cases the same by introducing a small perturbation into the equation itself. The perturbed model is

$$\frac{\partial z}{\partial t} = \nabla H(z, q) + \epsilon P,$$

(2.3)

where $0 < \epsilon \ll 1$ and $P$ is a perturbation. The parameters $q, r$ are allowed to vary on a slow time scale, $T = \epsilon t$. We restrict to perturbations which depend explicitly on time only through this slow time variable, $T$. In this case, ordinary differential equations governing the evolution of these parameters can be determined according to the following theorem.

**Theorem 2.1.** Given the perturbed Hamiltonian system (2.3). If

(i) the solitary wave solution, $u$, exists for the unperturbed system (2.3), $\epsilon = 0$, and is independent of $t$;
(ii) $J$ is invertible;
(iii) $\Delta H|_{z=u}$ is self-adjoint for all admissible $q$;
(iv) $\forall 1 \leq k \leq m, \exists 1 \leq j \leq s$ such that $(\partial/\partial q_k)\nabla H(z, q)|_{z=u} = -J^{-1}(\partial u/\partial r_j)$;

then letting $v = [r, q]^T \in \mathbb{R}^{s+m}$, the modulation equations are

$$\left(\sum_{i=1}^{s+m} \left[ J^{-1} \frac{\partial u}{\partial v_i} , \frac{\partial u}{\partial v_j} \right] \frac{du_i}{dT} \right) = \left( J^{-1} P , \frac{\partial u}{\partial v_j} \right).$$

(2.4)

The above equations are consistent with previous general results when applied to Hamiltonian systems [22]. The assumptions of theorem 2.1 may seem restrictive at first, but these conditions are frequently met in physical systems of interest. In all systems under consideration here, there does exist a solitary wave solution. These solutions generically depend on time, but for the case of a single solitary wave solution, transforming to the reference frame moving, rotating and/or precessing with the solitary wave can eliminate this explicit dependence on time. Such a transformation introduces parameters in $q$ and alters the Hamiltonian but leaves the Hamiltonian structure intact.

The second does offer a restriction. For instance, in the Korteweg–de Vries equation, $J$ does not admit a bounded inverse and correspondingly the modulation equations require additional considerations [28]. Nevertheless, formal calculations are possible and $J$ is frequently invertible for Hamiltonian systems (e.g. for NLS and the Landau–Lifshitz equation).

With appropriate restrictions on the Hamiltonian, the third assumption always holds. The self-adjoint property of the second variation essentially follows from the same calculation which proves the equality of mixed partial derivatives in finite-dimensional calculus. More care needs to be taken in the corresponding calculation on function spaces, but the Hamiltonians derived in physically relevant systems typically are well enough behaved.

The fourth assumption is restrictive and may seem obscure. However, the parameters of the soliton are often speeds or frequencies. These parameters are typically linked to initial positions
or initial phase values so that \( q \) and \( r \) have the same length \( (s = m) \). In such cases, the dependence of the soliton on the parameters in the laboratory frame will be in the form \( r + t q \). More generally, symmetry group arguments may be used to show that the soliton depends on the parameters in this way [29]. From this temporal dependence, the relations in assumption (iv) follow directly.

(a) Derivation of equation (2.4)

Theorem 2.1 relies on the following lemma.

**Lemma 2.2.** Let \( X \) be a Hilbert space. Let \( A \) be a linear operator mapping \( X \) to itself. Let \( f \in X \). Let \( A^\dagger \) be the adjoint of \( A \), i.e. the unique linear operator satisfying \( \langle A^\dagger x, y \rangle = \langle x, Ay \rangle \) for all \( x, y \in X \). Define \( \Upsilon : [0, \infty) \to X \) as the solution of the initial value problem

\[
\begin{align*}
\frac{\partial \Upsilon}{\partial t} &= A \Upsilon + f \\
\Upsilon(0) &= \Upsilon_0 \in X.
\end{align*}
\]

(2.5)

and

Let \( \mu_{-1} = 0 \) and \( \mu_i = \mu_{i-1} \) for \( 0 \leq i \leq N \), where \( N \) denotes the highest integer such that \( (A^\dagger)^N \) has non-trivial kernel. Then \( \Upsilon(t) \) will not be bounded in time unless \( \{\mu_{i-1}, \Upsilon_0\} + \{\mu_i, f\} = 0 \) for \( 0 \leq i \leq N \).

This lemma is a minor generalization of the solvability condition proved in [26]. There are a few key limitations which may not be clear upon first reading the statement of the lemma itself. First, \( A \) and \( f \) are assumed to be independent of time \( t \). Second, all assumptions of smoothness of \( f \) are bound up in the choice of \( X \) which is problem specific. In the context of Hamiltonian systems, \( X \) is given and the required smoothness of \( f \) is clear. In our intended application, equation (2.5) arises from a linearization of a nonlinear problem about a given state. In this case, \( A \) and \( f \) are given, but not \( X \). In order that lemma 2.2 applies, there must exist an \( X \) which makes \( A \) and \( f \) compatible, and it will be in that sense which \( \Upsilon(t) \) remains bounded in time. From here on out, we assume sufficient smoothness in our perturbation such that a Hilbert space is naturally chosen. For the perturbations we investigate in §5, this is the case. Finally, the details of defining the adjoint of the unbounded operator \( A \) are not considered here but can be handled in a standard manner (e.g. [26]).

The proof of theorem 2.1 proceeds by substituting the ansatz

\[
z = u(x; r(T), q(T)) + \epsilon u_1(x, t, T) + \mathcal{O}(\epsilon^2),
\]

(2.6)

into (2.3). Expanding in powers of \( \epsilon \), the first-order equation becomes

\[
\mathcal{O}(\epsilon) : \frac{\partial u_1}{\partial t} = J \Delta H(u, q)u_1 - \frac{\partial u}{\partial r} \frac{dr}{dT} - \frac{\partial u}{\partial q} \frac{dq}{dT} + P.
\]

(2.7)

Note that equation (2.7) is of the form in lemma 2.2 \((A = J \Delta H(u, q), f = P - (\partial u/\partial r) (dr/dT) - (\partial u/\partial q) (dq/dT))\). In order that the expansion in (2.6) remain asymptotically ordered, it is necessary that \( u_1(x, t, T) \) remain \( \mathcal{O}(1) \) for sufficiently long times. Lemma 2.2 thus gives a condition that must be satisfied. It remains to characterize the generalized nullspace of \((J \Delta H(u, q))^\dagger\). Note that since \( \Delta H(u, q) \) is self-adjoint, \((J \Delta H(u, q))^\dagger = -J \Delta H(u, q)\).

Differentiating (2.2) with respect to the parameter \( r_j \) for \( 1 \leq j \leq s \) and applying \( J^{-1} \) to the result yields \( \Delta H(u, q) (\partial u/\partial r_j) = 0 \). It follows that \( J^{-1}(\partial u/\partial r_j) \) is in the kernel of \((J \Delta H(u, q))^\dagger \) for all \( j \). Differentiating (2.2) with respect to the parameter \( q_k \) for \( 1 \leq k \leq m \) yields

\[
\Delta H(u, q) \frac{\partial u}{\partial q_k} + \frac{\partial}{\partial q_k} \nabla H(z, q) \bigg|_{z = u} = 0.
\]

(2.8)

Using assumption (iv), we can replace the second term in (2.8) so that there is some \( j \) with

\[
\Delta H(u, q) J^{-1} \frac{\partial u}{\partial q_k} = J^{-1} \frac{\partial u}{\partial r_j}.
\]

(2.9)
Hence, \( J^{-1}(\partial u/\partial q_k) \in \text{ker}(\Delta H(u, q)) \) and therefore in the generalized nullspace. These two sets of vectors do not necessarily characterize the full generalized nullspace; however, these offer a sufficient number of constraints to uniquely determine the modulation system. Requiring that \( f = P - (\partial u/\partial r)(d\mathbf{r}/dT) - (\partial u/\partial q)(dq/dT) \) be orthogonal to \( J^{-1}(\partial u/\partial q_k) \) and \( J^{-1}(\partial u/\partial r) \) yields equations (2.4). The modes \( J^{-1}(\partial u/\partial q_k) \) and \( J^{-1}(\partial u/\partial r) \) may not give rise to a complete characterization of the nullspace. As a result, equations (2.4) are only a necessary but not sufficient condition to prevent secular growth.

### 3. Approximate propagating droplet

Now that we have the general formulation of the soliton modulation equations for perturbed Hamiltonian systems in equation (2.4), we would like to apply them to the magnetic droplet soliton solution of equation (1.1). For this, we will need to compute derivatives of the droplet with respect to its parameters as well as associated inner products. This could be performed numerically with a ‘database’ of droplet solutions as in [15]. Here, we obtain an explicit, analytical formulation of the modulation equations in the strongly nonlinear, moving droplet regime. But before we can determine the modulation equations, we need an explicit representation of the propagating droplet itself. In this section, we derive an approximate solution to equation (1.1) when \( \epsilon = 0 \), a restriction we maintain for the remainder of this section. The solution describes a slowly moving droplet with frequency just above the Zeeman frequency. A droplet solution can be characterized by six parameters: its precession frequency \( \omega \) above the Zeeman frequency \( h_0 \) in these non-dimensional units, propagation velocity \( \mathbf{V} = [V_x, V_y]^T \), initial phase \( \Phi_0 \), and the coordinates of the droplet centre \( \xi = [\xi_x, \xi_y]^T = \mathbf{V} t + x_0 \).

Approximate droplet solutions have been found in two regimes. First, \( 0 < \omega - |\mathbf{V}|^2/4 \ll 1 \), near the linear (spin-wave) band edge corresponding to propagating, weakly nonlinear droplets approximated by the NLS Townes soliton [27]. While the Townes mode is unstable in two-dimensional NLS, it has been demonstrated via asymptotic calculations that the higher order Landau–Lifshitz corrections to NLS effectively stabilize the soliton [27]. The second regime where approximate droplets have been investigated is \( 0 < \omega \ll 1 \) with zero velocity corresponding to stationary, strongly nonlinear droplets approximated by a circular domain wall [30]. We will focus here on large amplitude propagating solitons where the magnetization is nearly reversed because experiments operate in this regime. Note, however, that the weakly nonlinear regime could also be studied. The defining equation for the droplet can be formulated as a boundary value problem by expressing the magnetization in spherical variables \( \mathbf{m} = [\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta)] \) in the frame moving and precessing with the soliton \( \Theta \rightarrow \Theta(x - \xi) \), \( \Phi \rightarrow \Phi_0 + (h_0 + \omega) t + \Phi(x - \xi) \):

\[
-\sin(\theta)\mathbf{V} \cdot \nabla \Theta = \nabla \cdot (\sin^2(\theta)\nabla \Phi),
\]

\[
\sin(\theta)(\omega - V \cdot \nabla \Phi) = -\nabla \Theta + \frac{1}{2}\sin(2\Theta)(1 + |\nabla \Phi|^2),
\]

and

\[
\lim_{|x| \to \infty} \nabla \Phi = -\frac{\mathbf{V}}{2}, \quad \lim_{|x| \to \infty} \Theta = 0.
\]

This problem can be further simplified by exploiting the invariance of equation (3.1) under rotation of the domain to align the \( x \)-axis with the propagation direction. In this coordinate system, \( \mathbf{V} = \mathbf{V} \hat{x} \). Adding the assumptions of small frequency and propagation speed, a simple correction to the known, approximate stationary droplet can be found (see the electronic supplementary material).

\[
\Theta = \cos^{-1}\left( \tanh\left( \rho - \frac{1}{\omega} \right) \right) + \mathcal{O}\left( \omega^2, V^2 \right)
\]

and

\[
\Phi = \Phi_0 + (h_0 + \omega) t - \frac{V}{\omega^2}\cos(\phi) + \mathcal{O}\left( \frac{V}{\omega} \right).
\]
Above, \((\rho, \varphi)\) are polar variables for the plane, whose origin is centred on the droplet. That is, 
\[ \rho = \sqrt{(x - \xi_x)^2 + (y - \xi_y)^2} \] and \(\varphi = \arctan((y - \xi_y)/(x - \xi_x))\). See figure 1 for a visualisation of this approximate solution. This approximation is valid so long as 
\[ 0 \leq |V| \ll \omega, \quad 0 < \omega \ll 1. \] (3.4)

As for the stationary case, the propagating droplet can be viewed as a precessing, circular domain wall with a radius that is the inverse of the frequency. The new term \(-V\cos(\varphi)/\omega^2\) reveals the deviation of the propagating droplet’s phase from spatial uniformity. While the relations in (3.4) may, at first, seem overly restrictive, we will show that important and practical information about propagating droplets can be obtained in this regime. This approximate solution offers both an error estimate and is amenable to further analysis in the context of the perturbed Landau–Lifshitz equation (1.1). Furthermore, it provides a significant improvement over the approximate droplets used in past numerical experiments [16] when the asymptotic relations (3.4) hold.

Another important property of equation (1.1) in the \(\epsilon = 0\) case is that it admits the conserved quantities
\[ N = \int_{\mathbb{R}^2} (1 - m_z) \, dx, \quad P = \int_{\mathbb{R}^2} \left( \frac{m_y \nabla m_x - m_x \nabla m_y}{1 + m_z} \right) \, dx \]
\[ \mathcal{E} = \frac{1}{2} \int_{\mathbb{R}^2} \left( |\nabla m|^2 + (1 - m_z^2) \right) \, dx + \frac{1}{2} h_0 N, \quad Q = \frac{1}{4\pi} \int_{\mathbb{R}^2} \left( \frac{\partial m}{\partial y} \times \frac{\partial m}{\partial x} \right) \, dx \] (3.5)

where \(N\) is the total spin, \(P\) is the momentum, \(\mathcal{E}\) is the total energy and \(Q\) is the topological charge. These quantities are not independent for the droplet itself, as the droplet is the energy minimizing solution constrained by the total spin and momentum. Using the approximate form (3.2) and (3.3) for the droplet, a map can be constructed between its parameters and the conserved quantities. Evaluating the integrals in equation (3.5) at the approximate droplet, we obtain
\[ N = \frac{2\pi}{\omega^2}, \quad P = \frac{2\pi}{\omega^3} V, \quad \mathcal{E} = \frac{\pi}{\omega^3} (|V|^2 + 4\omega^2 + h_0\omega), \quad Q = 0, \] (3.6)

where higher order terms in \(\omega\) and \(|V|\) have been neglected. These formulae extend the predictions for stationary droplets (e.g. [10]) and offer an analogy to classical particle dynamics. Rewriting \(\mathcal{E}\) in terms of the other conserved quantities, we obtain
\[ \mathcal{E} = \sqrt{2\pi} \left( \frac{1}{2} \frac{|P|^2}{N^{3/2}} + N^{1/2} \right) + \frac{1}{2} h_0 N. \] (3.7)

By analogy to classical systems, we can interpret \(\sqrt{2\pi} |P|^2/2N^{3/2}\) as the kinetic energy of the droplet, \(\sqrt{2\pi} N^{1/2}\) as the droplet’s potential energy due to precession, and \(h_0 N/2\) as the Zeeman energy of the droplet with the net dipole moment \(N\). Inspection of the kinetic energy term shows that
\[ m_{\text{eff}} = \frac{N^{3/2}}{\sqrt{2\pi}} = \frac{2\pi}{\omega^3}, \] (3.8)

serves as the effective mass for the droplet. Therefore, the \(0 < \omega \ll 1\) regime corresponds to droplets with large mass. This is a natural interpretation since it is the precession of the droplet that determines its size and prevents the structure from collapsing in on itself. On the other hand, equation (3.6) implies that the slowly propagating \(|V| \ll \omega\) regime supports droplets with up to \(|P| = O(1/\omega)\) momenta. We will return to this observation of an effective mass for the droplet in §5a, where we consider the dynamical equations induced by spatial inhomogeneity in an external magnetic field.

One description of the magnetic droplet is as a bound state of magnons [10]. It is then natural to interpret the potential energy \(\sqrt{2\pi} N^{1/2}\) as the energy released by decay into these constituent ‘subatomic particles’. The expressions (3.6) can also be used to verify the Vakhitov–Kolokolov...
soliton stability criteria [31,32] for a propagating droplet, namely that

$$N_\omega = -\frac{4\pi}{\omega^2} < 0, \quad N_\omega \nabla_V \cdot \mathcal{P} - \nabla_V N_\omega \cdot \mathcal{P}_\omega = -\frac{8\pi^2}{\omega^4} < 0,$$

as required. The Vakhitov–Kolokolov criterion has been numerically found to be satisfied across a large family of moving droplets [24].

While the moving droplet’s topological charge $Q$ is zero, its topological density $q$ (the integrand for $Q$ in (3.5)) is non-zero, $q = V \cdot \hat{\phi} \text{sech}^2(\rho - 1/\omega)/\omega^2$, as shown in figure 1c. Droplet motion introduces a non-zero local topological structure. A similar topological density, observed in numerical simulations of a NC-STO device supporting droplets, has been interpreted as a magnetic bubble–antibubble pair [33]. Droplets are inherently dynamic states in contrast to static magnetic bubbles [34]. It has been argued that a droplet could be deformed into a static bubble by the introduction of three-dimensional structure (non-zero film thickness) [12,15].

For the remainder of this work, we will use the approximate droplet in equations (3.2) and (3.3).

4. General modulation equations for propagating droplets

With the results of the previous sections, we now have developed sufficient tools to derive the droplet modulation equations. Previous attempts to do this have been limited either to a partial set of equations for $V$ and $\omega$ only, computed numerically [15], or stationary droplet equations [12]. The results of §§2 and 3 enable us to determine the slow evolution of all six soliton parameters due to the perturbation $p$ in equation (1.1). The calculation, see [35], requires some care in preserving the appropriate asymptotic relations (3.4). From expressions (2.4), (3.2) and (3.3), we obtain the droplet soliton modulation equations

$$\dot{\Phi} = \frac{1}{4\pi} \int_{\mathbb{R}^2} (V \cdot \hat{\rho}) \text{sech} \left( \rho - \frac{1}{\omega} \right) p_{\rho} \, dx + \frac{\omega}{4\pi} \int_{\mathbb{R}^2} \text{sech} \left( \rho - \frac{1}{\omega} \right) p_{\phi} \, dx,$$

$$\dot{\Theta} = \frac{V}{\epsilon} + \frac{\omega}{2\pi} \int_{\mathbb{R}^2} \text{sech} \left( \rho - \frac{1}{\omega} \right) \hat{\rho} p_{\rho} \, dx,$$

$$\dot{\omega} = -\frac{\omega^3}{4\pi} \int_{\mathbb{R}^2} \text{sech} \left( \rho - \frac{1}{\omega} \right) p_{\theta} \, dx$$

and

$$\dot{V} = -\frac{\omega^2}{2\pi} \int_{\mathbb{R}^2} \left( \frac{3}{2} V - \frac{V \cdot \hat{\phi}}{\rho \omega} \right) \text{sech} \left( \rho - \frac{1}{\omega} \right) p_{\phi} \, dx - \frac{\omega^3}{2\pi} \int_{\mathbb{R}^2} \text{sech} \left( \rho - \frac{1}{\omega} \right) \hat{\rho} p_{\phi} \, dx,$$

where the over dot denotes differentiation with respect to $T$. This general set of equations is the main result of this work. They are asymptotically valid when

$$0 < \epsilon \ll 1, \quad T \ll \epsilon^{-1}, \quad 0 \leq |V| \ll \omega \ll 1.$$

The perturbation components $p_{\rho}, p_{\phi}$ are to be evaluated with the approximate droplet solution (3.2) and (3.3). Some explanation of the perturbation components and the unit vectors $\hat{\rho}, \hat{\phi}$ is warranted. The magnetization $m$ has the unit sphere $S^2$ as its range. We can therefore define the standard, right-handed, orthonormal, spherical basis $\{\hat{r}, \hat{\phi}, \hat{\Theta}\}$ for $\mathbb{R}^3$ where $\hat{r} = m$ is the radial unit vector, $\hat{\phi}$ is the azimuthal unit vector and $\hat{\Theta}$ is the polar unit vector. The components of $p$ in this ‘magnetization centred’ basis are

$$0 = \hat{r} \cdot p, \quad p_{\phi} = \hat{\phi} \cdot p, \quad p_{\theta} = \hat{\Theta} \cdot p.$$

On the other hand, the domain $\mathbb{R}^2$ has the standard orthonormal, polar basis $\{\hat{r}, \hat{\phi}\}$, where $\hat{r}$ and $\hat{\phi}$ are the radial and azimuthal unit vectors centred on the droplet, respectively. This corresponds to the ‘domain centred’ basis. It is important not to confuse the domain $\mathbb{R}^2$ and range $S^2$ of $m$.

In this general formulation, we have neglected spatial inhomogeneity of the perpendicular magnetic field magnitude $h_0$. In §5a, we will incorporate inhomogeneity as a perturbation with non-zero $p_{\phi}$ component. Even with an inhomogeneous magnetic field, the total frequency of the
We see that variations in the initial phase $\Phi_0$ provide a higher order correction to the droplet frequency. Additionally, the second term on the right-hand side of equation (4.2) is a higher order correction to the droplet’s total velocity $\dot{\xi}$. These higher order corrections have proved to be of fundamental importance in the study of stationary droplets [12] and beyond, see, e.g., [36] for an application to NLS dark solitons.

While quite general, these equations do not treat all perturbations. It is important to note that the solvability condition which gives rise to these equations applies for those perturbations whose temporal dependence is on a slow time scale. In the sections that follow, we consider a range of physical perturbations that meet this criterion in order to demonstrate the versatility of this approach. However, some physical scenarios (such as an applied field varying rapidly in time) might not satisfy this assumption. Such perturbations may be regular perturbations and not induce dynamics within the family of solitons so they will not be discussed further.

5. Applications to perturbed systems

In this section, we analyse a range of perturbations to demonstrate the versatility of this framework as well as to provide physical insights into droplet dynamics.

(a) Slowly varying applied field

In practical applications, the magnetic field will typically have some spatial variation whose scale is much larger than the scale of the droplet, i.e. the exchange length divided by $\omega$. For this, we assume that $h_0 = h_0(\epsilon x, \epsilon t)$, $0 < \epsilon / \omega \ll 1$. This inhomogeneity is best treated by introducing an appropriate perturbation $p$ in equation (1.1). Expanding $h_0$ about the soliton centre, $\xi$,

$$h_0(\epsilon x, \epsilon t) = h_0(\epsilon \xi, \epsilon t) + \epsilon \tilde{\nabla} h_0 |_{x=\xi} \cdot (x - \xi) + O(\epsilon^2),$$

where $\tilde{\nabla}$ represents the gradient with respect to the slow variable $X = \epsilon x$. Inserting expansion (5.1) into the cross product $-m \times (h_0 \hat{z})$ from equation (1.1) introduces the perturbation

$$p_\Theta = 0 \quad \text{and} \quad p_\Phi = (\tilde{\nabla} h_0 \cdot \hat{\rho}) \rho.$$

Substituting these into equations (4.1)–(4.4) leads to Newton’s second law for the droplet centre

$$\frac{d^2 \xi}{dT^2} = \epsilon \frac{dV}{dT} = -\omega \nabla h_0. \quad (5.3)$$

Note that $\nabla$ here represents the gradient with respect to the fast variable $x$, distinguishing it from $\tilde{\nabla}$. The phase $\Phi_0$ and frequency $\omega$ are unchanged by the field gradient.

A favourable comparison of direct numerical simulations for equation (1.1) (see the electronic supplementary material) with the solution to (5.3) is shown in figure 2. We now demonstrate that the explicit equation (5.3) agrees with the previous result in [15] obtained by perturbing conservation laws and integrating the equations numerically. Previously, the non-trivial dynamical equation was $dP/dt = -N \nabla h_0$. We can transform this equation into equation (5.3) by using the explicit formulae in (3.6) for $N$ and $P$. Since $d\omega/dT = 0$ and $N$ depends only on $\omega$, $dN/dT = 0$. Then

$$\frac{dP}{dT} = \frac{N^{3/2}}{2\sqrt{\pi}} \frac{dV}{dT} = \frac{m_{\text{eff}}}{\epsilon} \frac{d^2 \xi}{dT^2} = -\frac{N}{\epsilon} \nabla h_0. \quad (5.4)$$

This is exactly (5.3). The particle-like droplet with mass $m_{\text{eff}}$ in equation (3.8) experiences a conservative force due to the potential $N h_0$. This interpretation is consistent with the analysis of the effective mass derived from the kinetic energy in §3. Furthermore, it demonstrates that a
The droplet in a magnetic field gradient behaves effectively like a single magnetic dipole with net dipole moment $N$.

The effect of an inhomogeneous magnetic field on a massive two-dimensional droplet is markedly different from its effect on a one-dimensional droplet [13] and a vortex [37]. A one-dimensional droplet experiences periodic, Bloch-type oscillations for a magnetic field with constant gradient, while a magnetic vortex exhibits motion perpendicular to the field gradient direction.

(b) Damping

In [15], it was observed that the droplet accelerates as it decays in the presence of damping alone. The framework presented here offers an analytical tool to understand this slightly counterintuitive result, namely that damping can cause the otherwise steady droplet to speed up. The relevant contributions to equation (1.1) are

$$
 p_\Theta = -(\omega + h_0 - V \cdot \nabla \Phi) \sin(\Theta) \quad \text{and} \quad p_\Phi = -V \cdot \nabla \Theta, \quad (5.5)
$$

where the small parameter $\epsilon$ is the Landau–Lifshitz magnetic damping parameter, usually denoted $\alpha$. In many practical situations, the damping parameter is quite small.

Evaluation of equations (4.1)–(4.4) with these perturbations yields two non-trivial equations

$$
 \frac{d\omega}{dT} = \omega^2 (\omega + h_0) \quad (5.6)
$$

and

$$
 \frac{dV}{dT} = \omega V (\omega + 2h_0). \quad (5.7)
$$

These equations are again consistent with the numerical, perturbed conservation law approach taken in [15] when evaluated at the approximate solution.

We observe that the right-hand sides of the modulation equations are both positive for $h_0 > -\omega/2$. Hence, the frequency and velocity increase. Equation (5.6) can be interpreted as a dynamical equation for the droplet’s mass $m_{\text{eff}}$ (equation (3.8)). The mass is decreasing at a fast rate so that the velocity increases. In light of the interpretation given in §3, even though the droplet is losing energy, it sheds mass fast enough that its acceleration is not a contradiction. In figure 3, we see good agreement between the modulation theory and full micromagnetic simulations.

As equation (5.6) decouples in this system, an analytical solution can be found. Elementary application of partial fractions yields an explicit solution in terms of the Lambert W-function;

Figure 2. Acceleration of the droplet due to the inhomogeneous magnetic field $h_0 = 0.5 - 10^{-4} x$ with $\omega(0) = 0.1$ and $|V(0)| = 0$. The exact solution to equation (5.3) (solid) compares favourably to direct numerical simulations of the PDE (dashed). (Online version in colour.)
Figure 3. The evolution of droplet frequency (a) and velocity (b) due to damping for both numerical solutions of equations (5.6) and (5.7) (solid) and direct numerical simulations of equation (1.1) (dashed) when $\epsilon = \alpha = 0.01, h_0 = 0.5, \omega(0) = 0.1$ and $|V(0)| = 0.01$. (Online version in colour.)

however, the analysis is significantly simplified when $h_0 = 0$. In this case, the analytical solution to equations (5.6) and (5.7) is

$$\omega(t) = \frac{\omega_0}{\sqrt{1 - 2\alpha \omega_0^2 t}}$$

and

$$V(t) = \frac{V_0}{\sqrt{1 - 2\alpha \omega_0^2 t}}$$

where $\omega_0$ is the initial precession frequency and $V_0$ the initial velocity. These expressions reveal two facts: a clear time of breakdown for modulation theory and the existence of an adiabatic invariant. Dividing equation (5.8) by the components of equation (5.9) demonstrates that the quantities $\omega/V_x$ and $\omega/V_y$ are constant in time.

(c) Nanocontact spin-torque oscillator and spatially inhomogeneous applied field

So far, the examples we have chosen to focus on have not included higher order contributions via the phase $\phi_0$ and the second term of the equation for the droplet centre $\xi$. But many perturbations and physical behaviours cannot be investigated without these higher order terms. Consider the more complex system of an NC-STO, in which a polarized spin current exerts a torque on the magnetization, the spin transfer torque [38,39]. This forcing can be confined to a localized region via a nanocontact [40–42]. Perturbations of this sort lead to dynamics within all the parameters of the droplet. In addition to spin torque, a droplet in an NC-STO also experiences damping, and it is precisely the balance between the two that leads to the stable droplet observed in experiments. Here, we consider the addition of weak spatial inhomogeneity of the applied magnetic field. For simplicity, we restrict our consideration to a constant magnetic field gradient.

This investigation has broader implications for the practical use and understanding of droplets in real devices. We show in this section that these three physical effects influence the system in competing ways, which can balance, allowing for the existence of stable droplets. Alternatively, a strong enough field gradient can push the droplet out of the NC-STO, giving rise to a previously unexplained drift instability [11]. As seen in §5b, damping decreases the effective mass of the droplet. In §5a, it was shown that a field inhomogeneity accelerates the droplet while leaving the mass of the droplet unaffected. The inclusion of forcing due to spin transfer torque in a nanocontact opposes both of these effects. The spin torque increases the droplet mass and generates an effective restoring force that centres the droplet in the nanocontact region [12].
Hence, there can exist a delicate balance between all of these effects: the NC-STO restoring force balancing the potential force due to the field gradient and the mass loss due to damping balancing the mass gain due to spin torque. Previous studies have been unable to identify when such a balance occurs and when it fails. Here, we analytically demonstrate stable droplets as fixed points of the modulation equations with all of these perturbations.

Because the perturbation components $p_\theta$ and $p_\phi$ appear linearly in the modulation equations (4.1)–(4.4), we can simply add the field inhomogeneity equation (5.2) and damping equation (5.5) perturbations to those due to spin torque [43]. Owing to the presence of three different perturbations, we no longer scale the perturbation $p$ in equation (1.1) by the single parameter $\epsilon$. Rather, we set $\epsilon = 1$ and introduce the small parameters in $p_\theta$ and $p_\phi$ directly. The perturbation components are

$$ p_\theta = -\alpha (\omega + h_0 - \mathbf{V} \cdot \nabla \Phi) \sin \Theta + \sigma \mathcal{H}(\rho_x - r) \sin \Theta $$

(5.10)

and

$$ p_\phi = (\nabla h_0 \cdot \hat{\rho}) \rho - \alpha \mathbf{V} \cdot \nabla \Theta. $$

(5.11)

The nanocontact where spin torque is active is assumed to be a circle with radius $\rho_x$. The coordinate $r$ in the argument of the Heaviside function $\mathcal{H}$ is measured from the centre of the nanocontact, which differs from the coordinates $\rho$ and $\varphi$ which are measured from the centre of the droplet. For simplicity, we have neglected the spin torque asymmetry that introduces another parameter into the analysis but does not appear to have a significant effect on the dynamics [11]. Experiments [6,9] and analysis [11,12] have shown that the ratio of damping, $\alpha$, to forcing strength, $\sigma$ (proportional to current), is roughly order $1$ for the existence of droplets to be satisfied. Thus $0 < \sigma \sim \alpha \ll 1$. The magnetic field is assumed to be linear

$$ h_0 = a + bx, \quad |b| \ll \omega. $$

(5.12)

We can restrict to droplet motion in the $\hat{x}$ direction only. Insertion of the perturbations in equations (5.10) and (5.11) into the modulation equations (4.1)–(4.4) results in the following system:

$$ \dot{\Phi} = \frac{abV}{2\omega^2} - \frac{\sigma V}{4\pi} \int_{|x|<\rho_x} \cos(\varphi) \text{sech}^2 \left( \rho - \frac{1}{\omega} \right) dx, $$

(5.13)

$$ \dot{\xi} = V - \frac{ab}{\omega} + \frac{\sigma \omega}{2\pi} \int_{|x|<\rho_x} \cos(\varphi) \text{sech}^2 \left( \rho - \frac{1}{\omega} \right) dx, $$

(5.14)

$$ \dot{\omega} = \alpha \varphi^2 (\omega + a) - \frac{\sigma \omega^3}{4\pi} \int_{|x|<\rho_x} \text{sech}^2 \left( \rho - \frac{1}{\omega} \right) dx $$

(5.15)

and

$$ \dot{V} = -b\omega + \alpha V \omega (\omega + 2a) - \frac{\sigma V \omega}{4\pi} \int_{|x|<\rho_x} \frac{(3\rho \omega + \cos(2\varphi) - 1)}{\rho} \text{sech}^2 \left( \rho - \frac{1}{\omega} \right) dx, $$

(5.16)

where we set $\xi = \xi_x$, $V = V_x$, and the over dot denotes differentiation with respect to $t$. None of the right-hand sides in the equations above depend explicitly on the parameter $\Phi_0$ so that the dynamics of the remaining parameters can be considered separately. We ignore the evolution of $\Phi_0$ for the remainder of the analysis noting that $\Phi_0$ corresponds to a small frequency shift as in equation (4.7) that can be obtained from the evolution of the other parameters by insertion into equation (5.13).

There is a complex interplay between the many small parameters in this problem. As we do not have access to an exact analytical solution, it is necessary that these perturbations dominate over the error terms in our approximate solution, while still remaining small. As we have $|V| \lesssim \omega^2$ to keep an overall consistent error estimate for the approximate droplet, we require that $a, \sigma \ll \omega$. The variation in the applied field, $b$, is more subtle and the appropriate scaling will be determined by directly computing fixed points.

The stationary droplet without a field gradient is stable when centred on the nanocontact [11,12]. This results from an analysis of the stationary modulation equations which exhibit an attractive, stable fixed point. Taking $\frac{dh_0}{dx} = b = 0$, the modulation equations (5.14)–(5.16) for
Thus, we observe that the droplet radius is approximately $\rho$ system of equations (5.14)–(5.16) can be written as very simple predictions in the regime of small field gradient. The key observation here is that the persistence of the droplet fixed point for very small $\sigma$ is increased, with $\omega = \omega_\sigma(\sigma)$ corresponding to the stable branch. For $\sigma$ sufficiently large, the stable branch quickly approaches

$$\omega_\sigma = \rho_\sigma^{-1} + \arctanh \left( \frac{2a\alpha}{\sigma} - 1 \right) \rho_\sigma^{-2} + O\left( \rho_\sigma^{-3} \right), \quad \rho_\sigma \gg 1, \quad 0 < \omega_\sigma - \rho_\sigma^{-1} \ll 1. \quad (5.18)$$

Thus, we observe that the droplet radius is approximately $\rho_\sigma$. Near the critical value $\sigma = 2a\alpha$, where the second term is small, the asymptotic form is

$$\omega_\sigma = \rho_\sigma^{-1} + \left( \frac{2a\alpha}{\sigma} - 1 \right) \rho_\sigma^{-2} + \left( \frac{2a\alpha + \ln 2}{\sigma} \right) \rho_\sigma^{-3} + O\left( \rho_\sigma^{-4} \right), \quad \left| \left| \frac{2a\alpha}{\sigma} - 1 \right| \right| = O\left( \rho_\sigma^{-1} \right). \quad (5.19)$$

Linearizing equations (5.14)–(5.16) about this fixed point, the Jacobian matrix is given by

$$J(0, \omega_\sigma, 0) = \begin{pmatrix} \lambda_1 & 0 & 1 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \quad (5.20)$$

$$\lambda_1 = -\frac{1}{2} \sigma \rho_\sigma \omega_\sigma \sech^2 \left( \rho_\sigma - \frac{1}{\omega_\sigma} \right), \quad (5.21)$$

$$\lambda_2 = -a\alpha \omega_\sigma + \lambda_1 + \frac{1}{2} \sigma \omega_\sigma \left( \tanh \left( \rho_\sigma - \frac{1}{\omega_\sigma} \right) + 1 \right) \quad (5.22)$$

and

$$\lambda_3 = -2a\alpha \rho_\sigma^{-2} + \lambda_2 - \lambda_1. \quad (5.23)$$

This linearization represents a generalization of that considered in [12], where the motion was restricted to $V = 0$. Since $\rho_\sigma > \omega_\sigma^{-1}$, we observe that all eigenvalues are negative when $\sigma > a\alpha$, so the fixed point is stable. The critical forcing value $\sigma = a\alpha$, below which the droplet may be unstable could be considered as an estimate for the minimum sustaining current of a droplet [11]. Note, however, that this is a dubious estimate due to $\omega_\sigma - \rho_\sigma^{-1}$ not being a small quantity. Using the approximation from equation (5.19), we find

$$\lambda_1 = -\frac{\sigma}{2} + O(\rho_\sigma^{-2}), \quad \lambda_2 \sim \lambda_1, \quad \lambda_3 = \left( -\alpha + \frac{\ln 2}{2\sigma} \right) \rho_\sigma^{-2} + O(\rho_\sigma^{-4}), \quad \left| \left| \frac{2a\alpha}{\sigma} - 1 \right| \right| = O\left( \rho_\sigma^{-1} \right). \quad (5.24)$$

We now turn our attention to the case of a small field gradient $0 < |b| \ll \omega$, where we observe the persistence of the droplet fixed point for very small $|b|$. These fixed points exist as a balance between the expulsive force provided by the field gradient and the attractive force provided by the nanocontact. This attraction manifests in the evolution of $\xi$ and so this balance can also be viewed as a balance between leading order effects (in $V$) and higher order effects (in $\xi$). Unlike the stationary fixed point, exact analytical expressions for the fixed point cannot be found since the droplet is no longer centred on the nanocontact ($\xi \neq 0$). Nevertheless, we can obtain the approximate form for these fixed points as follows. The structure of $J$ in equation (5.20) yields very simple predictions in the regime of small field gradient. The key observation here is that the system of equations (5.14)–(5.16) can be written as

$$\begin{pmatrix} \frac{\dot{\xi}}{\omega} \\ \frac{\dot{\omega}}{V} \end{pmatrix} = F(\xi, \omega, V) - b \begin{pmatrix} \frac{\alpha}{\omega} \\ 0 \end{pmatrix}. \quad (5.25)$$

Note that the corresponding equation in [12] has an error. There is a saddle node bifurcation as $\sigma$ is increased, with $\omega = \omega_\sigma(\sigma)$ corresponding to the stable branch. For $\sigma$ sufficiently large, the stable branch quickly approaches

$$\omega_\sigma = \rho_\sigma^{-1} + \arctanh \left( \frac{2a\alpha}{\sigma} - 1 \right) \rho_\sigma^{-2} + O\left( \rho_\sigma^{-3} \right), \quad \rho_\sigma \gg 1, \quad 0 < \omega_\sigma - \rho_\sigma^{-1} \ll 1. \quad (5.18)$$

Thus, we observe that the droplet radius is approximately $\rho_\sigma$. Near the critical value $\sigma = 2a\alpha$, where the second term is small, the asymptotic form is

$$\omega_\sigma = \rho_\sigma^{-1} + \left( \frac{2a\alpha}{\sigma} - 1 \right) \rho_\sigma^{-2} + \left( \frac{2a\alpha + \ln 2}{\sigma} \right) \rho_\sigma^{-3} + O\left( \rho_\sigma^{-4} \right), \quad \left| \left| \frac{2a\alpha}{\sigma} - 1 \right| \right| = O\left( \rho_\sigma^{-1} \right). \quad (5.19)$$

Linearizing equations (5.14)–(5.16) about this fixed point, the Jacobian matrix is given by

$$J(0, \omega_\sigma, 0) = \begin{pmatrix} \lambda_1 & 0 & 1 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \quad (5.20)$$

$$\lambda_1 = -\frac{1}{2} \sigma \rho_\sigma \omega_\sigma \sech^2 \left( \rho_\sigma - \frac{1}{\omega_\sigma} \right), \quad (5.21)$$

$$\lambda_2 = -a\alpha \omega_\sigma + \lambda_1 + \frac{1}{2} \sigma \omega_\sigma \left( \tanh \left( \rho_\sigma - \frac{1}{\omega_\sigma} \right) + 1 \right) \quad (5.22)$$

and

$$\lambda_3 = -2a\alpha \rho_\sigma^{-2} + \lambda_2 - \lambda_1. \quad (5.23)$$

This linearization represents a generalization of that considered in [12], where the motion was restricted to $V = 0$. Since $\rho_\sigma > \omega_\sigma^{-1}$, we observe that all eigenvalues are negative when $\sigma > a\alpha$, so the fixed point is stable. The critical forcing value $\sigma = a\alpha$, below which the droplet may be unstable could be considered as an estimate for the minimum sustaining current of a droplet [11]. Note, however, that this is a dubious estimate due to $\omega_\sigma - \rho_\sigma^{-1}$ not being a small quantity. Using the approximation from equation (5.19), we find

$$\lambda_1 = -\frac{\sigma}{2} + O(\rho_\sigma^{-2}), \quad \lambda_2 \sim \lambda_1, \quad \lambda_3 = \left( -\alpha + \frac{\ln 2}{2\sigma} \right) \rho_\sigma^{-2} + O(\rho_\sigma^{-4}), \quad \left| \left| \frac{2a\alpha}{\sigma} - 1 \right| \right| = O\left( \rho_\sigma^{-1} \right). \quad (5.24)$$

We now turn our attention to the case of a small field gradient $0 < |b| \ll \omega$, where we observe the persistence of the droplet fixed point for very small $|b|$. These fixed points exist as a balance between the expulsive force provided by the field gradient and the attractive force provided by the nanocontact. This attraction manifests in the evolution of $\xi$ and so this balance can also be viewed as a balance between leading order effects (in $V$) and higher order effects (in $\xi$). Unlike the stationary fixed point, exact analytical expressions for the fixed point cannot be found since the droplet is no longer centred on the nanocontact ($\xi \neq 0$). Nevertheless, we can obtain the approximate form for these fixed points as follows. The structure of $J$ in equation (5.20) yields very simple predictions in the regime of small field gradient. The key observation here is that the system of equations (5.14)–(5.16) can be written as

$$\begin{pmatrix} \frac{\dot{\xi}}{\omega} \\ \frac{\dot{\omega}}{V} \end{pmatrix} = F(\xi, \omega, V) - b \begin{pmatrix} \frac{\alpha}{\omega} \\ 0 \end{pmatrix}. \quad (5.25)$$
Figure 4. Fixed points from modulation theory, exact (solid) and approximate equation (5.26) (dashed), and direct numerical simulation of equation (1.1) (circles) when $\alpha = \sigma = 0.01, a = 0.5, \rho_* = 12$. In this case, the parameter $V$ cannot be extracted from direct numerical simulations without additional assumptions (see the electronic supplementary material). Accordingly, these data are not presented in (c). (Online version in colour.)

By virtue of the stationary fixed point, $F$ satisfies $F(0, \omega_0, 0) = 0$. We now seek a fixed point that slightly deviates from the stationary one according to $\xi = b\xi_1 + \cdots, \omega = \omega_* + b\omega_1 + \cdots$ and $V = bV_1 + \cdots$. Expanding and equating the right-hand side of equation (5.25) to zero gives the correction

$$\begin{pmatrix} \xi_1 \\ \omega_1 \\ V_1 \end{pmatrix} = J(0, \omega_0, 0)^{-1} \begin{pmatrix} \frac{\alpha}{\omega_*} \\ 0 \\ \frac{\omega_*}{\lambda_3} \end{pmatrix} \approx \begin{pmatrix} \frac{4\rho_*}{-2a\sigma + \sigma^2 \ln 2} \\ 0 \\ \frac{2\rho_*}{-2a + \sigma \ln 2} \end{pmatrix} , \quad (5.26)$$

where the approximations (5.19) and (5.24) were used to obtain the large $\rho_*$ estimate. These approximations are valid so long as $\sigma$ is more than $\rho_*^{-2}$ away from the critical value $2a/\ln 2 \approx 2.9a$. Otherwise, higher order terms in equation (5.24) would need to be considered.

As summarized in figure 4, these simple expressions make predictions in good agreement with the fixed points found by numerical continuation in $b$ and those observed in long-time micromagnetic simulations of equation (1.1) with perturbations (5.10) and (5.11). The Jacobian matrix of equations (5.14)–(5.16) can also be numerically evaluated, showing that all eigenvalues are negative, until continuation breaks down when one eigenvalue reaches zero. After this bifurcation, we do not find any fixed points. The condition of this eigenvalue reaching zero then corresponds exactly to the crossover where the attractive nanocontact is no longer strong enough to balance the expulsive force supplied by the field gradient. If we assume that a field gradient is strong enough to move the droplet an order one distance, still small relative to the droplet radius $\omega_*^{-1}$, then we obtain a typical field gradient scaling $b \approx a\sigma/2\rho_*$. This field gradient is very small. For the example studied here, $b \approx 10^{-6}$ compared to the NC-STO forcing magnitude $\sigma = 10^{-2}$. This demonstrates that droplet attraction due to spin torque is weak relative to droplet acceleration due to field inhomogeneity. A strong enough field gradient, on the scale of $a\sigma/2\rho_*$, can eject the droplet from the nanocontact, causing a drift instability previously observed in numerical simulations [11]. Additionally, the associated velocity scale from equation (5.26) is $V = bV_1 \approx a\sigma/(-2a + \sigma \ln 2)$ which is much smaller than $\omega$ as required for this order of accuracy of the approximate droplet.

6. Interacting droplets

An intriguing, indeed defining, aspect of solitary wave dynamics is their interaction behaviour. Maiden et al. [17] undertook a numerical investigation of two interacting droplets by varying droplet parameters and quantifying the properties of the solution post-interaction. It was found that the relative phase difference between the two droplets plays a fundamental role, controlling
whether the interaction is attractive or repulsive. The attractive interactions studied were strongly nonlinear, hence a perturbation theory would be insufficient to study the full complement of observed phenomena. Nevertheless, we can gain insight into the nature of the interaction (attractive/repulsive) by studying two well-separated droplets perturbatively, with the small parameter being the inverse of the droplet separation. This approach is well known and has been applied successfully to, for example, NLS-type models [23,44]. This approach has also been used in more general models in two and three dimensions [45].

In full generality, the perturbations arising from this analysis are complex. However, since the validity of these equations is strongly dependent on the separation of the two droplets, we only expect these equations to be valid over short time scales. Hence, we only seek to observe in full numerical simulations [17]. The initial configuration places one droplet on the left (subscripted 1) and another droplet (subscripted 2) a distance \( d \) away along the \( x \)-axis. We define the relative phase difference \( \Delta \Phi = \Phi_2 - \Phi_1 \), which will emerge as an important quantity in the modulation equations. Considering the modulation equations for two weakly interacting droplets with motion in the \( x \)-direction at the initial time only yields

\[
\Phi_{0,k} = -\frac{\omega}{2\pi} \cos(\Delta \Phi) \int_{\mathbb{R}^2} K_k(x) \, dx, \\
\dot{\xi}_k = \frac{\omega}{2\pi} (-1)^{k+1} \sin(\Delta \Phi) \int_{\mathbb{R}^2} K_k(x) \text{sech} \left( \rho - \frac{1}{\omega} \right) \cos \phi \, dx, \\
\dot{\omega}_k = -\frac{\omega^3}{4\pi} (-1)^{k+1} \sin(\Delta \Phi) \int_{\mathbb{R}^2} K_k(x) \text{sech} \left( \rho - \frac{1}{\omega} \right) \, dx
\]

and

\[
\dot{V}_k = \frac{\omega^3}{\pi} \cos(\Delta \Phi) \int_{\mathbb{R}^2} K_k(x) \cos \phi \, dx,
\]

where

\[
K_k(x) = \text{sech} \left( \tilde{\rho}_k - \frac{1}{\omega} \right) \text{sech} \left( \rho - \frac{1}{\omega} \right) \times \left[ 2\text{sech}^2 \left( \rho - \frac{1}{\omega} \right) - \omega \left( 1 - \tanh \left( \rho - \frac{1}{\omega} \right) \right) \right].
\]

The integration kernel \( K_k \) depends on the separation between the two droplets through \( \tilde{\rho}_k = \sqrt{(x + (-1)^k d)^2 + y^2} \). Using this framework, we can now offer some insight into the nature of two interacting droplets. Unlike the previous sections, the strongly nonlinear nature of droplet interaction at moderate time scales means that quantitative comparison between the modulation system and parameters extracted from direct numerical simulation is inappropriate. Instead, numerical evidence for this analysis is provided by comparison to Maiden et al. [17] and the analysis here is consistent with those observations.

(a) Attraction and repulsion

The attractive or repulsive nature of two droplets can be understood by considering equation (6.4). As \( \Delta \Phi \) varies, the sign of \( \cos(\Delta \Phi) \) is clear. Determining the initial direction of motion, right or left, of the droplet comes down to determining the sign of the integral term in (6.4). Figure 5a shows the numerical evaluation of the right-hand side of \( \dot{V}_1 \) (droplet on left) when \( \Delta \Phi = 1 < \pi/2 \), leading to positive values only. Thus, the left droplet experiences a positive acceleration to the right, towards the other droplet when \( |\Delta \Phi| < \pi/2 \). Since the kernel exhibits a symmetry with respect to droplet choice \( K_1(x,y) = K_2(-x,y) \), the integral in (6.4) for the right droplet, \( k = 2 \), has the opposite sign. The right droplet experiences a negative acceleration to the left when \( |\Delta \Phi| < \pi/2 \). Therefore, two droplets are attractive when \( |\Delta \Phi| < \pi/2 \), i.e. when they are sufficiently in phase. Similarly, when \( \pi/2 < |\Delta \Phi| < \pi \), the signs of \( \dot{V}_k \) are reversed and the droplets
move away from each other. Thus, two droplets are repulsive when they are sufficiently out of phase, precisely what was observed in [17].

As was noted in [17] by a nonlinear method of images, the attractive or repulsive nature of two droplets with the special initial values \( \Delta \Phi = 0 \) or \( \Delta \Phi = \pi \) describes the behaviour of a single droplet near a magnetic boundary with either a free spin (Neumann-type) boundary condition or a fixed spin (Dirichlet-type) boundary condition, respectively. The analysis presented here confirms this fact for any droplet that weakly interacts with a magnetic boundary. Such behaviour was observed in micromagnetic simulations of a droplet in an NC-STO, nanowire geometry [46].

(b) Asymmetry

Despite a highly symmetric initial condition, an asymmetry was observed in the so-called ‘head-on collisions’ of two droplets in [17]. The frequency equation (6.3) provides an explanation of this in the limit of very small velocities. Figure 5(b) contains the relevant information. \( \dot{\omega}_1 \) is always negative when \( \sin(\Delta \Phi) > 0 \). For the numerical experiments in [17] were done over the range \( \Delta \Phi = 0 \) to \( \Delta \Phi = \pi \), this was always the case. Symmetry breaking has been explained in one-dimensional systems [47] with a similar analysis to what is provided here for two-dimensional droplets.

Again using that \( K_1(x, y) = K_2(-x, y) \), it can be seen that the integrals involved in computing \( \dot{\omega}_1 \) and \( \dot{\omega}_2 \) are equal. Hence the sign of \( \dot{\omega}_k \) is determined by \((-1)^k+1\), and the signs of \( \dot{\omega}_1 \) and \( \dot{\omega}_2 \) will always be opposite. For the parameters discussed here, this means that the frequency decreases for the droplet on the left and increases on the right. This change in droplet structure is asymmetric because a reduced (increased) frequency implies larger (smaller) droplet mass and corresponds precisely with the observations of Maiden et al. [17].

(c) Acceleration

The discussion of attraction and repulsion in §6a suggests that the boundary between the two behaviours is \( \Delta \Phi = \pi/2 \). But this does not agree with numerical experiments where the crossover \( \Delta \Phi \) was found to vary with the initial droplet parameters [17]. To offer an explanation for this, we consider the total acceleration of the initial droplets, i.e. \( \ddot{\xi}_k \). This incorporates higher order information not included in \( \dot{V}_k \). Since the full modulation equations for interacting droplets when \( V \neq 0 \) are complex, we do not examine \( \ddot{\xi}_k \) for all values of \( \Delta \Phi \). However, at \( \Delta \Phi = \pi/2 \), we know \( \dot{V}_k = \Phi_0 = 0 \), (since \( \cos(\Delta \Phi) = 0 \)) and those terms will not contribute, which simplifies the calculation. Figure 6 shows the initial, total droplet acceleration \( \ddot{\xi}_1 \), evaluated numerically, as the initial frequency and separation are varied. The variable sign of this quantity as parameters change demonstrates that subtle, higher order effects cause the crossover value of \( \Delta \Phi \) to deviate from its nominal value \( \pi/2 \).
Figure 6. Numerical evaluation of $\xi_1$ initially for $\Delta \Phi = \pi/2$, variable droplet separation $d$ and frequency $\omega_0$. There is not one sign of acceleration, i.e. the left droplet can be repelled or attracted to the right droplet depending on the choice of parameters. (Online version in colour.)

7. Conclusion

The primary contribution of this work is a general framework for investigating perturbations of droplet solitons. Actual physical devices used to create and manipulate droplets are quite complex, incorporating a number of physical effects. Therefore, having a tractable, analytical theory to describe both the motion and precession of droplets due to physical perturbations is quite valuable. The examples presented here are meant to demonstrate the versatility and power of this tool. Additionally, the application of this theory to the NC-STO provides several insights into the behaviour of experimentally observed dissipative droplets. In particular, the dissipative droplet is shown to be robust in the presence of weak field gradients, but can be ejected from the nanocontact if the field gradient is too large, providing an explanation for a previously observed drift instability. These observations open possible mechanisms for generating a current of solitons which could serve as a mechanism for information transfer.

As demonstrated by examples in the preceding sections, many perturbations excite evolution of higher order parameters (overall phase and position). This subtle information proves to be of fundamental importance for several perturbations considered. As shown by our derivation of the modulation equations for a general class of Hamiltonian systems, the higher order parameter dynamics emerge when the generalized nullspace of the linearized evolution operator is incorporated. Owing to the existence of an approximate analytical form for the propagating droplet, we are able to characterize this nullspace and hence recover the droplet modulation equations in a convenient form.

A number of physical perturbations can now be investigated within this framework. Future developments of the modulation theory could be performed in the context of a different family of magnetic droplet solutions. One example is the weakly nonlinear droplet [27]. Micromagnetic simulations exhibited rotating, precessing localized waves in NC-STOs with non-trivial magnetostatic contributions [33]. The invariance of equation (1.1) when $\epsilon = 0$ with respect to rotation of the domain and an analysis of conserved quantities [37] suggests that there may be rotating and precessing solitary wave solutions. The modulation theory developed in this work could be extended to such solutions.

Competing interests. We declare we have no competing interests.

Funding. The authors gratefully acknowledge support through an NSF CAREER grant.

References

1. Uhlíř V. 2013 Dynamic switching of the spin circulation in tapered magnetic nanodisks. Nat. Nanotechnol. 8, 341–346. (doi:10.1038/nnano.2013.66)
2. Pulecio JF, Warnicke P, Pollard SD, Arena DA, Zhu Y. 2014 Coherence and modality of driven interlayer-coupled magnetic vortices. Nat. Commun. 5, 3760. (doi:10.1038/ncomms4760)
3. Nagaosa N, Tokura Y. 2013 Topological properties and dynamics of magnetic skyrmions. *Nat. Nanotechnol.* 8, 899–911. (doi:10.1038/nnano.2013.243)

4. Fert A, Cros V, Sampaio J. 2013 Skyrmions on the track. *Nat. Nanotechnol.* 8, 152–156. (doi:10.1038/nnano.2013.29)

5. Yu XZ, Tokunaga Y, Kaneko Y, Zhang WZ,Kimoto K, Matsui Y, Taguchi Y, Tokura Y. 2014 Biskyrmion states and their current-driven motion in a layered manganite. *Nat. Commun.* 5, 3198. (doi:10.1038/ncomms4198)

6. Mohseni SM *et al.* 2013 Spin torque–generated magnetic droplet solitons. *Science* 339, 1295–1298. (doi:10.1126/science.1230155)

7. Mohseni SM *et al.* 2014 Magnetic droplet solitons in orthogonal nano-contact spin torque oscillators. *Physica B* 435, 84–87. (doi:10.1016/j.physb.2013.10.023)

8. Chung S *et al.* 2014 Spin transfer generated magnetic droplet solitons (invited). *J. Appl. Phys.* 115, 172612. (doi:10.1063/1.4870696)

9. Macià F, Backes D, Kent AD. 2014 Stable magnetic droplet solitons in spin-transfer nanocontacts. *Nat. Nanotechnol.* 9, 992–996. (doi:10.1038/nnano.2014.255)

10. Kosevich AM, Ivanov BA, Kovalev AS. 1990 Magnetic solitons. *Phys. Rep.* 194, 117–238. (doi:10.1016/0370-1573(90)90130-T)

11. Hoefer MA, Silva TJ, Keller MW. 2010 Theory for a dissipative droplet soliton excited by a spin torque nanocontact. *Phys. Rev. B* 82, 054432. (doi:10.1103/PhysRevB.82.054432)

12. Bookman LD, Hoefer MA. 2013 Analytical theory of modulated magnetic solitons. *Phys. Rev. B* 88, 184401. (doi:10.1103/PhysRevB.88.184401)

13. Kosevich AM, Gann VV, Zhukov AI, Voronov VP. 1998 Magnetic soliton motion in a nonuniform magnetic field. *J. Exp. Theor. Phys.* 87, 401–407. (doi:10.1134/1.558674)

14. Babich IM, Kosevich AM. 2001 Relaxation of bloch oscillations of a magnetic soliton in a nonuniform magnetic field. *Low Temp. Phys.* 27, 35–39. (doi:10.1063/1.1344140)

15. Hoefer MA, Sommacal M, Silva TJ. 2012 Propagation and control of nanoscale magnetic-droplet solitons. *Phys. Rev. B* 85, 214433. (doi:10.1103/PhysRevB.85.214433)

16. Piette B, Zakrzewski WJ. 1998 Localized solutions in a two-dimensional Landau–Lifshitz model. *Physica D* 119, 314–326. (doi:10.1016/S0167-2789(98)00084-0)

17. Maiden MD, Bookman LD, Hoefer MA. 2014 Attraction, merger, reflection, and annihilation in magnetic droplet soliton scattering. *Phys. Rev. B* 89, 180409. (doi:10.1103/PhysRevB.89.180409)

18. Kivshar YS, Malomed BA. 1989 Dynamics of solitons in nearly integrable systems. *Rev. Mod. Phys.* 61, 763–915. (doi:10.1103/RevModPhys.61.763)

19. Sanchez A, Bishop AR. 1998 Collective coordinates and length-scale competition in spatially inhomogeneous soliton-bearing equations. *SIAM Rev.* 40, 579–615. (doi:10.1137/S0036144597317418)

20. Yang J. 2010 *Nonlinear waves in integrable and nonintegrable systems*. Philadelphia, PA: Society for Industrial and Applied Mathematics.

21. Kevrekidis PG, Frantzeskakis DJ, Carretero-González R. 2008 *Emergent nonlinear phenomena in Bose–Einstein condensates: theory and experiment*. Berlin, Germany: Springer.

22. Keener JP, McLaughlin DW. 1977 Solitons under perturbations. *Phys. Rev. A* 16, 777–790. (doi:10.1103/PhysRevA.16.777)

23. Ablowitz MJ, Horikis TP, Nixon SD, Zhu Y. 2009 Asymptotic analysis of pulse dynamics in mode-locked lasers. *Stud. Appl. Math.* 122, 411–425. (doi:10.1111/j.1467-9590.2009.00441.x)

24. Hoefer MA, Sommacal M. 2012 Propagating two-dimensional magnetic droplets. *Physica D* 241, 890–901. (doi:10.1016/j.physd.2012.02.003)

25. Gorschkov KA, Ostrovsky LA, Pelinovsky EN. 1974 Some problems of asymptotic theory of nonlinear waves. *Proc. IEEE* 62, 1511–1517. (doi:10.1109/PROC.1974.9657)

26. Weinstein MI. 1985 Modulational stability of ground states of nonlinear Schrödinger equations. *SIAM J. Math. Anal.* 16, 472–491. (doi:10.1137/0516034)

27. Ivanov BA, Zaspel CE, Yastremsky IA. 2001 Small-amplitude mobile solitons in the two-dimensional ferromagnet. *Phys. Rev. B* 63, 134413. (doi:10.1103/PhysRevB.63.134413)

28. Ablowitz MJ, Segur H. 1981 *Solitons and the inverse scattering transform*, vol. 4. Philadelphia, PA: SIAM.

29. Champneys AR, Sandstede B. 2007 Numerical computation of coherent structures. In *Numerical continuation methods for dynamical systems*, pp. 331–358. Berlin, Germany: Springer.

30. Ivanov BA, Stephanovich VA. 1989 Two-dimensional soliton dynamics in ferromagnets. *Phys. Lett. A* 141, 89–94. (doi:10.1016/0375-9601(89)90453-2)
31. Vakhitov NG, Kolokolov AA. 1973 Stationary solutions of the wave equation in a medium with nonlinearity saturation. *Radiophys. Quantum Electron.* 16, 783–789. (doi:10.1007/BF01031343)

32. Grillakis M, Shatah J, Strauss W. 1990 Stability theory of solitary waves in the presence of symmetry. II. *J. Funct. Anal.* 94, 308–348. (doi:10.1016/0022-1236(90)90016-E)

33. Finocchio G, Puliafito V, Komineas S, Torres L, Ozatay O, Hauet T, Azzerboni B. 2013 Nanoscale spintronic oscillators based on the excitation of confined soliton modes. *J. Appl. Phys.* 114, 163908. (doi:10.1063/1.4827384)

34. Leeuw FHD, Doel RVD, Enz U. 1980 Dynamic properties of magnetic domain walls and magnetic bubbles. *Rep. Prog. Phys.* 43, 689–783. (doi:10.1088/0034-4885/43/6/001)

35. Bookman LD. 2015 Approximate solitons of the Landau–Lifshitz. PhD thesis. North Carolina State University, USA.

36. Ablowitz MJ, Nixon SD, Horikis TP, Frantzeskakis DJ. 2011 Perturbations of dark solitons. *Proc. R. Soc. A* 467, 2597–2621. (doi:10.1098/rspa.2010.0663)

37. Papanicolaou N, Tomaras TN. 1991 Dynamics of magnetic vortices. *Nucl. Phys. B* 360, 425–462. (doi:10.1016/0550-3213(91)90410-Y)

38. Berger L. 1996 Emission of spin waves by a magnetic multilayer traversed by a current. *Phys. Rev. B* 54, 9353–9358. (doi:10.1103/PhysRevB.54.9353)

39. Slonczewski JC. 1996 Current-driven excitation of magnetic multilayers. *J. Magn. Magn. Mater.* 159, L1–L7. (doi:10.1016/0304-8853(96)00062-5)

40. Tsoi M, Jansen AGM, Bass J, Chiang W-C, Seck M, Tsoi V, Wyder P. 1998 Excitation of a magnetic multilayer by an electric current. *Phys. Rev. Lett.* 80, 4281–4284. (doi:10.1103/PhysRevLett.80.4281)

41. Slonczewski JC. 1999 Excitation of spin waves by an electric current. *J. Magn. Magn. Mater.* 195, L261–L268. (doi:10.1016/S0304-8853(99)00043-8)

42. Rippard WH, Pufall MR, Kaka S, Russek SE, Silva TJ. 2004 Direct-current induced dynamics in Co$_{90}$Fe$_{10}$/Ni$_{80}$Fe$_{20}$. *Phys. Rev. Lett.* 92, 027201. (doi:10.1103/PhysRevLett.92.027201)

43. Stiles MD, Miltat J. 2006 Spin transfer torque and dynamics. In *Spin dynamics in confined magnetic structures III* (eds B Hillebrands, A Thiaville), pp. 225–308. Berlin, Heidelberg: Springer.

44. Zhu Y, Yang J. 2007 Universal fractal structures in the weak interaction of solitary waves in generalized nonlinear Schrödinger equations. *Phys. Rev. E.* 75, 036605. (doi:10.1103/PhysRevE.75.036605)

45. Malomed BA. 1998 Potential of interaction between two- and three-dimensional solitons. *Phys. Rev. E* 58, 7928–7933. (doi:10.1103/PhysRevE.58.7928)

46. Iacocca E, Dumas RK, Bookman L, Mohseni M, Chung S, Hoefer MA, Åkerman J. 2014 Confined dissipative droplet solitons in spin-valve nanowires with perpendicular magnetic anisotropy. *Phys. Rev. Lett.* 112, 047201. (doi:10.1103/PhysRevLett.112.047201)

47. Khaykovich L, Malomed BA. 2006 Deviation from one dimensionality in stationary properties and collisional dynamics of matter-wave solitons. *Phys. Rev. A* 74, 023607. (doi:10.1103/PhysRevA.74.023607)