SHARP GAUSSIAN ESTIMATES FOR HEAT KERNELS
OF SCHröDINGER OPERATORS
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ABSTRACT. We characterize functions $V \leq 0$ for which the fundamental solution of the Schrödinger operator $\Delta + V$ is comparable with the Gauss-Weierstrass kernel uniformly in space and time. In dimension 4 and higher the condition for the sharp global Gaussian bounds is more restrictive than the condition of the boundedness of the Newtonian potential of $V$. This resolves the question of V. Liskevich and Y. Semenov posed in 1998. We also give specialized sufficient conditions for the comparability, and it turns out that the local $L^p$ integrability of $V$ for $p > 1$ is not necessary for the comparability.

1. INTRODUCTION AND MAIN RESULTS

Let $d = 1, 2, \ldots$. We consider the Gauss-Weierstrass kernel,

$$g(t, x, y) = (4\pi t)^{-d/2} e^{-|y-x|^2/(4t)}, \quad t > 0, \ x, y \in \mathbb{R}^d.$$ 

It is well known that $g$ is a time-homogeneous probability transition density and the fundamental solution of the equation $\partial_t = \Delta$. For Borel measurable function $V : \mathbb{R}^d \to \mathbb{R}$ we call $G$ the Schrödinger perturbation of $g$ by $V$, or the fundamental solution of $\partial_t = \Delta + V$, if the following Duhamel or perturbation formula holds for $t > 0$, $x, y \in \mathbb{R}^d$,

$$G(t, x, y) = g(t, x, y) + \int_0^t \int_{\mathbb{R}^d} G(s, x, z)V(z)g(t-s, z, y)dzds.$$ 

Under appropriate assumptions on $V$, explicit definition of $G$ may be given by means the Feynmann-Kac formula [7 Section 6], the Trotter formula [31, p. 467], the perturbation series [7], or by means of quadratic forms on $L^2$ spaces [12 Section 4]. In particular the assumption $V \in L^p(\mathbb{R}^d)$ with $p > d/2$ was used by Aronson [2], Zhang [31 Remark 1.1(b)] and by Dziubinski and Zienkiewicz [13]. Aizenman and Simon [1, 25] proposed functions $V(z)$ from the Kato class, which contains $L^p(\mathbb{R}^d)$ for every $p > d/2$ [1, Chapter 4], see also Chung and Zhao [11, Chapter 3, Example 2]. An enlarged Kato class was used by Voigt [27, in the study of Schrödinger semigroups on $L^1$ [27 Proposition 5.1]. For perturbations by time-dependent functions $V(u, z)$, Zhang [28, 30] introduced the so-called parabolic Kato condition. The condition was then generalized and employed by Schnaubelt and Voigt [23], Liskevich and Semenov [19], Milman and Semenov [21], Liskevich, Vogt and Voigt [20], and Gulisashvili and van Casteren [15].

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Given the function $V : \mathbb{R}^d \to \mathbb{R}$ we ask if there are numbers $0 < c_1 \leq c_2 < \infty$ such that the following two-sided bound holds,

$$(1) c_1 \leq \frac{G(t, x, y)}{g(t, x, y)} \leq c_2, \quad t > 0, \; x, y \in \mathbb{R}^d.$$ 

One can also ponder a weaker property–if for a given $T \in (0, \infty)$,

$$(2) c_1 \leq \frac{G(t, x, y)}{g(t, x, y)} \leq c_2, \quad 0 < t \leq T, \; x, y \in \mathbb{R}^d.$$ 

We call (1) and (2) sharp Gaussian estimates or bounds, respectively global (or uniform) and local in time. We observe that the inequalities in (1) and (2) are stronger than the plain Gaussian estimates:

$$c_1 \left(4\pi t\right)^{-d/2} e^{-\frac{|y-x|^2}{4t}} \leq G(t, x, y) \leq c_2 \left(4\pi t\right)^{-d/2} e^{-\frac{|y-x|^2}{4t}}, \quad x \in \mathbb{R}^d,$$

where $0 < c_1, c_1 \leq 1 \leq c_2, c_2 < \infty$, which can also be global or local in time.

Berenstein proved the plain Gaussian estimates for $V \in L^p$ with $p > d/2$ (see [18]). Simon [25, Theorem B.7.1] resolved them for $V$ in the Kato class, Zhang [30] and Milman and Semenov [21] applied the parabolic Kato class for this purpose. For further discussion we refer the reader to [19], [20], [21], [31]. We also refer to Bogdan and Szczypkowski [9, Section 1, 4] for a survey of the plain Gaussian bounds for Schrödinger heat kernels along with a streamlined approach, new results and explicit constants based on the so-called 4G inequality.

The plain Gaussian estimates are ubiquitous in analysis but (1) and (2) provide precious qualitative information, if they hold for $V$. It is intrinsically difficult to characterize (1) and (2) for those $V$ that take on positive values, while the case of $V \leq 0$ is more manageable. Arsen’ev proved (2) for $V \in L^p + L^\infty$ with $p > d/2$, $d \geq 3$. Van Casteren [26] proved (2) for the intersection of the Kato class and $L^{d/2} + L^\infty$ for $d \geq 3$ (see [21]). Arsen’ev also obtained (1) for $V \in L^p$ with $p > d/2$ under additional smoothness assumptions (see [18]). Zhang [31, Theorem 1.1] and Milman and Semenov [21, Theorem 1C, Remark (2)] gave sufficient integro-supremum conditions for (2) and (1) for general $V$ and characterized (2) and (1) for $V \leq 0$. It will be convenient to state the conditions by means of

$$S(V, t, x, y) = \int_0^t \int_{\mathbb{R}^d} \frac{g(s, x, z)g(t-s, z, y)}{g(t, x, y)} |V(z)| \; dz \; ds, \quad t > 0, \; x, y \in \mathbb{R}^d.$$ 

The motivation for using this quantity comes from Zhang [31, Lemma 3.1 and Lemma 3.2] and from Bogdan, Jakubowski and Hansen [7] ([1]). We often write $S(V)$ if we do not need to specify $t, x, y$. As explained in Section 3, $S(V)$ is the potential of $|V|$ for the so-called Gaussian bridges. We also note that [7] Section 6] uses $S(V)$ for general transition densities.

In the next two results we just compile [31] Theorem 1.1] with observations from [7] and [8]. For completeness, the proofs are given in Section 2.

Lemma 1.1. If $V \leq 0$, then (1) is equivalent to

$$(3) \sup_{t>0, x,y \in \mathbb{R}^d} S(V, t, x, y) < \infty.$$
If $V \leq 0$, then for each $T \in (0, \infty)$, (2) is equivalent to
\begin{equation}
\sup_{0 < t \leq T, x, y \in \mathbb{R}^d} S(V, t, x, y) < \infty.
\end{equation}

We say that $V$ satisfying (3) or (4) has bounded potential for bridges (is bridge-potential bounded) globally or locally in time, respectively.

Lemma 1.2. If for some $h > 0$ and $0 \leq \eta < 1$ we have
\begin{equation}
\sup_{0 < t \leq h, x, y \in \mathbb{R}^d} S(V^+, t, x, y) \leq \eta,
\end{equation}
and if $S(V^-)$ is bounded on bounded subsets of $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$, then
\begin{equation}
e^{-S(V^-, t, x, y)} \leq \frac{G(t, x, y)}{g(t, x, y)} \leq \left( \frac{1}{1 - \eta} \right)^{1 + t/h}, \quad t > 0, \ x, y \in \mathbb{R}^d.
\end{equation}

The conditions in Lemma 1.1 and 1.2 may be cumbersome to verify for specific functions. For this reason we propose simpler equivalent conditions for (1) and even simpler sufficient conditions for (1) and (2). For clarity we remark that $S(V)$ is unbounded for every nontrivial $V$ in dimensions $d = 1$ and 2. Therefore (1) is impossible for nontrivial $V \geq 0$ and nontrivial $V \leq 0$ in these dimensions. This is explained after Lemma 2.1 below.

For $d \geq 3$ and $x, y \in \mathbb{R}^d$ we define
\begin{equation}
K(V, x, y) = \int_{\mathbb{R}^d} |V(z)| K(z - x, y) \, dz,
\end{equation}
where
\begin{equation}
K(x, y) = \frac{e^{-((|x|+|y|)/2)^2}}{|x|^{d-2}} (1 + |x| |y|)^{d/2 - 3/2},
\end{equation}
and $x \cdot y$ is the usual scalar product. We denote
\begin{align*}
\|S(V)\|_\infty &= \sup_{t > 0, x, y \in \mathbb{R}^d} S(V, t, x, y), \\
\|K(V)\|_\infty &= \sup_{x, y \in \mathbb{R}^d} K(V, x, y).
\end{align*}

Confronted with $S(V)$, the quantity $K(V)$ has less arguments. However, the integro-supremum tests based on $K(V)$ and $S(V)$ appear equivalent.

Theorem 1.3. There are constants $M_1, M_2$ depending only on $d$, such that
\begin{equation}
M_1 \|K(V)\|_\infty \leq \|S(V)\|_\infty \leq M_2 \|K(V)\|_\infty.
\end{equation}

Here by constants we mean positive numbers. The proof of Theorem 1.3 is given in Section 3. By (7) and Lemma 1.1 we get the following result.

Corollary 1.4. If $V \leq 0$, then (1) holds if and only if $K(V)$ is bounded.

Similarly, sufficient smallness of $\|K(V)\|_\infty$ for general $V$ yields (1) by Lemma 1.2.

For $d \geq 3$ we let $C_d = \Gamma(d/2 - 1)/(4\pi^{d/2})$. The Newtonian potential of nonnegative function $f$ and $x \in \mathbb{R}^d$ is
\begin{equation}
-\Delta^{-1} f(x) := \int_0^\infty \int_{\mathbb{R}^d} g(s, x, z) f(z) \, dz ds = C_d \int_{\mathbb{R}^d} \frac{1}{|z - x|^{d-2}} f(z) \, dz.
\end{equation}
We note that for $d = 3$ the formula for $K$ simplifies and we easily obtain
\begin{equation}
||K(V)||_{\infty} = C_d^{-1}||\Delta^{-1}V||_{\infty}.
\end{equation}

Thus if $d = 3$ and $V \leq 0$, then the sharp global Gaussian bounds (1) are equivalent to the condition $||\Delta^{-1}V||_{\infty} < \infty$, see [21] Remark (3) on p. 4.

The main focus of the present paper is on the case of $d \geq 4$. The next estimate is a variant of [15] Corollary 1 and motivates our development:
\begin{equation}
C_d^{-1}||\Delta^{-1}V||_{\infty} \leq ||K(V)||_{\infty} \leq 2^{(d-3)/2} \left( C_d^{-1}||\Delta^{-1}V||_{\infty} + \kappa_d||V||_{d/2} \right).
\end{equation}

Here $||V||_{d/2} = (\int_{\mathbb{R}^d} |V(z)|^{d/2} dz)^{2/d}$ and $d \geq 4$. The proof of (9) and explicit value of $\kappa_d$ are given in Section 3. A long-standing open problem for (1) with $V \leq 0$ posed in 1998 by Liskevich and Semenov [15] p. 602] reads as follows: “The validity of the two-sided estimates for the case $d > 3$ without the additional assumption $V \in L^{d/2}$ is an open question.” In view of Theorem 1.3 and Lemma 1.1 the question is whether for $d \geq 4$ the finiteness of $||\Delta^{-1}V||_{\infty}$ implies the finiteness of $||S(V)||_{\infty}$ or $||K(V)||_{\infty}$. The following result is proved in Section 3.

**Proposition 1.5.** Let $d \geq 4$. For $z = (z_1, z_2, \ldots, z_d) \in \mathbb{R}^d$ we write $z = (z_1, z_2)$, where $z_2 = (z_2, \ldots, z_d) \in \mathbb{R}^{d-1}$. We define
\[ A = \{(z_1, z_2) \in \mathbb{R}^d : z_1 > 4, |z_2| \leq \sqrt{2}\}, \quad \text{and} \]
\[ V(z_1, z_2) = \frac{1}{z_1} 1_A(z_1, z_2). \]

Then $||\Delta^{-1}V||_{\infty} < \infty$ but $||K(V)||_{\infty} = \infty$. There is even a function $V \leq 0$ with compact support such that $||\Delta^{-1}V||_{\infty} < \infty$ but $||K(V)||_{\infty} = \infty$.

We conclude that for $d \geq 4$ neither finiteness nor smallness of $||\Delta^{-1}V||_{\infty}$ are sufficient for (1), so the answer to the question of Liskevich and Semenov is negative. The second question motivated by (9) is whether the finiteness of $||K(V)||_{\infty}$ implies that of $||V||_{d/2}$. The answer is also negative, as follows.

**Proposition 1.6.** For every $d \geq 3$ there is a function $V \leq 0$ such that (1) holds but $V \notin L^1(\mathbb{R}^d) \bigcup \bigcup_{p > 1} L^p_{\text{loc}}(\mathbb{R}^d)$.

In particular (1) may hold even if $||V||_{d/2} = \infty$. Proposition 1.6 is verified in Section 5 by means of explicit examples of functions $V$, which are highly anisotropic. They are constructed from tensor products of power functions and to study them we use in a crucial way the tensorization of the Gaussian-Weierstrass kernel and its bridges. This is the second main topic of the paper—in Theorem 1.9 below we give new sufficient conditions for the sharp Gaussian estimates, which show that $L^p$ integrability is not necessary for (1) or (2). We note in passing that local $L^1$ integrability is necessary for (2) if $V$ does not change sign, cf. Lemma 1.1 and 2.1.

The structure of the remainder of the paper is the following. In Section 2 we give definitions and preliminaries, in particular we prove Lemma 1.1 and 1.2. In Section 3 we prove Theorem 1.3 (9) and Proposition 1.10. In Theorem 1.9 of Section 4 we propose new sufficient conditions for (1) and (2), with emphasis on those functions $V$ which tensorize. In Section 5 we prove
Proposition 1.6 and give examples which illustrate and comment our results. Appendix A gives auxiliary results on inverse-Gaussian-type integral. We should also note that the present paper merges the results of the two preprints [3] and [6].

Here are a few more comments that relate our result to existing literature. First, in [22] Theorem 1C Milman and Semenov discuss (1) using
\[ e(V, 0) = \|\Delta^{-1}|V|\|_{\infty}, \quad \text{and} \]
\[ e_s(V, 0) = \sup_{\alpha \in \mathbb{R}^d} \|V(-\Delta + 2\alpha \cdot \nabla)^{-1}\|_{1 \to 1}. \]

The spatial anisotropy introduced by $\alpha \cdot \nabla$ has a similar role as that seen in the integral defining $S(V, t, x, y)$. In fact there are constants $c_1$, $c_2$ depending only on $d \geq 3$ such that
\[ c_1\|K(V)\|_{\infty} \leq e_s(V, 0) \leq c_2\|K(V)\|_{\infty}. \]

This follows from (18) and (19) below. For $d = 3$ we have $e(V, 0) = e_s(V, 0)$. On the contrary, for $d \geq 4$ by Proposition 1.5 there exists $V \leq 0$ such that $e(V, 0) < \infty$ but $e_s(V, 0) = \infty$.

Second, for $d = 3$ and $V \geq 0$ by [21] Theorem 1C (2), Remark (3) on p. 4, and the comments before Theorem 1B, $\|\Delta^{-1}V\|_{\infty} < 1$ suffices for (1).

Third, if $V \leq 0$, then the condition $\|\Delta^{-1}V\|_{\infty} < \infty$ characterizes the plain global Gaussian bounds, see [21] and [32], p. 556 and Corollary A. By (8), for $d = 3$ the plain global Gaussian bounds hold for $V$ if and only if the sharp global Gaussian bounds hold. In contrast, for $d \geq 4$ by Proposition 1.5 the plain global Gaussian bounds may occur in the absence of the sharp global Gaussian bounds (11).

2. Preliminaries

We let $\mathbb{N} = \{1, 2, \ldots\}$, $f^+ = \max\{0, f\}$ and $f^- = \max\{0, -f\}$. Recall that $d \in \mathbb{N}$ and $V$ is an arbitrary Borel measurable function: $\mathbb{R}^d \to \mathbb{R}$.

We start with the following observations on integrability and potential-boundedness (12) of functions $V$ which are bridges potential-bounded.

**Lemma 2.1.** If $S(V, t, x, y) < \infty$ for some $t > 0$, $x, y \in \mathbb{R}^d$, then $V \in L^1_{\text{loc}}(\mathbb{R}^d)$. If (1) holds, then
\[ \sup_{x \in \mathbb{R}^d} \int_0^T \int_{\mathbb{R}^d} g(s, x, z)|V(z)|\,dz\,ds < \infty. \]

If (3) even holds, then
\[ \sup_{x \in \mathbb{R}^d} \int_0^\infty \int_{\mathbb{R}^d} g(s, x, z)|V(z)|\,dz\,ds < \infty. \]

If $V \geq 0$, then (11) implies (3) and (2) implies (4).

**Proof.** The first statement follows, because $g(t, x, y)$ is locally bounded from below on $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$ (see [14] Lemma 3.7 for a quantitative general result). Since $\int_{\mathbb{R}^d} S(V, t, x, y)g(t, x, y)\,dy = \int_0^T \int_{\mathbb{R}^d} g(s, x, z)|V(z)|\,dz\,ds$, we see that (1) implies (11) and (3) implies (12). The last statement follows from the Duhamel formula. \[ \square \]
We see that (12) and thus also (3) fail for all nonzero \( V \) in dimensions \( d = 1 \) and \( d = 2 \), because then \( \int_0^\infty g(s, x, z) ds = \infty \). Consequently, (11) fails

for nontrivial \( V \leq 0 \) and for nontrivial \( V \geq 0 \) if \( d = 1 \) or \( 2 \), as noted in Section [1]

Following [7, 9] we shall study and use the following functions

\[
F(t) = \sup_{0 \leq s < t} f(s) = \sup_{0 \leq s < t} S(V, s, x, y), \quad t \in (0, \infty).
\]

We fix \( V \) and \( x, y \in \mathbb{R}^d \). For \( 0 < \varepsilon < t \), we consider

\[
S(V, t - \varepsilon, x, y) = \int_0^t \int_{\mathbb{R}^d} g(s, x, z)g(t - \varepsilon - s, z, y)\frac{|V(z)|}{g(t - \varepsilon, x, y)} dz ds.
\]

By Fatou’s lemma we get

\[
S(V, t, x, y) \leq \liminf_{\varepsilon \to 0} S(V, t - \varepsilon, x, y),
\]

meaning that \((0, \infty) \ni t \mapsto S(V, t, x, y)\) is lower semicontinuous on the left.

It follows that \( f \) is lower semi-continuous on the left, too. In consequence, \( f(t) \leq F(t) \) and \( F(t) = \sup_{0 \leq s \leq t} f(s) \) for \( 0 < t < \infty \).

We next claim that \( f \) is sub-additive, that is,

\[
(13) \quad f(t_1 + t_2) \leq f(t_1) + f(t_2), \quad t_1, t_2 > 0.
\]

This follows from the Chapman-Kolmogorov equations for \( g \). Indeed, we have

\[
S(V, t_1 + t_2, x, y) = I_1 + I_2,
\]

where

\[
I_1 = \int_0^{t_1} \int_{\mathbb{R}^d} g(s, x, z)g(t_1 + t_2 - s, z, y)\frac{|V(z)|}{g(t_1 + t_2, x, y)} dz ds
\]

\[
= \int_0^{t_1} \int_{\mathbb{R}^d} g(t_1, x, z)g(s - t_1, w, z)g(t_2, w, y)\frac{|V(z)|}{g(t_1 + t_2, x, y)} dz ds
\]

\[
\leq \int_{\mathbb{R}^d} g(t_1, x, w)S(V, t_1, x, w) dw \leq f(t_1),
\]

and

\[
I_2 = \int_{t_1}^{t_1 + t_2} \int_{\mathbb{R}^d} g(s, x, z)g(t_1 + t_2 - s, z, y)\frac{|V(z)|}{g(t_1 + t_2, x, y)} dz ds
\]

\[
= \int_{t_1}^{t_1 + t_2} \int_{\mathbb{R}^d} g(t_1, x, w)g(s - t_1, w, z)g(t_2, w, y)\frac{|V(z)|}{g(t_1 + t_2, x, y)} dz ds
\]

\[
\leq \int_{\mathbb{R}^d} g(t_1, x, w)S(V, t_2, w, y) dw \leq f(t_2).
\]

This yields (13).

**Lemma 2.2.** For all \( t, h > 0 \) we have \( f(t) \leq F(h) + t f(h)/h \).

**Proof.** Let \( k \in \mathbb{N} \) be such that \((k - 1)h < t \leq kh\), and let \( \theta = t - (k - 1)h \).

Then \( t = \theta + (k - 1)h \), and by (13) we get

\[
f(t) \leq f(\theta) + t f(h)/h \leq F(h) + t f(h)/h,
\]

since \( 0 < \theta \leq h \). □
Corollary 2.3. $F(t) \leq F(h) + t F(h)/h$ and $F(2t) \leq 2F(t)$ for $t, h > 0$.

Proof of Lemma 2.2. If $V \leq 0$ then $0 \leq G \leq g$ is constructed in [31, p. 470], and the Duhamel formula follows from the discussion after [31, (3.3)] and the finiteness of $S(V^-)$. Then the left-hand side of (15) follows from [31, pp. 467-469], or we can use [7, (41)], which follows therein from Jensen’s inequality and the second displayed formula on page 252 of [7]. In general, $G$ is obtained by applying the above procedure with $-V^-$, and then perturbing the resulting kernel by $V^+$, using the perturbation series, cf. [7], and then the Duhamel formula obtains without further conditions. We now prove the right hand side of (15), and without loss of generality we may assume that $V \geq 0$. For $0 < s < t, x, y \in \mathbb{R}^d$, we let $p_0(s, t, y) = g(t - s, x, y)$ and $p_n(s, t, y) = \int_t^s \int_{\mathbb{R}^d} p_{n-1}(s, u, z)V(z)p_0(u, z, t, y) dz du, n \in \mathbb{N}$. Let $Q : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ satisfy $Q(u, r) + Q(r, v) \leq Q(u, v, t > 0).$ By [16, Theorem 1] (see also [8, Theorem 3]) if there is $0 < \eta < 1$ such that

$$p_1(s, t, y) \leq [\eta + Q(s, t)]p_0(s, t, y),$$

then

$$\tilde{p}(s, t, y) := \sum_{n=0}^{\infty} p_n(s, t, y) \leq \left( \frac{1}{1-\eta} \right)^{\frac{Q(s, t)}{\eta}} p_0(s, t, y).$$

Corollary 2.3 and the assumptions of the lemma imply that (14) is satisfied with $\eta = F(h) < 1$ and $Q(s, t) = (t - s)F(h)/h$. Since $G(t, x, y) = \tilde{p}(0, x, t, y)$, the proof of (15) is complete (see also [7, (17)]).

Proof of Lemma 2.2. If (2) holds then Duhamel formula and nonnegativity of $G$ yield (11). Similarly, (11) implies (3). The reverse implications follow from (5). □

As a consequence of Corollary 2.3 we obtain the following result.

Corollary 2.4. Let $V \leq 0$ and $T > 0$. Then (2) holds if and only if

$$Ce^{-ct}g(t, x, y) \leq G(t, x, y), \quad t > 0, x, y \in \mathbb{R}^d,$$

for some constants $C > 0, c \geq 0$. In fact we can take

$$\ln C = \sup_{x, y \in \mathbb{R}^d} S(V, t, x, y) \quad \text{and} \quad c = \frac{1}{T} \sup_{x, y \in \mathbb{R}^d} S(V, T, x, y).$$

Proof. Obviously, (16) implies (2) for every fixed $T > 0$. Conversely, if (2) holds for fixed $T > 0$, then by Lemma 1.2 and 2.2 we have

$$\frac{G(t, x, y)}{g(t, x, y)} \geq e^{-S(V, t, x, y)} \geq e^{-f(t)} \geq e^{-f(T)}e^{-f(T)/T}.$$

□

We note in passing that the above proof shows that (2) is determined by the behavior of $\sup_{x, y \in \mathbb{R}^d} S(V, t, x, y)$ for small $t > 0$. We end our discussion by recalling the connection of $G$ to $\Delta + V$ aforementioned in Abstract. As it is
well known, and can be directly checked by using the Fourier transform or by arguments of the semigroup theory [4, Section 4],
\[
\int_0^\infty \int_{\mathbb{R}^d} g(u-s, x, z) \left[ \partial_\tau \phi(u, z) + \Delta \phi(u, z) \right] dz du = -\phi(s, x),
\]
for all \( s \in \mathbb{R}, x \in \mathbb{R}^d \) and for all \( \phi \in C^\infty_0(\mathbb{R} \times \mathbb{R}^d) \), the smooth compactly supported test functions on space-time. Similarly, if \( V \) satisfies the assumptions of Lemma 3.1, then by [31, Theorem 1.1] for all \( s \in \mathbb{R}, x \in \mathbb{R}^d, \phi \in C^\infty_0(\mathbb{R} \times \mathbb{R}^d) \),
\[
\int_0^\infty \int_{\mathbb{R}^d} G(u-s, x, z) \left[ \partial_\tau \phi(u, z) + \Delta \phi(u, z) + V(z) \phi(u, z) \right] dz du = -\phi(s, x).
\]
We refer to [8, Lemma 4] for a general approach to such identities.

3. Characterization of the sharp global Gaussian estimates

For \( t > 0, x, y \in \mathbb{R}^d \), we consider
\[
N(V, t, x, y) := \int_0^{t/2} \int_{\mathbb{R}^d} e^{-|z-y+(\tau/2)(y-x)|^2/(4\tau)} |V(z)| dz d\tau + \int_{t/2}^t \int_{\mathbb{R}^d} e^{-|z-y+(\tau/2)(y-x)|^2/(4(t-\tau))} |V(z)| dz d\tau = N(V, t, y, x),
\]
(17)

Clearly, \( N(V) = N(|V|) \) and \( S(V) = S(|V|) \). Because of the work of Zhang [31], \( N \) is a proxy for \( S \). Namely, by [31 Lemma 3.1, Lemma 3.2], there are constants \( m_1, m_2 \) depending only on \( d \) such that
\[
(L) \quad S(V, t, x, y) \geq m_1 N(V, t/2, x, y), \quad t > 0, x, y \in \mathbb{R}^d,
\]
\[
(U) \quad S(V, t, x, y) \leq m_2 N(V, t, x, y), \quad t > 0, x, y \in \mathbb{R}^d.
\]

In this section we prove our main result, i.e., Theorem 1.3. We start by using \( N(V, t) \), [14] and [14], to estimate \( S(V, t) \).

**Lemma 3.1.** Let \( t > 0 \). We have
\[
\int_0^{t/2} \int_{\mathbb{R}^d} e^{-|z-y+(\tau/2)(y-x)|^2/(4\tau)} |V(z)| dz d\tau \leq N(V, t)(x, y), \quad x, y \in \mathbb{R}^d,
\]
and
\[
\sup_{x, y} N(V, t)(x, y) \leq 2 \sup_{x, y} \int_0^{t/2} \int_{\mathbb{R}^d} e^{-|z-y+(\tau/2)(y-x)|^2/(4\tau)} |V(z)| dz d\tau.
\]

**Proof.** The first inequality follows by the definition of \( N(V, t)(x, y) \). For the proof of the second one we note that
\[
\int_{t/2}^t \int_{\mathbb{R}^d} e^{-|z-y+(\tau/2)(y-x)|^2/(4(t-\tau))} (t - \tau)^{d/2} |V(z)| dz d\tau
= \int_0^{t/2} \int_{\mathbb{R}^d} e^{-|z-x+(\tau/2)(x-y)|^2/(4\tau)} \tau^{d/2} |V(z)| dz d\tau.
\]
\( \square \)
We can now make connections to $e_*(V, 0)$, cf. \[10\]. Let
\[ J(x, y) = \int_0^\infty \tau^{-d/2} e^{-\frac{|x-\tau y|^2}{4\tau}} d\tau, \quad x, y \in \mathbb{R}^d. \]
By definition,
\[ e_*(V, 0) = \sup_{\alpha \in \mathbb{R}^d} \|(-\Delta + 2\alpha \cdot \nabla)^{-1}|V|\|_\infty \]
(18)
\[ = (4\pi)^{-d/2} \sup_{x, y \in \mathbb{R}^d} \int_{\mathbb{R}^d} J(z - x, y)|V(z)| dz. \]

Lemma 3.2. We have
\[ \sup_{t > 0, x, y \in \mathbb{R}^d} S(V, t, x, y) \geq m_1 (4\pi)^{d/2} e_*(V, 0), \]
and
\[ \sup_{t > 0, x, y \in \mathbb{R}^d} S(V, t, x, y) \leq 2 m_2 (4\pi)^{d/2} e_*(V, 0). \]

Proof. By (L) and Lemma 3.1
\[ \sup_{t > 0, x, y \in \mathbb{R}^d} S(V, t, x, y) \geq m_1 \sup_{t > 0, x, y \in \mathbb{R}^d} N(|V|, t/2)(x, y) \]
\[ \geq m_1 \sup_{t > 0, x, y \in \mathbb{R}^d} \int_0^{t/4} \int_{\mathbb{R}^d} e^{-|z - y + (2\tau/t)(y - x)|^2/(4\tau)} \tau^{d/2} |V(z)| dz d\tau \]
\[ = m_1 \sup_{t > 0, y, w \in \mathbb{R}^d} \int_0^{t/4} \int_{\mathbb{R}^d} e^{-|z - y - \tau w|^2/(4\tau)} \tau^{d/2} |V(z)| dz d\tau \]
\[ = m_1 \sup_{y, w \in \mathbb{R}^d} \int_{\mathbb{R}^d} J(z - y, w)|V(z)| dz. \]

By (L) and Lemma 3.1
\[ \sup_{t > 0, x, y \in \mathbb{R}^d} S(V, t, x, y) \leq m_2 \sup_{t > 0, x, y \in \mathbb{R}^d} N(V, t)(x, y) \]
\[ \leq 2 m_2 \sup_{t > 0, x, y \in \mathbb{R}^d} \int_0^{t/2} \int_{\mathbb{R}^d} e^{-|z - y + (\tau/t)(y - x)|^2/(4\tau)} \tau^{d/2} |V(z)| dz d\tau \]
\[ \leq 2 m_2 \sup_{t > 0, y, w \in \mathbb{R}^d} \int_0^{t/2} \int_{\mathbb{R}^d} e^{-|z - y + \tau w|^2/(4\tau)} \tau^{d/2} |V(z)| dz d\tau \]
\[ = 2 m_2 \sup_{y, w \in \mathbb{R}^d} \int_{\mathbb{R}^d} J(z - y, w)|V(z)| dz. \]

Proof of Theorem 3.3 We claim \[ \eqref{2} \] holds with $M_1 > 0$ that depends only on $d$, and $M_2 = m_2 2^d \int_0^\infty (1 \vee r)^{d/2-3/2} r^{-1/2} e^{-r} dr$. To this end, according to Lemma 3.2 we analyze
\[ J(z - x, y) = \int_0^\infty \tau^{-d/2} e^{-\frac{|z - x - \tau y|^2}{4\tau}} d\tau. \]
Obviously, $J = \infty$ if $d = 1$ or $d = 2$. For $d \geq 3$ we observe that
\[
\frac{|z - x - \tau y|^2}{4\tau} = \frac{1}{4} \left[ \frac{|z - x|}{\sqrt{\tau}} - \sqrt{\tau} |y| \right]^2 + \frac{1}{2} (|z - x||y| - (z - x) \cdot y),
\]
and thus
\[
J(z - x, y) = e^{-\frac{1}{2}(|z - x| - (z - x) \cdot y)} \int_0^\infty \tau^{-d/2} e^{-\frac{1}{4} [\sqrt{\tau}|y| - \frac{|z - x|}{\sqrt{\tau}}]^2} d\tau.
\]
Finally, by Theorem $A.5$ with $a = |z - x|/2$, $b = |y|/2$, $\beta = d/2$ and $c = 1$, (19)
\[
J(z - x, y) \approx K(z - x, y).
\]
This also gives the explicit constants, as a consequence of Remark $A.5$. For instance we can take $M_2 = 8m_2\sqrt{\pi}$ if $d = 3$.

\textbf{Proof of (9).} The left hand side inequality follows from the identity $K(V, x, 0) = C^{-1}_d(-\Delta^{-1})|V|(x)$. If $y = 0$, then the upper bound trivially holds. For $y \neq 0$ we consider two domains of integration. We have
\[
\int_{|z - x||y| \leq 1} K(z - x, y)|V(z)| dz \leq 2^{(d-3)/2} \int_{|z - x||y| \leq 1} \frac{1}{|z - x|^{d-2}}|V(z)| dz \leq \frac{2^{(d-3)/2}}{C_d} |\Delta^{-1}|V|_\infty.
\]
Furthermore, by a change of variables and the Hölder inequality,
\[
\int_{|z - x||y| \geq 1} K(z - x, y)|V(z)| dz \leq 2^{(d-3)/2} \int_{|z - x||y| \geq 1} e^{-\frac{1}{2}(|z - x||y| - (z - x) \cdot y)} |y|^{d-2} |V(z)| dz \leq 2^{(d-3)/2} \kappa_d |V|_{d/2},
\]
where
\[
\kappa_d = \left( \int_{|w| > 1} \left( e^{-\frac{1}{2}(|w| - w^{-1})|w| - (d-1)/2} \right)^{d/(d-2)} dw \right)^{(d-2)/d}.
\]
The finiteness of $\kappa_d$ follows from Lemma $A.6$ below.

\textbf{Proof of Proposition 1.2.} We use the notation introduced in the formulation of the theorem. First we prove that $\|K(V)\|_\infty = \infty$. Let $y = (1, 0) \in \mathbb{R}^d$, $x = 0$. Observe that for $z \in A$ we have
\[
0 \leq |z||y| - z \cdot y = |z| - z_1 = \frac{|z_2|^2}{\sqrt{z_1^2 + |z_2|^2} + z_1} \leq \frac{z_1}{\sqrt{z_1^2 + |z_2|^2} + z_1} \leq 1
\]
and thus also $z_1 \leq |z| \leq 2z_1$. Then,
\[
\|K(V)\|_\infty \geq \int_{\mathbb{R}^d} e^{-\frac{1}{2}|z||y| - z \cdot y} |V(z)| \left( 1 + \frac{|z||y|}{|z|^{d-2}} \right)^{d/2} dz \geq c \int_A \frac{1}{z_1 z_1^{d-2}} dz
\]
\[
= c \int_A \int_{|z_2| < \sqrt{z_1}} z_1^{-1 + 2 - d + \frac{d}{2} - \frac{3}{4}} dz_2 dz_1 = c_1 \int_1^\infty z_1^{-1 + 2 - d + \frac{d}{2} + \frac{3}{4} + (d-1)} dz_1 = c_1 \int_1^\infty \frac{1}{z_1} dz_1 = \infty.
\]
We now prove that \( \|\Delta^{-1}V\|_\infty < \infty \). By the symmetric rearrangement inequality (see [17, Chapter 3]) we have

\[
\sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} \frac{1}{|z - x|^{d-2}} |V(z)| \, dz = \sup_{x_1 \in \mathbb{R}} \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}^d} \frac{1}{|z_1 - x_1|^{d-2} + |z_2|^{d-2}} \frac{1}{z_1} \, dz_2 \, dz_1
\]

It suffices then to consider \( x = (x_1, 0, \ldots, 0) \) and we only need to show that the following three integrals are uniformly bounded for \( x_1 \geq 4 \). The first integral is

\[
I_1 = \int_{x_1 + \sqrt{x_1}} \int_{|z_2| < \sqrt{x_1}} \frac{1}{|z_1 - x_1|^{d-2} + |z_2|^{d-2}} \frac{1}{z_1} \, dz_2 \, dz_1 
\]

\[
\leq \int_{x_1 + \sqrt{x_1}} \int_{|z_2| < \sqrt{x_1}} \frac{1}{|z_1 - x_1|^{d-2} + |z_2|^{d-2}} \frac{1}{z_1} \, dz_2 \, dz_1 
\]

\[
c = \int_{x_1 + \sqrt{x_1}} \frac{1}{|z_1 - x_1|^{d-2} + |z_2|^{d-2}} \frac{1}{z_1} \, dz_1 = c \int_{x_1 + \sqrt{x_1}} \frac{1}{|z_1 - x_1|^{d-2}} \, dz_1 
\]

\[
\leq c' \int_{x_1} \frac{1}{|z_1 - x_1|^{d-2}} \, dz_1 \leq c'' < \infty.
\]

The second integral we consider is

\[
I_2 = \int_{x_1 - \sqrt{x_1}} \int_{|z_2| < \sqrt{x_1}} \frac{1}{|z_1 - x_1|^{d-2} + |z_2|^{d-2}} \frac{1}{z_1} \, dz_2 \, dz_1
\]

\[
\leq \int_{x_1 - \sqrt{x_1}} \int_{|z_2| < \sqrt{x_1}} \frac{1}{|z_1 - x_1|^{d-2} + |z_2|^{d-2}} \frac{1}{z_1} \, dz_2 \, dz_1 
\]

\[
= c \int_{x_1 - \sqrt{x_1}} \frac{1}{|z_1 - x_1|^{d-2}} \, dz_1 \leq c' < \infty.
\]

The remaining integral is

\[
I_3 = \int_{x_1 + \sqrt{x_1}} \int_{|z_2| < \sqrt{x_1}} \frac{1}{|z_1 - x_1|^{d-2} + |z_2|^{d-2}} \frac{1}{z_1} \, dz_2 \, dz_1
\]

\[
\leq \int_{|z_2| < 2\sqrt{x_1}} \frac{1}{|z_1 - x_1|^{d-2}} \frac{1}{z_1} \, dz_2 \, dz_1 
\]

\[
\leq 2 \int_{B(x, 3\sqrt{x_1})} \frac{1}{|z - x|^{d-2}} \frac{1}{x_1} \, dz 
\]

\[
\leq c < \infty.
\]

To prove the second statement of Proposition 1.6 for \( s > 0 \) we let \( d_s f(x) = s f(\sqrt{s} x) \). Note that the dilatation does not change the norms:

\[
\|\Delta^{-1} (d_s f)\|_\infty = \|\Delta^{-1} f\|_\infty, \quad \|K(d_s f)\|_\infty = \|K(f)\|_\infty.
\]

Moreover, \( \text{supp}(d_s f) \subseteq B(0, r/\sqrt{s}) \) if \( \text{supp}(f) \subseteq B(0, r) \), \( r > 0 \). Recall that \( V \leq 0 \) constructed above satisfies \( \|\Delta^{-1} V\|_\infty = C < \infty \) and \( \|K(V)\|_\infty = \infty \). Therefore \( \|\Delta^{-1} (V 1_{B_1})\|_\infty \leq C \) for every \( r > 0 \) and \( \|K(V 1_{B_1})\|_\infty \to \infty \) as \( r \to \infty \). For \( n \in \mathbb{N} \) we define

\[
V_n = d_{n^2} (V 1_{B_r(n)}),
\]

where \( r(n) \) is chosen such that \( \|K(V 1_{B_{r(n)}})\|_\infty \geq 4^n \). Also, \( \text{supp}(V_n) \subseteq B(0, 1) \). We define \( \bar{V} = \sum_{n=1}^{\infty} V_n / n^2 \). Then,

\[
\|K(\bar{V})\|_\infty \geq \|K(V_n)\|_\infty / 2^n \geq 2^n \to \infty,
\]
as $n \to \infty$, and

$$\|\Delta^{-1} \tilde{V}\|_\infty \leq \sum_{n=1}^{\infty} \|\Delta^{-1} V_n\|_\infty / 2^n \leq C.$$  

\[\square\]

Similarly, (11) fails for $-\varepsilon \tilde{V} \geq 0$ with any $\varepsilon > 0$, cf. Lemma 2.1.

4. Sufficient conditions for the sharp Gaussian estimates

Recall from [10, (2.5)] that for $p \in [1, \infty]$,

$$\|P_t f\|_\infty \leq C(d, p) t^{-d/(2p)} \|f\|_p, \quad t > 0,$$

where $P_t f(x) = \int_{\mathbb{R}^d} g(t, x, z) f(z) dz, f \in L^p(\mathbb{R}^d)$ and

$$C(d, p) = \begin{cases} (4\pi)^{-d/2}, & \text{if } p = 1, \\ (4\pi)^{-d/(2p)}(1 - p^{-1})(1-p^{-1})d/2, & \text{if } p \in (1, \infty]. \end{cases}$$

We will give an analogue for the bridges $T_{s,y}^t$. Here $t > 0$, $y \in \mathbb{R}^d$, and

$$T_{s,y}^t f(x) = \int_{\mathbb{R}^d} g(s, x, z) \frac{g(t-s, y)}{g(t, x, y)} f(z) dz, \quad 0 < s < t, \ x \in \mathbb{R}^d.$$

Clearly,

$$T_{s,y}^t f(x) = T_{t-s}^x f(y), \quad 0 < s < t, \ x, y \in \mathbb{R}^d.$$

By the Chapman-Kolmogorov equations (the semigroup property) for the kernel $g$, we have $T_{s,y}^t V = 1$. We also note that $S(V)$ is related to the potential (0-resolvent) operator of $T$ as follows,

$$S(V, t, x, y) = \int_0^t T_{s,y}^t |V|(x) ds .$$

Lemma 4.1. For $p \in [1, \infty]$ and $f \in L^p(\mathbb{R}^d)$ we have

$$\|T_{s,y}^t f\|_\infty \leq C(d, p) \left[ \frac{(t-s)s}{t} \right]^{-d/(2p)} \|f\|_p, \quad 0 < s < t, \ y \in \mathbb{R}^d.$$

Proof. We note that

$$g(s, x, z) g(t-s, y) \frac{g(t,x, y)}{(4\pi)^{-d/2} g(t, x, y)} = \left[ \frac{(t-s)s}{t} \right]^{-d/2} \exp \left[ -\frac{|z-x|^2}{4s} - \frac{|y-z|^2}{4(t-s)} + \frac{|y-x|^2}{4t} \right].$$

As in [29] (3.4), we have

$$|z-x|^2 + |y-z|^2 \geq |y-x|^2.$$  

(21)

Indeed, (21) obtains from by the triangle and Cauchy-Schwarz inequalities:

$$|y-x| \leq \sqrt{s} |z-x| + \sqrt{t-s} |y-z| \leq \sqrt{s} \left( \frac{|z-x|^2}{s} + \frac{|y-z|^2}{t-s} \right)^{1/2}.$$
Proposition 4.2. Let \( V : \mathbb{R}^d \to \mathbb{R} \) and \( p, q \in [1, \infty) \).

(a) If \( V \in L^p(\mathbb{R}^d) \), \( p > d/2 \) and \( c = C(d, p) \frac{\|V(1-d/(2p))\|}{\|V\|_p} \), then
\[
\sup_{x,y \in \mathbb{R}^d} S(V, t, x, y) \leq ct^{1-d/(2p)}, \quad t > 0.
\]

(b) If \( V \in L^p(\mathbb{R}^d) \cap L^q(\mathbb{R}^d) \) and \( q < d/2 < p \), then (3) holds.

Proof. Part (a) follows from Lemma 4.1 so we proceed to (b). For \( t > 2 \),
\[
(22) \quad \int_0^t T_s^{t,y} |V|(x) \, ds = \int_0^{t/2} T_s^{t,y} |V|(x) \, ds + \int_{t/2}^t T_s^{t,x} |V|(y) \, ds.
\]
Estimating the first term of the sum, by Lemma 4.1 we obtain
\[
\int_0^{t/2} T_s^{t,y} |V|(x) \, ds \leq c \|V\|_p \int_0^{t/2} \left( \frac{(t-s)s}{t} \right)^{-d/(2p)} ds + c \|V\|_q \int_1^{t/2} \left( \frac{(t-s)s}{t} \right)^{-d/(2p)} ds.
\]
(23)
\[
\leq c' \|V\|_p \int_0^{1} s^{-d/(2p)} ds + c' \|V\|_q \int_1^{\infty} s^{-d/(2p)} ds.
\]
By (22), the second term has the same bound. For \( t \in (0, 2) \) we use (a). \( \square \)

By Lemma 4.1 and 4.2 we get the following conclusion.

Corollary 4.3. Under the assumptions of Proposition 4.2(a), \( G \) satisfies the sharp local Gaussian bounds (2). If \( V \leq 0 \) and the assumptions of Proposition 4.2(b) hold, then \( G \) has the sharp global Gaussian bounds (1).

Recall that [13] Theorem 2 and [21] Remark (1) and (4) on p. 4 yield (1) for \( d \geq 4 \) if \( \|\Delta^{-1}V^{-} + \|V^{-}\|_{d/2} < \infty \), \( \|\Delta^{-1}V^{+}\|_{\infty} < 1 \) and \( \|V^{-}\|_{d/2} \) is small enough. We can reduce Proposition 4.2(b) to this result as follows.

Lemma 4.4. The assumptions of Proposition 4.2(b) necessitate that \( d \geq 3 \), \( V \in L^{d/2}(\mathbb{R}^d) \) and \( \|\Delta^{-1}V\|_{\infty} < \infty \).

Proof. Plainly, the assumptions of Proposition 4.2(b) imply \( d > 2 \) and \( V \in L^{d/2}(\mathbb{R}^d) \). We now verify that \( \|\Delta^{-1}V\|_{\infty} < \infty \). By Hölder’s inequality,
\[
\sup_{x \in \mathbb{R}^d} \int_{B(0,1)} \frac{|V(z+x)|}{|z|^{d-2}} \, dz \leq \|z|^{2-d} \|B_{(0,1)}(z)\|_{p'} \|V\|_p < \infty,
\]
\[
\sup_{x \in \mathbb{R}^d} \int_{B^c(0,1)} \frac{|V(z+x)|}{|z|^{d-2}} \, dz \leq \|z|^{2-d} \|B^c_{(0,1)}(z)\|_{q'} \|V\|_q < \infty,
\]
where $p', q'$ are the exponents conjugate to $p, q$, respectively.

In what follows, we propose suitable sufficient conditions for □ and □. We let $d_1, d_2 \in \mathbb{N}$ and $d = d_1 + d_2$.

**Remark 4.5.** The Gauss-Weierstrass kernel $g(t, x)$ in $\mathbb{R}^d$ can be represented as a tensor product:

$$g(t, x) = (4\pi t)^{-d_1/2}e^{-|x_1|^2/(4t)} (4\pi t)^{-d_2/2}e^{-|x_2|^2/(4t)},$$

where $x_1 \in \mathbb{R}^{d_1}, x_2 \in \mathbb{R}^{d_2}$ and $x = (x_1, x_2)$. The kernels of the bridges factorize accordingly:

$$g(s, x, z) g(t-s, y)$$

$$= \frac{(4\pi s)^{-d_1/2}e^{-|z_1|^2/(4s)} (4\pi (t-s))^{-d_1/2}e^{-|y_1-z_1|^2/(4(t-s))}}{(4\pi t)^{-d_1/2}e^{-|y_1-x_1|^2/(4t)}}$$

$$\times \frac{(4\pi s)^{-d_2/2}e^{-|z_2-x_2|^2/(4s)} (4\pi (t-s))^{-d_2/2}e^{-|y_2-z_2|^2/(4(t-s))}}{(4\pi t)^{-d_2/2}e^{-|y_2-x_2|^2/(4t)}}.$$

**Corollary 4.6.** Let $V_1: \mathbb{R}^{d_1} \to \mathbb{R}$, $V_2: \mathbb{R}^{d_2} \to \mathbb{R}$, and $V(x) = V_1(x_1)V_2(x_2)$, where $x = (x_1, x_2) \in \mathbb{R}^d$, $x_1 \in \mathbb{R}^{d_1}$ and $x_2 \in \mathbb{R}^{d_2}$. Assume that $V_1 \in L^\infty(\mathbb{R}^{d_1})$ and $\sup_{t>0, x_2, x_2 \in \mathbb{R}^{d_2}} S(V_2, t, x_2, y_2) < \infty$. Then □ holds.

**Proof.** In estimating $S(V, t, x, y)$ we first use the factorization of the bridges and the boundedness of $V_1$, and then the Chapman-Kolmogorov equations and the boundedness of $S(V_2)$. □

Let $p, p_1, p_2 \in [1, \infty]$.

**Definition 4.7.** We write $f \in L^{p_1}(\mathbb{R}^{d_1}) \times L^{p_2}(\mathbb{R}^{d_2})$ if there are $f_1 \in L^{p_1}(\mathbb{R}^{d_1})$ and $f_2 \in L^{p_2}(\mathbb{R}^{d_2})$, such that

$$f(x_1, x_2) = f_1(x_1)f_2(x_2), \quad x_1 \in \mathbb{R}^{d_1}, x_2 \in \mathbb{R}^{d_2}.$$ 

Clearly, $L^p(\mathbb{R}^{d_1}) \times L^p(\mathbb{R}^{d_2}) \subset L^p(\mathbb{R}^{d_1+d_2})$, in fact $\|f\|_p = \|f_1\|_p \|f_2\|_p$ if $f(x_1, x_2) = f_1(x_1)f_2(x_2)$.

**Lemma 4.8.** For $f(x_1, x_2) = f_1(x_1)f_2(x_2) \in L^{p_1}(\mathbb{R}^{d_1}) \times L^{p_2}(\mathbb{R}^{d_2})$, $0 < s < t$ and $y \in \mathbb{R}^d$, we have

$$\|T_{s,y}^t f\|_\infty \leq C(d_1, p_1) C(d_2, p_2) \left[ \frac{(t-s)s}{t} \right]^{-d_1/(2p_1)-d_2/(2p_2)} \|f_1\|_{p_1} \|f_2\|_{p_2}.$$

**Proof.** We proceed as in the proof of Lemma 4.1 using Remark 4.5 □

We extend Proposition 4.2 as follows.

**Theorem 4.9.** Let $d_1, d_2 \in \mathbb{N}$, $d = d_1 + d_2$, $V: \mathbb{R}^d \to \mathbb{R}$, $p_1, p_2 \in [1, \infty]$ and

$$\frac{d_1}{2p_1} + \frac{d_2}{2p_2} = 1.$$

(a) If $r \in (p_1, \infty]$ and $V \in L^r(\mathbb{R}^{d_1}) \times L^{p_2}(\mathbb{R}^{d_2})$, then

$$\sup_{x, y \in \mathbb{R}^d} S(V, t, x, y) \leq c t^{1-d_1/(2r)-d_2/(2p_2)},$$

where $c = C(d_1, r) C(d_2, p_2) \left[ \frac{\Gamma(1-d_1/(2r)-d_2/(2p_2))}{\Gamma(1-2-d_1/(2r)-d_2/(2p_2))} \right]^{1/2} \|V_1\|_r \|V_2\|_{p_2}$. 

□
(b) If $1 \leq q < p_1 < r \leq \infty$ and $V \in \left[L^q(\mathbb{R}^{d_1}) \cap L^r(\mathbb{R}^{d_1})\right] \times L^{p_2}(\mathbb{R}^{d_2})$, then $[\text{(I)}]$ holds.

**Proof.** We follow the proof of Proposition 4.2, replacing Lemma 4.1 by Lemma 4.8.

By Lemma 1.1 and 1.2 we get the following conclusion.

**Corollary 4.10.** Under the assumptions of Theorem 4.9(a), $G$ satisfies the sharp local Gaussian bounds $[\text{(II)}]$. If $V \leq 0$ and the assumptions of Theorem 4.9(b) hold, then $G$ has the sharp global Gaussian bounds $[\text{(I)}]$.

Clearly, if $|U| \leq |V|$, then $S(U) \leq S(V)$. This may be used to extend the conclusions of Theorem 4.9 and Corollary 4.10 beyond tensor products $V(x_1, x_2) = V_1(x_1)V_2(x_2)$.

5. Examples

Let $1_A$ denote the indicator function of $A$. In what follows, $G$ in $[\text{(I)}]$ is the Schrödinger perturbation of $g$ by $V$.

**Example 5.1.** Let $d \geq 3$ and $1 < p < \infty$. For $x_1 \in \mathbb{R}$, $x_2 \in \mathbb{R}^{d-1}$ we let $V(x_1, x_2) = -|x_1|^{-1/p}1_{|x_1|<1}1_{|x_2|<1}$. Then $[\text{(I)}]$ holds but $V \notin L^p_{loc}(\mathbb{R}^d)$.

Indeed, $V(x_1, x_2) = V_1(x_1)V_2(x_2)$, where

$$V_1(x_1) = -|x_1|^{-1/p}1_{|x_1|<1}, \quad x_1 \in \mathbb{R},$$

$$V_2(x_2) = 1_{|x_2|<1}, \quad x_2 \in \mathbb{R}^{d-1}.$$ 

Let

$$1 \leq q < p_1 < r < p,$$

and

$$p_2 = \frac{d-1}{2} \frac{p_1}{p_1 - 1/2}.$$ 

Since $d \geq 3$, $p_2 > 1$. In the notation of Theorem 4.9 we have $d_1 = 1$, $d_2 = d - 1$, and indeed $d_1/(2p_1) + d_2/(2p_2) = 1$. Since $V_1 \in L^r(\mathbb{R}) \cap L^q(\mathbb{R})$ and $V_2 \in L^{p_2}(\mathbb{R}^{d-1})$, the assumptions of Theorem 4.9(b) are satisfied, and $[\text{(I)}]$ follows by Corollary 4.10. Clearly, $V \notin L^p_{loc}(\mathbb{R}^d)$.

**Example 5.2.** For $d \geq 3$, $n = 2, 3, \ldots$, let $V_n(x) = |x_1|^{-1+1/n}1_{|x_1|<1}1_{|x_2|<1}$, where $x = (x_1, x_2)$, $x_1 \in \mathbb{R}$, $x_2 \in \mathbb{R}^{d-1}$. Let $a_n = \sup_{t>0, x,y \in \mathbb{R}^d} S(V_n, t, x, y)$,

$$V(x) = -\sum_{n=2}^{\infty} \frac{1}{n^2} \frac{V_n(x)}{a_n}, \quad x \in \mathbb{R}^d.$$ 

Then $[\text{(I)}]$ holds but $V \notin \bigcup_{p>1} L^p_{loc}(\mathbb{R}^d)$.

Indeed, $0 < a_n < \infty$ by Example 5.1, and so

$$\sup_{t>0, x,y \in \mathbb{R}^d} S(V, t, x, y) \leq \sum_{n=2}^{\infty} \frac{1}{n^2} < \infty.$$ 

This yields the global sharp Gaussian bounds. For $p > 1$ we let $m = \left[\frac{p}{p-1}\right]$, and we have $m \geq 2$, $\frac{m-1}{m}p \geq 1$. Then,

$$\int_{B(0,2)} |V(x)|^p \, dx \geq \left(\frac{1}{m^2 a_m}\right)^p \int_{|x_2|<1} \int_{|x_1|<1} |x_1|^{-\frac{m-1}{m}p} \, dx_1 \, dx_2 = +\infty.$$
Example 5.3. Let $d \geq 3$ and $V(x_1, x_2) = \frac{-1}{(|x_2| + 1)^2}$ for $x_1 \in \mathbb{R}^{d-3}$, $x_2 \in \mathbb{R}^3$. Then (1) holds but $V \notin L^1(\mathbb{R}^d)$.

Indeed, $V \notin L^1(\mathbb{R}^d)$. We let $V_2(x_2) = \frac{1}{(|x_2| + 1)}$, $x_2 \in \mathbb{R}^3$. By the symmetric rearrangement inequality [17, Chapter 3], in dimension $d = 3$ we have

$$0 \leq \Delta^{-1}V_2(\mathbb{R}^d) \leq C_3 \int_{\mathbb{R}^3} \frac{1}{|z|(|z| + 1)^3} dz < \infty.$$ 

By (7) and (8),

$$\sup_{t > 0, x_2, y_2 \in \mathbb{R}^3} S(V_2, t, x_2, y_2) < \infty.$$ 

By Corollary 4.6 and Lemma 1.1, we see that (1) holds for $V$.

**Proof of Proposition 5.3.** Add the functions from Example 5.2 and 5.3. \( \square \)

We can have nonnegative examples, too. Namely, let $V \leq 0$ be as in Proposition 5.3. Then $M = \sup_{t > 0, x, y \in \mathbb{R}^d} S(V, t, x, y) < \infty$. We let $V = |V|/(M + 1)$. Then $\tilde{V} \geq 0$, $\tilde{V} \notin L^1(\mathbb{R}^d) \cup \bigcup_{p > 1} L^p_{loc}(\mathbb{R}^d)$ and

$$\sup_{t > 0, x, y \in \mathbb{R}^d} S(\tilde{V}, t, x, y) = M/(M + 1) < 1.$$

Therefore (1) holds for $\tilde{V}$ with $h = \infty$ and $\eta = M/(M + 1)$, which yields (1).

Let $d_1, d_2 \in \mathbb{N}$, $d = d_1 + d_2$, $V_1: \mathbb{R}^{d_1} \to \mathbb{R}$, $V_2: \mathbb{R}^{d_2} \to \mathbb{R}$, and $V(x_1, x_2) = V_1(x_1) + V_2(x_2)$, where $x_1 \in \mathbb{R}^{d_1}$ and $x_2 \in \mathbb{R}^{d_2}$. Let $G(t, x_1, y_1), G_2(t, x_2, y_2)$ be the Schrödinger perturbations of the Gauss-Weierstrass kernels on $\mathbb{R}^{d_1}$ and $\mathbb{R}^{d_2}$ by $V_1$ and $V_2$, respectively. Then $G(t, (x_1, x_2), (y_1, y_2)) := G_1(t, x_1, y_1)$ $G_2(t, x_2, y_2)$ is the Schrödinger perturbation of the Gauss-Weierstrass kernel on $\mathbb{R}^d$ by $V$. Clearly, if the sharp Gaussian estimates hold for $G_1$ and $G_2$, then they hold for $G$. Our next example is aimed to show that such trivial conclusions are invalid for tensor products $V(x_1, x_2) = V_1(x_1)V_2(x_2)$.

Example 5.4. Let $\varepsilon \in [0, 1)$. For $x_1, x_2 \in \mathbb{R}^3$ let $V(x_1, x_2) = V_1(x_1)V_2(x_2)$, where

$$V_1(x) = V_2(x) = \frac{1 - \varepsilon}{2} |x|^{-1-\varepsilon} 1_{|x|<1}.$$ 

Then the fundamental solutions in $\mathbb{R}^3$ of $\partial_t = \Delta + V_1$ and $\partial_t = \Delta + V_2$ satisfy (1) and (2), but that of $\partial_t = \Delta + V$ in $\mathbb{R}^6$ satisfies neither (1) nor (2).

Indeed, by the symmetric rearrangement inequality [17, Chapter 3],

$$0 \leq -\Delta^{-1}V_1(x) \leq -\Delta^{-1}V_1(0) = \frac{1 - \varepsilon}{8\pi} \int_{|z|<1} \frac{1}{|z|^2} |z|^{-1-\varepsilon} dz = 1/2,$$

for all $x \in \mathbb{R}^3$. Thus, $\|\Delta^{-1}V_1\|_\infty = \|\Delta^{-1}V_2\|_\infty < \infty$. By the comment following (5), we get (1) for the fundamental solutions in $\mathbb{R}^3$ of $\partial_t = \Delta + V_1$ and $\partial_t = \Delta + V_2$. However, the fundamental solution in $\mathbb{R}^6$ of $\partial_t = \Delta + V$ fails even (2). Indeed, if we let $T \leq 1$, $a \in \mathbb{R}^6$, $|a| = 1$, and $c = \int_0^T p(s, 0, a) ds$, then

$$c(t, x) = \int_0^T \int_{|y|<t} \frac{|a|^2}{|y|^4} |y|^{-1-\varepsilon} \int_{|z|<1} |z|^{-1-\varepsilon} \frac{1}{|z|^2} |z|^{-1-\varepsilon} dz dy ds.$$
then by \[\text{[11] Lemma 3.5},
\]
\[
\int_0^T \int_{\mathbb{R}^d} g(s, 0, x) |V(x)| \, dx \, ds \geq \int_{\{x \in \mathbb{R}^d : |x|^2 \leq T\}} \int_0^T g(s, 0, x) \, ds \, |V(x)| \, dx
\]
\[
\geq c \int_{\{x \in \mathbb{R}^d : |x|^2 \leq T\}} \frac{1}{|x|^4} |V(x)| \, dx
\]
\[
\geq c \int_{\{x \in \mathbb{R}^d : |x|^2 < T/2\}} |V_1(x)| \int_{\{x \in \mathbb{R}^d : |x|^2 < T/2\}} \frac{|V_2(x_2)|}{(|x_1|^2 + |x_2|^2)^2} \, dx_2 \, dx_1
\]
\[
\geq c (1 - \varepsilon) \int_{\{x \in \mathbb{R}^d : |x|^2 < T/2\}} |V_1(x)| \int_{\{x \in \mathbb{R}^d : |x|^2 < T/2\}} \frac{|x_2|^{-1}}{(|x_1|^2 + |x_2|^2)^2} \, dx_2 \, dx_1
\]
\[
= \frac{c (1 - \varepsilon)}{2} \int_{\{x \in \mathbb{R}^d : |x|^2 < T/2\}} |V_1(x_1)| \frac{\pi T}{|x_1|^2 (T/2 + |x_1|^2)} \, dx_1
\]
\[
= \pi^2 c T (1 - \varepsilon)^2 \int_0^{T/2} \frac{r^{-1-\varepsilon}}{T/2 + r^2} \, dr = \infty.
\]

By Lemma \[\text{[11]}\] fails, and so does \[\text{[4]},\] according to the last sentence in Lemma \[\text{[1]}\]. Thus, the sharp Gaussian estimates may hold for the \( f^- \) Schrödinger perturbations of the Gauss-Weierstrass kernels by \( V_1 \) and \( V_2 \) but fail for \( f^+ \) the Schrödinger perturbation of the Gauss-Weierstrass kernel by \( V(x_1, x_2) = V_1(x_1) V_2(x_2) \). Considering \( V_1 \) and \( V_2 \), by a comment at the end of Section \[\text{[1]}\] we get a similar counterexample for perturbations by two nonnegative factors, because \( 1/2 < 1 \). Let us also remark that the sharp global Gaussian estimates may hold for \( V(x_1, x_2) = V_1(x_1) V_2(x_2) \) but fail for \( V_1 \) or \( V_2 \). Indeed, it suffices to consider \( V_1(x_1) = -1_{|x_1| < 1} \) on \( \mathbb{R}^3 \) and \( V_2 = 1 \) on \( \mathbb{R} \), and to apply Theorem \[\text{[1]}\]. We see that it is indeed the combined effect of the factors \( V_1 \) and \( V_2 \) that matters—as captured in Section \[\text{[4]}\].

APPENDIX A.

In this section we collect auxiliary calculations used in Section \[\text{[3]}\].

**Theorem A.1.** Let \( c > 0, \beta > 1 \) and
\[
f(a, b) = \int_0^\infty u^{-\beta} e^{-\left[\sqrt{ub - \frac{a^2}{4}}\right]^2} \, du, \quad a, b > 0.
\]
We have
\[
f(a, b) \approx \frac{(1 + 4ab)^{\beta-3/2}}{a^{2(\beta-1)}}.
\]

Here \( \approx \) means that the ratio of both sides is bounded above and below by constants depending only on \( \beta \) and \( c \).

**Lemma A.2.** Let \( \gamma > -1/2 \). Then
\[
h(x) = \int_0^\infty (x + s^2)^\gamma e^{-cs^2} \, ds \approx (1 + x)^\gamma, \quad x \geq 0.
\]

**Proof.** By putting \( r = s^2 \) we get
\[
h(x) = (1 + x)^\gamma \int_0^\infty \frac{(x + r)}{1 + x} \gamma r^{-1/2} e^{-cr} \, dr / 2,
\]
Since for all $x, r \geq 0$ we have

$$1 \vee r \geq \frac{x}{1 + x} + \frac{r}{1 + x} \geq \begin{cases} r/2, & \text{for } x \in (0, 1), \\ 1/2, & \text{for } x \geq 1, \end{cases}$$

the last integral in the above is comparable with a positive constant depending only on $\gamma$ and $c$. □

**Remark A.3.** If $\gamma \geq 0$, then $h(x) \leq C (1 + x)^\gamma$, $x \geq 0$, where $C = \frac{1}{2} \int_0^\infty (1 \vee r)^{\gamma r - 1/2} e^{-cr} \, dr$.

**Lemma A.4.** Let $c > 0$, $\beta > 1$ and

$$I_{\text{app}}(a, b) = \int_0^\infty \left( \frac{s + \sqrt{4ab + s^2}}{2a} \right)^{2(\beta-1)} e^{-cs^2} \frac{ds}{\sqrt{4ab + s^2}}, \quad a, b > 0.$$  

Then

$$I_{\text{app}}(a, b) \approx \frac{(1 + 4ab)^{\beta - 3/2}}{a^{2(\beta-1)}}.$$  

**Proof.** Observe that $0 \leq s \leq \sqrt{4ab + s^2}$. Thus with $h(x)$ and $\gamma = \beta - 3/2$ from Lemma A.2 we have

$$2^{-2(\beta-1)} a^{-2(\beta-1)} \leq \frac{I_{\text{app}}(a, b)}{h(4ab)} \leq a^{-2(\beta-1)}.$$  

The assertion follows by Lemma A.2. □

**Proof of Theorem A.1.** By substitution $u = (a/b)r$ we obtain

$$f(a, b) = (a/b)^{1-\beta} \int_0^\infty r^{-\beta+1} e^{-cab \left[ \sqrt{r} - \frac{1}{\sqrt{r}} \right]^2} \frac{dr}{r}.$$  

By change of variables from $r$ to $1/r$ we get

$$f(a, b) = (a/b)^{1-\beta} \int_0^\infty r^{\beta-1} e^{-cab \left[ \sqrt{r} - \frac{1}{\sqrt{r}} \right]^2} \frac{dr}{r}.$$  

Finally, we let $\sqrt{r} - 1/\sqrt{r} = s/\sqrt{ab}$, then $\left( \sqrt{r} - s/\sqrt{4ab} \right)^2 = 1 + s^2/(4ab)$.

Note that $\sqrt{r} > s/\sqrt{ab}$, hence

$$r = \left( s/\sqrt{4ab} + \sqrt{1 + s^2/(4ab)} \right)^2 = \left( s + \sqrt{4ab + s^2} \right)^2/(4ab),$$

and

$$dr = 2 \left( s + \sqrt{4ab + s^2} \right) \left( 1 + s/\sqrt{4ab + s^2} \right) ds/(4ab)$$

$$= 2 \left( s + \sqrt{4ab + s^2} \right)^2 ds / \left( 4ab \sqrt{4ab + s^2} \right)$$

$$= 2r \, ds / \sqrt{4ab + s^2}.$$  

This gives

$$f(a, b) = 2 \int_{-\infty}^\infty \left( \frac{s + \sqrt{4ab + s^2}}{2a} \right)^{2(\beta-1)} e^{-cs^2} \frac{ds}{\sqrt{4ab + s^2}}.$$
By splitting the last integral we have
\[
f(a, b) = 2 \int_0^\infty \left( \frac{s + \sqrt{4ab + s^2}}{2a} \right)^{2(\beta - 1)} e^{-c s^2} ds / \sqrt{4ab + s^2} + 2 \int_0^\infty \left( \frac{-s + \sqrt{4ab + s^2}}{2a} \right)^{2(\beta - 1)} e^{-c s^2} ds / \sqrt{4ab + s^2}.
\]

Since \( \beta > 1 \) and \( 0 \leq -s + \sqrt{4ab + s^2} \leq s + \sqrt{4ab + s^2} \), we have
\[
2 I_{\text{app}}(a, b) \leq f(a, b) \leq 4 I_{\text{app}}(a, b).
\]

The proof is ended by an application of Lemma A.4.

**Remark A.5.** Using (25), (24) and Remark A.3 we get an explicit constant in the upper bound in Theorem A.1 for \( \beta \geq 3/2 \):
\[
f(a, b) \leq C \frac{(1 + 4ab)^{\beta - 3/2}}{a^{2(\beta - 1)}},
\]
where
\[
C = 2 \int_0^\infty (1 + r)^{\beta - 3/2} r^{-1/2} e^{-cr} dr.
\]

In particular if \( \beta = 3/2 \), then \( C = \sqrt{4\pi/c} \).

We now verify the finiteness of \( \kappa_d \) from [3].

**Lemma A.6.** Let \( d \geq 3 \). Then,
\[
\int_{\mathbb{R}^d \setminus B(0,1)} e^{-(|w| - w_1)|w|^{-\beta}} dw < \infty \iff \beta > (d + 1)/2.
\]

**Proof.** We always have
\[
\int_{\{w \in \mathbb{R}^d \setminus B(0,1): \; w_1 \leq |w|/\sqrt{2}\}} e^{-(|w| - w_1)|w|^{-\beta}} dw < \infty,
\]
therefore we only need to characterize the finiteness of the complementary integral. We will follow the usual notation for spherical coordinates in \( \mathbb{R}^d \) [3]. In particular, \( w \cdot 1 = r \cos \varphi_1 \) and the Jakobian is \( r^{d-1} \prod_{k=1}^{d-2} \sin^k(\varphi_{d-1-k}) \).

We denote \( \varphi = \varphi_1 \), and we consider
\[
\int_1^\infty \int_0^{\pi/4} e^{-r(1-\cos \varphi)r^{-\beta+d-1}} \sin^{d-2} \varphi d\varphi dr = \int_0^{\pi/4} \frac{h(\varphi)}{(1-\cos \varphi)^{d-\beta}} d\varphi,
\]
where \( h(\varphi) = \int_{1-\cos \varphi}^\infty e^{-s\sin^2 \varphi} ds \). If \( \beta = d \), then \( h(\varphi) \approx 1 + |\log(1 - \cos \varphi)| \) and \( \int_0^{\pi/4} h(\varphi) \sin^{d-2} \varphi d\varphi < \infty \), as needed. If \( \beta > d \), then \( h(\varphi) \approx (1 - \cos \varphi)^{d-\beta} \), and the integral is finite, too. If \( \beta < d \), then \( h(\varphi) \approx 1 \) and
\[
\int_0^{\pi/4} \frac{\sin^{d-2} \varphi}{(1-\cos \varphi)^{d-\beta}} d\varphi \approx \int_0^{\pi/4} \varphi^{(d-2)-(d-\beta)} d\varphi,
\]
which converges if and only if \( \beta > (d + 1)/2 \).
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