Comment on “Quantum key distribution via quantum encryption” [Phys. Rev. A 64, 024302 (2001)]

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In the paper [Zhang, Li and Guo, Phys. Rev. A 64, 024302 (2001)], a quantum key distribution protocol based on quantum encryption was proposed, in which the quantum key can be reused. However, it is shown that, if Eve employs a special strategy to attack, this protocol becomes insecure because of the reused quantum key. That is, Eve can elicit partial information about the key bits without being detected. Finally, a possible improvement of the Zhang-Li-Guo protocol is proposed.

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\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig1.png}
\caption{The Zhang-Li-Guo protocol. Note that in this Comment, for simplicity, the operation \( R(\pi/4) \otimes R(\pi/4) \) or \( R(\pi/4) \otimes R(\pi/4) \otimes R(\pi/4) \) is not included in our figures.}
\end{figure}

In Ref.\textsuperscript{1}, Zhang, Li and Guo proposed a quantum key distribution protocol based on quantum encryption. This protocol employs previously shared EPR pairs as a quantum key to encode and decode the classical cryptography, and the quantum key is reusable. However, here we will show that, this protocol would become insecure if the quantum key is reused for more than two times.

For convenience, except for especial declarations, we use the same notations as in Ref.\textsuperscript{1}. Let us give a brief description of the Zhang-Li-Guo protocol firstly (see Fig. 1). At the beginning, Alice and Bob share some quantity of EPR pairs serving as the quantum key: \(|\Phi^+\rangle = 1/\sqrt{2}(|00\rangle + |11\rangle)\). To send the key bit (0 or 1) to Bob, Alice prepares a carrier particle \( \gamma \) in the corresponding state \(|\psi\rangle (|0\rangle \text{ or } |1\rangle)\), performs a controlled-NOT (CNOT) operation on \( \gamma \) and thus entangles this qubit to the previously shared Bell state. Then she transmits this qubit to Bob, from which Bob can obtain the key bit \( \psi \) by performing a CNOT operation and a measurement on it. Because every sending qubit is in a completely mixed state, Eve can not extract information about the key bit. Furthermore, to strengthen the security of this protocol, Alice and Bob perform a rotation

\[
R(\pi/4) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}
\]  \hspace{1cm} (1)

on their respective shared particles before encrypting each \( |\psi\rangle \).

It is well known that the shared particles in Bell state have strong quantum correlation (i.e., entanglement). It is this correlation that makes the quantum encryption secure. The author of Ref.\textsuperscript{1} argues that, because this correlation cannot be produced by LQCC and the eavesdropper cannot establish this correlation with the sender, the quantum key is reusable. However, they overlooked a fact that the sending qubit would bring Eve the chance to entangle her ancilla to the shared Bell state, which means that the eavesdropper can establish this correlation with the sender. As a result, this protocol becomes insecure when the quantum key is reused.

Now we come to Eve’s eavesdropping strategy. Consider a certain EPR pair shared by Alice and Bob, which will be used to encrypt \( \gamma_1, \gamma_2, \gamma_3, \ldots \) (the corresponding states are \(|\psi_1\rangle, |\psi_2\rangle, |\psi_3\rangle, \ldots \) respectively, where \( \psi_i = 0 \text{ or } 1 \)). Hereafter we use the term “the \( i \)-th round” to denote the processing procedures of \( \gamma_i \), and Alice and Bob’s operation \( R(\pi/4) \otimes R(\pi/4) \) is taken as the beginning of each round. Furthermore, we use \(|\phi_0\rangle_{A,B,E} \) and \(|\phi_1\rangle_{A,B,E} \) to denote the states shared by Alice, Bob and Eve at the beginning and the end of the \( i \)-th round, respectively. In addition, the subscriptions A, B and E represent the particles belong to Alice, Bob, and Eve respectively, and \( \gamma \) represents the sending particle. Suppose Eve prepares \(|0\rangle \) as her ancilla, the eavesdropping strategy can be described as follows:

(i) In the first round, Eve entangles her ancilla into the Bell state shared by Alice and Bob. More specifically, Eve intercepts the sending qubit and performs a CNOT operation on her ancilla, then resends the sending qubit to Bob (see Fig. 2). The initial state of Alice, Bob and Eve’s particles can be represented as

\[
|\phi_{10}\rangle_{A,B,E} = \frac{1}{\sqrt{2}}(|0,0,0\rangle + |1,1,0\rangle)_{A,B,E}.
\]  \hspace{1cm} (2)
Then the states at various stages in Fig. 2 are as follows:

$$|\Phi_0\rangle = \frac{1}{\sqrt{2}}(|0,0,\psi_1,0\rangle + |1,1,\psi_1,0\rangle)_{A,B,E}, \quad (3)$$

$$|\Phi_1\rangle = \frac{1}{\sqrt{2}}(|0,0,\psi_1,0\rangle + |1,1,\overline{\psi}_1,0\rangle)_{A,B,E}, \quad (4)$$

$$|\Phi_2\rangle = \frac{1}{\sqrt{2}}(|0,0,\psi_1,\psi_1\rangle + |1,1,\overline{\psi}_1,\overline{\psi}_1\rangle)_{A,B,E}, \quad (5)$$

$$|\Phi_3\rangle = \frac{1}{\sqrt{2}}(|0,0,\psi_1,\psi_1\rangle + |1,1,\overline{\psi}_1,\overline{\psi}_1\rangle)_{A,B,E}, \quad (6)$$

where the overline expresses bit flip, for example, $\overline{\psi}_1 = \psi_1 + 1 \text{ modulo } 2$.

In the last stage, when Bob performs his CNOT operation, he disentangles the sending qubit $|\psi_1\rangle$ and correctly gets the value of $\psi_1$, while the original Bell state has now been entangled with the state of Eve in the form of

$$|\phi_{11}\rangle_{A,B,E} = \frac{1}{\sqrt{2}}(|0,0,\psi_1\rangle + |1,1,\overline{\psi}_1\rangle)_{A,B,E}. \quad (7)$$

(ii) In the second round, Eve tries to avoid the detection and, at the same time, retain her entanglement with Alice and Bob. As was proved in Ref. [1], Eve cannot obtain information in this round. However, we will show that she can take some measures to avoid the detection.

Firstly, when Alice and Bob perform the operations $R(\pi/4) \otimes R(\pi/4)$ on their “Bell state”, Eve also performs $R(\pi/4)$ on her ancilla. As a result, the entangled state of Alice, Bob and Eve will be converted into

$$|\phi_{20}\rangle_{A,B,E} = R(\frac{\pi}{4})_{\otimes 3}|\phi_{11}\rangle_{A,B,E}$$

$$= \frac{1}{2} \left[ |0,0,0\rangle + (-1)^{\psi_1}|0,1,1\rangle + (-1)^{\psi_1}|1,0,1\rangle + (-1)^{\psi_1}|1,1,0\rangle \right]_{A,B,E}, \quad (8)$$

where the identity $R(\frac{\pi}{4})|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + (-1)^{\psi}|1\rangle)$ was used.

Afterwards, Eve intercepts the sending qubit, performs a CNOT operation on it, and then resends it to Bob (see Fig. 3). The states at various stages in Fig. 3 are as follows:

$$|\Psi_0\rangle = \frac{1}{2} \left[ |0,0,0\rangle + (-1)^{\psi_1}|0,1,1\rangle + (-1)^{\psi_1}|1,0,1\rangle + (-1)^{\psi_1}|1,1,0\rangle \right]_{A,B,E} \quad (9)$$

$$|\Psi_1\rangle = \frac{1}{2} \left[ |0,0,0\rangle + (-1)^{\psi_1}|0,1,1\rangle + (-1)^{\psi_1}|1,0,1\rangle + (-1)^{\psi_1}|1,1,0\rangle \right]_{A,B,E} \quad (10)$$

$$|\Psi_2\rangle = \frac{1}{2} \left[ |0,0,0\rangle + (-1)^{\psi_1}|0,1,1\rangle + (-1)^{\psi_1}|1,0,1\rangle + (-1)^{\psi_1}|1,1,0\rangle \right]_{A,B,E} \quad (11)$$

In the last stage, when Bob performs his CNOT operation, he disentangles the sending qubit $|\psi_2\rangle$ and correctly gets the value of $\psi_2$, while leaving the state

$$|\phi_{21}\rangle_{A,B,E} = \frac{1}{2} \left[ |0,0,0\rangle + (-1)^{\psi_1}|0,1,1\rangle + (-1)^{\psi_1}|1,0,1\rangle + (-1)^{\psi_1}|1,1,0\rangle \right]_{A,B,E}. \quad (12)$$

(iii) In the third round, Eve eavesdrops the key bit. Firstly, as in step (ii), Eve also performs $R(\pi/4)$ on her ancilla when Alice and Bob perform $R(\pi/4)$ on their respective particles. The entangled state will be changed into

$$|\phi_{30}\rangle_{A,B,E} = R(\frac{\pi}{4})_{\otimes 3}|\phi_{21}\rangle_{A,B,E}$$

$$= \frac{1}{2\sqrt{2}} \left[ |0,0,0\rangle - |1,0,1\rangle - |0,1,1\rangle + |1,1,0\rangle \right]_{A,B,E} \quad (14)$$

where $\alpha = 1 + (-1)^{\psi_1}$, $\beta = 1 - (-1)^{\psi_1}$.

Afterwards, Eve intercepts the sending qubit, performs a CNOT operation, a measurement and another CNOT operation on it, and then resends it to Bob (see Fig. 4).
The states at various stages in Fig. 4 are as follows:

\[
|\Omega_0\rangle = \frac{1}{\sqrt{2}} \left[ \alpha (|0, 0, \psi_3, 0\rangle - |1, 1, \psi_3, 1\rangle) 
- \beta (|0, 0, \psi_3, 1\rangle - |1, 1, \psi_3, 0\rangle) \right]_{A,B,\gamma,E} \tag{15}
\]

\[
|\Omega_1\rangle = \frac{1}{\sqrt{2}} \left[ \alpha (|0, 0, \psi_3, 0\rangle - |1, 1, \bar{\psi}_3, 1\rangle) 
- \beta (|0, 0, \psi_3, 1\rangle - |1, 1, \bar{\psi}_3, 0\rangle) \right]_{A,B,\gamma,E} \tag{16}
\]

\[
|\Omega_2\rangle = \frac{1}{\sqrt{2}} \left[ \alpha (|0, 0, \psi_3, 0\rangle - |1, 1, \bar{\psi}_3, 1\rangle) 
- \beta (|0, 0, \bar{\psi}_3, 1\rangle - |1, 1, \psi_3, 0\rangle) \right]_{A,B,\gamma,E} \tag{17}
\]

\[
|\Omega_3\rangle = \frac{1}{\sqrt{2}} \left[ \alpha (|0, 0, \psi_3, 0\rangle - |1, 1, \bar{\psi}_3, 1\rangle) 
- \beta (|0, 0, \psi_3, 1\rangle - |1, 1, \bar{\psi}_3, 0\rangle) \right]_{A,B,\gamma,E} \tag{18}
\]

\[
|\Omega_4\rangle = \frac{1}{\sqrt{2}} \left[ \alpha (|0, 0, \psi_3, 0\rangle - |1, 1, \psi_3, 1\rangle) 
- \beta (|0, 0, \psi_3, 1\rangle - |1, 1, \psi_3, 0\rangle) \right]_{A,B,\gamma,E} \tag{19}
\]

It can be seen that Eve disentangles the key qubit by a CNOT operation, and then restores the entangled state by another CNOT operation after a measurement. As a result, Eve obtains the measurement result \(\psi_3 + \psi_1\) (modulo 2) and Bob correctly gets the value of \(\psi_3\). At last, the entangled state of Alice, Bob and Eve can be written as

\[
|\phi_{31}\rangle_{A,B,E} = \frac{1}{\sqrt{2}} \left[ \alpha (|0, 0, 0\rangle - |1, 1, 1\rangle) 
- \beta (|0, 0, 1\rangle - |1, 1, 0\rangle) \right]_{A,B,E} \tag{20}
\]

(iv) In the fourth round, Eve uses a similar strategy as in the second round to avoid the detection, the only difference is that Eve has to perform an additional \(X\) operation on the sending qubit here (see Fig. 5). After their operation \(R(\frac{\pi}{4}) \otimes^3\), Alice, Bob and Eve change the entangled state into

\[
|\phi_{40}\rangle_{A,B,E} = R(\frac{\pi}{4}) \otimes^3 |\phi_{31}\rangle_{A,B,E}
= -\frac{1}{2} \left[ |0, 0, 1\rangle + (-1)^{\psi_3} |0, 1, 0\rangle 
+ (-1)^{\psi_1} |1, 0, 0\rangle + |1, 1, 1\rangle \right]_{A,B,E} \tag{21}
\]

Then Eve performs the operations as described in Fig. 5. The states at various stages are as follows:

\[
|\Theta_0\rangle = -\frac{1}{2} \left[ |0, 0, \psi_4, 1\rangle + (-1)^{\psi_4} |0, 1, \psi_4, 0\rangle 
+ (-1)^{\psi_1} |1, 0, \psi_4, 0\rangle + |1, 1, \psi_4, 1\rangle \right]_{A,B,\gamma,E} \tag{22}
\]

\[
|\Theta_1\rangle = -\frac{1}{2} \left[ |0, 0, \psi_4, 1\rangle + (-1)^{\psi_4} |0, 1, \psi_4, 0\rangle 
+ (-1)^{\psi_1} |1, 0, \psi_4, 0\rangle + |1, 1, \psi_4, 1\rangle \right]_{A,B,\gamma,E} \tag{23}
\]

\[
|\Theta_2\rangle = -\frac{1}{2} \left[ |0, 0, \psi_4, 1\rangle + (-1)^{\psi_4} |0, 1, \psi_4, 0\rangle 
+ (-1)^{\psi_1} |1, 0, \psi_4, 0\rangle + |1, 1, \psi_4, 1\rangle \right]_{A,B,\gamma,E} \tag{24}
\]

\[
|\Theta_3\rangle = -\frac{1}{2} \left[ |0, 0, \psi_4, 1\rangle + (-1)^{\psi_4} |0, 1, \psi_4, 0\rangle 
+ (-1)^{\psi_1} |1, 0, \psi_4, 0\rangle + |1, 1, \psi_4, 1\rangle \right]_{A,B,\gamma,E} \tag{25}
\]

It can be seen that, in the last stage, Bob correctly gets the value of \(\psi_4\), while leaving the state

\[
|\phi_{41}\rangle_{A,B,E} = -\frac{1}{2} \left[ |0, 0, 1\rangle + (-1)^{\psi_4} |0, 1, 0\rangle 
+ (-1)^{\psi_1} |1, 0, 0\rangle + |1, 1, 1\rangle \right]_{A,B,E} \tag{26}
\]

(v) In the fifth round, Eve uses the same strategy as in the third round to eavesdrop the key bit, that is, the strategy in step (iii). After their operation \(R(\frac{\pi}{4}) \otimes^3\), Alice, Bob and Eve change the entangled state into

\[
|\phi_{50}\rangle_{A,B,E} = R(\frac{\pi}{4}) \otimes^3 |\phi_{41}\rangle_{A,B,E}
= -\frac{1}{2} \left[ \alpha(|0, 0, 0\rangle + |1, 1, 1\rangle) 
+ \beta(|0, 0, 1\rangle + |1, 1, 0\rangle) \right]_{A,B,E} \tag{27}
\]
Then Eve performs the operations as described in Fig. 4. The states at various stages are as follows:

\[ |\Upsilon_0\rangle = -\frac{1}{2\sqrt{2}}[\alpha(|0,0,\psi_5,0\rangle + |1,1,\psi_5,1\rangle) + \beta(|0,0,\psi_5,1\rangle + |1,1,\psi_5,0\rangle)]_{A,B,\gamma,E}, \quad (28) \]

\[ |\Upsilon_1\rangle = -\frac{1}{2\sqrt{2}}[\alpha(|0,0,\psi_5,0\rangle + |1,1,\psi_5,1\rangle) + \beta(|0,0,\psi_5,1\rangle + |1,1,\psi_5,0\rangle)]_{A,B,\gamma,E}, \quad (29) \]

\[ |\Upsilon_2\rangle = -\frac{1}{2\sqrt{2}}[\alpha(|0,0,\psi_5,0\rangle + |1,1,\psi_5,1\rangle) + \beta(|0,0,\psi_5,1\rangle + |1,1,\psi_5,0\rangle)]_{A,B,\gamma,E}, \quad (30) \]

\[ |\Upsilon_3\rangle = -\frac{1}{2\sqrt{2}}[\alpha(|0,0,\psi_5,0\rangle + |1,1,\psi_5,1\rangle) + \beta(|0,0,\psi_5,1\rangle + |1,1,\psi_5,0\rangle)]_{A,B,\gamma,E}, \quad (31) \]

\[ |\Upsilon_4\rangle = -\frac{1}{2\sqrt{2}}[\alpha(|0,0,\psi_5,0\rangle + |1,1,\psi_5,1\rangle) + \beta(|0,0,\psi_5,1\rangle + |1,1,\psi_5,0\rangle)]_{A,B,\gamma,E}, \quad (32) \]

where \( \Upsilon_p \) corresponds to the state \( \Omega_p \) in Fig. 4 \((p = 0, 1, 2, 3, 4)\). It can be seen that Eve’s measurement result in this round is \( \psi_5 + \psi_1 \) (modulo 2).

Obviously, in the last stage, Bob correctly gets the value of \( \psi_5 \), while leaving the state

\[ |\phi_{51}\rangle_{A,B,E} = -\frac{1}{2\sqrt{2}}[\alpha(|0,0,0\rangle + |1,1,1\rangle) + \beta(|0,0,1\rangle + |1,1,0\rangle)]_{A,B,E}. \quad (33) \]

Comparing the state \( |\phi_{51}\rangle_{A,B,E} \) with \( |\phi_{11}\rangle_{A,B,E} \), we can verify that the two states are equivalent except for a global phase factor (i.e., -1). That is, from an observational point of view these two states are identical [2]. Therefore, in the following rounds, Eve can use the same strategy as in the steps from (ii) to (v) repeatedly.

Now let us give a concretely description of our eavesdropping strategy:

1. In the first round, Eve performs the operations as described in Fig. 2.

2. When Alice and Bob perform \( R(\varphi/4) \) on their respective particles at the beginning of every round (except for the first round), Eve also performs \( R(\varphi/4) \) on her ancilla;

3. From the second round to the fifth round, Eve performs the operations as described in Fig. 3, Fig. 4, Fig. 5 and Fig. 6 in turn;

4. In the following rounds, Eve performs the operations as described in item 3 repeatedly.

From the above analysis, we can see that in our eavesdropping strategy no error will be introduced to the key distribution between Alice and Bob, and Eve will obtain exactly the result of

\[ \psi_3 + \psi_1, \psi_5 + \psi_1, \psi_7 + \psi_1, \psi_9 + \psi_1, \ldots \]

distribution from which she can infer about half of the key bits by checking two possible values for \( \psi_1 \). It should be emphasized that there is another profitable fact for Eve. That is, at the end of QKD procedure, Alice and Bob will compare a subsequence of the key bits publicly to detect eavesdropping, which obviously leak useful information to Eve.

Now it is worthwhile to inspect the basic idea of our attack strategy. Though Eve cannot get information about the key bit in every even rounds (as proved in Ref. [1]), she can take some more clever measures to avoid the detection and retain her entanglement with Alice and Bob so that she can eavesdrop the key bit in the next round. Our attack strategy is exactly based on this fact. By our strategy, if the shared Bell states are reused for many times, Eve can obtain about half of the key bits without being detected by Alice and Bob. One may argue that the shared Bell states would not be reused for too many times without special treatments by Alice and Bob, such as quantum privacy amplification and entanglement purification [1]. However, from above analysis it can be seen that Eve needs only three rounds to elicit partial information about the key bits, which definitely forms a serious threat to the Zhang-Li-Guo protocol. In fact, the QKD protocols in Refs. [3, 4] have similar hidden troubles, see Refs. [3, 4] for details.

Before we conclude, let us give a discussion about the rotation

\[ R(\theta) = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}, \quad (34) \]

which plays an important role in the Zhang-Li-Guo protocol. Without Alice and Bob’s rotations at the beginning of every round, this QKD protocol would be insecure. For example, in this condition Eve can entangle her ancilla into the Bell state in the first round (as described in Fig. 2), and then elicit information about the key bits in the following rounds (as described in Fig. 4). As a result, Eve will obtain the result of

\[ \psi_2 + \psi_1, \psi_3 + \psi_1, \psi_4 + \psi_1, \psi_5 + \psi_1, \ldots \]

(To avoid confusion we call this attack strategy \( S_1 \), and call the strategy we showed in above paragraphs \( S_2 \).) Therefore, the rotations are necessary, and \( \pi/4 \) is selected as the rotation angle because it leads to the maximum error rate (i.e., 1/2) caused by Eve when she uses the strategy \( S_1 \). However, it is the selection of \( \theta = \pi/4 \) that makes the Zhang-Li-Guo protocol insecure against \( S_2 \). That is, the error rate caused by Eve is 0 when she uses the strategy \( S_2 \). Hereafter we use \( d_1 \) and \( d_2 \) to denote the error rate corresponding to \( S_1 \) and \( S_2 \), respectively. In fact, it is not difficult to prove that, if \( \theta \neq k\pi \pm \pi/4 \) \((k = 0, \pm 1, \pm 2, \ldots)\), it is impossible for Eve
to elicit information about the key bits without introducing disturbance (See the Appendix for details). Consequently, by altering $\theta$, we can modify the Zhang-Li-Guo protocol so that it can resist both $S_1$ and $S_2$.

As was given in Ref.[1], when Eve uses $S_1$ to attack, the error rate is $d_1 = 2 \cos^2 \theta \sin^2 \theta$. By similar deduction we can obtain the error rate when $S_2$ is used, i.e., $d_2 = \frac{1}{4}(\sin^2 \theta - \cos^2 \theta)^2$. Clearly, there is a trade-off between $d_1$ and $d_2$, which satisfy the relation of $d_1 + d_2 = 1/2$. That is, a greater $d_1$ results in a smaller $d_2$, and vice versa. It can be seen that $\theta = \pi/4$ is a extreme instance, where $d_1$ reaches its maximum value 1/2 but $d_2 = 0$. Therefore, we can select such a rotation angle (denoted as $\theta_0$) that $d_1 = d_2 = 1/4$, i.e., $2 \cos^2 \theta_0 \sin^2 \theta_0 = 1/4$. As a result, when we use $\theta_0$ instead of $\pi/4$ in the Zhang-Li-Guo protocol, it can resist both attack strategies (because either strategy will introduce an error rate of 1/4). We have to confess that this modification decreases the efficiency of eavesdropping detection. However, 1/4 is still a sufficient value for a detection probability. In fact, as far as the general intercept-resend strategy is concerned, the detection probability in BB84 protocol [2] is 1/4, too.

In summary, we have presented a special attack strategy to the Zhang-Li-Guo protocol [1], in which Eve can elicit partial information about the key bits without being detected when the quantum key is reused for more than two times. Furthermore, we have discussed about the relation between the security and the value of $\theta$, and pointed out that this QKD protocol would be secure if we use $\theta_0$ instead of $\pi/4$.

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APPENDIX

In this appendix we will show that when $\theta \neq k \pi \pm \pi/4$ ($k = 0, \pm 1, \pm 2, ...$), it is inevitable for Eve to introduce disturbance if she has entangled her ancilla into the Bell state in the first round.

Without loss of generality, suppose that in the first round Eve’s system has entangled with Alice and Bob’s key in the state

$$|\Lambda\rangle = \frac{1}{\sqrt{2}}(|00\rangle|\varphi_0\rangle + |11\rangle|\varphi_1\rangle)|_{A,B,E}. \quad (A.1)$$

where there is no restriction on the form of $|\varphi_0\rangle$ and $|\varphi_1\rangle$.

After Alice and Bob do a bilateral rotation $R(\theta)$, Alice does a CNOT operation on the sending qubit $|\psi_2\rangle$ and sends it out. Then Eve does a unitary transformation on the sending qubit and her own system. She expects that Alice and Bob cannot detect her existence (i.e., the error rate caused by her is 0). Assume that the unitary transformation has the universal form

$$U_{\gamma,E}|i\rangle|\varphi_j\rangle_E = (a_{ij}|0\rangle|\varphi_{aij}\rangle + b_{ij}|1\rangle|\varphi_{bij}\rangle)|_{\gamma,E}, \quad (A.2)$$

where $i, j = 0, 1$ and there is no restriction on the final state of $|\varphi\rangle_E$. At last, Bob receives the sending qubit and uses a CNOT operation to disentangle it from the shared state.

Suppose that the composite system $|\Lambda\rangle_{A,B,E} \otimes |\psi_2\rangle_{\gamma}$ is changed into $|\Delta\rangle$ after all the above operations, we can easily write the form of the state $|\Delta\rangle$. If the attack is successful, it requires that the sending qubit $|\psi_2\rangle$ is correctly disentangled by Bob. To satisfy this requirement, we obtain the following results:

When $\psi_2 = 0$, we get

$$b_{00} \cos^2 \theta |\varphi_{a00}\rangle + b_{01} \sin^2 \theta |\varphi_{a01}\rangle = 0, \quad (A.3)$$

$$-a_{00} \sin \theta \cos \theta |\varphi_{a00}\rangle + a_{01} \sin \theta \cos \theta |\varphi_{a01}\rangle = 0, \quad (A.4)$$

$$-b_{10} \sin \theta \cos \theta |\varphi_{b10}\rangle + b_{11} \sin \theta \cos \theta |\varphi_{b11}\rangle = 0, \quad (A.5)$$

$$a_{10} \sin^2 \theta |\varphi_{a10}\rangle + a_{11} \cos^2 \theta |\varphi_{a11}\rangle = 0. \quad (A.6)$$

When $\psi_2 = 1$, we get

$$a_{10} \cos^2 \theta |\varphi_{a10}\rangle + a_{11} \sin^2 \theta |\varphi_{a11}\rangle = 0, \quad (A.7)$$

$$b_{00} \sin^2 \theta |\varphi_{b00}\rangle + b_{01} \cos^2 \theta |\varphi_{b01}\rangle = 0. \quad (A.8)$$

where we omit two equations the same as Eqs. (A.3) and (A.4).

With the help of Eqs. (A.1)~(A.8), we then obtain two possible conditions: either (1) $|\varphi_0\rangle = |\varphi_1\rangle$, which means $|\Lambda\rangle$ is a product state of Eve’s ancilla and Alice and Bob’s Bell state; or (2) $\theta = k \pi \pm \pi/4$. This result implies that only when $\theta = k \pi \pm \pi/4$ Eve can entangle her ancilla into the Bell state without introducing any disturbance, which is the exact conclusion we want to prove.

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