Supergravity Instantons for $N = 2$ Hypermultiplets

Michael Gutperle$^1$ and Michał Spaliński$^2$

*Jefferson Physical Laboratory*
*Harvard University, Cambridge, MA 02138, USA*

**Abstract**

The dimensional reduction of eleven dimensional supergravity on a Calabi-Yau manifold gives $N = 2$ supergravity in five dimensions with $h_{1,1}$ vector and $h_{2,1} + 1$ hypermultiplets. In this paper instanton solutions are constructed which are responsible for non-perturbative corrections to the hypermultiplet moduli spaces. These instantons are wrapped Euclidean membranes and fivebranes. For vanishing fivebrane charge the BPS conditions for these solutions define a flow in the hypermultiplet moduli space and are isomorphic to the attractor equations for four dimensional black holes.

October 2000

1 gutperle@riemann.harvard.edu
2 mspal@schwinger.harvard.edu
1. Introduction

Non-perturbative effects in string theory and M-theory compactifications can often be described in terms of branes which wrap Euclidean cycles in the compactification manifold. An interesting area for investigating such effects is $N = 2$ supergravity in four or five dimensions. Membrane and fivebrane instantons will provide non-perturbative corrections to the metric on the moduli space of the hypermultiplets. Hypermultiplets of $N = 2$ supergravity parameterize quaternionic manifolds \[\mathcal{M}\]. The quaternionic geometry and the relation of the classical hyper and vector multiplet geometries via the c-map were discussed in \[\mathcal{M}\] (see also \[\mathcal{M}\]), the isometries of (special) quaternionic manifolds were discussed in \[\mathcal{M}\]. M-brane instanton effects were first discussed by Becker, Becker and Strominger \[\mathcal{M}\]. The study of such effects was continued in \[\mathcal{M}\], which in particular investigated charge quantization and the breaking of continuous isometries of the quaternionic manifold due to the interactions of the branes with the background.

\footnote{Some aspects of perturbative corrections to the universal hypermultiplet obtained by dimensional reduction of higher derivative terms in M-theory were discussed in \[\mathcal{M}\].}
to instantons and in [13] where supergravity instanton solutions carrying more than one charge were analyzed.

This paper generalizes the analysis of [12] [13], which focused on the universal hypermultiplet, to an arbitrary number of hypermultiplets. In a nice paper [14], Behrndt et al. used the c-map to relate four dimensional black hole solutions to four dimensional instanton solutions, which can be lifted to the five-dimensional solutions of this paper.

The action of the instantons depends on the charges and the values of complex structure moduli at infinity. Geometrically the action can be interpreted as the action of a Euclidean membrane wrapping on a supersymmetric three cycle. It turns out that for a large class the conditions on BPS-instantons are formally identical to the attractor equations for \( N = 2 \) black holes [15] as formulated by Sabra [16] [17]. The BPS instanton solution can be viewed as defining a flow of the hypermultiplet scalars. In the case of non-vanishing fivebrane charge we find that the action has the behavior characteristic of a non threshold bound state, generalizing the results of [13] obtained for the universal hypermultiplet.

2. Hypermultiplets in \( N = 2 \) Supergravity

The supergravity action splits into two parts: one dependent only on vector multiplets and the other only on hypermultiplets. At the two-derivative level these two parts are coupled only gravitationally. For the purposes pursued in this paper only the hypermultiplet part is required. Although its form is known in the literature, this section will present a derivation based on compactification of eleven dimensional supergravity on a Calabi-Yau threefold. This gives the action in five dimensions. The four dimensional action is essentially the same, since the hypermultiplets are unchanged by the dimensional reduction.

The bosonic part of the action of eleven dimensional supergravity [18] is given by

\[
S = \frac{1}{2k_{11}^2} \int d^{11} x \sqrt{-g} \left( R - \frac{1}{48} F_{MNPQ} F^{MNPQ} \right) - \frac{1}{12k_{11}^2} \int A \wedge F \wedge F .
\] (1)
The supersymmetry transformation of the gravitino in eleven dimensional supergravity is
\[ \delta \psi_M = \partial_M \epsilon + \frac{1}{4} \omega_M^{AB} \Gamma_{AB} \epsilon - \frac{1}{288} \left( \Gamma_M^{NPR} - 8 \delta_M^N \Gamma^{PQR} \right) \epsilon F_{NPR} \, . \] (2)

The notation here is that $A, B$ denote tangent space indices and $M, N$ denote world indices. Dimensional reduction of eleven dimensional supergravity on a Calabi-Yau manifold (with vanishing G-fluxes), produces ungauged five dimensional $N = 2$ supergravity with $h_{1,1}$ vector multiplets and $h_{2,1} + 1$ hyper multiplets [19]. The basic of special geometry are summarized in appendix A - the conventions used here will mainly follow [20].

The three form field strength of the eleven dimensional super gravity gives $2h_{2,1} + 2$ real scalars $\zeta^I, \tilde{\zeta}_I, I = 0, \cdots, h_{2,1}$.

\[ C = \sqrt{2}(\zeta^I \alpha_I + \tilde{\zeta}_J \beta^J) \, . \] (3)

The ansatz for the eleven dimensional metric is given by
\[ ds^2 = e^{-1/3\sigma(y^2)} ds^2_{CY}(z^i, \bar{z}^i) + e^{2/3\sigma(y^2)} g_{\mu\nu} dy^\mu dy^\nu \, , \] (4)

Here $ds^2_{CY}$ is the Calabi-Yau metric, $\sigma$ parameterizes the volume of the Calabi-Yau manifold and $z^i$ are the complex structure moduli. The Kähler moduli are related to vector multiplets. Since vector multiplets decouple from hypermultiplets they will not be discussed here. The coordinates of the five dimensional non-compact space are denoted by $y^\mu$.

Some details of the dimensional reduction are given in appendices A and B. The resulting action for $h_{2,1} + 1$ hypermultiplets is given by
\[ S = \frac{1}{2 k_5^3} \int d^5 x \sqrt{g} \left\{ \frac{1}{2} \left( \partial_\mu \sigma \partial^\mu \sigma \right) + g_{kj}(z) \partial_\mu z^k \partial^\mu z^j + \frac{1}{48} e^{-2\sigma} H_{\mu\nu\rho\lambda} H^{\mu\nu\rho\lambda} \right. \\
+ \frac{1}{24} \epsilon_{\mu\nu\rho\lambda} H^{\mu\nu\rho\lambda} (\zeta^I \partial^\alpha \bar{\zeta}_I - \bar{\zeta}^I \partial^\alpha \zeta^I) - e^{\sigma} \Im N_{IJ} \partial_\mu \zeta^I \partial^\mu \zeta^J \\
- e^{\sigma} (\Im N^{-1})^{IJ} \left( \Re N_{IK} \partial_\mu \zeta^K + \partial_\mu \bar{\zeta}_I \right) \left( \Re N_{IJL} \partial_\mu \zeta^L + \partial_\mu \bar{\zeta}_J \right) \right\} \, . \] (5)
Here $H$ is the field strength of the 3-form gauge field $C_{\mu\nu\rho}$ in five dimensions arising from the dimensional reduction of $C$, $g$ is the metric on the complex structure moduli space, and the matrix $\mathcal{N}$ is defined in appendix A. Note that $\text{Im}(\mathcal{N})_{I,J} < 0$ hence the moduli space metric is positive definite in Minkowski spacetime.

A three form potential in five dimensions is dual to a (pseudo)scalar $a$. After dualizing the action takes the form

$$S = \frac{1}{2}(\partial_\mu a \partial^\mu a) + g_k \bar{z}^k \partial_\mu z^L \partial^\mu \bar{z}^I$$

$$= \frac{1}{2} e^{2\sigma} (\partial_\mu a + \zeta^I \partial_\mu \bar{\zeta}_I - \bar{\zeta}_I \partial_\mu \zeta^I)^2 - e^{\sigma} \text{Im} \mathcal{N}_{I,J} \partial_\mu \zeta^I \partial^\mu \bar{\zeta}_J$$

$$- e^{\sigma} (\text{Im} \mathcal{N}^{-1})^{I,J} (\text{Re} \mathcal{N}_{IK} \partial_\mu \zeta^K + \partial_\mu \bar{\zeta}_I) (\text{Re} \mathcal{N}_{JL} \partial^\mu \zeta^L + \partial^\mu \zeta_J) \right\}$$

(6)

Where we set $2k_5^2 = 1$ for notational convenience. The hypermultiplet action (6) contains $4(h_2,1 + 1)$ real scalars. Of these, $(a, \sigma, \zeta^0, \bar{\zeta}_0)$ comprise the universal hypermultiplet and $(z^k, \bar{z}^k, \zeta^k, \bar{\zeta}_k k = 1, \cdots, h_2,1)$ make up the remaining hypermultiplets. It is a general consequence of $N = 2$ supergravity [4] that these hypermultiplets parameterize a quaternionic manifold, i.e. $4n$ dimensional manifold with holonomy $Sp(1) \times Sp(n)$. In [8] it was shown that the geometry defined by (6) is indeed quaternionic.

The focus of interest in the following are solutions of the five dimensional theory which are the analogs of the D-instanton solutions of ten dimensional type IIB supergravity [21][22]. This means that the solutions satisfy field equations of the Euclidean theory and are localized in all non-compact five dimensions. Since the metric in the non-compact dimensions is assumed to be flat and the vector multiplets are decoupled, one can conclude that the scalars in the vector multiplets must be constant.

An important consequence of the Euclidean continuation is that the sign of the “kinetic” terms for the scalars $\zeta_I, \bar{\zeta}_I, a$ in the action (6) is reversed. This rule for Euclidean continuation of pseudo-scalars follows from [21] dualizing the scalars to four forms (see also [23]). In the rest of the paper the Euclidean equations are simply obtained by replacing $\zeta_I \rightarrow i\zeta_I, \bar{\zeta}_I \rightarrow i\bar{\zeta}_I$ in the Minkowskian equations.
3. Isometries and Charges

The scalars $\zeta^I, \tilde{\zeta}^I$ and $a$ arise from the three form potential in eleven dimensions. The dualized action (6) is invariant under the following shift transformations

$$\zeta^I \rightarrow \zeta^I + \epsilon^I, \quad \tilde{\zeta}^I \rightarrow \tilde{\zeta}^I + \tilde{\epsilon}^I, \quad a \rightarrow a + \delta + \tilde{\epsilon}^I \zeta^I - \epsilon^I \tilde{\zeta}^I. \quad (7)$$

The currents associated with these shifts of are

$$j^I_{\mu} = e^\sigma \left( - Im N_{IJ} \partial \zeta^J - Re N_{IJ} (Im N^{-1})^J L (Re N_{LM} \partial_\mu \zeta^M + \partial_\mu \tilde{\zeta}_L) \right)$$

$$\tilde{j}^I_{\mu} = -2e^\sigma (Im N^{-1})^J L (Re N_{LM} \partial_\mu \zeta^M + \partial_\mu \tilde{\zeta}_L) - e^{2\sigma} (\partial_\mu a + \zeta^I \partial_\mu \tilde{\zeta}_I - \tilde{\zeta}^I \partial_\mu \zeta_I) \zeta^I,$$

$$\tilde{j}^5_{\mu} = e^{2\sigma} (\partial_\mu a + \zeta^I \partial_\mu \tilde{\zeta}_I - \tilde{\zeta}^I \partial_\mu \zeta_I).$$

The corresponding charges are

$$Q^I = \oint d\Sigma \mu j^I_{\mu}, \quad \tilde{Q}^I = \oint d\Sigma \mu \tilde{j}^I_{\mu}, \quad Q^5 = \oint d\Sigma \mu \tilde{j}^5_{\mu}. \quad (9)$$

Note that the charges $Q^I, \tilde{Q}^I$ are not invariant under the shifts; they transform as

$$Q_5 \rightarrow Q_5, \quad Q^I \rightarrow Q^I + \tilde{\epsilon}^I Q_5, \quad \tilde{Q}^I \rightarrow \tilde{Q}^I - \epsilon^I Q_5. \quad (10)$$

In general there will be additional isometries of the hypermultiplet moduli space which depend on the details of the geometry. For example, for quadratic prepotentials $F = 1/2 \sum_{I=0}^n (Z^I)^2$ (which however do not arise in Calabi-Yau compactifications) the geometry is the coset manifold $SU(2, n+1)/(U(2) \times SU(n+1))$ which the isometry group $SU(2, n+1)$. A detailed discussion of symmetries of special quaternionic manifolds can be found in (8) [10] [24].

4. Equations of Motion

The equations of motion for $\zeta^I, \tilde{\zeta}^I$ and $a$ are simply the conservation equations for the associated currents

$$\partial_\mu j^m_{\mu} = 0, \quad \partial_\mu \tilde{j}^m_{\mu} = 0, \quad I = 0, \cdots, h_2, \; ;$$

$$\partial_\mu j^5_{\mu} = 0. \quad (11)$$
The $\sigma$ equation of motion is given by (note that we display the Euclidean equations of motion)

$$\begin{align*}
-\partial^2 \sigma - e^{2\sigma} (\partial_a + \zeta^I \partial_I \tilde{\zeta}_I - \tilde{\zeta}^I \partial_I \zeta_I)^2 + e^\sigma Im N_{IJ} \partial_\mu \zeta^I \partial_\mu \zeta^J \\
+ e^\sigma (Im N^{-1})^{IJ} (Re N_{IK} \partial_\mu \zeta^K + \partial_\mu \tilde{\zeta}_I) (Re N_{JL} \partial_\mu \zeta^L + \partial_\mu \tilde{\zeta}_J) = 0.
\end{align*}$$

(12)

The equation of motion for the scalar $z^k$ is

$$\begin{align*}
\partial^2 z^k + \Gamma^k_{lm} \partial_\mu z^l \partial_\mu z^m + \frac{1}{2} e^\sigma g^{k\bar{l}} \partial_\mu \left\{ Im N_{IJ} \partial_\mu \zeta^I \partial_\mu \zeta^J \\
+ (Im N^{-1})^{IJ} (Re N_{IK} \partial_\mu \zeta^K + \partial_\mu \tilde{\zeta}_I) (Re N_{JL} \partial_\mu \zeta^L + \partial_\mu \tilde{\zeta}_J) \right\} = 0.
\end{align*}$$

(13)

The condition $R_{\mu\nu} = 0$ implies the vanishing of the corresponding components of the energy-momentum tensor, i.e.

$$\begin{align*}
\frac{1}{2} \partial_\mu \sigma \partial_\nu \sigma + g_{k\bar{l}}(z) \partial_\mu z^k \partial_\nu \bar{z}^\bar{l} + e^\sigma Im N_{IJ} \partial_\mu \zeta^I \partial_\nu \zeta^J \\
- \frac{1}{2} e^{2\sigma} (\partial_a + \zeta^I \partial_I \tilde{\zeta}_I - \tilde{\zeta}^I \partial_I \zeta_I) (\partial_a + \zeta^I \partial_I \tilde{\zeta}_I - \tilde{\zeta}^I \partial_I \zeta_I) \\
+ e^\sigma (Im N^{-1})^{IJ} (Re N_{IK} \partial_\mu \zeta^K + \partial_\mu \tilde{\zeta}_I) (Re N_{JL} \partial_\mu \zeta^L + \partial_\mu \tilde{\zeta}_J) = 0.
\end{align*}$$

(14)

Note that this condition implies the vanishing of the bulk part of the action for the Euclidean instanton solutions.

5. A Simple Solution

A simple solution is given by taking the $z^k$ to be constant and real (and as a consequence $N_{IJ}$ to be purely imaginary and negative definite). The fields $\zeta^I$ and $a$ can consistently be set to zero. The relevant part of the action is then given by

$$S = \int d^5x \sqrt{g} \left( \frac{1}{2} (\partial \sigma)^2 - e^\sigma (Im N^{-1})^{IJ} \partial_\mu \zeta^I \partial_\mu \zeta^J \right).$$

(15)

Passing to Euclidean space yields

$$S = \int d^5x \sqrt{g} \left( \frac{1}{2} (\partial \sigma)^2 + e^\sigma (Im N^{-1})^{IJ} \partial_\mu \zeta^I \partial_\mu \zeta^J \right).$$

(16)
The instanton solutions are assumed to be spherically symmetric in the five Euclidean dimensions. Specifically, the following ansatz is made:

$$\sigma = 2 \ln h, \quad \tilde{\zeta}_I = \alpha_I \frac{1}{h} + \text{const.}$$  \hspace{1cm} (17)

The function $h$ is harmonic in five dimensions

$$h(r) = e^{\sigma_\infty/2} + \frac{q}{3r^3}.$$  \hspace{1cm} (18)

The ansatz (17) solves the Euclidean equations of motion provided the constant vector $\alpha_I$ satisfies $(ImN^{-1})^{IJ}\alpha_I\alpha_J = -2$. One can calculate the non-vanishing charges carried by this solution:

$$\tilde{Q}^I = -Vol(S^4)q(ImN^{-1})^{IJ}\alpha_J.$$  \hspace{1cm} (19)

This leads to a relation between the quantity $q$ in (18) and the charges $\tilde{Q}^I$:

$$q = \alpha_I \oint d\Sigma^\mu j_\mu^I.$$  \hspace{1cm} (20)

The bulk part of the action for the instanton vanishes as discussed in [25][22], and the non-zero contribution comes solely from a boundary term:

$$S_{\text{inst}} = -\int d\Sigma^\mu \partial_\mu \sigma = e^{-\sigma_\infty/2} q.$$  \hspace{1cm} (21)

These solutions are the simplest generalizations of the single charged instanton solutions for the universal hypermultiplet found in [11][12][13] and correspond to Euclidean M2 branes wrapped on a three-cycle in the CY (the charge vector $\alpha_I$ is related to the homology of the three-cycle). This solution is in fact a BPS solution which preserves sixteen of the thirty two supersymmetries, as will be discussed in section 9.
6. Instanton Equations, Harmonic Functions and Geodesics

When the vector multiplets are neglected the bosonic $N = 2$ action is given by a quaternionic sigma model of the hypermultiplets couples to gravity. The action is has the form

$$ S = \int d^5 x \sqrt{g} (R - \frac{1}{2} G_{uv}(\phi) \partial_\mu \phi^u \partial^\mu \phi^v), $$

(22)

where $u, v = 1, \cdots, 4(h_2,1 + 1)$, so that the $\phi^u$ now denote all the hypermultiplet (pseudo)scalars and $G$ is the quaternionic metric determined by the action (6).

The solutions of interest here have the property that the five dimensional metric is flat, which implies that

$$ R_{\mu\nu} = \frac{1}{2} g_{uv} \partial_\mu \phi^u \partial^\mu \phi^v = 0. $$

(23)

A simple ansatz for finding such solutions was presented in [26][27][28]: the dependence of the scalar fields $\phi^u$ on the spacetime coordinates $x^\mu$ is through a scalar function $\sigma(x)$, i.e. $\phi^u(x) = \phi^u(\sigma(x))$. The equation of motion for $\phi$ then becomes

$$ \nabla^2 (\phi^u)' + \partial_\mu \sigma \partial^\mu \sigma [(\phi^u)'' + \Gamma^u_{vw} (\phi^v)'(\phi^w)'] = 0, $$

(24)

where $(\phi^k)' = \partial_\sigma \phi^k$. The first term in this equation is the spacetime Laplace operator acting on the scalar function $\sigma(x)$. Hence if $\sigma(x)$ is a harmonic function in spacetime the remaining part of (24) is nothing but the geodesic equation in the moduli space where the scalar $\sigma$ is now interpreted an affine parameter. Since one is seeking a solution where the spacetime is flat, the gravitational part of the equations of motion (23) implies $R_{\mu\nu} = 0$ which means that the geodesic is null.

This construction can be generalized by allowing the fields $\phi^k$ to depend on several harmonic maps $\sigma^a, a = 1, \ldots, n$. A solution to the equations of motion is then given by a totally geodesic null submanifold. Supersymmetry conditions impose further constraints on the solutions. For general moduli spaces finding such geodesic submanifolds is very complicated. However if the moduli space is a coset manifold one can apply the construction given in [28] to construct the solutions.
In the case of a single (universal) hypermultiplet, the moduli space is given by the coset $SU(2,1)/U(2)$, in toroidally compactified type II theories the scalar moduli space is given the well known cosets $G/H$, instanton solutions in these theories were discussed in [23]. Representing the elements of the coset $G/H$ as matrices $g$ the equation of motion for the scalars becomes

$$\partial^\mu(g^{-1}\partial_\mu g) = 0. \quad (25)$$

The matrix $g$ can be parameterized by [28][29]

$$g = a \exp(\sum_i b_i \sigma_i), \quad (26)$$

where $\sigma_i$ are harmonic function in the space time. The conditions that $\sigma_i$ parameterize a null geodesic submanifold translate into certain conditions on the matrices $b_i$ (see [28][29] for details). Hence finding the instanton solutions for hypermultiplets which are coset spaces can be reduced to solving algebraic matrix equations. However the hypermultiplet geometry for Calabi-Yau compactifications is in general not a coset manifold (since the associated prepotential is not quadratic). The techniques reviewed above might however still be useful if one can find a subspace of the full hypermultiplet moduli space which is a coset. In addition supersymmetry does not make a direct appearance in this discussion, yet as will become clear it provides further constraints on the velocity vectors of the geodesics. Therefore the general formalism might also be used to find non-supersymmetric solutions.

7. Supersymmetry

The supersymmetry transformations for the fermionic hyperino and gravitino fields can be derived by dimensional reduction of the eleven dimensional supersymmetry transformation rules (2). Some details of this can be found in appendix B. The transformation law of the gravitino is given by

$$\delta \bar{\psi}_\mu^A = (\partial_\mu + \frac{1}{4} \omega^{ab}_\mu \gamma_{ab})\epsilon^A + (Q_\mu)^A_B \epsilon^B, \quad (27)$$
where \((Q_\mu)_B^A\) is a composite \(Sp(1)\) gauge connection defined by

\[
(Q_\mu)_B^A = \left( \begin{array}{c} \frac{1}{4}(v_\mu - \bar{v}_\mu - \frac{Z N \partial_\mu Z - Z \partial_\mu \bar{Z}}{Z N Z}) \\ u_\mu & -\frac{1}{4}(v_\mu - \bar{v}_\mu - \frac{Z N \partial_\mu Z - Z \partial_\mu \bar{Z}}{Z N Z}) \end{array} \right).
\]

The supersymmetry transformations of the hyperinos are

\[
\delta \xi^I_1 = \epsilon^{1}_\mu \gamma_\mu \xi_1 - \epsilon^{2}_\mu \gamma_\mu \xi_2
\]

\[
\delta \xi^I_2 = \epsilon^{2}_\mu \gamma_\mu \xi_1 + \epsilon^{1}_\mu \gamma_\mu \xi_2.
\]

The vielbein components \(e^{I1}_\mu, e^{I2}_\mu\) are defined in terms of the scalar fields in the following way

\[
e^{I1}_\mu = \left( \begin{array}{c} u_\mu \\ E^A_\mu \end{array} \right), \quad e^{I2}_\mu = \left( \begin{array}{c} v_\mu \\ e^A_\mu \end{array} \right)
\]

where

\[
u_\mu = \frac{1}{2} \partial_\mu \sigma + \frac{i}{2} e^\sigma (\partial_\mu a + \zeta^I \partial_\mu \tilde{\zeta}_I - \tilde{\zeta}^I \partial_\mu \zeta_I)
\]

\[
e^A_\mu = e^A_i \partial_\mu z^i
\]

\[
E^A_\mu = e^{\sigma/2} e^{A_i} f^I_i (N^I_{J\mu} \partial_\mu \zeta^J + \partial_\mu \tilde{\zeta}_I)
\]

Here \(e^A_i\) is the vielbein associated with the metric on the moduli \(z^i\), i.e. \(\delta_{AB} e^A_i e^B_j = g_{ij}\). This form of the components agrees with the ones given in [6]. The components (30) can be combined into a quaternionic vielbein

\[
V^{\alpha A} = \begin{pmatrix} e^I_1 \\ e^I_2 \\ -\bar{e}^I_2 \\ \bar{e}^I_1 \end{pmatrix}, \quad \alpha = 1, \cdots, 2h_{2,1}, \quad A = 1, 2.
\]

The vielbein satisfies the reality constraint \((V^{\alpha A})^* = \epsilon_{ABC} \epsilon_{\alpha \beta} V^{\beta B}\) (this shows the connection with the notation used in [30][31]).

The action (6) can be expressed in the following way

\[
S = 2 \int d^5 x \sqrt{g} \left\{ \sum_I (e^{I1}_\mu e^{1I}_\mu + e^{2I}_\mu e^{2I}_\mu) \right\}
\]

\[
= 2 \int d^5 x \sqrt{g} \left\{ u_\mu u_\mu + v_\mu v_\mu + \sum_A (e^A_\mu e^A_\mu + E^A_\mu E^A_\mu) \right\},
\]
where the following useful identity of special geometry was employed

\[ f^I_i g^{\bar{j} \bar{k}} f^J_j = -\frac{1}{2} (Im\mathcal{N}^{-1})^{I\bar{J}} - \epsilon^K \bar{Z}^I Z^J. \]  

(34)

Note that this formula shows that $Im\mathcal{N}$ is negative definite since the Minkowskian metric has to be positive definite.

For a scalar field configuration that preserves half of the supersymmetries the variations $\delta \xi^k = 0$ have to vanish, for a pair of spinors $\epsilon^A$. This condition can be interpreted as the vanishing of a velocity vector defined by (29) for suitably chosen $\epsilon^A$. For rotationally invariant field configurations the hyperino transformations can be interpreted as $2 \times 2$ matrix equations and the BPS condition is that the determinant of these matrices vanishes

\[ \epsilon^{I\bar{I}} \bar{\epsilon}^{I\bar{I}} + \epsilon^{2I} \bar{\epsilon}^{2I} = 0, \quad I = 0, \ldots, h_{2,1}. \]  

(35)

This condition can be interpreted as follows: As discussed in section 7 the instanton solution is given by a geodesic submanifold, parameterized by a number of harmonic functions. The BPS condition implies that the submanifold is 'null' with respect to all the inner products defined in (35). Note that it follows that the solution satisfies (14) since this is simply the sum of (35) for $I = 0, \ldots, h_{2,1}$.

**8. BPS Solutions and Attractor Equations**

It is straightforward to check that for the simple solution carrying one charge described in section 6, only the $\delta \xi^0$ variation is nontrivial and that the solution preserves half the supersymmetries. It is also possible to write down solutions which carry arbitrary charges $Q^I$ and $\tilde{Q}^I$. For the moment however the five brane charge $Q_5$ will be set to zero. This is equivalent to imposing

\[ \partial_{\mu} \tilde{\phi} + \zeta^I \partial_{\mu} \tilde{\zeta}_I - \bar{\zeta}_I \partial_{\mu} \zeta^I = 0. \]  

(36)

A family of solutions depending on $2(h_{2,1} + 1)$ harmonic functions can be constructed. Let

\[ H_I = h_I + \frac{q_I}{3r^3}, \quad \tilde{H}_I = \tilde{h}_I + \frac{\tilde{q}_I}{3r^3}. \]  

(37)
In the solution is scalars $\zeta^I, \tilde{\zeta}^I$ are taken to satisfy the ansatz

$$
\partial_\mu \zeta^I = -e^{-\sigma} \left\{ (\text{Im}N^{-1})^{IJ} \partial_\mu H_J - (\text{Im}N^{-1})^{IJ} (\text{Re}N)_{JK} \partial_\mu \tilde{H}^K \right\},
$$

$$
\partial_\mu \tilde{\zeta}^I = -e^{-\sigma} \left\{ (\text{Im}N)^{IJ} \partial_\mu \tilde{H}^J - (\text{Re}N)(\text{Im}N^{-1})^{JK} \partial_\mu H_K + (\text{Re}N)(\text{Im}N^{-1})^{JK} (\text{Re}N)_{KL} \partial_\mu \tilde{H}^L \right\}.
$$

(38)

With this it is easy to see that the charges defined in the currents (8) for this solution are given by

$$
Q_5 = 0, \quad Q^I = \text{Vol}(S^4)q^I, \quad \tilde{Q}^I = \text{Vol}(S^4)\tilde{q}^I.
$$

(39)

The BPS condition following from the vanishing of the hyperino variation can be written as (using (29) and (31))

$$
\frac{1}{2} \partial_\mu \sigma \gamma^\mu \epsilon_2 - e^{\sigma/2} L^I (N_{IJ} \partial_\mu \zeta^J + \partial_\mu \tilde{\zeta}^I) \gamma^\mu \epsilon_1 = 0,
$$

$$
\partial_\mu z^i \gamma^\mu \epsilon_1 + e^{\sigma/2} g^{ij} f^I_j (N_{IJ} \partial_\mu \zeta^J + \partial_\mu \tilde{\zeta}^I) \gamma^\mu \epsilon_2 = 0.
$$

(40)

After continuation to Euclidean space the BPS equations for rotationally symmetric field configurations can be derived by choosing the spinors $\epsilon_1 = \pm \epsilon_2$. Using (38) one finds that (29) turns into

$$
\frac{1}{2} \frac{d\sigma}{dr} - L^I (q_I - N_{IJ} \tilde{q}^J) \frac{e^{-\sigma/2}}{r^4} = 0,
$$

$$
\frac{dz^i}{dr} + g^{ij} f^I_j (q_I - N_{IJ} \tilde{q}^J) \frac{e^{-\sigma/2}}{r^4} = 0.
$$

(41)

The equations (41) are exactly of the same form as the attractor equations for $N = 2$ black holes [15] as written in [30] [16] [17].

The solution is further specified by expressing the scalar fields $z^i$ in terms of the harmonic functions (37)

$$
i(Z^I - \tilde{Z}^I) = \tilde{H}^I, \quad i(F^I - \tilde{F}^I) = H_I.
$$

(42)

and the volume scalar $\sigma$ is given by

$$
\sigma = \ln i(Z^I F_I - Z^I \tilde{F}_I).
$$

(43)

---

4 This involves identifying $-\sigma/2$ with Sabra’s $U$ and the charges $q^I, \tilde{q}^I$ with dyonic charges of the black hole solution.
There is an additional constraint on the harmonic functions

$$H_I \partial_\mu \tilde{H}^I - \tilde{H}^I \partial_\mu H_I = 0$$ (44)

which guarantees the integrability of the condition (36) for the vanishing fivebrane charge.

In [17] Sabra has shown that the ansatz (42),(43) solves the attractor equations (41). From this it follows that the our ansatz defines a BPS configuration. Sabra’s calculation will not be repeated here. Instead, in appendix C it is demonstrated that the solution also satisfies the Euclidean equations of motion as expected.

Note that the attractor equation (41) together with (38) defines a flow of the hyper-multiplet scalars when $r$ varies from to $r = \infty$ to $r = 0$ for an instanton solutions with given charges $q_I, \tilde{q}^I$.

9. The Instanton Action

As discussed in section 8 the BPS equations (41) imply the vanishing of the bulk part of the action (14). It is straightforward to verify this directly along the lines of the calculations presented in appendix B.

The fact that the bulk contribution to the action vanishes for the Euclidean instanton solutions is a well known phenomenon [25][22]. The action comes from a boundary term (see [13]):

$$S_{inst} = - \oint d\Sigma^\mu \partial_\mu \sigma = |Z|_\infty e^{-\sigma_\infty/2}. \quad (45)$$

Here

$$Z = (L^I q_I - M_i \tilde{q}^I) \quad (46)$$

To obtain this the first equation in (41) was used to express the boundary term in terms of the asymptotic values of the $\sigma$ and $Z$.

There will also be a nontrivial phase dependence weighting the instanton amplitudes:

$$\exp(i\theta) = \exp(i\zeta^I q_I + i\tilde{\zeta}_I \tilde{q}^I). \quad (47)$$
The discussion of the phase is not complete without the investigation of the one loop determinants, but that lies beyond the scope of this paper.

The microscopic description of these instantons is well known: they correspond to M2 branes wrapped on supersymmetric three cycles (special Lagrangian submanifolds) \[11\]. The action is given by the world-volume action of the wrapped M2 brane on the three cycle in the Calabi-Yau manifold. Geometrically the action can therefore be expressed in the following way

\[
S_{\text{inst}} = e^K \int_{\Gamma} \Omega + i \int_{\Gamma} C, \tag{48}
\]

where \(\Gamma\) is the (special Lagrangian) three cycle on which the membrane instanton is wrapped and the subscript denotes that the fields are evaluated at asymptotic infinity. The cohomology class of the cycle is \(\Gamma = \tilde{q}' \alpha - q \beta\). Using \(\int_{\Gamma} \omega = \int_{\Gamma} \omega \wedge \Gamma \tag{45}\) and \(\tag{47}\) follow from \(\tag{48}\).

In this section we discussed solutions where \(Q_5 = 0\). If one relaxes this condition (and equivalently \(\tag{44}\)) the solutions become more complicated. The microscopic interpretation of such solutions is that of a ‘bound state’ of M5 and M2 branes wrapped on a CY (in flat space such configurations were discussed in \[32\]). These configurations might also be related to M5 branes with three form flux turned on \[33\][34\]. Solutions with \(Q_5\) for the universal hypermultiplet were discussed in \[13\].

In \[14\] the c-map was used to relate the stationary black hole solutions of \[33\] to instanton solutions in four dimensions. The main difference to the solution in section 9 is that the conditions \(\tag{36}\) and \(\tag{44}\) are relaxed:

\[
H_I \partial_\mu \tilde{H}^I - \tilde{H}^I \partial_\mu H_I = e^{2\sigma} (\partial_\mu a + \zeta^I \partial_\mu \tilde{\zeta}_I - \tilde{\zeta}^I \partial_\mu \zeta_I) \\
= \frac{Q_5}{r^4}. \tag{49}
\]

Even for these solutions the bulk part of the action of the instanton will vanish and the instanton action will be given by the boundary term \(\tag{45}\). Using the ansatz for the volume
scalar (43) gives
\[
\frac{1}{2} \partial_r \sigma = \frac{-i}{2} e^{-\sigma} (\partial_r \bar{Z} F_I + \bar{Z} \partial_r F_I - \partial_M Z^I F_I - \bar{Z}^I \partial_r F_I)
\]
\[
= \frac{1}{2} \frac{1}{r^4} e^{-\sigma/2} (\bar{q}^I M - q_I L^I + \bar{q}^I \bar{M}_I - q_I \bar{L}^I)
\]
\[
= \frac{1}{r^4} e^{-\sigma/2} \frac{Z + \bar{Z}}{2}.
\]

For \(Q_5 \neq 0\) one finds that \(Z \neq \bar{Z}\) in particular
\[
\frac{1}{r^4} (Z - \bar{Z}) = e^{-\sigma} (H_I \partial_r \bar{H}^I - \bar{H}^I \partial_r H_I) = e^{-\sigma} \frac{Q_5}{r^4}.
\]

Note that because of the analytic continuation of \(\tilde{\phi}\) in Euclidean space one has
\[
|Z|^2 = Re(Z)^2 + Im(Z)^2 = Re(Z)^2 - Q_5^2
\]
and the instanton action becomes
\[
S_{\text{inst}} = -\int d\Sigma^\mu \partial_\mu \sigma = \sqrt{e^{-\sigma}|Z|^2 + e^{-2\sigma} Q_5^2}.
\]

This generalizes the result (43) to the case \(Q_5 \neq 0\). The action (53) is characteristic of a non threshold bound state and has the same form as the action found in [13] for the universal hypermultiplet. Note that this is not surprising, since a bound state of fivebranes and membranes which preserves half the supersymmetry as to be non-threshold, since threshold bound state would preserve a quarter of the supersymmetry.

10. Fermionic Zero Modes

In addition to the vanishing of the hyperino variation (29) the variation of the gravitino (27) has to vanish for a BPS solution. Note that for the solution of section 9 the diagonal elements of the \(Sp(1)\) connection \(Q_A^B\) in (28) vanish. Using the hyperino BPS condition (40) the gravitino variation (27) becomes
\[
\delta \psi^{1,2}_\mu = \pm (\partial_\mu \epsilon - \frac{1}{2} \partial_\mu \sigma) \epsilon,
\]
Hence the supersymmetry parameter takes the form $\epsilon = e^{\sigma/2} \epsilon_0$, where $\epsilon_0$ is a constant spinor.

The instanton solution has five bosonic zero modes, which correspond to collective coordinates translating the center of the instanton. Since the instanton is a BPS object in a supersymmetric theory, there are also fermionic zero modes, corresponding to fermionic collective coordinates. The fermionic zero modes can be obtained by applying the broken supersymmetries on the bosonic field configuration. There are four broken supersymmetries and hence the instanton solutions will have (at least) four fermionic zero modes. In a path integral around the saddlepoint the integral over fermionic collective coordinates vanishes unless the zero modes are soaked up by field insertions. This leads to instanton induced interactions a la t’Hooft. In the present case the instanton will induce four-fermion terms like

$$\int d^5 x \, \Omega_{IJKL} \bar{\xi}^I \xi^J \bar{\xi}^K \xi^L e^{-S_{\text{inst}}}. \quad (55)$$

The completely symmetric tensor $\Omega_{IJKL}$ is related to a instanton modification of the $Sp(n)$ part of the curvature on the quaternionic geometry of the moduli space. Supersymmetry relates (55) to corrections to the moduli space metric.

For a complete calculation of such terms one would need a path integral formulation of M-theory which at the moment does not exist. In particular the calculation of fluctuation determinants is an important part of the calculation (see [38] for an eloquent discussion of this issue).

However the fact that terms like (55) will be induced and the 'semi-classical' contribution can be determined using that the broken supersymmetry is given by $\epsilon_1 = -\epsilon_2$ and the hyperino variation obtained from the broken supersymmetry is

$$\delta \xi_1^I = e_1^{1I} \gamma^M \epsilon_1, \quad \delta \xi_2^I = e_2^{1I} \gamma^\mu \epsilon_1. \quad (56)$$

\footnote{These are the 'center of mass' zero modes coming from the broken zero modes. In principle there could be additional zero modes, coming for example from moduli of the supersymmetric cycle. This is a very interesting question related to calculations of superpotentials for wrapped branes [36] [37]}
Using a similar reasoning as for $N = 2$ black holes one can use the BPS conditions \( (41) \) to estimate the behavior of the fermionic zero modes \( (56) \) as $r \to \infty$ and $r \to 0$. Using the fact that $|Z(r)| \to |Z|_{\infty}$ as $r \to \infty$ and that the scalar moduli approach a fixed point as $r \to 0$ one finds

$$
\delta \xi^I \sim \frac{1}{r}, \quad r \to 0, \quad \delta \xi^I \sim \frac{1}{r^4}, \quad r \to \infty.
$$

(57)

This implies that the fermionic zero modes will be normalizable and therefore be will be related to fermionic collective coordinates. Furthermore the instanton induced fermion four point vertex

$$
\int d^5 x_0 \delta \xi^I_1 (x - x_0) \delta \xi^J_2 (x - x_0) \delta \xi^K_1 (x - x_0) \delta \xi^L_2 (x - x_0)
$$

(58)

is finite, since the integrand will behave as $1/r^4$ as $r \to 0$ and as $1/r^{16}$ as $r \to \infty$. This integral determines the 'semiclassical' contribution to the four-fermion vertex \( (55) \), however the complicated form of the $e_\mu^{Ia}$ makes the closed evaluation of the integral difficult in general. In type II Calabi-Yau compactifications such instanton corrections can also be calculated at Gepner points using conformal field theory and boundary state techniques \[39\] \[40\].

11. Relation to Black Holes

In four dimensions the c-map \[3\] relates the special geometry of the N=2 vector multiplets to the quaternionic geometry of the hypermultiplets. This can be derived via dimensional reduction because on a circle a four dimensional vector field is related to two scalars by dualization. Hence in the compactified theory a $N = 2$ vector multiplet is on-shell equivalent to a $N = 2$ hypermultiplet.

In \[14\] the c-map was used to relate the most general stationary BPS black hole solutions found in \[17\] \[35\] \[16\] to D-instanton solutions in four dimensions. It was argued that the stationary black hole solutions can be reduced along the timelike Killing direction and a (formal) T-duality on the timelike orbit of this Killing vector relates the black holes...
to instantons. Since the hypermultiplet geometry is the same in four and five dimensions it is not surprising that the instanton solutions are also be solutions in five dimensions, as we showed in this paper. However the relation to the black hole solutions in five dimensions dimensions is lost.

Applying the point of view described in section 7 to these solutions it is clear that the harmonic functions \((37)\) parameterize a geodesic null submanifold, where the velocity vectors satisfy constraints imposed by supersymmetry.

In the previous section it was shown that the BPS conditions for instantons in five dimensions are isomorphic to the BPS flow equations of \(N = 2\) black holes in four dimensions. The instanton action is related to the asymptotic central charge and hence to the ADM mass of the four dimensional black hole, which depends on the charges and the asymptotic values of the complex structure moduli (but not on the asymptotic values of \(\zeta^I, \tilde{\zeta}^I\)). In the case of black holes the attractor mechanism resolves an important puzzle concerning the independence of the black hole entropy (given by the area of the horizon) from the value of the asymptotic moduli. It is an interesting question what the analogue of the area of the horizon and the entropy is for the D-instanton. Note that in the Einstein frame the instanton solution is flat. In the case of the ten dimensional D-instanton \([21]\) it was remarked that solution in the string frame has the interpretation of an Euclidean wormhole. It is quite likely that a similar interpretation of the instanton solution is possible here and that in the 'string'-frame the instanton solutions are a Euclidean wormhole and the size of the neck of the wormhole is related to \(Z(q^I, \tilde{q}_I)|_{fixed}\), which appeared in the formula for the area of the horizon of the black hole.

12. Gauging

In the case of instanton solutions of the Euclidean \(N = 2\) supergravity discussed in this paper only the hypermultiplets are nontrivial. Such solutions are saddlepoints of the action and are responsible for non-perturbative corrections to the geometry of the hypermultiplet moduli space. The solutions are governed by harmonic functions and can be
interpreted as null geodesic submanifolds in the moduli space manifold. Such an interpretation is only possible in the Euclidean theory where the moduli space metric components for the axion-like scalars $\zeta^I, \tilde{\zeta}_I$ change signature. In the original Minkowskian spacetime the moduli space metric is positive definite and the only null geodesics are given by constant maps, i.e. the hypermultiplet solutions are trivial. This implies that there are no nontrivial hypermultiplet solutions with all the other fields turned off. This conclusion not altered when the vector multiplets are taken into account due to the decoupling of vector and hypermultiplets at the two derivative level demanded by $N = 2$ supersymmetry. However if one considers gauged $N = 2$ supergravity one can obtain nontrivial flows involving the hypermultiplets. The reason why nontrivial BPS hypermultiplet dynamics are possible for gauged supergravity is that the supersymmetry transformation of the hyperinos is modified \[ \delta \zeta^A = V^A \gamma^\mu \partial_\mu \phi^u + i X^I k^u I \epsilon. \] Here $A$ and $\alpha$ are the $Sp(1)$ and $Sp(n_h)$ index of the hyperino respectively and $\phi^u$ denotes the scalars in the hypermultiplets and $V$ is the quaternionic vielbein which can be read of from (31). Here $X^I$ denotes the real scalars of the five dimensional (very) special geometry of the vector multiplets.

The new term in the supersymmetry transformation contains the Killing vector fields $k^I$ which generates $n_v + 1$ isometries of the quaternionic manifold $\phi^u \to \phi^u + k^u I$. The BPS condition following from (59) can be nontrivial in Minkowski space and for example allows for domain wall solutions where there are nontrivial flow equations governing the evolution of the hypermultiplets \[ 12, 13. \] The non-perturbative corrections to the hypermultiplet metrics can also be important since instanton effects might destroy isometries of the hypermultiplet moduli space which cannot be gauged. It would be very interesting to investigate this further.
13. Conclusions

In this paper Euclidean instanton solutions of $N = 2$ supergravity were found for theories with arbitrary number of hypermultiplets. The BPS conditions are isomorphic to the attractor equations for $N = 2, d = 4$ black holes. The instanton actions are basically given by the volume of the three cycle on which the membranes are wrapped. In the case of non vanishing fivebrane charge the action has a form reminiscent of a non threshold bound state. The investigations reported here were limited to supergravity: both the inclusion of higher derivative corrections like $R^4$ terms in the dimensional reduction of eleven dimensional supergravity along the lines of [45] would be very interesting. On a microscopic level it would be interesting to understand better the question of extra moduli associated with Euclidean wrapped membranes, if such extra moduli are not lifted by higher terms in the instanton action (which are analogous to superpotentials in spacetime filling branes [36]) such instantons would not contribute to corrections to hypermultiplet geometries. The best hope one might have to calculate instanton corrections is by heterotic-type II duality. However even though on the heterotic side the hypermultiplets geometry is determined at tree level the situation is complicated due to world sheet instanton corrections (see [46] for a discussion of these issues). The form of the instanton action for non vanishing fivebrane charge (53) generalizes the one found for the universal hypermultiplet [13] to any CY compactification, it would be very interesting to analyze this from the perspective of the world-volume theory of the fivebrane wrapped on the CY.

Acknowledgements

We are grateful to A. Strominger for very useful conversations. The work of M.G. is supported in part by the David and Lucile Packard Foundation. The work of M.S. was supported in part by NSF grants DMS-97-09694 and PHY-98-02709.
Appendix A. Special Geometry

A canonical cohomology basis \((\alpha_J, \beta^J), J = 0, \ldots, h_{2,1}\) of three forms satisfies

\[
\int \alpha_I \wedge \alpha_J = 0, \quad \int \beta^I \wedge \beta^J = 0,
\]

\[
\int \alpha_I \wedge \beta^J = -\int \beta^J \wedge \alpha_I = \delta^J_I. \tag{A.1}
\]

The periods over \(A_I, B_I\) cycles dual to the cohomology basis define projective coordinates of the moduli space of complex structure deformations:

\[
Z^I = \int_{A^I} \Omega, \quad F_I = \int_{B_I} \Omega, \tag{A.2}
\]

where \(F_I = \partial_I F\) and (if it exists) the prepotential \(F\) is a homogeneous polynomial in \(Z^I\) of weight two. The holomorphic three form can be expressed in the canonical cohomology basis in the following way

\[
\Omega = Z^I \alpha_I - F_I \beta^I. \tag{A.3}
\]

The metric for the the scalars \(z^k\) is determined by the Kähler potential

\[
K = -\ln(i(Z^I F_I - Z^I \bar{F}_I)). \tag{A.4}
\]

A variation of the holomorphic three form \(\Omega\) is given

\[
\partial_I \Omega = \Omega_I = -K_I \Omega + \Phi_I \tag{A.5}
\]

where \(\Phi_I\) is a \((2,1)\) form and

\[
K_I = -\frac{N_{IJJ} \bar{Z}^J}{Z^K N_{KLL} Z^L} \tag{A.6}
\]

and

\[
N_{IJJ} = (\text{Im} \, F)_{IJJ} \tag{A.7}
\]

It is convenient to define the covariantly constant sections \(L^I = e^{K/2} Z^I, M_I = e^{K/2} F_I\) which satisfy \(D_i L^I = (\partial_i - \frac{1}{2} \partial_i K) L^I = 0\). The following symplectic sections are defined as

\[
f^I_i = D_i L^I = e^{K/2} (\partial_i + \partial_i K) Z^I, \tag{A.8}
\]

\[
h_{iI} = D_i M_I = e^{K/2} (\partial_i + \partial_i K) F_I.
\]
which satisfy the following equations

\[ D_j f_i^I = g_{ij} L^I, \quad D_i h_I = g_{ij} M^I, \quad D_j f_i^I = iC_{ijk} g^{kI} f_i^I, \quad D_i h_{jI} = iC_{ijk} g^{kI} h_{jI} \] (A.9)

Note that \( D_i f_j^I = D_j f_i^I \), which follows from the total symmetry of \( C_{ijk} \).

To carry out the dimensional reduction procedure one needs the following relation expressing the property of the cohomology basis under Hodge duality

\[ \star \alpha_I = - (Im(N) + Re(N)Im(N)^{-1}Re(N))_{IJ} \beta^J + Re(N)Im(N)_{I}^{-1} J \alpha_J, \]
\[ \star \beta^I = Im(N)_{J}^{-1} I \alpha^J - Im(N)_{K}^{-1} IJ Re(N)_{JK} \beta^K, \] (A.10)

where

\[ N_{IJ} = \bar{F}_{IJ} + 2i N_{IK} Z^K N_{JL} Z^L / Z^P N_{PQ} Z^Q. \] (A.11)

This matrix satisfies the important relations

\[ N_{IJ} L^J = M_I \] (A.12)

and

\[ h_{iI} = \bar{N}_{IJ} f_i^J. \] (A.13)

Appendix B. Supersymmetry Transformations from Eleven Dimensions

In this appendix we indicate how to derive the supersymmetry transformation rules (29) for the hyperinos from the eleven dimensional ones (2) by dimensional reduction. On a Calabi-Yau threefold there are two covariantly constant spinors which are related to the existence of Killing spinors of unbroken N=2 spacetime supersymmetry. Using the commutation relation \( \{ \gamma_i, \gamma_j \} = 2g_{ij} \) one can define the two spinors by demanding

\[ \gamma_i \mid \Omega \rangle = 0, \ i = 1, 2, 3 \quad \gamma_i \mid \bar{\Omega} \rangle = 0, \ \bar{i} = 1, 2, 3 \] (B.1)

where \( \mid \Omega \rangle = 1/|\Omega|^2 \Omega^{ij} \gamma_{ij} \mid \Omega \rangle \). Decomposing the eleven dimensional gamma matrices as \( \Gamma_{\mu} = \gamma_\mu \otimes \gamma_7, \ \mu = 0, \cdots 3 \) and \( \Gamma_i = 1 \otimes \gamma_i, \ i = 1, \cdots 6 \). One can expand the Killing spinor
which parameterizes the unbroken $N = 2$ supersymmetry as $\eta = \epsilon_1 \otimes | \Omega \rangle + \epsilon_2 \otimes | \bar{\Omega} \rangle$ where $\epsilon_{1,2}$ are five dimensional (symplectically real) spinors. Massless fermions in the (non universal) hypermultiplets can be constructed using the covariantly constant spinors and harmonic $H^{2,1}$ and $H^{1,2}$ forms. The zero modes of the Dirac operator on the CY-manifold, transform as 3 and $\bar{3}$ under the $SU(3)$ holonomy respectively

$$\psi_{i}^{(k)} = h_{ij\bar{k}}^{(k)} \tilde{\gamma}^{\bar{j} k} | \Omega \rangle, \quad \psi_{j}^{(k)} = h_{ij\bar{k}}^{(k)} \tilde{\gamma}^{j k} | \Omega \rangle. \quad (B.2)$$

Here $h_{ij\bar{l}}^{(k)}$ and $h_{ijl}^{(k)}$ are the $H^{1,2}$ and $H^{2,1}$ forms and $\psi_{i}^{(k)}, \psi_{j}^{(k)}$ are identified with the hyperinos $\xi_{1}^{k}, \xi_{2}^{k}, k = 1, \cdots, h_{2,1}$.

An arbitrary three form $\Gamma$ can be decomposed in terms of the elements of $H^{3,0} \oplus H^{2,1} \oplus H^{1,2} \oplus H^{0,3}$ [17].

$$\Gamma = i \bar{Z}(\Gamma) | \Omega \rangle - ig^{i\bar{j}} \bar{D}_{\bar{j}} \bar{Z}(\Gamma) D_{i} | \Omega \rangle + ig^{i\bar{j}} D_{i} Z(\Gamma) D_{\bar{j}} | \Omega \rangle - iZ(\Gamma) \bar{\Omega}, \quad (B.3)$$

where $Z(\Gamma) = e^{K/2} \int \Gamma \wedge | \Omega \rangle$. For the field strength associated with the potential [3] $F_{\mu} = \partial_{\mu} \zeta^{I} \alpha_{I} + \partial_{\mu} \bar{\zeta}_{I} \beta^{I}$ gives

$$F_{\mu} = i \bar{L}^{I} (N_{IJ} \partial_{\mu} \zeta^{J} + \partial_{\mu} \bar{\zeta}) | \Omega \rangle - ig^{i\bar{j}} \bar{f}_{j}^{I} (N_{IJ} \partial_{\mu} \zeta^{J} + \partial_{\mu} \bar{\zeta}_{I}) D_{i} | \Omega \rangle + c.c. \quad (B.4)$$

The scalars $z^{i}$ are the moduli associated with complex structure deformation of the Calabi-Yau manifold. The spin connection $\omega_{\mu j}^{i}, \omega_{\mu j}^{i}$ can be expressed in terms of the Christoffel connections

$$\Gamma_{\mu i}^{j} = \frac{1}{2| \Omega |^{2}} \Omega^{j i k} h_{lkj}^{(i)} \partial_{\mu} z^{i}, \quad \Gamma_{\mu i}^{j} = \frac{1}{2| \Omega |^{2}} \Omega^{j i k} h_{lkj}^{(i)} \partial_{\mu} z^{i} \quad (B.5)$$

by appropriate multiplication with vielbeins. Here $| \Omega |^{2} = 1/3! \Omega_{ijk} \bar{\Omega}^{ijk}$ and $h_{i\bar{j}l}^{(k)}$ are the components of the $(2,1)$ form $D_{k} | \Omega \rangle$. Plugging the formulas for the spin connection and the gauge field given in this appendix into the supersymmetry transformation [2] of the eleven dimensional gravitino one can derive the form of the five dimensional supersymmetry transformation [31] where the vielbein $e^{I}$ with $I = 0$ is associated with $\Omega$ and $e^{k} k = 1, \cdots, h_{2,1}$ with $D_{k} | \Omega \rangle$ respectively.
Appendix C. Equations of Motion

In order to show that the ansatz satisfies the $\sigma$ equation of motion (12) one calculates

\[
\frac{1}{2} \nabla^2 \sigma = \frac{1}{2} \left( \frac{d^2 \sigma}{dr^2} + \frac{4}{r} \frac{d \sigma}{dr} \right) = -(L^I q_I - M_I q^I)^2 \frac{e^{-\sigma}}{r^8} - \frac{d}{dr} (L^I q_I - M_I q^I) \frac{e^{-\sigma/2}}{r^4} = -\left( (L^I q_I - M_I q^I)^2 - (f^I q_I - h_{iI} q^I) g^{ij} (\tilde{f}^I q_I - \tilde{h}_{iI} q^I) \right) \frac{e^{-\sigma}}{r^8} = \frac{1}{2} e^\sigma (N_{IK} \partial_\mu \zeta^K + \partial_\mu \tilde{\zeta}_I) (N_{IJ} q_I \zeta^L + \partial_\mu \tilde{\zeta}_J)
\]

(where repeated use has been made of (41) ). To show that (13) is satisfied it is convenient to start from

\[
\partial_\mu (g_{kl} \partial^\mu z^I) = \partial_\mu (\tilde{f}_k^l q_I - \tilde{h}_{kl} q^I) g^{il} (f^l q_I - \tilde{h}_{Iq} q^I) \frac{e^{-\sigma}}{r^8} + \partial_\mu (\tilde{f}_k^l q_I - \tilde{h}_{kl} q^I) g^{il} (f^l q_I - \tilde{h}_{Iq} q^I) \frac{e^{-\sigma}}{r^8} + (\tilde{f}_k^l q_I - \tilde{h}_{kl} q^I) (L^I q_I - M_I q^I) \frac{e^{-\sigma}}{r^8}
\]

together with

\[
(\partial_\mu g_{kl}) \partial_\mu z^I \partial_\mu z^J = -\partial_\mu \tilde{g}_{kl} (f^l q_I - h_{iI} q^I) (\tilde{f}_k^l q_I - \tilde{h}_{kl} q^I) \frac{e^{-\sigma}}{r^8}.
\]

Using the fact that $\partial_K \partial_\mu z^k - \partial_K \partial_\mu z^k = 0$ and several other identities given in appendix B one arrives at

\[
\partial_\mu (g_{kl} \partial_\mu z^I) - (\partial_m g_{kl}) \partial_\mu z^m \partial_\mu z^I = (\tilde{f}_k^l q_I - \tilde{h}_{kl} q^I) (L^I q_I - M_I q^I) \frac{e^{-\sigma}}{r^8} + \partial_\mu (\tilde{f}_k^l q_I - \tilde{h}_{kl} q^I) g^{lm} (f_m q_I - \tilde{h}_{mI} q^I) \frac{e^{-\sigma}}{r^8}.
\]

The other term in the equation of motion (13) is given by

\[
\frac{1}{2} e^\sigma \partial_K \left\{ \text{Im} N_{IJ} \partial_\mu \zeta^I \partial^\mu \zeta^J + (\text{Im} N^{-1})^{IJ} \left( \text{Re} N_{IK} \partial_\mu \zeta^K + \partial_\mu \tilde{\zeta}_I \right) \left( \text{Re} N_{KL} \partial_\mu \zeta^L + \partial_\mu \tilde{\zeta}_J \right) \right\} = \frac{1}{2} e^{-\sigma} r^8 \partial_K \left\{ (q_I - N_{IJ} q^I) (\text{Im} N^{-1})^{IK} (q_K - \bar{N}_{KL} q^L) \right\} = -e^{-\sigma} r^8 \partial_K \left\{ (f^I q_I - h_{Iq} q^I) g^{lm} (\bar{f}_m q_I - \bar{h}_{mI} q^I) + L^I (q_I - N_{IJ} q^I) \bar{L}^K (q_K - \bar{N}_{KL} q^L) \right\},
\]

(C.5)
where in the last line the identity (34) was used. The last term in the third line of (C.5) can be rewritten as follows

\[
\partial_{\bar{k}} \left\{ L^I (q_I - N_{IJ} \bar{q}^J) \bar{L}^K (q_K - \bar{N}_{KLM} \bar{q}^L) \right\} \\
= D_{\bar{k}} (L^I (q_I - N_{IJ} \bar{q}^J) ) \bar{L}^K (q_K - \bar{N}_{KLM} \bar{q}^L) + L^I (q_I - N_{IJ} \bar{q}^J) D_{\bar{k}} (\bar{L}^K (q_K - \bar{N}_{KLM} \bar{q}^L)) \\
= (L^I q_I - M_I \bar{q}^I) (\bar{f}^K_{\bar{k}} q_K - \bar{h}_{\bar{k}K} \bar{q}^K).
\]

(C.6)

Adding (C.4) and (C.5) and using (C.6) equation (13) follows.
References

[1] A. Strominger, “Loop corrections to the universal hypermultiplet,” Phys. Lett. B421 (1998) 139 [hep-th/9706195].
[2] I. Antoniadis, S. Ferrara, R. Minasian and K. S. Narain, “R**4 couplings in M- and type II theories on Calabi-Yau spaces,” Nucl. Phys. B507 (1997) 571
[3] H. Gunther, C. Herrmann and J. Louis, “Quantum corrections in the hypermultiplet moduli space,” Fortsch. Phys. 48 (2000) 119 [hep-th/9901137].
[4] J. Bagger and E. Witten, “Matter Couplings In N=2 Supergravity ,” Nucl. Phys. B222 (1983) 1.
[5] S. Cecotti, S. Ferrara and L. Girardello, “Geometry Of Type II Superstrings And The Moduli Of Superconformal Field Theories,” Int. J. Mod. Phys. A4 (1989) 2475.
[6] S. Ferrara and S. Sabharwal, “Quaternionic Manifolds For Type II Superstring Vacua Of Calabi-Yau Spaces,” Nucl. Phys. B332 (1990) 317.
[7] J. De Jaegher, B. de Wit, B. Kleijn and S. Vandoren, “Special geometry in hypermultiplets,” Nucl. Phys. B514 (1998) 553 [hep-th/9707262].
[8] B. de Wit, B. Kleijn and S. Vandoren, “Superconformal hypermultiplets,” Nucl. Phys. B568 (2000) 475 [hep-th/9909228].
[9] B. de Wit and A. Van Proeyen, “Isometries of special manifolds,” [hep-th/9505097].
[10] B. de Wit, F. Vanderseypen and A. Van Proeyen, “Symmetry structure of special geometries,” Nucl. Phys. B400 (1993) 463 [hep-th/9210068].
[11] K. Becker, M. Becker and A. Strominger, “Five-branes, membranes and nonperturbative string theory,” Nucl. Phys. B456 (1995) 130 [hep-th/9507158].
[12] K. Becker and M. Becker, “Instanton action for type II hypermultiplets,” Nucl. Phys. B551 (1999) 102 [hep-th/9901126].
[13] M. Gutperle and M. Spalinski, “Supergravity instantons and the universal hypermultiplet,” JHEP 0006 (2000) 037 [hep-th/0005068].
[14] K. Behrndt, I. Gaida, D. Lust, S. Mahapatra and T. Mohaupt, “From type IIA black holes to T-dual type IIB D-instantons in N = 2, D = 4 supergravity,” Nucl. Phys. B508 (1997) 659 [hep-th/9706096].
[15] S. Ferrara, R. Kallosh and A. Strominger, “N=2 extremal black holes,” Phys. Rev. D52 (1995) 5412 [hep-th/9508072].
[16] W. A. Sabra, “General static N = 2 black holes,” Mod. Phys. Lett. A12 (1997) 2585 [hep-th/9703101].
[17] W. A. Sabra, “Black holes in N = 2 supergravity theories and harmonic functions,” Nucl. Phys. B510 (1998) 247 [hep-th/9704147].
[18] E. Cremmer, B. Julia and J. Scherk, “Supergravity theory in 11 dimensions,” Phys. Lett. B76 (1978) 409.
[19] A. C. Cadavid, A. Ceresole, R. D'Auria and S. Ferrara, “Eleven-dimensional supergravity compactified on Calabi-Yau threefolds,” Phys. Lett. B357 (1995) 76 [hep-th/9506144].

[20] A. Ceresole, R. D'Auria and S. Ferrara, “The Symplectic Structure of N=2 Supergravity and its Central Extension,” in Nucl. Phys. Proc. Suppl. 46 (1996) 67 [hep-th/9509160].

[21] G. W. Gibbons, M. B. Green and M. J. Perry, “Instantons and Seven-Branes in Type IIB Superstring Theory,” Phys. Lett. B370 (1996) 37 [hep-th/9511080].

[22] M. B. Green and M. Gutperle, “Effects of D-instantons,” Nucl. Phys. B498 (1997) 195 [hep-th/9701093].

[23] E. Cremmer, I. V. Lavrinenko, H. Lu, C. N. Pope, K. S. Stelle and T. A. Tran, “Euclidean-signature supergravities, dualities and instantons,” Nucl. Phys. B534 (1998) 40 [hep-th/9803259].

[24] B. de Wit and A. Van Proeyen, “Symmetries Of Dual Quatunierionic Manifolds,” Phys. Lett. B252 (1990) 221.

[25] G. W. Gibbons, M. B. Green and M. J. Perry, “Instantons and Seven-Branes in Type IIB Superstring Theory,” Phys. Lett. B370 (1996) 37 [hep-th/9511080].

[26] G. Neugebauer and D. Kramer, Ann. der Physik (Leibzig) 24 (1962) 62.

[27] P. Breitenlohner, D. Maison and G. Gibbons, “Four-Dimensional Black Holes From Kaluza-Klein Theories,” Commun. Math. Phys. 120 (1988) 295.

[28] G. Clement and D. V. Galtsov, “Stationary BPS solutions to dilaton-axion gravity,” Phys. Rev. D54 (1996) 6136 [hep-th/9607043].

[29] K. S. Stelle, “BPS branes in supergravity,” [hep-th/9803116].

[30] P. Fre, “Supersymmetry and first order equations for extremal states: Monopoles, hyperinstantons, black holes and p-branes,” in Nucl. Phys. Proc. Suppl. 57 (1997) 52 [hep-th/9701054].

[31] A. Ceresole and G. Dall’Agata, “General matter coupled N = 2, D = 5 gauged supergravity,” [hep-th/0004111].

[32] M. B. Green, N. D. Lambert, G. Papadopoulos and P. K. Townsend, “Dyonic p-branes from self-dual (p+1)-branes,” Phys. Lett. B384 (1996) 86 [hep-th/9605146].

[33] D. Lust and A. Miemiec, “Supersymmetric M5-branes with H-field,” Phys. Lett. B476 (2000) 395 [hep-th/9912065].

[34] M. Marino, R. Minasian, G. Moore and A. Strominger, “Nonlinear instantons from supersymmetric p-branes,” JHEP 0001 (2000) 005 [hep-th/9911209].

[35] K. Behrndt, D. Lust and W. A. Sabra, “Stationary solutions of N = 2 supergravity,” Nucl. Phys. B510 (1998) 264 [hep-th/9705169].

[36] S. Kachru, S. Katz, A. Lawrence and J. McGreevy, “Open string instantons and superpotentials,” Phys. Rev. D62 (2000) 026001 [hep-th/9912151].
[37] I. Brunner and V. Schomerus, “On superpotentials for D-branes in Gepner models,” hep-th/0008194.
[38] J. A. Harvey and G. Moore, “Superpotentials and membrane instantons,” hep-th/9907026.
[39] M. Gutperle and Y. Satoh, Nucl. Phys. B543 (1999) 73 hep-th/9808080.
[40] M. Gutperle and Y. Satoh, “D0-branes in Gepner models and N = 2 black holes,” Nucl. Phys. B555 (1999) 477 hep-th/9902120.
[41] L. Andrianopoli, M. Bertolini, A. Ceresole, R. D’Auria, S. Ferrara, P. Fre and T. Magri, “N = 2 supergravity and N = 2 super Yang-Mills theory on general scalar manifolds: Symplectic covariance, gaugings and the momentum map,” J. Geom. Phys. 23 (1997) 111 hep-th/9605032.
[42] K. Behrndt and S. Gukov, “Domain walls and superpotentials from M theory on Calabi-Yau three-folds,” Nucl. Phys. B580 (2000) 225 hep-th/0001082.
[43] K. Behrndt, C. Herrmann, J. Louis and S. Thomas, “Domain walls in five dimensional supergravity with non-trivial hypermultiplets,” hep-th/0008112.
[44] M. B. Green, M. Gutperle and P. Vanhove, “One loop in eleven dimensions,” Phys. Lett. B409 (1997) 177 hep-th/9706175.
[45] I. Antoniadis, S. Ferrara, R. Minasian and K. S. Narain, “R**4 couplings in M- and type II theories on Calabi-Yau spaces,” Nucl. Phys. B507 (1997) 571 hep-th/9707013.
[46] P. S. Aspinwall, “Aspects of the hypermultiplet moduli space in string duality,” JHEP 9804 (1998) 019 hep-th/9802194.
[47] F. Denef, “Attractors at weak gravity,” Nucl. Phys. B547 (1999) 201 hep-th/9812049.