Generation of photon pairs in hyper-Raman Effects and its connection with transitions symmetries

Nicolae Enaki and Marina Turcan

Abstract. In this paper we propose the model interaction Hamiltonian which takes into consideration the four photon process in interaction of dressed field with $^{39}K$ atoms. Using the method of elimination of virtual states, we obtain this the effective interaction Hamiltonian which describe the simultaneously generation of photon pairs. The master equation for laser field is obtained, taking into account the good cavity limits in the process of two photon generation. The steady state solution of this equation for threshold case, which takes into account the quantum fluctuation and photon statistics, is proposed. The cross-correlation function between Stokes and anti-Stokes photon pairs is studied as function of the evolution of the system.

Institute of Applied Physics of Academy of Sciences of Moldova, Academiei str. 5, Chisinau, MD-2028, Republic of Moldova
E-mail: enache@asm.md

1. Introduction
The two-photon coherent generation of light is in the center of attention of many experimental and theoretical studies in the last time. The property of entanglement between the emitted photons in the processes of light generation has a great impact factor today in the applications in quantum information. As a rule this property are connected with generation processes, which take place in excited matter. The two-photon amplification of light is possible in multi-level atoms and open the new perspective of application the coherence and entanglement between the photon in the quantum communication.

The experimental realization of two-photons laser was proposed in the papers of J. Gauthier and coauthors [1]. The investigation of two-photon lasing in above experiments consists of spin-polarized and laser-driven $^{39}K$ atoms placed in a high-finesse transverse-mode-degenerate optical resonator and produces a beam with a power of $0.2$ Watt at a wavelength of 770 nm. In these experimental papers it was observed the complex dynamically instabilities of the state of polarization of the two-photon laser, which are made possible by the atomic Zeeman degeneracy. In the papers [1] the authors have considered laser amplification in a thermal vapor of potassium atoms $^{39}K$, when the laser frequency was tuned in the vicinity of the $4S_{1/2} \rightarrow 4P_{1/2}$ transition. The two-photon amplification arises in the system in the process of four-quantum Hyper-Raman scattering effect (see Fig. 1). In this four photons process, stimulated by dressed field with the frequency $\omega_d$, the atom makes a transition from the ground state $|g\rangle$ to the new quasi-stable state $|v\rangle$ by absorbing two photons from the dressing field with frequency $\omega_d$ and generating two new photons to the probe lasing with frequency $\omega_n$. Taking in to account the energy conservation law, we can found the frequency of new generated light $\omega_p = \omega_d - \Delta_{ge}/2$, where $\hbar \Delta_{ge}$ is the energetically distance between the ground $|g\rangle$ and excited $|e\rangle$ states respectively. The states $|g\rangle$
and $|e\rangle$ corresponded to the $4S_{1/2}(F=1)$ and $4S_{1/2}(F=2)$ hyperfine states of $^{39}K$ atom, respectively. The de-tuning is about $\Delta_{g}/2\pi = 462 MHz$. To obtain two-photon amplification, based on this stimulated emission process, need the condition for which the number of atoms in the ground state $N_{g}$ is larger then the number of atoms in excited $N_{e}$. The authors of the papers [2], [3] and [6] pointed out that $n-$photon Raman scattering processes occur in this system, in which the probe beam frequencies satisfies the condition $\omega = \omega_{d} - \Delta_{g}/n$ for $n = 1, 2, 3$... In the last time using the above condition the authors of the papers [4] and [5] have observed multi-photon Raman scattering in cooled $Rb$ atoms trapped in the potential wells of a $3D$, three-dimensional optical lattice. Following this experimental realization, we propose a model which takes into account the two-photon correlations between the generation and pump field in Hyper-Raman conversion. The model shows that the quantum correlation between Stokes and pump photons contains new peculiarities unstudied today. Following this experimental realization, we propose a model which takes into account the cooperativity between two-photon absorption and generation processes of photon pairs from the pump and radiation field in hyper-Raman lasing. The new master equation, which describes the correlation between the pump and generation photon pairs in the cavity is obtain and numerically solved. The possibility of two-photon generation of entangled light in multi-photon hyper-Raman process described in Fig. 1 and connection with the quantum properties of pump field is established. We have demonstrated that two-photon amplification arises in four-quantum Hyper-Raman scattering effect represented in Fig. 1 contains the amplitude and phase correlations with pump. In this four photons process stimulated by dressed field with the frequency $\omega_{d}$ the atom makes a transition from the ground state $|g\rangle$ to the new quasi-stable state $|e\rangle$ by absorbing two photons from the dressing field with

![Figure 1. The atomic pump of Hyper-raman scattering process with absorption of two-photon and emission of two Stokes photons](image-url)
frequency $\omega_d$ and generating two new photons to the probe lasing field with frequency $\omega_a$ via virtual intermediate states.

We propose the new interaction picture of hyper-Raman process in which two quanta from pumping field is absorbed an other two entangled photons are generated. Following the modern experimental description of two photon generation of light observed in papers [1], we obtained the effective interaction Hamiltonian for this process. The analytical expressions for two-photon correlation from atom driven in a hyper-Raman configuration are discussed. It is proposed novel two-photon entangled sources, which take into account the coherence and collective phenomena between the these states. The quantum propriety of realistic sources of powerful coherent bi-boson radiation (coherent entanglement of Stokes and anti-Stokes photons) is analyzed. As the lifetime of the atom in the cavity is considered smaller than the cooperative scattering time between the Stokes and anti-Stokes modes, we have eliminated the atomic variables and focused on the transformation of the Stokes mode in anti-Stokes mode and vice versa, as a function of the atomic inversion and initially prepared field states. Taking in to account the process, that takes place with the absorption (emission) of new quanta, we have introduced the bi-boson operators describing such a quantum transformation between the scattering photons into the two cavity modes. The master equation, which describes this process, gives us the possibility to discover the behavior of quantum fluctuations in stationary process of absorption and radiation of Stokes and anti-Stokes photons respectively. The stationary solution of this master equation reveal the strong quantum correlation between Stokes and anti-Stokes photons. This correlation function gives us the possibility to manipulate with quantum statistics of photons generated in the scattering field as a function of atomic inversion and cavity parameters, like de-tuning, losses, and quality factor of the cavity.

In the next section we derive the model Hamiltonian which describes the two-photon lasing effect, stimulated by dressed field. In the third section using the method of adiabatic elimination, it is derived the master equation for lasing field.

2. Model Hamiltonian and Master Equation
The effective Hamiltonian of potassium atom in interaction with cavity field and external laser field can be obtain from interaction diagrams of three level atom with bimodal electromagnetic field represented in Fig. 1. Let us consider the Hamiltonian of the single atom described by above diagram in interaction with external field. In the two cavity modes of this field we can introduce the creation and annihilation operators, $\hat{d}^\dagger$ and $\hat{d}$ of pump field, and generation field of hyper-Raman scattering described by other annihilation and creation operators $\hat{b}^+\text{and }\hat{\bar{b}}$

$$\hat{H} = \hat{H}_0 + \hat{H}_I,$$

where $\hat{H}_0$ and $\hat{H}_I$ are the free and interaction parts of the Hamiltonian, which can be represented in the following form

$$\hat{H}_0 = \hbar \omega_v \lvert v \rangle \langle v \rvert + \hbar \omega_e \lvert e \rangle \langle e \rvert + \hbar \omega_g \lvert g \rangle \langle g \rvert + \hbar \omega_a \hat{b}^+\hat{b} + \hbar \omega_d \hat{d}^+\hat{d} \quad (1)$$

and

$$\hat{H}_I = -\{[\mu_{ev,\mathbf{g}_a}\hat{b}^+(t)e^{-ik_{ax}} + (\mu_{ev,\mathbf{g}_d}\hat{d}^+(t)e^{-ik_{dx}})\lvert e \rangle \langle v \rvert + \mu_{ge,\mathbf{g}_a}\hat{d}^+(t)e^{-ik_{ax}} + (\mu_{ge,\mathbf{g}_d}\hat{b}^+e^{-ik_{dx}})\lvert g \rangle \langle v \rvert - \{[\mu_{ve,\mathbf{g}_a}\hat{b}^+e^{ik_{ax}} + (\mu_{ve,\mathbf{g}_d}\hat{d}^+e^{ik_{dx}})\lvert v \rangle \langle e \rvert + \mu_{vg,\mathbf{g}_a}\hat{d}^+e^{ik_{ax}} + (\mu_{vg,\mathbf{g}_d}\hat{b}^+e^{ik_{dx}})\lvert v \rangle \langle g \rvert \} \} \quad (2)$$

Here $\hbar \omega_v$, $\hbar \omega_e$ and $\hbar \omega_g$ are the energies of the virtual $\lvert v \rangle$, excited $\lvert e \rangle$ and ground $\lvert g \rangle$ states respectively, the expressions $\lvert e \rangle \langle v \rvert$ and $\lvert g \rangle \langle v \rvert$ represent the transition operators between upper
virtual state $|v\rangle$ and lower $|e\rangle, |g\rangle$ states accompanied with absorption and emission of the photons, $d^\dagger$ and $b^\dagger$, of external and scattering fields respectively; $k_d$ and $k_g$ are the photon wave vectors in the dressed and scattered modes respectively. The photon wave vectors of external and scattered modes are directed along the $z$ and $x$ axis in according with $\mu_{ev}$ and $\mu_{gv}$ are the dipole transition matrix elements between virtual $|v\rangle$ and lower $|e\rangle, |g\rangle$ states.

The interaction part of Hamiltonian has rapid oscillatory form. Indeed the terms like $\hat{b}^\dagger(t)|e(t)\rangle\langle v(t)|$ in the Hamiltonian (9) have the time dependent oscillatory functions like $i(\omega_a - \omega_v + \omega_e)\tau$, where $\omega_a - \omega_v + \omega_e \neq 0$. In order to obtain the model Hamiltonian, which describes the Hyper-Raman processes in which two-photon from dressed field are absorbed and other two new photons are generated, let us eliminate the operators of virtual atomic state $|v\rangle$ from the Hamiltonian (1). This procedure of elimination must continue till the situation in which the terms of interaction Hamiltonian includes the resonance diagrams like this $(\hat{d}(t))^2(\hat{b}^\dagger(t))^2|e(t)\rangle\langle g(t)|$ represented in the Fig.1. According to this, we solve the solution of Schrodinger equation $i\hbar\frac{\partial}{\partial t}|v(t)\rangle = -i\hbar \frac{\partial}{\partial t}|v(t)\rangle$ for representation of virtual state $|v\rangle$ through $|e\rangle$ and $|g\rangle$ states

$$\frac{\partial}{\partial t}|v(t)\rangle = i\omega_v|v(t)\rangle - \frac{i}{\hbar}[\mu_{ev}, g_a]\hat{b}^\dagger(t)e^{-i\omega_a x} + (\mu_{ev}, \mu_d)\hat{d}^\dagger(t)e^{-i\omega_d z}|e(t)\rangle$$

$$- \frac{i}{\hbar}[(\mu_{ev}, g_a)\hat{d}^\dagger(t)e^{-i\omega_a x} + (\mu_{ev}, \mu_d)\hat{d}^\dagger(t)e^{-i\omega_d z}]|g(t)\rangle.$$  

(3)

According to perturbation theory in first Born-Marcoff approximation, we can represent the solution of equation (3) the following form

$$|v(t)\rangle = |v(0)\rangle e^{i\omega_v t} + A^+_v(t, x, z)|e(t)\rangle + A^+_v(t, x, z)|g(t)\rangle.$$  

(4)

Here the coefficients $A^+_v(t, x, z)$ and $A^+_v(t, x, z)$ can be expressed in the Born-Marcoff approximation [9] through the operators of pump and scattering fields

$$A^+_v(t, x, z) = \frac{(\mu_{ev}, g_a)\hat{d}^\dagger(t)e^{-i\omega_a x} + (\mu_{ev}, \mu_d)\hat{b}^\dagger(t)e^{-i\omega_d z}}{\hbar(\omega_{va} - \omega_d)}, \quad \alpha = g, e.$$  

Introducing the vector $|v(t)\rangle$ in the interaction part of Hamiltonian (9), we obtain the new form of interaction Hamiltonian

$$H^{'IJ}_{v} = -[\mu_{ev}, g_a]\hat{b}^\dagger e^{-i\omega_a x} + (\mu_{ev}, \mu_d)\hat{d}^\dagger e^{-i\omega_d z}$$

$$\times [\hat{A}_{v, e}(t, x, z)|e(t)\rangle\langle e(t)| + \hat{A}_{v, g}(t, x, z)|e(t)\rangle\langle g(t)|]$$

$$+ [(\mu_{gv}, g_r)\hat{d}^\dagger e^{-i\omega_d z} + (\mu_{gv}, \mu_a)\hat{b}^\dagger e^{-i\omega_a x}$$

$$\times [\hat{A}_{v, g}(t, x, z)|g(t)\rangle\langle e(t)| + \hat{A}_{v, g}(t, x, z)|g(t)\rangle\langle g(t)|]$$

$$+ H.c.$$  

(5)

As the terms of this Hamiltonian remain rapid oscillatory functions, we must again eliminate the atomic virtual states from the Hamiltonian. In this case we again represent the solutions for excited $|e\rangle$ and ground states $|g\rangle$ through virtual state $|v\rangle$, using the Hamiltonian (2)

$$|e(t)\rangle = \exp[i\omega_v t]|e(0)\rangle - \frac{i}{\hbar} \int_0^t \exp[i\omega_v \tau][(\mu_{ev}, g_a)\hat{b}^\dagger(t - \tau)e^{-i\omega_a x}$$

$$+ (\mu_{ev}, \mu_d)\hat{d}^\dagger(t - \tau)e^{-i\omega_d z}]|v(t - \tau)\rangle;$$

$$|g(t)\rangle = \exp[i\omega_g t]|g(0)\rangle - \frac{i}{\hbar} \int_0^t \exp[i\omega_g \tau][(\mu_{vg}, g_a)\hat{b}(t - \tau)e^{i\omega_a x}$$

$$+ (\mu_{vg}, \mu_d)\hat{d}(t - \tau)e^{i\omega_d z}]|v(t - \tau)\rangle,$$
Now we can easily found the effective Hamiltonian which describes the Hyper-Raman resonances in fourth order of perturbation theory relative the small parameter $(\mu_{vea}, g)/\hbar$. For this we firstly represent the field and atomic operators through smooth and rapid oscillatory parts: $\hat{d}^\dagger(t) = \hat{d}^\dagger \exp(i\omega_d t), \hat{b}^\dagger(t) = \hat{b}^\dagger \exp(i\omega_b t)$ and $|\alpha > < \beta|(t) = |\alpha > < \beta| \exp[i\omega_{\alpha\beta} t]$, where $\alpha, \beta$ are $g$ and $e$ states. Integrating again the solutions for vectors $|e(t)\rangle$ and $|g(t)\rangle$ in Born-Marckoff approximation

$$|e(t)\rangle = \exp[i\omega_e t]|e(0)\rangle + \frac{(\mu_{vea}, g_d)\hat{d}(t)e^{ik_d x}}{\hbar(\omega_{ve} - \omega_d)} + \frac{(\mu_{vea}, g_b)\hat{b}(t)e^{ik_b x}}{\hbar(\omega_{ve} - \omega_b)}|v(t)\rangle,$$

$$|g(t)\rangle = \exp[i\omega_g t]|g(0)\rangle + \frac{(\mu_{veg}, g_d)\hat{d}(t)e^{ik_d x}}{\hbar(\omega_{vg} - \omega_d)} + \frac{(\mu_{veg}, g_b)\hat{b}(t)e^{ik_b x}}{\hbar(\omega_{vg} - \omega_b)}|v(t)\rangle,$$

we obtain the four photon transitions between the excited $|e\rangle$ and ground $|g\rangle$ states after introducing the solutions (4) in the right hand sides of expressions (6). After the substitution of the formal solutions (6) in the right hand side of the Hamiltonian (5) in the fourth order of small interaction parameter $(\mu_{vea}, g)/\hbar$, we obtain the following model Hamiltonian with smooth transitions diagrams between the real ground $|g\rangle$ and excited $|e\rangle$ states

$$\hat{H}_{eff} = \hbar \omega_{eg} \sum_{l=1}^{N} (|e\rangle \langle e|_{l} - |g\rangle \langle g|_{l})/2 + \hbar \omega_b \hat{b}^\dagger \hat{b} + \hbar \omega_d \hat{d}^\dagger \hat{d}$$

$$- \sum_{l=1}^{N} (q^* \hat{b}^\dagger(t) \hat{b}^\dagger(t)) \exp[2ik_a x_l - 2ik_d z_l]|e(t)\rangle \langle g(t)|_{l}$$

$$+ q^* \hat{d}^\dagger(t) \hat{d}^\dagger(t) \exp[-2ik_a x_l + 2ik_d z_l]|g(t)\rangle \langle e(t)|_{l}.$$  (7)

Here the interaction constant $q$ is presented through the small parameter in the following form

$$q = \left( \frac{1}{\omega_{ve} - \omega_a} + \frac{1}{\omega_{ve} - \omega_d} \right) \left[ \frac{(\mu_{vea}, g_d)(\mu_{vea}, g_b)^2}{\hbar^3(\omega_{ve} - \omega_d)(\omega_{vg} - \omega_a)} \right.$$

$$\left. + \frac{(\mu_{vea}, g_d)^2(\mu_{vea}, g_b)(\mu_{vea}, g_r)}{\hbar^3(\omega_{ve} - \omega_a)(\omega_{vg} - \omega_d)} \right].$$  (8)

In the interaction Hamiltonian (7) we have considered an atomic ensemble instead of one atom in the system. In four photon interaction with bimodal cavity field with frequencies $\omega_a$ and $\omega_d$, let us introduce the new bimodal field operators $J^- = \hat{d}^\dagger \hat{b}$, $J^+ = \hat{d} \hat{b}^\dagger$ and $J_z = (\hat{d}^\dagger \hat{d} - \hat{b}^\dagger \hat{b})/2$ and atomic operators $R_j^+ = |e\rangle \langle g|_j$ and $R_j^- = |g\rangle \langle e|_j$ and $R_{jz} = (|e\rangle \langle e|_j - |g\rangle \langle g|_j)/2$, which belong to $su(2)$ algebra. In this representation the model Hamiltonian (7) for the ensemble of atoms takes the following form

$$\hat{H}_I = \sum_{j=1}^{N} \hbar \omega \hat{R}_{jz} - \hbar \omega \hat{J}_z + \sum_{j=1}^{N} q(J^-)^2 e^{2ik_a x_j - 2ik_d z_j} R_j^+ + H.c.$$  (9)

Here the new operators satisfy the commutation relations belonging to $su(2)$ symmetry

$$[\hat{J}^+, \hat{J}^-] = 2\hat{J}_z, \quad [\hat{J}^z, \hat{J}^\pm] = \pm \hat{J}^\pm;$$

$$[\hat{R}_j^+, \hat{R}_j^-] = 2\hat{R}_{jz} \delta_{j,j'}, \quad [\hat{R}_{jz}, \hat{R}_{j'}^\pm] = \pm \hat{R}_{j'}^\pm \delta_{j,j'};$$

$\omega_0 = \omega_c - \omega_g$. In the next section the master equation approach for the cavity field is proposed in which the atomic variables is eliminated.
3. Master Equation for Hyper-Raman Generation Process and it’s solution

Using the method of the elimination of the operators of atomic subsystem from master equation [8] and considering that the polarization and inversion damping rates in the cavity are longer than the generation rate of anti-Stokes photons in scattering process, we can adiabatically eliminate the atomic variables from that equation. Following this method and taking in to account that, the atomic flux enters in the resonator in excited state, we can adiabatically eliminate the atomic operators $\hat{R}_t^+$ and $\hat{R}_t^-$ from Heisenberg equation of mean value of arbitrary field operator $\hat{O}(t)$

$$\frac{d}{dt} \langle \hat{O}(t) \rangle = i\tilde{\omega} \langle \hat{J}_z(t), \hat{O}(t) \rangle - \frac{iq}{\hbar} \sum_{l=1}^{N} \left\{ \langle \hat{R}_t^-(t) [\hat{J}^{+2}(t), \hat{O}(t)] \rangle + \langle [\hat{J}^{-2}(t), \hat{O}(t)] \hat{R}_t^+(t) \rangle \right\},$$

(10)

where $\hat{J}_z(t)$ is the photons inversion operator between the pump and anti-Stokes modes. In expression (10) we considered that the atomic subsystem has the dimension less than the emission wavelength of radiation so that the exponential function $\exp[-i2k_0z_1 + 2ik_qx_1] \approx 1$. In order to obtain an equation for cavity bimodal field, it is necessary to eliminate the free parts of atomic operators $\hat{R}_t^+(t) = \hat{R}_t^+(0) + \hat{R}_t^- (t)$ where

$$\hat{R}_t^+(t) = \hat{R}_t^+(0) \exp[i\omega_0t - \gamma_\perp t]$$

$$\hat{R}_t^-(t) = \frac{2qi}{\hbar} \int_0^t \! d\tau \hat{J}^{+2}(t - \tau) \hat{R}_\perp(t - \tau) \exp[i\omega_0\tau - \gamma_\perp \tau].$$

(11)

It is easy to understand that, when the atomic system is in excited states, the free part of the operators $\hat{R}_t^+(t)$ and $\hat{R}_t^-(t)$ is simply eliminated. When the atoms are in the excited state it is necessary to permute the operators $\hat{R}_t^+$ and $\hat{R}_t^-$ according to the definition of the anti-normal correlation product in the equation (10), so that the free part of these operators gives zero contributions $\langle \hat{R}_t^-(0) | e > = \hat{R}_t^- (0) | e > \exp[-i\omega_0 t] = 0: < e | \hat{R}_t^- (t) = < e | \hat{R}_t^-(0) \exp[i\omega_0 t] = 0$.

If $\hat{O}(t)$ is an arbitrary operator of cavity field subsystem, the mean values of the products are $\langle \hat{O}(t) \hat{R}_t^+(t) \rangle = \langle \hat{O}(t) \hat{R}_t^+(t) \rangle$. Let us represent solution of the Heisenberg operator (11) in the Born-Marcoff approximation, considering that the polarization de-phasing rate $\gamma_\perp$ is large than the Hyper-Raman scattering rate $\kappa_1$, $\kappa_1/\gamma_\perp < 1$

$$\hat{R}_j^+(t) \approx \hat{R}_j^+(0) \exp(i\tilde{\omega} t) + \frac{2qi\hat{R}_z(t)}{i(\tilde{\omega} - \omega_0) + \gamma_\perp \hbar} \hat{J}_z(t)^2, \hat{R}_j^-(t) = [\hat{R}_j^+(t)]^+.$$  

After the first elimination of free parts of these operators, the generalized equation (10) becomes

$$\frac{d}{dt} \langle \hat{O}(t) \rangle = i(\tilde{\omega} - \omega_0) \langle [\hat{J}_z(t), \hat{O}(t)] \rangle + \frac{2qi}{\hbar} \sum_{l=1}^{N} \left\{ \frac{\langle \hat{R}_l^+(t) \hat{J}^{-2}(t) [\hat{J}^{+2}(t), \hat{O}(t)] \rangle}{(i(\tilde{\omega} - \omega_0) - \gamma_\perp \hbar)} + \frac{\langle [\hat{J}^{-2}(t), \hat{O}(t)] \hat{J}^{+2}(t) \hat{R}_l^+(t) \rangle}{(i(\tilde{\omega} - \omega_0) + \gamma_\perp \hbar)} \right\}.$$  

(12)

As it was mentioned in papers [10],[8] the relative small second-order terms should be taken into account for the laser stabilization by stimulated emission. In this case, we must eliminate the atomic inversion $\hat{R}_z(t)$ from the generalized equation (12). Indeed, introducing the quasi-stationary solution of Heisenberg equation for $R_{sl}(t)$ operator

$$\hat{R}_s(t) = \frac{1}{2} \sum_{l=1}^{N} \frac{2q}{\hbar\gamma_\parallel} \left\{ \hat{R}_l^-(t) \hat{J}^{+2}(t) - \hat{J}^{-2}(t) \hat{R}_l^-(t) \right\}.$$  

(13)
in expression (12) and again eliminating the atomic operators $\hat{R}_i^+$ and $\hat{R}_i^-$ from the equation (13) we obtain for the following generalized equation for the bimodal cavity field operator $\hat{O}(t)$

$$\frac{d}{dt}\langle \hat{O}(t) \rangle = i\chi_{\perp}\langle [\hat{J}_z(t), \hat{O}(t)] \rangle + \kappa_1 \left\{ \langle [\hat{J}^{-2}(t)\hat{O}(t), \hat{J}^{+2}(t)] \rangle + H.c. \right\}$$

$$- \kappa_2 \frac{1 - \chi^2}{(1 + \chi^2)^2} \left\{ \langle [\hat{J}^{+2}(t)\hat{J}^{-2}(t)\hat{O}(t), \hat{J}^{+2}(t)] \rangle + H.c. \right\}$$

$$+ \frac{1}{1 + \chi^2} \left\{ \langle \hat{J}^{-2}(t)\hat{J}^{+2}(t)[\hat{O}(t), \hat{J}^{+2}(t)] \rangle + H.c. \right\}$$

$$+ \frac{2\chi}{(1 + \chi^2)^2} \left\{ \langle \hat{J}^{+2}(t)\hat{J}^{-2}(t)[\hat{O}(t), \hat{J}^{+2}(t)] \rangle - H.c. \right\}. \quad (14)$$

in the fourth order of perturbation theory relatively the small parameter $q/\hbar$. In this equation, the new constants are represented through the interaction parameter, $q^2$, detuning, $\omega - \omega_0$, number of atoms, $N$, polarization damping rates $\gamma_{\perp}$. Here $\kappa_1 = Nq^2/[\hbar^4\gamma_{\perp}(\chi^2 + 1)]$ is the explicit expression of emission rate in hyper-Raman scattering process, $\kappa_2 = 2q^4N/[\hbar^4\gamma_{\perp} \gamma_{\parallel}]$ describes the diffusion coefficient in the hyper-Raman generation process of anti-Stokes photons; $\chi = (\omega - \omega_0)\gamma_{\perp}^{-1}$ depends on the de-tuning between the transition frequency $\omega_0$ and $\omega = 2(\omega_d - \omega_s)$. The first term of equation (14) describes the rate of collective transformation of anti-Stokes photon pairs into Stokes photon pairs. The second order terms proportional to the diffusion coefficient in the generation process of anti-Stokes photons $\kappa_2$, correspond to nonlinear attenuation processes of the generation, which, together with first term in equation (14), has the stabilization limit of hyper-Raman scattering process.

Passing from field operator $\hat{O}(t)$ in Heisenberg picture to the density matrix $\hat{W}(t)$ in Schrodinger representation, $Sp\{\hat{W}(t), \hat{O}(0)\} = Sp\{\hat{W}(0), \hat{O}(t)\}$, we obtain the following master equation from equation (14)

$$\frac{d\hat{W}(t)}{dt} = -i\chi_{\perp} [\hat{J}_z, \hat{W}(t)] + \kappa_1 \left\{ [\hat{J}^{+2}, \hat{W}(t)\hat{J}^{-2}] + H.c. \right\}$$

$$- \kappa_2 \left\{ \frac{1 - \chi^2}{(1 + \chi^2)^2} \left\{ [\hat{J}^{+2}, \hat{W}(t)\hat{J}^{-2}\hat{J}^{+2}\hat{J}^{-2}] - \hat{W}(t)\hat{J}^{-2}\hat{J}^{+2}\hat{J}^{-2}\hat{J}^{+2} + H.c. \right\} \right\}$$

$$+ \frac{1}{1 + \chi^2} \left\{ [\hat{J}^{+2}, \hat{J}^{+2}\hat{W}(t)\hat{J}^{-2}\hat{J}^{-2}] + H.c. \right\}$$

$$+ \frac{2\chi}{(1 + \chi^2)^2} \left\{ [\hat{J}^{+2}, \hat{W}(t)\hat{J}^{-2}\hat{J}^{+2}\hat{J}^{-2}] - H.c. \right\}. \quad (15)$$

This equation describes the collective transformation of pairs of Stokes photons in anti-Stokes in Hyper-Raman generation process. Ignoring the losses from the cavity, we observe, that the steady state solution of this equation is applicable for the time interval $\gamma^{-1} < t < k^{-1}$, where $\gamma^{-1} \sim \gamma_{\parallel}^{-1}; \gamma_{\perp}^{-1}$.

Let us firstly consider that, at the initial time moment, we have 0 photons in the anti-Stokes mode and $n_{ph} = 2j$ photons in the pump field, so that the bimodal cavity fields is described by the state $|\Psi(0)\rangle = |0\rangle |n_{ph}\rangle_S$. Let us find a quantum solution for the master equation (15). As it is proposed in papers [11], [12], we can expand the density matrix (15) over the coherent states of generalized of the angular moment states. According to the $Q$- and $P$-diagonal representation [12], when the right side of master equation has a relatively biquadrate form of operators $\hat{J}^+$ and $\hat{J}^-$, the solution of master equation can be represented through diagonal bra-ket operators angular momentum states

$$W(t) = \sum_{m=-j}^{+j} P_m(t) |m, j\rangle \langle m, j| \quad \quad (16)$$
Here, the Hilbert’s vectors $|m,j\rangle$ belongs to the set of angular momentum states and obeys to $su(2)$ symmetry: $J^+|m,j\rangle = \sqrt{(j-m)(j+m+1)}|m+1,j\rangle$ and $J^-|m,j\rangle = \sqrt{(j-m+1)(j+m)}|m-1,j\rangle$. $P_m$ is the occupation probability of the state $|m,j\rangle$. By introducing the representation (16) in equation (15), it is easy to obtain the following system of equations for the probabilities $\{P_m\}$

$$
\frac{dP_m(t)}{dt} = 2\kappa_1\{P_{m-2}(t)(j-m+2)(j-m+1)(j+m)(j+m-1) \\
- P_m(t)(j-m-1)(j-m)(j+m+2)(j+m+1)\} \\
- 2\kappa_2(1-\chi^2) \frac{[P_{m-1}(t)(j-m+2)^2(j-m+1)^2(j+m)^2(j+m-1)^2]}{(1+\chi^2)^2} \\
- P_m(t)(j-m-1)^2(j-m)^2(j+m+2)^2(j+m+1)^2 \\
- 2\kappa_2 \frac{1+\chi^2}{2}(j-m+2)(j-m+1)(j+m-1)(j+m) \\
\times [P_{m-4}(t)(j-m+4)(j-m+3)(j+m-3)(j+m-2) \\
- P_{m-2}(t)(j+m+2)(j+m+1)(j-m-1)(j-m)], -j \leq m \leq j. 
$$

The set of functions $\{P_m(t)\}$ describes the probability of simultaneous existence of $(2j+1)$ Dicke [13] states in the hyper-Raman scattering process. The initial condition for the system of equations (17) is $P_{-j} = 1$, $P_{-j+1} = P_{-j+2} = ... = P_j = 0$.

The numerical behaviors of the system of equations (17) is graphically represented in Fig.2. The quantum correlations between number of photons in the pump and anti-Stokes photons play an important role in the hyper-Raman process. As results from the figures presented above, the transformation of 12 Stokes photons into 12 anti-Stokes photons, take places in a very interesting way. At earlier stage of radiation this correlation can be regarded as the probabilistic repartitions between Stokes and anti-Stokes photons: $(0,12); (2,10); (4,8); (6,6); (8,4); (10,2); (12,0)$, which are described by simultaneous absorption and generation of photon pairs $P_0(t), P_1(t), ... P_6(t)$ in the brackets, the first number $(n,m)$ represents the Stokes photons and the second number-the anti-Stokes photons. The sum of these probabilities is equal to unity: $\sum_{m=-j}^{+j} P_m(t) = 1$.

Let’s now study the correlations between Stokes and anti-Stokes photons in hyper-Raman process, which can be described by the following function

$$
K(t) = \langle \hat{d}^\dagger(t)\hat{d}(t)\hat{b}^\dagger(t)\hat{b}(t)\rangle - \langle \hat{d}^\dagger(t)\hat{d}(t)\rangle \langle \hat{b}^\dagger(t)\hat{b}(t)\rangle 
$$

According to the definition of the new field operators, the number of photons in Stokes and anti-Stokes modes can be represented through the generators of $su(2)$ algebra $J_z$ in the $\hat{d}^\dagger(t)\hat{d}(t) = j + \hat{J}_z(t)$ and $\hat{b}^\dagger(t)\hat{b}(t) = j - \hat{J}_z(t)$. According with this definition the correlation function can be represented through the quantum fluctuation of inversion operator $\hat{J}_z$

$$
K(t) = \langle \hat{J}_z^2(t)\rangle - (\langle \hat{J}_z(t)\rangle)^2. 
$$

We are interested in the behavior of the quantum correlations between bi-photons described by the function (18). In Fig.2B is plotted the time behavior of cross-correlation between the photon numbers absolute of Stokes and anti-Stokes photons $\Delta^2$ and relative square fluctuations $\delta^2$ during the transformation of 12 Stokes photons in anti-Stokes. From this Fig.2A, it follows that in the process of the stabilization of generation process, the square fluctuations remain positive function. The maximal value of quantum fluctuations corresponds to the equal value of photon pairs in Stokes and anti-stokes modes. In this case the transition corresponds to the maximal value of the fluctuations of the normal product of operators $\hat{J}^+$ and $\hat{J}^-$. This fluctuations is accompanied with simultaneous realization of the partition $(m,n)$ with the same magnitude of probabilities $P_m$ represented in Fig.2A.
Figure 2. A. The time dependence of probabilities $P_m(t)$ of transformation of pairs of pump photons in anti-stokes photon pairs for following parameters of the system $j = 6; \kappa_2/\kappa_1 = 0.00005; p = 100$.

B. The time dependence of quantum correlations between the Stokes and anti-Stokes photons in the hyper-Raman scattering. When the number $(n,m)$ of the distribution possibilities between Stokes and anti-Stokes modes archived the increase the fluctuations of cross-correlation function $K(t)$ archives the maximal value.

4. Conclusion
The dynamical transformation of Stokes photons in anti-Stokes and vice versa as the function of the preparation of atomic inversion and field in Hyper-Raman process was the subject of investigation of this article. The proposed $su(2)$ operators for the description of this transformation are similar to Dicke operators in super-radiance [13]. In this case, the role of the number of excited atoms is substituted by the number of Stokes photons, which are scattered into anti-Stokes photons in the process of hyper-Raman interaction. More than this nonlinear interaction is accompanied with annihilation of one pair of photons in Stokes mode.
and generation of a new pair in anti-Stokes field. The collective bi-operators are introduced which describe the process of generation (annihilation) of Stokes (or anti-Stokes) photons, accompanied by absorption of anti-Stokes (Stokes) photon pairs, respectively. The unique features of the field consisting of both Stokes and anti-Stokes photons have been studied by solving the master equation numerically. It has been found the correlation function between Stokes and anti-Stokes modes in the process of cooperative exchanges with pairs of photons in the hyper-Raman interaction between the modes through the atomic subsystem. This function achieved the maximal value when the mean number of photons in Stokes and anti-Stokes modes is approximatively equal.

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