PHYSICS AT LOW $x$ *

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* Review talk presented at the Workshop "QCD'94", 7-13 July 1994, Montpellier, France
The QCD expectations concerning the deep inelastic lepton - hadron scattering at low $x$ and their phenomenological implications for HERA are summarised. Theoretical predictions for the structure function $F_2(x, Q^2)$ based on the leading $\log 1/x$ resummation are presented and compared with the results obtained from the Altarelli-Parisi equations. Theoretical predictions are confronted with the recent data from HERA. The role of studying the final states in deep inelastic scattering for revealing the dynamics at low $x$ is emphasised and some dedicated measurements like deep inelastic plus jet events, transverse energy flow and dijet production in deep inelastic scattering are discussed.

In this talk we shall give an overview of the QCD expectations concerning the small $x$ behaviour of deep inelastic lepton-hadron scattering and will discuss their phenomenological implications for HERA. Besides discussing the deep inelastic structure functions we shall also consider specific measurements like deep inelastic scattering accompanied by the energetic jet, transverse energy flow and dijet production in deep inelastic scattering which should test the QCD predictions at small $x$ more directly and unambiguously.

The relevant framework for calculating the parton distributions in perturbative QCD in the small $x$ limit is the leading $\log(1/x)$ (LL1/$x$) approximation. It corresponds to the sum of those terms which contain the maximal power of $\ln(1/x)$ at each order of the perturbative expansion. The basic quantity in this approximation is the unintegrated gluon distribution $f(x, k^2)$ which is related in the following way to the conventional (scale dependent) gluon distribution $g(x, Q^2)$:

$$ xg(x, Q^2) = \int_0^{Q^2} \frac{dk^2}{k^2}f(x, k^2) \quad (1) $$

In the LL1/$x$ approximation the unintegrated gluon distribution satisfies the following equation [1]-[3]:

$$ -x \frac{\partial f(x, k^2)}{\partial x} = $$
\[
\frac{3 \alpha_s(k^2)}{\pi} k^2 \int_{k_0^2}^{\infty} \frac{dk'^2}{k'^2} \left[ \frac{f(x, k'^2) - f(x, k^2)}{|k'^2 - k^2|} \right] \\
+ \frac{f(x, k^2)}{\sqrt{4k'^4 + k^4}} \\
\equiv K_L \otimes f
\]  

(2)

which is called in the literature the Balitzkij-Fadin-Kuraev-Lipatov (BFKL) equation.

It corresponds to the sum of ladder diagrams but unlike the leading log\(Q^2\) approximation which gives the Altarelli-Parisi evolution equations (2) the transverse momenta of the gluons are not ordered along the chain. The kernel \(K_L\) of the eq.(2) contains both the real gluon emission term as well as the virtual corrections. The former corresponds to the term proportional to \(f(x, k'^2)\) while the latter correspond to terms proportional to \(f(x, k^2)\) in the integrand on the rhs. of the eq.(2). The virtual corrections correspond to the "gluon reggeisation" (1, 4) (or to the "non-Sudakov" form factor (2, 3)). The variable(s) \(k^2(k'^2)\) denote the transverse momenta squared of the gluons along the ladder. The parameter \(k_0^2\) is the infrared cut-off which is necessary if the running coupling constant effects are taken into account. More general treatment of the infrared region in the BFKL equation than simply imposing the lower limit cut-off \(k_0^2\) has been discussed in [2, 3].

When the running coupling effects are neglected (i.e. when one sets \(\alpha_s(k^2) = \bar{\alpha}_s\)) and when \(k_0^2 = 0\) then the equation (2) can be solved analytically and the leading small \(x\) behaviour of its solution is given by the following formula:

\[
f(x, k^2) \sim \frac{x^{-\lambda}}{\ln(1/x)^{1/2}} \exp \left( -\frac{\ln(k^2/\bar{k}^2)}{2\lambda' \ln(1/x)} \right)
\]

(3)

with

\[
\lambda = \frac{12 \ln(2)}{\pi} \bar{\alpha}_s
\]

(4)

\[
\lambda' = \frac{3 \bar{\alpha}_s}{\pi} 28 \zeta(3)
\]

(5)

where the Riemann zeta function \(\zeta(3) = 1.202\). The parameter \(\bar{k}\) is of nonperturbative origin and is fixed by the boundary condition for \(f(x, k^2)\) at \(x = x_0\) [8].

The solution (3) of the BFKL equation follows from the solution of the corresponding equation for the moment \(\bar{f}(n, k^2)\)

\[
\bar{f}(n, k^2) = \int_0^1 dx x^{n-2} f(n, k^2)
\]

(6)

of the distribution \(f(x, k^2)\)

\[
\bar{f}(n, k^2) = \bar{f}^0(n, k^2) + \frac{3\bar{\alpha}_s}{\pi(n-1)k^2} \int_0^{\infty} \frac{dk'^2}{k'^2} \left[ \frac{\bar{f}(n, k'^2) - f(n, k^2)}{|k'^2 - k^2|} \right] \\
+ \frac{f(n, k^2)}{\sqrt{4k'^4 + k^4}}
\]

(7)

where \(\bar{f}^0(n, k^2)\) is the moment of the suitably defined inhomogeneous term. This equation is diagonalised by the Mellin transform and its solution for the Mellin transform \(\bar{f}(n, \gamma)\) of the moment \(\bar{f}(n, k^2)\) is:

\[
\bar{f}(n, \gamma) = \frac{\bar{f}^0(n, \gamma)}{1 - \frac{3\bar{\alpha}_s}{\pi(n-1)} K_L(\gamma)}
\]

(8)

where

\[
K_L(\gamma) = 2\Psi(1) - \Psi(\gamma) - \Psi(1 - \gamma)
\]

(9)

is the Mellin transform of the kernel of the equation (3). The function \(\Psi(z)\) is
the logarithmic derivative of the Euler $\Gamma$ function. The Mellin transform $\tilde{f}(n, \gamma)$ is defined as below:

$$\tilde{f}(n, \gamma) = \int_0^\infty dk^2 (k^2)^{-\gamma-1} f(n, k^2)$$

(10)

The poles of $\tilde{f}(n, \gamma)$ in the $\gamma$ plane define the gluon anomalous dimensions $\bar{\gamma}(n, \bar{\alpha}_s)$. The leading twist anomalous dimension $\bar{\gamma}^{LT}(n, \bar{\alpha}_s)$ controls the $k^2$ dependence of $\tilde{f}(n, k^2)$ at large $k^2$

$$\tilde{f}(n, k^2) \sim (k^2)^{\bar{\gamma}^{LT}(n, \bar{\alpha}_s)}$$

(11)

It follows from the eq. (8) that the anomalous dimensions are the solutions of the following equation:

$$\frac{3\bar{\alpha}_s}{\pi(n-1)} K_L(\bar{\gamma}(n, \bar{\alpha}_s)) = 1$$

(12)

The solution of this equation allows to obtain the gluon anomalous dimensions as a power series of $\frac{3\bar{\alpha}_s}{\pi(n-1)}$ [11, 12]. For the leading twist anomalous dimension this power series corresponds to the leading log1/z expansion of the splitting function $P_{gg}(z, \alpha_s)$ which appears in the evolution equation for the gluon distribution. The leading twist quark anomalous dimension which includes the resummation of the powers of $\frac{3\bar{\alpha}_s}{\pi(n-1)}$ has recently been discussed in the ref. [13].

The exponent $\lambda$ controlling the small $x$ behaviour of $f(x, k^2)$ (cf. equations [3, 4]) is

$$\lambda = \frac{3\bar{\alpha}_s}{\pi} K_L(1/2)$$

(13)

The leading twist anomalous dimension has a branch point singularity at $n = 1 + \lambda$. We also have (cf. eqs. [12, 13]):

$$\bar{\gamma}^{LT}(n = 1 + \lambda, \bar{\alpha}_s) = \frac{1}{2}$$

(14)

The following properties of the solution of the BFKL equation summarised in the formula (8) should be noted:

1. It exhibits the Regge type $x^{-\lambda}$ increase with decreasing $x$ where the exponent $\lambda$ given by eq. (4) can have potentially large magnitude $\simeq 1/2$ or so. The quantity $1 + \lambda$ is equal to the intercept of the so called BFKL Pomeron which corresponds to the hard QCD interactions.

   Its potentially large magnitude ($\simeq 1.5$) should be contrasted with the intercept $\alpha_{soft} \approx 1.08$ of the (effective) ”soft” Pomeron which has been determined from the phenomenological analysis of the high energy behaviour of hadronic and photo-production total cross-sections [14].

2. It exhibits the $(k^2)^{1/2}$ growth with increasing $k^2$ modulated by the Gaussian distribution in $\ln(k^2)$ with its width increasing as $ln^{1/2}(1/x)$ with decreasing $x$. The Gaussian factor reflects the diffusion pattern of the BFKL equation [1, 8, 10]. The increase of the function $f(x, k^2)$ as $(k^2)^{1/2}$ is due to the fact that the leading twist anomalous dimension is equal to $1/2$ at the BFKL singularity (cf. eq. [14]). This shift of the anomalous dimension is the result of the (infinite) LL$(1/x)$ resummation and should be contrasted with the anomalous dimension calculated perturbatively retaining only finite number of terms [13].

If the running coupling effects are taken into account then the leading small $x$ behaviour is:

$$f(x, k^2) \sim x^{-\bar{\lambda}}$$

(15)

The exponent $\bar{\lambda}$ has to be now calculated numerically and is dependent upon the infrared cut-off $k^2_0$ [7, 9, 15, 16]. More recent
discussion of the BFKL equation is given in [17, 18].

The validity of the BFKL equation is, in principle, restricted to the region where
\[ \alpha_s \ln(1/x) \sim O(1) \] and
\[ \alpha_s \ln(Q^2/Q_0^2) \ll 1 \]
where \( Q_0^2 \) is some moderately large scale \( (Q_0^2 \gg \Lambda^2) \). Possible extension of the BFKL equation beyond this region is provided by the Marchesini equation [3] which treats both (large) logarithms \( \ln(1/x) \) and \( \ln(Q^2/Q_0^2) \) on equal footing. In the region of large values of \( x \) this equation becomes equivalent to the Altarelli-Parisi evolution equations.

For \( x g(x, Q^2) \) growing as \( x^{-\lambda} \) the transverse area \( S(x, Q^2) \) occupied by the gluons
\[ \tilde{S}(x, Q^2) = \text{const } x g(x, Q^2) \frac{\alpha_s(Q^2)}{Q^2} \] (16)
can, for sufficiently small value of \( x \) and for fixed \( Q^2 \) become comparable to the transverse area of a hadron \( S_H = \pi R_H^2 \) where \( R_H \) is the hadronic radius. When this happens (and in fact before this happens), the gluons can no longer be treated as free partons and their interaction leads to screening (or shadowing) effects [4, 5, 19]. The main effect of shadowing is to tame the indefinite increase of parton distributions.

One finds instead that at sufficiently small values of \( x \) and/or \( Q^2 \) the gluon distributions approach the so-called saturation limit \( x g_{sat}(x, Q^2) \) [4, 5]
\[ x g_{sat}(x, Q^2) = \frac{\text{const}}{\alpha_s(Q^2)} R_H^2 Q^2 \] (17)

In some models [20] the saturation limit contains some remnant weak \( x \) dependence. The most dramatic effect is the linear scaling violation exhibited by \( x g_{sat}(x, Q^2) \).

If one assumes that the gluons are not distributed uniformly within a hadron but are concentrated around the ”hot-spots” [21, 20] having their radius \( R_{h.s.} \) much smaller than the hadronic radius \( R_H \) then the shadowing effects are expected to be stronger. The saturation limit \( x g_{sat}(x, Q^2) \) is then controlled by the radius \( R_{h.s.} \) and not by \( R_H \).

The shadowing effects modify the BFKL equation by the non-linear terms [4, 5, 13, 23]:
\[ -x \frac{\partial f(x, k^2)}{\partial x} = K_L \otimes f - \frac{81\alpha_s^2(k^2)}{16R^2k^2} [x g(x, k^2)]^2 \] (18)
The equation (18) is called in the literature the Gribov, Levin, Ryskin (GLR) equation.

The second term in the right hand side of the eq. (18) which is quadratic in the gluon distribution describes the shadowing effects. They correspond to the QCD diagrams where two gluonic ladders merge into one. The parameter \( R \) describes the size of the region within which the gluons are concentrated.

The GLR equation has been derived assuming that the gluons within a hadron are uncorrelated and that the interaction between the gluonic ladders can be neglected. These assumptions have recently been shown to be unjustified [24]. The interaction between gluonic ladders can be approximately accounted for by introducing the appropriate enhancement factor in the nonlinear term in the eq. (18) [25].

It turns out that in the region of \( x \) and \( Q^2 \) which may be relevant for HERA the
shadowing effects are still relatively weak at least for \( R \sim 5\text{GeV}^{-1} \). The possibility of detecting the nonlinear shadowing terms in HERA has recently been discussed in detail in the ref. [26].

In Fig.1 we show the regions in the \((\ln 1/x, \ln(Q^2/\Lambda^2))\) plane where various dynamical effects and approximations discussed above are expected to play the dominant role.

We shall now discuss quantitative predictions for the structure function \( F_2(x, Q^2) \) at small \( x \) which will be directly based on the solution of the BFKL equation [3, 27, 28]. (For the first attempt to determine the structure function \( F_2(x, Q^2) \) from the solution of the BFKL equation see the ref. [29]). The relevant diagrams are shown in Fig.2 and their contribution to the structure function \( F_2(x, Q^2) \) can be written in the following factorisable form:

\[
F_2(x, Q^2) = \int_x^1 \frac{dx'}{x'} \int \frac{dk^2_T}{k^2_T} f(x/x', k^2_T) F_2^0(x', k^2_T, Q^2)
\]

(19)

where \( x/x' \) is the longitudinal momentum fraction carried by the gluon which couples to the \( q\bar{q} \) pair. Similar representation can be written for the longitudinal structure function \( F_L(x, Q^2) \). The function \( F_2^0 \) which denotes the quark box contribution to the photon-gluon subprocess shown in Fig.1 is given in ref. [27]. The function \( f(x/x', k^2_T) \) is the unintegrated gluon distribution corresponding to the sum of ladder diagrams in the lower part of the diagrams shown in Fig.2. In the leading \( \log(x'/x) \) approximation this distribution is given as the solution of the BFKL equation [2]. The \((x/x')^{-\lambda} \) behaviour of the unintegrated gluon distribution \( f \) (see the equations [3, 15]) generates the singular \( x^{-\lambda} \) behaviour of the structure function \( F_2(x, Q^2) \). We may also study the effects of shadowing in the structure functions by using the function \( f \) which is the solution of the GLR equation [18].

The \( Q^2 \) dependence of the structure function \( F_2(x, Q^2) \) reflects the \( k^2_T \) dependence of the function \( f(x, k^2_T) \) [28]. Thus, for instance, the increase of the unintegrated gluon distribution as \( (k^2_T)^{1/2} \) with increasing \( k^2_T \) (see the eq. (3)) implies increase of the structure function \( F_2(x, Q^2) \) as \((Q^2)^{1/2}\) with increasing \( Q^2 \).

The formula (19) is an example of the \( k_T \) factorisation theorem which is the basic tool for calculating the observable quantities (i.e. in this case the structure functions) at small \( x \) [30]. Its connection with the more conventional collinear factorisation has recently been discussed in detail in the ref. [31].

In Fig.3 we summarise the predictions for the structure function \( F_2(x, Q^2) \) based on the formula (19) [3, 27] (the curves marked as AKMS) and confront them with the recent data from HERA [36, 37, 38]. We also compare these predictions with structure functions which were obtained within the NLO Altarelli-Parisi formalism. Let us recall that in the latter case the small \( x \) behaviour depends upon the phenomenological input distributions at the reference scale and is therefore not constrained by a theory. One can mimick the BFKL behaviour at small \( x \) by assuming the \( x^{-\lambda} \) type extrapolation of the input parton distributions as it has been done in the refs. [32, 33, 34] (the curves MRS(A), [34]). In the next-to-leading approximation one neglects, of course, the effects
of the leading log\(^1/x\) resummation in the splitting (and coefficient) functions which are automatically taken care of in the calculation based on the \(k_T\) factorisation with the function \(f\) obtained from the solution of the BFKL equation. Systematic study of those effects within the Altarelli-Parisi formalism has however shown that their role is relatively small provided the starting distributions are sufficiently singular at small \(x\) \([12]\). This result explains possible origin of similarity between the AKMS and MRS curves.

In the plots in Fig.3 which correspond to \(Q^2 = 15\) \(\text{GeV}^2\) and to \(Q^2 = 30\) \(\text{GeV}^2\) we also show predictions obtained from the "dynamical model" of parton distributions \([35]\). In this model the parton distributions are generated radiatively from the valence-like input at the very low reference scale \(Q_0^2 \simeq 0.25\text{GeV}^2\). The strong increase of \(F_2(x, Q^2)\) with decreasing \(x\) at the HERA range (i.e. for \(Q^2 \sim 10\text{GeV}^2\)) comes now from the relatively large evolution length \(\xi(Q^2, Q_0^2)\)

\[
\xi(Q^2, Q_0^2) = \int_{Q_0^2}^{Q^2} \frac{dq^2 3\alpha_s(q^2)}{q^2} \pi^{10}
\]

since in this case \(F_2(x, Q^2)\) behaves at small \(x\) as below:

\[
F_2(x, Q^2) \sim \exp \left(2\sqrt{\xi(Q^2, Q_0^2)\ln(1/x)}\right)
\]

The structure function \(F_2(x, Q^2)\) which is generated radiatively in the leading log\(^2\) approximation from the non-singular input distributions at small \(x\), exhibits scaling in the two variables \(\sqrt{\ln(1/x)\ln\ln Q^2}\) and \(\sqrt{\ln(1/x)}/\ln\ln Q^2\). Exploration of this scaling property has recently been advocated as a possible discriminator between the BFKL-motivated or purely radiatively generated structure functions and evidence has been found for supporting the latter \([19]\). Above scaling property of structure functions may however be modified if the leading log\((1/x)\) terms are included in the splitting and coefficient functions. These terms play significant role in the case of the non-singular starting parton distributions \([12]\).

We notice from Fig.3 that in the region of \(x\) and \(Q^2\) relevant for HERA measurements the structure function \(F_2(x, Q^2)\) which was based on the solution of the BFKL equation is similar to that which was obtained within the Altarelli-Parisi evolution equations formalism in the NLO approximation. The inclusive quantity like \(F_2\) is not therefore the best discriminator for revealing the dynamical details at low \(x\).

One may however hope that this can be provided by studying the structure of the final states at small \(x\). It is expected in particular that absence of transverse momentum ordering which is the characteristic feature of the BFKL dynamics should reflect itself in the less inclusive quantities than the structure function \(F_2(x, Q^2)\).

The dedicated measurements of the low \(x\) physics i.e. the deep inelastic plus jet events, transverse energy flow in deep inelastic scattering, production of jets separated by the large rapidity gap and dijet production in deep inelastic scattering are summarised in Fig.4, \([40]\). The deep inelastic lepton-hadron scattering containing a measured jet (Fig. 3a) can provide a very clear test of the BFKL dynamics at small \(x\) \([41, 42]\). The idea is to study deep inelastic \((x, Q^2)\) events which contain an identified jet \((x_j, k_T^2)\)
where $x \ll x_j$ and $Q^2 \simeq k^2_{Tj}$. Since we choose events with $Q^2 \simeq k^2_{Tj}$ the QCD evolution (from $k^2_{Tj}$ to $Q^2$) is neutralised and attention is focussed on the small $x$, or rather small $x/x_j$ behaviour. The small $x/x_j$ behaviour of the jet production is generated by the gluon radiation as shown in the diagram of Fig.4a. Choosing the configuration $Q^2 \simeq k^2_{Tj}$ we eliminate by definition the gluon emission which corresponds to strongly ordered transverse momenta i.e. that emission which is responsible for the QCD evolution. The measurement of jet production in this configuration may therefore test more directly the $(x/x_j)^{-\lambda}$ behaviour which is generated by BFKL equation where the transverse momenta are not ordered.

Let us now discuss the feasibility of using the deep inelastic events which contain a measured jet to identify the singular $z^{-\lambda}$ type of behaviour at HERA [16,41,42].

One practical limitation is that jets can only be measured if they are emitted at sufficiently large angles ($\theta_j > 5^0$) to the proton beam direction in the HERA laboratory frame. Large $x_j$ jets are only emitted at small $\theta_j$; for a given $\theta_j$ we can reach larger $x_j$ by observing jets with larger $k^2_{Tj}$ but with a depleted event rate. In order to identify the BFKL $z^{-\lambda}$ behaviour we need deep inelastic + jet events, with $k^2_{Tj} \simeq Q^2$, over an interval of $z \equiv x/x_j$ which covers values of $z$ as small as is experimentally possible. As a compromise we select the region $x_j > 0.05$ and $x < 2 \times 10^{-3}$.

Fig.5 shows the predicted $x$ dependence of the deep inelastic+jet cross-section relevant for HERA [16,42]. (The cross-section shown in Fig.5 has been calculated assuming the following constraints: $x_j > 0.05$ and $\theta_j > 5^0$, $1/2Q^2 < k^2_{Tj} < 2Q^2$). The continuous curves give the values of the cross-section when the BFKL effects are included. These are to be contrasted with the dashed curves which show the values when the BFKL effect is neglected, that is when just the quark box approximation is used to evaluate the corresponding differential structure functions. The steep rise of the continuous curves with decreasing $x$ (i.e. decreasing $z \equiv x/x_j$) reflects the $z^{-\lambda}$ BFKL effect generated by the gluon radiation.

The recent H1 results concerning the deep inelastic plus jet events which were reported at this Workshop [33] are consistent with the increase of the cross-section with decreasing $x$ as predicted by the BFKL dynamics.

Conceptually similar process is that of the two-jet production separated by a large rapidity gap $\Delta y$ in hadronic collisions [43,44] or in photoproduction [45] as illustrated in the Fig.4c. Besides the characteristic $\exp(\lambda \Delta y)$ dependence of the two-jet cross-section one expects significant weakening of the azimuthal back-to-back correlations of the two jets. This is the direct consequence of the absence of transverse momentum ordering along the gluonic chain in the diagram of Fig.4c. The experimental data on the dijet production have recently been obtained at the Tevatron and were reported in this Workshop [16].

Another measurement which should be sensitive to the QCD dynamics at small $x$ is that of the transverse energy flow in deep inelastic lepton scattering [47,48]. The transverse energy flow in the central region away from the current jet and the proton remnants can be calculated from
the following formula [48]:

\[
\frac{\partial E_T}{\partial y} = \frac{1}{F_2} \int \frac{dk_j^2}{k_j} |k_j| \int \frac{d^2 k_p}{\pi k_p^2} \int \frac{d^2 k_\gamma}{k_\gamma^2} \left( \frac{3\alpha_s}{\pi} \frac{k_j^2}{k_\gamma^2} \right) F_2(x/x_j, k_j^2, Q^2) \delta^{(2)}(k_j - k_\gamma - k_p)
\] (22)

where the transverse momenta are defined in Fig.6a. The function \( F_2 \) describes the gluon radiation in the upper part of the diagram in Fig.6a. It satisfies the BFKL equation as does the unintegrated gluon distribution \( f \). The variable \( x_j \) is related to the rapidity \( y(cm) \) in the virtual photon-proton center-of-mass frame

\[
y(cm) = \frac{1}{2} \ln \left( \frac{x_j Q^2}{x k_j^2} \right)
\] (23)

The rapidity \( y \) in the HERA frame is approximately related to \( y(cm) \) by a simple boost

\[
y = y(cm) + \frac{1}{2} \ln \left( \frac{4x E_T^2}{Q^2} \right)
\] (24)

since at small \( x \) the two frames are approximately collinear. The formula (22) is only valid in a region where both \( x_j \) and \( x/x_j \) are sufficiently small for the BFKL equation for \( f(x_j, k_p^2) \) and for \( F_2(x/x_j, k_j^2, Q^2) \) to be valid (we assume \( x_j < 10^{-2} \) and \( x/x_j < 10^{-1} \)). In particular, for large \( x_j \), \( (x_j > 10^{-2}) \) the more appropriate way to calculate the energy flow is by the use of the following formula:

\[
\frac{\partial E_T}{\partial y} = \frac{1}{F_2} \int \frac{dk_j^2}{k_j} |k_j| \left( \frac{3\alpha_s}{\pi} \right) F_2(x/x_j, k_j^2, Q^2) \sum_a f_a(x_j, k_j^2)
\] (25)

which follows from the strong ordering, \( k_j^2 >> k_p^2 \), at the gluon emission vertex (cf. Fig.6b) where the "effective" parton combination \( \sum_a f_a \) can be set equal to \( g + 1/3(q + \bar{q}) \).

The integral in the equation (22) is weighted towards large values of the transverse momenta and so it is sensitive to the behaviour of the functions \( f \) and \( F \) for large \( k_p^2 \) and \( k_\gamma^2 \) respectively. In the central rapidity region away from the current jet and the proton remnants the BFKL dynamics does predict substantial amount of transverse energy which grows with decreasing \( x \).

For fixed QCD coupling \( \bar{\alpha}_s \) one gets the \( x^{-\epsilon} \) increase of transverse energy with decreasing \( x \) with \( \epsilon = (3\alpha_s/\pi)2(1/2) \) [48]. This increase is closely related to the diffusion pattern of the solutions of the BFKL equation for the functions \( f \) and \( F \) (cf. eq. (3)). It should be noted that without the Gaussian factors which broaden with decreasing \( x_j \) or \( x/x_j \) the integral defining the energy flow would be divergent.

Numerical estimate of the energy flow based on the BFKL equation with running coupling effects incorporated confirms the increase of the average \( E_T \) with decreasing \( x \) although this increase is weaker than for the fixed coupling case. The average value of \( E_T \) per unit rapidity which is generated by the BFKL dynamics has been estimated to be around 2 GeV. It is interesting to notice that the H1 collaboration has seen the excess of transverse energy in the forward region [34, 50] in com-
parison to the expectations based on the standard Monte Carlo models incorporating conventional QCD cascades. In Fig.7 we confront the theoretical predictions for the transverse energy flow based on the BFKL dynamics [49] with the experimental data. The histogram in Fig.7 is the LEPTO Monte Carlo estimate of the effects of radiation from the current jet and of hadronization.

Finally let us consider the production of dijets close to the photon fragmentation region and with small rapidity gap between the jets (Fig.4d). Absence of transverse momentum ordering within the BFKL gluon ladder modifies the theoretical expectations concerning production of such dijets in deep inelastic scattering when the dominant subprocess is that of the virtual photon-gluon fusion. In particular one gets significant broadening of the azimuthal distributions of the two jets which increases with decreasing $x$ as shown in Fig.8 [51]. The distributions plotted in Fig.8 are normalized to a common maximum. The weakening of the back-to-back correlations between the jets in photoproduction has recently been discussed in the ref. [52].

Other processes which are sensitive to the small $x$ physics are the deep inelastic diffractive and heavy quark production.

To sum up we have recalled in this talk the basic QCD expectations concerning the small $x$ physics and discussed their possible implications for deep inelastic lepton scattering. We have limited ourselves to large $Q^2$ region where perturbative QCD is expected to be applicable. Specific problems of the low $Q^2$, low $x$ region are discussed in the ref. [32].

We have shown that the gluon and sea quark distributions and so the deep inelastic structure functions should grow rapidly with decreasing $x$. This rapid $x^{-\lambda}$ type of increase with $\lambda \approx 1/2$ is the reflection of the BFKL Pomeron which originates from the gluon radiation.

This indefinite growth of parton distributions cannot go on forever and has to be eventually stopped by parton screening which leads to the parton saturation. Most probably however the saturation limit is still irrelevant for the small $x$ region which is now being probed at HERA.

The observed increase of $F_2(x, Q^2)$ with decreasing $x$ observed at HERA is in agreement with the predictions based on the BFKL equation. The explanations of this effect within the Altarelli-Parisi evolution equations are also possible.

Finally we have emphasised the role of studying the final states in deep inelastic scattering at small $x$ and have discussed jet production, transverse energy flow and dijet production. The small $x$ dynamics should also show up at other semihard processes like heavy quark production or deep inelastic diffraction.

Acknowledgments.
I wish to thank Stephan Narison for excellent organisation of the Workshop. I thank Adrian Askew, Barbara Badelek, Krzysztof Golec-Biernat, Dirk Graudenz, Alan Martin and Peter Sutton for numerous discussions and for very enjoyable research collaboration on problems presented in this talk. This research has been supported in part by the Polish Committee for Scientific Research Grant N0. 2 P302 062 04.
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Fig. 1 The regions in the \((\ln(1/x), \ln(Q^2/\Lambda^2))\) plane where various dynamical effects and approximations play dominant role.

Fig. 2 Diagrammatic representation of the gluon ladder contribution to deep inelastic scattering and of the \(k_T\) factorisation formula [13].
Fig. 3 The measurements of the $F_2$ at HERA shown together with the BFKL description (continuous curves) and the MRS(A) parton analysis fit (dashed curves). The curves marked as GRV at the plots for $Q^2 = 15 \text{ GeV}^2$ and for $Q^2 = 30 \text{ GeV}^2$ correspond to the predictions of the dynamical model of ref. [35].
Fig. 4 Diagrammatic representation of the processes testing the BFKL dynamics. (a) Deep inelastic scattering with the forward jet. (b) $E_T$ flow in deep inelastic scattering. (c) Production of jets separated by the large rapidity gap $\Delta y$. (d) Dijet production in deep inelastic scattering.

Fig. 5 The cross section $<\sigma>$, in pb, for deep inelastic plus jet events shown as the function of $x$ for three different bins of $Q^2$ [10, 12]. The continuous curves show $<\sigma>$ calculated with the inclusion of the BFKL soft gluon summation. The corresponding $<\sigma>$ values which were calculated neglecting the BFKL effects are shown as the dashed curves.
Fig. 6 (a) Diagrammatic representation of the formula (22) showing the gluon ladders which are resummed by the BFKL equation for $f$ and $F_2$. (b) The representation of formula (25), which is obtained by the simplification of (22) when $x_j$ is large.

Fig. 7 The data show the $E_T$ flow accompanying deep-inelastic events with $x < 10^{-3}$ observed by the H1 collaboration in the central region [37, 50]. The continuous curves show the BFKL predictions [13].

Fig. 8 The azimuthal distributions of the jets for deep inelastic dijet events [51].
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