PERIODIC HOMOGENIZATION OF SCHRÖDINGER TYPE EQUATIONS WITH RAPIDLY OSCILLATING POTENTIAL

LAZARUS SIGNING

Abstract. This paper is devoted to the homogenization of Schrödinger type equations with periodically oscillating coefficients of the diffusion term, and a rapidly oscillating periodic time-dependent potential. One convergence theorem is proved and we derive the macroscopic homogenized model. Our approach is the well known two-scale convergence method.

1. Introduction

Let us consider a smooth bounded open subset Ω of \( \mathbb{R}^N \) (the \( N \)-numerical space \( \mathbb{R}^N \) of variables \( x = (x_1, \ldots, x_N) \)), where \( N \) is a given positive integer, and let \( T \) and \( \varepsilon \) be real numbers with \( T > 0 \) and \( 0 < \varepsilon < 1 \). We consider the partial differential operator

\[
A^\varepsilon = - \sum_{i,j=1}^N \frac{\partial}{\partial x_i} \left( a^\varepsilon_{ij} \frac{\partial}{\partial x_j} \right)
\]

in \( \Omega \), where \( a^\varepsilon_{ij}(x) = a_{ij}(\frac{x}{\varepsilon}) \) \( (x \in \Omega) \), \( a_{ij} \in L^\infty(\mathbb{R}^N_y; \mathbb{R}) \) \( (1 \leq i,j \leq N) \) with

\[
a_{ij} = a_{ji},
\]

and the assumption that there exists a constant \( \alpha > 0 \) such that

\[
\text{Re} \sum_{i,j=1}^N a_{ij}(y,\tau) \overline{\zeta_j} \zeta_i \geq \alpha |\zeta|^2 \quad \text{for all } \zeta = (\zeta_j) \in \mathbb{C}^N
\]

for almost all \( y \in \mathbb{R}^N \), where \( \mathbb{R}^N_y \) is the \( N \)-numerical space \( \mathbb{R}^N \) of variables \( y = (y_1, \ldots, y_N) \), and where \(|·|\) denotes the Euclidean norm in \( \mathbb{C}^N \). Let us consider for fixed \( 0 < \varepsilon < 1 \), the following initial boundary value problem:

\[
\begin{align*}
&\frac{1}{\varepsilon} \frac{\partial u^\varepsilon}{\partial t} + A^\varepsilon u^\varepsilon + \frac{1}{\varepsilon} V^\varepsilon u^\varepsilon = f \quad \text{in } \Omega \times ]0,T[ \\
&u^\varepsilon = 0 \quad \text{on } \partial \Omega \times ]0,T[ \\
&u^\varepsilon(0) = u^0 \quad \text{in } \Omega,
\end{align*}
\]

where \( V^\varepsilon(x,t) = V\left(\frac{x}{\varepsilon},\frac{t}{\varepsilon}\right) \) is a real potential with \( V \in L^\infty(\mathbb{R}^N_y \times \mathbb{R}_\tau; \mathbb{R}) \), \( \mathbb{R}_\tau \) being the numerical space \( \mathbb{R} \) of variables \( \tau \), and where \( f \in L^2(0,T; L^2(\Omega)) \), \( u^0 \in L^2(\Omega) \). In view of (1.1)-(1.2), we will show later that the initial boundary value problem (1.3)-(1.5) admits a unique solution in \( C \left( [0,T]; H^1_0(\Omega) \right) \cap C^1 \left( [0,T]; L^2(\Omega) \right) \), provided some regularity assumptions on \( f \) and \( u^0 \), and some hypothesis on \( V \). The

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aim here is to investigate the limiting behaviour of $u_\varepsilon$ solution of \[1.3\]-\[1.5\] when $\varepsilon$ goes to zero, under the periodicity hypotheses on the coefficients $a_{ij}$ and the potential $\mathcal{V}$, and the assumption that the mean value of $\mathcal{V}$ is null.

The asymptotic analysis of boundary value problems with rapidly oscillating potential has been studied for the first time in the book of Bensoussan, Lions and Papanicolaou [2] using the asymptotic expansions. Recently, Allaire and Piatnitski in [11] have investigated the homogenization of a Schrödinger equation with a large periodic potential scaled as $\varepsilon^{-2}$, using the two-scale convergence method combined with the bloch waves decomposition.

This paper deals with the homogenization of an evolution problem with time-dependent coefficients and large rapidly oscillating potential via the two-scale convergence. Clearly, in our study we present another point of view concerning the asymptotic analysis of the Schrödinger model, when the potential is scaled as $\varepsilon^{-1}$. In this paper, the derived macroscopic homogenized model is given by \[3.31\]-\[3.39\], while the equations at the microscopic scale are given by \[3.27\]-\[3.28\] and the global equation (including the macroscopic and the microscopic scales) by \[3.15\].

This study is motivated by the fact that the asymptotic analysis of \[1.3\]-\[1.5\] is connected with the modelling of the wave function for a particle submitted to a potential. Let us note that the classical Schrödinger equation corresponds to the choice $A^\varepsilon = -\Delta$.

Unless otherwise specified, vector spaces throughout are considered over the complex field, $\mathbb{C}$, and scalar functions are assumed to take complex values. Let us recall some basic notation. If $X$ and $F$ denote a locally compact space and a Banach space, respectively, then we write $C(X; F)$ for continuous mappings of $X$ into $F$, and $B(X; F)$ for those mappings in $C(X; F)$ that are bounded. We shall assume $B(X; F)$ to be equipped with the supremum norm $\|u\| = \sup_{x \in X} \|u(x)\|$ (||-|| denotes the norm in $F$). For shortness we will write $C(X) = C(X; \mathbb{C})$ and $B(X) = B(X; \mathbb{C})$. Likewise in the case when $F = \mathbb{C}$, the usual spaces $L^p(X; F)$ and $L^p_{\text{loc}}(X; F)$ ($X$ provided with a positive Radon measure) will be denoted by $L^p(X)$ and $L^p_{\text{loc}}(X)$, respectively. Finally, the numerical space $\mathbb{R}^N$ and its open sets are each provided with Lebesgue measure denoted by $dx = dx_1 ... dx_N$.

The rest of the paper is organized as follows. Section 2 is devoted to some preliminary results on the two-scale convergence, whereas in Section 3 one convergence theorem is established for \[1.3\]-\[1.5\].

2. Preliminaries

We set $Y = (-\frac{1}{2}, \frac{1}{2})^N$, $Y$ considered as a subset of $\mathbb{R}^N_y$ (the space $\mathbb{R}^N_y$ of variables $y = (y_1, ..., y_N)$). We set also $Z = (-\frac{1}{2}, \frac{1}{2})^\tau$, $Z$ considered as a subset of $\mathbb{R}_\tau$ (the space $\mathbb{R}$ of variables $\tau$).

Let us first recall that a function $u \in L^1_{\text{loc}}(\mathbb{R}_y^N \times \mathbb{R}_\tau)$ is said to be $Y \times Z$-periodic if for each $(k, l) \in \mathbb{Z}^N \times \mathbb{Z}$ ($\mathbb{Z}$ denotes the integers), we have $u(y + k, \tau + l) = u(y, \tau)$ almost everywhere (a.e.) in $(y, \tau) \in \mathbb{R}^N \times \mathbb{R}$. If in addition $u$ is continuous, then the preceding equality holds for every $(y, \tau) \in \mathbb{R}^N \times \mathbb{R}$, of course. The space of all $Y \times Z$-periodic continuous functions on $\mathbb{R}^N_y \times \mathbb{R}_\tau$ is denoted by $C_{\text{per}}(Y \times Z)$; that of all $Y \times Z$-periodic functions in $L^p_{\text{loc}}(\mathbb{R}_y^N \times \mathbb{R}_\tau)$ (1 ≤ $p \leq \infty$) is denoted by $L^p_{\text{per}}(Y \times Z)$. $C_{\text{per}}(Y \times Z)$ is a Banach space under the supremum
norm on $\mathbb{R}^N \times \mathbb{R}$, whereas $L^p_{\text{per}}(Y \times Z)$ is a Banach space under the norm
\[
\|u\|_{L^p(Y \times Z)} = \left( \int_Y \int_Z |u(y,\tau)|^p \, dy \, d\tau \right)^{\frac{1}{p}} \quad (u \in L^p_{\text{per}}(Y \times Z)).
\]

The space $H^1_{\#}(Y)$ of $Y$-periodic functions $u \in H^1_{\text{loc}}(\mathbb{R}^N) = W^{1,2}_{\text{loc}}(\mathbb{R}^N)$ such that $\int_Y u(y) \, dy = 0$ will be of our interest in this study. Provided with the gradient norm,
\[
\|u\|_{H^1_{\#}(Y)} = \left( \int_Y |\nabla_y u|^2 \, dy \right)^{\frac{1}{2}} \quad (u \in H^1_{\#}(Y)),
\]
where $\nabla_y u = \left( \frac{\partial u}{\partial y_1}, \ldots, \frac{\partial u}{\partial y_N} \right)$, $H^1_{\#}(Y)$ is a Hilbert space. We will also need the space $L^2_{\text{per}}(Z; H^1_{\#}(Y))$ with the norm
\[
\|u\|_{L^2_{\text{per}}(Z; H^1_{\#}(Y))} = \left( \int_Z \int_Y |\nabla_y u(y,\tau)|^2 \, dy \, d\tau \right)^{\frac{1}{2}} \quad (u \in L^2_{\text{per}}(Z; H^1_{\#}(Y)))
\]
which is a Hilbert space.

Before we can recall the concept of two-scale convergence, let us introduce one further notation. The letter $E$ throughout will denote a family of real numbers $0 < \varepsilon < 1$ admitting 0 as an accumulation point. For example, $E$ may be the whole interval $(0,1)$; $E$ may also be an ordinary sequence $(\varepsilon_n)_{n \in \mathbb{N}}$ with $0 < \varepsilon_n < 1$ and $\varepsilon_n \to 0$ as $n \to \infty$. In the latter case $E$ will be referred to as a fundamental sequence.

Let $\Omega$ be a bounded open set in $\mathbb{R}^N$ and $Q = \Omega \times [0,T]$ with $T \in \mathbb{R}_+$, and let $1 \leq p < \infty$.

**Definition 2.1.** A sequence $(u_\varepsilon)_{\varepsilon \in E} \subset L^p(Q)$ is said to:

(i) weakly two-scale converge in $L^p(Q)$ to some $u_0 \in L^p(Q; L^p_{\text{per}}(Y \times Z))$ if as $E \ni \varepsilon \to 0$,

\[
\begin{align*}
\int_Q u_\varepsilon(x,t) \psi^\varepsilon(x,t) \, dx \, dt & \to \int \int_{Q \times Y \times Z} u_0(x,t,y,\tau) \psi(x,t,y,\tau) \, dx \, dt \, dy \, d\tau \\
\text{for all } \psi \in L^p(Q; C_{\text{per}}(Y \times Z)) \text{ if } \frac{1}{p} = 1 - \frac{1}{p} = \frac{1}{p} = 1 - \frac{1}{p} \quad (x,t) \in Q ;
\end{align*}
\]

(ii) strongly two-scale converge in $L^p(Q)$ to some $u_0 \in L^p(Q; L^p_{\text{per}}(Y \times Z))$ if the following property is verified:

\[
\left\{ \begin{array}{l}
\text{Given } \eta > 0 \text{ and } v \in L^p(Q; C_{\text{per}}(Y \times Z)) \text{ with} \\
\|u_0 - v\|_{L^p(Q \times Y \times Z)} \leq \frac{\eta}{2}, \text{ there is some } \alpha > 0 \text{ such that} \\
\|u_\varepsilon - v^\varepsilon\|_{L^p(Q)} \leq \eta \text{ provided } E \ni \varepsilon \leq \alpha.
\end{array} \right.
\]

We will briefly express weak and strong two-scale convergence by writing $u_\varepsilon \to u_0$ in $L^p(Q)$-weak 2-s and $u_\varepsilon \to u_0$ in $L^p(Q)$-strong 2-s, respectively.

**Remark 2.1.** It is of interest to know that if $u_\varepsilon \to u_0$ in $L^p(Q)$-weak 2-s, then \[2.1\] holds for $\psi \in C(\overline{Q}; L^\infty_{\text{per}}(Y \times Z))$. See \[8\], Proposition 10, for the proof.

For more details about the two-scale convergence the reader can refer to \[5\]. However, we recall below two fundamental results. First of all, let
\[
\mathcal{Y}(0,T) = \{ v \in L^2(0,T; H^1_0(\Omega)) : v' \in L^2(0,T; H^{-1}(\Omega)) \}.
\]
\( Y(0, T) \) is provided with the norm

\[
\|v\|_{Y(0, T)} = \left( \|v\|_{L^2(0, T; H^1_0(\Omega))}^2 + \|v\|_{L^2(0, T; H^{-1}(\Omega))}^2 \right)^{\frac{1}{2}} \quad (v \in Y(0, T))
\]

which makes it a Hilbert space.

**Theorem 2.1.** Assume that \( 1 < p < \infty \) and further \( E \) is a fundamental sequence. Let a sequence \((u_\varepsilon)_{\varepsilon \in E}\) be bounded in \( L^p(Q) \). Then, a subsequence \( E' \) can be extracted from \( E \) such that \((u_\varepsilon)_{\varepsilon \in E'}\) weakly-two-scale converges in \( L^p(Q) \).

**Lemma 2.1.** Let \((u_\varepsilon)_{\varepsilon \in E}\) be a bounded sequence in \( Y(0, T) \), where \( E \) is a fundamental sequence. There exists a subsequence \( E' \) extracted from \( E \) such that

\[
\begin{align*}
&\int_Q \frac{1}{\varepsilon} u_\varepsilon \psi dxdt \
&\rightarrow \int_Q \int_{Y \times Z} u_1(x,t,y,\tau) \psi(x,t,y,\tau) dxdt dyd\tau
\end{align*}
\]

for all \( \psi \in \mathcal{D}(Q) \odot (C_{\text{per}}(Y) \odot C_{\text{per}}(Z)) \) as \( E' \ni \varepsilon \rightarrow 0 \), where \( u_1 \in L^2\left(Q; L^2_{\text{per}}\left(Z; H^1_\#(Y)\right)\right) \).

**Proof.** As \((u_\varepsilon)_{\varepsilon \in E}\) is a bounded sequence in \( Y(0, T) \), thanks to Theorem 2.2 there exists a subsequence \( E' \) extracted from \( E \) and functions \( u_0 \in Y(0, T) \),

\[
u_1 \in L^2\left(Q; L^2_{\text{per}}\left(Z; H^1_\#(Y)\right)\right)
\]

such that

\[
u_\varepsilon \rightarrow u_0 \text{ in } Y(0, T) \text{-weak},
\]

\[
u_\varepsilon \rightarrow u_0 \text{ in } L^2(Q) \text{-weak } 2-s,
\]

\[
\frac{\partial u_\varepsilon}{\partial x_j} \rightarrow \frac{\partial u_0}{\partial x_j} + \frac{\partial u_1}{\partial y_j} \text{ in } L^2(Q) \text{-weak } 2-s \quad (1 \leq j \leq N),
\]

as \( E' \ni \varepsilon \rightarrow 0 \). Let \( \theta \in \mathcal{D}(Q) \odot C_{\text{per}}(Y) \odot C_{\text{per}}(Z) \). We have

\[
\frac{1}{\varepsilon} (\Delta_y \theta)^\varepsilon = \sum_{i=1}^N \frac{\partial}{\partial x_i} \left( \frac{\partial \theta}{\partial y_i} \right)^\varepsilon - \sum_{i=1}^N \left( \frac{\partial^2 \theta}{\partial x_i \partial y_i} \right)^\varepsilon,
\]

as is easily seen by observing that

\[
\frac{\partial \Phi}{\partial x_i} = \left( \frac{\partial \Phi}{\partial x_i} \right)^\varepsilon + \frac{1}{\varepsilon} \left( \frac{\partial \Phi}{\partial y_i} \right)^\varepsilon, \quad \Phi \in C^1\left(Q \times \mathbb{R}_y^N \times \mathbb{R}_\tau\right).
\]
Hence,

\begin{equation}
\int_Q \frac{1}{\varepsilon} u_\varepsilon (\Delta_y \theta)^{\varepsilon} \, dx \, dt = - \int_Q \nabla_x u_\varepsilon \cdot (\nabla_y \theta)^{\varepsilon} \, dx \, dt - \int_Q u_\varepsilon \sum_{i=1}^N \left( \frac{\partial^2 \theta}{\partial x_i \partial y_i} \right)^{\varepsilon} \, dx \, dt,
\end{equation}

where the dot denotes the Euclidean inner product. On the other hand, according to (2.3) and (2.4) we have

\begin{equation}
\int_Q u_\varepsilon \left( \frac{\partial^2 \theta}{\partial x_i \partial y_i} \right)^{\varepsilon} \, dx \, dt \to \int_Q u_0 \left( \int \int_{Y \times Z} \frac{\partial^2 \theta}{\partial x_i \partial y_i} \, dy \, d\tau \right) \, dx \, dt = 0
\end{equation}

and

\begin{equation}
\int_Q \nabla_x u_\varepsilon \cdot (\nabla_y \theta)^{\varepsilon} \, dx \, dt \to \int \int_{Q \times Y \times Z} (\nabla_x u_0 + \nabla_y u_1) \cdot \nabla_y \theta \, dx \, dt \, dy \, d\tau\nonumber
\end{equation}

as \( E' \ni \varepsilon \to 0 \). Therefore, on letting \( E' \ni \varepsilon \to 0 \) in (2.5), one has

\begin{equation}
\int_Q \frac{1}{\varepsilon} u_\varepsilon (\Delta_y \theta)^{\varepsilon} \, dx \, dt \to \int \int_{Q \times Y \times Z} u_1 \Delta_y \theta \, dx \, dt \, dy \, d\tau.
\end{equation}

With this in mind, let \( \psi \in \mathcal{D}(Q) \otimes (C^\infty_{\text{per}}(Y)/\mathbb{C}) \otimes C_{\text{per}}(Z) \), i.e.,

\[ \psi = \sum_{i \in I} \varphi_i \otimes \psi_i \otimes \chi_i \]

with \( \varphi_i \in \mathcal{D}(Q) \), \( \psi_i \in C^\infty_{\text{per}}(Y)/\mathbb{C} \) and \( \chi_i \in C_{\text{per}}(Z) \), where \( I \) is a finite set (depending on \( \psi \)). For any \( i \in I \), let \( \theta_i \in H^1(Y) \) such that \( \Delta_y \theta_i = \psi_i \). In view of the hypoellipticity of the Laplace operator \( \Delta_y \), the function \( \theta_i \) is of class \( C^\infty \), thus, it belongs to \( C^\infty_{\text{per}}(Y) \). Let

\[ \theta = \sum_{i \in I} \varphi_i \otimes \psi_i \otimes \chi_i. \]

We have \( \theta \in \mathcal{D}(Q) \otimes C^\infty_{\text{per}}(Y) \otimes C_{\text{per}}(Z) \) and \( \Delta_y \theta = \psi \). Hence, (2.2) follows and the lemma is proved. \( \square \)

3. CONVERGENCE OF THE HOMOGENIZATION PROCESS

3.1. Preliminary results. Let \( B^\varepsilon \) be the linear operator in \( L^2(\Omega) \) with domain

\[ D(B^\varepsilon) = \{ v \in H^1_0(\Omega) : A^\varepsilon v \in L^2(\Omega) \} , \]

defined by

\[ B^\varepsilon u = iA^\varepsilon u \quad \text{for all } u \in D(B^\varepsilon) . \]

In the sequel, we suppose that the coefficients \( (a_{ij})_{1 \leq i,j \leq N} \) verify

\begin{equation}
a_{ij} \in W^{1,\infty}(\mathbb{R}_y^N) \quad (1 \leq i, j \leq N) ,
\end{equation}

where \( W^{1,\infty}(\mathbb{R}_y^N) \) is the Sobolev space of functions in \( L^\infty(\mathbb{R}_y^N) \) with their derivatives of order 1. Then \( B^\varepsilon \) is of dense domain, and skew-adjoint since \( A^\varepsilon \) is self-adjoint (see [4] for more details). Consequently, \( B^\varepsilon \) is a \( m \)-dissipative operator in \( L^2(\Omega) \) by virtue of [4] Corollary 2.4.11. It follows by the Hille-Yosida-Philips theorem that \( B^\varepsilon \) is the generator of a contraction semi-group \( \{G^\varepsilon_t\}_{t \geq 0} \).

Now, let us check the existence and uniqueness for (1.3)-(1.5). The abstract evolution problem for (1.3)-(1.5) is given by

\begin{equation}
\begin{cases}
\varepsilon u'_e = B^\varepsilon u_e + F^\varepsilon \left( u_e \right) \text{ in } [0, T] \\
u_e(0) = u_0,
\end{cases}
\end{equation}

where \( u_0 \in L^2(\Omega) \).
where \( F_\varepsilon \) is defined in \( L^2 (0, T; L^2 (\Omega)) \) by
\[
F_\varepsilon (v) (t) = \frac{i}{\varepsilon} \nabla (t) v (t) - i f (t) \quad (t \in [0, T])
\]
for all \( v \in L^2 (0, T; L^2 (\Omega)) \). We have the following proposition.

**Proposition 3.1.** Suppose \( u^0 \in D (B^\varepsilon) \), \( f \in C ([0, T] ; L^2 (\Omega)) \) and
\[
(3.3) \quad \frac{1}{\varepsilon} \| \nabla^\varepsilon \|_{\infty} \leq \beta \quad \text{for all} \quad \varepsilon > 0,
\]
where \( \beta \) is a positive constant independent of \( \varepsilon \). Suppose further that \( T \) verifies
\[
(3.4) \quad \beta T < 1.
\]
Then, the abstract evolution problem (3.2) admits a unique solution \( u^\varepsilon \in C ([0, T] ; D (B^\varepsilon)) \cap C^1 ([0, T] ; L^2 (\Omega)) \).

**Proof.** Let us consider the mapping \( \Phi^\varepsilon \) of \( L^2 (0, T; L^2 (\Omega)) \) into \( L^2 (0, T; L^2 (\Omega)) \) defined by
\[
\Phi^\varepsilon (v) (t) = G^\varepsilon_t u^0 + \int_0^t G^\varepsilon_{t-s} F_\varepsilon (v) (s) \, ds \quad (t \in [0, T])
\]
for all \( v \in L^2 (0, T; L^2 (\Omega)) \) (we recall that \((G^\varepsilon_t)_{t \geq 0}\) is the contraction semi-group generated by \( B^\varepsilon \)). For fixed \( \varepsilon > 0 \), \( \Phi^\varepsilon \) is a contraction in \( L^2 (0, T; L^2 (\Omega)) \) with Lipschitz constant \( \beta T \). Indeed, for \( v \) and \( w \in L^2 (0, T; L^2 (\Omega)) \), one has,
\[
\| \Phi^\varepsilon (v) (t) - \Phi^\varepsilon (w) (t) \|_{L^2 (\Omega)} \leq \int_0^t \| F_\varepsilon (v) (s) - F_\varepsilon (w) (s) \|_{L^2 (\Omega)} \, ds \quad (t \in [0, T]).
\]
Moreover,
\[
\| F_\varepsilon (v) (s) - F_\varepsilon (w) (s) \|_{L^2 (\Omega)} \leq \beta \| v (s) - w (s) \|_{L^2 (\Omega)} \quad (s \in [0, T]),
\]
by virtue of (3.3). Its follows from the preceding inequalities that
\[
\| \Phi^\varepsilon (v) - \Phi^\varepsilon (w) \|_{L^2 (0, T; L^2 (\Omega))} \leq \beta T \| v - w \|_{L^2 (0, T; L^2 (\Omega))}
\]
with (3.4). Thus, for fixed \( \varepsilon > 0 \), there exists \( u^\varepsilon \in L^2 (0, T; L^2 (\Omega)) \) such that \( \Phi^\varepsilon (u^\varepsilon) = u^\varepsilon \) and \( u^\varepsilon \) is unique. On the other hand, we proceed as in [4, Lemma 4.1.1] and we see that, any solution \( u^\varepsilon \in C ([0, T] ; D (B^\varepsilon)) \cap C^1 ([0, T] ; L^2 (\Omega)) \) of (3.2) verifies \( \Phi^\varepsilon (u^\varepsilon) = u^\varepsilon \), and conversely. The proposition is proved.

Let us prove some estimates for (1.3)-(1.5).

**Lemma 3.1.** Suppose that the hypotheses of Proposition 3.1 are satisfied, and the coefficients \( a_{ij} \) \( (1 \leq i, j \leq N) \) are of the form
\[
(3.5) \quad a_{ij} = a \delta_{ij} \quad (1 \leq i, j \leq N),
\]
where \( a \in W^{1, \infty} (\mathbb{R}^N_y) \), and \( \delta_{ij} \) is the Kronecker symbol. Suppose further that
\[
(3.6) \quad f \text{ and } f' \in L^2 (0, T; L^2 (\Omega))
\]
and
\[
(3.7) \quad \frac{\partial \nabla}{\partial \tau} \in L^\infty (\mathbb{R}^N_y \times \mathbb{R}_T) \text{ with } \frac{1}{\varepsilon^2} \left\| \left( \frac{\partial \nabla}{\partial \tau} \right)^\varepsilon \right\|_{\infty} \leq c_0,
\]
c_0 being a constant independent of \( \varepsilon \). Then, there exists a constant \( c > 0 \) independent of \( \varepsilon \) such that the solution \( u_\varepsilon \) of \((1.3)-(1.5)\) verifies:

\begin{equation}
\|u_\varepsilon\|_{L^2(0,T;H^1(\Omega))} \leq c
\end{equation}

and

\begin{equation}
\|u'_\varepsilon\|_{L^2(0,T;H^{-1}(\Omega))} \leq c.
\end{equation}

Before the proof of this lemma, let us make some useful remarks. Let us put

\[ a^\varepsilon(u,v) = \sum_{i,j=1}^{N} a^\varepsilon_{ij} \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} \text{ for all } u,v \in H^1(\Omega). \]

**Remark 3.1.** As \( u_\varepsilon \in C([0,T];D(B^\varepsilon)) \cap C^1([0,T];L^2(\Omega)) \), the function \( t \to a^\varepsilon(u_\varepsilon(t),u_\varepsilon(t)) \) is bounded in \( C^1([0,T]) \) and

\[ \frac{d}{dt} a^\varepsilon(u_\varepsilon(t),u_\varepsilon(t)) = 2 \operatorname{Re}(A^\varepsilon u_\varepsilon(t),u'_\varepsilon(t)) \quad \text{for all } t \in [0,T]. \]

On the other hand, by \((3.7)\) we have

\[ \frac{1}{\varepsilon} \frac{d}{dt} (V^\varepsilon u_\varepsilon(t),u_\varepsilon(t)) = \frac{1}{\varepsilon^2} \left( \frac{\partial V^\varepsilon}{\partial \tau} u_\varepsilon(t),u_\varepsilon(t) \right) + \frac{2}{\varepsilon} \operatorname{Re}(V^\varepsilon u_\varepsilon(t),u'_\varepsilon(t)) \quad \text{for all } t \in [0,T], \]

where \((,\,)\) denotes the scalar product in \( L^2(\Omega) \). Further, by \((3.6)\) we have

\[ \frac{d}{dt}(f(t),u_\varepsilon(t)) = (f'(t),u_\varepsilon(t)) + (f(t),u'_\varepsilon(t)) \quad \text{for all } t \in [0,T]. \]

**Proof of Lemma 3.1.** Taking the scalar product in \( L^2(\Omega) \) of \((1.3)\) with \( u_\varepsilon \) yields

\[ i(u'_\varepsilon(t),u_\varepsilon(t)) + a^\varepsilon(u_\varepsilon(t),u_\varepsilon(t)) + \frac{1}{\varepsilon} (V^\varepsilon u_\varepsilon(t),u_\varepsilon(t)) = (f(t),u_\varepsilon(t)) \quad \text{for all } t \in [0,T]. \]

Using \((1.5)\), we see that \( t \to a^\varepsilon(u_\varepsilon(t),u_\varepsilon(t)) \) is a real values function. Thus, by the preceding equality we have

\[ \operatorname{Re}(u'_\varepsilon(t),u_\varepsilon(t)) = -\operatorname{Re}(if(t),u_\varepsilon(t)) \quad \text{for all } t \in [0,T], \]

i.e.,

\[ \frac{1}{2} \frac{d}{dt} \|u_\varepsilon(t)\|_{L^2(\Omega)}^2 = -\operatorname{Re}(if(t),u_\varepsilon(t)) \quad \text{for all } t \in [0,T]. \]

Integrating the preceding equality in \([0,t]\) with \( t \in [0,T] \) leads to

\begin{equation}
\|u_\varepsilon(t)\|_{L^2(\Omega)}^2 \leq \|u^0\|_{L^2(\Omega)}^2 + 2 \int_0^T \int_\Omega |f| |u_\varepsilon| \, dx \, dt.
\end{equation}

Moreover,

\[ 2 \int_0^T \int_\Omega |f| |u_\varepsilon| \, dx \, dt \leq \int_0^T \int_\Omega \left( 2T|f|^2 + \frac{1}{2T} |u_\varepsilon|^2 \right) \, dx \, dt. \]

Consequently, an integration on \([0,T]\) of \((3.10)\) leads to

\begin{equation}
\frac{1}{2} \|u_\varepsilon\|_{L^2(0,T;L^2(\Omega))}^2 \leq T \|u^0\|_{L^2(\Omega)}^2 + 2T^2 \|f\|_{L^2(0,T;L^2(\Omega))}^2.
\end{equation}

It follows from the preceding inequality that the sequence \((u_\varepsilon)_{\varepsilon>0}\) is bounded in \( L^2(0,T;L^2(\Omega)) \). Now, let us prove \((3.8)\). Taking the scalar product in \( L^2(\Omega) \) of \((1.3)\) with \( u'_\varepsilon \), one as

\[ i\|u'_\varepsilon(t)\|_{L^2(\Omega)}^2 + (A^\varepsilon u_\varepsilon(t),u'_\varepsilon(t)) + \frac{1}{\varepsilon} (V^\varepsilon u_\varepsilon(t),u_\varepsilon(t)) = (f(t),u'_\varepsilon(t)) \quad \text{for all } t \in [0,T]. \]
By the preceding equality we have,

\[ \text{Re} (A' u_\varepsilon (t), u'_\varepsilon (t)) + \frac{1}{\varepsilon} \text{Re} (V^*) u_\varepsilon (t), u_\varepsilon (t)) = \text{Re} (f(t), u'_\varepsilon (t)) \quad (t \in [0, T]). \]

Thus, using Remark 3.1 leads to

\[ \frac{1}{2} \frac{d}{dt} a^\varepsilon (u_\varepsilon (t), u_\varepsilon (t)) + \frac{1}{2} \frac{d}{dt} (V^* u_\varepsilon (t), u_\varepsilon (t)) = \frac{1}{2} \frac{d}{dt} (f(t), u_\varepsilon (t)) - \text{Re} (f'(t), u_\varepsilon (t)). \]

An integration on \([0, t]\) of (3.12) yields,

\[ \frac{1}{2} a^\varepsilon (u_\varepsilon (t), u_\varepsilon (t)) + \frac{1}{2} (V^* u_\varepsilon (t), u_\varepsilon (t)) - \frac{1}{2} a^\varepsilon (u_\varepsilon (0), u_\varepsilon (0)) - \frac{1}{2} (V^* (0) u_\varepsilon (0), u_\varepsilon (0)) = \frac{1}{2} \int_0^t (\frac{d}{dt} u^\varepsilon (s), u^\varepsilon (s)) ds + \text{Re} (f(t), u_\varepsilon (t)) - \text{Re} (f'(t), u_\varepsilon (t)) ds. \]

It follows from (1.2) and (3.10) that, by (3.7) we have

\[ \alpha \|u_\varepsilon (t)\|_{H^1_0 (\Omega)}^2 \leq \beta \|u_\varepsilon (t)\|_{L^2 (\Omega)}^2 + c_1 \|u_\varepsilon\|_{L^2 (\Omega)}^2 + \beta \|u_\varepsilon\|_{L^2 (\Omega)}^2 \leq c_0 \|u_\varepsilon\|_{L^2 (0,T;L^2 (\Omega))}^2 + 2 \|f\|_{L^2 (\Omega)} \|v\|_{L^2 (\Omega)} + 2 \|f\|_{L^2 (0,T;L^2 (\Omega))} \|v\|_{L^2 (0,T;L^2 (\Omega))}, \]

where \(c_1 = \sup_{1 \leq i \leq N} \|a_{ij}\|_{\infty}\). Integrating on \([0, T]\) the preceding inequality and using (3.11), we see that the sequence \((u'_\varepsilon)_{\varepsilon > 0}\) is bounded in \(L^2 (0, T; H^1_0 (\Omega))\), and (3.8) follows. Now, we can prove (3.9). By (1.3), we have

\[ i \int_0^T (u'_\varepsilon (t), \overline{v} (t)) dt + \int_0^T a^\varepsilon (u_\varepsilon (t), v (t)) dt + \frac{1}{\varepsilon} \int_0^T (V^* u_\varepsilon (t), v (t)) dt = \int_0^T (f (t), v (t)) dt \]

for all \(v \in L^2 (0, T; H^1_0 (\Omega))\). Hence,

\[ \left| \int_0^T (u'_\varepsilon (t), \overline{v} (t)) dt \right| \leq c_1 \|u_\varepsilon\|_{L^2 (0,T;H^1_0 (\Omega))} \|v\|_{L^2 (0,T;H^1_0 (\Omega))} + \beta c_2 \|u_\varepsilon\|_{L^2 (0,T;L^2 (\Omega))} \|v\|_{L^2 (0,T;H^1_0 (\Omega))} + c_2 \|f\|_{L^2 (0,T;L^2 (\Omega))} \|v\|_{L^2 (0,T;H^1_0 (\Omega))}, \]

where \(c_2\) is the constant in the Poincaré inequality. It follows from the preceding inequality that

\[ \|u'_\varepsilon\|_{L^2 (0,T;H^{-1} (\Omega))} \leq c_1 \|u_\varepsilon\|_{L^2 (0,T;H^1_0 (\Omega))} + \beta c_2 \|u_\varepsilon\|_{L^2 (0,T;L^2 (\Omega))} + c_2 \|f\|_{L^2 (0,T;L^2 (\Omega))}. \]

Then, by (3.11) and (3.8) we conclude that the sequence \((u'_\varepsilon)_{\varepsilon > 0}\) is bounded in \(L^2 (0, T; H^{-1} (\Omega))\). The lemma is proved. \(\Box\)

### 3.2. A convergence theorem.

Let us first introduce some functions spaces.

We consider the space

\[ \mathbb{F}_0^1 = \mathcal{Y} (0, T) \times L^2 (Q; L^2_{per} (Z; H^1_\# (Y))) \]

provided with the norm

\[ \|u\|_{\mathbb{F}_0^1} = \left( \|u_0\|_{\mathcal{Y}(0,T)}^2 + \|u_1\|_{L^2 (Q; L^2_{per} (Z; H^1_\# (Y)))}^2 \right)^{\frac{1}{2}} \quad (u = (u_0, u_1) \in \mathbb{F}_0^1), \]

which makes it Hilbert space. We consider also the space

\[ \mathcal{F}_0^\infty = \mathcal{D} (Q) \times [\mathcal{D} (Q) \otimes (C_{per} (Y) / \mathbb{C}) \otimes C_{per} (Z)] \]
which is a dense subspace of $F^1_0$. For $u = (u_0, u_1)$ and $v = (v_0, v_1) \in H^1_0(\Omega) \times L^2(\Omega; L^2_{\text{per}}(Z; H^1_\#(Y)))$, we set

$$a(u, v) = \sum_{i,j=1}^N \int \int_{\Omega \times \Omega} a_{ij}(y) \left( \frac{\partial u_0}{\partial x_j} + \frac{\partial u_1}{\partial y_j} \right) \left( \frac{\partial v_0}{\partial x_i} + \frac{\partial v_1}{\partial y_i} \right) dxdydr.$$ 

This defines a sesquilinear hermitian form on $[H^1_0(\Omega) \times L^2(\Omega; L^2_{\text{per}}(Z; H^1_\#(Y)))]^2$ which is continuous and verifies

$$\Re a(v, v) \geq \alpha \|v\|^2_{H^1_0(\Omega) \times L^2(\Omega; L^2_{\text{per}}(Z; H^1_\#(Y)))} \quad (v \in H^1_0(\Omega) \times L^2(\Omega; L^2_{\text{per}}(Z; H^1_\#(Y))))$$

according to (3.14). Further, we have the following lemma.

**Lemma 3.2.** Suppose that the coefficients $a_{ij} (1 \leq i, j \leq N)$ verify (3.15), and let $f \in L^2(0, T; L^2(\Omega))$ and $v \in L^\infty \left( \mathbb{R}^N \times \mathbb{R}; \mathbb{R} \right)$. Then the variational problem

$$\begin{cases}
\mathbf{u} = (u_0, u_1) \in F^1_0 \text{ with } u_0(0) = u^0 : \\
i \int_0^T \langle u_0'(t), \overline{v_0}(t) \rangle dt + \int_0^T a(u(t), v(t)) dt + \int \int_{\Omega \times \Omega} (u_0, v_0) V dx dy dt = 0
\end{cases}$$

for all $v = (v_0, v_1) \in F^1_0$. Taking in particular $v = \phi \otimes v_*$ with $\phi \in \mathcal{D}(0, T]$ and $v_* = (v_0, v_1) \in H^1_0(\Omega) \times L^2(\Omega; L^2_{\text{per}}(Z; H^1_\#(Y)))$ in (3.16), we obtain

$$i \langle z_0(t), \overline{\phi}(t) \rangle + a(z(t), v_*) + \int \int_{\Omega \times \Omega} (z_1(t) \overline{\phi} + z_0(t) \overline{\phi}) V dx dy dt = 0 \quad (t \in [0, T]).$$

for all $v_0 = (v_0, v_1) \in H^1_0(\Omega) \times L^2(\Omega; L^2_{\text{per}}(Z; H^1_\#(Y)))$. Thus, choosing $v_* = z(t)$ for $t \in [0, T]$ in the preceding equality yields,

$$i \langle z_0'(t), \overline{\phi}(t) \rangle + a(z(t), z(t)) + \int \int_{\Omega \times \Omega} (z_1(t) \overline{\phi} + z_0(t) \overline{\phi}) V dx dy dt = 0 \quad (t \in [0, T]).$$

But, according to (3.15), $t \rightarrow a(z(t), z(t))$ is a real valued function. Consequently, by the preceding equality we have

$$\Re \langle z_0'(t), \overline{\phi}(t) \rangle = 0 \quad (t \in [0, T]),$$

i.e.,

$$\frac{1}{2} \frac{d}{dt} \|z_0(t)\|^2_{L^2(\Omega)} = 0 \quad (t \in [0, T]).$$

Hence $z_0(t) = 0$ for all $t \in [0, T]$. Then, by (3.14) and (3.17) we see that $z(t) = 0$ for all $t \in [0, T]$, and the lemma follows. $\Box$
In the sequel the coefficients $a_{ij}$ ($1 \leq i, j \leq N$) are assumed to verify the periodicity hypothesis

$$a_{ij}(y + k) = a_{ij}(y) \quad \text{a.e. in } \mathbb{R}^N \quad (1 \leq i, j \leq N)$$

for all $k \in \mathbb{Z}^N$. Moreover, the potential $V$ is supposed to satisfy

$$V(y + k, \tau + l) = V(y, \tau) \quad \text{a.e. in } \mathbb{R}^N \times \mathbb{R}$$

for all $(k, l) \in \mathbb{Z}^N \times \mathbb{Z}$, and

$$\int_Y V(y, \tau) \, dy = 0 \quad \text{a.e. in } \mathbb{R}^N.$$ 

Therefore the functions $a_{ij}$ ($1 \leq i, j \leq N$) and $V$ are respectively $Y$-periodic and $Y \times Z$-periodic, where $Y = \left(-\frac{1}{2}, \frac{1}{2}\right)^N$ and $Z = \left(-\frac{1}{2}, \frac{1}{2}\right)$. Further, $V$ is of zero mean value.

**Theorem 3.1.** Suppose the hypotheses of Proposition 3.1 and Lemma 3.1 are satisfied. For fixed $\varepsilon > 0$, let $u_\varepsilon$ be the solution of (1.3)-(1.5). Then, as $\varepsilon \to 0$, we have:

$$u_\varepsilon \to u_0 \text{ in } \mathcal{Y}(0, T) \text{-weak},$$

$$u_\varepsilon \to u_0 \text{ in } L^2(Q) \text{-strong}$$

and

$$\frac{\partial u_\varepsilon}{\partial x_j} \to \frac{\partial u_0}{\partial x_j} + \frac{\partial u_1}{\partial y_j} \text{ in } L^2(Q) \text{-weak-2-s} \quad (1 \leq i, j \leq N),$$

where $u = (u_0, u_1) \in F_0$ is the unique solution of (3.15).

**Proof.** According to Lemma 3.1 the sequence $(u_\varepsilon)_{\varepsilon > 0}$ is bounded in $\mathcal{Y}(0, T)$. Hence, if $E$ is a fundamental sequence, by virtue of Theorem 2.2 there are some subsequence $E' \subseteq E$ and some vector function $u = (u_0, u_1) \in F_0$ such that (3.21)-(3.22) hold when $E' \ni \varepsilon \to 0$.

Thus, thanks to Lemma 3.2 the theorem is certainly proved if we can show that $u$ verifies (3.15).

Indeed, we begin by verifying that $u_0(0) = u^0$ (it is worth recalling that $u_0$ may be viewed as a continuous mapping of $[0, T]$ into $L^2(\Omega)$).

Let $v \in H_0^1(\Omega)$, and let $\varphi \in C^1([0, T])$ with $\varphi(T) = 0$. By integration by parts, we have,

$$\int_0^T \langle u'_\varepsilon(t), v \rangle \varphi(t) \, dt + \int_0^T \langle u_\varepsilon(t), v \rangle \varphi'(t) \, dt = -\langle u^0, v \rangle \varphi(0),$$

since $u_\varepsilon(0) = u^0$. In view of (3.21)-(3.22), we pass to the limit in the preceding equality as $E' \ni \varepsilon \to 0$. We obtain

$$\int_0^T \langle u'_0(t), v \rangle \varphi(t) \, dt + \int_0^T \langle u_0(t), v \rangle \varphi'(t) \, dt = -\langle u^0, v \rangle \varphi(0).$$

Since $\varphi$ and $v$ are arbitrary, we see that $u_0(0) = u^0$.

Finally, let us prove the variational equality of (3.15). Fix any arbitrary two functions $\psi_0 \in \mathcal{D}(Q)$ and $\psi_1 \in \mathcal{D}(Q) \otimes [C_{per}(Y) / \mathbb{C}] \otimes C_{per}(Z)$,
and let
\[ \psi_\varepsilon = \psi_0 + \varepsilon \psi_1, \text{ i.e., } \psi_\varepsilon(x,t) = \psi_0(x,t) + \varepsilon \psi_1 \left( x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon} \right) \text{ for all } (x,t) \in Q, \]
where \( \varepsilon > 0 \) is arbitrary. By (3.24), one as
\[ \int_0^T \langle u'_\varepsilon(t), \overline{\psi}_\varepsilon(t) \rangle dt + \int_0^T \varepsilon (u_\varepsilon(t), \psi_\varepsilon(t)) dt + \frac{1}{\varepsilon} \int_0^T (\nabla u_\varepsilon(t), \overline{\psi}_\varepsilon(t)) dt = \int_0^T (f(t), \psi_\varepsilon(t)) dt. \]
The aim is to pass to the limit in (3.24) as \( E' \ni \varepsilon \to 0 \). First, we have
\[ \int_0^T \langle u'_\varepsilon(t), \overline{\psi}_\varepsilon(t) \rangle dt = - \int_Q u_\varepsilon \nabla \overline{\psi}_0 \ dx dt = - \int_Q u_\varepsilon \left( \frac{\partial \overline{\psi}_0}{\partial t} + \varepsilon \left( \frac{\partial \overline{\psi}_1}{\partial t} \right) + \left( \frac{\partial \overline{\psi}_1}{\partial \tau} \right)^\varepsilon \right) dx dt. \]
Thus, in view of (3.22) (and using Definition 2.1), we have,
\[ \int_0^T \langle u'_\varepsilon(t), \overline{\psi}_\varepsilon(t) \rangle dt + \int_0^T a(u_\varepsilon(t), \psi_\varepsilon(t)) dt \to \int_0^T \langle u'_0(t), \overline{\psi}_0(t) \rangle dt \]
as \( E' \ni \varepsilon \to 0 \), since
\[ \int_Q \left( \int_{Y \times Z} \frac{\partial \overline{\psi}_1}{\partial \tau} \ dx \right) u_0 \ dx \ dt = 0 \]
by virtue of the \( Y \times Z \)-periodicity of \( \psi_1 \).
Next, we have
\[ \int_0^T a(u_\varepsilon(t), \psi_\varepsilon(t)) dt \to \int_0^T a(u_0(t), \phi(t)) dt \]
as \( E' \ni \varepsilon \to 0 \), where \( \phi = (\psi_0, \psi_1) \) (proceed as in the proof of the similar result in [8] p.179). On the other hand,
\[ \int_0^T \frac{1}{\varepsilon} (V \psi_\varepsilon(t), \nabla \overline{\psi}_\varepsilon(t)) dt = \frac{1}{\varepsilon} \int_Q V \psi_\varepsilon \overline{\psi}_0 \ dx \ dt + \int_Q V \psi_\varepsilon \overline{\psi}_1 \ dx \ dt. \]
In view of Lemma 2.1 and the density of \( (C_{\text{per}}(Y) / \mathbb{C}) \otimes C_{\text{per}}(Z) \) in \( L^2_{\text{per}}(Z; L^2_{\text{per}}(Y) / \mathbb{C}) \), we pass to the limit in (3.25) by virtue of (3.19) - (3.20). This yields,
\[ \int_0^T \frac{1}{\varepsilon} (V \psi_\varepsilon(t), \nabla \overline{\psi}_\varepsilon(t)) dt \to \int \int_{Q \times Y \times Z} (u_1 \overline{\psi}_0 + u_0 \overline{\psi}_1) V dx dt dy d\tau \]
as \( E' \ni \varepsilon \to 0 \). Hence, passing to the limit in (3.24) as \( E' \ni \varepsilon \to 0 \) leads to
\[ \int_0^T \langle u'_0(t), \overline{\psi}_0(t) \rangle dt + \int_0^T a(u_0(t), \phi(t)) dt + \int \int_{Q \times Y \times Z} (u_1 \overline{\psi}_0 + u_0 \overline{\psi}_1) V dx dt dy d\tau = \int_0^T (f(t), \psi_0(t)) dt \]
for all \( \phi = (\psi_0, \psi_1) \in \mathcal{F}_0^\infty \). Moreover, since \( \mathcal{F}_0^\infty \) is a dense subspace of \( \mathbb{F}^1_0 \), by (3.26) we see that \( u = (u_0, u_1) \) verifies (3.15). Thanks to the uniqueness of the solution for (3.15) and the fact that the sequence \( E \) is arbitrary, we have (3.21)-(3.23) as \( \varepsilon \to 0 \). The theorem is proved. \( \square \)
For further needs, we wish to give a simple representation of the function \( u_1 \) in Theorem 3.1. For this purpose, let us introduce the form \( \hat{a} \) on \( L^2_{\text{per}} \left( Z; H^1_Y \right) \times L^2_{\text{per}} \left( Z; H^1_Y \right) \) defined by

\[
\hat{a}(w,v) = \sum_{i,j=1}^N \int_{Y \times Z} a_{ij} \frac{\partial w}{\partial y_j} \frac{\partial v}{\partial y_i} dyd\tau
\]

for all \( w, v \in L^2_{\text{per}} \left( Z; H^1_Y \right) \). By virtue of (1.1)-(1.2) and (3.2), the sesquilinear form \( \hat{a} \) is continuous, hermitian and coercive with,

\[
\hat{a}(v,v) \geq \alpha \| v \|^2_{L^2_{\text{per}} \left( Z; H^1_Y \right)} \quad \text{for all } v \in L^2_{\text{per}} \left( Z; H^1_Y \right).
\]

Next, for any indice \( l \) with \( 1 \leq l \leq N \), we consider the variational problem

\[
(3.27) \quad \hat{a}(\chi^l, v) = \sum_{i=1}^N \int_{Y \times Z} a_{il} \frac{\partial \chi^l}{\partial y_i} dyd\tau \quad \text{for all } v \in L^2_{\text{per}} \left( Z; H^1_Y \right),
\]

which determines \( \chi^l \) in a unique manner. Further, let \( \eta \in L^2_{\text{per}} \left( Z; H^1_Y \right) \) be the unique function defined by

\[
(3.28) \quad \hat{a}(\eta, v) = \int \int_{Y \times Z} \nabla v dyd\tau \quad \text{for all } v \in L^2_{\text{per}} \left( Z; H^1_Y \right).
\]

**Lemma 3.3.** Under the hypotheses of Theorem 3.1, we have

\[
(3.29) \quad u_1(x,t,y,\tau) = -\sum_{j=1}^N \frac{\partial u_0}{\partial x_j}(x,t) \chi^j(y,\tau) + \eta(y,\tau) u_0(x,t)
\]

for almost all \((x,t,y,\tau) \in Q \times Y \times Z\).

**Proof.** In (3.15) choose the particular test function \( v = (v_0, v_1) \in F_0^3 \) with \( v_0 = 0 \) and \( v_1 = \varphi \otimes v \), where \( \varphi \in D(Q) \) and \( v \in L^2_{\text{per}} \left( Z; H^1_Y \right) \). This yields

\[
(3.30) \quad \hat{a}(u_1(x,t), v) = -\sum_{i,j=1}^N \frac{\partial u_0}{\partial x_j}(x,t) \int \int_{Y \times Z} a_{ij} \frac{\partial v}{\partial y_i} dyd\tau + u_0(x,t) \int \int_{Y \times Z} \nabla v dyd\tau
\]

almost everywhere in \((x,t) \in Q\) and for all \( v \in L^2_{\text{per}} \left( Z; H^1_Y \right) \). But it is clear that \( u_1(x,t) \) (for fixed \((x,t) \in Q\)) is the sole function in \( L^2_{\text{per}} \left( Z; H^1_Y \right) \) solving the variational equation (3.30). On the other hand, in view of (3.27)-(3.28) it is an easy matter to check that the right hand side of (3.29) solves the same variational equation. Hence the lemma follows immediately. \( \square \)

### 3.3. The macroscopic homogenized equation.

Our aim here is to derive the initial boundary value problem for \( u_0 \). To begin, for \( 1 \leq i, j \leq N \), let

\[
q_{ij} = \int_Y a_{ij} dy - \sum_{l=1}^N \int_{Y \times Z} a_{il} \frac{\partial \chi^l}{\partial y_i} dyd\tau,
\]
By (3.5) one has,

\[ q_{ij} = \delta_{ij} \int_Y d\eta - \int \int_{Y \times Z} a_{ij} \frac{\partial \eta_j}{\partial y_j} dyd\tau \]

and

\[ b_i = -\int \int_{Y \times Z} \chi^i \mathcal{V} dyd\tau - \int \int_{Y \times Z} a_i \frac{\partial \eta_i}{\partial y_i} dyd\tau. \]

Further, let

\[ \mu = \int \int_{Y \times Z} \eta \mathcal{V} dyd\tau. \]

To the coefficients \( q_{ij} \) we attach the differential operator \( Q \) on \( Q \) mapping \( D'(Q) \) into \( D'(Q) \) (\( D'(Q) \) being the usual space of complex distributions on \( Q \)) as

\[ Q u = -\sum_{i,j} q_{ij} \frac{\partial^2 u}{\partial x_j \partial x_i} \text{ for all } u \in D'(Q). \]

Let

\[ b = (b_i)_{i=1}^N. \]

We consider the following initial boundary value problem:

\begin{align*}
(3.31) & \quad \frac{i}{\partial t} u_0 + Q u_0 + b \nabla u_0 + \mu u_0 = f \text{ in } Q = \Omega \times [0,T[ \\
(3.32) & \quad u_0 = 0 \text{ on } \partial \Omega \times [0,T[ \\
(3.33) & \quad u_0(0) = u^0 \text{ in } \Omega.
\end{align*}

The initial boundary value problem (3.31)-(3.33) is the so-called macroscopic homogenized equation.

**Lemma 3.4.** Suppose the hypotheses of Proposition 3.1 and Lemma 3.1 are satisfied. Then, the initial boundary value problem (3.31)-(3.33) admits at most one weak solution \( u_0 \) in \( \mathcal{Y}(0,T) \).

**Proof.** It is an easy exercise to show that if \( u_0 \in \mathcal{Y}(0,T) \) verifies (3.31)-(3.33) then \( u = (u_0,u_1) \) [with \( u_1 \) given by (3.29)] satisfies (3.15). Hence, the unicity in (3.31)-(3.33) follows by Lemma 3.2.

**Theorem 3.2.** Suppose the hypotheses of Proposition 3.1 and Lemma 3.1 are satisfied. For \( \varepsilon > 0 \), let \( u_\varepsilon \in \mathcal{Y}(0,T) \) be defined by (1.3)-(1.5). Then, as \( \varepsilon \to 0 \), we have \( u_\varepsilon \to u_0 \) in \( \mathcal{Y}(0,T) \)-weak, where \( u_0 \) is the unique weak solution of (3.31)-(3.33) in \( \mathcal{Y}(0,T) \).

**Proof.** As in the proof of Theorem 3.1 from any fundamental sequence \( E \) one can extract a subsequence \( E' \) such that as \( E' \ni \varepsilon \to 0 \), we have (3.21)-(3.23), and further (3.26) holds for all \( \phi = (\psi_0,\psi_1) \in \mathcal{F}^0_\infty \), where \( u = (u_0,u_1) \in \mathcal{F}^1_0 \). Now, substituting (3.29) in (3.26) and then choosing therein the \( \phi \)'s such that \( \psi_1 = 0 \), a simple computation yields (3.31) with (3.32)-(3.33), of course. Hence the theorem follows by Lemma 3.4 and using of an obvious argument. □
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Current address: Department of Mathematics and Computer Science, University of Ngaoundere, P.O.Box 454 Ngaoundere (Cameroon)

E-mail address: lsigning@uiv-ndere.cm