Energy Conditions and Stability in generalized $f(R)$ gravity with arbitrary coupling between matter and geometry

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The energy conditions and the Dolgov-Kawasaki criterion in generalized $f(R)$ gravity with arbitrary coupling between matter and geometry are derived in this paper, which are quite general and can degenerate to the well-known energy conditions in GR and $f(R)$ gravity with non-minimal coupling and non-coupling as special cases. In order to get some insight on the meaning of these energy conditions and the Dolgov-Kawasaki criterion, we apply them to a class of models in the FRW cosmology and give some corresponding results.

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1. Introduction

Recent astrophysical observations$^{[1, 2]}$ have indicated that the expansion of our Universe is accelerating at the present time. In principle, this phenomenon can be explained by either dark energy (see, for instance, Ref.$^3$ for reviews), in which the reason of this phenomenon is due to an exotic component with large negative pressure, or modified theories of gravity$^4$. Unfortunately, up to now a satisfactory answer to the question that what dark energy is and where it came from has not yet to be obtained. Alternative to dark energy, modified theories of gravity is extremely attractive because the cosmic speed-up can be easily explained by the fact that some sub-dominant terms, like $1/R$, may become essential at small curvature. Under some additional conditions, the early-time inflation and late-time acceleration can be unified by different role of gravitational terms relevant at small and at large curvature.

$f(R)$ gravity is one of the competitive candidates in modified theories of gravity (see, for instance, Refs.$^5, 6$ for reviews). Here $f(R)$ is an arbitrary function of the Ricci scalar $R$. One can add any form of $R$ in it, such as $1/R^7$, $\ln R^8$, positive and negative powers of $R^9$, Gauss-Bonnet invariant$^{10}$, etc. It worth stressing that considering some additional conditions, the early-time inflation and late-time acceleration can be unified by different role of gravitational terms relevant at small and at large curvature. However, $f(R)$ gravity is not perfect because of containing a number of instabilities. For instance, the theory with $1/R$ may develop the instability$^{22}$. But by adding a term of $R^2$ to this specific $f(R)$ model, one can remove this instability$^{8, 9}$. For more general forms of $f(R)$, the stability condition $f'' \geq 0$ can be used to test $f(R)$ gravity models$^{25}$.

This paper is organized as follows. In section 2, we give some fundamental elements of generalized $f(R)$ gravity models with arbitrary matter-geometry coupling. In section 3, the well-known energy conditions, namely, the strong energy condition (SEC), the null energy condition (NEC), the weak energy condition (WEC) and the dominant energy...
condition (DEC), in the generalized $f(R)$ gravity models, will be derived. In order to get some insight on the meaning of these energy conditions, we apply them to a class of models. Furthermore we rewritten them in terms of parameters of the deceleration ($q$), the jerk ($j$) and the snap ($s$) and then use the rewritten WEC to restrict a special $f(R)$ model. The instability of generalized $f(R)$ gravity models with arbitrary matter-geometry coupling will be studied in section 4. Last section contains our summary.

2. GENERALIZED $f(R)$ GRAVITY MODELS WITH ARBITRARY MATTER-GEOMETRY COUPLING

A more general model of $f(R)$ gravity, in which the coupling style between matter and geometry is arbitrary and the Lagrangian density of matter only appears in coupling term, has been proposed in Ref.[11]. Its starting action is

$$S = \int \left[ \frac{1}{2} f_1(R) + G(L_m) f_2(R) \right] \sqrt{-g} d^4x,$$

where we have chosen $\kappa = 8\pi G = c = 1$. $f_i(R)$ ($i = 1, 2$) and $G(L_m)$ are arbitrary functions of the Ricci scalar $R$ and the Lagrangian density of matter respectively. When $f_2(R) = 1$ and $G(L_m) = L_m$, we obtain the general form of $f(R)$ gravity with non-coupling between matter and geometry. Furthermore, by setting $f_1(R) = R$, action (1) can be reduced to the standard General Relativity (GR).

Varying the action (1) with respect to the metric $g_{\mu\nu}$ yields the field equations

$$F_1(R) R_{\mu\nu} - \frac{1}{2} f_1(R) g_{\mu\nu} + (g_{\mu\nu} \Box - \nabla_\mu \nabla_\nu) F_1(R) = -2G(L_m) F_2(R) R_{\mu\nu} - 2(g_{\mu\nu} \Box - \nabla_\mu \nabla_\nu) G(L_m) F_2(R) - f_2(R) [K(L_m) L_m - G(L_m)] g_{\mu\nu},$$

where $\Box = g^{\mu\nu} \nabla_\mu \nabla_\nu$, $F_i(R) = df_i(R)/dR$ ($i = 1, 2$) and $K(L_m) = dG(L_m)/dL_m$ respectively. The energy-momentum tensor of matter is defined as:

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta (\sqrt{-g} L_m)}{\delta g^{\mu\nu}}.$$

In this class of models, the energy-momentum tensor of matter is generally not conserved due to the appearance of an extra force[12].

3. ENERGY CONDITIONS IN THE GENERALIZED $f(R)$ GRAVITY MODELS WITH ARBITRARY COUPLING BETWEEN MATTER AND GEOMETRY

3.1. The Raychaudhuri Equation

Many models of $f(R)$ gravity have been proposed, which can be restricted by imposing the so-called energy conditions[13]. These energy conditions were used in different contexts to derive general results that hold for a variety of situations. Under these energy conditions, one allows not only to establish gravity which remains attractive, but also to keep the demands that the energy density is positive and cannot flow faster than light. Below, we simply review the Raychaudhuri equation which is the physical origin of the NEC and the SEC[14].
In the case of a congruence of timelike geodesics defined by the vector field $u^\mu$, the Raychaudhuri equation is given by

$$\frac{d\theta}{d\tau} = -\frac{1}{3} \theta^2 - \sigma_\mu^\nu \sigma^{\mu\nu} + \omega_\mu^\nu \omega^{\mu\nu} - R_{\mu\nu} u^\mu u^\nu,$$  \hspace{1cm} (4)

where $R_{\mu\nu}$, $\theta$, $\sigma_{\mu\nu}$ and $\omega_{\mu\nu}$ are the Ricci tensor, the expansion parameter, the shear and the rotation associated with the congruence respectively. While in the case of a congruence of null geodesics defined by the vector field $k^\mu$, the Raychaudhuri equation is given by

$$\frac{d\theta}{d\tau} = -\frac{1}{2} \theta^2 - \sigma_\mu^\nu \sigma^{\mu\nu} + \omega_\mu^\nu \omega^{\mu\nu} - R_{\mu\nu} k^\mu k^\nu.$$  \hspace{1cm} (5)

From above expressions, it is clear that the Raychaudhuri equation is purely geometric and independent of the gravity theory. In order to constrain the energy-momentum tensor by the Raychaudhuri equation, one can use the Ricci tensor from the field equations of gravity to make a connection. Namely, through the combination of the field equations of gravity and the Raychaudhuri equation, one can obtain physical conditions for the energy-momentum tensor. Since $\sigma^2 \equiv \sigma_{\mu\nu} \sigma^{\mu\nu} \geq 0$ (the shear is a spatial tensor) and $\omega_{\mu\nu} = 0$ (hypersurface orthogonal congruence), from Eqs. (4) and (5), the conditions for gravity to remain attractive ($d\theta/d\tau < 0$) are

$$R_{\mu\nu} u^\mu u^\nu \geq 0 \quad \text{SEC},$$  \hspace{1cm} (6)

$$R_{\mu\nu} k^\mu k^\nu \geq 0 \quad \text{NEC}.$$  \hspace{1cm} (7)

Thus by means of the relationship (6) and Einstein’s equation, one obtains

$$R_{\mu\nu} u^\mu u^\nu = (T_{\mu\nu} - T g_{\mu\nu}) u^\mu u^\nu \geq 0,$$  \hspace{1cm} (8)

where $T_{\mu\nu}$ is the energy-momentum tensor and $T$ is its trace. If one considers a perfect fluid with energy density $\rho$ and pressure $p$,

$$T_{\mu\nu} = (\rho + p) U_\mu U_\nu - p g_{\mu\nu},$$  \hspace{1cm} (9)

the relationship (8) turns into the well-known SEC of Einstein’s theory, i.e.,

$$\rho + 3p \geq 0.$$  \hspace{1cm} (10)

Similarly, by using the relationship (7) and Einstein’s equation, one has

$$T_{\mu\nu} k^\mu k^\nu \geq 0.$$  \hspace{1cm} (11)

Then considering Eq.(9), the familiar NEC of general relativity can be reproduced as:

$$\rho + p \geq 0.$$  \hspace{1cm} (12)

### 3.2. Energy conditions

The Einstein tensor resulting from the field equations (2) is

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = T^{\mu\nu}_{\text{eff}},$$  \hspace{1cm} (13)
where the effective energy-momentum tensor $T_{\mu\nu}^{eff}$ is defined as follows:

$$T_{\mu\nu}^{eff} = \frac{1}{f_1 + 2Gf_2} \left\{ \frac{7}{2}g_{\mu\nu}[f_1 - (f_1' + 2Gf_2')R] - (g_{\mu\nu}\Box - \nabla_\mu \nabla_\nu)f_1' 
- 2(g_{\mu\nu}\Box - \nabla_\mu \nabla_\nu)Gf_2' - f_2(G'L_m - G)g_{\mu\nu} + 2G'T_{\mu\nu} \right\},$$  

(14)

where $f_i = f_i(R)$ ($i = 1, 2$), $G = G(L_m)$ and the prime denotes differentiation with respect to the Ricci scalar $R$ and the Lagrangian density $L_m$ respectively. Contracting the above equation, we have

$$T^{eff} = \frac{1}{f_1 + 2Gf_2} \left\{ 2[f_1 - (f_1' + 2Gf_2')R] - 3f_1' 
- 6Gf_2' - 4f_2(G'L_m - G) + 2f_2'G'T \right\},$$  

(15)

where $T = g^{\mu\nu}T_{\mu\nu}$. Thus, we can write $R_{\mu\nu}$ in terms of an effective stress-energy tensor and its trace, i.e.,

$$R_{\mu\nu} = T_{\mu\nu}^{eff} - \frac{1}{2}g_{\mu\nu}T^{eff}.$$  

(16)

In order to keep gravity attractive, besides the expressions (6) and (7), the following additional condition should be required

$$\frac{f_2G'}{f_1' + 2Gf_2'} > 0.$$  

(17)

Note that this condition is independent of the ones derived from the Raychaudhuri equation (i.e., the expressions (6) and (7)), and only relates to an effective gravitational coupling.

The FRW metric is chosen as:

$$ds^2 = dt^2 - a^2(t)ds_3^2,$$  

(18)

where $a(t)$ is the scale factor and $ds_3^2$ contains the spatial part of the metric. Using this metric, we can obtain $R = -6(2H^2 + \dot{H})$, where $H = \dot{a}(t)/a(t)$ is the Hubble expansion parameter, and $\Gamma^0_{\mu\nu} = a(t)\dot{a}(t)\delta_{\mu\nu}$ ($\mu, \nu \neq 0$) are the components of the affine connection.

By using the relationship (6) and Eq. (16), the SEC can be given as:

$$T_{\mu\nu}^{eff}u^\mu u^\nu - \frac{1}{2}T^{eff} \geq 0,$$  

(19)

where we have used the condition $g_{\mu\nu}u^\mu u^\nu = 1$. Taking the energy-momentum tensor $T_{\mu\nu}$ to be a perfect fluid (i.e., Eq. (9)) and considering the condition (17), we obtain

$$\rho + 3p - \frac{1}{f_2G'}[f_1 - (f_1' + 2Gf_2')R] + 3\frac{f_2G'}{f_1G'}(H\dot{R} + \dot{H}) 
+ 3\frac{f_2G'}{f_1G'}\dot{R}^2 + 6\frac{1}{f_2G'}(G'L_m f_2' + L_m'G'f_2' + 2f_2'RG'L_m) 
+ f_2''R^2G' + f_2'R\dot{G}) + 6\frac{H}{f_2G'}(G'L_m f_2'' + f_2''RG) 
+ \frac{1}{f_2'}(G'L_m - G) \geq 0,$$  

(20)

where the dot denotes differentiation with respect to cosmic time. This is the SEC in $f(R)$ gravity with arbitrary coupling between matter and geometry.

The NEC in $f(R)$ gravity with arbitrary coupling between matter and geometry can be expressed as:

$$T_{\mu\nu}^{eff}k^\mu k^\nu \geq 0.$$  

(21)
Thus by extending this approach to equivalent results can be obtained by taking the transformations coupling between matter and geometry, which are just the results given in Ref.\[15\]. By the same method as the SEC, the above relationship can be changed into 

\n
\[ \rho + p + \left( H \dot{R} + \ddot{R} \right) f_2^G + \frac{1}{f_2^G} \dot{R}^2 + \frac{2}{f_2^G} \left( G'' f_2^G \right) \frac{f_2^G}{G' L_m^2} + \left( \dot{L}_m G' f_2^G + 2 f_2^G \ddot{R} G' L_m + f_2^G \dot{R}^2 G + f_2^G \dot{R} G \right) - \frac{2 H}{f_2^G} \left( G' L_m f_2^G + f_2'' \ddot{R} G \right) \geq 0. \]  

\[ (22) \]

From above discussions, it is worth stressing that by taking \( G(L_m) = L_m \) and rescaling the function \( f_2(R) \) as \( 1 + \lambda f_2(R) \) in expressions (20) and (22), we can obtain the SEC and the NEC in \( f(R) \) gravity with non-minimal coupling between matter and geometry, which are just the results given in Ref.\[15\]. While by setting \( f_2(R) = 1 \) and \( G(L_m) = L_m \), we can derive the SEC and the NEC in \( f(R) \) gravity with non-coupling, which are just the same as the ones in Ref.\[16\]. Furthermore, when \( f_1(R) = R \), the SEC and the NEC in general relativity, i.e., \( \rho + 3p \geq 0 \) and \( \rho + p \geq 0 \), can be reproduced.

Note that the above expressions of the SEC and the NEC are directly derived from Raychaudhuri equation. However, equivalent results can be obtained by taking the transformations \( \rho \to \rho^{eff} \) and \( p \to p^{eff} \) into \( \rho + 3p \geq 0 \) and \( \rho + p \geq 0 \). Thus by extending this approach to \( \rho - p \geq 0 \) and \( \rho \geq 0 \), we will give the DEC and the WEC in \( f(R) \) gravity with arbitrary coupling between matter and geometry in the following.

By means of Eqs. (14) and (18), the effective energy density and the effective pressure can be derived as follows:

\n
\[ \rho^{eff} = \frac{1}{f_2^G + 2G f_2^G} \left( \frac{1}{2} \left[ f_2^G f_2'' + (f_2' + 2G f_2^G) R \right] - 3H \dot{R} f_2'' + 6H (G' L_m f_2^G) \right) - f_2 (G' L_m - G) + f_2 G' \rho, \]

\[ (23) \]

\[ p^{eff} = \frac{1}{f_2^G + 2G f_2^G} \left( -\frac{1}{2} \left[ f_2^G f_2'' + (f_2' + 2G f_2^G) R \right] + (2H \dot{R} + \ddot{R}) f_2'' + f_2'' R^2 \right. \]

\[ \left. + 2(G'' L_m f_2^G + \dot{L}_m G' f_2^G + 2 f_2'' \ddot{R} G' L_m + f_2'' \dot{R}^2 G + f_2'' \dot{R} G) - 4H (G' L_m f_2^G + f_2'' \ddot{R} G) + f_2 (G' L_m - G) + f_2 G' \rho \right). \]

\[ (24) \]

Then, the corresponding DEC and WEC in \( f(R) \) gravity with arbitrary coupling can be respectively written as:

\n
\[ \rho - p + \frac{1}{f_2^G} \left[ f_2^G f_2'' + (f_2' + 2G f_2^G) R \right] - (5H \dot{R} + \ddot{R}) f_2'' f_2^G - \frac{1}{f_2^G} \dot{R}^2 - \frac{1}{f_2^G} \left( G'' L_m^2 f_2^G + \dot{L}_m G' f_2^G + 2 f_2'' \ddot{R} G' L_m + f_2'' \dot{R}^2 G + f_2'' \dot{R} G \right) - \frac{10H}{f_2^G} (G' L_m f_2^G + f_2'' \ddot{R} G) - \frac{2}{G'} (G' L_m - G) \geq 0, \]

\[ (25) \]

\[ \rho + \frac{1}{f_2^G} \left[ f_2^G f_2'' + (f_2' + 2G f_2^G) R \right] - 3H \dot{R} f_2'' f_2^G - \frac{1}{f_2^G} \dot{R}^2 - \frac{1}{f_2^G} (G' L_m f_2^G + f_2'' \ddot{R} G) - \frac{2}{G'} (G' L_m - G) \geq 0. \]

\[ (26) \]

We show that by taking \( G(L_m) = L_m \) and rescaling the function \( f_2(R) \) as \( 1 + \lambda f_2(R) \), above expressions are the DEC and the WEC in \( f(R) \) gravity with non-minimal coupling between matter and geometry, which are just the same as the ones in Ref.\[15\]. While by setting \( f_2(R) = 1 \) and \( G(L_m) = L_m \), the results given by us are the DEC and the WEC in \( f(R) \) gravity with non-coupling, which are consistent with the results given in Ref.\[16\]. Furthermore, when \( f_1(R) = R \), the DEC and the WEC in general relativity, i.e., \( \rho - p \geq 0 \) and \( \rho \geq 0 \), can be reproduced.
3.3. Energy Conditions for a Class of Models

In order to get some insight on the meaning of the above energy conditions, we consider a specific type of models where \( f_1(R) \) and \( f_2(R) \) are taken as
\[
\begin{align*}
  f_1(R) &= R + \epsilon R^n, \\
  f_2(R) &= \alpha R^m,
\end{align*}
\]
In the FRW cosmology, the energy conditions can be written as
\[
\frac{\dot{\epsilon}}{\alpha} | R |^{n-1} \left\{ \frac{2\dot{\alpha}}{\epsilon} [G(L_m)C_m + A] | R |^{m-n} + C_n \right\} \geq B,
\]
where \( A, B \) and \( C_{m,n} \) depend on the energy condition under study and we take \( \dot{\epsilon} = (1)^n \epsilon \) and \( \dot{\alpha} = (1)^n \alpha \) due to the fact that for a FRW metric one has \( R < 0 \). For the SEC, one finds
\[
A^{SEC} = G'(L_m)[L_m + 3m \dot{R}^{-1}(L_m + H L_m) + 6m \dot{L}_m \dot{R}^{-2}(m - 1)] + 3mL_m^2 G''(L_m)R^{-1},
\]
\[
B^{SEC} = -\rho - 3p,
\]
\[
C^{SEC}_n = (n - 1)[3\dot{R}nR^{-2} + 1 + 3HnR^{-2}\dot{R} + 3nR^{-3}\dot{R}^2(n - 2)].
\]
For the NEC, one obtains
\[
A^{NEC} = [m \dot{L}_m G'(L_m)R^{-1} - Hm \dot{L}_m G'(L_m)R^{-1}]
+ 2m \dot{L}_m \dot{R}G''(L_m)R^{-2}(m - 1) + m \dot{L}_m^2 G''(L_m)R^{-1},
\]
\[
B^{NEC} = -\rho + p,
\]
\[
C^{NEC}_n = (n - 1)[\ddot{R}nR^{-2} + H \ddot{R}^{-2}\dot{R} + R^{-3}\dot{R}^2(n - 2)].
\]
For the DEC, one has
\[
A^{DEC} = G'(L_m)[- \dot{L}_m - mL_m \dot{R}^{-1} - 5Hm \dot{L}_m \dot{R}^{-1}
+ (1 - m)2m \dot{L}_m \dot{R}R^{-2}] - mL_m^2 G''(L_m)R^{-1},
\]
\[
B^{DEC} = -\rho - p,
\]
\[
C^{DEC}_n = (1 - n)[\ddot{R}nR^{-2} + 1 + 5HnR^{-2}\dot{R} + nR^{-3}\dot{R}^2(n - 2)].
\]
Finally, for the WEC, one gets

\[ A^{WEC} = -L_m - 6Hm\dot{L}_m R^{-1}, \quad (32a) \]

\[ B^{WEC} = -\rho, \quad (32b) \]

\[ C^{WEC}_n = (1 - n)(\frac{1}{2} + 3HnR^{-2}\dot{R}). \quad (32c) \]

Given these definitions, the study of all the energy conditions can be performed by satisfying the inequality (28). Note that all the energy conditions depend on the geometrical parameters. It means that for different models, the energy conditions can be satisfied by choosing them properly.

For models given by Eq.(27), the condition for keeping gravity attractive (GA), i.e. inequality (17), also can be obtained from inequality (28) by taking

\[ A^{GA} = \frac{1}{2m} | R |^{-m}, \quad (33a) \]

\[ B^{GA} = 0, \quad (33b) \]

\[ C^{GA}_n = nR^{-1}. \quad (33c) \]

This means that inequality (28) also stands for the condition that ensures gravity remains attractive for models given by Eq.(27).

In the following, we use energy conditions to restrict a special \( f(R) \) model also in the FRW cosmology. The Ricci scalar \( R \) and its derivatives can be expressed by the parameters of the deceleration \( q \), the jerk \( j \) and the snap \( s \)[17], namely,

\[ R = -6H^2(1 - q), \quad (34a) \]

\[ \dot{R} = -6H^3(j - q - 2), \quad (34b) \]

\[ \ddot{R} = -6H^4(s + q^2 + 8q + 6), \quad (34c) \]

where

\[ q = -\frac{1}{H^2} \frac{\ddot{a}}{a}, \quad j = \frac{1}{H^3} \frac{\dddot{a}}{a}, \quad and \quad s = \frac{1}{H^4} \frac{\dddot{a}}{a}. \quad (35) \]

Thus, the energy conditions (20), (22), (25) and (26) can be rewritten as:

\[ \rho + 3p - \frac{1}{f(R)} \left[ f_1 + 6H^2(f_1 + 2Gf_2^2)(1 - q) \right] - 18\frac{f_2''}{f_2^1} H^4(j + s + q^2 + 7q + 4) \\
+ 108\frac{f_2'''}{f_2^1} H^6(j - q - 2)^2 + 6\frac{1}{f_2^1} \left[ G''\dot{L}_m f_2^2 + \dot{L}_m G'f_2^2 - 12f_2''H^3(j - q - 2) \right] G' \dot{L}_m + 36f_2''H^6(j - q - 2)^2 - 6f_2''H^4(s + q^2 + 8q + 6)G + 6\frac{H}{f_2^1} \\
\left[ G'\dot{L}_m f_2^2 - 6f_2''H^3(j - q - 2)G \right] + \frac{2}{f_2^1}(G'\dot{L}_m - G) \geq 0, \quad \text{(SEC)} \quad (36a) \]
\[
\rho + p - 6H^4(j + s + q^2 + 7q + 4) \frac{f''}{f'_{2,GR}} + 36 \frac{f'''}{f'_{2,GR}} H^6(j - q - 2)^2 + \frac{2}{f'_{2,GR}} [G''L_m f'_2 + \ddot{L}_m G' f'_2 - 12 f''_2 H^3(j - q - 2) G' L_m + 36 f''_2 H^6(j - q)
- 2j^2 G - 6 f''_2 H^4(s + \sqrt{q} + 8q + 6G)] - \frac{2H}{f'_{2,GR}} [G' L_m f'_2 - 6 f''_2 H^3(j - q)
- 2jG] \geq 0, \quad (N \text{EC})
\]

\[
\rho - p + \frac{1}{f'_{2,GR}} [f_1 + 6H^2(f'_1 + 2G f'_2)(1 - q)] + 6H^4(j + s + q^2 + 3q)
- 4) \frac{f''}{f'_{2,GR}} - 36 \frac{f'''}{f'_{2,GR}} H^6(j - q - 2)^2 - \frac{2}{f'_{2,GR}} [G''L_m f'_2 + \ddot{L}_m G' f'_2 - 12 f''_2 H^3
(j - q - 2) G' L_m + 36 f''_2 H^6(j - q - 2) G - 6 f''_2 H^4(s + \sqrt{q} + 8q + 6G)]
- \frac{10H}{f'_{2,GR}} [G' L_m f'_2 - 6 f''_2 H^3(j - q - 2) G] - \frac{2}{f'_{2,GR}} (G' L_m - G) \geq 0, \quad (D \text{EC})
\]

\[
\rho + \frac{1}{f'_{2,GR}} [f_1 + 6H^2(f'_1 + 2G f'_2)(1 - q)] + 18H^4(j - q - 2) \frac{f''}{f'_{2,GR}} - 6H \frac{1}{f'_{2,GR}} [G' L_m f'_2 - 6 f''_2 H^3(j - q - 2) G] - \frac{2}{f'_{2,GR}} (G' L_m - G) \geq 0. \quad (W \text{EC})
\]

To exemplify how to use these energy conditions to constrain the \(f(R)\) theories of gravity, we consider a special model with \(f_1(R) = R, f_2(R) = \alpha R^n\) and \(G(L_m) = L_m = -\rho[18]\). Since there has been no reliable measurement for the snap parameter \((s)\) up to now, we only focus on the WEC. Under the requirement \(f'(R) > 0\) for all \(R\) and taking \(H_0 = 70.5[19]\), the WEC (36d) in this particular case is

\[
0.3Bn^2 - 0.3n(1 + B) + 1 \geq 0,
\]

where \(B = (j - q - 2)/(1 - q)^2\).

From the above expression, it is easy to see that the coefficient \(\alpha\) is arbitrary and the value of the index \(n\) depends on \(B\). Taking \(q_0 = -0.81 \pm 0.14\) and \(j_0 = 2.16^{+0.81}_{-0.74}[20]\) (the subscript 0 denotes the present value), we can give the present range of \(B\) is 0.03 ≤ \(B_0\) ≤ 0.5. By calculations and analysis, the results of the expression (37) are as follows: when the real solution exists, the range of \(B\) is either \(B \leq \frac{17 - 2\sqrt{70}}{3}\) or \(B \geq \frac{17 + 2\sqrt{70}}{3}\). Considering 0.03 ≤ \(B_0\) ≤ 0.5, we find the range of \(B\) is 0.03 ≤ \(B\) ≤ \(\frac{17 - 2\sqrt{70}}{3}\) and the index \(n\) are 7.36 ≤ \(n_+\) ≤ 30.716 and 3.617 ≤ \(n_-\) ≤ 5.263. However, when there is not any real solution, the range of \(B\) is \(\frac{17 - 2\sqrt{70}}{3}\) < \(B\) ≤ 0.5 and the index \(n\) can be taken as any real number.

We can point out that for the model of \(f_1(R) = R + \alpha R^n, f_2 = 1\) and \(G(L_m) = L_m\), the corresponding results to the WEC given by us are just the same as the ones in Ref.[16],

4. THE INSTABILITY OF GENERALIZED \(f(R)\) GRAVITY MODELS WITH ARBITRARY MATTER-GEOMETRY COUPLING

Modified gravity must be stable at the classical and quantum level. There are in principle several kinds of instabilities to consider[21]. Dolgov-Kawasaki instability[22] is one of them. Below, we will focus on this instability and generalize to \(f(R)\) gravity models with arbitrary matter-geometry coupling.

The trace of the field equation (2) is
\[ R + \frac{1}{f_1} \{ 2[f_1 - (f_1' + 2Gf_2')R] - 3\Box f_1' - 6\Box Gf_2' - 4f_2(G'L_m - G) \} = \frac{1}{f_1 + 2\epsilon\phi'} f_2 G'T, \]  
\tag{38} \]

where \( T = g^{\mu\nu} T_{\mu\nu} \). As usual, we take \( f_1(R) \) as \( f_1(R) = R + \epsilon \varphi(R) \), where \( \epsilon \) must be small to compatibility with Solar System experiment\([23]\). Following\([22]\), we expand the space-time quantities of interest as the sum of a background with constant curvature and a small perturbation: \( R = R_0 + R_1, \) \( T = T_0 + T_1, \) \( L = L_0 + L_1, \) and the space-time metric can locally be approximated by \( g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \), where \( \eta_{\mu\nu} \) is the Minkowski metric. In fact, this is a local expansion over small space-time regions that are locally flat. Accordingly, \( f_1(R) = R_0 + R_1 + \epsilon \varphi(R_0) + \epsilon \varphi'(R_0)R_1 + \ldots, \) \( f_1'(R) = 1 + \epsilon \varphi'(R_0) + \epsilon \varphi''(R_0)R_1 + \ldots \) and the linearized version of the trace equation (38) in the perturbations yields

\[
[6G(L_m)f''_0(R_0) + 3\epsilon \varphi''(R_0)]\tilde{R}_1 - 3\epsilon \varphi''(R_0)\nabla^2 R_1 + [12\tilde{L}_0G'(L_m)f''_2(R_0)
+12\tilde{L}_1G'(L_m)f''_2(R_0) + 12G(L_m)\tilde{R}_0 f''_2(R_0) + 6\epsilon \tilde{R}_0 \varphi'''(R_0)]\tilde{R}_1
-6\epsilon \varphi''(R_0)\nabla R_1 \cdot \nabla R_0 - [1 - 2G(L_m)f''_2(R_0) + \epsilon \varphi'(R_0) - R_0 \epsilon \varphi''(R_0)]
-3\tilde{R}_0 \epsilon \varphi''(R_0) + 3\epsilon \varphi''(R_0)\nabla^2 R_0] R_1 = 6f'_2(R_0)\nabla^2 G(L_m)
-f_2(R_0)(4L_1 - T_1)G'(L_m) - 6(L_0 + \tilde{L}_1)f'_2(R_0)G'(L_m) - [6G(L_m)f''_0(R_0)
+3\epsilon \varphi''(R_0)]\tilde{R}_0 - 12(\tilde{L}_0 + L_1)\tilde{R}_0 G'(L_m)f''_2(R_0) - 12\tilde{L}_0 \tilde{L}_1 f'_2(R_0)G''(L_m)
+3\epsilon \varphi''(R_0)]\tilde{R}_0, \]  
\tag{39} \]

where \( \nabla, \nabla^2 \) and overdot denote the gradient, Laplacian operators in Euclidean three-dimensional space and differentiation with respect to time, respectively, and the zero order equation

\[
f_2(R_0)T_0 G'(L_m) = -4f_2(R_0)G(L_m) - 2\epsilon \varphi(R_0) + 4G(L_m)R_0 f'_2(R_0)
+4f_2(R_0)L_0 G'(L_m) + 2R_0 \epsilon \varphi'(R_0) - R_0 - 2G(L_m)R_0 f'_2(R_0) - R_0 \epsilon \varphi'(R_0) \]  
\tag{40} \]

has been used. By further calculation, the effective mass \( m_{eff} \) of the dynamical degree of freedom \( R_1 \) can be given as

\[
m_{eff}^2 = [6G(L_m)f''_0(R_0) + 3\epsilon \varphi''(R_0)]^{-1}[2G(L_m)f'_2(R_0) - 1 - \epsilon \varphi'(R_0)]
+R_0 \epsilon \varphi''(R_0) + 3\tilde{R}_0 \epsilon \varphi''(R_0) - 3\epsilon \varphi''(R_0)\nabla^2 R_0]. \]  
\tag{41} \]

The dominant term on the right hand side is \([6G(L_m)f''_0(R_0) + 3\epsilon \varphi''(R_0)]^{-1} \) and the effective mass squared must be non-negative for stability. Therefore, \( f_1'(R) + 2G(L_m)f''_2(R) \geq 0 \) is the stability criterion for the generalized \( f(R) \) gravity models with arbitrary matter-geometry coupling against Dolgov-Kawasaki instabilities.

Note that by taking \( G(L_m) = L_m \) and rescaling the function \( f_2(R) \) as \( 1 + \lambda f_2(R) \), this criterion is the Dolgov-Kawasaki criterion in \( f(R) \) gravity with non-minimal coupling between matter and geometry, which is just the same as the one in Ref.\([24]\). While by setting \( f_2(R) = 1 \) and \( G(L_m) = L_m \), the results given by us is the Dolgov-Kawasaki criterion in \( f(R) \) gravity with non-coupling, which is consistent with the results given in Ref.\([25]\).

For models given by Eq.\((27)\), the Dolgov-Kawasaki criterion is

\[
\dot{\epsilon} n(n - 1) R^n + 2G(L_m) \dot{\alpha} m(m - 1) R^m \geq 0 \]  
\tag{42} \]

where

\[
\dot{\epsilon} = \begin{cases} 
(-1)^n \epsilon, & \text{if } R < 0 \\
\epsilon, & \text{if } R > 0 \end{cases}, \quad \dot{\alpha} = \begin{cases} 
(-1)^n \alpha, & \text{if } R < 0 \\
\alpha, & \text{if } R > 0 \end{cases}. \]  
\tag{43} \]
It is clear that the stability criterion of these models don’t relate to the values of $j$ and $s$, and the space-time only depends on $R$. When $n = m$ the inequality (42) gives, $\epsilon + 2G(L_m)\alpha \geq 0$.

It is worth stressing that the inequality (42) also can be obtained from the inequality (28) by taking

$$A^{DK} = 0,$$

(44a)

$$B^{DK} = 0,$$

(44b)

$$C_n^{DK} = n(n-1)\hat{\alpha} | R |^m G'(L_m).$$

(44c)

From the above discussions, we find that for models given by Eq.(27), the energy conditions, the Dolgov-Kawasaki criterion and the condition for attractive gravity have the the same type of inequalities, but note that they are independent each other.

From the inequality (42), we see that the viability of the model with respect to the Dolgov-Kawasaki instability criterion will depend not only on the value of the constants $\epsilon$ and $\alpha$, but also on the space-time metric under consideration. This fact will give further constraints on the Ricci scalar.

5. SUMMARY

So far, we have derived the energy conditions (SEC, NEC, DEC, WEC) in the generalized $f(R)$ gravity models with arbitrary coupling between matter and geometry. For the SEC and the NEC, the Raychaudhuri equation, which is the physical origin of them, has been used. From the derivation, we found that equivalent results can be obtained by taking the transformations $\rho \rightarrow \rho^{eff}$ and $p \rightarrow p^{eff}$ into $\rho + 3p \geq 0$ and $\rho + p \geq 0$. Thus by extending this approach to $\rho - p \geq 0$ and $\rho \geq 0$, the DEC and the WEC in the generalized $f(R)$ gravity models with arbitrary coupling between matter and geometry can be obtained. The condition to keep gravity attractive and the Dolgov-Kawasaki criterion in the generalized $f(R)$ gravity models have been also given, but the approaches of deriving them are different.

It is worth noting that the energy conditions and the Dolgov-Kawasaki criterion obtained in this paper are quite general, which include the corresponding results given in Refs.[15, 16, 24, 25] as well as in the general relativity (GR) as special cases.

Furthermore, in order to get some insight on the meaning of these energy conditions and the Dolgov-Kawasaki criterion, we have applied them to a class of models. In these models the energy conditions, the Dolgov-Kawasaki criterion and the condition for attractive gravity have the the same type of inequalities. By analysis, we find that the Dolgov-Kawasaki instability criterion depends not only on the value of the constants $\epsilon$ and $\alpha$ but also on the space-time metric under consideration.

In addition, we have considered the special model with $f_1(R) = R$, $f_2(R) = \alpha R^n$ and $G(L_m) = L_m = -\rho$. By virtue of the WEC and the present astrophysical observations, the values of parameters $\alpha$ and $n$ can be constrained in this model. Of course, we will continue to study other models of $f(R)$ gravity with arbitrary coupling between matter and geometry in our following investigations.
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