An enlargement of filtration formula with application to progressive enlargement with multiple random times

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Abstract

Given a reference filtration $\mathcal{F}$, we develop in this work a generic method for computing the semimartingale decomposition of $\mathcal{F}$-martingales in some specific enlargements of $\mathcal{F}$. This method is then applied to the study of progressive enlargement with multiple non-ordered random times, for which explicit decompositions can be obtained under the absolute continuity condition of Jacod.
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1 Introduction

In this paper, we work on a filtered probability space \((\Omega, \mathcal{F}, \mathbb{P})\) endowed with a filtration \(\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}\) satisfying the usual conditions with \(\mathcal{F}_\infty \subset \mathcal{F}\). For a semi-martingale \(X\) and a predictable process \(H\), we denote by \(H \cdot X\) the stochastic integral of \(H\) with respect to \(X\), whenever it is well defined. The optional (resp. predictable) \(\sigma\)-algebra generated by a filtration \(\mathbb{F}\) is denoted by \(\mathcal{O}(\mathbb{F})\) (resp. \(\mathcal{P}(\mathbb{F})\)). For the ease of language, for any \(\mathbb{F}\)-special semimartingale \(X\), the \(\mathbb{F}\)-predictable process of finite variation \(A\) in its \(\mathbb{F}\)-semimartingale decomposition \(X = M + A\) is called the \(\mathbb{F}\)-drift of \(X\).

Given a reference filtration \(\mathbb{F}\) and a filtration \(\mathbb{G}\) such that \(\mathbb{F} \subset \mathbb{G}\), one aim in the theory of enlargement is to study whether the hypothesis \((H')\) is satisfied between \(\mathbb{F}\) and \(\mathbb{G}\), i.e., whether any \(\mathbb{F}\)-martingale is a \(\mathbb{G}\)-semi-martingale. Traditionally, one attempts this directly by looking only at the filtrations \(\mathbb{F}\) and \(\mathbb{G}\). In the current literature, this direct approach has been used in the study of progressive (initial) enlargement of \(\mathbb{F}\) with a random time (a non negative random variable), including (but not limited to) the works of Jeulin [9, 10], Jeulin and Yor [15] and Jacod [8].

However, it is often difficult to study the hypothesis \((H')\) directly between \(\mathbb{F}\) and \(\mathbb{G}\), and, in some cases one can take advantage of the following result of Stricker [19].

Proposition 1.1. Let \(\mathbb{F}, \mathbb{G}\) and \(\hat{\mathbb{F}}\) be filtrations such that \(\mathbb{F} \subset \mathbb{G} \subset \hat{\mathbb{F}}\). If the hypothesis \((H')\) is satisfied between \(\mathbb{F}\) and \(\hat{\mathbb{F}}\), then the hypothesis \((H')\) is also satisfied between \(\mathbb{F}\) and \(\mathbb{G}\).

In addition, if the \(\hat{\mathbb{F}}\)-semimartingale decomposition of \(\mathbb{F}\)-martingales is known then (at least theoretically) one can make use of the \(\mathbb{G}\)-optional and \(\mathbb{G}\)-dual predictable projections to find the \(\mathbb{G}\)-semimartingale decomposition of \(\mathbb{F}\)-martingales.

In the current literature, Proposition 1.1 has been used in Jeanblanc and Le Cam [11], Callegaro et al. [1] and Kchia et al. [14], to study the relationship between the filtrations \(\mathbb{F} \subset \mathbb{F}' \subset \mathbb{G}'\), where \(\mathbb{F}'\) (resp. \(\mathbb{G}'\)) is the progressive (resp. initial) enlargement of \(\mathbb{F}\) with the random time \(\tau\).

If one assumes that the \(\mathbb{G}'\)-semimartingale decomposition of \(\mathbb{F}\) martingales is known, then essentially by exploiting the property that for any \(\mathbb{G}'\)-predictable process \(V^*\), the process \((V^* - V^*_\tau)1_{[\tau, \infty]}\) is \(\mathbb{F}'\)-predictable, one can derive the \(\mathbb{F}'\)-semimartingale decomposition of \(\mathbb{F}\)-martingales without calculating any \(\mathbb{F}'\)-dual predictable projection (see Lemma 2 and Theorem 3 in [14]).

In this paper, we also take advantage of the above Proposition to study the hypothesis \((H')\) between \(\mathbb{F}\) and any enlargement \(\mathbb{G}\) of \(\mathbb{F}\) that satisfies a specific structure (see Assumption 2.2). The filtration \(\mathbb{G}\) arises naturally while studying the hypothesis \((H')\) between \(\mathbb{F}\) and its progressive enlargement with minimum and maximum of two random times. However, we must stress that, although, we apply our result to the study of progressive enlargement with random times, our setup is different from the
usual progressive enlargement framework, and the progressive enlargement with a single random time cannot be retrieved from our setting as a specific case. Therefore, unlike the case studied in [1], [11] and [14], one cannot exploit the specific structure between \textit{progressive} and \textit{initial} enlargement with a single random time in computing the \(G\)-semimartingale decomposition. (for details, see section 2.2.)

The present work is divided into two parts. In section 2, a generic technique for computing the semimartingale decomposition of \(F\)-martingales in an enlargement of \(F\) satisfying a specific structure (namely a filtration \(G\) satisfying Assumption 2.2) is developed. We construct a filtration \(\hat{F}\) (\textit{the direct sum filtration}) that satisfies \(G \subset \hat{F}\) and we show that the hypothesis \((H')\) holds between \(F\) and \(G\) by showing that the hypothesis \((H')\) holds between \(F\) and \(\hat{F}\). The explicit \(G\)-decomposition formula is fostered in Theorem 2.13, which is obtained from projecting the \(\hat{F}\)-semimartingale decomposition computed in Theorem 2.3. In section 3, we apply the results of section 2 to the study of progressive enlargement of \(F\) with multiple non-ordered random times. Under the additional assumption that the joint \(F\)-conditional distribution of the given family of random times is absolutely continuous with respect to a non-atomic measure, the semimartingale decomposition is fully explicit (under mild integrability conditions). Progressive enlargement with multiple ordered random times have also been studied in [5] and [12], where the authors assume that the random times are ordered with random marks. Our approach is different from the method presented in [5] and [12], as it is generic and does not depend on the techniques developed in the literature on progressive enlargement with random times.

\section{The setup and the main results}

We work on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) endowed with a filtration \(\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}\) satisfying the usual conditions. Keeping Proposition 1.1 in mind, for a filtration \(\mathcal{G}\) that satisfies some specific assumptions, we construct in section 2.1, a filtration \(\hat{\mathbb{F}}\) such that \(\mathbb{F} \subset \mathcal{G} \subset \hat{\mathbb{F}}\) and that the hypothesis \((H')\) is satisfied between \(\mathbb{F}\) and \(\mathcal{G}\). In Theorem 2.3, we give the \(\hat{\mathbb{F}}\)-semimartingale decomposition of \(\mathbb{F}\)-local martingales. Then in section 2.2 the \(G\)-semimartingale decomposition of \(F\)-martingales is deduced from Theorem 2.3.

Before proceeding, let us introduce the following notion, which will be important throughout this section.

**Definition 2.1.** Given two \(\sigma\)-algebras \(\mathcal{K}, \mathcal{H}\) and a set \(D\), we write \(\mathcal{K} \cap D \subset \mathcal{H} \cap D\), if for every \(\mathcal{K}\)-measurable set \(X\), there exists a \(\mathcal{H}\)-measurable set \(Y\) such that \(X \cap D = Y \cap D\). We say that the \(\sigma\)-algebras \(\mathcal{K}\) and \(\mathcal{H}\) coincide on \(D\), if \(\mathcal{K} \cap D \subset \mathcal{H} \cap D\) and \(\mathcal{K} \cap D = \mathcal{H} \cap D\), in which case, we write \(\mathcal{K} \cap D = \mathcal{H} \cap D\).

Let us illustrate this notion in the classical cases of the initial enlargement \(\mathcal{G}^r = \)
setting, by using Assumption 2.2, we essentially partition the space \( \Omega \) using
\[
(\mathcal{G}_t^i)_{t \geq 0}
\]
of \( \mathbb{F} \) with the random variable \( \tau \) (this is the smallest filtration containing \( \mathbb{F} \)
which satisfies the usual conditions such that \( \tau \) is \( \mathcal{G}^\tau_0 \)-measurable) and of the progressive
enlargement \( \mathbb{F}^{\tau} = (\mathcal{F}_t^i)_{t \geq 0} \) of \( \mathbb{F} \) with the random time \( \tau \) (the smallest filtration
containing \( \mathbb{F} \) which satisfies the usual conditions such that \( \tau \) is a stopping times). In
the case of progressive enlargement with a random time, we know from Jeulin [10]
that for every \( t \), the \( \sigma \)-algebras \( \mathcal{G}_t^i \) and \( \mathcal{F}_t^i \) coincide on the set \( \{ \tau \leq t \} \) and that the
\( \sigma \)-algebras \( \mathcal{F}_t \) and \( \mathcal{F}_t^{\tau} \) coincide on the set \( \{ \tau > t \} \). This fact implies also that for
any \( \mathcal{P}(\mathbb{F}^{\tau}) \)-measurable random variable (process) \( V \) there exists a random variable
(process) \( K \) (resp. \( J \)) which is \( \mathcal{P}(\mathbb{F}) \) (resp. \( \mathcal{P}(\mathbb{G}^{\tau}) \)) measurable such that
\[
V1_{[0,\tau]} = K1_{[0,\tau]} \quad \text{and} \quad V1_{[\tau,\infty[} = J1_{[\tau,\infty[}.
\]
In other terms, the \( \sigma \)-algebras \( \mathcal{P}(\mathbb{F}^{\tau}) \) and \( \mathcal{P}(\mathbb{F}) \) coincide on the set \( [0, \tau] \), while
\( \mathcal{P}(\mathbb{F}^{\tau}) \) and \( \mathcal{P}(\mathbb{G}^{\tau}) \) coincide on \( [\tau, \infty[ \).

In this paper, we study the hypothesis \( (H') \) between \( \mathbb{F} \) and an enlargement of \( \mathbb{F} \)
denoted by \( \mathbb{G} = (\mathcal{G}_t)_{t \geq 0} \) which satisfies the following assumption.

**Assumption 2.2.** The filtration \( \mathbb{G} \) is such that there exists an \( \mathcal{F} \)-measurable partition
of \( \Omega \) given by \( \{ D_1, \ldots, D_k \} \) and a family of right-continuous filtrations \( \{ \mathbb{F}^1, \ldots, \mathbb{F}^k \} \)
where for every \( i = 1, \ldots, k \)

(i) \( \mathbb{F} \subset \mathbb{F}^i \) and \( \mathcal{F}_\infty^i \subset \mathcal{F}^i \),

(ii) for all \( t \geq 0 \), the \( \sigma \)-algebras \( \mathcal{G}_t \) and \( \mathcal{F}_t^i \) coincide on \( D_i \).

In such a case, we shall say that \( (\mathbb{F}, (\mathbb{F}^i)_{i=1,\ldots,k}, \mathbb{G}) \) satisfies Assumption 2.2 with
respect to the partition \( (D_1, \ldots, D_k) \).

The setting here is different from that of progressive enlargement studied in [1],
[11] and [14], and as mentioned above, we cannot retrieve from our framework, the
progressive enlargement with a single random time. This is because, in the case of
progressive enlargement with a single random time \( \tau \), for every \( t \geq 0 \), the space \( \Omega \)
is partitioned into \( \{ \tau > t \} \) and \( \{ \tau \leq t \} \), which are time dependent. Whereas, in our
setting, by using Assumption 2.2, we essentially partition the space \( \Omega \) using \( D_i \) for
\( i = 1, \ldots, k \), which are independent of time. In other words, unlike the progressive
enlargement case, we partition the product space \( \Omega \times [0, \infty[ \) only in ‘space’, rather
than in both ‘space’ and ‘time’.

On the other hand, although possible, we do not try to generalize the idea of
partitioning in both ‘space’ and ‘time’, to include the well studied initial and pro-
gressive enlargement setting (cf. Kchia and Protter [13]). Our purpose is different
and the fact that we do not partition in ‘time’ is used later in the computation of
\( \mathbb{G} \)-semimartingale decomposition.

Given a filtration \( \mathbb{G} \) satisfying Assumption 2.2, the goal here is to study the \( \mathbb{G} \)-
semimartingale decomposition of an \( \mathbb{F} \)-martingale \( M \), which for every \( i = 1, \ldots, k \)
has \( \mathbb{F}^i \)-semimartingale decomposition given by \( M = M^i + K^\tau \), where \( M^i \) is an \( \mathbb{F}^i \)-
local martingale and \( K^\tau \) is an \( \mathbb{F}^\tau \)-predictable process of finite variation. The main
idea of the paper is to construct from $\mathcal{G}$, a filtration $\hat{\mathcal{F}}$ (see (1)) such that we have $\mathcal{F} \subset \mathcal{G} \subset \hat{\mathcal{F}}$, and compute the $\mathcal{G}$-semimartingale decomposition of a $\mathcal{F}$-martingale from its $\hat{\mathcal{F}}$-semimartingale decomposition.

Before going into the technical details in the rest of the paper, we describe first our main results. Instead of working with the filtration $\mathcal{G}$ directly, the $\hat{\mathcal{F}}$-semimartingale decomposition of $M$ is easier to compute.

Theorem 2.3. If for every $i = 1, \ldots, k$, the $\mathcal{F}$-martingale $M$ is an $\mathcal{F}^i$-semimartingale with $\mathcal{F}^i$-semimartingale decomposition given by $M = M^i + K^i$, where $M^i$ is an $\mathcal{F}^i$-local martingale and $K^i$ is an $\mathcal{F}^i$-predictable process of finite variation, then under Assumption 2.2,

$$M - \sum_{i=1}^{k} \left( K^i \mathbf{1}_{D_i} + \frac{1}{N^-_i} \cdot \langle N^i, M \rangle^i \right)$$

is an $\hat{\mathcal{F}}$-local martingale.

Then we can retrieve the $\mathcal{G}$-semimartingale decomposition of $M$ by calculating the $\mathcal{G}$-optional projection of the above formula. We introduce, for every $i = 1, \ldots, k$,

$$\tilde{N}^i := \sigma^G(\mathbf{1}_{D_i}), \quad N^i := \sigma^{\mathcal{F}^i}(\mathbf{1}_{D_i})$$

$$\tilde{V}^i := K^i + \frac{1}{N^-_i} \cdot \langle N^i, M \rangle^i$$

where $\langle \cdot, \cdot \rangle^i$ is the $\mathcal{F}^i$-predictable bracket.

Theorem 2.4. Let $M$ be an $\mathcal{F}$-martingale, such that for every $i = 1, \ldots, k$, $M = M^i + K^i$, where $M^i$ is an $\mathcal{F}^i$-local martingale and $K^i$ an $\mathcal{F}^i$-predictable process of finite variation. Then under Assumption 2.2, for every $i = 1, \ldots, k$, the process $\tilde{N}^i$ belongs to $L^1(\psi(\tilde{V}^i))$ and

$$M - \sum_{i=1}^{k} \tilde{N}^i * \psi(\tilde{V}^i)$$

is a $\mathcal{G}$-local martingale. The operations $\psi$ and $*$ are defined below in Lemma 2.11 and (8) respectively.

Remark 2.5. We show in Lemma 2.14 that for every $i = 1, \ldots, k$, the process $\tilde{N}^i * \psi(\tilde{V}^i)$ is in fact equal to $(\tilde{V}^i \mathbf{1}_D)^{\psi,G}$. Furthermore, without going into all the details, we point out that the operations $\psi$ and $*$ are essentially introduced for technical reasons, namely, to deal with the cases where $\tilde{N}^i = \sigma^G(\mathbf{1}_{D_i})$ takes value zero. If one considers only $\tilde{N}^i * \psi(\tilde{V}^i)$ up to the first time $\tilde{N}^i$ hits zero, then $\tilde{N}^i * \psi(\tilde{V}^i)$ is simply the stochastic integral of $\tilde{N}^i$ against $\psi(\tilde{V}^i) = \frac{\psi,G(\tilde{V}^i \mathbf{1}_D)}{\psi,G(\mathbf{1}_D)}$. 

7
2.1 Direct Sum Filtration

We first construct a filtration $\hat{F}$, called the direct sum filtration, such that $\mathcal{G} \subset \hat{F}$ and study its properties. Let us define for every $t \geq 0$ the following family of sets

$$\hat{F}_t := \{ A \in \mathcal{F} | \forall i, \exists A_i^t \in \mathcal{F}_i^t \text{ such that } A \cap D_i = A_i^t \cap D_i \}. \quad (1)$$

The family $\hat{F} = (\hat{F}_t)_{t \geq 0}$ will be shown in Lemma 2.7 to be a right continuous filtration such that $\mathcal{F} \subset \mathcal{G} \subset \hat{F}$. In general, the inclusion $\mathcal{G} \subset \hat{F}$ is strict as for $i = 1, \ldots, k$, the sets $D_i$ are $\hat{F}_0$-measurable (hence $\hat{F}_t$-measurable) but not necessarily $\mathcal{G}_0$-measurable.

The constructed filtration $\hat{F}$ and the subsequent $\hat{F}$-semimartingale decomposition formula derived in Theorem 2.3 can be viewed as an extension of the study of initial enlargement in Meyer [16] and Yor [20]. In [16] and [20], the authors enlarged the base filtration $\mathcal{F}$ with a finite valued random variable $X = \sum_{i=1}^{k} a_i I_{D_i}$, where for all $i = 1, \ldots, k$, $a_i \in \mathbb{R}$ and $D_i = \{ X = a_i \}$. If we construct the filtration $\hat{F}$, by taking the partition of $\Omega$ to be $\{X = a_i\}_{i=1}^{k}$ and setting for all $i = 1, \ldots, k$, $\mathcal{F}_i^t = \mathcal{F}$, then we can recover from Proposition 2.9 the semimartingale decomposition result given in Meyer [16] and in Yor [20].

**Lemma 2.6.** For every $t \geq 0$ and $i = 1, \ldots, k$,

(i) the inclusion $D_i \subseteq \{ \mathbb{P}(D_i \mid \mathcal{F}_i^t) > 0 \}$ holds $\mathbb{P}$-a.s.

(ii) For any $\mathbb{P}$-integrable random variable $\eta$, one has

$$\mathbb{E}_{\mathbb{P}} \left( \eta 1_{D_i} \mid \hat{F}_t \right) = 1_{D_i} \frac{\mathbb{E}_{\mathbb{P}} \left( \eta 1_{D_i} \mid \mathcal{F}_i^t \right)}{\mathbb{P}(D_i \mid \mathcal{F}_i^t)}. \quad (2)$$

**Proof.** Let $t \geq 0$ be fixed and $i = 1, \ldots, k$.

(i) For $\Delta = \{ \mathbb{P}(D_i \mid \mathcal{F}_i^t) > 0 \}$, one has $\mathbb{E}_{\mathbb{P}}(1_{D_i} 1_{\Delta^c}) = 0$, which implies that $\mathbb{P}$-a.s. $D_i \subset \Delta$.

(ii) For $B \in \hat{F}_t$, by definition, there exists a set $B^i \in \mathcal{F}_i^t$ such that $B \cap D_i = B^i \cap D_i$. Then we have

$$\mathbb{E}_{\mathbb{P}}(\eta 1_{D_i} 1_B) = \mathbb{E}_{\mathbb{P}}(\eta 1_{D_i} 1_{B^i}) = \mathbb{E}_{\mathbb{P}}(1_{D_i} \frac{\mathbb{E}_{\mathbb{P}}(\eta 1_{D_i} \mid \mathcal{F}_i^t)}{\mathbb{P}(D_i \mid \mathcal{F}_i^t)} 1_{B^i}) = \mathbb{E}_{\mathbb{P}}(1_{D_i} \frac{\mathbb{E}_{\mathbb{P}}(\eta 1_{D_i} \mid \mathcal{F}_i^t)}{\mathbb{P}(D_i \mid \mathcal{F}_i^t)} 1_B).$$

**Lemma 2.7.** The family $\hat{F} = (\hat{F}_t)_{t \geq 0}$ is a right-continuous filtration and $\mathcal{G} \subset \hat{F}$.

**Proof.** It is not hard to check that the family $\hat{F}$ is a filtration and that $\mathcal{G}$ is a subfiltration of $\hat{F}$, therefore we only prove that $\hat{F}$ is right continuous.

To do that, we show for a fixed $t \geq 0$, that for every set $B \in \cap_{s \geq t} \hat{F}_s$, and for
every \( i = 1, \ldots, k \), there exists \( B_i \in \mathcal{F}_i^1 \) such that \( B \cap D_i = B_i \cap D_i \). The set \( B \) is \( \mathcal{F}_q \)-measurable for all rational number \( q \) strictly greater than \( t \), thus, for each rational \( q > t \) and each \( i = 1, \ldots, k \), there exists an \( \mathcal{F}_q^i \)-measurable set \( B_{i,q} \), such that \( B \cap D_i = B_{i,q} \cap D_i \). It is sufficient to set

\[
B_i := \bigcap_{n \ge 0} \bigcup_{q \in (t, t + \frac{1}{n}]} B_{i,q},
\]

which is \( \cap_{s>t} \mathcal{F}_s^i \) measurable and therefore \( \mathcal{F}_t^i \)-measurable by right continuity of \( \mathbb{F}^i \).

**Lemma 2.8.** Under Assumption 2.2, for every \( i = 1, \ldots, k \), the \( \sigma \)-algebras \( \mathcal{P}(\hat{\mathbb{F}}^i) \), \( \mathcal{P}(\mathbb{F}^i) \) and \( \mathcal{P}(\mathbb{G}) \) coincide on \( D_i \).

**Proof.** The fact that the \( \sigma \)-algebras \( \mathcal{P}(\hat{\mathbb{F}}^i) \) and \( \mathcal{P}(\mathbb{F}^i) \) coincide on \( D_i \) is a straightforward consequence of the definition of \( \hat{\mathbb{F}}^i \). On the other hand, \( \mathcal{P}(\mathbb{F}^i) \) and \( \mathcal{P}(\mathbb{G}) \) coincide on \( D_i \) due to (ii) of Assumption 2.2.

We are now in the position to compute the \( \hat{\mathbb{F}}^i \)-semimartingale decomposition of \( \mathbb{F}^i \)-martingales. Let us first introduce some useful notation. For each \( i = 1, \ldots, k \), we define an \( \mathbb{F}^i \) local martingale as \( N^i := \alpha_{\mathbb{F}^i}(\mathbbm{1}_{D_i}) \), where the process \( \alpha_{\mathbb{F}^i}(\mathbbm{1}_{D_i}) \) is the \( \text{cadlag} \) version of the \( \mathbb{F}^i \)-optional projection of the random variable \( \mathbbm{1}_{D_i} \). The processes \( N^i \) are bounded and therefore are \( BMO \)-martingales in the filtration \( \mathbb{F}^i \).

**Proposition 2.9.** Under Assumption 2.2, if \( M^i \) is an \( \mathbb{F}^i \)-local martingale, then

\[
\hat{M}^i := \left( M^i - \frac{\mathbbm{1}_{D_i}}{N^i_\infty}, \langle M^i, N^i \rangle^i \right) \mathbbm{1}_{D_i}
\]

is an \( \hat{\mathbb{F}}^i \)-local martingale.

**Proof.** We start by noticing that the \( \mathbb{F}^i \)-predictable bracket \( \langle M^i, N^i \rangle^i \) exists, since \( N^i \) is a \( BMO \)-martingale. Let \( (T_n)_{n \in \mathbb{N}^+} \) be a sequence of \( \mathbb{F}^i \)-stopping times which increases to infinity, such that, for each \( n \), the process \( \langle M^i, N^i \rangle_{T_n} \) is of integrable variation and that \( (M^i)_{T_n} \) is a uniformly integrable \( \mathbb{F}^i \)-martingale. For every \( n \in \mathbb{N}^+ \), we construct a sequence \( r_n \) of \( \mathbb{F}^i \)-stopping times by setting \( r_n := \inf \{ t > 0 : N^i_t \leq \frac{1}{n} \} \) and define \( S_{n,D_i} = (r_n \wedge T_n) \mathbbm{1}_{D_i} + \infty \mathbbm{1}_{D_i^c} \), which is a sequence of \( \hat{\mathbb{F}} \)-stopping times such that \( S_{n,D_i} \rightarrow \infty \) as \( n \rightarrow \infty \).

For every \( n \in \mathbb{N}^+ \) and any bounded elementary \( \hat{\mathbb{F}} \)-predictable process \( \hat{\xi} \),

\[
\mathbb{E}_{\hat{\mathbb{P}}}((\mathbbm{1}_{D_i} \hat{\xi} \cdot (M^i)^{S_{n,D_i}})_{\infty}) = \mathbb{E}_{\hat{\mathbb{P}}}((\mathbbm{1}_{D_i} \hat{\xi} \cdot (M^i)^{r_n \wedge T_n})_{\infty}) = \mathbb{E}_{\hat{\mathbb{P}}}((\hat{\xi} \mathbbm{1}_{[0,r_n \wedge T_n]} \cdot (N^i, M^i)^i)_{\infty}),
\]

where
where $\xi^i$ is the bounded $\mathbb{F}^i$-predictable process associated with the process $\hat{\xi}$ and the set $D_i$ by Lemma 2.8, and the second equality holds by integration by parts formula in $\mathbb{F}^i$. Next, we see that

$$
\mathbb{E}_P\left(\left(\xi^i 1_{[0,r_n \wedge T_n]} \cdot \langle N^i, M^i \rangle^i\right)_\infty\right) = \mathbb{E}_P\left(1_{D_i}(\xi^i(N^i)^{i-1} 1_{[0,r_n \wedge T_n]} \cdot \langle N^i, M^i \rangle^i\right)_\infty)
$$

where the first equality holds by taking the $\mathbb{F}^i$-dual predictable projection. This shows that

$$
\left(M^i - \frac{1_{D_i}}{N^i_i} \cdot \langle N^i, M^i \rangle^i\right) 1_{D_i}
$$

is an $\mathbb{F}$-local martingale.

Theorem 2.3 is now an immediate consequence of Proposition 2.9. From that, we see that if the hypothesis $(H')$ is satisfied between $\mathbb{F}$ and $\mathbb{F}^i$ for any $i$, then the hypothesis $(H')$ is satisfied between $\mathbb{F}$ and $\mathbb{F}$.

### 2.2 Computation of $\mathcal{G}$-semimartingale decomposition

Before proceeding with the computation of the $\mathcal{G}$-semimartingale decomposition of $\mathbb{F}$-martingales, we first summarize the current study and make some comparisons with the well studied cases of initial and progressive enlargement. Then we explain why our study is different from what has been previously done in the literature.

In our setting, we have the following relationships

$$
\mathbb{F} \subset \mathcal{G} \subset \mathbb{F} \quad \text{and} \quad \mathbb{F} \subset \mathbb{F}^i, \quad \forall i = 1, \ldots, k
$$

whereas, in the classic initial and progressive enlargement setting, we have $\mathbb{F} \subset \mathcal{G}^\tau \subset \mathbb{F}^\tau$.

Similar to [1], [11] and [14], where the $\mathbb{F}^\tau$-semimartingale decomposition of an $\mathbb{F}$-martingale is recovered from it’s $\mathcal{G}^\tau$-semimartingale decomposition, we retrieve the $\mathcal{G}$-semimartingale decomposition of an $\mathbb{F}$-martingale from its $\mathbb{F}$-semimartingale decomposition. However, the present work is different to the case of initial and progressive enlargement on two levels. First, on the level of assumptions: to show that the hypothesis $(H')$ holds between $\mathbb{F}$ and $\mathcal{G}$, we assume that, for $i = 1, \ldots, k$, the hypothesis $(H')$ holds between $\mathbb{F}$ and $\mathbb{F}^i$ instead of assuming directly that the hypothesis $(H')$ holds between $\mathbb{F}$ and $\mathbb{F}$ (In Jeanblanc and Le Cam [11], Callegaro et al. [1] and Kchia et al. [14], the largest filtration $\mathbb{F}$ is the initially enlarged filtration $\mathcal{G}^\tau$). Secondly, on the level of computation: in our setting, we cannot exploit the same techniques as the ones used in the case of initial and progressive enlargement with a single random time. To be more specific, suppose we know that the $\mathcal{G}^\tau$-semimartingale decomposition of an $\mathbb{F}$-martingale $M$ is given by $Y + A$, where $Y$
is (for ease of demonstration) a $\mathcal{G}^\tau$-martingale and $A$ is a $\mathcal{G}^\tau$-predictable process of finite variation. Then for all locally bounded $\mathcal{F}^\tau$-predictable process $V$,

$$E(V \cdot M) = E(V 1_{[0,\tau]} \cdot M) + E(V 1_{[\tau,\infty]} \cdot M).$$

The first term can be computed by the Jeulin-Yor formula, while for the second term, we have by assumption

$$E(V 1_{[\tau,\infty]} \cdot M) = E(V 1_{[\tau,\infty]} \cdot A).$$

Formally, in order to make sure that the finite variation part is $\mathcal{F}^\tau$-adapted, one should perform one more step in the calculation and take the $\mathcal{F}^\tau$-dual projection of $A$. However, the computation stops here, since in the case of initial and progressive enlargement, one can exploit the fact that the process $1_{[\tau,\infty]} \cdot A$ is $\mathcal{F}^\tau$-predictable (see Lemma 2. in [14]).

In our setting, suppose that the $\hat{\mathcal{F}}$-predictable process of finite variation in the $\hat{\mathcal{F}}$-semimartingale decomposition of the $\mathcal{F}$-martingale $M$ is known, which we again denoted by $A$. The differences here is that $1_{D_i} \cdot A$ is not necessarily $\mathcal{G}$-predictable and theoretically, in order to find the $\mathcal{G}$-semimartingale decomposition of $M$, one is forced to compute the $\mathcal{G}$-dual predictable projection of $A$. However in this paper, we work with the $\mathcal{G}$-optional projection. It is technically difficult and one soon realizes that the computation is not trivial and a large part of this subsection is devoted to overcome the technical difficulties associated with localization and integrability conditions that one encounters when computing the projections.

### 2.2.1 Stopping times and Increasing processes: general results

Before proceeding, we point out that for the purpose of this paper, increasing processes are positive and finite valued unless specified otherwise. In the following, we give two technical results (Lemma 2.10 and Lemma 2.11) for two given filtrations $\mathcal{K}$ and $\mathcal{H}$ satisfying the usual conditions and a subset $D$ of $\Omega$ such that

$$\mathcal{P}(\mathcal{K}) \cap D \subset \mathcal{P}(\mathcal{H}) \cap D. \quad (3)$$

We show how one can associate with every $\mathcal{K}$-stopping time (resp. $\mathcal{K}$-increasing process) an $\mathcal{H}$-stopping time (resp. $\mathcal{H}$-increasing process which can take value increasing) such that they are equal on the set $D$.

In Lemma 2.10 and Lemma 2.11, we set

$$\tilde{N} = \alpha^\mathcal{H}(1_D), \quad R_n = \inf \{t \geq 0 : \tilde{N}_t \leq \frac{1}{n}\} \quad (4)$$

for $n \in \mathbb{N}^+$ and $R := \sup_n R_n = \inf \{s : \tilde{N}_s \tilde{N}_{s-} = 0\}$. 

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Lemma 2.10. Assume that $\mathbb{H}$ and $\mathbb{K}$ satisfy (3). For any increasing sequence of $\mathbb{K}$-stopping times $(T_n)_{n \in \mathbb{N}^+}$, there exists an increasing sequence of $\mathbb{H}$-stopping times $(\hat{S}_n)_{n \in \mathbb{N}}$ such that $T_n \mathbb{1}_D = \hat{S}_n \mathbb{1}_D$. In addition, if $\sup_n T_n = \infty$, then
(i) $S := \sup_n \hat{S}_n$ is greater or equal to $R$.
(ii) $\cup_n \{R_n = R\} \subset \cup_n \{S_n \geq R\}$.

Proof. Let $(T_n)_{n \in \mathbb{N}^+}$ be an increasing sequence of $\mathbb{K}$-stopping times. For every $n$, there exists an $\mathbb{H}$-predictable process $H_n$ such that $\mathbb{1}_{(T_n, \infty)} \mathbb{1}_D = H_n \mathbb{1}_D$. Replacing $H_n$ by $H_n^+ \wedge 1$, we can suppose that $0 \leq H_n \leq 1$. Replacing $H_n$ by $\prod_{k=1}^n H_k$, we can suppose that $H_n \geq H_{n+1}$. Let $S_n = \inf\{t \geq 0 : H_n(t) = 1\}$. Then, $(\hat{S}_n)_{n \in \mathbb{N}}$ is an increasing sequence of $\mathbb{H}$-stopping times. We note that $S_n = T_n$ on the set $D$.
(i) If $\sup_n T_n = \infty$, by taking the $\mathbb{H}$-predictable projection, we see that
$$p_\mathbb{H}(\mathbb{1}_{[T_n, \infty]} \mathbb{1}_D) = \mathbb{1}_{[S_n, \infty]} \tilde{N}_-$$
then by using monotone convergence theorem and section theorem to the left-hand side, we obtain
$$0 = 1_{\cap_n [S_n, \infty]} \tilde{N}_-. $$
For arbitrary $\epsilon > 0$, we have from the above that $\tilde{N}_{(S+\epsilon)-} = 0$, where $S = \sup_n S_n$. The process $\tilde{N}$ is a positive $\mathbb{H}$-supermartingale, this implies $S + \epsilon \geq R$ and therefore $S \geq R$, since $\epsilon$ is arbitrary.

(ii) Suppose there exists some $k$ such that $R_k = R$, then $\tilde{N}_{R-} > 0$ and from $1_{\cap_n (S_n < R)} \tilde{N}_{R-} = 0$, one can there deduce that there exists $j$ such that $S_j \geq R$. $\square$

The goal in the following is to study the measures associated with finite variation process considered in different filtrations. As in Jacod [7], we use the concept of dominated measures, in order to define a measure associated with a process which is the difference of two increasing unbounded processes.

Lemma 2.11. There exists a map $\hat{V} \rightarrow \psi(\hat{V})$ from the space of $\mathbb{K}$-predictable increasing processes to the space of $\mathbb{H}$-predictable increasing processes, valued in $\mathbb{R}_+ \cup \{+\infty\}$, such that the following properties hold
(i) $1_{D}\hat{V} = 1_D \psi(\hat{V})$ and the support of the measure $d\psi(\hat{V})$ is contained in $\cup_n [0, R_n]$
(ii) for any $\mathbb{K}$-stopping time $\hat{U}$, there exists an $\mathbb{H}$-stopping time $U$ such that
$$\psi(1_{[0,\hat{U}]} \cdot \hat{V}) = 1_{[0,U]} \cdot \psi(\hat{V})$$
(iii) if the process $\hat{V}$ is bounded, then $\psi(\hat{V})$ is bounded by the same constant.

Proof. Let $\hat{V}$ be an increasing $\mathbb{K}$-predictable process. By Lemma 2.8, there exists an $\mathbb{H}$-predictable process $V$ such that $\hat{V} \mathbb{1}_D = V \mathbb{1}_D$. 

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(i) Given a $\mathcal{K}$-localizing sequence $(T_n)_{n \in \mathbb{N}^+}$ such that $\hat{V}^{T_n}$ is bounded, by Lemma 2.10 there exists an increasing sequence of $\mathcal{H}$-stopping times $(S_n)_{n \in \mathbb{N}^+}$ such that $\hat{V}^{T_n}1_D = V^{S_n}1_D$. By taking the $\mathcal{H}$-optional projection, we have

$$o_\mathcal{H}(\hat{V}^{T_n}1_D) = o_\mathcal{H}(V^{S_n}1_D) = V^{S_n} \hat{N}.$$ 

From Theorem 47, in Dellacherie and Meyer [2], the process $\hat{V}^{T_n}1_D$ is càdlàg, which implies that for all $n, k \in \mathbb{N}^+$, the process $V$ is càdlàg on $[0, S_n] \cap [0, R_k]$. Therefore $V$ is càdlàg on $\cup_k[0, R_k]$, since, due to the fact that $\sup_n S_n \geq R$, for all $k$, $[0, R_k] \subset \cup_n[0, S_n]$.

For every $n \in \mathbb{N}^+$ and any pair of rationals $s \leq t \leq R_n$, we have $\mathbb{P}$-a.s

$$V_t1_D = \hat{V}_t1_D \geq \hat{V}_s1_D = V_s1_D$$

and by taking the conditional expectation with respect to $\mathcal{H}_t-$, we have $\mathbb{P}$-a.s, the inequality $V_t\hat{N}_t \geq V_s\hat{N}_s$. Using the fact that the process $V$ is càdlàg on $[0, R_n]$, we can first take right-limit in $t$ to show that for all $t \in \mathbb{R}_+ \cap [0, R_n]$ and $s \in \mathbb{Q}_+ \cap [0, R_n]$ the inequality $V_t \geq V_s$ holds. Then by taking limit in $s$ we extend this inequality to all $s \leq t \leq R_n$.

From the above, one deduces that the process $V$ is càdlàg and increasing on $\cup_n[0, R_n]$. Then, one defines the increasing process $\psi(\hat{V})$, which may take the value infinity by setting

$$\psi(\hat{V}) := V1_{\cup_n[0, R_n]} + \lim_{s \uparrow R} V_s1_{(\hat{N}_{R_\infty} = 0, 0 < R < \infty)}1_{R_\infty} + V_R1_{R_\infty}1_{(\hat{N}_{R_\infty} = 0, 0 < R < \infty)}, \quad (5)$$

which is $\mathcal{H}$-predictable since $\{\hat{N}_{R_\infty} = 0, 0 < R < \infty\} \cap [R, \infty]$ is the intersection of the set $(\cup_n[0, R_n])^c$ and the complement of $[R, \infty] \cap \{\hat{N}_{R_\infty} > 0, 0 < R < \infty\}$, which are both $\mathcal{H}$-predictable. From (5), we see that the support of $d\psi(\hat{V})$ is contained in $\cup_n[0, R_n]$ and on the set $D$, we have $\sup_n R_n = R = \infty$ which implies $\hat{V}1_D = \psi(\hat{V})1_D$.

(ii) For any $\mathcal{K}$-stopping time $\hat{U}$ and any $\mathcal{K}$-predictable increasing process $\hat{V}$, we have from (i), the equality $\psi(1_{[0, U]} \cdot \hat{V})1_D = (1_{[0, U]} \cdot \hat{V})1_D$ and

$$(1_{[0, U]} \cdot \hat{V})1_D = 1_{[0, U]} \cdot (\hat{V}1_D) = 1_{[0, U]} \cdot (\psi(\hat{V})1_D) = (1_{[0, U]} \cdot \psi(\hat{V}))1_D$$

where the existence of the $\mathcal{H}$-stopping time $U$ in the first equality follows from Lemma 2.10 and the second equality follows from (i). By taking the $\mathcal{H}$-predictable projection, we conclude that for every $n \in \mathbb{N}^+$, the processes $1_{[0, U]} \cdot \psi(\hat{V})$ and $\psi(1_{[0, U]} \cdot \hat{V})$ are equal on $[0, R_n]$ and therefore on $\cup_n[0, R_n]$. This implies that the processes $1_{[0, U]} \cdot \psi(\hat{V})$ and $\psi(1_{[0, U]} \cdot \hat{V})$ are equal everywhere, since they do not increase on the complement of $\cup_n[0, R_n]$. 

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(iii) If the process $\hat{V}$ is bounded by $C$, then for every $n \in \mathbb{N}^+$, we have $V \leq \frac{\mathcal{G}(C_1 D)}{\mathcal{G}(1_D)} = C$ on every interval $[0, R_n]$ and therefore on $\cup_n [0, R_n]$. We deduce from (5) that on the set $\cup_n [0, R_n]$ the process $\psi(\hat{V})$ is bounded by $C$, which implies that $\psi(\hat{V})$ is bounded by $C$, since the support of $d\psi(\hat{V})$ is contained in the set $\cup_n [0, R_n]$. 

**Remark 2.12.** Our original goal in Lemma 2.11 was to define, starting from a $\mathbb{K}$-predictable process of finite variation $V = V_+ - V_-$, an $\mathbb{H}$-predictable process of finite variation by setting $\psi(\hat{V}) := \psi(\hat{V}_+) - \psi(\hat{V}_-)$. However, this is problematic since from (5) we see that in general the processes $\psi(\hat{V}_\pm)$ can take the value infinity at the same time. Therefore one can not make use of the usual definitions.

Following the assumption and notation of Lemma 2.11, one can associate with a given $\mathbb{K}$-predictable process $\hat{V}$ of finite variation ($\hat{V} = \hat{V}_+ - \hat{V}_-$), a pair of $\mathbb{H}$-predictable increasing processes $\psi(\hat{V}_+)$ and $\psi(\hat{V}_-)$. In order to treat the problem mentioned in the above remark, we define an auxiliary finite random measure $m$ on $\Omega \times \mathcal{B}(\mathbb{R}_+)$ by setting

$$dm := (1 + \psi(\hat{V}_+) + \psi(\hat{V}_-))^{-2}d(\psi(\hat{V}_+) + \psi(\hat{V}_-)).$$

(6)

Since the processes $\psi(\hat{V}_+)$ and $\psi(\hat{V}_-)$ can only take value infinity at the same time, we deduce that $d\psi(\hat{V}_\pm)$ is absolutely continuous with respect to $dm$, which is absolutely continuous with respect to $d(\psi(\hat{V}_+) + \psi(\hat{V}_-))$. By (i) of Lemma 2.11, this implies that the support of $m$ is contained in $\cup_n [0, R_n]$. Let us denote by $q^\pm$ the Radon-Nikodým density of $d\psi(\hat{V}_\pm)$ with respect to $m$ and introduce the following space of $\Omega \times \mathcal{B}(\mathbb{R}_+)$-measurable functions,

$$\{f : \forall t > 0, \int_{(0,t]} |f_s||q^+_s - q^-_s| m(ds) < \infty\}.$$  

(7)

We define a linear operator $\psi(\hat{V})$ on the above set (7), which maps $f$ to a process by setting for every $t \geq 0$,

$$f * \psi(\hat{V})_t := \int_{(0,t]} f_s(q^+_s - q^-_s) m(ds).$$

(8)

We shall call $\mathcal{L}^1(\psi(\hat{V}))$ the set (7) and an $\Omega \times \mathcal{B}(\mathbb{R})$-measurable function $f$ is said to be $\psi(\hat{V})$ integrable if $f \in \mathcal{L}^1(\psi(\hat{V}))$. One should point out that the measure $m$ is introduced to avoid the problem mentioned in Remark 2.12 and that the set defined in (7) and the map defined in (8) are essentially independent of the choice of $m$. 


2.2.2 The $G$-Semimartingale Decomposition

In this subsection, we place ourselves in the setting of section 2.1, and we are in position to derive the $G$-semimartingale decomposition of $F$-martingales. Let us first summarize the previous results and introduce new notations.

We suppose that for each $i = 1, \ldots, k$, the $F^i$-semimartingale decomposition of an $F$-martingale $M$ is given by $M = M^i + K^i$, where $M^i$ is an $F^i$-local martingale and $K^i$ an $F^i$-predictable process of finite variation. Then, from Proposition 2.9, for every $i = 1, \ldots, k$, the process

$$\widehat{M}^i := M\mathbb{1}_{D_i} - \mathbb{1}_{D_i} \left( K^i\mathbb{1}_{D_i} + \frac{\mathbb{1}_{D_i}}{N^i_i} \cdot \langle N^i, M \rangle^i \right)$$

is an $\widehat{F}$-local martingale. For simplicity, for every $i = 1, \ldots, k$, we adopt the following notation:

$$\tilde{N}^i := o, G(D_i), \quad N^i := o, F(D_i)$$
$$\tilde{V}^i := K^i + \frac{1}{N^i_i} \cdot \langle N^i, M \rangle^i.$$  

From Lemma 2.8, for every $i = 1, \ldots, k$, we have $\mathcal{P}(\widehat{F}) \cap D_i \subset \mathcal{P}(G) \cap D_i$ and one can apply Lemma 2.11 with the set $D_i$, the filtrations $\widehat{F}$ and $G$ and define the linear operator $\psi(\tilde{V}^i)$ as in (8) on the space of $\psi(\tilde{V}^i)$ integrable functions given by

$$\mathcal{L}^1(\psi(\tilde{V}^i)) := \{ f : \int_{\mathbb{R}_+} |f_s||q^i_s|^+ - q^i_s^-|dm^i_s < \infty \},$$

where the measure $m^i$ is constructed from the $\widehat{F}$-predictable process $\tilde{V}^i$ as shown in (6) and $q^i, \pm$ is the density of $\psi(\tilde{V}^i)$ with respect to $m^i$.

For an arbitrary filtration $K$, we write $X \overset{\text{K-mart}}{=} Y$, if $X - Y$ is a $K$-local martingale.

**Theorem 2.13.** Let $M$ be $F$-martingale, such that for every $i = 1, \ldots, k$, $M = M^i + K^i$, where $M^i$ is an $F^i$-local martingale and $K^i$ an $F^i$-predictable process of finite variation. Then under Assumption 2.2, for every $i = 1, \ldots, k$, the process $\tilde{N}^i$ defined in (10) belongs to $\mathcal{L}^1(\psi(\tilde{V}^i))$, where $\tilde{V}^i$ is defined in (11) and

$$M - \sum_{i=1}^k \tilde{N}^i \ast \psi(\tilde{V}^i)$$

is a $G$-local martingale, where $\ast$ is defined in (8).

**Proof.** It is sufficient to show that for any fixed $i = 1, \ldots, k$, the $G$-martingale $\tilde{N}^i$ is in $\mathcal{L}^1(\psi(\tilde{V}^i))$ and $M \tilde{N}^i - \tilde{N}^i \ast \psi(\tilde{V}^i)$ is an $G$-local martingale.
Before proceeding, we point out that the process $M$ is by Theorem 2.3 an $\widehat{F}$-semimartingale and therefore an $G$-semimartingale by the result of Stricker [19]. To see that $M$ is an $G$-special semimartingale, we note that there exists a sequence of $\overline{F}$-stopping times (therefore $G$-stopping times) which reduces $M$ to $H^1$, and one can check directly, that $M$ is an $G$-special semimartingale by using special semimartingale criteria such as Theorem 8.6 in [6]. The aim in the rest of the proof is to find explicitly the $G$-semimartingale decomposition of $M$.

The process $M\tilde{N}^i$ is an $G$-special semimartingale since $M$ is an $G$-special semimartingale and $\tilde{N}^i$ is a bounded $G$-martingale. Let us denote $B^i$ the unique $G$-predictable process of finite variation in the $G$-semimartingale decomposition of $M\tilde{N}^i$.

For every $n \in \mathbb{N}^+$, we define $R_n = \inf \{ t \geq 0, \tilde{N}^i_t \leq \frac{1}{n}\}$ and $R = \sup_n R_n = \inf \{ t \geq 0, \tilde{N}^i_t \tilde{N}^i = 0\}$. The method of the proof is to identify the process $B^i$ with $-\tilde{N}^i \ast \psi(\tilde{V}^i)$. To do that, it is sufficient to show that the two processes coincide on the sets $\cup_n [0, R_n] \cup [0, R] \setminus \cup_n [0, R_n]$; indeed $B^i$ and $\tilde{N}^i \ast \psi(\tilde{V}^i)$ are constant on $[R, \infty]$, as $M_t\tilde{N}^i_t = 0$ for $t > R$ and the support of $dm^i$ is in the complement of $[R, \infty]$ by (i) of Lemma 2.11.

On the set $\cup_n [0, R_n]$, let $(T_n)_{n \in \mathbb{N}^+}$ be a localizing sequence of $\widehat{F}$-stopping times such that the process $\tilde{M}^i$ defined in (9) stopped at $T_n$, i.e., $(\tilde{M}^i)^T_n$ is an $\widehat{F}$-martingale and $(\tilde{V}^i)^T_n$ is of bounded total variation, then

$$
(\tilde{M}^i)^T_n = M^{T_n} \mathbf{1}_{D_i} + \mathbf{1}_{[0,T_n]} \cdot \tilde{V}^i_t \mathbf{1}_{D_i} - \mathbf{1}_{[0,T_n]} \cdot \tilde{V}^i_t \mathbf{1}_{D_i} = M^{S_n} \mathbf{1}_{D_i} + \mathbf{1}_{[0,S_n]} \cdot \psi(\tilde{V}^i_t) \mathbf{1}_{D_i} - \mathbf{1}_{[0,S_n]} \cdot \psi(\tilde{V}^i_t) \mathbf{1}_{D_i}
$$

where the second equality holds from (i) of Lemma 2.11 with the existence of the sequence of $G$-stopping times $(S_n)_{n \in \mathbb{N}^+}$ given by Lemma 2.10. Since for every $n \in \mathbb{N}^+$, the $\widehat{F}$-adapted processes $\mathbf{1}_{[0,T_n]} \cdot \tilde{V}^i_t$ are bounded, by (ii) and (iii) of Lemma 2.11, we can conclude that the processes $\mathbf{1}_{[0,S_n]} \cdot \psi(\tilde{V}^i_t)$ are bounded $G$-predictable processes. Together with the property that the Lebesgue integral and the stochastic integral coincide when all quantities are finite, we obtain that

$$
\mathbf{1}_{[0,S_n]} \cdot \psi(\tilde{V}^i_t) - \mathbf{1}_{[0,S_n]} \cdot \psi(\tilde{V}^i_t) = \mathbf{1}_{[0,S_n]} \ast \psi(\tilde{V}^i_t),
$$

that is the left-hand side coincides with the application of $\psi(\tilde{V}^i_t)$ on $\mathbf{1}_{[0,S_n]}$ defined in (8). By taking the $G$-optional projection, we obtain

$$
M^{S_n} \tilde{N}^i \ast \psi(\tilde{V}^i_t) = -\tilde{N}^i \ast \psi(\tilde{V}^i_t) = -\tilde{N}^i \ast \left( \mathbf{1}_{[0,S_n]} \ast \psi(\tilde{V}^i_t) \right)
$$

where the second equality follows from integration by parts and Yoeurp’s lemma. For each $n \in \mathbb{N}^+$, from uniqueness of the $G$-semimartingale decomposition, we deduce that

$$
\mathbf{1}_{[0,S_n]} \ast B^i = -\tilde{N}^i \ast \left( \mathbf{1}_{[0,S_n]} \ast \psi(\tilde{V}^i_t) \right) = -(\tilde{N}^i \mathbf{1}_{[0,S_n]} \ast \psi(\tilde{V}^i_t)),
$$

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where the second equality follows from the property that the Lebesgue integral and the stochastic integral coincide when all quantities are finite. Then for any fixed \( t \geq 0 \) and \( H_s := 1_{\{q_s^{i,*} - q_s^i < 0\}} - 1_{\{q_s^{i,*} - q_s^i \geq 0\}} \),

\[
\int_{(0,t]} H_s 1_{s \leq S_n} dB_s^i = \int_{(0,t]} 1_{s \leq S_n} \tilde{N}_n^i |q_s^{i,*} - q_s^i| \, dm^i.
\]

To take limit in \( n \), one applies the dominated convergence theorem to the left-hand side and the Beppo-Levi (monotone convergence) theorem to the right. One then concludes that \( \tilde{N}_n^i \mathbf{1}_{[0,S_n]} \) is \( \psi(\tilde{V}^i) \) integrable as the limit on the left-hand side is finite. This implies that the jumps of the processes \( \tilde{N}_n^i \) is \( \hat{\psi}(\tilde{V}^i) \) integrable as the support of \( dm^i \) is contained in \( \cup_n [0,R_n] \) which is contained in \( \cup_n [0,S_n] \) by (i) of Lemma 2.10.

We have shown that \( \tilde{N}_n^i \in \mathcal{L}^1(\psi(\tilde{V}^i)) \) and therefore \( \tilde{N}_n^i * \hat{\psi}(\tilde{V}^i) \) is a process such that for all \( n \in \mathbb{N}^+ \), we have \( (\tilde{N}_n^i * \hat{\psi}(\tilde{V}^i))_{S_n} = (B^i)_{S_n} \). This implies \( \tilde{N}_n^i * \hat{\psi}(\tilde{V}^i) = B^i \) on \( \cup_n [0,S_n] \) and therefore on \( \cup_n [0,R_n] \).

On the set \( \{0,R\} \setminus \cup_n [0,R_n] \), one only needs to pay attention to the set \( F = \{ \forall n, R_n < R \} = \{ \tilde{N}_{R_n}^i = 0 \} \), since on the complement \( F^c \), the set \( \{0,R\} \setminus \cup_n [0,R_n] \) is empty. From Lemma 3.29 in He et al. [6], \( R_F = R_1 + \infty \mathbf{1}_{F^c} \) is a \( \mathcal{G} \)-predictable stopping time and on \( F \), we have \( \{0,R\} \setminus \cup_n [0,R_n] = [R_F] \). From the fact that \( \Delta(M \tilde{N}^i)_{R_F} = 0 \), Lemma 8.8 of [6] shows that \( \Delta B^i_{R_F} = 0 \). On the other hand, from (5), we deduce that the measure \( dm^i \), which is absolutely continuous with respect to \( \psi(\tilde{V}^i_+) + \hat{\psi}(\tilde{V}^i_-) \) has no mass at \( [R_F] \) and therefore \( \tilde{N}_n^i * \hat{\psi}(\tilde{V}^i) \) does not jump at \( [R_F] \). This implies that the jumps of the processes \( B^i \) and \( \tilde{N}_n^i * \hat{\psi}(\tilde{V}^i) \) coincide on \( [0,R] \setminus \cup_n [0,R_n] \).

We conclude this section with the following lemma, which appears to be very useful when one wants to compute the \( \mathcal{G} \)-drift in practice.

**Lemma 2.14.** For \( 1 \leq i \leq k \), the process \( \tilde{N}_n^i * \hat{\psi}(\tilde{V}^i) \) is the dual \( \mathcal{G} \)-predictable projection of \( \mathbf{1}_{\hat{D}_i} \tilde{V}^i \) and \( \tilde{N}_n^i \cdot \tilde{V}^i \) is the \( \mathcal{F}^i \) drift of the \( \mathcal{F}^i \)-special semimartingale \( \tilde{N}^i M \).

**Proof.** Similar to the proof of Theorem 2.13, for a fixed \( i = 1, \ldots, k \) and every \( n \in \mathbb{N}^+ \), we define \( R_n = \inf \{ t \geq 0, \tilde{N}^i \leq \frac{1}{n} \} \), \( R = \sup_n R_n = \inf \{ t \geq 0, \tilde{N}^i \tilde{N}^i = 0 \} \) and let \( (T_n)_{n \in \mathbb{N}^+} \) be a localizing sequence of \( \hat{\mathcal{F}} \)-stopping times (and \( (S_n)_{n \in \mathbb{N}^+} \) the corresponding \( \mathcal{G} \)-stopping times from Lemma 2.10) such that the process \( \tilde{M}^i \) defined in (9) stopped at \( T_n \), i.e., \( (\tilde{M}^i)^{T_n} \) is an \( \hat{\mathcal{F}} \)-martingale, and \( (\tilde{V}^i)^{T_n} \) is of bounded total variation. Again, the method of the proof is to identify the process \( (\tilde{V}^i \mathbf{1}_{D_i})_{p,G} \) with \( -\tilde{N}_n^i * \hat{\psi}(\tilde{V}^i) \).

By taking the \( \mathcal{G} \)-optional projection in (13), we see that for a fixed \( i = 1, \ldots, k \),

\[
M^{S_n \tilde{N}^i \mathbf{1}_{D_i}}_{\text{G-mart}} - o_{G}(\mathbf{1}_{[0,T_n]} \cdot \tilde{V}^i \mathbf{1}_{D_i})_{\text{G-mart}} \equiv -(\mathbf{1}_{[0,T_n]} \cdot \tilde{V}^i \mathbf{1}_{D_i})_{p,G} = -(\mathbf{1}_{[0,S_n]} \cdot \tilde{V}^i \mathbf{1}_{D_i})_{p,G} = -\mathbf{1}_{[0,S_n]} \cdot (\tilde{V}^i \mathbf{1}_{D_i})_{p,G},
\]
where the second and last equality follows from Corollary 5.31 and Corollary 5.24 in [6] respectively. From Theorem 2.13 and the uniqueness of $G$-special semimartingale decomposition, the process $\tilde{N}_i \ast \psi(V)$ is equal to $(\hat{V}^i 1_{D_i})^{p,G}$ on $[0,S_n]$ for all $n$ and therefore on $\cup_n[0,R_n]$. On the complement $(\cup_n[0,R_n])^c$, for every $C \in \mathcal{F}$ and every bounded $G$-predictable process $\xi$, by duality

$$\mathbb{E}_\mathbb{P}(1_C(\xi 1_{(\cup_n[0,R_n])^c}) \cdot (\hat{V}^i 1_{D_i})^{p,G}) = \mathbb{E}_\mathbb{P}((p,G(1_C)\xi 1_{\cap_n[0,R_n]} \cdot \hat{V}^i 1_{D_i})) = 0$$

where the last equality holds, since on the set $D_i$, $\sup_n R_n = \infty$. This implies $(\hat{V}^i 1_{D_i})^{p,G}$ also coincide with $\tilde{N}_i \ast \psi(V)$ on the $(\cup_n[0,R_n])^c$ and therefore everywhere.

To show for every $i = 1,\ldots,k$, the process $N_i \cdot \hat{V}^i$ is the $\mathbb{F}^i$-drift of the $\mathbb{F}^i$-special semimartingale $N^iM$, it is sufficient to apply the $\mathbb{F}^i$-integration by parts formula to $N^iM = N^i(M - K^i) + N^iK^i$. We see that

$$N^iM = \langle N^i, M - K^i \rangle^i + N_i^i \cdot K^i = \langle N^i, M \rangle^i + N_i^i \cdot K^i,$$

where the second equality follows from Yœurp's lemma. One can now conclude that the $\mathbb{F}^i$-drift of $N^iM$ is given by $N_i^i \cdot \hat{V}^i$ from the uniqueness of the $\mathbb{F}^i$-special semimartingale decomposition. \qed
3 Application to Multiple Random Times

Our goal here is to apply the previous methodology to enlargement of filtrations with random times.

3.1 Progressive enlargement with random times and their re-ordering

We introduce the following notations. For two elements $a, b$ in $[0, \infty]$ we denote

$$a \nmid b = \begin{cases} 
  a & \text{if } a \leq b, \\
  \infty & \text{if } a > b.
\end{cases}$$

Let $\xi = (\xi_1, \ldots, \xi_k)$ be an $k \in \mathbb{N}^*$ dimensional vector of random times. For $b \in [0, \infty]$, we write $\xi \nmid b = (\xi_1 \nmid b, \ldots, \xi_k \nmid b)$ and $\sigma(\xi \nmid b) = \sigma(\xi_1 \nmid b) \lor \cdots \lor \sigma(\xi_k \nmid b)$.

**Definition 3.1.** The progressive enlargement of $\mathbb{F}$ with the family of random times $\xi$ is denoted $\mathbb{F}^\xi = (\mathcal{F}_t^\xi)_{t \geq 0}$; this is the smallest filtration satisfying the usual conditions containing $\mathbb{F}$ making $\xi_1, \ldots, \xi_k$ stopping times. In other terms,

$$\mathcal{F}_t^\xi := \bigcap_{s \geq t} (\mathcal{F}_s \lor \sigma(\xi \nmid s)), \ t \in \mathbb{R}_+.$$

**Definition 3.2.** The initial enlargement of $\mathbb{F}$ with the family of random times $\xi$ is denoted $\mathbb{F}^{\sigma(\xi)} = (\mathcal{G}^\xi_{t \geq 0})$, this is the smallest filtration containing $\mathbb{F}$, satisfying the usual conditions, such that the random times $\xi = (\xi_1, \ldots, \xi_k)$ are $\mathcal{F}_0^{\sigma(\xi)}$ measurable. One has

$$\mathcal{F}_t^{\sigma(\xi)} = \bigcap_{s \geq t} (\mathcal{F}_s \lor \sigma(\xi)), \ t \in \mathbb{R}_+.$$

We need a sorting rule (cf. [18]). For any function $a$ defined on $\{1, \ldots, k\}$ taking values $\{a_1, \ldots, a_k\}$ in $[0, \infty]$, consider the points $(a_1, 1), \ldots, (a_k, k)$ in the space $[0, \infty] \times \{1, \ldots, k\}$. These points are two-by-two distinct. We order these points according to the alphabetic order in the space $[0, \infty] \times \{1, \ldots, k\}$. Then, for $1 \leq i \leq k$, the rang of $(a_i, i)$ in this ordering is given by

$$R^a(i) = R^{\{a_1, \ldots, a_k\}}(i) = \sum_{j=1}^{k} \mathbb{I}_{(a_j < a_i)} + \sum_{j=1}^{k} \mathbb{I}_{(j < i, a_j = a_i)} + 1.$$
The map \( i \in \{1, \ldots, k\} \to R^a(i) \in \{1, \ldots, k\} \) is a bijection. Let \( \rho^a \) be its inverse. Define \( \frac{\omega}{a} = a(\rho^a) \), where \( \frac{\omega}{a}(j) \) can be roughly qualified as the \( j \)th value in the increasing order of the values \( \{a_1, \ldots, a_k\} \). \( \frac{\omega}{a} \) is an non decreasing function on \( \{1, \ldots, k\} \) taking the same values as \( a \).

We consider \( \omega \) by \( \omega \) the random function \( a^\xi \) on \( \{1, \ldots, k\} \) taking the values \( \{\xi_1, \ldots, \xi_k\} \) and define the non decreasing function \( \frac{\omega}{a}^\xi \) as above.

**Lemma 3.3.** For any \( 1 \leq j \leq k \), there exists a Borel function \( s_j \) on \([0, \infty]^k \) such that

\[
\frac{\omega}{a}^\xi = s_j(\xi_1, \ldots, \xi_k).
\]

If the \( \xi_1, \ldots, \xi_k \) are stopping times with respect to some filtration, the random times \( \frac{\omega}{a}^\xi(1), \ldots, \frac{\omega}{a}^\xi(k) \) also are stopping times with respect to the same filtration.

**Proof.** This is a consequence of the following identity for any \( t \geq 0 \),

\[
\{\frac{\omega}{a}^\xi(j) \leq t\} = \bigcup_{I \subseteq \{1, \ldots, k\}} s_I t = j \{\xi_h \leq t, \forall h \in I\}.
\]

\( \square \)

We will call the random times \( \uparrow a^\xi(1), \ldots, \uparrow a^\xi(k) \) the increasing re-ordering of \( \{\xi_1, \ldots, \xi_k\} \), and we denote \( \xi(j) = \frac{\omega}{a}^\xi(j), 1 \leq j \leq k \), and \( \underline{\xi} = (\xi(1), \ldots, \xi(k)) \). We define then \( \underline{\mathbb{R}}^\xi \) and \( \underline{\mathbb{F}}^\xi \). Also we define \( \xi(0) = 0, \xi(k+1) = \infty \) to complete the increasing re-ordering (only when the length of the initial random vector \( \xi \) is \( k \)).

The following result will be useful.

**Lemma 3.4.** Suppose \( \tau \) and \( \xi \) are two \( k \)-dimensional vectors of random times. If a measurable set \( D \) is a subset of \( \{\tau = \xi\} \), then \( \mathcal{F}^\tau_t \cap D = \mathcal{F}^\xi_t \cap D \) for every \( t \geq 0 \).

**Proof.** By symmetry, it is enough to show that for all \( t \geq 0 \), we have \( \mathcal{F}^\tau_t \cap D \subseteq \mathcal{F}^\xi_t \cap D \). Fix \( t \geq 0 \), then for any \( A \in \mathcal{F}^\tau_t \) and all \( n \geq 1 \), there exist \( \mathcal{F}_{t+1/n} \otimes B(\mathbb{R}^d) \)-measurable function \( g_n \) such that \( 1_A = g_n(\tau \land (t + 1/n)) \). We have

\[
1_A 1_D = g_n(\tau \land (t + 1/n)) 1_D = g_n(\xi \land (t + 1/n)) 1_D
\]

and then

\[
A \cap D = \left\{ \lim_{n \to \infty} g_n(\xi \land (t + 1/n)) = 1 \right\} \cap D
\]

whilst \( \{\lim_{n \to \infty} g_n(\xi \land (t + 1/n)) = 1\} \) is \( \mathcal{F}^\xi_t \)-measurable by right continuity of \( \underline{\mathbb{R}}^\xi \). \( \square \)

The following notations will be used. Let \( n \) be an integer \( n \geq k \). Let \( I_k = \{1, \ldots, k\} \) and \( I_n = \{1, \ldots, n\} \). For a subset \( J \subset I_n \) denote \( \xi_J = (\xi_j : j \in J) \) in their natural order. In particular, if \( J = I_k \) we denote \( \xi_{I_k} \) be \( \xi_k \). For any injective map \( \varphi \) from \( I_k \) into \( I_n \) denote \( \xi_{\varphi} = (\xi_{\varphi(j)} : j \in I_k) \) and \( \xi_{\varphi} = (\xi_j : j \in I_n \setminus \varphi(I_k)) \) in their natural order.
3.2 Hypothesis \((H')\) for ordered random times

From now on we are given an \(n \in \mathbb{N}^*\) dimensional vector of random times \(\tau = (\tau_1, \ldots, \tau_n)\). Let \(k\) be an integer \(1 \leq k \leq n\). The aim of this section is to apply the results from section 2 to investigate the hypothesis \((H')\) between \(\mathbb{F}\) and \(\mathbb{F}^{\tau_k} = \mathbb{F}^{\tau(1), \ldots, \tau(k)}\).

Let \(I(k, n)\) denote the set of all injective functions \(\rho\) from \(I_k\) into \(I_n\). For \(\rho \in I(k, n)\), we introduce the set \(d_\rho := \{(\tau_1, \ldots, \tau_k) = (\tau_{\rho(1)}, \ldots, \tau_{\rho(k)})\}\). Let \(\delta\) be a strictly increasing map from \(I(k, n)\) into \(\mathbb{N}\). We define a partition of the space \(\Omega\) by setting

\[
D_\rho := d_\rho \cap (\cup_{d' < \delta(\rho)} d_{d'})^c.
\]

Following Lemma 3.4 we have

**Corollary 3.5.** The triplet \((\mathbb{F}, (\mathbb{F}^{\tau_\rho})_{\rho \in I(k, n)}, \mathbb{F}^{\tau_k})\) satisfies Assumption 2.2 with respect to the partition \((D_\rho)_{\rho \in I(k, n)}\).

With the family \((\mathbb{F}^{\tau_\rho})_{\rho \in I(k, n)}\) and the partition \((D_\rho)_{\rho \in I(k, n)}\), we can construct the filtration \(\mathbb{F}\) as in Corollary 3.5, which will be denoted by \(\mathbb{F}^{\tau_k} = (\mathbb{F}^{\tau_k}_t)_{t \in \mathbb{R}^+}\), by setting for every \(t \geq 0\),

\[
\mathbb{F}^{\tau_k}_t := \{A \in \mathbb{F} | \forall \rho \in I(k, n), \exists A_\rho \in \mathbb{F}^{\tau_\rho} \text{ such that } A \cap D_\rho = A_\rho \cap D_\rho\}. \tag{14}
\]

The \(\mathbb{F}^{\tau_k}\) and \(\mathbb{F}^{\tau_k}\)-semimartingale decompositions of \(\mathbb{F}\)-martingales are now readily available.

**Lemma 3.6.** Assume that for every \(\rho \in I(k, n)\), the \(\mathbb{F}\)-martingale \(M\) is an \(\mathbb{F}^{\tau_\rho}\)-semimartingale with \(\mathbb{F}^{\tau_\rho}\)-semimartingale decomposition given by \(M = M^\rho + K^\rho\), where \(M^\rho\) is an \(\mathbb{F}^{\tau_\rho}\)-local martingale and \(K^\rho\) is an \(\mathbb{F}^{\tau_\rho}\)-predictable process of finite variation. We denote

\[
\hat{N}^\rho := \alpha_{\mathbb{F}^{\tau_k}}(1_{D_\rho}), \quad N^\rho := \alpha_{\mathbb{F}^{\tau_\rho}}(1_{D_\rho})
\]

\[
\hat{V}^\rho := K^\rho + \frac{1}{N^\rho} \cdot \langle N^\rho, M \rangle^\rho,
\]

where the predictable bracket \(\langle ., . \rangle^\rho\) is computed with respect to the filtration \(\mathbb{F}^{\tau_\rho}\). Then

(i) the \(\mathbb{F}^{\tau_k}\)-semimartingale decomposition of \(M\) is given by

\[
M = \hat{M} + \sum_{\rho \in I(k, n)} 1_{D_\rho} \left( K^\rho 1_{D_\rho} + \frac{1_{D_\rho}}{N^\rho} \cdot \langle N^\rho, M \rangle^\rho \right)
\]

where \(\hat{M}\) is an \(\mathbb{F}^{\tau_k}\)-local martingale,

(ii) the \(\mathbb{F}^{\tau_k}\)-semimartingale decomposition of \(M\) is given by

\[
M = \hat{M} + \sum_{\rho \in I(k, n)} \hat{N}^\rho * \psi(\hat{V}^\rho)
\]

where \(\hat{M}\) is an \(\mathbb{F}^{\tau_k}\)-local martingale and the linear operator \(\psi(\hat{V}^\rho)\) is described in(8).
3.3 Computation under density hypothesis

The formula in Lemma 3.6 allows one to compute the $\mathbb{F}^{\tau_k}$-semimartingale decomposition of a $\mathbb{F}$-martingale $M$. Compared with classical result on this subject, the formula in Lemma 3.6 has the particularity to express the drift of $M$ in $\mathbb{F}^{\tau_k}$ as a ‘weighted average’ of its $\mathbb{F}^{\tau_\varrho}$-drifts when $\varrho$ runs over $I(k, n)$. Such a weighted average formula may be useful for model analysis and risk management. The rest of this paper is devoted to illustrate this weighted average by an explicit computation in the case of density hypothesis. To this end we develop techniques which can have their utility elsewhere.

Assumption 3.7. The conditional laws of the vector $\tau = (\tau_1, \ldots, \tau_n)$ satisfies the density hypothesis. In other terms, for any non negative Borel function $h$ on $\mathbb{R}_+^n$, for $t \in \mathbb{R}_+$, we have $\mathbb{P}$-almost surely,

$$
\mathbb{E}_\mathbb{P}(h(\tau) | \mathcal{F}_t) = \int_{\mathbb{R}_+^n} h(u)a_t(u)\mu^\otimes n(du),
$$

where $\mu$ is a non-atomic $\sigma$-finite measure on $\mathbb{R}_+$, and $a_t(\omega, u)$ is a non negative $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}_+^n)$ measurable function, called the conditional density function at $t$.

Remark that, according to [8, Lemme(1.8)], the density function $a_t(\omega, x)$ can be chosen everywhere càdlàg in $t \in \mathbb{R}_+$, with however a rougher measurability: $a_t$ being $\cap_{s>t}(\mathcal{F}_s \otimes \mathcal{B}(\mathbb{R}_+^n))$ measurable. Moreover, for fixed $x$, $a_t(x)$, $t \in \mathbb{R}_+$, is a ($\mathbb{P}, \mathbb{F}$) martingale. We assume this version of the density function in this section. Without loss of the generality, we assume

$$
\int_{\mathbb{R}_+^n} a_t(x)\mu^\otimes n(dx) = 1, \ t \in \mathbb{R}_+,
$$

everywhere. Thus the regular conditional laws, denoted by $\nu_t$, of the vector $\tau$ with respect to the $\sigma$-algebra $\mathcal{F}_t$ has $a_t$ as the density function with respect to $\mu^\otimes n$.

Lemma 3.8. Let $t \in \mathbb{R}_+$. For any bounded function $h$ on $\Omega \times \mathbb{R}_+^n$, $\cap_{s>t}(\mathcal{F}_s \otimes \sigma(\mathbb{R}_+^n))$ measurable, we have

$$
\mathbb{E}[h(\tau) | \mathcal{F}_t] = \int_{\mathbb{R}_+^n} a_t(x)h(x)\mu^\otimes n(dx).
$$

Proof. For any $s > t$, $h$ is $\mathcal{F}_s \otimes \sigma(\mathbb{R}_+^n)$ measurable. Let $B \in \mathcal{F}_t$.

$$
\mathbb{E}[\mathbb{1}_B h(\tau)] = \mathbb{E}[\int_{\mathbb{R}_+^n} \mathbb{1}_B h(x)a_s(x)\mu^\otimes n(dx)] = \mathbb{E}[\mathbb{1}_B \int_{\mathbb{R}_+^n} h(x)a_t(x)\mu^\otimes n(dx)].
$$

because, for every $x$, $a_t(x)$, $t \in \mathbb{R}_+$, is a $(\mathbb{P}, \mathbb{F})$ martingale, and $h(x)$ is $\mathcal{F}_t$ measurable, thanks to the right continuity of $\mathbb{F}$. The lemma is proved because $\int_{\mathbb{R}_+^n} h(x) a_t(x) \mu^\otimes n(dx)$ also is $\mathcal{F}_t$ measurable.

To make computations under Assumption 3.7, we introduce the following system of notations. For a subset $J \subset \{1, \ldots, n\}$ of cardinal $\#J = j \in \mathbb{N}$, for vectors $z \in \mathbb{R}^j$ and $y \in \mathbb{R}^{n-j}$, let $c_J(z, y)$ be the vector $x \in \mathbb{R}^n$ for which $x_J$ is given by $z$ in their natural order, and $x_{J^c}$ is given by $y$ in the natural order. For non negative Borel function $g$ we denote

$$g_J(z) = \int_{\mathbb{R}_+^{n-j}} g(c_J(z, y)) \mu^\otimes (n-j)(dy).$$

We check directly that $\tau_J$ satisfies the density hypothesis with density functions $(a_t)_J, t \in \mathbb{R}_+$. We denote by $a_{J,t}, t \in \mathbb{R}_+$, the càdlàg version of $(a_t)_J$ defined in [8, Lemme (1.8)]. Notice that, for all $t \in \mathbb{R}_+$, for almost all $\omega$, $(a_{J,t})_{(\omega, x)} = a_{J,t}(\omega, x)$ $\mu^\otimes n$-almost everywhere. Hence, if $p_J$ denotes the projection map $x \rightarrow x_J$ on $\mathbb{R}_+^n$, the conditional expectation of $g$ under $\nu_t(\omega)$ given $p_J = z$ is the function

$$\mathbb{E}^n [g|p_J = z] = \frac{(ga_{J,t})_J(z)}{a_{J,t}(z)} \mathbbm{1}_{a_{J,t}(z) > 0}.$$

For $x \in \mathbb{R}^n$, let $\overline{x} = (x(1), \ldots, x(n))$ be the increasing re-ordering of $x$ and let $\tau$ be the map $\tau(x) = \overline{x}$. Let $\mathcal{S}$ be the symmetric group on $I_n$. For $\pi \in \mathcal{S}$, we define the map $\pi(x) = (x_{\pi(1)}, \ldots, x_{\pi(n)})$ and

$$\overline{g}(x) = \sum_{\pi \in \mathcal{S}} g(\pi(x)).$$

We have the relationships $\pi^{-1}(\pi(x)) = x$, $\pi^{-1}(\mu^\otimes n) = \mu^\otimes n$ and $\overline{g}(\pi(x)) = \overline{g}(x)$. In particular, $\overline{g}(x) = \overline{g}(\tau(x))$. For non negative Borel function $h$ on $\mathbb{R}_+^n$, we compute

$$\mathbb{E}^\nu [gh(\tau)] = \int_{\mathbb{R}_+^n} (ga_t)(x) h(\tau(x)) \mu^\otimes n(dx)$$

because $\mu$ is atom-free

$$= \sum_{\pi \in \mathcal{S}} \int_{\mathbb{R}_+^n} (ga_t)(x) h(\tau(x)) \mathbbm{1}_{x(1) < \ldots < x(n)} \mu^\otimes n(dx)$$

$$= \sum_{\pi \in \mathcal{S}} \int_{\mathbb{R}_+^n} (ga_t)(x_{\pi(1)}, \ldots, x_{\pi(n)}) h(x_{\pi(1)}, \ldots, x_{\pi(n)}) \mathbbm{1}_{x(1) < \ldots < x(n)} \mu^\otimes n(dx)$$

$$= \sum_{\pi \in \mathcal{S}} \int_{\mathbb{R}_+^n} (ga_t)(\pi^{-1}(x)) h(x_{\pi(1)}, \ldots, x_{\pi(n)}) \mathbbm{1}_{x(1) < \ldots < x(n)} \mu^\otimes n(dx)$$

$$= \sum_{\pi \in \mathcal{S}} \int_{\mathbb{R}_+^n} (\overline{a_t})(x) h(x_{\pi(1)}, \ldots, x_{\pi(n)}) \mu^\otimes n(dx)$$

$$= \sum_{\pi \in \mathcal{S}} \int_{\mathbb{R}_+^n} \frac{1}{n!} (\overline{a_t})(\pi(x)) h(x_{\pi(1)}, \ldots, x_{\pi(n)} \mathbbm{1}_{x(1) < \ldots < x(n)} \mu^\otimes n(dx)$$

$$= \sum_{\pi \in \mathcal{S}} \int_{\mathbb{R}_+^n} \frac{1}{n!} (\overline{a_t})(x) h(\tau(x)) \mu^\otimes n(dx)$$

$$= \int_{\mathbb{R}_+^n} \frac{1}{n!} (\overline{a_t})(x) h(\tau(x)) \mu^\otimes n(dx).$$

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In particular, if \( g \equiv 1 \),
\[
\int_{\mathbb{R}_+^n} a_t(x) h(t(x)) \mu^{\otimes n}(dx) = \int_{\mathbb{R}_+^n} \frac{1}{n!} \overline{a_t}(x) h(t(x)) \mu^{\otimes n}(dx).
\]

Continuing the above computation with that property, we obtain
\[
\mathbb{E}^\nu [gh(t)] = \int_{\mathbb{R}_+^n} \frac{1}{n!} \overline{a_t}(x) \mu^{\otimes n}(x) \mathbb{I}_{[\overline{m}(x)>0]} h(t(x)) \mu^{\otimes n}(dx) = \int_{\mathbb{R}_+^n} a_t(x) \mu^{\otimes n}(x) \mathbb{I}_{[\overline{m}(x)>0]} h(t(x)) \mu^{\otimes n}(dx) = \mathbb{E}^\nu [\frac{\mu^{\otimes n}(x)}{\mu^{\otimes n}(x)} \mathbb{I}_{[\overline{m}(x)>0]} h(t)].
\]

This computation shows that
\[
\mathbb{E}^\nu [g|t = x] = \frac{\mu^{\otimes n}(x)}{\mu^{\otimes n}(x)} \mathbb{I}_{[\overline{m}(x)>0]}.
\]

Another consequence of the above computations is that, under Assumption 3.7, \( \tau \) satisfies also the density hypothesis with the density function with respect to \( \mu^{\otimes n} \) and to \( \mathcal{F}_t, t \in \mathbb{R}_+ \), given by \( \overline{a}_t(x) = \mathbb{1}_{\{x_1<x_2<\ldots<x_n\}} a_t(x) \) (cf. [4]).

### 3.3.1 Computing the \( \mathbb{F}^{\tau_{\psi}} \)-conditional expectation

We use the notation introduced in the previous subsections. Let \( \varphi \in \mathcal{I}(k, n) \) and \( T \subset I_k \) with \( j = \# T \). Note that, for \( t \in \mathbb{R}_+ \), the \( \sigma \)-algebra \( \mathcal{F}_t^{\sigma(\tau_{\psi}(T))} \) is generated by \( h(\tau_{\psi}(T)) \) where \( h \) runs over the family of all bounded functions on \( \Omega \times \mathbb{R}_+, \cap_{s \geq t} (\mathcal{F}_s \otimes \sigma(\mathbb{R}_+)) \) measurable.

**Lemma 3.9.** For any non negative \( \cap_{s \geq t} (\mathcal{F}_s \otimes \mathcal{B}(\mathbb{R}_+)) \) measurable function \( g \), for any bounded \( \mathbb{F}^{\sigma(\tau_{\psi}(T))} \) stopping time \( U \),
\[
\mathbb{E}_\varphi [g(\tau)|\mathcal{F}_{U}^{\sigma(\tau_{\psi}(T))}] = \frac{(ga_U)_{\psi(T)}(\tau_{\psi(T)})}{a_{\psi(T), U}(\tau_{\psi(T)})} \mathbb{I}_{[a_{\psi(T), U}(\tau_{\psi(T)}) > 0]}.
\]

If \( U \) is a bounded \( \mathbb{F}^{\sigma(\tau_{\psi}(T))} \) predictable stopping time, we also have
\[
\mathbb{E}_\varphi [g(\tau)|\mathcal{F}_{U-}^{\sigma(\tau_{\psi}(T))}] = \frac{(ga_{U-})_{\psi(T)}(\tau_{\psi(T)})}{a_{\psi(T), U-}(\tau_{\psi(T)})} \mathbb{I}_{[a_{\psi(T), U-}(\tau_{\psi(T)}) > 0]}.
\]

**Proof.** By monotone convergence theorem we only need to prove the lemma for \( 0 \leq g \leq 1 \). Let us show firstly the lemma for \( U = t \in \mathbb{R}_+ \). For a bounded function \( h \) on \( \Omega \times \mathbb{R}_+, \cap_{s \geq t} (\mathcal{F}_s \otimes \sigma(\mathbb{R}_+)) \) measurable, according Lemma 3.8,
\[
\mathbb{E}_\varphi [g(\tau)h(\tau_{\psi(T)})] = \mathbb{E}_\varphi [\int_{\mathbb{R}_+^n} g(x) h(x_{\psi(T)}) a_t(x) \mu^{\otimes n}(dx)] = \mathbb{E}_\varphi [\frac{(ga_t)_{\psi(T)}(\tau_{\psi(T)})}{a_{\psi(T), t}(\tau_{\psi(T)})} \mathbb{I}_{[a_{\psi(T), t}(\tau_{\psi(T)}) > 0]} h(\tau_{\psi(T)})] ;
\]
where the last equality comes from Lemma 3.8 applied with respect to $\tau_{\varrho(T)}$. The formula is proved for $U = t$, because $\frac{(g_{a t})_{\varrho(T)}(\tau_{\varrho(T)})}{a_{\varrho(T), t}(\tau_{\varrho(T)})} \mathbb{I}_{\{a_{\varrho(T), t}(\tau_{\varrho(T)}) > 0\}}$ is $\mathcal{F}_{t}^{\varrho(\tau_{\varrho(T)})}$ measurable.

For any $n \in \mathbb{N}$, for $(\omega, x) \in \Omega \times \mathbb{R}_{+}^{n}$, let $R^{n}(\omega, x) = \inf\{s \in \mathbb{Q}_{+} : a_{s}(\omega, x) > n\}$. Then, for $b \in \mathbb{R}_{+}$,

$$\{R^{n} \geq b\} = \{(\omega, x) \in \Omega \times \mathbb{R}_{+}^{n} : \forall s \in \mathbb{Q}_{+} \cap [0, b), a_{s}(\omega, x) \leq n\} \in \mathcal{F}_{b_{-}} \otimes \mathcal{B}(\mathbb{R}_{+}^{n}).$$

Applying the above formula at constant time $t$ to $g(\tau) \mathbb{I}_{\{t < R^{n}(\tau)\}}$, we can write

$$\mathbb{E}_{\varrho}[g(\tau) \mathbb{I}_{\{t < R^{n}(\tau)\}} | \mathcal{F}_{t}^{\varrho(\tau_{\varrho(T)})}] = \left(\frac{g \mathbb{I}_{\{t < R^{n}\}} a_{t}}{a_{\varrho(T), t}(\tau_{\varrho(T)})}\right) \mathbb{I}_{\{a_{\varrho(T), t}(\tau_{\varrho(T)}) > 0\}}.$$

Note that $g \mathbb{I}_{\{t < R^{n}\}} a_{t} \leq n$. By the dominated convergence theorem, for almost all $\omega$, the map

$$\left(\frac{g \mathbb{I}_{\{t < R^{n}\}} a_{t}}{a_{\varrho(T), t}(\tau_{\varrho(T)})}, t \in \mathbb{R}_{+}\right)$$

is right continuous. By [8, Lemme(1.8)], $\mathbb{I}_{\{a_{\varrho(T), t}(\tau_{\varrho(T)}) > 0\}}$ also is right continuous. Hence, the above formula can be extended to any bounded $\mathbb{F}^{\varrho(\tau_{\varrho(T)})}$ stopping time $U$:

$$\mathbb{E}_{\varrho}[g(\tau) \mathbb{I}_{\{U < R^{n}(\tau)\}} | \mathcal{F}_{U}^{\varrho(\tau_{\varrho(T)})}] = \left(\frac{g \mathbb{I}_{\{U < R^{n}\}} a_{U}}{a_{\varrho(T), U}(\tau_{\varrho(T)})}\right) \mathbb{I}_{\{a_{\varrho(T), U}(\tau_{\varrho(T)}) > 0\}}.$$

Note that, by dominated convergence theorem, $(g \mathbb{I}_{\{t < R^{n}\}} a_{t})_{\varrho(T)}(\tau_{\varrho(T)})$ has left limit $(g \mathbb{I}_{\{t < R^{n}\}} a_{t-})_{\varrho(T)}(\tau_{\varrho(T)})$. If $U$ is a bounded $\mathbb{F}^{\varrho(\tau_{\varrho(T)})}$ predictable stopping time, we also have

$$\mathbb{E}_{\varrho}[g(\tau) \mathbb{I}_{\{U \leq R^{n}(\tau)\}} | \mathcal{F}_{U-}^{\varrho(\tau_{\varrho(T)})}] = \left(\frac{g \mathbb{I}_{\{U \leq R^{n}\}} a_{U-}}{a_{\varrho(T), U-}(\tau_{\varrho(T)})}\right) \mathbb{I}_{\{a_{\varrho(T), U-}(\tau_{\varrho(T)}) > 0\}}.$$

Now let $n \uparrow \infty$ we prove the lemma. \hfill \Box

Notice that in the above formulas, we can remove the indicator $\mathbb{I}_{\{a_{\varrho(T), t}(\tau_{\varrho(T)}) > 0\}}$, because by [8, Corollaire(1.11)] the process $\mathbb{I}_{\{a_{\varrho(T), t}(\tau_{\varrho(T)}) = 0\}}$ is evanescent.

**Corollary 3.10.** For any Borel function $g$ on $\mathbb{R}_{+}^{n}$ such that $g(\tau)$ is integrable, the process $(g_{t})_{\varrho(T)}(\tau_{\varrho(T)}), t \in \mathbb{R}_{+}$, is càdlàg whose left limit is the process $(g_{t-})_{\varrho(T)}(\tau_{\varrho(T)}), t \in \mathbb{R}_{+}$.

**Remark 3.11.** Corollary 3.10 implies in particular that $(a_{t})_{\varrho(T)}, t \in \mathbb{R}_{+}$, is càdlàg so that $(a_{t})_{\varrho(T)}(\tau_{\varrho(T)}), t \in \mathbb{R}_{+}$, coincides with $a_{t, \varrho(T)}(\tau_{\varrho(T)}), t \in \mathbb{R}_{+}$. It is an important property in practice (for example, numerical implantation) because it gives a concrete way to compute $a_{t, \varrho(T)}(\tau_{\varrho(T)})$. 

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We define,
\[
\begin{align*}
\max x_{\varepsilon(T)} &:= \max_{i \in \mathbb{T}} x_{\varepsilon(i)}, \quad x \in \mathbb{R}_{+}^n, \\
\min x_{\varepsilon(I_k \setminus T)} &:= \min_{i \in I_k \setminus T} x_{\varepsilon(i)}, \quad x \in \mathbb{R}_{+}^n, \\
\mathcal{A}_{U,T,\varepsilon} &:= \{x \in \mathbb{R}_{+}^n : \max x_{\varepsilon(T)} \leq t, \min x_{\varepsilon(I_k \setminus T)} > t\}, \quad t \in \mathbb{R}_{+}.
\end{align*}
\]

Lemma 3.12. For any bounded \( \mathbb{F} \) stopping time \( U \) we have
\[
\mathcal{F}_U^{\tau_{\varepsilon}} = \mathcal{F}_U \vee \sigma(\tau_{\varepsilon} \uparrow U).
\]

For any subset \( T \) of \( I_k \), the process \( \mathbb{1}_{A_{U,T,\varepsilon}}(\tau) \), \( t \in \mathbb{R}_{+} \), is \( \mathbb{F}^{\tau_{\varepsilon}} \) optional and
\[
\mathcal{F}_U^{\tau_{\varepsilon}} \cap \{\tau \in \mathcal{A}_{U,T,\varepsilon}\} = (\mathcal{F}_U \vee \sigma(\tau_{\varepsilon} \uparrow U)) \cap \{\tau \in \mathcal{A}_{U,T,\varepsilon}\} = (\mathcal{F}_U \vee \sigma(\tau_{\varepsilon(T)})) \cap \{\tau \in \mathcal{A}_{U,T,\varepsilon}\}.
\]

Proof. We write
\[
\{\tau \in \mathcal{A}_{U,T,\varepsilon}\} = \{\forall i \in I, \tau_{\varepsilon(i)} \uparrow t < \infty, \forall i \in I_k \setminus T, \tau_{\varepsilon(i)} \uparrow t = \infty\},
\]
which is a \( \sigma(\tau_{\varepsilon} \uparrow t) \)-measurable set. Hence, the process \( \mathbb{1}_{A_{U,T,\varepsilon}}(\tau) \), \( t \in \mathbb{R}_{+} \), is \( \mathbb{F}^{\tau_{\varepsilon}} \) adapted. But it also is cádlág. The first assertion is proved.

Let \( \gamma = (\gamma_1, \ldots, \gamma_k) \) be the increasing re-ordering of \( \tau_{\varepsilon} \) and set \( \gamma_0 = 0, \gamma_{k+1} = \infty \). The density hypothesis with respect to \( \mathbb{F} \) holds for \( \tau_{\varepsilon} \), since it holds for \( \tau \). It is proved in [18] that the optional splitting formula holds with respect to \( \mathbb{F}^{\tau_{\varepsilon}} \). As a consequence, for \( 0 \leq j \leq k \),
\[
\mathcal{F}_U^{\tau_{\varepsilon}} \cap \{\gamma_j \leq U < \gamma_{j+1}\} = (\mathcal{F}_U \vee \sigma(\tau_{\varepsilon} \uparrow \gamma_j)) \cap \{\gamma_j \leq U < \gamma_{j+1}\} = (\mathcal{F}_U \vee \sigma(\tau_{\varepsilon} \uparrow U)) \cap \{\gamma_j \leq U < \gamma_{j+1}\}.
\]

Notice that
\[
\mathbb{1}_{\{\gamma_j \leq U < \gamma_{j+1}\}} = \sum_{T \subset I_k : \#T = j} \mathbb{1}_{A_{U,T,\varepsilon}}(\tau)
\]
is \( \mathcal{F}_U^{\tau_{\varepsilon}} \) as well as \( \mathcal{F}_U \vee \sigma(\tau_{\varepsilon} \uparrow U) \) measurable. Hence, we can take the union of the above identities to conclude
\[
\mathcal{F}_U^{\tau_{\varepsilon}} = \mathcal{F}_U \vee \sigma(\tau_{\varepsilon} \uparrow U).
\]

If the cardinal of \( T \) is equal to \( j \), we have \( \mathcal{A}_{U,T,\varepsilon} \subset \{\gamma_j \leq U < \gamma_{j+1}\} \) and the last claim of the lemma follows from the above identities together with the fact that \( \sigma(\tau_{\varepsilon} \uparrow U) \cap \mathcal{A}_{U,T,\varepsilon} = \sigma(\tau_{\varepsilon(T)}) \cap \mathcal{A}_{U,T,\varepsilon} \).

We introduce another notations \( U_{T,\varepsilon}(x) := \max x_{\varepsilon(T)} \) and \( S_{T,\varepsilon}(x) := \min x_{\varepsilon(I_k \setminus T)} \). Recall (cf. [9]) that, for any random time \( U \), \( \mathcal{F}_U \) (resp. \( \mathcal{F}_{U-} \)) denotes the \( \sigma \)-algebra generated by \( K_U \), where \( K \) denotes a \( \mathbb{F} \) optional (resp. predictable) process.
Lemma 3.13. For bounded $\mathbb{F}^\tau_e$ stopping time $U$, for any non negative $\mathcal{F}_U \otimes \mathcal{B}(\mathbb{R}^n)$ measurable function $g$, we have

$$\mathbb{E}_\phi(g(\tau) | \mathcal{F}_{U,e}^\tau) = \sum_{T \subset I_k} \mathbb{I}_{\{U,T,e(\tau) \leq U < S_T,e(\tau)\}} \frac{\mathbb{I}_{\{U < S_T,e\}} g_{\phi(U)}(T,e(T))}{\mathbb{I}_{\{U < S_T,e\}} g_{\phi(U)}(T,e(T))}. $$

Proof. By monotone convergence theorem we only need to prove the lemma for bounded $g$. By monotone class theorem, we only need to prove the lemma for bounded Borel function $g$ on $\mathbb{R}_+^n$. Let us firstly consider $U = t \in \mathbb{R}_+$. The parameters $U$, $T$ and $e$ being fixed, for simplicity we write $A$ instead of $A_{t,T,e}$. We also introduce

$$F = \{x \in \mathbb{R}_+^n : \min x_{(i_k \setminus T)} > t\}. $$

For any bounded $\mathcal{B}(\mathbb{R}^k)$-measurable function $h$, there exists a $\mathcal{B}(\mathbb{R}^{|T|})$-measurable function $h'$ such that $h(\tau \mid t) = h'(\tau_{\phi(T)})$ on $A$. Let $B \in \mathcal{F}_t$. With help of Lemma 3.9, we compute

$$\begin{align*}
\mathbb{E}_\phi \left( \mathbb{B} h(\tau \mid t) \mathbb{I}_A(\tau) \mathbb{E}[g(\tau)|\mathcal{F}_{t,e}^\tau] \right) &= \mathbb{E}_{\phi} \left( \mathbb{B} h'(\tau_{\phi(T)}) \mathbb{I}_A(\tau) g(\tau) \right) \\
&= \mathbb{E}_\phi \left( \mathbb{B} h'(\tau_{\phi(T)}) \mathbb{I}_{\max \tau_{\phi(T)} \leq t} \mathbb{I}_{\min \tau_{\phi(i_k \setminus T)} > t} g(\tau) \right) \\
&= \mathbb{E}_\phi \left( \mathbb{B} h'(\tau_{\phi(T)}) \mathbb{I}_{\max \tau_{\phi(T)} \leq t} \mathbb{E}_\phi \left[ \mathbb{I}_{\min \tau_{\phi(i_k \setminus T)} > t} g(\tau) | \mathcal{F}_{t,e}^\tau \right] \right) \\
&= \mathbb{E}_\phi \left( \mathbb{B} h'(\tau_{\phi(T)}) \mathbb{I}_{\max \tau_{\phi(T)} \leq t} \mathbb{I}_{\min \tau_{\phi(i_k \setminus T)} > t} \mathbb{I}_A(\tau) \right) \mathbb{E}_\phi \left[ \mathbb{I}_{\min \tau_{\phi(i_k \setminus T)} > t} g(\tau) | \mathcal{F}_{t,e}^\tau \right] \\
&= \mathbb{E}_\phi \left( \mathbb{B} h(\tau \mid t) \mathbb{I}_A(\tau) \mathbb{I}_{\max \tau_{\phi(T)} \leq t} \mathbb{I}_{\min \tau_{\phi(i_k \setminus T)} > t} \mathbb{I}_A(\tau) \right) \mathbb{I}_{\min \tau_{\phi(i_k \setminus T)} > t} g(\tau) | \mathcal{F}_{t,e}^\tau \right]
end{align*}$$

It is to note that the random variables

$$\mathbb{I}_A(\tau) \text{ and } \mathbb{I}_A(\tau) \mathbb{I}_{\max \tau_{\phi(T)} \leq t} \mathbb{I}_{\min \tau_{\phi(i_k \setminus T)} > t} \mathbb{I}_A(\tau) \mathbb{I}_{\min \tau_{\phi(i_k \setminus T)} > t} g(\tau) | \mathcal{F}_{t,e}^\tau \right]$$

are $\mathcal{F}_{t,e}^\tau = \mathcal{F}_t \vee \sigma(\tau_{\phi(T)} \mid t)$ measurable. By Lemma 3.12, the above computation implies that

$$\mathbb{I}_A(\tau) \mathbb{E}[g(\tau)|\mathcal{F}_{t,e}^\tau] = \mathbb{I}_A(\tau) \frac{\mathbb{I}_A(\tau) g_{\phi(U)}(T,e(T))}{\mathbb{I}_A(\tau) g_{\phi(U)}(T,e(T))}. $$

Recall that $\gamma = (\gamma_1, \ldots, \gamma_k)$ is the increasing re-ordering of $\tau_{\phi}$ and $\gamma_0 = 0$, $\gamma_{k+1} = \infty$. Notice also that, under Assumption 3.7, $\mathbb{P}(\tau_i = \tau_j) = 0$ for any pair of $i, j$ such that $i \neq j$. It results that

$$\sum_{T \subset I_k : \#T = j} \mathbb{I}_{A_{t,T,e}}(\tau) = \mathbb{I}_{\{\gamma_j \leq t < \gamma_{j+1}\}};$$
and 
\[
\mathbb{E}[g(\tau)|F^\tau_t] = \sum_{j=0}^{k} \mathbb{I}_{\{\tau_j \leq t < \tau_{j+1}\}} \mathbb{E}[g(\tau)|F^\tau_t],
\]
\[
\sum_{j=0}^{k} \sum_{T \subset I_k} \mathbb{I}_{\{T = j\}} \mathbb{E}[g(\tau)|F^\tau_t]
\]
\[
\sum_{T \subset I_k} \mathbb{I}_{\{U_{T,\varrho}(\tau) \leq \tau \}} \mathbb{E}[g(\tau)|F^\tau_t]
\]
\[
= \sum_{T \subset I_k} \mathbb{I}_{\{U_{T,\varrho}(\tau) \leq \tau \}} \mathbb{E}[g(\tau)|F^\tau_t] \cdot \frac{1}{(\mathbb{I}_{\{\tau \leq \eta\}} g a_U(\eta))_{\varrho}(\gamma(\eta))}.
\]

Applying Corollary 3.10, we extend this formula to any bounded $F^\tau_t$ stopping time $U$.

**Corollary 3.14.** For any bounded $F^\tau_t$ predictable stopping time $U$, for any non negative $F^\tau_t \otimes \mathcal{B}(\mathbb{R}^n)$ measurable function $g$, we have
\[
\mathbb{E}_F(g(\tau)|F^\tau_{U-}) = \sum_{T \subset I_k} \mathbb{I}_{\{U_{T,\varrho}(\tau) < \tau \}} \mathbb{E}_{\{U < \tau \}} \left( \mathbb{I}_{\{\tau \leq \eta\}} g a_U(\eta) \right)_{\varrho}(\gamma(\eta)) \cdot \frac{1}{(\mathbb{I}_{\{\tau \leq \eta\}} g a_U(\eta))_{\varrho}(\gamma(\eta))}.
\]

### 3.3.2 Computing the $F^\tau_t$-conditional expectation

The following lemma is straightforward.

**Lemma 3.15.** For $t \in \mathbb{R}_+$, for any non negative $F^\tau_t \otimes \mathcal{B}(\mathbb{R}^n)$ measurable function $g$, we have
\[
\mathbb{E}_F[g(\tau)|F^\tau_t] = \mathbb{E}_{\{\tau = \tau \}} \mathbb{E}_F[g(\tau)|\tau = \tau \} = \mathbb{E}_{\{\varrho(\tau) > 0\}} \frac{g a_U(\varrho(\tau))}{a_U(\varrho(\tau))} \mathbb{I}_{\{\tau = \tau \}} \mathbb{I}_{\{\varrho(\tau) > 0\}}.
\]

We will denote the last random variable by $\tilde{g}(\tau)$. We now can apply the results in subsection 3.3.1 on the vector $\tau$ with $\rho$ being the identity map in $I_k$. Then, $\mathbb{I}_{\{\tau \} \neq 0}$, only if $T$ is of the form $T = I_j = \{1, \ldots, j\}$ for $0 \leq j \leq k$ ($T = \emptyset$ if $j = 0$) and, in this case,
\[
\mathbb{I}_{\{\tau \} \neq 0} = \mathbb{I}_{\{\tau_j \leq \tau \}} \mathbb{I}_{\{\tau_{j+1} \}} = \mathbb{I}_{\{\tau_{j+1} \}} = \mathbb{I}_{\{\tau_{j+1} \}} = \mathbb{I}_{\{\tau_j \leq \tau \}} = \mathbb{I}_{\{\tau_j \leq \tau \}} = \mathbb{I}_{\{\tau_j \leq \tau \}}.
\]

Let $p_i$ be the projection $p_i(x) = x_i$. From Lemma 3.13 we obtain

**Lemma 3.16.** For any bounded $F^\tau_t$ stopping time $U$, for any non negative $F^\tau_t \otimes \mathcal{B}(\mathbb{R}^n)$ measurable function $g$, we have
\[
\mathbb{E}_F(g(\tau)|F^\tau_{U}) = \sum_{j=0}^{k} \mathbb{I}_{\{\tau_j \leq \tau \}} \mathbb{E}_{\{\tau \leq \tau \}} \left( \mathbb{I}_{\{\tau \leq \tau \}} g a_U(\tau) \right)_{\varrho}(\gamma(\tau)) \cdot \frac{1}{(\mathbb{I}_{\{\tau \leq \tau \}} g a_U(\tau))_{\varrho}(\gamma(\tau))}.
\]

For any bounded $F^\tau_t$ predictable stopping time $U$, for any non negative $F^\tau_t \otimes \mathcal{B}(\mathbb{R}^n)$ measurable function $g$, we have
\[
\mathbb{E}_F(g(\tau)|F^\tau_{U-}) = \sum_{j=0}^{k} \mathbb{I}_{\{\tau_j \leq \tau \}} \mathbb{E}_{\{\tau \leq \tau \}} \left( \mathbb{I}_{\{\tau \leq \tau \}} g a_U(\tau) \right)_{\varrho}(\gamma(\tau)) \cdot \frac{1}{(\mathbb{I}_{\{\tau \leq \tau \}} g a_U(\tau))_{\varrho}(\gamma(\tau))}.
\]
3.3.3 The \( \mathbb{F}^\tau \) drift computation

We fix a \( \varrho \in \mathcal{I}(k, n) \). We recall the notations \( T \subset I_k \), \( A_{s, T, \varrho} := \{ \max \tau_{\varrho(T)} \leq s, \min \tau_{\varrho(I_k \setminus T)} > s \} \), \( U_{T, \varrho}(x) := \max x_{\varrho(T)} \) and \( S_{T, \varrho}(x) := \min x_{\varrho(I_k \setminus T)} \), also \( (\gamma_1, \ldots, \gamma_k) \) representing the increasing re-ordering of \( (\tau_{\varrho(1)}, \ldots, \tau_{\varrho(k)}) \).

In next subsection we will compute the \( \mathbb{F}^\tau_k \)-semimartingale decomposition of a bounded \( \mathbb{F} \)-martingale \( M \) by first computing \( 1_{\mathcal{D}^\varrho \hat{V}^\varrho} \), which is the stochastic integral of \( 1_{\mathcal{D}^\varrho} N^\varrho \) against the \( \mathbb{F}^\tau \)-drift of \( N^\varrho M \) and then computing its dual \( \mathbb{F}^\tau_k \)-predictable projection. This subsection is devoted to the computation of the drift of \( N^\varrho M \).

We notice that this computation is an extension of the usual \( \mathbb{F}^\tau \) semimartingale decomposition formula for \( M \).

For a bounded Borel function \( g \) on \( \mathbb{R}_+^n \) we set

\[
L^g_t = \mathbb{E}_t \left( g(\tau) \mid \mathcal{F}^\tau_t \right), \quad t \in \mathbb{R}_+.
\]

A formula is given in Lemma 3.13 dealing with \( L^g \). But that formula is not adapted to the computation that we will do in this subsection. From [8, Théorème (2.5)], there exists a \( \mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+^n) \)-measurable process \( u^M_v(\omega, x) \) such that

\[
d \langle a(x), M \rangle_v = u^M_v(x) d \langle M, M \rangle_v,
\]

where the predictable bracket \( \langle M, M \rangle \) is calculated in the filtration \( \mathbb{F} \). The process \( u(x) \) is known to satisfy

\[
\int_0^t \frac{1}{a_{s-}(\tau)} |u^M_s(\tau)| d \langle M, M \rangle_s < \infty, \quad \forall t \in \mathbb{R}_+,
\]

(16)

(so that we assume that \( u^M_s(x) = 0 \) whenever \( a_{s-}(x) = 0 \)). But the computations below will require a stronger condition.

**Assumption 3.17.** There exists an increasing sequence \( (R_n)_{n \in \mathbb{N}} \) of bounded \( \mathbb{F} \) stopping times such that \( \sup_{n \in \mathbb{N}_+} R_n = \infty \) and

\[
\mathbb{E} \left[ \int_0^{R_n} \frac{1}{a_{s-}(\tau)} |u^M_s(\tau)| d \langle M, M \rangle_s \right] < \infty, \quad \forall n \in \mathbb{N}_+.
\]

Notice that the above inequality is equivalent to

\[
\mathbb{E} \left[ \int_0^{R_n} |u^M_s(x)| d \langle M, M \rangle_s \mu^{\otimes n}(dx) \right] < \infty.
\]

We give here a sufficient condition for Assumption 3.17 to hold for any bounded \( \mathbb{F} \) martingale \( M \).
Lemma 3.18. Suppose that there exists an increasing sequence \((R_n)_{n \in \mathbb{N}}\) of bounded \(\mathbb{F}\) stopping times such that

\[
\int \mathbb{E}[\sqrt{\langle a(x), a(x) \rangle} R_n] \mu^\otimes n(dx) < \infty.
\]

Then, for any bounded \(\mathbb{F}\) martingale \(M\),

\[
\mathbb{E}\left[ \int \int_{0}^{R_n} |u_s^M(x)| d\langle M, M \rangle_s \mu^\otimes n(dx) \right] < \infty
\]

for all \(n \in \mathbb{N}\), i.e. Assumption 3.17 holds.

Proof. Let \(H_s(x) = \text{sign} u_s(x)\). We have

\[
\int \mathbb{E}\left[ \int_{0}^{R_n} |a(x), H(x) \cdot M|_s \mu^\otimes n(dx) \right] = \int \sqrt{2} \mathbb{E}[\langle a(x), a(x) \rangle^12 R_n] \parallel H(x) \cdot M \parallel_{BMO} \mu^\otimes n(dx).
\]

We note that \(\parallel H(x) \cdot M \parallel_{BMO}\) is computed by its bracket (cf. [6, Theorem 10.9]) so that it is uniformly bounded by a multiple of \(\parallel M_{\infty} \parallel_{\infty}\). This boundedness together with the assumption of the lemma enables us to write

\[
\mathbb{E}\left[ \int \int_{0}^{R_n} |u_s^M(x)| d\langle M, M \rangle_s \mu^\otimes n(dx) \right] = \int \mathbb{E}\left[ \int_{0}^{R_n} H_s(x) u_s^M(x) d\langle M, M \rangle_s \mu^\otimes n(dx) \right] = \int \mathbb{E}\left[ \int_{0}^{R_n} d\langle a(x), H(x) \cdot M \rangle_s \mu^\otimes n(dx) \right] = \int \mathbb{E}\left[ \int_{0}^{R_n} d\langle a(x), H(x) \cdot M \rangle \right] \mu^\otimes n(dx) < \infty.
\]

\(\blacksquare\)

Theorem 3.19. Under Assumption 3.17, for any bounded Borel function \(g\) on \(\mathbb{R}^n_+\), the drift of \(L^g M\) in \(\mathbb{F}^\tau_v\) is given by

\[
\sum_{T \subset I_k} \int_0^t 1_{\{U_{T, \varphi}(\tau_v) < v \leq S_{T, \varphi}(\tau_v)\}} \left( \mathbb{1}_{\{v \leq S_{T, \varphi}(\tau_v)\}} g u^M_{e(T)}(\tau_v(T)) \right) \mathbb{1}_{\{v \leq S_{T, \varphi}(\tau_v)\}} d\langle M, M \rangle_v, \ t \in \mathbb{R}^+.
\]

Proof. Let \(T \subset I_k\) and \(j = \# T\). Let \(R\) be one of \(R_n\) in Assumption 3.17. We compute the following for \(s, t \in \mathbb{R}_+, s \leq t, B \in \mathcal{F}_s\) and \(h\) a bounded Borel function on \(\mathbb{R}^j_+\). Note that Fubini’s theorem can be applied because of, on the one hand, the boundedness of \(M\) and, on the other hand, of Assumption 3.17. As \(g, T,\) and \(t \in \mathbb{R}_+\) are known,
we simply write \( U(x) = U_{T,T}(x) \) and \( S(x) = S_{T,T}(x) \).

\[
\mathbb{E}_p \left( \int_0^t 1_B h(\tau_{\theta}(T)) \mathbb{I}_{\{s \leq \tau_{\theta}(T) \leq t\}} \mathbb{I}_{\{U(\tau_{\theta}(T)) < v \leq S(\tau_{\theta}(T)) \}} \mathbb{I}_{\{v \leq R \}} d(L^0 M)_v \right) = \mathbb{E}_p \left( 1_B h(\tau_{\theta}(T)) \mathbb{I}_{\{s \leq \tau_{\theta}(T) \leq t\}} \mathbb{I}_{\{U(\tau_{\theta}(T)) < v \leq S(\tau_{\theta}(T)) \}} \mathbb{I}_{\{v \leq R \}} \right) \left( L^0 M \right)_{\tau_{\theta}(T) \wedge \tau_{\theta}(T) \wedge R} - L^0 M_{(s \vee U(\tau_{\theta}(T))) \wedge (\tau_{\theta}(T) \wedge R)} \bigg) \\
= \mathbb{E}_p \left( 1_B h(\tau_{\theta}(T)) \mathbb{I}_{\{s \leq \tau_{\theta}(T) \leq t\}} \mathbb{I}_{\{U(\tau_{\theta}(T)) < v \leq S(\tau_{\theta}(T)) \}} \mathbb{I}_{\{v \leq R \}} \right) \left( M_{\tau_{\theta}(T) \wedge R} - M_{(s \vee U(\tau_{\theta}(T))) \wedge (\tau_{\theta}(T) \wedge R)} \bigg) \\
= \mathbb{E}_p \left( 1_B h(\tau_{\theta}(T)) \mathbb{I}_{\{s \leq \tau_{\theta}(T) \leq t\}} \mathbb{I}_{\{U(\tau_{\theta}(T)) < v \leq S(\tau_{\theta}(T)) \}} \mathbb{I}_{\{v \leq R \}} \right) \left( M_{\tau_{\theta}(T) \wedge R} - M_{(s \vee U(\tau_{\theta}(T))) \wedge (\tau_{\theta}(T) \wedge R)} \bigg) \\
= \mathbb{E}_p \left( 1_B h(\tau_{\theta}(T)) \mathbb{I}_{\{s \leq \tau_{\theta}(T) \leq t\}} \mathbb{I}_{\{U(\tau_{\theta}(T)) < v \leq S(\tau_{\theta}(T)) \}} \mathbb{I}_{\{v \leq R \}} \right) \left( M_{\tau_{\theta}(T) \wedge R} - M_{(s \vee U(\tau_{\theta}(T))) \wedge (\tau_{\theta}(T) \wedge R)} \bigg) \\
= \mathbb{E}_p \left[ 1_B \int h(x,\theta(T)) g(x)(M_{\tau_{\theta}(T) \wedge R} - M_{(s \vee U(\tau_{\theta}(T))) \wedge (\tau_{\theta}(T) \wedge R)} a_t(x) \mu_{\theta}(dx) \right] \\
= \int h(x,\theta(T)) g(x) \mathbb{E}_p \left[ 1_B \int h(x,\theta(T)) g(x)(M_{\tau_{\theta}(T) \wedge R} - M_{(s \vee U(\tau_{\theta}(T))) \wedge (\tau_{\theta}(T) \wedge R)} a_t(x) \mu_{\theta}(dx) \right] \\
\text{consequence of \cite{[8], Lemma(1.10)],} \\
= \mathbb{E}_p \left( 1_B h(\tau_{\theta}(T)) \mathbb{I}_{\{s \leq \tau_{\theta}(T) \leq t\}} \mathbb{I}_{\{U(\tau_{\theta}(T)) < v \leq S(\tau_{\theta}(T)) \}} \mathbb{I}_{\{v \leq R \}} \right) \int_0^t \mathbb{I}_{\{u \leq S(\tau_{\theta}(T)) \}} \frac{\mathbb{I}_{\{v \leq S\}} \mu_{\theta}(\mu_{\theta}(T) \wedge R)}{\mathbb{I}_{\{v \leq S\}} \mu_{\theta}(T)} d(M, M)_v \mathbb{I}_{\{v \leq R \}} \mathbb{I}_{\{v \leq S(\tau_{\theta}(T)) \}} \mathbb{I}_{\{v \leq R \}} \mathbb{I}_{\{v \leq S(\tau_{\theta}(T)) \}} \mathbb{I}_{\{v \leq R \}}
\]

Note that \( R = R_a \) tends to the infinity and the processes \( 1_B h(\tau_{\theta}(T)) \mathbb{I}_{\{s \leq \tau_{\theta}(T) \leq t\}} \) generate all bounded \( \mathbb{F}^\tau \) predictable processes on \((U(\tau_{\theta}), S(\tau_{\theta}))\). The above computation means that the drift of \( \mathbb{I}_{\{U(\tau_{\theta}), S(\tau_{\theta})\}} \cdot (L^0 M) \) is given by

\[
\int_0^t \mathbb{I}_{\{U(\tau_{\theta}) < v \leq S(\tau_{\theta})\}} \frac{\mathbb{I}_{\{u \leq S(\tau_{\theta})\}} \mathbb{I}_{\{t \leq S(\tau_{\theta})\}}}{\mathbb{I}_{\{u \leq S(\tau_{\theta})\}} \mathbb{I}_{\{t \leq S(\tau_{\theta})\}}} \frac{\mathbb{I}_{\{v \leq S(\tau_{\theta})\}}}{\mathbb{I}_{\{v \leq S(\tau_{\theta})\}}} d(M, M)_v \mathbb{I}_{\{v \leq R \}} \mathbb{I}_{\{v \leq S(\tau_{\theta})\}} \mathbb{I}_{\{v \leq R \}} \mathbb{I}_{\{v \leq S(\tau_{\theta})\}} \mathbb{I}_{\{v \leq R \}}
\]

The lemma follows because

\[
\sum_{T \in I_k} \mathbb{I}_{\{U_{\tau_{\theta}}, S_{\tau_{\theta}}\}} = \mathbb{I}_{[0,\infty]}.
\]

\[\square\]

3.3.4 The \( \mathbb{F}^\tau \)-semimartingale decomposition

Denote by \( p(\theta)_i \) the map \( x \rightarrow p_i(x,\theta) \).

**Theorem 3.20.** Under Assumption 3.17, the \( \mathbb{F}^\tau \)-drift of the bounded \( \mathbb{F} \)-martingale \( M \) is given by

\[
\sum_{\theta \in \Gamma^k} \sum_{j=0}^k \int_0^t \mathbb{I}_{\{\tau_{\theta}(j) < v \leq \tau_{\theta}(j+1) \}} \frac{\mathbb{I}_{\{v \leq p(\theta)_{j+1}\}} \mathbb{I}_{\{v \leq p(\theta)_{j+1}\}}}{\mathbb{I}_{\{v \leq p(\theta)_{j+1}\}} \mathbb{I}_{\{v \leq p(\theta)_{j+1}\}}} \mathbb{I}_{\{v \leq \tau_{\theta}(j) \}} \mathbb{I}_{\{v \leq \tau_{\theta}(j) \}} \mathbb{I}_{\{v \leq \tau_{\theta}(j) \}} \mathbb{I}_{\{v \leq \tau_{\theta}(j) \}} \mathbb{I}_{\{v \leq \tau_{\theta}(j) \}} \mathbb{I}_{\{v \leq \tau_{\theta}(j) \}} \mathbb{I}_{\{v \leq \tau_{\theta}(j) \}} \mathbb{I}_{\{v \leq \tau_{\theta}(j) \}} \mathbb{I}_{\{v \leq \tau_{\theta}(j) \}} \mathbb{I}_{\{v \leq \tau_{\theta}(j) \}} d(M, M)_v,
\]
for \( t \in \mathbb{R}_+ \), where \( \zeta_\psi(x) = \mathbb{I}_{\{x_{\psi(1)} \leq \ldots \leq x_{\psi(k)} < \min x_{I_{n}\setminus\{I_k\}}\}} \), and

\[
\tilde{a}_{v,-}^\psi(x) = \mathbb{I}_{\{x_1 < x_2 < \ldots < x_n\}} \sum_{\pi \in \mathbb{S}: \pi(\psi(i)) = i, \forall i \in I_k} a_{\pi,-}(\pi(x)), \quad x \in \mathbb{R}_+^n.
\]

**Proof.** According to Lemma 3.6, we only need to calculate the \( \mathbb{F}^{\mathfrak{F}_n} \)-predictable process of finite variation \( \tilde{N}^\psi \) \( \psi(\tilde{V}) \) for \( g \in T(k, n) \). Note that \( \zeta_\psi(\tau) = 1_{d_g} = 1_{D_g} \) because of Assumption 3.7. On the set \( \{x_{\psi(1)} < \ldots < x_{\psi(k)} < \min x_{I_{n}\setminus\{I_k\}}\} \), \( \{x : U_{T,\psi}(x_<) < v \leq S_{T,\psi}(x_<) = \emptyset \) if \( T \neq I_j \) with \( j = \#T \), while \( \{x : U_{I_j,\psi}(x_<) < v \leq S_{I_j,\psi}(x_<)\} = \{x : x_{\psi(j)} < v \leq x_{\psi(j+1)}\} \).

If we set \( g = \zeta_\psi \) in the above Theorem 3.19, we obtain the drift of \( N^\psi M \) in \( \mathbb{F}^{\mathfrak{F}_n} \).

According to Lemma 2.14, this drift process coincides with \( N^\psi \cdot \tilde{V}_\psi \). Consequently,

\[
\mathbb{I}_{D_g} \tilde{V}_\psi^t = \sum_{j=0}^{k} \int_0^t \frac{\zeta_\psi(\tau)}{N_{v,-}^\psi} \mathbb{I}_{\{(v < u, (I_{j+1}), (I_{j}), u, v)(\mathfrak{F}_v)\}} d\langle M, M \rangle_v
\]

for \( t \in \mathbb{R}_+ \). Note that \( N_{v,-}^\psi \) is computed by Corollary 3.14. On the set \( d_g \),

\[
N_{v,-}^\psi = \mathbb{E}(\zeta_\psi(\tau) | \mathcal{F}_v)
\]

\[
= \sum_{T \subseteq I_k} \mathbb{I}_{\{U_{T,\psi}(\tau) < v \leq S_{T,\psi}(\tau)\}} \frac{(1_{v < \min x_{I_{n}\setminus\{I_k\}}} \zeta_\psi u_{(I_{j+1})(I_{j})} \zeta_\psi(\tau))}{(1_{v < \min x_{I_{n}\setminus\{I_k\}}} \zeta_\psi(\tau))} d\langle M, M \rangle_v
\]

This yields

\[
\mathbb{I}_{D_g} \tilde{V}_\psi^t = \sum_{j=0}^{k} \int_0^t \zeta_\psi(\tau) \mathbb{I}_{\{(v < u, (I_{j+1}), (I_{j}), u, v)(\mathfrak{F}_v)\}} d\langle M, M \rangle_v
\]

By Lemma 2.14 we need to compute the \( \mathbb{F}^{\mathfrak{F}_n} \) dual predictable projection of the above process to obtain \( \tilde{N}^\psi \cdot \psi(\tilde{V}_\psi) \). Since \( \langle M, M \rangle \) is \( \mathbb{F} \)-predictable, it is enough to compute the predictable projection of the integrand. We compute firstly
Applying Lemma 3.16 the dual predictable projection $\tilde{\mathbb{N}}^\psi_t$ is given by

$$
\sum_{j=0}^{k} \int_0^t \mathbb{1}_{\tau(j) < v \leq \tau(j+1)} \left( \mathbb{1}_{\{v \leq p(j+1)\}} \zeta^\psi_{u^M_{v}}(\tau_j) \right) \left( \mathbb{1}_{\{v \leq p(j+1)\}} \zeta^{\bar{a}_{v^-}}_{u^-}(\tau_j) \right) d\langle M, M \rangle_v,
$$

for $t \in \mathbb{R}_+$. \hfill \Box

**Remark 3.21.** In the formula of the $\mathbb{F}^{\tau_k}$-drift of the bounded $\mathbb{F}$-martingale $M$, we see clearly the term

$$
\left( \mathbb{1}_{\{v \leq p(j+1)\}} \zeta^\psi_{u^M_{v}}(\tau_j) \right) \left( \mathbb{1}_{\{v \leq p(j+1)\}} \zeta^{\bar{a}_{v^-}}_{u^-}(\tau_j) \right)
$$

coming from the $\mathbb{F}^{\tau_0}$ decomposition of $M$, which appears in the formula with the weight

$$
\left( \mathbb{1}_{\{v \leq p(j+1)\}} \zeta^{\bar{a}^-}_{u^-}(\tau_j) \right) \left( \mathbb{1}_{\{v \leq p(j+1)\}} \zeta^{\bar{a}^-}_{u^-}(\tau_j) \right).
$$

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