Some remarks on critical sets of Laplace eigenfunctions
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Abstract. We study the set of critical points of a solution to $\Delta u = \lambda \cdot u$ and in particular components of the critical set that have codimension 1. We show, for example, that if a second Neumann eigenfunction of a simply connected polygon $P$ has infinitely many critical points, then $P$ is a rectangle.

1. Introduction
Let $u : \Omega \to \mathbb{R}$ satisfy $\Delta u = \lambda \cdot u$ where $\Omega \subset \mathbb{R}^d$ is an open set, $\Delta$ is the Laplacian, and $\lambda \in \mathbb{R}$. In this note, we study the critical set $\text{crit}(u) := \{\nabla u\}^{-1}(0)$ of $u$ and in particular the consequences of having a hypersurface contained in $\text{crit}(u)$. For example, we show that if a connected component $A$ of $\text{crit}(u)$ contains a hypersurface, then $A$ is a proper smooth hypersurface (Theorem 4), and we show that if $A$ is contained in a hyperplane or sphere, then $u$ depends only on the distance to the hyperplane or sphere (Theorem 11). We then derive consequences for second Neumann eigenfunctions on simply connected domains. For example, in §4 we prove the following:

**Theorem 1** (Compare Theorem 17 below). Let $P \subset \mathbb{R}^2$ be a bounded, simply connected, polygonal domain, and let $u : P \to \mathbb{R}$ be a second Neumann eigenfunction of the Laplacian. If the set of critical points lying in $P$ is not finite, then $P$ is a rectangle and $u$ is a multiple of $x \mapsto \cos(\sqrt{\lambda} \cdot \text{dist}(x, e))$ where $e$ is a side of $P$.

More generally, we show that if $u : \Omega \to \mathbb{R}$ is a second Neumann eigenfunction on a bounded, simply connected, open set $\Omega \subset \mathbb{R}^d$ with Lipschitz boundary, then $\text{crit}(u) \cap \Omega$ does not contain a hypersurface (Proposition 14). One wonders whether there exists a non-simply connected domain $\Omega \subset \mathbb{R}^d$ whose second Neumann eigenfunction contains a critical hypersurface.

Many of the results described here may be formulated so as to apply to Laplacians associated to a real-analytic metric on a real-analytic manifold. Moreover, proofs of some

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† Research partially supported by a Simons collaboration grant.
* partially supported by Ramanujan Fellowship of SERB, Govt. of India.

2010 Mathematics Subject Classification: Primary: 35P99. Secondary: 58J50.

Key words and phrases: Laplacian, eigenfunction, critical point.
results—for example Theorem 11 and Proposition 14—extend to the case of smooth metrics. It would be interesting to know exactly which results extend to smooth metrics. For example, Theorem 1 implies that a sequence of isolated critical points of $u$ does limit to the boundary of the polygon $P$. Does there exist a Lipschitz domain $\Omega \subset \mathbb{R}^2$ and a Neumann eigenfunction $u : \Omega \to \mathbb{R}$ so that the crit$(u)$ consists of infinitely many isolated points? Note that it was recently proved in [BLS20] that there exists a smooth metric on the 2-torus so that the associated Laplacian has a sequence $u_n$ of eigenfunctions with eigenvalues tending to infinity so that crit$(u_n)$ consists of infinitely many isolated points.

In connection to Proposition 14 we would like to mention the Schiffer’s conjecture. A variant of this conjecture says that a Neumann eigenfunction on a simply connected domain can have a loop in its critical set if and only if the domain is a disc and the loop is a distance circle [W76].

2. Critical hypersurfaces

The singular set, sing$(u)$, consists of critical points $x$ of $u$ such that $u(x) = 0$. The following is Lemma (1.9) in [HrtSmn89] specialized to the case of the Euclidean Laplacian. See also [CfrFrd85].

**Lemma 2.** Let $\Omega \subset \mathbb{R}^d$ be an open set and let $u : \Omega \to \mathbb{R}$ be a nonconstant Laplace eigenfunction. If $U$ is open and bounded with $\overline{U} \subset \Omega$, then sing$(u) \cap U$ is contained in the union of finitely many closed analytic submanifolds of codimension two.

For the convenience of the reader we provide the proof of [HrtSmn89]. Let $\partial_i$ denote the partial derivative $\partial / \partial x_i$, and for each multi-index $\alpha \in \mathbb{N}^d$, set $\partial^\alpha := \partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d}$.

**Proof.** Let $S_d$ be the set of points $x \in \Omega$ such that $\partial^\alpha u(x) = 0$ for each $|\alpha| < d$ and such that $\partial^\alpha u(x) \neq 0$ for some $\alpha$ with $|\alpha| = d$. Since $u$ is real-analytic, $\Omega$ equals the disjoint union $\bigcup_{d=0}^{\infty} S_d$. Because $\partial^\alpha u(x)$ is continuous and $\overline{U} \subset \Omega$ is compact, there exists $d_U$ such that $S_d \cap U = \emptyset$ for each $d \geq d_U$.

Each singular point lies in some $S_d$ with $2 \leq d \leq d_U$, and so to prove the claim, it suffices to show that for each $d \geq 2$, the set $S_d$ is contained in finitely many codimension two submanifolds. If $x \in S_d$ with $d \geq 2$, then $u(x) = 0$ and there exists $\alpha$ with $|\alpha| = d - 2$ so that Hess$(\partial^\alpha u)(x) \neq 0$. On the other hand, $\Delta \partial^\alpha u(x) = \lambda \partial^\alpha u(x) = 0$ and hence the trace of Hess$(\partial^\alpha u)(x) = 0$. Therefore, the rank of Hess$(\partial^\alpha u)(x)$ is at least two. In particular, there exist $j$ and $k$ so that the vectors $\nabla(\partial_j \partial^\alpha u)(x)$ and $\nabla(\partial_k \partial^\alpha u)(x)$ are nonzero and linearly independent. The analytic implicit function theorem then gives a neighborhood $V_x$ of $x$
such that $\partial_j \partial^a u^{-1}(0) \cap \partial_j \partial^a u^{-1}(0) \cap V_x$ is a codimension two analytic submanifold of $U$. Since $\overline{U}$ is compact, there exists a finite set $F \subset U$ such that $U = \cup_{x \in F} V_x$, and hence finitely many codimension two submanifolds cover $S_d$. □

The following is immediate.

**Corollary 3.** Let $\Omega \subset \mathbb{R}^d$ be an open set and let $u : \Omega \to \mathbb{R}$ be a nonconstant Laplace eigenfunction. Suppose that $A$ is a connected component of $\text{crit}(u)$. If there exists an open set $U$ such that $U \cap A$ is a hypersurface, then $A$ does not intersect $u^{-1}(0)$.

The Cauchy-Kovalevskaya theorem implies that each solution to $\Delta u = \lambda \cdot u$ is real-analytic. Thus, each partial derivative $\partial_j u$ is real-analytic, and therefore $\text{crit}(u)$ is a real-analytic variety. In general, every real-analytic variety $S$ possesses a stratification. In particular, there exists a (possibly disconnected) $m$-dimensional real-analytic submanifold $R \subset S$ so that the Hausdorff dimension of $S \setminus R$ is at most $m - 1$.\(^1\) In the case where $S$ is a connected component of $\text{crit}(u)$, the following implies that $S \setminus R$ is empty.

**Theorem 4.** Let $\Omega \subset \mathbb{R}^d$ be open, suppose that $u : \Omega \to \mathbb{R}$ solves $\Delta u = \lambda \cdot u$ with $\lambda \neq 0$, and let $A$ be a connected component of $\text{crit}(u)$. If there exists an open set $U \subset \Omega$ such that $U \cap A$ is a $d - 1$ dimensional manifold, then $A$ is a proper real-analytic manifold of dimension $d - 1$.

**Proof.** Let $E$ denote the subset of $A$ consisting of $x$ that have a neighborhood $W$ such that $A \cap W$ is a smooth hypersurface. The set $E$ is open. Since $A$ is closed and connected, it suffices to prove that $E$ is also closed.

Let $y$ lie in the closure of $E$. Then $y \in A$, and Corollary 3 implies that $u(y) \neq 0$. Hence since $\lambda \neq 0$, we have $\Delta u(y) \neq 0$. In particular, $\partial_j^2 u(y) \neq 0$ for some $j$. Because $y \in A$, we have $\partial_j u(y) = 0$. Therefore, the analytic inverse function theorem implies that there exists an open ball $B$ centered at the origin, a neighborhood $V$ of $y$, and an analytic diffeomorphism $\varphi : B \to V$ so that $(\partial_j u) \circ \varphi(z) = 0$ if and only if $z_j = 0$. The open set $V$ contains some $x \in E$, and hence there exists a neighborhood $W \subset V$ of $x$ such that $A \cap W$ is a hypersurface. It follows that $A \cap W = \varphi(z : z_j = 0) \cap W$. In particular, for each $k$, the restriction of $(\partial_k u) \circ \varphi(z)$ to $\{z : z_j = 0\} \cap \varphi^{-1}(W)$ vanishes identically. Since $(\partial_k u) \circ \varphi(z)$ is real-analytic, it vanishes on $B \cap \{z : z_j = 0\}$, and hence $A \cap V = \{x : \partial_j u(x) = 0\}$. Therefore $y \in E$, and $E$ is closed. □

\(^1\)See, for example, paragraph 3.4.10 in [Federer].
Proposition 5. Let \( \Omega \subset \mathbb{R}^d \) be a bounded open set with Lipschitz boundary, and let \( u : \Omega \to \mathbb{R} \) be a nonconstant Neumann eigenfunction. Suppose that \( M \subset \text{crit(}u\text{)} \) is a hypersurface. If \( U \) is a connected component of \( \Omega \setminus M \), then \( U \setminus u^{-1}(0) \) has at least two components.

Proof. Since \( \nabla u|_M = 0 \), the restriction \( u|_U \) is a Neumann eigenfunction on the bounded domain \( U \). Integration by parts gives \( \lambda \int_U u = \int_U (\Delta u) \cdot 1 = 0 \). Since \( u \) is nonconstant we have \( \lambda \neq 0 \), and hence \( u \) takes both positive and negative values. \( \square \)

Proposition 6. Let \( \Omega \subset \mathbb{R}^d \) be simply connected, bounded, and open with Lipschitz boundary, and let \( u : \Omega \to \mathbb{R} \) be a nonconstant Neumann eigenfunction. If \( \text{crit}(u) \) contains an analytic hypersurface \( M \), then \( \Omega \setminus u^{-1}(0) \) has at least three connected components.

Proof. Proposition 4 implies that \( M \) is a smooth proper hypersurface, and so Lemma 7 below implies that \( \Omega \setminus M \) has at least two connected components. In fact, since \( M \) is connected, \( \Omega \setminus M \) has exactly two connected components \( U_- \) and \( U_+ \) and \( U_- \cap U_+ = M \).

Proposition 5 implies that \( U_\pm \setminus u^{-1}(0) \) has (at least) two connected components.

By Corollary 3, the hypersurface \( M \) does not intersect \( u^{-1}(0) \). Thus, since \( M \) is connected, \( M \) intersects at most one connected component \( V \) of \( \Omega \setminus u^{-1}(0) \). Thus, there exists a connected component \( V_\pm \) of \( U_\pm \setminus u^{-1}(0) \) that is disjoint from \( V \).

Let \( \alpha : [-1,1] \to \Omega \) be a path that joins a point in \( V_- \) to a point in \( V_+ \). Since \( \alpha(\pm 1) \in U_\pm \) and \( U_- \) and \( U_+ \) are distinct connected components of \( \Omega \setminus M \) with \( U_- \cap U_+ = M \), there exists \( t \) so that \( \alpha(t) \in M \). In particular, since \( M \) is a smooth manifold, there exists \( s \) so that \( \alpha(s) \in V \). Since \( V \) is disjoint from both \( V_- \) and \( V_+ \), the components \( V_+ \) and \( V_- \) are distinct. Hence \( V_-, V_+, \) and \( V \) are three distinct components of \( \Omega \setminus u^{-1}(0) \). \( \square \)

The following is well-known. See [Sml69] or Theorem 4.4.6 in [Hirsch].

Lemma 7. Let \( \Omega \subset \mathbb{R}^d \) be a simply connected open set. If \( H \subset \Omega \) is a proper smooth hypersurface then \( \Omega \setminus H \) is not connected.

Proof. Near \( H \) we can find points \( x_\pm \) and a smooth arc \( \alpha : [-1,1] \to \Omega \) with \( h(\pm 1) = x_\pm \) so that \( \alpha^{-1}(H) = \{0\} \) and so that \( \alpha'(0) \) does not lie in the tangent space to \( H \).

Suppose that \( \Omega \setminus H \) were connected. Then there would exist an arc \( \beta \) joining \( x_+ \) to \( x_- \) so that \( H \cap \beta = \emptyset \). Since \( \Omega \) is simply connected, there exists a homotopy \( h : [-1,1] \times [0,1] \to \Omega \) with \( h(t,0) = \alpha(t), h(t,1) = \beta(t) \), and \( h(\pm 1, s) = x_\pm \) for each \( s \). By approximation, we may assume that \( h \) is smooth and transverse to \( H \) (see e.g. [Hirsch] Theorem 3.2.1).

\[ \text{See e.g. [Hirsch] Lemma 4.4.4.} \]
It follows that \( h^{-1}(H) \) is a 1-dimensional compact submanifold of \( S := [−1, 1] \times [0, 1] \) whose boundary lies in \( \partial S \). The point \((0, 0)\) is the only point that lies in both \( h^{-1}(H) \) and \( \partial S \). Hence the compact 1-manifold \( h^{-1}(H) \) has exactly one boundary component, a contradiction.

3. Invariant critical hypersurfaces

In this section, we suppose that a hypersurface component of the critical set is invariant under local isometries, and show that this forces a solution to \( \Delta u = \lambda u \) to depend only on the distance to the hypersurface.

Recall that a vector field is called a Killing field if and only if the local flow that it generates consists of local isometries. It follows that the Laplacian commutes with each Killing field. Each Killing field on \( \mathbb{R}^d \) is either rotational or constant.

Given an orientable hypersurface \( M \), there exists a smooth function \( r \) defined near \( M \) so that \( |r(x)| \) is the distance from \( x \) to \( M \). Let \( \partial r \) denote the gradient of \( r \). Note that \( \partial r \) is a unit vector field that is tangent to geodesics that meet \( M \) orthogonally. It is unique up to multiplication by \(-1\).

**Lemma 8.** If \( X \) is a Killing field that is tangent to \( M \), then \( [\partial r, X] = 0 \).

**Proof.** We claim that \( \partial r \) commutes with \( X \). Let \( \varphi_t \) denote the flow generated by \( X \), and let \( \psi_t \) denote the flow generated by \( \partial r \). Suppose \( x \in M \). Then for each sufficiently small \( t \) we have \( \text{dist}(x, \varphi_t(x)) = t \) and since \( \varphi_s \) is as isometry we have \( \text{dist}(\varphi_s(x), \varphi_s(\varphi_t(x))) = t \) for small \( s \). The unique geodesic that realizes the latter distance is a flow line of \( \partial r \) with length \( t \) and hence \( \varphi_t(\varphi_s(x)) = \varphi_s(\varphi_t(x)) \) for all small \( s, t \). The claim follows.

**Proposition 9.** Let \( \Omega \subset \mathbb{R}^d \) be open and connected, and let \( M \subset \Omega \) be an oriented hypersurface. Suppose that \( X \) is a Killing field on \( \Omega \) that is tangent to \( M \). If \( u : \Omega \rightarrow \mathbb{R} \) satisfies \( \Delta u = \lambda u \) and \( \nabla u \) vanishes on \( M \), then \( Xu \equiv 0 \) on \( \Omega \).

**Proof.** Because \( X \) is a Killing field, the function \( Xu \) is a Laplace eigenfunction. Since \( \nabla u = 0 \), we have \( Xu(x) = 0 \) and \( \partial_r u(x) = 0 \) for each \( x \in M \). Therefore, Lemma 8 implies \( \partial_r(Xu)(x) = X(\partial_r u)(x) = 0 \) for each \( x \in M \). Thus, since \( Xu \) vanishes on the hypersurface \( M \) we find that \( \nabla(Xu) = 0 \). Therefore, the claim follows from Lemma 10.

**Lemma 10.** Let \( \Omega \subset \mathbb{R}^d \) be an open set and let \( u : \Omega \rightarrow \mathbb{R} \) satisfy \( \Delta u = \lambda \cdot u \). If \( \text{sing}(u) \) contains a hypersurface, then \( u \equiv 0 \).
Proof. Both \( u \) and its normal derivative vanish along the hypersurface. The zero function lies in the kernel of \( \Delta - \lambda \cdot I \). Therefore the uniqueness theorem of Holmgren\(^3\) implies that \( u \equiv 0 \).

Let \( G \) be a group of isometries. We will say that the action of \( G \) is **codimension 1** if the typical orbit is a hypersurface. For example, the standard action of \( SO(n - k) \times \mathbb{R}^{k-1} \) on \( \mathbb{R}^{n-k} \times \mathbb{R}^{k-1} \) is codimension 1.

**Theorem 11.** Let \( \Omega \subset \mathbb{R}^d \) be open and connected, and let \( M \subset \Omega \) be a hypersurface that is contained in the typical orbit of a codimension one isometric group action. If \( M \subset \text{crit}(u) \) and \( \Delta u = \lambda u \), then \( u \) is invariant under the action.

**Proof.** Let \( x \in M \). Since \( G \cdot x \) contains \( M \) there are Killing fields \( X_1, \ldots, X_{n-1} \), such that the tangent space at \( x \) is spanned by \( \{X_1(x), \ldots, X_{n-1}(x)\} \). Proposition 9 implies that \( X_j u \equiv 0 \) for each \( j \). Since the \( X_j \) span the tangent space of an orbit, the function \( u \) is constant on the orbit.

**Example 12.** Suppose that the hypersurface \( M \) belongs to a hyperplane in \( \mathbb{R}^d \). The hyperplane is the orbit of an isometric action of \( \mathbb{R}^{n-1} \) via translations parallel to the hyperplane. Theorem 11 implies that \( u = v \circ r \) for some \( v : \mathbb{R} \to \mathbb{R} \). Since \( \Delta u = \lambda u \) and \( \nabla u = 0 \) along \( M \), we find that \( v \) is a multiple of \( t \mapsto \cos(\sqrt{\lambda} \cdot t) \).

**Example 13.** Suppose that the hypersurface \( M \) belongs to an \( n - 1 \) dimensional sphere in \( \mathbb{R}^d \). The sphere is the orbit of an isometric action of the orthogonal group \( O(n) \). Theorem 11 implies that \( u = v \circ r \) for some \( v : \mathbb{R} \to \mathbb{R} \) that satisfies an explicit second order ordinary differential equation. In particular, \( v \) is a Bessel function.

4. **Critical sets of second Neumann eigenfunctions**

In this section, we suppose that \( u \) is a second Neumann eigenfunction on a simply connected domain, and derive consequences.

**Proposition 14.** Let \( \Omega \subset \mathbb{R}^d \) be a simply connected, bounded, open set with Lipschitz boundary, and let \( u : \Omega \to \mathbb{R} \) be a nonconstant second Neumann eigenfunction. Then \( \text{crit}(u) \) does not contain a hypersurface.

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\(^3\)See, for example, Theorem 6.4.3 in [Taylor]. One can also use the unique continuation theorem of Hörmander [Hörmander] [Tr04].
Proof. Courant’s nodal domain theorem implies that the set $\Omega \setminus u^{-1}(0)$ has exactly two components. If $\text{crit}(u)$ were to contain an hypersurface, then Proposition 6 would imply that $\Omega \setminus u^{-1}(0)$ has at least three components.

If a portion of the boundary of $\Omega$ is sufficiently regular, then each Neumann eigenfunction extends in a $C^1$ fashion to a portion of the boundary, and hence one can extend the notion of critical point of $u$ on the boundary. We will let $\overline{\text{crit}}(u)$ denote the set of critical points of such an extension if it exists.

**Proposition 15.** Let $\Omega \subset \mathbb{R}^d$ be a bounded, simply connected domain with piecewise analytic boundary, and let $u : \Omega \to \mathbb{R}$ be a second Neumann eigenfunction. If $\overline{\text{crit}}(u)$ contains a hypersurface $M$ that lies in a hyperplane $H$, then there exists a domain $D \subset H$ such that $\Omega$ is the intersection of the cylinder over $D$ and the convex hull of $H \cup H'$ where $H'$ is a hyperplane parallel to $H$ such that $\text{dist}(H, H') = \pi / \sqrt{\lambda}$.

Proof. Proposition 14 implies that the hypersurface $M$ cannot lie in the interior of $\Omega$. Hence $M$ lies $\partial \Omega$. Because $u$ satisfies Neumann conditions, the eigenfunction $u$ extends via reflection to a neighborhood $U$ of a point of $H$, and hence we are in the situation of Example 12. In particular, $u$ is a multiple of $x \mapsto \cos(\sqrt{\lambda} \cdot \text{dist}(x, U))$. Therefore, since $u$ satisfies Neumann conditions, if the vector $v$ with footpoint $x$ is tangent to $\partial \Omega$, then either $v$ is orthogonal to $H$ or $\text{dist}(x, H) = k\pi / \sqrt{\lambda}$ for some $k \in \mathbb{Z}$. Proposition 14 implies that at most two distinct integers $k$ can appear and that they are successive integers. The claim follows.

A similar argument gives the following.

**Proposition 16.** Let $\Omega \subset \mathbb{R}^d$ be a bounded, simply connected domain with piecewise analytic boundary, and let $u : \Omega \to \mathbb{R}$ be a second Neumann eigenfunction. If $\overline{\text{crit}}(u)$ contains a hypersurface $M$ that lies in a hypersphere $S$, then there exists a domain $D \subset S$ such that $\Omega$ is the intersection of the cone on $D$ and a spherical shell whose boundary contains $S$.

The following finishes the proof of Theorem 1 in the introduction.

**Theorem 17.** Let $P \subset \mathbb{R}^2$ be a bounded polygonal domain, and let $u : P \to \mathbb{R}$ be a second Neumann eigenfunction. If $\overline{\text{crit}}(u)$ is not finite, then $P$ is a rectangle.

Proof. Via reflection across its sides, one may extend $u$ to an eigenfunction $\tilde{u}$ defined on an open set $\Omega$ that contains $P \setminus V$. Suppose that $\text{crit}(u)$ is not finite, and let $p$ be an accumulation point. Lemma 18 below implies that $p$ is not a vertex of $P$. Thus $p$ lies in the interior
of $\Omega$ and hence $p$ is a critical point of $\tilde{u}$. Because $|\nabla \tilde{u}|^2$ is real-analytic, the critical set is a locally finite graph.\textsuperscript{4} Thus, since $p$ is not an isolated critical point, there exists an arc in $\text{crit}(\tilde{u})$. Proposition 14 implies that this arc cannot intersect the interior of $P$. Because $\tilde{u}$ is defined on the exterior of $P$ via reflection in the sides, the arc also cannot intersect the exterior of $P$. Hence the arc lies in a side of $P$, and Proposition 15 implies that $P$ is a rectangle. \hfill $\square$

The following Lemma generalizes Proposition 5.6 in [JdgMnd20], a result that assumed the convexity of the polygonal domain.

**Lemma 18.** Let $P$ be a bounded, simply connected, polygonal domain\textsuperscript{5} and let $u : P \to \mathbb{R}$ be a second Neumann eigenfunction. No vertex $v$ of $P$ is an accumulation point of $\text{crit}(u)$.

**Proof.** Suppose to the contrary that a sequence of critical points converges to a vertex. By applying a planar isometry to the polygon, we may assume that the vertex is located at the origin and that the intersection of $P$ with a disc neighborhood of the origin has the form $\{re^{i\theta} \in \mathbb{C} : 0 < \theta < \beta, 0 < r < r_0\}$ where $\beta \neq \pi$ is the angle of the vertex.

The Neumann eigenfunction $u$ has the Fourier-Bessel expansion

$$u(re^{i\theta}) = \sum_{k=0}^{\infty} c_n \cdot r^{k \cdot \nu} \cdot g_{k \cdot \nu}(r^2) \cdot \cos(k \cdot \nu \cdot \theta).$$  \hfill (1)

where $\nu := \pi / \beta$, $g_{\nu}(r^2) := r^{-\nu} \cdot J_{\nu}(\sqrt{\lambda} \cdot r)$, and $J_{\nu}$ is the standard Bessel function (see e.g. §4 [JdgMnd20]). From the series expansion for $J_{\nu}$ [Lebedev], one finds that the function $g_{\nu}$ is entire and $\partial_{\nu}^k g_{\nu}(0) \neq 0$ for $k \in \mathbb{N}$.

Because $P$ is simply connected, Corollary 5.3 in [JdgMnd20] implies that either $c_0$ or $c_1$ is nonzero. Hence we will derive a contradiction by showing that both $c_0$ and $c_1$ equal zero.

Differentiation of (1) with respect to $\theta$ gives

$$\partial_{\theta} u(re^{i\theta}) = -\nu \cdot r^{\nu} \cdot \sin(\nu \cdot \theta) \cdot (c_1 \cdot g_{\nu}(r^2) + O(r^\nu)).$$  \hfill (2)

Let $r_n e^{i\theta_n}$ denote the sequence of critical points that converges to the origin. Then from (2) we have $c_1 \cdot g_{\nu}(r_n^2) = O(r_n^\nu)$. Thus, since $\nu > 0$ and $g_{\nu}(0) \neq 0$, it follows that $c_1 = 0$.

Let $k$ be the smallest integer greater than 1 such that $c_k \neq 0$. Since $c_1 = 0$, differentiation of (1) with respect to $r$ gives

$$\partial_{r} u(re^{i\theta}) = c_0 \cdot 2r \cdot g'_0(r^2) + c_k \cdot k \cdot r^{k-1} \cdot g_{k \cdot \nu}(r^2) \cdot \cos(k \cdot \nu \theta) + O(r^{k \cdot \nu + 1}) + O(r^{(k+1)\cdot \nu - 1}).$$  \hfill (3)

\textsuperscript{4}See, for example, the proof of Proposition 5 in [OtlRss09].

\textsuperscript{5}In particular, $P$ is an open set.
If \( re^{i\theta} \) is a critical point, then the left hand side vanishes. As \( r \) tends to zero, one of the first two terms on the right dominates. If \( kv - 1 > 1 \), then since \( r_n e^{i\theta_n} \) is a critical point and \( g_0'(0) \neq 0 \) we find that \( c_0 \cdot 2r_n \cdot g_0'(r_n^2) = o(r_n) \) and hence \( c_0 = 0 \). If \( kv - 1 < 1 \), then we find that \( c_k \cdot \cos(kv\theta_n) = o(1) \). Since \( c_k \neq 0 \), we have \( \cos(kv\theta_n) \to 0 \), and hence \( |\sin(kv\theta_n)| \to 1 \).

On the other hand, since

\[
\partial_\theta u \left( r \cdot e^{i\theta} \right) = -c_k \cdot \nu \cdot r^{kv} \cdot g_{kv}(r^2) \cdot \sin(vk\theta) + O \left( r^{(k+1)v} \right).
\]

we find that either \( c_k = 0 \) or \( \sin(kv\theta_n) \to 0 \), a contradiction in either case.

If \( kv - 1 = 1 \), then since \( k \geq 2 \) and \( 0 < \beta < 2\pi \) with \( \beta \neq \pi \), we find that \( k = 3 \) and \( \nu = 2/3 \) (\( \beta = 3\pi/2 \)). If we define

\[
f(s, t) = \sum_{k=0}^{\infty} c_k \cdot s^{2k} \cdot g_{2k}(s^6) \cdot \cos(k \cdot t)
\]

then from (1) we find that \( u(r \cdot e^{i\theta}) = f \left( r^\nu, \frac{2\theta}{3} \right) \). It is known that for \( \mu > 0 \), the Bessel function \( J_\mu \) is nonnegative and increasing on the interval \([0, \mu]\). It follows that for each \( k > 0 \), the function \( s \mapsto s^{2k} \cdot g_{2k}(s^6) \) is nonnegative and increasing on the interval \([0, (2/3\sqrt{2})^k] \). Since \( u \) is real-analytic on \( \Omega \), there exists \( s_0 < (2/3\sqrt{2})^{1/2} \), such that \( t \mapsto f(s_0, t) \) is continuously differentiable on \([0, \pi]\). In particular, \( \partial_t f(s_0, t) \) belongs to \( L^2([0, \pi]) \). Hence by Bessel’s inequality, the sum

\[
\sum_{k=0}^{\infty} k^2 \cdot |c_k|^2 \cdot s_0^{2k} \cdot \left| g_{2k}(s_0^6) \right|^2
\]

is finite. Thus, since \( \sum k^{-2} \) is finite, it follows from the Cauchy-Schwarz inequality that

\[
\sum_{k=0}^{\infty} |c_k| \cdot s_0^{2k} \cdot \left| g_{2k}(s_0^6) \right|
\]

is finite. Since for \( k > 0 \) the function \( s \mapsto s^{2k} \cdot g_{2k}(s^6) \) is nonnegative and increasing on the interval \([0, s_0]\), we find that the series in (5) is uniformly convergent on \([0, s_0]\), and thus by symmetry it is uniformly convergent on \([-s_0, s_0]\). It follows that \( f \) is analytic on \([-s_0, s_0] \times [0, \pi]\). In particular, \( f \) has at most finitely many critical points on \([0, s_0] \times [0, \pi]\). Therefore \( u \) does not have a sequence of critical points converging to the vertex.

\[\square\]

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\[6\]See, for example, §15.3 [Watson].
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