Abstract

We consider the KdV equation on a circle and its Euler-Poincaré reconstruction, which is reminiscent of the equation of motion for fluid particles. For periodic waves, the stroboscopic reconstructed motion is governed by an iterated map whose Poincaré rotation number yields the drift velocity. We show that this number has a geometric origin: it is the sum of a dynamical phase, a Berry phase, and an ‘anomalous phase’. The last two quantities are universal: they are solely due to the underlying Virasoro group structure. The Berry phase, in particular, was previously described in [1] for two-dimensional conformal field theories, and follows from adiabatic deformations produced by the propagating wave. We illustrate these general results with cnoidal waves, for which all phases can be evaluated in closed form thanks to a uniformizing map that we derive. Along the way, we encounter ‘orbital bifurcations’ occurring when a wave becomes non-uniformizable: there exists a resonance wedge, in the cnoidal parameter space, where particle motion is locked to the wave, while no such locking occurs outside of the wedge.
1 Introduction and summary of results

It is quite generally true that the state vector of a quantum system undergoing cyclic changes of reference frames picks up Berry phases [2, 3]. Typical examples of this behaviour include Thomas precession [4], a spin in a slowly rotating magnetic field [2, 5, 6], and its non-compact analogue [7] which appears in the quantum Hall effect [8]. In [1], such Berry phases were shown to arise in two-dimensional conformal field theories (CFTs) coupled to an environment that produces adiabatic conformal transformations. These phases can be computed exactly despite the infinite-dimensional parameter space, and coincide with ‘geometric actions’ of the Virasoro group [9]. From now on, we refer to them as Virasoro Berry phases. They are reminiscent of the response of a quantum Hall fluid to metric deformations, where the parameter space is infinite-dimensional as well [10].

The goal of this paper is to exhibit classical systems where Virasoro Berry phases are realized dynamically, i.e. without any implicit coupling to the ‘environment’.\footnote{We write ‘Berry phases’ despite the lack of quantization. ‘Geometric phases’ or ‘Hannay angles’ [11] would be more appropriate, but the distinction is unimportant as the classical geometric phase will coincide with its quantum analogue [1].} This
notably includes the Korteweg-de Vries (KdV) equation [12], or rather its Euler-Poincaré reconstruction [13–16], but it applies more generally to any Lie-Poisson equation based on the Virasoro group [17,18], such as the Hunter-Saxton and Camassa-Holm equations [19]. Indeed, reconstructed Lie-Poisson equations yield geodesics on Lie groups, and powerful geometric tools can then be used to predict universal properties of the reconstructed dynamics — such as Berry phases appearing when the system’s motion in momentum space is periodic. For example, the Lie-Poisson system of SO$(3)$ yields the standard Euler equations for the angular momentum of a rigid body. When angular momentum performs one period of its motion, the final orientation of the body in space differs from its initial one by a rotation whose angle is known as a Montgomery phase [20,21]; it is the sum of a dynamical phase and a geometric phase due to adiabatic rotations. The purpose of this paper is thus to describe the Virasoro analogue of Montgomery phases.

For the record, this is not the first time that geometric phases are found in the KdV equation: such phases were indeed found in [22] and reproduced, among other things, the standard phase shift occurring after the collision of two solitons. However, [22] crucially used the effective, finite-dimensional phase space description of KdV solitons, and the corresponding geometric phases are Hannay angles in a finite-dimensional parameter space. This is radically different from what we do here, since we, by contrast, explicitly use the infinite-dimensional nature of the Virasoro group and never rely on soliton dynamics per se. In this sense, there is, to our knowledge, no overlap between [22] and this work, other than the general context.

We now explain how Virasoro Berry phases can be observed through the motion of suitable (comoving) ‘fluid particles’, and how these phases can be computed. We then expose the plan of our work.

Summary of results. This work relies on a fair amount of symplectic geometry and Virasoro group theory, none of which is reviewed in a self-contained manner — we refer e.g. to [23] for an introduction to the former, and to [24,25] for the latter. Nevertheless, it is straightforward to describe the main aspects of our work with minimal technicalities. Namely, let $p(x,t)$ be a (spatially $2\pi$-periodic) wave profile that solves the KdV equation\(^2\)

$$\frac{\partial p}{\partial t} + 3p \frac{\partial p}{\partial x} - \frac{c}{12} \frac{\partial^3 p}{\partial x^3} = 0,$$

(1)

where $c \neq 0$ is a constant parameter (the Virasoro central charge). Suppose, then, that a particle on the line has a position $x(t)$ that satisfies

$$\frac{dx}{dt} = p(x(t),t),$$

(2)

with initial position $x(0) = x_0$ say. This particle could be, for example, a small fluid element in a shallow water channel supporting the wave $p$.\(^3\) Our goal is to find, analytically, general properties of the resulting solution $x(t)$, such as the drift velocity

$$v_{\text{Drift}} = \lim_{t \to +\infty} \frac{x(t) - x(0)}{t}.$$  

(3)

\(^2\)We choose $p$ to satisfy KdV, but virtually identical arguments apply to Camassa-Holm and Hunter-Saxton upon including a suitable inertia operator [17].

\(^3\)We will return to this interpretation repeatedly below, particularly in section 3.3. Up to a (crucial!) mismatch in reference frames, the interpretation of (2) as a fluid transport equation holds.
To ensure that the latter is a well-defined quantity, we add one extra condition: we require
the wave \( p \) to be periodic in time, i.e. \( p(x, t + T) = p(x, t) \) for some \( T > 0 \). Then, there
exists a (time-independent) diffeomorphism \( x \mapsto F(x) \) of \( \mathbb{R} \) such that, after \( N \) periods,
\[
x(NT) = F \circ F \circ \ldots \circ F(x_0) \equiv F^N(x_0).
\]

We can thus think of the ‘stroboscopic’ motion of particles at integer multiples of the
period \( T \) as a discrete-time dynamical system governed by the map \( F \). From that per-
spective, the drift velocity (3) reads
\[
v_{\text{Drift}} = \frac{\Delta \phi}{T}, \quad \Delta \phi \equiv \lim_{N \to +\infty} \frac{F^N(x_0) - x_0}{N}
\]
where \( \Delta \phi \) is the Poincaré rotation number of \( F \) [24, sec. 4.4.3]. It is easily read off by
integrating eq. (2) numerically over many periods. As we now explain, there is in fact a
way to predict the value of \( \Delta \phi \), analytically, using group theory and symplectic geometry.
This value involves, in particular, a Virasoro Berry phase.

To see where symplectic geometry plays a role, one has to think of (1) as a Lie-
Poisson equation for the Virasoro group. The phase space of any such system is the
cotangent bundle of a Lie group, where the cotangent part consists of ‘momenta’, while
the group manifold is a space of ‘positions’ or ‘configurations’. In the KdV case, for
instance, \( p(x, t) \) is a Virasoro momentum (which justifies our notation). By construction,
the motion of momenta determines that of configurations through Euler-Poincaré recon-
struction [13–15]. In the KdV case, this reconstruction turns out to precisely take the
form of eq. (2), as explained in greater detail in section 3.2. Importantly, periodic motion
of momenta does not, in general, imply periodicity of configurations. Instead, when the
system performs a loop in momentum space, its configuration typically traces an open
path, and the difference between the initial and final positions can be interpreted as an
(an)holonomy. The latter involves a Berry phase associated with adiabatic changes of
reference frames, exactly as in the aforementioned example of Montgomery phases [20,21].

For Lie-Poisson systems based on the Virasoro group, such as KdV, the holonomy in
the space of configurations is precisely the angle \( \Delta \phi \) of (5), except it can now be written
as a sum (63) whose schematic form is
\[
\Delta \phi = \text{Dynamical phase} + \underbrace{\text{Berry phase}} + \underbrace{\text{Anomalous phase}}
\]

In that expression, the first term, proportional to the period \( T \), is a dynamical phase, while
the second term is a Virasoro Berry phase associated with adiabatic diffeomorphisms [1].
The anomalous term is a contribution due to the Virasoro central extension and may be
seen as the integral of a Berry connection along the inverse of the reconstructed path.
Both the Berry phase and the anomalous term are universal: they solely follow from
Virasoro group theory and take the same form regardless of dynamics (though the path
\( p(x, t) \) that determines their value does, of course, depend on dynamics). Furthermore,
the dynamical phase and Berry phase are known functionals of \( p(x, t) \); the anomalous
term, on the other hand, is an implicit integral (62). All these functionals turn out to
simplify greatly for travelling waves \( p(x, t) = p(x - vt) \), which eventually yields eq. (89)
for \( \Delta \phi \). As a result, for cnoidal waves, the three terms of (6) can be evaluated analyt-
ically at any point in parameter space — they are displayed in eqs. (101)-(103). Their
sum coincides with the value (5) that can be computed by other means — thanks to a suitable uniformizing map that we derive —, and leads to the compact formula (99) for the drift velocity.

It should be noted that our derivation of eq. (6) for KdV rests on one key technical assumption: the profile $p(x,t)$ must be uniformizable, or amenable, in the sense that there exists a conformal transformation (i.e., a diffeomorphism of the circle) mapping it on some uniform, $x$-independent, profile $k$. This is generally not guaranteed, as there exist a great many Virasoro coadjoint orbits without uniform representative [26,27]. For any profile that does not satisfy the assumption of amenability, a notion of drift does exist in the sense of eqs. (3) and (5), but the corresponding $\Delta \phi$ is an integer multiple of $2\pi$ and cannot be written in the form (6). Following [28], we will show that such a regime occurs for cnoidal waves with sufficient pointedness: there exists a resonance wedge in the cnoidal parameter space where (6) does not apply, and, in that wedge, particle motion is ‘locked’ to the travelling wave: $v_{\text{Drift}} = v_{\text{Wave}}$. The transition along the wedge boundary is reminiscent of the SNIPER bifurcation of the Adler equation [29]. Outside of the wedge, cnoidal waves are amenable and eq. (6) applies, leading to a drift velocity $v_{\text{Drift}} \neq v_{\text{Wave}}$.

An important motivation for this work stems from fluid dynamics, where the KdV equation notoriously describes shallow water waves [30]. Indeed, in a comoving frame, the leading equation of motion for fluid particles in a two-dimensional channel supporting KdV waves is nearly identical to the reconstruction equation (2): the only difference is the presence of a (large) constant term on the right-hand side (see eq. (68) below and the surrounding discussion). The symplectic formula (6) for $\Delta \phi$, along with the drift velocity (3), thus suggests that Virasoro Berry phases contribute to the Stokes drift velocity of particles in shallow water [31], similarly to the crest slowdown phenomenon observed in wave breaking [32]. However, the seemingly innocuous change of reference frames that distinguishes eq. (2) from the actual equation of motion for fluid particles turns out to be crucial: it implies that Stokes drift in standard shallow water dynamics differs from the drift velocity introduced here, whose (subleading) effect is entirely washed out by the overwhelming, dominant contribution of the overall velocity $\sqrt{gh}$ of the comoving frame. More on that in section 3.3. Prospects for actual observations of Virasoro Berry phases are relegated to the conclusion of this paper.

Plan of the paper. This work is not self-contained: the necessary prerequisites include symplectic geometry [23] and Virasoro group theory [24,25], and the parts concerning cnoidal waves heavily rely on [28]. We will not review any of that content here, but we do adopt a logical flow that corresponds to the way one would naturally teach the subject. Accordingly, the structure is as follows.

First, section 2 contains general prerequisites in symplectic geometry. In it, we introduce Euler-Poincaré reconstruction and derive an abstract formula for the rotation $\Delta \phi$ associated with any periodic solution of a Lie-Poisson system based on a centrally extended group, provided the solution has a U(1) stabilizer in a suitable sense. This leads to eq. (48), which is critical to the rest of the paper (and is new to our knowledge, as it contains an ‘anomalous phase’ that appears to have been overlooked so far). In section 3 we apply this formula to any Lie-Poisson wave equation based on the Virasoro group, resulting in eq. (63) for $\Delta \phi$. We also establish the link between reconstruction and the equation of motion (2), hence between geometric phases and the drift velocity (3), and comment on the important difference between the latter notion and that of Stokes
Section 4 is devoted to the application of these arguments to travelling waves, and to a comparison between the geometric prediction (6) and the value of \( \Delta \phi \) computed analytically. To that end, we actually find a general formula for ‘uniformizing maps’ of travelling waves satisfying KdV, and deduce an exact expression for the solution of the equation of motion (2), from which the drift velocity (3) follows. As we shall see, the drift velocity is indeed perfectly predicted by the symplectic formula (6), but the values of cnoidal parameters strongly affect the drift velocity — in particular, waves located in a certain ‘resonance wedge’ produce particle motion that is locked to the wave, confirming the existence of ‘orbital bifurcations’ anticipated in [28]. Finally, we conclude in section 5 with a discussion of potential follow-ups of our work. For completeness, the appendix collects further details of group theory and symplectic geometry needed in section 2.

2 Reconstruction and Berry phases

This section is a mathematical prelude. We start by briefly reviewing general aspects of Lie groups and their relation to Lie-Poisson equations [18], then turn to the key method of Euler-Poincaré reconstruction [13–15], which will be instrumental for the entire paper. Following that, we derive general formulas for the reconstructed rotation angle \( \Delta \phi \) in Lie-Poisson systems with a \( U(1) \) stabilizer — first for generic groups, then for centrally extended ones. The former case includes Montgomery phases as an application [20,21], while the latter is crucial for the Virasoro group and the KdV equation.

2.1 Lie groups and Lie-Poisson equations

Lie-Poisson equations are Hamiltonian systems whose dynamics is almost entirely fixed by a parent Lie group. For instance, the group SO(3) of spatial rotations leads to the motion of free-falling rigid bodies, while the Virasoro group is associated with a host of non-linear wave equations that includes the KdV, inviscid Burgers, Hunter-Saxton and Camassa-Holm equations [17]. Here, as a preparation for KdV and its cousins, we recall the derivation of Lie-Poisson equations in a general group-theoretic setting. We refer to the appendix for the minimal necessary background on Lie groups and symplectic geometry; see also [18] for a pedagogical introduction.

Let \( G \) be a Lie group with algebra \( \mathfrak{g} \), whose dual space is \( \mathfrak{g}^* \). The adjoint representation of \( G \) on \( \mathfrak{g} \) is defined, for all \( g \in G \), by \( \text{Ad}_g(\xi) \equiv \partial_1|_0(g e^{t\xi} g^{-1}) \), where \( e^{t\xi} \) is the exponential of \( t\xi \in \mathfrak{g} \). For matrix groups, the right-hand side boils down to \( g \xi g^{-1} \). The dual of the adjoint is the coadjoint representation of \( G \), given for all \( g \in G, p \in \mathfrak{g}^*, \xi \in \mathfrak{g} \) by

\[
\langle g \cdot p, \xi \rangle \equiv \langle p, \text{Ad}_g^{-1}\xi \rangle. \tag{7}
\]

In what follows, the coadjoint representation will play a key role, so we reduce clutter by writing it as \( g \cdot p \), instead of the heavier notation \( \text{Ad}_g^*(p) \). The Lie-algebraic analogue of the coadjoint representation will be denoted as \( \text{ad}^* \) and is defined by the derivative of \( \text{Ad}^* \), that is, \( \text{ad}^*_\xi \equiv \partial_t|_0\text{Ad}_g^*\xi \). Using (7), this is equivalent to

\[
\langle \text{ad}^*_\xi p, \zeta \rangle \equiv -\langle p, [\xi, \zeta] \rangle \tag{8}
\]

where \( p \in \mathfrak{g}^* \) and \( \xi, \zeta \in \mathfrak{g} \), with \([\cdot, \cdot]\) the Lie bracket.
**Lie-Poisson equations.** As a starting point towards the Lie-Poisson construction, note that \( g^* \) can be seen as a phase space, since it can be endowed with a Poisson structure. Indeed, given any real function \( F \) on \( g^* \), its differential \( dF \) at a point \( p \) is a linear map from \( T_p g^* \cong g^* \) to \( \mathbb{R} \). Thus \( dF \) can be seen as an element of the Lie algebra \( g \cong (g^*)^* \), and one defines the *Kirillov-Kostant bracket* on \( g^* \) as\(^4\)

\[
\{ F, G \}(p) \equiv \langle p, [dF_p, dG_p] \rangle \overset{(117)}{=} -\langle \text{ad}^*_p F(p), dG_p \rangle.
\]

Once we think of \( g^* \) as a phase space, it is immediate to write down evolution equations: any Hamiltonian \( H \) on \( g^* \) determines the time-dependence of a function \( F \) according to

\[
\dot{F}(p) = \{ F, H \}(p) \overset{(9)}{=} \langle \text{ad}^*_p H(p), dF_p \rangle \equiv dF_p(\text{ad}^*_p H(p))
\]

where \( \dot{F} \equiv dF/dt \). The left-hand side can be written as \( \dot{F}(p) = dF(p)(\dot{p}) \), where \( \dot{p} \) is the vector field given by the Hamiltonian flow. Thus, removing the differential \( dF_p \) from both sides of (10), we read off the equation of motion

\[
\dot{p}(t) = \text{ad}^*_{\text{d}H_{p(t)}}(p(t)).
\]

Given \( H \), this yields a unique curve \( p(t) \) in phase space for any initial condition \( p(0) \).

To derive Lie-Poisson equations, one restricts attention to quadratic Hamiltonians. This requires an extra bit of terminology: by definition, an *inertia operator* is an invertible linear map

\[
\mathcal{I} : g \to g^* : \xi \mapsto \mathcal{I}(\xi)
\]

which is self-adjoint in the sense that \( \langle \mathcal{I}(\xi), \zeta \rangle = \langle \mathcal{I}(\zeta), \xi \rangle \), and positive definite in the sense that \( \langle \mathcal{I}(\xi), \xi \rangle > 0 \) for any non-zero \( \xi \in g \). Any such map defines a (positive-definite) quadratic Hamiltonian

\[
H(p) = \frac{1}{2} \langle p, \mathcal{I}^{-1}(p) \rangle
\]

and the associated evolution equation (11) reads

\[
\dot{p} = \text{ad}^*_{\mathcal{I}^{-1}(p)}(p).
\]

This is the *Lie-Poisson equation* of the group \( G \), given the inertia operator \( \mathcal{I} \). One can show that it is equivalent to a geodesic equation on \( G \) for the right-invariant metric induced by \( \mathcal{I} \) [18, sec. 4.3]. The point of ‘reconstruction’ will precisely be to recover a geodesic \( g(t) \equiv g_t \) in \( G \) from a solution \( p(t) \) of (14).\(^5\)

**Remarks on coadjoint orbits.** The name ‘inertia operator’ stresses that Lie-Poisson equations generalize the Euler equations of motion for free-falling rigid bodies. The latter have a configuration space \( G = \text{SO}(3) \) and an inertia tensor specified by their distribution of mass. The Lie algebras \( \mathfrak{so}(3) \) and its dual respectively consist of angular velocities and angular momenta, the two being related through the inertia tensor \( \mathcal{I} \). The time evolution of angular momentum, as seen from a (non-inertial) reference frame attached to the body,

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\(^4\)The bracket (9) is degenerate, so \( g^* \) is a Poisson manifold but *not* a symplectic manifold. Its symplectic leaves are coadjoint orbits of \( G \), which we will turn to shortly.

\(^5\)We write paths in \( G \) as \( g_t \) instead of \( g(t) \) for notational convenience: from section 3 onwards, each \( g_t \) will be a function \( g_t(x) \) of \( x \in \mathbb{R} \) and the subscript will stress the asymmetric roles of \( t \) and \( x \).
is given by eq. (14). By contrast, however, in any inertial frame, the angular momentum vector is constant.

This example illustrates a key general aspect of eq. (14). Namely, any solution \( p(t) \) of (14) is such that \( p(t_1) \) and \( p(t_2) \) are related by a change of reference frames, for all \( t_1, t_2 \), in the sense that \( p(t) = f_t \cdot p(0) \) for some path \( f_t \) in the group manifold. Thus, once an initial condition is fixed, the motion of \( p(t) \) takes place on a single coadjoint orbit of the group \( G \),

\[
\mathcal{O}_{p(0)} \equiv \{ f \cdot p(0) \mid f \in G \}.
\]  

In particular, there always exists a frame where the motion of \( p(t) \) is trivial, namely \( p(0) = f_t^{-1} \cdot p(t) = \text{const} \). For the Euler top, this is achieved in any inertial frame.

The orbit (15) is a submanifold of \( g^* \), so it is typically specified by a certain number of continuous parameters whose value remains constant in time. In that sense, the statement \( p(t) = f_t \cdot p(0) \) is a conservation law. This will allow us to fix a particular orbit representative, say \( k \in g^* \), and write time evolution as \( p(t) = g_t \cdot k \) for some path \( g_t \) in \( G \). Suitable choices of \( k \) will then greatly simplify the reconstruction equation for \( g_t \).

### 2.2 Euler-Poincaré reconstruction

As just reviewed, the dual \( g^* \) of an algebra \( g \) is a space of momenta endowed with a Poisson structure (9); the Lie-Poisson equation (14) describes a Hamiltonian system in that space. We now extend this picture by thinking of the group \( G \) as the configuration manifold of the system, with a phase space given by the cotangent bundle \( T^*G \cong G \times g^* \).

From that perspective, Lie-Poisson dynamics is a ‘reduction’ of more complete, parent dynamics in \( T^*G \). In the opposite direction, Euler-Poincaré reconstruction will lift the motion \( p(t) \) in \( g^* \) to a curve \( (g_t, p(t)) \) in \( T^*G \), with \( g_t \in G \) determined by \( p(t) \) (see fig. 1).

In the remainder of this section, we derive a general property of the reconstructed path \( g_t \) when \( p(t) \) is periodic, with \( p(T) = p(0) \) for some period \( T \). As we shall see, despite the periodicity of \( p(t) \), the curve \( g_t \) is generally not closed: \( g_T \neq g_0 \). This inequality will turn out to reflect a holonomy in the principal \( G \)-bundle \( T^*G \), and involves the sum of a dynamical phase and a Berry phase. Accordingly, the next few pages are crucial for the rest of the paper. We warn the reader that the discussion relies heavily on Lie group theory and symplectic geometry; some technical details are relegated to the appendix. For a pedagogical introduction, see e.g. [23]; see also [13,14] for a detailed account of Euler-Poincaré reconstruction in general, including a discussion of geometric phases.

**Defining Euler-Poincaré reconstruction.** We show in the appendix that the cotangent bundle \( T^*G \), i.e. the phase space of the reconstructed system, is a trivial bundle: it is equivalent to the product \( G \times g^* \). In particular, the symplectic form \( \omega = -\mathcal{L}_A \) of \( G \times g^* \) is obtained by pulling back the standard Liouville symplectic form of \( T^*G \), with\(^7\)

\[
\mathcal{A}_{(g,p)} = (\langle p, dg g^{-1} \rangle, 0)
\]

where \( dg g^{-1} = d(R_{g^{-1}}) \) is the right Maurer-Cartan form (and \( R_{g^{-1}} \) denotes right multiplication by \( g^{-1} \)). The one-form (16) is the group-theoretic version of what is commonly defined.

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\(^6\)This actually holds for any Hamiltonian in eq. (11), since the latter makes \( p(t) \) tangent to the orbit of \( p(t) \) regardless of \( H \). Quadratic Hamiltonians are special in that the reconstruction of (14) is a geodesic in \( G \) with respect to an invariant metric [18, sec. 4.3], which makes the dynamics more tractable.

\(^7\)See eq. (124) in the appendix.
written in mechanics as \( p \, dq \), with \( dq \) the Maurer-Cartan form of the group \( \mathbb{R} \). It is also the Berry connection that will eventually give rise to Berry phases in reconstructed dynamics (which is why we call it \( 'A' \)), so it is an essential object for all that follows.

Now consider the Lie-Poisson system (14) from the point of view of the full phase space \( T^*G \cong G \times \mathfrak{g}^* \). The key to Euler-Poincaré reconstruction is the fact, emphasized in section 2.1, that any curve \( p(t) \) which solves (14) lies entirely on a single coadjoint orbit of the group \( G \). Thus, the path in momentum space can be written as 

\[
p(t) = g_t \cdot k
\]

for some fixed coadjoint vector \( k \) and some path \( g_t \) in \( G \). Note that \( k \) need not coincide with \( p(0) = g_0 \cdot k \), as \( g_0 \) may well differ from the identity — indeed, this will exactly occur below. It is thus tempting to consider paths of the form \( (g_t, g_t \cdot k) \) in \( G \times \mathfrak{g}^* \), and declare that any such path is a reconstruction of \( g_t \cdot k \). However, this naïve definition suffers from a ‘gauge redundancy’: for any curve \( h_t \) in \( G \) such that \( h_t \cdot k = k \), one has 

\[
p(t) = g_t \cdot k = g_t h_t \cdot k \text{ even though the paths } g_t \text{ and } g_t h_t \text{ differ.}
\]

To fix this, one additionally requires \( g_t \) to be a geodesic in \( G \) with respect to a right-invariant metric determined by the inertia operator \( \mathcal{I} \), which turns out to produce the condition [14, sec. 13.5]

\[
p(t) = \mathcal{I}(g_t g_t^{-1})
\]

This is a generalization of the relation \( \mathbf{L} = \mathcal{I}(\omega) \) between angular momentum \( \mathbf{L} \equiv p \) and angular velocity \( \omega \equiv \dot{g} g^{-1} \). Given an initial condition \( g_0 \) such that \( g_0 \cdot k = p(0) \), the resulting unique solution \( (g_t, p(t)) = (g_t, g_t \cdot k) \) in \( G \times \mathfrak{g}^* \) is called an Euler-Poincaré reconstruction of \( p(t) \). In sections 2.3 and 2.4, we show how this definition leads to geometric phases when \( p(t) \) is periodic.

**Remarks.** Eq. (18) is consistent with the Lie-Poisson equation (14): writing \( p(t) = g_t \cdot k \) and omitting the dependence on time, the reconstruction condition (18) can be recast as

\[
g^{-1} \dot{g} = (\text{Ad}_{g^{-1}} \circ \mathcal{I}^{-1} \circ \text{Ad}_g^*)(k)
\]

Figure 1: A schematic picture of Euler-Poincaré reconstruction: a path \( p(t) \) in \( \mathfrak{g}^* \) is lifted to a pair of paths \((g_t, p(t))\) in \( G \times \mathfrak{g}^* \cong T^*G \).
where we temporarily reinstate the notation $\text{Ad}^*$ for the coadjoint representation, and both sides are now Lie algebra elements. Acting with them on $k$ through the coadjoint representation (8) of $\mathfrak{g}$, we find $\text{ad}^*_g^{-1}(k) = \text{Ad}^*_g^{-1} \frac{d}{dt}(g \cdot k) = \text{ad}^*_g^{-1} \text{(Ad}^{-1}_g(\mathcal{I}^{-1}(g \cdot k))(k) = \text{Ad}^*_g^{-1} \text{ad}^*_g^{-1}(g \cdot k)$, that is,

$$\frac{d}{dt}(g_t \cdot k) = \text{ad}^*_g^{-1}(g_t \cdot k)$$

(20)

which is indeed the Lie-Poisson equation (14). One may also ask the opposite question: does (14) imply the reconstruction formula (18)? The answer is nearly yes: eq. (20) does not imply (19), but it does imply that $g^{-1} \dot{g}$ and $(\text{Ad}^{-1}_g(\mathcal{I}^{-1}(g \cdot k))$ only differ by an element of the Lie algebra of the stabilizer of $k$. Eq. (18) sets that element to zero for any time $t$; choosing a different element would amount to a different gauge choice.

The rewriting (19) exhibits a general feature of Lie-Poisson equations: non-trivial dynamics only occurs when the inertia operator breaks $G$ symmetry, i.e. when in general

$$\text{Ad}^*_g \circ \mathcal{I} \circ \text{Ad}^{-1}_g \neq \mathcal{I}.\quad (21)$$

Indeed, suppose instead that $G$ symmetry were preserved, i.e. that the inequality (21) were replaced by an equality for all $g \in G$. Then the right-hand side of (19) would be a constant $\mathcal{I}^{-1}(k)$ and the solution of (19) would read

$$g_t = e^{t\mathcal{I}^{-1}(k)}.\quad (22)$$

In the case of the Euler top, this occurs when the tensor of inertia is proportional to the identity matrix, i.e. when the rigid body is isotropic. Eq. (22) then states that the top rotates around its axis without any precession.

### 2.3 Dynamical phase and Berry phase

Geometric phases appear when one performs a loop — a closed path — in a suitable parameter space [2, 5, 11]. In the case at hand, parameter space is momentum space (or rather a coadjoint orbit therein), so the question we wish to ask is: given a path $p(t)$ such that $p(T) = p(0)$, is the reconstructed path $(g_t, p(t))$ closed? If not, is there a way to measure the difference between the initial configuration, $g_0$, and the final one, $g_T$? As we now show, in anisotropic setups, the path $g_t$ is typically not closed even when $p(t)$ is (see fig. 2), and the degree to which it fails to close is the combination of a dynamical phase, proportional to the period $T$, and a Berry phase. In order to prove this, following [20], we will integrate the Liouville one-form (16) along a closed path in $G \times \mathfrak{g}^*$ given by Euler-Poincaré reconstruction. Then we will argue that this integral can be interpreted in two ways: first, as a Berry phase; secondly, as the sum of a dynamical phase and an observable rotation angle after one period. The centrally extended version of that argument is postponed to section 2.4.

**Integrating the Liouville one-form.** Let $p(t) = g_t \cdot k$ be a closed path, with period $T$, in the orbit $\mathcal{O}_k$ (notation as in (15)). At this stage we do not yet assume that $p(t)$ solves the Lie-Poisson equation (14), nor that $g_t$ satisfies the reconstruction condition (18). Instead, we introduce a loop in $G \times \mathfrak{g}^*$ given by

$$\gamma_t = (\bar{g}_t, \bar{g}_t \cdot k),\quad (23)$$

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Figure 2: The fate of fig. 1 when the path \( p(t) = g_t \cdot k \), in momentum space, is closed. Its reconstruction \( (g_t, g_t \cdot k) \) generally contains a curve \( g_t \) that does not close, corresponding to a non-trivial group element \( g_0^{-1} g_T \). To compensate this effect, we introduce the closed path \( \bar{g}_t \) defined in (24).

where \( \bar{g}_t \) is the concatenation of \( g_t \) with a curve \( h_t \) lying in the stabilizer of \( k \), chosen so that \( \bar{g}_T = \bar{g}_0 \).

\[
\bar{g}_t = \begin{cases} g_t & \text{for } 0 \leq t \leq T, \\ g_T h_t & \text{for } T \leq t \leq T'. \end{cases} \tag{24}
\]

Here \( h_t \cdot k = k \) for any \( t \) in the interval \([T, T']\); the starting point of \( h_t \) is \( h_T = I \), and its endpoint is \( h_T' = g_T^{-1} g_0 \), which indeed ensures that \( \bar{g}_T = \bar{g}_0 \). The fact that \( h_t \) fixes \( k \) also ensures that the momentum part of (23) is constant on the interval \([T, T']\), where it equals \( p(0) = p(T) = g_T \cdot k \). For simplicity, we assume from now on that the stabilizer of \( k \) is a \( U(1) \) group; this will be sufficient for any Lie-Poisson equation based on the Virasoro group, including the KdV equation.

Let us now integrate the Liouville one-form (16) along the closed curve (23):

\[
\oint_\gamma \mathcal{A} = \int_0^T dt \left\langle k, \text{Ad}_{g_t^{-1}} g^{-1} \right\rangle + \int_T^{T'} dt \left\langle k, \text{Ad}_{g_T^{-1}} d(R h^{-1} g_T^{-1})_{gT} d(L g_T)_{hT} \right\rangle. \tag{25}
\]

Here we split the integral in two pieces coming from the two parts of the path (23), and \( L_g \) \( (R_g) \) denotes left (right) multiplication by \( g \). In the first term, we relate the right Maurer-Cartan form to the left one: \( \text{Ad}_{g^{-1}} \dot{g} g^{-1} = g^{-1} \dot{g} \). In the second term, we simplify the integrand into \( \left\langle k, \dot{h} h^{-1} \right\rangle \) and use the fact that \( h \) stabilizes \( k \) to rewrite this as \( \left\langle h \cdot k, h^{-1} \dot{h} \right\rangle = \left\langle k, h^{-1} \dot{h} \right\rangle \), which yields

\[
\oint_\gamma \mathcal{A} = \int_0^T dt \left\langle k, g^{-1} \dot{g} \right\rangle + \int_T^{T'} dt \left\langle k, h^{-1} \dot{h} \right\rangle. \tag{26}
\]

This expression is a (known [13, 14, 20]) key result for the rest of this paper. We stress that, in order to derive it, we did not assume that the path \( p(t) = g_t \cdot k \) solves the Lie-Poisson equation (14); all we needed was that \( p(t) \) be closed, with period \( T \).
Still following [20], we shall now provide two interpretations of the integral (26). On the one hand, it will turn out to be the flux of a symplectic form through a surface enclosed by the path \( p(t) \), allowing us to think of it as a Berry phase associated with adiabatic changes of reference frames \( g_t \). On the other hand, when \( p(t) \) solves the Lie-Poisson equation (14) and provided \( g_t \) satisfies the reconstruction condition (18), eq. (26) will be the sum of a dynamical phase and a rotation angle \( \Delta \phi \) in the U(1) stabilizer of \( k \), eventually allowing us to express \( \Delta \phi \) as the sum of a geometric phase and a dynamical phase. Here is the detailed argument:

**Eq. (26) is a Berry phase.** To see this, we relate (26) to the symplectic structure of coadjoint orbits of \( G \). Indeed, the integral of \( \mathcal{A} \) along the path (23) can be written as the line integral of a one-form in the group manifold alone, without reference to \( g^* \):

\[
\oint_{\gamma} A = \oint_{(g, \bar{g}, k)} \langle (\bar{g} \cdot k, d\bar{g} \bar{g}^{-1}), 0 \rangle = \oint_{g} \langle k, \bar{g}^{-1} d\bar{g} \rangle = \int_{\Sigma_g} d\langle k, \bar{g}^{-1} d\bar{g} \rangle.
\]

In the last equality we used Stokes’ theorem, with \( \Sigma_g \) an oriented two-dimensional surface in \( G \) whose boundary is the closed path \( \bar{g} \). The integrand on the far right-hand side of this expression is a two-form on \( G \), and it can be shown (see e.g. [25, sec. 5.3.2]) that it coincides with the pullback of the Kirillov-Kostant symplectic form on \( \mathcal{O}_k \) by the projection \( \Pi : G \to \mathcal{O}_k : g \mapsto g \cdot k \). In formulas, this means that

\[
d\langle k, \bar{g}^{-1} d\bar{g} \rangle = -(\Pi^* \Omega)_g
\]

where the symplectic form \( \Omega \), defined on the coadjoint orbit of \( k \), is such that the Poisson bracket (9) reads \( \{ \mathcal{F}, \mathcal{G} \}(p) = \Omega_p (d\mathcal{F}_p, d\mathcal{G}_p) \) for any \( p \in \mathcal{O}_k \). Plugging (28) back into (27), we find

\[
\oint_{\gamma} A = - \int_{\Sigma_g} \Pi^* \Omega = - \int_{\Pi(\Sigma_g)} \Omega = - \int_{\Sigma_g} \Omega
\]

where \( \Sigma_{g \cdot k} \) is any surface in \( \mathcal{O}_k \) whose boundary is the curve \( g_t \cdot k = p(t) \). Thus, the integral (26) is the flux (29) of the Kirillov-Kostant symplectic form, and this flux, in turn, can be interpreted as a Berry phase. Indeed, it is often true that quantizing a coadjoint orbit \( \mathcal{O}_k \) produces unitary representations of \( G \) in which a coherent state, acted upon by transformations tracing a closed path \( \bar{g}_t \), has a Berry connection whose curvature is the symplectic form \( \Omega \) [1, 33]. The phase (29) is a classical analogue of that statement.

Note that eq. (29) is expressed solely in terms of the path \( p(t) = g_t \cdot k \) in momentum space — there is no longer any reference to the path \( g_t \) by itself. This fact will play an essential role for the evaluation of the geometric phase (26) in section 3: it implies that its value is independent of the choice of \( g_t \), as long as \( p(t) = g_t \cdot k \). In particular, this will allow us to evaluate (26) with relatively simple choices of paths, as opposed to the generally complicated paths produced by the reconstruction condition (18).

**Eq. (26) = dynamical phase + rotation.** So far, since introducing the path (23), we did not need to assume that \( p(t) \) solves the Lie-Poisson equation (14) or that \( g_t \) satisfies the reconstruction condition (18). We now enforce both of these assumptions and work out their consequences for the integral (26).

To begin, using eqs. (17)–(18), the integrand of the first term in (26) can be recast as

\[
\int_{\gamma} A = \langle k, g^{-1} \bar{g} \rangle = \langle g \cdot k, \bar{g} \rangle = \langle g \cdot k, \mathcal{I}^{-1}(g \cdot k) \rangle = \langle p, \mathcal{I}^{-1}(p) \rangle = 2H(p),
\]

where \( H(p) \) is the...
Hamiltonian (13). Since energy $E$ is conserved, $H(p)$ is constant and the first term of (26) becomes a dynamical phase:

$$\int_0^T dt \langle k, g^{-1} \dot{g} \rangle = 2ET. \quad (30)$$

On the other hand, the second part of (26) is a boundary term because the one-form $\langle p, h^{-1}dh \rangle$ is exact. In fact, it is essentially the difference between $g_0$ and $g_T$: since we assume that the stabilizer is $U(1)$, we may label its elements by an angle $\phi$ and write $h^{-1} \dot{h} = -\dot{\phi} \xi_0$, where $\xi_0 \in g$ generates the stabilizer. The normalization of $\xi_0$ is fixed so that $e^{2\pi \xi_0} = I$ be the identity in $G$, but $e^{t\xi_0} \neq I$ for any $t \in (0, 2\pi)$. Then the second integral in (26) is

$$\int_T^{T'} dt \langle k, h^{-1} \dot{h} \rangle = -\langle k, \xi_0 \rangle (\phi(T') - \phi(T)) \equiv -\langle k, \xi_0 \rangle \Delta \phi, \quad (31)$$

where $\Delta \phi$ is the angle of the rotation $g_0^{-1} g_T$. As a result, we can write the integral (26) as the sum of the dynamical phase (30) and the angle (31). Equivalently, upon rearranging the terms, one has

$$\Delta \phi = \frac{2ET}{\langle k, \xi_0 \rangle} - \frac{1}{\langle k, \xi_0 \rangle} \oint A. \quad (32)$$

Note that the value of $\Delta \phi$ depends on the normalization of $\xi_0$, but the product $\xi_0 \Delta \phi$ does not, so this ambiguity is merely a matter of ‘units’. In particular, if $\phi$ is normalized so that one turn corresponds to an angle $2\pi$ (as stated above eq. (31)), then the normalization of $\xi_0$ becomes fixed uniquely.

Formula (32) makes it manifest that the complete rotation $\Delta \phi$ is the sum of two very different contributions. The first, proportional to the energy $E$ and the period $T$, is a dynamical phase. The second is the integral (26); it is a geometric phase that coincides with the symplectic flux (29). We now apply this statement to centrally extended groups. As we shall see, the extension will affect the Berry phase formula (26) and contribute an extra term to the right-hand side of (32). Both modifications will have observable consequences in the KdV equation (and more generally in any Lie-Poisson equation for the Virasoro group).

### 2.4 Reconstruction for centrally extended groups

We are interested in the reconstructed dynamics of Lie-Poisson equations for the Virasoro group. The latter is a central extension of the group of diffeomorphisms of the circle, so we now describe the extended analogue of sections 2.2 and 2.3. We start with some general preliminaries on centrally extended groups and their Lie-Poisson equations, then briefly analyse their reconstruction, and finally write general formulas for the geometric phases of reconstructed dynamics.

**Central extensions and Lie-Poisson equations.** Let $\hat{G} = G \times \mathbb{R}$ be a central extension of a Lie group $G$. Its elements are pairs $(f, \alpha)$ with a group operation

$$(f, \alpha) \ast (g, \beta) = (fg, \alpha + \beta + C(f, g)) \quad (33)$$
where $C(f, g) \in \mathbb{R}$ is a cocycle.\(^8\) The corresponding Lie algebra is $\hat{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{R}$, and its dual space $\hat{\mathfrak{g}}^* = \mathfrak{g}^* \oplus \mathbb{R}$ consists of pairs $(p, c)$, where $p \in \mathfrak{g}^*$ and $c \in \mathbb{R}$, the latter being a central charge. The pairing between $\hat{\mathfrak{g}}$ and its dual reads $\langle (p, c), (\xi, \alpha) \rangle = \langle p, \xi \rangle + c\alpha$. As before, the coadjoint representation (7) will play a key role; it turns out to read

$$ (f, \alpha) \cdot (p, c) = \left( \text{Ad}^*_f(p) - \frac{c}{12} S[f^{-1}], c \right), $$ (34)

where the $\text{Ad}^*$ on the right is the coadjoint representation of $G$ (without central extension) and $S[f]$ is the Souriau cocycle associated with $C$, defined so that

$$ \langle S[f], \xi \rangle \equiv -12 \frac{d}{d\xi} \bigg|_{\xi=0} \left[ C(f, e^{\xi}) + C(f e^{\xi}, f^{-1}) \right]. $$ (35)

In the Virasoro group, $S[f]$ will be the Schwarzian derivative that plays an important role in CFT (see section 3.1). We also need the coadjoint representation of $\hat{\mathfrak{g}}$, obtained by differentiating (34) (or equivalently given by eq. (8)). One thus finds

$$ \hat{\text{ad}}^*_{\xi, \alpha}(p, c) = \left( \text{ad}^*_p + \frac{c}{12} s[x], 0 \right), $$ (36)

where $s[x] \equiv \partial_t |_0 S[e^{\xi}]$ is the infinitesimal Souriau cocycle.

In order to write the Lie-Poisson equation (14), we introduce a centrally extended inertia operator

$$ \hat{I} : \hat{\mathfrak{g}} \to \hat{\mathfrak{g}}^* : (\xi, \alpha) \mapsto \left( \mathcal{I}(\xi), J\alpha \right) $$ (37)

where $\mathcal{I}$ is an inertia operator (12) for $\mathfrak{g}$, while $J > 0$ is just a number.\(^9\) Using the coadjoint representation (36), the corresponding Lie-Poisson equation (14) reads

$$ (\dot{p}, \dot{c}) = \left( \text{ad}^*_{\mathcal{I}^{-1}(p)}(p) + \frac{c}{12} s[\mathcal{I}^{-1}(p)] \right), $$ (38)

In particular, the central charge $c$ is a fixed parameter — this really just follows from its being left invariant by the coadjoint representation (34).

**Reconstruction conditions.** Suppose we are given a solution $(p(t), c)$ of eq. (38). Euler-Poincaré reconstruction consists in finding a path $(g_t, \alpha_t)$ in $\hat{G}$ such that

$$ (p(t), c) = (g_t, \alpha_t) \cdot (k, c) $$ (39)

for some fixed coadjoint vector $k$, where the dot denotes the coadjoint representation (34) of $\hat{G}$. In addition, the path must be such that the reconstruction condition (18) holds. For a centrally extended group, this means that

$$ (p(t), c) = \hat{I} \left[ \partial_t \right] \left( (g_t, \alpha_t) * (g_t, \alpha_t)^{-1} \right) = \left( \mathcal{I}(\dot{g} g^{-1}), J\alpha + J \partial_t \right) C(g_t, g_t^{-1}). $$ (40)

On the one hand, this yields the expected reconstruction equation $p = \mathcal{I}(\dot{g} g^{-1})$, exactly as in the unextended case (18). On the other hand, it gives an ordinary differential equation for $\alpha_t$, which is readily solved thanks to the constancy of the central charge $c$ [16]:

$$ \alpha_t = \alpha_0 + \frac{ct}{J} - \int_0^t \frac{d\tau}{g_{\tau} g_{\tau}^{-1}} C(g_\tau, g_\tau^{-1}). $$ (41)

---

\(^8\)See e.g. [25, chap. 2] for this terminology. In short, $C$ is such that the product (33) is associative.

\(^9\)More generally, we could use a ‘non-diagonal’ inertia operator that mixes the central extension and the centreless algebra, but we do not need to consider such possibilities here.
This result will turn out to be crucial for the evaluation of $\Delta \phi$ in the reconstructed KdV equation. To lighten the notation, we introduce a one-form $d_1 C$ on $G$ defined by $(d_1 C)_g(\hat{g}) = \partial_r \big|_{\tau=t} C(g_r, g_r^{-1})$, whereby the last term of (41) becomes an integral of $d_1 C$ along the path $\hat{g}_t$.

**Geometric phases.** We now assume that $p(t)$ is a periodic solution of eq. (38), with period $T$ and central charge $c$, lying in a coadjoint orbit $\mathcal{O}_{(k,c)}$ with U(1) stabilizer.\(^{10}\) Our goal is to rewrite eqs. (26) and (32) in the centrally extended case. To do this, first note that the Berry connection (16) now gets an extra central contribution:

$$\hat{A}_{((g,\alpha),(p,c))} = \langle (p,c), d(g,\alpha) (g,\alpha)^{-1} \rangle = (p, dg^{-1}) + c(d\alpha + (d_1 C)_g).$$

(42)

Similarly to (24), we introduce a closed curve $(\tilde{g}_t, \tilde{\alpha}_t)$ in $\hat{G}$ by concatenating the path $(g,\alpha)$, which satisfies (39)-(40), with a path $(h,\beta)$ in the stabilizer of $k$ which ensures that $(\tilde{g}, \tilde{\alpha})$ closes. In particular, $\beta_T = 0$ and $\beta_{T'} = \alpha_0 - \alpha_T$. The integral of the Liouville one-form (42) along the path $\gamma = ((\tilde{g}, \tilde{\alpha}), (\tilde{g}, \tilde{\alpha}) \cdot (k,c))$ is then found to be [1]

$$\int_\gamma \hat{A} = \int_0^T dt \left[ \langle k, g^{-1} \hat{g} \rangle + c \partial_r |_{\tau=t} C(g_r^{-1}, g_r) \right] + \int_T^{T'} dt \langle k, h^{-1} \hat{h} \rangle.$$  

(43)

This Berry phase is a straightforward generalization of (26), involving just one extra contribution due to the central extension. As before, we stress that the value of that integral depends neither on the specific path $\hat{g}_t$, nor on the parametrization of time. It only depends on the image of the path $p(t) \in g^*$, as in (29). However, in order to interpret (43) as the sum of a dynamical phase and an observable rotation angle, we need to enforce the reconstruction conditions (40) and write

$$\int_\gamma \hat{A} = \int \langle (k,c), (g,\alpha)^{-1} d(g,\alpha) \rangle + \int \langle (k,c), (h^{-1} dh, d\beta) \rangle$$

(44)

where we have split the path $(\tilde{g}, \tilde{\alpha})$ into a piece $(g,\alpha)$ on the interval $[0,T]$ and a piece $(h,\beta)$ in the stabilizer. The first piece yields a dynamical phase analogous to (30):

$$\int \langle (k,c), (g,\alpha)^{-1} d(g,\alpha) \rangle = \int_0^T dt \langle p, (\hat{I}^{-1}(p,c)) \rangle = T \langle p, \hat{I}^{-1}(p) \rangle + \frac{\alpha^2}{J} = 2ET. \ (45)$$

The second piece, on the other hand, now contains an extra contribution with respect to the unextended expression (31). Indeed, it reads

$$\int \langle (k,c), (h^{-1} dh, d\gamma) \rangle = \int \langle k, h^{-1} dh \rangle - c \alpha_T + c \alpha_0 + c C(g_{T'}, g_0)$$

(46)

owing to the conditions $\beta_T = 0$ and $\beta_{T'} = \alpha_0 - \alpha_T$. Using now the solution (41) of the reconstruction conditions to evaluate $\alpha_T$, we can rewrite (44) as

$$\int_\gamma \hat{A} = 2T \left( E - \frac{\alpha^2}{2J} \right) + c \int_\gamma d_1 C + c C(g_{T'}, g_0) + \int \langle k, h^{-1} dh \rangle.$$  

(47)

\(^{10}\)Since the group is extended, the stabilizer of $(k,c)$ trivially contains all ‘central translations’ $(\mathbb{I},\alpha)$. We mod these out, so our ‘U(1) stabilizer’ only consists of group elements of the form $(h,0)$.  

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Finally, as in (31), we interpret the last term as a rotation angle: recall that we assumed the stabilizer of $k$ to be a U(1) group, generated by $\xi_0$. The result is

$$\langle k, \xi_0 \rangle \Delta \phi = 2T \left( E - \frac{c^2}{2J} \right) - \oint \wedge A + c \int_g d_t C + c C(g_t^{-1}, g_0).$$

(48)

This is the centrally extended version of eq. (32), and it takes the anticipated form (6):

(i) The first term is the expected dynamical phase, with a subtraction of $c^2/(2J)$ ensuring that $\Delta \phi$ does not depend on $J$ (since $E$ is given by (45)). This is as it should be, since $J$ does not affect the Lie-Poisson equation (38).

(ii) The second term is the Berry phase (43), given by a loop integral of the Liouville one-form. It is universal in the sense that it does not depend on the inertia operator.

(iii) The third and fourth terms are proportional to the central charge, are directly due to the extension $C$, and are also universal. In particular, the third term is a line integral of $c(d_t C)_g = -\langle (0,c), (g,0) d(g,0)^{-1} \rangle$. This is the Berry connection on the coadjoint orbit of $(0,c)$, evaluated at the point $g^{-1}$. Thus, the anomalous phase in (48) is akin to an ‘inverse Berry phase’, except that the path $(g_t^{-1}, 0) \cdot (0,c)$ is not closed in general.

We now apply the result (48) to Lie-Poisson equations based on the Virasoro group, taking KdV as our main example. In general, all three terms of (48) will be non-zero.

3 Geometric phases and drift in reconstructed KdV

This section is devoted to the first key statement of our work. Namely, we focus on wave profiles that are amenable (i.e. that can be mapped on a constant thanks to suitable diffeomorphisms) and satisfy a Lie-Poisson equation for the Virasoro group, such as KdV. We then show that, for periodic waves $p(x,t)$, the master equations (43) and (48) apply and correspond to a generally non-trivial one-period rotation of the reconstructed dynamics. The angle $\Delta \phi$ of that rotation is the sum of a dynamical phase, a Berry phase and an anomalous phase, all of which can be written explicitly as functionals of either the reconstructed path, or of its projection on the coadjoint orbit of $p$. The Berry and anomalous phases are universal (they follow solely from the Virasoro group structure), and the Berry phase in particular takes the form described in [1].

Before displaying these results, we briefly review some elementary properties of the Virasoro group and its relation to the KdV equation. For more background material, we refer e.g. to [28] and its appendix A; our notation and conventions will follow those of that paper. Much more detailed, pedagogical accounts of the Virasoro group and its coadjoint orbits [26] can be found e.g. in [18,24,25,27,34]. Finally, note that the application of the results of this section to travelling waves is postponed to section 4.

3.1 Virasoro group and KdV equation

Here we briefly recall the relation between the Virasoro group and the KdV equation, through the Lie-Poisson equations (14)-(38). For many more details on this relation and
its generalizations, see [18]. Up to a different choice of inertia operator, the same construction leads to the Hunter-Saxton and Camassa-Holm equations [17].

The Virasoro group, which we denote as \( \widetilde{\text{Diff}} S^1 \), is the central extension of the group \( \text{Diff} S^1 \) of diffeomorphisms of the circle. Accordingly, let \( x \in \mathbb{R} \) be a \( 2\pi \)-periodic coordinate. An element of the Virasoro group is a pair \((f, \alpha)\), where \( \alpha \) is a real number while the function \( f \in \text{Diff} S^1 \) is an (orientation-preserving) diffeomorphism, such that \( f(x + 2\pi) = f(x) + 2\pi \).\(^{11}\) For example, a rotation by \( \theta \) reads \( f(x) = x + \theta \), which we denote as \( \mathcal{R}_\theta(x) \) from now on (rotations will soon play a prominent role). The group law is, by definition, of the form (33):

\[
(f, \alpha) \ast (g, \beta) = (f \circ g, \alpha + \beta + \mathcal{C}(f, g)), \quad \mathcal{C}(f, g) \equiv -\frac{1}{48\pi} \int_0^{2\pi} dx \log (f' \circ g) \frac{g''}{g'}, \tag{49}
\]

where \( \circ \) denotes composition and \( \mathcal{C} \) is the Bott cocycle [35]. For future reference, note that \( \mathcal{C} \) vanishes on rotations: if either \( f \), or \( g \), or \( f \circ g \) is a rotation, then \( \mathcal{C}(f, g) = 0 \). We let \( \widetilde{\text{Vect}} S^1 \) denote the Lie algebra of \( \widetilde{\text{Diff}} S^1 \) — the Virasoro algebra. Its elements are pairs \((\xi, \alpha)\), where \( \xi = \xi(x) \partial_x \in \text{Vect} S^1 \) is a vector field on the circle and \( \alpha \in \mathbb{R} \) as before. Its dual space, \( (\widetilde{\text{Vect}} S^1)^* \), consists of pairs \((p, c)\), where \( p = p(x) dx^2 \) is a quadratic density and \( c \in \mathbb{R} \) is a central charge.\(^{12}\) The pairing between \( \text{Vect} S^1 \) and its dual is

\[
\langle (p, c), (\xi, \alpha) \rangle \equiv \frac{1}{2\pi} \int_0^{2\pi} dx \, p(x) \xi(x) + c \alpha. \tag{50}
\]

In two-dimensional CFT, \( p \) is interpreted as a (chiral component of the) stress tensor and (50) is the Noether charge of the conformal generator \( \xi \). In the KdV context and its cousins, \( p(x) \) is a wave profile, governed by a nonlinear evolution equation of the form (38). From the perspective of symplectic geometry, \( p(x) \) is thus a ‘momentum vector’, which justifies our notation.

We now derive KdV from Virasoro group theory. To begin, we need the coadjoint representation (34), which we write as \((f, \alpha) \cdot (p, c) = (f \cdot p, c)\) thanks to the fact that the central charge is invariant. Using the Bott cocycle (49) and the definition (35), one can then show [24, 25] that the term \( f \cdot p \) is given by

\[
(f \cdot p)(x) = \left[ (f^{-1})'(x) \right]^2 \, p(f^{-1}(x)) - \frac{c}{12} \left[ \frac{(f^{-1})'''}{(f^{-1})'} - \frac{3}{2} \left( \frac{(f^{-1})''}{(f^{-1})'} \right)^2 \right] \bigg|_x \tag{51}
\]

where \( f^{-1} \) is the inverse of \( f \), such that \( f^{-1}(f(x)) = f(f^{-1}(x)) = x \). This is the standard transformation law of the stress tensor under conformal transformations in any two-dimensional CFT [36, sec. 5.4]. In particular, the combination of derivatives of \( f^{-1} \) multiplying \( c/12 \) is the Schwarzian derivative of \( f^{-1} \): the Virasoro version of the Souriau cocycle (35). As a result, the coadjoint representation (36) of the Virasoro algebra reads

\[
\text{ad}^*_\xi(p, c) = \left( -\xi p' - 2\xi p + \frac{c}{12} \xi'', 0 \right). \tag{52}
\]

The vanishing second entry confirms that the central charge is constant in time, for any choice of the inertia operator. By contrast, \( p(x) \) transforms non-trivially under the

\(^{11}\)To be precise, what we are describing here are the universal covers of \( \text{Diff} S^1 \) and \( \widetilde{\text{Diff}} S^1 \).

\(^{12}\)Both vector fields and quadratic densities are \( 2\pi \)-periodic: \( \xi(x + 2\pi) = \xi(x), \, p(x + 2\pi) = p(x) \).
Virasoro group, so it will generally have a non-trivial time evolution. Specifically, we choose the inertia operator to be the simplest possible map of the form (37):

\[ \hat{T} : \hat{\text{Vect}} S^1 \rightarrow \hat{\text{Vect}} S^1^* : (\xi(x) \partial_x, \alpha) \mapsto (\xi(x) dx^2, J_\alpha), \tag{53} \]

where \( J \) is an arbitrary (and ultimately irrelevant) positive constant. This choice ensures that \( \hat{T} \) is invertible and self-adjoint (recall the definition around (12)), so it is indeed an inertia operator. It is also anisotropic in the sense of eq. (21), since the adjoint and coadjoint representations of Virasoro are inequivalent. More complicated inertia operators yield different wave equations, such as Hunter-Saxton and Camassa-Holm [17, 18]. For definiteness, we do not consider such more general cases, but our approach also applies to them up to straightforward modifications of all expressions involving \( \hat{T} \).

From now on, all integrals over \( x \) are implicitly evaluated on the interval \([0, 2\pi]\). The quadratic Hamiltonian (13) induced by the inertia operator (53) then reads

\[ H[p] = \frac{1}{4\pi} \int dx p(x)^2 + \frac{c^2}{2J}. \tag{54} \]

As \( p(x) dx^2 \) transforms according to eq. (51), this expression is manifestly not Virasoro-invariant. This implies that the resulting Lie-Poisson equation (14)-(38) is non-trivial; using (52), one finds indeed

\[ \dot{p} + 3pp' - \frac{c}{12} p''' = 0 \tag{55} \]

where, as in (38), the central charge \( c \) is a constant parameter. This is the Korteweg-de Vries equation (1) for the field \( p(x, t) \), derived here as a Lie-Poisson equation of Virasoro.

### 3.2 Reconstruction and phases for periodic waves

We now describe the reconstructed dynamics in the (cotangent bundle of the) Virasoro group when \( p(x, t) \) is a periodic solution of (55), say with period \( T \), so that \( p(x, t + T) = p(x, t) \). This is precisely the setup considered in section 2, so eqs. (43) and (48) will apply. At this stage, we adopt an abstract viewpoint without reference to particle motion, and without assuming that \( p(x, t) \) is a travelling wave — these problems will be addressed in sections 3.3 and 4, respectively. We refer again to [24, chap. 4-6] and [25, chap. 6-7] for the necessary background on the Virasoro group, especially its coadjoint orbits [26,27], which will now start playing an important role.

**Amenable profiles.** As in section 2, the motion of \( p(x, t) \) determines a path \( (g_t, \alpha_t) \) — actually a geodesic — in the Virasoro group, and our task is to find the difference between \( g_T \) and \( g_0 \) when \( p \) has period \( T \). Before doing that, however, we need to state one key simplifying assumption: from now on, we require the wave profile \( p(x, t) \) to be amenable, that is, conformally equivalent to a uniform (i.e. \( x \)-independent) field configuration \( k \). In other words, we assume that there exists a constant \( k \) and a diffeomorphism \( g_0 \in \text{Diff } S^1 \) such that, at time \( t = 0 \),

\[ p(x, 0) = (g_0 \cdot k)(x), \tag{56} \]

where the dot denotes the coadjoint action (51). One can show that the constant \( k \), provided it exists, is uniquely fixed by the wave \( p \).\(^{13}\) The map \( g_0 \) can then be seen as a

\(^{13}\)To find \( k \), one can evaluate the trace of the monodromy matrix of the Hill equation associated with \( p \) (see e.g. [34]). In section 4.3 we will give the explicit value of \( k \) for cnoidal waves [28].
In principle, the initial condition periodicity of the path (40) reads

\[ \text{periodic solution} \]

Let us now apply Euler-Poincaré reconstruction to a periodic solution of KdV. Geometric phases in KdV. Let us now apply Euler-Poincaré reconstruction to a periodic solution \((p(x,t), c)\) of KdV, assumed to be amenable. Given this wave, the reconstruction condition (40) reads

\[
(p(t), \frac{\mathcal{E}}{J}) = \partial_t \left[ (g_T, \alpha_t) \cdot (g_T, \alpha_t)^{-1} \right] = \left( \dot{g}_T \circ g_T^{-1}, \dot{\alpha}_t - \frac{1}{48\pi} \int dx \dot{g}_T \circ g_T^{-1} (g_T^{-1})'' \right)
\]

where we used the multiplication (49) of the Virasoro group. On the far right-hand side, the first entry yields the reconstruction condition that one would find, without central extension, in the group Diff\(S^1\):

\[ p(x,t) = \partial_t |_{\tau=t} g_T (g_T^{-1}(x)), \quad \text{i.e.} \quad \partial_t g_T^{-1}(x) + p(x,t) \partial_x g_T^{-1}(x) = 0. \]

In principle, the initial condition \(g_0\) is free, but we choose it to satisfy eq. (56). As for the path \(\alpha_t \in \mathbb{R}\), we can solve eq. (57) similarly to (41) and find [16]

\[
\alpha_t = \alpha_0 + \frac{ct}{J} + \frac{1}{48\pi} \int_0^t d\tau \int dx \dot{g}_T \circ g_T^{-1} (g_T^{-1})''.
\]

This will eventually contribute to \(\Delta \phi\) through the ‘anomalous phase’ of eq. (48).

As stressed earlier, choosing \(g_0\) to satisfy (56) for some constant \(k\), along with the periodicity of \(p\), ensures that \(g_0^{-1} \circ g_T\) is a rotation (since \(g_0 \cdot k = g_T \cdot k\), and \(k\) is only stabilized by rotations). We now use eq. (48) to compute the angle of that rotation as the sum of a dynamical phase, a Berry phase and an anomalous term:

(i) The dynamical phase (45) involves the energy, given by the Hamiltonian (54):

\[
E - \frac{c^2}{2J} = \frac{1}{4\pi} \int dx p(x,0)^2.
\]

\[ \text{14} \text{The discrete values } k = -n^2c/24, \text{ with integer } n, \text{ have a larger stabilizer (still including rotations), but we need not be concerned with these exceptional cases: our conclusions will apply even then.} \]
Since energy is conserved, one may evaluate it at any time \( t \); here we chose \( t = 0 \). As stressed below (48), the right-hand side of this expression is independent of \( J \). This is as it should be, since \( J \) does not affect dynamics and does not appear in the reconstruction condition (58).

(ii) The Berry phase (43) is, in fact, standard: as was shown in [9], line integrals of the symplectic potential can be computed in closed form; they are ‘geometric actions’ for the Virasoro group, and were later interpreted as Berry phases associated with adiabatic conformal transformations [1]. Importantly, these phases only depend on the image of the path \( p(t) \) in momentum space — not on the reconstructed path \( g_t \) in the group manifold. As a result, we are free to express the Berry phase in terms of any path \( f_t \) in Diff\( S^1 \) such that the curve traced by \( f_t \cdot k \) coincides with \( p(t) \). We choose such a path \( f \). Then, adapting the notation of [1] to the case at hand, eq. (43) becomes

\[
\oint_C \hat{A} = \int \frac{dt}{2\pi} \frac{\hat{f}}{f'} \left[ k + \frac{c}{24} \left( \frac{f''}{f'} \right) \right] - k f_0^{-1}(f_T(0))
\]

where it is understood that the integrals over \( t \) and \( x \) run from 0 to \( T \) and \( 2\pi \), respectively.

(iii) Finally, since \( g_t^{-1} \circ g_0 \) is a rotation and since the cocycle (49) vanishes on rotations, the last term of (48) does not contribute. The only non-zero contribution to the anomalous phase comes from the integral of the derivative of \( C \), namely

\[
c \int dt \partial_t C(g_{\tau}, g_{\tau}^{-1}) \overset{(49)}{=} -c \int \frac{dt}{48\pi} \hat{g} \circ g^{-1} (g^{-1})'' = -c \int \frac{dt}{48\pi} \frac{\partial_t g^{-1} (g^{-1})''}{(g^{-1})'} \frac{(g^{-1})''}{(g^{-1})} \frac{(g^{-1})''}{(g^{-1})'},
\]

where we used the properties \( \hat{g} \circ g^{-1} + \hat{g}' \circ g^{-1} \partial_x g^{-1} = 0 \) and \( \hat{g} \circ g^{-1} (g^{-1})' = 1 \), along with an integration by parts.

Combining eqs. (60), (61) and (62), we can finally write the angle \( \Delta \phi \), given by (48), as

\[
k \Delta \phi = \frac{T}{2\pi} \int dx \frac{p^2}{2} + k f_0^{-1}(f_T(0)) - \int \frac{dt}{2\pi} \frac{\hat{f}}{f'} \left[ k + \frac{c}{24} \left( \frac{f''}{f'} \right) \right] - \frac{c}{24} \int \frac{dt}{2\pi} \frac{\partial_t g^{-1} (g^{-1})''}{(g^{-1})'} \frac{(g^{-1})''}{(g^{-1})'}
\]

where we used \( \xi_0 = \partial_x \) to simplify \( (k, \xi_0) = k \). This takes the anticipated form (6) and contains two geometric phases: one is a Virasoro Berry phase [1], and the other is an ‘anomalous phase’ which we deliberately wrote in a way that exhibits its similarity with the Berry term. We stress, however, that the path \( g \) in the anomalous phase is the reconstructed curve that satisfies (58), whereas the Berry phase involves any path \( f_t \) such that the curve \( f_t \cdot k \) coincides with \( p(t) \). This distinction will allow us to evaluate the Berry phase easily for travelling waves, while the anomalous one will require a bit more work.

Note that the formula (63) for \( \Delta \phi \) is almost universal: aside from the model-dependent dynamical phase, it applies to any Lie-Poisson system based on the Virasoro group, such as the Hunter-Saxton and Camassa-Holm equations [19]. Note also that the overall factor \( k \) on the left-hand side of (63) implies that, at \( k = 0 \), the right-hand side of (63) vanishes. However, since \( c \neq 0 \) in general, one may well have \( \Delta \phi \neq 0 \) even for \( k = 0 \); cnoidal waves (section 4.3) will provide an explicit example of this.

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15What we call \( k \) here is denoted as \( h - c/24 \) in [1], where \( h \) is the weight of a primary state.
3.3 Drift velocity as a Poincaré rotation number

We now return to the reconstruction equation (58) to explain how the angle (63) can be observed by monitoring the motion of ‘fluid particles’ as defined by eq. (2). Again, we impose no restrictions on \( p(x, t) \) other than amenability and periodicity in space and time (so \( p(x, t) \) could, for instance, be a system of colliding periodic solitons with rational phase shift). At the end of this section, we will comment further on the (in)applicability of our approach to actual fluid dynamics, owing to a subtlety in reference frames that we already alluded to in the introduction. The application of our arguments to travelling waves is postponed to section 4.

Particle drift as reconstruction. Consider a ‘fluid particle’ on the real line whose position \( x(t) \) satisfies the equation of motion (2) in terms of the (given) wave profile \( p \). We claim that this equation is equivalent to the reconstruction condition (58).\(^{16}\) Indeed, let \( X(t, x_0) \) be the unique solution of (2) with initial condition \( X(0, x_0) = x_0 \). We can think of this solution as a time-dependent diffeomorphism \( g_t \), with an arbitrary initial configuration \( g_0 \), acting on a suitable starting point: \( X(t, x_0) \equiv g_t(g_0^{-1}(x_0)) \). Then, in terms of \( g_t \), eq. (2) becomes \( \dot{g}_t(g_0^{-1}(x_0)) = (p(t) \circ g_t)(g_0^{-1}(x_0)) \). Since this holds for all \( x_0 \), we may remove the argument \( g_0^{-1}(x_0) \) and deduce that \( g_t \) satisfies the reconstruction condition (58), as announced. Conversely, the condition (58) may thus be seen as an equation of motion for (comoving) fluid particles. This is true for any \( p \), but from now on we always let \( g_0 \) be a uniformizing map that satisfies eq. (56).

Relating reconstruction to particle motion suggests a way to observe the angle \( \Delta \phi \) computed in (63). Indeed, suppose one asks the following question: given a particle with initial position \( x_0 \) and equation of motion (2), what is the particle’s position after one period? We can certainly write \( x(t) = g_t(g_0^{-1}(x_0)) \) in terms of the reconstructed curve \( g_t \), since this is the unique solution of (2) with initial condition \( x_0 \). After one period, one has

\[
x(T) = g_0\left(g_0^{-1} \circ g_T(g_0^{-1}(x_0))\right).
\]

Now recall the crucial fact, due to the periodicity and amenability of \( p \), that \( g_0^{-1} \circ g_T \) is a rotation by \( \Delta \phi \) (given by eq. (63) when \( p \) solves KdV). As a result, after \( N \) periods,

\[
x(NT) = g_0(g_0^{-1}(x_0) + N\Delta \phi),
\]

and we may identify the map \( F \) in eq. (4) with the composition \( g_0 \circ R_{\Delta \phi} \circ g_0^{-1} \), where \( R_{\Delta \phi}(x) \equiv x + \Delta \phi \). The stroboscopic particle motion is thus a discrete-time dynamical system governed by iterations of the diffeomorphism \( F = g_0 \circ R_{\Delta \phi} \circ g_0^{-1} \). The latter is conjugate to rotation by \( \Delta \phi \), which allows us to exploit a key result on circle dynamics: the Poincaré rotation number of \( F \) \(^{17}\), as defined in the second equation of (5), coincides with \( \Delta \phi \).\(^{17}\) In terms of the particle’s position, we can write

\[
\Delta \phi = \lim_{N \to +\infty} \frac{x(NT) - x(0)}{N} \equiv v_{\text{Drift}} T,
\]

where the drift velocity is defined as in eq. (5). From that perspective, \( \Delta \phi \) is the average rotation angle of a particle during one period — which answers the question raised above.

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\(^{16}\)For the record, this is a well known result; see e.g. [16, sec. 2].

\(^{17}\)This is because the Poincaré rotation number is invariant under conjugation in \( \text{Diff} S^1 \) [24, sec. 4.4.3].
Note that $\Delta \phi$ is independent of the particle’s initial position $x(0)$, as is the drift velocity.

We have thus shown that ‘particle motion’ in the sense of eq. (2) provides a system whose late-time behaviour is directly sensitive to the angle (63) through the drift velocity (5)-(66). In particular, this velocity contains a contribution due to a Virasoro Berry phase [1], somewhat analogously to the crest slowdown found in [32] for breaking waves whose envelope is described by the nonlinear Schrödinger equation. Note that this prediction is independent of the uniformizing map $g_0$ that satisfies (56); in fact, that map is generally unknown (even when the value of $k$ is known for a given $p(x,t)$). In the next section we will study the drift velocity in travelling waves satisfying KdV, and in that case we will actually manage to find $g_0$ analytically.

Comparison with fluid dynamics. At this point, it is worth comparing our approach, and in particular the drift velocity defined in (5)-(66), to the Stokes drift of fluid particles in shallow water dynamics [31]. Indeed, within the KdV approximation of fluid mechanics in a shallow layer (see e.g. [30]), eq. (1) describes the slow time evolution of a right-moving wave $p(x,t)$. Here, $x$ is emphatically not a fix laboratory coordinate, but rather a (dimensionless) ‘lightcone’, or comoving, coordinate

$$x = X - Ct$$

where $X$ is a static laboratory coordinate, $t$ is the (dimensionless) slow time variable, and $C \gg 1$ is a dimensionless version of the standard velocity $\sqrt{gh}$ of gravity waves of average depth $h$ in a gravitational field $g$. In fact, $C \propto L^2/h^2$, where $L$ is the (dimensionful) wavelength. The KdV approximation then holds in the ‘non-relativistic’ limit $h/L \rightarrow 0$, where the velocity $C$ of (67) goes to infinity. In that limit, the leading velocity of fluid particles is purely horizontal and given by an equation of motion that closely resembles, yet is crucially different from, eq. (2) above. Indeed, in terms of the static laboratory coordinate $X$, particle motion reads

$$\dot{X}(t) = p(X(t) - Ct, t),$$

which only differs from (2) by the spatial argument of $p$. Equivalently, in terms of the comoving coordinate $x$, one has $\dot{x} = p(x,t) - C$, which obviously differs from eq. (2) by a dominant term.

One can then ask the same question as the one we raised above: given a periodic wave train, what is the drift velocity of $X(t)$? This is the velocity that would presumably be seen in a laboratory, and it is tempting to hope that it is related to the one we introduced (66). However, the latter was defined from the equation of motion (2) in the comoving (‘lightcone’) frame, and it is quite clear that the drift velocity in the laboratory frame, due to eq. (68), will take a very different form because of the extra dominant term $C \propto L^2/h^2$. For instance, at leading order in $h/L$, the particle satisfying (68) sees a fast average of the wave profile, and its position at time $t$ (assuming $t$ is of order one) is simply

$$X(t) \sim X(0) + \frac{t}{2\pi} \int_0^{2\pi} dx \, p(x,0) + O(h^2/L^2).$$

The drift velocity then coincides with the average of the wave profile (this average is constant along KdV time evolution), which is very different indeed from the prediction
Thus, while the Berry phases and drift studied here do have some similarities with fluid dynamics, they do not, ultimately, describe the same phenomenology.

4 Particle drift and phases of travelling waves

Travelling waves form a prominent class of solutions of the KdV equation (and of wave equations in general): they take the form \( p(x, t) = p(x - vt) \) for some velocity \( v \), so their shape is constant throughout time evolution. When the profile \( p(x) \) is \( 2\pi \)-periodic in space, such travelling waves are automatically time-periodic with period \( T = 2\pi/|v| \).

In this section, we study the reconstruction equations (2)-(58) for travelling waves that solve KdV. For amenable profiles, we show that these equations are integrable — they can be solved exactly in terms of known wave data —, and we build an explicit uniformizing map from which an exact expression for the drift velocity (5) follows. We then use this to derive a simplified formula for the rotation angle (63). Finally, we apply this to cnoidal waves and obtain a detailed picture of drift velocity throughout the cnoidal parameter space. In particular, we exhibit ‘orbital bifurcations’ that occur at the boundaries of a resonance wedge anticipated in [28]: in the wedge, particle motion is locked to the wave and \( v_{\text{Drift}} = v \), while no such locking occurs outside of the wedge and \( v_{\text{Drift}} \sim v/3 \) at large \( v \). Thus, the drift velocity emerges in this picture as a diagnostic of wave amenability, \( i.e. \) of the nature of Virasoro coadjoint orbits.

4.1 Exact reconstruction for travelling waves

Here we study the reconstruction equations (2)-(58), without any reference to geometric phases for now. Our goal is to show that, for amenable travelling waves, these equations can be solved exactly in terms of readily accessible wave data. An explicit expression for the drift velocity will follow. The comparison to geometric phases and formula (63) is postponed to section 4.2.

Amenable travelling waves. Consider a travelling wave \( p(x, t) = p(x - vt) \) that solves the KdV equation (1) with a non-zero velocity \( v \). Since \( \dot{p} = -vp' \), the KdV equation can be written entirely in terms of the time-independent profile \( p(x) \). Upon integrating KdV once, one finds that there must exist a constant \( A \) such that

\[
- vp + \frac{3}{2} p^2 - \frac{c}{12} p''' = A. \tag{70}
\]

Multiplying the latter equation by \( p' \) and integrating again, one concludes that there also exists a constant \( B \) such that

\[
- \frac{v}{2} p^2 + \frac{1}{2} p^3 - \frac{c}{24} p'^2 = Ap + B. \tag{71}
\]

These constants \( A, B \) will eventually turn out to be closely related to the drift velocity (5).

As before, we assume \( p(x) \) to be amenable in the sense that there exists a constant \( k \) and a map \( g_0 \in \text{Diff} S^1 \) satisfying eq. (56). We now show that this assumption, along
with the fact that \( p(x,t) \) is a travelling wave, yields a tremendous simplification of the reconstruction equation (58). Indeed, since \( p(x) = (g_0 \cdot k)(x) \), we can write
\[
p(x, t) = (\mathcal{R}_{vt} \circ g_0) \cdot k (x)
\]
where \( \mathcal{R}_\theta(x) \equiv x + \theta \) is a rotation by \( \theta \) (same notation as below (65)). On the other hand, we know, by definition of reconstruction, that \( p(x, t) = (g_t \cdot k)(x) \). Combining this with (72), it follows that
\[
(g_t^{-1} \circ \mathcal{R}_{vt} \circ g_0) \cdot k = k, \tag{73}
\]
i.e. the diffeomorphism \( g_t^{-1} \circ \mathcal{R}_{vt} \circ g_0 \) stabilizes \( k \). Since \( k \) is uniform, its stabilizer consists of rotations only,\(^{19}\) so there must exist a function \( \theta(t) \in \mathbb{R} \) such that
\[
g_t = \mathcal{R}_{vt} \circ g_0 \circ \mathcal{R}_{\theta(t)}. \tag{74}
\]
We have thus ‘factorized’ the dependence of \( g_t(x) \) on \( t \) and \( x \). Indeed, rewriting (74) as
\[
g_t^{-1}(x) = g_0^{-1}(x - vt) - \theta(t), \tag{75}
\]
the ‘advection’ form of eq. (58) yields \( \dot{\theta}(t) = [p(x - vt) - v] \times (g_0^{-1})'(x - vt) \), which holds for all \( x, t \). Since \( t \) and \( x - vt \) are independent coordinates on the plane, we may just as well rename \( x - vt \) into \( x \) and rewrite the equation for \( \theta(t) \) as
\[
\dot{\theta}(t) = [p(x) - v] (g_0^{-1})'(x). \tag{76}
\]
This is a crucial result, as we now explain.

**Uniformization and drift velocity.** Eq. (76) is an essential consequence of the result (74). Indeed, since the left- and right-hand sides of (76) depend separately on \( t \) and \( x \), we find several striking implications just by differentiating the equation. First, differentiating (76) with respect to \( x \), we conclude that there exists a constant \( \mathcal{V} \neq 0 \) such that
\[
(g_0^{-1})'(x) = \frac{\mathcal{V}}{p(x) - v}. \tag{77}
\]
We will soon see that \( \mathcal{V} \) contributes to the drift velocity of fluid particles — hence the notation. Since \( g_0 \in \text{Diff} S^1 \), eq. (77) readily implies that, if the profile \( p \) is amenable, then \( p(x) - v \) has no roots (so its sign is constant); in fact, we will show (right before section 4.2) that the implication also goes the other way around. Furthermore, the condition \( g_0^{-1}(x + 2\pi) = g_0^{-1}(x) + 2\pi \) implies that the constant \( \mathcal{V} \) is given by
\[
\mathcal{V} = \frac{2\pi}{\int_0^{2\pi} \frac{dx}{p(x) - v}}. \tag{78}
\]
Together with eq. (77), this determines the uniformizing map \( g_0^{-1} \) exactly and uniquely, up to an arbitrary rotation by \( \phi \):
\[
g_0^{-1}(x) = \phi + \int_0^x \frac{\mathcal{V} \, dy}{p(y) - v}. \tag{79}
\]
\(^{19}\)As before, we assume \( k \) to be generic, i.e. not of the form \( k = -n^2 c/24 \). See footnote 14.
Thus, we have found $g_0^{-1}$, hence $g_0$, in terms of the wave profile. We will use this below (section 4.3) to obtain an explicit expression for cnoidal uniformizing maps.

Secondly, differentiating eq. (76) with respect to $t$, we find $\dot{\theta} = 0$. In fact, owing to eq. (77), $\dot{\theta} = \mathcal{V}$, so $\theta(t) = \theta_0 + \mathcal{V}t$. The integration constant vanishes by virtue of eq. (74) and the initial condition $g_{x=0} = g_0$, so $\theta(t) = \mathcal{V}t$. From this we can deduce the exact reconstruction $g_t$, hence the solution $x(t)$ of (2), hence the drift velocity (3). Here we go: from (75) we read off

$$g_t^{-1}(x) = g_0^{-1}(x - vt) - \mathcal{V}t \quad \Leftrightarrow \quad g_t(x) = g_0(x + \mathcal{V}t) + vt,$$

which is an exact geodesic in the Virasoro group (with respect to the right-invariant metric induced by the inertia operator (53)). The ensuing particle motion reads

$$x(t) = g_t(g_0^{-1}(x_0)) = g_0(g_0^{-1}(x_0) + \mathcal{V}t) + vt.$$  (81)

Again, this is an exact solution of the equation of motion (2).

The drift velocity (66) can be obtained by computing the Poincaré rotation number of $g_0 \circ g_t^{-1}$: using (80), we find that $g_t^{-1} \circ g_0$ is a rotation by $-\mathcal{V}T \pm 2\pi$, where $\pm = \text{sign}(v)$. Since the rotation number is a conjugation-invariant homomorphism from $\text{Diff} S^1$ to $\mathbb{R}$ [24, sec. 4.4.3], we then have

Poincaré rotation number of $g_T \circ g_0^{-1} \equiv \Delta \phi = \pm 2\pi + \mathcal{V}T$,  (82)

where $g_T \circ g_0^{-1}$ was called $F$ in (4) and below (65). Thus, the drift velocity (5)-(66) is

$$v_{\text{Drift}} = \frac{\Delta \phi}{T} = v + \mathcal{V}.$$  (83)

This confirms that the normalization constant $\mathcal{V}$ from (77) is a velocity, as anticipated. All these expressions are exact. We shall apply them to cnoidal waves in section 4.3.

An accidental identity. We have just seen that amenable travelling waves solving KdV can be mapped on a constant profile $k$ through a diffeomorphism $g_0$ given by the inverse of (79), with $\mathcal{V}$ given by (78). We now show that this implies one constraint on the constants $A, B$ defined by (71). Indeed, recall that the coadjoint action of $g_0$ on $k$ is given by (51). Using eq. (77) for $(g_0^{-1})'$, we can thus rewrite $g_0 \cdot k$ as

$$(g_0 \cdot k)(x) = \frac{1}{[p(x) - v]^2} \left[ k\mathcal{V}^2 - \frac{c}{24} p'(x)^2 + \frac{c}{12} (p(x) - v) p''(x) \right].$$  (84)

Here the right-hand side is supposed to equal $p(x)$, which, at this stage, is not obvious at all. To make progress, we use the fact that $p$ solves the KdV equation, hence that eqs. (70)-(71) hold. This allows us to eliminate all derivatives of $p$ in eq. (84) and yields

$$g_0 \cdot k = p + \frac{1}{(p - v)^2} \left[ k\mathcal{V}^2 + B + Av \right].$$  (85)

Since the right-hand side must equal $p$, we conclude that the constants $A, B, k, \mathcal{V}$ are related through the following equation:

$$k\mathcal{V}^2 = -B - Av.$$  (86)
All coefficients here are determined by the wave profile $p(x - vt)$. Thus, we have now proven that any amenable travelling wave solution of the KdV equation satisfies identity (86). This can be used, for instance, to find $k$ once $A, B, V$ are known, or to find $V$ once $A, B, k$ are known. In practice, eq. (86) yields a consistency check that can be used once $A, B, k, V$ have been found by independent means. That will be our point of view below for cnoidal waves.

Incidentally, eq. (86) allows us to prove the following point, raised below eq. (77): if $p(x - vt)$ solves KdV, then $p(x)$ is amenable if and only if $p(x) - v$ has no roots. Indeed, we have already shown that amenability implies the absence of roots. Conversely, if $p(x) - v$ has no roots, then one can define a map $g_0 \in \text{Diff} S^1$ by eq. (79), and the resulting coadjoint action on any constant $k$ is given by eq. (85). Upon choosing $k$ to satisfy eq. (86), one finds $g_0 \cdot k = p$, proving that $p(x)$ is amenable.

4.2 Geometric phases of travelling waves

Having shown that particle motion is integrable for amenable travelling waves, we now return to eq. (63) for the rotation angle $\Delta \phi$ and use the properties of travelling waves to simplify it. We treat separately the dynamical and Berry phases on the one hand, and the anomalous phase on the other hand, then verify that the resulting prediction of $v_{\text{Drift}}$ is consistent with eq. (83).

**Dynamical and Berry phases.** For a travelling wave, the dynamical phase in (63) is readily evaluated as the integral of $p(x)^2$. As for the Berry phase, it is greatly simplified by the fact that the path $f_t(x)$ need not be the reconstructed one, $g_t(x)$. Owing to the fact that $p(x, t) = p(x - vt)$ is a travelling wave, we can thus choose $f_t(x) = g_0(x) + vt$, where $g_0$ satisfies (56). Upon plugging this into the Berry phase (61), one finds

$$\oint_{\gamma} \mathcal{A} = \pm \int \frac{dx}{g_0} \left[ k + \frac{c}{12} \left( \frac{g_0''}{g_0} - \frac{3}{2} \left( \frac{g_0''}{g_0} \right)^2 \right) \right] \mp 2\pi k = \pm \int dx \frac{p(x)}{p(x) - v} \mp 2\pi k, \quad (87)$$

where $\pm = \text{sign}(v)$ and we used the coadjoint representation (51) to recognize the integrand as $(g_0 \cdot k)(x) = p(x)$. Thus, up to a sign and a term $2\pi k$, the Berry phase is the zero-mode (the average) of the profile $p$.

**Anomalous phase.** The anomalous phase (62) explicitly depends on the reconstructed path $g_t$, so, in contrast to the dynamical and Berry phases, one really needs to solve eq. (58) in order to simplify it. Fortunately, we have already done that: we showed in section 4.1 that the equation of motion (2) can be integrated exactly for amenable travelling waves. Accordingly, we use the solution (80) to rewrite the anomalous phase (62) as

$$c \int_g \text{d}_1 \mathcal{C} = \frac{cT}{48\pi} \int dx \frac{V}{(g_0^{-1})' \left( (g_0^{-1})'' \right)'} = \frac{cT}{48\pi} \int dx \frac{p'(x)^2}{p(x) - v}, \quad (88)$$

where we also used eq. (77) to express $(g_0^{-1})'$ in terms of $p$. Combining this with the Berry phase (87) and the dynamical phase, we obtain an expression of $\Delta \phi$ that only involves the (time-independent) profile $p(x)$, without any other wave data. In practice, it is simpler to express the formula as a drift velocity (5) instead of $\Delta \phi$, so as to absorb
the awkward signs of eq. (87). The result reads:

$$v_{\text{Drift}} = \frac{\Delta \phi}{T} = \frac{1}{2\pi k} \int dx \, p(x)^2 + v - \frac{v}{2\pi k} \int dx \, p(x) + \frac{c}{48\pi k} \int dx \, \frac{p'(x)^2}{p(x) - v},$$

(89)

where we have grouped the various phases as in (6)-(63). Note that, from this perspective, the anomalous phase looks like a correction to the dynamical phase (both contribute terms that are not proportional to the velocity $v$, as opposed to the Berry phase). However, this is really specific to travelling waves: for other kinds of profiles $p(x, t)$, the simplification (88) would not hold.

**Consistency check.** At this point, one should compare the geometric phase prediction (89) to the previously derived exact result (83). Indeed, it is not obvious that these expressions coincide. The fact that they do follows from a series of formulas derived earlier: first, upon writing (89) as $v_{\text{Drift}} = v + \mathcal{V}$, one finds the condition

$$2\pi k \mathcal{V} = \int dx \left[ p(x) (p(x) - v) + \frac{c}{24} \frac{p'(x)^2}{p(x) - v} \right].$$

(90)

Proving this equality then proves that (89) and (83) coincide. To this end, one can use eq. (71) to express $p'(x)^2$ in terms of $p$, and eq. (70) to write the remaining $p^2$ term as a linear combination of constants, $p$ and $p''$. The latter does not contribute to the integral (by periodicity), and, after various cancellations, one ends up having to prove the identity

$$2\pi k \mathcal{V} = (-Av - B) \int \frac{dx}{p(x) - v}.$$

(91)

Owing to eq. (78), this is equivalent to the formula $-Av - B = k\mathcal{V}^2$, which we encountered in eq. (86). We conclude that eq. (89) does coincide, as expected, with eq. (83).

We have now completed a full conceptual circle: we first argued on symplectic grounds that particle motion, in the sense of eq. (2), has a drift velocity determined by the sum of phases (63). We then showed, independently, that the drift velocity is given by (83) for amenable travelling waves, and we just proved that this formula is consistent with the geometric phase prediction. In practice, it is much easier to compute the drift velocity using eq. (83), with $\mathcal{V}$ given by (78), than in terms of the phases (89). For travelling waves, the main virtue of (89) is that it neatly isolates the various geometric contributions to the drift velocity. For more complicated wave profiles, however, a simple formula such as (83) is generally not available, so one has to use the general geometric expression (63) to compute the drift velocity. We will consider such more general cases elsewhere. For now, we apply our results to cnoidal waves.

### 4.3 Drift in cnoidal waves and orbital bifurcations

Here, we apply the formulas of the previous pages to cnoidal waves — the periodic solitons of KdV. The key result in that context are explicit formulas for the cnoidal uniformizing map and drift velocity, both given in terms of an elliptic integral of the third kind. Importantly, not all cnoidal waves are amenable [28]: the profiles that have no uniform orbit representative span a ‘resonance wedge’ in the cnoidal parameter space,
Figure 3: The root wedge (94) and resonance wedge (95) in the \((m,V)\) plane. They partly overlap when \(m > 0.5\). For completeness, we also display the line \(V = (2 - m)/3\), along which \(k = 0\). As we shall see below, the contributions of dynamical, Berry and anomalous phases all diverge on that line, but these divergences cancel out so that the sum (89) is finite even when \(k = 0\). A more detailed picture can be found in fig. 7 of [28].

with \(v_{\text{Drift}} = v\) in the wedge. By contrast, outside of the wedge, \(v_{\text{Drift}} \neq v\) is such that \(v_{\text{Drift}} \sim v/3\) at large \(v\), corresponding to an average one-period rotation of \(\Delta \phi \sim \pm 2\pi/3\) (see eq. (100)). As we explain, all these results are consequences of the (symplectic) geometry of coadjoint orbits of the Virasoro group.

**Cnoidal waves.** A *cnoidal wave* (with \(2\pi\)-periodicity in space) is a travelling wave solution of the KdV equation (1). It is specified by two parameters: a ‘pointedness’ \(m \in [0,1)\) and a (rescaled) velocity \(V \in \mathbb{R}\). In these terms, the wave reads

\[
p(x,t) = \frac{cK(m)^2}{3\pi^2} \left[ \frac{V}{2} - \frac{m+1}{3} + m \sin^2 \left( \frac{K(m)}{\pi} (x - vt) \right)m \right]
\]

(92)

where \(K(m)\) is the complete elliptic integral of the first kind, \(\sin\) is the Jacobi elliptic sine, and the wave’s velocity is a function of \((m,V)\) given by

\[
v = \frac{cK(m)^2}{2\pi^2} V.
\]

(93)

Several qualitative aspects of the equation of motion (2) can be read off from simple properties of the profile (92). For example, \(\dot{x}(t)\) has a constant sign if and only if \(p(x)\) has no roots; such roots occur in the wedge

\[
\frac{2 - 4m}{3} < V < \frac{2 + 2m}{3}
\]

(root wedge).

Much more importantly, the results of section 4.1 imply that the key object is not quite \(p(x)\), but rather \(p(x) - v\): as shown below (77), \(p(x) - v\) has roots, if and only if \(p(x)\)
Figure 4: The uniform orbit representative \((96)\) of a cnoidal wave with parameters \((m, V)\); the black lines are level curves of \(k\). At fixed \(m\), \(k/c\) is a monotonously increasing function of \(V\) (and the dependence is roughly linear at large \(|V|\)). Note the wedge, with boundaries \(V = [(1 \pm 3)m - 2]/6\), in which \((96)\) becomes complex: all waves in the wedge are non-amenable. Adapted from fig. 4 in [28].

is non-amenable. Using \((92)\) and the velocity \((93)\), one readily sees that such roots only occur in the following resonance wedge:

\[
-\frac{m+1}{3} < V < \frac{2m-1}{3} \quad \text{(resonance wedge).} \tag{95}
\]

This is consistent with the classification of Virasoro orbits of cnoidal waves described in [28] (and closely related to the band structure of the Lamé equation [37]). Indeed, any profile with labels \((m, V)\) outside of the wedge \((95)\) is amenable, with a uniform orbit representative\(^{20}\)

\[
k = \frac{c}{6\pi^2}\left[K(m)\zeta(\wp^{-1}(V)) - \zeta(K(m))\wp^{-1}(V)\right]^2 \tag{96}
\]

that becomes complex (hence nonsensical) once \((m, V)\) enter into the resonance wedge. Here \(\wp^{-1}\) is the inverse Weierstrass function and \(\zeta\) is the Weierstrass zeta function (both specified by half-periods \(K(m)\) and \(iK(1-m)\)). On the boundaries of the wedge, where \(V = [(1 \pm 3)m - 2]/6\), the constant \((96)\) takes the ‘exceptional’ value \(k = -c/24\). See fig. 4 for a plot of \(k\) in the \((m, V)\) plane. For many more details about this, see [28]; for an introduction to elliptic functions, see \textit{e.g.} [37,38].

**Particle motion and drift.** The tools of section 4.1 readily apply to amenable cnoidal waves. Thus, the velocity \(\mathcal{V}\) defined by \((78)\) reads

\[
\mathcal{V} = -\frac{cK(m)^3}{3\pi^2} \frac{V + (m + 1)/3}{\Pi\left(\frac{m}{V + (m + 1)/3}\right)^m} \tag{97}
\]

\(^{20}\)Eq. \((96)\) is roughly the square of the crystal momentum for a state with energy \(\mathcal{E} \propto \text{cst} - V\) in a Lamé lattice. From that viewpoint, the resonance wedge is the Lamé band gap. See [28] for details.
Figure 5: A plot of the cnoidal boost \( g_0(x) \), the inverse of (98), for \( \phi = 0, m = 0.9, V = 1/3 \). On the right, we also represent the effect of such a map on uniformly distributed points on a circle with angular coordinate \( x \sim x + 2\pi \). The region with the highest density of points is \( x = \pi \), while the lowest density occurs at \( x = 0 \). These points respectively correspond to the maximum and minimum of \( p(x)/c \), when \( p(x) \) is a cnoidal profile (92).

where \( \Pi(x|m) \) is the complete elliptic integral of the third kind. As a result, the uniformizing map (79) can be written as

\[
g_0^{-1}(x) = \phi + \pi \frac{\Pi \left( m, \frac{m}{V+(m+1)/3} \right), \text{am} \left( K(m)x/\pi \right) | m \right)}{\Pi \left( m, \frac{m}{V+(m+1)/3} \right) | m \right)}
\]

where \( \Pi(x,y|m) \) is the incomplete elliptic integral of the third kind and \( \text{am}(x|m) \) is the Jacobi amplitude. This is an explicit ‘cnoidal boost’: by construction, \( g_0 \) maps a uniform profile \( k \) (‘at rest’) on a cnoidal (‘boosted’) one. An example of such a boost is plotted in fig. 5. The corresponding particle motion is given by eq. (81); see fig. 6 for a few examples. As is manifest there, the large-scale behaviour of \( x(t) \) is approximately linear in \( t \), with a drift velocity given by eq. (83):

\[
v_{\text{Drift}} = \frac{cK(m)^2}{\pi^2} \left[ \frac{V}{2} - \frac{K(m)}{3} \frac{m}{V+(m+1)/3 | m} \right] \]

The asymptotics of this expression at large \( |V| \) follow from \( \Pi(x|m) \sim K(m) + \mathcal{O}(x) \) for \( x \to 0 \), which yields

\[
v_{\text{Drift}} \sim \frac{v}{3} \quad \text{as} \quad v \to \pm \infty.
\]

This is the asymptotic formula announced above. Writing \( v_{\text{Drift}} = \Delta \phi/T \), it indicates an average rotation angle of \( \Delta \phi \sim \pm 2\pi/3 \) during each period.

We stress that eq. (99) only holds in the region of amenable cnoidal waves, i.e. outside of the resonance wedge (95). In order to obtain a complete picture of \( v_{\text{Drift}} \) as a function of \( (m, V) \), throughout the entire cnoidal parameter space, we therefore need to study the equation of motion (2) in the resonance wedge. This will be done below, and the resulting final shape of \( v_{\text{Drift}}(m, V) \) is displayed in fig. 8. For now, we study (99) from the point of view of geometric phases.
Figure 6: A few solutions of the equation of motion (2) for a cnoidal wave (92) at $c = 1, m = 0.9, V = 4/3$ (left panel) and $V = 2/3$ (right panel). Both sets of parameters are well outside of the resonance wedge (95), so cnoidal waves are amenable and the motion is given by eq. (81), with $g_0$ written in (98) and $V$ in eq. (97). Looking at the plots ‘from far away’, particle motion is approximately linear in time, with a drift velocity (99). In the right panel, note the regions where $\dot{x} < 0$: this is because $p(x)$ has roots for $(m, V) = (0.9, 2/3)$, which lies in the root wedge (94). Such a behaviour does not occur in the left panel, as $(m, V) = (0.9, 4/3)$ lies outside of the root wedge.

Geometric phases of cnoidal waves. As explained in section 4.2, eq. (83) for the drift velocity can be written in a ‘decomposed’ form (89) that exhibits the separate contributions of dynamical, Berry and anomalous phases. We now list the values of these three terms for cnoidal waves in terms of parameters $(m, V)$, being understood that all formulas only hold outside of the resonance wedge (95):

\[
\begin{align*}
\dot{v}_{\text{Dynamical}} &= \frac{c^2 K(m)^4}{9 \pi^3 k} \left[ \frac{V^2}{2} + V \left( \frac{4 - 2m}{3} - \frac{2E(m)}{K(m)} \right) + \frac{2 - 2m + 2m^2}{9} \right] \\
\dot{v}_{\text{Berry}} &= \frac{cVK(m)^2}{2 \pi^2} \left[ 1 - \frac{2cK(m)^2}{6 \pi^2 k} \left( \frac{V}{2} + \frac{2 - m}{3} - \frac{E(m)}{K(m)} \right) \right] \\
\dot{v}_{\text{Anomalous}} &= -\frac{c^2 m^2 K(m)^4}{144 \pi^4 k} \left[ 1 - \frac{V + \frac{m-2}{3}}{V + \frac{m+1}{3}} \frac{\Pi \left( \frac{m}{V+(m+1)/3} \right)|m|}{K(m)} \right],
\end{align*}
\]

where $K(m)$, $E(m)$ and $\Pi(x|m)$ are respectively complete elliptic integrals of the first, second and third kind, and $k$ is the function of $(m, V)$ written in eq. (96). It is not particularly illuminating to plot these velocities as functions of $(m, V)$. Their overwhelmingly dominant feature is a divergence on the line $V = (2 - m)/3$, where $k = 0$. This divergence cancels out when the three velocities are added together, since the total drift velocity (99) is finite even when $k = 0$ (as it should be); see fig. 8 below. Note also that the Berry velocity (102) is the only one that vanishes on the line $V = 0$.

As shown in section 4.2, the coincidence between formulas (83) and (89) for the drift velocity hinges on the identity (86) satisfied by amenable travelling waves. We now verify this identity for cnoidal waves: using the profile (92) and the definition (71) of the
Figure 7: A few solutions of (2) when \( p(x,t) \) is a cnoidal wave (92) at \( c = 1, m = 0.9, V = 0.1 \) (left panel) and \( V = -1/3 \) (right panel). These parameters lie in the resonance wedge (95), so eq. (81) does not apply and the plotted values \( x(t) \) were obtained by numerical integration of (2). Note the monotonous behaviour of \( x(t)/t \), radically different from the oscillating one in fig. 6. The rotation number \( \Delta \phi = \pm 2\pi \) is manifest.

The value of \(-B - Av\) follows. Upon plugging it in eq. (86) and using the value (96) of \( k \), one encounters the formula

\[
V^3 - \frac{m^2 - m + 1}{3} V - \frac{2m^3 - 3m^2 - 3m + 2}{27} = \left( \frac{K(m) \zeta(\wp^{-1}(V)) - \zeta(K(m)) \wp^{-1}(V)}{\Pi \left( \frac{m}{V + (m + 1)/3} \right)} \right)^2.
\]

This turns out to be a known, albeit somewhat obscure, identity that arises in the Lamé band structure: it coincides with eq. (7.20) of [39] upon identifying the crystal momentum as \( q(E) \equiv \frac{\pi}{K(m)} \sqrt{-6k/c} \) in terms of the uniform representative (96), along with \( e = V, e_1 = (2 - m)/3, e_2 = (2m - 1)/3 \) and \( e_3 = -(m + 1)/3 \).

Particle motion in the resonance wedge. The cnoidal waves whose parameters \((m, V)\) belong to the resonance wedge (95) are not amenable [28], so the solution of the equation of motion (2) is no longer given by eq. (81). It is, nonetheless, easy to compute the particle drift velocity as defined in (3). Indeed, for a travelling wave \( p(x, t) = p(x - vt) \), the equation of motion (2) can be recast as

\[
\dot{X}(t) = p(X(t)) - v
\]

in terms of \( X \equiv x - vt \). Thus, if \( p(X) - v \) has roots (which precisely occurs in the resonance wedge), then at least one of them, say \( X^* \), is a stable fixed point of the system.

\[\text{Up to permutation, those are the standard values of Weierstrass roots } e_i \text{ appropriate for Jacobi elliptic functions (see e.g. [28, app. B]). The left-hand side of (106) then coincides with } -(e_1 - e)(e_2 - e)(e_3 - e).\]
Figure 8: The cnoidal drift velocity given by (99) and (108), along with level curves of $v_{\text{Drift}}$, at $c = 1$. In the left panel, $v_{\text{Drift}}$ is plotted as a function of $(m, V)$, with the wave velocity given by (93), and the resonance wedge has straight boundaries as specified in (95). In the right panel, $V$ is traded for $v$, so that the boundaries of the resonance wedge are no longer straight lines, but the values $v_{\text{Drift}} = v$ in the wedge, and $v_{\text{Drift}} \sim v/3$, outside of the wedge, are apparent. Note that, despite the orbital bifurcations on the wedge boundaries, $v_{\text{Drift}}$ is a continuous function on the cnoidal parameter space.

(107). It follows that $X(t) \to X^*$ at late times, which is to say that $x(t) \sim X^* + vt$ at large $t > 0$. This is manifest in fig. 7, where we plot a few solutions of (2) for a wave in the resonance wedge. As a result, the drift velocity (3) is

$$v_{\text{Drift}} = v \quad \text{in the resonance wedge (95).}$$

This finally justifies the name ‘resonance wedge’: for parameters $(m, V)$ that satisfy (95), particle motion is ‘locked’ to the wave — it ‘resonates’. This is akin to the SNIPER bifurcation of the Adler equation [29]. The resulting, complete function $v_{\text{Drift}}(m, V)$, on the entire cnoidal parameter space, is shown in fig. 8.

We stress again that the result (108) could not have been deduced from a rotation angle such as (63). If anything, upon declaring that (108) is to be written in the form $v_{\text{Drift}} = \Delta \phi / T$, one would find a (piecewise) constant value

$$\Delta \phi = 2\pi \text{sign}(v) \quad \text{in the resonance wedge.}$$

This could not have emerged from the phase formula (63) since $k$, given by (96), is complex in the resonance wedge. Despite this, it is in fact possible to predict that
\( \Delta \phi / (2\pi) \) must be an integer in the resonance wedge by using the symplectic arguments of section 2 — the exact same line of thought that led to eq. (63). Indeed, we argued earlier that the equation of motion (2) is an Euler-Poincaré reconstruction equation for the Virasoro group, and this holds whether or not \( p(x,t) \) is amenable. Furthermore, when \( p(x,t) \) is periodic in time, it still remains true that the reconstructed path \( g_t \in \text{Diff} S^1 \) satisfies \((g_0^{-1} \circ g_T) \cdot k(x) = k(x)\) for a suitable orbit representative \( k(x) \). In the case of non-amenable orbits, \( k(x) \) cannot be a constant, but the fact remains that \( g_0^{-1} \circ g_T \) belongs to its stabilizer. And now, the key argument: for non-amenable Virasoro orbits, the stabilizer is not (conjugate to) a group of rotations \([18,24,25,27,34]\). Instead, the stabilizer is a non-compact group, isomorphic to \( \mathbb{R} \) (possibly up to further finite factors), and consists of circle diffeomorphisms whose rotation number vanishes modulo \( 2\pi \). It thus follows that

\[
\Delta \phi = \text{Rotation number of } g_T \circ g_0^{-1} = 2\pi n
\]  

(110)

for some integer \( n \), which confirms the observed result (109). In that sense, the drift velocity (108) is a diagnostic of the fact that cnoidal profiles in the resonance wedge are not amenable [28].

5 Conclusion and outlook

The purpose of this paper has been to initiate a geometric study of nonlinear wave equations, such as KdV, Hunter-Saxton or Camassa-Holm, from a point of view that strongly relies on group theory and symplectic geometry. This follows a long tradition in mathematics and physics [13,14,16,20,40] (see also the very recent [41]), but it seems to have been somewhat overlooked in the mainstream physics literature despite the relevance of geometric objects, such as Berry phases, in a plethora of systems ranging from condensed matter to nonlinear dynamics (see e.g. the classic [42] or the more contemporary [43], and references therein). Our goal has been to start filling that gap.

Specifically, we considered a spatially periodic KdV equation (1) and the equation of motion (2), seen as a model for a ‘fluid particle’ dragged along by the wave profile \( p(x,t) \). Using the fact that (2) effectively coincides with the equation for Euler-Poincaré reconstruction used in symplectic geometry, we predicted the value of the particle’s drift velocity in terms of a Poincaré rotation number \( \Delta \phi \). The latter turned out to be a sum of phases (6) that we wrote explicitly in eq. (63) and that crucially involves a Virasoro Berry phase in the sense of [1]. We then turned to travelling waves, for which the assumption of amenability yielded a striking simplification of the equation of motion (2). In fact, we showed that particle motion becomes integrable in that case, leading to the simple formula (83) for the drift velocity. An equivalent way to write that velocity in terms of geometric phases was displayed in eq. (89), and we showed that the two expressions, while not manifestly identical, do in fact coincide thanks to the KdV equation. Finally, we applied our tools to cnoidal waves and exhibited ‘orbital bifurcations’ occurring at the boundaries of the resonance wedge (95), inside of which particle motion is locked to the wave. These bifurcations are a direct observation of the sharp change in nature of Virasoro orbits of cnoidal waves, as investigated in [28].

As stressed in section 3.3, the equation of motion (2) whose drift we computed does not quite coincide with the actual equation of motion for a fluid particle in shallow water. In particular, while conceptually similar, it is not clear whether the drift velocity studied...
here has anything to do with the standard notion of Stokes drift [31]. A natural question that follows from our work, therefore, is whether any actual experiment could exhibit the rotation we studied, and in particular Virasoro Berry phases. For example, it is almost trivially true that a quantum-mechanical particle on a circle, subjected to the periodic Hamiltonian

$$\hat{H} = p(\hat{x}, t) \hat{P} - \frac{i\hbar}{2} p'(\hat{x}, t),$$

would have a wavefunction that rotates by an average angle $\Delta \phi$ during each period of the profile $p(x, t)$. This example, however, is somewhat artificial, as one is merely restating our construction in quantum language. It would be much more satisfactory to find a classical mechanical system that naturally reproduces the equation of motion (2) — for example, a plasma in one dimension subjected to a suitable external magnetic field. We hope to address this issue elsewhere.

Setting aside the issue of experimental signatures, one can think of numerous follow-ups of our work. For example, remaining within the confines of the KdV equation, it is natural to apply formula (63) to non-travelling waves, such as profiles containing multiple colliding cnoidal waves with a rational phase shift [44]. The phase shift being rational ensures that the profile as a whole is periodic in time. Indeed, in terms of the phase shift $\Delta \theta$ (which has a priori nothing to do with $\Delta \phi$!), the profile satisfies

$$p(x, t + \tau) = p(x - \Delta \theta, t)$$

for some quasi-period $\tau$. If the phase shift is rational in the sense that $\Delta \theta = 2\pi p/q$, with $p, q$ coprime integers, then the time period of the profile is $T = q\tau$, and our results of section 3 apply. It would be illuminating to display the resulting Berry phase spectrum (as a function of the wave profile parameters), especially as one might hope to even extend the picture, formally, to non-periodic profiles (for which $\Delta \theta$ is irrational). Other potential extensions of our paper include the quantum version of $\Delta \phi$, in the sense of the quantum KdV equation [45], and the effects of stochasticity. Regarding the latter, see e.g. [46] and references therein, or [47] for a language very close to ours.

The symplectic approach followed in this paper should apply to a host of other non-linear wave equations, and it would be interesting to investigate the analogue of our angle $\Delta \phi$ (and its observable effects) in such setups. As examples, we already mentioned the Hunter-Saxton and Camassa-Holm equations [19], for which the Berry phase and anomalous phase of eq. (63) would remain unchanged. Perhaps more interestingly, one could also extend the picture to Lie-Poisson wave equations not based on the Virasoro group, such as Hirota-Satsuma dynamics [48]. The nonlinear Schrödinger equation is also, in effect, a Lie-Poisson equation, albeit one based on the loop group of $\text{SO}(3)$ through its reformulation as a Landau-Lifschitz model [49]. We hope to contribute to some of these research avenues in the future.

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Appendix: Groups and geometry

In this appendix, we review some of the elementary material on Lie groups and symplectic
geometry needed in section 2. For a much broader and more pedagogical introduction,
we refer e.g. to [13,14,23].

**Lie groups.** Let $G$ be a Lie group with Lie algebra $\mathfrak{g} = T_I G$; we denote elements of the
former as $f$, $g$, etc. and those of the latter as $\xi$, $\zeta$, etc., while $I$ is the identity in $G$. The
group acts on its algebra through the adjoint representation\footnote{Here $e^{t\xi}$ is the exponential of $t\xi$, such that $e^0 = I$.}

$$ \text{Ad}_g(\xi) \equiv \frac{d}{dt} \bigg|_0 \left( g \cdot e^{t\xi} \cdot g^{-1} \right), \quad (113) $$

which for matrix groups boils down to $gXg^{-1}$. The corresponding infinitesimal action,
\textit{i.e.} the adjoint representation of the Lie algebra, coincides with the Lie bracket:

$$ \text{ad}_\xi \zeta \equiv \frac{d}{dt} \bigg|_0 \text{Ad}_e^{t\xi}(\zeta) = [\xi,\zeta], \quad (114) $$

which for matrix groups is just a commutator. In the context of Lie-Poisson evolution
equations (14), one thinks of the group manifold $G$ as the space of classical configura-
tions of a dynamical system, while the Lie algebra $\mathfrak{g}$ consists of all infinitesimal motions
(deformations) of $G$, such as angular velocities. Equivalently, $G$ is the group of all possible
changes of reference frames that a dynamical system is allowed to go through, while
the adjoint representation (113) says how the angular velocity varies under changes of
frames. For instance, the matrix group $G = \text{SO}(3)$ of spatial rotations consists of all possible
orientations of a rigid body (with respect to a reference frame whose origin lies at
the body’s centre of mass).

The Lie algebra $\mathfrak{g}$ is a vector space whose dual $\mathfrak{g}^*$ consists of elements that we write
as $p$, $q$, etc., each of which is a linear form on $\mathfrak{g}$,

$$ p : \mathfrak{g} \to \mathbb{R} : \xi \mapsto \langle p,\xi \rangle. \quad (115) $$

We refer to such maps as coadjoint vectors, and their transformation law under the action
of the group $G$ is the coadjoint representation defined in eq. (7):

$$ g \cdot p = \text{Ad}_g^* (p) \equiv p \circ \text{Ad}_g^{-1} \quad \forall \, p \in \mathfrak{g}^*. \quad (116) $$

This definition is dual to (113) and ensures that the pairing (115) is invariant under ‘changes of reference frames’ in the sense that $\langle \text{Ad}_g^* (p), \text{Ad}_g(\xi) \rangle = \langle p,\xi \rangle$. Analogously to
(114), the coadjoint representation of the Lie algebra is

$$ \text{ad}_\xi^* (p) \equiv \frac{d}{dt} \bigg|_0 \text{Ad}_e^{t\xi}(p) = -p \circ \text{ad}_\xi. \quad (117) $$
As mentioned in eq. (8), this is to say that \( \langle \text{ad}_p^\varepsilon \xi, \zeta \rangle = -\langle p, [\xi, \zeta] \rangle \) for all Lie algebra elements \( \xi, \zeta \) and any coadjoint vector \( p \). For any algebra admitting a non-degenerate invariant bilinear form, the adjoint and coadjoint representations are equivalent and their distinction is inconsequential. However, for the algebra of vector fields that are needed for KdV, adjoint and coadjoint representations differ, which is why the definitions (116)-(117) are important for our purposes.

The phase space \( T^*G \). What now follows is a technical preliminary to section 2.2. Given a Lie group \( G \), we show that its cotangent bundle \( T^*G = \bigsqcup_{g \in G} T^*_g G \) is a trivial bundle. This will allow us to think of the product \( G \times g^* \) as a symplectic manifold:

**Lemma:** \( T^*G \) is diffeomorphic to a direct product \( G \times g^* \).

**Proof:** The key point is that group translations can be used to map any tangent space \( T_g G \) on the Lie algebra \( g = T_e G \). Accordingly, any element of \( T^*_g G \), i.e. a map \( u : T_g G \to \mathbb{R} \), can be turned into a map from \( g \) to \( \mathbb{R} \), i.e. an element of \( g^* \). Concretely, let us write right translations in \( G \) as \( R_g : G \to G : f \mapsto fg \). Then define a map

\[
\Phi : T^*G \to G \times g^* : (g, u) \mapsto \left( g, \left( (R_g)^* u \right)_1 \right)
\]

where \( ( (R_g)^* u )_1 \) is a coadjoint vector such that \( \left( ((R_g)^* u)_1, \xi \right) = \langle u, d(R_g)_! \xi \rangle \) for any \( \xi \in g \). The map (118) is a smooth bijection whose inverse

\[
\Phi^{-1} : G \times g^* \to T^*G : (g, p) \mapsto \left( g, (R_{g^{-1}})_! p \right)
\]

is also smooth, so it is a diffeomorphism. This concludes the proof. \( \square \)

Having established that \( T^*G \cong G \times g^* \) is a trivial bundle, we now look at its symplectic form. First recall how the Liouville symplectic form is built on \( T^*G \): one has a projection \( \pi : T^* G \to G : (g, u) \mapsto g \) whose differential at \( (g, u) \) projects any tangent vector of \( T^* G \) on its part tangent to \( G \) alone:

\[
d\pi_{(g,u)} : T_{(g,u)} T^*G \to T_g G : (V,V) \mapsto V.
\]

The Liouville one-form on \( T^*G \) then reads

\[
\tilde{\mathcal{A}}_{(g,u)} \equiv u \circ d\pi_{(g,u)}, \quad \text{i.e.} \quad \tilde{\mathcal{A}}_{(g,u)}(V,V) = \langle u, V \rangle,
\]

and the symplectic form of \( T^*G \) is its exterior derivative: \( \omega = -d\tilde{\mathcal{A}} \). We put tildes on these objects because we are eventually interested in the counterpart of (121) in \( G \times g^* \), which will be tilde-free. To find this counterpart, we pullback the Liouville one-form thanks to the inverse diffeomorphism (119), finding

\[
[(\Phi^{-1})^* \mathcal{A}]_{(g,p)}(V,X) = \mathcal{A}_{(g,(R_{g^{-1}})_! p)_g)}(d(\Phi^{-1})_{(g,p)})(V,X)
\]

\[
\overset{(121)}{=} \langle (R_{g^{-1}})_! p, V \rangle = \langle p, d(R_{g^{-1}})_! V \rangle.
\]

Thus, if we introduce the right Maurer-Cartan form \( d(R_{g^{-1}})_! g \equiv dg g^{-1} \), and if we denote the (pulled-back) Liouville one-form on \( G \times g^* \) as \( \mathcal{A} \equiv (\Phi^{-1})^* \tilde{\mathcal{A}} \), then eq. (123) yields

\[
\mathcal{A}_{(g,p)} = \langle (p, dg g^{-1}), 0 \rangle.
\]

This is the result announced in eq. (16).
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