Transitive Lie algebroids - categorical point of view

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Introduction

Transitive Lie algebroids have specific properties that allow to look at the transitive Lie algebroid as an element of the object of a homotopy functor. Roughly speaking each transitive Lie algebroids can be described as a vector bundle over the tangent bundle of the manifold which is endowed with additional structures. Therefore transitive Lie algebroids admits a construction of inverse image generated by a smooth mapping of smooth manifolds. The construction can be managed as a homotopy functor from the category of smooth manifolds to the transitive Lie algebroids. The intention of this article is to make a classification of transitive Lie algebroids and on this basis to construct a classifying space. The realization of the intention allows to describe characteristic classes of transitive Lie algebroids form the point of view a natural transformation of functors similar to the classical abstract characteristic classes for vector bundles.

1 Definitions and formulation of the problem

Given smooth manifold $M$ let

$$E \xrightarrow{a} TM \xrightarrow{p_T} M$$

be a vector bundle over $TM$ with fiber $g$, $p_E = p_T \cdot a$. So we have a commutative diagram of two vector bundles

![Diagram](image-url)

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The diagram is endowed with additional structure (commutator braces) and then is called ([1], definition 3.3.1, [2], definition 1.1.1) transitive Lie algebroid

\[ A = \left\{ \begin{array}{c}
E \xrightarrow{\alpha} TM \\
p_E \\
M \xrightarrow{p_T} \{\bullet, \bullet\}
\end{array} \right\}. \]

Let \( f : M' \to M \) be a smooth map. Then one can define an inverse image (pullback) of the Lie algebroid ([1], page 156, [2], definition 1.1.4), \( \mathcal{A}^f \). This means that given the finite dimensional Lie algebra \( g \) there is the functor \( \mathcal{A} \) such that with any manifold \( M \) it assigns the family \( \mathcal{A}(M) \) of all transitive Lie algebroids with fixed Lie algebra \( g \).

In the dissertation [3] the following statement was proved: Each transitive Lie algebroid is trivial, that is there is a trivialization of vector bundles \( E, TM, \ker \alpha = \bar{g} \) such that

\[ E \approx TM \oplus \bar{g}, \]

and the Lie bracket is defined by the formula:

\[ [(X, u), (Y, v)] = ([X, Y], [u, v] + X(v) - Y(u)). \]

Then using the construction of pullback and the idea by Allen Hatcher [4] one can prove that the functor \( \mathcal{A} \) is homotopic functor. More exactly for two homotopic smooth maps \( f_0, f_1 : M_1 \to M_2 \) and for the transitive Lie algebroid

\( (E \xrightarrow{\alpha} TM \xrightarrow{p_T} M, \{\bullet, \bullet\}) \)

two inverse images \( \mathcal{A}^f_0(E) \) and \( \mathcal{A}^f_1(A) \) are isomorphic.

Hence there is a final classifying space \( B_g \) such that the family of all transitive Lie algebroids with fixed Lie algebra \( g \) over the manifold \( M \) has one-to-one correspondence with the family of homotopy classes of continuous maps \([M, B_g] \):

\[ \mathcal{A}(M) \approx [M, B_g]. \]

Using this observation one can describe the family of all characteristic classes of a transitive Lie algebroids in terms of cohomologies of the classifying space \( B_g \). Really, from the point of view of category theory a characteristic class \( \alpha \) is a natural transformation from the functor \( \mathcal{A} \) to the cohomology functor \( H^* \).

This means that for the transitive Lie algebroid \( E = (E \xrightarrow{\alpha} TM \xrightarrow{p_T} M, \{\bullet, \bullet\}) \) the value of the characteristic class \( \alpha(E) \) is a cohomology class

\[ \alpha(E) \in H^*(M), \]

such that for smooth map \( f : M_1 \to M \) we have

\[ \alpha(f_*^!(E)) = f^*(\alpha(E)) \in H^*(M_1). \]

Hence the family of all characteristic classes \( \{\alpha\} \) for transitive Lie algebroids with fixed Lie algebra \( g \) has a one-to-one correspondence with the cohomology group \( H^*(B_g) \).
On the base of these abstract considerations a natural problem can be formulated.

**Problem.** Given finite dimensional Lie algebra \( g \) describe the classifying space \( B_g \) for transitive Lie algebroids in more or less understandable terms.

Below we suggest a way of solution the problem and consider some trivial examples.

## 2 Description of transitive Lie algebroids using transition functions

Consider the trivial transitive Lie algebroids

\[ E \cong TM \oplus \bar{g}, \quad \bar{g} \cong M \times g, \]

and the Lie bracket is defined by the formula:

\[
[(X, u), (Y, v)] = ([X, Y], [u, v] + X(v) - Y(u)),
\]

where \( X, Y \in \Gamma^\infty(TM) \) are smooth vector fields, \( u, v \in \Gamma^\infty\bar{g} \) are smooth sections which are represented as smooth vector functions with values in the Lie algebra \( g \).

Consider a fiberwise isomorphism \( A : E \rightarrow E \) that is identical on the second summands and generates the Lie algebra homomorphism \( A : \Gamma^\infty(E) \rightarrow \Gamma^\infty(E) \). The isomorphism \( A \) can be written by formula:

\[
(v, Y) = A(u, X);
(v, X) = (\varphi(x)(u(x)) + \omega(X), X),
\]

where \( \varphi(x) : g \rightarrow g \) is a fiberwise map of the bundle \( \bar{g} \), and \( \omega \) is a differential form with values in \( g \). The isomorphism \( A \) can be expressed as a matrix

\[
\begin{pmatrix}
  v(x) \\
  Y \\
\end{pmatrix} =
\begin{pmatrix}
  \varphi(x) & \omega \\
  0 & 1 \\
\end{pmatrix}
\begin{pmatrix}
  u(x) \\
  X \\
\end{pmatrix}
\]

From the property of that \( A \) is a Lie algebra homomorphism:

\[ A([(X, u), (Y, v)]) = [A(X, u), A(Y, v)] \]

one has that

\[ \varphi(x)([u_1(x), u_2(x)]) = [\varphi(x)(u_1(x)), \varphi(x)(u_2(x))], \]

\[ d\omega(X_1, X_2) + [\omega(X_1), \omega(X_2)] = 0, \]

\[ d\varphi(X)(u) = [\varphi(u), \omega(X)]. \quad (1) \]

Consider an atlas of charts on the manifold \( M \), \( \{U_\alpha\}, \bigcup U_\alpha = M \), and the trivializations \( E_\alpha \cong TU_\alpha \otimes (U_\alpha \times g) \) of the Lie algebroid \( E \) over each chart \( U_\alpha \) with the Lie brackets defined by the formula

\[
[(X, u), (Y, v)] = ([X, Y], [u, v] + X(v) - Y(u)),
\]
for \( X, Y \in \Gamma^\infty(TU_\alpha), u, v \in \Gamma^\infty(U_\alpha \times g) \).

On the intersection of two charts \( U_{\alpha\beta} = U_\alpha \cap U_\beta \) we have the transition function

\[
\Phi_{\beta\alpha} = \Phi_{\beta} \Phi^{-1}_{\alpha} : T U_{\alpha\beta} \otimes (U_{\alpha\beta} \times g) \rightarrow T U_{\alpha\beta} \otimes (U_{\alpha\beta} \times g)
\]

which have the matrix form

\[
\begin{pmatrix}
  v(x) \\ Y
\end{pmatrix} = \Phi_{\beta\alpha}
\begin{pmatrix}
  u(x) \\ X
\end{pmatrix} = \begin{pmatrix}
  \varphi_{\beta\alpha}(x) & \omega_{\beta\alpha} \\
  0 & 1
\end{pmatrix}
\begin{pmatrix}
  u(x) \\ X
\end{pmatrix}.
\]

For another choice of trivializations \( \Phi'_\alpha \) the correspondent transition functions \( \Phi'_{\beta\alpha} \) satisfy the homology condition:

\[
\Phi'_{\beta\alpha} = H_{\beta} \cdot \Phi_{\beta\alpha} \cdot H^{-1}_{\alpha}
\]

\[
\begin{pmatrix}
  \varphi'_{\beta\alpha}(x) & \omega'_{\beta\alpha} \\
  0 & 1
\end{pmatrix} = \begin{pmatrix}
  \eta_{\beta}(x) & \mu_{\beta} \\
  0 & 1
\end{pmatrix}
\begin{pmatrix}
  \varphi_{\beta\alpha}(x) & \omega_{\beta\alpha} \\
  0 & 1
\end{pmatrix}
\begin{pmatrix}
  \eta^{-1}_{\alpha}(x) & -\eta^{-1}_{\alpha}\mu_{\alpha} \\
  0 & 1
\end{pmatrix}
\]

or

\[
\begin{pmatrix}
  \varphi'_{\beta\alpha}(x) & \omega'_{\beta\alpha} \\
  0 & 1
\end{pmatrix} = \begin{pmatrix}
  \eta_{\beta}(x)\varphi_{\beta\alpha}(x)\eta^{-1}_{\alpha}(x) & -\eta_{\beta}(x)\varphi_{\beta\alpha}(x)\eta^{-1}_{\alpha}(x)\mu_{\alpha} + \eta_{\beta}(x)\omega_{\beta\alpha} + \mu_{\beta} \\
  0 & 1
\end{pmatrix},
\]

or

\[
\varphi'_{\beta\alpha}(x) = \eta_{\beta}(x)\varphi_{\beta\alpha}(x)\eta^{-1}_{\alpha}(x),
\]

\[
\omega'_{\beta\alpha} = -\eta_{\beta}(x)\varphi_{\beta\alpha}(x)\eta^{-1}_{\alpha}(x)\mu_{\alpha} + \eta_{\beta}(x)\omega_{\beta\alpha} + \mu_{\beta}.
\]

The elements \( \eta_{\beta} \) and \( \mu_{\beta} \) satisfy similar conditions:

\[
\eta_{\beta}(x)([u_1(x), u_2(x)]) = [\eta_{\beta}(x)(u_1(x)), \eta_{\beta}(x)(u_2(x))],
\]

\[
d\mu_{\beta}(X_1, X_2) + [\mu_{\beta}(X_1), \mu_{\beta}(X_2)] = 0,
\]

\[
d\eta_{\beta}(X)(u) = [\eta_{\beta}(u), \mu_{\beta}(X)].
\]

### 3 Case of commutative Lie algebra \( g \)

In commutative case the conditions have for simple form:

\[
\varphi_{\beta\alpha}(x)([u_1(x), u_2(x)]) = [\varphi_{\beta\alpha}(x)(u_1(x)), \varphi_{\beta\alpha}(x)(u_2(x))],
\]

\[
d\omega_{\beta\alpha}(X_1, X_2) = 0,
\]

\[
d\varphi_{\beta\alpha}(X)(u) = 0.
\]

(2)
Hence

$$\varphi_{\beta\alpha}(x) = \text{const}.$$ 

This means that the vector bundle $\bar{g}$ is flat and the family $\omega = \{\omega_{\beta\alpha}\}$ defines a Čech cochain

$$\omega \in C^1(U, \Omega^1(\bar{g}))$$

in the bigraded Čech complex

$$C^{*,*} = \left\{ \bigoplus C^i(U, \Omega^j(\bar{g}); d = d' + d'') \right\}$$

where $U = \{U_\alpha\}$ is the atlas of charts.

One has

$$d'(\omega) = 0; \quad d''(\omega) = 0.$$ 

Hence $\omega$ defines cohomology class

$$[\omega] \in H^2(M; \bar{g}).$$ 

Therefore we have the following

**Theorem 1** The classification of all transitive Lie algebroids with fixed commutative Lie algebra $g$ over the manifold $M$ is determined by a flat Lie algebra bundle $\bar{g}$ over $M$ and a 2-dimensional cohomology class $[\omega] \in H^2(M; \bar{g})$.

## 4 Some general properties

In common case we can say that a little bit about the transition functions on the level of homology groups $H_*(g)$ of the Lie algebra $g$. Since each transition function $\varphi_{\beta\alpha}(x)$ is the homomorphism of the Lie algebra $g$, that is $\varphi_{\beta\alpha}(x) \in \text{Aut}(g)$, the cocycle $\{\varphi_{\beta\alpha}(x)\}$ generate associated bundles with fibers $H_*(g)$, say, $H_*(g)$, and bundles with fibers $H^*(g)$, $\overline{H^*(g)}$. The properties (1) imply that all bundles $H_*(g)$ and $\overline{H^*(g)}$ are flat. In particular the differential forms $\omega_{\beta\alpha} \in \Omega^1(U_{\alpha\beta}; \bar{g})$ generates the cocycle

$$\overline{\varpi} = \{\overline{\omega}_{\beta\alpha}\} \in C^1(U, \overline{H_1(g)}) = \bigoplus_{\alpha\beta} \Omega^1(U_{\alpha\beta}; \overline{H_1(g)}),$$

that is

$$d'(\overline{\varpi}) = 0, \quad d''(\overline{\varpi}) = 0.$$ 

This means that the cocycle $\overline{\varpi}$ induces a cohomology class

$$[\overline{\varpi}] \in H^2\left(M; \overline{H_1(g)}\right).$$

The foregoing consideration creates a conjecture that classification of the transitive Lie algebroid $E$ induces by two things: the Lie algebra bundle with
structural group \( \tilde{\text{Aut}}(g) \) with special topology and the cohomology class \([\varpi] \in H^2(M; \mathbb{H}_1(g))\). The special topology in the group \( \text{Aut}(g) \) is defined as a minimal topology, which is more fine topology than the classical topology in \( \text{Aut}(g) \) and such that all homomorphisms

\[
\text{Aut}(g) \to \text{Aut}(H_k(g))_{\text{discrete}}
\]

are continuous.

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\section*{Список литературы}

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