Scaling limit of $\mathcal{N} = 6$ superconformal Chern-Simons theories and Lorentzian Bagger-Lambert theories

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Abstract

We show that the $\mathcal{N} = 8$ superconformal Bagger-Lambert theory based on the Lorentzian 3-algebra can be derived by taking a certain scaling limit of the recently proposed $\mathcal{N} = 6$ superconformal $U(N) \times U(N)$ Chern-Simons-matter theories at level $(k, -k)$. The scaling limit (and In"o{n}"u-Wigner contraction) is to scale the trace part of the bifundamental fields as $X_0 \rightarrow \lambda^{-1}X_0$ and an axial combination of the two gauge fields as $B_\mu \rightarrow \lambda B_\mu$. Simultaneously we scale the level as $k \rightarrow \lambda^{-1}k$ and then take $\lambda \rightarrow 0$ limit. Interestingly the same constraint equation $\partial^2 X_0 = 0$ is derived by imposing finiteness of the action. In this scaling limit, M2-branes are located far from the origin of $\mathbb{C}^4/\mathbb{Z}_k$ compared to their fluctuations and $\mathbb{Z}_k$ identification becomes a circle identification. Hence the scaled theory describes $\mathcal{N} = 8$ supersymmetric theory of 2-branes with dynamical coupling. The coupling constant is promoted to a space-time dependent $SO(8)$ vector $X_0^I$ and we show that the scaled theory has a generalized conformal symmetry as well as manifest $SO(8)$ with the transformation of the background fields $X_0^I$.

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1 Introduction

Recently there has been a lot of activity in constructing superconformal Chern-Simons-matter gauge theories for multiple M2-branes. Bagger and Lambert, and Gustavsson discovered the $\mathcal{N} = 8$ superconformal $(2+1)$-dimensional field theories with $SO(8)$ global symmetries by exploiting the 3-algebra structures [1, 2].

The 3-algebra, or the quantization of the Nambu bracket, is a natural generalization of the ordinary Lie algebras and has been widely believed that it will play an important role in the quantization of membranes [4–9]. Furthermore the 3-algebraic structure is also expected to play an important role to construct M5 branes from M2-branes through the Basu-Harvey equation [3].

The quantization of the Nambu bracket is hard, if we impose the so-called Fundamental Identity (FI) on the 3-algebra and the antisymmetry of the structure constants, and there are only a few examples. The simplest one is the $\mathcal{A}_4$ algebra with four generators. The Bagger-Lambert theory based on the $\mathcal{A}_4$ algebra becomes an $\mathcal{N} = 8$ superconformal $SU(2) \times SU(2)$ Chern-Simons gauge theory [10]. By giving a vacuum expectation value (VEV) to the matter field, the theory is shown to describe low energy effective theory of D2-branes [11].

Another remarkable example is the 3-algebra with a Lorentzian metric [12–14]. It satisfies the FI and $\mathcal{N} = 8$ superconformal field theories based on this Lorentzian 3-algebra are constructed. Because of the Lorentzian signature, the model contains ghost fields and they may break the unitarity of the theory. The fields with a negative signature, however, are Lagrange multipliers and can be integrated out to give constraints on the “conjugate” fields. After the integration, the theory looks well-defined. (There are also attempts to kill the ghost fields by gauging shift symmetry [15, 16].) If we take a special solution to the constraint equation, the Lorentzian Bagger-Lambert theories are reduced to the action of the $N$ D2-branes in flat space. There are various solutions to the constraint equations and each of them give different looking Janus field theories [17–21] of D2-branes whose coupling is varying with the space-time [22].

There are various generalizations of the BL theories, including massive deformations [23–25] and models based on other 3-algebras [26]. These theories are also obtained from gauged supergravities in three dimensions by using the embedding tensor method [27]. For other related works, see [28–52].

Another very interesting proposal for multiple M2-branes actions was recently given by Aharony, Bergman, Jafferis and Maldacena (ABJM) [53]. They generalized the superconformal Chern-Simons matter theories [54, 55] to the $\mathcal{N} = 6$ superconformal $U(N) \times U(N)$ theories. The level of the Chern-Simons gauge theories is $(k, -k)$ and the theory is conjectured to describe the low energy limit of $N$ M2-branes probing a $\mathbb{C}^4/\mathbb{Z}_k$. Hence at large
$N$, it is dual to the M-theory on $AdS_4 \times S^7/\mathbb{Z}_k$. In the case of $SU(2) \times SU(2)$ gauge group, the theory is the same as the Bagger-Lambert theory based on the $A_4$ 3-algebra [10]. In this formulation by Aharony et.al. the 3-algebra structure does not seem to play any role and, for the general gauge groups, the relation to the BL theory is not clear. However, since the theory with a gauge group $U(N) \times U(N)$ is conjectured to describe $N$ M2-branes probing $\mathbb{C}^4/\mathbb{Z}_k$ and giving a VEV to the bifundamental field reduces the theory to a system of $N$ D2-branes, it must be related, by taking a certain scaling limit, to the Lorentzian Bagger-Lambert theory with gauge group $U(N)$ which can be also reduced to the system of $N$ D2-branes with a dynamical coupling.

There are various studies of the ABJM theory including an orbifolding [56], calculation of the index [57], and PP wave limit [58].

In this letter, we show that the Lorentzian Bagger-Lambert theory can be obtained by taking an appropriate scaling limit of the ABJM theory. We first scale the trace of the bifundamental fields $X_0$ (bosons and fermions) and an axial combination of the gauge fields $B_\mu = (A^{(L)}_\mu - A^{(R)}_\mu)/2$ as

$$X_0 \to \lambda^{-1}X_0, \quad B_\mu \to \lambda B_\mu.$$  \hfill (1.1)

The other fields are kept fixed. Simultaneously we scale $k \to \lambda^{-1}k$ and then take $\lambda \to 0$ limit (Inönü-Wigner contraction [60][1]). Many terms vanish in this limit, and the remaining terms give the action of the Lorentzian Bagger-Lambert theory whose Lagrange multiplier fields are integrated out. Interestingly divergent terms in $\lambda \to 0$ limit give constraints on the trace component of the bifundamental fields $X_0$. The constraints precisely agree with the constraints given by the Lagrange multiplier fields in the Lorentzian Bagger-Lambert theory.

The M2-branes described by the ABJM theory are expected to have the conformal and $SO(8)$ symmetries (for $k = 1, 2$). The scaling limit we consider in our paper corresponds to a limit of locating M2-branes far from the origin of the $\mathbb{Z}_k$ orbifold as well as taking $k \to \infty$. Hence from the M2-brane point of view the scaled theory must have (a kind of) conformal symmetry and (hidden) $SO(8)$ symmetry that the ordinary D2-branes are not expected to have. Since the coupling constant of the scaled theory of D2-branes is promoted to a $SO(8)$ vector $X_0^I(x)$, we show that the scaled theory has enhanced symmetries, i.e. generalized conformal symmetry and $SO(8)$ invariance, if we allow $X_0^I(x)$ to transform appropriately under these transformations. This generalized conformal symmetry is essentially the same as that proposed by Jevicki, Kazama and Yoneya [59] 10 years ago for general Dp-branes.

The paper is organized as follows. In section 2, we compare the gauge structures of the Lorentzian Bagger-Lambert theory and the ABJM theory for M2-branes. The gauge structure of the Lorentzian BL theory is obtained by taking an Inönü-Wigner contraction.

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1This scaling limit was also discussed to get the nonsemisimple algebra in [61].
of the ABJM theory. In section 3, we look at the scaling limit of the ABJM theory
and show that the scaled action gives the Lorentzian Bagger-Lambert theory. We also see
that the scaled theory of D2-branes with a dynamical coupling has a generalized conformal
symmetry. We end in section 4 with conclusions and discussions.

2 Gauge structures and Inönü-Wigner contraction
We first look at the gauge structures of the Lorentzian Bagger-Lambert theory [12–14].
The Bagger-Lambert theory [1,2] has a gauge symmetry generated by $\tilde{T}^{ab} X = [T^a, T^b, X]$.
Because of the fundamental identity
\[ [T^a, T^b, [T^c, T^d, X]] = [T^a, T^b, T^c, T^d] + [T^c, [T^a, T^b, T^d], X] + [T^c, T^d, [T^a, T^b, X]], \]
(2.1)
the following commutation relation holds:
\[ [\tilde{T}^{ab}, \tilde{T}^{cd}] X = [T^a, T^b, [T^c, T^d, X]] - [T^c, T^d, [T^a, T^b, X]] \]
\[ = [T^a, [T^b, [T^c, T^d, X]] + [T^c, [T^a, T^b, T^d], X] + [T^c, T^d, [T^a, T^b, X]], \]
(2.2)

The Lorentzian 3-algebra contains 2 extra generators $T^{-1}$ and $T^0$ in addition to the
generators of Lie algebra $T^i$. (Here we use the convention of [14].) The 3-algebra for them
is given by
\[ [T^{-1}, T^a, T^b] = 0, \]
(2.3)
\[ [T^0, T^i, T^j] = f^{ij} T^k, \]
(2.4)
\[ [T^i, T^j, T^k] = f^{ijk} T^{-1}, \]
(2.5)
where $a, b = \{-1, 0, i\}$. $T^i$ are generators of the ordinary Lie algebra with the structure
constant: $[T^i, T^j] = if^{ij} T^k$. This 3-algebra satisfies the fundamental identity. The metric
$h^{ab} = \text{tr} (T^a, T^b)$ is given by
\[ \text{tr} (T^{-1}, T^{-1}) = \text{tr} (T^{-1}, T^i) = 0, \quad \text{tr} (T^{-1}, T^0) = -1, \]
\[ \text{tr} (T^0, T^i) = 0, \quad \text{tr} (T^0, T^0) = 0, \quad \text{tr} (T^i, T^j) = h^{ij}. \]
(2.6)
Since the metric has a negative eigenvalue, the field associated with the generators $T^{-1}$
and $T^0$ become ghost modes.

The gauge generators of the Lorentzian 3-algebra can be classified into 3 classes:

\text{If we write the commutation relation as } [\tilde{T}^{ab}, \tilde{T}^{cd}] = f^{abc} \tilde{T}^{cd} + f^{ab} \tilde{T}^{ce}, \text{ it is not always associative. But when } \tilde{T}^{ab} \text{ acts on a field } X, \text{ associativity-violating terms (3-cocycles) vanish and it becomes an ordinary associative Lie algebra.}
\[ I = \{ T^{-1} \otimes T^a, a = 0, i \} \]
\[ A = \{ T^0 \otimes T^i \} \]
\[ B = \{ T^i \otimes T^j \}. \]

The generators in the class \( I \) vanish when they act on \( X \), hence we set these generators to zero in the following. Since the generators in the class \( B \) always appear as a combination with the structure constant, we define generators \( S^i \equiv f^{ij}_{\ k} T^j_0 \). Then they satisfy the algebra

\[ [\tilde{T}^{0i}, \tilde{T}^{0j}] = i f^{0ij}_{\ k} \tilde{T}^{0k}, \quad [\tilde{T}^{0i}, S^j] = i f^{ij}_{\ k} S^k, \quad [S^i, S^j] = 0. \tag{2.7} \]

The last commutator was originally proportional to the generators in the class \( I \). If we had kept these generators, the algebra would have become nonassociative. The algebra (2.7) is a semi direct sum of \( SU(N) \) (or \( U(N) \)) and translations. In the case of \( SU(2) \), it becomes the \( ISO(3) \) gauge group, which is the gauge group of the 3-dimensional gravity. The Lorentzian Bagger-Lambert theories have the above gauge symmetries and corresponding gauge fields \( \hat{A}_\mu \) and \( B_\mu \) as we will see in the next section.

On the other hand, the theory proposed by Aharony et.al. [53] is a Chern-Simons (CS) gauge theory with the gauge group \( U(N) \times U(N) \). They act on the bifundamental fields (e.g. \( X^I \)) from the left and the right as \( X \rightarrow UXV^\dagger \). If we write the generators as \( T^i_L \) and \( T^i_R \), the combination \( T^i = T^i_L + T^i_R \) and \( S^i = T^i_L - T^i_R \) satisfy the algebra

\[ [T^i, T^j] = i f^{ij}_{\ k} T^k, \quad [T^i, S^j] = i f^{ij}_{\ k} S^k, \quad [S^i, S^j] = i f^{ij}_{\ k} T^k. \tag{2.8} \]

By taking the Inönü-Wigner contraction, i.e. scaling the generators as \( S^i \rightarrow \lambda^{-1} S^i \) and taking \( \lambda \rightarrow 0 \) limit, the algebra (2.8) becomes the algebra (2.7) of the Lorentzian BL theory. Therefore it is tempting to think that the Lorentzian BL theory can be obtained by taking an appropriate scaling limit of the ABJM theory. In the next section, we see that it is indeed the case. Interestingly, even the constraint equations in the BL theory (obtained by integrating the Lagrange multiplier fields) can be derived from this scaling procedure.

3 Derivation of Lorentzian BL from ABJM

3.1 Lorentzian BL theory

We first give a quick summary of the Lorentzian BL theory. Bagger-Lambert theory is a \((2+1)\)-dimensional non-Abelian gauge theory with \( N = 8 \) supersymmetries. It contains 8 real scalar fields \( X^I = \sum_a X^I_a T^a \), \( I = 3, \ldots, 10 \), gauge fields \( A^\mu = \sum_{ab} A^\mu_{ab} T^a \otimes T^b, \mu = 0, 1, 2 \) with two internal indices and 11-dimensional Majorana spinor fields \( \Psi = \sum_a \Psi_a T^a \).
with a chirality condition $\Gamma_{\alpha \beta} = \Psi$. The action proposed by Bagger and Lambert is given by
\[
\mathcal{L} = -\frac{1}{2} \text{tr} (D^\mu X^I, D_\mu X^I) + \frac{i}{2} \text{tr} (\bar{\Psi}, \Gamma^\mu D_\mu \Psi) + \frac{i}{4} \text{tr} (\bar{\Psi}, \Gamma_{I J} [X^I, X^J, \Psi]) - V(X) + \mathcal{L}_{CS},
\]
(3.1)
where $D_\mu$ is the covariant derivative defined by
\[
(D_\mu X^I)_a = \partial_\mu X^I_a - f_{c d b}^a A_{\mu c d}(x) X^I_b.
\]
(3.2)
$V(X)$ is a the sextic potential term
\[
V(X) = \frac{1}{12} \text{tr} (\{X^I, X^J, X^K\}, \{X^I, X^J, X^K\}),
\]
(3.3)
and the Chern-Simons term for the gauge potential is given by
\[
\mathcal{L}_{CS} = \frac{1}{2} e^{\mu \nu \lambda} (f_{a b c d} a_{\mu b} A_{\nu c d} + \frac{2}{3} f_{c d a} g e^{f g b} A_{\mu a b} A_{\mu c d} A_{\lambda e f}).
\]
(3.4)

In the specific realization of the 3-algebra generated by $(T^{-1}, T^0, T^i)$, we can decompose the modes of the fields as
\[
X^I = X^I_0 T^0 + X^I_{-1} T^{-1} + X^I_i T^i,
\]
\[
\Psi = \Psi_0 T^0 + \Psi_{-1} T^{-1} + \Psi_i T^i.
\]
\[
A_\mu = T^{-1} \otimes A_{\mu(-1)} - A_{\mu(-1)} \otimes T^{-1} + A_{\mu 0} T^0 \otimes T^i - A_{\mu j 0} T^j \otimes T^0 + A_{\mu i j} T^i \otimes T^j.
\]
(3.7)

It will be convenient to define the following fields as in [14]
\[
\hat{X}^I = X^I_0 T^0, \quad \hat{\Psi} = \Psi_i T^i, \quad \hat{A}_\mu = 2 A_{\mu 0} T^0, \quad B_\mu = f^{i j k} A_{\mu i j} T^k.
\]
(3.8)
The gauge field $A_{\mu(-1)}$ is decoupled from the action and we drop it in the following discussions. The gauge field $\hat{A}_\mu$ is associated with the gauge transformation of the subalgebra $A$. Another gauge field $B_\mu$ will play a role of the $B$-field of the BF theory. With these expression the Bagger-Lambert action (3.1) can be rewritten as $\mathcal{L}_{BL} = \mathcal{L}_0 + \mathcal{L}_{gh}$ where
\[
\mathcal{L}_0 = \text{tr} \left[ -\frac{1}{2} (\hat{D}_\mu \hat{X}^I - B_\mu X^I_0)^2 + \frac{1}{4} (X^K_0)^2 (\hat{X}^I, \hat{X}^J)^2 - \frac{1}{2} (X^I_0 \hat{X}^I, \hat{X}^J)^2 \\
+ \frac{i}{2} \bar{\hat{\Psi}} \Gamma^\mu \hat{D}_\mu \hat{\Psi} + i \bar{\Psi}_0 \Gamma^\mu B_\mu \hat{\Psi} - \frac{1}{2} \bar{\Psi}_0 \hat{X}^I [\hat{X}^J, \Gamma_{I J} \hat{\Psi}] + \frac{1}{2} \bar{\Psi} X^I_0 [\hat{X}^J, \Gamma_{I J} \hat{\Psi}] \\
+ \frac{1}{2} e^{\mu \nu \lambda} \hat{F}_{\mu \nu} B_\lambda - \partial_\mu X^I_0 B_\mu \hat{X}^I \right],
\]
(3.9)
and
\[ \mathcal{L}_{gh} = (\partial_\mu X_I^I)(\partial^\mu X_I^I) - i\bar{\Psi}\gamma^\mu \partial_\mu \Psi. \] (3.10)

The fields \( X_{-1} \) and \( \Psi_{-1} \) are contained in \( \mathcal{L}_{gh} \) only.

The covariant derivative and the field strength
\[ \hat{D}_\mu X^I \equiv \partial_\mu \hat{X}^I + i[\hat{A}_\mu, \hat{X}^I], \quad \hat{D}_\mu \Psi \equiv \partial_\mu \hat{\Psi} + i[\hat{A}_\mu, \hat{\Psi}], \quad \hat{F}_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu + i[\hat{A}_\mu, \hat{A}_\nu] \] (3.11)
are the ordinary covariant derivative and field strength for the subalgebra \( \mathcal{A} \).

Here note that \( X_{-1}^I \) and \( \Psi_{-1} \) appear only linearly in the Lagrangian and thus they are Lagrange multipliers. By integrating out these fields, we have the following constraints:
\[ \partial^2 X_0^I = 0, \quad \Gamma^n \partial_\mu \Psi_0 = 0. \] (3.12)

Then the Lorentzian BL theory Lagrangian is described by the Lagrangian \( \mathcal{L}_0 \). The constraint equations (3.12) and the Lagrangian \( \mathcal{L}_0 \) are what we want to obtain from the ABJM theory by taking a scaling limit.

### 3.2 Classical Conformal symmetry of D2-branes with dynamical coupling

Before discussing the ABJM theory, we investigate the symmetry properties of the Lorentzian BL theory. As was shown in [12–14], the theory can be reduced to a system of D2-branes by integrating \( B_\mu \) fields. This is interpreted as giving a VEV to \( X_0^I \) field following [11], and a special solution \( X_0^I = \text{const.} \) to the constraint equation \( \partial^2 X_0^I = 0 \) was considered. In our previous paper [22], we revisited the constraint equation and considered a general solution with space-time dependent \( X_0^I(x) \) satisfying \( \partial^2 X_0^I = 0 \). Our interpretation is slightly different from the original one, and the field \( X_0^I \) is treated as a dynamical (but nonpropagating) field. In this subsection we show that if we consider the whole set of the solutions to the constraint equation the reduced action has a classical conformal symmetry as well as \( SO(8) \) symmetry.

For simplicity, we neglect the fermionic field here. By integrating the \( B_\mu \) gauge field the action becomes [22]
\[ S_0 = \int d^3x \ \text{tr} \left[ -\frac{1}{2}(\hat{D}_\mu Y^I)^2 + \frac{1}{4}X_0^2[Y^I, Y^J]^2 - \frac{1}{4(X_0)^2}(\hat{F}_{\mu\nu} + 2\epsilon_{\mu\nu\rho}Y_I \partial^\rho X_0^I)^2 \right], \] (3.13)
where \( X_0^2 \equiv \sum_I X_0^I X_0^I \) and we have defined a new scalar field \( Y^I = P_{IJ} \hat{X}^J \) with 7 degrees of freedom by using the projection operator
\[ P_{IJ}(x) = \delta_{IJ} - \frac{X_{0J}X_{0I}}{X_0^2}. \] (3.14)
Indices run $I, J = 0, \cdots, 8$ and $Y^I$ transforms as a vector of $SO(8)$. The field $X_0^I(x)$ is constrained to satisfy $\partial^2 X_0^I = 0$. If we pick up a specific solution $X_0^I = \nu \delta_{I0}$, the action is reduced to the familiar D2-brane effective action with a coupling constant given by $\nu$. Then $SO(8)$ symmetry is spontaneously broken to $SO(7)$. The conformal invariance is also broken. However if we consider whole set of solutions, $SO(8)$ invariance is restored in the action (3.13) with the background fields $X_0^I(x)$ although $Y^I$ has only 7 degrees of freedom.

Another important symmetry of the action is a conformal symmetry. The ordinary D2-brane action with a fixed coupling constant is not conformally invariant and the near horizon limit is not described by the AdS geometry. However, as discussed in a paper by Jevicki, Kazama and Yoneya [59], Dp brane theory has a generalized conformal symmetry if the coupling $g(x)$ is not constant and varies with space-time. Our reduced action for D2-branes (3.13) has exactly the same property. The coupling constant is no longer a constant and varies with space-time. A big difference, however, is that in our case the coupling constant $g$ is promoted to an $SO(8)$ vector $X^I_0$, which is a space-time dependent field satisfying the massless Klein-Gordon equation.

Under the dilation $x \rightarrow \exp(\epsilon)x$, each field transforms as $Y(x) \rightarrow Y'(x') = \exp(-\epsilon/2)Y(x)$, $X_0(x) \rightarrow X'_0(x') = \exp(-\epsilon/2)X_0(x)$ and $A_\mu(x) \rightarrow A'_\mu(x') = \exp(-\epsilon)A_\mu(x)$. It is easy to see that the action is invariant under the dilation. Special conformal transformations are more complicated. It is given by

$$\delta x^\mu = 2\epsilon \cdot xx^\mu - \epsilon^\mu x^2. \quad (3.15)$$

Writing an infinitesimal transformation for each field as $\delta Y = Y' - Y$, we define a special conformal transformation for each field as

$$\delta Y^I(x) = -\epsilon \cdot x Y^I(x) \quad (3.16)$$

$$\delta X_0^I(x) = -\epsilon \cdot x X_0^I(x) \quad (3.17)$$

$$\delta A_\mu(x) = -2\epsilon \cdot x A_\mu(x) - 2(x \cdot A \cdot \epsilon_\mu - \epsilon \cdot x A_\mu). \quad (3.18)$$

It is straightforward to show that the action is invariant under the special conformal transformation. It can be easily checked that the transformations preserve the condition $X_0 \cdot Y = 0$.

Finally we need to check that the transformation is closed within the constraint equation $\partial^2 X_0^I = 0$. From the transformation of $X_0^I$, we define the following transformation

\[\delta x^\mu = 2\epsilon \cdot xx^\mu - \epsilon^\mu x^2.\]

In the paper [16], it is discussed that the conformal invariance can be restored by sending the Yang-Mills coupling to infinity or integrating it over all values.

If $X_0^I$ is replaced by a single field $g(x)$, the transformation is the same as the generalized conformal transformation in [59]. Our scalar field $Y(x)$ corresponds to their $X(x)/g(x)$.

The kinetic term of the gauge fields is different from the ordinary one, but both of the ordinary type and ours are invariant under the same conformal transformations.
at the numerically same point as

\[ \delta X_0(x) = X'_0(x) - X_0(x) = \delta X_0(x) - \delta x^\mu \partial_\mu X_0(x). \] (3.19)

It is easy to see that if the original \( X_0^I(x) \) satisfies the constraint equation \( \partial^2 X_0(x) = 0 \), then the infinitesimal variation satisfies \( \partial^2 (\delta X_0) = 0 \) for both of the dilation and the special conformal transformations, which means that the transformed field also satisfies \( \partial^2 X_0'(x') = 0 \). Hence the classical conformal transformation is closed within the configurations of \( X_0 \) satisfying the constraint equation \( \partial^2 \). If we restrict the configurations of \( X_i^I \) that satisfy \( \partial X_i^I = 0 \), namely, to a set of constant vectors, the above special conformal transformations cannot be defined within the set. This indicates that taking into account the whole set of the constraint equation \( \partial^2 X_0^I = 0 \) as adopted in [22] is important in recovering the \( SO(8) \) superconformal symmetry. It is also interesting to note that \( \partial^2 (\delta X_0) = 0 \) holds only when \( p = 2 \). (Generalized conformal transformations for general \( p \) are given in [59].)

As we see later, the D2-brane action with a space-time dependent coupling is also derived from the M2-brane theory given by Aharony et al. by taking a certain scaling limit. This scaling limit corresponds to locating the M2-branes far from the origin of the orbifold and then taking the \( k \rightarrow \infty \) limit. It is natural from this brane picture that the model we considered in this subsection has a classical conformal symmetry as well as \( SO(8) \) symmetry.

More detailed studies of the conformal symmetries and the interpretation in the gravity side are discussed in a separate paper [64]. What we have suggested here is that if we allow the background fields \( X_0^I \) to transform under \( SO(8) \) and a special conformal transformation as an \( SO(8) \) vector and as in (3.17), the action (3.13) is invariant under them. Note also that the analysis here is just about the classical conformal invariance. It is interesting to see whether the conformal invariance can be preserved quantum mechanically.

\[ \text{In order to construct a set of solutions in which the conformal transformations are closed, it seems to be necessary to consider all the solutions to the constraint equation } \partial^2 X_0 = 0. \] Instead we can consider the following set of solutions studied by Verlinde [63]

\[ X_0^I(x) = \sum_i \frac{q_i^I}{|x - z_i|} \]

which satisfies the constraint equation with sources at \( x = z_i \):

\[ \partial^2 X_0^I = -4\pi \sum q_i^I \delta^3(x - z_i). \] These solutions are closed under the conformal transformation if we consider all set of \( q_i^I \) and \( z_i \). See [64] for details.
3.3 ABJM theory

The action of the ABJM theory is given by (we use the convention used in [56])

$$S = \int d^3 x \left[ -(D_\mu Z_A)^\dagger D^\mu Z^A - (D_\mu W^A)^\dagger D^\mu W_A + i \zeta_A^\dagger \Gamma^\mu D_\mu \zeta^A + i \omega^{A^\dagger} \Gamma^\mu D_\mu \omega_A \right]$$

$$+ S_{CS} - S_{V_f} - S_{V_b},$$

(3.20)

with $A = 1, 2$. This is an $\mathcal{N} = 6$ superconformal $U(N) \times U(N)$ Chern-Simons theory. $Z$ is a bifundamental field under the gauge group and its covariant derivative is defined by

$$D_\mu X = \partial_\mu X + i A^{(L)}_\mu X - i X A^{(R)}_\mu.$$  

(3.21)

The gauge transformations $U(N) \times U(N)$ act from the left and the right on this field as $Z \rightarrow UZV^\dagger$.

The level of the Chern-Simons gauge theories is $(k, -k)$ and the coefficients of the Chern-Simons terms for the two $U(N)$ gauge groups, $A^{(L)}_\mu$ and $A^{(R)}_\mu$, are opposite. Hence the action $S_{CS}$ is given by

$$S_{CS} = \int d^3 x \ 2K e^{\mu \lambda} \ tr \left[ A^{(L)}_\mu \partial_\nu A^{(L)}_\lambda + 2i \frac{1}{3} A^{(L)}_\mu A^{(L)}_\nu A^{(L)}_\lambda - A^{(R)}_\mu \partial_\nu A^{(R)}_\lambda - 2i \frac{1}{3} A^{(R)}_\mu A^{(R)}_\nu A^{(R)}_\lambda \right].$$

(3.22)

The potential term for bosons is given by

$$S_{V_b} = - \frac{1}{48K^2} \int d^3 x \ tr \left[ Y^A Y_A^\dagger Y_B^\dagger Y_C^\dagger + Y_A^\dagger Y_B^\dagger Y_C^\dagger Y^C + 4Y_A^\dagger Y_B^\dagger Y_C^\dagger Y^C \right].$$

(3.23)

and for fermions by

$$S_{V_f} = \frac{i}{4K} \int d^3 x \ tr \left[ Y_A^\dagger Y_B^\dagger Y_C^\dagger Y_D^\dagger - Y_A^\dagger Y_B^\dagger Y_C^\dagger Y_D^\dagger \right].$$

(3.24)

$Y^A$ and $\psi_A$ ($A = 1 \cdots 4$) are defined by

$$Y^C = \{ Z^A, W^{A\dagger} \}, \quad \psi_C = \{ \epsilon_{AB} e^{-i\pi/4}, \epsilon_{AB} \omega^{A\dagger} e^{-i\pi/4} \},$$

(3.25)

where the index $C$ runs from 1 to 4. The $SU(4)$ R-symmetry of the potential terms is manifest in terms of $Y^A$ and $\psi_A$.

The ABJM theory is similar to the Lorentzian BL theory, but different in the following points. First the gauge group is $U(N) \times U(N)$ while it is a semi direct product of $U(N)$ and translations in the BL theory. Accordingly the matter fields are in the bifundamental representation in the ABJM theory. Furthermore the BL theory contains an extra field
\(X_0\) and \(\Psi_0\) associated with the generator \(T_0\), and they are required to obey the constraint equations (3.12).

The bosonic potential terms in both theories are sextic, but the potential in the BL theories contains two \(X^I_0\) fields and four adjoint matter fields \(\hat{X}^I\) while the potential terms in the ABJM theory are written in the product of six bifundamental matter fields \(Y\). Hence it is natural to think that the trace part of \(Y\) will play a role of \(X^0\) in the Lorentzian BL theory. We will see that, if we separate the matter field \(Y\) into a trace and a traceless part, the potential terms coincide in a certain scaling limit.

### 3.4 Scaling limit of ABJM theory

In order to take a scaling limit, we first recombine the gauge fields as

\[
\hat{A}_\mu = \frac{A^{(L)}_\mu + A^{(R)}_\mu}{2}, \quad B_\mu = \frac{A^{(L)}_\mu - A^{(R)}_\mu}{2},
\]

then the gauge transformations corresponding to \(\hat{A}_\mu\) and \(B_\mu\) are \(Z \to e^{i\sigma_a T^a} Z e^{-i\sigma_b T^b}\) and \(Z \to e^{i\sigma_a T^a} Z e^{i\sigma_b T^b}\) respectively. They are vectorial and axial gauge transformations. Matter fields are in the adjoint representation for the \(\hat{A}_\mu\) gauge fields. Hence the \(U(1)\) part of \(\hat{A}_\mu\) decouples from the matter sector.

The covariant derivative can be written in terms of \(\hat{A}_\mu\) and \(B_\mu\) as

\[
D_\mu Z = \partial_\mu Z + i[\hat{A}_\mu, Z] + i\{B_\mu, Z\} = \hat{D}_\mu Z + i\{B_\mu, Z\},
\]

where \(\hat{D}_\mu\) is the covariant derivative with respect to the gauge field \(\hat{A}_\mu\). \(S_{CS}\) can be written in terms of \(\hat{A}_\mu\) and \(B_\mu\) as

\[
S_{CS} = \int d^3x \ 4K \epsilon^{\mu\nu\rho} \text{tr} [B_\mu \hat{F}_{\mu\nu} + \frac{2}{3} B_\mu B_\nu B_\rho],
\]

where \(\hat{F}_{\mu\nu}\) is field strength of \(\hat{A}_\mu\).

The gauge fields \(\hat{A}_\mu\), \(B_\mu\) are associated with the gauge transformations generated by \(T^i\) and \(S^i\) in (2.8). Hence in order to take the Inönü-Wigner contraction to obtain the gauge structure of the Lorentzian BL theory (2.7), we need to rescale the gauge field \(B_\mu\) as \(B^\mu \to \lambda B^\mu\) and take the \(\lambda \to 0\) limit. Simultaneously we need to scale the coefficient \(K\) by \(\lambda^{-1}K\). Since the coefficient \(K\) is proportional to the level of the Chern-Simons theory \(k\) as \(K = k/8\pi\), the scaling limit corresponds to taking the large \(k\) limit. In this scaling limit, the cubic term of the \(B_\mu\) fields vanishes and the Chern-Simons action coincides with the BF-type action in the Lorentzian BL theory:

\[
S_{CS} \to \int d^3x \ 4K \epsilon^{\mu\nu\rho} \text{tr} B_\mu \hat{F}_{\mu\nu}. \tag{3.29}
\]
In order to match the covariant derivatives in the Lorentzian BL action (3.9) and in the ABJM theory (3.27), we separate the bifundamental fields into the trace and the traceless part, and scale them differently. We write the matter fields \( Y^A \) as
\[
Y^A_{ij} = Y^A_0 \delta_{ij} + \tilde{Y}^A_a T^a_{ij},
\]
where \( T^a \) is the generator of \( SU(N) \).

Now we perform the following rescaling:
\[
B_\mu \rightarrow \lambda B_\mu,
Y^A_0 \rightarrow \lambda^{-1} Y^A_0,
\psi^A_0 \rightarrow \lambda^{-1} \psi^A_0,
K \rightarrow \lambda^{-1} K,
\]
where \( Y^A_0 \) and \( \psi^A_0 \) is the trace part of \( Y^A \) and \( \psi^A \). All the other fields are kept fixed. Then take the \( \lambda \rightarrow 0 \) limit. If we take the scaling limit, we can show that the covariant derivatives in both theories exactly match.

In the following we consider the ABJM theory with the \( SU(N) \times SU(N) \) gauge group. In the presence of the \( U(1) \times U(1) \) group, a little more care should be taken for the scaling of the \( U(1) \) part of the \( B_\mu \) gauge field.

In taking the above scaling limit, many terms vanish. The kinetic term of the ABJM action becomes
\[
\text{tr} \left[ -\frac{1}{\lambda^2} \partial_\mu Y^A_{0A} \partial^\mu Y^A_0 + \frac{1}{\lambda^2} \psi^A_0 \Gamma^\mu \partial_\mu \psi^A_0 + 2(i \partial_\mu Y^A_{0A} B^\mu Y^A + \text{h.c.}) - (\hat{D}_\mu \tilde{Y}^A + 2i \tilde{B}_\mu Y^A_0) (\hat{D}^\mu \tilde{Y}^A + 2i \tilde{B}^\mu Y^A_0) + i \tilde{\psi}^A_0 \Gamma^\mu \hat{D}_\mu \tilde{\psi}^A - 2 \tilde{\psi}^A_0 \Gamma^\mu \tilde{B}_\mu \psi^A_0 - 2 \psi^A_0 \Gamma^\mu \tilde{B}_\mu \tilde{\psi}^A \right].
\]

The first and the second terms are divergent for small \( \lambda \). In order to make the action finite, we need to impose that the trace part of the bifundamental fields must satisfy the constraint equations
\[
\partial^2 Y^I_0 = 0, \quad \Gamma^\mu \partial_\mu \psi^A_0 = 0
\]
in the \( \lambda \rightarrow 0 \) limit. They are precisely the same constraint equations (3.12) in the BL theory.

In the Lorentzian BL theory, the constraints are obtained by integrating out the Lagrange multiplier fields \( X_{-1} \) and \( \Psi_{-1} \). Here they arise from a condition that the action should be finite in the scaling limit.

The other terms in (3.32) are finite in the scaling limit and it can be easily shown that they are precisely the same kinetic terms as that of the Lorentzian Bagger-Lambert theory (after a redefinition of the gauge field \( 2B_\mu \rightarrow B_\mu \) and setting \( K = 1/2 \)).
trace part of the bifundamental fields is identified with the fields $X_0$ associated with one of the extra generators $T^0$ in the Lorentzian Bagger-Lambert theory. This is the reason why we have used the same convention with subscript 0 for both of the trace part of the bifundamental fields and the field associated with the generator $T^0$.

Now let us check the potential terms. The potential terms of the ABJM theory are invariant under the $SU(4)$ symmetries but not under full $SO(8)$. By decomposing the matter fields $Y^A$ into the trace part $Y_0^A$ and the traceless part $\tilde{Y}^A$, the bosonic sextic potential becomes a sum of $V_B = \sum_{n=0}^{6} V_B^{(n)}$, where $V_B^{(n)}$ contains $n$ $Y_0$ fields and $(6 - n)$ $\tilde{Y}$ fields. Since the coefficient of the bosonic potential is proportional to $K^{-2}$, $V_B^{(n)}$ term scales as $\lambda^{2-n}$. It can be easily checked that the coefficients of $V_B^{(n)}$ vanishes for $n > 3$. On the other hand, the potential terms $V_B^{(n)}$ for $n < 2$ vanish in the scaling limit of $\lambda \to 0$. Hence the only remaining term in the scaling limit is $V_B^{(2)}$. This part of the potential has the full $SO(8)$ symmetry and becomes identical with the potential in the Lorentzian BL theory. In order to see that the BL potential is obtained, we assume that only the field $Z^1$ has the trace part for simplicity. Let us write the 4 complex scalar field $Y^A$ by 8 real scalar fields as

\begin{align}
Z^1 &= X^1_0 + iX^5_0 + i\tilde{X}^1_0 T^a - \tilde{X}^5_0 T^a, \\
Z^2 &= i\tilde{X}^2_0 T^a - X^6_0 T^a, \\
W_1^\dagger &= i\tilde{X}^3_0 T^a - X^7_0 T^a, \\
W_2^\dagger &= i\tilde{X}^4_0 T^a - X^8_0 T^a.
\end{align}

Substituting them into $S_{V_b}$ and taking the scaling limit, we can obtain the following bosonic potential:

$$S_{V_b} = -\frac{1}{8K^2} \int d^3x \ tr \left( (X^1_0)^2 + (X^5_0)^2 \right) [P_I, P_J][P^I, P^J].$$

$P^I$ is defined by

\begin{align}
P^I &\equiv (P^1, \tilde{X}^2, \tilde{X}^3, \tilde{X}^4, \tilde{X}^6, \tilde{X}^7, \tilde{X}^8), \\
&= \left( \frac{1}{2}(\tilde{Y}^A + \tilde{Y}^A_\dagger), \frac{1}{2i}(\tilde{Y}^B - \tilde{Y}^B_\dagger) \right), \\
\tilde{Y}^A &\equiv (P^1, Z^2, W_1^\dagger, W_2^\dagger), \\
P^1 &\equiv \frac{X^1_0 \tilde{X}^5 - X^5_0 \tilde{X}^1}{\sqrt{(X^1_0)^2 + (X^5_0)^2}}.
\end{align}

We can rewrite it as,

$$S_{V_b} = -\frac{1}{8K^2} \int d^3x \ tr \left[ \frac{1}{4} (X^0_0)^2 \left( [\tilde{X}^I, \tilde{X}^J] \right)^2 - \frac{1}{2} \left( X_0^I [\tilde{X}^I, \tilde{X}^J] \right)^2 \right].$$
where we have used $X^I_0 = (X^1_0, 0, 0, X^5_0, 0, 0, 0)$. This is the potentials for bosons in the Lorentzian BL theory [3.9]. It is straightforward to see that the complete potential of the BL theory can be obtained by considering general $X^I_0$ and the full $SO(8)$ invariance is restored.

It should be noted that the above potential term is written in terms of the commutators. This shows that, if we replace more than two bosons by their trace components, the potential vanishes. This assures that the would-be divergent terms $V_B^{(n)}$ for $n > 3$ vanish and the only remaining term in the scaling limit is given by the above potential.

Finally consider the fermion potential. We expand the potential as $V_f = \sum_{n=0}^{4} V_f^{(n)}$ where $V_f^{(n)}$ contains $n$ trace parts and $(4-n)$ traceless parts. Since the coefficient of the fermion potential is proportional to $1/K$, $V_f^{(n)}$ scales as $\lambda^{1-n}$. $V_f^{(n)}$ for $n > 1$ diverges in the scaling limit and their coefficients must vanish. $V_f^{(0)}$ vanishes in the scaling limit $\lambda \rightarrow 0$. Hence the only remaining finite terms are $V_f^{(1)}$. In the following we look at the potential term with one of the bosons replaced by the trace part $X^I_0$. Such a term can be written as

$$S_{V_f} = \frac{i}{2K} X^I_0 \text{tr} \left[-\psi_1^\dagger \tilde{X}^0, \psi_1 + \psi_2^\dagger \tilde{X}^0, \psi_2 + \psi_3^\dagger \tilde{X}^0, \psi_3 + \psi_4^\dagger \tilde{X}^0, \psi_4ight]$$

$$\quad + \psi_1^\dagger [Y_2, \psi_2] + \psi_2^\dagger [Y_2, \psi_1] + \psi_3^\dagger [Y_2, \psi_4] + \psi_4^\dagger [Y_2, \psi_3]$$

$$\quad + \psi_1^\dagger [Y_3, \psi_3] + \psi_2^\dagger [Y_3, \psi_1] + \psi_4^\dagger [Y_3, \psi_2] + \psi_3^\dagger [Y_3, \psi_4]$$

$$\quad + \psi_1^\dagger [Y_4, \psi_4] + \psi_2^\dagger [Y_4, \psi_1] + \psi_3^\dagger [Y_4, \psi_3] + \psi_4^\dagger [Y_4, \psi_2] \right]$$

$$\quad + \frac{i}{2K} X^I_0 \text{tr} \left[\psi_1^\dagger [\tilde{X}^1, \psi_1] - \psi_2^\dagger [\tilde{X}^1, \psi_2] - \psi_3^\dagger [\tilde{X}^1, \psi_3] - \psi_4^\dagger [\tilde{X}^1, \psi_4]$$

$$\quad - \psi_1^\dagger [i Y_2, \psi_2] + \psi_2^\dagger [i Y_2, \psi_1] + \psi_3^\dagger [i Y_2, \psi_4] - \psi_4^\dagger [i Y_2, \psi_3]$$

$$\quad - \psi_1^\dagger [i Y_3, \psi_3] + \psi_2^\dagger [i Y_3, \psi_1] + \psi_4^\dagger [i Y_3, \psi_2] - \psi_3^\dagger [i Y_3, \psi_4]$$

$$\quad - \psi_1^\dagger [i Y_4, \psi_4] + \psi_2^\dagger [i Y_4, \psi_1] + \psi_3^\dagger [i Y_4, \psi_3] - \psi_4^\dagger [i Y_4, \psi_2] \right].$$

(3.37)

Here for simplicity we have assumed that the trace part of the boson $X^I_0$ is nonvanishing for $I = 1, 5$. This can be done by using the original $SU(4)$ symmetry. Note again that these potential terms are written as a form of commutators.

To get the 3-dimensional Majorana fermion as the BL theory, we rewrite the $SU(4)$ complex fermion in terms of the real variables\(^7\).

$$\psi_1 = i \chi_1 - \chi_5, \quad \psi_2 = i \chi_2 - \chi_6,$$

$$\psi_3 = i \chi_3 - \chi_7, \quad \psi_4 = i \chi_4 - \chi_8,$$

(3.38)

where $\chi_I$ are real 2-component spinors. We also expand the complex bosons as the real

\(^7\) When we give a VEV to the $X^5_0$ part only, we will get 7 $\Gamma$ matrices as in [62]. In our case we need 8 $\Gamma$ matrices and their antisymmetrized-products because we give a VEV to a more general direction.
ones (3.33). Then the fermion potential \((3.37)\) becomes by using the \(8 \times 8\) \(\Gamma\) matrice as

\[
S_{V_f} = -\frac{1}{2K} \text{tr} \tilde{\Psi}^I X^I_0 [\tilde{X}^J, \Gamma_{IJ} \Psi],
\]

where the indices \(I, J\) run from 1 to 8 and \(X^I_0 = (X^1_0, 0, 0, 0, X^5_0, 0, 0, 0)\). The explicit forms of the \(\Gamma\) matrices are given in the Appendix A. This fermion potential has the same \(SO(8)\) invariant form as that of the Lorentzian BL action (3.9). In the same fashion as the bosonic potential, the full \(SO(8)\) invariance can be seen easily by considering the general \(X^I_0\).

4 Conclusions and Discussions

In this paper, we have shown that the Lorentzian Bagger-Lambert theory is derived by taking a scaling limit of the \(N = 6\) superconformal Chern-Simons field theories proposed by Aharony et.al. In the scaling limit the trace components of the matter fields are taken to be large compared to the fluctuating traceless components. Hence the M2-branes are located far from the origin of the \(C^4/Z_k\) orbifold (or in the sufficiently low energy). Large values of the trace components means that they have a classical VEV and branes are fluctuating around them. The \(U(N) \times U(N)\) gauge group is broken to the diagonal \(U(N)\) and the axial \(U(N)\) symmetries become translational symmetries by the Inönü-Wigner contraction. Simultaneously in order to keep the action finite we need to scale the level \(k\) of the Chern-Simons gauge theories to infinity before taking the large \(N\) limit. This makes the \(Z_k\) identification of the orbifold to become a continuous circle identification. Since the radius of the M theory is proportional to \(1/k\), the gravity dual is reduced to \(d = 10\). Hence we conclude that the Bagger-Lambert theory based on the Lorentzian 3-algebra is a theory of multiple D2-branes.

However there is a subtlety to interpret the scaled theory as a \(d = 10\) IIA superstring on \(AdS_4 \times CP^3\). The Lorentzian BL theory is obtained by the \(k \rightarrow \infty\) limit before taking the large \(N\) limit. The ABJM theory is conjectured to be dual to the M-theory on \(AdS_4 \times S^7/Z_k\) and the radius of \(AdS_4\) is proportional to \(R/l_p \sim (kN)^{1/6}\) in the unit of the Planck scale \(l_p\). Hence in \(k \rightarrow \infty\) limit the radius becomes large and the \(d = 11\) supergravity is a good approximation. On the other hand, the compactification radius is proportional to \(R/kl_p \propto (Nk)^{1/6}/k\) and becomes 0 in the limit. Then it is tempting to think that the scaled theory is described by type IIA superstring on \(AdS_4 \times CP^3\). However, in the string scale \(l_s\) the radius of \(AdS_4\) is given by \(R/l_s \propto (N/k)^{1/4}\) and becomes 0. Hence the gravity dual of the scaled theory is more appropriately interpreted as \(d = 11\) supergravity in \(AdS_4 \times CP^3\) space-time rather than a type IIA supergravity.
We have discussed that the scaled theory has $SO(8)$ invariance if we consider rotations of the background fields $X_0^I(x)$. Since our scaling limit corresponds to locating the M2-branes far from the orbifold singularity, the recovery of the $SO(8)$ invariance is natural. In the field theory side we have explicitly checked the recovery of the $SO(8)$ invariance in the potentials of bosons and fermions. It is interesting that the symmetry is enhanced from $\mathcal{N} = 6$ to $\mathcal{N} = 8$ by taking the scaling limit.

We have also investigated a generalized conformal symmetry of the scaled theory. The familiar D2-brane action has a fixed and space-time independent coupling constant, and both of the conformal symmetry and $SO(8)$ invariance of the M2-branes are broken by the VEV of the M2-branes. On the contrary, our D2-brane action has a classical conformal symmetry and $SO(8)$ invariance if we allow the background fields $X_0^I$ to transform under them. This became possible by promoting the coupling constant to a space-time dependent $SO(8)$ vector field $X_0^I$. Discussions on the gravity side are given in [64]. In particular, it is important to clarify where we can get the same type of constraint equations in the gravity side and see how a dual geometry of D2-branes can acquire the generalized conformal symmetry.

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**A The Gamma matrices**

The explicit forms of the antisymmetrized products of the $8 \times 8$ $\Gamma$ matrices we have used in (3.39) are given as $\Gamma_{IJ} = \mathbb{I}_{2 \times 2} \otimes \gamma_{IJ}$ where

\[
\begin{align*}
\gamma_{12} &= \begin{pmatrix}
i\sigma^2 & \sigma^2 \\
-\sigma^2 & i\sigma^2
\end{pmatrix}, \\
\gamma_{13} &= \begin{pmatrix}
\mathbb{I} & -\sigma^3 \\
-\sigma^3 & \mathbb{I}
\end{pmatrix}, \\
\gamma_{14} &= \begin{pmatrix}
i\sigma^2 & -\sigma^1 \\
\sigma^1 & i\sigma^2
\end{pmatrix}, \\
\gamma_{15} &= \begin{pmatrix}
\sigma^3 & -\mathbb{I} \\
-\mathbb{I} & -\sigma^3
\end{pmatrix}, \\
\gamma_{16} &= \begin{pmatrix}
\sigma^1 & -i\sigma^2 \\
-i\sigma^2 & \sigma^1
\end{pmatrix}, \\
\gamma_{17} &= \begin{pmatrix}
\sigma^3 & \mathbb{I} \\
-\mathbb{I} & \sigma^3
\end{pmatrix},
\end{align*}
\]
\[\gamma_{18} = \begin{pmatrix} \sigma^1 & -\sigma^1 \\ i\sigma^2 & \sigma^1 \end{pmatrix}, \quad \gamma_{52} = \begin{pmatrix} \sigma^1 & -i\sigma^2 \\ -\sigma^1 & -i\sigma^2 \end{pmatrix},\]

\[\gamma_{53} = \begin{pmatrix} \sigma^1 & \sigma^3 \\ -\sigma^3 & \sigma^1 \end{pmatrix}, \quad \gamma_{54} = \begin{pmatrix} \sigma^1 & -\sigma^1 \\ i\sigma^2 & i\sigma^2 \end{pmatrix},\]

\[\gamma_{56} = \begin{pmatrix} i\sigma^2 & i\sigma^2 \\ i\sigma^2 & \sigma^1 \end{pmatrix}, \quad \gamma_{57} = \begin{pmatrix} -\sigma^3 & \sigma^3 \\ \sigma^3 & \sigma^1 \end{pmatrix},\]

\[\gamma_{58} = \begin{pmatrix} -\sigma^1 & \sigma^1 \\ i\sigma^2 & -i\sigma^2 \end{pmatrix}\] (A.1)

and \(\mathbb{I}_{2 \times 2}\) is a 2 \(\times\) 2 identity matrix. We have also defined

\[\Gamma^0 = i\sigma^2 \otimes \mathbb{I}_{8 \times 8}.\] (A.2)

The \(i\sigma^2\) was used to contract the indices of the 2-component spinor \(\chi\) and it is the 3 dimensional \(\gamma^0\) matrix (see the Appendix of [56]). \(\mathbb{I}_{8 \times 8}\) is an 8 \(\times\) 8 identity matrix. They satisfy the following consistency relations as \(\Gamma_{12}\Gamma_{13} + \Gamma_{13}\Gamma_{12} = -(\Gamma_2\Gamma_3 + \Gamma_3\Gamma_2) = 0\). At this stage, there is an ambiguity to determine the \(\Gamma\) matrices, but the explicit forms of \(\Gamma_I\) are not necessary here. To fix the ambiguity, we need to consider more general VEVs of \(X^I_0\).

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