Inducing charges and currents from extra dimensions

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Abstract

In a particular variant of Kaluza-Klein theory, the so-called induced-matter theory (IMT), it is shown that any configuration of matter may be geometrically induced from a five-dimensional vacuum space. By using a similar approach we show that any distribution of charges and currents may also be induced from a five-dimensional space. Whereas in the case of IMT the geometry is Riemannian and the fundamental equations are the five-dimensional Einstein equations in vacuum, here we consider a Minkowskian geometry and five-dimensional Maxwell equations in vacuum.

1 Introduction

Certainly it is not easy to assess the weighty and considerable influence that the ideas put forward by Kaluza-Klein theory \textsuperscript{[1,2]} in the early twentieth century have had on the development of modern theories of physics, particularly in their quest for unification of the fundamental interactions of matter. In Kaluza-Klein theory one would say that a certain kind of unification of gravity and electromagnetism is achieved. This accomplishment, however, has a price: we need to introduce an extra (as yet unseen) dimension to ordinary spacetime. On the other hand, this article allowed T. Kaluza and O. Klein to devise a mechanism through which an extra dimensionality of space combines with curvature in such a way that electromagnetic phenomena may be looked upon as a manifestation of pure geometry.

The idea that we might live in a five-dimensional (5D) space instead of the traditional four-dimensional (4D) picture and that some observable physical ef-
f e c t s m a y b e a t t r i but e d t o t h e e x i s t e n c e o f an e x t r a d i m e n s i o n w a s s e m i n a l. I t 
end u r e s u n t i l t h e s e d a y s a n d l i e s a t t h e c o r e o f m a n y c u r r e n t r e s e a r c h w h i c h, 
one w o u l d s a y, c o n s t i t u t e t h e ' m a i n s t r e a m ' o f t h e o r e t i c a l p a r t i c l e p h y s i c s a n d 
c o s m o l o g y [3]. I n s p i r e d b y t h e v e - d i m e n s i o n a l Kal u z a-K l e i n t h e o r y t h e r e a p-
pe a r e d, i n t h e s i x t i e s a n d s e v e n t i e s, a n u m b e r o f h i g h e r - d i m e n s i o n a l t h e o r i e s 
s u c h a s e l e v e n - d i m e n s i o n a l s u p e r g r a v i t y a n d t e n - d i m e n s i o n a l s u p e r s t r i n g s, a l l 
of t h e m a i m i n g a t a g e n e r a l s c h e m e o f u n i c a t i o n [3]. M o r e r e c e n t l y, a n o t h e r 
hi g h e r - d i m e n s i o n a l m o d e l, t h e s o - c a l l e d b r a n e w o r l d s c e n a r i o h a s e m e r g e d, a c-
c o r d i n g t o w h i c h o u r 4D s p a c e t i m e i s v i e w e d a s h y p e r s u r f a c e ( t h e b r a n e ) 
is t r i c a l l y e m b e d d e d i n a v e - d i m e n s i o n a l E i n s t e i n s p a c e ( t h e b u l k ) [4,5]. 
The o r i g i n a l v e r s i o n o f K a l u z a-K l e i n t h e o r y a s s u m e s , a s a p o s t u l a t e, t h a t t h e 
f i f t h d i m e n s i o n i s c o m p a c t. R e c e n t l y, h o w e v e r, a n o n - c o m p a c t i o n l a p p r o a c h t o 
K a l u z a-K l e i n g r a v i t y, k n o w n a s I n d u c e d- M a t t e r t h e o r y ( I M T) h a s b e e n p r o-
p o s e d b y W e s s o n a n d c o l a b o r a t o r s [6,7,8]. T h e b a s i c p r i n c i p l e o f t h e I M T 
ap p r o a c h i s t h a t a l l c l a s s i c a l p h y s i c a l q u a n t i t i e s , s u c h a s m a t t e r d e n s i t y a n d 
p r e s s u r e , a r e s u s c e p t i b l e o f a g e o m e t r i c a l i n t e r p r e t a t i o n. M o r e o v e r, i t i s a s s e r t e d 
that o n l y o n e e x t r a d i m e n s i o n i s s u f f i c i e n t t o e x p l a i n a l l t h e p h e n o m e n o l o g i c a l 
pr o p e r t i e s o f m a t t e r a n d f i n a l l y, g e n e r a t e d b y p u r e g e o m e t r i c a l m e a n s. M u c h i n t h e 
s a m e s p i r i t o f K a l u z a-K l e i n t h e o r y, W e s s o n s ' s p r o p o s a l a l s o a s s u m e s t h a t t h e 
fi n d a m e n t a l v e - d i m e n s i o n a l s p a c e i n w h i c h o u r u s u a l s p a c e t i m e i s e m b e d d e d, s h o u l d 
be a s o l u t i o n o f t h e v e - d i m e n s i o n a l v a c u u m E i n s t e i n e q u a t i o n s
\[ R_{ab} = 0; \]

S u r e l y a l o n g t h e h i s t o r y o f p h y s i c s t h e r e h a v e b e e n m a n y a t t e m p t s t o f o r m u-
l a t e a u n i f i e d e l d t h e o r y w h i c h w o u l d n o t r e s o r t t o e x t r a d i m e n s i o n s. I n f a c t, 
r e c o u r s e t o a n u n o b s e r v e d e x t r a d i m e n s i o n w a s b a s i c a l l y t h e c a u s e o f E i n s t e i n ' s 
re l u c t a n c e t o a c c e p t t h e p l a u s i b i l i t y o f K a l u z a-K l e i n t h e o r y [9]. A l t e r n a t i v e w a y s 
t o g e o m e t r i z e t h e e l e c t r o m a g n e t i c e l d i n t h e u s u a l f o u r - d i m e n s i o n a l s p a c e t i m e 
h a v e b e e n p r o p o s e d s i n c e t h e e a r l y d a y s o f g e n e r a l r e l a t i v i t y. A m o n g t h e s e s, f o r 
in s t a n c e, t h e w o r k o f W e y l, w h o i n 1918 p r o p o s e d a k i n d o f g e n e r a l i z a t i o n 
of R i e m a n n g e o m e t r y [10], n o t t o m e n t i o n t h e a t t e m p t s m a d e b y E i n s t e i n, 
E d d i t t o n, S c h r ö d i n g e r a n d m a n y o t h e r s [11].

A n a s p e c t o f K a l u z a-K l e i n t h e o r y t h a t p e r h a p s h a s g o n e u n o b s e r v e d c o n-
c e r n s t h e r o l e p l a y e d b y t h e g e o m e t r y u p o n w h i c h t h e u n d e r l y i n g f i n d a m e n-
t a l h i g h e r - d i m e n s i o n a l t h e o r y i s b a s e d. I n t h e c a s e o f K a l u z a-K l e i n t h i s f i n d a m e n-
t a l t h e o r y i s g e n e r a l r e l a t i v i t y, r e c a s t e d a s v e - d i m e n s i o n a l R i e m a n n s p a c e. H e r e l i e s t h e m y s t e r i o u s p o w e r o f t h e K a l u z a-K l e i n p r o g r a m. I n d e e d, w e r e n o t 
fo r t h e r i c h n e s s R i e m a n n g e o m e t r y h a s c o m p a r a t i v e l y t o a g e o m e t r i c b a c k-
g r o u n d, t h e r e w o u l d h a r d l y b e e n o u t e n d a b l e d e g r e e s o f f r e e d o m c a p a b l e o f 
ac c o m d a t i n g i n t h e i r g e o m e t r i c a l s t r u c t u r e n o n - g r a v i t a t i o n a l e l d s s u c h a s t h e 
e l e c t r o m a g n e t i c e l d. T h i s u n d e r s t a n d i n g l e a d s t o t h e e p i s t e m o l o g i c a l q u e s-
t i o n o f w h a t r e a l l y i s t h e r o l e a n d t h e p o w e r o f t h e e x t r a d i m e n s i o n a l h y p o t h e s i s,
taken by itself, in our quest for formulating unifying theories. In other words, to what extent is the apparent success of Kaluza-Klein theory (or its extension to more general gauge fields [12]) a consequence of the extra-dimensional hypothesis alone?

In this paper we shall try to give a partial answer to the above question by examining a theory in which the original Kaluza-Klein scheme has been modified. That is, instead of assuming the (vacuum) Einstein field equations as the fundamental equations, that are capable of generating electrodynamics in 4D, or matter in the Induced-Matter theory approach, we promote the (vacuum) Maxwell equations in 5D to the level of fundamentalequations and explore their potentiality to generate new physics in 4D. For simplicity we shall work in a five-dimensional space background. It is a curious question whether the Kaluza-Klein mechanism or the Wesson’s procedure for generating matter from the extra dimensions still works. In other words, we would like to answer the question: When the fundamental 5D equations are Maxwell equations, instead of Einstein equations, is it possible to “induce” electric charge and currents in 4D from pure vacuum in 5D?

2 The five-dimensional field equations

We start by assuming that our fundamental space is a 5-dimensional Lorentzian manifold $M^5$ with metric $\eta_{ab} = \text{diag}(+,-,-,-,-)$. We also take the view, according to modern embedding theories, that $M^5$ is foliated by a family of hypersurfaces $g$ de ned by $l = \text{const}$ and that our ordinary spacetime consists of one of these hypersurfaces $g_0 (l = 0)$, which is then embedded in $M^5$. In the fundamental space $M^5$ we shall postulate the existence of an electromagnetic field, which is described by a 5-dimensional potential vector defined on $M^5$ and denoted by $A^a = (A^0; A^1; \ldots; A^4)$: We then define the 5-dimensional electromagnetic field tensor $F_{ab} = \partial_a A_b - \partial_b A_a$ and postulate that the dynamics of $A^a$ in $M^5$ is governed by the vacuum field equations

$$\square F^{ab} = 0$$

where $F^{ab} = \eta^{ac}n^{bd}F_{cd}$.

Let us consider the equations (1) separately by first taking $b = 0$, and then putting $b = 4$. We then have

$$\square (\triangledown A \cdot \triangledown A) = \square A + \frac{\partial^2 A}{\partial l^2}$$

\footnote{Throughout Latin indices take value in the range $(0,1,\ldots,4)$, while Greek indices run from $(0,1,2,3)$. The operation of raising and lowering indices is done with the help of the 5D and 4D metrics $(\eta_{ab}; \eta^{ab})$ and their inverses $(\eta_{ab}; \eta^{ab})$. We shall also denote the $0^\text{th}$ coordinate $y^0$ by $l$ and the rest four coordinates $y$ (the “spacetime” coordinates) by $x$, that is, $y^a = (x^1; x^2; x^3; y^0)$, with $0 = y^0; 1; 2; 3)$.}

In order to simplify the model we shall neglect gravity, hence $\square$ will be identi ed to 4D Minkowski spacetime.
and

\[ \frac{\partial^{2}A}{\partial l^{2}} = \partial \left( \partial A \right) ; \tag{3} \]

with \( \partial \) denoting the usual d'Alembertian operator of 4-dimensional
spacetime. By proceeding in analogy with Kaluza-Klein theory (or with the
I M T approach) we shall consider the above equations taken at \( l = 0 \) and then
regard Eq. (2) as the ordinary 4D Maxwell equations, where the right-hand side
has been formally identified with a 4-dimensional current density vector \( j \).[18]

The following question now arises: Can we induce any arbitrary four-dimen-
sional current density \( j = j(x) \) in the same way as one can induce any arbitrary
energy-momentum tensor in the IMT approach? In the case of the Induced-
M after approach this question was resolved by translating the problem into
a geometrical language and employing the Campbell-Maganna theorem, which
in essence asserts that any \( n \)-dimensional Riemannian manifold is locally em-
beddable into a \( (n + 1) \)-dimensional Riemannian manifold \([14,15,16]\). Here we
shall try a more direct approach by investigating the system of partial dif-
ferential equations defined by (2) and (3), and look for solutions \( A = A(x;l),
= (x;l) \) for a given set of prescribed functions \( j = j(x) \). Let us note that
since \( j \) is to be interpreted physically as a current density it has to satisfy the
equation of continuity \( \partial j = 0 \).

For simplicity let us first set \( = 0 \) (we shall see later that this is a possible
choice). Then (2) becomes

\[ \frac{\partial^{2}A}{\partial l^{2}} = \partial \left( \partial A \partial A \right) \tag{4} \]

We note that according to the Cauchy-Kowalevskaya theorem \([17]\) the equation
(4) admits an analytical solution in a neighbourhood of \( l = 0 \).

It turns out that an explicit solution of (4) may be found in the form of the
power series

\[ A(x;l) = \sum_{n} a_{(n)}(x) l^{n} \]

where the coefficients \( a_{(n)} \) are calculated by a process of repeated iteration, as
we show below. If (4) is to induce the 4D Maxwell equations for an arbitrary
prescribed current density \( j = j(x) \) in the hypersurface \( l = 0 \), then we must have

\[ \frac{\partial^{2}A}{\partial l^{2}} = \frac{4}{c} j(x) \tag{5} \]

Now the third derivative may be obtained by taking the derivative of (3) with
respect to \( l \). In this way we have:

\[ \frac{\partial^{3}A}{\partial l^{3}} = \partial \left( \partial A \partial A \partial A \right) \tag{6} \]

\(^{3}\text{In this work we are employing Gaussian units.}\)
Clearly the coefficient $\frac{\partial^2 A}{\partial l^2}$ must be chosen in such a way to satisfy (3). Thus, let us set
$$\frac{\partial^2 A}{\partial l^2} = 0,$$
where $(x)$ is an arbitrary four-vector with null divergence ($\frac{\partial}{\partial l} = 0$). With this choice it follows that
$$\frac{\partial^3 A}{\partial l^3} = 0.$$

The fourth derivative, on the other hand, gives:
$$\frac{\partial^4 A}{\partial l^4} = \frac{\partial^2 A}{\partial l^2} = \frac{\partial^2 A}{\partial l^2} = 0,$$
From (5) and recalling that $\frac{\partial}{\partial l} = 0$, we have
$$\frac{\partial^4 A}{\partial l^4} = \frac{4}{c} j \frac{\partial}{\partial l}.$$

Repeating the same argument it is easy to verify that $\frac{\partial^5 A}{\partial l^5} = 0$, $\frac{\partial^5 A}{\partial l^5} = \frac{4}{c} (j)$, and so on. Therefore, we conclude that $A$ $(x;l)$, solution of (4), will be given by
$$A (x;l) = a (x) + \frac{4}{2! c} j (x) l^1 + \frac{1}{3!} (x) l^3 + \frac{1}{4!} \frac{4}{c} j (x) l^4 + \cdots$$

It is important to note that $A$ $(x;l)$ satisfy the equations (2) and (4) for all orders of $l$. Note also that $a (x)$ is nothing more than the potential induced on the hypersurface $l = 0$ and that $a (x)$ satisfies the 4D Maxwell equations with the source $j (x)$. Moreover, we have usual gauge freedom in the induced 4D electrodynamics since the 4D electromagnetic field tensor $F (x) = \frac{\partial}{\partial l} A (x;0) - \frac{\partial}{\partial l} A (x;0)$ is clearly invariant under transformations of the type $a (x) = a (x) + \frac{\partial}{\partial l} f (x)$.

If, for some reason, we would like to have an even solution $A$ $(x;l)$ in the $l$ coordinate (in order to have $Z_2$ symmetry, for instance), then we can choose $a (x) = 0$. In this case the series reduces to
$$A (x;l) = a (x) + \frac{4}{c} \frac{X}{\sum_{n=1}^{\infty} \frac{1}{(2n)!}} \frac{1}{n} j \frac{\partial^2 A}{\partial l^2},$$
where the use of the symbol $n$ means that the d'Alembertian operator has been applied $n$ times.

We conclude then that if we are given any analytical four-dimensional vector function $j = j (x)$, describing a certain physical distribution of charge and current, then we can produce a vector potential $A^x = (A (x;l); j (x;l))$, solution of the vacuum Maxwell equations in 5D, with $A (x;l)$ given by (5) and an arbitrary function $\frac{\partial}{\partial l} A (x;l)$, which for simplicity we have chosen to be $= 0$, such that when $A^x (x;l)$ is restricted to the hypersurface $l = 0$,
it generates or induces $j = j(x)$. One would say then that, similarly to the generation of matter in the Induced-Matter approach from a vacuum higher-dimension space, any physical configuration of charge and current may also be viewed as having a purely geometric origin due to extra dimensionality of space.

### 3 Final remarks

In this paper we basically have tried to answer the following question: What is the potentiality of Maxwell equations to generate new physics in four-dimensional spacetime from a vacuum higher-dimensional vacuum? It is well known the mechanism by which the Kaluza-Klein theory can generate, via the Einstein equations, the gravitational and gauge fields in four dimensions from a higher-dimensional vacuum space. We know that in the Induced-Matter approach, which also employs the Einstein equations in vacuum, matter can be viewed as being generated from an extra dimension by purely geometric means. If we replace the Einstein equations by the vacuum Maxwell equations as the fundamental equations of our higher-dimensional theory, apparently there are not enough degrees of freedom in a four-dimensional Minkowski geometry capable of generating a metric field describing gravitation plus the four-dimensional electromagnetic field. Nevertheless we discover quite surprisingly that, analogously to what happens in the case of the induced-matter approach, although no other field is generated, charges and currents may also be viewed as induced by the extra dimension.

We would like to mention that when the present article was almost ready we became aware of a recent paper by Liko, in which the idea of inducing an effective electromagnetic current from higher dimension is also taken up. In this approach the mentioned author considers a 4D electromagnetic tensor $F$ modified by the presence of an extra term depending non-linearly on the electromagnetic potentials $A$ which are supposed to be functions of the $5\text{th}$ coordinate, while the 4D Maxwell equations are derived from the variation of a four-dimensional action built with $F$. In this case the induced current does not follow directly from the 5D Maxwell equations in vacuum.

One basic assumption in Kaluza-Klein theory is that the vacuum is the space $\mathbb{M}^4 \times S^1$, i.e. the product of 4D Minkowski spacetime with a circle of radius $R$. In the case of 4D induced electromagnetism since we do not need the assumption of compactness of the $5\text{th}$ dimension, the vacuum should be considered as the vacuum in a four-dimensional Minkowski space $\mathbb{M}^4$. As was pointed out by G. Ross and Perry, both vacua are classically stable, although it has been argued by Witten that the ground state $\mathbb{M}^4 \times S^1$ is unstable against a process of semi-classical barrier penetration.

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4In this respect it should be mentioned the work of G. Nordström, which was the first attempt to formulate a unified theory of gravitation and electromagnetism by postulating the existence of a $5\text{th}$ dimension. However, in his theory spacetime is at and gravitation was described by means of a scalar field.
Finally, the question of inducing electromagnetic charges and currents from a higher-dimensional vacuum space may be given a more geometrical approach by reducing it to an initial value problem for the electromagnetic field, much in the same way as the Induced-Matter theory may be formulated in the light of the Campbell-Magaard theorem [22]. In a quantum context it is worth mentioning that the idea of a 4D electromagnetic field embedded in a 5D pure vacuum has been considered very recently by Raya, Aguiar and Bellini in connection with gravitoelectromagnetic effects [23].

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