ON ISOTROPIC DIVISORS ON IRREDUCIBLE SYMPLECTIC MANIFOLDS

DAISUKE MATSUSHITA

Abstract. Let $X$ be an irreducible symplectic manifold and $L$ a divisor on $X$. Assume that $L$ is isotropic with respect to the Beauville-Bogomolov quadratic form. We define the rational Lagrangian locus and the movable locus on the universal deformation space of the pair $(X, L)$. We prove that the rational Lagrangian locus is empty or coincide with the movable locus of the universal deformation space.

1. Introduction

We start with recalling the definition of an irreducible symplectic manifold.

**Definition 1.1** ([Bea83 Théorèm 1]). Let $X$ be a compact Kähler manifold. The manifold $X$ is said be irreducible symplectic if $X$ satisfies the following three properties.

1. $X$ carries a symplectic form.
2. $X$ is simply connected.
3. $\dim H^0(X, \Omega^2_X) = 1$.

Together with Calabi-Yau manifolds and complex tori, irreducible symplectic manifolds form a building block of a compact Kähler manifold with $c_1 = 0$. It is shown in [Mat01], [Mat99] and [Hwa08] that a fibre space structure of an irreducible symplectic manifold is very restricted. To state the result, we recall the definition of a Lagrangian fibration.

**Definition 1.2.** Let $X$ be an irreducible symplectic manifold and $L$ a line bundle on $X$. A surjective morphism $g : X \to S$ is said to be Lagrangian if a general fibre is connected and Lagrangian. A dominant map $g : X \dasharrow S$ is said to be rational Lagrangian if there exists a birational map $\phi : X \dasharrow X'$ such that the composite map $g \circ \phi^{-1} : X' \to S$ is Lagrangian. We say that $L$ defines a Lagrangian fibration if the linear system $|L|$ defines a Lagrangian fibration. Also we say that $L$ defines a rational Lagrangian fibration if $|L|$ defines a rational Lagrangian fibration.

**Theorem 1.1** ([Mat99, Mat01] and [Hwa08]). Let $X$ be a projective irreducible symplectic manifold. Assume that $X$ admits a surjective morphism $g : X \to S$ over a smooth projective manifold $S$. Assume that $0 < \dim S < \dim X$ and $g$ has connected fibres. Then $g$ is Lagrangian and $S \cong \mathbb{P}^{1/2 \dim X}$.

It is a natural question when a line bundle $L$ defines a Lagrangian fibration. If $L$ defines a rational Lagrangian fibration, then $L$ is isotropic with respect to the Beauville-Bogomolov quadratic form. Moreover the first Chern class $c_1(L)$ of $L$ belongs to the biational Kähler cone which is defined in [Huy03b Definition 4.1].

**Conjecture 1.1** (D. Huybrechts and J. Sawon). Let $X$ be an irreducible symplectic manifold and $L$ a line bundle on $X$. Assume that $L$ is isotropic with respect to the Beauville-Bogomolov quadratic form on $H^2(X, \mathbb{C})$. We also assume that $c_1(L)$

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belongs to the birational Kähler cone of $X$. Then $L$ will define a rational Lagrangian fibration.

At that moment, partial results are known about Conjecture [11]. We could consult [AC08], [COP10], [Mat08] and [Ver10]. In this note, we consider the above conjecture by a different approach. To state the result, we recall the basic facts of a deformation of a pair which consists of a symplectic manifold and a line bundle.

**Definition 1.3.** Let $X$ be a Kähler manifold and $L$ a line bundle on $X$. A deformation of the pair $(X, L)$ consists of a smooth morphism $\mathfrak{X} \to S$ over a smooth manifold $S$ with a reference point $o$ and a line bundle $\mathfrak{L}$ on $\mathfrak{X}$ such that the fibre $\mathfrak{X}_o$ at $o$ is isomorphic to $X$ and the restriction $\mathfrak{L}|_{\mathfrak{X}_o}$ is isomorphic to $L$.

If $X$ is an irreducible symplectic manifold, it is known that there exists the universal deformation of deformations of a pair $(X, L)$.

**Proposition 1.1** ([Huy99 (1.14)]). Let $X$ be an irreducible symplectic manifold and $L$ a line bundle on $X$. We also let $\mathfrak{X} \to \text{Def}(X)$ be the Kuranishi family of $X$. Then there exists a smooth hypersurface $\text{Def}(X, L) \subset \text{Def}(X)$ such that the restriction $\mathfrak{X}_L := \mathfrak{X} \times_{\text{Def}(X)} \text{Def}(X, L) \to \text{Def}(X, L)$ forms the universal family of deformations of the pair $(X, L)$. Namely, $\mathfrak{X}_L$ carries a line bundle $\mathfrak{L}$ and every deformation $\mathfrak{X}_S \to S$ of $(X, L)$ is isomorphic to the pull back of $(\mathfrak{X}_L, \mathfrak{L})$ via a uniquely determined map $S \to \text{Def}(X, L)$.

Now we can state the result.

**Theorem 1.2.** Let $X$ be an irreducible symplectic manifold and $L$ a line bundle on $X$. We also let $\pi : \mathfrak{X}_L \to \text{Def}(X, L)$ be the universal family of deformations of the pair $(X, L)$ and $\mathfrak{L}$ the universal bundle. We denote by $q$ the Beauville-Bogomolov form on $H^2(X, \mathbb{C})$. Assume that $q(L) = 0$. We define the locus of movable $\text{Def}(X, L)_{\text{mov}}$ by

$$\{ t \in \text{Def}(X, L); c_1(\mathfrak{L}_t) \text{ belongs to the birational Kähler cone of } X. \}$$

We also define more two subsets of $\text{Def}(X, L)$. The first is the locus of rational Lagrangian fibration $V$ which is defined by

$$\{ t \in \text{Def}(X, L); \mathfrak{L}_t \text{ defines a rational Lagrangian fibration over the projective space. } \}$$

The second is the locus of Lagrangian fibration $V_{\text{reg}}$ which is defined by

$$\{ t \in \text{Def}(X, L); \mathfrak{L}_t \text{ defines a Lagrangian fibration over the projective space. } \}$$

Then $V = \emptyset$ or $V = \text{Def}(X, L)_{\text{mov}}$. Moreover if $V \neq \emptyset$, $V_{\text{reg}}$ is a dense open subset of $\text{Def}(X, L)$ and $\text{Def}(X, L) \backslash V_{\text{reg}}$ is contained in a union of countably hypersurfaces of $\text{Def}(X, L)$.

**Remark 1.1.** Professors L. Kamenova and M. Verbitsky obtained $V_{\text{reg}}$ is an dense open set of $\text{Def}(X, L)$ under the assumption $V_{\text{reg}} \neq \emptyset$ in [KV12 Theorem 3.4].

To state an application of Theorem [12] we need the following two definitions.

**Definition 1.4.** Two compact Kähler manifolds $X$ and $X'$ are said to be deformation equivalent if there exists a proper smooth morphism $\pi : \mathfrak{X} \to S$ over a smooth connected complex manifold $S$ such that both $X$ and $X'$ form fibres of $\pi$.

**Definition 1.5.** An irreducible symplectic manifold $X$ is said to be of $K3^{[n]}$-type if $X$ is deformation equivalent to the $n$-pointed Hilbert scheme of a $K3$ surface. An irreducible symplectic manifold $X$ is also said to be of type generalized Kummer if $X$ is deformation equivalent to a generalized Kummer variety which is defined in [Bea85 Théorème 4].
It was shown in [Mar13], [BM13] and [Yos12] that if $X$ is isomorphic to the $n$-pointed Hilbert scheme of a $K3$ surface or a generalized Kummer variety, then Conjecture 1.1 holds. To combine these results and Theorem 1.2, we obtain the following result.

**Corollary 1.1.** Let $X$ be an irreducible symplectic manifold of type $K3^{[n]}$ or of type generalized Kummer. We also let $L$ be a line bundle $L$ on $X$ which is not trivial, isotropic with respect to the Beauville-Bogomolov quadratic form on $H^2(X, \mathbb{C})$ and $c_1(L)$ belongs to the birational Kähler cone of $X$. Then $L$ defines a rational Lagrangian fibration over the projective space.

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2. **Birational correspondence of deformation families**

In this section we study a relationship between deformation families. We start with introducing the following Lemma.

**Lemma 2.1** ([Huy99, Lemma 2.6]). Let $X$ and $X'$ be irreducible symplectic manifolds. Assume that there exists a bimeromorphic map $\phi : X \dashrightarrow X'$. Then $\phi$ induces an isomorphism

$$\phi_* : H^2(X, \mathbb{C}) \cong H^2(X', \mathbb{C})$$

which compatible with the Hodge structures and the Beauville-Bogomolov quadratic forms.

We consider the relationship of the Kuranishi families of bimeromorphic irreducible symplectic manifolds.

**Proposition 2.1.** Let $X$ and $X'$ are irreducible symplectic manifolds. We denote by $\pi : \mathcal{X} \to \text{Def}(X)$ the universal family of deformations of $X$. We also denote by $\pi' : \mathcal{X}' \to \text{Def}(X')$ the universal family of deformations of $X'$. Assume that $X$ and $X'$ are bimeromorphic. Then there exist dense open subset $U$ of $\text{Def}(X)$ and $U'$ of $\text{Def}(X')$ which satisfy the following three properties.

(1) The set $\text{Def}(X) \setminus U$ is contained in a union of countably hypersurfaces in $\text{Def}(X)$ and $\text{Def}(X') \setminus U'$ is also contained in a union of countably hypersurfaces in $\text{Def}(X')$.

(2) They satisfy the following diagram:

$$\xymatrix{ \mathcal{X} \times_{\text{Def}(X)} U \ar[r]^{\phi} \ar[d] & \mathcal{X}' \times_{\text{Def}(X')} U' \ar[d] \\
U \ar[r]^{\varphi} & U', }$$

where $\phi$ and $\varphi$ are isomorphic.

(3) Let $s$ be a point of $U$ and $s'$ the point $\varphi(s)$. We also let $\phi_s : \mathcal{X}_s \cong \mathcal{X}'_{s'}$ be the restriction of the isomorphism $\phi : \mathcal{X} \times_{\text{Def}(X)} U \cong \mathcal{X}' \times_{\text{Def}(X')} U'$ in the above diagram to the fibre $\mathcal{X}_s$ at $s$ and the fibre $\mathcal{X}'_{s'}$ at $s'$. We denote by $\eta$ a parallel transport in the local system $\mathcal{R}^2\pi_*\mathbb{C}$ along a path from the reference point to $s$. We also denote by $\eta'$ a parallel transport in the local system
$R^2\pi'_s\mathbb{C}$ along a path from the reference point to $s'$. Then the composition of the isomorphisms

$$H^2(X, \mathbb{C}) \cong H^2(\mathbb{X}_s, \mathbb{C}) \overset{\eta}{\cong} H^2(\mathbb{X}'_s, \mathbb{C}) \overset{\eta^{-1}}{\cong} H^2(X', \mathbb{C})$$

coincides with $\phi_s$, which is the isomorphism induced by $\phi : X \dashrightarrow X'$.

Proof. The proof of this proposition is a mimic of the proof of [Huy99, Theorem 5.9]. The proof consists of two steps. First, we show that there exist open sets $U$ of $\text{Def}(X)$ and $U'$ of $\text{Def}(X')$ which satisfy the assertions (2) and (3) of Proposition 2.4. Since $X$ and $X'$ are bimeromorphic, we have a deformation $\mathbb{X}_S \rightarrow S$ of $X$ and a deformation $\mathbb{X}_S' \rightarrow S$ of $X'$ over a small disk $S$ which are isomorphic to each other over the punctured disk $S \setminus 0$ by [Huy99, Theorem 4.6]. By the universality, $\mathbb{X}_S \rightarrow S$ is isomorphic to the base change $\mathbb{X} \rightarrow \text{Def}(X)$ by a uniquely determined morphism $S \rightarrow \text{Def}(X)$. The family $\mathbb{X}_S \rightarrow S$ is also isomorphic to the base change of $\mathbb{X}' \rightarrow \text{Def}(X')$ by a uniquely determined morphism $S \rightarrow \text{Def}(X')$. Thus there exist points $t \in \text{Def}(X)$ and $t' \in \text{Def}(X')$ such that the fibres $\mathbb{X}_t$ and $\mathbb{X}'_{t'}$ are isomorphic. Let $\eta$ be a parallel transportation of $R^2\pi_s\mathbb{C}$ along a path from the reference point to $t$ and $\eta'$ a parallel transportation of $R^2\pi'_s\mathbb{C}$ along a path from the reference point to $t'$. To consider the composition of the isomorphisms

\begin{equation}
H^2(X, \mathbb{C}) \overset{\eta}{\cong} H^2(\mathbb{X}_t, \mathbb{C}) \overset{\eta}{\cong} H^2(\mathbb{X}'_{t'}, \mathbb{C}) \overset{\eta^{-1}}{\cong} H^2(X', \mathbb{C})
\end{equation}

we need more information of the construction of the two families $\mathbb{X}_S \rightarrow S$ and $\mathbb{X}_S' \rightarrow S$. By the last paragraph of the proof of [Huy99, Theorem 4.6], the construction is due to [Huy99, Proposition 4.5]. According to [Huy99, Proposition 4.5], $\mathbb{X}_S \rightarrow S$ is constructed as follows. Let $H'$ be an ample divisor on $X'$ and $H := (\phi_s)^{-1}H'$. We consider a deformation $\pi_S : (\mathbb{X}_S, S) \rightarrow (X, H)$. Then the closure of the image of the rational map $\mathbb{X}_S \dashrightarrow \text{Proj}(\mathbb{H}(\pi_S, S))$ gives the desired deformation $\mathbb{X}_S \rightarrow S$. Hence we have a birational map $\mathbb{X}_S \dashrightarrow \mathbb{X}_S$ which commutes with the two projections. Moreover the restriction of this birational map coincides with $\phi$. Thus the composition of the isomorphisms $\phi$ coincides with $\phi_s$. By [Bea83, Théorème 5 (b)], we can extend the isomorphism $\mathbb{X}_t \cong \mathbb{X}'_{t'}$ over open sets of $\text{Def}(X)$ and $\text{Def}(X')$, that is, there exist open sets $U$ of $\text{Def}(X)$ and $U'$ of $\text{Def}(X')$ such that the restriction families $\mathbb{X} \times_{\text{Def}(X)} U$ and $\mathbb{X}' \times_{\text{Def}(X')} U'$ are isomorphic and this isomorphism is compatible with the two projections $\mathbb{X} \rightarrow \text{Def}(X)$ and $\mathbb{X}' \rightarrow \text{Def}(X')$. By this construction, the restriction of the isomorphism $\phi : \mathbb{X} \times_{\text{Def}(X)} U \cong \mathbb{X}' \times_{\text{Def}(X')} U'$ to the fibres satisfies the assertion (3) of Proposition 2.4.

Next we show that $U$ and $U'$ satisfies the assertion (1) of Proposition 2.1. Let $s$ be a point of $U$. By [Huy99, Theorem 4.3], the fibres $\mathbb{X}_s$ and $\mathbb{X}'_s$ are bimeromorphic. If $\dim H^{1,1}(\mathbb{X}_s, \mathbb{Q}) = 0$, then $\mathbb{X}_s$ and $\mathbb{X}'_s$ carries neither curves nor effective divisors. Thus $\mathbb{X}_s$ and $\mathbb{X}'_s$ are isomorphic by [Huy03b, Proposition 2.1] and $s \in U$. Thus if $s \in U \setminus U$ then $\dim H^{1,1}(\mathbb{X}_s, \mathbb{C}) \geq 1$. This implies that $U \setminus U$ is contained in a union of countably hypersurfaces. \(\square\)

For the proof of Theorem 1.2, we need also a correspondence of deformation families of pairs. Before we state the assertion, we give a proof of the following Lemma.

**Lemma 2.2.** Let $X$ and $X'$ be irreducible symplectic manifold. Assume that there exists a bimeromorphic map $\phi : X \dashrightarrow X'$. We also assume $\dim H^{1,1}(\mathbb{X}, \mathbb{Q}) = 1$ and $q(\beta) \geq 0$ for every element $\beta$ of $H^{1,1}(\mathbb{X}, \mathbb{Q})$, where $q_X$ is the Beauville-Bogomolov quadratic form on $H^2(X, \mathbb{C})$. Then $X$ and $X'$ are isomorphic.

**Proof.** Since $X$ and $X'$ are bimeromorphic, we have an isomorphism $\phi : H^2(X, \mathbb{C}) \cong H^2(X', \mathbb{C})$.
by Lemma 2.1. Since $\phi_s$ respects the Beauville-Bogomolov quadratics and the Hodge structures, $\dim H^{1,1}(X', \mathbb{Q}) = 1$ and $H^{1,1}(X', \mathbb{Q})$ is generated by a class $\gamma \in H^{1,1}(X', \mathbb{Q})$ such that $q_X(\gamma) \geq 0$, where $q_X$ is the Beauville-Bogomolov quadratic form on $H^2(X', \mathbb{C})$. Let $\mathcal{C}_X$ and $\mathcal{C}_X'$ be the positive cones in $H^{1,1}(X, \mathbb{R})$ and $H^{1,1}(X', \mathbb{R})$, respectively. By [Huy99, Corollary 7.2], $\mathcal{C}_X$ and $\mathcal{C}_X'$ coincide with the Kähler cones of $X$ and $X'$, respectively. Since $\phi_s$ maps $\mathcal{C}_X$ to $\mathcal{C}_X'$, $\phi_s$ is Kähler for all Kähler class of $H^{1,1}(X, \mathbb{R})$. By [Fuj81, Corollary 3.3], $\phi$ can be extended to an isomorphism.

Now we can state a correspondence of deformation families of pairs.

**Proposition 2.2.** Let $X$ are irreducible symplectic manifolds and $L$ a line bundle on $X$. We also let $X'$ are irreducible symplectic manifold and $L'$ a line bundle on $X'$. We denote the universal family of deformations of the pair $(X, L)$ by $(\mathcal{X}_L, \mathcal{L})$ and the parametrizing space by $\text{Def}(X, L)$. We also denote by the universal family of deformations of the pair $(X', L')$ by $(\mathcal{X}'_{L'}, \mathcal{L}')$ and the parameter space by $\text{Def}(X', L')$. Assume that there exists a birational map $\phi : X \dashrightarrow X'$ such that $\phi_* L \cong L'$ and $q_X(L) \geq 0$, where $q_X$ is the Beauville-Bogomolov quadratic form on $H^2(X, \mathbb{C})$. Then we have the followings.

1. There exist open subsets $U_L$ of $\text{Def}(X, L)$ and $U'_{L'}$ of $\text{Def}(X', L')$ such that they satisfy the following diagram

$$\mathcal{X}_L \times_{\text{Def}(X, L)} U_L \cong \mathcal{X}'_{L'} \times_{\text{Def}(X', L')} U'_{L'},$$

where $\varphi$ is the isomorphism in the diagram of the assertion (2) of Proposition 2.1. Moreover $\text{Def}(X, L) \setminus U_L$ is contained in a union of countably hypersurfaces of $\text{Def}(X, L)$ and $\text{Def}(X, L) \setminus U'_{L'}$ is also contained in a union of countably hypersurfaces of $\text{Def}(X, L)$.

2. For every point $s \in U_L$, $(\phi_s)_* \mathcal{L}_s \cong \mathcal{L}'_{s'}$, where $s = \varphi(s)$ and $\phi_s$ is the restriction of the isomorphism $\mathcal{X}_L \times_{\text{Def}(X, L)} U_L \to \mathcal{X}'_{L'} \times_{\text{Def}(X', L')} U'_{L'}$ to the fibres $\mathcal{X}_L, s$ and $\mathcal{X}'_{L'}, s'$.

**Proof.** We use the same notation in the statements and the proof of Proposition 2.1. If $U \cap \text{Def}(X, L) \neq \emptyset$, then $U_L := U \cap \text{Def}(X, L)$ and $U'_{L'} := U' \cap \text{Def}(X', L')$ satisfies the assertion (1) and every point $s \in U_L$ satisfy the assertion (2) because the restricted isomorphism satisfies the assertion (3) of Proposition 2.1. Let $s$ be a point of $\text{Def}(X, L)$ such that $\dim H^{1,1}(X_s, \mathbb{Q}) = 1$, where $X_s$ is the fibre at $s$. We will prove that $s \in U$. Since $U$ is dense and open, there exists a small disk $S$ of $\text{Def}(X)$ such that $s \in S$ and $S \setminus \{s\} \subset U$. We denote $\varphi(s)$ by $s'$ and $\varphi(S)$ by $S'$. If we consider the base changes $\mathcal{X} \to \text{Def}(X)$ by $S$ and $\mathcal{X}' \to \text{Def}(X')$ by $S'$, we obtain the following diagram:

$$\mathcal{X}_{S \setminus \{s\}} \cong \mathcal{X}'_{S' \setminus \{s'\}},$$

$$S \setminus s \cong S' \setminus \{s\}$$

By [Huy99, Theorem 4.3], there exists a birational map $\mathcal{X}_s \dashrightarrow \mathcal{X}'_{s'}$. By the definition of the Beauville-Bogomolov quadratic form [Bea83, Page 772], the function $\text{Def}(X, L) \ni s \mapsto q_X(L_s) \in \mathbb{Z}$
is constant, where $q_X$ stands for the Beauville-Bogomolov quadratic form on $H^2(X, \mathcal{C})$. Thus we have $q_X(L) = q_X(\mathcal{L}_s) \geq 0$. By Lemma 2.2, $\phi_s$ is an isomorphism. This implies that $s \in U$.

3. Proof of Theorem

We start with giving a numerical criterion of existence of Lagrangian fibrations.

**Lemma 3.1.** Let $X$ be an irreducible symplectic manifold and $L$ a line bundle on $X$. The linear system $|L|$ defines a Lagrangian fibration over the projective space if and only if $L$ is nef and $L$ has the following property:

$$\dim H^0(X, L^\otimes k) = \dim H^0(\mathbb{P}^{1/2 \dim X}, \mathcal{O}(k))$$

for every positive integer $k$.

**Proof.** If $|L|$ defines a Lagrangian fibration over the projective space, it is trivial that $L$ is nef and the dimension of global sections of $L^\otimes k$ satisfies the equation (2) by Definition 1.2. Thus we prove that $|L|$ defines a Lagrangian fibration under the assumption that $L$ is nef and $\dim H^0(X, L^\otimes k)$ satisfy the equation (3). By the assumption, the linear system $|L|$ defines a rational map $X \dashrightarrow \mathbb{P}^{1/2 \dim X}$. Let $\nu: Y \rightarrow X$ be a resolution of indeterminacy and $g: Y \rightarrow \mathbb{P}^{1/2 \dim X}$ is the induced morphism. Comparing $\nu^* L$ and $g^* \mathcal{O}(1)$, we have

$$\nu^* L \cong g^* \mathcal{O}(1) + F,$$

where $F$ is a $\nu$-exceptional divisor. By multiplying the both hand sides, we have

$$k \nu^* L \cong g^* \mathcal{O}(k) + kF.$$

If $F \neq 0$, then the above isomorphism and the equality (2) implies that $L$ is not semiample. By the assumption, $L$ is nef. If $L^{1/2 \dim X} \neq 0$, then $L$ is also big and $\dim H^0(X, L^\otimes k)$ does not satisfy the equation (2). Thus $L^{1/2 \dim X} = 0$. By [Fuj87, Theorem 4.7], we obtain

$$e_X g(kL + \alpha)^{1/2 \dim X} = (kL + \alpha)^{\dim X},$$

where $e_X$ is the Beauville-Bogomolov quadratic form on $H^2(X, \mathcal{C})$, $e_X$ is the positive constant of $X$ and $\alpha$ is a Kähler class of $H^{1,1}(X, \mathcal{C})$. Comparing the degrees of both hand sides of the above equation, we obtain that the numerical Kodaira dimension $\nu(L)$ is $(1/2) \dim X$. By the equation (2), the Kodaira dimension $\kappa(L)$ is also equal to $(1/2) \dim X$. Since $K_X$ is trivial, the equality $\nu(L) = \kappa(L)$ implies that $L$ is semiample by [Kaw85, Theorem 6.1] and [Fuj11, Theorem 1.1]. Thus $F = 0$ and the linear system $|L|$ defines the morphism $f: X \rightarrow \mathbb{P}^{1/2 \dim X}$. The linear system $|L|$ defines a morphism

$$f_l: X \rightarrow \text{Proj} \oplus_{m \geq 0} H^0(X, L^\otimes ml) \cong \mathbb{P}^{n+l \choose n} - 1.$$

This morphism has connected fibres if $l$ is sufficiently large. By the above expression, $f_l$ is the composition of $f$ and the Veronese embedding. This implies that $f$ has connected fibres.

We introduce a criterion which asserts locally freeness of direct images of line bundles.

**Lemma 3.2.** Let $\pi: \mathcal{X}_S \rightarrow S$ be a smooth morphism over a small disk $S$ with the reference point $o$. We also let $\mathcal{L}_S$ be a line bundle on $\mathcal{X}_S$. Assume that $\mathcal{X}_S$ and $\mathcal{L}_S$ satisfy the following conditions.

1. The canonical bundle of every fibre is trivial.
For every point $t$ of $S \setminus \{o\}$, the restriction $\mathcal{L}_{S,t}$ of $\mathcal{L}$ to the fibre $\mathfrak{X}_{S,t}$ at $t$ is semiample.

(3) The restriction $\mathcal{L}_{S,o}$ of $\mathcal{L}$ to the fibre $\mathfrak{X}_{S,o}$ at $o$ is nef. Then the higher direct images $R^q\pi_*\mathcal{L}_{S,o}^\otimes k$ are locally free for all $q \geq 0$ and $k \geq 1$. Moreover the morphisms

$$R^q\pi_*\mathcal{L}_{S,o}^\otimes k \otimes k(o) \rightarrow H^q(\mathfrak{X}_{S,o}, \mathcal{L}_{S,o})$$

are isomorphic for all $q \geq 0$ and $k \geq 1$.

Proof. The first part is a special case of [Nak87] Corollary 3.14. By the criteria of cohomological flatness in [BS76] page 134, if $R^q\pi_*\mathcal{L}_{S,o}^\otimes k$ is locally free and the morphism $\mathcal{L}_{S,o}$ is not nef, then the morphism

$$R^{q-1}\pi_*\mathcal{L}_{S,o}^\otimes k \otimes k(o) \rightarrow H^{q-1}(\mathfrak{X}_{S,o}, \mathcal{L}_{S,o})$$

is also isomorphic for every $k \geq 1$. If $q \geq \dim \mathfrak{X}_{S,o} + 1$, the both hand sides of the morphism $\mathcal{L}_{S,o}$ are zero. By a reverse induction, we obtain the last part of the assertions of Lemma. □

We need one more lemma to prove Theorem 1.2.

Lemma 3.3. Let $X$ be an irreducible symplectic manifold. Assume that $X$ is not projective. Then a line bundle $L$ such that $q_X(L) = 0$ is nef, where $q_X$ is the Beauville-Bogomolov quadratic form on $H^2(X, \mathbb{C})$.

Proof. Assume that $L$ is not nef. By [Huy99] Theorem 7.1, there exists a line bundle $M$ on $X$ such that $q_X(M, L) < 0$ and $q_X(M, \alpha) \geq 0$ for all Kähler class $\alpha \in H^2(X, \mathbb{C})$. If we choose a suitable rational number $\lambda$, we have $q_X(L + \lambda M) > 0$. This implies that $X$ is projective by [Huy03a] Theorem 2. That is a contradiction. □

Now we prove that if $V_{reg} \neq \emptyset$ then $V_{reg}$ is a dense open subset of $\text{Def}(X, L)$.

Lemma 3.4. We use the same notation as in Theorem 1.2. If $V_{reg} \neq \emptyset$, $V_{reg}$ is dense and open in $\text{Def}(X, L)$. Moreover $\text{Def}(X, L) \setminus V_{reg}$ is contained in a countably union of hypersurfaces of $\text{Def}(X, L)$.

Proof. Let $t$ be a point of $V_{reg}$ and we denote by $\mathfrak{X}_t$ the fibre at $t$ and by $\mathcal{L}_t$ the restriction of $\mathcal{L}$ to $\mathfrak{X}_t$. First we prove that $V_{reg}$ is open. By the definition of $V_{reg}$ in Theorem 1.2, the linear system $|\mathcal{L}_t|$ defines a Lagrangian fibration $f_t : \mathfrak{X}_t \rightarrow \mathbb{P}^{1/2 \dim \mathfrak{X}_t}$. Let us consider the Relay spectral sequence

$$E_2^{p,q} = H^p(\mathbb{P}^{1/2 \dim \mathfrak{X}_t}, R^q(f_t)_*\mathcal{O}(1)) \Rightarrow E^{p+q} = H^{p+q}(\mathfrak{X}_t, (f_t)_*\mathcal{O}(1)).$$

The edge sequence of the above spectral sequence is

$$0 \rightarrow H^1(\mathbb{P}^{1/2 \dim \mathfrak{X}_t}, \mathcal{O}(1)) \rightarrow H^1(\mathfrak{X}_t, f_t^*\mathcal{O}(1)) \rightarrow H^0(\mathbb{P}^{1/2 \dim \mathfrak{X}_t}, R^1(f_t)_*\mathcal{O}_{\mathfrak{X}_t} \otimes \mathcal{O}(1)).$$

By [Mat05] Theorem 1.2,

$$R^1(f_t)_*\mathcal{O}_{\mathfrak{X}_t} \cong \mathcal{O}_{\mathbb{P}^{1/2 \dim \mathfrak{X}_t}}.$$ 

Since $H^1(\mathbb{P}^{1/2 \dim \mathfrak{X}_t}, \mathcal{O}_{\mathbb{P}^{1/2 \dim \mathfrak{X}_t}}(1)) = 0$, we have $H^1(\mathfrak{X}_t, f_t^*\mathcal{O}(1)) = H^1(\mathfrak{X}_t, \mathcal{L}_t) = 0$. By [BS76] Corollary III. 3.9], $\pi_*\mathcal{L}$ is locally free in an open neighbourhood of $t$ and the morphism

$$\pi_*\mathcal{L} \otimes k(t) \rightarrow H^0(\mathfrak{X}_t, \mathcal{L}_t)$$

is surjective over an open neighbourhood of $t$. This implies that $V_{reg}$ is open.

Next we prove that $\text{Def}(X, L) \setminus V_{reg}$ is contained in a union of countably hypersurfaces of $\text{Def}(X, L)$. Since a union of real codimension two subsets cannot
separate two non-empty open subsets, this implies that $V_{\text{reg}}$ is dense. Let $t'$ be a point of the closure of $V_{\text{reg}}$ such that $\dim H^{1,1}(X_{t' \tau}, \mathbb{Q}) = 1$, where $X_{t' \tau}$ is the fibre at $t'$. We denote by $\Sigma_{t'}$ the restriction of $\Sigma$ to $X_{t' \tau}$. By the definition of the Beauville-Bogomolov quadratic in [Bea83, page 772], the function

$$\text{Def}(X, L) \ni t \mapsto q_{X_t}(\Sigma_t) \in \mathbb{Z}$$

is a constant function, where $q_{X_t}$ is the Beauville-Bogomolov quadratic form on $H^2(X_t, \mathbb{C})$. Thus $q_{X_{t'}}(\Sigma_{t'}) = 0$. Since $H^{1,1}(X_{t' \tau}, \mathbb{Q})$ is spanned by $\Sigma_{t'}$, $X_{t' \tau}$ is not projective by [Huy03a] Theorem 2. Thus $\Sigma_{t'}$ is nef by Lemma 3.3. We choose a small disk $S$ in $\text{Def}(X, L)$ such that $t' \in S$ and $S \setminus \{t'\} \subset V_{\text{reg}}$. We also consider the restriction family $\pi_S : \mathcal{X}_L \times_{\text{Def}(X, L)} S \to S$. Then $\Sigma_{t'}$ is free for every point $t''$ of $S \setminus \{t'\}$ and $k \geq 1$, where $\Sigma_{t''}$ is the restriction of $\Sigma$ to the fibre $X_{t''}$ at $t''$. By Lemma 3.2 $(\pi_S)_* \mathcal{L}^\otimes k \otimes \mathcal{O}(t'') \to H^0(\Sigma_{t''}, \mathcal{L}^\otimes k)$ is bijective for every $k \geq 1$. By Lemma 3.1 $t' \in V_{\text{reg}}$. Let $W$ be the subset of $\text{Def}(X, L)$ defined by

$$W := \{t \in \text{Def}(X, L); \dim H^{1,1}(X_t, \mathbb{Q}) \geq 2\}. \quad \Box$$

By the above argument, $\text{Def}(X, L) \setminus V_{\text{reg}} \subset W$. By [Huy99] (1.14), $W$ is contained in a union of countably hypersurfaces of $\text{Def}(X, L)$ and we are done.

We give a proof of Theorem 1.2

**Proof of Theorem 1.2** The proof consists of three parts. We start with proving the following Claim.

**Claim 3.1.** If $V \neq \emptyset$, then $V_{\text{reg}} \neq \emptyset$.

**Proof.** We may assume that the reference point $o$ of $\text{Def}(X, L)$ is contained in $V$. By Definition 1.2 there exists a birational map $\phi : X' \dashrightarrow X'$ such that the linear system $|\phi_* L|$ defines a Lagrangian fibration $X' \to \mathbb{P}^{1/2 \dim X}$. Let $L' := \phi_* L$ and $(X'_{t'}, \mathcal{L}')$ be the universal family of deformations of the pair $(X', L')$. Let $V'_{\text{reg}}$ be the locus of Lagrangian fibration of $\text{Def}(X', L')$. Then the reference point $o'$ of $\text{Def}(X', L')$ is contained in $V'_{\text{reg}}$. By Lemma 3.3 $V'_{\text{reg}}$ is a dense open set of $\text{Def}(X', L')$. By Proposition 2.2 we also have dense open sets $U'_L$ of $\text{Def}(X', L')$ and $U_L$ of $\text{Def}(X, L)$ which satisfy the following diagram:

$$\begin{array}{ccc}
\mathcal{X}_L \times_{\text{Def}(X, L)} U_L & \xrightarrow{\approx} & \mathcal{X}'_{L'} \times_{\text{Def}(X', L')} U'_L \\
U_L & \xrightarrow{\approx} & U'_L
\end{array}$$

By the assertion (2) of Proposition 2.2 $\varphi^{-1}(U'_L \cap V'_{\text{reg}}) \subset V_{\text{reg}}$. Since $U'_L \cap V'_{\text{reg}} \neq \emptyset$, we obtain $V_{\text{reg}} \neq \emptyset$. \Box

By Claim 3.1 and Lemma 3.3 $\text{Def}(X, L)$ coincides with the closure of $V_{\text{reg}}$ under the assumption that $V \neq \emptyset$.

**Claim 3.2.** Assume that the reference point $o$ of $\text{Def}(X, L)$ is contained in the closure of $V_{\text{reg}}$ and $L$ is nef. Then $o \in V_{\text{reg}}$.

**Proof.** By the assumption that $o \in V_{\text{reg}}$, we choose a small disk $S$ in $\text{Def}(X, L)$ which has the following properties:

1. $o \in S$.
2. $S \setminus \{o\} \subset V_{\text{reg}}$.
Let \( \pi_S : X_L \times \text{Def}(X, L) S \rightarrow S \) be the restriction family and \( \mathcal{L}_S \) the restriction of the universal bundle \( \mathcal{L} \) to \( X_L \times \text{Def}(X, L) S \). Then \( \pi_S : X_L \times \text{Def}(X, L) S \rightarrow S \) and \( \mathcal{L}_S \) satisfy all assumptions of Lemma 3.2. Hence \( \pi, \mathcal{L}_S^\otimes k \) are locally free and the morphisms
\[
\pi_* \mathcal{L}_S^\otimes k \otimes k(s) \rightarrow H^0(\mathcal{X}_s, \mathcal{L}_s^\otimes k)
\]
are isomorphic for all \( k \geq 0 \) and all points \( s \in S \). This implies that the pair \( (X, L) \) satisfies the all assumptions of Lemma 3.1 and we obtain \( o \in V_{\text{reg}} \).

\[\text{Claim 3.3.} \quad \text{Assume that the reference point} \ o \ \text{of Def}(X, L) \ \text{is contained in the} \ \text{closure of} \ V_{\text{reg}} \ \text{and} \ c_1(L) \ \text{belongs to the birational Kähler cone. Then} \ o \ \in V. \]

\[\text{Proof.} \quad \text{We remark that} \ X \ \text{is projective by Lemma 3.3.} \ \text{We consider the same restriction family} \ \pi : X_L \times \text{Def}(X, L) S \rightarrow S \ \text{in the proof of Claim 3.2.} \ \text{By the upper semicontinuity of the function}
\]
\[s \in S \mapsto \dim H^0(\mathcal{X}_s, \mathcal{L}_s),\]
\[\mathcal{L}_s = L \ \text{is effective. By} \ [MZ09, \text{Theorem 1.2}], \ \text{there exists a birational map} \ \phi : X' \dashrightarrow X' \ \text{such that} \ L' = \phi_\ast L. \ \text{By Proposition 3.2.4, we have dense open sets} \ U'_L \ \text{of Def}(X', L') \ \text{and} \ U_L \ \text{of Def}(X, L) \ \text{which satisfy the following diagram:}
\]
\[
\begin{array}{ccc}
X_L \times \text{Def}(X, L) U_L & \overset{\sim}{\longrightarrow} & X'_L \times \text{Def}(X', L') U_{L'} \\
\downarrow & & \downarrow \\
\mathcal{U} & \overset{\varphi}{\cong} & \mathcal{U}_{L'}
\end{array}
\]

Let \( V'_{\text{reg}} \) be the locus of Lagrangian fibrations of \( \text{Def}(X', L') \). Then \( V'_{\text{reg}} \neq \emptyset \) because the image \( \varphi(V_{\text{reg}} \cap U_L) \) is contained in \( V'_{\text{reg}} \) by Proposition 3.2.2 (2). By Lemma 3.3.4, \( V'_{\text{reg}} \) is dense. Hence the reference point \( o' \) of \( \text{Def}(X', L') \) is contained in the closure of \( V'_{\text{reg}} \). By Claim 3.2.2, \( o' \in V'_{\text{reg}} \). This implies that \( o \in V \).

We finish the proof of Theorem 1.2. If \( V \neq \emptyset \), \( V_{\text{reg}} \) is open and dense in \( \text{Def}(X, L) \) by Claim 3.3.1. Thus every point \( s \) of \( \text{Def}(X, L)_{\text{mov}} \) is contained in the closure of \( V_{\text{reg}} \). Then \( s \in V \) by Claim 3.3 and we are done.

\[\text{Proof of Corollary 1.1.} \ \text{We use the same notation of Theorem 1.2 and Corollary 1.1.} \ \text{We also define the subset} \ \Lambda \ \text{of Def}(X) \ \text{by}
\]
\[\Lambda := \{ s \in \text{Def}(X); \ \mathcal{X}_s \ \text{is isomorphic to the} \ n\text{-pointed Hilbert scheme of} \ K3 \ \text{or a generalized Kummer variety} \}
\]
If \( \Lambda \cap \text{Def}(X, L)_{\text{mov}} \neq \emptyset \), then the restriction of the universal bundle \( \mathcal{L}_s \) to the fibre \( \mathcal{X}_s \) at \( s \in \Lambda \cap \text{Def}(X, L)_{\text{mov}} \) defines a rational Lagrangian fibration by [BM13, Conjecture 1.4, Theorem 1.5], [Mar13, Theorem 1.3] and [Yos12, Proposition 3.36].

First we prove that \( \Lambda \cap \text{Def}(X, L) \) is dense in \( \text{Def}(X, L) \). The subset \( \Lambda \) is dense in \( \text{Def}(X) \) by [MM12, Theorem 1.1, Theorem 4.1]. Moreover, by [Bea83, Théorème 6, 7], each irreducible component of \( \Lambda \) forms a smooth hypersurface of \( \text{Def}(X) \). Therefore \( \Lambda \cap \text{Def}(X, L) \) is dense in \( \text{Def}(X, L) \). Next we will prove the following Lemma.

\[\text{Lemma 3.5.} \ \text{Under the same assumptions and notation of Theorem 1.2, the closure} \ \text{of} \ \text{Def}(X, L) \ \text{\setminus} \ \text{Def}(X, L)_{\text{mov}} \ \text{is a proper closed subset of} \ \text{Def}(X, L). \]

\[\text{Proof.} \ \text{We derive a contradiction assuming that the closure of} \ \text{Def}(X, L) \ \text{\setminus} \ \text{Def}(X, L)_{\text{mov}} \ \text{coincides with} \ \text{Def}(X, L). \ \text{For a point} \ s \ \text{of} \ \text{Def}(X, L) \ \text{\setminus} \ \text{Def}(X, L)_{\text{mov}}, \ \text{we denote by} \ \mathcal{L}_s \ \text{the restriction of the universal bundle} \ \mathcal{L} \ \text{to the fibre} \ \mathcal{X}_s \ \text{at} \ s. \ \text{We will prove that} \ \mathcal{L}_s \ \text{is big. By Corollary [Huy99, Corollary 3.10], the interior of the positive} \]

cone of an irreducible symplectic manifold is contained in the effective cone. By the assumption, $L$ belongs to the closure of the positive cone of $X$. Hence $\mathcal{L}_s$ also belongs to the closure of the positive cone of $X_s$. Thus $\mathcal{L}_s$ is pseudo-effective. By [MZ09] Theorem 3.1, we obtain the $q$-Zariski decomposition

$$\mathcal{L}_s = P_s + N_s$$

By [MZ09] Theorem 3.1 (I) (iii)]

$$0 = q_X(\mathcal{L}_s) = q_X(P_s + N_s) = q_X(P_s) + q_X(N_s).$$

Since $\mathcal{L}_s$ does not belong to the birational Kähler cone of $X_s$, $N \neq 0$. This implies that $q_X(P_s) > 0$. We deduce $P_s$ is big by [Huy99] Corollary 3.10. Hence $\mathcal{L}_s$ is also big.

Let us consider the following function

$$\text{Def}(X, L) \ni s \mapsto h_n(s) := \dim H^0(X_s, \mathcal{L}_s^n) \in \mathbb{Z}.$$ 

By the upper semicontinuity of $h_n(s)$, there exists an open set $W$ of $\text{Def}(X, L)$ such that for every point $s$ of $W,$

$$h_n(t) \geq h_n(s)$$

for all points $t \in \text{Def}(X, L).$ By the assumption that the closure of $\text{Def}(X, L) \setminus \text{Def}(X, L)_{\text{mov}}$ coincides with $\text{Def}(X, L),$ $W \cap (\text{Def}(X, L) \setminus \text{Def}(X, L)_{\text{mov}}) \neq \emptyset.$ In the first half of the proof of this Lemma, we had proved that $\mathcal{L}_s$ is big for every point $s \in \text{Def}(X, L) \setminus \text{Def}(X, L)_{\text{mov}}.$ This implies that $\mathcal{L}_t$ is big for every point $\text{Def}(X, L).$ Let $t$ be a point of $\text{Def}(X, L)$ such that $\dim H^1(X, \mathbb{Q}) = 1.$ Then $\mathcal{L}_t$ is nef by Lemma [5.3]. Since $\mathcal{L}_t$ is nef and big, the higher cohomologies of $\mathcal{L}_t$ vanish. By the Riemann-Roch formula in [Huy99] (1.11), we obtain

$$\dim H^0(X_t, \mathcal{L}_t^m) = \frac{\dim X_t}{2} \sum_{j=0}^{\dim X_t/2} m^j \chi(\mathcal{L}_t^j) = \chi(O_{X_t}),$$

because $q_X(\mathcal{L}_t) = q_X(L) = 0.$ That is a contradiction. □

We finish the proof of Corollary [1.2]. If $\Lambda \cap \text{Def}(X, L)_{\text{mov}} = \emptyset,$ $\text{Def}(X, L) \setminus \text{Def}(X, L)_{\text{mov}}$ contains dense subsets of $\text{Def}(X, L).$ This contradicts Lemma [5.3]. □

References

[AC08] Ekaterina Amerik and Frédéric Campana. Fibrations méromorphes sur certaines variétés à fibré canonique trivial. Pure Appl. Math. Q., 4(2, part 1):509–545, 2008.

[Bea83] Arnaud Beauville. Variétés Kähleriennes dont la première classe de Chern est nulle. J. Differential Geom., 18(4):755–782 (1984), 1983.

[Bea85] Arnaud Beauville. Variétés Kähleriennes compactes avec $\chi(X, \mathcal{O}_X) = 0$. Differential Geom. J., 28(1-2):49–55, 1987.

[BM13] Arend Bayer and Emanuele Macri. MMP for moduli of sheaves on $K3$s via wall-crossing: nef and movable cones, lagrangian fibrations, 2013, arXiv:1301.6968.

[COP10] Frédéric Campana, Keiji Oguiso, and Thomas Peternell. Non-algebraic hyperkähler manifolds. J. Differential Geom., 85(3):397–424, 2010.

[Fuj81] Akira Fujiki. A theorem on bimeromorphic maps of Kähler manifolds and its applications. Publ. Res. Inst. Math. Sci., 17(2):735–754, 1981.

[Fuj87] Akira Fujiki. On the de Rham cohomology group of a compact Kähler symplectic manifold. In Algebraic geometry, Sendai, 1985, volume 10 of Adv. Stud. Pure Math., pages 105–165, North-Holland, Amsterdam, 1987.

[Fuj11] Osamu Fujino. On Kawamata’s theorem. In Classification of algebraic varieties, EMS Ser. Congr. Rep., pages 305–315. Eur. Math. Soc., Zürich, 2011.

[Huy99] Daniel Huybrechts. Compact hyper-Kähler manifolds: basic results. Invent. Math., 135(1):63–113, 1999.
ON ISOTROPIC DIVISORS ON IRREDUCIBLE SYMPLECTIC MANIFOLDS

[Huy03a] Daniel Huybrechts. Erratum: “Compact hyper-Kähler manifolds: basic results” [Invent. Math. 135 (1999), no. 1, 63–113; MR1664696 (2000a:32039)]. Invent. Math., 152(1):209–212, 2003.

[Huy03b] Daniel Huybrechts. The Kähler cone of a compact hyperkähler manifold. Math. Ann., 326(3):499–513, 2003.

[Hwa08] Jun-Muk Hwang. Base manifolds for fibrations of projective irreducible symplectic manifolds. Invent. Math., 152(1):209–212, 2003.

[Kaw85] Y. Kawamata. Pluricanonical systems on minimal algebraic varieties. Invent. Math., 79(3):625–644, 1985.

[KV12] Ljudmila Kamenova and Misha Verbitsky. Families of lagrangian fibrations on hyperkähler manifolds, 2012, arXiv:1206.4838.

[Mar13] Eyal Markman. Lagrangian fibrations of holomorphic-symplectic varieties of \(k3[n]\)-type, 2013, arXiv:1301.6584.

[Mat99] Daisuke Matsushita. On fibre space structures of a projective irreducible symplectic manifold. Topology, 38(1):79–83, 1999.

[Mat01] Daisuke Matsushita. Addendum: “On fibre space structures of a projective irreducible symplectic manifold” [Topology 38 (1999), no. 1, 79–83; MR1644091 (99f:14054)]. Topology, 40(2):431–432, 2001.

[Mat05] Daisuke Matsushita. Higher direct images of dualizing sheaves of Lagrangian fibrations. Amer. J. Math., 127(2):243–259, 2005.

[Mat08] Daisuke Matsushita. On nef reductions of projective irreducible symplectic manifolds. Math. Z., 258(2):267–270, 2008.

[MM12] E. Markman and S. Mehrotra. Hilbert schemes of K3 surfaces are dense in moduli. ArXiv e-prints, December 2012, 1201.0031.

[MZ09] Daisuke Matsushita and De-Qi Zhang. Zariski f-decomposition and lagrangian fibration on hyperkähler manifolds. Mathematical Research Letters 20 (2013) 951 - 959, 2009, arXiv:0907.5311.

[Nak87] Noboru Nakayama. The lower semicontinuity of the plurigenera of complex varieties. In Algebraic geometry, Sendai, 1985, volume 10 of Adv. Stud. Pure Math., pages 551–590. North-Holland, Amsterdam, 1987.

[Ver10] Misha Verbitsky. HyperKähler SYZ conjecture and semipositive line bundles. Geom. Funct. Anal., 19(5):1481–1493, 2010.

[Yos12] Kota Yoshioka. Bridgeland’s stability and the positive cone of the moduli spaces of stable objects on an abelian surface, 2012, arXiv:1206.4838.