On the rotation of ONC stars in the Tsallis formalism context

B. B. Soares(a) and J. R. P. Silva

Department de Física, Universidade do Estado do Rio Grande do Norte - Mossoró, Brazil

received 4 July 2011; accepted in final form 10 August 2011
published online 12 September 2011

PACS 97.10.Kc – Stellar rotation
PACS 97.10.Yp – Star counts, distribution, and statistics
PACS 98.20.-d – Stellar clusters and associations

Abstract – The theoretical distribution function of projected stellar rotational velocities is derived in the context of the Tsallis formalism. The distribution is used to estimate the average $\langle \sin i \rangle$ for a stellar sample from the Orion Nebula Cloud (ONC), producing an excellent result when compared with observational data. In addition, the value of the parameter $q$ obtained from the distribution of the projected rotational velocities reinforces the idea that there is a relation between this parameter and the age of the cluster.

Copyright © EPLA, 2011

Introduction. – Stellar rotation is one of the most challenging problems in astrophysics because it is a complex physical phenomenon. Stellar rotation has its origin in stellar formation, when the angular momentum of the progenitor cloud is transferred to the new stars. At this stage, the stars can suffer the influence of several physical processes, which can also affect the stellar rotation. Such processes include proper orbital motion of the cloud, innate differences in internal pressure due to the anisotropy of the cloud and the action of the local galactic magnetic field on the parent cloud, as well as shock waves from supernova explosions. These processes make the initial range of rotation rates difficult to understand, but when evolution of rotation rates is considered along with stellar evolution, processes such as magnetic braking and structural changes within the star (e.g., development of a radiative core) are also important.

The projected rotational velocity, $V \sin i$, where $V$ is the true (equatorial) rotational velocity and $i$ is the angle between the axis of rotation and the line of sight, can be derived from the Doppler broadening of the spectral lines of the star [1–3]. However, it is usually difficult to know the inclination of the rotation axis of a single star, being possible only in special cases, e.g., when the rotational period can be measured (spotted stars). Another special case is in binary systems with short orbital periods in which the orbital motion is synchronized with the rotation and the rotational and orbital axes are parallel [4].

Even given the difficulty in determining the equatorial velocity, one can derive some global properties of the true rotation from the frequency distribution of $V \sin i$. However, this requires some assumptions about the frequency of a given angle of inclination of the rotational axis for the group of stars. In general it is assumed that the axes of rotation of the stars are randomly oriented (e.g., [5–8]). In this context, the relation between the observed and true rotational velocity for a sample of stars is always $\langle V \sin i \rangle / \langle V \rangle = \pi / 4$ [9], independently of the environment of stellar formation.

The hypothesis of randomly oriented axes was examined in several works and, although it has not been ruled out, is not always in agreement with the observed velocity distributions (see discussion in [10], sect. 3.1.2; [11], sect. VI). In addition, later studies showed a tendency for stars of a given spectral type to present rotation correlated with the galactic coordinates as well as with the galactocentric distance (e.g., [12,13]). These studies open up the possibility of considering a non-random distribution for the inclination of the rotational axes, even if only in special cases. In fact, several cases in which a power law fits better than a Gaussian function (e.g., [14–19]) show that the main issue may not just be deciding which mathematical function can be used but what statistical theory one should consider.

It is now well established that a considerable number of stochastic phenomena are better described by power-law distributions. For example, one could mention the solar neutrino problem [20], peculiar velocities of galaxies [21] and self-gravitating polytropic systems [22]. Fundamentally all the systems whose entropies are more appropriately addressed to $q$-entropy [23], postulated as

$$S_q \equiv \frac{k}{q-1} \left(1 - \sum_{i=1}^{W} \rho_i^q \right),$$

(1)

(a) E-mail: brauliosoares@uern.br
where \( p_i \) denotes the probabilities of the microscopic (individual) configurations, could be included in the list of stochastic phenomena. In the limit \( q = 1 \), the \( q \)-entropy recovers the standard Boltzmann form. That statistic is therefore a generalization of Boltzmann statistics. Usually named nonextensive statistics, Tsallis thermostatistics states that the entropy of the sum of two systems is not simply the sum of individual entropies, but provides for cases where such a sum may result in a greater or lesser entropy.

The main point of the considerations presented so far is that, as well as the exponential distribution considered in study as Chandrasekhar and Münch [9] and Deutsch [24], we are often faced with stable power-law distributions (the \( q \)-exponentials in eq. (4)), since such distributions are ubiquitous in physical phenomena. The ubiquity is due to the fact that such distributions naturally obey the generalized central limit theorem (cf. [25]). Thus, Gaussian-type diffusion and anomalous diffusion, for example, may be unified into a single scenario which is the nonextensive statistics framework [26–29].

In this paper we present a new way to derive the distribution function of the projected rotational velocities in the context of nonextensive statistics. The distribution function is tested with a sample of stellar rotation from the ONC and accurately reproduces the observational results. The paper is organized as follows. In the second section we derive the power-law distribution proposed for this study and the general equations of its moments. In the third section we present the work sample and the method used to determine the average \( \langle \sin i \rangle \) from the observational sample, and discuss the results. Finally the last section summarizes the main results.

The power-law distribution function. – The rotational kinetic energy of a star with mass \( M \) and radius \( R \) rotating as a rigid body around its own axis is \( E_{\text{rot}} = I\omega^2 \), where \( I = 0.5MR^2 \) is the moment of inertia and \( \omega = VR^{-1} \) is the angular velocity of the rotating star. Defining \( \omega_{\text{min}} = (V \sin i)R^{-1} \), it follows that the minimal rotational energy \( E_{\text{min}} \) of the star is given by

\[
E_{\text{min}} = I\omega_{\text{min}}^2. \tag{2}
\]

Consider a group of stars with similar mass, radius and age, such that the rotational energy \( E_{\text{min}} \) depends only on \( V \sin i \). Suppose further that the energy \( E_{\text{min}} \) is distributed according to the function \( \varphi(E_{\text{min}}) \). The number of stars with energy between \( E_{\text{min}} \) and \( E_{\text{min}} + dE_{\text{min}} \) is

\[
dN(E_{\text{min}}) = p(E_{\text{min}}) dE_{\text{min}}, \tag{3}
\]

where \( p(E_{\text{min}}) \) is the probability density of stars with rotational energy \( E_{\text{min}} \).

It is well known from observational studies, that the probability density of observed rotational velocities vanishes for large \( V \sin i \). Thus it is reasonable to take the distribution function proposed by Deutsch [30] as an inspiration and propose that the probability density decrease according to a generalized exponential law. We are interested in deriving the distribution of the rotational velocities in the context of the Tsallis formalism. We therefore propose that the probability density is governed by the nonlinear differential equation, \( dp/dx = -\beta p^q \), whose solution is given by

\[
p(x) = [1 - (1 - q)\beta x]^{1/(1-q)} \tag{4}
\]

This function is the \( q \)-deformation of the usual exponential function \( \exp(-\beta x) \) and reduces to it in the limit \( q \to 1 \) (cf. [31]). Accordingly, we can rewrite eq. (3) as

\[
dN(E_{\text{min}}) = [1 - (1 - q)\beta E_{\text{min}}]^{1/(1-q)} dE_{\text{min}}, \tag{5}
\]

from which it follows that

\[
dN(y) = \frac{2}{5} M R y \left[ 1 - (1 - q) \frac{y^2}{\sigma_y^2} \right]^{1/(1-q)} dy, \tag{6}
\]

where \( y = V \sin i \) and \( \sigma_y = (\beta M/5)^{-1/2} \) is the half-width of the distribution.

Equation (6) gives the number of stars with a given observed rotational velocity \( V \sin i \) from a sample of stars with similar mass, radius and age. It follows, in the formalism proposed by Tsallis, that the distribution function of the observed rotation for that group of stars is given, apart from a normalization constant, by

\[
\varphi_q(y) = y \left[ 1 - (1 - q) \frac{y^2}{\sigma_y^2} \right]^{1/(1-q)}. \tag{7}
\]

The above equation has also been derived in a different way by Soares et al. [14].

In accordance with Kraft [32], we can write the distribution function of true rotational velocities, apart from a normalization constant, as

\[
F_q(V) = V^2 \left[ 1 - (1 - q) \frac{V^2}{\sigma_V^2} \right]^{1/(1-q)}, \tag{8}
\]

where \( \sigma_V \) is the half-width of the distribution. Since the function (4) recovers the usual exponential in the limit \( q \to 1 \), eqs. (7) and (8) recover the standard form of \( \varphi(y) \) and \( F(V) \) [24,32], respectively, when \( q \to 1 \).

Moments of the distribution. The \( r \)-th moment of the distribution \( F_q(V) \) is calculated by making

\[
\langle V^r \rangle = \int_0^{V_{\text{max}}} V^r F_q(V) \, dV, \tag{9}
\]

where \( V_{\text{max}} = \sigma_V / \sqrt{1-q} \) for \( q < 1 \), or \( V_{\text{max}} = \infty \) for \( q > 1 \), in order to ensure the positivity of the function. Hence eq. (9) gives

\[
\langle V^r \rangle = \frac{2}{\sqrt{\pi}} \left( \frac{\sigma_V}{\sqrt{1-q}} \right)^r \frac{\Gamma \left( \frac{r+3}{2} \right) \Gamma \left( \frac{2-q}{2} \right)}{\Gamma \left( \frac{2-q}{2} + \frac{r+3}{2} \right)} \tag{10}
\]
Rotation of ONC stars in Tsallis formalism

for \( q < 1 \), and

\[
\langle V' \rangle = \frac{2}{\sqrt{\pi}} \left( \frac{\sigma_V}{\sqrt{1-q}} \right)^r \frac{\Gamma \left( \frac{r+3}{2} \right) \Gamma \left( \frac{1}{q-1} - \frac{r+3}{2} \right)}{\Gamma \left( \frac{5-3q}{2q-2} \right)} (11)
\]

for \( q \geq 1 \). The moments of order \( r = 1, r = 2 \) and \( r = 3 \) in eqs. (10) and (11) give the mean, mean-square deviation, and asymmetry of \( F_q(V) \) and \( \varphi_q(y) \). It is important to notice that in the limit \( q \to 1 \) all \( q \)-dependent terms in eq. (15) tend to \( \Gamma(\frac{1}{q-1}) \) and therefore

\[
\lim_{q \to 1} \left( \frac{y'}{V'} \right) = \frac{\sqrt{\pi}}{2} \frac{\Gamma \left( \frac{5}{2} + 1 \right)}{\Gamma \left( \frac{5}{2} + \frac{3}{2} \right)}, (16)
\]

where we have assumed \( \sigma_y = \sigma_V \). Equation (16) is precisely the relation between moments found by Chandrasekhar and Münch \([9], \text{eq. 18} \). The proposed general power-law function (7) can describe distributions of projected rotational velocities for stars from different populations, depending on the value of the parameter \( q \). As we will discuss later, the parameter \( q \) constitutes a link between the theoretical distribution and the observational one.

**Confronting model with observational data.**

The sample. Our sample was selected from the data provided by Rhode et al. \([10] \) to match the following criteria: projected rotational velocity, \( \sin i \), and true rotational velocity, \( V \), greater than \( \sim 11 \text{ km/s} \), in addition to \( \sin i \leq 1 \). Applying these criteria we obtained a sample of 86 stars whose \( \sin i \) parameters are accurately determined. Stellar radii are estimated from the luminosities, \( L \), effective temperatures, \( T_\text{eff} \) displayed in \([10] \) (table 1), according to the relation \( R \sim L^{1/2} T^{-2} \). The stellar parameters \( L \) and \( T_\text{eff} \) are uncertain by about 0.2 and 0.02 dex, respectively (see Hillenbrand \([33] \)). Adding these uncertainties in quadrature, we have determined an error \( \lesssim 0.11 \text{ dex for } R \). As the absolute error in the logarithm is the relative error in the argument, we have a relative error in \( R \) less than 11%. The masses and spectral types for stars in our sample can be found in \([33] \).

The main features of the stars in our sample are: i) stellar age less than 1–2 Myr; ii) spectral types ranging from G6 to M5, with 89% of the stars ranging from K0 to M4; iii) stellar masses covering a range between 0.1 \( M_\odot \) and 2.7 \( M_\odot \), with median 0.3 \( M_\odot \); and iv) stellar radius, \( R \), ranging from about 1.2 \( R_\odot \) to 8.2 \( R_\odot \), with a median of 2.4 \( R_\odot \). Thus our sample consists of stars whose main parameters are very similar, as required for this study.

The projected rotational velocities were obtained from the high-dispersion spectra of stars by using the Fourier cross-correlation method. The smallest \( V \sin i \) value that is taken to be reliably measured is 11.0 km/s. For a complete discussion on the observational procedure and error analysis of \( V \sin i \) data, the reader is referred to \([10] \). The rotational velocities have been calculated from measurements of the stellar rotational period, \( P \), and radius, \( R \), using the equation \( V = 2\pi R P^{-1} \). The relative error in rotational velocities was calculated by adding in quadrature the uncertainties from the radii and from the periods. The rotational velocities have uncertainties around 11%. The main source of uncertainty in these values is

19001-p3
the stellar radius inasmuch as the periods have a typical accuracy of 1% or better (cf. [10], see also Choi and Herbst [34]).

The average \( \langle \sin i \rangle \). In this section we show that function (7) reproduces accurately the average inclination angles of the rotational axes, \( \langle \sin i \rangle \), for stars in the sample. First we have determined the values of the parameters \( q \) and \( \sigma_y \) of the curve that best fits the distribution of the projected rotational velocities. In order to avoid biases due to the choice of the bin when using histograms to represent frequency distributions, we fit the integral of eq. (7) to the cumulative distribution of the projected rotational velocities. Figure 1 shows the cumulative distribution and the integral of eq. (7) that best fits the distribution. We have found \( q = 1.33 \pm 0.03 \) and \( \sigma_y = 20.9 \pm 0.88 \). To adjust the curve to the distribution of \( V \sin i \) we have used a Levenberg-Marquardt nonlinear least squares algorithm. The fit parameters are \( \chi^2/\text{dof} \approx 2.01 \times 10^{-3} \), for a number of degrees of freedom \( \text{dof} = 84 \), and probability \( P > 99\% \). The region of low rotational velocities, by reason of lack of measurements of \( V \sin i \), presents the largest differences between observational data and the model. This lack of the data is due to the impossibility for detecting precise rotational velocities below \( 11 \text{ km/s} \) in the observational campaign of Rhode et al. [10]. The observed average sine of the inclination of the rotational axes of the ONC stars is \( \langle \sin i \rangle = 0.64 \pm 0.07 \). This average was calculated by using the values of \( V \sin i \) and \( V \) of our sample. The uncertainty in this value is mainly due to the error of 11%, related to the rotational velocity, \( V \). Assuming a random distribution of stellar rotational axes, the mean value is expected to be \( \pi/4 = 0.79 \), which is very different from the observed value. This is also displayed in fig. 2, where we represent the distribution of \( V \sin i \) as a function of \( V \). In this figure, the majority of points are well distributed around the curve \( \langle V \sin i \rangle / V = 0.64 \), but the curve \( \langle V \sin i \rangle / V = \pi/4 \) is located near the upper limit of the set of points. Rhode et al. [10] discuss this discrepancy between the observed average of \( \sin i \) and theoretical expected value. They consider that a possible explanation is that the rotational axes in young clusters such as ONC could be aligned rather than oriented randomly. In fact, the stars in association are formed from a cloud of interstellar material in a relatively short interval of time. In the formation process, the angular momentum of the parent interstellar cloud is transferred to the newly formed stars. It is therefore reasonable to assume that the orientation of the angular momentum of these stars initially reflect the momentum of the rotating progenitor cloud. Thus, an association of very young stars tends to have a preferential orientation of axes of rotation, although this feature can be lost gradually over time (cf. [35,36]).

By using eq. (15) with parameter \( q = 1.33 \), as obtained from the fit of the \( V \sin i \) cumulative distribution, we calculate the expected value for the average \( \langle \sin i \rangle \) as \( 0.62 \pm 0.04 \). This value is approximately equal to the average of \( \sin i \) observed, as shown in fig. 2, where the curve \( \langle V \sin i \rangle / V = 0.62 \) almost coincides with the observed one \( \langle V \sin i \rangle / V = 0.64 \). The expected average, calculated
in accordance with our model, has a connection with the observational data through the parameter $q$, which is derived from the distribution of the projected rotational velocities. It is therefore understandable that our proposal provides an expected average value of $\langle \sin i \rangle$ close to the observed one. Our model is particularly important in cases where only the projected rotational velocities are available, because it can provide an average $\langle \sin i \rangle$ closer to the true observed value than the standard model can. The standard relation $\langle V \sin i \rangle / \langle V \rangle = \pi / 4$ is appropriate only in cases where there is no preferential orientation of stellar angular momenta. The relation expressed in eq. (15) can be used in any situation, even if the distribution is completely random ($q = 1$).

In line with the discussion above, if one assumes that the younger the cluster the greater the tendency for a preferential orientation of stellar angular momenta, the parameter $q$ can be associated with the degree of randomness of the angular momentum orientations. It can also be related to the age of the cluster. Using eq. (7), Soares et al. [14] have analyzed the distribution of a sample of projected rotational velocities of stars in the Pleiades cluster (115My) and have found $q = 1.38$. Utilizing the same method, Santoro [19] has found $q = 1.51$ for the Hyades cluster (665My). These results, in combination with the present study, reinforce the idea that the parameter $q$ is related to the age of the clusters. However, a detailed study with stellar rotational data from clusters of different ages using the methodology presented here in this work is needed in order to determine whether such a relation exists. In order to confirm this relation, it is necessary to verify that $q = 1$ for the oldest clusters.

Conclusions. – We have derived the distribution function of the projected rotational velocity in the context of the Tsallis formalism. The distribution function is $q$-dependent and recovers Deutsch’s function in the limit $q \to 1$. The parameter $q = 1.33 \pm 0.03$ was determined from a sample of 86 stellar projected rotational velocities from the ONC. The expected average $\langle \sin i \rangle = 0.62 \pm 0.04$ was calculated using the relation between the first moments of the theoretical distribution functions of the observed, $V \sin i$, and the true, $V$, rotational velocities in the Tsallis formalism. The result reproduces accurately the average $\langle \sin i \rangle = 0.64 \pm 0.07$ from the observational data. The procedure presented in this work constitutes an efficient method to determine the relation between the moments of the distributions of the observed and the true stellar rotational velocities. We suggest and discuss the possible existence of a relation between the parameter $q$ and the degree of randomness of the angular momentum orientations of stars. We also suggest the possible existence of a relation between the parameter $q$ and the age of the stellar clusters. However, a detailed study involving clusters of different ages is needed in order to determine whether any relation exists. This study is being developed for a forthcoming paper.

***

This study was partially funded by the Programa Institutos Nacionais de Ciência e Tecnologia (MCT-CNPq-Edital No. 015/2008). We should like to thank Dr T. DUMELOW for critically reading the manuscript and the anonymous referee for constructive comments and suggestions that helped to improve this paper.

REFERENCES

[1] SHAIN G. and STRUVE O., Mon. Not. R. Astron. Soc., 89 (1929) 222.
[2] SLETTEBAK A., Astron. J., 110 (1949) 498.
[3] HUANG S.-S., Astrophys. J., 118 (1953) 285.
[4] TASSOUL J.-L. and TASSOUL M., Astrophys. J., 395 (1992) 259.
[5] VAN DIEN E., J. R. Astron. Soc. Can., 42 (1948) 249.
[6] BROWN A., Astrophys. J., 111 (1950) 366.
[7] BÖHM K.-H., Z. Astrophys., 30 (1952) 117.
[8] BERNACCA P. L., in Stellar Rotation, Proceedings of the IAU colloquium, edited by SLETTEBAK A. (D. Reidel, Dordrecht-Holland) 1970, p. 227.
[9] CHANDRASEKHAR S. and MÜCH G., Astrophys. J., 111 (1950) 142.
[10] RHODE K. L., HERBST W. and MATHIEU R. D., Astron. J., 122 (2001) 3258.
[11] BERNACCA P. L. and PERINOTTO M., Astron. Astrophys., 33 (1974) 443.
[12] BURKI G. and MAEDER A., Astron. Astrophys., 57 (1977) 401.
[13] DE MEDEIROS J. R., CARVALHO J. C., SOARES B. B., DA ROCHA C. and MAIA M. R. G., Astron. Astrophys., 358 (2000) 113.
[14] SOARES B. B., CARVALHO J. C., DO NASCIMENTO J. D. jr. and DE MEDEIROS J. R., Physica A, 364 (2006) 413.
[15] CARVALHO J. C., SOARES B. B., CANTO MARTINS B. L. et al., Physica A, 384 (2007) 507.
[16] CARVALHO J. C., SILVA R., DO NASCIMENTO J. D. jr. and DE MEDEIROS J. R., EPL, 84 (2008) 59001.
[17] CARVALHO J. C., DO NASCIMENTO J. D. jr., SILVA R. and DE MEDEIROS J. R., Astrophys. J., 696 (2009) 48.
[18] CARVALHO J. C., SILVA R., DO NASCIMENTO J. D. jr., SOARES B. B. and DE MEDEIROS J. R., EPL, 91 (2010) 69002.
[19] SANTORO L., in Proceedings of the Annual Meeting of the French Society of Astronomy and Astrophysics, edited by BOISSIER S., HEYDARI-MALAYERI M., SAMADI R. and VALLIS-GABAUD D. (Marseille, France) 2010, p. 387.
[20] KANIAKADIS G., LAVAGNO A. and QUARATI P., Phys. Lett. B, 369 (1996) 308.
[21] LAVAGNO A., KANIAKADIS G., REGO-MONTEIRO M. et al., Astrophys. Lett. Commun., 35 (1998) 449.
[22] PLASTINO A. R. and PLASTINO A., Phys. Lett. A, 174 (1993) 384.
[23] TSALLIS C., J. Stat. Phys., 52 (1988) 479.
[24] DEUTSCH A. J., in Stellar Rotation, Proceedings of the IAU colloquium, edited by SLETTEBAK A. (D. Reidel, Dordrecht-Holland) 1970, p. 207.
[25] TSALLIS C., LEVY S. V. F., SOUZA A. M. C. and MAYNARD R., Phys. Rev. Lett., 75 (1995) 3589.
[26] Curado E. M. F. and Tsallis C., J. Phys. A, 24 (1991) L69; 24 (1991) 3187; 25 (1992) 1019.

[27] Tsallis C., Mendes R. S. and Plastino A. R., Physica A, 261 (1998) 534.

[28] Prato D. and Tsallis C., Phys. Rev. E, 60 (1999) 2308.

[29] Tsallis C., Borges E. P. and Baldovin F., Physica A, 305 (2002) 1.

[30] Deutsch A. J., in The Magnetic and Related Stars, edited by Cameron R. C. (Mono Book Corp., Baltimore) 1967, p. 181.

[31] Borges E. P., J. Phys. A: Math. Gen., 31 (1998) 5281.

[32] Kraft R. P., Astrophys. J., 142 (1965) 681.

[33] Hillenbrand L. A., Astrophys. J., 113 (1997) 1733.

[34] Choi P. and Herbst W., Astrophys. J., 111 (1996) 283.

[35] Vant Veer F., Astron. Astrophys., 44 (1975) 437.

[36] Struve O., Stellar Evolution, an Exploration from the Observatory (Princeton University Press, Princeton) 1950.