On the Cheeger-Gromoll metric

Melek ARAS*
Giresun, Turkey

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Abstract

The purpose of this paper is to investigate applications the covariant derivatives, killing vector fields and to calculate the components of the curvature tensor $CGR$ of the Cheeger-Gromoll metric with respect to adapted frames in a the Riemannian manifold to its tangent bundle $T(M_n)$.

Keywords: Covector field; Levi-Civita connections; Killing vector field.

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1. Introduction

Let $(M_n, g)$ be a Riemannian manifold and $T (M_n)$ its tangent bundle with the projection $\pi : T (M_n) \rightarrow M_n$ [8]. In the present paper $\mathcal{I}_{p,q} (M_n)$ is the set of all tensor fields of type $(p, q)$ on $(M_n)$.

Cheeger and Gromoll studied complete manifolds of non-negative curvature in [2]. In spired by the paper [2] of Cheeger and Gromoll, Musso and Tricerri defined the metric $CG_g$ which they called the Cheeger – Gromoll metric on tangent bundle of Riemannian manifold in [4]. The Levi-Civita connection of $CG_g$ are calculated by A.A. Salimov and S. Kazimova in [5] with respect to the adapted frame. In [1] M. Abbas and M. Sarih studied killing vector field on tangent bundles with Cheeger – Gromoll metric. In [6] Sekizawa calculated Levi-Civita connection and curvature tensor of the metric $CG_g$ (for more details see [3]).

Let there be given a vector field $X = X^i \partial_i$ and covector field $g_X = g_{ji} X^j dx^i$ in $U \subset M_n$. Then $\gamma g_X \in \mathcal{I}_{0}^0 (M_n)$ is a function on $\pi^{-1} : (M_n) \rightarrow T (M_n)$ defined by $\gamma g_X = y^i g_{ji} X^j$ with respect to the induced coordinates $(x^i, y^i)$ (where $\pi$ is the naturel projection $\pi : T (M_n) \rightarrow M_n$) [8]. Now, denote by $r$ the norm a vector $y = x^i = y^i$, i.e., $r^2 = g_{ij} y^j y^i$. The metric $CG_g$ on $T (M_n)$ is given by

$$
\begin{align*}
CG_g (H X, H Y) &= V (g (X, Y)) \\
CG_g (H X, V Y) &= 0, \\
CG_g (V X, V Y) &= \frac{1}{\alpha} \left[ V (g (X, Y)) + (\gamma g_X) (\gamma g_Y) \right],
\end{align*}
$$

*Department of Mathematics, Faculty of Arts and Sciences, Giresun University, 28049 Giresun, Turkey
E-mail: melekaras25@hotmail.com; melek aras@giresun.edu.tr
for all vector field \( X, Y \in \mathcal{X}(M_n) \), where \( V (g(X, Y)) = (g(X, Y)) \circ \pi \) and \( \alpha = 1 + r^2 \).

Then the special frame is called the adapted frame. The Cheeger–Gromoll metric \( CG \) has components

\[
CG_{g_{\alpha\beta}} = \begin{pmatrix} g_{ji} & 0 \\ \frac{1}{\alpha}(g_{ji} + g_{ja}y^a y^j) & 0 \end{pmatrix}
\]

with respect to the adapted frame \( \{X_{(j)}, X_{(i)}\} \). The Levi-Civita connections of the \( CG \) \[5\] are

\[
\begin{align*}
CG\Gamma^h_{ji} &= \Gamma^h_{ji}, \quad CG\Gamma^h_{ji} = -\frac{1}{2} R^h_{jik} y^k, \quad CG\Gamma^h_{ji} = -\frac{1}{2} R^h_{jik} y^k, \\
CG\Gamma^h_{ji} &= \Gamma^h_{ji}, \quad CG\Gamma^h_{ji} = -\frac{1}{2} R^h_{jik} y^k, \quad \Gamma^h_{ji} = 0, \quad \Gamma^h_{ji} = 0, \\
CG\Gamma^h_{ji} &= -\frac{1}{\alpha} \left( y_j \delta_i^h + y_i \delta_j^h \right) + \frac{1+\alpha}{\alpha} g_{ji} y^h - \frac{1}{\alpha} g_{ij} y^h
\end{align*}
\]

with respect to the adapted frame, where \( y_j = g_{ji} y^i, R^h_{jik} = g^{hk} g_{ja} R^k_{ja} \).

2. **The metric Cheeger–Gromoll in adapted frames**

Let \( X \) be a vector field in \( T(M_n) \) and \( (X^\alpha) = \begin{pmatrix} X^h \\ X^\gamma \end{pmatrix} \) its components with respect to the adapted frame. Then the covariant derivative \( \nabla X \) has components

\[
CG \nabla_\beta X^\alpha = D_\beta X^\alpha + CG \Gamma^\alpha_{\beta\delta} X^\delta,
\]

\( CG \Gamma^\alpha_{\beta\delta} \) being Levi-Civita connections of the metric \( CG \) with respect to the adapted frame.

Now let us consider the covariant derivatives of vertical lift, complete lift and horizontal lift. Then we have respectively components

\[
CG \nabla^V_\beta X^\alpha = \begin{pmatrix} CG \nabla^V_i X^h & CG \nabla^V_i X^h \\ CG \nabla^V_i X^\gamma & CG \nabla^V_i X^\gamma \end{pmatrix}
\]

\[
\begin{align*}
CG \nabla^V_i X^h &= -\frac{1}{2\alpha} R^h_{ikm} y^k X^m \\
CG \nabla^V_i X^\gamma &= 0 \\
CG \nabla^V_i X^\gamma &= \nabla^V_i X^h \\
CG \nabla^V_i X^\gamma &= \left[ -\frac{1}{\alpha} \left( y_i \delta^h_m + y_m \delta^h_i \right) + \frac{1+\alpha}{\alpha} g_{im} y^h - \frac{1}{\alpha} y_i y_m y^h \right] X^h
\end{align*}
\]

\[
CG \nabla^C_\beta X^\alpha = \begin{pmatrix} CG \nabla^C_i X^h & CG \nabla^C_i X^h \\ CG \nabla^C_i X^\gamma & CG \nabla^C_i X^\gamma \end{pmatrix},
\]

\[
\begin{align*}
CG \nabla^C_i X^h &= \nabla^C_i X^h - \frac{1}{2\alpha} R^h_{ikm} y^k X^m \\
CG \nabla^C_i X^h &= -\frac{1}{2\alpha} R^h_{ikm} y^k X^m \\
CG \nabla^C_i X^\gamma &= \nabla^C_i X^h - \frac{1}{2\alpha} R^h_{ikm} y^k X^m \\
CG \nabla^C_i X^\gamma &= \nabla^C_i X^h + \left[ -\frac{1}{\alpha} \left( y_i \delta^h_m + y_m \delta^h_i \right) + \frac{1+\alpha}{\alpha} g_{im} y^h - \frac{1}{\alpha} y_i y_m y^h \right] X^h
\end{align*}
\]

\[5\]
Proposition 1  The Complete and horizontal lifts of a vector field in $M_n$ to $T(M_n)$ with the Cheeger – Gromoll metric $CGg$ are parallel if and only if the given vector field in $M_n$ is parallel.

We now consider the vertical, complete and horizontal lifts of a vector field in $M_n$ with local components $X^h$ in $M_n$ to $T(M_n)$ and compute components of the associated covector fields of $X$ with respect to the metric $CGg$. Then we obtain respectively

\[
\begin{align*}
\{ V & X_B = (0, \frac{1}{\alpha} (X_i + g_{is}X_{iy}y^i)) \\
C & X_B = (X_i, \frac{1}{\alpha} (\nabla X_i + g_{is}\nabla X_{iy}y^i)) \\
H & X_B = (X_i, 0)
\end{align*}
\]

with respect to the adapted frame, where $X_i = g_{is}X^s$ are local components of the associated covector field $X^*$ in $M_n$. Thus we see that the vertical, complete and horizontal lifts of the associated covector field $X^*$ have respectively covariant derivatives with components

\[
CG\nabla^V_{\beta} X_\gamma = \begin{pmatrix} CG\nabla^V_i X_j & CG\nabla^V_j X_i & CG\nabla^V_j X_i \end{pmatrix}
\]

\[
CG\nabla^V_i X_j = -\frac{1}{2\alpha} R_{ijk}^h y^k (X_h + g_{hs}X_{iy}y^i)
\]

\[
CG\nabla^V_j X_i = \frac{1}{\alpha} (\nabla_i X_j + \nabla_i X_{ijy}y^i)
\]

\[
CG\nabla^V_i X_i = 0
\]

\[
CG\nabla^V_i X_\gamma = [-\left(y_i\delta_j^h + y_j\delta_i^h\right) + (1+\alpha) g_{ij}y_j^h - y_iy_j^h] \frac{(X_h + g_{hs}X_{iy}y^i)}{\alpha^2}
\]

\[
CG\nabla^C_{\beta} X_\gamma = \begin{pmatrix} CG\nabla^C_i X_j & CG\nabla^C_j X_i & CG\nabla^C_j X_i \end{pmatrix}
\]

\[
CG\nabla^C_i X_j = \begin{align*}
\nabla_i X_j - \frac{1}{2\alpha} R_{ijk}^h y^k (\nabla X_h + g_{hs}\nabla X_{iy}y^i)
\end{align*}
\]

\[
CG\nabla^C_j X_i = \frac{1}{\alpha} (\nabla_i \nabla X_j y^m + (g_{js} \nabla_i \nabla X_{iy}y^m)) - \frac{1}{2\alpha} R_{imj}^k y^k X_m
\]

\[
CG\nabla^C_i X_i = -\frac{1}{2\alpha} R_{ij}^k y^k X_m
\]

\[
CG\nabla^C_j X_\gamma = \frac{1}{\alpha} (\nabla_i X_j + g_{js}\nabla_i X_{iy}y^i)
\]

\[
+ \left(y_i\delta_j^h + y_j\delta_i^h\right) - (1+\alpha) g_{ij}y_j^h + y_iy_j^h \frac{\left(\nabla X_h + g_{hs}X_{iy}y^i\right)}{\alpha^2}
\]

Thus, taking account of the fact that $\nabla_i X^h = 0$ implies $R_{imk}^h X_m = 0$, we have
$$CG \nabla^H_{\beta} X_\gamma = \left( \begin{array}{c} \nabla_i X_j - \frac{1}{2\alpha} R^m_{ijk} y^k X_m \\ \frac{1}{2\alpha} R^m_{ijk} y^k X_m \end{array} \right)$$ (10)

with respect to the adapted frame. Thus the rotations of $^H X, ^C X$ and $^V X$
have respectively components of the form

$$CG \nabla^H_{\beta} X_\gamma - CG \nabla^H_{\gamma} X_\beta = \left( \begin{array}{c} \nabla_i X_j - \nabla_j X_i \\ 0 \\ 0 \end{array} \right),$$ (11)

$$CG \nabla^C_{\beta} X_\gamma - CG \nabla^C_{\gamma} X_\beta = \left( \begin{array}{ccc} A & B \\ C & D \end{array} \right)$$ (12)

$$CG \nabla^V_{\beta} X_\gamma - CG \nabla^V_{\gamma} X_\beta = \left( \begin{array}{ccc} A' & B' \\ C' & D' \end{array} \right)$$ (13)

From (12), we see, If complete lift of the associated covector field of
$X$ is closed in $T(M_n)$, then

$$\nabla_i X_j - \nabla_j X_i = 0$$ (14)

Further, if the conditions (14) are satisfied, we deduce that

$$R^h_{ijk} (\nabla X_i + g_{hs} \nabla X_s) = 0.$$ (15)

From there we have

**Proposition 2** The complete lift of the associated covector field of $X$ is closed in $T(M_n)$ if the associated covector field of $X$ is closed and the second covariant derivative of $X$ vanishes in $M_n$.

From (11), we see that the horizontal lift of the associated covector field of
$X$ is closed in $T(M_n)$ if and only if the horizontal lift of the associated covector field of $X$ is closed.
We now compute the Lie derivatives of the metric $G_g$ with respect to $C\ X$ and $\ H\ X$, by making use of (7). The Lie derivatives of the metric $G_g$ with respect to $V\ X$, $C\ X$ and $\ H\ X$ have respectively components

$$L_{C\ X}G_g = G_g\nabla^C_{\beta}X_\gamma + G_g\nabla^C_{\gamma}X_\beta = \left( \begin{array}{cc} A_1 & B_1 \\ C_1 & D_1 \end{array} \right)$$

$$A_1 = (\nabla_iX_j + \nabla_jX_i)$$
$$B_1 = \frac{1}{\alpha} \left[ \nabla_i\nabla_nX_jy^n + (g_{js}\nabla_i\nabla_nX_t + g_{is}\nabla_j\nabla_nX_t)y^s y^n - R^m_{ikj}\ y^kX_m \right]$$
$$C_1 = -\frac{1}{\alpha} R^m_{ikj}\ y^kX_m$$
$$D_1 = \frac{1}{\alpha} \left[ (\nabla_iX_j - \nabla_jX_i) + (g_{js}\nabla_iX_t - g_{is}\nabla_jX_t)y^s y^n \right] + \left[ (y_i\delta^m_j + y_j\delta^m_i) - (1 + \alpha) g_{ij}y^n + y_iy_jy^m \right] \frac{2(\nabla X_m + g_{mst}X_yy^s)}{\alpha^2}$$

(15)

$$L_{H\ X}G_g = G_g\nabla^H_{\beta}X_\gamma + G_g\nabla^H_{\gamma}X_\beta = \left( \begin{array}{cc} (\nabla_iX_j + \nabla_jX_i) & -\frac{1}{\alpha} R^m_{ikj}\ y^kX_m \\ -\frac{1}{\alpha} R^m_{ikj}\ y^kX_m & 0 \end{array} \right)$$

(16)

with respect to the adapted frame in $T\ (M_n)$.

Since we have

$$\nabla_i\nabla_nX_j + (g_{js}\nabla_i\nabla_nX_t + g_{is}\nabla_j\nabla_nX_t) - R^m_{ikj}\ X_m = 0$$

as a consequence of $L_X g_{ji} = \nabla_iX_j + \nabla_jX_i = 0[7]$, we conclude by means of (15) that the complete lift $C\ X$ is a Killing vector field in $T\ (M_n)$ if and only if is a Killing vector field in $M_n$.

We next have

$$R^m_{ikj}\ X_m = 0$$

as a consequence of the vanishing of the second covariant derivative of $X$. Conversely, the conditions

$$\nabla_iX_j + \nabla_jX_i = 0 \quad \text{and} \quad R^m_{ikj}\ X_m = 0$$

imply that the second covariant derivative of $X$ vanishes. From these results, we have

**Proposition 3** Necessary and sufficient conditions in order that (a) the complete lift to $T\ (M_n)$, with metric $G_g$, of a vector field $X$ in $M_n$ be a Killing vector field in $T\ (M_n)$ is that $X$ is a Killing vector field with vanishing covariant derivative in $M_n$, (b) the horizontal lift to $T\ (M_n)$, with metric $G_g$, of a vector field $X$ in $M_n$ be a Killing vector field in $T\ (M_n)$ is that $X$ is a Killing vector field with vanishing second covariant derivative in $M_n$. 

5
We now calculate the components of the curvature tensor \( CGR \) of \( T(M_n) \) with the metric Cheeger–Gromoll metric \( CG \). Components of the curvature tensor \( CGR \) with respect to the adapted frame are given by

\[
CGR_{\alpha\beta\gamma} = D_\alpha CG \Gamma^\delta_{\beta\gamma} - D_\beta CG \Gamma^\delta_{\alpha\gamma} + CG \Gamma^\delta_{\alpha\beta} CG \Gamma^\epsilon_{\delta\epsilon} - CG \Gamma^\delta_{\beta\epsilon} CG \Gamma^\epsilon_{\alpha\gamma} - \Omega^\epsilon_{\alpha\beta} CG \Gamma^\gamma_{\epsilon\gamma}
\]

(17)

where \( D_\alpha \) are local vector fields and \( \Omega^\epsilon_{\alpha\beta} \) are components of the non-holonomic object. From (17) we have

\[
\begin{align*}
CGR_{hijk} &= R_{hijk} + \frac{1}{4\alpha} \left( R_{imn}^h R_{jkl}^n - R_{ijnm}^h R_{ikl}^n \right) y^m y^l - \frac{1}{2\alpha} R_{jnm}^n R_{h_{kmi}}^y y^m y^s \\
CGR_{jk}^n &= \frac{1}{2} \left( \nabla_j R_{h_{kim}}^n - \nabla_i R_{jkm}^n \right) y^m \notag \\
CGR_{jk}^n &= R_{h_{jmn}^k} + \frac{1}{4\alpha} \left( R_{h_{jnm}}^n R_{kli}^n - R_{h_{jnm}}^n R_{kli}^n \right) y^m y^l - \frac{1}{2\alpha} R_{lmm}^n R_{h_{jkm}^l} y^m \notag \\
CGR_{jk}^n &= 0, CGR_{jk}^n = 0, CGR_{jk}^n = 0, CGR_{jk}^n = 0 \\
CGR_{jk}^n &= -\frac{1}{2\alpha} \left( \nabla_j R_{h_{km}}^n \right) y^m \\
CGR_{jk}^n &= \frac{1}{2} R_{jik}^h + \frac{1}{4\alpha} R_{h_{jnm}}^n R_{kli}^m y^m y^l + \frac{1}{2\alpha} \left( y_i^h\delta_n^k + y_n^h\delta_i^k \right) + \frac{1}{2\alpha} g_{kn} y^m - \frac{1}{\alpha} y^k y^h y^n \notag \\
\end{align*}
\]

(18)

with respect to the adapted frame. Thus we have

**Proposition 4** The tangent bundle \( T(M_n) \) over a Riemannian manifold \( M_n \) is locally flat with respect to the metric \( CG \) if and only if \( M_n \) is locally flat.

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