CUBIC ARC-TRANSITIVE k-CIRCULANTS

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Abstract. For an integer $k \geq 1$, a graph is called a $k$-circulant if its automorphism group contains a cyclic semiregular subgroup with $k$ orbits on the vertices. We show that, if $k$ is even, there exist infinitely many cubic arc-transitive $k$-circulants. We conjecture that, if $k$ is odd, then a cubic arc-transitive $k$-circulant has order at most $6k^2$. Our main result is a proof of this conjecture when $k$ is squarefree and coprime to 6.

1. Introduction

All graphs in this paper are finite, simple and connected. A permutation group is called semiregular if its only element fixing a point is the identity. For an integer $k \geq 1$, a graph is called a $k$-circulant if its automorphism group contains a cyclic semiregular subgroup with $k$ orbits on the vertices.

Clearly, every graph is a $k$-circulant for some $k$, for example if $k$ is the order of the graph. Moving beyond this trivial observation is often quite difficult: whether every vertex-transitive graph is a $k$-circulant for some other $k$ is a famous open problem (see [2, 20]). This question has been settled in the affirmative for graphs of valency at most four [8, 22].

On the other hand, studying $k$-circulants for fixed $k$ often yields interesting results. For example, 1-circulants, usually called simply circulants, are exactly Cayley graphs on cyclic groups. These graphs have been intensively studied. The family of 2-circulants (sometimes called bicirculants) has also attracted some attention.

In many cases, additional symmetry conditions are imposed on the graphs. In particular, cubic arc-transitive $k$-circulants have been the focus of some recent investigation. (A graph is called arc-transitive if its automorphism group acts transitively on ordered pairs of adjacent vertices. A graph is cubic if each of its vertices has degree 3.) It is a rather easy exercise to show that a cubic arc-transitive circulant is isomorphic to either $K_4$ or $K_{3,3}$. The classification of cubic arc-transitive bicirculants can be deduced from [12, 21, 23], while cubic arc-transitive $k$-circulants for $k \in \{3, 4, 5\}$ are classified in [14, 16]. Rather than describe these classifications in detail, we would simply like to point out one striking feature: for $k = 2$ or $k = 4$, there exist infinitely many cubic arc-transitive $k$-circulants, whereas for $k \in \{1, 3, 5\}$, there are only finitely many. This immediately suggests the following question.

Question 1.1. Given a positive integer $k$, does there exist infinitely many cubic arc-transitive $k$-circulants?

Investigating this question is the main topic of this paper. Our first result is that the answer is positive when $k$ is even.

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Proposition 1.2. If \( k \) is an even positive integer, then there exist infinitely many cubic arc-transitive \( k \)-circulants.

We then turn our attention to the case when \( k \) is odd. We are unable to settle Question 1.1 in full generality in this case, but we prove the following, which is our main result.

Theorem 1.3. If \( k \) is a squarefree positive integer coprime to 6, then a cubic arc-transitive \( k \)-circulant has order at most \( 6k^2 \).

We would like to note that the methods used in the proof of Theorem 1.3 can, with some more effort, yield a complete classification of cubic arc-transitive \( k \)-circulants, for \( k \) squarefree coprime to 6. We also show that the bound of \( 6k^2 \) in Theorem 1.3 is best possible.

Proposition 1.4. If \( k \) is an odd positive integer, then there exists a cubic arc-transitive \( k \)-circulant of order \( 6k^2 \).

Finally, in view of Theorem 1.3, Proposition 1.4 and computational evidence gathered from the census of cubic arc-transitive graphs of order at most 10000 [1][5], we would like to propose the following conjecture which would completely settle Question 1.1.

Conjecture 1.5. If \( k \) is an odd positive integer, then a cubic arc-transitive \( k \)-circulant has order at most \( 6k^2 \).

2. Proof of Proposition 1.2

Let \( k = 2m \) be an even positive integer and let \( p \) be a prime with \( p \equiv 1 \pmod{3} \) and \( p \nmid m \). Let

\[ G = \langle u, v, w, x \mid u^m, v^m, w^p, x^2, [u, v], [u, w], [v, w], u^x, v^x, w^x w \rangle. \]

In other words, \( G \) is the generalised dihedral group on the abelian group \( A := \langle u, v, w \rangle \cong \mathbb{Z}_m \times \mathbb{Z}_p \). Since \( p \equiv 1 \pmod{3} \), there exists a positive integer \( \alpha \) satisfying \( \alpha^3 + \alpha + 1 \equiv 0 \pmod{p} \).

Let \( \gamma \) be the automorphism of \( G \) satisfying \( w^y = v, v^y = u^{-1}v^{-1}, w^y = w^\alpha \), and \( x^2 = x \), and let \( s = uvx \). Note that \( \gamma \) has order 3 while \( s \) is an involution.

Let \( S = \{s, ss^2, ss^3\} \), let \( R = \langle S \rangle \) and let \( \Gamma = \text{Cayley}(R, S) \). It is clear that \( \Gamma \) is connected and, since \( s^3 \neq s \), we have \( |S| = 3 \) and thus \( \Gamma \) is cubic. Since \( \gamma \) has order 3, \( S \) is the orbit of \( s \) under \( \langle \gamma \rangle \) which implies that \( \gamma \) induces an automorphism of \( \Gamma \) fixing the identity but acting transitively on \( S \). In particular, \( \Gamma \) is arc-transitive.

Let \( a = ss^y \) and \( b = ss^{y^2} \). Note that \( a, b \in A \) thus \( a^s = a^x = a^{-1} \) and \( b^s = b^x = b^{-1} \). In particular, \( R = \langle a, b, s \rangle = \langle a, b \rangle \rtimes \langle s \rangle \). Now, a simple calculation yields that \( a = u^{-1}w^{1-\alpha} \) and \( b = u^2v^{1-\alpha} \). In particular, \( a^m = w^{(1-\alpha)m} \). Since \( p \) divides neither \( m \) nor \( 1 - \alpha \), it follows that \( w \in \langle a \rangle \). This implies that \( \langle a, b \rangle = \langle uw^{-1}, u^2v \rangle \times \langle w \rangle = \langle uw^{-1}, w^3 \rangle \times \langle w \rangle \). In particular, \( |A : \langle a, b \rangle| = 3 \) if \( 3 \) divides \( m \) and \( \langle a, b \rangle = A \) otherwise. Similarly, \( |G : R| = 3 \) if \( 3 \) divides \( m \) and \( G = R \) otherwise.

Let \( C \) be the group generated by \( a^3w \) if \( 3 \) divides \( m \) and by \( uvw \) otherwise. Note that \( C \leq \langle a, b \rangle \leq R \) and thus \( C \) is a semiregular subgroup of \( G \). Since \( p \) is coprime to \( m \), \( C \) has order \( \frac{m}{3} \) if \( 3 \) divides \( m \), and \( mp \) otherwise. In both cases, \( C \) has \( 2m = k \) orbits on the vertices of \( \Gamma \) and thus \( \Gamma \) is a \( k \)-circulant. To conclude the proof, it suffices to note that there are infinitely many primes \( p \) with \( p \equiv 1 \pmod{3} \), and \( p \nmid m \).
3. Proof of Proposition 1.3

Let

\[ R = \langle u, v, x, y \mid u^k, v^k, x^3, y^2, [u, v], u^x v^{-1}, v^x u, u^y v, v^y u, x^3, y \rangle. \]

Note that \( R = \langle u, v \rangle \times \langle x, y \rangle \cong \mathbb{Z}_2^2 \times \text{Sym}(3) \). Let

\[ \delta = \begin{cases} 
1, & \text{if } 3 \text{ divides } k \\
0, & \text{otherwise},
\end{cases} \]

and let \( \sigma \) be the automorphism of \( R \) such that \( u^\sigma = u^{-1}v^{-1}, v^\sigma = u, x^\sigma = (uv)^\delta x, y^\sigma = x^2 y \), and let \( G = R \rtimes \langle \sigma \rangle \). Note that one needs to check that \( \sigma \) is indeed an automorphism of \( R \). This is obvious when \( \delta = 0 \) as, in this case, \( \sigma \) acts on \( R \) as conjugation by \( x^{-1} \). Furthermore, even when \( \delta = 1 \), \( \sigma \) still acts on \( \langle u, v \rangle \times \langle y \rangle \) as conjugation by \( x^{-1} \), hence one needs only check that the relations involving \( x \) are preserved by \( \sigma \). This is a straightforward computation. For example:

\[ (u^\sigma)^{x^2} = (u^{-1}v^{-1})^{(uv)^\delta x} = (u^{-1}v^{-1}) x = u = v^\sigma. \]

It is also easy to check that \( x^{\sigma^3} = x \) and thus \( \sigma \) has order 3. Let \( s = uwy \), let \( S = \{s, s^3, s^6\} \), and let \( \Gamma = \text{Cay}(R, S) \). Note that \( s \) is an involution and \( s^\sigma = v^{-1}x^2 y \neq s \) hence \( S \) is an inverse-closed set of cardinality 3 and \( \Gamma \) is a cubic graph of order \( |R| = 6k^2 \). Since \( S \) is a \( \langle \sigma \rangle \)-orbit, \( \sigma \) induces an automorphism of \( \Gamma \) fixing the identity but acting transitively on \( S \). In particular, \( \Gamma \) is \( G \)-arc-transitive.

We now show that \( \Gamma \) is connected or, equivalently, that \( S \) generates \( R \). Recall that \( s^\sigma = v^{-1}x^2 y \). An easy computation yields \( s^\sigma^3 = u^{\delta - 1}xy \). A more elaborate computation then yields that \( ss^\sigma ss^\sigma^2 = u^{\delta - \delta} \in \langle S \rangle \). From the definition of \( \delta \), it follows that \( u \in \langle S \rangle \), but then \( u^\sigma = v^{-1} \in \langle S \rangle \) and we easily conclude that \( S \) generates \( R \). It remains to exhibit a semiregular cyclic subgroup of \( G \) of order 6k.

Suppose first that 3 does not divide \( k \) and let \( C = \langle uv^{-1}yx, x \rangle \). An easy computation shows that \( uv^{-1}, y \) and \( x \) pairwise commute and have orders \( k \), 2, and 3, respectively. Since \( k \) is coprime to 6, it follows that \( |C| = 6k \). It remains to show that \( C \) is semiregular. Since \( |G_v| = 3 \), it suffices to show that \( \langle x \sigma \rangle \) is semiregular but, in fact, \( \langle x \sigma \rangle \) is central in \( G \) and thus must intersect every point-stabiliser trivially.

Suppose now that 3 divides \( k \) and let \( C = \langle y \sigma \rangle \). As \( y \sigma \) is not contained in \( R \) which is a normal subgroup of index 3 in \( G \), we get that \( |y \sigma| = 3[(y \sigma)^3]| \). A computation yields that \( (y \sigma)^3 = v^{-1}xy \). Similarly, \( (y \sigma)^3 \) is not contained in \( \langle u, v \rangle \times \langle x \rangle \) which is a normal subgroup of index 2 in \( R \), and thus \( |(y \sigma)^3| = 2[(y \sigma)^6] \). Now, \( (y \sigma)^6 = (v^{-1}xy)^2 - v^{-2}u^{-1} \) and \( |v^{-2}u^{-1}| = k \) and thus \( |C| = |y \sigma| = 6k \). It remains to show that \( C \) is semiregular but, since \( |G_v| = 3 \), it suffices to show that \( (y \sigma)^{2k} \) is semiregular. Since 3 divides \( k \), \( C \) is normal of index 3 in \( G \), we find that \( (y \sigma)^{2k} \) is contained in the regular group \( R \). This completes the proof.

4. Preliminaries to the proof of Theorem 1.5

We start with some notation and definitions. Let \( G \) be a group of automorphisms of a graph \( \Gamma \). We denote by \( G_v \) the stabiliser of \( G \) in the vertex \( v \), by \( \Gamma(v) \) the neighbourhood of \( v \), and by \( G^{t}(v) \) the permutation group induced by the action of \( G_v \) on \( \Gamma(v) \). We say that \( \Gamma \) is \( G \)-vertex-transitive (\( G \)-arc-transitive, respectively) if \( G \) is transitive on the set of vertices (arcs, respectively) of \( \Gamma \), and that it is \( G \)-locally-transitive if \( G^{t}(v) \) is transitive for every vertex \( v \).

A \( t \)-arc of \( \Gamma \) is a sequence of \( t + 1 \) vertices such that any two consecutive vertices in the sequence are adjacent, and with any repeated vertices being more than 2
steps apart. We say that \( \Gamma \) is \((G, t)\)-arc-transitive if \( G \) is transitive on the set of \( t \)-arcs of \( \Gamma \).

Given an integer \( n \) and a prime \( p \), we will sometimes denote by \( n_p \) the \( p \)-part of \( n \) (that is, the largest power of \( p \) dividing \( n \)) and by \( n_{p'} \) the \( p' \)-part (that is, \( n/n_p \)).

Given a graph \( \Gamma \) and \( N \) a group of automorphisms of \( \Gamma \), the quotient graph \( \Gamma/N \) is the graph whose vertices are the \( N \)-orbits, and with two such \( N \)-orbits \( v^N \) and \( u^N \) adjacent whenever there is a pair of vertices \( v' \in v^N \) and \( u' \in u^N \) that are adjacent in \( \Gamma \). If the natural projection \( \pi : \Gamma \to \Gamma/N \) is a local bijection (that is, if \( \pi_{|\Gamma(v)} : \Gamma(v) \to (\Gamma/N)(v^N) \) is a bijection for every vertex \( v \) of \( \Gamma \)) then \( \Gamma \) is called a regular \( N \)-cover of \( \Gamma/N \). Such covers have many important properties that will be used repeatedly, most of which are folklore. (See [24, Lemma 3.2] for example.)

We now collect a few results that will be useful in the proof.

**Lemma 4.1.** Let \( p \) be an odd prime, let \( P \) be a \( p \)-group with a maximal cyclic subgroup and let \( X \) be the group generated by elements of order \( p \) in \( P \).

1. If \( P = X \), then \( P \) is elementary abelian.
2. If \( X \) is cyclic, then so is \( P \).
3. If \( P \) is cyclic of order at least \( p^2 \), then an automorphism of \( P \) of order 2 centralising the maximal subgroup of \( P \) must centralise \( P \).

**Proof.** These results easily follow from the classification of \( p \)-groups with a cyclic maximal subgroup (see for example [17, Section 5.3]). \( \square \)

**Lemma 4.2.** Let \( G \) be a group with a normal subgroup \( N \) and let \( T \) be a perfect group acting on \( G \) and centralising \( N \). If \( T \) acts trivially on \( G/N \), then it acts trivially on \( G \).

**Proof.** Since \( T \) acts trivially on \( G/N \), we have \([G,T] \leq N \) and thus \([G,T,T] \leq [N,T] = 1 \). Similarly, \([T,G,T] = 1 \). By the three subgroups lemma, it follows that \([T,T,G] = 1 \) and, since \( T \) is perfect, \([T,G] = 1 \). \( \square \)

**Lemma 4.3.** Let \( \Gamma \) be a graph with every vertex having odd valency and let \( C \) be a semiregular cyclic group of automorphisms of \( \Gamma \). If \( C \) has an odd number of orbits, then \( C \) has even order and the unique involution of \( C \) reverses some edge of \( \Gamma \).

**Proof.** Let \( (C_1, \ldots, C_k) \) be an ordering of the orbits of \( C \) and let \( A = \{a_{ij}\} \) be the \( k \times k \) matrix such that \( a_{ij} \) is the number of vertices of \( C_j \) adjacent to a given vertex of \( C_i \). It is not hard to see that this is independent of the choice of vertex, hence \( A \) is well-defined and, moreover, \( a_{ij} = a_{ji} \), hence \( A \) is symmetric.

By hypothesis, \( k \) is odd and the sum of every row and column is odd. In particular, the sum of all the entries of \( A \) is odd. On the other hand, \( A \) is symmetric and thus the sum of the non-diagonal entries is even. This shows that at least one diagonal entry of \( A \), say \( a_{nn} \), must be odd.

Let \( X \) be the graph induced on \( C_n \). Since \( C \) is semiregular, it acts regularly on \( X \) and we can view \( X \) as a Cayley graph \( \text{Cay}(C,S) \). Since \( X \) has odd valency, \(|S|\) is odd, \(|C|\) is even and \( S \) contains the unique involution of \( C \). The result follows. \( \square \)

**Lemma 4.4.** Let \( \Gamma \) be a \( G \)-arc-transitive group and let \( N \) be a normal subgroup of \( G \). If \( N \) contains an element reversing some edge of \( \Gamma \), then \( \Gamma \) is \( N \)-vertex-transitive.

**Proof.** Let \( e \) be an edge of \( \Gamma \). Since \( N \) is normal in the arc-transitive group \( G \), \( N \) must contain an element reversing \( e \). In particular, the endpoints of \( e \) are in the same \( N \)-orbit. By connectedness, \( N \) is vertex-transitive. \( \square \)

**Lemma 4.5.** Let \( G \) be a transitive permutation group, let \( N \) be a normal subgroup of \( G \) and let \( C \) be a semiregular subgroup of \( G \) with \( k \) orbits. If \(|N| \) is coprime to
$|G_v|$, then the induced action of $C$ on the $N$-orbits is semiregular with $k'$ orbits, where $k'$ divides $k$.

**Proof.** Since $|N|$ is coprime to $|G_v|$, $N_v = 1$ and thus $|N| = |v^N|$ for every point $v$. It follows that $|(G/N)_{v^N}| = \frac{|G/N|}{|v^N|} = \frac{|G|}{|N|} = |G_v|$. 

Let $c \in C$ such that $Nc$ (viewed as an element of $G/N$) fixes some $v^N$. For the first part, it suffices to show that $cN$ is trivial. Note that, by the previous paragraph, the order of $Nc$ divides $|G_v|$. On the other hand, since $v^N$ is fixed by $Nc$, $v^N$ can be partitioned in $(c)$-orbits, but these all have the same size, namely $|c|$, and thus $|c|$ divides $|N|$. It follows that the order of $Nc$ divides both $|G_v|$ and $|N|$ but these are coprime and thus $Nc$ is trivial.

As for the second claim, $k = \frac{|G|}{|v^N|}$ while $k' = \frac{|G|}{|v^N|} = \frac{|G|}{|v^N|} \frac{|v^N|}{|N|} = \frac{|v^N|}{|N|}$. Recall that $|N| = |v^N|$ and thus $\frac{k}{k'} = \frac{|v^N|}{|v^N|}$ which is an integer. \hfill \Box

**Lemma 4.6.** Let $\Gamma$ be a $G$-arc-transitive graph. If $G$ has a normal semiregular subgroup with at most two orbits on vertices, then the subgroup of $G$ fixing a vertex and all its neighbours is trivial.

**Proof.** If $G$ has a normal regular group, then the result follows by [13] Lemma 2.1]. Otherwise, it is not hard to see that $\Gamma$ must be bipartite and the result follows by applying [18] Lemma 2.4] with $X = G$ and $N$ the bipartition-preserving subgroup of $G$. \hfill \Box

**Theorem 4.7.** [23][24] Let $\Gamma$ be a cubic graph. If $\Gamma$ is $G$-arc-transitive, then it is $(G, t+1)$-arc-regular for some $0 \leq t \leq 4$. Moreover, the structure of $G_v$ is uniquely determined by $t$ and is as in Table 1.

| $t$ | $G_v$ |
|-----|-------|
| 0   | $Z_3$ |
| 1   | Sym(3) |
| 2   | $Sym(3) \times Z_2$ |
| 3   | Sym(4) |
| 4   | $Sym(4) \times Z_2$ |

**Table 1.** Vertex-stabilisers in cubic $(t+1)$-arc-regular graphs

**Proposition 4.8.** [19] Corollary 4.6] Let $\Gamma$ be a cubic $(G, t+1)$-arc-transitive graph. If $G$ is soluble, then $t \leq 2$. Moreover, if $t = 2$, then $\Gamma$ is a regular cover of $K_{3,3}$.

**Lemma 4.9.** Let $\Gamma$ be a cubic $G$-arc-transitive graph and let $N$ be a normal subgroup of $G$ that is locally-transitive on $\Gamma$. If $|N_v| \leq 12$, then $|G_v| \leq 12$.

**Proof.** Suppose, by contradiction, that $|G_v| > 12$. By Theorem 4.7 $G_v$ is isomorphic to either $Sym(4)$ or $Sym(4) \times Z_2$. Now, $N_v$ is a normal subgroup of $G_v$ of order divisible by 3. Since $|N_v| \leq 12$, it is not hard to check that this implies $N_v \cong Alt(4)$. Since $N_v^{G(v)}$ is a quotient of $N_v$ with order divisible by 3, we have that $N_v^{G(v)}$ is regular of order 3. As $N$ is normal in a vertex-transitive group, this holds for every vertex, but this implies that $N_v$ itself has order 3, a contradiction. \hfill \Box

**Lemma 4.10.** Let $\Gamma$ be a $G$-locally-transitive cubic graph. If $N$ is a normal subgroup of $G$ such that $G/N$ is insoluble, then $N$ has at least three orbits and is semiregular on the vertices of $\Gamma$. In particular, $\Gamma$ is a regular cover of $G/N$.

**Proof.** If $N$ has at most two orbits on vertices, then $|G : N|$ divides $2|G_v|$. Since $|G_v|$ is a $\{2, 3\}$ group, so is $G/N$ and thus $G/N$ is soluble, a contradiction. If follows that $N$ has at least three orbits on vertices and, since $G$ is locally-primitive, $N$ must be semiregular. \hfill \Box
Note that, for an integer \(n\), the property of having a cyclic group of index dividing \(n\) is inherited by normal subgroups and quotients. This fact will be used repeatedly throughout the paper.

**Lemma 4.11.** Let \(G\) be a cubic \((G, t + 1)\)-arc-regular graph such that \(G\) is insoluble and let \(S\) be the soluble radical of \(G\).

1. If \(C\) is a semiregular cyclic subgroup of \(G\) with an odd number of orbits, then \(|C \cap S|\) is odd and \(|G/S:CS/S|_{2}|2| = 2^t\).

2. If a Sylow 2-subgroup of \(G\) has a cyclic subgroup of index at most \(2^t\), then \(G/S\) is almost simple.

**Proof.** We first prove (1). Suppose, by contradiction, that \(|C \cap S|\) is even. This implies that \(S\) contains the unique involution of \(C\). By Lemmas 4.8 and 4.9, it follows that \(S\) is vertex-transitive, contradicting Lemma 4.10. We conclude that \(|C \cap S|\) is odd. Note that \(|G/S:CS/S| = |G|/|CS| = |G||C \cap S|/|S||C| = 3 \cdot 2^k|C \cap S|/|S|\), where \(k\) is the number of orbits of \(C\). Since \(|C \cap S|\) is odd it follows that \(|G/S:CS/S|_{2}|2| = 2^t\). This concludes the proof of (1).

We now prove (2). By Theorem 4.7 we have \(0 \leq t \leq 4\). By Lemma 4.10 \(\Gamma\) is a regular cover of \((\Gamma/S)\) and \(G_\circ \cong (G/S)_\circ\). Let \(N\) be the socle of \(G/S\). Write \(N = T_1 \times \cdots \times T_m\), such that the \(T_i\)'s are nonabelian simple and ordered such that the exponent of their Sylow 2-subgroups is non-increasing. We suppose that \(m \geq 2\) and will obtain a contradiction.

Let \(N_2\) be a Sylow 2-subgroup of \(N\). Recall that the Sylow 2-subgroup of a nonabelian simple group is never cyclic and, in particular, has order at least 4. Thus, any cyclic subgroup of \(N_2\) has index at least \(2|T_2| \cdots |T_m|\). On the other hand, \(N_2\) has a cyclic subgroup of index at most \(2^t\). It follows that \(2^t \geq 2|T_2| \cdots |T_m| \geq 2 \cdot 4^{m-1}\). Since \(t \leq 4\), we have \(m = 2, |T_1| \leq 8\) and \(t \geq 3\).

If \(N\) has at least three orbits on the vertices of \((\Gamma/S)/\Gamma\), then \((\Gamma/S)/\Gamma\) is a regular cover of \((\Gamma/S)/\Gamma\). By the Schreir Conjecture, \((G/S)/\Gamma\) is soluble and thus \(t \leq 2\) by Proposition 4.8, a contradiction. It follows that \(N\) has at most two orbits. If \(N\) is semiregular, then it follows by Lemma 4.9 that \(t \leq 1\). We may thus assume that \(N\) is locally-transitive.

We may thus apply Lemma 4.10 to conclude that \((\Gamma/S)/\Gamma\) is a regular cover of \((\Gamma/S)/\Gamma\). In particular, \(N/\Gamma_1\) is locally-transitive. Since \(N/\Gamma_1 \cong T_2, |T_2| \leq 8\) and \((\Gamma/S)/\Gamma_1\) has even order, we find that \(|(N/\Gamma_1)_{2}| \leq 4\), where \(\pi\) is a vertex in \((\Gamma/S)/\Gamma_1\), and thus \(|N_{\circ\pi}| = |(N/\Gamma_1)_{\pi}| \leq 12\). By Lemma 4.9, this implies \(|G_{\circ\pi} = |(G/S)_{\circ\pi}| \leq 12\) and thus \(t \leq 2\), a contradiction.

**Proposition 4.12.** Let \(t\) be an integer with \(0 \leq t \leq 4\), let \(k\) be a squarefree positive integer coprime to 6 and let \(\overline{G}\) be an almost simple group with order divisible by 3.

If \(\overline{G}\) has a cyclic subgroup \(\overline{C}\) of even order and index dividing \(3 \cdot 2^k\), then \(\overline{G}, |\overline{C}|\) and \(\log_2 |\overline{G} : \overline{C}|_2\) are given in Table 2.

**Proof.** Let \(T\) be the socle of \(\overline{G}\). Note that \(|T : T \cap \overline{C}|\) divides \(3 \cdot 2^k\), this will play a crucial role.

If \(T \cong \text{Alt}(n)\), then \(n < 9\) since the Sylow 3-subgroup of \(\overline{G}\) contains a cyclic subgroup of index dividing 3. The cases \(n \in \{5, 6, 7, 8\}\) yield rows (1 \(- 6\) of Table 2.

Suppose now that \(T\) is a sporadic simple group (including the Tits group). By considering the order of elements in \(T\) (see [9]), one can check that \(T\) does not have a cyclic subgroup of index dividing \(3 \cdot 2^k\) unless \(T\) is isomorphic to the Mathieu group \(M_{11}\) or the Janko group \(J_1\). Both of these have trivial outer automorphism group, hence \(\overline{G} = T\) and it is easy to check that \(\overline{C}\) must be as in rows (7) and (8) of Table 2.
From now on, we may thus assume that $T$ is a simple group of Lie type, of characteristic $r$, say. We record the order and a crude upper bound on the exponent of a Sylow $r$-subgroup of $T$ in Table 3. The orders can be found in [6, p xvi], while bounds for exponents are obtained by first taking the smallest dimension $n$ of an irreducible representation of $T$ (or some covering group) over a field of characteristic $r$ from [15, Table 5.4C], and then using the fact that an $r$-element in $\text{GL}(n, r')$ has order at most $r^e$ where $e = \lceil \log_r n \rceil \leq (n + 1)/2$ (see [14, §16.5] for example).

| $G$ | $|C|$ | $\log_2 |G : C|_2$ | Upper bound on $t$ | Upper bound on $\log_2 |S|_2$ |
|-----|-----|----------------|----------------|----------------|
| (1) $\text{Alt}(5)$ | 2 | 1 | 1 | 0 |
| (2) $\text{Sym}(5)$ | 2, 4 or 6 | 1 or 2 | 2 | 1 |
| (3) $\text{Sym}(6)$ | 6 | 3 | 3 | 0 |
| (4) $\text{Aut}(\text{Sym}(6))$ | 6 | 4 | 4 | 0 |
| (5) $\text{Alt}(7)$ | 6 | 2 | 3 | 1 |
| (6) $\text{Sym}(7)$ | 6 or 12 | 2 or 3 | 4 | 2 |
| (7) $\text{M}_{11}$ | 6 | 3 | 3 | 0 |
| (8) $J_1$ | 2, 6 or 10 | 2 | 2 | 0 |
| (9) $\text{Aut}(2B_2(8))$ | 4 or 12 | 4 | 0 | $-$ |
| (10) $\text{PSL}(2, 2^4)$ | 2 | 3 | 1 | $-$ |
| (11) $\text{PSL}(2, 2^4).2$ | 2, 4, 6 or 10 | 3 or 4 | 2 | $-$ |
| (12) $\text{PGL}(2, 2^4)$ | 4, 8 or 12 | 3 or 4 | 2 | $-$ |
| (13) $\text{PSL}(2, 2^5)$ | 2 | 4 | 1 | $-$ |
| (14) $\text{PGL}(2, 2^5)$ | 2 or 10 | 4 | 1 | $-$ |
| (15) $\text{PSL}(2, r)$, $r \geq 7$ | $(r+1)/2$ | $\geq 1$ | 3 | 2 |
| (16) $\text{PGL}(2, r)$, $r \geq 7$ | $r+1$ | $\geq 1$ | 3 | 2 |
| (17) $\text{PSL}(2, 2^2)$, $r \geq 5$ | $2r$ | $\geq 3$ | 4 | 1 |
| (18) $\text{PGL}(2, 2^2)$, $r \geq 5$ | $2r$ | $\geq 4$ | 4 | 0 |

Table 2.

| $T$ | $|T|_r$ | Upper bound on $r$-exponent | Condition |
|-----|--------|-----------------------------|-----------|
| $\text{PSL}(n, r')$ | $r^{f_n(n-1)/2}$ | $r^{(n+1)/2}$ | $n \geq 2$ |
| $\text{PSU}(n, r')$ | $r^{f_n(n-1)/2}$ | $r^{(n+1)/2}$ | $n \geq 3$ |
| $\text{PSp}(n, r')$ | $r^{f_{(n+1)/2}}$ | $r^{(n+1)/2}$ | $n \geq 4$, even |
| $\text{P}O(n, r')$ | $r^{(n+1)/2}$ | $r^{(n+1)/2}$ | $n \geq 7$, $n r$ odd |
| $\text{P}O'(n, r')$ | $r^{f_{(n+1)2/4}}$ | $r^{(n+1)/2}$ | $n \geq 8$, $n$ even |
| $E_6(r')$ | $r^{120f}$ | $r^8$ | $n \geq 2$ |
| $E_7(r')$ | $r^{61f}$ | $r^6$ | $n \geq 3$ |
| $E_8(r')$ | $r^{36f}$ | $r^5$ | $n \geq 4$, even |
| $2E_6(r')$ | $r^{36f}$ | $r^5$ | $n \geq 7$, $n r$ odd |
| $F_4(r')$ | $r^{24f}$ | $r^5$ | $n \geq 8$, $n$ even |
| $2F_4(2^{2m+1})$ | $2^{12f}$ | $2^5$ | $m \geq 2$ |
| $G_2(r')$ | $r^{6f}$ | $r^4$, $r$ odd | $n \geq 2$ |
| $G_2(r')$ | $r^{6f}$ | $r^3$, $r = 2$, $f \geq 2$ | $r \geq 2$, $f \geq 2$ |
| $\text{P}G_2(3^{2m+1})$ | $3^{3(2m+1)}$ | $3^2$ | $m \geq 2$ |
| $\text{P}G_2(3^{2m+1})$ | $2^{2(2m+1)}$ | $2^2$ | $m \geq 2$ |
| $3D_4(r')$ | $r^{12f}$ | $r^3$ | $m \geq 2$ |

Table 3. Orders and exponents of Sylow $r$-subgroups of simple groups of Lie type of characteristic $r$. 

CUBIC ARC-TRANSITIVE $&$-CIRCULANTS
Recall that $|T : T \cap C|$ divides $3 \cdot 2^f k$. In particular, a Sylow $r$-subgroup of $T$ must contain a cyclic subgroup of index at most $r$ if $r$ is odd and at most $16$ if $r = 2$. Using this fact and Table 3 we deduce that $T$ is isomorphic to one of $\text{PSp}(4, 2)$, $\text{PSU}(4, 2)$, $\text{PSL}(2, 4)$, $\text{PSL}(2, 3), 2B_3(8)$, $\text{PSU}(3, r^f)$, or $\text{PSL}(n, r^f)$ with $n \leq 3$.

It can be checked that $\text{PSU}(4, 2)$ and $\text{PSL}(4, 2)$ do not contain a cyclic subgroup of index dividing $3 \cdot 2^f k$, whereas the case $T \cong \text{PSp}(4, 2) \cong \text{Sym}(6)$ has already been dealt with. The group $2B_3(8)$ has order coprime to $3$ but its automorphism group yields row (9) of Table 2.

Suppose now that $T$ is isomorphic to $\text{PSL}(3, r^f)$ or $\text{PSU}(3, r^f)$. A Sylow $r$-subgroup of $T$ has order $r^3 f$ and exponent $2^2$ if $r = 2$, and $r$ otherwise. It follows that $r = 2$ and $f \leq 2$. It can be checked that no example arises when $f = 2$, while $\text{PSU}(3, 2)$ is soluble. Finally, we will deal with $T \cong \text{PSL}(3, 2) \cong \text{PSL}(2, 7)$ as part of our next and last case.

It remains to deal with the case $T \cong \text{PSL}(2, r^f)$. Since $\text{PSL}(2, 2)$ and $\text{PSL}(2, 3)$ are soluble, $\text{PSL}(2, 4) \cong \text{PSL}(2, 5) \cong \text{Alt}(5)$ and $\text{PSL}(2, 9) \cong \text{Alt}(6)$, we may assume that $r^f \geq 7$ and $r^f \neq 9$. The Sylow $r$-subgroup of $T$ has order $r^f$ and exponent $r$. In particular, $f \leq 2$ unless $r = 2$ in which case $f \leq 5$.

It can be checked that when $r = 2$ and $f \in \{3, 4, 5\}$, the examples that arise are in rows (10–14) of Table 2.

Suppose now that $f = 1$. In particular, $r$ is odd and $G = \text{PSL}(2, r)$ or $G = \text{PGL}(2, r)$. The orders of maximal cyclic subgroups of $\text{PSL}(2, r)$ are $(r + 1)/2$, $(r - 1)/2$ and $r$, while the orders of maximal cyclic subgroups of $\text{PGL}(2, r)$ are $(r + 1)$, $(r - 1)$ and $r$. Since $|C|$ is even, we get rows (15) and (16) of Table 2.

Finally, suppose that $f = 2$ and $r \geq 5$. Since $k$ is squarefree and $r^2$ divides $|\text{PSL}(2, r^2)|$, $r$ must divide $|\text{PSL}(2, r^2) \cap C|$. On the other hand, a Sylow $r$-subgroup $S$ of $\text{PSL}(2, r^2)$ is elementary abelian hence $|\text{PSL}(2, r^2) \cap C| = r$. Moreover, for each element $c$ of order $r$ in $S$, the centraliser of $c$ in $\text{PGL}(2, r^2)$ is $S$. Since $|C|$ is even, it follows that $\text{PGL}(2, r^2) \leq G$ and $|C| = 2r$. Note that $|\text{PGL}(2, r^2)| \geq 2^4$ and $|\text{PGL}(2, r^2)|_2 \geq 2^5$. This gives rows (17) and (18) of Table 2.

\section{Proof of Theorem 1.3}

In view of the statement of Theorem 1.3 we will consider the following hypothesis.

\textbf{Hypothesis 5.1.} Let $k \geq 5$ be a squarefree integer coprime to 6 and let $\Gamma$ be a cubic $(G, t + 1)$-arc-regular graph such that $C$ is semiregular with $k$ orbits.

Our goal is to show that $\Gamma$ has order at most $6k^2$. We introduce the following notation which we will use whenever we assume Hypothesis 5.1.

\textbf{Notation.} For a prime $p$ dividing $|G|$, we denote by $P_p$ a Sylow $p$-subgroup of $G$ and by $C_p$ a Sylow $p$-subgroup of $C$ contained in $P_p$. (Note that we may have $C_p = 1$.) Let $c$ be the unique involution in $C$. ($C$ has even order since $\Gamma$ does but $k$ is odd.)

We denote by $S$ the soluble radical of $G$ and write $G = G/S$ and $C = CS/S$. Let $T$ be the socle of $C$.

We first note a few obvious facts about $G$ and $C$ that will be very useful.

\textbf{Lemma 5.2.} Assuming Hypothesis 5.1, the following holds.

\begin{enumerate}
\item $|G| = 3 \cdot 2^f k |C|.$
\item $|P_2 : C_2| = 2^f.$
\item For every odd prime $p$, we have that $|P_p : C_p|$ divides $p$.
\end{enumerate}

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\begin{enumerate}
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\item $|P_2 : C_2| = 2^f.$
\item For every odd prime $p$, we have that $|P_p : C_p|$ divides $p$.
\end{enumerate}
5.1. $G$ Soluble. We first focus on the case when $G$ is soluble.

**Lemma 5.3.** Assume Hypothesis [7] If $G$ is soluble, then $t \leq 1$ and, for every prime $p$, we have $|P_p : C_p| \leq p$.

**Proof.** By Lemma [5.2] it suffices to show that $t \leq 1$. Suppose that $t \geq 2$. By Proposition [1.3] $t = 2$ and $\Gamma$ is a regular cover of $K_{3,3}$.

Since $G$ is soluble, $\Gamma$ is a regular cover of $\Gamma^*$ which is itself a regular $\mathbb{Z}^p$-cover of $K_{3,3}$ for some prime $q$ and some integer $a \geq 1$. Since $t = 2$, it follows by [10] Theorem 1.1 and [9] Theorem 4.1] that $a \geq 4$, or $q = 3$ and $a \neq 2$.

By Lemma [5.2] $P_q$ has a cyclic subgroup of index dividing $q$ or $4$. In particular, every elementary abelian section of $P_q$ has rank at most 2, unless $q = 2$, in which case it has rank at most 3. By the previous paragraph, we get that $q = 3$ and $a = 1$ and, by [10] Theorem 1.1], $\Gamma^*$ is isomorphic to the Pappus graph. Since $t = 2$ and the Pappus graph is 3-arc-regular, Aut$(\Gamma^*)$ is a quotient of $G$. This is a contradiction because the Sylow 3-subgroup of Aut$(\Gamma^*)$ does not have a cyclic maximal subgroup. □

**Lemma 5.4.** Assume Hypothesis [7] and let $p \geq 5$ be a prime dividing the order of $\Gamma$. If $P_p$ is normal in $G$ then

1. $c$ does not centralise $P_p$, and
2. $|P_p : C_p| = p$ and $|C_p| \leq p$.

**Proof.** We first prove (1). Suppose, by contradiction, that $c$ centralises $P_p$. Let $Z$ be the centraliser of $P_p$ in $G$. This is a normal subgroup of $G$. By the Schur-Zassenhaus Theorem, we can write $Z = Z(P_p) \times Y$ where $Y$ is a $p'$-group. Note that $Y$ is characteristic in $Z$ and thus normal in $G$. Since $p$ is odd, we have $c \in Y$. By Lemmas [1.3] and [4.4] it follows that $Y$ is transitive on the vertices of $\Gamma$, a contradiction, as $p$ divides the order of $\Gamma$. This concludes the proof of (1).

We now prove (2). By (1), $P_p \nsubseteq C$ and thus Lemma [5.2] implies $|P_p : C_p| = p$. In particular, $P_p$ is a $p$-group with a cyclic maximal subgroup.

Let $X$ be the group generated by elements of order $p$ in $P_p$. This is a characteristic subgroup of $P_p$ and thus normal in $G$. Suppose first that $X = P_p$. The result then follows by Lemma [4.1][4].

Suppose next that $C_pX < P_p$. Since $C_p$ is maximal in $P_p$, this implies that $X \leq C_p$. It follows by Lemma [4.1][3] that $P_p$ is cyclic. By (1), $c$ centralises $C_p$ but not $P_p$ and thus $|P_p| = p$ by Lemma [4.1][5].

From now, we assume that $X < P_p = C_pX$. This implies that $1 \neq P_p/X \leq CX/X$ and thus $P_p/X$ is a non-trivial normal Sylow $p$-subgroup of $G/X$. By Lemma [4.5] Hypothesis [5.1] is satisfied with $(k, \Gamma, G, C)$ replaced by $(k', \Gamma/X, G/X, CX/X)$ for some divisor $k'$ of $k$. In particular, $p$ divides the order of $\Gamma/X$ and we may apply (1) to conclude that $cX$ does not centralise $P_p/X$, contradicting the fact that $P_p/X \leq CX/X$. □

**Theorem 5.5.** Assume Hypothesis [5.1]. If $G$ is soluble, then $\Gamma$ has order at most $6k^2$.

**Proof.** By Lemma [5.3] every Sylow $p$-subgroup of $G$ is metacyclic. It follows by [3] Theorem 1 that $G = N \rtimes A$, where $A$ is a Hall $(2, 3)$-subgroup of $G$ and $N$ has a normal series

$$1 = N_0 \unlhd N_1 \unlhd \cdots \unlhd N_n = N$$

where $N_{i+1}/N_i \cong P_{p_i}$. For every $i \in \{0, \ldots, n\}$, $|N_i|$ is coprime to 6 and thus semiregular. In particular, $\Gamma$ is a regular cover of $\Gamma/N_i$ and, by Lemma [4.5] $CN_i/N_i$ is semiregular and has $n_i$ orbits on $\Gamma/N_i$ for some divisor $\kappa_i$ of $k$. It follows that $(\Gamma/N_i, G/N_i)$ satisfies Hypothesis [5.1] with $(k, \Gamma, G, t, C)$ replaced by
(κ_i, Γ/N_i, G/N_i, t, CN_i/N_i). Note that N_{i+1}/N_i is a normal Sylow p_i-subgroup of G/N_i and we may thus apply Lemma 5.4 to conclude that |C_{p_i}| ≤ |P_{p_i} : C_{p_i}| = k_{p_i}.
Finally, k is coprime to 6 but G/N_n = G/N ∼= A is a {2, 3}-group and thus κ_n = 1. Hence Γ/N is a cubic arc-transitive circulant and thus has order at most 6. It follows that |C_2||C_3| ≤ 6 hence |C| ≤ 6k, which concludes the proof.

5.2. G not soluble. We now consider the remaining case, namely when G is not soluble.

Lemma 5.6. Assume Hypothesis 5.7. If G is insoluble, then |C|, |G|, log_2 |C| : |G| and upper bounds for t and log_2 |S| are as in rows (1–8) or (15–18) of Table 3.

Proof. By Lemma 5.10 Γ/S is a cubic (G, t + 1)-arc-regular graph. In particular, 3 divides |G|. By Lemma 4.11 |C ∩ S| is odd, and thus C is an almost simple group. Recall that |G : C| = 3 · 2^t. By Proposition 4.12, |C| and log_2 |S| are as in one of the rows of Table 2.

We now compute upper bounds on t and record them in Table 3. We do this by using the fact that the isomorphism type of the vertex-stabiliser G_φ is uniquely determined by t (see Theorem 4.7). For example, Alt(7) does not contain a subgroup isomorphic to Sym(4) × Z_2 and thus t ≤ 3 when G ∼= Alt(7). The fact that PSL(2, r) does not contain a subgroup isomorphic to Sym(4) × Sym(2) follows from Dickson’s classification of the subgroups of PSL(2, r) [7].

We then combine this upper bound on t with Lemma 5.11(1) to obtain an upper bound on log_2 |S|, which we also record in Table 3. (When log_2 |C| : |G| > t, we obtain a contradiction and record this as a –.)

Theorem 5.7. Assume Hypothesis 5.7. If G is insoluble, then Γ has order at most 6k^2.

Proof. By Lemma 5.6 |C|, |G|, log_2 |C| : |G| and upper bounds for t and |S| are as in Table 3. Write G = S.T.A. Note that G ∼= T.A and we can read off A from Table 2. In fact, |A| ≤ 2, unless G ∼= PGL(2, r^2), in which case |A| = 4. We denote by G_infty the last term of the derived series of G. By the Schreier conjecture, G_infty ∼= Y.T for some normal subgroup Y of S. Let

1 = S_0 ⊂ S_1 ⊂ · · · ⊂ S_n = S

be a maximal characteristic series for S. For every i ∈ {0, . . . , n − 1}, let φ_i : G → Aut(S_{i+1}/S_i) and let K_i be the kernel of φ_i in G_infty.

Suppose that φ_i(G_infty) is insoluble for some i. Since S_{i+1}/S_i is characteristically simple, it is elementary abelian, say S_{i+1}/S_i ∼= Z_p^a. By Table 2 |S| ≤ 4. Together with Lemma 5.2 this implies that a ≤ 2. Since φ_i(G_infty) is insoluble, a = 2, p ≥ 5 and Aut(S_{i+1}/S_i) ∼= GL(2, p).
By Dickson’s classification of subgroups of PSL(2, p) [7], either SL(2, p) ≤ φ_i(G_infty) or SL(2, 5) ≤ φ_i(G_infty) ≤ SL(2, 5) ⊂ Z_{p−1}. In the latter case, C ∼= Alt(5) and |S| is even, contradicting Table 2. Thus SL(2, p) ≤ φ_i(G_infty) and T = PSL(2, p). By [1], Table 1, an extension of Z_p^2 by SL(2, p) splits hence G contains a group of order p^3 and exponent p as a section, contradicting Lemma 5.2.

It follows that G_infty/K_i ∼= φ_i(G_infty) is soluble for every i. Since G_infty is perfect, it follows that G_infty = K_i and thus φ_i(G_infty) = 1. Since this is true for every i ∈ {0, . . . , n − 1}, it follows by Lemma 4.2 and induction that G_infty ≤ C_G(S) and G_infty ∩ S ≤ Z(G_infty). On the other hand, Z(G_infty) is an abelian normal subgroup of G hence Z(G_infty) ≤ S and thus G_infty ∩ S = Z(G_infty). In particular, G_infty/Z(G_infty) = G_infty/(G_infty ∩ S) ∼= G_infty/S/S = T. Since T is simple, we conclude that G_infty is quasisimple. In particular, Y is a subgroup of the Schur multiplier of T.
We want to show that the order of $\Gamma$ is at most $6k^2$. This is equivalent to $|C| \leq 6k = \frac{6|G|}{|G|_c}$, and thus to $|G| \geq 2^{-1}|C|^2$. On the other hand, $|C| = |G|/|S|$ but $|S|$ is odd by Lemma LI.1, hence $|C| \leq |G|/|S|_{2'}$. Since $|G| = |G/|S| \geq |G/|S|_{2'}$, it thus suffices to show that

$$|G/|S|_{2'}|S|_{2'} \geq 1$$

Suppose first that $\Gamma$ is semiregular on the vertices of $\Gamma$. This implies that $|G/\Gamma|_2 \geq 2^t$. On the other hand, $G/\Gamma \cong (S/Y).\psi$ hence $|G/\Gamma|_2 \leq |S|_{2'}|A|$. Combining this with Lemma LI.1, we get $|G/\Gamma|_2(\Gamma/\psi|_2 \leq 2^t|A| \leq |G/\Gamma|_2|A|$ and thus $\Gamma/\psi|_2 \leq |\psi|$. By running through Table 2, we find that $\psi \cong \text{PGL}(2, r)$ with $r \geq 5$ and $|\psi| = |\Gamma/\psi|_2 = 2$. (Note that this includes the case $\psi \cong \text{Sym}(5)$.) Using the previous inequalities, this implies that $|G/\Gamma|_2 = 2^t$ and thus $\Gamma$ has an odd number of orbits. Since $\Gamma$ is semiregular, it follows that it is transitive hence $|(S/Y).\psi| = |G/\Gamma| = |G|_c = 3 \cdot 2^t$ and Lemma LI.1 implies $t \leq 1$. Since $|\psi| = 2$, we have $t = 1$ and $|S|/Y = 3$. On the other hand, since the Schur multiplier of $\text{PSL}(2, r)$ has order 2, we see that $|Y|_{2'} = 1$ and thus $|S|_{2'} = 3$. Now, $\Gamma = (r + 1)r(r - 1)$ while $|\psi| \leq |\psi| + 1$ hence (1) is satisfied.

We may thus assume that $\Gamma$ is not semiregular on the vertices of $\Gamma$. In particular, $\Gamma$ is locally transitive and has at most two orbits on the vertices of $\Gamma$. It follows that $G/\Gamma$ is a 2-group hence so is $S/Y$ and thus $|S|_{2'} = |Y|_{2'}$. Now, by considering the Schur multiplier of $T$, we find that $|Y|_{2'} = 1$ unless $T$ is isomorphic to $\text{Alt}(6)$ or $\text{Alt}(7)$, when we may have $|Y|_{2'} = 3$. It is then a matter of routine to go through Table 2 and verify that (1) is satisfied.

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