THE EXPONENTIAL BEHAVIOR OF A STOCHASTIC CAHN-HILLIARD-NAVIER-STOKES MODEL WITH MULTIPLICATIVE NOISE

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Abstract. In this article, we study the stability of weak solutions to a stochastic version of a coupled Cahn-Hilliard-Navier-Stokes model with multiplicative noise. The model consists of the Navier-Stokes equations for the velocity, coupled with an Cahn-Hilliard model for the order (phase) parameter. We prove that under some conditions on the forcing terms, the weak solutions converge exponentially in the mean square and almost surely exponentially to the stationary solutions. We also prove a result related to the stabilization of these equations.

1. Introduction. The incompressible Navier-Stokes (NS) equation governs the motions of single-phase fluids such as air or water. On the other hand, we are faced with the difficult problem of understanding the motion of binary fluid mixtures, that is fluids composed by either two phases of the same chemical species or phases of different composition. Diffuse interface models are well-known tools to describe the dynamics of complex (e.g., binary) fluids, [17]. For instance, this approach is used in [4] to describe cavitation phenomena in a flowing liquid. The model consists of the NS equation coupled with a phase-field system, [8, 17, 16, 18]. In the incompressible isothermal case, neglecting chemical reactions and other forces, the model reduces to an evolution system which governs the fluid velocity $v$ and the order parameter $\phi$. This system can be written as a NS equation coupled with a convective Allen-Cahn equation or a Cahn-Hilliard equation, [17, 16]. The associated initial and boundary value problem was studied in [17, 16], in which the authors proved that the system generated a strongly continuous semigroup on a suitable phase space.

The long-time behavior of flows is a very interesting and important problem in the theory of fluid dynamic. As the vast literature [27, 15, 3, 21, 23, 25, 20, 19, 11, 10] shows, the problem has been receiving very much attention over the last three decades. Another interesting question is to analyze the effects produced on a deterministic system by some stochastic or random disturbances appearing in the model. This problem has been studied for the NS model, [12, 13, 26]. In [12], the authors studied the stability of the stationary solutions of the stochastic 2D NS equations. In particular, they proved that the weak solutions converge

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exponentially in the mean square and almost surely exponentially to the stationary solutions under some restrictions on the viscosity and the forcing terms. In [13], the authors generalized to the results of [12] to a class of dissipative nonlinear systems that include the 3D lagrangian average NS equations.

Our work is motivated by the above references. We study the stability of weak solutions to the stochastic 2D CH-NS model with multiplicative noise. One of our goals is to investigate the exponential convergence in mean square and almost surely of the weak probabilistic solutions to stationary solutions. Using techniques similar to that of [12, 13], we prove that the exponential convergence in mean square and almost surely of the weak probabilistic solutions to stationary solutions hold, provided some reasonable restrictions on the forcing terms $g_1$, $g_2$, as well as some physical parameters such as the viscosity $\nu$ and the physical constant $\epsilon > 0$, which is related to the thickness of the interface separating the two fluids. Under similar restrictions on the noise term $g_2$ and the physical parameters $\nu$ and $\epsilon$, we also show that there is a stabilization effect of the stochastic perturbation. It is worth mentioning that our model does not fall into the general framework considered in [13], since the coupling between the Navier-Stokes and the Cahn-Hilliard systems introduces in the model a highly nonlinear term that makes the analysis of the problem more involved.

The article is divided as follows. In the next section, we introduce the stochastic 2D CH-NS model and its mathematical setting. The third section studies the stability of weak solutions. As in [12], applying the Itô formula, we study the stability of stationary solutions to the stochastic 2D CH-NS model. We also prove in the fourth section a result related to the stabilization of these equations.

2. The stochastic CH-NS model and its mathematical setting.

2.1. Governing equations. In this article, we consider a coupled CH-NS model with multiplicative noise. More precisely, we assume that the domain $\mathcal{M}$ of the fluid is a bounded domain in $\mathbb{R}^2$. Then, we consider the system

$$
\begin{aligned}
\frac{\partial v}{\partial t} - \nu \Delta v + (v \cdot \nabla)v + \nabla p - K\mu \nabla \phi &= g_1^1(v, \phi) + g_1^2(t, v, \phi) \dot{W}_t^1, \\
\text{div} v &= 0, \\
\frac{\partial \phi}{\partial t} + v \cdot \nabla \phi - \Delta \mu &= g_2^1(v, \phi) + g_2^2(t, v, \phi) \dot{W}_t^2, \\
\mu &= -\epsilon \Delta \phi + \alpha f(\phi),
\end{aligned}
$$

(2.1)
in $\mathcal{M} \times [0, T]$.

In (2.1), the unknown functions are the velocity $v = (v_1, \cdots, v_d)$ of the fluid, its pressure $p$ and the order (phase) parameter $\phi$. The external volume force $g_1(v, \phi) \equiv (g_1^1(v, \phi), g_1^2(v, \phi))$ is given. The term $g_2(t, v, \phi) \dot{W}_t \equiv (g_2^1(t, v, \phi) \dot{W}_t^1, g_2^2(t, v, \phi) \dot{W}_t^2)$ represents random external forces depending eventually on $(v, \phi)$ where $\dot{W}_t \equiv (\dot{W}_t^1, \dot{W}_t^2)$ denotes the time derivative of a cylindrical Wiener process. The quantity $\mu$ is the variational derivative of the following free energy functional

$$
\mathcal{F}_1(\phi) = \int_{\mathcal{M}} \left( \frac{\epsilon}{2} |\nabla \phi|^2 + \alpha F(\phi) \right) ds,
$$

(2.2)

where, e.g., $F(x) = \int_0^x f(\zeta) d\zeta$. Here, the constants $\nu > 0$ and $K > 0$ correspond to the kinematic viscosity of the fluid and the capillarity (stress) coefficient respectively. Here $\epsilon, \alpha > 0$ are two physical parameters describing the interaction between the two phases. In particular, $\epsilon$ is related with the thickness of the interface separating the two fluids. Hereafter, as in [17] we assume that $\epsilon \leq \alpha$. 
A typical example of potential $F$ is that of logarithmic type. However, this potential is often replaced by a polynomial approximation of the type $F(x) = \gamma_1 x^4 - \gamma_2 x^2$, $\gamma_1, \gamma_2$ being positive constants. As noted in [16], (2.1) can be replaced by

$$
\frac{\partial v}{\partial t} - \nu \Delta v + (v \cdot \nabla)v + \nabla \hat{p} = -K \text{div} (\nabla \phi \otimes \nabla \phi) + g_1^2(v, \phi) + g_2^2(t, v, \phi) W_t^1, \tag{2.3}
$$

where $\hat{p} = p - \mathcal{K}(\frac{1}{2} |\nabla \phi|^2 + \alpha F(\phi))$, since $\mathcal{K} \mu \nabla \phi = \nabla(\mathcal{K}(\frac{1}{2} |\nabla \phi|^2 + \alpha F(\phi))) - \mathcal{K} \text{div} (\nabla \phi \otimes \nabla \phi)$. The stress tensor $\nabla \phi \otimes \nabla \phi$ is considered the main contribution modeling capillary forces due to surface tension at the interface between the two phases of the fluid.

Regarding the boundary conditions for the model, as in [16] we assume that the boundary conditions for $\phi$ are the natural no-flux condition

$$
\partial_n \phi = \partial_n \mu = 0, \quad \text{on } \partial M \times [0, T], \tag{2.4}
$$

where $\partial M$ is the boundary of $M$ and $\eta$ is the outward normal to $\partial M$. These conditions ensure the mass conservation. In fact, from $\partial_n \mu = 0$ on $\partial M \times [0, T]$, we have the conservation of the following quantity

$$
\langle \phi(t) \rangle = \frac{1}{|M|} \int_M \phi(x, t) dx, \tag{2.5}
$$

where $|M|$ stands for the Lebesgue measure of $M$. More precisely, we have

$$
\langle \phi(t) \rangle = \langle \phi(0) \rangle, \quad \forall t \in [0, T]. \tag{2.6}
$$

Concerning the boundary condition for $v$, we assume the Dirichlet (no-slip) boundary condition

$$
v = 0, \quad \text{on } \partial M \times (0, \infty). \tag{2.7}
$$

Therefore we assume that there is no relative motion at the fluid-solid interface.

The initial condition is given by

$$(v, \phi)(0) = (v_0, \phi_0) \text{ in } M. \tag{2.8}$$

2.2. Mathematical setting. We first recall from [16] a weak formulation of (2.1), (2.4), (2.7)-(2.8). Hereafter, we assume that the domain $M$ is bounded with a smooth boundary $\partial M$ (e.g., of class $C^2$). We also assume that $f \in C^2(\mathbb{R})$ satisfies

$$
\begin{cases}
\lim_{|r| \to +\infty} f'(r) > 0, \\
|f'(r)| \leq c_f(1 + |r|^k), \quad \forall r \in \mathbb{R},
\end{cases} \tag{2.9}
$$

where $c_f$ is some positive constant and $k \in [1, +\infty)$ is fixed. It follows from (2.9) that

$$
|f(r)| \leq c_f(1 + |r|^{k+1}), \quad \forall r \in \mathbb{R}. \tag{2.10}
$$

If $X$ is a real Hilbert space with inner product $(\cdot, \cdot)_X$, we will denote the induced norm by $| \cdot |_X$, while $X^*$ will indicate its dual. We set

$$
V_1 = \{ u \in C^\infty_c(M) : \quad \text{div } u = 0 \text{ in } M \}. 
$$

We denote by $H_1$ and $V_1$ the closure of $V_1$ in $(L^2(M))^2$ and $(H^1_0(M))^2$ respectively. The scalar product in $H_1$ is denoted by $(\cdot, \cdot)_{L^2}$ and the associated norm by $| \cdot |_{L^2}$. Moreover, the space $V_1$ is endowed with the scalar product

$$
((u, v)) = \sum_{i=1}^{2} (\partial_x u, \partial_x v)_{L^2}, \quad \|u\| = (\langle (u, u) \rangle)^{1/2}. 
$$
We now define the operator $A_0$ by

$$A_0 v = \mathcal{P} \Delta v, \quad \forall v \in D(A_0) = H^2(\mathcal{M}) \cap V_1,$$

where $\mathcal{P}$ is the Leray-Helmholtz projector in $L^2(\mathcal{M})$ onto $H_1$. Then, $A_0$ is a self-adjoint positive unbounded operator in $H_1$ which is associated with the scalar product defined above. Furthermore, $A_0^{-1}$ is a compact linear operator on $H_1$ and $|A_0 \cdot |_{L^2}$ is a norm on $D(A_0)$ that is equivalent to the $H^2-$norm.

Hereafter, we set

$$H_2 = L^2(\mathcal{M}), \quad V_2 = H^1(\mathcal{M}), \quad H = H_1 \times H_2, \quad V = V_1 \times V_2.$$  \hfill (2.11)

Then we introduce the linear nonnegative unbounded operator on $L^2(\mathcal{M})$

$$A_1 \phi = -\Delta \phi, \quad \forall \phi \in D(A_1) = \{ \phi \in H^2(\mathcal{M}), \quad \partial_\nu \phi = 0, \quad \text{on } \partial \mathcal{M} \},$$

and we endow $D(A_1)$ with the norm $|A_1 \cdot |_{L^2} + |\cdot |_{L^2}$, which is equivalent to the $H^2-$norm. Also we define the linear positive unbounded operator on the Hilbert space $L^2_0(\mathcal{M})$ of the $L^2-$ functions with null mean

$$B_n \phi = -\Delta \phi, \quad \forall \phi \in D(B_n) = D(A_1) \cap L^2_0(\mathcal{M}).$$ \hfill (2.13)

Note that $B_n^{-1}$ is a compact linear operator on $L^2_0(\mathcal{M})$. More generally, we can define $B_n^s$, for any $s \in \mathbb{N}$, noting that $|B_n^{s/2} \cdot |_{L^2}$, $s > 0$, is an equivalent norm to the canonical $H^s-$ norm on $D(B_n^{s/2}) \subset H^s(\mathcal{M}) \cap L^2_0(\mathcal{M})$. Also note that $A_1 = B_n$ on $D(B_n)$. If $\phi$ is such that $\phi - \langle \phi \rangle \in D(B_n^{s/2})$, we have that $|B_n^{s/2} (\phi - \langle \phi \rangle) |_{L^2} + |\langle \phi \rangle |_{L^2}$ is equivalent to the $H^s-$norm. Moreover, we set $H^{-s}(\mathcal{M}) = (H^s(\mathcal{M}))^*$, whenever $s < 0$.

We introduce the bilinear operators $B_0, B_1$ (and their associated trilinear forms $b_0, b_1$) as well as the coupling mapping $R_0$, which are defined from $D(A_0) \times D(A_0)$ into $H$, $D(A_0) \times D(A_1)$ into $L^2(\mathcal{M})$, and $L^2(\mathcal{M}) \times (D(A_1) \cap H^3(\mathcal{M}))$ into $H_1$, respectively. More precisely, we set

$$B_0(u, v) = \int_{\mathcal{M}} [(u \cdot \nabla)v] \cdot w \, dx = b_0(u, v, w), \quad \forall u, v, w \in D(A_0),$$

$$B_1(u, \phi, \rho) = \int_{\mathcal{M}} [(u \cdot \nabla)\phi] \rho \, dx = b_1(u, \phi, \rho), \quad \forall u \in D(A_0), \quad \phi, \rho \in D(A_1),$$

$$R_0(\mu, \phi, w) = \int_{\mathcal{M}} \mu [\nabla \phi \cdot w] \, dx = b_1(w, \phi, \mu), \quad \forall w \in D(A_0), \quad \phi \in D(A_1) \cap H^3(\mathcal{M}), \quad \mu \in L^2(\mathcal{M}).$$ \hfill (2.14)

Note that

$$R_0(\mu, \phi) = \mathcal{P} \mu \nabla \phi.$$

We recall that $B_0, B_1$ and $R_0$ satisfy the following estimates

$$|B_0(u, v)|_{V_1^*} \leq c |u|_{L^2}^{1/2} |v|_{L^2}^{1/2} \|v\|_1, \quad \forall u, v \in V_1,$$

$$|B_0(u, v)|_{L^2} \leq c |u|_{L^2}^{1/2} |v|_{L^2}^{1/2} \|v\|^{1/2} |A_0 v|_{L^2}^{1/2}, \quad \forall u \in V_1, \quad v \in D(A_0),$$ \hfill (2.15)

$$|B_1(u, \phi)|_{V_1^*} \leq c |u|_{L^2}^{1/2} |\phi|^{1/2} \|\phi\|_1, \quad \forall u \in V_1, \quad \phi \in V_2,$$

$$|B_1(u, \phi)|_{L^2} \leq c |u|_{L^2}^{1/2} |\phi|^{1/2} \|\phi\|^{1/2} |A_1 \phi|_{L^2}^{1/2}, \quad \forall u \in V_1, \quad \phi \in D(A_1),$$ \hfill (2.16)

$$|R_0(A_1 \phi, \rho)|_{V_1^*} \leq c |A_1 \phi|_{L^2}^{1/2} |\phi|^{1/2} \|\rho\|_1, \quad \forall \phi \in D(A_1), \quad \rho \in V_2,$$

$$|R_0(A_1 \phi, \rho)|_{L^2} \leq c |\rho|^{1/2} |A_1 \phi|_{L^2}^{1/2} |\phi|^{1/2} |\rho|_1, \quad \forall \phi \in D(A_1), \quad \rho \in D(A_1^{3/2}).$$ \hfill (2.17)
We recall that (due to the mass conservation) we have
\[\langle \phi(t) \rangle = \langle \phi(0) \rangle = M_0, \forall t > 0.\] (2.18)

Thus, up to a shift of the order parameter field, we can always assume that the mean of \(\phi\) is zero at the initial time and, therefore it will remain zero for all positive times. Hereafter, we assume that
\[\langle \phi(t) \rangle = \langle \phi(0) \rangle = 0, \forall t > 0.\] (2.19)

We set
\[\mathcal{H} = H_1 \times D(A_1^{1/2}).\] (2.20)

The space \(\mathcal{H}\) is a complete metric space with respect to the norm
\[\| (v,\phi) \|_\mathcal{H}^2 = |v|_{L^2}^2 + |\nabla \phi|_{L^2}^2.\] (2.21)

We define the Hilbert space \(\mathcal{U}\) by
\[\mathcal{U} = V_1 \times D(A_1^{3/2}),\] (2.22)

endowed with the scalar products whose associated norm is
\[\| (v,\phi) \|_\mathcal{U}^2 = \| v \|^2 + |A_{1/2}^3 \phi|_{L^2}^2.\] (2.23)

We will denote by \(\lambda_1 > 0\) a positive constant such that
\[\lambda_1 \langle (w,\psi) \rangle_\mathcal{H} \leq \| (w,\psi) \|_\mathcal{U}^2, \forall (w,\psi) \in \mathcal{U}.\] (2.24)

We will also denote by \(c\) a generic positive constant that depends on the domain \(\mathcal{M}\). To simplify the notations, we set (without loss of generality) \(K = 1\).

Let \((\Omega, \mathcal{P}, \mathcal{F})\) be a probability space on which an increasing and right continuous family \(\{\mathcal{F}_t\}_{t \in [0,\infty)}\) of complete sub \(\sigma\)-algebra of \(\mathcal{F}\) is defined. Let \(\beta_n(t) (n = 1,2,3,\cdots)\) be a sequence of real valued one-dimensional standard Brownian motions mutually independent on \((\Omega, \mathcal{P}, \mathcal{F})\).

We set
\[\dot{W}_t(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} \beta_n(t)e_n, \quad t \geq 0,\] (2.25)

where \(\lambda_n (n = 1,2,3,\cdots)\) are nonnegative real numbers such that \(\sum_{n=1}^{\infty} \lambda_n < \infty\), and \(\{e_n\} (n = 1,2,3,\cdots)\) is a complete orthogonal basis in the real and separable Hilbert space \(K\). Let \(Q \in L(K,K)\) be the operator defined by \(Qe_n = \lambda_n e_n\). The above \(K\)-valued stochastic process \(W(t)\) is called a \(Q\)-Wiener process.

Thus, we consider the stochastic coupled CH-NS model written in the following abstract mathematical setting:
\[
\begin{align*}
&\frac{dv}{dt} + \nu A_0 v + B_0(v,v) - R_0 (\epsilon A_1 \phi, \phi) = g_1^1 (v,\phi) + g_1^2 (t,v,\phi) \dot{W}_1^1 \quad \text{in} \ V_1^*, \\
&\frac{d\phi}{dt} + A_1 \mu + B_1 (v,\phi) = g_2^1 (v,\phi) + g_2^2 (t,v,\phi) \dot{W}_2^2, \quad \mu = \epsilon A_1 \phi + f(\phi) \quad \text{in} \ V_2^*, \\
&(v,\phi)(0) = (v_0,\phi_0) \in \mathcal{H},
\end{align*}
\] (2.26)
or equivalently
\[
\begin{aligned}
v(t) + \int_0^t (vA_0v(s) + B_0(v(s), v(s)))ds &= v_0 + \int_0^t R_0(\varepsilon A_1\phi(s), \phi(s))ds \\
+ \int_0^t g_1^1(v, \phi)(s)ds + \int_0^t g_2^1(s, v(s), \phi(s))dW^1_s, \\
\phi(t) &= \phi_0 + \int_0^t \langle A_1\mu(s) + B_1v(s), \phi(s) \rangle ds,
\end{aligned}
\]
\[
\begin{aligned}
\mu &= \varepsilon A_1\phi + f(\phi), \\
P - a.s., \text{ and for all } t \in [0, T], \text{ where }
\end{aligned}
\]
\[
g_1 \equiv (g_1^1, g_1^2) : U \to U^*, \quad g_2 \equiv (g_2^1, g_2^2) : [0, \infty) \times U \to L(K, \mathcal{H}).
\]

**Remark 2.1.** In the formulation (2.26) or (2.27), the term \(\mu \nabla \phi\) is also bounded in \(H\). This is justified since \(f(\phi)\) is the gradient \(F(\phi)\) and can be incorporated into the pressure gradient, see [16] for details.

**Definition 2.1.** A stochastic process \((v, \phi)(t), t \geq 0\) is said to be a weak solution to (2.26) or (2.27) if

i) \((v, \phi)(t) = \mathcal{J}_t -\) adapted,

ii) \((v, \phi)(t) \in L^2(0, T; \mathcal{H}) \cap L^2(0, T; U)\) almost surely for all \(T > 0\),

iii) \((v, \phi)\) satisfies (2.27) as an identity in \(U^*\), almost surely, for \(t \in [0, \infty)\).

Note that (2.27) implies that almost surely, \((v, \phi) \in C(0, T; U^*)\) and since \((v, \phi)(\cdot)\) is also bounded in \(\mathcal{H}\), as in [27, 28] we can check that \((v, \phi)\) is almost surely in \(C(0, T; \mathcal{H}_{weak})\), the space of \(\mathcal{H}\)-valued weakly continuous functions on \([0, T]\).

In the deterministic case, the weak formulation of (2.26) was proposed and studied in [7, 5, 6, 17, 16] (see also [2, 1, 9]), where the existence and uniqueness results for weak and strong solutions were proved in the deterministic case.

In [22], the author studied a stochastic CH-NS model in a two or three-dimensional domain. Using a Galerkin approximation, he proved the existence of weak solutions.

We will use the notation
\[
\|g_2(t, v, \phi)\|^2_{L^2(\mathcal{H})} \equiv tr(g_2(t, v, \phi)Qg_2(t, v, \phi)^*)
\]
\[
\langle (x_1, x_2), (y_1, y_2) \rangle = \langle x_1, y_1 \rangle + \langle x_2, y_2 \rangle, \quad \forall (x_1, x_2), (y_1, y_2) \in \mathcal{H}.
\]  
(2.29)

We assume that for any \((v_1, \phi_1), (v_2, \phi_2) \in \mathcal{U}\), \(g_2(\cdot; (v_1, \phi_1))\) is \(\mathcal{F}_t -\)progressively measurable, \(d\mathbb{P} \oplus dt\) a.e. in \(\Omega \times (0, T)\) and

\[
\begin{aligned}
\|g_1(v_1, \phi_1) - g_1(v_2, \phi_2)\|^2_{L^2(\mathcal{H})} &\leq L_1\|(v_1, \phi_1) - (v_2, \phi_2)\|_{\mathcal{H}}, \\
\|g_2(t, v_1, \phi_1) - g_2(t, v_2, \phi_2)\|^2_{L^2(\mathcal{H})} &\leq L_2\|(v_1, \phi_1) - (v_2, \phi_2)\|_{\mathcal{H}},
\end{aligned}
\]  
(2.30)

where \(L_1 > 0\), \(L_2 > 0\) are fixed.

Moreover, we also assume that for all \(T > 0\), we have

\[
g_2(\cdot, (0, 0)) \in L^4(\Omega, L^2(0, T; L^2(K, \mathcal{H}))), \quad (v_0, \phi_0) \in L^4(\Omega, \mathcal{F}_0, \mathbb{P}, \mathcal{H}).
\]  
(2.31)

The following result is proved in [22, 24].

**Theorem 1.** We assume that the hypotheses (2.30)-(2.31) above are satisfied. Then, there exists a unique probabilistic weak solution \(\{\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \in [0, T]}, W, v, \phi\}\) to problem (2.26) or (2.27) such that

\[
(v, \phi) \in L^4(\Omega, C(0, T; \mathcal{H})) \cap L^4(\Omega, L^2(0, T; L^2(0, T; U))) \quad \forall T > 0.
\]
Proof. The proof of the existence of weak solution is given in [22], using a Galerkin approximation as well as some compactness results. The proof of the uniqueness is based on the pathwise uniqueness, which implies the uniqueness of weak solution, is given in [24].

In this article, we study the exponential stability of the weak solutions to (2.26).
Hereafter, we assume that \( f \) satisfies the additional condition
\[
\langle \alpha A_1 f(\psi), A_1 \psi \rangle = \langle \alpha A_1^{1/2} f(\psi), A_1^{1/2} \psi \rangle \geq -\kappa_0 |A_1^{3/2} \psi|_{L^2}^2, \quad \forall \psi \in D(A_1^{3/2}),
\]
\[
\langle \alpha A_1 f(\phi_1) - A_1 f(\phi_2), A_1(\phi_1 - \phi_2) \rangle \geq -\kappa_0 |A_1^{3/2}(\phi_1 - \phi_2)|_{L^2}^2, \quad \forall \phi_1, \phi_2 \in D(A_1^{3/2}),
\]
where \( \kappa_0 > 0 \) is a fixed constant.

We also set
\[
\alpha_0 = \max(1, \epsilon), \quad \alpha_1 \equiv \min(\nu, \epsilon^2 - \epsilon \kappa_0) > 0.
\]

3. The exponential stability of solutions. In this section we discuss the moment exponential stability and almost sure exponential stability of weak solutions to (2.26) assuming that they exist. We discuss the long-time behavior of the weak solutions \((v, \phi)(t)\) under some conditions. As in [12], applying the Itô formula, we study the stability of stationary solutions to (3.1) below. Let us note that the coupling of the Cahn-Hilliard equation and the Navier-Stokes system introduces in the model a highly nonlinear term that makes the analysis of the problem more involved than the Navier-Stokes system studied for instance in [12]. The existence of weak solutions to the stochastic version of the coupled CH-NS model is proved in [22].

We now consider the following stationary equation
\[
\begin{cases}
\nu A_0 v^* + B_0(v^*, v^*) - R_0(\epsilon A_1 \phi^*, \phi^*) = g_1^1(v^*, \phi^*), \\
\epsilon A_1^2 \phi^* + \alpha A_1 f(\phi^*) + B_1(v^*, \phi^*) = g_1^2(v^*, \phi^*).
\end{cases}
\]

3.1. Existence and uniqueness of stationary solution.

Let \( \{(w_i, \psi_i), i = 1, 2, 3, \cdots \} \subset U \) be an orthonormal basis of \( \mathcal{H} \), where \( \{w_i, i = 1, 2, \cdots \} \) are eigenvectors of \( A_0 \) and \( A_1 \) respectively. We set \( \mathcal{H}_m = \text{span}\{\{(w_i, \psi_i), \cdots (w_m, \psi_m)\}\} \). For fixed \((U, \Phi) \in \mathcal{U}_m \), We consider the following approximating problems: find \((v_m, \phi_m) \in \mathcal{U}_m \) such that
\[
\begin{cases}
\nu A_0 v_m + B_0(U, v_m) - R_0(\epsilon A_1 \phi_m, \Phi) = g_1^1(U, \Phi), \\
\epsilon A_1^2 \phi_m + \alpha A_1 f(\phi_m) + B_1(v_m, \Phi) = g_1^2(U, \Phi).
\end{cases}
\]

It is clear (using the Lax-Milgram theorem) that for \((U, \Phi) \in \mathcal{U}_m \), there exists a unique solution \((v_m, \phi_m) \) to (3.2). Define \( T_m : \mathcal{U}_m \to \mathcal{U}_m \) the linear operator given by \( T_m(U, \Phi) = (v_m, \phi_m) \).

We will see that for each \( m \) we may apply a fixed point theorem to the map \( T_m \) (restricted to a suitable subset \( \mathcal{U}_m \subset \mathcal{U}_m \)) to ensure that we can obtain the existence of a solution \((v_m, \phi_m) \) to (3.2).

Lemma 1. We assume that \( g_1 = (g_1^1, g_1^2) \) satisfies (2.30)-(2.31). Then any solution \((v_m, \phi_m) \) to (3.2) satisfies the estimate
\[
\|(v_m, \phi_m)\|_U \leq \alpha_1^{-1} (\alpha_0 \|g_1(0,0)\|_{U^*} + \alpha_0 L_1 \|(U, \Phi)\|_U),
\]
where \( \alpha_1 \) is given by (2.33).
Proof. If \((v_m, \phi_m) \in \mathcal{U}\) is a solution to (3.2), we can easily check that
\[
\nu \|v_m\|^2 + \epsilon^2 |v_m|^{3/2} \phi_m^2 + \alpha \epsilon (A_1 f(\phi_m), A_1 \phi_m) = \langle g_1^1(U, \Phi, v_m) + g_2^1(U, \Phi, \epsilon A_1 \phi_m), (3.4)
\]
which gives (see (2.30)-(2.31))
\[
\nu \|v_m\|^2 + \epsilon^2 |v_m|^{3/2} \phi_m^2 - \epsilon \kappa_0 |v_m|^{3/2} \phi_m^2 \leq \alpha \epsilon L_1 \|v_m, \phi_m\|_{\mathcal{U}} \|U, \Phi\|_{\mathcal{U}} (3.5)
\]
and
\[
\nu \|v_m\|^2 + (\epsilon^2 \epsilon \kappa_0) |v_m|^{3/2} \phi_m^2 \leq (\alpha \epsilon g_1(0, 0) \|U, \Phi\|_{\mathcal{U}} + \epsilon a L_1 \|U, \Phi\|_{\mathcal{U}}) \|(v_m, \phi_m)\|_{\mathcal{U}}. (3.6)
\]
It follows that
\[
\alpha_1 \|(v_m, \phi_m)\|_{\mathcal{U}}^2 \leq (\alpha_a g_1(0, 0) \|U, \Phi\|_{\mathcal{U}} + \epsilon \alpha L_1 \|U, \Phi\|_{\mathcal{U}}) \|(v_m, \phi_m)\|_{\mathcal{U}}, (3.7)
\]
which gives
\[
\|(v_m, \phi_m)\|_{\mathcal{U}} \leq \alpha_1^{-1} (\alpha a g_1(0, 0) \|U, \Phi\|_{\mathcal{U}} + \epsilon \alpha L_1 \|U, \Phi\|_{\mathcal{U}}), (3.8)
\]
where \(\alpha_1 > 0\) is given by (2.33).
\[
\Box
\]
\textbf{Theorem 2.} Suppose that \(g_1\) satisfies (2.30)-(2.31) and \(f\) satisfies (2.32). We also assume that
\[
\alpha_1 - \alpha a L_1 > 0. (3.9)
\]
Then there exists at least one solution to (3.2).

\textbf{Proof.} Recall that \((v_m, \phi_m)\) satisfies the a priori estimates (3.3). Let \(K_0 > 0\) such that \(K_0 (\alpha_1 - \alpha a L_1) \geq \|g_1(0, 0)\|_{\mathcal{U}}\). Then from (3.8) we can check that
\[
\|(v_m, \phi_m)\|_{\mathcal{U}} \leq K_0 \text{ provided that } \|(U, \Phi)\|_{\mathcal{U}} \leq K_0.
\]
Now let
\[
\wedge_m = \{(U, \Phi) \in \mathcal{U}_m, \|(U, \Phi)\|_{\mathcal{U}} \leq K_0\}. (3.10)
\]
Then \(\wedge_m\) is a compact and convex subset of \(\mathcal{U}_m\). It is also clear that \(T_n\) maps \(\wedge_m\) into itself. To prove the existence of solution, we apply the Brouwer fixed point theorem to the restriction of \(T_m\) to \(\wedge_m\). Therefore it remains to check that \(T_m\) is continuous.

For the continuity of \(T_m\), we proceed as follows. Let \((v_1, \phi_1) = T_m(U_1, \Phi_1)\) and \((v_2, \phi_2) = T_m(U_2, \Phi_2)\), where \((U_1, \Phi_1), (U_2, \Phi_2) \in \mathcal{U}_m\). Let \((w_1, \psi_1) = (v_1, \phi_1) - (v_2, \phi_2), (U, \Phi) = (U_1, \Phi_1) - (U_2, \Phi_2)\). Then from (3.2) can check that we \((w, \psi)\) satisfies
\[
\left\{ \begin{array}{l}


\nu A_0 w + B_0(U, v_1) + B_0(U, w) - R_0(\epsilon A_1 \phi_1, \Phi) - R_0(\epsilon A_1 \psi, \Phi) \\

= g_1^1(U_1, \Phi_1) - g_1^1(U_2, \Phi_2), \\

\epsilon A_1^2 \psi + \alpha A_1 f(\phi_1) - \alpha A_1 f(\phi_2) + B_1(v_1, \Phi) + B_1(w, \Phi) = \epsilon g_1^1(U_1, \Phi_1) - \epsilon g_1^1(U_2, \Phi_2).
\end{array} \right. (3.11)
\]
Note that
\[
\left\{ \begin{array}{l}

R_0(\epsilon A_1 \psi, \Phi, w) = \langle B_1(w, \Phi, \epsilon A_1 \psi), \\

\|g_1^1(U_1, \Phi_1) - g_1^1(U_2, \Phi_2, w)\| + \|g_1^2(U_1, \Phi_1) - g_1^2(U_2, \Phi_2, \epsilon A_1 \psi)\| \\

\leq \alpha a L_1 \|(U, \Phi)\|_{\mathcal{U}} \|(w, \psi)\|_{\mathcal{U}} \\

|\langle B_0(U, v_1, w)\rangle| \leq c \|U\| \|(w)\|_{\mathcal{U}} \\
\end{array} \right. (3.12)
\]
We derive that and (3.15) is proved.

which gives (assuming (3.9))

This gives

solution (\ref{2.0}).

Moreover if

extract a subsequence (still) denoted (\ref{2.0}). Therefore we can

derive that

It follows that there exists a fixed point \((v_m, \phi_m)\) of \(T_m\) in \(\land_m\). Therefore we can

extract a subsequence (still) denoted \((v_m, \phi_m)\) that converges to \((v^*, \phi^*)\) strongly in \(\mathcal{U}\). Using the same argument as in \cite{17}, we can prove that \((v^*, \phi^*)\) is a weak solution to \eqref{3.1}.

3.2. Some a priori estimates on \((v^*, \phi^*)\). We derive some explicit a priori estimates in the \(\mathcal{U}\)-norm.

**Theorem 3.** We assume that \eqref{2.30}-\eqref{2.31} and \eqref{2.32} are satisfied. Then any solution \((v^*, \phi^*)\) to \eqref{3.1} satisfies the following estimate:

\[
\|(v^*, \phi^*)\|_{\mathcal{U}} \leq c\|g_1(0,0)\|_{\mathcal{U}^*} \equiv K_1.
\]

Moreover if

\[
\alpha_1 - c(2K_1 + L_1a_0) > 0,
\]

then the solution to \eqref{3.1} is unique.

**Proof.** To prove \eqref{3.15}, by multiplying \eqref{3.1} by \(v^*\) and \eqref{3.1} by \(\epsilon A_1 \phi^*\), to derive that

\[
\nu \|v^*\|^2 + \epsilon^2 \|A_1^{3/2} \phi^*\|^2_{L^2} + \langle \alpha A_1^{1/2} f(\phi^*), \epsilon A_1^{3/2} \phi^* \rangle
\]

\[
\quad = \langle g_1(v^*, \phi^*), v^* \rangle + \langle g_1(v^*, \phi^*), \epsilon A_1 \phi^* \rangle.
\]

This gives

\[
\nu \|v^*\|^2 + \epsilon^2 \|A_1^{3/2} \phi^*\|^2_{L^2}
\]

\[
\quad \leq \alpha_0\|g_1(0,0)\|_{\mathcal{U}^*} \|(v^*, \phi^*)\|_{\mathcal{U}} + \alpha_0 L_1 \|(v^*, \phi^*)\|_{\mathcal{U}} \|(v^*, \phi^*)\|_{\mathcal{U}}
\]

\[
\quad \leq \alpha_0\|g_1(0,0)\|_{\mathcal{U}^*} \|(v^*, \phi^*)\|_{\mathcal{U}} + \alpha_0 L_1 \|(v^*, \phi^*)\|_{\mathcal{U}}
\]

which gives (assuming \eqref{3.9})

\[
\|(v^*, \phi^*)\|_{\mathcal{U}} \leq \alpha_0\|g_1(0,0)\|_{\mathcal{U}^*} \|(v^*, \phi^*)\|_{\mathcal{U}}.
\]

We derive that

\[
\|(v^*, \phi^*)\|_{\mathcal{U}} \leq (\alpha_1 - \alpha_0 L_1)^{-1} g_1(0,0) \|\mathcal{U}^*\| \|(v^*, \phi^*)\|_{\mathcal{U}},
\]

and \eqref{3.15} is proved.

For the uniqueness, let \((v_1^*, \phi_1^*), (v_2^*, \phi_2^*)\) be two solutions and \((w, \psi) = (v_1^*, \phi_1^*) - (v_2^*, \phi_2^*)\). Then \((w, \psi)\) satisfies

\[
\begin{cases}
\nu A_0 w + B_0(w, v_1^*) + B_0(v_2^*, w) - R_0(\epsilon A_1 \phi_2^*, \psi) - R_0(\epsilon A_1 \psi, \phi_1^*) \\
\quad = g_1^2(v_1^*, \phi_1^*) - g_1^2(v_2^*, \phi_2^*),
\end{cases}
\]

\[
\epsilon A_1^2 \psi + \alpha A_1 f(\phi_1^*) - \alpha A_1 f(\phi_2^*) + B_1(v_2^*, \psi) - B_1(v_1^*, \phi_1^*) = g_1^2(v_1^*, \phi_1^*) - g_1^2(v_2^*, \phi_2^*).
\]

Note that

\[
\begin{align*}
\langle R_0(\epsilon A_1 \psi, \phi_1^*), w \rangle &= \langle B_1(w, \phi_1^*), \epsilon A_1 \psi \rangle, \\
\langle B_0(w, v_1^*), w \rangle &\leq c\|v_1^*\|_w, \\
\langle B_0(\epsilon A_1 \phi_2^*, \psi), w \rangle &\leq c\|A_1 \psi\|_{L^2} \|A_1 \phi_2^*\|_{L^2}, \\
\langle B_1(v_2^*, \Phi), \epsilon A_1 \psi \rangle &\leq c\|A_1 \psi\|_{L^2} \|v_2^*\|.
\end{align*}
\]
Multiplying \((3.21)_1\) and \((3.21)_2\) by \(w\) and \(\epsilon A_1 \psi\) respectively and using \((3.22)-(3.23)\) yields

\[

\nu \|w\|^2 + (\epsilon^2 - \kappa_0 \epsilon) |A_1^{3/2} \psi|^2 \leq c \left(\|v^*_1\| + |A_1 \phi^*_2|_{L^2} + \|\phi^*_2\| + \alpha_0 L_1 \right) \|(w, \psi)\|^2_{H_t}
\]

which gives

\[

(\alpha_1 - (\alpha_0 L_1 + 2K_1)) \|(w, \psi)\|^2_{H_t} \leq 0,
\]

and \(\|(w, \psi)\|_{H_t} = 0\) assuming \((3.16)\), and the theorem is proved. \(\square\)

**Remark 3.1.** We note that conditions \((3.9)\) and \((3.16)\) are satisfied if \(\nu > 0\) and \(\epsilon > 0\) are large enough, and properly chosen as follows for instance:

- Step 1: Take \(\nu > 0\) and \(\epsilon > 0\) large enough such that \(\alpha_1 = \min(\nu, \epsilon^2 - \kappa_0) = \epsilon^2 - \kappa_0 > 0\).
- Step 2: Then, \((3.9)\) becomes

\[

\alpha_1 - \alpha_0 L_1 = \epsilon^2 - \kappa_0 \epsilon - \alpha_0 L_1 > 0,
\]

which is satisfied for \(\epsilon > 0\) large enough, where \(\alpha_0\) is given by \((2.33)\).

- Step 3: For condition \((3.16)\), we first note that we can choose the constant \(K_1\) in \((3.15)\) independent of \(\nu\) and \(\epsilon\), when \(\nu > 0\) and \(\epsilon > 0\) are large enough. In fact, since \((\alpha_1 - \alpha_0 L_1)^{-1} = (\epsilon^2 - \kappa_0 - \alpha_0 L_1)^{-1}\) goes to zero as \(\epsilon\) goes to \(\infty\), we can always assume that

\[

(\alpha_1 - \alpha_0 L_1)^{-1} < 1, \quad \text{for } \nu > 0, \epsilon > 0 \text{ large enough.}
\]

- Step 4: Therefore, we have

\[

\alpha_1 - c(K_1 + \alpha_0 L_1) = \epsilon^2 - \kappa_0 \epsilon - \epsilon(K_1 + \alpha_0 L_1) > 0,
\]

is satisfied for \(\nu > 0, \epsilon > 0\) large enough, since \(K_1 > 0\) is assumed to be a constant.

### 3.3. Stability of the steady state solutions

We study in this section the stability of the steady state solutions. We assume that \(\nu\) and \(\epsilon\) are large enough so that \((3.1)\) has a unique solution \((v^*, \phi^*)\), See Remark above. We first recall from [14] some preliminary definitions.

**Definition 3.1.** We say that a weak solution \((v, \phi)(t)\) to \((2.26)\) converges to \((v^*, \phi^*) \in H\) exponentially in the mean square if there exists \(\eta > 0\) and \(M_0 = M_0((v, \phi)(0)) > 0\) such that

\[

E[(v, \phi)(t) - (v^*, \phi^*)]^2_H \leq M_0 e^{-\eta t}, \quad t \geq 0.
\]

If \((v^*, \phi^*)\) is a solution to \((3.1)\), we say that \((v^*, \phi^*)\) is exponentially stable in the mean square provided that every weak solution to \((2.26)\) converges to \((v^*, \phi^*)\) exponentially in the mean square with the same exponential order \(\eta > 0\).

**Definition 3.2.** We say that a weak solution \((v, \phi)(t)\) to \((2.26)\) converges to \((v^*, \phi^*) \in H\) almost surely exponentially if there exists \(\eta > 0\) such that

\[

\lim_{t \to \infty} \frac{1}{t} \log |(v, \phi)(t) - (v^*, \phi^*)|_H \leq -\eta.
\]

If \((v^*, \phi^*)\) is a solution to \((3.1)\), we say that \((v^*, \phi^*)\) is almost surely exponentially stable provided that every weak solution to \((2.26)\) converges to \((v^*, \phi^*)\) almost surely exponentially with the same constant \(\eta > 0\).
Theorem 4. Let \((v^*, \phi^*) \in \mathcal{U}\) be the unique solution to (3.1). We assume that \(g_1\) satisfies (2.30)-(2.31) and \(g_2\) satisfies
\[
\|g_2(t, v, \phi)\|_{L^2(H)}^2 \leq \varphi(t) + (\zeta + \delta(t))(v, \phi)(t) - (v^*, \phi^*)_H^2,
\]  
where \(\zeta > 0\) is a constant and \(\varphi(t), \delta(t)\) are nonnegative integrable functions such that there exist real numbers \(\rho > 0, M_\delta \geq 1, M_\varphi \geq 1\) with
\[
\varphi(t) \leq M_\varphi e^{-\rho t}, \quad \delta(t) \leq M_\delta e^{-\rho t}, \quad t \geq 0.
\]  
We also assume that
\[
\lambda_1^{-1}\zeta + (c_1 + c_2\epsilon)\|(v^*, \phi^*)\|_\mathcal{U} + 2a_0L_1 - 2a_1 < 0,
\]  
where \(c_1\) and \(c_2\) are defined by (3.37) below.

Then any weak solution \((v, \phi)(t)\) to (2.26) converges to \((v^*, \phi^*)\) exponentially in the mean square. More precisely, there exist real numbers \(\eta \in (0, \rho), M_0 \equiv M_0((v, \phi)(0)) > 0\) such that
\[
\mathbb{E}[(v, \phi)(t) - (v^*, \phi^*)]^2_{\mathcal{H}} \leq M_0 e^{-\eta t}, \quad \forall t > 0.
\]  
Proof. First we choose \(\eta \in (0, \rho)\) such that
\[
\lambda_1^{-1}(\zeta + \eta) + (c_1 + c_2\epsilon)\|(v^*, \phi^*)\|_\mathcal{U} + 2a_0L_1 - 2a_1 < 0. \tag{3.32}
\]  
Let us set
\[
(w, \psi) = (v, \phi) - (v^*, \phi^*).
\]  
Applying Itô’s formula to \(e^{\eta t}|(w, \psi)(t)|_{\mathcal{H}}^2\) gives
\[
e^{\eta t}|(w, \psi)(t)|_{\mathcal{H}}^2 = \left|(w, \psi)(0)\right|_{\mathcal{H}}^2 + \int_0^t e^{\eta s}|(w, \psi)(s)|_{\mathcal{H}}^2 ds - 2\nu \int_0^t e^{\eta s} \langle A_0v(s), w(s) \rangle ds
- 2\epsilon^2 \int_0^t e^{\eta s} \langle \phi(s), A_1v(s) \rangle ds - 2 \int_0^t e^{\eta s} \langle B_0(v(s), w(s)) - R_0(\epsilon A_1\phi(s), \phi(s)), w(s) \rangle ds
- 2 \int_0^t e^{\eta s} \langle B_1(v(s), \phi(s)) + \alpha A_1f(\phi(s)), \epsilon A_1\psi(s) \rangle ds
+ 2 \int_0^t e^{\eta s} \langle g_1(v(s), \phi(s)), w, \epsilon A_1\psi(s) \rangle ds + 2 \int_0^t e^{\eta s} \langle g_2(v(s), \phi(s)), w, \epsilon A_1\psi(s) \rangle dW_s
+ \int_0^t e^{\eta s} \|g_2(s, v(s), \phi(s))\|_{L^2(H)}^2 ds.
\]  
We also know that \((v^*, \phi^*)\) satisfies
\[
\int_0^t e^{\eta s} \langle \nu A_0v^* + B_0(v^*, v^*) - R_0(\epsilon A_1\psi^*, \phi^*), w(s) \rangle ds
+ \int_0^t e^{\eta s} \langle \epsilon A_1^2\phi^* + B_1(v^*, \phi^*) + \alpha A_1f(\phi^*), \epsilon A_1\psi(s) \rangle ds
= \int_0^t e^{\eta s} \langle g_1(v^*, \phi^*), w(s) \rangle ds + \int_0^t e^{\eta s} \langle g_2(v^*, \phi^*), \epsilon A_1\psi(s) \rangle ds
\]  
(3.34)
Using (3.33)-(3.34), we derive that

\[
e^{nt}E[(w, \psi)(t)]^2_{H_t} = E[(w, \psi)(0)]^2_{H_0} + \int_0^t \eta e^{ns}E[(w, \psi)(s)]^2_{H_s}ds - 2\alpha \int_0^t e^{ns}E\|w(s)\|^2ds
- 2\alpha \int_0^t e^{ns}E[A_1^{\frac{3}{2}}(\psi(s))_2^2ds - 2\alpha \int_0^t e^{ns}Eb_1(w, \psi, \epsilon A_1 \psi^*)ds
- 2\alpha \int_0^t e^{ns}Eb_1(v^*, \psi, \epsilon A_1 \psi^*)ds - 2\alpha \int_0^t e^{ns}E\langle A_1 f(\phi)(s) - A_1 f(\phi^*), \epsilon A_1 \psi(s)\rangle ds
+ \int_0^t e^{ns}E\|g_2(s, v(s), \phi(s))\|^2_{L^2(\Omega)}ds + 2\alpha \int_0^t e^{ns}E\langle g_1(v(s), \phi(s)) - g_1(v^*, \phi^*), (w, \epsilon A_1 \psi(s))\rangle ds.
\]

(3.35)

Note that

\[
2|b_0(w, v^*, w)| \leq C_1\|v^*\|\|w\|^2 \leq C_1\|(v^*, \phi^*)\|\|E(w, \psi)\|_{H_t}^2,
2|b_1(v^*, \psi, \epsilon A_1 \psi)| \leq C_1\|v^*\|\|A_1 \psi\|_{L^2}^2 \leq C_1\|v^* \phi^*\|\|E(w, \phi)\|_{H_t}^2,
2|b_1(w, \psi, \epsilon A_1 \phi^*)| \leq C_1\|w\|\|A_1 \phi^*\|_{L^2}^2 \leq C_1\|v^* \phi^*\|\|E(w, \phi)\|_{H_t}^2.
\]

(3.36)

It follows that

\[
2|b_0(w, v^*, w)| + |b_1(v^*, \psi, \epsilon A_1 \psi)| + 2|b_1(w, \psi, \epsilon A_1 \phi^*)| - \epsilon A_1 f(\phi) - A_1 f(\phi^*, \epsilon A_1 \psi(s)) \leq \epsilon \alpha_0 A_1^{\frac{3}{2}} \psi^2_{L^2(\Omega)} \leq \epsilon \alpha_0 \|(w, \phi)\|_{H_t}^2,
\]

\[
\langle g_1(v(s), \phi(s)) - g_1(v^*, \phi^*), (w, \epsilon A_1 \psi)\rangle \leq \alpha_0 L_1 \|(w, \phi)\|_{H_t}^2.
\]

(3.37)

for some \(c_1 > 0\) and \(c_2 > 0\), independent of \(nuo\) and \(\epsilon\).

We derive from (3.33)-(3.36) that

\[
e^{nt}E[(w, \psi)(t)]^2_{H_t} \leq E[(w, \psi)(0)]^2_{H_0} + \int_0^t \eta e^{ns}E[(w, \psi)(s)]^2_{H_s}ds
- 2\alpha \int_0^t e^{ns}E[(w, \psi)(s)]^2_{H_s}ds + [(c_1 + \epsilon c_2)]\|(v^*, \phi^*)\|_{H_t}^2 + 2\alpha L_1 + \epsilon \alpha_0 \|(w, \phi)\|_{H_t}^2.
\]

(3.38)

We recall that

\[
- 2\alpha_1 + [(c_1 + \epsilon c_2)]\|(v^*, \phi^*)\|_{H_t}^2 + 2\alpha L_1 + \lambda_1^{-1}(\eta + \zeta) < 0.
\]

(3.39)

It follows from (3.38)-(3.39) that

\[
e^{nt}E[(w, \psi)(t)]^2_{H_t} \leq E[(w, \psi)(0)]^2_{H_0} + \int_0^t e^{ns}(\varphi(s) + \delta(s))(w, \psi)(s)]^2_{H_s}ds.
\]

(3.40)

Using the Gronwall lemma, we derive that there exists \(M_0 > 0\) such that

\[
E[((w, \psi)(t)]^2_{H_t} \leq M_0 e^{-\eta t}, \forall t > 0,
\]

which proves (3.31).

\[\square\]

**Theorem 5.** The hypotheses are the same as in Theorem 4. Then any weak solution \((v, \phi)(t)\) to (2.26) converges to the stationary solution \((v^*, \phi^*)\) of (3.1) almost surely exponentially.
Proof. Let $N$ be a positive integer. By the Itô formula, for any $t \geq N$ we have
\begin{align}
|\langle w, \psi \rangle(t)|^2_H &= |\langle w, \psi \rangle(N)|^2_H - 2\nu \int_N^t \|w(s)\|^2 ds - 2\epsilon^2 \int_N^t |A_1^{3/2}\psi(s)|^2_{L^2} ds \\
&\quad - 2 \int_N^t (b_0(\psi_*, w) - b_1(\psi_*, \psi, \epsilon A_1 \psi)) ds + 2 \int_N^t b_1(\psi, \epsilon A_1 \phi^*) ds \\
&\quad - 2\alpha \epsilon \int_N^t \langle A_1 f(\phi) - A_1 f(\phi^*), A_1 \psi \rangle ds + 2 \int_N^t g_1(v(s), \phi(s)) ds \\
&\quad - g_1(\psi^*, \phi^*), (w, \epsilon A_1 \psi(s)) ds + 2 \int_N^t \|g_2(s, v(s), \phi(s))\|^2_{L^2(H)} ds \\
&\quad + 2 \int_N^t \langle (w, \epsilon A_1 \psi(s), g_2(s, v(s), \phi(s)) dW_s(s) \rangle
\end{align}

By the Burkholder-Davis-Gundy lemma, we have
\begin{align}
2 \mathbb{E} \left[ \sup_{N \leq t \leq N+1} \int_N^t \langle (w, \epsilon A_1 \psi(s), g_2(s, v(s), \phi(s)) dW_s(s) \rangle \right]^{1/2} \\
\leq \eta_1 \left[ \mathbb{E} \left( \int_N^t \|g_2(s, v(s), \phi(s))\|^2_{L^2(H)} ds \right) \right]^{1/2} \\
\leq \eta_1 \left[ \mathbb{E} \left( \sup_{N \leq t \leq N+1} |\langle w, \psi \rangle(t)|^2_H \right) \right]^{1/2} \\
\leq \eta_2 \int_N^t \mathbb{E} \left[ g_2(s, v(s), \phi(s))\|^2_{L^2(H)} ds + \frac{1}{2} \mathbb{E} \sup_{N \leq t \leq N+1} |\langle w, \psi \rangle(t)|^2_H,
\end{align}

where $\eta_1 > 0, \eta_2 > 0$ are some constants.

Therefore as in (3.33)-(3.38), we obtain that
\begin{align}
\mathbb{E} \left[ \sup_{N \leq t \leq N+1} |\langle w, \psi \rangle(t)|^2_H \right] &\leq \mathbb{E} |\langle w, \psi \rangle(N)|^2_H - 2\alpha \int_N^{N+1} \mathbb{E} |\langle w, \psi \rangle(s)|^2_H ds \\
&\quad + \left[ \lambda_1 + (c_1 + c_2) \right] \|\psi^*\|_U + 2\alpha_0 L_1 + \epsilon \kappa_0 \int_N^{N+1} \mathbb{E} |\langle w, \psi \rangle(s)|^2_H ds \\
&\quad + \eta_0 \int_N^{N+1} \mathbb{E} \|g_2(s, v(s), \phi(s))\|^2_{L^2(H)} ds \\
&\quad + \frac{1}{2} \mathbb{E} \sup_{N \leq t \leq N+1} |\langle w, \psi \rangle(t)|^2_H,
\end{align}

for some $\eta_0 > 0$.

It follows (3.28), (3.32) and (3.44) that
\begin{align}
\frac{1}{2} \mathbb{E} \sup_{N \leq t \leq N+1} |\langle w, \psi \rangle(t)|^2_H \\
\leq \mathbb{E} |\langle w, \psi \rangle(N)|^2_H + \eta_0 \int_N^{N+1} (\varphi(s) + (\zeta + \delta(s)) \mathbb{E} |\langle w, \psi \rangle(s)|^2_H) ds.
\end{align}

Since
\begin{align}
\varphi(t) \leq M_\varphi e^{-\rho t}, \quad \delta(t) \leq M_\delta e^{-\rho t}, \quad \eta \in (0, \rho), \quad M_\varphi \geq 1, \quad M_\delta \geq 1,
\end{align}

it follows from Theorem 3.4 that there exist $M_1 = M_1((v, \phi)(0)) \geq 1$ such that
\begin{align}
\mathbb{E} \left( \sup_{N \leq t \leq N+1} |\langle w, \psi \rangle(t)|^2_H \right) \leq M_1 e^{-\eta N},
\end{align}
Theorem 6. Let \((v^*, \phi^*) \in \mathcal{U}\) be the unique solution to (3.1). Furthermore, we assume that
\[
g_2(v^*, \phi^*) = 0, \forall t \geq 0, \quad \|g_2(t, v_1, \phi_1) - g_2(v_2, \phi_2)\|_{L^2(\mathcal{H})} \leq c_2|v_1, \phi_1 - (v_2, \phi_2)|_{\mathcal{H}}, \forall (v_1, \phi_1), (v_2, \phi_2) \in \mathcal{H}. \quad (3.48)
\]
If
\[
-2\alpha_1 + c_2 \lambda_1^{-1} + (c_1 + c\epsilon_2)\|(v^*, \phi^*)\|_{\mathcal{U}} + 2\alpha_0 L_1 < 0, \quad (3.49)
\]
then any weak solution to (2.26) converges to \((v^*, \phi^*)\) exponentially in the mean square. That is, there exists \(\eta > 0\) such that
\[
\mathbb{E}|(v, \phi)(t) - (v^*, \phi^*)|^2_{\mathcal{H}} \leq \mathbb{E}|(v_0, \phi_0) - (v^*, \phi^*)|^2_{\mathcal{H}}e^{-\eta t}, \forall t \geq 0. \quad (3.50)
\]
Moreover, the path-wise exponential stability with probability one of \((v^*, \phi^*)\) also holds true.

Proof. We start with the equality
\[
\begin{align*}
v(t) - v^* = v(0) - v^* & - \int_0^t [\nu A_0(v - v^*) + B_0(v, v) - B_0(v^*, v^*)] ds \\
& + \int_0^t \left[R_0(\epsilon A_1 \phi, \phi - R_0(\epsilon A_1 \phi^*, \phi^*)) + g_1^1(v, \phi) - g_1^1(v^*, \phi^*)\right] ds \\
& + \int_0^t (g_2^1(s, v, \phi) - g_2^1(s, v^*, \phi^*))d\tilde{W}_t^1, \\
\phi(t) - \phi^* = \phi(0) - \phi^* & - \epsilon \int_0^t A_1^2(\phi - \phi^*)ds - \int_0^t [(B_1(v, \phi) - B_1(v^*, \phi^*)) ds \\
& - \alpha \int_0^t [A_1 f(\phi) - A_1 f(\phi^*)] ds + \int_0^t (g_2^1(v, \phi) - g_2^1(v^*, \phi^*))d\tilde{W}_t^2 \\
& + \int_0^t (g_1^1(v, \phi) - g_1^1(v^*, \phi^*))ds.
\end{align*}
\]

Let \(\eta > 0\) small enough and fixed later. By the Itô formula, we have
\[
\begin{align*}
\mathbb{E}e^{\eta t}|(w, \psi)(t)|^2_{\mathcal{H}} &= \mathbb{E}|(w, \psi)(0)|^2_{\mathcal{H}} + \int_0^t \eta e^{\eta s}\mathbb{E}|(w, \psi)(s)|^2_{L^2} ds - 2\nu \int_0^t e^{\eta s}\mathbb{E}|w(s)|^2 ds \\
& - 2\epsilon \int_0^t e^{\eta s}\mathbb{E}|A_1^{3/2} \psi(s)|^2_{L^2} ds - 2\int_0^t e^{\eta s}\mathbb{E}b_0(w, v^*, w) ds \\
& + 2\epsilon \int_0^t e^{\eta s}\mathbb{E}b_1(w, \psi, A_1 \phi^*) ds - 2\epsilon \int_0^t e^{\eta s}\mathbb{E}b_1(v^*, \psi, A_1 \psi) ds \\
& + 2 \int_0^t e^{\eta s}\mathbb{E}(g_1(v(s), \phi(s)) - g_1(v^*, \phi^*), (w, \epsilon A_1 \psi)(s)) ds \\
& + \int_0^t e^{\eta s}\mathbb{E}|g_2(v(s), \phi(s)) - g_2(v^*, \phi^*)|^2_{L^2(\mathcal{H})} ds \\
& - 2 \int_0^t e^{\eta s}\mathbb{E}(\alpha A_1 f(\phi) - \alpha A_1 f(\phi^*), \epsilon A_1 \psi(s)) ds.
\end{align*}
\]

(3.52)
It follows from (3.52) and (2.15)-(2.17) that

\[
Ee^{\eta t}|(w, \psi)(t)|^2_{L^2_H} \leq E|w, \psi)(0)|^2_{L^2_H} + \int_0^t \eta e^{\eta s}E|w, \psi)(s)|^2_{L^2_H} ds
\]

\[
+ \left[ -2\alpha_1 + c_2 \lambda_1^{-1} + (c_1 + \epsilon c_2)\|v^*, \phi^*\|_{U} + 2\epsilon\kappa_0 \right] \int_0^t e^{\eta s}E|w, \psi)(s)|^2_{L^2_H} ds
\]

\[
\leq E|w, \psi)(0)|^2_{L^2_H} + (\eta - \kappa_2 \lambda_1) \int_0^t e^{\eta s}|w, \psi)(s)|^2_{L^2_H} ds \leq E|w, \psi)(0)|^2_{L^2_H},
\]

where

\[
\kappa_2 \equiv -2\alpha_1 + c_2 \lambda_1^{-1} + (c_1 + \epsilon c_2)\|v^*, \phi^*\|_{U} < 0,
\]

and \( \eta \) is chosen such that

\[
\eta + \kappa_2 \lambda_1 < 0.
\]

It follows from (3.53) that

\[
Ee^{\eta t}|(w, \psi)(t)|^2_{L^2_H} \leq E|(w, \psi)(0)|^2_{L^2_H},
\]

and the proof of the first part of the theorem follows as that of Theorem 3.4. The rest of the theorem is proved using a similar method to the one in the proof of Theorem 5.

**Theorem 7.** We assume that there exists a constant \( \zeta > 0 \) such that

\[
\|g_2(t, v, \phi)\|_{L^2(\mathcal{H})}^2 \leq \varphi(t) + (\zeta + \delta(t))(v, \phi)|_{L^2_H},
\]

where \( \varphi(t), \delta(t) \) satisfy (3.29). We also suppose that \( g_1 : [0, \infty) \times \mathcal{U} \to \mathcal{U}^* \) satisfies

\[
\langle g_1(t, v, \phi), (v, \epsilon A_1 \phi) \rangle \leq \alpha(t) + (c_3 + \beta(t))(v, \phi)|_{L^2_H},
\]

where \( c_3 > 0, \alpha(t), \beta(t) \) are integrable functions such that there exist real numbers \( \rho > 0, M_\alpha \geq 1, M_\beta \geq 1, \) with

\[
\alpha(t) \leq M_\alpha e^{-\rho t}, \beta(t) \leq M_\beta e^{-\rho t}, \ t \geq 0.
\]

Furthermore, let

\[
2\alpha_1 > \zeta \lambda_1^{-1} + 2c_3 \lambda_1^{-1}.
\]

Then any weak solution \((v, \phi)(t)\) to (2.26) converges to zero almost surely exponentially.

**Proof.** Let \( \eta \in (0, \rho) \) be such that

\[
2\alpha_1 > \lambda_1^{-1}(\zeta + \eta) + 2c_3 \lambda_1^{-1} + 2\epsilon\kappa_0.
\]
Then we have
\[
E e^{\eta t} |(v, \phi)(t)|_H^2 = E|(v, \phi)(0)|_H^2 + \int_0^t \eta e^{\eta s} E|(v, \phi)(s)|_H^2 ds \\
-2\nu \int_0^t e^{\eta s} \|v(s)\|^2_2 ds - 2\nu \int_0^t e^{\eta s} E|A^{1/2}_1 \phi(s)|^2_2 ds - 2\epsilon \int_0^t e^{\eta s} E(A_1 \phi, \epsilon A_1 \phi) ds \\
+ 2 \int_0^t e^{\eta s} (g_1(v(s), \phi(s)), (v, \epsilon A_1 \phi) ds) + \int_0^t e^{\eta s} \|g_2(s, v(s), \phi(s))\|^2_{L^2(H)} ds
\]
\[
\leq E|(v, \phi)(0)|_H^2 + (-2\alpha_1 + \lambda_1^{-1}(\zeta + \eta) + 2c_3 \lambda_1^{-1}) \int_0^t e^{\eta s} E|(v, \phi)(s)|_H^2 ds \\
+ \int_0^t e^{\eta s} (2\alpha_1 + \varphi(s) + (\beta(s) + \delta(s))(v, \phi)(s)|^2_2) ds
\]
\[
\leq E|(v, \phi)(0)|_H^2 + \int_0^t e^{\eta s} (2\alpha_1 + \varphi(s) + (\beta(s) + \delta(s))(v, \phi)(s)|^2_2) ds,
\]
(3.61)

which gives
\[
E e^{\eta t} |(v, \phi)(t)|_H^2 \leq E|(v, \phi)(0)|_H^2 + \int_0^t e^{\eta s} \varphi(s) + (2\alpha_1 + (2\beta_1 + \delta(s))) E|(v, \phi)(s)|_H^2 ds.
\]
(3.62)

By the Gronwall lemma, we obtain that any weak solution to (2.26) converges to zero exponentially in the mean square. We can then finish the proof using the same method as in the proof of Theorem 5.

\[\square\]

4. Stabilization of the 2D CH-NS model (2.26). Hereafter, we briefly discuss the stabilization of the 2D CH-NS model (2.26). As noted in [12, 13], in order to produce a stabilization effect, it is enough to consider a one dimensional Wiener process for that purpose.

Hereafter, we suppose that
\[
g_2(t, v, \phi) = \sigma(v - v^* - \phi^*), \forall (v, \phi) \in H,
\]
for some \(\sigma \in \mathbb{R}\). We also assume that
\[
|g_1(v_1, \phi_1) - g_1(v_2, \phi_2)|_H \\
\leq L_1 |(v_1, \phi_1) - (v_2, \phi_2)|_H, \forall (v_1, \phi_1), (v_2, \phi_2) \in H, \ g_1(0, 0) \neq 0.
\]
(4.1)

\textbf{Lemma 2.} Let \((v^*, \phi^*) \in U\) be the unique solution to (3.1). If \(g_1\) satisfies (4.1) and
\[
2\alpha_1 - [(c_1 + c_2\epsilon)(v^*, \phi^*)]_U + 2\alpha_0 L_1 > 0,
\]
(4.2)
where \(L_1\) is the Lipschitz constant of \(g_1\) given in (4.1), then the stationary solution \((v^*, \phi^*)\) to (3.1) is exponentially stable.

\textbf{Proof.} We will only sketch the proof as it is similar to the proof of Theorem 10.2 of [27]. Let \((v, \phi)\) be a solution to the deterministic system:
\[
\begin{cases}
\frac{dv}{dt} + \nu A_0 v + B_0(v, v) - R_0(\epsilon A_1 \phi, \phi) = g_1^1(v, \phi), \\
\frac{d\phi}{dt} + A_1 \mu + B_1(v, \phi) = g_2^2(v, \phi), \quad \mu = \epsilon A_1 \phi + \alpha f(\phi), \\
(v, \phi)(0) = (v_0, \phi_0).
\end{cases}
\]
(4.3)

Let
\[
(w, \psi) = (v, \phi) - (v^*, \phi^*).
\]
Then \((w, \psi)\) satisfies
\[
\begin{cases}
\frac{dw}{dt} + \nu A_0 w + B_0(v, w) + B_0(w, v^*) - R_0(\epsilon A_1 \phi^*, \psi) - R_0(\epsilon A_1 \psi, \phi^*) \\
= g_1^1(v, \phi) - g_1^1(v^*, \phi^*), \\
\frac{d\psi}{dt} + \epsilon A_1^2 \psi + B_1(w, \psi) + B_1(v^*, \psi) + \alpha A_1 f(\phi) - \alpha A_1 f(\phi^*) \\
= g_1^2(v, \phi) - g_1^2(v^*, \phi^*), \\
(w, \psi)(0) = (v_0, \phi_0) - (v^*, \phi^*).
\end{cases}
\tag{4.4}
\]

Let
\[ y = |(w, \psi)|^2_H. \]

Then, multiplying \((4.4)_1\) by \(w\), \((4.4)_2\) by \(\epsilon A_1 \psi\) and adding the resulting equalities and using \((2.15)-(2.17)\) gives
\[
\frac{dy}{dt} + 2\alpha_1 \| (w, \psi) \|^2_H \leq [(c_1 + c_2 \epsilon) \| (v^*, \phi^*) \|_U + 2\alpha_0 L_1] \| (w, \psi) \|^2_H. \tag{4.5}
\]

Assuming that
\[ \sigma_0 = 2\alpha_1 - [(c_1 + c_2 \epsilon) \| (v^*, \phi^*) \|_U + 2\alpha_0 L_1] > 0, \tag{4.6} \]
we derive that
\[ \frac{dy}{dt} + \kappa_2 y \leq 0, \tag{4.7} \]
where
\[ \kappa_2 = \lambda_1 \sigma_0 > 0. \tag{4.8} \]

It follows that
\[ y(t) \leq y(0) e^{-\kappa_2 t}, \quad \forall t \geq 0, \tag{4.9} \]
and the lemma is proved. \(\square\)

If the Lipschitz constant \(L_1\) of \(g_1\) is sufficiently large such that \(\kappa_2 < 0\), then we do not know if \((v^*, \phi^*)\) is exponentially stable or not. However, the following result related to the stabilization of the 2D CH-NS systems holds true.

**Theorem 8.** We assume that \(g_1\) satisfies (4.1). Let \((v^*, \phi^*) \in \mathcal{U}\) be the unique solution to (3.1). Let \(\kappa_2 < 0\), where \(\kappa_2\) is given by (4.8). Assume that \(\sigma\) is any real number such that
\[ \lambda_1 \kappa_2 + \sigma^2 > 0. \tag{4.10} \]

Then there exists \(\Omega_0 \subset \Omega, \mathcal{P}(\Omega_0) = 0\), such that for \(\omega \notin \Omega_0\), there exists \(T(\omega) > 0\) such that
\[ |(v, \phi)(t) - (v^*, \phi^*)|^2_H \leq |(v, \phi)(0) - (v^*, \phi^*)|^2_H e^{-\eta t}, \quad \forall t \geq T(\omega), \tag{4.11} \]

where \(\eta > 0\) is given below and \((v, \phi)(t)\) is any weak solution to (2.26) with the function \(g_2\) given by
\[ g_2(t, x, y) = \sigma(x - v^*, y - \phi^*), \quad \forall (x, y) \in \mathcal{H}. \tag{4.12} \]

**Proof.** By avoiding some technicalities, the result is easily proved as follows (a more rigorous proof is given in the Appendix). Let
\[ (w, \psi)(t) = (v, \phi)(t) - (v^*, \phi^*). \]
Applying the Itô formula to \(|(w, \psi)(t)|^2_{\mathcal{H}}\), we derive as in (3.33)-(3.35) that

\[
|\langle w, \psi \rangle(t)|^2_{\mathcal{H}} = |\langle w, \psi \rangle(0)|^2_{\mathcal{H}} - 2\nu \int_0^t \|w(s)\|^2 ds - 2\nu^2 \int_0^t |A_{1}^{3/2}\psi(s)|^2_{L^2} ds
- 2\nu \int_0^t b_0(w, v^*, w)ds - 2 \int_0^t b_1(v^*, \psi, \epsilon A_1 \psi)ds + 2 \int_0^t b_1(w, \psi, \epsilon A_1 \phi^*)ds
- 2\alpha \int_0^t \langle A_1 f(\phi) - A_1 f(\phi^*), \epsilon A_1 \psi \rangle ds + \int_0^t \|g_2(s, v(s), \phi(s))\|^2_{L^2(\mathcal{H})} ds
+ 2 \int_0^t \langle (w, \epsilon A_1 \psi), g_2(s, v, \phi) dW_s(s) \rangle + 2 \int_0^t \langle g_1(v, \phi) - g_1(v^*, \phi^*), (w, \epsilon A_1 \psi) \rangle ds.
\]

Using (3.36), we also have

\[
- 2\nu \|w(s)\|^2 - 2\nu^2 |A_1 \psi(s)|_{L^2}^2 + 2 |b_0(w, v^*, w)| + 2 |b_1(v^*, \psi, \epsilon A_1 \psi)|
+ 2 |b_1(w, \psi, \epsilon A_1 \phi^*)| - 2\alpha \int_0^t \langle A_1 f(\phi) - A_1 f(\phi^*), \epsilon A_1 \psi \rangle ds
\leq [-2\alpha_1 + (c_1 + c_2 \epsilon)] \|w^*, \phi^* \|_{L^2} \| (w, \psi)(s) \|^2_{\mathcal{H}}
\leq [-2\alpha_1 + (c_1 + c_2 \epsilon)] \|w^*, \phi^* \|_{L^2} \lambda_1 \| (w, \psi)(s) \|^2_{\mathcal{H}}.
\]

Let

\[
2\eta = \lambda_1 \kappa_2 + \sigma^2 > 0,
\]

where \(\kappa_2\) is given by (4.8).

It follows from (5.5)-(4.15) that

\[
\log |\langle w, \psi \rangle(t)|^2_{\mathcal{H}}
= \int_0^t \frac{1}{|\langle w, \psi \rangle(t)|^2_{\mathcal{H}}} \left( -2\nu \|w(s)\|^2 - 2\nu^2 |A_1^{3/2}\psi(s)|^2_{L^2} + \sigma^2 |(w, \psi)(s)|^2_{\mathcal{H}} \right) ds
+ \log |\langle w, \psi \rangle(0)|^2_{\mathcal{H}}
- 2 \int_0^t \frac{2}{|\langle w, \psi \rangle(s)|^2_{\mathcal{H}}} \left( b_0(w, v^*, w) + b_1(v^*, \psi, \epsilon A_1 \psi) - b_1(w, \psi, \epsilon A_1 \phi^*) \right) ds
+ \int_0^t \frac{2}{\| (w, \psi)(s) \|^2_{\mathcal{H}}} \left( g_1(v, \phi) - g_1(v^*, \phi^*), (w, \epsilon A_1 \psi) \right) - \alpha \langle A_1 f(\phi) - A_1 f(\phi^*), \epsilon A_1 \psi \rangle ds
+ 2 \int_0^t \frac{\sigma |(w, \psi)(s)|^2_{\mathcal{H}} dW_s(s) - 1}{2} \int_0^t \frac{4\sigma^2 |(w, \psi)(s)|^4_{\mathcal{H}} ds}{| (w, \psi)(s) |^4_{\mathcal{H}}}
\leq \log |\langle w, \psi \rangle(0)|^2_{\mathcal{H}} - 2\eta t + 2\sigma W(t).
\]

Since almost surely we have

\[
\lim_{t \to \infty} \frac{W(t)}{t} = 0,
\]

we can find \(\Omega_0 \subset \Omega\) with \(P(\Omega_0) = 0\) such that for each \(\omega \notin \Omega_0\), there exists \(T(\omega) > 0\) such that for all \(t \geq T(\omega)\), we have

\[
\frac{2\sigma W(t)}{t} \leq \eta.
\]

Therefore, we obtain that for \(t \geq T(\omega)\) we derive from (5.7) that

\[
\log |\langle w, \psi \rangle(t)|^2_{\mathcal{H}} \leq \log |\langle w, \psi \rangle(0)|^2_{\mathcal{H}} - \eta t,
\]

which proves (4.11).
Remark 4.1. Note that conditions (3.30), (3.49), (4.2) and (4.10) are satisfied if $\nu > 0, \epsilon > 0$ are properly chosen and large enough. In fact, for $\alpha = \epsilon^2 - \kappa_0 \epsilon$, (see Remark ), condition (3.30) becomes

$$
\lambda_1^{-1} \zeta + (c_1 + c_2 \epsilon) \|(v^*, \phi^*)\|_U + 2\alpha L_1 - 2(\epsilon^2 - \kappa_0 \epsilon) < 0,
$$

which is satisfied for $\epsilon > 0$ large enough. A similar observation is true for (3.49), (4.2) and (4.10).

5. Appendix: Another proof of Theorem 8. In this part, following similar steps as in [13], we give another (more rigorous) proof of Theorem 4.2.

Hereafter, we assume without loss of generality that

$$(v_0, \phi_0) - (v^*, \phi^*) \neq 0, \quad (5.1)$$

for any $\omega \in \Omega$. Otherwise, we can start as in the first lines of the Proof of Theorem 4.3 in [13].

We define

$$
\tau_0(\omega) = \inf\{t \geq 0; (v(t, \omega), \phi(t, \omega)) - (v^*, \phi^*) = (0, 0)\},
$$

$$(u(t), \psi(t)) = (v(t \wedge \tau_0), \phi(t \wedge \tau_0)) - (v^*, \phi^*). \quad (5.2)$$

If $\tau_0(\omega)$ is finite then

$$(v(\tau_0(\omega), \omega), \phi(\tau_0(\omega), \omega)) - (v^*, \phi^*) = (0, 0). \quad (5.3)$$

It is easy to see that $(u(t), \psi(t))$ satisfies the same equation as $(v(t, \omega), \phi(t, \omega)) - (v^*, \phi^*)$, and consequently by uniqueness

$$(u(t), \psi(t)) = (v(t \wedge \tau_0), \phi(t \wedge \tau_0)) - (v^*, \phi^*) = (v(t, \omega), \phi(t, \omega)) - (v^*, \phi^*) \quad (5.2)$$

for all $t \geq 0$, a.s., i.e. if we denote

$$
\bar{\Omega} = \{\omega; \tau_0(\omega) < \infty\},
$$

then

$$(v(t, \omega), \phi(t, \omega)) - (v^*, \phi^*) = (v(\tau_0(\omega), \omega), \phi(\tau_0(\omega), \omega)) - (v^*, \phi^*) = (0, 0), \quad (5.3)$$

for all $t \geq \tau_0(\omega)$, a.s. in $\bar{\Omega}$.

Consider now the sequence of $F_t$ stopping times $\{\tau_n, n \geq 1\}$ defined by

$$
\tau_n(\omega) = \inf\{t \geq 0 : \| (v(t, \omega), \phi(t, \omega)) - (v^*, \phi^*) \|_U \leq \frac{1}{n} \}. \quad (5.4)
$$

This is an increasing sequence almost surely convergent to $\tau_0$. Denote

$$(v_n, \phi_n)(t) = (v(t \wedge \tau_n), \phi(t \wedge \tau_n)) - (v^*, \phi^*). \quad (5.4)$$
Applying the Ito formula to $|(v_n, \phi_n)|_{H}^2$, we derive that

$$
|\langle v_n, \phi_n \rangle (t)|_{H}^2 = |\langle v_n, \phi_n \rangle (0)|_{H}^2 - 2 \nu \int_{0}^{t} 1_{[0, \tau_n]} \|v_n(s)\|^2 ds \\
- 2c^2 \int_{0}^{t} 1_{[0, \tau_n]} A_1^{3/2} \phi_n(s) |v_n(s)|^2 ds - 2 \int_{0}^{t} 1_{[0, \tau_n]} b_0(v_n, v^*, v_n) ds \\
- 2 \int_{0}^{t} 1_{[0, \tau_n]} b_1(v^*, \phi_n, \epsilon_A \phi_n) ds + 2 \int_{0}^{t} 1_{[0, \tau_n]} b_1(v_n, \phi_n, \epsilon_A \phi^*) ds \\
- 2\alpha \int_{0}^{t} 1_{[0, \tau_n]} (A_1 f(\phi) - A_1 f(\phi^*), \epsilon_A \phi_n) ds + \int_{0}^{t} 1_{[0, \tau_n]} \|g_2(s, v(s), \phi(s))\|^2_{L^2(H)} ds \\
+ 2 \int_{0}^{t} 1_{[0, \tau_n]} (g_1(v, \phi) - g_1(v^*, \phi^*), (v_n, \epsilon_A \phi_n)) ds.
$$

(5.5)

Using (3.36), we also have

$$
-2\nu \|v_n(s)\|^2 - 2c^2 |A_1 \phi - n(s)|^2_{L^2} + 2|b_0(v_n, v^*, v_n)| + 2|b_1(v^*, \phi_n, \epsilon_A \phi_n)| + 2|b_1(v_n, \phi_n, \epsilon_A \phi^*)| - 2\alpha \int_{0}^{t} (A_1 f(\phi) - A_1 f(\phi^*), \epsilon_A \phi_n) ds
$$

\leq [-2\alpha + (c_1 + c_2\epsilon) \||v^*, \phi^*|_{L^2}|||v_n, \phi_n(s)|_{H}]^2 \\
\leq [-2\alpha + (c_1 + c_2\epsilon) \||v^*, \phi^*|_{L^2}||\lambda_1 \||v_n, \phi_n(s)|_{H}]^2.
$$

(5.6)

Let us denote

$$
\Omega_n = \{ \omega \in \Omega, |\langle v_0, \phi_0 \rangle - (v^*, \phi^*)|_{H} \leq \frac{1}{n} \}.
$$

It is clear that

a) $\tau_n(\omega) = 0$ and $\langle v_n, \phi_n \rangle (t, \omega) = (v_0, \phi_0) - (v^*, \phi^*)$ if $\omega \in \Omega_n$.

b) $|\langle v_n, \phi_n \rangle (t, \omega)|_{H} \geq \frac{1}{n} \text{if } \omega \in \Omega \setminus \Omega_n$.

Then

$$
|\langle v_n, \phi_n \rangle (t, \omega) > 0,
$$

for all $\omega \in \Omega$, for all $t > 0$.

Let us take any function $g_n \in C^2(\mathbb{R})$ such that

$$
g_n(r) = log r
$$

for all $r \geq \frac{1}{\sigma}$.

Applying the Ito formula to $g_n(|\langle v_n, \phi_n(t, \omega) \rangle |_{H}^2)$, we obtain

$$
\log (|\langle v_n, \phi_n \rangle (t)|_{H}^2) = \\
\int_{0}^{t} \frac{1}{2} |\langle v_n, \phi_n \rangle (t)|_{H}^2 \left( -2\nu \|v_n(s)\|^2 - 2c^2 |A_1^{3/2} \phi_n(s)|^2_{L^2} + \sigma^2 |(v_n, \phi_n(s))|^2_{H} \right) ds \\
+ \log (|\langle v_n, \phi_n \rangle (0)|_{H}^2) \\
- \frac{1}{2} \int_{0}^{t} |\langle v_n, \phi_n \rangle (s)|_{H}^2 \left( b_0(v_n, v^*, v_n) + b_1(v^*, \phi_n, \epsilon_A \phi_n) - b_1(v_n, \phi_n, \epsilon_A \phi^*) \right) ds \\
+ \int_{0}^{t} |\langle v_n, \phi_n \rangle (s)|_{H}^2 \left( (g_1(v, \phi) - g_1(v^*, \phi^*), (v_n, \epsilon_A \phi_n)) \\
- \alpha (A_1 f(\phi) - A_1 f(\phi^*), \epsilon_A \phi_n) \right) ds \\
+ 2 \int_{0}^{t} \sigma |\langle v_n, \phi_n \rangle (s)|_{H}^2 dW_s(s) - \frac{1}{2} \int_{0}^{t} \frac{2\sigma^2 |\langle v_n, \phi_n \rangle (s)|_{H}^4}{|\langle v_n, \phi_n \rangle (s)|_{H}^4} ds
$$

\leq \log (|\langle v_n, \phi_n \rangle (0)|_{H}^2) - 2\eta t + 2\sigma W_t(t).
$$

(5.7)
Since almost surely we have
\[ \lim_{t \to \infty} \frac{W_n(t)}{t} = 0, \]
we can find \( \Omega_0 \subset \Omega \) with \( \mathcal{P}(\Omega_0) = 0 \) such that for each \( \omega \notin \Omega_0 \), there exists \( T(\omega) > 0 \) such that for all \( t \geq T(\omega) \), we have
\[ \frac{2\sigma W_n(t)}{t} \leq \eta. \] (5.8)
Therefore, we obtain that for \( t \geq T(\omega) \) we derive from (5.7) that
\[ \log |(v_n, \phi_n)(t)|^2_{\mathcal{H}} \leq \log |(w_n, \phi_n)(0)|^2_{\mathcal{H}} - \eta t, \] (5.9)
or, in other words,
\[ |(v_n, \phi_n)(t)|^2_{\mathcal{H}} \leq e^{-\eta t}|(w_n, \phi_n)(0)|^2_{\mathcal{H}}, \] (5.10)
i.e.
\[ |(v(t \land \tau_n), \phi(t \land \tau_n)) - (v^*, \phi^*)|^2_{\mathcal{H}} \leq e^{-\eta t}|(v_0, \phi_0) - (v^*, \phi^*)|^2_{\mathcal{H}}. \] (5.11)
Letting \( n \to \infty \), we deduce that
\[ |(v(t \land \tau_0), \phi(t \land \tau_0)) - (v^*, \phi^*)|^2_{\mathcal{H}} \leq e^{-\eta t}|(v_0, \phi_0) - (v^*, \phi^*)|^2_{\mathcal{H}}. \] (5.12)
for all \( t \geq T(\omega) \).
Therefore
\[ |(v(t), \phi(t)) - (v^*, \phi^*)|^2_{\mathcal{H}} \leq e^{-\eta t}|(v_0, \phi_0) - (v^*, \phi^*)|^2_{\mathcal{H}}. \] (5.13)
for all \( t \geq T(\omega) \), a.s. in \( \Omega \setminus \tilde{\Omega} \).
On \( \Omega \), we have
\[ (v(t, \omega), \phi(t, \omega)) = (v(\tau_0(\omega), \omega), \phi(\tau_0(\omega), \omega)) - (v^*, \phi^*) = (0, 0), \] (5.14)
for all \( t \geq \tau_0(\omega) \), a.s. in \( \tilde{\Omega} \).

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