Sampling generalized cat states with linear optics is probably hard

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Boson-sampling has been presented as a simplified model for linear optical quantum computing. In the boson-sampling model, Fock states are passed through a linear optics network and sampled via number-resolved photodetection. It has been shown that this sampling problem likely cannot be efficiently classically simulated. This raises the question as to whether there are other quantum states of light for which the equivalent sampling problem is also computationally hard. We present evidence, without using a full complexity proof, that a very broad class of quantum states of light — arbitrary superpositions of two or more coherent states — when evolved via passive linear optics and sampled with number-resolved photodetection, likely implements a classically hard sampling problem.

I. INTRODUCTION

Linear optical quantum computing (LOQC) [1, 2] has become a leading candidate for the implementation of large-scale universal quantum computation [3]. While LOQC is possible in principle, the technological requirements are daunting, requiring technologies that are not readily available today, such as fast feed-forward and optical quantum memory. Thus, the search for simplified, more technologically realistic models for LOQC is a priority. One recent proposal, by Aaronson & Arkhipov [4], known as boson-sampling, significantly simplifies the requirements for LOQC, allowing a type of non-universal quantum computation that implements a classically hard algorithm using technologies that are, for the larger part, available today. Numerous elementary experimental demonstrations have recently been performed [5–9].

In the boson-sampling model, we simply input n copies of the single-photon Fock state into a passive linear optics network, comprised of beamsplitters and phase-shifters, and perform number-resolved photodetection at the output. This yields a sampling problem, which is believed to be classically hard to simulate [4]. Thus, the full model requires only single-photon Fock state preparation, passive linear optics, and photodetection, technologies that are all available today on a small scale. Note that number-resolving photo-detectors are not required in the limit of large n as there is likely only zero or one photon per output mode.

In boson-sampling, the classical hardness of the sampling problem relates to the computational complexity of calculating the amplitudes in the output superposition.

In the case of Fock state inputs, the output amplitudes are related to matrix permanents, which reside in the complexity class #P-complete, believed to be classically hard to solve.

The classical hardness of this Fock state sampling raises the question as to whether there are other quantum states of light, which also yield classically hard sampling problems. It was shown recently by Seshadreesan et al. [10] that photon-added coherent states (PACS) are an example of such states. It is known that passive linear optics may be efficiently simulated with Gaussian inputs and non-adaptive measurements [11, 12]. However, the more general question as to which quantum states of light may be efficiently simulated with number-resolved measurements is an open question.

Here we consider a more generalized boson-sampling device where the input states are not Fock states, but rather superpositions of coherent states. This is a very broad class of continuous-variable optical states.

We first consider the small amplitude limit, and show that small amplitude cat states necessarily yield a computationally hard problem. In fact, in the zero amplitude limit, cat sampling reduces to ideal boson-sampling. For small, but non-zero amplitudes, treating the non-single-photon-number terms as an error model places a bound on the cat state amplitudes for the problem to be provably hard. We then consider the general case and demonstrate that the output state is a highly entangled superposition of an exponential number of terms, where the amplitude of each term is related to a permanent-like combinatoric problem, which, via brute-force, requires exponential resources to calculate. Combined, this provides strong evidence that such generalized optical sampling problems might in general be implementing classically hard problems. Without a full and sufficient complexity proof we cannot determine the complexity class

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of this scheme, but based on the evidence it likely resides in a classically hard class comparable to ideal boson-sampling.

While such quantum states of light may be more challenging to prepare than Fock states, addressing this question sheds light on what makes a quantum optical system classically hard to simulate, and may provide motivation for developing technologies for preparing quantum states of light beyond Fock states.

II. REVIEW OF BOSON-SAMPLING

We begin by reviewing the boson-sampling model. First we prepare an \( m \)-mode state, where the first \( n \) modes are prepared with the single photon Fock state, and the remainder with the vacuum state,

\[
|\psi_{\text{in}}\rangle = |1, \ldots, 1, 0_{n+1}, \ldots, 0_m\rangle = a_1^\dagger \cdots a_n^\dagger |1, \ldots, 1, 0_m\rangle, \quad (1)
\]

where \( a_i^\dagger \) is the photon creation operator on the \( i \)th mode, and \( m = O(n^2) \). Next we propagate this state through a passive linear optics network, which implements a unitary map on the photon creation operators,

\[
\hat{U} : \hat{a}_i^\dagger \rightarrow \sum_{j=1}^m U_{i,j} \hat{a}_j^\dagger. \quad (2)
\]

The output state can be expressed in a photon occupation number representation as,

\[
|\psi_{\text{out}}\rangle = \sum_S \gamma_S |n_1^{(S)} \cdots n_m^{(S)}\rangle, \quad (3)
\]

where \( S \) are the different photon number configurations, \( n_i^{(S)} \) is the number of photons in mode \( i \) associated with configuration \( S \), and \( \gamma_S \in \mathbb{C} \) are the respective amplitudes. The number of configurations scales exponentially with \( n \), \( |S| = (\binom{n+m-1}{m}) \). The total photon-number is conserved, thus \( \sum_i n_i^{(S)} = n \) for all \( S \). Performing number-resolving photon measurements, which are described by projection operators \( \Pi(n) = |n\rangle \langle n| \), we sample from the distribution \( P_S = |\gamma_S|^2 \), each time obtaining an \( m \)-fold coincidence measurement outcome of a total of \( n \) photons.

The sampling amplitudes are related to matrix permanents, \( \gamma_S \propto \text{Per}(U_S) \), where \( U_S \) is an \( n \times n \) sub-matrix of \( U \) as a function of the respective configuration \( S \). The best known classical algorithm for calculating matrix permanents is by Ryser [13], requiring \( O(2^n n^2) \) time steps. Because this requires exponential time to evaluate, sampling from the distribution \( P_S \) is believed to be a classically hard problem. Importantly, boson-sampling does not allow us to calculate matrix permanents as this would require an exponential number of measurements.

The boson-sampling model is illustrated in Fig. 1.

![FIG. 1: The boson-sampling model for quantum computation.](image)

A series of single-photon and vacuum states are prepared, \( |1, \ldots, 1, 0, \ldots, 0\rangle \), and passed through a linear optics network, \( \hat{U} \). The experiment is repeated many times and each time the output distribution is measured via number-resolved photodetection, sampling from the distribution \( P_S \).

III. CAT STATES

‘Cat states’ is a generic term for arbitrary superpositions of macroscopic states. In quantum optics, this is generally understood to mean a superposition of two coherent states, potentially with large amplitudes. This is the definition we will use in this work. Two illustrative examples are the ‘even’ (+) and ‘odd’ (−) cat states, so-called because they contain only even or odd photon-number terms respectively,

\[
|\text{cat}_{\pm}\rangle = \mathcal{N}_\pm (|\alpha\rangle \pm |-\alpha\rangle), \quad (4)
\]

where,

\[
\mathcal{N}_\pm = \frac{1}{\sqrt{2(1 \pm e^{-2|\alpha|^2})}}. \quad (5)
\]

The normalization factor arises because \( |\alpha\rangle \) and \( |-\alpha\rangle \) are not orthogonal, except in the large \( \alpha \) limit.

In this paper we will consider arbitrary superpositions of an arbitrary number of coherent states, in which case a general cat is of the form,

\[
|\text{cat}\rangle = \sum_{j=1}^t \lambda_j |\alpha_j\rangle. \quad (6)
\]

IV. CAT STATE SAMPLING IN THE SMALL AMPLITUDE LIMIT

The odd cat state has the property that in the limit of small amplitude it identically approaches the single-photon state,

\[
\lim_{\alpha \to 0} |\text{cat}_-\rangle = |1\rangle. \quad (7)
\]

Furthermore, the vacuum state (we require \( O(n^2) \) vacuum states to be consistent with the boson-sampling model) is given by a trivial cat state containing only a
single term in the superposition \((t = 1)\) with the respective amplitude \(\alpha = 0\). Alternately, the vacuum state can be regarded as the zero amplitude limit of the even cat state,

\[
\lim_{\alpha \to 0} |\text{cat}_+= 0\rangle.
\]  

(8)

Thus, it is immediately clear that in the small amplitude limit, cat state sampling reduces to ideal boson-sampling, using an appropriate configuration of odd and even cat states, which is a provably hard problem. Specifically, to implement exact boson-sampling with cat states, we choose our input state to be,

\[
|\psi_{\text{in}}\rangle = \lim_{\alpha \to 0} (|\text{cat}_-\rangle_1 \cdots |\text{cat}_-\rangle_n |\text{cat}_+\rangle_{n+1} \cdots |\text{cat}_+\rangle_m).
\]  

(9)

In App. B we present an example of this reduction in the case of Hong-Ou-Mandel interference to explicitly demonstrate that small amplitude cats behave as single photons. This demonstrates that in certain regimes, cat state sampling reproduces single-photon statistics.

Having established that cat sampling reduces to boson-sampling in the zero amplitude limit, the obvious next question is “what if the amplitude is small but non-zero?” It was shown by Aaronson & Arkhipov that boson-sampling, when corrupted by erroneous samples, remains computationally hard provided that the error rate scales as \(1/\text{poly}(n)\). If we consider a small, but non-zero, amplitude odd cat state, we can treat the non-single-photon terms (which scale as a function of \(\alpha\)) as erroneous terms, which must be kept below the \(1/\text{poly}(n)\) bound. Specifically,

\[
|\text{cat}_-\rangle = \gamma_1(\alpha)|1\rangle + \gamma_3(\alpha)|3\rangle + \gamma_5(\alpha)|5\rangle + \ldots
\]  

(10)

where \(\gamma_i(\alpha)\) defines the odd photon-number distribution and follows from Eq. 24. The two underbraced components represent the desired single-photon term and the remaining photon-number terms, which are treated as errors. In the limit of small \(\alpha\) the errors vanish, and \(\lim_{\alpha \to 0} \gamma_i(\alpha) = \delta_{i,1}\).

Thus, it follows that for non-zero, but sufficiently small \(\alpha\), cat sampling remains computationally hard. In App. C we show that the bound on the amplitude of the cat states for a provably hard sampling problem to take place is,

\[
\alpha^{2n} \text{csch}^n(\alpha^2) > 1/\text{poly}(n).
\]  

(11)

We have established that cat state boson-sampling is a provably computationally hard problem in two regimes: (1) when the amplitude approaches zero, in which case we reproduce ideal boson-sampling, and (2) for small amplitudes, in which case the non-single-photon-number terms may be regarded as errors, which remains a computationally hard problem, subject to the bound given in Eq. 11. Having established this, the remainder of this paper is dedicated to the completely general case, whereby the terms in the cat states may have arbitrary amplitudes, potentially at a macroscopic scale.

V. GENERAL CAT STATE BOSON-SAMPLING

Let the input state to our generalized boson-sampling model comprise \(m\) arbitrary superpositions of \(t\) coherent states, which we will refer to as generalized cat states,

\[
|\psi_{\text{in}}\rangle = \bigotimes_{i=1}^{m} \sum_{j=1}^{t} \lambda_i^{(j)} |\alpha_i^{(j)}\rangle,
\]  

(12)

where \(|\alpha_i^{(j)}\rangle\) is the coherent state of amplitude \(\alpha \in \mathbb{C}\) in the \(i\)th mode, and \(\lambda_i^{(j)} \in \mathbb{C}\) is the amplitude of the \(j\)th term of the superposition in the \(i\)th mode\(^1\). It should be noted here that, in line with traditional boson-sampling, we can choose a number of the modes to be the vacuum. This is achieved by setting \(\lambda_i^{(j)} = 1\) and \(\alpha_i^{(j)} = 0\), with \(\alpha_{i}^{(j)} = 0\).

Expanding this expression yields a superposition of multi-mode coherent states of the form,

\[
|\psi_{\text{in}}\rangle = \sum_{i=1}^{t} \lambda_i^{(1)} \cdots \lambda_i^{(m)} |\alpha_i^{(1)}, \ldots, \alpha_i^{(m)}\rangle,
\]  

(13)

where \(\ell\) is shorthand for \(\{t_1, \ldots, t_m\}\). We propagate this state through the passive linear optics network, \(\hat{U}\). Such a unitary network has the property that a multi-mode coherent state is mapped to another multi-mode coherent state,

\[
\hat{U}|\alpha^{(1)}, \ldots, \alpha^{(m)}\rangle \rightarrow |\beta^{(1)}, \ldots, \beta^{(m)}\rangle,
\]  

(14)

where the relationship between the input and output amplitudes is given by (see App. A),

\[
\beta^{(j)} = \sum_{k=1}^{m} U_{j,k} \alpha^{(k)}.
\]  

(15)

\(\hat{U}\) acts on each term in the superposition of Eq. 13 independently. Thus, the output state will be of the form,

\[
|\psi_{\text{out}}\rangle = \hat{U} |\psi_{\text{in}}\rangle = \sum_{i=1}^{t} \sum_{\ell=1}^{m} \lambda_{i,\ell}^{(1)} \cdots \lambda_{i,\ell}^{(m)} |\beta_{\ell}^{(1)}, \ldots, \beta_{\ell}^{(m)}\rangle.
\]  

(16)

The number of terms in the output superposition is \(t^m\), scaling exponentially with the number of modes, provided \(t > 1\).

Our goal is to sample this distribution using number-resolved photodetectors, which are described by the measurement projectors,

\[
\hat{N}_i(n) = |n\rangle_i \langle n|_i,
\]  

(17)

\(^1\) Continuous superpositions are a simple generalization of our formalism, and with this generalization arbitrary states could be expressed as continuous superpositions of coherent states.
where \( n \) is the photon-number measurement outcome on the \( i \)th mode. Multi-mode measurements are described by the projectors,

\[
\hat{\Pi}(S) = \hat{\Pi}_1(S_1) \otimes \cdots \otimes \hat{\Pi}_m(S_m),
\]

where \( S = \{S_1, \ldots, S_m\} \) is the multi-mode measurement signature, with \( S_i \) photons measured in the \( i \)th mode. The sample probabilities are given by,

\[
P_S = \langle \psi_{\text{out}} | \hat{\Pi}(S) | \psi_{\text{out}} \rangle.
\]

In the case of continuous-variable states, the number of measurement signatures, \(|S|\), is unbounded as the photon-number is undefined, unlike Fock states where the total photon-number is conserved.

The full model is illustrated in Fig. 2.

![FIG. 2: The model for generalized boson-sampling with superpositions of coherent states — ‘cat states’. The input state to each mode is an arbitrary superposition of coherent states, some of which are set to the vacuum. Following the application of a linear optics network, the distribution is sampled via number-resolved photo-detection.](image)

We argue that this sampling problem is likely to be classically hard if three criteria are satisfied:

1. There must be an exponential number of terms in the output distribution. This rules out brute-force simulation by explicitly calculating the state vector.

2. The terms in the superposition must be entangled, such that the distribution cannot be trivially sampled by independently sampling each mode.

3. Each of the amplitudes in the output distribution must be related to a computationally hard problem. This ensures that classical simulation of the individual amplitudes is not efficient.

We have chosen these criteria as they are general properties that classically hard problems are known to exhibit, but we do not prove that these criteria are sufficient to establish whether a problem is likely to be classically hard.

Criteria (1) is achieved by virtue of our choice of input state — there are \( 2^m \) terms in the output distribution.

It is easily seen that criteria (2) holds in general. As a simple example, consider the input state,

\[
|\psi_{\text{in}}\rangle = \mathcal{N}^2(|\alpha\rangle + |-\alpha\rangle) \otimes (|\alpha\rangle + |-\alpha\rangle) = |\text{cat, cat}\rangle, \quad (20)
\]
a tensor product of two cat states.

Passing this separable two-mode state through a 50/50 beamsplitter given by the Hadamard matrix,

\[
\hat{H} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},
\]

gives rise to the output state,

\[
|\psi_{\text{out}}\rangle = \hat{H}|\psi_{\text{in}}\rangle = |\text{cat}', 0\rangle + |0, \text{cat}'\rangle,
\]

where \(|\text{cat}'\rangle = \mathcal{N}^2(|\sqrt{2} \alpha\rangle + |\sqrt{2} \alpha\rangle)\) is a cat state. This is a path-entangled superposition of a cat state across two modes. Thus, while Eq. 14 demonstrates that a unitary network maps a tensor product of coherent states to a tensor product of coherent states, such a network will in general generate path-entanglement when the input state is a tensor product of superpositions of coherent states. Note the structural similarity between cat state interference and two-photon Hong-Ou-Mandel (HOM) [14] -type interference. In the case of HOM interference we have \(\hat{H}|1, 1\rangle = (|2, 0\rangle + |0, 2\rangle)/\sqrt{2}\), whereas for cat states we have \(\hat{H}|\text{cat, cat}\rangle = |\text{cat}', 0\rangle + |0, \text{cat}'\rangle\).

It was recently and independently reported by Jiang et al. [15] that linear optics networks fed with nonclassical pure states of light almost always generates modal entanglement, consistent with our observation here. This ensures that the output state to our generalized boson-sampling device is highly entangled, thus satisfying criteria (2). However, Jiang et al. have no discussion at all about our hardness criteria (3); they do not connect their states to a computationally hard problem. Thus their work provides a sufficient but not necessary proof of computational hardness. It is important, as in our work here, to examine such non-classical input states individually and make the case for the sufficient criteria (3). For example it is well known from the Gottesman-Knill theorem that some systems with exponentially large Hilbert spaces that satisfy our criteria (1) and (2) can nevertheless be efficiently simulated. An example is the circuit model for quantum computation that deploys only gates from the Clifford algebra.

Finally let us consider criteria (3). Let the expansion for a coherent state be,

\[
|\alpha\rangle = \sum_{n=0}^{\infty} f_n(\alpha) |n\rangle, \quad (23)
\]
in the photon-number basis, where,

\[
f_n(\alpha) = e^{-\frac{|\alpha|^2}{2}} \frac{\alpha^n}{\sqrt{n!}}, \quad (24)
\]
is the amplitude of the \( n \)-photon term. Then,

\[
\langle n | \alpha \rangle = f_n(\alpha). \quad (25)
\]

Thus, acting the measurement projector for configuration \( S \), Eq. 18, on the output state, Eq. 16, we obtain,

\[
\hat{\Pi}(S) |\psi_{\text{out}}\rangle = \gamma_S |S_1, S_2, \ldots, S_m\rangle, \quad (26)
\]
where,

\[ \gamma_S = \sum_{t=1}^{t} \left( \prod_{j=1}^{m} \lambda_{j,t} f_{S_j} \left( \sum_{k=1}^{m} U_{j,k} \alpha_{t,k}^{(k)} \right) \right), \quad (27) \]

and the sampling probability takes the form \( P_S = |\gamma_S|^2 \).

We can group the terms under the product and label them \( A_{j,t}^{(S)} \). Then the amplitudes are given by,

\[ \gamma_S = \sum_{t=1}^{t} \prod_{j=1}^{m} A_{j,t}^{(S)}, \quad (28) \]

which has the same analytic structure as the permanent when \( t = m \). Via brute force, evaluating this expression requires summing a number of terms exponential in \( m \), provided \( t > 1 \). Given that Eq. 28 has the same analytic form as the matrix permanent, which is known to be classically hard, this implies a striking similarity between cat state sampling and Fock state sampling, with the constraint that \( A \) is of a form whose permanent is not trivial.

In the original proof by Aaronson & Arkhipov, it is required that \( U \) is Haar-random. It is an open question as to whether \( A \) can be made Haar-random in the presented generalized boson-sampling model.

In the trivial case of \( t = 1 \) this expression simplifies to,

\[ \gamma_S = \prod_{j=1}^{m} f_{S_j} \left( \sum_{k=1}^{m} U_{j,k} \alpha_{1,k}^{(k)} \right), \quad (29) \]

which evaluates in polynomial time. In this case the input state is simply a tensor product of coherent states, and the runtime is consistent with the known result that simulating coherent states is trivial as the tensor product structure allows sampling to proceed by independently sampling each mode, each of which is an efficient sampling problem. However, when \( t > 1 \) the complexity of evaluating Eq. 27 grows exponentially.

VI. PREPARING CAT STATES

Finally, we will discuss the prospects for experimentally preparing cat states of the form used in our derivation. There exists a significant number of schemes for generating a finite number of superpositions of coherent states all of which are extremely difficult to scale to higher order cat states. For example, superpositions of coherent states of equal amplitude but different phases can be produced by a sequence of quantum nondemolition (QND) measurements in the photon number implemented via the interaction of a strong Kerr nonlinearity [16, 17]. The way to understand this is that QND measurements of photon number cause uncontrollable but discrete jumps in the phase of the state producing the superposition [16]. This type of superposition generation has been carried out in microcavities where the cavity Q factor enhances the nonlinearity. Outside of a cavity, for the propagating coherent states we need here, this scheme is impractical. A more reasonable approach is to use strong Kerr nonlinearities together with coupled Mach-Zehnder interferometers [17] but this is also impractical as outside the cavity a strong Kerr would require a coherent Electromagnetically Induced Transparency effect in an atomic gas cloud and even there too in practice the nonlinearities are too weak for our purposes.

In a similar way that measurements of photon number can produce discrete coherent state superpositions in phase; measurements of the phase can produce discrete coherent state superpositions in amplitude. This can be understood via the number-phase uncertainty relation. Any improved knowledge of the phase of a state induces kicks in the number and vice versa. In this way, by combining such different measurements, one can produce discrete superpositions in both phase and amplitude; approaching the arbitrary superpositions of coherent states we require here. Exactly such a scheme was proposed by Jeong et al. in 2005 [18, 19]. By combining both types of detection schemes, even with detectors of non-unit efficiency, they show that a large number of propagating superpositions of coherent states may be thus produced. These states then could be used in proof-of-principle experiments for our protocol outlined here.

VII. CONCLUSION

We have presented evidence that a linear optics network, fed with arbitrary superpositions of coherent states, and sampled via number-resolved photodetection, is likely to be a classically hard problem. While we have not presented a full complexity proof, our argument is based on three realistic criteria for computational hardness of the sampling problem. In the case of input states comprising superpositions of coherent states, these three criteria are satisfied. In the case of (3), we find that the amplitudes are related to a permanent-like function of a matrix, strikingly similar to ideal boson-sampling. Furthermore, we show that,

1. In the limit of small amplitude odd cat states, cat sampling reduces to ideal boson-sampling.

2. When the amplitude is increased away from zero, cat sampling remains hard for sufficiently low amplitudes, by treating the non-single-photon-number terms as an error model.

Combined, these observations present strong evidence that such a generalized sampling problem is likely classically hard to simulate.

Because coherent states form an over-complete basis, any pure optical state can be expressed in terms of coherent states, suggesting that most quantum states of light may yield hard sampling problems. This observation further motivates interest in developing sources for
quantum states of light other than Fock states.

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Appendix A: Propagating multi-mode coherent states through passive linear optics networks

The unitary map describing a passive linear optics network is given by,

$$
\hat{U} : \hat{a}_i^{\dagger} \rightarrow \sum_{j=1}^{m} U_{i,j} \hat{a}_j^{\dagger},
$$

(A1)

and taking the Hermitian conjugate yields,

$$
\hat{U} : \hat{a}_i \rightarrow \sum_{j=1}^{m} U_{j,i}^{\dagger} \hat{a}_j.
$$

(A2)

A coherent state can be expressed in terms of a displacement operator acting on the vacuum state,

$$
|\alpha^{(i)}\rangle_i = \hat{D}_i(\alpha^{(i)}|0\rangle_i,
$$

(A3)

where the displacement operator may be expressed in terms of creation and annihilation operators as,

$$
\hat{D}_i(\alpha^{(i)}) = \exp(\alpha^{(i)} \hat{a}_i^{\dagger} - \alpha^{(i)*} \hat{a}_i).
$$

(A4)

Applying the unitary map Eqs. A1 & A2, we obtain,

$$
\hat{U} \hat{D}_i(\alpha_i) = \exp\left(\alpha^{(i)} \sum_{j=1}^{m} U_{i,j} \hat{a}_j^{\dagger} - \alpha^{(i)*} \sum_{j=1}^{m} U_{j,i}^{\dagger} \hat{a}_j\right).
$$

(A5)

Let,

$$
\hat{U} |\alpha^{(1)}, \ldots, \alpha^{(m)}\rangle = |\beta^{(1)}, \ldots, \beta^{(m)}\rangle.
$$

(A6)

Then,

$$
\hat{U} |\alpha^{(1)}, \ldots, \alpha^{(m)}\rangle = \hat{D}_1(\alpha^{(1)}) \ldots \hat{D}_m(\alpha^{(m)}) |0_1, \ldots, 0_m\rangle.
$$

(A7)

For each term,

$$
\hat{U} \hat{D}_i(\alpha^{(i)}) = \prod_{j=1}^{m} \exp\left(\alpha^{(i)} U_{i,j} \hat{a}_j^{\dagger} - \alpha^{(i)*} U_{j,i}^{\dagger} \hat{a}_j\right)
= \prod_{j=1}^{m} \hat{D}_j(U_{i,j} \alpha^{(i)}).
$$

(A8)
Thus,
\[
\hat{U} \prod_{i=1}^{m} |\alpha(i)\rangle_i = \hat{U} \prod_{i=1}^{m} \hat{D}(\alpha(i))|0\rangle_i
\]
\[
= \prod_{i=1}^{m} \prod_{j=1}^{m} \hat{D}_j(U_{i,j}\alpha(i))|0\rangle
\]
\[
= \bigotimes_{j=1}^{m} \sum_{i=1}^{m} U_{i,j}\alpha(i)
\]
\[
= \bigotimes_{j=1}^{m} |\beta(j)\rangle_j.
\]  \hspace{1cm} (A9)

And,
\[
\beta(j) = \sum_{i=1}^{m} U_{i,j}\alpha(i),
\]  \hspace{1cm} (A10)
as per Eq. 15. It is interesting to note the similarity between Eq. A10 and Eq. A2. 

Appendix B: Reproducing Hong-Ou-Mandel interference using small amplitude odd cat states

We begin with our generalized cat state result from Eq. 27,
\[
\gamma_s = \sum_{t=1}^{2} \left( \prod_{j=1}^{m} \lambda_{t_j}^{(j)} f_{S_j}(\beta_{t_j}^{(j)}) \right),
\]  \hspace{1cm} (B1)
and input the odd cat state which has the form
\[
|\text{cat}_-\rangle = \frac{|\alpha\rangle - |-\alpha\rangle}{\sqrt{2(1 - \exp[-2\alpha^2])}}.
\]  \hspace{1cm} (B2)

When considering the specific example of $|\text{cat}_-\rangle$ the $\lambda_{t_j}^{(j)}$ of Eq. B1 goes to $(-1)^{t_j}$. Eq. B1 then becomes,
\[
\gamma_s = \sum_{t=1}^{2} \left( \prod_{j=1}^{m} (-1)^{t_j} \frac{f_{S_j}(\beta_{t_j}^{(j)})}{\sqrt{2(1 - \exp[-2\alpha^2])}} \right).
\]  \hspace{1cm} (B3)

Since the $\beta_{t_j}^{(j)}$'s in Eq. B3 depend on $\alpha$, we substitute the argument of $f_{S_j}$ using Eq. 24,
\[
\gamma_s = \frac{1}{\left(\sqrt{2(1 - \exp[-2\alpha^2])}\right)^m} \times \sum_{t=1}^{2} \left( \prod_{j=1}^{m} (-1)^{t_j} \exp \left[ -\frac{|\beta_{t_j}^{(j)}|^2}{2} \right] \frac{(\beta_{t_j}^{(j)})^{S_j}}{\sqrt{S_j!}} \right).
\]  \hspace{1cm} (B5)

Next we take a first order approximation. Since $\alpha$ is small, the exponential in the numerator goes to one while the exponential in the denominator goes to $\exp(x) \approx 1 + x$ because otherwise this would diverge. This yields,
\[
\gamma_s \approx \frac{1}{(2\alpha)^m \sqrt{S_1! \cdots S_m!}} \sum_{t=1}^{2} \left( \prod_{j=1}^{m} (-1)^{t_j} \left( \frac{\beta_{t_j}^{(j)} S_j}{\sqrt{S_j!}} \right) \right)
\]
\[
= \frac{1}{(2\alpha)^m \sqrt{S_1! \cdots S_m!}} \sum_{t=1}^{2} \left( \prod_{j=1}^{m} (-1)^{t_j} (\beta_{t_j}^{(j)}) S_j \right)
\]
\[
= \frac{1}{(2\alpha)^m \sqrt{S_1! \cdots S_m!}} \sum_{t=1}^{2} (-1)^{t} \prod_{j=1}^{m} (\beta_{t_j}^{(j)}) S_j.
\]  \hspace{1cm} (B6)

In the limit of small $\alpha$ we know that the odd cat state reduces to a single photon Fock state. Here we consider the case of a cat state being inputted into the first two modes and let the unitary be the Hadamard gate. In small $\alpha$ this corresponds to inputting a single photon Fock state into the first two modes and interfering them in a single 50/50 beamsplitter. Therefore, the corresponding bunching in the output modes would to be expected. In this section we show that our expression of Eq. B6 does show the expected bunching.

We begin by putting an odd cat state $|\text{cat}_-\rangle$ with $t = 2$ terms into the first $m = 2$ modes. Then Eq. B6 becomes,
\[
\gamma_s \approx \frac{1}{(2\alpha)^2 \sqrt{S_1! S_2!}} \sum_{t_{12}=1}^{2} (-1)^{t_{12}} \prod_{j=1}^{2} (\beta_{t_{12}, t_{22}}^{(j)}) S_j
\]
\[
= \frac{1}{(2\alpha)^2 \sqrt{S_1! S_2!}} \sum_{t_{12}=1}^{2} (-1)^{t_{12}} (\beta_{t_{12}, t_{22}}^{(1)}) S_1 (\beta_{t_{12}, t_{22}}^{(2)}) S_2
\]
\[
= \frac{1}{(2\alpha)^2 \sqrt{S_1! S_2!}} \left[ (\beta_{11,11}^{(1)}) S_1 (\beta_{11,11}^{(2)}) S_2 - (\beta_{11,22}^{(1)}) S_1 (\beta_{11,22}^{(2)}) S_2 
\right.
\]
\[
- (\beta_{22,11}^{(1)}) S_1 (\beta_{22,11}^{(2)}) S_2 + (\beta_{22,22}^{(1)}) S_1 (\beta_{22,22}^{(2)}) S_2 \right].
\]  \hspace{1cm} (B7)

Now to calculate the $\beta_{t_j}^{(j)}$'s for this case we first take the tensor product between the first two modes. Ignoring the normalization factor this yields,
\[
|\text{cat}_-\rangle = (|\alpha\rangle - |\alpha\rangle) \otimes (|\alpha\rangle - |\alpha\rangle)
\]
\[
= |\alpha, \alpha\rangle - |\alpha, -\alpha\rangle - |\alpha, \alpha\rangle + |\alpha, -\alpha\rangle,
\]  \hspace{1cm} (B8)

Next we pass them through a Hadamard gate,
\[
U|\text{cat}_-\rangle = |\sqrt{2} \alpha, 0\rangle - |0, \sqrt{2} \alpha\rangle - |0, -\sqrt{2} \alpha\rangle + |\sqrt{2} \alpha, 0\rangle.
\]  \hspace{1cm} (B9)
Now we read off the $\beta_i^{(j)}$s to be

\[
\begin{align*}
\beta_{1,1}^{(1)} & = \sqrt{2}\alpha \\
\beta_{1,2}^{(1)} & = 0 \\
\beta_{2,1}^{(1)} & = 0 \\
\beta_{2,2}^{(1)} & = -\sqrt{2}\alpha \\
\beta_{1,1}^{(2)} & = 0 \\
\beta_{1,2}^{(2)} & = \sqrt{2}\alpha \\
\beta_{2,1}^{(2)} & = -\sqrt{2}\alpha \\
\beta_{2,2}^{(2)} & = 0.
\end{align*}
\]  
\tag{B10}
\]

Now Eq. B7 becomes,

\[
\begin{align*}
\gamma_s & = \frac{1}{(2\alpha)^2} \left[ (\sqrt{2}\alpha)^{S_1} (0)^{S_2} - (0)^{S_1} (\sqrt{2}\alpha)^{S_2} \\
& - (0)^{S_1} (-\sqrt{2}\alpha)^{S_2} + (-\sqrt{2}\alpha)^{S_1} (0)^{S_2} \right]. 
\end{align*}
\tag{B11}
\]

Because we are dealing in the limit of small $\alpha$, a non-zero number arbitrarily close to zero raised to a zero power is one, so the terms $0^{S_j} = \delta_{S_j,0}$. Now Eq. B11 becomes,

\[
\begin{align*}
\gamma_s & = \frac{1}{(2\alpha)^2} \left[ (\sqrt{2}\alpha)^{S_1} \delta_{S_2,0} - (\sqrt{2}\alpha)^{S_2} \delta_{S_1,0} \\
& - (-\sqrt{2}\alpha)^{S_2} \delta_{S_1,0} + (-\sqrt{2}\alpha)^{S_1} \delta_{S_2,0} \right]. 
\end{align*}
\tag{B12}
\]

For this example we know that there are three possible signature outcomes. We expect that the configuration $S_1 = S_2 = 1$ is not possible due to HOM photon bunching and thus in this case $\gamma_s = 0$. For configurations $S_1 = 0$ and $S_2 = 2$ or $S_1 = 2$ and $S_2 = 0$ we would expect a non-zero configuration amplitude of $\gamma_s = 1/2$ in each case. Next, we will show that this is indeed the case.

1. **Configuration $S_1 = S_2 = 1$**

With configuration $S_1 = S_2 = 1$ Eq. B12 becomes,

\[
\begin{align*}
\gamma_s & \approx \frac{1}{4\alpha^2} \left[ (\sqrt{2}\alpha)\delta_{1,0} - (\sqrt{2}\alpha)\delta_{1,0} \\
& - (-\sqrt{2}\alpha)\delta_{1,0} + (-\sqrt{2}\alpha)\delta_{1,0} \right] \\
& = 0,
\end{align*}
\]  
\tag{B13}
\]

which vanishes as expected.

2. **Configuration $S_1 = 0$ and $S_2 = 2$**

With configuration $S_1 = 0$ and $S_2 = 2$ Eq. B12 becomes,

\[
\begin{align*}
\gamma_s & = \frac{1}{(2\alpha)^2} \left[ (\sqrt{2}\alpha)^0 \delta_{2,0} - (\sqrt{2}\alpha)^0 \delta_{0,0} \\
& - (\sqrt{2}\alpha)^0 \delta_{2,0} + (\sqrt{2}\alpha)^0 \delta_{0,0} \right] \\
& = \frac{1}{4\alpha^2} \left[ 2\alpha^2 - 2\alpha^2 \right] \\
& = -\frac{1}{\sqrt{2}},
\end{align*}
\tag{B14}
\]

and the corresponding classical probability is $1/2$ as expected.

3. **Configuration $S_1 = 2$ and $S_2 = 0$**

With configuration $S_1 = 2$ and $S_2 = 0$ Eq. B12 becomes,

\[
\begin{align*}
\gamma_s & = \frac{1}{(2\alpha)^2} \left[ (\sqrt{2}\alpha)^2 \delta_{0,0} - (\sqrt{2}\alpha)^2 \delta_{0,0} \\
& - (\sqrt{2}\alpha)^2 \delta_{0,0} + (\sqrt{2}\alpha)^2 \delta_{0,0} \right] \\
& = \frac{1}{4\alpha^2} \left[ 2\alpha^2 + 2\alpha^2 \right] \\
& = \frac{1}{\sqrt{2}},
\end{align*}
\tag{B15}
\]

again with classical probability $1/2$ as expected.

Thus, our result generalizes to the expected results for passing a single photon Fock state inputted in modes one and two through a Hadamard gate. This shows that our cat state generalization works for the odd cat state in the limit of small $\alpha$, which is equivalent to Aaronson &Arkhipov’s boson-sampling.

**Appendix C: Non-zero amplitude odd cat states as an error model**

Consider the odd cat state from Eq. B2. The amplitude of the single-photon component is given by,

\[
\gamma_1 = \frac{f_1(\alpha) - f_1(-\alpha)}{\sqrt{2}(|1,\ldots,1,0,\ldots,0\rangle \text{ term is then given by},}
\]

where $f_n(\alpha)$ is defined in Eq. 24.

The probability of having sampled from the $|1,\ldots,1,0,\ldots,0\rangle$ term is then given by,

\[
P = \gamma_1^{2n} = \alpha^{2n} \text{csch}^n(\alpha^2) \tag{C2}
\]

Based on the error bound of Aaronson & Arkhipov, this in turn requires that,

\[
\alpha^{2n} \text{csch}^n(\alpha^2) > 1/\text{poly}(n) \tag{C3}
\]
in order for the sampling problem to be in a regime which is provably computationally hard.