ON ORBITAL VARIETY CLOSURES IN \( \mathfrak{sl}_n \)

I. INDUCED DUFLO ORDER

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ABSTRACT. For a semisimple Lie algebra \( \mathfrak{g} \) the orbit method attempts to assign representations of \( \mathfrak{g} \) to (coadjoint) orbits in \( \mathfrak{g}^* \). Orbital varieties are particular Lagrangian subvarieties of such orbits leading to highest weight representations of \( \mathfrak{g} \). In \( \mathfrak{sl}_n \) orbital varieties are described by Young tableaux. Inclusion relation on orbital variety closures defines a partial order on Young tableaux. Our aim is to describe this order. The paper is devoted to the combinatorial description of induced Duflo order on Young tableaux (the order generated by inclusion of generating subspaces of orbital varieties). This is a very interesting and complex combinatorial question.

This is the first paper in the series. In Part II and Part III we use repeatedly the results of the paper as a basis for further study of orbital variety closures.

1. Introduction

This is the first paper in the series of three papers devoted to the study of orbital variety closures in \( \mathfrak{sl}_n \). They are referred to as Part I, Part II and Part III respectively.

1.1. The orbital varieties derive from the works of N. Spaltenstein [13] and [14], and R. Steinberg [15] and [16] during their studies of unipotent variety of a complex semi-simple group \( G \). Let \( B \) be the variety of Borel subgroups of \( G \) on which \( G \) acts by conjugation. For a fixed unipotent \( u \in G \) let \( B_u \) be the subvariety of \( B \) containing \( u \) or equivalently the variety of flags in \( G/B \) fixed by \( u \) for some fixed Borel subgroup \( B \). Spaltenstein and Steinberg studied the irreducible components of this variety.

Orbital varieties are the translation of these components from unipotent variety of \( G \) to nilpotent cone of \( \mathfrak{g} = \text{Lie}(G) \). We give their description in the next subsection.

1.2. Let \( G \) be a connected semisimple finite dimensional complex algebraic group. Let \( \mathfrak{g} \) be its Lie algebra and \( U(\mathfrak{g}) \) be the enveloping algebra of \( \mathfrak{g} \). Consider the adjoint action of \( G \) on \( \mathfrak{g} \). Fix some triangular decomposition \( \mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}^- \). A \( G \) orbit \( \mathcal{O} \) in \( \mathfrak{g} \) is called nilpotent if it consists of nilpotent elements, that is if \( \mathcal{O} = G x \) for some \( x \in \mathfrak{n} \). The intersection \( \mathcal{O} \cap \mathfrak{n} \) is reducible in general. Its irreducible components are called orbital varieties associated to \( \mathcal{O} \). Orbital varieties play a key role in the study of primitive...
ideals in $U(\mathfrak{g})$. They also play an important role in Springer’s Weyl group representations, described in terms of $B_u$.

1.3. The first role above can be detailed as follows. Since $\mathfrak{g}$ is semisimple we can identify $\mathfrak{g}^*$ with $\mathfrak{g}$ through the Killing form. This identification gives an adjoint orbit a symplectic structure. Let $\mathcal{V}$ be an orbital variety associated to $O$. By [14] and [15] one has $\dim \mathcal{V} = \frac{1}{2} \dim O$. Moreover as it was pointed out in [6] this implies that an orbital variety is a Lagrangian subvariety of the nilpotent orbit it is associated to. Following the orbit method one would like to attach an irreducible representation of $U(\mathfrak{g})$ to $\mathcal{V}$. This should be a simple highest weight module. Combining the results of A. Joseph and T. A. Springer one obtains a one to one correspondence between the set of primitive ideals of $U(\mathfrak{g})$ containing the augmentation ideal of its centre (thus corresponding to integral weights) and the set of orbital varieties in $\mathfrak{g}$ corresponding to Lusztig’s special orbits (see for example [1]). The picture is especially beautiful for $\mathfrak{g} = \mathfrak{sl}_n$. In this case all orbits are special and by [9] the associated variety of a simple highest (integral) weight module is irreducible. By [11] and [6] in general the orbital variety closures are just the irreducible components of an associated variety of a simple highest weight module. Thus for $\mathfrak{g} = \mathfrak{sl}_n$ orbital variety closures are these associated varieties and therefore give a natural geometric understanding of the classification of primitive ideals. This makes their study especially interesting.

1.4. Orbital varieties are very interesting objects from algebro-geometric point of view as well. Given an orbital variety $\mathcal{V}$ one can easily find $m_\mathcal{V}$ – the nilradical of the smallest dimension containing $\mathcal{V}$ (see 2.1.7). Consider an orbital variety closure as an algebraic variety in the affine linear space $m_\mathcal{V}$. Then vast majority of orbital varieties are not complete intersections. So orbital varieties are examples of algebraic varieties which are both Lagrangian subvarieties and not complete intersections.

1.5. Orbital varieties still remain rather mysterious objects. The only general description was given by R. Steinberg [15]. It is explained in detail in 2.1.2 and 2.1.3. Briefly speaking let $B$ be the standard Borel subgroup of $G$, i.e. such that $\text{Lie}(B) = b = \mathfrak{h} \bigoplus \mathfrak{n}$. $B$ acts by conjugation on $\mathfrak{n}$ and its subsets. Let $W$ be the Weyl group for the pair $(\mathfrak{g}, \mathfrak{h})$. Then by [15] there exists a surjection $\phi : w \mapsto \overline{V}_w$ from the Weyl group onto the set of orbital variety closures defined by $\phi(w) = B(\mathfrak{n} \cap \mathfrak{w} \mathfrak{n}) =: \overline{V}_w$. The fibers of this mapping, namely $\phi^{-1}(\overline{V}) = \{ w \in W : \overline{V}_w = \overline{V} \}$ are called geometric cells.

This description is not very satisfactory from the geometric point of view since a $B$ invariant subvariety generated by a linear space is a very complex object.

1.6. On the other hand there exist a very nice combinatorial characterization of orbital varieties in $\mathfrak{sl}_n$ in terms of Young tableaux. The detailed description of this characterization is given in 2.4.5, 2.4.6, 2.4.17 and 2.4.18. It is defined by Robinson-Schensted procedure giving a bijection from the symmetric group $S_n$ onto the pairs of standard Young tableaux of the same shape $w \mapsto (T(w), Q(w))$. Let us identify $W$ with $S_n$ (see
Then geometric cells are given by Young tableaux as follows \( \phi^{-1}(\mathcal{V}_w) = \{ y : T(y) = T(w) \} \). We will denote \( T_{\mathcal{V}_w} := T(w) \).

1.7. The description of an orbital variety closure has both geometric and combinatorial parts. The geometric component is whether an orbital variety closure is a union of orbital varieties. The combinatorial part is to describe orbital variety closure in terms of manipulations on Young tableaux.

It is noted in 4.1.1 that the projections on the Levi factor of standard parabolic subalgebras of \( g \) preserve orbital variety closures. Using this fact together with computations in low rank cases we show in Part III that if \( g \) has factors not of type \( A_n \) then a closure of orbital variety is not necessarily a union of orbital varieties and includes some varieties of smaller dimensions.

The same argument does not work for \( \mathfrak{sl}_n \) and there we conjecture that an orbital variety closure is a union of orbital varieties. This conjecture is supported by computations for \( n \leq 6 \). As well it is true for some special cases. In particular this is true for orbital varieties of nilpotent order 2 as it is shown in [10], and for orbital varieties whose closure is a nilradical of some standard parabolic subalgebras as it is shown in Part II.

1.8. To specify the combinatorial part let us define the geometric order and the notion of a geometric descendant. Given orbital varieties \( \mathcal{V}, \mathcal{W} \) we define the geometric order by \( \mathcal{V} \leq G \mathcal{W} \) if \( \mathcal{W} \subset \overline{\mathcal{V}} \). We say that \( \mathcal{W} \neq \mathcal{V} \) is a geometric descendant of \( \mathcal{V} \) if \( \mathcal{V} \leq G \mathcal{W} \) and for any \( \mathcal{Y} \) such that \( \mathcal{V} \leq G \mathcal{Y} \leq \mathcal{W} \) one has \( \mathcal{Y} = \mathcal{V} \) or \( \mathcal{Y} = \mathcal{W} \).

Remark. The above definition of geometric order seems to be reverse to the obvious one defined by the inclusion of closures. Initially this order was introduced for inclusions of primitive ideals described in 1.3. In \( \mathfrak{sl}_n \) as it is shown in 9 \( I_w \subset I_y \) implies \( \mathcal{V}_{w^{-1}} \supset \mathcal{V}_{y^{-1}} \). So that the “right” definition for primitive spectrum induces the “reverse” definition for orbital varieties. We will discuss the relationship between ordering of orbital varieties and ordering of primitive spectrum in detail in Part III.

We wish to describe the set of descendants of an orbital varieties in terms of Young tableaux and in particular to determine whether \( \mathcal{V} \leq G \mathcal{W} \) can be described by regarding their Young tableaux.

Defining the order on nilpotent orbits (resp. a descendant of a nilpotent orbit) exactly in the same manner as for orbital varieties we can ask the same questions about nilpotent orbits. The construction of Gerstenhaber described in 2.3.2 gives very elegant combinatorial answers to both questions in terms of Young diagrams.

We would like to find similar answers for orbital variety closures. As we show this is much more complex. For example given an orbital variety \( \mathcal{V} \), let \( \mathcal{O}_\mathcal{V} = G \mathcal{V} \) be the nilpotent orbit, \( \mathcal{V} \) is associated to. Then, as we show in Part II, \( \mathcal{W} \) being a geometric descendant of \( \mathcal{V} \) does not imply necessarily that \( \mathcal{O}_\mathcal{W} \) is a descendant of \( \mathcal{O}_\mathcal{V} \).
1.9. Let us consider another order relation on orbital varieties, which had long been thought to be the same as the geometric order.

Let \( R \subset \mathfrak{h}^* \) denote the set of non-zero roots, \( R^+ \) the set of positive roots corresponding to \( \mathfrak{n} \) in the triangular decomposition of \( \mathfrak{g} \) and \( \Pi \subset R^+ \) the resulting set of simple roots. Each \( w \) in \( W \) is a product of fundamental reflections \( s_\alpha : \alpha \in \Pi \). We denote by \( \ell(w) \) the minimal length of any such expression for \( w \). Consider the order generated by the following preorder. For \( s_\alpha \), \( \alpha \in \Pi \) and \( w \in W \) put

\[
D_w \leq w s_\alpha \quad \text{if} \quad \ell(w s_\alpha) = \ell(w) + 1. \quad (*)
\]

We call it the (right) Duflo order. This is also known as the weak (right) Bruhat order. We prefer the former nomenclature in the present context since it was Duflo who first discovered the implication of the (left) Duflo order for the primitive spectrum.

In 2.1.6 we induce the Duflo order to orbital varieties, geometric cells and Young tableaux in the obvious manner and call it the induced Duflo order. It is weaker than the geometric order, that is \( V_D \leq W \) implies \( V_G \leq W \).

As we show in Part III the induced Duflo order coincides with the geometric order up to \( n = 5 \) and it is strictly weaker than the geometric order for \( n \geq 6 \).

1.10. Part I is devoted to the combinatorial description of the induced Duflo order on Young tableaux. This is a very interesting and complex combinatorial question. In Part II and Part III we use repeatedly this description as a basis for further study of orbital variety closures.

All nilradicals of standard parabolic subgroups are orbital variety closures. They are called also Richardson orbital varieties. They are the simplest examples of orbital variety closures since they are linear subspaces of \( \mathfrak{n} \). For Richardson orbital variety \( \mathcal{V} \) the question whether \( \mathcal{V}^G \leq \mathcal{W} \) has a very simple answer in combinatorics of Young tableaux. Consider the invariant \( \tau(T) \) determined in 2.4.14. Then \( \mathcal{V}^G \leq \mathcal{W} \) if and only if \( \tau(T_{\mathcal{W}}) \supset \tau(T_{\mathcal{V}}) \). The description of the set of descendants of Richardson orbital variety \( \mathcal{V} \) is a much more delicate problem. We give the full solution to this problem in Part II.

In Part III we study geometric properties of orbital variety closures. In particular we use the Vogan \( T_{\alpha,\beta} \) operators to strengthen the induced Duflo order. This gives a combinatorial description of the geometric order up to at least \( n = 10 \). In Part III we also discuss in detail a connection between primitive ideals and orbital varieties mentioned briefly in 1.8 and use it to study the properties of orbital varieties as well as primitive ideals in \( \mathfrak{sl}_n \).

1.11. Let us describe in more detail the results of Part I. Since the induced Duflo order is generated by 1.9 (\( * \)) we call \( \mathcal{W} = \mathcal{V}_{w s_\alpha} \) (resp. Young tableau \( S = T(ws_\alpha) \)) a Duflo offspring of \( \mathcal{V} = \mathcal{V}_w \) (resp. of \( T = T(w) \)) if \( \alpha \in \Pi \) and \( \ell(ws_\alpha) = \ell(w) + 1 \) (see 2.5.2).
We define the Duflo descendant of a given orbital variety or Young tableau with respect to the induced Duflo order exactly in the same manner as we have defined the geometric descendant of a given orbital variety in [1.8] (see [2.5.1]). It is obvious that the set of Duflo descendants of a given orbital variety or Young tableau is a subset of its set of Duflo offsprings. But it is much easier to describe the set of Duflo offsprings than the set of Duflo descendants since the first set has a definitive combinatorial definition.

We show that the induced Duflo order can be completely described by the natural ordering on $S_2$ and the Robinson-Schensted insertion of an element $a$ into a Young tableau $T$ from the above $T \mapsto (T \downarrow a)$ and from the left $T \mapsto (a \Rightarrow T)$. Precisely, in [3.4.5] we show that for any $S$ in the set of Duflo offsprings of $T$ there exists $a$ such that $T = (T' \downarrow a)$ or $T = (a \Rightarrow T')$ and $S = (S' \downarrow a)$ or respectively $S = (a \Rightarrow S')$ where $S'$ is an offspring of $T'$. We also provide in [3.3.3] an exact way to compute the set of offsprings of a Young tableau of size $n$ from the knowledge of sets of offsprings of Young tableaux of size $n - 1$. It involves the shuffling of numbered boxes in a manner prescribed by the Robinson-Schensted procedure and, technically, is the most difficult theorem of this work.

1.12. Using the results described in [1.11] we show that the set of offsprings is preserved under projections on a Levi factor (see [1.1.3]) and under Robinson-Schensted embeddings $\downarrow$ and $\Rightarrow$ (see [4.1.4]).

In [4.1.2] we show as well that the geometric order is preserved under projections on a Levi factor. In Part III we show that $\downarrow$ and $\Rightarrow$ preserve the geometric order as well.

However as we show in [4.1.2] neither projections nor embeddings preserve the set of Duflo or geometric descendants. These facts again underline the difficulty of constructing the set of descendants.

Using the results on embeddings and projections we show in [4.1.8] that the induced Duflo order is compatible with the order on nilpotent orbits in the following strong sense: if $\mathcal{O}_1$, $\mathcal{O}_2$ are nilpotent orbits and $\mathcal{O}_1 \subset \overline{\mathcal{O}_2}$ then for every orbital variety $\mathcal{V}_2 \subset \mathcal{O}_2$ there exist an orbital variety $\mathcal{V}_1 \subset \mathcal{O}_1$ such that $\mathcal{V}_2 \leq \mathcal{V}_1$. This property is important in consideration of an orbital variety closure and fails to be true outside of $\mathfrak{sl}_n$ as we show in Part III.

1.13. The body of the paper consists of three sections.

In section 2 we explain all the background in geometry of orbital varieties and combinatorics of Young tableaux essential in the subsequent analysis. I hope this part makes the paper self-contained.

In section 3 we work out the machinery for recursive construction of the set of Duflo offsprings with the main results stated in [3.3.3] and [3.4.6].
Finally section 4 is devoted to the study of the properties of the induced Duflo order resulting immediately from section 3. Further properties of the induced Duflo order are studied in Part II.

In the end one can find the index of notation in which symbols appearing frequently are given with the subsection where they are defined. We hope that this will help the reader to find his way through the paper.

Acknowledgments. I would like to express my deep gratitude to A. Joseph for introducing the world of orbital varieties to me, for posing the problems, suggesting ideas underlying this research and many fruitful discussions through the various stages of this work. I would also like to thank V. Hinich for fruitful discussions.

2. Combinatorics of Symmetric group

2.1. Steinberg map and Duflo order on Weyl groups.

2.1.1. Recall the notation from 1.2. Let $\mathcal{N} = \mathbf{G}(\mathfrak{n})$ denote the nilpotent cone in $\mathfrak{g}$. For $u \in \mathcal{N}$ let $\mathcal{O}_u$ be the nilpotent orbit it defines, that is the orbit of $u$ under the coadjoint action of $\mathbf{G}$. We define an order relation on the set of nilpotent orbits by

$$\mathcal{O}_u \geq \mathcal{O}_v \text{ if } \mathcal{O}_u \subseteq \mathcal{O}_v.$$

Let $\mathcal{O}_u \neq \mathcal{O}_v$ and $\mathcal{O}_u \geq \mathcal{O}_v$. We call $\mathcal{O}'$ a descendant of $\mathcal{O}$ if for any $\mathcal{O}''$ such that $\mathcal{O} \geq \mathcal{O}'' \geq \mathcal{O}'$ one has $\mathcal{O}'' = \mathcal{O}$ or $\mathcal{O}'' = \mathcal{O}'$.

2.1.2. An orbital variety $\mathcal{V}$ associated to a nilpotent orbit $\mathcal{O}$ is an irreducible component of $\mathcal{O} \cap \mathfrak{n}$. Let $\mathbf{V}$ denote the set of all orbital varieties of $\mathfrak{g}$ and $\mathcal{W}$ the Weyl group for the pair $(\mathfrak{g}, \mathfrak{h})$. We describe first the Steinberg map of $\mathcal{W}$ onto $\mathbf{V}$. Let $R \subset \mathfrak{h}^*$ denote the set of non-zero roots, $R^+$ the set of positive roots corresponding to $\mathfrak{n}$ in the triangular decomposition of $\mathfrak{g}$ and $\Pi \subset R^+$ the resulting set of simple roots. Let $X_\alpha = \mathbb{C}x_\alpha$ denote the root subspace corresponding to $\alpha \in R$.

Then $\mathfrak{n} = \bigoplus_{\alpha \in R^+} X_\alpha$ (resp. $\mathfrak{n}^- = \bigoplus_{\alpha \in -R^+} X_\alpha$).

2.1.3. Given $S, T \subset R$ and $w \in \mathcal{W}$ we set $S \cap^w T := S \cap w(T) = \{ \alpha \in S : \alpha \in w(T) \}$. Then set

$$\mathfrak{n} \cap^w \mathfrak{n} := \bigoplus_{\alpha \in R^+ \cap R^+} X_\alpha.$$

This is a subspace of $\mathfrak{n}$. For each closed, irreducible subgroup $\mathbf{H}$ of $\mathbf{G}$ let $\mathbf{H}(\mathfrak{n} \cap^w \mathfrak{n})$ be the set of $\mathbf{H}$ conjugates of $\mathfrak{n} \cap^w \mathfrak{n}$. It is an irreducible locally closed subvariety. Since there are only finitely many nilpotent orbits in $\mathfrak{g}$ it follows that there exists a unique nilpotent orbit which we denote by $\mathcal{O}_w$ such that $\overline{\mathbf{G}(\mathfrak{n} \cap^w \mathfrak{n})} = \overline{\mathcal{O}_w}$. A result of Steinberg [15] asserts that
\[ V_w := \overline{B(n \cap^w n)} \cap O_w \] is an orbital variety and that the map \( \varphi : w \mapsto V_w \) is a surjection of \( W \) onto \( V \).

2.1.4. Recall the notion of geometric order and geometric descendant from 1.8. Set \( V_2 \leq G V_1 \) if \( V_2 \leq V_1 \) and \( V_2 \neq V_1 \).

The partial order relation on the corresponding Weyl group \( W \) defined by inclusion of orbital variety closures takes the form

\[ w \leq y \quad \text{if} \quad B(n \cap^y n) \subset B(n \cap^w n). \]

We decompose \( W \) according to this relation into the cells:

\[ C_w = \{ y \in W : V_y = V_w \}. \]

We call these the (right) geometric cells of \( W \) following [15], and [1, 6.8].

Set \( C_y \leq G C_w \) (resp. \( C_y \leq C_w \)) if \( V_y \leq V_w \) (resp. \( V_y \leq G V_w \)). We call \( C_w \) a geometric descendant of \( C_y \) if \( V_y \) is a geometric descendant of \( V_w \).

As well we decompose \( W \) into double geometric cells according to nilpotent orbits as follows

\[ C^d_w = \{ y \in W : O_y = O_w \}. \]

One sees immediately that \( C^d_w \) is the union of geometric cells whose orbital varieties are attached to \( O_w \).

2.1.5. Recall \( \ell(w) \) from 1.9. Set \( S(w) := R^+ \cap^w R^- \), then a classical result, described, for instance in [2, §2.2] provides

\textbf{Lemma.} Take \( x, y \in W \) and set \( w = yx \). Then \( \ell(w) = \ell(x) + \ell(y) \) if and only if

\[ S(y) \subset S(w) \quad (\ast) \]

Moreover if (\ast) holds there are exactly \( \ell(x) \) roots \( \beta \in R^+ \) such that \( y(\beta) \in R^+ \) and \( w(\beta) \in R^- \).

It is known that the map \( w \mapsto S(w) \) is injective. Hence we may define a partial order relation on \( W \) by \( y \leq D w \) when (\ast) holds. It is called the Duflo order. It is generated by the preorder defined in 1.9.

Note that if \( y \leq D w \) then \( R^+ \cap^y R^+ \supset R^+ \cap^w R^+ \) thus \( V_y \supset V_w \) just by the inclusion of generating subspaces.
2.1.6. We induce Duflo order on orbital varieties (resp. on geometric cells) by the following. Set $\mathcal{V}_y \leq \mathcal{V}_w$ (resp. $\mathcal{C}_y \leq \mathcal{C}_w$) if there exist a chain $y = x_0^1, x_0^2, x_1^1, x_1^2, \ldots, x_k^1, x_k^2 = w$ such that $\mathcal{C}_{x_i^1} = \mathcal{C}_{x_i^2}$ for $0 \leq i \leq k$ and $x_i^2 \leq x_{i+1}^1$ for $0 \leq i \leq k - 1$. Set $\mathcal{V}_y \leq \mathcal{V}_w$ if $\mathcal{V}_y \neq \mathcal{V}_w$ and $\mathcal{V}_y \neq \mathcal{V}_w$. In the same fashion we define $\mathcal{C}_y \leq \mathcal{C}_w$.

By note 2.1.5 we get that the induced Duflo order is weaker than the geometric order, that is for cells $\mathcal{C}_y, \mathcal{C}_w$ the relation $\mathcal{C}_y \leq \mathcal{C}_w$ implies $\mathcal{C}_y \leq \mathcal{C}_w$.

2.1.7. Recall the standard Borel subalgebra $\mathfrak{b}$ from 1.3. For $\alpha \in \Pi$ let $\mathfrak{p}_\alpha$ be the standard parabolic subgroup with Lie $(\mathfrak{p}_\alpha) = \mathfrak{p}_\alpha := \mathfrak{b} \oplus X_\alpha$.

Given an orbital variety $\mathcal{V}$, let $\mathcal{P}_\mathcal{V}$ be its stabilizer in $\mathcal{G}$. This is a standard parabolic (that is $\mathcal{P}_\mathcal{V} \supset \mathfrak{b}$) subgroup of $\mathcal{G}$. Let $\mathfrak{m}_\mathcal{V}$ be the maximal subalgebra of $\mathfrak{g}$ stabilized by $\mathcal{P}_\mathcal{V}$. This is a nilradical and the linear subspace of $\mathfrak{n}$ of minimal possible dimension containing $\mathcal{V}$.

Take $w \in W$, $\mathcal{V} \in \mathcal{V}$, a standard parabolic subgroup $\mathcal{P}$ and a standard parabolic subalgebra $\mathfrak{p} = \text{Lie}(\mathcal{P})$. Define their $\tau$-invariants to be

\begin{align*}
\tau(w) &:= \Pi \cap S(w), \\
\tau(\mathcal{P}) &:= \{ \alpha \in \Pi : \mathcal{P}_\alpha \subset \mathcal{P} \}, \\
\tau(\mathfrak{p}) &:= \{ \alpha \in \Pi : \mathfrak{p}_\alpha \subset \mathfrak{p} \}, \\
\tau(\mathcal{V}) &:= \{ \alpha \in \Pi : \mathcal{P}_\alpha(\mathcal{V}) = \mathcal{V} \}.
\end{align*}

Note that $\mathcal{P}$ (resp. $\mathfrak{p}$) is uniquely determined by its $\tau$-invariant. One has (see [6, §9])

\textbf{Lemma.} $\tau(w) = \tau(\mathcal{V}_w) = \tau(\mathcal{P}_\mathcal{V}_w)$.

Therefore $\tau(\mathcal{C}_w) := \tau(w)$ is well defined.

2.1.8. Given $\mathcal{I} \subset \Pi$, let $\mathcal{P}_\mathcal{I}$ denote the unique standard parabolic subgroup of $\mathcal{G}$ such that $\tau(\mathcal{P}_\mathcal{I}) = \mathcal{I}$. Let $\mathcal{M}_\mathcal{I}$ be the unipotent radical of $\mathcal{P}_\mathcal{I}$ and $\mathcal{L}_\mathcal{I}$ a Levi factor. Let $\mathfrak{p}_\mathcal{I}$, $\mathfrak{m}_\mathcal{I}$, $\mathcal{L}_\mathcal{I}$ denote the corresponding Lie algebras. Set $\mathcal{B}_\mathcal{I} := \mathcal{B} \cap \mathcal{L}_\mathcal{I}$ and $\mathfrak{n}_\mathcal{I} := \mathfrak{n} \cap \mathcal{L}_\mathcal{I}$. We have decompositions $\mathcal{B} = \mathcal{M}_\mathcal{I} \ltimes \mathcal{B}_\mathcal{I}$ and $\mathfrak{n} = \mathfrak{n}_\mathcal{I} \oplus \mathfrak{m}_\mathcal{I}$. They define projections $\mathcal{B} \to \mathcal{B}_\mathcal{I}$ and $\mathfrak{n} \to \mathfrak{n}_\mathcal{I}$ which we denote by $\pi_\mathcal{I}$.

Set $W_\mathcal{I} := \{ w : \alpha \in \mathcal{I} \}$. To be a parabolic subgroup of $W$, set $D_\mathcal{I} := \{ w \in W : w(\alpha) \in R^+ \forall \alpha \in \mathcal{I} \}$. Set $R^+_\mathcal{I} = R^+ \cap \text{span}(\mathcal{I})$. A well-known result described for example in [2, 2.5.8] gives

\textbf{Lemma.} Each $w \in W$ has a unique expression of the form $w = w_\mathcal{I} d_\mathcal{I}$ where $d_\mathcal{I} \in D_\mathcal{I}$, $w_\mathcal{I} \in W_\mathcal{I}$ and $\ell(w) = \ell(w_\mathcal{I}) + \ell(d_\mathcal{I})$. Moreover

$$R^+_\mathcal{I} \cap w^+ R^+ = R^+_\mathcal{I} \cap w^+ R^+_\mathcal{I}.$$
The decomposition $W = W_\pi \times D_\pi$ explained in the lemma defines a projection $\pi_\pi : W \rightarrow W_\pi$.

Set $D_j^{-1} := \{ f^{-1} : f \in D_j \}$. Applying the lemma to $w^{-1}$ we get that each $w \in W$ has a unique expression of the form $w = f_j w_j$ where $w_j \in W_j$, $f_j \in D_j^{-1}$ and $\ell(w) = \ell(w_j) + \ell(f_j)$.

2.2. $S_n$ as a Weyl group of $\mathfrak{sl}_n$.

2.2.1. From now and on we consider only $\mathfrak{sl}_n$. It is convenient to replace $\mathfrak{sl}_n$ by $\mathfrak{g} = \mathfrak{gl}_n$. This obviously makes no difference when we consider nilpotent cone and adjoint action (conjugation) by $G = \text{GL}_n$ on it. Let $\mathfrak{n}$ be the subalgebra of strictly upper-triangular matrices and let $\mathfrak{n}^-$ be the subalgebra of strictly lower-triangular matrices. Let $\mathcal{B}$ be the (Borel) subgroup of upper-triangular matrices in $G$. All parabolic subgroups we consider further are standard, that is contain $\mathcal{B}$. Let $e_{i,j}$ be the matrix having 1 in the $i,j$-th entry and 0 elsewhere. Set $\mathcal{B} = \{ e_{i,j} \}_{i,j=1}^n$ which is a basis of $\mathfrak{g}$.

Take $i < j$ and let $\alpha_{i,j}$ be the root corresponding to $e_{i,j}$. Set $\alpha_{j,i} = -\alpha_{i,j}$. We write $\alpha_{i,i+1}$ simply as $\alpha_i$. Then $\Pi = \{ \alpha_i \}_{i=1}^{n-1}$. Moreover $\alpha_{i,j} \in R^+$ exactly when $i < j$. One has

$$\alpha_{i,j} = \begin{cases} 
\sum_{k=i}^{j-1} \alpha_k & \text{if } i > j \\
- \sum_{k=j}^{i-1} \alpha_k & \text{if } i > j
\end{cases}$$

For each $\alpha \in \Pi$, let $s_\alpha \in W$ be the corresponding reflection and set $s_i = s_{\alpha_i}$.

2.2.2. We represent every element of the symmetric group $S_n$ in word form

$$w = [a_1, a_2, \ldots, a_n], \quad \text{where } a_i = w(i). \quad (*)$$

We identify $W$ with $S_n$ by taking $s_i$ to be the elementary permutation interchanging $i, i+1$. We consider multiplication from right to left that is given $w$ from $(*)$ one has $s_i w(j) = s_i (a_j)$. For example $s_1 s_2 = [2, 3, 1]$.

Remark. In our notation if $w = [a_1, \ldots, a_i, a_{i+1}, \ldots, a_n]$ then $w s_i$ is obtained from $w$ by interchanging $a_i$ and $a_{i+1}$ that is $w s_i = [a_1, \ldots, a_{i+1}, a_i, \ldots, a_n]$.

Definition. A word (of length $k$) is an ordered array $[a_1, \ldots, a_k]$, where all $a_i$ are distinct and $\{ a_i \}_{i=1}^k \subset \{ j \}_{j=1}^n$, $n > k$. If $\{ a_i \}_{i=1}^k = \{ j \}_{j=1}^k$ the word is called standard.

Expression $(*)$ for $w$ is a standard word of length $n$.

2.2.3. Given $w = [a_1, \ldots, a_n]$. Set $p_w(i) := j$ if $a_j = i$, that is $p_w(i)$ is the place (index) of $i$ in the word form of $w$. One has $w(p_w(i)) = w(j) = a_j = i$, that is $p_w(i) = w^{-1}(i)$. As well one has

Lemma. $w(\alpha_{i,j}) = \alpha_{w(i), w(j)}$. 

Proof.
It is enough to show only for $i < j$.

One has

$$s_k(m) = \begin{cases} 
m & \text{if } m \neq k, k + 1 \\
k + 1 & \text{if } m = k \\
k & \text{if } m = k + 1 \end{cases}$$

As well

$$s_k(\alpha_{m,m+1}) = s_k(\alpha_m) = \begin{cases} 
-\alpha_k & \text{if } m = k \\
\alpha_{k-1,k+1} & \text{if } m = k - 1 \\
\alpha_{k,k+2} & \text{if } m = k + 1 \\
\alpha_m & \text{if } m \neq k - 1, k, k + 1 \end{cases}$$

Thus $s_k(\alpha_{m,m+1}) = \alpha_{s_k(m),s_k(m+1)}$. Applying this to $\alpha_{i,j}$ we get

$$s_k(\alpha_{i,j}) = s_k(\sum_{m=i}^{j-1} \alpha_m) = \alpha_{s_k(i),s_k(j)}.$$

Now we can show lemma by induction on $\ell(w)$. Indeed let $w = s_k y$ where $\ell(w) = \ell(y) + 1$ then

$$w'(\alpha_{i,j}) = s_k y(\alpha_{i,j}) = s_k(\alpha_{y(i),y(j)})$$

by induction hypothesis

$$= \alpha_{s_k(y(i)),s_k(y(j))}$$

$$= \alpha_{w(i),w(j)}$$

by the previous computations

\[\blacksquare\]

2.2.4. Recall the notation $S(w)$ from \[2.1.5\]. We get from \[2.2.3\]

Corollary (\cite[2.3]{7}). Take $i < j$. Then $\alpha_{i,j} \in S(w)$ if and only if $p_w(i) > p_w(j)$.

Proof.
Indeed $S(w) = \{ \alpha \in R^+ : \alpha \in w(R^-) \} = \{ \alpha \in R^+ : w^{-1}(\alpha) \in R^- \}$ and by \[2.2.3\] for $i < j$ one has $w^{-1}(\alpha_{i,j}) \in R^-$ if and only if $w^{-1}(i) > w^{-1}(j)$ which is equivalent to $p(i) > p(j)$.

\[\blacksquare\]

Remark 1. In particular $\alpha_i \in \tau(w)$ exactly when $i + 1$ comes before $i$ in the word form of $w$. This is of course well-known.

Remark 2. As a corollary of \[2.2.3\] \[2.1.5\] and \[2.2.4\] we get that $w \preceq w_{i}$ if and only if $a_i < a_{i+1}$. This is also well-known and given for example in \cite[pp. 73-74]{5}.
Remark 3. As a corollary of 2.2.2, 2.1.5 and 2.2.4 we get that $S_n$ has the unique longest element $w_o = [n, n-1, \ldots, 2, 1]$. For any $w \in S_n$ one has $w \leq w_o$. This is also well-known.

2.2.5. Introduce the following useful notational conventions.

Given a word $w = [a_1, \cdots, a_n]$ we denote by $\langle w \rangle := \{a_i\}_{i=1}^n$ the set of its entries.

(i) Given $m \in \langle w \rangle$, set $(w - m)$ to be a word obtained from $w$ by deleting $m$, that is if $m = a_i$, then $(w - m) := [a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n]$.

(ii) For $i < j$ set $w_{[i,j]} := [a_i, a_{i+1}, \ldots, a_j]$.

(iii) Set $\overline{w}$ to be the word with order reverse to the order of $w$, that is if $w = [a_1, \cdots, a_n]$ then $\overline{w} := [a_n, \cdots, a_1]$. We call $\overline{w}$ the reversal of $w$.

(iv) Given words $x = [a_1, \cdots, a_k]$, $y = [b_1, \cdots, b_l]$. If $\langle x \rangle \cap \langle y \rangle = \emptyset$ we define a colligation $[x, y] = [a_1, \cdots, a_k, b_1, \cdots, b_l]$.

Given a fixed set $E$ of $n$ distinct positive integers we let $S_n$ or $S_E$ denote the set of words $w$ such that the set of its entries $\langle w \rangle = E$. Let $E = \{a_i\}_{i=1}^n$, where $a_i < a_{i+1}$ for all $i \colon 1 \leq i < n$. Taking $i$ instead of $a_i$ as a corresponding entry in the word we get a bijection $\phi : S_E \rightarrow S_n$. This bijection is constructed as a composition of bijections $\phi_{i}^{-1}$, where $\phi_j$ is induced from $\phi_j : E \rightarrow E'$ defined by

$$
\phi_j(a_i) := \begin{cases} a_i & \text{if } a_i < j \\ a_i + 1 & \text{otherwise.} \end{cases}
$$

Thus if $j \not\in E$ we define $\phi^{-1} : E \rightarrow E'$ by

$$
\phi^{-1}_j(a_i) := \begin{cases} a_i & \text{if } a_i < j \\ a_i - 1 & \text{if } a_i > j. \end{cases}
$$

For $m < n$ we consider $S_m$ as a subgroup of $S_n$ by the following identification

$$S_m = \{ w \in S_n : a_i = w(i) = i \ \forall \ i > m \}$$

Set

$$s_{i,j}^< := \begin{cases} s_is_{i+1} \cdots s_j & \text{if } 1 \leq i \leq j \\ \text{Id} & \text{otherwise} \end{cases}$$

$$s_{i,j}^> := \begin{cases} s_is_{i-1} \cdots s_j & \text{if } i \geq j \geq 1 \\ \text{Id} & \text{otherwise} \end{cases}$$

2.2.6. One has

Lemma. (i) Given $w \in S_n$. Let $p_w(n) = i$. Then $w = y s_{n-1,i}^>$ where $y = (w - n)$, $y \in S_{n-1}$. In this $\ell(w) = n - i + \ell(y)$ and

$$S(w) = S(y) \cup \{a_{a_k,n}\}_{k=i+1}^n.$$
(ii) Given \( w \in S_n \). Let \( w(n) = i \). Then \( w = s_{i,n-1}^r y \) where \( y = \phi_i^{-1}(w_{<1,n-1>}^<), \ y \in S_{n-1} \). In this \( \ell(w) = n - i + \ell(y) \) and

\[
S(w) = \{\alpha_{\phi_j(k),\phi_i(k)} : \alpha_{j,k} \in S(y)\} \bigcup \{a_{i,j}^n \}_{j=i+1}^n.
\]

**Proof.**

Applying remark 2.2.2 consequently to multiplication of \( y = [a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n] \) by \( s_{n-1,i}^r \) we get

\[
y s_{n-1,i}^r = [a_1, \ldots, a_n, n] s_{n-1}^r s_{n-2,i}^r = [a_1, \ldots, n, a_n] s_{n-2,i}^r = \cdots = [a_1, \ldots, a_{i-1}, n, a_{i+1}, \ldots, a_n].
\]

By 2.1.8 one has \( \ell(w) = n - i + \ell(y) \) and by 2.2.4 we get

\[
S(w) = S(y) \bigcup \{\alpha_{j,n} : p_{w(j)} > p_{w(n)}\} = S(y) \bigcup \{\alpha_{a,k}^n \}_{k=1}^n.
\]

(ii) is obtained by applying (i) to \( w^{-1} \) and again by applying 2.2.4 for the computation of \( S(w) \).

\[
\square
\]

2.2.7. There are two obvious subgroups of \( S_n \) isomorphic to \( S_{n-1} : S_{n-1} = W_J \) where \( J = \{\alpha_i\}_{i=1}^{n-2} \) and \( S_{n-1}' = W_{J'} \) where \( J' = \{\alpha_i\}_{i=2}^{n-1} \). In the notation of 2.2.5 one has \( S_{n-1}' = \phi_1(S_{n-1}) \) where we add 1 to the first place in all the words of \( \phi_1(S_{n-1}) \). Note that we get in such a way that \( \phi_1(s_i) = s_{i+1} \) and \( \phi_1(\alpha_i) = \alpha_{i+1} \). Lemma 2.2.6 can be reformulated for \( S_{n-1}' \) as follows

**Lemma.**

(i) Given \( w \in S_n \). Let \( p_w(1) = i \). Then \( w = y s_{i,n-1}^r \) where \( y = (w - 1), \ y \in S_{n-1}' \). In this \( \ell(w) = i - 1 + \ell(y) \) and \( S(w) = S(y) \bigcup \{\alpha_{a,k}^1 \}_{k=1}^{i-1} \).

(ii) Given \( w \in S_n \). Let \( w(1) = i \). Then \( w = s_{i,n-1}^r y \) where \( y = \phi_i^{-1}(\phi_{i+1}(w_{<2,n>^<})) \). In this \( \ell(w) = i - 1 + \ell(y) \) and \( S(w) = \{\alpha_{\phi_{i+1}(j),\phi_{i+1}(k)} : \alpha_{j,k} \in S(y)\} \bigcup \{a_{i,j}^1 \}_{j=i+1}^i \).

2.2.8. We will also need.

**Lemma.** Take \( y_1, y_2 \in S_{n-1} \) and \( i : 1 \leq i \leq n - 1 \). If \( y_1 \leq y_2 \) then \( s_{i,n-1}^\leq y_1 \leq s_{i,n-1}^\leq y_2 \).

**Proof.**

This follows from the addition of lengths criterion in 2.1.8 using 2.2.6 (ii).

\[
\square
\]

2.3. Young Diagrams and nilpotent orbits.
2.3.1. Let us define Young diagrams corresponding to the partitions. Let \( \lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0) \) be a partition of \( n \). Set \( \lambda^* := \{\lambda_1^* \geq \lambda_2^* \geq \cdots \geq \lambda_l^* > 0\} \) to be the dual partition, that is \( \lambda_i^* = \sharp\{j \mid \lambda_j \geq i\} \). For example \( k = \lambda_1^* \).

We define the corresponding Young diagram \( D_\lambda \) of \( \lambda \) to be an array of \( k \) rows of boxes starting on the left with the \( i \)-th row containing \( \lambda_i \) boxes.

For example given \( \lambda = (4, 3, 1) \) then \( \lambda^* = (3, 2, 2, 1) \) and the corresponding Young tableau is

\[
D_\lambda = \begin{array}{cccc}
\square & \square & \square & \square \\
\square & \square & \square & \square \\
\square & \square & \square & \square \\
\square & \square & \square & \square \\
\end{array}
\]

The set of Young diagrams with \( n \) boxes is denoted by \( D_n \).

Recall order relation on nilpotent orbits defined in 2.1.1. In case \( g = \mathfrak{sl}_n \) the nilpotent orbits and the above order relation on them have nice and simple combinatorial descriptions. In this case \( G = SL_n \) acts on \( g \) by conjugation. For \( u \in g \) its \( G \) orbit is determined uniquely by the Jordan form of \( u \). If \( u \in \mathcal{N} \) all the eigenvalues of \( u \) are zero and the Jordan form of \( u \) is determined only by the length of its Jordan blocks. Let us write the Jordan blocks of \( u \) in decreasing order and denote the length of the \( i \)-th block by \( \lambda_i \). The resulting partition \( \lambda = (\lambda_1, \cdots, \lambda_k) \) is denoted by \( J(u) \). For example

\[
u = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad J(u) = (4, 2, 1, 1)
\]

The map \( u \mapsto J(u) \) gives a bijection of \( \mathcal{N}/G \) onto \( D_n \). Given \( J(u) = \lambda \) we also write \( \mathcal{O}_\lambda := \mathcal{O}_u \) and \( D_u := D_\lambda \).

2.3.2. Define an order relation on Young diagrams as follows. Let \( \lambda = (\lambda_1, \cdots, \lambda_k) \) and \( \mu = (\mu_1, \cdots, \mu_j) \) be partitions of \( n \) with corresponding diagrams \( D_\lambda, D_\mu \in D_n \). If \( j \neq k \) complete the partition with the lesser number of parts by adding the appropriate number of 0’s. In this manner we can consider that both partitions have \( \max(j, k) \) elements. Define \( D_\lambda \geq D_\mu \) if for each \( i : 1 \leq i \leq \max(j, k) \) one has

\[
\sum_{m=1}^{i} \lambda_m \leq \sum_{m=1}^{i} \mu_m .
\]
The following result of Gerstenhaber (see [4 §3.10] for example) describes the closure of a nilpotent orbit.

**Theorem.** Let $\mu$ be a partition of $n$ and $O_\mu$ be the corresponding nilpotent orbit in $\mathfrak{sl}_n$. One has

$$O_\mu = \bigcup_{\lambda \mid D_\lambda \geq D_\mu} O_\lambda .$$

In particular

$$O_\lambda \geq O_\mu \iff D_\lambda \geq D_\mu .$$

This describes the order relation on nilpotent orbits through the combinatorics of Young diagrams.

2.3.3. From theorem 2.3.2 we easily obtain

**Corollary.** Let $O_\lambda$ be a descendant of $O_\mu$. Let $D_\lambda$, $D_\mu$ be the corresponding Young diagrams. Then $D_\lambda$ is obtained from $D_\mu$ in one of two ways:

(i) There exists $i$ such that $\mu_i - \mu_{i+1} \geq 2$. Then $\lambda_j = \mu_j$ for $j \neq i, i+1$ and

$$\lambda_i = \mu_i - 1, \quad \lambda_{i+1} = \mu_{i+1} + 1 .$$

(ii) There exists $i$ such that $\mu_{i+1} = \mu_{i+2} = \cdots = \mu_{i+k} = \mu_i - 1$ for some $k \geq 1$ and $\mu_{i+k+1} = \mu_i - 2$. Then $\lambda_j = \mu_j$ for $j \neq i, i+k+1$ and

$$\lambda_i = \mu_i - 1, \quad \lambda_{i+k+1} = \mu_i - 1 .$$

The above result can be described pictorially as follows.

In the first case $D_\lambda$ is obtained from $D_\mu$ by pushing one box down one row (and possible across several columns). For example

$$D_\mu = \begin{array}{|c|c|c|} \hline & & X \\
\hline & & \\
\hline \end{array}, \quad D_\lambda = \begin{array}{|c|c|c|} \hline & & X \\
\hline & & \\
\hline \end{array} .$$

In the second case diagram $D_\lambda$ is obtained from $D_\mu$ by pushing one box across one column (and possible down several rows). For example

$$D_\mu = \begin{array}{|c|c|c|} \hline & & X \\
\hline & & \\
\hline \end{array}, \quad D_\lambda = \begin{array}{|c|c|c|} \hline & & X \\
\hline & & \\
\hline \end{array} .$$
In these cases we say that \( D_\lambda \) is a descendant of \( D_\mu \).

### 2.4. Young Tableaux and orbital varieties.

#### 2.4.1. Fill the boxes of Young diagram \( D_\lambda \) with \( n \) distinct positive integers. If the entries increase in rows from left to right and in columns from top to bottom we call such an array a Young tableau or simply a tableau. If the numbers in a Young tableau form the set of integers from 1 to \( n \), then the tableau is called a standard Young tableau. For example

\[
T = \begin{array}{cccc}
1 & 2 & 4 & 8 \\
3 & 5 & 9 & \\
6 & & & \\
7 & & & \\
\end{array}
\]

is a standard Young tableau.

Let \( T_n \) denote the set of all standard Young tableaux of size \( n \). The shape of a Young tableau \( T \) is defined to be the Young diagram, denoted \( \text{sh}(T) \), from which it was built.

#### 2.4.2. We will use the following notation for Young tableaux. Let \( T \) be a Young tableau and let \( T^i_j \) for \( i, j \in \mathbb{N} \) denote the entry on the intersection of \( i \)-th row and \( j \)-th column. It is sometimes convenient (for example in defining the Robinson - Schensted insertion) to assume that each row and column is completed to semi-infinite length by insertion of \( \infty \) i.e. if the length of \( i \)-th row or \( j \)-th column in the corresponding diagram is \( k \) then we set

\[
T^i_{k+1} = T^i_{k+2} = \cdots = \infty, \quad T^k_j = T^{k+1}_j = \cdots = \infty
\]

Given \( u \) an entry of \( T \) set \( r_T(u) \) to be the number of the row, \( u \) belongs to and \( c_T(u) \) to be the number of the column, \( u \) belongs to. That is if \( u = T^i_j \) then \( r_T(u) = i \) and \( c_T(u) = j \). The hook number of the \( ij \)-th entry of \( T \) is defined by \( h(T^i_j) := 1 + (\lambda^* - j) + (\lambda - i) \).

Let \( T^i \) (resp. \( T_i \)) denote the \( i \)-th row (resp. column) of \( T \), that is

\[
T^i := (T^i_1, \ldots), \quad T_i := \left( T^1_i, \ldots \right)
\]

We let \( |T^i| \) (resp. \( |T_i| \)) denote the number of finite elements in the row \( T^i \) (resp. column \( T_i \)) and \( \omega^i(T) \) (resp. \( \omega_i(T) \)) denote the largest finite entry of \( T^i \) (resp. \( T_i \)).

We consider a tableau as a matrix \( T := (T^i_1) \) and write \( T \) by rows or by columns :

\[
T = \begin{pmatrix}
T^1_1 \\
\vdots \\
T^m_i
\end{pmatrix} = (T_1, \ldots, T_i).
\]
We set $T_{i,j}$, $i < j$ to be the subtableau of $T$ consisting of rows from $i$ to $j$, that is

$$T_{i,j} = \begin{pmatrix} T^i \\ \vdots \\ T^j \end{pmatrix},$$

and $T^{i,\infty}$ to be the subtableau of $T$ consisting of all $T^j$ such that $j \geq i$.

We set $T_{i,j}$, $i < j$ to be the subtableau of $T$ consisting of columns from $i$ to $j$, that is

$$T_{i,j} = (T_i, \cdots, T_j)$$

and $T_{i,\infty}$ to be a subtableau consisting of all $T_j$ such that $j \geq i$.

Writing $T = T_{1,l}^{1,k}$ designates that $T$ has $l$ columns and $k$ rows.

For each tableau $T$ let $T^\dag$ denote the transposed tableau, that is

$$\text{if } T = \begin{pmatrix} T^1 \\ \vdots \\ T^m \end{pmatrix} = (T_1, \cdots, T_l), \text{ then } T^\dag = (T^1, \cdots, T^m) = \begin{pmatrix} T_1 \\ \vdots \\ T_l \end{pmatrix}.$$ 

Note that $\text{sh} (T^\dag) = \text{sh} (T)^\ast$.

Given a fixed set $E$ of $n$ distinct positive integers, let $T_n$ or $T_E$ be the set of Young tableaux $T$ such that its set of entries $<T>$ is $E$. Then $T_E$ is in bijection with $T_n$ exactly the same way as $S_E$ with $S_n$. We define $\phi_j$, $\phi_j^{-1}$, $\phi$ to be the maps induced on $T_n$ (or $T_E$) by their action on $S_n$ (or $S_E$) defined in 2.2.5.

To each row $T^i = (T^i_1, \cdots, T^i_k, \cdots)$ with $|T^i| = k$ and column $T_j = (T^1_j, \cdots, T^m_j, \cdots)$ with $|T_j| = m$ we associate words, denoted by $[T^i]$ and $[T_j]$, given by

$$[T^i] = [T^1_i, \cdots, T^k_i], \quad [T_j] = [T^1_j, \cdots, T^m_j].$$

For simplicity of notation we will omit square brackets inside a colligation. For example $[T^i, T_j]$ means $[[T^i], [T_j]]$.

2.4.3. A row (resp. column) of $T$ is determined by the set of its entries since these must increase from left to right (resp. from top to bottom).

Let $T = (T_1, T_2, \ldots, T_l)$, $S = (S_1, S_2, \ldots, S_{l'})$ be Young tableaux given by their columns. Assume that $T, S$ have no common entries. Then we define $(T, S)$ to be the array whose rows are the same as the rows of $(T_1, T_2, \ldots, T_l, \tilde{S}_1, S_2, \ldots, S_{l'})$, that is $r_{(T,S)}(T^i_j) = i$ and $r_{(T,S)}(S^j_{l'}) = i$, and ordered in the increasing order. Of course this involves the shuffling of numbered boxes within a row.

**Lemma** ( [2.7]). $(T, S)$ is a Young tableau.

If the entries of $S$ all exceed those of $T$ then one only needs to shift numbered boxes (to the left).
Taking $T$ and $S$ by rows instead of columns, that is

$$ T = \begin{pmatrix} T_1 \\ \vdots \\ T_k \end{pmatrix}, \quad S = \begin{pmatrix} S_1 \\ \vdots \\ S_{k'} \end{pmatrix} $$

we define $\left( \begin{array}{c} T \\ S \end{array} \right)$ in a fashion similar to $(T, S)$. One has $\left( \begin{array}{c} T \\ S \end{array} \right)^\dagger = (T^\dagger, S^\dagger)$.

Note that $T_{i,i+1} = (T_iT_{i+1})^-$, for example.

2.4.4. Given $D_\lambda \in D_n$ with $\lambda = (\lambda_1, \cdots, \lambda_j)$ we define a corner box (or simply, a corner) of the Young diagram to be a box with no neighbours to right and below.

For example in $D$ below all the corner boxes are labeled by $X$.

\[
D = \begin{array}{ccc}
X & & \\
X & & \\
X & & \\
\end{array}
\]

The entry of a Young tableau in a corner is called a corner entry. Take $D_\lambda$ with $\lambda = (\lambda_1, \cdots, \lambda_k)$. Then there is a corner entry $\omega^i(T)$ at the corner $c = c(i, \lambda_i)$ with coordinates $(i, \lambda_i)$ iff $\lambda_{i+1} < \lambda_i$. We order corners by ordering their first coordinate, i.e. for $c = c(i, \lambda_i)$, $c' = c(j, \lambda_j)$ one has $c < c'$ iff $i < j$.

2.4.5. We now define the insertion algorithm. Consider a row $R = (a_1, a_2, \cdots)$ completed by $\infty$. Given $j \in \mathbb{N}^+$, $j \not< R >$ or $j = \infty$. Let $a_i$ be the first smallest integer greater or equal to $j$ (possibly $\infty$) in $< R >$ and set

$$(R \downarrow j) := \begin{cases} (a_1, \cdots, a_{i-1}, j, a_{i+1}, \cdots), & j_R = a_i \quad \text{if } j \neq \infty \\ R & \text{otherwise} \end{cases}$$

The inductive extension of this operation to a Young tableau $T$ with $m$ rows for $j \not< T >$ given by

$$ (T \downarrow j) = \begin{pmatrix} (T_1 \downarrow j) \\ (T_{2,m} \downarrow j_{r_1}) \end{pmatrix} $$

is called the insertion algorithm.

Note that the shape of $(T \downarrow j)$ is the shape of $T$ obtained by adding one new corner. The entry of this corner is denoted by $j_T$. 
2.4.6. Let $w = [a_1, a_2, \ldots, a_n]$. According to Robinson-Schensted procedure we associate an ordered pair of Young tableaux $(T(w), Q(w))$ to $w$. The procedure is fully explained in [8, 5.1.4] or in [11, 2.5]. As well we explain it in 2.4.18. In what follows we call it RS procedure. Here we explain only the inductive procedure of constructing the first tableau $T(w)$.

1. Set $1T(w) = (a_1)$.
2. Set $j+1T(w) = (jT(w) \downarrow a_{j+1})$.
3. Set $T(w) = nT(w)$.

As an example we take $w = [2, 5, 1, 4, 3]$

This gives

\[
1T(w) = \begin{array}{c}
2 \\
\end{array}
\quad 2T(w) = \begin{array}{c}
2 \\
5 \\
\end{array}
\quad 3T(w) = \begin{array}{c}
1 \\
5 \\
2 \\
\end{array}
\quad 4T(w) = \begin{array}{c}
1 \\
4 \\
2 \\
5 \\
\end{array}
\quad 5T(w) = \begin{array}{c}
1 \\
3 \\
2 \\
4 \\
5 \\
\end{array}
\]

The result due to Robinson and Schensted (see 2.4.18) implies the map $\varphi: w \mapsto T(w)$ is a surjection of $S_n$ (resp. $S_E$) onto $T_n$ (resp. $T_E$).

Recall the notion of geometric cells from 2.1.4. By R. Steinberg [15] the fibres of $\varphi$ provide a partition of $S_n$ into geometric cells, that is

**Theorem.** For any $w \in S_n$ one has $C_w = \{y \in S_n : T(y) = T(w)\}$.

We generalize this description to $S_E$ by taking $C_w := \{y \in S_E : T(y) = T(w)\}$. As well we define $C_T := \{y \in S_n$ (or $S_E) : T(y) = T\}$. We also denote by $T_C$ the tableau corresponding to the cell $C$.

2.4.7. RS procedure together with lemma 2.2.6 (ii) implies

**Proposition.** For $y \in S_n$ one has

$$T(s_{i,n}^\leq y) = (\phi_i(T(y)) \downarrow i) \quad \text{where} \quad 1 \leq i \leq n + 1$$

**Proof.**

Set $w = s_{i,n}^\leq y$. By 2.2.6 (ii) $w(n + 1) = i$ and $y = \phi_i^{-1}(w_{<i,n>})$. Hence by RS procedure

$$T(w) = (nT(w) \downarrow i) = (T(\phi_i(y)) \downarrow i) = (\phi_i(T(y)) \downarrow i).$$
2.4.8. Let us describe a few algorithms connected to RS procedure which we use for proofs and constructions.

First let us describe some operations for rows and tableaux. Consider a row \( R = (a_1, \ldots) \).

(i) Set \((R - a_i) := (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots)\).

(ii) For \( j \in \mathbb{N}, \ j \not\in < R > \) let \( a_i \) be the greatest element of \(< R >\) smaller than \( j \) and set

\[ (R + j) := (a_1, \ldots, a_j, a_{j+1}, \ldots). \]

(iii) We define a pushing up operation. Again let \( j \in \mathbb{N}, \ j \not\in < R > \) and \( j > a_1 \). Let \( a_i \) be the greatest entry of \( R \) smaller than \( j \) and set :

\[ (R \uparrow j) := (a_1, \ldots, a_{i-1}, j, a_{i+1}, \ldots), \quad j^R := a_i. \]

The last operation is extended to a Young tableau \( T \) by induction on the number of rows. Let \( T^m \) be the last row of \( T \) and assume \( T^m_{1,m} < j \). Then

\[ (T \uparrow j) = \left( \begin{array}{c} (T^1, m-1 \uparrow j^{\uparrow m}) \\ (T^m \uparrow j) \end{array} \right) \]

We denote by \( j^T \) the element pushed out from the first row of the tableau in the last step.

2.4.9. The pushing up operation gives us a procedure of deleting a corner inverse to the insertion algorithm. This is also described in [8, 5.1.4] or [11, 2.8].

As a result of insertion we get a new tableau with a shape obtained from the old one just by adding one corner. As a result of deletion we get a new tableau with the shape obtained from the old one by extracting one corner.

Let \( T \) be a tableau. Recall the definition of \( \omega^i(T) \) from \([2.4.2]\). Assume \( \lambda_i > \lambda_{i+1} \) and let \( c = c(i, \lambda_i) \) be the corner of \( T \) on the \( i \)-th row. To delete the corner \( c \) we delete \( \omega^i(T) \) from the row \( T^i \) and push it up through the tableau \( T^{1,i-1} \). The position of the displaced boxes can be joined to form a segment \( s_c(T) \) in \( \text{sh}(T) \). The element pushed out from the tableau is denoted by \( c^T \). This is written

\[ (T \uparrow c) := \left( \begin{array}{c} (T^{1,i-1} \uparrow \omega^i(T)) \\ (T^i \setminus \omega^i(T)) \\ T^{i+1, \infty} \end{array} \right) \]

For example

\[
\left( \begin{array}{ccc}
1 & 3 \\
2 & 4 \\
5 \\
\uparrow c(3,1)
\end{array} \right) = \left( \begin{array}{cc}
1 & 4 \\
2 & 5 \\
\uparrow \end{array} \right), \quad c^T = 3, \quad s_c(T) = \left( \begin{array}{c}
\uparrow \\
\uparrow \\
\uparrow
\end{array} \right).
\]
Note that insertion and deletion are indeed inverse since for any $T \in \mathbf{T}_n$

\[(T \uparrow c) \downarrow c^\uparrow) = T \quad \text{and} \quad ((T \downarrow j) \uparrow j_T) = T \quad \text{(for } j \not\in < T >).\]  

Note that sometimes we will write $(T \uparrow a)$ where $a$ is a corner entry just as we have written above.

2.4.10. Inserting, deleting, adding and pushing up operations are defined for rows. For our further analysis we need to define these operations also for columns. The most simple way to define them is by the operation of transposition ($\dagger$). Given a column $C$; given $a \in < C >$ and $j \in \mathbb{N}$, $j \not\in < C >$, or $j = \infty$ set

(i) $(C - a) := (C^\dagger - a)^\dagger$;  
(ii) $(C + j) := (C^\dagger + j)^\dagger$;  
(iii) $(j \rightarrow C) := (C^\dagger \downarrow j)^\dagger$ and $c_j := j_{c^\dagger}$;  
(iv) if $j > C^1$ then $(C \leftarrow j) := (C^\dagger \uparrow j)^\dagger$ and $c_j := j_{c^\dagger}$;

We also extend the operations to tableaux:

\[(j \Rightarrow T) := (T \uparrow j)^\dagger, \quad \tau j := j_{\tau^\dagger}; \quad (T \Leftarrow c) := (T^\dagger \uparrow c)^\dagger, \quad \tau c := c_{\tau^\dagger}. \] (*)

2.4.11. Recall the notion of $h(T^i_j)$ from 2.4.2. Note that if $h(T^i_j) > 1$ and $T^i_{j+1}$ (resp. $T^i_{j+1}$) is defined that is $T^i_{j+1} \neq \infty$ (resp. $T^i_{j+1} \neq \infty$) then $h(T^i_{j+1}) < h(T^i_j)$ (resp. $T^i_{j+1} < h(T^i_j)$).

Let us describe the jeu de taquin (see [12]) which removes $T^i_j$ from $T$. The resulting tableau is denoted by $T - T^i_j$ and is obtained by the following procedure recursive on $h(T^i_j)$.

(1) If $h(T^i_j) = 1$ then

\[(T - T^i_j) = \begin{pmatrix} T^{1,i-1} \\ (T^i - T^i_j) \\ T^{i+1,\infty} \end{pmatrix} \]

(2) If $h(T^i_j) > 1$, then

(i) If $T^i_{j+1} > T^i_{j+1}$ or $\lambda_i = j$, set

\[(T - T^i_j) = \begin{pmatrix} T^{1,i-1} \\ (T^i \uparrow T^i_{j+1}) \\ (T^{i+1,m} - T^i_{j+1}) \end{pmatrix} \]

(ii) If $T^i_{j+1} < T^i_{j+1}$ or $\lambda^*_j = i$, set

\[(T - T^i_j) = (T_{1,j-1}, (T_j \Leftarrow T^i_{j+1}), (T_{j+1,\lambda_i} - T^i_{j+1})) \]

The result due to M. P. Schützenberger [12] gives

**Theorem.** If $T$ is a Young tableau then $(T - T^i_j)$ is a Young tableau.
As an example we take

\[
T = \begin{pmatrix}
1 & 2 & 5 \\
3 & 4 \\
6
\end{pmatrix}
\]

Then

\[
(T - 6) = \begin{pmatrix}
1 & 2 & 5 \\
3 & 4 \\
6
\end{pmatrix}, \quad (T - 5) = \begin{pmatrix}
1 & 2 \\
3 & 4 \\
6
\end{pmatrix}, \quad (T - 4) = \begin{pmatrix}
1 & 2 & 5 \\
3 \\
6
\end{pmatrix}
\]

\[
(T - 3) = \begin{pmatrix}
4 \\
1 & 2 & 5 \\
6
\end{pmatrix}, \quad (T - 2) = \begin{pmatrix}
3 \\
1 & 4 & 5 \\
6
\end{pmatrix}, \quad (T - 1) = \begin{pmatrix}
2 & 4 & 5 \\
3 \\
6
\end{pmatrix}
\]

2.4.12. The importance of removing the largest entry and the smallest one is provided by the following

**Proposition** (\([8, \text{p. 60}])

Consider \(w = [a_1, \ldots, a_n]\) and set \(M := \max < w >\). Then \(T(w - M) = (T(w) - M)\).

and by the following theorem due to M. P. Schützenberger (see \([8, 5.1.4])

**Theorem.** Consider \(w = [a_1, \ldots, a_n]\) and set \(m := \min < w >\). Then \(T(w - m) = (T(w) - m)\).

2.4.13. 

**Corollary.**

(i) Let \(w \in S_n\) have a decomposition \(w = ys_{n-1,i}^\ge\) where \(y \in S_{n-1}\). If \(T(w) = T\) then \(T(y) = (T - n)\).

(ii) Let \(w \in S_n\) have a decomposition \(w = ys_{i,i}^\le\) where \(y \in S'_{n-1}\). If \(T(w) = T\) then \(T(y) = (T - 1)\).

**Proof.**

Indeed, this is straightforward from 2.4.12 and lemmas 2.2.6 (i), 2.2.7 (i).

2.4.14. Recall the notion of \(\tau\)-invariant from 2.1.7. Given \(T \in T_n\) set

\[
\tau(T) := \{ \alpha_i : r_T(i + 1) > r_T(i) \}
\]

By 2.2.4 Remark 1, 2.1.7 and 2.4.6 one has

**Lemma.** For \(w \in S_n\) set \(T = T(w)\). Then \(\tau(w) = \tau(T) = \tau(V_T) = \tau(C_T)\).
2.4.15. After Schensted - Schützenberger (see 5.4.1) one has

**Theorem.** $T^+(w) = T(w)$, $\forall w \in S_n$.

2.4.16. Recall projections $\pi_i$ from 2.1.8. We consider these projections for $g = sl_n$ and $\mathcal{I} \subset \Pi$ defined as follows. For $1 \leq i < j \leq n$ set

\[ \langle i, j \rangle := \{ r \in \mathbb{N} | i \leq r \leq j \}, \quad \Pi_{i,j} := \{ \alpha_r | r, r + 1 \in \langle i, j \rangle \} \quad \text{and} \quad \pi_{i,j} := \pi_{\Pi_{i,j}}. \]

Define $\pi_{i,j} : T_n \to T_{\langle i,j \rangle}$, through the jeu de taquin applied to the entries of $T$ not lying in $\langle i,j \rangle$ (taken in any order). Note that the order of elimination is indeed not important. Denote $T_{\langle i,j \rangle} := \pi_{i,j}(T)$. Set $D_{\langle i,j \rangle} := sh(T_{\langle i,j \rangle})$.

We can represent each Young tableau as a chain of Young diagrams by the following. Take $D_\lambda \in D_n, D_\mu \in D_m, n < m$. We set $D_\lambda \subset D_\mu$ if $\lambda_i \leq \mu_i, \forall i$. Let $C_n$ denote the set of all decreasing chains of Young diagrams:

\[ C_n := \{ (D_n, \cdots, D_1) : D_i \in D_i \text{ such that } D_{i-1} \subset D_i \} \]

We define a map $\psi : T_n \to C_n$ by $\psi(T) = (D_T^{<1,n>}, D_T^{<1,n-1>}, \cdots, D_T^{<1,1>})$.

For example, if

\[ T = \begin{array}{ccc}
1 & 2 & 5 \\
3 & 4 & \\
6 & & \\
\end{array} \]

Then

\[ \psi(T) = \left( \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}

\end{array}
\end{array}
\end{array}
\end{array}
\end{array}, \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}

\end{array}
\end{array}
\end{array}
\end{array}
\end{array}, \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}

\end{array}
\end{array}
\end{array}
\end{array}
\end{array}, \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}

\end{array}
\end{array}
\end{array}
\end{array}, \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}

\end{array}
\end{array}
\end{array}
\end{array}, \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}

\end{array}
\end{array}
\end{array}
\end{array}\end{array}\right). \]

We reconstruct $T$ from this chain by inserting $i$ into the box deleted on going from $D_T^{<1,i>}$ to $D_T^{<1,i-1>}$. It follows easily that $\psi$ is a bijection.

2.4.17. The interpretation of $\psi$ from 2.4.16 given by Spaltenstein in [13] provides us a description of the connection between orbital varieties and Young tableaux somewhat different from Steinberg’s construction described in 2.1.3 and 2.4.6.

Let $0 = V_0 \subset V_1 \subset \cdots \subset V_n = \mathbb{C}^n$ be the standard flag invariant under $n$. Consider $x \in n$ and let $x_i = x|_{V_i}$ be the restriction of $x$ to $V_i$. Then $C = (D_{x_n}, D_{x_{n-1}}, \cdots, D_{x_1})$ is a decreasing chain of Young diagrams and so defines a standard Young tableau $T(x) = \psi^{-1}(C)$ . Set $\eta : n \to T_n$ by $\eta(x) = T(x)$. Consider $T \in T_n$ and denote the fiber of $\eta$ in $n$ by

\[ n(T) = \{ x \in n : \eta(x) = T \}. \]

By [13] one has
**Theorem.** Every $n(T)$ is dense in a unique orbital variety $V \subset \mathfrak{sl}_n$, and every orbital variety is obtained in that way.

2.4.18. Recall RS procedure from 2.4.6. Given $w \in S_n$ let $C(w) := (\text{sh}(nT(w)), \ldots, \text{sh}(1T(w)))$ and set $Q(w) := \psi^{-1}(C(w))$. Define $\theta(w) := (T(w), Q(w))$. Set

$$\bar{T}_n := \{(T, S) \in T_n \times T_n : \text{sh}(T) = \text{sh}(S)\}$$

It was shown by Robinson and Schensted (see [8, 5.1.4] or in [11, 2.5])

**Theorem.** $\theta : S_n \rightarrow \bar{T}_n$ is a bijection. Moreover $\theta(w^{-1}) = (Q(w), T(w))$.

In particular one has $|C_w|$ is the number of standard tableaux of $\text{sh}(T(w))$. Thus for any $y \in C_w$ one has $|C_y| = |C_w|$.

### 2.5. Duflo descendant and Duflo offspring of a Young cell.

2.5.1. Recall the induced Duflo order from 2.1.6. We can speak about ordering of cells in $S_n$ as well as about ordering of cells in $S_n$. We set $C \leq_D C'$ if $\phi(C) \leq_D \phi(C')$. We define the induced Duflo order on tableaux as well by $T \leq_D S$ if $C_T \leq_D C_S$.

We define a Duflo descendant of an orbital variety exactly as we have defined a geometric descendant of an orbital variety in 1.4.3 only with respect to Duflo order.

Given orbital varieties $V_1$ and $V_2$ such that $V_1 \leq_D V_2$.

**Definition.** We call $V_2$ a Duflo descendant of $V_1$ if for any orbital variety $W$ one has, $V_1 \leq_D W \leq_D V_2$ implies $W = V_1$ or $W = V_2$.

In that case we call $C_{V_2}$ (resp. $T_{V_2}$) a Duflo descendant of $C_{V_1}$ (resp. $T_{V_1}$).

2.5.2. Given $y \in S_n$ we call $ys_i$ an offspring of $y$ if $y \leq_D ys_i$. We induce this definition to the cells.

**Definition.** $C_2$ is called a Duflo offspring of $C_1$ if there exist $y \in C_1$ and $i : 1 \leq i \leq n - 1$ such that $y \leq_D ys_i$ and $ys_i \in C_2$.

In that case we call $V_{C_2}$ (resp. $T_{C_2}$) a Duflo offspring of $V_{C_1}$ (resp. $T_{C_1}$).

Let $D(T)$ (resp. $D(C)$) denote the set of all offsprings of a given tableau $T$ (resp. of a given cell $C$).

Note that each non-trivial cell (i.e. $C \neq \{[1, 2, \ldots, n]\}, \{[n, n - 1, \ldots, 1]\}$) is an offspring of itself. For uniqueness we define $T([1, 2, \ldots, n]) \in D(T([1, 2, \ldots, n]))$ and $D(T([n, n - 1, \ldots, 1])) := \{T([n, n - 1, \ldots, 1])\}$. 

Note also that some offsprings of $C$ can be an offspring of another offspring of $C$ as it is shown by the

**Example.** Consider the following cells in $S_4$

$$C_1 = \{s_2, s_2s_1, s_2s_3\},$$
$$C_2 = \{s_2s_1s_3, s_2s_1s_3s_2\},$$
$$C_3 = \{s_2s_1s_2, s_2s_1s_2s_3, s_2s_1s_2s_3s_2\}.$$

Then $C_2, C_3 \in D(C_1)$ since $s_2s_1s_3$, $s_2s_1s_3s_2$ and $s_2s_1s_2s_3s_2$. As well $C_3 \in D(C_2)$ since $s_2s_1s_3s_2 < s_2s_1s_3s_2 = s_2s_1s_2s_3s_2$.

Obviously the set of Duflo descendants of $C$ is a subset of Duflo offsprings of $C$.

### 3. Description of Induced Duflo Order

#### 3.1. Decomposition of a cell.

3.1.1. Recall the notation from 2.2.5 and 2.4.4. Consider a standard Young tableau $T \in T_n$ with $m$ corners $\{c_i\}_{i=1}^m$ where $c_1 < c_2 < \cdots < c_m$. The deletion of a corner $c_i$ by the algorithm described in 2.4.9 gives rise to the tableau $(T \uparrow \! \! \! \uparrow c_i)$ and the pushed out element $p_i = c_j^T$ where $(T \uparrow \! \! \! \uparrow c_i) > = \{j\}_{j=1}^n, j \neq p_i$. By 2.4.2 there exist a bijection $\phi : T_{n-1} \rightarrow T_{n-1}$ such that $\phi(T \uparrow \! \! \! \uparrow c_i)$ is a standard Young tableau (in our case $\phi = \phi_{p_i}^{-1}$ in notation of 2.2.5). In what follows we denote $C_{(T \uparrow \! \! \! \uparrow c_i)} := C_{\phi(T \uparrow \! \! \! \uparrow c_i)}$ just for simplicity of notation. This is a cell in $S_{n-1}$.

**Proposition.** For each standard Young tableau $T \in T_n$ one has that $C_T$ is a disjoint union:

$$C_T = \bigcup_{i=1}^m s_{p_i, n-1}^\prec C_{(T \uparrow \! \! \! \uparrow c_i)}$$

**Proof.**

Take $w \in C_T$. Let $w(n) = p$. Then by 2.2.6(ii) $w = s_{p, n-1}^\prec y$ with $y \in S_{n-1}$. By RS procedure $T = (n-1, T(w) \downarrow p)$. Moreover by procedure 2.4.9 one has $(T \uparrow \! \! \! \uparrow p_r) = n-1T(w) = \phi_p(T(y))$. Consequently $y \in C_{(T \uparrow \! \! \! \uparrow p_r)}$. Writing $p_r = c_j$ we have $c_j^T = (p_r)^T = p$ and this establishes the inclusion $\subset$.

Conversely to any corner $c$ of $T$ and $y \in C_{(T \uparrow \! \! \! \uparrow c)}$ we obtain that $T(s_{c^T, n-1}^\prec y) = (T \uparrow \! \! \! \uparrow c) \downarrow c^T) = T$ and this establishes the reverse inclusion.

$\blacksquare$
For example take

\[
T = \begin{array}{ccc}
1 & 2 & 5 \\
3 & 4 & \\
6 & 
\end{array}
\]

Then

\[
\phi_1(T \uparrow c_1) = \begin{array}{ccc}
1 & 2 & \\
3 & 4 & 5 \\
\end{array}, \quad \phi_2(T \uparrow c_2) = \begin{array}{ccc}
1 & 3 & 4 \\
2 & & 5 \\
\end{array}, \quad \phi_2(T \uparrow c_3) = \begin{array}{ccc}
1 & 3 & 4 \\
2 & 5 & \\
\end{array}.
\]

Hence \( C_T = s_{i,5}^{-}C_{(T \uparrow c_1)} \prod s_{2,5}^{-}C_{(T \uparrow c_2)} \prod s_{2,5}^{-}C_{(T \uparrow c_3)} \).

3.1.2. Let \( C \) be a cell of \( S_{n-1} \) and put \( T = T_C \). By 2.4.7 \( s_{i,n-1}^{-}C \) lies in a unique cell \( C_{(\phi_i(T) \downarrow i)} \) of \( S_n \). When it does not cause an ambiguity we shall denote this cell also by \( s_{i,n-1}^{-}C \).

3.1.3. Recall the notion of \( \omega_i(T) \), \( \omega_i^j(T) \) from 2.4.2, \( j_T \) from 2.4.8 and \( c_T \), \( s_{c}(T) \) from 2.4.9.

**Lemma.** For a given tableau \( T \) and two numbers \( j_1, j_2 \in \mathbb{N} \) such that \( j_1, j_2 \not\in \langle T \rangle \) and \( j_1, j_2 > \omega_1(T) \) one has that \( j_1 \leq j_2 \) implies

\[
j_1^T \leq j_2^T \tag{*}
\]

In particular given \( c, c' \) two corners of \( T \) one has:

(i) \( c < c' \) implies \( c^T \geq c'^T \).

(ii) The segments \( s_{c}(T), s_{c'}(T) \) cannot cross, but coalesce exactly when equality holds in (i).

(iii) Let \( c < c' \) and \( c^T = c'^T \) then for all the corners \( c'' : c < c'' < c' \) one has \( c''^T = c^T \).

**Proof.**

For a given row \( R = (a_1, \cdots) \) and two numbers \( j_1, j_2 \in \mathbb{N} \setminus \langle R \rangle \) such that \( j_1, j_2 > a_i \) one has that \( (R \uparrow j_i) \) is defined for \( i = 1, 2 \) and \( j_1 \leq j_2 \) implies

\[
j_1^R \leq j_2^R.
\]

Using this result inductively on rows we obtain (*).

(i). Set \( c = c(r, s) \) and \( p_r = c^{r,\infty} \). Since \( p_r \in \langle T^r \rangle \) one has \( p_r \leq \omega^r(T) \). Hence by (*) one gets \( c^r = p_r^{\omega^r-1} \leq (\omega^r(T))^{\omega^r-1} = c^r \).

(ii) and (iii) follow from (i).
For example consider $T$ from 3.1.1. One has $c_1^r = 5$, $c_2^r = c_3^r = 2$ and $s_{c_2}(T)$ coalesces with $s_{c_2}(T)$.

3.1.4. We can change the order of deletions of corners if their segments do not coalesce. This may be precisely expressed as follows.

**Lemma.** Given a tableau $T$ and two distinct numbers $j_1, j_2 \in \mathbb{N}$ such that $j_1, j_2 \notin \langle T \rangle$ and $\omega_j(T) < j_1 < j_2$. If $j_1^r \neq j_2^r$ then

$$((T \uparrow j_1) \uparrow j_2) = ((T \uparrow j_2) \uparrow j_1).$$

(*)

In particular

(i) if $c, c'$ are two corners of $T$ such that $c^r \neq c'^r$ then

$$((T \uparrow c) \uparrow c') = ((T \uparrow c') \uparrow c).$$

**Proof.**

Given a row $R = (a_1, \cdots)$ and two numbers $j_1, j_2 \notin \langle R \rangle$ such that $a_1 < j_1 < j_2$. Set $a_{i_k} = j^r_k$ for $k = 1, 2$. If $i_1 < i_2$ then $((R \uparrow j_1) \uparrow j_2) = ((R \uparrow j_2) \uparrow j_1)$. Using this result inductively on rows we obtain (*).

To show (i) let $c < c'$, $c = c(r, s)$. Consider the procedure of pushing out from $T^{r, \infty}$. If $c^r > c'^r$ then in particular $j_r = c'^{s-1} < \omega^r(T)$ and $(T \uparrow c)^{r, \infty} = (T^{r, \infty} - \omega^r(T))$. Hence

$$((T \uparrow c) \uparrow c') = \left(\frac{(T^{1, r-1} \uparrow \omega^r(T)) \uparrow j_r}{(T^{r, \infty} - \omega^r(T)) \uparrow c'}\right) \stackrel{(*)}{=} \left(\frac{(T^{1, r-1} \uparrow j_r) \uparrow \omega^r(T)}{(T^{r, \infty} \uparrow c') - \omega^r(T)}\right) = ((T \uparrow c') \uparrow c).$$

\[\blacksquare\]

As an example let us consider once more $T$ from 3.1.1. One has

$$\begin{pmatrix}
1 & 2 & 5 \\
3 & 4 \\
6
\end{pmatrix}
\uparrow c(1, 3) \uparrow c(2, 2)
= \begin{pmatrix}
1 & 2 \\
3 & 4 \\
6
\end{pmatrix}
\uparrow c(2, 2) = \begin{pmatrix}
1 & 4 \\
3 \\
6
\end{pmatrix}.$$

On the other hand

$$\begin{pmatrix}
1 & 2 & 5 \\
3 & 4 \\
6
\end{pmatrix}
\uparrow c(2, 2) \uparrow c(1, 3)
= \begin{pmatrix}
1 & 4 & 5 \\
3 \\
6
\end{pmatrix}
\uparrow c(1, 3) = \begin{pmatrix}
1 & 4 \\
3 \\
6
\end{pmatrix}.$$
3.1.5. Let $c = c(r, s)$ be some corner of $T$ and let $\hat{T} = (T \uparrow c)$.

**Lemma.** Let $c' = c(p, t)$ be a corner of $\hat{T}$. One has

$$c'^T \begin{cases} > c^T, & \text{if } p < r \\ < c^T, & \text{otherwise} \end{cases}.$$ 

**Proof.**

The proofs for $p < r$ and $p \geq r$ are the same so we prove the lemma only for $p < r$.

Let us show this by induction on $r$. $r \geq p$ implies that $r \geq 2$. If $r = 2$ then one has $c(r, s) = \omega^2(T)$ so that $c'^T < \omega^2(T)$. On the other hand by pushing up process $\omega^2(T) \in \hat{T}^1$, thus

$$c'^T = \omega^1(\hat{T}) \geq \omega^2(T) > c^T.$$ 

Assume that this is true for $r < q$ and show it for $r = q$. If $p = 1$ then exactly as in previous case

$$c'^T = \omega^1(\hat{T}) \geq c^{r, \infty} > c^T.$$ 

If $p > 1$ then by induction assumption one has $c^{r, \infty} < c'^{r, \infty}$ and by pushing up process $c^{r, \infty} \in \hat{T}^1$. By pushing up process one has

$$c'^T = (c'^{r, \infty})^{\hat{T}^1} \geq c^{r, \infty} > c^T.$$ 

■

3.1.6. Let $T = T_{1, i}^{1, k}$ then $c_1 = c(r, l)$ for some $r \geq 1$. In particular one always has $c_1^T = \omega^1(T)$.

3.2. **Standard Young tableaux and canonical elements.**

3.2.1. We need to construct some “canonical” representatives of a given cell for simplification of proofs and calculations. Recall the notion of $\overline{w}$ and of colligation from 2.2.3.

We need the following very simple but important construction.

**Lemma.** Consider $T = T_{1, 1}^{1, k}$.

(i) Take $w' \in T^{2, \infty}$ and set $w = [w', T^1]$. Then $T(w) = T$.

(ii) Take $y' \in T_{1,1-i}$ and set $y = [y', T_i]$. Then $T(y) = T$.

**Proof.**

(i) By RS algorithm $T(w) = ((\cdots (T^{2, \infty} \downarrow T_i^j) \downarrow \cdots) \downarrow T_i^j)$. In that case roughly speaking the insertion of $T_i^j$ just pushes down the $i$-th column $T_i$. Formally $T_i^j < T_i^{j+1}$ and $T_i^{j} < T_i^j < T_i^{j+1}$ for all $j \geq 1$, $i \geq 2$ gives $T(w) = T$. 


(ii) Again using RS algorithm we get that $T(y) = ((\cdots (T_{1, l-1} \downarrow \omega_l(T)) \downarrow \cdots) \downarrow T^1_i)$.
Note that $T^j_i > T^j_{i-1}$ for all $j : j \geq 1$ and all $i : 1 \leq i \leq j$, and also $T^j_i > T^{j-1}_i$ for all $j : j \geq 2$. Hence $T(y) = T$.

3.2.2. In what follows we frequently use the two “canonical” words defined below so we introduce special notations for them. For a given $T = T^{1,k}_{1,l}$ set $w_r(T) := [T^k, \cdots , T^1]$, $w_c(T) := [T^1, \cdots , T^l]$.

For example for $T$ from 3.1.1 one has $w_r(T) = [6, 3, 4, 1, 2, 5]$ and $w_c(T) = [6, 3, 1, 4, 2, 5]$.

The inductive use of lemma 3.2.1 provides the

**Corollary.** $T(w_r(T)) = T(w_c(T)) = T$.

**Proof.**

Let us prove the statement for $w_r(T)$ by the induction on the number of rows in $T$.

If $T$ has only one row that is $T = T^1 = (T^1_1, \cdots , T^1_l)$ then $w_r(T) = [T^1_1, \cdots , T^1_l]$ and by RS procedure $T(w_r(T)) = T$.

Now assume that the statement is true for a tableau with $k-1$ rows and show that it is true for a tableau with $k$ rows. In that case $w_r(T) = [T^k, \cdots , T^2, T^1]$. Consider the word $w_r(T^{2,\infty}) = [T^k, \cdots , T^2]$. By the induction hypothesis we obtain that $T(w_r(T^{2,\infty})) = T^{2,\infty}$. Then by lemma 3.2.1 $T(w_r(T)) = T$.

Exactly the same way we obtain $T(w_c(T)) = T$.

3.2.3. Lemma 3.2.1 provides us also the following

**Corollary.** Given a Young tableau $T = T^{1,k}_{1,l}$

(i) For $i : 2 \leq i \leq k$ let $w' \in C_{T^i,\infty}$ and set $w = [w', w_r(T^{i-1})]$. Then $T(w) = T$.
(ii) For $i : 1 \leq i \leq l-1$ let $x' \in C_{T^i,i}$ and set $x = [x', w_c(T_{i+1,l})]$. Then $T(x) = T$.
(iii) For $i : 2 \leq i \leq l$ let $y' \in C_{T^i,\infty}$ and set $y = [w_c(T_{i-1,i}), y']$. Then $T(y) = T$.
(iv) For $i : 1 \leq i \leq k-1$ let $z' \in C_{T^i,1}$ and set $z = [w_r(T^{i+1,\infty}), z']$. Then $T(z) = T$.
(v) For $i : 2 \leq i \leq l$ let $w \in C_{T^i,\infty}$ and $z' \in C_{T^1,1}$. Set $w = [w', z']$. Then $T(w) = T$.
(vi) For $i : 1 \leq i \leq k-1$ let $x' \in C_{T^1,i}$ and $y' \in C_{T^1,i+1,\infty}$. Set $x = [x', y']$. Then $T(x) = T$.

**Proof.**

(i), (ii) generalize corollary 3.2.2 and are proved similarly. Part (iii) is obtained from (i) by 2.4.1b. Indeed if $T(y') = T_{i,\infty}$, then by 2.4.16 $T(y') = (T_{i,\infty})^1 = (T^i)^{i-\infty}$. Now by (i)
one has $T([y, T_{i-1}, \ldots, T_1]) = T^\dagger$. Using again \textbf{2.4.15} we obtain
\[
T([T_1, \ldots, T_{i-1}, y]) = T^\dagger([y, T_{i-1}, \ldots, T_1]) = T.
\]
Part (iv) is similarly obtained from part (ii).

To show (v) we use the following easy property of RS procedure. If $s, t$ are two words such that $T(s) = T(t)$ and $q$ is a word such that $< q > \cap < s >= \emptyset$ then $T([s, q]) = T([t, q])$.

By \textbf{3.2.2} one has $T(w') = T^{s, \infty} = T([w_r(T^{i,m})])$. Thus $T(w) = T([w', z']) = T([w_r(T^{i,m}), z']) = T$ just by (iv).

Part (vi) is similarly obtained from part (iii).

\[\square\]

For example consider
\[
T = \begin{array}{cccccc}
1 & 2 & 3 & 7 & 11 \\
4 & 5 & 8 \\
6 & 10 \\
9
\end{array}
\]

One has
\[
T^{3, \infty} = \begin{array}{ccc}
6 & 10 \\
9
\end{array} \quad \text{and} \quad T^{1,2} = \begin{array}{cccccc}
1 & 2 & 3 & 7 & 11 \\
4 & 5 & 8
\end{array}
\]
so that $w' = [9, 10, 6] \in C_T^{3, \infty}$ and $z' = [4, 1, 5, 2, 3, 8, 7, 11] \in C_T^{1,2}$. Hence $w = [w', w_r(T^{1,2})] = [9, 10, 6, 4, 5, 8, 1, 2, 3, 7, 11]$ as well as $z = [w_r(T^{3, \infty}), z'] = [9, 6, 10, 4, 1, 5, 2, 3, 8, 7, 11]$ and $v = [w', z'] = [9, 10, 6, 4, 1, 5, 2, 3, 8, 7, 11]$ belong to $C_T$.

\textbf{3.2.4.} Let us note that the converse to lemma 3.2.1(i) is also true, moreover

\textbf{Lemma.} Given $T = T_{1,t}^{1,k}$. Let $x$ (resp. $y$) be a word such that $T = T([T_1, x])$ (resp. $T = T([y, T_1])$). Then $T(x) = T_{2, \infty}$ (resp. $T(y) = T^{2, \infty}$).

\textbf{Proof.}

By \textbf{2.4.13} it is enough to show one of the assertions. The proof is based on a counting principle. Let us show it for $x$.

One has $|T_1| = k$. Recall the notion of $Q(w)$ from \textbf{2.4.18}. Consider the following subsets of $C_T$

(i) $U = \{w \in C_T : w = [T_1, w']\}$.
(ii) $V = \{[T_1, w] : T(w) = T_{2, \infty}\}$.
(iii) $W = \{w \in C_T : (Q(w))_1 = (1, \cdots, m)\}$.  
Applying corollary 3.2.3 (iii) to \(i = 2\) we obtain \(V \subset U\). On the other hand by RS procedure
\[
(Q([T_1, x]))_1 = \begin{pmatrix}
1 \\
\vdots \\
1
\end{pmatrix}
\]
hence \(U \subset W\).

Yet by 2.4.18
\[
|V| = |\{Q \in T_{n-m} : sh(Q) = sh(T_{2,\infty})\}|,
\]
\[
|W| = |\{Q \in T_{\{l+1, \ldots, n\}} : sh(Q) = sh(T_{2,\infty})\}|,
\]
which provides \(U = V\).

The same argument gives us that the assertions converse to corollary 3.2.3 (i), (iii) are also true.

The converse of lemma 3.2.1 (ii) is false in general as it is shown by the

**Example.**

Consider
\[
T = \begin{array}{ccc}
1 & 2 \\
& & 3
\end{array}
\]

Note that \(w = [1, 3, 2] = [1, 3, T_2] \in \mathcal{C}_T\), and
\[
T([1, 3]) = \begin{array}{ccc}
1 & 3 \\
& & 1
\end{array} = \neq \begin{array}{ccc}
1 & 3 \\
& & 1
\end{array}
\]

Respectively the assertions converse to corollary 3.2.3 (ii), (iv), (v), (vi) are false in general.

3.2.5. Now we continue the description of representatives for a given cell. Given \(T = T_{1,l} \in \mathbb{T}_E\) and \(h \in \mathbb{N} : h \notin E\). We want to describe special words connected to an insertion of \(h\) into \(T\).

**Lemma.** Take \(T, h\) as above. Assume that \(T_{i-1}^i < h < T_i^i\) for some \(i : 1 \leq i \leq l\). Then

(i) \((T \downarrow h) = T([w, T_i^i, (T^1 \downarrow h)])\) for any word \(w \in \mathcal{C}_{T^2, \infty}\).

(ii) For all \(j \geq i\) one has \((T \downarrow h) = T([y_j, h, z_{j+1}])\) for any words \(y_j \in \mathcal{C}_{T_{1,j}}\), \(z_{j+1} \in \mathcal{C}_{T_{j+1, \infty}}\).

(iii) In particular \((T \downarrow h) = T([w_c(T_{1,j}), h, w_c(T_{j+1, \infty})])\) for all \(j : i \leq j \leq l\).

**Proof.**

Set \(U := (T \downarrow h)\).
(i) To show (i) we recall that by the insertion algorithm

\[ U = (T \downarrow h) = \left( \begin{array}{c} T_1 \downarrow h \\ T^1_2 \downarrow \vdots \downarrow T_i \end{array} \right). \]

Set \( w' = [w, T^1_1] \). Since \( T(w) = T^{2,\infty} \) by hypothesis, the insertion algorithm gives \( T(w') = (T^{2,\infty} \downarrow T^1_1) = U^{2,\infty} \), hence lemma 3.2.1 gives \( T([w, T^1_1, (T^1_1 \downarrow h)]) = U \).

(ii) Since \( h < T^1_i < T^2_i < \cdots \) the insertion of \( h \) into \( T \) does not affect \( T_{i+1,\infty} \), so \( U_{i+1,\infty} = T_{i+1,\infty} \). That means that

\[ U_{i,j} = (T_{1,j} \downarrow h) \text{ for every } j \geq i. \]

Hence for every \( j : i \leq j \leq l \) the word \( y'_j = [y_j, h] \) satisfies \( T(y'_j) = (T_{1,j} \downarrow h) = U_{1,j} \). Then corollary 3.2.3(vi) gives \( T([y_j, h, z_{j+1}]) = U \).

\[ \blacksquare \]

3.3. Inductive definition of the set of offsprings.

3.3.1. Consider \( w \in S_n \). Write \( w = [a_1, \ldots, a_n] \) and recall that by 2.2.6(ii) \( w = s_{a_n}^{\leq a_{n-1}} y \), where \( y = \omega_{a_n}^{-1}([a_1, \ldots, a_{n-1}]) \), \( y \in S_{n-1} \). By 3.1.2 given a cell \( C \subset S_{n-1} \) we can define a cell \( s_{i,n-1}^C \) in \( S_n \).

By 2.2.2 Remark and 2.2.4 Remark 2 for \( i : 1 \leq i \leq n-2 \) one has \( w \overset{D}{\prec} w_s \) iff \( y \overset{D}{\prec} y_s \). Thus

**Lemma.** If \( C_1, C_2 \) are cells of \( S_{n-1} \) satisfying \( C_1 \overset{D}{\leq} C_2 \) then \( s_{i,n-1}^C \overset{D}{\leq} s_{i,n-1}^C \) for all \( i : 1 \leq i \leq n-1 \).

**Proof.**

By definition of induced Duflo order it is enough to show this only for the case when \( C_2 \) is an offspring of \( C_1 \) which is straightforward corollary of 2.2.8.

\[ \blacksquare \]

It can occur that \( C_1 \overset{D}{\prec} C_2 \) in \( S_{n-1} \) and yet \( s_{i,n-1}^C \overset{D}{\leq} s_{i,n-1}^C \). This is shown by the

**Example.**

Consider \( S_2 = \{ C_1 = \{[1,2]\}, C_2 = \{[2,1]\} \} \), it is obvious that \( C_1 \overset{D}{\prec} C_2 \). Now consider sets \( s_2 C_1 = \{[1,3,2]\}, s_2 C_2 = \{[3,1,2]\} \). One has

\[ T([1,3,2]) = T([3,1,2]) = \begin{array}{c} 1 \\ 2 \\ 3 \end{array}. \]

Thus \( s_2 C_1 = s_2 C_2 \).
3.3.2. Let \( C_T \) be a cell in \( S_n \). By proposition 3.1.1 it can be considered as a disjoint union
\[
C_T = \bigcup_{i=1}^{m} s_{p_i,n-1} \mathcal{C}(T \upharpoonright c_i)
\]
where \( p_i = c_i^T \).

Consider right multiplication by \( s_j \in \Pi_{n-2} \) on some element \( w \in C_T \). Since \( w = s_{p_i,n-1}^c y \) for \( y \in \mathcal{C}(T \upharpoonright c_i) \) one has \( \mathcal{C}_{ws_j} = s_{p_i,n-1}^c \mathcal{C}_{ys_j} \). Recall the notion of \( \mathcal{D}(C_T) \) from 2.5.2. Set for \( T \in \mathcal{T}_{n-1} \)
\[
s_{i,n-1}^{-1} \mathcal{D}(C_T) := \{ s_{i,n-1}^{-1} C \mid C \in \mathcal{D}(C_T) \} = \{ \mathcal{C}(\phi_i(T') \downarrow i) \mid T' \in \mathcal{D}(T) \}.
\]
Set
\[
\mathcal{D}_o(C_T) := \bigcup_{i=1}^{m} s_{p_i,n-1}^{-1} \mathcal{D}(C(T \upharpoonright c_i)) \quad \text{and} \quad \mathcal{D}_o(T) := \bigcup_{i=1}^{m} \{(S \downarrow p_i) \mid S \in \mathcal{D}(T \upharpoonright c_i)\}.
\]
these are subsets of respectively \( \mathcal{D}(C_T) \) and \( \mathcal{D}(T) \).

Set
\[
\mathcal{D}_n(C_T) := \mathcal{D}(C_T) \setminus \mathcal{D}_o(C_T) \quad \text{and} \quad \mathcal{D}_n(T) := \mathcal{D}(T) \setminus \mathcal{D}_o(T).
\]
Clearly if \( \mathcal{C}' \in \mathcal{D}_n(C_T) \) then \( \mathcal{C}' = \mathcal{C}_{ws_{n-1}} \) where \( w \in C_T \) and \( ws_{n-1} > w \). Summarizing

**Corollary.** \( \mathcal{D}(C_T) = \mathcal{D}_o(C_T) \bigsqcup \mathcal{D}_n(C_T) \).

3.3.3. Recall the notion of \( \omega^i(T) \) from 2.4.2 and of \( c_T \) from 2.4.9. Take some \( T \in \mathcal{T}_n \). Let \( \{c_i\}_{i=1}^{m} \) be the set of its corners.

If \( |T^1| > |T^2| \) then \( c_1 \) is in the first row. In that case consider an array
\[
S := \begin{pmatrix}
T^1 - \omega^1(T) \\
T^{2,\infty} \downarrow \omega^1(T)
\end{pmatrix}
\]
and define the following conditions

(i) \( |T^1| \geq |T^2| + 2 \);
(ii) \( |T^1| = |T^2| + 1 \) and \( \omega^i(T) < \omega^2(T) \);

Set
\[
S_T(c_1) := \begin{cases}
S, & \text{if } T \text{ satisfies (i) or (ii)}; \\
\emptyset, & \text{otherwise}.
\end{cases}
\]

If there exist \( c_i \) such that \( c_i^T = \omega_1(T) \) then consider an array
\[
S := \begin{pmatrix}
(T \upharpoonright c_i)^1 \\
(T \upharpoonright c_i)^{2,\infty} \downarrow \omega^1(T)
\end{pmatrix}
\]
and define
\[
S_T(c_i) := \begin{cases}
S, & \text{if } c_i^T = \omega_1(T) \text{ and } S \text{ is standard with } \text{sh} (S) > \text{sh} (T) \\
\emptyset, & \text{otherwise}.
\end{cases}
\]
Set $\mathcal{D}_n'(T) := \{S_T(c_i)\}_{i=1}^m$.

**Theorem.** $\mathcal{D}(T) = \mathcal{D}_o(T) \cup \mathcal{D}_n'(T)$.

This theorem provides a justification to the rough idea that induced Duflo order is given by lowering numbered boxes in the sense of RS procedure.

Indeed if $c_1$ is in the first row then $S$ defined in $(\ast)$ is a tableau if and only if $T$ satisfies (i) or (ii). In that case $S_T(c_1)$ is obtained by lowering the (numbered) box from the first row of $T$ to the second row by RS procedure.

If $c_i = c(j, |T_j|)$ where $j > 1$ then the procedure described by $(\ast\ast)$ is a more complicated shuffle in which a corner of $T$ is deleted pushing out the last (numbered) box of the first row of $T$ which is then inserted into $(T \uparrow c_i)^{2,\infty}$.

We further note that any $T' \in \mathcal{D}_o(T)$ is obtained by first deleting some corner $c$ of $T$ then computing $T'' \in \mathcal{D}(T \uparrow c)$ and finally computing $T' = (T'' \downarrow c^T)$. Thus 3.3.2 and 3.3.3 together describe the offsprings of $\mathcal{C}_T$ by a precisely determined process of deleting corners and lowering (possibly with shuffling) numbered boxes.

For example let us compute $\mathcal{D}(T)$ for $T = \[
\begin{array}{cccc}
1 & 2 & 6 & 7 \\
3 & 5 \\
4
\end{array}
\]$

Note that $S_T(c) \neq \emptyset$ only for $c = c_1 = c(1, 4)$. Hence $\mathcal{D}_n'(T) = \{S_T(c_1)\}$ where $S_T(c_1) = \[
\begin{array}{cccc}
1 & 2 & 6 \\
3 & 5 & 7 \\
4
\end{array}
\]$

Now one has

\[
(T \uparrow c_1) = \[
\begin{array}{cccc}
1 & 2 & 6 \\
3 & 5 \\
4
\end{array}
\], \quad (T \uparrow c_2) = \[
\begin{array}{cccc}
1 & 5 & 6 & 7 \\
3 \\
4
\end{array}
\], \quad (T \uparrow c_3) = \[
\begin{array}{cccc}
1 & 3 & 6 & 7 \\
4 & 5
\end{array}
\]$

Applying further our algorithm to $(T \uparrow c_1)$, $(T \uparrow c_2)$, $(T \uparrow c_3)$ we get that $\mathcal{D}_o(\mathcal{C}_T)$ is described by the following set of tableaux:

\[
\begin{array}{cccc}
1 & 2 & 6 \\
3 & 5 & 7 \\
4
\end{array}
\], \quad \[
\begin{array}{cccc}
1 & 2 & 6 \\
3 & 5 \\
4 & 7
\end{array}
\], \quad \[
\begin{array}{cccc}
1 & 2 & 7 \\
3 & 5 \\
4 & 6
\end{array}
\].
Then \[ T \] only rather few two-fold deletions intervene. We describe the results in terms of Young tableaux. Set \( \ddots \) to consider only those \( w \) with \( a_n > a_{n-1} \). Assume that \( w \) is such a word. One has \( ws_{n-1} = [a_1, a_2, \ldots, a_n, a_{n-1}] \).

Let \( T = T(w) \) be a tableau with \( l \) columns (that is \( |T^1| = l \)) and with \( m \) corners. It determines the set \( \{(T \uparrow c_i)\}_{i=1}^m \). Each one of the \( (T \uparrow c_i) \) determines in turn the set of \( \{((T \uparrow c_i) \uparrow c'_j)\}_{j=1}^{m_i} \). It is obvious that one has \( T_{n-2}(w) = ((T \uparrow c_i) \uparrow c'_j) \) for some \( c_i, c'_j \), and \( a_{n-1} = c'_j(T \uparrow c_i), a_n = c_i T \). Hence the offsprings obtained by right multiplication by \( s_{n-1} \) can be computed from considering two-fold deletions of corners. We show that only rather few two-fold deletions intervene. We describe the results in terms of Young tableaux. Set \( \hat{T} = n_{-2}T(w) \) which we saw was some \( ((T \uparrow c_i) \uparrow c'_j) \). Set \( \hat{T} = n_{-1}T(w) \).

Then \( \hat{T} = (\hat{T} \downarrow a_{n-1}) = (\hat{T} \downarrow c'_j(T \uparrow c_i)) = (T \uparrow c_i) \). The resulting offsprings are described in the following

**Lemma.** Let \( T \) be a tableau described above, such that \( |T^1| = l \). Consider \( w \in C_T, w = [a_1, \ldots, a_n] \) with \( a_n > a_{n-1} \). Set \( \hat{T} = n_{-2}T(w) \) and \( \hat{T} = n_{-1}T(w) \). Set \( l' = |\hat{T}^1| \). Then \( T \leq T(ws_{n-1}) \) and one of the following holds:

(i) If there exist a pair \( i, j \) with \( 1 < i < j \leq l' \) or \( 1 < i = j \leq l' - 2 \) such that \( T^1_{i-1} < a_{n-1} < \hat{T}^1_i \) and \( T^1_{j-1} < a_n < \hat{T}^1_j \), then there exist \( y \in C_T, \alpha_\epsilon \in \Pi_{n-2} \) such that \( T(ws_{n-1}) = T(y_{se}) \).

(ii) If \( a_{n-1} < \hat{T}^1_{l'} < a_n \) then \( T(ws_{n-1}) = T \).

(iii) (a) If \( a_{n-1} > \hat{T}^1_{l'} \) then \( l = l' + 2 \);
    (b) If \( \hat{T}^1_{l-1} < a_{n-1} \) and \( a_n < \hat{T}^1_{l'} \) then \( l = l' + 1 \);
    (c) If \( \hat{T}^1_{l-2} < a_{n-1} \) and \( a_n < \hat{T}^1_{l-2} \), then \( l = l' \).

In all the three cases (a), (b) and (c) \( T^1_{l-1} = a_{n-1}, \omega^i(T) = T^1_i = a_n \) and

\[
T(ws_{n-1}) = \left( \hat{T}^{1}_{l, \omega^i(T)} \right)_{\hat{T}^{-1,2}_{l,\omega^i(T)}}.
\]
Proof.

Set $S = T(w s_{n-1})$. Let us choose “good” elements in $C_T$ and $C_S$.

Assume that there exist $j < j'$ such that $\dot{T}_{j-1} < a_n < \ddot{T}_j$. Then since $a_{n-1} < a_n$ there must exist $i \leq j$ such that $\dot{T}_{i-1} < a_{n-1} < \ddot{T}_i$. By lemma 3.2.3(iii) we have $\dot{T} = T(x)$ where $x = ([w_c(\dot{T}_{1,j}), a_{n-1}, w_c(\ddot{T}_{j+1}, \infty)])$. Then $T = (T(x) \downarrow a_n)$. Note that $\dot{T}_{1,j} = (\ddot{T}_{1,j} \downarrow a_{n-1}) = T([w_c(\ddot{T}_{1,j}), a_{n-1}])$. Then by lemma 3.2.3(ii) we can write $T = (T(x) \downarrow a_n) = T(y)$ where

$$y = \begin{cases} [w_c(\ddot{T}_{1,j}), a_{n-1}, a_n, w_c(\ddot{T}_{j+1}, \infty)], & \text{if } i < j, \\ [w_c(\ddot{T}_{1,j}), a_{n-1}, a_n, w_c(\ddot{T}_{j+2}, \infty)], & \text{if } i = j. \end{cases}$$

Note that in case $i = j$ we obtain that $\dot{T}^1 = (\ddot{T}^1 \downarrow a_{n-1}) = (\ddot{T}^1_{1,j-1}, a_{n-1}, \ddot{T}^1_{j+1}, \cdots)$, so that $a_{n-1} < a_n < \ddot{T}^1_{j+1}$. That is why we can insert $a_n$ only after inserting the column $\ddot{T}^1_{j+1}$.

Using once more lemma 3.2.3(ii) for the case $i = j$ in definition of $y$ we can also write $T = T(z)$ where

$$z = \begin{cases} [w_c(\ddot{T}_{1,j}), a_{n-1}, a_n, w_c(\ddot{T}_{j+2}, \infty)], & \text{if } i < j, \\ [w_c(\ddot{T}_{1,j+1}), a_{n-1}, a_n, w_c(\ddot{T}_{j+2}, \infty)], & \text{if } i = j. \end{cases} \quad (\ast)$$

In exactly the same way we obtain $S = ((\ddot{T} \downarrow a_n) \downarrow a_{n-1}) = T(u)$ where

$$u = \begin{cases} [w_c(\ddot{T}_{1,j}), a_{n-1}, a_n, w_c(\ddot{T}_{j+2}, \infty)], & \text{if } i < j, \\ [w_c(\ddot{T}_{1,j+1}), a_{n-1}, a_n, w_c(\ddot{T}_{j+2}, \infty)], & \text{if } i = j. \end{cases} \quad (\ast\ast)$$

We now describe how cases (i)-(iii) result.

(i) (a) If $i < j < j'$ or $i = j < j' - 2$ then the hypotheses of (i) are fulfilled. In this case the above computation shows that the conclusion of (i) is obtained. Indeed denote by $e$ the place of $a_{n-1}$ in $z$ and note that $e < n - 1$. Then (\ast) and (\ast\ast) gives $S = T(z s_e)$.

(b) To complete case (i) it remains to study the case $j = j'$ and $i < j'$. By the insertion algorithm $T^1 = (\ddot{T}^1_{1,i-1}, a_{n-1}, \cdots, \ddot{T}^1_{j-1}, a_n)$ and

$$T(w) = \left( \langle \ddot{T}^{2,\infty} \downarrow \ddot{T}^1_i \rangle \downarrow \ddot{T}^1_{j'} \right).$$

On the other hand by the insertion algorithm $S^1 = T^1$ and

$$S = \left( \langle \ddot{T}^{2,\infty} \downarrow \ddot{T}^1_i \rangle \downarrow \ddot{T}^1_i \right).$$

Set $y = [w_c(\ddot{T}^{2,\infty}), \ddot{T}^1_i, \ddot{T}^1_{j'}, T^1]$ and denote by $e$ the place of $\ddot{T}^1_i$ in $y$. Then $S = T(y s_e)$ which is conclusion of (i).
(ii) If \( a_{n-1} < \tilde{T}_l < a_n \) then the hypotheses of (ii) are fulfilled. In this case

\[
T = ((\tilde{T} \downarrow a_{n-1}) \downarrow a_n) = \left( \tilde{T}^1 + a_n \right)_{\tilde{T}^2,\infty} \\
S = ((\tilde{T} \downarrow a_n) \downarrow a_{n-1}) = \left( \tilde{T}^1 + a_n \right)_{\tilde{T}^2,\infty} \downarrow a_{n-1} \\
= \left( \tilde{T}^1 + a_n \right)_{\tilde{T}^2,\infty} = T.
\]

This is just the conclusion of (ii).

(iii) (a) If \( i = j = l' + 1 \) then \( \tilde{T}_l < a_{n-1} \) which are hypotheses of iii (a). In that case

\[
T^1 = ((\tilde{T}^1 \downarrow a_{n-1}) \downarrow a_n) = (\tilde{T}^1 + a_{n-1} + a_n) \text{ so that}
\]

\[
T = \left( T^1_{\tilde{T}^2,\infty} \right).
\]

Hence \( l = l' + 2, T^3_{l-1} = a_{n-1}, \tilde{T}^1 = (\tilde{T}^1 + a_{n-1}) = (T^1 - a_n) \) and \( \omega^1(T) = a_n \). Thus

\[
S = ((\tilde{T} \downarrow \omega^1(T)) \downarrow T^3_{l-1}) = \left( \tilde{T}^1_{\tilde{T}^2,\infty} \downarrow \omega^1(T) \right)
\]

exactly as in conclusion of (iii).

(b) If \( j = l' \) and \( i = l' \) then \( \tilde{T}^3_{l-1} < a_{n-1}, a_n < \tilde{T}^3_{l'} \) which are the hypotheses of iii (b). In that case we get

\[
T = ((\tilde{T} \downarrow a_{n-1}) \downarrow a_n) = \left( \tilde{T}^1 + a_n \right)_{\tilde{T}^2,\infty}
\]

where \( \tilde{T}^1 = (\tilde{T}^1 \downarrow a_{n-1}), \tilde{T}^2,\infty = (\tilde{T}^2,\infty \downarrow \tilde{T}_l). \) Hence \( T^1 = (\tilde{T}^1, \ldots, \tilde{T}^3_{l-1}, a_{n-1}, a_n) \) and \( T^2,\infty = \tilde{T}^2,\infty. \) In particular \( l = l' + 1 \) and \( T^3_{l-1} = a_{n-1}, \omega^1(T) = a_n. \) Then

\[
n_{n-1}T(ws_{n-1}) = (\tilde{T} \downarrow \omega^1(T)) = \left( \tilde{T}^1_{\tilde{T}^2,\infty} \downarrow \omega^1(T) \right)
\]

which provides us the conclusion of (iii), namely,

\[
S = (n_{n-1}T(ws_{n-1}) \downarrow T^3_{l-1}) = \left( \tilde{T}^1_{\tilde{T}^2,\infty} \downarrow T^3_{l-1} \downarrow \omega^1(T) \right) = \left( \tilde{T}^1_{\tilde{T}^2,\infty} \downarrow \omega^1(T) \right).
\]

(c) It remains to consider \( i = j = l' - 1 \). Here \( \tilde{T}^3_{l'-2} < a_{n-1}, a_n < \tilde{T}^3_{l'-1} \) which are the hypotheses of iii(c). In that case

\[
\tilde{T} = (\tilde{T} \downarrow a_{n-1}) = \left( \tilde{T}^1_{\tilde{T}^2,\infty} \downarrow a_{n-1} \right)
\]
and so \( \hat{T}_1 = (\hat{T}_1 \downarrow a_{n-1}) = (\hat{T}_i_1, \ldots, \hat{T}_{l'-1}, a_{n-1}, \hat{T}_{l'}) \) Note that \( a_{n-1} < a_n < \hat{T}_{l'} \) which gives \( a_n = \omega^i(T) \) and \( l = l' \). Now consider \( S \). By RS algorithm

\[
S = \begin{pmatrix}
\hat{T}_1 \downarrow a_{n-1} \\
\hat{T}_{2, \infty} \downarrow \omega^i(T)
\end{pmatrix}
\]

where \( (\hat{T}_1 \downarrow \omega^i(T)) = (\hat{T}_i_1, \ldots, \hat{T}_{l'-2}, \omega^i(T), \hat{T}_{l'}, \infty \ldots) \). This again provides us the conclusion of (iii), namely,

\[
\begin{pmatrix}
\hat{T}_1 \downarrow a_{n-1} \\
\hat{T}_{2, \infty} \downarrow \omega^i(T)
\end{pmatrix} = \begin{pmatrix}
\hat{T}_1 \downarrow \omega^i(T) \\
\hat{T}_{2, \infty} \downarrow \omega^i(T)
\end{pmatrix}.
\]

\[\Box\]

3.3.5. Now we are ready to prove theorem 3.3.3.

**Proof.**

Let \( \mathcal{D}''(T) = \{T(ws_{s_n-1})\} \) for \( w \) satisfying conditions of 3.3.4 (iii). It is immediate from lemma 3.3.4 \( \mathcal{D}_n(T) \subset \mathcal{D}''(T) \subset \mathcal{D}(T) \). Thus we must show that \( \mathcal{D}''(T) = \mathcal{D}'_n(T) \).

First we show that condition 3.3.3 (i) is satisfied if and only if there exists \( w \in \mathcal{C}_T \) satisfying 3.3.4 (iia) and in this case \( T(ws_{s_n-1}) = S_T(c_i) \). Then we show that if condition 3.3.3 (ii) is satisfied then there exists \( w \in \mathcal{C}_T \) satisfying 3.3.4 (iib) and if there exists \( w \in \mathcal{C}_T \) satisfying 3.3.4 (iii) or condition 3.3.3 (ii) is satisfied and in both cases \( T(ws_{s_n-1}) = S_T(c_i) \). Finally we show that for any \( c_i \neq c(1,l) \) one has \( S_T(c_i) \neq \emptyset \) if and only if there exists \( w \in \mathcal{C}_T \) satisfying 3.3.4 (c) and in this case \( T(ws_{s_n-1}) = S_T(c_i) \). This completes the proof of 3.3.3.

(i) If there exists \( w \in \mathcal{C}_T \) satisfying 3.3.4 (iia) then \( \hat{T}_1 = (\hat{T}_1, a_{n-1}), T^1 = (\hat{T}_1, a_{n-1}, a_n) \) and \( T^2, \infty = \hat{T}_{2, \infty} = \hat{T}_{2, \infty} \). In particular \( |T^1| = |\hat{T}_1| + 2 \geq |T^2| + 2 = |T^2| + 2 \). Hence 3.3.3 (i) is satisfied. As well in this case

\[
S = T(ws_{s_n-1}) = \begin{pmatrix}
\hat{T}_1 \\
\hat{T}_{2, \infty} \downarrow \omega^i(T)
\end{pmatrix} = \begin{pmatrix}
T^1 - \omega^i(T) \\
T^2, \infty \downarrow \omega^i(T)
\end{pmatrix} = S_T(c_i)
\]

On the other hand if hypotheses of 3.3.3 (i) is satisfied then \( w(T) \) satisfies 3.3.4 (a).

(ii) If there exists \( w \in \mathcal{C}_T \) satisfying 3.3.4 (iib) then \( \hat{T}_1 = (\hat{T}_1, \ldots, \hat{T}_{l'-1}, a_{n-1}), T^1 = (\hat{T}_1, a_n) \) and \( T^2, \infty = \hat{T}_{2, \infty} \). In particular \( |T^1| = |\hat{T}_1| + 1 = l' + 1 \) and \( \omega^i(T) \geq \hat{T}_{l'} \). Note that \( |T^2| = |T^2| \leq |T^1| = l' \). If \( |T^2| = l' \) then \( |T^1| = |T^2| + 1 \) so that 3.3.3 (ii) is satisfied. If \( |T^2| < l' \) then \( |T^1| \geq |T^2| + 2 \) so that 3.3.3 (i) is satisfied. In both cases

\[
S = T(ws_{s_n-1}) = \begin{pmatrix}
\hat{T}_1 \\
\hat{T}_{2, \infty} \downarrow \omega^i(T)
\end{pmatrix} = \begin{pmatrix}
T^1 - \omega^i(T) \\
T^2, \infty \downarrow \omega^i(T)
\end{pmatrix} = S_T(c_i)
\]
On the other hand if \(3.3.3\) (ii) is satisfied then \(w_r(T)\) satisfies \(3.3.4\) iii (b). Indeed in that case by RS procedure \(\hat{T}^1_{r-1} = T^1_{r-1} < T^1_r = a_{n-1}\) and \(T^1_r = \omega^2(T) > \omega^1(T) = a_n\).

(iii) If there exists \(w \in C_T\) satisfying \(3.3.4\) iii(c) then \(a_n = \omega^1(T)\) and \(\hat{T} = (T \uparrow c_i)\) where \(c_i\) is the corner of \((\omega^1(T))\). In that case

\[
S = T(ws_{n-1}) = \left( \hat{T}^1_{\uparrow \downarrow \omega^1(T)} \right) \left( \left( T_{\uparrow c_i}^1 \downarrow \omega^1(T) \right) \right) = S_T(c_i).
\]

On the other hand let us show that if there exists \(c_i\) such that \((c_i)^T = \omega^1(T)\) for some \(c_i \neq c(1, l)\) and \(3.3.4\) iii (c) is not satisfied then \(sh(T) \neq sh(S)\) where

\[
S = \left( \left( T_{\uparrow c_i}^1 \downarrow \omega^1(T) \right) \right).
\]

Set \(c_i = c(r, s), \hat{T} = (T \uparrow c_i)\). Let us show that we can reformulate conditions \(3.3.4\) iii (c) as follows:

\[
c_i^T = \omega^1(T) \quad \text{and} \quad \omega^1(T) < \begin{cases} 
\omega^2(\hat{T}) \\
(\omega^r(\hat{T}))^{T^2,r-1}
\end{cases}
\]

if \(r = 2\), \(r > 2\).

Indeed, conditions \(3.3.4\) iii(c) mean that \(c_i^T = \omega^1(T)\) and there exist a corner \(c'\) of \(\hat{T} = (T \uparrow c_i)\) such that \(c'_{\downarrow 2} = a_{n-1} = T^1_{l-1}\). Put \(\hat{T} = (\hat{T} \uparrow c')\). Note that \(\hat{T}^1_{l-1} = c'^{T^2,\infty}\). By \(3.3.4\) iii(c) one has that \(\hat{T}^1_{l-1} > \omega^1(T), T^1_{l-1}\), i.e.

\[
c_i^T < \omega^1(T) \quad \text{and} \quad c'^{T^2,\infty} > \omega^1(T).
\]

Note that by \(3.1.3\) (i), (iii) for every corner \(c' = c'(r', s')\)

\[
c'^T = \begin{cases} 
\omega^1(\hat{T}) \\
\leq T^1_{l-1}
\end{cases}
\]

if \(r' < r\), \(r' \geq r\).

Furthermore applying \(3.1.3\) (i) to two corners of \(\hat{T}\) which are as well corners of \(\hat{T}^{T^2,\infty}\) we get \(c' > c''\) implies \(c'^{T^2,\infty} \leq c''^{T^2,\infty}\). Hence we can choose the smallest \(c' = c'(r', s')\) such that \(r' \geq r\). Note also that \(|\hat{T}^r| = s - 1\) hence applying \(3.1.6\) to \(\hat{T}^{r,\infty}\) one has \(c'^{T^{r,\infty}} = \omega^r(\hat{T})\) thus by \(3.1.3\) (ii) \(c'^{T^2,\infty} = \omega^r(\hat{T})^{T^2,r-1}\). This together with (***) gives (****).

Hypotheses (****) are not satisfied only in two cases: if \(s = 1\) or \(s > 1\) and

\[
\omega^1(T) > \begin{cases} 
\omega^2(\hat{T}^2) \\
\omega^r(\hat{T})^{T^2,r-1}
\end{cases}
\]

if \(r = 2\), \(r > 2\).

If \(s = 1\) then \(sh(T) = (l, \cdots, |T^k-1|, 1)\) and \(sh(S)\) is obtained by adding one box to \(sh(\hat{T}) = (l, \cdots, |T^k-1|)\) so that \(sh(S) = (l, \cdots, |T^i| + 1, \cdots)\) for some \(i : 1 \leq i \leq k\). Hence \(sh(S) \leq sh(T)\) so that \(sh(T) \neq sh(S)\) as required.
Assume that $s > 1$ and $r = 2$. Then
\[
S = \left( \begin{array}{c}
\hat{T}^1 \\
(\hat{T}^2 + \omega^i(T)) \\
T^{3,\infty}
\end{array} \right)
\]
and $\text{sh}(S) = \text{sh}(T)$, so that $\text{sh}(T) \not\subset \text{sh}(S)$ as required.

Finally consider the case $s > 1$ and $r > 2$ and $\omega^i(T) > \omega^r(\hat{T}) T^{2,r-1}$ (* * *). Assume that $\text{sh}(S) > \text{sh}(T)$. Then $\text{sh}(S^{1,r}) = \text{sh}(\hat{T}^{1,r})$. In particular putting $S' = (\hat{T}^{2,r} \downarrow \omega^i(T))$ one gets $\text{sh}(S') = (|\hat{T}^2|, \ldots, |\hat{T}^r|, 1)$. Hence $c(r + 1, 1)S' = \omega^i(T)$ and $T^{2,r} = (S' \uparrow c(r + 1, 1))$. One has
\[
\omega^r(\hat{T}) T^{2,r-1} = c'(r, s - 1) T^{2,r} = c'(r, s - 1)^{(S' \uparrow c(r + 1, 1))} > c(r + 1, 1)S' = \omega^i(T)
\]
where inequality is obtained by 3.1.5. This contradicts to (* * *). Hence $\text{sh}(S) \not\subset \text{sh}(T)$ as required.

Note that the set $D_n(T)$ may contain a few offsprings of type $S_T(c_i)$ for $i > 1$. Let us illustrate this by

**Example.**

Let us regard $S_{10}$ and consider the following tableau:

\[
T = \begin{array}{cccc}
1 & 2 & 3 & 4 \\
5 & 6 & 9 & 10 \\
7 & 8 \\
\end{array}
\]

Deletion of the corners provides us $(T \uparrow c_i) = \hat{T}'$, $c'_1 = 4$, $(T \uparrow c_2) = \hat{T}'', c'_2 = 4$, where

\[
\hat{T}' = \begin{array}{cccc}
1 & 2 & 3 & 10 \\
5 & 6 & 9 \\
7 & 8 \\
\end{array}, \quad \hat{T}'' = \begin{array}{cccc}
1 & 2 & 3 & 6 \\
5 & 8 & 9 & 10 \\
7 \\
\end{array}.
\]

A straightforward checking provides $D_n'(T) = \{S_T(c_1), S_T(c_2)\}$ where

\[
S_T(c_1) = \begin{array}{cccc}
1 & 2 & 3 & 10 \\
4 & 6 & 9 \\
5 & 8 \\
7 \\
\end{array}, \quad S_T(c_2) = \begin{array}{cccc}
1 & 2 & 3 & 6 \\
4 & 8 & 9 & 10 \\
5 \\
7 \\
\end{array}.
\]

Note also that any $S \in D_n(T)$ is obtained by $S = (S' \downarrow 4)$, where $S' \in D(T') \cup D(T'')$, in particular $4 \in <S^1>$. Hence in that case $D_n(T) = D_n'(T)$. 
3.4. The set of offsprings as an induction of two subgroups.

3.4.1. In the previous section we have described the set of offsprings of a cell in \( S_n \) as formed from induced sets of offsprings of certain cells in \( S_{n-1} \) together with some new offsprings obtained by action of \( s_{n-1} \). Recall the notation \( S'_{n-1} \) from 2.2.7. Here we show that a set of offsprings can be represented as a union of induced sets of offsprings from \( S_{n-1} \) and \( S'_{n-1} \) which are two isomorphic subgroups of \( S_n \).

3.4.2. Recall \( w_o \) from 2.2.4 Remark 3. Let \( \hat{w}_o \) denote the unique largest element of the subgroup \( S_{n-1} \) of \( S_n \). Recall the notation \( T \) and \( s_{i,j}^\circ \) from 2.2.5. One has

**Lemma.**

(i) For any \( y \in S_n \) one has \( \overline{y} = yw_o \).
(ii) \( w_o \) can be decomposed as \( w_o = y^{-1}\overline{y} \) for any \( y \in S_n \).
(iii) In particular \( w_o = w_o s_{n-1,1}^\circ \).
(iv) For any \( y \in S_{n-1} \) set \( \hat{y} = y\hat{w}_o \). Then \( w_o \) can be decomposed as \( w_o = y^{-1}s_{n-1,1}^\circ \phi_1(\hat{y}) \).

**Proof.**

Let \( y = [a_1, \ldots, a_n] \) then

\[
\overline{y} = [a_n, \ldots, a_1] = [a_1, \ldots, a_n][n, \ldots, 1]
\]

which provides (i).

(ii) follows from (i).

To show (iii) note that \( \hat{w}_o = \hat{w}^{-1} = [n-1, \ldots, 1, n] \) and \( \overline{w_o} = [n, 1, 2, \ldots, n-1] = s_{n-1,1}^\circ \). Thus (iii) is obtained by applying (ii) to \( y = \hat{w}_o \).

To show (iv) note that by (ii) \( \hat{w}_o = y^{-1}\hat{y} \). Then by (iii) \( w_o = \hat{w}_o s_{n-1,1}^\circ = y^{-1}\hat{y}s_{n-1,1}^\circ \).

Further note that for any \( j : 1 \leq j \leq n-2 \) one has

\[
s_j s_{n-1,1} = s_{n-1} \cdots s_j s_{j+1} \cdots s_1 = s_{n-1,1}^\circ s_{n-1,1} \phi_1(s_j),
\]

which implies \( y s_{n-1,1}^\circ = s_{n-1,1}^\circ \phi_1(y) \) and completes the proof.

**Remark.** Let \( y = [a_1, \ldots, a_{n-1}, n] \in S_{n-1} \) then by (i) \( \hat{y} = [a_{n-1}, \ldots, a_1, n] \) and by 2.2.7 \( \phi_1(\hat{y}) = [1, a_{n-1}+1, \ldots, a_1+1] \).

3.4.3. Recall the notation \( (T \equiv c) \) and \( Tc \) from 2.4.10.

Given a cell \( C \) in \( S_{n-1} \) let \( \phi_1(C) \) be its displacement into \( S'_{n-1} \), that is \( \phi_1(C) = \{ \phi_1(y) : y \in C \} \).

**Proposition.** Consider a standard Young tableau \( T \in T_n \) with \( m \) corners \( \{ c_i \}_{i=1}^m \), and set \( q_i = T(c_i) \). Then

\[
C_T = \prod_{i=1}^m s_{q_{i-1,1}}^\circ \phi_1(C(T \equiv c_i)).
\]
Given $T \in T_n$. By 2.4.15 $T^\dagger(y) = T(\overline{y})$. Further by 3.4.2(i) $y = \overline{y}w_o$. Thus we can represent $C_T$ as $C_T = C_{T^\dagger}w_o$. Then by proposition 3.1.1

$$C_T = \prod_{i=1}^{m} s_{q_i,n-1}^< C_{(T^\dagger \cap c_i)}w_o. \quad (*)$$

Note that

$$C_{(T^\dagger \cap c)}w_o = s_{n-1,1}^> \phi_1(C_{(T \cap c)}). \quad (**)$$

Indeed for any $y \in C_{(T^\dagger \cap c)}$ one has by 3.4.2(iv)

$$yw_o = yy^{-1}s_{n-1,1}^> \phi_1(\overline{y}) = s_{n-1,1}^> \phi_1(\overline{y}).$$

Again by 2.4.15 applied to $S_{n-1}$ one has that $y \in C_{(T^\dagger \cap c)}$ iff $\overline{y} \in C_{(T \cap c)}$ and by 2.4.10 (* $C_{(T^\dagger \cap c)} = C_{(T \cap c)}$ which completes the proof of (**)).

Finally $s_{q_i,n-1}^< s_{n-1,1}^> = s_{q_i-1,1}^>$ which combined with (*) proves the proposition.

3.4.4. Let us describe a few simple properties connected to insertion and deletion by columns. Recall the notation from 2.4.10.

Let $T$ be a non-standard Young tableau and choose distinct $i, j \in \mathbb{N} \setminus \langle T \rangle$.

**Lemma.** Take $y$ such that $T(y) = T$. Then

(i) $(i \Rightarrow T) = T([i, y])$.

(ii) (($i \Rightarrow T) \iff T i) = T$ and $(T c \Rightarrow (T \iff c)) = T$ for any corner $c$ of $T$.

(iii) $(i \Rightarrow T) \downarrow j = (i \Rightarrow (T \downarrow j)) = T([i, y, j])$.

(iv) If $c, c'$ are distinct corners of $T$ then $((T \iff c) \uparrow c') = ((T \uparrow c') \iff c)$ and $c(T \iff c) = c^T, (T \iff c) = T$.\n
(v) If $c = c(r, s)$ is a corner of $T$ such that $|T^r| - |T^{r+1}| > 1$ then $c' = c(r, s - 1)$ is a corner of $(T \iff c)$ with an entry $T_s^c$ and $c'' = c(r, s - 1)$ is a corner of $(T \uparrow c)$. In that case one has $((T \iff c) \uparrow c') = ((T \uparrow c) \iff c'')$, $c_T = c(T \iff c)$, $T_c = (T \iff c)c''$.

**Proof.**

(i) By RS procedure one has $(T(y) \downarrow i) = T([y, i])$. By the Schensted -Schützenberger theorem one has $(T(y) \downarrow i)^\dagger = T([i, \overline{y}])$. Hence

$$(i \Rightarrow T(y)) = (T^\dagger \downarrow i)^\dagger = T^\dagger([\overline{y}, i]) = T([i, y])$$

as required.

(ii) is obvious and is similar to 2.4.15 (*)

(iii) Using twice (i) and RS insertion we get $(i \Rightarrow T) \downarrow j = (T([i, y]) \downarrow j) = T([i, y, j]) = (i \Rightarrow (T([y, j]) = (i \Rightarrow (T \downarrow j)).$
(iv), (v) are obtained as follows. Set \( p = c^T, p' = T'c', T' = ((T \uparrow c) \iff c'). \) One has
\[
(p' \Rightarrow (T' \downarrow p))^{(ii)} = ((p' \Rightarrow T') \downarrow p) = ((T \uparrow c) \downarrow p)^{2.4.15 T}
\]
On the other hand
\[
T'' = ((T \iff c') \uparrow c) = ((p' \Rightarrow (T' \downarrow p)) \iff c') \uparrow c) = ((T' \downarrow p) \uparrow c)^{2.4.15 T'}
\]
\[
\blacksquare
\]
3.4.5. The knowledge of sets of offsprings for \( S_{n-1} \) combined with the above two corner deletions \( \uparrow, \iff \) and two insertion procedures \( \downarrow, \Rightarrow \) determines the set of offsprings of a cell in \( S_n \). Indeed by proposition 3.4.3
\[
\mathcal{C}_T = \prod_{i=1}^{m} s_{q_{i-1},1}^{p_i} \phi_1 (\mathcal{C}_{(T \downarrow c_i)})
\]
and \( \phi_1 (\mathcal{C}_{(T \downarrow c_i)}) \) is a cell in \( S'_{n-1} \). So each \( w \in \mathcal{C} \) can be written as \( w = s_{q_{i-1},1}^{p_i} y \) for some \( y \in \phi_1 (\mathcal{C}_{(T \downarrow c_i)}) \). Then right multiplication of \( w \) by \( s_{n-1} \) can be considered as multiplication of the corresponding \( y \in \phi_1 (\mathcal{C}_{(T \downarrow c_i)}) \) and then replacement of the resulting cell \( \mathcal{C}_y s_{n-1} \in S'_{n-1} \) by the cell \( s_{q_{i-1},1}^{p_i} \mathcal{C}_y s_{n-1} \). This means that offsprings obtained by action of \( s_{n-1} \) can be read off from sets of offsprings in \( S'_{n-1} \). Summarizing

**Theorem.** Given a cell \( C_T \) in \( S_n \), with corners \( c_i, i = 1, 2, \ldots, m \). We can write
\[
C_T = \prod_{i=1}^{m} s_{p_{i,n-1}}^{c_i} \mathcal{C}_{(T \downarrow c_i)} = \prod_{i=1}^{m} s_{q_{i-1},1}^{p_i} \phi_1 (\mathcal{C}_{(T \downarrow c_i)})
\]
where \( p_i = c^T, q_i = T c_i \). Then
\[
\mathcal{D}(C_T) = \left\{ \bigcup_{i=1}^{m} \{s_{p_{i,n-1}}^{c_i} \mathcal{D}(\mathcal{C}_{(T \downarrow c_i)})\} \right\} \cup \left\{ \bigcup_{i=1}^{m} \{s_{q_{i-1},1}^{p_i} \phi_1 (\mathcal{D}(\mathcal{C}_{(T \downarrow c_i)})\} \right\}.
\]
3.4.6. For the completion we add the description of the set of offsprings for a given cell \( \mathcal{C}_T \) provided by the decomposition
\[
\mathcal{C}_T = \prod_{i=1}^{m} s_{q_{i-1},1}^{p_i} \phi_1 (\mathcal{C}_{(T \downarrow c_i)})
\]
alogous to that in theorem 3.3.3.

We need the following very simple corollary of the Schensted - Schützenberger theorem

**Lemma.** Given \( T, S \in \mathbf{T}_n \). Then \( T \ prec S \) iff \( S \ prec T \).

**Proof.**
By \( 2.1.15 \) \( T^{\uparrow} = T(\uparrow) \). By \( 2.1.3 \) for \( w, y \in S_n \) one has \( y \ prec w \) iff \( S(y) \subset S(w) \). It then suffices to note that by \( 2.2.4 \) \( S(x) = R^+ \setminus S(x), \) for all \( x \in W. \)
3.4.7. Let $T = T_{1,k}^{1}$ be a tableau with $m$ corners. It is immediate from lemma 3.4.6 that

$$\mathcal{D}(\mathcal{C}_T) = \{ \mathcal{C}_{S^\dagger} \mid S \in \mathcal{D}(\mathcal{C}_S) \}$$

Set

$$\mathcal{D}_v(\mathcal{C}_T) := \bigcup_{i=1}^{m} \{ s_{q_i - 1}^\geq \hat{\phi}_1(\mathcal{D}(\mathcal{C}_T \rightleftharpoons c_i)) \}$$
and

$$\mathcal{D}_v(T) := \bigcup_{i=1}^{m} \{ (q_i \Rightarrow S) : S \in \mathcal{D}(T \leftleftharpoons c_i) \}$$

For any corner $c_i \neq c_{(k, 1)}$ set $\hat{T}_{2,\infty}(c_i) := (T_{2,\infty} \leftleftharpoons c_i)$ and $d_i := t_{2,\infty} c_i$.

(i) If $d_i > T_{1}^k$ then set

$$\mathcal{T}S(c_i) = ((T_{1} + d_i), \hat{T}_{2,\infty}(c_i))$$

(ii) If $T_{1}^{k-1} < d_i < T_{1}^k$ then set

$$S =: (d_i \Rightarrow (T_{1}, \hat{T}_{2,\infty}(c_i)))$$

Set

$$\mathcal{T}S(c_i) = \begin{cases} S & \text{if } \text{sh } S > \text{sh } T \\ \emptyset & \text{otherwise} \end{cases}$$

As well set $\mathcal{T}S(c_i) := \emptyset$ if $d_i < T_{1}^{k-1}$ or $c_i = c_{(k, 1)}$. Set $\mathcal{D}_1(T) = \{ \mathcal{T}S(c_i) \}_{i=1}^{m}$.

**Theorem.** $\mathcal{D}(T) = \mathcal{D}_v(T) \cup \mathcal{D}_1(T)$.

**Proof.**

3.4.3 together with 3.3.1 and 3.4.6 show that $\mathcal{D}_v(T) \subset \mathcal{D}(T)$. To get all the set $\mathcal{D}(T)$ we must add to $\mathcal{D}_v(T)$ the offsprings, obtained by the right multiplication of the elements of $\mathcal{C}$ by $s_i$. Again by 3.4.6 $\mathcal{C}_T \leq \mathcal{C}_S$ if $S^\dagger \leq T^\dagger$. Note that $\overline{y} s_{n-1} = s_i \overline{y}$. Hence to complete the proof we must show the bijection $(T, \mathcal{T}S(c_i))^\dagger = (S_T(c_j), T')$.

Note that (i) corresponds 3.3.3(i) or (ii). Indeed, in this case

$$T' := \mathcal{T}S(c_i)^\dagger = \left( \begin{array}{c} (T_{1} + d_i) \\ \hat{T}_{2,\infty}^{\dagger} \end{array} \right), \quad \omega^i(T') = d_i$$

so that condition 3.3.3(i) or (ii) holds for $T'$ and

$$T'^\dagger = \left( \begin{array}{c} (T'^{\dagger} - \omega^i(T')) \\ (T_{2,\infty} \downarrow \omega^i(T')) \end{array} \right) = S_T(c_i).$$

A similar computation shows that $T$ satisfies (ii) iff $T' = \mathcal{T}S(c_i)^\dagger$ is such that $T'^\dagger = S_T(c_j)$ as defined by 3.3.3(**).
4. Closures of orbital varieties in \( sl_n \) and Induced Duflo Order

4.1. Projections and embeddings of Duflo offsprings.

4.1.1. Recall the notation from 2.1.8 and the notation of \( \pi_{i,j} \) from 2.4.16. For \( w \in W \) set \( w_I := \pi_I(w) \). This element can be regarded as an element of \( W_I \) and as an element of \( W \). Let \( C_{w_I} \) denote its cell in \( W \) and \( C^I_{w_I} \) denote its cell in \( W_I \). Respectively let \( V_{w_I} \) be the corresponding orbital variety in \( g \) and \( V^I_{w_I} \) be the corresponding orbital variety in \( l_I \).

All the projections are in correspondence on orbital varieties and cells, namely

**Theorem.** Let \( g \) be a reductive algebra. Let \( I \subset \Pi \).

(i) For every \( w \in W \) one has \( \pi_I(C(w)) \subset \pi_I(C(w_I)) = C^I(w_I) \).

(ii) For every orbital variety \( V_w \subset g \) one has \( \pi_I(V_w) = V^I_{w_I} \).

(iii) If \( g = sl_n \) and \( I = \Pi_{<i,j>} \) then one also has \( \pi_{i,j}(T(w)) = T(\pi_{i,j}(w)) \) where \( \pi_{i,j}(w) \) is considered as an element of \( S_{<i,j>} \).

**Proof.**

We prove (i), (ii) simultaneously. Note that lemma 2.1.8 gives

\[
\mathfrak{n}_x \cap \mathfrak{n} = \mathfrak{n}_x \cap \mathfrak{n}_x \mathfrak{n}_x.
\]

This implies that the projections \( \pi_I : \mathfrak{B} \to \mathfrak{B}_I, \pi_I : \mathfrak{n} \to \mathfrak{n}_x \) and \( \pi_I : W \to W_I \) satisfy the following condition

\[
\pi_I(\mathfrak{B}(\mathfrak{n} \cap \mathfrak{n})) = \pi_I(\mathfrak{B}(\mathfrak{n} \cap \pi_I(\mathfrak{n})) = B_I(\mathfrak{n}_x \cap \mathfrak{n}_x).
\]

By the continuity of the map \( \pi_I : \mathfrak{n} \to \mathfrak{n}_x \) in Zariski topology and (\( \ast \)) we have

\[
\pi_I(\mathfrak{B}(\mathfrak{n} \cap \mathfrak{n})) \subset \mathfrak{B}_I(\mathfrak{n}_x \cap \mathfrak{n}_x)
\]

(\( \ast \ast \))

Let us show the opposite inclusion. One has

\[
\mathfrak{B}(\mathfrak{n} \cap \mathfrak{n}) \supset \mathfrak{B}(\mathfrak{n}_x \cap \mathfrak{n}_x) \supset \mathfrak{B}_I(\mathfrak{n}_x \cap \mathfrak{n}_x)
\]

Using the fact that \( \pi_I|_{\mathfrak{n}_x} = \text{id} \) we get

\[
\pi_I(\mathfrak{B}(\mathfrak{n} \cap \mathfrak{n})) \supset \pi_I(\mathfrak{B}_I(\mathfrak{n}_x \cap \mathfrak{n}_x)) = \mathfrak{B}_I(\mathfrak{n}_x \cap \mathfrak{n}_x),
\]

which together with (\( \ast \ast \)) provides (ii).

Applying (ii) we get that for any \( y \in C(w) \), that is such that \( \mathfrak{B}(\mathfrak{n} \cap \mathfrak{n}) = \mathfrak{B}(\mathfrak{n} \cap \mathfrak{n}) \), one has

\[
\mathfrak{B}_I(\mathfrak{n}_x \cap \mathfrak{n}_x) = \mathfrak{B}_I(\mathfrak{n}_x \cap \mathfrak{n}_x)
\]

so that

\[
\pi_I(C(w)) \subset C_I(w_I).
\]

(\( \ast \ast \))

On the other hand consider \( y_I \in C_I(w_I) \). One has that

\[
\mathfrak{B}_I(\mathfrak{n}_x \cap \mathfrak{n}_x) = \mathfrak{B}_I(\mathfrak{n}_x \cap \mathfrak{n}_x)
\]
and by \[2.2.2\] one has that for any \(u \in W\)
\[
\cap^u n = (n_x \cap^u n_x) \oplus m_x.
\]

Note also that \(m_I\) is \(B_I\) stable. Thus
\[
B(n \cap^u I n) = \mu_I(B((n_x \cap^u n_x) \oplus m_x))
\]
by \((**\ast)\)
\[
= \mu_I(B(n_x \cap^u x n_x) + m_x)
\]
by \(B_I\) stability
\[
= \mu_I(B(n_x \cap^u x n_x) + m_x)
\]
by hypothesis
\[
= B(n \cap^u I n)
\]
as above.

Hence \(C_I(w_x) \subseteq C(w_x)\) which provides together with \((**\ast)\) the assertion (i).

(iii) is a straightforward corollary of \(2.1.8\) and \(2.4.13\).

4.1.2. As an immediate corollary of 4.1.1 (ii) we get that \(\pi_I\) respects the geometric order, namely

**Corollary.** Let \(g\) be a reductive algebra. Let \(I \subseteq \Pi\).

(i) If \(V, W\) are orbital varieties of \(g\) such that \(\overline{V} \subseteq W\) then \(\overline{\pi_I(V)} \subseteq \overline{\pi_I(W)}\).

(ii) If \(y, w \in W\) are such that \(\overline{y} \subseteq \overline{w}\) then \(\overline{\pi_I(y)} \subseteq \overline{\pi_I(w)}\).

4.1.3. Recall \(T^{<i,j>}\) from \(2.4.16\)

Induced Duflo order is also preserved under the projection \(\pi_I\) which is obvious from \(2.1.8\). Moreover if \(g = sl_n\) one has

**Proposition.** Let \(T, S \in T_n\). If \(S \in D(T)\) then for any \(i, j : 1 \leq i < j \leq n\) one has \(S^{<i,j>} \in D(T^{<i,j>}))\).

**Proof.**

Since \(S\) is an offspring of \(T\) there exists \(w \in C_T\) and \(\alpha_m \in \Pi_{n-1}\) such that \(ws_m \geq w\) and \(T(ws_m) = S\). It is enough to show the proposition for \(\pi_{1,n-1}\) and \(\pi_{2,n}\). Applying induction to these two cases we get the result for any \(\pi_{i,j}\). The proofs for \(\pi_{1,n-1}\) and \(\pi_{2,n}\) are exactly the same so we will show the proposition only for \(\pi_{1,n-1}\). By \(2.4.13\) (i) \(w = y s^m_{n-1,i}\) where \(y \in C(T_{n-1})\). One has

\[
w s_m = \begin{cases} 
ys_{m-1,i} & \text{if } m < i - 1 \\
ys_{n-1,i-1} & \text{if } m = i - 1 \\
ys_{m-1} s_{n-1,i} & \text{if } m > i
\end{cases}
\]

which implies the result. Note that in case \(m = i - 1\) we get \(S^{<1,n-1>} = T^{<1,n-1>}\) but by our convention in \(2.5.2\) every tableau is an offspring of itself.
4.1.4. Offsprings are also preserved under embeddings \( \downarrow, \Rightarrow: T_n \rightarrow T_{n+1} \). Formally

**Proposition.** Let \( T, S \in T_n \). If \( S \in \mathcal{D}(T) \) then \( (\phi_i(S) \downarrow i) \in \mathcal{D}(\phi_i(T) \downarrow i) \) and \( (i \Rightarrow \phi_i(S)) \in \mathcal{D}(i \Rightarrow \phi_i(T)) \).

**Proof.**

Since \( S \) is an offspring of \( T \) there exists \( w \in C_T \) and \( \alpha_m \in \Pi_{n-1} \) such that \( ws_m \overset{D}{\geq} w \) and \( T(ws_m) = S \). By 2.4.7 for any \( y \in S_n \) one has \( (\phi_i(T(y)) \downarrow i) = T(s_{i,n}^y) \) hence the first statement is just the reformulation of 3.3.1.

Using the duality \( T^\dagger(y) = T(\bar{y}) \) we get by 2.4.7 that for any \( y \in S_n \) one has \( (i \Rightarrow \phi_i(T(y))) = T(s_{i,n}^y) \) hence the second statement is equivalent to the first by 3.4.6.

4.1.5. Note that the last proposition provides us the following

**Corollary.** Given \( S, T \in T_n \) (resp. \( T_n \))

(i) Suppose \( T^1 = S^1 \). If \( S^{2,\infty} \) is an offspring of \( T^{2,\infty} \) then \( S \) is an offspring of \( T \).

(ii) Suppose \( T_1 = S_1 \). If \( S_{2,\infty} \) is an offspring of \( T_{2,\infty} \) then \( S \) is an offspring of \( T \).

**Proof.**

It is enough to show only (i) since (ii) is equivalent to (i) by duality \( T^\dagger(y) = T(\bar{y}) \) and 3.4.6.

If \( S^{2,\infty} \) is an offspring of \( T^{2,\infty} \) then 3.2.1 (i) and inductive use of 4.1.4 gives that

\[
S = ((\cdots (S^{2,\infty} \downarrow T_1^{\dagger}) \cdots) \downarrow \omega^1(T))
\]

is an offspring of \( T = ((\cdots (T^{2,\infty} \downarrow T_1^{\dagger}) \cdots) \downarrow \omega^1(T)) \).

4.1.6. The converse to corollary 4.1.5 is not true as it is shown by the following

**Example.** Regard \( T_{10} \)

\[
T = \begin{pmatrix}
1 & 2 & 3 & 9 \\
4 & 5 & 6 \\
7 & 8 & 10
\end{pmatrix}
\]
One can easily check that $T = T(w)$ where $w = [4, 7, 8, 5, 6, 10, 9, 1, 2, 3]$. By Remark 2 $w_s \geq w$ and

$$S = T(ws) = \begin{array}{ccc}
1 & 2 & 3 \\
4 & 5 & 8 \\
6 & 10 & \\
7 & & \\
\end{array}$$

Set

$$\hat{T} := T^{2,\infty} = \begin{array}{ccc}
4 & 5 & 6 \\
7 & 8 & 10 \\
\end{array}$$

and

$$\hat{S} := S^{2,\infty} = \begin{array}{ccc}
4 & 5 & 8 \\
6 & 10 & \\
7 & & \\
\end{array}.$$  

Let us show that $\hat{S} \not\geq \hat{T}$. Indeed $\text{sh} (\hat{S})$ is a descendant of $\text{sh} (\hat{T})$ by 2.3.3. So that if $\hat{S} \geq \hat{T}$ it is an offspring of $\hat{T}$. The only corner of $\hat{T}$ is $c(2, 3)$. Since $c^\hat{T} = 6$ and $6 \not< \hat{S}^1$ we must have $\hat{S} \in D_n(\hat{T}) = \{S_T(c_i)\}$. The straightforward computation gives that $S \neq S_T(c_i)$ thus $\hat{S} \not\geq \hat{T}$.

**Remark.** This example is important since in Part III we show that $\phi(\hat{T}) \subset \phi(\hat{S})$ which shows that induced Duflo order is strictly weaker than geometric order.

4.1.7. The following lemma describes the possible first rows of offsprings of a given $T$.

**Lemma.** If $S$ is an offspring of $T$ and $S^1 \neq T^1$ then $S^1$ is one of the follows

(i) $S^1 = (T^1 - T^1_j)$ for some $j$.
(ii) $S^1 = ((T^1 - T^1_i) + b)$ for some $j$ and $b > T^1_{j+1}$.
(iii) $S^1 = (T^1 \uparrow b)$.

In particular given $T$, $S$ then $T \not\leq S$ implies $T^1_j \leq S^1_j$.

**Proof.**

We prove this by induction on $n$. It holds for $n = 2$. Suppose it is true for $T_{n-1}$ and take $T \in T_n$. Set $l = |T^1|$.  

If $n$ occurs in the corner $c = c(i, j), i > 1$ of $T$ then $(T - n)^1 = T^1$. By 4.1.3 $(S - n)$ is an offspring of $(T - n)$ hence by the induction hypothesis the pair $(T - n)$, $(S - n)$ satisfies one of the conclusions. Suppose this pair satisfies (ii) or (iii). Then $|(S - n)^1| = l$ and since $|S^1| \leq l$ one has that $S^1 = (S - n)^1$ hence (ii) or (iii) is satisfied for $T, S$. Now suppose the pair $(T - n)$, $(S - n)$ satisfies (i). Then either $S^1 = (S - n)^1$ and $T, S$ satisfy (i) or $S^1 = ((S - n)^1 + n) = ((T - T^1_j) + n)$ and (ii) for $b = n$ is satisfied for $T, S$. 
Suppose \( n \) occurs in the corner \( c(1, l) \) of \( T \). By theorem 3.3.3 either there exist a corner \( c \) such that \( S = (\hat{S} \downarrow c^T) \) where \( \hat{S} \in \mathcal{D}(T \uparrow c) \) or \( T \) satisfies 3.3.3 (i) and \( S = S_T(c_1) \). If \( S = S_T(c_1) \) then \( S^1 = (T^1 - n) \) so (i) is satisfied. Otherwise by the induction hypothesis the pair \( \hat{T}, \hat{S} \) satisfies one of the conclusions. If \( c = c(1, l) \) then \( T^1 = (\hat{T}^1 + n) \), \( S^1 = (\hat{S}^1 + n) \), hence \( T, S \) satisfies the same conclusion as \( \hat{T}, \hat{S} \). It remains to consider \( c > c(1, l) \).

Suppose the pair \( \hat{T}, \hat{S} \) satisfies (i), this is \( \hat{S}^1 = (\hat{T}^1 - \hat{T}^1_j) \). Then

\[
S^1 = ((\hat{T}^1 - \hat{T}^1_j) \downarrow c^T) = \begin{cases} (T^1 - T^1_j) & \text{if } c^T \neq T^1_j, \\ (T^1 - T^1_{j+1}) & \text{if } c^T = T^1_j. \end{cases}
\]

In both cases the pair \( T, S \) satisfies (i). Finally suppose the pair \( \hat{T}, \hat{S} \) satisfies (ii) or (iii). Then \( |\hat{S}^1| = |T^1| = l \) and furthermore \( \hat{S}^1 \geq T^1_j \), \( \forall j \). So in particular \( \omega^i(\hat{S}) = n \) and \( c(1, l) \) is a corner of \( \hat{S} \). Now \( \hat{S} \in \mathcal{D}(\hat{T}) \) implies by 4.1.3 that \( (\hat{S} - n) \in \mathcal{D}(\hat{T} - n) \). Further by 4.1.3 \( S' = ((\hat{S} - n) \downarrow c^T) = (\hat{S} - n) \) is an offspring of \( T' = ((\hat{T} - n) \downarrow c^T) = (\hat{T} - n) \). Since \( |S^1| = |T^1| = l - 1 \) one has the pair \( T', S' \) satisfies (ii) or (iii) by induction hypothesis. Note that \( T^1 = (T^1 + n) \), \( S^1 = (S^1 + n) \), hence the pair \( T, S \) satisfies the same conclusion as \( T', S' \).

For example let

\[
T = \begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 & 8 \\
7 & 9 & 10 & & & & \\
\end{array}
\]

Then \( S \) is an offspring described in (i), \( U \) is an offspring described in (ii) and \( V \) is an offspring described in (iii) of lemma 4.1.7 where

\[
S = \begin{array}{cccc}
1 & 3 & 4 & 5 & 6 & 8 \\
2 & 7 & 9 & 10 & & & \\
\end{array},
\]

\[
U = \begin{array}{ccccc}
1 & 2 & 3 & 4 & 5 & 8 & 10 \\
6 & 9 & & & & & \\
7 & & & & & & & \\
\end{array}, \quad V = \begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 & 10 \\
7 & 8 & & & & & \\
9 & & & & & & & \\
\end{array}
\]

4.1.8. This lemma provides

**Proposition.** Given two diagrams \( D_1 < D_2 \) and a Young tableau \( T \) such that \( \text{sh}(T) = D_1 \) there exists \( S \) such that \( T \triangleright S \) and \( \text{sh}(S) = D_2 \).

**Proof.**
It is enough to show that for any $D_2$ descendant of $D_1$ there exist $S$ of shape $D_2$ such that $S \geq T$. $\text{sh}(S)$ being a descendant of $\text{sh}(T)$ forces $S$ to be an offspring of $T$.

We prove the statement by induction on $n$. For $S_2, S_3$ the statement is trivial. Suppose it holds for $n' < n$ and take $D_1, D_2 \in \mathcal{D}_n$.

If $D_1 = (n)$, $D_2 = (n-1, 1)$ then the assertion is trivial. Hence we can assume that $D_1 = (l_1, l_2, \ldots, l_k)$ where $l_1 < n$ and $D_2 = (\mu_1, \ldots)$.

If $\mu_1 = l_1$ then by induction hypothesis we can find an offspring $S'$ of $T_2^2, \infty$ such that $\text{sh}(S') = D_2^2, \infty$. By 4.1.7 one has $S_j^2 \geq T_j^2 > T_j^1$ hence

$$S = \left( \begin{array}{c} T_1^1 \\ T_2^2 + T_1^1 \\ T_3^3, \infty \end{array} \right).$$

is a standard Young tableau and by 4.1.3 $S$ is an offspring of $T$ and $\text{sh}(S) = D_2$.

We are reduced to the case $\mu_1 = l_1 - 1$. In that case we get a result with the help of case by case analysis.

(i) If $l_1 - l_2 \geq 2$ then $\mu_1 = l_1 - 1$, $\mu_2 = l_2 + 1$, $\mu_i = l_i$ for $i \geq 3$. We will show that in that case there exist $T_i^1$ such that

$$S = \left( \begin{array}{c} T_1^1 - T_i^1 \\ T_2^2 + T_i^1 \\ T_3^3, \infty \end{array} \right).$$

Indeed this is trivially true for $T_2$, $T_3$. Assume that this is true for $T_n'$ where $n' \leq n - 1$ and show for $T_n$.

If $D_1$ has more than 2 rows let $t := n - l_1 - l_2$. Consider $T^{1,2}$. By induction hypothesis there exist $y \in \mathcal{C}_T$ and $s_m$ such that $y s_m \geq y$ and

$$T(y s_m) = \left( \begin{array}{c} T_1^1 - T_i^1 \\ T_2^2 + T_i^1 \end{array} \right) := \hat{S}$$

Note that for any $j \hat{S}_j^2 \leq T_j^2$ thus $S = \left( \begin{array}{c} \hat{S}_j^1 \\ T_3^3, \infty \end{array} \right)$ is a standard tableau. By 3.2.3

(iv) $T = T([w_r(T^3, \infty), y])$ and $S = T([w_r(T^3, \infty), y s_m]) = T([w_r(T^3, \infty), y] s_{m+t})$. Hence $S \in \mathcal{D}(T)$.

Now consider that $D_1$ has only two rows.

a) If $\omega^1(T) = n$ then $T$ satisfies 3.3.3 (i) and

$$S_T(c_i) = \left( \begin{array}{c} T_1^1 - n \\ T_2^2 + n \end{array} \right)$$

satisfies $(*)$. 
b) If \( \omega^2 = n \) then consider \( \hat{T} = ((T \uparrow c(2, l_2)) \uparrow c(1, l_1)) \). One has
\[
\hat{T} = \left(\begin{array}{c}
T_1^1 - \omega^1(T) \\
T_2^2 - n
\end{array}\right)
\]
so that \( \text{sh}(\hat{T}) = (l_1 - 1, l_2 - 1) \) thus by induction hypothesis \( \hat{T} \) has an offspring \( \hat{S} \) of the form
\[
\hat{S} = \left(\begin{array}{c}
T_1^1 - \omega^1 - T_1^1 \\
T_2^2 - n + T_1^1
\end{array}\right)
\]
Further by 4.1.3
\[
S = ((\hat{S} \uparrow n) \uparrow \omega^1(T)) = \left(\begin{array}{c}
T_1^1 - T_1^1 \\
T_2^2 + T_1^1
\end{array}\right)
\]
is an offspring of \( T \). This completes case (i).

(ii) The case when \( l_2 = \cdot \cdot \cdot = l_j = l_i - 1, l_{j+1} = l_i - 2 \) for \( j \geq 2 \) and \( \mu_1 = \cdot \cdot \cdot = \mu_{j+1} = l_i - 1, \mu_i = l_i \) for \( i > j + 1 \) is proved in the same manner as (i).

a) First let us show that if \( D_1 \) has \( j + 1 \) rows then there exist an offspring \( S \) of shape \( \mu = (l_i - 1, \cdot \cdot \cdot, l_i - 1) \) such that
\[
S_{i+1}^j \leq T_{i+1}^j \quad \text{for all } i < l_i - 1 \quad \text{and} \quad S_{i-1}^m \leq T_{i-1}^m \quad \text{for all } m \leq j. \quad (*)
\]
Indeed this is true for \( T_3 \) we can assume that it is true for \( T_E, |E| < n \).

1. If \( \omega^i(T) > \omega^2(T) \) then consider
\[
T' = \left(\begin{array}{c}
T_2^2 + \omega^1(T) \\
T_3^3, \infty
\end{array}\right)
\]
By induction hypothesis there exists a required \( S' \). Then by lemma 4.1.7
\[
S_i^1 \geq T_i^1 \quad \text{for every } 1 \leq i \leq l - 1 \quad \text{so that}
\]
\[
S = \left(\begin{array}{c}
T_1^1 - \omega_1(T) \\
S_i^1
\end{array}\right)
\]
is a standard Young tableau. Using subsequently 4.1.4 we get that \( S = (\cdot \cdot \cdot (S' \downarrow T_i^2) \cdot \cdot \cdot \downarrow T_m^m) \) is an offspring of \( T = (\cdot \cdot \cdot (T' \downarrow T_i^1) \cdot \cdot \cdot \downarrow T_m^m) \) satisfying (*)

2. If \( \omega^i(T) < \omega^2(T) \) and \( \omega^j(T) < \omega^{j+1}(T) \) then consider
\[
T' = (\cdot \cdot \cdot (T \uparrow c(j + 1, l_i - 2)) \uparrow c(j, l_i - 1)) \cdot \cdot \cdot \uparrow c(1, l_i))
\]
this is \( T \) without segment \( s_{c(j+1, l_i - 2)} \). By induction hypothesis there exist an offspring \( S' \) of \( T' \) holding (*) for \( l_i' = l_i - 1 \). Then note that \( \omega^{j+1}(T) = \max < T \) hence \( \omega^{j+1}(T) > \omega^{j+1}(S') \) which together with (*) provides that
\[
S = (\cdot \cdot \cdot (S' \downarrow \omega^{j+1}(T)) \cdot \cdot \cdot \downarrow \omega^1(T)) = \left(\begin{array}{c}
\omega^1(T) \\
\vdots \\
\omega^{j+1}(T)
\end{array}\right)
\]
is an offspring of \( T \) satisfying (*).
3. If $\omega^1(T) < \omega^2(T)$ and $\omega^j(T) > \omega^{j+1}(T)$ then considering

$$T' = (\cdots((T \leftarrow c(j, l_1 - 1)) \leftarrow c(j + 1, l_1 - 2)) \cdots \leftarrow c(j + 1, 1)) = \begin{pmatrix} T^{1,j-1} \\
T^j - \omega^j(T) \end{pmatrix}$$

that is $T$ without segment of $T^\dagger : s_{c(l_1-1,j)}$ we get the result in the same way as in 2.

b) If $D_1$ has more than $j + 1$ rows then by induction hypothesis there exist $S'$ an offspring of $T^{1,j+1}$ satisfying $(\ast)$. In particular

$$S = \begin{pmatrix} S' \\
T_{j+2,\infty} \end{pmatrix}$$

is a standard tableau and the same considerations as in (i) provide that $S$ is an offspring of $T$.

This proposition shows that for every two nilpotent orbits $O_1, O_2 \in \mathfrak{sl}_n$ such that $O_2 \subsetneq O_1$ and every orbital variety $V_1$ attached to $O_1$ one can find $V_2$ attached to $O_2$ such that $V_1 \not\leq_D V_2$ and in particular $V_1 \not\leq_G V_2$.

4.2. Offsprings and descendants.

4.2.1. By 4.1.3 and 4.1.4 both projections and embeddings preserve Duflo offsprings. By 4.1.1 projections respects geometric order as well. As we will show in Part III embeddings also respect geometric order. But neither of them preserve Duflo (and geometric) descendants. First of all it is obvious from 3.3.3 that there exist $T, S$ such that $S$ is a descendant of $T$ (in both geometric and Duflo order) but $\pi_{i,j}(S) = \pi_{i,j}(T)$ for some $i, j$ (resp. $(\phi_i(S) \downarrow i) = (\phi_i(T) \downarrow i)$ for some $i$). As we show by the examples in the subsections below there exist $T, S$ such that $S$ is a descendant of $T$ (in both geometric and Duflo order) and $\pi_{i,j}(S) > \pi_{i,j}(T)$ (resp. $(\phi_i(S) \downarrow i) > (\phi_i(T) \downarrow i)$ for some $i$) but $\pi_{i,j}(S)$ (resp. $(\phi_i(S) \downarrow i)$) is not a descendant of $\pi_{i,j}(T)$ (resp. $(\phi_i(T) \downarrow i)$).

4.2.2. Let us first show that $S$ being a descendant of $T$ does not imply that $(\phi_i(S) \downarrow i)$ is a descendant of $(\phi_i(T) \downarrow i)$. The first such example occurs in embedding from $T_3$ into $T_4$. Consider

$$T = \begin{array}{c}
1 \\
2 \\
3 \\
\end{array}$$

and

$$S = \begin{array}{c}
1 \\
2 \\
3 \\
\end{array}$$
Note that $S > T$ in both geometric and Duflo order because $\nu_S = \{0\}$ and $S$ is a descendant of $T$ since $O_S$ is a descendant of $O_T$. Now consider

$$T' = (T \downarrow 4) = \begin{array}{c} 1 \\ 2 \\ 4 \\ 3 \end{array}, \quad S' = (S \downarrow 4) = \begin{array}{c} 1 \\ 4 \\ 2 \\ 3 \end{array} \quad \text{and} \quad P' = \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array}.$$ 

By 3.3.3 $P' = S_{T'}(c_1)$ and $S' = S_{P'}(c_1)$. Thus $T' < P' < S'$ (both in geometric and Duflo orders) so that $S'$ is not a descendant of $T'$.

4.2.3. Now let us show that $S$ being a descendant of $T$ does not imply $\pi_{2,n}(S)$ is a descendant of $\pi_{2,n}(T)$. The first such example occurs in projection from $T_5$ onto $T_4$. Consider

$$T = \begin{array}{c} 1 \\ 3 \\ 5 \\ 2 \\ 4 \end{array} \quad \text{and} \quad S = \begin{array}{c} 1 \\ 3 \\ 2 \\ 4 \\ 5 \end{array}.$$ 

Note that $T = T(s_1 s_3)$ and $S = T(s_1 s_3 s_4 s_3)$ so that $T^D < S$. Note also that if there exists $P$ such that $T^G \leq P^G \leq S$ then $\pi_{1,4}(T)^G \leq \pi_{1,4}(P)^G \leq \pi_{1,4}(S)$. Since $\pi_{1,4}(T) = \pi_{1,4}(S)$ one has $\pi_{1,4}(P) = \pi_{1,4}(T)$ which implies just by the shape consideration that $P = T$ or $P = S$. We get that $S$ is a descendant of $T$ (both in geometric and Duflo orders). Now consider

$$T' = \phi^{-1}_{1,2}T_5(T) = \begin{array}{c} 1 \\ 2 \\ 4 \\ 3 \end{array}, \quad S' = \phi^{-1}_{1,2}T_5(S) = \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \quad \text{and} \quad P' = \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array}.$$ 

By 3.3.3 one has $T'^D < P'^D < S'$.

**Remark.** $S$ is obtained from $T$ just by moving the box with the maximal number down to the first possible place. As we show in Part III such $S$ is always a descendant of $T$ (both in geometric and Duflo orders).

4.3. **Definition of Induced Duflo Order.**

4.3.1. One of the first general questions that arises about induced Duflo order is the following. Let $C_1$, $C_2$ be two cells such that $C_1^D \leq C_2^D$, can we always find representatives $x \in C_1$, $y \in C_2$ such that $x^D \leq y^D$? The answer is negative (which is natural enough) and we show this in the corresponding...
Example. Regard $S_5$. Consider

$$T = \begin{pmatrix} 1 & 2 & 5 \\ 3 & 4 \end{pmatrix}, \quad C_T = \left\{ [3, 1, 4, 2, 5], [3, 4, 1, 2, 5], [3, 1, 4, 5, 2], \right.\left. [3, 4, 1, 5, 2], [3, 4, 5, 1, 2] \right\}.$$ 

Consider $x = [3, 4, 1, 2, 5] \in C_T$, $s_3 x = [3, 4, 2, 1, 5] \overset{\text{D}}{>} x$. Hence $S = T(s_3 x) \overset{\text{D}}{>} T$ and

$$S = \begin{pmatrix} 1 & 4 & 5 \\ 2 & \end{pmatrix}, \quad C_S = \left\{ [3, 2, 1, 4, 5], [3, 2, 4, 1, 5], [3, 2, 4, 5, 1] \right\}.$$ 

Consider $y = [3, 2, 1, 4, 5] \in C_S$, $s_4 y = [3, 2, 1, 5, 4] s_4 \overset{\text{D}}{>} y$. Hence $U = T(s_4 y) \overset{\text{D}}{>} S \overset{\text{D}}{>} T$ and

$$U = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 \end{pmatrix}, \quad C_U = \left\{ [3, 2, 1, 5, 4], [3, 2, 5, 1, 4], [3, 5, 2, 1, 4] \right\}.$$ 

Since $t = [3, 1, 4, 2, 5]$ is a minimal element of $C_T$ and $z = [3, 5, 2, 4, 1]$ is a maximal element of $C_U$ it is enough to show that $t \nless z$. Indeed using 2.2.4 one gets

$$n \cap^t n = X_{1,2} \oplus X_{1,4} \oplus X_{1,5} \oplus X_{2,5} \oplus X_{3,4} \oplus X_{3,5} \oplus X_{4,5}$$
$$n \cap^z n = X_{2,4} \oplus X_{3,4} \oplus X_{3,5}$$

4.3.2. The previous example shows that in some sense the original definition of induced Duflo order is not good. Theorem 3.4.5 shows that induced Duflo order is the minimal partial order generalized by embeddings $\downarrow$ and $\Rightarrow$ from the natural order on $T_2$. 

INDEX OF NOTATION

| 1.1 | G, g |
| 1.2 | n, n−, h, O |
| 1.3 | V |
| 1.5 | B, W |
| 2.2.4 | $S'_n$ |
| 2.3.1 | $\lambda, \lambda^*, D_\lambda, D_n, J(u), O_\lambda, D_u$ |
| 2.3.2 | $D_\lambda \geq D_\mu$ |
| 2.4.4 | $T_n, \text{sh}(T)$ |
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