On the motion of three-dimensional compressible isentropic flows with large external potential forces and vacuum

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Abstract. We study the global existence and uniqueness of classical solutions to the three-dimensional compressible isentropic Navier-Stokes equations with vacuum and external potential forces which could be arbitrarily large provided the initial data is of small energy and the unique steady state is strictly away from vacuum. In particular, the solution may have large oscillations and contain vacuum states. For the case of discontinuous initial data, we also prove the global existence of weak solutions. The large-time behavior of the solution is obtained simultaneously. It is worthwhile mentioning that the compatibility condition on the initial data and the regularity condition of the external potential forces in the present paper are much weaker than those assumed in the existing literature.

Keywords. Compressible Navier-Stokes equations; Large external forces; Vacuum; Large oscillations; Global solutions; Large-time behavior

1 Introduction

The motion of three-dimensional viscous compressible isentropic flows occupying a domain \( \Omega \subset \mathbb{R}^3 \) is governed by the compressible Navier-Stokes equations:

\[
\begin{align*}
\rho_t + \text{div}(\rho u) &= 0, \\
(\rho u)_t + \text{div}(\rho u \otimes u) + \nabla P(\rho) &= \mu \Delta u + (\mu + \lambda) \nabla \text{div} u + \rho \tilde{f},
\end{align*}
\]

where the unknown functions \( \rho \geq 0, u = (u^1, u^2, u^3) \) and \( P(\rho) = A \rho^\gamma \) (\( A > 0, \gamma > 1 \)) are the fluid density, velocity and pressure, respectively, and \( \tilde{f} = \tilde{f}(x) \) is the known external force. The viscosity coefficients \( \mu \) and \( \lambda \) satisfy the physical restrictions \( \mu > 0 \) and \( 3\lambda + 2\mu \geq 0 \).

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Let $\Omega = \mathbb{R}^3$ and $\rho_\infty > 0$ be a fixed positive constant. For the external force in the form:

$$\tilde{f} = \nabla f \quad \text{with} \quad f = f(x)$$

we look for the solutions, $(\rho(x,t), u(x,t))$, to the Cauchy problem of (1.1), (1.2) with the far field behavior

$$(\rho, u)(x,t) \to (\rho_\infty, 0) \quad \text{as} \quad |x| \to \infty, \quad t > 0,$$

and the initial data

$$(\rho, u)(x,0) = (\rho_0, u_0)(x), \quad x \in \mathbb{R}^3.$$

The equations (1.1), (1.2) describe the conservation laws of mass and momentum, respectively. There has been a lot of literature on the existence and the large-time behavior of solutions to the compressible Navier-Stokes equations. The one-dimensional problem has been extensively studied, see [1, 19] and the references therein. For the multi-dimensional case, the local existence and uniqueness of classical solutions was proved in [29, 30] in the absence of vacuum, and recently, the local strong solutions were studied in [2, 4, 30] for the case that the initial density need not be positive and may vanish in open sets. The global classical solutions were first obtained by Matsumura-Nishida [25] when the initial data are close to a non-vacuum equilibrium in Sobolev space $H^3$, see also [26] for the exterior problem. Later, Hoff [10, 11] considered the weak solutions without vacuum and external forces for the discontinuous initial data. The global theory for the multi-dimensional compressible Navier-Stokes equations with “large data” is more delicate. Vaigant and Kazhikov [33] studied the global existence of classical solutions in two-dimensional periodic domain when the viscosity coefficients depend on density in a very specific way and the initial density is strictly positive. One of the most important breakthrough about the global theory of “large data” is the work of Lions [24] (see also Feireisl et al. [7, 8]), who first proved the global existence of weak solutions (the so-called “finite energy weak solutions”) to the initial/initial-boundary value problem of (1.1), (1.2) with generally large initial data when the initial energy is finite and the adiabatic exponent $\gamma$ is suitably large (i.e. $\gamma > 3/2$). Recently, under the additional assumptions that the viscosity coefficients $\mu$ and $\lambda$ satisfy $\mu > \max\{4\lambda, -\lambda\}$ and that the far field density is away from vacuum, Hoff et al. (cf. [12, 14]) obtained a new type of global weak solutions with small energy, which have extra regularity information compared with those large weak ones constructed by Lions [24] and Feireisl [7, 8]. More recently, Huang-Li-Xin [17] established the global existence and uniqueness of classical solutions to the Cauchy problem for the isentropic compressible Navier-Stokes equations in three-dimensional space with smooth initial data which are of small energy but possibly large oscillations; in particular, the initial density is allowed to vanish, even has compact support.

It has been mentioned in many papers (see, e.g. [9, 22, 27]) that the large external forces will significantly affect the dynamic motion of compressible flows and cause some serious difficulties in the mathematical study. In the following, we briefly recall some recent progress on the multi-dimensional compressible Navier-Stokes equations subject to external forces. Indeed, when both the initial perturbations and the external forces are sufficiently small, there have been many studies on the global existence and the large-time behavior of the compressible Navier-Stokes equations, see, for example, [5, 6, 20, 32, 34], and among others. For large external forces, Feireisl and Petzeltová [9], Novotný and Straškraba [28] proved for different boundary conditions that if the adiabatic exponent $\gamma$ is larger than $3/2$ and there exists a unique steady state, then the density of any global weak solution converges to the steady state density in some $L^q$-norm as time goes to infinity. Under the assumptions that the adiabatic exponent $\gamma$ is close to 1 and the external forces satisfy some decay properties in the far field, Matsumura and Yamagata [27]
studied the global existence and large-time behavior of weak solutions to the Cauchy problem of (1.1), (1.2) when the initial perturbations are suitably small in \( L^2 \cap L^\infty \) for density (away from vacuum) and in \( H^1 \) for velocity. Recently, under the same smallness assumptions on the initial perturbations, Li and Matsumura [22] succeeded in removing the smallness condition on \(|\gamma - 1|\) and the decay assumptions on the external force, and thus, improved the Matsumura-Yamagata’s result [27].

It is worth noting that among the papers [10, 11, 22, 27] mentioned above, the initial perturbation of density around a given positive state is small in \( L^\infty \), which in particular implies that the density is uniformly away from vacuum. However, as emphasized in many papers related to compressible fluid dynamics [2–4, 35], the possible presence of vacuum is one of the major difficulties when the problems of global existence, uniqueness and regularity of solutions to the compressible Navier-Stokes equations are concerned.

Thus, our main aim in this paper is to establish global well-posedness theorem and to study the large-time behavior of solutions for the compressible Navie-Stokes equations (1.1), (1.2) with large external forces and vacuum states.

Before stating the main results, we explain the notations and conventions used throughout this paper. For simplicity, we denote

\[
\int f \, dx = \int_{\mathbb{R}^3} f \, dx.
\]

For \( 1 \leq p \leq \infty \) and integer \( k \geq 0 \), we adopt the following simplified notations for the standard homogeneous and inhomogeneous Sobolev spaces:

\[
\begin{align*}
L^p &= L^p(\mathbb{R}^3), \\
W^{k,p} &= W^{k,p}(\mathbb{R}^3), \\
H^k &= W^{k,2}, \\
D^1 &= \{ u \in L^6(\mathbb{R}^3) \mid \|\nabla u\|_{L^2} < \infty \}, \\
D^{1,p} &= \{ u \in L^1_{\text{loc}}(\mathbb{R}^3) \mid \|\nabla u\|_{L^p} < \infty \}.
\end{align*}
\]

We first study the stationary problem of (1.1)–(1.5). In view of [22, Remark 2.1], it is known that the smooth steady solution, \((\rho_s(x), u_s(x))\), is unique and \( u_s \equiv 0 \). Hence, we infer from (1.2) that the steady state density \( \rho_s(x) \) satisfies

\[
\nabla P(\rho_s) = \rho_s \nabla f, \quad \rho_s(x) \rightarrow \rho_\infty \quad \text{as} \quad |x| \rightarrow \infty, \tag{1.6}
\]

which implies that \( \rho_s \) is uniquely determined by

\[
\int_{\rho_\infty}^{\rho_s} \rho^{-1} P'(\rho) \, d\rho = f(x).
\]

In order to avoid the vacuum states in \( \rho_s \), we suppose that

\[
-\int_0^{\rho_\infty} \frac{P'(\rho)}{\rho} \, d\rho < \inf_{x \in \mathbb{R}^3} f(x) \leq \sup_{x \in \mathbb{R}^3} f(x) < \int_{\rho_\infty}^{\rho_\infty} \frac{P'(\rho)}{\rho} \, d\rho. \tag{1.7}
\]

Thus, it follows from (1.6) and (1.7) that

**Proposition 1.1** Assume that \( f \in H^2 \) satisfy (1.7). Then the stationary problem (1.6) has a unique solution \( \rho_s = \rho_s(x) \) satisfying

\[
\rho_s - \rho_\infty \in H^2, \quad 0 < \rho \leq \inf_{x \in \mathbb{R}^3} \rho_s(x) \leq \sup_{x \in \mathbb{R}^3} \rho_s(x) \leq \bar{\rho} < \infty, \tag{1.8}
\]
where $\bar{\rho}, \check{\rho}$ are two positive constants depending only on $A, \gamma, \rho_\infty$, $\sup_{x \in \mathbb{R}^3} f(x)$, and $\inf_{x \in \mathbb{R}^3} f(x)$. Furthermore, if $f \in W^{2,q}$ with some $q \in (3, 6)$, then

$$\|\nabla \rho_s\|_{H^1 \cap W^{1,q}} \leq C, \quad (1.9)$$

where $C$ is a positive constant depending only on $A, \gamma, \rho_\infty$, $\inf_{x \in \mathbb{R}^3} f(x)$, and $\|f\|_{H^2 \cap W^{2,q}}$.

**Remark 1.1** For $P(\rho) = A \rho^\gamma$ with $A > 0, \gamma > 1$, it is easily seen from (1.6) and (1.7) that

$$\rho_s(x) = \left( \rho_\infty^{\gamma - 1} + \frac{\gamma - 1}{A \gamma} f(x) \right)^{\frac{1}{\gamma - 1}},$$

which, together with $f \in H^2$, implies that (1.8) holds provided

$$\inf_{x \in \mathbb{R}^3} f(x) > - \frac{A \gamma}{\gamma - 1} \rho_\infty^{\gamma - 1}.$$

**Remark 1.2** To study the large-time behavior of global weak solutions, the authors in [22, 27] technically required that $f \in H^3$ which particularly yields $\nabla \rho_s \in L^\infty$ and plays a key role in their analysis. Here we only assume that $f \in H^2$ which is much weaker than that in [22, 27] and is only used to guarantee (1.8). The additional condition $f \in W^{2,q} \supset H^3$ will be used to derive the high order estimates needed for the global classical existence.

With the steady state $(\rho_s, u_s)$ at hand, we can define the initial energy $C_0$ as follows:

$$C_0 \triangleq \int_{\mathbb{R}^3} \left( G(\rho_0) + \frac{1}{2} \rho_0|u_0|^2 \right) dx, \quad (1.10)$$

where $G(\cdot)$ is the potential energy density defined by

$$G(\rho) \triangleq \int_{\rho_s}^{\rho} \int_{\rho_s}^{\rho} \frac{P'(\xi)}{\xi} d\xi d\rho = \int_{\rho_s}^{\rho} \frac{P(\xi) - P(\rho_s)}{\xi^2} d\xi. \quad (1.11)$$

Now, our first result concerning the global existence of classical solution of (1.1)–(1.5) can be formulated as follows.

**Theorem 1.1** For $q \in (3, 6)$, let $f \in H^2 \cap W^{2,q}$ satisfy (1.7) and $\rho_s = \rho_s(x)$ be the steady state density of (1.6). For given positive numbers $M$ ($M$ may be arbitrarily large) and $\bar{\rho} \geq \check{\rho} + 1$, assume that the initial data $(\rho_0, u_0)$ satisfy

$$\begin{align*}
(\rho_0 - \rho_\infty, P(\rho_0) - P(\rho_\infty)) &\in H^2 \cap W^{2,q}, \quad u_0 \in H^2, \\
0 &\leq \inf_{x \in \mathbb{R}^3} \rho_0(x) \leq \sup_{x \in \mathbb{R}^3} \rho_0(x) \leq \bar{\rho}, \quad \|\nabla u_0\|_{L^2}^2 \leq M,
\end{align*} \quad (1.12)$$

and that the compatibility condition

$$- \mu \Delta u_0 - (\lambda + \mu) \text{div} u_0 + \nabla P(\rho_0) = \rho_0^{1/2} g$$

holds for some $g \in L^2$. Then there exists a positive constant $\varepsilon > 0$, depending only on $\mu, \lambda, A, \gamma, \rho_\infty, \bar{\rho}, M, \inf_{x \in \mathbb{R}^3} f(x)$, and $\|f\|_{H^2 \cap W^{2,q}}$, such that if the initial energy satisfies

$$C_0 \leq \varepsilon, \quad (1.15)$$
the Cauchy problem (1.1)–(1.5) has a unique global classical solution \((\rho, u)\), defined on \(\mathbb{R}^3 \times (0, T]\) for any \(0 < T < \infty\) and satisfying

\[
0 \leq \rho(x, t) \leq 2\bar{\rho} \quad \text{for all } x \in \mathbb{R}^3, \ t \geq 0,
\]

and

\[
\begin{cases}
(\rho - \rho_{\infty}, P(\rho) - P(\rho_{\infty})) \in C([0, T]; H^2 \cap W^{2,q}), \\
u \in C([0, T]; H^2) \cap L^\infty(\tau, T; H^3 \cap W^{3,q}), \\
u_t \in L^\infty(\tau, T; H^2) \cap H^1(\tau, T; H^1),
\end{cases}
\]

for any \(0 < \tau < T < \infty\). Moreover, one has the following large-time behavior:

\[
\lim_{t \to \infty} \left( \|\rho(\cdot, t) - \rho_s\|_{L^p} + \|\nabla u(\cdot, t)\|_{L^r} \right) = 0,
\]

with any \(p \in (2, \infty), \ r \in [2, 6)\).

Remark 1.3 It is clear from (1.17) that the solution obtained in Theorem 1.1 becomes a classical one for any positive time (\([15, \text{Remark 1.1}]\)). Moreover, although the solution has small energy, its oscillations could be arbitrarily large and the interior vacuum states are allowed.

Remark 1.4 Recently, in the absence of external force, Huang-Li-Xin \([17]\) (see also \([4]\)) proved the global existence of classical solutions of the Cauchy problem of (1.1), (1.2) with smooth non-negative initial density under the following compatibility conditions

\[
-\mu \Delta u_0 - (\lambda + \mu)\nabla \text{div} u_0 + \nabla P(\rho_0) = \rho_0 g
\]

with \(g \equiv g(x)\) satisfying

\[
\rho_0^{1/2} g \in L^2, \quad \nabla g \in L^2,
\]

which play a key role in the analysis of \([4, 17]\). It is worthwhile noting that the compatibility conditions (1.20), (1.21) are much stronger than that in (1.14).

It is well-known that the discontinuous solutions (namely, weak solutions) are fundamental and important in both the physical and mathematical theory. So, our second aim is to study the global weak solutions (see Definition 1.1) of (1.1)–(1.5).

Definition 1.1 A pair of functions \((\rho, u)\) is called a weak solution of (1.1)–(1.5), provided that

\[
\rho - \rho_{\infty} \in L^\infty_{\text{loc}}(0, \infty; L^2 \cap L^\infty), \quad u \in L^\infty_{\text{loc}}(0, \infty; L^4),
\]

and that for all test functions \(\psi \in \mathcal{D}(\mathbb{R}^3 \times (\infty, \infty))\) and \(j = 1, 2, 3\),

\[
\int_0^\infty \int \rho_0 \psi(\cdot, 0) dx + \int_0^\infty \int (\rho u_t + \rho u \cdot \nabla \psi) dx dt = 0,
\]

\[
\int_0^\infty \int \rho_0 u_0^j \psi(\cdot, 0) dx + \int_0^\infty \int (\rho u_t^j + \rho u^j u \cdot \nabla \psi + P(\rho) \psi x_j) dx dt = \int_0^\infty \int (\mu \nabla u^j \cdot \nabla \psi + (\mu + \lambda) \text{div} u \psi x_j) dx dt - \int_0^\infty \int \rho f x_j \psi dx dt.
\]
The global existence of weak solutions of (1.1)–(1.5) with discontinuous initial data can be stated as follows.

**Theorem 1.2** Let \( f \in H^2 \) satisfy (1.7) and \( \rho_s = \rho_s(x) \) be the steady state density of (1.6). For given positive numbers \( M \) (\( M \) may be arbitrarily large) and \( \bar{\rho} \geq \bar{\rho} + 1 \), assume that the initial data \((\rho_0, u_0)\) satisfy

\[
\begin{cases}
(\rho_0 - \rho_\infty, P(\rho_0) - P(\rho_\infty)) \in L^2 \cap L^\infty, & u_0 \in H^1, \\
0 \leq \inf_{x \in \mathbb{R}^3} \rho_0(x) \leq \sup_{x \in \mathbb{R}^3} \rho_0(x) \leq \bar{\rho}, & \|\nabla u_0\|_{L^2} \leq M.
\end{cases}
\]

(1.22)

Then there exists a positive constant \( \varepsilon > 0 \), depending only on \( \mu, \lambda, A, \gamma, \rho_\infty, \bar{\rho}, M, \inf_x f(x) \), and \( \|f\|_{H^2} \), such that if

\[
C_0 \leq \varepsilon,
\]

(1.23)

the Cauchy problem (1.1)–(1.5) has a global weak solution \((\rho, u)\) on \( \mathbb{R}^3 \times (0, \infty) \) in the sense of Definition 1.1 satisfying

\[
0 \leq \rho(x, t) \leq 2\bar{\rho} \quad \text{for all} \quad x \in \mathbb{R}^3, \ t \geq 0,
\]

(1.24)

\[
\rho - \rho_\infty \in C([0, \infty); L^2), \quad \rho u \in C([0, \infty); H^{-1}), \quad \nabla u \in L^2(0, \infty; L^2)
\]

(1.25)

and for any \( p \in (2, \infty) \),

\[
\lim_{t \to \infty} (\|\rho(\cdot, t) - \rho_s\|_{L^p} + \|u(\cdot, t)\|_{L^p \cap L^\infty}) = 0.
\]

(1.26)

**Remark 1.5** Theorem 1.2 extends those results in [10, 12, 22, 27] to the case that both the vacuum states and the large external forces are involved. Moreover, the regularity condition \( f \in H^2 \) on the external force is much weaker than the one \( f \in H^3 \) which is technically needed in [22, 27]. Indeed, more regularities of the solutions away from \( t = 0 \) can be obtained (cf. [10]).

**Remark 1.6** Similar to [17], the condition \( \|\nabla u_0\|_{L^2} \leq M \) in both (1.13) and (1.22) can be replaced by \( \|u_0\|_{H^\beta} \leq M \) with any \( \beta \in (1/2, 1] \).

We now comment on the analysis of this paper. Note that the compatibility condition (1.14) is much weaker than those in [4], i.e., (1.20), (1.21) (see Remark 1.4), we cannot apply the local existence theorem of classical solution in [4] to the problem considered. Indeed, we shall split the proof of Theorem 1.1 into three steps. Roughly speaking, we first use the well-known Matsumura-Nishida’s theorem (see Lemma 2.5) to guarantee the local existence of classical solutions with strictly positive initial density, then extend the local classical solutions globally in time just under the condition that the initial energy is suitably small (see Proposition 5.1), and finally let the lower bound of the initial density go to zero. So, to this end, we need some global a priori estimates which are independent of the lower bound of density. It turns out that the key issue in the proof is to derive both the time-independent upper bound of density and the time-dependent higher order estimates of \((\rho, u)\). To do this, we will borrow some ideas due from [10, 11, 17, 22]. However, because of the arbitrariness of external forces, the presence of vacuum states and the weaker compatibility condition (1.14), some new difficulties arise and the methods therein cannot be applied directly.
First, similar to that in [10, 17], we begin our proof with the careful initial layer analysis. To do this, we technically need the following modified “effective viscous flux”:

\[
F \triangleq \rho_s^{-1} [(\lambda + 2\mu)\text{div}u - (P(\rho) - P(\rho_s))],
\]

which was introduced by Li-Matsumura [22] and is different from the ones in [10, 11, 17, 27]. Basing on (1.2), (1.27) and the standard \( L^r \)-estimates of elliptic system, one can derive some subtle connections among the modified “effective viscous flux”, the gradient and material derivative of the velocity, and the pressure (see Lemma 3.3), which are important in the entire analysis, particularly in closing the time-independent energy estimates stated in Lemma 3.2. To obtain such connections, we need to deal with some difficulties induced by the large external forces. This will be done by making a full use of the mathematical structure of the steady state density (see (1.6)) and adopting an idea due to Huang-Li-Xin [16] (see (3.29) below), which enable us to control the terms associated with the pressure and the large external force by the deviation from the steady state density \( \rho \) to \( \rho_s \) only belong to \( H^2 \), which also makes the analysis here need to be more careful than that in [22].

However, unlike that in [22], it seems difficult to use directly this modified “effective viscous flux” \( F \) in (1.27) to prove the uniform upper bound of the density \( \rho \) since we only assume that \( f \in H^2 \) which implies that \( \nabla \rho_s \in L^s \) with any \( s \in [2, 6] \). Indeed, we overcome this difficulty by replacing \( F \) by the following standard “effective viscous flux” (see Lemma 3.8):

\[
\tilde{F} \triangleq (\lambda + 2\mu)\text{div}u - (P(\rho) - P(\rho_s)),
\]

which is in a similar form as the one defined in [10, 11, 17, 24]. A key observation is that for \( r, r_1 \in (3, 6], r_2 \in (6, \infty) \) and \( 1/r_1 + 1/r_2 = 1/r \), one has

\[
\|\nabla \tilde{F}\|_{L^r} \leq C \left( \|\rho\tilde{u}\|_{L^r} + \|\rho - \rho_s\|_{L^r} \right) \leq C \left( \|\rho\tilde{u}\|_{L^r} + \|\nabla f\|_{L^{r_1}} \|\rho - \rho_s\|_{L^{r_2}} \right),
\]

where “ \( \cdot \)” denotes the material derivative. Estimate (1.29) shows that we only need that \( \rho_s \) satisfies \( \nabla \rho_s \in L^2 \cap L^6 \). This is the main difference between \( F \) and \( \tilde{F} \) defined respectively in (1.27) and (1.28). By virtue of (1.29), we can apply the Sobolev embedding inequality to derive a desired estimate of \( \|F\|_{L^\infty} \), and thus, prove the pointwise boundedness of density by using the Zlotnik inequality (see Lemma 2.2).

There are also some new difficulties lying in the proof of the higher order time-dependent estimates. Indeed, to achieve the estimates on the derivatives of the solutions, we first prove the important estimates on the gradients of the density and velocity by solving a logarithm Gronwall inequality in a similar manner as that in [17, 18]. As a result, one can also easily obtain the \( L^2 \)-estimates for the second-order derivatives of density, pressure and velocity. However, due to the weaker compatibility condition in (1.14) (cf. (1.20), (1.21)), the method used in [17] cannot be applied any more to obtain further estimates needed for the existence of classical solutions. In fact, instead of the \( L^2 \)-method, we succeed in obtaining these classical estimates by deriving some desired \( L^q \)-estimates \( (3 < q < 6) \) on the higher-order time-space derivatives of the density and velocity, basing on some careful initial-layer analysis (see Lemmas 4.4, 4.6).

The rest of this paper is organized as follows. In Sect. 2, we first recall some known inequalities and facts which will be frequently used in our analysis. In Sect. 3, we derive the key a priori estimates of the weighted estimates on the gradient and the material derivative of the
Then there exists a positive constant $C$ satisfies the following conditions:

**Lemma 2.4**

The density is only nonnegative. Indeed, we have depending only on $q$ such that

$$
\|f\|_{L^6} \leq C \|\nabla f\|_{L^2},
$$

(2.1)

$$
\|g\|_{L^\infty} \leq C \|g\|_{L^q}^{q(r-3)/(3r+q(r-3))} \|\nabla g\|_{L^r}^{3r/(3r+q(r-3))}.
$$

(2.2)

The following Zlotnik inequality, whose proof can be found in [36], will be used later to prove the uniform-in-time upper bound of the density.

**Lemma 2.2**

Assume that the function $y \in W^{1,1}(0, T)$ solves the ODE system:

$$
y' = g(y) + b(t) \quad \text{on} \quad [0, T], \quad y(0) = y_0,
$$

where $b \in W^{1,1}(0, T)$ and $g \in C(\mathbb{R})$. If $g(\infty) = -\infty$ and

$$
b(t_2) - b(t_1) \leq N_0 + N_1(t_2 - t_1)
$$

for all $0 \leq t_1 \leq t_2 \leq T$ with some positive constants $N_0$ and $N_1$, then one has

$$
y(t) \leq \max\{y_0, \xi^*\} + N_0 < +\infty \quad \text{on} \quad [0, T],
$$

(2.4)

where $\xi^* \in \mathbb{R}$ is a constant such that

$$
g(\xi) \leq -N_1 \quad \text{for} \quad \xi \geq \xi^*.
$$

(2.5)

In order to obtain the time-dependent estimates of $\|\nabla u\|_{L^\infty}$ and $\|\nabla \rho\|_{L^2 \cap L^6}$, we need the following Beale-Kato-Majda-type inequality, the proof of which can be found in [18].

**Lemma 2.3**

For $3 < q < \infty$, assume that $\nabla u \in L^2 \cap D^{1,\alpha}$. Then there exists a constant $C > 0$, depending only on $q$, such that

$$
\|\nabla u\|_{L^\infty} \leq C \left(\|\text{div} u\|_{L^\infty} + \|\nabla \times u\|_{L^\infty}\right) \ln (e + \|\nabla^2 u\|_{L^q}) + C \|\nabla u\|_{L^2} + C.
$$

(2.6)

Since there is no vacuum state in the far field, it is easy to show that the density is only nonnegative. Indeed, we have

**Lemma 2.4**

Let $\rho_s = \rho_s(x)$ be the steady state density as in Proposition 1.1. Assume that $(\rho, v)$ satisfies the following conditions:

$$
0 \leq \rho \leq 2\tilde{\rho}, \quad (\rho - \rho_s, \rho^{1/2} v) \in L^2, \quad \nabla v \in L^2.
$$

Then there exists a positive constant $C$, depending only on $\rho, \tilde{\rho}$ and $\hat{\rho}$, such that

$$
\|v\|_{L^2} \leq C \left(\|\rho^{1/2} v\|_{L^2} + \|\rho - \rho_s\|_{L^2}^{2/3} \|\nabla v\|_{L^2}\right).
$$

(2.7)
Proof. Indeed, it is easy to see that
\[
\int |v|^2 dx \leq C(\rho) \int \rho^2 |v|^2 dx \leq C(\rho) \left( \int \rho^2 |v|^2 dx + \int (\rho - \rho_s)^2 |v|^2 dx \right)
\]
\[
\leq C(\rho, \tilde{\rho}) \int \rho |v|^2 dx + C(\rho) \left( \int |\rho - \rho_s|^3 dx \right)^{2/3} \left( \int |v|^6 dx \right)^{1/3}
\]
\[
\leq C(\rho, \tilde{\rho}) \|g^{1/2} v\|_{L^2}^2 + C(\rho, \tilde{\rho}) \|\rho - \rho_s\|_{L^2}^{4/3} \|\nabla v\|_{L^2}^2,
\]
which immediately proves (2.7). □

We end this section with the following local well-posedness theorem of classical solution to the problem (1.1)–(1.5) when the initial density is strictly away from vacuum (see, e.g. [29], and especially Matsumura-Nishida [25, Theorem 5.2]).

Lemma 2.5 Assume that the initial data \((\rho_0, u_0)\) satisfies
\[
(\rho_0 - \rho_\infty, u_0) \in H^3, \quad \inf_{x \in \mathbb{R}^3} \rho_0(x) > 0.
\]
Then there exist a small time \(T_0 > 0\) and a unique classical solution \((\rho, u)\) to the Cauchy problem (1.1)–(1.5) on \(\mathbb{R}^3 \times (0, T_0]\) such that
\[
\inf_{(x, t) \in \mathbb{R}^3 \times [0, T_0]} \rho(x, t) \geq \frac{1}{2} \inf_{x \in \mathbb{R}^3} \rho_0(x),
\]
and
\[
\begin{align*}
\rho - \rho_\infty &\in C([0, T_0]; H^3) \cap C^1([0, T_0]; H^2), \\
u &\in C([0, T_0]; H^3) \cap C^1([0, T_0]; H^1) \cap L^2(0, T_0; H^4),
\end{align*}
\]
where \(T_0 > 0\) may depend on \(\inf_{x \in \mathbb{R}^3} \rho_0(x)\).

3 Time-independent a priori estimates

This section is concerned with the time-independent (weighted) energy estimates and the uniform upper bound of density, which are essential for the proofs of Theorems 1.1 and 1.2. To do this, we assume that \((\rho, u)\), defined over \((0, T)\) with some positive \(T > 0\), is a smooth solution of the Cauchy problem (1.1)–(1.5). For simplicity, we introduce the following functionals:
\[
\Phi_1(T) \triangleq \sup_{0 \leq t \leq T} \sigma \int |\nabla u|^2 dx + \int_0^T \sigma \int \rho |\dot{u}|^2 dx dt,
\]
\[
\Phi_2(T) \triangleq \sup_{0 \leq t \leq T} \sigma^3 \int \rho |\dot{u}|^2 dx + \int_0^T \sigma^3 \int |\nabla \dot{u}|^2 dx dt
\]
and
\[
\Phi_3(T) \triangleq \sup_{0 \leq t \leq T} \int |\nabla u|^2 dx,
\]
where \(\sigma(t) \triangleq \{1, t\}\), and the symbol “˙” denotes the material derivative \(\dot{v} = v_t + u \cdot \nabla v\).

The main purpose of this section is to prove the following key a priori estimates.
Proposition 3.1 Assume that the conditions of Theorem 1.2 are satisfied. There exist two positive constants \( \tilde{\varepsilon} \) and \( K \), depending only on \( \mu, \lambda, A, \gamma, \rho, \bar{\rho}, \tilde{\rho}, \inf_{x \in \mathbb{R}^3} f(x), \| f \|_{H^2} \) and \( M \), such that if \((\rho, u)\) is a smooth solution of (1.1)–(1.5) satisfying

\[
\begin{align*}
0 &\leq \rho(x, t) \leq 2\tilde{\rho} \quad \text{for all} \quad (x, t) \in \mathbb{R}^3 \times [0, T], \\
\Phi_1(T) + \Phi_2(T) &\leq 2C_0^{1/2} \quad \text{and} \quad \Phi_3(\sigma(T)) \leq 3K,
\end{align*}
\] (3.4)

then one has

\[
\begin{align*}
0 &\leq \rho(x, t) \leq \frac{7}{4}\tilde{\rho} \quad \text{for all} \quad (x, t) \in \mathbb{R}^3 \times [0, T], \\
\Phi_1(T) + \Phi_2(T) &\leq C_0^{1/2} \quad \text{and} \quad \Phi_3(\sigma(T)) \leq 2K,
\end{align*}
\] (3.5)

provided that the initial energy \( C_0 \) defined in (1.10) satisfies

\[
C_0 \leq \tilde{\varepsilon}.
\] (3.6)

Proof. Proposition 3.1 follows directly from the following Lemmas 3.5, 3.6, and 3.8 with \( K \) and \( \tilde{\varepsilon} \) being the positive constants as in Lemmas 3.5 and 3.8, respectively. \( \square \)

Remark 3.1 We assume throughout this section that the initial data and the external force only satisfy the conditions of Theorem 1.2, and hence, the uniform-in-time estimates derived in this section can be used to study the existence and large-time behavior of global weak solutions as stated in Theorem 1.2.

For notational convenience, throughout this section we denote by \( C \) or \( C_i \) (\( i = 1, 2, \ldots \)) the generic positive constants which may depend on \( \mu, \lambda, A, \gamma, \rho, \bar{\rho}, \tilde{\rho}, \inf_{x \in \mathbb{R}^3} f(x), \| f \|_{H^2} \) and \( M \), but not on \( T \). We also sometimes write \( C(\alpha) \) to emphasize that \( C \) relies on \( \alpha \).

We start the proof with the following standard energy estimate.

Lemma 3.1 Let \((\rho, u)\) be a smooth solution to (1.1)–(1.3) on \( \mathbb{R}^3 \times (0, T] \). Then,

\[
\sup_{0 \leq t \leq T} \left( \int \frac{1}{2} \rho |u|^2 + G(\rho) \right) dx + \int_0^T \left( \mu |\nabla u|^2 + (\mu + \lambda) (\text{div} u)^2 \right) dx dt \leq C_0,
\] (3.7)

where \( G(\rho) \) is the potential energy density defined in (1.11).

Proof. Thanks to (1.6), the momentum equation (1.2) can be written as

\[
\rho u_t + \rho u \cdot \nabla u + \left( \nabla P(\rho) - \rho \rho_s^{-1} \nabla P(\rho_s) \right) = \mu \Delta u + (\mu + \lambda) \nabla (\text{div} u),
\]

which, multiplied by \( u \) and integrated by parts over \( \mathbb{R}^3 \times (0, t) \), yields

\[
\begin{align*}
\frac{1}{2} \int_0^t \rho |u|^2 dx dt &+ \int_0^t \int u \cdot \left( \nabla P(\rho) - \rho \rho_s^{-1} \nabla P(\rho_s) \right) dx ds \\
&+ \int_0^t \int (\mu |\nabla u|^2 + (\mu + \lambda) (\text{div} u)^2) dx ds = 0.
\end{align*}
\] (3.8)

After integrating by parts, one infers from (1.1) that

\[
\int_0^t \int u \cdot \left( \nabla P(\rho) - \rho \rho_s^{-1} \nabla P(\rho_s) \right) dx ds
\]
\[
\begin{align*}
&= \int_0^t \int \rho u \cdot \nabla \left( \int_{\rho_s}^\rho \frac{P'(\xi)}{\xi} d\xi \right) dx ds \\
&= - \int_0^t \int \text{div}(\rho u) \left( \int_{\rho_s}^\rho \frac{P'(\xi)}{\xi} d\xi \right) dx ds \\
&= \int_0^t \int \rho t \left( \int_{\rho_s}^\rho \frac{P'(\xi)}{\xi} d\xi \right) dx ds = \int G(\rho) dx \bigg|_0^t,
\end{align*}
\]

which, inserted into (3.8), leads to the desired estimate in (3.7). □

It is clear that for all \(0 \leq \rho \leq 2\tilde{\rho}\) and \(\rho \leq \rho_s \leq \tilde{\rho}\), there are positive constants \(C_1\) and \(C_2\) depending only on \(\rho\), \(\tilde{\rho}\), and \(\tilde{\rho}\), such that

\[C_1(\rho - \rho_s)^2 \leq G(\rho) \leq C_2(\rho - \rho_s)^2,\]

so that, it readily follows from (3.7) that

\[
\sup_{0 \leq t \leq T} \|\rho - \rho_s\|_{L^2} \leq CC_0. \tag{3.9}
\]

The next lemma is concerned with the temporary weighted \(L^2\)-estimates on the gradient and the material derivatives of the velocity, the proof of which will be concluded in Lemma 3.6 below. The idea of the proof mainly comes from \([10, 17, 22]\). However, due to the arbitrariness of external potential forces, the presence of vacuum states, and especially, the weaker regularity assumption of the external forces (i.e. \(f \in H^2\)), the analysis here needs to be more careful.

**Lemma 3.2** Let \((\rho, u)\) with \(\rho \in [0, 2\tilde{\rho}]\) be a smooth solution of (1.1)–(1.5) on \(\mathbb{R}^3 \times (0, T]\). Then there exists a positive constant \(C\), depending on \(\tilde{\rho}\), such that

\[
\Phi_1(T) \leq C \left( C_0 + \int_0^T \sigma \|\nabla u\|_{L^3}^3 dt \right), \tag{3.10}
\]

\[
\Phi_2(T) \leq C \left( C_0 + \Phi_1(T) + \int_0^T \sigma^3 \|\nabla u\|_{L^4}^4 dt \right). \tag{3.11}
\]

**Proof.** It follows from (1.2) and (1.6) that

\[
\rho \dot{u} - \mu \Delta u - (\mu + \lambda) \nabla \text{div} u + \nabla (P(\rho) - P(\rho_s)) = (\rho - \rho_s) \nabla f. \tag{3.12}
\]

For \(m \geq 0\), multiplying (3.12) by \(\sigma^m \dot{u}\) in \(L^2\) gives

\[
\begin{align*}
\sigma^m \int \rho |\dot{u}|^2 dx &= \mu \sigma^m \int \Delta u \cdot \dot{u} dx + (\mu + \lambda) \sigma^m \int \nabla \text{div} u \cdot \dot{u} dx \\
&\quad - \sigma^m \int \dot{u} \cdot \nabla (P(\rho) - P(\rho_s)) dx + \sigma^m \int (\rho - \rho_s) \dot{u} \cdot \nabla f dx \\
&\triangleq \sum_{i=1}^4 I_i, \tag{3.13}
\end{align*}
\]

where the right-hand side can be estimated term by term as follows. First, by the definition of the material derivative “\(\cdot\)” and integration by parts, we easily get that

\[
I_1 = -\frac{\mu}{2} \left( \sigma^m \|\nabla u\|_{L^2}^2 \right)_t + \frac{\mu}{2} \sigma^m \|\nabla u\|_{L^2}^2 - \mu \sigma^m \int \partial_i u^j \partial_i (u^k \partial_k u^j) dx
\]
so that, using (1.8), (2.1) and (3.9), we obtain after integrating by parts that

\[
\rho = 4 \leq d\frac{dt}{dt} = \int \int \mu + m - C\sigma \leq \int \int \sigma \leq \int \int \frac{m}{\sigma} \leq \int \int \frac{\sigma}{L^2} + C\sigma^m \leq \int \int \frac{\sigma}{L^3},
\]  

(3.14)

and similarly,

\[
I_2 \leq -\frac{\mu + \lambda}{2} (\sigma^m \leq \int \int \sigma^m \leq \int \int \sigma \leq \int \int \frac{m}{\sigma} \leq \int \int \frac{\sigma}{L^2} + C\sigma^m \leq \int \int \frac{\sigma}{L^3}.
\]  

(3.15)

Secondly, noticing that (1.1) implies

\[
P(\rho_t) + \text{div}(P(\rho)u) + (\rho P'(\rho) - P(\rho))\text{div}u = 0,
\]  

(3.16)

so that, using (1.8), (2.1) and (3.9), we obtain after integrating by parts that

\[
I_3 = \int [\sigma^m \leq \int \int \sigma^m \leq \int \int \sigma \leq \int \int \frac{m}{\sigma} \leq \int \int \sigma \leq \int \int \frac{m}{\sigma} \leq \int \int \frac{\sigma}{L^2} + C\sigma^m \leq \int \int \frac{\sigma}{L^3}],
\]  

(3.17)

Analogously, using the fact that \( \rho_t + \text{div}((\rho - \rho_s)u) + \text{div}(\rho_s u) = 0 \) due to (1.1), we have from (1.8), (2.1) and (3.9) that

\[
I_4 = \int [\sigma^m \leq \int \int \sigma^m \leq \int \int \sigma \leq \int \int \frac{m}{\sigma} \leq \int \int \sigma \leq \int \int \frac{m}{\sigma} \leq \int \int \frac{\sigma}{L^2} + C\sigma^m \leq \int \int \frac{\sigma}{L^3}]
\]  

(3.18)

Combining (3.13)–(3.15), (3.16) and (3.18) leads to

\[
\left(\frac{\mu}{2}\sigma^m \leq \int \int \sigma^m \leq \int \int \sigma \leq \int \int \frac{m}{\sigma} \leq \int \int \sigma \leq \int \int \frac{m}{\sigma} \leq \int \int \frac{\sigma}{L^2} \right)_t + \int \sigma^m \rho |\dot{u}|^2 dx
\]

\[
= \frac{d}{dt} \int [\sigma^m \leq \int \int \sigma^m \leq \int \int \sigma \leq \int \int \frac{m}{\sigma} \leq \int \int \sigma \leq \int \int \frac{m}{\sigma} \leq \int \int \frac{\sigma}{L^2} + C\sigma^m \leq \int \int \frac{\sigma}{L^3}],
\]  

(3.19)
where the first term on the right-hand side can be estimated as follows:

\[
\left| \int \sigma^m (\text{div}(P(\rho) - P(\rho_s)) + (\rho - \rho_s) u \cdot \nabla f) \, dx \right| \\
\leq C \sigma^m (\|\rho - \rho_s\|_{L^2} \|\nabla u\|_{L^2} + \|\rho - \rho_s\|_{L^2} \|u\|_{L^2} \|\nabla f\|_{L^2}) \\
\leq \frac{\mu}{4} \sigma^m \|\nabla u\|^2_{L^2} + C \sigma^m C_0.
\]

Thus, choosing \( m = 1 \) in (3.19), integrating it over \((0, T)\) and using (3.1) one gets (3.10).

To prove (3.11), operating \( \sigma^m \dot{u}^j (\partial_t + \text{div}(u \cdot \cdot \cdot )) \) to both sides of (3.2) and integrating the resulting equations over \( \mathbb{R}^3 \), we obtain after summing them up that

\[
\sum_{i=1}^{3} L_i \triangleq \sigma^m \int \dot{u}^j \left[ \partial_t (\rho \dot{u}^j) + \partial_k (\rho u^k \dot{u}^j) \right] \, dx - \mu \sigma^m \int \dot{u}^j \left[ \Delta u^j + \partial_k (u^k \Delta u^j) \right] \, dx \\
- (\mu + \lambda) \sigma^m \int \dot{u}^j \left[ \partial_j \text{div} u + \partial_k (u^k \partial_j u) \right] \, dx \\
= -\sigma^m \int \dot{u}^j \left[ (\partial_j P(\rho))_t + \partial_k \left( u^k \partial_j (P(\rho) - P(\rho_s)) \right) \right] \, dx \\
+ \sigma^m \int \dot{u}^j \left[ \rho \partial_j f + \partial_k \left( u^k (\rho - \rho_s) \partial_j f \right) \right] \, dx \triangleq \sum_{i=1}^{2} R_i. \tag{3.20}
\]

We now estimate each term in (3.20). First, by (1.1) one easily gets that

\[
L_1 = \frac{1}{2} \frac{d}{dt} \int \sigma^m \rho |\dot{u}|^2 \, dx - \frac{m}{2} \sigma^{-1} \sigma' \int \rho |\dot{u}|^2 \, dx. \tag{3.21}
\]

Recalling the definition of “\( \cdot \)”, we deduce from integration by parts that

\[
L_2 = \mu \sigma^m \int \left( |\nabla \dot{u}|^2 - \partial_k \dot{u}^j \partial_k (u \cdot \nabla u^j) + \partial_k \dot{u}^j u^k \Delta u^j \right) \, dx \\
= \mu \sigma^m \int \left( |\nabla \dot{u}|^2 - \partial_k \dot{u}^j \partial_k u^j u^j - \partial_k \dot{u}^j u^j \partial_k u^j - \partial_k \dot{u}^j u^j \partial_k \partial_l u^j \right) \, dx \\
= \mu \sigma^m \int \left( |\nabla \dot{u}|^2 - \partial_k \dot{u}^j \partial_k u^j \partial_k u^j + \partial_k \dot{u}^j \partial_k u^j \partial_k \partial_l u^j \right) \, dx \\
\geq \frac{7}{8} \mu \sigma^m \int |\nabla \dot{u}|^2 \, dx - C \sigma^m \int |\nabla u|^4 \, dx, \tag{3.22}
\]

where the Cauchy-Schwarz inequality was also used in the last inequality. Analogously,

\[
L_3 \geq (\mu + \lambda) \sigma^m \int (\text{div} \dot{u})^2 \, dx ds - C \sigma^m \int |\nabla \dot{u}| |\nabla u|^2 \, dx \\
\geq \sigma^m \int \left( (\mu + \lambda) (\text{div} \dot{u})^2 - \frac{\mu}{8} |\nabla \dot{u}|^2 \right) \, dx - C \sigma^m \int |\nabla u|^4 \, dx. \tag{3.23}
\]

In view of (3.16), we obtain after integrating by parts that

\[
R_1 = -\sigma^m \int \text{div} \dot{u} (\partial_t \text{div}(P(\rho)u) + (\rho P'(\rho) - P(\rho)) \text{div} u) \, dx \\
- \sigma^m \int \left( u^k \partial_k \text{div} \dot{u} + \partial_k \dot{u}^j \partial_j u^k \right) (P(\rho) - P(\rho_s)) \, dx
\]

\[
= -\sigma^m \int \text{div} \dot{u} \left[ (\rho P'(\rho) - P(\rho)) \text{div} u + \text{div} (P(\rho)u) \right] \, dx
- \sigma^m \int \partial_k \dot{u}^j \partial_j u^k (P(\rho) - P(\rho_s)) \, dx
\leq \frac{\mu}{8} \sigma^m \| \nabla \dot{u} \|^2_{L^2} + C \sigma^m \left( \| \nabla u \|^2_{L^2} + \| u \|^2_{L^6} \right)
\leq \frac{\mu}{8} \sigma^m \| \nabla \dot{u} \|^2_{L^2} + C \sigma^m \| \nabla u \|^2_{L^2},
\]
where we have also used (1.8) and (2.1). Similarly, by (1.1) we have
\[
R_2 = \sigma^m \int \rho u \cdot \nabla (\dot{u}^j \partial_j f) \, dx - \sigma^m \int \partial_k \dot{u}^j \left( u^k (\rho - \rho_s) \partial_j f \right) \, dx
\leq C \sigma^m \left( \| \nabla \dot{u} \|_{L^2} \| \nabla f \|_{L^3} \| u \|_{L^6} + \| \rho \dot{u} \|_{L^3} \| u \|_{L^6} \| \nabla^2 f \|_{L^2} \right)
\leq \frac{\mu}{8} \sigma^m \| \nabla \dot{u} \|^2_{L^2} + C \sigma^m \left( \| \rho^{1/2} \dot{u} \|^2_{L^2} + \| \nabla u \|^2_{L^2} \right),
\]
where we have also used Cauchy-Schwarz inequality and the following estimate:
\[
\| \rho \dot{u} \|_{L^3} \leq C \| \rho^{1/2} \dot{u} \|_{L^2}^{1/2} \| \dot{u} \|_{L^6}^{1/2} \leq C \| \rho^{1/2} \dot{u} \|_{L^2}^{1/2} \| \nabla \dot{u} \|_{L^2}^{1/2}.
\]
Putting the estimates of (3.21)–(3.25) into (3.20) gives
\[
\left( \sigma^m \| \rho^{1/2} \dot{u} \|^2_{L^2} \right) + \mu \sigma^m \| \nabla \dot{u} \|^2_{L^2}
\leq C \sigma^m \left( \| \nabla u \|^2_{L^2} + \| \nabla u \|^2_{1, 1} \right) + C \left( m \sigma^{m-1} \sigma' + \sigma^m \right) \| \rho^{1/2} \dot{u} \|^2_{L^2}.
\]
Thus, choosing \( m = 3 \) in (3.26) and integrating it over \((0, T)\) lead to (3.11) immediately.

Next, we prove some important connections among the modified "effective viscous flux", the density, and the gradient and material derivative of the velocity, which are crucial for our further analysis. To do this, we need to make a full use of the mathematical structure of steady state density \( \rho_s \) of (1.6) and the standard \( L^p \)-estimate of elliptic system, basing on a key observation due to Huang-Li-Xin [16] (see (3.29) below).

Lemma 3.3 Let \( f \in H^2 \) satisfy (1.7) and \( \rho_s = \rho_s(x) \) be the steady state density of (1.6). Assume that \( (\rho, u) \) with \( \rho \in [0, 2\bar{\rho}] \) is a smooth solution of (1.7)–(1.5) and \( F = F(x, t) \) is the modified "effective viscous flux" defined by (1.27). Then there exists a positive constant \( C \), depending only on \( \bar{\rho}, \bar{\rho}, \inf_{x \in \mathbb{R}^3} f(x) \) and \( \| f \|_{H^2} \), such that
\[
\| \nabla F \|_{L^2} + \| \nabla (\rho_s^{-1} \text{curl} u) \|_{L^2} \leq C \left( \| \rho \dot{u} \|_{L^2} + \| \nabla u \|_{L^3} + \| \rho - \rho_s \|_{L^6} \right),
\]
\[
\| \nabla u \|_{L^6} \leq C \left( \| \rho \dot{u} \|_{L^2} + \| \nabla u \|_{L^2} + \| \rho - \rho_s \|^2_{L^6} + \| \rho - \rho_s \|_{L^6} \right).
\]

Proof. The proofs are analogous to those in [10][17], however, since \( \rho_s - \rho_\infty \notin H^3 \), we modify the proof of \( \| \nabla F \|_{L^2} \) slightly. Indeed, as observed in [16], one can utilize (1.6) to get that
\[
\rho_s^{-1} (\nabla P(\rho) - \rho \nabla f)
= \rho_s^{-1} [\nabla (P(\rho) - P(\rho_s)) - \rho^{-1}(\rho - \rho_s)\nabla P(\rho_s)]
= \nabla [\rho_s^{-1} (P(\rho) - P(\rho_s))] + \rho_s^{-2} [P(\rho) - P(\rho_s) - (\rho - \rho_s)P'(\rho_s)] \nabla \rho_s.
\]
Thus, by virtue of (1.2), (1.27) and (3.29), we deduce
\[\rho^{-1}_s \dot{\rho} u - \nabla F + \mu \text{curl} (\rho^{-1}_s \text{curl} u)\]
\[= - [\lambda (\dot{\rho} + 2\mu) (\text{div} u) \nabla \rho^{-1}_s - \mu \nabla \rho^{-1}_s \times (\text{curl} u)] + \left[ P(\rho) - P(\rho_s) - P'(\rho_s)(\rho - \rho_s) \right] \nabla \rho^{-1}_s\]
\[\triangleq G_1 + G_2, \quad (3.30)\]
which gives
\[\Delta F = \text{div} (\rho^{-1}_s \dot{\rho} u - G_1 - G_2) \quad (3.31)\]
and
\[\mu \Delta (\rho^{-1}_s \text{curl} u) = \mu \text{div} (\rho^{-1}_s \text{curl} u) - \mu \text{curl} \text{curl} (\rho^{-1}_s \text{curl} u)\]
\[= \mu \nabla (\text{curl} u \cdot \nabla \rho^{-1}_s) - \text{curl} (\rho^{-1}_s \dot{\rho} u - G_1 - G_2). \quad (3.32)\]
Noticing that
\[\|G_1\|_{L^2} + \|G_2\|_{L^2} \leq C \left( \|\nabla \rho\|_{L^6} \|\nabla u\|_{L^3} + \|\nabla \rho_s\|_{L^6} \|\rho - \rho_s\|^2\|L^3\| \right)\]
\[\leq C \left( \|\nabla u\|_{L^3} + \|\rho - \rho_s\|_{L^6}^2 \right),\]
we apply the standard \(L^2\)-estimate to (3.31) and (3.32) to obtain
\[\|\nabla F\|_{L^2} + \|\nabla (\rho^{-1}_s \text{curl} u)\|_{L^2}\]
\[\leq C \left( \|\dot{\rho} u\|_{L^2} + \|G_1\|_{L^2} + \|G_2\|_{L^2} + \|\nabla \rho_s\|_{L^6} \|\nabla u\|_{L^3} \right)\]
\[\leq C \left( \|\dot{\rho} u\|_{L^2} + \|\nabla u\|_{L^3} + \|\rho - \rho_s\|_{L^6}^2 \right),\]
which gives (3.27).

On the other hand, using (1.27), (2.1), and (3.27), we find
\[\|\nabla u\|_{L^6} \leq C \left( \|\text{div} u\|_{L^6} + \|\text{curl} u\|_{L^6} \right)\]
\[\leq C \left( \|F\|_{L^6} + \|\rho - \rho_s\|_{L^6} + \|\rho^{-1}_s \text{curl} u\|_{L^6} \right)\]
\[\leq C \left( \|\nabla F\|_{L^2} + \|\rho - \rho_s\|_{L^6} + \|\nabla (\rho^{-1}_s \text{curl} u)\|_{L^2} \right)\]
\[\leq C \left( \|\dot{\rho} u\|_{L^2} + \|\rho - \rho_s\|_{L^6} + \|\nabla u\|_{L^2} \right) + \frac{1}{2} \|\nabla u\|_{L^6},\]
where we have used the fact that \(\|\nabla u\|_{L^6}^2 \leq C \|\nabla u\|_{L^2} \|\nabla u\|_{L^6}\) and Cauchy-Schwarz inequality in the last inequality. This proves (3.28), and thus, the proof of Lemma 3.3 is completed. \(\square\)

With the help of (3.4) and Lemma 3.3 we can prove

**Lemma 3.4** Let \((\rho, u)\) with \(\rho \in [0, 2\tilde{\rho}]\) be a smooth solution of (1.1)–(1.5) on \(\mathbb{R}^3 \times (0, T]\). Then there exists a positive constant \(\varepsilon_0 > 0\), depending on \(\tilde{\rho}\), such that
\[\int_0^T \sigma^3 \left( \|\nabla u\|_{L^4}^4 + \|\rho - \rho_s\|_{L^4}^4 + \|F\|_{L^4}^4 + \|\text{curl} u\|_{L^4}^4 \right) \, dt \leq C C_0. \quad (3.33)\]
provided \(\Phi_1(T) + \Phi_2(T) \leq 2C_0^{1/2} \) and \(C_0 \leq \varepsilon_0\).
Proof. Direct calculations from (1.1) and (1.27) show
\[
(r - \rho_s)_t + \frac{\rho_s}{2\mu + \lambda} (P(r) - P(\rho_s)) = -u \cdot \nabla (r - \rho_s) - (r - \rho_s) \text{div} u - u \cdot \nabla \rho_s - \frac{\rho_s^2}{2\mu + \lambda}.
\]
Multiplying this by \(4(r - \rho_s)^3\) in \(L^2\) and integrating by parts, we find
\[
\frac{d}{dt} \int (r - \rho_s)^4 \, dx + \frac{4}{2\mu + \lambda} \int \rho_s (P(r) - P(\rho_s)) (r - \rho_s)^3 \, dx 
\leq C \int ((r - \rho_s)^4 |\nabla u| + |r - \rho_s|^3 |u| |\nabla \rho_s| + |r - \rho_s|^3 |F|) \, dx
\leq C ||r - \rho_s||^2_{L^4}(\|\nabla u\|_{L^2}^2 + \|u\|_{L^3} \|\nabla \rho_s\|_{L^3}) + C ||r - \rho_s||^3_{L^4} \|F\|_{L^4}
\leq \delta ||r - \rho_s||^4_{L^4} + C(\delta) \left( \|\nabla u\|_{L^2}^2 + \|F\|_{L^4}^2 \right), \quad \delta > 0,
\]
where we have used (1.8), (2.1) and Cauchy-Schwarz inequality. Due to (1.8), one has
\[
\rho_s (P(r) - P(\rho_s)) (r - \rho_s)^3 = (r - \rho_s)^4 \rho_s \int_0^1 P'(\alpha \rho + (1 - \alpha) \rho_s) \, d\alpha
\geq C(r - \rho_s)^4
\]
for some positive constant \(C\) depending only on \(A, \gamma, \rho, \bar{\rho}\) and \(\bar{\rho}\). Hence, choosing \(\delta > 0\) suitably small, multiplying (3.34) by \(\sigma^3\) and integrating it over \((0, T)\), we infer from (3.7) and (3.9) that
\[
\int_0^T \sigma^3 \|r - \rho_s\|^4_{L^4} \, dt \leq CC_0 + C \int_0^T \sigma^3 \|F\|^4_{L^4} \, dt.
\]
By (1.27), we have
\[
\|\nabla u\|_{L^4} \leq C (\|\text{div} u\|_{L^4} + \|\text{curl} u\|_{L^4}) \leq C (\|F\|_{L^4} + \|r - \rho_s\|_{L^4} + \|\rho_s^{-1} \text{curl} u\|_{L^4}),
\]
which, combining with (3.35) and (3.27), yields that
\[
\int_0^T \sigma^3 \left(\|\nabla u\|^4_{L^4} + \|r - \rho_s|^4_{L^4} + \|F|^4_{L^4}\right) \, dt
\leq CC_0 + C \int_0^T \sigma^3 \left(\|F|^4_{L^4} + \|\rho_s^{-1} \text{curl} u|^4_{L^4}\right) \, dt
\leq CC_0 + C \int_0^T \sigma^3 \left(\|\nabla u|^2_{L^2} \|\nabla F|^2_{L^2} + \|\rho_s^{-1} \text{curl} u\|_{L^2} \|\nabla (\rho_s^{-1} \text{curl} u)|^2_{L^2}\right) \, dt
\leq CC_0 + C \int_0^T \sigma^3 \left(\|\nabla u|^2_{L^2} + C_0^{1/2}\right) \left(\|\rho_s^{-1} \text{curl} u|^2_{L^2} + \|\rho - \rho_s\|^2_{L^4}\right) dt
\leq CC_0 + C \int_0^T \left(\sigma^{1/2} \|\nabla u|^2_{L^2} + C_0^{1/2}\right) \left(\sigma^3 \|\rho_s^{-1} \text{curl} u|^2_{L^2}\right)^{1/2} \sigma \|\rho u\|^2_{L^2} dt
\leq CC_0 + C \int_0^T \|\nabla u|^2_{L^2} \left(\sigma^{1/2} \|\nabla u|^2_{L^2}\right)^{5/2} \|\rho - \rho_s\|^2_{L^4} + \sigma^3 C_0^{1/2} \|\rho - \rho_s\|^2_{L^4} \, dt
+ CC_0^{1/2} \int_0^T \sigma^3 \left(\|\nabla u|^4_{L^4} + \|\rho - \rho_s\|^4_{L^4}\right) \, dt
\leq CC_0 + CC_0^{1/2} \int_0^T \sigma^3 \left(\|\nabla u|^4_{L^4} + \|\rho - \rho_s\|^4_{L^4}\right) \, dt,
where in the last inequality we have used $\Phi_1(T) + \Phi_2(T) \leq 2C_0^{1/2}$. This directly gives (3.33) provided $C_0 \leq \varepsilon_0 \triangleq \min\{1, (2C)^{-2}\}$.

**Lemma 3.5** Let $(\rho, u)$ with $\rho \in [0, 2\tilde{\rho}]$ be a smooth solution of $(1.1)$–$(1.5)$ on $\mathbb{R}^3 \times (0, T)$. Then there exist positive constants $K$ and $\varepsilon_1$, depending on $\tilde{\rho}$ and $M$, such that

$$\Phi_3(\sigma(T)) + \int_0^{\sigma(T)} \|\rho^{1/2} \dot{u}\|_{L_2}^2 dt \leq 2K,$$

(3.36)

provided $\Phi_3(\sigma(T)) \leq 3K$ and $C_0 \leq \varepsilon_1$.

**Proof.** Choosing $m = 0$ in (3.19) and integrating it over $[0, \sigma(T)]$, we deduce from (3.7), (3.9), (3.28) and Cauchy-Schwarz inequality that

$$\Phi_3(\sigma(T)) + 2 \int_0^{\sigma(T)} \|\rho^{1/2} \dot{u}\|_{L_2}^2 dt$$

$$\leq C(\varepsilon_0 \triangleq \min\{1, (2C)^{-2}\}) + 2 \int_0^{\sigma(T)} \|\rho^{1/2} \dot{u}\|_{L_2}^2 dt$$

$$\leq C(\varepsilon_0 \triangleq \min\{1, (2C)^{-2}\}) + 2 \int_0^{\sigma(T)} \|\rho^{1/2} \dot{u}\|_{L_2}^2 dt$$

$$\leq \Phi_3(\sigma(T)) + \int_0^{\sigma(T)} \|\rho^{1/2} \dot{u}\|_{L_2}^2 dt$$

with a positive constant $K \triangleq C(\varepsilon_0 \triangleq \min\{1, (2C)^{-2}\})$ depending on $\tilde{\rho}, C_0$ and $M$. As a result, we immediately obtain (3.36) provided $\Phi_3(\sigma(T)) \leq 3K$ and $C_0 \leq \varepsilon_1 \triangleq (9CK)^{-1}$.

**Lemma 3.6** Assume that $(\rho, u)$ satisfying (3.4) with $K > 0$ as in Lemma 3.5 is a smooth solution of $(1.1)$–$(1.5)$ on $\mathbb{R}^3 \times (0, T)$. Then, there exists a positive constant $\varepsilon_2 > 0$, depending on $\tilde{\rho}$ and $M$, such that

$$\Phi_1(T) + \Phi_2(T) \leq C_0^{1/2},$$

(3.37)

provided $C_0 \leq \varepsilon_2$.

**Proof.** By (3.14), (3.19) and (3.28), we find

$$\sigma \|\nabla u\|_{L_6}^{2/3} \leq C\sigma (\|\rho \dot{u}\|_{L_6} + \|\nabla u\|_{L_6} + \|\rho - \rho_s\|_{L_6} + \|\rho - \rho_s\|_{L_6})^{2/3}$$

$$\leq C \left( C_0^{1/6} + C_0^{2/9} + C_0^{1/9} \right) \leq CC_0^{1/9}.$$

Thus, using (3.7), (3.9), (3.28), (3.33) and (3.36), one derives from (3.10) and (3.11) that

$$\Phi_1(T) + \Phi_2(T) \leq CC_0 + C \int_{\sigma(T)}^T \sigma \|\nabla u\|_{L_3}^3 dt + C \int_0^{\sigma(T)} \sigma \|\nabla u\|_{L_3}^3 dt$$

$$\leq CC_0 + C \int_{\sigma(T)}^T (\|\nabla u\|_{L_6}^2 + \sigma^3 \|\nabla u\|_{L_4}^4) dt + C \int_0^{\sigma(T)} \sigma \|\nabla u\|_{L_6}^2 \|\nabla u\|_{L_6}^{3/2} dt$$
\[
\begin{align*}
&\leq CC_0 + CC_0^{1/9} \int_0^{\sigma(T)} \|\nabla u\|^7_{L^6} (\|\rho \dot{u}\|_{L^2} + \|\nabla u\|_{L^2} + \|\rho - \rho_s\|^2_{L^6} + \|\rho - \rho_s\|^4_{L^6})^{5/6} dt \\
&\leq CC_0 + CC_0^{2/3} \left( \int_0^{\sigma(T)} (\|\rho \dot{u}\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\rho - \rho_s\|^2_{L^6} + \|\rho - \rho_s\|^4_{L^6}) dt \right)^{5/12} \\
&\leq C_1 C_0^{2/3} \leq C_0^{1/2},
\end{align*}
\]
provided \(C_0 \leq \varepsilon_2 \triangleq \min\{\varepsilon_0, \varepsilon_1, C_1^{-6}\}. The proof of Lemma 3.6 is thus complete.

To complete the proof of Proposition 3.1 it remains to prove the uniform upper bound of the density. To do this, we first need the following refined estimates.

**Lemma 3.7** Assume that \((\rho, u)\) satisfying (3.4) with \(K > 0\) as in Lemma 3.5 is a smooth solution of (1.1)–(1.5) on \(\mathbb{R}^3 \times (0, T)\). Then
\[
\sup_{0 \leq t \leq T} \|\nabla u\|^2_{L^2} + \int_0^T \|\rho^{1/2} \dot{u}\|^2_{L^2} dt \leq C,
\]
(3.38)
\[
\sup_{0 \leq t \leq T} \left( \sigma \|\rho^{1/2} \dot{u}\|^2_{L^2} \right) + \int_0^T \sigma \|\nabla \dot{u}\|^2_{L^2} dt \leq C,
\]
(3.39)
provided \(C_0 \leq \varepsilon_2\).

**Proof.** We only have to prove (3.39) since the estimate (3.38) is an immediate result of (3.36) and (3.37). Choosing \(m = 1\) in (3.26) and integrating it over \((0, T)\) yields that
\[
\sup_{0 \leq t \leq T} \left( \sigma \|\rho^{1/2} \dot{u}\|^2_{L^2} \right) + \int_0^T \sigma \|\nabla \dot{u}\|^2_{L^2} dt \\
\leq C \left( C_0 + \int_0^T \|\rho^{1/2} \dot{u}\|^2_{L^2} dt + \int_0^T \sigma \|\nabla u\|^4_{L^4} dt \right) \\
\leq C + C \int_0^{\sigma(T)} \sigma \|\nabla u\|^2_{L^2} \|\nabla u\|^3_{L^6} dt \\
\leq C + C \int_0^{\sigma(T)} \sigma \left( \|\rho \dot{u}\|^3_{L^2} + \|\nabla u\|^3_{L^2} + \|\rho - \rho_s\|^6_{L^6} + \|\rho - \rho_s\|^3_{L^6} \right) dt \\
\leq C + C \sup_{0 \leq t \leq T} \left( \sigma^{1/2} \|\rho^{1/2} \dot{u}\|^2_{L^2} \right),
\]
where we have used (3.9), (3.33), (3.28) and (3.38). Combining this with Young inequality immediately leads to (3.39).

We are now ready to prove the uniform upper bound of the density, which is in fact the key to extend the local smooth solution to be global and will be proved by modifying the arguments in [16, 17, 21] basing on the Zlotnik inequality (cf. Lemma 2.2) and the standard “effective viscous flux” \(\tilde{F}\) (see (1.28)) in a similar form as the one in [10, 17, 24].

**Lemma 3.8** Assume that \((\rho, u)\) satisfying (3.4) with \(K > 0\) as in Lemma 3.5 is a smooth solution of (1.1)–(1.5) on \(\mathbb{R}^3 \times (0, T)\). Then there exists a positive constant \(\tilde{\varepsilon} > 0\), depending on \(\tilde{\rho}\) and \(M\), such that for all \((x, t) \in \mathbb{R}^3 \times [0, T]\),
\[
\rho(x, t) \leq \frac{7}{4} \tilde{\rho},
\]
provided \(C_0 \leq \tilde{\varepsilon}\).
Proof. Let $\tilde{F}$ be the standard “effective viscous flux” defined by (1.28). Then (1.1) can be written in the form:

$$ D_t \rho = g(\rho) + b'(t), $$

where

$$ D_t \triangleq \partial_t + u \cdot \nabla, \quad g(\rho) \triangleq \frac{A \rho}{\lambda + 2\mu} (\rho^\gamma - \rho_0^\gamma), \quad b(t) \triangleq -\frac{1}{\lambda + 2\mu} \int_0^t \rho \tilde{F} d\tau. $$

In order to apply Lemma 2.2, we have to deal with $b(t)$ since $\lim_{\rho \to +\infty} g(\rho) = -\infty$. To do this, we first observe from (1.2) and (1.6) that

$$ \Delta \tilde{F} = \text{div}(\rho \tilde{u} - (\rho - \rho_s) \nabla f). \quad (3.40) $$

Applying the standard $L^p$-estimate to the elliptic problem (3.40) gives

$$ \|\nabla \tilde{F}\|_{L^4} \leq C \|\rho \tilde{u}\|_{L^4} + C \|(\rho - \rho_s) \nabla f\|_{L^4} $$

$$ \leq C \|\rho \tilde{u}\|_{L^2}^{1/4} \|\rho \tilde{u}\|_{L^4}^{3/4} + C \|\rho - \rho_s\|_{L^2} \|\nabla f\|_{L^6} $$

$$ \leq C \sigma^{-1/8} \|\nabla \tilde{u}\|_{L^2}^{3/4} + CC_0^{1/12}, \quad (3.41) $$

where we have also used (1.8), (3.9) and (3.39). It thus follows from (1.28), (2.2), (3.9), (3.38) and (3.41) that

$$ \|\tilde{F}\|_{L^\infty} \leq C \|\tilde{F}\|_{L^2}^{1/7} \|\nabla \tilde{F}\|_{L^4}^{6/7} $$

$$ \leq C \left( \|\nabla u\|_{L^2}^{1/7} + \|\rho - \rho_s\|_{L^2}^{1/7} \right) \left( \sigma^{-1/8} \|\nabla \tilde{u}\|_{L^2}^{3/4} + C_0^{1/12} \right)^{6/7} $$

$$ \leq C \sigma^{-3/28} \|\nabla \tilde{u}\|_{L^2}^{9/14} + CC_0^{1/14}. \quad (3.42) $$

For $0 \leq t_1 < t_2 \leq \sigma(T) \leq 1$, we deduce from (3.37), (3.39) and (3.42) that if $C_0 \leq \varepsilon_2$, then

$$ |b(t_2) - b(t_1)| $$

$$ \leq C \int_0^{\sigma(T)} \|\tilde{F}\|_{L^\infty} dt $$

$$ \leq CC_0^{1/14} + C \int_0^{\sigma(T)} \sigma^{-4/7} (\sigma \|\nabla u\|_{L^2}^2)^{1/4} \left( \sigma^3 \|\nabla \tilde{u}\|_{L^2}^2 \right)^{1/4} dt $$

$$ \leq CC_0^{1/14} + C \left( \int_0^{\sigma(T)} \sigma \|\nabla u\|_{L^2}^2 dt \right)^{1/4} \left( \int_0^{\sigma(T)} \sigma^3 \|\nabla \tilde{u}\|_{L^2}^2 dt \right)^{1/4} $$

$$ \leq CC_0^{1/14} + C \Phi_2^{1/14}(T) \leq CC_0^{1/28}. $$

Therefore, for any $t \in [0, \sigma(T)]$, one can choose $N_0, N_1$ in (2.3) and $\xi^*$ in (2.5) as follows:

$$ N_0 = CC_0^{1/28}, \quad N_1 = 0, \quad \xi^* = \bar{\rho}. $$

Then, due to the facts that $\underline{\rho} \leq \rho_s \leq \bar{\rho}$ (see Proposition 1.1) and

$$ g(\xi) \leq -\frac{A \xi}{\lambda + 2\mu} (\xi^\gamma - \bar{\rho}^\gamma) \leq -N_1 = 0, \quad \forall \xi \geq \xi^* = \bar{\rho}, $$

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it follows from (2.4) that (keeping in mind that $0 \leq \rho_0 \leq \bar{\rho}$ and $\bar{\rho} \geq \bar{\rho} + 1$)

$$\sup_{0 \leq t \leq \sigma(T)} \|\rho(t)\|_{L^\infty} \leq \max\{\bar{\rho}, \bar{\rho}\} + N_0 \leq \bar{\rho} + CC_0^{1/28} \leq \frac{3}{2} \bar{\rho},$$ (3.43)

provided the initial energy $C_0$ is chosen to be such that

$$C_0 \leq \min\{\varepsilon_2, \varepsilon_3\} \quad \text{with} \quad \varepsilon_3 \triangleq \left(\frac{\bar{\rho}}{2C}\right)^{28}.$$ 

For any $\sigma(T) \leq t_1 < t_2 \leq T$, we have from (3.9), (3.37) and (3.38) that

$$|b(t_2) - b(t_1)| \leq C \int_{t_1}^{t_2} \|\ddot{F}\|_{L^\infty} dt \leq C \int_{t_1}^{t_2} \|\ddot{F}\|^{1/7}_{L^2} \|\nabla \ddot{F}\|^{6/7}_{L^4} dt \leq CC_0^{1/14}(t_2 - t_1) + C \int_{t_1}^{t_2} \|\rho\|_{L^6} \|\dot{u}\|_{L^6}^{3/4} \|\rho - \rho_s\|_{L^2} \|\nabla f\|_{L^6}^{6/7} dt \leq \left(CC_0^{1/14} + \frac{A}{2(2\mu + \lambda)}\right)(t_2 - t_1) + C \int_{\sigma(T)}^{T} \sigma^3(\|\rho^{1/2}\ddot{u}\|_{L^2} + \|\rho\dot{u}\|_{L^6}) dt \leq \frac{A}{2\mu + \lambda}(t_2 - t_1) + CC_0^{1/2},$$

where in the last inequality we have chosen $C_0$ to be such that

$$C_0 \leq \min\{\varepsilon_2, \varepsilon_3, \varepsilon_4\} \quad \text{with} \quad \varepsilon_4 \triangleq \left(\frac{A}{2C(2\mu + \lambda)}\right)^{14}.$$ 

Therefore, for any $t \in [\sigma(T), T]$, we can choose $N_0$ and $N_1$ in (2.3) and $\xi^*$ in (2.5) as follows:

$$N_0 = CC_0^{1/2}, \quad N_1 = \frac{A}{2\mu + \lambda}, \quad \xi^* = \bar{\rho} + 1.$$ 

Then it is easy to check that for any $\xi \geq \xi^*$,

$$g(\xi) \leq -\frac{A\xi}{2\mu + \lambda}(\xi^\gamma - \bar{\rho}^\gamma) \leq -N_1 = -\frac{A}{2\mu + \lambda},$$

and hence, one has from (2.4) and (3.43) that

$$\sup_{\sigma(T) \leq t \leq T} \|\rho(t)\|_{L^\infty} \leq \max\left\{\frac{3}{2} \bar{\rho}, \bar{\rho} + 1\right\} + N_0 \leq \frac{3}{2} \bar{\rho} + CC_0^{1/2} \leq \frac{7}{4} \bar{\rho},$$ (3.44)

provided the initial energy $C_0$ satisfies

$$C_0 \leq \bar{\varepsilon} \triangleq \min\{\varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5\} \quad \text{with} \quad \varepsilon_5 \triangleq \left(\frac{\bar{\rho}}{4C}\right)^2.$$ 

The combination of (3.43) with (3.44) proves Lemma 3.8. □
4 Time-dependent higher-order estimates

In this section, we prove the higher-order estimates of the smooth solution \((\rho, u)\) to (1.1)-(1.5), which are needed for the existence of classical solutions. From now on, we always assume that the initial energy \(C_0\) satisfies (3.6) and that the conditions of Theorem 1.1 are satisfied. We also denote by \(C\) the various positive constants which may depend on \(\rho_0, u_0, \|g\|_{L^2}, f, \lambda, \mu, A, \gamma, \bar{\rho}, \tilde{\rho}, M, \) and \(T\) as well.

We begin with the \(L^2\)-estimate on the material derivative of the velocity.

Lemma 4.1 For any given \(T > 0\), there exists a positive constant \(C(T)\) such that

\[
\sup_{0 \leq t \leq T} \|\rho^{1/2} \dot{u}\|^2_{L^2} + \int_0^T \|\nabla \dot{u}\|^2_{L^2} dt \leq C(T).
\]

**Proof.** Choosing \(m = 0\) in (3.26) and integrating it over \((0, T)\), we have

\[
\sup_{0 \leq t \leq T} \int \rho |\dot{u}|^2 dx - \int \rho |\dot{u}|^2(x, 0) dx + \int_0^T \|\nabla \dot{u}\|^2_{L^2} dt
\]

\[
\leq C \int_0^T \left( \|\nabla u\|^2_{L^2} + \|\rho^{1/2} \dot{u}\|^2_{L^2} \right) dt + C \int_0^T \|\nabla u\|^4_{L^1} dt
\]

\[
\leq C + C \int_0^T \|\nabla u\|^3_{L^6} \|\nabla u\|^2_{L^6} dt
\]

\[
\leq C + C \sup_{0 \leq t \leq T} \|\rho^{1/2} \dot{u}\|^2_{L^2},
\]

where we have used (3.7), (3.9), (3.28) and (3.38). Combining this with Young inequality gives (4.1), since the compatibility condition (1.14) implies that \(\sqrt{\rho} \dot{u}|_{t=0} = \sqrt{\rho_0}(\nabla f - g) \in L^2\) is well defined.

Next, similar to that in [17,18], we utilize the Beale-Kato-Majda-type inequality (see Lemma 2.3) to prove the important estimates on the gradients of \((\rho, u)\).

Lemma 4.2 There exists a positive constant \(C = C(T)\) such that

\[
\sup_{0 \leq t \leq T} (\|\nabla \rho\|_{L^2 \cap L^6} + \|\nabla u\|_{H^1}) + \int_0^T \|\nabla u\|_{L^\infty} dt \leq C(T).
\]

**Proof.** Since \(\mathcal{L} = -\mu \Delta - (\mu + \lambda) \nabla \text{div}\) is a strong elliptic operator (see [3] for instance), applying the \(L^p\)-estimate of elliptic system to (1.2) gives that for \(2 \leq p \leq 6\),

\[
\|\nabla^2 u\|_{L^p} \leq C (\|\rho \dot{u}\|_{L^p} + \|\nabla \rho\|_{L^p} + \|\nabla f\|_{L^p}).
\]

By integration by parts, we easily derive from (1.1) that

\[
\frac{d}{dt} \|\nabla \rho\|_{L^p} \leq C \|\nabla u\|_{L^\infty} \|\nabla \rho\|_{L^p} + C \|\nabla^2 u\|_{L^p}
\]

\[
\leq C (1 + \|\nabla u\|_{L^\infty}) \|\nabla \rho\|_{L^p} + C (1 + \|\rho \dot{u}\|_{L^p}),
\]

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where (4.3) was used in the second inequality. Using (3.38), (4.3), and (2.1), we deduce from (2.6) that
\[
\|\nabla u\|_{L^\infty} \leq C \left( \|\text{div} u\|_{L^\infty} + \|\text{curl} u\|_{L^\infty} \right) \ln (e + \|\rho u\|_{L^6} + \|\nabla \rho\|_{L^6}) + C
\]
\[
\leq C + C \left( \|\text{div} u\|_{L^\infty} + \|\text{curl} u\|_{L^\infty} \right) \ln (e + \|\nabla \dot{u}\|_{L^2})
\]
\[
+ C \left( \|\text{div} u\|_{L^\infty} + \|\text{curl} u\|_{L^\infty} \right) \ln (e + \|\nabla \rho\|_{L^6}).
\] (4.5)

Set
\[
\Phi(t) \triangleq e + \|\nabla \rho\|_{L^6}, \quad \Psi(t) \triangleq 1 + \left( \|\text{div} u\|_{L^\infty} + \|\text{curl} u\|_{L^\infty} \right) \ln (e + \|\nabla \dot{u}\|_{L^2}) + \|\nabla \dot{u}\|_{L^2}.
\]
Then it follows from (4.4) with \(p = 6\) and (4.5) that
\[
\Phi'(t) \leq C \Psi(t) \Phi(t) \ln \Phi(t),
\]
which, together with the fact that \(\Phi(t) > 1\), implies
\[
\frac{d}{dt} \ln \Phi(t) \leq C \Psi(t) \ln \Phi(t).
\] (4.6)

Applying the standard \(L^p\)-estimate to (3.31) and (3.32) yields that
\[
\|\nabla F\|_{L^6} + \|\nabla (\rho_s^{-1} \text{curl} u)\|_{L^6} \leq C \left( \|\rho u\|_{L^5} + \|G_1\|_{L^6} + \|G_2\|_{L^6} + \|\nabla u \nabla \rho_s^{-1}\|_{L^6} \right)
\]
\[
\leq C \left( \|\nabla \dot{u}\|_{L^2} + \|\nabla \rho_s\|_{L^\infty} \|\nabla u\|_{L^6} + \|\nabla \rho_s\|_{L^6} \right)
\]
\[
\leq C \left( \|\nabla \dot{u}\|_{L^2} + 1 \right),
\]
where we have also used (1.9), (3.9), (3.28), (3.38), (4.1) and Lemma 2.1. Thus,
\[
\int_0^T \Psi(t) \, dt \leq C \int_0^T \left( \|\nabla \dot{u}\|^2_{L^2} + \|\text{div} u\|^2_{L^\infty} + \|\text{curl} u\|^2_{L^\infty} \right) \, dt
\]
\[
\leq C + C \int_0^T \left( \|P(\rho) - P(\rho_s)\|^2_{L^\infty} + \|F\|^2_{L^\infty} + \|\rho_s^{-1} \text{curl} u\|^2_{L^\infty} \right) \, dt
\]
\[
\leq C + C \int_0^T \left( \|F\|^2_{L^2} + \|\text{curl} u\|^2_{L^2} + \|\nabla F\|^2_{L^6} + \|\nabla (\rho_s^{-1} \text{curl} u)\|^2_{L^6} \right) \, dt
\]
\[
\leq C + C \int_0^T \|\nabla \dot{u}\|^2_{L^2} \, dt \leq C,
\] (4.7)

and consequently, it follows from (4.6) and Gronwall’s inequality that
\[
\sup_{0 \leq t \leq T} \|\nabla \rho(t)\|_{L^6} \leq \sup_{0 \leq t \leq T} \Phi(t) \leq C.
\] (4.8)

As a result of (4.5), (4.7) and (4.8), we obtain
\[
\int_0^T \|\nabla u\|_{L^\infty} \, dt \leq C + C \int_0^T \Psi(t) \, dt \leq C.
\] (4.9)

Using (4.1) and (4.9), we infer from (4.3) with \(p = 2\) and Gronwall’s inequality that
\[
\sup_{0 \leq t \leq T} \|\nabla \rho(t)\|_{L^2} \leq C,
\]
which, together with (4.1) and (4.3), also implies that \(\|\nabla u\|_{H^1} \leq C\). The proof of Lemma 4.2 is therefore completed.

By virtue of Lemmas 4.1 and 4.2, we easily obtain
Lemma 4.3. For any given $T > 0$, it holds that

\[ \sup_{0 \leq t \leq T} \| \rho^{1/2} u_t \|^2_{L^2} + \int_0^T \| \nabla u_t \|^2_{L^2} dt \leq C(T), \]  
\[ (4.10) \]

\[ \sup_{0 \leq t \leq T} (\| \nabla \rho \|_{H^1} + \| \nabla P \|_{H^1}) + \int_0^T \| \nabla^2 u \|^2_{H^1} dt \leq C(T), \]  
\[ (4.11) \]

\[ \sup_{0 \leq t \leq T} (\| \rho_t \|_{H^1} + \| P_t \|_{H^1}) + \int_0^T (\| \rho_{tt} \|^2_{L^2} + \| P_{tt} \|^2_{L^2}) dt \leq C(T). \]  
\[ (4.12) \]

**Proof.** First, (4.10) follows directly from Lemmas 4.1 and 4.2.

Next, we prove (4.11). Since $P(\rho) = A\rho^\gamma$ satisfies

\[ P_t + u \cdot \nabla P + \gamma P \text{div} u = 0, \]  
\[ (4.13) \]

from which and (1.1) we have by (4.2) and direct computations that

\[ \frac{d}{dt} (\| \nabla^2 \rho \|^2_{L^2} + \| \nabla^2 P \|^2_{L^2}) \leq C \| \nabla u \|_{L^\infty} (\| \nabla^2 \rho \|^2_{L^2} + \| \nabla^2 P \|^2_{L^2}) + C \| \nabla^3 u \|_{L^2} (\| \nabla^2 \rho \|^2_{L^2} + \| \nabla^2 P \|^2_{L^2}) \]  
\[ + C \| \nabla^2 u \|_{L^3} (\| \nabla \rho \|_{L^6} + \| \nabla P \|_{L^6}) (\| \nabla^2 \rho \|^2_{L^2} + \| \nabla^2 P \|^2_{L^2}) \leq C (1 + \| \nabla u \|_{L^\infty}) (\| \nabla^2 \rho \|^2_{L^2} + \| \nabla^2 P \|^2_{L^2}) + C \| \nabla^2 u \|^2_{H^1}. \]  
\[ (4.14) \]

Using (4.2), (4.10) and Lemma 2.1, we deduce from (4.2) and the standard $L^2$-estimate of elliptic system that

\[ \| \nabla^2 u \|_{H^1} \leq C (\| \rho u \|_{H^1} + \| \rho u \cdot \nabla u \|_{H^1} + \| \nabla P \|_{H^1} + \| \rho \nabla f \|_{H^1}) \]  
\[ \leq C (1 + \| \nabla u \|_{L^2} + \nabla \| \rho \|_{L^6} \| u \|_{L^6} + \| \nabla^2 P \|_{L^2}) \]  
\[ \leq C (1 + \| \nabla u \|_{L^2} + \| \nabla^2 P \|_{L^2}). \]  
\[ (4.15) \]

Thus, putting (4.15) into (4.14) and using Gronwall inequality, we immediately arrive at (4.11) since it holds that $\| \nabla u \|_{L^\infty} + \| \nabla u \|_{L^2} \leq C(T)$ due to (4.2) and (4.10).

Finally, we prove (4.12). Thanks to (4.2) and (4.11), it follows from (1.1) and (4.13) that

\[ \| \rho_t \|_{L^2} + \| P_t \|_{L^2} \leq C \| u \|_{L^\infty} (\| \nabla \rho \|_{L^2} + \| \nabla P \|_{L^2}) + C \| \nabla u \|_{L^2} \leq C, \]  
\[ (4.16) \]

\[ \| \nabla \rho_t \|_{L^2} + \| \nabla P_t \|_{L^2} \leq C (\| \nabla^2 u \|_{L^2} + C \| u \|_{L^\infty}) (\| \nabla^2 \rho \|_{L^2} + \| \nabla^2 P \|_{L^2}) \]  
\[ + C \| \nabla u \|_{L^3} (\| \nabla \rho \|_{L^6} + \| \nabla P \|_{L^6}) \leq C, \]  
\[ (4.17) \]

where Sobolev inequalities were used to get that $\| u \|_{L^\infty} + \| \nabla u \|_{L^3} \leq C \| \nabla u \|_{H^1} \leq C$. Moreover, since (4.13) implies

\[ P_{tt} + u_t \cdot \nabla P + u \cdot \nabla P_t + \gamma P_t \text{div} u + \gamma P \text{div} u_t = 0, \]

one obtains after using (4.2), (4.10), (4.11), (4.16), and (4.17) that

\[ \int_0^T \| P_{tt} \|^2_{L^2} dt \leq C \int_0^T (\| u_t \|_{L^6} \| \nabla P \|_{L^3} + \| \nabla P_t \|_{L^2} + \| \nabla u \|_{L^\infty} \| P_t \|_{L^2} + \| \nabla u_t \|_{L^2})^2 dt \]  
\[ \leq C + C \int_0^T (\| \nabla u \|^2_{H^1} + \| \nabla u_t \|^2_{L^2}) dt \leq C. \]  
\[ (4.18) \]
In the same way, one also has \( \|\rho t\|_{L^2} \in L^2(0, T) \). So, combining this with \((4.16)-(4.18)\) completes the proof of \((4.12)\). 

In order to prove the solution obtained is indeed a classical one on the time-interval \([\tau, T]\) for any \( 0 < \tau < T < \infty \), we need some further estimates on the higher order derivatives of \((\rho, u)\). However, due to the weaker compatibility condition \((1.14)\) (cf. \((1.20), (1.21)\)), the methods used in \([17]\) cannot be applied any more. To overcome this difficulty, we need the following initial-layer analysis.

**Lemma 4.4** Let \( \sigma \triangleq \min\{1, t\} \). Then it holds for any given \( T > 0 \) that
\[
\sup_{0 \leq t \leq T} \sigma \left( \|\nabla u\|_{H^2}^2 + \|\nabla u_t\|_{L^2}^2 \right) + \int_0^T \sigma \left( \|\rho^{1/2}u_{tt}\|_{L^2}^2 + \|\nabla u_t\|_{H^1}^2 \right) dt \leq C(T). \tag{4.19}
\]

**Proof.** First, differentiating \((1.2)\) with respect to \( t \) gives
\[
\mu \Delta u_t + (\mu + \lambda) \nabla \text{div} u_t = \rho u_{tt} - \nabla f + \rho u_t \cdot \nabla u + \rho u_t \cdot \nabla u_t + \nabla P_t. \tag{4.20}
\]

Multiplying \((4.20)\) by \( u_{tt} \) and integrating the resulting equality over \( \mathbb{R}^3 \) yield
\[
\frac{1}{2} \frac{d}{dt} \int \left[ \mu |\nabla u_t|^2 + (\mu + \lambda)(\text{div} u_t)^2 \right] dx + \int \rho |u_t|^2 dx = \int \left[ (\rho_t \nabla f - \rho_t u \cdot \nabla u - \rho u \cdot \nabla u_t - \rho u_t \cdot \nabla u_t - \nabla P_t) \cdot u_{tt} \right] dx + \frac{d}{dt} \int \left[ -\frac{1}{2} \rho_t |u_t|^2 + (\rho_t \nabla f - \rho_t u \cdot \nabla u - \nabla P_t) \cdot u_t \right] dx \]
\[
- \int \left[ (\rho_t \nabla f - \rho_t u \cdot \nabla u - \rho_t u \cdot \nabla u_t) \right] \cdot u_{tt} dx + \frac{1}{2} \int \rho |u_{tt}|^2 dx - \int P_t \text{div} u_t dx - \int (\rho u_t \cdot \nabla u + \rho u \cdot \nabla u_t) \cdot u_{tt} dx \]
\[
\triangleq \frac{d}{dt} I_0 + \sum_{i=1}^{4} I_i. \tag{4.21}
\]

Then, we estimate each term on the right-hand side of \((4.21)\). Using \((1.1)\) and integrating by parts, we see from \((4.2)\) and \((4.10)-(4.12)\) that
\[
I_0 = \int \left[ -\rho u \cdot \nabla u_t \cdot u_t + \rho_t (\nabla f - u \cdot \nabla u) \cdot u_t + P_t \text{div} u_t \right] dx \]
\[
\leq C \|u\|_{L^\infty} \|\rho^{1/2}u_t\|_{L^2} \|\nabla u_t\|_{L^2} + C \|\rho_t\|_{L^2} \|\nabla f\|_{L^3} \|u_t\|_{L^6} + C \|\rho_t\|_{L^2} \|u\|_{L^\infty} \|\nabla u\|_{L^3} \|u_t\|_{L^6} + C \|P_t\|_{L^2} \|\nabla u_t\|_{L^2} \]
\[
\leq \frac{\mu}{4} \|\nabla u_t\|_{L^2}^2 + C.
\]

Using \((4.2)\) and \((4.10)-(4.12)\) again, one infers that
\[
I_1 \leq C \|\rho u_t\|_{L^2} \|\nabla f\|_{L^3} \|u_t\|_{L^6} + C \|\rho u\|_{L^2} \|u\|_{L^\infty} \|\nabla u\|_{L^3} \|u_t\|_{L^6} \]
\[
+ C \|\rho_t\|_{L^2} \|u_t\|_{L^6} \|\nabla u\|_{L^6} + C \|\rho_t\|_{L^6} \|u\|_{L^6} \|\nabla u_t\|_{L^2} \|u_t\|_{L^6} \]
\[
\leq C \|\rho u_t\|_{L^2} \|\nabla u_t\|_{L^2} + C \|\nabla u_t\|_{L^2}^2 \leq C \left( \|\rho u_t\|_{L^2}^2 + \|\nabla u_t\|_{L^2}^2 \right).
\]
By (1.1) and integration by parts, we have from (4.10) and (4.12) that

\[ I_2 = \int (\rho u_t) \cdot \nabla u_t \cdot u_t dx = \int (\rho_t u \cdot \nabla u_t + \rho u_t \cdot \nabla u_t \cdot u_t) dx \]

\[ \leq C \| \rho_t \|_{L^6} \| u_t \|_{L^\infty} \| \nabla u_t \|_{L^2} \| u_t \|_{L^6} + C \| \rho u_t \|_{L^3} \| \nabla u_t \|_{L^2} \| u_t \|_{L^6} \]

\[ \leq C \| \nabla u_t \|^2_{L^2} + C \| \rho u_t \|_{L^3}^{1/2} \| u_t \|_{L^6}^{1/2} \| \nabla u_t \|_{L^2} \leq C \left(1 + \| \nabla u_t \|^2_{L^2}\right). \]

Obviously, \( I_3 \leq C \left(\| P_t \|^2_{L^2} + \| \nabla u_t \|^2_{L^2}\right). \) Finally, it follows from (4.12) and Lemma 2.1 that

\[ I_4 \leq C \left(\| u_t \|_{L^6} \| \nabla u_t \|_{L^3} + \| u_t \|_{L^\infty} \| \nabla u_t \|_{L^2} \right) \| \rho^{1/2} u_{tt} \|_{L^2} \]

\[ \leq \frac{1}{2} \| \rho^{1/2} u_{tt} \|^2_{L^2} + C \| \nabla u_t \|^2_{L^2}. \]

Thus, putting the estimates of \( I_i \) into (4.21), multiplying the resulting inequality by \( \sigma(t) \) and integrating it over \([0,T]\), we infer from (4.10), (4.12) and Gronwall’s inequality that

\[ \sup_{0 \leq t \leq T} \sigma \left(\| \nabla u \|^2_{H^1} + \| \nabla u_t \|^2_{L^2}\right) + \int_0^T \sigma \| \rho^{1/2} u_{tt} \|^2_{L^2} dt \leq C, \]

which, together with (4.11) and (4.15), gives

\[ \sup_{0 \leq t \leq T} \sigma \left(\| \nabla u \|^2_{H^1} + \| \nabla u_t \|^2_{L^2}\right) + \int_0^T \sigma \| \rho^{1/2} u_{tt} \|^2_{L^2} dt \leq C. \]  \hspace{1cm} (4.22)

Finally, applying the standard \( L^2 \)-estimate to the elliptic system (4.20) together with (4.12) and (4.12) gives

\[ \| \nabla^2 u_t \|_{L^2} \leq C \| \mu \Delta u_t + (\mu + \lambda) \nabla \text{div} u_t \|_{L^2} \]

\[ \leq C \| \rho u_{tt} - \rho_t \nabla f + \rho u_{tt} + \rho u_t \cdot \nabla u + \rho u_t \cdot \nabla u + \rho \u \cdot \nabla u_t + \nabla P_t \|_{L^2} \]

\[ \leq C \left( \| \rho^{1/2} u_{tt} \|_{L^2} + \| \rho_{tt} \|_{L^2} + \| \rho_t \|_{L^3} \| u_t \|_{L^6} + \| \rho_t \|_{L^6} \| u_t \|_{L^\infty} \| \nabla u \|_{L^6} \right) \]

\[ + \| u_t \|_{L^6} \| \nabla u \|_{L^3} + \| u_t \|_{L^\infty} \| \nabla u_t \|_{L^2} \| \nabla P_t \|_{L^2} \]

\[ \leq C \left(1 + \| \nabla u_t \|_{L^2} + \| \rho^{1/2} u_{tt} \|_{L^2}\right), \]  \hspace{1cm} (4.23)

from which and (4.22) it follows that

\[ \int_0^T \sigma \| \nabla^2 u_t \|^2_{L^2} dt \leq C. \]

This, together with (4.22), finishes the proof of (4.19).

The next lemma is concerned with the \( W^{1,q} \)-estimate \((q \in (3,6))\) on the gradients of density and pressure, which in particular indicates the Hölder continuity of \((\nabla \rho, \nabla P)\).

**Lemma 4.5** Let \( q \in (3,6) \) be as in Theorem 1.1. For any given \( T > 0 \), it holds that

\[ \sup_{0 \leq t \leq T} (\| \nabla \rho \|_{W^{1,q}} + \| \nabla P \|_{W^{1,q}}) + \int_0^T (\| u_t \|_{W^{1,q}}^{p_0} + \| \nabla^2 u \|_{W^{1,q}}^{p_0} ) dt \leq C(T), \]  \hspace{1cm} (4.24)

where

\[ p_0 \triangleq (9q - 6)/(10q - 12) \in (1, 4q/(5q - 6)). \]  \hspace{1cm} (4.25)
Proof. Applying the differential operator $\nabla^2$ to both sides of (4.13), multiplying the resulting equations by $q|\nabla^2 P(\rho)|^{q-2}\nabla^2 P(\rho)$, and integrating it by parts over $\mathbb{R}^3$, one deduces from (4.12), (4.11) and Lemma 2.1 that

$$
\frac{d}{dt} \|\nabla^2 P\|_{L^q}^q \leq C \left( \|\nabla u\|_{L^\infty} \|\nabla^2 P\|_{L^q} + \|\nabla P\|_{L^\infty} \|\nabla^2 u\|_{L^q} + \|\nabla^2 u\|_{W^{1,q}} \right) \|\nabla^2 P\|_{L^2}^{q-1} \\
\leq C \left( 1 + \|\nabla u\|_{H^2} \right) \left( 1 + \|\nabla^2 P\|_{L^q}^q + C \|\nabla^2 u\|_{W^{1,q}} \|\nabla^2 P\|_{L^2}^{q-1} \right).
$$

The similar estimate also holds for $\|\nabla^2 \rho\|_{L^q}$. Therefore,

$$
\frac{d}{dt} \left( \|\nabla^2 \rho\|_{L^q}^q + \|\nabla^2 P\|_{L^q}^q \right) \leq C \left( 1 + \|\nabla u\|_{H^2} \right) \left( 1 + \|\nabla^2 \rho\|_{L^q}^q + \|\nabla^2 P\|_{L^q}^q \right) \\
+C \|\nabla^2 u\|_{W^{1,q}} \left( \|\nabla^2 \rho\|_{L^q}^{q-1} + \|\nabla^2 P\|_{L^2}^{q-1} \right).
$$

(4.26)

Applying the standard $W^{1,p}$-estimate to the elliptic system (1.2) yields that

$$
\|\nabla^2 u\|_{W^{1,q}} \leq C \left( \|\rho u_t\|_{W^{1,q}} + \|\rho \cdot \nabla u\|_{W^{1,q}} + \|\nabla P\|_{W^{1,q}} + \|\rho \nabla f\|_{W^{1,q}} \right) \\
\leq C \left( 1 + \|u_t\|_{W^{1,q}} + \|\nabla \rho\|_{L^\infty} \|u_t\|_{L^\infty} + \|\nabla u\|_{W^{1,q}} \right) \\
+ \|\nabla \rho\|_{L^\infty} \|\nabla u\|_{L^q} + \|\nabla \rho\|_{L^\infty} \|\nabla u\|_{L^q} + \|\nabla P\|_{W^{1,q}}) \\
\leq C \left( 1 + \|\nabla u\|_{H^2} + \|u_t\|_{W^{1,q}} + \|\nabla \rho\|_{W^{1,q}} + \|\nabla P\|_{W^{1,q}} \right),
$$

(4.27)

where we have also used (4.2), (4.11) and Lemma 2.1. Putting (4.27) into (4.26) gives

$$
\frac{d}{dt} \left( \|\nabla^2 \rho\|_{L^q}^q + \|\nabla^2 P\|_{L^q}^q \right) \leq C \left( 1 + \|\nabla u\|_{H^2} + \|u_t\|_{W^{1,q}} \right) \left( 1 + \|\nabla^2 \rho\|_{L^q}^q + \|\nabla^2 P\|_{L^q}^q \right).
$$

(4.28)

Using (2.1), (2.7), (3.9) and (4.10), we find that

$$
\int_0^T \|u_t\|_{L^2}^2 dt \leq C \int_0^T \|u_t\|_{H^1}^2 dt \leq C \int_0^T \left( \|u_t\|_{L^2}^2 + \|\nabla u_t\|_{L^2}^2 \right) dt \\
\leq C \int_0^T \left( \|\rho^{1/2} u_t\|_{L^2}^2 + \|\nabla u_t\|_{L^2}^2 \right) dt \leq C.
$$

(4.29)

Note that $4q/(5q - 6) \in (1, 4/3)$ for $q \in (3, 6)$. So, for $p_0$ as in (4.25), we obtain by Lemma 2.1 Hölder inequality and (4.19) that

$$
\int_0^T \|\nabla u_t\|_{L^2}^{p_0} dt \leq C \int_0^T \|\nabla u_t\|_{L^2}^{p_0(6-q)/(2q)} \|\nabla u_t\|_{L^6}^{p_0(3q-6)/(2q)} dt \\
\leq C \int_0^T \sigma^{-p_0/2} (\|\nabla u_t\|_{L^2}^2)^{p_0(6-q)/(4q)} (\|\nabla u_t\|_{H^1}^2)_{p_0(3q-6)/(4q)} dt \\
\leq C \left( \sup_{0 \leq t \leq T} \sigma \|\nabla u_t\|_{L^2}^2 \right)^{p_0(6-q)/(4q)} \int_0^T \sigma^{-p_0/2} (\|\nabla u_t\|_{H^1}^2)^{p_0(3q-6)/(4q)} dt \\
\leq C \left( \int_0^T \sigma^{-2pq/(4q-3q-6)} dt \right)^{(4q-p_0(3q-6))/(4q)} \left( \int_0^T \sigma \|\nabla u_t\|_{H^1}^2 dt \right)^{p_0(3q-6)/(4q)} \\
\leq C,
$$

(4.30)

since $0 < 2pq/(4q - p_0(3q - 6)) < 1$ and $0 < p_0(3q - 6)/(4q) < 1$. 

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The combination of (4.29) with (4.30) shows that for \( q \in (3, 6) \),
\[
\int_0^T \| u_t \|_{W^{1,q}}^q dt \leq C. \tag{4.31}
\]
Thus, by (4.11), (4.31) and Gronwall’s inequality, one sees from (4.28) that
\[
\sup_{0 \leq t \leq T} (\| \nabla \rho \|_{W^{1,q}} + \| \nabla P \|_{W^{1,q}}) \leq C,
\]
which, combining with (4.11), (4.27), and (4.31), finishes the proof of Lemma 4.5.

Finally, we still need the following lemma, which implies that \( u_t \) and \( \nabla^2 u \) are Hölder continuous away from \( t = 0 \).

**Lemma 4.6** For any given \( T > 0 \), it holds that
\[
\sup_{0 \leq t \leq T} \sigma \left( \| \rho^{1/2} u_{ttt} \|_{L^2} + \| \nabla^2 u_t \|_{L^2} + \| \nabla^2 u \|_{W^{1,q}} \right) + \int_0^T \sigma^2 \| \nabla u_{ttt} \|_{L^2}^2 dt \leq C(T). \tag{4.32}
\]

**Proof.** Differentiating (4.20) with respect to \( t \) gives
\[
\rho u_{ttt} + \rho \cdot \nabla u_{tt} - \mu \Delta u_{tt} - (\mu + \lambda) \nabla \text{div} u_{tt} = 2 \text{div}(\rho u_t) u_t + \text{div}(\rho u_t) u_t - 2(\rho u_t) \cdot \nabla u_t - (\rho u_t + 2\rho_t u_t) \cdot \nabla u - \rho u_{ttt} \cdot \nabla u - \nabla P + \rho_t \nabla f,
\]
which, multiplied by \( u_{ttt} \) in \( L^2 \) and integrated by parts over \( \mathbb{R}^3 \), yields
\[
\frac{1}{2} \frac{d}{dt} \int \rho |u_{ttt}|^2 dx + \int (\mu |\nabla u_{ttt}|^2 + (\mu + \lambda) |\text{div} u_{ttt}|^2) dx
\]
\[
= -4 \int \rho u \cdot \nabla u_{tt} \cdot u_{ttt} dx - \int (\rho u_t) \cdot (\nabla (u_t \cdot u_{tt}) + 2 \nabla u_t \cdot u_{tt} dt) dx
\]
\[
- \int (\rho u_t + 2\rho_t u_t) \cdot \nabla u \cdot u_{ttt} dx - \int \rho u_{ttt} \cdot \nabla u \cdot u_{tt} dx
\]
\[
+ \int P_{tt} \text{div} u_{ttt} dx + \int \rho_t \nabla f \cdot u_{ttt} dx \overset{6}{=} \sum_{i=1}^6 J_i. \tag{4.33}
\]

The right-hand side of (4.33) can be estimated term by term as follows, using Lemma 2.1

Cauchy-Schwarz inequality and the estimates obtained.
\[
J_1 \leq \| \rho^{1/2} u_{ttt} \|_{L^2} \| \nabla u_{ttt} \|_{L^2} \leq \delta \| \nabla u_{tt} \|_{L^2}^2 + C(\delta) \| \rho^{1/2} u_{tt} \|_{L^2}^2,
\]
\[
J_2 \leq C \left( \| \rho u_t \|_{L^3} + \| \rho_t \|_{L^3} \right) \left( \| \nabla u_t \|_{L^2} \| u_{tt} \|_{L^6} + \| u_t \|_{L^6} \| \nabla u_t \|_{L^2} \right)
\]
\[
\leq C \left( 1 + \| \rho u_t \|_{L^2}^{1/2} \| \nabla u_t \|_{L^2}^{1/2} \right) \| \nabla u_t \|_{L^2} \| \nabla u_{tt} \|_{L^2}
\]
\[
\leq \delta \| \nabla u_{ttt} \|_{L^2}^2 + C(\delta) \left( 1 + \| \nabla u_t \|_{L^2} \right),
\]
\[
J_3 \leq C \left( \| \rho u_t \|_{L^2} \| u_t \|_{L^\infty} \| \nabla u_t \|_{L^2} + \| \rho_t \|_{L^2} \| u_t \|_{L^6} \| \nabla u_t \|_{L^3} \right) \| u_{tt} \|_{L^6}
\]
\[
\leq \delta \| \nabla u_{ttt} \|_{L^2}^2 + C(\delta) \left( \| \rho u_t \|_{L^2} + \| \nabla u_t \|_{L^2} \right),
\]
\[
J_4 \leq C \| \rho^{1/2} u_{ttt} \|_{L^2} \| \nabla u \|_{L^2} \| u_{ttt} \|_{L^6} \leq \delta \| \nabla u_{tt} \|_{L^2}^2 + C(\delta) \| \rho^{1/2} u_{tt} \|_{L^2}^2
\]
and
\[ J_5 + J_6 \leq C \left( \|P_t\|_{L^2} + \|\rho_t\|_{L^2} \right) \left( \|\nabla u_t\|_{L^2} + \|\nabla f\|_{L^3} \|u_t\|_{L^6} \right) \leq \delta \|\nabla u_t\|_{L^2}^2 + C(\delta) \left( \|P_t\|_{L^2}^2 + \|\rho_t\|_{L^2}^2 \right). \]

Putting the estimates of \(J_1, \ldots, J_6\) into (4.33), multiplying it by \(\sigma^2\) and integrating the resulting relation over \((0, T)\), we have by choosing \(\delta > 0\) small enough that
\[
\sup_{0 \leq t \leq T} \left( \sigma^2 \|\rho^{1/2} u_t\|_{L^2}^2 \right) + \int_0^T \sigma^2 \|\nabla u_t\|_{L^2}^2 dt \leq C + C \int_0^T \left( \sigma \|\rho^{1/2} u_t\|_{L^2}^2 + \sigma^2 \left( \|P_t\|_{L^2}^2 + \|\rho_t\|_{L^2}^2 + \|\nabla u_t\|_{L^2}^2 \right) \right) dt \leq C + C \sup_{0 \leq t \leq T} \left( \sigma^{1/2} \|\nabla u_t\|_{L^2}^2 \right) \int_0^T \|\nabla u_t\|_{L^2}^2 dt \leq C, \tag{4.34}
\]
where we have used (4.10), (4.12) and (4.19).

As a result of (4.34), (4.19) and (4.23), we also see that
\[
\sigma \|\nabla^2 u_t\|_{L^2} \leq C \sigma \left( 1 + \|\nabla u_t\|_{L^2} + \|\rho^{1/2} u_t\|_{L^2} \right) \leq C, \quad \forall t \in [0, T]. \tag{4.35}
\]
and thus, it follows from (4.27), (4.19) and (4.24) that for any \(t \in [0, T]\),
\[
\sigma \|\nabla^2 u\|_{W^{1,q}} \leq C \sigma \left( 1 + \|\nabla u\|_{H^2} + \|u_t\|_{W^{1,q}} + \|\nabla\rho\|_{W^{1,q}} + \|\nabla P\|_{W^{1,q}} \right) \leq C \left( 1 + \sigma \|u_t\|_{H^2} \right) \leq C, \tag{4.36}
\]
where we have used (2.7), (3.9), (4.10) and (4.19) to get that \(\sigma \|u_t\|_{L^2} \leq C\). Lemma 4.6 now readily follows from (4.34)–(4.36). \(\square\)

5 Proofs of Theorems 1.1 and 1.2

With all the a priori estimates at hand, we are now ready to prove our main results. First of all, we prove the global well-posedness of classical solutions to the problem (1.1)–(1.5) provided the initial density is strictly away from vacuum and the initial energy is small.

**Proposition 5.1** Let \(C_0\) be the initial energy defined by (1.10). For given numbers \(M > 0\) (not necessarily small) and \(\tilde{\rho} > 0\), assume that \(\rho_0, u_0\) and \(f\) satisfy
\[
\begin{align*}
&f \in H^3, \quad (\rho_0 - \rho_\infty, u_0) \in H^3, \\
&\inf_{x \in \mathbb{R}^3} \rho_0(x) \leq \sup_{x \in \mathbb{R}^3} \rho_0(x) < \tilde{\rho}, \quad \|\nabla u_0\|_{L^2} \leq M, \tag{5.1}
\end{align*}
\]
where \(\tilde{\rho} > 0\) is the same one as in Proposition 3.1. Then for any \(0 < T < \infty\), there exists a unique classical solution \((\rho, u)\) of (1.1)–(1.3) on \(\mathbb{R}^3 \times (0, T]\) satisfying (2.9), (2.10) with \(T_0\) replaced by any \(T > 0\). Moreover, all the uniform-in-time estimates in Sect. 3 hold for \((\rho, u)\).
Proof. The standard local existence result (i.e. Lemma 2.5) shows that the Cauchy problem (1.1)–(1.5) has a unique local classical solution \((\rho, u)\), defined up to a positive \(T_0\) which may depend on \(\inf_{x \in \mathbb{R}^3} \rho_0(x)\) and satisfying (2.9), (2.10).

In view of (5.1) and the definitions of \(\Phi_i(T)\) \((i = 1, 2, 3)\), we know that
\[
\Phi_1(0) = \Phi_2(0) = 0, \quad \Phi_3(0) \leq M \quad \text{and} \quad \rho_0 \leq \tilde{\rho}.
\]
Thus, there exists a positive \(T_1 \in (0, T_0]\) such that (3.4) holds for \(T = T_1\).

Set
\[
T_* = \sup \{ T \mid (3.4) \text{ holds} \}.
\]
(5.2)

It is clear that \(T_* \geq T_1 > 0\).

We claim that \(T_* = \infty\).
(5.3)

Otherwise, \(T_* < \infty\). Then it follows from Proposition 3.1 that (3.5) holds for any \(0 < T < T_*\), and furthermore, the estimates in Lemmas 4.1–4.6 are also valid for all \(0 < T < T_*\).

In the next, for the sake of simplicity, we denote by \(\tilde{C}\) the positive constant which may depend on the lower bound of the initial density (i.e. \(\inf_{x \in \mathbb{R}^3} \rho_0(x)\)) and \(T_*\). We shall prove that there exists a positive constant \(\tilde{C} > 0\) such that for any \(T \in (0, T_*]\),
\[
\sup_{0 \leq t \leq T} \| (\rho - \rho_\infty, u) \|_{H^3} \leq \tilde{C}.
\]
(5.4)

This in particular implies that
\[
\| (\rho - \rho_\infty, u)(\cdot, T_*) \|_{H^3} < \infty, \quad \inf_{x \in \mathbb{R}^3} \rho(x, T_*) > 0.
\]

So, Lemma 2.5 together with (3.5), yields that there exists some \(T_{**} > T_*\) such that (3.4) holds for \(T = T_{**}\), which contradicts (5.2). Hence, (3.5) holds. Thus, \((\rho, u)\) is in fact the unique classical solution of (1.1)–(1.5) on \(\mathbb{R}^3 \times (0, T_*]\) for all \(0 < T < T_* = \infty\).

We are now in a position of proving (5.4). To do so, we first note that under the conditions of (5.1) \(_1\) and (5.1) \(_2\), it holds that (keeping in mind that \(\rho_0 > 0\))
\[
u_t(\cdot, 0) = \nabla f - u_0 \cdot \nabla u_0 + \rho_0^{-1} (\mu \Delta u_0 + (\mu + \lambda) \nabla \text{div} u_0 - P(\rho_0)) \in H^1.
\]
(5.5)

Thus, similar to the proof of (4.22), by (5.5) one easily deduces that
\[
\sup_{0 \leq t \leq T} (\| \nabla u \|_{H^2}^2 + \| \nabla u_t \|_{L^2}^2) + \int_0^T \| \rho^{1/2} u_{tt} \|_{L^2}^2 dt \leq \tilde{C}.
\]
(5.6)

Similar to the derivation of (4.23), one gets by using (5.6) that
\[
\int_0^T \| \nabla^2 u_t \|_{L^2}^2 dt \leq \tilde{C} \int_0^T \left( 1 + \| \nabla u_t \|_{L^2}^2 + \| \rho^{1/2} u_{tt} \|_{L^2}^2 \right) dt \\
\leq \tilde{C} + \tilde{C} \int_0^T \| \rho^{1/2} u_{tt} \|_{L^2}^2 dt \leq \tilde{C}.
\]
(5.7)

Moreover, using the standard \(H^2\)-theory of elliptic system, one deduces from Lemmas 2.1–4.3 and (5.6) that
\[
\| \nabla^2 u \|_{H^2} \leq \tilde{C} \| \mu \Delta u + (\mu + \lambda) \nabla \text{div} u \|_{H^2}
\]
\begin{align*}
&\leq \tilde{C}\|\rho u_t + \rho u \cdot \nabla u + \nabla P - \rho \nabla f\|_{H^2} \\
&\leq \tilde{C}(\|\rho u_t\|_{L^2} + \|\nabla \rho\|_{H^1} \|u_t\|_{L^\infty} + \|\nabla \rho\|_{L^3} \|\nabla u_t\|_{L^6} + \|\nabla u_t\|_{H^1} \\
&\quad + \|\rho u\|_{H^2} \|\nabla u\|_{H^2} + \|\nabla P\|_{H^2} + \|\nabla f\|_{H^2} + \|\nabla \rho\|_{H^1} \|\nabla f\|_{H^2}) \\
&\leq \tilde{C} (1 + \|\nabla^3 \rho\|_{L^2} + \|\nabla^2 P\|_{L^2} + \|\nabla \rho\|_{H^1} ).
\end{align*}

Applying the differential operator \(\nabla^3\) to both sides of (1.1) and (4.13), and multiplying the resulting equations by \(\nabla^3 \rho\) and \(\nabla^3 P(\rho)\) in \(L^2\), respectively, one easily obtains after integrating by parts and using Lemmas 2.1, 4.1–4.3, (5.6) and (5.10) that

\[
\frac{d}{dt}(\|\nabla^3 \rho\|^2_{L^2} + \|\nabla^3 P\|^2_{L^2}) \leq \tilde{C} (1 + \|\nabla^2 u\|^2_{H^2} + \|\nabla^3 \rho\|^2_{L^2} + \|\nabla^3 P\|^2_{L^2}).
\]

Combining (5.7), (5.8) and (5.9), we conclude from Gronwall’s inequality that

\[
\sup_{0 \leq t \leq \tau} (\|\nabla^3 \rho\|^2_{L^2} + \|\nabla^3 P\|^2_{L^2}) \leq \tilde{C}.
\]

So, Lemmas 4.1–4.3, together with (5.6) and (5.10), lead to (5.14). The proof of Proposition 5.1 is therefore complete.

**Proof of Theorem 1.1.** Let \((\rho_0, u_0)\) be the initial data satisfying the conditions (1.12)–(1.14) in Theorem 1.1. Assume that the initial energy \(C_0\) satisfies

\[
C_0 \leq \bar{\varepsilon} \triangleq \bar{\varepsilon}/2,
\]

where \(\bar{\varepsilon} > 0\) is the same one as in Proposition 3.1.

To apply Proposition 5.1, we define the smooth approximate data as follows:

\[
\rho^\delta,\eta_0 = \frac{j_\delta * \rho_0 + \eta}{1 + \eta}, \quad g^\delta,\eta = j_\delta * g, \quad f^\delta,\eta = j_\delta * f,
\]

where \(0 < \delta, \eta < 1\), \(j_\delta(x)\) is the standard mollifier with width \(\delta\), and “ * ” denotes the usual convolution operator. As that in [2], let \(u^\delta,\eta_0\) be the unique solution to the elliptic problem:

\[
\begin{cases}
-\mu \Delta u^\delta,\eta_0 - (\mu + \lambda) \nabla \text{div} u^\delta,\eta_0 = -\nabla P(\rho^\delta,\eta_0) + \left(\rho^\delta,\eta_0\right)^{1/2} g^\delta,\eta, \\
u^\delta,\eta(x) \to 0 \quad \text{as} \quad |x| \to \infty.
\end{cases}
\]

(5.13)

Also let \(\rho^\delta,\eta \triangleq \rho^\delta,\eta(x)\) be the unique solution of (1.6) with \(f\) being replaced by \(f^\delta,\eta\). Obviously, \(\rho^\delta,\eta\) satisfies (1.8), (1.9). Moreover, it is easy to check that

\[
\begin{cases}
0 < \frac{\eta}{1+\eta} \leq \rho^\delta,\eta \leq \bar{\rho} < \infty, \\
(\rho^\delta,\eta - \rho_\infty, g^\delta,\eta, u^\delta,\eta, f^\delta,\eta, \rho^\delta,\eta) \in H^\infty, \\
\lim_{\delta,\eta \to 0} \left(\|\rho^\delta,\eta - \rho_0\|_{H^2 \cap W^{2,q}} + \|u^\delta,\eta - u_0\|_{H^2}\right) = 0, \\
\lim_{\delta,\eta \to 0} \left(\|g^\delta,\eta - g\|_{L^2} + \|(f^\delta,\eta - f, \rho^\delta,\eta - \rho_\delta)\|_{H^2 \cap W^{2,q}}\right) = 0.
\end{cases}
\]

(5.14)

The initial norm of the mollified data \((\rho^\delta,\eta_0, u^\delta,\eta_0)\) now reads

\[
C_0^\delta,\eta \triangleq \int \left(G(\rho^\delta,\eta_0) + \frac{1}{2} \rho^\delta,\eta_0 |u^\delta,\eta_0|^2\right) dx,
\]
where $G(\cdot)$ is the function defined in (1.11) with $\rho_s$ replaced by $\rho^{\delta, \eta}_s$.

By (5.14), we have
\[
\lim_{\eta \to 0} \lim_{\delta \to 0} C_0^{\delta, \eta} = C_0.
\]
So, there exist positive constants $\eta_0 \in (0, 1)$ and $\delta_0(\eta)$ such that
\[
C_0^{\delta, \eta} \leq C_0 + \tilde{\varepsilon}/2 \leq \tilde{\varepsilon},
\]
provided that
\[
0 < \eta < \eta_0 \quad \text{and} \quad 0 < \delta < \delta_0(\eta).
\]

Let $\delta, \eta$ satisfy (5.16). Then it follows from (5.15) and Proposition 5.1 that there exists a unique smooth solution $(\rho^{\delta, \eta}, u^{\delta, \eta})$ of (1.1), (1.2) with the mollified data $\rho^{\delta, \eta}_0, u^{\delta, \eta}_0, f^{\delta, \eta}$ and $g^{\delta, \eta}$ on $\mathbb{R}^3 \times (0, T]$ for all $T > 0$. Moreover, Proposition 5.1 and Lemmas 4.1–4.6 independent of $\delta$ and $\eta$, hold for $(\rho^{\delta, \eta}, u^{\delta, \eta})$.

Now, passing to the limit first $\delta \to 0$, then $\eta \to 0$, we deduce from standard arguments that there exists a unique solution $(\rho, u)$ to the origin problem (1.1)–(1.5) on $\mathbb{R}^3 \times (0, T]$ for all $T > 0$ satisfying
\[
\begin{cases}
0 \leq \rho(x, t) \leq 2\tilde{\rho} \quad \text{for all} \quad x \in \mathbb{R}^3, t \geq 0, \\
(\rho - \rho_{\infty}, P(\rho) - P(\rho_{\infty})) \in L^\infty(0, T; H^2 \cap W^{2, q}), \\
u \in L^\infty(0, T; H^2) \cap L^2(0, T; H^3) \cap L^\infty(\tau, T; H^3 \cap W^{3, q}), \\
u_t \in L^\infty(\tau, T; H^2) \cap H^1(\tau, T; H^1),
\end{cases}
\]
for any $0 < \tau < T < \infty$. Note that, the uniqueness of $(\rho, u)$ satisfying (5.17) can be proved in a standard way as that in [2].

To complete the proof of the first part of Theorem 1.1 we still need to show that
\[
(\rho - \rho_{\infty}, P(\rho) - P(\rho_{\infty})) \in C([0, T]; H^2 \cap W^{2, q}), \quad u \in C([0, T]; H^2). \tag{5.18}
\]

The proof of (5.18) is similar to that in [15], and we sketch it here for completeness. First, by virtue of (4.2), (4.10)–(4.12) and (4.24), it is easy to get that
\[
\begin{cases}
(\rho - \rho_{\infty}, P(\rho) - P(\rho_{\infty})) \in C([0, T]; H^1 \cap W^{1, q}), \\
(\rho - \rho_{\infty}, P(\rho) - P(\rho_{\infty})) \in C([0, T]; H^2 \cap W^{2, q} - \text{weak}), \\
u \in C([0, T]; H^1 \cap W^{1, q}).
\end{cases}
\]

Denote $D_{ij} \triangleq \partial^2_{ij}$ with $i, j = 1, 2, 3$. Note that it holds in $\mathcal{D}'(\mathbb{R}^3 \times (0, T))$ that
\[
\partial_t D_{ij} \rho + \text{div}(u D_{ij} \rho) = -\text{div}(\rho D_{ij} u) - \text{div}(\partial_i \rho \cdot \partial_j u + \partial_j \rho \cdot \partial_i u).
\]
Let $j_{\nu}(x)$ be the standard mollifying kernel with width $\nu$, and set $\rho^{\nu} \triangleq \rho * j_{\nu}$. Then,
\[
\partial_t D_{ij} \rho^{\nu} + \text{div}(u D_{ij} \rho^{\nu}) = -\text{div}(\rho D_{ij} u) * j_{\nu} - \text{div}(\partial_i \rho \cdot \partial_j u + \partial_j \rho \cdot \partial_i u) * j_{\nu} + R_{\nu},
\]
where $R_{\nu} \triangleq \text{div}(u D_{ij} \rho^{\nu}) - \text{div}(u D_{ij} \rho) * j_{\nu}$ satisfies (cf. [23, Lemma 2.3])
\[
\int_0^T \| R_{\nu} \|_{L^2 \cap L^4}^p dt \leq C \int_0^T \| u \|_{W^{1, \infty}}^p \| D_{ij} \rho \|_{L^2 \cap L^4}^p dt \leq C,
\]
(5.21)
with $p_0 > 1$ as in (4.25).

Multiplying (5.20) by $q|D_{ij}\rho^\nu|^{q-2}D_{ij}\rho^\nu$ and integrating by parts over $\mathbb{R}^3$, we obtain
\[
\frac{d}{dt}\|D_{ij}\rho^\nu\|_L^q = -(q-1) \int |D_{ij}\rho^\nu|q\text{div}u dx - q \int \text{div}(\rho D_{ij}u) * j_\nu |D_{ij}\rho^\nu|^{q-2}D_{ij}\rho^\nu dx \\
- q \int (\text{div}(\partial_i \rho \cdot \partial_j u + \partial_j \rho \cdot \partial_i u) * j_\nu) |D_{ij}\rho^\nu|^{q-2}D_{ij}\rho^\nu dx \\
+ q \int R_\nu |D_{ij}\rho^\nu|^{q-2}D_{ij}\rho^\nu dx,
\]
which, combining with (4.11), (4.24) and (5.21), yields
\[
\sup_{0 \leq t \leq T} \|\nabla^2 \rho^\nu(\cdot, t)\|_L^q + \int_0^T \left| \frac{d}{dt}\|\nabla^2 \rho^\nu\|_L^q \right|^{p_0} dt \\
\leq C + C \int_0^T (\|\nabla u\|_{W^{2,q}}^{p_0} + \|R_\nu\|_{L^q}^{p_0}) dt \leq C.
\]
This, together with Ascoli-Arzela theorem, gives
\[
\|\nabla^2 \rho^\nu(\cdot, t)\|_L^q \to \|\nabla^2 \rho(\cdot, t)\|_L^q \text{ in } C([0, T]), \quad \text{as } \nu \to 0,
\]
which particularly implies
\[
\|\nabla^2 \rho(\cdot, t)\|_L^q \in C([0, T]). \tag{5.22}
\]
Similarly, one can also obtain that
\[
\|\nabla^2 \rho(\cdot, t)\|_{L^2} \in C([0, T]). \tag{5.23}
\]
So, it readily follows from (5.22), (5.23) and (5.19) that
\[
\nabla^2 \rho \in C([0, T]; L^2 \cap L^q). \tag{5.24}
\]
In the exactly same way, we also have
\[
\nabla^2 P(\rho) \in C([0, T]; L^2 \cap L^q). \tag{5.25}
\]
It follows from (1.2) that
\[
(\rho u_t)_t = (\mu \Delta u_t + (\mu + \lambda)\nabla \text{div}u_t) + \rho_t \nabla f - \nabla P_t - (\rho u \cdot \nabla u)_t.
\]
Thus, it is easily seen from (4.2), (4.10) and (4.12) that
\[
\|(\rho u_t)_t\|_{H^{-1}} \leq C (1 + \|\nabla u_t\|_{L^2}),
\]
and hence,
\[
(\rho u_t)_t \in L^2(0, T; H^{-1}).
\]
This, combining with the fact that $\rho u_t \in L^2(0, T; H^1)$ due to (4.10) and (4.11), leads to
\[
\rho u_t \in C([0, T]; L^2). \tag{5.26}
\]
Thus, by virtue of (5.19), (5.21) and (5.25), one easily gets
\[
u \in C([0, T]; H^2), \tag{5.27}
\]
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since it holds that
\[ \mu \Delta u + (\mu + \lambda) \nabla \text{div} u = \rho u_t + \rho u \cdot \nabla u + \nabla P(\rho) - \rho \nabla f. \]

Combining (5.19), (5.24), (5.25), and (5.27), we finish the proof of (5.18).

To complete the proof of Theorem 1.1, it remains to prove (1.18). To do this, we first deduce in a manner similar to the derivation of (3.34) that
\[
\int_1^\infty \left| \frac{d}{dt} \| \rho - \rho_s \|_{L^4}^4 \right| dt \
\leq C \int_1^\infty \left( \| \nabla u \|_{L^2}^2 + \| \rho - \rho_s \|_{L^4}^4 + \| F \|_{L^4}^4 \right) dt \leq C,
\]
where we have used (3.37) and (3.33). Thus, it follows from (3.33) and (5.28) that
\[ \| \rho - \rho_s \|_{L^4} \to 0 \quad \text{as} \quad t \to \infty. \]

This, together with (3.9), Proposition 1.1 and the interpolation inequality, immediately gives
\[ \| \rho - \rho_\infty \|_{L^p} \to 0 \quad \text{as} \quad t \to \infty, \quad \forall \ p \in (2, \infty). \] (5.29)

To study the large-time behavior of the velocity, we set
\[ M(t) \equiv \frac{\mu}{2} \| \nabla u \|_{L^2}^2 + \frac{\mu + \lambda}{2} \| \text{div} u \|_{L^2}^2. \]

Then, multiplying (3.12) by \( \dot{u} \) in \( L^2 \) and integrating by parts, we obtain
\[ |M'(t)| \leq C \int \rho |\dot{u}|^2 dx + C \| \nabla u \|_{L^3}^3 + C \| \nabla \dot{u} \|_{L^2}. \] (5.30)

Here, we have used (3.14), (3.15) and the following estimates:
\[ \left| \int \dot{u} \cdot \nabla (P(\rho) - P(\rho_s)) dx \right| \leq C \| \rho - \rho_s \|_{L^2} \| \nabla \dot{u} \|_{L^2} \leq C \| \nabla \dot{u} \|_{L^2} \]
and
\[ \left| \int (\rho - \rho_s) \dot{u} \cdot \nabla f dx \right| \leq C \| \rho - \rho_s \|_{L^2} \| \dot{u} \|_{L^6} \| \nabla f \|_{L^3} \leq C \| \nabla \dot{u} \|_{L^2} . \]

Thus, it follows from (3.33), (3.37), (3.39) and (5.30) that
\[
\int_1^\infty |M'(t)|^2 dt \leq C \int_1^\infty \left( \| \rho^{1/2} \dot{u} \|_{L^2}^2 + \| \nabla u \|_{L^4}^4 + \| \nabla \dot{u} \|_{L^2}^2 \right) dt \\
\leq C \int_1^\infty \left( \| \rho^{1/2} \dot{u} \|_{L^2}^2 + \| \nabla u \|_{L^4}^4 + \| \nabla \dot{u} \|_{L^2}^2 \right) dt \leq C.
\] (5.31)

Due to (5.38) and (3.7), one has
\[
\int_1^\infty |M(t)|^2 dt \leq C \sup_{t \geq 1} \| \nabla u \|_{L^2}^2 \int_1^\infty \| \nabla u \|_{L^2}^2 dt \leq C,
\]
from which and (5.31), we know that
\[ \| \nabla u(t) \|_{L^2} \to 0 \quad \text{as} \quad t \to \infty. \] (5.32)
As a result, we also have
\[ \int \rho^{1/2}|u|^4 \, dx \leq \left( \int \rho |u|^2 \, dx \right)^{1/2} \|u\|_{L^6} \leq C \|\nabla u\|_{L^2} \rightarrow 0 \quad \text{as} \quad t \rightarrow \infty. \]
This, together with (5.29) and (5.32), proves (1.18). Therefore, the proof of Theorem 1.1 is complete. \(\square\)

**Proof of Theorem 1.2.** Define the approximate data \((\rho^{\delta,\eta}, u^{\delta,\eta}, f^{\delta,\eta})\) as follows:
\[ \rho_0^{\delta,\eta} = j_\delta * \rho_0 + \eta, \quad u_0^{\delta,\eta} = j_\delta * u_0, \quad f^{\delta,\eta} = j_\delta * f. \]
Then, we can apply Proposition 5.1 to obtain a global smooth solution \((\rho^{\delta,\eta}, u^{\delta,\eta})\) of the Cauchy problem (1.1), (1.2) with the mollified data \(\rho_0^{\delta,\eta}, u_0^{\delta,\eta}\) and \(f^{\delta,\eta}\), which satisfies the uniform bounds in Proposition 3.1. Now, the remaining arguments to obtain the global weak solution and its asymptotic behavior are almost the same as that of [12]. The proof of Theorem 1.2 is thus finished. \(\square\)

**References**

[1] Antontsev, S.N., Kazhikhov, A.V., Monakhov, V.N.: Boundary Value Problems in Mechanics of Nonhomogeneous Fluids. Amsterdam, New York: North-Holland, 1990

[2] Choe, H., Kim, H.: Strong solutions of the Navier-Stokes equations for isentropic compressible fluids. J. Diff. Eqs. 190, 504-523 (2003)

[3] Cho, Y., Choe, H. J., Kim, H.: Unique solvability of the initial boundary value problems for compressible viscous fluid. J. Math. Pures Appl. 83, 243-275 (2004)

[4] Cho, Y., Kim, H.: On classical solutions of the compressible Navier-Stokes equations with nonnegative initial densities. Manuscript Math. 120, 91-129 (2006)

[5] Duan, R. J., Liu, L. X., Ukai, S., Yang, T.: Optimal \(L^p-L^q\) convergence rates for the compressible Navier-Stokes equations with potential force. J. Differential Eqs. 238, 220-233 (2007)

[6] Duan, R. J., Ukai, S., Yang, T., Zhao, H. J.: Optimal convergence rates for the compressible Navier-Stokes equations with potential forces. Math. Models Meth. Appl. Sci. 17, 737-758 (2007)

[7] Feireisl, E.: Dynamics of Viscous Compressible Fluids. Oxford: Oxford University Press, 2003

[8] Feireisl, E., Novotny, A., Petzeltová: On the existence of globally defined weak solutions to the Navier-Stokes equations. J. Math. Fluid Mech. 3(4), 358-392 (2001)

[9] Feireisl, E., Petzeltová, H.: Large-time behaviour of solutions to the Navier-Stokes equations of compressible flow. Arch. Rational Mech. Anal. 150(1), 77-96 (1999)

[10] Hoff, D.: Global solutions of the Navier-Stokes equations for multidimensional compressible flow with discontinuous initial data. J. Diff. Eqs. 120, 215-254 (1995)
[11] Hoff, D.: Strong convergence to global solutions for multidimensional flows of compressible, viscous fluids with polytropic equations of state and discontinuous initial data. Arch. Rational Mech. Anal. 132, 1-14 (1995)

[12] Hoff, D. Compressible flow in a half-space with Navier boundary conditions. J. Math. Fluid Mech. 7 (2005), no. 3, 315-338.

[13] Hoff, D.; Santos, M. M. Lagrangean structure and propagation of singularities in multidimensional compressible flow. Arch. Rational Mech. Anal. 188 (2008), no. 3, 509-543.

[14] Hoff, D.; Tsyganov, E. Time analyticity and backward uniqueness of weak solutions of the Navier-Stokes equations of multidimensional compressible flow. J. Differ. Eqs. 245 (2008), no. 10, 3068-3094.

[15] Huang, X., Li, J.: Global classical and weak solutions to the three-dimensional full compressible Navier-Stokes system with vacuum and large oscillations, http://arxiv.org/abs/1107.4655

[16] Huang, F., Li, J., Xin, Z. P.: Convergence to equilibria and blowup behavior of global strong solutions to the Stokes approximation equations for two-dimensional compressible flows with large data. J. Math. Pures Appl. 86(6), 471-491 (2006)

[17] Huang, X., Li, J., Xin, Z. P.: Global well-posedness of classical solutions with large oscillations and vacuum to the three-dimensional isentropic compressible Navier-Stokes equations. Commun. Pure Appl. Math., in press.

[18] Huang, X, Li, J. Xin, Z. P.: Serrin-type criterion for the three-dimensional viscous compressible flows. SIAM J. Math. Anal. 43 (4), 1872-1886 (2011)

[19] Ya. Kanel’: On a model system of equations of one-dimensional gas motion. Diff. Eqns. 4 (1968), 374-380.

[20] O. A. Ladyzenskaja, V. A. Solonnikov and N. N. Uralceva, Linear and Quasilinear Equations of Parabolic Type. Amer. Math. Soc., Providence, RI, 1968

[21] Li, J, Xin, Z. P.: Some uniform estimates and blowup behavior of global strong solutions to the Stokes approximation equations for two-dimensional compressible flows. J. Diff. Eqs. 221(2), 275-308 (2006)

[22] Li J., Matsumura, A.: On the Navier-Stokes equations for three-dimensional compressible barotropic flow subject to large external potential forces with discontinuous initial data. J. Math. Pures Appl. 95(5), 495-512 (2011)

[23] Lions, P. L.: Mathematical Topics in Fluid Mechanics. Vol. 1. Incompressible Models. Oxford: Oxford Science Publication, 1998

[24] Lions, P. L.: Mathematical Topics in Fluid Mechanics. Vol. 2. Compressible Models. Oxford: Oxford Science Publication, 1998

[25] Matsumura, A., Nishida, T.: The initial value problem for the equations of motion of viscous and heat-conductive gases. J. Math. Kyoto Univ. 20, 67-104 (1980)
[26] Matsumura, A., Nishida, T.: Initial boundary value problems for the equations of motion of compressible viscous and heat-conductive fluids. Commun. Math. Phys. 89, 445-464 (1983)

[27] Matsumura A., Yamagata N.: Global weak solutions of the Navier-Stokes equations for multidimensional compressible flow subject to large external potential forces. Osaka J. Math. 38(2), 399-418 (2001)

[28] Novotny, A., Straskraba, I.: Stabilization of weak solutions to compressible Navier-Stokes equations. J. Math. Kyoto Univ. 40(2), 217-245 (2000)

[29] Nash, J.: Le Problème de Cauchy pour les équations différentielles d’un fluide général. Bull. Soc. Math. France 90, 487-497 (1962)

[30] Salvi, R., Straskraba, I.: Global existence for viscous compressible fluids and their behavior as $t \to \infty$. J. Fac. Sci. Univ. Tokyo Sect. IA. Math. 40, 17-51 (1993)

[31] Serrin, J. On the uniqueness of compressible fluid motion. Arch. Rational. Mech. Anal. 3 (1959), 271-288.

[32] Shibata, Y., Tanaka, K.: On the steady flow of compressible viscous fluid and its stability with respect to initial disturbance. J. Math. Soc. Japan 55(3), 797-826 (2003)

[33] Vaigant, V. A., Kazhikhov, A.: On the existence of global solutions to two dimensional Navier-Stokes equations of a compressible viscous fluid (in Russian). Sibirski Mat. Z. 36(6), 1283-1316 (1995)

[34] Valli, A.: Periodic and stationary solutions for compressible Navier-Stokes equations via a stability method. Ann. Scuola. Norm. Super. Pisa CI. Sci. IV 10, 607-647 (1983)

[35] Xin, Z. P.: Blowup of smooth solutions to the compressible Navier-Stokes equations with compact density. Comm. Pure Appl. Math. 51, 229-240 (1998)

[36] Zlotnik, A. A.: Uniform estimates and stabilization of symmetric solutions of a system of quasilinear equations. Diff. Eqs. 36, 701-716 (2000)