ABSTRACT

We show that the coefficients of a decomposition into an irreducible components of the tensor powers of level \( r \) symmetric algebra of adjoint representation coincide with the Verlinde numbers. Also we construct (for \( sl(2) \)) the representations of a general linear group those dimensions are given by corresponding Verlinde’s numbers.

In this note we sum up some results dealing with a level \( r \) fusion algebra \( F_r(\mathfrak{g}) \) for the semisimple Lie algebra \( \mathfrak{g} \). In fact we will try to describe in details the fusion algebra for \( \mathfrak{g} = gl(n) \) and even for \( \mathfrak{g} = sl(2) \) leaving the general case as a collection of likelihood conjectures. Our leading idea is to model the Kostant results \([Ko1]\), \([Ko2]\) about the structure of the symmetric algebra of adjoint representation of the Lie algebra \( \mathfrak{g} \) for the case of quantum deformation of universal enveloping algebra \( U_q(\mathfrak{g}) \) at root of unity: \( q^r = 1 \).

§1. Multiplication in the \( sl(2) \) fusion algebra. The Bethe ansatz approach.

We start with a consideration of the Lie algebra \( \mathfrak{g} = sl(2) \). In this case a level \( r \) fusion algebra \( F_r \) is defined as a finite dimensional algebra over real numbers \( \mathbb{R} \) with generators \( \{v_j \mid j = 0, 1/2, 1, 3/2, \ldots, r - 2/2\} \) and the following multiplication rules (the level \( r \) Clebsch-Gordan series):

\[
v_{j_1} \otimes v_{j_2} = \sum_{j = |j_1 - j_2|, j - j_1 - j_2 \in \mathbb{Z}} v_j.
\]

Let us remark that the fusion rules (1.1) correspond to decomposing the tensor product \( V_{j_1} \otimes V_{j_2} \) of restricted representations \([Ro]\) \( V_{j_1} \) and \( V_{j_2} \) of the Hopf algebra \( U_q(sl(2)) \), when \( q = \exp(\frac{2\pi i}{r}) \), into the irreducible parts (see e.g. \([Lu]\)). It is
Anatol N. Kirillov also well-known (see e.g. [Ka]) that the fusion algebra \( F(g) \) is a commutative and associative one.

**Theorem 1.1.** Let us consider a decomposition of a product \( v_{j_1} \otimes \ldots \otimes v_{j_l} \) in the fusion algebra \( F_r \):

\[
v_{j_1} \otimes v_{j_2} \otimes \ldots \otimes v_{j_l} = \sum_k a_j(k)v_k.
\]

Then

\[
a_j(k) = \sum_{\{\nu\}} \prod_{n \geq 1} \left( P_{n,r}(\nu,j) + m_{n}(\nu) \right), \tag{1.2}
\]

where a summation in (1.2) is taken over all partitions \( \nu = (\nu_1 \geq \nu_2 \geq \cdots \geq 0) \) of the number \( \sum_{s=1}^l j_s - k \) such that for all \( n \geq 1 \) the following inequalities are valid

\[
P_{n,r}(\nu,j) := \sum_{s=1}^l \min (n, 2j_s) - 2Q_n(\nu) - \max (2k + n - r + 2, 0) \geq 0. \tag{1.3}
\]

Here

\[
Q_n(\nu) := \sum_{i \geq 1} \min (n, \nu_i) = \sum_{i \leq n} \nu'_i,
\]

\[
m_n(\nu) := \nu'_n - \nu'_{n+1}
\]

be the number of parts of the partition \( \nu \) which are equal to \( n \) and

\[
\binom{m+n}{n} = \frac{(m+n)!}{m!n!}
\]

be a binomial coefficient.

A proof of Theorem 1.1 is based on an investigation of the Bethe equations for RSOS models [BR]. Note that validity of the equalities (1.2) for fixed \( j_1, \ldots, j_l \) and for all \( 0 \leq k \leq r - 2 \), are equivalent to a combinatorial completeness (e.g. [Ki]) of the Bethe ansatz for RSOS models (for the case \( g = sl(2) \)).

Now let us consider a simple example: \( r = 5, \; j_1 = \ldots = j_l = 1 \). We have two “even” representations in the fusion algebra \( F_5 \), namely, \( V_0 \) and \( V_1 \) and \( V_0 \otimes V_1 = V_0 + V_1 \). Let us put \( V_i^\otimes = a_i V_0 + b_i V_1 \). Then it is easy to see that \( a_{l+1} = b_l \) and \( b_{l+1} = a_l + b_l = b_l + b_{l-1} \). Consequently, we have \( a_l = F_{l-1} \) and \( b_l = F_l \), where \( F_l \) be the \( l \)-th Fibonacci number. Hence from Theorem 1.1 we obtain

**Corollary 1.2.** Let \( F_l \) be the \( l \)-th Fibonacci number. Then we have

\[
1) \quad F_{l-1} = \sum_{\{\nu\}} \prod_{n \geq 1} \left( P_{n,5}(\nu,\bullet) + m_{n}(\nu) \right), \tag{1.4}
\]
where a summation in (1.4) is taken over all partitions \( \nu \) such that
\[
\begin{align*}
  i) & \quad |\nu| = l \\
  ii) & \quad P_{n,5}(\nu, \bullet) := \ell \min (n, 2) - 2Q_n(\nu) - \max (n - 3, 0) \geq 0.
\end{align*}
\]

where a summation in (1.5) is taken over all partitions \( \nu \) such that
\[
\begin{align*}
  i) & \quad |\nu| = l - 1 \\
  ii) & \quad \bar{P}_{n,5}(\nu, \bullet) := \ell \min (n, 2) - 2Q_n(\nu) - \max (n - 1, 0) \geq 0.
\end{align*}
\]

Let us underline that formulae (1.4) and (1.5) give the different expressions for the Fibonacci numbers. For example, a formula (4) gives for \( F_7 \) the following expression in terms of rigid configurations
\[
\begin{array}{cccc}
0 & 4 & 0 & 0 \\
5 & 1 & 6 & 1 + 8 + 4 = 13
\end{array}
\]
where as the formula (1.5) gives the following one
\[
\begin{array}{cccc}
0 & 4 & 0 & 0 \\
1 & 3 & 0 & 1 + 8 + 4 = 13.
\end{array}
\]

It is possible to give the bijective proofs for identities (1.4) and (1.5). Furthermore, using (1.4) and (1.5) one can construct two Fibonacci lattice (e.g. [St1], [St2]). However, we do not assume to give here any combinatorial details about very interesting combinatorial objects related with fusion algebra: restricted Young tableaux, restricted Kostka-Foulkes polynomials, restricted Littlewood-Richardson rule. All these things deserve a separate publication.

Now let us return back to our example concerning with the fusion algebra \( F_5(sl(2)) \). It seems a very interesting task to find the natural \( q \)-analogs for identities (1.4) and (1.5). We leave for another publication an exact construction of such \( q \)-analogs, but here let us consider the following well-known \( q \)-analogue of the Fibonacci numbers \( F_l(q) \). Namely, let us define
\[
F_0(q) = 0, \quad F_1(q) = 1, \quad F_{l+1}(q) = qF_l(q) + F_{l-1}(q), \quad l \geq 1.
\]

It is easy to see that
\[
\sum_{l \geq 1} F_l(q) \cdot t^l = \frac{t}{1 - qt - t^2}, \quad \text{and}
\]
\[
F_l(q) = \det | \delta_{i,j} + \delta_{i,j-1} - q \delta_{i,j+1} |_{1 \leq i,j \leq l-1}.
\]
In order to understand better the relations between the $q$-Fibonacci numbers (1.6) and fusion algebra $\tilde{F}_5$, let us introduce an algebra $\tilde{F}_5 = \{v_0, v_1 \mid v_1 \cdot v_1 = v_0 + qv_1\}$.

Then it is easy to see that

$$v_1^l = F_l(q) \cdot v_0 + F_l(q) \cdot v_1.$$  \hfill (1.7)

It is possible to rewrite (1.7) as follows

$$\begin{pmatrix} F_l \\ F_{l+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & q \end{pmatrix} \begin{pmatrix} F_{l-1} \\ F_l \end{pmatrix}.$$  

Note that the matrices $\begin{pmatrix} 0 & 1 \\ 1 & x \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 1 & y \end{pmatrix}$ do not commute (if $x \neq y$):

$$\begin{pmatrix} 0 & 1 \\ 1 & x \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & y \end{pmatrix} = \begin{pmatrix} 1 & y \\ x & 1 + xy \end{pmatrix} \neq \begin{pmatrix} 0 & 1 \\ 1 & y \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & x \end{pmatrix}.$$  

However, it is easy to check that if we put $h(x) = \begin{pmatrix} 1 & x \\ x & 1 + x \end{pmatrix}$ then $h(x)$ and $h(y)$ are commute. Now let us define a vacuum vector $|0> : = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and consider the following vectors

$$h_n(x)|0> := h(x_1)h(x_2)\ldots h(x_n)|0> = \begin{pmatrix} a_n(x) \\ b_n(x) \end{pmatrix},$$  \hfill (1.8)

$$s_n(x)|0> := 2^n h_n\left(\frac{x}{2}\right)|0> = \begin{pmatrix} c_n(x) \\ d_n(x) \end{pmatrix}.$$  

From what it was said above one can deduce that $a_n(x)$ and $b_n(x)$ are to be the symmetric functions on $x = (x_1, \ldots, x_n)$. It is a simple exercise to decompose these symmetric functions into a linear combination of the Schur functions $s_\lambda(x)$.

**Exercise 1.** Let us check:

$$a_n(x) = 1 + \sum_{k=1}^{n-1} F_k \cdot s_{(1^k+1)}(x),$$

$$b_n(x) = \sum_{k=1}^{n} F_k \cdot s_{(3^k)}(x),$$

$$c_n(x) = 2^n + \sum_{k=1}^{n-1} 2^{n-1-k} F_k s_{(1^k+1)}(x),$$  \hfill (1.9)

$$d_n(x) = \sum_{k=1}^{n} 2^{n-k} F_k s_{(1^k)}(x).$$
Now let us calculate the values of $a_n(1)$ and $b_n(1)$. For this goal we consider the following elements in the fusion algebra $\mathcal{F}_5(sl(2))$

\[
H(x) = v_0 + x v_1 = h(x)|0>, \quad H := H(1),
\]

\[
S(x) = 2v_0 + x v_1 = \left(\begin{array}{c} 2 \\ x \\ 2 + x \end{array}\right)|0>, \quad S := S(1).
\]

We must compute the coefficients of decompositions

\[
H^n := a_nv_0 + b_nv_1,
\]

\[
S^n := c_nv_0 + d_nv_1.
\]

Here we give an answer only for $a_n$ and $c_n$.

From a definition it is clear that

\[
a_{n+1} = a_n + b_n, \quad b_{n+1} = a_n + 2b_n, \quad c_{n+1} = 2c_n + d_n, \quad d_{n+1} = c_n + 3d_n.
\]

Consequently, if we put $\varphi(t) = \sum_{n \geq 1} a_nt^n$ and $\psi(t) = \sum_{n \geq 1} c_nt^n$ then

\[
\varphi(t) = \frac{t - t^2}{1 - 3t + t^2}, \quad \psi(t) = \frac{2t - 5t^2}{1 - 5t + 5t^2}.
\]  \hspace{1cm} (1.10)

Using the generating functions (1.10), one can easily find ($n > 0, \quad k = 3$):

\[
a_n = a_n(1) = \frac{2}{k+2} \sum_{m=0}^{k} \sin^2 \left(\frac{(m+1)\pi}{k+2}\right) \left(2 \cos \left(\frac{(m+1)\pi}{k+2}\right)\right)^{-2(n-1)},
\]

\[
c_n = c_n(1) = \frac{1}{2} \sum_{m=0}^{k} \left(\sqrt{\frac{4}{k+2}} \sin \left(\frac{(m+1)\pi}{k+2}\right)\right)^{-2(n-1)}.
\]

In particular, we have the following equalities

\[
c_n = 2^n + \sum_{k=1}^{n-1} 2^{n-1-k} \left(\begin{array}{c} n \\ k+1 \end{array}\right) F_k,
\]

\[
a_n = 1 + \sum_{k=1}^{n-1} \left(\begin{array}{c} n \\ k+1 \end{array}\right) F_k,
\]

\[
d_n = \sum_{k=1}^{n} \left(\begin{array}{c} n \\ k \end{array}\right) F_k.
\]

Let us sum up the results of our computations in the fusion algebra $\mathcal{F}_5$. First of all, it was shown (for $r = 5$) that a multiplicity of a representation $V_0$ in the $n$-fold restricted tensor product $\hat{S}^\otimes n$ of the level $r$ symmetric algebra $S(= 2V_0 + V_1)$ is equal to the Verlinde number $V(r - 2, n)$:

\[
[V_0 : \hat{S}^\otimes n] = V(r - 2, n) := \frac{1}{2} \sum_{m=0}^{r-2} (S_0m)^{-2(n-1)},
\]
where
\[ S_{jm} = \sqrt{\frac{4}{r}} \sin \left( \frac{(j + 1)(m + 1)\pi}{r} \right). \]

After this we proved an existence of a \( gl(n) \)-module \( V(0) \) such that \( \dim V(0) = V(r - 2, n) \) and computed its character:
\[ \text{ch} V(0) = \sum_{\lambda} a\lambda W\lambda, \]
where \( a\lambda \in \mathbb{Z}^+, \ l(\lambda) \leq n, \ l(\lambda') \leq \frac{r - 3}{2} \) and \( W\lambda \) be an irreducible representation of \( gl(n) \) with the highest weight \( \lambda \).

\section{Verlinde character.}

In this section we study a decomposition of a product of some distinguish elements in the fusion algebra \( F_r := F_r(sl(2)) \). Thus we start with a definition of these elements. Let us define a level \( r \) symmetric algebra \( S_r \) and a module of level \( r \) harmonic polynomials \( H_r \) as follow
\[ S_r = \sum_{k=0}^{r-3} \left( \frac{r-1}{2} - k \right) x^k \cdot V_k \in F_r[x], \]
\[ H_r = \sum_{k=0}^{r-3} x^k \cdot V_k \in F_r[x]. \quad (2.1) \]

In sequel we will assume that \( r \equiv 1(\text{mod } 2) \). Our nearest aim is to find the coefficients in the following decompositions
\[ S_r^\otimes n = \sum_{k=0}^{r-3} a_{k,n}(x) \cdot V_k, \quad (2.2) \]
\[ H_r^\otimes n = \sum_{k=0}^{r-3} b_{k,n}(x) \cdot V_k. \]

But at the beginning we find the values \( a_{k,n}(1) \) and \( b_{k,n}(1) \).

\textbf{Theorem 2.1.} We have
\[ a_{0,n}(1) = \frac{1}{2} \sum_{m=0}^{r-2} \left( \sqrt{\frac{4}{r}} \sin \frac{(m + 1)\pi}{r} \right)^{2-2n}, \quad (2.3) \]
\[ b_{0,n}(1) = \frac{2}{r} \sum_{m=0}^{r-2} \sin^2 \frac{(m + 1)\pi}{r} \left( 2 \cos \frac{(m + 1)\pi}{r} \right)^{2-2n}. \]
Sketch of a proof. We start with a solution of a local problem, namely, we want to find a connection matrix $M(x)$ (resp. $N(x)$) such that
\begin{align}
\mathcal{S}_r^\otimes(n+1) &= M(x) \cdot \mathcal{S}_r^\otimes n, \\
\mathcal{H}_r^\otimes(n+1) &= N(x) \cdot \mathcal{H}_r^\otimes n.
\end{align}

**Proposition 2.2.** The connection matrices $M(x) = (m_{ij}(x))$ and $N(x) = (n_{ij}(x))$ have the following matrix elements ($1 \leq i, j \leq \frac{r-1}{2}$)
\begin{align}
m_{ij}(x) &= x^{i-j} \left\{ \sum_{k=0}^{\min(2i-2, r-2j)} \left( \frac{r-1}{2} + i - j - k \right) x^k \right\}, \\
n_{ij}(x) &= x^{i-j} \left\{ \sum_{k=0}^{\min(2i-2, r-2j)} x^k \right\}.
\end{align}

The statement of Proposition 2.2 may be verified by direct computation using the definitions (2.1).

**Corollary 2.3.** We have
\begin{align}
m_{ij}(1) &= \frac{1}{2} (2i - 1)(r - 1 - 2j), \quad \text{if} \quad 1 \leq i \leq j \leq \frac{r-1}{2}, \\
\text{and} \quad m_{ij}(1) &= m_{ji}(1) \quad \text{if} \quad i \geq j.
\end{align}

The next step of a proof of Theorem 2.1 is the following observation.

**Proposition 2.4.** Matrix $M(1)$ admits a decomposition
\[ M(1) = T \cdot C^{-1}, \]
where $C = (c_{ij})$, $c_{ij} = 2\delta_{i,j} - \delta_{i,j+1} - \delta_{i,j-1}$ ($1 \leq i, j \leq \frac{r-1}{2}$) be the Cartan matrix of type $A_{r-1}$ and $T = (t_{ij})$ be a lower triangle matrix with elements
\begin{align}
t_{11} &= \frac{r+1}{2} = \det C, \\
t_{ij} &= r \quad \text{and} \quad t_{ij} = -\left( \frac{r+1}{2} - j \right), \quad \text{if} \quad 2 \leq j \leq \frac{r-1}{2}, \\
t_{ij} &= 0 \quad \text{otherwise}.
\end{align}

Using Proposition 2.4 we may find the spectra of matrix $M(1)$ and solve a recurrence relation (2.4) for $\mathcal{S}_r^\otimes n$. As a result, we obtain the first formula of (2.3).

Now we want to construct the representations $\mathcal{V}(k)$ and $\mathcal{W}(k)$, $1 \leq k \leq \frac{r-1}{2}$, of the Lie algebra $\mathfrak{gl}(n)$ such that
\begin{align}
\dim \mathcal{V}(k) &= a_{k,n}(1), \quad \dim \mathcal{W}(k) = b_{k,n}(1).
\end{align}
For this aim we use the following observation
Proposition 2.5. If the matrices \( M(x) \) and \( N(x) \) are defined by the formulae (2.5) and (2.6), then they satisfy the commutation relations

\[
M(x) \cdot M(y) = M(y) \cdot M(x),
\]
\[
N(x) \cdot N(y) = N(y) \cdot N(x),
\]

Finally, let us define a vacuum vector \( |0\rangle = (1, 0, \ldots, 0)^t \in R^{r-1} \) and consider the following vectors

\[
M(x_1) \ldots M(x_n) |0\rangle = (\chi_{1,n}(x), \ldots, \chi_{r-1,n}(x))^t,
\]
\[
N(x_1) \ldots N(x_n) |0\rangle = (\varphi_{1,n}(x), \ldots, \varphi_{r-1,n}(x))^t.
\]

It is clear from our construction that

\[
\chi_{k,n}(1) = a_{k,n}(1) \quad \text{and} \quad \varphi_{k,n}(1) = b_{k,n}(1),
\]
\[
\chi_{k,n}(x, \ldots, x) = a_{k,n}(x) \quad \text{and} \quad \varphi_{k,n}(x, \ldots, x) = b_{k,n}(x).
\]

Theorem 2.6. The symmetric functions \( \chi_{k,n}(x) \) and \( \varphi_{k,n}(x) \), \( 1 \leq k \leq \frac{r-1}{2} \), may be expressed as the linear combinations of the Schur functions with positive integer coefficients, i.e.

\[
\chi_{k,n}(x) = \sum_\lambda a_{\lambda} s_\lambda(x),
\]

where for all partitions \( \lambda \) we have \( a_\lambda \in Z^+ \) and if \( a_\lambda \neq 0 \) then \( l(\lambda) \leq n \) and \( l(\lambda') \leq \frac{r-3}{2} \).

It seems an interesting task to define a natural action of the Lie group \( GL(n) \) on the space \( H^0(M_n, L^{\otimes(r-2)}) \) such that

\[
\text{char } H^0(M_n, L^{\otimes(r-2)}) = \chi_{0,n}(x).
\] (2.7)

Probably, such action may be extracted from [Kh].

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