Einstein Equations in the Null Quasi-spherical Gauge

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Abstract

Properties of the Einstein equations in a coordinate gauge based on expanding null hypersurfaces foliated by metric 2-spheres are described. This null quasi-spherical (NQS) gauge leads to particularly simple analyses of the characteristic structure of the equations and of the propagation of gravitational shocks, and clarifies the geometry of timelike boundary condition. A feature of the NQS gauge is the use of the standard $\bar{\partial}$ (“edth”) operator on $S^2$ to express angular derivatives, and the consequent use of spin-weighted spherical harmonic decompositions of the metric fields.

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The objectives of numerical relativity in particular, force us to address the problem of constructing explicit formulations of the full Einstein equations. There are two main avenues of approach, respectively using either a 3+1 ADM representation [1], which leads to a Cauchy initial value problem, or a characteristic coordinate, an approach first considered by Bondi [2,3].

Both the second order and the recently popular first order formulations of the 3+1 equations involve coordinate gauge freedoms which must be eliminated through gauge-fixing conditions which typically involve subsidiary differential equations. First order formulations in particular introduce large numbers of auxiliary variables to parameterise the field, and these variables must satisfy additional compatibility relations. Whilst these features do not present difficulties of a conceptual or theoretical nature, they do cause difficulties in the practical numerical formulation of the equations — in disentangling geometric from coordinate gauge and numerical algorithm effects [4], in formulating the numerical algorithm to ensure that the constraints and compatibility relations are preserved by the evolution [4,5], and in simply constructing codes handling large number of variables.

Characteristic methods [9], [10] on the other hand typically involve very little coordinate freedom and lead to formulations of the equations involving a small number of variables, with few compatibility conditions. However, the advantage of simpler representations of the equations is balanced by the disadvantage that characteristic coordinates are generically not globally applicable, for either topological or caustic reasons. However, a characteristic-based coordinate system can be very useful when available — for example, in describing asymptotic structure [2], or for describing discontinuity properties of vacuum spacetimes [11].

The Newman-Penrose formulation in terms of null tetrad based connection ("spin") coefficients [12] can be used to describe the characteristic structure of the Einstein equations, either by regarding the equations as determining the propagation of the spin coefficients [13,14], or by introducing coordinates and determining the spin coefficients in terms of the metric parameters (eg. [10]). The former ("pure NP") technique introduces numerous compatibility differential relationships amongst the spin coefficients, which suggests that
more variables have been introduced than are necessary to fully describe the connection. The latter approach bridges the gap between coordinate- and connection-based techniques, and is the starting point for this paper.

Ideally, a “good” parameterisation of the Einstein equations will have limited or no gauge freedom (to suppress unphysical gauge modes in the numerical evolution, for example \[4\]), with parameters free of constraints and having a direct relation to known radiation parameters, and leading to simple expressions for the Einstein equations, to facilitate analysis of the local existence, decay etc.

The null quasi-spherical gauge presented here is a modification of the usual characteristic formulations \[2\], \[10\] which satisfies many of these criteria. It combines a characteristic gauge (exploiting the ode structure of the Einstein on a null hypersurface \[1\]) with the reduced gauge-dependence of the null tetrad methods \[12\], but avoids the complications of the spin coefficient integrability conditions. The quasi-spherical condition means that all derivatives can be presented using radial and time partial derivatives, combined with canonical differential operators on \(S^2\) with the standard metric.

An interesting feature is the central role played by the NQS metric field \(\beta\), which determines in particular both the shear \(\sigma_{NP}\) of the outgoing null curves of the null coordinate, and the intrinsic geometry of the null hypersurfaces. Given \(\beta\) on one null hypersurface, the remaining NQS parameters are determined by integrating a system of radial ordinary differential equations which generalise the usual null Raychaudhuri equation (which is just the first equation \(8\) of the system). Heuristically, \(\beta\) may be regarded as describing the ingoing gravitational radiation passing through the outgoing null hypersurfaces.

The radial system of equations for the metric parameters is of course well-known in similar situations \[2,10\] and can be understood in terms of the NP formulation of the equations for Einstein tensor terms \(\Phi_{00}, \Phi_{01}, \Phi_{02}\) and \(\Phi_{11} + 3\Lambda\), which can be expressed solely in terms of derivatives tangent to the null hypersurfaces of the NP spin coefficients.

However, closer examination of the explicit NQS coordinate representations shows that the hypersurface equations may be expressed in terms of hypersurface-tangential derivatives
of the metric functions and just a subset of the connection parameters (see Eqns. (8–11) below). In all, we obtain a complete parameterisation of both the metric and the connection in terms of just 12 real variables, made up of 2 real and 2 complex variables for each of the metric \((u, v, \beta, \gamma)\) and the connection \((H, J, Q, K)\). Not only is this cheaper than the NP parameterisation (which requires 12 complex variables for the connection alone), but it is also explicit in the sense that there are no auxiliary compatibility conditions. The resulting expressions for the Einstein tensor (8–11), (16–17), (18) and Weyl tensor (22–26) components are comparable in simplicity with the usual NP expressions, yet have the advantage that they are completely explicit and involve fewer variables.

The reduced set \((H, J, K, Q)\) of connection parameters was found by careful examination of the explicit expressions for the hypersurface Einstein tensor components in the NQS metric. It seems quite likely that a similar examination using instead other characteristic-based metric forms (eg. Bondi, Newman-Unti) will uncover other reduced (“beyond NP”) parameterisations for the connection.

Because of the very explicit form of the Einstein equations in the NQS gauge, it enables us to give simple descriptions of several relatively well-known computations, such as: the null hypersurface matching conditions and the propagation laws for gravitational (vacuum) shock waves \([11, 13]\); a description of the free boundary data for the Einstein equations on a timelike boundary surface; and the relationship between the outgoing Einstein tensor component and the geodesic parameterisation of the outgoing null rays. Two applications which are not discussed here, but which are quite feasible within the NQS framework, are the rederivation of the linearised Einstein equations \([16, 18]\), and the asymptotic form of a vacuum NQS metric. The application which motivated the present study, namely the numerical evolution of of a vacuum black hole spacetime in the NQS gauge, will be described elsewhere \([19, 20]\).
The central idea is to assume that the radial function foliates the null hypersurfaces by metric spheres, of area \(4\pi r^2\). A similar foliation was exploited in \cite{21} to construct spatial initial data satisfying the Hamiltonian constraint. Denoting the null coordinate by \(z\) (cf. \cite{11}) and letting \(\vartheta, \varphi\) denote standard polar coordinates on the spheres, the most general compatible metric (assuming that \(r\) is a valid coordinate) is

\[
ds_{NQS}^2 = -2u \, dz (dr + v \, dz) + 2 |r \bar{\theta} + \beta \, dr + \gamma \, dz|^2,
\]

where we use a complex notation for simplicity, with

\[
\theta = \frac{1}{\sqrt{2}} (d\vartheta + i \sin \vartheta \, d\varphi),
\]

\[
\beta = \frac{1}{\sqrt{2}} (\beta^1 - i \beta^2), \quad \gamma = \frac{1}{\sqrt{2}} (\gamma^1 - i \gamma^2).
\]

Thus \(u, v\) and \(\beta, \gamma\) are metric parameters, with \(u, v\) real and \(u > 0\), and we may regard \(\beta, \gamma\) either as spin-1 (complex) fields or as vector fields tangent to the spheres, via \(\beta \sim \beta^1 \partial_{\theta} + \beta^2 \csc \vartheta \partial_{\varphi}\).

The metric form Eqn. (1) differs from other parameterisations of the metric with a characteristic coordinate \cite{3, 4, 12, 11, 22} principally in the choice of radial coordinate. For example, Bondi-Sachs use a luminosity parameter determined by the volume form in the angular directions \cite{23} (with the consequent disadvantage that the luminosity coordinate depends strongly on the specific choice of angular coordinate labelling of the outgoing null curves), and Newman-Unti use the geodesic affine parameter (and thus the radial coordinate is not determined uniquely from the intrinsic induced metric on the null hypersurfaces).

\[1\]Our metric has signature +2, the expressions below for the NP spin coefficients have been adjusted to give the usual signs, and the curvature tensor sign convention has been chosen so that \(G_{\ell\ell} = 2\Phi_{00}\) is non-negative for physically reasonable matter.
However, the NQS radius and radial curves are determined by the metric $S^2$ foliation and standard polar coordinates, so that the NQS radial curves $(z, \vartheta, \varphi) = \text{const.}$ do not in general coincide with the null generating curves.

For the purposes of this paper it is sufficient to regard the metric form (1) as an ansatz (assumption) rather than as a coordinate condition. The important question of whether this ansatz is generic, in the sense that coordinates satisfying the NQS conditions may be found in all metrics in some open subset of the space of “all” metrics, does not yet have a complete answer. Some of the evidence which suggests that the NQS coordinates are generically available is discussed below; we note also that the gauge used by Regge, Wheeler and Zerilli \cite{16,17} in the study of the linearisation of the Einstein equations about Schwarzschild, is just a linearised NQS gauge. Thus the NQS gauge could be used to extend the Regge-Wheeler-Zerilli perturbation analysis to include nonlinear effects. (The RWZ equations in the NQS gauge were rederived in \cite{18}.

The complex notation suggests encoding the angular derivatives using the differential operator $\bar{\partial}$ ("edth" \cite{24,25}), defined on a spin-$s$ field $\phi$ by

$$\bar{\partial}\phi = \frac{i}{\sqrt{2}} \sin^s \vartheta \left( \frac{\partial}{\partial \vartheta} - \frac{i}{\sin \vartheta} \frac{\partial}{\partial \varphi} \right) (\sin^{-s} \vartheta \phi).$$  \hspace{1cm} (2)

All the standard differential operators on $S^2$ may be expressed in terms of $\bar{\partial}$, for example:

$$\text{div} \beta = \bar{\partial} \bar{\beta} + \bar{\partial} \beta = \beta_{1}^{1} + \beta_{2}^{2},$$

$$\text{curl} \beta = i (\bar{\partial} \beta - \bar{\partial} \bar{\beta}) = \beta_{1;2} - \beta_{2;1},$$

$$\Delta \phi = (\bar{\partial} \bar{\partial} + \bar{\partial} \bar{\partial}) \phi.$$

Here the indices refer to the $S^2$-orthonormal frame $\partial_\vartheta$, $\csc \vartheta \partial_\varphi$, and the standard $S^2$ covariant derivative. The spin-weighted spherical harmonics $Y_{s \ell m}$ are then eigenfields of the Laplacian, satisfying $\Delta Y_{s \ell m} = (s^2 - l(l+1)) Y_{s \ell m}$, $|s| \leq l$, and $Y_{s \ell m}$ may be taken proportional to $\bar{\partial}^s Y_{\ell m}$ for $s \geq 0$ and $\bar{\partial}^{-s} Y_{\ell m}$ for $s < 0$, where $Y_{\ell m}$ are the usual spherical harmonic functions.

To compute the curvature of $ds^2_{\text{NQS}}$ we introduce the vector frame $(\ell, n, m, \bar{m})$, null with respect to $ds^2_{\text{NQS}}$:
\begin{align*}
\ell &= \partial_r - r^{-1}\partial_\beta, \\
n &= u^{-1}(\partial_z - r^{-1}\partial_\gamma - v(\partial_r - r^{-1}\partial_\beta)), \\
m &= \frac{1}{r\sqrt{2}}(\partial_\theta - i \csc \vartheta \partial_\varphi),
\end{align*}

and its associated dual (null) coframe $dr + v dz, u dz, r \vartheta + \bar{\beta} dr + \bar{\gamma} dz$ and $r \bar{\vartheta} + \beta dr + \gamma dz$.

Note that $\partial_\beta$ denotes the angular directional derivative $\beta^1 \partial_\vartheta + \beta^2 \csc \vartheta \partial_\varphi$, which coincides with the $S^2$-covariant derivative $\nabla_\beta = \bar{\beta} \partial_\vartheta + \beta \partial_\varphi$ when acting on functions.

It will be useful to introduce the differential operators

\begin{align*}
\mathcal{D}_r := \partial_r - r^{-1}\nabla_\beta &= \partial_r - r^{-1}(\beta \partial_\vartheta + \beta \varphi), \\
\mathcal{D}_z := \partial_z - r^{-1}\nabla_\gamma &= \partial_z - r^{-1}(\gamma \partial_\vartheta + \gamma \varphi),
\end{align*}

which have a natural interpretation as spin weight 0 operators. Since we consider here only integer spins, we may use the complexified cotangent bundle $T^{(1,0)}_\mathbb{C}S^2$ as a model for the spin-1 line bundle $\mathcal{L}^1$. From $\mathcal{L}^1$ we construct the spin-$s$ complex line bundle $\mathcal{L}^s$ over $\mathbb{R} \times \mathbb{R}^+ \times S^2$, whose sections are just spin-$s$ fields depending also on the parameters $(z, r)$.

Then the action of $\mathcal{D}_r, \mathcal{D}_z$ on spin-$s$ fields is defined via the standard covariant derivative on $S^2$, extended naturally to $T^{(1,0)}_\mathbb{C}S^2$ and $\mathcal{L}^s$.

The structure of the hypersurface equations motivates the introduction of the NQS connection parameters $H, J, K, Q, Q^\pm$, defined by:

\begin{align*}
H &= u^{-1}(2 - \text{div}\beta), \\
J &= \text{div}\gamma + v(2 - \text{div}\beta), \\
K &= v\bar{\beta} - \bar{\gamma}, \\
Q &= r\mathcal{D}_z\beta - r\mathcal{D}_r\gamma + \gamma, \\
Q^\pm &= u^{-1}(Q \pm \partial_\vartheta).
\end{align*}

These have well-defined spin-weights under $S^2$ frame rotations: $u, v, H, J$ have spin 0 (and are real), $\beta, \gamma, Q, Q^\pm$ have spin 1, and $K$ has spin 2.
The connection $\hat{\nabla}$ of $ds^2_{NQS}$ may be described in terms of the NQS parameters either via the Cartan connection coefficients $\omega_{abc} = g(a, \hat{\nabla}_c b)$, $a, b, c = \ell, n, m, \bar{m}$, or via the Newman-Penrose spin coefficients [12]:

$$
\kappa_{NP} = \omega_{\ell m \ell} = 0
$$

$$
\sigma_{NP} = \omega_{\ell m m} = r^{-1} \partial \beta
$$

$$
\rho_{NP} = \omega_{\ell \bar{m} \bar{m}} = - \frac{1}{2} r^{-1} (2 - \text{div} \beta)
$$

$$
\epsilon_{NP} = \frac{1}{2} (\omega_{\ell n \ell} + \omega_{m n \ell}) = \frac{1}{2} u^{-1} D_r u + i r^{-1} \left( \frac{1}{\sqrt{2}} \cot \vartheta \Im(\beta) + \frac{1}{4} \text{curl} \beta \right)
$$

$$
\gamma_{NP} = \frac{1}{2} (\omega_{\ell m n} + \omega_{m n m})
$$

$$
= \frac{1}{2} u^{-1} D_r v + i r^{-1} u^{-1} \left( \frac{1}{\sqrt{2}} \cot \vartheta \Im(\gamma - v \beta) + \frac{1}{4} (\text{curl} \gamma - v \text{curl} \beta) \right)
$$

$$
\tilde{\alpha}_{NP} + \beta_{NP} = \tilde{\pi}_{NP} = \omega_{\ell m m} = \omega_{m n \ell} = \frac{1}{2} r^{-1} Q^+
$$

$$
\tilde{\alpha}_{NP} - \beta_{NP} = \omega_{m \bar{m} m} = - \frac{1}{\sqrt{2}} r^{-1} \cot \vartheta
$$

$$
\tau_{NP} = \omega_{\ell m n} = \frac{1}{2} r^{-1} Q^-
$$

$$
\tilde{\lambda}_{NP} = \omega_{m m n} = r^{-1} u^{-1} K
$$

$$
\mu_{NP} = \omega_{\bar{m} m m} = - \frac{1}{2} r^{-1} u^{-1} J
$$

$$
\bar{v}_{NP} = \omega_{m m m} = r^{-1} u^{-1} \partial v
$$

As noted above, the NQS parameters $(u, v, \beta, \gamma)$ and $(H, J, K, Q)$ form a more efficient and complete representation of the metric and connection than the NP coefficients $\alpha_{NP}, \ldots, \tau_{NP}$, in part because they are defined in terms of a specific coordinate-adapted frame $(\ell, n, m, \bar{m})$. The “compatibility” conditions are also much simpler than those required of the NP coefficients. Clearly, $H, J, K, Q$ are determined uniquely from the metric parameters $u, v, \beta, \gamma$. Conversely, if $\beta, H, J, K$ are given then the remaining metric parameters $(u, v, \gamma)$ may be reconstructed as follows: $u$ is found algebraically from $\text{div} \beta$ and $H$; $\gamma$ is found by solving

$$
L_\beta \gamma := \partial \gamma + \frac{\partial \beta}{2 - \text{div} \beta} \text{div} \gamma = J \frac{\partial \beta}{2 - \text{div} \beta} - K \quad (5)
$$

on each $S^2$; and finally $v = (J - \text{div} \gamma)/(2 - \text{div} \beta)$. Observe that the reconstruction process
is local to the null hypersurfaces, since $D_z$ derivatives are not involved, and that $Q$ is not used — it is used instead to determine $\partial \beta / \partial z$.

If $\beta = 0$ then $L_\beta = L_0 = \partial$, an elliptic operator on $S^2$ acting on spin-1 fields, which is surjective and has 6-dimensional kernel consisting of the $l = 1$ spin-1 spherical harmonics $Y_{1m}$, $m = -1, 0, 1$. Consequently if $\beta$ is small (and we will show elsewhere that the pointwise bound

$$|\partial \beta| < 3^{-1/2}(2 - \text{div}\beta)$$

is sufficient), then (3) is solvable for $\gamma$, uniquely if the $l = 1$ components of $\gamma$ are prescribed. The kernel of $L_0$ may be identified with the infinitesimal Lorentz transformations and conformal motions of $S^2$ — this suggests we may interpret this ambiguity in the reconstruction of $\gamma$ as a gauge degeneracy.

Observe that the hypersurfaces are expanding, $\rho_{NP} < 0$, exactly when $\text{div}\beta < 2$, and this condition follows from (3). The deformation term $\partial \beta/(2 - \text{div}\beta)$ in $L_\beta$ (5) is equal to $-\frac{1}{2} \sigma_{NP}/\rho_{NP}$, which is an invariant of the null geometry (type I in the terminology of [15]).

To understand the genericity of the NQS ansatz, consider the effect of an infinitesimal variation $h_{ab}$ of the null metric $ds^2_{NQS}|_N$ on a null hypersurface $N$. A calculation shows that the metric $S^2$ condition is preserved (at the linearised level) if the accompanying coordinate deformation vector $X = r(\zeta \bar{m} + \bar{\zeta} m) + r f \ell$ satisfies

$$L_\beta \zeta = -\frac{1}{2} h_0 \frac{\partial \beta}{2 - \text{div}\beta} - \frac{1}{2} h_2,$$

where $h_0 = h_{ab} m^a \bar{m}^b$, $h_2 = h_{ab} m^a m^b$, and

$$f = -\frac{\text{div}\zeta + \frac{1}{2} h_0}{2 - \text{div}\beta}.$$

Thus if the size condition (3) holds then the linearisation of the NQS gauge can be enforced by solving (4) for $\zeta$; moreover, the existence of NQS coordinates in axially symmetric Bondi metrics was established in [18]. Consequently it is plausible to regard the NQS gauge as a coordinate condition rather than an ansatz. Note also that the NQS parameters $u, v, \beta, \gamma$
have the degrees of freedom \((6 = 10 - 4)\) expected of a general metric after removing all coordinate degeneracies.

The characteristic structure of the Einstein equations \([9]\) becomes particularly clear when they are expressed in NQS coordinates using the connection quantities \(H, J, K, Q\). Let \(G_{ab} = R_{ab} - \frac{1}{2} R g_{ab}\) be the Einstein tensor, normalised by \(G_{\ell\ell} = \ell^a \ell^b G_{ab} = 2\Phi_{00}\). Then \(G_{\ell\ell}, G_{\ell m}, G_{\ell n}, G_{mm}\) are given by the hypersurface equations \([9]\)

\[
rD_r H = \left( \frac{1}{2} \text{div} \beta - \frac{2|\beta|^2}{2 - \text{div} \beta} \right) H, \tag{8}
\]

\[
rD_r Q^- = (\overline{\beta} - u H) Q^- + \overline{Q^-} \beta + 2 \overline{\beta} \beta + u \overline{\beta} - H \overline{\beta} u + 2r^2 G_{\ell m}, \tag{9}
\]

\[
rD_r J = -(1 - \text{div} \beta) J + u - \frac{1}{2} u |Q^+|^2 - \frac{1}{2} u \text{div}(Q^+) - ur^2 G_{\ell n}, \tag{10}
\]

\[
rD_r K = \left( \frac{1}{2} \text{div} \beta + i \text{curl} \beta \right) K - \frac{1}{2} J \overline{\beta} + \frac{1}{2} u \overline{\beta} Q^+ + \frac{1}{4} u (Q^+)^2 + \frac{1}{2} ur^2 G_{mm}. \tag{11}
\]

These formulae provide the justification for introducing the specific forms \([1]\) for the NQS connection parameters \((H, J, K, Q)\).

If \(\beta\) is known on a null hypersurface \(\mathcal{N}\) then \((8-11)\) form a system of differential equations for fields on \(S^2\) along the integral curves of the null generators \(\ell\) of \(\mathcal{N}\). Although the general ode structure is well known \([3]\), the form of the RHS is considerably simpler than those found in other gauges \([2], [10], [23]\). Note that the NP equivalents of \((8-11)\) are easily determined: for example, Eqn. \((8)\) is equivalent to the NP propagation identity (with \(\kappa_{NP} = 0\) and \(G_{\ell\ell} = 2\Phi_{00}\))

\[
D \rho_{NP} = \rho_{NP}^2 + |\sigma_{NP}|^2 + 2 \Re(\epsilon_{NP})\rho_{NP} + \Phi_{00}, \tag{12}
\]

which is just the null Raychaudhuri equation.

The remaining Einstein equations may be handled via the second Bianchi identity \([2, 3]\).

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**Lemma 1** Let \(\mathcal{N}\) be an expanding null hypersurface with generating null vector \(\ell\), and let \(T\) be a hypersurface which intersects each of the null curves of \(\mathcal{N}\) exactly once. Suppose that \(F_{ab}\) is a symmetric 2-tensor satisfying the conservation law \(F_{ab}^{\ell b} = 0\), and
\[ F_{\ell\ell} = F_{\ell n} = F_{\ell m} = F_{mn} = 0, \]

and such that \( F_{nm} = F_{nm} = 0 \) on \( \mathcal{T} \). Then \( F_{ab} = 0 \) everywhere on \( \mathcal{N} \).

**Proof:** Writing out \( F_{ab}^i{}^b = 0 \) in NP form, using \( F_{\ell\ell} = F_{\ell n} = F_{\ell m} = F_{mm} = 0 \) and the bicharacteristic condition \( \kappa_{NP} = 0 \), gives \[ 0 = \text{Re}\rho_{NP} F_{m\bar{m}}, \quad \tag{13} \]

\[
D_{\ell}(F_{nm}) = (2\rho_{NP} + \bar{\rho}_{NP} - 2\bar{\varepsilon}_{NP})F_{nm} + \sigma_{NP} F_{nm}
+ D_m F_{m\bar{m}} + (\bar{\pi}_{NP} - \tau_{NP})F_{m\bar{m}}, \quad \tag{14} \]

\[
D_{\ell}(F_{mn}) = 2\text{Re}(\rho_{NP} - 2\bar{\varepsilon}_{NP})F_{nn} + D_m F_{nm} + D_m F_{nm}
- \text{Re}\mu_{NP} F_{m\bar{m}} + \text{Re}((2\beta_{NP} + 2\bar{\pi}_{NP} - \tau_{NP})F_{m\bar{m}}). \quad \tag{15} \]

Since the expansion \( \rho_{NP} \neq 0 \) by hypothesis, Eqn. \((13)\) gives \( F_{m\bar{m}} = 0 \); eliminating \( F_{m\bar{m}} \) turns Eqn. \((14)\) into a linear homogeneous ode along the integral curves, which gives \( F_{nm} = 0 \) by the initial condition and uniqueness. Finally, Eqn. \((15)\) now reduces to a linear homogeneous equation for \( F_{nn} \) with only the zero solution satisfying the initial condition. QED.

When applied to the Einstein tensor, \( F_{ab} = G_{ab} \), the lemma shows that in order to show that the Einstein equations \( G_{ab} = 0 \) are everywhere satisfied, it suffices to find metric parameters satisfying only the hypersurface equations \((8-11)\) everywhere, and the boundary (subsidiary \((4)\)) equations \( G_{nn} = G_{nm} = 0 \) on one hypersurface \( \mathcal{T} \) transverse to the expanding null foliation. Observe that the signature of the hypersurface \( \mathcal{T} \) is immaterial; all that is needed is that \( \mathcal{T} \) be transverse to the outgoing foliation.

The following explicit NQS expressions for \( G_{nm} \) and \( G_{nn} \) show that the boundary equations may be used to constrain the initial data on \( \mathcal{T} \) for the hypersurface equations \((8-11)\):

\[
\begin{align*}
\text{r } D_z (J/u) &= v^2 r D_r (J/(uv)) + \frac{1}{2}(\text{div}\gamma - v \text{ div}\beta)J/u \\
&+ 2u^{-1}|K|^2 - \nabla Q^+ v - \Delta v + ur^2 G_{nn} \quad \tag{16} \\
\text{r } D_z Q^+ &= (v r D_r + J - v \bar{\partial}\beta + \bar{\partial}\xi)Q^+ - K Q^+ + 2\bar{\partial}K \\
&+ 2u^{-1}r D_r(u\bar{\partial}v) - (2 + i \text{ curl}\beta)\bar{\partial}v + \bar{\partial}J - 2u^{-1}J\bar{\partial}u - 2ur^2 G_{nm}. \quad \tag{17}
\end{align*}
\]
The terms on the right hand sides of (16,17) are determined on a single null hypersurface from the hypersurface equations and a gauge choice for $\gamma_{l=1}$, thereby determining the evolution terms $\partial(J/u)/\partial z$ and $\partial(Q^+)/\partial z$. This constrains $(J/u)$ and $Q^+$ on $\mathcal{T}$ — clearly when $\mathcal{T}$ is taken to be a level set of $r$ (so $\partial_z$ is tangent to $\mathcal{T}$), whilst for more general transverse hypersurfaces $\mathcal{T}$, using knowledge of $\partial(J/u)/\partial r$ and $\partial(Q^+)/\partial r$.

The remaining Einstein tensor equation $G_{m\bar{m}} = 0$ is called the trivial equation in [9] by virtue of (13). In the NQS parameterisation, $G_{m\bar{m}}$ is given explicitly by

$$ur^2G_{m\bar{m}} = rD_rJ - \frac{1}{2}u^2\text{div}J - u|Q^+|^2 + \frac{1}{2}u\text{div}Q^+ + K\delta\beta + K\bar{\delta}\bar{\beta} + \bar{Q}^+\partial u + Q^+\bar{\partial}u$$

$$+ r^2(vD_r - D_z)(u^{-1}D_ru) + u^{-1}r^2D_r(uD_rv).$$

(18)

II. SOME APPLICATIONS

The simplicity of the explicit NQS expressions for the Einstein tensor suggests many possible applications, and in the following we outline three: the formal structure of the null-timelike boundary value problem [2]; matching conditions across a null hypersurface and the propagation of impulsive waves [11]; and the relation between the null intrinsic geometry (type I [15]) and the class of affine null geodesic parameterisations.

A formal solution algorithm for the Einstein equations in the NQS gauge starts with $\beta$ defined on a null hypersurface $N_z$. Specifying $H, J, K, Q^-$ on the intersection $\mathcal{T} \cap N_z$ with a transverse hypersurface gives boundary (initial) conditions for the hypersurface equations, with $\beta|_{N_z}$ providing source terms. Solving the hypersurface equations for $H, J, K, Q^-$ on $N_z$ and imposing a gauge fixing condition on the $l = 1$ components of $\gamma$ (eg. $\gamma_{l=1} = 0$) then determines the remaining metric parameters $u, v, \gamma$. The definition Eqn. (4) of $Q^-$ determines $\partial\beta/\partial z$ on $N_z$, which is used to evolve $\beta$ to the “next” null hypersurface $N_{z+\Delta z}$. The boundary equations (16,17) determine new boundary data for $J/u, Q^+$, which are combined with arbitrary boundary data for $u, K$ on $\mathcal{T}$ and the new seed field $\beta|_{N_{z+\Delta z}}$ to repeat the process. Practical experience with a direct numerical implementation of this
procedure has been very good, which suggests that it might be possible to prove existence
theorems by this approach.

The freedom in the choice of $u$ on $\mathcal{T} \cap \mathcal{N}_z$ should be interpreted as the freedom in
choosing the outgoing null hypersurfaces. This can be seen as follows: at the linearised
level, the coordinate tangent vector $\partial_z$ points from one null hypersurface to the “next” —
the $z$-translations are diffeomorphisms preserving the null foliation, and generated by the
vector field $\partial_z$. Adding a component tangential to the null hypersurfaces does not affect this
property. Since the NQS null vector $\ell = \partial_r - r^{-1} \beta$ depends only on the intrinsic (type I)
geometry of $\mathcal{N}_z$, the equivalence class of spacetime evolution vectors $W$ preserving a given
null foliation may be characterised by the inner product $g(W, \ell)$. But for the deformation
vector $W = \partial_z$, this inner product is determined by $u$ since $u = -g(\partial_z, \ell)$. Hence the choice
of $u > 0$ on $\mathcal{T}$ determines the intersection 2-surfaces $\mathcal{N}_z \cap \mathcal{T}$, which in turn generate the null
hypersurfaces $\mathcal{N}_z$; in other words, the boundary freedom in $u$ is a gauge (NQS coordinate)
freedom.

The boundary data for $K$ (essentially the incoming shear $\bar{\lambda}_{NP}$) is unconstrained and
may be considered as representing the outgoing gravitational radiation injected into the
spacetime from the region beyond the boundary $\mathcal{T}$. In a similar fashion, we may regard the
shear vector $\beta$ (which is arbitrarily specifiable on one outgoing surface $\mathcal{N}_z$) as measuring the
incoming radiation.

Since $\beta$ acts as a source for the remaining fields on the null hypersurface, we might expect
NQS metrics with $\beta$ identically zero to play a distinguished role. Such metrics admit a null
geodesic congruence which is expanding, twist free and shear free, and thus correspond to
the Robinson-Trautman spacetimes [26]. The NQS parameterisation of Robinson-Trautman
metrics with $S^2$ cross-sections is described in [27] and the remaining NQS gauge freedoms
are classified in [28].

NQS coordinates may also be used to describe the junction conditions for null hyper-
surfaces. The NQS approach has the advantage of simplicity, when compared with general
coordinate methods [11,29] which must separately consider the effects of coordinate changes
on the discontinuity fields. On the other hand, the reliance on a special coordinate system apparently limits the applicability of the NQS analysis.

Let $M^\pm$ be two spacetimes with NQS coordinates and having null boundary pieces $\mathcal{N}^\pm$ (one past, the other future), together with an identification map $\mathcal{N}^+ \simeq \mathcal{N}^- \simeq \mathcal{N}$. We use the notation $[\cdot]$ to denote the discontinuity of parameters across the matching surface, e.g. $[\beta] = \beta_{\mathcal{N}^+} - \beta_{\mathcal{N}^-}$. Supposing that the matching is an isometry along $\mathcal{N}$, i.e. $[\beta] = 0$ (this is the most natural matching criterion), we consider the effect of requiring that the Einstein equations be satisfied in the weak (distributional) sense.

Let $T$ be a hypersurface transverse to $\mathcal{N}$ in $M = M^+ \cup M^-$. Since we may regard $u|_T$ as gauge, it is reasonable to require also that $[u] = 0$ on $T \cap \mathcal{N}$. Likewise we assume that the $l = 1$ spherical harmonic components of $\gamma$ are gauged to zero. The boundary equations (16),(17) show that $[J/u] = 0$, $[Q^+] = 0$ on $T \cap \mathcal{N}$ since their right hand sides are bounded on $\mathcal{N}$, from which it follows (using the definitions (3)) that $[H] = [J] = [Q^-] = 0$ on $T \cap \mathcal{N}$. Considering the jump in (8) across $\mathcal{N}$ shows that $[H]$ satisfies the ordinary differential equation

$$rD_r[H] = \left(\tfrac{1}{2}\text{div}\beta - \frac{2[\partial\beta]^2 + r^2G_{\ell\ell}}{2 - \text{div}\beta}\right)[H],$$

(19)

along the null generating curves of $\mathcal{N}$, and uniqueness with the initial data $[H] = 0$ on $T \cap \mathcal{N}$ shows that $[H] = 0$ on $\mathcal{N}$. Since $[\text{div}\beta] = 0$ we have $[u] = 0$ also. Similar arguments using Eqn. (8) show that $[Q^-] = 0$ (and thus $[Q] = [Q^+] = 0$) on $\mathcal{N}$, and using Eqn. (10), that $[J] = 0$.

However, the discontinuity $[K]$ on $T \cap \mathcal{N}$ is not constrained in similar fashion by gauge or boundary equation considerations, and thus Eqn. (11) yields a formula for the propagation along $\mathcal{N}$:

$$rD_r[K] = (\tfrac{1}{2}\text{div}\beta + i\text{curl}\beta)[K].$$

(20)

Using the gauge condition $[\gamma]|_{t=1} = 0$ on $\mathcal{N}$ to invert $\mathcal{L}_\beta$, from Eqn. (5) we recover the discontinuities across $\mathcal{N}$ of the remaining metric coefficients,
\[ [\gamma] = \mathcal{L}_\beta^{-1}[K], \quad [v] = -\frac{\text{div}[\gamma]}{2 - \text{div}\beta}, \]  
\[ (21) \]

and we find \( \partial\beta/\partial z \neq 0 \), using \( [Q] = 0 \) and Eqn. (4). From the NQS formulas for the Weyl curvature spinors

\[ r^2u^{-1}\Psi_0 = (rD_r + 1 - 2\bar{\partial}\beta)(\bar{\partial}\beta/u), \]
\[ (22) \]

\[ 4r^2\Psi_1 = (rD_r - \bar{\partial}\beta)Q^- - \bar{\partial}\text{div}\beta \]
\[ - (3\bar{Q} + \bar{\partial}u)u^{-1}\bar{\partial}\beta + 2\bar{\partial}\bar{\partial}\beta, \]
\[ (23) \]

\[ 2r^2\Psi_2 = \frac{1}{3}r^2(2G_{\ell n} + G_{m\bar{m}}) - (1 - \frac{1}{2}HJ) \]
\[ - 2u^{-1}\bar{K}\bar{\partial}\beta - \frac{1}{2}(\bar{\partial}\bar{Q}^+ - \bar{\partial}Q^+), \]
\[ (24) \]

\[ 4r^2u^2\bar{\Psi}_3 = uv(rD_r - \bar{\partial}\beta)Q^+ - u(rD_z - \bar{\partial}\gamma)Q^+ \]
\[ + 2(rD_r - 1 - \frac{1}{2}i\text{curl}\beta)(u\bar{\partial}v) \]
\[ - u\bar{\partial}J - (3\bar{Q} - \bar{\partial}u)K - 2u\bar{\partial}K, \]
\[ (25) \]

\[ r^2u\Psi_4 = v^2(rD_r + 1 - 2\bar{\partial}\beta)(K/(uv)) \]
\[ - (rD_z - 2\bar{\partial}\gamma)(K/u) + Q^+\bar{\partial}v + \bar{\partial}\bar{\partial}v, \]
\[ (26) \]

we see easily that \( \Psi_0, \Psi_1, \Psi_2 \) and \( \Psi_3 \) will be bounded, but \( \Psi_4 \) will have a \( \delta \)-function component \( -r^{-1}u^{-2}[K]\delta(z) \), giving an impulsive gravitational wave \[15\] propagating along \( \mathcal{N} \).

Although the resulting metric is not continuous in NQS coordinates, there is a (non-\( C^1 \)) coordinate change which makes the metric continuous, since the boundary identification is an isometry \[30\]. There are two interesting points about this NQS construction: it provides a geometric coordinate condition which does not detect the “optimal” metric regularity (\( C^0 \)), and secondly, it provides strong evidence for the existence of vacuum spacetime metrics admitting particular coordinates in which the metric is not continuous, but such that the full curvature tensor, computed in those coordinates, is well defined as a distribution. Such discontinuities are well beyond the reach of the usual Sobolev space approaches to local existence for the Cauchy problem for the Einstein equations, for which \( g \in H^2 \) seems to be the best plausibly possible (and \( g \in H^{5/2+\epsilon} \) is the best proven result at this time).
The difference between characteristic and 3 + 1 approaches is again underscored by noting that whereas the Einstein equations across a null matching hypersurface may be satisfied with unbounded curvature components, determined by the shock transport law (20), imposing the Einstein equations across a smooth spacelike matching hypersurface forces the metric to be $C^2$ in Gaussian coordinates [29].

From the dual reformulations of Eqs. (10), (11),

\begin{align*}
 rD_z(uH) &= rD_r(uvH) + ur^2G_{tn} \\
 &- u(1 + Hv - HJ) + \frac{1}{2}u(|Q^-|^2 - \text{div}Q^-), \\
 rD_z(\beta) &= rD_r(v\beta) - \frac{1}{2}ur^2G_{mm} \\
 &+ (v + i\text{curl}\gamma - i\text{curl}\beta) \, \beta \\
 &+ \frac{1}{2}(uHK + J\beta) - \frac{1}{4}u(Q^-)^2 + \frac{1}{2}u\beta Q^-,
\end{align*}

which correspond to interchanging the role of the inward and outward null directions, we see that the conditions \( \text{div}[\beta] = 0, [\beta] = 0 \) are required by the Einstein equations. Thus the isometry assumption \([\beta] = 0\) amounts to a constraint on the remaining, purely rotational, component of \([\beta]\).

Because the NQS foliation is intrinsic to the null hypersurfaces (determined by the type I geometry, in the terminology of [15]), the interaction between the Einstein tensor and the null geodesic parameterisation may be simply explained. It is easy to see that $u^{-1}\ell$ is a null geodesic vector field, and thus the geodesic parameterisation is determined by the null type I geometry (ie. $\beta$) and $u$, which is in turn determined from $G_{\ell\ell} = 2\Phi_{00}$ by Eqn. (8), up to a choice of initial condition which amounts to fixing the scaling of the affine parameter. Thus, the effect of matter (as described by $G_{\ell\ell} \geq 0$) is to slow down the null geodesic parameterisation, when compared to the parameterisation on an isometric null hypersurface in vacuum. This leads to a redshift of outgoing photons, as seen by an observer at constant NQS radius.
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