A Field-Theoretic Approach to Connes’ Gauge Theory on $M_4 \times Z_2$

Hiromi Kase$^1$, Katsusada Morita$^2$ and Yoshitaka Okumura$^3$

$^1$Department of Physics, Daido Institute of Technology, Nagoya 457-0811, Japan
$^2$Department of Physics, Nagoya University, Nagoya 464-8602, Japan
$^3$Department of Natural Sciences, Chubu University, Kasugai, 487-0027, Japan

Abstract

Connes’ gauge theory on $M_4 \times Z_2$ is reformulated in the Lagrangian level. It is pointed out that the field strength in Connes’ gauge theory is not unique. We explicitly construct a field strength different from Connes’ one and prove that our definition leads to the generation-number independent Higgs potential. It is also shown that the nonuniqueness is related to the assumption that two different extensions of the differential geometry are possible when the extra one-form basis $\chi$ is introduced to define the differential geometry on $M_4 \times Z_2$. Our reformulation is applied to the standard model based on Connes’ color-flavor algebra. A connection between the unimodularity condition and the electric charge quantization is then discussed in the presence or absence of $\nu_R$. 

1
§1. Introduction

Connes’ interpretation of the standard model in non-commutative geometry (NCG) is based on the assumption that an algebra underlies the gauge symmetry. This assumption armed with the mathematical apparatus of NCG allows us to define the Yang-Mills (YM) gauge theory on general manifold, either continuous or discrete. It is remarkable that the spontaneously broken gauge theory observed in Nature belongs to Connes’ YM on a discrete manifold. Thus the non-commutative one-form on a two-sheeted Minkowski space-time $M_4 \times Z_2$ combines the YM gauge fields with the Higgs one in the standard model and determines dynamics in the bosonic sector through Connes’ field strength $G$.

In view of the mathematical niceties involved it is important to extract a physical information as simple as possible. We, therefore, feel it worthwhile looking for a more accessible way of reformulating Connes’ YM without the axioms of NCG, which would disclose important features in the theory from physical point of view. In this paper we continue our previous work to derive the non-commutative one-form on $M_4 \times Z_2$ in the Lagrangian formulation.

During our investigation we find that the field strength in Connes’ YM is not unique. The nonuniqueness is totally unrelated with Connes’ ambiguity problem but rather originates from the different associative products of the Dirac matrices. We explicitly construct a different field strength $F$ than Connes’ one $G$ by introducing a new associative product of the Dirac matrices including $\gamma_5$. The new field strength leads to the generation-number independent Higgs potential, while the quartic coupling constant derived from $G$ depends on the generation number. We shall show a close connection of the nonuniqueness with the extended differential formalism.

The plan of this paper is as follows. In the next section we present a field-theoretic approach to Connes’ YM by taking into account of two different field strengths. In §3 we review two possible extensions of the ordinary differential geometry and show that they precisely lead to the two different field strengths. We shall derive in §4 Asquish’s representation of Connes’ color-flavor algebra of the standard model using the double sum prescription and discuss its consequence regarding the electric charge quantization in the presence or absence of $\nu_R$. The final section is devoted to discussion.
§2. A field-theoretic approach to Connes’ YM on $M_4 \times Z_2$

Suppose that an algebra $\mathcal{A}$ underlies the gauge symmetry. To explain what this means in our methodology we remark that, although an arbitrary element of the algebra $\mathcal{A}$ never defines the symmetry transformation, it is possible to regard the Hilbert space of spinors $\psi$ as an $\mathcal{A}$-module such that the gauge group is given by the unitary group, $G = \mathcal{U}(\mathcal{A}) = \{ g \in \mathcal{A}; gg^\dagger = g^\dagger g = 1 \}$. To meet this condition $\mathcal{A}$ must be a local, unital and involutive algebra. We are then tempted to consider the local non-symmetry transformations

$$\begin{align*}
\psi &\rightarrow \rho(b_i)\psi, \\
\bar{\psi} &\rightarrow \bar{\psi}\rho(a_i),
\end{align*}$$

(2.1)

where $\rho$ is the $\ast$-preserving representation of the algebra $\mathcal{A}$. The linearity of the algebra fits the concept of generation in a neat way. Along with the transformations (2.1) we take the sum over the index $i$ in the Lagrangian level provided that

$$\sum_i \rho(a_i)\rho(b_i) = 1(\equiv 1_{\text{dim} \rho}),$$

(2.2)

which expresses the unity decomposition and leaves $\bar{\psi}\psi$ invariant, so that we end up with the covariant derivative $D_0 + A$ with the YM gauge field

$$A = \sum_i \rho(a_i)[D_0, \rho(b_i)] \equiv \sum_i a_i[D_0, b_i], ~ D_0 = i\partial \otimes 1_{\text{dim} \rho}, ~ \gamma^0 A^\dagger \gamma^0 = A,$$

(2.3)

where $1_n$ denotes $n$-dimensional unit matrix and $\text{dim} \rho$ is the dimension of $\rho$. Here and hereafter we omit the notation $\rho$ for simplicity unless necessary.

Since the non-commutative one-form (2.3) depends on the Dirac matrices, there must be an ambiguity in extracting the curvature to be identified with the YM field strength.

---

*) Recall that the unity has an infinite variety of decompositions. For instance, the unit matrix

$$\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
$$

equals

$$\begin{pmatrix}
\alpha^* & -\beta \\
\beta^* & \alpha
\end{pmatrix}
$$

with $|\alpha|^2 + |\beta|^2 = 1$ or the sum of terms

$$\begin{pmatrix}
\gamma & 0 \\
0 & \gamma^*
\end{pmatrix}
+ \begin{pmatrix}
0 & \delta \\
-\delta^* & 0
\end{pmatrix}
+ \begin{pmatrix}
-\gamma & 0 \\
0 & -\gamma^*
\end{pmatrix}
+ \begin{pmatrix}
c & -ic \\
-ic^* & c
\end{pmatrix}
+ \begin{pmatrix}
d & id \\
-id & d
\end{pmatrix}
$$

(for real $c, d$ with $cd = 1/2$) and so on. The first form defines $SU(2)$, whereas the second sum contains non-unimodular matrices. In fact, all matrices in the second sum belongs to the algebra $\mathcal{H}$ of the real quaternions.

**) Connes’ original definition $A = \sum_i a_i[D_0, b_i]$ has nothing to do with the transformations (2.1). In our interpretation which may also be regarded as a mnemonic one without NCG mathematics, $a_i$ and $b_i$ are only the transformation parameters, not treated as the canonical variables, but the connection $A = \sum_i a_i[D_0, b_i]$ is assumed to be a field variable, as in Connes’ YM, which is promoted to be a quantum field.
We compare it with Connes’ ambiguity in defining the field strength based on the sum (2.3). The latter ambiguity arises from the fact that the exterior derivative $dA$ in Connes’ field strength

$$G = dA + A^2, \quad dA \equiv \sum_i [D_0, a_i][D_0, b_i]$$

(2.4)

may not vanish even for $A = \sum_i a_i[D_0, b_i] = 0$. To see this note that $G = F - 1/4 \otimes X$, where

$$F = d \wedge A + A \wedge A, \quad d \wedge A \equiv \sum_i [D_0, a_i] \wedge [D_0, b_i]$$

(2.5)

is the YM field strength $F = (D_0 + A) \wedge (D_0 + A) = -(1/4)[\gamma^\mu, \gamma^\nu]F_{\mu\nu}$ with the wedge product of the Dirac matrices

$$\gamma^\mu \wedge \gamma^\nu = \frac{1}{2} (\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu) = \frac{1}{2} [\gamma^\mu, \gamma^\nu],$$

(2.6)

and $X = C + A_\mu A^\mu$ with $C = \sum_i \partial_\mu a_i \partial^\mu b_i$. Consequently, $G|_{A=\sum_i a_i[D_0, b_i]=0} = -1/4 \otimes C$. (One may replace $C$ with $C' = -\sum_i a_i \partial^2 b_i$.) This implies\(^\dagger\)\(^\#\) that the field strength in Connes’ YM is to be defined as an equivalence class, $[G] = [G']$ if $G = G' + \sum_i [D_0, a_i][D_0, b_i]$ with $\sum_i a_i[D_0, b_i] = 0, a_i, b_i \in \mathcal{A}$. Thus $[G] = F$ because the subtracted piece must be covariant. In other words, if we define the field strength in Connes’ YM using the wedge product (2.6), we directly obtain the YM field strength. In this sense Connes’ ambiguity is related to the ambiguity alluded to above. As we shall see later, this is no longer the case if Higgs is generated.

There is an alternative method\(^\ddagger\) to achieve the result $[G] = F$. Although $X$ is gauge-covariant, $C$ is not covariant and has no kinetic energy term in the bosonic Lagrangian defined by the trace of the square of $G$

$$\mathcal{L}_B = -\frac{1}{8} \text{Tr} \frac{1}{g^2} G^2 = -\frac{1}{8} \text{Tr} \frac{1}{g^2} F^2 - \frac{1}{2} \text{tr} \frac{1}{g^2} X^2,$$

(2.7)

where $\text{Tr}$ also includes the trace over Dirac matrices. If we vary $A_\mu$ and $C$ independently, we can eliminate the auxiliary field $C$ through its equation of motion $X = 0$. Then the bosonic Lagrangian (2.7) is reduced to the YM one

$$\mathcal{L}_{YM} = -\frac{1}{8} \text{Tr} \frac{1}{g^2} F^2 = -\frac{1}{8} \text{Tr} \frac{1}{g^2} [G]^2.$$  

(2.8)

If the fermion mass matrix $M$ is not gauge-invariant and fermions exist in chiral multiplets, we use the chiral decomposition of the spinors so that the free Dirac operator
reads

\[ D = D_0 + i\gamma_5 M, \quad D_0 = \begin{pmatrix} i\gamma \otimes 1_{nL} & 0 \\ 0 & i\gamma \otimes 1_{nR} \end{pmatrix} \otimes 1_{N_g}, \quad M = \begin{pmatrix} 0 & M_1 \\ M_1^\dagger & 0 \end{pmatrix}. \quad (2.9) \]

We then obtain the gauge-invariant Dirac Lagrangian \( \mathcal{L}_D = \sum_i (\bar{\psi} \gamma^\mu \partial_\mu \psi) = \bar{\psi} (D + A) \psi \) with \( \gamma^0 A^\dagger \gamma^0 = A \), where use has been made of the condition (2.2) and

\[ A = \sum_i a_i [D, b_i] \]

\[ = A + i\gamma_5 \Phi, \quad A = \sum_i a_i [D_0, b_i] = i\gamma^\mu A_\mu, \quad A_\mu = \sum_i a_i \partial_\mu b_i, \quad \Phi = \sum_i a_i [M, b_i]. \]

To define the curvature from the non-commutative one-form (2.10) there again occur two kinds of ambiguity, one intrinsic in Connes’ YM and the other coming from the different multiplication rules of the Dirac matrices containing \( \gamma_5 \). The first ambiguity is well-known. We shall argue in the next section that there are only two possible definitions. In this section we first consider Connes’ field strength and then our new field strength which is obtained by generalizing the wedge product (2.6) to include \( \gamma_5 \).

Connes’ field strength is given by

\[ G = dA + A^2, \quad dA \equiv \sum_i [D, a_i][D, b_i]. \quad (2.11) \]

As noted before \( dA \) may not vanish even when \( A = 0 \). This makes it necessary to subtract a matrix \( \langle G \rangle \) from \( G \) where \( \langle G \rangle \) is the matrix which is of the same form as \( dA \) with the constraint \( A = 0 \). We have seen above that, for \( A \), this is equivalent to putting \( X = 0 \) in Eq. (2.7) to obtain Eq. (2.8) by the variational principle. The computation involved is not so simple for \( A \). A detailed mathematical account was given in Refs. 2), 4) and 6).

The gauge-invariant bosonic Lagrangian is

\[ \mathcal{L}_B = -\frac{1}{8} \text{Tr} \frac{1}{g^2} G^2 = \mathcal{L}_{YM} + \frac{1}{2} \text{tr} \frac{1}{g^2} (D^\mu H)(D\mu H) - \frac{1}{2} \text{tr} \frac{1}{g^2} Y^2, \]

\[ \begin{cases} 
D\mu H = [\partial_\mu + A_\mu, H], & H = \Phi + M, \\
Y = X + Y_0, \\
Y_0 = H^2 - M^2 - \sum_i a_i [M^2, b_i]. \end{cases} \quad (2.12) \]

\(^*) \text{As before the field strength is defined as an equivalence class, the equivalence being given by } G \sim G' \text{ if } G' = G + \sum_i [D, a_i][D, b_i] \text{ with } \sum_i a_i [D, b_i] = 0, \ a_i, b_i \in A.\)
See Ref. 7) for the most general gauge-invariant Lagrangian. The potential term \( V = \text{tr}(1/g^2)Y^2 \) (except for the factor \( \frac{1}{2} \)) is evaluated\(^*\) for the Glashow-Weinberg-Salam model in the leptonic sector \((n_L = n_R = 2)\) by assuming the flavor algebra \( A = C^\infty(M_4) \otimes (H \oplus C)\), whose unitary group is \( U(C^\infty(M_4) \otimes (H \oplus C)) = \text{Map}(M_4, SU(2) \times U(1)) \). Writing \( A = \sum_i \rho(a_i^1, b_i^1) [D, \rho(a_i^2, b_i^2)] \), we assume the following representation of \( A \) on the chiral spinor
\[
\rho(a, b) = \begin{pmatrix} a & 0 \\ 0 & B \end{pmatrix} \otimes 1_{N_g}, \quad B = \begin{pmatrix} b & 0 \\ 0 & b^* \end{pmatrix},
\]
where \( a = a(x) \in C^\infty(M_4) \otimes H \), \( b = b(x) \in C^\infty(M_4) \otimes C \) and \( * \) denotes the complex conjugation so that the left-handed and right-handed spinors belong to doublet \( \psi_L = \begin{pmatrix} \nu \\ e \end{pmatrix}_L \) and singlet \( \psi_R = \begin{pmatrix} \nu_R \\ e_R \end{pmatrix} \), respectively, in \( N_g \) generations. Only doublets and singlets appear in this model, while the nonzero Abelian charge is quantized to be \( \pm 1 \).\(^*\)\(^*\)\(^*\) (In the present model \( Y = 0 \) for \( \psi_L \), \( Y = +1 \) for \( \nu_R \), and \( Y = -1 \) for \( e_R \) provided that the hypercharge of Higgs doublet is normalized to be \( +1 \).) Choosing the mass matrix as
\[
M = \begin{pmatrix} 0 & M_1 \\ M_1^\dagger & 0 \end{pmatrix}, \quad M_1 = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}, \quad m_{1,2} : N_g \times N_g,
\]
and putting \( 1/g^2 = \begin{pmatrix} (1/g_1^2) \otimes 1_{2N_g} \\ 0 \\ (1/g_2^2) \otimes 1_{2N_g} \end{pmatrix} \), we find after making Connes’ subtraction or resorting to the auxiliary field method
\[
V = K(\phi^\dagger \phi - 1)^2, \quad K = \frac{1}{2g_1^2} \text{tr}_g(m_1m_1^\dagger + m_2m_2^\dagger)^2 + \frac{1}{g_2^2} \text{tr}_g[(m_1^\dagger m_1)^2 + (m_2^\dagger m_2)^2], \quad (2.13)
\]
where \( \phi = \begin{pmatrix} \phi_+ \\ \phi_0 \end{pmatrix} \) is the normalized Higgs field and \( \text{tr}_g \) denotes the trace in the generation space with \( \text{tr}_g f_L^2 = \text{tr}_g f^2 - (1/N_g)(\text{tr}_g f)^2 \).

We note that \( K = 0 \) for \( N_g = 1 \) or \( N_g > 1 \) with the degenerate mass spectrum. For \( N_g > 1 \) with non-degenerate mass spectrum \( K \) is positive. We can take the vacuum expectation value\(^***\)\(^\dagger\) of the normalized Higgs field to be \( \langle \phi \rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \).

\(^*\) Similar quantization was also pointed out by Hayakawa\(^\dagger\) in non-commutative QED.

\(^*\) The correct hypercharge (\( Y \)) assignment of chiral leptons will be discussed in the section 4.

\(^***\) By canonically normalizing the Higgs kinetic energy term, the vacuum expectation value of the rescaled Higgs field is proportional to the quantity \( \sqrt{\text{tr}_g(m_1m_1^\dagger + m_2m_2^\dagger)} \). In the standard model this implies that the electroweak scale is essentially governed by the top mass, which is not inconsistent with experiment. The same remark will also apply for the potential Eq. (2-13) below.
Next comes a generalization of the wedge product (2.6) of the Dirac matrices to include $\gamma^5 = \gamma_5$. A naive generalization dismisses $\gamma^5$ when it appears twice since $\gamma^5 \wedge \gamma^5 = (1/2)[\gamma^5, \gamma^5] = 0$. To avoid this we assign the ‘grade’ of $\gamma^A(A = \mu, 5)$ by $\epsilon_\mu = 0(\mu = 0, 1, 2, 3)$ and $\epsilon_5 = 1$ such that the wedge product of $\gamma^5$ by itself is converted into the anticommutator $\gamma^5 \wedge \gamma^5 = (1/2)(\gamma^5 \gamma^5 - (-1)^{1+1}\gamma^5 \gamma^5) = (1/2)\{\gamma^5, \gamma^5\} = 1$. This would give a sensible definition of the field strength different from Connes’ one.

We found that the following definition works. (Capital letters $A, B, C$ take the values $0, 1, 2, 3, 5$.)

$$f \wedge \gamma^A \equiv f \gamma^A = \gamma^A f = \gamma^A \wedge f \text{ for any complex number or function } f,$$

$$\gamma^A \wedge \gamma^B = \tilde{A}_2[\gamma^A \gamma^B] \equiv \frac{1}{2!}(\gamma^A \gamma^B - (-1)\epsilon_{A+B}\gamma^B \gamma^A),$$

$$\gamma^A \wedge \gamma^B \wedge \gamma^C = \tilde{A}_3[\gamma^A \gamma^B \gamma^C] \equiv \frac{1}{3!}[\gamma^A \gamma^B \gamma^C + (-1)\epsilon_{A+B+C}\gamma^B \gamma^C \gamma^A$$

$$+(-1)\epsilon_{C-(A+B)}\gamma^C \gamma^A \gamma^B - (-1)\epsilon_{A+B} \gamma^B \gamma^A \gamma^C$$

$$-(-1)\epsilon_{A-B+C}(A+B)\gamma^C \gamma^B \gamma^A - (-1)\epsilon_{C-B} \gamma^A \gamma^B \gamma^C]$$

(2.14)

where $\tilde{A}_n[\gamma^{A_1}\gamma^{A_2}\ldots\gamma^{A_n}]$ denotes the graded antisymmetrization among the indices $A_1, A_2, \ldots, A_n$, as indicated above for $n = 2, 3$. The number of the indices $\mu = 0, 1, 2, 3$ is restricted to less than or equal to 4. We also define

$$(\gamma^A \wedge \gamma^B) \wedge \gamma^C = \tilde{A}_3[(\gamma^A \wedge \gamma^B) \gamma^C],$$

$$\gamma^A \wedge (\gamma^B \wedge \gamma^C) = \tilde{A}_3[\gamma^A(\gamma^B \wedge \gamma^C)].$$

(2.15)

It is easy to check the associativity

$$(\gamma^A \wedge \gamma^B) \wedge \gamma^C = \gamma^A \wedge (\gamma^B \wedge \gamma^C) = \gamma^A \wedge \gamma^B \wedge \gamma^C.$$

(2.16)

In terms of the wedge product of the Dirac matrices defined above we introduce the new field strength

$$F = d \wedge A + A \wedge A, \ d \wedge A \equiv \sum_i [D, a_i] \wedge [D, b_i].$$

(2.17)

Connes’ ambiguity problem still remains but, since $F = G|_{\chi = 0}$, we find the different result from Eq. (2.13)

$$V = K' (\phi^\dagger \phi - 1)^2, \hspace{1cm} K' = \frac{1}{2g_1^2} tr_g (m_1 m_1^\dagger + m_2 m_2^\dagger)^2 + \frac{1}{g_2^2} tr_g [(m_1^\dagger m_1)^2 + (m_2^\dagger m_2)^2].$$

(2.18)

The Higgs potential is generation-number independent for Eq. (2.17).
§3. Extended differential formalism of Connes’ YM on $M_4 \times Z_2$

One may inquire why there are two different field strengths for the non-commutative one-form. In this section we shall give a non-commutative differential geometric reason using the extended differential formalism with the extra one-form basis $\chi$. In this formalism, in addition to the ordinary exterior derivative $d$ with $d\psi = \partial_\mu \psi d\hat{x}^\mu$, the hat indicating the dimensionless coordinates, we define the extra exterior derivative $d_\chi$ by

$$d_\chi \psi = M \psi \chi,$$  \hfill (3.1)

where $M$ is the mass matrix in Eq.(2.9). From the free Dirac Lagrangian in the form

$$L_D = i \langle \tilde{\psi}, (d_\psi) \rangle, \quad \tilde{\psi} = \gamma_\mu \psi d\hat{x}^\mu - \gamma_5 \psi \chi, \quad \chi^\dagger = -\chi,$$

where $d = d + d_\chi$ is the generalized exterior derivative and $\langle d\hat{x}^\mu, d\hat{x}^{\nu} \rangle = \eta^{\mu\nu}, \langle d\hat{x}^\mu, \chi \rangle = \langle \chi, d\hat{x}^\mu \rangle = 0, \langle \chi, \chi \rangle = -1$, we follow the prescription in §2 to obtain

$$L_D = i \sum_i \langle \{\rho_i^\dagger (a_i) \tilde{\psi}, (d (\rho(b_i) \psi)) \rangle = i \langle \psi, (d + A) \psi \rangle,$$  \hfill (3.2)

where we have assumed the Leibniz rule

$$d_\chi (f \psi) = (d_\chi f) \psi + f (d_\chi \psi), \quad f = \rho (b_i),$$  \hfill (3.3)

used the condition (2.2) and defined the generalized gauge field $A$ by

$$A = \sum_i a_i db_i.$$  \hfill (3.4)

This is the non-commutative one-form in the present notation. In general, using the Leibniz rule (below) an arbitrary $n$-form is written as $\sum_j a_{0,j}^i da_i^j \wedge da_2^j \wedge \cdots \wedge da_n^j$, where $a_{0}^j(i = 0, 1, \cdots, n)$ are 0-forms similar to the function $f$.

From Eqs.(3.1) and (3.3) we derive the action of $d_\chi$ on $f$

$$d_\chi f = [M, f] \chi, \quad d_\chi (f h) = (d_\chi f) h + f (d_\chi h).$$  \hfill (3.5)

The antisymmetry $d\hat{x}^\mu \wedge d\hat{x}^{\nu} = -d\hat{x}^\nu \wedge d\hat{x}^\mu, \quad d\hat{x}^\mu \wedge \chi = -\chi \wedge d\hat{x}^\mu$ ensures the nilpotency $d^2 = 0$ and the relation $(dd_\chi + d_\chi d) f = (dd_\chi + d_\chi d) \psi = 0$.

\footnote{In this section we use the usual notation $\wedge$ for the exterior product.}
There are two options to go further. One is to assume the antisymmetry also for the extra one-form basis
\[ \chi \wedge \chi = 0. \tag{3.6} \]
The other instead assumes the symmetry
\[ \chi \wedge \chi \neq 0. \tag{3.7} \]
We now show that these alternatives lead to the field strength, \( G \) of Eq.(2.11), and the field strength, \( F \) of Eq.(2.17), respectively.

Let us first consider the symmetric case (3.7). We define the action of the operator \( d \) on the \( n \)-form through
\[ d(\sum_j a_j^0 d a_j^1 \wedge d a_j^2 \wedge \cdots \wedge d a_j^n) = \sum_j d a_j^0 \wedge d a_j^1 \wedge d a_j^2 \wedge \cdots \wedge d a_j^n. \]
Then \( d \) is ‘nilpotent’ in the sense that \( d(d a) = (d1) \wedge (d a) = 0 \) because \( d1 = 0 \) due to the Leibniz rule. However, this definition leads to an ambiguity \( d(a_0 d a_1) = d a_0 \wedge d a_1 \neq 0 \) even when \( a_0 d a_1 = 0 \).

We next define the field strength in this case by the two-form
\[ F = d \wedge A + A \wedge A, \quad d \wedge A \equiv \sum_i d a_i \wedge d b_i, \tag{3.8} \]
which turns out to be given by
\[ F = F + D H \wedge \chi + Y_0 \chi \wedge \chi, \tag{3.9} \]
where \( F = d \wedge A + A \wedge A, \) \( D H = d H + [A, H] \) and \( Y_0 \) is given by Eq.(2.12). The bosonic Lagrangian
\[ \mathcal{L}_B = -\langle \langle \frac{1}{g^2} F, F \rangle \rangle, \quad F = F + D H \wedge \chi + Y_0 \chi \wedge \chi \]
is evaluated by taking the inner product of the two-form basis and performing the trace over the 2-dimensional chiral space. The result is the same as in Eq.(2.12) with \( Y \to Y_0 \).

We recover the generation-number independent Higgs potential (2.18) in this case.

Next we consider the antisymmetric case (3.6). In this case the operator \( d_\chi \) is automatically nilpotent. Thus the operator \( d \) is also nilpotent, \( d^2 = 0 \), so that \( d(\sum_j a_j^0 d a_j^1 \wedge d a_j^2 \wedge \cdots \wedge d a_j^n) = \sum_j d a_j^0 \wedge d a_j^1 \wedge d a_j^2 \wedge \cdots \wedge d a_j^n \) holds true. Consequently, \( d(a_0 d a_1) = 0 \) if \( a_0 d a_1 = 0 \). At first sight there seems to arise no ambiguity problem encountered in the symmetric case. However, the two-form field strength now lacks the Higgs potential generating term at all.
\[ F = F + D H \wedge \chi. \tag{3.10} \]
We are then led to add a zero-form piece to the two-form \((3.10)\) to define the field strength by the Clifford product

\[
G = d \vee A + A \vee A = F + F_0,
\]

\[
F_0 = \langle d, A \rangle + \langle A, A \rangle \equiv \sum_i \langle da_i^\dagger, db_i \rangle.
\]

This time the ambiguity problem reappears because \(\langle d, A \rangle\) may not vanish even when \(A = 0\). The zero-form piece \(F_0 = Y\) is given by Eq.\((2.12)\). Using the fact that two-form and zero-form are orthogonal, the bosonic Lagrangian becomes

\[
\mathcal{L}_B = -\langle \langle \frac{1}{g^2} G, G \rangle \rangle = -\langle \langle \frac{1}{g^2} F, F \rangle \rangle - V, \quad V = \text{tr} \frac{1}{g^2} Y^2,
\]

where \(F\) is defined by Eq.\((3.10)\). We thus obtain the same result as in the previous section using Connes’ field strength.

\section*{§4. Double sum prescription and the standard model}

In this section we shall derive Asquish’s representation\(^{11}\) of Connes’ color-flavor algebra of the standard model

\[
\mathcal{A} = C^\infty(M_4) \otimes (H \oplus C \oplus M_3(C)),
\]

whose unitary group is \(U(\mathcal{A}) = \text{Map}(M_4, U(3) \times SU(2) \times U(1))\), from our formulation using the double sum prescription.\(^{12}\) Here \(M_3(C)\) denotes the set of \(3 \times 3\) complex matrices.

The algebra \((4.1)\) is represented on the doubled spinor

\[
\psi = \begin{pmatrix} \psi \\ \psi^c \end{pmatrix}, \quad \psi^c = C \bar{\psi},
\]

where \(\psi\) stands for the total fermion field

\[
\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}, \quad \psi_L = \begin{pmatrix} q_L \\ l_L \end{pmatrix}, \quad \psi_R = \begin{pmatrix} u_R \\ d_R \\ \nu_R \\ e_R \end{pmatrix}.
\]
We omit the color and generation indices for simplicity. The free massive Dirac Lagrangian is written in the present case \( (n_L = n_R = 8) \) as

\[
L_D = \frac{1}{2} \bar{\Psi} D \Psi, \quad D = D_0 + i\gamma_5 M, \quad D_0 = i\partial_0 \otimes 1_{32N_g}, \quad M = \begin{pmatrix} M & 0 \\ 0 & M^* \end{pmatrix}.
\] (4.4)

We choose the fermion mass matrix as

\[
M_1 = \begin{pmatrix} M_q \otimes 1_3 & 0 \\ 0 & M_l \end{pmatrix}, \quad M_q = \begin{pmatrix} M_u & 0 \\ 0 & M_d \end{pmatrix}, \quad M_l = \begin{pmatrix} M_\nu & 0 \\ 0 & M_e \end{pmatrix}.
\] (4.5)

This choice is dictated by the global color symmetry and the electric charge conservation. We assume Dirac mass \( M_\nu \) for neutrinos.

The product of the \( \ast \)-preserving representations \( \rho_{1,2} \) is written as

\[
\rho(a, b, c) = \rho_1(a, b, c)\rho_2(a, b, c) = \rho_2(a, b, c)\rho_1(a, b, c),
\]

\[
\rho_1(a, b, c) = \begin{pmatrix} \rho_w(a, b) & 0 \\ 0 & \rho_s(b', c) \end{pmatrix}, \quad \rho_w(a, b) = \begin{pmatrix} a \otimes 1_4 & 0 \\ 0 & B \otimes 1_4 \end{pmatrix} \otimes 1_{N_g},
\]

\[
\rho_2(a, b, c) = \begin{pmatrix} \rho_s^*(b', c) & 0 \\ 0 & \rho_w^*(a, b) \end{pmatrix}, \quad b' = b \text{ or } b^*,
\] (4.6)

where \( (a, b, c) \) are the element of the algebra \( \mathbb{F}[1] \) with \( c = c(x) \in C^\infty(M_4) \otimes M_3(\mathbb{C}) \). The commutativity \( \rho_1\rho_2 = \rho_2\rho_1 \) demands that \( \rho_s(b', c) \) does not depend on \( a \). Connes took\(^3\) \( b' = b \) for the case of massless neutrinos. On the other hand, Asquish\(^11\) found for either massless or massive neutrinos that the case \( b' = b^* \) is also allowed from Poincaré duality.

We shall now derive Asquish's representation and discuss implication of it for the electric charge quantization.

To this purpose we generalize the prescription in §2 as follows. To simplify the notation let \( a \in \mathcal{A} \) such that

\[
\rho(a) = \rho_1(a)\rho_2(a) = \rho_2(a)\rho_1(a),
\]

\[
\rho_1(a) = \begin{pmatrix} \rho_w(a) & 0 \\ 0 & \rho_s(a) \end{pmatrix}, \quad \rho_2(a) = \begin{pmatrix} \rho_w^*(a) & 0 \\ 0 & \rho_s^*(a) \end{pmatrix}.
\] (4.7)

To make Eq. (4.4) gauge-invariant under the gauge transformation

\[
\Psi \rightarrow \rho(g)\Psi = \rho_1(g)\rho_2(g)\Psi, \quad \bar{\Psi} \rightarrow \bar{\Psi}\rho^\dagger(g) = \bar{\Psi}\rho_1^\dagger(g)\rho_2^\dagger(g), \quad g \in \mathcal{U}(\mathcal{A}),
\] (4.8)
we consider the non-symmetry transformations

$$\Psi \rightarrow \rho_1(b_i)\rho_2(b_j)\Psi, \quad \bar{\Psi} \rightarrow \bar{\Psi}\rho_1(a_i)\rho_2(a_j), \quad a_i, a_j, b_i, b_j \in A$$  \hspace{1cm} (4.9)

and take the double sum\textsuperscript{2} over the indices \(i\) and \(j\) after substituting Eq. (4.9) into the free massive Dirac Lagrangian (4.4) to maintain the equal-time anticommutation relations by the condition

$$\sum_{i,j} \rho_1(a_i)\rho_2(a_j)\rho_1(b_i)\rho_2(b_j) = \left(\sum_i \rho_1(a_i)\rho_1(b_i)\right)\left(\sum_j \rho_2(a_j)\rho_2(b_j)\right) = 1.$$  \hspace{1cm} (4.10)

Take, for instance, \(a_1^\dagger = b_1 = g_1 \in U(A)\) and \(a_{i\neq 1} = b_{i\neq 1} = 0\) inside the first round bracket. Then the first factor equals unity, implying the second factor to be equal to 1. Consequently, we have the general conditions

$$\sum_i \rho_1(a_i)\rho_1(b_i) = \sum_j \rho_2(a_j)\rho_2(b_j) = 1.$$  \hspace{1cm} (4.11)

We then get the result

$$\mathcal{L}_D = \frac{1}{2} \overline{\Psi}(D_0 + i\gamma_5M + A)\Psi, \quad A = \sum_{i,j} \rho_1(a_i)\rho_2(a_j)[D, \rho_1(b_i)\rho_2(b_j)] \equiv \tilde{A} + \tilde{A}^c,$$

$$\tilde{A} = \sum_i \rho_1(a_i)[D, \rho_1(b_i)], \quad \tilde{A}^c = \sum_j \rho_2(a_j)[D, \rho_2(b_j)],$$  \hspace{1cm} (4.12)

where we have assumed that

$$[[M, \rho_1(b_i)], \rho_2(b_j)] = 0.$$  \hspace{1cm} (4.13)

It turns out that this is equivalent to the condition\textsuperscript{1} from Poincaré duality. Putting

$$A = \sum_i \rho_w(a_i)[D_0, \rho_w(b_i)], \quad \Phi = \sum_i \rho_w(a_i)[M, \rho_w(b_i)],$$

$$A^c = \sum_j \rho_s(a_j)[D_0, \rho_s(b_j)], \quad \Phi^c = \sum_j \rho_s(a_j)[M, \rho_s(b_j)],$$  \hspace{1cm} (4.14)

the gauge-invariant Dirac Lagrangian (4.12) becomes with \(A^{c*} \equiv i\gamma^\mu A^{c*}_\mu\)

$$\mathcal{L}_D = \bar{\psi}(D + A + A^{c*} + i\gamma_5(\Phi + \Phi^{c*}))\psi, \quad D = D_0 + i\gamma_5M, \quad D_0 = i\bar{\psi} \otimes 1_{\dim\rho_w}. \hspace{1cm} (4.15)$$

\textsuperscript{1}) The condition (4.13) is assumed only for Dirac mass terms. Majorana mass terms for neutrinos do not obey this condition and lead to Higgs triplet and singlet.
We are now in a position to determine the representation \( \rho_s(b', c) \) based on Eq. (4.13) which means
\[
[[M, \rho_w(a, b)], \rho_s^*(b', c)] = 0. \tag{4.16}
\]
The reasoning is the same as in Ref. 11. Using the mass matrix (4.5) and writing
\[
\rho_w(a, b) = \begin{pmatrix} \rho_{wL}(a) & 0 \\ 0 & \rho_{wR}(b) \end{pmatrix} \otimes 1_{N_g},
\]
\[
\rho_s^*(b', c) = \begin{pmatrix} \rho_{sL}(b', c) & 0 \\ 0 & \rho_{sR}(b', c) \end{pmatrix} \otimes 1_{N_g},
\]
we obtain the equation
\[
(M_1 \rho_w - \rho_{wL} M_1) \rho_{sR} - \rho_{sL} (M_1 \rho_w - \rho_{wL} M_1) = 0.
\]
(It is enough to consider the case \( N_g = 1 \).) Since only \( \rho_{wL} \) depends on \( a \) this is equivalent to two conditions
\[
\rho_{wL} M_1 \rho_{sR} - \rho_{sL} \rho_{wL} M_1 = 0,
\]
\[
M_1 \rho_{wR} \rho_{sR} - \rho_{sL} M_1 \rho_{wR} = 0. \tag{4.17}
\]
The commutativity \( \rho_1 \rho_2 = \rho_2 \rho_1 \) implies \([\rho_{wL}, \rho_{sL}] = [\rho_{wR}, \rho_{sR}] = 0 \) so that Eq. (4.17) is reduced to a single equation
\[
M_1 \rho_{sR} - \rho_{sL} M_1 = 0,
\]
from which we deduce that
\[
\rho_{sL}(b', c) = \rho_{sR}(b', c) = \begin{pmatrix} 1_2 \otimes c & 0 \\ 0 & b^* 1_2 \end{pmatrix} \otimes 1_{N_g}.
\]
Consequently, we obtain
\[
\rho_s(b', c) = \begin{pmatrix} 1_2 \otimes c^* & 0 \\ b' 1_2 & 1_2 \otimes c^* \\ 0 & b' 1_2 \end{pmatrix} \otimes 1_{N_g}, \quad b' = b \text{ or } b^* \tag{4.18}
\]
This implies that $\Phi^c = 0$ in Eq. (4.14) so that the strong force associated with $\rho_s(b', c)$ is vectorial. In other words, the gauge-invariant Dirac Lagrangian (4.14) becomes

$$\mathcal{L}_D = \bar{\psi}(D_0 + A + A^c + i\gamma_5H)\psi, \quad H = M + \Phi.$$  

(4.19)

It can be shown that this is the well-known standard model Dirac Lagrangian. We do not feel it necessary to discuss the bosonic sector any more. Rather we focus upon the charge quantization problem in the light of Eq. (4.18).

The only ambiguity in our derivation of Eq. (4.19) is the appearance of $b'$ in the representation (4.18). We shall now show that the correct hypercharge assignment is obtained irrespective of the choice $b' = b$ or $b^*$. The gauge transformation

$$\psi \rightarrow g\psi = \rho_w(a, b)\rho_s^*(b', c)\psi, \quad (4.20)$$

where $(a, b, c)$ is the element of $\mathcal{U}(\mathcal{A})$, namely, $a = u \in SU(2), b = e^{i\alpha}, c = e^{i\beta}v, v \in SU(3)$, $\alpha$ and $\beta$ being real, contains two $U(1)$ factors. If the gauge transformation (4.20) is unimodular

$$\det\rho_w(a, b)\rho_s^*(b', c) = 1, \quad (a, b, c) \in \mathcal{U}(\mathcal{A}), \quad (4.21)$$

the $\alpha$ and $\beta$ are related such that only $U(1)_Y$ survives with $\text{tr}Y = 0$ per generation. The unimodularity condition (4.21) implies $\det\rho_s(b', c) = 1$ for unitary $(b', c)$, leading to $3\beta = \pm\alpha = 0$, the sign depending on $b' = b$ or $b^*$. The minus sign ($b' = b$) implies the usual assignment of the hypercharge (the coefficient of $\alpha^*$), $Y(l_L) = -1, Y(\nu_R) = 0, Y(e_R) = -2, Y(q_L) = 1/3, Y(u_R) = 4/3$ and $Y(d_R) = -2/3$. On the other hand, the plus sign ($b' = b^*$) gives a different set of values, $Y(l_L) = +1, Y(\nu_R) = +2, Y(e_R) = 0, Y(q_L) = -1/3, Y(u_R) = 2/3$ and $Y(d_R) = -4/3$. It can be shown, however, that the renaming $l_L = \begin{pmatrix} \nu \\ e \end{pmatrix}_L \rightarrow l^c_L = \begin{pmatrix} e^c \\ -\nu^c \end{pmatrix}_L$ and $l_R = \begin{pmatrix} \nu_R \\ e_R \end{pmatrix} \rightarrow l^c_R = \begin{pmatrix} e^c_R \\ -\nu^c_R \end{pmatrix}$ together with $M_\nu \leftrightarrow M^*_e$ and $\rho_s \rightarrow \rho_s^*$ and similarly for quarks converts the second solution $Q(\nu) = 1, Q(e) = 0, Q(u) = 1/3, Q(d) = -2/3$ to the first one $Q(e^c) = 1, Q(\nu^c) = 0, Q(d^c) = 1/3, Q(u^c) = -2/3$, where $Q(f)$ denotes the electric charge of the fermion $f$. That is, the electric charge quantization is linked to the single unimodularity condition (4.21) for the case of massive neutrinos.*

*) The hypercharge of Higgs doublet is normalized to be +1.

**) This conclusion solely depends on Asquish's representation and remains true even if our double sum prescription turns out to be wrong.
In contrast, the case is not true for massless neutrinos. In fact, we should replace Eqs. (4.6) and (4.18) with

$$\rho_w(a, b) = \begin{pmatrix} a \otimes 1_4 & 0 & 0 \\ 0 & B \otimes 1_3 & 0 \\ 0 & 0 & b^* \end{pmatrix} \otimes 1_{N_g},$$

$$\rho_s(b, c) = \begin{pmatrix} 1_2 \otimes c^* & 0 \\ b'1_2 & 0 \\ 1_2 \otimes c^* & b' \end{pmatrix} \otimes 1_{N_g}, \quad b' = b \text{ or } b^*.$$ (4.22)

The unimodularity condition (4.21) then leads to $3\beta - \alpha = 0$ for $b' = b$ but to $6\beta + \alpha = 0$ for $b' = b^*$. The case $b' = b$ implies the usual assignment of the hypercharge. However, the case $b' = b^*$ gives the anomaly-non-free solution $Y(l_L) = +1, Y(e_R) = 0, Y(q_L) = -1/6, Y(u_R) = 5/6$ and $Y(d_R) = -7/6$. To summarize we have found that the single unimodularity condition (4.21) leads to the anomaly-free solution provided that $\nu_R$ exist in each generation.

§5. Discussion

The present paper concerned with a field-theoretic prescription for Connes’ YM on $M_4 \times Z_2$ which derives Higgs from the Dirac operator but does not assume Higgs as an input element of the theory. Our reformulation based on the local non-symmetry transformations greatly simplifies Connes’ mathematical presentation and achieves the unification of the gauge and Higgs fields without the axioms of NCG.

Incidentally, we also found that the field strength in Connes’ gauge theory is not unique and there are two definitions possible. Connes’ definition leads to the generation-number dependent Higgs potential, while our definition yields the generation-number independent Higgs potential.

It can be shown that in the standard model only Higgs doublet, triplet and singlet are allowed in our formulation because our method generates only Higgs coupled to chiral fermions. (Higgs triplet and singlet can appear only for massive neutrinos with Majorana masses.) It is an open question whether or not our method is generalizable to describe
GUT which contains Higgs without Yukawa coupling to chiral fermions. We postpone this problem to a future work.

Acknowledgements

One (K.M.) of the authors is grateful to Professor S. Kitakado for useful discussions and continuous encouragement.

References

[1] A. Connes, p.9 in The Interface of Mathematics and Particle Physics, Clarendon Press, Oxford, 1990.
    A. Connes and J. Lott, Nucl. Phys. Proc. Suppl. 18B(1990), 29.
[2] A. Connes, Noncommutative Geometry, Academic Press, New York, 1994.
[3] A. Connes, J. Math. Phys. 36(1995), 6194; Commun. Math. Phys. 1182(1996), 155.
[4] J. C. Várilly and J. M. Gracia-Bondia, J. Geom. Phys. 12(1993), 223.
    T. Schücker and J.-M. Zylinski, J. Geom. Phys. 16(1995), 207.
[5] A. H. Chamseddine, G. Felder and J. Fröhlich, Phys. Lett. B296(1992), 109; Nucl. Phys. B395(1993), 672.
[6] As a recent review paper see, for instance,
    C. P. Martín, J. M. Gracia-Bondia, J. S. Várilly, Phys. Rep. 294(1998), 363.
[7] H. Kase, K. Morita and Y. Okumura, Prog. Theor. Phys. 101(1999), 1093.
[8] A. Sitarz, Phys. Lett. 308(1993), 787; J. Geom. Phys. 15(1995), 123.
    H.-G. Ding, H.-Y. Guo, J.-M. Li and K. Wu, Z. Phys. C64(1994), 521.
    B. Chen, T. Saito, H.-B. Teng, K. Uehara and K. Wu, Prog. Theor. Phys. 95(1996), 1173.
[9] K. Morita, and Y. Okumura, Phys. Rev. 50(1994), 1016.
[10] K. Morita, Prog. Theor. Phys. 96(1996), 801.
[11] B. Asquish, Phys. Letters 366(1996), 220.
[12] H. Kase, K. Morita and Y. Okumura, Prog. Theor. Phys. 102(1999), 1027.
[13] Y. Okumura, Journ. Math. Phys. 41(2000), 1788.
[14] M. Hayakawa, ‘Perturbative analysis on infrared and ultraviolet aspects of non-commutative QED on $R^4$’, hep-th/9912167.