Kinetical foundations of non conventional statistics

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(October 27, 2018)

Abstract

After considering the kinetical interaction principle (KIP) introduced in ref. Physica A 296, 405 (2001), we study in the Boltzmann picture, the evolution equation and the H-theorem for non extensive systems. The $q$-kinetics and the $\kappa$-kinetics are studied in detail starting from the most general non linear Boltzmann equation compatible with the KIP.

PACS number(s): 05.10.Gg, 05.20.-y

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I. INTRODUCTION

It is common in equations of Boltzmann type to treat the collisions as local and instantaneous. The spectral particle function in energy and momentum is given by the Dirac function $\delta(\epsilon - p^2/2m)$.

Assumptions to set Boltzmann equation are based on three main features:

1. Triple collisions are neglected, BBGKY hierarchy is truncated at the level of two-particle distribution function. One is limited to study dilute gases of neutral particles with short-range forces of interaction range much shorter than average particle separation and mean-free particle path between collisions. All the processes are governed by binary collisions;

2. Two-particle distribution function is factorizable with one-particle distribution, i.e. two-body collisions are statistically independent. One does not retain information from previous encounters and memory about dynamical correlations; Boltzmann equation can be solved with a class of dynamical information.

3. The particle kinetics underlying the Boltzmann equation is linear, i.e. the transition probability of a particle from one site to another is given by factors proportional to the distribution function $f$ rather than to powers of $f$ (as will be discussed in the case of non linear kinetics).

A non local and non instantaneous treatment of binary collisions, as required in many different physical problems related, for instance, to electron transport, Lorentz dense gas, rate properties in non ideal plasma and heavy-ion reactions, is strictly related to quasi-particle features and a Lorentz profile $\delta_{\gamma}(\epsilon, p)$ of the spectral particle distribution in energy and momentum can be assumed. A complete and unique formulation and treatment of Boltzmann equation, in this case, is not yet available, particularly for systems far from equilibrium (which concept of quasi-particle works better in kinetic equations for non equilibrium systems?).
With the purpose of treating the Boltzmann equation in this context in the near future, in the present work we limit ourselves to discuss a generalization of the Boltzmann equation in order to treat non-extensive systems composed by particles whose kinetics is non linear and subject to binary, local and instantaneous collisions.

This is achieved by employing the kinetic interaction principle (KIP) which is local and instantaneous but contains a generalized exclusion-inclusion principle which eventually takes into account of a non linear kinetics. The KIP has been introduced and described in refs. [1,2]. It permits to treat different statistical distributions already known in literature in a unified way and in time dependent conditions.

The nonextensive statistics introduced by Tsallis [3] has been considered in time dependent conditions in the framework of the Fokker-Planck pictures in refs. [4–8]. Recently, the Tsallis kinetics has been studied also in the Boltzmann picture in refs. [1,9].

In the present effort, starting from the KIP and after introducing the nonlinear kinetics in the Boltzmann picture we consider in detail the Tsallis statistics and the $\kappa$-statistics [1,2,10]

**II. KINETICAL INTERACTION PRINCIPLE**

Let us consider a particle system in the Boltzmann picture where only point, binary collisions occur: $(A + A_1 \rightarrow A' + A'_1)$. The most general non linear kinetics is described through the evolution equation:

$$
\left[ \frac{\partial}{\partial t} + v \frac{\partial}{\partial x} - \frac{1}{m} \frac{\partial V(x)}{\partial x} \frac{\partial}{\partial v} \right] f(t, x, v)
= \int d^n v' d^n v_1 d^n v_1' \left[ \pi(t, x, v' \rightarrow v, v_1' \rightarrow v_1) - \pi(t, x, v \rightarrow v', v_1 \rightarrow v_1') \right],
$$

(1)

The transition probabilities are defined according to the *Kinetical Interaction Principle* (KIP) [1,2] by means of:

$$
\pi(t, x, v \rightarrow v', v_1 \rightarrow v_1') = T(t, x, v, v', v_1, v_1') \gamma(f, f') \gamma(f_1, f_1'),
$$

(2)

being $f = f(t, x, v), f' = f(t, x, v'), f_1 = f(t, x, v_1), f_1' = f(t, x, v_1')$ while the function $\gamma(f, f')$ assumes the form:
\[ \gamma(f, f') = a(f) b(f') c(f, f') . \] (3)

The first factor \( a(f) \) is an arbitrary function of the particle population of the starting site and satisfies the condition \( a(0) = 0 \). The second factor \( b(f') \) is an arbitrary function of the arrival site particle population obeying the condition \( b(0) = 1 \). The third factor \( c(f, f') = c(f', f) \) takes into account that the populations of the two sites, namely \( f \) and \( f' \), can eventually affect the transition, collectively and symmetrically.

The KIP imposes the following form to the Boltzmann equation:

\[
\frac{\partial}{\partial t} + \mathbf{v} \frac{\partial}{\partial x} - \frac{1}{m} \left( \frac{\partial V(x)}{\partial x} \right) \frac{\partial}{\partial v} \] \[ f(t, x, v) = \int_\mathbb{R} d^n v' d^n v_1 d^n v_1' T(t, x, v, v', v_1, v_1') c(f, f') c(f_1, f'_1) \times \left[ a(f') b(f) a(f'_1) b(f_1) - a(f) b(f') a(f_1) b(f'_1) \right] . \] (4)

After introducing the two functions:

\[ B(f, f_1, f', f'_1) = c(f, f') c(f_1, f'_1) b(f) b(f_1) b(f') b(f'_1) , \] (5)

and

\[ \kappa(f) = \frac{a(f)}{b(f)} , \] (6)

one can write Eq. (4) under the form:

\[
\frac{\partial}{\partial t} + \mathbf{v} \frac{\partial}{\partial x} - \frac{1}{m} \left( \frac{\partial V(x)}{\partial x} \right) \frac{\partial}{\partial v} \] \[ f(t, x, v) = \int_\mathbb{R} d^n v' d^n v_1 d^n v_1' T(t, x, v, v', v_1, v_1') B(f, f_1, f', f'_1) \times \left\{ \exp [\ln \kappa(f') + \ln \kappa(f'_1)] - \exp [\ln \kappa(f) + \ln \kappa(f_1)] \right\} . \] (7)

We remark that this evolution equation describes a very large class of non linear generalized kinetics which includes also the standard linear kinetics obtained after posing \( a(f) = f \), \( b(f') = 1 \) and \( c(f, f') = 1 \).

We recall that in the frame of the present non linear kinetics the mean value of a given quantity \( A \) is defined through:

\[ \langle A \rangle = \int_\mathbb{R} d^n x d^n v A f \quad ; \quad \int_\mathbb{R} d^n x d^n v f = 1 . \] (8)
III. GENERALIZED ENTROPY

From Eq.(7), in stationary conditions, we have:

$$\ln \kappa(f_s) + \ln \kappa(f_{is}) = \ln \kappa(f'_s) + \ln \kappa(f'_{is})$$

(9)
	herefore the quantity \(\ln \kappa(f_s)\) is the collisional invariant of the system. Taking into account that during the collisions the particle number, energy and momentum are conserved, we can express the collisional invariant as [1]:

$$\ln \kappa(f_s) = -\beta \left[ \frac{1}{2} m v^2 + V(x) - \mu' \right]$$

(10)

with \(\beta = 1/k_B T\). This last equation, which defines the stationary distribution of the system, can be written also under the form:

$$\kappa(f_s) = \frac{1}{Z} \exp \left\{ -\beta \left[ \frac{1}{2} m v^2 + V(x) - \mu \right] \right\}$$

(11)

with \(\beta \mu = \beta \mu' + \ln Z\) and \(Z\) given through \(\int d^nv \int d^n x f_s = 1\).

Let us consider the functional \(\mathcal{K}(t)\):

$$\mathcal{K}(t) = -k_B \int_R d^n x d^n v \int df \ln \frac{\kappa(f)}{\kappa(f_s)}$$

(12)

Of course, the stationary distribution \(f_s\) can be obtained also by maximizing the functional \(\mathcal{K}(t)\):

$$\frac{\delta \mathcal{K}(t)}{\delta f} = 0 \implies f = f_s$$

(13)

In ref. [2] it has been shown that, if the condition:

$$\frac{d\kappa(f)}{df} \geq 0$$

(14)

is satisfied and \(f\) obeys the Boltzmann equation [3], the quantity \(-\mathcal{K}(t)\) is a Lyapunov functional:

$$\frac{d\mathcal{K}(t)}{dt} \geq 0$$

$$\mathcal{K}(t) \leq \mathcal{K}(\infty)$$

(15)

(16)
In order to introduce the entropy $S(t)$ of the system, we pose:

$$K(t) = S(t) - k_B \beta (E - \mu') ,$$

and observe that the energy $E$ of the system:

$$E = \int_R d^n x d^n v \left[ \frac{1}{2} m v^2 + V(x) \right] f$$

is a conserved quantity: $dE/dt = 0$, as well as the particle number $\int_R d^n x d^n v f = 1$. By comparing the two expressions of the functional $K(t)$ given by Eq.s (12) and (17) we obtain that:

$$S = -k_B \int_R d^n x d^n v \int df \kappa(f) .$$

The entropy $S(t)$ obeys the H-theorem:

$$\frac{dS(t)}{dt} \geq 0 ,$$

$$S(t) \leq S(\infty) ,$$

as can be verified immediately if we take into account Eq.s (15)-(17).

IV. TSALLIS KINETICS

Let us consider the kinetics defined by fixing $\kappa(f)$ through:

$$\ln[Z \kappa(f)] = \ln_q(Z f) ,$$

where $\ln_q(x) = (x^{1-q} - 1)/(1 - q)$ is the Tsallis logarithm and $f$ is a normalized function: $\int_R d^n x d^n v f = 1$. The evolution equation (7) becomes:

$$f(t, x, v) = \int_R d^n v' d^n v_1 d^n v_1 \times
\left\{ (Z' Z'_1)^{-1} \exp[\ln_q(Z' f') + \ln_q(Z'_1 f'_1)] - (Z Z_1)^{-1} \exp[\ln_q(Z f) + \ln_q(Z_1 f_1)] \right\} .$$
where \( Z, Z', Z_1 \) and \( Z'_1 \) are the partition functions related to the distributions \( f, f', f_1 \) and \( f'_1 \) respectively.

Being \( d\kappa(f)/df \geq 0 \) for the system described by Eq. (23), the H-theorem is satisfied and its stationary distribution is given by:

\[
f_s = \frac{1}{Z} \exp_q \left\{ -\beta \left[ \frac{1}{2} m v^2 + V(x) - \mu \right] \right\},
\]

where \( \exp_q(x) = [1 + (1 - q)x]^{1/(1-q)} \) is the Tsallis exponential while the partition function is \( Z = \int_{\mathcal{R}} d^nx \, d^nv \, \exp_q \left\{ -\beta \left[ m v^2/2 + V(x) - \mu \right] \right\} \).

The above Eq. (24) is the Tsallis distribution and can be obtained also from the variational principle, defined by means of Eqs. (12) and (13), which now assumes the form:

\[
\frac{\delta}{\delta f} k_B \int_{\mathcal{R}} d^nx \, d^nv \left[ -\frac{1}{2 - q} \frac{(Z f)^{1-q} - 1}{1 - q} + \frac{1}{2 - q} - \beta \frac{1}{2} m v^2 + \beta \mu \right] f = 0.
\]

From Eq. (25) it results clear that the entropy of the system is given by:

\[
S = -\frac{k_B}{2 - q} \ln_q(Z f) + \frac{k_B}{2 - q},
\]

and can be written also as:

\[
S = \frac{1}{2 - q} S_{2-q}^{(T)} [Z f] + \frac{k_B}{2 - q},
\]

being the Tsallis entropy defined through:

\[
S_q^{(T)} [f] = -k_B \int_{\mathcal{R}} d^nx \, d^nv \, \frac{f - f^q}{1 - q}.
\]

V. THE \( \kappa \)-KINETICS

In ref. [1] it has been proposed the following one parameter deformation of the logarithm function:

\[
\ln_{(\kappa)} (x) = \frac{x^\kappa - x^{-\kappa}}{2 \kappa},
\]
which reduces to the standard logarithm as $\kappa \to 0$, obeys the scaling law $\ln_{(\kappa)}(x^m) = m \ln_{(m\kappa)}(x)$ and presents the following power law asymptotic behaviour:

\[
\ln_{(\kappa)}(x) \sim \frac{1}{|2\kappa|} x^{-|\kappa|},
\]

\[
\ln_{(\kappa)}(x) \sim \frac{1}{|2\kappa|} x^{|\kappa|}.
\]

We consider the kinetics defined by fixing $\kappa(f)$ as follows:

\[
\ln [Z \kappa(f)] = \ln_{(\kappa)}(Z f).
\]

The evolution equation (32) becomes:

\[
\left[ \frac{\partial}{\partial t} + v \frac{\partial}{\partial x} - \frac{1}{m} \frac{\partial V(x)}{\partial x} \frac{\partial}{\partial v} \right] f(t, x, v) = \int_{\mathcal{R}} d^n v' d^n v_1 d^n v'_1 T(t, x, v, v', v_1, v'_1) B(f, f_1, f', f'_1)
\]

\[
\times \left\{ (Z' Z'_1)^{-1} \exp [\ln_{(\kappa)}(Z' f') + \ln_{(\kappa)}(Z'_1 f'_1)]
\right.
\]

\[
\left. - (Z Z_1)^{-1} \exp [\ln_{(\kappa)}(Z f) + \ln_{(\kappa)}(Z_1 f_1)] \right\},
\]

and reduces to the standard Boltzmann equation if we pose $\kappa \to 0$ and $B(f, f_1, f', f'_1) = 1$.

We observe that $d\kappa(f)/df \geq 0$ for $\forall \kappa \in \mathbb{R}$ so that the H-theorem still holds.

The stationary distribution of Eq. (33) is given by:

\[
f_s = \frac{1}{Z} \exp_{(\kappa)} \left[ -\beta \left( \frac{1}{2} m v^2 + V(x) - \mu \right) \right],
\]

with $Z = \int_{\mathcal{R}} d^n x d^n v \exp_{(\kappa)} \left\{ -\beta \left[ m v^2 / 2 + V(x) - \mu \right] \right\}$, being the $\kappa$-exponential defined as the inverse function of $\kappa$-logarithm:

\[
\exp_{(\kappa)}(x) = \left( \sqrt{1 + \kappa^2 x^2} + \kappa x \right)^{1/\kappa}.
\]

The $\kappa$-exponential reduces to the standard exponential as $\kappa \to 0$, obeys the scale law $[\exp_{(\kappa)}(x)]^m = \exp_{(\kappa/m)}(m x)$ and shows a power law asymptotic behaviour

\[
\exp_{(\kappa)}(x) \sim |2\kappa x|^{\pm 1/|\kappa|}.
\]
It is easy to verify that the distribution (34) can be obtained after maximization under the appropriate constraint of the entropy:

\[
S_\kappa = -\frac{k_B}{2\kappa} \int_R d^nx d^n v \left( \frac{Z^\kappa}{1 + \kappa} f^{1+\kappa} - \frac{Z^{-\kappa}}{1 - \kappa} f^{1-\kappa} \right),
\]

which reduces to the standard Shannon entropy \( S_0 = -k_B \int d^nx d^n v f \ln(Z f) \) as the deformation parameter \( \kappa \to 0 \). The entropy \( S_\kappa \) and the entropy \( S \) defined in Eq. (19) are connected through \( S_\kappa = S - k_B \ln Z \). The variational equation (13) reproducing the distribution (34) can be written under the form:

\[
\delta \frac{S_\kappa - k_B \beta E + k_B \beta \mu}{\delta f} = 0.
\]

VI. CONCLUSION

In the present contribution, we have studied the kinetics of a non linear system in the frame of the Boltzmann picture. The main points are the following:

The introduction of the KIP, given through a particular expression of the transition probability (3), permits to describe the time evolution of a non linear system and imposes, by means of the functional \( \kappa(f) \) in Eq. (6), its steady state as stationary solution of the evolution equation.

The KIP imposes the entropy form of the non linear system. Its expression is given in Eq. (19), and, as shown in Eq. (20), obeys to the H-theorem whenever the condition \( d\kappa(f)/df \geq 0 \) is satisfied.

Finally, we have discussed within the formalism here developed, as a working example, the well known Tsallis statistics and the \( \kappa \)-statistics, already introduced and described by one of us in ref. [1].

The evolution equation (23) is very close to the equation recently proposed by Lima, Silva, and Plastino [9] within a kinetic foundations of Tsallis thermostatistics. In some points we differ from them. These authors consider a hard sphere particle gas and a \( q \)-collisional term.
written in terms of a difference of two $q$-exponent correlation functions (before and after collision). Our collisional terms is a function deduced from KIP and is a difference of two standard exponents.

In ref. [9] is reported that other possibilities also leading to Tsallis distribution can be obtained if $\exp_q(x)$ is substituted by other positive increasing functions $F_q(x)$ such that $\lim_{q \to 1} F_q(x) = \exp(x)$. An attempt to unify the two derivations of Tsallis evolution equation is in progress.
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