RSOS models and Jantzen-Seitz representations of Hecke algebras at roots of unity.

Omar Foda*, Bernard Leclerc†, Masato Okado‡, Jean-Yves Thibon§ and Trevor A. Welsh*

Abstract

A special family of partitions occurs in two apparently unrelated contexts: the evaluation of 1-dimensional configuration sums of certain RSOS models, and the modular representation theory of symmetric groups or their Hecke algebras $H_m$. We provide an explanation of this coincidence by showing how the irreducible $H_m$-modules which remain irreducible under restriction to $H_{m-1}$ (Jantzen-Seitz modules) can be determined from the decomposition of a tensor product of representations of $\hat{\mathfrak{sl}}_n$.

1 Introduction

The solution of a class of “restricted-solid-on-solid” (RSOS) models by the corner transfer matrix method leads to the evaluation of weighted sums of combinatorial objects called paths [1]. The Kyoto group realized that these combinatorial sums are branching functions of the affine Lie algebra $\hat{\mathfrak{sl}}_2$ [3], and was able to define similar models associated with other affine Lie algebras, in particular $\hat{\mathfrak{sl}}_n$ [16].

For the models associated with the cosets $(\hat{\mathfrak{sl}}_m)_1 \times (\hat{\mathfrak{sl}}_n)_1 / (\hat{\mathfrak{sl}}_n)_2$, a different description of the branching functions as generating series of certain sets of partitions has been obtained in [3], and was used to derive fermionic expressions for the configuration sums.

It turns out that exactly the same partitions arise in the modular representation theory of the symmetric groups: as conjectured by Jantzen and Seitz [14] and established recently by Kleshchev [20], such partitions label the irreducible representations of a symmetric group $S_m$ over a field of characteristic $n$ which remain irreducible under restriction to $S_{m-1}$.

The aim of this Letter is to provide an explanation of this seemingly mysterious coincidence.

The first point is to replace symmetric groups in characteristic $n$ by Hecke algebras of type $A$ over $\mathbb{C}$ at an $n$th root of unity. Indeed, the representation theories of $F_n(S_m)$ and $H_m(\sqrt{n})$ have many formal similarities, but the consideration of Hecke algebras removes the restriction of $n$ being a prime, which does not appear on the statistical mechanics side.

Moreover, a connection between the representation theory of $H_m(\sqrt{n})$ and the level 1 representations of the quantum affine algebra $U_q(\hat{\mathfrak{sl}}_n)$ has been pointed out in [22]. Building on a conjecture of [22], recently proved by Ariki and Grojnowski, we show that the Jantzen-Seitz type problem for $H_m(\sqrt{n})$ is equivalent to the decomposition via crystal bases of tensor products of level 1 $\hat{\mathfrak{sl}}_n$-modules. Using the results of [1], we can then characterize the Hecke algebra modules of Jantzen-Seitz type and explain the occurrence of their partition labels in the configuration sums of RSOS-models.

*Department of Mathematics, University of Melbourne, Parkville, Victoria 3052, Australia.
†Département de Mathématiques, Université de Caen, BP 5186, 14032 Caen Cedex, France.
‡Department of Mathematical Sciences, Faculty of Engineering Science, Osaka University, Osaka 560, Japan.
§Institut Gaspard Monge, Université de Marne-la-Vallée, 93166 Noisy-le-Grand Cedex, France.
As an application, we express the generating function of the Jantzen-Seitz partitions having a given \( n \)-core in terms of branching functions of \( \widehat{sl}_n \). This is to be compared with a well-known result on blocks of Hecke algebras. Indeed, the blocks of \( H_m(\sqrt{n}) \) are labelled by \( n \)-cores, and the dimension of a block is the number of \( n \)-regular partitions of \( m \) with the corresponding \( n \)-core. Using a formula first proved in the fifties by Robinson (in the symmetric group case) one can compute the generating series of the dimensions of all blocks labelled by a given \( n \)-core, and recognize the string function of the level 1 \( \widehat{sl}_n \)-modules. Our result shows that some branching functions of \( \widehat{sl}_n \) other than the level 1 string function arise in a natural way in the representation theory of \( H_m(\sqrt{n}) \).

## 2 Characters and branching functions of \( \widehat{sl}_n \)

Using the notion of paths, it was shown in [5] that the characters of the integrable highest weight modules of \( \widehat{sl}_n \) may be obtained by enumerating certain coloured multipartitions. In this Letter, we are interested only in the case when the highest weight is of level one, and therefore the multipartitions are simply partitions.

As is usual, we define a partition \( \lambda \) of \( m \) to be a sequence

\[
\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)
\]

such that \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \) and \( \lambda_1 + \lambda_2 + \cdots + \lambda_k = m \). If \( i > k \), we understand that \( \lambda_i = 0 \). The set of all partitions of \( m \) is denoted \( \Pi(m) \) and we write

\[
\Pi = \bigcup_{m \geq 0} \Pi(m).
\]

Occasionally, it will be convenient to use the exponent notation

\[
\lambda = (\lambda_1^{a_1}, \lambda_2^{a_2}, \ldots, \lambda_r^{a_r})
\]

where here \( \lambda_1 > \lambda_2 > \cdots > \lambda_r > 0 \), and \( a_i > 0 \) specifies the multiplicity of \( \lambda_i \) in \( \lambda \). If \( a_i < n \) for \( 1 \leq i \leq r \) then we say that the partition \( \lambda \) is \( n \)-regular. The set of all \( n \)-regular partitions of \( m \) is denoted \( \Pi_n(m) \), and we define

\[
\Pi_n = \bigcup_{m \geq 0} \Pi_n(m).
\]

The Young diagram associated with the partition \( \lambda \) is an array of \( k \) left-adjusted rows of nodes (or boxes) in which the \( i \)th row contains \( \lambda_i \) nodes. For example if \( \lambda = (4, 3, 1) \), the corresponding Young diagram is:

\[
F^{(4,3,1)} = 
\begin{array}{ccc}
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\end{array}
\]

We will not distinguish between a partition and its Young diagram. The partition \( \lambda' = (\lambda'_1, \lambda'_2, \ldots) \) conjugate to \( \lambda \) is defined such that \( \lambda'_j \) is the length of the \( j \)th column of \( \lambda \), reading from left to right.

A coloured partition is a Young diagram in which each node \( \gamma \) is filled by its colour charge \( c(\gamma) \) given by \( c(\gamma) = (j - i) \mod n \), when \( \gamma \) is the node at the intersection of the \( i \)th row and the \( j \)th column. For example, in the case \( n = 3 \), the coloured partition \( \lambda = (5, 5, 4, 1, 1) \) appears as follows:

\[
\begin{array}{c}
1 & 1 & 2 & 2 & 1 \\
2 & 1 & 2 & 2 & 1 \\
1 & 1 & 1 & 1 & 1 \\
\end{array}
\]
Let \( m_i \) be the number of nodes of \( \lambda \) with colour charge \( i \). The energy of \( \lambda \) is defined by 
\[
E(\lambda) = m_0,
\]
and its weight by \( \text{wt}(\lambda) = \Lambda_0 - \sum_{i=1}^{n-1} m_i \alpha_i \). (We use freely the standard notations for roots, weights, etc. of \( \mathfrak{sl}_n \), see e.g. \[3\]) Then, the formal character of the basic representation \( V(\Lambda_0) \) of \( \mathfrak{sl}_n \) is
\[
\text{ch } V(\Lambda_0) = \sum_{\Lambda \in \Pi_n} e^{\text{wt}(\lambda)}.
\]

A realisation of \( V(\Lambda_0) \) that has a basis naturally indexed by the set of \( n \)-regular partitions \( \Pi_n \) was given in \[3\].

In \[15\], it was also shown how to describe combinatorially the branching functions of tensor products of irreducible \( \mathfrak{sl}_n \)-modules. The branching function \( b_{\Lambda',\Lambda}(q) \) is defined such that, for each \( i \geq 0 \), the multiplicity of the module \( V(\Lambda' - i\delta) \) in the tensor product \( V(\Lambda) \otimes V(\Lambda') \) is given by the coefficient of \( q^i \) in \( b_{\Lambda',\Lambda}(q) \). Therefore, in terms of characters, we have
\[
\text{ch } (V(\Lambda)) \text{ch } (V(\Lambda')) = \sum_{\Lambda'' \in P^+} b_{\Lambda',\Lambda}(e^{-\delta}) \text{ch } (V(\Lambda'')).
\]

The result of \[15\] relies on the notion of a path which we describe in the case where \( \Lambda' = \Lambda_0 \). A path \( p \) is a sequence \( p = (p_0, p_1, \ldots) \), where \( p_k \in P \), the weight lattice of \( \mathfrak{sl}_n \). Given a dominant integral weight \( \Lambda \) of level \( l \) for some \( l \geq 0 \), and \( \lambda \in \Pi_n \), a path \( p = p(\lambda) \) is determined as follows. For \( k \geq \lambda_1 \), define \( p_k = \Lambda + \Lambda_k \). Then for \( 0 < k \leq \lambda_1 \), recursively obtain \( p_{k-1} \) from \( p_k \) by
\[
p_{k-1} = p_k - \epsilon_{k-1} - \Lambda_k',
\]
where \( \epsilon_i = \Lambda_{i+1} - \Lambda_i \) and as usual the indices are understood modulo \( n \). We write \( \lambda \in \mathcal{Y}(\Lambda, \Lambda_0) \) if and only if all the coordinates \( p_k \) of the path \( p(\lambda) \) are dominant weights.

**Theorem 2.1** \[15\]
\[
b_{\Lambda',\Lambda_0}(q) = \sum_{\substack{\lambda \in \mathcal{Y}(\Lambda, \Lambda_0) \\text{wt}(\lambda) = \Lambda' - \Lambda \pmod{\delta}}} q^{E(\lambda)}.
\]

In the case when \( \Lambda \) is of level one, the partitions appearing in this sum were characterised in \[2\]. We define the set \( \text{FOW}(n,j) \subset \Pi_n \) as follows. Let \( \lambda = (\lambda_1^a, \lambda_2^a, \ldots, \lambda_r^a) \in \Pi_n \). Then \( \lambda \in \text{FOW}(n,j) \) if and only if either \( r = 1 \) or
\[
a_i + \lambda_i - \lambda_{i+1} + a_{i+1} = 0 \pmod{n},
\]
for \( i = 1, 2, \ldots, r-1 \) and \( j = (\lambda_1 - a_1) \pmod{n} \).

**Theorem 2.2** \[2\]
\[
\mathcal{Y}(\Lambda_j, \Lambda_0) = \text{FOW}(n, j).
\]
If we now define \( \text{FOW}(n,j,k) \) to be the subset of \( \text{FOW}(n,j) \) comprising those partitions \( \lambda \) for which \( \text{wt}(\lambda) = \Lambda_k + \Lambda_{j-k} - \Lambda_j \pmod{\delta} \), then Theorem 2.1 immediately yields the following:

**Corollary 2.3** \[2\] Let \( 0 \leq j, k < n \). Then the branching function \( b_{\Lambda_j,\Lambda_0}(q) \) is given by:
\[
b_{\Lambda_j,\Lambda_0}(q) = \sum_{\lambda \in \text{FOW}(n,j,k)} q^{E(\lambda)}.
\]

By means of this expression, the following was obtained:
The set of partitions $\text{FOW}(n,j,k)$ has a definite meaning in modular representation theory. Indeed, $\bigcup_{j,k} \text{FOW}(n,j,k)$ labels the modular representations of $\mathfrak{S}_m$ in characteristic $n$ that remain irreducible under reduction to $G_{m-1}$ \cite{4, 27}.

However, in the case of $\text{FOW}(n,j,k)$, $n$ can be any positive integer, whereas in the case of $\mathfrak{S}_m$, it has to be a prime number. This difference can be removed by working in the context of Hecke algebras at an $n$th root of unity, where the Jantzen-Seitz problem still makes sense.

The Hecke algebra $H_m(v)$ of type $A_{m-1}$, is the $\mathbb{C}(v)$-algebra generated by the elements $T_1, \ldots, T_{m-1}$ subject to the relations

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1};$$

$$T_i T_j = T_j T_i \quad |i - j| > 1;$$

$$T_i^2 = (v - 1) T_i + v.$$ 

In the case $v = 1$, $H_m(v)$ may be identified with the group algebra of the symmetric group $\mathfrak{S}_m$, through identifying $T_i$ with the simple transposition $(i, i+1) \in \mathfrak{S}_m$. In fact, for generic values of $v$, $H_m(v)$ is isomorphic to the group algebra of the symmetric group $\mathfrak{S}_m$. Hence, it is semisimple, and its irreducible representations are parametrised by $\Pi(m)$. A convenient realisation of the representation labelled by $\lambda$ is the Specht module $S^\lambda$ in which the entries of the representation matrices of the generators are elements of $\mathbb{Z}[v]$.

In the non-generic case when $v$ is a primitive $n$th root of unity, $H_m(v)$ is no longer semisimple in general, and its representations are not necessarily completely reducible. The full set of irreducible representations is indexed by $\Pi_n(m)$ (see \cite{4}). The irreducible module labelled by $\mu \in \Pi_n(m)$ is denoted by $D^\mu$.

Define $\mathcal{J}S(n)$ to be the set of all partitions that label the irreducible representations of $H_m(\sqrt[n]{T})$ which remain irreducible under reduction to $H_{m-1}(\sqrt[n]{T})$. One of the aims of this work is to show that the partitions in $\mathcal{J}S(n)$ are defined by the same conditions as in the case of symmetric groups, and to explain why

$$\mathcal{J}S(n) = \bigcup_{j,k} \text{FOW}(n,j,k).$$

To tackle this problem, it is convenient to consider, as was done in \cite{24, 3}, the Grothendieck group $G_0(H_m(\sqrt[n]{T}))$ of the category of finitely generated $H_m(\sqrt[n]{T})$-modules. The elements of $G_0(H_m(\sqrt[n]{T}))$ are classes $[M]$ of modules, where $[M_1] = [M_2]$ if and only if the simple composition factors of $M_1$ and $M_2$ occur with identical multiplicities (the order of the composition factors in the series is disregarded). The sum is defined by $[M] + [N] = [M \oplus N]$. It is known that this is a free abelian group with basis the set $\{D^\mu\}$ of classes of irreducible $H_m(\sqrt[n]{T})$-modules. Then define $\mathcal{G}(n)$ as the direct sum $\mathcal{G}(n) = \bigoplus_m G_0(H_m(\sqrt[n]{T}))$.

As observed in \cite{22}, $\mathcal{G}(n)$ can be identified with the basic representation $V(\Lambda_0)$ of $\tilde{\mathfrak{sl}}_n$, the action of the Chevalley generators $e_i$, $f_i$ being given by the $i$-restriction and $i$-induction operators, as defined in the fifties by G. de B. Robinson in the case of symmetric groups.
As an integrable highest weight module of an affine algebra, \( \mathcal{G}(n) \) has a canonical basis in the sense of Lusztig and Kashiwara. It turns out that this canonical basis coincides with the natural basis given by the classes \([D(\mu)]\) of the irreducible \( H_m(\sqrt{\delta})\)-modules.

To prove this, and to compute explicitly the canonical basis, one considers the Fock space representation \( \mathcal{F} \) of \( \hat{\mathfrak{sl}}_n \), which contains \( V(\Lambda_0) \) as its highest irreducible component:

\[
\mathcal{F} = V(\Lambda_0) \oplus V_{\text{low}}.
\]

The standard basis \((v_\lambda)\) of \( \mathcal{F} \) is labelled by all \( \lambda \in \Pi \) and one has an \( \hat{\mathfrak{sl}}_n \)-homomorphism

\[
d: \mathcal{F} \rightarrow V(\Lambda_0) \simeq \mathcal{F}/V_{\text{low}} \quad \quad \quad \quad v_\lambda \mapsto v_\lambda \mod V_{\text{low}}
\]

If \( \mathcal{F} \) is identified with the Grothendieck ring of all \( H_m(v) \) for a generic \( v \) by writing \([S^\lambda] = v_\lambda\), then \( d \) coincides with the decomposition map of modular representation theory. In order to introduce the canonical basis, one needs to \( q \)-deform the picture and to consider the \( q \)-Fock space representation of \( U_q(\hat{\mathfrak{sl}}_n) \).

### 4 The Fock space representation of \( U_q(\hat{\mathfrak{sl}}_n) \)

The \( q \)-Fock space \( \mathcal{F} \) of \( U_q(\hat{\mathfrak{sl}}_n) \) has been described in [11] using a \( q \)-analogue of the Clifford algebra. Another realization was given in [23] in terms of semi-infinite \( q \)-wedge products. Let \((v_\lambda)\) denote the standard basis of weight vectors of \( \mathcal{F} \). The Fock space \( \mathcal{F} \) affords an integrable representation of \( U_q(\hat{\mathfrak{sl}}_n) \) whose decomposition into irreducible highest weight modules is given by

\[
\mathcal{F} = \bigoplus_{k \geq 0} V(\Lambda_0 - k\delta)^{\oplus p(k)}.
\]

In [23], the lower crystal basis of \( \mathcal{F} \) and its crystal graph structure was described. Let \( A \subset \mathbb{Q}(q) \) denote the ring of rational functions without a pole at \( q = 0 \). The lower crystal basis at \( q = 0 \) of \( \mathcal{F} \) is the pair \((L, B)\) where \( L \) is the lower crystal lattice given by

\[
L = \bigoplus_{\lambda \in \Pi} A v_\lambda,
\]

and \( B \) is the basis of the \( \mathbb{Q} \)-vector space \( L/qL \) given by

\[
B = \{v_\lambda \mod qL \mid \lambda \in \Pi\}.
\]

The action of Kashiwara’s operators \( \tilde{f}_i \) on the element \( v_\lambda \in B \) corresponds to adding to \( \lambda \) a certain node of colour \( i \) which is called a good addable \( i \)-node. Likewise the action of \( \tilde{e}_i \) corresponds to the removal from \( \lambda \) of a certain node of colour \( i \) which is called a good removable \( i \)-node. As observed in [22], these nodes are precisely those used by Kleshchev in his modular branching rule for symmetric groups [21], whence the terminology.

For each \( \lambda \in \Pi \), the largest integer \( k \) such that \( \tilde{e}_i^k(\lambda) \neq 0 \) \( (resp \, \tilde{f}_i^k(\lambda) \neq 0) \) is denoted by \( \varepsilon_i(\lambda) \) \( (resp \, \varphi_i(\lambda)) \). The crystal graph \( \Gamma_\lambda \) of \( \mathcal{F} \) is disconnected and reflects the decomposition \((\emptyset)\). The connected component of the empty partition is the crystal graph of \( V(\Lambda_0) \) and its vertices are labelled by \( \Pi_n \).

We denote by \((G(\mu), \mu \in \Pi_n)\) the lower global basis of \( V(\Lambda_0) \). It is the \( \mathbb{Q}[q, q^{-1}] \)-basis of the integral form \( V_q(\Lambda_0) \) of \( V(\Lambda_0) \) characterized by

\[
G(\mu) \equiv v_\mu( \mod qL), \quad \overline{G(\mu)} = G(\mu).
\]

Here, for \( v = xv_0 \in V(\Lambda_0) \), \( \overline{v} \) is defined by \( \overline{v} = \overline{v}v_0 \), where \( x \mapsto \overline{x} \) is the \( \mathbb{Q}(q) \)-ring automorphism of \( U_q(\hat{\mathfrak{sl}}_n) \) given by

\[
\overline{q} = q^{-1}, \quad \overline{q^n} = q^{-n}, \quad \overline{e_i} = e_i, \quad \overline{f_i} = f_i,
\]
for all \( h \in \mathcal{P}^\vee \) and \( 0 \leq i < n \).

The upper global crystal basis \( \{ G^{\text{up}}(\mu), \mu \in \Pi_n \} \) of \( V(\Lambda_0) \) is the basis adjoint to \( \{ G(\mu) \} \) with respect to the inner product \( \langle \cdot, \cdot \rangle \) defined by

\[
\langle v_0, v_0 \rangle = 1 \quad \langle q^h v, v' \rangle = \langle v, q^h v' \rangle, \quad \langle e_i v, v' \rangle = \langle v, f_i v' \rangle
\]

for all \( v, v' \in V(\Lambda_0), h \in \mathcal{P}^\vee \) and \( 0 \leq i < n \).

From now on we fix \( n \geq 2 \) and write \( H_m \) instead of \( H_m(\sqrt{t}) \). It has been conjectured in [22] and proved in [2] that at \( q = 1 \), the upper global crystal basis of the \( U_q(\mathfrak{sl}_n) \)-module \( V(\Lambda_0) \) coincides in the identification \( V(\Lambda_0) \cong \mathcal{G}(n) \) with the basis \( \{ [D^\lambda] \} \) of \( \mathcal{G}(n) \). This was also proved by Grojnowski, using the results of [1]. This implies that

**Theorem 4.1** If \( \lambda \in \Pi_n(m) \) and the coefficients \( c_{\lambda \mu}(q) \) are defined by:

\[
\left( \sum_{i=0}^{n-1} e_i \right) G^{\text{up}}(\lambda) = \sum_{\mu \in \Pi_n(m-1)} c_{\lambda \mu}(q) G^{\text{up}}(\mu),
\]

then,

\[
[D^{\lambda \downarrow} H_m^\downarrow] = \bigoplus_{\mu \in \Pi_n(m-1)} c_{\lambda \mu}(1) [D^\mu].
\]

5 Jantzen-Seitz modules of Hecke algebras

We are now in position to prove the following result.

**Theorem 5.1** Let \( \lambda = (\lambda_1^a, \ldots, \lambda_r^c) \in \Pi_n(m) \). Then \( D^{\lambda \downarrow} H_m^\downarrow \) is irreducible, i.e. \( \lambda \in JS(n) \), if and only if either \( r = 1 \) or \( a_i + \lambda_i - \lambda_{i+1} + a_{i+1} \equiv 0 \pmod{n} \) for \( i = 1, 2, \ldots, r - 1 \).

**Proof:** By Theorem 4.1, \( D^{\lambda \downarrow} \) is irreducible if and only if \( e_i G^{\text{up}}(\lambda) = G^{\text{up}}(\nu) \) for some \( \nu \). It is known (see [18], eqs. 5.3.8. - 5.3.10.) that

\[
e_i G^{\text{up}}(\lambda) = [\varepsilon_i(\lambda)]_q G^{\text{up}}(\tilde{\varepsilon}_i \lambda) + \sum_{\mu \neq \lambda} E^i_{\lambda \mu}(q) G^{\text{up}}(\mu) \tag{2}
\]

where \( E^i_{\lambda \mu}(q) \) is a Laurent polynomial invariant under \( q \mapsto q^{-1} \) such that

\[
E^i_{\lambda \mu}(q) \in q^{2-\varepsilon_i(\lambda)} \mathbb{Z}[q].
\]

If \( \varepsilon_i(\lambda) = 1 \), then \( [\varepsilon_i(\lambda)]_q = 1 \), and \( E^i_{\lambda \mu}(q) = 0 \) since for all \( k \) the coefficients of \( q^k \) and \( q^{-k} \) in \( E^i_{\lambda \mu} \) are equal. Therefore \( e_i G^{\text{up}}(\lambda) = G^{\text{up}}(\nu) \) where \( \nu = \tilde{\varepsilon}_i \lambda \). On the other hand, if \( e_i G^{\text{up}}(\lambda) = G^{\text{up}}(\nu) \) for some \( \nu \), then (2) implies that \( \varepsilon_i(\lambda) = 1 \) so that \( \nu = \tilde{\varepsilon}_i \lambda \).

Hence we have obtained that \( D^{\lambda \downarrow} \) is irreducible if and only if \( \varepsilon_i(\lambda) = 1 \) for some \( i \in \{0, 1, \ldots, n - 1\} \) and \( \varepsilon_j(\lambda) = 0 \) for \( j \neq i \).

Using crystal graph theory (see e.g. Lemma 5.1 of [18]), we see that the coefficient of \( z^d \) in the branching function \( b^{\lambda \downarrow}_{\Lambda_0}(z) \) is the number of vertices \( b \in B \) for which \( wt(b) = \Lambda - d \delta \) and \( \varepsilon_j(b) \leq \langle \lambda', h_j \rangle \) for \( j = 0, 1, \ldots, n - 1 \). Therefore, in the case \( \lambda' = \Lambda_i \), the coefficient of \( z^d \) in \( b^{\lambda \downarrow}_{\Lambda_0}(z) \) is given by the number of vertices \( b \) of the crystal graph of \( V(\Lambda_0) \) such that

\[
wt(b) = \Lambda - \Lambda_i - d \delta, \quad \varepsilon_i(b) \leq 1, \quad \varepsilon_j(b) = 0 \quad (j \neq i).
\]

It follows from this discussion that the partitions \( \lambda \) such that \( D^{\lambda \downarrow} \) is irreducible can be identified with the set of vertices \( b \) of \( B \) which contribute to the tensor product branching function \( b^{\lambda \downarrow}_{\Lambda_0}(z) \) for some \( i \) and some \( \Lambda \). Note that this branching function is non-zero only if \( \Lambda = \Lambda_k \downarrow \Lambda_{i-k} \) for some \( k \). Then, by [2,3], the Theorem is proved.
Example 5.2 The argument considered in the above proof may be illustrated by reference to the $\emptyset$-connected component of the crystal graph $\Gamma_3$ of $\mathfrak{sl}_3$ given (up to weight 8) in Fig. 1. Here, those partitions $\lambda$ for which $\lambda \in JS(n)$ have been highlighted with an asterisk. As in the above proof, these partitions correspond to the nodes $b$ in this crystal graph for which, for some $j$, $\varepsilon_j(b) \leq 1$ and $\varepsilon_i(b) = 0$ for all $i \neq j$.

Now define the set $JS(n, \mu, d)$ to be the subset of $JS(n)$ comprising those partitions with $n$-core $\mu$ and $n$-weight $d$ (see [13, 7] for definitions of $n$-core and $n$-weight). Then define the generating series

$$\chi_{n, \mu}(z) = \sum_{d \geq 0} \#JS(n, \mu, d)z^d.$$ 

Theorem 5.3 (i) If $\lambda \in JS(n)$ then the $n$-core $\mu$ of $\lambda$ is a rectangular partition $\mu = (k^l)$ such that $k + l \leq n$ (it is assumed here that if either $k = 0$ or $l = 0$ then $(k^l)$ means the empty partition).

(ii) If $k \neq 0 \neq l$ then

$$\chi_{n, (k^l)}(z) = z^{-s}b_{\Lambda_{k-l} \Lambda_0}(z),$$

where $s = \min(k, l)$.

(iii) $\chi_{n, \emptyset}(z) = \left\{ \sum_{k=0}^{n-1} b_{\Lambda_k + \Lambda_0}(z) \right\} - (n - 1)$.

Proof: In the tensor product $V(\Lambda) \otimes V(\Lambda_0)$, all highest weights are of the form $\Lambda_k + \Lambda_{-l} - e\delta$ with $k - l = i \mod n$ (for later convenience we use $-l$ and not $l$ here). We take $0 \leq k, l < n$ here and can also assume that $k \leq -l \mod n$ whereupon $k + l \leq n$, and $l = 0$ only if $k = 0$. By definition, the multiplicity of $V(\Lambda_k + \Lambda_{-l} - e\delta)$ is given by the coefficient of $z^e$ in $b_{\Lambda_k, \Lambda_0}(z)$ and hence, by Corollary 23, the number of partitions $\lambda \in JS(n)$ for which $\operatorname{wt}(\lambda) = \Lambda_k + \Lambda_{-l} - \Lambda_i - e\delta$. We claim that the $n$-core of such a $\lambda$ is the rectangular partition $\mu = (k^l)$, for which we calculate $\operatorname{wt}(\mu) = \Lambda_k + \Lambda_{-l} - \Lambda_k - s\delta = \Lambda_k + \Lambda_{-l} - \Lambda_i - s\delta$, where $s = \min(k, l)$ is the multiplicity of the colour charge 0 in $\mu$. This follows from the fact that for every string of weights $\Lambda, \Lambda - \delta, \Lambda - \epsilon, \ldots$ of the $\mathfrak{sl}_n$-module $V(\Lambda_0)$ (where $\Lambda + \delta$ is not a weight of $V(\Lambda_0)$), those partitions having these weights have the same $n$-core, and this $n$-core has weight $\Lambda$. Then since $\mu = (k^l)$ with $k + l \leq n$ is manifestly an $n$-core, it follows that it is the $n$-core of $\lambda$, hence proving part (i).

Part (ii) follows immediately since in the case $k \neq 0$ (so that $l \neq 0$), the partitions enumerated by the branching function $b_{\Lambda_k + \Lambda_{-l} \Lambda_0}$ are precisely those elements $\lambda \in JS(n)$ having weight $\operatorname{wt}(\lambda) = \Lambda_k + \Lambda_{-l} - \Lambda_k \epsilon - e\delta$, for some $e$, and hence $n$-core $(k^l)$.

Finally, for $k = 0$ and arbitrary $l$, each partition counted by $b_{\Lambda_{-l} \Lambda_0}$ has empty $n$-core, and hence contributes to $\chi_{n, \emptyset}$. However, the empty partition occurs for each $l$, hence an adjustment of $n - 1$ is needed after summing over all $l$. No other partition $\lambda$ is repeated since, as indicated by Theorem 22, the $b_{\Lambda_{-l} \Lambda_0}$ to which it contributes is uniquely determined by $-l \mod n = (\lambda_1 - a_1) \mod n$. (The summation over $-l$ is replaced by one over $k$ to give the final result).
Figure 1.
Example 5.4 To illustrate this result, consider again the case \( n = 3 \), where we have the following branching functions (to three terms):

\[
\begin{align*}
  b^{2A_0}_{2A_0} &= 1 + q^2 + \cdots; \\
  b^{A_1 + A_2}_{2A_0} &= q + 2q^2 + 2q^3 + \cdots; \\
  b^{A_1}_{A_1, A_0} &= 1 + q + 2q^2 + \cdots; \\
  b^{2A_2}_{A_1, A_0} &= q + q^2 + 2q^3 + \cdots; \\
  b^{A_2 + A_0}_{A_2, A_0} &= 1 + q + 2q^2 + \cdots; \\
  b^{A_2}_{A_2, A_0} &= q + q^2 + 2q^3 + \cdots. 
\end{align*}
\]

These are calculated using Corollary 2.3 which leads to the enumeration of the nodes of Fig. 1. labelled by asterisks. The only rectangular 3-cores are \( \emptyset, (1), (2) \) and \((1^2)\). Using Theorem 5.3, we thus obtain:

\[
\begin{align*}
  \chi_{n, \emptyset} &= 1 + 2q + 5q^2 + \cdots; \\
  \chi_{n, (1)} &= 1 + 2q + 2q^2 + \cdots; \\
  \chi_{n, (2)} &= 1 + q + 2q^2 + \cdots; \\
  \chi_{n, (1^2)} &= 1 + q + 2q^2 + \cdots. 
\end{align*}
\]

These correspond to the following sets:

\[
\begin{align*}
  JS(3, 0, 0) &= \{\emptyset\}; \\
  JS(3, 0, 1) &= \{(3), (21)\}; \\
  JS(3, 0, 2) &= \{(6), (51), (3^2), (41^2), (321)\}; \\
  JS(3, (1), 0) &= \{(1)\}; \\
  JS(3, (1), 1) &= \{(4), (2^2)\}; \\
  JS(3, (1), 2) &= \{(7), (43)\}; \\
  JS(3, (2), 0) &= \{(2)\}; \\
  JS(3, (2), 1) &= \{(5)\}; \\
  JS(3, (2), 2) &= \{(8), (321^2)\}; \\
  JS(3, (1^2), 0) &= \{(1^2)\}; \\
  JS(3, (1^2), 1) &= \{(32)\}; \\
  JS(3, (1^2), 2) &= \{(62), (44)\}. 
\end{align*}
\]

6 Discussion

We posed and solved the Jantzen-Seitz problem for Hecke algebras of type A. The solution is obtained by mapping the problem to a problem in exactly solvable lattice models, namely that of characterising the space of states of a certain class of restricted solid-on-solid model in terms of the states of the corresponding (unrestricted) solid-on-solid models. The latter was solved in \[9\]. The relationship between the two problems is based on the fact that both can be formulated in the same language of representation theory of \( q \)-affine algebra. The background to these two problems is discussed in greater detail in \[7\].

In a forthcoming paper \[8\], we plan to discuss the Jantzen-Seitz problem in the context of type-B Hecke algebras, and more generally, of Ariki-Koike (cyclo-tomic) Hecke algebras.

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Note added — A forthcoming preprint \[25\], contains (among other things) an elementary purely combinatorial proof of part (i) of \[4\].
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