DISCRETE HESSIAN COMPLEXES IN THREE DIMENSIONS

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ABSTRACT. One conforming and one non-conforming virtual element Hessian complexes on tetrahedral grids are constructed based on decompositions of polynomial tensor space. They are applied to discretize the linearized time-independent Einstein-Bianchi system.

1. Introduction

Let \( \Omega \) be a bounded Lipschitz domain. The Hessian complex, also known as grad-grad complex, in three dimensions reads as \([8, 38]\)

\[
\begin{align*}
P_1(\Omega) & \subset \mathbf{hess} \subset \mathbf{curl} \subset \mathbf{div} \subset L^2(\Omega; \mathbb{R}^3) \to 0,
\end{align*}
\]

where \( P_1(\Omega) \) is linear polynomial space, \( H^2(\Omega) \) and \( L^2(\Omega; \mathbb{R}^3) \) are standard Sobolev spaces, \( H(\mathbf{curl}, \Omega; \mathbb{S}) \) is the space of symmetric matrices whose \( \mathbf{curl} \) is in \( L^2(\Omega) \), and \( H(\mathbf{div}, \Omega; \mathbb{T}) \) is the space of trace-free matrices whose \( \mathbf{div} \) is in \( L^2(\Omega) \). In this paper, given a mesh of domain \( \Omega \), we shall construct discrete Hessian complexes with conforming or non-conforming virtual element spaces and apply to solve the linearized Einstein-Bianchi (EB) system \([39]\).

Finding finite elements with second derivatives, symmetry, or trace-free leads to higher number of degrees of freedom. To avoid this issue, Arnold and Quenneville-Belair \([39]\) use multipliers to impose the weak \( H^2 \) and weakly symmetry and obtain an optimal order discretization of EB system. In \([30]\) Hu and Liang construct the first finite element Hessian complexes in three dimensions. The lowest order complex starts with the \( P_9 \mathcal{C}^1 \)-element by Zhang \([42]\) and consists of \( P_7 \) for \( H(\mathbf{curl}, \Omega; \mathbb{S}) \) and \( P_6 \) for \( H(\mathbf{div}, \Omega; \mathbb{T}) \). These spaces are used in the mixed form of the linearized Einstein-Bianchi system. Although the practical significance may be limited due to the high polynomial degree of the elements, the work \([30]\) is the first construction of conforming discrete Hessian complexes consisting of finite element spaces in \( \mathbb{R}^3 \), and it motivates us to the development of simpler methods.

We shall use ideas of virtual element methods (VEMs) to construct discrete Hessian complexes with fewer degrees of freedom. The virtual element was a generalization of the finite element on a general polytope in \([12, 13]\) and can be also thought of as variational framework for the mimetic finite difference methods \([17, 33]\). Compared with the standard finite element methods mainly working on tensorial/simplicial meshes, VEMs have a variety of distinct advantages.

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The VEMs are, foremost, highly adaptable to the polygonal/polyhedral, or even anisotropic quadrilateral/hexahedral meshes. For problems with complex geometries, this leads to great convenience in mesh generation, e.g., discrete fracture network simulations [16], and the elliptic interface problems in three dimensions [24]. Another trait of VEMs is its astoundingly painless construction of smooth elements for high-order elliptic problems. For instance, the $H^2$-conforming VEMs has been constructed in [18, 4, 10] which shows a simple and elegant construction readily to be implemented. A uniform construction of the $H^m$-nonconforming virtual elements of any order $k$ and $m$ on any shape of polytope in $\mathbb{R}^n$ with constraint $k \geq m$ has been developed in [23, 31]. One more merit is that the virtual element space can devised to be structure preserving, such as the harmonic VEM [25, 34] and the divergence-free Stokes VEMs [11, 41]. VEMs for de Rham complex [14] and Stokes complexes [15] has been also constructed recently.

In the construction VEM space, the most subtle and key component is the well-posedness of a local problem with non-zero Dirichlet boundary condition. Take $H^2$-conforming VEM space as an example. Given data $(f, g_1, g_2)$, we consider the biharmonic equation with Dirichlet boundary condition on a polyhedral domain $K$

$$\Delta^2 v = f \text{ in } K, \quad v = g_1, \partial_n v = g_2 \text{ on } \partial K.$$  

When $g_1 = g_2 = 0$, the existence and uniqueness is a consequence of the Lax-Milligram lemma on $H^2_0(K)$. The classical way to deal with non-zero Dirichlet boundary condition $(g_1, g_2)$ is to find a lifting $v^1 \in H^2(K)$ with $v^1 = g_1, \partial_n v^1 = g_2$ and change (5.11) to homogenous boundary condition with modified source $f - \Delta^2 v^1$. Such lifting is guaranteed by trace theorems of Sobolev spaces which is usually established for smooth domains. For polyhedral domains, however, compatible conditions [29] are needed. For $H^2$-functions, $(g_2|_F \mathbf{n}_F + \nabla F(g_1|_F))|_e$ should be single-valued across each edge $e$ of the polyhedron $K$.

For vector function spaces, characterization of the trace spaces is even harder as tangential and normal components of the trace should be treated differently. We refer to [15, Appendix A] for the discussion of the well-posedness of the biharmonic problem of vector functions with a non-homogeneous boundary conditions, and refer to [19] and references therein for the trace of $H(\text{curl}, \Omega)$, where variants of $H^{1/2}(\partial K)$ are introduced. Specifically lifting for traces of $H(\text{curl}, \Omega)$ is explicitly constructed in [40] which is highly non-trivial.

We are not able to characterize the trace space of $H(\text{curl}, \Omega; \mathbb{S})$ and therefore construct its VEM space for general polyhedral meshes using local problems. Instead we still consider tetrahedron element $K$ and combine finite element and virtual element spaces. We first establish polynomial complex and corresponding Koszul complex which leads to the decomposition

$$P_k(K; \mathbb{S}) = \nabla^2 P_{k+2}(K) \oplus \text{sym}(x \times P_{k-1}(K; \mathbb{T})) \quad k \geq 1.$$  

Based on this decomposition, we can construct a virtual element space

$$\Sigma(K) = \nabla^2 W(K) \oplus \text{sym}(x \times V(K)),$$

where $W(K)$ is an $H^2$-conforming VEM space and $V(K) = P_2(K; \mathbb{T})$ is a $H(\text{div})$-conforming finite element space. Degrees of freedom for $\Sigma(K)$ is carefully chosen so that the resulting global space $\Sigma_h$ is $H(\text{curl})$-conforming and its $L^2$-projection to $P_3(\mathbb{S})$ is computable. Although there are non-polynomial shape functions, the trace
\( n \times \tau \) on each triangular face is always polynomial. Our construction is different with the approach in [30], where characterization of bubble functions is crucial.

Our \( H^2 \)-conforming virtual element \( W(K) \) is slightly different of those constructed in [18, 4, 10]. Again we take the advantage of \( K \) being a tetrahedron to construct an element so that when restricted to each face, \( v|_F \in \mathbb{P}_5(F) \) is an Argyris element [5] and \( (\partial_n v)|_F \in \mathbb{P}_4(F) \) is a Hermite element [26].

The \( H(\text{div}, \Omega; T) \) finite element \( V_h(K) = \mathbb{P}_2(K; T) \) is a low order version of finite element spaces construct in [30] which requires \( \mathbb{P}_k \) for \( k \geq 3 \).

The four local spaces \((W(K), \Sigma(K), V(K), Q(K))\) will contain polynomial spaces \((\mathbb{P}_5, \mathbb{P}_3, \mathbb{P}_2, \mathbb{P}_1)\) with 12 non-polynomial shape functions added in \( W(K) \) and \( \Sigma(K) \). The dimensions are \((68, 132, 80, 12)\) which are more tractable for implementation.

We show the constructed discrete spaces forms a discrete Hessian complex

\[
\begin{align*}
P_1(\Omega) \xrightarrow{\nabla^2} W_h \xrightarrow{\text{curl}} \Sigma_h \xrightarrow{\text{div}} V_h \xrightarrow{\text{div}} Q_h \rightarrow 0,
\end{align*}
\]

Optimal order discretization of the linearized Einstein-Bianchi (EB) system is obtained consequently.

We further present a lower order nonconforming Hessian complex with fewer degrees of freedom \((28, 92, 80, 12)\). Only the space \( \Sigma_h \) is non-conforming, while the rest discrete spaces are still conforming. By adding jumps of the tangential trace, we obtain a first order non-conforming discretization of EB system.

During the construction, integration by parts is indispensable and therefore the dual complex: div-div complex as well as its polynomial versions are also presented. Finite elements for div-div complex are recently constructed in [21, 22].

The rest of this paper is organized as follows. Some matrix and vector operations are present in Section 2. In Section 3 Hessian complex and divdiv complex are presented. Several polynomial complexes are explored in Section 4. A conforming and non-conforming virtual element Hessian complex are constructed in Section 5 and Section 6 respectively. In Section 7, the conforming and non-conforming virtual element Hessian complexes are adopted to discretize the linearized EB system. Conclusion and future work is outlined in the last section.

2. Matrix and Vector Operations

In this section, we shall survey operations for vectors and matrices. Some of them are standard but some are not well documented in the literature. A shorter version tailored to div div complex is presented in [22].

2.1. Vector operations. For three vectors \( a, b, c \in \mathbb{R}^3 \), we denote the mixed product

\[
(a, b, c) := (a \times b) \cdot c,
\]

which is cyclic invariant in the sense that

\[
(a, b, c) = (b, c, a) = (c, a, b).
\]

For the cross product, we denote by

\[
(a, b, c)_\times := (a \times b) \times c = (a \cdot c) b - (b \cdot c) a,
\]

which satisfies the Jacobi identity if we identify the cross product as the Poisson bracket

\[
(a, b, c)_\times + (b, c, a)_\times + (c, a, b)_\times = 0.
\]
In these operations, row or column vectors are mixed. For the cross product, the result is a row or column vector depending on the second variable. For example, if \( a \) is a row vector and \( b \) is a column vector. Then \( b \times a := b^\top \times a \) is a row vector.

Given a plane \( F \) with normal vector \( n \), for a vector \( v \in \mathbb{R}^3 \), we introduce two vectors on the plane \( F \):

\[
\Pi_F v := (n \times v) \times n, \quad \Pi_F^\perp v := n \times v.
\]

The vector \( \Pi_F^\perp v \) is a rotation of \( \Pi_F v \) by \( 90^\circ \) counter-clockwise with respect to \( n \). We have the orthogonal decomposition

\[
v = (v \cdot n)n + \Pi_F v.
\]

For this reason, we call \( \Pi_F v \) the tangential component of \( v \). We call \( \Pi_F^\perp v \) the tangential trace of \( v \) as it will appear in the integration by parts, e.g. (5.25).

We treat Hamilton operator \( \nabla = (\partial_1, \partial_2, \partial_3)^\top \) as a column vector. When applied to functions, \( \nabla \) is distributive but not commutative nor associative, i.e.,

\[
\nabla(f + g) = \nabla f + \nabla g, \quad \nabla f \neq f \nabla, \quad (\nabla f)h \neq \nabla(fh).
\]

Define

\[
\nabla_F := n \times \nabla, \quad \nabla_F := \Pi_F \nabla = (n \times \nabla) \times n.
\]

\( \nabla_F \) is the surface Hamilton operator and \( \nabla_F^\perp \) is its rotation. In the special case \( n = (0,0,1)^\top \),

\[
\nabla_F = (\partial_1, \partial_2, 0)^\top, \quad \nabla_F^\perp = (-\partial_2, \partial_1, 0)^\top,
\]

which can be identified as two dimensional gradient and curl on the plane. For a scalar function \( v \), which is differentiable in a neighborhood of \( F \) in space,

\[
\nabla_F v = \Pi_F(\nabla v), \quad \nabla_F^\perp v = n \times \nabla v,
\]

are the surface gradient of \( v \) and surface curl, respectively. For a vector function \( v \), the surface divergence operator is

\[
\text{div}_F v := \nabla_F \cdot v = \nabla_F \cdot (\Pi_F v) = (n \times \nabla) \cdot (n \times v) = \nabla_F^\perp \cdot (n \times v).
\]

By the cyclic invariance of the mix product, the surface rot operator is

\[
\text{rot}_F v := \nabla_F^\perp \cdot v = (n \times \nabla) \cdot v = n \cdot (\nabla \times v).
\]

Although we define the surface differentiation through the projection of differentiation of a function defined in space, it can be verified that the definition is intrinsic in the sense that it depends only on the function value on the surface \( F \).

2.2. Matrix-vector operations. The matrix-vector product \( Ab \) can be interpret as the inner product of \( b \) with the row vectors of \( A \). We thus define the dot operator

\[
b \cdot A := Ab.
\]

Namely the vector inner product is applied row-wise to the matrix. Similarly we can define the cross product row-wise \( b \times A \). Here rigorously speaking when a column vector \( b \) is treat as a row vector, notation \( b^\top \) should be used. In most places, however, we will scarify this precision for the ease of notation.

When the vector is on the right of the matrix, the operation is defined column-wise. That is

\[
A \cdot b := b^\top A = (A^\top b)^\top = (b \cdot A^\top)^\top, \quad A \times b := -(b \times A^\top)^\top.
\]
By moving the column operation to the right, the product operation is consistent with the transpose operator. For the transpose of product of two objects, we take transpose of each one, switch their order, and add a negative sign if it is the cross product. Note that we do not have the symmetry or skew-symmetry of the product involving matrices. Namely

\[ \mathbf{A} \cdot \mathbf{b} \neq \mathbf{b} \cdot \mathbf{A}, \quad \mathbf{A} \times \mathbf{b} \neq -\mathbf{b} \times \mathbf{A}. \]

For two vectors \( \mathbf{b}, \mathbf{c} \) and matrix \( \mathbf{A} \), we define the following products

\[ \mathbf{b} \cdot \mathbf{A} \cdot \mathbf{c} := \mathbf{b}^\top \mathbf{A}^\top \mathbf{c} = \begin{pmatrix} \mathbf{b} \cdot \mathbf{A} \end{pmatrix} \cdot \mathbf{c} = \mathbf{b} \cdot \begin{pmatrix} \mathbf{A} \cdot \mathbf{c} \end{pmatrix}, \]
\[ \mathbf{b} \times \mathbf{A} \times \mathbf{c} := \begin{pmatrix} \mathbf{b} \times \mathbf{A} \end{pmatrix} \times \mathbf{c} = \mathbf{b} \times \begin{pmatrix} \mathbf{A} \times \mathbf{c} \end{pmatrix}. \]

Also thanks to the column operation, these triple products are associative. That is the ordering of performing the products does not matter.

The projection \( \Pi_F \) can be also defined for a matrix either row-wise or column-wise. Note that

\[ \Pi_F \mathbf{A} = -\mathbf{n} \times (\mathbf{n} \times \mathbf{A}) \neq \mathbf{n} \times \mathbf{A} \times \mathbf{n}. \]

Apply these matrix-vector operations with the Hamilton operator \( \nabla \), we get row-wise differentiation

\[ \nabla \cdot \mathbf{A}, \quad \nabla \times \mathbf{A}, \]

and column-wise differentiation

\[ \mathbf{A} \cdot \nabla, \quad \mathbf{A} \times \nabla. \]

We then define the following second order differentiation applied to matrices

\[ \text{div div} \mathbf{A} := \nabla \cdot \mathbf{A} \cdot \nabla := \nabla \cdot (\nabla \cdot \mathbf{A}^\top), \]
\[ \text{inc} \mathbf{A} := \nabla \times \mathbf{A} \times \nabla := -(\nabla \times (\nabla \times \mathbf{A}^\top))^\top, \]
\[ \nabla \cdot \mathbf{A} \times \nabla := -\nabla \times (\nabla \cdot \mathbf{A}) = -\nabla \cdot (\nabla \times \mathbf{A}^\top)^\top. \]

2.3. Coupling of differentiation and algebraic operations. Denote the space of all \( 3 \times 3 \) matrices by \( \mathbb{M} \), all symmetric \( 3 \times 3 \) matrices by \( \mathbb{S} \), all skew-symmetric \( 3 \times 3 \) matrices by \( \mathbb{K} \), and all trace-free \( 3 \times 3 \) matrices by \( \mathbb{T} \). For any matrix \( \mathbf{B} \in \mathbb{M} \), we can decompose it into symmetric and skew-symmetric part as

\[ \mathbf{B} = \text{sym}(\mathbf{B}) + \text{skw}(\mathbf{B}) := \frac{1}{2}(\mathbf{B} + \mathbf{B}^\top) + \frac{1}{2}(\mathbf{B} - \mathbf{B}^\top). \]

We can also decompose it into a direct sum of a trace-free matrix and a diagonal matrix as

\[ \mathbf{B} = \text{dev} \mathbf{B} + \frac{1}{3} \text{tr}(\mathbf{B}) \mathbf{I} := (\mathbf{B} - \frac{1}{3} \text{tr}(\mathbf{B}) \mathbf{I}) + \frac{1}{3} \text{tr}(\mathbf{B}) \mathbf{I}. \]

As the skew-symmetric matrix is always trace-free, we can combine the two decompositions as:

\[ \mathbf{B} = \text{devsym} \mathbf{B} + \text{skw} \mathbf{B} + \frac{1}{3} \text{tr}(\mathbf{B}) \mathbf{I}. \]

For a vector function \( \mathbf{u} = (u_1, u_2, u_3)^\top \), \( \text{curl} \mathbf{u} = \nabla \times \mathbf{u} \) and \( \text{div} \mathbf{u} = \nabla \cdot \mathbf{u} \) are standard differential operations. The gradient \( \nabla \mathbf{u} := \mathbf{u} \otimes \nabla = \mathbf{u} \nabla^\top \) is a matrix
\begin{align*}
\begin{pmatrix}
\nabla u_1 \\
\nabla u_2 \\
\nabla u_3
\end{pmatrix}
\end{align*}
Its symmetric part is defined as
\begin{align*}
\text{def } u := \text{sym } \nabla u = \frac{1}{2}(\nabla u + (\nabla u)^\top) = \frac{1}{2}(u\nabla^\top + \nabla u^\top).
\end{align*}
In the last identity notation \( u \nabla^\top \) is used to emphasize the symmetry form. Similarly we can define \text{sym } \text{curl } A
\begin{align*}
\text{sym } \text{curl } A = \frac{1}{2}(\nabla \times A + (\nabla \times A)^\top) = \frac{1}{2}(\nabla \times A - A^\top \times \nabla).
\end{align*}
We define an isomorphism of \( \mathbb{R}^3 \) and the space of skew-symmetric matrices as follows: for a vector \( \omega = (\omega_1, \omega_2, \omega_3)^\top \in \mathbb{R}^3 \),
\begin{align*}
\text{mspn } \omega := [\omega]_x := \begin{pmatrix}
0 & -\omega_3 & \omega_2 \\
\omega_3 & 0 & -\omega_1 \\
-\omega_2 & \omega_1 & 0
\end{pmatrix} \in \mathbb{K}.
\end{align*}
Obviously \( \text{mspn} \) is a bijection. Its inverse is denoted by \( \text{vspan} \) satisfying
\begin{align*}
\text{mspan } \text{vspan}(Z) = Z \quad Z \in \mathbb{K},
\end{align*}
and
\begin{align*}
\text{vspan } \text{mspn}(\omega) = \omega \quad \omega \in \mathbb{R}^3.
\end{align*}
For two vectors \( u, v \), one can easily verify that
\begin{align*}
u \times v = [u]_x v = (\text{mspn } u)v.
\end{align*}
Using the cross product and \([\cdot]_x\), we have the following identity
\begin{align}
\text{skw}(\nabla u) = \frac{1}{2}[\nabla \times u]_x, \quad \text{tr}(\nabla u) = \nabla \cdot u.
\end{align}
Consequently we can write the decomposition for the matrix \( \nabla u \) as
\begin{align}
\nabla u = \text{def } u + \frac{1}{2}[\nabla \times u]_x = \text{dev def } u + \frac{1}{2}\text{mspn}(\nabla \times u) + \frac{1}{3}(\nabla \cdot u)I.
\end{align}
We list the following identities which can be verified by direct calculation.
\begin{enumerate}
\item \begin{align}
\text{skw}(\nabla u) = \frac{1}{2}(\text{mspn } \nabla \times u) \quad \forall u \in H^1(\mathbb{R}^3),
\end{align}
\item \begin{align}
\text{skw}(\nabla \times A) = \frac{1}{2}\text{mspn }[(A \cdot \nabla)^\top - \nabla (\text{tr}(A))] \quad \forall A \in H^1(\mathbb{M}),
\end{align}
\item \begin{align}
\text{div } \text{mspn } u = -\nabla \times u \quad \forall u \in H^1(\mathbb{R}^3),
\end{align}
\item \begin{align}
2\text{div } \text{vspan } \text{skw } A = \text{tr } \text{curl } A \quad \forall A \in H^1(\mathbb{M}),
\end{align}
\item \begin{align}
\nabla \times (uI) = -\text{mspn}(\nabla u) \quad \forall u \in H^1(\mathbb{R}).
\end{align}
\end{enumerate}
More identities involving the matrix operation and differentiation are summarized in [8].
3. Two Hilbert Complexes for Tensors

In this section we shall present two Hilbert complexes for tensors: Hessian complex and divdiv complex. They are dual to each other. When constructing VEM or FEM spaces, Hessian complex will be used for the construction of shape function spaces and divdiv complex for the degrees of freedom. For the completeness we shall prove the exactness of these two complexes following [38] and refer to [8] for a systematical way of deriving complexes from complexes.

Recall that a Hilbert complex is a sequence of Hilbert spaces $V_i$ connected by a sequence of closed densely defined linear operator $d_i$

$$0 \rightarrow V_1 \xrightarrow{d_1} V_2 \xrightarrow{d_2} \ldots V_n \xrightarrow{d_n} V_{n+1} \rightarrow 0,$$

satisfying the property $\text{img}(d_i) \subset \ker(d_{i+1})$, i.e., $d_{i+1} \circ d_i = 0$. In this paper, we shall consider domain complexes only, i.e., dom$(d_i) = V_i$. The complex is called an exact sequence if $\text{img}(d_i) = \ker(d_{i+1})$ for $i = 1, \ldots, n$. We usually skip the first 0 in the complex and use the embedding operator to indicate $d_1$ is injective. We refer to [6] for background on Hilbert complexes.

3.1. Hessian complex. The Hessian complex in three dimensions reads as [8, 38]

$$\begin{align*}
P_1(\Omega) & \xrightarrow{\subset} H^2(\Omega) \xrightarrow{\text{hess}} H(\text{curl}, \Omega; \mathbb{S}) \xrightarrow{\text{curl}} H(\text{div}, \Omega; \mathbb{T}) \xrightarrow{\text{div}} L^2(\Omega; \mathbb{R}^3) \xrightarrow{\subset} 0.
\end{align*}$$

Lemma 3.1. Assume $\Omega$ is a bounded and strong Lipschitz domain in $\mathbb{R}^3$. It holds

$$\text{div} H^1(\Omega; \mathbb{T}) = L^2(\Omega; \mathbb{R}^3).$$

Proof. First consider $v = \nabla w \in L^2(\Omega; \mathbb{R}^3)$ with $w \in H^1(\Omega)$. There exists $\phi \in H^2(\Omega; \mathbb{R}^3)$ satisfying $2 \text{div} \phi = -3w$. Take $\tau = wI + \text{curl} \text{mspn} \phi \in H^1(\Omega; \mathbb{M})$. It is obvious that $\text{div} \tau = \text{div}(wI) = v$. It follows from (2.6) that

$$\text{tr} \tau = 3w + \text{tr} \text{curl} \text{mspn} \phi = 3w + 2 \text{div} \text{vspan} \text{skw} \text{mspn} \phi = 3w + 2 \text{div} \phi = 0.$$

Next consider general $v \in L^2(\Omega; \mathbb{R}^3)$. There exists $\tau_1 \in H^1(\Omega; \mathbb{M})$ satisfying $\text{div} \tau_1 = v$. Then there exists $\tau_2 \in H^1(\Omega; \mathbb{T})$ satisfying $\text{div} \tau_2 = \frac{1}{3} \nabla(\text{tr} \tau_1)$. Now take $\tau = \text{dev} \tau_1 + \tau_2 \in H^1(\Omega; \mathbb{T})$. We have

$$\text{div} \tau = \text{div} \text{dev} \tau_1 + \text{div} \tau_2 = \text{div} \text{dev} \tau_1 + \frac{1}{3} \nabla(\text{tr} \tau_1) = \text{div} \tau_1 = v.$$

Thus (3.2) follows. \hfill \Box

Lemma 3.2. Assume $\Omega$ is a bounded and topologically trivial strong Lipschitz domain in $\mathbb{R}^3$. It holds

$$\text{curl} H^1(\Omega; \mathbb{S}) = H(\text{div}, \Omega; \mathbb{T}) \cap \ker(\text{div}).$$

Proof. For any $\tau \in H(\text{div}, \Omega; \mathbb{T}) \cap \ker(\text{div})$, there exists $\sigma_1 \in H^1(\Omega; \mathbb{M})$ such that $\tau = \text{curl} \sigma_1$.

Thanks to (2.6), we have

$$2 \text{div} \text{vspan} \text{skw} \sigma_1 = \text{tr} \text{curl} \sigma_1 = \text{tr} \tau = 0.$$

Hence there exists $v \in H^2(\Omega; \mathbb{R}^3)$ such that $\text{vspan} \text{skw} \sigma_1 = \frac{1}{2} \text{curl} v$. Then apply mspan and use (2.3) to get

$$\text{skw} \sigma_1 = \frac{1}{2} \text{mspn} \text{curl} v = \text{skw}(\nabla v).$$
Taking $\sigma = \sigma_1 - \nabla v$, we have $\sigma \in H^1(\Omega; \mathbb{S})$ and $\mathbf{curl} \sigma = \tau$. \hfill \Box

**Theorem 3.3.** Assume $\Omega$ is a bounded and topologically trivial strong Lipschitz domain in $\mathbb{R}^3$. Then (3.1) is a complex and exact sequence.

**Proof.** It is obvious that (3.1) is a complex and $H^2(\Omega) \cap \ker(\mathbf{hess}) = \mathbb{P}_1(\Omega)$. As results of (3.2) and (3.3), we have

$\mathbf{div} \mathbf{H}(\mathbf{div}, \Omega; \mathbb{T}) = L^2(\Omega; \mathbb{R}^3)$, \hspace{1em} $\mathbf{curl} \mathbf{H}(\mathbf{curl}, \Omega; \mathbb{S}) = \mathbf{H}(\mathbf{div}, \Omega; \mathbb{T}) \cap \ker(\mathbf{div})$.

We only need to prove $\mathbf{H}(\mathbf{curl}, \Omega; \mathbb{S}) \cap \ker(\mathbf{curl}) = \mathbf{hess} H^2(\Omega)$. For any $\sigma \in \mathbf{H}(\mathbf{curl}, \Omega; \mathbb{S}) \cap \ker(\mathbf{curl})$, there exists $v \in H^1(\Omega; \mathbb{R}^3)$ such that

$$\sigma = \nabla v.$$ 

Since $\sigma$ is symmetric, by (2.3), we have

$$\text{mspn} \nabla \times v = 2 \text{skw}(\nabla v) = 2 \text{skw}(\sigma) = 0,$$

which means $\nabla \times v = 0$. Hence there exists $w \in H^2(\Omega)$ that $v = \nabla w$ and consequently $\sigma = hess w \in hess H^2(\Omega)$. \hfill \Box

As a result of the Hessian complex (3.1), we have the Poincaré inequality [8]

(3.4) \hspace{1em} $\|\tau\|_0 \lesssim \|\mathbf{curl} \tau\|_0$

for any $\tau \in \mathbf{H}(\mathbf{curl}, \Omega; \mathbb{S})$ satisfying

$$\langle \tau, \nabla^2 w \rangle = 0 \quad \forall \, w \in H^2(\Omega).$$

When $\Omega \subset \mathbb{R}^2$, the Hessian complex in two dimensions becomes

$$\begin{array}{c}
\mathbb{P}_1(\Omega) \hookrightarrow H^2(\Omega) \to \mathbf{H}(\mathbf{curl}, \Omega; \mathbb{S}) \to \mathbf{L}^2(\Omega; \mathbb{R}^2) \to 0,
\end{array}$$

which is the rotation of the elasticity complex [27, 9].

### 3.2. divdiv complex

The div div complex in three dimensions reads as [8, 38]

(3.5) \hspace{1em} $\begin{array}{c}
\mathbb{H}_1(\Omega; \mathbb{R}^3) \to H^1(\Omega; \mathbb{R}^3) \to H(\mathbf{sym} \mathbf{curl}, \Omega; \mathbb{T}) \to H(\mathbf{div} \mathbf{div}, \Omega; \mathbb{S}) \to L^2(\Omega) \to 0,
\end{array}$

For completeness, we prove the exactness of the complex (3.5).

**Theorem 3.4.** Assume $\Omega$ is a bounded and topologically trivial strong Lipschitz domain in $\mathbb{R}^3$. Then (3.5) is a complex and exact sequence.

**Proof.** Any skew-symmetric $\mathbf{\tau}$ can be written as $\mathbf{\tau} = \text{mspn} \mathbf{v}$ with $\mathbf{v} = \text{vspan} \mathbf{\tau}$. Assume $\mathbf{v} \in \mathcal{C}^2(\Omega; \mathbb{R}^3)$, it follows from (2.5) that

(3.6) \hspace{1em} $\mathbf{div} \mathbf{div} \mathbf{v} = \mathbf{div} \text{mspn} \mathbf{v} = -\mathbf{div} \mathbf{curl} \mathbf{v} = 0$.

Since $\mathbf{div} \mathbf{div} \mathbf{\tau} = 0$ for any smooth skew-symmetric tensor field $\mathbf{\tau}$, we obtain

$$\mathbf{div} \mathbf{div} \mathbf{H}(\mathbf{div} \mathbf{div}, \Omega; \mathbb{S}) = \mathbf{div} \mathbf{div} \mathbf{H}(\mathbf{div} \mathbf{div}, \Omega; \mathbb{M}) = L^2(\Omega).$$

For any $\mathbf{\tau} \in \mathcal{C}^3(\Omega; \mathbb{T})$, by applying operators on both sides,

$$\mathbf{div} \mathbf{div} \mathbf{sym} \mathbf{curl} \mathbf{\tau} = \frac{1}{2} \nabla \cdot (\nabla \times \mathbf{\tau} - \mathbf{\tau}^T \times \nabla) \cdot \nabla = 0.$$
Hence $\text{div div} \text{sym curl} H(\text{sym curl}, \Omega; T) = 0$. For any $v \in C^2(\Omega; \mathbb{R}^3)$, it holds from (2.7) that
\[
\text{sym curl dev grad } v = \text{sym curl} \left( \text{grad } v - \frac{1}{3}(\text{div } v) I \right) = -\frac{1}{3} \text{sym mspn}((\text{div } v) I) = \frac{1}{3} \text{sym mspn}(\text{grad}(\text{div } v)) = 0.
\]
We get $\text{sym curl dev grad } H^1(\Omega; \mathbb{R}^3) = 0$. As a result, (3.5) is a complex.

For any $\sigma \in H(\text{div div}, \Omega; S) \cap \ker(\text{div div})$, there exists $v \in L^2(\Omega; \mathbb{R}^3)$ such that
\[
\text{div } \sigma = \text{curl } v = -\text{div}(\text{mspn } v).
\]
Hence there exists $\tau \in H^1(\Omega; M)$ such that
\[
\sigma = -\text{mspn } v + \text{curl } \tau.
\]
By the symmetry of $\sigma$, we have $\sigma = \text{sym curl } \tau$. Noting that
\[
\text{sym curl } ((\text{tr } \tau) I) = -\text{sym mspn} \text{grad}(\text{tr } \tau) = 0,
\]
it follows $\sigma = \text{sym curl } \text{dev } \tau$. Thus
\[
H(\text{div div}, \Omega; S) \cap \ker(\text{div div}) = \text{sym curl } H(\text{sym curl}, \Omega; T).
\]

For any $\tau \in H(\text{sym curl}, \Omega; T) \cap \ker(\text{sym curl})$, by the fact that $\text{tr } \tau = 0$, we have from (2.4) that
\[
\text{curl } \tau = \text{skw curl } \tau = \frac{1}{2} \text{mspn}(\text{div } \tau^T - \text{grad}(\text{tr } \tau)) = \frac{1}{2} \text{mspn}(\text{div } \tau^T).
\]
Then
\[
\text{curl}(\text{div } \tau^T) = -\text{div}(\text{mspn } \text{div } \tau^T) = -2\text{div}(\text{curl } \tau) = 0.
\]
Thus there exists $w \in L^2_0(\Omega)$ satisfying $\text{div } \tau^T = 2 \text{grad } w$, which implies
\[
\text{curl } \tau = \text{mspn } \text{grad } w = -\text{curl}(w I).
\]
Hence there exists $v \in H^1(\Omega; \mathbb{R}^3)$ such that $\tau = -w I + \text{grad } v$. Since $\tau$ is trace-free, we achieve
\[
\tau = \text{dev } \tau = \text{dev } \text{grad } v,
\]
which means $H(\text{sym curl}, \Omega; T) \cap \ker(\text{sym curl}) = \text{dev } \text{grad } H^1(\Omega; \mathbb{R}^3)$. Therefore the complex (3.5) is exact. $\square$

When $\Omega \subset \mathbb{R}^2$, the $\text{div div}$ complex in two dimensions becomes (cf. [20])
\[
\begin{array}{ccccccccccc}
\text{RT} & \longrightarrow & H^1(\Omega; \mathbb{R}^2) & \longrightarrow & H(\text{div div}, \Omega; S) & \longrightarrow & L^2(\Omega) & \longrightarrow & 0.
\end{array}
\]

4. Polynomial Complexes for Tensors

In this section we consider Hessian and $\text{div div}$ polynomial complex on a bounded and topologically trivial domain $D \subset \mathbb{R}^3$ in this section. Without loss of generality, we assume $(0, 0, 0) \in D$.

Given a non-negative integer $k$, let $\mathbb{P}_k(D)$ stand for the set of all polynomials in $D$ with the total degree no more than $k$, and $\mathbb{P}_k(D; X)$ denote the tensor or vector version. Let $\mathbb{H}_k(D) := \mathbb{P}_k(D)/\mathbb{P}_{k-1}(D)$ be the space of homogeneous polynomials of degree $k$. Denote by $Q^D_k$ the $L^2$-orthogonal projector onto $\mathbb{P}_k(D)$, and $Q^D_k$ the tensor or vector version.
4.1. De Rham and Koszul polynomial complexes. First we recall the polynomial de Rham complex

\[(4.1) \quad \mathbb{R} \leftrightarrow P_{k+1}(D) \xrightarrow{\nabla} P_k(D; \mathbb{R}^3) \xrightarrow{\nabla \times} P_{k-1}(D; \mathbb{R}^3) \xrightarrow{\nabla \cdot} P_{k-2}(D) \rightarrow 0,\]

and the Koszul complex going backwards

\[(4.2) \quad P_{k+1}(D) \xleftarrow{\nabla} P_k(D; \mathbb{R}^3) \xleftarrow{\nabla \times} P_{k-1}(D; \mathbb{R}^3) \xleftarrow{\nabla} P_{k-2}(D) \leftrightarrow 0.\]

Those two complexes can be combined into one

\[(4.3) \quad \mathbb{R} \xrightarrow{C} P_{k+1}(D) \xrightarrow{\nabla} P_k(D; \mathbb{R}^3) \xrightarrow{\nabla \times} P_{k-1}(D; \mathbb{R}^3) \xrightarrow{\nabla \cdot} P_{k-2}(D) \xrightarrow{\nabla} 0.\]

We refer to [7] for a systematical derivation of (4.1)-(4.2) and focus on two decompositions of vector polynomial spaces based on (4.3).

The first one is, for an integer \(k \geq 1\),

\[(4.4) \quad P_k(D; \mathbb{R}^3) = \nabla P_{k+1}(D) \oplus \mathbf{x} \times P_{k-1}(D; \mathbb{R}^3),\]

which leads to

\[P_k(D; \mathbb{R}^3) = \nabla H_{k+1}(D) \oplus \mathcal{ND}_{k-1},\]

where

\[\mathcal{ND}_{k-1} := P_{k-1}(D; \mathbb{R}^3) \oplus \mathbf{x} \times H_{k-1}(D; \mathbb{R}^3) = P_{k-1}(D; \mathbb{R}^3) + \mathbf{x} \times P_{k-1}(D; \mathbb{R}^3)\]

is the first family of Nédélec element [36]. Note that the component \(\mathbf{x} \times H_{k-1}(D; \mathbb{R}^3)\) can be also written as \(\ker(\mathbf{x} \cdot) \cap H_k(D; \mathbb{R}^3)\) by the exactness of the Koszul complex (4.2), which unifies the notation in both two and three dimensions.

The second decomposition is, for an integer \(k \geq 1\),

\[(4.5) \quad P_k(D; \mathbb{R}^3) = \nabla \times P_{k+1}(D; \mathbb{R}^3) \oplus \mathbf{x} P_{k-1}(D),\]

which leads to

\[P_k(D; \mathbb{R}^3) = \nabla \times H_{k+1}(D; \mathbb{R}^3) \oplus \mathcal{RT}_{k-1},\]

where

\[\mathcal{RT}_{k-1} := P_{k-1}(D; \mathbb{R}^3) \oplus \mathbf{x} H_{k-1}(D) = P_{k-1}(D; \mathbb{R}^3) + \mathbf{x} P_{k-1}(D)\]

is Raviart-Thomas face element in three dimensions [37].

4.2. Hessian polynomial complexes. By the Euler’s formula, for an integer \(k \geq 0\),

\[(4.6) \quad \mathbf{x} \cdot \nabla q = k q \quad \forall \ q \in H_k(D).\]

Due to (4.6), for any \(q \in P_k(D)\) satisfying \(\mathbf{x} \cdot \nabla q + q = 0\), we have \(q = 0\). And

\[(4.7) \quad P_k(D) \cap \ker(\mathbf{x} \cdot \nabla) = P_0(D),\]

\[(4.8) \quad P_k(D) \cap \ker(\mathbf{x} \cdot \nabla + \ell) = 0\]

for any positive integer \(\ell\).

Lemma 4.1. The operator \(\text{div} : \text{div}(P_k(D; \mathbb{R}^3) \mathbf{x}) \rightarrow P_k(D; \mathbb{R}^3)\) is bijective.
Proof. Since \( \text{div} \text{dev}(P_k(D; \mathbb{R}^3)x^\top) \subseteq \mathbb{P}_k(D; \mathbb{R}^3) \) and 
\[
\dim \text{dev}(P_k(D; \mathbb{R}^3)x^\top) = \dim \mathbb{P}_k(D; \mathbb{R}^3),
\]
it suffices to show that \( \text{div} : \text{dev}(P_k(D; \mathbb{R}^3)x^\top) \rightarrow \mathbb{P}_k(D; \mathbb{R}^3) \) is injective.

For any \( q \in \mathbb{P}_k(D; \mathbb{R}^3) \) satisfying \( \text{div} \text{dev}(qx^\top) = 0 \), we have 
\[
\text{div}(qx^\top) - \frac{1}{3} \nabla(x^\top q) = \text{div}(\text{dev}(qx^\top)) = 0.
\]
Multiplying (4.9) by \( x^\top \) from the left, we obtain 
\[
\frac{2}{3}(x \cdot \nabla)(x^\top q) = x^\top \left( \text{div}(qx^\top) - \frac{1}{3} \nabla(x^\top q) \right) = 0.
\]
By (4.7), we have \( x^\top q \in \mathbb{P}_0(D) \). In turn, it follows from (4.9) that \( (x \cdot \nabla + 3)q = \text{div}(qx^\top) = 0 \), which together with (4.8) gives \( q = 0 \). \( \square \)

**Lemma 4.2.** For \( k \in \mathbb{N}, k \geq 2 \), the polynomial complex
\[
(4.10) \quad \mathbb{P}_1(D) \xrightarrow{\text{hess}} \mathbb{P}_{k+1}(D) \xrightarrow{\text{curl}} \mathbb{P}_{k-1}(D; \mathbb{T}) \xrightarrow{\text{div}} \mathbb{P}_{k-2}(D; \mathbb{R}^3) \rightarrow 0
\]
is exact.

**Proof.** It is obvious \( \nabla^2(\mathbb{P}_{k+2}(D)) \subseteq \mathbb{P}_k(D; \mathbb{S}) \cap \ker(\text{curl}) \). By identity (2.6),
\[
\text{tr}(\text{curl}\tau) = 2 \text{div}(\text{vspan}(\text{skew}\ \tau)) \quad \forall \ \tau \in H^1(D; \mathbb{M}),
\]
we have \( \text{curl}(\mathbb{P}_k(D; \mathbb{S})) \subseteq \mathbb{P}_{k-1}(D; \mathbb{T}) \cap \ker(\text{div}) \). Therefore (4.10) is a complex.

We then verify this complex is exact. By the polynomial version of de Rham complex (4.1), we have \( \text{hess}\mathbb{P}_{k+2}(D) = \mathbb{P}_k(D; \mathbb{S}) \cap \ker(\text{curl}) \), and 
\[
\dim \text{curl}\mathbb{P}_k(D; \mathbb{S}) = \dim \mathbb{P}_k(D; \mathbb{S}) - \dim \text{hess}\mathbb{P}_{k+2}(D) = \frac{1}{6}k(k+1)(5k+19).
\]
Thanks to Lemma 4.1, we get \( \text{div}\mathbb{P}_{k-1}(D; \mathbb{T}) = \mathbb{P}_{k-2}(D; \mathbb{R}^3) \). And then 
\[
\dim(\mathbb{P}_{k-1}(D; \mathbb{T}) \cap \ker(\text{div})) = \dim \mathbb{P}_{k-1}(D; \mathbb{T}) - \dim \mathbb{P}_{k-2}(D; \mathbb{R}^3) = \dim \text{curl}\mathbb{P}_k(D; \mathbb{S}),
\]
which means \( \mathbb{P}_{k-1}(D; \mathbb{T}) \cap \ker(\text{div}) = \text{curl}\mathbb{P}_k(D; \mathbb{S}) \). Therefore the complex (4.10) is exact. \( \square \)

Define operator \( \pi_1 : C^1(D) \rightarrow \mathbb{P}_1(D) \) as 
\[
\pi_1 v := v(0, 0, 0) + x^\top(\nabla v)(0, 0, 0).
\]
It is exactly the first order Taylor polynomial of \( v \) at \((0, 0, 0)\). Obviously 
\[
(4.11) \quad \pi_1 v = v \quad \forall \ v \in \mathbb{P}_1(D).
\]

We present the following Koszul-type complex associated to the Hessian complex.

**Lemma 4.3.** For \( k \in \mathbb{N}, k \geq 2 \), the polynomial complex
\[
(4.12) \quad 0 \xrightarrow{} \mathbb{P}_{k-2}(D; \mathbb{R}^3) \xrightarrow{\text{dev}(x^\top x)} \mathbb{P}_{k-1}(D; \mathbb{T}) \xrightarrow{\text{sym}(x \times \tau)} \mathbb{P}_k(D; \mathbb{S}) \xrightarrow{x^\top x} \mathbb{P}_{k+2}(D) \xrightarrow{\pi_1} \mathbb{P}_1(D)
\]
is exact.
Proof. For any \( v \in \mathbb{P}_{k-2}(D; \mathbb{R}^3) \), it follows
\[
\text{sym}(x \times (\text{dev}(v x^T))) = \text{sym}(x \times (v x^T)) - \frac{1}{3}(x^Tv)\text{sym}(x \times I) = 0.
\]

For any \( \tau \in \mathbb{P}_{k-1}(D; \mathbb{T}) \), we have
\[
x^T(\text{sym}(x \times \tau))x = x^T(x \times \tau)x = 0.
\]
And it’s trivial that \( \pi_1(x^T \tau x) = 0 \) for any \( \tau \in \mathbb{P}_k(D; \mathbb{S}) \). Thus (4.12) is a complex.

Next we prove that the complex (4.12) is exact. By the Taylor’s theorem, we get \( \mathbb{P}_{k+2}(D) \cap \ker(\pi_1) = x^T \mathbb{P}_k(D; \mathbb{S})x \), and
\[
\dim x^T \mathbb{P}_k(D; \mathbb{S})x = \dim \mathbb{P}_{k+2}(D) - 4 = \frac{1}{6}(k+5)(k+4)(k+3) - 4.
\]

For any \( \tau \in \mathbb{P}_k(D; \mathbb{S}) \) satisfying \( x^T \tau x = 0 \), there exists \( q \in \mathbb{P}_k(D; \mathbb{R}^3) \) such that
\[
\tau x = q \times x = (\text{mspn} q)x, \text{ that is } (\tau - \text{mspn} q)x = 0.
\]
As a result, there exists \( \varsigma \in \mathbb{P}_k(D; \mathbb{M}) \) such that
\[
\tau = \text{mspn} q + x \times \varsigma.
\]
From the symmetry of \( \tau \), we obtain
\[
\tau = \text{sym}(\text{mspn} q + x \times \varsigma) = \text{sym}(x \times \varsigma) = \text{sym}(x \times \text{dev} \varsigma) \in \text{sym}(x \times \mathbb{P}_{k-1}(D; \mathbb{T})).
\]
Hence
\[
\dim \text{sym}(x \times \mathbb{P}_{k-1}(D; \mathbb{T})) = \mathbb{P}_k(D; \mathbb{S}) - \dim x^T \mathbb{P}_k(D; \mathbb{S})x = \frac{1}{6}k(k+1)(5k+19).
\]

Since \( \dim \text{dev}(\mathbb{P}_{k-2}(D; \mathbb{R}^3)x^T) = \dim \mathbb{P}_{k-2}(D; \mathbb{R}^3) \), we have
\[
\dim \mathbb{P}_{k-1}(D; \mathbb{T}) = \dim \text{dev}(\mathbb{P}_{k-2}(D; \mathbb{R}^3)x^T) + \dim \text{sym}(x \times \mathbb{P}_{k-1}(D; \mathbb{T})).
\]
Thus the complex (4.12) is exact. \( \square \)

Combining the two complexes (4.10) and (4.12) yields
\[
\begin{array}{c}
\mathbb{P}_1(D) \xrightarrow{\sigma_1} \mathbb{P}_{k+2}(D) \xrightarrow{\text{hess}} \mathbb{P}_k(D; \mathbb{S}) \xrightarrow{\text{curl}} \mathbb{P}_{k-1}(D; \mathbb{T}) \xrightarrow{\text{div}} \mathbb{P}_{k-2}(D; \mathbb{R}^3) \rightarrow 0.
\end{array}
\]

Unlike the Koszul complex for vectors functions, we do not have the identity property applied to homogenous polynomials. Fortunately decomposition of polynomial spaces using Koszul and differential operators still holds.

It follows from (4.11) and the complex (4.12) that
\[
\mathbb{P}_{k+2}(D) = x^T \mathbb{P}_k(D; \mathbb{S})x \oplus \mathbb{P}_1(D), \quad k \geq 0.
\]

Then we give the following decompositions for the polynomial tensor spaces \( \mathbb{P}_k(D; \mathbb{S}) \) and \( \mathbb{P}_{k-1}(D; \mathbb{T}) \).

Lemma 4.4. For \( k \in \mathbb{N} \), we have the decompositions
\[
\begin{align*}
\mathbb{P}_k(D; \mathbb{S}) &= \text{hess} \mathbb{P}_{k+2}(D) \oplus \text{sym}(x \times \mathbb{P}_{k-1}(D; \mathbb{T})) & k \geq 1, \\
\mathbb{P}_{k-1}(D; \mathbb{T}) &= \text{curl} \mathbb{P}_k(D; \mathbb{S}) \oplus \text{dev}(\mathbb{P}_{k-2}(D; \mathbb{R}^3)x^T) & k \geq 2.
\end{align*}
\]

Proof. Noting that the dimension of space in the left hand side is the summation of the dimension of two subspaces in the right hand side in (4.15) and (4.16), we only need to prove the sum is direct. The direct sum of (4.16) follows from Lemma 4.1.

For any \( \tau = \nabla^2 q \) with \( q \in \mathbb{P}_{k+2}(D) \) satisfying \( \tau \in \text{sym}(x \times \mathbb{P}_{k-1}(D; \mathbb{T})) \), it follows from the fact \( (x \cdot \nabla)x = x \) that
\[
(x \cdot \nabla)(x \cdot \nabla q - q) = (x \cdot \nabla)(x \cdot \nabla q) - x \cdot \nabla q = x^T((x \cdot \nabla)\nabla q) = x^T(\nabla^2 q)x = 0.
\]
Applying (4.7) to get \( x \cdot \nabla q - q \in P_0(K) \), which together with (4.6) gives \( q \in P_1(D) \). Thus the decomposition (4.15) holds.

When \( D \subset \mathbb{R}^2 \), the Hessian polynomial complex in two dimensions

\[
\begin{array}{cccccc}
P_1(D) & \subset & P_{k+2}(D) & \xrightarrow{\text{hess}} & P_k(D; \mathbb{S}) & \xrightarrow{\text{rot}} P_{k-1}(D; \mathbb{R}^2) & \to 0 \\
\end{array}
\]

has been proved in [21], which is a rotation of the elasticity polynomial complex [9].

4.3. Divdiv Polynomial complexes. We now move to divdiv polynomial complexes derived in [21, 22]. For completeness, we include proofs here.

**Lemma 4.5.** For \( k \in \mathbb{N}, k \geq 2 \), the polynomial complex

\[
\begin{array}{cccccc}
RT & \subset & P_{k+2}(D; \mathbb{R}^3) & \xrightarrow{\text{dev grad}} & P_{k+1}(D; \mathbb{T}) & \xrightarrow{\text{sym curl}} P_k(D; \mathbb{S}) & \xrightarrow{\text{div div}} P_{k-2}(D) & \to 0 \\
\end{array}
\]

is exact.

**Proof.** It follows from (3.6) that

\[ \text{div div } P_k(D; \mathbb{S}) = \text{div } \text{div } P_{k+2}(D; \mathbb{R}^3) = P_{k-2}(D). \]

For any \( q \in P_{k+2}(D; \mathbb{R}^3) \cap \ker(\text{dev grad}), \) we have \( \nabla q = \frac{1}{5}(\text{div } q) I \). Hence

\[ -\text{mspn}(\nabla \text{div } q) = \text{curl}(\text{div } q) I = 3 \text{curl}(\nabla q) = 0, \]

which means \( \text{div } q \in P_0(D) \) and \( q \in P_1(D; \mathbb{R}^3) \). We conclude \( q \in RT \) from the fact that \( \nabla q \) is the identity matrix multiplied by a constant.

For any \( \tau \in P_{k+1}(D; \mathbb{T}) \cap \ker(\text{sym curl}), \) there exists \( v \in H^1(D; \mathbb{R}^3) \) satisfying \( \tau = \text{dev grad } v \), i.e. \( \tau = \nabla v - \frac{1}{3}(\text{div } v) I \). Then

\[ \text{mspn}(\nabla \text{div } v) = -\text{curl}(\text{div } v) I = 3 \text{curl}(\tau - \nabla v) = 3 \text{curl } \tau \in P_k(D; \mathbb{K}), \]

from which we get \( \text{div } v \in P_{k+1}(D), \) and thus \( \nabla v \in P_{k+1}(D; \mathbb{S}) \). As a result \( v \in P_{k+2}(D; \mathbb{R}^3) \). And we also have

\[
\dim \text{sym curl } P_{k+1}(D; \mathbb{T}) = \dim P_{k+1}(D; \mathbb{T}) - \dim \text{dev grad } P_{k+2}(D; \mathbb{R}^3)
\]

\[ = \frac{1}{6}(5k^3 + 36k^2 + 67k + 36), \]

and

\[
\dim P_k(D; \mathbb{S}) \cap \ker(\text{div div}) = \frac{1}{6}(5k^3 + 36k^2 + 67k + 36).
\]

Finally we conclude \( P_k(D; \mathbb{S}) \cap \ker(\text{div div}) = \text{sym curl } P_{k+1}(D; \mathbb{T}) \), and (4.19) and (4.20). Therefore the complex (4.18) is exact.

Define operator \( \pi_{RT} : C^1(D; \mathbb{R}^3) \to RT \) as

\[ \pi_{RT} v := v(0, 0, 0) + \frac{1}{3}(\text{div } v)(0, 0, 0) x. \]

We have the following Koszul-type complex.

**Lemma 4.6.** For \( k \in \mathbb{N}, k \geq 2 \), the polynomial complex

\[
\begin{array}{cccccc}
\mathbb{R}^T & \subset & P_{k-2}(D) & \xrightarrow{x \times x^T} & P_k(D; \mathbb{S}) & \xrightarrow{x \times} P_{k+1}(D; \mathbb{T}) & \xrightarrow{\pi_{RT}} P_{k+2}(D; \mathbb{R}^3) & \to 0 \\
\end{array}
\]

is exact.
Proof. Since 
\[ \text{tr}(x \times \tau) = 2x^T \text{vspan} (\text{skw} \tau) \quad \forall \tau \in L^2(D; \mathbb{M}), \]
thus (4.21) is a complex.

For any \( v \in \mathbb{P}_{k+2}(D; \mathbb{R}^3) \) satisfying \( \pi_{RT} v = 0 \), since \( v(0, 0, 0) = 0 \), there exist \( \tau_1 \in \mathbb{P}_{k+1}(D; \mathbb{T}) \) and \( q \in \mathbb{P}_{k+1}(D) \) such that \( v = \tau_1 x + q \). Noting that
\[ \text{div}(\tau_1 x) = x^T \text{div}(\tau_1^T) + \text{tr} \tau_1 = x^T \text{div}(\tau_1^T), \]
we have
\[ \pi_{RT}(q x) = \pi_{RT} v - \pi_{RT}(\tau_1 x) = 0, \]
which indicates \( (\text{div}(q x))(0, 0, 0) = 0 \) and thus \( q(0, 0, 0) = 0 \). Hence there exists \( q_1 \in \mathbb{P}_k(D; \mathbb{R}^3) \) such that \( q = q_1^T x \). Taking \( \tau = \tau_1 + \frac{2}{3} xq_1^T - \frac{4}{3} q_1^T xI \in \mathbb{P}_{k+1}(D; \mathbb{T}) \), we get
\[ \tau x = \tau_1 x + qx_1^T x = \tau_1 x + q x = v. \]
Hence \( \mathbb{P}_{k+2}(D; \mathbb{R}^3) \cap \ker(\pi_{RT}) = \mathbb{P}_{k+1}(D; \mathbb{T}) x \) holds. Apparently
\[ (4.22) \quad \pi_{RT} v = v \quad \forall \ v \in \mathbb{RT}. \]
Namely \( \pi_{RT} \) is a projector. Consequently, the operator \( \pi_{RT} : \mathbb{P}_{k+1}(D; \mathbb{R}^3) \to \mathbb{RT} \) is surjective. And we have
\[ \dim(\mathbb{P}_{k+1}(D; \mathbb{T}) \cap \ker(x)) = \dim(\mathbb{P}_{k+1}(D; \mathbb{T}) - \dim(\mathbb{P}_{k+1}(D; \mathbb{T}) x)
\]
\[ = \dim(\mathbb{P}_{k+1}(D; \mathbb{T}) - \dim(\mathbb{P}_{k+2}(D; \mathbb{R}^3) + 4
\]
\[ (4.23) \quad = \frac{1}{6}(5k^3 + 36k^2 + 67k + 36). \]

For any \( \tau \in \mathbb{P}_k(D; \mathbb{S}) \) satisfying \( x \times \tau = 0 \), there exists \( v \in \mathbb{P}_{k-1}(D; \mathbb{R}^3) \) such that \( \tau = v x^T \). By the symmetry of \( \tau \), it follows
\[ x \times (x v^T) = x \times (v x^T)^T = x \times \tau^T = 0, \]
which indicates \( x \times v = 0 \). Then there exists \( q \in \mathbb{P}_{k-2}(D) \) satisfying \( v = q x \). Hence \( \tau = q x x^T \). Therefore \( \mathbb{P}_k(D; \mathbb{S}) \cap \ker(x \times) = \mathbb{P}_{k-2}(D) x x^T \). And we have
\[ \dim(x \times \mathbb{P}_k(D; \mathbb{S})) = \dim(\mathbb{P}_k(D; \mathbb{S}) - \dim(\mathbb{P}_k(D; \mathbb{S}) \cap \ker(x \times))
\]
\[ = \dim(\mathbb{P}_k(D; \mathbb{S}) - \dim(\mathbb{P}_{k-2}(D) x x^T)
\]
\[ = \frac{1}{6}(5k^3 + 36k^2 + 67k + 36), \]
which together with (4.23) implies \( \mathbb{P}_{k+1}(D; \mathbb{T}) \cap \ker(x \times) = x \times \mathbb{P}_k(D; \mathbb{S}) \). Therefore the complex (4.21) is exact. \( \square \)

Those two complexes (4.18) and (4.21) are connected as
\[ (4.24) \quad \mathbb{RT} \xrightarrow{\pi_{RT}} \mathbb{P}_{k+2}(D; \mathbb{R}^3) \xrightarrow{\text{dev grad}} \mathbb{P}_{k+1}(D; \mathbb{T}) \xrightarrow{\text{sym curl}} \mathbb{P}_k(D; \mathbb{S}) \xrightarrow{\text{div}} \mathbb{P}_{k-2}(D) \xrightarrow{\nabla} 0. \]

It follows from (4.22) and the complex (4.21) that
\[ \mathbb{P}_k(D; \mathbb{R}^3) = x \cdot \mathbb{P}_{k-1}(D; \mathbb{T}) \oplus \mathbb{RT} \quad k \geq 1. \]
We then move to the space \( \mathbb{P}_{k+1}(D; \mathbb{T}) \).

Lemma 4.7. We have the decomposition
\[ (4.25) \quad \mathbb{P}_k(D; \mathbb{T}) = x \times \mathbb{P}_{k-1}(D; \mathbb{S}) \oplus \text{grad} \mathbb{P}_{k+1}(D; \mathbb{R}^3) \quad k \geq 1. \]
Proof. Since the dimension of space in the left hand side is the summation of the dimension of the two spaces in the right hand side in (4.25), we only need to prove that the sum in (4.26) is the direct sum. For any \( q \in \mathbb{P}_{k+2}(D; \mathbb{R}^3) \) satisfying \( q(0, 0, 0) = 0 \) and \( \text{dev grad } q \in x \times \mathbb{P}_k(D; \mathbb{S}) \), we have \( (\text{dev grad } q)x = 0 \), that is
\[
(x \cdot \nabla)q = \frac{1}{3}(\text{div } q)x.
\]
By (4.6), there exist \( p_1, p_2, p_3 \in \mathbb{P}_{k+1}(D) \) such that \( q = (p_1 x_1, p_2 x_2, p_3 x_3)^T \). Then
\[
x \cdot \nabla p_i + p_i = \frac{1}{3} \text{div } q,
\]
which indicates \( p_1 = p_2 = p_3 \). Hence it follows from the last identity that \( x \cdot \nabla p_1 = 0 \), which combined with (4.7) gives \( p_1 = c \) for some constant \( c \). Thus \( q = cx \), and \( \text{dev grad } q = 0 \). This ends the proof. \qed

Finally we present a decomposition of space \( \mathbb{P}_k(D; \mathbb{S}) \) based on (4.24).

**Lemma 4.8.** We have the decomposition
\[
\mathbb{P}_k(D; \mathbb{S}) = \text{sym curl } \mathbb{P}_{k+1}(D; \mathbb{T}) \oplus xx^T \mathbb{P}_{k-2}(D) \quad k \geq 2.
\]

**Proof.** Since \( \text{div}(xx^T q) = (\text{div}(xq) + qx) \) and \( \text{div}(xq) = (x \cdot \nabla)q + 3q \), we get
\[
\text{div} xx^T q = \text{div}((x \cdot \nabla + 4)q) = (x \cdot \nabla + 3)(x \cdot \nabla + 4)q.
\]
Hence \( \text{div} xx^T q = (k + 4)(k + 3)q \) for any \( q \in \mathbb{P}_{k}(D) \). Writing \( \mathbb{P}_{k-2}(D; \mathbb{R}^3) = \bigoplus_{i=0}^{k-2} \mathbb{H}_i(D; \mathbb{R}^3) \), we conclude \( \text{div} : xx^T \mathbb{P}_{k-2}(D) \to \mathbb{P}_{k-2}(D; \mathbb{R}^3) \) is a bijection.

To prove (4.26), we first note that the dimension of space in the left hand side is the summation of the dimension of the two subspaces in the right hand side. So it suffices to prove the sum is a direct sum. Assume \( q \in \mathbb{P}_{k-2}(D) \) satisfies \( xx^T q \in \text{sym curl } \mathbb{P}_{k+1}(D; \mathbb{T}) \), which means
\[
\text{div} xx^T q = 0.
\]
Thus from (4.27) and (4.8), \( q = 0 \). \qed

When \( D \subset \mathbb{R}^2 \), the divdiv polynomial complex in two dimensions
\[
RT \subset \mathbb{P}_{k+1}(D; \mathbb{R}^2) \xrightarrow{\text{sym curl}} \mathbb{P}_k(D; \mathbb{S}) \xrightarrow{\text{div}} \mathbb{P}_{k-2}(D) \rightarrow 0
\]
has been proved in [21] and used to construct the first finite element divdiv complex in two dimensions.

5. **Conforming Virtual Element Hessian Complex**

In this section we shall construct virtual element and finite element spaces and obtain a discrete Hessian complex:
\[
\mathbb{P}_1(\Omega) \xrightarrow{\nabla^2} \mathbb{W}_h \xrightarrow{\text{curl}} \mathbb{V}_h \xrightarrow{\text{div}} \mathbb{Q}_h \xrightarrow{\text{dev grad}} 0,
\]
where
- \( \mathbb{W}_h \) is an \( H^2(\Omega) \)-conforming virtual element space containing piecewise \( \mathbb{P}_5 \) polynomials;
- \( \mathbb{V}_h \) is an \( H(\text{curl}, \Omega; \mathbb{S}) \)-conforming virtual element space containing piecewise \( \mathbb{P}_3 \) polynomials;
• \( \mathbf{V}_h \) is an \( H(\text{div}, \Omega; \mathbb{T}) \)-conforming finite element space containing piecewise \( \mathbb{P}_2 \) polynomials;
• \( Q_h \) is piecewise \( \mathbb{P}_1(\mathbb{R}^3) \) polynomial which is obviously conforming to \( L^2(\Omega) \).

The domain \( \Omega \) is decomposed into a triangulation \( \mathcal{T}_h \) consisting of tetrahedrons. That is each element \( K \in \mathcal{T}_h \) is a tetrahedron. Extension to general polyhedral meshes will be explored in a future work.

In \cite{30}, a finite element Hessian complex has been constructed and the lowest polynomial degree for \((W_h, \mathbf{Q}_h, \mathbf{V}_h, Q_h)\) is \((9, 7, 6, 5)\) and ours is \((5, 3, 2, 1)\) with a few additional virtual shape functions.

For each element \( K \in \mathcal{T}_h \), denote by \( \mathbf{n}_K \) the unit outward normal vector to \( \partial K \), which will be abbreviated as \( \mathbf{n} \) for simplicity. Let \( \mathcal{F}_h, \mathcal{F}_h^1, \mathcal{E}_h \) and \( \mathcal{V}_h \) be the union of all faces, interior faces, edges and vertices of the partition \( \mathcal{T}_h \), respectively. For any \( F \in \mathcal{F}_h \), fix a unit normal vector \( \mathbf{n}_F \). For any \( e \in \mathcal{E}_h \), fix a unit tangent vector \( \mathbf{t}_e \) and two unit normal vectors \( \mathbf{n}_{e,1} \) and \( \mathbf{n}_{e,2} \), which will be abbreviated as \( \mathbf{n}_1 \) and \( \mathbf{n}_2 \) without causing any confusions. For \( K \) being a polyhedron, denote by \( \mathcal{F}(K) \), \( \mathcal{E}(K) \) and \( \mathcal{V}(K) \) the set of all faces, edges and vertices of \( K \), respectively. For any \( F \in \mathcal{F}_h \), let \( \mathcal{E}(F) \) and \( \mathcal{V}(F) \) be the set of all edges and vertices of \( F \), respectively. And for each \( e \in \mathcal{E}(F) \), denote by \( \mathbf{n}_{F,e} \) the unit vector being parallel to \( F \) and outward normal to \( \partial F \).

5.1. \( H(\text{div}) \)-conforming element for trace-free tensors. We chose \( \mathbb{P}_2(K; \mathbb{T}) \) as the shape function space. Its trace \( \mathbf{v} \mathbf{n} \) on each face \( F \) is in \( \mathbb{P}_2(F; \mathbb{R}^3) \). In the classic \( H(\text{div}) \) element for vector functions, such trace can be determined by the face moments \( \int_F \mathbf{v} \mathbf{n} \mathbf{q} \) for \( \mathbf{q} \in \mathbb{P}_2(F; \mathbb{R}^3) \). For the tensor polynomial with additional structure, e.g., here is the trace-free, face moments cannot reflect to this property. One fix is to introduce the nodal continuity of each component of the tensor so that the structure of the tensor is utilized.

We first consider the following degrees of freedom for a quadratic polynomial on a triangle.

**Lemma 5.1.** Let \( F \in \mathcal{F}(K) \) be a triangular face and \( v \in \mathbb{P}_2(F) \). If
\[
v(a_1) = v(a_2) = v(a_3) = 0, \quad (v, q)_F = 0 \quad \forall \ q \in \mathbb{P}_1(F)
\]
with \( a_1, a_2 \) and \( a_3 \) being the vertices of triangle \( F \), then \( v = 0 \).

**Proof.** Let \( (\lambda_1, \lambda_2, \lambda_3) \) be the barycentric coordinate of point \( \mathbf{x} \) with respect to \( F \). Since \( v(a_1) = v(a_2) = v(a_3) = 0 \), we have \( v = c_1 \lambda_2 \lambda_3 + c_2 \lambda_3 \lambda_1 + c_3 \lambda_1 \lambda_2 \), where \( c_1, c_2 \) and \( c_3 \) are constants. Now taking \( q = \lambda_i \) with \( i = 1, 2, 3 \), we obtain
\[
\frac{1}{60} |F| \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.
\]

Noting that the coefficient matrix is invertible, it follows \( c_1 = c_2 = c_3 = 0 \). \( \square \)

Next we use the quadratic \( H(\text{div}; \mathbb{T}) \) bubble function introduced in \cite{30} to characterize the interior part. Denote by
\[
\mathbb{B}_2(K; \mathbb{T}) = \mathbb{P}_2(K; \mathbb{T}) \cap H_0(\text{div}; \mathbb{T}).
\]
In [30], a constructive characterization of \( \mathbb{B}_2(K; \mathbb{T}) \) is given by

\[
\mathbb{B}_2(K; \mathbb{T}) := \sum_{i=1}^{4} \sum_{1 \leq j,l \leq 4} \text{span}\{\lambda_j \lambda_l \mathbf{n}_j t_{j,l}^T\},
\]

where \((\lambda_1, \lambda_2, \lambda_3, \lambda_4)\) be the barycentric coordinate of point \(x\) with respect to \(K\), and \(t_{j,l} := x_l - x_j\) with \(\mathcal{V}(K) := \{x_1, x_2, x_3, x_4\}\). That is on each face we use the normal vector and an edge vector to form a traceless matrix and extend to the whole element by the scalar edge bubble function. Therefore \(\dim \mathbb{B}_2(K; \mathbb{T}) = 3 \times 4 = 12\).

Now we define a lower order \(H(\text{div})\)-conforming finite element for trace-free tensors. Take \(P_2(K; \mathbb{T})\) as the space of shape functions. The degrees of freedom are given by

\[
\begin{align*}
(5.3) & \quad \mathbf{v}(\delta) \quad \forall \, \delta \in \mathcal{V}(K), \\
(5.4) & \quad (\mathbf{v} \mathbf{n}, \mathbf{q})_F \quad \forall \, \mathbf{q} \in P_1(F; \mathbb{R}^3), \, F \in \mathcal{F}(K), \\
(5.5) & \quad (\mathbf{v}, \mathbf{q})_K \quad \forall \, \mathbf{q} \in P_0(K; \mathbb{T}) \oplus (\mathbb{B}_2(K; \mathbb{T}) \cap \ker(\text{div})).
\end{align*}
\]

We can also replace the degrees of freedom (5.5) by

\[
(5.6) \quad (\mathbf{v}, \mathbf{q})_K \quad \forall \, \mathbf{q} \in \mathbb{B}_2(K; \mathbb{T}).
\]

Thanks to the explicit formulation of bubble functions (5.2), the implementation using (5.6) will be easier. On the other hand, (5.5) will be helpful when defining spaces for \(H(\text{curl}, K; \mathbb{S})\).

**Lemma 5.2.** The degrees of freedom (5.3)-(5.5) are unisolvent for \(P_2(K; \mathbb{T})\).

**Proof.** First of all both \(\dim \mathbb{P}_2(K; \mathbb{T})\) and the number of the degrees of freedom (5.3)-(5.5) are 80.

Take any \(\mathbf{v} \in \mathbb{P}_2(K; \mathbb{T})\) and suppose all the degrees of freedom (5.3)-(5.5) vanish. Applying Lemma 5.1 to each component of \(\mathbf{v} \mathbf{n}\), we get \(\mathbf{v} \in \mathbb{B}_2(K; \mathbb{T})\). It follows from the integration by parts and the first part of the degrees of freedom (5.5) that \(\text{div}\mathbf{v} = 0\), i.e. \(\mathbf{v} \in \mathbb{B}_2(K; \mathbb{T}) \cap \ker(\text{div})\). Finally we arrive at \(\mathbf{v} = 0\) by using the second part of the degrees of freedom (5.5).

The global finite element space is

\[
V_h := \{v_h \in H(\text{div}, \Omega; \mathbb{T}) : v_h|_K \in \mathbb{P}_2(K; \mathbb{T}) \forall \, K \in \mathcal{T}_h, \, \text{the degrees of freedom (5.3)-(5.4) are single-valued}\},
\]

For \(\mathbf{v} \in V_h\), by Lemma 5.1, the trace \(\mathbf{v} \mathbf{n}|_F \in \mathbb{P}_2(F)\) is determined uniquely by the degree of freedom (5.3)-(5.4). Therefore \(V_h \subset H(\text{div}, \Omega; \mathbb{T})\) is a conforming finite element space.

**Remark 5.3.** We have from [30] that

\[
\text{div} \mathbb{B}_2(K; \mathbb{T}) = \mathbb{P}_2^+(K; \mathbb{R}^3),
\]

where \(\mathbb{P}_2^+(K; \mathbb{R}^3) := \{\mathbf{v} \in \mathbb{P}_2(K; \mathbb{R}^3) : (\mathbf{v}, \mathbf{q})_K = 0 \forall \, \mathbf{q} \in \mathbb{R}^T\}\). Based on this, we can define a reduced \(H(\text{div})\)-conforming element for trace-free tensors. To be specific, we can reduce the shape functions space \(\mathbb{P}_2(K; \mathbb{T})\) to

\[
V''(K) := \{\mathbf{v} \in \mathbb{P}_2(K; \mathbb{T}) : \text{div}\mathbf{v} \in \mathbb{R}^T\},
\]
whose dimension is 72. And the degrees of freedom are given by
\[ v(\delta) \quad \forall \delta \in \mathcal{V}(K), \]
\[ (vn, q)_F \quad \forall q \in P_1(F; \mathbb{R}^3), F \in \mathcal{F}(K), \]
\[ (v, q)_K \quad \forall q \in B_2(K; T) \cap \ker(\text{div}). \]

5.2. $H^2$-conforming virtual element. To define an $H^2$-conforming virtual element in three dimensions, we shall adapt two dimensional $H^2$-conforming virtual elements constructed in [18, 4] and three dimensional $C^1$ virtual element in [10].

Define an $H^2$-conforming virtual element space on tetrahedron $K$
\[ \tilde{W}(K) := \{ v \in H^2(K) : \Delta^2 v \in P_1(K), \text{both} \ v|_{\partial K} \text{and} \ \nabla v|_{\partial K} \text{are continuous}, \]
\[ v|_F \in P_3(F), \partial_n v|_F \in P_4(F) \text{for each} \ F \in \mathcal{F}(K) \}. \]

The space of degrees of freedom $\mathcal{N}(K)$ consists of
\[ v(\delta), \nabla v(\delta), \nabla^2 v(\delta) \quad \forall \delta \in \mathcal{V}(K), \]
\[ \int_e \partial_n v \, ds \quad \forall e \in \mathcal{E}(K), i = 1, 2, \]
\[ (\partial_n v, q)_F \quad \forall q \in P_1(F), F \in \mathcal{F}(K), \]
\[ (v, q)_K \quad \forall q \in P_1(K). \]

The space $\tilde{W}(K)$ is not empty as $P_3(K) \subset \tilde{W}(K)$. Its dimension is, however, not so clear from the definition. There is a compatible condition given implicitly in the definition of the local space $\tilde{W}(K)$. As the trace of a function in $H^2(K)$, the boundary value $v|_{\partial K}$ and $\partial_n v|_{\partial K}$ are compatible in the sense that $\nabla v|_F = \nabla_S v + (\partial_n v)|_F n_F$ should be continuous on edges [29]. The degrees of freedom $\nabla^2 v(\delta)$ is also questionable for a function $v \in H^2(K)$ only. In the classic finite element space, this is not an issue as shape functions are polynomials.

For a more rigorous verification of unisolvence, we introduce data space
\[ \mathcal{D}(K) = \{ (f, v_0, v_1, v_2, u_e, u_F) : f \in P_1(K), v_0 \in P_0(\mathcal{V}(K)), v_1 \in P_0(\mathcal{V}(K), \mathbb{R}^3), \]
\[ v_2 \in P_0(\mathcal{V}(K), S), u_e \in P_0(\mathcal{E}(K), \mathbb{R}^2), u_F \in P_1(\mathcal{F}(K)) \}. \]

Obviously $\dim \mathcal{D}(K) = 68 = \dim \mathcal{N}(K)$. For function $v \in \tilde{W}(K) \cap C^2(K)$, the mapping
\[ (\Delta^2 v, v(\delta), \nabla v(\delta), \nabla^2 v(\delta), \int_e \partial_n v \, ds, \partial_n v|_F) \quad \forall \delta \in \mathcal{V}(K), e \in \mathcal{E}(K), F \in \mathcal{F}(K) \]
defines from $\tilde{W}(K) \cap C^2(K) \rightarrow \mathcal{D}(K)$.

Let $P_k(\partial K)$ be the function space which is continuous on the boundary $\partial K$ and its restriction to each face is a polynomial of degree at most $k$. Given a data $(f, v_0, v_1, v_2, u_e, u_F) \in \mathcal{D}(K)$, using $(v_0, v_1, v_2, u_e)$, we can determine a $P_3(F)$ Argyris element [5] and consequently define a function $g_1 \in P_3(\partial K)$. Similarly using $(v_1, v_2, u_e, u_F)$, we can determine a $P_4(F)$ Hermite element [26] and consequently a function $g_2 \in P_4(\partial K)$. By the unisolvence of the Argyris element and Hermite element in two dimensions, we know $(g_1, g_2)$ is uniquely determined by $(v_0, v_1, v_2, u_e, u_F)$ and $(g_2|_F n_F + \nabla_F (g_1|_F))|_e$ is single-valued across each edge $e \in \mathcal{E}(K)$.
Given data \((f, g_1, g_2)\), we consider the biharmonic equation with Dirichlet boundary condition
\begin{equation}
\Delta^2 v = f \text{ in } K, \quad v = g_1, \partial_n v = g_2 \text{ on } \partial K.
\end{equation}
As \(g_1, g_2\) are compatible in the sense \(g_2 \| n + \nabla \partial_K (g_1) \in \mathbb{P}_3(\partial K)\), by the trace theorem of \(H^2(K)\) on polyhedral domains [29], there exists \(v^b \in H^2(K)\) such that
\begin{equation}
v^b|_{\partial K} = g_1, \quad \partial_n v^b|_{\partial K} = g_2.
\end{equation}
Indeed \(v^b\) can be chosen as a polynomial in \(\mathbb{P}_3(K)\) using the \(C^1\) finite element constructed in [42]. Then consider biharmonic equation with homogenous boundary condition
\begin{equation}
\Delta^2 v^0 = f - \Delta^2 v^b \text{ in } K, \quad v^0 = 0, \partial_n v^0 = 0 \text{ on } \partial K.
\end{equation}
The existence and uniqueness of \(v^0\) is guaranteed by the Lax-Milligram lemma. Setting \(v = v^b + v^0\) gives a solution to (5.11). The uniqueness of the solution to (5.11) is trivial.

Therefore we have constructed an embedding \(\mathcal{L} : D(K) \rightarrow \tilde{W}(K)\) and \(\mathcal{L}\) is injective. We shall chose \(W(K) = \mathcal{L}(D(K))\) and by construction \(\mathcal{L} : D(K) \rightarrow W(K)\) is a bijection and thus \(\dim W(K) = 68\). Functions in \(W(K)\) are defined as solutions to local PDEs which may still not be smooth enough to take nodal values of the Hessian.

To be consistent with finite element notation, we still use the form \(\nabla^2 v(\delta)\) but understand it with the help of \(\mathcal{L}\). For \(v \in W(K)\), \(L^{-1} v = (f, v_0, v_1, v_2, u_e, u_F) \in D(K)\). We define \(\nabla^2 v(\delta) \in W'(K)\) by
\begin{equation}
\nabla^2 v(\delta) := v_2.
\end{equation}
That is we understand \(\nabla^2 v\) as a functional defined on \(W(K)\) which will match the vertex value of the hessian if \(v\) is smooth enough. Other degrees of freedom (5.7)-(5.9) can be understood in a similar fashion. The interior moment (5.10) keeps unchanged and the relation of (5.10) and \(f \in L^{-1} v\) is discussed below.

**Lemma 5.4.** The degrees of freedom (5.7)-(5.10) are unisolvent for \(W(K)\).

**Proof.** First of all \(\dim W(K) = \dim N(K) = 68\). Take any \(v \in W(K)\) and suppose all the degrees of freedom (5.7)-(5.10) vanish. By the unisolvence of the Argyris element and Hermite element in two dimensions, we have \(v \in H^2_0(K)\). It follows from the integration by parts that
\[
\|\nabla^2 v\|_{0,K}^2 = (\Delta^2 v, v)_{0,K} = 0,
\]
as \(\Delta^2 v \in \mathbb{P}_1(K)\) and the vanishing degree of freedom (5.10). Thus \(v = 0\).

As \(\dim \mathbb{P}_5 = 56\), there are 12 shape functions in \(W(K)\) are non-polynomials and thus are treated as virtual. The \(L^2\)-projection of \(\nabla^2 v\) to \(\mathbb{P}_3(K, S)\) can be computed by degrees of freedom using the Green’s identity [22]: for \(\tau \in \mathbb{P}_3(K, S), v \in W(K)\)
\[
(\nabla^2 v, \tau)_K = (\text{div} \, \text{div} \, \tau, v)_K + \sum_{F \in \mathcal{F}(K)} \sum_{e \in \mathcal{E}(F)} (n^T_F \tau n, v)_e
\]
\begin{equation}
+ \sum_{F \in \mathcal{F}(K)} \left[ (n^T_F \tau n, \partial_n v)_F - (2 \text{div} F(\tau F) + \partial_n(n^T F \tau n), v)_F \right].
\end{equation}
As $\text{div} \, \tau \in \mathcal{P}_1(K)$, the first term can be computed by (5.10). On the boundary, $v|_F$ is a $\mathcal{P}_3(F)$ Argyris element, and $\partial_n v|_F$ is a $\mathcal{P}_4(F)$ Hermite element and thus all boundary terms are computable. In particular by choosing $\tau \in \nabla^2 \mathcal{P}_5(K)$, we can compute an $H^2$-projection of $v$ to $\mathcal{P}_5(K)$, that is $\Pi^k v \in \mathcal{P}_5(K)$ is determined by
\[
(\nabla^2 \Pi^k v, \nabla^2 q)_K = (\nabla^2 v, \nabla^2 q)_K \quad \forall \, q \in \mathcal{P}_5(K),
\]
\[
(\Pi^k v, q)_K = (v, q)_K \quad \forall \, q \in \mathcal{P}_1(K).
\]

We have the following properties of $\Pi^k$
\[
\Pi^k q = q \quad \forall \, q \in \mathcal{P}_5(K),
\]
(5.15) $h_K^i |v - \Pi^k v|_{i,K} \lesssim h_K^2 \inf_{q \in \mathcal{P}_5(K)} |v - q|_{2,K} \quad \forall \, v \in H^2(K), i = 0, 1, 2.$

Remark 5.5. The lowest order $C^1$ macro-element on the Alfeld split following the approach in this paper. [28, 2, 32] has the same number of degrees of freedom and same degrees of freedom on boundary as (5.7)-(5.10). We can construct a conforming macro-element Hessian element space $\mathcal{F}_h(\Omega)$ with the Alfeld split following the approach in this paper. □

For any $F \in \mathcal{F}(K)$, both $v|_F$ and $\partial_n v|_F$ are determined by the degrees of freedom (5.7)-(5.9) on the face $F$. Thus we can define the $H^2$-conforming virtual element space
\[
W_h := \{ v_h \in H^2(\Omega) : v_h|_K \in W(K) \text{ for each } K \in \mathcal{T}_h, \text{ the degrees of freedom } (5.7)-(5.9) \text{ are single-valued} \}.
\]

Let $I_h^5 : H^4(\Omega) \to W_h$ be the nodal interpolation operator with respect to the degrees of freedom (5.7)-(5.10). By standard techniques, it holds
\[
h_K^i |v - I_h^5 v|_{i,K} \lesssim h_K^5 |v|_{5,K} \quad \forall \, v \in H^5(\Omega), i = 0, 1, 2.
\]

5.3. $H(\text{curl})$-conforming element for symmetric tensors. Motivated by the decomposition (4.15), we take the space of shape functions
\[
\Sigma(K) := \nabla^2 W(K) \oplus \text{sym}(\mathbf{x} \times \mathcal{P}_2(K; \mathbb{T})).
\]
The degrees of freedom are given by
\[
\text{curl} \tau(\delta) \quad \forall \, \delta \in \mathcal{V}(K),
\]
(5.17)
\[
\tau(\delta) \quad \forall \, \delta \in \mathcal{V}(K),
\]
(5.18)
\[
(\tau \cdot q)_e \quad \forall \, q \in \mathcal{P}_1(e; \mathbb{R}^3), e \in \mathcal{E}(K),
\]
(5.19)
\[
(\mathbf{n} \times \tau \times \mathbf{n}, q)_F \quad \forall \, q \in \mathcal{P}_0(F; \mathbb{S}), F \in \mathcal{F}(K),
\]
(5.20)
\[
(\Pi_F \tau \cdot \mathbf{n}, q)_F \quad \forall \, q \in \mathcal{P}_0(F; \mathbb{R}^2) \oplus \mathcal{P}_1(F) \mathbf{x}, F \in \mathcal{F}(K),
\]
(5.21)
\[
(\text{curl} \tau, q)_K \quad \forall \, q \in \mathcal{P}_2(K; \mathbb{T}) \cap \ker(\text{div}),
\]
(5.22)
\[
(\tau, \mathbf{x} \mathbf{x}^T q)_K \quad \forall \, q \in \mathcal{P}_1(K).
\]

From the decomposition (4.15), we know that $\mathcal{P}_3(K; \mathbb{S}) \subset \Sigma(K)$. The dimension of the space is
\[
\dim \Sigma(K) = \dim W(K) - 4 + \dim \text{sym}(\mathbf{x} \times \mathcal{P}_2(K; \mathbb{T})) = 64 + 68 = 132.
\]
The number of the degrees of freedom (5.17)-(5.22) is
\[
32 + 24 + 36 + 12 + 20 + 4 + 4 = 132,
\]
which agrees with $\dim \Sigma(K)$. In (5.20)-(5.21) we separate the trace $\boldsymbol{n} \times \boldsymbol{\tau}$ into tangential-tangential part and tangential-normal part. As $\boldsymbol{n} \times \boldsymbol{\tau} \times \boldsymbol{n}|_F$ is a $2 \times 2$ symmetric tensor, the number of degrees of freedom (5.20) is $3 \times 4 = 12$. Most of the shape functions are polynomials except 12 non-polynomial ones in the form $\nabla^2 v$ for some $v \in W(K)$ and $\nabla^2 v(\delta)$ should be understood in the sense of (5.13).

Although there are non-polynomial shape functions, the trace $\boldsymbol{n} \times \boldsymbol{\tau}$ on each face is always polynomial and determined by (5.17)-(5.21).

**Lemma 5.6.** For each $F \in \mathcal{F}(K)$ and any $\boldsymbol{\tau} \in \Sigma(K)$, $\boldsymbol{n} \times \boldsymbol{\tau}|_F \in \mathbb{P}_3(F; \mathbb{M})$ is determined by the degrees of freedom (5.17)-(5.21) on face $F$.

**Proof.** First of all, we show although $\nabla^2 v$ on face $F$ is a tetrahedron and $\delta \in \mathcal{V}(F)$. Due to the unisolvence of the second-type Nédélec element $[37]$, we get $\Pi_F \tau \cdot \boldsymbol{n} = 0$.

For the tangential-tangential part, as $\Pi_F \tau \Pi_F$ of the Hessian complex (4.17) in two dimensions, there exists $w_F \in \mathbb{P}_3(F)$ such that $\Pi_F \tau \Pi_F = \nabla_F^2 w_F$ and $w_F(\delta) = 0$ for each $\delta \in \mathcal{V}(F)$. Then we get from the vanishing degrees of freedom (5.18)-(5.21) that

$$
(\Pi_F \tau \cdot \boldsymbol{n}, q)_F = 0 \quad \forall q \in \mathbb{P}_2(F; \mathbb{R}^2),
$$

where we use the decomposition $\mathbb{P}_2(F; \mathbb{R}^2) = \nabla_F^2 \mathbb{P}_3(F) \oplus x \mathbb{P}_1(F)$ which is a two dimensional version of (4.5). Due to the unisolvence of the second-type Nédélec element $[37]$, we get $\Pi_F \tau \cdot \boldsymbol{n} = 0$.

To show the unisolvence, we adapt the unisolvence proof of three dimensional $H(\text{curl})$-conforming virtual element in $[14]$. We take the advantage of the fact that $K$ is a tetrahedron and $\text{curl} \Sigma(K)$ is polynomial. The approach of using local problems is troublesome as for symmetric matrices, the well-posedness of curl – div...
system with non-homogenous Dirichlet boundary condition is unclear. A crucial and missing part is the characterization of the trace of $H(\text{curl}, \Omega; \mathcal{S})$.

**Lemma 5.7.** The degrees of freedom (5.17)-(5.23) are unisolvent for $\Sigma(K)$.

*Proof.* Take any $\tau \in \Sigma(K)$ and suppose all the degrees of freedom (5.17)-(5.23) vanish. We are going to prove $\tau = 0$.

With vanishing degrees of freedom (5.17)-(5.21), we have proved that $\tau \in H_0(\text{curl}, K; \mathcal{S})$. Then $\text{curl} \tau \in \mathbb{B}_2(K, \mathbb{T}) \cap \ker(\text{div})$, together with the vanishing degree of freedom (5.22) implies $\text{curl} \tau = 0$.

Using integration by parts, with $\tau \times n|_{\partial K} = 0$ and $\text{curl} \tau = 0$,

$$
(\tau, \text{sym} \text{curl} \sigma)_K = (\text{curl} \tau, \sigma)_K - (n \times \tau, \sigma)_{\partial K},
$$

we conclude that $\tau \perp \text{sym} \text{curl} \sigma$ for any $\sigma \in H(\text{sym} \text{curl}; \mathcal{M})$.

Use the fact $\text{div} \text{div} \tau \in \mathbb{P}_1(K)$ and $\text{div} \text{div} : xx^\top \mathbb{P}_1(K) \to \mathbb{P}_1(K)$ is bijection, cf. Lemma 4.8, we can find a polynomial $xx^\top q$ with $q \in \mathbb{P}_1(K)$ such that $\text{div} \text{div}(\tau - xx^\top q) = 0$ and thus $\tau = xx^\top q + \text{sym} \text{curl} \sigma$ for some $\sigma \in H(\text{sym} \text{curl}; \mathcal{M})$.

Then

$$
(\tau, \tau)_K = (\tau, xx^\top q + \text{sym} \text{curl} \sigma)_K = 0,
$$

which implies $\tau = 0$. \hfill \Box

We now discuss how to compute the $L^2$-projection of an element $\tau \in \Sigma(K)$ to $\mathbb{P}_3(K; \mathcal{S})$. By Lemma 5.6, we can determine the piecewise polynomial $n \times \tau$ on the boundary and $(\text{curl} \tau)n|_F$. Together with (5.17), $\text{curl} \tau \in \mathbb{P}_2(K; \mathbb{T})$ is determined. Then, using (5.25), we can compute the $L^2$-projection to the subspace $\text{sym} \text{curl} \mathbb{P}_4(K; \mathbb{T})$. Use the degree of freedom (5.23), we can compute the $L^2$-projection to the subspace $xx^\top \mathbb{P}_1(K)$. Finally, recalling that $\mathbb{P}_3(K; \mathcal{S}) = xx^\top \mathbb{P}_1(K) \oplus \text{sym} \text{curl} \mathbb{P}_4(K; \mathcal{S})$, the $L^2$-projection to $\mathbb{P}_3(K; \mathcal{S})$ will be obtained by combining the projection to each subspace and an orthogonalization step.

Let the global finite element spaces

$$
\Sigma_h := \{\tau_h \in L^2(\Omega; \mathcal{S}) : \tau_h|_K \in \Sigma(K) \quad \forall K \in \mathcal{T}_h, \text{ the degrees of freedom (5.17)-(5.21) are single-valued}\}.
$$

It follows from Lemma 5.6 that $\Sigma_h \subset H(\text{curl}, \Omega; \mathcal{S})$.

For any sufficiently smooth and symmetric tensor $\tau$ defined on tetrahedron $K$, let $I^*_K \tau \in \Sigma(K)$ be the nodal interpolation of $\tau$ based on the degrees of freedom (5.17)-(5.22). We have

$$
I^*_K \tau = \tau \quad \forall \tau \in \Sigma(K),
$$

(5.26) $\|\tau - I^*_K \tau\|_{0,K} + h_K\|\text{curl}(\tau - I^*_K \tau)\|_{0,K} \lesssim h_K^4|\tau|_{4,K} \quad \forall \tau \in H^4(K; \mathcal{S})$.

For any sufficiently smooth and symmetric tensor $\tau$ defined on $\Omega$, let $I^*_h \tau \in \Sigma_h$ be defined by $(I^*_h \tau)|_K := I^*_K (\tau|_K)$ for each $K \in \mathcal{T}_h$.

If $\tau \in H^4(K; \mathcal{S})$ satisfying $\text{curl} \tau \in \mathbb{P}_2(K; \mathbb{T})$, due to Lemma 5.38 in [35] and Lemma 4.7 in [3], the interpolation $I^*_K \tau$ is well-defined, and it follows from the integration by parts and Lemma 5.2 that

$$
\text{curl}(I^*_K \tau) = \text{curl} \tau.
$$

Moreover, by the scaling argument we have

$$
\|\tau - I^*_K \tau\|_{0,K} + h_K\|I^*_K \tau|_{1,K} \lesssim h_K\|\tau|_{1,K}.
$$

(5.28)
5.4. **Discrete conforming Hessian complex.** In this subsection we will prove the conforming virtual element Hessian complex (5.1) in three dimensions.

The polynomial space for \( L^2(\Omega) \) is simply discontinuous \( \mathbb{P}_1 \)

\[
Q_h := \{ q_h \in L^2(\Omega; \mathbb{R}^3) : q_h|_K \in \mathbb{P}_1(K; \mathbb{R}^3) \quad \forall \ K \in T_h \}.
\]

**Lemma 5.8.** It holds

\[
(5.29) \quad \text{div} V_h = Q_h.
\]

**Proof.** It is apparent that \( \text{div} V_h \subseteq Q_h \). Conversely taking any \( p_h \in Q_h \), by (3.2) there exists \( v \in H^1(\Omega; \mathbb{T}) \) such that \( \text{div} v = p_h \). Choose \( v_h \in V_h \) determined by

\[
v_h(\delta) = 0 \quad \forall \ \delta \in V_h,
\]

\[
(v_h \cdot n, q)_F = (vn, q)_F \quad \forall \ q \in \mathbb{P}_1(F; \mathbb{R}^3), \ F \in \mathbb{F}_h,
\]

\[
(v_h, q)_K = (v, q)_K \quad \forall \ q \in P_0(K; \mathbb{T}) \oplus (\mathbb{B}_2(K; \mathbb{T}) \cap \ker(\text{div})), \ K \in T_h.
\]

It follows from the integration by parts that

\[
(\text{div}(v_h - v), q_h) = 0 \quad \forall \ q_h \in Q_h.
\]

Therefore \( \text{div} v_h = \text{div} v = q_h \). \( \square \)

**Lemma 5.9.** Assume \( \Omega \) is a topologically trivial domain. Then we have the discrete Hessian complex

\[
(5.30) \quad \mathbb{P}_1(\Omega) \xrightarrow{\subset} W_h \xrightarrow{\nabla^2} \Sigma_h \xrightarrow{\text{curl}} V_h \xrightarrow{\text{div}} Q_h \xrightarrow{0}.
\]

**Proof.** It is easy to see that (5.30) is a complex as all discrete spaces are conforming. We check the exactness of this complex. First of all, \( W_h \cap \ker(\nabla^2) = \mathbb{P}_1(\Omega) \). Then

\[
\dim \nabla^2 W_h = \dim W_h - 4 = 10\#\mathcal{V}_h + 2\#\mathcal{E}_h + 3\#\mathcal{F}_h + 4\#\mathcal{T}_h - 4.
\]

For any \( \boldsymbol{\tau}_h \in \Sigma_h \cap \ker(\text{curl}) \), there exists \( w \in H^2(\Omega) \) satisfying \( \boldsymbol{\tau}_h = \nabla^2 w \). On each element \( K \), we have \( \nabla^2 (w|_K) \in \nabla^2 W(K) \), which means \( w|_K \in W(K) \). Noting that \( \nabla^2 w \) is single-valued at each vertex in \( \mathcal{V}_h \). Then \( w \in W_h \). This indicates \( \Sigma_h \cap \ker(\text{curl}) = \nabla^2 W_h \), and

\[
\dim \text{curl}\Sigma_h = \dim \Sigma_h - \dim \nabla^2 W_h
\]

\[
= 14\#\mathcal{V}_h + 6\#\mathcal{E}_h + 8\#\mathcal{F}_h + 8\#\mathcal{T}_h - (10\#\mathcal{V}_h + 2\#\mathcal{E}_h + 3\#\mathcal{F}_h + 4\#\mathcal{T}_h - 4)
\]

\[
= 4\#\mathcal{V}_h + 4\#\mathcal{E}_h + 5\#\mathcal{F}_h + 4\#\mathcal{T}_h + 4.
\]

On the other side, it holds from (5.29) that

\[
\dim V_h \cap \ker(\text{div}) = \dim V_h - \dim Q_h = 8\#\mathcal{V}_h + 9\#\mathcal{F}_h.
\]

Hence we acquire from the Euler’s formula that

\[
\dim V_h \cap \ker(\text{div}) - \dim \text{curl}\Sigma_h = 4(-\#\mathcal{T}_h + \#\mathcal{F}_h - \#\mathcal{E}_h + \#\mathcal{V}_h - 1) = 0,
\]

which yields \( V_h \cap \ker(\text{div}) = \text{curl}\Sigma_h \).

\( \square \)

**Remark 5.10.** When the topology of \( \Omega \) is non-trivial, it is assumed to be captured by the triangulation \( T_h \). As all discrete spaces are conforming, the co-homology groups defined by the Hessian complex is preserved in the discrete Hessian complex.
5.5. **Discrete Poincaré inequality.** Due to the exactness of the discrete Hessian complex, we have the following discrete Poincaré inequality.

**Lemma 5.11.** Assume $\Omega$ is a topologically trivial domain. For any $\tau_h \in \Sigma_h$ satisfying

$$(\tau_h, \nabla^2 w_h) = 0 \quad \forall \ w_h \in W_h,$$

it holds the discrete Poincaré inequality

$$\|\tau_h\|_0 \lesssim \|\text{curl } \tau_h\|_0.$$  \hfill (5.31)

Consequently

$$\|\tau_h\|_0 \leq \|\text{curl } \tau_h\|_0 + \sup_{w_h \in W_h} \frac{(\tau_h, \nabla^2 w_h)}{\|w_h\|^2} \quad \forall \ \tau_h \in \Sigma_h.$$  \hfill (5.32)

**Proof.** Since $\text{curl } \tau_h \in H(\text{div}, \Omega; T)$, by (3.3) there exists $\tau \in H^1(\Omega; S)$ such that

$$\text{curl } \tau = \text{curl } \tau_h, \quad \|\tau\|_1 \lesssim \|\text{curl } \tau_h\|_0.$$  

By (5.27), we have

$$\text{curl}(I_h^\ast \tau) = \text{curl } \tau = \text{curl } \tau_h.$$  

It follows from the complex (5.30) that $\tau_h - I_h^\ast \tau \in \nabla^2 W_h$. Hence we obtain from (5.28) that

$$\|\tau_h\|_0^2 = (\tau_h, \tau_h) = (\tau_h, I_h^\ast \tau) \leq \|\tau_h\|_0 \|I_h^\ast \tau\|_0 \lesssim \|\tau_h\|_0 \|\tau\|_1,$$

which means (5.31).

For a general $\tau_h \in \Sigma_h$, by the exact sequence (5.30), we have the $L^2$-orthogonal Helmholtz decomposition

$$\tau_h = \nabla^2 v_h + \tau_h^0,$$

and $\tau_h^0 + \nabla^2 W_h$ whose $L^2$-norm can be controlled by (5.31) $\|\tau_h^0\|_0 \lesssim \|\text{curl } \tau_h^0\|_0 = \|\text{curl } \tau_h\|_0$. The first part $\nabla^2 v_h$ is the $L^2$-projection of $\tau_h$ to $\nabla^2 W_h$ and thus

$$\|\nabla^2 v_h\|_0 = \sup_{w_h \in W_h/\mathbb{P}_1(\Omega)} \frac{(\nabla^2 v_h, \nabla^2 w_h)}{\|\nabla^2 w_h\|_0} = \sup_{w_h \in W_h/\mathbb{P}_1(\Omega)} \frac{(\tau_h, \nabla^2 w_h)}{\|\nabla^2 w_h\|_0}.$$  

Then we use Poincaré inequality

$$\|w_h\|_0 \lesssim \|\nabla^2 w_h\|_0 \quad \forall w_h \in W_h/\mathbb{P}_1(\Omega)$$

to finish the proof. \hfill $\square$

### 6. A Lower Order Non-conforming Discrete Hessian complex

In this section we shall construct a discrete Hessian complex:

$$\mathbb{P}_1(\Omega) \hookrightarrow W_h \xrightarrow{\nabla^2} \Sigma_h \xrightarrow{\text{curl}_h} V_h \xrightarrow{\text{div}} Q_h \rightarrow 0.$$  \hfill (6.1)

with fewer dimensions for $W_h$ and $\Sigma_h$. In (6.1), $W_h$ is an $H^2$-conforming virtual element spaces containing piecewise $\mathbb{P}_3$ polynomials. Spaces $V_h$ and $Q_h$ remain unchanged, i.e., $V_h$ is the $\mathbb{P}_2 H(\text{div}, T)$-conforming finite element space and $Q_h$ is discontinuous $\mathbb{P}_1$ polynomial. Only the space $\Sigma_h$ is non-conforming and the operator $\text{curl}_h$ is applied element-wise.
6.1. \textit{H$^2$-conforming virtual element.} Let $F \in \mathcal{F}(K)$ be a triangular face. Define \textit{H$^2$-conforming virtual element space} (cf. \cite{18})

\[ \tilde{W}^\Delta(F) := \{ v \in H^2(F) : \text{both } v|_{\partial F} \text{ and } \nabla_F v|_{\partial F} \text{ are continuous, } \Delta_F^2 v \in \mathbb{P}_0(F), \ v|_e \in \mathbb{P}_3(e), \partial_n v|_e \in \mathbb{P}_2(e) \text{ for each } e \in \mathcal{E}(F) \}. \]

The degrees of freedom for $\tilde{W}^\Delta(F)$ can be chosen as (cf. \cite{18})

\[ (6.2) \quad v(\delta), \nabla_F v(\delta) \quad \forall \delta \in \mathcal{V}(F), \]

\[ (6.3) \quad \int_e \partial_{n_F} v \, ds \quad \forall e \in \mathcal{E}(F), \]

\[ (6.4) \quad \int_F v \, ds. \]

The degrees of freedom (6.2)-(6.4) are unisolvent in $\tilde{W}^\Delta(F)$ and a proof can be found in \cite[Proposition 4.2]{18}.

We can remove the face average degrees of freedom (6.4) and consider (cf. \cite{18})

\[ W^\Delta(F) := \{ v \in \tilde{W}^\Delta(F) : Q^v_{\partial F} \Pi^\Delta_K v = Q^v_{\partial F} v \}. \]

where the \textit{H$^2$-projection} $\Pi^\Delta_K : H^2(F) \to \mathbb{P}_3(F)$ is determined by

\[ (\nabla_F^2 \Pi^\Delta_K v, \nabla_F^2 q)_F = (\nabla_F^2 v, \nabla_F^2 q)_F \quad \forall q \in \mathbb{P}_3(F), \]

\[ \Pi^\Delta_K v(\delta) = v(\delta) \quad \forall \delta \in \mathcal{V}(F). \]

Recall the Green’s formula derived in \cite{21}: for $\tau \in C^2(F; \mathbb{S})$ and $v \in H^2(F)$

\[ (\text{div}_F \text{div}_F \tau, v)_F = (\tau, \nabla^2_F v)_F - \sum_{e \in \mathcal{E}(F)} \sum_{\delta \in \partial e} \text{sign}_{e,\delta}(t^e_\tau \tau n_e)(\delta)v(\delta) \]

\[ - \sum_{e \in \mathcal{E}(F)} [(n_e^\tau \tau n_e, \partial_n v)_e - (\partial_i(t^e_\tau \tau n_e) + n^\tau_e \text{div}_F \tau, v)_e], \]

where

\[ \text{sign}_{e,\delta} := \begin{cases} 1, & \text{if } \delta \text{ is the end point of } e, \\ -1, & \text{if } \delta \text{ is the start point of } e. \end{cases} \]

The projection $\Pi^\Delta_K v$ and $Q^v_{\partial F} \nabla_F^2 v$ are computable using degrees of freedom (6.2)-(6.3) only. The degrees of freedom for $W^\Delta(F)$ can be thus chosen as (6.2)-(6.3).

By the definition of $W^\Delta(F)$, we have $\mathbb{P}_3(F) \subset W^\Delta(F)$.

We can directly define a \textit{H$^2$-conforming virtual element space} by bi-harmonic extension, i.e., $\Delta^2_F v = 0$. Following \cite{1}, the current definition enable us to compute the average $Q^F_{\partial F} v$ for $v \in W^\Delta(F)$ which will be used later cf. (6.8).

Now we define an \textit{H$^2$-conforming virtual element space} in three dimensions

\[ W(K) := \{ v \in H^2(K) : \text{both } v|_{\partial K} \text{ and } \nabla v|_{\partial K} \text{ are continuous, } \Delta^2 v = 0, \ v|_F \in W^\Delta(F), \partial_n v|_F \in \mathbb{P}_2(F) \text{ for each } F \in \mathcal{F}(K) \}. \]

Our space is slightly different with \cite{10}. The degrees of freedom are given by

\[ (6.6) \quad v(\delta), \nabla v(\delta) \quad \forall \delta \in \mathcal{V}(K), \]

\[ (6.7) \quad \int_e \partial_{n_e} v \, ds \quad \forall e \in \mathcal{E}(K), i = 1, 2. \]
It is straightforward to count the dimension of the degrees of freedom is 28. More precisely, we can follow Section 5.2 to identify $W(K)$ as $L(D(K))$ with $D(K) = \mathbb{P}_0(\mathcal{V}(K)) \times \mathbb{P}_0(\mathcal{E}(K)) : \mathbb{R}^2$ and conclude $\text{dim} W(K) = 28$.

Given $(v_0, v_1, v_2) \in D(K)$, we will determine an element $v \in W(K)$ as follows. First restrict to each edge $e$, we can use $(v_0, v_1 \cdot t_e)$ to determine a Lagrange element $v|_e \in \mathbb{P}_3(e)$. Similarly using $(v_1, v_2)$, we can compute a Lagrange element $\partial_{n_F} v|_e \in \mathbb{P}_2(e)$ and by the unisolvence of $W^\Delta(F)$, we can determine a function $v|_F \in W^\Delta(F)$. Second restrict to each face $F$, by expressing $n_F = c_1 n_1 + c_2 n_2$, we have $\int_e \partial_{n_F} v \, ds = (c_1, c_2) \cdot v_e$. Together with $v_1 \cdot n_F$ at vertices, we can determine a Lagrange element $\partial_{n} v|_F \in \mathbb{P}_2(F)$ as $F$ is a triangle. Finally with $v|_F$ and $\partial_{n} v|_F$, we use bi-harmonic extension to get the function in $W(K)$. The biharmonic equation with non-zero Dirichlet boundary condition is well-posed since although the vanishing degrees of freedom (6.7) that $(\nabla v)|_e = 0$. Then we obtain from the vanishing degrees of freedom (6.6) that $(\nabla v)|_e = 0$. Thus $v|_F \in H^\Delta(F)$ with all the degrees of freedom (6.2)-(6.4) vanishing. Thus $v|_F = 0$.

\begin{lemma}
For each $F \in \mathcal{F}(K)$ and any $v \in W(K)$, both $v|_F$ and $\partial_{n_F}v|_F$ are determined by the degrees of freedom (6.6)-(6.7) on the face $F$.
\end{lemma}

\begin{proof}
Assume all the degrees of freedom (6.6)-(6.7) on face $F$ are zeros. By the fact $\partial_{n_F}v|_F \in \mathbb{P}_2(F)$, we get from the vanishing degrees of freedom (6.6)-(6.7) that $\partial_{n_F}v|_F = 0$. Similarly it follows from the definition of $\Pi_2^K$ that $\Pi_2^K v|_F = 0$. Hence $v|_F \in W^\Delta(F)$ with all the degrees of freedom (6.2)-(6.4) vanishing. Thus $\Pi_2^K v = 0$.
\end{proof}

\begin{lemma}
The degrees of freedom (6.6)-(6.7) are unisolvent for $W(K)$.
\end{lemma}

\begin{proof}
Apparentaly the number of the degrees of freedom (6.6)-(6.7) is 28, which is same as $\text{dim} W(K)$. Now take any $v \in W(K)$ and suppose all the degrees of freedom (6.6)-(6.7) vanish. Since $v|_e \in \mathbb{P}_3(e)$ for each edge $e \in \mathcal{E}(K)$, it follows from the vanishing degrees of freedom (6.6) that $v|_e = 0$. Then we obtain from the vanishing degrees of freedom (6.7) that $(\nabla v)|_e = 0$. Thus $v|_F \in H^\Delta_0(F)$ and $\partial_{n} v|_F \in H^\Delta_0(F)$ for each face $F \in \mathcal{F}(K)$. By the definition of $W(K)$, $v \in H^\Delta_0(K)$. As a result $v = 0$.
\end{proof}

We now discuss what we can compute using degrees of freedom (6.6)-(6.7). Define the local projector $\Pi^K_2 : H^2(K) \rightarrow \mathbb{P}_3(K)$ as follows

$$\langle \nabla^2 \Pi^K_2 v, \nabla^2 q \rangle_K = \langle \nabla^2 v, \nabla^2 q \rangle_K \quad \forall \, q \in \mathbb{P}_3(K),$$

$$\langle \Pi^K_2 v, \delta \rangle_K = \langle v, \delta \rangle_K \quad \forall \, \delta \in \mathcal{V}(K).$$

Apparently we have $\Pi^K_2 q = q$ for any $q \in \mathbb{P}_3(K)$, and

$$\|\Pi^K_2 v\|_{2,K} \leq \|v\|_{2,K} \quad \forall \, v \in H^2(K).$$
For any $v \in H^2(K)$ and $\tau \in \mathbb{P}_1(K; \mathbb{S})$, it follows from the integration by parts that
\[
(\nabla^2 v, \tau)_K = (\nabla v, \tau n)_{\partial K} - (\nabla v, \text{div}\tau)_K
\]
\[
= (\nabla v, \tau n)_{\partial K} + \sum_{F \in \mathcal{F}(K)} (\nabla_F v, \tau n)_F - (v, n^T \text{div}\tau)_{\partial K}
\]
\[
= (\nabla v, \tau n)_{\partial K} + \sum_{F \in \mathcal{F}(K)} \sum_{e \in \mathcal{E}(F)} (v, n_e \tau n)_e
\]
\[
- \sum_{F \in \mathcal{F}(K)} (v, \text{div}_F(\tau n) + n^T \text{div}\tau)_F
\]
\[
= (\nabla v, \tau n)_{\partial K} + \sum_{F \in \mathcal{F}(K)} \sum_{e \in \mathcal{E}(F)} (v, n_e \tau n)_e
\]
\[
- \sum_{F \in \mathcal{F}(K)} (Q_0^F v, \text{div}_F(\tau n) + n^T \text{div}\tau)_F.
\]
(6.8)

Hence both $\Pi_K^2 v$ and $Q_0^K (\nabla^2 v)$ are computable by using the degrees of freedom (6.6)-(6.7) for $v \in W(K)$.

We define the global virtual element space
\[
W_h := \{ v_h \in L^2(\Omega) : v_h|_K \in W(K) \quad \forall K \in \mathcal{T}_h, \text{ the degrees of freedom } (6.9)-(6.12) \text{ are single-valued} \}.
\]

We have $W_h \subset H^2(\Omega)$ due to Lemma 6.1.

6.2. $H(\text{curl})$-nonconforming element for symmetric tensors. We take the space of shape functions
\[
\Sigma(K) := \nabla^2 W(K) \oplus \text{sym}(x \times \mathbb{P}_2(K; \mathbb{T})).
\]

As $\mathbb{P}_3(K; \mathbb{S}) \subset W(K)$ and the decomposition (4.15), we have $\mathbb{P}_1(K; \mathbb{S}) \subset \Sigma(K)$ and
\[
\dim \Sigma(K) = \dim W(K) - 4 + \dim \text{sym}(x \times \mathbb{P}_2(K; \mathbb{T})) = 24 + 68 = 92.
\]

The degrees of freedom are given by
\[
\text{curl} \tau(\delta) \quad \forall \delta \in \mathcal{V}(K),
\]
\[
(\tau t, q)_e \quad \forall q \in \mathbb{P}_1(e; \mathbb{R}^3), e \in \mathcal{E}(K),
\]
\[
\int_F n \times \tau \, ds \quad \forall F \in \mathcal{F}(K),
\]
\[
(\text{curl} \tau, q)_K \quad \forall q \in \mathbb{P}_2(K; \mathbb{T}) \cap \ker(\text{div}).
\]

Apparently (6.9) and (6.12) are motivated by the degrees of freedom (5.3) and (5.5) for $H(\text{div}, \Omega; \mathbb{T})$ element.

Lemma 6.3. The degrees of freedom (6.9)-(6.12) are unisolvent for $\Sigma(K)$.

Proof. The number of the degrees of freedom (6.9)-(6.12) is
\[
32 + 36 + 20 + 4 = 92,
\]
which agrees with $\dim \Sigma(K)$.

Take any $\tau \in \Sigma(K)$ and suppose all the degrees of freedom (6.9)-(6.12) vanish. Applying the integration by parts on each face $F \in \mathcal{F}(K)$, for any $q \in \mathbb{P}_1(F; \mathbb{R}^3)$, we get from the vanishing degrees of freedom (6.10) that
\[
((\text{curl} \tau)n, q)_F = ((n \times \nabla) \cdot \tau, q)_F = -(\text{div}_F(n \times \tau), q)_F = 0.
\]
And similarly it follows
\[(\text{curl } \tau, q)_K = 0 \quad \forall \ q \in \mathbb{P}_0(K; \mathbb{T}).\]

Due to Lemma 5.2, we obtain from the vanishing degrees of freedom (6.9) and (6.12), and the fact \(\text{curl } \tau \in \mathbb{P}_2(K; \mathbb{T})\) that \(\text{curl } \tau = 0\). Hence there exists \(v \in W(K)\) such that \(\tau = \nabla^2 v\). Then we acquire from (6.10) that

\[(6.13) \quad (\partial_i (\nabla v), q)_e = 0, \quad \forall \ q \in \mathbb{P}_1(e; \mathbb{R}^3), e \in \mathcal{E}(K).\]

Since \(v|_e \in \mathbb{P}_3(e)\) for each edge \(e \in \mathcal{E}(K)\), it follows from (6.13) that \(v|_e \in \mathbb{P}_1(e)\). Hence we can choose \(v \in W(K)\) such that \(v|_e = 0\). Then \(\nabla v(\delta) = 0\) for each vertex \(\delta \in \mathcal{V}(K)\). Applying the integration by parts to (6.13), we achieve

\[(\nabla v, q)_e = 0, \quad \forall \ q \in \mathbb{P}_0(e; \mathbb{R}^3), e \in \mathcal{E}(K),\]

which indicates \(\nabla v|_e = 0\). Therefore \(v = 0\) follows from Lemma 6.2. \(\square\)

Define the global finite element spaces
\[\Sigma_h := \{\tau_h \in L^2(\Omega; \mathbb{S}) : \tau_h|_K \in \Sigma(K) \quad \forall \ K \in \mathcal{T}_h, \text{ the degrees of freedom } (6.9)-(6.11) \text{ are single-valued}\},\]

Space \(\Sigma_h\) is not \(H(\text{curl}, \Omega; \mathbb{S})\)-conforming but \(\text{curl}_h \Sigma_h\) is \(H(\text{div}, \Omega; \mathbb{T})\)-conforming. The edge and face degrees of freedom (6.10)-(6.11) are not enough to determine \(\mathbf{n} \times \tau\) which contains a polynomial in \(\mathbb{P}_3(F; \mathbb{R}^3)\). This is partially due to the missing of degrees of freedom \(\tau(\delta)\). On the other hand, the dimension of the local space is reduced from 132 to 92.

Now we illustrate \(Q^K_h \tau\) of an element \(\tau \in \Sigma(K)\) is computable by using the degrees of freedom (6.9)-(6.12). Similarly as the conforming one in last section, \(\text{curl } \tau \in \mathbb{P}_2(K; \mathbb{T})\) is computable. Since \(\text{curl } : \text{sym}(\mathbf{x} \times \mathbb{P}_2(K; \mathbb{T})) \rightarrow \mathbb{P}_3(K; \mathbb{S})\) is bijective, we can find a unique polynomial \(\tau_1 \in \text{sym}(\mathbf{x} \times \mathbb{P}_2(K; \mathbb{T}))\) such that \(\text{curl } \tau_1 = \text{curl } \tau\). Then there exists a \(w \in W(K)\), and \(\tau = \tau_1 + \nabla^2 w\).

To determine \(w\), by (6.6)-(6.7), it suffices to know \(w(\delta), \nabla w|_e\) using degrees of freedom for \(\tau\). First of all, we are free to add \(\mathbb{P}_1(K)\) polynomial to \(w\) and thus can choose \(w\) with property \(w(\delta) = 0\) for all \(\delta \in \mathcal{V}(K)\). Then \(\partial_\mathcal{E} w = Q_1^K(\tau t) - Q_1^K(\tau_1 t)\) is computable on each edge \(e \in \mathcal{E}(K)\), which implies \(w|_e \in \mathbb{P}_3(e)\) is computable using function value \(w\) and second order derivative \(\partial_\mathcal{E} w\) at vertices. Then

\[\nabla w(\delta) = \frac{n_1}{t_1 \cdot n_1}(\partial_{t_1} w)(\delta) + \frac{n_2}{t_2 \cdot n_2}(\partial_{t_2} w)(\delta) + \frac{n_3}{t_3 \cdot n_3}(\partial_{t_3} w)(\delta)\]

is determined, where \(t_1, t_2, t_3\) are the unit tangential vector of edges sharing vertex \(\delta\), \(n_i = t_j \times t_k\) with \((ijk)\) being a permutation cycle of (123). Noting that \(w \in W(K) \subset H^2(K)\), the vector field \(\nabla w|_{\mathcal{E}(K)}\) is continuous and piecewise quadratic. Hence \(\nabla w|_{\mathcal{E}(K)}\) is computable. And \(\partial_\mathcal{F} w|_F \in \mathbb{P}_2(F)\) is determined by \(\partial_\mathcal{F} w|_{\partial F}\) for each face \(F \in \mathcal{F}(K)\). By the Green’s identity (6.8), \(Q^K_1(\nabla w)\) is computable, and then \(Q^K_1 \tau = Q^K_1(\tau - \tau_1) + Q^K_2(\nabla^2 w)\) is computable.

We then compute polynomial projections of the tangential trace \(\mathbf{n} \times \tau\) on each face. As \(\tau_1\) is a polynomial, it suffices to compute \(\Pi_F \nabla^2 w\). Using (5.24), the tangential-normal part is \(\nabla_F (\partial_\mathcal{F} w|_F)\) which is a \(\mathbb{P}_1\) polynomial. The tangential-tangential component \(\nabla_F^2 (\tau|_F)\) is non-polynomial but its projection \(Q_F^1 \nabla_F^2 (\tau|_F)\) is computable using (6.5).
6.3. **Discrete nonconforming Hessian complex.** We now construct a discrete Hessian complex in three dimensions.

**Lemma 6.4.** Assume $\Omega$ is a contractible domain. Then we have the discrete Hessian complex

\[
\begin{array}{cccccc}
P_1(\Omega) & \subset & W_h & \nabla^2 & \nabla \tau & \nabla \psi & \div \nabla \psi & Q_h & \rightarrow & 0.
\end{array}
\]

**Proof.** It is easy to see that (6.14) is a complex and $W_h \cap \ker(\nabla^2) = P_1(\Omega)$. Then

\[
\dim \nabla^2 W_h = \dim W_h - 4 = 4\#V_h + 2\#E_h - 4.
\]

For any $\psi_h \in V_h \cap \ker(\div)$, by (3.3) there exists $\tau \in H^1(\Omega; \mathbb{S})$ such that $\psi_h = \nabla \tau$. Then it follows from (5.27) that

\[
\nabla \psi_h(\nabla \tau) = \nabla \tau = \psi_h,
\]

i.e. $V_h \cap \ker(\div) = \nabla \Sigma_h$. And we have

\[
\dim \nabla \Sigma_h = \dim V_h - \dim Q_h = 8\#V_h + 9\#F_h,
\]

\[
\dim \Sigma_h \cap \ker(\nabla \Sigma_h) = \dim \Sigma_h - \dim \nabla \Sigma_h = 6\#E_h - 4\#F_h + 4\#T_h.
\]

Hence we acquire from the Euler's formula that

\[
\dim \Sigma_h \cap \ker(\nabla \Sigma_h) - \dim \nabla^2 W_h = 4(\#T_h - \#F_h + \#E_h - \#V_h + 1) = 0,
\]

which yields $\Sigma_h \cap \ker(\nabla \Sigma_h) = \nabla^2 W_h$. \qed

According Remark (5.3), the last two spaces in (6.14) can further reduced to $V^+(K)$ and $RT(K)$.

### 7. Discretizations for the Linearized Einstein-Bianchi system

In this section we will apply the constructed conforming and non-conforming virtual Hessian complex to discretize the time-independent linearized Einstein-Bianchi system.

**7.1. Linearized Einstein-Bianchi system.** Consider the time-independent linearized Einstein-Bianchi system [39]: find $\sigma \in H^2(\Omega)$, $E \in H(\nabla \psi, \Omega; \mathbb{S})$ and $B \in L^2(\Omega; \mathbb{T})$ such that

\[
\begin{align*}
(\sigma, \tau) - (E, \nabla^2 \tau) &= 0 \quad \forall \, \tau \in H^2(\Omega), \\
(\nabla^2 \sigma, v) + (B, \nabla \psi) &= (f, v) \quad \forall \, v \in H(\nabla \psi, \Omega; \mathbb{S}), \\
(B, \psi) - (\nabla \psi - \nabla \psi) &= 0 \quad \forall \, \psi \in L^2(\Omega; \mathbb{T}),
\end{align*}
\]

where $f \in L^2(\Omega; \mathbb{S})$. Here following [30, 39] we switch the notation and use $\sigma, \tau$ for functions in $H^2$ and $E, v$ for functions in $H(\nabla \psi, \Omega; \mathbb{S})$.

To show the well-posedness of the linearized Einstein-Bianchi system (7.1)-(7.3), we introduce the product space

\[
\mathcal{X} = H^2(\Omega) \times H(\nabla \psi, \Omega; \mathbb{S}) \times L^2(\Omega; \mathbb{T})
\]

and the bilinear form $A(\cdot, \cdot) : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$ as

\[
A(\sigma, E, B; \tau, v, \psi) := (\sigma, \tau) - (E, \nabla^2 \tau) - (\nabla^2 \sigma, v) - (B, \psi) - (\nabla \psi, \psi).
\]

It is easy to prove the continuity

\[
A(\sigma, E, B; \tau, v, \psi) \lesssim (\|\sigma\|_2 + \|E\|_{H(\nabla \psi)} + \|B\|_0)(\|\tau\|_2 + \|v\|_{H(\nabla \psi)} + \|\psi\|_0)
\]

and the inf-sup condition

\[
\inf_{\psi} \sup_{(\sigma, E, B)} A(\sigma, E, B; \tau, v, \psi) > 0
\]
for any $\sigma, \tau \in H^2(\Omega)$, $E, v \in H(\text{curl}, \Omega; S)$ and $B, \psi \in L^2(\Omega; T)$. The well-posedness of (7.1)-(7.3) is then derived from the following inf-sup condition.

Lemma 7.1. For any $\sigma \in H^2(\Omega)$, $E \in H(\text{curl}, \Omega; S)$ and $B \in L^2(\Omega; T)$, it holds

\begin{equation}
\|\sigma\|_2 + \|E\|_{H(\text{curl})} + \|B\|_0 \lesssim \sup_{(\tau, v, \psi) \in X} \frac{A(\sigma, E, B; \tau, v, \psi)}{\|\tau\|_2 + \|v\|_{H(\text{curl})} + \|\psi\|_0}.
\end{equation}

Proof. For ease of presentation, let

$$\alpha = \sup_{(\tau, v, \psi) \in X} \frac{A(\sigma, E, B; \tau, v, \psi)}{\|\tau\|_2 + \|v\|_{H(\text{curl})} + \|\psi\|_0}.$$ 

Then it follows from the Poincaré inequality that

$$\|E\|_0 \lesssim \|\text{curl}E\|_0 + \sup_{\tau \in H^2(\Omega)} \frac{(E, \nabla^2\tau)}{\|\tau\|_2} \leq \|\text{curl}E\|_0 + \|\sigma\|_0 + \sup_{\tau \in H^2(\Omega)} \frac{(E, \nabla^2\tau) - (\sigma, \tau)}{\|\tau\|_2} \leq \|\text{curl}E\|_0 + \|\sigma\|_0 + \alpha.$$ (7.6)

On the other side, we have

$$A(\sigma, E, B; \sigma, -E - \nabla^2\sigma, \frac{1}{2}(B - \text{curl}E)) = \|\sigma\|_0^2 + \|\sigma\|_0^2 + \frac{1}{2}\|B\|_0^2 + \frac{1}{2}\|\text{curl} E\|_0^2.$$ 

Hence we get from the definition of $\alpha$ and (7.6) that

$$\|\sigma\|_0^2 + \|\sigma\|_0^2 + \frac{1}{2}\|B\|_0^2 + \frac{1}{2}\|\text{curl} E\|_0^2$$

$$\leq \alpha(\|\sigma\|_0^2 + \|\text{curl}E\|_0^2 + \frac{1}{2}\|B - \text{curl}E\|_0^2)$$

$$\lesssim \alpha(\|\sigma\|_0^2 + \|\text{curl}E\|_0^2 + \|B\|_0^2)$$

$$\lesssim \alpha(\|\sigma\|_0^2 + \|\text{curl}E\|_0^2 + \|B\|_0^2 + \alpha^2),$$

which yields

$$\|\sigma\|_0^2 + \|\text{curl}E\|_0^2 + \|B\|_0^2 \lesssim \alpha.$$ 

Finally the inf-sup condition (7.5) follows from the last inequality and (7.6). \qed

As a result of (7.4) and the inf-sup condition (7.5), the variational formulation (7.1)-(7.3) of the linearized Einstein-Bianchi system is well-posed, and

$$\|\sigma\|_2 + \|E\|_{H(\text{curl})} + \|B\|_0 \lesssim \|f\|_{(H(\text{curl}),\Omega;S)}.$$ 

It follows from (7.3) that $B = \text{curl}E$, so the linearized Einstein-Bianchi system (7.1)-(7.3) is equivalent to find $E \in H(\text{curl}, \Omega; S)$ and $\sigma \in H^2(\Omega)$ such that

\begin{align}
\begin{split}
a(E, v) + b(v, \nabla^2\sigma) &= (f, v) \quad \forall \ v \in H(\text{curl}, \Omega; S), \quad \text{(7.7)} \\
b(E, \nabla^2\tau) - c(\sigma, \tau) &= 0 \quad \forall \ \tau \in H^2(\Omega), \quad \text{(7.8)}
\end{split}
\end{align}

where

$$a(E, v) = (\text{curl}E, \text{curl}v), \quad b(E, \nabla^2\tau) = (E, \nabla^2\tau), \quad c(\sigma, \tau) = (\sigma, \tau).$$

Then the inf-sup condition (7.5) is equivalent to

\begin{equation}
\|\sigma\|_2 + \|E\|_{H(\text{curl})} \lesssim \sup_{\tau \in H^2(\Omega)} \frac{a(E, v) + b(v, \nabla^2\sigma) + b(E, \nabla^2\tau) - c(\sigma, \tau)}{\|\tau\|_2 + \|v\|_{H(\text{curl})}}.
\end{equation}
for any $\sigma \in H^2(\Omega)$ and $E \in H(\text{curl}, \Omega; \mathbb{S})$.

7.2. Conforming Discretization. With conforming subspaces $W_h$ and $\Sigma_h$, we could directly consider the Galerkin approximation of (7.7)-(7.8). However, as pointwise information of functions in virtual element spaces are not available, the $L^2$-inner product $(\cdot, \cdot)$ involved in $b(\cdot, \cdot)$ and $c(\cdot, \cdot)$ are not computable.

We will replace them by equivalent and accurate approximations which can be thought of as numerical quadrature. First introduce two stabilizations

$$S^0_K(\sigma, \tau) := h_K(\sigma, \tau)_{\partial K} + h_K^1(\partial_{\nu} \sigma, \partial_{\nu} \tau)_{\partial K},$$

$$S^1_K(E, v) := h_K^2(\text{curl} E, \text{curl} v)_K + h_K(n \times E, n \times v)_{\partial K},$$

which are computable as all integrands are polynomials.

**Lemma 7.2.** For each $K \in T_h$, we have

$$S^0_K(\tau, \tau) \approx \|\tau\|^2_{0, K} \quad \forall \tau \in W(K) \cap \ker(Q^K_1),$$

$$S^1_K(v, v) \approx \|v\|^2_{0, K} \quad \forall v \in \Sigma(K) \cap \ker(Q^K_3).$$

**Proof.** By the norm equivalence on the finite dimensional spaces and the scaling argument, it is sufficient to prove $S^0_K(\cdot, \cdot)$ and $S^1_K(\cdot, \cdot)$ are squared norms for the spaces $W(K) \cap \ker(Q^K_1)$ and $\Sigma(K) \cap \ker(Q^K_3)$, respectively.

Assume $\tau \in W(K) \cap \ker(Q^K_1)$ and $S^0_K(\tau, \tau) = 0$. Then $\tau \in H^0_0(K)$. By the integration by parts and the definition of $W(K)$, it follows

$$\|\nabla^2 \tau\|^2_{0, K} = (\Delta^2 \tau, \tau)_K = (\Delta^2 \tau, Q^K_1 \tau)_K = 0,$$

which results in $\tau = 0$. Hence $S^0_K(\cdot, \cdot)$ is a squared norm for the space $W(K) \cap \ker(Q^K_1)$.

Assume $v \in \Sigma(K) \cap \ker(Q^K_3)$ and $S^1_K(v, v) = 0$. Apparently $v \in H^0(\text{curl}, \Omega; \mathbb{S}) \cap \ker(\text{curl})$. Then there exists $w \in W(K) \cap H^0_0(K)$ satisfying $v = \nabla^2 w$. Since $v \in \ker(Q^K_3)$, we get

$$(w, \text{div} \text{div} q)_K = (\nabla^2 w, q)_K = 0 \quad \forall q \in P_3(K; \mathbb{S}),$$

which together with complex (4.18) implies

$$(w, q)_K = 0 \quad \forall q \in P_1(K).$$

Therefore $w = 0$ and $v = 0$. 

With these two stabilizations, define local bilinear forms

$$b_K(E, v) := (Q^K_3 E, Q^K_3 v)_K + S^1_K(E - Q^K_3 E, v - Q^K_3 v),$$

$$c_K(\sigma, \tau) := (\Pi^K \sigma, \Pi^K \tau)_K + S^0_K(\sigma - \Pi^K \sigma, \tau - \Pi^K \tau).$$

It is obvious that

$$b_K(E, q) = (E, q)_K \quad \forall E \in H^1(K; \mathbb{S}) \cup \Sigma(K), q \in P_3(K; \mathbb{S}),$$

$$c_K(\sigma, q) = (\Pi^K \sigma, q)_K \quad \forall \sigma \in H^2(K) \cap W(K), q \in P_3(K).$$

And we obtain from (7.10) and (7.11) that

$$b_K(v, v) \approx \|v\|^2_{0, K} \quad \forall v \in \Sigma(K),$$

$$c_K(\tau, \tau) \approx \|\tau\|^2_{0, K} \quad \forall \tau \in W(K).$$
Then we have from the Cauchy-Schwarz inequality that

\begin{align}
    b_h(E, v) & \lesssim \|E\|_{0,K} \|v\|_{0,K} \quad \forall \ E, v \in \Sigma(K), \\
    c_K(\sigma, \tau) & \lesssim \|\sigma\|_{0,K} \|\tau\|_{0,K} \quad \forall \ \sigma, \tau \in W(K).
\end{align}

We propose the following conforming mixed virtual element method for the variational formulation (7.7)-(7.8): find \( E_h \in \Sigma_h \) and \( \sigma_h \in W_h \) such that

\begin{align}
    a(E_h, v_h) + b_h(v_h, \nabla^2 \sigma_h) &= (f, Q_h v_h) \quad \forall \ v_h \in \Sigma_h, \\
    b_h(E_h, \nabla^2 \tau_h) - c_h(\sigma_h, \tau_h) &= 0 \quad \forall \ \tau_h \in W_h,
\end{align}

where \( Q_h v_h \in L^2(\Omega; S) \) is given by \((Q_h v_h)|_K := Q^K_h(v_h|_K)\) for each \( K \in T_h \) and

\[
    b_h(E_h, \nabla^2 \tau_h) := \sum_{K \in T_h} b_K(E_h|_K, \nabla^2 \tau_h|_K), \quad c_h(\sigma_h, \tau_h) := \sum_{K \in T_h} c_K(\sigma_h|_K, \tau_h|_K).
\]

For any \( E_h, v_h \in \Sigma_h \) and \( \sigma_h, \tau_h \in W_h \), it follows from (7.16)-(7.17) that

\[
    A_h(E_h, \sigma_h; v_h, \tau_h) \leq (\|E_h\|_{H(curl)} + \|\sigma_h\|_2)(\|v_h\|_{H(curl)} + \|\tau_h\|_2),
\]

where

\[
    A_h(E_h, \sigma_h; v_h, \tau_h) := a(E_h, v_h) + b_h(v_h, \nabla^2 \sigma_h) + b_h(E_h, \nabla^2 \tau_h) - c_h(\sigma_h, \tau_h).
\]

**Lemma 7.3.** Assume \( \Omega \) is a topologically trivial domain. For any \( v_h \in \Sigma_h \) satisfying

\[
    b_h(v_h, \nabla^2 \tau_h) = 0 \quad \forall \ \tau_h \in W_h,
\]

it holds the discrete Poincaré inequality

\[
    \|v_h\|_0 \lesssim \|\text{curl} v_h\|_0.
\]

Consequently

\[
    \|v_h\|_0 \lesssim \|\text{curl} v_h\|_0 + \sup_{\tau_h \in W_h} \frac{b_h(v_h, \nabla^2 \tau_h)}{\|\tau_h\|_2}.
\]

**Proof.** Due to the proof of Lemma 5.11, there exists \( v \in H^1(\Omega; S) \) such that \( v_h - I_h v \in \nabla^2 W_h \), and

\[
    \text{curl} v = \text{curl} v_h, \quad \|v\|_1 \lesssim \|\text{curl} v_h\|_0.
\]

Hence we obtain from (7.14), (7.16) and (5.28) that

\[
    \|v_h\|_0^2 = b_h(v_h, v_h) = b_h(v_h, I_h v) \lesssim \|v_h\|_0 \|I_h v\|_0 \lesssim \|v_h\|_0 \|v\|_1,
\]

which means (7.21). Applying the similar argument as in proof of Lemma 5.11, (7.22) follows from (7.21).

We then prove the discrete inf-sup condition.

**Lemma 7.4.** For any \( E_h \in \Sigma_h \) and \( \sigma_h \in W_h \), it holds

\[
    \|E_h\|_{H(curl)} + \|\sigma_h\|_2 \lesssim \sup_{\tau_h \in W_h} \frac{A_h(E_h, \sigma_h; v_h, \tau_h)}{\|v_h\|_{H(curl)} + \|\tau_h\|_2}.
\]
Proof. For ease of presentation, let
\[
\alpha = \sup_{v_h \in \mathcal{S}_h, \tau_h \in W_h} \frac{A_h(E_h, \sigma_h; v_h, \tau_h)}{\|v_h\|_{H(\text{curl})} + \|\tau_h\|_2}.
\]
Since
\[
\sup_{\tau_h \in W_h} \frac{b_h(E_h, \nabla^2 \tau_h)}{\|\tau_h\|_2} = \sup_{\tau_h \in W_h} \frac{b_h(E_h, \nabla^2 \tau_h) - c_h(\sigma_h, \tau_h) + c_h(\sigma_h, \tau_h)}{\|\tau_h\|_2} \leq \|\sigma_h\|_0 + \alpha,
\]
we get from the discrete Poincaré inequality (7.22) that
\[
(7.24) \quad \|E_h\|_0 \lesssim \|\text{curl} E_h\|_0 + \|\sigma_h\|_0 + \alpha.
\]
On the other side, by the fact \(a(E_h, \nabla^2 \sigma_h) = 0\) we have
\[
A_h(E_h, \sigma_h; E_h + \nabla^2 \sigma_h, -\sigma_h) = a(E_h, E_h) + b_h(\nabla^2 \sigma_h, \nabla^2 \sigma_h) + c_h(\sigma_h, \sigma_h),
\]
which combined with (7.14)-(7.15) implies
\[
\|\sigma_h\|_2^2 + \|\text{curl} E_h\|_0^2 \lesssim \alpha(\|\sigma_h\|_2 + \|E_h + \nabla^2 \sigma_h\|_{H(\text{curl})}) \lesssim \alpha(\|\sigma_h\|_2 + \|E_h\|_{H(\text{curl})}).
\]
Hence
\[
\|\sigma_h\|_2^2 + \|\text{curl} E_h\|_0^2 \lesssim \alpha^2 + \alpha \|E_h\|_0.
\]
Finally combining the last inequality and (7.24) gives (7.23).

From now on we always denote by \(E_h \in \Sigma_h\) and \(\sigma_h \in W_h\) the solution of the mixed method (7.18)-(7.19).

Lemma 7.5. Assume \(E \in H^4(\Omega; \mathbb{S})\) and \(\sigma \in H^5(\Omega)\). Then
\[
(7.25) \quad b_h(v_h, \nabla^2 I_h^\alpha \sigma) - b(v_h, \nabla^2 \sigma) \lesssim h^3 \|v_h\|_0 \|\sigma\|_5,
\]
\[
(7.26) \quad b_h(I_h^\alpha E, \nabla^2 \tau_h) - b(E, \nabla^2 \tau_h) \lesssim h^4 \|E\|_{1,\tau_h} \|\tau_h\|_2,
\]
\[
(7.27) \quad (\sigma, \tau_h) - c_h(I_h^\alpha, \sigma, \tau_h) \lesssim h^4 \|\sigma\|_4 \|\tau_h\|_2.
\]

Proof. For each \(K \in \mathcal{T}_h\), we acquire from (7.12), (7.16) and (5.16) that
\[
b_K(v_h, \nabla^2 I_h^\alpha \sigma) - (v_h, \nabla^2 \sigma)_K
\]
\[
= b_K(v_h, \nabla^2 I_h^\alpha \sigma - Q_3^K(\nabla^2 \sigma)) - (v_h, \nabla^2 \sigma - Q_3^K(\nabla^2 \sigma))_K
\]
\[
\lesssim \|v_h\|_0, K \|\nabla^2 I_h^\alpha \sigma - Q_3^K(\nabla^2 \sigma)\|_{0, K} + \|v_h\|_0, K \|\nabla^2 \sigma - Q_3^K(\nabla^2 \sigma)\|_{0, K}
\]
\[
\lesssim \|v_h\|_0, K (\|\sigma - I_h^\alpha \sigma\|_{2, K} + \|\nabla^2 \sigma - Q_3^K(\nabla^2 \sigma)\|_{0, K}) \lesssim h^2 \|v_h\|_0, K \|\sigma\|_{5, K}.
\]
Thus
\[
b_h(v_h, \nabla^2 I_h^\alpha \sigma_h) - b(v_h, \nabla^2 \sigma) = \sum_{K \in \mathcal{T}_h} \left( b_K(v_h, \nabla^2 I_h^\alpha \sigma_h) - (v_h, \nabla^2 \sigma)_K \right)
\]
\[
\lesssim h^4 \|v_h\|_0 \|\sigma\|_5,
\]
i.e. (7.25).

Similarly it holds from (7.12), (7.16) and (5.26) that
\[
b_K(I_h^\alpha E, \nabla^2 \tau_h) - (E, \nabla^2 \tau_h)_K = b_K(I_h^\alpha E - Q_3^K E, \nabla^2 \tau_h) - (E - Q_3^K E, \nabla^2 \tau_h)_K
\]
\[
\lesssim \|I_h^\alpha E - Q_3^K E\|_{0, K} + \|E - Q_3^K E\|_{0, K} \|\tau_h\|_2, K
\]
\[
\lesssim h^4 \|E\|_{4, K} \|\tau_h\|_2, K.
\]
which yields (7.26).

Employing (7.13), (7.17), (5.15) and (5.16), we get

\[
\sigma_h - c_K (I_h \sigma^\Delta, \tau_h) = (\sigma - \Pi^K \sigma, \tau_h)_K - c_K (I_h \sigma^\Delta - \Pi^K \sigma, \tau_h)_K + (\Pi^K \sigma, \tau_h - \Pi^K \tau_h)_K
\]

\[
= (\sigma - \Pi^K \sigma, \tau_h)_K - c_K (I_h \sigma^\Delta - \Pi^K \sigma, \tau_h)_K + (\Pi^K \sigma - Q^K \Pi^K \sigma, \tau_h - \Pi^K \tau_h)_K
\]

\[
\lesssim (\|\sigma - \Pi^K \sigma\|_{0,K} + \|I_h \sigma - \Pi^K \sigma\|_{0,K}) \|\tau_h\|_{0,K}
\]

\[
+ \|\Pi^K \sigma - Q^K \Pi^K \sigma\|_{0,K} \|\tau_h - \Pi^K \tau_h\|_{0,K}
\]

\[
\lesssim (\|\sigma - \Pi^K \sigma\|_{0,K} + \|\tau_h\|_{0,K} + h^2 K^4 |\sigma|_{2,K} |\tau_h|_{2,K} + h^2 K^4 |\tau_h|_{2,K} + h^2 K^4 |\tau_h|_{2,K} + h^4 |\tau_h|_{2,K}).
\]

Therefore (7.27) is true.

\[\square\]

**Theorem 7.6.** Let \( E_h \in \Sigma_h \) and \( \sigma_h \in \Sigma_h \) be the solution of the mixed method (7.18)-(7.19) and let \( E \) and \( \sigma \) be the solution of (7.7)-(7.8). Assume \( E \in H^4(\Omega; \mathbb{S}) \), \( \sigma \in H^3(\Omega) \) and \( f \in H^3(\Omega; \mathbb{S}) \). We have

\[
\|E - E_h\|_{H(\text{curl})} + \|\sigma - \sigma_h\|_2 \lesssim h^3 (|E|_4 + \|\sigma\|_5 + |f|_3).
\]

**Proof.** Take any \( v_h \in \Sigma_h \) and \( \tau_h \in \Sigma_h \). We get from the variational formulation (7.7)-(7.8) and estimates (7.25)-(7.27) that

\[
A_h(I_h \sigma^\Delta; v_h, \tau_h) - (f, v_h)
\]

\[
= a(I_h \sigma - E, v_h) + b_h(v_h, \nabla^2 I_h \sigma) - b(v_h, \nabla^2 \sigma)
\]

\[
+ b_h(I_h \sigma, \nabla^2 \tau_h) - b(E, \nabla^2 \tau_h) + (\sigma, \tau_h) - c_h(I_h \sigma, \tau_h)
\]

\[
\lesssim h^3 |v|_4 \|\text{curl} v_h\|_0 + h^3 \|v_h\|_0 |\sigma|_5 + h^4 |E|_4 |\tau_h|_2 + h^4 \|\sigma\|_4 |\tau_h|_2.
\]

Since

\[
(f, v_h - Q_h f, v_h) = (f - Q_h f, v_h) \leq \|f - Q_h f\|_0 \|v_h\|_0 \lesssim h^3 |f|_3 \|v_h\|_0,
\]

we achieve from the mixed method (7.18)-(7.19) that

\[
A_h(I_h \sigma^\Delta; v_h, \tau_h)
\]

\[
= A_h(I_h \sigma^\Delta; v_h, \tau_h) - (f, Q_h v_h)
\]

\[
\lesssim h^3 |E|_4 \|\text{curl} v_h\|_0 + h^3 (|\sigma|_5 + |f|_3) \|v_h\|_0 + h^4 |E|_4 |\tau_h|_2 + h^4 \|\sigma\|_4 |\tau_h|_2.
\]

Now it follows from the inf-sup condition (7.23) that

\[
\|I_h \sigma^\Delta - E_h\|_{H(\text{curl})} + \|I_h \sigma - \sigma_h\|_2 \lesssim \sup_{\tau_h \in \Sigma_h} \frac{A_h(I_h \sigma^\Delta - E_h, I_h \sigma - \sigma_h; v_h, \tau_h)}{\|v_h\|_{H(\text{curl})} + \|\tau_h\|_2}
\]

\[
\lesssim h^3 (|E|_4 + \|\sigma\|_5 + |f|_3).
\]

Thus we acquire from (5.26) and (5.16) that

\[
\|E - E_h\|_{H(\text{curl})} + \|\sigma - \sigma_h\|_2
\]

\[
\leq \|E - I_h \sigma^\Delta\|_{H(\text{curl})} + \|\sigma - I_h \sigma\|_2 + \|I_h \sigma - E_h\|_{H(\text{curl})} + \|I_h \sigma - \sigma_h\|_2
\]

\[
\lesssim h^3 (|E|_4 + \|\sigma\|_5 + |f|_3),
\]

as required. \[\square\]
7.3. Nonconforming discretization. In this sub-section we will apply the constructed nonconforming virtual element Hessian complex in Section 6 to discretize the linearized Einstein-Bianchi system. Throughout this subsection, $W(K), W_h$ and $\Sigma(K), \Sigma_h$ are spaces constructed in Section 6. On each element, the dimension of $(W(K), \Sigma(K))$ is (28, 92) which is more tractable for the implementation.

First introduce two stabilizations

\[ S_K^0(\sigma, \tau) := h_K^2 \sum_{e \in \mathcal{E}(K)} (\sigma, \tau)_e + h_K^4 \sum_{e \in \mathcal{E}(K)} (\nabla \sigma, \nabla \tau)_e, \]

\[ S_K^1(E, v) := h_K^2 (\text{curl}E, \text{curl}v)_K + h_K(Q_0^F(n \times E), Q_0^F(n \times v))_{\partial K} + h_K^2 \sum_{e \in \mathcal{E}(K)} (Q_1^F(E_t), Q_1^F(v_t))_e, \]

which are computable using degrees of freedom. By the norm equivalence on the finite dimensional spaces and the scaling argument, we get

(7.28) \[ S_K^0(\tau, \tau) \approx \|\tau\|^2_{0,K} \quad \forall \tau \in W(K) \cap \ker(\Pi_K^2), \]

(7.29) \[ S_K^1(\tau, \tau) \approx \|\tau\|^2_{0,K} \quad \forall \tau \in W(K) \cap \ker(\Pi_K^2). \]

With these two stabilizations, define local bilinear forms

\[ b_K(E, v) := (Q_h^E, Q_h^v)_K + S_K^0(E - Q_h^K, v - Q_h^v) \]

\[ c_K(\sigma, \tau) := (\Pi_K^2 \sigma, \Pi_K^2 \tau)_K + S_K^0(\sigma - \Pi_K^2 \sigma, \tau - \Pi_K^2 \tau). \]

It is obvious that

\[ b_K(E, q) = (E, q)_K \quad \forall E \in H^1(K; \mathbb{R}) \cup \Sigma(K), q \in \mathbb{P}_1(K; \mathbb{R}), \]

\[ c_K(\sigma, q) = (\Pi_K^2 \sigma, q)_K \quad \forall \sigma \in H^2(K) \cap W(K), q \in \mathbb{P}_3(K). \]

And we obtain from (7.28) and (7.29) that

(7.30) \[ b_K(v, v) \lesssim \|v\|^2_{0,K} \quad \forall \sigma \in \Sigma(K), \]

(7.31) \[ c_K(\sigma, \tau) \lesssim \|\sigma\|_{0,K} \|\tau\|_{0,K} \quad \forall \sigma, \tau \in W(K). \]

Then we have from the Cauchy-Schwarz inequality that

(7.32) \[ a_h(E_h, v_h) + b_h(v_h, \nabla^2 \sigma_h) = (f, Q_h v_h) \quad \forall v_h \in \Sigma_h, \]

(7.33) \[ b_h(E_h, \nabla^2 \tau_h) - c_h(\sigma_h, \tau_h) = 0 \quad \forall \tau_h \in W_h, \]

where $Q_h v_h \in L^2(\Omega; \mathbb{R})$ is given by $(Q_h v_h)|_K := Q_h^K(v_h|_K)$ for each $K \in T_h$, and

\[ a_h(E_h, v_h) := \sum_{K \in T_h} (\text{curl}E_h, \text{curl}v_h)_K + \sum_{F \in F_h} h_F^{-1}(Q_1^F(n_F \times E_h), Q_1^F(n_F \times v_h))_F, \]

\[ b_h(E_h, \nabla^2 \tau_h) := \sum_{K \in T_h} b_K(E_h|_K, \nabla^2 \tau_h|_K), \]

\[ c_h(\sigma_h, \tau_h) := \sum_{K \in T_h} c_K(\sigma_h|_K, \tau_h|_K). \]
Here we add \( \|\mathbf{n}_F \times \mathbf{E}_h\|_F \) the jump of \( \mathbf{n}_F \times \mathbf{E}_h \) across \( F \) and more precisely its computable projection to \( P_1(F) \). We modify the norm for non-conforming space \( \Sigma_h \) accordingly
\[
\|v_h\|^2_{\text{curl}, h} := a_h(v_h, v_h) + \|v_h\|^2_0 \quad v_h \in \Sigma_h.
\]
For any \( E_h, v_h \in \Sigma_h \) and \( \sigma_h, \tau_h \in W_h \), it follows from (7.30)-(7.31) that
\[
A_h(E_h, v_h; \sigma_h, \tau_h) \leq (\|E_h\|_{\text{curl}, h} + \|\sigma_h\|_2)(\|v_h\|_{\text{curl}, h} + \|\tau_h\|_2),
\]
where
\[
A_h(E_h, v_h; \sigma_h, \tau_h) := a_h(E_h, v_h) + b_h(v_h, \nabla^2 \sigma_h) + b_h(E_h, \nabla^2 \tau_h) - c_h(\sigma_h, \tau_h),
\]
Employing the same argument as the proof of (7.23), we have the discrete inf-sup condition
\[
\|E_h\|_{\text{curl}, h} + \|\sigma_h\|_2 \lesssim \sup_{E_h, \sigma_h \in \Sigma_h, \tau_h \in W_h} \frac{A_h(E_h, \sigma_h; v_h, \tau_h)}{\|v_h\|_{\text{curl}, h} + \|\tau_h\|_2} \quad \forall E_h \in \Sigma_h, \sigma_h \in W_h.
\]

**Theorem 7.7.** Let \( E_h \in \Sigma_h \) and \( \sigma_h \in W_h \) be the solution of the mixed method (7.32)-(7.33) and let \( E \) and \( \sigma \) be the solution of (7.7)-(7.8). Assume \( E \in H^3(\Omega; S) \), \( \sigma \in H^3(\Omega) \) and \( f \in H^1(\Omega; S) \). We have
\[
\|E - E_h\|_{\text{curl}, h} + \|\sigma - \sigma_h\|_2 \lesssim h(\|E\|_3 + \|\sigma\|_3 + |f|_1).
\]

**Proof.** Following the proof of Theorem 7.6, we have
\[
\|E - E_h\|_{\text{curl}, h} + \|\sigma - \sigma_h\|_2 \lesssim h(\|E\|_2 + \|\sigma\|_3 + |f|_1) + \sup_{v_h \in \Sigma_h} \frac{a(E, v_h) + (v_h, \nabla^2 \sigma) - (f, v_h)}{\|v_h\|_{\text{curl}, h}}.
\]
On the other hand, applying the integration by parts give parts
\[
a(E, v_h) + (v_h, \nabla^2 \sigma) - (f, v_h) = \sum_{F \in F_h^e} (\text{curl} E, [\mathbf{n}_F \times v_h])_F
\]
\[
= \sum_{F \in F_h^e} (\text{curl} E, [\mathbf{n}_F \times v_h] - Q^F_1([\mathbf{n}_F \times v_h]))_F + \sum_{F \in F_h^e} (\text{curl} E, Q^F_1([\mathbf{n}_F \times v_h]))_F
\]
\[
= \sum_{F \in F_h^e} (\text{curl} E - Q^F_1(\text{curl} E), [\mathbf{n}_F \times v_h])_F + \sum_{F \in F_h^e} (\text{curl} E - Q^F_1(\text{curl} E), Q^F_1([\mathbf{n}_F \times v_h]))_F
\]
\[
\lesssim h(\|E\|_3\|v_h\|_0 + h|E|_2\|v_h\|_{\text{curl}, h}) + h\|E\|_3\|v_h\|_{\text{curl}, h}.
\]
Thus (7.34) follows. \( \square \)

8. Conclusion and Future Work

We have constructed one conforming and one non-conforming virtual element Hessian complexes on tetrahedral grids in this paper. Both discrete Hessian complexes start from \( H^2 \)-conforming virtual elements, and end with discontinuous \( P_1(\Omega; \mathbb{R}^3) \) element. The middle \( H(\text{curl}, \Omega; S) \) virtual elements are constructed by mixing the hessian of \( H^2 \)-conforming virtual elements and the pulling back
of $H(\text{div}, \Omega; \mathbb{T})$-conforming finite element through operator $\text{sym}(\mathbf{x} \times \cdot)$, which is inspired by the decomposition of the space of symmetric tensors

$$P_k(K; \mathbb{S}) = \text{hess} P_{k+2}(K) \oplus \text{sym}(\mathbf{x} \times P_{k-1}(K; \mathbb{T})).$$

The local dimensions of the four spaces in the conforming virtual element Hessian complex are $(68, 132, 80, 12)$, while the non-conforming version is $(28, 92, 80, 12)$. We have applied these two discrete Hessian complexes to discretize the linearized time-independent Einstein-Bianchi system.

In future work we will explore the following two topics:

1. Construct a family of conforming virtual element Hessian complexes on general polyhedral meshes, in which the lowest order virtual element Hessian complex has fewer degrees of freedom;
2. Construct a lower order non-conforming finite element or virtual element Hessian complex, which makes the discretization of the linearized Einstein-Bianchi system easily implemented.

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