LOCAL STRUCTURE OF ALGEBRAIC MONOIDS

MICHEL BRION

ABSTRACT. We describe the local structure of an irreducible algebraic monoid $M$ at an idempotent element $e$. When $e$ is minimal, we show that $M$ is an induced variety over the kernel $MeM$ (a homogeneous space) with fibre the two-sided stabilizer $Me$ (a connected affine monoid having a zero element and a dense unit group). This yields the irreducibility of stabilizers and centralizers of idempotents when $M$ is normal, and criteria for normality and smoothness of an arbitrary monoid $M$. Also, we show that $M$ is an induced variety over an abelian variety, with fiber a connected affine monoid having a dense unit group.

0. Introduction

An algebraic monoid is an algebraic variety equipped with an associative product map, which is a morphism of varieties and admits an identity element. Algebraic monoids are closely related to algebraic groups: the group $G$ of invertible elements of any irreducible algebraic monoid $M$ is a connected algebraic group, open in $M$. Thus, $M$ is an equivariant embedding of its unit group $G$ with respect to the action of $G \times G$ via left and right multiplication; this embedding has a unique closed orbit, the kernel of the monoid.

This relationship takes a particularly precise form in the case of affine (or, equivalently, linear) monoids and groups. Indeed, by work of Vinberg and Rittatore, the affine irreducible algebraic monoids are exactly the affine equivariant embeddings of connected linear algebraic groups. Furthermore, any irreducible algebraic monoid having an affine unit group is affine (see [Vi95, Ri98, Ri06]).

Affine irreducible algebraic monoids have been intensively investigated, primarily by Putcha and Renner (see the books [Pu88, Re05]). The idempotents play a fundamental role in their theory: for instance, the kernel contains idempotents, and these form a unique conjugacy class of the unit group.

From the viewpoint of algebraic groups, the idempotents are exactly the limit points of multiplicative one-parameter subgroups. It follows easily that every irreducible algebraic monoid having a reductive unit group is unit regular, that is, any element is the product of a unit and an idempotent. Such reductive monoids are of special interest (see the above references); their study has applications to compactifications...
of reductive groups (see [Ti03]) and to degenerations of varieties with group actions (see [AB04]).

In contrast, little was known about the non-affine case until the recent classification of normal algebraic monoids by Rittatore and the author (see [BR07]). Loosely speaking, any such monoid is induced from an abelian variety, with fibre a normal affine monoid. This result extends, and builds on, Chevalley’s structure theorem for connected algebraic groups (see [Ch60, Co02]); it holds in arbitrary characteristics, like most of Putcha and Renner’s results. More generally, any normal equivariant embedding of a homogeneous variety under an arbitrary algebraic group is induced from an abelian variety, with fibre a normal equivariant embedding of a homogeneous variety under an affine group; see [Br07], which also contains examples showing that the normality assumption cannot be omitted.

In the present paper, we obtain a classification of all irreducible algebraic monoids in the spirit of [BR07]: they are also induced from abelian varieties, but fibres are allowed to be connected affine monoids having a dense unit group (Theorem 3.2.1). This answers a longstanding question of Renner, see [Re84]. Also, we characterize the irreducible algebraic monoids having a prescribed unit group $G$, as those equivariant embeddings $X$ of $G$ such that the Albanese morphism $\alpha_X$ is affine (Corollary 3.3.3).

Our approach differs from those of [BR07, Br07]; it relies on a local structure theorem for an irreducible algebraic monoid $M$ at an idempotent $e$ (Theorem 2.2.1). Loosely speaking again, an open neighbourhood of $e$ in $M$ is an induced variety over an open subvariety of the product $MeM$, with fibre the two-sided stabilizer $M_e = \{ x \in M \mid xe = ex = e \}$. Note that $M_e$ is a closed submonoid of $M$ with the same identity element, and the zero element $e$; we show that $M_e$ is affine and connected, and its unit group $G_e$ is dense (Lemma 3.1.4).

When $e$ lies in the kernel, our local structure theorem takes a global form: the whole variety $M$ is induced over the kernel $MeM$, with fibre $M_e$ (Corollary 2.3.2). As a direct consequence, the normality or smoothness of $M$ is equivalent to that of $M_e$.

This raises the question of classifying all smooth monoids having a zero element; such a monoid is isomorphic (as a variety) to an affine space, by Corollary 3.1.5. Another open problem arising from our local structure theorem is the classification of all connected algebraic monoids having a dense unit group and a zero element; to make this problem tractable, one may assume that the unit group is reductive.

Our results are obtained over an algebraically closed field of characteristic zero. They may be adapted to arbitrary characteristics, by considering group schemes and monoid schemes at appropriate places.
For example, the stabilizer $M_e$ should be understood as a closed sub-monoid scheme of $M$; this subscheme turns out to be reduced in characteristic zero (Remark 2.2.2), but this fails in positive characteristics, e.g., for certain non-normal affine toric varieties.

This may be a motivation for developing a theory of monoid schemes; note that, unlike group schemes, many monoid schemes over a field of characteristic zero are not reduced. For example, view the affine plane $\mathbb{A}^2$ as a monoid with product $(x_1, y_1) \cdot (x_2, y_2) = (x_1x_2, x_1y_2 + x_2y_1)$ and unit $(1, 0)$. Then the closed subscheme with ideal $(x^2, xy)$ is a non-reduced submonoid, the affine line with a fat point at the origin.

This paper is organized as follows. We begin by gathering some basic definitions and results on algebraic varieties, algebraic groups and induced varieties. In Section 1, we study various stabilizers and centralizers associated with idempotents in affine irreducible algebraic monoids. This builds on work of Putcha (exposed in [Pu88, Chapter 6]), but we have modified some of his terminology in order to comply with standard conventions in algebraic geometry and algebraic groups. Section 2 is devoted to the local structure of affine irreducible algebraic monoids, with applications to criteria for normality or smoothness, and to the irreducibility of stabilizers and centralizers in normal monoids. In the final Section 3, we obtain our classification theorem and derive some consequences, e.g., all irreducible algebraic monoids are quasi-projective varieties.

**Notation and conventions.** We consider algebraic varieties over an algebraically closed field $k$ of characteristic zero; morphisms are understood to be $k$-morphisms. By a variety, we mean a separated reduced scheme of finite type over $k$; in particular, varieties are not necessarily irreducible. A point will always mean a closed point.

An algebraic group $G$ is a group scheme of finite type over $k$; then $G$ is a smooth variety, as $k$ has characteristic 0. Also recall that $G$ is affine if and only if it is linear, i.e., isomorphic to a closed subgroup of some general linear group.

Given an arbitrary algebraic group $G$ and a closed subgroup $H \subset G$, there exists a quotient morphism $q : G \to G/H$, where $G/H$ is a quasi-projective variety, and $q$ is a principal $H$-bundle. Furthermore, $q$ is affine if and only if $H$ is affine.

More generally, given a variety $Y$ where $H$ acts algebraically, consider the product $G \times Y$ where $H$ acts via $h \cdot (g, y) = (gh^{-1}, hy)$. If $Y$ is quasi-projective, then there exists a quotient morphism

$$q_Y : G \times Y \to (G \times Y)/H =: G \times^H Y,$$

where $G \times^H Y$ is a quasi-projective variety, and $q_Y$ is a principal $H$-bundle. The induced variety $G \times^H Y$ is equipped with a $G$-action and a morphism

$$p_Y : G \times^H Y \to G/H$$
such that $p_Y$ and $q_Y$ are $G$-equivariant; the (scheme-theoretic) fibre of $p_Y$ at the base point of $G/H$ is $H$-equivariantly isomorphic to $Y$.

Moreover, $p_Y$ is affine if and only if $Y$ is affine (for these facts, see [Se58a, Proposition 4] and [MFK94, Proposition 7.1]. In particular, $G \times^H Y$ is affine whenever $G/H$ and $Y$ are both affine.

1. Stabilizers and centralizers

1.1. Stabilizers. Throughout this section, $M$ denotes an irreducible affine algebraic monoid, $1 \in M$ the identity element, and $G = G(M)$ the unit group. Then $G \times G$ acts on $M$ via $(x, y) \cdot z = xzy^{-1}$ (the two-sided action); we also have the left action of $G$ on $M$ via $x \cdot y = xy$, and the right action via $x \cdot y = yx^{-1}$.

We fix an idempotent $e \in M$ and denote by

\begin{equation}
\ell_e : M \longrightarrow M, \quad x \longmapsto ex
\end{equation}

the left multiplication by $e$. Clearly, $\ell_e$ is a retraction of the variety $M$ onto the closed subvariety $eM = \{x \in M \mid ex = x\}$. Likewise, the right multiplication by $e$,

\begin{equation}
r_e : M \longrightarrow M, \quad x \longmapsto xe
\end{equation}

is a retraction of $M$ onto $Me = \{x \in M \mid xe = x\}$. We also have a retraction of varieties

\begin{equation}
t_e : M \longrightarrow eMe, \quad x \longmapsto exe.
\end{equation}

Moreover, $eMe = \{x \in M \mid exe = x\} = eM \cap Me$ is a closed irreducible submonoid of $M$ with identity element $e$.

We put

\begin{equation}
M^\ell_e := \{x \in M \mid xe = e\}
\end{equation}

(the left stabilizer of $e$ in $M$) and

\begin{equation}
G^\ell_e := G \cap M^\ell_e.
\end{equation}

Clearly, $M^\ell_e$ is a closed submonoid of $M$ with identity element 1 and unit group $G^\ell_e$. Moreover, $G^\ell_e$ is dense in $M^\ell_e$ by [Pu88, Theorem 6.11]. Note that $M^\ell_e$ is the set-theoretic fibre of $r_e$ at $e$. In fact, $M^\ell_e$ is also the scheme-theoretic fibre, as we shall see in Remark 2.1.3 (i).

Likewise, the right stabilizer of $e$ in $M$,

\begin{equation}
M^r_e := \{x \in M \mid ex = e\},
\end{equation}

is a closed submonoid of $M$ with identity element 1, and dense unit group

\begin{equation}
G^r_e := G \cap M^r_e.
\end{equation}

The (two-sided) stabilizer of $e$ in $M$,

\begin{equation}
M_e := \{x \in M \mid xe = ex = e\} = M^\ell_e \cap M^r_e,
\end{equation}

is given by the intersection of the left and right stabilizers of $e$.
is also a closed submonoid of $M$ with identity element 1, zero element $e$, and unit group

\[(1.1.9)\quad G_e := G \cap M_e.\]

Moreover, $G_e$ is dense in $M_e$ by [Pu88, Theorem 6.11] again.

Our notation for $M_e$ and $G_e$ differs from that of Putcha in [Pu88]: his $M_e$ and $G_e$ are the irreducible components of ours that contain 1.

Also, note that $M^l_e$, $M^r_e$ and $M_e$ are generally reducible; equivalently, $G^l_e$, $G^r_e$ and $G_e$ are generally non-connected. This happens for many non-normal affine toric varieties, regarded as commutative monoids; see [Pu88, Example 6.12] for an explicit example.

Yet the stabilizers in $M$ are always connected, as follows from the existence of a multiplicative one-parameter subgroup of $G$ with limit point $e$:

**Lemma 1.1.1.** (i) There exists a homomorphism of algebraic groups $\theta : \mathbb{G}_m \to G_e$ which extends to a morphism of varieties $\overline{\theta} : \mathbb{A}^1 \to M$ such that $\overline{\theta}(0) = e$. In particular, the closure of $G^0_e$ in $M$ contains $e$.

(ii) The fibres of $\ell_e$ and $r_e$ are connected.

(iii) $M^l_e$, $M^r_e$ and $M_e$ are connected.

**Proof.** (i) By [Pu88, Corollary 6.10], there exists a maximal torus $T \subset G$ such that $e$ lies in $\overline{T}$ (the closure of $T$ in $M$). Since $\overline{T}$ is a (possibly non-normal) toric variety, there exist a one-parameter subgroup $\theta : \mathbb{G}_m \to T$ and an element $t \in T$ such that $\theta$ extends to a morphism $\overline{\theta} : \mathbb{A}^1 \to M$ such that $\overline{\theta}(0) = te$. Then $\overline{\theta}$ is a homomorphism of monoids, so that $\overline{\theta}(0)$ is idempotent. Since $t$ and $e$ commute, it follows that $te = e$. Moreover, $\theta(x)e = \theta(x)\overline{\theta}(0) = \overline{\theta}(0) = e = e\theta(x)$ for all $x \in k^*$.

(ii) Clearly, $\ell_e$ is invariant under the left action of $G^l_e$. In particular, each fibre $\ell_e^{-1}(y)$, $y \in eM$, is stable under left multiplication by $G^0_e$. So, for any $x \in \ell_e^{-1}(y)$, the closure of the orbit $G^0_e x$ is an irreducible subvariety of $\ell_e^{-1}(y)$ containing both points $x$ and $ex = y$. It follows that $\ell_e^{-1}(y)$ is connected.

(iii) The connectedness of $M^r_e$ (resp. $M^l_e$) follows from (ii). To show the connectedness of $M_e$, note as above that the closure of any orbit $G^0_e x$ is an irreducible subvariety of $M_e$ containing both points $x$ and $ex = e$. □

### 1.2. The centralizer.

Let

\[(1.2.1)\quad C_M(e) := \{x \in M \mid xe = ex\},\]

this is the centralizer of $e$ in $M$. Clearly, $C_M(e)$ is a closed submonoid of $M$ with identity element 1 and unit group

\[(1.2.2)\quad C_G(e) := G \cap C_M(e).\]
Moreover, $C_G(e)$ is connected by [Pu88, Theorem 6.16]. But the example below (a variant of [Pu88, Example 6.15]) shows that $C_M(e)$ is generally reducible; in other words, $C_G(e)$ may not be dense in $C_M(e)$.

**Example 1.2.1.** Let $V$, $W$ be vector spaces of dimensions $m, n \geq 2$. Consider the multiplicative monoids $\text{End}(V)$, $\text{End}(W)$ and the map

$$\varphi : \text{End}(V) \times \text{End}(W) \rightarrow \text{End}(V \otimes W), \quad (A, B) \mapsto A \otimes B.$$ 

Then $\varphi$ is a homomorphism of monoids, and is the invariant-theoretical quotient by the $\mathbb{G}_m$-action via $t \cdot (x, y) = (tx, t^{-1}y)$. Thus, the image of $\varphi$ is a closed normal submonoid,

$$M := \text{End}(V) \otimes \text{End}(W) \subset \text{End}(V \otimes W).$$

Its unit group is the quotient of $\text{GL}(V) \times \text{GL}(W)$ by $\mathbb{G}_m$ embedded via $t \mapsto (t \text{id}_V, t^{-1} \text{id}_W)$.

Given two idempotents $e \in \text{End}(V)$ and $f \in \text{End}(W)$, the idempotent $e \otimes f \in M$ satisfies

$$C_M(e \otimes f) = \{ x \otimes y \in M \mid xe \otimes yf = ex \otimes fy \}.$$ 

It follows easily that

$$C_G(e \otimes f) = C_{\text{GL}(V)}(e) \otimes C_{\text{GL}(W)}(f),$$

while

$$C_M(e \otimes f) \supset (1 - e) \otimes \text{End}(W).$$

Thus, $C_M(e \otimes f)$ is reducible whenever $e, f \neq 0, 1$.

However, the centralizers in $M$ are always connected, as shown by the following:

**Lemma 1.2.2.** (i) The morphism

$$\tau_e : C_M(e) \rightarrow eMe, \quad x \mapsto xe = ex = exe$$

is a retraction of algebraic monoids.

(ii) The fibres of $\tau_e$ are connected. In particular, $C_M(e)$ is connected.

(iii) We have an exact sequence of algebraic groups

$$1 \rightarrow G_e \rightarrow C_G(e) \xrightarrow{\tau_e} G(eMe) \rightarrow 1.$$ 

(iv) The normalizer $N_G(G_e)$ equals $C_G(e)$.

**Proof.** (i) is straightforward.

(ii) Note that $\tau_e$ is invariant under the (left or right) action of $G_e \subset C_G(e)$. So the assertion follows by arguing as in the proof of Lemma 1.1.1(ii).

(iii) Clearly, $\tau_e$ restrict to a homomorphism $C_G(e) \rightarrow G(eMe)$ with kernel $G_e$. This homomorphism is surjective by [Pu88, Remark 1.3(ii), Theorem 6.16].

(iv) $C_G(e)$ normalizes $G_e$ by (1.2.4). Conversely, if $x \in G$ normalizes $G_e$, then it normalizes $M_e$ (the closure of $G_e$ in $M$). Thus, $x$ commutes with the zero element $e$ of $M_e$. \qed
Next, we consider the action of \( G \) on \( M \) by conjugation. Then the isotropy group of \( e \) is \( C_G(e) \), so that the conjugacy class of \( e \) is isomorphic to \( G/C_G(e) \).

**Lemma 1.2.3.** The \( G \)-conjugacy class of \( e \) is closed in \( M \). In particular, the variety \( G/C_G(e) \) is affine.

**Proof.** We adapt a classical argument for the closedness of semi-simple conjugacy classes in affine algebraic groups. Let \( B \) be a Borel subgroup of \( G \). Since \( G/B \) is complete, it suffices to check that the \( B \)-conjugacy class of \( e \) is closed in \( M \). We may assume that \( B \) contains a maximal torus \( T \) such that \( e \in T \), see the proof of Lemma 1.1.1. Then \( T \) centralizes \( e \), so that the \( B \)-conjugacy class of \( e \) is an orbit of the unipotent radical of \( B \); hence this class is closed in the affine variety \( M \), by \[Ro61\]. \( \square \)

### 1.3. Left and right centralizers.

Let

\[
C^\ell_M(e) := \{ x \in M \mid xe = exe \},
\]

the left centralizer of \( e \) in \( M \). For any \( x, y \in C^\ell_M(e) \), we have

\[
xye = xeye = exye = exye.
\]

Thus, \( C^\ell_M(e) \) is a closed submonoid of \( M \) with identity element 1. The unit group of \( C^\ell_M(e) \) equals

\[
C^\ell_G(e) := G \cap C^\ell_M(e)
\]

(indeed, this is a closed submonoid of \( G \), and hence a subgroup by \[Re05\] 3.5.1 Exercises 1 and 2]). Moreover, \( C^\ell_G(e) \) is connected by \[Pu88\] Theorem 6.16. However, \( C^\ell_M(e) \) is generally reducible. For instance, with the notation of Example 1.2.1, we have \( C^\ell_G(e \otimes f) = C^\ell_G(e) \otimes C^\ell_G(f) \) while \( C^\ell_M(e \otimes f) \supset \text{End}(V)(1 - e) \otimes \text{End}(W) \).

We now extend the statement of Lemma 1.2.2 to left centralizers:

**Lemma 1.3.1.** (i) \( C^\ell_M(e) \) is the preimage of \( eMe \) under the morphism \( r_e \) of (1.1.2). Moreover, \( r_e \) restricts to a retraction of algebraic monoids

\[
\rho_e : C^\ell_M(e) \longrightarrow eMe, \quad x \longmapsto xe = exe.
\]

(ii) The fibres of \( \rho_e \) are connected. In particular, \( C^\ell_M(e) \) is connected.

(iii) We have an exact sequence

\[
1 \longrightarrow G^\ell_e \longrightarrow C^\ell_G(e) \longrightarrow \rho_e, \quad G(eMe) \longrightarrow 1
\]

and the equality

\[
C^\ell_G(e) = G^\ell_GC_G(e).
\]

(iv) The normalizer \( N_G(G^\ell_e) \) equals \( C^\ell_G(e) \).
normalizes to $C$. Both varieties 1.2.2; this implies the second statement.

Lemma 1.3.3. (i) The variety $G^r_e/G_e$ is isomorphic to $C^r_G(e)/C_G(e)$. (ii) Both varieties $G^r_e/G_e$ and $C^r_G(e)/G_e$ are affine.

Proof. (i) We have

$$G^r_e/G_e = G^r_e/(G^r_e \cap C_G(e)) \cong G^r_eC_G(e)/C_G(e) = C^r_G(e)/C_G(e),$$

where the latter equality follows from (1.3.6).

(ii) $C^r_G(e)/C_G(e)$ is closed in $G/C_G(e)$, and hence is affine by Lemma 1.2.3. Moreover, $C^r_G(e)/G_e$ is an induced variety over the homogeneous space $C^r_G(e)/C_G(e)$, with fibre $C_G(e)/G_e$. The latter is affine by Lemma 1.2.2; this implies the second statement.

Similar assertions hold for the right centralizer of $e$ in $M$ resp. $G$,

(1.3.7) $C^r_M(e) := \{ x \in M \mid ex = ex \}$, $C^r_G(e) := G \cap C^r_M(e)$.

(Here again, our notation differs from that of Putcha: his $C^r_M(e)$ is our $C^r_M(e)$.) In particular, the morphism $\ell_e$ of (1.1.1) yields an exact sequence of algebraic groups

(1.3.8) $1 \longrightarrow G^r_e \longrightarrow C^r_G(e) \longrightarrow \lambda_e G(eMe) \longrightarrow 1$

and the equality

(1.3.9) $C^r_G(e) = G^r_eC_G(e)$.

This implies readily the following description of the stabilizer of $e$ in $M \times M$,

$$(M \times M)_e := \{(x, y) \in M \times M \mid xe = ey \}.$$  

and of its stabilizer in $G \times G$,

$$(G \times G)_e = \{(x, y) \in G \times G \mid xey^{-1} = e \}.$$  

Note that $(M \times M)_e$ is a closed submonoid of the product monoid $M \times M$, with identity element $(1, 1)$ and unit group $(G \times G)_e$.

Lemma 1.3.3. (i) $(M \times M)_e = \{(x, y) \in C^r_M(e) \times C^r_M(e) \mid \rho_e(x) = \lambda_e(y)\}$. 

(ii) The two projections \( M \times M \to M \) yield surjective morphisms \( (M \times M)_e \to C_M^e(e) \), \( (M \times M)_e \to C_M^r(e) \) with connected fibres. In particular, \( (M \times M)_e \) is connected.

(iii) The two projections \( G \times G \to G \) yield exact sequences
\[
1 \to G_e^r \to (G \times G)_e \to C_G^r(e) \to 1,
\]
\[
1 \to G_e^l \to (G \times G)_e \to C_G^r(e) \to 1.
\]

**Remark 1.3.4.** If \( G \) is reductive, then \( C_G^r(e) \) and \( C_G^r(e) \) are opposite parabolic subgroups of \( G \), with common Levi subgroup \( C_G(e) \); moreover, the unipotent radical \( R_{uG}(C_G^r(e)) \) is contained in \( G_e^l \) (see [Re05, Theorem 4.5]).

In view of [1.2.4], it follows that \( G_e \) and \( G(eMe) \) are reductive. Moreover, \( G_e^r \) is the semi-direct product of \( R_{uG}(C_G^r(e)) \) with \( C_G(e) \cap G_e^r = G_e \). Likewise, \( G_e^l \) is the semi-direct product of \( R_{uG}(C_G^r(e)) \) with \( G_e \).

The stabilizer \( (G \times G)_e \) is described in [AB04, Section 3], in the more general setting of stable reductive varieties.

2. **The local structure of affine irreducible monoids**

2.1. **Local structure for the left action.** Throughout this section, we maintain the notation and assumptions of Section 1. We first record the following consequence of a result of Putcha:

**Lemma 2.1.1.** (i) The product \( C_G^r(e)e \) is an open affine subvariety of \( Me \), isomorphic to \( C_G^r(e)/G_e \).

(ii) The product \( C_G^r(e)G_e^l \) is an open affine subvariety of \( G \), isomorphic to \( C_G^r(e) \times C_G^l(e) \) where \( G_e \) acts on \( C_G^r(e) \times C_G^l(e) \) via \( x \cdot (y, z) = (yx^{-1}, xz) \).

**Proof.** By [Pu88, Theorem 6.16], \( Me \) is contained in \( C_G^r(e) \), the closure of \( C_G^r(e) \) in \( M \). Thus, \( Me \subset C_G^r(e)e \), that is, \( C_G^r(e)e = Me \). So \( C_G^r(e)e \) is dense in \( Me \). But \( C_G^r(e)e \) is an orbit, and hence is open in \( Me \); the isotropy group of \( e \) is \( C_G^r(e) \cap G_e^l = G_e \). Together with Lemma 1.3.2 this implies (i).

Note that
\[
C_G^r(e)e \cong C_G^r(e)/G_e \cong C_G^r(e)/(C_G^r(e) \cap G_e^l) \cong C_G^r(e)G_e^l/G_e^l
\]
is an open affine subvariety of \( G \) since \( r_e|_G : G \to G e \) is affine (as its source is affine), this implies (ii). \( \square \)

Likewise, the product \( eC_G^l(e) \) is an open affine subvariety of \( eM \), isomorphic to \( C_G^l(e)/G_e \). Also, combining Lemmas 1.3.1 and 2.1.1 we see that the product map \( G_e^r \times C_G(e) \times G_e^l \to G \) induces an isomorphism
\[
G_e^r \times C_G^l(e) G_e^l \cong C_G^r(e)G_e^l = C_G^r(e)C_G^l(e) = C_G^r(e)C_G^r(e),
\]
and the right-hand side is an open affine subvariety of \( G \).

Next, we show that an affine neighbourhood of \( e \) in \( M \) is an induced variety relative to the left action of \( C_G^r(e) \):
Proposition 2.1.2.  (i) The subvariety

\[ M_0^r := \{ x \in M \mid xe \in C_G^r(e) \} \]

is open in \( M \), affine, stable under the two-sided action of \( C_G^r(e) \times C_G^r(e) \) on \( M \), and contains \( M_e^\ell \).

(ii) The product map \( C_G^r(e) \times M_e^\ell \to M \) induces an isomorphism

\[ f^r : C_G^r(e) \times G^e M_e^\ell \to M_0^r, \]

equivariant under the two-sided action of the subgroup \( C_G^r(e) \times G^e \subset C_G^r(e) \times C_G^r(e) \).

(iii) The scheme-theoretic intersection \( Me \cap M_e^\ell \) consists of the (reduced) point \( e \).

Proof. (i) Note that \( M_0^r \) is the preimage of \( C_G^r(e) \subset Me \) under the morphism \( r_e \) of \( (1.1.2) \). Since that morphism is affine, and \( C_G^r(e) \subset Me \) is open and affine (by Lemma \( 2.1.1 \)), \( M_0^r \) is open and affine as well.

Clearly, \( M_0^r \) contains \( M_e^\ell \) and is stable under \( C_G^r(e) \). To show the stability under \( C_G^r(e) \), consider \( x \in M_0^r \) and \( g \in C_G^r(e) \). Then

\[ xge = xge \in C_G^r(e)CeG(e) = C_G^r(e), \]

as \( ege \in eC_G(e) \) by \( (1.3.9) \).

(ii) Since \( r_e \) is equivariant under \( C_G^r(e) \), the natural map

\[ C_G^r(e) \times G^e r_e^{-1}(e) \to r_e^{-1}(C_G^r(e)) = M_0^r \]

is an isomorphism, where \( r_e^{-1}(e) \) denotes the scheme-theoretic fibre. So it suffices to check the equality

\[ (2.1.3) \quad M_e^\ell = r_e^{-1}(e). \]

Clearly, \( M_e^\ell \) is contained in \( r_e^{-1}(e) \) as its maximal closed reduced subscheme. Moreover, \( M_e^\ell \) is stable under the left action of \( G_e \). So \( C_G^r(e) \times G^e M_e^\ell \) is a closed subscheme of \( C_G^r(e) \times G^e r_e^{-1}(e) \), and both have the same closed points. But \( C_G^r(e) \times G^e r_e^{-1}(e) \) is an open subscheme of \( M \), and hence is reduced; this implies \( (2.1.3) \).

(iii) By (ii), \( f^r \) restricts to an isomorphism

\[ C_G^r(e) \times G^e (Me \cap M_e^\ell) \cong Me \cap M_0^r. \]

Moreover, \( e \) is the unique closed point of \( Me \cap M_e^\ell \). Since \( Me \cap M_0^r \) is an irreducible variety, it follows that \( Me \cap M_e^\ell = \{ e \} \) as schemes, by arguing as in the proof of (ii).

\[ \square \]

Remarks 2.1.3.  (i) As shown in the above proof, \( M_e^\ell \) is the scheme-theoretic fibre of \( r_e \) at \( e \). Also, \( M_e^\ell \) may be regarded as a slice at \( e \) to the orbit \( C_G^r(e) \), or to its closure \( Me \).

(ii) The right action of \( C_G^r(e) \) on the open subvariety

\[ (2.1.4) \quad M_0^r := \{ x \in M \mid ex \in eC_G^r(e) \} \]
is described in similar terms.

(iii) By the argument of Proposition 2.1.2, the product of $M$ induces an open immersion $G \times^{G_e} M_e \to M$; this yields a local structure result for the left action of $G$. However, the orbit $Ge \cong G/G_e$ is generally not affine; this happens, for example, if $M = \text{End}(V)$ and $e \neq 0, 1$. As a consequence, the variety $G \times^{G_e} M_e$ is generally not affine either.

2.2. Local structure for the two-sided action. We now show that an affine neighbourhood of $e$ in $M$ is an induced variety relative to the two-sided action of $C_r G(e) \times C_l G(e)$.

**Theorem 2.2.1.** (i) The subvariety

\[(2.2.1) \quad M_0 := \{x \in M \mid xe \in C_r G(e) e \text{ and } ex \in eC_l G(e)\} = M_0^r \cap M_0^l \]

is open in $M$, affine, stable under the two-sided action of $C_r G(e) \times C_l G(e)$ on $M$, and contains $M_e$.

(ii) The product map $C_r G(e) \times M_e \times C_l G(l) \to M$ induces an isomorphism

\[(2.2.2) \quad f : C_r G(e) \times M_e \times C_l G(e) \to M_0^r ,\]

equivariant under $C_r G(e) \times C_l G(e)$.

(iii) The scheme-theoretic intersection $MeM \cap M_e$ consists of the (reduced) point $e$.

**Proof.** (i) follows from Proposition 2.1.2 together with the fact that the intersection of any two affine open subvarieties is affine.

(ii) By Proposition 2.1.2 again, the natural map

\[C_r G(e) \times M_e \cap M_0^r \to M_0^r \]

is an isomorphism. Thus, it suffices to show that the natural map

\[M_e \times^{C_r G(e)} G_e \to M_e^r \cap M_0^r \]

is an isomorphism.

Let $x \in M_e \cap M_0$, then $xe = e$ and $ex = eg$ for some $g \in C_r G(e)$. Thus, $e = exe = ege = ge$, that is, $g \in G_e$. Hence

\[M_e^r \cap M_0 = \{x \in M_e^r \mid ex \in eG_e\} .\]

Since $G_e^r$ is open and dense in $M_e^r$, the product $eG_e^r \cong G_e^r / G_e$ is open and dense in $eM_e^r$. Thus, the natural map

\[(M_e^r \cap M_e) \times^{G_e} G_e^r \to M_e^r \cap M_0 \]

is an isomorphism, where $M_e^r \cap M_e^r$ denotes the scheme-theoretic intersection. The latter intersection equals $M_e$ as a set, and hence as a scheme by the argument of Proposition 2.1.2. This yields the desired isomorphism.

(iii) By the argument of Proposition 2.1.2 again, it suffices to check that $MeM \cap M_e = \{e\}$ as sets. For this, recall that $M$ is isomorphic to a closed submonoid of the multiplicative monoid $\text{End}(V)$, where $V$
is a finite-dimensional vector space; see [Pu88, Theorem 3.15]. Let $x \in MeM \cap M_e$. Then $\text{rk}(x) \leq \text{rk}(e)$ and $x = e + y$ where $y \in \text{End}(V)$ satisfies $ye = ey = 0$; thus, $\text{rk}(x) = \text{rk}(e) + \text{rk}(y)$. It follows that $y = 0$, and $x = e$.

**Remarks 2.2.2.** (i) $M_e$ may be regarded as a slice at $e$ to the orbit $C^r_G(e)eG^r_e$, or to its closure $MeM$. Moreover, $M_e$ (regarded as a closed subscheme of $M$) is reduced and equals the scheme-theoretic intersection of $M^r_e$ and $M^r_e$.

(ii) One may wonder whether this local structure result extends to the two-sided action of the whole group $G \times G$. The answer is positive for reductive monoids and minimal idempotents, by a corollary of the Luna slice theorem (see [AB04, Lemma 4.3]). However, the answer is generally negative: if $M_0$ is a $G \times G$-stable neighbourhood of $e$ admitting an equivariant morphism $f$ to the orbit $GeG \cong (G \times G)/(G \times G)_e$, then $M_0$ contains the open orbit $G \cong (G \times G)/\text{diag}(G)$. Thus, the isotropy group, $\text{diag}(G)$, is contained in a conjugate of $(G \times G)_e$ in $G \times G$. But this does not hold in general, e.g., when $M = \text{End}(V)$ and $e \neq 0,1$.

The left and right actions do not play symmetric roles in the statement of Theorem 2.2.1. We now reformulate this result in a symmetric way, and apply it to the local structure of the centralizer of $e$.

**Corollary 2.2.3.** (i) With the notation of Theorem 2.2.1 the product of $M$ induces isomorphisms

$$C_G(e) \times^{G_e} M_e \cong C_M(e) \cap M_0$$

and

$$G^r_e \times^{G_e} (C_M(e) \cap M_0) \times^{G_e} G^r_e \cong M_0.$$

(ii) $C_M(e) \cap M_0$ is irreducible. In particular, $C_M(e)$ is irreducible at $e$.

**Proof.** (i) Let $g \in C^r_G(e)$, $x \in M_e$ and $h \in G^r_e$ be such that $gxh \in C_M(e)$. Then

$$ge = gxhe = egxh = exe = ge$$

so that $ge = egeh = ege$. Thus, $g \in C^r_G(e) \cap C^r_G(e) = C_G(e)$. It follows that $e = eh$, that is, $h \in G^r_e \cap G^r_e = G_e$. Combined with Theorem 2.2.1 this implies the first assertion. The second assertion is a consequence of that theorem in view of the isomorphism

$$C^r_G(e) \cong G^r_e \times^{G_e} C_G(e),$$

which follows in turn from (1.3.9).

(ii) By (i), $C_G(e)M_e$ is an open neighborhood of $e$ in $C_M(e)$. Moreover, $C_G(e)$ is dense in $C_G(e)M_e$, since $G_e$ is dense in $M_e$. Thus, $C_G(e)M_e = C_M(e) \cap M_0$ is irreducible.

Similar arguments yield:
Corollary 2.2.4. (i) With the notation of Theorem 2.2.1, the product of \( M \) induces isomorphisms
\[
C_G^r(e) \times G_e \cong C_M^r(e) \cap M_0
\]
and
\[(G \times G)_e \times G_e \times G_e (M_e \times M_e) \cong (M_0 \times M_0)_e.
\]
(ii) \( C_M^r(e) \cap M_0 \) is irreducible. In particular, \( C_M^r(e) \) is irreducible at \( e \).

Another geometric consequence of Proposition 2.1.2 and Theorem 2.2.1 is the following normality criterion:

Corollary 2.2.5. If \( M \) is normal at \( e \), then:
(i) The stabilizers \( M^e, M^r_e, M_e \) are irreducible and normal. In particular, \( G^e, G^r_e, G_e \) are connected.
(ii) The two-sided stabilizer \( (G \times G)_e \) is connected as well.
(iii) \( C_M^r(e), C_M^l(e), C_M^r(e) \) and \( (M \times M)_e \) are normal at \( e \).

Conversely, if one of the varieties \( M^e, M^r_e, M_e, C_M^r(e), C_M^l(e), C_M^r(e), (M \times M)_e \) is normal at \( e \), then \( M \) is also normal at \( e \).

Proof. (i) Denote by
\[
\nu : \tilde{M}^e \rightarrow M^e
\]
the normalization map of \( M^e \). Then the left action of \( G_e \) on \( M^e \) lifts to an action on \( \tilde{M}^e \). This yields a finite morphism
\[
\varphi : C_G^r(e) \times G_e \tilde{M}^e \rightarrow C_G^r(e) \times G_e M^e
\]
which restricts to an isomorphism over a dense open subvariety. Since \( C_G^r(e) \times G_e M^e \) is irreducible and normal (by the normality of \( M \) and Proposition 2.1.2), \( \varphi \) is an isomorphism. Thus, \( \nu \) is an isomorphism, that is, \( M^e \) is normal; this variety is also connected by Lemma 1.1.1 and hence irreducible. It follows that \( G_e^e \) is irreducible as well.

The same argument shows that \( M^r_e \) is normal. Likewise, the normality of \( M_e \) follows from Lemma 1.1.1 and Theorem 2.2.1.

(ii) follows from the connectedness of \( G_e \) in view of Lemma 1.3.3.

(iii) is a consequence of the normality of \( M_e \) together with Corollaries 2.2.3 and 2.2.4.

The converse statement is proved similarly. \( \square \)

Remark 2.2.6. Assume that \( G \) is reductive. Then the natural map
\[
R_u(C_G^r(e)) \times (C_G(e) \times G_e M_e) \times R_u(C_G^l(e)) \rightarrow M
\]
is an open immersion with image \( M_0 \). This statement follows from Theorem 2.2.1 combined with Remark 1.3.4; alternatively, this may be deduced from a local structure theorem for actions of reductive groups, see [T103, Section 6] or [AB04, Lemma 2.8].

Also, note that each orbit of \( M \) for the (left or right) \( G \)-action contains an idempotent. Hence the above statement describes the local structure of \( M \) at an arbitrary point.
2.3. The case of a minimal idempotent. In this subsection, we assume that the idempotent $e$ is **minimal**, that is, $e$ is the unique idempotent of $eMe$; equivalently, $e$ lies in the kernel $\ker(M)$, the unique closed orbit of $G \times G$ in $M$. Hence

$$\ker(M) = GeG = MeM.$$  \hfill (2.3.1)

Moreover, $eMe$ is an algebraic group with identity element $e$; the $G$-conjugates of $e$ are exactly the minimal idempotents of $M$ (for these results, see [Pu88, Chapter 6] and [Hu05, Section 1]). Combined with (1.2.4), it follows that

$$eMe = eGe = eC_G(e) = C_G(e)e.$$  \hfill (2.3.2)

Furthermore,

$$G = C^r_G(e)C^\ell_G(e)$$  \hfill (2.3.3)

by [Pu88, Theorem 6.30 and Corollary 6.34]. In view of Lemma 1.3.1, (iii), this implies in turn:

$$G = C^r_G(e)G^\ell_e = C^r_eC^\ell_G(e).$$  \hfill (2.3.4)

We now show that the open subvarieties that occur in Proposition 2.1.2 and Theorem 2.2.1 are all equal to $M$:

**Lemma 2.3.1.** (i) $Me = C^r_G(e)e$; equivalently, $M = C^r_G(e)M^\ell_e$. Likewise, $eM = eC^\ell_G(e)$ and $M = M^\ell_eC^r_G(e)$.

(ii) $M^r_e = G^r_e$; equivalently, $M^r_e = G^r_G(e)M$. Likewise, $eM^\ell_e = eG^\ell_e$ and $M^\ell_e = M\ell_G(e)$.

(iii) $M = C^r_G(e)M^\ell_e G^\ell_e = C^\ell_G(e)M^r_e G^\ell_e$ and $\ker(M) = C^r_G(e)eG^\ell_e = C^\ell_G(e)eG^\ell_e$.

(iv) $M^r_e = M^\ell_0 = M_0 = M$.

**Proof.** (i) By (2.3.1), (2.3.2) and (2.3.4), $Me = MeMe = GeGe = GeC_G(e) = Ge = C^r_G(e)G^\ell_e = C^\ell_G(e)e$.

(ii) Let $x \in M^r_e$, then $xe \in C^r_G(e)e = C_G(e)G^\ell_e$. Write accordingly $xe = ghe$, then

$$e = ex = exe = eghe = gehe = ge.$$  

Thus, $g \in G_e$ and $x \in G^r_e$.

(iii) follows from (i) and (ii) together with (2.3.1); likewise, (iv) follows from (i) and (iii). \hfill $\square$

Together with Theorem 2.2.1 and Corollaries 2.2.3 and 2.2.4, this lemma implies the following global structure result:

**Corollary 2.3.2.** For any minimal idempotent $e$, the product of $M$

$$C^r_G(e) \times G^e M^r_e \times G^e G^\ell_e \cong M, \quad M^\ell_e \times G^e G^\ell_e \cong M^\ell_e,$$

$$C^r_G(e) \times G^e M^r_e \cong C^r_M(e), \quad C_G(e) \times G^e M_e \cong C_M(e),$$
and \((G \times G)_e \times^{G_e \times G_e} (M_e \times M_e) \cong (M \times M)_e\).

Also, \(M\) is normal if and only if it is normal at some minimal idempotent, since the normal locus is stable under the two-sided \(G \times G\)-action. Together with Corollaries \ref{cor:2.2.5} and \ref{cor:2.3.2} this implies in turn:

**Corollary 2.3.3.** Let \(e\) be a minimal idempotent of an irreducible algebraic monoid \(M\). Then the following assertions are equivalent:

(i) \(M\) is normal.

(ii) All the varieties \(M_e, M^\ell_e, M^r_e, C_M(e), C^\ell_M(e), C^r_M(e)\) and \((M \times M)_e\) are irreducible and normal.

(iii) At least one of these varieties is normal at \(e\).

3. **The structure of irreducible monoids**

3.1. **Local structure.** In this subsection, we extend most results of the previous sections to an arbitrary (possibly non-affine) irreducible algebraic monoid \(M\) with unit group \(G\).

As in \cite{BR07}, which treats the case where \(M\) is normal, our main tool is a theorem of Chevalley: there exists a unique exact sequence of connected algebraic groups

\[1 \rightarrow G_{\text{aff}} \rightarrow G \rightarrow \mathcal{A}(G) \rightarrow 0,\]

where \(G_{\text{aff}}\) is affine and \(\mathcal{A}(G)\) is an abelian variety (see \cite{Ch60}, and \cite{Co02} for a modern proof). It follows that \(G_{\text{aff}}\) is the maximal closed connected affine subgroup of \(G\), while the quotient morphism \(\alpha_G : G \rightarrow \mathcal{A}(G)\) is the Albanese morphism of the variety \(G\) (the universal morphism to an abelian variety, see \cite{Se58b}).

Denote by \(M_{\text{aff}}\) the closure of \(G_{\text{aff}}\) in \(M\). Clearly, \(M_{\text{aff}}\) is a submonoid of \(M\) with identity element \(1\) and unit group \(G_{\text{aff}}\). In fact, \(M_{\text{aff}}\) is affine by \cite{Ri06, Theorem 2}; as a consequence, \(M_{\text{aff}}\) is the maximal closed irreducible affine submonoid of \(M\). Moreover, the natural map

\[\pi : G \times^{G_{\text{aff}}} M_{\text{aff}} \rightarrow M, \quad (g, x)_{G_{\text{aff}}} \mapsto gx\]

is birational (since \(\pi\) restricts to an isomorphism \(G \times^{G_{\text{aff}}} G_{\text{aff}} \rightarrow G\)) and proper (since \(G/G_{\text{aff}} \cong \mathcal{A}(G)\) is complete). It follows that \(\pi\) is surjective, that is,

\[M = GM_{\text{aff}}.\]

Let \(C\) denote the centre of \(G\); then \(G = CG_{\text{aff}}\) (see e.g. \cite{Se58a, Lemme 2}). As a consequence,

\[G = C^0G_{\text{aff}} \quad \text{and} \quad M = C^0M_{\text{aff}},\]

where \(C^0\) denotes the neutral component of \(C\). In particular,

\[C^0/(C^0 \cap G_{\text{aff}}) \cong G/G_{\text{aff}} \cong \mathcal{A}(G),\]
and the natural map
\[(3.1.6) \quad \pi : C^0 \times C^0 \rightarrow M_{\text{aff}} \rightarrow M\]
is proper and birational. This yields the following generalization of [BR07 Corollary 2.4]:

**Lemma 3.1.1.** Any idempotent of $M$ is contained in $M_{\text{aff}}$.

**Proof.** Given $x \in M$, the (set-theoretical) fibre of $\pi$ at $x$ may be identified with the subvariety
\[
\{ z(C^0 \cap G_{\text{aff}}) \mid z \in C^0, \ z^{-1}x \in M_{\text{aff}} \} \subset C^0/(C^0 \cap G_{\text{aff}}) \cong \mathcal{A}(G).
\]
If $x$ is idempotent, then the above subvariety is a closed subsemigroup of $\mathcal{A}(G)$, and hence is a group by [Re05, 3.5.1 Exercises 1 and 2]. It follows that $x \in M_{\text{aff}}$. \[\square\]

We now choose an idempotent $e \in M_{\text{aff}}$ and define the stabilizers $M_e^\ell, M_e^r, M_e \subset M$ and $G_e^\ell, G_e^r, G_e \subset G$ as in Section 1.1. Then, $M_e^\ell$ is a submonoid of $M$ with identity element 1 and unit group $G_e^\ell$, and likewise for $M_e^r, M_e$.

**Lemma 3.1.2.** The stabilizers $G_e^\ell, G_e^r$ and $G_e$ are affine.

**Proof.** Recall that $G_e^\ell$ is the isotropy group of the point $e \in M$ for the left $G$-action. Since this action is faithful, $G_e^\ell$ is affine by [Ma63 Lemma p. 54]. So $G_e^r$ and $G_e = G_e^e \cap G_e^r$ are affine as well. \[\square\]

**Remarks 3.1.3.** (i) The two-sided stabilizer $(G \times G)_e$ is not necessarily affine, as it contains $C^0$ embedded diagonally in $G \times G$.

(ii) In general, the stabilizers are not contained in $G_{\text{aff}}$, as shown by [BR07 Example 2.7]. Specifically, let $A$ be a non-trivial abelian variety, $F \subset A$ a non-trivial finite subgroup, and $M$ the commutative monoid obtained from the product monoid $A \times A^3$ by identifying the points $(x,0)$ and $(x+f,0)$, for all $x \in A$ and $f \in F$. Then $G = A \times \mathbb{G}_m$, $G_{\text{aff}} = \mathbb{G}_m$, and the image of $(0,0)$ in $M$ is an idempotent with stabilizer

$$G_e = G_e^e = G_e^r = F \times \mathbb{G}_m,$$

which strictly contains $G_{\text{aff}}$.

Similarly, we may define the centralizers $C_M^\ell(e), C_M^r(e), C_M(e) \subset M$ and $C_G^\ell(e), C_G^r(e), C_G(e) \subset G$ as in Section 1.2. Then

$$C_M^\ell(e) = C^0 C_M^\ell(e), \quad C_M^r(e) = C^0 C_M^r(e), \quad C_M(e) = C^0 C_M(e)$$
and likewise

$$C_G^\ell(e) = C^0 C_G^\ell(e), \quad C_G^r(e) = C^0 C_G^r(e), \quad C_G(e) = C^0 C_G(e).$$

In particular, these closed subvarieties are all connected, and

$$C_G^\ell(e) = G_e^r C_G(e), \quad C_G^r(e) = G_e^\ell C_G(e).$$
Moreover, $M \ell e = C^0 G_{\text{aff}} e$ contains $C^0 C_G^r(e) e = C_G^r(e) e$ as a dense open subvariety, by Lemma 2.1.1.

Thus, all the statements of Proposition 2.1.2 and Theorem 2.2.1 hold in this setting, except for the affineness of $M_0^\ell$, $M_0^r$ and $M_0$; the proofs are exactly the same. Corollaries 2.2.3, 2.2.4 and 2.2.5 hold as well, since their proofs do not use any assumption of affineness. Combined with the following result, this reduces the local structure of irreducible algebraic monoids to that of connected affine monoids having a dense unit group.

**Lemma 3.1.4.** The stabilizers $M_\ell^e$, $M_r^e$ and $M^e$ are affine and connected. Their unit groups $G_\ell^e$, $G_r^e$, $G^e$ are dense.

**Proof.** With the notation of Proposition 2.1.2, the preimage

$$(f^r)^{-1}(G \cap M_0^\ell) = C_G^r(e) \times^{G^e} (G \cap M_\ell^e) = C_G^r(e) \times^{G^e} C_r^G e$$

is dense in $C_G^r(e) \times^{G^e} M_\ell^e$, as $G$ is dense in $M$. It follows that $G_\ell^e$ is dense in $M_\ell^e$. Since $G_\ell^e$ is affine, this implies the affineness of $M_\ell^e$ in view of [Ri06, Theorem 3].

The connectedness of $M_\ell^e$ is obtained by arguing as in the proof of Lemma 1.1.1.

Likewise, the desired properties of $M_e$ follow from the statement of Theorem 2.2.1. □

These considerations yield the following smoothness criterion:

**Corollary 3.1.5.** Let $e$ be an idempotent of an irreducible algebraic monoid $M$. Then $e$ is a smooth point of $M$ if and only if the variety $M_e$ is an affine space.

**Proof.** By Theorem 2.2.1, $M$ is smooth at $e$ if and only if $M_e$ is smooth at $e$. So the assertion follows from the existence of an attractive $\mathbb{G}_m$-action on $M_e$ with fixed point $e$.

Specifically, let $\theta : \mathbb{G}_m \rightarrow G$ be as in Lemma 1.1.1. Then the $\mathbb{G}_m$-action on $G$ via $t \cdot x = \theta(t) x$ extends to an action of the multiplicative monoid $\mathbb{A}^1$ on $M_e$, such that $0 \cdot x = ex = e$ for all $x \in M_e$. This yields a positive grading of the algebra of regular functions on the affine variety $M_e$. Now the graded version of Nakayama’s lemma implies our assertion. □

**Remark 3.1.6.** The above smoothness criterion raises the question of classifying algebraic monoid structures on a given affine $n$-space, having the origin as their zero element. When the unit group is reductive, such structures correspond bijectively to decompositions of $n$ into a sum of squares of positive integers, as the corresponding monoids are just products of matrix monoids.

Indeed, if $M$ is a smooth monoid with reductive unit group $G$ and zero element 0, then the variety $M$ is equivariantly isomorphic to the $G \times G$-module $T_0 M$, as follows from the Luna slice theorem. This
yields a $G \times G$-equivariant isomorphism $\varphi : M \to \prod_{i=1}^{m} \text{End}(V_i)$, where $V_1, \ldots, V_m$ are simple $G$-modules; as a consequence, $\varphi$ is an isomorphism of monoids. Thus, $G$ is identified to an open subgroup of the product $\prod_{i=1}^{m} \text{GL}(V_i)$, and hence to the whole product. (This is also proved in [Ti03, Section 11], via a representation-theoretic argument.)

In the case that $e$ is a minimal idempotent of $M_{\text{aff}}$, the subvariety $e G e G = C^0 G_{\text{aff}}^e G_{\text{aff}}$ is the unique closed orbit of $G_{\text{aff}} \times G_{\text{aff}}$ in $M_{\text{aff}}$. As $\pi$ is proper, it follows that $G e G = C^0 G_{\text{aff}}^e G_{\text{aff}}$ is the unique closed $G \times G$-orbit in $M$. In other words, $G e G$ is the kernel of $M$. Then all the statements of Section 2.3 hold, with exactly the same proofs.

Also, note that the minimal idempotents of $M$ are exactly those of $M_{\text{aff}}$ (by Lemma 3.1.1); they form a unique conjugacy class of $G_{\text{aff}}$, or, equivalently, of $G$ by (3.1.4).

Since the smooth locus of $M$ is stable under the two-sided action of $G \times G$, we see that $M$ is smooth if and only if $M_e$ is an affine space for some minimal idempotent $e$.

3.2. Global structure. By the main result of [BR07], the map $\pi$ of (3.1.2) is an isomorphism whenever $M$ is normal, and then $M_{\text{aff}}$ is normal as well. In other words, any normal monoid is an induced variety over an abelian variety, with fibre a normal affine monoid.

This statement does not extend to arbitrary irreducible monoids, in view of [BR07, Example 2.7]. Yet we show that any such monoid is an induced variety over an abelian variety, with fibre a connected affine monoid having a dense unit group:

**Theorem 3.2.1.** Let $M$ be an irreducible algebraic monoid, and $G$ its unit group. Then there exists a closed submonoid $N \subset M$ satisfying the following properties:

(i) $N$ is affine, connected, and contains $1$.

(ii) The unit group $H := G(N)$ is dense in $N$, and contains $G_{\text{aff}}$ as a subgroup of finite index. In particular, $M_{\text{aff}}$ is the irreducible component of $N$ containing $1$.

(iii) The canonical map

$$\varphi : G \times^H N \longrightarrow M, \quad (g,n)H \longmapsto gn$$

is an isomorphism of varieties.

Moreover, the projection $p : G \times^H N \to G/H$ is identified with the Albanese morphism of the variety $M$. In particular, $H$ and $N$ are uniquely determined by $M$.

**Proof.** We begin with the proof of the final assertion: we assume that $M = G \times^H N$ where $H$ and $N$ satisfy (i)–(iii), and show that $p$ equals the Albanese morphism $\alpha_M : M \to \mathcal{A}(M)$. The latter morphism is uniquely determined up to a translation in $\mathcal{A}(M)$; we normalize it by imposing that $\alpha_M(1) = 0$ (the origin of the abelian variety $\mathcal{A}(M)$).
Consider a morphism (of varieties)
\[ \alpha : G \times^H N \to A \]
where \( A \) is an abelian variety. The restriction of \( \alpha \) to the neutral component \( H^0 \subset N \) is a morphism from a connected affine algebraic group to an abelian variety, and hence is constant (as follows e.g. from \cite[Corollary 3.9]{Mi86}). Thus, \( \alpha \) is constant on every irreducible component of \( N \). Since \( N \) is connected, \( \alpha \) maps \( N \) to a point; likewise, it maps each fibre of \( p \) (that is, each translate \( gN \) in \( G \times^H N \)) to a point. Together with Zariski’s Main Theorem, this implies that \( \alpha \) factors as \( p \) followed by a morphism \( G/H \to A \). This proves the desired equality.

In particular, \( N \) is identified with the fibre of \( \alpha_M \) at 0. We now show that this fibre satisfies the properties (i)--(iii).

By rigidity, the restriction \( \alpha_M|_G \) is a homomorphism of algebraic groups (see e.g. \cite[Corollary 3.6]{Mi86}). Thus, \( \alpha_M \) is a homomorphism of algebraic monoids. In particular, \( N \) is a closed submonoid of \( M \) containing 1.

Moreover, \( \alpha_M|_G \) factors through a unique homomorphism \( \mathcal{A}(G) \to \mathcal{A}(M) \), which is surjective as \( G \) is dense in \( M \). Since \( \mathcal{A}(G) = G/G_{\text{aff}} \), we may identify \( \mathcal{A}(M) \) with the homogeneous space \( G/H \), where \( H \) is a closed subgroup of \( G \) containing \( G_{\text{aff}} \). This identifies \( M \) with \( G \times^H N \), equivariantly for the right \( G \)-action on \( M \).

Since \( M \) is connected, it follows that \( H \) acts transitively on the connected components of \( N \). Let \( N' \subset N \) be the connected component containing 1, and \( H' \subset H \) its stabilizer. Then the canonical map \( H \times^{H'} N' \to N \) is an isomorphism, as \( H/H' \) is identified with the set of connected components of \( N \). Thus, the analogous map \( G \times^{H'} N' \to M \) is an isomorphism as well. Moreover, since \( H' \) has finite index in \( H \), and \( G/H \) is complete, it follows that \( G/H' \) is complete as well. Thus, the composite map \( M \cong G \times^{H'} N' \to G/H' \) factors through a \( G \)-equivariant morphism \( G/H \to G/H' \). This implies that \( H = H' \) and \( N = N' \), i.e., \( N \) is connected.

Likewise, since \( G \) is dense in \( M \), it follows that \( H \) is dense in \( N \). To complete the proof, it suffices to show that the quotient \( H/G_{\text{aff}} \) is finite. Indeed, this implies that \( H \) is affine and, in turn, that \( N \) is affine in view of \cite[Theorem 3]{Ri06}.

The finiteness of \( H/G_{\text{aff}} \) is equivalent to the assertion that the canonical homomorphism
\[ G/G_{\text{aff}} = \mathcal{A}(G) \to \mathcal{A}(M) = G/H \]
has a finite kernel, and hence to the existence of a \( G \)-equivariant morphism
\[ \psi : M \to \mathcal{A}(G)/F, \]
where \( F \subset \mathcal{A}(G) \) is a finite subgroup.
To construct such a morphism $\psi$, choose a minimal idempotent $e \in M$ and recall that $Me = Ge \cong G/G^e_e$ (see Lemma 2.3.1). This yields a $G$-equivariant morphism $\gamma : M \rightarrow G/G^e_e$. Now let

$$\psi : M \rightarrow A(G/G^e_e)$$

be the composition of $\gamma$ with the Albanese morphism of $G/G^e_e$. Then $\psi$ is $G$-equivariant. Moreover, $A(G/G^e_e)$ is the quotient of $A(G)$ by the image of the subgroup $G^e_e$, and the latter image is a finite group (as $G^e_e$ is affine by Lemma 3.1.2).

□

Remark 3.2.2. We may define a natural structure of algebraic monoid on $G \times H N$ so that the map $\varphi$ of (3.2.1) is an isomorphism of algebraic monoids. Indeed, the canonical map

$$C^0 \times C^0 \cap H N \rightarrow G \times H N$$

is an isomorphism, as $G = C^0 H \cong C^0 \times C^0 \cap H$. Moreover, $C^0 \times C^0 \cap H N$ is the quotient of the product monoid $C^0 \times N$ by the central subgroup $C^0 \cap H$, embedded via $x \mapsto (x, x^{-1})$.

Alternatively, one may observe that the $H$-action on $N$ by conjugation extends uniquely to a $G$-action, where $C^0$ acts trivially (since $C^0 \cap H$, a central subgroup of $H$, acts trivially on $N$ by conjugation). Thus, one may form the semi-direct product of monoids $G \times N$: its product is given by

$$(x, a) \cdot (y, b) = (xy, a^y b)$$

where $a^z$ denotes the conjugate of $a \in N$ by $z \in G$ (see [Re05, Example 3.7]). Then one checks that this product induces a unique product on $G \times H N$ such that the quotient map $G \times N \rightarrow G \times H N$ is a homomorphism of monoids.

The above construction is an analogue for algebraic monoids of the induction of varieties with group actions.

3.3. Some applications. We begin by stating two direct consequences of Theorem 3.2.1, first obtained in [BR07] for normal monoids:

Corollary 3.3.1. Any irreducible algebraic monoid is quasi-projective.

Corollary 3.3.2. The category of irreducible algebraic monoids is equivalent to the category having as objects the triples $(G, H, N)$, where $G$ is a connected algebraic group, $H \subset G$ is a closed subgroup containing $G_{aff}$ as a subgroup of finite index, and $N$ is a connected affine algebraic monoid with unit group $H$, dense in $N$.

The morphisms from such a triple $(G, H, N)$ to a triple $(G', H', N')$ are the pairs $(\varphi, \psi)$, where $\varphi : G \rightarrow G'$ is a homomorphism of algebraic groups such that $\varphi(H) \subset H'$, and $\psi : N \rightarrow N'$ is a homomorphism of algebraic monoids such that $\varphi|_H = \psi|_H$.

Another consequence is a characterization of monoids among (possibly non-normal) equivariant embeddings of algebraic groups:
Corollary 3.3.3. Let $G$ be a connected algebraic group and let $X$ be a $G \times G$-equivariant embedding of $G$. Then $X$ admits a (unique) structure of algebraic monoid if and only if $\alpha_X$ is affine.

Proof. If $X$ is an irreducible algebraic monoid, then its Albanese morphism is affine by Theorem 3.2.1. For the converse, arguing as in the proof of that theorem, one shows that $\mathcal{A}(X) \cong G/H$, where $H \subset G$ is a closed subgroup containing $G_{\text{aff}}$; moreover, $\alpha_X$ is $G$-equivariant. Thus, $X \cong G \times^H Y$, where $Y$ is an equivariant embedding of the (possibly non-connected) algebraic group $H$. Moreover, $Y$ is affine by assumption, and hence is an algebraic monoid. In particular, its unit group $H$ is affine, so that $H/G_{\text{aff}}$ is finite. As in Remark 3.2.2 the induced variety $X$ is then an algebraic monoid. □

Next, we show how to recover the main result of [BR07] (Theorem 4.1 and its proof):

Corollary 3.3.4. For any irreducible algebraic monoid $M$, the morphism $\pi : G \times G_{\text{aff}} M_{\text{aff}} \to M$ of (3.1.2) is finite.

In particular, $M$ is normal if and only if the associated triple satisfies: $H = G_{\text{aff}}$ and $N = M_{\text{aff}}$ is normal.

Proof. Since $\pi$ is proper and $G$-equivariant, and $M = GN$, the finiteness of $\pi$ is equivalent to the finiteness of its restriction to $\pi^{-1}(N)$. But

$$\pi^{-1}(N) = H \times G_{\text{aff}} M_{\text{aff}}$$

by Theorem 3.2.1. Furthermore, $\pi|_{\pi^{-1}(N)}$ factors as the closed embedding

$$H \times G_{\text{aff}} M_{\text{aff}} \to H \times G_{\text{aff}} N$$

(corresponding to the inclusion of $M_{\text{aff}}$ into $N$), followed by the isomorphism

$$H \times G_{\text{aff}} N \cong (H/G_{\text{aff}}) \times N$$

(since $N$ is $H$-stable), followed in turn by the projection

$$(H/G_{\text{aff}}) \times N \to N,$$

a finite morphism.

Since $\pi$ is birational and finite, it is an isomorphism whenever $M$ is normal, by Zariski’s Main Theorem; it then follows that $M_{\text{aff}}$ is normal as well. Moreover, $H = G_{\text{aff}}$ and $N = M_{\text{aff}}$ by the uniqueness statement in Theorem 3.2.1. The converse is obvious. □

Finally, one may show as in [BR07] Theorem 5.3 that any irreducible algebraic monoid $M$ has a faithful representation by endomorphisms of a homogeneous vector bundle over an abelian variety (the Albanese variety of $M$.)
References

[AB04] V. Alexeev and M. Brion, Stable reductive varieties I: Affine varieties, Invent. math. 157 (2004), 227–274.

[Br07] M. Brion, Some basic results on actions of non-affine algebraic groups, arXiv: math.AG/0702518.

[BR07] M. Brion and A. Rittatore, The structure of normal algebraic monoids, Semigroup Forum 74 (2007), 410–422.

[Ch60] C. Chevalley, Une démonstration d’un théorème sur les groupes algébriques, J. Math. Pures Appl. (9) 39 (1960), 307–317.

[Co02] B. Conrad, A modern proof of Chevalley’s theorem on algebraic groups, J. Ramanujam Math. Soc. 17 (2002), 1–18.

[Hu05] W. Huang, The kernel of a linear algebraic semigroup, Forum Math. 17 (2005), 851–869.

[Ma63] H. Matsumura, On algebraic groups of birational transformations, Atti Acad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) 34 (1963), 151–155.

[Mi86] J. S. Milne, Abelian Varieties, in: Arithmetic Geometry (G. Cornell and J. H. Silverman, eds.), 103–150, Springer-Verlag, New York, 1986.

[MFK94] D. Mumford, J. Fogarty and F. Kirwan, Geometric Invariant Theory, 3rd enlarged edition, Ergeb. Math. 36, Springer-Verlag, 1994.

[Pu88] M. S. Putcha, Linear Algebraic Monoids, London Math. Soc. Lecture Note Series 133, Cambridge University Press, Cambridge, 1988.

[Re84] L. E. Renner, Quasi-affine algebraic monoids, Semigroup Forum 30 (1984), 167–176.

[Re05] L. E. Renner, Linear Algebraic Monoids, Invariant Theory and Algebraic Transformation Groups, V, Encyclopædia Math. Sci. 134, Springer-Verlag, Berlin, 2005.

[Ri98] A. Rittatore, Algebraic monoids and group embeddings, Transform. Groups 3 (1998), 375–396.

[Ri06] A. Rittatore, Algebraic monoids with affine unit group are affine, Transform. Groups 12 (2007), 601–605.

[Ro61] M. Rosenlicht, On quotient varieties and the affine embedding of certain homogeneous spaces, Trans. Amer. Math. Soc. 101 (1961), 211–223.

[Se58a] J.-P. Serre, Espaces fibrés algébriques, Séminaire C. Chevalley (1958), Exposé No. 1, Documents Mathématiques 1, Soc. Math. France, Paris, 2001.

[Se58b] J.-P. Serre, Morphismes universels et variété d’Albanese, Séminaire Chevalley (1958–1959), Exposé No. 10, Documents Mathématiques 1, Soc. Math. France, Paris, 2001.

[Ti03] D. A. Timashev, Equivariant compactifications of reductive groups, Russ. Acad. Sci. Sb. Math. 194 (2003), No. 4, 589–616.

[Vi95] E.B. Vinberg, On reductive algebraic semigroups, in: Lie groups and Lie algebras: E.B. Dynkin’s seminar, Amer. Math. Soc. Transl. Ser. 2 169 (1995), 145–182.

Université de Grenoble I, Département de Mathématiques, Institut Fourier, UMR 5582 du CNRS, 38402 Saint-Martin d’Hères Cedex, France

E-mail address: Michel.Brion@ujf-grenoble.fr