Non-homogeneous boundary value problems for coupled KdV-KdV systems posed on a quarter plane

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Abstract

In this article, we study an initial-boundary-value problem of a coupled KdV-KdV system on the half line $\mathbb{R}^+$ with non-homogeneous boundary conditions:

\[
\begin{aligned}
&u_t + v_x + uu_x + v_{xxx} = 0, \\
v_t + u_x + (vu)_x + u_{xxx} = 0, \\
&u(x,0) = \phi(x), \quad v(x,0) = \psi(x), \\
u(0,t) = h_1(t), \quad v(0,t) = h_2(t), \quad v_x(0,t) = h_3(t),
\end{aligned}
\]

It is shown that the problem is locally unconditionally well-posed in $H^s(\mathbb{R}^+) \times H^s(\mathbb{R}^+)$ for $s > -\frac{3}{4}$ with initial data $(\phi, \psi)$ in $H^s(\mathbb{R}^+) \times H^s(\mathbb{R}^+)$ and boundary data $(h_1, h_2, h_3)$ in $H^{\frac{3s}{4}}(\mathbb{R}^+) \times H^{\frac{3s}{4}}(\mathbb{R}^+) \times H^{s}(\mathbb{R}^+)$. The approach developed in this paper can also be applied to study more general KdV-KdV systems posed on a quarter plane.

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1 Introduction

The coupled KdV-KdV system
\[
\begin{align*}
    u_t + v_x + uu_x + \frac{1}{6}v_{xxx} &= 0, \\
    v_t + u_x + (vu)_x + \frac{1}{6}u_{xxx} &= 0,
\end{align*}
\]  

(1.1)
is a special case of a broad class of Boussinesq systems or the so-called abcd systems,
\[
\begin{align*}
    u_t + v_x + (uv)_x + av_{xxx} - bu_{xxt} &= 0, \\
    v_t + u_x + vv_x + cu_{xxx} - dv_{xxt} &= 0,
\end{align*}
\]

(1.2)
where the constant coefficients \(a, b, c, d\) satisfy
\[
a + b = \frac{1}{2} \left( \eta^2 - \frac{1}{3} \right), \quad c + d = \frac{1}{2} \left( 1 - \eta^2 \right) \geq 0, \quad a + b + c + d = \frac{1}{3}.
\]

This four-parameter family of Boussinesq system was derived by Bona, Chen and Saut \[BCS02, BCS04\] from the two-dimensional Euler equations for free-surface flow to describe the two-way propagation of small-amplitude, long wavelength, gravity waves on the surface of water in a canal which arise also when modeling the propagation of long-crested waves on large lakes or the ocean. Compared to those uni-directional models, such as the KdV equation,
\[
    u_t + uu_x + u_{xxx} = 0,
\]
the Boussinesq equation,
\[
    u_{tt} + uu_x + (u^2)_{xx} - u_{xxxx} = 0,
\]
and the BBM equation,
\[
    u_t + uu_x - u_{xxt} = 0,
\]
the bi-directional systems (1.2) are more interesting due to their wider range of potential in application. Since significant amount of theoretical study has been conducted on the single equation models, the rich technique used there will guide our research on the systems due to their formally equivalence and structure similarities.

The Cauchy problem of the system (1.1) posed either on \(\mathbb{R}\) or on the periodic domain \(\mathbb{T}\) has been well studied. In particular, the Cauchy problem of the system (1.1) is known to be analytically well-posed in the space \(H^s(\mathbb{R})\) for any \(s > -\frac{3}{4}\) but related bilinear estimates fails in \(H^s(\mathbb{R})\) for any \(s < -\frac{3}{4}\). (See e.g. \[AC08, BGK10, YZ22\].)

In this paper, we are concerned with the initial-boundary-value problem (IBVP) of the coupled KdV-KdV system posed on the quarter plane,
\[
\begin{align*}
    u_t + v_x + uu_x + v_{xxx} &= 0, \\
    v_t + u_x + (vu)_x + u_{xxx} &= 0, \\
    u(x, 0) = \phi(x), \quad v(x, 0) = \psi(x), \\
    u(0, t) = h_1(t), \quad v(0, t) = h_2(t), \quad v_x(0, t) = h_3(t),
\end{align*}
\]

(1.3)
for its well-posedness in the space $H^s(\mathbb{R}^+) \times H^s(\mathbb{R}^+)$. 

For the convenience of adopting results and techniques obtained in the investigation of KdV type equations, the new variables $U = \frac{1}{4}(u + v)$ and $V = \frac{1}{4}(u - v)$ are introduced so that the linear parts of the new equation are KdV type on $U$ and $V$ respectively. The system on $U$ and $V$ reads as

$$
\begin{align*}
&\begin{cases}
U_t + U_x + U_{xxx} + 3UU_x + (UV)_x - VV_x = 0, \\
V_t - V_x - V_{xxx} - UU_x + (UV)_x + 3VV_x = 0, \\
U(x, 0) = \frac{1}{4}(\phi + \psi), \\
V(x, 0) = \frac{1}{4}(\phi - \psi), \\
U(0, t) = \frac{1}{4}(h_1 + h_2), \\
V(0, t) = \frac{1}{4}(h_1 - h_2), \\
V_x(0, t) - U_x(0, t) = -\frac{1}{2}h_3,
\end{cases} \\
x, t > 0.
\end{align*}
$$

For ease of notations, we write $(U, V)$ as $(u, v)$, then this reduces to study the following IBVP,

$$
\begin{align*}
&\begin{cases}
u_t + v_{xxx} + u_x + 3uu_x + (uv)_x - vv_x = 0, \\
u_t - v_{xxx} - v_x - uu_x + (uv)_x + 3vv_x = 0, \\
u(x, 0) = \phi(x), \\
v(x, 0) = \psi(x), \\
u(0, t) = h_1(t), \\
v(0, t) = h_2(t), \\
v_x(0, t) - u_x(0, t) = h_3(t),
\end{cases} \\
x, t > 0.
\end{align*}
$$

(1.4)

which is equivalent to the IBVP (1.3) for its well-posedness in the space $H^s(\mathbb{R}^+) \times H^s(\mathbb{R}^+)$. 

**Definition 1.1.** For given $s \in \mathbb{R}$, the IBVP (1.4) is said to be locally well-posed in the space

$$
H^s_x(\mathbb{R}^+) := H^s(\mathbb{R}^+) \times H^s(\mathbb{R}^+)
$$

if for any $r > 0$, there exists a $T > 0$ depending only on $r$ and $s$ such that for any naturally compatible\footnote{For the concept of compatibility, the reader can refer to Definition 1.9} $(\phi, \psi) \in H^s_x(\mathbb{R}^+)$, and

$$
\bar{h} = (h_1, h_2, h_3) \in H^s_r(\mathbb{R}^+) := H^{\frac{s+1}{4}}(\mathbb{R}^+) \times H^{\frac{s+1}{4}}(\mathbb{R}^+) \times H^{\frac{s}{4}}(\mathbb{R}^+),
$$

with

$$
\|\phi, \psi\|_{H^s_x(\mathbb{R}^+)} + \|\bar{h}\|_{H^s_r(\mathbb{R}^+)} \leq r,
$$

the IBVP (1.4) admits a unique mild solution $(u, v) \in C([0, T]; H^s_x(\mathbb{R}^+))$ and, moreover, the solution map is continuous in the corresponding spaces. If the solution map is real analytic instead, then the IBVP (1.4) is said to be locally analytically well-posed.

The concept of the mild solution of the IBVP (1.4) is as described below.

**Definition 1.2.** Let $s \in \mathbb{R}$ and $T > 0$ be given.

(i) A function pair $(u, v) \in C([0, T]; H^s_x(\mathbb{R}^+)) \cap C^1([0, T]; H^0_x(\mathbb{R}^+))$ is said to be a strong solution of the IBVP (1.3) if

$$
(u, v, v_x)|_{x=0} \in H^s_r(0, T)
$$

and the equations in (1.3) holds for any $t \in (0, T)$ and for any $x \in \mathbb{R}^+$, a.e..
(ii) A function pair \( (u, v) \in C([0, T]; H_x^s(\mathbb{R}^+)) \) is said to be a **mild solution** of the IBVP (1.4) if there exists a sequence of strong solutions \( (u_n, v_n) \in C([0, T]; H_x^s(\mathbb{R}^+)) \) such that

\[
\lim_{n \to \infty} (u_n, v_n) \to (u, v) \quad \text{in} \quad C([0, T]; H_x^s(\mathbb{R}^+))
\]

and

\[
\lim_{n \to \infty} (h_{1,n}, h_{2,n}, h_{3,n}) \to (h_1, h_2, h_3) \quad \text{in} \quad H_t^s(0, T)
\]

with

\[
(h_{1,n}, h_{2,n}, h_{3,n}) = (u_n, v_n, \partial_x v_n)|_{x=0}.
\]

We intend to find those values of \( s \) for which the IBVP (1.4) is locally well-posed in the space \( H_x^s(\mathbb{R}^+) \). The following theorem is the main result of this paper.

**Theorem 1.3.** The IBVP (1.4) is locally analytically well-posed in the space \( H_x^s(\mathbb{R}^+) \), with any naturally compatible data \( (\phi, \psi) \in H_x^s(\mathbb{R}^+) \) and \( \vec{h} = (h_1, h_2, h_3) \in H_t^s(\mathbb{R}^+) \), for any \( s > -\frac{3}{4} \).

To prove Theorem 1.3 which is a result on local well-posedness, we use scaling argument to introduce

\[
u^\beta(x, t) = \beta u(\beta^\frac{1}{3} x, \beta^\frac{2}{3} t) \quad \text{and} \quad \psi^\beta(x, t) = \beta \psi(\beta^\frac{1}{3} x, \beta^\frac{2}{3} t),
\]

where \( \beta \in (0, 1) \) is a parameter. Then (1.4) becomes

\[
\begin{cases}
    u_t + u_{xxx} + \beta u_x + 3uu_x + (u^3v)_x - v^3 = 0, \\
    v_t - v_{xxx} - \beta v_x - u^3u_x + (u^3v)_x + 3v^3v_x = 0, \\
    u(x, 0) = \phi^\beta(x), \quad v(x, 0) = \psi^\beta(x), \\
    u(0, t) = h_1^\beta(t), \quad v(0, t) = h_2^\beta(t), \quad v_x(0, t) - u_x(0, t) = h_3^\beta(t),
\end{cases}
\]

where

\[
\begin{align*}
(\phi^\beta(x) &= \beta \phi(\beta^\frac{1}{3} x), \\
\psi^\beta(x) &= \beta \psi(\beta^\frac{1}{3} x), \\
h_1^\beta(t) &= \beta h_1(\beta^\frac{2}{3} t), \\
h_2^\beta(t) &= \beta h_2(\beta^\frac{2}{3} t), \\
h_3^\beta(t) &= \beta^\frac{2}{3} h_3(\beta^\frac{2}{3} t).
\end{align*}
\]

It is easy to check that the compatibility of \( (\phi, \psi) \) and \( (h_1, h_2, h_3) \) yields the compatibility of \( (\phi^\beta, \psi^\beta) \) and \( (h_1^\beta, h_2^\beta, h_3^\beta) \). On the other hand, when \( -\frac{3}{4} < s \leq 3 \),

\[
\| (\phi, \psi) \|_{H_x^s(\mathbb{R}^+)} \lesssim \beta^s \| (\phi^\beta, \psi^\beta) \|_{H_x^s(\mathbb{R}^+)} \quad \text{and} \quad \| (h_1^\beta, h_2^\beta, h_3^\beta) \|_{H_t^s(\mathbb{R}^+)} \lesssim \beta^s \| (h_1, h_2, h_3) \|_{H_t^s(\mathbb{R}^+)},
\]

which implies

\[
\lim_{\beta \to 0^+} \| (\phi^\beta, \psi^\beta) \|_{H_x^s(\mathbb{R}^+)} + \| (h_1^\beta, h_2^\beta, h_3^\beta) \|_{H_t^s(\mathbb{R}^+)} = 0.
\]

Thus, in order to prove Theorem 1.3 it is sufficient to establish Theorem 1.4 below for the following IBVP:

\[
\begin{cases}
    u_t + u_{xxx} + \beta u_x + 3uu_x + (u^3v)_x - v^3 = 0, \\
    v_t - v_{xxx} - \beta v_x - u^3u_x + (u^3v)_x + 3v^3v_x = 0, \\
    u(x, 0) = \phi(x), \quad v(x, 0) = \psi(x), \\
    u(0, t) = h_1(t), \quad v(0, t) = h_2(t), \quad v_x(0, t) - u_x(0, t) = h_3(t),
\end{cases}
\]

**Theorem 1.4.** For given \( s > -\frac{3}{4} \) and \( T > 0 \), there exists an \( \epsilon > 0 \) depending only on \( T \) and \( s \) such that for any \( \beta \in (0, 1] \) and for any naturally compatible data \( (\phi, \psi) \in H_x^s(\mathbb{R}^+) \) and \( \vec{h} = (h_1, h_2, h_3) \in H_t^s(\mathbb{R}^+) \)
with

$$\|\phi, \psi\|_{H^s_x(R^+)} + \|\delta\|_{H^t_x(R^+)} \leq \epsilon,$$

the IBVP \(\{L\}\) admits a unique mild solution \((u, v) \in C([0, T]; H^s_x(R^+))\) and, moreover, the solution map is real analytic in the corresponding spaces.

The theorems will be proved using the approach similar to that developed in Bona-Sun-Zhang [BSZ02], Colliander-Kenig [CK02] and Holmer [Hol06] for the KdV equation but with some modifications to handle the bi-directional system \(\{L\}\).

We first introduce some types of Bourgain spaces. Consider the Cauchy problem for the linear KdV equation posed on \(\mathbb{R}\),

$$w_t + \alpha w_{xx} + \beta w_x = 0, \quad w(x, 0) = w_0, \quad x \in \mathbb{R}, \ t \in \mathbb{R},$$

where \(\alpha \in \mathbb{R}\backslash\{0\}\) and \(\beta \in \mathbb{R}\). Denote by \(W^{\alpha, \beta}_R\) the semigroup operator associated to \(\{L\}\). The solution of \(\{L\}\) is given explicitly by

$$W^{\alpha, \beta}_R(t)w_0(x) = \int_{\mathbb{R}} e^{i\xi x} e^{i\alpha^\beta \xi t} \hat{w}_0(\xi) d\xi, \quad (1.8)$$

where \(\phi^{\alpha, \beta}\) is a cubic polynomial:

$$\phi^{\alpha, \beta}(\xi) := \alpha \xi^3 - \beta \xi$$

and \(\hat{w}_0(\xi)\) is the Fourier transform of \(w_0(x)\).

**Definition 1.5.** For \(\alpha, \beta, s, b \in \mathbb{R}\), the Fourier restriction space \(X^{\alpha, \beta}_{s,b}\) is defined to be the completion of the Schwarz class \(S(\mathbb{R}^2)\) under the norm

$$\|w\|_{X^{\alpha, \beta}_{s,b}} = \|(\xi)^s (\tau - \phi^{\alpha, \beta}(\xi))^b \hat{w}(\xi, \tau)\|_{L^2_{\xi, \tau}(\mathbb{R}^2)},$$

where \(\phi^{\alpha, \beta}\) is as defined in \(\{L\}\) and \(\hat{w}\) represents the space-time Fourier transform of \(w\).

When \(b > \frac{1}{2}\), it follows directly from the Sobolev embedding that \(X^{\alpha, \beta}_{s,b} \subset C_t(\mathbb{R}; H^b(\mathbb{R}))\). But the estimate on the boundary integral operator (see e.g. Lemma 3.3 and 3.9) forces \(b < \frac{1}{2}\), and then the space \(X^{\alpha, \beta}_{s,b}\) may not be in \(C_t(\mathbb{R}; H^b(\mathbb{R}))\). In addition, the bilinear estimates fail. In order to overcome these difficulties, we introduce some modified Fourier restriction spaces \(X^{\alpha, \beta}_{s,b,\sigma}\) and \(Y^{\alpha, \beta}_{s,b,\sigma}\). The spaces \(X^{\alpha, \beta}_{s,b,\sigma}\) will be used to justify the bilinear estimates and \(Y^{\alpha, \beta}_{s,b,\sigma}\) will serve as suitable solution spaces.

**Definition 1.6.** For \(\alpha, \beta, s, b, \sigma \in \mathbb{R}\) and \(\sigma > \frac{1}{2}\), let \(X^{\alpha, \beta}_{s,b,\sigma}\) be the completion of the Schwarz class \(S(\mathbb{R}^2)\) under the norm

$$\|w\|_{X^{\alpha, \beta}_{s,b,\sigma}} = \|\mathbbm{1}_{\{|\xi|\leq 3+|\tau|\}} (\xi)^s (\tau - \phi^{\alpha, \beta}(\xi))^b \hat{w}(\xi, \tau)\|_{L^2_{\xi, \tau}(\mathbb{R}^2)},$$

where \(\mathbbm{1}\) means the characteristic function. In addition, let

$$X^{\alpha, \beta}_{s,b,\sigma} = X^{\alpha, \beta}_{s,b} \cap X^{\alpha, \beta}_{s,b,\sigma} \quad \text{and} \quad Y^{\alpha, \beta}_{s,b,\sigma} = X^{\alpha, \beta}_{s,b,\sigma} \cap C_t(\mathbb{R}, H^b(\mathbb{R})).$$

The norm of the space \(X^{\alpha, \beta}_{s,b,\sigma}\) is defined to be

$$\|w\|_{X^{\alpha, \beta}_{s,b,\sigma}} = \|w\|_{X^{\alpha, \beta}_{s,b}} + \|w\|_{X^{\alpha, \beta}_{s,b,\sigma}}$$

$$\sim \|(\xi)^s \left(\mathbbm{1}_{\{|\xi|\leq 3+|\tau|\}} (\tau - \phi^{\alpha, \beta}(\xi))^b \hat{w}(\xi, \tau)\right)\|_{L^2_{\xi, \tau}(\mathbb{R}^2)},$$

\(\{L\}\)
where \( L = \tau - \phi^{\alpha,\beta}(\xi) \) and the norm of the space \( Y^{\alpha,\beta}_{s,b,\sigma} \) is defined to be

\[
\|w\|_{Y^{\alpha,\beta}_{s,b,\sigma}} = \|w\|_{X^{\alpha,\beta}_{s,b,\sigma}} + \sup_{t \in \mathbb{R}} \|w(x,t)\|_{H^s_x(R)} .
\] (1.12)

From the above definition, we can see that when \( b \leq \frac{1}{2} \), the space \( X^{\alpha,\beta}_{s,b,\sigma} \) provides more regularity in time than the space \( X^{\alpha,\beta}_{s,b} \) in the low frequency region and this improvement is essential to establish the bilinear estimates in Proposition 1.12. Note that for any \( b \leq \sigma \),

\[
\|w\|_{X^{\alpha,\beta}_{s,b,\sigma}} \leq \|w\|_{X^{\alpha,\beta}_{s,b}} .
\] (1.13)

Besides, when \( |\xi| \leq 1 \), \( e^{|\xi|} \leq 3 + |\tau| \). Thus

\[
\|w\|_{X^{\alpha,\beta}_{s,b,\sigma}} \geq \|\langle \xi \rangle^s \left[(L)^b + \mathbb{1}_{\{|\xi| \leq 1\}} (L)^\sigma \right] \hat{w}(\xi,\tau)\|_{L^2_{\xi,\tau}(\mathbb{R}^d)} ,
\] (1.14)

where \( L = \tau - \phi^{\alpha,\beta}(\xi) \).

Finally, for technical reasons, for \( \sigma > \frac{1}{2} \), we define the \( Z^{\alpha,\beta}_{s,\sigma-1} \) space to be the completion of the Schwarz class \( S(\mathbb{R}^d) \) with respect to the following norm.

\[
\|w\|_{Z^{\alpha,\beta}_{s,\sigma-1}} := \|\langle \tau \rangle^{\frac{s}{2}+\frac{1}{\sigma} - \sigma} \langle \tau - \phi^{\alpha,\beta}(\xi)\rangle^{\sigma-1} \hat{w}(\xi,\tau)\|_{L^2_{\xi,\tau}(\mathbb{R}^d)} .
\] (1.15)

This space is an intermediate space that is only used to estimate the inhomogeneous term in the Duhamel formula, see Lemma 2.6. The definition of this space is a modification of the \( Y_{s,\sigma-1} \) space defined in Section 5.3 in [Hol06].

Let \( \Omega_T := \mathbb{R}^+ \times (0,T) \) for given \( T > 0 \); define a restricted version of the Bourgain space \( X_{s,b} \) to the domain \( \Omega_T \) as follows:

\[
X^{\alpha,\beta}_{s,b}(\Omega_T) = X^{\alpha,\beta}_{s,b}\big|_{\Omega_T}
\]

with the quotient norm

\[
\|u\|_{X^{\alpha,\beta}_{s,b}(\Omega_T)} = \inf_{w \in X^{\alpha,\beta}_{s,b}} \{\|w\|_{X^{\alpha,\beta}_{s,b}} : w(x,t) = u(x,t) \text{ on } \Omega_T\}
\]

for any given function \( u(x,t) \) defined on \( \Omega_T \). The spaces \( X^{\alpha,\beta}_{s,b,\sigma}(\Omega_T), Y^{\alpha,\beta}_{s,b,\sigma}(\Omega_T) \) and \( Z^{\alpha,\beta-1}_{s,b,\sigma}(\Omega_T) \) are defined similarly.

Let us denote that

\[
F(u,v) := -3uv_x - (uv)_x + vv_x, \quad G(u,v) := uu_x - (uv)_x - 3vv_x,
\]

the IBVP 1.5 will be converted to the following system of nonlinear integral equations

\[
\begin{cases}
\begin{align*}
\frac{d}{dt} u(x,t) &= W^{1,\beta}_{\mathbb{R}^+}(t) \phi(x) + \int_0^t W^{1,\beta}_{\mathbb{R}^+}(t - \tau) F(u,v)(\tau)d\tau + W^{1,\beta}_{\text{bdr}}[h_1](x,t), \\
\frac{d}{dt} v(x,t) &= W^{-1,-\beta}_{\mathbb{R}^+}(t) \psi(x) + \int_0^t W^{-1,-\beta}_{\mathbb{R}^+}(t - \tau) G(u,v)(\tau)d\tau + W^{-1,-\beta}_{\text{bdr}}[h_2, h_3 + u(x,0,\tau)](x,t)
\end{align*}
\end{cases}
\] (1.16)

for \( x > 0, t > 0 \), where \( W^{1,\beta}_{\mathbb{R}^+}(t) \) and \( W^{-1,-\beta}_{\mathbb{R}^+}(t) \) are the \( C^0 \) semigroups associated to the IBVPs of the
linear KdV equation posed on $\mathbb{R}^+$,

\[
\begin{aligned}
\begin{cases}
    w_t + w_{xxx} + \beta w_x = 0, & x \in \mathbb{R}^+, \ t \in \mathbb{R}^+, \\
    w|_{x=0} = 0, \ w|_{t=0} = \phi \in H^s(\mathbb{R}^+),
\end{cases}
\end{aligned}
\tag{1.17}
\]

and

\[
\begin{aligned}
\begin{cases}
    w_t - w_{xxx} - \beta w_x = 0, & x \in \mathbb{R}^+, \ t \in \mathbb{R}^+, \\
    w|_{x=0} = 0, \ w_x|_{x=0} = 0, \ w|_{t=0} = \psi \in H^s(\mathbb{R}^+),
\end{cases}
\end{aligned}
\tag{1.18}
\]

respectively, $W_{bd_{t}}^{1,\beta}(t)$ and $W_{bd}_{t}^{-1,-\beta}(t)$ are the boundary integral operators associated with the IBVPs of the linear KdV equation posed on the half line $\mathbb{R}^+$

\[
\begin{aligned}
\begin{cases}
    u_t + u_{xxx} + \beta u_x = 0, & x \in \mathbb{R}^+, \ t \in \mathbb{R}^+, \\
    u(x,0) = 0, \ u(0,t) = h_1(t),
\end{cases}
\end{aligned}
\tag{1.19}
\]

and

\[
\begin{aligned}
\begin{cases}
    v_t - v_{xxx} - \beta v_x = 0, & x \in \mathbb{R}^+, \ t \in \mathbb{R}^+, \\
    v(x,0) = 0, \ v(0,t) = h_2(t), \ v_x(0,t) = h_3(t),
\end{cases}
\end{aligned}
\tag{1.20}
\]

respectively. Here for simplicity we have assume that $\phi(0) = h_1(0) = \psi(0) = h_2(0) = 0$ if $\frac{1}{2} < s < \frac{3}{2}$ and $h_3(0) = h_3(0) = \psi'(0) = \psi'(0) \in H^s(\mathbb{R}^+)$ if $\frac{1}{2} < s < 3$. We will first establish the following conditional well-posedness.

**Theorem 1.7** (Conditional Well-posedness).

Let $-\frac{3}{4} < s < 3$, $T > 0$ and $0 \leq \beta \leq 1$ be given. There exists a $r > 0$ such that for any naturally compatible $(\phi, \psi) \in \mathcal{H}_s^2(\mathbb{R}^+)$, $\vec{h} = (h_1, h_2, h_3) \in \mathcal{H}_s^2(\mathbb{R}^+)$ related to the IBVP (1.5) with

\[
\|\phi, \psi\|_{\mathcal{H}_s^2(\mathbb{R}^+)} + \|\vec{h}\|_{\mathcal{H}_s^2(\mathbb{R}^+)} \leq r,
\]

the system of the integral equations (SIE) (1.16) admits a unique solution

\[
(u,v) \in \mathcal{Y}_\sigma(\Omega_T) := Y_{s, \frac{1}{2}, \beta}^{1, \beta}(\Omega_T) \times Y_{s, \frac{1}{2}, \beta}^{-1, -\beta}(\Omega_T)
\]

for some $\sigma > \frac{1}{2}$, and, moreover, the solution map is real analytic in the corresponding spaces.

**Remark 1.8.** The well-posedness presented in Theorem 1.7 is conditional in the sense that the solution $(u,v) \in C([0, T]; \mathcal{H}_s^2(\mathbb{R}^+))$ of (1.16) is in fact a restriction for a function defined on $\mathbb{R} \times \mathbb{R}$ to the region $(0, T) \times \mathbb{R}^+$, and the uniqueness of the solution holds in the space $\mathcal{Y}_s(\Omega_T)$ rather than in the space $C([0, T]; \mathcal{H}_s^2(\mathbb{R}^+))$, which leads to a serious issue: If a solution of the IBVP (1.5) is found in a different approach, will it be the same as that presented by Theorem 1.7?

In this paper, we will prove Theorem 1.4 to give a positive answer to the above question. The justification of Theorem 1.4 follows from Theorem 1.7, and the following two further properties which will be established in Proposition 5.8 and Proposition 5.9 in Section 5.

(i) the solution of SIE (1.16) is a mild solution of the IBVP (1.5)

(ii) for given initial data $(\phi, \psi)$ and the boundary data $\vec{h}$, the IBVP (1.5) admits at most one mild solution.
Theorem 1.7 will be proved by the standard contraction mapping principle. We first need to investigate the following two linear IBVPs,

\[
\begin{aligned}
    & u_t + u_{xxx} + \beta u_x = f, \\
    & u(x,0) = p(x), \\
    & u(0,t) = a(t)
\end{aligned}
\]

and

\[
\begin{aligned}
    & v_t - v_{xxx} - \beta v_x = g, \\
    & v(x,0) = q(x), \\
    & v(0,t) = b_1(t), \\
    & v_x(0,t) = b_2(t).
\end{aligned}
\]

**Definition 1.9.** Let \(s \in (-\frac{3}{4}, 3]\), \((p,q) \in \mathcal{H}_s^2(\mathbb{R}^+)\) and \((a,b_1,b_2) \in \mathcal{H}_s^2(\mathbb{R}^+)\). Then

- the data \(p,a\) is said to be compatible for (1.21) if \(p(0) = a(0)\) when \(s > \frac{1}{2}\);
- the data \((q,b_1,b_2)\) is said to be compatible for (1.22) if they satisfy \(q(0) = b_1(0)\) when \(s > \frac{1}{2}\) and further satisfy \(q'(0) = b_2(0)\) when \(s > \frac{3}{2}\). The compatibility for (1.22) is understood in a similar way.

The following two estimates for their solutions are key tools in showing Theorem 1.7.

**Proposition 1.10.** Let \(s \in (-\frac{3}{4},3]\), \(T > 0\) and \(0 < \beta \leq 1\). Assume \(p \in \mathcal{H}_s^s(\mathbb{R}^+)\) and \(a \in \mathcal{H}_s^{\frac{s+1}{2}}(\mathbb{R}^+)\) are compatible for (1.21). Then there exists \(\sigma_1 = \sigma_1(s) > \frac{1}{2}\) such that for any \(\sigma \in (\frac{1}{2}, \sigma_1)\) and for any \(f \in X_{s,\sigma}^{1,\beta} \cap Z_{s,\sigma}^{1,\beta}\), the equation (1.21) has a solution up to time \(T\). More precisely, there exists a function

\[
\bar{u} := \Gamma_\beta^s(f,p,a),
\]

defined on \(\mathbb{R} \times \mathbb{R}\), belongs to \(Y_{s,\sigma}^{1,\beta} \cap C^1([0,T]; \mathcal{H}_s^{\frac{s+1}{2}}(\mathbb{R}^+))\) for \(j = 0, 1\), and its restriction \(\bar{u}|_{\mathbb{R}_+^* \times [0,T]}\) solves (1.21) on \(\mathbb{R}_+^* \times [0,T]\). In addition, \(\bar{u}\) satisfies the following estimates with \(C = C(s,\sigma)\).

\[
\begin{aligned}
    \|\bar{u}\|_{Y_{s,\sigma}^{1,\beta}} & \leq C \left( \|f\|_{X_{s,\sigma}^{1,\beta}} + \|f\|_{Z_{s,\sigma}^{1,\beta}} + \|p\|_{\mathcal{H}^s} + \|a\|_{\mathcal{H}^{\frac{s+1}{2}}} \right), \\
    \sup_{x \geq 0} \|\partial_x^{j+1}\bar{u}\|_{\mathcal{H}^{\frac{s+1}{2}}(\mathbb{R}^+)} & \leq C \left( \|f\|_{X_{s,\sigma}^{1,\beta}} + \|f\|_{Z_{s,\sigma}^{1,\beta}} + \|p\|_{\mathcal{H}^s} + \|a\|_{\mathcal{H}^{\frac{s+1}{2}}} \right), \quad j = 0, 1.
\end{aligned}
\]

**Proposition 1.11.** Let \(s \in (-\frac{3}{4},3]\), \(T > 0\) and \(0 < \beta \leq 1\). Assume \(g \in \mathcal{H}_s^s(\mathbb{R}^+)\), \(b_1 \in \mathcal{H}_s^{\frac{s+1}{2}}(\mathbb{R}^+)\) and \(b_2 \in \mathcal{H}_s^{\frac{s+1}{2}}(\mathbb{R}^+)\) are compatible for (1.22). Then there exists \(\sigma_2 = \sigma_2(s) > \frac{1}{2}\) such that for any \(\sigma \in (\frac{1}{2}, \sigma_2)\) and for any \(g \in X_{s,\sigma}^{-1,\beta} \cap Z_{s,\sigma}^{-1,\beta}\), the equation (1.22) has a solution up to time \(T\). More precisely, there exists a function

\[
\bar{v} := \Gamma_\beta^s(g,q,b_1,b_2),
\]

defined on \(\mathbb{R} \times \mathbb{R}\), belongs to \(Y_{s,\sigma}^{-1,\beta} \cap C^1([0,T]; \mathcal{H}_s^{\frac{s+1}{2}}(\mathbb{R}^+))\) for \(j = 0, 1\), and its restriction \(\bar{v}|_{\mathbb{R}_+^* \times [0,T]}\) solves (1.22) on \(\mathbb{R}_+^* \times [0,T]\). In addition, \(\bar{v}\) satisfies the following estimates with \(C = C(s,\sigma)\).

\[
\begin{aligned}
    \|\bar{v}\|_{Y_{s,\sigma}^{-1,\beta}} & \leq C \left( \|g\|_{X_{s,\sigma}^{-1,\beta}} + \|g\|_{Z_{s,\sigma}^{-1,\beta}} + \|p\|_{\mathcal{H}^s} + \|b_1\|_{\mathcal{H}^{\frac{s+1}{2}}} + \|b_2\|_{\mathcal{H}^{\frac{s+1}{2}}} \right), \\
    \sup_{x \geq 0} \|\partial_x^{j+1}\bar{v}\|_{\mathcal{H}^{\frac{s+1}{2}}(\mathbb{R}^+)} & \leq C \left( \|g\|_{X_{s,\sigma}^{-1,\beta}} + \|g\|_{Z_{s,\sigma}^{-1,\beta}} + \|p\|_{\mathcal{H}^s} + \|b_1\|_{\mathcal{H}^{\frac{s+1}{2}}} + \|b_2\|_{\mathcal{H}^{\frac{s+1}{2}}} \right), \quad j = 0, 1.
\end{aligned}
\]
The following bilinear estimates will also be the key ingredients in proving Theorem 1.7.

**Proposition 1.12.** Let $-\frac{3}{2} < s \leq 3$, $\alpha \neq 0$ and $|\beta| \leq 1$. Then there exists $\sigma_0 = \sigma_0(s, \alpha) > \frac{1}{2}$ such that for any $\sigma \in (\frac{1}{2}, \sigma_0)$, the following bilinear estimates hold for any $w_1, w_2$ with $C = C(s, \alpha, \sigma)$.

\[
\begin{align*}
\|\partial_x(w_1 w_2)\|_{X^{s,\beta}_{\sigma,-1}} + \|\partial_x(w_1 w_2)\|_{Z^{s,\beta}_{\sigma,-1}} &\leq C\|w_1\|_{X^{s,\beta}_{\sigma}}\|w_2\|_{X^{s,\beta}_{\sigma}}, \\
\|\partial_x(w_1 w_2)\|_{X^{s,\beta}_{\sigma,-1}} + \|\partial_x(w_1 w_2)\|_{Z^{s,\beta}_{\sigma,-1}} &\leq C\|w_1\|_{X^{s,\beta}_{\sigma}}\|w_2\|_{X^{s,\beta}_{\sigma}}, \\
\|\partial_x(w_1 w_2)\|_{X^{s,\beta}_{\sigma,-1}} + \|\partial_x(w_1 w_2)\|_{Z^{s,\beta}_{\sigma,-1}} &\leq C\|w_1\|_{X^{s,\beta}_{\sigma}}\|w_2\|_{X^{s,\beta}_{\sigma}}.
\end{align*}
\]

We point out that the approach developed in this paper for the IBVP (1.3) can also be applied to study the IBVPs of general coupled KdV-KdV systems,

\[
\begin{align*}
\{ & u_t + a_{11} u_{xxx} + a_{12} v_{xxx} + b_{11} u_x + b_{12} v_x = c_{11} u u_x + c_{12} v v_x + d_{11} u_x v + d_{12} u v_x, \\
v_t + a_{21} u_{xxx} + a_{22} v_{xxx} + b_{22} v_x + b_{21} u_x = c_{21} u u_x + c_{22} v v_x + d_{21} u_x v + d_{22} u v_x,
\end{align*}
\]

(1.29)

that includes, in particular, the well-known

- **Majda-Biello system:**

\[
\begin{align*}
& u_t + u_{xxx} = -v v_x, \\
v_t + a_2 v_{xxx} = -(uv)_x,
\end{align*}
\]

(1.30)

with $a_2 \neq 0$, which was proposed by Majda and Biello in [MB03] as a reduced asymptotic model to study the nonlinear resonant interactions of long wavelength equatorial Rossby waves and barotropic Rossby waves;

- **Hirota-Satsuma system:**

\[
\begin{align*}
& u_t + a_1 u_{xxx} = -6a_1 u u_x + c_{12} v v_x, \\
v_t + v_{xxx} = -3u v_x,
\end{align*}
\]

(1.31)

with $a_1 \neq 0$, which was proposed by Hirota and Satsuma in [HS81] to describe the interaction of two long waves with different dispersion relations;

- **Gear-Grimshaw system:**

\[
\begin{align*}
& u_t + u_{xxx} + \sigma_3 v_{xxx} = -u u_x + \sigma_1 v v_x + \sigma_2 (uv)_x, \\
& \rho_1 v_t + \rho_2 \sigma_3 u_{xxx} + v_{xxx} + \sigma_4 v_x = \rho_2 \sigma_2 u u_x - v v_x + \rho_2 \sigma_1 (uv)_x,
\end{align*}
\]

(1.32)

with $\sigma_i \in \mathbb{R}(1 \leq i \leq 4)$ and $\rho_1, \rho_2 > 0$, which was derived by Gear and Grimshaw in [GG84] (also see [BPST92] for the explanation about the physical context) as a model to describe the strong interaction of two-dimensional, weakly nonlinear, long, internal gravity waves propagating on neighboring pycnoclines in a stratified fluid, where the two waves correspond to different modes.

The Cauchy problem of the general systems (1.29) posed either on $\mathbb{R}$ or on the torus $T$ has been well studied in the literature for is well-posedness in the space $H^s(\mathbb{R}) \times H^s(\mathbb{R})$ or $H^s(T) \times H^s(T)$. The interested readers are referred to [YZ22a, YZ22b] and the references therein for an overall review of this study. For the IBVPs of the system (1.29) posed on $\mathbb{R}^+$, we can have the well-posedness results similar to that present in Theorem 1.3 for the IBVP (1.3) using the same approach.
To end our introduction we list two such theorems for the IBVPs of Gear-Grimshaw system (1.32) posed on \( \mathbb{R}^+ \). Note first the matrix

\[
A = \begin{pmatrix}
\frac{1}{\rho_1 \sigma_3} & \sigma_3 \\
\frac{1}{\rho_1} & \lambda
\end{pmatrix}.
\]

possesses two nonzero real eigenvalues \( \lambda_1 \leq \lambda_2 \) when \( \rho_1 > 0, \rho_2 > 0 \) and \( \rho_2 \sigma_3^2 \neq 1 \). Moreover,

\[
\begin{cases}
\lambda_1 < 0 < \lambda_2 & \text{if } \rho_2 \sigma_3^2 > 1, \\
0 < \lambda_1 < \lambda_2 & \text{if } \rho_2 \sigma_3^2 < 1, \\
\lambda_1 = \lambda_2 = 1 & \text{if } \sigma_3 = 0, \rho_1 = 1.
\end{cases}
\]

Thus, in the case of \( \rho_2 \sigma_3^2 > 1 \), for the IBVP of the Gear-Grimshaw system (1.32) posed on \( \mathbb{R}^+ \), we need to impose three boundary conditions as in the case of the IBVP (1.3),

\[
\begin{align*}
&u_t + u_{xxx} + \sigma_3 v_{xxx} = -u_x + \sigma_1 v_x + \sigma_2 (uv)_x, \quad x, t > 0 \\
&\rho_1 v_t + \rho_2 \sigma_3 u_{xxx} + v_{xxx} + \sigma_4 v_x = \rho_2 \sigma_2 u u_x - v v_x + \rho_2 \sigma_1 (uv)_x, \quad x, t > 0 \\
u(0, t) = h_1(t), &\quad v(0, t) = h_2(t), &\quad v_x(0, t) = h_3(t), \quad t > 0.
\end{align*}
\] (1.33)

But in the case of \( \rho_2 \sigma_3^2 < 1 \), for the IBVP of the Gear-Grimshaw system (1.32) posed on \( \mathbb{R}^+ \), we only need to impose two boundary conditions,

\[
\begin{align*}
&u_t + u_{xxx} + \sigma_3 v_{xxx} = -u_x + \sigma_1 v_x + \sigma_2 (uv)_x, \quad x, t > 0 \\
&\rho_1 v_t + \rho_2 \sigma_3 u_{xxx} + v_{xxx} + \sigma_4 v_x = \rho_2 \sigma_2 u u_x - v v_x + \rho_2 \sigma_1 (uv)_x, \quad x, t > 0 \\
u(0, t) = h_1(t), &\quad v(0, t) = h_2(t), \quad t > 0.
\end{align*}
\] (1.34)

We first recall a sharp well-posedness result due to Yang-Zhang [YZ22a] for the Cauchy problem of the Gear-Grimshaw system (1.32) posed on \( \mathbb{R} \).

**Theorem 1.13** (Local Well-posedness of Gear-Grimshaw system, [YZ22a]). _The Cauchy problem of the Gear-Grimshaw system (1.32) is locally analytically well-posed in the space \( \mathcal{H}^s(\mathbb{R}) \) for any

\( (i) \) \( s > -\frac{3}{2} \) if \( \sigma_3 = 0 \) and \( \rho_1 = 1 \);

\( (ii) \) \( s > -\frac{3}{2} \) if \( \rho_2 \sigma_3^2 > 1 \);

\( (iii) \) \( s \geq 0 \) if \( \rho_2 \sigma_3^2 < 1 \) but (1.35) below is not satisfied;

\( (iv) \) \( s \geq \frac{3}{4} \) and

\[
\rho_2 \sigma_3^2 \leq \frac{9}{25}, \quad \rho_1 = \frac{1}{2} \left( \alpha \pm \sqrt{\alpha^2 - 4} \right)
\] (1.35)

with

\[
\alpha = \frac{17 - 25 \rho_2 \sigma_3^2}{4}.
\]

Based on Theorem 1.13 and the method we developed to prove Theorem 1.3 we can also obtain the following well-posedness results concerning the Gear-Grimshaw systems.
Theorem 1.14. If
\[ \rho_1 > 0, \quad \rho_2 > 0, \quad \rho_2 \sigma_3^2 > 1, \]
then the IBVP (1.33) is locally analytically well-posed in the space \( \mathcal{H}^s(\mathbb{R}^+) \), with any naturally compatible data \((\phi, \psi) \in \mathcal{H}_x^s(\mathbb{R}^+) \) and \( \vec{h} = (h_1, h_2, h_3) \in \mathcal{H}_t^s(\mathbb{R}^+) \), for any \( s > -\frac{3}{4} \).

Theorem 1.15. Assume
\[ \rho_1 > 0, \quad \rho_2 > 0, \quad \rho_2 \sigma_3^2 < 1. \]
The IBVP (1.34) is locally analytically well-posed in the space \( \mathcal{H}^s(\mathbb{R}^+) \), with any naturally compatible data
\[ (\phi, \psi) \in \mathcal{H}_x^s(\mathbb{R}^+), \quad (h_1, h_2) \in H^{\frac{s+1}{2}}(\mathbb{R}^+) \times H^{\frac{s+1}{3}}(\mathbb{R}^+) , \]
for any
(i) \( s > -\frac{3}{4} \) if \( \sigma_3 = 0 \) and \( \rho_1 = 1 \);
(ii) \( s \geq 0 \) if \( \rho_2 \sigma_3^2 < 1 \) but (1.35) is not satisfied;
(iii) \( s \geq \frac{3}{4} \) if (1.35) holds.

Finally, we remark that the current paper only studies the 1D Boussinesq systems, however, the 2D Boussinesq systems are also of great importance and has attracted significant attention, see e.g. [BCL05, BLS08, DMS07, MSZ12, SWX17, LPS12]. These earlier works mainly focus on the well-posedness of the 2D Boussinesq systems on the space-time region \( \mathbb{R}^2 \times (0, T) \), it might be an interesting future task to study their well-posedness on the octant space-time region \( (\mathbb{R}^+)^2 \times (0, T) \).

The remaining part of paper is organized as follows.

• In Section 2, we present some results for the Cauchy problem of the KdV equation and reversed KdV equation posed on the whole line \( \mathbb{R} \) which will play important roles later in dealing with the initial boundary value problems of the coupled KdV systems.

• In Section 3, we investigate the following two linear IBVPs,
\[
\begin{align*}
& \begin{cases}
  u_t + u_{xxx} + \beta u_x = f, \\
  u(x, 0) = p(x), \\
  u(0, t) = a(t)
\end{cases}, \quad x, t > 0, \\
& \begin{cases}
  v_t - v_{xxx} - \beta v_x = g, \\
  v(x, 0) = q(x), \\
  v(0, t) = b_1(t), \quad v_x(0, t) = b_2(t).
\end{cases}
\end{align*}
\]
and present the proofs of Proposition 1.10 and Proposition 1.11.

• In Section 4, various bilinear estimates presented in Proposition 1.12 will be established.

• Section 5 is devoted to the proof of Theorem 1.3, the main result of this paper.
2 Preliminaries

Throughout this paper, $\eta(t)$ is fixed to be a bump function in $C_0^\infty(\mathbb{R})$ which satisfies $0 \leq \eta \leq 1$ and

$$
\eta(t) = \begin{cases} 
1 & \text{if } |t| \leq 1, \\
0 & \text{if } |t| \geq 2.
\end{cases} \quad (2.1)
$$

The notation $a+ \epsilon$ denotes $a + \epsilon$ for any $\epsilon > 0$. Similarly, $a- \epsilon$ denotes $a - \epsilon$ for any $\epsilon > 0$.

For a function $f$ in both $x$ and $t$ variables, we use $\mathcal{F}_x f$ (or $\hat{f}$) to denote its space-time Fourier transform. Meanwhile, we use $\mathcal{F}_t f$ or $\hat{f}_t$ to denote its Fourier transform with respect to $t$. If a function $h$ is only in $x$ variable, then we use $\hat{h}_x$ (or just $\hat{h}$ when there is no ambiguity) to denote its Fourier transform. Similar notations are applied to functions which are only in $t$ variable.

For any $s \in \mathbb{R}$, we denote $E_s$ to be an extension operator from $H^s(\mathbb{R}^+)$ to $H^s(\mathbb{R})$ such that

$$
\|E_s f\|_{H^s(\mathbb{R})} \leq C \|f\|_{H^s(\mathbb{R}^+)} \quad \forall f \in H^s(\mathbb{R}^+) \quad (2.2)
$$

where $C$ is a constant which only depends on $s$.

**Lemma 2.1.** Let $h \in H^{\frac{5}{2}}(\mathbb{R}^+)$ and

$$
h^*(x) := \begin{cases} 
h(x) & \text{if } x \geq 0, \\
0 & \text{if } x < 0.
\end{cases} \quad (2.3)
$$

Assume one of the following conditions.

(1) $-\frac{5}{2} < s < -\frac{1}{2}$;

(2) $\frac{1}{2} < s < \frac{7}{2}$ and $h(0) = 0$.

Then $h^* \in H^{\frac{s+1}{2}}(\mathbb{R})$ and there exists $C = C(s)$ such that

$$
\|h^*\|_{H^{\frac{s+1}{2}}(\mathbb{R})} \leq C \|h\|_{H^{\frac{s+1}{2}}(\mathbb{R}^+)} \quad (2.4)
$$

The result in Lemma 2.1 is standard, see e.g. Lemma 2.1(i)(ii) in [ET16] or [JK95].

For any $\beta > 0$, define the function $P_\beta : \mathbb{R} \to \mathbb{R}$ as

$$
P_\beta(\mu) = \mu^3 - \beta \mu. \quad (2.5)
$$

**Lemma 2.2.** Let $s \in \mathbb{R}$ and $0 < \beta \leq 1$. If a function $M : \mathbb{R} \to \mathbb{C}$ satisfies

$$
|M(\mu)| \leq C |(P_\beta)'(\mu)|^{\frac{1}{2}} |\mu|^{s+1}, \quad \forall \mu \in \mathbb{R}, \quad (2.6)
$$

for some constant $C = C(s)$. Then there exists $C_1 = C_1(s)$ such that

$$
\|M(\mu)\hat{f}(P_\beta(\mu))\|_{L^2(\mathbb{R})} \leq C_1 \|f\|_{H^{\frac{s+1}{2}}(\mathbb{R})} \quad \forall f \in H^{\frac{s+1}{2}}(\mathbb{R}). \quad (2.7)
$$
Proof. First, it follows from (2.6) that
\[
\left\| M(\mu) \hat{f}(P_\beta(\mu)) \right\|_{L^2}^2 = \int_{\mathbb{R}} |M(\mu)|^2 |\hat{f}(P_\beta(\mu))|^2 d\mu \\
\lesssim \int_{\mathbb{R}} |(P_\beta)'(\mu)|^2 (\mu)^{2(s+1)} |\hat{f}(P_\beta(\mu))|^2 d\mu.
\]

Then applying the change of variable \( \xi = P_\beta(\mu) = \mu^3 - \beta \mu \) yields
\[
\left\| M(\mu) \hat{f}(P_\beta(\mu)) \right\|_{L^2}^2 \lesssim \int_{\mathbb{R}} |\xi|^{2(s+1)} |\hat{f}(\xi)|^2 d\xi,
\]
which implies (2.7).
\( \square \)

Based on Lemma 2.1 and Lemma 2.2, we immediately obtain the following conclusion.

**Corollary 2.3.** Let \( a \in H^{\frac{s+1}{2}}(\mathbb{R}^+) \) and denote \( a^\ast \) to be the zero extension of \( a \) as defined in (2.3). Assume one of the following conditions.

1. \(-\frac{5}{2} < s < \frac{1}{2};\)
2. \(\frac{1}{2} < s < \frac{7}{2} \) and \( a(0) = 0.\)

If a function \( M : \mathbb{R} \to \mathbb{C} \) satisfies
\[
|M(\mu)| \leq C |(P_\beta)'(\mu)|^{\frac{1}{2}} (\mu)^{s+1}, \quad \forall \mu \in \mathbb{R},
\]
then there exists \( C = C(s) \) such that
\[
\left\| M(\mu) a^\ast (P_\beta(\mu)) \right\|_{L^2(\mathbb{R})} \leq C \|a\|_{H^{\frac{s+1}{2}}(\mathbb{R}^+)}, \quad \forall a \in H^{\frac{s+1}{2}}(\mathbb{R}^+),
\]
where \( a^\ast \) is the Fourier transform of \( a^\ast \).

Next, we present some results for the Cauchy problem of the KdV equation and the reversed KdV equation posed on the whole line \( \mathbb{R} \) which will play important roles later in dealing with the initial boundary value problems of the coupled KdV systems.

**Lemma 2.4.** Let \( s \in \mathbb{R}, \alpha \in \mathbb{R} \setminus \{0\}, |\beta| \leq 1 \) and \( \frac{1}{2} < \sigma \leq 1. \) Then for any \( w_0 \in H^s(\mathbb{R}) \) and \( F \in X^{\alpha,\beta}_{s,\sigma}, \) both \( \eta(t)W_{\alpha,\beta}^{-1}(t)w_0 \) and \( \eta(t) \int_0^t W_{\alpha,\beta}^{-1}(t - \tau)F(\cdot, \tau) d\tau \) belong to \( Y^{\alpha,\beta}_{s,\sigma}. \) In addition, there exists \( C = C(s,\alpha,\sigma) \) such that
\[
\left\| \eta(t)W_{\alpha,\beta}^{-1}(t)w_0 \right\|_{Y^{\alpha,\beta}_{s,\sigma}} \leq C \|w_0\|_{H^s(\mathbb{R})}
\]
and
\[
\left\| \eta(t) \int_0^t W_{\alpha,\beta}^{-1}(t - \tau)F(\cdot, \tau) d\tau \right\|_{Y^{\alpha,\beta}_{s,\sigma}} \leq C \|F\|_{X^{\alpha,\beta}_{s,\sigma-1}}.
\]

**Proof.** Recalling (1.13), the norm \( \| \cdot \|_{Y^{\alpha,\beta}_{s,\sigma}} \) is bounded by \( \| \cdot \|_{X^{\alpha,\beta}_{s,\sigma}}, \) so it suffices to prove
\[
\left\| \eta(t)W_{\alpha,\beta}^{-1}(t)w_0 \right\|_{X^{\alpha,\beta}_{s,\sigma}} \leq C \|w_0\|_{H^s(\mathbb{R})}
\]
(2.10)
Lemma 2.6. Let
\[ \eta(t) \int_0^t W^\alpha_\beta_R(t - \tau) F(\cdot, \tau) \, d\tau \leq C \| F \|_{X^\alpha_\beta_{s,\sigma - 1}}. \] (2.11)

The proofs for (2.10) and (2.11) follow directly from Lemma 3.1 and Lemma 3.3 in [KPV93].

Lemma 2.5. Let \( s \in \mathbb{R}, \alpha \in \mathbb{R}\setminus\{0\} \) and \( |\beta| \leq 1 \). Then for any \( w_0 \in H^s(\mathbb{R}) \),
\[ \eta(t)[W^\alpha_\beta_R(t)w_0] \in C^1_x(\mathbb{R}; H^{s + 1 - \frac{j}{2}}_t(\mathbb{R})) \]
for \( j = 0, 1 \). In addition, there exists \( C = C(s, \alpha) \) such that
\[ \sup_{x \in \mathbb{R}} \| \eta(t) \partial_x^j [W^\alpha_\beta_R(t)w_0] \|_{H^{s + 1 - \frac{j}{2}}_t(\mathbb{R})} \leq C \| w_0 \|_{H^s(\mathbb{R})}, \quad j = 0, 1. \]

Proof. The proof is standard, see e.g. [KPV91] or Lemma 5.5 in [Hol06].

Lemma 2.6. Let \( s \in \mathbb{R}, \alpha \in \mathbb{R}\setminus\{0\}, \ |\beta| \leq 1 \) and \( \frac{1}{2} < \sigma \leq 1 \). Then for any \( F \in X^\alpha_\sigma_{s,\sigma - 1} \cap Z^\alpha_\sigma_{s,\sigma - 1} \)
\[ \eta(t) \int_0^t W^\alpha_\beta(t - \tau) F(\cdot, \tau) \, d\tau \in C^1_x(\mathbb{R}; H^{s + 1 - \frac{j}{2}}_t(\mathbb{R})) \]
for \( j = 0, 1 \). In addition, there exists \( C = C(s, \alpha, \sigma) \) such that
\[ \sup_{x \in \mathbb{R}} \| \eta(t) \partial_x^j W^\alpha_\beta(t - \tau) F(\cdot, \tau) \|_{H^{s + 1 - \frac{j}{2}}_t(\mathbb{R})} \leq C \left( \| F \|_{X^\alpha_\sigma_{s,\sigma - 1}} + \| F \|_{Z^\alpha_\sigma_{s,\sigma - 1}} \right), \quad j = 0, 1. \] (2.12)

Similar results can be found in [Hol06] or [CK02], but the current result is slightly stronger in order to prove the bilinear estimates (4.3), (4.5) and (4.7) in the case \( s = 3 \).

Proof. We first consider the case when \( j = 0 \). Meanwhile, without loss of generality, we just assume \( \alpha = 1 \) and \( \beta = 0 \) since the argument for general \( \alpha \) and \( \beta \) is similar to that for \( \alpha = 1 \) and \( \beta = 0 \). As a result, (2.12) is reduced to
\[ \sup_{x \in \mathbb{R}} \| w \|_{H_t^{s + 1 - \frac{j}{2}}(\mathbb{R})} \leq C \left( \| F \|_{X^1_{s,\sigma - 1}} + \| F \|_{Z^1_{s,\sigma - 1}} \right), \]
where \( w(x, t) = \eta(t) \int_0^t W_1^1_0(t - \tau) F(\cdot, \tau) \, d\tau \). By the definition of the semigroup \( W_1^1_0 \) in (1.7), we find
\[ w(x, t) = \eta(t) \int_0^t \int_{\mathbb{R}} e^{i\xi x} e^{i\xi^3(t - \tau)} \mathcal{F}_x F(\xi, \tau) \, d\xi \, d\tau. \]

By regarding \( \mathcal{F}_x F \) as the inverse temporal Fourier transform of \( \hat{F} \) which represents the space-time Fourier transform of \( F \), we know \( \mathcal{F}_x F(\xi, \tau) = \int_{\mathbb{R}} e^{i\tau \xi} \hat{F}(\xi, \tau_1) \, d\tau_1 \). By plugging this expression into the above formula for \( w \) and changing the order of integration, we obtain
\[ w(x, t) = \eta(t) \int_{\mathbb{R}} \int_{\mathbb{R}} e^{i\xi t} \hat{F}(\xi, \tau_1) \left( \int_0^t e^{i\xi^3(t - \tau)} e^{i\tau_1} \, d\tau \right) d\tau_1 \, d\xi. \]

By direct computation of the integral \( \int_0^t e^{i\xi^3(t - \tau)} e^{i\tau_1} \, d\tau \), we deduce that
\[ w(x, t) = (-i) \eta(t) \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{e^{i\xi t} \hat{F}(\xi, \tau_1)}{\tau_1 - \xi^3} (e^{i\tau_1} - e^{i\xi t}) \, d\tau_1 \, d\xi. \]

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By the properties of Fourier transform,
\[ \mathcal{F}_x w(\xi, t) = -i \int_{\mathbb{R}} \frac{\hat{F}(\xi, \tau_1)}{\tau_1 - \xi^3} \eta(t) (e^{i\tau_1 t} - e^{i\xi^3 t}) \, d\tau_1 \]
and
\[ \hat{w}(\xi, \tau) = \mathcal{F}_t [\mathcal{F}_x w(\xi, \cdot)](\tau) = -i \int_{\mathbb{R}} \frac{\hat{F}(\xi, \tau_1)}{\tau_1 - \xi^3} [\hat{\eta}(\tau - \tau_1) - \hat{\eta}(\tau - \xi^3)] \, d\tau_1. \quad (2.14) \]
Noticing that
\[ \sup_{x \in \mathbb{R}} \|w\| H^\frac{s+1}{2} \lesssim \sup_{x \in \mathbb{R}} \left\{ \|\langle \tau \rangle^{\frac{s+1}{2}} \mathcal{F}_x w(x, \tau)\|_{L^2_x} \right\} \lesssim \left\| \langle \tau \rangle^{\frac{s+1}{2}} \int \hat{w}(\xi, \tau) \, d\xi \right\|_{L^2_x}, \]
a by duality, in order to justify \((2.13)\), it is equivalent to prove that for any \(g = g(\tau) \in L^2(\mathbb{R})\) with \(\|g\|_{L^2(\mathbb{R})} = 1\),
\[ \int_{\mathbb{R}^2} |g(\tau)| \langle \tau \rangle^{\frac{s+1}{2}} |\hat{w}(\xi, \tau)| \, d\xi \, d\tau \lesssim \|F\|_{X_{s,\sigma}^{1,0}} + \|F\|_{Z_{1,\sigma}^{1,0}}. \]
Recalling the formula \((2.14)\) for \(\hat{w}\), it suffices to establish
\[ \int_{\mathbb{R}^3} |g(\tau)| \langle \tau \rangle^{\frac{s+1}{2}} |\hat{\eta}(\tau - \tau_1) - \hat{\eta}(\tau - \xi^3)| \frac{\|\hat{F}(\xi, \tau_1)\|}{|\tau_1 - \xi^3|} \, d\tau_1 \, d\xi \, d\tau \lesssim \|F\|_{X_{s,\sigma}^{1,0}} + \|F\|_{Z_{1,\sigma}^{1,0}}. \quad (2.15) \]
Due to the definitions \((1.9)\) and \((1.15)\), we can rewrite \(\|F\|_{X_{s,\sigma}^{1,0}}\) and \(\|F\|_{Z_{1,\sigma}^{1,0}}\) as weighted \(L^2\) norms of \(\hat{F}\). More precisely, by defining
\[ u(\xi, \tau) = \langle \xi \rangle^s \langle \tau - \xi^3 \rangle^{\sigma-1} \hat{F}(\xi, \tau) \quad \text{and} \quad v(\xi, \tau) = \langle \tau \rangle^{\frac{s+1}{2} - \sigma} \langle \tau - \xi^3 \rangle^{\sigma-1} \hat{F}(\xi, \tau), \quad (2.16) \]
then \(\|F\|_{X_{s,\sigma}^{1,0}} = \|u\|_{L^2(\mathbb{R}^2)}\), \(\|F\|_{Z_{1,\sigma}^{1,0}} = \|v\|_{L^2(\mathbb{R}^2)}\) and \(\hat{F}(\xi, \tau) = \langle \xi \rangle^{-s} \langle \tau - \xi^3 \rangle^{1-\sigma} u(\xi, \tau)\). Therefore, \((2.15)\) reduces to
\[ \int_{\mathbb{R}^3} |g(\tau)| \langle \tau \rangle^{\frac{s+1}{2}} |\hat{\eta}(\tau - \tau_1) - \hat{\eta}(\tau - \xi^3)| \frac{|\xi^{-s} \langle \tau_1 - \xi^3 \rangle^{1-\sigma} u(\xi, \tau_1)|}{|\tau_1 - \xi^3|} \, d\tau_1 \, d\tau \, d\xi \lesssim \|u\|_{L^2} + \|v\|_{L^2}. \quad (2.17) \]
• Case 1: \(|\tau_1 - \xi^3| \leq 1\).

Making the change of variable \(\tau - \xi^3 \rightarrow \rho\) and \(\tau_1 - \xi^3 \rightarrow \rho_1\), the left hand side of \((2.17)\) becomes
\[ \int_{\mathbb{R}} \int_{|\rho_1| \leq 1} \int |g(\rho + \xi^3)| |\rho + \xi^3|^{\frac{s+1}{2}} |\hat{\eta}(\rho - \rho_1) - \hat{\eta}(\rho)| \frac{(\xi - s) \langle \rho_1 \rangle^{1-\sigma} |u(\xi, \rho_1 + \xi^3)|}{|\rho_1|} \, d\rho_1 \, d\rho \, d\xi. \quad (2.18) \]
Since \(\hat{\eta}\) is a Schwarz function and \(|\rho_1| \leq 1\),
\[ \frac{|\hat{\eta}(\rho - \rho_1) - \hat{\eta}(\rho)|}{|\rho_1|} \lesssim_m \frac{1}{|\rho|^{m}}, \quad \forall \ m \geq 0. \quad (2.19) \]
On the other hand, by applying the following basic inequality:
\[ \langle a + b \rangle^r \leq \langle a \rangle^{r_1} \langle b \rangle^{r_2}, \quad \forall \ a, b, r \in \mathbb{R}, \quad (2.20) \]
we have
\[ \langle \rho + \xi^3 \rangle^{\frac{s+1}{s}} \leq \langle \rho \rangle^{\frac{(s+1)}{s}} \langle \xi^3 \rangle^{\frac{s+1}{s}} \sim \langle \rho \rangle^{\frac{(s+1)}{s}} \langle \xi \rangle^{s+1}. \] (2.21)

Taking \( m = \frac{|s+1|}{s} + 2 \), then we infer from (2.19) and (2.21) that
\[ \langle \rho + \xi^3 \rangle^{\frac{s+1}{s}} \frac{|\hat{\eta}(\rho - \rho_1) - \hat{\eta}(\rho)|}{|\rho_1|} \lesssim \frac{(\xi)^{s+1}}{(\rho)^2}. \]

Consequently,
\[ \text{RHS of (2.18)} \lesssim \int \int \int_{|\rho_1| \leq 1} \frac{\langle \xi \rangle}{(\rho)^2} g(\rho + \xi^3) |u(\xi, \rho_1 + \xi^3)| d\rho_1 d\rho d\xi, \]

which can be rewritten as
\[ \int \frac{1}{(\rho)^2} \int \langle \xi \rangle g(\rho + \xi^3) \left( \int_{|\rho_1| \leq 1} |u(\xi, \rho_1 + \xi^3)| d\rho_1 \right) d\xi d\rho. \] (2.22)

Since \( \|g\|_{L^2} = 1 \), \( \|\langle \xi \rangle g(\rho + \xi^3)\|_{L^2} \) is bounded by a universal constant (independent of \( \rho \)). On the other hand,
\[ \int_{|\rho_1| \leq 1} |u(\xi, \rho_1 + \xi^3)| d\rho_1 \lesssim \|u(\xi, \rho_1)\|_{L^2_{\rho_1}(R)}. \]

Thus, it follows from Hölder’s inequality that
\[ \int \langle \xi \rangle g(\rho + \xi^3) \int_{|\rho_1| \leq 1} |u(\xi, \rho_1 + \xi^3)| d\rho_1 d\xi \lesssim \|u\|_{L^2(R^2)}. \]

Hence, (2.22) \( \lesssim \|u\|_{L^2(R^2)}. \)

- Case 2: \( |\tau_1 - \xi^3| > 1 \).

In this case, \( |\tau_1 - \xi^3| \sim |\tau_1 - \xi^3| \). So
\[ \text{LHS of (2.17)} \lesssim I + II, \]

where
\[ I = \int \int_{R^3} |g(\tau)| \langle \tau \rangle^{\frac{s+1}{s}} \frac{|\hat{\eta}(\tau - \tau_1)|}{\langle \tau \rangle} \frac{|u(\xi, \tau_1)|}{\langle \tau_1 - \xi^3 \rangle \sigma} d\tau_1 d\tau d\xi, \] (2.23)
\[ II = \int \int_{R^3} |g(\tau)| \langle \tau \rangle^{\frac{s+1}{s}} \frac{|\hat{\eta}(\tau - \xi^3)|}{\langle \tau \rangle} \frac{|u(\xi, \tau_1)|}{\langle \tau_1 - \xi^3 \rangle \sigma} d\tau_1 d\tau d\xi. \] (2.24)

Let’s first estimate II. Since \( \sigma > \frac{1}{2} \), it follows from Hölder’s inequality that
\[ \int \frac{|u(\xi, \tau_1)|}{\langle \tau_1 - \xi^3 \rangle \sigma} d\tau_1 \lesssim \|u(\xi, \tau_1)\|_{L^2_{\tau_1}(R)}. \]

We also notice that
\[ \langle \tau \rangle^{\frac{s+1}{s}} \leq \langle \tau - \xi^3 \rangle^{\frac{|s+1|}{s}} \langle \xi^3 \rangle^{\frac{s+1}{s}} \sim \langle \tau - \xi^3 \rangle^{\frac{|s+1|}{s}} \langle \xi \rangle^{s+1}. \]
Therefore,

\[
\text{RHS of (2.24)} \lesssim \int_{\mathbb{R}} \langle \xi \rangle \|u(\xi, \tau_1)\|_{L_t^2} \left( \int_{\mathbb{R}} \|g(\tau)\| (\tau - \xi^3)^{\frac{1}{2} + 1} \|\hat{u}(\tau - \xi^3)\| d\tau \right) d\xi. \tag{2.25}
\]

By the change of variable \(\tau - \xi^3 \to \rho\) and by exchanging the order of integration, we find

\[
\text{RHS of (2.25)} = \int_{\mathbb{R}} \langle \rho \rangle \left( \frac{1}{2 + 1} \right) \|\hat{u}(\rho)\| \left( \int_{\mathbb{R}} \|g(\rho + \xi^3)\| \langle \xi \rangle \|u(\xi, \tau_1)\|_{L_t^2} d\xi \right) d\rho. \tag{2.26}
\]

Recalling the argument after (2.22) in Case 1, we have shown that \(\|g(\rho + \xi^3)\|_{L_t^2}\) is bounded by a universal constant (independent of \(\rho\)), so

\[
\int_{\mathbb{R}} \|g(\rho + \xi^3)\| \langle \xi \rangle \|u(\xi, \tau_1)\|_{L_t^2} d\xi \lesssim \|u\|_{L_t^2(\mathbb{R}^2)}.
\]

Hence, the right hand side of (2.26) \(\lesssim \|u\|_{L_t^2(\mathbb{R}^2)}\).

Next, we turn to estimate \(I\) as in (2.23) and we divide its proof further into two cases.

- Case 2.1: \(\langle \tau_1 \rangle \sim \langle \xi^3 \rangle\).
  
  In this case,

  \[
  \langle \tau \rangle^{\frac{1}{2} + 1} \lesssim \langle \tau - \tau_1 \rangle^{\frac{1}{2} + 1} \langle \tau_1 \rangle^{\frac{1}{2} + 1} \sim \langle \tau - \tau_1 \rangle^{\frac{1}{2} + 1} \langle \xi^3 \rangle^{\frac{1}{2} + 1}.
  \]

  So

  \[
  I \lesssim \int_{\mathbb{R}} \int_{\mathbb{R}} |g(\tau)| \langle \tau - \tau_1 \rangle^{\frac{1}{2} + 1} |\hat{u}(\tau - \tau_1)| \frac{\langle \xi \rangle |u(\xi, \tau_1)|}{\langle \tau_1 - \xi^3 \rangle^\sigma} d\tau d\xi d\tau_1
  = \int_{\mathbb{R}} \int_{\mathbb{R}} |g(\tau)| \langle \tau - \tau_1 \rangle^{\frac{1}{2} + 1} |\hat{u}(\tau - \tau_1)| \left( \int_{\mathbb{R}} \frac{\langle \xi \rangle |u(\xi, \tau_1)|}{\langle \tau_1 - \xi^3 \rangle^\sigma} d\xi \right) d\tau d\tau_1. \tag{2.27}
  \]

  By using Hölder’s inequality,

  \[
  \int_{\mathbb{R}} \frac{\langle \xi \rangle |u(\xi, \tau_1)|}{\langle \tau_1 - \xi^3 \rangle^\sigma} d\xi \leq C \|u(\xi, \tau_1)\|_{L_t^2(\mathbb{R})},
  \]

  where \(C\) is a constant only depending on \(\sigma\). Thus, it follows from (2.27) that

  \[
  I \lesssim \int_{\mathbb{R}} \|u(\xi, \tau_1)\|_{L_t^2} \int_{\mathbb{R}} |g(\tau)| \langle \tau - \tau_1 \rangle^{\frac{1}{2} + 1} |\hat{u}(\tau - \tau_1)| d\tau_1 d\tau
  = \int_{\mathbb{R}} \|u(\xi, \tau_1)\|_{L_t^2} \int_{\mathbb{R}} |g(\tau + \tau_1)| (\tau)^{\frac{1}{2} + 1} |\hat{u}(\tau)| d\tau d\tau_1. \tag{2.28}
  \]

  Exchanging the order of integration, we obtain

  \[
  I \lesssim \int_{\mathbb{R}} \langle \tau \rangle^{\frac{1}{2} + 1} |\hat{u}(\tau)| \left( \int_{\mathbb{R}} \|u(\xi, \tau_1)\|_{L_t^2} |g(\tau + \tau_1)| d\tau_1 \right) d\tau. \tag{2.29}
  \]

  Applying Hölder’s inequality yields \(I \lesssim \|u\|_{L_t^2(\mathbb{R}^2)}\).

- Case 2.2: \(\langle \xi^3 \rangle \ll \langle \tau_1 \rangle\) or \(\langle \xi^3 \rangle \gg \langle \tau_1 \rangle\). Recalling the definition (2.16) for \(u\) and \(v\), we know
Remark 2.7.

Now we finish the proof for \( j = 0 \). The case \( j = 1 \) can be justified in a similar way (just changing \( s \) to be \( s - 1 \) in the above argument), so we omit its proof.

\[ I = \iint |g(\tau)| (\tau)^{\frac{s+1}{2}} |\tilde{\eta}(\tau - \tau_1)| (\tau_1)^{-\frac{s}{2} - \frac{1}{2} + \sigma} \left| \frac{v(\xi, \tau_1)}{\langle \tau_1 - \xi \rangle^\sigma} \right| d\tau_1 d\xi. \]

Since \( (\tau)^{\frac{s+1}{2}} \leq (\tau - \tau_1)^{\frac{s+1}{2}} \), then

\[ I \lesssim \int_{\mathbb{R}} \int_{\mathbb{R}} |g(\tau)| (\tau - \tau_1)^{\frac{s+1}{2}} |\tilde{\eta}(\tau - \tau_1)| (\tau_1)^{-\frac{s}{2} - \frac{1}{2} + \sigma} \left( \int_{A_{\tau_1}} \left| \frac{v(\xi, \tau_1)}{\langle \tau_1 - \xi \rangle^\sigma} \right| d\xi \right) d\tau_1 d\tau, \tag{2.30} \]

where

\[ A_{\tau_1} = \{ \xi \in \mathbb{R} : (\xi)^3 \ll (\tau_1) \ or \ (\xi)^3 \gg (\tau_1) \}. \]

By Hölder’s inequality,

\[ \int_{A_{\tau_1}} \left| \frac{v(\xi, \tau_1)}{\langle \tau_1 - \xi \rangle^\sigma} \right| d\xi \lesssim (\tau_1)^{\frac{1}{2} - \sigma} \|v(\xi, \tau_1)\|_{L^2_z(\mathbb{R})}. \]

As a result, it follows from (2.30) that

\[ I \lesssim \int_{\mathbb{R}} \int_{\mathbb{R}} |g(\tau)| (\tau - \tau_1)^{\frac{s+1}{2}} |\tilde{\eta}(\tau - \tau_1)| \|v(\xi, \tau_1)\|_{L^2_z} d\tau_1 d\tau. \tag{2.31} \]

Then similar to the argument in (2.28) and (2.29), we conclude

\[ \text{RHS of (2.31)} \lesssim \|v\|_{L^2(\mathbb{R}^2)}. \tag{2.32} \]

Now we finish the proof for \( j = 0 \). The case \( j = 1 \) can be justified in a similar way (just changing \( s \) to be \( s - 1 \) in the above argument), so we omit its proof.

**Remark 2.7.**

- If \(-1 \leq s \leq 3\sigma - 1\), then

\[ \sup_{x \in \mathbb{R}} \left\| \eta(t) \int_0^t W_{R,t}^{\alpha,\beta}(t - \tau) F(\cdot, \tau) d\tau \right\|_{H^\sigma_x(\mathbb{R})} \leq C\|F\|_{X^\alpha,\beta_{\sigma - 1}}. \]

But if \( s > 3\sigma - 1 \), the above relation fails and we have to add the term \( \|F\|_{Z^\alpha,\beta_{\sigma - 1}} \). The reason is as follows. Choosing \( N \gg 1 \), \( u(\xi, \tau_1) = 1_{B_N}(\xi, \tau_1) \), where

\[ B_N = \{ (\xi, \tau_1) : |\xi| \leq 1, N - 1 \leq \tau_1 \leq N \}. \]

Define \( g(\tau) = 1_{\{N-1 \leq \tau \leq N\}} \). Then it follows from (2.23) that

\[ I \sim \int_{|\xi| \leq 1} \int_{N-1 \leq \tau \leq N} \int_{N-1 \leq \tau_1 \leq N} N^{\frac{s+1}{2}} N^{-\sigma} d\tau_1 d\tau d\xi \sim N^{s+\frac{1}{2} - \sigma}. \]

So in order to have \( I \lesssim \|u\|_{L^2} \sim 1 \), \( s \) can be at most \( 3\sigma - 1 \).

On the other hand, recall the definition of \( \|F\|_{Z^\alpha,\beta_{\sigma - 1}} \) which is

\[ \|F\|_{Z^\alpha,\beta_{\sigma - 1}} = \|\langle \tau \rangle^{\frac{s+1}{2} - \sigma} (\tau - \phi^{\alpha,\beta}(\xi))^{\sigma - 1} \hat{F}(\xi, \tau)\|_{L^2_{\xi,\tau}}. \]
The power \( \frac{s}{3} + \frac{1}{2} - \sigma \) is also optimal. In fact, if the \( \| \cdot \|_{Z_{\alpha,\beta}^{s, \sigma - 1}} \) norm is defined as

\[
\| F \|_{Z_{\alpha,\beta}^{s, \sigma - 1}} := \| \langle \tau \rangle^{\rho} (\tau - \phi^{\alpha,\beta}(\xi))^{\sigma - 1} \tilde{F}(\xi, \tau) \|_{L^2_{\xi, \tau}},
\]

for some \( \rho \in \mathbb{R} \), then

\[
I = \iint |g(\tau)| \langle \tau \rangle^{\frac{s+1}{3}} |\hat{\eta}(\tau - \tau_1)\rangle \langle \tau_1 \rangle^{-\rho} \left| \frac{\langle \xi \rangle^{\sigma - 1}}{\langle \tau_1 - \xi \rangle^{\frac{1}{3}}} \right| \hat{F}(\xi, \tau_1) \rangle \langle \tau_1 \rangle \, d\tau_1 \, d\tau \, d\xi.
\]

For \( N \gg 1 \), define \( v(\xi, \tau_1) = 1_{C_N}(\xi, \tau_1) \), where

\[
C_N = \left\{ (\xi, \tau_1) : N - 1 \leq \tau_1 \leq N, \frac{1}{4} \leq \xi \leq \frac{1}{2} N^{\frac{1}{3}} \right\}.
\]

Then \( \| v \|_{L^2} \sim N^{\frac{\rho}{2}} \). Define \( g(\tau) = 1_{\{N-1 \leq \tau \leq N\}} \). Then

\[
I \sim \int_{\frac{1}{4} N^{\frac{1}{3}} \leq \xi \leq \frac{1}{2} N^{\frac{1}{3}}} \int_{N-1 \leq \tau \leq N} \int_{N-1 \leq \tau_1 \leq N} N^{\frac{s+1}{3}} \, d\tau_1 \, d\tau \, d\xi \sim N^{\frac{s+2}{3} - \rho - \sigma}.
\]

In order to bound \( I \) by \( \| v \|_{L^2} \), it has to satisfy \( \frac{s+2}{3} - \rho - \sigma \leq \frac{1}{6} \), that is \( \rho \geq \frac{s}{3} + \frac{1}{2} - \sigma \).

• Similarly, if \( 0 \leq s \leq 3\sigma \), then

\[
\sup_{x \in \mathbb{R}} \left\| \eta(t) \partial_x \left( \int_0^t W_R^{\alpha,\beta}(t - \tau) F(\cdot, \tau) \, d\tau \right) \right\|_{H^2(\mathbb{R})} \leq C \| F \|_{X_{0,\sigma}^{s, \beta}}.
\]

But if \( s > 3\sigma \), the above relation fails and we have to add the term \( \| F \|_{Z_{0,\sigma}^{s, \beta}} \).

### 3 Linear problems

In this section we study the linear IBVPs,

\[
\begin{align*}
\left\{ u_t + u_{xxx} + \beta u_x &= f, \\
u(x, 0) &= p(x), \quad x, t > 0 \\
u(0, t) &= a(t),
\right.
\end{align*}
\]

and

\[
\begin{align*}
v_t - v_{xxx} - \beta v_x &= g, \\
v(x, 0) &= q(x), \quad x, t > 0, \\
v(0, t) &= b_1(t), \quad v_x(0, t) = b_2(t),
\end{align*}
\]

and present the proofs of Proposition 1.10 and Proposition 1.11.
3.1 KdV flow traveling to the right

Consideration in this subsection is given to the linear IBVP

\[
\begin{aligned}
&u_t + u_{xxx} + \beta u_x = f, \\
&u(x, 0) = p(x), \quad x, t > 0 \\
&u(0, t) = a(t),
\end{aligned}
\]  

(3.3)

which describes a linear dispersive wave traveling to the right. We assume that \( p \in H^s(\mathbb{R}^+), \ a \in H^{\frac{s+1}{2}}(\mathbb{R}^+), \) and \( f \) is a function defined on \( \mathbb{R}^2 \). The function space that \( f \) lives in will be specified later. In addition, the data \( p \) and \( a \) are assumed to be compatible, that is \( p(0) = a(0) \) if \( s > \frac{1}{2} \).

The solution \( u \) of the IBVP (3.3) can be written as \( u = u_1 + u_2 \) with

\[
u_1 = \Phi_{\beta}^{1,\beta}(f, p) := W_{R}^{1,\beta}(E_s p) + \int_0^t W_{R}^{1,\beta}(t - \tau) f(\cdot, \tau) d\tau,
\]

(3.4)

where \( E_s p \) being the extension of \( p \) as at (2.2), and

\[
u_2 = W_{bdr}^{1,\beta}(a - q),
\]

where \( q(t) := u_1(0, t) \) and \( W_{bdr}^{1,\beta}(a) \) is the solution operator of the linear IBVP

\[
\begin{aligned}
&z_t + z_{xxx} + \beta z_x = 0, \\
&z(x, 0) = 0, \quad x, t > 0 \\
&z(0, t) = a(t),
\end{aligned}
\]

(3.5)

and is called the boundary integral operator associated to (3.5).

**Lemma 3.1.** Let \( s \in (-\frac{3}{4}, 3], \ 0 < \beta \leq 1 \) and \( \frac{1}{2} < \sigma \leq 1 \). For any \( p \in H^s(\mathbb{R}^+) \) and \( f \in X_{s,\sigma}^{1,\beta} \bigcap Z_{s,\sigma}^{1,\beta} \), the function

\[
\widehat{\eta} := \eta(t) \Phi_{\beta}^{1,\beta}(f, p)
\]

belongs to \( Y_{s,\sigma}^{1,\beta} \bigcap C_j^0(\mathbb{R}; H_{t}^{\frac{s+1}{2}-j}(\mathbb{R})) \) for \( j = 0, 1 \). In addition, the following estimates hold with some constant \( C = C(s, \sigma) \).

\[
\|\widehat{\eta}\|_{Y_{s,\sigma}^{1,\beta}} \leq C(\|f\|_{X_{s,\sigma}^{1,\beta}} + \|p\|_{H^{s}(\mathbb{R}^+)})
\]

(3.6)

\[
\sup_{x \in \mathbb{R}} \|\partial_x^j \widehat{\eta}\|_{H_t^{\frac{s+1}{2}-j}(\mathbb{R})} \leq C(\|f\|_{X_{s,\sigma}^{1,\beta}} + \|f\|_{Z_{s,\sigma}^{1,\beta}} + \|p\|_{H^{s}(\mathbb{R}^+)})
\]

(3.7)

**Proof.** (3.6) follows from (2.2) and Lemma 2.4. (3.7) follows from (2.2), Lemma 2.5 and Lemma 2.6. \( \square \)

For any \( \beta > 0 \), let \( P_\beta \) be as given in (2.5) and define the function \( R_\beta : \mathbb{R} \to \mathbb{C} \) by

\[
R_\beta(\mu) := \frac{\sqrt{3\mu^2 - 4\beta}}{2} = \begin{cases} 
\frac{i \sqrt{4\beta - 3\mu^2}}{2} & \text{if } \mu^2 \leq \frac{4}{3}\beta, \\
\frac{\sqrt{3\mu^2 - 4\beta}}{2} & \text{if } \mu^2 > \frac{4}{3}\beta.
\end{cases}
\]

(3.8)
Lemma 3.2. The boundary integral operator $W_{bdr}^{1,\beta}$ associated to (3.5) has the following explicit integral representation
\[
W_{bdr}^{1,\beta}(a)(x,t) = \frac{1}{\pi} \text{Re} \int_{\sqrt{3}}^{\infty} e^{iP_{\beta}(\mu)t} e^{-i\mu x/2} e^{-R_{\beta}(\mu)x} a^*[P_{\beta}(\mu)] dP_{\beta}(\mu), \quad x,t \geq 0, \tag{3.9}
\]
where $a^*$ is the zero extension of $a$ from $\mathbb{R}^+$ to $\mathbb{R}$.

The proof of this lemma is standard based on the Laplace transform method in [BSZ02] and is therefore omitted.

Next, we will extend $W_{bdr}^{1,\beta}(a)$ to the whole plane $\mathbb{R}^2$ with some good properties. According to (3.8), when $\sqrt{3} \leq \mu \leq \sqrt{3}/2$, $R_{\beta}(\mu)$ is a pure imaginary number, so (3.9) is also well-defined for $x,t < 0$. But when $\mu > \sqrt{3}/2$, if we do not modify the definition of $W_{bdr}^{1,\beta}(a)$ for $x < 0$, then $e^{-R_{\beta}(\mu)x} \to +\infty$ as $R_{\beta}(\mu)x \to -\infty$, which makes it difficult to control $W_{bdr}^{1,\beta}(a)$.

To resolve this issue, our first strategy is to multiply a cut-off function (see (3.12)) to eliminate the part where $R_{\beta}(\mu)x \leq -1$, this idea comes from [ET16]. Let $\psi \in C^\infty(\mathbb{R})$ such that $0 \leq \psi \leq 1$ and
\[
\psi(x) = \begin{cases} 
1 & \text{if } x \geq 0, \\
0 & \text{if } x \leq -1.
\end{cases} \tag{3.10}
\]

Define $\Phi_{bdr}^{1,\beta}(a) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by
\[
\Phi_{bdr}^{1,\beta}(a)(x,t) = \frac{1}{\pi} \text{Re} \int_{\sqrt{3}}^{\infty} e^{iP_{\beta}(\mu)t} e^{-i\mu x/2} K_{\beta}(x,\mu) a^*[P_{\beta}(\mu)] dP_{\beta}(\mu), \tag{3.11}
\]
where
\[
K_{\beta}(x,\mu) := \begin{cases} 
e^{-R_{\beta}(\mu)x} & \text{if } \sqrt{3} \leq \mu \leq \sqrt{3}/2, \\
e^{-R_{\beta}(\mu)x} \psi(R_{\beta}(\mu)x) & \text{if } \mu > \sqrt{3}/2.
\end{cases} \tag{3.12}
\]

It is easily seen that $\Phi_{bdr}^{1,\beta}(a)$ is well-defined on $\mathbb{R} \times \mathbb{R}$. In addition, when $x \geq 0$, $K_{\beta}(x,\mu) = e^{-R_{\beta}(\mu)x}$, so $\Phi_{bdr}^{1,\beta}(a)$ matches $W_{bdr}^{1,\beta}(a)$ on $\mathbb{R}^+ \times \mathbb{R}^+$. Hence, $\Phi_{bdr}^{1,\beta}(a)$ is an extension of $W_{bdr}^{1,\beta}(a)$ to $\mathbb{R} \times \mathbb{R}$. Moreover, if $s > 1/2$, it enjoys some desired properties after being localised in time.

Lemma 3.3. Let $s \in (-1/2, 3]$, $0 < \beta \leq 1$ and $1/2 < \sigma \leq \min \{1, 2s+7/4\}$. For any $a \in H^{1,\beta}(\mathbb{R}^+)$ that is compatible with (3.3), the function
\[
\widetilde{u}_2 := \eta(t) \Phi_{bdr}^{1,\beta}(a)
\]
equals $W_{bdr}^{1,\beta}(a)$ on $\mathbb{R}^+_t \times [0,1]$ and belongs to $Y^{1,\beta}_{s,\sigma} \cap C^j_y \bigcup H^{s+1-j}_x (\mathbb{R})$ for $j = 0, 1$. In addition, the following estimates hold with some constant $C = C(s, \sigma)$.
\[
\|\widetilde{u}_2\|_{Y^{1,\beta}_{s,\sigma}} \leq C\|a\|_{H^{1,\beta}(\mathbb{R}^+)},
\]
\[
\sup_{x \in \mathbb{R}} \|\partial_t^j \widetilde{u}_2\|_{H^{s+1-j}_x (\mathbb{R})} \leq C\|a\|_{H^{1,\beta}(\mathbb{R}^+)}, \quad j = 0, 1. \tag{3.14}
\]

Although the extension $\Phi_{bdr}^{1,\beta}$ has a very simple expression, the estimate (3.13) fails for $s < -1/2$, see the proof of Lemma 3.3. So in order to deal with the case when $-3/4 < s \leq -1/2$, we have to use another extension of $W_{bdr}^{1,\beta}(a)$. In fact, we will combine the extension $\Phi_{bdr}^{1,\beta}(a)$ and the construction in [BSZ06] to achieve the goal. Since the construction in [BSZ06] is too technical to be written as an explicit expression,
we will first state the result and then provide more details later in the proof.

**Lemma 3.4.** Let \(-\frac{3}{4} < s \leq -\frac{1}{2}\), \(0 < \beta \leq 1\) and \(a \in H^{s+1,\beta} (\mathbb{R}^+).\) Then there exist \(\sigma_1 = \sigma_1(s) > \frac{1}{2}\) and an extension \(\Psi_{\text{bdr}}^{1,\beta}(a)\) of \(W_{\text{bdr}}^{1,\beta}(a)\) such that for any \(\sigma \in \left(\frac{1}{2}, \sigma_1\right],\) the function

\[
\tilde{u}_2 := \eta(t)\Psi_{\text{bdr}}^{1,\beta}(a)
\]

equals \(W_{\text{bdr}}^{1,\beta}(a)\) on \(\mathbb{R}^+_0 \times [0, 1]\) and belongs to \(Y_{s,\frac{1}{2},\sigma} \cap C^j_x(\mathbb{R}^+_0; H^{s+1,\beta}_t (\mathbb{R}))\) for \(j = 0, 1.\) In addition, the following estimates hold with some constant \(C = C(s, \sigma)\).

\[
\|\tilde{u}_2\|_{Y_{s,\frac{1}{2},\sigma}} \leq C\|a\|_{H^{s+1,\beta} (\mathbb{R}^+)} \tag{3.15}
\]

\[
\sup_{x \geq 0} \|\partial_x^j \tilde{u}_2\|_{H^{s+1,\beta}_t (\mathbb{R})} \leq C\|a\|_{H^{s+1,\beta} (\mathbb{R}^+)}, \quad j = 0, 1. \tag{3.16}
\]

**Remark 3.5.** On the one hand, the extension function \(\eta(t)\Psi_{\text{bdr}}^{1,\beta}(a)\) in Lemma 3.3 is shown to live in \(C^2_x (\mathbb{R}; H^{s+1,\beta}_t (\mathbb{R}))\) and the estimate (3.14) is valid for any \(x \in \mathbb{R}.\) On the other hand, the extension function \(\eta(t)\Psi_{\text{bdr}}^{1,\beta}(a)\) in Lemma 3.4 is only proved to live in \(C^2_x (\mathbb{R}^+_0; H^{s+1,\beta}_t (\mathbb{R}))\) and the estimate (3.16) is only justified for \(x \geq 0.\)

Assuming Lemma 3.2 and Lemma 3.4 hold, we will first carry out the proof for Proposition 1.10. The proofs of Lemma 3.3 and Lemma 3.4 will be presented afterwards.

**Proof of Proposition 1.10.**

Without loss of generality, we assume \(T = 1.\) Then, we first deal with the case when \(s \in \left(-\frac{1}{2}, 3\right].\) Choose \(\sigma_1(s) = \min \left\{1, \frac{2s+1}{12}\right\}\) and consider any \(\sigma \in \left(\frac{1}{2}, \sigma_1(s)\right].\) By Lemma 3.1 and Lemma 3.3, define

\[
\tilde{u} = \Gamma^+_\beta(f, p, a) := \eta(t)\Phi^{1,\beta}_R(f, p) + \eta(t)\Phi_{\text{bdr}}^{1,\beta}(a - \eta(t)\Phi^{1,\beta}_R(f, p)|_{x=0}).
\]

Then \(\tilde{u}\) is defined on \(\mathbb{R} \times \mathbb{R}\) and solves (1.21) on \(\mathbb{R}^+_0 \times [0, 1].\) Furthermore, it follows from (3.6) and (3.13) that there exists a constant \(C = C(s, \sigma)\) such that

\[
\|\Gamma^+_\beta(f, p, a)\|_{Y^{1,\beta}_{s,\frac{1}{2},\sigma}} \leq C \left(\|f\|_{X_{s,\sigma}^{1,\beta}} + \|p\|_{H^s} + \|a - \eta(t)\Phi^{1,\beta}_R(f, p)|_{x=0}\|_{H^{s+1}_t}\right) \tag{3.17}
\]

By (3.7) with \(j = 0,

\[
\|\eta(t)\Phi^{1,\beta}_R(f, p)|_{x=0}\|_{H^{s+1}_t} \leq C \left(\|f\|_{X_{s,\sigma}^{1,\beta}} + \|f\|_{Z_{s,\sigma}^{1,\beta}} + \|p\|_{H^s}\right).
\]

Combining the above two estimates yields (1.24). Next, by similar argument and using (3.7) and (3.14), we can justify (1.25) as well. Finally, it also follows from Lemma 3.1 and Lemma 3.3 that \(\tilde{u}\) belongs to \(Y_{s,\frac{1}{2},\sigma} \cap C^j_x(\mathbb{R}^+_0; H^{s+1,\beta}_t (\mathbb{R}))\) for \(j = 0, 1.\)

Now we treat the case when \(s \in (-\frac{3}{4}, -\frac{1}{2}).\) The proof for this case is almost the same as the above case except we replace \(\Phi^{1,\beta}_R\) by \(\Psi_{\text{bdr}}^{1,\beta}\) and replace Lemma 3.3 by Lemma 3.4. The proof of Proposition 1.10 is thus complete. \(\square\)

**Proof of Lemma 3.3**
We will first establish the estimates (3.13) and (3.14) for $s \in (-\frac{1}{2}, \frac{1}{2})$. Then the corresponding estimates in the case of $s = \frac{1}{2}$ can be deduced by interpolating between the case $s = 0$ and the case $s = 3$.

Fix $s \in (-\frac{1}{2}, \frac{1}{2})$ and $0 < \beta \leq 1$. Define $h_\beta$ such that $\widehat{h_\beta}(\mu) = (P_\beta)'(\mu)\mu^s a^\ast [P_\beta(\mu)]$. Then

$$
\Phi_{bd}(a)(x,t) = \frac{1}{\pi} \Re \int_{\pi} e^{iP_\beta(\mu)t} e^{-i\mu x/2} K_\beta(x,\mu) \widehat{\mu}(\mu) \, d\mu,
$$

where $K_\beta$ is as defined in (3.12). Since $|(P_\beta)'(\mu)| = |3\mu^2 - \beta| \lesssim (\mu)^2$, then

$$
(\mu)^s |\widehat{h_\beta}(\mu)| \lesssim |(P_\beta)'(\mu)|^{1/2}(\mu)^{-s+1} a^\ast [P_\beta(\mu)].
$$

Since $a \in H^{\frac{1+s}{2}}(\mathbb{R}^+)$, it then follows from Corollary 2.3 that $h_\beta \in H^s(\mathbb{R})$ and

$$
\|h_\beta\|_{H^s(\mathbb{R})} \lesssim \|a\|_{H^{\frac{s+1}{2}}(\mathbb{R}^+)}. 
$$

Inspired by the above observation, we define an operator $T_\beta$ as

$$
[T_\beta(h)](x,t) := \int_{\pi} e^{iP_\beta(\mu)t} e^{-i\mu x/2} K_\beta(x,\mu) \hat{h}(\mu) \, d\mu. 
$$

(3.17)

Recalling $Y_{s,\frac{1}{2},\sigma}^1 = X_{s,\frac{1}{2}}^1 \cap \Lambda_{s,\frac{1}{2}}^1 \cap C_t(\mathbb{R},H^s(\mathbb{R}))$, so the proof of Lemma 3.3 reduces to establishing the following estimates for the operator $T_\beta$.

$$
\|\eta(t)T_\beta(h)\|_{Y_{s,\frac{1}{2}}^1} \leq C\|h\|_{H^s(\mathbb{R})}, 
$$

(3.18)

$$
\|\eta(t)T_\beta(h)\|_{\Lambda_{s,\frac{1}{2}}^1} \leq C\|h\|_{H^s(\mathbb{R})}, 
$$

(3.19)

$$
\sup_{t \in \mathbb{R}} \|\eta(t)T_\beta(h)\|_{H^j_t(\mathbb{R})} \leq C\|h\|_{H^s(\mathbb{R})}, 
$$

(3.20)

$$
\sup_{x \in \mathbb{R}} \|\partial_x^j [\eta(t)T_\beta(h)]\|_{H^{s-1-j}_x(\mathbb{R})} \leq C\|h\|_{H^s(\mathbb{R})}, \quad j = 0,1. 
$$

(3.21)

Before showing those estimates hold, we introduce some notations. Recalling the formula (3.12), we have

$$
K_\beta(x,\mu) = \begin{cases} 
\ e^{-i\sqrt{4\beta-3\mu^2}x/2} & \text{if } \sqrt{\beta} \leq \mu \leq \sqrt{\frac{4}{3}}\beta, \\
\ k(R_\beta(\mu)x) & \text{if } \mu > \sqrt{\frac{4}{3}}\beta,
\end{cases}
$$

where

$$
k(y) := e^{-y}\psi(y), \quad \forall y \in \mathbb{R}. 
$$

(3.22)

It is easily seen that $k$ is a real-valued Schwarz function on $\mathbb{R}$. We decompose $T_\beta(h)$ as $T_\beta(h) = T_{\beta,1}(h) + T_{\beta,2}(h)$, where

$$
[T_{\beta,1}(h)](x,t) := \int_{\pi} e^{iP_\beta(\mu)t} e^{-i(\mu+\sqrt{4\beta-3\mu^2})x/2} \hat{h}(\mu) \, d\mu, 
$$

(3.23)

$$
[T_{\beta,2}(h)](x,t) := \int_{\pi} e^{iP_\beta(\mu)t} e^{-i\mu x/2} k(R_\beta(\mu)x) \hat{h}(\mu) \, d\mu. 
$$

(3.24)
Proof of (3.18).

First, we discuss the estimate for $T_{\beta,1}(h)$ and will actually prove a stronger result:

$$\|\eta(t)T_{\beta,1}(h)\|_{X^1_{1,1}} \lesssim \|h\|_{H^s(\mathbb{R})}.$$  \hfill (3.25)

By definition,

$$\|\eta(t)T_{\beta,1}(h)\|_{X^1_{1,1}} = \|\langle \xi \rangle^3 \langle \tau - \phi^{1,\beta}(\xi) \rangle F[\eta T_{\beta,1}(h)](\xi,\tau)\|_{L^2_{\xi,\tau}},$$

where $F[\eta T_{\beta,1}(h)]$ means the space-time Fourier transform of $\eta(t)T_{\beta,1}(h)$. Since $\langle \tau - \phi^{1,\beta}(\xi) \rangle \lesssim \langle \xi \rangle^3$, it suffices to justify

$$\|\langle \xi \rangle^6 \langle \tau \rangle F[\eta T_{\beta,1}(h)](\xi,\tau)\|_{L^2_{\xi,\tau}} \lesssim \|h\|_{H^s(\mathbb{R})}.$$  \hfill (3.26)

By direct computation,

$$F[\eta T_{\beta,1}(h)](\xi,\tau) = \frac{\sqrt{\beta}}{\sqrt{\pi}} \int \hat{\eta}(\tau - P_\beta(\mu))\delta\left(\xi + \frac{\mu + \sqrt{4\beta - 3\mu^2}}{2}\right)\hat{h}(\mu) \, d\mu,$$

where $\delta$ represents the Dirac delta function such that $\int_{\mathbb{R}} f(x)\delta(x) \, dx = f(0)$ for any test function $f$ in the sense of distribution. By duality, we have

$$\|\langle \xi \rangle^6 \langle \tau \rangle F[\eta T_{\beta,1}(h)](\xi,\tau)\|_{L^2_{\xi,\tau}} = \sup_{\|g\|_{L^2_{\xi,\tau}} = 1} \int_{\mathbb{R}} \langle \xi \rangle^6 \langle \tau \rangle \int_{\mathbb{R}} \hat{g}(\tau - P_\beta(\mu))\delta\left(\xi + \frac{\mu + \sqrt{4\beta - 3\mu^2}}{2}\right)\hat{h}(\mu) \, d\mu \, d\xi \, d\tau$$

$$= \sup_{\|g\|_{L^2_{\xi,\tau}} = 1} \int_{\mathbb{R}} \langle \tau \rangle \int_{\mathbb{R}} \langle \xi \rangle^6 \hat{g}(\tau - P_\beta(\mu))\delta\left(\xi + \frac{\mu + \sqrt{4\beta - 3\mu^2}}{2}\right) d\xi \, d\tau \, d\mu. \hfill (3.27)$$

Taking advantage of the properties of the $\delta$ function, we obtain

$$\text{RHS of (3.27)} \lesssim \sup_{\|g\|_{L^2_{\xi,\tau}} = 1} \int_{\mathbb{R}} \|\hat{h}(\mu)\|\langle \mu + \sqrt{4\beta - 3\mu^2} \rangle^6 \int_{\mathbb{R}} |\hat{g}(\tau - P_\beta(\mu))| d\tau \, d\mu.$$  \hfill (3.28)

Since $\mu$ is between $\sqrt{\beta}$ and $\sqrt{4\beta/3}$, then $\langle \mu + \sqrt{4\beta - 3\mu^2} \rangle$ is bounded and $\langle \tau \rangle \sim \langle \tau - P_\beta(\mu) \rangle$. Hence, by Hölder’s inequality,

$$\text{RHS of (3.27)} \lesssim \sup_{\|g\|_{L^2_{\xi,\tau}} = 1} \int_{\mathbb{R}} \|\hat{h}(\mu)\| \|g\left(-\frac{\mu + \sqrt{4\beta - 3\mu^2}}{2}, \tau\right)\|_{L^2_{\xi,\tau}} d\mu.$$  \hfill (3.29)
Thus, the proof of (3.26) reduces to show for any \( \| g \|_{L^2_{\tau}} = 1 \),

\[
\sqrt{\frac{\beta^3}{3}} \int_{\sqrt{\beta}} |\hat{h}(\mu)| \left\| g \left( -\frac{\mu + \sqrt{4 \beta - 3 \mu^2}}{2}, \tau \right) \right\|_{L^2_{\xi}} d\mu \lesssim \| h \|_{H^s(\mathbb{R})}.
\]  

(3.28)

Since

\[
\left| \frac{d}{d\mu} \left( \mu + \sqrt{4 \beta - 3 \mu^2} \right) \right| = \frac{3 \mu - \sqrt{4 \beta - 3 \mu^2}}{\sqrt{4 \beta - 3 \mu^2}} \sim \frac{\mu}{\sqrt{4 \beta - 3 \mu^2}},
\]

then

\[
\text{LHS of (3.28)} \lesssim \sqrt{\frac{\beta^3}{3}} \int_{\sqrt{\beta}} |\hat{h}(\mu)| \left( \left| \frac{d}{d\mu} \left( \mu + \sqrt{4 \beta - 3 \mu^2} \right) \right| \right)^{1/2} \left\| g \left( -\frac{\mu + \sqrt{4 \beta - 3 \mu^2}}{2}, \tau \right) \right\|_{L^2_{\xi}} d\mu.
\]  

(3.29)

Applying Hölder’s inequality again and taking advantage of the fact that \( \| g \|_{L^2_{\xi}} = 1 \), we obtain

\[
\text{RHS of (3.29)} \lesssim \left( \int_{\sqrt{\beta}} |\hat{h}(\mu)|^2 \frac{\sqrt{4 \beta - 3 \mu^2}}{\mu} d\mu \right)^{1/2} \lesssim \left( \int_{\sqrt{\beta}} |\hat{h}(\mu)|^2 d\mu \right)^{1/2} \lesssim \| h \|_{H^s(\mathbb{R})}.
\]

Next we discuss the estimate for \( T_{\beta,2}(h) \) which is further decomposed into two parts: \( T_{\beta,2}(h) = T_{\beta,3}(h) + T_{\beta,4}(h) \), where

\[
[T_{\beta,3}(h)](x,t) := \int_{\sqrt{\beta}} e^{iP_{h}(\mu)\mu} e^{-i\mu x/2} k(R_{\beta}(\mu)x) \hat{h}(\mu) d\mu,
\]  

(3.30)

\[
[T_{\beta,4}(h)](x,t) := \int_{4}^{\infty} e^{iP_{h}(\mu)\mu} e^{-i\mu x/2} k(R_{\beta}(\mu)x) \hat{h}(\mu) d\mu.
\]  

(3.31)

The reason of introducing such a decomposition is to make sure that \( R_{\beta}(\mu) \) has a positive lower bound in \( T_{\beta,4}(h) \). For example, with the choice of 4, we have \( R_{\beta}(\mu) > 3 \) for any \( \mu \geq 4 \) since \( 0 < \beta \leq 1 \). The choice of the cut-off number 4 is flexible, as long as it is away from \( \sqrt{\frac{1}{3}} \beta \).

For the estimate on \( T_{\beta,3} \), we will also prove a stronger result:

\[
\| \eta(t) T_{\beta,3}(h) \|_{X_{1,1}^{1,\beta}} \lesssim \| h \|_{H^s(\mathbb{R})}.
\]  

(3.32)

Similar to the argument above, it suffices to verify

\[
\left\| (\xi^6(\tau)) \mathcal{F} \left[ \eta T_{\beta,3}(h) \right] (\xi,\tau) \right\|_{L^2_{\xi,\tau}} \lesssim \| h \|_{H^s(\mathbb{R})}.
\]  

(3.33)
By direct computation,

\[ \mathcal{F}[\eta(t)T_{\beta,\beta}(h)](\xi, \tau) = \int_{\mathbb{R}^3} \tilde{\eta}(\tau - P_{\beta}(\mu)) \frac{1}{R_{\beta}(\mu)} \tilde{k}\left(\frac{\xi + \mu/2}{R_{\beta}(\mu)}\right) \tilde{h}(\mu) \, d\mu. \]  

(3.34)

For \( \sqrt{\frac{4}{3}} \beta \leq \mu \leq 4 \), we have \( \langle \tau \rangle \sim \langle \tau - P_{\beta}(\mu) \rangle \), so \( \| \langle \tau \rangle \tilde{\eta}(\tau - P_{\beta}(\mu)) \|_{L^2} \leq C \). Then it follows from (3.34) and the Minkowski’s inequality that

\[ \text{LHS of (3.33)} \lesssim \left\| \langle \xi \rangle^6 \int_{\mathbb{R}^3} \frac{1}{R_{\beta}(\mu)} \left| \tilde{k}\left(\frac{\xi + \mu/2}{R_{\beta}(\mu)}\right)\right| \tilde{h}(\mu) \, d\mu \right\|_{L^2_s}. \]  

(3.35)

By duality, it suffices to show that for any \( g = g(\xi) \) with \( \|g\|_{L^2_s} = 1 \), the following estimate holds.

\[ \int_{\mathbb{R}} |g(\xi)| \int_{\mathbb{R}^3} \left| \langle \xi \rangle^6 \tilde{k}\left(\frac{\xi + \mu/2}{R_{\beta}(\mu)}\right)\right| \tilde{h}(\mu) \, d\mu \, d\xi \lesssim \|h\|_{H^s(\mathbb{R})}. \]

Exchanging the order of integration, that is to show

\[ \int_{\mathbb{R}^3} \frac{\tilde{h}(\mu)}{R_{\beta}(\mu)} \int_{\mathbb{R}} \left| \langle \xi \rangle^6 \tilde{k}\left(\frac{\xi + \mu/2}{R_{\beta}(\mu)}\right)\right| d\xi \, d\mu \lesssim \|h\|_{H^s(\mathbb{R})}. \]  

(3.36)

By the change of variable \( \xi \rightarrow y := \frac{\xi + \mu/2}{R_{\beta}(\mu)} \),

\[ \text{LHS of (3.36)} = \int_{\mathbb{R}^3} \tilde{h}(\mu) \int_{\mathbb{R}} \left| g\left(R_{\beta}(\mu)y - \frac{\mu}{2}\right)\right| \left| \langle R_{\beta}(\mu)y - \frac{\mu}{2}\rangle^6 \tilde{k}(y)\right| \, dy \, d\mu. \]  

(3.37)

Since \( \sqrt{4/3} \beta \leq \mu \leq 4 \), \( \langle R_{\beta}(\mu)y - \frac{\mu}{2}\rangle \lesssim \langle y \rangle \), then

\[ \text{RHS of (3.37)} \lesssim \left( \int_{A_1} + \int_{A_2} + \int_{A_3} \right) \tilde{h}(\mu) \left| g\left(R_{\beta}(\mu)y - \frac{\mu}{2}\right)\right| \langle y \rangle^6 |\tilde{k}(y)| \, dy \, d\mu, \]  

(3.38)

where

\[ A_1 := \{ \mu, y : \sqrt{4/3} \beta \leq \mu \leq 4\sqrt{3}, \, y \in \mathbb{R} \}, \]

\[ A_2 := \{ \mu, y : 4\sqrt{3} \leq \mu \leq 4, \, |y - \frac{1}{\sqrt{3}}| \lesssim \frac{\beta^2}{\mu^2} \}, \]

\[ A_3 := \{ \mu, y : 4\sqrt{3} \leq \mu \leq 4, \, |y - \frac{1}{\sqrt{3}}| \gg \frac{\beta^2}{\mu^2} \}. \]

- **Contribution on \( A_1 \):** Since \( \|g\|_{L^2} = 1 \), it follows from the Hölder’s inequality that

\[ A_1 \text{ part of (3.38)} = \int_{\sqrt{4/3} \beta}^{4\sqrt{3}} \tilde{h}(\mu) \left( \int_{\mathbb{R}} \left| g\left(R_{\beta}(\mu)y - \frac{\mu}{2}\right)\right| \langle y \rangle^6 |\tilde{k}(y)| \, dy \right) \, d\mu \lesssim \int_{\sqrt{4/3} \beta}^{4\sqrt{3}} \frac{\tilde{h}(\mu)}{\sqrt{R_{\beta}(\mu)}} \, d\mu, \]
Applying the Hölder’s inequality again, we deduce
\[ \int_{4\sqrt{3}}^{4\sqrt{9\beta}} \frac{\hat{h}(\mu)}{R_\beta(\mu)} \, d\mu \lesssim \left( \int_{4\sqrt{3}}^{4\sqrt{9\beta}} |\hat{h}(\mu)|^2 \, d\mu \right)^{\frac{1}{2}} \left( \int_{4\sqrt{3}}^{4\sqrt{9\beta}} \frac{d\mu}{R_\beta(\mu)} \right)^{\frac{1}{2}} \lesssim \left( \int_{4\sqrt{3}}^{4\sqrt{9\beta}} |\hat{h}(\mu)|^2 \, d\mu \right)^{\frac{1}{2}} \lesssim \|h\|_{H^s(\mathbb{R})}. \]

This completes the desired estimate on $A_1$. We point out that this argument does not work for $4\sqrt{3} \leq \mu \leq 4$ since
\[ \int_{4\sqrt{3}}^{4} \frac{d\mu}{R_\beta(\mu)} \sim \int_{4\sqrt{3}}^{4} \frac{d\mu}{\mu} \sim \ln \left( \frac{1}{\beta} \right), \]
which tends to $+\infty$ as $\beta \to 0^+$. 

- **Contribution on $A_2$:** Since $k$ is a Schwarz function, $(y)^6|\hat{k}(y)|$ is bounded, we thus conclude
\[ A_2 \text{ part of (3.38)} \lesssim \int_{4\sqrt{3}}^{4} |\hat{h}(\mu)| \int_{|y-\frac{\mu}{2}| \lesssim \frac{1}{2\pi}} |g \left( R_\beta(\mu)y - \frac{\mu}{2} \right)| \, dy \, d\mu. \]  

By Hölder’s inequality and the fact that $\|g\|_{L^2} = 1$, we know
\[ \int_{|y-\frac{\mu}{2}| \lesssim \frac{1}{2\pi}} |g \left( R_\beta(\mu)y - \frac{\mu}{2} \right)| \, dy \lesssim \frac{1}{\sqrt{R_\beta(\mu)}} \sqrt{\frac{3}{\mu}} \sim \frac{\sqrt{3}}{\mu^{3/2}}, \]
where the last inequality is due to $\mu \geq 4\sqrt{3}$. As a result,
\[ \text{RHS of (3.39)} \lesssim \sqrt{\beta} \int_{4\sqrt{3}}^{4} \left( \int_{4\sqrt{3}}^{\mu^{3/2}} \frac{d\mu}{\mu} \right)^{\frac{1}{2}} \left( \int_{4\sqrt{3}}^{\mu^{3/2}} |\hat{h}(\mu)|^2 \, d\mu \right)^{\frac{1}{2}} \lesssim \left( \int_{4\sqrt{3}}^{\mu^{3/2}} |\hat{h}(\mu)|^2 \, d\mu \right)^{\frac{1}{2}} \lesssim \|h\|_{H^s(\mathbb{R})}. \]

- **Contribution on $A_3$:** for any $(\mu, y) \in A_3$, we claim
\[ \left| R_\beta(\mu)y - \frac{1}{2} \right| \sim |y - \frac{1}{\sqrt{3}}|. \]  

We will first use (3.41) to finish the estimate on $A_3$. The justification of (3.41) will be given after that. Thanks to (3.41),
\[ A_3 \text{ part of (3.38)} \lesssim \int_{\mathbb{R}} (y)^6|\hat{k}(y)| \left| y - \frac{1}{\sqrt{3}} \right|^{-\frac{1}{2}} \int_{4\sqrt{3}}^{4} |\hat{h}(\mu)| \left| R_\beta(\mu)y - \frac{1}{2} \right|^{1/2} \left| g \left( R_\beta(\mu)y - \frac{\mu}{2} \right) \right| \, d\mu \, dy. \]

By Hölder’s inequality and $\|g\|_{L^2} = 1$,
\[ \int_{4\sqrt{3}}^{4} |\hat{h}(\mu)| \left| R_\beta(\mu)y - \frac{1}{2} \right|^{1/2} \left| g \left( R_\beta(\mu)y - \frac{\mu}{2} \right) \right| \, d\mu \lesssim \left( \int_{4\sqrt{3}}^{4} |\hat{h}(\mu)|^2 \, d\mu \right)^{\frac{1}{2}} \lesssim \|h\|_{H^s(\mathbb{R})}. \]
Hence,

\[ A_3 \text{ part of } (3.38) \lesssim \left( \int_{\mathbb{R}} |k(y)|^6 \left| y - \frac{1}{\sqrt[3]{3}} \right|^{-\frac{1}{2}} dy \right) \| h \|_{H^s(\mathbb{R})} \lesssim \| h \|_{H^s(\mathbb{R})}. \]

It thus remains to verify (3.41) on \( A_3 \). Since \( \mu \geq 4\sqrt{3} \),

\[ R'_\beta(\mu) = \frac{3\mu}{2\sqrt{3}\mu^2 - 4\beta} \sim 1. \]

Consequently,

\[ |R'_\beta(\mu)y - \frac{1}{2}| \sim |y - \frac{1}{2R'_\beta(\mu)}| = \left| y - \frac{1}{\sqrt[3]{3}} + \left( \frac{1}{\sqrt[3]{3}} - \frac{\sqrt{3}\mu^2 - 4\beta}{3\mu} \right) \right|. \]  

(3.42)

By direct computation,

\[ 0 < \frac{1}{\sqrt[3]{3}} - \frac{\sqrt{3}\mu^2 - 4\beta}{3\mu} \leq \frac{\beta}{\mu^2}. \]

Since \( |y - \frac{1}{\sqrt[3]{3}}| \gg \frac{\beta}{\mu^2} \) for any \( (\mu, y) \in A_3 \), it then follows from (3.42) that \( |R'_\beta(\mu)y - \frac{1}{2}| \sim |y - \frac{1}{\sqrt[3]{3}}| \).

Next, the focus is turned to the estimate on \( \| \eta(t)T_{3,4}(h) \|_{X^s,4} \). By direct computation,

\[ \mathcal{F}[\eta(t)T_{3,4}(h)](\xi, \tau) = \int_{4}^{\infty} \hat{\eta}(\tau - P_\beta(\mu)) \frac{1}{R_\beta(\mu)} \hat{k}\left( \frac{\xi + \mu/2}{R_\beta(\mu)} \right) \hat{h}(\mu) \, d\mu. \]

Since \( k \) is a Schwarz function, so is \( \hat{k} \), hence,

\[ \left| \hat{k}\left( \frac{\xi + \mu/2}{R_\beta(\mu)} \right) \right| \lesssim \frac{R^6(\mu)}{R^6(\mu) + |\xi + \mu/2|^6} \sim \frac{\mu^6}{\mu^6 + |2\xi + \mu|^6}. \]

As a result,

\[ |\mathcal{F}[\eta(t)T_{3,4}(h)](\xi, \tau)| \lesssim \int_{4}^{\infty} \hat{\eta}(\tau - P_\beta(\mu)) \frac{\mu^6|\hat{h}(\mu)|}{\mu^6 + |2\xi + \mu|^6} \, d\mu. \]  

(3.43)

On the other hand, for any \( \mu \geq 4 \),

\[ \langle \tau - \phi^{1,\beta}(\xi) \rangle \lesssim \langle \tau - P_\beta(\mu) \rangle \langle P_\beta(\mu) - \phi^{1,\beta}(\xi) \rangle \lesssim \langle \tau - P_\beta(\mu) \rangle (\mu + |\xi|^3). \]

So it follows from (3.43) that

\[ \langle \xi \rangle^s (\tau - \phi^{1,\beta}(\xi))^{\frac{1}{2}} |\mathcal{F}[\eta(t)T_{3,4}(h)](\xi, \tau)| \lesssim \int_{4}^{\infty} \langle \tau - P_\beta(\mu) \rangle^{\frac{1}{2}} \hat{\eta}(\tau - P_\beta(\mu)) \left| \frac{\langle \xi \rangle^s(\mu + |\xi|)^{\frac{1}{2}}}{\mu^6 + |2\xi + \mu|^6} \mu^5|\hat{h}(\mu)| \right| \, d\mu. \]  

(3.44)

It remains to estimate the \( L^2_{t,\tau} \) norm of the right hand side of (3.44). First, for any fixed \( \mu \geq 4 \), by dividing \( \mathbb{R} \) into \( \{ \xi : |\xi| \geq \mu \} \) and \( \{ \xi : |\xi| < \mu \} \), and by utilizing the assumption \(-\frac{1}{2} < s \leq 3\), we attain

\[ \left\| \frac{\langle \xi \rangle^s(\mu + |\xi|)^{\frac{1}{2}}}{\mu^6 + |2\xi + \mu|^6} \right\|_{L^2_{t,\tau}} \lesssim C\mu^{s-4}. \]  

(3.45)
Thus, by (3.44) and Minkowski’s inequality,
\[
\left\| (\xi)^s (\tau - \phi^{1,\beta}(\xi))^{\frac{1}{2}} \mathcal{F}[\eta(t)T_{\beta,4}(h)](\xi, \tau) \right\|_{L^2_\xi} \lesssim \int_4^\infty \langle \tau - P_\beta(\mu) \rangle^\frac{1}{2} |\hat{\eta}(\tau - P_\beta(\mu))| \mu^{s+1} |\hat{h}(\mu)| \, d\mu.
\] (3.46)

Note that \( P_\beta(\mu) = \mu^3 - \beta \mu \) is increasing on \([4, \infty)\), so its inverse function is well-defined which is denoted as \( P_\beta^{-1} \). By making the change of variable \( y = P_\beta(\mu) \), we have \( y \sim \mu^3 \) and
\[
\text{RHS of (3.46)} \lesssim \int_{P_\beta(4)}^\infty \langle \tau - y \rangle^\frac{1}{2} |\hat{\eta}(\tau - y)| y^{-\frac{1}{2}} |\hat{h}(P_\beta^{-1}(y))| \, dy.
\] (3.47)

Since \( \eta \) is a Schwarz function, \( \langle \cdot \rangle^\frac{1}{2} \hat{\eta}(\cdot) \) is integrable. Then it follows from the Young’s inequality that
\[
\left\| \text{RHS of (3.47)} \right\|_{L^2_\xi} \lesssim \left( \int_4^\infty \mu^{2s} |\hat{\eta}(\mu)|^2 \, d\mu \right)^\frac{1}{2} = \| \eta \|_{H^s(\mathbb{R})}.
\] (3.48)

Finally, making the change of variable \( \mu = P_\beta^{-1}(y) \) yields
\[
\text{RHS of (3.48)} \lesssim \left( \int_4^\infty \mu^{2s} |\hat{\eta}(\mu)|^2 \, d\mu \right)^\frac{1}{2} = \| h \|_{H^s(\mathbb{R})}.
\] (3.49)

Combining (3.46)-(3.49), we conclude \( \| \eta(t) T_{\beta,4}(h) \|_{X^{1,\beta}_{\lambda,\tau}} \lesssim \| h \|_{H^s(\mathbb{R})} \).

**Proof of (3.19).**

Noticing for any function \( f \), \( \| f \|_{A^{1,\beta}_{1,\tau}} \lesssim \| f \|_{X^{1,\beta}_{1,\tau}} \). Then according to (3.25) and (3.32), we have
\[
\| \eta(t) T_{\beta,1}(h) \|_{A^{1,\beta}_{1,\tau}} + \| \eta(t) T_{\beta,3}(h) \|_{A^{1,\beta}_{1,\tau}} \\
\lesssim \| \eta(t) T_{\beta,1}(h) \|_{X^{1,\beta}_{1,\tau}} + \| \eta(t) T_{\beta,3}(h) \|_{X^{1,\beta}_{1,\tau}} \lesssim \| h \|_{H^s(\mathbb{R})}.
\]

It thus remains to estimate \( \| \eta(t) T_{\beta,4}(h) \|_{A^{1,\beta}_{1,\tau}} \). On the one hand, it follows from (3.43) that
\[
| \mathcal{F}[\eta(t)T_{\beta,4}(h)](\xi, \tau) | \lesssim \int_4^\infty |\hat{\eta}(\tau - P_\beta(\mu))| \mu^{-1} |\hat{h}(\mu)| \, d\mu.
\]

On the other hand, it is easily seen that for any \( \epsilon_1 > 0 \),
\[
\left\| \mathbb{I}_{\{s+|\xi| \leq 3+|\tau|\}} (\xi)^s (\tau - \phi^{1,\beta}(\xi))^\sigma \right\|_{L^2_\xi} \lesssim \epsilon_1 (\tau)^{s+\epsilon_1}.
\]

The specific value of \( \epsilon_1 \) will be chosen later in (3.51) and it only depends on \( s \). Hence,
\[
\left\| \mathbb{I}_{\{s+|\xi| \leq 3+|\tau|\}} (\xi)^s (\tau - \phi^{1,\beta}(\xi))^\sigma \mathcal{F}[\eta(t)T_{\beta,4}(h)](\xi, \tau) \right\|_{L^2_\xi} \lesssim \int_4^\infty (\tau)^{s+\epsilon_1} |\hat{\eta}(\tau - P_\beta(\mu))| \mu^{-1} |\hat{h}(\mu)| \, d\mu.
\] (3.50)

Noticing
\[
(\tau)^{s+\epsilon_1} \lesssim (\tau - P_\beta(\mu))^{s+\epsilon_1} \mu^{3(s+\epsilon_1)},
\]

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so
\[
\text{RHS of (3.50) } \lesssim \int_1^\infty |\tau - P_\beta(\mu)|^{s+\epsilon_1} |\tilde{h}(\tau - P_\beta(\mu))| \mu^{3(s+\epsilon_1)-1} |\tilde{h}(\mu)| d\mu.
\]

Comparing to (3.46), as long as \(3(\sigma + \epsilon_1) - 1 \leq s + 1\), that is \(\sigma \leq \frac{s+2}{3}-\epsilon_1\), we can finish the proof by similar argument as that after (3.46). Choosing
\[
\epsilon_1 = \frac{2s+1}{12}
\]
fulfills this requirement since \(\sigma \leq \frac{2s+7}{12}\).

\[\square\]

**Proof of (3.20).**

Since \(\|f\|_{L^\infty_t H^s_x} \lesssim \|f\|_{X^{1,\beta}_{2,1}} \lesssim \|f\|_{X^{1,\beta}_{3,1}}\) for any function \(f\), then again according to (3.25) and (3.32), we conclude
\[
\|\eta(t)T_{\beta,1}(h)\|_{L^\infty_t H^s_x} + \|\eta(t)T_{\beta,3}(h)\|_{L^\infty_t H^s_x} \\
\lesssim \|\eta(t)T_{\beta,1}(h)\|_{X^{1,\beta}_{2,1}} + \|\eta(t)T_{\beta,3}(h)\|_{X^{1,\beta}_{3,1}} \lesssim \|h\|_{H^s(\mathbb{R})}.
\]

It thus remains to estimate \(\|\eta(t)T_{\beta,4}(h)\|_{L^\infty_t H^s_x}\). We first consider the case when \(t = 0\) which reduces to the proof of \(\|T_{\beta,4}(h)\|_{H^s_x} \lesssim \|h\|_{H^s(\mathbb{R})}\). By direct computation,
\[
\|T_{\beta,4}(h)\|_{H^s_x} = \left\| \int_4^\infty \hat{\xi}^s \left( \frac{\xi + \mu/2}{R_\beta(\mu)} \right) \frac{\hat{h}(\mu)}{R_\beta(\mu)} d\mu \right\|.
\]

Applying duality,
\[
\|T_{\beta,4}(h)\|_{H^s_x} = \sup_{\|g\|_{L_x^2} = 1} \int_4^\infty g(\xi) \hat{\xi}^s \left( \frac{\xi + \mu/2}{R_\beta(\mu)} \right) \frac{\hat{h}(\mu)}{R_\beta(\mu)} d\mu d\xi.
\]

Exchanging the order of integration and shifting \(\xi\), we obtain
\[
\int_4^\infty \int_4^\infty g(\xi) \hat{\xi}^s \left( \frac{\xi + \mu/2}{R_\beta(\mu)} \right) \frac{\hat{h}(\mu)}{R_\beta(\mu)} d\mu d\xi \\
= \int_4^\infty \int_4^\infty g\left( \xi - \frac{\mu}{2} \right) \left( \xi - \frac{\mu}{2} \right)^s \left( \frac{\xi}{R_\beta(\mu)} \right) \frac{\hat{h}(\mu)}{R_\beta(\mu)} d\xi d\mu.
\]

Define
\[
B_1 = \left\{ (\mu, \xi) \in \mathbb{R}^2 : \mu \geq 4, |\xi - \frac{\mu}{2}| \lesssim 1 \right\},
\]
\[
B_2 = \left\{ (\mu, \xi) \in \mathbb{R}^2 : \mu \geq 4, |\xi - \frac{\mu}{2}| \gg 1 \right\}.
\]

**Contribution on \(B_1\):** since \(|\xi - \frac{\mu}{2}| \lesssim 1\) and \(\hat{k}\) is bounded,
\[
\text{Contribution on } B_1 \lesssim \int_4^\infty \left( \int_{|\xi - \frac{\mu}{2}| \lesssim 1} \left| g\left( \xi - \frac{\mu}{2} \right) \right| d\xi \right) \frac{\hat{h}(\mu)}{R_\beta(\mu)} d\mu.
\]

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Since \( \|g\|_{L^2} = 1 \) and \( R_{\beta}(\mu) \sim \mu \) for \( \mu \geq 4 \),
\[
\text{RHS of (3.54)} \lesssim \int_4^{\infty} \frac{\hat{h}(\mu)}{\mu} \, d\mu \sim \int_4^{\infty} \frac{\langle \mu \rangle^{s} |\hat{h}(\mu)|}{\mu^{1+s}} \, d\mu.
\]

By applying Hölder’s inequality, we deduce that the above integral is bounded by \( \|h\|_{H^s(\mathbb{R})} \) thanks to the assumption \( s > -\frac{1}{2} \).

- **Contribution on** \( B_2 \): by the change of variable \( \xi = R_{\beta}(\mu)y \), we have

  Contribution on \( B_2 \lesssim \int_{\mathbb{R}} \int_{\mathbb{R}} 1_{B_3}(\mu, y) \left| g \left( R_{\beta}(\mu)y - \frac{\mu}{2} \right) \right| \left| \left( R_{\beta}(\mu)y - \frac{\mu}{2} \right)^s \hat{h}(\mu) \right| dy \, d\mu, \) (3.55)

  where

  \[
  B_3 := \left\{ (\mu, y) : \mu \geq 4, \left| R_{\beta}(\mu)y - \frac{\mu}{2} \right| \gg 1 \right\}.
  \]

  Switching the order of integration yields

  \[
  \text{RHS of (3.55)} \leq \int_{\mathbb{R}} |\hat{h}(\mu)| \left( \int_{\mathbb{R}} 1_{B_3}(\mu, y) \left| g \left( R_{\beta}(\mu)y - \frac{\mu}{2} \right) \right| \left| \left( R_{\beta}(\mu)y - \frac{\mu}{2} \right)^s \hat{h}(\mu) \right| d\mu \right) dy. \] (3.56)

  For any \( (\mu, y) \in B_3, \mu \geq 4 \) and \( |R_{\beta}(\mu)y - \frac{\mu}{2}| \gg 1 \). Then we claim that

  \[
  \left| R_{\beta}(\mu)y - \frac{\mu}{2} \right| \sim \left| \mu \left( y - \frac{1}{\sqrt{3}} \right) \right| \quad \text{and} \quad \left| R_{\beta}'(\mu)y - \frac{1}{2} \right| \sim \left| y - \frac{1}{\sqrt{3}} \right| \gg \frac{1}{\mu}, \] (3.57)

  Let’s first use (3.57) to finish the proof, the justification of (3.57) will be provided afterwards.

  Since \( \|g\|_{L^2} = 1 \) and \( \frac{d}{d\mu} \left[ R_{\beta}(\mu)y - \frac{\mu}{2} \right] = R_{\beta}'(\mu)y - \frac{1}{2} \), it follows from Hölder’s inequality that

  \[
  \int_{\mathbb{R}} 1_{B_3}(\mu, y) \left| g \left( R_{\beta}(\mu)y - \frac{\mu}{2} \right) \right| \left| \left( R_{\beta}(\mu)y - \frac{\mu}{2} \right)^s \hat{h}(\mu) \right| d\mu
  \]

  \[
  \lesssim \left( \int_{\mathbb{R}} 1_{B_3}(\mu, y) \left( \left| R_{\beta}(\mu)y - \frac{\mu}{2} \right|^{2s} |\hat{h}(\mu)|^2 \right)^{\frac{1}{2}} d\mu \right)^{\frac{1}{2}}. \] (3.58)

  Based on (3.57),

  \[
  \text{RHS of (3.58)} \lesssim \left( \int_{\mathbb{R}} 1_{B_3}(\mu, y) \left| y - \frac{1}{\sqrt{3}} \right|^{2s-1} \mu^{2s} |\hat{h}(\mu)|^2 \, d\mu \right)^{\frac{1}{2}} \lesssim \left| y - \frac{1}{\sqrt{3}} \right|^{s-\frac{1}{2}} \|h\|_{H^s(\mathbb{R})}.
  \]

  Then we infer from (3.56) that

  \[
  \text{RHS of (3.55)} \lesssim \left( \int_{\mathbb{R}} |\hat{h}(\mu)||y - \frac{1}{\sqrt{3}}|^{s-\frac{1}{2}} \, dy \right) \|h\|_{H^s(\mathbb{R})} \lesssim \|h\|_{H^s(\mathbb{R})},
  \]

  where the last inequality is due to \( s > -\frac{1}{2} \).
Now it remains to verify (3.57). In fact, since $R_\beta(\mu) \sim \mu$,
\[
\left| R_\beta(\mu)y - \frac{\mu}{2} \right| \sim \left| \frac{\mu}{2 R_\beta(\mu)} \left( R_\beta(\mu)y - \frac{\mu}{2} \right) \right| = \left| \mu \left( y - \frac{1}{\sqrt{3}} \right) + \left( \frac{\mu}{\sqrt{3}} - \frac{\mu^2}{2 R_\beta(\mu)} \right) \right|, \]
Noticing
\[
\left| \frac{\mu}{\sqrt{3}} - \frac{\mu^2}{2 R_\beta(\mu)} \right| \approx \frac{1}{\mu} \leq 1,
\]
then it follows from $\left| R_\beta(\mu)y - \frac{\mu}{2} \right| \gg 1$ that $\left| R_\beta(\mu)y - \frac{\mu}{2} \right| \sim \left| \mu \left( y - \frac{1}{\sqrt{3}} \right) \right|$. On the other hand,
\[
\left| R'_\beta(\mu)y - \frac{1}{2} \right| = \left| \frac{3\mu}{4 R_\beta(\mu)} - \frac{1}{2} \right| = \left| \frac{3\mu}{4 R_\beta(\mu)} \right| \left| y - \frac{2R_\beta(\mu)}{3\mu} \right| 
\sim \left| \left( y - \frac{1}{\sqrt{3}} \right) + \left( \frac{1}{\sqrt{3}} - \frac{2R_\beta(\mu)}{3\mu} \right) \right|.
\]
Since we have already shown that $\left| \mu \left( y - \frac{1}{\sqrt{3}} \right) \right| \sim \left| R_\beta(\mu)y - \frac{\mu}{2} \right| \gg 1$, then $\left| y - \frac{1}{\sqrt{3}} \right| \gg \frac{1}{\mu}$. In addition, it is readily seen that
\[
\left| \frac{1}{\sqrt{3}} - \frac{2R_\beta(\mu)}{3\mu} \right| \leq \frac{1}{\mu^2} \leq \frac{1}{\mu}.
\]
Hence, $\left| R'_\beta(\mu)y - \frac{1}{2} \right| \sim \left| y - \frac{1}{\sqrt{3}} \right|$. Therefore, (3.57) is justified.

Having established the case when $t = 0$, the general case follows easily. Actually, for any $t$, by defining $f$ such that $\hat{f}(\mu) = e^{i P_\beta(\mu) t} \hat{h}(\mu)$, we have $[T_{\beta,A}(f)](x,0) = [T_{\beta,A}(h)](x,t)$. So it follows from the $t = 0$ case that
\[
\| [T_{\beta,A}(h)](x,t) \|_{H^2} = \| [T_{\beta,A}(f)](x,0) \|_{H^2} \lesssim \| f \|_{H^s} = \| h \|_{H^s}.
\]

Remark: we emphasize that the above proof for (3.20) can not be extended to $s < -\frac{1}{2}$. Actually, if we choose $g(\xi) = 1_{\{|\xi| \leq 1\}}$ in (3.52), then
\[
\text{RHS of (3.52)} \sim \int_4^\infty \frac{\hat{h}(\mu)}{R_\beta(\mu)} d\mu \sim \int_4^\infty \frac{\hat{h}(\mu)}{\mu} d\mu.
\]
In order to prove
\[
\int_4^\infty \frac{\hat{h}(\mu)}{\mu} d\mu \lesssim \| h \|_{H^s}, \quad \forall h \in H^s(\mathbb{R}),
\]
s has to be at least $-\frac{1}{2}$. However, by the scaling $\hat{h}_\lambda := \hat{h}(\lambda \mu)$ and sending $\lambda \to 0^+$, we can see $s \geq -\frac{1}{2}$.

Proof of (3.24). We first consider the case when $j = 0$. For any fixed $x \in \mathbb{R}$, taking the Fourier transform of $\eta T_\beta(h)$ (see (3.17) for the expression of $T_\beta(h)$) with respect to $t$ leads to
\[
\mathcal{F}_t[\eta T_\beta(h)](x,\tau) = \int_{\sqrt{3}}^\infty \hat{\eta}(\tau - P_\beta(\mu)) e^{-i \mu x/2} K_\beta(x,\mu) \hat{h}(\mu) d\mu.
\]
Since $|e^{-i\mu x/2}K_\beta(x,\mu)| \leq C$ for a universal constant $C$,

$$|\mathcal{F}_x[\eta T_\beta(h)](x,\tau)| \lesssim \int_{\sqrt{\beta}}^{\infty} |\hat{\eta}(\tau - P_\beta(\mu))| |\hat{h}(\mu)| \, d\mu,$$

Noticing that

$$\langle \tau \rangle^{s+1} \leq \langle \tau - P_\beta(\mu) \rangle^{s+1} \langle P_\beta(\mu) \rangle^{s+1},$$

so

$$\langle \tau \rangle^{s+1} |\mathcal{F}_x[\eta T_\beta(h)](x,\tau)| \lesssim \int_{\sqrt{\beta}}^{\infty} \langle \tau - P_\beta(\mu) \rangle^{s+1} |\hat{\eta}(\tau - P_\beta(\mu))| \langle P_\beta(\mu) \rangle^{s+1} |\hat{h}(\mu)| \, d\mu$$

$$= \int_{\sqrt{\beta}}^{\infty} \hat{f}_s(\tau - P_\beta(\mu)) \langle P_\beta(\mu) \rangle^{s+1} |\hat{h}(\mu)| \, d\mu,$$

where $f_s$ is defined such that $\hat{f}_s(\cdot) = \langle \cdot \rangle^{s+1} \hat{\eta}(\cdot)$. Then we split this integral into $\int_{\sqrt{\beta}}^{4}$ and $\int_{4}^{\infty}$.

- On $[\sqrt{\beta}, 4]$, it follows from the fact $\hat{f}_s \in L^2$ and the Minkowski’s inequality that

$$\left\| \int_{\sqrt{\beta}}^{4} \hat{f}_s(\tau - P_\beta(\mu)) \langle P_\beta(\mu) \rangle^{s+1} |\hat{h}(\mu)| \, d\mu \right\|_{L^2_x} \lesssim \int_{\sqrt{\beta}}^{4} \langle P_\beta(\mu) \rangle^{s+1} |\hat{h}(\mu)| \, d\mu \lesssim \|h\|_{H^s(\mathbb{R})}.$$  

- On $[4, \infty)$, $P_\beta(\mu) \sim \mu^3$ and $P_\beta^{-1}$ is well-defined. So by the change of variable $y = P_\beta(\mu)$,

$$\int_{4}^{\infty} \hat{f}_s(\tau - P_\beta(\mu)) \langle P_\beta(\mu) \rangle^{s+1} |\hat{h}(\mu)| \, d\mu \lesssim \int_{P_\beta(4)}^{\infty} \hat{f}_s(\tau - y) \langle y \rangle^{s+1} |\hat{h}(P_\beta^{-1}(y))| \, dy.$$

Then it follows from the fact $\hat{f}_s \in L^1$ and the Young’s inequality that

$$\left\| \int_{P_\beta(4)}^{\infty} \hat{f}_s(\tau - y) \langle y \rangle^{s+1} |\hat{h}(P_\beta^{-1}(y))| \, dy \right\|_{L^2_x} \lesssim \left\| \mathbb{1}_{y > P_\beta(4)} \langle y \rangle^{s+1} |\hat{h}(P_\beta^{-1}(y))| \right\|_{L^2_x}.$$  

Finally, making the change of variable $\mu = P_\beta^{-1}(y)$ yields

$$\left\| \mathbb{1}_{y > P_\beta(4)} \langle y \rangle^{s+1} |\hat{h}(P_\beta^{-1}(y))| \right\|_{L^2_x} \lesssim \|h\|_{H^s(\mathbb{R})}.$$  

This completes the proof for the case $j = 0$.

Next, for the case $j = 1$,

$$\mathcal{F}_x(\partial_x[\eta T_\beta(h)])(x,\tau) = \int_{\sqrt{\beta}}^{\infty} \hat{\eta}(\tau - P_\beta(\mu)) \partial_x [e^{-i\mu x/2}K_\beta(x,\mu)] \hat{h}(\mu) \, d\mu.$$

Noticing there exists a universal constant $C$ such that

$$\left| \partial_x [e^{-i\mu x/2}K_\beta(x,\mu)] \right| \lesssim C \mu, \quad \forall \mu \geq 4.$$

Consequently,

$$|\mathcal{F}_x(\partial_x[\eta T_\beta(h)])(x,\tau)| \lesssim \int_{\sqrt{\beta}}^{\infty} |\hat{\eta}(\tau - P_\beta(\mu))| \mu |\hat{h}(\mu)| \, d\mu.$$  

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Define \( h_1 \) such that \( \hat{h}_1(\mu) = 1_{\{\mu > \sqrt{3}\}} \mu [\hat{h}(\mu)] \). Then by following the same argument as that for the case \( j = 0 \), we obtain
\[
\sup_{x \in \mathbb{R}} \| \partial_x [\eta \mathcal{T}_\beta(h)] \|_{H^1_x(\mathbb{R})} \lesssim \| h_1 \|_{H^s-1(\mathbb{R})} \lesssim \| h \|_{H^s(\mathbb{R})}.
\]

Thus, the proof of Lemma 3.3 is completed.

**Proof of Lemma 3.4**

Recall the definition of \( W^{1,\beta}_{bdr} \) in (3.9), we write
\[
W^{1,\beta}_{bdr}(a)(x,t) = \frac{1}{\pi} Re \int_{\sqrt{\pi}}^{1} e^{iP_\beta(\mu)t} e^{-im\alpha x/2} e^{-R_\beta(\mu)\sigma} \hat{a}^* [P_\beta(\mu)] dP_\beta(\mu)
\]
\[
+ \frac{1}{\pi} Re \int_{\sqrt{\pi}}^{1} e^{iP_\beta(\mu)t} e^{-im\alpha x/2} e^{-R_\beta(\mu)\sigma} \hat{a}^* [P_\beta(\mu)] dP_\beta(\mu)
\]
\[
:= I_1(x,t) + I_2(x,t), \quad \forall x, t \geq 0.
\]

Notice that one can extend \( I_1(x,t) \) for all \( x \) and \( t \) with its extension chosen as
\[
\mathcal{E} I_1(x,t) := \frac{1}{\pi} Re \int_{\sqrt{\pi}}^{1} e^{iP_\beta(\mu)t} e^{-im\alpha x/2} K_\beta(x,\mu) \hat{a}^* [P_\beta(\mu)] dP_\beta(\mu), \quad \forall x, t \in \mathbb{R}
\] (3.59)

where \( K_\beta \) is as defined in (3.12). For this part, it is very similar to the operator \( \Phi^{1,\beta}_{bdr} \) in Lemma 3.3 but with \( \mu \) restricted on \([\sqrt{3}, 4]\). On this small interval \([\sqrt{3}, 4]\), the proof of Lemma 3.3 also works for \(-\frac{3}{4} < s \leq -\frac{1}{2}\) and the extension for this part satisfies all the desired estimates.

Unfortunately, the same extension fails to work if \( \mu \) is large. So we need a more subtle extension for \( I_2(x,t) \). For the convenience of statement and the latter generalization in Section 3.2 we introduce the following operator \( \mathcal{I}_{\beta,m} \) for \( 0 < \beta \leq 1 \) and \( m \in \mathbb{R} \).
\[
[\mathcal{I}_{\beta,m}(f)](x,t) = Re \int_{\sqrt{\pi}}^{1} e^{iP_\beta(\mu)t} e^{-im\alpha x/2} e^{-R_\beta(\mu)\sigma} \hat{f}(P_\beta(\mu)) dP_\beta(\mu), \quad \forall x \geq 0, t \in \mathbb{R}
\] (3.60)

Then
\[
I_2(x,t) = \frac{1}{\pi} \mathcal{I}_{\beta,\frac{3}{2}}(a^*) (x,t), \quad \forall x, t \geq 0.
\]

So in order to extend \( I_2 \), it suffices to extend \( \mathcal{I}_{\beta,\frac{3}{2}}(a^*) \). By the following Claim 3.6 we can choose the extension to be \( \frac{1}{2} \mathcal{E} \mathcal{I}_{\beta,\frac{3}{2}}(a^*) \) to prove Lemma 3.4. Therefore, the justification of Lemma 3.4 reduces to establish the following result.

**Claim 3.6.** Let \(-\frac{3}{4} < s \leq -\frac{1}{2}, 0 < \beta \leq 1, m \neq 0 \) and \( f \in H^{s+1/2}(\mathbb{R}) \). Then there exist \( \sigma_1 = \sigma_1(s) > \frac{1}{2} \) and an extension \( \mathcal{E} \mathcal{I}_{\beta,m}(f) \) of \( \mathcal{I}_{\beta,m}(f) \) from \( \mathbb{R}_0^+ \times \mathbb{R} \) to \( \mathbb{R} \times \mathbb{R} \) such that for any \( \sigma \in \left( \frac{1}{2}, \sigma_1 \right) \) and for \( j = 0, 1 \),
\[
\eta(t) \mathcal{E} \mathcal{I}_{\beta,m}(f) \in Y^{1,\beta}_{s+1/2,\sigma} \bigcap C^j_w(\mathbb{R}_0^+; H^{s+1/2}_{t,s+1/2}(\mathbb{R})).
\]

In addition, the following estimates hold with some constant \( C = C(s, \sigma, m) \).
\[
\| \eta(t) \mathcal{E} \mathcal{I}_{\beta,m}(f) \|_{Y^{1,\beta}_{s+1/2,\sigma}} \leq C \| f \|_{H^{s+1/2}_{t,s+1/2}(\mathbb{R})},
\] (3.61)
\[
\sup_{x \geq 0} \| \partial_x [\eta(t) \mathcal{E} \mathcal{I}_{\beta,m}(f)] \|_{H^{s+1/2}_{t,s+1/2}(\mathbb{R})} \leq C \| f \|_{H^{s+1/2}_{t,s+1/2}(\mathbb{R})}, \quad j = 0, 1.
\] (3.62)
Proof of Claim 3.6. Without loss of generality, we assume $m = \frac{1}{2}$, the general case is similar. First, we notice that when $x \geq 0$, any extension of $I_{\beta, \frac{1}{2}}(f)$ from $\mathbb{R}_+^+ \times \mathbb{R} \times \mathbb{R}$ is equal to $I_{\beta, \frac{1}{2}}(f)$ on $\mathbb{R}_0^+ \times \mathbb{R}$, so the LHS of (3.62) is the same for all extensions. Define $h$ such that

$$\hat{h}(\mu) = \hat{f}(P_\beta(\mu))(P_\beta)'(\mu)\mathbbm{1}_{\mu > 0}.$$  

Then $\|h\|_{H^r(\mathbb{R})} \lesssim \|f\|_{H^{\frac{r+1}{2}}(\mathbb{R})}$. Meanwhile, it follows from (3.60) that

$$[I_{\beta, \frac{1}{2}}(f)](x, t) = \text{Re} \int_4^\infty e^{ip_\beta(\mu)t}e^{-i\mu x/2}e^{-R_\beta(\mu)x}\hat{h}(\mu) d\mu, \quad \forall x \geq 0, t \in \mathbb{R}.$$  

Recalling the definition (3.31) for the operator $\mathcal{T}_{\beta, \frac{1}{2}}(h)$, we see that $I_{\beta, \frac{1}{2}}(f)$ agrees with $\text{Re}(\mathcal{T}_{\beta, \frac{1}{2}}(h))$ when $x \geq 0$. So $\text{Re}(\mathcal{T}_{\beta, \frac{1}{2}}(h))$ is an extension of $I_{\beta, \frac{1}{2}}(f)$ from $\mathbb{R}_0^+ \times \mathbb{R}$ to $\mathbb{R} \times \mathbb{R}$. As a result, for any other extension $\mathcal{E}[I_{\beta, \frac{1}{2}}(f)],$

$$\sup_{x \geq 0} \left\| \partial_x^j \left[ \eta(t)\mathcal{E}[I_{\beta, \frac{1}{2}}(f)] \right] \right\|_{H^r_x(\mathbb{R})} = \sup_{x \geq 0} \left\| \partial_x^j \left[ \eta(t)\text{Re}(\mathcal{T}_{\beta, \frac{1}{2}}(h)) \right] \right\|_{H^r_x(\mathbb{R})}, \quad j = 0, 1.$$

Although the value of $s$ is required to be greater than $-1/2$ in Lemma 3.3, the proof of (3.21) is valid for any $s$ in $(-\frac{3}{4}, -\frac{1}{2}]$. So we conclude $\eta(t)\mathcal{T}_{\beta, \frac{1}{2}}(h) \in C^j([\mathbb{R}^+_0; H^{\frac{r+1}{2}}(\mathbb{R})])$ and

$$\sup_{x \geq 0} \left\| \partial_x^j \left[ \eta(t)\mathcal{T}_{\beta, \frac{1}{2}}(h) \right] \right\|_{H^r_x(\mathbb{R})} \leq C\|h\|_{H^r(\mathbb{R})} \leq C\|f\|_{H^{\frac{r+1}{2}}(\mathbb{R})}.$$  

Then it remains to find an extension $\mathcal{E}[I_{\beta, \frac{1}{2}}(f)]$ which belongs to $Y^{1, \beta}_{s, \frac{1}{2}, \sigma}$ and satisfies (3.61).

Since our construction is very similar to that in [BSZ06], we will only briefly illustrate how to apply that strategy to the current problem. In [BSZ06], they provided an extension for the integral $I_2(x, t)$, see page 16 in Section 2.2 in [BSZ06]. Essentially, the term $I_2(x, t)$ in [BSZ06] can be rewritten as

$$I_2(x, t) = \text{Re} \int_4^\infty e^{i(\mu^3 - \mu)t}e^{-(\sqrt{3\mu^2 - 4\mu} + i\tilde{\mu})x}\hat{h}^*(\mu^3 - \mu) d(\mu^3 - \mu),$$  

where $h \in H^{\frac{r+1}{2}}(\mathbb{R}^+)$ and $h^*$ is the zero extension of $h$ from $\mathbb{R}^+$ to $\mathbb{R}$. By Lemma 2.1, $h^* \in H^{\frac{r+1}{2}}(\mathbb{R})$ and this is the only property needed in [BSZ06] to carry out the extension which satisfies the property (3.61). See the extension $BI_{m_1}$ on page 21, Theorem 3.1 on page 22, and Lemma 3.10 on page 36 in [BSZ06].

So after replacing $h^*$ by $f$, the method also applies to extend $I_{1, \frac{1}{2}}(f)$ to satisfy (3.61). Thus, Claim 3.6 is justified when $\beta = 1$ and $m = \frac{1}{2}$. For general $\beta \in (0, 1)$, it follows from $\mu \geq 4$ that $P_\beta(\mu) \sim \mu^3$ and $R_\beta(\mu) \sim \mu$, so the influence of $\beta$ can be ignored. In other words, Claim 3.6 can also be justified in a similar way. □

Hence, the proof of Lemma 3.4 is completed.
3.2 KdV flow traveling to the left

In this subsection we consider the linear IBVP

\[
\begin{aligned}
v_t - v_{xxx} - \beta v_x &= g, \\
v(x, 0) &= q(x), \\
v(0, t) &= b_1(t), \\
v_x(0, t) &= b_2(t), 
\end{aligned}
\tag{3.63}
\]

which describes a linear dispersive wave traveling to the left, where \( q \in H^s(\mathbb{R}^+) \), \( b_1 \in H^{\frac{3}{2}+\varepsilon}(\mathbb{R}^+) \), \( b_2 \in H^\varepsilon(\mathbb{R}^+) \), and \( g \) is a function defined on \( \mathbb{R}^2 \). The function space that \( g \) lives in will be specified later.

In addition, the data \( q, b_1 \) and \( b_2 \) are assumed to be compatible, that is they satisfy \( q(0) = b_1(0) \) if \( s > \frac{1}{2} \) and further satisfy \( q'(0) = b_2(0) \) if \( s > \frac{3}{2} \).

The solution \( v \) of (3.63) can be decomposed into \( v = v_1 + v_2 \) with

\[
v_1 = \Phi_R^{-1,-\beta}(g, q) := W_R^{-1,-\beta}(E_s q) + \int_0^t W_R^{-1,-\beta}(t - \tau)g(\cdot, \tau)\,d\tau.
\tag{3.64}
\]

where \( E_s q \) is defined as in (2.2), and

\[
v_2 = W_{bdr}^{-1,-\beta}(b_1 - q_1, b_2 - q_2),
\]

where \( q_1 := v_1(0, t), q_2(t) := \partial_x v_1(0, t) \), and \( W_{bdr}^{-1,-\beta}(b_1, b_2) \) denotes the solution operator (also called the boundary integral operator) associated to the IBVP

\[
\begin{aligned}
(v_2)_t - (v_2)_{xxx} - \beta(v_2)_x &= 0, \\
v_2(x, 0) &= 0, \\
v_2(0, t) &= b_1(t), \\
(v_2)_x(0, t) &= b_2(t).
\end{aligned}
\tag{3.65}
\]

**Lemma 3.7.** Let \( s \in (-\frac{3}{4}, 3], 0 < \beta \leq 1 \) and \( \frac{1}{2} < \sigma \leq 1 \). For any \( q \in L^s(\mathbb{R}^+) \) and \( g \in X_{s,\sigma-1}^{-1,-\beta} \cap Z_{s,\sigma-1}^{-1,-\beta} \), the function

\[
\tilde{v}_1 := \eta(t)\Phi_R^{-1,-\beta}(g, q)
\]

belongs to \( Y_{s,\frac{1}{2},\sigma}^{-1,-\beta} \cap C^j_t(\mathbb{R}; H_{s,\frac{1}{2},\sigma}^{\frac{3}{2}+\varepsilon}((\mathbb{R})) \) for \( j = 0, 1 \). In addition, the following estimates hold with some constant \( C = C(s, \sigma) \).

\[
\| \tilde{v}_1 \|_{Y_{s,\frac{1}{2},\sigma}^{-1,-\beta}} \leq C \left( \| g \|_{X_{s,\sigma-1}^{-1,-\beta}} + \| q \|_{H^s} \right),
\]

\[
\sup_{x \in \mathbb{R}} \| \partial_x^j \tilde{v}_1 \|_{H_{s,\frac{1}{2},\sigma}^{\frac{3}{2}+\varepsilon}((\mathbb{R}))} \leq C \left( \| g \|_{X_{s,\sigma-1}^{-1,-\beta}} + \| g \|_{Z_{s,\sigma-1}^{-1,-\beta}} + \| q \|_{H^s} \right), \quad j = 0, 1.
\]

**Proof.** (3.66) follows from (2.2) and Lemma 2.4. While (3.67) follows from (2.2) Lemma 2.5 and Lemma 2.6.

For any \( \beta > 0 \), recall the definitions (2.5) and (3.8) for \( P_\beta \) and \( R_\beta \).

**Lemma 3.8.** The boundary integral operator \( W_{bdr}^{-1,-\beta}(b_1, b_2) \) associated to (3.65) has the explicit integral representation

\[
W_{bdr}^{-1,-\beta}(b_1, b_2)(x, t) = \frac{1}{\pi} \text{Re} \int_\mathbb{R} e^{iP_\beta(\mu)t} \left( e^{-i\mu x} A - e^{i\mu x} e^{-R_\beta(\mu)x} B \right) dP_\beta(\mu), \quad x, t \geq 0,
\]

\[
\tag{3.68}
\]
where

\[
\begin{align*}
A &= A(\mu) = \frac{R_\beta(\mu) - i\mu/2}{R_\beta(\mu) - 3\mu i/2} \hat{b}_1^1[P_\beta(\mu)] + \frac{1}{R_\beta(\mu) - 3\mu i/2} \hat{b}_2^1[P_\beta(\mu)], \\
B &= B(\mu) = \frac{i\mu}{R_\beta(\mu) - 3\mu i/2} \hat{b}_1^2[P_\beta(\mu)] + \frac{1}{R_\beta(\mu) - 3\mu i/2} \hat{b}_2^2[P_\beta(\mu)].
\end{align*}
\]  

(3.69)

\(b_1^1\) and \(b_2^2\) are the zero extensions of \(b_1\) and \(b_2\) from \(\mathbb{R}_+^+\) to \(\mathbb{R}\), respectively.

The proof of this lemma is standard based on the Laplace transform method in [BSZ02] and is omitted.

Then similar to the discussion after Lemma 3.2, we want to extend \(W_{\beta, bdr}^{-1, -\beta}(b_1, b_2)\) to the whole plane \(\mathbb{R}^2\) with some specific properties. Recalling the function \(K_\beta(x, \mu)\) in (3.12), we define \(\Phi_{bdr, \beta}^{-1, -\beta}(b_1, b_2) : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}\) by

\[
\Phi_{bdr, \beta}^{-1, -\beta}(b_1, b_2)(x, t) = \frac{1}{\pi} Re \int_{\sqrt{\beta}}^{\infty} e^{iP_\beta(\mu)t} \left[ e^{-i\mu x} A - e^{i\mu x} K_\beta(x, \mu)B \right] dP_\beta(\mu),
\]  

(3.70)

where \(A\) and \(B\) are as defined in (3.69). It is easily seen that \(\Phi_{bdr, \beta}^{-1, -\beta}(b_1, b_2)\) is well-defined on \(\mathbb{R} \times \mathbb{R}\). In addition, when \(x, t \geq 0\), \(K_\beta(x, \mu) = e^{-R_\beta(\mu)x}\), so \(\Phi_{bdr, \beta}^{-1, -\beta}(b_1, b_2)\) matches \(W_{\beta, bdr}^{-1, -\beta}(b_1, b_2)\) on \(\mathbb{R}_+^+ \times \mathbb{R}_+^+\). Hence, \(\Phi_{bdr, \beta}^{-1, -\beta}(b_1, b_2)\) is an extension of \(W_{\beta, bdr}^{-1, -\beta}(b_1, b_2)\) to \(\mathbb{R} \times \mathbb{R}\). Moreover, if \(s > -\frac{1}{2}\), then it satisfies some good properties after being localised in time.

**Lemma 3.9.** Let \(s \in (-\frac{1}{2}, 3], 0 < \beta \leq 1\) and \(\frac{1}{2} < \sigma \leq \min \{1, \frac{2s+7}{13}\}\). For any \(b_1 \in H^{\frac{s+1}{2}}(\mathbb{R}_+)\) and \(b_2 \in H^{\frac{s+1}{2}}(\mathbb{R}_+)\) that are compatible with (3.65), the function

\[
\tilde{\nu}_2 := \eta(t)\Phi_{bdr, \beta}^{-1, -\beta}(b_1, b_2),
\]

equals \(W_{\beta, bdr}^{-1, -\beta}(b_1, b_2)\) on \(\mathbb{R}_+^+ \times [0, 1]\) and belongs to \(Y^{-1, -\beta}_{s, \frac{1}{2}, \sigma}(\mathbb{R}; H^{\frac{s+1}{2}}(\mathbb{R}))\) for \(j = 0, 1\). In addition, the following estimates hold with some constant \(C = C(s, \sigma)\).

\[
\begin{align*}
\|\tilde{\nu}_2\|_{Y^{-1, -\beta}_{s, \frac{1}{2}, \sigma}} &\leq C \left( \|b_1\|_{H^{\frac{s+1}{2}}(\mathbb{R}_+^+)} + \|b_2\|_{H^{\frac{s+1}{2}}(\mathbb{R}_+^+)} \right), \\
\sup_{x \in \mathbb{R}} \|\partial_x^j \tilde{\nu}_2\|_{H^{\frac{s+1}{2}}(\mathbb{R})} &\leq C \left( \|b_1\|_{H^{\frac{s+1}{2}}(\mathbb{R}_+^+)} + \|b_2\|_{H^{\frac{s+1}{2}}(\mathbb{R}_+^+)} \right), \quad j = 0, 1.
\end{align*}
\]  

(3.71)

(3.72)

**Proof.** We will first establish the estimates (3.71) and (3.72) for \(s \in (-\frac{1}{2}, 3] - \{\frac{1}{2}, \frac{3}{2}\}\). Then the corresponding estimates in the case of \(s = \frac{1}{2}\) or \(s = \frac{3}{2}\) can be deduced by interpolating between the case \(s = 0\) and the case \(s = 3\).

Fix \(s \in (-\frac{1}{2}, 3] - \{\frac{1}{2}, \frac{3}{2}\}\) and \(0 < \beta \leq 1\). Define \(f_\beta\) and \(g_\beta\) such that \(\tilde{f}_\beta(\mu) = (P_\beta)'(\mu)\) and \(\tilde{g}_\beta(\mu) = (P_\beta)'(\mu)B\), where \(A\) and \(B\) are as defined in (3.69). Then

\[
\Phi_{bdr, \beta}^{-1, -\beta}(b_1, b_2)(x, t) = \int_{\sqrt{\beta}}^{\infty} e^{iP_\beta(\mu)t} \left[ e^{-i\mu x} \tilde{f}_\beta(\mu) - e^{i\mu x/2} K_\beta(x, \mu)\tilde{g}_\beta(\mu) \right] d\mu.
\]
Since $0 < \beta \leq 1$ and $(P_\beta)'(\mu) = 3\mu^2 - \beta$, then $|(P_\beta)'(\mu)| \sim \mu$ and $|R_\beta(\mu) - 3\mu i/2| \sim \mu$. Therefore,

$$
\left\| \frac{(P_\beta)'(\mu) R_\beta(\mu) - i\mu/2}{R_\beta(\mu) - 3\mu i/2} \right\| \lesssim \| (P_\beta)'(\mu) \|^{1/2}(\mu),
\left\| \frac{i\mu}{R_\beta(\mu) - 3\mu i/2} \right\| \lesssim \| (P_\beta)'(\mu) \|^{1/2}(\mu),
\left\| \frac{1}{R_\beta(\mu) - 3\mu i/2} \right\| \lesssim \| (P_\beta)'(\mu) \|^{1/2}.
$$

(3.73)

Hence, it follows from (3.73) and (3.69) that

$$
\left| \hat{f}_\beta(\mu) \right| + \left| \hat{g}_\beta(\mu) \right| \lesssim \| (P_\beta)'(\mu) \|^{1/2} \left( (\mu) \hat{b}_1[\hat{P}_\beta(\mu)] + \hat{b}_2[\hat{P}_\beta(\mu)] \right).
$$

Therefore,

$$
\langle \mu \rangle^s \left( \left| \hat{f}_\beta(\mu) \right| + \left| \hat{g}_\beta(\mu) \right| \right) \lesssim \| (P_\beta)'(\mu) \|^{1/2} \left( (\mu)^{s+1} \hat{b}_1[\hat{P}_\beta(\mu)] + \langle \mu \rangle^s \hat{b}_2[\hat{P}_\beta(\mu)] \right).
$$

Since $b_1 \in H^{\frac{s+1}{2}}(\mathbb{R}^+)$ and $b_2 \in H^{\frac{s}{2}}(\mathbb{R}^+)$ are compatible with (3.65), it then follows from Corollary 2.3 that $f_\beta, g_\beta \in H^s(\mathbb{R})$ and

$$
\| f_\beta \|_{H^s(\mathbb{R})} + \| g_\beta \|_{H^s(\mathbb{R})} \lesssim \| b_1 \|_{H^{\frac{s+1}{2}}(\mathbb{R}^+)} + \| b_2 \|_{H^{\frac{s}{2}}(\mathbb{R}^+)}.
$$

Inspired by the above observation, we define an operator $\mathcal{L}_\beta$ as

$$
[\mathcal{L}_\beta(f, g)](x, t) := \int_{\sqrt{\beta}}^{\infty} e^{iP_\beta(\mu)t} \left[ e^{-iux} \hat{f}(\mu) - e^{iux/2} K_\beta(x, \mu) \hat{g}(\mu) \right] d\mu.
$$

(3.74)

Recalling $Y_{\beta, \gamma, \sigma}^{-1,-\beta} = X_{\beta, \gamma, \sigma}^{-1,-\beta} \cap \Lambda_{\beta, \gamma, \sigma}^{-1,-\beta} \cap C_\ell(\mathbb{R}, H^s(\mathbb{R}))$, the proof of Lemma 3.3 reduces to establishing the following estimates for the operator $\mathcal{L}_\beta$.

$$
\| \eta(t) \mathcal{L}_\beta(f, g) \|_{X_{\beta, \gamma}^{-1,-\beta}} \leq C(\| f \|_{H^s} + \| g \|_{H^s}),
$$

(3.75)

$$
\| \eta(t) \mathcal{L}_\beta(f, g) \|_{X_{\beta, \gamma, \sigma}^{-1,-\beta}} \leq C(\| f \|_{H^s} + \| g \|_{H^s}),
$$

(3.76)

$$
\sup_{\tau \in \mathbb{R}} \| \eta(t) \mathcal{L}_\beta(f, g) \|_{H^s(\mathbb{R})} \leq C(\| f \|_{H^s} + \| g \|_{H^s}),
$$

(3.77)

$$
\sup_{x \in \mathbb{R}} \| \partial_x^j [\eta(t) \mathcal{L}_\beta(f, g)] \|_{H^s_x(\mathbb{R})} \leq C(\| f \|_{H^s} + \| g \|_{H^s}), \quad j = 0, 1.
$$

(3.78)

To justify these estimates, we write $\mathcal{L}_\beta(f, g) = \mathcal{L}_{\beta,1}(f) - \mathcal{L}_{\beta,2}(g)$, where

$$
[\mathcal{L}_{\beta,1}(f)](x, t) := \int_{\sqrt{\beta}}^{\infty} e^{iP_\beta(\mu)t} e^{-iux} \hat{f}(\mu) d\mu,
$$

$$
[\mathcal{L}_{\beta,2}(g)](x, t) := \int_{\sqrt{\beta}}^{\infty} e^{iP_\beta(\mu)t} e^{iux/2} K_\beta(x, \mu) \hat{g}(\mu) d\mu.
$$

- Firstly, we define $h$ such that $\hat{h}(\mu) = 1_{(\mu > \sqrt{\beta})}(\mu)$. Then $\| h \|_{H^s} \leq \| f \|_{H^s}$ and it follows from (1.7) that

$$
[\mathcal{L}_{\beta,1}(f)](x, t) = \left[ W_R^{-1,-\beta}(-t)h \right](\mu), \quad \forall x, t \in \mathbb{R}.
$$

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Noticing that for any function \( F(x, t) \), the \( X^{-1, \beta}_{s, 1/2} \), \( A^{-1, \beta}_{s, 1/2} \), \( L_t^\infty \mathcal{H}_x^{s} \) and \( L_t^\infty \mathcal{H}_x^{s+1,i} \) norms are preserved under the operation \( F(x, t) \to F(-x, -t) \), so the estimates (3.75)-(3.78) for \( \mathcal{L}_{\beta, 1}(f) \) follow from Lemma 2.4 Lemma 2.5 and the fact \( \|h\|_{H^s} \leq \|f\|_{H^s} \).

- Secondly, the operator \( \mathcal{L}_{\beta, 2} \) is very similar to \( \mathcal{T}_{\beta} \) (see (3.17)). So by analogous arguments, the estimates (3.75)-(3.78) for \( \mathcal{L}_{\beta, 2}(g) \) can also be established.

\[
\Phi_{-1, \beta, \sigma} \leq -1, \quad \text{with some constant} \quad \sigma.
\]

Similar to Lemma 3.3, the estimate (3.71) fails for \( s < -\frac{1}{2} \), so in order to handle the case when \( -\frac{3}{4} < s \leq -\frac{1}{2} \), we have to adopt another extension for \( W_{bdr}^{-1, \beta}(b_1, b_2) \). Again, we will combine the above extension \( \Phi_{-1, \beta, \sigma} \) as in (3.70) and the construction in [BSZ06] to achieve the goal.

**Lemma 3.10.** Let \( -\frac{3}{4} < s < -\frac{1}{2} \), \( 0 < \beta \leq 1 \), \( b_1 \in H^{\frac{1}{2}+\epsilon}(\mathbb{R}^+) \) and \( b_2 \in H^{-\frac{1}{2}}(\mathbb{R}^+) \). Then there exist \( \sigma_2 = \sigma_2(s) > \frac{1}{2} \) and an extension \( \Psi_{bdr}^{-1, \beta}(b_1, b_2) \) of \( W_{bdr}^{-1, \beta}(b_1, b_2) \) such that for any \( \sigma \in \left( \frac{1}{2}, \sigma_2 \right) \), the function

\[
\tilde{v}_2 := \eta(t) \Psi_{bdr}^{-1, \beta}(b_1, b_2)
\]
equals \( W_{bdr}^{-1, \beta}(b_1, b_2) \) on \( \mathbb{R}_+^n \times [0, 1] \) and belongs to \( Y^{-1, \beta}_{s, 2} \sigma \cap C^{1}_x(\mathbb{R}^+; H^{\frac{1}{2}+\epsilon}(\mathbb{R})) \) for \( j = 0, 1 \). In addition, the following estimates hold with some constant \( C = C(s, \sigma) \).

\[
\|\tilde{v}_2\|_{Y^{-1, \beta}_{s, 2} \sigma} \leq C \left( \|b_1\|_{H^{\frac{1}{2}+\epsilon}(\mathbb{R}^+)} + \|b_2\|_{H^{-\frac{1}{2}}(\mathbb{R})} \right), \tag{3.79}
\]

\[
\sup_{x \geq 0} \left\| \partial_x^j \tilde{v}_2 \right\|_{H^{\frac{1}{2}+\epsilon}(\mathbb{R})} \leq C \left( \|b_1\|_{H^{\frac{1}{2}+\epsilon}(\mathbb{R}^+)} + \|b_2\|_{H^{-\frac{1}{2}}(\mathbb{R})} \right), \quad j = 0, 1. \tag{3.80}
\]

**Proof.** Recall the definition of \( W_{bdr}^{-1, \beta} \) in (3.68) and write

\[
W_{bdr}^{-1, \beta}(b_1, b_2)(x, t) = \frac{1}{\pi} \Re \int_{\sqrt{\beta}}^{\infty} e^{iP_\beta(\mu)t} e^{-iux} A dP_\beta(\mu)
\]

\[
- \frac{1}{\pi} \Re \int_{\sqrt{\beta}}^{\infty} e^{iP_\beta(\mu)t} e^{-iux} e^{-R_\beta(\mu)x} B dP_\beta(\mu)
\]

\[
- \frac{1}{\pi} \Re \int_{\sqrt{1}}^{\infty} e^{iP_\beta(\mu)t} e^{-iux} e^{-R_\beta(\mu)x} B dP_\beta(\mu)
\]

\[
:= I_1(x, t) - I_2(x, t) - I_3(x, t), \quad \forall x, t \geq 0.
\]

where \( A \) and \( B \) are defined as in (3.69). For \( I_1(x, t) \), it is actually well defined for any \( x, t \in \mathbb{R} \), so the trivial extension

\[
\mathcal{E}I_1(x, t) = \frac{1}{\pi} \Re \int_{\sqrt{\beta}}^{\infty} e^{iP_\beta(\mu)t} e^{-iux} A dP_\beta(\mu), \quad \forall x, t \in \mathbb{R},
\]

is enough, see the proof in Lemma 3.9. For \( I_2(x, t) \), we define the extension to be

\[
\mathcal{E}I_2(x, t) = \frac{1}{\pi} \Re \int_{\sqrt{\beta}}^{\infty} e^{iP_\beta(\mu)t} e^{-iux} K_\beta(x, \mu) B dP_\beta(\mu), \quad \forall x, t \in \mathbb{R},
\]

where \( K_\beta \) is as defined in (3.12). Then again, by similar argument as that in the proof of Lemma 3.9, this extension works. For \( I_3(x, t) \), due to the formula (3.69) for \( B \), we define \( \tilde{h} \) such that

\[
\tilde{h}(\lambda) = 1_{\{\lambda > R_\beta(4)\}} \left( \frac{2\mu}{R_\beta(\mu) - 3\mu i/2} \tilde{g}_1(\lambda) + \frac{1}{R_\beta(\mu) - 3\mu i/2} \tilde{g}_2(\lambda) \right).
\]
where $\mu = P_{\beta}^{-1}(\lambda)$ for $\lambda \geq P_{\beta}(4)$. This choice of $h$ implies $\hat{h}(P_{\beta}(\mu)) = 1_{\mu \geq 4}B$. So $I_3(x, t)$ can be rewritten as

$$I_3(x, t) = \frac{1}{\pi} Re \int_4^{\infty} e^{iP_{\beta}(\mu)t} e^{i\mu x/2} e^{-R_{\beta}(\mu)x} \hat{h}(P_{\beta}(\mu)) dP_{\beta}(\mu) = \frac{1}{\pi} \mathcal{I}_{\beta, -\frac{1}{2}}(h)(x, t),$$

where the operator $\mathcal{I}_{\beta, -\frac{1}{2}}(h)$ is defined as in (3.60). Then it follows from Claim 3.6 that the extension $\mathcal{E}I_3(x, t) := \frac{1}{\pi} \left[ \mathcal{E} \mathcal{I}_{\beta, -\frac{1}{2}}(h) \right](x, t)$ satisfies the requirement.

We now advance to present the proof of Proposition 1.11.

**Proof of Proposition 1.11**

Without loss of generality, we assume $T = 1$. We then first deal with the case when $s \in (-\frac{1}{2}, 3]$. Choose $\sigma_2(s) = \min \left\{ 1, \frac{2s + 7}{12} \right\}$ and consider any $\sigma \in (\frac{1}{2}, \sigma_2(s)]$. According to the operators $\Phi_R^{-1, -b}$ and $\Phi_{bdr}^{-1, -b}$ in Lemma 3.7 and Lemma 3.9 respectively, we define

$$\bar{v} = \Gamma_{\beta}(g, q, b_1, b_2)$$

$$:= \eta(t) \Phi_R^{-1, -b}(g, q) + \eta(t) \Phi_{bdr}^{-1, -b} \left( b_1 - \eta(t) \Phi_{bdr}^{-1, -b}(g, q) \right)_{x=0} b_2 - \partial_x \left[ \eta(t) \Phi_{bdr}^{-1, -b}(g, q) \right]_{x=0} \right).$$

Then $\bar{v}$ is defined on $\mathbb{R} \times \mathbb{R}$ and solves (1.20) on $\mathbb{R}_0^+ \times [0, 1]$. Furthermore, it follows from (3.66) and (3.71) that there exists a constant $C = C(s, \sigma)$ such that

$$\|\Gamma_{\beta}^{-1, -b}(g, q, b_1, b_2)\|_{Y^{-1, -b}} \leq C \left( \|g\|_{X_s^{1, -b}} + \|q\|_{H^1} + \|b_1 - \eta(t) \Phi_{bdr}^{-1, -b}(g, q)\|_{x=0} \right)_{H_s^1} + \|b_2 - \partial_x \left[ \eta(t) \Phi_{bdr}^{-1, -b}(g, q) \right]_{x=0} \right)_{H_s^1} \).$$

(3.81)

By (3.67) in Lemma 3.7 we have

$$\|\partial_x^j [\eta(t) \Phi_R^{-1, -b}(g, q)]\|_{x=0} \leq C \left( \|g\|_{X_s^{1, -b}} + \|g\|_{Z_s^{1, -b}} + \|q\|_{H^1} \right), \quad j = 0, 1. \quad (3.82)$$

Combining (3.81) and (3.82) yields (1.27). Next, by similar argument and using (3.67) and (3.72), we can justify (1.28) as well. Finally, it follows from Lemma 3.7 and Lemma 3.9 that $\bar{v}$ belongs to $Y_{s, \frac{1}{2}, \sigma}^{-1, -b} C_t (\mathbb{R}_0^+; H_s^1) \cap C_t^j (\mathbb{R}^+; H_s^{\frac{3}{4} + j/2})$ for $j = 0, 1, \ldots$

Now we treat the case when $s \in (-\frac{3}{4}, -\frac{1}{2}]$. The proof for this case is almost the same as the above case except we replace $\Phi_{bdr}^{-1, -b}$ by $\Psi_{bdr}^{-1, -b}$ and replace Lemma 3.9 by Lemma 3.10.

**4 Bilinear Estimates**

This section is intended to prove Proposition 1.12. For the sake of clarity, we split the proof into three parts, see the following Proposition 4.2-Proposition 4.4. Since $b \leq \frac{1}{2}$ is required to deal with the boundary integral operators, it brings two issues. First, the bilinear estimate may not be justified in the space $X_{s, b}^{1, \beta}$ via the classical methods as in [Bou93, KPV93, KPV96]. Secondly, $X_{s, b}^{1, \beta}$ fails to be in $C_t (\mathbb{R}; H_s^1)$, so living in the space $X_{s, b}^{1, \beta}$ may not guarantee the continuity of the flow of the solution. In order to overcome these two difficulties, we take advantage of the modified Fourier restriction spaces as defined in (1.11) and (1.12) to settle these issues. On the other hand, as it is well-known that the resonance function plays an essential role in the bilinear estimates, so we first write out its definition, see e.g. [Tao01].

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Definition 4.1 ([Tao01]). Let \((\alpha_1, \beta_1), (\alpha_2, \beta_2), (\alpha_3, \beta_3)\) be a triple in \(((\mathbb{R}\setminus\{0\}) \times \mathbb{R})^3\). Define the resonance function \(H\) associated to this triple by

\[
H(\xi_1, \xi_2, \xi_3) = \sum_{i=1}^{3} \phi^{\alpha_i, \beta_i}(\xi_i), \quad \forall (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 \text{ with } \sum_{i=1}^{3} \xi_i = 0, \tag{4.1}
\]

where \(\phi\) is defined as in (4.8).

Proposition 4.2. Let \(-\frac{3}{4} < s \leq 3, \alpha \neq 0\) and \(|\beta| \leq 1\). Then there exists \(\sigma_0 = \sigma_0(s, \alpha) > \frac{1}{2}\) such that for any \(\sigma \in (\frac{1}{2}, \sigma_0]\), the following bilinear estimates

\[
\|\partial_x (w_1 w_2)\|_{X^{\alpha, \beta}_{s, \sigma, -1}} \leq C \|w_1\|_{X^{\alpha, \beta}_{s, \sigma, -1}} \|w_2\|_{X^{\alpha, \beta}_{s, \sigma, -1}} \tag{4.2}
\]

\[
\|\partial_x (w_1 w_2)\|_{Z^{\alpha, \beta}_{s, \sigma, -1}} \leq C \|w_1\|_{X^{\alpha, \beta}_{s, \sigma, -1}} \|w_2\|_{X^{\alpha, \beta}_{s, \sigma, -1}} \tag{4.3}
\]

hold for any \(w_1, w_2 \in X^{\alpha, \beta}_{s, \sigma}\) with some constant \(C = C(s, \alpha, \sigma)\).

Proposition 4.3. Let \(-\frac{3}{4} < s \leq 3, \alpha \neq 0\) and \(|\beta| \leq 1\). Then there exists \(\sigma_0 = \sigma_0(s, \alpha) > \frac{1}{2}\) such that for any \(\sigma \in (\frac{1}{2}, \sigma_0]\), the following bilinear estimates

\[
\|\partial_x (w_1 w_2)\|_{X^{-\alpha, -\beta}_{s, \sigma, -1}} \leq C \|w_1\|_{X^{-\alpha, -\beta}_{s, \sigma, -1}} \|w_2\|_{X^{-\alpha, -\beta}_{s, \sigma, -1}} \tag{4.4}
\]

\[
\|\partial_x (w_1 w_2)\|_{Z^{-\alpha, -\beta}_{s, \sigma, -1}} \leq C \|w_1\|_{X^{-\alpha, -\beta}_{s, \sigma, -1}} \|w_2\|_{X^{-\alpha, -\beta}_{s, \sigma, -1}} \tag{4.5}
\]

hold for any \(w_1, w_2 \in X^{\alpha, \beta}_{s, \sigma}\) with some constant \(C = C(s, \alpha, \sigma)\).

Proposition 4.4. Let \(-\frac{3}{4} < s \leq 3, \alpha \neq 0\) and \(|\beta| \leq 1\). Then there exists \(\sigma_0 = \sigma_0(s, \alpha) > \frac{1}{2}\) such that for any \(\sigma \in (\frac{1}{2}, \sigma_0]\), the following bilinear estimates

\[
\|\partial_x (w_1 w_2)\|_{X^{\alpha, \beta}_{s, \sigma, -1}} \leq C \|w_1\|_{X^{\alpha, \beta}_{s, \sigma, -1}} \|w_2\|_{X^{\alpha, \beta}_{s, \sigma, -1}} \tag{4.6}
\]

\[
\|\partial_x (w_1 w_2)\|_{Z^{\alpha, \beta}_{s, \sigma, -1}} \leq C \|w_1\|_{X^{\alpha, \beta}_{s, \sigma, -1}} \|w_2\|_{X^{\alpha, \beta}_{s, \sigma, -1}} \tag{4.7}
\]

hold for any \(w_1 \in X^{\alpha, \beta}_{s, \sigma}\) and \(w_2 \in X^{-\alpha, -\beta}_{s, \sigma}\) with some constant \(C = C(s, \alpha, \sigma)\).

The proofs for these three Propositions are similar while Proposition 4.4 is slightly more difficult since \(w_1\) and \(w_2\) live in different spaces. As a result, we will only verify Proposition 4.4. Before the proof, we will first present some elementary technical tools.

Lemma 4.5. Let \(\rho_1 \geq \rho_2 > \frac{1}{2}\) and \(\rho_3 \in \mathbb{R}\) satisfy

\[
\begin{cases}
\rho_3 = \rho_2 & \text{if } \rho_1 > 1, \\
\rho_3 < \rho_2 & \text{if } \rho_1 = 1, \\
\rho_3 = \rho_1 + \rho_2 - 1 & \text{if } \frac{1}{2} < \rho_1 < 1,
\end{cases}
\]

then there exists \(C = C(\rho_1, \rho_2, \rho_3)\) such that for any \(a, b \in \mathbb{R}\),

\[
\int_{-\infty}^{\infty} \frac{dx}{(x-a)^{\rho_1} (-x-b)^{\rho_2}} \leq \frac{C}{(a+b)^{\rho_3}}. \tag{4.8}
\]
The proof for this lemma is standard (see e.g. [ET16, KPV96]) and therefore omitted, we just want to remark that \( (a + b) = ((x - a) + (-x - b)) \).

**Lemma 4.6.** If \( \rho > \frac{1}{2} \), then there exists \( C = C(\rho) \) such that for any \( \sigma_i \in \mathbb{R} \) (\( 0 \leq i \leq 2 \)) with \( \sigma_2 \neq 0 \),

\[
\int_{-\infty}^{\infty} \frac{dx}{|\sigma_2 x^2 + \sigma_1 x + \sigma_0|^\rho} \leq \frac{C}{|\sigma_2|^{1/2}}.
\]  

(4.9)

Similarly, if \( \rho > \frac{1}{2} \), then there exists \( C = C(\rho) \) such that for any \( \sigma_i \in \mathbb{R} \) (\( 0 \leq i \leq 3 \)) with \( \sigma_3 \neq 0 \),

\[
\int_{-\infty}^{\infty} \frac{dx}{|\sigma_3 x^3 + \sigma_2 x^2 + \sigma_1 x + \sigma_0|^\rho} \leq \frac{C}{|\sigma_3|^{1/2}}.
\]  

(4.10)

**Proof.** We refer the reader to the proof of Lemma 2.5 in [BOP97] where (4.10) was proved. The similar argument can also be applied to obtain (4.9). \( \square \)

For the proof of the bilinear estimate, it is usually beneficial to reduce it to an estimate of some weighted convolution of \( L^2 \) functions as pointed out in [Tao01, CKS+03]. The next lemma illustrates this in a general situation. For the convenience of notations, we denote \( \xi = (\xi_1, \xi_2, \xi_3) \), \( \vec{\tau} = (\tau_1, \tau_2, \tau_3) \) and

\[
A := \{ (\vec{\xi}, \vec{\tau}) \in \mathbb{R}^6 : \sum_{i=1}^{3} \xi_i = \sum_{i=1}^{3} \tau_i = 0 \}.
\]  

(4.11)

**Lemma 4.7.** Let \( (\alpha_i, \beta_i) \in (\mathbb{R} \setminus \{0\}) \times \mathbb{R} \) for \( 1 \leq i \leq 3 \). For \( s \in \mathbb{R} \) and \( \sigma > \frac{1}{2} \), the bilinear estimate

\[
\| \partial_x (w_1 w_2) \|_{X_{s, \sigma}^2} \leq C \| w_1 \|_{X_{s, \sigma}^{\alpha_1, \beta_1}} \| w_2 \|_{X_{s, \sigma}^{\alpha_2, \beta_2}}, \quad \forall w_1 \in X_{s, \sigma}^{\alpha_1, \beta_1}, \ w_2 \in X_{s, \sigma}^{\alpha_2, \beta_2},
\]

is equivalent to

\[
\int_A \frac{\xi_3 \langle \xi_3 \rangle^s \prod_{i=1}^{3} f_i(\xi_i, \tau_i)}{(\xi_1)^s (\xi_2)^s M_1 M_2 \langle \tau_3 \rangle^{1-\sigma}} \leq C \prod_{i=1}^{3} \| f_i \|_{L^2_{\xi \tau}}^s, \quad \forall f_i \in L^2(\mathbb{R} \times \mathbb{R}), \quad i = 1, 2, 3,
\]

where

\[
M_1 = \langle L_1 \rangle^{\frac{1}{2}} + \mathbb{1}_{\{ |\xi| \leq 3 |\xi_1| \}} \langle L_1 \rangle^{\sigma}, \quad M_2 = \langle L_2 \rangle^{\frac{1}{2}} + \mathbb{1}_{\{ |\xi_2| \leq 3 |\tau_2| \}} \langle L_2 \rangle^{\sigma},
\]

and

\[
L_i = \tau_i - \phi^{\alpha_i, \beta_i}(\xi_i), \quad i = 1, 2, 3.
\]

Similarly, the bilinear estimate

\[
\| \partial_x (w_1 w_2) \|_{Z_{s, \sigma}^2} \leq C \| w_1 \|_{Z_{s, \sigma}^{\alpha_1, \beta_1}} \| w_2 \|_{Z_{s, \sigma}^{\alpha_2, \beta_2}}, \quad \forall w_1 \in Z_{s, \sigma}^{\alpha_1, \beta_1}, \ w_2 \in Z_{s, \sigma}^{\alpha_2, \beta_2},
\]

is equivalent to

\[
\int_A \frac{\xi_3 \langle \xi_3 \rangle^{\frac{3}{2} + \frac{1}{2} - \sigma} \prod_{i=1}^{3} f_i(\xi_i, \tau_i)}{(\xi_1)^s (\xi_2)^s M_1 M_2 \langle \tau_3 \rangle^{1-\sigma}} \leq C \prod_{i=1}^{3} \| f_i \|_{L^2_{\xi \tau}}^s, \quad \forall f_i \in L^2(\mathbb{R} \times \mathbb{R}), \quad i = 1, 2, 3.
\]

**Proof.** By duality and Plancherel theorem, the proof is standard and thus omitted. \( \square \)
Now we present the proof of Proposition 4.4

**Proof of Proposition 4.4**

The triple \(((\alpha_1, \beta_1), (\alpha_2, \beta_2), (\alpha_3, \beta_3))\) associated to the bilinear estimates (4.6) and (4.7) is

\[
(\alpha_1, \beta_1) = (\alpha, \beta), \quad (\alpha_2, \beta_2) = (-\alpha, -\beta), \quad (\alpha_3, \beta_3) = (\alpha, \beta).
\]

Define the set \(A\) as in (4.11). For any \((\xi, \tau) \in A\), let

\[
L_1 := \tau_1 - \phi^{\alpha, \beta}(\xi_1), \quad L_2 := \tau_2 - \phi^{-\alpha, -\beta}(\xi_2), \quad L_3 := \tau_3 - \phi^{\alpha, \beta}(\xi_3).
\]

According to Definition 4.1 the resonance function \(H = H(\xi_1, \xi_2, \xi_3)\) has the property that

\[
H = -\sum_{i=1}^{3} L_i
\]

and has the following formula.

\[
H(\xi_1, \xi_2, \xi_3) = -\alpha \xi_2 (2\xi_2^2 + 3\xi_1 \xi_2 + 3\xi_1^2) + 2\beta \xi_2, \quad \forall \sum_{i=1}^{3} \xi_i = 0. \tag{4.13}
\]

If \((\xi_3, \tau_3)\) is fixed, then by substituting \((\xi_2, \tau_2) = -(\xi_1 + \xi_3, \tau_1 + \tau_3)\), \(L_1 + L_2\) can be viewed as a function in \(\xi_1:\)

\[
L_1 + L_2 = P(\xi_1) := -2\alpha \xi_1^3 - 3\alpha \xi_3 \xi_1^2 + (-3\alpha \xi_3^2 + 2\beta) \xi_1 + \phi^{-\alpha, -\beta}(\xi_3) - \tau_3. \tag{4.14}
\]

Taking the derivative of \(P\) with respect to \(\xi_1\) yields

\[
P'(\xi_1) = -3\alpha (2\xi_1^2 + 2\xi_3 \xi_1 + \xi_3^2) + 2\beta. \tag{4.15}
\]

Similarly, if \((\xi_1, \tau_1)\) is fixed, then by substituting \((\xi_3, \tau_3) = -(\xi_1 + \xi_2, \tau_1 + \tau_2)\), \(L_2 + L_3\) can be viewed as a function in \(\xi_2:\)

\[
L_2 + L_3 = Q(\xi_2) := 2\alpha \xi_2^3 + 3\alpha \xi_1 \xi_2^2 + (3\alpha \xi_1^2 - 2\beta) \xi_2 + \phi^{\alpha, \beta}(\xi_1) - \tau_1. \tag{4.16}
\]

Taking the derivative of \(Q\) with respect to \(\xi_2\) yields

\[
Q'(\xi_2) = 3\alpha (2\xi_2^2 + 2\xi_1 \xi_2 + \xi_1^2) - 2\beta. \tag{4.17}
\]

Finally, if \((\xi_2, \tau_2)\) is fixed, then by substituting \((\xi_1, \tau_1) = -(\xi_2 + \xi_3, \tau_2 + \tau_3)\), \(L_3 + L_1\) can be viewed as a function in \(\xi_3:\)

\[
L_3 + L_1 = R(\xi_3) := 3\alpha \xi_2 \xi_3^2 + 3\alpha \xi_2^2 \xi_3 + \phi^{\alpha, \beta}(\xi_2) - \tau_2. \tag{4.18}
\]

Taking the derivative of \(R\) with respect to \(\xi_3\) yields

\[
R'(\xi_3) = 3\alpha \xi_2 (2\xi_3 + \xi_2). \tag{4.19}
\]

These formulas \((4.14) - (4.19)\) will play an important role in this section.

On the other hand, since \(|\beta| \leq 1\) and \(2\xi_2^2 + 3\xi_1 \xi_2 + 3\xi_1^2 \geq \frac{1}{2}(\xi_1^2 + \xi_2^2)\), it follows from the formula (4.13)
that

$$|H(\xi_1, \xi_2, \xi_3)| \geq_\alpha |\xi_2|(|\xi_1^2 + \xi_3^2|)$$

if $|\xi_1| + |\xi_2| \geq C$ for some constant $C = C(\alpha)$. Furthermore, since $\sum_{i=1}^{3} \xi_i = 0$, then

$$|H(\xi_1, \xi_2, \xi_3)| \geq_\alpha |\xi_2| \sum_{i=1}^{3} (\xi_i)^2 \quad (4.20)$$

if $\sum_{i=1}^{3} |\xi_i| \geq C_1$ for some constant $C_1 = C_1(\alpha)$. Analogously, it follows from (4.15) and (4.17) that there exists a constant $C_2 = C_2(\alpha)$ such that if $\sum_{i=1}^{3} |\xi_i| \geq C_2$, then

$$|P'(\xi_1)| \geq_\alpha \sum_{i=1}^{3} (\xi_i)^2 \quad \text{and} \quad |Q'(\xi_2)| \geq_\alpha \sum_{i=1}^{3} (\xi_i)^2. \quad (4.21)$$

Define

$$C^* = \max\{4, C_1, C_2\} \quad (4.22)$$

and fix it in the rest of this section. Now we finish the preparation for the proofs of (4.6) and (4.7). Next, we will deal with (4.6) first and then (4.7). For convenience of notations, the dependence of constants on $s, \alpha, \sigma$ may not be explicitly shown.

**Proof of (4.6)**

By Lemma 4.7, it suffices to prove for any $\{f_i\}_{i=1,2,3} \subset L^2(\mathbb{R} \times \mathbb{R})$,

$$\int_A \frac{\xi_3(\xi_3)^s \prod_{i=1}^{3} f_i(\xi_i, \tau_i)}{\langle \xi_1 \rangle^{\sigma} \langle \xi_2 \rangle^s M_1 M_2(L_3)^{1-\sigma}} \leq C \prod_{i=1}^{3} \|f_i\|_{L^2_{\xi, \tau}},$$

where $A$ is defined as in (4.11),

$$M_1 = \langle L_1 \rangle^{\frac{3}{4}} + \mathbb{1}_{\{c|\xi_1| \leq 3 + |\tau_1|\}} \langle L_1 \rangle^\sigma \quad \text{and} \quad M_2 = \langle L_2 \rangle^{\frac{3}{4}} + \mathbb{1}_{\{c|\xi_2| \leq 3 + |\tau_2|\}} \langle L_2 \rangle^\sigma. \quad (4.23)$$

Noticing $\langle \xi_3 \rangle \leq \langle \xi_1 \rangle \langle \xi_2 \rangle$, it suffices to consider the case when $s$ is close to $-\frac{3}{4}$. Without loss of generality, we assume $-\frac{3}{4} < s \leq -\frac{9}{10}$ and denote $\rho = -s$. Then $\frac{9}{10} \leq \rho < \frac{3}{4}$ and it reduces to show

$$\int_A \frac{|\xi_3|^{\rho} \langle \xi_1 \rangle^{\rho} \langle \xi_2 \rangle^s \prod_{i=1}^{3} |f_i(\xi_i, \tau_i)|}{\langle \xi_3 \rangle^{\rho} M_1 M_2(L_3)^{1-\sigma}} \leq C \prod_{i=1}^{3} \|f_i\|_{L^2_{\xi, \tau}}. \quad (4.24)$$

Define

$$\sigma_0 = \frac{3}{12} - \frac{\rho}{9} \quad (4.25)$$

Then (4.24) will be justified for any $\sigma \in \left\{\frac{1}{2}, \sigma_0\right\}$. The choice of $\sigma_0$ in (4.25) implies

$$\sigma \leq \min\left\{\frac{\rho + 1}{3}, \frac{7}{12} - \frac{\rho}{9}\right\}. \quad (4.26)$$

Usually, the most difficult part to handle is when the resonance function $H$ is small. Inspired by the estimate (4.20), we will divide the proof into three cases. The first case is when $\sum_{i=1}^{3} |\xi_i|$ is small which does not guarantee the estimate (4.20). The second case is when $\sum_{i=1}^{3} |\xi_i|$ is large enough but $|\xi_2|$ is very

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small, so although the estimate (4.20) holds, the lower bound in (4.20) may be too small. It is this step that forces to use the modified Fourier restriction space. The last case is when \( \sum_{i=1}^{3} |\xi_i| \) is large enough and \( |\xi_2| \) is not too small, so not only the estimate (4.20) holds, but also the lower bound in (4.20) is effective.

- **Case 1:** \( \sum_{i=1}^{3} |\xi_i| \leq C^* \).

  In this case, the proof of (4.24) reduces to

  \[
  \int \frac{\prod_{i=1}^{3} |f_i(\xi_i, \tau_i)|}{\langle L_1 \rangle^\frac{1}{2} \langle L_2 \rangle^\frac{1}{2} \langle L_3 \rangle^{1-\sigma}} \leq C \prod_{i=1}^{3} \|f_i\|_{L^2}. \tag{4.27}
  \]

  In the following, we will adopt the Cauchy-Schwarz inequality argument as in KPV96.

  LHS of (4.27) = \( \int \int |\xi_3| \left( \int \int \frac{|f_1 f_2|}{L_3} d\tau_1 d\xi_1 \right) d\tau_3 d\xi_3 \)

  \[
  \leq \int \int \frac{|f_3|}{L_3} \left( \int \int \frac{d\tau_1 d\xi_1}{L_1} \right)^{\frac{1}{2}} \left( \int \int f_1^2 f_2^2 d\tau_1 d\xi_1 \right)^{\frac{1}{2}} d\tau_3 d\xi_3. \tag{4.28}
  \]

  If

  \[
  \sup_{\xi_3, \tau_3} \frac{1}{(L_3)^{2(1-\sigma)}} \int \int \frac{d\tau_1 d\xi_1}{L_1 \langle L_2 \rangle} \leq C, \tag{4.29}
  \]

  then it follows from (4.28) and Cauchy-Schwarz inequality that

  RHS of (4.28) \( \leq \int \int |f_3| \left( \int \int f_1^2 f_2^2 d\tau_1 d\xi_1 \right)^{\frac{1}{2}} d\tau_3 d\xi_3 \leq \prod_{i=1}^{3} \|f_i\|_{L^2}. \tag{4.30} \]

  So it remains to verify (4.29). Noticing \( L_2 = \tau_2 - \phi^{-\alpha, -\beta}(\xi_2) = -\tau_1 - \tau_3 - \phi^{\alpha, \beta}(\xi_1 + \xi_3) \), then it follows from Lemma 4.5 that

  LHS of (4.29) \( \leq \sup_{\xi_3, \tau_3} \frac{1}{(L_3)^{2(1-\sigma)}} \int \int \frac{d\xi_1}{L_1 + L_2}^{1-\epsilon} \]

  for any \( \epsilon > 0 \). Now we choose \( \epsilon = 2\sigma - 1 \) so that

  LHS of (4.29) \( \leq \sup_{\xi_3, \tau_3} \frac{1}{(L_3)^{2(1-\sigma)}} \int \int \frac{d\xi_1}{L_1 + L_2}^{2-2\sigma}. \tag{4.31} \]

  Recalling (4.14), \( L_1 + L_2 = P(\xi_1) \) is a cubic polynomial in \( \xi_1 \), so it follows from \( 2 - 2\sigma > \frac{1}{3} \) and Lemma 4.6 that the right hand side of (4.31) is bounded.

- **Case 2:** \( \sum_{i=1}^{3} |\xi_i| > C^* \) and \( |\xi_2| \leq 1 \).

  In this case, \( |\xi_1| \sim |\xi_3| \geq 1 \) since \( C^* \geq 4 \). In addition, thanks to the modified Fourier restriction space, we now have \( M_2 \geq \langle L_2 \rangle^\sigma \). So the proof of (4.24) reduces to show

  \[
  \int \frac{|\xi_3| \prod_{i=1}^{3} |f_i(\xi_i, \tau_i)|}{\langle L_1 \rangle^\frac{1}{2} \langle L_2 \rangle^\sigma \langle L_3 \rangle^{1-\sigma}} \leq C \prod_{i=1}^{3} \|f_i\|_{L^2}. \tag{4.32}
  \]
– **Case 2.1:** $\langle L_1 \rangle \leq \langle L_3 \rangle$.

In this case, it follows from $\sigma > \frac{1}{2}$ and $\frac{1}{2} + (1 - \sigma) = \sigma + \left(\frac{3}{2} - 2\sigma\right)$ that

$$
\frac{1}{\langle L_1 \rangle^{\frac{1}{2}}\langle L_3 \rangle^{1-\sigma}} \leq \frac{1}{\langle L_1 \rangle^{\sigma}\langle L_3 \rangle^{\frac{1}{2}-2\sigma}}.
$$

So in order to prove (4.32), it suffices to establish

$$
\int \left| \xi_3 \| f_3 \| \right| \frac{\| f_1 f_2 \|}{\langle L_3 \rangle^{\frac{1}{2}-2\sigma} (\langle L_1 \rangle\langle L_2 \rangle)^{\sigma}} \leq C \prod_{i=1}^{3} \| f_i \|_{L^2}.
$$

Analogous to the Cauchy-Schwarz argument in Case 1, it remains to verify

$$
\sup_{\xi_3, \tau_3} \frac{\xi_3^2}{\langle L_3 \rangle^{3-4\sigma}} \int \frac{d\tau_1 d\xi_1}{\langle L_1 \rangle^{\sigma}\langle L_2 \rangle^{\sigma}} \leq C.
$$

It then follows from $2\sigma > 1$ and Lemma 14.8 that

$$
\int \frac{d\tau_1 d\xi_1}{\langle L_1 \rangle^{\sigma}\langle L_2 \rangle^{\sigma}} \lesssim \int \frac{d\xi_1}{\langle P(\xi_1) \rangle^{2\sigma}},
$$

where $P(\xi_1) = L_1 + L_2$ is as defined in (4.14). Thus, it suffices to establish

$$
\sup_{\xi_3, \tau_3} \frac{\xi_3^2}{\langle L_3 \rangle^{3-4\sigma}} \int \frac{d\xi_1}{\langle P(\xi_1) \rangle^{2\sigma}} \leq C.
$$

Since $\sum_{i=1}^{3} |\xi_i| > C^*$, it follows from (4.21) that $|P'(\xi_1)| \gtrsim \sum_{i=1}^{3} \xi_i^2$. Therefore,

$$
\text{LHS of (4.33)} = \sup_{\xi_3, \tau_3} \frac{\xi_3^2}{\langle L_3 \rangle^{3-4\sigma}} \int \frac{|P'(\xi_1)|}{\langle P(\xi_1) \rangle^{2\sigma}} d\xi_1
\lesssim \int \frac{|P'(\xi_1)|}{\langle P(\xi_1) \rangle^{2\sigma}} d\xi_1 \leq C.
$$

– **Case 2.2:** $\langle L_3 \rangle \leq \langle L_1 \rangle$.

In this case,

$$
\frac{1}{\langle L_1 \rangle^{\frac{1}{2}}\langle L_3 \rangle^{1-\sigma}} \leq \frac{1}{\langle L_1 \rangle^{\frac{1}{2}}\langle L_3 \rangle^{\frac{1}{2}-2\sigma}}.
$$

In order to prove (4.32), it suffices to justify

$$
\int \left| \frac{f_1}{\langle L_1 \rangle^{\frac{1}{2}-2\sigma} (\langle L_2 \rangle\langle L_3 \rangle)^{\sigma}} \right| \frac{\| f_2 f_3 \|}{\langle L_3 \rangle^{\frac{1}{2}-2\sigma} (\langle L_2 \rangle\langle L_3 \rangle)^{\sigma}} \leq C \prod_{i=1}^{3} \| f_i \|_{L^2}.
$$

Similiar as before, it reduces to show

$$
\sup_{\xi_1, \tau_1} \frac{1}{\langle L_1 \rangle^{\frac{1}{2}}\langle L_3 \rangle^{1-\sigma}} \int \frac{\xi_3^2}{\langle P(\xi_2) \rangle^{2\sigma}} d\xi_2 \leq C,
$$

where $Q(\xi_2) = L_2 + L_3$ is as defined in (4.16). Recalling (4.21) again yields $|Q'(\xi_2)| \gtrsim \sum_{i=1}^{3} \xi_i^2$, which can be used to prove (4.35) in the same way as that for (4.34) in Case 2.1.

– **Case 3:** $\sum_{i=1}^{3} |\xi_i| > C^*$ and $|\xi_2| > 1$. 

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In this case, \(|\xi_2| \sim \langle \xi_2 \rangle\), so it suffices to prove
\[
\int \frac{|\xi_3| \langle \xi_1 \rangle^\rho |\xi_2|^\rho \prod_{i=1}^{3} |f_i(\xi_i, \tau_i)|}{\langle \xi_3 \rangle^\rho (L_1)^{\frac{1}{2}} (L_2)^{\frac{1}{2}} (L_3)^{1-\sigma}} \leq C \prod_{i=1}^{3} \|f_i\|_{L^2_{\xi_i}}.
\]  
(4.36)

Define
\[
\text{MAX} = \max\{\langle L_1 \rangle, \langle L_2 \rangle, \langle L_3 \rangle\}.
\]  
(4.37)

Since \(H = -\sum_{i=1}^{3} L_i\) and \(\sum_{i=1}^{3} |\xi_i| > C^\ast\), it follows from (4.20) that
\[
\text{MAX} \gtrsim \langle H \rangle \gtrsim |\xi_2| \sum_{i=1}^{3} \langle \xi_i \rangle^2.
\]

Next, we will further decompose the proof into three cases depending on which \(L_i\) equals \text{MAX}.

- **Case 3.1:** \(\langle L_1 \rangle = \text{MAX}.

  In this case,
  \[
  \langle L_1 \rangle \gtrsim |\xi_2| \sum_{i=1}^{3} \langle \xi_i \rangle^2.
  \]

  Meanwhile, it follows from \(\frac{1}{2} + \frac{1}{2} + (1 - \sigma) = (2 - 3\sigma) + \sigma + \sigma\) that
  \[
  \frac{1}{\langle L_1 \rangle^{\frac{1}{2}} (L_2)^{\frac{1}{2}} (L_3)^{1-\sigma}} \leq \frac{1}{(L_1)^{2-3\sigma} (L_2) (L_3)}.
  \]

  Then similar as before, it suffices to establish
  \[
  \sup_{\xi_1, \tau_1} \frac{\langle \xi_1 \rangle^{2\rho}}{(L_1)^{4-6\sigma}} \int \frac{|\xi_2|^{2\rho} \xi_3^2}{(L_3)^{2\rho} (Q(\xi_2))^{2\sigma}} d\xi_2 \leq C,
  \]  
(4.38)

  where \(Q(\xi_2) = L_2 + L_3\) is as defined in (4.16). Since \(\sum_{i=1}^{3} |\xi_i| > C^\ast\), then \(|Q'(\xi_2)| \gtrsim \sum_{i=1}^{3} \langle \xi_i \rangle^2\). Hence,
  \[
  \frac{\langle \xi_1 \rangle^{2\rho} |\xi_2|^{2\rho} \xi_3^2}{(L_3)^{2\rho}} \lesssim \langle L_1 \rangle^{2-2\rho} |Q'(\xi_2)|.
  \]

  As a result,
  \[
  \text{LHS of (4.38)} \lesssim \sup_{\xi_1, \tau_1} \frac{\langle L_1 \rangle^{2-2\rho}}{(L_1)^{4-6\sigma}} \int |Q'(\xi_2)| d\xi_2 \lesssim \sup_{\xi_1, \tau_1} \frac{\langle L_1 \rangle^{2-2\rho}}{(L_1)^{4-6\sigma}}.
  \]  
(4.39)

  Thanks to (4.26), the right hand side of (4.39) is bounded.

- **Case 3.2:** \(\langle L_2 \rangle = \text{MAX}.

  Similar to Case 3.1, we have \(\langle L_2 \rangle \gtrsim |\xi_2| \sum_{i=1}^{3} \langle \xi_i \rangle^2\) and it suffices to justify
  \[
  \sup_{\xi_2, \tau_2} \frac{|\xi_2|^{2\rho}}{(L_2)^{4-6\sigma}} \int \frac{\langle \xi_1 \rangle^{2\rho} \xi_3^2}{(L_3)^{2\rho} (R(\xi_3))^{2\sigma}} d\xi_3 \leq C,
  \]  
(4.40)

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where $R(\xi_3) = L_3 + L_1$ is as defined in (4.18).

If $|2\xi_3 + \xi_2| \geq \frac{1}{10} (\xi_1)$, then it follows from (4.19) that $|R'(\xi_3)| \gtrsim |\xi_2(\xi_1)|$, so

$$\frac{|\xi_2|^{2p} (\xi_1)^{2p} \xi_3^2}{(\xi_3)^{2p}} \lesssim (L_2)^{2-2p} |R'(\xi_3)|.$$ 

Consequently,

$$\text{LHS of (4.40)} \lesssim \sup_{\xi_2, \tau_2} \frac{(L_2)^{2-2p}}{(L_2)^{4-6\sigma}} \int \frac{|R'(\xi_3)|}{(R(\xi_3))^{2\sigma}} d\xi_3 \lesssim \sup_{\xi_2, \tau_2} \frac{(L_2)^{2-2p}}{(L_2)^{4-6\sigma}} \leq C,$$ 

(4.41)

where the last inequality is due to (4.26).

If $|2\xi_3 + \xi_2| \leq \frac{1}{10} (\xi_1)$, then $|\xi_1| \sim |\xi_2| \sim |\xi_3|$ and

$$\frac{|\xi_2|^{2p} (\xi_1)^{2p} \xi_3^2}{(\xi_3)^{2p}} \lesssim |\xi_2| \frac{1}{2} (L_2)^{2p + \frac{1}{2}}.$$ 

As a result,

$$\text{LHS of (4.40)} \lesssim \sup_{\xi_2, \tau_2} \frac{|\xi_2|^{2p} (L_2)^{2p + \frac{1}{2}}}{(L_2)^{4-6\sigma}} \int \frac{d\xi_3}{(R(\xi_3))^{2\sigma}}.$$ 

(4.42)

It then follows from (4.18) and Lemma 4.6 and that

$$\int \frac{d\xi_3}{(R(\xi_3))^{2\sigma}} \lesssim \frac{1}{|\xi_1|^{\frac{1}{2}}},$$

so

$$\text{RHS of (4.42)} \lesssim \sup_{\xi_2, \tau_2} \frac{(L_2)^{2p + \frac{1}{2}}}{(L_2)^{4-6\sigma}} \leq C,$$

where the last inequality is due to (4.26).

- **Case 3.3:** $\langle L_3 \rangle = \text{MAX}.$

Similar to Case 3.1, $\langle L_3 \rangle \gtrsim |\xi_2| \sum_{i=1}^{3} |\xi_i|^2$ and

$$\frac{1}{(L_1)^{\frac{1}{2}} (L_2)^{\frac{1}{2}} (L_3)^{1-\sigma}} \leq \frac{1}{(L_3)^{2-3\sigma} (\langle L_1 \rangle (\langle L_2 \rangle)^{\sigma}}.$$ 

So it reduces to prove

$$\sup_{\xi_2, \tau_3} \frac{|\xi_3|^2}{(\xi_3)^{2p} (L_3)^{4-6\sigma}} \int \frac{|\xi_1|^{2p} |\xi_3|^{2p}}{(P(\xi_1))^{2\sigma}} d\xi_1 \leq C.$$ 

(4.43)

Since $\sum_{i=1}^{3} |\xi_i| > C^*$, it follows from (4.21) that $|P'(\xi_1)| \gtrsim \sum_{i=1}^{3} |\xi_i|^2$. Hence,

$$\frac{\xi_3^2 (\xi_1)^{2p} |\xi_2|^{2p}}{(\xi_3)^{2p}} \lesssim (L_3)^{2-2p} |P'(\xi_1)|.$$
Consequently,

\[
\text{LHS of (4.43)} \lesssim \sup_{\xi_3, \tau_3} \frac{(L_3)^{2-2\rho}}{(L_3)^{1-6\sigma}} \int A \frac{|P'(\xi_1)|}{(P(\xi_1))^{2\sigma}} d\xi_1 \\
\lesssim \sup_{\xi_3, \tau_3} \frac{(L_3)^{2-2\rho}}{(L_3)^{1-6\sigma}} \leq C,
\]

where the last inequality is because of (4.26).

**Proof of (4.7)**

Thanks to Lemma 4.7, it suffices to prove

\[
\int_A \left| \xi_3 \right| \left\langle \xi_1 \right\rangle^{\rho/3} \left\langle \xi_2 \right\rangle^{\rho/3} M_1 M_2 (L_3)^{(1-\sigma)/3} \ll \frac{\sigma}{\rho} \sum_{i=1}^{3} \left| f_i (\xi_i, \tau_i) \right| \left\langle \xi_1 \right\rangle^{\rho/3} \left\langle \xi_2 \right\rangle^{\rho/3} \leq C, \tag{4.44}
\]

where \(M_1, M_2\) are as defined in (4.23). It suffices to consider the case \(-\frac{3}{4} < s \leq -\frac{9}{16}\) and the case \(s = 3\) since

\[
\left( \frac{\langle \tau_3 \rangle}{\langle \xi_1 \rangle \langle \xi_2 \rangle} \right)^s \leq \left( \frac{\langle \tau_3 \rangle}{\langle \xi_1 \rangle \langle \xi_2 \rangle} \right)^{-\frac{9}{16}} + \left( \frac{\langle \tau_3 \rangle}{\langle \xi_1 \rangle \langle \xi_2 \rangle} \right)^3
\]

for any \(-\frac{9}{16} < s < 3\).

For the case of \(-\frac{3}{4} < s \leq -\frac{9}{16}\), let \(\rho = -s\). Then \(\frac{9}{16} \leq \rho < \frac{3}{4}\) and (4.44) becomes

\[
\int_A \frac{\left| \xi_3 \right| \left( \langle \tau_3 \rangle^{-\frac{9}{16}} \right)^s \prod_{i=1}^{3} \left| f_i (\xi_i, \tau_i) \right|}{\langle \xi_1 \rangle^{\frac{\rho}{3}} \langle \xi_2 \rangle^{\frac{\rho}{3}} M_1 M_2 (L_3)^{(1-\sigma)/3}} \leq C \prod_{i=1}^{3} \left\| f_i \right\|_{L^2_{\xi_1}}, \tag{4.45}
\]

Denote \(\sigma_0 = \frac{7}{12} - \frac{\rho}{9}\) as in (4.23). Then (4.45) will be proved for any \(\sigma \in (\frac{1}{2}, \sigma_0]\). Noticing that this choice of \(\sigma_0\) not only implies (4.26) but also guarantees

\[
\sigma \leq \frac{2}{3} - \frac{2\rho}{9}. \tag{4.46}
\]

Firstly, in the region where \(\langle \tau_3 \rangle \gtrsim \langle \xi_3 \rangle^{3}\), we have

\[
\frac{1}{\langle \tau \rangle^{\frac{s}{2} + \sigma - \frac{9}{16}}} \lesssim \frac{1}{\langle \xi_3 \rangle^{\rho/2}},
\]

so (4.45) holds as a corollary of (4.24). Then it suffices to consider the region where \(\langle \tau_3 \rangle \ll \langle \xi_3 \rangle^{3}\). Because of this simplification, we can assume

\[
|\xi_3| \gg 1, \quad |\xi_3| > C^*, \quad |L_3| = |\tau_3 - \phi^{\alpha, \beta}(\xi_3)| \sim |\xi_3|^3. \tag{4.47}
\]

Consequently, it follows from (4.20) and (4.21) that

\[
|H| \sim |\xi_2| \sum_{i=1}^{3} \xi_i^2, \quad |P'(\xi_1)| \gtrsim \sum_{i=1}^{3} \xi_i^2, \quad |Q'(\xi_2)| \gtrsim \sum_{i=1}^{3} \xi_i^2. \tag{4.48}
\]
Thanks to (4.47) and (4.48), we will actually prove (4.49) which is a stronger estimate than (4.45).

\[
\int \frac{|\xi_3|^{\rho}\langle \xi_2\rangle^\rho \prod_{i=1}^3 |f_i(\xi_i, \tau_i)|}{\langle L_1 \rangle^\frac{1}{2} \langle L_2 \rangle^\frac{1}{2} \langle L_3 \rangle^{1-\sigma}} \leq C \prod_{i=1}^3 \|f_i\|_{L^2_{\xi_i}}.
\] (4.49)

Since \( \langle L_3 \rangle \sim |\xi_3|^3 \), the estimate on the region where \(|\xi_3|\) is large should be easier to justify. So the following proof is divided into two parts depending on how large \(|\xi_3|\) is comparing to \(\min\{|\xi_1|, |\xi_2|\}\).

- **Case 1: \( |\xi_3| \gtrsim \min\{|\xi_1|, |\xi_2|\} \).**
  In this case, \(|\xi_3| \gtrsim \sum_{i=1}^3 |\xi_i|\) and \(|\xi_3|^{\rho}\langle \xi_2\rangle^\rho \lesssim |\xi_3|^{1+2\rho}\). Then similar to Case 1 in the proof of (4.6), it remains to establish

\[
\sup_{\xi_3, \tau_3} \frac{|\xi_3|^{2+4\rho}}{\langle L_3 \rangle^{2-2\sigma}} \int \frac{d\xi_3}{\langle L_1 + L_2 \rangle^{2-2\sigma}} \leq C.
\] (4.50)

Since \(|\xi_3| \gtrsim \sum_{i=1}^3 |\xi_i|\), it follows from (4.48) that

\[
\left| \sum_{i=1}^3 L_i \right| = |H| \lesssim |\xi_3|^3 \sim |L_3|.
\]

As a result, \(|L_1 + L_2| \lesssim |L_3|\), which implies

\[
\frac{1}{\langle L_3 \rangle^{2-2\sigma}} \frac{1}{\langle L_1 + L_2 \rangle^{2-2\sigma}} \lesssim \frac{1}{\langle L_3 \rangle^{4-6\sigma}} \frac{1}{\langle L_1 + L_2 \rangle^{2-2\sigma}}.
\]

Hence, the proof of (4.50) reduces to that of

\[
\sup_{\xi_3, \tau_3} \frac{|\xi_3|^{2+4\rho}}{\langle L_3 \rangle^{4-6\sigma}} \int \frac{d\xi_3}{\langle P(\xi_1) \rangle^{2\sigma}} \leq C,
\] (4.51)

where \(P(\xi_1) = L_1 + L_2\). Thanks to (4.48), \(|P'(\xi_1)| \gtrsim |\xi_3|^2\). Therefore,

\[
\int \frac{d\xi_1}{\langle P(\xi_1) \rangle^{2\sigma}} \lesssim \frac{1}{\xi_3^2} \int \frac{|P'(\xi_1)|}{\langle P(\xi_1) \rangle^{2\sigma}} d\xi_1 \lesssim \frac{1}{\xi_3^2}.
\]

Consequently,

\[
\text{LHS of (4.51)} \lesssim \sup_{\xi_3, \tau_3} \frac{|\xi_3|^{4\rho}}{\langle L_3 \rangle^{4-6\sigma}}.
\] (4.52)

Recalling \( \langle L_3 \rangle \sim |\xi_3|^3 \), so the right hand side of (4.52) is bounded because of (4.46).

- **Case 2: \( |\xi_3| \ll \min\{|\xi_1|, |\xi_2|\} \).**
  In this case, we have

\[
|\xi_1| \sim |\xi_2| \gg |\xi_3|.
\] (4.53)

So it follows from (4.48) that

\[
\left| \sum_{i=1}^3 L_i \right| = |H| \gg |\xi_3|^3 \sim |L_3|,
\]

which implies \(|L_1 + L_2| \gg |L_3|\). Recalling \(\text{MAX} = \{\langle L_1 \rangle, \langle L_2 \rangle, \langle L_3 \rangle\}\) as defined in (4.37), so either \(\langle L_1 \rangle = \text{MAX}\) or \(\langle L_2 \rangle = \text{MAX}\). We will only prove for the case \(\langle L_1 \rangle = \text{MAX}\) since the other case is
analogous. So next we assume \( \langle L_1 \rangle = \text{MAX} \), which implies
\[
\frac{1}{\langle L_1 \rangle^{\frac{1}{2}}} \frac{1}{\langle L_2 \rangle^{\frac{1}{2}}} \langle \xi \rangle^{1-\sigma} \leq \frac{1}{\langle L_1 \rangle^{2-3\sigma}} \frac{1}{\langle L_2 \rangle^\sigma} \langle \xi \rangle^\sigma.
\]

Meanwhile, recalling the relation (4.13), so the estimate (4.49) reduces to
\[
\int \frac{|x|^{2p+1}|f_1| |f_2 f_3|}{\langle L_1 \rangle^{2-3\sigma} \langle L_2 \rangle^\sigma} \leq C \prod_{i=1}^3 \|f_i\|_{L_{\xi}^2}.
\]

Similar to Case 3.2 in the proof of (4.6), it remains to justify
\[
sup_{\xi_1, \tau_1} |\xi_1|^{4p+2} \int \frac{1}{\langle Q(\xi_2) \rangle^{2\sigma}} d\xi_2 \leq C,
\]
where \( Q(\xi_2) = L_2 + L_3 \) is as defined in (4.10). Since (4.48) shows \( |Q'(\xi_2)| \gtrsim |\xi_1|^2 \), then
\[
\text{LHS of (4.54)} \lesssim |\xi_1|^{4p} \int \frac{|Q'(\xi_2)|}{\langle Q(\xi_2) \rangle^{2\sigma}} d\xi_2 \lesssim |\xi_1|^{4p} \langle L_1 \rangle^{4-6\sigma}.
\]

Since \( \langle L_1 \rangle = \text{MAX} \gtrsim |\xi_2|^{\frac{3}{2}} \approx |\xi_1|^3 \), then \( |\xi_1|^{4p} \lesssim \langle L_1 \rangle^{4-6\sigma} \) due to (4.46).

For the case of \( s = 3 \), (4.44) becomes
\[
\int \frac{|\xi_3| |\tau_3|^{\frac{3}{2} - \sigma} \prod_{i=1}^3 |f_i(\xi, \tau_1)|}{\langle \xi_1 \rangle^{3} \langle \xi_2 \rangle^3 M_1 M_2} \leq C \prod_{i=1}^3 \|f_i\|_{L_{\xi}^2},
\]
where
\[
M_1 = \langle L_1 \rangle^{\frac{1}{2}} + \mathbb{1}_{\{e<|\xi_1|\leq|\tau_1|\}} \langle L_1 \rangle^{\sigma} \quad \text{and} \quad M_2 = \langle L_2 \rangle^{\frac{1}{2}} + \mathbb{1}_{\{e<|\xi_2|\leq|\tau_2|\}} \langle L_2 \rangle^{\sigma}
\]
are defined as in (4.23). Define
\[
\sigma_0 = \frac{7}{12} - \frac{1}{9} = \frac{25}{48}.
\]

Then (4.24) can be verified for \( \rho = \frac{9}{16} \) (i.e. \( s = -\frac{9}{16} \)), and therefore also holds for \( \rho = -3 \) (i.e. \( s = 3 \)).

Taking advantage of this result, we will justify (4.55) for any \( \sigma \in \left( \frac{1}{2}, \frac{25}{48} \right) \).

Firstly, if \( \langle \tau_3 \rangle \lesssim \langle \xi_3 \rangle^3 \), then (4.55) holds as a corollary of (4.24), so we assume \( \langle \tau_3 \rangle \gg \langle \xi_3 \rangle^3 \) in the following. In particular, this assumption implies \( \langle \tau_3 \rangle \sim \langle L_3 \rangle \). Then (4.55) reduces to
\[
\int \frac{|\xi_3| |\tau_3|^{\frac{3}{2}} \prod_{i=1}^3 |f_i(\xi, \tau_1)|}{\langle \xi_1 \rangle^{3} \langle \xi_2 \rangle^3 M_1 M_2} \leq C \prod_{i=1}^3 \|f_i\|_{L_{\xi}^2}.
\]

The key observation in the rest proof is
\[
\langle \xi_1 \rangle^3 + \langle \xi_2 \rangle^3 + \langle L_1 \rangle + \langle L_2 \rangle \geq \langle \tau_3 \rangle.
\]

Based on this observation, we divide the proof into two cases.

- **Case 1:** \( \langle \xi_1 \rangle^3 \gtrsim |\tau_3| \) or \( \langle \xi_2 \rangle^3 \gtrsim |\tau_3| \).

  We will only prove for the case \( \langle \xi_1 \rangle^3 \gtrsim |\tau_3| \) since the other case is similar. Under this assumption,
we have $(\tau_3)^{1/2} \lesssim (\xi_1)^{3/2}$. Moreover, since $|\xi_3| \lesssim (\xi_1)(\xi_2)$, it suffices to show

$$\int \prod_{i=1}^{3} \frac{|f_i(\xi_i, \tau_i)|}{\langle \xi_2 \rangle^3 \langle L_1 \rangle^{\frac{1}{2}} \langle L_2 \rangle^{\frac{1}{2}}} \leq C \prod_{i=1}^{3} \|f_i\|_{L^2},$$  \hspace{1cm} (4.58)

in order to prove (4.56). Similar as before, it remains to establish

$$\sup_{\xi_3, \tau_3} \int \int \frac{d\tau_2 d\xi_2}{\langle \xi_2 \rangle^3 \langle L_1 \rangle \langle L_2 \rangle} \leq C.$$

By Lemma 4.5

$$\int \frac{d\tau_2}{\langle L_1 \rangle \langle L_2 \rangle} \lesssim \frac{1}{\langle L_1 + L_2 \rangle^2} \leq 1,$$

so

$$\text{LHS of (4.59)} \lesssim \sup_{\xi_3, \tau_3} \int \frac{d\xi_2}{\langle \xi_2 \rangle^4} \lesssim 1.$$

- Case 2: $(\xi_1)^3 \ll |\tau_3|$, $(\xi_2)^3 \ll |\tau_3|$ and $\langle L_1 \rangle \geq \langle L_2 \rangle$.

The assumptions in this case and the key observation (4.57) together imply

$$\langle L_1 \rangle \geq \frac{1}{4} (\tau_3), \quad |\tau_1| \sim \langle L_1 \rangle \quad \text{and} \quad |\tau_1| \gg (\xi_1)^3.$$  \hspace{1cm} (4.60)

- Case 2.1: $e^{\xi_1} \leq 3 + |\tau_1|$.

Thanks to the definition of $M_1$, this case implies

$$M_1 \geq \langle L_1 \rangle^\sigma \gtrsim |\tau_3|^\frac{1}{2} \langle L_2 \rangle^{-\frac{1}{2}}.$$

So

$$\text{LHS of (4.56)} \lesssim \int \frac{\langle \xi_3 \rangle \prod_{i=1}^{3} |f_i(\xi_i, \tau_i)|}{\langle \xi_1 \rangle^3 \langle \xi_2 \rangle^3 \langle L_2 \rangle^\sigma}.$$

Since $|\xi_3| \lesssim (\xi_1)(\xi_2)$, it then remains to establish

$$\sup_{\xi_3, \tau_3} \int \int \frac{d\tau_2 d\xi_2}{\langle \xi_2 \rangle^4 \langle L_2 \rangle^{2\sigma}} \leq C,$$

which is obvious due to $\sigma > \frac{1}{2}$.

- Case 2.2: $e^{\xi_1} > 3 + |\tau_1|$.

In this case, $M_1 = \langle L_1 \rangle^{\frac{1}{2}} \gtrsim |\tau_3|^\frac{1}{2}$ and $|\xi_1| > \ln(3 + |\tau_1|)$. In addition, because of (4.60), we have

$$|\xi_1| \gtrsim \ln(3 + |L_1|) \geq \ln(3 + |L_2|).$$  \hspace{1cm} (4.61)

As a result,

$$\frac{|\xi_3| \langle \tau_3 \rangle^{\frac{1}{2}}}{\langle \xi_1 \rangle^3 \langle \xi_2 \rangle^3 M_1} \lesssim \frac{1}{\langle \xi_1 \rangle^2 \langle \xi_2 \rangle^2} \lesssim \frac{1}{[\ln(3 + |L_2|)]^2 \langle \xi_2 \rangle^2}.$$
Hence,
\[
\text{LHS of (4.56)} \lesssim \int \frac{\prod_{i=1}^{3} |f_i(\xi_i, \tau_i)|}{\langle \xi_2 \rangle^2 \langle L_2 \rangle \frac{1}{2} \left[ \ln(3 + |L_2|) \right]^2}.
\]

Similar as before, it remains to show
\[
\sup_{\xi_3} \int \frac{d\tau_2 d\xi_2}{\langle \xi_2 \rangle^4 \langle L_2 \rangle \left[ \ln(3 + |L_2|) \right]^2} \leq C.
\]

Noticing
\[
\int_{\mathbb{R}} \frac{d\tau_2}{\langle L_2 \rangle \left[ \ln(3 + |L_2|) \right]^2} = 2 \int_{0}^{\infty} \frac{dx}{(1 + x) \left[ \ln(3 + x) \right]^2} < \infty,
\]
so
\[
\text{LHS of (4.62)} \lesssim \int \frac{d\xi_2}{\langle \xi_2 \rangle^4} \leq C.
\]

## 5 Well-posedness

In this section we present the proofs of Theorem 1.7 and Theorem 1.3, the main results of the paper.

**Proof of Theorem 1.7.** Without loss of generality, we assume \( T = 1 \), let \( s \in (-\frac{3}{4}, 3] \) and \( \beta \in (0, 1) \) be given. Define
\[
\sigma = \min \{ \sigma_1(s), \sigma_2(s), \sigma_0(s, 1), \sigma_0(s, -1) \},
\]
where \( \sigma_1(s) \) and \( \sigma_2(s) \) are the thresholds as in Proposition 1.10 and Proposition 1.11 and \( \sigma_0(s, 1) \) and \( \sigma_0(s, -1) \) are the thresholds as in Proposition 1.12 when \( \alpha = 1 \) or \(-1\). Then \( \sigma > \frac{1}{2} \) and \( \sigma \) only depends on \( s \), so in the following, the dependence of any constant on \( \sigma \) will be considered as the dependence on \( s \). The choice of \( r \) will be determined later, see (5.6).

Define the space \( \mathcal{Y} = Y^{1, \beta}_{s, \frac{1}{2}, \sigma}(\Omega_1) \times Y^{-1, -\beta}_{s, \frac{1}{2}, \sigma}(\Omega_1) \) equipped with the product norm. Denote
\[
E_0 = \| (p, q) \|_{H^{\sigma}_1(\mathbb{R}^+)} + \| (a, b, c) \|_{H^{\sigma}_3(\mathbb{R}^+)}.
\]

Then it follows from the assumption that \( E_0 \leq r \). Define
\[
\mathcal{B}_{C^*} = \left\{ (u, v) \in \mathcal{Y} : \| (u, v) \|_{\mathcal{Y}} \leq C^* \right\}.
\]

We will choose suitable \( C^* \) and \( r \) to guarantee the existence of a solution in the space \( \mathcal{B}_{C^*} \). For any \( (u, v) \in \mathcal{B}_{C^*} \), denote
\[
f(u, v) = -3uu_x - (uv)_x + vv_x, \\
g(u, v) = uu_x - (uv)_x - 3vv_x.
\]

Then we infer from Proposition 1.12 that \( f \in X^{1, \beta}_{s, \sigma-1}(\Omega_1) \cap Z^{1, \beta}_{s, \sigma-1}(\Omega_1) \) and \( g \in X^{-1, -\beta}_{s, \sigma-1}(\Omega_1) \cap Z^{-1, -\beta}_{s, \sigma-1}(\Omega_1) \).

Since the data \((p, q)\) and \((a, b, c)\) are compatible for (1.5), then it implies \( p \) and \( a \) are compatible for (1.21). So according to Proposition 1.10, we define \( \tilde{u} = \Gamma^+_\beta(f, p, a) \). By the properties of \( \Gamma^+_\beta(f, p, a) \) stated in Proposition 1.10, \( \tilde{u}_x(0, t) \) is well-defined and belongs to \( H^{\sigma}_3(\mathbb{R}) \). Then again the compatibility of \((p, q)\) and \((a, b, c)\) for (1.5) implies the compatibility of \( q \) and \( b, c + \tilde{u}_x|_{x=0} \). So based on Proposition 1.11 we
define \( \tilde{v} = \Gamma^- (g, q, b, c + \tilde{u}_x | x = 0) \). Combining \( \Gamma^+ \) and \( \Gamma^- \) together, we define \( \Gamma(u, v) = (\tilde{u}, \tilde{v}) \), where

\[
\begin{align*}
\tilde{u} &= \Gamma^+(f, p, a), \\
\tilde{v} &= \Gamma^- (g, q, b, c + \tilde{u}_x | x = 0).
\end{align*}
\]  

(5.2)

We will prove \( \Gamma \) is a contraction mapping in \( B_{C^*} \) so that its fixed point \( (u, v) \) are the desired functions in Theorem 1.7.

It will be shown first that \( \Gamma \) maps the closed ball \( B_{C^*} \) into itself for suitable \( C^* \). For any \( (u, v) \in B_{C^*} \), it follows from (1.24) in Theorem 1.10 and (1.27) in Theorem 1.11 that

\[
\left\|
\tilde{\nu}
\right\|_{Y^{1, \beta}_{s, \frac{1}{2}, \sigma} (\Omega_t)} \leq C_1 \left( \|f\|_{X^{1, \beta}_{s, \frac{1}{2}, \sigma} (\Omega_t)} + \|f\|_{Z^{2, \beta}_{s, \frac{1}{2}, \sigma} (\Omega_t)} + \|g\|_{X^{1, \beta}_{s, \frac{1}{2}, \sigma} (\Omega_t)} + \|g\|_{Z^{1, \beta}_{s, \frac{1}{2}, \sigma} (\Omega_t)} + \|\tilde{u}_x | x = 0\|_{H^\sigma (\Omega_t)} \right),
\]

(5.3)

where \( C_1 \) is a constant that only depends on \( s \). Adding these two together and recalling the definition of \( E_0 \), we obtain

\[
\left\|
(\tilde{u}, \tilde{v})
\right\|_Y \leq C_1 \left( E_0 + \|f\|_{X^{1, \beta}_{s, \frac{1}{2}, \sigma} (\Omega_t)} + \|f\|_{Z^{2, \beta}_{s, \frac{1}{2}, \sigma} (\Omega_t)} + \|g\|_{X^{1, \beta}_{s, \frac{1}{2}, \sigma} (\Omega_t)} + \|g\|_{Z^{1, \beta}_{s, \frac{1}{2}, \sigma} (\Omega_t)} + \|\tilde{u}_x | x = 0\|_{H^\sigma (\Omega_t)} \right). 
\]

(5.4)

Since \( f \) and \( g \) are defined as in (5.1), we apply Proposition 1.12 to conclude that

\[
\begin{align*}
\|f\|_{X^{1, \beta}_{s, \frac{1}{2}, \sigma} (\Omega_t)} + \|f\|_{Z^{2, \beta}_{s, \frac{1}{2}, \sigma} (\Omega_t)} &\leq C_2 (\|u\|_{X^{1, \beta}_{s, \frac{1}{2}, \sigma} (\Omega_t)} + \|v\|_{Z^{1, \beta}_{s, \frac{1}{2}, \sigma} (\Omega_t)})^2, \\
\|g\|_{X^{1, \beta}_{s, \frac{1}{2}, \sigma} (\Omega_t)} + \|g\|_{Z^{1, \beta}_{s, \frac{1}{2}, \sigma} (\Omega_t)} &\leq C_2 (\|u\|_{X^{1, \beta}_{s, \frac{1}{2}, \sigma} (\Omega_t)} + \|v\|_{Z^{1, \beta}_{s, \frac{1}{2}, \sigma} (\Omega_t)})^2,
\end{align*}
\]

where \( C_2 \) is a constant which only depends on \( s \). Putting these two estimates into (5.4), we find

\[
\left\|
(\tilde{u}, \tilde{v})
\right\|_Y \leq C_3 \left( E_0 + (\|u\|_{X^{1, \beta}_{s, \frac{1}{2}, \sigma} (\Omega_t)} + \|v\|_{Z^{1, \beta}_{s, \frac{1}{2}, \sigma} (\Omega_t)})^2 \right) \leq C_3 (E_0 + (u, v)^2) \]

(5.5)

where the constant \( C_3 \) only depends on \( s \). Since \( (u, v) \in B_{C^*} \), \( (u, v) \|_Y \leq C^* \). Hence,

\[
\left\|
(\tilde{u}, \tilde{v})
\right\|_Y \leq C_3 \left[ E_0 + (C^*)^2 \right] \leq C_3 [r + (C^*)^2].
\]

Choosing

\[
C^* = 8C_3 \sqrt{r},
\]

then

\[
\left\|
(\tilde{u}, \tilde{v})
\right\|_Y \leq C_3 r + 64C_3^3 \sqrt{r}^2.
\]
Define
\[ r = \frac{1}{64 C_3^2}. \]  
(5.6)
Then \( r \) only depends on \( s \) and it follows from (5.5) that \( \| (\tilde{u}, \tilde{v}) \|_Y \leq C^*/4 \), which implies \( (\tilde{u}, \tilde{v}) \in \mathcal{B}_{C^*} \).

Next for any \( (u_j, v_j) \in \mathcal{B}_{C^*}, j = 1, 2 \), similar argument as above yields
\[ \| \Gamma(u_1, v_1) - \Gamma(u_2, v_2) \|_Y \leq \frac{1}{2} \| (u_1, v_1) - (u_2, v_2) \|_Y. \]

We have thus shown that \( \Gamma \) is a contraction on \( \mathcal{B}_{C^*} \), which implies \( \Gamma \) has a fixed point \( (u, v) \in \mathcal{B}_{C^*} \).

Proposition 5.2 (Existence and uniqueness of classical solutions).

Let \( r > 0 \) be given. There exists a \( T > 0 \) such that for compatible set
\[ (p, q) \in \mathcal{H}_3^3(\mathbb{R}^+), \quad (a, b, c) \in \mathcal{H}_3^3(\mathbb{R}^+) \]
with
\[ \| ((p, q))\|_{\mathcal{H}_3^3(\mathbb{R}^+)} + \| (a, b, c) \|_{\mathcal{H}_3^3(\mathbb{R}^+)} \leq r \]
the IBVP (1.4) admits a unique strong solution \( (u, v) \) such that both \( u \) and \( v \) belong to \( C^2(0, T; L^2(\mathbb{R}^+)) \cap C(0, T; H^3(\mathbb{R}^+)) \).

Now we turn to consider the unconditional well-posedness for the IBVP (1.5) and present the proof of Theorem 1.3.

**Proof of Theorem 1.3**

Note first that by scaling argument, Theorem 1.7 can be restated as the following theorem.

**Theorem 5.1** (Conditional Well-posedness). Let \(-\frac{3}{4} < s \leq 3\), \( r > 0 \) and \( 0 \leq \beta \leq 1 \) be given. There exist \( \sigma = \sigma(s) > \frac{1}{2} \) and \( T = T(s, r) > 0 \) such that for any naturally compatible \( (\phi, \psi) \in \mathcal{H}_s^3(\mathbb{R}^+) \) and \( \tilde{h} = (h_1, h_2, h_3) \in \mathcal{H}_s^3(\mathbb{R}^+) \) related to the IBVP (1.5) with
\[ \| (\phi, \psi) \|_{\mathcal{H}_s^3(\mathbb{R}^+)} + \| \tilde{h} \|_{\mathcal{H}_s^3(\mathbb{R}^+)} \leq r, \]
the system of the integral equations (SIE) (1.16) admits a unique solution
\[ (u, v) \in \mathcal{Y}_\sigma(\Omega_T) := Y_{s, \frac{1}{2}, \sigma}^1(\Omega_T) \times Y_{s, \frac{1}{2}, \sigma}^{-1, -\beta}(\Omega_T). \]
Moreover, the solution map is real analytic in the corresponding spaces.

**Proposition 5.2** (Existence and uniqueness of classical solutions).

Let \( r > 0 \) be given. There exists a \( T > 0 \) such that for compatible set
\[ (p, q) \in \mathcal{H}_3^3(\mathbb{R}^+), \quad (a, b, c) \in \mathcal{H}_3^3(\mathbb{R}^+) \]
with
\[ \| ((p, q))\|_{\mathcal{H}_3^3(\mathbb{R}^+)} + \| (a, b, c) \|_{\mathcal{H}_3^3(\mathbb{R}^+)} \leq r \]
the IBVP (1.4) admits a unique strong solution \( (u, v) \) such that both \( u \) and \( v \) belong to \( C^2(0, T; L^2(\mathbb{R}^+)) \cap C(0, T; H^3(\mathbb{R}^+)) \).
Proof. The existence of the classical solution \((u, v)\) follows directly from Theorem \ref{thm:existence} with \(s = 3\), and the uniqueness of the solution can be proved using the standard energy estimate method.

We then show that the solution \((u, v)\) given in Theorem \ref{thm:existence} is a mild solution of the IBVP \ref{eq:ibvp}. First of all, note that by the standard extension arguments, the solutions given in Theorem \ref{thm:existence} possess the uniqueness of the solution can be proved using the standard energy estimate method.

**Proposition 5.3.** \textbf{[Blow-up alternative]} Let \(\frac{-3}{4} < s \leq 3\) be given. For any given

\[
(p, q) \in \mathcal{H}_s^x(\mathbb{R}^+), \quad (a, b, c) \in \mathcal{H}_s^r(\mathbb{R}^+),
\]

there exists a \(T_{max}^s > 0\) such that the corresponding solution \((u, v)\) is \(C(0, T_{max}^s; H^s(\mathbb{R}^+))\) and

\[
either T_{max}^s = +\infty \text{ or } \lim_{t \to T_{max}^s} (\|u(\cdot, t)\|_{H^s(\mathbb{R}^+)} + \|v(\cdot, t)\|_{H^s(\mathbb{R}^+)}) = +\infty.
\]

It can be readily checked that \(T_{max}^{s_1} \geq T_{max}^{s_2}\) for \(s_1 < s_2\). We then propose that \(T_{max}^{s_1} = T_{max}^{s_2}\).

**Proposition 5.4.** For \(-\frac{3}{4} < s_1 < s_2 \leq 3\), and for any given \((p, q) \in \mathcal{H}_s^x(\mathbb{R}^+)\) and \((a, b, c) \in \mathcal{H}_s^r(\mathbb{R}^+)\), one has

\[
T_{max}^{s_2} = T_{max}^{s_1},
\]

where \(T_{max}^{s_1}\) and \(T_{max}^{s_2}\) are the lifespans of the solutions, corresponding to \(s_2\) and \(s_1\) respectively, determined in Proposition \ref{prop:5.3}.

Before stating the proof, we present some smoothing properties based on the linear estimate in Lemma \ref{lem:linear} and the bilinear estimates \ref{eq:bilinear1}-\ref{eq:bilinear7}, the proofs of the following Lemma \ref{lem:5.5}-Lemma \ref{lem:5.7} will be shown in Appendix A.

**Lemma 5.5.** Let \(-\frac{3}{4} < s_1 < s_2 \leq 3\), \(0 < T \leq 1\), \(\alpha \neq 0\) and \(|\beta| \leq 1\). Then there exists \(\epsilon_0 = \epsilon(s_1, s_2, \alpha) > 0\) such that for any \(\sigma \in \left(\frac{1}{2}, \frac{1}{2} + \epsilon_0\right]\), the following bilinear estimate

\[
\left\| \eta \left(\frac{t}{T}\right) \int_0^t W_R^{\alpha, \beta}(t - \tau)(w_1 w_2)_x \, d\tau \right\|_{X_{s_2, \frac{1}{2}, \sigma}}^2 
\leq C T^{\epsilon_0} \left(\|w_1\|_{X_{s_1, \frac{1}{2}, \sigma}} \cdot \|w_2\|_{X_{s_2, \frac{1}{2}, \sigma}} + \|w_1\|_{X_{s_2, \frac{1}{2}, \sigma}} \cdot \|w_2\|_{X_{s_1, \frac{1}{2}, \sigma}}\right),
\]

holds for any \(w_1, w_2 \in X_{s_1, \frac{1}{2}, \sigma}\) with some positive constant \(C = C(s_1, s_2, \alpha, \epsilon_0, \sigma)\).

**Lemma 5.6.** Let \(-\frac{3}{4} < s_1 < s_2 \leq 3\), \(0 < T \leq 1\), \(\alpha \neq 0\) and \(|\beta| \leq 1\). Then there exists \(\epsilon_0 = \epsilon(s_1, s_2, \alpha) > 0\) such that for any \(\sigma \in \left(\frac{1}{2}, \frac{1}{2} + \epsilon_0\right]\), the following bilinear estimate

\[
\left\| \eta \left(\frac{t}{T}\right) \int_0^t W_R^{\alpha, -\beta}(t - \tau)(w_1 w_2)_x \, d\tau \right\|_{X_{s_2, \frac{1}{2}, \sigma}}^2 
\leq C T^{\epsilon_0} \left(\|w_1\|_{X_{s_1, \frac{1}{2}, \sigma}} \cdot \|w_2\|_{X_{s_2, \frac{1}{2}, \sigma}} + \|w_1\|_{X_{s_2, \frac{1}{2}, \sigma}} \cdot \|w_2\|_{X_{s_1, \frac{1}{2}, \sigma}}\right),
\]

holds for any \(w_1, w_2 \in X_{s_1, \frac{1}{2}, \sigma}\) with some positive constant \(C = C(s_1, s_2, \alpha, \epsilon_0, \sigma)\).
Lemma 5.7. Let $-\frac{3}{4} < s_1 < s_2 \leq 3, 0 < T \leq 1, \alpha \neq 0$ and $|\beta| \leq 1$. Then there exists $\epsilon_0 = \epsilon(s_1, s_2, \alpha) > 0$ such that for any $\sigma \in \left(\frac{1}{2}, \frac{3}{2} + \epsilon_0\right]$, the following bilinear estimate

$$\left\| \frac{T}{T} \int_0^T W_{R}^{\alpha, \beta}(t-t)(w_1 w_2) \right\| \leq CT^\epsilon \left( \|w_1\|_{X^{s_1, \frac{1}{2}, \sigma}} \|w_2\|_{X^{s_1, \frac{1}{2}, \sigma}} + \|w_1\|_{X^{s_1, \frac{1}{2}, \sigma}} \|w_2\|_{X^{s_1, \frac{1}{2}, \sigma}} \right). \quad (5.9)$$

holds for any $w_1 \in X^{s_1, \frac{1}{2}, \sigma}$ and $w_2 \in X^{s_1, \frac{1}{2}, \sigma}$ with some positive constant $C = C(s_1, s_2, \alpha, \epsilon_0, \sigma)$.

Now we are ready to justify Proposition 5.4.

Proof of Proposition 5.4. For $s = s_1$ or $s_2$, denote

$$Y_s = Y^{1,1}_{s, \frac{1}{2}, \sigma} \times Y^{-1,-1}_{s, \frac{1}{2}, \sigma}.$$

If

$$t^* := T_{\max}^{s_2} < T_{\max}^{s_1},$$

then by Proposition 5.3

$$r := \sup_{0 \leq t \leq t^*} \left( \|u(\cdot, t)\|_{H^1(R^+)} + \|v(\cdot, t)\|_{H^1(R^+)} \right) < +\infty, \quad (5.10)$$

and

$$\lim_{t \to t^*} \left( \|u(\cdot, t)\|_{H^2(R^+)} + \|v(\cdot, t)\|_{H^2(R^+)} \right) = +\infty. \quad (5.11)$$

According to Theorem 5.1 and its proof, there exists some $T = T(s_2, r) > 0$ such that for any $\delta \in (0, T)$,

$$\| (u, v) \|_{Y_{s_2}(R^+ \times (t^* - \delta, t^*))} \leq \alpha_{r, s_1} \left( \| (p, q) \|_{H^{s_1}(R^+)} + \| (a, b, c) \|_{H^{s_1}(R^+)} \right).$$

In addition, according to Lemma 5.5, 5.7 one also has,

$$\sup_{t^* - \delta < t < t^* - \delta} \| (u, v) \|_{H^2(R^+) \times H^2(R^+)} \leq \| (u, v) \|_{Y_{s_2}(R^+ \times (t^* - \delta, t^*))} \leq C_1 \left( \| (p, q) \|_{H^{s_1}(R^+)} + \| (a, b, c) \|_{H^{s_1}(R^+)} \right) + C_2 \delta^\epsilon \left( \|u\|_{X^{1,1}_{s_1, \frac{1}{2}, \sigma}} + \|v\|_{X^{-1,-1}_{s_1, \frac{1}{2}, \sigma}} \right),$$

where $\epsilon_0 > 0$ and $\sigma \in \left(\frac{1}{2}, \frac{3}{2} + \epsilon_0\right]$ are some constants which only depends on $s_1$ and $s_2$. Hence, we can choose $\delta$ small enough such that

$$C_2 \delta^\epsilon \left( \|u\|_{X^{1,1}_{s_1, \frac{1}{2}, \sigma}} + \|v\|_{X^{-1,-1}_{s_1, \frac{1}{2}, \sigma}} \right) \leq \frac{1}{2}.$$

This yields

$$\sup_{t^* - \delta < t < t^* - \delta} \| (u, v) \|_{H^2(R^+) \times H^2(R^+)} \leq 2C_1 \left( \| (p, q) \|_{H^{s_1}(R^+)} + \| (a, b, c) \|_{H^{s_1}(R^+)} \right),$$

which contradicts with (5.11).

Therefore, $T_{\max}^{s_2} = T_{\max}^{s_1}$ and the proof is complete.
Next, we show that the solution obtained from Theorem 5.1 is, in fact, a mild solution.

**Proposition 5.8** (Existence of the mild solution).

Fix \(s \in (-\frac{3}{4}, 3]\) and \(r > 0\). Then there exist \(\sigma = \sigma(s) > \frac{1}{2}\) and \(T = T(s, r) > 0\) such that for any compatible data \((p, q) \in \mathcal{H}^s_x(\mathbb{R}^+)\) and \((a, b, c) \in \mathcal{H}^r_t(\mathbb{R}^+)\) with

\[
\| (p, q) \|_{\mathcal{H}^s_x(\mathbb{R}^+)} + \| (a, b, c) \|_{\mathcal{H}^r_t(\mathbb{R}^+)} \leq r,
\]

the IBVP (I4) admits a mild solution

\[
(u, v) \in C(0, T; H^s(\mathbb{R}^+)) \times C(0, T; H^s(\mathbb{R}^+))
\]

that satisfies

\[
u \in Y^{1, 1}_{s, \frac{1}{2}, \sigma}(\Omega_T), \quad v \in Y^{-1, -1}_{s, \frac{1}{2}, \sigma}(\Omega_T),
\]

and

\[
\| (u, v) \|_{Y^s} \leq \alpha_{r, s} \left( \| (p, q) \|_{\mathcal{H}^s_x(\mathbb{R}^+)} + \| (a, b, c) \|_{\mathcal{H}^r_t(\mathbb{R}^+)} \right),
\]

where \(\alpha\) is a continuous function from \(\mathbb{R}^+\) to \(\mathbb{R}^+\) depending only on \(r\) and \(s\).

**Proof.** According to Theorem 5.1 for given \((p, q) \in \mathcal{H}^s_x(\mathbb{R}^+)\) and \((a, b, c) \in \mathcal{H}^r_t(\mathbb{R}^+)\), one can obtain a solution \((u, v) \in Y^{1, 1}_{s, \frac{1}{2}, \sigma}(\Omega_T) \times Y^{-1, -1}_{s, \frac{1}{2}, \sigma}(\Omega_T)\). We will show that such a solution \((u, v)\) is, in fact, a mild solution of IBVP (I4).

Let us choose sequences

\[
(p_n, q_n) \in \mathcal{H}^2_x(\mathbb{R}^+), \quad (a_n, b_n, c_n) \in \mathcal{H}^3_t(\mathbb{R}^+)
\]

such that

\[
\lim_{n \to \infty} (p_n, q_n) = (p, q) \quad \text{in} \quad \mathcal{H}^s_x(\mathbb{R}^+), \quad \lim_{n \to \infty} (a_n, b_n, c_n) = (a, b, c) \quad \text{in} \quad \mathcal{H}^r_t(\mathbb{R}^+).
\]

Since the solution map in Theorem 5.1 is continuous, there exist solutions \(\{(u_n, v_n)\}_{n \geq 1}\) under the initial-boundary conditions \(\{(p_n, q_n, a_n, b_n, c_n)\}_{n \geq 1}\) such that \(\{(u_n, v_n)\}_{n \geq 1}\) lie in \(C(0, T; H^s(\mathbb{R}^+)) \times C(0, T; H^s(\mathbb{R}^+))\)

and

\[
\lim_{n \to \infty} u_n = u, \quad \lim_{n \to \infty} v_n = v \quad \text{in} \quad C(0, T; H^s(\mathbb{R}^+)).
\]

On the other hand, according to Proposition 5.3 and 5.4 one has

\[
u_n \in C(0, T_{n,\max}^3; H^3(\mathbb{R}^+)), \quad \text{for} \quad n = 1, 2, \ldots,
\]

Since \(T_{n,\max}^3 \geq T\) for \(n = 1, 2, \ldots\), it leads to \(u_n, v_n \in C(0, T; H^3(\mathbb{R}^+))\). The proof is now complete. \(\square\)

Finally, we show that the mild solution of IBVP (I4) is unique.

**Proposition 5.9** (Uniqueness of the mild solution). For fixed \(s \in (-\frac{3}{4}, 3]\), given any \((p, q) \in \mathcal{H}^s_x(\mathbb{R}^+)\) and \((a, b, c) \in \mathcal{H}^r_t(\mathbb{R}^+)\), the IBVP (I4) admits at most one mild solution \((u, v)\).

**Proof.** For given

\[
(p, q) \in \mathcal{H}^s_x(\mathbb{R}^+), \quad (a, b, c) \in \mathcal{H}^r_t(\mathbb{R}^+),
\]

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let us assume that there are two mild solutions for the IBVP (1.4), denoted as,

\[(u^{(1)}, v^{(1)}) \quad \text{and} \quad (u^{(2)}, v^{(2)}),\]

with 

\[u^{(1)}, v^{(1)} \in C(0, T; H^s(\mathbb{R}^+)), \quad u^{(2)}, v^{(2)} \in C(0, T_2; H^s(\mathbb{R}^+)),\]

for some \(T_1, T_2 > 0\). Without loss of generality, one can set \(T := T_1 \leq T_2\). According to the definition of mild solutions, one can find two sequences of classic solutions of the IBVP (1.4),

\[(u^{(1)}_n, v^{(1)}_n), (u^{(2)}_n, v^{(2)}_n) \in C(0, T; H^3(\mathbb{R}^+)) \times C(0, T; H^3(\mathbb{R}^+)),\]

with

\[
\begin{align*}
(u^{(1)}_n(x, 0) &= p^{(1)}_n(x), \quad v^{(1)}_n(x, 0) = q^{(1)}_n(x), \\
\dot{v}^{(1)}_n(0, t) &= a^{(1)}_n(t), \quad v^{(1)}_n(0, t) = b^{(1)}_n(t), \quad \partial_x (v^{(1)}_n - u^{(1)}_n)(0, t) = c^{(1)}_n(t) \end{align*}
\]

and

\[
\begin{align*}
(u^{(2)}_n(x, 0) &= p^{(2)}_n(x), \quad v^{(2)}_n(x, 0) = q^{(2)}_n(x), \\
\dot{v}^{(2)}_n(0, t) &= a^{(2)}_n(t), \quad v^{(2)}_n(0, t) = b^{(2)}_n(t), \quad \partial_x (v^{(2)}_n - u^{(2)}_n)(0, t) = c^{(2)}_n(t),
\end{align*}
\]

for \(n = 1, 2, \ldots\), such that

\[
\lim_{n \to \infty} u^{(1)}_n = u^{(1)}, \quad \lim_{n \to \infty} v^{(1)}_n = v^{(1)}, \quad \lim_{n \to \infty} u^{(2)}_n = u^{(2)}, \quad \lim_{n \to \infty} v^{(2)}_n = v^{(2)}, \quad \text{in} \quad C(0, T; H^s(\mathbb{R}^+)),
\]

\[
\lim_{n \to \infty} (p^{(1)}_n, q^{(1)}_n) = (p, q), \quad \lim_{n \to \infty} (p^{(2)}_n, q^{(2)}_n) = (p, q) \quad \text{in} \quad \mathcal{H}_x^s(\mathbb{R}^+),
\]

and

\[
\lim_{n \to \infty} (a^{(1)}_n, b^{(1)}_n, c^{(1)}_n) = (a, b, c), \quad \lim_{n \to \infty} (a^{(2)}_n, b^{(2)}_n, c^{(2)}_n) = (a, b, c) \quad \text{in} \quad \mathcal{H}_x^s(\mathbb{R}^+).
\]

We then denote

\[
(\tilde{u}, \tilde{v}) \in \left[ C(0, T_3; H^s(\mathbb{R}^+)) \right]^2,
\]

\[
(\tilde{u}^{(1)}_n, \tilde{v}^{(1)}_n) \in \left[ C(0, T_{n, max}^3; H^3(\mathbb{R}^+)) \right]^2,
\]

\[
(\tilde{u}^{(2)}_n, \tilde{v}^{(2)}_n) \in \left[ C(0, T_{n, max}^3; H^3(\mathbb{R}^+)) \right]^2,
\]

to be the solutions provided by Theorem 5.1 with conditions corresponding to

\[
(p, q, a, b, c), \quad (p^{(1)}_n, q^{(1)}_n, a^{(1)}_n, b^{(1)}_n, c^{(1)}_n), \quad (p^{(2)}_n, q^{(2)}_n, a^{(2)}_n, b^{(2)}_n, c^{(2)}_n),
\]

respectively. We define \(T^* := \min\{T, T_3\}\). According to Proposition 5.3 we can infer that \(T_{n, max}^3 \geq T^*\) and then

\[
(a^{(1)}_n, v^{(1)}_n) = (\tilde{u}^{(1)}_n, \tilde{v}^{(1)}_n), \quad (u^{(2)}_n, v^{(2)}_n) = (\tilde{u}^{(2)}_n, \tilde{v}^{(2)}_n), \quad \text{in} \quad \left[ C(0, T^*; H^3(\mathbb{R}^+)) \right]^2,
\]

due to the uniqueness of the classic solution of the IBVP (1.4). Moreover, according to the continuity of the solution map in Theorem 5.1 one has

\[
\lim_{n \to \infty} (\tilde{u}^{(1)}_n, \tilde{v}^{(1)}_n) = (\tilde{u}, \tilde{v}), \quad \lim_{n \to \infty} (\tilde{u}^{(2)}_n, \tilde{v}^{(2)}_n) = (\tilde{u}, \tilde{v}), \quad \text{in} \quad \left[ C(0, T^*; H^s(\mathbb{R}^+)) \right]^2.
\]

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Therefore, we can infer that

\[ (u^{(1)}, v^{(1)}) = (\hat{u}, \hat{v}) = (u^{(2)}, v^{(2)}) \quad \text{in } [C(0, T^+; H^s(\mathbb{R}^+))]^2. \]

Finally, we can use continuity property of the solutions \((u^{(1)}, v^{(1)}), (u^{(2)}, v^{(2)})\) and \((\hat{u}, \hat{v})\) to show \(T_1 = T_2 = T_3.\)

The proof of Theorem 1.3 is thus completed when \(s \in (-\frac{3}{4}, 3].\) If \(s > 3,\) then the conclusion also holds by using similar but actually simpler argument since the data has higher regularity now.

## A Proofs of Lemma 5.5–Lemma 5.7

The proofs for Lemma 5.5, Lemma 5.6, and Lemma 5.7 are similar, so we will only prove Lemma 5.7. In order to make the argument more clearly, we first present several auxiliary results. We start with the following Lemma A.1 which is a refinement of Proposition 4.4 on the temporal regularity of the term \((uv)_x.\)

**Lemma A.1.** Let \(-\frac{3}{4} < s \leq 3, \alpha \neq 0 \text{ and } |\beta| \leq 1.\) Then there exists \(\epsilon_0 = \epsilon_0(s, \alpha) > 0\) such that for any \(\sigma \in (\frac{1}{2}, \frac{1}{2} + \epsilon_0],\)

\[
\|(uv)_x\|_{X^{\alpha, \beta}_{s+1, \frac{1}{2}}^{1+\epsilon_0}} \leq C\|u\|_{X^{\alpha, \beta}_{s+1, \frac{1}{2}}} \|v\|_{X^{\alpha, -\beta}_{s, \frac{1}{2}}} \quad \text{on } (0, T),
\]

where \(C = C(s, \alpha, \sigma).\)

**Proof.** By duality, it suffices to prove

\[
\iint_{\mathbb{R}^2} \xi \hat{u} \hat{w} d\xi d\tau \leq C\|u\|_{X^{\alpha, \beta}_{s+1, \frac{1}{2}}} \|v\|_{X^{\alpha, -\beta}_{s, \frac{1}{2}}} \|w\|_{X^{\alpha, \beta}_{s+1, \frac{1}{2}+\epsilon_0}},
\]

for any \(u \in X^{\alpha, \beta}_{s+1, \frac{1}{2}}, v \in X^{\alpha, -\beta}_{s, \frac{1}{2}},\) and \(w \in X^{\alpha, \beta}_{s+1, \frac{1}{2}+\epsilon_0} \text{ on } (0, T),\) Similar to the proof of Proposition 4.4 we denote

\[
L_1 = \tau_1 - \phi^{\alpha, \beta}(\xi_1), \quad L_2 = \tau_2 - \phi^{-\alpha, -\beta}(\xi_2), \quad L_3 = \tau_3 - \phi^{\alpha, \beta}(\xi_3),
\]

and

\[
M_1 = \langle L_1 \rangle + \mathbb{1}_{\{\xi_1 \leq 3+|\tau_1|\}} \langle L_1 \rangle, \quad M_2 = \langle L_2 \rangle + \mathbb{1}_{\{\xi_2 \leq 3+|\tau_2|\}} \langle L_2 \rangle.
\]

In addition, we define

\[
\begin{align*}
&f_1(\xi_1, \tau_1) = \langle \xi_1 \rangle^{\rho} M_1 \hat{w}(\xi_1, \tau_1), \\
&f_2(\xi_2, \tau_2) = \langle \xi_2 \rangle^{\rho} M_2 \hat{v}(\xi_2, \tau_2), \\
&f_3(\xi_3, \tau_3) = \langle \xi_3 \rangle^{-s} \langle L_3 \rangle^{1-s-\epsilon_0} \hat{w}(\xi_3, \tau_3).
\end{align*}
\]

Then it reduces to justify

\[
\iint_{A} \frac{|\xi_3|^{\rho} \langle \xi_1 \rangle^{\rho} \langle \xi_2 \rangle^{\rho} \prod_{i=1}^3 |f_i(\xi_i, \tau_i)|}{\langle \xi_3 \rangle^{\rho} M_1 M_2 (L_3)^{1-s-\epsilon_0}} \leq C \prod_{i=1}^3 \|f_i\|_{L_{\xi, \tau}^2},
\]

where \(\rho = -s\) and the set \(A\) is as defined in (4.11). In the remaining proof, instead of introducing \(\sigma_0\) as in (4.25), we define

\[
\epsilon_0 = \frac{1}{16} - \frac{1}{12} \rho.
\]

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Then for any $\sigma \in \left(\frac{1}{2}, \frac{1}{2} + \epsilon_0\right]$, Lemma A.2 can be proved in an analogous way as the proof of (A.2) in Proposition 4.2.

Next, we will take advantage of Proposition A.1 to deduce a slightly stronger estimate since the spatial regularity requirement on the right-hand side of (A.3) is weaker than that of (A.1).

Lemma A.2. Let $-\frac{3}{2} < s_1 \leq s_2 \leq 3, \alpha \neq 0, |\beta| \leq 1$. Then there exists $\epsilon_0 = \epsilon_0(s_1, s_2, \alpha) > 0$ such that for any $\sigma \in \left(\frac{1}{2}, \frac{1}{2} + \epsilon_0\right]$,

$$\|(uv)_x\|_{X^{s_2, -\sigma - 1 + \epsilon_0}} \leq C \left( \|u\|_{X^{s_1, \frac{1}{2} - \sigma}} \|v\|_{X^{s_2, \frac{1}{2} + \sigma}} + \|u\|_{X^{s_1, \frac{1}{2} - \sigma}} \|v\|_{X^{s_2, \frac{1}{2} + \sigma}} \right),$$

(A.3)

where $C = C(s_1, s_2, \alpha, \sigma)$.

Proof. By duality, it suffices to prove

$$\iint_{\mathbb{R}^2} \xi \tilde{w} \psi \xi \tilde{w} d\xi d\tau \leq C \left( \|u\|_{X^{s_1, \frac{1}{2} - \sigma}} \|v\|_{X^{s_2, \frac{1}{2} + \sigma}} + \|u\|_{X^{s_1, \frac{1}{2} - \sigma}} \|v\|_{X^{s_2, \frac{1}{2} + \sigma}} \right) \|w\|_{X^{s_2, 1 - \sigma - \epsilon_0}},$$

for any $u \in X^{s_1, \frac{1}{2} - \sigma}, v \in X^{s_2, \frac{1}{2} + \sigma}$ and $w \in X^{s_2, 1 - \sigma - \epsilon_0}$. Denote $L_1, L_2, L_3, M_1, M_2$ in the same way as the above proof for Lemma A.1 and define

$$f_1(\xi_1, \tau_1) = (\xi_1)^{s_1} M_1 \tilde{w}(\xi_1, \tau_1), \quad f_2(\xi_2, \tau_2) = (\xi_2)^{s_2} M_2 \tilde{w}(\xi_2, \tau_2), \quad f_3(\xi_3, \tau_3) = (\xi_3)^{-s_2} (L_3)^{1 - \sigma - \epsilon_0} \tilde{w}(\xi_3, \tau_3).$$

Denote $\rho = -s_1$ and $r = s_2 - s_1 \geq 0$. Then similar to (A.2), it reduces to show

$$\left(\xi_3\right)^r \left|\xi_3\right| \left|\xi_3\right| \left|\xi_3\right| \left|f_1(\xi_1, \tau_1)\right| \left|f_2(\xi_2, \tau_2)\right| \left|f_3(\xi_3, \tau_3)\right| \left(\xi_3\right)^{s_3} M_1 M_2 (L_3)^{1 - \sigma - \epsilon_0} \leq C \left( \|f_1\|_{L^{2^*}_{\xi_1}} \|f_2\|_{L^{2^*}_{\xi_2}} + \|f_3\|_{L^{2^*}_{\xi_1}} \|f_2\|_{L^{2^*}_{\xi_2}} \right) \|f_3\|_{L^{2^*}_{\xi_1}}. \quad (A.4)$$

Since $r \geq 0$ and $\sum_{i=1}^{3} \xi_i = 0$, then

$$\left(\xi_3\right)^r = \left(\xi_1 + \xi_2\right)^r \leq 2^r \left( \left(\xi_1\right)^r + \left(\xi_2\right)^r \right).$$

Consequently,

$$\text{LHS of (A.4) } \leq 2^r (I + II),$$

where

$$I = \int_{\mathbb{R}^3} \left|\xi_3\right| \xi_1 \left|\xi_2\right| \left|f_1(\xi_1, \tau_1)\right| \left|f_2(\xi_2, \tau_2)\right| \left|f_3(\xi_3, \tau_3)\right| \left(\xi_3\right)^{s_3} M_1 M_2 (L_3)^{1 - \sigma - \epsilon_0} \left(\xi_3\right)^{s_3} M_1 M_2 (L_3)^{1 - \sigma - \epsilon_0},$$

$$II = \int_{\mathbb{R}^3} \left|\xi_3\right| \left|\xi_2\right| \left|f_1(\xi_1, \tau_1)\right| \left|f_2(\xi_2, \tau_2)\right| \left|f_3(\xi_3, \tau_3)\right| \left(\xi_3\right)^{s_3} M_1 M_2 (L_3)^{1 - \sigma - \epsilon_0}.$$
According to Lemma A.1

\[ I \leq C \| (\xi)^{-1} f_1 \|_{L^2_x} \| f_2 \|_{L^2_x} \| f_3 \|_{L^2_x}, \]
\[ II \leq C \| f_1 \|_{L^2_x} \| (\xi)^{-1} f_2 \|_{L^2_x} \| f_3 \|_{L^2_x}. \]

Thus, (A.4) is verified. \( \square \)

Recalling the cut-off function \( \eta \in C_0^\infty(\mathbb{R}) \) which satisfies \( \eta(t) = 1 \) on \((-1, 1)\) and \( \text{supp} \eta \subset (-2, 2) \), the following is a classical estimate on the Fourier restriction norms when the time is localized.

\textbf{Lemma A.3.} For any \( s \in \mathbb{R}, -\frac{1}{2} < b_1 \leq b_2 < \frac{1}{2}, \alpha \neq 0, |\beta| \leq 1 \) and \( 0 < T \leq 1 \), there exists a constant \( C = C(s, b_1, b_2) \) such that

\[ \| \eta \left( \frac{t}{T} \right) w \|_{X^{s, b_1}} \leq CT^{b_2 - b_1} \| w \|_{X^{s, b_2}}, \quad \forall w \in X^{s, b_2}, \quad (A.5) \]

The proof of this estimate is well-known, see e.g. Lemma 2.11 in [Tao06]. Combining Lemma A.2 and Lemma A.3 yields the outcome below.

\textbf{Lemma A.4.} Let \( -\frac{3}{4} < s_1 \leq s_2 \leq 3, \alpha \neq 0, |\beta| \leq 1 \) and \( 0 < T \leq 1 \). Then there exists \( \epsilon_0 = \epsilon_0(s_1, s_2, \alpha) > 0 \) such that for any \( \sigma \in (\frac{1}{2}, \frac{1}{2} + \epsilon_0) \),

\[ \| \eta \left( \frac{t}{T} \right) (uv)_x \|_{X^{s_2, \sigma - 1}} \leq CT^{s_1 - s_2} \| (uv)_x \|_{L^2_t}, \quad (A.6) \]

where \( C = C(s_1, s_2, \alpha, \sigma) \).

\textbf{Proof.} Firstly, we choose \( \epsilon_0 \) as that in Lemma A.2. Then for any \( \sigma \in (\frac{1}{2}, \frac{1}{2} + \epsilon_0) \), we choose \( w = (uv)_x, s = s_2, b_1 = \sigma - 1 \) and \( b_2 = \sigma - 1 + \epsilon_0 \) in (A.5) to obtain

\[ \| \eta \left( \frac{t}{T} \right) (uv)_x \|_{X^{s_2, \sigma - 1}} \leq CT^{s_1 - s_2} \| (uv)_x \|_{L^2_t}. \]

Then we can apply Lemma A.2 to verify (A.6). \( \square \)

Similar to Lemma A.3, the estimate for modified Fourier restriction norms of the localized (in time) Duhamel term associated with the semigroup operator \( W^{\alpha, \beta}_R \) can also be established.

\textbf{Lemma A.5.} Let \( s \in \mathbb{R}, \alpha \neq 0, |\beta| \leq 1, \frac{1}{2} < \sigma < 1 \) and \( 0 < T \leq 1 \). Then there exists a constant \( C = C(s, \alpha, \sigma) \) such that

\[ \| \eta \left( \frac{t}{T} \right) \int_0^t W^{\alpha, \beta}_R (t - \tau) F d\tau \|_{X^{s - \sigma, \frac{1}{2} \sigma}} \leq C \| \eta \left( \frac{t}{2T} \right) F \|_{X^{s, \sigma - 1}}. \quad (A.7) \]

\textbf{Proof.} Define \( g(x, t) = \eta \left( \frac{t}{T} \right) F(x, t) \). Then it is easily seen that

\[ \text{LHS of (A.7)} = \| \eta \left( \frac{t}{T} \right) \int_0^t W^{\alpha, \beta}_R (t - \tau) g d\tau \|_{X^{s - \sigma, \frac{1}{2} \sigma}}. \]
Then similar to Lemma 2.4 (also see Lemma 2.1 in [GTV97]), we have
\[
\left\| \eta \left( \frac{t}{T} \right) \int_0^t W^\alpha_\beta_R (t - \tau) g \, d\tau \right\|_{X^\alpha_\beta_{s, \frac{1}{2}, \sigma}} \leq C \left\| g \right\|_{X^\alpha_\beta_{s, \sigma - 1}} = C \left\| \eta \left( \frac{t}{2T} \right) F \right\|_{X^\alpha_\beta_{s, \sigma - 1}}.
\]

Now we have developed all the tools needed to justify Lemma 5.7.

**Proof of Lemma 5.7.** Firstly, we choose \( \epsilon_0 = \epsilon_0(s_1, s_2, \alpha) > 0 \) as that in Lemma A.4. Then for any \( \sigma \in \left( \frac{1}{2}, \frac{1}{2} + \epsilon_0 \right) \), we apply Lemma A.5, with \( s = s_2 \) and \( F = (w_1 w_2)_x \), to conclude that
\[
\left\| \eta \left( \frac{t}{T} \right) \int_0^t W^\alpha_\beta_R (t - \tau)(w_1 w_2)_x \right\|_{X^\alpha_\beta_{s_2, \frac{1}{2}, \sigma}} \leq C \left\| \eta \left( \frac{t}{2T} \right)(w_1 w_2)_x \right\|_{X^\alpha_\beta_{s_2, \sigma - 1}}.
\]
Then it follows from Lemma A.4 that
\[
\left\| \eta \left( \frac{t}{2T} \right)(w_1 w_2)_x \right\|_{X^\alpha_\beta_{s_2, \sigma - 1}} \leq C T^{\epsilon_0} \left( \left\| w_1 \right\|_{X^\alpha_\beta_{s_1, \frac{1}{2}, \sigma}} \left\| w_2 \right\|_{X^{-\alpha, -\beta}_{s_2, \frac{1}{2}, \sigma}} + \left\| w_1 \right\|_{X^\alpha_\beta_{s_2, \frac{1}{2}, \sigma}} \left\| w_2 \right\|_{X^{-\alpha, -\beta}_{s_1, \frac{1}{2}, \sigma}} \right),
\]
where \( C = C(s_1, s_2, \alpha, \sigma) \). Combining the above two estimates leads to (5.9) in Lemma 5.7.

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**Conflict of interest statement**

Declarations of interest: none.

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