INTEGRAL INEQUALITIES OF HERMITE-HADAMARD TYPE FOR $(\alpha, m)$-GA-CONVEX FUNCTIONS

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Abstract. In the paper, the authors introduce a notion “$(\alpha, m)$-GA-convex functions” and establish some integral inequalities of Hermite-Hadamard type for $(\alpha, m)$-GA-convex functions.

1. Introduction

In [8, 11], the concepts of $m$-convex functions and $(\alpha, m)$-convex functions were introduced as follows.

Definition 1.1 ([11]). A function $f : [0, b] \to \mathbb{R}$ is said to be $m$-convex for $m \in (0, 1]$ if the inequality

$$f(\alpha x + m(1 - \alpha)y) \leq \alpha f(x) + m(1 - \alpha)f(y)$$

(1.1)

holds for all $x, y \in [0, b]$ and $\alpha \in [0, 1]$.

Definition 1.2 ([8]). For $f : [0, b] \to \mathbb{R}$ and $(\alpha, m) \in (0, 1]^2$, if

$$f(tx + m(1-t)y) \leq t^\alpha f(x) + m(1-t^\alpha)f(y)$$

(1.2)

is valid for all $x, y \in [0, b]$ and $t \in [0, 1]$, then we say that $f(x)$ is an $(\alpha, m)$-convex function on $[0, b]$.

Hereafter, a few of inequalities of Hermite-Hadamard type for the $m$-convex and $(\alpha, m)$-convex functions were presented, some of them can be recited as following theorems.

Theorem 1.1 ([3, Theorems 2.2]). Let $I \supset \mathbb{R}_0 = [0, \infty)$ be an open interval and let $f : I \to \mathbb{R}$ be a differentiable function on $I$ such that $f' \in L([a, b])$ for $0 \leq a < b < \infty$, where $L([a, b])$ denotes the set of all Lebesgue integrable functions on $[a, b]$. If $|f'(x)|^q$ is $m$-convex on $[a, b]$ for some given numbers $m \in (0, 1]$ and $q \geq 1$, then

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \frac{b-a}{4} \min \left\{ \left[ \frac{|f'(a)|^q + m|f'(b)/m|^q}{2} \right]^{1/q} \cdot \left[ \frac{m|f'(a/m)|^q + |f'(b)|^q}{2} \right]^{1/q} \right\}.$$ 

(1.3)

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Theorem 1.2 ([3, Theorem 3.1]). Let $I \supset [0, \infty)$ be an open interval and let $f : I \to (-\infty, \infty)$ be a differentiable function on $I$ such that $f^{\prime} \in L([a, b])$ for $0 \leq a < b < \infty$. If $|f^{\prime}(x)|^q$ is $(\alpha, m)$-convex on $[a, b]$ for some given numbers $m, \alpha \in (0, 1]$, and $q \geq 1$, then

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(x) \, dx \right| \leq \frac{b - a}{2} \left( \frac{1}{2} \right)^{1-1/q} \times \min \left\{ \left[ v_1 |f^{\prime}(a)|^q + v_2 m \left| f^{\prime} \left( \frac{b}{m} \right) \right|^q \right]^{1/q}, \left[ v_2 m \left| f^{\prime} \left( \frac{a}{m} \right) \right|^q + v_1 |f^{\prime}(b)|^q \right]^{1/q} \right\},$$

where

$$v_1 = \frac{1}{(\alpha + 1)(\alpha + 2)} \left( \alpha + \frac{1}{2^\alpha} \right)$$

and

$$v_2 = \frac{1}{(\alpha + 1)(\alpha + 2)} \left( \frac{\alpha^2 + \alpha + 2}{2} - \frac{1}{2^\alpha} \right).$$

For more information on Hermite-Hadamard type inequalities for various kinds of convex functions, please refer to the monograph [5], the recently published papers [1, 2, 4, 6, 7, 12, 13], and closely related references therein.

In this paper, we will introduce a new concept “$(\alpha, m)$-geometric-arithmetically convex function” (simply speaking, $(\alpha, m)$-GA-convex function) and establish some integral inequalities of Hermite-Hadamard type for $(\alpha, m)$-GA-convex functions.

2. A Definition and a Lemma

Now we introduce the so-called $(\alpha, m)$-GA-convex functions.

Definition 2.1. Let $f : [0, b] \to \mathbb{R}$ and $(\alpha, m) \in [0, 1]^2$. If

$$f(x^\lambda y^{m(1-\lambda)}) \leq \lambda^\alpha f(x) + m(1 - \lambda^\alpha) f(y)$$

for all $x, y \in [0, b]$ and $\lambda \in [0, 1]$, then $f(x)$ is said to be a $(\alpha, m)$-geometric-arithmetically convex function or, simply speaking, an $(\alpha, m)$-GA-convex function. If (2.1) is reversed, then $f(x)$ is said to be a $(\alpha, m)$-geometric-arithmetically concave function or, simply speaking, a $(\alpha, m)$-GA-concave function.

Remark 2.1. When $m = \alpha = 1$, the $(\alpha, m)$-GA-convex (concave) function defined in Defintion 2.1 becomes a GA-convex (concave) function defined in [9, 10].

To establish some new Hermite-Hadamard type inequalities for $(\alpha, m)$-GA-convex functions, we need the following lemma.

Lemma 2.1. Let $f : I \subseteq \mathbb{R}_+ = (0, \infty) \to \mathbb{R}$ be a differentiable function and $a, b \in I$ with $a < b$. If $f^{\prime}(x) \in L([a, b])$, then

$$\frac{b^2 f(b) - a^2 f(a)}{2} - \int_a^b x f(x) \, dx = \frac{\ln b - \ln a}{2} \int_0^1 a^{3(1-t)b^3 t} f^{\prime}(a^{1-t}b^t) \, dt. \tag{2.2}$$

Proof. Let $x = a^{1-t}b^t$ for $0 \leq t \leq 1$. Then

$$(\ln b - \ln a) \int_0^1 a^{3(1-t)b^3 t} f^{\prime}(a^{1-t}b^t) \, dt = \int_a^b x^2 f^{\prime}(x) \, dx.$$
Lemma 2.1 is thus proved. \hfill \Box

3. Inequalities of Hermite-Hadamard Type

Now we turn our attention to establish inequalities of Hermite-Hadamard type for \((\alpha, m)\)-GA-convex functions.

**Theorem 3.1.** Let \(f : \mathbb{R}_0 = [0, \infty) \to \mathbb{R}\) be a differentiable function and \(f' \in L([a, b])\) for \(0 < a < b < \infty\). If \(|f'|^q\) is \((\alpha, m)\)-GA-convex on \([0, \max\{a^{1/m}, b\}]\) for \((\alpha, m) \in (0, 1]^2\) and \(q \geq 1\), then

\[
\left| \frac{b^2 f(b) - a^2 f(a)}{2} - \int_a^b x f(x) \, dx \right| \leq \frac{\ln b - \ln a}{2} \left[ \int_0^1 a^{3(1-t)} b^{3t} |f'(a^{1/m})|^q \right]^{1/q} \times \left\{ m[L(a^3, b^3) - G(\alpha, 3)] |f'(a^{1/m})|^q + G(\alpha, 3) |f'(b)|^q \right\}^{1/q},
\]

where

\[
G(\alpha, \ell) = \int_0^1 t^\alpha a^{\ell(1-t)} b^{\ell t} \, dt
\]

for \(\ell \geq 0\) and

\[
L(x, y) = \frac{y - x}{\ln y - \ln x}
\]

for \(x, y > 0\) with \(x \neq y\).

**Proof.** Making use of the \((\alpha, m)\)-GA-convexity of \(|f'(x)|^q\) on \([0, \max\{a^{1/m}, b\}]\), Lemma 2.1, and Hölder inequality yields

\[
\left| \frac{b^2 f(b) - a^2 f(a)}{2} - \int_a^b x f(x) \, dx \right| \leq \frac{\ln b - \ln a}{2} \left[ \int_0^1 a^{3(1-t)} b^{3t} |f'(a^{1/m})|^q \right]^{1/q} \times \left\{ m[L(a^3, b^3) - G(\alpha, 3)] |f'(a^{1/m})|^q + G(\alpha, 3) |f'(b)|^q \right\}^{1/q}
\]

as a result, the inequality (3.1) follows. The proof of Theorem 3.1 is complete. \hfill \Box

**Corollary 3.1.1.** Under the conditions of Theorem 3.1, if \(q = 1\), then

\[
\left| \frac{b^2 f(b) - a^2 f(a)}{2} - \int_a^b x f(x) \, dx \right| \leq \frac{\ln b - \ln a}{2} \left\{ m[L(a^3, b^3) - G(\alpha, 3)] |f'(a^{1/m})| + G(\alpha, 3) |f'(b)| \right\}.
\]
Corollary 3.1.2. Under the conditions of Theorem 3.1, if $\alpha = 1$, then

$$\left| \frac{b^2 f(b) - a^2 f(a)}{2} - \int_a^b x f(x) \, dx \right| \leq \frac{(b^3 - a^3)^{1-1/q} \times m[L(a^3, b^3) - a^3] f'(a^{1/m}) |q} + [b^3 - L(a^3, b^3)] f'(b) q^{1/q}}{6}. \quad (3.5)$$

Proof. This follows from the fact that

$$G(1, 3) = \int_0^1 ta^{3(1-t)} b^{3t} \, dt = \frac{b^3 - L(a^3, b^3)}{3(\ln b - \ln a)}.$$

The proof of Corollary 3.1.2 is complete. \qed

Corollary 3.1.3. Under the conditions of Theorem 3.1, we have

$$\left| \frac{b^2 f(b) - a^2 f(a)}{2} - \int_a^b x f(x) \, dx \right| \leq \frac{\ln b - \ln a}{2} [L(a^3, b^3)]^{1-1/q}$$

$$\times \left( \frac{1}{\alpha + 1} \right)^{1/q} \left\{ m [(\alpha + 1)L(a^3, b^3) - b^3] f'(a^{1/m}) q^q + b^3 q f'(b) q^{1/q} \right\}. \quad (3.6)$$

and

$$\left| \frac{b^2 f(b) - a^2 f(a)}{2} - \int_a^b x f(x) \, dx \right| \leq \frac{\ln b - \ln a}{2} L(a^3, b^3) |f'(b)|. \quad (3.7)$$

Proof. Using $\left( \frac{b}{a} \right)^3 \leq \left( \frac{b}{a} \right)$ for $t \in [0, 1]$ in (3.2) gives

$$G(\alpha, 3) = a^3 \int_0^1 t^\alpha \left( \frac{b}{a} \right)^{3t} \, dt \leq \frac{b^3}{\alpha + 1}.$$ Substituting this inequality into (3.1) yields (3.6). Utilizing $t^\alpha \leq 1$ for $t \in [0, 1]$ in (3.2) reveals

$$G(\alpha, 3) \leq \int_0^1 a^{3(1-t)} b^{3t} \, dt = L(a^3, b^3).$$ Combining this inequality with (3.1) yields (3.7). Corollary 3.1.3 is thus proved. \qed

Theorem 3.2. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function and $f' \in L([a, b])$ with $0 < a < b < \infty$. If $|f'|^q$ is $(\alpha, m)$-GA-convex on $[0, \max\{a^{1/m}, b\}]$ for $(\alpha, m) \in (0, 1]^2$ and $q > 1$, then

$$\left| \frac{b^2 f(b) - a^2 f(a)}{2} - \int_a^b x f(x) \, dx \right| \leq \frac{\ln b - \ln a}{2} \left( \frac{1}{\alpha + 1} \right)^{1/q}$$

$$\times \left[ L(a^{3q/(q-1)}, b^{3q/(q-1)}) \right]^{1-1/q} \left[ ||f'(b)||^q + \alpha m |f'(a^{1/m})|^q \right]^{1/q}, \quad (3.8)$$

where $L$ is defined by (3.3).

Proof. Since $|f'(x)|^q$ is $(\alpha, m)$-GA-convex on $[0, \max\{a^{1/m}, b\}]$, from Lemma 2.1 and Hölder inequality, we have

$$\left| \frac{b^2 f(b) - a^2 f(a)}{2} - \int_a^b x f(x) \, dx \right| \leq \frac{\ln b - \ln a}{2} \int_0^1 a^{3(1-t)} b^{3t} |f'(a^{1-t}b^t)| \, dt$$

$$\leq \frac{\ln b - \ln a}{2} \left[ \int_0^1 a^{3q/(q-1)(1-t)} b^{3q/(q-1)t} \, dt \right]^{1-1/q} \left[ \int_0^1 |f'(a^{1/m})^{m(1-t)} b^t|^q \, dt \right]^{1/q}$$. 

Theorem 3.2. Let \( f : \mathbb{R}_0 \to \mathbb{R} \) be a differentiable function and \( f' \in L([a, b]) \) for \( 0 < a < b < \infty \). If \( |f'|^q \) is \((\alpha, m)\)-GA-convex on \([0, \max\{a^{1/m}, b\}]\) for \( q > 1 \) and \((\alpha, m) \in (0, 1]^2\), then

\[
\begin{align*}
\frac{b^2 f(b) - a^2 f(a)}{2} - \int_a^b x f(x) \, dx &\leq \frac{\ln b - \ln a}{2} \left[ \frac{b^{3q/(q-1)} - a^{3q/(q-1)}}{\ln b^{3q/(q-1)} - \ln a^{3q/(q-1)}} \right]^{1-1/q} \\
& \times \left[ \int_0^1 (t^\alpha |f'(b)|^q + m(1-t^\alpha)|f'(a^{1/m})|^q) \, dt \right]^{1/q} \\
= \frac{\ln b - \ln a}{2} [L(a^{3q/(q-1)}, b^{3q/(q-1)})]^{1-1/q} \left[ \frac{1}{\alpha + 1} |f'(b)|^q + \frac{\alpha m}{\alpha + 1} |f'(a^{1/m})|^q \right]^{1/q}.
\end{align*}
\]

The proof of Theorem 3.2 is complete. \( \square \)

Corollary 3.3.1. Under the conditions of Theorem 3.2, if \( \alpha = 1 \), then

\[
\begin{align*}
\frac{b^2 f(b) - a^2 f(a)}{2} - \int_a^b x f(x) \, dx &\leq \frac{\ln b - \ln a}{2} \left[ \frac{b^{3q/(q-1)} - a^{3q/(q-1)}}{\ln b^{3q/(q-1)} - \ln a^{3q/(q-1)}} \right]^{1-1/q} \\
& \times \left[ \int_0^1 (t^\alpha |f'(b)|^q + m(1-t^\alpha)|f'(a^{1/m})|^q) \, dt \right]^{1/q} \\
= \frac{\ln b - \ln a}{2} [L(a^{3q/(q-1)}, b^{3q/(q-1)})]^{1-1/q} \left[ \frac{1}{\alpha + 1} |f'(b)|^q + \frac{\alpha m}{\alpha + 1} |f'(a^{1/m})|^q \right]^{1/q}.
\end{align*}
\]
**Theorem 3.4.** Let \( f : \mathbb{R}_0 \to \mathbb{R} \) be a differentiable function and \( f' \in L([a, b]) \) for \( 0 < a < b < \infty \). If \( |f'|^q \) is \((\alpha, m)\)-GA-convex on \([0, \max\{a^{1/m}, b\}]\) for \( q > 1 \), \( q > p > 0 \), and \((\alpha, m) \in (0, 1)^2\), then

\[
\left| \frac{b^2 f(b) - a^2 f(a)}{2} - \int_a^b x f(x) \, dx \right| \leq \frac{\ln b - \ln a}{2} \left[ L(a^{3(q-p)/(q-1)}, b^{3(q-p)/(q-1)}) \right]^{1-1/q}
\]

\[
\times \left\{ m \left[ L(a^{3p}, b^{3p}) - G(\alpha, 3p) \right] |f'(a^{1/m})|^q + G(\alpha, 3p) |f'(b)|^q \right\}^{1/q},
\]

where \( G \) and \( L \) are respectively defined by (3.2) and (3.3).

**Proof.** Since \( |f'(x)|^q \) is \((\alpha, m)\)-GA-convex on \([0, \max\{a^{1/m}, b\}]\), from Lemma 2.1 and H"older inequality, we have

\[
\left| \frac{b^2 f(b) - a^2 f(a)}{2} - \int_a^b x f(x) \, dx \right| \leq \frac{\ln b - \ln a}{2} \left[ \int_0^1 a^{3(q-p)/(q-1)(1-t)} b^{3(q-p)/(q-1)t} \, dt \right]^{1-1/q}
\]

\[
\times \left[ \int_0^1 a^{3p(1-t)} b^{3p(1-t)} \left( f' \left( \left( a^{1/m} \right)^{m(1-t)} b^t \right) \right)^q \, dt \right]^{1/q}
\]

\[
\leq \frac{\ln b - \ln a}{2} \left[ \frac{b^{3(q-p)/(q-1)} - a^{3(q-p)/(q-1)}}{\ln b^{3(q-p)/(q-1)} - \ln a^{3(q-p)/(q-1)}} \right]^{1-1/q}
\]

\[
\times \left[ \int_0^1 a^{3p(1-t)} b^{3p(1-t)} \left( t^{\alpha} |f'(b)|^q + m(1-t^{\alpha}) |f'(a^{1/m})|^q \right) \, dt \right]^{1/q}
\]

\[
= \frac{\ln b - \ln a}{2} \left[ L(a^{3(q-p)/(q-1)}, b^{3(q-p)/(q-1)}) \right]^{1-1/q}
\]

\[
\times \left\{ m \left[ L(a^{3p}, b^{3p}) - a^{3p} \right] |f'(a^{1/m})|^q + b^{3p} - L(a^{3p}, b^{3p}) \right\}^{1/q}.\]

The proof of Theorem 3.4 is complete. \( \square \)

**Corollary 3.4.1.** Under the conditions of Theorem 3.4, if \( \alpha = 1 \), then

\[
\left| \frac{b^2 f(b) - a^2 f(a)}{2} - \int_a^b x f(x) \, dx \right| \leq \frac{\ln b - \ln a}{2} \left( \frac{1}{3p} \right)^{1/q}
\]

\[
\times \left[ L(a^{3(q-p)/(q-1)}, b^{3(q-p)/(q-1)}) \right]^{1-1/q}
\]

\[
\times \left\{ m \left[ L(a^{3p}, b^{3p}) - a^{3p} \right] |f'(a^{1/m})|^q + b^{3p} - L(a^{3p}, b^{3p}) \right\}^{1/q}.\]

**Proof.** By

\[
G(1, 3p) = \int_0^1 ta^{3p(1-t)} b^{3pt} \, dt = \frac{b^{3p} - L(a^{3p}, b^{3p})}{\ln b^{3p} - \ln a^{3p}},
\]

Corollary 3.4.1 can be proved easily. \( \square \)

**Theorem 3.5.** Let \( f, g : \mathbb{R}_0 \to \mathbb{R}_0 \) and \( fg \in L([a, b]) \) for \( 0 < a < b < \infty \). If \( f^q(x) \) is \((\alpha_1, m_1)\)-GA-convex on \([0, \max\{a^{1/m_1}, b\}]\) and \( g^q(x) \) is \((\alpha_2, m_2)\)-GA-convex on \([0, \max\{a^{1/m_2}, b\}]\) for \( q > 1 \), \( (\alpha_1, m_1) \), and \((\alpha_2, m_2) \in (0, 1]^2\), then

\[
\int_a^b f(x) g(x) \, dx \leq (\ln b - \ln a)[L(a, b)]^{1-1/q} \left\{ m_1 m_2 [L(a, b) - G(\alpha_1, 1) - G(\alpha_2, 1) + G(\alpha_1 + \alpha_2, 1)] f^q(a^{1/m_1}) g^q(a^{1/m_2}) + m_1 G(a, 1) - G(\alpha_1, 1) - G(\alpha_1 + \alpha_2, 1)] f^q(a^{1/m_1}) g^q(b)
\]
where \( G \) and \( L \) are respectively defined by (3.2) and (3.3).

**Proof.** Using the \( (\alpha_1, m_1) \)-GA-convexity of \( f^q(x) \) and the \( (\alpha_2, m_2) \)-GA-convexity of \( g^q(x) \), we have

\[
f^q(a^{1-t}b^t) \leq t^{\alpha_1} f^q(b) + m_1(1 - t^{\alpha_1}) f^q(a^{1/m_1})
\]

and

\[
g^q(a^{1-t}b^t) \leq t^{\alpha_2} g^q(b) + m_2(1 - t^{\alpha_2}) g^q(a^{1/m_2})
\]

for \( 0 \leq t \leq 1 \). Letting \( x = a^{1-t}b^t \) for \( 0 \leq t \leq 1 \) and using Hölder’s inequality figure out

\[
\int_a^b f(x)g(x) \, dx = (\ln b - \ln a) \int_0^1 a^{1-t}b^t f(a^{1-t}b^t) g(a^{1-t}b^t) \, dt
\]

\[
\leq (\ln b - \ln a) \left( \int_0^1 a^{1-t}b^t dt \right)^{1-1/q} \left\{ \int_0^1 a^{1-t}b^t [f(a^{1-t}b^t)g(a^{1-t}b^t)]^q \, dt \right\}^{1/q}
\]

\[
\leq (\ln b - \ln a) \left( \int_0^1 a^{1-t}b^t dt \right)^{1-1/q} \left\{ \int_0^1 a^{1-t}b^t [t^{\alpha_1} f^q(b)
\]

\[
+ m_1(1 - t^{\alpha_1}) f^q(a^{1/m_1}) [t^{\alpha_2} g^q(b) + m_2(1 - t^{\alpha_2}) g^q(a^{1/m_2})] \, dt \right\}^{1/q}
\]

\[
= (\ln b - \ln a)[L(a, b)]^{1-1/q} \left\{ \int_0^1 a^{1-t}b^t [t^{\alpha_1} + m_1(1 - t^{\alpha_1})] f^q(a^{1/m_1})g^q(a^{1/m_2}) \, dt \right\}^{1/q}
\]

\[
+ m_2(1 - t^{\alpha_1})(1 - t^{\alpha_2}) f^q(a^{1/m_1})g^q(a^{1/m_2})] \, dt \right\}^{1/q}
\]

\[
= (\ln b - \ln a)[L(a, b)]^{1-1/q} \left\{ m_1 m_2[L(a, b) - G(\alpha_1, 1)
\]

\[
+ G(\alpha_2, 1) + G(\alpha_1, \alpha_2, 1)] f^q(a^{1/m_1})g^q(a^{1/m_2}) \right\}^{1/q}
\]

\[+ m_1[G(\alpha_2, 1) - G(\alpha_1, \alpha_2, 1)] f^q(a^{1/m_1})g^q(a^{1/m_2}) \right\}^{1/q}.
\]

The proof of Theorem 3.5 is complete. \( \square \)

**Corollary 3.5.1.** Under the conditions of Theorem 3.5,

1. if \( q = 1 \), then

\[
\int_a^b f(x)g(x) \, dx \leq (\ln b - \ln a) \left\{ m_1 m_2[L(a, b) - G(\alpha_1, 1) - G(\alpha_2, 1)
\]

\[
+ G(\alpha_1, \alpha_2, 1)] f^q(a^{1/m_1})g^q(a^{1/m_2}) + m_1[G(\alpha_2, 1) - G(\alpha_1, \alpha_2, 1)] f^q(a^{1/m_1})g(b)
\]

\[
+ m_2[G(\alpha_1, 1) - G(\alpha_2, 1)] f^q(b)g^q(a^{1/m_2}) + G(\alpha_1, \alpha_2, 1) f^q(b)g^q(b) \right\}, \quad (3.15)
\]

2. if \( q = 1 \) and \( \alpha_1 = \alpha_2 = m_1 = m_2 = 1 \), then

\[
\int_a^b f(x)g(x) \, dx \leq \frac{1}{\ln b - \ln a} \left\{ [2L(a, b) - a(\ln b - \ln a) - 2a] f(a)g(a) + [a + b}
\]

\[
- a^2 \ln b + b^2 \ln a + 2ab - a^2 - b^2 \right\}.
\]
(3) if \( \alpha_1 = \alpha_2 = m_1 = m_2 = 1 \), then
\[
\int_a^b f(x)g(x) \, dx \leq \frac{[L(a, b)]^{1 - 1/q}}{(\ln b - \ln a)^{2/(q-1)}} \left\{ [2L(a, b) - a(\ln b - \ln a) - 2a]f^q(a) g^q(a) + [a + b - 2L(a, b)][f^q(a) g^q(b) + f^q(b) g^q(a)]ight. \\
\left. + [2L(a, b) + b(\ln b - \ln a) - 2b]f^q(b) g^q(b) \right\}^{1/q}. 
\]

**Theorem 3.6.** Let \( f, g : \mathbb{R}_0 \to \mathbb{R}_0 \) and \( f, g \in L([a, b]) \) for \( 0 < a < b < \infty \). If \( f^q(x) \) is \((\alpha_1, m_1)\)-GA-convex on \([0, \max\{a^{1/m_1}, b\}]\) and \( g^{q/(q-1)}(x) \) is \((\alpha_2, m_2)\)-GA-convex on \([0, \max\{a^{1/m_2}, b\}]\) for \( q > 1 \), \((\alpha_1, m_1)\), and \((\alpha_2, m_2)\) \in \((0, 1]^2\), then
\[
\int_a^b f(x)g(x) \, dx \leq (\ln b - \ln a) \left\{ m_1 f^q(a^{1/m_1}) L(a, b) \\
+ G(\alpha_1, 1) [f^q(b) - m_1 f^q(a^{1/m_1})] \right\}^{1/q} \left\{ m_2 g^{q/(q-1)}(a^{1/m_2}) L(a, b) \\
+ G(\alpha_2, 1) [g^{q/(q-1)}(b) - m_2 g^{q/(q-1)}(a^{1/m_2})] \right\}^{1 - 1/q},
\]
where \( G \) and \( L \) are respectively defined by (3.2) and (3.3).

**Proof.** By the \((\alpha_1, m_1)\)-GA-convexity of \( f^q(x) \) and the \((\alpha_2, m_2)\)-GA-convexity of \( g^{q/(q-1)}(x) \), we have
\[
f^q(a^{-t}b^t) \leq t^{\alpha_1} f^q(b) + m_1 (1 - t^{\alpha_1}) f^q(a^{1/m_1})
\]
and
\[
g^{q/(q-1)}(a^{-1} b^t) \leq t^{\alpha_2} g^{q/(q-1)}(b) + m_2 (1 - t^{\alpha_2}) g^{q/(q-1)}(a^{1/m_2})
\]
for \( t \in [0, 1] \). Letting \( x = a^{1-t} b^t \) for \( 0 \leq t \leq 1 \) and employing Hölder’s inequality yield
\[
\int_a^b f(x)g(x) \, dx \leq \left\{ \int_a^b f^q(x) \, dx \right\}^{1/q} \left\{ \int_a^b g^{q/(q-1)}(x) \, dx \right\}^{1 - 1/q} \\
= (\ln b - \ln a) \left\{ \int_0^1 a^{-t} b^t f^q(a^{-1} b^t) \, dt \right\}^{1/q} \left\{ \int_0^1 a^{-1} b^t g^{q/(q-1)}(a^{1-t} b^t) \, dt \right\}^{1 - 1/q} \\
\leq (\ln b - \ln a) \left\{ \int_0^1 a^{-t} b^t \left[ t^{\alpha_1} f^q(b) + m_1 (1 - t^{\alpha_1}) f^q(a^{1/m_1}) \right] \, dt \right\}^{1/q} \\
\times \left[ \int_0^1 a^{-1} b^t \left[ t^{\alpha_2} g^{q/(q-1)}(b) + m_2 (1 - t^{\alpha_2}) g^{q/(q-1)}(a^{1/m_2}) \right] \, dt \right\}^{1 - 1/q} \\
= (\ln b - \ln a) \left\{ m_1 f^q(a^{1/m_1}) L(a, b) + G(\alpha_1, 1) [f^q(b) - m_1 f^q(a^{1/m_1})] \right\}^{1/q} \\
\times \left\{ m_2 g^{q/(q-1)}(a^{1/m_2}) L(a, b) \\
+ G(\alpha_2, 1) [g^{q/(q-1)}(b) - m_2 g^{q/(q-1)}(a^{1/m_2})] \right\}^{1 - 1/q}.
\]
The proof of Theorem 3.6 is complete. \( \square \)

**Corollary 3.6.1.** Under the conditions of Theorem 3.6, if \( \alpha_1 = \alpha_2 = m_1 = m_2 = 1 \), then
\[
\int_a^b f(x)g(x) \, dx \leq \{ f^q(b) L(a, b) - a \} + \{ b - L(a, b) f^q(b) \}^{1/q}
\]
Under the conditions of Theorem 3.7, if Corollary 3.7.1. The proof of Theorem 3.7 is complete. 

\[ \int_a^b f(x)g(x) \, dx \geq (\ln b - \ln a) \{ m_1 m_2[L(a, b) - G(\alpha_1, 1) - G(\alpha_2, 1) \\
+ G(\alpha_1 + \alpha_2, 1)]f(a^{1/\alpha_1})g(a^{1/\alpha_2}) + m_1[G(\alpha_2, 1) - G(\alpha_1 + \alpha_2, 1)]f(a^{1/\alpha_1})g(b) \\
+ m_2[G(\alpha_1, 1) - G(\alpha_1 + \alpha_2, 1)]f(b)g(a^{1/\alpha_2}) G + G(\alpha_1 + \alpha_2, 1)f(b)g(a^{1/\alpha_2}) \} \] 

where \( G \) and \( L \) are respectively defined by (3.2) and (3.3).

**Proof.** Since \( f(x) \) is \((\alpha_1, m_1)\)-GA-concave on \([0, \max\{a^{1/m_1}, b\}]\) and \( g(x) \) is \((\alpha_2, m_2)\)-GA-concave on \([0, \max\{a^{1/m_2}, b\}]\), we have 

\[ f(a^{-1-t}b^t) \geq t^{\alpha_1} f(b) + m_1(1 - t^{\alpha_1}) f(a^{1/\alpha_1}) \]

and 

\[ g(a^{-1-t}b^t) \geq t^{\alpha_2} g(b) + m_2(1 - t^{\alpha_2}) g(a^{1/\alpha_2}) \]

for \( t \in [0, 1] \). Further letting \( x = a^{-1-t}b^t \) for \( 0 \leq t \leq 1 \) and utilizing Hölder’s inequality reveal 

\[ \int_a^b f(x)g(x) \, dx \geq (\ln b - \ln a) \left\{ \int_0^1 a^{-1-t}b^t f(a^{-1-t}b^t) g(a^{-1-t}b^t) \, dt \right\} \]

\[ \geq (\ln b - \ln a) \left\{ \int_0^1 a^{-1-t}b^t \left[ t^{\alpha_1} f(b) + m_1(1 - t^{\alpha_1}) f(a^{1/\alpha_1}) \right] \right. \times \left. \left[ t^{\alpha_2} g(b) + m_2(1 - t^{\alpha_2}) g(a^{1/\alpha_2}) \right] \, dt \right\} \]

\[ = (\ln b - \ln a) \int_0^1 a^{-1-t}b^t \left[ t^{\alpha_1+\alpha_2} f(b)g(b) + m_1(1 - t^{\alpha_1}) t^{\alpha_2} f(a^{1/\alpha_1})g(b) \\
+ m_2 t^{\alpha_1}(1 - t^{\alpha_2}) f(b)g(a^{1/\alpha_2}) + m_1 m_2(1 - t^{\alpha_1})(1 - t^{\alpha_2}) g(a^{1/\alpha_2}) f(a^{1/\alpha_1}) \right] \, dt \]

The proof of Theorem 3.7 is complete. \qed

**Corollary 3.7.1.** Under the conditions of Theorem 3.7, if \( \alpha_1 = \alpha_2 = m_1 = m_2 = 1 \), we have 

\[ \int_a^b f(x)g(x) \, dx \geq (\ln b - \ln a) \left\{ [2L(a, b) - a(\ln b - \ln a) - 2a] f(a)g(a) + [a + b \\
- 2L(a, b)] [f(a)g(b) + f(b)g(a)] + [2L(a, b) + b(\ln b - \ln a) - 2b] f(b)g(b) \right\} \] (3.21)
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