LOWER BOUNDS FOR NODAL SETS OF DIRICHLET AND NEUMANN EIGENFUNCTIONS

SINAN ARITURK

Abstract. Let $\varphi$ be a Dirichlet or Neumann eigenfunction of the Laplace-Beltrami operator on a compact Riemannian manifold with boundary. We prove lower bounds for the size of the nodal set $\{\varphi = 0\}$.

1. Introduction

Let $(M,g)$ be a compact smooth Riemannian manifold with boundary. Let $\Delta$ be the Laplace-Beltrami operator. Let $\lambda \geq 1$ and let $\varphi$ be an eigenfunction of $-\Delta$, i.e. a smooth real-valued function on $M$ with

$$-\Delta \varphi = \lambda \varphi$$

over the interior of $M$. We will assume that $\varphi$ is a Dirichlet eigenfunction, meaning

$$\varphi \bigg|_{\partial M} = 0$$

or a Neumann eigenfunction, meaning

$$\partial_{\nu} \varphi \bigg|_{\partial M} = 0$$

where $\nu$ is the outward unit normal vector on $\partial M$ and $\partial_{\nu}$ is the corresponding directional derivative. Define the nodal set

$$Z = \left\{ x \in M : \varphi(x) = 0, x \notin \partial M \right\}$$

Let $n$ be the dimension of $M$ and let $\mathcal{H}$ be the $(n-1)$-dimensional Hausdorff measure on $M$. We will prove lower bounds for $\mathcal{H}(Z)$.

We use the notation $A \lesssim B$ to mean there is a positive constant $C$, independent of $\lambda$ and $\varphi$, such that $A \leq CB$.

Theorem 1.1. If $\varphi$ is a Neumann eigenfunction, then

$$\lambda^{\frac{n-2}{n}} \lesssim \mathcal{H}(Z)$$

If $\varphi$ is a Dirichlet eigenfunction and $n \leq 3$, then

$$\lambda^{\frac{n-2}{n}} \lesssim \mathcal{H}(Z)$$

If the boundary is strictly geodesically concave and $\varphi$ is a Dirichlet eigenfunction, then for $n \leq 4$,

$$\lambda^{\frac{n-4}{n}} \lesssim \mathcal{H}(Z)$$
If \((M, g)\) is a compact real analytic Riemannian manifold with boundary, then Donnelly and Fefferman \([2]\) proved that
\[
\lambda^{1/2} \lesssim \mathcal{H}(Z) \lesssim \lambda^{1/2}
\]
If \((M, g)\) is a compact smooth Riemannian manifold without boundary, then Colding and Minicozzi \([1]\) proved that
\[
\lambda^{\frac{2}{n}} \lesssim \mathcal{H}(Z)
\]
This same result was later obtained by Hezari and Sogge \([6]\). Their argument was based on the identity
\[
\lambda \int_M |\varphi| \, dV = 2 \int_Z |\nabla \varphi| \, dS
\]
where \(dV\) is the Riemannian volume measure and \(dS\) is the Riemannian surface measure on \(Z\). This identity had been proven by Sogge and Zelditch \([10]\), who also showed that
\[
\lambda^{-\frac{n+1}{n}} \lesssim \int_M |\varphi| \, dV
\]
Hezari and Sogge \([6]\) proved that
\[
\int_Z |\nabla \varphi|^2 \, dS \lesssim \lambda^{3/2}
\]
and then used (1.2), (1.3), and (1.4) to obtain the bound (1.1).

We will prove analogues of (1.2), (1.3), and (1.4) for a compact smooth Riemannian manifold with boundary. This will enable us to establish Theorem 1.1. In particular, we will prove the following.

**Theorem 1.2.** If \(\varphi\) is a Dirichlet or Neumann eigenfunction, then
\[
\lambda \int_M |\varphi| \, dV = \int_{\partial M} |\partial_n \varphi| \, dS + 2 \int_Z |\nabla \varphi| \, dS
\]
More generally, for any function \(f\) in \(C^2(M)\),
\[
\int_M \left( (\Delta + \lambda) f \right) |\varphi| \, dV = \int_{\partial M} f |\partial_n \varphi| \, dS + \int_{\partial M} |\varphi| |\partial_n f| \, dS + 2 \int_Z f |\nabla \varphi| \, dS
\]
For a Neumann eigenfunction, the first term on the right side is zero, and this identity is the same as (1.2). For a Dirichlet eigenfunction, the integral over \(\partial M\) is an additional obstacle and causes the argument to break down in higher dimensions.

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## 2. Proofs

Define
\[
P = \left\{ x \in M : \varphi(x) > 0, x \notin \partial M \right\}
\]
and
\[
N = \left\{ x \in M : \varphi(x) < 0, x \notin \partial M \right\}
\]
We can write \(M\) as a disjoint union
\[
M = P \cup N \cup \partial M \cup Z
\]
Define
\[ \Omega = \left\{ x \in M : \varphi(x) = 0 \right\} \]
and
\[ \Sigma = \left\{ x \in \Omega : \nabla \varphi(x) = 0 \right\} \]

**Lemma 2.1.** If \( \varphi \) is a Dirichlet or Neumann eigenfunction, then \( \mathcal{H}(\Omega) < \infty \), and the Hausdorff dimension of \( \Sigma \) is at most \( n - 2 \). If \( \varphi \) is a Neumann eigenfunction, then the Hausdorff dimension of \( \Omega \cap \partial M \) is at most \( n - 2 \).

**Proof.** Fix a point \( p \) in \( M \). To prove the first statement, it suffices to find a neighborhood \( U \) of \( p \) in \( M \) such that \( \mathcal{H}(\Omega \cap U) < \infty \) and \( \Sigma \cap U \) has Hausdorff dimension at most \( n - 2 \).

If \( \varphi(p) \neq 0 \), then finding such a neighborhood \( U \) is trivial. So we assume \( \varphi(p) = 0 \). By Donnelly and Fefferman [2], the eigenfunction \( \varphi \) only vanishes to finite order at \( p \). If \( p \) is in the interior of \( M \), we use geodesic normal coordinates about \( p \). Then by Hardt and Simon [5], we can obtain \( U \).

If \( p \) is on the boundary \( \partial M \), then we use boundary normal coordinates \((x_1, \ldots, x_n)\) about \( p \). These are defined by first letting \((x_1, \ldots, x_{n-1})\) be geodesic normal coordinates on \( \partial M \) about \( p \), with respect to the metric on \( \partial M \) induced by \( g \). Then for fixed \( x_1, \ldots, x_{n-1} \), the curves \( x_n \to (x_1, \ldots, x_n) \), for \( x_n \geq 0 \), are geodesics in \( M \) which intersect \( \partial M \) normally.

These coordinates are well-defined near \( p \) and allow us to identify some neighborhood of \( p \) with \( B_+ = \left\{ x \in \mathbb{R}^n : |x| < \varepsilon, x_n \geq 0 \right\} \) for some small \( \varepsilon > 0 \). Here the point \( p \) is being identified with the origin in \( \mathbb{R}^n \). Let \( g_{ij} \) be the Riemannian metric on \( B_+ \). Let

\[ B = \left\{ x \in \mathbb{R}^n : |x| < \varepsilon \right\} \]

We extend the metric \( g_{ij} \) to \( B \) so that it is even in the \( x_n \)-variable. Let \( g^{ij} \) be the metric, defined so that the matrix \([g^{ij}]\) is the inverse matrix of \([g_{ij}]\). Define

\[ J = \left( \det[g_{ij}] \right)^{1/2} \]

The functions \( g_{ij}, g^{ij}, \) and \( J \) are Lipschitz continuous and bounded on \( B \). If \( \varphi \) is a Dirichlet eigenfunction, extend \( \varphi \) to \( B \) so that it is odd in the \( x_n \)-variable. If \( \varphi \) is a Neumann eigenfunction, extend \( \varphi \) to \( B \) so that it is even in the \( x_n \)-variable. Then the extended function \( \varphi \) is in \( C^1(B) \cap H^2(B) \). Let \( \psi \) be a smooth function on \( \mathbb{R}^2 \) with compact support contained strictly inside \( B \). By Green’s identity,

\[ \sum_{i,j=1}^n \int_B (D_j \varphi)(D_i \psi) J g^{ij} \, dx = \int_B \lambda \varphi \psi J \, dx \]

That is,

\[ \left( \sum_{i,j=1}^n D_i J g^{ij} D_j \varphi \right) + \lambda J \varphi = 0 \]

We can write this equation as

\[ \left( \sum_{i,j=1}^n J g^{ij} D_i D_j \varphi + (D_i J g^{ij}) D_j \varphi \right) + \lambda J \varphi = 0 \]
Now by Hardt and Simon [5], we can obtain $U$.

It remains to prove the second statement. Fix a point $p$ in $(\Omega \setminus \Sigma) \cap \partial M$. It suffices to show that there is a neighborhood $V$ of $p$ in $\partial M$ such that the Hausdorff dimension of $(\Omega \setminus \Sigma) \cap V$ is at most $n - 2$. The set $\Omega \setminus \Sigma$ is a hypersurface with normal vector $\nabla \varphi(p)$ at $p$. Since $\varphi$ is a Neumann eigenfunction and $\nabla \varphi(p) \neq 0$, the sets $\Omega \setminus \Sigma$ and $\partial M$ intersect transversally, which yields $V$. □

In particular, it follows that $\partial P$ is smooth almost everywhere, with respect to $\mathcal{H}$, so the divergence theorem and Green’s identities hold on $P$. See, e.g., Evans and Gariepy [3]. Let $\eta$ be the outward unit normal on $\partial P$, defined at these smooth points, and let $\partial \eta$ be the corresponding directional derivative. On $Z \setminus \Sigma$, we have

$$
\eta = -\frac{\nabla \varphi}{|\nabla \varphi|}
$$

At any point on $\partial M \cap \partial P$ where $\eta$ is defined, we have

$$
\eta = \nu
$$

**Proof of Theorem 1.2.** By Green’s identity,

$$
\int_P (\Delta + \lambda) f |\varphi| dV = \int_P (\Delta + \lambda) f \varphi dV
$$

$$
= \int_P f (\Delta + \lambda) \varphi dV - \int_{\partial P} f \partial_{\eta} \varphi dS + \int_{\partial P} \varphi \partial_{\eta} f dS
$$

$$
= -\int_{\partial P \cap \partial M} f \partial_{\eta} \varphi dS - \int_Z f \partial_{\eta} \varphi dS + \int_{\partial P \cap \partial M} \varphi \partial_{\eta} f dS
$$

$$
= \int_{\partial P \cap \partial M} f |\partial_{\nu} \varphi| dS + \int_Z f |\nabla \varphi| dS + \int_{\partial P \cap \partial M} |\varphi| \partial_{\nu} f dS
$$

The last equality holds because $-\partial_{\eta} \varphi = |\partial_{\nu} \varphi|$ over $\partial P \cap M$ and $-\partial_{\eta} \varphi = |\nabla \varphi|$ over $\partial P \cap M$. We can similarly obtain

$$
\int_N (\Delta + \lambda) f |\varphi| dV = \int_{\partial N \cap M} f |\partial_{\nu} \varphi| dS + \int_Z f |\nabla \varphi| dS + \int_{\partial N \cap M} |\varphi| \partial_{\nu} f dS
$$

Now

$$
\int_M (\Delta + \lambda) f |\varphi| dV = \int_P (\Delta + \lambda) f |\varphi| dV + \int_N (\Delta + \lambda) f |\varphi| dV
$$

$$
= \int_{\partial M} f |\partial_{\nu} \varphi| dS + \int_{\partial M} |\varphi| \partial_{\nu} f dS + 2 \int_Z f |\nabla \varphi| dS
$$

□

The following lemma is an analogue of (1.3).

**Lemma 2.2.** If $\varphi$ is a Dirichlet or a Neumann eigenfunction, then

$$
\lambda_{\text{Dir}} \lesssim ||\varphi||_{L^1(M)}
$$

If the boundary is strictly geodesically concave and $\varphi$ is a Dirichlet eigenfunction, then

$$
\lambda_{\text{Dir}}^{1-n} \lesssim ||\varphi||_{L^1(M)}
$$
Proof. Fix $p$ satisfying $2 < p < \frac{2(n+1)}{n-1}$. Then, by Smith [7],

$$\|\varphi\|_{L^p(M)} \lesssim \lambda^{\frac{(n-1)(p-2)}{2p}}$$

(2.1)

If the boundary is strictly geodesically concave and $\varphi$ is a Dirichlet eigenfunction, then by Grieser [4] and Smith-Sogge [8],

$$\|\varphi\|_{L^p(M)} \lesssim \lambda^{\frac{(n-1)(p-2)}{2p}}$$

(2.1)

Let $\theta = \frac{p-2}{2(p-1)}$. By Hölder’s inequality,

$$1 = \|\varphi\|_{L^2(M)} \leq \|\varphi\|_{L^\theta(M)} \|\varphi\|^{1-\theta}_{L^p(M)}$$

The estimates now follow. \qed

Remark. On the flat unit disc $\{|x| \leq 1\}$ in $\mathbb{R}^2$, there are whispering gallery modes, which are concentrated near the boundary. It follows from Grieser [4] that Lemma 2.2 is sharp for these eigenfunctions. However, for $n \geq 3$, Smith and Sogge [9] conjectured that 2.1 can be strengthened to

$$\|\varphi\|_{L^p(M)} \lesssim \lambda^{\frac{(3n-2)(p-2)}{24p}}$$

(2.2)

Applying Hölder’s inequality as above would then yield

$$\lambda^\frac{3n-2}{24n} \lesssim \|\varphi\|_{L^1(M)}$$

The following lemma is an analogue of (1.4).

Lemma 2.3. If $\varphi$ is a Dirichlet or Neumann eigenfunction, then

$$\int_Z |\nabla \varphi|^2 dS \lesssim \lambda^{3/2}$$

Proof. This will follow from the identity

$$-\int_M \text{sgn} (\varphi) \text{div} (|\nabla \varphi| \nabla \varphi) dV = \int_{\partial M} |\partial_\nu \varphi|^2 dS + 2 \int_Z |\nabla \varphi|^2 dS$$

We first prove this identity. Note that $-\partial_\eta \varphi = |\nabla \varphi|$ over $Z \setminus \Sigma$. If $\varphi$ is a Dirichlet eigenfunction, then we also have $|\nabla \varphi| = -\partial_\eta \varphi = |\partial_\nu \varphi|$ at any point on $\partial P \cap \partial M$ where $\eta$ is defined. By the divergence theorem,

$$-\int_P \text{div} (|\nabla \varphi| \nabla \varphi) dV = -\int_{\partial P} |\nabla \varphi| \partial_\eta \varphi dS$$

$$= \int_{\partial P \cap \partial M} |\partial_\nu \varphi|^2 dS + \int_Z |\nabla \varphi|^2 dS$$

Similarly,

$$\int_N \text{div} (|\nabla \varphi| \nabla \varphi) dV = \int_{\partial N \cap \partial M} |\partial_\nu \varphi|^2 dS + \int_Z |\nabla \varphi|^2 dS$$

Adding these equations establishes the identity. Now we have

$$\int_Z |\nabla \varphi|^2 dS \lesssim \int_M \text{div} (|\nabla \varphi| \nabla \varphi) dV$$

$$\lesssim \|\varphi\|_{H^2(M)} \|\varphi\|_{H^1(M)}$$

$$\lesssim \lambda^{3/2}$$

\qed
For a Dirichlet eigenfunction, we also need the following lemma.

**Lemma 2.4.** If \( \varphi \) is a Dirichlet eigenfunction, then

\[
\left( \int_{\partial M} |\partial_\nu \varphi|^2 dS \right)^{1/2} \lesssim \lambda^{1/2}
\]

This lemma follows from a much more general result obtained by Tataru [11]. There is also the following short proof.

**Proof.** Let \( X \) be a smooth first-order differential operator on \( M \) with \( X = \partial_\nu \) over \( \partial M \). Then, by Green’s identity,

\[
\int_M u[X, \Delta] u \, dV = -\lambda \int_M uX u \, dV - \int_M u \Delta X u \, dV
\]

\[
= \int_M (\Delta u)(X u) \, dV - \int_M u \Delta X u \, dV
\]

\[
= \int_{\partial M} (\partial_\nu u)(X u) \, dS
\]

\[
= \int_{\partial M} |\partial_\nu u|^2 \, dS
\]

Since \([X, \Delta]\) is a second-order differential operator, this yields

\[
\int_{\partial M} |\partial_\nu u|^2 \, dS = \int_M u[X, \Delta] u \, dV
\]

\[
\lesssim \|u\|_{L^2(M)}^2 \|u\|_{H^2(M)}
\]

\[
\lesssim \lambda
\]

□

We can now prove Theorem 1.1.

**Proof of Theorem 1.1.** First assume \( \varphi \) is a Neumann eigenfunction. By Theorem 1.2 and Lemma 2.3,

\[
\lambda \int_M |\varphi| \, dV = 2 \int_Z |\nabla \varphi| \, dS \lesssim \mathcal{H}(Z)^{1/2} \lambda^{3/4}
\]

We can rewrite this as

\[
\lambda^{1/2} \left( \int_M |\varphi| \, dV \right)^2 \lesssim \mathcal{H}(Z)
\]

So by Lemma 2.2,

\[
\lambda^{\frac{n-2}{2}} \lesssim \mathcal{H}(Z)
\]

Now assume \( \varphi \) is a Dirichlet eigenfunction. By Theorem 1.2, Lemma 2.3, and Lemma 2.4,

\[
\lambda \int_M |\varphi| \, dV = \int_{\partial M} |\partial_\nu \varphi| \, dS + 2 \int_Z |\nabla \varphi| \, dS \lesssim \lambda^{1/2} + \mathcal{H}(Z)^{1/2} \lambda^{3/4}
\]

We can rewrite this as

\[
\lambda^{1/2} \left( \int_M |\varphi| \, dV \right)^2 \lesssim \mathcal{H}(Z) + \lambda^{-1/2}
\]

Now applying Lemma 2.2 yields the desired estimates. □
Remark. If (2.2) is true, then we would have a better lower bound for the \( L^1 \) norm of \( \varphi \). If \( \varphi \) is a Neumann eigenfunction, this would yield

\[
\lambda^{\frac{3n}{12}} \lesssim \mathcal{H}(Z)
\]

The same bound would hold if \( \varphi \) is a Dirichlet eigenfunction and \( n \leq 4 \).

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