A Number Theoretic Interpolation Between Quantum and Classical Complexity Classes

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Abstract

We reveal a natural algebraic problem whose complexity appears to interpolate between the well-known complexity classes \( \text{BQP} \) and \( \text{NP} \):

\( \ast \) Decide whether a univariate polynomial with exactly \( m \) monomial terms has a \( p \)-adic rational root.

In particular, we show that while \((\ast)\) is doable in quantum randomized polynomial time when \( m = 2 \) (and no classical randomized polynomial time algorithm is known), \((\ast)\) is nearly \( \text{NP} \)-hard for general \( m \): Under a plausible hypothesis involving primes in arithmetic progression (implied by the Generalized Riemann Hypothesis for certain cyclotomic fields), a randomized polynomial time algorithm for \((\ast)\) would imply the widely disbelieved inclusion \( \text{NP} \subseteq \text{BPP} \). This type of quantum/classical interpolation phenomenon appears to new.

1 Introduction and Main Results

Thanks to quantum computation, we now have exponential speed-ups for important practical problems such as Integer Factoring and Discrete Logarithm \[\text{Sho97}\]. However, a fundamental open question that remains is whether there are any \( \text{NP-complete} \) problems admitting exponential speed-ups via quantum computation. Succinctly, this is the \( \text{NP} \subseteq \text{BQP} \) question, and a positive answer would imply that quantum computation can provide efficient algorithms for a myriad of problems that have occupied practitioners in optimization and computer science for decades \[\text{BV97}\]. (The classic reference \[\text{GJ79}\] lists dozens of such problems, and we briefly review the aforementioned complexity classes in Section 2 below.) However, the truth of the inclusion \( \text{NP} \subseteq \text{BQP} \) is currently unknown and widely disbelieved (as of early 2006).

We present an algebraic approach to this question by illustrating a problem, involving sparse polynomials over \( \mathbb{Q}_p \) (the \( p \)-adic rationals), whose complexity appears to interpolate between the complexity classes \( \text{BQP} \) and \( \text{NP} \). Our results thus suggest that sparse polynomials can shed light on the difference between \( \text{BQP} \) and \( \text{NP} \). Indeed, one consequence of our results is a new family of problems which admit (or are likely to admit) \( \text{BQP} \) algorithms. Also, in addition to providing a new complexity limit for factoring polynomials over \( \mathbb{Q}_p \), we can address questions posed earlier by Cox \[\text{Cox04}\], and Karpinski and Shparlinski \[\text{KS99}\], regarding sparse polynomials over finite fields.

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Let us first review some necessary terminology: For any ring \( R \) containing the integers \( \mathbb{Z} \), let \( \text{FEAS}_R \) — the \textit{R-feasibility problem} — denote the problem of deciding whether a given system of polynomials \( f_1, \ldots, f_k \) chosen from \( \mathbb{Z}[x_1, \ldots, x_n] \) has a root in \( R^n \). Observe then that \( \text{FEAS}_R \) and \( \text{FEAS}_\mathbb{Q} \) are respectively the central problems of algorithmic real algebraic geometry and algorithmic arithmetic geometry (see Section 1.1 below for further details).

To measure the “size” of an input polynomial in our complexity estimates, we will essentially count just the number of bits needed to write down the coefficients and exponents in its monomial term expansion. This is the \textbf{sparse} input size, as opposed to the “dense” input size used frequently in computational algebra.

**Definition 1** Let \( f(x) := \sum_{i=1}^{m} c_i x^{a_i} \in \mathbb{Z}[x_1, \ldots, x_n] \) where \( x^{a_i} := x_1^{a_{i1}} \cdots x_n^{a_{in}} \), \( c_i \neq 0 \) for all \( i \), and the \( a_i \) are distinct. We call such an \( f \) an \textbf{n-variate m-nomial}. Also let
\[
\text{size}(f) := \sum_{i=1}^{m} (1 + \lceil \log_2(2 + |c_i|) \rceil + \lceil \log_2(2 + |a_{i1}|) \rceil + \cdots + \lceil \log_2(2 + |a_{in}|) \rceil),
\]
and \( \text{size}_p(f) := \text{size}(f) + \log(2 + p) \). (We also extend size, and thereby \( \text{size}_p \), additively to polynomial systems.) Finally, for any collection \( \mathcal{F} \) of polynomial systems with integer coefficients, let \( \text{FEAS}_R(\mathcal{F}) \) denote the natural restriction of \( \text{FEAS}_R \) to inputs in \( \mathcal{F} \).

Observe that \( \text{size}(a + bx^{99} + cx^d) = \Theta(\log d) \) if we fix \( a, b, c \), so the degree of a polynomial can sometimes be exponential in its sparse size. Since it is not hard to show that \( \text{FEAS}_\mathbb{Q}(\mathcal{U}_2) \in \text{P} \) when \( p \) is fixed (cf. Section 3 below), it will be more natural to take the size of an input prime \( p \) into account as well.

**Definition 2** Let \( \bigcup_{p \text{ prime}} \text{FEAS}_{\mathbb{Q}}(\mathcal{F}) \) denote the union of problems \( \bigcup_{p \text{ prime}} \text{FEAS}_{\mathbb{Q}_p}(\mathcal{F}) \), so that a prime \( p \) is also part of the input, and the underlying input size is \( \text{size}_p \). Also let \( Q_n \) denote the product of the first \( n \) primes and define \( \mathcal{U}_m := \{ f \in \mathbb{Z}[x_1] \mid f \text{ is an m-nomial} \} \).

Observe that \( \mathbb{Z}[x_1] \) is thus the disjoint union \( \bigcup_{m \geq 0} \mathcal{U}_m \). Our results will make use of the following plausible number-theoretic hypothesis.

**Flat Primes Hypothesis (FPH)** Following the notation above, there are absolute constants \( C' \geq C \geq 1 \) such that for any \( n \in \mathbb{N} \), the set \( \{ 1 + kQ_n \mid k \in \{1, \ldots, 2^{nC} \} \} \) contains at least \( \frac{2^{Cn}}{n^{C'}} \) primes.

Assumptions at least as strong as FPH are routinely used, and widely believed, in the cryptology and algorithmic number theory communities (see, e.g., [Mil76, Mih94, Koi97, Roj01a, Hal05]). In particular, we will see in Section 2.1 below how FPH is implied by the Generalized Riemann Hypothesis (GRH) for the number fields \( \{ \mathbb{Q}(\omega_{Q_n}) \}_{n \in \mathbb{N}} \), where \( \omega_M \) denotes a primitive \( M^{th} \) root of unity\(^1\), but can still hold under certain failures of the latter hypotheses.

**Theorem 1** Following the notation above, \( \text{FEAS}_{\mathbb{Q}_{\text{primes}}}(\mathcal{U}_2) \in \text{BQP} \). However, assuming the truth of FPH, if \( \text{FEAS}_{\mathbb{Q}_{\text{primes}}}(\mathbb{Z}[x_1]) \in \mathcal{C} \) for some complexity class \( \mathcal{C} \), then \( \text{NP} \subseteq \text{BQP} \cup \mathcal{C} \). In particular, assuming the truth of FPH, \( \text{FEAS}_{\mathbb{Q}_{\text{primes}}}(\mathbb{Z}[x_1]) \in \text{BQP} \implies \text{NP} \subseteq \text{BQP} \).

\(^1\)i.e., a complex number \( \omega_M \) with \( \omega_M^{-1} = 1 \); and \( \omega_M^{d} = 1 \implies M \mid d \)
Recall that a univariate polynomial has a root in a field \( K \) iff it possesses a degree 1 factor with coefficients in \( K \). Independent of its connection to quantum computing, Theorem 1 thus provides a new complexity limit for polynomial factorization over \( \mathbb{Q}_p[x_1] \). In particular, Theorem 1 shows that finding even just the low degree \((p\text{-adic})\) factors for sparse polynomials (with \( p \) varying) is likely not doable in randomized polynomial time. This complements Chistov’s earlier deterministic polynomial time algorithm for dense polynomials and fixed \( p \) [Chi91]. Theorem 1 also provides an interesting contrast to earlier work of Lenstra [Len99a], who showed that one can at least find all low degree factors (in \( \mathbb{Q}[x_1] \)) of a sparse polynomial in polynomial time.

**Remarks 1** While it has been known since the late 1990’s that \( \text{FEAS}_{\mathbb{Q}\text{primes}} \in \text{EXPTIME} \) [MW96, MW97] (relative to our notion of input size), we are unaware of any earlier algorithms yielding \( \text{FEAS}_{\mathbb{Q}\text{primes}}(\mathcal{F}) \in \text{BQP} \), for any non-trivial family of polynomial systems \( \mathcal{F} \). Also, while it is not hard to show that \( \text{FEAS}_{\mathbb{Q}\text{primes}} \) is NP-hard from scratch, there appear to be no earlier results indicating the smallest \( n \) such that \( \text{FEAS}_{\mathbb{Q}\text{primes}}(\mathbb{Z}[x_1, \ldots, x_n]) \) is NP-hard. ◦

As for the quantum side of Theorem 1, the author is unaware of any other natural algebraic problem that interpolates between \( \text{BQP} \) and \( \text{NP} \) in the sense above. Moreover, since the exact complexity of the problems \( \{\text{FEAS}_{\mathbb{Q}\text{primes}}(\mathcal{U}_m)\}_{m \geq 3} \) is currently unknown, a \( \text{BQP} \) algorithm for any of these problems would yield a new family of algebraic problems — distinct from Integer Factoring or Discrete Logarithm — admitting an exponential quantum speed-up over classical methods.

The only other problem known to interpolate between \( \text{BQP} \) and some classical complexity class arises from very recent results on the complexity of approximating a certain braid invariant — the famous Jones polynomial, for certain classes of braids, evaluated at an \( n^{th} \) root of unity — and involves a complexity class (apparently) higher than \( \text{NP} \). In brief: (1) seminal work of Freedman, Kitaev, Larsen, and Wang shows that such approximations can simulate any \( \text{BQP} \) computation, already for \( n=5 \) [FKW02, FLW02], (2) [AJL05] gives a \( \text{BQP} \) algorithm that computes an additive approximation for arbitrary \( n \), and (3) [YW06] shows that for arbitrary \( n \), computing the most significant bit of the absolute value of the Jones polynomial is \( \text{PP} \)-hard. (Recall that \( \text{BQP} \cup \text{NP} \cup \text{coNP} \subseteq \text{PP} \).) Our results thus provide a new alternative source for quantum/classical complexity interpolation.

Let \( \text{FEAS}_{\mathbb{F}\text{primes}} \) denote the obvious finite field analogue of \( \text{FEAS}_{\mathbb{Q}\text{primes}} \). While we do not yet know whether \( \text{FEAS}_{\mathbb{Q}\text{primes}}(\mathcal{U}_2) \) is \( \text{BQP} \)-complete in any rigorous sense, we point out that \( \text{FEAS}_{\mathbb{Q}\text{primes}}(\mathcal{U}_2) \) is polynomial-time equivalent to \( \text{FEAS}_{\mathbb{F}\text{primes}}(\mathcal{U}_2) \) (cf. Section 3 below), and the inclusion \( \text{FEAS}_{\mathbb{F}\text{primes}}(\mathcal{U}_2) \subseteq \text{BPP} \) is a well-known, decades-old open problem from algorithmic number theory (see, e.g., [BS96, Ch. 7] and [Gao05]). Note also that the \( \text{BQP} \)-completeness of Integer Factoring and Discrete Logarithm are open questions as well.

One can also naturally ask if detecting a degenerate root in \( \mathbb{Q}_p \) for \( f \) (i.e., a degree 1 factor over \( \mathbb{Q}_p \), whose square also divides \( f \)) is as hard as detecting arbitrary roots in \( \mathbb{Q}_p \). Via our techniques, we can easily prove essentially the same complexity lower-bound as above for the latter problem.
Corollary 1 Using size$_p(f)$ as our notion of input size, suppose we can decide for any input prime $p$ and $f \in \mathbb{Z}[x_1]$ whether $f$ is divisible by the square of a degree 1 polynomial in $\mathbb{Q}_p[x_1]$, within some complexity class $\mathcal{C}$. Then, assuming the truth of FPH, $\text{NP} \subseteq \mathcal{C} \cup \text{BPP}$.

Let $\mathbb{F}_p$ denote the finite field with $p$ elements. Corollary 1 then complements an analogous earlier result of Karpinski and Shparlinski (independent of the truth of FPH) for detecting degenerate roots in $\mathbb{C}$ and the algebraic closure of $\mathbb{F}_p$.

Note also that while the truth of GRH usually implies algorithmic speed-ups (in contexts such as primality testing [Mil76], complex dimension computation [Koi97], detection of rational points [Roj01b], or class group computation [Hal05]), Theorem 1 and Corollary 1 instead reveal complexity speed-limits implied by GRH.

1.1 Open Questions and the Relevance of Ultrametric Complexity

Complexity results over one ring sometimes inspire and motivate analogous results over other rings. An important early instance of such a transfer was the work of Paul Cohen, on quantifier elimination over $\mathbb{R}$ and $\mathbb{Q}_p$ [Coh69]. To close this introduction, let us briefly review how results over $\mathbb{Q}_p$ can be useful over $\mathbb{Q}$, and then raise some natural questions arising from our main results.

First, recall that the decidability of $\text{FEAS}_Q$ is a major open problem: decidability for the special case of cubic polynomials in two variables would already be enough to yield significant new results in the direction of the Birch-Swinnerton-Dyer conjecture (see, e.g., [Sil96, Ch. 8]), and the latter conjecture is central in modern number theory (see, e.g., [HS00]). The fact that $\text{FEAS}_Z$ is undecidable is the famous negative solution of Hilbert’s Tenth Problem, due to Matiyasevich and Davis, Putnam, and Robinson [Mat73, DLPvG00], and is sometimes taken as evidence that $\text{FEAS}_Q$ may be undecidable as well (see also [Poo03]).

From a more positive direction, much work has gone into using $p$-adic methods to find an algorithm for $\text{FEAS}_Q(\mathbb{Z}[x, y])$ (i.e., deciding the existence of rational points on algebraic curves), via extensions of the Hasse Principle$^2$ (see, e.g., [Poo01a, Poo06]). Algorithmic results over the $p$-adics are also central in many other computational results: polynomial time factoring algorithms over $\mathbb{Q}[x_1]$ [LLL82], computational complexity [Roj02], and elliptic curve cryptography [Lau04].

Our results thus provide another step toward understanding the complexity of solving polynomial equations over $\mathbb{Q}_p$, and reveal yet another connection between quantum complexity and number theory. Let us now consider some possible extensions of our results.

Question 1 Is $\text{FEAS}_{\text{primes}}(\mathbb{Z}[x_1])$ $\text{NP}$-hard?

Question 2 Given a prime $p$ and an $f \in \mathbb{F}_p[x_1]$, is it $\text{NP}$-hard to decide whether $f$ is divisible by the square of a degree 1 polynomial in $\mathbb{F}_p[x_1]$ (relative to $\text{size}_p(f)$)?

$^2$The Hasse Principle is the assumption that an equation $F(x_1, \ldots, x_n) = 0$ having roots in $\mathbb{Q}_p^n$ for all primes $p$ must have a root in $\mathbb{Q}^n$ as well. The Hasse Principle is a theorem for quadratic polynomials, is conjectured to hold for equations defining smooth plane curves, but fails in subtle ways for cubic polynomials (see, e.g., [Poo01a]).
David A. Cox asked the author whether $\text{FEAS}_{F_p\text{primes}}(\mathbb{Z}[x_1]) \in P$ around August 2004 [Cox04], and Erich Kaltofen posed a variant of Question 1 — $\text{FEAS}_{F_p\text{primes}}(U_3) \in P$ — a bit earlier in [Kal03]. Karpinski and Shparlinski raised Question 2 toward the end of [KS99]. Since Hensel’s Lemma (cf. Section 2 below) allows one to find roots in $\mathbb{Q}_p$ via computations in the rings $\mathbb{Z}/p^l\mathbb{Z}$, Theorem 1 thus provides some evidence toward positive answers for Questions 1 and 2. Note in particular that a positive answer to Question 1 would provide a definitive complexity lower bound for polynomial factorization over $F_p[x_1]$, since randomized polynomial time algorithms (relative to the dense encoding) are already known (e.g., Berlekamp’s algorithm [BS96 Sec. 7.4]).

On a more speculative note, one may wonder if quantum computation can produce new speed-ups by circumventing the dependence of certain algorithms on GRH. This is motivated by Hallgren’s recent discovery of a BQP algorithm for deciding whether the class number of a number field of constant degree is equal to a given integer [Hal05]: The best classical complexity upper bound for the latter problem is $\text{NP} \cap \text{coNP}$, obtainable so far only under the assumption of GRH [BvS89, McC89]. Unfortunately, the precise relation between BQP and $\text{NP} \cap \text{coNP}$ is not clear. However, could it be that quantum computation can expunge the need for GRH in an even more direct way? For instance:

**Question 3** Is there a quantum algorithm which generates, within a number of qubit operations polynomial in $n$, a prime of the form $k Q_n + 1$ with probability $> \frac{2}{3}$?

Indeed, it is natural to try to remove the dependence of our main results on the hypothesis FPH. Here is one possible route.

**Question 4** Let $\text{FEAS}_{\mathbb{Q}_{p\text{primeideals}}}^\prime$ denote the obvious generalization of $\text{FEAS}_{\mathbb{Q}_{\text{primes}}}^\prime$ to arbitrary finite algebraic extensions of the fields $\{\mathbb{Q}_p\}_{p \text{ a prime}}$. Then $\text{FEAS}_{\mathbb{Q}_{p\text{primeideals}}}(\mathbb{Z}[x_1])$ is $\text{NP}$-hard, independent of FPH.

We are currently pursuing a solution to the last question. In particular, it appears likely that $\text{FEAS}_{\mathbb{Q}_{p\text{primeideals}}}(U_2) \in \text{BQP}$.

Our main results are proved mostly in Section 3 after the development of some necessary theory in Section 2 below. For the convenience of the reader, we recall the definitions of all relevant complexity classes and review certain types of Generalized Riemann Hypotheses.

## 2 Background and Ancillary Results

Recall the containments of complexity classes $P \subseteq \text{BPP} \subseteq \text{BQP} \subseteq \text{PP} \subseteq \text{PSPACE}$ and $P \subseteq \text{NP} \cap \text{coNP} \subseteq \text{NP} \cup \text{coNP} \subseteq \text{PP}$, and the fact that the properness of every preceding containment is a major open problem [Pap95, BV97]. We briefly review the definitions of the aforementioned complexity classes below (see [Pap95, BV97] for a full and rigorous treatment):

- **P** The family of decision problems which can be done within (classical) polynomial-time.
BPP The family of decision problems admitting (classical) randomized polynomial-time algorithms that terminate with an answer that is correct with probability at least $\frac{3}{2}$.

BQP The family of decision problems admitting quantum randomized polynomial-time algorithms that terminate with an answer that is correct with probability at least $\frac{3}{2}$ [BV97].

NP The family of decision problems where a ‘‘Yes’’ answer can be certified within (classical) polynomial-time.

coNP The family of decision problems where a ‘‘No’’ answer can be certified within (classical) polynomial-time.

PP The family of decision problems admitting (classical) randomized polynomial-time algorithms that terminate with an answer that is correct with probability strictly greater than $\frac{1}{2}$.

PSPACE The family of decision problems solvable within polynomial-time, provided a number of processors exponential in the input size is allowed.

Now recall that 3CNFSAT is the famous seminal NP-complete problem [GJ79] which consists of deciding whether a Boolean sentence of the form $B(X) = C_1(X) \land \cdots \land C_k(X)$ has a satisfying assignment, where $C_i$ is of one of the following forms: $X_i \lor X_j \lor X_k$, $\neg X_i \lor X_j \lor X_k$, $\neg X_i \lor \neg X_j \lor X_k$, $\neg X_i \lor \neg X_j \lor \neg X_k$, $i, j, k \in [3n]$, and a satisfying assignment consists of an assignment of values from $\{0, 1\}$ to the variables $X_1, \ldots, X_{3n}$ which makes the equality $B(X) = 1$ true. Each $C_i$ is called a clause.

We will need a clever reduction from 3CNFSAT to feasibility testing for univariate polynomial systems over certain fields.

Definition 3 Letting $Q_n$ denote the product of the first $n$ primes, let us inductively define a homomorphism $P_n = \text{the (nth) Plaisted morphism}$ — from certain Boolean polynomials in the variables $X_1, \ldots, X_n$ to $\mathbb{Z}[x_1]$, as follows: (1) $P_n(0) := 1$, (2) $P_n(x_i) := x_i^{Q_n/p_i} - 1$, (3) $P_n(\neg B) := \frac{x_1^{Q_n} - 1}{P_n(B)}$, for any Boolean polynomial $B$ for which $P_n(B)$ has already been defined, (4) $P_n(B_1 \lor B_2) := \text{lcm}(P_n(B_1), P_n(B_2))$, for any Boolean polynomials $B_1$ and $B_2$ for which $P_n(B_1)$ and $P_n(B_2)$ have already been defined.

Lemma 1 For all $n \in \mathbb{N}$ and all clauses $C(X_i, X_j, X_k)$ with $i, j, k \leq n$, we have size($P_n(C)$) = $O(n^2)$. Furthermore, if $K$ is any field possessing $Q_n$ distinct $Q_n^{\text{th}}$ roots of unity, then a 3CNFSAT instance $B(X) := C_1(X) \land \cdots \land C_k(X)$ has a satisfying assignment iff the zero set in $K$ of the polynomial system $F_B := (P_n(C_1), \ldots, P_n(C_k))$ has a root $\zeta$ satisfying $\zeta^{Q_n} - 1$.

\[ \frac{3}{2} \] It is easily shown that we can replace $\frac{3}{2}$ by any constant strictly greater than $\frac{1}{2}$ and still obtain the same family of problems [Pap95].

\[ \frac{1}{2} \] Throughout this paper, for Boolean expressions, we will always identify 0 with ‘‘False’’ and 1 with ‘‘True’’.
David Alan Plaisted proved the special case $K = \mathbb{C}$ of the above lemma in [Pla84]. His proof extends with no difficulty whatsoever to the more general family of fields detailed above. Other than a slightly earlier (and independent) observation of Kaltofen and Koiran [KK05], we are unaware of any other variant of Plaisted’s reduction involving a field other than $\mathbb{C}$.

Let us recall a version of Hensel’s Lemma sufficiently general for our proof of Theorem 3 along with a useful characterization of certain finite rings. Recall that for any ring $R$, $R^*$ is the group of multiplicatively invertible elements of $R$.

Hensel’s Lemma (See, e.g., [RobOil, Pg. 48].) Suppose $f \in \mathbb{Z}_p[x]$ and $x \in \mathbb{Z}_p$ satisfies $f(x) \equiv 0 \pmod{p^k}$ and ord$_p f'(x) < \frac{k}{2}$. Then there is a root $\zeta \in \mathbb{Z}_p$ of $f$ with $\zeta \equiv x \pmod{p^{\ell - \ord_p f'(x)}}$ and ord$_p f'(\zeta) = \ord_p f'(x)$.

Lemma 2 Given any cyclic group $G$, $a \in G$, and an integer $d$, the equation $x^d = a$ has a solution iff the order of $a$ divides $\frac{\#G}{\gcd(d, \#G)}$. In particular, $F_q^*$ is cyclic for any prime power $q$, and $(\mathbb{Z}/p^d\mathbb{Z})^*$ is cyclic for any $(p, \ell)$ with $p$ an odd prime or $\ell \leq 2$. Finally, for $\ell \geq 3$, $(\mathbb{Z}/2^\ell\mathbb{Z})^* \cong \{-1, 1\} \times \{1, 5, 5^2, 5^3, \ldots, 5^{2\ell-2-1} \pmod{2^\ell}\}$.

The last lemma is standard (see, e.g., [BS96, Ch. 5]).

We will also need the following result on an efficient randomized reduction of $\mathsf{FEAS}_K(\mathbb{Z}_p[x])$ to $\mathsf{FEAS}_K(\mathbb{Z}_p[x]^2)$. Recall that $\mathbb{C}_p$ — the $p$-adic complex numbers — is the metric closure of the algebraic closure of $\mathbb{Q}_p$, and $\mathbb{C}_p$ is algebraically closed.

Lemma 3 Suppose $f_1, \ldots, f_k \in \mathbb{Z}_p[x] \setminus \{0\}$ are polynomials of degree $\leq d$, with $k \geq 3$. Also let $Z_K(f_1, \ldots, f_k)$ denote the set of common zeroes of $f_1, \ldots, f_k$ in some field $K$. Then, if $a = (a_1, \ldots, a_k)$ and $b = (b_1, \ldots, b_k)$ are chosen uniformly randomly from $\{1, \ldots, 18dk^2\}^{2k}$, we have

$$\Pr \left( Z_K \left( \sum_{i=1}^k a_if_i, \sum_{i=1}^k b_if_i \right) = Z_K(f_1, \ldots, f_k) \right) \geq \frac{8}{9}$$

for any $K \in \{\mathbb{C}, \mathbb{C}_p\}$.

While there are certainly earlier results that are more general than Lemma 3 (see, e.g., [GH93, Sec. 3.4.1] or [Koi97, Thm. 5.6]), Lemma 3 is more direct and self-contained for our purposes. For the convenience of the reader, we provide its proof.

Proof of Lemma 3 Assume $f_i(x) := \sum_{j=0}^d c_{i,j}x^j$ for all $i \in \{1, \ldots, k\}$. Let $W := \left( \bigcup_{i=1}^k Z_K(f_i) \right) \setminus Z_K(f_1, \ldots, f_k)$ and $\varphi(u, \zeta) := \sum_{i=1}^k u_if_i(\zeta)$ for any $\zeta \in W$. Note that $\#W \leq kd$ and that for any fixed $\zeta \in W$, the polynomial $\varphi(u, \zeta)$ is linear in $u$ and not identically zero. By Schwartz’s Lemma [Sch80], for any fixed $\zeta \in W$, there are at most $kN^{k-1}$ points $u \in \{1, \ldots, N\}^k$ with $\varphi(u, \zeta) = 0$. So then, there at most $dk^2N^{k-1}$ points $u \in \{1, \ldots, N\}^k$ with $\varphi(u, \zeta) = 0$ for some $\zeta \in W$.

Clearly then, the probability that a uniformly randomly chosen pair $(a, b) \in \{1, \ldots, N\}^{2k}$ satisfies $\varphi(a, \zeta) = \varphi(b, \zeta) = 0$ for some $\zeta \in W$ is bounded above by $\frac{2dk^2}{N^{k-1}}$. So taking $N = 18dk^2$ we are done.

2.1 Review of Riemann Hypotheses

Primordial versions of the connection between analysis and number theory are not hard to derive from scratch and have been known at least since the 19th century. For example, letting
\(\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}\) denote the usual Riemann zeta function (for any real number \(s > 1\)), one can easily derive with a bit of calculus (see, e.g., \[TF00\, pp. 30–32\]) that

\[
\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1 - \frac{1}{p^s}}, \text{ and thus } \frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s},
\]

where \(\Lambda\) is the classical Mangoldt function which sends \(n\) to \(\log p\) or 0, according as \(n = p^m\) for some prime \(p\) (and some positive integer \(m\)) or not. For a deeper connection, recall that \(\pi(x)\) denotes the number of primes (in \(\mathbb{N}\)) \(\leq x\) and that the Prime Number Theorem (PNT) is the asymptotic formula \(\pi(x) \sim \frac{x}{\log x}\) for \(x \to \infty\). Remarkably then, the first proofs of PNT, by Hadamard and de la Vallée-Poussin (independently, in 1896), were based essentially on the fact that \(\zeta(\beta + i\gamma)\) has no zeroes on the vertical line \(\beta = 1\).

More precisely, writing \(\rho = \beta + i\gamma\) for real \(\beta\) and \(\gamma\), recall that \(\zeta\) admits an analytic continuation to the complex plane sans the point 1 \([TF00, \text{Sec. 2}]\). In particular, the only zeroes of \(\zeta\) outside the critical strip \(\{\rho = \beta + i\gamma \mid 0 < \beta < 1\}\) are the so-called trivial zeroes \((-2, -4, -6, \ldots\)\). Furthermore the zeroes of \(\zeta\) in the critical strip are symmetric about the critical line \(\beta = \frac{1}{2}\) and the real axis. The Riemann Hypothesis (RH), from 1859, is then the following assertion:

\[\text{(RH) All zeroes } \rho = \beta + i\gamma \text{ of } \zeta \text{ with } \beta > 0 \text{ lie on the critical line } \beta = \frac{1}{2}.\]

Among a myriad of hitherto unprovably sharp statements in algorithmic number theory, it is known that RH is true \(\iff\) \(\left| \pi(x) - \int_{2}^{x} \frac{dt}{\log t} \right| = O(\sqrt{x} \log x)\) \([TF00]\). In particular, RH is widely agreed to be the most important problem in modern mathematics. Since May 24, 2000, RH even enjoys a bounty of one million US dollars thanks to the Clay Mathematics Foundation.

Let us now consider the extension of RH to primes in arithmetic progressions: For any primitive \(M\)th root of unity \(\omega_M\), define the (cyclotomic) Dedekind zeta function via the formula \(\zeta_{\mathbb{Q}(\omega_M)}(s) := \sum_a \frac{1}{(\mathcal{N}a)^s}\), where \(a\) ranges over all nonzero ideals of \(\mathbb{Z}[\omega_M]\) (the ring of algebraic integers in \(\mathbb{Q}(\omega_M)\)). \(\mathcal{N}\) denotes the norm function, and \(s > 1\) \([BS96]\). Then, like \(\zeta\), the function \(\zeta_{\mathbb{Q}(\omega_M)}\) also admits an analytic continuation to \(\mathbb{C} \setminus \{1\}\) (which we’ll also call \(\zeta_{\mathbb{Q}(\omega_M)}\)), \(\zeta_{\mathbb{Q}(\omega_M)}\) has trivial zeroes \((-2, -4, -6, \ldots\)\), and all other zeroes of \(\zeta_{\mathbb{Q}(\omega_M)}\) lie in the critical strip \((0, 1) \times \mathbb{R}\) \([LO77]\). (The zeroes of \(\zeta_{\mathbb{Q}(\omega_M)}\) in the critical strip are also symmetric about the critical line \(\frac{1}{2} \times \mathbb{R}\) and the real axis.) We then define the following statement:

\[\text{(GRH}_{\mathbb{Q}(\omega_M)}\text{)}^7 \text{ For any primitive } M\text{th root of unity } \omega_M, \text{ all the zeroes } \rho = \beta + i\gamma \text{ of } \zeta_{\mathbb{Q}(\omega_M)} \text{ with } \beta > 0 \text{ lie on the critical line } \beta = \frac{1}{2}.\]

In particular, letting \(\pi(x, M)\) denote the number of primes \(p\) congruent to 1 mod \(M\) satisfying \(p \leq x\), it is known that GRH\(_{\mathbb{Q}(\omega_M)}\) is true \(\iff\) \(\left| \pi(x, M) - \frac{1}{\varphi(M)} \int_{2}^{x} \frac{dt}{\log t} \right| = O(\sqrt{x} (\log x + \log M))\),

\[^5\text{Shikau Ikehara later showed in 1931 that PNT is in fact equivalent to the fact that } \zeta \text{ has no zeroes on the vertical line } \beta = 1\text{ (the proof is reproduced in } [DMcK72]).\)

\[^6\text{We’ll abuse notation henceforth by letting } \zeta \text{ denote the analytic continuation of } \zeta \text{ to } \mathbb{C} \setminus \{1\}.\)

\[^7\text{There is definitely conflicting notation in the literature as to what the “Extended” Riemann Hypothesis or “Generalized” Riemann Hypothesis are. We thus hope to dissipate any possible confusion via subscripts clearly declaring the field we are working with.}\)
where $\varphi(M)$ is the number of $k \in \{1, \ldots, M - 1\}$ relatively prime to $M$. (This follows routinely from the conditional effective Chebotarev Theorem of [LO77 Thm. 1.1], taking $K = \mathbb{Q}$ and $L = \mathbb{Q}(\omega_M)$ in the notation there. One also needs to recall that the discriminant of $\mathbb{Q}(\omega_M)$ is bounded from above by $M^{\varphi(M)}$ [BS96 Ch. 8, pg. 260].)

From the very last estimate, an elementary calculation shows that FPT is implied by the truth of the hypotheses $\{GRH_{\mathbb{Q}(\omega_M)}\}_{n \in \mathbb{N}}$. However, we point out that FPT can still hold even in the presence of infinitely many non-trivial zeta zeroes off the critical line. For instance, if we instead make the weaker assumption that there is an $\epsilon > 0$ such that all the non-trivial zeroes of $\{\zeta_{\mathbb{Q}(\omega_M)}\}_{n \in \mathbb{N}}$ have real part $\leq \frac{1}{2} + \epsilon$, then one can still prove the weaker inequality $|\pi(x, M) - \frac{1}{\varphi(M)} \int_2^x \frac{dt}{\log t}| = O\left(x^{\frac{1}{2} + \epsilon} (\log x + \log M)\right)$ (see, e.g., [BGMcI91]).

Another elementary calculation then shows that this looser deviation bound still suffices to yield FPT. In fact, one can even have non-trivial zeroes of $\zeta_{\mathbb{Q}(\omega_M)}$ approach the line $\{\beta = 1\}$ arbitrarily closely, provided they do not approach too quickly as a function of $n$. (See [Roj06] for further details.)

3 The Proofs of Our Main Results

3.1 The Univariate Threshold Over $\mathbb{Q}_p$: Proving Theorem 1

The first assertion rests upon a quantum algorithm for finding the multiplicative order of an element of $(\mathbb{Z}/p^l\mathbb{Z})^*$ (see [Sho97, BL95]), once we make a suitable reduction from FEAS$_{\mathbb{Q}(\text{primes})}$. The second assertion relies on properties of primes in specially chosen arithmetic progressions, via our generalization (cf. Section 2) of an earlier trick of Plaisted [Pla84].

**Proof of the First Assertion:** First note that it clearly suffices to show that we can decide (with error probability $\frac{1}{3}$, say) whether the polynomial $f(x) := x^d - \alpha$ has a root in $\mathbb{Q}_p$, using a number of qubit operations polynomial in $\log \alpha + \log d$. (This is because we can divide by a suitable constant, and arithmetic over $\mathbb{Q}$ is doable in polynomial time.) The case $\alpha = 0$ always results in the root 0, so let us assume $\alpha \neq 0$. Clearly then, any $p$-adic root $\zeta$ of $x^d - \alpha$ satisfies $\ord_p \zeta = \ord_p \alpha$. Since we can compute $\ord_p \alpha$ and reductions of integers mod $d$ in P [BS96 Ch. 5], we can then clearly assume that $d|\ord_p \alpha$ (for otherwise, there can be no root over $\mathbb{Q}_p$). Moreover, by rescaling $x$ by an appropriate power of $p$, we can assume further that $\ord_p \alpha = 0$.

Now note that $f'(x) = dx^{d-1}$ and thus $\ord_p f'(x) = \ord_p (d)$. So by Hensel’s Lemma, it suffices to decide whether the mod $p^\ell$ reduction of $f$ has a root in $\mathbb{Z}/p^\ell\mathbb{Z}$, for $\ell = 1 + 2\ord_p d$. (Note in particular that $\text{size}(p^\ell) = O(\log(p) \log(d))$ which is polynomial in our notion of input size.) By Lemma 2 we can easily decide the latter feasibility problem, given the multiplicative order of $\alpha$ in $(\mathbb{Z}/p^\ell\mathbb{Z})^*$; and we can do the latter in BQP by Shor’s seminal algorithm for computing order in a cyclic group [Sho97 pp. 1498–1501], provided $p^\ell \not\in \{8, 16, 32, \ldots\}$. So the first assertion is proved for $p^\ell \not\in \{8, 16, 32, \ldots\}$.

To dispose of the remaining cases $p^\ell \in \{8, 16, 32, \ldots\}$, write $\alpha = (-1)^a 5^b$ and observe that such an expression is unique, by the last part of Lemma 2. The first part of Lemma 2 then easily yields that $x^d - \alpha$ has a root iff

$(a \text{ odd } \implies d \text{ is odd}) \wedge (\text{the order of } 5^b \text{ divides } \frac{2^e - 2}{\gcd(2^e, 2^e - 2)})$. 


In particular, we see that \( x^d - \alpha \) always has a root when \( d \) is odd, so we can assume henceforth that \( d \) is even.

Letting \( b \) be the order of \( 5^b \), it is then easy to check that the order of \( \alpha \) is either \( b \) or \( 2b \), according as \( a \) is even or odd. Moreover, since \( d \) is even, we see that \( x^d - \alpha \) can have no roots in \((\mathbb{Z}/2^e\mathbb{Z})^*\) when \( a \) is odd. So we can now reduce the feasibility of \( x^d - \alpha \) to two order computations as follows: Compute, now via Boneh and Lipton's quantum algorithm for order computation in Abelian groups [BL95 Thm. 2], the order of \( \alpha \) and \( -\alpha \). Observe then that \( a \) is odd iff the order of \( \alpha \) is larger (and then \( x^d - \alpha \) has no roots in \((\mathbb{Z}/2^e\mathbb{Z})^*\)), so we can assume henceforth that \( \alpha \) has the smaller order. To conclude, we then declare that \( x^d - \alpha \) has a root in \( \mathbb{Q}_2 \) iff the order of \( \alpha \) divides \( \frac{2^e-2}{d} \). This last step is correct, thanks to the first part of Lemma 2, so we are done.

**Proof of the Second Assertion:** First note that \( \text{size}(Q_n) = O(n \log n) \), via the Prime Number Theorem. Observe then that the truth of FPH implies that we can efficiently find a prime \( p \) of the form \( kQ_n + 1 \), with \( k \in \{1, \ldots, 2^nC\} \), via random sampling, as follows: Pick a uniformly randomly integer from \{1, \ldots, 2^nC\} and using, say, the famous polynomial-time AKS primality testing algorithm [AKS02], verify whether \( kQ_n + 1 \) is prime. We repeat this, no more than \( 9nC' \) times, until we've found a prime.

Via the elementary estimate \((1 - \frac{1}{B})^t < \frac{1}{2^t} \), valid for all \( B, t > 1 \), we then easily obtain that our method results in a prime with probability at least \( \frac{8}{9} \). Since \( \text{size}(1 + 2^nCQ_n) = O(\log(2^nCQ_n)) = O(nC + n \log n) \), it is clear that our simple algorithm requires a number of bit operations just polynomial in \( n \). Moreover, the number of random bits needed is clearly \( O(nC) \).

Having now probabilistically generated a prime \( p = 1 + kQ_n \), Lemma 1 then immediately yields the implication \( \text{FEAS}_{\mathbb{Q}}(US) \in \mathcal{C} \implies \text{NP} \in \mathcal{C} \cup \text{BPP} \)”, where \( US := \{ (f_1, \ldots, f_k) \mid f_i \in \mathbb{Z}[x_1], \ k \in \mathbb{N} \} \): Indeed, if \( \text{FEAS}_{\mathbb{Q}}(US) \in \mathcal{C} \) for some complexity class \( \mathcal{C} \), then we could combine our hypothetical \( \mathcal{C} \) algorithm for \( \text{FEAS}_{\mathbb{Q}}(US) \) with our randomized prime generation routine (and the Plaisted morphism for \( K = \mathbb{Q}_p \)) to obtain an algorithm with complexity in \( \mathcal{C} \cup \text{BPP} \) for any \( 3\text{CNFSAT} \) instance.

So now we need only show that this hardness persists if we reduce \( US \) to systems consisting of just one univariate sparse polynomial. Clearly, we can at least reduce to pairs of polynomials via Lemma 3 so now we need only reduce from pairs to singletons.

Toward this end, suppose \( a \in \mathbb{Z} \) is a non-square mod \( p \) and \( p \) is odd. Clearly then, the only root in \( \mathbb{F}_p \) of (the mod \( p \) reduction of) the quadratic form \( q(x, y) := x^2 - ay^2 \) is \( (0, 0) \). Furthermore, by considering the valuations of \( x \) and \( y \), it is also easily checked that the only root of \( q \) in \( \mathbb{Q}_p \) is \( (0, 0) \). Thus, given any \( (f, g) \in \mathbb{Z}[x_1]^2 \), we can form \( q(f, g) \) (which has size \( O(\text{size}(f) + \text{size}(g) + \text{size}(p)) \)) to obtain a polynomial time reduction of \( \text{FEAS}_{\mathbb{Q}}(\mathbb{Z}[x_1]^2) \) to \( \text{FEAS}_{\mathbb{Q}}(\mathbb{Z}[x_1]) \), assuming we can find a quadratic non-residue efficiently. (If \( p = 2 \) then we can simply use \( q(x, y) := x^2 + xy + y^2 \) and then there is no need at all for a quadratic non-residue.) However, this can easily be done by picking two random \( a \in \mathbb{F}_p \). With probability at least \( \frac{3}{4} \), at least one of these numbers will be a quadratic non-residue (and this can be checked in \( \text{P} \) by computing \( a^{(p-1)/2} \) via recursive squaring). So we are done. ■
3.2 Detecting Square-Freeness: Proving Corollary 1

Given any \( f \in \mathbb{Z}[x_1] \), observe that \( f \) has a root in \( \mathbb{Q}_p \) iff \( f^2 \) is divisible by the square of a degree 1 polynomial in \( \mathbb{Q}_p[x_1] \). Moreover, since \( \text{size}(f^2) = O(\text{size}(f)^2) \), we thus obtain a polynomial-time reduction of \( \text{FEAS}_{\mathbb{Q}_{\text{prime}}}(\mathbb{Z}[x_1]) \) to the problem considered by Corollary 1. So we are done. ■

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