Abstract. The hidden supersymmetry of the monopole found by De Jonghe et al. is generalized to a spin \( \frac{1}{2} \) particle in the combined field of a Dirac monopole plus a \( \lambda^2/r^2 \) potential [considered before by D’Hoker and Vinet], and related to the operator introduced by Biedenharn a long time ago in solving the Dirac-Coulomb problem. Explicit solutions are obtained by diagonalizing the Biedenharn operator.

1. Introduction

In [1], De Jonghe et al. found that a spin-\( \frac{1}{2} \) particle of mass, here chosen \( m = \frac{1}{2} \), and charge \( q \) in the field of a Dirac monopole of unit strength, described by the Pauli Hamiltonian

\[
H_D = \left( \vec{D}^2 - q \frac{\vec{\sigma} \cdot \vec{r}}{r^3} \right),
\]

admitted a ‘hidden’ supercharge \( \tilde{Q}_D \) given by

\[
\tilde{Q}_D = (\vec{\sigma} \cdot \vec{\ell} + 1),
\]

where \( \vec{\ell} \) is the [non-conserved] orbital part of the angular momentum, \( \vec{\ell} = \vec{r} \times \vec{\pi}, \vec{\pi} = \vec{p} - q\vec{\lambda} \).

Their discussion is based on the study of Killing-Yano tensors. The supercharge \( \tilde{Q}_D \) anticommutes with the supercharge \( Q_D \) found by D’Hoker and Vinet [2], where

\[
Q_D = \vec{\sigma} \cdot \vec{\pi}.
\]

These supercharges form, together with the [conserved] total angular momentum

\[
\vec{J} = \vec{L} + \frac{\vec{\sigma}}{2}, \quad \vec{L} = \vec{\ell} - q\vec{r}/r,
\]

a closed, non-linear algebra [1].

In this Letter, we (i) generalize the result of De Jonghe et al.; (ii) relate it to earlier work of Dirac [3], Biedenharn [4], and Berrondo and McIntosh [5]; (iii) use it to solve the system, and (iv) discuss a full Minkowski space generalisation.

Very recently [6], Plyushchay discussed related problems, but from a rather different viewpoint: while our results here are derived from supersymmetric quantum mechanics, he uses pseudoclassical mechanics with anticommuting (Grassmann) variables. See also the comments on [1] in [7].
2. The generalized monopole system

Let $\vec{A}$ denote the vector potential of the Dirac monopole of unit strength. Let $\lambda$ be a positive constant, $q > 0$ a half-integer, and consider, setting $A_4 \equiv \lambda/r$ and $\vec{A} \equiv q \vec{A}_D$, the gauge field $A_\alpha, (\alpha = 1, \ldots, 4)$ on $\mathbb{R}^4$. We use the Dirac matrices

\[ \vec{\gamma} = \begin{pmatrix} -i \vec{\sigma} & -i \vec{\sigma} \end{pmatrix}, \quad \gamma^4 = \begin{pmatrix} 1_2 \\ -1_2 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} 1_2 \\ -1_2 \end{pmatrix}. \]

When restricted to fields which do not depend on $x^4$, the associated Dirac operator $D \equiv \gamma^\mu D_\mu, D_\mu \equiv \partial_\mu - i A_\mu$, reads

\[ D = \begin{pmatrix} T^\dagger T \\ \vec{\sigma} \cdot \vec{r} + i \lambda \end{pmatrix}. \]

$D$ anticommutes with the chirality operator $\gamma^5$.

The square of the Dirac operator,

\[ H = \begin{pmatrix} H_1 \\ H_0 \end{pmatrix} = \begin{pmatrix} T^\dagger T \\ \vec{r} \cdot \vec{r} + q \vec{\sigma} \cdot \vec{r} + \lambda^2 \vec{r}^3 \end{pmatrix}, \]

is a supersymmetric Hamiltonian [8]: the non-zero-energy parts of the chiral sectors, defined as the eigensectors of $\gamma^5$, are intertwined by the unitary transformations

\[ U = T \frac{1}{\sqrt{H_1}} \quad \text{and} \quad U^{-1} = U^\dagger = \frac{1}{\sqrt{H_1}} T^\dagger, \quad U H_1 U^\dagger = H_0. \]

The partner Hamiltonians $H_1$ and $H_0$, have therefore the same positive spectra, and the spectrum of $\hat{D}$ can be obtained from that of $\hat{D}^2$. For further analysis, it is convenient to set

\[ \sigma_r = \vec{\sigma} \cdot (\vec{r}/r), \quad z = \vec{\sigma} \cdot \vec{\ell} + 1. \]

Note that $\sigma_r^2 = 1$, and that $z$ anticommutes with $\sigma_r$ and $\vec{\sigma} \cdot \vec{\pi}$, $\{ z, \sigma_r \} = 0$ and $\{ z, \vec{\sigma} \cdot \vec{\pi} \} = 0$. Also $z$ is equal to the supercharge $\vec{Q}_D$ of (1.2). For $\lambda \neq 0, z$ satisfies the relations $zT = -T^\dagger z, zT^\dagger = -Tz, zH_1 = H_0 z$ and $zH_0 = H_1 z$. Therefore, the operator

\[ \kappa = \begin{pmatrix} iz \\ -iz \end{pmatrix} \equiv i \begin{pmatrix} \vec{\sigma} \cdot \vec{\ell} + 1 \\ -\vec{\sigma} \cdot \vec{\ell} - 1 \end{pmatrix} \]

commutes with the Dirac operator $\hat{D}$, and hence also with its square. Using $(\vec{\sigma} \cdot \vec{L})^2 = \vec{L}^2 + i \vec{\sigma} \cdot (\vec{L} \times \vec{L}) = \vec{L}^2 - \vec{\sigma} \cdot \vec{L}$, one proves furthermore that $\kappa^2 = z^2 = \vec{z}^2 = J^2 + 1/4 - q^2$. Thus, since the eigenvalues of $\vec{J}^2$ are $j(j+1), j = q - 1/2, q + 1/2, \ldots$, the operators $z$ (and $\kappa$) have irrational eigenvalues,

\[ \kappa = \pm \sqrt{(j + 1/2)^2 - q^2}. \]
The operator $K$ is hermitian because $j \geq q - 1/2$. For the lowest allowed value $j = q - 1/2$, however, $\kappa$ vanishes and $K$ is not invertible. The operator $K$ has been used by Dirac in the study of the relativistic hydrogen atom [3] long time ago; its form adapted to the monopole, (2.7), was found by Berrondo and McIntosh [5]. It is more convenient to use, however, the hermitian Biedenharn operator [4]

$$
(2.8) \quad \Gamma = - \left( \vec{\sigma} \cdot \vec{\ell} + 1 + \lambda \sigma_r \right) \equiv - \begin{pmatrix} y \\ x \end{pmatrix}.
$$

Here $y = z + \lambda \sigma_r$ and $x = z - \lambda \sigma_r$, so that $x \sigma_r = -\sigma_r y$. The eigenvalues of $\Gamma$,

$$
(2.9) \quad \gamma = \pm \sqrt{(j + 1/2)^2 + \lambda^2 - q^2}, \quad \text{(sign } \gamma = -\text{sign } \kappa)\n$$

are in general still irrational. However, owing to the presence of $\lambda^2$, the operator $\Gamma$ is invertible whenever $\lambda^2 > 0$. The clue is that, in terms of $\Gamma$, $\vec{\phi}^2$ becomes simply

$$
(2.10) \quad \vec{\phi}^2 \equiv \begin{pmatrix} H_1 \\ H_0 \end{pmatrix} = -\left( \partial_r + \frac{1}{r} \right)^2 + \frac{\Gamma(\Gamma + 1)}{r^2}.
$$

Here $p_r = -i(1/r)\partial_r, r = -i(\partial_r + 1/r)$ is the hermitian operator conjugate to $r$. This is conveniently checked by writing, using the radial form $\vec{\sigma} \cdot \vec{\sigma} = i\sigma_r(\partial_r + 1/r - z/r)$. The supercharges $T$ and $T^\dagger$ as

$$
(2.11) \quad T = -i\sigma_r \left( \partial_r + \frac{1}{r} - \frac{y}{r} \right) = -i \left( \partial_r + \frac{1}{r} + \frac{y}{r} \right) \sigma_r,
$$

$$
T^\dagger = -i\sigma_r \left( \partial_r + \frac{1}{r} - \frac{x}{r} \right) = -i \left( \partial_r + \frac{1}{r} + \frac{x}{r} \right) \sigma_r.
$$

The self-adjointness of $\vec{\phi}^2$ requires $|\lambda| \geq 3/2$ [8]. It follows from our previous formulae that $\Gamma$ anticommutes with $\vec{\phi}$ and commutes therefore with its square. Hence the shifted operators $x$ and $y$ commute with the partner Hamiltonians, $[x, H_0] = 0 = [y, H_1]$.

In conclusion, our 4-component operators satisfy the non-linear algebra

$$
(2.12) \quad \vec{\phi}^2 = H, \quad \{ \Gamma, \vec{\phi} \} = 0, \quad [\Gamma, H] = 0, \quad \Gamma^2 = J^2 + 1/4 + \lambda^2 - q^2,
$$

$$
[J, \Gamma] = 0, \quad [\vec{J}, \vec{\phi}] = 0, \quad [\vec{J}, H] = 0, \quad [J_i, J_j] = i\epsilon_{ijk} J_k,
$$

which only differs from that found for the 2-components objects of [1] in the appearance of $\lambda^2$ in $\Gamma^2$. Note that it is now $\vec{\phi}$ which plays the rôle of $Q_D$, and $\Gamma$ plays that of $\vec{Q}_D$.

3. Explicit solutions

A look at (2.10) shows that the Biedenharn operator $\Gamma$ plays clearly a rôle analogous to angular momentum. Since $\Gamma$ and $J$ commute, they can be simultaneously diagonalized. A convenient basis is found as follows [9]. Let us first assume that $j \geq q + 1/2$, and let $L_\pm = j \pm 1/2$ be the orbital angular momentum quantum number. Consider first the two-component spinors

$$
(3.1) \quad \varphi^\mu_{\pm} = \sqrt{\frac{L_\pm + 1/2 \pm \mu}{2L_\pm + 1}} Y^{\mu-1/2}_{L_\pm} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \pm \sqrt{\frac{L_\pm + 1/2 \pm \mu}{2L_\pm + 1}} Y^{\mu+1/2}_{L_\pm} \begin{pmatrix} 0 \\ 1 \end{pmatrix},
$$
where the \( Y \)'s are the monopole harmonics, defined in \([10]\). The \( \varphi \)'s satisfy
\[
\vec{J}^2 = j(j+1), \quad J_3 = \mu, \quad (\mu = -j, \cdots, j),
\]
\( \vec{L}^2 = L_\pm (L_\pm + 1) \), and the action of \( \sigma_r \) upon the
\( \varphi \)'s can be obtained from \([11]\). Then, dropping the upper index \( \mu \), and using
\[
\vec{\sigma} \cdot \vec{\ell} = \vec{J}^2 - \vec{L}^2 - 3/4 + q\vec{\sigma} \cdot \vec{r}/r,
\]
it may be shown that the 2-spinors
\[
(3.2) \quad \chi_\pm = \frac{q}{j + 1/2 + |\kappa|} \varphi_\pm \mp \varphi_\mp
\]
satisfy
\[
z \chi_\pm = \pm |\kappa| \chi_\pm \text{ and } \sigma_r \chi_\pm = \chi_{\mp}, \text{ as well as } \vec{J}^2 = j(j+1), \quad J_3 = \mu, \quad (\mu = -j, \cdots, j).
\]
Hence, defining \( \gamma \) by \( \gamma^2 = \kappa^2 + \lambda^2 \), we have
\[
(3.3) \quad \phi_+ = (|\kappa| + |\gamma|) \chi_+ - \lambda \chi_- \quad \phi_- = \lambda \chi_+ + (|\kappa| + |\gamma|) \chi_-
\]
diagonalize \( x \) and \( y \), \( x\phi_\pm = \mp |\gamma| \phi_\pm \) and \( y\Phi_\pm = \mp |\gamma| \Phi_\pm \). The operator \( \sigma_r = \vec{\sigma} \cdot \vec{r}/r \)
interchanges the \( x \) and \( y \) eigenspinors, \( \sigma_r \phi_\pm = \Phi_\mp \), a result which also follows directly
from \( x\sigma_r = -\sigma_r y \).

When \( j = q - \frac{1}{2} \), there are no \( L_- = q-1 \) states, though: we only have \( 2(q-1) + 1 = 2q \)
states with \( L = L_+ = q \), namely
\[
(3.4) \quad \varphi^0_+ = \sqrt{\frac{q + 1/2 + \mu}{2q + 1}} Y^\mu_{q-1/2}(1) + \sqrt{\frac{q + 1/2 - \mu}{2q + 1}} Y^\mu_{q+1/2}(0),
\]
Thus, for \( j = q - \frac{1}{2} \), no \( \varphi_- \) is available, and \( \chi^0_+ = -\chi^0_- = \varphi^0_+ \) is annihilated by \( z \).
Therefore, there are no \( \phi_- \)-states in the \( \gamma^5 = -1 \) sector, and no \( \Phi_+ \) states in the \( \gamma^5 = 1 \)
sector. In each \( \gamma^5 \) sector, \( 3.4 \) yields in turn \( (2q) \) states, namely
\[
(3.5) \quad \phi^0_+ = -\Phi^0_- \propto \varphi^0_+.
\]
They are eigenvectors of \( x \) and \( y \) with eigenvalues \( \pm \lambda \), respectively, and are still inter-
changed by \( \sigma_r \). In conclusion, the eigenfunctions of \( \vec{J}^2 \) are
\[
(3.6) \quad \begin{cases}
\Psi_\pm = u_\pm \begin{pmatrix} \Phi_\pm \\ 0 \end{pmatrix} & \quad \gamma^5 = 1 \\
\psi_\pm = u_\pm \begin{pmatrix} 0 \\ \phi_\pm \end{pmatrix} & \quad \gamma^5 = -1
\end{cases}
\quad \text{for } j \geq q + 1/2,
\]
\[
\begin{cases}
\Psi^0_- = u^0_- \begin{pmatrix} \Phi_- \\ 0 \end{pmatrix} & \quad \gamma^5 = 1 \\
\psi^0_+ = u^0_+ \begin{pmatrix} 0 \\ \phi_+ \end{pmatrix} & \quad \gamma^5 = -1
\end{cases}
\quad \text{for } j = q - 1/2.
\]
where \( u_\pm \) solves the radial equation
\[
(3.7) \quad \left[ -\left( \partial_r + \frac{1}{r} \right)^2 + \frac{\gamma(\gamma + 1)}{r^2} - E \right] u_\pm(r) = 0.
\]
There are no bound states; the scattering states involve Bessel functions:

\[ u_{\pm}(r) = \frac{1}{\sqrt{r}} J_{|\gamma + \frac{1}{2}|}(\sqrt{E} r). \]

4. Symmetries

A spin 0 particle in the field of a Dirac monopole has an \( o(2, 1) \) symmetry, generated by \( H \equiv H_0 = \vec{\pi}^2 \) and by dilations and expansions [12],

\[ D = tH - 1/2 \{ \vec{\pi} \cdot \vec{r} \} \quad K = -\frac{1}{2} t^2 H + tD + \frac{1}{2} r^2. \]

This symmetry has been extended to the Pauli Hamiltonian (1.1) with formally the same generators (4.1), with \( H \equiv H_D \) replacing \( H_0 \) [2]. The supercharge \( Q_D \) is a square-root of \( H_D \). Commuting \( Q_D \) with the expansion, \( K \), yields a new fermionic generator, namely

\[ S = i[Q_D, K] = \frac{1}{\sqrt{2}} \vec{\sigma} \cdot (\vec{r} - \vec{\pi} t), \]

and it is then readily proved that the bosonic operators \( H_D, D, K \) close, with \( Q \) and \( S \) into an \( osp(1/1) \) superalgebra. [2, 8, 13]. Now remarkably

\[ i[Q_D, S] - \frac{1}{2} = z, \]

and \( z^2 \) is a Casimir operator of this \( osp(1/1) \) [8].

The same bosonic \( o(2, 1) \) symmetry arises for the generalized monopole system (2.4). The Dirac operator \( Q \equiv \slashed{D} \) is a square-root of \( H \) by construction. However,

\[ Q^* = \gamma^5 Q \equiv \begin{pmatrix} \vec{\sigma} \cdot \vec{\pi} - i\frac{\lambda}{r} \\ -\vec{\sigma} \cdot \vec{\pi} - i\frac{\lambda}{r} \end{pmatrix}. \]

is a new square-root, \( \{Q^*, Q^*\} = H \). Commuting \( K \) with \( Q \) and with \( Q^* \) yields

\[ S = \gamma^5 \vec{\gamma} \cdot \vec{r} - tQ \quad \text{and} \quad S^* = -i\gamma^5 S. \]

In this way, we get two, independent, super-extensions of the bosonic \( o(2, 1) \). The two \( osp(1/1) \)'s do not close yet: the “mixed” anticommutators between the \( Q \)-type and \( S \)-type charges yield a new bosonic charge, namely

\[ Y = \{Q, S^*\} = -\{Q^*, S\} = \gamma^5 (z + \frac{1}{2}) - \lambda \sigma_r, \]

that commutes with the other bosonic charges. The four operators \( H, D, K, Y \) do close.
finally with the four fermionic charges $Q, Q^*, S, S^*$,

\[
\begin{align*}
\{Q, D\} &= iQ, & \{Q^*, D\} &= iQ^*, \\
\{Q, K\} &= -iS, & \{Q^*, K\} &= -iS^*, \\
\{Q, H\} &= 0, & \{Q^*, H\} &= 0, \\
\{Q, Y\} &= -iQ^*, & \{Q^*, Y\} &= iQ, \\
\{S, D\} &= -iS, & \{S^*, D\} &= -iS^*, \\
\{S, K\} &= 0, & \{S^*, K\} &= 0, \\
\{S, Y\} &= -iS^*, & \{S^*, Y\} &= iS, \\
\{Q, Q\} &= H, & \{Q^*, Q^*\} &= H, \\
\{S, S\} &= 2K, & \{S^*, S^*\} &= 2K, \\
\{Q, Q^*\} &= 0, & \{S, S^*\} &= 0, \\
\{Q, S\} &= -D, & \{Q^*, S^*\} &= -D, \\
\{Q, S^*\} &= Y, & \{Q^*, S\} &= -Y.
\end{align*}
\]

which are the commutation relations of the osp(1/2) superalgebra, to which spin adds an extra $o(3)$ [8]. Now the Casimir of $osp(1/2)$ is the square of

\[
i[Q, S] - \frac{1}{2} = i[Q^*, S^*] - \frac{1}{2} = \Gamma,
\]

which provides a nice interpretation for the Biedenharn operator $\Gamma$. Similar algebras were studied in [14].

5. Particular cases

(i) For $\lambda = 0$, we have $Q = Q^\dagger = Q_D$, $H_1 = H_0 = H_D$, the Pauli Hamiltonian in a pure monopole field [2]. The 4-component Hamiltonian is simply $\text{diag}(H_D, H_D)$; the Biedenharn and the Dirac operators are related as $\Gamma = -i\gamma^4K$. In this case, we recover the formulæ in [1, 2].

(ii) Another particular value is $\lambda = \pm q$, when the situation is similar to that in Taub-NUT space [15]: the spin drops out in one of the chiral sectors, while the Pauli term gets doubled in the other. For $\lambda = q$, for example, the Hamiltonian (2.4) reduces to

\[
H = \begin{pmatrix} H_1 \\ H_0 \end{pmatrix} = \begin{pmatrix} \vec{\pi}^2 + \frac{q^2}{r^2} - 2q\frac{\vec{\sigma} \cdot \vec{r}'}{zr^3} \\ \vec{\pi}^2 + \frac{q^2}{r^2} \end{pmatrix}.
\]

Here $H_0$ describes a spin 0 particle in the combined field of a Dirac monopole and of an inverse-square potential, while $H_1$ corresponds to a particle with anomalous gyromagnetic
ratio 4. The Biedenharn operator has half-integer eigenvalues, $\gamma = \pm (j + \frac{1}{2})$. Note that the $\gamma^5 = -1$ eigenspinors now reduce to those in Eq. (3.1), $\phi_\gamma \propto \varphi_\gamma$.

Assume first that $j \geq q + 1/2$. Since $\gamma (\gamma + 1)$ is now the same for $-|\gamma|$ as for $|\gamma| - 1$, these values lead to identical solutions. Therefore, in each $\gamma^5$ sector, the energy levels are two-fold degenerate. The numerator of the $r^{-2}$ term reads

\[
(5.2) \quad \begin{cases}
(j + \frac{3}{2})(j + \frac{1}{2}) & \text{for } \gamma > 0 \\
(j + \frac{1}{2})(j - \frac{1}{2}) & \text{for } \gamma < 0
\end{cases}
\]

so that we indeed get the same equation with $\gamma > 0$ for $j$, as with $\gamma < 0$ for $(j - 1)$, provided that $(j - 1)$ states do exist. In both cases, (3.8) yields $r^{-1/2} J_{j+1}(\sqrt{Er})$.

The two-fold degeneracy is hence also explained by an extra $o(3)$ symmetry in addition to the rotational symmetry: spin is trivially conserved for $H_0$, and this is exported to $H_1$ by supersymmetry. The extra $o(3)$ symmetry is generated hence by the spin vectors

\[
(5.3) \quad \vec{S}_0 = \frac{1}{2} \vec{\sigma} \quad \text{for } H_0, \quad \vec{S}_1 = U^\dagger \vec{S}_0 U \quad \text{for } H_1,
\]

where $U$ and $U^{-1} = U^\dagger$ are the intertwiners of (2.4). The two-fold degeneracy corresponds precisely to this $o(3)$ symmetry. For $j = q - 1/2$, half of the states are missing.

The system admits further symmetries. Firstly, $H_0$ admits the non-relativistic conformal $o(2,1)$ symmetry [10]; supersymmetry exports this to the partner Hamiltonian $H_1$. The symmetries combine with $\vec{P}$ and $-i \gamma^5 \vec{P}$ into an $osp(2,1)$ superalgebra [8].

(iii) Replacing the scalar potential $\lambda/r$ by $q(1 - 1/r)$ — which corresponds to the long-range limit of the scalar field of a self-dual “BPS” monopole [14] — we get

\[
(5.4) \quad \begin{pmatrix}
H_1 \\
H_0
\end{pmatrix} = \begin{pmatrix}
\vec{\pi}^2 - \frac{2q^2}{r} + \frac{q^2}{r^2} + q^2 - 2q \frac{\vec{\sigma} \cdot \vec{r}}{r^3} \\
\vec{\pi}^2 - \frac{2q^2}{r} + \frac{q^2}{r^2} + q^2
\end{pmatrix}.
\]

The “lower” Hamiltonian, $H_0$, has again gyromagnetic ratio 0 yielding an extra $o(3)$ symmetry. Its properties have been thoroughly studied by McIntosh and Cisneros, and by Zwanziger [17], who found that it admits a Kepler-type dynamical symmetry. Its superpartner $H_1$ describes a spinning particle of anomalous gyromagnetic ratio 4: this is the ‘dyon’ of D’Hoker and Vinet [18]. Then supersymmetry can be used to transfer the symmetries of $H_0$ to $H_1$ [16]; the system can be solved using the Biedenharn method [9].

6. Minkowski space extension

In this section we turn to the Dirac-equation in Minkowski space-time, with a combined Coulomb-monopole and massless scalar background. This relativistic system of equations has properties very similar to the Euclidean Dirac problem considered above, and can be solved by analogous methods.

In Minkowski space-time, with metric $\eta_{\mu\nu} = \text{diag}(-1,1,1,1)$, we use Dirac matrices with the properties $\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}$ and $\gamma_\mu^\dagger = -\gamma_0 \gamma_\mu \gamma_0$. 

Our starting point is the Dirac equation
\[(i\slashed{D} + m + g\varphi)\psi = 0 .\] (6.1)

Here \(\varphi = -\tilde{g}/r\) is a dynamical scalar background and \(m\) the mass (which can be taken as the vacuum expectation value of the scalar field). Eq. (6.1) describes the motion in the long-range field of a Julia-Zee dyon [19]. The covariant derivatives contain electromagnetic background potentials corresponding to a Coulomb field
\[D_0 = \partial_0 + iq\phi \quad \phi = \tilde{q}/r ,\]
and the field of a magnetic monopole taken, as in Sec. 2, to be of unit strength.

Multiplication by \((-i\slashed{D} + m + g\varphi)\) gives a Klein-Gordon type equation, which on stationary states \(\psi(r,t) = \exp(-iEt)\psi_E(r)\) takes the form
\[\left[-(E - q\phi)^2 - (\tilde{\nabla} - iq\tilde{A})^2 + (m + g\varphi)^2 - i\sigma^{\mu\nu}F_{\mu\nu} - ig\gamma_\mu\partial^\mu \varphi\right]\psi_E = 0.\] (6.2)

The generalized Klein-Gordon operator can be block-diagonalized in \((2 \times 2)\) blocks, with 2-component eigenspinors \(\psi_{\pm}\) satisfying
\[\left[-(E - q\phi)^2 - (\tilde{\nabla} - iq\tilde{A})^2 + (m + g\varphi)^2 - \sigma \cdot (q\tilde{B} \pm \tilde{\nabla}\Lambda)\right]\psi_{E,\pm} = 0,\] (6.3)
with \(\Lambda = \sqrt{g^2\varphi^2 - q^2\phi^2}.\) Note that the square root is real for \(g^2\varphi^2 \geq q^2\phi^2\), and imaginary for \(g^2\varphi^2 < q^2\phi^2\). For the case of the Coulomb and scalar potentials this becomes
\[\Lambda = \frac{\lambda}{r}, \quad \lambda = \sqrt{g^2\tilde{g}^2 - q^2\tilde{q}^2}.\] (6.4)

Defining the operators \(\tilde{\ell}\) and \(\tilde{J}\) as in Sec. 2, we may cast (6.3) into the form
\[\left[-(\partial_r + \frac{1}{r})^2 + \frac{1}{r^2}(\tilde{J}^2 - \frac{3}{4} - \tilde{\ell} \cdot \tilde{\sigma} - q^2 \mp \lambda\sigma_r) - (E - \frac{q\tilde{q}}{r})^2 + (m - \frac{g\tilde{g}}{r})^2\right]\psi_{E,\pm} = 0.\] (6.5)

To make contact with the Biedenharn operator and the work of Sec. 2, we note that the operators \(y\) and \(x\) of (2.8) occur as blocks in (6.5). Writing here \(\Gamma_+ = -y\) and \(\Gamma_- = -x\), we have
\[\Gamma_{\pm} (\Gamma_{\pm} + 1) = \tilde{\ell}^2 + \lambda^2 - (q \pm \lambda)\sigma_r ,\] (6.6)
and can hence rewrite (6.5) in the standard form
\[\left[-(\partial_r + \frac{1}{r})^2 + \frac{1}{r^2}\Gamma_{\pm} (\Gamma_{\pm} + 1) + \frac{2k}{r} + \varepsilon\right]\psi_{E,\pm} = 0 .\] (6.7)

Here the constants \(k\) and \(\varepsilon\) are given by \(k = q\tilde{q}E - g\tilde{g}m\) and \(\varepsilon = m^2 - E^2\). Note the symmetry under the simultaneous exchange of \((E, q\tilde{q}) \leftrightarrow i(m, g\tilde{g})\). The eigenvalues of \(\Gamma_{\pm}\) are the ones given in (2.10): here
\[\gamma = \pm \sqrt{(j + \frac{1}{2})^2 + g^2\tilde{g}^2 - q^2(\tilde{q}^2 + 1)} .\] (6.8)
Introducing the notation \( l_\gamma \), where \( l_\gamma = \gamma \) for \( \gamma \geq 0 \) and \( l_\gamma = -(1 + \gamma) \) for \( \gamma < 0 \), we see that the eigenstates of \( \Gamma_\pm \) satisfy the equation

\[
\left[ -\left( \partial_r + \frac{1}{r} \right)^2 + \frac{1}{r^2} l_\gamma (l_\gamma + 1) + \frac{2k}{r} + \varepsilon \right] \psi_{E,l(\gamma)} = 0. \tag{6.9}
\]

The spectrum of eigenvalues for bound states is well-known from atomic physics:

\[ \varepsilon = k^2/n^2_\gamma, \quad n_\gamma = 1 + l_\gamma + N, \quad N = 0, 1, 2, \ldots. \tag{6.10} \]

However, in this case the bound-state energy eigenvalues themselves then are given by

\[ E(j,N) = m \left( g\tilde{g}q\tilde{q} \pm n_\gamma \sqrt{n^2_\gamma + q^2\tilde{q}^2 - g^2\tilde{g}^2} \right) . \tag{6.11} \]

For the ground state \( j = 0, N = 0 \), the wave equation factorizes as expected on the basis of supersymmetric quantum mechanics:

\[
\left( -\partial_r + \frac{\gamma_g - 1}{r} + \frac{k_g}{\gamma_g} \right) \left( \partial_r + \frac{\gamma_g + 1}{r} + \frac{k_g}{\gamma_g} \right) \psi_0 = 0 , \tag{6.12}
\]

with \( \gamma_g \) and \( k_g \) the ground state values of \( \gamma \) and \( k \).

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