A novel approach to solve eigenvalue and forced response problems in waveguide theory by means of bi-orthogonality relations

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Abstract. In this paper recent advances on solving challenging problems in dynamics of multimodal symmetric waveguides (having identical properties of positive/negative going waves) using bi-orthogonality are summarised. It has already been shown by the authors that bi-orthogonality, valid for arbitrarily complicated unbounded symmetric waveguides, greatly facilitates analysis of their forced response in any excitation conditions. Recent findings have shown that bi-orthogonality is a powerful tool also for bounded waveguides and by direct use it resolves the classical Boundary Integral Equations (BIE) into a set of individual relations between modal amplitudes at the boundaries. Moreover, bi-orthogonality permits to uniquely identify special sets of boundary conditions for which the eigenfrequency spectra may be found directly from the dispersion diagrams. These solutions are available simply by drawing horizontal lines specifying an inverse of the wave length of interest in the \((\Omega, k)\)-dispersion diagram and so the eigenfrequency spectrum may be read directly as intersections of these lines with dispersion curves. This paper therefore serves to promote this method, discuss applications, new possibilities, physical interpretation and its generalisation to a broader class of waveguide problems.

1. Introduction

In this paper the intend is to highlight and illustrate how to use bi-orthogonality relations to solve both eigenvalue and forced response problems in the waveguide theory. Despite their great advantages they appear surprisingly vague in literature but has, however, been used in e.g. [1–3] to find analytical solutions to classical waveguide forcing problems such as beams, plates and springs. In a recent paper by the authors, [4], the derivation of bi-orthogonality was generalised to cover any symmetric waveguide (defined by having a dispersion equation formulated only in even powers of the wavenumbers) in whichever realm of physics.

The paper, [4], addressed the advantages of using the bi-orthogonality relation for a cylindrical fluid-filled shell. Here we highlight these benefits for any symmetric waveguide and discuss also the possibilities to extend and generalise bi-orthogonality to a broader class of waveguide problems.

In classical waveguide theory a tool much used is the reciprocity relation, [5]. It holds for any two solutions (free waves or expansions) labelled \(n\) and \(j\). The drawback of reciprocity for
some engineering problems is that it satisfies also the trivial case when \( n = j \); making its use sometimes limited. In Sec. 2 we thus illustrate how to split the reciprocity relation for any two non-identical free waves into the pair of relations that excludes the trivial case \( n = j \). These relations are in accordance with \([4]\) denoted orthogonality relations as they are valid for any two free waves not identical i.e. only when \( n \neq j \). Then, we deduce from the orthogonality relations similar relations between modal coefficients to exclude also the case \( n = -j \). These relations are denoted (also in accordance with \([4]\)) bi-orthogonality relations since they are valid for any two wave-pairs not identical i.e. when \( n^2 \neq j^2 \). The bi-orthogonality and orthogonality relations are both much stronger than the reciprocity relation for free waves, yet bi-orthogonality still holds some additional advantages over the orthogonality relation as we shall see in the following. In Sec. 3 we illustrate the advantages obtained by applying (bi-)orthogonality for solving both partially and fully bounded free/forced waveguide problems. Then in Sec. 4 we discuss the possibilities of generalising (bi-)orthogonality to a broader class of problems before concluding in Sec. 5. Finally, it is not the intend of this paper to derive all equations rigorously but rather to emphasise the simplicity and strength of deriving and using the (bi-)orthogonality relation. For details regarding the derivations the reader is directed to \([4]\).

2. (Bi-)orthogonality of free waves

The derivation of the (bi-)orthogonality relation is straightforward and relies simply upon subtraction of two reciprocity relations written, respectively, for the free waves; \((n,j)\) and \((-n,j)\), \(-n\) being the ‘twin’ wave of \( n \) (moving in opposite direction). To illustrate the simple mathematics required for deriving the (bi-)orthogonality relation without actually carrying out any rigorous derivations, let us consider a symmetric waveguide which may be one- or multi-directional but has at least one direction bounded. This may be written on the general form of Eq.(1). Mathematical details can be found in \([4]\).

\[
Lu = 0 \quad \text{on} \quad V \in \mathbb{R}^n
\]  

(1)

with \( u \) being the field variables and \( V \) the volume when \( n = 3 \). If the waveguide is symmetric then \( L \) is a symmetric operator and thus obey the condition;

\[
\left\langle Lu^{(n)}, u^{(j)} \right\rangle_V = \left\langle Lu^{(j)}, u^{(n)} \right\rangle_V
\]

(2)

where \( L \) and \( u \) is, respectively, a matrix and vector for multi-dimensional problems and \( \left\langle \cdot \right\rangle_V \) imply the inner product over some volume i.e. \( \left\langle \cdot \right\rangle_V = \int_V \cdot \, dV \). Then applying ‘by parts integration’ we simply arrive at;

\[
\left\langle Lq^{(n)}, q^{(j)} \right\rangle_\partial V - \left\langle Lq^{(j)}, q^{(n)} \right\rangle_\partial V = 0
\]

(3)

since the volume integrals eventually cancel so that only the inner product over the bounding surfaces remain (indicated by subscript \( \partial V \)). Also this equation is very well known as it is simply the reciprocity relation, see e.g. \([5]\), and thus our starting point. Here it is important to note that the inner product also encounter hypothetical ‘open’ surfaces in the waveguide direction.

The modal state vectors, \( Lq^{(n/j)} \) and \( q^{(j/n)} \), constitute, in general terms, what is known as, respectively, the generalised forces and displacements. Then, when \( L \) is symmetric each of these generalised forces/displacements are found to have certain Class properties; defined by being either even (Class A) or odd (Class B) with respect to their wavenumbers. Arranging the modal state vectors of forces/displacements in terms of their Class properties rather than their physical ones, the reciprocity relation in Eq.(3) is rewritten to that of Eq.(4).

\[
\left\langle C_A^{(j)}, C_B^{(n)} \right\rangle_{\partial V} = \left\langle C_A^{(n)}, C_B^{(j)} \right\rangle_{\partial V}
\]

(4)
where inherently $C^{(j)}_A$ and $C^{(n)}_B$ include both generalised forces and displacements but on the contrary belong to both the same Class and state, respectively, A/B and $j/n$. From this it is very simple to show from the difference between the two reciprocity relations for $(n, j)$ and $(-n, j)$ using the Class properties, that;

$$\left< C^{(j)}_A, C^{(n)}_B \right>_\partial V = \left< C^{(n)}_A, C^{(j)}_B \right>_\partial V = 0 \quad n \neq j \quad (5)$$

This relation is the orthogonality relation valid for all waves not identical. From this we may continue and straightforwardly derive the bi-orthogonality relation as;

$$\left< \bar{C}^{(n)}_A, \bar{C}^{(j)}_B \right> = \left< \bar{C}^{(j)}_A, \bar{C}^{(n)}_B \right> = 0 \quad n^2 \neq j^2 \quad (6)$$

where the bar indicate modal coefficients independent of the waveguide direction(s) and the inner product defined accordingly over the remaining coordinates i.e. some hypothetical 'open' surface. Remark also that the bi-orthogonality relation is confined to $n^2 \neq j^2$, meaning that identical waves travelling in opposite directions (denoted wave pairs) are not bi-orthogonal and bi-orthogonality thus relate distinct pairs of $\pm$ waves.

In simple words, for any symmetric one- or multi-directional waveguide, the orthogonality relation is the inner product of generalised forces/displacements over some bounding surface including the hypothetical 'open' boundaries in the waveguide direction, also denoted 'stations' in [6]. The bi-orthogonality relation, on the other hand, is the inner product over some hypothetical 'open' surface crossing the waveguide direction i.e. not in the waveguide coordinate(s). This is usually associated with a cross-section or line (depending on the waveguide dimensions). For a fluid-filled cylindrical shell with preferred axial waveguide direction this corresponds to integration over a cross-section of the pipe through which the waves propagate. In other words, the net energy flow through any cross-section caused by interacting wave pairs must balance.

3. Application of (bi-)orthogonality in waveguide problems

As mentioned initially the (bi-)orthogonality offers many advantages in solving, for instance, unbounded and 'bounded' waveguide problems. Moreover, the relation helps also to reveal analytical solutions, even for some of the most advanced cases of wave propagation and standing waves.

3.1. Wave propagation – Forcing problems

For waveguide problems where the propagation of energy (energy flow) along different transmission paths is of prime interest the (bi-)orthogonality relation reveal explicit analytical formula for the modal amplitudes; even when the waveguide supports an infinite number of waves as is the case for most engineering waveguides.

Following the same derivation steps as for the (bi-)orthogonality relation in Sec. 2, however, with $Lu^{(j)} = f$ and $Lu^{(n)} = 0$, where $n$ still is a free wave while $j$ denote the complete solution (eigenfunction expansion), then Eq.(2) becomes;

$$\left< Lu^{(j)}, u^{(n)} \right>_V - \left< Lu^{(n)}, u^{(j)} \right>_V = \left< f, u^{(n)} \right>_V \quad (7)$$

We can then show, using Eq.(5), that the modal amplitudes may be found individually as;

$$U^{(n)} = -\frac{1}{2} \left< f, \tilde{u}^{(n)} \right>_V \frac{\left< \bar{C}^{(n)}_A, \bar{C}^{(n)}_B \right>}{\partial V} \quad \{ n \in \mathbb{Z} \mid n \neq 0 \} \quad (8)$$
where \(^\sim\) indicate coefficients with the modal amplitudes, \(U^{(n)}\), sorted out and \(\text{(n)}\) the index representing the free waves (wavenumbers) i.e. \(n = 0\) is not eligible. The derivation above is slightly more complicated than for the (bi-)orthogonality relation itself as it requires additional use of Class properties and Eq.(6), but is, otherwise, straightforwardly derived by alternately subtracting and adding the two reciprocity relations as discussed with Eq.(4). From more detailed studies, as those carried out in [4], it becomes obvious that this equation is simply an energy balance between the ejected energy and the total transmitted energy from all paths.

In contrast, conventional methods require formulation and solution of an equation system based on a truncated expansion of eigenfunctions. Additional equations are here formulated, for example, using Galerkin Orthogonalisation or the like. Obviously, the magnitudes of the amplitudes depend in this case on the truncation order, which is indeed not the true when using the bi-orthogonality method as seen from Eq.(8).

Another interesting aspect of the bi-orthogonality relation is its close relation to the physical properties of a waveguide as suggested already by Eq.(8). Though it require rather tedious derivations and analysis it can be shown to be directly related to the time-averaged total energy flow known from [7], seen here in the left of Eq.(9). The relation between the time-averaged energy flow with \(\exp(-i\omega t)\) and bi-orthogonality follows as:

\[
N \sum = \frac{1}{2} \text{Re} \left( \langle Lq, q^* \rangle \right) \quad \Rightarrow \quad N \sum = \frac{1}{2} \omega \text{Im} \left( \langle Lq, q^* \rangle \right) = \frac{1}{2} \omega \text{Im} \left( \langle C_A^{(n)}, C_B^{(j)} \rangle \right)
\]

i.e. correct structure and definition of the state vectors replaces conjugation of velocities. Note that the inner product follow Eq.(6) and thus acquire no integration in the unbounded coordinate(s). From more detailed studies of this relation one may in fact conclude that bi-orthogonality is a generalisation of the total energy, left in Eq.(9). This can be argued from the simple fact that while \(\text{Im} \left( \langle Lq, q^* \rangle \right) = \text{Im} \left( \langle C_A^{(n)}, C_B^{(j)} \rangle \right)\) from Eq.(9);

\[
\text{Re} \left( \langle Lq, q^* \rangle \right) \neq \text{Re} \left( \langle C_A^{(n)}, C_B^{(j)} \rangle \right)
\]

Since \(\text{Re} \left( \langle Lq, q^* \rangle \right)\) is generally accepted as a pseudo-energy of no physical relevance nor use, it is natural to consider the energy version of the bi-orthogonality relation as a generalisation because its real part, \(\text{Re} \left( \langle C_A^{(n)}, C_B^{(j)} \rangle \right)\), is indeed meaningful as it is essential for the correctness of the modal amplitudes in Eq.(8).

Further, from Eq.(9) direct use of bi-orthogonality surprisingly proofs linearity of the total energy flow. In addition, we find from the Class properties of \(C_A^{(n)}\), that only propagating waves produce a net energy flow (i.e. having non-zero imaginary parts) and so the total energy reduce from

\[
N \sum = \frac{1}{2} \omega \sum_n \sum_j \text{Im} \left( \langle Lq^{(n)}, q^{*(j)} \rangle \right) \overset{\text{to}}{\Rightarrow} N \sum = \frac{1}{2} \omega \sum_{n=1}^\hat{N} \text{Im} \left( \langle \bar{C}_A^{(n)}, \bar{C}_B^{(n)} \rangle \right)
\]

where \(\hat{N}\) is the set of propagating waves travelling in the positive direction. By similar means one can reduce also the equations for energy flow of each individual transmission path, see [4] for details.

Finally, as a consequence of Eq.(8) the convergence pattern has changed conveniently compared with convergence of the conventional methods, [4]. Since we find the amplitudes 'one-by-one' directly from the modal energy balance in Eq.(8), we no longer require continuity to be strictly
satisfied for any truncation order but instead have continuity satisfied in the limit as the solution converges. Conveniently, this reveals the accuracy (convergence) of a truncated solution directly from the energy flow graphs, where all transmission paths except the loaded one(s) must obey continuity i.e. depart from zero. This gives rise to deriving exact error measures which may be found in [4]. In Fig. 1 the energy flow plot is shown for a monopole load in the fluid of a fluid-filled cylindrical shell. Details to be found in [4]. In (a) it appears clearly that the unloaded transmission paths of the unconverged solution does not obey continuity across the excitation point, \( x = 0 \), nor does the loaded component carry all load at the excitation point and so the accuracy is immediately deemed poor.

3.2. Standing waves – Eigenfrequency analysis

For fully bounded problems, also known as boundary value problems, application of the (bi-)orthogonality relation also ease significantly the solution. In this study the prime interest is standing waves typically known as eigenfrequency/eigenmode analysis or forced response. Today there exist hundreds of methods to assess such properties, however, we shall focus here on one of the more classical ones; the Boundary Integral Equations Method (BIEM), and show how application of (bi-)orthogonality resolves the tedious Boundary Integral Equations (BIE) into the simplest of algebraic modal boundary identities.

The equation governing the BIE’s is in structural dynamics known as Somigliana’s identity, while it is known in acoustics as the Kirchhoff integral. In this framework the attempt is to apply Green’s solution: \( LG = \delta(x-x_0) \), for a waveguide to deduce, in first instance, the eigenfrequency solution for the ‘bounded’ waveguide i.e. \( Lu = 0 \), bounded by any set of boundary conditions at any station of the original (unbounded) waveguide. Details of Green’s solution as well as Somigliana’s identity may be found in many classical texts such as [8–13]. Without much further detail we derive Somigliana’s identity to Eq.(12) from a system on similar form as the general one in Eq.(1).

\[
\langle u(x), \delta^0F(x-x_0) \rangle_V = \langle g^0F(x, x_0), Lq(x) \rangle_{\partial V} - \langle Lg^0F(x, x_0), q(x) \rangle_{\partial V} \tag{12}
\]

where the generalised forces/displacements of Green’s solution depend also on the excitation point of the delta function, \( x_0 \), while \( 0^F \) indicates the different loading cases required by Green’s
solution.

For simplicity let us consider a one-directional waveguide with preferred Cartesian propagation direction, \( x_m \). Green’s solution, \( g \), to this problem is found straightforwardly following Sec. 3.1, while the solution \( q \) is yet unknown. To find this solution we typically set up the BIE’s (by letting \( x_0 \) alternately towards the boundaries) together with a number of boundary conditions and solve the integral equations using any preferred approximation method – usually the Boundary Element Method (BEM). However, since the only difference between the unbounded and bounded problem is that the bounded satisfy additional boundary conditions, they must satisfy the same governing equation, Eq.(1), so that \( q \) can be expanded on the same set of eigenfunctions (modes) as Green’s solution.

Omitting details of derivation it is possible to once again apply Class properties, (bi-)orthogonality and, in addition, the analytical solution from Eq.(8) to arrive at the novel boundary identity, Eq.(13), immediately from the BIE’s.

\[
U^{(n)}_b = U^{(n)}_a \exp(k^{(n)}|a-b|) \quad \{n \in \mathbb{Z} \mid n \neq 0\}
\]  

where \( k^{(n)} \) is the \( n \)th wavenumber corresponding to the \( n \)th eigenfunction of the governing equation and \( U^{(n)}_{a/b} \) unknown modal boundary amplitudes at \( a \) and \( b \). Thus, the tedious (boundary) integral equations from Eq.(12) have been resolved to the above simple algebraic identity. This means that we are left with satisfying only the boundary conditions at \( a \) and \( b \) expressed as;

\[
\mathcal{L}q_a(\hat{x}) - Z_a(\hat{x})q_a(\hat{x}) = 0 \quad \mathcal{L}q_b(\hat{x}) - Z_b(\hat{x})q_b(\hat{x}) = 0
\]  

where \( \mathcal{L}q_{a/b} \) and \( q_{a/b} \) are the generalised boundary forces/displacements at \( x_m = a \) and \( x_m = b \), \( \hat{x} \) the remaining coordinates not in the waveguide direction and \( Z \) the prescribed boundary impedance functions (scalar or diagonal matrix depending on the problem).

Omitting details we may formulate the boundary conditions at each boundary into a unified boundary condition on modal form using a modal projection technique. From this we can formulate a well-posed algebraic equation system to be solved straightforwardly by any mathematical program.

Further, there exist two special cases of boundary conditions for any such ‘bounded’ symmetric waveguide where analytical closed form solutions can be found; revealed only by way of the modal projection method and bi-orthogonality. These boundary conditions are assembled according to the Class properties, recall Eq.(4), i.e.

Case 1: \( \mathcal{C}_A(x_m = a) = \mathcal{C}_A(b) = 0 \) or \( \mathcal{C}_B(a) = \mathcal{C}_B(b) = 0 \)

Case 2: \( \mathcal{C}_A(x_m = a) = \mathcal{C}_B(b) = 0 \) or vice versa

and are meaningful not only for this problem but hold also special properties for periodic structures, see e.g. [14]. Now, using this modal projection method together with bi-orthogonality it is straightforward to show that for these two cases the eigenfrequency equations factorise and become as simple as;

\[
\prod_{n=1}^{\hat{N}} \left( e^{2k^{(n)}L} - 1 \right)^2 = 0 \quad \text{Case 1:} \quad \prod_{n=1}^{\hat{N}} \left( e^{2k^{(n)}L} + 1 \right)^2 = 0 \quad \text{Case 2:}
\]  

Recall from Eq.(11) that \( \hat{N} \) constitute the set of propagating waves in the positive direction. From the factorisation of these eigenfrequency equations it is obvious that each propagating
wave produce independent spectra, meaning that waves are uncoupled so that no conversion or interaction happens at the boundaries. By this we simply find the spectrum of eigenfrequencies by substituting, respectively for Case 1 and 2, $k_\omega = \frac{i m \pi}{L}$ and $k_\omega = \frac{i (2m-1) \pi}{2L}$ into the dispersion equation, which we may then solve with arbitrary accuracy using, for instance, the novel Finite Product Method, [15; 16]. Alternatively, these $k_\omega$-solutions may also be visualised in the dispersion diagrams as the intersection points between the propagating wave branches, $(n)$, and the horizontal lines governed by the $k_\omega$-solutions. This is seen in Fig. 2.

\[7 \pi \]

- Prop. waves
- Case 1
- Case 2

Figure 2. Dispersion diagram for a fluid-filled shell vibrating in bending mode. The horizontal lines constitute each a spectrum of $\Omega_{mn}$ eigenfrequencies in that intersection with the propagating wave branches $(n)$ constitute eigenfrequencies for Case 1 boundary conditions (dashed) and Case 2 (dash-dot).

3.3. Standing waves – Forcing problems

In forcing (inhomogeneous) boundary value problems the derivation procedure is analogue to the above and so both the boundary identities (denoted the inhomogeneous boundary identity) and formulation of the equation system follows directly from the latter; the difference being that the system assumes a right-hand-side given by the inhomogeneous boundary identity. Then the response to a given driving function is found simply by inversion of the equation system.

In summary we have shown that the bi-orthogonality relation is indeed strong, in particular, to solve problems in the field of waveguides where their application, without much effort, help to prove: 1) that modal amplitudes of the forced response may be found ‘one-by-one’, Eq.(8), 2) that bi-orthogonality relations are closely linked to the energy flow and prove linearity thereof, Eq.(11), 3) that relations between amplitudes of free waves at the boundaries are factorised regardless of boundary conditions, Eq.(13), and 4) that bi-orthogonality relations convert integral equations into algebraic ones and provide analytical elementary eigenfrequency solutions on the closed form for two special sets of boundary conditions, Eq.(15).

Furthermore, bi-orthogonality relations apply equally well to both forcing (Sec. 3.1 and 3.3) and eigenvalue (Sec. 3.2) problems for partially (wave propagation) as well as fully bounded (standing waves) problems.
4. Future work: Generalisation of (bi-)orthogonality and its applications
In this section we discuss the possibilities of generalising the bi-orthogonality relation and its applications to a broader class of problems. So far, we have been restricted only to symmetric waveguide problems i.e. a bounded structure (or equivalents in other realms of physics) with at least one direction unbounded and with a dispersion equation (or characteristic equation) formulated in $k^2$ (even orders). In structural dynamics this covers, for instance, one- and multi-directional waveguides such as pipes, plates, layers, springs, laminates etc. Beside the many applications just in structural dynamics it covers a huge range of problems in physics, some of the main areas being: vibro-acoustics, electro-magnetics, optics, acoustics, quantum mechanics etc. However, there is still a class of problems within the same areas of physics to which bi-orthogonality does not yet apply. In general, the class comprises unsymmetric waveguides. Nevertheless, there are multiple and vastly different ways to govern the unsymmetry of a waveguide. This has been verified in an analytical waveguide example with an orthotropic cylindrical shell where bi-orthogonality is seen to vanish as soon as the waveguide assumes orthotropy; controlled here by a fibre angle, $\alpha$. This causes the dispersion equation to acquire also odd powers of $k$; having it exhibit an unsymmetric behaviour. Nevertheless, further studies involving the energy flow through the different transmission paths (corresponding to the structural ones in Fig. 1) reveal that for a given load, components other than the loaded one carry energy at the excitation point. Based on the discussion with Fig. 1 (elaborated in [4]) could this suggest that there exist a more appropriate (or general) set of 'generalised' forces/displacements in which the 'load carrying property' of Fig. 1 is preserved and in which the symmetry properties are eventually recovered? If such generalised forces/displacements exist, will it in this case correspond to a simple coordinate transformation through the angle $\alpha$? And if so how does this help to explain how to transform unsymmetric problems governed by a non-conservative system? Given the derivation of bi-orthogonality for symmetric waveguides, there are grounds for pessimism here when it comes to its generalisation for unsymmetric problems. Nevertheless, hope persist as the physical interpretation from Sec. 2 and 3 (discussed in detail in [4]) suggest otherwise, because there are no obvious reasons as to why, for instance, orthogonality of waves should suddenly vanish or why free waves at different boundaries can suddenly not be transferred individually to other boundaries using their eigenfunctions and thereby cease to obey the boundary identities.

Thus, the future work indeed covers by which means unsymmetric waveguides may be transformed or viewed as symmetric ones. If possible it means that much more complicated problems can be solved easily by transformation into a solution space where analytical expressions exist. These possibilities are believed to justify the effort of finding such transformations.

5. Conclusion
In this paper we have illustrated and emphasised the strength of applying the bi-orthogonality relation to solve both partially and fully bounded eigenvalue problems. Derivation of the bi-orthogonality (and orthogonality) relation relies only on the reciprocity relation and Class properties of the generalised forces and displacements, which is a virtue of symmetric waveguide problems. Therefore, it is indeed simple. Then, by application of the bi-orthogonality relation in the forced waveguide problem we showed that amplitudes may always be found individually and that bi-orthogonality conveniently replicates the conventional definition of total energy so that it may be perceived as a generalisation hereof. From this, showing linearity of the total energy is a trivial exercise.
By similar application of bi-orthogonality to the Boundary Integral Equations we have shown that they conveniently resolve from tedious integral equations to simple algebraic modal boundary identities. Then using these together with the modal projection method we convert any set of boundary conditions into a well-posed algebraic equation system. For two special sets of boundary conditions the equation system factorises completely and analytical closed form solutions appear. These solutions may be visualised in the dispersion diagram as intersections of propagating wave branches and horizontal lines (multiples of inverse length of a structure, Fig.2). Further, we conveniently find, using this method, that solution accuracy may be evaluated directly either through visual inspection of energy flow graphs (Fig.1) or by exact error measures.

Finally, we have discussed how bi-orthogonality and its applications for symmetric waveguides may be generalised to unsymmetric waveguides. Even though bi-orthogonality is seen to vanish in the original solution space, the physical interpretation suggests that they are likely to exist in a transformed solution space.

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