A NOVEL COLLOCATION APPROACH TO SOLVE A NONLINEAR STOCHASTIC DIFFERENTIAL EQUATION OF FRACTIONAL ORDER INVOLVING A CONSTANT DELAY

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\textbf{Abstract.} In present work, a step-by-step Legendre collocation method is employed to solve a class of nonlinear fractional delay differential equations (FSDDEs). The step-by-step method converts the nonlinear FSDDE into a non-delay nonlinear fractional stochastic differential equation (FSDE). Then, a Legendre collocation approach is considered to obtain the numerical solution in each step. By using a collocation scheme, the non-delay nonlinear FSDE is reduced to a nonlinear system. Moreover, the error analysis of this numerical approach is investigated and convergence rate is examined. The accuracy and reliability of this method is shown on three test examples and the effect of different noise measures is investigated. Finally, as an useful application, the proposed scheme is applied to obtain the numerical solution of a stochastic SIRS model.

1. \textbf{Introduction.} There are many phenomena in physics, chemistry, and engineering that appear randomly and are described by stochastic processes [13, 8]. Stochastic behavior arises naturally in many different phenomena that are affected by random perturbations, such as biology [2, 43, 22], population dynamics [37, 33], the movement of ions in materials [45], optimal option pricing in finance [32, 11, 20] and various engineering problems [23, 18]. Because of the importance of stochastic differential equations (SDEs) in modeling applications where significant uncertainty is present, theoretical and computational studies of scientists have been quickly increasing.

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The theory of delay differential equations has many applications in various fields such as electronics, financial mathematics and biology. In nature, there are mechanisms that are associated with time delay, which means that the future conditions of the system depend on its past. In cases where some kind of uncertainty is considered in these systems, they are modeled as delay stochastic differential or integro-differential equations. Chen et al. [12] introduced a stochastic susceptible, infective and removed (SIR) model and investigated its stability. Lian et al. [25] surveyed a stochastic delay Gilpin–Ayala competition model where is an ecological competitive system. Moreover, various researches are proposed by scientists based on stochastic delay differential equations [27, 36, 28, 9].

In recent years, more researches have been done in the field of fractional stochastic differential equation (FSDE) as a special case of SDEs. The definition of fractional order operators can preserve hereditary and memory traits of a considered functions in a real problem. This property helps researchers to propose more accurate models of various phenomena [34, 6, 3, 5, 16, 7, 21]. Thus, many problems in physics, economics and other sciences have been modeled as fractional stochastic differential equations [38, 44, 1, 4]. In [14] the global existence and uniqueness of solutions for the FSDEs in the Caputo sense are described. In [26] the existence, uniqueness and well-posedness of solution for a partial FSDE are considered. Also, in [30] the unique solvability of a fractional stochastic delay differential equation (FSDDE) is proved.

Hereafter, the main aim is to propose a step-by-step collocation scheme based on the Legendre polynomials to solve the following class of nonlinear FSDDEs

\[
D_0^\alpha u(t) = P(t, u(t), u(t - \tau)) + H(t, u(t))\dot{B}(t), \quad t \in (0, T],
\]

subject to the initial condition

\[
u(t) = \eta(t), \quad t \in [-\tau, 0].
\]

\[\dot{B}(t) := \frac{dB(t)}{dt}\]

is a white noise that \(B(t), t \in [0, T]\) denotes Brownian motion adapted to a filtration \(\mathcal{F} = \{\mathcal{F}_t\}_{t \in [0, T]}\) on the probability space \((\Omega, \mathcal{F}, \mathbb{P})\). Also, \(D_0^\alpha[\cdot]\) is the Caputo fractional derivative defined as [34]:

\[
D_0^\alpha u(t) = \frac{1}{\Gamma(1 - \alpha)} \int_0^t \frac{u'(s)}{(t - s)^\alpha} \, ds, \quad 0 < \alpha < 1,
\]

where \(\Gamma(\cdot)\) shows the Gamma function. The functions \(P\) and \(H\) are known measurable functions which satisfy the following Lipschitz conditions:

\[
\|P(t, u, \tilde{u}) - P(t, v, \tilde{v})\| \leq \rho_u \|u - v\| + \rho_{\tilde{u}} \|\tilde{u} - \tilde{v}\|,
\]

\[
\|H(t, u) - H(t, v)\| \leq \rho_H \|u - v\|,
\]

where \(\rho_u\), \(\rho_{\tilde{u}}\) and \(\rho_H\) are positive real constants and the unknown stochastic function \(u(t)\) should be determined. In addition, \(\tau\) represents the delay and \(\eta(t)\) denotes the history function on \([-\tau, 0]\).

If \(\tau \to 0\), the delay term in Eq. (1) will be removed. Also, when \(\alpha \to 1\), Eq. (1) becomes a classical nonlinear differential equation with a multiplicative white noise in the following form:

\[
du(t) = p(t, u(t))dt + h(t, u(t))dB(t).
\]

Let \(u^m(t)\) denotes the solution to the integer order SDE

\[
du^m(t) = p(t, u^m(t))dt + h(t, u^m(t))dB^m(t),
\]
where $B^m(t)$ represents the $m$th term of a sequence of piecewise linear approximations to the Brownian motion $B(t)$ on $[0, T]$ and $p$ and $h$ are smooth functions. Then, $u^m(t)$ converges in the mean, as $m \to \infty$, to the solution of the Stratonovich SDE

$$du(t) = p(t, u(t))dt + h(t, u(t)) \circ dB(t),$$

and does not converge to the Itô’s interpretation of the corresponding stochastic equation (6). The famous Wong-Zakai theorem [41, 42] provides a crucial insight in the theory of SDEs. Due to this theorem, if $p(t, u), h(t, u) \text{ and } h'(t, u) = \frac{\partial h(t, u)}{\partial u}$ be continuous real functions and $p(t, u), h(t, u) \text{ and } h'(t, u)$ satisfy the Lipschitz conditions with respect to $u$, then the sequence of solutions $u^m(t)$ of (7) converges in the mean to $u(t)$ as $m \to \infty$, where $u(t)$ is the unique solution of the SDE

$$du(t) = \left( p(t, u(t)) + \frac{1}{2} h'(t, u(t))h(t, u(t)) \right) dt + h(t, u(t))dB(t).$$

This equation is to be interpreted as a SDE.

Usually the analytical solutions of the equations in the form (1) are not in hand. Hence, numerical schemes are utilized to find suitable approximate solutions. Wang et al. in [40] considered $\theta$-Maruyama method to solve the nonlinear SDDEs, in the case $\alpha = 1$. Also, the problem (1)-(2) has been considered in [31] and a B-spline collocation approach is proposed to obtain the numerical solution. Furthermore, finite difference methods [17, 35], finite element methods [39, 15], wavelets Galerkin methods [19], operational matrix method based on hat functions [29] and mean square scheme [24] are some of the other algorithms that are used for solving the problems related to SDEs.

The organization of this work is as follows: the Legendre polynomials and their properties are described in Section 2. Section 3 presents the step-by-step Legendre collocation scheme. Section 4 discusses the error analysis of the proposed method. The results of the numerical examples are provided in Section 5. In Section 6, as an application, the proposed scheme is used to find the numerical solution of a stochastic SIRS model. Finally, a conclusion is drawn in Section 7.

2. Preliminary concepts.

**Definition 2.1.** The Riemann-Liouville fractional integration of order $0 < \alpha < 1$ is defined as [34]

$$\mathcal{D}_0^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s)ds.$$

**Definition 2.2.** ([10]) The Legendre polynomials (LPs) are defined on $[-1, 1]$ as:

$$\hat{\theta}_i(t) = \frac{(-1)^i}{2^i i!} \frac{d^i}{dt^i} \left( (1-t)^i (1+t)^i \right).$$

This function satisfies the recurrence relation

$$\hat{\theta}_{i+1}(t) = \frac{2i+1}{i+1} \hat{\theta}_i(t) - \frac{i}{i+1} \hat{\theta}_{i-1}(t), \quad i = 1, 2, \ldots$$

in which $\hat{\theta}_0(t) = 1$ and $\hat{\theta}_1(t) = t$.

**Definition 2.3.** The shifted LPs are defined on $[a, b]$ by

$$\theta_i^{a,b}(t) = \hat{\theta}_i \left( \frac{2}{b-a} (t-b) + 1 \right), \quad i = 0, 1, 2, \ldots$$
Also, an explicit analytic form of this polynomial is as:

\[ \theta_{a,b}^i(t) = \sum_{r=0}^{i} \sum_{m=j=0}^{i} Z_{r,m,j} t^r, \]

where

\[ Z_{r,m,j} = \frac{(-1)^{r-j+m} 2^m j^m (i-r)! (i-2r-m)! (m-j)!}{(b-a)^m (i-r)! (i-2r-m)! (m-j)!}, \]

which satisfy

\[ \theta_{a,b}^i(a) = (-1)^i, \quad \theta_{a,b}^i(b) = 1. \]

**Remark 1.** The shifted LPs \( \theta_{a,b}^i(t) \) for \( i = 0, 1, 2, \ldots \) are orthogonal on \([a,b]\) according to the unit weight function. Also, their orthogonality condition is satisfied for these polynomials as follows

\[ \langle \theta_{a,b}^i(t), \theta_{a,b}^j(t) \rangle_w = \int_a^b \theta_{a,b}^i(t) \theta_{a,b}^j(t) dt = \frac{2}{2i+1} \delta_{i,j}. \]

\( \delta_{i,j} \) is the Kronecker function and \( \langle \cdot, \cdot \rangle_w \) is weighted inner product.

A square integrable function \( f \) can be expanded based on the shifted LPs as:

\[ f(t) = \sum_{i=0}^{\infty} c_{a,b}^i \theta_{a,b}^i(t), \]

where the coefficients \( c_i \) are given by

\[ c_{a,b}^i = \frac{2i+1}{2} \int_a^b f(t) \theta_{a,b}^i(t) dt, \quad i = 0, 1, 2, \ldots \]

By truncating the infinite series in (10), we can estimate \( f(t) \) as follows:

\[ f(t) \simeq \sum_{i=0}^{n} c_{a,b}^i \theta_{a,b}^i(t) \triangleq C_{a,b}^T \Theta_{a,b}(t), \]

where

\[ C_{a,b} = \begin{bmatrix} c_{a,b}^0, \ldots, c_{a,b}^n \end{bmatrix}^T, \]

\[ \Theta_{a,b}(t) = \begin{bmatrix} \theta_{a,b}^0(t), \ldots, \theta_{a,b}^n(t) \end{bmatrix}^T. \]

**Definition 2.4.** Consider the bounded interval \( \Omega := [a, b] \subset \mathbb{R} \) and the weight function \( w \). Let \( \mathbf{m} \in \mathbb{Z}^+ \). The Sobolev space \( H^m_w(\Omega) \) is defined by

\[ H^m_w(\Omega) = \left\{ f \in L^2_w(\Omega) : f^{(r)}(t) \in L^2_w(\Omega), r = 0, \ldots, \mathbf{m} \right\}. \]

**Remark 2.** The Sobolev space \( H^m_w(\Omega) \) is accompanied with the weighted inner product

\[ \langle f, g \rangle_{\mathbf{m},w} = \sum_{r=0}^{\mathbf{m}} \int_\Omega w(s) f^{(r)}(s) g^{(r)}(s) ds, \]

and the and associated norm

\[ \| f \|_{H^m_w(\Omega)}^2 = \sum_{r=0}^{\mathbf{m}} \| f^{(r)} \|_{L^2_w(\Omega)}^2. \]
Moreover, the associated semi-norm is defined as follows

\[ |f|_{H_w^m, N}^2 = \sum_{r = \min(n, N + 1)}^n \|f^{(r)}\|_{L_w^2(\Omega)}^2, \]

where \( N \) is the number of nodal bases.

**Lemma 2.5.** ([10]) Assume that \( w(t) = 1 \) and \( f \in H_w^m(\Omega) \). Also, let for a continuous function \( f \), \( I_n f \) denotes the Legendre interpolant. Then, the truncation error \( f - I_n f \) satisfies

\[ \|f - I_n f\|_{L_w^2(\Omega)} \leq \lambda h^{\min(\tilde{m}, n) - \tilde{m}} |f|_{H_w^m(\Omega)}, \]

where \( h = b - a \) and the constant \( \lambda \) is independent of both \( n \) and \( h \), although they depend as above on \( \tilde{m} \).

### 3. Description of the method.

Now, a steps-collocation technique is described to solve the problem (1)-(2). It is clear that, for \( t \in [0, \tau] \), the nonlinear FSDDE (1) is equivalent to the following nonlinear non-delay FSDE

\[ D_{0,t}^\alpha u(t) = P(t, u(t), \eta(t - \tau)) + H(t, u(t))\dot{B}(t), \quad t \in (0, \tau], \quad u(0) = \eta(0). \]

To obtain a numerical solution of (14)-(15), we will use the Legendre collocation method. Hence, we consider the solution \( U^1_n(t) \) of (14), as

\[ U^1_n(t) = \sum_{i=0}^n c^0_{i,\tau} \theta^0_{i,\tau}(t) = C^T_{0,\tau} \Theta^0_{0,\tau}(t), \]

According to Eqs. (14) and (16), we have

\[ C^T_{0,\tau} D_{0,\tau}^\alpha(t) = P(t, C^T_{0,\tau} \Theta_{0,\tau}(t), \eta(t - \tau)) + H(t, C^T_{0,\tau} \Theta_{0,\tau}(t))\dot{B}(t), \]

where \( D_{0,\tau}^\alpha(t) \) is Caputo’s fractional derivative of \( \Theta_{0,\tau}(t) \) and is obtained from the explicit form (9) as follows:

\[ D_{0,\tau}^\alpha(t) = \left[ 0, \vartheta^\alpha_{0,\tau}(t), ..., \vartheta^{\alpha, n}_{0,\tau}(t) \right]^T, \]

where

\[ \vartheta^{\alpha, i}_{0,\tau}(t) = \sum_{r=0}^i \sum_{m=0}^m \sum_{j=0}^m \Gamma(i + 1) \Gamma(i + 1 - \alpha) Z_{r,m,i}^0 \tau^j t^{i-j}. \]

Also, from initial condition (15) and Eq. (16), we have

\[ C^T_{0,\tau} \Theta_{0,\tau}(0) = \eta(0). \]

The quadrature nodes for discrete shifted Legendre basis obtain by

\[ t^0_{0,\tau} = 0, \quad t^0_i = \tau, \quad \forall i = 1, ..., n - 1, \quad t^0_i \text{ is roots of } \partial_t \theta^0_{n,\tau}(t), \]

where \( \partial_t [\cdot] := \frac{\partial}{\partial t} [\cdot] \). Therefore, considering (17) and (19) at the collocation nodes \( t_i, i = 0, ..., n, \) results

\[ A_{0,\tau}^T C_{0,\tau} = P_{0,\tau} + B_{0,\tau} H_{0,\tau}, \]
where
\[ A_{0,\tau} = \left[ \Theta_{0,\tau}(t_0^{0,\tau}), D_{0,\tau}^0(t_1^{0,\tau}), \ldots, D_{0,\tau}^n(t_n^{0,\tau}) \right], \quad (22) \]
\[ P_{0,\tau} = \left[ \eta(t_0^{0,\tau}), P_1^{0,\tau}, \ldots, P_n^{0,\tau} \right]^T, \quad (23) \]
\[ H_{0,\tau} = \left[ 0, H_1^{0,\tau}, \ldots, H_n^{0,\tau} \right]^T, \quad (24) \]

where
\[ P_{i}^{0,\tau} := P\left(t_i^{0,\tau}, C_{0,\tau}^T \Theta_{0,\tau}(t_i^{0,\tau}), \eta(t_i^{0,\tau} - \tau) \right), \]
\[ H_{i}^{0,\tau} := H(t_i^{0,\tau}, C_{0,\tau}^T \Theta_{0,\tau}(t_i^{0,\tau})), \]

and
\[ B_{0,\tau} = \text{diag}(1, b_1^{0,\tau}, \ldots, b_n^{0,\tau}), \quad b_i^{0,\tau} := B(t_i^{0,\tau}) - B(t_{i-1}^{0,\tau}). \quad (25) \]

Now, the unknown coefficients \( c_i^{0,\tau}, i = 0, 1, \ldots, n \), can be determined by solving the system of nonlinear algebraic equations (21) with a iterative numerical approach, such as Newton’s method. Thus, we achieve an approximate solution \( \tilde{U}_n^i(t) \) for (14)-(15) (or (1)-(2) on \([0, \tau]\)). Generally, to get a numerical estimation of (1)-(2) on \([ (j-1)\tau, j\tau] \), for \( j = 2, \ldots, M \) and \( M = \left\lceil \frac{T}{\tau} \right\rceil \), we need to solve the following nonlinear equation
\[ D_{0,\tau}^n u_j(t) = P(t, u_j(t), u_{j-1}(t - \tau)) + H(t, u_j(t)) \dot{B}(t), \quad (26) \]
\[ u_j(0) = u_{j-1}(\tau), \quad (27) \]

where \( u_j(t) := u((j-1)\tau + t) \).

To this end, the numerical solution \( \tilde{U}_n^i(t) \) of (26) is considered as:
\[ \tilde{U}_n^i(t) = \sum_{i=0}^{n} c_i^{(j-1)\tau,j\tau} \theta_{(j-1)\tau,j\tau}(t) = C_{(j-1)\tau,j\tau}^T \Theta_{(j-1)\tau,j\tau}(t). \quad (28) \]

Similarly, by using the proposed method, we have
\[ A_{(j-1)\tau,j\tau}^n C_{(j-1)\tau,j\tau} = P_{(j-1)\tau,j\tau} + B_{(j-1)\tau,j\tau} H_{(j-1)\tau,j\tau}, \quad (29) \]

where
\[ A_{(j-1)\tau,j\tau} = \left[ \Theta_{(j-1)\tau,j\tau}(t_0^{(j-1)\tau,j\tau}), D_{(j-1)\tau,j\tau}^0(t_1^{(j-1)\tau,j\tau}), \ldots, D_{(j-1)\tau,j\tau}^n(t_n^{(j-1)\tau,j\tau}) \right], \quad (30) \]
\[ P_{(j-1)\tau,j\tau} = \left[ \tilde{U}_n^{-1}(t_0^{(j-1)\tau,j\tau}), P_1^{(j-1)\tau,j\tau}, \ldots, P_n^{(j-1)\tau,j\tau} \right]^T, \quad (31) \]
\[ H_{(j-1)\tau,j\tau} = \left[ 0, H_1^{(j-1)\tau,j\tau}, \ldots, H_n^{(j-1)\tau,j\tau} \right]^T, \quad (32) \]

where
\[ D_{(j-1)\tau,j\tau} = \left[ 0, \theta_{(j-1)\tau,j\tau}^{0,1}, \ldots, \theta_{(j-1)\tau,j\tau}^{n,i}, \ldots, \theta_{(j-1)\tau,j\tau}^{0,n} \right]^T, \quad (33) \]
\[ \theta_{(j-1)\tau,j\tau}^{0,i}(t) = \sum_{r=0}^{\left\lceil \frac{t}{\alpha} \right\rceil} \sum_{m=0}^{\left\lceil \frac{t}{\alpha} \right\rceil} \frac{\Gamma(i + 1)}{\Gamma(i + 1 - \alpha)} z_{r,m,i}^{(j-1)\tau,j\tau} t^i - \alpha, \]
\[ P_{i}^{(j-1)\tau,j\tau} := P\left(t_i^{(j-1)\tau,j\tau}, C_{(j-1)\tau,j\tau}^T \Theta_{(j-1)\tau,j\tau}(t_i^{(j-1)\tau,j\tau}), \tilde{U}_n^{-1}(t_i^{(j-1)\tau,j\tau} - \tau) \right), \]
\[ H_{i}^{(j-1)\tau,j\tau} := H\left(t_i^{(j-1)\tau,j\tau}, C_{(j-1)\tau,j\tau}^T \Theta_{(j-1)\tau,j\tau}(t_i^{(j-1)\tau,j\tau}) \right). \]
and

\[ B_{(j-1)\tau,j\tau} = \text{diag}(1, b_1^{(j-1)\tau,j\tau}, ..., b_n^{(j-1)\tau,j\tau}), \]  

\[ b_i^{(j-1)\tau,j\tau} := B(t_i^{(j-1)\tau,j\tau}) - B(t_{i-1}^{(j-1)\tau,j\tau}), \]

in which the quadrature nodes \( t_i^{(j-1)\tau,j\tau} \) obtain by

\[ t_{0}^{(j-1)\tau,j\tau} = (j-1)\tau, \quad t_n^{(j-1)\tau,j\tau} = j\tau, \]

\[ \forall i = 1, ..., n, \quad t_i^{(j-1)\tau,j\tau} \text{ is roots of } \partial_i^\theta_n^{(j-1)\tau,j\tau}(t). \]

So, for each \( j = 2, ..., M \), the unknown coefficients \( c_i^{(j-1)\tau,j\tau}, i = 0, 1, ..., n \), can be identified by solving the nonlinear system (29). After applying these schemes, \( u(t) \) is calculated on \([0, T]\), as follows

\[ u(t) \simeq \Upsilon_n(t) = \begin{cases} \Upsilon_n^1(t), & t \in [0, \tau], \\ \vdots & \vdots \\ \Upsilon_n^{j-1}(t), & t \in [(j-1)\tau, j\tau], \\ \vdots & \vdots \\ \Upsilon_n^M(t), & t \in [(M-1)\tau, T]. \end{cases} \]  

4. **Error analysis.** In the rest, the convergence analysis of the obtained numerical approximation will be examined.

**Theorem 4.1.** Suppose that \( u_j(t) \) at the \( j \)-th step, is the exact solution of (26) and \( u_j(t) \in H_{\text{loc}}^\infty(\Omega) \) where \( \Omega = [(j-1)\tau, j\tau] \) and \( j, m \geq 1 \). Moreover, let \( \Upsilon_n^j(t) \) is the numerical estimation of (26) obtained by the presented step-by-step collocation method and \( R_n^j(t) \) is the residual term for this numerical solution. Thus, we have

\[ E\|R_n^j(t)\|_{L^2(\Omega)} \leq (\gamma_j + \zeta_j\|B(t)\|_{L^2(\Omega)})^\tau^{\min(\tilde{m}, n)}n^{-\tilde{m}}|u_j|_{H_{\text{loc}}^\infty(\Omega)} + \mu_j^{\tau^{\min(\tilde{m}, n)}n^{-\tilde{m}}}|u_j|_{H_{\text{loc}}^\infty(\Omega)}^\tau, \]

where \( \gamma_j, \zeta_j \) and \( \mu_j \) are positive constants independent of both \( n \) and \( \tau \), although they depend as above on \( \tilde{m} \).

**Proof.** First, by the definition 2.1, Eq. (26) is rewritten as

\[ u_j(t) = u_{j-1}(\tau) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}P(s, u_j(s), u_{j-1}(s-\tau))ds \\ + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}H(s, u_j(s))dB(s). \]

So, \( \Upsilon_n^j(t) \) satisfies the following equation

\[ \Upsilon_n^j(t) = \Upsilon_n^{j-1}(\tau) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}P(s, \Upsilon_n^j(s), \Upsilon_n^{j-1}(s-\tau))ds \\ + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}H(s, \Upsilon_n^j(s))dB(s) + R_n^j(t). \]

\( R_n^j(t) \) is the residual term that satisfies the relation

\[ R_n^j(t) = -e_n^j(t) + e_n^{j-1}(\tau) + \Lambda_n^j[P](t) + \Lambda_n^j[H](t), \]
in which \( e_j^n(t) = u_j(t) - \mathcal{U}_n(t) \),
\[
\Lambda_j^n[P](t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left( P(s, u_j(s), u_{j-1}(s-\tau)) - P(s, \mathcal{U}_n^j(s), \mathcal{U}_n^{j-1}(s-\tau)) \right) ds,
\]
and
\[
\Lambda_j^n[H](t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left( H(s, u_j(s)) - H(s, \mathcal{U}_n^j(s)) \right) dB(s).
\]
Thus, we have
\[
\mathbb{E}[R_j^n(t)]_{L_2^u(\Omega)} \leq \mathbb{E}[e_j^n(t)]_{L_2^u(\Omega)} + \mathbb{E}[\varepsilon_j^n(t)]_{L_2^u(\Omega)} + \mathbb{E}[\Lambda_j^n[P](t)]_{L_2^u(\Omega)} + \mathbb{E}[\Lambda_j^n[H](t)]_{L_2^u(\Omega)}.
\]
By using Lemma 2.5, we obtain
\[
\mathbb{E}[e_j^n(t)]_{L_2^u(\Omega)} \leq \lambda_j \tau^{\min(m,n)} n^{-m} |u_j| H_\text{w}^m(\Omega), \quad \mathbb{E}[\varepsilon_j^n(t)]_{L_2^u(\Omega)} \leq \tilde{\lambda}_j \tau^{\min(m,n)} n^{-m} |u_{j-1}| H_\text{w}^m(\Omega),
\]
where the constants \( \lambda_j \) and \( \tilde{\lambda}_j \) are independent of both \( n \) and \( \tau \), although they depend as above on \( m \). The function \( P \) satisfies the Lipschitz condition (4), thus by using Eq.(41) and Lemma 2.5, we obtain
\[
\mathbb{E}[\Lambda_j^n[P](t)]_{L_2^u(\Omega)} \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \mathbb{E}[P(s, u_j(s), u_{j-1}(s-\tau)) - P(s, \mathcal{U}_n^j(s), \mathcal{U}_n^{j-1}(s-\tau))] ds
\]
\[
\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left\{ \rho_s \mathbb{E}[e_j^n(s)]_{L_2^u(\Omega)} + \rho_t \mathbb{E}[\varepsilon_j^n(t-\tau)]_{L_2^u(\Omega)} \right\} ds
\]
\[
\leq \frac{T^n}{\Gamma(\alpha+1)} \left\{ \rho_s \lambda_j \tau^{\min(m,n)} n^{-m} |u_j| H_\text{w}^m(\Omega) + \rho_t \tilde{\lambda}_j \tau^{\min(m,n)} n^{-m} |u_{j-1}| H_\text{w}^m(\Omega) \right\}.
\]
Also, the function \( H \) satisfies the Lipschitz condition (5), thus by using Eq.(42) and Lemma 2.5, we obtain
\[
\mathbb{E}[\Lambda_j^n[H](t)]_{L_2^u(\Omega)} \leq \frac{1}{\Gamma(\alpha)} \int_0^t |t-s|^{\alpha-1} \mathbb{E}[H(s, u_j(s)) - H(s, \mathcal{U}_n^j(s))]_{L_2^u(\Omega)} dB(s)
\]
\[
\leq \frac{1}{\Gamma(\alpha)} \left( \int_0^t |t-s|^{\alpha-1} dB(s) \right) \left\{ \rho_s \mathbb{E}[e_j^n(s)]_{L_2^u(\Omega)} \right\},
\]
since, \( 0 < s < t < T \), we have
\[
\mathbb{E}[\Lambda_j^n[H](t)]_{L_2^u(\Omega)} \leq \frac{T^n}{\Gamma(\alpha)} |B(t)|_{L_2^u(\Omega)} \rho_s \lambda_j \tau^{\min(m,n)} n^{-m} |u_j| H_\text{w}^m(\Omega).
\]
So, Eqs. (43)-(47) result
\[
\mathbb{E}[R_j^n(t)]_{L_2^u(\Omega)} \leq (\gamma_j + \zeta_j |B(t)|_{L_2^u(\Omega)}) \tau^{\min(m,n)} n^{-m} |u_j| H_\text{w}^m(\Omega)
\]
\[
+ \mu_j \tau^{\min(m,n)} n^{-m} |u_{j-1}| H_\text{w}^m(\Omega),
\]
where \( \gamma_j = (1 + \frac{T^n}{\Gamma(\alpha+1)} \rho_s \lambda_j, \zeta_j = \frac{T^n}{\Gamma(\alpha)} \rho_s \lambda_j \) and \( \mu_j = (1 + \frac{T^n}{\Gamma(\alpha+1)} \rho_t \tilde{\lambda}_j).
\]
\textbf{Corollary 1.} By using Theorem 4.1, it can be concluded that for \( j \geq 1 \), \( \mathbb{E}[R_j^n(t)]_{L_2^u(\Omega)} \) tends to zero, when \( n \to \infty \).
Algorithm.
Input: $T, \tau \in \mathbb{R}^+, n \in \mathbb{Z}^+, \alpha \in (0, 1)$, functions $P, H, \eta$ and Brownian motion process $B(t)$.
Step 1: Compute the shifted Legendre polynomials $\theta^a_b(t)$ from Definition 2.3.
Step 2: Compute the vector of shifted Legendre polynomials $\Theta_a(t)$ from Eq. (13).
Step 3: Compute the collocation points $t^{0, \tau}_i$ for $i = 0, \ldots, n$ of the domain $[0, \tau]$ from Eq. (20).
Step 4: Compute the matrices $A_{0, \tau}$ and $B_{0, \tau}$ from Eqs. (22) and (25).
Step 5: Compute the vector $D_{0, \tau}(t)$ from Eq. (18) and the vectors $P_{0, \tau}$ and $H_{0, \tau}$ from Eqs. (23) and (24).
Step 6: Solve the nonlinear system $A_{0, \tau}C_{0, \tau} = P_{0, \tau} + B_{0, \tau}H_{0, \tau}$ and obtain the unknown vector $C_{0, \tau}$ by using Step 4 and Step 5.
Step 7: Let $U_{0, \tau}(t) := C_{0, \tau} \eta(t)$ on the interval $[0, \tau]$.
Step 8: Start temporal loop for $j = 2, \ldots, M$ where $M = \left\lfloor T \right\rfloor$.
Step 8.1: Compute the collocation points $t^{(j-1)\tau,j \tau}_i$ for $i = 0, \ldots, n$ of the domain $[(j-1)\tau, j \tau]$ from Eqs. (35)-(36).
Step 8.2: Compute the matrices $A_{(j-1)\tau,j \tau}$ and $B_{(j-1)\tau,j \tau}$ from (30) and (34).
Step 8.3: Compute the vectors $P_{(j-1)\tau,j \tau}$, $H_{(j-1)\tau,j \tau}$ and $D_{(j-1)\tau,j \tau}(t)$ from (31)-(33).
Step 8.4: Solve the nonlinear system
$$A_{(j-1)\tau,j \tau} C_{(j-1)\tau,j \tau} = P_{(j-1)\tau,j \tau} + B_{(j-1)\tau,j \tau} H_{(j-1)\tau,j \tau}$$
and obtain the unknown vector $C_{(j-1)\tau,j \tau}$ by using Step 8.2 and Step 8.3.
Step 8.5: Let $U_{0, \tau}(t) := C_{(j-1)\tau,j \tau}^T \eta_{(j-1)\tau,j \tau}(t)$ on $[(j-1)\tau, j \tau]$.
Step 9: Post-processing the results.
Output: The approximate solution: $u(t) \simeq U_{0, \tau}(t)$ from (37).

5. Illustrative examples. In this section, we survey some numerical applications to show the ability of the suggested method for finding the solution of nonlinear FSDDEs. In the described method, the discretized Brownian motion $B(t)$ at the discrete points $t^{i, j \tau}_{i}$ will be considered.

Set $t^0, \tau = 0$. We have $t^{(j-1)\tau,j \tau}_i < t^{(j-1)\tau,j \tau}_{i+1}$ for each value $j = 1, \ldots, M$, and $i < r$. Also, suppose $B^j_{i} = B(t^{(j-1)\tau,j \tau}_i)$ and

$$for \quad j = 1, \ldots, M : \quad \Delta^j_{i} = t^{(j-1)\tau,j \tau}_i - t^{(j-1)\tau,j \tau}_{i-1}.$$

Let $B^0_{i} = t^0, \tau$ with the probability 1.

For $0 \leq t^{(j-1)\tau,j \tau}_i < t^{(j-1)\tau,j \tau}_{i+1} \leq T$, the increment $B^j_{i} - B^j_{i+1}$ is a random variable with normal distribution that its mean is zero and the variance is $t^{(j-1)\tau,j \tau}_i - t^{(j-1)\tau,j \tau}_{i+1}$. Also, for $0 \leq i_1 < i_2 < r_1 < r_2 \leq T$, the increments $B^j_{i_2} - B^j_{i_1}$ and $B^j_{i_2} - B^j_{i_1}$ are independent. Thus, from these conditions, we have

$$for \quad j = 1, \ldots, M : \quad B^j_{i} = B^j_{i-1} + dB^j_{i},$$

where each $dB^j_{i}$ is an independent random variable with the distribution $\sqrt{\Delta^j_{i}} N(0, 1)$ where $N(0, 1)$ is the standard normal distribution. Thus, we evaluate the numerical solution $u(t)$ along $\bar{P}$ discretized Brownian paths. The average of $u(t)$ over these paths is considered.
Table 1. The $l_{\infty}$-norm and $l_2$-norm errors, convergence orders and CPU-time for Example 1.

| n  | $\|{\mathcal{E}}_n\|_\infty$ | CO | $\|{\mathcal{E}}_n\|_2$ | CO | CPU-time(s) |
|----|-----------------|----|-----------------|----|-------------|
| 6  | $5.9498 \times 10^{-2}$ | 0  | $1.1604 \times 10^{-2}$ | 0  | 5.149  |
| 9  | $1.1548 \times 10^{-6}$ | 26.7585 | $1.1768 \times 10^{-7}$ | 28.3589 | 7.982  |
| 12 | $7.8465 \times 10^{-11}$ | 33.3590 | $4.2500 \times 10^{-12}$ | 35.5559 | 12.986  |

The $l_2$-norm and the $l_{\infty}$-norm errors are evaluated using the following definition:

$$\|{\mathcal{E}}_n\|_\infty = \max_{t \in [0,T]} |u(t) - {\mathcal{U}}_n(t)|,$$

$$\|{\mathcal{E}}_n\|_2 = \left( \int_0^T |u(t) - {\mathcal{U}}_n(t)|^2dt \right)^{\frac{1}{2}},$$

where $u(t)$ and ${\mathcal{U}}_n(t)$ are, respectively, the exact solution and the approximate solution defined in (37). The convergence order will be determined with the relation:

$$\text{CO} = \log_{\frac{\|{\mathcal{E}}_{n_1}\|_r}{\|{\mathcal{E}}_{n_2}\|_r}},$$

where $r = 2, \infty$. The codes are written in Matlab software and the computations are performed on a machine using a 1.70 GHz processor.

Example 1. Consider the nonlinear FSDDE

$$D_0^\tau u(t) = u^2(t) - \sin(u(t - \tau)) + \varepsilon \sqrt{t} u(t) \hat{B}(t) + f(t),$$

subject to

$$u(0) = 0, \quad t \in [-\tau, 0],$$

where $\varepsilon$ is a positive constant, $\alpha \in (0, 1)$ and

$$f(t) = \begin{cases} \gamma(11)^{\tau(11) - \alpha} t^{10 - \alpha} - \varsigma^2 t^{20} - \varepsilon \varsigma t^{10} \sqrt{\tau} \hat{B}(t), & t \in [0, \tau], \\ \gamma(11)^{\tau(11) - \alpha} t^{10 - \alpha} - \varsigma^2 t^{20} + \sin(\varsigma (t - \tau)^{10}) - \varepsilon \varsigma t^{10} \sqrt{\tau} \hat{B}(t), & t \in (\tau, 1]. \end{cases}$$

For this problem, the exact solution is

$$u(t) = \begin{cases} 0, & t \in [-\tau, 0], \\ \varsigma t^{10}, & t \in (0, 1]. \end{cases}$$

Figure 1 shows the exact solution, numerical solution and absolute error of $u(t)$ and Figure 2 shows the absolute error for $t \in (0, \tau)$ (left) and $t \in (\tau, 1)$ (right) for $\hat{P} = 40$ discretized Brownian paths, when $\varepsilon = 1$, $\varsigma = 2$, $\tau = 0.5$, $\alpha = 0.75$, $T = 1$ and $n = 10$. Figure 3 displays the numerical and exact solutions of the problem for numerous $\hat{P}$ when $\varepsilon = 2$, $\varsigma = 1$, $\tau = 0.25$, $n = 6$ and $\alpha = 0.75$. Table 1 shows the $l_2$-norm and $l_{\infty}$-norm errors, convergence orders and CPU-time for $\varepsilon = 1$, $\varsigma = 1$, $\alpha = 0.55$, $\tau = 0.25$, $\hat{P} = 80$ and various values of $n$. Finally, Figures 4 represents the numerical estimations for several values of $\varepsilon$ when $\alpha = 0.55$, $\varsigma = 1$, $\tau = 0.2$, $\hat{P} = 30$ and $n = 8$.

Example 2. Consider the nonlinear FSDDE

$$D_0^\tau u(t) = u(t) - u^2(t - \tau) + u(t) \hat{B}(t) + f(t),$$

with the initial condition:

$$u(0) = 0, \quad t \in [-\tau, 0],$$
where $\alpha \in (0, 1)$ and
\[
f(t) = \begin{cases} 
\frac{\pi^\alpha t^{1-\alpha}}{\Gamma(2-\alpha)} \mathcal{F}_2^1([1; [1 - \frac{\alpha}{2}, \frac{3-\alpha}{2}], -\frac{1}{4}\pi^2t^2]) & t \in [0, \tau], \\
-\alpha \sin(\pi t)(1 + \dot{B}(t)) & \quad \text{and} \\
\frac{\pi^\alpha t^{1-\alpha}}{\Gamma(2-\alpha)} \mathcal{F}_2^1([1; [1 - \frac{\alpha}{2}, \frac{3-\alpha}{2}], -\frac{1}{4}\pi^2t^2]) + \alpha^2 \sin^2(\pi(t - \tau)) - \alpha \sin(\pi t)(1 + \dot{B}(t)) & t \in (\tau, T],
\end{cases}
\]
where $F_p^q([a_1, ..., a_p]; [b_1, ..., b_q]; t)$ is the hypergeometric function. With this assumptions, the exact solution is

$$u(t) = \begin{cases} 0, & t \in [-\tau, 0], \\ \alpha \sin(\pi t), & t \in (0, T]. \end{cases}$$

Figure 5 shows the exact and numerical solutions, and absolute errors along $\tilde{P} = 40$ discretized Brownian paths, when $\tau = 0.5$, $T = 4$, $\alpha = 0.75$ and $n = 12$. Figure 6 displays the exact and numerical solutions for numerous $\alpha$ with $\tau = 0.5$, $T = 2$, $n = 10$ and $\tilde{P} = 20$. Table 2 displays the $l_\infty$-norm error, CPU-time and convergence orders and for $\alpha = 0.25$ and $\alpha = 0.75$ when, $\tau = 0.2$, $T = 1$, $\tilde{P} = 20$ and several values of $n$. Also, Figures 7 demonstrates the phase-space diagram $u(t)$ vs. $u(t-\tau)$ when, $\tau = 0.2$, $T = 2$, $\tilde{P} = 15$ and $n = 10$.

**Example 3.** Consider the nonlinear FSDDE

$$D^\alpha_{0,t} u(t) = -\sigma u(t) + \frac{\hat{g}u(t-\tau)}{1 + [u(t-\tau)]^m} + \varepsilon u(t)\dot{B}(t),$$

with initial condition:

$$u(0) = u_0, \quad t \in [-\tau, 0],$$

**Table 2.** Absolute errors, CPU-time and convergence orders for Example 2.

| $n$  | $\|E_n\|_\infty$ | CO | $\|E_n\|_\infty$ | CO | CPU-time(s) |
|------|-------------------|----|-------------------|----|-------------|
| 6    | $5.7280 \times 10^{-6}$ | -- | $1.4068 \times 10^{-4}$ | -- | 77.011 |
| 9    | $3.4219 \times 10^{-9}$ | 18.3071 | $1.9357 \times 10^{-7}$ | 16.2495 | 135.44 |
| 12   | $1.1281 \times 10^{-11}$ | 19.8650 | $9.6833 \times 10^{-10}$ | 18.4155 | 178.156 |

**Figure 4.** The exact and numerical solutions in Example 1 for several values of $\varepsilon$.

**Figure 5.** The exact and numerical solutions in Example 1 for several values of $\varepsilon$.

**Figure 6.** The exact and numerical solutions in Example 1 for several values of $\varepsilon$.

**Figure 7.** The exact and numerical solutions in Example 1 for several values of $\varepsilon$. 


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Figure 5. The exact and numerical solutions (up) and absolute error (down) in Example 2 for $\alpha = 0.75$ on the domain $[0, 4]$.

Figure 6. The exact and numerical solutions in Example 2 for numerous values of $\alpha$.

where $\alpha \in (0, 1)$, $\sigma > 0$, $\tilde{\sigma}$ and $\varepsilon$ are real parameters and $m$ is an even positive integer. This equation with $\alpha = 1$ is the Mackey-Glass equation with a multiplicative noise input that used for describing the stochastic growth of density of blood cells [40]. Figure 8 shows the approximate solution along $\bar{P} = 50$ discretized Brownian paths with $\varepsilon = 1$ and $\alpha = 1$ and Figure 9 demonstrates the phaspace diagram $u(t)$ vs. $u(t - \tau)$ with $\varepsilon = 2$, $\alpha = 0.75$ and $\bar{P} = 100$, when $\sigma = 3$, $\tilde{\sigma} = 1$, $m = 2$, $\varepsilon = 0.15$. 

0 0.5 1 1.5 2 2.5 3 3.5 4
$t$

-0.5
0
0.5
1

0 0.2 0.4 0.6 0.8 1 1.2 1.4 1.6 1.8 2
$t$

-0.5
0
0.5
1

$\bar{P} = 50$

Exact and Numerical solution

-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
1

$10^{-7}$

Absolute Error

0 0.5 1 1.5 2 2.5 3 3.5 4
$t$

-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
1

$\alpha = 0.99$

$\alpha = 0.75$

$\alpha = 0.55$

$\alpha = 0.36$

$\alpha = 0.15$

$\varepsilon = 0.15$

$\varepsilon = 0.55$

$\varepsilon = 0.99$

$\varepsilon = 0.36$

$\varepsilon = 0.15$

$\varepsilon = 0.36$

$\varepsilon = 0.15$
$\tau = 1$, $T = 2$, $u_0 = 1$ and $n = 7$. The results are compared with $\theta$-Maruyama methods [40] with $\theta = 0.5$ and $\theta = 1$ in Figure 8. Figure 10 represents the obtained solutions along $P = 1$ and $P = 100$ discretized Brownian paths with $\alpha = 0.55$ and $\varepsilon = 0.2$ and Figure 11 displays the numerical solution for several $\alpha$, with $\varepsilon = 0.2$ and $P = 150$, when $\sigma = 1$, $\tilde{\sigma} = 2$, $m = 10$, $\tau = 5$, $T = 5$, $u_0 = 0.5$ and $n = 12$. 

![Figure 7](image)

**Figure 7.** The phase-space diagram of Example 2 with $\alpha = 0.99$.

![Figure 8](image)

**Figure 8.** The numerical solution obtained by $\theta$-Maruyama methods [40] (up) and the proposed method (down) with $\alpha = 1$ for Example 3.
6. **Application to a stochastic SIRS model.** In this section, we apply the proposed method for solving the stochastic SIRS model. The corresponding SIRS
Table 3. The parameter values in the stochastic SIRS model.

| Parameter | Value | Parameter | Value |
|-----------|-------|-----------|-------|
| Λ         | 1.8   | μ₂        | 0.5   |
| β         | 0.2   | μ₃        | 0.5   |
| β̃        | 0.1   | γ         | 0.3   |
| μ₁        | 0.85  | γ̃        | 0.25  |

The model is given by [46]

\[
\begin{align*}
\frac{dS(t)}{dt} &= \left(\Lambda - \mu_1 S(t) - \beta \frac{S(t)I(t)}{1 + \beta I(t)} + \gamma e^{-\mu_3 \tau} I(t - \tau)\right) dt - \sigma_1 S(t) dB_1(t) \\
& \quad - \sigma_4 S(t)I(t) \frac{dB_4(t)}{1 + \beta I(t)}, \\
\frac{dI(t)}{dt} &= \left(\beta \frac{S(t)I(t)}{1 + \beta I(t)} - (\mu_2 + \gamma) I(t)\right) dt - \sigma_2 I(t) dB_2(t) + \sigma_4 \frac{S(t)I(t)}{1 + \beta I(t)} dB_4(t), \\
\frac{dR(t)}{dt} &= \left(\gamma I(t) - \mu_3 R(t) - \gamma e^{-\mu_3 \tau} I(t - \tau)\right) dt - \sigma_3 R(t) dB_3(t),
\end{align*}
\]

where \(S(t)\) is the number of population susceptible to the disease, \(I(t)\) denotes the number of infective members and the number of members who have been removed from the possibility of infection through a temporal immunity are denoted by \(R(t)\). Other parameters are defined in [46]. For numerical simulation, the constant parameters in this stochastic SIRS system are described in Table 3.

Suppose \(S(0) = 1.15, I(0) = 1.5, R(0) = 2.7, \tau = 0.2\) and \(n = 8\). First, let \(\sigma_1 = \sigma_2 = \sigma_3 = \sigma_4 = \sigma\). Figure 12 shows the graphs of trajectories of \(S(t), I(t)\) and \(R(t)\) for the deterministic SIRS model with \(\sigma = 0\) (blue) and the stochastic SIRS model (red) with \(\sigma = 0.5\) (up) and \(\sigma = 1.2\) (down). Also, Figure 13 displays the trajectories of \(S(t), I(t)\) and \(R(t)\) for different values of \(\bar{P}\) with \(\sigma_1 = 1.5, \sigma_2 = 0.5, \sigma_3 = 0.85\) and \(\sigma_4 = 0.75\).

7. Conclusion. In this work, a numerical technique for solving nonlinear fractional stochastic differential equations involving a constant delay was introduced. A step-by-step collocation method based on the Legendre polynomials was employed to determine the approximate solution of the considered problem. An error analysis of the discussed method was thoroughly provided. Furthermore, some numerical illustrative applications were investigated to authenticate the ability and effectiveness of the examined numerical scheme for the studied stochastic equation.

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Figure 12. Graphs of the trajectories of $S(t)$, $I(t)$ and $R(t)$ for the deterministic SIRS model with $\sigma = 0$ (blue) and the stochastic SIRS model (red) with $\sigma = 0.5$ (up) and $\sigma = 1.2$ (down).

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