Complete non-orientable minimal surfaces in $\mathbb{R}^3$ and asymptotic behavior

Antonio Alarcón and Francisco J. López

Abstract In this paper we give new existence results for complete non-orientable minimal surfaces in $\mathbb{R}^3$ with prescribed topology and asymptotic behavior.

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1. Introduction

Non-orientable surfaces appear quite naturally in the origin itself of Minimal Surface theory and present a rich and interesting geometry.

This is part of a series of papers devoted to exploit the Runge-Mergelyan type approximation theorem for non-orientable minimal surfaces, furnished by the authors in [3], in order to construct non-orientable minimal surfaces in $\mathbb{R}^3$ with involved geometries.

The first main result of this paper concerns complete non-orientable minimal surfaces in $\mathbb{R}^3$ spanning a finite collection of closed curves.

Theorem 1.1. Let $S_0$ be an open non-orientable smooth surface with finite topology.

Then there exist a relatively compact domain $S$ in $S_0$ and a continuous map $X: \overline{S} \to \mathbb{R}^3$ such that $S$ is homeomorphic to $S_0$, the Hausdorff dimension of $X(\overline{S} \setminus S)$ equals 1, and the restriction $X|_S: S \to \mathbb{R}^3$ is a complete minimal immersion.

Furthermore, the flux of the immersion $X|_S$ can be arbitrarily prescribed.

A map $X$ as those given by Theorem 1.1 is said to be a non-orientable compact complete minimal immersion. The ones in the above theorem are the first examples of such immersions in the literature.

We point out that our method does not give control over the topology of $\overline{S} \setminus S$. In particular, we can not ensure that $\overline{S} \setminus S$ consists of a finite collection of Jordan curves (see Remark 3.2 below).

In the orientable setting, compact complete minimal immersions of the disc into $\mathbb{R}^3$ were constructed by Martín and Nadirashvili [10]; examples with arbitrary finite topology were given later by Alarcón [1]. Other related results can be found in [8, 2]. The construction...
methods used in [10, 1] are refinements of Nadirashvili’s technique for constructing complete bounded minimal surfaces in $\mathbb{R}^3$; see [14]. In the context of null holomorphic curves in $\mathbb{C}^3$ (i.e., holomorphic immersions from Riemann surfaces into $\mathbb{C}^3$ whose real and imaginary parts are conformal minimal immersions), Alarcón and López [6] gave compact complete examples with any given finite topological type. Their method, which relies on the Runge-Mergelyan theorem for null curves in $\mathbb{C}^3$ (see [5]), is the inspiration of our proof.

Compact complete minimal surfaces in $\mathbb{R}^3$ are interesting objects since they lie in the intersection of two well known topics on minimal surface theory: the Plateau problem (dealing with the existence of compact minimal surfaces spanning a given family of closed curves in $\mathbb{R}^3$) and the Calabi-Yau problem (concerning the existence of complete minimal surfaces in bounded regions of $\mathbb{R}^3$). See the already cited sources and references therein for a more detailed discussion.

The second main result of this paper regards with complete non-orientable minimal surfaces in $\mathbb{R}^3$ properly projecting into planar convex domains.

**Theorem 1.2.** Let $S$ be an open non-orientable smooth surface (possibly with infinite topology) and let $D \subset \mathbb{R}^2$ be a convex domain.

Then there exists a complete minimal immersion $X = (X_1, X_2, X_3): S \to \mathbb{R}^3$ such that $(X_1, X_2)(S) \subset D$ and $(X_1, X_2): S \to D$ is a proper map.

Furthermore, the flux of the immersion $X$ can be arbitrarily prescribed.

The problem of whether there exist minimal surfaces in $\mathbb{R}^3$ with hyperbolic conformal structure and properly projecting into $\mathbb{R}^2$ was proposed by Schoen and Yau [15]. (Recall that an open Riemann surface is said to be hyperbolic if it carries non-constant negative subharmonic functions; otherwise it is said to be parabolic.) This question was settled in the affirmative by the authors in both the orientable and the non-orientable settings [5, 4, 3]. More specifically, such surfaces with any given conformal structure and flux map were provided. Theorem 1.2 shows that the corresponding result for complete non-orientable surfaces and convex domains of $\mathbb{R}^2$ holds as well; cf. [7] for the orientable case. On the other hand, Ferrer, Martín, and Meeks [9] provided complete non-orientable minimal surfaces, with arbitrary topology, properly immersed in any given convex domain $\Omega$ of $\mathbb{R}^3$; however, if the domain $\Omega$ is a right cylinder over a convex domain $D$ of $\mathbb{R}^2$, their method does not provide any information about the projection of the surface into $D$.

Although our techniques are inspired by those already developed in the orientable setting (cf. [6, 7]), the non-orientable character of the surfaces requires a much more careful discussion. Indeed, every non-orientable minimal surface $S$ in $\mathbb{R}^3$ can be represented by a triple $(\mathcal{N}, \mathcal{J}, X)$, where $\mathcal{N}$ is an open Riemann surface, $\mathcal{J}: \mathcal{N} \to \mathcal{N}$ is an antiholomorphic involution without fixed points, and $X: \mathcal{N} \to \mathbb{R}^3$ is a conformal minimal immersion satisfying

\[(1.1) \quad X \circ \mathcal{J} = X\]

and $S = X(\mathcal{N})$; see Subsec. 2.2 for details. The moduli space of open Riemann surfaces admitting an antiholomorphic involution without fixed points is real analytic and rather subtle; as a matter of fact this condition implies not only topological restrictions on the surfaces but also conformal ones. Moreover, the required compatibility (1.1) with respect to the antiholomorphic involution makes the construction of non-orientable minimal surfaces a much more involved problem. In order to overcome these difficulties, we exploit the Runge-Mergelyan theorem for non-orientable minimal surfaces [3] (see Theorem 2.12...
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below). This flexible tool enables us to obtain the examples in Theorems 1.1 and 1.2 as limit of sequences of compact non-orientable minimal surfaces (with non-empty boundary), considerably simplifying the construction methods in Sec. 3 and Sec. 4.

Outline of the paper. In Sec. 2 we introduce the background and notation about Riemann surfaces, non-orientable minimal surfaces, and convex domains, that will be needed throughout the paper. In particular, we state the Runge-Mergelyan theorem for non-orientable minimal surfaces [3]; see Theorem 2.12. With this approximation result in hand, Theorems 1.1 and 1.2 are proved in Sec. 3 and Sec. 4, respectively; see the more general Theorems 3.1 and 4.1.

2. Preliminaries

We denote by $\| \cdot \|$, $\langle \cdot, \cdot \rangle$, and $\text{dist}(\cdot, \cdot)$ the Euclidean norm, metric, and distance in $\mathbb{R}^n$, $n \in \mathbb{N}$. Given a compact topological space $K$ and a continuous map $f: K \to \mathbb{R}^n$, we denote by

$$\| f \|_{0,K} := \max \{ \| f(p) \| : p \in K \} \quad (2.1)$$

the maximum norm of $f$ on $K$. The corresponding space of continuous functions on $K$ will be endowed with the $C^0$ topology associated to $\| \cdot \|_{0,K}$.

Given a topological surface $N$, we denote by $bN$ the (possibly non-connected) 1-dimensional topological manifold determined by its boundary points. Open connected subsets of $N \setminus bN$ will be called domains. Proper connected topological subspaces of $N \setminus bN$ being compact surfaces with boundary will be said regions. For any subset $A \subset N$, we denote by $A^\circ, \overline{A}, \text{Fr}A = \overline{A} \setminus A^\circ$, the interior, the closure, and the topological frontier of $A$ in $N$, respectively. Given subsets $A, B$ of $N$, we say that $A \preceq B$ if $\overline{A}$ is compact and $\overline{A} \subset B^\circ$.

2.1. Riemann surfaces and non-orientability. A Riemann surface $N$ is said open if it is non-compact and $bN = \emptyset$. For such $N$, we denote by $\partial$ the global complex operator given by $\partial|_U = \frac{\partial}{\partial z}$ for any conformal chart $(U, z)$ on $N$.

Riemann surfaces are orientable; the conformal structure of a Riemann surface induces a (positive) orientation on it. The natural notion of non-orientable Riemann surface is described as follows; see [11, 3] for a detailed exposition of this issue.

Definition 2.1. By a non-orientable Riemann surface we mean an orbit space $N/\mathcal{I}$; where $N$ is an open Riemann surface and $\mathcal{I}: N \to N$ is an antiholomorphic involution without fixed points. Therefore, a non-orientable Riemann surface is identified with an open Riemann surface $N$ equipped with an antiholomorphic involution $\mathcal{I}$ without fixed points.

In this setting, $N$ is the two-sheets conformal orientable cover of $N/\mathcal{I}$. We denote by $\pi: N \to N/\mathcal{I}$ the natural projection. Further, $N$ carries conformal Riemannian metrics $\sigma_N^2, \sigma_N^2$ such that $\mathcal{I}^*(\sigma_N^2) = \sigma_N^2$.

From now on in this section, let $N, \mathcal{I}, \pi, \text{and } \sigma_N^2, \sigma_N^2$ be as in Def. 2.1.

Definition 2.2. A subset $A \subset N$ is said to be $\mathcal{I}$-invariant if $\mathcal{I}(A) = A$.

For an $\mathcal{I}$-invariant set $A \subset N$, a map $f: A \to \mathbb{R}^n, n \in \mathbb{N}$, is said to be $\mathcal{I}$-invariant if $f \circ \mathcal{I}|_A = f$. 
Let $\Gamma \subset \mathcal{N}$ be an $\mathcal{I}$-invariant subset consisting of finitely many pairwise disjoint smooth Jordan curves $\gamma_j, j = 1, \ldots, k$. For any $\epsilon > 0$ we denote by

$$T_\epsilon(\Gamma) := \{ P \in N : \text{dist}_{\sigma_N^2}(P, \Gamma) < \epsilon \};$$

where $\text{dist}_{\sigma_N^2}$ means Riemannian distance in $(\mathcal{N}, \sigma_N^2)$. Notice that $T_\epsilon(\Gamma) \subset N$ is an $\mathcal{I}$-invariant set. If $\epsilon$ is small enough, the exponential map

$$F : \Gamma \times [-\epsilon, \epsilon] \to \overset{\epsilon}{\Gamma}, \quad F(P, t) = \exp_P(t \, n(P)),$$

is a diffeomorphism and $T_\epsilon(\Gamma) = F(\Gamma \times (-\epsilon, \epsilon))$; where $n$ is an $\mathcal{I}$-invariant normal field along $\Gamma$ in $(\mathcal{N}, \sigma_N^2)$. In this setting, $T_\epsilon(\Gamma)$ is said to be a metric tubular neighborhood of $\Gamma$ (of radius $\epsilon$). Furthermore, if $\pi_\Gamma : \Gamma \times (-\epsilon, \epsilon) \to \Gamma$ denotes the projection $\pi_\Gamma(P, t) = P$, we denote by $\pi_\Gamma : T_\epsilon(\Gamma) \to \Gamma$, $\pi_\Gamma(Q) := \pi_\Gamma(F^{-1}(Q))$, the natural orthogonal projection. Since $\Gamma, n, \text{ and } \sigma_N^2$ are $\mathcal{I}$-invariant, then $F \circ (\mathcal{I} \times \text{Id})|_{\Gamma \times [-\epsilon, \epsilon]} = \mathcal{I} \circ F$ and $\pi_\Gamma \circ \mathcal{I}|_{T_\epsilon(\Gamma)} = \mathcal{I} \circ \pi_\Gamma$.

**Definition 2.3.** A domain $U \subset \mathcal{N}$ is said to be bordered if it is the interior of a compact Riemann surface $\overline{U} \subset \mathcal{N}$ with smooth boundary. In this case $bU = \overline{U} \setminus U \subset \mathcal{N}$ consists of finitely many closed Jordan curves.

If $\mathcal{N}$ is of finite topology, we denote by $\mathscr{B}_3(\mathcal{N})$ the family of $\mathcal{I}$-invariant bordered domains $U \Subset \mathcal{N}$ such that $\mathcal{N}$ is a topological tubular neighborhood of $U$. (The latter means that $\mathcal{N} \setminus \overline{U}$ has no relatively compact connected components and consists of finitely many open annuli.)

### 2.2. Non-orientable minimal surfaces.

In this subsection we describe the Weierstrass representation for non-orientable minimal surfaces (see [11]), and introduce some notation.

An $\mathcal{I}$-invariant conformal minimal immersion $X : \mathcal{N} \to \mathbb{R}^3$ induces a conformal minimal immersion $\overline{X} : \mathcal{N}/\mathcal{I} \to \mathbb{R}^3$, satisfying $X = \overline{X} \circ \pi$. In this sense, $X(\mathcal{N})$ is an immersed non-orientable minimal surface in $\mathbb{R}^3$. Conversely, any immersed non-orientable minimal surface in $\mathbb{R}^3$ comes in this way.

Let $X = (X_j)_{j=1,2,3} : \mathcal{N} \to \mathbb{R}^3$ be an $\mathcal{I}$-invariant conformal minimal immersion. Denote by $\phi_j = \partial X_j, j = 1, 2, 3$, and $\Phi = \partial X \equiv (\phi_j)_{j=1,2,3}$. The 1-forms $\phi_j$ are holomorphic, have no real periods, and satisfy

\begin{equation}
\sum_{j=1}^{3} \phi_j^2 = 0
\end{equation}

and

\begin{equation}
\mathcal{I}^* \Phi = \overline{\Phi}.
\end{equation}

The intrinsic metric in $\mathcal{N}$ is given by

\begin{equation}
ds^2 = \sum_{j=1}^{3} |\phi_j|^2;
\end{equation}

hence

\begin{equation}
\sum_{j=1}^{3} |\phi_j|^2 \text{ vanishes nowhere on } \mathcal{N}.
\end{equation}
The triple $\Phi$ is said to be the \textit{Weierstrass representation} of $X$.

Conversely, any vectorial holomorphic 1-form $\Phi = (\phi_j)_{j=1,2,3}$ on $\mathcal{N}$ without real periods, enjoying (2.2), (2.3), and (2.5), determines an $\mathcal{I}$-invariant conformal minimal immersion $X: \mathcal{N} \to \mathbb{R}^3$ by the expression

$$X = \Re \int \Phi,$$

where $\Re$ means real part. Cf. [11].

The following notation will be required later on.

**Definition 2.4.** For any $\mathcal{I}$-invariant subset $A \subset \mathcal{N}$, we denote by $M_{\mathcal{I}}(A)$ the space of $\mathcal{I}$-invariant conformal minimal immersions of $\mathcal{I}$-invariant open domains $W \subset \mathcal{N}$, containing $A$, into $\mathbb{R}^3$ by the expression

$$X = \Re \int \Phi,$$

where $\Re$ means real part. Cf. [11].

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where $\Re$ means real part. Cf. [11].
(b) $R_S := \overline{S'}$ is non-empty and consists of a finite collection of pairwise disjoint compact regions in $\mathcal{N}$ with $C^0$ boundary.

(c) $C_S := S \setminus R_S$ consists of a finite collection of pairwise disjoint analytical Jordan arcs.

(d) Any component $\alpha$ of $C_S$ with an endpoint $P \in R_S$ admits an analytical extension $\beta$ in $\mathcal{N}$ such that the unique component of $\beta \setminus \alpha$ with endpoint $P$ lies in $R_S$.

**Figure 2.1.** An $\mathcal{I}$-admissible set $S \subset \mathcal{N}$.

An $\mathcal{I}$-invariant compact subset $S \subset \mathcal{N}$ enjoying (b), (c), and (d), is $\mathcal{I}$-admissible if and only if $i_*: \mathcal{H}_1(S, \mathbb{Z}) \to \mathcal{H}_1(\mathcal{N}, \mathbb{Z})$ is a monomorphism; where $\mathcal{H}_1(\cdot, \mathbb{Z})$ means first homology group, $i: S \to \mathcal{N}$ denotes the inclusion map, and $i_*$ is the induced group morphism.

From now on in this section, let $S \subset \mathcal{N}$ be an $\mathcal{I}$-admissible set.

**Definition 2.8.** We say that an $\mathcal{I}$-invariant map $X: S \to \mathbb{R}^3$ is an $\mathcal{I}$-invariant generalized minimal immersion, and write $X \in \mathcal{M}_{g, \mathcal{I}}(S)$, if it meets the following requirements:

- $X|_{R_S} \in \mathcal{M}_2(R_S)$ (see Def. 2.4); hence it extends as an $\mathcal{I}$-invariant conformal minimal immersion $X_0$ to an open domain $V$ in $\mathcal{N}$ containing $R_S$.
- For any component $\alpha$ of $C_S$ and any open analytical Jordan arc $\beta$ in $\mathcal{N}$ containing $\alpha$, $X|_{\alpha}$ is a regular curve admitting a smooth extension $X_\beta$ to $\beta$ such that $X_\beta|_{V \cap \beta} = X_0|_{V \cap \beta}$.

Notice that $X|_S \in \mathcal{M}_{g, \mathcal{I}}(S)$ for all $X \in \mathcal{M}_2(S)$.

Let $X \in \mathcal{M}_{g, \mathcal{I}}(S)$, and let $\varpi$ be a smooth 3-dimensional real 1-form on $C_S$. This means that $\varpi = (\varpi_j)_{j=1,2,3}$, where $\varpi_j$ is a real smooth 1-form on $C_S$, $j = 1, 2, 3$. For any $\alpha \subset C_S$ we write $\varpi|_{\alpha} = \varpi(\alpha(s))\, ds$, where $s$ is the arc-length parameter of $X \circ \alpha$. By definition, $\varpi$ is said to be a mark along $C_S$ with respect to $X$ if for any arc $\alpha \subset C_S$ the following conditions hold:

- $\varpi(\alpha(s)) \in \mathbb{R}^3$ is a smooth unitary vector field along $\alpha$ orthogonal to $(X \circ \alpha)'(s)$.
- $\varpi$ extends smoothly to any open analytical arc $\beta$ in $\mathcal{N}$ containing $\alpha$.
- $\varpi(\beta(s))$ is unitary, orthogonal to $(X \circ \beta)'(s)$, and tangent to $X(R_S)$ at $\beta(s)$ for any $\beta(s) \in \beta \cap R_S$, where as above $s$ is the arc-length parameter of $(X \circ \beta)(s)$.

Let $n: R_S \to S^2$ denote the Gauss map of the (oriented) conformal minimal immersion $X|_{R_S}$. The mark $\varpi$ is said to be orientable with respect to $X$ if the orientations at the two
endpoints of each arc in $C_S$ agree, that is to say, if there exists $\delta \in \{-1, 1\}$ such that for any regular embedded curve $\alpha \subset S$ and arc-length parametrization $(X \circ \alpha)(s)$ of $X \circ \alpha$,

$$(X \circ \alpha)'(s_0) \times \varpi(\alpha(s_0)) = \delta n(\alpha(s_0))$$

for all $s_0 \in \alpha^{-1}(C_S \cap R_S)$.

Orientable marks along $C_S$ with respect to $X$ always exist since $\mathcal{N}$ is orientable. An orientable mark $\varpi$ with respect to $X$ is said to be positively oriented if $\delta = 1$. Obviously, if $\varpi$ is orientable with respect to $X$ then either $\varpi$ or $-\varpi$ is positively oriented.

In the sequel we will only consider orientable marks.

If $\varpi$ is a positively oriented mark along $C_S$ with respect to $X$, we denote by $n_{\varpi}: S \to S^2 \subset \mathbb{R}^3$ the map given by $n_{\varpi}|_{R_S} = n$ and $(n_{\varpi} \circ \alpha)(s) := (X \circ \alpha)'(s) \times \varpi(\alpha(s))$, where $\alpha$ is any component of $C_S$ and $s$ is any arc-length parameter of $X \circ \alpha$. By definition, $n_{\varpi}$ is said to be the (generalized) Gauss map of $X$ associated to the orientable mark $\varpi$.

**Definition 2.9.** We denote by $\mathcal{M}^*_\mathcal{N}(S)$ the space of marked immersions $X_{\varpi} := (X, \varpi)$, where $X \in \mathcal{M}^*_\mathcal{N}(S)$ and $\varpi$ is a positively oriented mark along $C_S$ with respect to $X$ satisfying the following properties:

- $\mathcal{J}^*(\varpi) = -\varpi$, or equivalently,

$$n_{\varpi} \circ \mathcal{J} = -n_{\varpi}.$$ 

- If $\text{st} : S^2 \to \mathbb{C}$ is the stereographic projection, the function $\text{st} \circ n_{\varpi} : S \to \mathbb{C}$, which is holomorphic on an open neighborhood of $R_S$, is smooth in an analogous way to Def. 2.8.

A 1-form $\theta$ on $S$ is said to be of type $(1, 0)$ if for any conformal chart $(U, z)$ in $\mathcal{N}$, $\theta|_{U \cap S} = h(z)dz$ holds for some function $h: U \cap S \to \mathbb{C}$. Finite sequences $\Theta = (\theta_1, \ldots, \theta_n)$, where $\theta_j$ is a $(1, 0)$-type 1-form for all $j \in \{1, \ldots, n\}$, are said to be $n$-dimensional vectorial $(1, 0)$-forms on $S$. The space of continuous $n$-dimensional $(1, 0)$-forms on $S$ will be endowed with the $C^0$ topology induced by the norm

$$\|\Theta\|_{0, S} := \|\Theta\|_{\sigma_{\mathcal{N}}(0, S)} = \max_{S} \left\{ \left( \sum_{j=1}^{n} \frac{\|\theta_j\|_{\sigma_{\mathcal{N}}(0, S)}}{\sigma_{\mathcal{N}}^2} \right)^{1/2} \right\}.$$ 

**Definition 2.10.** For every $X_{\varpi} \in \mathcal{M}^*_\mathcal{N}(S)$, we denote by $\partial X_{\varpi}$ the complex vectorial 1-form on $S$ given by

$$\partial X_{\varpi}|_{R_S} = \partial(X|_{R_S}), \quad \partial X_{\varpi}(\alpha'(s)) = dX(\alpha'(s)) + t\varpi(s);$$

where $t = \sqrt{-1}$,

- $dX$ denotes the vectorial 1-form of type $(1, 0)$ on $C_S$ given by

$$dX|_{\alpha \cap U} = (X \circ \alpha)'(x)dz|_{\alpha \cap U},$$

for any component $\alpha$ of $C_S$, where $(U, z = x + iy)$ is any conformal chart on $\mathcal{N}$ satisfying that $z(\alpha \cap U) \subset \mathbb{R}$ (the existence of such a conformal chart is guaranteed by the analyticity of $\alpha$), and

- $s$ is the arc-length parameter of $X|_{\alpha}$ for which $\{dX(\alpha'(s_1)), \varpi(s_1)\}$ are positive, where $s_1$ and $s_2$ are the values of $s$ for which $\alpha(s) \in bR_S$.

In the setting of Def. 2.10, writing $\partial X_{\varpi} = (\partial_j)_{j=1,2,3}$, it follows that

$$\sum_{j=1}^{3} \partial_j^2$$

vanishes everywhere on $S$. 

\[ \sum_{j=1}^{3} |\phi_j|^2 \text{ vanishes nowhere on } S, \] and
\[ \mathcal{H}(\partial X_w) = \partial X_w. \]

For these reasons the triple \( \partial X_w \) is said the **generalized Weierstrass representation** of \( X_w \).

For \( F \in \mathcal{M}_2(S) \), we denote by \( \omega F \) the conormal field of \( F \) along \( C_S \). Notice that \( \omega F \) satisfies (2.7) and \( (\partial F)|_S = \partial F_{\omega F} \); where \( F_{\omega F} := (F|_S, \omega F) \in \mathcal{M}_{b,3}^*(S) \).

The space \( \mathcal{M}_{b,3}^*(S) \) is naturally endowed with the following \( C^1 \) topology:

**Definition 2.11.** Let \( W \) be an \( \mathcal{H} \)-invariant open domain in \( N \) containing \( S \).

- Given \( X_w, Y_\xi \in \mathcal{M}_{b,3}^*(S) \), we set
  \[ \|X_w - Y_\xi\|_{1,S} := \|X - Y\|_{0,S} + \|\partial X_w - \partial Y_\xi\|_{0,S} \text{ (see (2.1) and (2.8)).} \]

- Given \( F, G \in \mathcal{M}_2(S) \), we set
  \[ \|F - X_w\|_{1,S} := \|F_{\omega F} - X_w\|_{1,S} \quad \text{and} \quad \|F - G\|_{1,S} := \|F_{\omega F} - G_{\omega G}\|_{1,S}. \]

- We will say that \( X_w \in \mathcal{M}_{b,3}^*(S) \) can be approximated in the \( C^1 \) topology on \( S \) by \( \mathcal{H} \)-invariant conformal minimal immersions in \( \mathcal{M}_3(W) \) if for any \( \epsilon > 0 \) there exists \( Y \in \mathcal{M}_3(W) \) such that \( \|Y - X_w\|_{1,S} < \epsilon \).

If \( X_w \in \mathcal{M}_{b,3}^*(S) \), then the group homomorphism

\[ (2.9) \quad p_{X_w} : \mathcal{H}_1(S, \mathbb{Z}) \rightarrow \mathbb{R}^3, \quad p_{X_w}(\gamma) = 3 \int_{\gamma} \partial X_w, \]

is said to be the **generalized flux map** of \( X_w \). Notice that \( p_{X_w} \) satisfies (2.6). Obviously, \( p_{X_wY} = p_Y|_{\mathcal{H}_1(S, \mathbb{Z})} \) provided that \( X = Y|_S \) for some \( Y \in \mathcal{M}_2(S) \).

The following Runge-Mergelyan type approximation result for non-orientable minimal surfaces plays a fundamental role in this paper.

**Theorem 2.12** ([3]). Let \( S \subset N \) be an \( \mathcal{H} \)-admissible subset (see Def. 2.7), let \( X_w \in \mathcal{M}_{b,3}^*(S) \) (see Def. 2.9), and let \( p : \mathcal{H}_1(N, \mathbb{Z}) \rightarrow \mathbb{R}^3 \) be a group homomorphism such that

- \( p(\mathcal{J}_s(\gamma)) = -p(\gamma) \) for all \( \gamma \in \mathcal{H}_1(N, \mathbb{Z}) \), and
- \( p|_{\mathcal{H}_1(S, \mathbb{Z})} \) equals the generalized flux map \( p_{X_w} \) of \( X_w \).

Write \( X_w = (X = (X_j)_{j=1,2,3}, \omega) \), \( \partial X_w = (\phi_j)_{j=1,2,3} \), and \( p = (p_j)_{j=1,2,3} \).

Then the following assertions hold:

(I) \( X_w \) can be approximated in the \( C^1 \) topology on \( S \) (see Def. 2.11) by \( \mathcal{H} \)-invariant conformal minimal immersions \( Y \in \mathcal{M}_3(N) \) with flux map \( p_Y = p \).

(II) If \( \phi_3 \) does not vanish everywhere on \( S \) and extends to \( N \) as a holomorphic 1-form without real periods, vanishing nowhere on \( C_S \), and satisfying \( p_3(\gamma) = 3 \int_{\gamma} \phi_3 \) for all \( \gamma \in \mathcal{H}_1(N, \mathbb{Z}) \), then \( X_w \) can be approximated in the \( C^1 \) topology on \( S \) by \( \mathcal{H} \)-invariant conformal minimal immersions \( Y = (Y_j)_{j=1,2,3} \in \mathcal{M}_3(N) \) with flux map \( p_Y = p \) and third coordinate \( Y_3 = X_3 \).
2.4. Convex domains and Hausdorff distance. A convex domain \( D \subset \mathbb{R}^n, \) \( D \neq \mathbb{R}^n, \) \( n \geq 2, \) is said to be regular (resp., analytic) if its frontier \( \text{Fr}D = \partial D \setminus D \) is a regular (resp., analytic) hypersurface of \( \mathbb{R}^n. \) Moreover, \( D \) is said to be strictly convex if \( \text{Fr}D \) contains no straight segments.

For any couple of compact subsets \( K \) and \( O \) in \( \mathbb{R}^n, \) the Hausdorff distance between \( K \) and \( O \) is given by
\[
\delta^H(K, O) := \max \left\{ \sup_{x \in K} \inf_{y \in O} \|x - y\|, \sup_{y \in K} \inf_{x \in O} \|x - y\right\}.
\]

A sequence \( \{K^j\}_{j \in \mathbb{N}} \) of (possibly unbounded) closed subsets of \( \mathbb{R}^n \) is said to converge in the Hausdorff topology to a closed subset \( K^0 \) of \( \mathbb{R}^n \) if \( \{K^j \cap B\}_{j \in \mathbb{N}} \to K^0 \cap B \) in the Hausdorff distance for any closed Euclidean ball \( B \subset \mathbb{R}^n. \) If \( K^j \subset K^{j+1} \subset K^0 \) for all \( j \in \mathbb{N} \) and \( \{K^j\}_{j \in \mathbb{N}} \to K^0 \) in the Hausdorff topology, then we write \( \{K^j\}_{j \in \mathbb{N}} \nearrow K^0. \)

**Theorem 2.13** ([13, 12]). Let \( B \subset \mathbb{R}^n \) be a (possibly neither bounded nor regular) convex domain. Then there exists a sequence \( \{D^j\}_{j \in \mathbb{N}} \) of bounded strictly convex analytic domains in \( \mathbb{R}^n \) with \( \overline{D^j} \nearrow B. \)

3. Compact complete non-orientable minimal immersions

In this section we prove Theorem 1.1, which is a particular instance of Theorem 3.1 below.

Recall that we have fixed \( \mathcal{N}, \mathcal{I}, \pi, \) and \( \sigma^2_{\mathcal{N}} \) as in Def. 2.1; see Remark 2.6. Throughout this section \( \mathcal{N} \) is assumed to be of finite topology. See Def. 2.3 and Def. def:M(A) for notation.

**Theorem 3.1.** Let \( U \in \mathcal{B}_3(\mathcal{N}) \), let \( K \subset U \) be a compact set, and let \( X \in \mathcal{M}_3(\overline{U}). \)

Then, for any \( \epsilon > 0 \) there exists an \( \mathcal{I}\)-invariant domain \( M \in \mathcal{N} \) and a continuous \( \mathcal{I}\)-invariant map \( Y : \overline{M} \to \mathbb{R}^3, \) enjoying the following properties:

- \( K \subset M \subset U \) and the inclusion map \( \mathcal{H}_3(M, \mathbb{Z}) \hookrightarrow \mathcal{H}_3(\mathcal{N}, \mathbb{Z}) \) is an isomorphism.
- \( Y|_M : M \to \mathbb{R}^3 \) is a complete conformal \( \mathcal{I}\)-invariant minimal immersion.
- \( \|Y - X\|_{1,K} < \epsilon. \)
- \( \|Y - X\|_{0,M} < \epsilon \) and the Hausdorff distance \( \delta^H(X(\overline{bU}), Y(\text{Fr}M)) < \epsilon. \)
- If \( \text{Fr}M = \Gamma \cup \mathcal{I}(\Gamma) \) with \( \Gamma \cap \mathcal{I}(\Gamma) = \emptyset, \) then \( Y|_\Gamma : \Gamma \to \mathbb{R}^3 \) is injective.
- The Hausdorff dimension of \( Y(\text{Fr}M) \) equals 1.
- The flux map \( \mathcal{F}_Y \) of \( Y \) equals the one \( \mathcal{F}_X \) of \( X. \)

**Remark 3.2.** Theorem 3.1 does not insure that \( M \) is a bordered domain in \( \mathcal{N}. \) In particular, we cannot guarantee that \( \text{Fr}M \) consists of a finite collection of pairwise disjoint Jordan curves; the same happens to \( Y(\text{Fr}M). \)

As usual in this kind of constructions, the map \( Y \) in Theorem 3.1 will be constructed in a recursive procedure; the key tool in this process is Lemma 3.6 below. Most of the technical arguments in the proof of Lemma 3.6 are contained in the following

**Lemma 3.3.** Let \( U \in \mathcal{B}_3(\mathcal{N}), \) let \( K \subset U \) be a compact set, let \( \mathcal{T} \in \mathcal{N} \setminus K \) be a metric tubular neighborhood of \( \overline{bU} \) in \( \mathcal{N}, \) and let \( \mathcal{P} : \mathcal{T} \to \overline{bU} \) be the orthogonal projection. Let \( X \in \mathcal{M}_3(\overline{U}), \) let \( \mathfrak{F} : \overline{bU} \to \mathbb{R}^3 \) be an \( \mathcal{I}\)-invariant analytical map, and let \( \mu > 0 \) such that
\[
(3.1) \quad \|X - \mathfrak{F}\|_{0,\overline{bU}} < \mu.
\]
Then, for any $\rho > 0$ and any $\epsilon > 0$ there exist $V \in \mathcal{B}(N)$ and $Y \in \mathcal{M}(\overline{V})$ enjoying the following properties:

(i) $K \subset V \subset U$ and $b\overline{V} \subset T$.
(ii) If $b\overline{V} = \Gamma \cup J(\Gamma)$ with $\Gamma \cap J(\Gamma) = \emptyset$, then $Y|_\Gamma : \Gamma \to \mathbb{R}^3$ is an embedding.
(iii) $\text{dist}_Y(K, b\overline{V}) > \rho$.
(iv) $\|Y - X\|_{1,K} < \epsilon$.
(v) $\|Y - \mathcal{F} \circ \mathcal{P}\|_{0, b\overline{V}} < \sqrt{4\rho^2 + \mu^2} + \epsilon$.
(vi) The flux map $p_Y$ of $Y$ equals the one $p_X$ of $X$.

**Proof.** Let $\epsilon_0 > 0$.

Since $U \in \mathcal{B}(N)$, then

$$b\overline{U} = \bigcup_{i=1}^j (\beta_i \cup \mathcal{I}(\beta_i)),$$

where $\{\beta_i\}_{i=1}^j$ are pairwise disjoint smooth Jordan curves with $\beta_i \cap \mathcal{I}(\beta_j) = \emptyset$ for all $i, j \in \{1, \ldots, j\}$. Denote by $\beta = \bigcup_{i=1}^j \beta_i$. Obviously, $b\overline{U} = \beta \cup \mathcal{I}(\beta)$ and $\beta \cap \mathcal{I}(\beta) = \emptyset$.

For any $P \in \beta$ we choose a simply connected open neighborhood $O_P$ of $P$ in $\overline{U} \cap T$ meeting the following requirements:

(A1) $\mathcal{P}(Q) \subset O_P \cap \beta_i$ for all $Q \in O_P$ and $P \in \beta_i$, $i = 1, \ldots, i$.
(A2) $\max \{\|X(Q_1) - X(Q_2)\|, \|\mathcal{F}(\mathcal{P}(Q_1)) - \mathcal{F}(\mathcal{P}(Q_2))\|\} < \epsilon_0$, for all $\{Q_1, Q_2\} \subset O_P$ and $P \in \beta_i$, $i = 1, \ldots, i$.
(A3) $\|X - \mathcal{F} \circ \mathcal{P}\|_{0, O_P} < \mu$ for all $P \in \beta$.
(A4) $O_P \cap \mathcal{I}(O_P) = \emptyset$ for all $P \in \beta$.

Notice that (A3) is ensured by hypothesis (3.1), provided that $O_P$ is chosen small enough. To guarantee (A2), just take $O_P$ sufficiently small and use the continuity of $X$, $\mathcal{F}$, and $\mathcal{P}$.

Set

$$\mathcal{O} = \{O_P : P \in \beta\},$$

and observe that $\mathcal{O} \cap \beta := \{O_P \cap \beta : P \in \beta\}$ is an open covering of $\beta$. Choose $M \in \mathcal{B}(N)$ satisfying that

(3.2) $K \subset M \Subset U$, $\overline{U} \setminus M \subset \bigcup_{P \in \beta} (O_P \cup \mathcal{I}(O_P))$, and $\mathcal{P}|_{b\overline{M}} : b\overline{M} \to b\overline{U}$ is one to one.

For instance, one can take $M$ as the complement in $\overline{U}$ of a sufficiently small metric tubular neighborhood of $b\overline{U}$.

Since $U, M \in \mathcal{B}(N)$ and (3.2), then

(3.3) $\overline{U} \setminus M = \bigcup_{i=1}^j (A_i \cup \mathcal{I}(A_i)) \subset T$;

where $\{A_i\}_{i=1}^j$ are pairwise disjoint compact annuli with $A_i \cap \mathcal{I}(A_j) = \emptyset$ for all $i, j$, and $\beta_i \subset A_i$. $i = 1, \ldots, i$. Denote $\alpha_i = A_i \cap b\overline{M}$, $i = 1, \ldots, i$, $\alpha = \bigcup_{i=1}^j \alpha_i$, and $A = \bigcup_{i=1}^j A_i$.

Obviously, $b\overline{M} = \alpha \cup \mathcal{I}(\alpha)$, $\alpha \cap \mathcal{I}(\alpha) = \emptyset$, $\overline{U} \setminus M = A \cup \mathcal{I}(A)$, $A \cap \mathcal{I}(A) = \emptyset$, and $A \subset \bigcup_{P \in \beta} O_P$. In particular, $\mathcal{O} \cap A := \{O_P \cap A : P \in \beta\}$ is an open covering of $A$.

Denote by $\mathbb{Z}_n$ the additive cyclic group of integers modulus $n$, $n \in \mathbb{N}$. Since $\mathcal{O} \cap A$ is an open covering of the compact set $\alpha$ in $A$, then there exist $j \in \mathbb{N}$, $j \geq 3$, and a family of compact Jordan arcs $\{\alpha_{i,j}(i, j) \in I = \{1, \ldots, i\} \times \mathbb{Z}_n\}$ meeting the following requirements:

(B1) $\cup_{j \in \mathbb{Z}_n} \alpha_{i,j} = \alpha_i$ for all $i \in \{1, \ldots, i\}$. 

(B2) \( \alpha_{i,j} \) and \( \alpha_{i,j+1} \) have a common endpoint \( Q_{i,j} \) and are otherwise disjoint for all \( (i,j) \in I \).

(B3) \( \alpha_{i,j} \cap \alpha_{i,k} = \emptyset \) for all \( (i,j) \in I \) and \( k \in \mathbb{Z} \setminus \{j, j + 1\} \).

(B4) \( \alpha_{i,j} \cup \alpha_{i,j+1} \subset O_{i,j} \in \Omega \) for all \( (i,j) \in I \).

Up to suitably trimming the \( O_{i,j} \)'s we can further assume that

(B5) \( O_{i,j-1} \cap O_{i,j} \cap O_{i,j+1} \) is connected for any \( (i,j) \in I \). Observe that \( Q_{i,j} \in O_{i,j-1} \cap O_{i,j} \cap O_{i,j+1} \neq \emptyset \) by (B2) and (B4).

For any \( (i,j) \in I \), choose a point \( P_{i,j} \in O_{i,j-1} \cap O_{i,j} \) and set

\[
e^{3}_{i,j} := \begin{cases} X(P_{i,j}) - \hat{\mathfrak{V}}(P_{i,j}) \over \| X(P_{i,j}) - \hat{\mathfrak{V}}(P_{i,j}) \| & \text{if } X(P_{i,j}) - \hat{\mathfrak{V}}(P_{i,j}) \neq 0 \\
\text{any vector in } \mathbb{S}^2 & \text{if } X(P_{i,j}) - \hat{\mathfrak{V}}(P_{i,j}) = 0.
\end{cases}
\]

Observe that \( e^{3}_{i,j} \in \mathbb{S}^2 \) and the orthogonal projection of \( X(P_{i,j}) - \hat{\mathfrak{V}}(P_{i,j}) \) into the orthogonal complement of \( e^{3}_{i,j} \) in \( \mathbb{R}^3 \) equals 0.

For any \( (i,j) \in I \) choose \( e^{1}_{i,j}, e^{2}_{i,j}, e^{3}_{i,j} \) an orthonormal basis of \( \mathbb{R}^3 \), and denote by \( B_{i,j} \in O(3, \mathbb{R}) \) the orthogonal matrix of change of coordinates in \( \mathbb{R}^3 \) from the canonical basis to the basis \( \{e^{1}_{i,j}, e^{2}_{i,j}, e^{3}_{i,j}\} \); i.e.,

\[
B_{i,j} = \left( e^{1}_{i,j} T, e^{2}_{i,j} T, e^{3}_{i,j} T \right)^{-1},
\]

where \( T \) means transpose.

Let \( \{r_{i,j} : (i,j) \in I \} \) be a family of pairwise disjoint analytical compact Jordan arcs in \( A \) meeting the following requirements:

(C1) \( r_{i,j} \subset O_{i,j-1} \cap O_{i,j} \cap O_{i,j+1} \) for all \( (i,j) \in I \).

(C2) \( r_{i,j} \) has initial point \( Q_{i,j} \), final point \( \hat{\mathfrak{V}}(Q_{i,j}) \), and it is otherwise disjoint from \( \alpha_{i} \cup \beta_{i} \), for all \( (i,j) \in I \).

(C3) The set \( S := \overline{M} \cup \bigcup_{(i,j) \in I} (r_{i,j} \cup \mathfrak{J}(r_{i,j})) \subset \overline{U} \subset \mathcal{N} \) is \( \mathfrak{J} \)-admissible in the sense of Def. 2.7.

See Fig. 3.1. For instance, one can take \( r_{i,j} = \mathfrak{P}^{-1}(\mathfrak{V}(Q_{i,j})) \cap (\overline{U} \setminus M) \) for all \( (i,j) \in I \); see (3.2).

Properties (C1) and (C2) are possible thanks to (3.2), (A1), (B2), (B4), and (B5). Notice that \( r_{i,j} \cap \mathfrak{J}(r_{i,j}) = \emptyset \) for all \( (i,j) \in I \); see (A4) and (C1).

The first main step in the proof of Lemma 3.3 consists of deforming \( X \) over \( \bigcup_{(i,j) \in I} (r_{i,j} \cup \mathfrak{J}(r_{i,j})) \). To do this we first extend \( X|_{\mathfrak{M}} \) to an \( \mathfrak{J} \)-invariant generalized minimal immersion \( \hat{X} \in \mathcal{M}_{\mathfrak{R}, \mathfrak{J}}(S) \) (see Def. 2.8) enjoying the following properties:

(a) \( \hat{X}|_{\mathfrak{M}} = X \).

(b) \( \| \hat{X}(P) - \hat{X}(Q) \| < e_0 \) for all \( \{P, Q \} \subset r_{i,j} \), for all \( (i,j) \in I \).

(c) If \( \Upsilon \subset r_{i,j} \) is a Borel measurable subset, then

\[
\min \left\{ \ell(\pi_{i,j}(\hat{X}(\Upsilon))) , \ell(\pi_{i,j+1}(\hat{X}(\Upsilon))) \right\} + \min \left\{ \ell(\pi_{i,j}(\hat{X}(r_{i,j} \setminus \Upsilon))) , \ell(\pi_{i,j+1}(\hat{X}(r_{i,j} \setminus \Upsilon))) \right\} > 2 \rho,
\]
where $\ell$ denotes Euclidean length in $\mathbb{R}^3$ and

\begin{equation}
\pi_{i,j}: \mathbb{R}^3 \to \text{span}\{e^3_{i,j}\} \leq \mathbb{R}^3 \text{ the orthogonal projection,}
\end{equation}

for all $(i,j) \in I$.

To construct $\tilde{X}$ we first define it over each arc $r_{i,j}$ to be highly oscillating in the direction of both $e^3_{i,j}$ and $e^3_{i,j+1}$ (property (c)), but with small diameter in $\mathbb{R}^3$ (property (b)). We then define $\tilde{X}$ over each arc $\mathcal{I}(r_{i,j})$ just to be $\mathcal{I}$-invariant.

Theorem 2.12 applied to any marked immersion $\tilde{X}_{\omega} = (\tilde{X}, \omega) \in \mathcal{M}_{g,\omega}^*(S)$ and $p = p_{\tilde{X}_{\omega}} = p_X: \mathcal{H}(N, \mathbb{Z}) \to \mathbb{R}^3$ (recall that $M, U \in \mathcal{B}_2(N)$ and see property (a) of $\tilde{X}$), furnishes an $\mathcal{I}$-invariant conformal minimal immersion $F \in \mathcal{M}_2(U)$ satisfying:

\begin{itemize}
\item [(D1)] $\|F - X\|_{1, \tilde{M}} < \epsilon_0$.
\item [(D2)] $\|F - X\|_{0, S} < 2\epsilon_0$.
\item [(D3)] If $\mathcal{Y} \subset r_{i,j}$ is a Borel measurable subset, then
\begin{align*}
\min \left\{ \ell(\pi_{i,j}(F(\mathcal{Y}))) \right\} + \\
\min \left\{ \ell(\pi_{i,j+1}(F(\mathcal{Y}))) \right\} > 2\rho,
\end{align*}

for all $(i,j) \in I$; see (3.6).
\item [(D4)] The flux map $p_F$ of $F$ equals the one $p_X$ of $X$.
\end{itemize}

Take into account properties (a), (b), and (c) of $\tilde{X}$. For (D2), recall that $\tilde{X}(Q_{i,j}) = X(Q_{i,j})$ for all $(i,j) \in I$ and use properties (C1), (A2), and (b).

By continuity of $F$, there exists $W \in \mathcal{B}_2(N)$ such that:

\begin{itemize}
\item [(E1)] $K \subset M \Subset W \subset U$.
\item [(E2)] $S \subset \overline{W}$ and $S \cap \overline{bW} = \cup_{(i,j) \in I} \{ \mathcal{P}(Q_{i,j}), \mathcal{I}(Q_{i,j}) \}$; recall (C2).
\item [(E3)] $\overline{W} \setminus M = \cup_{(i,j) \in I} \left( (\tilde{\alpha}_{i,j} \cup \tilde{\mathcal{I}}_{i,j}) \cup (\tilde{\gamma}_{i,j} \cup \tilde{\mathcal{I}}_{i,j}) \right)$, where $\tilde{\alpha}_{i,j}$ and $\tilde{\gamma}_{i,j}$ are simply connected compact neighborhoods of $\alpha_{i,j}$ and $\gamma_{i,j}$, respectively, in $\overline{W} \setminus M$, such that
\begin{itemize}
\item $\tilde{\alpha}_{i,j} \cap \tilde{\gamma}_{i,j} = \emptyset$ for all $(i,j) \in I$ and $k \in \mathbb{Z}_1 \setminus \{j - 1, j\}$.
\item $\tilde{\alpha}_{i,j} \cap \tilde{\gamma}_{i,k} = \emptyset$ for all $(i,j) \in I$ and $k \in \mathbb{Z}_1 \setminus \{j\}$.
\end{itemize}
\item [(E4)] $\|F - X\|_{1, \overline{W}} < \epsilon_0$.
\item [(E5)] $\|F - X\|_{0, \overline{W}} < 2\epsilon_0$.
\item [(E6)] Denote by $\gamma_{i,j}$ the piece of $b\overline{W}$ connecting $\mathcal{P}(Q_{i,j-1})$ and $\mathcal{P}(Q_{i,k})$, and containing $\mathcal{P}(Q_{i,k})$ for no $k \in \mathbb{Z}_1 \setminus \{j - 1, j\}$. Then $\gamma_{i,j}$ is split into three connected sub-arcs $\gamma_{i,j}^{-1} \subset \text{Fr}(\tilde{\gamma}_{i,j-1})$, $\Gamma_{i,j}$, and $\gamma_{i,j}^{-1} \subset \text{Fr}(\tilde{\gamma}_{i,j})$, such that $\gamma_{i,j}^{-1} \cap \gamma_{i,j}^{-1} = \emptyset$, $\Gamma_{i,j}$ has a common point with $\gamma_{i,j}^{-1}$ and a common point with $\gamma_{i,j}^{-1}$, and the following assertion holds:
\begin{itemize}
\item For any arc $\sigma \subset \tilde{r}_{i,j}$ connecting $\tilde{\alpha}_{i,j} \cup \tilde{\alpha}_{i,j+1}$ and $\gamma_{i,j}^{-1} \cup \gamma_{i,j+1}^{-1}$, if $\sigma = \sigma_0 \cup \sigma_1$, where $\sigma_k$ is a collection of subarcs of $\sigma$ contained in the closure of the connected component of $\tilde{r}_{i,j} \setminus r_{i,j}$ intersecting $\alpha_{i,j+k}$, $k = 0, 1$, then
\begin{equation}
\ell(\pi_{i,j}(F(\sigma_0))) + \ell(\pi_{i,j+1}(F(\sigma_1))) > 2\rho \text{ for all } (i,j) \in I.
\end{equation}
\end{itemize}
\item [(E7)] $(\partial F B_{i,j}^T)_{3}$ vanishes nowhere on $b\overline{W}$ for all $(i,j) \in I$; here $(\cdot)_3$ means third coordinate in $\mathbb{R}^3$ and $B_{i,j}$ is given by (3.5).
\end{itemize}

See Fig. 3.1.
Denote by $\Omega_{i,j}$ the closure of the connected component of $W \setminus S$ bounded by $r_{i,j-1}$, $\alpha_{i,j}$, $\alpha_{i,j+1}$, and $\gamma_{i,j}$, for all $(i,j) \in I$. Observe that $\Omega_{i,j}$ is a closed disc for all $(i,j) \in I$ and $W = M \cup (\bigcup_{(i,j) \in I} \Omega_{i,j} \cup J(\Omega_{i,j})))$. See Fig. 3.1.

Let $\eta: \{1, \ldots, i\} \to \{1, \ldots, i\} \times \mathbb{Z}$ be the bijection $\eta(k) = (E(k-1)) + 1, k-1$, where $E(\cdot)$ means integer part.

The second main step in the proof of Lemma 3.3 consists of deforming $F$ over each disc $\Omega_{\eta(k)}$, $k \in \{1, \ldots, i\}$. For that, let us recursively construct a sequence $\{F_0 = F, F_1, \ldots, F_i\} \subset \mathcal{M}_3(\bar{U})$ satisfying the following properties for all $k \in \{0, 1, \ldots, i\}$:

(F.1k) $\|F_k - X\|_{1,\mathcal{M}} < \epsilon_0$.
(F.2k) $\|F_k - X\|_{0,\mathcal{S}((\bigcup_{\eta=1}^{k+1} (\Omega_{\eta})) \cup (\Omega_{\eta}))} < 2\epsilon_0$.
(F.3k) $\langle F_k - F_{k-1}, e_{\eta(k)} \rangle = 0$, $k \geq 1$.
(F.4k) $\|F_k(Q) - X(Q)\| > 2\rho + 1$, $\forall Q \in \Gamma_{\eta}$, $\forall a \in \{1, \ldots, k\}$, $k \geq 1$.
(F.5k) For any arc $\sigma \subset \tilde{\Gamma}_{\eta(a)}$ connecting $\tilde{\alpha}_{\eta(a)} \cup \tilde{\alpha}_{\eta(a)+0,1}$ and $\gamma_{\eta(a)}^{-1} \cup \gamma_{\eta(a)+0,1}$, then
\[
\ell(\pi_{\eta(a)}(F_k(\sigma \cap \Omega_{\eta(a)}))) + \ell(\pi_{\eta(a)}(0,1)(F_k(\sigma \cap \Omega_{\eta(a)+(0,1)}))) > 2\rho \quad \forall a \in \{1, \ldots, i\}.
\]

Recall that $\pi_{\eta(a)}: \mathbb{R}^2 \to \text{span}\{e_{\eta(a)}^3\}$ is the orthogonal projection; see (3.6).
(F.6k) $\|F_k - F_{k-1}\|_{1,\mathcal{W}((\bigcup_{\eta} (\Omega_{\eta})) \cup (\bigcup_{\eta} (\Omega_{\eta})))} < \epsilon_0/ij$, $k \geq 1$.
(F.7k) $(\partial F_k B_{\eta(a)}^T)_{\partial a}$ vanishes nowhere on $b\bar{W}$ for all $a \in \{1, \ldots, i\}$.
(F.8k) The flux map $p_{F_k}$ of $F_k$ equals the one $p_X$ of $X$.

Indeed, observe that (F.1k)=(E4), (F.2k) is implied by (E5), (F.3k)=(E6), and (F.7k)=(E7), whereas (F.3k), (F.4k), and (F.6k) make no sense. Finally (F.8k) follows from (D4). Reason by induction and assume that we already have $F_0, \ldots, F_{k-1}$, for some $k \in \{1, \ldots, i\}$, satisfying the corresponding properties. Let us construct $F_k$.

Denote $G = (G_1, G_2, G_3) := F_{k-1} B_{\eta(k)}^T \in \mathcal{M}_3(\bar{U})$. Obviously, $G \in \mathcal{M}_3(\bar{U})$ and
\[
G_3 = (F_{k-1}, e_{\eta(k)}^3)
\]
(see. (3.5)). Denote
\[
S_k := \mathcal{M} \cup \left( \bigcup_{\eta \neq k} (\Gamma_{\eta} \cup J(\Gamma_{\eta})) \cup \left( \bigcup_{i,j \neq k} (\Omega_{\eta} \cup J(\Omega_{\eta})) \right) \right)
\]
and observe that $S_k$ is $\mathcal{I}$-admissible (Def. 2.7). Observe also that $S_k$ has exactly three connected components, which are

$$\mathcal{M} \cup \left( \bigcup_{a \neq k} \Omega_{\eta(a)} \cup \mathcal{I}(\Omega_{\eta(a)}) \right) = W \setminus (\Omega_{\eta(k)} \cup \mathcal{I}(\Omega_{\eta(k)})), \quad \Gamma_{\eta(k)}, \text{ and } \mathcal{I}(\Gamma_{\eta(k)}); \text{ see (E3)}.$$

Extend $G|_{S_k \setminus (\Gamma_{\eta(k)} \cup \mathcal{I}(\Gamma_{\eta(k)}))}$ to an $\mathcal{I}$-invariant generalized minimal immersion $\widehat{G} = (\widehat{G}_1, \widehat{G}_2, \widehat{G}_3) \in \mathcal{M}_{\mathcal{I},3}(S_k)$ (Def. 2.8) such that $\widehat{G}_3 = G_3|_{S_k}$ and

$$(3.8) \quad \| \widehat{G}(Q) - X(Q)B^T_{\eta(k)} \| > 2p + 1 \text{ for all } Q \in \Gamma_{\eta(k)}.$$ Take for instance $\widehat{G}|_{\Gamma_{\eta(k)}} = (x_0, y_0, 0) + G|_{\Gamma_{\eta(k)}}$ for any constant $(x_0, y_0) \in \mathbb{R}^2$ with sufficiently large norm. Then define $\widehat{G}|_{\partial(\Gamma_{\eta(k)})}$ to be $\mathcal{I}$-invariant.

In view of (F.7k−1), assertion (II) in Theorem 2.12 applied to any marked immersion $\widehat{G}_\omega = (\widehat{G}, \omega) \in \mathcal{M}_{\mathcal{I},3}(S_k)$ such that $(\partial \widehat{G}_\omega)_3 = (\partial G_3)|_{S_k}$ and $\mathcal{P} = \mathcal{P}_{\widehat{G}_\omega} = \mathcal{P}_G; \quad H_1(N, \mathcal{Z}) \to \mathbb{R},$ provides an $\mathcal{I}$-invariant conformal minimal immersion $\widehat{G} = (\widehat{G}_1, \widehat{G}_2, \widehat{G}_3) \in \mathcal{M}_{\mathcal{I}}(\overline{U})$ such that

$$(3.9) \quad \widehat{G}_3 = G_3$$ and

$$(3.10) \quad \text{the flux map } \mathcal{P}_G \text{ of } \widehat{G} \text{ equals the one } \mathcal{P}_G \text{ of } G.$$ Furthermore, if the approximation of $\widehat{G}$ by $\tilde{G}$ in $S_k$ is close enough, then $F_k := \widehat{G}(B^T_{\eta(k)})^{-1} \in \mathcal{M}_3(\overline{U})$ meets conditions (F.1k)–(F.7k).

Indeed, observe that (3.5) implies that $\widehat{G}_3 = (F_k, e_{\eta(k)})$; hence (3.9) and (3.7) ensure (F.3k). Since $\widehat{G}$ agrees with $G$ on $S_k \setminus (\Gamma_{\eta(k)} \cup \mathcal{I}(\Gamma_{\eta(k)})) = W \setminus (\Omega_{\eta(k)} \cup \mathcal{I}(\Omega_{\eta(k)})) \supset S \cup \left( \bigcup_{a=k+1} \Omega_{\eta(a)} \cup \mathcal{I}(\Omega_{\eta(a)}) \right),$ then (F.1k−1) and (F.2k−1) guarantee (F.1k) and (F.2k), respectively, whereas (F.6k) directly follows. Likewise, (3.8) guarantees (F.4k). Finally, since $\|F_k - F_{k-1}\|_{\mathcal{M}(\Omega_{\eta(k)} \cup \mathcal{I}(\Omega_{\eta(k)}))} \approx 0$ and $\pi_{\eta(k)} \circ F_k = \pi_{\eta(k)} \circ F_{k-1}$ (see (F.3k) and (3.6)), then (F.5k−1) and (F.7k−1) ensure (F.5k) and (F.7k), respectively. Finally, (3.10) and (F.8k−1) give (F.8k).

This concludes the construction of the sequence $\{F_0 = F, F_1, \ldots, F_{i_k}\} \subset \mathcal{M}_3(\overline{U})$.

Label $H := F_{i_k} \in \mathcal{M}_3(\overline{U})$. Let us prove that

Claim 3.4. $\text{dist}(\overline{M}, b\overline{W}) > 2p$.

Proof. Consider a connected curve $\sigma$ in $\overline{U}$ with initial point $Q \in \overline{M}$ and final point $P \in b\overline{W}$. It suffices to show that $\ell(H(\sigma)) > 2p$. Assume without loss of generality that $\langle \sigma \setminus \{P, Q\} \cap (\overline{M} \cup b\overline{W}) = \emptyset$ and, up to possibly replacing $\sigma$ by $\mathcal{I}(\sigma)$, that $\sigma \subset \bigcup_{k=1}^i \Omega_{\eta(k)}$. Let us distinguish cases.

Assume $P \in \Gamma_{\eta(k)}$ for some $k \in \{1, \ldots, i\}$. Then there exists a point $Q_0 \in \sigma \cap (r_{\eta(k)}(0, 1) \cup a_{\eta(k)}(0, 1) \cup r_{\eta(k)}(1))$ and it follows that

$$(F.4_i),(A_2),(F.2_i) \quad \ell(H(\sigma)) \geq \|H(P) - H(Q_0)\| \geq \|H(\mathcal{X}(P)) - X(P) - X(Q_0)\| - \|H - X\| \geq 2p + 1 - \epsilon_0 - 2\epsilon_0 > 2p.$$

For the latter inequality we assume from the beginning $\epsilon_0 < 1/3$. 

Assume now that \( P \in \Gamma_{\eta(k)} \) for no \( k \in \{1, \ldots, i\} \). In this case there exists \( k \in \{1, \ldots, i\} \) such that \( P \in \gamma_{\eta(k)}^1 \cup \gamma_{\eta(k)}^{-1}+0,1) \subseteq \tilde{r}_{\eta(k)}. \) Therefore, there exists a connected sub-arc \( \tilde{\sigma} \subseteq \gamma_{\eta(k)} \cap \tilde{r}_{\eta(k)} \) connecting \( \alpha_{\eta(k)} \cup \tilde{\alpha}_{\eta(k)}+0,1) \neq \emptyset \) and \( P \). Then (F.5ii) gives \( \ell(H(\sigma)) > \ell(H(\tilde{\sigma})) > 2\rho \). This proves the claim \( \square \)

Let us now prove the following

**Claim 3.5.** The inequality

\[
\|H(Q) - \tilde{g}(\mathcal{P}(Q))\| < \sqrt{4\rho^2 + \mu^2 + \epsilon},
\]

where \( \rho, \mu, \epsilon, \) and \( \tilde{g} \) are the data given in the statement of Lemma 3.3, is satisfied for any \( Q \in W \setminus M \) such that \( \text{dist}_H(Q, M) < 2\rho \).

**Proof.** Choose \( Q \in W \setminus M \) with \( \text{dist}_H(M, Q) < 2\rho \). In view of Claim 3.4 and up to possibly replacing without loss of generality \( Q \) by \( \mathcal{P}(Q) \), there exist \( k \in \{1, \ldots, i\} \) and \( P \in r_{\eta(k)} - 0,1) \cup \alpha_{\eta(k)} \cup r_{\eta(k)} \subseteq S \) such that

\[
Q \in \Omega_{\eta(k)} \setminus \gamma_{\eta(k)} \quad \text{and} \quad \text{dist}_H(P, Q) < 2\rho.
\]

By Pitagoras Theorem,

\[
\|H(Q) - \tilde{g}(\mathcal{P}(Q))\| = \sqrt{\langle H(Q) - \tilde{g}(\mathcal{P}(Q)), e_{\eta(k)}^3 \rangle^2 + \|\Theta_{\eta(k)}(H(Q) - \tilde{g}(\mathcal{P}(Q)))\|^2},
\]

where \( \Theta_{\eta(k)} : \mathbb{R}^3 \to \text{span}\{e_{\eta(k)}^1, e_{\eta(k)}^2, \} \) is the orthogonal projection (see (3.5)).

On the one hand,

\[
\|H(Q) - \tilde{g}(\mathcal{P}(Q))\| \leq \|H(Q) - F_k(Q)\| + \|((F_k - F_{k-1})(Q), e_{\eta(k)}^3)\|
\]

\[
+ \|F_{k-1}(Q) - X(Q)\| + \|X(Q) - \tilde{g}(\mathcal{P}(Q))\|;
\]

hence, in view of (F.6a), (a = k + 1, \ldots, i), (F.3k), (F.2k−1), and (A.3),

\[
\|H(Q) - \tilde{g}(\mathcal{P}(Q))\| < \epsilon_0 + \mu = \mu = \mu + 3\epsilon_0.
\]

On the other hand,

\[
\|\Theta_{\eta(k)}(H(Q) - \tilde{g}(\mathcal{P}(Q)))\| \leq \|H(Q) - H(P)\| + \|H(P) - X(P)\| + \|X(P) - X(P_{\eta(k)})\| \]

\[
+ \|\Theta_{\eta(k)}(X(P_{\eta(k)}) - \tilde{g}(\mathcal{P}(P_{\eta(k)})))\| + \|\tilde{g}(\mathcal{P}(P_{\eta(k)})) - \tilde{g}(\mathcal{P}(Q))\|;
\]

hence, using (3.11), (F.2ii), (A.2), (3.4), and again (A.2),

\[
\|\Theta_{\eta(k)}(H(Q) - \tilde{g}(\mathcal{P}(Q)))\| < 2\rho + 2\epsilon_0 + \epsilon_0 + 0 + \epsilon_0 = 2\rho + 4\epsilon_0.
\]

Combining (3.12), (3.13), and (3.14) one obtains

\[
\|H(Q) - \tilde{g}(\mathcal{P}(Q))\| < \sqrt{(\mu + 3\epsilon_0)^2} + (2\rho + 4\epsilon_0)^2 < \sqrt{4\rho^2 + \mu^2 + \epsilon}
\]

as claimed (the latter inequality is satisfied provided that \( \epsilon_0 \) is chosen small enough from the beginning). \( \square \)
Choose a bordered domain $V \in \mathcal{B}_3(\mathcal{N})$ satisfying that
\begin{equation}
M \subset V \subset W
\end{equation}
and
\begin{equation}
\rho < \text{dist}_H(Q, \overline{M}) < 2\rho \ \forall Q \in b\overline{V}.
\end{equation}
Existence of such a $V$ is guaranteed by Claim 3.4. Moreover, up to a slight deformation of the domain $V$, it can be ensured in addition that
\begin{equation}
H|_\Gamma : \Gamma \rightarrow \mathbb{R}^3 \text{ is injective,}
\end{equation}
where $\Gamma := b\overline{V} \cap (\bigcup_{i=1}^b A_i)$ (see (3.3)). In particular, $b\overline{V} = \Gamma \cup \mathcal{J}(\Gamma)$ and $\Gamma \cap \mathcal{J}(\Gamma) = \emptyset$.

The domain $V$ and the map $Y := H|_{\overline{V}} : \mathcal{M}_3(\overline{V})$ satisfy the conclusion of Lemma 3.3. Indeed, property (i) follows from (3.15), (3.2), and (3.3); (ii) = (3.17); (iii) is implied by (3.16); (iv) is given by (F.1$_i$) and (3.2); (v) follows from Claim 3.5 and (3.16); and (vi) = (F.8$_i$).

This concludes the proof of Lemma 3.3.

The following application of Lemma 3.3 is the key tool in this section.

**Lemma 3.6.** Let $U \in \mathcal{B}_3(\mathcal{N})$, let $K \subset U$ be a compact set, and let $X \in \mathcal{M}_3(\overline{V})$.

Then, for any $\epsilon > 0$ there exist $V \in \mathcal{B}_3(\mathcal{N})$ and $Y \in \mathcal{M}_3(\overline{V})$ enjoying the following properties:

1. $K \subset V \subset U$.
2. If $b\overline{V} = \Gamma \cup \mathcal{J}(\Gamma)$ with $\Gamma \cap \mathcal{J}(\Gamma) = \emptyset$, then $Y|_\Gamma : \Gamma \rightarrow \mathbb{R}^3$ is an embedding.
3. $\text{dist}_Y(K, b\overline{V}) > 1/\epsilon$.
4. $\|Y - X\|_{1, K} < \epsilon$.
5. $\|Y - X\|_{0, b\overline{V}} < \epsilon$.
6. The Hausdorff distance $d^H(Y(b\overline{V}), X(b\overline{V})) < \epsilon$.
7. The flux map $p_Y$ of $Y$ equals the one $p_X$ of $X$.

**Proof.** Let $\rho_1 > 0$.

Let $\{\rho_n\}_{n \in \mathbb{N}}$ and $\{\mu_n\}_{n \in \mathbb{N}}$ be the sequences of positive numbers given by
\begin{equation}
\rho_n = \rho_1 + \sum_{j=2}^{n} \frac{a}{j} \quad \text{and} \quad \mu_n = \sqrt{\mu_{n-1}^2 + 4 \left( \frac{a}{n} \right)^2 + \frac{a}{n^2}}, \quad \forall n \geq 2,
\end{equation}
where $a > 0$ and $\mu_1 > 0$ are small enough constants so that
\begin{equation}
\mu_n < \epsilon/2 \quad \forall n \in \mathbb{N}.
\end{equation}

Call $U_0 := U$. Let $T_0$ be a metric tubular neighborhood of $b\overline{U_0}$ in $\mathcal{N}$ disjoint from $K$ and denote by $\Psi_0 : T_0 \rightarrow b\overline{U_0}$ the natural projection.

A standard recursive application of Lemma 3.3 gives a sequence $\{\Xi_n = (U_n, T_n, Y_n)\}_{n \in \mathbb{N}}$, where $U_n \in \mathcal{B}_3(\mathcal{N})$, $T_n$ is a metric tubular neighborhood of $b\overline{U_n}$ in $\mathcal{N}$, and $Y_n \in \mathcal{M}_3(\overline{T_n})$, satisfying the following conditions:

1. If $b\overline{U_n} = \Gamma \cup \mathcal{J}(\Gamma)$ with $\Gamma \cap \mathcal{J}(\Gamma) = \emptyset$, then $Y_n|_\Gamma : \Gamma \rightarrow \mathbb{R}^3$ is an embedding.
2. $K \subset U_n \subset U_{n-1} \subset U_0$ and $T_n \subset T_{n-1} \subset T_0$, for all $n \geq 1$.
3. $\rho_n < \text{dist}_{Y_n}(K, b\overline{U_n})$. 

\[ \]
Therefore, the Maximum Principle for harmonic maps ensures that 
\[
(2) \quad \|Y_n - X\|_{1,K} < \epsilon	on K.
\]
(3) The Hausdorff distance
\[
\|X - X \circ \Psi_0 \circ \cdots \circ \Psi_{n-1}\|_{0,\partial \overline{M}_n} < \mu_n
\]
and (vii) the flux map \(p_{Y_n}\) of \(Y_n\) equals the one \(p_X\) of \(X\).

See [6, Proof of Claim 4.2] for details; here we use Lemma 3.3 instead of [6, Lemma 3.1].

Choose \(k \in \mathbb{N}\) such that
\[
\rho_k > 1/\epsilon,
\]
recalling that \(\{\rho_n\}_{n \in \mathbb{N}} \rightarrow +\infty\). The bordered domain \(V := U_k \in \mathcal{B}_3(\mathcal{N})\) and the map 
\(Y := Y_k \in \mathcal{M}_3(\overline{V})\) satisfy the conclusion of the lemma.

Indeed, (i) is implied by (2); (ii) is (1); (iii) is ensured by (3) and (3\(_k\)); (iv) is (5); and (vii) is (6\(_k\)). In order to check (v), observe that
\[
\|Y - X\|_{0,\partial V} \leq \|Y_k - X \circ \Psi_0 \circ \cdots \circ \Psi_{k-1}\| + \|X \circ \Psi_0 \circ \cdots \circ \Psi_{k-1} - X\| < \mu_k + \mu_k < \epsilon.
\]
Therefore, the Maximum Principle for harmonic maps ensures that \(\|Y - X\|_{0,\partial V} < \epsilon\). The same argument gives property (vi).

This concludes the proof of Lemma 3.6. \(\square\)

We are now ready to prove the main result in this section. Since the proof of Theorem 3.1 relies in a standard recursive application of Lemma 3.6, we will omit some of the details. We refer to the proof of [6, Theorem 5.1] for a careful exposition.

Before going into the proof we need the following notation. For any \(k \in \mathbb{N}\), any compact set \(K \subset \mathcal{N}\), and any continuous injective map \(f:\) \(\mathcal{V} \rightarrow \mathbb{R}^3\), denote by
\[
\Psi(K, f, k) = \frac{1}{2k^2} \inf \left\{ \|f(P) - f(Q)\|: P, Q \in K, \text{dist}_{\sigma_K}^2(P, Q) > \frac{1}{k} \right\} > 0,
\]
where \(\text{dist}_{\sigma_K}^2(\cdot, \cdot)\) denotes the intrinsic distance in \(\mathcal{N}\) with respect to the conformal Riemannian metric \(\sigma_K\); see Remark 2.6.

**Proof of Theorem 3.1.** Let \(\epsilon_1\) and \(a\) be numbers with \(0 < \epsilon_1 < a/2\).

Choose an \(\mathcal{I}\)-invariant bordered domain \(M_1 \in \mathcal{B}_3(\mathcal{N})\) satisfying the following properties:

(i) \(K \subset M_1 \Subset \cup \mathcal{V}\).
(ii) If \(b\overline{M}_1 = \Gamma \cup \mathcal{I}(\Gamma)\) and \(\Gamma \cap \mathcal{I}(\Gamma) = \emptyset\), then \(X|_{\Gamma}: \Gamma \rightarrow \mathbb{R}^3\) is an embedding.
(iii) The Hausdorff distance \(d_\mathcal{H}(X(b\overline{M}_1), X(b\overline{\mathcal{V}))) < \epsilon_1\).

Denote \(X_1 := X|_{\overline{M}_1} \in \mathcal{B}_3(\overline{M}_1)\). A standard recursive application of Lemma 3.6 provides a sequence \(\{\Theta_n = (M_n, X_n, \mathcal{T}_n, \epsilon_n, \tau_n)\}_{n \in \mathbb{N}}\), where \(M_n \in \mathcal{B}_3(\mathcal{N})\), \(X_n \in \mathcal{M}_3(\overline{M}_n)\), \(\mathcal{T}_n\) is a metric tubular neighborhood of \(b\overline{M}_n\) in \(\overline{M}_n\), \(0 < \epsilon_n < a/2^n\), and \(\tau_n > 0\), for all \(n \in \mathbb{N}\), satisfying the following properties:

(1) If \(\overline{\mathcal{V}} = \Omega \cup \mathcal{I}(\Omega)\) with \(\Omega \cap \mathcal{I}(\Omega) = \emptyset\), then \(X_n|_{\Omega}: \Omega \rightarrow \mathbb{R}^3\) is an embedding.
(2) \(K \subset M_n^{n-1} \setminus \mathcal{T}_{n-1} \Subset M_n \setminus \mathcal{T}_n \Subset M_n \setminus M_{n-1} \Subset M\) for all \(n \geq 2\).
(3) \(\|X_n - X_{n-1}\|_{0,\overline{n-1}\setminus \mathcal{T}_{n-1}} < \epsilon_n\) and \(\|X_n - X_{n-1}\|_{0,\overline{M}_n} < \epsilon_n\) for all \(n \geq 2\).
(4n) dist\(_{X_n}(K, \overline{\mathcal{T}_n}) > 1/\epsilon_n\) for all \(n \geq 2\).

(5n) The Hausdorff distance \(\delta^H(X_n(b\overline{M}_n), X_{n-1}(b\overline{M}_{n-1})) < \epsilon_n\) for all \(n \geq 2\).

(6n) There exist \(a_n := i \cdot E((\tau_n)^{n+1})\) points \(x_{n,1}, \ldots, x_{n,a_n}\) in \(X_n(b\overline{M}_n) \subset \mathbb{R}^3\) such that

\[
\delta(X_n(\overline{\mathcal{T}}_n), \{x_{n,1}, \ldots, x_{n,a_n}\}) < \left(1/\tau_n\right)^n,
\]

where \(E(\cdot)\) means integer part and \(2i\) is the number of ends of \(\mathcal{N}'\).

\[(7n) \quad \epsilon_n < \min\left\{\epsilon_{n-1}, \theta_{n-1}, \frac{1}{n^2(\tau_n-1)^n}, \Psi(\mathcal{T}_{n-1}, X_{n-1}(\overline{\mathcal{T}}_{n-1}), n)\right\}, \quad \text{where}
\]

\[
\theta_{n-1} = 2^{-n} \min \left\{ \min_{M_{k-1}\backslash \mathcal{T}_{k-1}} \left\| \frac{\partial X_k}{\sigma N} \right\| : k = 1, \ldots, n-1 \right\} > 0, \quad n \geq 2.
\]

(8n) \(\tau_n \geq \tau_{n-1} + 1 \geq n\) for all \(n \geq 2\).

(9n) \(||\text{dist}^2_{\mathcal{T}_n}(\cdot, b\overline{M}_n)||_0 < \epsilon_n\); see Remark 2.6.

(10n) The flux map \(p_{X_n}\) of \(X_n\) equals the one \(p_X\) of \(X\).

See [6, Proof of Claim 5.2] for details on how to construct such a sequence; here we use Lemma 3.6 instead of [6, Lemma 4.1].

Set \(\mathcal{N}_n = M_n \setminus \overline{\mathcal{T}}_n\) for all \(n \in \mathbb{N}\) and define

\[
M := \bigcup_{n \in \mathbb{N}} \mathcal{N}_n.
\]

From (2n) and (9n), \(n \in \mathbb{N}\) we obtain that \(\overline{M} = \cap_{n \in \mathbb{N}} M_n\) and the inclusion map \(\mathcal{H}_1(M, \mathbb{Z}) \rightarrow \mathcal{H}^1(\mathcal{N}, \mathbb{Z})\) is a homeomorphism.

In view of (2n), (3n), (5n), and (7n), \(n \in \mathbb{N}\), the sequence \(\{X_n|_{\overline{M}}\}_{n \in \mathbb{N}}\) uniformly converges to an \(\mathcal{N}\)-invariant continuous map

\[
Y : \overline{M} \rightarrow \mathbb{R}^3
\]

such that \(\max\left\{\|Y - X\|_{1,K}, \|Y - X\|_{0,\overline{M}}\right\} < a\). Moreover, (3n) and (7n), \(n \in \mathbb{N}\), ensure that \(Y|_{M}\) is an \(\mathcal{N}\)-invariant conformal minimal immersion, which is complete by (4n), \(n \in \mathbb{N}\), and its flux map \(p_Y\) equals the one \(p_X\) of \(X\) by (10n), \(n \in \mathbb{N}\). Finally, properties (3n) and (7n), \(n \in \mathbb{N}\), ensure that \((Y|_{\mathcal{F}M})^{-1}(Y(P)) = \{P, \mathcal{F}(P)\}\) for all \(P \in \mathcal{F}M\), whereas (6n), (7n), and (9n), \(n \in \mathbb{N}\), guarantee that the Hausdorff dimension of \(Y(\mathcal{F}M)\) equals to 1, provided that \(a\) is taken small enough from the beginning. (See [6, Proof of Theorem 5.1] for details.)

Therefore, the domain \(M\) and the map \(Y\) satisfy the conclusion of Theorem 3.1, provided that \(a\) is chosen sufficiently small. \(\square\)

4. Complete non-orientable minimal surfaces properly projecting into planar convex domains

Recall that we have fixed \(\mathcal{N}, \mathcal{F},\) and \(\pi\) as in Def. 2.1 (see Remark 2.6); in particular \(\mathcal{N}\) is an open Riemann surface possibly with infinite topology.

In this section we prove Theorem 1.2 in the introduction. We actually prove the following more precise result.
**Theorem 4.1.** Let $D \subset \mathbb{R}^2$ be a convex domain (possibly neither bounded nor smooth), let $U \Subset \mathcal{N}$ be a Runge connected $\mathcal{I}$-invariant bordered domain, and let $X = (X_1, X_2, X_3) \in \mathcal{M}_3(\overline{U})$ such that

\begin{equation}
(X_1, X_2)(\overline{U}) \subset D.
\end{equation}

Let also $p : \mathcal{H}_4(\mathcal{N}, \mathbb{Z}) \to \mathbb{R}^3$ be a group morphism satisfying

\begin{equation}
p|_{\mathcal{H}_4(U, \mathbb{Z})} = p_X \quad \text{and} \quad p(\mathcal{I}_4(\gamma)) = -p(\gamma) \quad \forall \gamma \in \mathcal{H}_4(\mathcal{N}, \mathbb{Z}).
\end{equation}

(See (2.6).)

Then for any $\epsilon > 0$ there exist a Runge $\mathcal{I}$-invariant domain $M \subset \mathcal{N}$ and $Y = (Y_1, Y_2, Y_3) \in \mathcal{M}_3(M)$ enjoying the following properties:

- $U \Subset M$ and $M$ is homeomorphic to $\mathcal{N}$.
- $||Y - X||_{1, \overline{U}} < \epsilon$.
- $Y : M \to \mathbb{R}^3$ is complete.
- $(Y_1, Y_2)(M) \subset D$ and $(Y_1, Y_2) : M \to D$ is a proper map.
- The flux map $p_Y$ of $Y$ equals $p$.

If $D = \mathbb{R}^2$, the above theorem is already known; furthermore, in this particular case one can choose $M = \mathcal{N}$ (see [3]).

The following result contains most of the technical arguments in the proof of Theorem 4.1. Lemma 3.6 will play an important role in its proof.

**Lemma 4.2.** Let $L \Subset D \subset \mathcal{B} \Subset \mathcal{A} \subset \mathbb{R}^2$ be bounded smooth convex domains, i.e. with smooth frontier. Let $U \Subset \hat{U} \Subset \mathcal{N}$ be Runge connected $\mathcal{I}$-invariant bordered domains, and let $X = (X_1, X_2, X_3) \in \mathcal{M}_3(\overline{U})$ such that

\begin{equation}
(X_1, X_2)(b \overline{U}) \subset D \setminus \overline{\mathcal{B}}.
\end{equation}

Then, for any $\epsilon > 0$ there exist a Runge $\mathcal{I}$-invariant bordered domain $V \Subset \mathcal{N}$ and $Y = (Y_1, Y_2, Y_3) \in \mathcal{M}_3(\overline{V})$ satisfying the following properties:

- (i) $U \Subset V \Subset \hat{U}$ and $V \setminus \overline{U}$ consists of a finite collection of open annuli.
- (ii) $||Y - X||_{1, \overline{U}} < \epsilon$.
- (iii) $(Y_1, Y_2)(b \overline{V}) \subset \mathcal{A} \setminus \overline{\mathcal{B}}$.
- (iv) $(Y_1, Y_2)(\overline{V} \setminus U) \subset \mathcal{A} \setminus \overline{\mathcal{B}}$.
- (v) $\text{dist}_{\overline{V}}(b \overline{U}, b \overline{V}) > 1/\epsilon$.
- (vi) The flux map $p_Y$ of $Y$ equals the one $p_X$ of $X$.

**Proof.** Let $\epsilon_0 > 0$.

Since $X \in \mathcal{M}_3(\overline{U})$, there exists an $\mathcal{I}$-invariant bordered domain $M_0 \Subset \mathcal{N}$ such that $X \in \mathcal{M}_3(M_0)$ and the following properties hold:

- (A1) $U \Subset M_0 \Subset \hat{U}$.
- (A2) $M_0 \setminus \overline{U}$ consists of a finite collection of open annuli.
- (A3) $(X_1, X_2)(\overline{M_0} \setminus U) \subset D \setminus \overline{\mathcal{B}}$; see (4.3).

We now use Lemma 3.6 to get an $\mathcal{I}$-invariant bordered domain $M \Subset \mathcal{N}$ and an $\mathcal{I}$-invariant conformal minimal immersion $F = (F_1, F_2, F_3) \in \mathcal{M}_3(M)$, such that:

- (B1) $U \Subset M \Subset M_0 \Subset \hat{U}$; see (A1).
(B2) $M \setminus \overline{U}$ consists of a finite collection of open annuli; see (A2).
(B3) $\|F - X\|_1, \overline{U} < \epsilon_0$.
(B4) $(F_1, F_2)(\overline{M} \setminus U) \subset D \setminus \overline{Z}$; see (A3) and Lemma 3.6 (v).
(B5) $\text{dist}_F(\overline{U}, b\overline{M}) > 1/\epsilon_0$.
(B6) The flux map $p_F$ of $F$ equals the one $p_X$ of $X$.

Write $$\overline{bM} = \bigcup_{i=1}^{j}(\alpha_i \cup \mathcal{J}(\alpha_i));$$
where $\{\alpha_i\}_{i=1}^{j}$ are pairwise disjoint smooth Jordan curves with $\alpha_i \cap \mathcal{J}(\alpha_i) = \emptyset$. Denote $\alpha = \bigcup_{i=1}^{j} \alpha_i$. It follows that $\overline{bM} = \alpha \cup \mathcal{J}(\alpha)$ and $\alpha \cap \mathcal{J}(\alpha) = \emptyset$.

Since $\mathcal{L}$ is convex, (B4) ensures that for any $P \in \alpha$ there exist a line $l_P$ in $\mathbb{R}^2$ and an open neighborhood $O_P$ of $P$ in $\alpha$ such that
\begin{equation}
((F_1, F_2)(Q) + l_P) \cap \overline{Z} = \emptyset \quad \forall Q \in O_P.
\end{equation}

Since $\alpha$ is compact, then there exist $i \in \mathbb{N}$, $j \geq 3$, and a family of compact Jordan arcs $\{\alpha_{i,j} : (i, j) \in I = \{1, \ldots, i\} \times \mathbb{Z}_3\}$ meeting the following requirements:

\begin{enumerate}
\item \(\bigcup_{j \in \mathbb{Z}_3} \alpha_{i,j} = \alpha_i\) for all $i \in \{1, \ldots, i\}$.
\item $\alpha_{i,j}$ and $\alpha_{i,j+1}$ have a common endpoint $Q_{i,j}$ and are otherwise disjoint for all $(i, j) \in I$.
\item $\alpha_{i,j} \cap \alpha_{i,k} = \emptyset$ for all $(i, j) \in I$ and $k \in \mathbb{Z}_3 \setminus \{j, j + 1\}$.
\item $\alpha_{i,j} \subset O_{R_{i,j}}$ for a point $R_{i,j} \in \alpha_i$ for all $(i, j) \in I$.
\end{enumerate}

For any $(i, j) \in I$ label $l_{i,j} := l_{R_{i,j}}, O_{i,j} := O_{R_{i,j}}$, and denote by $u_{i,j}$ the unitary vector in $\mathbb{R}^2$ orthogonal to $l_{i,j}$ and pointing to the connected component of $\mathbb{R}^2 \setminus ((F_1, F_2)(R_{i,j}) + l_{i,j})$ disjoint from $\overline{Z}$. Set
$$e_{i,j} := (u_{i,j}, 0) \in \mathbb{S}^2$$
and denote by $\pi_{i,j} : \mathbb{R}^2 \rightarrow \text{span}\{u_{i,j}\} \subset \mathbb{R}^2$ the orthogonal projection.

For any $(i, j) \in I$ choose $\{e_{1,i,j}, e_{2,i,j}, e_{3,i,j}\} \subset \mathbb{R}^3$ such that $\{e_{1,i,j}, e_{2,i,j}, e_{3,i,j}\}$ is an orthonormal basis of $\mathbb{R}^3$, and denote
\begin{equation}
B_{i,j} = \begin{pmatrix}
e_{1,i,j}^T, & e_{2,i,j}^T, & e_{3,i,j}^T\end{pmatrix}^{-1}.\end{equation}

Let $\{r_{i,j} : (i, j) \in I\}$ be a family of pairwise disjoint analytical compact Jordan arcs in $M_0 \setminus M$ meeting the following requirements:

\begin{enumerate}
\item $r_{i,j} \subset O_{i,j} \cap O_{i,j+1}$ for all $(i, j) \in I$.
\item $r_{i,j}$ has initial point $Q_{i,j}$ and is otherwise disjoint from $\overline{bM}$ for all $(i, j) \in I$. Denote by $P_{i,j}$ the other endpoint of $r_{i,j}$ for all $(i, j) \in I$.
\item The set
$$S := \overline{M} \cup \bigcup_{(i, j) \in I} (r_{i,j} \cup \mathcal{J}(r_{i,j}))) \subset M_0 \subset \mathcal{N}$$
is $\mathcal{J}$-admissible in the sense of Def. 2.7.
\end{enumerate}

See Fig. 4.1.

We first deform $F$ over the arcs $r_{i,j}$ and $\mathcal{J}(r_{i,j})$, $(i, j) \in I$.

Extend $F|_{\overline{M}}$ to an $\mathcal{J}$-invariant generalized minimal immersion $\hat{F} = (\hat{F}_1, \hat{F}_2, \hat{F}_3) \in \mathcal{M}_{q,\mathcal{J}}(S)$ such that (see Def. 2.8)
(E1) \( \pi_{i,a}(\widehat{F}_1, \widehat{F}_2)(r_{i,j}) \cap \pi_{i,a}(\overline{Z}) = \emptyset \) for all \( (i,j) \in I \) and \( a \in \{j, j+1\} \).
(E2) \( \pi_{i,a}(\widehat{F}_1, \widehat{F}_2)(P_{i,j}) \neq \pi_{i,a}(\overline{X}) = \emptyset \) for all \( (i,j) \in I \) and \( a \in \{j, j+1\} \).

For instance, one can take \( \pi_{i,j} \) to be \( C^0 \) close to a long enough straight segment in \( \mathbb{R}^2 \) with initial point \( (F_1, F_2)(Q_{i,j}) \) and directed by a vector \( \hat{u}_{i,j} \in \mathbb{R}^2 \) with \( \hat{u}_{i,j}, u_{i,a} > 0 \) for \( a = j, j+1 \). Such a vector \( \hat{u}_{i,j} \) exists since \( u_{i,j} \neq -u_{i,j+1} \); take into account that \( Q_{i,j} \in O_{i,j} \cap O_{i,j+1} \) by (C4).

Applying Theorem 2.12 to any marked immersion \( \widehat{F}_{\varnothing} = (\widehat{F}, \varnothing) \in \mathcal{M}^*_{g,3}(S) \) and \( p = p_{\widehat{F}_{\varnothing}} = p_X : \mathcal{H}\{N, \mathbb{Z}\} \rightarrow \mathbb{R}^3 \) (see Def. 2.9 and (B6)), one obtains an \( \mathcal{H} \)-invariant conformal minimal immersion \( H = (H_1, H_2, H_3) \in \mathcal{M}_3(\overline{M}_0) \) satisfying:

(F1) \( ||H - F||_{1, \overline{M}} < \varepsilon_0 \).
(F2) \( ||H - \widehat{F}||_{0, S} < \varepsilon_0 \).
(F3) \( (H_1, H_2)(\overline{M} \setminus U) \subset \mathcal{D} \setminus \overline{L} \); see (B4).
(F4) \( \text{dist}_H(U, b\overline{M}) > 1/\varepsilon_0 \); see (B5).
(F5) \( \pi_{i,j}((H_1, H_2)(r_{i,j-1} \cup \alpha_{i,j} \cup r_{i,j})) \cap \pi_{i,j}(\overline{L}) = \emptyset \) for all \( (i,j) \in I \); see (C4), (4.4), and (E1).
(F6) \( \pi_{i,j}((H_1, H_2)(\{P_{i,j-1}, P_{i,j}\})) \cap \pi_{i,j}(\overline{X}) = \emptyset \) for all \( (i,j) \in I \); see (E2).
(F7) \( \text{The flux map} \ p_H \) of \( H \) equals the one \( p_X \) of \( X \).

By continuity of \( H \), there exists an \( \mathcal{H} \)-invariant bordered domain \( W \subset \mathcal{N} \) satisfying the following properties:

(G1) \( M \Subset W \Subset \overline{M}_0 \).
(G2) \( S \subset \overline{W} \) and \( S \cap b\overline{W} = \bigcup_{(i,j) \in I} \{P_{i,j}, \mathcal{I}(P_{i,j})\} \).
(G3) \( W \setminus M \) consists of a finite collection of open annuli.
(G4) \( \pi_{i,j}((H_1, H_2)(\Omega_{i,j})) \cap \pi_{i,j}(\overline{L}) = \emptyset \), where \( \Omega_{i,j} \) is the closure of the connected component of \( W \setminus S \) bounded by \( r_{i,j-1}, \alpha_{i,j}, r_{i,j} \), and the piece \( \beta_{i,j} \) of \( b\overline{W} \) which connects \( P_{i,j-1} \) and \( P_{i,j} \) and is otherwise disjoint from \( S \); see (F5).
(G5) \( \pi_{i,j}((H_1, H_2)(\overline{\Gamma_{i,j}})) \cap \pi_{i,j}(\overline{X}) = \emptyset \), where \( \Gamma_{i,j} \) is a compact sub-arc of \( \beta_{i,j} \setminus \{P_{i,j-1}, P_{i,j}\} \); see (F6).
(G6) \( (\partial H B^3_{\Gamma_{i,j}}) \) vanishes nowhere on \( b\overline{W} \) for all \( (i,j) \in I \); here \((\cdot)_3\) denotes third coordinate in \( \mathbb{R}^3 \) and \( B_{\Gamma_{i,j}} \) is given by (4.5).

See Fig. 4.1.

![Figure 4.1. \( W \setminus M \)](image-url)
Let \( \eta : \{1, \ldots, \iota\} \to \{1, \ldots, \iota\} \times \mathbb{Z}_i \) be the bijection \( \eta(k) = (E(k-1) + 1, k-1) \), where \( E(\cdot) \) means integer part.

We now recursively construct a sequence \( \{H^0 = H, H^1, \ldots, H^\iota\} \subset \mathcal{M}_2 (\mathcal{M}_0) \), \( H^k = (H_i^k, H_j^k, H^k) \) for all \( k \in \{0, 1, \ldots, \iota\} \), satisfying the properties for all \( k \in \{0, 1, \ldots, \iota\} \):

1. \( \|H^k - F\|_{1,\mathcal{M}} < \epsilon_0 \).
2. \( \|H^k - H^{k-1}\|_{1,\mathcal{W} \setminus (\Omega_{\eta(k)} \cup \bar{\Omega}(\Omega_{\eta(k)}))} < \epsilon_0/i, k \geq 1 \).
3. \( \langle H^k - H^{k-1}, e^k_\eta(k) \rangle = 0, k \geq 1 \).
4. \( \text{dist}_{H^k}(\mathcal{M}, \partial \mathcal{M}) > 1/\epsilon_0 \).
5. \( \pi_{\eta(a)}(H^k, H^k) \cap \eta_{\eta(a)}(\mathcal{F}) = \emptyset \) for all \( a \in \{1, \ldots, \iota\} \).
6. \( \pi_{\eta(a)}(H^k, H^k) \cap \eta_{\eta(a)}(\mathcal{F}) = \emptyset \) for all \( a \in \{1, \ldots, \iota\} \).
7. \( \langle H^k, H^k \rangle_{\Gamma_{\eta(a)}} \cap \mathcal{F} = \emptyset \) for all \( a \in \{1, \ldots, \iota\} \).
8. \( \partial(\delta(H^k B^T_{ij}) \cap \mathcal{F}) \) vanishes nowhere on \( b\mathcal{W} \) for all \((i, j) \in I \) see (4.5).
9. \( \text{The flux map } p^{F_k} \text{ of } F_k \text{ equals the one } p_X \text{ of } X \).
10. \( \langle H^k, H^k \rangle_{\mathcal{M} \setminus \mathcal{U}} \subset \mathcal{D} \setminus \mathcal{F} \).

Indeed, observe that (1) holds (F1), (4) holds (F4), (5) holds (G4), (6) holds (G5), (8) holds (G6), (9) holds (F7), and (10) holds (F3), whereas (2), (3), and (7) make no sense. Reason by induction and assume that we already have \( H^0, \ldots, H^{k-1}, \) for some \( k \in \{1, \ldots, \iota\} \), satisfying the corresponding properties. Let us construct \( H^k \).

Denote \( G = (G_1, G_2, G_3) := H^{k-1} B^T_{\eta(k)} \in \mathcal{M}_2 (\mathcal{M}_0) \), where \( B_{\eta(k)} \) is the orthogonal matrix (4.5). It follows that \( G \in \mathcal{M}_2 (\mathcal{M}_0) \) and

\[ G_3 = \langle H^{k-1}, e_3^{\eta(k)} \rangle. \]

Denote

\[ S_k := \mathcal{M} \cup \left( \Gamma_{\eta(k)} \cup \mathcal{I}(\Gamma_{\eta(k)}) \right) \cup \left( \bigcup_{a \neq k} \Omega_{\eta(a)} \cup \mathcal{I}(\Omega_{\eta(a)}) \right) \]

and observe that \( S_k \) is \( \mathcal{I} \)-admissible (Def. 2.7). Extend \( G_{|S_k \setminus (\Gamma_{\eta(k)} \cup \mathcal{I}(\Gamma_{\eta(k)}))} \) to an \( \mathcal{I} \)-invariant generalized minimal immersion \( \tilde{G} = (\tilde{G}_1, \tilde{G}_2, \tilde{G}_3) \in \mathcal{M}_2 (S_k) \) (cf. Def. 2.8) such that \( \tilde{G}_3 = G_3_{|S_k} \) and

\[ \pi_{\eta(k)} \left( \left( \tilde{G}_i B^T_{\eta(k)} \right)^{-1} \right)_1, \left( \tilde{G}_i B^T_{\eta(k)} \right)^{-1} \left( \Gamma_{\eta(k)} \right) \cap \mathcal{F} = \emptyset, \]

where \( \cdot \) means \( j \)-th coordinate in \( \mathbb{R}^3 \). For instance, choose \( \tilde{G}_{\Gamma_k} = G_{\Gamma_k} + (x_0, y_0, 0) \) where \( (x_0, y_0) \in \mathbb{R}^2 \) is a constant with large enough norm.

In view of (8.4), one can apply Theorem 2.12 (II) to any marked immersion \( \hat{G}_\mathcal{M} = (\hat{G}, \mathcal{V}) \in \mathcal{M}_2 (S_k) \) such that \( \delta \hat{G}_\mathcal{M} = \hat{G}_\mathcal{M} \mathcal{I} \) and \( p \hat{G}_\mathcal{M} : \mathcal{H}_1 (N, \mathcal{V}) \to \mathbb{R}^3 \), obtaining an \( \mathcal{I} \)-invariant conformal minimal immersion \( \hat{G} = (\hat{G}_1, \hat{G}_2, \hat{G}_3) \in \mathcal{M}_2 (\mathcal{M}_0) \) such that \( \hat{G}_3 = G_3 \) and the flux map \( p_\mathcal{G} \) of \( \hat{G} \) equals the one \( p_G \) of \( G \). It is now straightforward to check that, if the approximation of \( \hat{G} \) by \( \hat{G} \) in \( S_k \) is close enough, then \( H^k := \hat{G}_i B^T_{\eta(k)} \) satisfies properties (1)–(10). This concludes the construction of the sequence \( H^k, \ldots, H^\iota \). Since \( D \subset \mathcal{F} \).
and $H^i$ is $\mathcal{I}$-invariant, then there exists an $\mathcal{I}$-invariant bordered domain $V \Subset \mathcal{N}$ such that
\begin{equation}
M \Subset V \Subset W; \quad V \setminus \overline{M} \text{ consists of } i \text{ open annuli,}
\end{equation}
and
\begin{equation}
(H^1_i, H^2_i)(bV) \subset \mathcal{A} \setminus \overline{\mathcal{A}}.
\end{equation}
In particular, by the Convex Hull property of minimal surfaces,
\begin{equation}
(H^1_i, H^2_i)(\overline{V}) \subset \mathcal{A}.
\end{equation}

Set $Y := H^i|_{\overline{\mathcal{A}}} \in \mathcal{M}_3(\overline{V})$ and notice that $Y$ and $V$ satisfy the conclusion of the lemma provided that $\epsilon_0$ is chosen small enough from the beginning. Indeed, Lemma 4.2(i) follows from (B1), (B2), and (4.6); (ii) is implied by (B1), (B3), and (1) (9); (iii) is ensured by (4.8), (10), (5), and the fact that $\overline{V} \setminus \overline{M} \subset W \setminus \overline{M} = \bigcup_{a=1}^{\infty} (\Omega_{\mathcal{I}(a)} \cup \mathcal{I}(\Omega_{\mathcal{I}(a)}))$; (v) is guaranteed by (4) and (4.6); and (vi) = (9).

This proves the lemma. \hfill \square

We can now prove the main result in this section. It will follow from a standard recursive application of Lemma 4.2.

**Proof of Theorem 4.1.** Let $\epsilon_0 > 0$.

By (4.1) and Theorem 2.13, we can take a sequence $\{D^j\}_{j \in \mathbb{N} \cup \{0\}}$ of bounded smooth convex domains in $\mathbb{R}^2$, such that $D^{j-1} \Subset D^j \Subset D$ for all $j \in \mathbb{N}$,
\begin{equation}
\{\overline{D}^j\} \not\subset D,
\end{equation}
and
\begin{equation}
(X_1, X_2)(bU) \subset D^1 \setminus \overline{D}^0.
\end{equation}

Call $U_0 := U$, and take also a sequence $\{U_j\}_{j \in \mathbb{N}}$ of Runge connected $\mathcal{I}$-invariant bordered domains in $\mathcal{N}$, satisfying:

(a) $U_{j-1} \Subset U_j$ for all $j \in \mathbb{N}$.
(b) The Euler characteristic $\chi(U_j \setminus \overline{U}_{j-1}) \in \{0, -2\}$ for all $j \in \mathbb{N}$.
(c) $\mathcal{N} = \bigcup_{j \in \mathbb{N}} U_j$.

Such a sequence is constructed in [3, Remark 5.8].

Call $M_0 := U_0$ and $Y^0 = (Y^0_1, Y^0_2, Y^0_3) := X$. Let us construct a sequence $\{(\epsilon_j, M_j, Y^j)\}_{j \in \mathbb{N}}$, where $\epsilon_j > 0$, $M_j$ is a Runge connected $\mathcal{I}$-invariant bordered domain in $\mathcal{N}$, and $Y^j = (Y^j_1, Y^j_2, Y^j_3) \in M_3(\overline{M_j})$ enjoy the following properties:

(1) $M_{j-1} \Subset M_j \Subset U_j$ for all $j \in \mathbb{N}$.
(2) The inclusion map $M_j \hookrightarrow U_j$ induces an isomorphism $\mathcal{H}_1(M_j, \mathbb{Z}) \rightarrow \mathcal{H}_1(U_j, \mathbb{Z})$ for all $j \in \mathbb{N}$.
(3) $\|Y^j - Y^{j-1}\|_{1, M_{j-1}} < \epsilon_j < \epsilon_0/2^j$ for all $j \in \mathbb{N}$.
(4) $(Y^j_1, Y^j_2, (bM_j) \subset D^{j+1} \setminus \overline{D}^j$ for all $j \in \mathbb{N}$.
(5) $(Y^j_1, Y^j_2)(\overline{M_j} \setminus M_{j-1}) \subset D^j \setminus \overline{D}^{j-1}$ for all $j \in \mathbb{N}$.
(6) $d_{\mathcal{H}_1}(\overline{M_{j-1}}, bM_j) > 1/\epsilon_j > 2^j/\epsilon_0$.
(7) The flux map $p_{Y^j}$ of $Y^j$ equals $p_{\mathcal{H}_1}(M_j, \mathbb{Z})$. 

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Indeed, observe that $M_0$ and $Y^0$ satisfy (2), (4), and (7) by the fact $(M_0, Y^0) = (U_0, X)$, (4.10), and (4.2), respectively, whereas the remaining requirements make no sense for $j = 0$. Reason by induction and assume that we already have $(\varepsilon_{j-1}, M_{j-1}, Y^{j-1})$ for some $j \in \mathbb{N}$, satisfying the corresponding properties, and let us construct $(\varepsilon_j, M^j, Y^j)$.

Choose $\varepsilon_j < \varepsilon_0/2^j$.

If the Euler characteristic $\chi(U_j \setminus \overline{U}_{j-1}) = 0$, then we directly obtain $M_j$ and $Y^j$ as the resulting data to apply Lemma 4.2 to

$$\left( L, D, \mathcal{B}, b, U, \overline{U}, X, \varepsilon \right) = \left( D^{j-1}, D^j, D^{j+1}, M_{j-1}, U_j, Y^{j-1}, \varepsilon_j \right).$$

Otherwise, properties (b) and (2) ensure that the Euler characteristic $\chi(U_j \setminus \overline{U}_{j-1}) = \chi(U_j \setminus \overline{M}_{j-1}) = -2$. In this case, by elementary topological arguments, there exists a compact Jordan arc $\gamma \subset U_j$ such that:

- $\gamma$ has its endpoints in $b\overline{M}_{j-1}$ and is otherwise disjoint from $\overline{M}_{j-1}$.
- $\gamma \cap \mathcal{I}(\gamma) = \emptyset$.
- $S' := \overline{M}_{j-1} \cup \gamma \cup \mathcal{I}(\gamma)$ is an $\mathcal{I}$-admissible subset in $\mathcal{N}$ (see Def. 2.7).
- The Euler characteristic $\chi(U_j \setminus S_j) = 0$.

(Cf. [3, Remark 5.8].)

Extend $Y^{j-1}$ to an $\mathcal{I}$-invariant generalized minimal immersion $\hat{Y} = (\hat{Y}_1, \hat{Y}_2, \hat{Y}_3) \in \mathcal{M}_{b, \mathcal{I}}(S)$ satisfying $\hat{Y}(\gamma) \subset \mathcal{D}^j \setminus \mathcal{D}^{j-1}$; existence of such extension is guaranteed by (4). Applying Theorem 2.12 (I) to any marked immersion $\hat{Y}_\varpi = (\hat{Y}, \varpi) \in \mathcal{M}_{b, \mathcal{I}}^*(S)$, such that the generalized flux map $\mathcal{P}_{\hat{Y}_\varpi}$ of $\hat{Y}_\varpi$ equals $\mathcal{P}_{|\mathcal{H}_1(U_j, Z)}$ (see (2.9)), we obtain an $\mathcal{I}$-invariant bordered domain $W$ and $F = (F_1, F_2, F_3) \in \mathcal{M}_{\mathcal{I}}(W)$ meeting the following requirements:

- $\overline{M}_{j-1} \subset S \subset W \subset U_j$ and the Euler characteristic $\chi(U_j \setminus W) = 0$.
- $\|F - Y^{j-1}\|_{1, \mathcal{M}_{j-1}} \approx 0$.
- $(F_1, F_2)(W \setminus \overline{M}_{j-1}) \subset \mathcal{D}^j \setminus \mathcal{D}^{j-1}$.
- The flux map $\mathcal{P}_F$ of $F$ equals $\mathcal{P}_{|\mathcal{H}_1(U_j, Z)}$.

This reduces the construction of the triple $(\varepsilon_j, M_j, Y^j)$ to the already done in the case when $\chi(U_j \setminus \overline{U}_{j-1}) = 0$, concluding the construction of the desired sequence $\{(\varepsilon_j, M_j, Y^j)\}_{j \in \mathbb{N}}$.

Set

$$M := \bigcup_{j \in \mathbb{N}} M_j \subset \mathcal{N},$$

which is a Runge $\mathcal{I}$-invariant domain, homeomorphic to $\mathcal{N}$, satisfying $U \in M$; take into account properties (1) and (2), $j \in \mathbb{N}$. By (3), $j \in \mathbb{N}$, the sequence $\{Y^j\}_{j \in \mathbb{N}}$ converges uniformly in compact subsets of $M$ to an $\mathcal{I}$-invariant conformal minimal immersion $Y = (Y_1, Y_2, Y_3) \in \mathcal{M}_{\mathcal{I}}(M)$ with $\|Y - X\|_{1, \mathcal{I}} < \varepsilon$, provided that $\varepsilon_j$ is chosen small enough for each $j \in \mathbb{N}$. Furthermore, (6), $j \in \mathbb{N}$, ensure that $Y$ is complete, whereas (4) and (5), $j \in \mathbb{N}$, imply that $(Y_1, Y_2)(M) \subset D$ and $(Y_1, Y_2): M \to D$ is a proper map. Finally, the flux map $\mathcal{P}_Y$ of $Y$ equals $\mathcal{P}$ by (7), $j \in \mathbb{N}$.

This concludes the proof. \qed
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