PROPERTY (τ) IN POSITIVE CHARACTERISTIC

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Abstract. We prove a quantitative equidistribution statement for certain adelic homogeneous subsets in positive characteristic. As an application, we describe a proof of property (τ) for arithmetic groups in this context.

Contents

1. Introduction 1
2. Notation and preliminaries 3
3. Construction of a good place 6
4. Deduction of Theorem 1.2 from Theorem 1.1 10
5. Effective generation of a group at a good place 11
6. Non-divergence of unipotent orbits 14
7. Invariant subgroups for the measure \( \mu \) 17
References 24

1. Introduction

Let \( X = L/\Lambda \) be the quotient of a locally compact group \( L \) by a lattice \( \Lambda \subset L \). Any subgroup \( M \subset L \) acts on \( X \) by left multiplication. A homogeneous measure on \( X \) is, by definition, a probability measure \( \mu \) that is supported on a single closed orbit \( Y = M g \Lambda \) of its stabilizer \( M Y = \text{Stab}(\mu) \). A homogeneous set is the support of a homogeneous measure.

Motivated by problems in number theory, the limiting behavior of a sequence \( \{ \mu_i \} \) of homogeneous measures has been extensively studied. When \( L \) is a Lie group and \( \text{Stab}(\mu_i) \) is generated by unipotent subgroups, Mozes and Shah \( [25] \) used Ratner’s celebrated measure classification theorems \( [31] \) and the Linearization techniques of Dani and Margulis \( [7] \) to obtain a very satisfactory classification theorem for the limiting measure, see also \( [12, 15, 9] \).

More recently, the quantitative aspects of this problem have attracted considerable attention. Indeed, in a landmark paper, Einsiedler, Margulis, and Venkatesh \( [11] \) proved an effective equidistribution theorem for homogeneous measures when \( M \) and \( L \) are semisimple Lie groups and \( \Lambda \) is a congruence lattice, under some additional conditions; and in a followup work Einsiedler, Margulis, Venkatesh and the first named author \( [10] \) extended this result to certain adelic periods in the number field setting.

In this paper, we consider homogeneous measures in positive characteristic setting. Let us fix some notation in order to state our main results. Let \( F \) be a

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global function field, and let $G$ be an absolutely almost simple, simply connected $F$-group. Throughout the paper, we assume $G$ does not admit non-standard isogenies. This condition is always satisfied if $\text{char}(F) > 3$. More precisely, $G$ admits non-standard isogenies only in the following cases: if $\text{char}(F) = 2$ and $G$ is of type $B_n, C_n, (n \geq 2)$ or of type $F_4$, another case is if $\text{char}(F) = 3$ and $G$ is of type $G_2$, see [27] §1 and references therein.

Put $G' = G \times G$, and let
\[
H = \{(h, h) : h \in G\} \subset G'
\]
be the diagonal embedding of $G$ in $G'$.

Throughout the paper, $\Sigma$ denotes the set of all places of $F$. For each place $v \in \Sigma$, let $F_v$ be the completion of $F$ at $v$, and let $\mathbb{A}_F$ (or simply $\mathbb{A}$ if there is no confusion) denote the ring of adeles over $F$.

Let $G' = G'(\mathbb{A})$ and $X = G'/G(F)$. We let $m_X$ denote the $G'$-invariant probability measure on $X$. Let $i : H \to G'$ be the above inclusion. For every $g \in G'$,
\[
Y_g := gi(H(\mathbb{A})/H(F))
\]
is a closed orbit in $X$ equipped with a probability measure $\mu_g$ which is invariant under $H_g := gH(\mathbb{A})g^{-1}$. Throughout the paper, we drop $i$ from the notation and simply write $Y_g = gH(\mathbb{A})/H(F)$.

We will use the following notion of complexity which was introduced and studied in [10]. Let $\Omega_0 \subset G'$ be a compact open subgroup. Given $(Y_g, \mu_g)$ as above, define
\[
\text{vol}(Y_g) = m_g(\Omega_0 \cap H_g)^{-1}
\]
where $m_g$ denotes a Haar measure on $H_g$ which projects to $\mu_g$. Note that a different choice of $\Omega_0$ only changes the above function by a multiplicative constant, see [10] §2.3 for a proof.

The following is the main theorem of this paper.

**Theorem 1.1.** There exists $\kappa > 0$ so that the following holds. Let $g \in G'$, and let $(Y_g, \mu_g)$ be as above. Then
\[
\left| \int_{Y_g} f \, d\mu_g - \int f \, dm_X \right| \ll \text{vol}(Y_g)^{-\kappa} S(f), \quad f \in C_c^\infty(X),
\]
where $S(f)$ denotes a certain adelic Sobolev norm, and the implied multiplicative constant depends on $X$. The exponent $\kappa$ depends only on the type of $G$ and $F$. If $X$ is compact, then $\kappa$ depends only on the type of $G$.

Similar to the proofs in [11] [10, our proof relies on uniform spectral gap. However, our treatment deviates from loc. cit. in that instead of using effective ergodic theorems for the action of unipotent subgroups, we use averages over expanding pieces of root subgroups, see [5,2]. This idea (which is due to Margulis, see [24]) makes the analysis in the case at hand more transparent. We also use the work of Prasad [29]; albeit our application here is less intricate than in [10], in particular, our treatment does not rely on [2], as $H \subset G'$ is simply connected and embedded.

As it was mentioned our proof relies on uniform spectral gap. However, it also allows us to give an independent proof of property $(\tau)$ in all cases except for groups of type $A_1$. That is, if we assume property $(\tau)$ for groups of type $A_1$, Drinfeld [8],
and use the uniform estimates on decay of matrix coefficients for groups with Kazhdan’s property $(T)$, Oh [20], we can deduce property $(\tau)$ in all other cases as well as our theorem.

In the number field setting, the proof of property $(\tau)$ has a long history and was completed thanks to several deep contributions, see [17, 20, 32, 16, 4, 5, 14]. In the positive characteristic setting, the existing literature is far less satisfactory beyond groups with property $(T)$, the aforementioned work of Drinfeld [5], and the work of Lafforgue [21, 22]. However, the statement itself is agreed upon by the experts. We hope the following theorem will contribute to filling this lacuna in the literature.

**Theorem 1.2** (Property $\tau$). Let the notation and assumptions be as above, in particular, we assume that $G$ does not admit non-standard isogenies.

Let $v \in \Sigma$ be such that $G$ is isotropic over $F_v$. Then $L^2(G(\mathcal{A})/G(F))$, the orthogonal complement of $G(\mathcal{A})$-invariant functions, is isolated from the trivial representation as a representation of $G(F_v)$. Moreover, this isolation (spectral gap) is independent of the $F$-form of $G$.

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**2. Notation and preliminaries**

Let $F$ be a global function field. Let $\Sigma$ be the set of places on $F$, and let $\mathcal{A}$ be the ring of adeles over $F$. For each place $v \in \Sigma$, let $F_v$ be the completion of $F$ at $v$. Let $\mathfrak{o}_v$ be the ring of $v$-integers in $F_v$; $k_v$ denotes the residue field of $\mathfrak{o}_v$ and $\mathcal{w}_v$ is a uniformizer of $\mathfrak{o}_v$. We will denote by $|x|_v$ the absolute value on $F_v$. Note that $\mathfrak{o}_v$ is the maximal compact subring of $F_v$. Let $B_v(a, r) = \{b \in F_v : |a - b|_v \leq r\}$.

As in the introduction, let $G$ be an absolutely almost simple, simply connected $F$-group which does not admit non-standard isogenies. This condition is always satisfied if char($F$) > 3. More precisely, $G$ admits non-standard isogenies only if char($F$) = 3 and $G$ is of type $G_2$ or char($F$) = 2 and $G$ is of type $B_n, B_n, (n \geq 2)$ or of type $F_4$, see [27, §1] and references therein.

Put $G' = G \times G$. Let

$$H = \{(h, h) : h \in G\} \subset G'$$

be the diagonal embedding of $G$ in $G'$. We also write $G^1 = G \times \{1\}$ and $G^2 = \{1\} \times G$. For every group $L$, we put $L = L(\mathcal{A})$. We will often simplify the notation and denote $G^1$ simply by $G$, but $G^2$ will always be denoted as $G^2$.

Abusing the notation slightly, for every $g \in G'$ and a subset $B \subset G$, we let

$$\Delta_g(B) = g\{(g, g) : (g, 1) \in B\}g^{-1}.$$ 

We adapt a similar notation for subsets of $G(F_v)$ for all $v \in \Sigma$.

For $i = 1, 2$, let $\pi^i$ denote the projection from $G'$ onto the $i$-th component. Denote by $i^1 : G \rightarrow G'$ the map $g \mapsto (g, 1)$ and $i^2$ is the map $g \mapsto (1, g)$. For every $v \in \Sigma$, we let $\pi_v : G' \rightarrow G'(F_v)$ denote the natural projection, and $\iota_v : G'(F_v) \rightarrow G'$ denotes the natural embedding in the $v$-th place. Put $\pi^i_v := \pi^i \circ \pi_v$ and $\iota_v = \iota_v \circ i^v$.

For $L = G', G', H$ and every $v \in \Sigma$, let $L_v = \iota_v(L(F_v))$. Given $g = (g_v) \in G'$ and $w \in \Sigma$, we will let

$$g^w = \iota_w \circ \pi_w(g).$$
That is, \( g_v^w = g_w \) and \( g_v^w = 1 \) for all \( v \neq w \).

Let \( \rho : G \to \text{SL}_N \) be an embedding defined over \( F \). For any \( v \in \Sigma \), let 
\[
K_v = \iota_v^1(\rho^{-1}(\text{SL}_N(\mathfrak{a}_v))) \subset G_v
\]
and let \( K \subset G \) denote the product of \( K_v \) for all \( v \). For \( m \geq 1 \), let
\[
K_v[m] := \ker(K_v \to \text{SL}_N(\mathfrak{a}_v/\mathfrak{w}_w^m \mathfrak{a}_v));
\]
note that \( K_v = K_v \). Let 
\[
K_v' := \{(g^1, g^2) : (g^1, 1) \in K_v\}.
\]
Define \( K' \) and \( K'_v[m] \) accordingly. We choose \( \Omega_0 \) in \([10, \text{Lemma 2.2}]\) to be \( K' \). For any \( L \subset K_v \), \( v \in \Sigma \), and \( m \geq 0 \) we defined \( L[m] := L \cap K[m] \).

For any subspace \( \mathfrak{h} \subset \mathfrak{g}_w \) we write \( \mathfrak{h}[0] \) for the preimage of the \( \mathfrak{a}_w \)-integral \( N \times N \) matrices under the restriction of the differential \( D\rho : \mathfrak{g} \to \mathfrak{sl}_N \) to \( \mathfrak{h} \). More generally, we write \( \mathfrak{h}[m] \) for the preimage of the matrices all of whose entries have valuation at least \( m \).

Recall that \( X = G'/G'(F) \) and \( Y_g = gH/H(F) \) where \( g \in G' \). The homogeneous set \( Y_g \) is equipped with the \( H_g = gHg^{-1} \)-invariant probability measure \( \mu_g \).

**Lemma 2.1.** We have \( \text{Stab}(\mu_g) = H_g \cdot \mathcal{C}(F) \), where \( \mathcal{C} \) denotes the center of \( G' \).

**Proof.** By virtue of \([10, \text{Lemma 2.2}]\), we have 
\[
\text{Stab}(\mu_g) = gHN(F)g^{-1}
\]
where \( N \) denotes the normalizer of \( H \) in \( G' \). Since \( H = \{(h, h) : h \in G\} \) and \( G \) is absolutely, almost simple, we conclude that \( N = H \cdot \mathcal{C} \) which implies the claim. \( \square \)

**Lemma 2.2.** For every \( g \in G' \), there exists some \( \hat{g} \in G^2 \) so that \( Y_g = Y_{\hat{g}} \).

**Proof.** Let \( g = (g^{(1)}, g^{(2)}) \), then 
\[
Y_g = \{(g^{(1)}, g^{(2)})(h, h) : h \in G(\mathbb{A})\} = \{(g^{(1)} h, g^{(2)} (g^{(1)})^{-1} g^{(1)} h) : h \in G(\mathbb{A})\} = \{(h, g^{(2)} (g^{(1)})^{-1} h) : h \in G(\mathbb{A})\}.
\]
The claim thus holds with \( \hat{g} = (1, (g^{(2)} (g^{(1)})^{-1})_{v}) \). \( \square \)

In view of this lemma, from this point until the end of the paper, we will always assume \( g \in G^2 \).

### 2.1. Properties of a split Lie algebra

Let \( \mathfrak{g} \) be the Lie algebra of \( G \) and for any \( v \in \Sigma \), let \( \mathfrak{g}_v := \text{Lie}(G) \otimes F_v \). We fix \( w \) such that \( G \) is \( F_w \)-split. In particular, \( G_w \) is a Chevalley group (see \([35, \text{§3.4.2}]\)). Let \( \Phi \) be the set of roots for \( G_w \). Let \( p_w := \text{char}(k_w) \).

Let \( \mathfrak{a} \) be a maximal split torus of \( \mathfrak{g}_w \). Then, \( \mathfrak{g} \) can be decomposed as \( \mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\lambda \in \Phi} \mathfrak{g}_\lambda \), where for any \( \lambda \in \mathfrak{a}^* \)
\[
(3) \quad \mathfrak{g}_\lambda := \{v \in \mathfrak{g} : \forall a \in \mathfrak{a}, \text{ad}(a)v = \lambda(a)v\},
\]
and \( \Phi \subset \mathfrak{a}^* \) is the set of roots for \( \mathfrak{g}_w \), i.e., the set of non-zero characters for which \([33, \text{§2.6}]\).

For any \( \lambda \in \Phi \), there exists a one-dimensional Lie algebra \( \mathfrak{g}_\lambda \subset \mathfrak{g} \), and a one-dimensional unipotent subgroup \( U_\lambda = \{u_\lambda(r)\} \subset G_w \) such that \( \mathfrak{g}_\lambda \) is the Lie algebra
of $U_{\lambda}$ (see \cite{3} §2.3)). Set $L_{\lambda} = (U_{\lambda}, U_{-\lambda})$. Then, for any $t$ we can also denote by $a_{\lambda}(t)$ the unique diagonal element in $L_{\lambda}$ such that

$$a_{\lambda}(t)u_{\lambda}(1)a_{\lambda}(t^{-1}) = u(t^2).$$

Fix two opposite Borel subgroups $B$ and $B^{-}$ in $G_{w}$. Denote by $U$ and $U^{-}$ the unipotent radicals of $B$ and $B^{-}$, respectively, and by $A = B \cap B^{-}$ a maximal split torus of $G_{w}$. For every nonnegative integer $m$, let

$$U_{m} = U \cap K_{w}[m], \quad U_{m}^{-} = U^{-} \cap K_{w}[m], \quad \text{and} \quad A_{m} = A \cap K_{w}[m].$$

Then, according to \cite{34} §3.1.1 for any $m \in \mathbb{N}$, we have

$$(4) \quad K_{w}[m] = U_{m}A_{m}U_{m}.$$ 

Fix an ordering $\lambda_{1}, \ldots, \lambda_{d}$ on $\Phi^{+}$ once and for all; we use the ordering $-\lambda_{1}, \ldots, -\lambda_{d}$ on $\Phi^{-}$. In view of \cite{34} §3.1.1 again, the product map from $\prod_{i=1}^{d} U_{\pm \lambda_{i}}[m]$ to $U^{\pm}[m]$ is a bijection. For every $\lambda \in \Phi$ and every $g \in U^{\pm}[1]$, we define $g_{\lambda}$ to be the $\lambda$-coordinate of $g$ in this identification. Using \cite{4}, we can define $g_{\lambda}$ for any $\lambda \in \Phi$ and $g = u^{-}\alpha^{+} \in K_{w}[1]$ by taking $g_{\lambda} = u_{\lambda}^{\pm}$.

For every non-negative integer $m$, we also define

$$A_{-m} = \{ a \in A : aU_{2m}a^{-1} \subset U_{0} \text{ and } a^{-1}U_{2m}a \subset U_{0} \}.$$ 

**Lemma 2.3.** There exists some $m_{0} \in \mathbb{N}$ (depending only on the dimension of $G$) such that the following holds. For every $g \in K_{w}[2m_{0} + 1]$, there exists $\lambda \in \Phi$, so that

$$\left| \{ g' \in K_{w}A_{-m_{0}}K_{w} : \|g'gg'^{-1}\lambda - I\| \geq q_{w}^{-m_{0}}\|g'gg'^{-1} - I\| \} \right| \geq q_{w}^{-3m_{0}} \dim G.$$ 

**Proof.** We first note that for all $m \in \mathbb{N}$, all $g' \in K_{w}A_{-m}K_{w}$ and all $g \in K_{w}[2m+1]$, we have

$$g'gg'^{-1} \in K_{w}[1].$$

Therefore $(g'gg'^{-1})_{\lambda}$ is defined.

Let us begin with the following claim: one can assume

$$(5) \quad \|g_{\lambda} - I\| \geq q_{w}^{-c}\|g - I\| \quad \text{for some } \lambda \in \Phi$$

where $c \in \mathbb{N}$ depends only on $\dim G$.

To see \cite{5}, note that since $g \in K_{w}[1]$, we can write $g = g^{-}g^{0}g^{+}$ where $g^{0} \in A_{1}$, $g \in U_{1}$ and $g^{-} \in U_{1}^{-}$. Suppose now that \cite{5} fails with $c = 1$, then $\|g^{-} - I\| > \|g^{+} - I\|$. Let $\lambda \in \Phi$ be a simple root so that $|\lambda(g^{0})|$ is maximal. The map

$$r \mapsto (u_{\lambda}(r)g u_{\lambda}(-r))_{\lambda}$$

is a rational function $\frac{f_{2}(r)}{f_{1}(r)}$ so that coefficients of $f_{1}$ are bounded by $\ll \|g - I\|$ and $|f_{2}(r) - 1| \leq 1/2$ for all $|r|_{w} \leq 1$. Therefore, there exists some $r \in F_{w}$ with $|r|_{w} = 1$ so that if we put $\tilde{g} = u_{\lambda}(r)g u_{\lambda}(-r)$, then

$$\|\tilde{g}_{\lambda} - I\| \gg \|\tilde{g} - I\|.$$ 

Now suppose the claim is proved for $\tilde{g}$. For $\beta > 0$ and $\ast \in G$ put

$$E_{\beta}(\ast) = \{ g' \in K_{w}A_{-m_{0}}K_{w} : \|g'\ast g'^{-1}\lambda - I\| \geq \beta\|g'\ast g'^{-1} - I\| \}.$$ 

Then,

$$E_{\beta}(\tilde{g})u_{\lambda}(r) \subset E_{\beta}(g) \quad \text{and} \quad |E_{\beta}(\tilde{g})u_{\lambda}(r)| = |E_{\beta}(\tilde{g})| \geq \beta^{3}\dim G.$$
We thus assume that (3) is satisfied, and will show the lemma holds with $\lambda$ and $m_0 = c + 1$. For simplicity in the notation, let us assume $\lambda \in \Phi^+$, the argument when $\lambda \in \Phi^-$ is similar. Put $\bar{g} = a_\lambda(\varphi_w^{-c-1})g_\lambda(\varphi_w^{c+1})$. Then $\bar{g} \in K_w[1]$. Using (4) and the decomposition of $U_1$ into one-dimensional unipotent subgroups, we can write

$$g = g^0 h_1 u_\lambda(s') h_2, \quad h_1, h_2 \in U_1.$$ 

We now want to look at the decomposition of $\bar{g}$ into a diagonal element and elements of the one dimensional unipotent groups. Conjugation by $a_\lambda(\varphi_w^{-c-1})$ expend $u_\lambda(s')$ by $q_w^{2c+2}$, and for each other root $\sigma \neq \lambda$, it expand $g_{\sigma}$ by at most $q_w^{c+1}$. we conclude from (5) that $s = q_w^{2c+2}s'$ satisfies $u_\lambda(s) = \bar{g}_\lambda$. Moreover, for $k$ such that $|s|_w = q^{-k}$, we have

$$q_w^{-c-1} \leq |s|_w = q_w^{-k} \leq q_w^{2c+2}q_w^{-2m-1} = q_w^{-1},$$

and $\bar{g}^0 \in K_w[k+1]$, $\bar{g}_{\sigma} \in K_w[k+1]$ for all $\sigma \neq \lambda$.

Since $K_w[k+1]$ is a normal subgroup of $K_w$, we have $\bar{g} = h\bar{u}_\lambda(s)$ where $h \in K_w[k+1]$. Moreover, for every $g_2 \in K_w[k+1]$, we have

$$g_2\bar{g}^{-1} = h''u_\lambda(s)$$

where $h'' \in K_w[k+1]$.

Altogether, there is some $g_1 \in K_wA_{-c-1}K_w = K_wA_{-m_0}K_w$

so that for every $g_2 \in K_w[m_0 + 1]$ we have $g_2g_1g_2^{-1}g_2^{-1} \in K_w[1]$ and

$$\|(g_2g_1g_2^{-1}g_2^{-1})_\lambda - I\| \geq q_w^{-m_0}\|g - I\|.$$

The proof is complete. 

\[ \square \]

3. Construction of a Good Place

Recall that $Y_g = gH(\mathbb{A})/H(F)$, where $g = (1, (g_v))$. In this section we show the existence of a place $w$ where the group

$$H_{g,w} = \Delta_g(G_w) = gH_wg^{-1}$$

has controlled geometry.

Recall the definition of $K_v$ and $K'_v$ from (2). For every $v \in \Sigma$, let

$$K_v,Y_g := H_{g,v} \cap K'_v = gH_vg^{-1} \cap K'_v = \Delta_g(K_v) \cap K'_v.$$ 

In this section, we will simply write $Y$ for $Y_g$, and denote $K_v,Y_g$ by $K_v,Y$.

The following is the main result of this section.

Proposition 3.1 (Existence of a good place). There exists a place $w$ of $F$ such that

1. $G$ is split over $F_w$.
2. $\pi_w(K_w')$ and $\pi_w(K_{w,Y})$ are hyperspecial subgroup of $G'(F_w)$ and $\pi_w(H_g)$, respectively,
3. $g_w \in K_w'$, see (3) for the notation.
4. $q_w \ll \log(\text{vol} Y)^2$ where the implied constant depends on $G$.

Let us begin by recalling some standard facts from Bruhat-Tits theory.
3.1. Bruhat-Tits theory. Let $G, F$ as in our setting (see [32]), and let $v \in \Sigma$. Then

1. For any point $x$ in the Bruhat-Tits building of $G(F_v)$, there exists a smooth affine group scheme $\mathfrak{G}_v(x)$ over $\mathfrak{o}_v$, unique up to isomorphism, such that: its generic fiber is $G(F_v)$, and the compact open subgroup $\mathfrak{G}_v(x)(\mathfrak{o}_v)$ is the stabilizer of $x$ in $G(F_v)$, see [34] 3.4.1.

2. If $G$ is split over $F_v$ and $x$ is a special point, then the group scheme $\mathfrak{G}_v(x)$ is a Chevalley group scheme with generic fiber $G$, see [34] 3.4.2.

3. $\text{red}_v : \mathfrak{G}_v(x)(\mathfrak{o}_v) \to \mathfrak{G}_v(x)(k_v)$, the reduction mod $\mathfrak{o}_v$ map, is surjective, which follows from the smoothness above, see [34] 3.4.4.

4. $\mathfrak{G}_v(x)$ is connected and semisimple if and only if $x$ is a hyperspecial point. Stabilizers of hyperspecial points in $G(F_v)$ will be called hyperspecial subgroups, see [34] 3.8.1 and [30] 2.5.

If $G$ is quasi-split over $F_v$ and splits over $F_v$ (the maximal unramified extension of $F_v$), then hyperspecial vertices exist; and they are compact open subgroups with maximal volume. Moreover a theorem of Steinberg implies that $G$ is quasi-split over $F_v$ for all $v$, see [34] 1.10.4.

It is known that for almost all $v$, the groups $K_v$ are hyperspecial, see [34] 3.9.1 (and [32] for the definition of $K_v$). We also recall that: for almost all $v$ the group $G$ is quasi-split over $F_v$, see [30] Thm. 6.7.

3.2. Adelic volumes and Tamagawa number. Fix an algebraic volume form $\omega$ on $G$ defined over $F$. The form $\omega$ determines a Haar measure on each vector space $\mathfrak{g}_v := \text{Lie}(G) \otimes F_v$ which also gives rise to a normalization of the Haar measure on $G(F_v)$. We denote both these measures by $|\omega_v|$, and denote by $|\omega_h|$ the product measure on $G(A_F)$. Then

\[ |\omega_h|(G(A_F))/G(F)) = D_F^{\frac{\dim G}{2}} \tau(G), \]

where $\tau(G)$ is the Tamagawa number of $G$, and $D_F$ is the discriminant of $F$. In the case at hand $G$ is simply connected, it was recently proved that $\tau(G) = 1$, see [6] §1.3.

3.3. The quasisplit form. The volume formula (7) relates the Haar measure on $Y$ to the algebraic volume form $\omega$ (and the field $F$). However, the volume of our homogeneous set $Y$ as a subset of $X$ depends heavily on the amount of distortion coming from $g$.

Following [30] Sect. 0.4], we let $\mathcal{G}$ denote a simply connected algebraic group defined and quasi-split over $F$ which is an inner form of $G$. Let $L$ be the field associated to $\mathcal{G}$ as in [30] Sect. 0.2], it has degree $[L : F] \leq 3$. We note that $\mathcal{G}$ should be thought of as the least distorted version of $G$.

Let $\omega^0$ be a differential form on $\mathcal{G}$ corresponding to $\omega$. This can be described as follows: Let $\varphi : G \to \mathcal{G}$ be an isomorphism defined over some Galois extension of $F$. We choose $\omega^0$ so that $\omega' = \varphi^*(\omega^0)$, it is defined over $F$. It is shown in [30] Sect.

\[ \text{In most cases } L \text{ is the splitting field of } \mathcal{G} \text{ except in the case where } \mathcal{G} \text{ is a trality form of } D_4 \text{ where it is a degree 3 subfield of the degree 6 Galois splitting field with Galois group } S_3. \]
2.0–2.1] that, up to a root of unity of order at most 3, this is independent of the choice of $\varphi$.

As was done in [30], we now introduce local normalizing parameters $c_v$ which scale the volume form $\omega^0$ to a more canonical volume form on $G(F_v)$.

For every place $v \in \Sigma$, choose an $o_v$-structure on $G$, i.e., a smooth affine group scheme over $o_v$ with generic fiber $G$ as in [30 Sect. 1.2].

Let $\{P_v \subset G(F_v)\}$ denote a coherent collection of parahoric subgroups with “maximal volume”, see [30, Sect. 1.2] for an explicit description. Let us recall that by a coherent collection we mean that $\prod_{v \in \Sigma} P_v$ is a compact open subgroup of $G(A_{\overline{F}})$. Note that any $F$-embedding of $G$ into $GL_N$ gives rise to a coherent family of compact open subgroups of $G(A_{\overline{F}})$ which at almost all places satisfies the above requirements on $P_v$, see §3.1. At the other places we may choose $P_v$ as above and then use (1) in §3.1 to define the $o_v$-structure on $G(F_v)$. Let us also remark that maximality of the volume implies that the corresponding parahoric is either hyperspecial (if $G$ splits over an unramified extension) or special with maximum volume (otherwise).

This allows us, in particular, to speak of “reduction modulo $\varpi_v$”. For every place $v$ of $F$, we let $\overline{M}_v$ denote the reductive quotient of $\text{red}_v(P_v)$; this is a reductive group over the residue field.

3.4. Product formula. Let us use the abbreviation $D_{L/F} = D_L D_F^{-[L:F]}$ for the norm of the relative discriminant of $L/F$, see [30 Thm. A]. It is shown in [30 Thm. 1.6] that

$$\prod_{v \in \Sigma} c_v = D_{L/F}' \cdot A,$$

where $A > 0$ depends only on $G$ over $\overline{F}$ and $[F : \mathbb{Q}]$, $s(\mathcal{G}) = 0$ when $G$ splits over $F$ in which case $L = F$, and $s(\mathcal{G}) \geq 5$ otherwise; these constants depend only on the root system of $\mathcal{G}$.

It should be noted that the parameters $c_v$ were defined using $G$ and $\omega^0$ but will be used to renormalize $\omega_v$ on $G$, and hence on $H$.

Abusing the notation, we will speak of hyperspecial subgroups of $G_w$, $G'_w$, and $H_{g,w}$ in the remaining parts of this section.

Let $\Sigma' \subset \Sigma$ denote the set of places where $G$ is isomorphic to $G$ over $F_v$ and that $K_v$ is hyperspecial. Then $\Sigma \setminus \Sigma'$ is finite.

Lemma 3.2. Let $v \in \Sigma'$. Then

(1) $c_v |\omega_v|(K_{v,Y}) \leq 1$.
(2) Assume $g^v \notin K'_v$. Then

$$c_v |\omega_v|(K_{v,Y}) \leq 1/2$$

Proof. In view of the definition of $K_{v,Y}$, see (9), we have

$$K_{v,Y} = \Delta_g(K_v) \cap K'_v.$$
To see the second claim, first note that if \( g^v \) does not normalize \( K'_v \), then \( g_v \) does not normalize \( \pi^2_v(K_v) \), where \( \pi^2_v : G_v \rightarrow G(F_v) \) is the natural projection, see \( \S 2 \). In particular,

\[
[\pi^2_v(K_v) : \pi^2_v(K_v) \cap g_v^{-1}\pi^2_v(K_v)g_v] \geq 2.
\]

Now since \( v \in \Sigma' \), we have \( \pi^2_v(K_v) \) is a hyperspecial vertex in \( G(F_v) \), \( \pi^2_v(K_v) \) equals its normalizer in \( G(F_v) \); this and the above imply part (2) in view of \( \S 3 \).

We recall the argument that \( \pi^2_v(K_v) \) equals its own normalizer for the convenience of the reader; \( \pi^2_v(K_v) \) is the stabilizer of a hyperspecial point, \( x_v \), say. Since \( g_v \) normalizes \( \pi^2_v(K_v) \), \( \pi^2_v(K_v) \) also stabilizes of \( g_vx_v \). If \( g_vx_v \neq x_v \), then \( \pi^2_v(K_v) \) stabilizes the geodesic between these two points which is a contradiction. \( \square \)

### 3.5. The volume of a homogeneous set.

In view of our definition of volume and taking into account the choice of \( \Omega_0 \), the equation (7) implies that

\[
\text{vol}(Y) = D^\frac{1}{2} \dim F \prod_{v \in \Sigma} (|\omega_v|(K_{v,Y}))^{-1},
\]

where \( |\omega_v| \) is as before. This and (8) imply that

\[
\text{vol}(Y) = AD^\frac{1}{2}s(\phi) D^\frac{1}{2} \dim G \prod_{v \in \Sigma} (c_v|\omega_v|(K_{v,Y}))^{-1}.
\]

Let us note the rather trivial consequence\(^2\) \( \text{vol}(Y) \gg 1 \) of (11). Below we will assume implicitly \( \text{vol}(Y) \geq 2 \) (which we may achieve by replacing \( \Omega_0 \) by a smaller neighborhood at one place in a way that depends only on \( G \)).

**Proof Proposition 3.1** Recall that \( \Sigma' \subset \Sigma \) denotes the set of places where \( G \) is isomorphic to \( G \) over \( F_v \) and that \( K_v' \) and hence \( K'_v = \{(g_1, g_2) : (g_i, 1) \in K_v \} \) is hyperspecial. Furthermore, if \( g^v \in K'_v \), then

\[
K_{v,Y} = \Delta_g(K_v) \cap K'_v
\]

is hyperspecial in \( H_{g,v} \).

Now recall from Lemma 3.2 that for all places \( v \in \Sigma' \), we have \( c_v|\omega_v|(K_{v,Y}) \leq 1 \). Moreover, if \( v \in \Sigma' \) but \( g^v \notin K'_v \), then

\[
c_v|\omega_v|(K_{v,Y}) \leq 1/2.
\]

By Chebotarev density theorem, the set

\[
\Sigma'' = \{v \in \Sigma' : G \text{ is split over } F_v\}
\]

has positive density, \( c > 0 \) say, which depends only on \( G \).

Let \( B \) be a constant depending on \( G \) which will be explicated later. Assume now (for a contradiction) that for every \( v \in \Sigma'' \) with \( q_v \leq Bc^{-1}(\log(\text{vol}(Y)))^2 \), we have \( c_v|\omega_v|(K_{v,Y})^{-1} \geq 2 \).

Recall also that we are assuming \( \text{vol}(Y) \geq 2 \). The above discussion, (11), and the prime number theorem imply that

\[
\text{vol}(Y) \geq B'2B \log(\text{vol}(Y))^2/(\log B - \log c + 2 \log \log(\text{vol}(Y)))
\]

where \( B' \) depends only on \( G \). Now if \( B \gg G 1 \), then

\[
B'2B \log(\text{vol}(Y))^2/(\log B - \log c + 2 \log \log(\text{vol}(Y))) > \text{vol}(Y).
\]

\(^2\)This would also follow trivially from the definition if only we would know that the orbit intersects a fixed compact subset.
This contradiction finishes the proof. \[ \square \]

4. Deduction of Theorem 1.2 from Theorem 1.1

In this section, we complete the proof of Theorem 1.2. The proof relies on Theorem 4.1 below, which in turn is a consequence of Theorem 1.1. It should be noted, however, that the proof of Theorem 1.1 requires a certain uniform decay of matrix coefficients for $H_{g,w}$ where $w$ is a good place, see §4.1. In the setting of Theorem 4.1, the group $H_{g,w}$ has Kazhdan’s property (T), thus the required uniform rate follows from the work of Oh [26].

As before, we let $\pi^1$ denote the projection onto the first factor. Throughout this section, the space $G(\mathbb{A})/G(F)$ is equipped with the $G(\mathbb{A})$-invariant probability measure, we thus drop this measure from the notation.

Theorem 4.1. Assume the absolute rank of $G$ is at least 2. There exists $\kappa_1 > 0$ which satisfies the following. Let $f_1, f_2 \in L_0^2(G(\mathbb{A})/G(F))$ — the orthogonal complement of $G(\mathbb{A})$-invariant functions — be $\pi^1(K_v)$-finite functions. Then

$$|\langle g.f_1, f_2 \rangle| \leq \dim(\pi^1(K_v)).f_1^{1/2} \dim(\pi^1(K_v)).f_2^{1/2} \Xi_{G(F_v)}(g)^{\frac{\kappa_1}{2}} \|f_1\|_2 \|f_2\|_2,$$

where $g \in G(F_v)$, $\langle \pi^1(K_v).f \rangle$ is the linear span of $\pi^1(K_v).f$, and $\Xi_{G(F_v)}$ is the Harish-Chandra spherical function of $G(F_v)$.

Let us first assume Theorem 4.1 and finish the proof of Theorem 1.2.

Proof of Theorem 1.2. First note that the theorem in the case where $G$ is of type $A_1$ follows from §4. We may, therefore, assume that the absolute rank of $G$ is at least 2.

By [23, see also [1] Lemma 3.22], we may restrict ourselves to smooth functions in $L_0^2(G(\mathbb{A})/G(F))$. Now by [23 Thm. 2 and Cor.], Theorem 1.2 is equivalent to the existence of a uniform decay rate for matrix coefficients for $\pi^1(K_v)$-finite functions in $L_0^2(G(\mathbb{A})/G(F))$ as claimed in Theorem 4.1. \[ \square \]

Proof of Theorem 4.1. Let $g = (1,(g_u)) \in G'$ where $g_v = g$ and $g_u = 1$ for all $u \neq v$.

In view of our definition of the volume of a homogeneous set, there exist positive constants $\kappa_2$ and $\kappa_3$ (depending only on the root system of $G(F_v)$) such that

$$\|g\|_{K_v}^{\kappa_2} \ll \vol(Y_g) \ll \|g\|_{K_v}^{\kappa_3}.$$

Let $w$ be a place obtained by applying Proposition 3.1 to $Y_g$. In particular,

1. $G$ is split over $F_w$.
2. $K'_v$ and $K_{w,y}$ are hyperspecial subgroup of $G_w$ and $H_w$, respectively.
3. $q_w \ll \log(\vol Y_g)^2$ where the implied constant depends on $G$.

Then, by [26 Thm. 1.1], and our assumption that the absolute rank of $G$ is at least two, the group $G_w$ has property (T). Therefore, there exists a constant $\kappa_4$, (which depends only on the type of $G$) so that for all $K_w$-finite functions $f_1, f_2 \in L_0^2(G(\mathbb{A})/G(F))$ and all $h_w \in G_w$, we have

$$|\langle h_w.f_1, f_2 \rangle| \ll \dim(\pi_1(K_w)).f_1^{1/2} \dim(\pi_1(K_w)).f_2^{1/2} \|f_1\|_2 \|f_2\|_2 \|h_w\|_{K_w}^{\kappa_4},$$

where the implicit constant depends on $G_w$. If $\text{char}(F) > 2$ this is stated in [26 Thm. 1.1]. A careful examination of the proof shows that (13) holds when $\text{char}(F) = 2$ as well, see [26 §4.7].
Using [13] as an input in the proof of Theorem 4.1, we conclude from Theorem 4.1 that if \( f = f_1 \otimes f_2 \) with \( f_i \in C_c(G(\mathbb{A})/G(F)) \) for \( i = 1, 2 \), then
\[
\left| \int_X f \, d\mu_{g_X} - \int_X f_1 \otimes f_2 \, d\mu_X \right| = \left| \langle g, f_1, f_2 \rangle_{G(\mathbb{A})/G(F)} - \int_{G(\mathbb{A})/G(F)} f_1 \int_{G(\mathbb{A})/G(F)} f_2 \right| \ll \|g\|^{-\kappa} S(f).
\]

The implied multiplicative constant depends on \( G(\mathbb{A})/G(F) \) and so also on the \( F \)-structure of \( G \). We note however, that thanks to [23], the above upgrades to a uniform effective bound on the decay of the matrix coefficients as in Theorem 4.1 with \( \kappa \) independent of the \( F \)-form of \( G \). This implies Theorem 4.1. \( \square \)

5. Effective generation of a group at a good place

In this section, we fix a good place \( w \) and show some useful properties of \( G_w \), see \( \S 2 \) for the definition and the proof of the existence of a good place. In particular, \( G \) is split over \( F_w \) and we can apply the notation and results from \( \S 2.1 \).

Recall that \( \mathfrak{o}_w \) is the maximal compact subring of \( F_w \), and \( \varpi_w \) is a uniformizer of \( \mathfrak{o}_w \), and \( \pi_1(K_w) = G(\mathfrak{o}_w) \). For any \( L > 0 \) and \( R \subset G \), denote by \( R[L] \) the set of elements in \( R \) whose entries vanish in \( \mathfrak{o}_w/\varpi_w^L \mathfrak{o}_w \). For any \( L > 0 \) and \( \tau \subset \mathfrak{g} \) denote by \( \tau[L] \) the set of elements in \( \tau \) whose entries have valuation at least \( L \).

For every \( \lambda \in \Phi \), let \( A_\lambda \) be the scheme theoretic split torus in \( (U_\lambda, U_-) \equiv SL_2 \) which we realize as a closed \( \mathbb{Z} \)-subscheme of \( G \). Put
\[
A'_{\lambda} = \Delta_g(A_\lambda) \subset H_g
\]
the elements in \( A'_{\lambda} \) will be denoted by \( a'_{\lambda}(t) \). Similarly, put \( U'_{\pm \lambda} = \Delta_g(U_{\pm \lambda}) \subset H_g \) and their elements by \( u'_{\pm \lambda}(s) \).

Note that for every \( g \in G_w \), \( t \in F_w^\times \), and \( s \in F_w \), we have
\[
a'_{\lambda}(t)g a'_{\lambda}(t)^{-1} = a_{\lambda}(t) g a_{\lambda}(t)^{-1} \quad \text{and} \quad u'_{\lambda}(s) g u'_{\lambda}(-s) = u_{\lambda}(s) g u_{\lambda}(-s).
\]

For \( b \in F_w \) and \( r > 0 \), let
\[
B(b, r) := \{ s \in F_w : |b - s|_w < r \}.
\]

**Lemma 5.1.** There exists an absolute constant \( \kappa_5 \) such that the following holds.

1. Assume that \( \text{char}(F) > 2 \). For any \( \lambda \in \Phi \) and every \( |s|_w = 1 \), we have
\[
U_{\lambda, \kappa_5} \subset \{ u_\lambda(r^2 - s^2) : |r|_w = 1 \}.
\]

2. Assume that \( \text{char}(F) = 2 \). Then
\[
\{ u_\lambda(c^2) : |c|_w \leq 1 \} \subset \{ u_\lambda(r^2 - s^2) : |r|_w = 1 \}.
\]

**Proof.** Let us first assume that \( \text{char}(F) > 2 \). Let \( \psi_s : F_w^2 \to F_w \) be defined by \( \psi_s(t, r) = t - (r^2 - s^2) \). Then,
\[
\frac{\partial \psi_s}{\partial r}(0, 1) = -2 \neq 0.
\]

Hence, by the implicit function theorem (see [20 §4]), there exists \( 0 < \alpha < 1 \), such that for all \( t \in B(0, \alpha^2) \) there exists \( r \in B(s, \alpha) \) so that \( \psi_s(t, r) = 0 \), i.e., \( t = r^2 - s^2 \). This implies the first claim with \( \kappa_5 = \alpha^2 \).

If \( \text{char}(F) = 2 \), then \( r^2 - s^2 = (r - s)^2 \) which implies the claim. \( \square \)
Lemma 5.2. Recall that $G$ does not admit any non-standard isogenies. There exist $k_0$ and $\ell \geq 1$ with the following property. Let $\lambda \in \Phi$. Then, there exist $h_1, \ldots, h_\ell \in H_{g,w}$ with $\|h_i\| \leq 1$ so that every $g \in K_w^{k_0}$ can be written as

$$g = g_1 g_2 \cdots g_\ell,$$

where for any $1 \leq i \leq \ell$, $g_i \in h_i U_\lambda[0] h_i^{-1}$.

Proof. Clearly it suffices to prove this for a fixed $\lambda \in \Phi$. First note that since $G = G^1$ (see [2]), for every $h = (h^1, h^2) \in H_{g,w}$, we have $h U_\lambda h^{-1} = (h^1,1) U_\lambda (h^1,1)^{-1} \subset G_w$ and $\Ad(h) u_\lambda = \Ad((h^1,1)) u_\lambda \subset g_w$.

Since $G$ is simply connected and does not admit any non-standard isogenies, there exist $h'_1, \ldots, h'_{\dim G} \in H_{g,w}$ so that the $F_w$-span of $\{\Ad(h'_i)u_\lambda\}$ equals $g_w$, see [27, Prop. 1.11] and references therein. Moreover, since $K_w$ is Zariski dense in $G_w$, we can choose $h'_i$ so that $\|h'_i\| \leq 1$ for all $i$.

In consequence, by [3, Prop. 1.16] applied with $\{h'_i U_\lambda h_i'^{-1} : 1 \leq i \leq \dim G\}$ and $G_w$, we conclude that the product map

$$\varphi : \prod_{j=1}^\ell U_j \to G_w$$

is a separable surjective morphism, where $\ell$ is absolute and $U_j \in \{h'_i U_\lambda h_i'^{-1} : 1 \leq i \leq \dim G\}$ for every $1 \leq j \leq \ell$.

Thus, by the implicit function theorem (see [20, §3]), there exists $k_0$ so that $G_w^{k_0} \subset \varphi\left(\prod_{j=1}^\ell U_j[0]\right)$ where $U_j[0] := h'_i U_\lambda[0] h_i'^{-1}$. The proof is complete. \qed

Remark 5.3. Lemma 5.2 is the only place in the paper where the assumption that $G$ does not admit non-standard isogenies is used.

We also need the following lemma, which deals with the case that the characteristic of $F$ equals 2, see Lemma 5.1.

Lemma 5.4. Assume $\text{char}(F) = 2$ and let $\tau_0 \in \mathfrak{a}_w$ be nonzero. Let $\lambda \in \Phi$. There exists an absolute constant $\ell'$ with the following property. For every $r \in B(0,1)$,

$$u_\lambda(r) = h_1 \cdots h_{\ell'}$$

where for all $1 \leq i \leq \ell'$, either $h_i \in \{u_\lambda(\tau_0 \alpha^2) : |\tau_0 \alpha^2| \leq |\tau_0|^{-1}\}$ or $h_i \in H_{g,w}$ with $\|h_i\| \leq 1$.

Proof. We give a hands on proof based on direct computations. Let us begin with the following observation: Let $\varphi_\lambda : \SL_2 \to \langle U_\lambda, U_{-\lambda} \rangle$ be the natural isomorphism which maps the upper (resp. lower) triangular unipotent subgroup of $\SL_2$ to $U_\lambda$ (resp. $U_{-\lambda}$). There exist $0 < \kappa_7 < 1$ and $h_1, h_2, h_3 \in H_{g,w}$ with $\|h_i\| \leq 1$ so that every element

$$h \in \varphi_\lambda \left( \begin{pmatrix} \tau_0 & 0 \\ 0 & 1 \end{pmatrix} \SL_2(\mathfrak{a}_w^{\kappa_7}) \begin{pmatrix} \tau_0^{-1} & 0 \\ 0 & 1 \end{pmatrix} \right) =: L_\lambda, \tau_0^{\kappa_7}$$

can be written as a product

$$(16) \quad h = h_1 \cdots h_6$$

where for every $1 \leq j \leq 6$, $h_j \in h_i \{u_\lambda(\tau_0 \alpha^2) : |\tau_0 \alpha^2| \leq 1\} h_i^{-1}$ for some $1 \leq i \leq 3$.
This is a special case of Lemma 5.2 applied with $\text{SL}_2$ over $F_w^2$ and the root subgroup $\{u_\lambda(r) : r \in F_w^2\}$.

We now turn to the proof of the lemma. Let $|a|, |c| \leq |a_0|$ and put $r' = \frac{r - 1}{1 - r a^2}$, then
\[
\begin{pmatrix}
  r' & 1 \\
  1 & 0 \\
\end{pmatrix}
\begin{pmatrix}
  a^2 & 0 \\
  r_0^{-1} c^2 & a^{-2} \\
\end{pmatrix}
\begin{pmatrix}
  0 & 1 \\
  1 & r' \\
\end{pmatrix}
= \begin{pmatrix}
  a^{-2} & 0 \\
  0 & a^2 \\
\end{pmatrix}
\begin{pmatrix}
  a^2 r' a^2 + r' a^{-2} + r_0^{-1} c^2 \\
  r_0^{-1} c^2 \\
\end{pmatrix}.
\]

This implies that
\[
\Delta_g\left(\varphi_\lambda\left(\begin{pmatrix} r' & 1 \\ 1 & 0 \end{pmatrix}\right)\right) \varphi_\lambda\left(\begin{pmatrix} a^2 & 0 \\ r_0^{-1} c^2 & a^{-2} \end{pmatrix}\right) \Delta_g\left(\varphi_\lambda\left(\begin{pmatrix} 0 & 1 \\ 1 & r' \end{pmatrix}\right)\right) = \varphi_\lambda\left(\begin{pmatrix} a^{-2} & 0 \\ 0 & a^2 \\
  r' a^2 + r' a^{-2} + r_0^{-1} c^2 \\
  r_0^{-1} c^2 \end{pmatrix}\right).
\]

Now, we have
\[
\begin{pmatrix} a^2 & 0 \\ 0 & a^{-2} \end{pmatrix}
\begin{pmatrix} a^{-2} & 0 \\ r' a^2 + r' a^{-2} + r_0^{-1} c^2 \end{pmatrix}
\begin{pmatrix} 1 & \tau_0^{-1} (r_0^{-1} c^2)^2 \\ 0 & 1 \end{pmatrix}
= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix};
\]

applying $\varphi_\lambda$, we conclude that
\[
\varphi_\lambda\left(\begin{pmatrix} a^2 & 0 \\ 0 & a^{-2} \end{pmatrix}\right) \varphi_\lambda\left(\begin{pmatrix} a^{-2} & 0 \\ r' a^2 + r' a^{-2} + r_0^{-1} c^2 \end{pmatrix}\right) u_\lambda(\tau_0^{-1} (r_0^{-1} c^2)^2) = u_\lambda(r),
\]
where we also used $r = (1 + a^4)r'$.

The claim follows from these computations and (16). \qed

Lemma 5.5. There exist $C_1$ so that the following holds. Let $m_1 \in \mathbb{N}$, $m \geq 3 m_1$, and $\lambda \in \Phi$. Let $g \in K_w[m]$ satisfy $\|g_\lambda - I\| \geq q_w^{-m_1} \|g - I\|$. Let $m_g \geq m$ be so that $g \in K_w[m_g] \setminus K_w[m_g + 1]$. For every $q_w^{m_1} \leq |t|_w \leq q_w^{(m_g - m_1)/2}$, and all $|s|_w \leq 1$, we have
\[
\|a_\lambda(t) u_\lambda(s) g u_\lambda(-s) a_\lambda(t^{-1}) - a_\lambda(t) g a_\lambda(t^{-1})\| \leq C_1 q_w^{m_1} \max\{|s|_w, |t|_w^{-1}\}.
\]

If we further assume that $|t|_w = q_w^{(m_g - m_1)/2} \geq C_2 q_w^{2 m_1 + 3}$ and $|s|_w \leq C_1^{-1} q_w^{-2 m_1 - 3}$, we have
\[
\|a_\lambda(t) u_\lambda(s) g u_\lambda(-s) a_\lambda(t^{-1}) - I\| \geq q_w^{-m_1 - 2},
\]
where the implied constant is absolute.

Proof. First note that in view of (16), we may replace $a'_\lambda$ and $u'_\lambda$ with $a_\lambda$ and $u_\lambda$, respectively.

The proof is based on the following observation: let $g \in K_w[m']$ for some $m' \geq 1$, then for all $s \in B_F(0, 1)$ we have
\[
u_\lambda(s) g u_\lambda(-s) = gh(s, g)
\]
where $\|h(s, g)\| \ll \|g - I\| |s|_w$ and the implied constant is absolute.

Let $g$ and $t$ be as in the statement, and let $s \in B_w(0, 1)$. Conjugating the above with $a_\lambda(t)$, we conclude that
\[
a_\lambda(t) u_\lambda(s) g u_\lambda(-s) a_\lambda(t^{-1}) = a_\lambda(t) g a_\lambda(t^{-1}) h_t(s, g)
\]
where $h_t(s, g) = a_\lambda(t) h(s, g) a_\lambda(t^{-1})$, hence,
\[
\|h_t(s, g) - I\| \ll q_w^{m_1} \|g - I\| |s|_w \ll |s|_w
\]
for an absolute implied constant.
Recall the condition \(|g_\lambda - I| \geq q_w^{-m_1} \|g - I\|\). Moreover, note that the conjugation by \(a_\lambda(t)\) has weight 2 for \(U_\lambda\) and weight \(\leq 1\) otherwise. We thus conclude that
\[
\|a_\lambda(t)g_\lambda a_\lambda(t^{-1}) - a_\lambda(t)g_\lambda a_\lambda(t^{-1})\| \leq q_w^{-m_1} |t|^{-1}
\]
This and (17) finish the proof of the first claim.

To see the second claim first note that \(a_\lambda(t)g_\lambda a_\lambda(t^{-1}) = \nu_\lambda(\tau)\) where \(|\tau| \geq q_w^{-m_1-2}\); moreover, if \(|t|_w = q_w^{(m_w - m_1)/2} \geq C_1 q_w^{2m_1+3}\) and \(|s|_w \leq C_1 q_w^{-2m_1-3}\), then
\[
C_1 q_w^{m_1} \max\{|s|_w, |t|_w^{-1}\} \leq q_w^{-m_1-3}.
\]
The second claim thus follows from the first claim and the above.

Recall from (3) that we fixed a Haar measure \(\prod c_w[\omega_w]\) on \(H_g\), and in particular, \(c_w[\omega_w]\) on \(H_{g,w}\). Unless otherwise is stated explicitly, \(c_w[\omega_w]\) is the measures of reference on \(H_{g,w}\) throughout the paper.

**Lemma 5.6.** Let \(m_0\) be as in Lemma 2.3. Let \(m \geq 6m_0 + 1\). For \(i = 1, 2\), let
\[
E_i \subset K_w^{A_{-m_0}K_w}
\]
be two subsets so that \(|K_w^{A_{-m_0}K_w} \setminus E_i| \leq q_w^{-4m_0 \dim G} G
\]
Then for every \(g' \in K'_w[m]\), there exist \(\alpha_i \in \Delta_g(E_i)\) and \(\lambda \in \Phi\) so that
\[
\alpha_2 g' \alpha_1^{-1} = g \in K_w[m - 2m_0] \subset G_w \times \{1\}
\]
where the implied constant depends only on \(\dim G\).

**Proof.** Let us write \(g' = h\tilde{g}\) for some \(h \in H_{g,w} \cap K'_w[m]\) and \(\tilde{g} \in K_w[m] = G_w \cap K'_w[m]\). If \(\alpha \in \Delta_g(K_w^{A_{-m_0}K_w})\), then
\[
g' \alpha^{-1} = (h\alpha^{-1})g\alpha^{-1}.
\]
Note that since \(\tilde{g} \in K_w[m]\), for all \(\alpha\) as above we have \(\alpha g\alpha^{-1} \in K_w[m - 2m_0]\). We will find \(\alpha_1, \alpha_2 \in \Delta_g(E_i)\) so that \(\alpha_2 g' \alpha_1^{-1} = \alpha_1 g\alpha_1^{-1}\).

Since \(\tilde{g} \in K_w[m] \subset K_w[2m_0 + 1]\), in view of Lemma 2.3 there exists \(\lambda \in \Phi\) so that
\[
|E_{\tilde{g}}| \geq q_w^{-3m_0 \dim G}
\]
where \(E_{\tilde{g}} = \{y \in K_w A_{-m_0} K_w : \|(y\tilde{g}y^{-1})\lambda - I\| \geq q_w^{-m_0} \|y\tilde{g}y^{-1} - I\|\}\).

Since \(h \in K'_w \cap H_{g,w}\), we have
\[
h \Delta_g(K_w^{A_{-m_0}K_w}) = \Delta_g(K_w^{A_{-m_0}K_w}).
\]
In particular, the map \(f : \Delta_g(E_1) \to \Delta_g(K_w^{A_{-m_0}K_w})\) defined by \(\alpha \mapsto h\alpha^{-1}\) is measure-preserving.

Altogether, and in view of our assumption on the relative measures of \(E_1\) and \(E_2\), we may choose \(\alpha_1 \in \Delta_g(E_1) \cap \Delta_g(E_{\tilde{g}})\) with \(\alpha_2 = f(\alpha_1)^{-1} \in \Delta_g(E_2)\). The lemma follows.

6. **Non-divergence of unipotent orbits**

In this section we show that when \(X\) is not compact, the cusps have small measure with respect to \(\mu_g\). We follow the proof of [11, Lemma 3.6.1], which relies on the non-divergence of unipotent flows.
Let \( F \) be a finite field such that \( F \) is a finite separable extension of \( \tilde{F} := F(T) \), the field of rational functions in one variable over \( F \), see e.g. [37, Ch. III]. Let \( \mathcal{G} = \text{Res}_{F/\tilde{F}}(G') \). Then \( \mathcal{G} \) is a semisimple \( \tilde{F} \)-group and

\[
\mathcal{G}(k_{\tilde{F}})/\mathcal{G}(\tilde{F}) = G'(k_{\tilde{F}})/G'(F).
\]

Fix a good place \( w \in \Sigma \), and let \( \tilde{w} \) be a place of \( \tilde{F} \) so that \( w|\tilde{w} \). We note that since \( w \) is a good place for \( G' \), the group \( \mathcal{G} \) is isotropic over \( \tilde{F}_w \) (however, it need not be split).

Let \( K'(\tilde{w}) := \prod_{i | \tilde{w}} K'_i \). Since \( \mathcal{G} \) is simply connected, it follows from strong approximation (see [28]) that one can write

\[
\mathcal{G}'(\tilde{A}) = K'(\tilde{w}) \left( \prod_{i | \tilde{w}} G_i \right) \mathcal{G}'(F) = K'(\tilde{w}) \mathcal{G}(\tilde{F}_w) \mathcal{G}(\tilde{F}).
\]

Let \( \Gamma = \mathcal{G}(\tilde{F}) \cap K'(\tilde{w}) \) and \( \mathcal{G}_w = \mathcal{G}(\tilde{F}_w); \) note that \( \Gamma \) is a congruence subgroup of \( \mathcal{G}_w \). The above implies that

\[
K'(\tilde{w}) \backslash \mathcal{G}'(\tilde{A}) \backslash \mathcal{G}'(F) \cong \mathcal{G}_w / \Gamma.
\]

Let \( \mathcal{O} \) be the ring of \( \tilde{w} \)-integers in \( \tilde{F} \); note that \( \mathcal{O} \) is a PID. Let \( \tilde{g} = \text{Lie}(\mathcal{G}_w) \), and let \( \tilde{g}_\mathcal{O} \) be an \( \mathcal{O} \)-lattice in the \( \tilde{g} \) with the property that \([ \tilde{g}_\mathcal{O} , \tilde{g}_\mathcal{O} ] \subseteq \tilde{g}_\mathcal{O} \). As it is done in [11], for any \( x \in X \), we set

\[
\text{ht}(x) = \sup \{ \| \text{Ad}(g)u \|^{-1} : x = g\Gamma, \ g \in \mathcal{G}_w, \ u \in \tilde{g}_\mathcal{O} \setminus \{0\} \},
\]

and

\[
\mathcal{G}(R) := \{ x \in X : \text{ht}(x) \leq R \}.
\]

Lemma 6.1. For every \( m \in \mathbb{N} \), let

\[
Q_m' := \{ ((g^0_1, g^0_2) \in K' : (g^0_1, g^0_2) \in \pi_w(K'_w[m])) \}.
\]

There exists \( \kappa_8 \) so that for all \( x \in X \) with \( m \leq \text{ht}(x)^{-1} \), the map \( g \mapsto gx \) is injective on \( Q_m' \).

Proof. Fix some \( m \in \mathbb{N} \) and assume that \( g_1 x = g_2 x \) for \( g_1, g_2 \in Q_m' \). Fix \( \tilde{g} \in \mathcal{G}_w = \mathcal{G}(\tilde{F}_w) \) such that the projection to the \( \tilde{w} \) place of \( x \) is \( \tilde{g}\Gamma \). Then, \( g_1 \tilde{g} \tilde{g} \Gamma = g_2 \tilde{g} \tilde{g} \Gamma \), and so \( g_2^{-1} g_1 \tilde{g} \tilde{g} \) fixes \( L_x := \text{Ad}(g)\tilde{g}_\mathcal{O} \). Note that \( L_x \) is an \( \mathcal{O} \)-lattice in the \( \tilde{g} \). By the definition of the height, for all \( u \in L_x \) we have \( \| u \| \geq \text{ht}(x)^{-1} \).

Since the covolume of \( L_x \) is independent of \( x \) (and equals the covolume of \( \tilde{g}_\mathcal{O} \)), by lattice reduction theory, one can find a basis \( u_1, \ldots , u_d \) (where \( d \leq 2N^2 \)) of \( L_x \) such that \( \| u_i \| \ll \text{ht}(x)^d \), see [18, Thm. 1.2] and references therein.

Thus, by choosing the constants \( m_{[\kappa_8]} \) sufficiently large, we get

\[
\| g_1^{-1} g_1 \tilde{g}_1 \tilde{g}_1 u_i - u_i \| < \text{ht}(x)^{-1} \text{ for all } 1 \leq i \leq d.
\]

In that case, the vector \( g_2^{-1} g_1 \tilde{g}_1 \) fixing the lattice \( L_x \) setwise implies that it is, in fact, fixing \( L_x \) pointwise. Thus, \( g_2^{-1} g_1 \tilde{g}_1 \) belongs to the center of \( \mathcal{G}_w \). Let \( g' \in G' \) be so that \( x = g' G'(F) \). Then, we have \( g_2^{-1} g_1 = g' \gamma g'^{-1} \) where \( \gamma = (\gamma^1, \gamma^2) \in G'(F) \) and \( \gamma^1 \) embeds diagonally in \( G(A) \). Now since \( g_2^{-1} g_1 \) is central, we conclude that \( (\gamma^1, \gamma^2) \) is central, which is impossible if \( m \) is big enough.

Proposition 6.2 (Non-divergence estimate). There are positive constants \( \kappa_9 \) and \( \kappa_{10} \), depending on \( N \) and \([F : \tilde{F}]\), so that for any \( g \in G' \) we have

\[
\mu_g (X \setminus \mathcal{G}(R)) \ll \mu_{[\kappa_9]} R^{-[\kappa_{10}]}.
\]
recall that \( p_w = \text{char}(k_w) \).

Proof. The proof is similar to that of [11 Lemma 3.6.1] (see [11 App. B]) where one replaces the use of the non-divergence result by Kleinbock and Margulis [19]

Following [13 §4], for a discrete \( O \)-module \( \Delta = \oplus_{i=1}^r O u_i \), we denote \( \|\Delta\| = \|u_1 \land \cdots \land u_r\|_v \). Then, \( \|\Delta\| \) is a "norm like" function in the language of [13] that plays the role of the covolume of \( \Delta \) in the proof (see the proof of Lemma 4.3 and properties N1-N3 in [13]). We refer to \( \Delta \) as an \( O \)-arithmetic lattice in

\[
V = \oplus_{i=1}^r F\hat{w} u_i \subset \hat{g}.
\]

Let \( h \in \hat{G}_{\hat{w}} \). A subspace \( V \subset \hat{g} \) is called \( h \)-rational if \( V \cap \text{Ad}_h \hat{g}_O \) is an \( O \)-arithmetic lattice in \( V \); we let \( \|V\|_h = \|V \cap \text{Ad}_h \hat{g}_O\| \). The first step is to show that for any \( h \) there are no \( h \)-rational subspaces which are \( H_w \) invariant of low norm.

Let \( U = \{u(t)\} \) be a one parameter \( \tilde{F}_{\hat{w}} \)-unipotent subgroup of \( H_w \), and use \( \text{Ad}_{\tilde{g}(u(t))} \) in the definition of \( h(t) \) in the Proof of [11 Lemma 3.6.1], i.e. \( h(t) = \text{Ad}_{\tilde{g}(u(t))} |V\in\text{SL}(V) \), where \( V \) is an \( h \)-rational space for some \( h \in \hat{G}_{\hat{w}} \).

Now, one can follow the arguments in [11 Proof of Lemma 3.6.1] using the above notations, Lemma 6.1 [13 Thm. 5.2], and [13 Property N2] to show that there exist positive constants \( c, \kappa \) such that for any \( x \in \pi_v(Hg\Gamma) \)

\[
\text{(18)} \quad \text{there is no } x \text{-rational, } H_w \text{-invariant proper subspace of norm } \leq cp_w^\kappa.
\]

where \( c \) is an absolute constant and \( p_w \) depends only on \( \dim G \).

Put \( \xi = cp_w^\kappa \). Since the number of \( x \)-rational proper subspaces of norm at most \( \xi \) is finite and \( U \subset H_w \), a.e. \( h \in H_w \) has the property that \( hU h^{-1} \) does not leave invariant any proper \( x \)-rational subspace of norm \( \leq \xi \). Alternatively, we may also conclude for a.e. \( h \in H_w \cap K_v \) that \( U \) does not leave any proper \( h x \)-rational subspace of norm \( \leq \xi \) invariant.

Since \( \mu_g \) is \( H \)-invariant and \( H \)-ergodic, by Mautner phenomenon \( \mu_g \) is also \( U \)-ergodic. This also implies that the \( U \)-orbit of \( h x \) equidistributes with respect to \( \mu_g \) for a.e. \( h \). We choose \( h \in H_w \cap K_v \) so that both of the above properties hold for \( x' = h x \).

Let \( h' \) be such that \( x' = h' T_{\hat{w}} \). Then, for any \( h' T_{\hat{w}} \)-rational subspace \( V \), if we let

\[
\psi_V(t) := \left| \left| \text{Ad}_{u(t)}^{-1} V \right|_{u(t)^{-1} h'} = \left| \left| \text{Ad}_{u(t)}^{-1} (V \cap \text{Ad}_{h'} \hat{g}_O) \right|_v \right|,
\]

then either \( \psi_V \) is unbounded or equals a constant \( \geq \xi \). Thus, by [13 Thm 4.4] there exists a positive constant \( \kappa_{11} \) so that

\[
\mu_g(\{t : |t|_\hat{w} \leq r, x'u(t) \notin \mathcal{E}(\varepsilon^{-1})\}) \ll \frac{\kappa_{11}}{\xi^{\alpha}} \mu_g(\{t : |t|_\hat{w} \leq r\}),
\]

for all large enough \( r \) and \( \varepsilon > 0 \), where \( \alpha = \frac{\kappa_{10}}{\kappa_{11}} \) only depends on the degree of the polynomials appearing in the matrix entries for the elements of the one-parameter unipotent subgroup \( U \). The lemma now follows as the \( U \)-orbit equidistributes with respect to \( \mu_g \). □
7. Invariant subgroups for the measure \( \mu \)

Recall that \( \mu \) is an \( H_g \)-invariant measure on \( Y_g = gH(\mathbb{A})/H(F) \). In this section, we will use tools from homogeneous dynamics combined with algebraic properties of good places which have been established in previous sections to prove Theorem \[1.1\]. When \( X \) is not compact, the proof will also use the non-divergence result proved in Lemma \[6.2\].

7.1. Adelic Sobolev norms. Let \( C^\infty(X) \) denote the space of functions which are invariant by a compact open subgroup of \( G(\mathbb{A}) \). We follow \[10\] App. A] to define Sobolev norms on \( C^\infty(X) \), and present some of its properties.

First, we define a system of projections \( \text{pr}_v[m] \) for any unitary \( G(F_v) \)-representation for each place \( v \) such that \( \sum_{m \geq 0} \text{pr}_v[m] = 1 \). For any \( v \in \Sigma \), let \( \text{Av}_v[m] \) be the projection on \( K_v[m] \)-invariant vectors — this is simply given by averaging over the compact open subgroup \( K_v[m] \) equipped with the probability Haar measure. Put \( \text{pr}_v[0] := \text{Av}_v[0] \) and \( \text{pr}_v[m] := \text{Av}_v[m] - \text{Av}_v[m-1] \) for \( m \geq 1 \).

Let \( \mathcal{M} \) be the set of functions mapping \( \Sigma \) to the non-negative integers, such that almost every \( v \) is mapped to zero. For any \( \mathbf{m} \in \mathcal{M} \) let

\[
\| \mathbf{m} \| := \prod_v q_v^{m_v}, \quad \text{and} \quad \text{pr}[\mathbf{m}] := \prod_v \text{pr}_v[m_v].
\]

Then \( \text{pr}[\mathbf{m}] \) acts on any unitary \( G(\mathbb{A}) \)-representation, and \( \sum_{\mathbf{m}} \text{pr}[\mathbf{m}] f = f \) for any \( f \in C^\infty(X) \). Moreover, for any \( f \in C^\infty(X) \) and \( \mathbf{m} \in \mathcal{M} \), the function \( \text{pr}[\mathbf{m}] f \) is invariant under \( K_\mathbf{m} := \prod_v K_v[m_v] \).

Given an integer \( d \geq 0 \) we define a degree \( d \) Sobolev norm by

\[
S_d(f)^2 := \sum_{\mathbf{m}} \| \mathbf{m} \|^d \| \text{pr}[\mathbf{m}] (1 + \text{ht}(x))^d f(x) \|_2^2,
\]

where \( \text{ht}(\cdot) \) is as defined in \[6\]. For a compactly supported smooth function on \( X \) any of these Sobolev norms is finite. It is also easy to see that

\[
S_d(f) \leq S_{d'}(f) \quad \text{if} \quad d < d'.
\]

We claim that in a similar way to \[10\] App. A], we have the following:

S0. For any \( d \), \( S_d \) is a pre-Hilbert norm on \( C^\infty_c(X) \).

S1. There exists \( d_0 \), depending on \( N \), and \( [F : F(\mathbb{A})] \) (see \[6\] for the definition of \( F(\mathbb{A}) \)) such that for all \( d \geq d_0 \), we have \( \| f \|_{L^\infty} \ll_d S_d(f) \).

S2. Given \( d_0 \), there are \( d > d' > d_0 \) and an orthonormal basis \( \{ e_k \} \) of the completion of \( C^\infty_c(X) \) with respect to \( S_d \) which is orthogonal with respect to \( S_{d'} \) such that

\[
\sum_k S_{d'}(e_k)^2 < \infty \quad \text{and} \quad \sum_k \frac{S_{d_0}(e_k)^2}{S_{d'}(e_k)^2} < \infty.
\]

S3. Let \( g \in G(\mathbb{A}) \) and \( f \in C^\infty_c(X) \). We write \( g.f \) for the action of \( g \) on \( f \), and \( \| g \| = \prod_v \| g_v \| \). Then, for every \( d \geq 0 \)

\[
S_d(g.f) \ll \| g \|^{4N_d} S_d(f),
\]

where the implied constant is absolute. If in addition \( g \in K \), then we have \( S_d(g.f) = S_d(f) \).

Now let us fix a good place \( w \). Note that for every \( \lambda \in \Phi \), we have \( \| u_\lambda'(s) \| \leq (1 + |s|_w)^N \) for all \( s \in F_w \).
S4. For any $d \geq d_0$, any $r \geq 0$, any $g \in K_w[r]$, and any $f \in C_c^\infty(X)$, we have
\[ \|g \cdot f - f\|_\infty \leq q_w^{-r}S_d(f). \]

S5. There exist $d_0, M$, which depend only on $N$, so that the following holds.
For every $L$, let $AV_L$ be the operation of averaging over $K_w[L]$. Let $\lambda \in \Phi$ and $s \in F_w$. Put $T_s = AV_L \ast \delta_{\phi(s)} \ast AV_L$ where $\ast$ denotes convolution of operators. For all $x \in X$, all $f \in C_c^\infty(X)$, and $d \geq d_0$ we have
\[ \left| T_s f(x) - \int f \, dm_x \right| \ll q_w^{(d+2)l}ht(x)^d \|T_s\|_{2,0}S_d(f), \]
where $\|T_s\|_{2,0}$ denotes the operator norm of $T_s$ on $L^2_0(X, m_X)$. Moreover, we have
\[ \|T_s\|_{2,0} \ll (1 + |s|_w)^{-1/2M} q_w^{2dL}. \]

S6. There exist $d_0, M$, which depend only on $N$, so that for all $d \geq d_0$ and all $\lambda \in \Phi$, we have
\[ \left| (u_\lambda(s)f_1, f_2)_{L^2_0(\mu)} - \int f_1 \, d\mu \int f_2 \, d\mu \right| \ll (1 + |s|_w)^{-1/2M}S_d(f_1)S_d(f_2). \]

Indeed, the above properties can be shown arguing in a similar way to [10, App. A4]. We only present a brief guided tour of the proof, and the implied changes in the statements. In the proof of property S1., Lemma [6.1] replaces [10, Lemma 7.2], and so the implied constant only depends on $N$ and $[F : F]$. In the proof of property S3., for any place $v$ and $g \in G_v$, the inequality
\[ ht(gx) \ll \|g\|_{2N}^2 ht(x) \]
follows from the definition of $ht(x)$ (see [8]), and so (except possibly a different power in the result) the proof is the same. Properties S5. and S6. follow from $1/M$-temperedness (see [23, Thm. 2]) of the natural representation of $G(F_w)$ on $L_0^2(X, m_X)$ and $L_0^2(X, \mu)$, respectively. In the case where $G$ is type $A_1$, this follows from property (7), [8]. In all other cases, $G$ has absolute rank at least 2, and $w$ is so that $G$ is $F_w$-split (see [33]). Therefore, $G(F_w)$ has property $(T)$ and the desired $1/M$-temperedness follows from [26].

7.2. Generic points. Let $\lambda \in \Phi$. Let $n \in \mathbb{N}$. For any $x \in X$ and $t = \varpi^{-n}$, define
\[ D_n f(x) := q_w^n \int_{B(0, q_w^{-n})} f(a_\lambda'(t)u_\lambda'(s)x) \, ds - \int_X f \, d\mu; \]
see the beginning of [85] more specifically [12], for the definition of $a_\lambda'(t)$ and $u_\lambda'(s)$. In particular, we have $a_\lambda'(t), u_\lambda'(s) \in H_{\varphi, w} \subset H_{\varphi}$ for all $t, s, \lambda$. Therefore, they preserve the measure $\mu$.

A point $x \in X$ is called $n_0$-generic for $\lambda$ with respect to the Sobolev norm $S_d$, if for any integer $\ell \geq n_0$, we have
\[ |D_\ell f(x)| \leq q_w^{-\ell/8M}S_d(f) \]
where $M$ is as in (22).

Proposition 7.1. If $d_0$ is chosen large enough, depending on $N$, then for all $d \geq d_0$ the $\mu$-measure of points which are not $n_0$-generic for some $\lambda \in \Phi$ with respect to $S_d$ is \( \ll q_w^{-n_0/5M} \).
Proof. Let $\lambda \in \Phi$ be arbitrary. For simplicity in the notation, we will write $a'(t)$ and $u'(s)$ for $a'_\lambda(t)$ and $u'_\lambda(s)$, respectively.

Let $\ell \geq n_0$ be an integer and $d_0$ satisfy \(\text{(22)}\). Let $f$ be a fixed function in $C_0^\infty(X)$. Let us write $B = B(0, q_w^{-\ell})$. Put

$$A_1 = \{(s_1, s_2) \in B \times B : |s_1 - s_2|_w < q_w^{-11\ell/10}\}$$

and $A_2 = B \times B \setminus A_1$.

For $(s_1, s_2) \in B \times B$, put $F(s_1, s_2) = \langle a'(t)u'(s_1 - s_2)a'(t^{-1})f, f \rangle_{\mu}$. If $(s_1, s_2) \in A_2$ and $|t|_w \geq q_w^t$, then $a'(t)u'(s_1 - s_2)a'(t^{-1}) = u'(\tau)$ with $|\tau| \geq q_w^{0.0\ell}$. Hence, for any $(s_1, s_2) \in A_2$ and $|t|_w \geq q_w^t$, by \(\text{(22)}\) we have

$$|F(s_1, s_2) - \left(\int_X f \, d\mu\right)^2| \ll q_w^{-9\ell/20M}S_{d_0}(f)^2.$$

Since the Lebesgue measure of $A_1$ is $q_w^{-21\ell/10}$, using the definition of $D_\ell f(x)$ and the Fubini’s theorem, now we have

$$\int_X |D_\ell f(x)|^2 \, d\mu = q_w^{2\ell}\int_B \int_B F(s_1, s_2) \, ds_1 \, ds_2 - \left(\int_X f \, d\mu\right)^2$$

$$\leq q_w^{2\ell}\left(\int_{A_1} F(s_1, s_2) \, ds_1 \, ds_2 + \int_{A_2} F(s_1, s_2) \, ds_1 \, ds_2\right) - \left(\int_X f \, d\mu\right)^2$$

$$\ll 3d_w^{-9\ell/20M}S_{d_0}(f)^2$$

Using Markov’s inequality we arrive at

\(\text{(25)}\)

$$\mu\left\{x \in X : |D_\ell f(x)| > \varepsilon\right\} \ll 3\varepsilon^{-2}q_w^{-9\ell/20M}S_{d_0}(f)^2.$$

To conclude, we use property S2. of the Sobolev norms. That is, there are $d > d' > d_0$ and an orthonormal basis $\{e_k\}$ of the completion of $C_0^\infty(X)$ with respect to $S_d$ which is orthogonal with respect to $S_{d'}$ so that

\(\text{(26)}\)

$$\sum_k S_{d'}(e_k)^2 < \infty \quad \text{and} \quad \sum_k \frac{S_{d_0}(e_k)^2}{S_{d'}(e_k)^2} < \infty.$$ 

Put $c = (\sum_k S_{d'}(e_k)^2)^{-1/2}$ and define

\(\text{(27)}\)

$$X' = \bigcup_{\ell \geq n_0, i \geq 1} \left\{x \in X : |D_\ell e_i(x)| \geq cq_w^{-\ell/8M}S_{d'}(e_i)\right\}.$$

Using \(\text{(25)}\) and \(\text{(26)}\), we have

\(\text{(28)}\)

$$\mu(X') \ll q_w^{-n_0/5M}.$$

Let $f \in C_0^\infty(X)$, and write $f = \sum f_k e_k$ and suppose $x \notin X'$. Let $\ell \geq n_0$, then using the triangle inequality for $D_\ell$ we obtain

$$|D_\ell f(x)| \leq \sum_k |f_k||D_\ell e_k(x)| \leq cq_w^{-\ell/8M}\sum_k |f_k|S_{d'}(e_k)$$

$$\leq cq_w^{-\ell/8M}\left(\sum_k |f_k|^2\right)^{1/2}\left(\sum_k S_{d'}(e_k)^2\right)^{1/2}$$

$$= q_w^{-\ell/8M}S_d(f),$$

where the last inequality follows from the definition of $c$ and since $\{e_i\}$ is an orthonormal basis with respect to $S_d$.

This completes the proof in view of \(\text{(28)}\) as $\lambda$ was arbitrary. \(\square\)
For the rest of this section, we fix some \( d \geq d_0 \) so that Proposition \( \ref{prop:7.1} \) holds, and write \( S = S_d \).

For every \( m \), let \( Q_m = \{(g_w, 1) \in K : g_w \in \pi^1_w(K_w[m])\} \).

**Proposition 7.2.** There exists \( \kappa_{12} \) such that for all \( c \in \mathbb{N} \) and all large enough \( m \geq 3c \), the following holds. Let \( g \in Q_m \) satisfy \( g_w \neq 1 \) and

\[
\|g_w - 1\| \geq q^{-c}_w \|g_w - 1\|.
\]

Let \( \lambda \in \Phi \). Suppose \( y_1, y_2 \in X \) are two \( \lfloor m/10 \rfloor \)-generic points for the measure \( \mu \) with respect to \( S \) satisfying that \( y_2 = g y_1 \). Then,

\[
|u_\lambda(\tau).u(f) - u(\tau)| \ll q^{-\kappa_{12}}_w S(f)
\]

for all \( \tau \in B(0,1) \) and all \( f \in C_c^\infty(X) \).

**Proof.** As in the proof of Proposition \( \ref{prop:7.1} \) we write \( a'(t), u'(s) \) for \( a_\lambda(t), u_\lambda(s) \); we also write \( a(t), u(s) \) for \( a_\lambda(t), u_\lambda(s) \). Recall from \( \ref{eq:15} \) that

\[
a(t)g a(t^{-1}) = a'(t)g a'(t^{-1}) \text{ and } u(s)g u(-s) = u'(s)g u'(-s).
\]

Let us also recall that \( a(t) \) and \( u'(s) \) belong to \( H_y \), hence, they leave \( \mu \) invariant; while \( a(t), u(s) \in G_w \) and we do not have any a priori information regarding there action on \( \mu \).

For all \( s \) and \( t \), put

\[
g_{s,t} = a'(t)u'(s)g u'(-s)a'(t^{-1});
\]

note that \( g_{s,t,v} = g_v \) for all \( v \neq w \).

Apply Lemma \( \ref{lem:5.5} \) with \( m_1 = c \), and let \( m \geq 3c \). Let us write \( \ell = \lfloor (m_g - c)/2 \rfloor \) where \( g_w \in K_w[m_g] \setminus K_w[m_g + 1] \). Let \( |t|_w = q^{-2}_w \); we will always assume \( |s|_w \leq |t|_w^{-1} \). By Lemma \( \ref{lem:5.5} \) we have

\[
\|g_{s,t,w} - a(t)g_{s,w}a(t^{-1})\| \leq C_1^{-\ell} q_1^{-c};
\]

moreover, if \( |t|_w = q^{\ell}_w \geq C_2 q_1^{2c+3} \), then

\[
\|g_{s,t,w} - I\| \geq q^{-2}_w.
\]

Recall now that \( y_i, i = 1, 2 \), is \( \lfloor m/10 \rfloor \)-generic for \( \mu \) w.r.t. \( S \). This, together with our choice of the integer \( \ell \geq \lfloor m/10 \rfloor \), implies that

\[
|q^{\ell}_w \int_{B_w(0,q^{\ell}_w)} f(a'(t)u'(s)g y_i) \, ds - \int_X f \, d\mu| \leq q^{-\ell/8M}_w S(f)
\]

for \( i = 1, 2 \).

Since \( y_2 = g y_1 \), we get that \( a'(t)u'(s)y_2 = g_{s,t}a'(t)u'(s)y_1 \). In particular,

\[
f(a'(t)u'(s)y_2) = f(g_{s,t}a'(t)u'(s)y_1).
\]

Applying \( \ref{eq:31} \) with \( y_2 \), we thus conclude that

\[
|q^{\ell}_w \int_{B_w(0,q^{\ell}_w)} f(g_{s,t}a'(t)u'(s)y_1) \, ds - \int_X f \, d\mu| \leq q^{-\ell/8M}_w S(f).
\]

Put \( \hat{g} = a(t)^{g} a(t^{-1}) \) where \( g_{\hat{w}} = g_w \) for all \( v \neq w \), and \( g_{w,\lambda} = g_{w,\lambda} \). Then \( \ref{eq:32} \) and \( \ref{eq:29} \) imply that if \( m \geq 5c \), then

\[
|q^{\ell}_w \int_{B_w(0,q^{\ell}_w)} f(\hat{g} a(t)^{g} a(t^{-1})y_1) \, ds - \int_X f \, d\mu| \ll q^{-\ell/9M}_w S(f).
\]
The above, (31), applied with $y_1$ and $\hat{g}^{-1}.f$, and property S3, yield
\begin{equation}
|\hat{g}.\mu(f) - \mu(f)| \ll q_w^{-\ell/9M}S(f).
\end{equation}

Recall that $\|\hat{g}_{w,\lambda}^{-1}f\| \geq q_w^{-c}\|\hat{g}_{w}^{-1}\|$. Therefore, if we write $u(\tau_0) = a(t)\hat{g}_{w,\lambda}a(t^{-1})$, then $q_w^{-c-2} \leq |\tau_0| \leq 1$.

For every $\rho$, put $\hat{g}_\rho := a'(\rho)\hat{g}_\alpha(\rho^{-1})$. Then $\hat{g}_{\rho} = \hat{g}_\rho$ and $\hat{g}_{w,\rho} = u(\rho^2\tau_0)$. Moreover, we have
\begin{equation}
|\hat{g}_{\rho}.\mu(f) - \mu(f)| = |\hat{g}.\mu(a'(\rho^{-1}).f) - \mu(a'(\rho^{-1}).f)| \ll q_w^{-\ell/9M}S(a'(\rho^{-1}).f).
\end{equation}

where we use the fact that $\mu$ is $H_g$-invariant in the first equality and (33) in the second line. In view of property S3., we have $S(a'(\rho^{-1}).f) \ll \max\{|\rho|^{4Nd}_w, |\rho|^{-4Nd}_w\}S(f)$. This and (34), imply
\begin{equation}
|\hat{g}_{\rho}.\mu(f) - \mu(f)| \ll q_w^{-\ell/9M}\max\{|\rho|^{4Nd}_w, |\rho|^{-4Nd}_w\}S(f).
\end{equation}

Recall now that $\hat{g}^{-1}\hat{g}_\rho = u((\rho^2 - 1)\tau_0)$. Altogether we conclude that
\begin{equation}
|u((\rho^2 - 1)\tau_0).\mu(f) - \mu(f)| \ll q_w^{-\ell/9M}S(f).
\end{equation}

for all $|\rho|_w = 1$. We now consider two cases:

**Case 1:** $\text{char}(F) > 2$. Then by Lemma 5.1, $B(0, \overline{K}) \subset \{\rho^2 - 1 : |\rho|_w = 1\}$. This and the fact that $|\tau_0| \geq q_w^{-c-2}$, imply that
\begin{equation}
|u(\tau).\mu(f) - \mu(f)| \ll q_w^{-\ell/9M}S(f)
\end{equation}

for all $\tau \in B(0, q_w^{-c-3})$. Using $u(\rho^2\tau) = a'(\rho)u(\tau)a'(\rho^{-1})$ and arguing as in (34) again, we conclude that so long as $m$ (and hence $\ell$) is large enough,
\begin{equation}
|u(\tau).\mu(f) - \mu(f)| \ll q_w^{-\ell/10M}S(f)
\end{equation}

for all $\tau \in B(0, 1)$; the proof is complete in this case.

**Case 2:** $\text{char}(F) = 2$. Then by Lemma 5.1 we have
\begin{equation}
|u(\tau).\mu(f) - \mu(f)| \ll q_w^{-\ell/9M}S(f)
\end{equation}

for all $\tau = \tau_0\alpha^2$ and $\alpha \in B(0, 1)$. Using $u(\rho^2\tau) = a'(\rho)\mu(\tau)a'(\rho^{-1})$ and arguing as in (34) again, we conclude that so long as $m$ is large enough,
\begin{equation}
|u(\tau).\mu(f) - \mu(f)| \ll q_w^{-\ell/10M}S(f)
\end{equation}

for all $\tau = \tau_0\alpha^2$ with $|\alpha| \leq q_w^c$ (recall that $q_w^{-c-2} \leq |\tau_0| \leq 1$).

Now by Lemma 5.4, for every $r \in B(0, 1)$
\begin{equation}
u_\alpha(r) = h_1 \cdots h_{\ell'}
\end{equation}

where $\ell'$ is absolute and $h_i \in H_{g, w}$ with $\|h_i\| \leq 1$ or $h_i \in \{u\alpha(\tau_0) : |\tau_0\alpha^2| \leq |\tau_0|^{-1}\}$. This and (35) complete the proof in this case as well.

Recall that $\text{vol}_{G'}$ denotes our fixed Haar measure on $G'$. Also recall that for every $m$, we let
\begin{equation}
Q'_m = \left\{ ((g^1_w, g^2_w)) \in K' : (g^1_w, g^2_w) \in \pi_w(K'_w[m]) \right\}.
\end{equation}
Lemma 7.3. Assume $n$ is so that
\[
\text{vol}_{G'}(Q'_n) < \frac{\text{vol}(Y)}{4(\#C(F))} \text{vol}(Y),
\]
where $C$ denotes the center of $G'$. There exist points $y_1, y_2$ and $\lambda \in \Phi$ so that
\begin{enumerate}
\item $y_1, y_2$ are $[n/20]$-generic for $\lambda$ and the measure $\mu$ w.r.t. $S$.
\item $y_2 = g y_1$ where $g \in Q_{n-m_0} \setminus \text{Stab}(\mu)$. Moreover, $g_w \neq 1$ and we have
\[
\|g_{w,\lambda} - I\| \geq q_w^{-m_0} \|g_w - I\|,
\]
where $m_0$ is as in Lemma 2.3.
\end{enumerate}

Proof. We may assume $\text{vol}(Y)$ is large enough so that $n$ may be chosen with $[n/20] \geq 2m_0 + 1$. Let
\[
Y'_\text{good}(R) = \mathcal{S}(R) \cap \{y \in Y : y \text{ is } [n/20]\text{-generic}\}.
\]
Then by Propositions 7.1 and 6.2 if $R \gg_{q_w} 1$ is large enough, we have
\[
\mu(Y'_\text{good}(R)) \geq 1 - q_w^{-11m_0 \dim G}.
\]
Fix one such $R$ for the rest of the argument, and put $Y'_\text{good} = Y'_\text{good}(R)$.
Let $H = \Delta_y(K_{g,\lambda}A_{-m_0}K_w)$. Put
\[
Y_H = \{(h, y) \in H \times Y'_\text{good} : hy \in Y'_\text{good}\}.
\]
Note that for every $h \in H$ and all $y \in Y'_\text{good} \cap h^{-1}Y'_\text{good}$, we have $(h, y) \in Y_H$. Since $H \subset H_{g,w}$ leaves $\mu$ invariant, it follows from this observation and (36) that for every $h \in H$, the fiber $\{y : (h, y) \in Y_H\}$ has measure $\geq 1 - 2q_w^{-11m_0 \dim G}$. Therefore, by Fubini’s theorem, there exists a subset $Y'_\text{good} \subset Y'_\text{good}$ with
\[
\mu(Y'_\text{good}) \geq 1 - q_w^{-5m_0 \dim G}
\]
so that if for every $y \in Y'_\text{good}$ we put $H_y \equiv \{h \in H : hy \in Y'_\text{good}\}$, then
\[
|H \setminus H_y| < q_w^{-5m_0 \dim G}.
\]
Let $\{Q'_n,x_i : i \in I\}$ be a a disjoint covering of $\mathcal{S}(R)$, then $\#I \leq \text{vol}_{G'}(Q'_n)^{-1}$, moreover, since $\mu(Y'_\text{good}) \geq 0.9$, there exists some $i_0$ so that
\[
\mu(Q'_n,x_{i_0} \cap Y'_\text{good}) \geq \frac{1}{2 \cdot (\#I)}.
\]

Recall now from Lemma 2.1 that $\text{Stab}(\mu) = H \cdot C(F)$. We now claim that there exist $y'_1, y'_2 \in Q'_n,x_{i_0} \cap Y'_\text{good}$ so that $y'_2 = g'y'_1$ for some $g' \notin \text{Stab}(\mu)$. Assume to the contrary that for every $y, y' \in Q'_n,x_{i_0} \cap Y'_\text{good}$, we have $y' = hy$ where $h \in \text{Stab}(\mu)$. Then, since $b \mapsto bx$ is injective for all $x \in \mathcal{S}(R)$ and all $b \in Q'_n$, see Lemma 6.1, we have
\[
\mu(Q'_n,x_{i_0} \cap Y'_\text{good}) \leq (\#C(F)) \cdot m(Q'_n \cap H).
\]
From this we conclude that
\[
\text{vol}_{G'}(Q'_n) \leq (\#I)^{-1} \leq 2\mu(Q'_n,x_{i_0} \cap Y'_\text{good}) \leq 2 \cdot (\#C(F)) \cdot m(Q'_n \cap H) \leq 2 \cdot (\#C(F)) \cdot \text{vol}(Y)^{-1}
\]
which contradicts our assumption and proves the claim.

We now use Lemma 5.6 to move $g'$ to $g \in Q_{m-2m_0}$ while keeping points generic; moreover, we need $g_w \neq 1$ and that $\|g_{w,\lambda} - I\| \gg \|g_w - I\|$ for some $\lambda \in \Phi$. 

Let $E_1, E_2 \subset K_w A_{-m_0} K_w$ be so that

$$\Delta_g(E_i) = H_{g_i}'$$ and $|K_w A_{-m_0} K_w \setminus E_i| \leq q_w^{-5 m_0 \dim G}.$

For $i = 1, 2$, let $\alpha_i \in H_{g_i}'$ and $\lambda \in \Phi$ satisfy the conclusion of Lemma 5.6 and put $g := \alpha_2 g' \alpha_1^{-1}$. Note that $g \equiv \lambda \bmod{2 m_0}$, since $g_v = g'_v$ for any $v \neq w$, and by Lemma 7.6 we have

$$\alpha_2 g' \alpha_1^{-1} = g_w \in K_w [n - 2 m_0] \subset G_w \times \{1\}$$
satisfies $||g_w, \lambda - I|| \gg ||g_w - I||$. Moreover, since $\alpha_i \in \Stab(\mu)$ and $g' \notin \Stab(\mu)$, we have $g \notin \Stab(\mu)$.

Let $y_i = \alpha_i y'_i$. Since $\alpha_i \in H_{g_i}'$, we have $y_i \in \Stab(\mu)$. Furthermore, since $y'_2 = g' y'_1$,

$$y_2 = \alpha_2 y'_2 = \alpha_2 g' \alpha_1^{-1} \alpha_1 y'_1 = g y_1.$$

It only remains to show that $g_w \neq 1$. Assume to the contrary that $g_w = 1$. Then for every $\ell \geq \lceil n/20 \rceil$ and $i = 1, 2$, we have

$$\left| q_w^\ell \int_{B_w(a_0, q_w^{-\ell})} f(a'_\lambda(t) u'_\lambda(s) y_1) \, ds - \int_X f \, d\mu \right| \leq q_w^{-\ell/8M} S(f)$$

for all $f \in C_c^\infty(X)$. Now since $g_w = 1$ and $a'_\lambda(t), u'_\lambda(t) \in H_{g, w} \subset G'_w$, we have

$$a'_\lambda(t) u'_\lambda(s) y_2 = a'_\lambda(t) u'_\lambda(s) g y_1 = g a'_\lambda(t) u'_\lambda(s) y_1$$

for all $t, s$. This and the above imply that $\mu(g^{-1} \cdot f) = \mu(f)$. That is, $g \in \Stab(\mu)$ which is a contradiction. The proof is complete. \hfill $\square$

**Corollary 7.4.** There exists some $\kappa_{13}$ and $L$ so that

$$|\mu(b, f) - \mu(f)| \ll \text{vol}(Y)^{-\kappa_{13}} S(f),$$

for every $b \in K_w [L]$ and every $f \in C_c^\infty(X)$.

**Proof.** Since $K_w [L] = K_w [L] : (H_g \cap K_w [L])$ for every $L \geq 0$ and $H_g$ leaves $\mu$ invariant, it suffices to find $\kappa_{13}$ and $L$ so that the claim holds for all $b \in K_w [L]$.

Let $n \in \mathbb{N}$ be so that

$$\text{vol}_{G'}(Q'_n)^{-1} \leq \frac{\text{vol}(Y)}{4(\#C(F))} \text{vol}(Y),$$

see Lemma 7.3. Then by that lemma, there exist $y_1, y_2 = g y_1$, and $\lambda \in \Phi$ which are both $\lfloor n/20 \rfloor$-generic for $\lambda$ and the measure $\mu$, so that $g \in Q_{n - 2 m_0}, g_w \neq 1$, and

$$||g_w, \lambda - I|| \geq q_w^{-5 m_0} ||g_w - I||.$$

Let $m = n - 2 m_0$, Assuming $n$ is large enough, we have $\lfloor m/10 \rfloor \geq \lceil n/20 \rceil$. Apply Lemma 7.2 with $c = m_0$ and these $y_1, y_2 = g y_1$, and $m$. We thus conclude that

$$|u_\lambda(\tau), \mu(f) - \mu(f)| \ll q_w^{-\kappa_{12}} S(f)$$

for all $\tau \in B(0, 1)$ and all $f \in C_c^\infty(X)$.

In view of Lemma 5.2 there exist $h_1, \ldots, h_\ell \in H_g$ so that every $b \in K_w \cap G$ may be written as $b = h_1 \cdots b_\ell$, where $b_i \in h_i U_{\lambda i} [0] h_i^{-1}$ for every $1 \leq i \leq \ell$.

Since $h_i \in H_g$ leave the measure $\mu$ invariant, and we may choose $h_i$ so that $||h_i|| \leq 1$, the above and (37) imply the claim with $L = \kappa_{12}$ and $\kappa_{13} = \kappa_{12}$. \hfill $\square$
Proof of Theorem 7.1. The proof goes along the same lines as the proof of [10] Thm. 1.5; we recall the details for the convenience of the reader.

Let $f \in C^\infty(X)$. Let $L$ be as in Corollary 7.3 and let $\mathcal{A}_v$ be the operation of averaging over $K_w[L]$. Let us also fix some $\lambda \in \Phi$ and let $\delta_{u^*_\lambda(s)}$ be the delta mass at $u^*_\lambda(s)$ for some $s \in F_w$. Let $T_s = \mathcal{A}_v \ast \delta_{u^*_\lambda(s)} \ast \mathcal{A}_v$, where $\ast$ denotes the convolution operation. In view of Corollary 7.3 and the fact that $u^*_\lambda(s) \in H_2$ leaves $\mu$ invariant, we conclude that

\begin{equation}
|\mu(f) - \mu(T_s f)| \leq \mathop{\text{vol}}(Y)^{-s}(S(\delta_{u^*_\lambda(s)} \ast \mathcal{A}_v * f) + S(f)).
\end{equation}

Therefore,

\begin{equation}
|\mu(f) - m_X(f)| \leq |\mu(T_s f) - m_X(f)| + |\mathop{\text{vol}}(Y)^{-s}(S(\delta_{u^*_\lambda(s)} \ast \mathcal{A}_v * f) + S(f))|
\end{equation}

\begin{equation}
\leq |\mu(T_s f) - m_X(f)| + |\mathop{\text{vol}}(Y)^{-s}(1 + |s|_w)^{4d} S(f)|
\end{equation}

where we used $S(\delta_{u^*_\lambda(s)} \ast \mathcal{A}_v * f) \leq (1 + |s|_w)^{4d} S(f)$. We now estimate the first term on the right side of the above. First recall from S5. that

\[ |T_s f(x) - \int f \, dm_X| \leq q_w^{(d+1)L} h(x)^d \|T_s\|_{2,0} S(f), \]

where $\|T_s\|_{2,0}$ denotes the operator norm of $T_s$ on $L^2_0(X, m_X)$. Therefore,

\[ |\mu(T_s f) - m_X(f)| = \left| \int T_s f - m_X(f) \, d\mu \right| \leq \left| \int_{\Theta(R)} T_s f - m_X(f) \, d\mu \right| + \left| \int_{X \setminus \Theta(R)} T_s f - m_X(f) \, d\mu \right| \]

\[ \leq q_w^{(d+1)L} R^d \|T_s\|_{2,0} S(f) + q_w^* R^{-s} \|f\|_\infty \]

\[ \leq \left( 1 + |s|_w \right)^{-1/2M} q_w^{(3d+2)L} R^d + q_w^* R^{-s} S(f), \]

in the last inequality we used $\|T_s\|_{2,0} \leq (1 + |s|_w)^{-1/2M} q_w^{2dL}$, see (21).

Optimizing $|s|_w$ and $R$, and using the fact that $q_w \ll (\log \mathop{\text{vol}}(Y))^2$, we get the theorem from the above and (29). \qed

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