Observer-based boundary control of distributed port-Hamiltonian systems

Jesús Toledo*, Yongxin Wu*, Héctor Ramírezb, Yann Le Gorrec*

*FEMTO-ST Institute, Univ. Bourgogne Franche-Comté, ENSMM, CNRS, 24 rue Savary, F-25000 Besançon, France. jesus.toledo@femto-st.fr, yongxin.wu@femto-st.fr, legorrec@femto-st.fr.

bDepartment of Electronic Engineering, Universidad Técnica Federico Santa María, Avenida España 1680, Valparaíso, Chile. hector.ramirez@usm.cl

Abstract

An observer-based boundary controller for infinite-dimensional port-Hamiltonian systems defined on 1D spatial domains is proposed. The design is based on an early-lumping approach in which a finite-dimensional approximation of the infinite-dimensional system derived by spatial discretization is used to design the observer and the controller. As long as the finite-dimensional approximation approaches the infinite-dimensional model, the performances also do. The main contribution is a constructive method which guarantees that the interconnection between the controller and the infinite-dimensional system is asymptotically stable. A Timoshenko beam model has been used to illustrate the approach.

Keywords: infinite-dimensional systems, port-Hamiltonian systems, boundary control systems, Luenberger observer, state feedback.

1. Introduction

Boundary control systems (BCS) (Fattorini 1968) are a class of control systems where the dynamics are described by partial differential equations (PDEs) with actuation and measurement situated at the boundaries of the spatial domain. Motivated by technological advances, these type of systems have been of great interest for engineers and mathematicians during the last decades since a large class of physical processes can be represented as BCS. This is for instance the case of beams and waves in mechanical systems, heat bars and bed reactors in chemical systems or telegraph equations in electronic systems, among others (Curtain and Zwart 2012).

Recently, the control of BCS has been addressed by using the framework of port-Hamiltonian systems (PHS) (van der Schaft and Maschke 2002, Le Gorrec et al. 2005). Boundary controlled PHS (BC-PHS) are an extension of the Hamiltonian formulation of mechanical systems to open multi-physical systems (Duindam et al. 2009). This formalism has been proven to be particularly suitable for the modeling and control of complex physical systems, such as systems described by infinite-dimensional or non-linear models. The stability, stabilization and control synthesis of BC-PHS have been addressed in (Villegas et al., 2009, Ramirez et al., 2014, Augner and Jacoby 2014, Macchelli et al., 2017, Ramirez et al., 2017). More recently, the framework has been extended to deal with robust and adaptive regulation (Macchelli and Califano 2018, Humaloja and Paunonen 2018).

In the case of observer-based control design there are generally two approaches. The first one is the late-lumping approach in which the observer is designed from the infinite-dimensional PHS and actuation and measurement are situated at the boundaries. The second one is the early-lumping approach. In this case, the system is first approximated and a finite-dimensional observer is implemented on the reduced order system. This approach has been proven to be particularly suitable for the modeling and control of complex physical systems, such as systems described by infinite-dimensional or non-linear models. The stabilizability and control synthesis of BC-PHS have been addressed in (Villegas et al., 2009, Ramirez et al., 2014, Augner and Jacoby 2014, Macchelli et al., 2017, Ramirez et al., 2017). More recently, the framework has been extended to deal with robust and adaptive regulation (Macchelli and Califano 2018, Humaloja and Paunonen 2018).

2. Background on port-Hamiltonian systems

2.1. Some notation

In this paper, $\mathbb{M}_n(\mathbb{R})$ denotes the space of real $n \times n$ matrices and $I$ denotes the identity matrix of appropriate dimensions.
By $\langle \cdot , \cdot \rangle_L$, or only $\langle \cdot \rangle$, we denote the standard inner product on $L_2(a, b; \mathbb{R}^p)$ and the Sobolev space of order $p$ is denoted by $H^p(a, b; \mathbb{R}^n)$. A detailed description of the class of boundary control systems under consideration can be found in [Le Gorrec et al., 2005].

Theorem 2. (Le Gorrec et al., 2005) Let $W$ be a $(n \times 2n)$-matrix. If $W$ has full rank and satisfies $W \eta \in \mathbb{R}^n$, then the state variable $z(t)$ is usually proportional to the stored energy of the system, i.e., the controller and observer are designed on a finite-dimensional approximation of $[1]$. The following theorem ensures the existence and uniqueness of solutions of $[1]$.

**Definition 1.** Let $L_z \in H^1(a, b; \mathbb{R}^n)$. Then, the boundary port variables associated with $[1]$ are the vectors $f_0$ and $e_0 \in \mathbb{R}^n$, defined by

$$
\begin{bmatrix} f_0(t) \\ e_0(t) \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} P_1 & -P_1 \end{bmatrix} \begin{bmatrix} L(b)z(b, t) \\ L(a)z(a, t) \end{bmatrix}.
$$

(2)

Note that the port-variables are nothing else than a linear combination of the boundary variables. We also define the matrix $\Sigma \in M_{2n}(\mathbb{R})$ as follows

$$
\Sigma = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}.
$$

(3)

The following theorem ensures the existence and uniqueness of solutions of $[1]$.

**Theorem 2.** (Le Gorrec et al., 2005) Let $W$ be a $(n \times 2n)$-real matrix. If $W$ has full rank and satisfies $W \Sigma W^T \geq 0$, then the system $[1]$ with input

$$
u(t) = W \begin{bmatrix} f_0(t) \\ e_0(t) \end{bmatrix}
$$

(4)

is a BCS on $Z$. Furthermore, the operator $\mathcal{A} = P_1 \frac{d}{dt}(\mathcal{L}z) + (P_0 - G_0)\mathcal{L}z$ with domain

$$
D(\mathcal{A}) = \left\{ L_z \in H^1(a, b; \mathbb{R}^n) \left| \begin{bmatrix} f_0(t) \\ e_0(t) \end{bmatrix} \in \ker W \right. \right\}
$$

generates a contraction semigroup on $Z$.

Let $\tilde{W}$ be a full rank matrix of size $n \times 2n$ with $[w \ w^T]^T$ invertible and $P_{\tilde{W}}$ given by

$$
P_{\tilde{W}} = \begin{bmatrix} W \Sigma W^T & W \Sigma \tilde{W}^T \\ W \Sigma W^T & W \Sigma \tilde{W}^T \end{bmatrix}^{-1}
$$

(5)

Define the output of the system as the linear mapping $C : L^2(a, b; \mathbb{R}^n) \rightarrow \mathbb{R}^n$

$$
y(t) = Cz(t, t) = \tilde{W} \begin{bmatrix} f_0(t) \\ e_0(t) \end{bmatrix}
$$

(6)

Then, for $u \in C^2(0, \infty; \mathbb{R}^n)$, $L_z(\cdot, 0) \in H^1(a, b; \mathbb{R}^n)$ and $u(0) = W \begin{bmatrix} f_0(0) \\ e_0(0) \end{bmatrix}$ the following balance equation is satisfied

$$
\frac{1}{2} \frac{d}{dt} \| z(\cdot, t) \|^2_{L_z} = \frac{1}{2} \left( \langle u(t) \rangle \right)^T P_{\tilde{W}} \begin{bmatrix} f_0(t) \\ e_0(t) \end{bmatrix}
$$

(7)

**Remark 3.** The matrix

$$
\begin{pmatrix} W & \Sigma \end{pmatrix} = \begin{pmatrix} W \Sigma W^T & W \Sigma \tilde{W}^T \\ W \Sigma W^T & W \Sigma \tilde{W}^T \end{pmatrix}
$$

(8)

is invertible if and only if $[w \ w^T]^T$ is invertible.

In this work, we shall consider an early-lumping approach, i.e., the controller and observer are designed on a finite-dimensional approximation of $[1]$. The following assumption is considered

**Assumption 4.** There exists the following finite-dimensional approximation of $[1]$

$$
P \begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \end{bmatrix} = \begin{bmatrix} (J - R)Qx(t) + Bu(t) \\ B^TQx(t) \end{bmatrix}
$$

(9)

where $x \in \mathbb{R}^n$, with $n$ given by the order of the approximation, $J = -J^T$, $R = R^T \geq 0$, $Q = Q^T > 0$ all of them in $M_n(\mathbb{R})$ and $B \in \mathbb{R}^{m \times n}$. Furthermore, we assume $[2]$ to be controllable and observable. For simplicity, we shall define $A = (J - R)Q$ and $C = B^TQ$ and we will refer to the system $(A, B, C)$ as the approximated model of $[1]$.

**Remark 5.** Approximation schemes which preserve the port-Hamiltonian structure of the original system using mixed finite elements or finite differences on staggered grids for instance, can be found in [Seidtja et al., 2012]. The achievable closed-loop performances depend on the quality of the approximated model. The order of the approximation $n$ has then to be chosen large enough such that in the frequency range of interest the approximated system poles behave similar to the original ones.
3. The observer-based controller

The main objective of this work is to design a finite-dimensional controller that achieves some desired performances on the finite-dimensional system [9], while ensuring closed-loop stability when applied to the infinite-dimensional system [1]. The considered controller is an observer-based state feedback

\[ u(t) = r(t) - K \hat{x}(t) \]

where \( \hat{x} \in \mathbb{R}^n \), \( r \in \mathbb{R}^n \) and the Luenberger observer

\[ \dot{\hat{x}}(t) = A \hat{x}(t) + B u(t) + L(y(t) - C \hat{x}(t)) \]

with matrices \( K \in \mathbb{R}^{n \times n} \), \( L \in \mathbb{R}^{n \times n} \) to be designed and \((A, B, C)\) defined in [9]. Note that, \( n_c \) is the size of the observer given by the chosen discretization scheme and \( n \) is the number of boundary variables.

Several issues can arise when using an early-lumping approach to design the control, the most critical one being the loss of stability when the controller is applied on the infinite-dimensional system. It is known as the spillover effect (Bontsema and Curtain [1988]). Consider the following illustrative example.

Example 6. Consider the 1D wave equation with unitary parameters and Neumann boundary control. The system can be written (see Jacob and Zwart [2012] for more details) in the form [1] with

\[
P_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad P_0 = G_0 = 0, \quad L = I_2.
\]

This model is discretized by using finite differences on staggered grids in order to preserve the structure of the system. Consider \( n_c = 59 \) elements for the discretization. \( G_0 = 0 \) implies \( R = 0 \) thus all the eigenvalues of \( A \) are on the imaginary axis as shown in Figure 2(a). \((A, B)\) is controllable and \((A, C)\) observable, hence \( K \) and \( L \) can be designed such that \( A_k = A - BK \) and \( A_L = A - LC \) are Hurwitz. Using for instance the Linear Quadratic Regulator (LQR) method the closed-loop eigenvalues can be assigned as in Figure 2(a).

The question that naturally arises is if the same control law, i.e. the same choice of matrices \( K \) and \( L \), preserves the stability when applied on the infinite-dimensional system. The answer in general is no. In this particular case for instance, when increasing the order of the discretized model to \( n_c = 67 \), the closed-loop system turns unstable as shown in Figure 7(b).

In what follows we start from the achievable closed-loop performances on the finite-dimensional system i.e. an appropriate choice of \( K \), and design the observer gain such that the Luenberger observer [11] is a strictly positive real PHS. Then we show that since [10] corresponds to a power preserving interconnection between the infinite-dimensional system and the dynamic boundary controller, the closed-loop system is asymptotically stable.

3.1. Some technical results

Before presenting the main result we give the following definitions, lemma, corollary and theorem which are instrumental in the proof.

Definition 7. A \( n \times n \) transfer matrix \( G(s) \) is positive real (PR) if \( G(s) + G(s) G^T(s) \geq 0 \) for all \( s \) such that \( \text{Re}(s) > 0 \).

Definition 8. A \( n \times n \) transfer matrix \( G(s) \) is strictly positive real (SPR) if there exists a scalar \( \varepsilon > 0 \) such that \( G(s - \varepsilon) \) is PR.

Lemma 9. (Lefschetz-Kalman-Yakubovich) (Tao and Ioannou [1988]) Assume for the system \((A, B, C, D)\) that \((A, B)\) is controllable and \((A, C)\) is observable. Then, the transfer matrix \( G(s) = C(sI - A)^{-1}B + D \) is SPR if and only if there exist real matrices \( P = P^T > 0 \), \( \Gamma, W_1 \) and a scalar \( \varepsilon > 0 \) such that

\[
PA + A^T P = -\Gamma^T \Gamma - \varepsilon P \tag{12a}
\]

\[
C - B^T P = W_1^T \Gamma \tag{12b}
\]

\[
D + D^T = W_1^T W_1 \tag{12c}
\]

Corollary 10. The system \((A, B, C, D)\) with \( A = (J - R)Q \), \( C = B^T Q \) and \( D = 0 \) is strictly positive real if \( J = -J^T \), \( R = R^T > 0 \) and \( Q = Q^T > 0 \).

Proof. From Lemma 9 choose \( P = Q \) and \( W_1 = 0 \), then (12c) is trivial, (12b) is \( C = B^T Q \) and (12a) becomes

\[
\Gamma^T \Gamma = 2Q R Q - \varepsilon Q \tag{13}
\]

then, for \( R > 0 \) there exists a constant \( \varepsilon > 0 \) such that the right hand side is positive definite, giving a solution for \( \Gamma \), for using instance Cholesky factorization. See Corollary 7.2.9 in (Horn and Johnson [2012]).

The next theorem assures the stability of [11] interconnected in a power preserving way with a SPR controller.
**Theorem 11.** [Villegas, 2007, Ch. 5.1.2] Consider \( u(t) \) defined according to Theorem 2 and \( y(t) \) such that
\[
\frac{1}{2} \frac{d}{dt} \| e(t) \|^2 = u^T(t) y(t),
\]
Equation (14) i.e. \( W \Sigma W^T = \bar{W} \Sigma \bar{W}^T = 0 \) and \( \bar{W} \Sigma W^T = I \). Consider also a finite-dimensional controller with input \( u_i(t) \) and output \( y_i(t) \) such that its transfer matrix is SPR. Then, the closed-loop system with the passive interconnection
\[
\begin{align*}
\dot{x}(t) &= (J_c - R_c)Q_c x(t) + B_i u_i(t) + B r(t) \\
\dot{y}(t) &= B_i Q_c \dot{x},
\end{align*}
\]
Proof. The control scheme of Definition 12 is asymptotically stable and converges to zero when \( r \) and \( y \) are both zero, i.e. \( u_c = y = 0 \), the controller being controllable. The system (9) being observable the only equilibrium point is zero.

**Remark 17.** A simple choice for \( R_c \) is \( R_c = \alpha I \) for some \( \alpha > 0 \) small enough such that the matrix (19) has no pure imaginary eigenvalues.

**Proposition 14.** The interconnection of the system (9) with the observer based controller (10)-(11) is equivalent to the control by interconnection with the SPR-PH controller of Definition 12 if the following matching conditions are satisfied
\[
\begin{align*}
(J_c - R_c)Q_c &= A - BK - LC \quad (18) \\
B_i Q_c &= K \\
B_r &= L.
\end{align*}
\]
Proof. The matching equations (18) are directly obtained replacing (19) in (11) and identifying with (16) in order to get a passive and colocated dynamic controller.

**3.3. Main result**

The following proposition is the main contribution of this work. It is based on two main assumptions.

**Assumption 15.** The matrix \( K \) has been designed such that \( A - BK \) is Hurwitz by using traditional methods such as LQR design, pole-placement or LMI passivity based control, such as for instance in [Prajna et al., 2002].

**Assumption 16.** The matrix \( R_c \) is chosen such that the following matrix
\[
H_M = \begin{pmatrix} A_K & 2R_c \\ -C_K & -A_K^T \end{pmatrix}
\]
with
\[
A_K = A - BK, \quad C_K = -(K^T C + C^T K),
\]
have no pure imaginary eigenvalues.

**Remark 17.** A simple choice for \( R_c \) is if \( R_c = \alpha I \) for some \( \alpha > 0 \) small enough such that the matrix (19) has no pure imaginary eigenvalues.

**Proposition 18.** Under Assumptions 15 and 16 there exists a matrix \( Q_c = Q_c^T > 0 \), solution of the algebraic Riccati equation (ARE)
\[
A_K^T Q_c + Q_c A_K + 2Q_c R_c Q_c + C_K = 0,
\]
such that the matching equations (18) are satisfied with
\[
\begin{align*}
J_c &= \frac{1}{2} \left[ A_K Q_c^{-1} - Q_c^{-1} A_K^T - Q_c^{-1}(K^T C + C^T K) Q_c^{-1} \right] \\
B_c &= Q_c^{-1} K^T \\
L &= B_c.
\end{align*}
\]
Furthermore, the matrix \( A - LC \) is Hurwitz.
Proof. From [Kosmidou 2007] it is known that if the Hamiltonian matrix \( A \) has no pure imaginary eigenvalues then there exists a solution \( Q_c = Q^T_c > 0 \) for (21). Hence we only need to prove that (21) is compatible with the matching equation (18) for \( J_c \) and \( L \) as in (23). Since \( Q_c \) is invertible and solution of (21) we have

\[
R_c = -\frac{1}{2}\left[ Q_c^{-1}A_c^T + A_cQ_c^{-1} + Q_c^{-1}C_cQ_c^{-1} \right]
= -\frac{1}{2}\left[ Q_c^{-1}A_c^T + A_cQ_c^{-1} - Q_c^{-1}(K^TCP + C^TK)Q_c^{-1} \right]
\]

(23)

Then using (22) and (23) we have

\[
(J_c - R_c)Q_c = \frac{1}{2}(2A_cQ_c^{-1} - 2Q_c^{-1}K^TCPQ_c^{-1})Q_c
= A_c - Q_c^{-1}K^TCPQ_c^{-1}
= A_c - LC
= A - BK - LC
\]

(24)

which correspond to (18). From Theorem 13 the closed-loop system

\[
\frac{d}{dt}\left( x \right) = \begin{pmatrix} A & -BK \\ B & C \end{pmatrix}(J_c - R_c)Q_c(x) + \begin{pmatrix} B \\ 0 \end{pmatrix}r
\]

(25)

is asymptotically stable. Applying the following transformation

\[
\begin{pmatrix} \dot{x} \\ \dot{\tilde{x}} \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\begin{pmatrix} x \\ \tilde{x} \end{pmatrix}
\]

the closed-loop system (25) can be written

\[
\frac{d}{dt}\left( \begin{pmatrix} \dot{x} \\ \dot{\tilde{x}} \end{pmatrix} \right) = \begin{pmatrix} A_K & BK \\ BK & -A_c \end{pmatrix}\begin{pmatrix} \dot{x} \\ \dot{\tilde{x}} \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix}r
\]

(26)

with \( A_K = A - BK, B_c = L \) and \( A_c = (J_c - R_c)Q_c = A - BK - LC \) or equivalently

\[
\frac{d}{dt}\left( \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} \right) = \begin{pmatrix} A - BK & BK \\ 0 & A - LC \end{pmatrix}\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix}r
\]

(27)

Since \( A_K \) is Hurwitz, and the closed-loop system asymptotically stable, \( A - LC \) is also Hurwitz. ■

Theorem 19. Let’s consider the infinite-dimensional system (1) with \( u = -K\tilde{x} \) and \( x \) solution of the dynamic equation (16) in Proposition 18. The closed-loop system is asymptotically stable.

Remark 20. One special case of Proposition 18 is proposed in [Wu et al. 2018] in the context of reduced order control of finite-dimensional PHS. There the matrix \( K \) obtained by a LQR method and the matrix \( Q_c = Q(\cdot) \).

4. Example: the Timoshenko beam

We consider the boundary control of a Timoshenko beam clamped at the left side and controlled through force and torque at the right side. Both longitudinal and angular velocities at the right side are used for control purposes. The port Hamiltonian formulation of the Timoshenko beam can be found in [Macchelli and Melchiorri 2004]. It can be written in the form (1) with

\[
P_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad P_0 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},
\]

where \( T(\zeta) \) is the shear modulus, \( \rho(\zeta) \) the mass per length unit, \( EI(\zeta) \) the product of the Young’s modulus of elasticity \( E \) and the moment of inertia of a cross section \( I \), and \( I_p(\zeta) \) the moment of inertia of a cross section.

The state variables are: the shear displacement, the transverse momentum distribution, the angular displacement and the angular momentum distribution defined respectively by \( z_1(\zeta,t) = \frac{d}{dt}(\zeta,t) = \frac{\partial}{\partial t}(\zeta,t), z_2(\zeta,t) = \rho(\zeta)\frac{\partial}{\partial t}(\zeta,t), z_3(\zeta,t) = \frac{\partial}{\partial \zeta}(\zeta,t), z_4(\zeta,t) = I_p(\zeta)\frac{\partial}{\partial \zeta}(\zeta,t) \), where \( w(\zeta,t) \) and \( \phi(\zeta,t) \) are respectively the transverse displacement of the beam and the rotation angle of a neutral fiber of the beam. Note that, \( T(\zeta)z_1(\zeta,t) \) is the shear force, \( \frac{1}{E}z_2(\zeta,t) \) the longitudinal velocity, \( EI(\zeta)z_3(\zeta,t) \) the torque and \( \frac{1}{I_p(\zeta)}z_4(\zeta,t) \) the angular velocity.

We choose as inputs and outputs

\[
u(t) = \begin{pmatrix} \frac{1}{\rho(\zeta)}z_2(a,t) \\ \frac{1}{T(b)}z_1(a,t) \\ T(b)z_1(b,t) \\ EI(b)z_3(b,t) \end{pmatrix}, \quad y(t) = \begin{pmatrix} -T(a)z_1(a,t) \\ -EI(a)z_3(a,t) \\ \frac{1}{I_p(a)}z_4(a,t) \\ \frac{1}{I_p(b)}z_4(b,t) \end{pmatrix}
\]

The total energy of the beam is defined as

\[
H(t) = \frac{1}{2}\|z(\zeta,t)\|_2^2 = \frac{1}{2} \int_a^b \left[ T(\zeta)z_2(\zeta,t) \right] L(z_1(\zeta,t),t) \partial \zeta.
\]

and satisfies

\[
\frac{d}{dt}H(t) = y^T(t)u(t).
\]

4.1. Discretization

The infinite-dimensional system is discretized using finite differences on staggered grids [Trenchant et al. 2018] considering 20 elements per state variable, which leads to \( n_c = 80. \)
The finite-dimensional model is then given by

$$J = \begin{bmatrix} 0 & D & 0 & -F \\ -D^T & 0 & 0 & 0 \\ 0 & 0 & 0 & D \\ F^T & 0 & -D^T & 0 \end{bmatrix}$$

$$R = 0, \quad Q = h \begin{bmatrix} Q_1 & 0 & 0 & 0 \\ 0 & Q_2 & 0 & 0 \\ 0 & 0 & Q_3 & 0 \\ 0 & 0 & 0 & Q_4 \end{bmatrix}$$

where $Q_i$, $i \in \{1, \cdots, 4\}$ are diagonal matrices containing the evaluation of $T(\zeta)$, $\frac{1}{\sin(\zeta)}$, $EI(\zeta)$ and $\frac{1}{\sin(\zeta)}$ respectively, at the specific points chosen for the discretization.

$$D = \frac{1}{h^2} \begin{bmatrix} 1 & 0 & \cdots & 0 \\ -1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}, \quad F = \frac{1}{2h} \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

and

$$B = \begin{bmatrix} b_{11} & b_{12} & 0 & 0 \\ 0 & 0 & b_{23} & 0 \\ 0 & b_{32} & 0 & 0 \\ 0 & 0 & b_{43} & b_{44} \end{bmatrix}$$

with

$$b_{11} = \frac{1}{h} \begin{bmatrix} -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad b_{12} = \frac{1}{2} \begin{bmatrix} -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

$$b_{23} = \frac{1}{h} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}, \quad b_{43} = \frac{1}{2} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

$$b_{32} = b_{11}, \quad b_{44} = b_{23}.$$ 

The state variables are

$$x(t) = \begin{bmatrix} x_1^d(t)^T \\ x_2^d(t)^T \\ x_3^d(t)^T \\ x_4^d(t)^T \end{bmatrix}^T$$

where $x_i^d(t) \in \mathbb{R}^{20}$, $i \in \{1, \cdots, 4\}$ and the $j$-th component of $x_1^d$, $x_2^d$, $x_3^d$ and $x_4^d$ correspond respectively to the approximation of $z_1(j(0.5)h)$, $z_2(jh)$, $z_3(j(0.5)h)$ and $z_4(jh)$, with $h = \frac{k}{200}$,$\quad a = 0.0146$, and $b = a = 0.0010$ with $a = 0$. The beam is clamped at the left side and force and torque actuators at the right side are considered. $Tz_1(b, t)$ and $EIz_3(b, t)$ respectively. Hence $b_{11} = b_{12} = b_{32} = 0$, which give pairs $(A, B)$ controllable and $(A, C)$ observable. In this case $C = B^TQ$.

4.2. The observer-based state feedback design

We use the parameters of Table 1.

![Image](image.png)

Figure 3: (a): $\lambda(A)$: eigenvalues of the discretized model with $n_c = 80$, $[A_{KL}]$: eigenvalues of $A - BK$ and $[A_{KL}]$: eigenvalues of $A - LC$. (b): $[A_{KL}]$: discretized model eigenvalues with $n_c = 200$, $[A_{KL}]$: closed-loop eigenvalues.

| Parameters | Values | Unit |
|------------|--------|------|
| $\bar{T}$ | $3.4531 \times 10^3$ | Pa |
| $\rho$ | 0.0643 | kg.m$^{-1}$ |
| $EI$ | 37.0116 | Pa.m$^4$ |
| $I_p$ | $2.1485 \times 10^{-6}$ | Kg.m$^2$ |
| $[a, b]$ | [0, 0.3] | m |

Table 1: Plant parameters.

Two different state feedbacks $K$ minimizing the cost function

$$J_{LQR} = \int_0^{\infty} \{x^TQ_{LQR}x + u^TR_{LQR}u + 2x^TN_{LQR}u\} dt$$

are designed using the Matlab Control System Toolbox `lqr.m`. In both designs the ARE algorithm proposed in [Lanzon et al. 2008] has been used to solve (21).

For the first design, the matrix $K$ is performed choosing $Q_{LQR} = 0.8I_{n_c}, R_{LQR} = 10I_4$ and $N_{LQR} = 0$, while the matrix $L$ is designed following Proposition 18 with $R_c = 10I_{n_c}$. The eigenvalues of the matrices $A, A - BK$ and $A - LC$ are shown in Figure 3(a). The eigenvalues of the closed-loop system using the same controller on a higher order discretization of the beam, choosing $n_c = 200$, are given in Figure 3(b).

For the second design, the matrix $K$ is performed choosing $Q_{LQR} = 0.8I_{n_c}, R_{LQR} = 1.33I_4$ and $N_{LQR} = 0$, while the matrix $L$ is designed following Proposition 18 with $R_c = 4I_{n_c}$. The eigenvalues of the matrices $A, A - BK$ and $A - LC$ are shown in Figure 3(a). The eigenvalues of the closed-loop system using the same controller on a higher order discretization of the beam,
choosing \( n_c = 200 \), are given in Figure 4 (b). In both cases, the closed-loop system remains stable and the high frequency modes are not destabilized. Even if \( n_c \to \infty \), the closed-loop eigenvalues do not cross the imaginary axis.

### 4.3. Simulations

Simulations are performed using Matlab over the time interval \( t \in [0, 0.4] \) s. The simulation starts from the initial condition \( z_1(\zeta, 0) = 0.2896 \times 10^{-4}, z_2(\zeta, 0) = 0, z_3(\zeta, 0) = -0.2702\zeta + 0.0811 \) and \( z_4(\zeta, 0) = 0 \) corresponding to the equilibrium position associated to a force of 10N applied at the end tip of the beam. The initial condition for the observer is set to zero, i.e. \( \dot{\hat{x}}(0) = 0 \). An external force \( r(t) = 100N \) is applied at \( t = 0.2s \) at the end of the tip to modify the equilibrium position. In the first simulation the observer-based controller of size 80 and designed on the discretized model for 20 elements is applied to the large scale system obtained considering 50 discretization elements, i.e. 200 state variables. Figure 5 shows the time responses for the two controllers proposed in subsection 4.2, where \( w_1(b, t) \) and \( w_2(b, t) \) are the end tip displacements of the beam for the first and second design respectively. Note that, before \( t = 0.2s \) the convergence to the null equilibrium is due to the observer and state feedback dynamics. After the step at \( t = 0.2s \), the convergence is mostly due to the state feedback as the observer already converged to the system state over the considered range of frequencies.

The estimated values are given by \( \hat{w}_1(b, t) \) and \( \hat{w}_2(b, t) \), the error is obtained by \( \tilde{w}_i(b, t) = w_i(b, t) - \hat{w}_i(b, t) \) and it is shown in Figure 6 for both designs.

In Figure 7 is given the evolution of the norm of the error between the system state, that has been used for control design, and the observer state, with respect to time when considering different initial conditions. Figure 7 illustrates the convergence rate of the observer on the reduced order system.
Since the controller is designed based on a finite-dimensional approximation $P$ of the system, but at the end, has to be implemented on the infinite-dimensional system $P$, it is interesting to compare the behavior of both closed-loop systems. For that purpose, we denote by $w^P(b,t)$ the end tip displacement when applying the controller to the finite-dimensional model $P$ used for the design (20 elements) and by $w^P(b,t)$ the end tip displacement when applying the controller to a higher dimensional model stemming from a fine approximation of the infinite-dimensional model $P$ (obtained for 50 elements). The approximation error $e_i(b,t) = w^P_i(b,t) - w^P_i(b,t)$ is shown in Figure 8 for both designs. We can notice that the approximation error increases for high frequencies signals, meaning performances cannot be guaranteed over all frequencies. Yet the error remains small with respect to the higher order approximation.

Finally, a new simulation is done using the second design for $t \in [0,0.2]$ with the same initial conditions than before and an external force step applied at $t = 0.1s$. The deformation of the beam along the space and over time is shown in Figure 9. The oscillations occurring during the first 0.1s are due to the observer convergence since the system and the observer do not have the same initial conditions.

Figure 10 shows the evolution of the system and observer energies with respect to time when considering different initial conditions. We can see that the observer energy function converges to the plant energy function, and at the same time the control brings the closed-loop energy function to zero.

5. Conclusion

An observer based boundary controller has been proposed for a class of boundary controlled PHS defined on a 1D spatial domain. The design is based on an early-lumping approach in which a finite-dimensional PHS approximation of the infinite-dimensional system is used to design the observer and the controller. The main contribution is a constructive method that guarantees that the finite-dimensional dynamic boundary controller is a strictly positive real PHS. This guarantees that the interconnection between the controller and the infinite-dimensional system is asymptotically stable. As soon as the finite-dimensional approximation of the system that is used for the observer design is close to the infinite-dimensional system over the considered range of frequencies, the closed-loop performances on the infinite-dimensional system are close to the ones obtained on the finite-dimensional approximation. The stabilization of a Timoshenko beam with force and torque actuators and collocated measurements (velocities) has been used to illustrate the approach.

Acknowledgements

This work has been supported by the French-German ANR-DFG INFIDHEM project ANR-16-CE92-0028 and the the EIPHI Graduate School (contract ANR-17-EURE-0002). The second author has received funding from Bourgogne-Franche-Comté Region ANER 2018Y-06145. The third author acknowledges Chilean FONDECYT 1191544 and CONICYT BASAL FB0008 projects. The fourth author has received funding from the European Unions Horizon 2020 research and innovation programme under the Marie Sklodowska-Curie grant agreement No 765579.
References

Augner, B., Jacob, B., 2014. Stability and stabilization of infinite-dimensional linear port-hamiltonian systems. Evolution Equations and Control Theory 3, 207–229.

Bontsema, J., Curtain, R.F., 1988. A note on spillover and robustness for flexible systems. IEEE Transactions on Automatic Control 33, 567–569.

Curtain, R.F., Zwart, H., 2012. An introduction to infinite-dimensional linear systems theory. volume 21. Springer Science & Business Media.

Duindam, V., Macchelli, A., Stramigioli, S., Bruynincks, H., 2009. Modeling and control of complex physical systems: the port-Hamiltonian approach. Springer Science & Business Media.

Fattorini, H., 1968. Boundary control systems. SIAM Journal on Control 6, 349–385.

Guo, B.Z., Xu, C.Z., 2007. The stabilization of a one-dimensional wave equation by boundary feedback with noncollocated observation. IEEE Transactions on Automatic Control 52, 371–377.

Horn, R.A., Johnson, C.R., 2012. Matrix analysis. Cambridge university press.

Humaloja, J., Paunonen, L., 2018. Robust regulation of infinite-dimensional port-Hamiltonian systems. IEEE Transactions on Automatic Control 63, 1480–1486. doi:10.1109/TAC.2017.2748055

Jacob, B., Zwart, H.J., 2012. Linear port-Hamiltonian systems on infinite-dimensional spaces. volume 223. Springer Science & Business Media.

Kosmidou, O.I., 2007. Generalized Riccati equations associated with guaranteed cost control: An overview of solutions and features. Applied Mathematics and Computation 191, 511–520.

Lanzon, A., Feng, Y., Anderson, B.D., Rotkowitz, M., 2008. Computing the positive stabilizing solution to algebraic Riccati equations with an indefinite quadratic term via a recursive method. IEEE Transactions on Automatic Control 53, 2280–2291.

Le Gorrec, Y., Zwart, H., Maschke, B., 2005. Dirac structures and boundary control systems associated with skew-symmetric differential operators. SIAM journal on control and optimization 44, 1864–1892.

Macchelli, A., Califano, F., 2018. Dissipativity-based boundary control of linear distributed port-Hamiltonian systems. Automatica 95, 54 – 62. doi:10.1016/j.automatica.2018.05.029

Macchelli, A., Le Gorrec, Y., Ramirez, H., Zwart, H., 2017. On the synthesis of boundary control laws for distributed port-Hamiltonian systems. IEEE Transactions on Automatic Control 62, 1700–1713. doi:10.1109/TAC.2016.2595263

Macchelli, A., Melchiorri, C., 2004. Modeling and control of the Timoshenko beam. The distributed port Hamiltonian approach. SIAM Journal on Control and Optimization 43, 743–767.

Meurer, T., 2013. On the extended Luenberger-type observer for semilinear distributed-parameter systems. IEEE Transactions on Automatic Control 58, 1732–1743.

Prajna, S., van der Schaft, A., Meinsma, G., 2002. An LMI approach to stabilization of linear port-controlled Hamiltonian systems. Systems & control letters 45, 371–385.

Ramirez, H., Le Gorrec, Y., Macchelli, A., Zwart, H., 2014. Exponential stabilization of boundary controlled port-Hamiltonian systems with dynamic feedback. IEEE Transactions on Automatic Control 59, 2849–2855. doi:10.1109/TAC.2014.2315754

Ramirez, H., Zwart, H., Le Gorrec, Y., 2017. Stabilization of infinite dimensional port-Hamiltonian systems by nonlinear dynamic boundary control. Automatica 85, 61 – 69. doi:https://doi.org/10.1016/j.automatica.2017.07.048

van der Schaft, A., Maschke, B.M., 2002. Hamiltonian formulation of distributed-parameter systems with boundary energy flow. Journal of Geometry and Physics 42, 166–194.

Seslija, M., van der Schaft, A., Scherpen, J.M., 2012. Discrete exterior geometry approach to structure-preserving discretization of distributed-parameter port-hamiltonian systems. Journal of Geometry and Physics 62, 1509–1531.

Tao, G., Ioannou, P., 1988. Strictly positive real matrices and the Lefschetz-Kalman-Yakubovich lemma. IEEE Transactions on Automatic Control 33, 1183–1185.

Trenchant, V., Ramirez, H., Kotyczka, P., Le Gorrec, Y., 2018. Finite differences on staggered grids preserving the port-Hamiltonian structure with application to an acoustic duct. Journal of Computational Physics 373, 673–697.

Villegas, J.A., Zwart, H., Le Gorrec, Y., Maschke, B., 2009. Exponential stability of a class of boundary control systems. IEEE Transactions on Automatic Control 54, 142–147.

Wu, Y., Hamroun, B., Le Gorrec, Y., Maschke, B., 2018. Reduced order LQG control design for port Hamiltonian systems. Automatica 95, 86–92.