ON A POSSIBLE SCHEME OF $q$-DEFORMATION OF
MANY-BOSON SYSTEMS IN TIME-DEPENDENT
VARIATIONAL METHOD

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Abstract

A possible scheme of $q$-deformation, which was recently developed by the
present authors, is reviewed by stressing the starting idea.
I. INTRODUCTION

Time-dependent variational method may be one of the powerful methods for investigating
time-evolution of quantum mechanical systems in the framework of a possible approximation.
In this method, a trial state for the variation is prepared as a function of variational
parameters. If a certain condition is introduced, the time-dependent variational method can
be formulated in the framework of classical Hamiltonian mechanics \cite{1}.

For many-boson systems consisting of a kind of boson operator, the simplest trial state
may be boson coherent state. Ref. \cite{1} tells us that the boson coherent state is an eigenstate
of boson annihilation operator with the eigenvalue which is regarded as a canonical variable
in classical mechanics. Further, we know that the boson coherent state consists of the states
with the boson numbers from 0 to \( \infty \) in a fixed superposition. In order to give variety
to the trial state, the boson coherent state must be generalized. This generalized boson
coherent state consists of the states of the boson numbers from 0 to \( \infty \) with a superposition
different from the boson coherent state. As for the idea for the generalization, there exists
boson annihilation like operator, the eigenstate of which is just the generalized boson coherent
state and the eigenvalue is the variational parameter. The above idea can be realized
through introducing a function of the boson number operator. This boson annihilation like
operator and its hermit conjugate form \( q \)-deformation of the boson system. By adopting
various function of the boson number operator, we are able to obtain various types of the
\( q \)-deformation. The concept of the \( q \)-deformation serves us an interesting viewpoint for un-
derstanding dynamics of many-body systems described by boson operators. In this sense,
it may be interesting to investigate the \( q \)-deformation of many-boson systems in relation to
the generalized boson coherent states as the trial states in the time-dependent variational
method.

The aim of this report is to present the starting idea for the \( q \)-deformation based on
the above-mentioned scheme which was recently developed by the present authors \cite{2}. Of
course, the background of the present report is colored with the papers mainly published
by the Coimbra group \cite{3}. Of course, the paper by Penson and Solomon \cite{4} also gave us
independent influence on the present work.

In §§2 and 3, the time-dependent variational method is recapitulated. Section 4 is a
central part, in which three forms of generalized boson coherent states and \( q \)-deformation
are discussed. In §5, the Holstein-Primakoff boson representation for the \( su(2) \)- and \( su(1, 1) \)-
algebras is shown as a result of the \( q \)-deformation. In §6, as a simple example, damped and
amplified oscillation is discussed in the language of the \( q \)-deformation. Finally, in §7, two
remarks are mentioned.

II. TIME-DEPENDENT VARIATIONAL METHOD FORMULATED IN TERMS
OF CLASSICAL CANONICAL VARIABLES

First, let us recapitulate the time-dependent variational method formulated in Ref. \cite{1}.
For simplicity, a trial state for the time-dependent variation is parametrized in terms of two
real parameters \((p, q)\). The trial state is denoted by \(|p, q\rangle\). Let \(|p, q\rangle\) obey the following
condition :
\[ \langle p, q | i \partial_q | p, q \rangle = p + \partial_q S(p, q) , \quad \langle p, q | i \partial_p | p, q \rangle = \partial_p S(p, q) . \] (2.1)

Next, the following quantity is introduced:

\[ L = \langle p, q | i \partial_t - \hat{H} | p, q \rangle . \] (2.2)

Here, \( \hat{H} \) denotes the Hamiltonian under investigation. With the use of the condition (2.1), \( L \) can be rewritten as

\[ L = p\dot{q} - H(p, q) + \dot{S}(p, q) , \]
\[ H(p, q) = \langle p, q | \hat{H} | p, q \rangle . \] (2.3)

The time-dependent variational method is formulated in the following form:

\[ \delta \int L dt = 0 . \] (2.5)

The relation (2.5), together with the form (2.3), gives us the following equation:

\[ \dot{q} = \partial_p H , \quad \dot{p} = -\partial_q H . \] (2.6)

The relation (2.6) is nothing but the Hamilton’s equation of motion. Of course, \( L \) and \( H \) denote the Lagrangian and the Hamiltonian, respectively, and \( (p, q) \) can be regarded as the canonical variable in classical mechanics. By solving Eq.(2.4) under an appropriate initial condition, the time-dependence of \( (p, q) \) is determined. Then, \( |p, q\rangle \) is obtained as a function of \( t \) and the time-evolution of the system under investigation is described in the framework of the time-dependent variational method.

III. BOSON COHERENT STATE AS A TRIAL STATE FOR THE VARIATION

Instead of the variable \( (p, q) \), the following ones denoted as \( (z, z^*) \) is useful in some cases:

\[ p = i(z^* - z)/\sqrt{2} , \quad q = (z^* + z)/\sqrt{2} . \] (3.1)

Then, for the case \( S = pq/2 \), the condition (2.1) can be rewritten as

\[ \langle p, q | \partial_z | p, q \rangle = +z^*/2 , \quad \langle p, q | \partial_{z^*} | p, q \rangle = -z/2 , \]
\[ S = i(z^{*2} - z^2)/4 . \] (3.3)

The Lagrangian \( L \) can be expressed in the form

\[ L = i(z^* \dot{z} - \dot{z}^* z) - H(z, z^*) . \] (3.4)

The time-dependent variation gives us

\[ iz \dot{z} = +\partial_{z^*} H , \quad iz^* = -\partial_z H . \] (3.5)

As a possible trial state, let us adopt the boson coherent state in the form
\[ |p, q\rangle = |c\rangle = \left(\sqrt{\Gamma}\right)^{-1} \exp(\gamma \hat{c}^* ) |0\rangle . \] (3.6)

Here, \( \Gamma \) denotes the normalization constant:
\[ \Gamma = \exp(|\gamma|^2) . \] (3.7)

The state \( |c\rangle \) satisfies
\[ \hat{\gamma} = \hat{c} , \quad \hat{\gamma}|c\rangle = \gamma |c\rangle . \] (3.8)

The condition (3.2) can be rewritten as
\[ \langle c|\partial_z|c\rangle = (\gamma^* \cdot \partial_z \gamma - \gamma \cdot \partial_z \gamma^*)/2 = +z^*/2 , \]
\[ \langle c|\partial_z^*|c\rangle = (\gamma^* \cdot \partial_z^* \gamma - \gamma \cdot \partial_z^* \gamma^*)/2 = -z/2 . \] (3.9)

The condition (3.4) gives us
\[ \gamma = z , \quad \gamma^* = z^* . \] (3.10)

The parameter \( (\gamma, \gamma^*) \) can be regarded as boson type canonical variable.

**IV. THREE FORMS OF GENERALIZED BOSON COHERENT STATES AND \( Q \)-DEFORMATION**

In order to generalize the trial states from the boson coherent state, let us introduce the following three states:
\[ |c_0\rangle = \left(\sqrt{\Gamma_0}\right)^{-1} \exp(\gamma_0 \hat{c}^* f_0(\hat{N})) |0\rangle , \] (4.1)
\[ |c_P\rangle = \left(\sqrt{\Gamma_P}\right)^{-1} \hat{P}_{n^0} \cdot \exp(\gamma_P \hat{c}^* f_P(\hat{N})) |0\rangle , \] (4.2)
\[ |c_Q\rangle = \left(\sqrt{\Gamma_Q}\right)^{-1} \hat{Q}_{n^0} \cdot \exp(\gamma_Q \hat{c}^* f_Q(\hat{N})) |0\rangle , \] (4.3)
\[ \hat{N} = \hat{c}^* \hat{c} , \] (4.4)
\[ \hat{P}_{n^0} = \sum_{n=0}^{n_0^0} |n\rangle\langle n| , \quad \hat{Q}_{n^0} = 1 - \hat{P}_{n^0} , \] (4.5)
\[ |n\rangle = \left(\sqrt{n!}\right)^{-1} (\hat{c}^*)^n |0\rangle . \] (4.6)

The function \( f_R(\hat{N}) \) \( (R = 0, P, Q) \) is defined by the relation
\[ f_R(\hat{N}) |n\rangle = f_R(n) |n\rangle . \] (4.7)

For the function \( f_R(n) \), the following conditions are imposed:
\[ f_0(0) = 1 , \quad f_0(n) > 0 \quad (n = 1, 2, 3, \cdots) , \] (4.8)
\[ f_P(0) = 1 , \quad f_P(n) > 0 \quad (n = 1, 2, \cdots, n^0 - 1) , \]
\[ f_P(n) ; \text{arbitrary} \quad (n = n^0, n^0 + 1, \cdots) , \] (4.9)
\[ f_Q(n) = 1 \quad (n = 0, 1, \cdots, n^0) , \]
\[ f_Q(n) > 0 \quad (n = n^0 + 1, n^0 + 2, \cdots) . \] (4.10)
Next, the following operators are introduced:

\[ \hat{\gamma}_0 = f_0(\hat{N})^{-1} \hat{c}, \quad \hat{\gamma}_P = \hat{P}_\alpha f_P(\hat{N})^{-1} \hat{c}, \quad \hat{\gamma}_Q = \hat{Q}_\alpha f_Q(\hat{N})^{-1} \hat{c}. \]  

They obey the relations

\[ \hat{\gamma}_0 |c_0\rangle = \gamma_0 |c_0\rangle, \quad \hat{\gamma}_P |c_P\rangle = \gamma_P \hat{P}_\alpha |c_P\rangle, \quad \hat{\gamma}_Q |c_Q\rangle = \gamma_Q |c_Q\rangle. \]

The forms (4.12) permit us to call the states \(|c_0\rangle, |c_P\rangle\) and \(|c_Q\rangle\) the generalized coherent states. The operator \((\hat{\gamma}_0, \hat{\gamma}_0^*)\) can be regarded as \(q\)-deformed boson operator which is characterized by the function \(f_0(\hat{N})\). It is justified through the following relations:

\[ |n\rangle = \left(\sqrt{|n|_q}\right)^{-1} (\hat{\gamma}_0^*)^n |0\rangle. \quad (n = 0, 1, 2, \cdots) \]

\[ \hat{\gamma}_0 |0\rangle = 0, \quad (4.13) \]

\[ \hat{\gamma}_0^* \hat{\gamma}_0 = [\hat{N}]_q, \quad \hat{\gamma}_0 \hat{\gamma}_0^* = [\hat{N} + 1]_q, \quad (4.14) \]

\[ \hat{\gamma}_0 |n\rangle = \sqrt{|n|_q}|n-1\rangle, \quad \hat{\gamma}_0^* |n\rangle = \sqrt{|n+1|_q}|n+1\rangle, \quad (4.15) \]

\[ [\hat{N}, \hat{\gamma}_0] = -\hat{\gamma}_0, \quad [\hat{N}, \hat{\gamma}_0^*] = +\hat{\gamma}_0^*, \quad \hat{N}|n\rangle = n|n\rangle. \quad (4.16) \]

The quantities \([n]_q\) and \([x]_q\) are defined as

\[ [n]_q! = n!(f_0(0) \cdots f_0(n-1))^{-2}, \quad [0]_q! = 1, \quad (4.17) \]

\[ [x]_q = x f_0(x - 1)^{-2}, \quad (x = \hat{N}, \hat{N} + 1, n, n + 1), \quad [0]_q = 0. \quad (4.18) \]

From the relation (4.14), the following relation is derived:

\[ \hat{\gamma}_0 \hat{\gamma}_0^* - (f_0(\hat{N} - 1)^2 f_0(\hat{N}^{-2}) + F(\hat{N}))(\hat{\gamma}_0^* \hat{\gamma}_0) = f_0(\hat{N})^{-2} - F(\hat{N})\hat{N} f_0(\hat{N} - 1)^{-2}, \quad F(\hat{N}) : \text{arbitrary}. \quad (4.19) \]

Especially, the commutation relation for \(\hat{\gamma}_0\) and \(\hat{\gamma}_0^*\) is given in the form

\[ [\hat{\gamma}_0, \hat{\gamma}_0^*] = [\hat{N} + 1]_q - [\hat{N}]_q = \Delta_N(\hat{\gamma}_0^* \hat{\gamma}_0). \quad (4.20) \]

Here, \(\Delta_N(\hat{\gamma}_0^* \hat{\gamma}_0)\) denotes the difference with \(\Delta N = 1\).

We show three concrete examples:

1. The most popular form:

\[ f_0(n) = \sqrt{(n+1)(q - q^{-1})/(q^{n+1} - q^{-(n+1)})}, \quad (4.21) \]

\[ \hat{\gamma}_0 \hat{\gamma}_0^* - q^{-1} \hat{\gamma}_0^* \hat{\gamma}_0 = q^{\hat{N}}, \quad F(\hat{N}) = q^{-1} - f_0(\hat{N} - 1)^2 f_0(\hat{N})^{-2}. \quad (4.22) \]

2. The form presented by Penson and Solomon [4]:

\[ f_0(n) = q^{n/2}, \quad (4.23) \]

\[ \hat{\gamma}_0 \hat{\gamma}_0^* - q^{-1} \hat{\gamma}_0^* \hat{\gamma}_0 = q^{-\hat{N}}, \quad F(\hat{N}) = 0. \quad (4.24) \]
(3) A possible modification given by the present authors:

\[ f_0(n) = \sqrt{(n+1)(1-q^{2(n+1)})}, \]  

\[ \hat{\gamma}_0\hat{\gamma}_0^* - q^{-2}\hat{\gamma}_0^2 = 1, \]

\[ F(N) = q^{-1} - f_0(N-1)^2f_0(N)^{-2}. \]  

We can formulate classical counterpart of the \( q \)-deformation. For the state \( |c_0\rangle \), the following relation can be derived:

\[ \langle c_0|\partial_z|c_0\rangle = (\gamma_0^*\partial_z\gamma_0 - \gamma_0\partial_z\gamma_0^*)/2 \cdot (\Gamma_0'/\Gamma_0) = +z^*/2, \]

\[ \langle c_0|\partial_{z^*}|c_0\rangle = (\gamma_0^*\partial_{z^*}\gamma_0 - \gamma_0\partial_{z^*}\gamma_0^*)/2 \cdot (\Gamma_0'/\Gamma_0) = -z/2, \]

\[ \Gamma_0' = d\Gamma_0/d|\gamma_0|^2. \]  

The condition \( (4.27) \) gives us

\[ z = \gamma_0\sqrt{\Gamma_0'/\Gamma_0}, \quad z^* = \gamma_0^*\sqrt{\Gamma_0'/\Gamma_0}. \]  

Since \( \Gamma_0 \) is a function of \( |\gamma_0|^2 \), \( (\gamma_0,\gamma_0^*) \) can be expressed in terms of \( (z,z^*) \). Then, the Hamilton's equation of motion is derived:

\[ i\hat{z} = \partial_{z^*}H, \quad i\hat{z}^* = -\partial_zH, \quad H = \langle c_0|\hat{H}|c_0\rangle. \]

However, actually, in many cases, it is impossible to express \( (\gamma_0,\gamma_0^*) \) in terms of \( (z,z^*) \).

The state \( |c_0\rangle \) satisfies the relations

\[ \langle c_0|\gamma_0|c_0\rangle = \gamma_0, \quad \langle c_0|\hat{\gamma}_0|c_0\rangle = \gamma_0^*, \]

\[ \langle c_0|\hat{\gamma}_0^*\gamma_0|c_0\rangle = \gamma_0^*\gamma_0. \]  

Further, \( \langle c_0|\hat{N}|c_0\rangle \) is given as

\[ \langle c_0|\hat{N}|c_0\rangle = \gamma_0^*\gamma_0 \cdot \Gamma_0'/\Gamma_0 = z^*z (= N). \]  

With the use of the relation \( (4.30) \), the following relations are derived:

\[ [N,\gamma_0]_P = -\gamma_0, \quad [N,\gamma_0^*]_P = \gamma_0^*, \]

\[ [\gamma_0,\gamma_0^*]_P = d_N(\gamma_0^*\gamma_0). \]  

Here, \( d_N(\gamma_0^*\gamma_0) \) denotes the differential with respect to \( N \) and \( [A,B]_P \) expresses the Poisson bracket:

\[ [A,B]_P = \partial_zA \cdot \partial_{z^*}B - \partial_{z^*}A \cdot \partial_zB. \]

Thus, the following correspondence is obtained:

\[ \hat{\gamma}_0 \sim \gamma_0, \quad \hat{\gamma}_0^* \sim \gamma_0^*, \quad \hat{N} \sim N. \]  

The Hamilton's equation is written as

\[ i\hat{\gamma}_0 = +\partial_{\gamma_0}H \cdot (N')^{-1}, \quad i\hat{\gamma}_0^* = -\partial_{\gamma_0}H \cdot (N')^{-1}, \]

\[ N' = dN/d|\gamma_0|^2. \]
V. THE HOLSTEIN-PRIMAKOFF BOSON REPRESENTATION FOR THE $SU(2)$- AND $SU(1,1)$-ALGEBRAS AS $Q$-DEFORMATION

Under appropriate choices of $f_R(n)$ ($R = 0, P, Q$), the $q$-deformation leads us to the Holstein-Primakoff boson representation for the $su(2)$- and $su(1,1)$-algebras. We show three cases.

(1) $f_0(n) = \left(\sqrt{1 + n/n^0}\right)^{-1} : (n^0 : \text{positive})$

\[ \gamma_0 = (\sqrt{n^0})^{-1} \cdot \hat{T}_- , \quad \hat{T}_- = \sqrt{n^0 + \hat{N} \hat{c}} , \]
\[ \gamma^*_0 = (\sqrt{n^0})^{-1} \cdot \hat{T}_+ , \quad \hat{T}_+ = \hat{c}^* \sqrt{n^0 + \hat{N}} , \]
\[ [\gamma_0, \gamma^*_0] = 2(n^0)^{-1} \cdot \hat{T}_0 , \quad \hat{T}_0 = \hat{N} + n^0/2 . \] (5.1)

The form of $\hat{T}_{\pm,0}$ is identical to the Holstein-Primakoff boson representation of the $su(1,1)$-algebra.

(2) $f_P(n) = \left(\sqrt{1 - n/n^0}\right)^{-1} : (n^0 : \text{positive})$

\[ \gamma_P = (\sqrt{n^0})^{-1} \cdot \hat{P}_n^0 \cdot \hat{S}_- \cdot \hat{P}_n^0 , \quad \hat{S}_- = \sqrt{n^0 - \hat{N} \hat{c}} , \]
\[ \gamma^*_P = (\sqrt{n^0})^{-1} \cdot \hat{P}_n^0 \cdot \hat{S}_+ \cdot \hat{P}_n^0 , \quad \hat{S}_+ = \hat{c}^* \sqrt{n^0 - \hat{N}} , \]
\[ [\gamma^*_P, \gamma_P] = 2(n^0)^{-1} \cdot \hat{P}_n^0 \cdot \hat{S}_0 \cdot \hat{P}_n^0 , \quad \hat{S}_0 = \hat{N} - n^0/2 . \] (5.2)

The form of $\hat{S}_{\pm,0}$ is identical to the Holstein-Primakoff boson representation of the $su(2)$-algebra.

(3) $f_Q(n) = \left(\sqrt{n/n^0 - 1}\right)^{-1} : (n^0 : \text{positive})$

\[ \gamma_Q = (\sqrt{n^0})^{-1} \cdot \hat{Q}_n^0 \cdot \hat{T}_- \cdot \hat{Q}_n^0 , \quad \hat{T}_- = \sqrt{\hat{N} - n^0 \hat{c}} , \]
\[ \gamma^*_Q = (\sqrt{n^0})^{-1} \cdot \hat{Q}_n^0 \cdot \hat{T}_+ \cdot \hat{Q}_n^0 , \quad \hat{T}_+ = \hat{c}^* \sqrt{\hat{N} - n^0} , \]
\[ [\gamma_Q, \gamma^*_Q] = 2(n^0)^{-1} \cdot \hat{Q}_n^0 \cdot \hat{T}_0 \cdot \hat{Q}_n^0 , \quad \hat{T}_0 = \hat{N} - n^0/2 . \] (5.3)

The form of $\hat{T}_{\pm,0}$ is identical to the second Holstein-Primakoff boson representation of the $su(1,1)$-algebra.

VI. SIMPLE EXAMPLE — DAMPED AND AMPLIFIED OSCILLATION —

(1) The $su(1,1)$-algebraic model [3]:

The Hamiltonian for this model consists of

\[ \hat{H} = \hat{K}_b - \hat{K}_a + \hat{V}_{ab} , \]
\[ \hat{K}_b = e \cdot (\hat{b}^* \hat{b}) + f \cdot (\hat{b}^* \hat{b})^2 , \]
\[ \hat{K}_a = e \cdot (\hat{a}^* \hat{a}) + f \cdot (\hat{a}^* \hat{a})^2 , \]
\[ \hat{V}_{ab} = -ig \cdot (\hat{b}^* \hat{a}^* - \hat{a} \hat{b}) . \] (6.1)
Here, \((\hat{b}, \hat{b}^*)\) and \((\hat{a}, \hat{a}^*)\) denote two kinds of bosons and \((e, f, g)\) are constants characterizing the Hamiltonian. The Hamiltonian (6-1) can be expressed in the form

\[
\hat{H}^{(0)}_{su(1,1)} = 2(e - f) \cdot (\hat{T} - 1/2) + 4f \cdot (\hat{T} - 1/2) \cdot \hat{T}_0 - ig \cdot (\hat{T}_+ - \hat{T}_-) .
\]

(6-2)

The set \((\hat{T}_\pm, 0)\) obeys the \(su(1, 1)\)-algebra and commutes with \(\hat{T}\):

\[
\hat{T}_- = \hat{a} \hat{b} , \quad \hat{T}_+ = \hat{b}^* \hat{a}^* , \quad \hat{T}_0 = (\hat{a}^* \hat{a} + \hat{b} \hat{b}^*)/2 ,
\]

\[
\hat{T} = (\hat{b} \hat{b}^* - \hat{a}^* \hat{a})/2 .
\]

(6-3)

(6-4)

Since \(\hat{T}\) commutes with \((\hat{T}_\pm, 0)\), \(\hat{T}\) gives us a constant of motion. Then, in the space where the eigenvalue of \(\hat{T}\) is equal to \(T\), the Hamiltonian (6-2) can be written as

\[
\hat{H}_{su(1,1)} = 2(e - f) \cdot (T - 1/2) + 4f \cdot (T - 1/2) \cdot \hat{T}_0 - ig \cdot (\hat{T}_+ - \hat{T}_-) .
\]

(6-5)

The Hamiltonian (6-3) enables us to describe the damped and the amplified oscillation. The boson \((\hat{a}, \hat{a}^*)\) plays a role of phase space doubling in the sense of the thermo field dynamics formalism presented by Umezawa et al. [6] and it does not mean any physical object. Therefore, the Hamiltonian (6-3) does not mean the total energy of the system and, then, it may be very difficult to draw a picture of motion induced by the Hamiltonian (6-3). Of course, the results obtained in this model are quite interesting.

(2) The \(q\)-deformed model:

This model starts in the following Hamiltonian:

\[
\begin{align*}
\hat{H} &= \hat{K}_d + \hat{K}_e + \hat{V}_{cd} , \\
\hat{K}_d &= \omega \cdot (\hat{d}^* \hat{d}) , \\
\hat{K}_e &= \epsilon \cdot (\hat{c}^* \hat{c}) = \epsilon \cdot \hat{c}^* \cdot [2/(1 + [\hat{c}, \hat{c}^*])] \cdot \hat{c} , \\
\hat{V}_{cd} &= -i \eta \cdot (\hat{c}^* \hat{d} - \hat{d}^* \hat{c}) .
\end{align*}
\]

(6-6)

Here, \((\hat{d}, \hat{d}^*)\) and \((\hat{c}, \hat{c}^*)\) denote two kinds of bosons and \((\omega, \epsilon, \eta)\) are constants characterizing the Hamiltonian.

Let Hamiltonian given in the relation (6-4) deform by the function \(f_0(n) = (\sqrt{1 + n}/n^0)^{-1}\) for the boson \((\hat{c}, \hat{c}^*)\):

\[
\hat{H}_{def}^{(0)} = \omega \cdot (\hat{d}^* \hat{d}) + \epsilon \cdot (\hat{c}^* \hat{c}) - i \eta \left( \hat{c}^* \sqrt{1 + \hat{N}/n^0} \cdot \hat{d} - \hat{d}^* \cdot \sqrt{1 + \hat{N}/n^0} \cdot \hat{c} \right) .
\]

(6-7)

For the Hamiltonian (6-7), the following picture can be drawn: The external environment, for example, such as the heat bath, is described by the boson \((\hat{d}, \hat{d}^*)\). The oscillation described by the boson \((\hat{c}, \hat{c}^*)\) is damped and amplified due to the interaction with the external environment which is assumed to be extremely big system. Therefore, \((\hat{d}, \hat{d}^*)\) has no fluctuation around the equilibrium value and, then, \((\hat{d}, \hat{d}^*)\) can be replaced by the time-independent \(c\)-number \((\delta, \delta^*)\). It may be performed by calculating the expectation value of \(\hat{H}_{def}^{(0)}\) for the boson coherent state for \((\hat{d}, \hat{d}^*)\).

With the aid of the above picture, the following Hamiltonian is derived:

\[
\begin{align*}
\hat{H}_{def} &= \omega \cdot |\delta|^2 + \epsilon \cdot (\hat{c}^* \hat{c}) - i \eta \cdot \sqrt{|\delta|^2/n^0} \cdot [\hat{c}^* e^{-i\phi} \sqrt{n^0 + \hat{N}} - \sqrt{n^0 + \hat{N}} \cdot \hat{c}^* e^{i\phi}] , \\
\delta &= |\delta| e^{-i\phi} , \\
\delta^* &= |\delta| e^{i\phi} .
\end{align*}
\]

(6-8)
Further, the following relations are set up:

\[ \zeta = \eta \cdot \sqrt{|\delta|^2 / n^0} , \]
\[ \tilde{c} = \tilde{c} e^{i\phi} , \quad \tilde{c}^* = \tilde{c}^* e^{-i\phi} , \quad \tilde{N} = \tilde{c}^* \tilde{c} = \tilde{N} , \quad (6.9) \]
\[ \tilde{T}_- = \sqrt{n^0 + \tilde{N}} \tilde{c} , \quad \tilde{T}_+ = \tilde{c} \sqrt{n^0 + \tilde{N}} , \quad \tilde{T}_0 = \tilde{N} + n^0 / 2 . \quad (6.10) \]

Then, the Hamiltonian \( \hat{H}_{\text{def}} \) can be expressed as

\[ \hat{H}_{\text{def}} = (\omega - \epsilon / 2) \cdot n^0 + \epsilon \cdot \tilde{T}_0 - i\zeta \cdot (\tilde{T}_+ - \tilde{T}_-) . \quad (6.11) \]

The relation (6.10) shows us that \( (\tilde{T}_\pm, 0) \) is the Holstein-Primakoff boson representation of the \( su(1,1) \)-algebra and the formal structure of \( \hat{H}_{\text{def}} \) is completely the same as the \( \hat{H}_{su(1,1)} \) given in the relation (5.5).

**VII. CONCLUDING REMARKS**

As for concluding remarks, we mention two points (A) and (B).

(A) The \( q \)-deformation of the \( su(2) \)- and the \( su(1,1) \)-algebra in two kinds of boson operators \( (\hat{a}, \hat{a}^*) \) and \( (\hat{b}, \hat{b}^*) \):

In the above cases, the deformations are characterized by functions \( f_0(x) \) and \( g_0(x) \) in the form

\[ [x]_f = x f_0(x - 1)^{-2} , \quad [x]_g = x g_0(x - 1)^{-2} . \quad (7.1) \]

(1) The \( su(2) \)-algebra and its \( q \)-deformation:

\[ \hat{S}_0^0 = \hat{b}^* \hat{a}^* , \quad \hat{S}_0^0 = \hat{a}^* \hat{b}^* , \quad \hat{S}_0 = (\hat{a}^* \hat{a} - \hat{b}^* \hat{b}) / 2 , \quad (7.2) \]
\[ \hat{S} = (\hat{b}^* \hat{b} + \hat{a}^* \hat{a}) / 2 , \quad (7.3) \]
\[ \hat{S}_- = \sqrt{[\hat{S} + \hat{S}_0 + 1]_f [\hat{S} - \hat{S}_0]_g} \left( \sqrt{\hat{S} + \hat{S}_0 + 1} \right)^{-1} \hat{S}_0 \left( \sqrt{\hat{S} - \hat{S}_0 + 1} \right)^{-1} , \]
\[ \hat{S}_+ = \left( \sqrt{\hat{S} - \hat{S}_0 + 1} \right)^{-1} \hat{S}_0^0 \left( \sqrt{\hat{S} + \hat{S}_0 + 1} \right)^{-1} \sqrt{[\hat{S} + \hat{S}_0 + 1]_f [\hat{S} - \hat{S}_0]_g} , \]
\[ [2 \hat{S}_0]_q = [\hat{S} + \hat{S}_0]_f [\hat{S} - \hat{S}_0 + 1]_g - [\hat{S} + \hat{S}_0 + 1]_f [\hat{S} - \hat{S}_0]_g , \quad (7.4) \]

(2) The \( su(1,1) \)-algebra and its \( q \)-deformation:

\[ \hat{T}_0^0 = \hat{b} \hat{a} , \quad \hat{T}_0^0 = \hat{a}^* \hat{b}^* , \quad \hat{T}_0 = (\hat{a}^* \hat{a} + \hat{b}^* \hat{b}) / 2 , \quad (7.5) \]
\[ \hat{T} = (\hat{b} \hat{b}^* - \hat{a}^* \hat{a}) / 2 , \quad (7.6) \]
\[ \hat{T}_- = \sqrt{[\hat{T}_0 + \hat{T}]_f [\hat{T}_0 + \hat{T} - 1]_g} \left( \sqrt{(\hat{T}_0 + \hat{T})(\hat{T}_0 - \hat{T} + 1)} \right)^{-1} \hat{T}_0^0 , \]
\[ \hat{T}_+ = \hat{T}_0^+ \left( \sqrt{(\hat{T}_0 + \hat{T})(\hat{T}_0 - \hat{T} + 1)} \right)^{-1} \sqrt{[\hat{T}_0 + \hat{T}]_f [\hat{T}_0 + \hat{T} - 1]_g} , \]
\[ [2 \hat{T}_0]_q = [\hat{T}_0 - \hat{T} + 1]_f [\hat{T}_0 + \hat{T}]_g - [\hat{T}_0 - \hat{T}]_f [\hat{T}_0 + \hat{T} - 1]_g . \quad (7.7) \]

(3) The most popular form:
\[ f_0(n) = g_0(n) = \sqrt{(n + 1)(q - q^{-1})/(q^{n+1} - q^{-(n+1)})}. \] (7.8)

For example, \([2\hat{S}_0]_q\) and \([2\hat{T}_0]_q\) are given in the form

\[
[2\hat{S}_0]_q = (q^{2\hat{S}_0} - q^{-2\hat{S}_0})/(q - q^{-1}), \tag{7.9}
\]

\[
[2\hat{T}_0]_q = (q^{2\hat{T}_0} - q^{-2\hat{T}_0})/(q - q^{-1}). \tag{7.10}
\]

(B) The generalized Schwinger boson representation for the \(su(M+1)\)- and the \(su(N,1)\)-algebra and its \(q\)-deformation:

With the aid of Ref. [7], it may be possible to generalize the \(q\)-deformed algebra in two kinds of boson operators. Certainly, we have the \(q\)-deformation of the \(su(2)\)- and the \(su(1,1)\)-algebra in four kinds of boson operators [8]. It is an example of the \(q\)-deformation of the generalized Schwinger boson representation. With the help of this form, for example, thermal effects on the pairing rotation, the paring vibration and the intrinsic structure may be described.

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