A NOTE ON VASSILIEV INVARIANTS OF QUASIPOSITVE KNOTS

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Abstract. It has been known that any Alexander polynomial of a knot can be realized by a quasipositive knot. As a consequence, the Alexander polynomial cannot detect quasipositivity. In this paper we prove a similar result about Vassiliev invariants: for any oriented knot $K$ and any natural number $n$ there exists a quasipositive knot $Q$ whose Vassiliev invariants of order less than or equal to $n$ coincide with those of $K$.

A quasipositive braid is a product of conjugates of a positive standard generator of the braid group. If a link can be realized as the closure of a quasipositive braid then we call it quasipositive. When Lee Rudolph introduced quasipositive links (in [8]), he showed that they could be realized as transverse $\mathbb{C}$-links, i.e. as transverse intersections of complex plane curves with the standard sphere $S^3 \subset \mathbb{C}^2$. Here a complex plane curve is any set $f^{-1}(0) \subset \mathbb{C}^2$, where $f(z, w) \in \mathbb{C}[z, w]$ is a non-constant polynomial. Conversely, every transverse $\mathbb{C}$-link is a quasipositive link, as was recently proved by M. Boileau and S. Orevkov (in [2]). We shall use yet another description of quasipositive knots which is based upon Seifert diagrams in order to prove the following result.

Theorem 1. For any oriented knot $K$ and any natural number $n$ there exists a quasipositive knot $Q$ whose Vassiliev invariants of order less than or equal to $n$ coincide with those of $K$.

This is related to a result of Lee Rudolph ([7]), who showed that any Alexander polynomial can be realized by a quasipositive knot. We also mention that theorem 1 was formulated as a question by A. Stoimenow in [9].

The proof of theorem 1 is based upon a construction of Y. Ohyama, who showed that any finite number of Vassiliev invariants can be realized by an unknotting number one knot (see [4]). His construction involves certain $C_n$-moves, which were defined by K. Habiro in [3], see also [5]. A special $C_n$-move is defined diagrammatically in figure 1. It takes place in a section with $2(n+1)$ endpoints or $(n+1)$ strands, respectively. The strands are numbered from 1 to $n+1$ and are all connected outside the indicated section, since they belong to one knot $K$. Going along $K$ according to its orientation, starting at the first strand, we encounter the other strands in a certain order which...
depends on how the strands are connected outside the indicated section. This order defines a permutation, say $\sigma \in S_n$, of the numbers $2, 3, \ldots, n+1$.

In [6], Y. Ohyama and T. Tsukamoto explain the effect of a $C_n$-move on Vassiliev invariants of order $n$. Their result ([6], theorem 1.2) implies the following:

(1) A $C_n$-move does not change the values of Vassiliev invariants of order less than $n$.

(2) Let $K$ and $\tilde{K}$ be two knots which differ by one $C_n$-move, and $v_n$ any Vassiliev invariant of order $n$. Then $|v_n(K) - v_n(\tilde{K})|$ depends only on the permutation $\sigma \in S_n$ defined by the cyclic order of the $(n+1)$ strands of the section where the $C_n$-move takes place.

Proof of theorem 1. Starting from the diagram of the positive twist knot $5_2$ shown in figure 2, we construct a quasipositive knot $Q$ with the desired properties by applying several $C_n$-moves, $2 \leq i \leq n$, step by step.

In the first step, we construct a quasipositive knot $Q_1$ whose Vassiliev invariants of order two (the Casson invariant) equals that of $K$. Choose natural numbers $a$ and $b$, such that $v_2(K) = 2 + a - b$. Here $v_2(K)$ is the Casson invariant of $K$. Using these two numbers, we define a knot $Q_1$ diagrammatically, as shown in figure 3.

By construction, we have

$$v_2(Q_1) = 2 + a - b = v_2(K).$$
This follows easily by one application of the following relation for the Casson invariant of knots:

\[ v_2(\uparrow) - v_2(\downarrow) = \text{lk}(\downarrow). \]

Indeed, a crossing change at the clasp on the left side of the diagram of \( Q_1 \) produces a trivial knot, and the linking number \( \text{lk} \) of the corresponding link equals \( 2 + a - b \). Moreover, the Seifert diagram of \( Q_1 \) at the bottom of figure 3 is quasipositive. Here a Seifert diagram is quasipositive if its set of crossings can be partitioned into single crossings and pairs of crossings, such that the following three conditions are satisfied.

1. Each single crossing is positive.
2. Each pair of crossings consists of one positive and one negative crossing joining the same two Seifert circles.
3. A pair of crossings does not separate other pairs of crossings. More precisely, going from one crossing of a pair to its opposite counterpart along a Seifert circle, one cannot meet only one crossing of a pair. Such pairs of crossings are called conjugating pairs of crossings.

In [1], we proved that a quasipositive diagram represents a quasipositive knot. Hence \( Q_1 \) is a quasipositive knot.

In the second step, we arrange the Vassiliev invariants of order three. Since 'all' the Vassiliev invariants of order less than or equal to two of \( Q_1 \) and \( K \) coincide (i.e. \( v_2(Q_1) = v_2(K) \)), we conclude that \( Q_1 \) and \( K \) are related by a sequence of \( C_3 \)-moves. This is K. Habiro's result for \( n = 2 \) (see [3]). Let \( K_1 = Q_1, K_2, \ldots, K_l = K \) be a sequence of knots, such that two succeeding knots are related by a \( C_3 \)-move. Our aim is to replace this sequence of knots by a sequence of quasipositive knots \( \tilde{K}_1 = Q_1, \tilde{K}_2, \ldots, \tilde{K}_l \), such that

\[ v_3(\tilde{K}_{i+1}) - v_3(\tilde{K}_i) = v_3(K_{i+1}) - v_3(K_i), \]
\[1 \leq i \leq l-1.\] By Ohyama and Tsukamoto’s result, \(|v_3(K_2) - v_3(Q_1)|\) depends only on the permutation \(\sigma \in S_3\) defined by the cyclic order of the four strands of the section where the \(C_3\)-move takes place, as explained above. From this viewpoint, i.e. if we are only interested in the change of the Vassiliev invariants of order three, there are only finitely many combinatorial patterns of \(C_3\)-moves. A ’standard’ pattern of a \(C_3\)-move can be applied inside a local box on the right side of the diagram of \(Q_1\), as shown in figure 4.

\[\text{Figure 4.}\]

Moreover, we can choose a quasipositive representative for this pattern, i.e. a representative whose Seifert diagram (inside the box) is quasipositive, see figure 5. Here we remark that the two segments above and below the cross-shaped Seifert circle belong to the same Seifert circle since they are connected outside the local box.

\[\text{Figure 5.}\]

However, this standard pattern corresponds to one specific permutation \(\sigma \in S_3\). In order to get patterns corresponding to other permutations, we have to permute the strands inside the local box, as shown by two examples on the left side of figure 6. We observe that all these patterns have quasipositive representatives. They are depicted on the right side of figure 6, together with their Seifert diagrams.

Thus we can replace the knot \(K_2\) by a quasipositive knot \(\tilde{K}_2\), such that
\[
v_3(\tilde{K}_2) - v_3(Q_1) = \pm(v_3(K_2) - v_3(Q_1)).
\]
If the sign of this difference is wrong (i.e. ‘−’), we may arrange it to be ‘+’ by changing four crossings between two strands inside the local box, see figure 4, where the four crossings are encircled. This modified pattern has the inverse effect on Vassiliev invariants of order three, as follows from Ohyama and Tsukamoto’s calculation ([6], proof of theorem 1.2).

Likewise, we can replace all $C_3$-moves of the sequence $K_1 = Q_1, K_2, \ldots, K_l = K$ by $C_3$-moves that take place in a clearly arranged box and preserve the quasipositivity of the knot $Q_1$. In this way, we obtain a sequence of quasipositive knots $\tilde{K}_1 = Q_1, \tilde{K}_2, \ldots, \tilde{K}_l$ and end up with a quasipositive knot $Q_2 := \tilde{K}_l$ whose Vassiliev invariants of order two and three coincide with those of $K$.

At this point we merely sketch how the process continues: in the $i$-th step, we arrange the Vassiliev invariants of order $i + 1$ and define a quasipositive knot $Q_i$ whose Vassiliev invariants of order less than or equal to $i + 1$ coincide with those of $K$. For this purpose, we need only observe that every combinatorial pattern of a $C_{i+1}$-move has a quasipositive representative: the heart of such a representative consists of $i^2 + i$ conjugating pairs of arcs. This is illustrated for a $C_4$-move in figure 7.

At last, the quasipositive knot $Q := Q_{n-1}$ has the required properties. □
Remarks.

(1) All the quasipositive knots $Q_i$ can be unknotted by a single crossing change at the clasp that appears on the left side of their defining diagram (see e.g. figure 3). In particular, the unknotting number of $Q$ is one, unless $Q$ happens to be the trivial knot.

(2) By theorem 1 and Habiro’s result, we conclude that every knot can be transformed into a quasipositive knot by a finite sequence of $C_n$-moves, for any fixed natural number $n$. It would be interesting to have a direct proof for this fact, which in turn implies theorem 1. This would possibly simplify the construction of the desired quasipositive knots.

(3) The knot $Q$ might even be chosen to be strongly quasipositive. However, we do not know how to prove that.

Acknowledgements. Alexander Stoimenow pointed out an erroneous statement in the first version of this paper. I would like to thank him for that and for other useful remarks.

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