AN ORDERING ON GREEN’S FUNCTION AND A LYAPUNOV–TYPE INEQUALITY FOR A FAMILY OF NABLA FRACTIONAL BOUNDARY VALUE PROBLEMS

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Abstract. In this article, we consider a family of two-point Riemann–Liouville type nabla fractional boundary value problems involving a fractional difference boundary condition. We construct the corresponding Green’s function and deduce its ordering property. Then, we obtain a Lyapunov-type inequality using the properties of the Green’s function, and illustrate a few of its applications.

1. Introduction

In this article, we construct the Green’s function $G(b, \beta; t, s)$ of the following two-point nabla fractional boundary value problem

$$
\begin{align*}
&\left( \nabla_a^\alpha u \right)(t) + h(t) = 0, \quad t \in \mathbb{N}_{a+2}^b, \\
&u(a) = 0, \quad \left( \nabla_a^\beta u \right)(b) = 0.
\end{align*}
$$

(1.1)

Here $1 < \alpha < 2$, $0 \leq \beta \leq 1$, $a, b \in \mathbb{R}$ with $b - a \in \mathbb{N}_2$, $h : \mathbb{N}_{a+2}^b \to \mathbb{R}$, $\nabla_a^\alpha$ and $\nabla_a^\beta$ are the Riemann–Liouville type $\alpha$th and $\beta$th-order nabla difference operators, respectively. Observe that the pair of boundary conditions in (1.1) reduces to conjugate [6, 12, 20], right-focal [18] and right-focal type [19] boundary conditions as $\beta \to 0^+$, $\beta \to 1^-$ and $\beta \to (\alpha - 1)$, respectively. In Section 3, we obtain an ordering property on $G(b, \beta; t, s)$ with respect to $b$ and $\beta$.

Lately, there has been an increased interest in establishing Lyapunov-type inequalities for discrete fractional boundary value problems. For the first time, Ferreira [10] deduced a Lyapunov-type inequality for a discrete boundary value problem involving the Riemann–Liouville type $\alpha$th-order $(1 < \alpha \leq 2)$ forward difference operator. Following Ferreira’s work, authors of [8, 11] established Lyapunov-type inequalities for various classes of delta fractional boundary value problems. In this line, Ikram [16]...
developed Lyapunov-type inequalities for certain nabla fractional boundary value problems of Caputo type. Recently, the author [18, 19] obtained Lyapunov-type inequalities for the nabla fractional difference equation
\[(\nabla^\alpha_a u)(t) + q(t)u(t) = 0, \quad t \in \mathbb{N}^b_{a+2},\]
associated with two-point conjugate (C), left focal (LF), right focal (RF), left-focal type (LFT) and right-focal type (RFT) boundary conditions:

(C) \[u(a) = u(b) = 0;\]

(LF) \[(\nabla u)(a+1) = u(b) = 0;\]

(RF) \[u(a) = (\nabla u)(b) = 0;\]

(LFT) \[(\nabla^{\alpha-1}_a u)(a+1) = u(b) = 0;\]

(RF) \[u(a) = (\nabla^{\alpha-1}_a u)(b) = 0.\]

Motivated by these developments, in this article, we obtain a Lyapunov-type inequality for the two-point nabla fractional boundary value problem
\[
\begin{cases}
(\nabla^\alpha_a u)(t) + q(t)u(t) = 0, & t \in \mathbb{N}^b_{a+2}, \\
u(a) = 0, & (\nabla^\beta_b u)(b) = 0,
\end{cases}
\]
where \(q : \mathbb{N}^b_{a+2} \to \mathbb{R},\) and demonstrate a few of its applications.

2. Preliminaries

Denote the set of all real numbers by \(\mathbb{R}.\) Define
\[
\mathbb{N}_a := \{a, a+1, a+2, \ldots\} \quad \text{and} \quad \mathbb{N}^b_a := \{a, a+1, a+2, \ldots, b\}
\]
for any \(a, b \in \mathbb{R}\) such that \(b - a \in \mathbb{N}_1.\) Assume that empty sums and products are taken to be 0 and 1, respectively.

**DEFINITION 2.1.** (See [7]) The backward jump operator \(\rho : \mathbb{N}_a \to \mathbb{N}_a\) is defined by
\[
\rho(t) = \max\{a, (t-1)\}, \quad t \in \mathbb{N}_a.
\]

**DEFINITION 2.2.** (See [22, 23]) The Euler gamma function is defined by
\[
\Gamma(z) := \int_0^\infty t^{z-1}e^{-t}dt, \quad \Re(z) > 0.
\]
Using the reduction formula
\[
\Gamma(z+1) = z\Gamma(z), \quad \Re(z) > 0,
\]
the Euler gamma function can be extended to the half-plane \(\Re(z) \leq 0\) except for \(z \neq 0, -1, -2, \ldots\
DEFINITION 2.3. (See [14]) For $t \in \mathbb{R} \setminus \{\ldots, -2, -1, 0\}$ and $r \in \mathbb{R}$ such that $(t + r) \in \mathbb{R} \setminus \{\ldots, -2, -1, 0\}$, the generalized rising function is defined by

$$t^r = \frac{\Gamma(t + r)}{\Gamma(t)}.$$ 

Also, we use the convention that if $t \in \{\ldots, -2, -1, 0\}$ and $r \in \mathbb{R}$ such that $(t + r) \in \mathbb{R} \setminus \{\ldots, -2, -1, 0\}$, then

$$t^r := 0.$$

DEFINITION 2.4. (See [7]) Let $u : \mathbb{N}_a \rightarrow \mathbb{R}$ and $N \in \mathbb{N}_1$. The first order backward (nabla) difference of $u$ is defined by

$$(\nabla u)(t) := u(t) - u(t - 1), \quad t \in \mathbb{N}_{a+1},$$

and the $N$th-order nabla difference of $u$ is defined recursively by

$$(\nabla^N u)(t) := \left( \nabla (\nabla^{N-1} u) \right)(t), \quad t \in \mathbb{N}_{a+N}.$$ 

DEFINITION 2.5. (See [14]) Let $u : \mathbb{N}_{a+1} \rightarrow \mathbb{R}$ and $N \in \mathbb{N}_1$. The $N$th-order nabla sum of $u$ based at $a$ is given by

$$(\nabla_a^{-N} u)(t) := \frac{1}{(N-1)!} \sum_{s=a+1}^{t} (t \rho(s))^{-N-1} u(s), \quad t \in \mathbb{N}_a,$$

where by convention $(\nabla_a^{-N} u)(a) = 0$. We define $(\nabla_a^{-0} u)(t) = u(t)$ for all $t \in \mathbb{N}_{a+1}$.

DEFINITION 2.6. (See [14]) Let $u : \mathbb{N}_{a+1} \rightarrow \mathbb{R}$ and $\nu > 0$. The $\nu$th-order nabla sum of $u$ based at $a$ is given by

$$(\nabla_a^{-\nu} u)(t) := \frac{1}{\Gamma(\nu)} \sum_{s=a+1}^{t} (t \rho(s))^{-\nu-1} u(s), \quad t \in \mathbb{N}_a,$$

where by convention $(\nabla_a^{-\nu} u)(a) = 0$.

DEFINITION 2.7. (See [14]) Let $u : \mathbb{N}_{a+1} \rightarrow \mathbb{R}$, $\nu > 0$ and choose $N \in \mathbb{N}_1$ such that $N - 1 < \nu \leq N$. The Riemann–Liouville type $\nu$th-order nabla difference of $u$ is given by

$$(\nabla_a^\nu u)(t) := \left( \nabla^N (\nabla_a^{-(N-\nu)} u) \right)(t), \quad t \in \mathbb{N}_{a+N}.$$ 

THEOREM 2.1. (See [2]) Assume $u : \mathbb{N}_a \rightarrow \mathbb{R}$, $\nu > 0$, $\nu \notin \mathbb{N}_1$, and choose $N \in \mathbb{N}_1$ such that $N - 1 < \nu < N$. Then,

$$(\nabla_a^\nu u)(t) = \frac{1}{\Gamma(-\nu)} \sum_{s=a+1}^{t} (t \rho(s))^{-\nu-1} u(s), \quad t \in \mathbb{N}_{a+1}.$$
THEOREM 2.2. (See [14]) Let $\nu, \mu > 0$ and $u : \mathbb{N}_a \to \mathbb{R}$. Then,

$$\left( \nabla^\nu_a \left( \nabla^{-\mu} u \right) \right)(t) = \left( \nabla^{\nu-\mu} a u \right)(t), \quad t \in \mathbb{N}_a.$$ 

THEOREM 2.3. (See [14, 17]) We observe the following properties of gamma and generalized rising functions.

1. $\Gamma(t) > 0$ for all $t > 0$.
2. $t^{\alpha} \Gamma(t) = t^{\alpha+\beta}.$
3. If $t \leq r$, then $t^{\alpha} \leq r^{\alpha}$.
4. If $\alpha < t \leq r$, then $r^{\alpha} \leq t^{\alpha}.$
5. $\nabla(t + \alpha)^\beta = \beta(t + \alpha)^{\beta-1}.$
6. $\nabla(\alpha - t)^\beta = -\beta(\alpha - \rho(t))^{\beta-1}.$

THEOREM 2.4. (See [14]) Let $\nu \in \mathbb{R}^+$ and $\mu \in \mathbb{R}$ such that $\mu, \mu + \nu$ and $\mu - \nu$ are nonnegative integers. Then,

$$\nabla^{-\nu}_a (t - a)^\mu = \frac{\Gamma(\mu + 1)}{\Gamma(\mu + \nu + 1)}(t - a)^{\mu-\nu}, \quad t \in \mathbb{N}_a,$$

$$\nabla^{\nu}_a (t - a)^\mu = \frac{\Gamma(\mu + 1)}{\Gamma(\mu - \nu + 1)}(t - a)^{\mu+\nu}, \quad t \in \mathbb{N}_a.$$ 

THEOREM 2.5. (See [14]) Assume $\nu > 0$ and $N - 1 < \nu \leq N$. Then, a general solution of

$$\left( \nabla^{\nu}_a u \right)(t) = 0, \quad t \in \mathbb{N}_{a+N},$$

is given by

$$u(t) = C_1(t - a)^{\nu-1} + C_2(t - a)^{\nu-2} + \ldots + C_N(t - a)^{\nu-N}, \quad t \in \mathbb{N}_a,$$

where $C_1, C_2, \ldots, C_N \in \mathbb{R}$.

3. Properties of Green’s function

First, we deduce the unique solution of (1.1).

THEOREM 3.1. The discrete boundary value problem (1.1) has the unique solution

$$u(t) = \sum_{s=a+2}^{b} G(b, \beta; t, s) h(s), \quad t \in \mathbb{N}_a, \quad (3.1)$$
where the Green’s function $G(b, \beta; t, s)$ is given by

$$G(b, \beta; t, s) = \begin{cases} \frac{1}{\Gamma(\alpha)} \frac{(b-s+1)^{\alpha-1}}{(b-a)^{\alpha-\beta-1}}(t-a)^{\alpha-1}, & t \in \mathbb{N}_a^0, \\ \frac{1}{\Gamma(\alpha)} \left[ \frac{(b-s+1)^{\alpha-1}}{(b-a)^{\alpha-\beta-1}}(t-a)^{\alpha-1} - (t-s+1)^{\alpha-1} \right], & t \in \mathbb{N}_b^0. \end{cases}$$  

(3.2)

**Proof.** Applying $\nabla_a^{-\alpha}$ on both sides of (1.1) and using Theorem 2.5, we have

$$u(t) = -\left( \nabla_a^{-\alpha} h \right)(t) + C_1(t-a)^{\alpha-1} + C_2(t-a)^{\alpha-2}, \quad t \in \mathbb{N}_a,$$

for some $C_1, C_2 \in \mathbb{R}$. Using $u(a) = 0$ in (3.3), we get $C_2 = 0$. Applying $\nabla_a^{\beta}$ on both sides of (3.3) and using Theorems 2.2 and 2.4, we have

$$\left( \nabla_a^{\beta} u \right)(t) = -\left( \nabla_a^{\beta-a} h \right)(t) + C_1 \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}(t-a)^{\alpha-\beta-1}, \quad t \in \mathbb{N}_a.$$

(3.4)

Using $\left( \nabla_a^{\beta} u \right)(b) = 0$ in (3.4), we get

$$C_1 = \frac{1}{(b-a)^{\alpha-\beta-1}\Gamma(\alpha)} \sum_{s=a+1}^{b} (b-s+1)^{\alpha-\beta-1} h(s).$$

Substituting the values of $C_1$ and $C_2$ in (3.3), we have

$$u(t) = \left( t-a \right)^{\alpha-1} \frac{(b-a)^{\alpha-\beta-1}}{\Gamma(\alpha)} \sum_{s=a+1}^{b} (b-s+1)^{\alpha-\beta-1} h(s) - \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{b} (t-s+1)^{\alpha-1} h(s)$$

$$= \frac{1}{\Gamma(\alpha)} \sum_{s=a+2}^{t} \left[ \frac{(b-s+1)^{\alpha-\beta-1}}{(b-a)^{\alpha-\beta-1}}(t-a)^{\alpha-1} - (t-s+1)^{\alpha-1} \right] h(s)$$

$$+ \frac{1}{\Gamma(\alpha)} \sum_{s=t+1}^{b} \left[ \frac{(b-s+1)^{\alpha-\beta-1}}{(b-a)^{\alpha-\beta-1}}(t-a)^{\alpha-1} \right] h(s)$$

$$= \sum_{s=a+2}^{b} G(b, \beta; t, s)(t, s) h(s).$$

The proof is complete. $\square$

**Remark 1.** Observe that

1. $G(b, \beta; t, a+1) = 0$ for $t \in \mathbb{N}_a^b$.

2. $G(b, \beta; a, s) = 0$ for $s \in \mathbb{N}_a$.

Brackins [6], Gholami et al. [12] and the author [18, 19, 20] have derived the Green’s functions $G(b, 0; t, s)$, $G(b, 1; t, s)$ and $G(b, \alpha-1; t, s)$ of the two-point nabla fractional boundary value problem associated with conjugate, right-focal and right-focal type boundary conditions, respectively, and also obtained a few properties.
THEOREM 3.2. (See [6, 12, 18, 19, 20]) \( G(b_0; t, s), G(b_1; t, s) \) and 
\( G(b, \alpha - 1; t, s) \) are nonnegative for \((t, s) \in \mathbb{N}_a^b \times \mathbb{N}_a^b\).

Next, we obtain a few properties of \( G(b, \beta; t, s) \).

**Lemma 3.3.** If \( 0 \leq \beta_1 < \beta_2 \leq 1 \), then \( G(b, \beta_1; t, s) < G(b, \beta_2; t, s) \) for \((t, s) \in \mathbb{N}_a^{b+1} \times \mathbb{N}_a^{b+2}\).

**Proof.** Using (2) of Theorem 2.3, we rewrite \( G(b, \beta_1; t, s) \) in terms of \( G(b, \beta_2; t, s) \) as follows:

\[
G(b, \beta_1; t, s) = \begin{cases} 
\frac{1}{\Gamma(\alpha)} \frac{(b-a+\alpha-\beta_1-1)\beta_1\beta_2}{(b-s+\alpha-\beta_1)\beta_1\beta_2(\beta_1-\beta_2)}(t-a)^{\alpha-1}, & t \in \mathbb{N}_a^{b(s)}, \\
\frac{1}{\Gamma(\alpha)} \frac{(b-a+\alpha-\beta_1-1)\beta_1\beta_2}{(b-s+\alpha-\beta_1)\beta_1\beta_2(\beta_1-\beta_2)}(t-a)^{\alpha-1} - (t-s+1)^{\alpha-1}, & t \in \mathbb{N}_s^b.
\end{cases}
\]

Since \( \beta_2 - \beta_1 < (b-s+\alpha-\beta_1) < (b-a+\alpha-\beta_1-1) \), from (4) of Theorem 2.3, we have

\[
(b-a+\alpha-\beta_1-1)\beta_1\beta_2 < (b-s+\alpha-\beta_1)\beta_1\beta_2,
\]

implying that

\[
G(b, \beta_1; t, s) < G(b, \beta_2; t, s), \quad (t, s) \in \mathbb{N}_a^{b+1} \times \mathbb{N}_a^{b+2}.
\]

The proof is complete. \( \square \)

**Theorem 3.4.** \( G(b, \beta; t, s) \geq 0 \) for \((t, s) \in \mathbb{N}_a^b \times \mathbb{N}_a^{b+2}\).

**Proof.** The proof follows from Remark 1, Theorem 3.2 and Lemma 3.3. \( \square \)

**Lemma 3.5.** Assume \( b_1 < b_2 \) and \((t, s) \in \mathbb{N}_a^{b+1} \times \mathbb{N}_a^{b+2}\).

1. If \( 0 \leq \beta < (\alpha - 1) \), then \( G(b_1, \beta; t, s) < G(b_2, \beta; t, s) \).
2. If \( (\alpha - 1) < \beta \leq 1 \), then \( G(b_1, \beta; t, s) > G(b_2, \beta; t, s) \).
3. If \( \beta = (\alpha - 1) \), then \( G(b, \beta; t, s) \) is independent of \( b \).

**Proof.** Consider

\[
\nabla_b[G(b, \beta; t, s)] = \frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)} \nabla_b \left[ \frac{(b-s+1)^{\alpha-\beta-1}}{(b-a)^{\alpha-\beta-1}} \right]
\]

\[
= \frac{(b-a)^{\alpha-\beta-2}(b-s+1)^{\alpha-\beta-2}(t-a)^{\alpha-1}(s-a-1)(\alpha-\beta-1)}{(b-a)^{\alpha-\beta-1}(b-a-1)^{\alpha-\beta-1}\Gamma(\alpha)}
\]

\[
= \frac{(b-s+1)^{\alpha-\beta-2}(t-a)^{\alpha-1}}{(b-a-1)^{\alpha-\beta}\Gamma(\alpha)}(s-a-1)(\alpha-\beta-1).
\]
Clearly, \((s - a - 1) > 0\), \(\Gamma(\alpha) > 0\) and it follows from (1) of Theorem 2.3 that

\[
(t - a)^{\alpha - 1} = \frac{\Gamma(t - a + \alpha - 1)}{\Gamma(t - a)} > 0,
\]

\[
(b - s + 1)^{\alpha - \beta - 2} = \frac{\Gamma(b - s + \alpha - \beta - 1)}{\Gamma(b - s + 1)} > 0,
\]

and

\[
(b - a - 1)^{\alpha - \beta} = \frac{\Gamma(b - a + \alpha - \beta - 1)}{\Gamma(b - a - 1)} > 0.
\]

Thus, if \(0 \leq \beta < (\alpha - 1)\), then \(\nabla_b [G(b, \beta; t, s)] > 0\) implying that (1) follows. If \((\alpha - 1) < \beta \leq 1\), then \(\nabla_b [G(b, \beta; t, s)] < 0\) implying that (2) follows. If \(\beta = (\alpha - 1)\), then \(\nabla_b [G(b, \beta; t, s)] = 0\) implying that \(G(b, \beta; t, s)\) is independent of \(b\). The proof is complete.  \(\square\)

DEFINITION 3.1. Denote by

\[
H(b, \beta; s) = \frac{(b - s + 1)^{\alpha - \beta - 1}}{(b - a)^{\alpha - \beta - 1}}, \quad s \in \mathbb{N}_a^b.
\]

REMARK 2. We have

\[
H(b, \beta; s) = \frac{\Gamma(b - s + \alpha - \beta) \Gamma(b - a)}{\Gamma(b - s + 1) \Gamma(b - a + \alpha - \beta - 1)}, \quad s \in \mathbb{N}_a^b.
\]

(i) It follows from (1) of Theorem 2.3 that \(H(b, \beta; s) > 0\) for \(s \in \mathbb{N}_a^b\).

(ii) Since \((b - s + 1) < (b - a)\), from (3) of Theorem 2.3, we have

\[
(b - s + 1)^{\alpha - 1} < (b - a)^{\alpha - 1},
\]

implying that \(H(b, 0; s) < 1\).

(iii) Since \((2 - \alpha) < (b - s + 1) < (b - a)\), from (4) of Theorem 2.3, we have

\[
(b - a)^{\alpha - 2} < (b - s + 1)^{\alpha - 2},
\]

implying that \(H(b, 1; s) > 1\).

LEMMA 3.6. If \(0 \leq \beta_1 < \beta_2 \leq 1\), then \(H(b, \beta_1; s) < H(b, \beta_2; s)\) for \(s \in \mathbb{N}_a^b\).

Proof. Using (2) of Theorem 2.3, we rewrite \(H(b, \beta_1; s)\) in terms of \(H(b, \beta_2; s)\) as follows:

\[
H(b, \beta_1; s) = \frac{(b - s + 1)^{\alpha - \beta_1 - 1}}{(b - a)^{\alpha - \beta_1 - 1}} = \frac{(b - a + \alpha - \beta_1 - 1)^{\beta_1 - \beta_2}}{(b - s + \alpha - \beta_1)^{\beta_1 - \beta_2}} \frac{(b - s + 1)^{\alpha - \beta_2 - 1}}{(b - a)^{\alpha - \beta_2 - 1}}
\]

\[
= \frac{(b - a + \alpha - \beta_1 - 1)^{\beta_1 - \beta_2}}{(b - s + \alpha - \beta_1)^{\beta_1 - \beta_2}} H(b, \beta_2; s).
\]
It follows from (3.5) that
\[ H(b, \beta_1; s) < H(b, \beta_2; s), \quad s \in \mathbb{N}_{a+2}^b. \]
The proof is complete. □

**Lemma 3.7.** Assume \( s \in \mathbb{N}_{a+2}^b. \)
1. If \( 0 \leq \beta < (\alpha - 1) \), then \( H(b, \beta; s) < 1. \)
2. If \( (\alpha - 1) < \beta \leq 1 \), then \( H(b, \beta; s) > 1. \)
3. If \( \beta = (\alpha - 1) \), then \( H(b, \beta; s) = 1. \)

**Proof.**
1. Since \( (b-s+1) < (b-a) \), from (3) of Theorem 2.3, we have
\[ (b-s+1)^{\alpha-\beta-1} < (b-a)^{\alpha-\beta-1}, \]
implying that \( H(b, \beta; s) < 1. \)
2. Since \(- (\alpha - \beta - 1) < (b-s+1) < (b-a)\), from (4) of Theorem 2.3, we have
\[ (b-a)^{\alpha-\beta-1} < (b-s+1)^{\alpha-\beta-1}, \]
implying that \( H(b, \beta; s) > 1. \)
3. The proof of (3) is trivial. □

**Lemma 3.8.** Assume \( b_1 < b_2. \)
1. If \( 0 \leq \beta < (\alpha - 1) \), then \( H(b_1, \beta; s) < H(b_2, \beta; s) \) for \( s \in \mathbb{N}_{a+2}^b. \)
2. If \( (\alpha - 1) < \beta \leq 1 \), then \( H(b_1, \beta; s) > H(b_2, \beta; s) \) for \( s \in \mathbb{N}_{a+2}^b. \)

**Proof.** Consider
\[ \nabla_b [H(b, \beta; s)] = \nabla_b \left[ \frac{(b-s+1)^{\alpha-\beta-1}}{(b-a)^{\alpha-\beta-1}} \right] = \frac{(b-a)^{\alpha-\beta-1}(b-s+1)^{\alpha-\beta-1}(s-a-1)(\alpha-\beta-1)}{(b-a)^{\alpha-\beta-1}(b-a-1)^{\alpha-\beta-1}} = \frac{(b-s+1)^{\alpha-\beta-2}(s-a-1)(\alpha-\beta-1)}{(b-a-1)^{\alpha-\beta}}. \]
Clearly, \( (s-a-1) > 0 \) and it follows from (1) of Theorem 2.3 that
\[ (b-s+1)^{\alpha-\beta-2} = \frac{\Gamma(b-s+\alpha-\beta-1)}{\Gamma(b-s+1)} > 0, \]
and

\[(b - a - 1)\varphi = \frac{\Gamma(b - a + \alpha - \beta - 1)}{\Gamma(b - a - 1)} > 0.\]

Thus, if \(0 \leq \beta < (\alpha - 1)\), then \(\nabla_b[H(b, \beta; s)] > 0\) implying that (1) follows. If \((\alpha - 1) < \beta \leq 1\), then \(\nabla_b[H(b, \beta; s)] < 0\) implying that (2) follows. The proof is complete. \(\square\)

**Theorem 3.9.** The maximum of the Green’s function \(G(b, \beta; t, s)\) defined in (3.2) is given by

\[
\max_{(t, s) \in \mathbb{N}_{a+1} \times \mathbb{N}_{a+2}} G(b, \beta; t, s) = \begin{cases} 
\Omega, & 0 \leq \beta < (\alpha - 1), \\
\max\{\Omega, \Lambda - 1\}, & (\alpha - 1) < \beta \leq 1,
\end{cases}
\]

where

\[
\Omega = G\left(b, \beta; \frac{(a + b + 3)(\alpha - \beta - 1) + b\beta}{2\alpha - 2 - \beta}\right) - 1, \quad \left(\frac{(a + b + 3)(\alpha - \beta - 1) + b\beta}{2\alpha - 2 - \beta}\right),
\]

and

\[
\Lambda = G\left(b, \beta; \frac{(a + b + 3)(\alpha - \beta - 1) + b\beta + 1}{2\alpha - 2 - \beta}\right), \quad \left(\frac{(a + b + 3)(\alpha - \beta - 1) + b\beta + 1}{2\alpha - 2 - \beta}\right).
\]

**Proof.** Assume \((t, s) \in \mathbb{N}_{a+1} \times \mathbb{N}_{a+2}\). First, we show that for any fixed \(s \in \mathbb{N}_{a+2}\), \(G(b, \beta; t, s)\) is an increasing function of \(t\) between \(a + 1\) and \(s - 1\). Consider the first order nabla difference of \(G(b, \beta; t, s)\) with respect to \(t\).

\[
\nabla_t[G(b, \beta; t, s)] = \frac{H(b, \beta; s)}{\Gamma(\alpha)} \nabla_t(t - a)^{\alpha - 1} = \frac{H(b, \beta; s)(t - a)^{\alpha - 2}}{\Gamma(\alpha - 1)}
\]

\[
= \frac{H(b, \beta; s)(t - a + \alpha - 2)}{\Gamma(\alpha - 1)} \frac{1}{\Gamma(\alpha - 1)} (t - a).
\]

It follows from Remark 2 and (1) of Theorem 2.3 that \(\nabla_t[G(b, \beta; t, s)] > 0\) implying that \(G(b, \beta; t, s)\) is an increasing function of \(t\) between \(a + 1\) and \(s - 1\). Next, we show that for any fixed \(s \in \mathbb{N}_{a+2}\), \(G(b, \beta; t, s)\) is a decreasing function of \(t\) between \(s\) and \(b\). Consider the first order nabla difference of \(G(b, \beta; t, s)\) with respect to \(t\).

\[
\nabla_t[G(b, \beta; t, s)] = \frac{1}{\Gamma(\alpha)} \left[H(b, \beta; s)\nabla_t(t - a)^{\alpha - 1} - \nabla_t(t - s + 1)^{\alpha - 1}\right]
\]

\[
= \frac{1}{\Gamma(\alpha - 1)} \left[H(b, \beta; s)(t - a)^{\alpha - 2} - (t - s + 1)^{\alpha - 2}\right]
\]

\[
= \frac{H(b, \beta; s)(t - a)^{\alpha - 2}}{\Gamma(\alpha - 1)} \left[1 - \frac{H(t, 1; s)}{H(b, \beta; s)}\right].
\]
Clearly, $\Gamma(\alpha - 1) > 0$ and it follows from (3.6) that
\[
\frac{H(b, \beta; s)(t - a)^{\alpha - 2}}{\Gamma(\alpha - 1)} > 0.
\]
We consider two different cases based on $\alpha$ and $\beta$.

(i) Suppose $0 \leq \beta \leq (\alpha - 1)$. Since $t \in \mathbb{N}^b_s$ and $s \in \mathbb{N}^b_{a+2}$, from Remark 2 and Lemma 3.7, we obtain
\[
H(t, 1; s) > 1 \text{ and } H(b, \beta; s) < 1,
\]
implying that $\nabla_t [G(b, \beta; t, s)] < 0$.

(ii) Suppose $(\alpha - 1) < \beta \leq 1$. Since $t \in \mathbb{N}^b_s$ and $s \in \mathbb{N}^b_{a+2}$, from Lemmas 3.6 and 3.8, we have
\[
H(t, 1; s) > H(t, \beta; s) > H(b, \beta; s),
\]
implying that $\nabla_t [G(b, \beta; t, s)] < 0$.

Thus, $G(b, \beta; t, s)$ is a decreasing function of $t$ between $s$ and $b$. Therefore, we have demonstrated that for any fixed $s \in \mathbb{N}^b_{a+2}$, $G(b, \beta; t, s)$ increases from $G(b, \beta; a+1, s)$ to $G(b, \beta; s-1, s)$ and then decreases from $G(b, \beta; s, s)$ to $G(b, \beta; b, s)$. Now, we examine $G(b, \beta; t, s)$ to determine whether the maximum for a fixed $t$ will occur at $(s-1, s)$ or $(s, s)$. We have
\[
G(b, \beta; s-1, s) = \frac{H(b, \beta; s)(s - a - 1)^{\alpha - 1}}{\Gamma(\alpha)}
\]
and
\[
G(b, \beta; s, s) = \frac{H(b, \beta; s)(s - a)^{\alpha - 1}}{\Gamma(\alpha)} - 1.
\]
We consider two different cases based on $\alpha$ and $\beta$.

(i) Suppose $0 \leq \beta \leq (\alpha - 1)$. Consider
\[
G(b, \beta; s-1, s) - G(b, \beta; s, s) = \frac{H(b, \beta; s)}{\Gamma(\alpha)} [(s - a - 1)^{\alpha - 1} - (s - a)^{\alpha - 1}] + 1
\]
\[
= -\frac{H(b, \beta; s)}{\Gamma(\alpha)} \nabla_s [(s - a)^{\alpha - 1}] + 1
\]
\[
= -\frac{H(b, \beta; s)}{\Gamma(\alpha - 1)} (s - a)^{\alpha - 2} + 1. \tag{3.8}
\]
Using Lemma 3.7 in (3.8), we obtain
\[
G(b, \beta; s-1, s) - G(b, \beta; s, s) \geq -\frac{1}{\Gamma(\alpha - 1)} (s - a)^{\alpha - 2} + 1.
\]
Now we wish to maximize the expression
\[
(\text{the expression if }
\]
the equation implying that so we consider solution and thus the expression maximum of the expression and thus the expression \(G(\alpha - 1, b + s + \alpha - \beta) - (\alpha - \beta - 1)(s - a + \alpha - 3)\).

In this expression, \(\Gamma(\alpha) > 0,\)
\[
(b - s + 2)\alpha - \beta = \frac{\Gamma(b - s + \alpha - \beta)}{\Gamma(b - s + 2)} > 0,
\]
\[
(s - a - 1)\alpha = \frac{\Gamma(s - a + \alpha - 3)}{\Gamma(s - a - 1)} > 0,
\]
and
\[
(b - a)\alpha - \beta = \frac{\Gamma(b - a + \alpha - \beta - 1)}{\Gamma(b - a)} > 0.
\]
The equation \((\alpha - 1)(b - s + \alpha - \beta) - (\alpha - \beta - 1)(s - a + \alpha - 3) = 0\) has the solution
\[
s = \frac{(a + b + 3)(\alpha - \beta - 1) + b\beta}{2\alpha - 2 - \beta},
\]
so we consider
\[
s = \left[ \frac{(a + b + 3)(\alpha - \beta - 1) + b\beta}{2\alpha - 2 - \beta} \right].
\]
If
\[
s \leq \left[ \frac{(a + b + 3)(\alpha - \beta - 1) + b\beta}{2\alpha - 2 - \beta} \right],
\]
the expression \((\alpha - 1)(b - s + \alpha - \beta) - (\alpha - \beta - 1)(s - a + \alpha - 3)\) is positive, and thus the expression \((b - s + 1)\alpha - \beta - 1(s - a - 1)\alpha - 1\) is increasing. If
\[
s \geq \left[ \frac{(a + b + 3)(\alpha - \beta - 1) + b\beta}{2\alpha - 2 - \beta} \right],
\]
the expression \((\alpha - 1)(b - s + \alpha - \beta) - (\alpha - \beta - 1)(s - a + \alpha - 3)\) is negative, and thus the expression \((b - s + 1)\alpha - \beta - 1(s - a - 1)\alpha - 1\) is decreasing. Hence the maximum of the expression \((b - s + 1)\alpha - \beta - 1(s - a - 1)\alpha - 1\) occurs at
\[
s = \left[ \frac{(a + b + 3)(\alpha - \beta - 1) + b\beta}{2\alpha - 2 - \beta} \right].\]
Thus, we have
\[
\max_{(t,s)\in \mathbb{N}^b_{a+1}\times \mathbb{N}^b_{a+2}} G(b,\beta; t, s) = \max_{s\in \mathbb{N}^b_{a+2}} G(b,\beta; s-1, s) = \Omega. \tag{3.9}
\]

(ii) Suppose \((\alpha - 1) < \beta \leq 1\). First, we maximize \(G(b,\beta; s, s)\) for \(s \in \mathbb{N}^b_{a+2}\). Consider the first order nabla difference of \(G(b,\beta; s, s)\) with respect to \(s\).

\[
\nabla_s [G(b,\beta; s, s)] = \frac{1}{\Gamma(\alpha)(b-a)^{\alpha-1}} \nabla_s [(b-s+1)^{\alpha-\beta-1}(s-a)^{\alpha-1}]
\]

\[
= \frac{(b-s+2)^{\alpha-\beta-2}(s-a)^{\alpha-2}}{\Gamma(\alpha)(b-a)^{\alpha-1}} \times [(\alpha - 1)(b-s + \alpha - \beta) - (\alpha - \beta - 1)(s-a + \alpha - 2)].
\]

In this expression, \(\Gamma(\alpha) > 0\),

\[
(b-s+2)^{\alpha-\beta-2} = \frac{\Gamma(b-s+\alpha-\beta)}{\Gamma(b-s+2)} > 0,
\]

\[
(s-a)^{\alpha-2} = \frac{\Gamma(s-a+\alpha-2)}{\Gamma(s-a)} > 0,
\]

and

\[
(b-a)^{\alpha-\beta-1} = \frac{\Gamma(b-a+\alpha-\beta-1)}{\Gamma(b-a)} > 0.
\]

The equation \((\alpha - 1)(b-s + \alpha - \beta) - (\alpha - \beta - 1)(s-a + \alpha - 2) = 0\) has the solution

\[
s = \frac{(a+b+3)(\alpha - \beta - 1) + b\beta + 1}{(2\alpha - 2 - \beta)},
\]

so we consider

\[
s = \frac{(a+b+3)(\alpha - \beta - 1) + b\beta + 1}{(2\alpha - 2 - \beta)}.
\]

If

\[
s \leq \frac{(a+b+3)(\alpha - \beta - 1) + b\beta + 1}{(2\alpha - 2 - \beta)},
\]

the expression \((\alpha - 1)(b-s + \alpha - \beta) - (\alpha - \beta - 1)(s-a + \alpha - 2)\) is positive, and thus the expression \((b-s+1)^{\alpha-\beta-1}(s-a)^{\alpha-1}\) is increasing. If

\[
s \geq \frac{(a+b+3)(\alpha - \beta - 1) + b\beta + 1}{(2\alpha - 2 - \beta)},
\]

the expression \((\alpha - 1)(b-s + \alpha - \beta) - (\alpha - \beta - 1)(s-a + \alpha - 2)\) is negative, and thus the expression \((b-s+1)^{\alpha-\beta-1}(s-a)^{\alpha-1}\) is decreasing. Hence the maximum of the expression \((b-s+1)^{\alpha-\beta-1}(s-a)^{\alpha-1}\) occurs at

\[
s = \frac{(a+b+3)(\alpha - \beta - 1) + b\beta + 1}{(2\alpha - 2 - \beta)}.\]
Thus, from (3.9), we have

$$
\max_{(t,s) \in \mathbb{N}_a^b \times \mathbb{N}_{a+2}^b} G(b, \beta; t, s) = \max \left\{ \max_{s \in \mathbb{N}_{a+2}^b} G(b, \beta; s-1, s), \max_{s \in \mathbb{N}_{a+2}^b} G(b, \beta; s, s) \right\}
= \max \{\Omega, \Lambda - 1\}.
$$

The proof is complete. \(\square\)

**Theorem 3.10.** The following inequality holds for \(G(b, \beta; t, s):\)

$$
\max_{t \in \mathbb{N}_a^b} \sum_{s=a+2}^b G(b, \beta; t, s) = \frac{(b-a-1)^{\alpha}}{(\alpha - \beta)\Gamma(\alpha)}.
$$

**Proof.** Consider

$$
\sum_{s=a+2}^b G(b, \beta; t, s)
= \sum_{s=a+2}^t G(b, \beta; t, s) + \sum_{s=t+1}^b G(b, \beta; t, s)
= \frac{1}{\Gamma(\alpha)} \sum_{s=a+2}^t \left[ \frac{(b-s+1)^{\alpha-1}}{(b-a)^{\alpha-1}}(t-a)^{\alpha-1} - (t-s+1)^{\alpha-1} \right]
+ \frac{1}{\Gamma(\alpha)} \sum_{s=t+1}^b \frac{(b-s+1)^{\alpha-1}}{(b-a)^{\alpha-1}}(t-a)^{\alpha-1}
= \frac{\Gamma(\alpha - \beta)(t-a)^{\alpha-1}}{\Gamma(\alpha)(b-a)^{\alpha-1} \Gamma(\alpha - \beta)} \sum_{s=a+2}^b \frac{(b-s+1)^{\alpha-1}}{\Gamma(\alpha)} - \sum_{s=a+2}^t \frac{(t-s+1)^{\alpha-1}}{\Gamma(\alpha)}
= \frac{(t-a)^{\alpha-1}}{(\alpha - \beta)\Gamma(\alpha)(b-a)^{\alpha-1} \Gamma(\alpha - \beta)} (b-a-1)^{\alpha-1} - (t-a-1)^{\alpha-1}
$$

We now find the maximum of this expression with respect to \(t \in \mathbb{N}_a^b\). Since

$$
\frac{(t-a-1)^{\alpha}}{\Gamma(\alpha + 1)} = \frac{\Gamma(t-a+\alpha-1)}{\Gamma(t-a-1)\Gamma(\alpha + 1)} \geq 0, \quad t \in \mathbb{N}_a^b,
$$

we have

$$
\max_{t \in \mathbb{N}_a^b} \sum_{s=a+2}^b G(b, \beta; t, s) = \max_{t \in \mathbb{N}_a^b} \frac{(b-a-1)(t-a)^{\alpha-1}}{(\alpha - \beta)\Gamma(\alpha)} = \frac{(b-a-1)^{\alpha}}{(\alpha - \beta)\Gamma(\alpha)}.
$$

The proof is complete. \(\square\)

We are now able to formulate a Lyapunov-type inequality for the discrete boundary value problem (1.2).
THEOREM 3.11. If (1.2) has a nontrivial solution, then

\[
\sum_{s=a+2}^b |q(s)| \geq \begin{cases} 
\frac{1}{\Omega}, & 0 \leq \beta \leq (\alpha - 1), \\
\frac{1}{\max\{\Omega, \Lambda-1\}}, & (\alpha - 1) < \beta \leq 1.
\end{cases}
\]

Proof. Let \( \mathfrak{B} \) be the Banach space of functions \( u : \mathbb{N}_a^b \rightarrow \mathbb{R} \) endowed with norm

\[
\|u\| = \max_{t \in \mathbb{N}_a^b} |u(t)|.
\]

It follows from Theorem 3.1 that a solution to (1.2) satisfies the equation

\[
u(t) = \sum_{s=a+2}^b G(b, \beta; t, s)q(s)u(s).
\]

Hence,

\[
\|u\| = \max_{t \in \mathbb{N}_a^b} \left| \sum_{s=a+2}^b G(b, \beta; t, s)q(s)u(s) \right| = \max_{t \in \mathbb{N}_a^b} \left| \sum_{s=a+2}^b G(b, \beta; t, s)q(s)u(s) \right|
\]

\[
\leq \max_{t \in \mathbb{N}_a^b+1} \left[ \sum_{s=a+2}^b G(b, \beta; t, s)|q(s)||u(s)| \right] \leq \|u\| \left[ \max_{t \in \mathbb{N}_a^b} \sum_{s=a+2}^b G(b, \beta; t, s)|q(s)| \right]
\]

\[
\leq \|u\| \left[ \max_{(t,s) \in \mathbb{N}_a^b+1 \times \mathbb{N}_a^b+2} G(b, \beta; t, s) \right] \sum_{s=a+2}^b |q(s)|,
\]

or, equivalently,

\[
1 \leq \left[ \max_{(t,s) \in \mathbb{N}_a^b+1 \times \mathbb{N}_a^b+2} G(b, \beta; t, s) \right] \sum_{s=a+2}^b |q(s)|.
\]

An application of Theorem 3.9 yields the result. \( \square \)

Now, we discuss two applications of Theorem 3.11. First, we obtain a criterion for the nonexistence of nontrivial solutions of (1.2).

THEOREM 3.12. Assume \( 1 < \alpha < 2 \) and

\[
\sum_{s=a+2}^b |q(s)| \leq \begin{cases} 
\Omega, & 0 \leq \beta \leq (\alpha - 1), \\
\max\{\Omega, \Lambda - 1\}, & (\alpha - 1) < \beta \leq 1.
\end{cases}
\]

Then, the discrete fractional boundary value problem (1.2) has no nontrivial solution on \( \mathbb{N}_a^b \).

Next, we estimate a lower bound for eigenvalues of the eigenvalue problem corresponding to (1.2).
THEOREM 3.13. Assume $1 < \alpha < 2$ and $u$ is a nontrivial solution of the eigenvalue problem
\[
\begin{aligned}
&\left( \nabla^\alpha_a u \right)(t) + \lambda u(t) = 0, \quad t \in \mathbb{N}_{a+2}^b, \\
u(a) = 0, \quad \left( \nabla^\beta_a u \right)(b) = 0,
\end{aligned}
\tag{3.11}
\]
where $u(t) \neq 0$ for each $t \in \mathbb{N}_{a+2}^{b-1}$. Then,
\[
|\lambda| \geq \begin{cases} 
\frac{1}{\Omega}, & 0 \leq \beta \leq (\alpha - 1), \\
\frac{1}{\max\{\Omega, \Lambda - 1\}}, & (\alpha - 1) < \beta \leq 1.
\end{cases}
\tag{3.12}
\]

Conclusion

In this article we established a Lyapunov-type inequality for (1.2) using the properties of the corresponding Green’s function. This inequality is a generalization of those Lyapunov-type inequalities obtained in [18, 19]. Two applications are provided to illustrate the applicability of established results.

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