Lipschitz Metric for the Novikov Equation

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Abstract

We consider the Lipschitz continuous dependence of solutions for the Novikov equation with respect to the initial data. In particular, we construct a Finsler type optimal transport metric which renders the solution map Lipschitz continuous on bounded sets of $H^1(\mathbb{R}) \cap W^{1,4}(\mathbb{R})$, although it is not Lipschitz continuous under the natural Sobolev metric from an energy law due to the finite time gradient blowup. By an application of Thom’s transversality theorem, we also prove that when the initial data is in an open dense subset of $H^1(\mathbb{R}) \cap W^{1,4}(\mathbb{R})$, the solution is piecewise smooth. This generic regularity result helps us extend the Lipschitz continuous metric to the general weak solutions. Our method of constructing the metric can be used to treat other kinds of quasi-linear equations, provided a good knowledge about the energy concentration.

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1. Introduction

Many evolutionary partial differential equations (PDEs) have the general form

\[ u_t + Lu = 0, \quad u(0) = u_0. \]

Classical well-posedness theory suggests the existence of a continuous semigroup of solutions, at least for a short time. For a large class of semi-linear PDEs, basic techniques such as Picard and Duhamel iteration can be applied to obtain Lipschitz continuity of the semigroup, that is, for any pair of solutions \( u, v \), it holds that

\[
\frac{d}{dt}\|u(t) - v(t)\| \lesssim \|u(t) - v(t)\|
\]

for a suitable (Sobolev) norm. Typical examples include the Korteweg-de Vries (KdV) equation, the nonlinear Schrödinger (NLS) equation, the semi-linear wave equation, and so on.

On the other hand, a noteworthy exception is provided by many quasi-linear equations. Due to the dominating nonlinearity, the initial information of the data can determine the later dynamics in a substantial way. In particular, solutions with smooth initial data can lose regularity in finite time, and one cannot in general expect (1.1) to hold true under the natural energy norms.

In this paper, we would like to address the issue of the Lipschitz continuity of the flow map using the following quasi-linear equation, namely the Novikov equation

\[
u_t - u_{xxt} + 4u^2u_x = 3uu_xu_{xx} + u^2u_{xxx}.\]

(1.2)

This equation was derived by Novikov [21] in a symmetry classification of nonlocal PDEs with cubic nonlinearity, and can in some sense be related to the well-known Camassa–Holm equation [8,12]. In fact writing the Novikov equation (1.2) in the weak form

\[
u_t + u^2u_x + \partial_x(1 - \partial_x^2)^{-1} \left( u^3 + \frac{3}{2}uu_x^2 \right) + (1 - \partial_x^2)^{-1} \left( \frac{1}{2}u_x^3 \right) = 0,\]

(1.3)

one may recognize the similarities with the Camassa–Holm equation, which, in a nonlocal form, reads

\[
u_t + uu_x + \partial_x(1 - \partial_x^2)^{-1} \left( u^2 + \frac{1}{2}u_x^2 \right) = 0.\]

(1.4)

Analytically, the Novikov equation also shares many properties in common with the Camassa–Holm equation, among which the two most remarkable features are the breaking waves and peakons. From examining the weak formulation (1.3), a transport theory can be applied to derive the blow-up criterion which asserts that singularities are caused by the focusing of characteristics. This in combined with