Alternative algebras with hyperbolic unit loops

Juriaans, S. O., Polcino Milies, C., Souza Filho, A.C.

Abstract

Let $K$ be a quadratic extensions of the field of rational numbers. We investigate the structure of an alternative finite dimensional $K$-algebra $A$ subject to the condition that for some $Z$-order $\Gamma \subset A$, the loop of units of $U(\Gamma)$ does not contain a free abelian subgroup of rank two. As a result, we give a complete classification of the finite and infinite $RA$-loops $L$ for which $KL$ has this property. In particular if $K = \mathbb{Q}(\sqrt{-d})$, we show that $L$ is the Cayley loop and $d \equiv 7 \pmod{8}$ is positive and square free. The complete classification for group rings is still an open problem.

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1 Introduction

Group rings $\mathbb{Z}G$ whose unit groups $U(\mathbb{Z}G)$ are hyperbolic were characterized in [8] in case $G$ is polycyclic-by-finite. A similar question was considered for $RG$, $R$ being the ring of algebraic integers of $K = \mathbb{Q}(\sqrt{-d})$ and $G$ a finite group (see [9]). In [6, 7], these results were extended to associative algebras $A$ of finite dimension over the rational numbers containing a $Z$-order $\Gamma \subset A$ whose unit group $U(\Gamma)$ is hyperbolic. An algebra $A$ with this property is said to have the hyperbolic property. Using these general results, the finite semigroups $S$ and the field $K = \mathbb{Q}(\sqrt{-d})$ such that $KS$ has the hyperbolic property were classified.

In this paper we study the same problem in the context of non-associative algebras, in special those which are loop algebras. A loop $L$ is a nonempty set with a closed binary operation $\cdot$ relative to which there is a two-sided identity element and such that the right and left translation maps $R_x(g) := g \cdot x$ and $L_x(g) := x \cdot g$ are bijections. $L$ is said to be hyperbolic if it does not contain a free abelian subgroup of rank two. This definition is an extension of the notion of hyperbolic group defined by Gromov [5] via the Flat Plane Theorem [2, Corollary III.1.3.10.(2)].

Here we characterize the $RA$-loops $L$ and the rings of integers $\mathfrak{o}_K$ of $K = \mathbb{Q}(\sqrt{-d})$ such that $U(\mathfrak{o}_KL)$ has the hyperbolic property.

In section 2, we fix notation and give definitions. For fields $K \subset \mathbb{Q}(\sqrt{-d})$, we focus on the alternative algebras of finite dimension over $K$ with the hyperbolic property and prove that the Cayley-Dickson algebra over $K$, where $d \equiv 7 \pmod{8}$ is a positive integer, has the hyperbolic property. In section 3, we prove a structure theorem for finite dimensional alternative algebras with the hyperbolic property. As a consequence we obtain that, for these algebras, the radical associates with the whole algebra. In the last section we present the main result, giving a full classification of

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those $RA$-loops $L$ whose unit loop $U(\mathbb{Z}L)$ is hyperbolic and extend this also to $\mathfrak{o}_gL$. It is important to notice that this problem is not yet completely settled for groups.

2 The Hyperbolic Property

$D = \{ d \in \mathbb{Z} \setminus \{ -1, 0 \} : c^2 \mid d, c \in \mathbb{Z}, c^2 \neq 1 \}$ denotes the set of square free integers. For a field $K$ let $H(K) = \left\{ \alpha, \beta \in K : \alpha, \beta \in K \right\}$ be the generalized quaternion algebra over $K$, i.e., $H(K) = K[i, j : i^2 = -\alpha, j^2 = -\beta, ij = -ji =: k]$. The set $\{1, i, j, k\}$ is a $K$-basis of $H(K)$. Such an algebra is a totally definite quaternion algebra, which we will denote $K$. Thus the result follows from [6, Lemma 3.17].

Let $K$ be a field, of characteristic zero, and let $K$ be an alternative finite dimension $K$-algebra. We say $K$ has the hyperbolic property if there exists a $\mathbb{Z}$-order $\Gamma \subset K$ whose unit loop $U(\Gamma)$ is a hyperbolic loop.

For an associative finite dimensional $\mathbb{Q}$-algebra this property was coined the hyperbolic property (see [6]). We will use this name also in the non-associative setting.

Proposition 2.2. Let $\mathfrak{A}$ be an alternative finite dimension $\mathbb{Q}$-algebra such that $\mathfrak{A} \cong \mathfrak{S} \oplus \mathfrak{R}$, with $\mathfrak{R}$ being the radical of $\mathfrak{A}$. If $\mathfrak{A}$ has the hyperbolic property, then $\mathfrak{R}$ is 2-nilpotent. Furthermore, there exists $j_0 \in \mathfrak{R}$ such that $j_0^2 = 0$ and $\mathfrak{A} \cong \langle j_0 \rangle$ is the $\mathbb{Q}$-linear span of $j_0$ over $\mathbb{Q}$.

Proof. Let $x, y \in \mathfrak{A}$, by Artin’s Theorem the subalgebra generated by $x, y$ is an associative algebra. Thus the result follows from [6, Lemma 3.2 and Corollary 3.3] □

Definition 2.3. Let $\mathcal{A}$ be an algebra over a field $K$. An involution is a $K$-linear map $*: \mathcal{A} \to \mathcal{A}$ $a^* := \overline{a}$ satisfying $(a \cdot b)^* = b^* \cdot a^*$ and $(a^*)^* = a$. The map $n: \mathcal{A} \to K$, $n(a) := a \cdot \overline{a}$, is called a norm on $\mathcal{A}$.

We recall the Cayley-Dickson duplication process: Let $\mathcal{A}$ be a given $K$-algebra, with $\text{char}(K) \neq 2$, $\alpha \in K$ and $x$ an indeterminate over $\mathcal{A}$, such that $x^2 = \alpha$. The composition algebra $\mathfrak{A} = (\mathcal{A}, \alpha)$, is the algebra whose elements are of the form $a + bx$, where $a, b \in \mathcal{A}$, with operations defined as follows:

\begin{align*}
(+): & (a_1 + b_1x) + (a_2 + b_2x) := (a_1 + a_2) + (b_1 + b_2)x; \\
(\cdot): & (a_1 + b_1x) \cdot (a_2 + b_2x) := (a_1a_2 + \alpha b_2b_1) + (b_2a_1 + b_1a_2)x
\end{align*}

On $\mathfrak{A}$, we have a natural involution defined by $a + bx := \overline{a} - bx$.

The algebra $\mathfrak{A} = (K, \alpha, \beta, \gamma)$ is the composition algebra $(\mathcal{A}, \gamma)$, where $\mathcal{A} = (K, \alpha, \beta)$ is the generalized quaternion algebra $H(K) = (\alpha, \beta)$, with $\alpha, \beta \in K$. Writing $\mathfrak{A} = \{ u + vz : u, v \in \mathcal{A} \}$ we have that $B = \{ 1, x, y, xy \} \cup \{ z, xz, yz, (xy)z \}$ is a $K$-basis of $\mathfrak{A}$ with $x^2 = \alpha, y^2 = \beta, z^2 = \gamma$. Moreover $n(a_1 + a_2x + a_3y + a_4xy + a_5z + a_6xz + a_7yz + a_8xyz + a_9(axyz)(xy)z) = a_1^2 - a_2^2\alpha - a_3^2\beta + a_4^2\beta\gamma - a_5^2\gamma + a_6^2\alpha - a_7^2\alpha\gamma - a_8^2\beta\gamma - a_9^2(axyz)\alpha\beta\gamma$ is a norm.
Lemma 2.4. Let $\mathcal{A}$ be the Cayley-Dickson algebra $(\mathbb{K}, -1, -1)$, $\mathbb{K} = \mathbb{Q}(\sqrt{-d})$, $d \in \mathcal{D}$ be a quadratic rational extension and $\mathcal{O}_K$ the ring of algebraic integers of $\mathbb{K}$. The algebra $\mathcal{A}$ has the hyperbolic property if, and only if, $d$ is positive and $d \equiv 7 \pmod{8}$.

Proof. 

Since the quaternion algebra $H(\mathbb{K}) \cong (\mathbb{K}, -1, -1)$ over $\mathbb{K}$ is a subalgebra of $\mathcal{A}$, if $\mathcal{A}$ has the hyperbolic property, then for all $\mathbb{Z}$-order $\Gamma \subset H(\mathbb{K})$, the group $\mathcal{U}(\Gamma)$ does not contain a free abelian subgroup of rank two, in particular $\mathbb{Z}^2 \not\subset \mathcal{U}(H(\mathcal{O}_K))$. Therefore, by [9, Theorem 4.7], $d \equiv 7 \pmod{8}$ and $d > 1$. 

Conversely, since $d \equiv 7 \pmod{8}$ and $d$ is positive we claim that $\mathcal{A}$ is a division algebra. In fact, suppose $\mathcal{A}$ splits. By [3, Theorem I.3.4], $\mathcal{A}$ splits if, and only if, the equation $x^2 + y^2 = -z^2$ has non trivial solution in $\mathbb{K}$ and this yields also that $(\mathbb{K}, -1, -1)$ splits, but $(\mathbb{K}, -1, -1) \cong H(\mathbb{K})$, and the quaternion algebra $H(\mathbb{K})$ over $\mathbb{K}$ is a division ring if, and only if, $d$ is positive and $d \equiv 7 \pmod{8}$, contradicting the fact that $(\mathbb{K}, -1, -1)$ splits. Suppose there exists a $\mathbb{Z}$-order $\Gamma \subset \mathcal{A}$ with $\mathbb{Z}^2 \not\rightarrow \Gamma$. Hence there exists $u, v \in \mathcal{U}(\Gamma)$ such that $\langle u, v \rangle \cong \mathbb{Z}^2$. Let $\mathbb{L} := \mathbb{K}[u, v]$ be the ring generated by $\{u, v\}$ over $\mathbb{K}$, since $[\mathcal{A} : \mathbb{Q}] = 16$ and $\mathcal{A}$ is diassociative $\mathbb{L} = \mathbb{K}(u, v)$ is a field and there exists $\beta \in \mathcal{A}$ such that $\mathbb{L} = \mathbb{K}(\beta)$. Obviously $\beta$ is not central in $\mathcal{A}$, because $\mathbb{K} \subset \mathbb{L}$, hence there exist $\gamma \in \mathcal{A}, \gamma \beta \neq \beta \gamma$, therefore the algebra $H := L(\gamma) = K(\beta, \gamma)$ is a division ring. We claim that $H \cong H(\mathbb{K})$ the quaternion algebra over $\mathbb{K}$; in fact the algebra $F = \mathbb{Q}(\beta, \gamma)$ is a division ring. Computing the degree $[H : \mathbb{Q}] = [H : \mathbb{K}][\mathbb{K} : \mathbb{Q}] = [H : \mathbb{K}] \cdot 2$, but $\mathcal{A} \supseteq H$ because $H$ is associative, hence $[H : \mathbb{Q}]$ is at most 8. Since $F$ is a division ring $[F : \mathbb{Q}] \geq 4$. Since $u, v \notin \mathbb{K}$ thus $\beta \notin \mathbb{K}$, also $\mathbb{K}$ is the center of $H$ hence $\gamma \notin \mathbb{K}$; clearly $[H : \mathbb{K}] = [F : \mathbb{Q}]$ thus $[F : \mathbb{Q}] = 4$ and $F$ is a quaternion algebra over $\mathbb{Q}$, therefore $H \cong H(\mathbb{K})$. This last condition shows that there exists $\Gamma' \subset H$ and $\mathbb{Z}^2 \not\rightarrow \mathcal{U}(\Gamma')$, but by [9, Theorem 4.7] $H$ has the hyperbolic property, a contradiction. \hfill \Box

Definition 2.5 (Alternative Totally Definite Octonion Algebra). An alternative division algebra $\mathfrak{A}$ whose center is a field $\mathbb{K}$ is called an alternative totally definite octonion algebra if $\mathbb{K}$ is totally real and $\mathcal{B} = \{1, x, y, xy \} \cup \{z, xz, yz, (xy)z\}$ is a $\mathbb{K}$-basis of $\mathfrak{A}$, with $x^2 = -\alpha$, $y^2 = -\beta$, $z^2 = -\gamma$ and $\alpha, \beta, \gamma \in \mathbb{K}$ all totally positive elements. In this case we write $\mathfrak{A} := \mathbb{K}(-\alpha, -\beta, -\gamma)$.

One should compare our definition with those in [3] and [11, Chapter 3, Section 21]. An example of such an algebra is $(\mathbb{Q}, -1, -1, -1)$, which is non-split.

The alternative totally definite octonion algebra $\mathfrak{A} := \mathbb{K}(-\alpha, -\beta, -\gamma)$ is non-split: since $\alpha, \beta, \gamma$ are totally positive and $\mathbb{K}$ is totally real, the equation $x^2 + \alpha y^2 + \beta z^2 + \alpha \beta w^2 + \alpha \beta \gamma t^2 = 0, x, y, z \in \mathbb{K}$, has only the trivial solution. Thus, by [3, Theorem 3.4], $\mathfrak{A}$ is non-split.

Next we give a characterization of the alternative totally definite octonion algebras which is a naturally extension of [11, Lemma 21.3].

Theorem 2.6. Let $\mathfrak{A}$ be a non commutative alternative division algebra, finite dimensional over its center $\mathbb{K}$. Suppose that $\mathbb{K}$ is a number field, $\mathfrak{A} \subset \mathbb{K}$ its ring of algebraic integers and $\mathcal{O}$ a maximal order in $\mathfrak{A}$. Then the following are equivalent.

1. $SL_1(\mathcal{O})$, the loop of units in $\mathcal{O}$ having reduced norm 1, is finite;
2. $|\mathcal{U}(\mathcal{O}) : \mathcal{U}(\mathcal{O}_K)| < \infty$;
3. $\mathfrak{A}$ is an alternative totally definite octonion algebra.
Proof. (1) \(\Rightarrow\) (2): The reduced norm \(\eta_1\) induces a map \(\varphi : U(\mathcal{D})/U(\mathfrak{o}_K) \rightarrow U(\mathfrak{o}_K)/(U(\mathfrak{o}_K))^2\). If \(xU(\mathfrak{o}_K) \in \ker(\varphi)\), then \(\varphi(xU(\mathfrak{o}_K)) = \eta_1(x)(U(\mathfrak{o}_K))^2 \in (U(\mathfrak{o}_K))^2\) and \(\eta_1(x) := \lambda^2 \in (U(\mathfrak{o}_K))^2 \cap \mathbb{K}\). Define \(z := \lambda^{-1}x \in U(\mathcal{D})\), clearly \(\eta_1(z) = 1\) thus \(\lambda^{-1}x \in SL_1(\mathcal{D})\) and the kernel of \(\varphi\) is \(SL_1(\mathcal{D})U(\mathfrak{o}_K)/U(\mathfrak{o}_K)\). Since we are dealing with finitely generated abelian groups we have that \(|U(\mathcal{D}) : U(\mathfrak{o}_K)| < \infty\).

(2) \(\Rightarrow\) (3): Let \(D\) be a division ring which is maximal in \(A\). Since \(|U(\mathcal{D}) : U(\mathfrak{o}_K)| < \infty\), if \(\Gamma \subset D\) is a \(\mathbb{Z}\)-order, then \(|U(\Gamma) : U(\mathfrak{o}_K)| < \infty\). By [11, Lemma 21.3], \(D\) is a totally definite quaternion algebra and \(\mathbb{K}\) is totally real. Let \(x \in A\) and \(B = (K, x)\). If \(D\) is a maximal subalgebra of \(A\) with \(B \subset D\), then \(B\) must be a quadratic field. Let \(D_0 < A\) be a maximal subalgebra and \(x_0 \in A \setminus D_0\). Then \(A = (D_0, x_0)\) and we may suppose that \(x_0^2 \in \mathbb{K}\). Since \(D_0\) is a totally definite quaternion algebra, then \(D_0 = (\omega_{0, b_0})\). Thus \(E := (\mathbb{K}, -a_0, -x_0)\), with \(x_0, a_0 \in \mathbb{K}\), is a totally definite quaternion algebra, hence \(A = E + E_{j_0}\), with \(j_0^2 = -b_0 \in \mathbb{K}\), is a totally definite octonion algebra.

(3) \(\Rightarrow\) (1) is a consequence of [11, Lemma 21.3].

Let \(P\) be a a theoretical group property. Recall that a group \(G\) is virtually \(P\) if it has a subgroup of finite index with the property \(P\). Also, if \(G\) and \(H\) are commensurable groups, then there exists subgroups \(K \leq G\) and \(L \leq H\), both of finite index, which are isomorphic.

Lemma 2.7. Let \(A = \mathbb{K}(\alpha, -\beta, -\gamma)\) be an alternative totally definite octonion algebra over a number field \(\mathbb{K}\), and \(\mathcal{D} \subset A\) a maximal \(\mathbb{Z}\)-order of \(A\). The unit loop \(U(\mathcal{D})\) is a hyperbolic loop if and only if \(\mathbb{K} \in \{Q, Q(\sqrt{-d}) : -d > 1\ a\ square\ free\ integer\}\).

Proof. If \(U(\mathcal{D})\) is a hyperbolic loop, then \(Z^2 \not\subset U(\mathcal{D})\). Suppose \(U(\mathcal{D})\) is finite. Let \(O \subset \mathbb{K}\) be an order of \(\mathbb{K}\), \(U(O) \subset U(\mathcal{D})\) is a finite subgroup. Since \(\mathbb{K}\) is totally real, by Dirichlet’s Unit Theorem, \(\mathbb{K} = Q\). Suppose \(U(\mathcal{D})\) is infinite. For \(a, b \in \{\alpha, \beta, \gamma\}, a \neq b\) we have, since \(\mathbb{K}\) is totally real, that the algebra \(A = \mathbb{K}(a, b)\) is a totally definite quaternion algebra. Let \(\mathfrak{o}_A \subset A\) be a \(\mathbb{Z}\)-order, then either \(|U(\mathfrak{o}_A)| = \infty\ or \mathbb{K} = Q\). In either case, since \(U(\mathfrak{o}_A)\) and \(U(\mathcal{D})\) are commensurable, \(Z^2 \not\subset U(\mathfrak{o}_A)\). By [11, item (b) of Lemma 21.3], \(|U(\mathfrak{o}_A) : U(O)|\) is finite, therefore \(U(O)\) is virtually cyclic, thus \(\mathbb{K} = Q(\sqrt{-d})\ where the integer \(-d > 1\).

Conversely, if \(\mathbb{K} = Q\), then \(U(\mathfrak{o}_K)\) is finite and, by the item (2) of the last theorem, \(U(\mathcal{D})\) is finite. If \(\mathbb{K} = Q(\sqrt{-d}), -d > 1\), then, by Dirichlet Unit Theorem, \(U(\mathfrak{o}_K) \cong Z\) and item (2) of last theorem yields that \(Z^2 \not\subset U(\mathcal{D})\). Hence \(U(\mathcal{D})\) is a hyperbolic loop.

3 A Structure Theorem

In this section we prove a structure theorem for alternative algebra with the hyperbolic property. This is not only a crucial step in the classification of the RA-loops whose integral loop algebra has the hyperbolic property but it should also be of independent interest.

Theorem 3.1. Let \(A\) be an alternative algebra of finite dimension over \(Q\), \(A_i\) a simple epimorphic image of \(A\), and \(\Gamma_i \subset A_i\) a \(\mathbb{Z}\)-order. Then

1. The algebra \(A\) has the hyperbolic property, is semi-simple and without non-zero nilpotent elements if, and only if, \(A = \bigoplus A_i\),

and for at most one index \(i_0\) the loop \(U(\Gamma_{i_0})\) is infinite and hyperbolic.
2. The algebra \( \mathfrak{A} \) has the hyperbolic property, is semi-simple with non-zero nilpotent element if, and only if,

\[
\mathfrak{A} = (\oplus \mathcal{A}_i) \oplus M_2(\mathbb{Q})
\]

and for all \( i \) the loop \( \mathcal{U}(\Gamma_i) \) is finite.

3. The algebra \( \mathfrak{A} \) has the hyperbolic property and is non-semi-simple with central radical \( J \) if, and only if,

\[
\mathfrak{A} = (\oplus \mathcal{A}_i) \oplus J, \dim_{\mathbb{Q}}(J) = 1
\]

and for all \( i \) the loop \( \mathcal{U}(\Gamma_i) \) is finite.

4. The algebra \( \mathfrak{A} \) has the hyperbolic property and is non-semi-simple with non-central radical if, and only if,

\[
\mathfrak{A} = (\oplus \mathcal{A}_i) \oplus T_2(\mathbb{Q}).
\]

For each of the items (1), (2), (3) and (4), at least one index \( j \) is such that \( \mathcal{A}_j \) is an alternative totally definite octonion algebra over \( \mathbb{Q} \). The components \( \mathcal{A}_i \) are one of the following algebras.

i. \( \mathbb{Q} \)

ii. a quadratic imaginary extension of \( \mathbb{Q} \)

iii. a totally definite quaternion algebra over \( \mathbb{Q} \)

iv. an alternative totally definite octonion algebra over \( \mathbb{Q} \) whose center, except for the index \( i_0 \) of item (1), has finitely many units.

Furthermore, in the decompositions of (1), (2), (3) and (4), every simple epimorphic image of \( \mathfrak{A} \) in the direct sum is an ideal of \( \mathfrak{A} \). It follows also that the radical associates with the whole algebra.

Now, we will work toward a proof of this theorem, proving it at the end of this section.

From now on, \( \mathbb{K} \) denotes the quadratic extension \( \mathbb{Q}\sqrt{-d} \), where \( d \in \mathcal{D} \). Let \( F \) be a field of characteristic \( \text{char}(F) \neq 2 \). The Cayley-Dickson algebras we refer here are 8-dimensional algebras constructed in the previous section. We shall start to look at the algebra \( (\mathbb{K}, -1, -1, -1) \) and to one of its \( \mathbb{Z} \)-order \( (\mathcal{O}_\mathbb{K}, -1, -1, -1) \).

A Cayley-Dickson algebra \( \mathcal{A} \) is a simple non-associative alternative ring which may have zero-divisors. If \( \mathcal{A} \) does not split, then it is said to be a division ring.

If \( \mathcal{R} \) is a ring, then we denote by \( \mathcal{Z}(\mathcal{R}) \), the \( \mathcal{Z} \)-orn vector matrix algebra over \( \mathcal{R} \). This is a split simple alternative algebra.

Clearly if \( \{ \theta_1, \theta_2 \} \) is a \( \mathbb{Z} \)-independent set of commuting nilpotent elements then \( \langle 1 + \theta_1, 1 + \theta_2 \rangle \cong \mathbb{Z}^2 \). We will use this in our next result.

**Proposition 3.2.** The \( \mathcal{Z} \)-orn vector matrix algebra over \( \mathbb{Q} \), \( \mathcal{Z}(\mathbb{Q}) \), does not have the hyperbolic property.

**Proof.** \( \Lambda = \mathcal{Z}(\mathbb{Z}) \) is a \( \mathbb{Z} \)-order of \( \mathcal{Z}(\mathbb{Q}) \) and if \( e_1 := (1, 0, 0) \) and \( e_2 := (0, 1, 0) \) then

\[
\theta_1 := \begin{pmatrix} 0 & e_1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \theta_2 := \begin{pmatrix} 0 & e_2 \\ 0 & 0 \end{pmatrix}.
\]

are 2-nilpotent element which are \( \mathbb{Z} \)-independent. The result now follows. \( \square \)
For a commutative and associative unital ring $R$, a loop $L$ and a group $G$, the loop ring $RG$ and the group ring $RG$ have been objects of intensive research (see [3, chap. III], [11, chap. 1], [12]).

We will concentrate on $RA$-loops, i.e., a loop $L$ whose loop algebra $RL$ over some commutative, associative and unitary ring $R$ of characteristic not equal to 2 is alternative, but not associative (see [3]).

$RA$-loops are Moufang Loops, i.e., loops satisfying any one of the following Moufang identities:

1. $((xy)x)z = x(y(xz))$ the left Moufang identity;
2. $((xy)z)y = x(y(zy))$ the right Moufang identity;
3. $(xy)(xz) = (x(yz))x$ the middle Moufang identity;

The following duplication process of a group results in Moufang loops. It turns out that all $RA$-loops are obtained in this way. Let $G$ be a nonabelian group, $g_0 \in Z(G)$ be a central element, $*: G \to G$ be an involution such that $g_0^* = g_0$ and $gg^* \in Z(G)$, for all $g \in G$, and $u$ be an indeterminate. The set $L = G \cup Gu =: M(G, *, g_0)$, with the operations

1. $(g)(hu) = (hg)u$;
2. $(gu)h = (gh^*)u$;
3. $(gu)(hu) = g_0h^*g$, 

is a Moufang Loop (see [3]).

A Hamiltonian loop is a non-associative loop $L$ whose subloops are all normal. A theorem of Norton gives a complete characterization of these loops ([3, Theorem II.8]). In what follows $E$ stands for an elementary abelian 2-group and $Q_8$ stands for the quaternion group of order 8.

**Proposition 3.3.** Let $G$ be a Hamiltonian 2-group and $L = M(G, *, g_0)$. Then $L$ is an $RA$-loop which is a Hamiltonian 2-loop and $U(L) = L$.

**Proof.** This is well known and [4, Theorem 2.3] is a good reference. \( \square \)

**Lemma 3.4.** Let $L$ be a finite $RA$-loop. If the algebra $QL$ has non-zero nilpotent elements, then $QL$ has a simple epimorphic image which is isomorphic to Zorn’s matrix algebra over $Q$.

**Proof.** This is well known and follows from [3, Corollary VI.1.3] and [3, Corollary VI.4.8]. \( \square \)

The last fact that we need is that alternative algebras are diassociative (see [9]).

**Proof.** (of Theorem 5.1): Suppose $A$ has the hyperbolic property, is semi-simple without non-trivial nilpotent elements. Let $\Gamma \subseteq A$ be a $Z$-order. We may suppose that $\Gamma = \bigoplus \Gamma_i$, where each $\Gamma_i \subseteq A_i$ is a $Z$-order, and so $U(\Gamma) = \prod U(\Gamma_i)$. If $U(\Gamma)$ is infinite, then is each group $U(\Gamma_i)$, thus by Theorem 2.6 and [11, Lemma 21.2], the components $A_i$ are determined. If $U(\Gamma)$ is infinite, then there exists a unique index $i_0$ such that $U(\Gamma_{i_0})$ is infinite, otherwise $U(\Gamma)$ is not hyperbolic. By the hypothesis we have that $U(\Gamma_{i_0})$ is a hyperbolic loop. Conversely, let $\Gamma \subseteq A$ be a $Z$-order and let $\sigma_{A_i}$ be the ring of algebraic integers of $A_i$. By hypothesis, $\Gamma_0 = \bigoplus \sigma_{A_i}$ is such that $U(\Gamma_0)$ is hyperbolic. Since $U(\Gamma)$ and $U(\Gamma_0)$ are commensurable, we have that $U(\Gamma)$ is a hyperbolic loop.

To prove item (2), we suppose that $A$ has the hyperbolic property, is semi-simple with non-trivial nilpotent elements. If $A_i$ is non-associative, then, by Proposition 3.2 each $A_i$ is a division algebra. Since the algebra $A$ has a non-zero nilpotent element, clearly there exists exactly one component
There exists a unique index $i$ that $U_i$ contains a copy of $A_i$. We observe that $GL_2(\mathbb{Z})$ is an infinite hyperbolic group. It now easily follows that $A$, whose components $A_i$ are prescribed, has the hyperbolic property.

Item (3): Proposition 2.2 assures that the radical $J$ has dimension 1 over $\mathbb{Q}$, $J = (j_0)_{\mathbb{Q}}$. Since $J$ is central and $(1 + j_0) \cong \mathbb{Z}$ it clearly follows that for each $A_i$ any $\mathbb{Z}$-order $\Gamma_i \subset A_i$. We must have that $U(\Gamma_i)$ is finite and $A_i$ is as described. The converse is also obvious.

Item (4): By [10, Theorem 3.18], $\mathfrak{K} \cong \mathfrak{S} \oplus \mathfrak{R}$ whose $\mathfrak{S} \cong \bigoplus_{i=1}^{N} A_i$. Assume $\mathfrak{K}$ has the hyperbolic property, then $\mathfrak{S} = \mathbb{Q}j_0$, where $j_0^{2} = 0$. Let $\{e_i/e_i \in A_i\}$ be the set of primitive central idempotents of $\mathfrak{K}$. For each idempotent $e_i$, $e_i \cdot j_0 \in \mathfrak{K}$ and hence, $e_i \cdot j_0 = \lambda_i j_0$. Since $e_i = e_i^2$ and $A_i$ is associative, we have $e_i \cdot (e_i \cdot j_0) = e_i \cdot j_0 = \lambda_i e_i$, also $e_i \cdot (e_i \cdot j_0) = e_i \cdot (\lambda_i j_0) = \lambda_i^2 e_i$, thus $\lambda_i^2 = \lambda_i$. Since $A_i$ is unitary, $1 = e_1 + \cdots + e_N$, we have that $j_0 = j_0(\lambda_1 + \cdots + \lambda_N)$ and hence $\sum \lambda_i = 1$. So, there exists a unique index $i$ such that $e_i \cdot j_0 = j_0$. Similarly, there exists a unique index $J$ such that $j_0 \cdot e_J = j_0$. Reordering indexes, we have that $e_1 \cdot j_0 = j_0 \cdot e_N = j_0$. Let $M$ be the annihilator of $j_0$ in $A_1$. It is easily seen that $M$ is closed under addition, multiplication and left and right multiplication by any element of $A_1$. To see this use the fact that, the radical being one dimensional, $a \cdot j_0 = j_0$, for some $\lambda \in \mathbb{Q}$, $\forall a \in A_1$. By [3, Proposition 1.11], $(xy)[x, y, z] = y(x[x, y, z])$, let $x \in A_1$, $y \in M$ and $z = j_0$, then $(xy)^2 \cdot j_0 = \lambda_j(j_0)j_0 = 0$. Since $[xy, xy, j_0] = 0$, we have that $(\lambda_j)^2 = \lambda_j(xy)^2 = 0$. Hence $\lambda_{xy} = 0$ and thus $xy \in M$. Similarly we prove that $yx \in M$. Using this, it is easily seen that associators belong to $M$ and hence $M$ is an ideal. It follows that either $M = A_1$ or $M = (0)$ and hence $M = A_1$.

Observe that if $a \in A_1$ and $a \cdot j_0 = \lambda_j j_0$ then $a - \lambda_j e_1 \in M$. It follows that $A_i = M \oplus \mathbb{Q} e_1$. From this and the fact that $M = 0$ if follows that $A_i$ is one dimensional. Similarly we prove that $A_N$ is one dimensional.

If $A_i$ is non-associative, then, by Lemma 3.4 and Proposition 3.2, $A_i$ contains no nilpotent elements. Consequently, for some $\mathbb{Z}$-order $\Gamma \subset QL$, we would have that $\mathbb{Z}^2 \not\cong U(\Gamma)$, a contradiction.

Finally, if $G$ is a Hamiltonian 2-group, then, by Proposition 3.3, $U(\mathbb{Z}L)$ is trivial.

4 RA-Loops with Hyperbolic Unit Loop $U(RL)$

In this section we classify the RA-loops $L$ and the ring of algebraic integers $\mathfrak{a}_K$ of a field $K$, such that the loop of units loop $U(RL)$ is a hyperbolic loop. We start to look at integral loop rings of finite RA-loops.

Lemma 4.1. Let $L$ be a finite RA-loop. The loop $U(\mathbb{Z}L)$ is hyperbolic if, and only if, $U(\mathbb{Z}L)$ is trivial.

Proof. As we saw before, there exists a non-abelian finite group $G$ such that $L = M(G, \ast, g_0) = G \cup G u$. Since $U(\mathbb{Z}L)$ is hyperbolic we have that $\mathbb{Z}^2 \not\cong U(\mathbb{Z}G)$. $G$ being finite and [8, Theorem 3.2] implies that $G \in \{S_3, D_4, C_3 \rtimes C_4, C_4 \rtimes C_4\} \cup \{M : M \text{ is a 2-Hamiltonian group}\}$. By [1, Theorem 3.1], $G' \cong C_2$ and hence $G \not\cong \{S_3, C_3 \rtimes C_4\}$. We also have that $G \not\cong \{D_4, C_4 \rtimes C_4\}$, since if this were the case then the algebra $QL$ would contain nilpotent elements and thus, by Lemma 3.4, $QL$ would contain a copy of Zorn’s matrix algebra. Consequently, for some $\mathbb{Z}$-order $\Gamma \subset QL$, we would have that $\mathbb{Z}^2 \not\cong U(\Gamma)$, a contradiction.

Finally, if $G$ is a Hamiltonian 2-group, then, by Proposition 3.3, $U(\mathbb{Z}L)$ is trivial.
We can now characterize RA-loops whose integral loop ring has the hyperbolic property.

**Theorem 4.2.** Let $L$ be an RA-loop. The loop $U(\mathbb{Z}L)$ is hyperbolic if, and only if, one of the following holds.

1. $L$ is a finite loop
2. $L$ is a loop whose center is virtually cyclic. $T(G)$, the torsion subloop of $G$ is abelian with its exponent dividing 4 or 6 and $T(L)$ is a Hamiltonian 2-loop (it can be a group) whose subgroups are all normal in $L$.

In either case we have that $U_1(\mathbb{Z}L) = L$.

**Proof.** The previous lemma deals with the finite case and so we only have to deal with the case when $L$ is infinite. Obviously we may suppose that $L$ is finitely generated and hence its torsion subloop, $T(L)$, is finite. It is known that the center $Z(L)$ is a finitely generated abelian group, and so $Z(L) \cong T(Z(L)) \times F$, where $T(Z(L))$ is a finite abelian group and $F$ an abelian torsion free group, (see [1], Lemma 2.1). $U(\mathbb{Z}L)$ being hyperbolic, gives us that $L$ is hyperbolic and thus $F$, and hence $Z(L)$, is virtually cyclic.

Choose an element $z_0 \in Z(L)$ of infinite order. If $U(Z(T(L)))$ is non-trivial then it contains an element $y_0$ of infinite order and hence $\langle x_0, y_0 \rangle$ is a copy of $\mathbb{Z}^2$. Hence we must have that $U(Z(T(L)))$ is trivial and, by the previous lemma, $T(L)$ is a hamiltonian 2-loop or 2-group. In particular $Z(T(L))$ does not contain nilpotent elements and hence all subgroups of $T(L)$ are normal in $L$ (this is a standard proof in group rings). So we proved have that $L$ is a finitely generated RA-loop whose torsion subloop $T(L)$ is a Hamiltonian 2-loop and all its subloops are normal in $L$. Therefore, by [3, Proposition X.1.3], $U(\mathbb{Z}L) = L[U(\mathbb{Z}(T(L)))] = LT(L) = L$, i.e., $U(\mathbb{Z}L)$ is trivial. Since $[L : Z(L)] = 8$ it follows that the unit group is also virtually cyclic.

We now look at the case $RL$, with $R = \sigma_K$ is the ring of algebraic integers of a quadratic extension.

**Theorem 4.3.** Let $L$ be a finite RA-loop and let $R = \sigma_K$ be the ring of algebraic integers of $K = \mathbb{Q}(\sqrt{-d})$, $d \in D$. The loop of units $U_1(RL)$ is hyperbolic if, and only if, $L = M_{16}(\mathbb{Q}s)$ and $d \in \mathbb{Z}^+$ with $d \equiv 7 \pmod{8}$.

**Proof.** Clearly $\mathbb{Z}L \subset \sigma_KL$ and thus $U_1(\mathbb{Z}L)$ is also hyperbolic. By the Lemma 4.1, $U(\mathbb{Z}L)$ is trivial and $L \cong M_{16}(\mathbb{Q}s) \times E \times A$. Since the hyperbolic loop $U(\sigma_KL) \supset U(\sigma_K\mathbb{Q}s)$, we have that $U(\sigma_K\mathbb{Q}s)$ is a hyperbolic group. Therefore $d \equiv 7 \pmod{8}$ and $K$ is an imaginary extension of $\mathbb{Q}$ (see [9, Theorem 4.7]). It follows that $E = A = 1$.

Conversely, it is well known that $\mathbb{K}L = \mathbb{K}(M_{16}(\mathbb{Q}s)) \cong 8 \cdot \mathbb{K} \oplus (\mathbb{K}, -1, -1, -1)$ ( [3, Corollary VII.2.3]). By Lemma 4.4, $(\mathbb{K}, -1, -1, -1)$ has the hyperbolic property. Since orders in $\mathbb{K}L$ have commensurable unit loops, we have that $U_1(\sigma_KL)$ is hyperbolic.

**Theorem 4.4.** Let $L$ be an RA-loop and $\sigma_K$ be the ring of algebraic integers of $K = \mathbb{Q}(\sqrt{-d})$, $d \in D$. The loop of units $U_1(\sigma_KL)$ is hyperbolic if, and only if, $L$ and $d$ are as follows:

1. $L = M_{16}(\mathbb{Q}s)$ and $d \equiv 7 \pmod{8}$, $d > 0$.
2. $L$ is an infinite virtually cyclic loop whose torsion subloop are all normal. Furthermore, $T(L)$ is an abelian group of exponent dividing 2, if $d > 0$, 4 if $d = 1$ and 6 if $d = 3$. 

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In each case we have that $U_1(o_K L) = L$.

Proof. The finite case is settled by the previous theorem and so we may suppose that $L$ is infinite and finitely generated. In particular its torsion subloop is finite.

Since $\mathbb{Z}L \subset o_K L$, we have that $U_1(ZL)$ is hyperbolic and hence, by Theorem 4.2, $T(L)$ is either a Hamiltonian Moufang 2-loop or an abelian group of exponent dividing 4 or 6, $L$ is virtually cyclic and has a central trivial unit $z_0$ of infinite order.

If $T(L)$ is a loop, then $U_1(o_K T(L))$ is hyperbolic. By Theorem 4.2, $T(L) = M_{16}(Q_8)$ and since $H(o_K)$ has non-trivial units of infinite order it follows that there exists $u_0 \in U(o_K T(L))$ of infinite order (see also [9, Theorem 5.4] or [12, Theorem 1.8.6]). Hence $\langle z_0, u_0 \rangle \cong \mathbb{Z}^2$, a contradiction and therefore $T(L)$ is a group. Theorem 4.2 guarantees that $T(L)$ is an abelian group of exponent dividing 4 or 6. By hypothesis and choice of $z_0$ we have that $U_1(o_K T(L))$ is trivial. In particular we have that $T(L)$ is an abelian group of exponent dividing 6 and $d = 3$ (see [9, Theorem 3.7]).

Conversely, if $T(L)$ is one of the groups of item (2), then $U_1(o_K T(L))$ is trivial ( [9, Theorem 3.7]). As we already observed in Theorem 4.2 we must have that $U_1(o_K L) = L$ and hence it is hyperbolic. □

In the proof of the previous theorem, we claimed the existence of a unit $u \in U_1(o_K M_{16}(Q_8))$ of infinite order which is given by [9, Theorem 5.4]. In fact, let $\epsilon := x + y\sqrt{d}$ be the fundamental invertible of $o_K$. We provide two explicit examples:

1. Taking $d = 7$ we have that $\epsilon = 8 + 3\sqrt{7}$. Take $u = 24\sqrt{7} - (24\sqrt{7})i - 63j + 64k$; then $u$ is a unit of infinite order and of augmentation 1 (see ( [12], Proposition 1.8.2);

2. For $d = 39$ we have that $\epsilon = 25 + 4\sqrt{39}$ and $v = 2\sqrt{39} - (2\sqrt{39})i - 12j + 13k$ has infinite order.

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Universidade de São Paulo, Instituto de Matemática e Estatística (IME-USP)
Caixa Postal 66281, São Paulo, CEP 05315-970 - Brasil
E-mails: ostanley@usp.br and polcino@ime.usp.br –

Universidade de São Paulo, Escola de Artes, Ciências e Humanidades (EACH-USP),
Rua Arlindo Béttio, 1000, Ermelindo Matarazzo, São Paulo, CEP 03828-000 - Brasil
E-mail: acsouzafilho@ime.usp.br