Topological phases from higher gauge symmetry in 3+1D

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We propose an exactly solvable Hamiltonian for topological phases in 3 + 1 dimensions utilising ideas from higher lattice gauge theory, where the gauge symmetry is given by a finite 2-group. We explicitly show that the model is a Hamiltonian realisation of Yetter’s homotopy 2-type topological quantum field theory whereby the groundstate projector of the model defined on the manifold \(M^3\) is given by the partition function of the underlying topological quantum field theory for \(M^3 \times [0,1]\). We show that this result holds in any dimension and illustrate it by computing the ground state degeneracy for a selection of spatial manifolds and 2-groups. As an application we show that a subset of our model is dual to a class of Abelian Walker-Wang models describing 3 + 1 dimensional topological insulators.

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Topological phases of matter have received considerable interest recently due to their practical applications related to various quantum Hall effect phenomena 14 and for the realisation of topological quantum computation 15 16. Topological phases of matter have an underlying effective (infra-red limit) description given by a Topological Quantum Field Theory (TQFT) 17 18 19. Such theories are independent of the metric structure of space time, so low-energy physical processes are insensitive to local perturbations. Amplitudes of these physical processes are global quantities, topological invariants of the configuration space. In a canonical approach, the Hamiltonian is a sum of mutually commuting constraints, so the groundstate space is their joint eigenspace 5, 10, 11. The degeneracy of the groundstate and types, fusion and braiding of possibly exotic excitations fully characterize such a theory 12 13. An important property of TQFTs is negative corrections to the experimentally observable entanglement entropy (due to global constraints on the correlations 5, 10).

In 2+1D, all phases of quantum matter with topological order are described by Chern-Simons-Witten/BF theories 17, (twisted) quantum double (QD) models 18 and Levin Wen string nets 19. In 3+1D there are few known examples of TQFTs. As such there is limited knowledge of the kinds of observables and quasi-excitations which could be expected to exist in 3+1D topological phases of matter. So far in 3+1D only the Dijkgraaf-Witten topological gauge theory 19 20 which describe symmetry protected topological phases and the Crane-Yetter TQFT 21 22 which describe topological insulators have been studied in the physical literature, (see also the very recent construction of TQFT based on a G crossed braided fusion category 23). Both TQFTs give observables which depend on at most the fundamental group and the signature of the space-time. They both support quasi-excitations given by point-particles with charge like quantum numbers and fermionic/bosonic mutual statistics and loop ex-
citations which carry both charge and flux like quantum numbers. In this article we will describe a Hamiltonian formalism for a third type of TQFT, the Yetter homotopy 2-type TQFT. Yetter’s TQFT uses a 2-group (equivalently a crossed module) to define a TQFT utilising the ideas of topological higher lattice gauge theory. Unlike the previous theories, such a theory is also sensitive to the homotopy 2-type information of the space-time, e.g. the second homotopy group. This feature is likely to be necessary to find non-trivial (meaning neither bosonic nor fermionic) representations of the loop braid group since the evolution of loop excitations are world-sheets.

The aim of this work is to better understand candidate theories for topological phases of matter in 3+1D space-time. Many questions understood in the 2+1D case are still open in 3+1D. For example, what is the relation between the ground-state degeneracy and the number of quasi-particle excitations? In 2+1D it is known that there is a 1-1 correspondence between the ground-state degeneracy and the number of irreducible quasi-particle excitations, but the analogue in 3+1D with the ground-state degeneracy on the 3-torus is still open. For example, what is the relation between the ground-state degeneracy and the number of irreducible quasi-particle excitations? In 2+1D it is known that there exist a 1-1 correspondence between the ground-state degeneracy and the number of irreducible quasi-particle excitations, but the analogue in 3+1D with the ground-state degeneracy on the 3-torus and the number of irreducible excitations is known not to hold. Another related question is what are the topological quantum numbers needed to classify the ground-states and quasi-excitations in 3+1D. Such quantum numbers are expected to be related to the generators of the mapping class group of the 3-torus SL(3, Z) but a full proof is still lacking. In this paper we will outline a model of topological phases of matter in 3+1D with a 2-group gauge symmetry. (In an accompanying article, we explicitly verify the mathematical consistency of such models, and also further discuss the topological observables and quantum numbers of the model.)

Defining Hamiltonian formalisms for TQFTs in 2+1D and 3+1D has been an effective strategy for understanding the spectrum of observables and physical manifestations of topological phases of matter. This is the approach taken in this manuscript, where we will present a Hamiltonian formalism for the Yetter TQFT in 3+1D. The structure of the text is as follows. We introduce the basic components of Higher Lattice Gauge Theory (HLGT) in section II A, including the definition of a crossed module (section II A). This serves as a framework for the remaining structure. We will then outline the Hamiltonian model in section II B. Section IV explains why our model is a Hamiltonian formalism of the 4D Yetter TQFT. Finally we describe an inclusion of a class of our Hamiltonian models into the class of Walker-Wang model (which form a Hamiltonian presentation of the Crane-Yetter TQFT) in section V.

II. HIGHER LATTICE GAUGE THEORY

In this section we will establish the lattice formulation of higher gauge theories. These are more complicated than ordinary lattice gauge theory. Instead of a group (the gauge group) on lattice edges, here we need two groups, the group of holonomies of ordinary and higher gauge fields. The latter types of holonomies sit on plaquettes and can be thought to arise from surface integral of a non-Abelian 2-connection. Beside the two groups, the physical edge/plaquette geometry induces two maps between them, which satisfy certain compatibility conditions. The collection of this data is called crossed module (crossed modules are actually equivalent to 2-group) and it replaces the notion of the gauge group in ordinary gauge theories. Just as the structure of the gauge group ensures that gauge-invariant and measurable quantities are independent of the choices made when defining the holonomy and the way holonomies are composed, it is the structure encoded in a crossed module which takes care of the same independence in a higher gauge theory.

For the proof of independence, some algebraic topology is needed. The proof will be published in a companion paper. Here we lean instead on existing work and only sketch the internal consistency of the theory we derive our Hamiltonian model from.

A. Crossed modules

Let $G$ and $E$ be groups, $\partial : E \rightarrow G$ a group homomorphism and $\triangleright$ an action of $G$ on $E$ by automorphisms (i.e., the maps $G \times E \rightarrow E$, $(g, e) \mapsto g \triangleright e$ are homomorphisms for both variables). If the Peiffer conditions

$$\partial(g \triangleright e) = g\partial(e)g^{-1} \quad \forall g \in G, \forall e \in E, \tag{1}$$
$$\partial(e) \triangleright f = efe^{-1} \quad \forall e, f \in E. \tag{2}$$

are satisfied then the tuple $(G, E, \partial, \triangleright)$ is called a crossed module.

An example is $(G, G, \text{id}, \triangleright)$ with $g \triangleright h = ghg^{-1}$, the double DG of the group $G$. Another example is $(G, \text{AUT}(G), \text{ad}, \triangleright)$, where AUT$(G)$ is the automorphism group of $G$. Here ad sends a $g \in G$ to conjugation by $g$, and the action is simply by evaluation of AUT$(G)$. If $V$ is a representation of a group $G$ then we can build a crossed module $(G, V, \partial, \triangleright)$ where $V$ is a group as a vector space, $\partial(V) = \{1_G\}$ and where $\triangleright$ is the given action of $G$ on $V$.

Now, we need a lattice, encoding the physical space of the theory.

B. Lattices and lattice paths

Given a manifold $x$ we write $bd(x)$ for the boundary and $(x)$ for $x \setminus bd(x)$. Given a set $K$ of subsets of a set we write $|K|_u$ for the union. Given a d-manifold $M$, a lattice $L$ for $M$ is a set of subsets $L^i$, for each $i = 0, 1, 2, 3$, where $x \in L^i$ is a closed topological i-disk embedded in $M$, satisfying the following requirements, with $M^i := |\bigcup_{j=0}^i L^j|_u$. 

For $i = 1, 2, 3$ and for $x, y \in L^i$ we have $bd(x) \subset M^{i-1}$; $(x) \cap y = \emptyset; (x) \cap M^{i-1} = \emptyset$.

Finally, either $d \leq 3$ and $M^3 = M$ or an extension of $L$ exists so that $M^d = M$, with all additional cells in $L^4$ or above.

- An element in $L^0$ is a point of $M$, called a vertex.
- An element in $L^1$ is called an edge or a track.
- An element in $L^2$ is called a face or a plaquette.
- An element in $L^3$ is called a blob.

A lattice for $M$ is essentially the same as a regular CW-complex decomposition for $M$ (a CW-complex is said to be regular if each attaching map is an embedding). Examples are triangulations and cubulations. To describe field configurations succinctly, we need to give extra structure to the lattice. Let us assume that the lattice does not contain 1-gons. For every element of $p \in L^2$ we distinguish the smallest vertex $v_0(p)$ and fix an orientation for $p$ according to which, for $p$ with $n > 2$ boundary edges, $v_1(p) < v_{n-1}(p)$ for the two neighbours of $v_0(p)$. The default target $t_p$ of $p$ is the edge whose source is $v_0(p)$, target is $v_{n-1}(p)$, the default source $s_p$ of $p$ is the path with consecutive boundary vertices $v_0(p), v_1(p), v_2(p), \ldots, v_{n-1}(p)$. In figures, if the target path of a face $p$ is an edge then we indicate the target edge by a double arrow, thus the default case is:

For simplicity here we exclude lattices with 2-gons in $L^2$. Let us call the lattice with the chosen total order dressed lattice.

A simple path from vertex $v$ to vertex $v'$ in $L$ is a path in the 1-skeleton $M^1$ without repeated vertices. Thus a simple path is a 1-disk in $M^1$ with its boundary decomposed into an ordered pair of vertices (the source and target). Similarly a 2-path is a disk surface $P$ in $M^2$ with $bd(P)$ decomposed into an ordered pair of simple paths.

C. Gauge fields

Given a crossed module $(G, E, \partial, \triangleright)$ and a dressed lattice $L$, a gauge field configuration is an analogue of a conventional one, which is encoded by a map $L^1 \to G$ assigning an element of gauge group $G$ to each edge of $L$. Beside 1-holonomies associated to directed paths, also 2-holonomies are associated to surfaces between two paths with common source and target. Here it is encoded by functions $L^1 \to G, i \mapsto g_i$ and $L^2 \to E, p \mapsto \epsilon_p$. More precisely, we associate an element of $G$ to each oriented edge and an element of $E$ to each face with reference source and target. We call them the 1- and 2-holonomy with given source and target, respectively. Let us assume that a specific oriented edge $i$ is determined by its vertices $v, w$ (with $v < w$). Then we may write $i = vw$, and $g_i = g_{vw}$.

We denote a complete ‘colouring’ of the dressed lattice with such reference-oriented data by $L_c$ and write $G^L = G^{L^1} \times E^{L^2}$ for the full set of colorings.

Given $i = vw \in L^1$, the 1-holonomy $g_{vw}$ with $w > v$ is given by $g_{vw} = g_{vw}^{-1}$. The 1-holonomy along a path in the 1-skeleton of the lattice is given by the multiplication of the group elements of subpaths (and hence eventually of $g_i$s, or their inverses, depending on the direction of the edge with respect to that of the path) along the path. Note that 1-holonomy is well-defined by (existence of inverses and) associativity of $G$. Similarly, we can compose 2-holonomies of disk-surfaces, where the target of one coincides with the source of the next. By (non-obvious) analogy with 1-holonomy, 2-path 2-holonomy is well-defined by the crossed module axioms. Below we explain the analogue of inverses and (more briefly) associativity.

First we will establish some notation. In figures, 2-holonomies of faces and other disk-surfaces will be depicted by double arrows, which point toward the reference target edge, just as for the 2-paths themselves. For a face $p$ we will call the 1-holonomy $g_{s_p}$ of the source $s_p$ the source of the 2-holonomy $e_p$. For example in the next figure we say that the 2-holonomy associated to the triangle have source $g_1g_2$ and target $g_3$.

Note, that our conventions for composing 1-holonomies throughout the paper is $g_1g_2$ for consecutive edges 1 and 2.

In the following we will adopt a restriction on the lattice: we assume that there can be at most one edge between two vertices and that a face is determined by its boundary vertex set. This is not strictly necessary, but the notation becomes simpler, we will use $vw$ for the unique edge oriented from vertex $v$ to $w$ and $wwu$ for the unique triangle with distinct boundary vertices $v, w, u$.

The terminology of source and target above comes from the axioms of 2-categories, which is the language used in Pfeiffer, for example, to define higher lattice gauge theories. In this paper, we will not define 2-categories, but simply write down rules from that formalism which we can use to define our model.

We are now ready to compute the 2-holonomy of an arbitrary 2-path. As already noted, it is intrinsic to the
notion of a gauge field that changing the direction of an edge is equivalent to changing the associated group element to its inverse. One can compute the 1-holonomy along a path in the 1-skeleton of the lattice by using these transformations to ensure that target of the 1-holonomy of an edge in the path coincides with the source of the next.

The 2-holonomy of a 2-path is constructed from the reference 2-holonomies of its plaquettes using a set of rules relating the reference 2-holonomy of each \( p \in L^2 \) with the 2-holonomy at \( p \) with different source and target. This way one multiplies all 2-holonomies of the elements of \( L^2 \) that are parts of the surface transformed appropriately so that the target of one is the source of the next.

The fact that the procedure is consistent and independent of the choices made will be explained in a companion paper\(^3\) (using the language of crossed modules of groupoids equivalent to that of 2-groupoids). Here we only illustrate the composition rules in indicative cases.

For each face \( p \in L^2 \) we define an 1-holonomy operator

\[
H_1(p) \equiv \partial e_p g_s p g_t^{-1} \tag{4}
\]

It is also called the fake curvature and it corresponds to the curvature 1-form\(^3\) of higher gauge theory\(^2\). In what follows, we will only consider configurations (unless otherwise stated) where \( H_1(p) = 1 \in G \) for \( p \in L^2 \). This is needed for consistency of the lattice formulation of 2d holonomy.

Let us write down the multiplication convention and the rules of changing source and target of 2-holonomies. We can multiply (compose) the 2-holonomy \( e \) with \( e' \) if \( t_e = s_{e'} (= g_2 \) in the figure):

\[
\begin{align*}
\begin{array}{c}
\begin{array}{c}
g_1 \downarrow e \\
g_2 \downarrow e' \\
g_3 \\
\end{array}
\end{array}
\end{align*}
= \begin{align*}
\begin{array}{c}
\begin{array}{c}
g_1 \downarrow e \\
g_3 \\
\end{array}
\end{array}
\end{align*}
\tag{5}
\]

We use the convention to multiply the group elements from right to left \( \tilde{e} = e' \cdot e \). The ‘whiskering’ rules\(^3\) for changing the source and target of a 2-holonomy are as follows:

- We can switch source and target of the 2-holonomy by changing \( e \) to \( e^{-1} \).
- We can change the direction of the source \( g_s \in G \) and target \( g_t \in G \) of the 2-holonomy simultaneously by changing \( e \) to \( g_s^{-1} \cdot e^{-1} \).
- We can change the source of both \( s_e \) and \( t_e \) simultaneously and also the target of both as shown in the figure.

Note that the fake flatness \( H_1(p) = 1 \in G \) (the lhs. is defined by (1)) of the face \( p \) is a crucial condition for consistency of the above: changing the basepoint (the source of the source and target) of the 2-holonomy around the boundary of face \( p \) back to the beginning gives

\[
\begin{align*}
g_n g_{n-1}^{-1} \cdots g_1^{-1} & \triangleright e_p = \partial e_p \triangleright e_p = e_p e_p e_p^{-1} = e_p
\end{align*}
\]

by the second Peiffer condition of crossed modules\(^2\).

Note that the fake flatness \( H_1(p) = 1 \in G \) (the lhs. is defined by (1)) of the face \( p \) is a crucial condition for consistency of the above: changing the basepoint (the source of the source and target) of the 2-holonomy around the boundary of face \( p \) back to the beginning gives

\[
\begin{align*}
\begin{array}{c}
\begin{array}{c}
g_1 \downarrow e \\
g_2 \\
g_3 \downarrow e^{-1} \\
g_2 \downarrow e \\
g_3 \downarrow e^{-1} \\
\end{array}
\end{array}
\end{align*}
= \begin{align*}
\begin{array}{c}
\begin{array}{c}
g_1 \downarrow e \\
g_3 \downarrow e^{-1} \\
g_2 \downarrow e \\
\end{array}
\end{array}
\end{align*}
\tag{7}
\]

The next example illustrates the composition of 2-holonomies associated to faces with a common boundary edge, where the condition of matching first target and second source is not satisfied.
Using the whiskering rules above, we can change the target and source of the top 2-holonomy \( e_q \) to \( t(e'_q) = g_{v_0 v_k} g_{v_k v_{n-1}} v_{n-1} \) and \( s(e'_q) = g_{v_0 v_k} h g_{v_k v_{n-1}} \), respectively, by changing \( e_q \) to \( e'_q \equiv g_{v_0 v_k} e^{-1} \), so the holonomy of the big disk \( r = p \cup q \) with \( s(r) = g_{v_0 v_k} h g_{v_k v_{n-1}} \) and \( t(r) = g_{v_0 v_{n-1}} \) is
\[
e_r = e_p ( g_{v_0 v_k} e^{-1} ) .
\]

Note, that \( g_{v_{n-1}} \) for a non-adjacent vertex pair \((v, w)\) stands for the 1-holonomy along the boundary of the face according to its circular orientation.

We could use the whiskering rule to the direction opposite to the orientation of \( p \), which results in
\[
(g_{v_0 v_k} \triangleright e_q^{-1}) e_p ,
\]
where \( g_{v_0 v_k} \triangleright e_q^{-1} \) denotes the 1-holonomy from \( v \) to \( w \) along the boundary of \( p \) in direction opposite to its circular orientation \((g_{v_0 v_k} = g_{v_0 v_{n-1}} g_{v_{n-1} v_{n-2}} \cdots g_{v_k v_{k+1}})\). Due to the fake flatness condition the expressions \([6] \) and \([9] \) agree:
\[
(g_{v_0 v_k} \triangleright e_q^{-1}) e_p = (\partial e_p g_{v_0 v_k} e^{-1}) e_p = e_p ( g_{v_0 v_k} e^{-1} ) e_p = e_p ( g_{v_0 v_k} e^{-1} ) e_p = e_p ( g_{v_0 v_k} e^{-1} ) e_p .
\]

In the first equality we used fake flatness \( \partial e_p = g_{v_0 v_k} g_{v_k v_{n-1}} \), in the second we used \([6] \).

Finally we give the formula for the 2-holonomy operator \( H_2(P) : G^L \to E \) in the case of a reference tetrahedron \( P \in L^2 \):
\[
P = \{ [abcd], s(P) = t(P) = ab, a < b < c < d \}
\]
\[
H_2(P) = e_{acde} e_{abc} (g_{ab} \triangleright e_{bca}) e_{abc}^{-1} .
\]

Using the multiplication convention we read the formula off from the figure
\[
H_2(P) = e_{acde} e_{abc} (g_{ab} \triangleright e_{bca}) e_{abc}^{-1} .
\]

Note that choosing the basepoint to be the lowest ordered of the constituting vertices \( a \), and the direction of the 2-holonomy associated to the triangle \( acd \) to be that of the 2-holonomy associated to the tetrahedron, makes the latter unambiguously defined. This fact follows from the work\([23] \) (as discussed there, it can be considered as a consequence of the Coherence Theorem for 2-categories).

III. THE HAMILTONIAN MODEL

Recall that given a crossed module \( G = (G, E, \partial, \triangleright) \) and a dressed lattice \( L \) we have a set \( G^L \) of gauge field configurations or ‘colourings of \( L \).’ We write \( \mathcal{H}^L \) for the ‘large’ Hilbert space which has basis \( G^L \) as a \( \mathbb{C} \)-vector space, and has the natural delta-function scalar product. We write \( | \otimes_{v \in L^1} g_v \otimes_{p \in L^2} e_p \rangle \) for a colouring regarded as an element of \( \mathcal{H}^L \).

We also have the subset \( G_p^L \) of \( G^L \) of fake flat colourings. The Hilbert space \( \mathcal{H} \) is defined using \( G_p^L \) as basis. These are the colourings satisfying the constraint at every face \( p \in L^2 \) that the fake curvature \([4] \) vanishes:
\[
H_1(p) = \partial e_p g_{v_p} e_p^{-1} =
\]
\[
\partial e_p g_{v_0 v_1} g_{v_1 v_2} g_{v_2 v_3} \cdots g_{v_{n-2} v_{n-1}} g_{v_{n-1} v_0} = 1_G .
\]

For example for a fake-flat colouring of \( L \) a ‘triangle’ we may choose \( e_p \in E \), \( g_{v_0 v_1}, g_{v_1 v_2} \in G \) arbitrarily, but then \( g_{v_2 v_3} = e_p g_{v_0 v_1} g_{v_1 v_2} \) is fixed. Thus \( \text{dim}(\mathcal{H}) = |G|^2 |E| \) here. The scalar product for \( \mathcal{H} \) is the one induced by that of \( \mathcal{H}^L \).

In the following we will often use a simplified notation \([L, L] \) for a basis element of the Hilbert space, with \( L \) denoting the dressed lattice and \( c \) its colouring. If clear from the context, we will also use this simplified notation for various dimensions and Hilbert spaces, for example for the QD model\([3] \) which corresponds to the finite ‘group’ crossed module with \( G \) and a dressed lattice \( L \), and a ‘group’ crossed module with \( G \) and a dressed lattice \( L \). We are now going to define operators in \( \text{End}(\mathcal{H}^L) \). We will show in Section IIIB that they restrict to \( \text{End}(\mathcal{H}) \). We will show how the 1- and 2-holonomy transform under their action, and hence show that they are gauge transformations. We adopt the latter terminology now.

An intuitive way to think about these operators is as follows. A vertex or 1-gauge transformation at vertex \( v \) is the analogue of ordinary \( G \) gauge transformation: edge labels change as they do in ordinary gauge theory. There, the 1-holonomies corresponding to boundaries of faces, also called Wilson loops, transform by conjugation, their traces are observables.

In higher gauge theory however, the ‘1-holonomy’ is already different: Compare \( g_{ab} g_{bc} g_{ca} \) with \( \partial(\partial_{abc}) g_{ab} g_{bc} g_{ca} \).
Since the 2-holonomy does not “really” (i.e. physically) change under 1-gauge transformations, the face labels are invariant except when the vertex \( v \) is the base-
point of the face. An edge transformation is a “pure” \( E \)
2-gauge transformation: it changes the 2-holonomy associated
to each face \( p \) adjacent to the edge. The action
on the face 2-holonomy \( e_p \) is the composition of an auxiliary
face 2-holonomy also adjacent to the edge, where the
source and target is appropriately modified to be composable
with \( e_p \). Here the quantities, which transform in a
covariant way, are the 2-holonomies associated to bound-
aries of blobs. The edge label to which the edge transformation
is associated also changes such that the auxiliary
bigon composed of the edge and the transformed edge is
fakley.

The transformation properties of 1-holonomies associated
to faces, \( H_1(p) \), and 2-holonomies associated
to blobs, \( H_2(P) \), will be discussed in the next subsection.
There the reader can also find figures illustrating the ef-
gect of the gauge transformations on a reference triangle.
The explicit transformation formulas for the 1-holonomy
of a triangle face and the 2-holonomy of a tetrahedron
are given in Appendix B.

Let us recall first the definition of left and right multi-
multiplication operators for a group \( G \)

\[
L^0: G \to G, \quad h \mapsto gh \\
R^0: G \to G, \quad h \mapsto hg^{-1}
\]

linearly extended to the group algebra \( CG \). In the follow-
ing we will use the notation \( L^0_i \) \( (R^0_i) \), \( i \in L^1, g \in G \) for
the linear operator in \( \text{End}(H^L) \), which acts as left (right)
multiplication by \( g \) on the tensor factor \( CG \) of \( H^L \)
corresponding the edge \( i \) and identity on all other factors —
i.e. ‘locally’ at \( i \). Similarly \( L^p_i \) \( (R^p_i) \) stands for the same

\[
\text{type of local operators in } \text{End}(H^L) \text{ acting on the tensor}
\text{factor } LE \text{ corresponding to the face } p \text{ and identity on all other factors.}
\]

We will also use \( g \triangleright_p (\cdot) \) for the op-
erator acting on the tensor factor \( E \) of \( H^L \) corresponding
to the face \( p \) as \( e_p \mapsto g \triangleright e_p \). The gauge transformation
associated to vertex \( v \) is defined by

\[
A^g_v = \prod_{i \in \ast(v)} L^0_i \prod_{p \in \ast^r(v)} L^p_i(p)
\]

where \( \ast(v) \) (\( \ast^r(v) \)) is the set of reference edges (faces)
adjacent to the vertex \( v \), respectively; the terms in the
product are defined as follows:

\[
L^0_i(i) = \begin{cases} 
L^0_i, & \text{if } v = s(i); \\
R^0_i, & \text{if } v = t(i).
\end{cases}
\]

\[
L^p_i(p) = g \triangleright_p (\cdot), \text{ if } v = v_0(p)
\]

and both families of operators act as identity in all other
cases. Note that all factors in the product of the
expression of \( A^g_v \) act on different tensor factors and their action
depend only on the parameter \( g \in G \), so they commute:

\[
A^g_v \otimes A^h_v = A^{gh}_v,
\]

\[
A^g_v \otimes A^h_v = A^h_v \otimes A^g_v, \quad \text{if } v \neq v',
\]

\[
A^f_i \otimes A^g_i = A^f_i \otimes A^g_i, \quad \text{if } i \neq i',
\]

\[
A^g_v \otimes A^f_v = A^f_v \otimes A^g_v, \quad \text{if } v = s(i).
\]

The proof of these identities are given in Appendix B.

\[
A^g_v \otimes A^h_v = A^{gh}_v,
\]

\[
A^g_v \otimes A^h_v = A^h_v \otimes A^g_v, \quad \text{if } v \neq v',
\]

\[
A^f_i \otimes A^g_i = A^f_i \otimes A^g_i, \quad \text{if } i \neq i',
\]

\[
A^g_v \otimes A^f_v = A^f_v \otimes A^g_v, \quad \text{if } v = s(i).
\]

The proof of these identities are given in Appendix B.

B. Covariance and 2-flatness constraints

We will now define the operators enforcing ‘2-flatness’
of boundaries of blobs \( x \in L^3 \).

At this point we restrict the lattice to be a triangu-
lation. The reason is of technical nature: for triangulations
all blobs are tetrahedra and we have an essentially unique
expression for their 2-holonomy given by \( \text{[12]} \).

Even for a cubic lattice, we would need to write a formula with
several distinct cases depending on the order of the ver-
tices on the boundary of cubes. From a physical point
of view, however, restricting ourselves to triangulations is not severe. (Nevertheless this restriction will be eliminated in our companion paper \cite{4} where we will prove that any lattice can be used.)

The linear operators enforcing 2-flatness (trivial 2-holonomy) of a tetrahedron $P(=bd(x))$ with $x \in L^3$; and fake-flatness, regarded as elements of $\text{End}(H^L)$ act on basis elements in $G^L$ by

$$B_P = \delta H_2(P), 1; \quad B_p = \delta H_1(p), 1.$$  \hfill (19)

Let us now check how the holonomies $H_1(p) = (\partial e_{abc}) g_{ab} g_{bc} g_{ac}^{-1}$ and $H_2(P) = e_{acd} e_{abc} (g > e_{bcd}) e_{abd}^{-1}$ transform under gauge transformations. The six gauge transformations ‘touching’ a triangle with vertices $a < b < c$ are depicted below. Caveat: Here and hereafter, we omit to record the changes to the configuration on the rest of the lattice.

\begin{align}
A_{e=0}^g | \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 a
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
 b
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
 c
\end{array}
\end{array}
\end{array} &= \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 a
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
 b
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
 c
\end{array}
\end{array}
\end{array} \quad \text{(20)}
\end{align}

\begin{align}
A_{e=0}^g | \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 a
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
 b
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
 c
\end{array}
\end{array}
\end{array} &= \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 a
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
 b
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
 c
\end{array}
\end{array}
\end{array} \quad \text{(21)}
\end{align}

\begin{align}
A_{e=0}^g | \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 a
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
 b
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
 c
\end{array}
\end{array}
\end{array} &= \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 a
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
 b
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
 c
\end{array}
\end{array}
\end{array} \quad \text{(22)}
\end{align}

\begin{align}
A_{e=0}^g | \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 a
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
 b
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
 c
\end{array}
\end{array}
\end{array} &= \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 a
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
 b
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
 c
\end{array}
\end{array}
\end{array} \quad \text{(23)}
\end{align}

\begin{align}
A_{e=0}^g | \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 a
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
 b
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
 c
\end{array}
\end{array}
\end{array} &= \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 a
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
 b
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
 c
\end{array}
\end{array}
\end{array} \quad \text{(24)}
\end{align}

\begin{align}
A_{e=0}^g | \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 a
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
 b
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
 c
\end{array}
\end{array}
\end{array} &= \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 a
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
 b
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
 c
\end{array}
\end{array}
\end{array} \quad \text{(25)}
\end{align}

Now we can compute the transformations of the 1- and 2-holonomy under all possible gauge transformations of the
triangle (three vertex gauge transformations and three edge gauge transformations) and the tetrahedron (four vertex gauge transformation and six edge gauge transformations). The formulas are given in Appendix B. An immediate corollary of them is that denoting the transformed quantities by tilde we have

\[ H_1(p) = 1 \iff \tilde{H}_1(p) = 1, \quad p \in L^2 \]  

(26)

Another consequence is

\[ [B_p, A^\mu_v] = [B_p, A^\mu_i] = 0. \]  

(27)

By (26), operators \( A^\mu_v, A^\mu_i \) restrict to elements of \( \text{End}(\mathcal{H}) \) (preserve \( H_1(p) = 1 \) with \( p \in L^2 \)). When restricted we also have

\[ H_2(P) = 1 \iff \tilde{H}_2(P) = 1, \quad P = \partial x, \quad x \in L^3 \]  

(28)

Thus, when restricted to \( \mathcal{H} \), they are gauge transformations (transform covariantly).

Hereafter we understand the operators as restricted to \( \mathcal{H} \). Consequently the commutation relations

\[ [B_p, A^\mu_v] = [B_p, A^\mu_i] = 0 \]  

(29)

hold for all possible values of the parameters.

C. The Hamiltonian

Now we define the vertex operators \( A_v \) and edge operators \( A_i \) as

\[ A_v = \frac{1}{|G|} \sum_{g \in G} A^g_v, \quad A_i = \frac{1}{|E|} \sum_{e \in E} A^e_i. \]

Using relation (13) we can verify that \( A_v \) is a projector operator, that is \( A^2_v = A_v \):

\[ A^2_v = \frac{1}{|G|^2} \sum_{g, h \in G} A^g_v A^h_v = \frac{1}{|G|^2} \sum_{g, h \in G} A^{gh}_v = \frac{1}{|G|^2} \sum_{g', h' \in G} A^{g'h'}_v = A_v. \]

Similarly, using (15) one obtains \( A^2_i = A_i \). Also, they satisfy the following commutation relations

\[ [A_v, A_i] = 0 \]  

(30)

\[ [A_v, A_{v'}] = 0 \]  

(31)

\[ [A_i, A_{v'}] = 0 \]  

(32)

for any vertices \( v, v' \) and edges \( i, i' \). In fact, the relations (31) and (32) follow immediately from relations (14) and (16), respectively. In the same way, (30) follows from (17) if \( v \neq s(i) \). If \( v = s(i) \), using (18) we obtain

\[ \sum_{e \in E, g \in G} A^g_v A^e_{i'} = \sum_{e' \in E, g' \in G} A^{g'e'}_v A^g_i \]

where in the last equality we used the fact that the map \( g \mapsto \cdot \) is a bijection. Then it is clear that (30) holds. Let us now consider the multiplicative operators \( B_p = \delta_{H^2(p), 1} \) defined above. It is clear that they mutually commute and they are projections. It is also clear that they commute with the projections \( A_v \) and \( A_i \) due to (29). Now we can write down a Hamiltonian of higher lattice gauge theory in terms of mutually commuting operators in \( \text{End}(\mathcal{H}) \)

\[ H = -\sum_v A_v - \sum_i A_i - \sum_p B_p \]  

(33)

where the summations run over vertices \( v \in L^0 \), edges \( i \in L^1 \) and blob boundaries \( P = \partial x, x \in L^3 \).

Note that one can define a Hamiltonian on \( \mathcal{H}^e \) with the same groundstate sector by including the 1-flatness constraint operators:

\[ H' = -\sum_v A_v - \sum_i A_i - \sum_p B_p \prod_p B_p - \sum_p B_p \]  

(34)

where summation and product over \( p \) mean over elements of \( p \in L^2 \). The reason for the modified form of the \( B_p \) operators is that without the multiplier that enforces fake-flatness, they would not commute with the gauge transformations (as can be seen from the transformation of \( H_2(p) \) under \( A^e_{g'} \) of the tetrahedron \( P = a < b < c < d \), see Appendix B. This way \( H' \) is also a sum of mutually commuting projections and the groundstates of \( H \) and \( H' \) agree.

In the restriction to a two-manifold and crossed module \( \mathcal{G} = (G, 1_G, \partial, \cdot) \) we note the model \( H' \) reproduces the Kitaev quantum double\(^{23}\) Hamiltonian for group \( G \). The second term is zero due to the triviality of \( E \) and the third term does not enter the equation as there are no blobs in \( d = 2 \), so we have

\[ H = -\sum_v A_v - \sum_p B_p. \]  

(35)

From this connection, the first term in (33) can be seen naively as the Gauss constraint and the last term as the magnetic constraint. Caveat: Unless fake-flatness constraints are imposed, the 2-holonomy of a blob is ambiguous: the formula (12) depends on the choice of the composition of the 2-holonomies of the boundary faces of the blob\(^{31}\).

IV. RELATION TO THE 4D YETTER TQFT

A. Ground state projection as a 4D state sum

In this chapter we will relate our Hamiltonian model in three space dimensions to the Yetter TQFT in four dimensions. The fact that one can associate a \( d \)-dimensional Hamiltonian lattice model to a \( D = d + 1 \) dimensional TQFTs is not new. For \( d = 2 \) Kádár et al. showed\(^{20}\) that the Levin-Wen model\(^{10}\) is the Hamiltonian

\[ \sum_{e \in E, g \in G} A^g_v A^e_{i'} = \sum_{e' \in E, g' \in G} A^{g'e'}_v A^g_i \]

...
version of the Turaev-Viro (TV) TQFT\cite{TV} in that the groundstate projection of the former defined on the two dimensional lattice $L$ for $M^2$ is given by the TV state sum for the $M^2 \times [0,1]$. The rigorous proof was given by Kirillov\cite{Kirillov}.

A subset of the Levin-Wen Hamiltonian models was shown\cite{Levin-Wen} to be the the dual lattice description of those corresponding to BF gauge theories: the Quantum Double models of Kitaev\cite{Kitaev}. Here duality means Fourier transformation on the gauge group, as a result of which states are labeled by irreducible representations instead of group elements. Hence, the above statement has to hold in the dual description as well. We will state and prove it in this section and generalise to the case of our 3D model: its ground state projection is given by the appropriate 4D Yetter TQFT amplitude.

The general correspondence we will investigate has been known qualitatively. The $2+1$ BF-theory action with gauge group $SU(2)$ is equivalent to the Einstein-Hilbert one for Euclidean signature and zero cosmological constant as shown by e.g., Ooguri and Sasakura\cite{Ooguri-Sasakura}. On the other hand, in 1969, Ponzano and Regge derived the gravity action from the asymptotic form of the Wigner-Racah coefficients\cite{Ponzano-Regge}. These results were the motivation for several works in the $2+1$ quantum gravity literature, where the details of the correspondence were well understood for the case of $SU(2)$\cite{Adamo-Barron}: the Hilbert space is the state space of canonical quantum gravity, the TQFT is the corresponding state sum or spin foam model.

We will sketch the derivation of the TQFT state sum from a general BF gauge theory. Then we replace the continuous gauge group with a finite one and work out the correspondence for $d = 2, 3$ for lattice and higher lattice gauge theory (LGT, HLGT) and show the pattern, which arises for arbitrary dimension.

### B. Ordinary Pure Lattice gauge theory based on a finite group

Let us first consider the theory defined on an oriented manifold $M^D$ of dimension $D$ by the action with a compact Lie group $G$ and its Lie algebra $\mathfrak{g}$

$$S[B,A] = \int_{M^D} \text{tr} (B \wedge F(A))$$

with $F(A) = dA + [A,A]$ being the $(\mathfrak{g}\text{-valued})$ curvature of the connection $A$; $B$ a locally $\mathfrak{g}$ valued (D-2)-form; and $\text{tr}$ the Cartan-Killing form of $\mathfrak{g}$. The partition function (a map that associates a scalar to each manifold $M^D$) is defined formally in terms of the path integral

$$Z_{BF}(M^D) = \int DBDA e^{iS[B,A]}$$

with $DA$ and $DB$ standing for some measure over the space of connections and the $B$-field. To make sense of this formal expression there is a standard discretization procedure, see e.g., Oeckl\cite{Oeckl} or Baez\cite{Baez}. Here we will only sketch the procedure and write down the discrete version of the partition function. Let $\Delta$ be a dressed lattice of $M^D$ and $\Delta$ the dual complex (whose $k \leq D$ dimensional simplices are in one-to-one correspondence with the $D-k$ simplices of $\Delta$). We dress $\Delta$ similarly to before: we orient edges (which are dual to $(D-1)$-simplices of $\Delta$) and give circular orientation to faces and distinguish basepoints in each face. A gauge configuration is an assignment of a group element to each edge. For a face $p$ we define the exponentiated curvature by $g_p = \prod_{\epsilon \in \partial p} g_{\epsilon}$: the multiplication is done in the order along the chosen cyclic orientation starting at $v_0(p)$. The $B$-field, locally being a $D-2$ form, is naturally associated to dual faces. $F_p$ is the curvature valued associated to the dual face $p$. Now, the integral $\int dB_p e^{iB_p F_p}$ vanishes, unless the curvature vanishes, thus it can be replaced by $\delta_{g_p,1} = \delta_{g_p,1}$ in terms of the chosen variables: the vanishing of the curvature in the dual face $p$ is equivalent to the trivial holonomy along the boundary of the face. The expression for the discretized path integral reads

$$Z_{LGT}(M^D, \Delta) = \int \prod_{i \in \Delta^1} dg_i \prod_{p \in \Delta^2} \delta_{g_p,1} \ . \ (36)$$

where $\Delta^1$ ($\Delta^2$) is the set of edges (faces) of $\Delta$ (see\cite{LGT} for details of the discretization procedure). For compact Lie groups the measure $dg_p$ is the Haar-measure on $G$. The partition function is the sum of all colouring subject to the constraint that the holonomy $g_p$ around each face is trivial: the underlying connection is flat.

At this point there is no difference between using $\Delta$ or $\Delta$ for edge and face coloring. We will use the former.

Note that we still do not know whether $Z_{LGT} < \infty$ in general, but we are interested here in replacing $G$ with a finite group\cite{FiniteGroups}. For a finite group $G$, we will choose the measure $\int dg = \sum_{g \in G}$. The partition function needs to be normalized so that it is independent of the lattice used:

$$Z_{LGT}(M^D, \Delta) = |G|^{-|\Delta^0|} \{\text{admissible colourings of } \Delta\} , \ (37)$$

where admissibility means that all faces are flat: $\prod_{p \in \Delta^2} \delta_{g_p,1} = 1$ and the multiplicative factor ensures independence on the lattice. This formula holds in arbitrary dimension (it is proved in more general settings by Porte\cite{Porte} and Yetter\cite{Yetter} and Faria Martins / Porte\cite{Faria-Martins}.

Generalising to manifolds with boundary, we can seek to satisfy the (weak) cobordism property of a TQFT (partition functions become partition vectors with composition by dot product — see e.g. Martin\cite{Martin} (§2.1, §10.2)). For this the above definition has to be generalised. Let the boundary be a closed $d = D-1$ dimensional manifold with a lattice $\Delta^{(b)} \subset \Delta$. A boundary colouring is denoted by $\Delta^{(b)} (\text{so } c \text{ stands for a set } \{g_i \in G, i \in \Delta^{(b)}\})$. Let us call a colouring of $\Delta$ $c$-admissible if it restricts to $\Delta^{(b)}$ and if the flatness constraints for all faces are satisfied: $\prod_{p \in \Delta^2} \delta_{g_p,1} = 1$. The partition vector is a vector with
components indexed by the possible boundary colourings \(\Delta^{(b)}\), and a component then reads
\[
Z_{LGT}(M^D, \Delta, \Delta^{(b)}_c) = |G|^{\left|\Delta^0\right| - |\Delta^{(b)}_c|/2} \quad \{c\text{-admissible colourings of } \Delta \}\tag{38}
\]
This way the cobordism property of a TQFTs holds as follows. Let \(N^d\) be a closed submanifold of a manifold \(M^D\), such that \(N^d\) separates \(M^D\) into two unconnected components. Then write \(M^D_1\) and \(M^D_2\) for the closures of the two components of \(M^D \setminus N^d\). The manifolds \(M^D_1\) and \(M^D_2\) (with lattices \(\Delta^1, \Delta^2\), say) have homeomorphic and oppositely oriented boundaries \(N^d\) (with lattice \(L \subset \Delta^1, \Delta^2\)). Then the partition functions satisfy the cobordism property of TQFTs. That is, changing the notation as \(\int \prod_{l \in L^1} dg_i \to \sum_c\), we can write
\[
Z_{LGT} \left( (M^D \cup_{N^d} M^D_2), \Delta_1 \cup \Delta_2 \right) = \sum_c Z_{LGT}(M^D_1, \Delta_1, IC_c) Z_{LGT}(M^D_2, \Delta_2, IC_c)
\]
where \(M_1 \cup_{N^d} M_2\) denotes \(M^D\) (we think of it here as obtained from gluing \(M_1\) and \(M_2\) along \(N^d\)) and the prime in \(\sum_c\) indicates that flatness constraints have to be inserted for all \(p \in L^2\).

1. 2+1D lattice gauge theory with a finite group

Recall the 2+1D Kitaev QD Hamiltonian from \[35\]. The ground state projection reads
\[
P^K_{gs} = \prod_{v \in L^0} A_v \prod_{p \in L^2} B_p.
\]

**Theorem IV.1.** Let \(M^2\) be a 2-manifold with lattice \(L\) and let \(\Delta\) be the 3-dimensional lattice of \(M^2 \times [0,1]\) that restricts to \(L_0 \approx L_1 \approx L\) at the boundaries \(M^2 \times \{0\}\) and \(M^2 \times \{1\}\). Let the internal edge set be \(\{v \times [0,1]\}_{v \in L^0}\). Let \(L^1_j\) refer to \(L_j\) and \(L_{jc}\) the edge colourings of \(L^1_j\). Finally, we consider the QD Hilbert space based on \(L^1_j\) with states \(|L_{jc}\rangle\) and identify the Hilbert space bases on \(L_{j1}, j = 0,1\). Then
\[
Z_{LGT}(M^2 \times [0,1], \Delta, L_{0c} \cup L_{1c}) = \langle L_{1c} \mid P^K_{gs} \mid L_{0c} \rangle \tag{39}
\]

The proof is written in Appendix \[C\]. Note, that we made a choice for using the edge and face set of \(\Delta\) for defining the partition function. An alternative approach, more standard in the realm of the Turaev-Viro model\[34,49,53\], makes use of the edge and face set of the dual lattice \(\Delta^\ast\). Then, the minimal lattice, which restricts to \(L^1\) at the boundary is three translated copies of \(L\) connected with vertical edge\[43\]. That way, no Dirac-deltas are needed for boundary faces as those in the middle layer of \(L\) enforce flatness for the gauge equivalent boundaries \(L_{ic}\) and the proposition is stated identically to the Fourier dual (Turaev-Viro vs. Levin-Wen) case.

C. Higher lattice gauge theory based on a finite crossed module

Fix now a crossed module \((G, E, \partial, \triangleright)\) and a four manifold \(M^4\). Let us consider the theory given by the BFCG action\[39\]
\[
S[A, B, C, \Sigma] = \int_{M^4} (\text{tr}_g(B \wedge F_A) + \text{tr}_g(C \wedge G_\Sigma))
\]
where \(B\) is a \(G\)-valued 2-form, \(F_A = dA + [A, A]\) is the curvature of the connection \(A\), \(C\) is an \(E\)-valued 1-form and \(G_\Sigma = d\Sigma + A \triangleright \Sigma\) the curvature 3-form of the two-connection \(\Sigma\) corresponding to the gauge group \(E\). We will use the following form of the partition function
\[
Z_{BFCG}(M^4) = \int DA DB DC DG e^{\delta S[A, B, C, G]}
\]
whose discretized form defined on the dressed lattice \(\Delta\) of \(M^4\) is given by
\[
Z_{HGT}(M^4) = \int \prod_{i \in L^1} dg_i \prod_{p \in L^2} \delta_{H_1(p),1} \prod_{t \in L^3} \delta_{H_2(t),1}.
\]

For a finite group we can rewrite it analogously to \[37\] in LGT. Let a colouring be called admissible if all Dirac-delta constraints are satisfied: all faces are fake-flat and all blobs are 2-flat. We do the substitutions \(\int dg_i \to \sum_{g_i \in G}\) and \(\int dp \to \sum_{e_p \in E}\) and write
\[
Z_{HGT}(M^4) = \frac{|E|^{|\Delta^1| - |\Delta^0|}}{|G|^{\left|\Delta^0\right|}} |\{\text{admissible colourings of } \Delta \}| \tag{40}
\]
where the multiplicative factor ensures independence on the lattice. This is proved to be the same in arbitrary dimensions\[39,52,54\]. For manifolds with boundary, we need to modify the above similarly to the LGT case. Let the manifold \(M^4\) with boundary have a lattice decomposition \(\Delta\), and let \(\Delta^{(b)} \subset \Delta\) denote the boundary lattice. Let \(\Delta^{(b)}_c\) be a colouring of \(\Delta^{(b)}\): \(\{g_i \in G, i \in \Delta^{(b)}, e_p \in E, p \in \Delta^{(b)}\}\). We call a colouring of \(\Delta\) \(c\)-admissible if it is admissible in \(\Delta\) and restricts to \(\Delta^{(b)}_c\) on \(\Delta^{(b)}_c\). The components of the partition vector read:
\[
Z_{HGT}(M^4, \Delta, \Delta^{(b)}_c) = \frac{|E|^{|\Delta^1| - |\Delta^0|}}{|G|^{\left|\Delta^0\right|}} \frac{|\{\text{admissible colourings of } \Delta \}|}{|\{\text{c-admissible colourings of } \Delta \}|} \tag{40}
\]
This does not depend on the lattice decomposition of \(M^4\) extending a given lattice decomposition of the boundary\[53\].
Theorem IV.2. Let $\Delta_L \equiv \Delta$ be the lattice of $M^3 \times [0, 1]$ which restricts to $L_0 \simeq L_1 \simeq L$ at boundaries $M^3 \times \{0\}$ and $M^3 \times \{1\}$ and the internal edge set is $\{e \times [0, 1]\}_{e \in L_0}$. Let $L_j$ refer to $L_j$ and $L_{je}$ the colorings of those, where fake flatness is assumed for each face $p \in L_j^2$, $j = 1, 2$. Finally, we consider the Hilbert space $\mathcal{H}$ defined in section III based on $L$ with states $|L_c\rangle$ and identify the Hilbert spaces based on $L_j$, $j \in \{0, 1\}$ with it and consider the projection $\mathcal{P}^{B}_{gs}$ to the groundstate defined by

$$\mathcal{P}_{gs}^{B} = \prod_{v \in L^0} A_v \prod_{v \in L^1} A_i \prod_{p \in L^3} B_p$$

Then

$$Z_{\text{HGT}}(M^3 \times [0, 1], L_{0c}, L_{1c}) = \langle L_{1c}|\mathcal{P}^{B}_{gs}|L_{0c}\rangle \ . \quad (41)$$

In words, the groundstate projection of our 3D Hamiltonian model associated to $M^3$ is given by the Yetter TQFT amplitude on $M^3 \times [0, 1]$. The proof is in Appendix D.

D. Hamiltonians corresponding to lattice gauge theories

We will look at the correspondence in arbitrary dimension $d \geq 1$ for both ordinary and higher lattice gauge theories. We can observe the following.

- The fake flatness constraints of internal faces and 2-flatness of internal blobs of the $d + 1$ dimensional prism lattice are equivalent to the bottom and top layer of the prism being connected by gauge transformations. For ordinary gauge theory, the latter is equivalent to the flatness of internal faces.

- The 2-flatness (flatness) constraints of internal faces of the boundary lattice are the magnetic operators of the Hamiltonian (in ordinary gauge theory, respectively).

As a consequence, the Hamiltonian in the Hilbert space associated to the $d$ dimensional lattice $L$, whose groundstate projections are given by the corresponding $d + 1$ dimensional partition function is given by the following table, where the sign $-||-$ means the same formula as on its left for all terms to the right starting from the sign $[24]$

$$
\begin{array}{c|ccc|c}
 d & 1 & 2 & 3 & \cdots \\
 \hline 
 \text{LGT} & -\sum_{v \in L^0} A_v & -\sum_{v \in L^0} A_v - \sum_{p \in L^2} B_p & -||- & \cdots \\
 \text{HLGT} & -\sum_{v \in L^0} A_v - \sum_{i \in L^1} A_i & -\sum_{v \in L^0} A_v - \sum_{i \in L^1} A_i & -\sum_{v \in L^0} A_v - \sum_{i \in L^1} A_i - \sum_{p \in L^3} B_p & -||- & \cdots \\
\end{array}
$$

E. The ground state degeneracy

We compute here the groundstate degeneracy (GSD) for a few examples. This is given by the trace of the groundstate projection. By virtue of the theorems, it can also be computed from the invariant

$$\text{Tr}(\mathcal{P}^{B}_{gs}) = Z_{\text{HGT}}(M^d \times S^1) = |G|^{-|L^0|}|E|^{-|L^1|} \times \quad (42)$$

$$|\{\text{admissible colouring of } \Delta_L\}|$$

with $L$ being the lattice of $M^d$ and $L^i$ referring to its set of $i$-dimensional cells as before and $\Delta_L$ is the prism lattice whose top $L_1$ and bottom $L_0$ are identified. It applies to LGT too with the obvious modifications.

- $d = 1$. Here a minimal lattice of $S_1$ is the lattice with one edge with its source and target vertex identified. Let us first consider the case of ordinary lattice gauge theory; i.e $E$ is the trivial group. The GSD is by definition the number of gauge equivalence classes of admissible colourings of $S^1$. This clearly coincides with the number of conjugacy classes of $G$. We can also obtain this GSD as $Z_{\text{LGT}}(S^1 \times S^1)$, which is $|G|^{-1}$ times the number of colourings of the lattice of the torus; explicitly: $|G|^{-1} \{\phi : \pi_1(T^2) \rightarrow G\} = |\{(g, h) \in G^2 : gh = hg\}|$. So in this case, the equality (42) boils down to the well established fact from group theory that the number of conjugacy classes of a finite group equals the order of the group times its commuting fraction, the probability that two elements commute; i.e. $\frac{|G|}{|G|^{-1}} = \text{number of conjugacy classes of } G$.

In the general HGT case and also for $S^1$, looking at the lhs of (42), the GSD can be expressed as the
number of conjugacy classes of $G/\partial(E)$. The rhs. of (42) explicitly is: \(\frac{1}{|G|} \sum_{e \in \ker(\partial)} |\{(g, h, e) \in G \times G \times E : \partial(e) = [g, h]\}|.\) It is amusing the check these two coincide, which follows from the group theory fact stated in the above paragraph.

• $d = 2$. Here, the ordinary gauge theory model is well studied, the dimension of the ground-state for the manifold $M^2$ and gauge group $G$ is $|\{(\phi : \pi_1(M^2) \to G)\}/\sim|$, where $\sim$ means modulo an overall conjugation $\phi(g) \mapsto h\phi(g)h^{-1}, g \in \pi_1(M^2), h \in G$. That is, the GSD is the number of gauge equivalent classes of flat connections. For $M^2 = T^2$, this is well known to coincide with the number of irreps of the double DG. Computing the GSD from the partition function gives $\frac{1}{|G|} |\{(g_1, g_2, g_3) \in G^3, [g_i, g_j] = 1\}|$. Recalling that the irreps of the DG are in one-to-one correspondence of the irreps of the centraliser subgroups of the representatives of conjugacy classes of $G$, the equality is clear.

For HGT, consider $S^2$, with a cell decomposition with one single vertex and a unique 2-cells. (This would be a trivial case to consider for ordinary gauge theory.) In this case the GSD can be computed, looking at the lhs of (42) as being the cardinality of the set of orbits of the action of $G$ on $\ker(\partial)$. Computing the GSD from the rhs of (42) yields $\frac{1}{|G|} \sum_{e \in \ker(\partial)} |\{g \in G : g \circ e = e\}|$. Elementary tools from group actions tell us that these two coincide.

• $d = 3$. Let us consider the $T^3$ case, using the obvious cell decomposition of the cube, and then identifying sides. Fake flatness of the three distinct faces and 2-flatness of the cube read $([g_1, g_2] \equiv g_1g_2g_1^{-1}g_2^{-1}, g_1, g_2 \in G)$:

\[
[x, y] = \partial f, \quad [x, z] = \partial e, \quad [y, z] = \partial k,
\]

\[
f(yxy^{-1} \triangleright k^{-1}) (y \triangleright e) (yz^{-1} \triangleright f^{-1}) k = e
\]

Let the subset of $G^3 \times E^3$ defined by the joint solutions of the above equations be denoted by $S$ and consider the equivalence relation $\cong$ in $S$ generated by (with $a \in G$ and $e_x, e_y, e_z \in E$):

\[
(x, y, z, e, f, k)
\]

\[\cong (axa^{-1}, aya^{-1}, aza^{-1}, a \triangleright e, a \triangleright f, a \triangleright k),\]

\[\cong (x, \partial e y, z, e, (a \triangleright e) f e_y^{-1}, e_y k (z \triangleright e_y^{-1})),\]

\[\cong (x, y, \partial e z, k, y \triangleright e_z) e_z^{-1}, (x \triangleright e_z) k e_z^{-1}),\]

\[\cong (\partial e x, x, y, z, e_y e_x (z \triangleright e_x^{-1}), e_x f (y \triangleright e_x^{-1}) k),\]

The GSD is $|S/\cong|$. For instance consider the crossed module $DG = (G, G, \text{id}, \text{ad})$, (where ad stands for the conjugation action). Here GSD = 1, and this is easily computable from the rhs. of (42).

Another easily computable example is $(\mathbb{Z}_2, \mathbb{Z}_{2r}, \text{sgn}, \text{id})$, where sgn is the parity ($r \in \mathbb{N}_+$), and id denotes the trivial action. Here GSD = $r^3$. This is easily seen from the rhs. of (42): $x, y, z \in G$ are arbitrary, they determine $(e, f, k)$ via the fake flatness constraints and the 2-flatness of the cube holds by construction. We have three more cubes based on 2- and 3-faces, whose faces are again pairwise identified. The identical argument applies: the new face labels are determined by the fake flatness of the sides, so the four edges labels are arbitrary and all 6 face labels are determined by the commutators. So we have $|G|^4$ admissible colourings, $|L_0| = 1$ and $|L_1| = 3$. Another way to infer that GSD = $r^3$ is the following. Yetter’s state sum $Z_{\text{HGT}}(W)$, where $W$ is a closed manifold, depends only on the weak homotopy type of the underlying crossed module.[23] The crossed module $(\mathbb{Z}_2, \mathbb{Z}_{2r}, \text{sgn}, \text{id})$ is weak equivalent to $(\{0\}, \mathbb{Z}_r)$, where we consider the constant map $\partial : \mathbb{Z}_r \to \{0\}$. In general, considering a crossed module of the form $E = (1_E, E, \partial, \triangleright)$, where $\partial$ and $\triangleright$ are trivial maps, we have that $Z_{\text{HGT}}(S^1 \times S^1 \times S^1 \times S^1) = |E|^3$. The number of admissible colouring is $|E|^4$ since we can colour the 2-cells of the 4-cube with faces identified as we please.

V. RELATION TO WALKER WANG MODELS

In this section we discuss the relation between the Walker-Wang model[22] and our model. In particular we outline a duality map between our model with the finite crossed module $E = (1_E, E, \partial, \triangleright)$, where $\partial : E \to \mathbb{I}_E$ and $\triangleright$ is the identity and the Walker-Wang model based on the symmetric fusion category $\mathcal{M}(E)$, where $E$ is any finite Abelian group.

A. Walker-Wang Model

To begin, we briefly outline the Walker-Wang model[22]. The Walker-Wang model is a 3+1D model of string-net condensation with groundstates proposed to describe time-reversal invariant topological phases of matter in the bulk and chiral anyon theories on the boundary[23]. Such models are believed to be the Hamiltonian realisation of the Crane-Yetter-Kauffman TQFT[21] state sum models analogous to the relation between our model and the Yetter’s homotopy 2-type TQFT[23].

The Walker-Wang model is specified by two pieces of input data, a unitary braided fusion category (UBFC) $\mathcal{C}$ and a cubulation $C$ of a 3-manifold $M^3$. In the following we will define the generic model on a trivalent
The Walker-Wang model is defined on the trivalent cubic graph $\Gamma$ (see fig. 2) with directed edges. The Hilbert space has an orthonormal basis given by colourings of the directed edges of $\Gamma$ by labels from $\mathcal{L} = \{1, a, b, c, \cdots \}$. For each edge label $a \in \mathcal{L}$ there is a conjugate label $a^* \in \mathcal{L}$ which satisfy the relation $a^{**} = a$. We define the states such that reversing the direction of an edge and conjugating the edge label gives the same state of the Hilbert space as the original configuration. The label set $\mathcal{L}$ has a unique element $1 \in \mathcal{L}$ we call the vacuum which satisfies the relation $1 = 1^*$.

To specify the Hamiltonian we introduce the fusion algebra of the label set $\{1, a, b, c, \cdots \}$. A fusion rule is an associative, commutative product of labels such that for $a, b, c \in \mathcal{L}$, $a \otimes b = \sum_c N^c_{ab} c$. Here $N^c_{ab} \in \mathbb{Z}^+$ is a non-negative integer called the fusion multiplicity. In the following we will restrict to the case of “multiplicity free” which is the restriction $N^c_{ab} \in \{0, 1\}$ $\forall a, b, c \in \mathcal{L}$. The fusion multiplicities satisfy the following relations

$$\begin{align*}
N^c_{ab} &= N^c_{ba}, \\
N^1_{ab} &= \delta_{ab}, \\
N^0_{ab} &= \delta_{ab}, \\
\sum_{x \in \mathcal{L}} N^x_{ab} N^d_{xc} &= \sum_{x \in \mathcal{L}} N^d_{ax} N^x_{cd}.
\end{align*}$$

Given the label set and fusion algebra we define $d : \mathcal{L} \rightarrow \mathbb{R}$ such that $\forall a \in \mathcal{L}$, $d : a \rightarrow d_a$ and $d_\delta = d_a$. We will refer to $d_a$ as the quantum dimension of the label $a$. The quantum dimensions are required to satisfy

$$d_a d_b = \sum_c N^c_{ab} d_c.$$  \hfill (47)

Additionally we define $\alpha_i = sgn(d_i) \in \{\pm 1\}$ which satisfies

$$\alpha_i \alpha_j \alpha_k = 1$$ \hfill (48)

if $N^i_{jk}$.

Given the fusion algebra and quantum dimensions we define the $6j$-symbols which enforce the associativity of fusion of processes. The $6j$-symbols are a map $F : \mathcal{L}^6 \rightarrow \mathbb{C}$ which satisfy the following relations

$$\begin{align*}
F^{ijm}_{jm} &= \frac{v_{mj}}{v_{ij}} N^m_{ij}, \\
F^{ijm}_{jk} &= F^{km}{}_{jm} = F^{nk}{}_{jm}, \\
\sum_n F^{mlq}_{kp} F^{njq} F^{sr}{}_{ln} &= F^{qip} F^{rjq} F^{sir}, \\
\sum_n F^{mlq}_{kp} F^{qip} F^{rjq} F^{sir} &= \delta_{qip} \delta_{mlq} \delta_{k-ip}.
\end{align*}$$

where $v_a = \sqrt{d_a}$.

The final piece of data required to define the Walker-Wang model is the braiding relations or $R$-matrices. The $R$-matrices are a map $R : \mathcal{L}^2 \rightarrow \mathbb{C}$ which are required to satisfy the Hexagon equations which ensure the compatibility of braiding and fusion. The Hexagon equations are as follows

$$\begin{align*}
\sum_g R_{bc} R_{cg} F^{abg} F^{ecf} &= R_{ad} R_{bc} F^{acd} R_{bf}, \\
\sum_g F^{e-cd} R_{bd} F^{abc} &= R_{ac} F^{ecb} R_{bf}.
\end{align*}$$

The data $(\mathcal{L}, N, d, F, R)$ forms a UBFC. Examples of solutions to the above data are representations of a finite group or a quantum group.

Using the above data we can write down the Walker-Wang Hamiltonian. The Hamiltonian is of the following form

$$H = -\sum_{v \in \Gamma} A_v - \sum_{p \in \Gamma} B_p$$ \hfill (54)
where \( \Gamma \) is the directed, trivalent graph on which the model is defined and the \( v \) and \( p \) are the vertices and plaquettes of the graph. The plaquettes are defined with reference to the original square faces of \( C \) before the vertex resolution. The term \( A_v \) is the vertex operator and acts on the 3-edges adjacent to a vertex. We define the action of \( A_v \) on states as follows

\[
A_v \left| \begin{array}{c} a \\ e \\ b \end{array} \right> = \delta_{abc} \left| \begin{array}{c} a \\ e \\ b \end{array} \right> \tag{55}
\]

where \( \delta_{abc} = 1 \) if \( N_{ab}^c \geq 1 \) and \( \delta_{abc} = 0 \) else.

The plaquette operator \( B_p \) has a slightly more complicated form in terms of the 6j-symbols and \( R \)-matrices. Using Fig. 2 as the basis, \( B_p \) has the following form

\[
B_p^a = \sum_{a',b',c',d',e',f',g',h',i',j'} R_{a'a}^d R_{b'b}^{d'} R_{c'c}^{d''} R_{e'e}^{f'} R_{f'f}^{g'} R_{g'g}^{h'} R_{h'h}^{i'} R_{i'i}^{j'} \times |a',b',c',d',e',f',g',h',i',j'> (a,b,c,d,e,f,g,h,i,j) \tag{56}
\]

\[
B_p = \sum_{n \in \mathcal{E}} \frac{d_n}{|E|^2} B_p^n. \tag{57}
\]

We define the inner product of such states by

\[
\langle a, b, c, \ldots | a', b', c', \ldots \rangle = \delta_{aa'} \delta_{bb'} \delta_{cc'} \cdots. \tag{58}
\]

### B. The Symmetric Braided Fusion Category \( \mathcal{M}(\mathcal{E}) \)

Utilising the work of Bantay one can define a UBFC for every finite crossed module. Following this construction we will define the symmetric braided fusion category \( \mathcal{M}(\mathcal{E}) \) induced from the data of the finite crossed module \( \mathcal{E} = (1_E, E, \partial, \triangleright) \), where \( E \) is any finite Abelian group and \( \partial \) and \( \triangleright \) are trivial.

The label set of \( \mathcal{M}(\mathcal{E}) \) is given by elements of \( E \), with the vacuum label given by the identity element of \( E \) and \( a^* = a^{-1}. \) The quantum dimension \( d_n = 1 \) for all \( a \in E \) and \( D^2 = |E| \). The fusion multiplicities are multiplicity free with \( N_{ab}^c = \delta_{a+b,c} \) such that the fusion rules are given by the group composition rules (we use + for the group composition as \( E \) is an Abelian group) and \( a \otimes b = a + b \) for all \( a, b \in E \). We list the data of \( \mathcal{M}(\mathcal{E}) \) below.

\[
\mathcal{L} = \text{underlying set of } E
\]

\[
a \otimes b = a + b
\]

\[
d_n = 1 \quad \forall a \in \mathcal{L}
\]

\[
D^2 = |E|
\]

\[
N_{ab}^c = \delta_{a+b,c}
\]

\[
F_{klm}^{ij} = \delta_{i+j,m-1} \delta_{l+k,l,m} \delta_{l+i,n-1} \delta_{j+k,n} \]

\[
R_{i+j}^k = \delta_{i+j,k}
\]  \tag{59}

### C. Walker-Wang Models for \( \mathcal{M}(\mathcal{E}) \)

Utilising \( \mathcal{M}(\mathcal{E}) \) as defined in the previous section as the input data of the Walker-Wang model we may write the terms of the Hamiltonian as follows. The vertex operator acts on basis elements as

\[
A_v \left| \begin{array}{c} a \\ e \\ b \end{array} \right> = \delta_{a+b+c,0} \left| \begin{array}{c} a \\ e \\ b \end{array} \right> \tag{60}
\]

which energetically penalises configurations of labels around vertices which do not fuse to the identity object.

To define the plaquette operator we first choose an orientation of the plaquette (although the action of \( B_p \) is independent of the choice taken). In the following we choose an anti-clockwise convention and define \( \{ e^+\} \in p \) as the set of edges with direction parallel (anti-parallel) to the choice of orientation. We may then write the plaquette operator for \( n \in E \) as follows

\[
B_p^n = \left( \prod_{v \in p} A_v \right) \prod_{e^+ \in p} \Sigma_{e}^{n} \prod_{e^- \in p} \Sigma_{e}^{-n} \tag{61}
\]

where \( \Sigma_e^n \) acts on the label \( l \) of edge \( e \) such that \( \Sigma_e^n : l \mapsto l + n \). The operators \( \Sigma_e^n \) commute for all edges and \( \Sigma_{e^-}^m \Sigma_{e^+}^n = \Sigma_{e^+}^{m+n} \). The operator \( B_p \) in the Hamiltonian is then defined as

\[
B_p = \frac{1}{|E|} \sum_{n \in \mathcal{E}} B_p^n. \tag{62}
\]

As such an operator symmetrises over all group elements the action on basis states is independent of orientation convention for the plaquette.

As the model based on \( \mathcal{M}(\mathcal{E}) \) does not have any strict dependency on the trivalent lattice we may equally well resolve the trivalent vertices and define the model on a cubic lattice without changing the dynamics of the model. Under such a transformation the vertex operator becomes

\[
A_v \left| \begin{array}{c} a \\ e \\ b \end{array} \right> = \delta_{a+b+c+d+e+f,0} \left| \begin{array}{c} a \\ e \\ b \end{array} \right> \tag{63}
\]

while the plaquette operator takes the same form with the trivalent vertex operators replaced with the 6-valent counterpart.

### D. Yetter Model for \( \mathcal{E} \) on Cubic Lattice

As mentioned previously in the text the Yetter model can be equally be defined on any cellular decomposition of a 3-manifold. In this section we will outline the model with crossed module of the form of \( \mathcal{E} \) on the cubic lattice and show by considering the dual of the model that such
a model is equivalent to the Walker-Wang model of $\mathcal{M}(\mathcal{E})$ on the cubic lattice.

We begin by defining the Yetter model $\mathcal{E}$ on the cubic lattice following the general procedure outlined in section III. The first step is define an orientation to each square face of the lattice in analogy to the orientation of edges which is inherited from the vertex ordering. Following the previous definitions we choose to orient faces from the lowest ordered vertex on each face which we call the basepoint. We then assign the orientation relative to the two adjacent vertices to the basepoint such that the orientation points to the lowest ordered vertex adjacent to the basepoint. This is demonstrated in the left hand side of equation (64) where the face carries the group element $e \in \mathcal{E}$ and vertex $a$ is the basepoint and the orientation is given by the relation $a < i < j$. Reversing the orientation of the face replaces the face label with its inverse as shown in the right hand side of equation (64).

$$a^e_{ij} = a^e_{ij}^{-1}$$ (64)

Using the above conventions for the sign of face labels we can now define the 2-flatness condition of a cubic cell. As $\mathcal{E}$ is Abelian and only assigns the identity element to edges, the computation of the 2-holonomy is much simpler than in the general setting. In order to calculate the 2-holonomy of a cubic cell we fix a convention of defining the orientation of faces from either inside or outside of the cubic cell, in the following we choose outside the cell (the flatness condition is independent of such a choice). We then compose the group elements on faces of the cubic with the convention that if the orientation is clockwise we compose the element of the face and if the orientation is anticlockwise we compose the inverse of the face label. We notate this process by introducing the variable $\epsilon \in \{\pm 1\}$ where $\epsilon_f = +1(-1)$ if the face $f$ has clockwise (anti-clockwise) orientation such that the 2-holonomy $H_2$ on the cube can be written as

$$H_2 = \sum_{f\in\text{cube}} \epsilon_f^e$$ (65)

and the 2-flatness condition becomes

$$\sum_{f\in\text{cube}} \epsilon_f^e = 1_E$$ (66)

We now define the 2-gauge transformation on the cubic lattice. We may neglect the 1-gauge transform as the 1-gauge group is trivial for the crossed module $\mathcal{E}$. The 2-gauge transformation acts on the four faces adjacent to an edge. We notate the 2-gauge transformation as $A_h^{ij}$ on the edge $ij$ where $h \in \mathcal{E}$ is the gauge parameter. The gauge transformation has the action of multiplying the faces adjacent to the edge by either $h$ or $h^{-1}$ depending on whether the direction of the edge is parallel or anti-parallel to the orientation of the adjacent edges. An example is shown in equation (67).

$$A_h^{ij} : e_1 \mapsto e_1 - h e_2 + h e_3 - h e_4$$ (67)

1. Model on the Dual Lattice

After defining the Yetter model for crossed module $\mathcal{E}$ on the cubic lattice, we will now define the model on the dual cubulation. We define dualisation by a map which takes the n-cells of a cellular decomposition of a d-manifold to the (d-n)-cells of the dual cellulation. We will make the assumption that we are working with a cubulation such that the dual cell decomposition is also a cubulation, such a restriction is for ease of presentation and the arguments follow straightforwardly outside of such a restriction. In this case the cubes (3-cells) are taken to vertices (0-cells) of the new cellulation, square faces (2-cells) are taken to edges (1-cells) and edges (1-cells) are taken to faces (2-cells). In this way we can canonically map the Yetter model with degrees of freedom on faces to a dual lattice where the face labels are now on edges. Examples are shown in figure 3 where black edges are of the original lattice and blue are dual.
Cubic cells becomes a vertex condition on the dual lattice. Thus we see that the 2-flatness condition on the section, on the dual lattice this constraint becomes the condition discussed in the previous section, on the dual lattice this constraint becomes the condition.

\[
\prod_{\tilde{e} \in \ast (\tilde{v})} g_{\tilde{e}}^{e_{\tilde{e}}} = 1_E
\]  

(69)

where \(\tilde{e}\) and \(\tilde{v}\) are the dual edges and vertices respectively, \(\ast (\tilde{v})\) is the set of dual edges adjacent to \(\tilde{v}\) and \(e_{\tilde{e}} \in \pm 1\) is +1 when \(\tilde{v}\) is the target of \(\tilde{e}\) and −1 when \(\tilde{v}\) is the source. Thus we see that the 2-flatness condition on the cubic cells becomes a vertex condition on the dual lattice.

The action of \(B_c^{(2)}\) on states of the dual lattice is shown below.

\[
B_c^{(2)} \left| c \begin{array}{c} \epsilon_a \ \epsilon_d \ \epsilon_f \\ b \\ \epsilon_e \\ \epsilon_b \end{array} \right\rangle = \delta_{a+b+c+d+e+f,0} \left| c \begin{array}{c} \epsilon_a \ \epsilon_d \ \epsilon_f \\ b \\ \epsilon_e \\ \epsilon_b \end{array} \right\rangle
\]  

(70)

We note \(B_c^{(2)}\) now has the same action as the vertex operator in the Walker-Wang model for the group \(\mathcal{M} (\mathcal{E})\).

We now consider the edge gauge transformation. On the dual lattice this operator acts on the four edges bounding a plaquette on the dual lattice \(\tilde{p}\). As with the plaquette operator \(B_p\) in the Walker-Wang model we define the operator \(A_{\tilde{p}}^h\) as the gauge transformation on the dual plaquette \(\tilde{p}\) by taking an anti-clockwise orientation around the plaquette and define \(e^+(\tilde{e})\) by whether the dual edge \(\tilde{e}\) is parallel (anti-parallel) to the orientation convention for the plaquette. We then define \(A_{\tilde{p}}^h\) as

\[
A_{\tilde{p}}^h = \prod_{\tilde{e}^+ \in \tilde{p}} \Sigma_{\tilde{e}}^h \prod_{\tilde{e}^- \in \tilde{p}} \Sigma_{\tilde{e}}^{-h}
\]  

(71)

where \(\Sigma_{\tilde{e}}^h\) is defined as previously and

\[
A_{\tilde{p}} = \frac{1}{|E|} \sum_{h \in E} A_{\tilde{p}}^h.
\]  

(72)

E. Comparison of Models

Using the discussion outlined in the previous sections we now compare the Yetter model with input \(\mathcal{E}\) and the Walker-Wang model with input \(\mathcal{M} (\mathcal{E})\). Both models are defined on a cubic lattice \(\Gamma\) with a local Hilbert space defined by \(\mathcal{H} = \otimes_{e \in \Gamma} \mathbb{C}^{|\mathcal{E}|}\) with edge labels indexed by the group \(E\). The Hamiltonian for the Yetter and Walker-Wang models can respectively be written as follows.

\[
H_{Yetter} (\mathcal{E}) = - \sum_{v \in \Gamma} A_v - \sum_{p \in \Gamma} \left( \frac{1}{|E|} \sum_{h \in E} \prod_{e^+ \in p} \Sigma_{e}^h \prod_{e^- \in p} \Sigma_{e}^{-h} \right)
\]

\[
H_{WW} (\mathcal{M} (\mathcal{E})) = - \sum_{v \in \Gamma} A_v
\]

\[
\sum_{p \in \Gamma} \left( \frac{1}{|E|} \sum_{h \in E} \prod_{e^+ \in p} \Sigma_{e}^h \prod_{e^- \in p} \Sigma_{e}^{-h} \right)(\prod_{v \in p} A_v)
\]

(73)

Where \(A_v = B_c^{(2)}\) is defined on basis states as previous but we reproduce for convenience

\[
A_v \left| c \begin{array}{c} \epsilon_a \ \epsilon_d \ \epsilon_f \\ b \\ \epsilon_e \\ \epsilon_b \end{array} \right\rangle = \delta_{a+b+c+d+e+f,0} \left| c \begin{array}{c} \epsilon_a \ \epsilon_d \ \epsilon_f \\ b \\ \epsilon_e \\ \epsilon_b \end{array} \right\rangle
\]  

Comparing the two equations above the only difference is in the definition of the second term which acts
VI. DISCUSSION AND OUTLOOK

Here we comment on our main results and discuss the open questions naturally raised by our construction.

The main result of our manuscript is section [11]. In this section we outline a large class of exactly soluble Hamiltonian models for topological phases in 3+1D. Such models utilise the conventions of higher lattice gauge theory to define a topological lattice model on a simplicial triangulation of a closed, compact 3-manifold M. The algebraic data of the model is defined by a crossed module \( G = (G, E, \partial, \triangledown) \) (equivalently a 2-group). Each edge of the triangulation is “coloured” by a group element \( g \in G \) as in topological gauge theory models [11,15,20,51] while additionally faces of the triangulation are “coloured” with an element \( e \in E \). In a companion paper [33] we will further describe the mathematical consistency of our model.

In an additional article [20] we will present results further generalising the model in order to include crossed module cohomology utilising the work of Faria Martins and Porter [51]. To extend the model we introduce a 4-cocycle \( \omega \in H^4(BG, U(1)) \), where \( BG \) is the classifying space of \( G \). Such a cocycle adds a \( U(1) \) valued phase to the vertex and gauge transformations while the flatness conditions remain unchanged. The value of \( \omega \) is determined by considering the gauge transformations as 4-simplices connecting the original lattice colouring to the gauge transformed colouring. \( \omega \) is then defined by the 4-cocycle of such a complex. Such a phase generalises the model by allowing for groundstates of the model which are not in an equal superposition of basis states as in the current manuscript.

In section [14] we established the relation between our model and the Yetter Homotopy 2-type TQFT (Yetter TQFT) [25]. Specifically we showed that the groundstate projector of our model for \( M^3 \) is given by the Yetter partition function \( Z_{\text{Yetter}} \) defined on the 4-manifold \( M^3 \times [0,1] \). An intriguing consequence of the proof and the fact that the model can be defined in arbitrary dimension \( d > 0 \) is that this results also holds for arbitrary dimensions \( d > 0 \), but for \( d < 3 \), the Hamiltonian does not contain magnetic operators (there are no blobs in a \( d < 2 \) dimensional boundary). A direct consequence of this is that the groundstate degeneracy (GSD) can be obtained by the relation \( \text{GSD} = Z_{\text{Yetter}}(M^d \times S^1) \). We illustrate the formula with several examples in different dimensions for both ordinary and higher gauge theory.

In section [14] we described a duality between our model with the crossed module \( E = (1_E, E, \partial, \triangledown) \) and the Walker-Wang model [22] with symmetric-braided fusion category \( M(E) \). The duality is established using the results of Bantay [55] to relate the algebraic data of a crossed module and a braided fusion category. One could expect a further generalisation of such results by noting that the crossed module \( M = (G, G, 1_G, A_d) \) defines a modular tensor category, the quantum double \( D(G) \) of the group \( G \). This observation is seemingly justified by the fact that both models give rise to a unique groundstate on all 3-manifolds [22] although further work is needed to establish such a connection.

Further to the results in this paper, another avenue for exploration would be to classify the excitation spectrum of the model. There are two complimentary approaches to such a classification. The first is to consider local operators of the model [25]. Using this approach we expect there to be four distinct types of excitations. The four classes of excitations should be point particles and extended line like excitations carrying 1-gauge and 2-gauge charges respectively. Additionally we expect there to be closed loop like excitations and membrane type excitations. The second approach is by considering the quantum numbers associated with the groundstates of our model. In section [16] we discussed the groundstate degeneracy which is a topological observable of the theory which is independent of local mutations of \( M \) which keep the global topology intact, eg. Pachner moves. In general one can consider other topological observables associated with \( M \) which come from global transformations of \( M \) which keep the global topology invariant. Such global transformations are indexed by the mapping class group (MCG) of \( M \). The associated observables should give a full classification of quantum numbers in our model. This approach has already been utilised in [20,31,51] for 3+1D topological phases using projective representations of \( \text{MCG}(T^3) = SL(3, \mathbb{Z}) \) to understand the quantum numbers for topological gauge theories. \( SL(3, \mathbb{Z}) \) has two generators given by the \( S \) and \( T \)-matrices. We call eigenstates of the \( T \) matrix, \( \{|\psi_j\rangle\} \) the quasi-particle basis. The eigenvalues of \( T \) in such a basis then give the topological spin of the quasi-excitations associated with the groundstate. The exchange statistics of such excitations may be calculated by considering the overlap of each basis state with the \( S \)-matrix such that the exchange statistics are given by matrix elements \( S_{ij} = \langle \psi_j | S | \psi_i \rangle \).

We expect the loop-like excitations of our model to form a representation of the loop-braid group. Note that Yetter’s TQFT have been shown to give non-trivial invariants of knotted-surfaces in 3+1D space-time [64,65] and furthermore an embedded 1+1D TQFT for links in \( S^3 \) and their cobordisms [59].

Another possible generalisation which should be explored in the future is to consider 3-manifolds with boundary. It is known that BF like theories such as...
the Walker-Wang model with boundaries reproduce chiral anyon theories on 2-dimensional boundaries. Such a relation is suggestive of the boundaries of our model could support non-trivial anyon models.

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Appendix A: Algebra of gauge transformations

In this appendix the proof of the relations [13] to 18 of gauge transformations is presented. First we remember that the operators $R^i_v$ and $L^i_v$ are representations of the group $G$, that is, for all $g, h \in G$ we have $R^g_i R^h_i = R^{gh}_i$ and $L^g_i L^h_i = L^{gh}_i$. Then it is clear that $L^g_i(i) = L^g_v(i) L^h_v(i)$. Also, since $g \triangleright (h \triangleright \cdot) = gh \triangleright \cdot$, one obtains $L^g_v(p) = L^g_v(p)L^h_v(p)$. Using these properties we deduce that

\[
\prod_{i \in s(v)} L^g_i(i) \prod_{p \in s(\triangleright)} L^g_v(p) = L^g_v(p)
\]

\[
\prod_{i \in s(v)} L^g_i(i) L^h_i(i) = L^g_v(p) L^h_v(p) = \prod_{i \in s(v)} L^g_i(i) \prod_{p \in s(\triangleright)} L^g_v(p) L^h_v(p),
\]

where in the last equality we used the fact that all the operators in the middle two products commute pairwise, because each one of them acts non-trivially on only a distinct edge or face label. Thus we proved (13).

Now we prove the identity (15). First note that $A^e_i A^f_i$ acts as $g_i \triangleright \partial(e) \partial(f) g_i$ on the edge label of edge $i$, while $A^f_i A^e_i$ acts as $g_i \triangleright \partial(e f) g_i$. Both act trivially on the other edge labels and, since $\partial(e f) = (\partial e) (\partial f)$, they have the same action on all edge labels. To prove that they coincide also on face labels, we must consider two cases, (i) $i = v_k v_{k+1}$, and (ii) $i = v_{k+1} v_k$. Also we can suppose that the face $p$ is adjacent to the edge $i$, otherwise both operators act trivially on the edge label of this face.

(i) If $i = v_k v_{k+1}$, $A^e_i A^f_i$ acts as $e_p \mapsto e_p (g_{v_k v_{k+1}} \triangleright f^{-1})(g_{v_{k+1} v_k} \triangleright e^{-1})$ and $A^f_i A^e_i$ acts as $e_p \mapsto e_p (g_{v_{k+1} v_k} \triangleright e f^{-1})$. As $g \triangleright \cdot$ is a homomorphism, we have that the actions coincide.

(ii) If $i = v_{k+1} v_k$, $A^e_i A^f_i$ acts as $e_p \mapsto (g_{v_{k+1} v_k} \triangleright e)(g_{v_{k+1} v_k} \triangleright f)e_p$ and $A^f_i A^e_i$ acts as $e_p \mapsto (g_{v_{k+1} v_k} \triangleright e f)e_p$. As before, the two sides agree.

So we proved (15).

In order to verify (14) we deduce first that $[L^g_v(i), L^h_v(i)] = 0$ and $[L^g_v(p), L^h_v(p)] = 0$, if $i \neq v'$. The first relation holds because both operators can act non-trivially only on the edge $i$ and only if one of the vertices is the source and the other is the target of $i$. However, the operator associated to the source of $i$ is a left multiplication operator and the other associated to the target of $i$ is a right multiplication operator, and these actions are obviously commutative. The second relation follows more easily because at most one of the operators can act non-trivially on a face label. The validity of relation (14) is now a consequence of the fact that all operators in the definitions of $A^e_i$ and $A^f_i$ commute pairwise.

To prove (10) we note that $A^e_i$ and $A^f_i$ commute on edge labels because each one of them acts non-trivially on only one edge label and they are distinct. They also commute on a face label if the associated face is not adjacent to both edges, since in this case at least one of them acts trivially on such face label. If the face is adjacent to both edges $i$ and $i'$ and they are oppositely oriented then one is a left action and the other is a right one, so they commute. Assume that $s(i) < t(i)$ and $s(i') < t(i')$. Assume without loss of generality that $s(i') > s(i)$. Then $A^e_i A^f_i$ acts as $e_p \mapsto e_p (g_{v_{0} v_{1} (i') \triangleright f^{-1}})(g_{v_{0} v_{1} (i) \triangleright e^{-1}})$. The action of $A^f_i A^e_i$ reads:

\[
e_p \mapsto \begin{cases} e_p (g_{v_{0} v_{1} (i) \triangleright e^{-1}})(g_{v_{0} v_{1} (i') \triangleright f^{-1}}) & \text{if } i \neq v, \\
e_p (g_{v_{0} v_{1} (i) \triangleright e^{-1}})(g_{v_{0} v_{1} (i') \triangleright f^{-1}}) & \text{if } i = v \end{cases}
\]

where we used the homomorphism property of $\triangleright$ in the first and last equation and the second Peiffer condition in the second. The other case ($s(i) > t(i)$ and $s(i') > t(i')$) is a similar computation.

Let us consider now the proof of the identities (17) and (18). First we note that if $i$ and $v$ are not adjacent to a given face, then $A^e_i$ and $A^f_i$ commute on all edge labels of edges adjacent to this face and on the face label of this face, because in such case at least one of the operators is the identity operator. Therefore, for the rest of the proof we consider a face adjacent to both $i$ and $v$ (if it exists). Note that $A^e_i$ and $A^f_i$ commute on edge labels (on face labels) if $v \neq s(i)$ and $v \neq t(i)$ (if $v \neq v_0$), since in this case $A^e_i$ acts trivially on edge labels (on face labels, respectively). Thus we need to verify the identities for the cases: (i) $v = v_0$ and $i = v_0 v_1$, (ii) $v = v_0$ and $i = v_0 v_{n-1}$, (iii) $v = v_0$, $i \neq v_0 v_1$ and $i \neq v_0 v_{n-1}$, (iv) $v \neq v_0$ and $v = s(i)$, and (v) $v \neq v_0$ and $v = t(i)$. Furthermore, it is enough to verify (iii) only on face labels and (iv),(v) only on edge labels. Now we have
(i) \( A_0^{p,e}A_0^{e} \) acts as \( g_i \mapsto \partial(g \triangleright e)(gg_i) \) and \( e_p \mapsto (g \triangleright e_p)(g \triangleright e^{-1}) \) and \( A_0^{p,e}A_0^{e} \) acts as \( g_i \mapsto g \partial(e)g_i \) and \( e_p \mapsto g \triangleright (e_p e^{-1}) \). They agree since \( g \partial(e)g_i = g \partial(e)g_i = \partial(g \triangleright e)(gg_i) \) and \( g \triangleright (e_p e^{-1}) = (g \triangleright e_p)(g \triangleright e^{-1}) \). Thus by the previous item the maps agree on edge labels and on face labels by the homomorphism property of \( \triangleright \).

(ii) \( A_0^{p,e}A_0^{e} \) acts as \( g_i \mapsto \partial(g \triangleright e)(gg_i) \) and \( e_p \mapsto (g \triangleright e)(g \triangleright e_p) \) and \( A_0^{p,e}A_0^{e} \) acts as \( g_i \mapsto g \partial(e)g_i \) and \( e_p \mapsto g \triangleright (e e_p) \). Thus by the previous item the maps agree on edge labels and on face labels by the homomorphism property of \( \triangleright \).

(iii) For \( s(i) < t(i) \) the operator \( A_0^{p,e}A_0^{e} \) acts as \( e_p \mapsto (g \triangleright e_p)(g g_{\alpha \beta \gamma}(i) \triangleright e^{-1}) \) and \( A_0^{p,e}A_0^{e} \) acts as \( e_p \mapsto g \triangleright (e_p g_{\alpha \beta \gamma}(i) \triangleright e^{-1}) \). For \( s(i) > t(i) \) the operator \( A_0^{p,e}A_0^{e} \) acts as \( e_p \mapsto (g g_{\alpha \beta \gamma}(i) \triangleright \partial \triangleright e)(g \triangleright e_p) \) whereas \( A_0^{p,e}A_0^{e} \) acts as \( e_p \mapsto g \triangleright (g g_{\alpha \beta \gamma}(i) \triangleright \partial(\triangleright e)) \). For both cases equality is clear by the homomorphism property of \( \triangleright \).

(iv) \( A_0^{p,e}A_0^{e} \) acts as \( g_i \mapsto \partial(g \triangleright e)(gg_i) \) and \( A_0^{p,e}A_0^{e} \) acts as \( g_i \mapsto g \partial(e)g_i \). As \( \partial(g \triangleright e)(gg_i) = g g_{\partial(e)}g g_i = g g_{\partial(e)}g g_i \), they coincide.

(v) \( A_0^{p,e}A_0^{e} \) and \( A_0^{p,e}A_0^{e} \) act both as \( g_i \mapsto \partial(e)g_i g^{-1} \).

This ends the proof of the relations \( \Delta \) to \( \Delta' \).

Appendix B: Transformation properties of 1- and 2-holonomies

In this section we compute the transformation properties of the 1-holonomy of a reference triangle with labels given by the lhs. of (20) and a reference tetrahedron with labels given by (11).

\[
\begin{align*}
H_1 & \xrightarrow{A_0^{p,e}} \partial(g \triangleright e_{abc})gg_{ab}g_{bc}g_{ac}^{-1}g^{-1} = g(\partial g_{abc})gg_{ab}g_{bc}g_{ac}^{-1}g^{-1} = g H_1g^{-1} \\
A_0^{p,e} & \xrightarrow{A_0^{p,e}} (\partial g_{abc})gg_{ab}g^{-1}g_{bc}g_{ac}^{-1} = H_1 \\
A_0^{p,e} & \xrightarrow{A_0^{p,e}} (\partial g_{abc})gg_{ab}g^{-1}g_{bc}g_{ac}^{-1} = H_1 \\
A_0^{p,e} & \xrightarrow{A_0^{p,e}} \partial(e_{abc})e^{-1}(\partial g_{abc})gg_{ab}g_{bc}g_{ac}^{-1} = \partial(e_{abc})e^{-1}(\partial g_{abc})gg_{ab}g_{bc}g_{ac}^{-1} = H_1 \\
A_0^{p,e} & \xrightarrow{A_0^{p,e}} \partial(e_{abc})e^{-1}(\partial g_{abc})gg_{ab}g_{bc}g_{ac}^{-1} = \partial(e)g_{abc}g_{ab}g_{bc}g_{ac}^{-1} = H_1 \\
A_0^{p,e} & \xrightarrow{A_0^{p,e}} \partial(e_{abc})g_{abc}g_{ab}g_{bc}g_{ac}^{-1}e_{abc}^{-1} = H_1 \\
H_2 & \xrightarrow{A_0^{p,e}} (g \triangleright e_{acd})(g \triangleright e_{abc})(g g_{ab}g^{-1}e_{acd}(g \triangleright e_{abc}) = g \triangleright H_2 \\
A_0^{p,e} & \xrightarrow{A_0^{p,e}} e_{acd}e_{abc}(g g_{ab}g^{-1}e_{acd}(g \triangleright e_{abc})e_{abcd}^{-1} = H_2 \\
A_0^{p,e} & \xrightarrow{A_0^{p,e}} H_2 \\
A_0^{p,e} & \xrightarrow{A_0^{p,e}} e_{acd}e_{abc}e^{-1}(\partial g_{abc})gg_{ab}g_{bc}g_{ac}^{-1}e_{abcd}^{-1} = e_{acd}e_{abc}e^{-1}(\partial g_{abc})gg_{ab}g_{bc}g_{ac}^{-1}e_{abcd}^{-1} =
\end{align*}
\]

In the last equation the substitution \( e' = g_{abc} \triangleright e \) has been made and \( \approx \) means equality in case when \( H_{abc} \) is 1. Note that the very last relation shows that the 2-holonomy does not transform covariantly if fake flatness of the boundary faces is not imposed.

Appendix C: Proof of Theorem IV.1

A three cell of \( \Delta \) is a prism based on a face of \( L_j \).

The figure shows a part of the complex. The lattice \( \Delta \) consists of \( L_0 \), \( L_1 \) with the coloring given by \( \{ 0 \} \in L_0 \), \( \{ 1 \} \in L_1 \), \( \{ 0 \} \in L \), oriented edges in \( L_0, L_1 \) and vertical edges assumed to be oriented towards \( L^0 \) (downwards in the figure) connecting corresponding vertices of \( L_j \), respectively.

We have a Dirac delta \( \delta_p \) in the partition function for each face. In particular, for an internal face \( p \) (this is a rectangle connecting corresponding edges of \( L_j \)) the term \( \delta_p \) enforces \( g_i \equiv g_{i(i)}g_{i(i)}^{-1} \). Taking
all such faces into account, identifying the colouring \( L_{0c} \) with \( |L_{0c}| \) we have\(^2\)

\[
\prod_{p \in \Delta^{(1)}} \delta_{g_p,1} = \langle L_{1c} \rangle \prod_{v \in L^0} A^0_v |L_{0c}\rangle
\]

and consequently we can write

\[
\frac{1}{{|G|L_0^0}} \prod_{v \in L^0} \sum_{g_v \in G} \prod_{p \in \Delta^{(1)}} \delta_{g_p,1}
\]

\[
= \langle L_{1c} \rangle \prod_{v \in L^0} \frac{1}{|G|} \sum_{g_v \in G} A^0_v |L_{0c}\rangle = \langle L_{1c} \rangle \prod_{v \in L^0} A^0_v |L_{0c}\rangle
\]

(C1)

(C2)

Now, let us compute the prefactor in the definition \(^3\) for the lattice: \(|G|^{-\frac{|L_0|}{2}} \prod_{p \in \Delta^{(1)}} \delta_{g_p,1} \) is missing. This enforces flatness on the faces \( p \in L_0^0 \cup L_1^0 \). It agrees with the action of the operator \( \prod_{p \in L^2} B_p \) on \( |L_{0c}\rangle \) times that on \( |L_{1c}\rangle \) (note that the lattices \( L_0 \) and \( L_1 \) are identified, but the states \( |L_{0c}\rangle \) and \( |L_{1c}\rangle \) are different). However, the operators \( B_p, p \in L^2 \) and \( A_v, v \in L^0 \) commute for any pair of labels and are also self-adjoint, so we can simply insert the factor \( \prod_{p \in L^2} B_p \) anywhere in the scalar product, using the fact that \( B^2 = B_p, p \in L^2 \), thus inserting \( B_p \) only once is sufficient.

\[ \square \]

**Appendix D: Proof of Theorem IV.2**

Consider that \( M^1 \times [0,1] \) has the product lattice decomposition \( \Delta \). Let us choose a total order on \( \Delta^0 \) in the following way. The Hilbert space is associated to a dressed \( L \), i.e., \( L^0 \) has a total order. Let the total orders in \( L_0^0 \) agree with that on \( L^0 \) and let any vertex in \( L_0^0 \) be smaller than any other in \( L_0^0 \). Then denoting the colour of the internal edge connecting the vertices in \( L_0^0 \) and \( L_0^0 \) corresponding to \( v \in L^0 \) by \( g_v \) and the colour of the internal face connecting edges in \( L_0^1 \) and \( L_0^1 \) corresponding to \( i \in L^1 \) by \( e_i \), the following equality is true.

\[
\prod_{p \in \Delta^{(1)}} \delta_{H_2(p),1} \prod_{p \in \Delta^{(2)}} \delta_{H_1(p),1} = \\
\langle L_{c}^{(1)} | \prod_{v \in L^0} A^0_v \prod_{i \in L^1} A\epsilon_i |L_{c}^{(0)}\rangle
\]

(D1)

To justify it, let us consider the rectangle \( p \) depicted in Fig. 4 with boundary edges \( g^0_1, g^0_2, g^0_i, g^1_1 \). The constraint \( \delta_{H_1(p),1} \) enforces \( \partial e_i = g^0_1 g^1_{gi} (g^0_1)^{-1} \). Equivalently

\[
g^0_1 = g^0_{si} \partial e_i g^1_{gi} (g^0_1)^{-1}
\]

(D2)

which is precisely the image of \( g^0_1 \) under the product of gauge transformations with parameters \( g^0_{si}, g^1_{gi}, e_i \) at vertices \( s(i), t(i) \) and edge \( i \), respectively such that the edge transformation acts first. Note that no other generators in the product act on the edge colour \( g_i \). Let us recall that we identify \( L_{ic} \) with the vector \( |L_{ic}\rangle \) and colours on \( \Delta^{(i)} \) are identified with parameters of gauge transformation on the Hilbert space.

Now consider \( p^0 \in L_{0c} \), coloured with \( e^0_p \) corresponding to the 2-holonomy based at \( v_0(p^0) \) and denote the boundary edge set by \( L_{p^0} \). Assume that \( p^0 \) is an n-gon. Denote the disk (depicted as a bucket in Fig. 4) by \( p^0 \), consisting of the rectangles bounded by the edges \( \{g^0_1, g^0_{si}, g^1_{gi}, g^0_1\}, i \in bd(p^0) \) and \( p^0 \). We can compute the 2-holonomy of this disk based at \( v_0(p) \), which we denote by \( e^0_p \). We have to show that this (again via the identification of boundary colourings with basis vectors in the Hilbert space and internal colours as parameters of gauge transformations) agrees with the action of the product \( \prod_{v \in L^0} A^0_v \prod_{i \in L^1} A\epsilon_i \) on \( e^0_p \) where \( g_v \) is the colour of the vertical edge pointing toward the vertex \( v \) and \( e_i \) is the colour of the vertical rectangle based at edge \( i \).

Introducing the notation \( \mathcal{D} \) for the \( i \)-th edge starting from \( v_0(p) \) in the circular order the action of the product of edge transformations reads

\[
e^0_p \mapsto e^0_p \mathcal{D} = e^0_p
\]

\[
\rho_{ps} \langle \mathcal{D} | e^0_p \rangle |g_{v_0,s(n-1)} e^{\pm 1}_{n-1} (g_{v_0,s(n-2)} e^{\pm 1}_{n-2}) \ldots (g_{v_0,s(2)} e^{\pm 1}_{2}) \rangle e^{\pm 1}_1 \ldots
\]

(D3)

where \( e^{\pm 1}_i \) means \( e_i \) (\( e_i^{-1} \)) if \( i \) is oriented opposite according to the orientation of \( bd(p) \), respectively. Now we apply the vertex transformations. We will show that (i) the rhs. of \( (D3) \) is unchanged under a vertex gauge transformation \( A^{v,v} \) with \( v \neq v_0 \) and (ii) that it changes as \( \mathcal{A} \mapsto \mathcal{A} g_v \mathcal{A}^\dagger \) for \( v = v_0 \). This is how the face holonomy \( e^0_p \) should change under \( \prod_{v \in L^0} A^0_v \). Consequently, the 2-holonomy \( e^0_p \) of the disk \( p^0 \) is given by the image of \( e^0_p \) under gauge transformation.
It is easy to see that \( g_{v_0,s(k)} \) changes only if \( v = s(k) \) since then \( g_{v_0,s(k)} \) switches \( g_{v_0,s(k)}^{-1} \), which induces \( g_{v_0,s(k)} \rightarrow g_{v_0,s(k)}g^{-1} \), which changes only for \( s(k) = v \) as \( e_k \rightarrow g \rightarrow e_k \). Altogether

\[
 g_{v_0,s(k)} \leftrightarrow e_k \rightarrow g_{v_0,s(k)}g^{-1} \rightarrow (g \rightarrow e_k) = g_{v_0,s(k)} \rightarrow e_k 
\]

so (i) is proved. The action of \( A_{v_0}^{e_0} \) is \( e_0 \rightarrow g \rightarrow e_0 \) and \( e_1 \rightarrow g \rightarrow e_1 \) on the two faces with basepoint \( v_0 \) and all pairness in \( [D\text{3}] \) transform as \( (.) \rightarrow g \rightarrow (.) \) since \( g_{v_0,s(k)} \rightarrow g_{v_0,g_{v_0,s(k)}} \). Hence, since \( (.) \rightarrow g \rightarrow (.) \) is a homomorphism, we showed (ii), so the claim is proved. Note that the edge labels change as \( g_{s(i)} \rightarrow \partial e_i g_{s(i)}^{-1} \rightarrow g_{s(i)} \partial e_i g_{s(i)}^{-1} \) under \( \prod_j A_j^{e_j} \) and \( \prod_j A_j^{e_j} \), respectively, in accordance with [D2]. Another remark is that the order of the terms on the rhs of (D1) depends on the total order on \( \Delta^0 \). In particular, if we had chosen \( n_{s(i)}^0 > s(i) \) the fake flatness condition would have read \( \delta e_i = g_{s(i)} \partial e_i g_{s(i)}^{-1} \) equivalent to the order \( A_{v_0}^{e_0} \) of action of gauge transformations. Once the integration is done, this dependence will of course disappear, equivalently the vertex projections defined as averaged gauge transformations mutually commute.

Now, consider the 2-sphere \( S_{p_0} \) bounded by the disks \( p_0 \) and \( p^0 \) coloured by \( e_0^{p_0} \) and \( e_0^{p^0} \), respectively. They have the same boundary \( b\delta(p^0) \), and they are oriented oppositely. Hence, \( \delta H_2(S_{p_0}) = 1 \) iff \( e_0^{p^0} = e_0^{p^0} \). This completes the proof of (D1).

Let us now determine the multiplicative factors from the definition of the partition vector from (8) for the lattice at hand. We have only boundary vertices \( \Delta^0 = L_0^0 \cup L_1^0 \) and the edge set is in one-to-one correspondence with \( L^0 \cup (L_0^1 \cup L_1^1) \) such that the first factor is for internal edges and the second for external ones, respectively. The multiplicative factor determined from (8) turns out to be \( |G|^{-|L^0||E|-|L^1|} \). This means that we can assign a factor of \( |G|^{-1} \) to each term \( \sum g \in L^0 \) \( A_{e_0} \) for \( v \in L^0 \) and a factor of \( |E|^{-1} \) to \( \sum_{i=1}^n A_{e_i} \) for \( i \in L^1 \). So, similarly to the ordinary gauge theory case, we found

\[
 |G|^{-|L^0||E|-|L^1|} \prod_{v \in L^0} \prod_{i \in L^1} \sum_{g \in G} \delta_{H_2(v,P)} \sum_{\tilde{p} \in \Delta^0} \hat{\delta}_{H_2(\tilde{p})} = \left\langle L^0, A_{\Delta^0}, \prod_{i \in L^1} A_{e_i}, L^1 \right\rangle 
\]

The last step is the identification of the flatness constraints of the blobs in \( L_0^1 \cup L_1^1 \) with the \( B_P \) operators. This goes parallel to the ordinary lattice gauge theory case. □

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This group is the 3+1D analogue of the braid group, which governs particle statistics in 2+1D.

In the differential formulation of higher gauge theory, the equations of motion analogous to the vanishing of the field strength in ordinary gauge theory has an additive contribution of the derivative of the map $\theta$. So whenever the latter is non-trivial the equation $H_1(p) = 1$ is not equivalent to flatness of the 1-connection, hence the adjective "fake".

The most studied case of a compact Lie group is $SU(2)$, and finiteness requires gauge fixing.

One considers $\hat{L} \times [0,1]$ and constructs the dual complex of this. It will have a vertex in the middle of each prism connected vertically to the middle points of $\hat{L}_i$, middle points of neighbor prisms are connected and the duals of $\hat{L}_i$ are the original graphs $L_i$ at the boundary.

The Hamiltonian for $d = 3$ HLGTT in the table differs from $\frac{1}{2}$ by an unimportant constant.

Every 3-manifold has a presentation in terms of a cubulation, in other words in terms of a partition into 3-dimensional cubes, which only intersect along a common face. However in some cases the valence of some of the edges of a cubulation may be different of 4, and therefore some vertices may not be six-valent. For some manifolds these features are not avoidable; see by a result of Hatcher et al.

Twisted Higher Symmetry Topological Phases, Bullivant et al.

Note, that $|L_0\rangle$ is a basis element, not a generic state in the Hilbert space.