Shape-preserving properties of a new family of generalized Bernstein operators

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Abstract
In this paper, we introduce a new family of generalized Bernstein operators based on $q$ integers, called ($\alpha, q$)-Bernstein operators, denoted by $T_{n,\alpha,q}(f)$. We investigate a Kovovkin-type approximation theorem, and obtain the rate of convergence of $T_{n,\alpha,q}(f)$ to any continuous functions $f$. The main results are the identification of several shape-preserving properties of these operators, including their monotonicity- and convexity-preserving properties with respect to $f(x)$. We also obtain the monotonicity with $n$ and $q$ of $T_{n,\alpha,q}(f)$.

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1 Introduction
A generalization of Bernstein polynomials based on $q$-integers was proposed by Lupaş in 1987 in [1]. However, the Lupaş $q$-Bernstein operators are rational functions rather than polynomials. In 1997, Phillips [2] proposed the Phillips $q$-Bernstein polynomials, and for decades thereafter the application of $q$ integers in positive linear operators became a hot topic in approximation theory, such as generalized $q$-Bernstein polynomials [3–6], Durrmeyer-type $q$-Bernstein operators [7–9], Kantorovich-type $q$-Bernstein operators [10–13], etc. As we know, $q$ integers play important roles not only in approximation theory, but also in CAGD. Based on the Phillips $q$-Bernstein polynomials [2], which are generalizations of Bernstein polynomials, generalized Bézier curves and surfaces were introduced in [14–16]. In [14], Oruç and Phillips constructed $q$-Bézier curves using the basis functions of Phillips $q$-Bernstein polynomials. Dişibüyük and Oruç [15, 16] defined the $q$ generalization of rational Bernstein–Bézier curves and tensor product $q$-Bernstein–Bézier surfaces. Moreover, Simeonov et al. [17] introduced a new variant of the blossom, the $q$ blossom, which is specifically adapted to developing identities and algorithms for $q$-Bernstein bases and $q$-Bézier curves. In 2014, Han et al. [18] proposed a generalization of $q$-analog Bézier curves with one shape parameter, and established degree evaluation and de Casteljau algorithms and some other properties. In 2016, Han et al. [19] introduced a new generalization of weighted rational Bernstein–Bézier curves based on $q$ integers, and investigated the generalized rational Bézier curve from a geometric point of view, obtaining degree evaluation and de Casteljau algorithms, etc.
Recently, Chen et al. [20] introduced a new family of $\alpha$-Bernstein operators, and investigated some approximation properties, such as the rate of convergence, Voronovskaja-type asymptotic formulas, etc. They also obtained the monotonic and convex properties. For $f(x) \in [0,1]$, $n \in \mathbb{N}$, and any fixed real $\alpha$, the $\alpha$-Bernstein operators they introduced are defined as

$$T_{n,\alpha} = \sum_{i=0}^{n} f_i P_{n,i}^{(\alpha)}(x),$$

(1)

where $f_i = f(\frac{i}{n})$. For $i = 0,1,\ldots, n$, the $\alpha$-Bernstein polynomial $p_n^{(\alpha)}(x)$ of degree $n$ is defined by $p_{1,0}^{(\alpha)}(x) = 1 - x$, $p_{1,1}^{(\alpha)}(x) = x$ and

$$p_{n,i}^{(\alpha)}(x) = \left[ \binom{n-2}{i} (1-\alpha)x + \binom{n-2}{i-2} (1-\alpha)(1-x) + \binom{n}{i} \alpha x(1-x) \right] x^{i-1}(1-x)^{n-1-i},$$

(2)

where $n \geq 2$.

Motivated by above research, in this paper we propose the $q$ analogue of $\alpha$-Bernstein operators, called $(\alpha, q)$-Bernstein operators, which are defined as

$$T_{n,q,\alpha}(f;x) = \sum_{i=0}^{n} f_i P_{n,i}^{(\alpha,q)}(x),$$

(3)

where $q \in (0,1]$, $f_i = f(\frac{i}{[n]_q})$, $i = 0,1,\ldots,n$, $p_{1,q,0}^{(\alpha)}(x) = 1 - x$, $p_{1,q,1}^{(\alpha)}(x) = x$, and

$$p_{n,q,i}^{(\alpha)}(x) = \left[ \binom{n-2}{i} \frac{1}{q} (1-\alpha)x + \binom{n-2}{i-2} (1-\alpha) q^{n-i-2}(1-q^{n-i-1}x) \right] x^{i-1}(1-x)^q x^{n-1}(1-x)^{n-i-1}(n \geq 2).$$

(4)

By simple computations, we can also express the $(\alpha, q)$ operators (3) as

$$T_{n,q,\alpha}(f;x) = (1-\alpha) \sum_{i=0}^{n-1} g_i \binom{n-1}{i} q \left(1-x\right)^{n-1-i} + \alpha \sum_{i=0}^{n} f_i \binom{n}{i} x^i (1-x)^{n-i},$$

(5)

where

$$g_i = \left( 1 - \frac{q^{n-1-i}[i]_q}{[n-1]_q} \right) f_i + \frac{q^{n-1-i}[i]_q}{[n-1]_q} f_{i+1}.$$

(6)

Here, we mention some definitions based on $q$ integers, the details of which can be found in [21, 22]. For any fixed real number $0 < q \leq 1$ and each non-negative integer $k$, we denote
$q$-integers by $[k]_q$, where

$$[k]_q := \begin{cases} \frac{1-q^k}{1-q}, & q \neq 1, \\ k, & q = 1. \end{cases}$$

Also, $q$-factorial and $q$-binomial coefficients are defined as follows:

$$[k]_q! := \begin{cases} [k]_q [k-1]_q \cdots [1]_q, & k = 1, 2, \ldots, \\ 1, & k = 0, \end{cases}$$

$$\binom{n}{k}_q := \frac{[n]_q!}{[k]_q! [n-k]_q!} \quad (n \geq k \geq 0).$$

The $q$-analog of $(1 + x)^n$ is defined by $(1 + x)_q^n := \prod_{i=0}^{n-1} (1 + q^i x)$. The $q$ derivative and $q$ derivative of the product are defined as $D_q f(x) := \frac{df(x)}{dx^{(q)}}$ and $D_q (f(x)g(x)) := f(qx)D_q g(x) + g(x)D_q f(x)$, respectively. We also have $D_q x^n = [n]_q x^{n-1}$ and $D_q (1-x)_q^n = \frac{[-n]_q (1-qx)^{n-1}}{q^n}.

The rest of this paper is organized as follows. In the next section, we give some basic properties of the operators $T_{n,q,a}(f)$, such as the moments and central moments for proving the convergence theorems, the forward difference form of $T_{n,q,a}(f)$ for proving shape-preserving properties, etc. In Sect. 3, we obtain the convergence property and the rate of convergence theorem. In Sect. 4, we investigate some shape-preserving properties, such as monotonicity- and convexity-preserving properties with respect to $f(x)$, and also we study the monotonicity with $n$ and $q$ of $T_{n,q,a}(f)$.

2 Auxiliary results

For proving the main results, we require the following lemmas.

**Lemma 2.1** We have the following equalities:

$$T_{n,q,a}(1;x) = 1, \quad T_{n,q,a}(t;x) = x. \quad (7)$$

**Proof** By (5), we have

$$T_{n,q,a}(1;x) = (1 - \alpha) \sum_{i=0}^{n-1} \binom{n-1}{i}_q x^i (1-x)^{n-1-i}_q + \alpha \sum_{i=0}^{n} \binom{n}{i}_q x^i (1-x)^{n-i}_q$$

$$= 1.$$
produced linear functions; that is, in

We can obtain (8) easily by [2]. Next, in order to prove (9), we use induction on \( r \). It is clear that (9) holds for \( r = 0 \). Let us assume that (9) holds for some \( r = k \geq 0 \). For \( r = k + 1 \), we have

\[
\Delta_{q}^{k+1}g_l = \Delta_{q}^{k}g_{l+1} - q^k \Delta_{q}^{k}g_l
\]

\[
= \left( 1 - q^{n-i-2}[i + 1]_{q} \right) \Delta_{q}^{i+1}g_{l+1} + \frac{q^{n-i-2-k}[i + k + 1]_{q}}{[n-1]_{q}} \Delta_{q}^{i}g_{l+2}
\]

Lemma 2.1 is proved. \( \square \)

Remark 2.2 From Lemma 2.1, we know that the \((\alpha, q)\)-Bernstein operators \( T_{n,\alpha,\mu}(f; x) \) reproduce linear functions; that is,

\[
T_{n,\alpha,\mu}(at + b; x) = ax + b,
\]

for all real numbers \( a \) and \( b \).

We immediately obtain Lemma 2.3 from (5) and Lemma 2.1.

Lemma 2.3 For all functions \( f \) and \( g \) defined in \([0, 1]\), \( x \in [0, 1] \), real numbers \( \lambda, \mu \) defined in \([0, 1]\), and \( q \in (0, 1) \), the following statements hold true.

(i) Endpoint interpolation: \( T_{n,\alpha,\mu}(f; 0) = f(0) \) and \( T_{n,\alpha,\mu}(f; 1) = f(1) \).

(ii) Linearity: \( T_{n,\alpha,\mu}(\lambda f + \mu g; x) = \lambda T_{n,\alpha,\mu}(f; x) + \mu T_{n,\alpha,\mu}(g; x) \).

(iii) Non-negative: For \( 0 \leq \alpha \leq 1 \) and \( 0 < q \leq 1 \), if \( f \) is non-negative on \([0, 1]\), so is \((\alpha, q)\)-Bernstein operators \( T_{n,\alpha,\mu}(f; x) \).

(iv) Monotone: For fixed \( 0 \leq \alpha \leq 1 \) and \( 0 < q < 1 \), if \( f \geq g \), then \( T_{n,\alpha,\mu}(f; x) \geq T_{n,\alpha,\mu}(g; x) \).

Lemma 2.4

(i) The \((\alpha, q)\)-Bernstein operators may be expressed in the form

\[
T_{n,\alpha,\mu}(f; x) = \sum_{r=0}^{n} \left( 1 - \alpha \right) \left( \frac{n-1}{r} \right) \Delta_{q}^{r}g_0 + \alpha \left( \frac{n}{r} \right) \Delta_{q}^{r}f_0 \]

where \([n-1]_{q} = 0\), \( \Delta_{q}^{r}f_{j} = \Delta_{q}^{r-1}f_{j+1} - q^{r-1}f_{j} \), \( r \geq 1 \), with \( \Delta_{q}^{0}f_{j} = f_{j} = f\left( \frac{j}{[n]_{q}} \right) \).

(ii) The higher-order forward difference of \( g \) may be expressed in the form

\[
\Delta_{q}^{r}g_{i} = \left( 1 - q^{n-i-1}[i]_{q} \right) \Delta_{q}^{r}f_{i} + \frac{q^{n-i-r}[i + r]_{q}}{[n-1]_{q}} \Delta_{q}^{r}f_{i+1},
\]

where \( \Delta_{q}^{0}g_{i} = g_{i} \), which is defined in (6).

Proof We can obtain (8) easily by [2]. Next, in order to prove (9), we use induction on \( r \). It is clear that (9) holds for \( r = 0 \). Let us assume that (9) holds for some \( r = k \geq 0 \). For \( r = k + 1 \), we have
Proof. Indeed, from (9) and Lemma 2.4, we have

\[
- q^i \left[ \left( 1 - q^{n-i-1} \right) \frac{\Delta_q^k f_i}{[n-1]_q} + \frac{q^{n-i-1}[i+k]_q}{[n-1]_q} \Delta_q^k f_{i+1} \right] = \left[ 1 - \frac{q^{n-i-2}(1 + q[i]_q)}{[n-1]_q} \right] \frac{\Delta_q^k f_{i+1}}{[n-1]_q} - \left( 1 - \frac{q^{n-i-1} [i]_q}{[n-1]_q} \right) q^k \Delta_q^k f_i
\]

This shows that (9) holds when \( k \) is replaced by \( k + 1 \), and this completes the proof of Lemma 2.4. \( \square \)

Since \( f_{\frac{j}{q}, \frac{j-1}{q}, \frac{j-2}{q}, \ldots, \frac{j-k}{q}} = \frac{[n]_q! \Delta_q f_j}{[n-k]_q!} = \nu_{(k)}(\xi) \), where \( \xi \in \left[ \frac{j}{q}, \frac{j-k}{q} \right] \), the \( q \)-differences of the monomial \( x^k \) of order greater than \( k \) are zero. We see from Lemma 2.4 that, for all \( n \geq k \), \( T_{n,q,a}(x^k) \) is a polynomial of degree \( k \). Actually, the \( (\alpha, q) \)-Bernstein operators are degree-reducing on polynomials; that is, if \( f \) is a polynomial of degree \( m \), and then \( T_{n,q,a}(f) \) is a polynomial of degree \( \leq \min\{m, n\} \). In particular, we have the following results.

Lemma 2.5. Letting \( f(t) = t^k \), \( n-1 \geq k \geq 2 \), we have

\[
T_{n,q,a}(x^k) = a_k x^k + a_{k-1} x^{k-1} + \cdots + a_1 x + a_0,
\]

where \( a_k = \frac{[k]_q^{(k-1)}}{[n-k]_q! [n]_q!} \frac{[n]_q! \Delta_q f_j}{[n-k]_q!} \).

Proof. Indeed, from (9) and \( \Delta_q^k f_j = \frac{q^{i(k-1)} \cdot [j]_q^{(k)}}{k [n]_q^k} \), we have

\[
\Delta_q^k f_0 = \Delta_q^k f_0 + \frac{q^{n-1-k} [k]_q}{[n-1]_q} \Delta_q^k f_1, \quad \Delta_q^k f_0 = \frac{q^{(k-1)} [k]_q^k}{[n]_q^k}, \quad \Delta_q^k f_1 = \frac{q^{(k-1)} [k]_q^k}{[n]_q^k}.
\]

Thus, we obtain

\[
\Delta_q^k f_0 = \left( 1 + \frac{q^{n-1-k} [k]_q}{[n-1]_q} \right) \frac{q^{(k-1)} [k]_q^k}{[n]_q^k} = \frac{[n-1+k]_q}{[n-1]_q} \frac{q^{(k-1)} [k]_q^k}{[n]_q^k}.
\]
Hence, using (8), we have

\[
a_k = \left(1 - \alpha\right) \frac{n - 1}{k} \frac{[n - 1 + k]_q}{[n - 1]_q} + \alpha \frac{n}{k} \frac{q^{\frac{k-1}{2}}[k]_q!}{[n]_q^2}.
\]

We then obtain the proof of Lemma 2.5 by simple computations.

Lemma 2.6 The following equalities hold true:

\[
T_{n,q,a}(t^2; x) = x^2 + \frac{x(1 - x)}{[n]_q} + \frac{(1 - \alpha)q^{n-1}[2]_q x(1 - x)}{[n]_q^2}, \quad (10)
\]

\[
T_{n,q,a}((t - x)^2; x) = \frac{x(1 - x)}{[n]_q} + \frac{(1 - \alpha)q^{n-1}[2]_q x(1 - x)}{[n]_q^2}. \quad (11)
\]

Proof For \( f(t) = t^2 \), we have \( \Delta^0_{q} f_0 = f_0 = 0 \), \( \Delta^1_{q} f_0 = f_1 - f_0 = \frac{1}{[n]_q}, \Delta^1_{q} f_1 = f_2 - f_1 = \frac{2q^{n-2} + q^n}{[n]_q^2}, \Delta^2_{q} f_0 = \Delta^1_{q} f_1 - q\Delta^0_{q} f_0 = \Delta^2_{q} f_1 = f_3 - [2]_q f_2 + qf_0 = \frac{q^2 + q^n}{[n]_q^2}, \) and \( \Delta^2_{q} f_1 = f_3 - [2]_q f_2 + qf_1 = \frac{q^2 + q^n}{[n]_q^2}. \) By (9), we have \( \Delta^0_{q} g_0 = 0 \), and

\[
\begin{align*}
\Delta^1_{q} g_0 &= \Delta^1_{q} f_0 + \frac{q^{n-2}}{[n - 1]_q} \Delta^1_{q} f_1 = \frac{1}{[n]_q^2} + \frac{2q^{n-1} + q^n}{[n - 1]_q [n]_q^2}, \\
\Delta^2_{q} g_0 &= \Delta^2_{q} f_0 + \frac{q^{n-3}[2]_q}{[n - 1]_q} \Delta^2_{q} f_1 = \frac{q^2}{[n]_q^2} + \frac{2q^n(q^n + q^{n+1})}{[n - 1]_q [n]_q^2}.
\end{align*}
\]

From (8), we have

\[
T_{n,q,a}(t^2; x)
= (1 - \alpha)\Delta^0_{q} g_0 + \alpha \Delta^0_{q} f_0 + \left[(1 - \alpha)[n - 1]_q \Delta^1_{q} g_0 + \alpha[n]_q \Delta^1_{q} f_0\right] x
+ \left[(1 - \alpha)\frac{n - 1}{[n]_q} + \frac{2q^{n-1} + q^n}{[n]_q^2} \right] \frac{q^2}{[n]_q^2} + \frac{2q^n(x^n + x^{n+1})}{[n]_q^2}
+ \left[(1 - \alpha)[n - 1]_q \frac{2q^n}{[n]_q^2} + \alpha[n]_q \frac{q^2}{[n]_q^2} \right] x^2
\]

\[
= x^2 + \frac{x(1 - x)}{[n]_q} + \frac{(1 - \alpha)q^{n-1}[2]_q x(1 - x)}{[n]_q^2}.
\]

Hence, (10) is proved. Finally, using Lemma 2.1, we obtain

\[
T_{n,q,a}((t - x)^2; x) = T_{n,q,a}(t^2; x) - 2xT_{n,q,a}(t; x) + x^2 T_{n,q,a}(1; x) = T_{n,q,a}(t^2; x) - x^2.
\]

Then (11) is proved by (10). This completes the proof of Lemma 2.6.
3 Convergence properties

We now state the well-known Bohman–Korovkin theorem, followed by a proof based on that given by Cheney [23].

**Theorem 3.1** Let \( \{L_n\} \) denote a sequence of monotone linear operators that map a function \( f \in C[a,b] \) to a function \( L_n f \in C[a,b] \), and let \( L_n f \to f \) uniformly on \([a,b]\) for \( f = 1, t \) and \( t^2 \). Then \( L_n f \to f \) uniformly on \([a,b]\) for all \( f \in C[a,b] \).

Theorem 3.1 leads to the following theorem on the convergence of \((\alpha, q)\)-Bernstein operators.

**Theorem 3.2** Let \( q := \{q_n\} \) denote a sequence such that \( q_n \in (0,1) \) and \( \lim_{n \to \infty} q_n = 1 \). Then, for any \( f \in C[0,1] \) and \( \alpha \in [0,1] \), \( T_{n,\alpha,q} f \) converges uniformly to \( f(x) \) on \([0,1]\).

**Proof** From Lemma 2.1, we see that \( T_{n,\alpha,q} f \) vanishes for \( f(t) = 1 \) and \( f(t) = t \). Since \( \lim_{n \to \infty} q_n = 1 \), we see from (10) that \( T_{n,\alpha,q} f \) converges uniformly to \( f(x) \) for \( f(t) = t^2 \) as \( n \to \infty \). It also follows that \( T_{n,\alpha,q} \) is a monotone operator by Lemma 2.3; the proof is then completed by applying the Bohman–Korovkin theorem 3.1. \( \square \)

As we know, the space \( C[0,1] \) of all continuous functions on \([0,1]\) is a Banach space with sup-norm \( \|f\| := \sup_{x \in [0,1]} |f(x)| \). Letting \( f \in C[0,1] \), the Peetre K-functional is defined by \( K_2(f; \delta) := \inf_{g \in C^2[0,1]} \{\|f - g\| + \delta \|g''\|\}, \) where \( \delta > 0 \) and \( C^2[0,1] := \{g \in C[0,1] : g', g'' \in C[0,1]\} \). By [24], there exists an absolute constant \( C > 0 \), such that

\[
K_2(f; \delta) \leq C \omega_2(f; \sqrt{\delta}),
\]

where \( \omega_2(f; \delta) := \sup_{0 < h \leq \delta} \sup_{x \in [x+2h, x+2h]} |f(x + 2h) - 2f(x + h) + f(x)| \) is the second-order modulus of smoothness of \( f \in C[0,1] \).

**Theorem 3.3** For \( f \in C[0,1], \alpha \in [0,1], q \in (0,1), \) we have

\[
|T_{n,\alpha,q} f(x) - f(x)| \leq C \omega_2 \left( f; \sqrt{\frac{2[n]_q + (1 - \alpha) 2[2]_q q^{\alpha - 1}}{4[n]_q}} \right),
\]

where \( C \) is a positive constant.

**Proof** Letting \( g \in C^2[0,1], x, t \in [0,1] \), by Taylor’s expansion we have

\[
g(t) = g(x) + g'(x)(t - x) + \int_x^t (t - u) g''(u) \, du.
\]

Using Lemma 2.1, we obtain

\[
T_{n,\alpha,q} g(x) = g(x) + T_{n,\alpha,q} \left( \int_x^t (t - u) g''(u) \, du; x \right).
\]

Thus, we have

\[
|T_{n,\alpha,q} g(x) - g(x)| = \left| T_{n,\alpha,q} \left( \int_x^t (t - u) g''(u) \, du; x \right) \right|
\]
\[ \leq T_{n,q,a} \left( \int_0^t (t-u)|g''(u)| \, du \right) \]
\[ \leq T_{n,q,a}((t-x)^2; x) \|g''\| \]
\[ \leq \frac{[n]_q + (1-\alpha)q^{n+1}[2]_q}{4[n]_q^2} \|g''\|. \quad (13) \]

However, using Lemma 2.1, we have
\[ |T_{n,q,a}(f;x)| \leq \|f\|. \quad (14) \]

Now, (13) and (14) imply
\[ |T_{n,q,a}(f;x) - f(x)| \leq |T_{n,q,a}(f-g;x) - (f-g)(x)| + |T_{n,q,a}(g;x) - g(x)| \]
\[ \leq 2\|f-g\| + \frac{[n]_q + (1-\alpha)q^{n+1}[2]_q}{4[n]_q^2} \|g''\|. \]

Hence, taking the infimum on the right-hand side over all \( g \in C^2[0,1] \), we obtain
\[ |T_{n,q,a}(f;x) - f(x)| \leq 2K_2 \left( f; \frac{[n]_q + (1-\alpha)q^{n+1}[2]_q}{8[n]_q} \right). \]

By (12), we obtain
\[ |T_{n,q,a}(f;x) - f(x)| \leq C_2 \left( f; \frac{\sqrt{2[n]_q + (1-\alpha)2[2]_qq^{n+1}}}{4[n]_q} \right), \]

where \( C \) is a positive constant. Theorem 3.3 is proved.

**Remark 3.4** Letting \( q := \{q_n\} \) denote a sequence such that \( q_n \in (0,1) \) and \( \lim_{n \to \infty} q_n = 1 \), we know that, under the conditions of theorem 3.3, the convergence rate of the operators \( T_{n,q,a}(f) \) to \( f \) is \( 1/\sqrt{n} \) as \( n \to \infty \). This convergence rate can be improved depending on the choice of \( q \), at least as fast as \( 1/\sqrt{n} \).

**Example 3.5** Letting \( f(x) = 1 - \cos(4e^x) \), the graphs of \( f(x) \) and \( T_{n,q,0.5}(f;x) \) with different values of \( n \) and \( q \) are shown in Fig. 1. Figure 2 shows the graphs of \( f(x) \) and \( T_{10,0.9,0.6}(f;x) \) with \( \alpha = 0.6 \) and \( \alpha = 0.9 \).

### 4 Shape-preserving properties

The \((\alpha,q)\)-Bernstein operators \( T_{n,q,a}(f;x) \) have a monotonicity-preserving property.

**Theorem 4.1** Let \( f \in C[0,1] \). If \( f \) is a monotonically increasing or monotonically decreasing function on \([0,1]\), so are all its \((\alpha,q)\)-Bernstein operators for fixed \( q \in (0,1) \) and \( \alpha \in [0,1] \).

**Proof** From (5), we have
\[ T_{n+1,q,a}(f;x) = (1-\alpha) \sum_{i=0}^{n} g_i \binom{n}{i} x^i (1-x)^{n-i}_q + \alpha \sum_{i=0}^{n+1} f_i \binom{n+1}{i} x^i (1-x)^{n+1-i}_q, \]
Figure 1 Convergence of $T_{n,q}(f; x)$ to $f(x)$ for fixed $\alpha = 0.9$

Figure 2 Convergence of $T_{n,q}(f; x)$ to $f(x)$ for fixed $q = 0.9$
where \( f_i = \frac{[i]_q}{[n]_q} \times g_i = (1 - \frac{q^{n-i}[i]_q}{[n]_q}) f_i + \frac{q^{n-i}[i]_q}{[n]_q} f_{i+1} \). Then the \( q \) derivative of \( T_{n,1,q,a}(f;x) \) is

\[
D_q[T_{n,1,q,a}(f;x)] = (1 - \alpha) \sum_{i=0}^{n} g_i \left[ \begin{array}{c} n \\ i \\ \end{array} \right] D_q[x(1-x)^{n-i}] + \alpha \sum_{i=0}^{n+1} f_i \left[ \begin{array}{c} n+1 \\ i \\ \end{array} \right] D_q[x(1-x)^{n+1-i}],
\]

and we denote the first and second parts of the right-hand side of the last equation by \( \Lambda_1 \) and \( \Lambda_2 \), respectively. We then have

\( \Lambda_1 \)

\[
= (1 - \alpha) \sum_{i=0}^{n} g_i \left[ \begin{array}{c} n \\ i \\ \end{array} \right] \left[ [i]_q x^{i-1} (1-qx)^{n-i} - [n-i]_q x^i (1-qx)^{n-i+1} \right]
\]

\[
= (1 - \alpha) [n]_q \sum_{i=1}^{n} g_i \left[ \begin{array}{c} n-1 \\ i-1 \\ \end{array} \right] q^{i-1} (1-qx)_q^{n-i} - \sum_{i=0}^{n-1} g_i \left[ \begin{array}{c} n-1 \\ i \\ \end{array} \right] x^i (1-qx)_q^{n-i+1}
\]

\[
= (1 - \alpha) [n]_q \sum_{i=0}^{n+1} \left[ \begin{array}{c} n-1 \\ i \\ \end{array} \right] x^i (1-qx)_q^{n-i+1} \Delta_q g_i.
\]

Using (9), we obtain

\[
\Delta_1 q g_i = \left( 1 - \frac{q^{n-i}[i]_q}{[n]_q} \right) \Delta_1 q f_i + \frac{q^{n-i+1}[i+1]_q}{[n]_q} \Delta_1 q f_{i+1}.
\]

Thus, we have

\[
\Lambda_1 = (1 - \alpha) \sum_{i=0}^{n-1} \left[ ([n]_q - q^{n-i}[i]_q) \Delta_1 q f_i + q^{n-i+1}[i+1]_q \Delta_1 q f_{i+1} \right] \left[ \begin{array}{c} n-1 \\ i \\ \end{array} \right]_q
\]

\[
\times x^i (1-qx)_q^{n-i+1}.
\] (15)

Similarly, we can obtain

\[
\Lambda_2 = \alpha [n+1]_q \sum_{i=0}^{n} \left[ \begin{array}{c} n \\ i \\ \end{array} \right] x^i (1-qx)_q^{n-i} \Delta_1 q f_i.
\] (16)

Therefore, by using (15) and (16), the derivative of \((\alpha, q)\)-Bernstein operators \(T_{n,q,a}(f;x)\) may be expressed in the form

\[
D_q[T_{n,q,a}(f;x)]
\]

\[
= (1 - \alpha) \sum_{i=0}^{n-1} \left[ ([n]_q - q^{n-i}[i]_q) \Delta_1 q f_i + q^{n-i+1}[i+1]_q \Delta_1 q f_{i+1} \right] \left[ \begin{array}{c} n-1 \\ i \\ \end{array} \right]_q
\]

\[
\times x^i (1-qx)_q^{n-i+1} + \alpha [n+1]_q \sum_{i=0}^{n} \left[ \begin{array}{c} n \\ i \\ \end{array} \right] x^i (1-qx)_q^{n-i} \Delta_1 q f_i.
\]
Since if \( f \) is monotonically increasing on \([0, 1]\), the forward differences \( \triangle^1_q f \) and \( \triangle^1_q f_{i+1} \) are non-negative, and so is \( D_q [T_{n,q,a}(f; x)] \). Hence, (\( \alpha, q \))-Bernstein operators \( T_{n,q,a}(f; x) \) are monotonically increasing on \([0, 1]\) for fixed \( q \in (0, 1) \) and \( \alpha \in [0, 1] \). On the contrary, if \( f \) is monotonically decreasing on \([0, 1]\), then operators \( T_{n,q,a}(f; x) \) are monotonically decreasing on \([0, 1]\) for fixed \( q \in (0, 1) \) and \( \alpha \in [0, 1] \). Theorem 4.1 is proved. \( \square \)

The (\( \alpha, q \))-Bernstein operators \( T_{n,q,a}(f; x) \) have a convexity-preserving property

**Theorem 4.2** Let \( f \in C[0, 1] \). If \( f \) is convex on \([0, 1]\), so are all of its (\( \alpha, q \))-Bernstein operators \( T_{n,q,a}(f; x) \) for fixed \( q \in (0, 1) \) and \( \alpha \in [0, 1] \).

**Proof** From (5), we obtain

\[
T_{n,2,q,a}(f; x) = (1 - \alpha) \sum_{i=0}^{n+1} \frac{n+1}{i} \binom{n+1}{i} x^i (1-x)^{n-i+1} + \alpha \sum_{i=0}^{n+2} f_i \binom{n+2}{i} x^i (1-x)^{n+2-i},
\]

where \( f_i = \frac{[3]}{[2]} \binom{3}{2} g_i = (1 - \frac{q^{n+1}-1}{q^n+1}) f_{i+1} + \frac{q^{n+1}+1}{[n+1]} f_{i+1} \). The \( q \)-derivative of \( T_{n,2,q,a}(f; x) \) can easily obtained by the proof theorem 4.1, which may be expressed as

\[
D_q[T_{n,2,q,a}(f; x)] = (1 - \alpha)[n+1] \sum_{i=0}^{n} \binom{n}{i} \binom{n+1}{i} x^i (1-x)^{n-i} (g_{i+1} - g_i)
\]

\[+ \alpha [n+2] \sum_{i=0}^{n+1} \binom{n+1}{i} x^i (1-x)^{n+1-i} (f_{i+1} - f_i).
\]

Then we have

\[
D_q^2[T_{n,2,q,a}(f; x)] = (1 - \alpha)[n+1] \sum_{i=0}^{n} \binom{n}{i} \binom{n+1}{i} x^i (1-x)^{n-i} (g_{i+1} - g_i) D_q x^i (1-x)^{n-i} + \alpha [n+2] \sum_{i=0}^{n+1} \binom{n+1}{i} x^i (1-x)^{n+1-i} (f_{i+1} - f_i) D_q x^i (1-x)^{n+1-i}.
\]

By some easy computations, we obtain

\[
D_q^2[T_{n,2,q,a}(f; x)] = (1 - \alpha)[n+1] \sum_{i=0}^{n-1} \binom{n-1}{i} \binom{n+1}{i} x^i (1-x)^{n-i-1} \triangle_q^2 g_i
\]

\[+ \alpha [n+2] \sum_{i=0}^{n} \binom{n}{i} \binom{n+1}{i} x^i (1-x)^{n-i} \triangle_q^2 f_i
\]

where \( \triangle_q^2 g_i = (1 - \frac{q^{n-i+1}}{[n]} \triangle_q^2 f_i + \frac{q^{n-i+1}[n+1]}{[n+1]} \triangle_q^2 f_{i+1} \). By the connection between the second-order \( q \) differences and convexity, we know that \( \triangle_q^2 f_i \) and \( \triangle_q^2 f_{i+1} \) are all non-negative since
Theorem 4.3 For $0 < q_1 \leq q_2 \leq 1$, $\alpha \in [0,1]$ and for $f(x)$ convex on $[0,1]$, then $T_{n,q_2,\alpha}(f;x) \leq T_{n,q_1,\alpha}(f;x)$.

Proof In the following main proof of our results, we must introduce a linear polynomial function:

$$g(x) = \frac{f_{i+1} - f_i}{\binom{i+1}{q} - \binom{i}{q}} \left( x - \frac{\binom{i}{q}}{\binom{n}{q}} \right) + f_i,$$

(17)

where $\frac{\binom{i}{q}}{\binom{n}{q}} \leq x < \frac{\binom{i+1}{q}}{\binom{n}{q}}$. Then it is straightforward to check that $g_i = g\left(\frac{\binom{i}{q}}{\binom{n}{q}}\right)$. Since $f$ is convex on $[0,1]$, the intrinsic linear polynomial function $g(x)$ must be convex on $[0,1]$ as well. Therefore, by the classical results of $q$-Bernstein operators (see [3]), we note that

$$T_{n,q,\alpha}(f;x) = (1-\alpha)B^n_{n-1}(g;x) + \alpha B^n_{n}(f;x).$$

(18)

We have $B^n_{n-1}(g;x) \leq B^n_{n-1}(g;x)$ and $B^n_{n}(f;x) \leq B^n_{n}(f;x)$, and the desired result is obvious. Theorem 4.3 is proved.

Finally, if $f(x)$ is convex, we give the monotonicity of $(\alpha, q)$-Bernstein operators $T_{n,q,\alpha}(f;x)$ with $n$.

Theorem 4.4 If $f(x)$ is convex on $[0,1]$, for fixed $q \in (0,1)$ and $\alpha \in [0,1]$, we have

$$T_{n-1,\alpha,q}(f;x) - T_{n,q,\alpha}(f;x) \geq 0 \quad (n \geq 2).$$

Proof Combining (17) and (18), and the fact that if $f$ and $g$ are convex on $[0,1]$, then

$$B^n_{n-2}(g;x) \geq B^n_{n-1}(g;x), \quad B^n_{n-1}(f;x) \geq B^n_{n}(f;x)$$

(see [25]). The desired result is obvious.

Example 4.5 Letting the convex function $f(x) = 1 - \sin(\pi x)$, $x \in [0,1]$, the graphs of $f(x)$ and $T_{n,0.9,0.9}(f;x)$ with different values of $n = 10, 15, 20, 30$ are shown in Fig. 3. Figure 4 shows the graphs of $f(x) = 1 - \sin(\pi x)$ and $T_{10,0.9,0.9}(f;x)$ with $q = 0.6, 0.7, 0.8, 0.9$.

5 Conclusion

In this paper, we proposed a new family of generalized Bernstein operators, named $(\alpha, q)$-Bernstein operators, and denoted by $T_{n,q,\alpha}(f)$. We study the rate of convergence of these operators, investigate their monotonicity-, convexity-preserving properties with respect to $f(x)$, and also obtain their monotonicity with $n$ and $q$ of $T_{n,q,\alpha}(f)$.
Figure 3 Monotonicity of $T_{n,q}(f;x)$ in the parameter $n$.

Figure 4 Monotonicity of $T_{n,q}(f;x)$ in the parameter $q$. 
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