General Interpolation and Strong Amalgamation for Contiguous Arrays

Silvio Ghilardi\textsuperscript{1}, Alessandro Gianola\textsuperscript{2}, Deepak Kapur\textsuperscript{3}, and Chiara Naso\textsuperscript{1}

\textsuperscript{1} Dipartimento di Matematica, Università degli Studi di Milano (Italy)
\textsuperscript{2} Faculty of Computer Science, Free University of Bozen-Bolzano (Italy)
\textsuperscript{3} Department of Computer Science, University of New Mexico (USA)

\texttt{gianola@inf.unibz.it}

Abstract. Interpolation is an essential tool in software verification, where first-order theories are used to constrain datatypes manipulated by programs. In this paper, we introduce the datatype theory of contiguous arrays with maxdiff, where arrays are completely defined in their allocation memory and for which maxdiff returns the max index where they differ. This theory is strictly more expressive than the array theories previously studied. By showing via an algebraic analysis that its models strongly amalgamate, we prove that this theory admits quantifier-free interpolants and, notably, that interpolation transfers to theory combinations. Finally, we provide an algorithm that significantly improves the ones for related array theories: it relies on a poly-size reduction to general interpolation in linear arithmetics, thus avoiding impractical full terms instantiations and unbounded loops.

1 Introduction

Craig Interpolation Theorem \cite{13} is a well-known result in first logic that, given an entailment between two logical formulae $\alpha$ and $\beta$, establishes the existence of a third formula $\gamma$ that shares its non-logical symbols with both $\alpha$ and $\beta$ and such that it is entailed by $\alpha$ and entails $\beta$. Studying interpolation has a long-standing tradition in non-classical logics and in algebraic logic. Nevertheless, interpolation has been obtaining an increasing attention in automated reasoning and formal verification since the seminal works by McMillan \cite{23,24}. Indeed, specifically in infinite-state model checking, where an exhaustive, explicit exploration of the state space is not possible, computing interpolants has been proven to be a useful method for practical and efficient approximations of preimage computation. In the context of software verification, the initial configurations and the transitions relation are usually represented symbolically by means of logical formulae, which gives the possibility of implicitly encoding the execution traces of the system. More precisely, if $T$ is the first order theory that constraints the state space, one can symbolically express via a suitable $T$-inconsistent formula the fact that the system, starting from its initial configuration, cannot reach in $n$-steps an error configuration. Through this inconsistency, interpolants are then extracted from the symbolic representations of these ‘error’ traces with the goal of helping
the search of (safety) invariants of the modeled system: interpolants can be successfully used to refine and improve the construction of the candidate safety invariants.

Model-checking applications usually require that such computed interpolants are not arbitrary but present specific shapes so as to guarantee their concrete usability. Since in many cases studied in software verification the underlying theories have a decidable quantifier-free fragment (but are undecidable or have prohibitive complexity outside), the most natural choice is to consider quantifier-free interpolants. However, even in case $\alpha$ and $\beta$ are quantifier-free, Craig’s Theorem does not guarantee that an interpolant $\gamma$ is quantifier-free too. Indeed, this property, called ‘quantifier-free interpolation’, does not hold in general for arbitrary first order theories. It is then a non-trivial (and, very often, challenging) problem to prove that useful theories admit quantifier-free interpolation.

In this paper, we are interested in studying the problem of quantifier-free interpolation for an expressive datatype theory that strictly extends the well-studied McCarthy’s theory of arrays with extensionality. The original theory was introduced by McCarthy in [22]; however, in [20] it is shown that quantifier-free interpolation fails for this theory. Moreover, although its quantifier-free fragment is decidable, it is well-known that this theory in its full generality is undecidable [5]. Nonetheless, in the same paper, the authors studied a significant decidable fragment, the so-called ‘array property fragment’, which strictly extends the quantifier-free one. The array property fragment is expressive enough to formalize several benchmarks; however, as proved in [19], it is not closed under interpolation. Thus, a particularly challenging but interesting problem is that of identifying expressive extensions of the quantifier-free fragment of arrays that are still decidable but also enjoy interpolation: this is what we attack in this contribution.

A first attempt in this direction is in [7], where a variant of McCarthy’s theory was introduced by Skolemizing the axioms of extensionality. This variant turned out to enjoy quantifier-free interpolation [7, 35]. However, this Skolem function $\text{diff}$ is generic because its semantic interpretation is undetermined. Moreover, all the array theories mentioned so far allow unlimited out-of-bound write operations and so cannot express the notion of array length, which is fundamental when formalizing the real behavior of programs. In this respect, there are two possible variants that can be considered: (i) the weak length $|\_\_|$ formalizes the minimal interval $[0, |a|]$ of indexes outside which the array $a$ is undefined ($a$ can be undefined also in some location inside $[0, |a|]$); (ii) the (strong) length $\lvert\_\rvert$ represents the exact interval $[0, |a|]$ where the array $a$ is fully defined. Strong length is essential for the faithful logical formalization of benchmarks coming from software verification, such as C programs included in the SV-COMP competition [4].

These are the main reasons why in [17] the theory has been further enriched. There, the semantics of $\text{diff}$, called maxdiff, is uniquely determined in the models of the theory and is more informative: it returns the biggest index where two different arrays differ. In this theory, weak length can be defined: this is no-
table, since it represents a first step toward capturing real program arrays. The main contribution of [17] is to show that this enriched theory has quantifier-free interpolants and its quantifier-free fragment is decidable. Still, some expressive limitations (shared with the previous literature) persist: arrays are not forced to be completely defined inside their allocation interval (when an array satisfies this property, we call it ‘contiguous’), because they might contain undefined values in some location. Hence, strong length cannot be defined. Moreover, although in [17] a complete terminating procedure for computing interpolants is provided, a complexity upper bound is given only in the simple basic case where indexes are just linear orders: for more complex arithmetical theories of indexes, no complexity analysis is carried out and the algorithm becomes quite impractical, since it requires to fully instantiate universal quantifiers coming from the theory axioms with index terms of arbitrarily large size.

In this paper, we overcome all those limitations. For that purpose, we introduce the very expressive theory of contiguous arrays with maxdiff $\text{CARD}(T_I)$ (parameterized over an index theory $T_I$), which improves and strictly extends the theory presented in [17] by requiring arrays to be all contiguous. This makes the theory more adequate to represent arrays used in common programming languages: for instance, strong length is now definable. Moreover, in contrast to [17] where only amalgamation is shown, we prove here a strong amalgamation result, when $\text{CARD}(T_I)$ is enriched with ‘constant arrays’ of a fixed length with a default value in all their locations. Notably, this not only yields that quantifier-free interpolants exist, but also that interpolation is preserved under disjoint signatures combinations and holds in presence of free function symbols (see the definition of ‘general interpolation’ below). This result is completely novel and particularly challenging to be proven, since it requires a sophisticated model-theoretic machinery and a careful algebraic analysis of the class of all models. We also radically re-design the interpolation algorithm, avoiding the use of unbounded loops and of impractical full instantiation routines. Our new algorithm reduces the computation of interpolants of a jointly unsatisfiable pair of constraints to a polynomial size instance of the same problem in the underlying index theory enriched with unary function symbols. As such, the new algorithm becomes part of the hierarchical interpolation algorithms family [31] and in particular formally resembles the algorithm presented in [35] for array theory enriched with the basic $\text{diff}$ symbol. We underline that one aspect making our problems technically more challenging than similar problems investigated in the literature is the fact that we handle a combination with very expressive index theories: such a combination is non-disjoint because the total orderings on indexes enter into the specification of the maxdiff and length axioms for arrays.

1.1 Plan of the paper

In the following, we call $\mathcal{EUF}$ the theory of equality and uninterpreted symbols. We introduce two novel theories in Section 3 $\text{CARD}(T_I)$, i.e., the theory of contiguous arrays with maxdiff, and $\text{CARDC}(T_I)$, which is an extension of $\text{CARD}(T_I)$ also containing ‘constant arrays’ of a fixed length with a default
value (called ‘el’) in all locations. The main technical results are that, for every index theory $T_I$:

(i) $\text{CARD}(T_I)$ has general quantifier-free interpolation;

(ii) $\text{CARD}(T_I)$ enjoys quantifier-free interpolation and such interpolants can be computed hierarchically by relying on a black-box interpolation algorithm for the weaker theory $T_I \cup \text{EUF}$ (which has quantifier free interpolation because $T_I$ is strongly amalgamable, see Theorem 1).

Result (i) is proved semantically, i.e., we show the equivalent strong amalgamation property (see Section 2 for the definitions). The semantic proof requires dedicated constructions (Section 5), relying on some important facts about models and their embeddings (Section 4).

The fact that $\text{CARD}(T_I)$ has interpolants follows from the results in Section 5 (where we prove that this theory is amalgamable). Result (ii) is proved last (Section 7); we first need an investigation on the solvability of the $\text{SMT}(\text{CARD}(T_I))$ problem (Section 6).

We supply here some intuitions about our interpolation algorithm from Section 7. The algorithm computes an interpolant out of a pair of (suitably preprocessed) mutually unsatisfiable quantifier-free formulae $A^0, B^0$. We call common the variables occurring in both $A^0$ and $B^0$. The existence of quantifier-free interpolants intuitively means that there are two reasoners, one for $A^0$ and one for $B^0$, the first (the second, resp.) of which operates on formulae involving only variables from $A_0$ ($B_0$, resp.). The reasoners discover the inconsistency of $A^0 \land B^0$ by exchanging information on the common language, i.e., by communicating each other only the entailed quantifier-free formulae over the common variables. The information exchange is hierarchical, i.e., it is limited to $T_I \cup \text{EUF}$-formulae: literals from the richer language of $\text{CARD}(T_I)$ and outside the language of $T_I \cup \text{EUF}$ can contribute to the information exchange only via instantiation of the universal quantifiers in suitable $T_I \cup \text{EUF}$-formulae given in Section 3; these formulae, as proved in Lemmas 2 and 3, supply equivalent definitions of such literals. In contrast to [17], instantiations of universal quantifiers is limited to variables and constants for efficiency.

The main problem is to show that the above limited information exchange is sufficient. This is the case thanks to the fact that the the algorithm manipulates iterated diff operators $\text{35}, \text{17}$ (formally defined in Section 3) and it gives names to all such operators when applied to common array variables. Both the production of names for iterated diff-terms and the variable instantiations of the universal quantifiers in the equivalent universal $T_I \cup \text{EUF}$-formulae need in principle to be repeated infinitely many times; what we prove (this is the content of our main Theorem 7 below) is that a pre-determined polynomial size subset of such manipulations is sufficient for the $T_I \cup \text{EUF}$-interpolation module to produce the interpolant we are looking for.4

4 One could reformulate this fact using the $W$-separability framework from [35]; however, using this framework would not sensibly modify the proof of Theorem 7, so we preferred for space reasons and for simplicity to supply proofs within standard direct terminology.
Related work. We already mentioned the related work on first-order theories axiomatizing arrays [22,20,17], which our theories of contiguous arrays strictly extend. Since we adopt a hierarchical approach, our method is closely related to hierarchical interpolation, where interpolants are computed by reduction to a base theory treated as black-box. A non-exhaustive summary of this literature is given by the approach in [28,29], where in the context of linear arithmetic general interpolation is reduced to constraint solving, the one based on local extensions in [30,31,32,31] and the one based on \(W\)-compatibility and finite instantiations of [34,35].

2 Formal Preliminaries

We assume the usual syntactic (e.g., signature, variable, term, atom, literal, formula, and sentence) and semantic (e.g., structure, sub-structure, truth) notions of first-order logic. The equality symbol “\(=\)” is in all signatures. Notations like \(E(x)\) mean that the expression (term, literal, formula, etc.) \(E\) contains free variables only from the tuple \(x\). A ‘tuple of variables’ is a list of variables without repetitions and a ‘tuple of terms’ is a list of terms (possibly with repetitions). These conventions are useful for substitutions: we use them when denoting with \(\phi(t/x)\) (or simply with \(\phi(t)\)) the formula obtained from \(\phi(x)\) by simultaneous replacement of the ‘tuple of variables’ \(x\) with the ‘tuple of terms’ \(t\). A constraint is a conjunction of literals. A formula is universal (existential) iff it is obtained from a quantifier-free formula by prefixing it with a string of universal (existential, resp.) quantifiers.

Theories and satisfiability modulo theory. A theory \(T\) is a pair \((\Sigma, Ax_T)\), where \(\Sigma\) is a signature and \(Ax_T\) is a set of \(\Sigma\)-sentences, called the axioms of \(T\) (we shall sometimes write directly \(T\) for \(Ax_T\)). The models of \(T\) are those \(\Sigma\)-structures in which all the sentences in \(Ax_T\) are true. A \(\Sigma\)-formula \(\phi\) is \(T\)-satisfiable (or \(T\)-consistent) if there exists a model \(M\) of \(T\) such that \(\phi\) is true in \(M\) under a suitable assignment \(a\) to the free variables of \(\phi\) (in symbols, \((M, a) \models \phi\)); it is \(T\)-valid (in symbols, \(T \models \varphi\)) if its negation is \(T\)-unsatisfiable or, equivalently, \(\varphi\) is provable from the axioms of \(T\) in a complete calculus for first-order logic.

A theory \(T = (\Sigma, Ax_T)\) is universal iff all sentences in \(Ax_T\) are universal. A formula \(\varphi_1\) \(T\)-entails a formula \(\varphi_2\) if \(\varphi_1 \rightarrow \varphi_2\) is \(T\)-valid (in symbols, \(\varphi_1 \vdash_T \varphi_2\) or simply \(\varphi_1 \vdash \varphi_2\) when \(T\) is clear from the context). If \(\Gamma\) is a set of formulæ and \(\phi\) a formula, \(\Gamma \vdash_T \phi\) means that there are \(\gamma_1, \ldots, \gamma_n \in \Gamma\) such that \(\gamma_1 \land \cdots \land \gamma_n \vdash_T \phi\). The satisfiability modulo the theory \(T\) (SMT(\(T\)) problem amounts to establishing the \(T\)-satisfiability of quantifier-free \(\Sigma\)-formulæ (equivalently, the \(T\)-satisfiability of \(\Sigma\)-constraints). Some theories have special names, which are becoming standard in SMT-literature, we shall recall some of them during the paper. As already mentioned, we shall call \(EUF(\Sigma)\) (or just \(EUF\)) the pure equality theory in the signature \(\Sigma\). A theory \(T\) admits quantifier-elimination iff for every formula \(\phi(x)\) there is a quantifier-free formula \(\phi'(x)\) such that \(T \vdash \phi \leftrightarrow \phi'\).
Embeddings and sub-structures

The support of a structure $M$ is denoted with $|M|$. For a (sort, constant, function, relation) symbol $\sigma$, we denote as $\sigma^M$ the interpretation of $\sigma$ in $M$. Let $M$ and $N$ be two $\Sigma$-structures; a $\Sigma$-embedding (or, simply, an embedding) $\mu: M \rightarrow N$ is an injective function from $|M|$ into $|N|$ that preserves and reflects the interpretation of functions and relation symbols (see, e.g., [10] for the formal definition). If such an embedding is a set-theoretical inclusion, we say that $M$ is a substructure of $N$ or that $N$ is a superstructure of $M$. As it is known, the truth of a universal (resp. existential) sentence is preserved through substructures (resp. superstructures).

Given a signature $\Sigma$ and a $\Sigma$-structure $M$, we indicate with $\Delta_\Sigma(M)$ the diagram of $M$: this is the set of sentences obtained by first expanding $\Sigma$ with a fresh constant $\overline{a}$ for every element $a$ from $|M|$ and then taking the set of ground $\Sigma \cup |M|$-literals which are true in $M$ (under the natural expanded interpretation mapping $\overline{a}$ to $a$). An easy but nevertheless important basic result (to be frequently used in our proofs), called Robinson Diagram Lemma [10], says that, given any $\Sigma$-structure $N$, there is an embedding $\mu : M \rightarrow N$ iff $N$ can be expanded to a $\Sigma \cup |M|$-structure in such a way that it becomes a model of $\Delta_\Sigma(M)$.

Combinations of theories. A theory $T$ is stably infinite iff every $T$-satisfiable quantifier-free formula (from the signature of $T$) is satisfiable in an infinite model of $T$. By compactness, it is possible to show that $T$ is stably infinite iff every model of $T$ embeds into an infinite one (see, e.g., [10]). Let $T_i$ be a stably-infinite theory over the signature $\Sigma_i$ such that the SMT($T_i$) problem is decidable for $i = 1, 2$ and $\Sigma_1$ and $\Sigma_2$ are disjoint (i.e., the only shared symbol is equality). Under these assumptions, the Nelson-Oppen combination result [26] says that the SMT problem for the combination $T_1 \cup T_2$ of the theories $T_1$ and $T_2$ is decidable. Nelson-Oppen result trivially extends to many-sorted languages.

Interpolation properties. In the introduction, we roughly stated Craig’s interpolation theorem [10]. In this paper, we are interested to specialize this result to the computation of quantifier-free interpolants modulo (combinations of) theories.

Definition 1. [Plain quantifier-free interpolation] A theory $T$ admits (plain) quantifier-free interpolation iff for every pair of quantifier-free formulae $\phi, \psi$ such that $\psi \land \phi$ is $T$-unsatisfiable, there exists a quantifier-free formula $\theta$, called an interpolant, such that: (i) $\psi$ $T$-entails $\theta$, (ii) $\theta \land \phi$ is $T$-unsatisfiable, and (iii) only the variables occurring in both $\psi$ and $\phi$ occur in $\theta$.

In verification, the following extension of the above definition is considered more useful.

Definition 2. [General quantifier-free interpolation] Let $T$ be a theory in a signature $\Sigma$; we say that $T$ has the general quantifier-free interpolation property iff for every signature $\Sigma'$ (disjoint from $\Sigma$) and for every pair of ground $\Sigma \cup \Sigma'$-formulae $\phi, \psi$ such that $\phi \land \psi$ is $T$-unsatisfiable there is a ground formula $\theta$ such

\[\text{By this (and similar notions) we mean that } \phi \land \psi \text{ is unsatisfiable in all } \Sigma' \text{-structures whose } \Sigma \text{-reduct is a model of } T.\]
that: (i) $\phi$ $T$-entails $\theta$; (ii) $\theta \land \psi$ is $T$-unsatisfiable; (iv) all relations, constants and function symbols from $\Sigma'$ occurring in $\theta$ also occur in $\phi$ and $\psi$.

By replacing free variables with free constants, it is easily seen that the general quantifier-free interpolation property (Definition 2) implies the plain quantifier-free interpolation property (Definition 1); the converse implication does not hold, however (a counterexample can be found in this paper too, see Example 1 below).

**Amalgamation and strong amalgamation.** Interpolation can be characterized semantically via amalgamation.

**Definition 3.** A universal theory $T$ has the amalgamation property iff, given models $M_1$ and $M_2$ of $T$ and a common submodel $A$ of them, there exists a further model $M$ of $T$ (called $T$-amalgam) endowed with embeddings $\mu_1 : M_1 \rightarrow M$ and $\mu_2 : M_2 \rightarrow M$ whose restrictions to $|A|$ coincide.

A universal theory $T$ has the strong amalgamation property [21] if the above embeddings $\mu_1, \mu_2$ and the above model $M$ can be chosen so to satisfy the following additional condition: if for some $m_1 \in |M_1|, m_2 \in |M_2|$ we have $\mu_1(m_1) = \mu_2(m_2)$, then there exists an element $a$ in $|A|$ such that $m_1 = a = m_2$.

The first point of the following theorem is an old result due to [3]; the second point is proved in [8] (where it is also suitably reformulated for theories which are not universal):

**Theorem 1.** Let $T$ be a universal theory. Then

(i) $T$ has the amalgamation property iff it admits quantifier-free interpolants;

(ii) $T$ has the strong amalgamation property iff it has the general quantifier-free interpolation property.

We underline that, in presence of stable infiniteness, strong amalgamation is a modular property (in the sense that it transfers to signature-disjoint unions of theories), whereas amalgamation is not (see again [8] for details). As a special case, since $\mathcal{EUF}$ has strong amalgamation and is stably infinite, the following result follows:

**Theorem 2.** If $T$ is stably infinite and has strong amalgamation, so does $T \cup \mathcal{EUF}$.

### 3 Arrays with MaxDiff

The McCarthy theory of arrays [22] has three sorts $\text{ARRAY}$, $\text{ELEM}$, $\text{INDEX}$ (called “array”, “element”, and “index” sort, respectively) and two function symbols $\text{rd}$ (“read”) and $\text{wr}$ (“write”) of appropriate arities; its axioms are:

- $\forall y, i, e. \ \text{rd}(\text{wr}(y, i, e), i) = e$
- $\forall y, i, j, e. \ i \neq j \rightarrow \text{rd}(\text{wr}(y, i, e), j) = \text{rd}(y, j)$. 


Arrays *with extensionality* have the further axiom

\[ \forall x, y. x \neq y \rightarrow (\exists i. rd(x, i) \neq rd(y, i)), \]

called the ‘extensionality’ axiom. This theory is not universal and does not have quantifier-free interpolants. Here, we want to introduce a variant of this theory where Axiom (1) is skolemized via a function \( \text{diff} \) with a precise semantic interpretation: it returns the biggest index where two different arrays differ. We first need the notion of index theory.

**Definition 4.** [7] An index theory \( T_I \) is a mono-sorted theory (\( \text{INDEX} \) is its sort) satisfying the following conditions:
- \( T_I \) is universal, stably infinite and has the general quantifier-free interpolation property (i.e., it is strongly amalgamable, see Theorem 7);
- \( \text{SMT}(T_I) \) is decidable;
- \( T_I \) extends the theory \( TO \) of linear orderings with a distinguished element 0.

We recall that \( TO \) is the theory whose only proper symbols (beside equality) are a binary predicate \( \leq \) and a constant 0 subject to the axioms saying that \( \leq \) is reflexive, transitive, antisymmetric and total. Thus, the signature of \( T_I \) contains at least the binary relation symbol \( \leq \) and the constant 0. In the paper, when we speak of a \( T_I \)-term, \( T_I \)-atom, \( T_I \)-formula, etc. we mean a term, atom, formula in the signature of \( T_I \). Below, we use the abbreviation \( i < j \) for \( i \leq j \land i \neq j \). The constant 0 is used to separate ‘positive’ indexes - those satisfying \( 0 \leq i \) - from the remaining ‘negative’ ones.

Examples of index theories are \( TO \) itself, integer difference logic \( \text{IDL} \), integer linear arithmetic \( \text{LIA} \), and real linear arithmetic \( \text{LRA} \). In order to match the requirements of Definition 4 one need however to make a careful choice of the language (see [8] for details): most importantly, notice that integer (resp., real) division by all positive integers should be added to the language of \( \text{LIA} \) (resp. \( \text{LRA} \)). For most applications, \( \text{IDL} \) (which is the theory of integer numbers with 0, ordering, successor and predecessor) is sufficient as in this theory one can model counters for scanning arrays.

Given an index theory \( T_I \), we can now introduce our *contiguous array theory with maxdiff C.ARD(\( T_I \))* (parameterized by \( T_I \)) as follows. We still have three sorts \( \text{ARRAY}, \text{ELEM}, \text{INDEX} \); the language includes the symbols of \( T_I \), the read and write operations \( \text{wr}, \text{rd} \), a binary function \( \text{diff} \) of type \( \text{ARRAY} \times \text{ARRAY} \rightarrow \text{INDEX} \), a unary function \( |\cdot| \) of type \( \text{ARRAY} \rightarrow \text{INDEX} \), as well as constant \( \bot \) of sort \( \text{ELEM} \). The constant \( \bot \) models an undefined value; the term \( \text{diff}(x, y) \) returns the maximum index where \( x \) and \( y \) differ and returns 0 if \( x \) and \( y \) are equal.\(^6\)

The term \( |a| \) indicates the (strong) length of \( a \), meaning that \( a \) is allocated in the interval \( [0, |a|] \) and undefined outside. Formally, the axioms of \( \text{C.ARD}(T_I) \) include, besides the axioms of \( T_I \), the following ones:

\[ \forall y, i, e, \ |\text{wr}(y, i, e)| = |y| \quad (2) \]

\(^6\) Notice that it might well be the case that \( \text{diff}(x, y) = 0 \) for different \( x, y \), but in that case 0 is the only index where \( x, y \) differ.
\[ \forall y, i, \ wr(y, i, \bot) = y \]  
(3)

\[ \forall y, i, e, (e \neq \bot \land 0 \leq i \leq |y|) \rightarrow rd(wr(y, i, e), i) = e \]  
(4)

\[ \forall y, i, j, e, i \neq j \rightarrow rd(wr(y, i, e), j) = rd(y, j) \]  
(5)

\[ \forall y, i, rd(y, i) \neq \bot \leftrightarrow 0 \leq i \leq |y| \]  
(6)

\[ \forall y, |y| \geq 0 \]  
(7)

\[ \forall y, \ diff(y, y) = 0 \]  
(8)

\[ \forall x, y, x \neq y \rightarrow rd(x, diff(x, y)) \neq rd(y, diff(x, y)) \]  
(9)

\[ \forall x, y, i, diff(x, y) < i \rightarrow rd(x, i) = rd(y, i) \]  
(10)

\[ \bot \neq el. \]  
(11)

Axiom (11) prevents the ELEM sort to contain just \( \bot \) (thus trivializing a model). Since an array \( a \) is fully allocated only in the interval \([0, |a|]\), any reading or writing attempt outside that interval should produce some runtime error; similarly, it should be impossible to overwrite \( \bot \) inside that interval. In our declarative context, there is nothing like a ‘runtime error’, so we assume that such illegal operations simply do not produce any effect. However, when applying the theory to code annotations, the verification conditions should include that no memory violation like the above ones occur (that is, when, e.g., a term like \( rd(b, i) \) occurs, it should be accompanied by the proviso annotation \( 0 \leq i \leq |a| \), etc.).

As we shall see the above theory enjoys amalgamation (i.e., plain quantifier-free interpolation) but not strong amalgamation (i.e., it lacks the general quantifier-free interpolation). To restore it, it is sufficient to add some (even limited) support for constant arrays: we call the related theory \( \text{CARDC}(T_I) \).

The extension is interesting by itself, because it increases the expressivity of the language: in \( \text{CARDC}(T_I) \), applying the \( wr \) operation to terms \( \text{Const}(i) \), one can encode all finite lists (if \( T_I \) has a reduct to \( \text{IDL} \)). Formally, \( \text{CARDC}(T_I) \) has an additional unary function \( \text{Const} : \text{INDEX} \rightarrow \text{ARRAY} \), constrained by the following axioms:

\[ \forall i, |\text{Const}(i)| = \max(i, 0). \]  
(12)

\[ \forall i, j, (0 \leq j \land j \leq |\text{Const}(i)|) \rightarrow rd(\text{Const}(i), j) = el. \]  
(13)

(we assume without loss of generality that \( \max \) is a symbol of \( T_I \) - in fact it is definable in it).

The following easy facts will be often used in our proofs:

**Lemma 1.** The following formulae are \( \text{CARD}(T_I) \)-valid

\[ |a| \neq |b| \rightarrow \text{diff}(a, b) = \max(|a|, |b|) \]  
(14)

\[ \max(\text{diff}(a, b), \text{diff}(b, c)) \geq \text{diff}(a, c) \]  
(15)

The next lemma follows from the axioms of \( \text{CARD}(T_I) \):
Lemma 2. An atom like \( a = b \) is equivalent (modulo \( \text{CARD}(T_1) \)) to

\[
\text{diff}(a, b) = 0 \land rd(a, 0) = rd(b, 0).
\]

(16)

An atom like \( a = \text{wr}(b, i, e) \) is equivalent (modulo \( \text{CARD} \)) to the conjunction of the following formulae

\[
(e \neq \bot \land 0 \leq i \leq |b|) \rightarrow rd(a, i) = e
\]

\[
(i < 0 \lor i > |b| \lor e = \bot) \rightarrow rd(a, i) = rd(b, i)
\]

\[
\forall h \ (h \neq i \rightarrow rd(a, h) = rd(b, h)).
\]

(17)

An atom of the kind \( |a| = i \) is equivalent to:

\[
i \geq 0 \land \forall h \ (rd(a, h) \neq \bot \leftrightarrow 0 \leq h \leq i).
\]

(18)

Lemma 3. An atom like \( \text{Const}(i) = a \) is equivalent (modulo \( \text{CARDC}(T_1) \)) to

\[
|a| = i \land \forall h \ (0 \leq h \leq i \rightarrow rd(a, h) = el).
\]

(19)

Similarly to [35] and [17], we now introduce iterated \text{diff} operations, that will be used in our interpolation algorithm. In fact, in addition to \( \text{diff} := \text{diff}_1 \) we need an operator \( \text{diff}_2 \) that returns the last-but-one index where \( a, b \) differ (0 if \( a, b \) differ in at most one index), an operator \( \text{diff}_3 \) that returns the last-but-two index where \( a, b \) differ (0 is they differ in at most two indexes), etc. Our language is already sufficiently expressive for that. Indeed, given array variables \( a, b \), we define by mutual recursion the sequence of array terms \( b_1, b_2, \ldots \) and of index terms \( \text{diff}_1(a, b), \text{diff}_2(a, b), \ldots \):

\[
b_1 := b; \quad \text{diff}_1(a, b) := \text{diff}(a, b_1);
b_{k+1} := \text{wr}(b_k, \text{diff}_k(a, b), rd(a, \text{diff}_k(a, b))); \quad \text{diff}_{k+1}(a, b) := \text{diff}(a, b_{k+1});
\]

A useful fact is that formulae like \( \bigwedge_{j<l} \text{diff}_j(a, b) = k_j \) can be eliminated in favor of universal clauses in a language whose only symbol for array variables is \( rd \). In detail:

Lemma 4. A formula like

\[
\text{diff}_1(a, b) = k_1 \land \cdots \land \text{diff}_l(a, b) = k_l
\]

(20)
is equivalent modulo $\text{CARD}(T_I)$ to the conjunction of the following seven formulae:

\[
\begin{align*}
&k_1 \geq k_2 \land \ldots \land k_{l-1} \geq k_l \land k_l \geq 0 \\
\land_{j<l} (k_j > k_{j+1} \rightarrow rd(a, k_j) \neq rd(b, k_j)) \\
\land_{j<l} (|a| = |b| \land k_j = k_{j+1}) \rightarrow k_j = 0 \\
\land_{j<l} (rd(a, k_j) = rd(b, k_j) \rightarrow k_j = 0)
\end{align*}
\]

(21)

\[
\forall h \ (h > k_l \rightarrow rd(a, h) = rd(b, h) \lor h = k_1 \lor \ldots \lor h = k_{l-1})
\]

\[
\begin{align*}
|a| > |b| & \rightarrow (k_1 = k_l \land k_l = |a|) \\
|b| > |a| & \rightarrow (k_1 = k_l \land k_l = |b|).
\end{align*}
\]

\section{Embeddings}

In this section we present some useful facts about embeddings that will be crucial in the proofs throughout the paper.

We first introduce the third array theory $\text{AR}_{\text{ext}}(T_I)$, which is weaker than $\text{CARD}(T_I)$, lacks the $\text{diff}$ symbol and axiom (10) is replaced by the following extensionality axiom:

\[
\forall x, y, x \neq y \rightarrow (\exists i, rd(x, i) \neq rd(y, i)).
\]

(22)

Notice that $\text{AR}_{\text{ext}}(T_I) \subseteq \text{CARD}(T_I) \subseteq \text{CARDC}(T_I)$ (the inclusion holds both for signatures and for axioms). To simplify the statements of some lemmas below, let us also introduce the theory $\text{CARC}_{\text{ext}}(T_I)$: this theory is obtained from $\text{AR}_{\text{ext}}(T_I)$ by adding the function symbol $\text{Const}$ to the signature and the sentences (12), (13) to the axioms.

We now discuss the class of models of $\text{AR}_{\text{ext}}(T_I)$ and we clarify the important features of embeddings between such models. A model $M$ of $\text{AR}_{\text{ext}}(T_I)$ is functional when the following conditions are satisfied:

(i) $\text{ARRAY}^M$ is a subset of the set of all positive-support functions from $\text{INDEX}^M$ to $\text{ELEM}^M$ (a function $a$ is positive-support iff there exists an index $|a|$ such that $|a| \geq 0$ and, for every $j$, $a(j) \neq \bot$ iff $j \in [0, |a|]$);

(ii) $rd$ is function application;

(iii) $wr$ is the point-wise update operation inside the interval $[0, |a|]$ (i.e., function $wr(a, i, e)$ returns the same values as function $a$, except at the index $i$ and only in case $i \in [0, |a|]$: in this case it returns the element $e$);

(iv) if $M$ is also a model of $\text{CARC}_{\text{ext}}(T_I)$, then the set $\text{ARRAY}^M$ contains the positive-support functions with value $e^M$ inside their support.

Because of the extensionality axiom (22), it can be shown that every model of $\text{AR}_{\text{ext}}(T_I)$ or of $\text{CARC}_{\text{ext}}(T_I)$ is isomorphic to a functional one. For an array $a \in \text{INDEX}^M$ in a functional model $M$ and for $i \in \text{INDEX}^M$, since $a$ is a function, we interchangeably use the notations $a(i)$ and $rd(a, i)$. 
Let $a, b$ be elements of $\text{ARRAY}^M$ in a model $M$. We say that $a$ and $b$ are 
cardinality equivalent iff $|a| = |b|$ and $\{i \in \text{INDEX}^M \mid M \models \text{rd}(a, i) \neq \text{rd}(b, i)\}$ is 
finite. This relation in $M$ is an equivalence, that we denote as $\sim_M$ or simply as 
$\sim$. We also write $M \models a \sim b$ to say that $a \sim_M b$ holds.

**Lemma 5.** Let $N, M$ be models of $\text{AR}_{\text{ext}}(T_I)$ such that $M$ is a substructure 
of $N$. For every $a, b \in \text{ARRAY}^M$, we have that 

$$M \models a \sim b \iff N \models a \sim b.$$ 

In a functional model $M$ of $\text{AR}_{\text{ext}}(T_I)$, we say that $\text{diff}(a, b)$ is defined iff 
there is a maximum index where $a, b$ differ (or if $a = b$). If in the model $M$ the 
index sort $\text{INDEX}$ is interpreted as the set of the integers, with standard ordering, 
then for any two positive-support functions $a, b$, we have that $\text{diff}(a, b)$ is 
defined. However, this will not be the case if the index sort $\text{INDEX}$ is interpreted, 
e.g., in some non-standard model of the integers. We must take into considerations 
these models too, since we want to prove amalgamation. For this purpose, 
we need to build amalgams for all models of the theory (only in that case in 
fact, amalgamation turns out to be equivalent to quantifier-free interpolation). 
Thus, we are forced to take into consideration below also phenomena that might 
arise only in non-standard models.

An embedding $\mu : M \rightarrow N$ between $\text{AR}_{\text{ext}}(T_I)$-models (or of $\text{CARC}_{\text{ext}}(T_I)$- 
models) is said to be $\text{diff}$-faithful iff, whenever $\text{diff}(a, b)$ is defined, so is 
$\text{diff}(\mu(a), \mu(b))$ and it is equal to $\mu(\text{diff}(a, b))$. Since there might not be a 
maximum index where $a, b$ differ, in principle it is not always possible to expand 
a functional model of $\text{AR}_{\text{ext}}(T_I)$ to a functional model of $\text{CARD}(T_I)$, if the set 
of indexes remains unchanged. Indeed, in order to do that in a $\text{diff}$-faithful way, 
one needs to explicitly add to $\text{INDEX}^M$ new indexes including at least the ones 
representing the missing maximum indexes where two given array differ. This 
idea leads to Theorem 3 below, which is the main result of the current section.

**Theorem 3.** For every index theory $T_I$, every model $M$ of $\text{AR}_{\text{ext}}(T_I)$ (resp. of 
$\text{CARC}_{\text{ext}}(T_I)$) has a $\text{diff}$-faithful embedding into a model of $\text{CARD}(T_I)$ (resp. 
of $\text{CARDC}(T_I)$).

**Proof.** It is sufficient to well-order the pairs $a, b \in \text{INDEX}^M$ such that $\text{diff}(a, b)$ 
is not defined in $M$, apply to each pair the construction of the next lemma (taking 
unions at limit ordinals), and then repeat the whole construction $\omega$ times, taking 
union again.

**Lemma 6.** Let $M$ be a model of $\text{AR}_{\text{ext}}(T_I)$ (resp. of $\text{CARC}_{\text{ext}}(T_I)$) and let 
a, $b \in \text{ARRAY}^M$ be such that $\text{diff}^M(a, b)$ is not defined. Then there are a model $N$ 
of $\text{AR}_{\text{ext}}(T_I)$ (resp. of $\text{CARC}_{\text{ext}}(T_I)$) and a $\text{diff}$-faithful embedding $\mu : M \rightarrow N$ 
such that $\text{diff}^N(a, b)$ is defined.

**Proof.** We can assume that $\text{ELEM}^M$ has at least an element $e$, different from 
$\perp^M, \text{ELEM}^M$ (for details, see Lemma [10] in the appendix). We must have $|a| = |b|$, 
otherwise $\text{diff}(a, b)$ is defined and it is $\max(|a|, |b|)$ according to Lemma [1].
Let $I = \{ i \in \text{INDEX}^M \mid a(i) \neq b(i) \}$ be the set of indices without maximum element (hence infinite) where they differ. Let $\downarrow I := \{ j \in \text{INDEX}^M \mid \exists i \in I, j \leq i \} \supseteq I$. The condition

$$(+) \quad \exists i \in I \forall j \in I (j \geq i \rightarrow x(j) = el)$$

cannot be satisfied both for $x = a$ and $x = b$ (see the appendix). In case one of them satisfies it, we assume it is $b$.

Let $\Delta$ be the Robinson diagram of the $T_I$-reduct of $M$ and let $k_0$ be a new constant; let us introduce the set

$$\Delta' := \Delta \cup \{ i < k_0 \mid i \in \downarrow I \} \cup \{ k_0 < i \mid i \in \text{INDEX}^M \setminus \downarrow I \}.$$

By compactness theorem and since $I$ is infinite, the set $\Delta'$ turns out to be consistent (see again the appendix for details).

By Robinson Diagram Lemma, there exists a model $A$ of $T_I$ extending the $T_I$-reduct of $M$; such $A$ contains in its support an element $k_0$ such that

$$\forall i \in \downarrow I, i < k_0,$$

$$\forall i \in \text{INDEX}^M \setminus \downarrow I, k_0 < i.$$

We now take $\text{ELEM}^N = \text{ELEM}^M$, $\text{INDEX}^N = \text{INDEX}^A$; we let also $\text{ARRAY}^N$ to be the set of all positive-support functions from $\text{INDEX}^N$ into $\text{ELEM}^N$ (notice that this $N$ is trivially also a model of $\text{CARC}^{\text{ext}}(T_I)$). We observe that $k_0 < |a|^M$ and recall that $|a|^M = |b|^M$.

Let us now define the embedding $\mu : M \rightarrow N$: at the level of the sorts $\text{INDEX}$ and $\text{ELEM}$, we use inclusions. For the $\text{ARRAY}$ sort, we need to specify the value $\mu(c)(k)$ for $c \in \text{ARRAY}^M$ and $k \in \text{INDEX}^N \setminus \text{INDEX}^M$ (for the other indices we keep the old $M$-value to preserve the read operation). Our definition for $\mu$ must preserve the maxdiff index (whenever already defined in $M$) and must guarantee that $\text{diff}^N(\mu(a), \mu(b)) = k_0$ (by construction, we have $k_0 > 0$). For a generic array $c \in \text{ARRAY}^M$, we operate as follows:

1. if $|c|^M < k_0$ we put $\mu(c)(k_0) = \bot^M$, otherwise:
2. if the condition $(\star)$ below holds, we put $\mu(c)(k_0) = e$,
3. if such condition does not hold, we put $\mu(c)(k_0) = e \cdot M$.

The condition $(\star)$ is specified as follows:

$$(\star) \quad \text{there is } i \in I \text{ such that for all } j \in I, j \geq i \text{ we have } c(j) = a(j).$$

For all the remaining indexes $k \in \text{INDEX}^N \setminus (\text{INDEX}^M \cup \{ k_0 \})$:

$$\mu(c)(k) = \begin{cases} \bot^M, & \text{if } k \notin [0, |c|^M] \\ e \cdot M, & \text{if } k \in [0, |c|^M]. \end{cases} \quad (23)$$

Notice that we have $\mu(a)(k_0) = e \neq e \cdot M = \mu(b)(k_0)$ (because $I$ is infinite and does not have maximum, hence condition $(\star)$ holds for $a$ but not for $b$). In addition:
for all $i \in \text{INDEX}^M$ such that $k_0 < i$, we have $i \notin \downarrow I$ (according to the construction of $k_0$) and consequently $i \notin I$, that is $a(i) = b(i)$;

• for all $i \in \text{INDEX}^N \setminus (\text{INDEX}^M \cup \{k_0\})$ such that $k_0 < i$, since we have $|a|^M = |b|^M$, we get

\[ \mu(a)(i) = \bot^M \iff \mu(b)(i) = \bot^M, \]
\[ \mu(a)(i) = \text{el}^M \iff \mu(b)(i) = \text{el}^M. \]

Hence, we can conclude that \( \text{diff}^N(\mu(a), \mu(b)) \) is defined and equal to $k_0$.

We only need to check that our $\mu$ preserves $rd, |−|, wr$, constant arrays and \( \text{diff} \) (whenever defined). For preservation of constant arrays, we need to prove only that $\mu(\text{Const}(i))(k_0) = \text{el}^M$ in case $k_0 < i$: this is clear, because $a$ does not satisfy $(\oplus)$, hence $(\ast)$ does not hold for $\text{Const}(i)$. The other cases are proved in the appendix.

5 Strong Amalgamation for $\text{CARDC}(T_I)$

In this section, we prove that the most expressive theory of the paper $\text{CARDC}(T_I)$ has strong amalgamation. However, we also show that this is not the case for $\text{CARD}(T_I)$ (even if it is amalgamable). We recall that strong amalgamation holds for models of $T_I$ (see Definition 1): this observation is crucial for the following.

Strong amalgamation of $\text{CARD}(T_I)$ will be proved in two steps. First, we provide the amalgam construction for $\text{CARDC}(T_I)$, where we also notice that the same arguments can be used to prove that $\text{CARD}(T_I)$ has amalgamation too. Then, after exhibiting a counterexample showing that the strong amalgamation fails for $\text{CARD}(T_I)$, we check that the amalgam construction for $\text{CARDC}(T_I)$ satisfies the condition for being a $\text{CARDC}(T_I)$-strong amalgam.

5.1 Amalgam constructions

Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two models of $\text{CARDC}(T_I)$ (resp. of $\text{CARD}(T_I)$); we want to amalgamate them over their common substructure $\mathcal{A}$ and let $f_i$ be the embedding of $\mathcal{A}$ into $\mathcal{M}_i$ (we assume that $f_i$ is just inclusion for the INDEX and ELEM components). We can assume w.l.o.g. that our models are all functional and, by applying renaming, that

\[ \text{(INDEX}^{\mathcal{M}_1} \setminus \text{INDEX}^\mathcal{A}) \cap (\text{INDEX}^{\mathcal{M}_2} \setminus \text{INDEX}^\mathcal{A}) = \emptyset \]
\[ \text{(ELEM}^{\mathcal{M}_1} \setminus \text{ELEM}^\mathcal{A}) \cap (\text{ELEM}^{\mathcal{M}_2} \setminus \text{ELEM}^\mathcal{A}) = \emptyset. \]

We build the amalgamated model in two steps. We first embed $\mathcal{M}_1$ and $\mathcal{M}_2$, via the embeddings $\mu_i$ ($i=1,2$), into a model $\mathcal{M}$ of $\text{CARD}_{\text{ext}}(T_I)$ (resp., of $\text{AR}_{\text{ext}}(T_I)$) in a diff-faithful way. Then $\mathcal{M}$ is embedded, via another diff-faithful embedding $\mu'$ into a model $\hat{\mathcal{M}}$ of $\text{CARDC}(T_I)$ (resp., of $\text{CARD}(T_I)$): $\mu'$ is guaranteed
to exist by Theorem 3.

\[
\begin{align*}
M_1 & \xrightarrow{f_1} A \xleftarrow{\mu_i} M \\
M_2 & \xrightarrow{f_2} A \xleftarrow{\mu'_{\tilde{i}}} M
\end{align*}
\]

**Construction of \(\mu_i\)**

We build the model \(M\) and the two \textit{diff}-faithful embeddings \(\mu_i : M_i \to M\) such that \(\mu_1 \circ f_1 = \mu_2 \circ f_2\).

We let \(\text{INDEX}^M\) be a strong amalgam of \(\text{INDEX}^{M_1}\) and \(\text{INDEX}^{M_2}\) (\(T_I\) enjoys strong amalgamation), whereas we let \(\text{ELEM}^M = \text{ELEM}^{M_1} \cup \text{ELEM}^{M_2}\). Let \(\text{ARRAY}^M\) be the set of all positive-support functions from \(\text{INDEX}^M\) into \(\text{ELEM}^M\).

The \text{INDEX} and \text{ELEM} components of the embeddings \(\mu_i\) will be inclusions. The definition of the value of \(\mu_i(a)(k)\), for \(a \in \text{ARRAY}^M\) and \(k \in \text{INDEX}^M\), is given by cases as follows:

- if \(k \in \text{INDEX}^M\), we put \(\mu_i(a)(k) = a(k)\);
- if \(k \in \text{INDEX}^{M_{3-i}} \setminus \text{INDEX}^M\) let \((2\ast)\) be the relation:

\[
\begin{align*}
\mu_i(a)(k) &= \begin{cases} 
  f_{3-i}(c)(k), & \text{if } (2\ast) \text{ holds} \\
  \bot^M, & \text{if } (2\ast) \text{ does not hold } \& \ k \notin [0, |a|^M_i] \\
  \text{el}^M, & \text{if } (2\ast) \text{ does not hold } \& \ k \in [0, |a|^M_i]
\end{cases}
\end{align*}
\]

- if \(k \notin \text{INDEX}^M \cup \text{INDEX}^{M_{3-i}}\), we put

\[
\mu_i(a)(k) = \begin{cases} 
  \bot^M, & \text{if } k \notin [0, |a|^M_i] \\
  \text{el}^M, & \text{if } k \in [0, |a|^M_i]
\end{cases}
\]

We need to prove that the functions \(\mu_i\): (i) are well-defined, (ii) are injective, (iii) preserve \(|-|\), (iv) preserve \(rd\) and \(wr\), (v) preserve \text{diff}, (vi) satisfy the condition \(\mu_1 \circ f_1 = \mu_2 \circ f_2\), (vii) preserve constant arrays (this point is not needed for \(\text{CARD}(T_I)\)). These checks require a careful analysis of all the cases: all the details can be found in the appendix.

As a consequence of these constructions, we get the following result for both \(\text{CARD}(T_I)\) and \(\text{CARD}(T_I)\).

**Theorem 4.** \(\text{CARD}(T_I)\) and \(\text{CARD}(T_I)\) enjoy the amalgamation property.

\(^7\) When we write \(k > \text{diff}^M(b, f_i(c))\) we mean in fact that \(k > \mu_i(\text{diff}^M(b, f_i(c)))\) (this relation is meant to hold in \(M\)). Our simplified notation is justified by the fact that \(\mu_i\) is inclusion for \(\text{INDEX}\) sort.
5.2 The $\textit{CARDC}(T_1)$-amalgam is strong

We now prove the main result of the section, i.e., strong amalgamation for $\textit{CARDC}(T_1)$. Unfortunately, this property does not hold for $\textit{CARD}(T_1)$: as it is shown in the example below, we need constant arrays in the language (recall that strong amalgamation is equivalent to general quantifier-free interpolation, see Theorem 1(ii)).

**Example 1.** Consider the following two formulae (where $P$ is a free predicate symbol):

\[
\begin{align*}
(A) & \quad |a| = 0 \land rd(a, 0) = e \land P(a) \\
(B) & \quad |b| = 0 \land rd(b, 0) = e \land \neg P(b).
\end{align*}
\]

The conjunction $(A) \land (B)$ is inconsistent because $a$ and $b$ are in fact the same array (because of Axioms [6] and [9]). However, the only common variable is $c$; to get the interpolant, we can use $P(wr(\text{Const}(0), 0, e))$, but then it is clear that the language lacking constant arrays does not suffice.

**Theorem 5.** $\textit{CARDC}(T_1)$ has strong amalgamation.

**Proof.** We keep the same notation and construction as in the proof of Theorem 4. However, it can be proved (for details, see Lemma 7 in the appendix) that all arrays $a \in \text{ARRAY}^M_i$ (for $i = 1, 2$) whose length belongs to $\text{INDEX}^A$ can be assumed to be s.t. one of the following two conditions are satisfied:

1. there exists $c \in \text{ARRAY}^A$ with $f_i(c) \sim^M_i a$;
2. there exists $k_a \in \text{INDEX}^M_i \setminus \text{INDEX}^A$ such that $a(k_a)$ is an element from $\text{ELEM}^M_i \setminus \text{ELEM}^A$ different from all the $f_i(c)(k_a)$, varying $c \in \text{ARRAY}^A$.

Let $a_i \in \text{ARRAY}^M_i$ ($i = 1, 2$) be s. t. $\forall k \in \text{INDEX}^M_i$ we have

\[
(24) \quad \mu_1(a_1)(k) = \mu_2(a_2)(k).
\]

Notice that, since $T_1$ has the strong amalgamation property and the $\mu_i$ preserve length, this can only happen if $|a_1| = |a_2|$ belongs to $\text{INDEX}^A$. We look for some $c \in \text{ARRAY}^A$ such that $a_1 = f_1(c)$; since $\mu_2$ is injective this would entail $a_2 = f_2(c)$ because $\mu_2(a_2) = \mu_1(a_1) = \mu_1(f_1(c)) = \mu_2(f_2(c))$, implying that $M$ is a strong amalgam, as requested.

We separate two cases: (i) one of the arrays $a_1, a_2$ satisfy condition 2; (ii) both arrays $a_1, a_2$ satisfy condition 1.

(i) This case is impossible: this is proved in the appendix, by exploiting condition 2 and the definition of $\mu_2$ via rule (2*).

(ii) Hence, we have $\mu_1(a_1) = \mu_2(a_2)$ only when both $a_1, a_2$ satisfy condition 1.

Let $c_i \in \text{ARRAY}^A$ ($i = 1, 2$) be the arrays s.t. $f_i(c_i) \sim^M_i a_i$. In the appendix, we prove how to build, from $c_2$, a $c \in \text{ARRAY}^A$ such that $f_1(c) = a_1$.

Strong amalgamation corresponds to general quantifier-free interpolation (Theorem 1), hence we obtain that:

**Corollary 1.** The theory $\textit{CARDC}(T_1)$ has the general quantifier-free interpolation property.
6 Satisfiability

We now address the problem of checking satisfiability of quantifier-free formulæ in our theories. Decidability of the SMT(CARD(T)) and SMT(CARDC(T))-problems, at least in the relevant case where T is any fragment of Presburger arithmetics, can be solved by reduction to the satisfiability problem for the so-called ‘array-property fragment’ of \( \mathcal{L} \): the reduction can be obtained by eliminating the wr, \([\cdot,\cdot]\), diff and Const symbols in favor of universally quantified formulæ belonging to that fragment (see Lemmas 2314). However, we now supply a decision procedure for quantifier-free formulæ, since this will be useful for the interpolation algorithm in Section 7.

A flat literal \( L \) is a formula of the kind \( x_0 = f(x_1,\ldots,x_n) \) or \( x_1 \neq x_2 \) or \( R(x_1,\ldots,x_n) \) or \( \neg R(x_1,\ldots,x_n) \), where the \( x_i \) are variables, \( R \) is a relation symbol, and \( f \) is a function symbol. If \( I \) is a set of \( T \)-terms, an \( I \)-instance of a universal formula of the kind \( \forall \phi \) is a formula of the kind \( \phi(t/i) \) for some \( t \in I \).

**Definition 5.** A pair of sets of quantifier-free CARD(T)-formulæ \( \Phi = (\Phi_1, \Phi_2) \) is a separated pair iff

1. \( \Phi_1 \) contains equalities of the form \( |a| = i, \text{diff}_k(a,b) = i \) and \( a = \text{wr}(b,i,e) \); moreover if it contains the equality \( \text{diff}_k(a,b) = i \), it must also contain an equality of the form \( \text{diff}_j(a,b) = j \) for every \( i < k \); finally, if \( \Phi_1 \cup \Phi_2 \) contains occurrences of an array variable \( a \), \( \Phi_1 \) must contain also an equality of the form \( |a| = i \);
2. \( \Phi_2 \) contains Boolean combinations of \( T \)-atoms and of atoms of the forms:

\[
\text{rd}(a,i) = \text{rd}(b,j), \quad \text{rd}(a,i) = e, \quad e_1 = e_2,
\]

where \( a, b, i, j, e, e_1, e_2 \) are variables or constants of the appropriate sorts.

\( \Phi \) is said to be finite iff \( \Phi_1 \) and \( \Phi_2 \) are both finite.

Notably, in a separated pair \( \Phi = (\Phi_1, \Phi_2) \), if we introduce a unary function symbol \( f_\alpha \) for every array variable \( a \) and rewrite \( \text{rd}(a,i) \) as \( f_\alpha(i) \), it turns out that the formulæ from \( \Phi_2 \) can be seen as formulæ of the combined theory \( T \cup \mathcal{U} \mathcal{F} \). \( T \cup \mathcal{U} \mathcal{F} \) is decidable in its quantifier-free fragment and admits quantifier-free interpolation because \( T \) is an index theory (see Nelson-Oppen results 26 and Theorems 12): we adopt a hierarchical approach (similarly to 3133) and we rely on satisfiability and interpolation algorithms for such a theory as black boxes.

**Definition 6.** Let \( I \) be a set of \( T \)-terms and let \( \Phi = (\Phi_1, \Phi_2) \) be a separated pair: \( \Phi(I) = (\Phi_1(I), \Phi_2(I)) \) is the smallest separated pair satisfying the following conditions:

1. \( \Phi_1(I) \) is equal to \( \Phi_1 \) and \( \Phi_2(I) \) contains \( \Phi_2 \);
2. if \( \Phi_1 \) contains the atom \( a = \text{wr}(b,i,e) \) then \( \Phi_2(I) \) contains all the \( I \)-instances of the formulæ 17 (with the terms \( |a|, |b| \) replaced by the index constants \( i, j \) such that \( |a| = i, |b| = j \in \Phi_1 \), respectively);
3. if \( \Phi_1 \) contains the atom \( |a| = i \), then \( \Phi_2(I) \) contains all the \( I \)-instances of the formulæ 18.
if $\Phi_1$ contains the conjunction $\bigwedge_{i=1}^l \text{diff}_i(a,b) = k$, then $\Phi_2(I)$ contains the formulae (21) (with the terms $|a|, |b|$ replaced by the index constants $i, j$ such that $|a| = i, |b| = j \in \Phi_1$, respectively).

A separated pair $\Phi$ is 0-instantiated iff $\Phi = \Phi(I)$, where $I$ is the set of index variables occurring in $I$.

We say that a separated pair $\Phi = (\Phi_1, \Phi_2)$ is $\text{CARD}(T_I)$-satisfiable iff so it is the formula $\bigwedge \Phi_1 \land \bigwedge \Phi_2$.

Lemma 7. Let $\phi$ be a quantifier-free formula; then it is possible to compute in linear time a finite separation pair $\Phi = (\Phi_1, \Phi_2)$ such that $\phi$ is $\text{CARD}(T_I)$-satisfiable iff so is $\Phi$.

Lemma 8. The following conditions are equivalent for a finite 0-instantiated separation pair $\Phi = (\Phi_1, \Phi_2)$:

(i) $\Phi$ is $\text{CARD}(T_I)$-satisfiable;

(ii) $\bigwedge \Phi_2$ is $T_I \cup \text{EU}$-satisfiable.

From Lemmas 7 and 8 we get the following result:

Theorem 6. The $\text{SMT}(\text{CARD}(T_I))$ problem is decidable for every index theory $T_I$.

Regarding complexity for the $\text{SMT}(\text{CARD}(T_I))$ problem, notice that the satisfiability of the quantifier-free fragment of common index theories (like $\text{IDL}$, $\text{LIA}$, $\text{LRA}$) is decidable in NP; hence, for such index theories, an NP bound for our $\text{SMT}(\text{CARD}(T_I))$-problems is easily obtained, because 0-instantiation is clearly finite and polynomial (all strings of universal quantifiers to be instantiated have length one).

The above decidability and complexity results apply also to $\text{CARDC}(T_I)$: one only simply has to allow the $\Phi_1$-component of a separation pair to contain also atoms of the form $\text{const}(i) = a$ and Definition 6 to require that $\Phi_2(I)$ contains all the $I$-instances of the formulae (19) in case $\text{const}(i) = a \in \Phi_1$.

7 The interpolation algorithm

Since amalgamation is equivalent to quantifier-free interpolation for universal theories such as $\text{CARD}(T_I)$ and $\text{CARDC}(T_I)$ (thanks to Theorem 1), Theorem 4 guarantees that $\text{CARD}(T_I)$ and $\text{CARDC}(T_I)$ admit quantifier-free interpolation. However, the proof of Theorem 4 is not constructive; hence, in order to compute an interpolant for an unsatisfiable conjunction like $\psi(x, y) \land \phi(y, z)$, one needs in principle to enumerate all quantifier-free formulae $\theta(y)$ that are consequences of $\phi$ and are inconsistent with $\psi$. Since the quantifier-free fragments of $\text{CARD}(T_I)$ and $\text{CARDC}(T_I)$ are decidable, this is an effective procedure and, considering the fact that interpolants of jointly unsatisfiable pairs of formulae exist, it also terminates. However, this type of algorithm is not practical. In this section, we
provide a better and more practical algorithm that relies on a hierarchical reduction to \( T_I \cup \text{EUF} \). Our algorithm works for \( \text{CARD}(T_I) \) only; for \( \text{CARDC}(T_I) \), we make some comments in Section 8.

Our problem is the following: given two quantifier-free formulae \( A_0 \) and \( B_0 \) such that \( A_0 \land B_0 \) is not satisfiable (modulo \( \text{CARD}(T_I) \)), to compute a quantifier-free formula \( C \) such that

1. \( \text{CARD}(T_I) \models A_0 \rightarrow C \);
2. \( \text{CARD}(T_I) \models C \land B_0 \rightarrow \bot \);
3. \( C \) contains only the variables (of sort \( \text{INDEX}, \text{ARRAY}, \text{ELEM} \)) which occur both in \( A_0 \) and in \( B_0 \).

Below, we work with ground formulae over signatures expanded with free constants instead of quantifier-free formulae. We use letters \( A, B, \ldots \) for finite sets of ground formulae; the logical reading of a set of formulae is the conjunction of its elements. For a signature \( \Sigma \) and a set \( A \) of formulae, \( \Sigma^A \) denotes the signature \( \Sigma \) expanded with the free constants occurring in \( A \). Let \( A \) and \( B \) be two finite sets of ground formulae in the signatures \( \Sigma^A \) and \( \Sigma^B \), resp., and \( \Sigma^C := \Sigma^A \cap \Sigma^B \). We ‘color’ a term, a literal, or a formula \( \varphi \) by calling it:

- \( \text{AB-common} \) iff it is defined over \( \Sigma^C \);
- \( \text{A-local} \) (resp. \( \text{B-local} \)) if it is defined over \( \Sigma^A \) (resp. \( \Sigma^B \));
- \( \text{A-strict} \) (resp. \( \text{B-strict} \)) iff it is \( \text{A-local} \) (resp. \( \text{B-local} \)) but not \( \text{AB-common} \);
- \( \text{strict} \) if it is either \( \text{A-strict} \) or \( \text{B-strict} \).

There are a number of manipulations that can be freely applied to a jointly unsatisfiable pair \( A, B \) without compromising the possibility of extracting an interpolant out of them. A list of such manipulations (called ‘metarules’) is supplied in [7], [8]. Here we need to introduce only some of them:

1. we can add to \( A \) an \( \text{A-local} \) quantifier-free formula entailed by \( A \) (similarly we can add to \( B \) a \( \text{B-local} \) quantifier-free formula entailed by \( B \)): the interpolant computed after such a transformation is trivially an interpolant for the original pair too;
2. we can pick an \( \text{A-local} \) term \( t \) and a fresh constant \( x \) (to be considered \( \text{A-strict} \) from now on) and add to \( A \) the equality \( x = t \); again, the interpolant computed after such a transformation is trivially an interpolant for the original pair too (the same observation extends to \( B \));
3. we can pick an \( \text{AB-common} \) term \( t \) and a fresh constant \( x \) (to be considered \( \text{AB-common} \) from now on) and add to both \( A \) and \( B \) the equality \( x = t \): in this case, if \( \theta \) is the interpolant computed after such a transformation, then \( \theta(t/x) \) is an interpolant for the original pair.

We shall often apply the above metarules (i)-(ii)-(iii) in the sequel.

### 7.1 The Algorithm

*We reduce the problem of finding an interpolant of an unsatisfiable pair \((A^0, B^0)\) to an analogous polynomial size problem in the weaker theory \( T_I \cup \text{EUF} \).*

Our unsatisfiable pair \((A^0, B^0)\) needs to be preprocessed. Using the procedure in the proof of Lemma 7, we can suppose that both \( A^0 \) and \( B^0 \) are given in the form of finite separated pairs. In fact, the procedure of Lemma 7 just introduces
constants in order to explicitly name terms, so that it fits within the above explained remarks (see metarules (ii)-(iii)). The newly introduced constants are colored \( A \)-strict, \( B \)-strict or \( AB \)-common depending on the color of the terms they name. Notice that because of this preprocessing, for every \( A \)-strict (resp. \( B \)-strict, \( AB \)-common) array constant \( a \), in \( A^0 \) (resp. \( B_0 \), \( A^0 \land B^0 \)) there is an atom of the kind \(|a| = l_a\).

To sum up, \( A^0 \) is of the form \( \bigwedge A_1^0 \land \bigwedge A_2^0 \) and \( B^0 \) is of the form \( \bigwedge B_1^0 \land \bigwedge B_2^0 \), for separated pairs \((A_1^0, A_2^0)\) and \((B_1^0, B_2^0)\).

Our interpolation algorithm consists of three transformation steps (all of them fit our metarules (i)-(ii)-(iii)). We let \( N_A \) (resp. \( N_B \)) be the number of \( A \)-local (resp. \( B \)-local) index constants occurring within a \( ur \) symbol in \( A^0 \) (resp. \( B^0 \)); we let also \( N \) be equal to \( 1 + \max(N_A, N_B) \).

\textbf{Step 1.} This transformation must be applied for every pair of distinct \( AB \)-common \( \texttt{ARRAY}\)-constants \( c_1, c_2 \). The transformation picks fresh \( \texttt{INDEX}\) constants \( k_1, \ldots, k_N \) (to be colored \( AB \)-common) and adds the atoms \( \texttt{diff}_n(c_1, c_2) = k_n \) (for all \( n = 1, \ldots, N \)) to both sets \( A_1 \) and \( B_1 \). This transformation fits metarule (iii).

\textbf{Step 2.} We apply 0-instantiation, that is we replace \( A \) with \( A(I_A) \) and \( B \) with \( B(I_B) \), where \( I_A \) is the set of \( A \)-local index constants and \( B \) is the set of \( B \)-local index constants (see Definition 5). This transformation fits metarule (i).

\textbf{Step 3.} As proved in Theorem 7 below, at this step \( A_2 \land B_2 \) is \( T_I \uplus \mathcal{EUF}\)-inconsistent; since \( T_I \uplus \mathcal{EUF} \) has quantifier-free interpolation by Theorem 2 we can compute an interpolant \( \theta \) of the jointly unsatisfiable pair \( A_2, B_2 \). To get our desired \( \texttt{CARD}(T_I)\)-interpolant, we only have to replace back in it the fresh \( AB \)-common constants introduced by our transformations by the \( AB \)-common terms they name.

\textit{Example 2.} We let

\[
\begin{align*}
A^0 &\equiv \{ \texttt{diff}(a_1, a_2) = j, \texttt{diff}(a_1, c_1) = j_1, \texttt{diff}(a_2, c_2) = j_2 \} \\
B^0 &\equiv \{ j < l, j_1 < l, j_2 < l, rd(c_1, l) \neq rd(c_2, l) \}
\end{align*}
\]

In the preprocesing step, we must add the atoms \(|a_1| = l_{a_1}, |a_2| = l_{a_2}\) to \( A^0 \) and \(|c_1| = l_{c_1}, |c_2| = l_{c_2}\) to both \( A^0 \) and \( B^0 \). Since \( N_A = N_B = 0 \), we have \( N = 1 \); Step 1 adds the \( AB \)-common atom \( \texttt{diff}(c_1, c_2) = k_1 \). Step 2 makes the required 0-instantiations producing the 0-instantiated separated pair \( (A, B) \).

From such instantiations, we can conclude that \( A_1 \) entails \( k_1 \leq \max(j_1, j_2, j) \); this is \( T_I \uplus \mathcal{EUF}\)-inconsistent with \( B_1 \) as \( B_1 \) contains, in addition to \( j < l, j_1 < l, j_2 < l, rd(c_1, l) = rd(c_2, l) \), also

\[
k_1 < l \rightarrow rd(c_1, l) = rd(c_2, l)
\]

by (21). Thus \( k_1 \leq \max(j_1, j_2, j) \) is a \( T_I \uplus \mathcal{EUF}\)-interpolant. Using the recover instruction of meta-rule (iii), we get \( \texttt{diff}(c_1, c_2) \leq \max(j_1, j_2, j) \) as an \( \texttt{CARD}(T_I)\)-interpolant.
Example 3. Let $A^0$ be
\[
\begin{align*}
\{ \text{diff}(a, c_1) = i_1, \text{diff}(b, c_2) = i_1, a_1 = wr(b, i_1, e_1), |a| = k, \\
a = wr(a_1, i_3, c_3), |a_1| = k, |b_1| = k, |c_1| = k, |c_2| = k \}
\end{align*}
\]
and let $B^0$ be \{rd($c_1, i_1)$ \neq rd($c_2, i_2)$, $i_1 < i_2$, $i_2 < i_3$, $i_3 < k$, $|c_1| = k$, $|c_2| = k$\}. We do not need any preprocessing here; since $N = 3$, Step 1 adds the $AB$-common atoms
\[
\begin{align*}
\text{diff}_1(c_1, c_2) = k_1, \text{diff}_2(c_1, c_2) = k_2, \text{diff}_3(c_1, c_2) = k_3.
\end{align*}
\]
Step 2 produces a separated pair $(A, B)$ such that $A_2 \land B_2$ is $T_I \cup \mathcal{EF}$-inconsistent (inconsistency can be tested via an SMT-solver like z3 [13] or MATHSAT [3]). The related $T_I \cup \mathcal{EF}$-interpolant (once $k_1$, $k_2$ and $k_3$ are replaced by $\text{diff}_1(c_1, c_2), \text{diff}_2(c_1, c_2)$ and $\text{diff}_3(c_1, c_2)$, respectively) gives our $\mathcal{CARD}(T_I)$-interpolant.

Example 4. This is the classical example (due to R. Jhala) showing that $\mathcal{AR}_{\text{ext}}$ does not have quantifier-free interpolation (one needs $\text{diff}$ in the signature to recover it). Let $A^0$ be \{c_1 = wr(c_2, i, e)\} and $B^0$ be \{c_1 \neq i_2, \text{rd}(c_1, i_1) \neq \text{rd}(c_2, i_2), \text{rd}(c_1, i_2) \neq \text{rd}(c_2, i_2)\}. Preprocessing adds the $AB$-common literals $|c_1| = l_{c_1}, |c_2| = l_{c_2}$ to both $A^0_1$ and $B^0_1$. Step 1 introduces the $AB$-common atoms
\[
\begin{align*}
\text{diff}_1(c_1, c_2) = k_1, \text{diff}_2(c_1, c_2) = k_2.
\end{align*}
\]
In the 0-instantiated separated pair $(A, B)$ produced by Step 2, from [17] and [21], we realize that
\[
k_2 = 0 \land (k_1 = k_2 \lor \text{rd}(c_1, k_2) = \text{rd}(c_2, k_2))
\]
is $T_I \cup \mathcal{EF}$-implied by $A_1$ and $T_I \cup \mathcal{EF}$-inconsistent with $B_1$. To get an $\mathcal{CARD}(T_I)$-interpolant, it is enough to replace $k_1, k_2$ respectively by $\text{diff}_1(c_1, c_2), \text{diff}_2(c_1, c_2)$ in [26].

Theorem 7. The above rule-based algorithm computes a quantifier-free interpolant for every $\mathcal{CARD}(T_I)$-mutually unsatisfiable pair $A^0, B^0$ of quantifier-free formulae.

Proof. We only need to prove that Step 3 really applies.

Suppose not; let $A = (A_1, A_2)$ and $B = (B_1, B_2)$ be the separated pairs obtained after applications of Steps 1 and 2. If Step 3 does not apply, then $A_2 \land B_2$ is $T_I \cup \mathcal{EF}$-consistent. We claim that $(A, B)$ is $\mathcal{CARD}(T_I)$-consistent (contradicting that $(A^0, B^0) \subseteq (A, B)$ was $\mathcal{CARD}(T_I)$-inconsistent).

Let $\mathcal{M}$ be a $T_I \cup \mathcal{EF}$-model of $A_2 \land B_2$. $\mathcal{M}$ is a two-sorted structure (the sorts are $\mathsf{INDEX}$ and $\mathsf{ELEM}$) endowed for every array constant $d$ occurring in $A \cup B$ of a function $d^\mathcal{M} : \mathsf{INDEX}^\mathcal{M} \rightarrow \mathsf{ELEM}^\mathcal{M}$. In addition, $\mathsf{INDEX}^\mathcal{M}$ is a model of $T_I$. We list the properties of $\mathcal{M}$ that comes from the fact that our Steps 1-2 have been applied; below, we denote by $k^\mathcal{M}$ the element of $\mathsf{INDEX}^\mathcal{M}$ assigned to an index constant $k$ (thus, if, e.g., $k, l$ are index constants, $\mathcal{M} \models k = l$ is the same as $k^\mathcal{M} = l^\mathcal{M}$).
(a) we have that $\mathcal{M} \models \bigwedge A_1 (I_A)$ (where $I_A$ is the set of $A$-local constants) and $\mathcal{M} \models \bigwedge B_1 (I_B)$ (where $I_A$ is the set of $B$-local constants): this is because $(A_1, A_2)$ and $(B_1, B_2)$ are 0-instantiated by Step 2;

(b) for $AB$-common array variables $c_1, c_2$, we have that $A_1 \cap B_1$ contains a literal of the kind $\text{diff}_n(c_1, c_2) = k_n$ for $n \leq N$; suppose that $\mathcal{M} \models l_{c_1} = l_{c_2}$ and that $k$ is an index constant such that $\mathcal{M} \models k \neq l$ for all $AB$-common index constant $l$; then, we can have $\mathcal{M} \models c_1(k) \neq c_2(k)$ only when $\mathcal{M} \models k < k_N$: this is because Step 1 has been applied and because of (a).

We expand $\mathcal{M}$ to an $\mathcal{A}R_{ext}(T_I)$-structure $\mathcal{N}$ and endow it with an assignment to our $A$-local and $B$-local variables, in such a way that all $\text{diff}$ operators mentioned to our $A$-local and $B$-local variables are defined and all formulae in $A, B$ are true. In view of Theorem 6, this structure can be expanded to the desired full model of $\mathcal{C}ARDS(T_I)$.

We take $\text{INDEX}^N$ and $\text{ELEM}^N$ to be equal to $\text{INDEX}^M$ and $\text{ELEM}^M$; the $T_I$-reduct of $\mathcal{N}$ will be equal to the $T_I$-reduct of $\mathcal{M}$ and we let $x^N = x^M$ for all index and element constants occurring in $A \cup B$. $\text{ARRAY}^N$ is the set of all positive support functions from $\text{INDEX}^N$ into $\text{ELEM}^N$. The interpretation of $A$-local and $B$-local constants of sort $\text{ARRAY}$ is more subtle. We need a détour to introduce it.

Let $k$ be an index constant s.t. $\mathcal{M} \models k \neq l$ for all $A$-local index constant $l$; we introduce an equivalence relation $\equiv_k$ on the set of $A$-local array variables as follows: $\equiv_k$ is the smallest equivalence relation that contains all pairs $(a_1, a_2)$ such that $\mathcal{M} \models l_{a_1} = l_{a_2}$ and moreover an atom of one of the following two kinds belongs to $A^0_1$: (I) $a_1 = \text{wr}(a_2, i, c)$; (II) $\text{diff}(a_1, a_2) = l$, for an $l$ such that $\mathcal{M} \models l < k$.

Claim: if $c_1, c_2$ are $AB$-common and $c_1 \equiv_k c_2$, then $c_1^M(k^M) = c_2^M(k^M)$. The claim is proved by preventively showing, for every $A$-local constants $a_1, a_2$ such that $a_1 \equiv_k a_2$, that the number of the $A$-local constants $j$ such that $\mathcal{M} \models k < j$ and $a_1^M(j^M) \neq a_2^M(j^M)$ is less or equal to $N_A < N$. In the proof (see the appendix for details), properties (a) and (b) above and the fact that $N := 1 + \max(N_A, N_B)$ play a crucial role.

In order to interpret $A$-local constants of sort $\text{ARRAY}$, we assign to an $A$-local constant $a$ of sort $\text{ARRAY}$ the function $a^N$ defined as follows for every $i \in \text{INDEX}^N$:

(†-i) if $i$ is equal to $k^M$, where $k$ is an $A$-local constant, then $a^N(i) := a^M(k^M)$;

(†-ii) if $i$ is different from $k^M$ for every $A$-local constant $k$, but $i$ is equal to $k^M$ for some (necessarily $B$-strict) index constant $k$ and there is an $AB$-common array variable $c$ such that $c \equiv_k a$, then $a^N(i) = c^M(k^M)$ (this is well-defined thanks to the claim);

(†-iii) in the remaining cases, $a^N(i)$ is equal to $\text{et}^M$ or $\bot^M$ depending whether $\mathcal{M} \models 0 \leq i \wedge i \leq l_a$ holds or not.

It can be now shown that

(*) $A$ all $a^N$ are positive-support functions and all formulae from $A_1 \cup A_2$ are true in $\mathcal{N}$.

In fact, formulae in $A_2$ are Boolean combinations of $A$-local atoms of the kind $\{\square 25\}$; these are $T_I \cup \text{EUF}$-atoms and, due to their shape, each of them

\[ \text{Recall that } l_{c_1}, l_{c_2} \text{ are the } AB\text{-common constants such that the literals } |c_1| = l_{c_1}, |c_2| = l_{c_2} \text{ belongs to } A_1 \cap B_1. \]
is true in $M$ iff it is true in $N$. Moreover, formulae in $A_1$ are all $wr$, $diff$ and $|\cdot|$-atoms. They are true in $N$ because of the 0-instantiation performed by Step 2 (see (a) above). Full details are in the appendix.

The assignments to the $B$-local array variables $b$ are defined analogously, so that

\[ \text{(⋆-$B$) all } b^N \text{ are positive-support functions and all formulae from } B_1 \cup B_2 \text{ are true in } N. \]

There is however one important point to notice. For all $AB$-common constants $c$ of sort ARRAY, our specification of $c^M$ does not depend on the fact that we use the above definition for $A$-local or for $B$-local array constants: to see this, notice that $c \equiv_k c$ holds in case (†-ii) is applied. This remark concludes the proof.

It is not difficult to see that the quantifier-free $T_I \cup EUF$-interpolation problem generated by our algorithm is of polynomial size. Notice however that $T_I \cup EUF$-interpolation is at least exponential.

8 Further Related Work and Conclusions

We introduced two theories of arrays, namely the theory $CARD(T_I)$ of contiguous arrays with maxdiff and its extension $CARDC(T_I)$ that also supports ‘constant’ arrays. These theories are strictly more expressive than McCarthy’s theory and the other variants studied in the literature: notably, strong length of arrays is definable, and inside it arrays are fully defined in every memory location. We proved that $CARDC(T_I)$ admits general interpolation by showing that its models are strongly amalgamable; the existence of (plain) amalgams also implies that $CARD(T_I)$ has (plain) interpolants. We also studied the SMT problem for $CARD(T_I)$ and showed through instantiations techniques that it is decidable. Finally, we provided a general algorithm for computing $CARD(T_I)$ quantifier-free interpolants that relies on a polynomial reduction to the problem of computing general interpolants for the index theory. Differently from previous algorithms, this procedure avoids full instantiation of terms.

One future research direction regards the implementation of this procedure, which is still missing. In the last decade, some implemented approaches have been introduced to compute interpolants for different theories, by relying on different techniques. For the significant theory of EUF, in [15] the DPT prover, which implements a method that exploits colored congruence graphs for extracting interpolants, has been shown to produce simpler and smaller interpolants than other solvers. For more complex theories, in [25] McMillan proposed an interpolating proof calculus to compute interpolants via refutational proofs obtained from the z3 SMT-solver. It is worth mentioning his approach because it takes advantages from the flexibility of the z3 solver to deal with several theories and their combination: it makes use of a secondary interpolation engine in order to ‘fill the gaps’ of refutational proofs introduced by theory lemmas, which are specific formulae derived by the satellites theories encoded in z3. This secondary
engine only needs an interpolation algorithm for QF_UFLIA. This approach can be used to compute interpolants for array theories, but since they use quantified formulæ, the method can generate quantified formulæ.

Concerning in particular array theories, another notable approach for computing interpolants is due to the authors of [18], which exploited the proof tree preserving interpolation scheme from [12] to construct interpolants via a resolution proof. This approach is capable of handling, following the nomenclature of our paper, AB-common literals but not AB-common terms: for this reason, weakly equivalences between arrays are there introduced in order to cope with those cases. This method supports the use of the diff operation between arrays in order to compute quantifier-free interpolants, but its semantic interpretation is undetermined as in [7].

In [9], the authors presented AXDInterpolator [1], an implementation of the interpolation algorithm from [17], which allows the user to choose z3, MATHSAT, or SMTINTERPOL [11] as the underlying interpolation engines. In order to show its feasibility, it was tested against a benchmark based on C programs from the ReachSafety-Arrays and MemSafety-Arrays tracks of SV-COMP [4]. Since many C programs from [4] require the usage of array length (and, in particular, strong length) we plan to develop a tool that implements the new algorithm for contiguous arrays presented in this paper.

There is still a question concerning our interpolation algorithm that needs to be investigated: extending the algorithm to the theory CARDC(T₁) (with constant arrays in the language). In order to handle constant arrays, the construction of Theorem 7 is still appropriate, except for the fact that condition (†-ii) should not be applied to define \( \text{Const}(i)^N \) when \( i \) is an A-strict constant such that \( i^M = j^M \) for some AB-common \( j \). To avoid this, one could introduce right after Step 1 some form of guessing for equalities between index constants: however, such a guessing (based on colorings) would create branches and consequently would not produce a polynomial instance of a \( T₁ \cup EUF \)-interpolation problem in Step 3. This issue needs further analysis.

Finally, although quite challenging, it would be interesting to extend our interpolation results also to array theories combined with cardinality constraints, similar to those introduced, e.g., in [2], [27].

References

1. AXDInterpolator, 2021. Accessed: 2021-10-12.
2. Francesco Alberti, Silvio Ghilardi, and Elena Pagani. Cardinality constraints for arrays (decidability results and applications). Formal Methods Syst. Des., 51(3):545–574, 2017.
3. Paul D. Bacsich. Amalgamation properties and interpolation theorems for equational theories. Algebra Universalis, 5:45–55, 1975.
4. Dirk Beyer. Software Verification: 10th Comparative Evaluation (SV-COMP 2021). In Proc. of TACAS 2021, volume 12652 of LNCS, pages 401–422, Berlin, Heidelberg, 2021. Springer.
5. Aaron R. Bradley, Zohar Manna, and Henny B. Sipma. What’s decidable about arrays? In Proc. of VMCAI 2006, volume 3855 of LNCS, pages 427–442, Berlin, Heidelberg, 2006. Springer.

6. Roberto Bruttomesso, Alessandro Cimatti, Anders Frøzén, Alberto Griggio, and Roberto Sebastiani. The MathSAT 4 SMT solver. In Proc. of CAV 2008, volume 5123 of LNCS, pages 299–303, Berlin, Heidelberg, 2008. Springer.

7. Roberto Bruttomesso, Silvio Ghilardi, and Silvio Ranise. Quantifier-free interpolation of a theory of arrays. Log. Methods Comput. Sci., 8(2), 2012.

8. Roberto Bruttomesso, Silvio Ghilardi, and Silvio Ranise. Quantifier-free interpolation in combinations of equality interpolating theories. ACM Trans. Comput. Log., 15(1):5:1–5:34, 2014.

9. José Abel Castellanos Joo, Silvio Ghilardi, Alessandro Gianola, and Deepak Kapur. Axinterop: A tool for computing interpolants for arrays with maxdiff. In Proc. of SMT 2021, volume 2908, pages 40–52. CEUR Workshop Proceedings, 2021.

10. C. C. Chang and H. Jerome Keisler. Model Theory. North-Holland, Amsterdam-London, third edition, 1990.

11. Jürgen Christ, Jochen Hoenicke, and Alexander Nutz. SMTInterpol: An interpolating SMT solver. In Proc. of SPIN 2012, volume 7385 of LNCS, pages 248–254, Berlin, Heidelberg, 2012. Springer.

12. Jürgen Christ, Jochen Hoenicke, and Alexander Nutz. Proof tree preserving interpolation. In Proc. of TACAS 2013, volume 7795 of LNCS, pages 124–138, Berlin, Heidelberg, 2013. Springer.

13. William Craig. Three uses of the Herbrand-Gentzen theorem in relating model theory and proof theory. J. Symbolic Logic, 22:269–285, 1957.

14. Leonardo De Moura and Nikolaj Bjørner. Z3: An Efficient SMT Solver. In Proc. of the TACAS 2008, volume 4963 of LNCS, pages 337–340, Berlin, Heidelberg, 2008. Springer.

15. Alexander Fuchs, Amit Goel, Jim Grundy, Sava Krstic, and Cesare Tinelli. Ground interpolation for the theory of equality. Log. Methods Comput. Sci., 8(1), 2012.

16. Silvio Ghilardi. Model theoretic methods in combined constraint satisfiability. J. Autom. Reasoning, 33(3-4):221–249, 2004.

17. Silvio Ghilardi, Alessandro Gianola, and Deepak Kapur. Interpolation and amalgamation for arrays with maxdiff. In Proc. of FOSSACS 2021, volume 12650 of LNCS, pages 268–288, Berlin, Heidelberg, 2021. Springer.

18. Jochen Hoenicke and Tanja Schindler. Efficient interpolation for the theory of arrays. In Proc. of IJCAR 2018, volume 10900 of LNCS, pages 549–565, Berlin, Heidelberg, 2018. Springer.

19. Jochen Hoenicke and Tanja Schindler. Interpolation and the array property fragment. CoRR, abs/1904.11381, 2019.

20. Deepak Kapur, Rupak Majumdar, and Calogero G. Zarba. Interpolation for Data Structures. In Proc. of SIGSOFT/FSE 2006, pages 105–116. ACM, 2006.

21. Emil W. Kiss, László Mátrki, Péter Pröhle, and Walter Tholen. Categorical algebraic properties. A compendium on amalgamation, congruence extension, epimorphisms, residual smallness, and injectivity. Studia Sci. Math. Hungar., 18(1):79–140, 1982.

22. John McCarthy. Towards a Mathematical Science of Computation. In Proc. of IFIP Congress 1962, pages 21–28. North-Holland, 1962.

23. Kenneth L. McMillan. Interpolation and sat-based model checking. In Proc. of CAV 2003, volume 2725 of LNCS, pages 1–13, Berlin, Heidelberg, 2003. Springer.
24. Kenneth L. McMillan. Lazy abstraction with interpolants. In Proc. of CAV 2006, volume 4144 of LNCS, pages 123–136, Berlin, Heidelberg, 2006. Springer.
25. Kenneth L. McMillan. Interpolants from Z3 proofs. In Proc. of FMCAD 2011, pages 19–27. FMCAD Inc., 2011.
26. Greg Nelson and Derek C. Oppen. Simplification by Cooperating Decision Procedures. ACM Trans. Program. Lang. Syst., 1(2):245–257, 1979.
27. Rodrigo Raya and Viktor Kunkak. Np satisfiability for arrays as powers. In Proc. of VMCAI 2022, volume 13182 of LNCS, pages 301–318, Berlin, Heidelberg, 2022. Springer.
28. Andrey Rybalchenko and Viorica Sofronie-Stokkermans. Constraint solving for interpolation. In Proc. of VMCAI 2007, volume 4349 of LNCS, pages 346–362, Berlin, Heidelberg, 2007. Springer.
29. Andrey Rybalchenko and Viorica Sofronie-Stokkermans. Constraint solving for interpolation. J. Symb. Comput., 45(11):1212–1233, 2010.
30. Viorica Sofronie-Stokkermans. Interpolation in local theory extensions. In Proc. of IJCAR 2006, volume 4130 of LNCS, pages 235–250, Berlin, Heidelberg, 2006. Springer.
31. Viorica Sofronie-Stokkermans. Interpolation in local theory extensions. Log. Methods Comput. Sci., 4(4), 2008.
32. Viorica Sofronie-Stokkermans. On interpolation and symbol elimination in theory extensions. In Proc. of IJCAR 2016, volume 9706 of LNCS, pages 273–289, Berlin, Heidelberg, 2016. Springer.
33. Viorica Sofronie-Stokkermans. On interpolation and symbol elimination in theory extensions. Log. Methods Comput. Sci., 14(3), 2018.
34. Nishant Totla and Thomas Wies. Complete instantiation-based interpolation. In Proc. of POPL 2013, pages 537–548. ACM, 2013.
35. Nishant Totla and Thomas Wies. Complete instantiation-based interpolation. J. Autom. Reasoning, 57(1):37–65, 2016.
A  Proofs of results from Section 4

Lemma 5. Let \( \mathcal{N}, \mathcal{M} \) be models of \( \mathcal{AR}_{\text{ext}}(T_I) \) such that \( \mathcal{M} \) is a substructure of \( \mathcal{N} \). For every \( a, b \in \text{ARRAY}^M \), we have that

\[
\mathcal{M} \models a \sim b \text{ iff } \mathcal{N} \models a \sim b.
\]

Proof. The left-to-right side is trivial because if \( \mathcal{M} \models a \sim b \) then \( a \) and \( b \) have equal length in \( \mathcal{M} \) and in \( \mathcal{N} \) too because length is preserved; moreover, \( \mathcal{M} \models a = \text{wr}(b, I, E) \), where \( I \equiv i_1, \ldots, i_n \) is a list of costants (naming elements of \( \mathcal{M} \)) of sort \( \text{INDEX} \), \( E \equiv e_1, \ldots, e_n \) is a list of costants (naming elements of \( \mathcal{M} \)) of sort \( \text{ELEM} \), and \( \text{wr}(b, I, E) \) abbreviates the term \( \text{wr}(\text{wr}(\cdots \text{wr}(b, i_1, e_1) \cdots, i_n, e_n)) \).

Thus, also \( \mathcal{N} \models a = \text{wr}(b, I, E) \) because \( \mathcal{M} \) is a substructure of \( \mathcal{N} \). Vice versa, suppose that \( \mathcal{M} \not\models a \sim b \). This means that either \( |a| \neq |b| \) or that there are infinitely many \( i \in \text{INDEX}^M \) such that \( rd^M(a, i) \neq rd^M(b, i) \). Since \( \mathcal{M} \) is a substructure of \( \mathcal{N} \), these conditions hold in \( \mathcal{N} \) too.

Lemma 9. Let \( \mathcal{M} \) be a model of \( \mathcal{AR}_{\text{ext}}(T_I) \) and let \( a, a', b, b' \in \text{ARRAY}^M \); if \( a \sim_M a', b \sim_M b' \) and \( \text{diff}_k(a', b') \) is defined for every \( k \), then \( \text{diff}(a, b) \) is also defined.

Proof. Notice first that, from \( a \sim a' \) and \( b \sim b' \), it follows that \( |a| = |a'| \) and \( |b| = |b'| \). In case \( |a| \neq |b| \), we have that \( \text{diff}(a, b) = \max\{|a|, |b|\} \) (see Lemma 11), which implies that \( \text{diff}(a, b) \) is defined. Hence, the relevant case is when we have \( l = |a| = |b| = |a'| = |b'| \) and \( a' \neq b' \) (if \( a' \sim b' \), then we have also \( a \sim b \) and the maximum of the finitely many indexes where \( a, b \) differ is \( \text{diff}(a, b) \)). Then for the infinitely many indexes \( j_k = \text{diff}_k(a', b') \) we have \( a'(j_k) \neq b'(j_k) \); for at least one of such \( j_k \) we must also have \( a(j_k) \neq b(j_k) \) because \( a \sim a' \) and \( b \sim b' \). Consider now the indexes in \( [j_k, l] \); in this interval, the pair of arrays \( a, b \) differs on at least one but at most finitely many indices (because \( a, a' \) differs on finitely many indices there and so do the pairs \( b, b' \) and \( a', b' \)), so the biggest one such index will be \( \text{diff}(a, b) \).

Lemma 10. Let \( \mathcal{M} \) be a model of \( \mathcal{AR}_{\text{ext}}(T_I) \). There exist a model \( \mathcal{N} \) of \( \mathcal{AR}_{\text{ext}}(T_I) \) and a \( \text{diff}-\text{faithful} \) embedding \( \mu: \mathcal{M} \rightarrow \mathcal{N} \) such that the restriction of \( \mu \) to the sort \( \text{ELEM} \) is not surjective. In addition, if \( \mathcal{M} \) is a model of \( \mathcal{CARC}_{\text{ext}}(T_I), \mathcal{CARD}(T_I) \) or of \( \mathcal{CARDC}(T_I) \), so it is \( \mathcal{N} \).

Proof. To build \( \mathcal{N} \) it is sufficient to put:

\[
\begin{align*}
\text{INDEX}^N &= \text{INDEX}^M, \\
\text{ELEM}^N &= \text{ELEM}^M \cup \{e\} \text{ where } e \notin \text{ELEM}^M, \\
\text{ARRAY}^N &= \text{ARRAY}^M \text{ of the positive-support functions } a: \text{INDEX}^N \rightarrow \text{ELEM}^N \text{ for which there exist } a' \in \text{ARRAY}^M \text{ such that } a \sim a'.
\end{align*}
\]
Now notice that if $\text{diff}$ is totally defined in $\mathcal{M}$, so it is in $\mathcal{N}$. In fact, this follows from the definition of $\text{ARRAY}^\mathcal{N}$ and Lemma 8 if $a \sim a'$, $b \sim b'$ and $\text{diff}_k(a', b')$ is defined for every $k$, then $\text{diff}(a, b)$ is also defined by the previous lemma. The claim is proved in the same way for all the above mentioned array theories (notice that in case the signature includes the $\text{Const}$ symbol, $\mu$ trivially preserves it).

**Lemma 6** Let $\mathcal{M}$ be a model of $\text{AR}_{\text{ext}}(T_I)$ (resp. of $\text{CARC}_{\text{ext}}(T_I)$) and let $a, b \in \text{ARRAY}^\mathcal{M}$ be such that $\text{diff}^\mathcal{M}(a, b)$ is not defined. Then there are a model $\mathcal{N}$ of $\text{AR}_{\text{ext}}(T_I)$ (resp. of $\text{CARC}_{\text{ext}}(T_I)$) and a $\text{diff}$-faithful embedding $\mu : \mathcal{M} \rightarrow \mathcal{N}$ such that $\text{diff}^\mathcal{N}(a, b)$ is defined.

**Proof.** Thanks to Lemma 10, we can assume that $\text{ELEM}^\mathcal{M}$ has at least an element $e$ (different from $\bot^\mathcal{M}, \text{el}^\mathcal{M}$). Notice that we must have $|a| = |b|$, otherwise $\text{diff}(a, b)$ is defined and it is $\max(|a|, |b|)$ according to Lemma 11.

Let $I = \{i \in \text{INDEX}^\mathcal{M} \mid a(i) \neq b(i)\}$ be the set of indices without maximum element (hence infinite) where they differ. Let $\downarrow I := \{j \in \text{INDEX}^\mathcal{M} \mid \exists i \in I, j \leq i\} \supseteq I$. Notice that the condition

$$(+) \ "\exists i \in I \forall j \in I (j \geq i \rightarrow x(j) = el)"$$

cannot be satisfied both for $x = a$ and $x = b$: indeed, if this were the case, assuming w.l.o.g. that $i_a \leq i_b$ (where $i_a$ and $i_b$ are the witnesses for the existentially quantified index $i$ in $(+)$ for $x = a$ and $x = b$ respectively), we would have that $a(j) = e\text{el}^\mathcal{M} = b(j)$ for all $j \geq i_b$, $j \in I$, which is a contradiction with the definition of $I$. In case one of them satisfies it, we assume it is $b$.

Let $\Delta$ be the Robinson diagram of the $T_I$-reduct of $\mathcal{M}$ and let $k_0$ be a new constant; let us introduce the set

$$\Delta' := \Delta \cup \{i < k_0 \mid i \in \downarrow I\} \cup \{k_0 < i \mid i \in \text{INDEX}^\mathcal{M} \downarrow \downarrow I\}.$$ 

By the compactness theorem for first order logic and since $I$ is infinite, the set $\Delta'$ turns out to be consistent. In fact, if $\Delta'$ were inconsistent, then there would exist a finite subset of it not admitting a model. However, a finite subset of $\Delta'$ can contain constraints only for a finite number of index constants $d$ occurring in $\Delta$, $i \in \downarrow I$, $i' \in \text{INDEX}^\mathcal{M} \downarrow I$ and $k_0$. Such constraints can be verified inside the $T_I$-reduct of $\mathcal{M}$ itself: to interpret the additional constant $k_0$, it is sufficient to use the fact that $I$ contains arbitrarily large indexes and the fact that the definition of $\downarrow I$ implies that

$$\forall i \in \downarrow I, \forall j \in \text{INDEX}^\mathcal{M} \downarrow I, \ i < j.$$ 

By Robinson Diagram Lemma, there exists a model $\mathcal{A}$ of $T_I$ extending the $T_I$-reduct of $\mathcal{M}$; such $\mathcal{A}$ contains in its support an element $k_0$ such that

$$\forall i \in \downarrow I, \ i < k_0,$$

$$\forall i \in \text{INDEX}^\mathcal{M} \downarrow I, \ k_0 < i.$$
We now take $\text{ELEM}^N = \text{ELEM}^M$, $\text{INDEX}^N = \text{INDEX}^A$; we let also $\text{ARRAY}^N$ to be the set of all positive-support functions from $\text{INDEX}^N$ into $\text{ELEM}^N$ (notice that this $N$ is trivially also a model of $\text{CARD}_{\text{ext}}(T_I)$). We observe that $k_0 < |a|^M$ and recall that $|a|^M = |b|^M$.

Let us now define the embedding $\mu : M \rightarrow N$; at the level of the sorts $\text{INDEX}$ and $\text{ELEM}$, we use inclusions. For the $\text{ARRAY}$ sort, we need to specify the value $\mu(c)(k)$ for $c \in \text{ARRAY}^M$ and $k \in \text{INDEX}^N \setminus \text{INDEX}^M$ (for the other indices we keep the old $M$-value to preserve the read operation). Our definition for $\mu$ must preserve the maxdiff index (whenever already defined in $M$) and must guarantee that $\text{diff}^N(\mu(a), \mu(b)) = k_0$ (by construction, we have $k_0 > 0$). For a generic array $c \in \text{ARRAY}^M$, we operate as follows:

1. if $|c|^M < k_0$ we put $\mu(c)(k_0) = \bot^M$, otherwise:
2. if the condition $(\star)$ below holds, we put $\mu(c)(k_0) = e$,
3. if such condition does not hold, we put $\mu(c)(k_0) = e^M$.

The condition $(\star)$ is specified as follows:

\[(\star) \quad \text{there is } i \in I \text{ such that for all } j \in I, j \geq i \text{ we have } c(j) = a(j).\]

For all the remaining indexes $k \in \text{INDEX}^N \setminus (\text{INDEX}^M \cup \{k_0\})$ we put

$$
\mu(c)(k) = \begin{cases}
\bot^M, & \text{if } k \notin [0, |c|^M] \\
e^M, & \text{if } k \in [0, |c|^M]
\end{cases}
$$

Notice that we have $\mu(a)(k_0) = e \neq e^M = \mu(b)(k_0)$ (the last equality holds because $I$ is infinite and does not have maximum, hence condition $(\star)$ holds for $a$ but not for $b$). In addition:

\[- \text{for all } i \in \text{INDEX}^M \text{ such that } k_0 < i, \text{ we have } i \notin I, \text{ according to the construction of } k_0 \text{ and consequently } i \notin I, \text{ that is } a(i) = b(i);\]

\[- \text{for all } i \in \text{INDEX}^N \setminus (\text{INDEX}^M \cup \{k_0\}) \text{ such that } k_0 < i, \text{ since we have } |a|^M = |b|^M, \text{ we get } \]

$$
\mu(a)(i) = \bot^M \text{ iff } \mu(b)(i) = \bot^M,
$$

$$
\mu(a)(i) = e^M \text{ iff } \mu(b)(i) = e^M.
$$

Hence, we can conclude that $\text{diff}^N(\mu(a), \mu(b))$ is defined and equal to $k_0$.

We only need to check that our $\mu$ preserves $\text{rd}, |\cdot|, \text{wr}$, constant arrays and $\text{diff}$ (whenever defined).

The operation $\text{rd}$ is preserved because $\mu$ acts as an inclusion for indexes and elements and because we have $\mu(c)(k) = c(k)$ if $k \in \text{INDEX}^M$.

Concerning length, we have $|\mu(c)|^N = |c|^M$ because of $(23)$ and because of the above definition of $\mu(c)(k_0)$ (recall that $k_0 > 0$).

Concerning write operation, we prove that for all $c \in \text{ARRAY}^M$, $i \in \text{INDEX}^M \cap [0, |c|]$ and $e' \in \text{ELEM}^M \setminus \{\bot^M\}$ we have

$$
\mu(\text{wr}(c, i, e')) = \text{wr}(\mu(c), i, e').
$$

Remember that we have $|\text{wr}(c, i, e')| = |c| = |\mu(c)|$. 
For $k \neq i$ in $\text{INDEX}^M$

$$\mu(wr(c, i, e'))(k) = wr(c, i, e')(k) = c(k)$$

$$\mu(wr(c, i, e'))(k) = c(k);$$

For $k = i$

$$\mu(wr(c, i, e'))(i) = wr(c, i, e')(i) = e'$$

$$\mu(wr(c, i, e'))(i) = e';$$

For $k = k_0 > |c|$ the claim follows immediately from the definition;

For $k = k_0 < |c|$, $(\ast)$ holds for $c$ iff it holds for $wr(c, i, e')$, because $I$ is infinite. Hence we have

$$\mu(wr(c, i, e'))(k_0) = e \text{ iff } \mu(c)(k_0) = e.$$

For $k \in \text{INDEX}^N \setminus (\text{INDEX}^M \cup \{k_0\})$, the claim is clear from the definition and from the fact that $wr$ preserves length.

For constant arrays, we must show only that $\mu(\text{Const}(i))(k_0) = el^M$ in case $k_0 < i$: this is clear, because $a$ does not satisfy (+), hence $(\ast)$ does not hold for $\text{Const}(i)$.

Let us now finally consider the $\text{diff}$ operation and let us prove that if $\text{diff}^M(c_1, c_2)$ is defined, then $\text{diff}^N(\mu(c_1), \mu(c_2))$ is also defined and equal to it. Assume that $\text{diff}^M(c_1, c_2)$ is defined; since $\mu$ preserve length, the only relevant case, in view of Lemma 1, is when we have $|c_1| = |c_2|$: since the values of $c_1, c_2$ on indexes from $\mathcal{M}$ are preserved, taking in mind (23) (in particular, that for $k \in \text{INDEX}^N \setminus (\text{INDEX}^M \cup \{k_0\})$ we have $\mu(c_1)(k) = \mu(c_2)(k)$), we only have to exclude that we have

$$\text{diff}^M(c_1, c_2) < k_0 \text{ and } \mu(c_1)(k_0) \neq \mu(c_2)(k_0)$$

If this is the case, we have, e.g., $\mu(c_1)(k_0) = e \neq el^M = \mu(c_2)(k_0)$ ($k_0 < |c_1| = |c_2|$), which implies that $(\ast)$ holds for $c_1$ but not for $c_2$. However, it cannot be that $(\ast)$ holds for only one among $c_1, c_2$. The reason for this is as follows. Indeed, if $i \in I$ is the index that witnesses $(\ast)$ for $c_1$, then $c_1(j) = a(j)$ for all indexes $j \in I$ such that $j \geq i$ (which are infinitely many). Since $I$ is infinite and without maximum and since $\text{diff}^M(c_1, c_2) < k_0$, we must have $\text{diff}^M(c_1, c_2) \in I$ by the definition of $\Delta$, so there must be infinitely many indices in $I$ bigger than $\text{diff}^M(c_1, c_2)$ and arbitrarily large, which means in particular that there exists an index $i' \in I$ such that $i' \geq i$ and $i' > \text{diff}(c_1, c_2)$. This $i'$ witnesses $(\ast)$ for $c_2$, as wanted. This concludes the proof.

B Proofs of results from Section 5

We prove here that the model $\mathcal{M}$ introduced in Section 5 is a CARDC($T_1$)-amalgam (and also a CARD($T_1$)-amalgam) for the models $\mathcal{M}_1$ and $\mathcal{M}_2$ with the common substructure $\mathcal{A}$. 
Requirements check for the amalgamated model

Proof. We need to prove that the functions \( \mu_i \): (i) are well-defined, (ii) are injective, (iii) preserve \(|-|\), (iv) preserve \(rd\) and \(wr\), (v) preserve \(diff\), (vi) satisfy the condition \( \mu_1 \circ f_1 = \mu_2 \circ f_2 \), (vii) preserve constant arrays (for the statement about \( CARDC(T_i) \)).

(i) Since \( INDEX^M \) is a strong amalgam of \( INDEX^{M_1} \) and \( INDEX^{M_2} \), the case distinctions we made for defining \( \mu_i(a)(k) \) are non-overlapping and exhaustive. We now show that if, for \( i = 1, 2 \) and \( k \in INDEX^{M_{3-i}} \setminus INDEX^A \) and \( a \in ARRAY^{M_i} \), the relation \((2\ast)\) holds relatively to two different pairs of arrays \((c_1, b_1), (c_2, b_2)\) from \( ARRAY^A \times ARRAY^{M_i} \), then we nevertheless have \( f_{3-i}(c_1)(k) = f_{3-i}(c_2)(k) \) (this proves the consistency of the definition). For symmetry, let us consider only the case \( i = 1 \). Since the index ordering is total, let us suppose that we have for instance

\[
k > \text{diff}^{M_1}(b_1, f_1(c_1)) \geq \text{diff}^{M_1}(b_2, f_1(c_2)). \tag{27}
\]

By the transitivity of \( \sim^{M_1} \), \( b_1 \) and \( b_2 \) differ on finitely many indices, hence we can consider the finite sets

\[
J := \{ j \in INDEX^A \mid b_1(j) \neq b_2(j), j > \text{diff}^{M_1}(b_1, f_1(c_1)) \}
\]

\[
E := \{ b_1(j) \mid j \in J \} \subseteq ELEM^A.
\]

Let now pick \( c := wr(c_2, J, E) \); then \( c \sim^A c_2 \). Since \( f_2 \) is an embedding, we have \( f_2(c)(k) = f_2(c_2)(k) \) for all \( k \in INDEX^{M_2} \setminus INDEX^A \). Suppose we have also

\[
\text{diff}^A(c_1, c) \leq \text{diff}^{M_1}(f_1(c_1), b_1).
\tag{28}
\]

Then

\[
\text{diff}^A(c_1, c) \leq \text{diff}^{M_1}(f_1(c_1), b_1) < k
\]

and consequently also the desired equality

\[
f_2(c_1)(k) = f_2(c)(k) = f_2(c_2)(k)
\]

follows.

In order to prove \([28]\), we consider \( j \in INDEX^A \) with \( j > \text{diff}^{M_1}(f_1(c_1), b_1) \) and show that we have \( c(j) = c_1(j) \). Suppose not, i.e. that \( c(j) \neq c_1(j) \); then we cannot have \( j \in J \) otherwise, by definition of \( c \) and \( E \), we would have \( c(j) = b_1(j) = f_1(c_1)(j) = c_1(j) \) \( (f_1 \) is inclusion for indexes and \( J \subseteq INDEX^A \), contradiction. Hence, we have \( j \notin J \), so \( c(j) = c_2(j) \) and \( b_1(j) = b_2(j) \). Now remember that

\[
j > \text{diff}^{M_1}(b_1, f_1(c_1)) \geq \text{diff}^{M_1}(b_2, f_1(c_2)),
\]

hence

\[
c_1(j) = b_1(j), \ c_2(j) = b_2(j)
\]

\[
c(j) = c_2(j) = b_2(j) = b_1(j) = c_1(j)
\]

thus getting an absurdity.
(ii) Injectivity of $\mu_1$ and $\mu_2$ is immediate.

(iii) In order to prove that $|\cdot|$ is preserved, it is sufficient to show that for every $a \in \text{ARRAY}^{M_1}$ and for all $k \in \text{INDEX}^M$, we have

$$\mu_1(a)(k) \neq \bot \iff 0 \leq k \leq |a|^M_1.$$ 

The only relevant case is when $k \in \text{INDEX}^{M_2} \setminus \text{INDEX}^A$ and $(2*)$ holds. In such a case, we have two possibilities:

1. $|b|^M_1 = |c|^A$: in this case, since $b \sim^{M_1} a$, we have $|a|^M_1 = |b|^M_1 = |c|^A = |f_2(c)|^M_2$ ($f_2$ is an embedding), thus getting what we need for $k$;
2. $|b|^M_1 \neq |c|^A$: in this case $k > \text{diff}^{M_1}(b, f_1(c)) = \max\{|b|^M_1, |c|^A\}$ by Lemma [1]. Since $a \sim b$ implies $|a| = |b|$, the definition of $\mu_1$ produces $\mu_3(a)(k) = f_2(c)(k) = \bot$ (the last identity holds because $k > |c|^A$), which is as desired because $k > |b|^M_1 = |a|^M_1$ too.

(iv) The fact that $\text{rd}$ and $\text{wr}$ operations are preserved is easy (notice in particular that, if $(2*)$ holds for $a$ via the pair $(c, b)$, then the same pair guarantees $(2*)$ for arrays of the kind $\text{wr}(a, i, e)$).

(v) Again we limit to the case of $\mu_1$ for symmetry. We need to show that for every $a_1, a_2 \in \text{ARRAY}^{M_1}$, we have $\mu_1(\text{diff}^{M_1}(a_1, a_2)) = \text{diff}^M(\mu_1(a_1), \mu_1(a_2))$.

Notice that, if we call $j$ the index $\text{diff}(a_1, a_2) \in \text{INDEX}^{M_1}$, by definition of $\mu_1$ on array applied to indexes in $\text{INDEX}^{M_1}$, we have that $\mu_1(a_1)(j) = a_1(j) \neq a_2(j) = \mu_1(a_2)(j)$. Hence, in order to conclude, it is sufficient to show that, given $k \in \text{INDEX}^M$ such that $k > \text{diff}^{M_1}(a_1, a_2)$, the equality $\mu_1(a_1)(k) = \mu_1(a_2)(k)$ holds. Notice first that we can always reduce to one of the following three cases

(a) $|a_1| < |a_2| = \text{diff}^{M_1}(a_1, a_2)$;
(b) $\text{diff}^{M_1}(a_1, a_2) < |a_1| = |a_2|$;
(c) $|a_1| = |a_2| = \text{diff}^{M_1}(a_1, a_2)$.

We now show that $\mu_1(a_1)(k) = \mu_1(a_2)(k)$.

- If $k \in \text{INDEX}^{M_1}$:

$$\mu_1(a_1)(k) = a_1(k) = a_2(k) = \mu_1(a_2)(k).$$

- If $k \notin \text{INDEX}^{M_1} \cup \text{INDEX}^{M_2}$, we analyze the three cases separately:

  Case (a) : then $k \notin [0, |a_i|]$ for $i = 1, 2$. We have $\mu_1(a_1)(k) = \bot = \mu_1(a_2)(k)$;

  Case (b) : we have

  $$\mu_1(a_1)(k) = \bot \iff \mu_1(a_2)(k) = \bot$$

  $$\mu_1(a_1)(k) = el \iff \mu_1(a_2)(k) = el.$$ 

  Case (c) : similarly to (a), we have $k \notin [0, |a_i|]$ for $i = 1, 2$.

  - If $k \in \text{INDEX}^{M_2} \setminus \text{INDEX}^A$ and $(2*)$ does not hold neither for $a_1$ nor for $a_2$, the argument is the same as in the previous case.

    Otherwise, suppose that $(2*)$ holds for, say, $a_1$ as witnessed by the pair $(c_1, b_1)$. Then we get $\mu_1(a_1)(k) = f_2(c_1)(k)$. We prove that $(2*)$ holds for $a_2$ too and that we have $\mu_1(a_1)(k) = f_2(c_1)(k) = \mu_1(a_2)(k)$. 

Since \( b_1 \) and \( a_1 \) differ on finitely many indices inside \( \mathcal{M}_1 \), we can consider the finite sets
\[
I := \{ i \in \text{INDEX}^{\mathcal{M}_1} \mid b_1(i) \neq a_1(i), i > \text{diff}^{\mathcal{M}_1}(a_1, a_2) \},
\]
\[
E := \{ b_1(i) \mid i \in I \} \subseteq \text{ELEM}^{\mathcal{M}_1}
\]
and the array \( \hat{b} := \text{wr}(a_2, I, E) \); for this array, we obviously have \( a_2 \sim^{\mathcal{M}_1} \hat{b} \). If we also have
\[
\text{diff}^{\mathcal{M}_1}(b_1, \hat{b}) \leq \text{diff}^{\mathcal{M}_1}(a_1, a_2)
\]
then we get: \( k > \text{diff}^{\mathcal{M}_1}(a_1, a_2) \) e \( k > \text{diff}^{\mathcal{M}_1}(b_1, f_1(c_1)) \) (the latter is from \((2\ast)\)). Hence:
\[
k > \max\{ \text{diff}^{\mathcal{M}_1}(a_1, a_2), \text{diff}^{\mathcal{M}_1}(b_1, f_1(c_1)) \} \geq
\]
\[
\geq \max\{ \text{diff}^{\mathcal{M}_1}(b_1, \hat{b}), \text{diff}^{\mathcal{M}_1}(b_1, f_1(c_1)) \} \geq
\]
\[
\geq \text{diff}(\hat{b}, f_1(c_1))
\]
(the last disequality holds because of the ‘triangular disequality’ of Lemma [1]). Hence we obtain \((2\ast)\) for \( a_2 \) via the pair given by \( c := c_1 \) and \( b := \hat{b} \) (because we have \( a_2 \sim^{\mathcal{M}_1} \hat{b} \) and \( k > \text{diff}(\hat{b}, f_1(c_1)) \)) and consequently \( \mu_1(a_2)(k) = f_2(c_1(k)) \).

It remains to prove \((29)\); to this aim, let us pick \( j \in \text{INDEX}^{\mathcal{M}_1} \) such that \( j > \text{diff}(a_1, a_2) \) and let us show that \( b_1(j) = \hat{b}(j) \). If this is not the case, i.e., if \( b_1(j) \neq \hat{b}(j) \), then according to the definition of \( \hat{b} \) we have \( \hat{b}(j) = a_2(j) \) and \( j \notin I \). Hence \( \hat{b}(j) = a_2(j) = a_1(j) = b_1(j) \) (the last identity holds because \( j \notin I \) and \( j > \text{diff}(a_1, a_2) \)), absurd.

\( \text{(vi)} \) In order to prove \( \mu_1 \circ f_1 = \mu_2 \circ f_2 \), let us consider \( c \in \text{ARRAY}^{\mathcal{M}} \), let us put \( a_i = f_i(c) \) for \( i = 1, 2 \) and let us check that
\[
\mu_1(a_1)(k) = \mu_2(a_2)(k)
\]
holds for all \( k \in \text{INDEX}^{\mathcal{M}} \).

- Case \( k \in \text{INDEX}^{\mathcal{M}} \): we have
  \[
  \mu_1(a_1)(k) = a_1(k) = f_1(c)(k) = c(k)
  \]
  \[
  \mu_2(a_2)(k) = a_2(k) = f_2(c)(k) = c(k).
  \]
- Case \( k \in \text{INDEX}^{\mathcal{M}_1} \setminus \text{INDEX}^{\mathcal{M}} \): clearly \((2\ast)\) holds for \( a_2 \) with \( c := c \) and \( b := a_2 \), consequently
  \[
  \mu_1(a_1)(k) = a_1(k) = f_1(c)(k)
  \]
  \[
  \mu_2(a_2)(k) = f_1(c)(k).
  \]
Here we assume that $\mathcal{M}_2 \setminus \text{INDEX}^A$: clearly (2*) holds for $a_1$ with $c := c$ and $b := a_1$, consequently

$$\mu_1(a_1)(k) = f_2(c)(k)$$
$$\mu_2(a_2)(k) = a_2(k) = f_2(c)(k).$$

Case $k \notin (\text{INDEX}^{\mathcal{M}_1} \cup \text{INDEX}^{\mathcal{M}_2})$ and $k \in [0, |c|]$: we have

$$\mu_1(a_1)(k) = e = \mu_2(a_2)(k).$$

Case $k \notin (\text{INDEX}^{\mathcal{M}_1} \cup \text{INDEX}^{\mathcal{M}_2})$ and $k \notin [0, |c|]$: we have

$$\mu_1(a_1)(k) = \bot = \mu_2(a_2)(k).$$

This completes our case analysis.

(vii) Here we assume that $\mathcal{M}_1, \mathcal{M}_2$ are models of $\text{CARDC}(T_I)$; we need to show, e.g., that $\mu_1(\text{Const}^{\mathcal{M}_1}(i))(k) = e$ for every $k$ such that $0 \leq k \leq i$. Now, if $i \in \text{INDEX}^A$ this is obvious, because $\text{Const}^{\mathcal{M}_1}(i) = f_1(\text{Const}^{\mathcal{A}}(i))$. Hence suppose that $i \in \text{INDEX}^{\mathcal{M}_1} \setminus \text{INDEX}^A$: the only possibly problematic case is when $k \in \text{INDEX}^{\mathcal{M}_1} \setminus \text{INDEX}^A$ and (2*) applies, as witnessed by a pair $(b, c)$ for $\text{Const}^{\mathcal{M}_1}(i)$. But we have $|b| = |\text{Const}^{\mathcal{M}_1}(i)| = i$ and $|f_1(c)| \neq i$ (because $|f_1(c)| = |c| \notin \text{INDEX}^A$). Then, according to (2*) and recalling Lemma 11, we have $k > \text{diff}^{\mathcal{M}_1}(b, f_1(c)) = \max\{|b|, |f_1(c)|\} = \max\{i, |f_1(c)|\} \geq i$, contradicting the choice of $k$.

To prove strong amalgamation for $\text{CARDC}(T_I)$ we need a couple of lemmas.

**Lemma 11.** Every model $\mathcal{M}$ of $\text{CARDC}(T_I)$ can be embedded into a model $\mathcal{N}$ such that $\text{INDEX}^\mathcal{N}$ is infinite.

**Proof.** This is basically due to the fact that $T_I$ is stably infinite. So let us first embed the $T_I$-reduct of $\mathcal{M}$ into an infinite model $\mathcal{A}$ of $T_I$. We define $\mathcal{N}$ as follows.

We let $\text{ELEM}^\mathcal{N}$ be equal to $\text{ELEM}^\mathcal{M}$ and the $T_I$-reduct of $\text{INDEX}^\mathcal{N}$ be equal to $\mathcal{A}$. We let $\text{ARRAY}^\mathcal{N}$ be the set of positive support functions from $\text{INDEX}^\mathcal{N}$ to $\text{ELEM}^\mathcal{N}$ (the model so built will then be embedded into a full model of $\text{CARDC}(T_I)$ using Theorem 3). We only need to define the embedding $\mu : \mathcal{M} \rightarrow \mathcal{N}$. This embedding will be the identity for $\text{INDEX}$ and $\text{ELEM}$ sorts; for arrays, we let $\mu(a)(k)$ be equal to $a(k)$ for $k \in \text{INDEX}^\mathcal{M}$ and for $k \notin \text{INDEX}^\mathcal{M}$, we put $\mu(a)(k)$ equal to $e\mu^\mathcal{M}$ or $\bot^\mathcal{M}$ depending whether we have $k \in [0, |a|]$ or not. The proof that $\mu$ preserves all operations is easy.

Let us call an element $i \in \text{INDEX}^\mathcal{M}$ of a model $\mathcal{M}$ of $\text{CARDC}(T_I)$ finite iff the set $\{j \in \text{INDEX}^\mathcal{M} \mid 0 \leq j \leq i\}$ is finite. $\text{Fin}(\mathcal{M})$ denotes the set of finite elements of $\mathcal{M}$.

**Lemma 12.** Let $\mathcal{A}, \mathcal{M}$ be models of $\text{CARDC}(T_I)$ and let $f : \mathcal{A} \rightarrow \mathcal{M}$ be an embedding. Then there exist a third model $\mathcal{N}$ of $\text{CARDC}(T_I)$ and an embedding $\nu : \mathcal{M} \rightarrow \mathcal{N}$, such that for every $a \in \text{ARRAY}^\mathcal{N}$ with $|a| \ominus \nu(f(\text{INDEX}^\mathcal{A}))$ one of the following conditions hold:
We define a step as shown below). The required by the lemma will be a chain unions of the \( \mu \) contains an element \( k \) such that for all \( M \) has a model \( B \) has an \( M \) that is \( \mu \)-consistent, which is a contradiction. Otherwise, one take a well ordering of the arrays, apply the construction below by tranfinite induction and repeat it \( \omega \)-times. The union of the chain so built will have the required properties.

Let \( a \in \text{ARRAY}^M \) be such that \( |a| \) is from \( A \) (i.e. such that \( |a| = f(i) \) for some \( i \in \text{INDEX}^A \)) and such that there does not exist \( c \in \text{ARRAY}^A \) such that \( f(c) \sim^A a \). Then \( |a| \) is not finite, because otherwise we would have that \( a \sim^M f(\text{Const}(i)) \).

Consider the diagram \( \Delta \) of the \( T_I \)-reduct of \( M \) and let \( k_a \) a fresh constant; the set
\[
\Delta' := \Delta \cup \{ i < k_a \mid i \in \text{Fin}(M) \} \cup \\
\{ i > k_a \mid i \in \text{INDEX}^M \setminus \text{Fin}(M) \}.
\]

is consistent. Suppose that \( \Delta' \) is inconsistent. By compactness, we would have that there exists a finite subset \( \Delta'_0 \) of \( \Delta' \) which is inconsistent too. \( \Delta'_0 \) would involve finitely many finite indexes \( i_1 < \cdots < i_n \) and finitely many infinite indexes \( j_1 < \cdots < j_m \) and a finite subset \( \Delta_0 \) of \( \Delta \).

If \( m = 0 \), since \( \text{INDEX}^M \) is infinite, there exist an element \( i' \in \text{INDEX}^M \) large enough, so as to get \( i_1 < \cdots < i_n < i' \) in \( M \). If \( m > 0 \), then already in \( \text{INDEX}^M \) there exists an element \( i' \) such that \( i_1 < \cdots < i_n < i' \) and such that \( i_1 < j_1 < \cdots < j_m \) hold in \( M \); otherwise \( j_1 \) would be a finite index. This element \( i' \) can interpret the constant \( k_a \). In both cases, we conclude that \( \Delta'_0 \) would be consistent, which is a contradiction.

By Robinson Diagram Lemma, \( \Delta \) has a model \( B \) extending the \( T_I \)-reduct of \( M \).

We let now \( \text{ELEM}^N = \text{ELEM}^M \), \( \text{INDEX}^N = \text{INDEX}^B \) and we let \( \text{ARRAY}^N \) to be the set of positive-support functions from \( \text{INDEX}^N \) into \( \text{ELEM}^N \) (then, in view of Theorem 3, \( N \) can be embedded into a full model of \( \text{CARD}(T_I) \)). This model contains an element \( k_a \) such that for all \( i \in \text{INDEX}^M \) we have that \( k_a \neq i \) and \( i < k_a \) iff \( i \in \text{Fin}(M) \). In particular, \( k_a < |a| \) (because \( |a| \) is infinite).

Thanks to Lemma 11 we can freely suppose that \( \text{ELEM}^N = \text{ELEM}^M \) has an element \( e \) not belonging to \( f(\text{ELEM}^A) \) (in particular \( e \neq f(e \text{L}^M) = e \text{L}^N = e \text{L}^M \)). We build \( \mu \) as required by the statement of the lemma (more precisely, the \( \nu \) required by the lemma will be a chain unions of the \( \mu \)'s built at each transfinite step as shown below). The \( \text{INDEX} \) and \( \text{ELEM} \)-components of \( \mu \) will be inclusions. We define \( \mu(b)(k) \) for all \( b \in \text{ARRAY}^M \). If \( k \in \text{INDEX}^M \), we obviously put \( \mu(b)(k) = b(k) \); in the other cases, the definition is as follows:

1. if \( b \not\in^M a \), then \( \mu(b)(k) = e \text{L}^N \) or \( \mu(b)(k) = \bot^N \), depending on whether \( k \in [0,|b|] \) or not;

---

9 Notice that this is the only argument in the whole strong amalgamation proof requiring the fact that we have \( \text{Const} \) in the language.
2. if \( b \sim_{\mathcal{M}} a \) and \( k \neq k_a \), then again \( \mu(b)(k) = c \ell^N \) or \( \mu(b)(k) = \bot_{\mathcal{M}} \), depending on whether \( k \in [0,|b|] \) or not;
3. if \( b \sim_{\mathcal{M}} a \) and \( k = k_a \), then \( \mu(b)(k) = e \).

We need to show that \( \mu \) preserves \( rd, wr, \cdot, |\cdot| \), constant arrays and \( \text{diff} \). Preservation of \( rd, wr, \cdot, |\cdot| \) are easy: constant arrays are preserved, because we cannot have \( \text{Const}^\mathcal{M}(i) \sim_{\mathcal{M}} a \), otherwise \( |a| = i \), which cannot be because \( |a| \) is an element from \( \text{INDEX}^A \) by hypothesis, so that we would have \( f(\text{Const}^\mathcal{A}(i)) = \text{Const}^\mathcal{M}(i) \sim_{\mathcal{M}} a \), contradiction. For preservation of \( \text{diff} \), the problematic case would be the case in which we have \( k_a > \text{diff}(b_1, b_2) \), \( b_1 \sim_{\mathcal{M}} a \) and \( b_2 \not\sim_{\mathcal{M}} a \). However, this is impossible because \( \text{diff}(b_1, b_2) \in \text{INDEX}^\mathcal{M} \) and the fact that we have \( k_a > \text{diff}(b_1, b_2) \) implies that \( \text{diff}(b_1, b_2) \) is finite, which would entail either \( b_1 \sim_{\mathcal{M}} b_2 \) or \( |b_1| \neq |b_2| \): in the former case, we would have \( b_2 \sim_{\mathcal{M}} a \) and in the latter \( k_a > \text{diff}(b_1, b_2) = \max(|b_1|, |b_2|) = \max(|a|, |b_2|) \geq |a| \).

We finally notice that \( k_a \) satisfies the requirements of the lemma. First, \( k_a \not\in \mu(\text{INDEX}^\mathcal{M}) \) and \( \mu(a)(k_a) = c \not\in \mu(f(\text{ELEM}^\mathcal{A})) \) hold by construction. Moreover, since for every \( c \in \text{ARRAY}^A \) we have \( \mu(f(c))(k_a) = c \ell^N \) or \( \mu(f(c))(k_a) = \bot_{\mathcal{M}} \) (depending whether \( k_a \in [0,|c|] \) holds or not), in any case we see that \( \mu(f(c))(k_a) \neq \mu(a)(k_a) \).

**Theorem 5** \( \text{CARDC}(T_1) \) enjoys the strong amalgamation property.

**Proof.** We keep the same notation and construction as in the proof of Theorem 4. However, thanks to Lemma 12 we can now suppose (for \( i = 1,2 \)) that all arrays \( a \in \text{ARRAY}^\mathcal{M}_i \) whose length belongs to \( \text{INDEX}^A \) are such that one of the following two conditions are satisfied:

1. there exists \( c \in \text{ARRAY}^A \) with \( f_i(c) \sim_{\mathcal{M}_i} a \);
2. there exists \( k_a \in \text{INDEX}^\mathcal{M}_i \setminus \text{INDEX}^A \) such that \( a(k_a) \) is an element from \( \text{ELEM}^\mathcal{M}_i \setminus \text{ELEM}^A \) different from all the \( f_i(c)(k_a) \), varying \( c \in \text{ARRAY}^A \).

Let \( a_i \in \text{ARRAY}^\mathcal{M}_i \ (i = 1,2) \) be such that \( \forall k \in \text{INDEX}^\mathcal{M} \) we have

\[
\mu_1(a_1)(k) = \mu_2(a_2)(k) \tag{24}
\]

Notice that, since \( T_1 \) has the strong amalgamation property and the \( \mu_i \) preserve length, this can only happen if \( |a_1| = |a_2| \) belongs to \( \text{INDEX}^A \). We look for some \( c \in \text{ARRAY}^A \) such that \( a_1 = f_1(c) \); since \( \mu_2 \) is injective this would entail \( a_2 = f_2(c) \) because

\[
\mu_2(a_2) = a_1(a_1) = a_1(f_1(c)) = \mu_2(f_2(c)),
\]

implying that \( \mathcal{M} \) is a strong amalgam, as requested.

We separate two cases: (i) one of the arrays \( a_1, a_2 \) satisfy the above condition 2; (ii) both arrays \( a_1, a_2 \) satisfy the above condition 1.

(i) We show that this case is impossible. Suppose, e.g., that \( a_1 \) satisfies condition 2 in \( \mathcal{M}_1 \). Then there exists an index \( k_{a_1} \) in \( \text{INDEX}^\mathcal{M}_1 \setminus \text{INDEX}^A \) such that \( a(k_{a_1}) \) is an element from \( \text{ELEM}^\mathcal{M}_1 \setminus \text{ELEM}^A \) which is different from all the \( f_1(c)(k_{a_1}) \),
varying \( c \in \text{ARRAY}^A \). Since we must have \( \mu_1(a_1)(k_{a_1}) = \mu_2(a_2)(k_{a_1}) \) and \( \mu_1(a_1)(k_{a_1}) \) does not belong to \( \text{ELEM}^{M_2} \) (recall that \( \text{ELEM}^{M_1} \cap \text{ELEM}^{M_2} = \text{ELEM}^A \)), the value of \( a_2 \) for the index \( k_{a_1} \) \((0 < k_{a_1} < |a_1| = |a_2|)\) is built according to the rule \((2*)\), because otherwise \( \mu_2(a_2)(k_{a_1}) \) would be equal to some element in \( \text{ELEM}^{M_2} \). Let \((c, b)\) the pair such that

\[
c \in \text{ARRAY}^A, b \in \text{ARRAY}^{M_2}, \ b \sim^{M_2} a_2, \ k_{a_1} > \text{diff}^{M_2}(b, f_2(c))
\]

Then we have

\[
\mu_1(a_1)(k_{a_1}) = a_1(k_{a_1}) \neq f_1(c)(k_{a_1}) = \mu_2(a_2)(k_{a_1})
\]

contradiction.

(ii) Hence we can have \( \mu_1(a_1) = \mu_2(a_2) \) only when both \( a_1, a_2 \) satisfy condition \( 1 \) above. Let us call \( c_i \in \text{ARRAY}^A \) \((i = 1, 2)\) the arrays such that \( f_i(c_i) \sim^{M_i} a_i \). Then, the pair \((c_1, f_1(c_1))\) witnesses \((2*)\) for \( a_1 \) and for every positive index \( k \in \text{INDEX}^{M_1} \setminus \text{INDEX}^A \) (and similarly for \( a_2 \)). We look for \( c \in \text{ARRAY}^A \) such that \( f_1(c) = a_1 \). Let us consider the following relations coming from \[(24)\] and from the definition of \( \mu_i \):

\[
\forall k \in \text{INDEX}^A, \ a_1(k) = a_2(k)
\]

\[
\forall k \in \text{INDEX}^{M_1} \setminus \text{INDEX}^A, \ a_1(k) = f_1(c_2)(k)
\] (31)

\[
\forall k \in \text{INDEX}^{M_2} \setminus \text{INDEX}^A, \ f_2(c_1)(k) = a_2(k).
\]

where we used that \( \mu_2(a_2)(k) = f_1(c_2)(k) \) if \( k \in \text{INDEX}^{M_1} \setminus \text{INDEX}^A \), and \( \mu_1(a_1)(k) = f_2(c_1)(k) \) if \( k \in \text{INDEX}^{M_2} \setminus \text{INDEX}^A \).

Let us now consider the sets (they are finite because \( f_2(c_2) \sim^{M_2} a_2 \))

\[
J = \{ j \in \text{INDEX}^A \mid c_2(j) \neq a_2(j) \} \subseteq \text{INDEX}^A
\]

\[
E = \{ a_2(j) \mid j \in J \} \subseteq \text{ELEM}^A,
\]

and let us put \( c = wr(c_2, J, E) \); we check that \( c \) is such that \( f_1(c) = a_1 \).

- If \( k \in \text{INDEX}^A \):

\[
f_1(c)(k) = c(k) = wr(c_2, J, E)(k) = a_2(k) = a_1(k)
\]

because \( f_1 \) preserves \( rd \), by the definition of \( J \) and because of the equalities \[(31)\]:

- If \( k \in \text{INDEX}^{M_1} \setminus \text{INDEX}^A \):

\[
f_1(c)(k) = f_1(wr(c_2, J, E))(k) = wr(f_1(c_2), J, E)(k) = f_1(c_2)(k) = a_1(k)
\]

by the definition of \( c \), the fact that \( f_1 \) is an embedding, because \( J \subseteq \text{INDEX}^A \) (hence \( k \notin J \)) and because of the equalities \[(31)\].

\[\text{Notice that if } k \in \text{INDEX}^{M_1} \setminus \text{INDEX}^A \text{ is positive, then } k > \text{diff}(f_1(c), f_1(c)) = 0.\]
Lemma 7. Let \( \phi \) be a quantifier-free formula; then it is possible to compute in linear time a finite separation pair \( \Phi = (\Phi_1, \Phi_2) \) such that \( \phi \) is \( \text{CARD}(T_I) \)-satisfiable iff so is \( \Phi \).

Proof. We first flatten all atoms from \( \phi \) by repeatedly abstracting out subterms (to abstract out a subterm \( t \), we introduce a fresh variable \( x \) and update \( \phi \) to \( x = t \land \phi(x/t) \)); then we remove all atoms of the kind \( a = b \) occurring in \( \phi \) by replacing them by the equivalent formula (16), namely

\[
\text{diff}(a, b) = 0 \land \text{rd}(a, 0) = \text{rd}(b, 0).
\]

Then we abstract out all terms of the kind \( \text{wr}(b, i, e), \text{diff}(a, b) \) and \( |a| \), so that \( \phi \) has now the form \( \Phi_1 \land \Phi_2 \), where \( \Phi_2 \) does not contain \( \text{wr}, \text{diff}, |−| \)-symbols and \( \Phi_1 \) is a conjunction of atoms of the form \( a = \text{wr}(b, i, e), i = \text{diff}(a, b), j = |a| \). Finally, we add to \( \Phi_1 \) the missing atoms of the kind \( |a| = i \) required by Definition 5.\(^{11}\)

Lemma 8. The following conditions are equivalent for a finite 0-instantiated separation pair \( \Phi = (\Phi_1, \Phi_2) \):

(i) \( \Phi \) is \( \text{CARD}(T_I) \)-satisfiable;

(ii) \( \land \Phi_2 \) is \( T_I \cup \text{EUF} \)-satisfiable.

Proof. (i) \( \Rightarrow \) (ii) is clear.

To prove (ii) \( \Rightarrow \) (i), let \( A \) be the model witnessing the satisfiability of \( \land \Phi_2 \) in \( T_I \cup \text{EUF} \) and let \( I \) be the set of all index variables occurring in \( \Phi_1 \cup \Phi_2 \). According to the definition of a separated pair, for every array variable \( a \) occurring in \( \Phi_1 \cup \Phi_2 \) there is an index variable \( l_a \) such that:

\[
A \models f_a(i) \neq \bot \iff 0 \leq i \leq l_a.
\]

for all \( i \in I \) (here \( f_a \) is the unary function symbol replacing \( a \) in \( T_I \cup \text{EUF} \)).

The standardization \( A' \) of \( A \) is the \( T_I \cup \text{EUF} \)-model obtained from \( A \) by modifying the values \( a(k) \) (for all array variables \( a \) occurring in \( \Phi_2 \) and for all indexes \( k \in \text{INDEX}_A^4 \) different from the elements assigned in \( A \) to the variables in \( I \)) in such a way that we have

\[
A' \models f_a(k) = \bot \iff (l_a < k \lor k < 0),
\]

\[
A' \models f_a(k) = \bot \iff 0 \leq k \leq l_a.
\]

The standardization \( A' \) of \( A \) is still a model of \( \land \Phi_2 \). However, in \( A' \) now also the formulae (18, 21) and (17) hold: this is because the \( I \)-instantiations of the

\(^{11}\) The transformation of Lemma 4 does not introduce in \( \Phi_1 \) any formula of the kind \( \text{diff}_n(a, b) = k_n \) (for \( n > 1 \)). These formulae will however be introduced by the Step 1 of the interpolation algorithm of Section 7.
universal index quantifiers occurring in such formulae were taken care in $A$ and their truth value is not modified passing to $A'$, whereas the construction of $A'$ takes care of the instantiations outside $I$.

Let us now define an $CARD(T_I)$-model $M$ satisfying $\Phi$. We first build a structure $N$ where $\text{diff}$ may not be totally defined. We let $\text{INDEX}^N = \text{INDEX}^{A'}$ and $\text{ELEM}^N = \text{ELEM}^{A'}$; we take as $\text{ARRAY}$ the set of all positive-support functions from $\text{INDEX}^N$ into $\text{ELEM}^N$: this includes all functions of the form $f_a$. In addition, if $\bigwedge_{n=1}^l \text{diff}_n(a_1,a_2) = k_n \in \Phi_1$, then the related iterated maxdiff’s are defined in $N$ and we have $N \models \bigwedge_{n=1}^l \text{diff}_n(a_1,a_2) = k_n$ by the above construction. Thus $\Phi$ holds in $N$ and in order to obtain our final $M$ we only need to apply Theorem $3$.

### D Proofs from Section $7$

**Theorem $7$** The above rule-based algorithm computes a quantifier-free interpolant for every $CARD(T_I)$-mutually unsatisfiable pair $A^0, B^0$ of quantifier-free formulae.

**Proof.** We only need to prove that Step 3 really applies.

Suppose not; let $A = (A_1, A_2)$ and $B = (B_1, B_2)$ be the separated pairs obtained after applications of Steps 1 and 2. If Step 3 does not apply, then $A_2 \land B_2$ is $T_I \cup EUF$-consistent. We claim that $(A,B)$ is $CARD(T_I)$-consistent (contradicting that $(A^0, B^0) \subseteq (A,B)$ was $CARD(T_I)$-inconsistent).

Let $\mathcal{M}$ be a $T_I \cup EUF$-model of $A_2 \land B_2$. $\mathcal{M}$ is a two-sorted structure (the sorts are $\text{INDEX}$ and $\text{ELEM}$) endowed for every array constant $d$ occurring in $A \cup B$ of a function $d^\mathcal{M} : \text{INDEX}^{\mathcal{M}} \rightarrow \text{ELEM}^{\mathcal{M}}$. In addition, $\text{INDEX}^{\mathcal{M}}$ is a model of $T_I$.

We list the properties of $\mathcal{M}$ that come from the fact that our Steps 1-2 have been applied (below, we denote by $k^\mathcal{M}$ the element of $\text{INDEX}^{\mathcal{M}}$ assigned to an index constant $k$):

(a) we have that $\mathcal{M} \models \bigwedge A_1(I_A)$ (where $I_A$ is the set of $A$-local constants) and $\mathcal{M} \models \bigwedge B_1(I_B)$ (where $I_A$ is the set of $B$-local constants): this is because $(A_1, A_2)$ and $(B_1, B_2)$ are 0-instantiated by Step 2;\(^\text{[12]}\)

(b) for $AB$-common array variables $c_1, c_2$, we have that $A_1 \cap B_1$ contains a literal of the kind $\text{diff}_n(c_1, c_2) = k_n$ for $n \leq N$; suppose that $\mathcal{M} \models c_{l_1} = c_{l_2}$\(^\text{[13]}\) and that $k$ is an index constant such that $\mathcal{M} \models k \neq l$ for all $AB$-common index constant $l$; then, we can have $\mathcal{M} \models c_1(k) \neq c_2(k)$ only when $\mathcal{M} \models k < k_N$: this is because Step 1 has been applied and because of (a).

We expand $\mathcal{M}$ to an $\text{AR}_{ext}(T_I)$-structure $N$ and endow it with an assignment to our $A$-local and $B$-local variables, in such a way that all $\text{diff}$ operators

\(^{12}\) Thus if e.g. $k,l$ are index constants, $\mathcal{M} \models k = l$ is the same as $k^\mathcal{M} = l^\mathcal{M}$.

\(^{13}\) The sets $A_1(I_A), B_1(I_B)$ are introduced in Definition $6$.

\(^{14}\) Recall that $l_{c_1}, l_{c_2}$ are the $AB$-common constants such that the literals $|c_1| = l_{c_1}, |c_2| = l_{c_2}$ belongs to $A_1 \cap B_1$. 

According to the definition of reflexive-symmetric-transitive closure, if $M \models \equiv_k$ holds, this means by Lemma 4 that we have $k \leq k$ follows. Let us consider the atoms $\equiv_k$ on the set of $A$-local array variables as follows: $\equiv_k$ is the smallest equivalence relation that contains all pairs $(a_1, a_2)$ such that $M \models l_{a_1} = l_{a_2}$ and moreover an atom of one of the following two kinds belongs to $A_1^a$: (I) $a_1 = wr(a_2, i, e)$; (II) $\text{diff}(a_1, a_2) = l$, for an $l$ such that $M \models l < k$.

Claim: if $c_1, c_2$ are $AB$-common and $c_1 \equiv_k c_2$, then $c_1^M(k^M) = c_2^M(k^M)$.

Proof of the Claim. The claim is proved by preferentially showing, for every $A$-local constants $a_1, a_2$ such that $a_1 \equiv_k a_2$, that the number of the $A$-local constants $j$ such that $M \models j < j$ and $a_1^M(j^M) \neq a_2^M(j^M)$ is less or equal to $N_A < N$. This is easily shown by induction on the length of the finite sequence witnessing $a_1 \equiv_k a_2$. We apply this observation to the $AB$-common array variables $c_1, c_2$ and let us consider the atoms $\text{diff}(c_1, c_2) = k_1, \ldots, \text{diff}_n(c_1, c_2) = k_N \in A_1$. Now $k_1, \ldots, k_N$ are all $AB$-common (hence also $A$-local) constants, moreover $N > N_A$ and the number of the $A$-local constants above $k^M$ where $c_1, c_2$ differ is at most $N_A$. According to (b) above, if for absurdity $c_1^M(k^M) = c_2^M(k^M)$ does not hold, then we have $M \models k < k_N$. Since $M \models k \geq 0$ (otherwise $c_1^M(k^M) = c_2^M(k^M)$ follows), this means by Lemma 4 that we have $M \models k_1 \cdots > k_N > 0$. However $k_1, \ldots, k_N$ are all $A$-local constants above $k$, their number is bigger than $N_A$, hence we must have $M \models c_i^M(k^M) = c_j^M(k^M)$ for some $i = 1, \ldots, N$; the latter implies $M \models k_1 = \cdots = k_N = 0$ by Lemma 4 absurd.

In order to interpret $A$-local constants of sort $\text{ARRAY}$, we assign to an $A$-local constant $a$ of sort $\text{ARRAY}$ the function $a^N_X$ defined as follows for every $i \in \text{INDEX}^X$: (i-i) if $i$ is equal to $k^M$, where $k$ is an $A$-local constant, then $a^N_X(i) := a^M_X(k^M)$; (i-ii) if $i$ is different from $k^M$ for every $A$-local constant $k$, but nevertheless $i$ is equal to $k^M$ for some (necessarily $B$-strict) index constant $k$ and there

15 According to the definition of reflexive-symmetric-transitive closure, if $a_1 \equiv_k a_2$ holds then there are $d_0, \ldots, d_n$ such that $d_0 = a_1, d_n = a_2$ and for each $j < n$, we have that either $(d_j, d_{j+1})$ or $(d_{j+1}, d_j)$ satisfies the above requirements: the induction is on such $n$. Notice that the statement is not entirely obvious because the number of the $A$-local index constants is much bigger than $N_A$ (for instance, it includes the $AB$-common constants introduced in Step 1). However induction is easy: it goes through the atoms occurring in the input set $A_1^a$ and uses (a). The required observations are the following: if $d_j = wr(d_{j+1}, i, e) \in A_1^a$, then the only $A$-local constant where $d_j^M$ and $d_{j+1}^M$ can differ is $i$; if $\text{diff}(d_j, d_{j+1}) = l \in A_1^a$ and $M \models l < k$, then $d_j^M$ and $d_{j+1}^M$ cannot differ on any $A$-local constant above $k^M$. Iterating these observations during induction, the claim is clear: we can collect at most the set of the $A$-local constants occurring in $A_1^a$ within a wr symbol.
is an $AB$-common array variable $c$ such that $c \equiv_k a$, then $a^N(i)$ is equal to $c^M(k^M)$; 

(†-iii) in the remaining cases, $a^N(i)$ is equal to $el^M$ or $\bot^M$ depending whether $\mathcal{M} \models 0 \leq i \land i \leq l_a$ holds or not.

Notice that $a^N(i)$ is univocally specified in case (†-ii) because of the above claim. We now show that

(*) all $a^N$ are positive-support functions and all formulae from $A_1 \cup A_2$ are true in $\mathcal{N}$.

Recall in fact that formulae in $A_2$ are Boolean combinations of $A$-local atoms of the kind (23): these are $T_I \cup \mathcal{EUF}^*$-atoms and, due to their shape, each of them is true in $\mathcal{M}$ iff it is true in $\mathcal{A}$ (this is because rd functions are applied only to $A$-local index constants, so that the modifications we introduced for passing from $a^M$ to $a^N$ does not affect truth of these atoms). Concerning formulæ in $A_1$, these are all $wr, \text{diff}$ and $\{\cdot\}$-atoms16 The reason why they are true in $\mathcal{N}$ is the 0-instantiation performed by Step 2 (see (a) above). For example, consider an atom of the kind $|a| = l_a$ appearing in $A_1$: since formulæ $\mathcal{L}$ have been instantiated with all the $A$-local index constants via Step 2, for all $A$-local index constant $h$, we have $\mathcal{M} \models rd(a, h) \neq \bot$ iff $\mathcal{M} \models 0 \leq h \leq i$. Now, thanks to definition (†), for all the elements $h \in \text{INDEX}^N$ we have an analogous result: in case $h$ is equal to $k^M$ for some $A$-local constant $k$, we employ definition (†-i), otherwise we use (†-ii) or (†-iii) (for (†-ii), notice that $a \equiv_k c$ implies that $\mathcal{M} \models l_a = l_c$ and 0-instantiation guarantees that $c^M(k^M)$ is equal to $\bot^M$ or to $el^M$ depending whether $\mathcal{M} \models 0 \leq k \land k \leq l_c$ holds or not). Hence we get that $\mathcal{A}$ satisfies formula $\mathcal{L}$, since the universal quantifier has been instantiated in all possible ways. Thus $\mathcal{A} \models |a| = l_a$, by Lemma 2. The other cases are similar: notice in particular that if $a = wr(a', i, e) \in A_1$ then $a \equiv_k a'$. If $\text{diff}(a, a') = i \in A_1$, the relevant case is when $\mathcal{M} \models i < k$ and $\mathcal{M} \models l_{a_1} = l_{a_2}$, but in this case we have $a \equiv_k a'$ too.

The assignments to the $B$-local array variables $b$ are defined analogously, so that

(*) all $b^N$ are positive-support functions and all formulae from $B_1 \cup B_2$ are true in $\mathcal{N}$.

There is however one important point to notice: for all $AB$-common constants $c$ of sort ARRAY, our specification of $c^M$ does not depend on the fact that we use the above definition for $A$-local or for $B$-local array variables; to see this, we only have to notice that $c \equiv_k c$ holds in case (†-ii) is applied. This remark concludes our proof.

---

16 In addition, we have $\text{diff}_n$-atoms for $n > 1$, but these are all $AB$-common atoms that were not part of the initial pair $A^0, B^0$. In fact they are also true in $\mathcal{N}$, but strictly speaking we do not need to check this fact to get the absurdity that $A^0 \land B^0$ is $\text{CARD}(T_1)$-consistent.