Newtonian and Post Newtonian
Expansionfree Fluid Evolution in $f(R)$
Gravity

M. Sharif *and H. Rizwana Kau sar†
Department of Mathematics, University of the Punjab,
Quaid-e-Azam Campus, Lahore-54590, Pakistan.

Abstract
We consider a collapsing sphere and discuss its evolution under the vanishing expansion scalar in the framework of $f(R)$ gravity. The fluid is assumed to be locally anisotropic which evolves adiabatically. To study the dynamics of the collapsing fluid, Newtonian and post Newtonian regimes are taken into account. The field equations are investigated for a well-known $f(R)$ model of the form $R + \delta R^2$ admitting Schwarzschild solution. The perturbation scheme is used on the dynamical equations to explore the instability conditions of expansion-free fluid evolution. We conclude that instability conditions depend upon pressure anisotropy, energy density and some constraints arising from this theory.

Keywords: $f(R)$ gravity; Instability; Newtonian and post Newtonian regimes.
PACS: 04.50.Kd

1 Introduction
To accommodate observational data with theoretical predictions, one needs to introduce the dark energy (DE) contributions in the most successful gravitational theory of General Relativity (GR). On the other hand, modified
theories of gravity may provide a cosmological accelerating mechanism without introducing any extra DE contribution. Modifications of GR by $f(R)$ models are able to mimic the standard ΛCDM cosmological evolution. These models may have vacuum solutions with null scalar curvature that allow to recover some GR solutions. These may also lead to the existence of some new solutions particularly in spherically symmetric scenario.

In the last decade, some $f(R)$ models were considered to modify GR at small scales to explain inflation, e.g., $f(R) \propto R^2$ but failed to explain the late time acceleration. A model like $f(R) \propto \frac{1}{R}$ was proposed to explain this acceleration but attained no interest due to conflict with solar system tests (Chiba 2003; Dolgov and Kawasaki 2003). A cosmological viable model needs to satisfy the evolution of big-bang nucleosynthesis, radiation and matter dominated eras. Also, they must provide cosmological perturbations compatible with cosmological constraints from cosmic microwave background and large scale structure. The problem of cosmological perturbations in this modified theory and their consequences have widely been discussed in literature, e.g., (Hwang and Noh 2006; Carroll et al. 2006; Carloni et al. 2008; Tsujikawa et al. 2008). Since the lagrangian $R + f(R)$ is analytic at $R = 0$, so the Schwarzschild and other important GR solutions (without cosmological constant) are also the solutions of $f(R)$ gravity (de la Cruz-Dombriz and Dobado 2006).

It is known that $f(R)$ models reveal black hole solutions as in GR, therefore it is quite natural to discuss the question of black hole features and dynamics of the gravitational collapse in this modified theory. When we take conformal transformation of $f(R)$ action, it is found that Schwarzschild solution is the only static spherically symmetric solution for a model of the form $R + \delta R^2$ (Whitt 1984). Also, the uniqueness theorems of spherically symmetric solutions for general polynomial action in arbitrary dimension were proposed by using conformal transformation (Mignemi and Wiltshire 1992). Work on black hole solutions in $f(R)$ gravity has been carried out by many authors, e.g., (Moon et al. 2011; de la Cruz-Dombriz et al. 2009; Mazharimousavi et al. arXiv/1105.3659v3)

It is found that evolution of expansionfree spherically symmetric distribution is consistent with the existence of a vacuum cavity within the matter distribution (Herrera and Santos 2004; Herrera et al. 2008 ). Skripkin model is the first example of expansionfree evolution (Skripkin 1960). This model corresponds to evolution of spherically symmetric perfect fluid distribution with constant energy density. We have explored this model in $f(R)$ gravity
(Sharif and Kausar 2011) and effects of $f(R)$ DE on the dynamics of dissipative fluid collapse (Sharif and Kausar 2010; Sharif and Kausar 2011). Also, solution of the field equations for Bianchi models and spherically symmetric dust solution are obtained in this theory (Sharif and Kausar 2011; Sharif and Kausar 2011; Sharif and Kausar 2011).

The physical applications of such models lie at the core of astrophysical background where a cavity within the fluid distribution is present. It may help to study the formation of voids on the cosmological scales (Liddle and Wands 1991). Also, the expansionfree fluid evolution causes a blowup of shear scalar which results the appearance of a naked singularity (Joshi et al. 2002). It is interesting to note that expansionfree models defines the two hypersurfaces, one separating the fluid distribution externally from the Schwarzschild vacuum solution while other one is the boundary between the central Minkowskian cavity and the fluid. Furthermore, expansionfree models require anisotropy in pressure and inhomogeneity in energy density.

The existence of stellar model can be assured if it is stable against fluctuations. The problem of dynamical instability is closely related to the structure formation and evolution of self-gravitating objects. A pioneer work in this direction was done by Chandrasekhar (Chandrasekhar 1964). Afterwards, this issue has been investigated by many authors for adiabatic, non-adiabatic, anisotropic and shearing viscous fluids (Herrera et al. 1989; Chan et al. 1989; Chan et al. 1994; Chan et al 1993; Herrera et al, arXiv/1010.1518). The dynamical instability of collapsing fluids can be well discussed in term of adiabatic index $\Gamma$. It measures the variation of pressure with respect to a given variation of energy density and defines the range of instability. In GR, the dynamical instability is independent of this $\Gamma$ factor (Herrera et al, arXiv/1010.1518) in the Newtonian and post Newtonian regimes. In a recent paper (Sharif and Kausar 2011), we have investigated the dynamical instability of expansionfree fluid evolution in $f(R)$ gravity by assuming all the higher order curvature terms on the matter side. This provides the effects of $f(R)$ DE on the instability conditions.

In this paper, we treat the field equation in the usual fourth order form and discuss instability conditions of expansionfree fluid collapse in metric $f(R)$ theory. Here we use Ricci scalar to evaluate time dependent part of the perturbed quantities. We consider locally anisotropic fluid distribution inside the collapsing sphere. The format of the paper is as follows. In section 2, we formulate the field equations and dynamical equations for spherically symmetric spacetime in $f(R)$ gravity by taking locally anisotropy in the pressure.
Section 3 is devoted to study the perturbation scheme on physical quantities and the metric coefficients by assuming that fluid initially is in static equilibrium. A well-known physical \( f(R) \) model is considered and perturbed on the same pattern. In section 4, Newtonian and post Newtonian regimes are taken into account. Discussion is made on the dynamical instability conditions of the expansion free fluid evolution in the Newtonian approximation. The last section 5 concludes the main results of the paper.

2 Field Equations and Dynamical Equations

We take spherically symmetric distribution of anisotropic collapsing fluid in co-moving coordinates. The line element is given by

\[
ds^2 = A^2(t, r) dt^2 - B^2(t, r) dr^2 - C^2(t, r) (d\theta^2 + \sin^2 \theta d\phi^2). \tag{2.1}
\]

This yields two radii: one is the areal radius \( C(r, t) \) measuring the radius from spherical surface while other is the proper radius found from \( \int B(t, r) dr \). Also, there is a hypersurface \( \Sigma(e) \) separating interior spacetime from the exterior one. The energy-momentum tensor for locally anisotropic fluid can be written as

\[
T_{\alpha\beta} = (\rho + p_\perp) u_\alpha u_\beta - p_\perp g_{\alpha\beta} + (p_r - p_\perp) \chi_\alpha \chi_\beta, \quad (\alpha, \beta = 0, 1, 2, 3), \tag{2.2}
\]

where \( \rho \) is the energy density, \( p_\perp \) the tangential pressure, \( p_r \) the radial pressure, \( u_\alpha \) the four-velocity of the fluid and \( \chi_\alpha \) is the unit four-vector along the radial direction. These quantities satisfy the relations

\[
u^\alpha u_\alpha = 1, \quad \chi^\alpha \chi_\alpha = -1, \quad \chi^\alpha u_\alpha = 0 \tag{2.3}
\]

which are obtained from the following definitions in co-moving coordinates

\[
u^\alpha = A^{-1} \delta_0^\alpha, \quad \chi^\alpha = B^{-1} \delta_1^\alpha. \tag{2.4}
\]

The acceleration \( a_\alpha \) and the expansion \( \Theta \) are given by

\[a_\alpha = u_{\alpha;\beta} u^\beta, \quad \Theta = u^\alpha_{,\alpha}. \tag{2.5}\]

Using Eqs. (2.4) and (2.5), it follows that

\[a_1 = -\frac{A'}{A}, \quad a^2 = a^\alpha a_\alpha = \left( \frac{A'}{AB} \right)^2, \quad \Theta = \frac{1}{A} \left( \frac{\dot{B}}{B} + 2 \frac{\dot{C}}{C} \right), \tag{2.6}\]
where dot and prime represent derivatives with respect to $t$ and $r$ respectively.

The Einstein-Hilbert action in GR is

$$S_{EH} = \frac{1}{2\kappa} \int d^{4}x \sqrt{-g} R,$$  \hspace{1cm} (2.7)

where $\kappa$ is the coupling constant, $R$ is the Ricci scalar and $g$ is the metric tensor. For $f(R)$ theory of gravity, it is modified as follows

$$S_{\text{modif}} = \frac{1}{2\kappa} \int d^{4}x \sqrt{-g} f(R),$$  \hspace{1cm} (2.8)

where $f(R)$ is the general function of the Ricci scalar. Varying this action with respect to the metric tensor, we obtain the following field equations

$$F(R) R_{\alpha\beta} - \frac{1}{2} f(R) g_{\alpha\beta} - \nabla_{\alpha} \nabla_{\beta} F(R) + g_{\alpha\beta} \Box F(R) = \kappa T_{\alpha\beta},$$  \hspace{1cm} (2.9)

where $F(R) \equiv df(R)/dR$. For spherically symmetric spacetime, these equations become

$$\frac{AA''}{B^2} - \frac{\dot{B}}{B} + \frac{\dot{A}}{AB} - \frac{AA'B'}{B^3} - \frac{2\dot{C}}{C} + \frac{2\dot{AC}'}{AC'} + \frac{2AA'C'}{B^2C} - \frac{A^2 f(R)}{2}$$

$$- \frac{A^2 F''}{B^2 F} - \frac{\dot{F}}{F} \left( \frac{-\dot{B}}{B} + \frac{2\dot{C}}{C} \right) - \frac{F' A^2}{FB^2} \left( \frac{-B'}{B} + \frac{2C''}{C} \right) = \kappa \rho A^2,$$  \hspace{1cm} (2.10)

$$- 2 \left( \frac{\dot{C}}{C} - \frac{\dot{CA'}}{CA} - \frac{\dot{BC}'}{BC} \right) - \frac{\dot{F}'}{F} + \frac{A' \dot{F}}{AF} + \frac{\dot{B} F'}{BF} = 0,$$  \hspace{1cm} (2.11)

$$- \frac{2B B''}{A^2 F} + \frac{\dot{F} B^2}{FA^2} \left( \frac{\dot{A}}{A} + \frac{2\dot{C}}{C} \right) + \frac{F'}{F} \left( \frac{A'}{A} + \frac{2C''}{C} \right) = \kappa \rho_{\perp} B^2,$$  \hspace{1cm} (2.12)

$$\frac{\dot{C}}{CA^2} - \frac{\dot{A} C}{CA^3} - \frac{A' C'}{B^2 AC} - \frac{C''}{B^2 C} + \frac{\dot{C} B}{A^2 BC} + \frac{B'C'}{B^3 C} + \frac{1}{C^2}$$

$$+ \frac{\dot{C}^2}{A^2 C^2} - \frac{C'^2}{B^2 C^2} + \frac{f(R)}{2} \frac{-\dot{F}}{FA^2} + \frac{F''}{FB^2} \frac{\dot{F}}{A^2} \left( \frac{\dot{A}}{A} - \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right)$$

$$+ \frac{F'}{B^2} \left( \frac{A'}{A} - \frac{B'}{B} + \frac{C'}{C} \right) = \kappa \rho_{\perp}.$$  \hspace{1cm} (2.13)
The Ricci scalar curvature is given by
\[
R = 2 \left[ \frac{A''}{AB^2} - \frac{\dot{B}}{A^2B} + \frac{\ddot{A}}{A^3B} - \frac{A'B'}{AB^3} + \frac{2\dot{C}}{CA^2} + \frac{2\dot{A}C}{AB^2} \\
+ \frac{2C''}{CB^2} - \frac{2\dot{C}B}{CA^2B} - \frac{2C'B'}{CB^3} - \frac{1}{C^2} - \frac{\dot{C}}{A^2C^2} + \frac{C''}{B^2C^2} \right].
\] (2.14)

For the exterior spacetime to \( \Sigma^{(e)} \), we take the Schwarzschild spacetime in the form
\[
ds^2 = \left(1 - \frac{2M}{r}\right) d\nu^2 + 2 dr d\nu - r^2 (d\theta^2 + \sin^2 \theta d\phi^2),
\] (2.15)
where \( M \) represents the total mass and \( \nu \) is the retarded time.

In order to study the properties of collapsing process, we formulate the dynamical equations. For this purpose, we use the Misner-Sharp mass function defined as follows (Misner and Sharp 1964)
\[
m(t, r) = \frac{C}{2} (1 + g^{\mu\nu}C_{,\mu}C_{,\nu}) = \frac{C}{2} \left( 1 + \frac{\dot{C}^2}{A^2} - \frac{C''}{B^2} \right).
\] (2.16)

This equation provides the total energy inside a spherical body of radius \( C \).

From the continuity of the first and second fundamental forms, the matching of the adiabatic sphere to the Schwarzschild spacetime on the boundary surface, \( \Sigma^{(e)} \), yields
\[
M_{\Sigma^{(e)}} = m(t, r).
\] (2.17)

The proper time and radial derivatives are given by
\[
D_T = \frac{1}{A} \frac{\partial}{\partial t}, \quad D_C = \frac{1}{C'} \frac{\partial}{\partial r},
\] (2.18)
where \( C \) is the areal radius of the spherical surface. The velocity of the collapsing fluid is defined as
\[
U = D_T C = \frac{\dot{C}}{A}
\] (2.19)
which is always negative. Using this expression, Eq. (2.16) implies that
\[
E \equiv \frac{C'}{B} = \left[ 1 + U^2 + \frac{2m}{C} \right]^{1/2}.
\] (2.20)
The dynamical equations can be obtained from the non-trivial contracted components of the Bianchi identities. Consider the following two equations

\[ T_{\alpha\beta} u_{\alpha} = 0, \quad T_{\alpha\beta} \chi_{\alpha} = 0 \quad (2.21) \]

which yield

\[ \dot{\rho} + (\rho + p_r) \frac{\dot{B}}{B} + 2(\rho + p_\perp) \frac{\dot{C}}{C} = 0, \quad (2.22) \]

\[ p'_r + (\rho + p_r) \frac{A'}{A} + 2(p_r - p_\perp) \frac{C'}{C} = 0. \quad (2.23) \]

### 3 The \( f(R) \) Model and Perturbation Scheme

In this section, we consider a particular \( f(R) \) model and apply perturbation scheme on all the above equations. Consider the following well-known \( f(R) \) model

\[ f(R) = R + \delta R^2, \quad (3.1) \]

where \( \delta \) is any positive real number. The stability criteria for this model corresponds to \( f''(R) > 0 \). For \( \delta = 0 \), GR is recovered in which black holes are stable classically but not quantum mechanically due to Hawking radiations. Since such features are also found in \( f(R) \) gravity, hence the classical stability condition for the Schwarzschild black hole can be enunciated as \( f''(R) > 0 \) (Sotiriou and Faraoni 2010).

The purpose of introducing perturbation scheme is to analyze the instability conditions of the dynamical equation. We assume that initially all the quantities have only radial dependence, i.e., fluid is in static equilibrium. Then these quantities have time dependence as well in their perturbation, i.e.,

\[ A(t, r) = A_0(r) + \epsilon T(t) a(r), \quad (3.2) \]

\[ B(t, r) = B_0(r) + \epsilon T(t) b(r), \quad (3.3) \]

\[ C(t, r) = C_0(r) + \epsilon T(t) c(r), \quad (3.4) \]

\[ \rho(t, r) = \rho_0(r) + \epsilon \bar{\rho}(t, r), \quad (3.5) \]

\[ p_r(t, r) = p_{r0}(r) + \epsilon \bar{p}_r(t, r), \quad (3.6) \]

\[ p_\perp(t, r) = p_{\perp0}(r) + \epsilon \bar{p}_\perp(t, r), \quad (3.7) \]

\[ m(t, r) = m_0(r) + \epsilon \bar{m}(t, r), \quad (3.8) \]

\[ \Theta(t, r) = \epsilon \bar{\Theta}(t, r). \quad (3.9) \]
Also, the Ricci scalar in \( f(R) \) model leads to

\[
R(t, r) = R_0(r) + \epsilon T(t) e(r), \tag{3.10}
\]

\[
f(R) = R_0(1 + 2\delta R_0) + \epsilon T(t) e(r)(1 + 2\delta R_0), \tag{3.11}
\]

\[
F(R) = (1 + 2\delta R_0) + 2\epsilon \delta T(t) e(r). \tag{3.12}
\]

where \( 0 < \epsilon \ll 1 \). We choose radial part of the areal radius as the Schwarzschild coordinate, i.e., \( C_0(r) = r \).

Using the above values in (2.10)-(2.13), the static configuration turns out to be

\[
\frac{A''_0}{A_0} - \frac{A'_0 B'_0}{A_0 B_0} + \frac{2A'_0}{A_0 r} - \frac{B^2_0}{2} \frac{R_0(1 + 2\delta R_0)}{1 + 2\delta R_0}
- \frac{2\delta R'_0}{1 + 2\delta R_0} + \frac{2\delta R''_0}{1 + 2\delta R_0} \left( \frac{-B'_0}{B_0} + \frac{2}{r} \right) = \kappa \rho_0 B^2_0, \tag{3.13}
\]

\[
- \frac{A''_0}{A_0} + \frac{A'_0 B'_0}{A_0 B_0} + \frac{2B'_0}{B_0 r} + \frac{B^2_0}{2} \frac{R_0(1 + 2\delta R_0)}{1 + 2\delta R_0}
+ \frac{2\delta R'_0}{1 + 2\delta R_0} \left( \frac{A'_0}{A_0} + \frac{2}{r} \right) = \kappa p_r B^2_0, \tag{3.14}
\]

\[
- \frac{A'_0}{A_0 r} + \frac{B'_0}{B_0 r} - \frac{B^2_0}{r^2} - \frac{1}{r^2} + \frac{B^2_0}{2} \frac{R_0(1 + 2\delta R_0)}{1 + 2\delta R_0} + \frac{2\delta R''_0}{1 + 2\delta R_0}
+ \frac{2\delta R'_0}{1 + 2\delta R_0} \left( \frac{A'_0}{A_0} - \frac{B'_0}{B_0} + \frac{1}{r} \right) = \kappa p_\perp B^2_0. \tag{3.15}
\]

The static part of the Ricci scalar takes the form

\[
R_0(r) = \frac{2}{B^2_0} \left[ \frac{A''_0}{A_0} - \frac{A'_0 B'_0}{A_0 B_0} - \frac{2B'_0}{B_0 r} - \frac{B^2_0}{r^2} + \frac{1}{r^2} \right]. \tag{3.16}
\]

Using the perturbed quantities given in Eqs. (3.11)-(3.12) along with (3.13)-(3.15).
in the field equations, it follows that

\[
\frac{aA''}{A_0} + a'' - 2bA' - \frac{a'B'}{A_0B_0} - \frac{aB''}{B_0} - \frac{b'A'}{B_0} - \frac{bB'}{A_0B_0} + \frac{3bB'_0A'_0}{B_0^2} + \frac{2aA'_0}{B_0r} + \frac{2a'}{r}
\]

\[
- \frac{2A'_0b}{B_0r} - \frac{2A'_0}{r^2} - \frac{2\bar{c}'A'_0}{r} - \frac{aB''_0(1 + \delta R_0)}{A_0(1 + 2\delta R_0)} - \frac{2eB^2_0}{2A_0} - \frac{e\delta A_0B_0R_0(1 + \delta R_0)}{(1 + 2\delta R_0)^2}
\]

\[
+ \frac{2\delta}{1 + \delta R_0} \left[ - \frac{2a}{A_0} + \frac{2b}{A_0} - \frac{e''}{2} + 2\delta e\delta R'_0A_0B_0^2 + \frac{2aB'_0R'_0}{A_0B_0} - \frac{2bA_0B'_0R'_0}{B_0^2} \right]
\]

\[
+ \frac{a'_0B'_0}{B_0}(1 + 2\delta R_0)B_0 + \frac{4aR'_0}{rA_0} + \frac{4bA_0R'_0}{B_0r} + \frac{2a' A'_0}{r} + \frac{4e\delta A_0R'_0}{r(1 + 2\delta R_0)}
\]

\[
+ A_0R'_0 \left( \frac{b}{B_0} \right)' - 2A_0R''_0 \left( \frac{\bar{c}}{r} \right) - \frac{\bar{c}}{r} = \frac{\kappa \bar{\rho} A_0B^2_0}{T} + 2\kappa \rho_0 aB^2_0.
\]

\[
(3.17)
\]

\[
\left( \frac{\bar{c}}{r} \right)' - \frac{b}{B_0r} - \frac{\bar{c} A'_0}{rA_0} - \frac{\delta}{1 + 2\delta R_0} \left[ - e' + e\frac{A'_0}{A_0} + \frac{bR'_0}{B_0} \right] = 0,
\]

\[
(3.18)
\]

\[
\frac{aA''}{A_0} - \frac{a''}{A_0} + \frac{aB'}{A_0B_0} + \frac{bA'_0}{A_0B_0} + \frac{2B'}{B_0} + \frac{2\bar{c}'B'_0}{B_0r} - \frac{2bB'_0}{B_0r^2} - \frac{2eB''}{r^2B_0} = \frac{eB^2_0}{2}(1 + 2\delta R_0) + 2\delta R'_0 \left\{ \left( \frac{a}{A_0} \right)' + 2 \left( \frac{\bar{c}}{r} \right)' \right\}
\]

\[
+ \frac{\bar{c}}{r} = \frac{\kappa \bar{\rho} B^2_0}{T} + \kappa bB_0\rho_0,
\]

\[
(3.19)
\]

\[
\frac{1}{r} \left\{ \left( \frac{a}{A_0} \right)' + \left( \frac{b}{B_0} \right)' \right\} + \left( \frac{\bar{c}}{r} \right)' \left\{ \frac{A'_0}{A_0} + \frac{B'_0}{B_0} \right\} + \left( \frac{\bar{c}}{r} \right)'' + \frac{2}{r^2} \left( \frac{b}{B_0} + \frac{\bar{c}}{r} \right)
\]

\[
+ \frac{2B^2_0}{r^2} \left( \frac{b}{B_0} - \frac{\bar{c}}{r} \right) - \frac{aA'_0}{rA_0} - \frac{4b}{B_0r} \left( \frac{A'_0}{A_0} + \frac{B'_0}{B_0} \right) - \frac{bB_0R_0(1 + \delta R_0)}{1 + 2\delta R_0}
\]

\[
+ \frac{2\delta R'_0}{1 + 2\delta R_0} \left\{ \left( \frac{a}{A_0} \right)' - \left( \frac{b}{B_0} \right)' + \frac{2\delta}{r^2} \left( \frac{b}{B_0} + \frac{\bar{c}}{r} \right) \right\} - \frac{eB^2_0}{2} \left( \frac{A'_0}{A_0} - \frac{B'_0}{B_0} + \frac{2}{r} \right)
\]

\[
+ \frac{2\delta}{1 + 2\delta R_0} \left( \frac{e}{1 + 2\delta R_0} \right)' + \frac{2\delta}{1 + 2\delta R_0} \left( \frac{A'_0}{A_0} - \frac{B'_0}{B_0} + \frac{2}{r} \right)
\]

\[
- \frac{\bar{T} B^2_0}{T A_0^2} \left( \frac{\bar{c}}{r} + \frac{2c\delta}{1 + 2\delta R_0} \right) = \frac{\kappa \bar{\rho} B^2_0}{T} + \kappa bB_0\rho_0.
\]

\[
(3.20)
\]
We can write the perturbed configuration of the Ricci scalar curvature, by using Eq. (3.16), as follows

\[
- \frac{1}{A_0 B_0} \left( a'' - \frac{a A_0''}{A_0} - \frac{2b A_0''}{B_0} \right) - \frac{2}{r B_0^3} \left( b' + \bar{c}' B_0' - \frac{\bar{c} B_0'}{r} - \frac{3b B_0'}{B_0} \right) + \frac{e}{2} + \frac{\bar{c}}{r^3} - \frac{1}{A_0 B_0^3} \left( a' B_0' + b A_0' - \frac{a A_0' B_0'}{A_0} - \frac{3b A_0' B_0'}{B_0} \right) + \frac{2\bar{c}''}{r B_0^2} + \frac{1}{B_0^2 r^2} \left( \frac{c'}{B_0} - \frac{b}{r} - \frac{\bar{c}}{r} \right) - \frac{\ddot{T}}{T} \left( \frac{b}{B_0} - \frac{2\bar{c}}{r} \right) \frac{1}{A_0^2} = 0. \tag{3.21}
\]

This equation can also be written as

\[
\ddot{T}(t) - \alpha(r) T(t) = 0, \tag{3.22}
\]

where

\[
\alpha(r) = \left[ - \frac{1}{A_0 B_0} \left( a'' - \frac{a A_0''}{A_0} - \frac{2b A_0''}{B_0} \right) - \frac{2}{r B_0^3} \left( b' + \bar{c}' B_0' - \frac{\bar{c} B_0'}{r} - \frac{3b B_0'}{B_0} \right) + \frac{e}{2} + \frac{\bar{c}}{r^3} - \frac{1}{A_0 B_0^3} \left( a' B_0' + b A_0' - \frac{a A_0' B_0'}{A_0} - \frac{3b A_0' B_0'}{B_0} \right) + \frac{1}{B_0^2 r^2} \left( \frac{c'}{B_0} - \frac{b}{r} - \frac{\bar{c}}{r} \right) + \frac{2\bar{c}''}{r B_0^2} \right] \frac{A_0^2}{\left( \frac{b}{B_0} - \frac{2\bar{c}}{r} \right)}. \tag{3.23}
\]

For the sake of instability region, we assume that all the functions involved in the above equation are such that \( \alpha \) remains positive. Consequently, the solution of Eq. (3.22) becomes

\[
T(t) = -e^{\sqrt{\alpha}}. \tag{3.24}
\]

Here we assume that the system starts collapsing at \( t = -\infty \) such that \( T(-\infty) = 0 \), keeping it in static position. Afterwards, it goes on collapsing with the increase of \( t \).

Applying static and non-static perturbation scheme respectively on dy-
namical equations (2.22) and (2.23), we have

\[ p'_{\rho_0} + (\rho_0 + p_{\rho_0}) \frac{A'_0}{A_0} + \frac{2}{r} (p_{\rho_0} - p_{\perp}) = 0, \]

(3.25)

\[ \frac{1}{A_0} \left[ \dot{\bar{\rho}} + (\rho_0 + p_{\rho_0}) \frac{\dot{T} B_0}{b} + 2(\rho_0 + p_{\perp}) \frac{\dot{\bar{c}}}{r} \right] = 0, \]

(3.26)

\[ \frac{1}{B_0} \left[ \bar{\rho}' + (\rho_0 + p_{\rho_0}) T \left( \frac{a}{A_0} \right)' + (\bar{\rho} + \bar{p}_r) \frac{A'_0}{A_0} \right. \]

\[ + 2(p_{\rho_0} - p_{\perp}) T \left( \frac{\bar{c}}{r} \right)' \left. + \frac{2}{r} (\bar{p}_r - \bar{p}_{\perp}) \right] = 0. \]

(3.27)

Integrating Eq. (3.26) with respect to time, it follows that

\[ \bar{\rho} = - \left[ (\rho_0 + p_{\rho_0}) \frac{b}{B_0} + 2(\rho_0 + p_{\perp}) \frac{\bar{c}}{r} \right] T. \]

(3.28)

Perturbation on Eq. (2.16) yields

\[ m_0 = \frac{r}{2} \left( 1 - \frac{1}{B_0^2} \right), \]

(3.29)

\[ \bar{m} = - \frac{T}{B_0^2} \left[ r \left( \frac{\bar{c}' - b}{B_0} \right) + (1 - B_0^2) \frac{\bar{c}}{2} \right]. \]

(3.30)

In order to relate \( \bar{\rho} \) and \( \bar{p}_r \) for the static spherically symmetric configuration, we may assume an equation of state of Harrison-Wheeler type as follows (Chan et al. 1973; Wheeler et al. 1965)

\[ \bar{p}_r = \Gamma \frac{p_{\rho_0}}{\rho_0 + p_{\rho_0}} \bar{\rho}, \]

(3.31)

where \( \Gamma \) is the adiabatic index. It measures the variation of pressure for a given variation of density. We take it constant throughout the fluid evolution.

4 Expansionfree Newtonian and Post Newtonian Regimes

Here we assume expansionfree (\( \Theta = 0 \)) evolution of anisotropic fluid and develop dynamical equations. Since the expansion scalar describes the rate
of change of small volumes of the fluid, so the expansionfree fluid evolution of spherically symmetric distribution is consistent with the formation of a vacuum cavity within the fluid. The Minkowski spacetime is supposed to present inside the cavity. Using Eq. (3.9) in (2.6), we have

\[ \bar{\Theta} = \frac{\dot{T}}{A_0} \left( \frac{b}{B_0} + \frac{2c}{r} \right). \] (4.1)

The expansionfree condition \((\bar{\Theta} = 0)\) implies that

\[ \frac{b}{B_0} = -\frac{2c}{r}. \] (4.2)

Before implementing the above result on the last section, let us identify the terms differentiating Newtonian (N), post Newtonian (pN) and post post Newtonian (ppN) regimes. These terms will be considered to develop the dynamical equations which help to understand the instability conditions of the expansionfree fluid evolution. For N approximation, we assume

\[ \rho_0 \gg p_r, \quad \rho_0 \gg p_\perp. \] (4.3)

For the metric coefficients expanded up to pN approximation, we take

\[ A_0 = 1 - \frac{Gm_0}{c^2}, \quad B_0 = 1 + \frac{Gm_0}{c^2}, \] (4.4)

where \(G\) is the gravitational constant and \(c\) is the speed of light. Adding Eqs. (3.13) and (3.14) and using Eq. (3.29), it follows that

\[ \frac{A'_0}{A_0} = \frac{\kappa r^3 (\rho_0 + p_r)(1 + 2\delta R_0) + 2\delta R_0^\prime r^2 (r - 2m_0) + 2m_0 (1 + 2\delta R_0 + \delta R_0^\prime r)}{2r(r - 2m_0)(1 + 2\delta R_0 + \delta R_0^\prime r)}. \] (4.5)

Inserting this equation in (3.25), we have first dynamical equation in relativistic units as follows

\[ p'_{r_0} = -\frac{(\rho_0 + p_r)}{2r(r - 2m_0)(1 + 2\delta R_0 + \delta R_0^\prime r)} \left[ \kappa r^3 (\rho_0 + p_r)(1 + 2\delta R_0) \right. \]
\[ + \left. 2\delta R_0^\prime r^2 (r - 2m_0) + 2m_0 (1 + 2\delta R_0 + \delta R_0^\prime r) \right] \]
\[ + \frac{2}{r} (p_\perp - p_r). \] (4.6)
In view of dimensional analysis, this equation can be written in c.g.s. units as follows

\[
p'_r = \frac{-\left(\rho_0 + c^{-2}p_{r0}\right)}{2rc^{-2}(r - 2Gc^{-2}m_0)(1 + 2\delta R_0 + \delta R'_0)}[G\kappa r^3(\rho_0 + c^{-2}p_{r0})(1 + 2\delta R_0) + 2\delta R'_0 r^2(r - 2Gc^{-2}m_0) + 2Gc^{-2}m_0(1 + 2\delta R_0 + \delta R'_0)]
\]

[4.7]

Expanding up to \(c^{-4}\) order, we get

\[
p'_r = \frac{2}{r}(p_{\perp 0} - p_{r0}) - \frac{\rho_0}{r^3}(r + Gm_0)[G\kappa r^3\rho_0(1 + 2\delta R_0) + 2\delta R'_0 r^3](1 - 2\delta R_0)
\]

- \(\delta R'_0 r\) - \(\frac{G}{c^4r^2}\left[\rho_0(r + Gm_0)\{p_{r0}\kappa r^3(1 + 2\delta R_0) - 4\delta R'_0 m_0 r^2\}
\]

[4.8]

\[
+ 2m_0(1 + 2\delta R_0 + \delta R'_0 r) - 2\rho_0 Gm_0^2 r^2\{G\kappa \rho_0(1 + 2\delta R_0) + 2\delta R'_0\}
\]

[4.8]

\[
+ r^3\{r + Gm_0\}p_{r0}\{\kappa \rho_0(1 + 2\delta R_0)}\}(1 - 2\delta R_0 - \delta R'_0 r)
\]

[4.8]

\[
- \frac{G}{c^4r^2}\left[\rho_0(r + Gm_0)\{p_{r0}\kappa r^3(1 + 2\delta R_0) - 4\delta R'_0 m_0 r^2 + 2m_0(1 + 2\delta R_0)
\]

[4.8]

\[
+ \delta R'_0 r\} + r_0 Gm_0^2 r^3\{G\kappa \rho_0(1 + 2\delta R_0) + 2\delta R'_0\} + 2\rho_0 G^2 m_0^2\{\kappa r^3 p_{r0}
\]

[4.8]

\[
\times (1 + 2\delta R_0) - 4\delta R'_0 m_0 r^2 + 2m_0(1 + 2\delta R_0 + \delta R'_0 r)](1 - 2\delta R_0 - \delta R'_0 r).
\]

It is worth mentioning here that the order of \(c\) differentiates the terms belonging to \(N, pN\) and \(ppN\) regimes, i.e.,

- terms of order \(c^0\) correspond to \(N\)-approximation,
- terms of order \(c^{-2}\) correspond to \(pN\)-approximation,
- terms of order \(c^{-4}\) correspond to \(ppN\)-approximation.

Simplification of Eq.(4.8) yields the following table

**Table 1. N, pN and ppN Terms**

| N terms | \(p_{r0}, p_{\perp 0}, \rho_0^2(1 + 2\delta R_0), m_0 \rho_0^2(1 + 2\delta R_0), \rho_0 \delta R'_0, m_0 \rho_0 \delta R'_0\) |
|---------|----------------------------------------------------------------------------------------------------------------------------------|
| pN terms | \(p_{r0} p_{r0}(1 + 2\delta R_0), p_{r0} \rho_0 m_0(1 + 2\delta R_0), m_0 \rho_0^2(1 + 2\delta R_0), m_0 \rho_0 \delta R'_0, m_0 \rho_0 \delta R'_0, m_0 \rho_0 \delta R'_0, p_{r0} m_0(1 + 2\delta R_0 + \delta R'_0)\) |
| ppN terms | \(p_{r0}^2(1 + 2\delta R_0), p_{r0} \rho_0 \rho_0 (1 + 2\delta R_0), m_0 \rho_0 \rho_0 (1 + 2\delta R_0), m_0 \rho_0 \rho_0 \delta R'_0, m_0 \rho_0 \rho_0 \delta R'_0, p_{r0} \rho_0 m_0(1 + 2\delta R_0 + \delta R'_0)\) |

13
Using the expansionfree condition (4.2), Eq.(3.28) becomes

\[ \bar{\rho} = 2(p_{r0} - p_{\perp0}) \frac{T\bar{c}}{r}. \] (4.12)

This shows that the perturbed energy density depends on the static pressure anisotropy, hence supporting the expansionfree condition. Inserting Eq.(4.12) in (3.31), we have

\[ \bar{p}_r = 2\Gamma \frac{p_{r0}}{\rho_0 + p_{r0}} (p_{r0} - p_{\perp0}) \frac{T\bar{c}}{r}. \] (4.13)

From Eq.(3.25), we can write

\[ \frac{A'_0}{A_0} = -\frac{1}{\rho_0 + p_{r0}} \left[ p'_{r0} + \frac{2}{r} (p_{r0} - p_{\perp0}) \right]. \] (4.14)

Also, from Eq.(3.29), we obtain

\[ \frac{B'_0}{B_0} = \frac{-m_0}{r(r - 2m_0)}. \] (4.15)

Notice that the expressions of \( \bar{p}_r \) and \( \bar{p}_{\perp0} \) in Eqs.(4.12)-(4.14) are of ppN order approximation. In order to discuss the instability conditions up to pN order, we neglect these quantities.

Substituting the value of \( \bar{p}_{\perp} \) from Eq.(3.20) in dynamical equation (3.27) and using Eqs.(3.24) and (4.2), it follows that

\[ \kappa (\rho_0 + p_{r0}) r \left( \frac{a}{A_0} \right)' + 2\kappa r (p_{r0} - p_{\perp0}) \left( \frac{\bar{c}}{r} \right)' - 8\kappa p_{\perp0} \frac{\bar{c}}{r} - \frac{2}{B_0^2} \left[ \frac{1}{r} \left( \frac{a}{A_0} \right)' \right] ' + \left( \frac{\bar{c}}{r} \right)' \left( \frac{A'_0}{A_0} + \frac{B'_0}{B_0} - \frac{2}{r} \right) + \left( \frac{\bar{c}}{r} \right)'' - \frac{2c}{r^3} (1 + 3B_0^2) + \frac{a A'_0}{r A_0} + \frac{8\bar{c}}{r^2} \left( \frac{A'_0}{A_0} + \frac{B'_0}{B_0} \right) \\
+ e = \frac{2R_0(1 + \delta R_0)}{1 + 2\delta R_0} \left( \frac{2\bar{c}}{r} - \frac{\delta c}{1 + 2\delta R_0} \right) + \frac{4\delta R'_0}{B_0^2(1 + 2\delta R_0)} \left\{ \left( \frac{a}{A_0} \right)' + 3 \left( \frac{\bar{c}}{r} \right)' \right\} \\
+ \frac{4\delta}{B_0^2(1 + 2\delta R_0)} \left( \frac{e''}{1 + 2\delta R_0} - \frac{4\delta c R'_0}{1 + 2\delta R_0} \right) - \frac{4\delta}{B_0^2(1 + 2\delta R_0)} \left( \frac{e}{1 + 2\delta R_0} \right)' \left( \frac{A'_0}{A_0} - \frac{B'_0}{B_0} + \frac{1}{r} \right) \\
+ \frac{2\alpha}{A_0^2} \left( \frac{\bar{c}}{r} + \frac{2e\delta}{1 + 2\delta R_0} \right) = 0. \] (4.16)
This shows that the general dependence of radial function affects dynamics of collapsing fluid. For the sake of simplicity, we assume

\[
a(r) = a_0 + a_1 r, \quad \tilde{c}(r) = c_0 + c_1 r, \quad e(r) = e_0 + e_1 r,
\]

where the quantities with subscript "0" and "1" are arbitrary constants. Using these values in Eq. (4.16), it follows that

\[
\begin{align*}
\kappa (\rho_0 + p_r 0 ) r & \frac{a_1}{A_0} - \kappa r (\rho_0 + p_r 0 ) (a_0 + a_1 r) \frac{A_0'}{A_0^3} - 2 \kappa (p_r 0 + 3 p_{10}) \frac{c_0}{r} \\
- 8 \kappa p_{10} c_0 & - \frac{2}{B_0^2} \left[ \frac{a_1}{A_0} - \frac{(a_0 + a_1)}{r} \left( A_0 - 1 \right) \frac{A_0'}{A_0^2} - \frac{c_0}{r^2} \left( \frac{A_0'}{A_0} + \frac{B_0'}{B_0} - \frac{2}{r} \right) \right] \\
+ \frac{4 c_0}{r^3} & + \frac{2 c_1}{r^2} + \frac{6}{r^3} (c_0 + c_1 r) B_0^2 + \frac{8}{r^2} (c_0 + c_1 r) \left( \frac{A_0'}{A_0} + \frac{B_0'}{B_0} \right) \right] + e_0 + e_1 r \\
+ \frac{2 R_0 (1 + \delta R_0)}{1 + 2 \delta R_0} & \left[ \frac{2 (c_0 + c_1 r)}{r} + \frac{\delta (e_0 + e_1 r)}{1 + 2 \delta R_0} \right] + \frac{4 \delta R_0'}{B_0^2 (1 + 2 \delta R_0)} \\
\times \left[ \frac{a_1}{A_0} - \frac{(a_0 + a_1 r)}{r} \right] & \frac{A_0'}{A_0^2} - \frac{3 c_0}{r^2} - \frac{8 \delta^2 R_0 (e_0 + e_1 r)}{B_0^2 (1 + 2 \delta R_0)^2} - \frac{4 \delta}{B_0^2} \left( e_0 + e_1 r \right) \\
\times \left( \frac{A_0'}{A_0} - \frac{B_0'}{B_0} + \frac{1}{r} \right) & \frac{2 \alpha}{A_0^2} \left[ \frac{c_0 + c_1 r}{r} + \frac{2 (e_0 + e_1 r) \delta}{1 + 2 \delta R_0} \right] = 0.
\end{align*}
\]

Inserting Eqs. (4.13), (4.14) and (4.15) up to pN order (with \( c = G = 1 \)) in the above equation, we obtain

\[
\begin{align*}
\kappa (\rho_0 + p_r 0) a_1 r & + \kappa m_0 (\rho_0 + p_r 0) + \kappa r (a_0 + a_1 r) \left( 1 + \frac{m_0}{r} \right) \\
\times \left[ p_r 0 + \frac{2}{r} (p_r 0 - p_{10}) \right] & - 2 \kappa (p_r 0 + 3 p_{10}) \frac{c_0}{r} - 8 \kappa p_{10} c_1 - 2 \left( 1 - \frac{m_0}{r} \right) \\
\times \left[ a_1 + \frac{a_1 m_0}{r^2} - \frac{1}{\rho_0} \left( \frac{p_r 0 + \frac{2}{r} (p_r 0 - p_{10})}{1 - \frac{p_r 0}{\rho_0}} \right) \right] & \left( 1 - \frac{p_r 0}{\rho_0} \right) \left( 1 + \frac{m_0}{r} \right) \left( \frac{a_0}{r} \right) \\
+ \frac{10 c_0}{r^3} & + \frac{8 c_1}{r^2} + \frac{12 m_0}{r^4} (c_0 + c_1 r) + \frac{c_0}{r^2 \rho_0} \left\{ p_r 0 + \frac{2}{r} (p_r 0 - p_{10}) \right\} \\
+ \left( 1 - \frac{p_r 0}{\rho_0} \right) & + \frac{2 m_0 \rho_0}{r^4} \left( 1 + \frac{2 m_0}{r} \right) - \frac{2 c_0}{r^3} - \left( \frac{a_0}{r} + a_1 \right) \frac{1}{\rho_0} \left\{ p_r 0 + \frac{2}{r} (p_r 0 \right\} \\
- p_{10}) & \left( 1 - \frac{p_r 0}{\rho_0} \right) - \frac{8 (c_0 + c_1 r)}{r^2 \rho_0} \left\{ p_r 0 + \frac{2}{r} (p_r 0 - p_{10}) \right\} \left( 1 - \frac{p_r 0}{\rho_0} \right)
\end{align*}
\]
In view of Table 1 and neglecting terms with $p_{r0}/\rho_0$ being of ppN order, Eq. (4.19) reduces to the following equation in the Newtonian regime

$$\kappa(r + m_0)\rho_0 + \kappa p_{r0}(3a_1r + 2a_0 - \frac{2c_0}{r}) + 2\kappa p_{\perp 0}(a_0 + a_1r + \frac{3c_0}{r} - 8c_1)
+ \kappa r(a_0 + a_1r)|p'_{r0}| + e_0 + e_1r - 2a_1 - \frac{20c_0}{r^3} - \frac{16c_1}{r^2} + \frac{2}{r}
+ \frac{m_0}{r^2} \left[ a_1r + \frac{2c_0}{r}(4r + 5) + \frac{8c_1}{r}(r + 1) \right]
+ \frac{2R_0(1 + \delta R_0)}{1 + 2\delta R_0} \left[ \frac{2(c_0 + c_1r)}{r} + \frac{\delta(e_0 + e_1r)}{1 + 2\delta R_0} \right]
+ \frac{4\delta R_0'}{1 + 2\delta R_0} \left[ a_1 - \frac{3c_0}{r^2} + \frac{m_0}{r^2}(2 - a_1r) \right] + \frac{4\delta}{r} \left( \frac{e_0 + e_1r}{1 + 2\delta R_0} \right)'
+ 2\alpha \left( 1 + \frac{2m_0}{r} \right) \left[ c_0 + c_1r \right] \left[ \frac{2(e_0 + e_1r)}{1 + 2\delta R_0} \right] = 0.$$ (4.20)

In general, the instability range depends upon the index $\Gamma$ as it measures the compressibility of the fluid. However, the above equation is independent of $\Gamma$ which shows that instability region totally depends upon the pressure anisotropy, energy density, chosen $f(R)$ model and arbitrary constants. Notice that independence of $\Gamma$ factor indicates that under expansionfree condition, fluid evolves without being compressed. In this way, the given $f(R)$ model shows the consistency of the physical results with expansionfree condition.
In order to satisfy the instability conditions of expansionfree fluids, we need to keep all the terms positive in Eq. (4.20). Here we assume that all the arbitrary constants and dynamical quantities are positive whereas \( p'_{r0} < 0 \) shows that pressure decreases during collapsing process. To keep all the terms in dynamical equations positive in the Newtonian regime, we need to satisfy the following constraints arising from different terms

\[
\frac{3c_0}{r^2} < a_1 < \frac{2}{r}, \quad (4.21)
\]

\[
a_0 + a_1 r + \frac{3c_0}{r} > 8c_1, \quad (4.22)
\]

\[
2a_0 + 3a_1 r > \frac{2c_0}{r}, \quad (4.23)
\]

\[
0 < a_1 r + \frac{10c_0}{r^2} + \frac{8c_1}{r} < 1. \quad (4.24)
\]

Thus the system would be unstable in N-approximation as long as the above inequalities are satisfied. The dynamical equation \((4.19)\) for pN regime becomes

\[
3\kappa (\rho_0 + p_{r0})a_1 r + \kappa m_0 (\rho_0 + p_{r0}) + \kappa r (a_0 + a_1 r) \left(1 + \frac{m_0}{r}\right) \times \left[p_{r0}' + \frac{2}{r}(p_{r0} - p_{r0})\right] - 2\kappa (p_{r0} - 3p_{r0}) \frac{c_0}{r} - 8\kappa p_{r0} c_1 - 2 \left(1 - \frac{m_0}{r}\right)
\]

\[
\times \left[a_1 + \frac{a_1 m_0}{r} + \frac{10c_0}{r^3} + \frac{8c_1}{r^2} + \frac{c_0}{r^2} \left\{ \frac{m_0}{r^2} \left(1 + \frac{2m_0}{r}\right) - \frac{2}{r}\right\} + \frac{8m_0 (c_0 + c_1 r)}{r^4}\right]
\]

\[
\times \left[1 + \frac{2m_0}{r}\right] + e_0 + e_1 r + \frac{2R_0 (1 + \delta R_0)}{1 + 2\delta R_0} \left[\frac{2(c_0 + c_1 r)}{r} + \frac{\delta (c_0 + c_1 r)}{1 + 2\delta R_0}\right]
\]

\[
+ \frac{4\delta R_0'}{1 + 2\delta R_0} \left(1 - \frac{2m_0}{r}\right) \left[a_1 + \frac{a_1 m_0}{r} - \frac{3c_0}{r^2}\right] + \frac{8\delta^2 R_0'' (e_0 + e_1 r)}{(1 + 2\delta R_0)^2}
\]

\[
\times \left(1 - \frac{2m_0}{r}\right) + 4\delta \left(\frac{e_0 + e_1 r}{1 + 2\delta R_0}\right) \left(1 - \frac{2m_0}{r}\right) \left(\frac{m_0}{r^2} + \frac{1}{r}\right)
\]

\[
+ 2\alpha \left(1 + \frac{2m_0}{r}\right) \left[\frac{e_0 + c_1 r}{r} + \frac{2(e_0 + e_1 r) \delta}{1 + 2\delta R_0}\right] = 0. \quad (4.25)
\]

It is remarked that only relativistic effects are taken into account at pN regime, however, the dependence of instability condition remains the same. Also, the index \( \Gamma \) does not involve, so instability conditions depend on the same parameters and constants as that in the N-approximation.
5 Summary

This paper is devoted to investigate the dynamical instability conditions of the expansion-free fluid evolution for a particular model \( f(R) = R + \delta R^2 \) in \( f(R) \) gravity. To make consistency with the physical application of the vanishing \( \Theta \), we have considered the locally anisotropic fluid with inhomogeneous energy density. Perturbation analysis is used in \( N \) and \( pN \) approximations and terms differentiated in both the regimes depending upon the order of \( c \).

Expansion-free models may help to study the formation of voids. Voids are the spongelike structures and occupying 40%-50% volume of the universe. Observations suggest very different sizes of the voids, i.e., mini-voids (Tikhonov et al. 2006) to super-voids (Rudnick et al. 2007). As concerned to the shapes of the voids, they are neither empty nor spherical. However, for the sake of investigations, they are usually described as vacuum spherical cavities surrounding by the fluid. The assumption of spherically symmetric spacetime outside the cavity is justified for cavities with sizes of the order of 20 Mpc or smaller, as the observed universe would be inhomogeneous on scale less than 150-300 Mpc.

The adiabatic index \( \Gamma \) measures the variation of pressure, its value defines the range of instability. For example, for a Newtonian perfect fluid, the system is unstable for \( \Gamma < 4/3 \). We have found that like GR (Herrera et al. arXiv/1010.1518), our results are also independent of the \( \Gamma \) factor. This shows the consistency of expansion-free condition with \( f(R) \) gravity as this requires that fluid should evolve without compressibility. Moreover, the instability range depends upon the radial anisotropy of pressure, energy density and some constraints on the constants arising from the positivity of the dynamical equation. Equation (4.20) holds the instability requirement as long as the inequalities (4.21)-(4.24) are satisfied.

It is worthwhile to mention here that the chosen \( f(R) \) model is the only model admitting Schwarzschild solution. We have used the perturbed Ricci scalar to get the time dependent part of the perturbed metric coefficients. In this way, dynamics of the gravitational collapse comprises the scalar curvature in its evolution.

Acknowledgment

We would like to thank the Higher Education Commission, Islamabad,
Pakistan for its financial support through the *Indigenous Ph.D. 5000 Fellowship Program Batch-III*.

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