On special values at integers of $L$-functions of Jacobi theta products of weight 3

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Abstract

In this paper, we consider $L$-functions of modular forms of weight 3, which are products of the Jacobi theta series, and express their special values at $s = 3, 4$ in terms of special values of Kampé de Fériet hypergeometric functions. Moreover, via $L$-values, we give some relations between special values of Kampé de Fériet hypergeometric functions and generalized hypergeometric functions.

1 Introduction and Main Results

Let $f$ be a modular form of weight $k$ with $q$-expansion $f(q) = \sum_{n=0}^{\infty} a_n q^n$ ($q = e^{2\pi i \tau}$, $\text{Im}(\tau) > 0$). Then its $L$-function $L(f, s) = \sum_{n=1}^{\infty} a_n/n^s$ converges absolutely on $\text{Re}(s) > k + 1$ ($\text{Re}(s) > k/2 + 1$ if $f$ is a cusp form). When the Fricke involution image $f^\#$ of $f$ is also a modular form, then $L(f, s)$ is meromorphically continued to the whole complex plane with a possible simple pole at $s = k$, and is entire when $f^\#(0) = 0$ (see [15, Theorem 3.2]). In this paper, we consider the case when $f(q)$ is a product of the Jacobi theta series

$$
\theta_2(q) := \sum_{n \in \mathbb{Z}} q^{(n+\frac{1}{2})^2}, \ \theta_3(q) := \sum_{n \in \mathbb{Z}} q^n, \ \theta_4(q) := \sum_{n \in \mathbb{Z}} (-1)^n q^{n^2},
$$

which are modular forms of weight $1/2$, and satisfies the condition $f^\#(0) = 0$.

In [13], by an analytic method, Rogers and Zudilin expressed $L(f, 2)$ for some theta products $f(q)$ of weight 2 in terms of special values of generalized hypergeometric functions

$$
p+1 F_p \left[ \begin{array}{c} a_1, a_2, \ldots, a_{p+1} \\ b_1, b_2, \ldots, b_p \end{array} | z \right] := \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_{p+1})_n}{(b_1)_n \cdots (b_p)_n} \frac{z^n}{(1)_n},
$$

where $(a)_n := \Gamma(a + n)/\Gamma(a)$ denotes the Pochhammer symbol. Other known results of hypergeometric expressions of $L$-values are the following.

1. Otsubo [10] expressed $L(f, 2)$ for some theta products $f(q)$ of weight 2 in terms of $3F_2(1)$ via regulators.

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2. Rogers [11], Rogers-Zudilin [13], Zudilin [19] and the author [8] expressed \( L(f, 2) \) for some theta products \( f(q) \) of weight 2 in terms of \( {}_3F_2(1) \) by using the Rogers-Zudilin method. Furthermore, Zudilin [19] expressed \( L(f, 3) \) for the theta product which corresponds to the elliptic curve of conductor 32 in terms of \( {}_4F_3(1) \).

3. Rogers-Wan-Zucker [12] expressed \( L(f, 2) \) (resp. \( L(f, 3), L(f, 4) \)) for some quotients \( f(q) \) of the Dedekind eta function \( \eta(q) := q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) \) of weight 3 (resp. 4, 5) in terms of special values of generalized hypergeometric functions or the gamma function by an analytic method. The author [9] expressed \( L(f, 1) \) (hence the values at 2 by the functional equations) for some theta products \( f(q) \) of weight 3 in terms of \( {}_3F_2(1) \) by using the Rogers-Zudilin method.

4. Samart [14] expressed \( L(f, 3) \) for some eta quotients \( f(q) \) of weight 3 in terms of \( {}_3F_4(1) \) via Mahler measures.

In this paper, we consider the following normalized Jacobi theta products of weight 3

\[
f(q) = \frac{1}{16} \theta_2^4(q) \theta_4^2(q), \quad g(q) = \frac{1}{16} \theta_2^4(q) \theta_4^2(q^2).
\]

We remark that \( f(q) \) is an Eisenstein series twisted by some Dirichlet characters and \( g(q) \) is the cusp form corresponding to the Kummer K3 surface defined by \( z^2 = x(x^2 - 1)y(y^2 - 1) \) (cf. [18, Theorem 7.4]).

The aim of this paper is to express \( L(f, n) \) and \( L(g, n) \) for \( n = 3, 4 \) in terms of special values of the Kampé de Fériet hypergeometric function [1, 17]

\[
F^{A;B;B'}_{C;D;D'} \left( a_1, \ldots, a_A, b_1, \ldots, b_B, b'_1, \ldots, b'_{B'}; c_1, \ldots, c_C, d_1, \ldots, d_D, d'_1, \ldots, d'_{D'} \middle| x, y \right) := \sum_{m,n=0}^{\infty} \prod_{i=1}^{A} (a_i)_{m+n} \prod_{i=1}^{B} (b_i)_m \prod_{i=1}^{B'} (b'_i)_m \prod_{i=1}^{C} (c_i)_{m+n} \prod_{i=1}^{D} (d_i)_m \prod_{i=1}^{D'} (d'_i)_n (1)_m (1)_n,
\]

which is a two-variable generalization of generalized hypergeometric functions.

The main results are the following.

**Theorem 1.** 1.

\[
L(f, 3) = \frac{\pi^2}{96} F^{1;2;2}_{1;1;1} \left( \frac{2}{3}, \frac{1}{2}, \frac{1}{2} \middle| 1, 1 \right).
\]

2.

\[
L(g, 3) = \frac{\pi^3}{128} F^{1;2;2}_{1;1;1} \left( \frac{2}{3}, \frac{1}{2}, \frac{1}{2} \middle| 1, 1 \right).
\]

**Theorem 2.** 1.

\[
L(f, 4) = \frac{\pi^3}{288} \left( 3F^{1;3;2}_{1;2;1} \left( \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \middle| 1, 1 \right) + F^{1;3;2}_{1;2;1} \left( \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \middle| 1, 1 \right) \right).
\]

2.

\[
L(g, 4) = \frac{\pi^4}{768} \left( 2F^{1;3;2}_{1;2;1} \left( \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \middle| 1, 1 \right) + F^{1;3;2}_{1;2;1} \left( \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \middle| 1, 1 \right) \right).
\]
We remark that the double series $F_{\alpha;B;C}^{A:B+1:C+1}(x,y)$ converges absolutely on $|x| \leq 1$ and $|y| \leq 1$ when the parameters satisfy the following conditions [7, Theorem 1]

\[
\begin{align*}
\text{Re} & \left( \sum_{i=1}^{A} c_i + \sum_{i=1}^{B} d_i - \sum_{i=1}^{A} a_i - \sum_{i=1}^{B+1} b_i \right) > 0, \\
\text{Re} & \left( \sum_{i=1}^{A} c_i + \sum_{i=1}^{C} d'_i - \sum_{i=1}^{A} a_i - \sum_{i=1}^{B+1} b'_i \right) > 0, \\
\text{Re} & \left( \sum_{i=1}^{A} c_i + \sum_{i=1}^{B} d_i + \sum_{i=1}^{C} d'_i - \sum_{i=1}^{A} a_i - \sum_{i=1}^{B} b_i - \sum_{i=1}^{C+1} b'_i \right) > 0.
\end{align*}
\]

We prove Theorems 1 and 2 by using the Rogers-Zudilin method. The strategy of the method is as follows. For $h \in \mathbb{Z}_{\geq 1}$, the value $L(h, n)$ is obtained by the Mellin transformation of $h(q)$

\[
L(h, n) = \frac{(-1)^{n-1}}{\Gamma(n)} \int_0^1 (h(q) - a_0)(\log q)^{n-1} \frac{dq}{q}.
\]

(1)

The key to express the value $L(h, n)$ in terms of special values of hypergeometric functions is the following transformation formulas

\[
\theta_2^2(q) = 2F_1 \left[ \frac{1}{2}, \frac{1}{2} \right| \alpha \right], \quad \theta_3^2(q) = \frac{dq}{\alpha(1-\alpha)}
\]

(2)

where $\alpha := \theta_2^2(q)/\theta_3^2(q)$. The former is [3, p.101, Entry 6], and the latter follows from the former and [2, p.87, Entry 30]. By these formulas, we reduce the integral (1) to an integral of the form

\[
\int_0^1 P(\alpha) \sum_{p=1}^{n} F_p \left[ \frac{a_1, a_2, \ldots, a_{p+1}}{b_1, \ldots, b_p} \right] \frac{dq}{\alpha(1-\alpha)}.
\]

where $P(\alpha)$ is a polynomial in $\alpha^k (1 - \alpha)^l$ for various $k$ and $l$. Then, by simple computations, we obtain hypergeometric expressions of $L$-values.

Finally, we remark that we have simpler hypergeometric expressions of the $L$-values $L(f, 3)$, $L(f, 4)$ and $L(g, 3)$. Since $f(q)$ is the Eisenstein series twisted by the Dirichlet characters $\chi_{-4}(n) := \text{Im}(i^n)$ and $\psi(n) := (-1)^{n-1}$ [5, p. 281, Lemma 3.32 (3.85)]

\[
f(q) = \frac{1}{16} \theta_2^2(q)\theta_3^2(q) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^2 q^n}{1 + q^{2n}} = \sum_{n,k=1}^{\infty} \psi(n)n^2\chi_{-4}(k)q^{nk},
\]

we have

\[
L(f, s) = \sum_{n,k=1}^{\infty} \psi(n)n^2\chi_{-4}(k) = L(\psi, s - 2)L(\chi_{-4}, s),
\]

where $L(\chi, s)$ is the Dirichlet $L$-function associated to a Dirichlet character $\chi$. We know $L(\psi, 1) = \log 2$ and $L(\chi_{-4}, 3) = \pi^3/32$, hence we obtain

\[
L(f, 3) = \frac{\pi^3 \log 2}{32}.
\]
Moreover, the value $L(f, 4)$ can be expressed in terms of $_5F_4$, since we have
$L(\psi, 2) = \pi^2/12$ and

$$L(\chi_{-4}, 4) = _5F_4 \left[ \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 1 \right] - 1 = _5F_4 \left[ \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 1 \right] - \frac{1}{81} _5F_4 \left[ \frac{4}{3}, \frac{4}{3}, \frac{4}{3}, 1 \right],$$

which easily follows from

$$2n + 1 = \frac{3}{2} n, \quad 4n + 1 = \frac{5}{2} n, \quad 4n + 3 = \frac{7}{2} n.$$

Similarly, we have a simpler expression of $L(g, 3)$ [14, Corollary 1.3]

$$L(g, 3) = \frac{\pi^3}{1024} \left( 48 \log 2 - _5F_4 \left[ \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, 1, 1 \right] \right).$$

Therefore we obtain via $L$-values the following reduction formulas for Kampé de Fériet hypergeometric functions.

**Corollary 3.** 1.

$$F_{1;1;1}^{1;2;2} \left[ \frac{2}{5}; \frac{1}{2}; \frac{1}{2}; 1 \right] = 3\pi \log 2.$$

2.

$$8F_{1;1;1}^{1;2;2} \left[ \frac{3}{2}; \frac{3}{2}; \frac{3}{2}; \frac{1}{2}; 1 \right] = 48 \log 2 - _5F_4 \left[ \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, 1, 1 \right].$$

3.

$$\frac{\pi}{24} \left( 3F_{1;2;1}^{1;3;2} \left[ \frac{1}{2}; \frac{1}{2}; \frac{1}{2}; \frac{1}{2}; 1 \right] + F_{1;2;1}^{1;3;2} \left[ \frac{1}{2}; \frac{3}{4}; \frac{3}{4}; \frac{3}{4}; 1 \right] \right)$$

$$= _5F_4 \left[ \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1 \right] - 1.$$

The author does not know how to derive these formulas directly. It is new that the value $L(g, 4)$ is expressed in terms of special values of hypergeometric functions. Similarly to the results above, one might be able to express the value $L(g, 4)$ in terms of special values of generalized hypergeometric functions.

### 2 Proof of Theorem 1

We first show the following integral expressions of the $L$-values $L(f, 3)$ and $L(g, 3)$.

**Proposition 4.** 1.

$$L(f, 3) = \frac{\pi^2}{8} \int_0^1 \theta_2^4(q) \theta_4^2(q) \sum_{r,k=1}^{\infty} \frac{q^{(2r-1)(2k-1)}}{2r-1} dq.$$ 

2.

$$L(g, 3) = \frac{\pi^2}{16} \int_0^1 \theta_2^4(q) \theta_4^2(q) \sum_{r,k=1}^{\infty} \frac{q^{2(r-1/2)(k-1/2)}}{2r-1} dq.$$
Proof. We prove the formula for \( L(f, 3) \) only. Similar computation leads to the remaining formula.

By (1), we have

\[
L(f, 3) = \frac{1}{2} \int_0^1 \frac{1}{16} \quad \frac{\theta_2^4(q)\theta_4^2(q)(\log q)^2 dq}{q}
\]

By changing the variable \( q = e^{-\pi u} \), we have

\[
L(f, 3) = \frac{\pi^3}{32} \int_0^\infty \theta_2^4(e^{-\pi u})\theta_4^2(e^{-\pi u})u^2 du.
\]

If we use the involution formula for \( \theta_4(q) \) [4, p.40, (2.3.3)], the Lambert series expansions of \( \theta_2^2(q) \) and \( \theta_4^2(q) \) [5, p.177, Theorem 3.10 (3.15) and p.196, Theorem 3.26 (3.69)]

\[
\sqrt{u}\theta_4(e^{-\pi u}) = \theta_2(e^{-\frac{\pi}{2}u}), \quad (3)
\]

\[
\theta_2^2(q) = 4 \sum_{n,k=1}^\infty \chi_{-4}(n)q^{n(k-1/2)}, \quad \chi_{-4}(n) := \text{Im}(i^n), \quad (4)
\]

\[
\theta_4^2(q) = 16 \sum_{r,s=1}^\infty (2r-1)q^{(2r-1)(2s-1)}, \quad (5)
\]

and the substitution \( u \mapsto nu/(2r-1) \), then we obtain

\[
L(f, 3) = 2\pi^3 \int_0^\infty \left( \sum_{n,s=1}^\infty \chi_{-4}(n)n^2e^{-\pi un(2s-1)} \right) \left( \sum_{r,k=1}^\infty e^{-\frac{\pi(2r-1)(k-1/2)}{2r-1}} \right) u du.
\]

We know that the first series in the integral above is the theta product [5, Lemma 3.32 (3.84)]

\[
\sum_{n,s=1}^\infty \chi_{-4}(n)n^2q^{n(2s-1)} = \frac{1}{4} \theta_2^2(q^2)\theta_4^2(q^2).
\]

By this identity and (3), we have

\[
L(f, 3) = \frac{\pi^3}{2} \int_0^\infty \theta_2^4(e^{-2\pi u})\theta_4^2(e^{-2\pi u}) \left( \sum_{r,k=1}^\infty e^{-\frac{\pi(2r-1)(k-1/2)}{2r-1}} \right) u du
\]

\[
= \frac{\pi^3}{16} \int_0^\infty \theta_2^4(e^{-\pi u})\theta_4^2(e^{-\pi u}) \left( \sum_{r,k=1}^\infty e^{-\frac{\pi(2r-1)(k-1/2)}{2r-1}} \right) du / u^2.
\]

If we use the substitutions \( u \mapsto 1/u \), \( q = e^{-\pi u} \) and \( q \mapsto q^2 \), then we obtain the formula.

\[
\square
\]

The series in the integrals in Proposition 4 are hypergeometric functions.

Lemma 5. 1.

\[
\sum_{r,k=1}^\infty \frac{q^{(2r-1)(2k-1)}}{2r-1} = \frac{\alpha}{16} \quad _2F_1 \left[ \begin{array}{c} 1, 1 \\ \frac{1}{2} | \alpha \end{array} \right].
\]
\[ \sum_{r,k=1}^{\infty} \frac{q^{2(r-1/2)(k-1/2)}}{2r-1} = \frac{\alpha^{1/2}}{4} \binom{\frac{1}{2} \cdot \frac{1}{2}}{1 \cdot \frac{1}{2}}. \]

**Proof.** We prove these hypergeometric expressions by using the transformation formulas (2).

By (5), we have
\[ \sum_{r,k=1}^{\infty} \frac{q^{2r-1)(2k-1)}}{2r-1} = \int_0^q \sum_{r,k=1}^{\infty} (2k-1)q^{2(r-1)(2k-1)} \frac{dq}{q} = \frac{1}{16} \int_0^q \theta_2^4(q) \frac{dq}{q}. \]

If we use (2), then the integral above is equal to
\[ \frac{1}{16 \int_0^\alpha} \frac{d\alpha}{\alpha(1 - \alpha)} = -\frac{1}{16} \log(1 - \alpha). \]

Since we know
\[ -\log(1 - \alpha) = a_2 F_1 \left[ \frac{1}{2}, 1 \right] \alpha, \]
we obtain the first formula.

Similarly, we have
\[ \sum_{r,k=1}^{\infty} \frac{q^{2(r-1/2)(k-1/2)}}{2r-1} = \frac{1}{32} \int_0^q \theta_2^4(q^2) \frac{dq}{q} = \frac{1}{8} \int_0^q \theta_2^3(q) \theta_3^2(q) \frac{dq}{q}. \]

Here we used \(2\theta_2(q^2) \theta_3(q^2) = \theta_2^3(q)\) \([3, \text{ p. } 40, \text{ Entry } 25 (iv)]\) for the last equality. Then, by (2), we obtain
\[ \int_0^q \theta_2^3(q) \theta_3^2(q) \frac{dq}{q} = \int_0^\alpha \frac{d\alpha}{\alpha(1 - \alpha)}. \]

By the integral representation of hypergeometric functions [16, (1.6.6)]
\[ a_2 F_1 \left[ \frac{a, b}{c} \right] = \frac{\Gamma(c)}{\Gamma(c - b) \Gamma(b)} \int_0^1 t^b (1 - t)^{c-b} (1 - zt)^{-a} \frac{dt}{t(1-t)}, \]
we have
\[ \int_0^\alpha \frac{d\alpha}{\alpha(1 - \alpha)} = \frac{1}{2} \int_0^1 \theta_2^4(1-t)(1-\alpha t)^{-1} \frac{dt}{t(1-t)} = \frac{1}{2} \frac{\Gamma \left( \frac{1}{2} \right) \Gamma \left( \frac{1}{2} \right)}{\Gamma \left( \frac{3}{2} \right)} a_2 F_1 \left[ \frac{1}{2}, 1 \right] \alpha, \]

hence we obtain the second formula.

**Proof of Theorem 1.** By Lemma 5 and the transformation formulas (2), we have
\[ L(f, 3) = \frac{\pi^2}{128} \int_0^1 \alpha^2 (1 - \alpha)^{1/2} \binom{1/2}{1/2} \frac{d\alpha}{\alpha(1 - \alpha)}, \]
\[ L(g, 3) = \frac{\pi^2}{64} \int_0^1 \alpha^{3/2} (1 - \alpha)^{1/2} \binom{1/2}{1/2} \frac{d\alpha}{\alpha(1 - \alpha)}. \]
If we use the series expansions of hypergeometric functions and integrate term-by-term, we obtain

\[ L(f, 3) = \frac{\pi^2}{128} \int_0^1 \frac{\alpha^2 (1 - \alpha)^{1/2} \, \, _2F_1 \left[ \frac{1}{2}, 1, 2 \right] \, \, _2F_1 \left[ \frac{1}{2}, 1, 2 \right]}{\alpha (1 - \alpha)} \, \, d\alpha \]

\[ = \frac{\pi^2}{128} \sum_{m, n = 0}^{\infty} \frac{(1)^2_m}{(2)_m (1)^2_n} \int_0^1 \alpha^{2+m+n} (1 - \alpha)^{1/2} \, \, _2F_1 \left[ \frac{1}{2}, 1, 2 \right] \, \, d\alpha \]

\[ = \frac{\pi^2}{128} \sum_{m, n = 0}^{\infty} \frac{(1)^2_m}{(2)_m (1)^2_n} \Gamma (2 + m + n) \Gamma \left( \frac{1}{2} \right) \]

\[ = \frac{\pi^2}{96} \, _1F_1;2;2 \left[ \frac{2}{5}, 2, 1; 1, 1 \right]. \]

By similar computations, we have the hypergeometric expression of the value \( L(g, 3) \).

\[ \square \]

3 Proof of Theorem 2

Similarly to the computations in the proof of Proposition 4, we obtain the following integral expressions of the \( L \)-values \( L(f, 4) \) and \( L(g, 4) \).

Proposition 6. 1.

\[ L(f, 4) = \frac{\pi^4}{48} \int_0^1 (2\theta_4^2 (q^2) - \theta_4^2 (q)) \left( \sum_{n, r = 1}^{\infty} \frac{\chi_1(n)}{(2r - 1)^2} q^{n(r - 1/2)} \right) \, \, dq. \]

2.

\[ L(g, 4) = \frac{\pi^4}{48} \int_0^1 (2\theta_4^2 (q^4) - \theta_4^2 (q^2)) \left( \sum_{n, r = 1}^{\infty} \frac{\chi_1(n)}{(2r - 1)^2} q^{n(r - 1/2)} \right) \, \, dq. \]

Proof. We prove the formula for \( L(f, 4) \) only. The formula for \( L(g, 4) \) is obtained by similar computations.

By (1), we have

\[ L(f, 4) = \frac{1}{6} \int_0^1 \frac{1}{16} \theta_4^2 (q) \theta_4^2 (q) (\log q)^3 \, \, dq = \frac{\pi^4}{96} \int_0^1 \theta_4^2 (e^{-\pi u}) \theta_4^2 (e^{-\pi u}) u^3 \, \, du. \]

By (3), (4), (5) and the variable transformation \( u \mapsto (k - 1/2)u/(2r - 1) \), we have

\[ L(f, 4) = \frac{\pi^4}{12} \sum_{s, k = 1}^{\infty} (2k - 1)^3 e^{-\pi (2s - 1)(2k - 1)} \left( \sum_{n, r = 1}^{\infty} \frac{\chi_1(n)}{(2r - 1)^2} e^{-\pi n(2r - 1)} \right) u^2 \, \, du. \]

The first series in the integral is a theta product.

Lemma 7.

\[ \sum_{s, k = 1}^{\infty} (2k - 1)^3 q^{(2s - 1)(2k - 1)} = \frac{\theta_4^2 (q^{1/2}) - 8 \theta_4^2 (q)}{256}. \]
Proof. We have
\[
\sum_{s,k=1}^{\infty} (2k-1)^3 q^{(2s-1)(2k-1)} = \sum_{s,k=1}^{\infty} (k^3 q^{sk} - 9k^3 q^{2sk} + 8k^3 q^{sk})
\]
\[= \frac{M(q) - 9M(q^2) + 8M(q^4)}{240},\]
where
\[M(q) := 1 + 240 \sum_{s,k=1}^{\infty} k^3 q^{sk}.\]
We know that \(\theta_2(q)\) has the connection with \(M(q)\) [5, p.207, Theorem 3.39 (3.101)]
\[\theta^8_2(q^{1/2}) = \frac{16}{15} (M(q) - M(q^2)).\]
By this identity, we obtain the lemma.

If we use the lemma above and (3), we have
\[
L(f,4) = \frac{\pi^4}{3072} \int_0^{\infty} (\theta^8_2(e^{-u/4}) - 8\theta^8_2(e^{-u/2})) \left( \sum_{n,r=1}^{\infty} \frac{\chi_{n-4}(n)}{(2r-1)^2} e^{-\pi n(2r-1)/u} \right) u^2 du
\]
\[= \frac{\pi^4}{24} \int_0^{\infty} (\theta^8_4(e^{-4\pi/u}) - \theta^8_4(e^{-2\pi/u})) \left( \sum_{n,r=1}^{\infty} \frac{\chi_{n-4}(n)}{(2r-1)^2} e^{-\pi n(2r-1)/u} \right) u^2 du.
\]
By changing the variables \(u \mapsto 1/u\) and \(q = e^{-2\pi u}\), we obtain the proposition.

We remark that the series in the integrals in Proposition 6 can be expressed in terms of generalized hypergeometric functions [6, (2.2)]
\[
\sum_{n,r=1}^{\infty} \frac{\chi_{n-4}(n)}{(2r-1)^2} q^{n(2r-1)/2} = \frac{\alpha^{1/2}}{4} \left[ F_2 \left[ \begin{array}{c} 1 \frac{3}{2}, \frac{3}{2} \\ 1 \right] \right] \alpha = \frac{1}{2} F_1 \left[ \begin{array}{c} 1 \frac{3}{2} \frac{3}{2} \\ 1 \right] \alpha
\]
(6)

Proof of Theorem 2. By (2) and (6), we obtain
\[
L(f,4) = \frac{\pi^3}{192} \int_0^{1} ((\alpha^{1/2} + \alpha^{3/2})(1-\alpha) F_2 \left[ \begin{array}{c} 1 \frac{3}{2}, \frac{3}{2} \\ 1 \frac{3}{2} \frac{3}{2} \\ 1 \right] \right] \frac{d\alpha}{\alpha(1-\alpha)},
\]
\[
L(g,4) = \frac{\pi^3}{384} \int_0^{1} ((1-\alpha)^{1/2} + (1-\alpha)^{3/2}) F_2 \left[ \begin{array}{c} 1 \frac{3}{2}, \frac{3}{2} \\ 1 \frac{3}{2} \frac{3}{2} \\ 1 \right] \alpha
\]
Here we used
\[2\theta^8_4(q^2) - \theta^8_4(q) = (1+\alpha)(1-\alpha)\theta^8_4(q),\]
\[2\theta^8_4(q^4) - \theta^8_4(q^2) = \frac{1}{2} ((1-\alpha)^{1/2} + (1-\alpha)^{3/2})\theta^8_4(q),\]
which follow from the formulas [4, p. 34, (2.1.7i), (2.1.7ii)]
\[2\theta^8_4(q^2) = \theta^8_4(q) + \theta^8_4(q), \quad \theta_3(q)\theta_4(q) = \theta^2_4(q^2).
\]
Then the hypergeometric expressions can be proved by interchanging of the order of summation and integration.

8
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