On the Robustness of Noise-Blind Low-Rank Recovery from Rank-One Measurements

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Abstract

We prove new results about the robustness of well-known convex noise-blind optimization formulations for the reconstruction of low-rank matrices from underdetermined linear measurements. Our results are applicable for symmetric rank-one measurements as used in a formulation of the phase retrieval problem.

We obtain these results by establishing that with high probability rank-one measurement operators defined by i.i.d. Gaussian vectors exhibit the so-called Schatten-1 quotient property, which corresponds to a lower bound for the inradius of their image of the nuclear norm (Schatten-1) unit ball.

We complement our analysis by numerical experiments comparing the solutions of noise-blind and noise-aware formulations. These experiments confirm that noise-blind optimization methods exhibit comparable robustness to noise-aware formulations.

Keywords: low-rank matrix recovery, phase retrieval, quotient property, noise-blind, robustness, nuclear norm minimization

1 Introduction

1.1 Motivation and Literature Overview

Initiated by the seminal works introducing the idea of compressive sensing [13, 14, 25], the problem of recovering structured data from an underdetermined system random linear measurements has been subject of intensive study in mathematics, signal processing, and computer science in recent years.

Two classes of structural models that have proven particularly useful are variants of sparsity, where the signal is assumed to have only few significant coefficients in an appropriate basis, and low-rank models, where a matrix-valued signal is assumed to be well-approximated by a matrix of rank much less than the dimension. In both cases, the measurements are of the form

\[ \mathcal{A}X + \omega = b, \]

where \( \mathcal{A} \) is a (random) linear operator, which maps the unknown structured object \( X \) to the observed measurements \( b \) and \( \omega \) is a noise. One typically assumes that the number of measurement is much smaller than the signal dimension, which is why such measurement scenarios (especially in combination with a sparsity assumption) are commonly referred to as compressive sensing.

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It is not difficult to show that for both the sparsity and the low-rank model, the object that is extremal with regard to the underlying structure, i.e., has fewest non-zero coefficients or smallest rank, respectively, is indeed the correct solution in many cases if there is no additive noise.

That is, in case of sparse recovery with $\omega = 0$ one can often recover $X$ by solving

$$\min_{\mathcal{Z}} \|\mathcal{Z}\|_{\ell_0} \text{ such that } \mathcal{A}\mathcal{Z} = \mathcal{b}$$

with $\|\mathcal{Z}\|_{\ell_0}$ denoting the number of non-zero entries of $\mathcal{Z}$, and for noiseless low-rank matrix recovery $X$ can typically be recovered via the problem

$$\min_{\mathcal{Z}} \text{rank } \mathcal{Z} \text{ such that } \mathcal{A}\mathcal{Z} = \mathcal{b}.$$ 

Both these problems, however, are non-convex optimization problems and, in general, NP-hard [52, 27].

However, tractable reconstruction of $X$ via appropriate convex relaxations is provably possible [31] if $\mathcal{A}$ possesses the null space property [20] or the restricted isometry property [16]. If $X$ is sparse, one can reconstruct it in the noiseless case ($\omega = 0$) as a minimizer of

$$\min_{\mathcal{Z}} \|\mathcal{Z}\|_{\ell_1} \text{ such that } \mathcal{A}\mathcal{Z} = \mathcal{b}, \quad (2)$$

which can be efficiently solved numerically [26]. If $X$ is of low rank, one can find it via the second order cone problem

$$\min_{\mathcal{Z}} \|\mathcal{Z}\|_{S_1} \text{ such that } \mathcal{A}\mathcal{Z} = \mathcal{b}, \quad (3)$$

where the nuclear norm $\| \cdot \|_{S_1}$ of a matrix is defined to be the sum of its singular values.

Subsequent works in the field gave an answer to the questions that followed:

Q1 Will the reconstruction work if the assumption of sparsity or low rank is only approximately satisfied [14, 12]?

Q2 Can $X$ be reconstructed robustly in the presence of noise with known strength as constraints in (2) or (3) are no longer valid [14, 12]?

Q3 What if the strength is unknown or underestimated [65, 6]?

Q4 Can one blindly apply (2) or (3) in the presence of noise and still obtain reasonable reconstruction [65, 24, 36]?

The answer to the first two questions is “yes”, if $\mathcal{A}$ possesses the robust null space property [31], which is known to be fulfilled for many random matrices with high probability, mostly matrices with independent rows.

The last two questions, however, cannot be answered affirmatively in general if $\mathcal{A}$ fulfills only a suitable restricted isometry property [7, 12] or robust null space property [20, 39] with respect to the underlying structure. On the other hand, if in addition $\mathcal{A}$ fulfills the so-called quotient property [65, 31] (see Section 1.5 below), the last two questions can be answered with “yes” in the case of the sparse recovery problem, i.e., if $X$ is a sparse vector. For low-rank matrix recovery answers to question Q3 are only available in limited classes of scenarios and Q4 has hardly been studied in the literature. For sparse recovery, in contrast, these questions are much better understood, in parts also because the quotient property is closely related to well-studied questions in the field of high-dimensional geometry [33].
As the null space property, the quotient property is typically established via random constructions, and the choice of the random distribution of $\mathcal{A}$ is critical. In contrast to the null space property, however, these constructions often rely on the independence of the columns rather than the rows. The quotient property adapted for $\mathcal{A}$ has been studied in a number of works over the recent years, extending the seminal results for Gaussian matrices [33] to matrices with i.i.d. subgaussian entries such as Bernoulli [48], Weibull matrices [30], log-concave distributions [21], and, more recently, even more heavy-tailed distributions including Cauchy random vectors [36].

When the random matrix has independent entries (and, consequently, both independent rows to yield the quotient property and independent rows to yield the null space property), these results can be combined with results about the null space property to yield noise-blind recovery guarantees, e.g., for Gaussian [65] and subgaussian random matrices [24]. As such bounds require a logarithmic number of finite moments [51], the resulting guarantees for noise-blind sparse recovery require slightly stronger conditions on the tail decay of the random matrix entries than the quotient property by itself. Similar results about null space and quotient properties hold for random measurement operators based on independent matrices with i.i.d. subgaussian entries and can be used to characterize the performance of (3) for low-rank matrix recovery.

For most application scenarios of sparse recovery, the assumption of measurement operators with independent entries is too restrictive. In more realistic measurement scenarios, one encounters structural constraints imposed by the application. Examples for sparse recovery include Fourier structure [58], combinations of Fourier and wavelet structure [1, 43] as motivated, e.g., by magnetic resonance imaging [50] and convolutional structure as motivated, e.g., by channel identification [37, 56, 41]. Such structured scenarios are also preferred for reasons of reduced computational complexity [55], but their theoretical analysis pose considerable challenges as compared to unstructured random ones [17, 57].

Structural constraints are even more significant in the low-rank matrix recovery as the matrices that constitute the linear operator $\mathcal{A}$ of (1) are in many applications not modeled very accurately by random matrices with i.i.d. entries. For example, in recommender systems, it is common to model a user-product matrix as approximately low-rank [40], which is known only through a subset of its entries, often assumed to be revealed at random locations [11, 17]. This corresponds to a low-rank matrix recovery problem with an operator $\mathcal{A}$ whose constituent matrices do not have i.i.d. entries, but are outer products of random standard basis vectors. Other relevant structured measurement scenarios for low-rank matrix recovery include symmetric rank-one measurements (see also Section 1.2 below), as related to phase retrieval [8], often in combination with additional structure such as coded diffraction patterns [10] or non-symmetric rank-one and low-rank measurements as encountered in blind deconvolution [2] and blind demixing [38], and various structural models related to quantum state tomography [35, 28, 49].

Despite this large variety of relevant structured measurement scenarios, the study of the quotient property with structure is only in its beginnings. To our knowledge, there are two main contributions to mention here: [6] has provided such an estimate for sparse recovery with Fourier structure, and [49] observed for low-rank matrix recovery from Pauli measurements the quotient property directly follows from orthogonality.

One should mention though that solving (2) is not the only way to attempt noise-blind recovery. For measurement systems satisfying the restricted isometry property, recovery guarantees have been established for various greedy and greedy-type methods such as orthogonal matching pursuit [62] and compressive sampling matching pursuit [53], which do not require a prior knowledge of the noise level, but instead require an upper bound on
the sparsity level, which may also not be available in all cases. Furthermore, the square root lasso, a variant of (2) given by the minimization problem

$$\min_Z \|Z\|_1 + \lambda \|b - AZ\|_2$$

has been shown to yield recovery with a parameter $\lambda$ independent of the noise level under restricted eigenvalue conditions [4], and also under the robust null space property [54].

For the low-rank matrix recovery problem, the restricted isometry property has mainly been established for unstructured measurement systems (with the notable exception of Pauli measurements as relevant in quantum information theory [49]), so the guarantees for greedy and greedy-type methods have limited applicability. While an approach analogous to the square root lasso also allows for noise-blind recovery when the noise is random [32] – for example for matrix completion – no analysis based on the robust null space property analogous to [54] is available yet. For the phase retrieval problem, where the solution is known to be positive semidefinite (see below for more details), it has been shown that noise-blind recovery at near-optimal rate can be achieved by completely ignoring the nuclear norm objective of (3) and just optimizing the data fidelity over the positive semidefinite cone [23, 9, 59].

Despite the availability of alternative approaches for a number of scenarios, however, we feel that understanding the quotient property of structured random measurement systems is of interest both from the mathematical point of view – given the fundamental role of algorithms (2) and (3) as a benchmark – and from the viewpoint of applications, given that these approaches form the basis of competitive solution approaches such as Iteratively Reweighted Least Squares [22, 29, 46, 45], and hence their understanding sheds light on these methods as well.

1.2 Rank-One Measurements of Low-Rank Matrices

In this article, we study the recovery of low-rank matrices from measurements $\mathcal{A}$ described by random, symmetric rank one matrices, which constitutes a structured random measurement system that arises in many applications.

Such measurements arise, for instance, in covariance estimation [47, 3, 18], where the goal is to recover the covariance matrix $X$ of a random distribution from a quadratic sketches, and in the recovery of images from phaseless measurements [12, 15, 9, 44, 39], which is relevant for example in Fourier ptychographic microscopy [60]: If an image is represented by a vector $x \in \mathbb{C}^n$, phaseless, noisy measurements reflecting only the squared magnitude $|a_j^* x|^2$ of the scalar products $a_j^* x$ are measured such that

$$b_j = \mathcal{A}(xx^*)_j + w_j = \langle A_j, xx^* \rangle_F + w_j$$

for all $j = 1, \ldots, m$ where the measurement matrices $A_j := a_j a_j^*$ are Hermitian rank-one since for each $j$,

$$b_j - w_j = |a_j^* x|^2 = a_j^* (xx^*) a_j = \text{tr}(a_j a_j^* xx^*) = \langle a_j a_j^*, xx^* \rangle_F = \langle A_j, xx^* \rangle_F = \langle A_j, X \rangle_F$$

with $X = xx^*$ being a Hermitian rank-one matrix [8]. Beyond that, we also consider the recovery of Hermitian matrices $X$ of small rank $r > 1$.

The randomness of such measurements is described by the random distribution of the vectors $a_j$. In this paper, we study the case of independent standard complex Gaussian random vectors $\{a_j : j = 1, \ldots, m\}$.

For this measurement model, we briefly review prior results addressing the questions Q1-Q4 with respect to the tractable program based on nuclear norm minimization (3): Let
\( \mathcal{H}_n \) denote set of Hermitian \((n \times n)\) matrices. It has been shown that for this model, the solution of (3), i.e., the nuclear norm minimizer

\[
\Delta(b) := \arg \min \left\{ \|Z\|_{S_1}, \ Z \in \mathcal{H}_n \text{ such that } \mathcal{A}Z = b \right\},
\]

provides good reconstruction even if \( X \) is only approximately low-rank [44, 39], resulting in a positive answer to \( Q_1 \). Similar results [15, 18] have been obtained for the optimization problem \( \Delta \geq 0 \), a variant of \( \Delta \) with an additional positive semidefinite constraint on \( Z \). The positive semidefinite constraint significantly reduces the search space. However, one no longer minimizes over a subspace of matrices, but a convex cone.

In [44] authors introduce the recovery method

\[
\Delta_{q,\eta}(b) := \arg \min \left\{ \|Z\|_{S_1}, \ Z \in \mathcal{H}_n \text{ such that } \|\mathcal{A}Z - b\|_q \leq \eta \right\},
\]

a variant of \( \Delta \) where its equality constraint is replaced by an inequality constraint on the residual error in terms of \( \|\cdot\|_q \)-norm. If \( \eta \) is chosen compatibly with the noise level \( \|\omega\|_{\ell_q} \), i.e., if we encounter noise of known level, [44] shows that \( \Delta_{q,\eta} \) robustly reconstructs \( X \), answering \( Q_2 \) affirmatively. However, an analysis of the performance of \( \Delta \) in the presence of noise, i.e., an answer to \( Q_4 \) has not been achieved.

Analogous results for rank-one and general semidefinite matrices \( X \) have been established in [15] and [18], respectively, and have been extended to simultaneous stability in terms of both measurement noise and structure violation later developed in [39]. We review these results in detail in Section 1.5.

An alternative approach for general Hermitian low-rank matrices is to minimize an \( S_1 \)-norm Lasso-style [61] objective

\[
\Delta_{q,\mu}^{\text{Lasso}}(b) := \arg \min_{Z \in \mathcal{H}_n} \left\{ \|\mathcal{A}Z - b\|_q^2 + \mu \|Z\|_{S_1} \right\},
\]

as considered, e.g., in [12, 49]. It is not hard to show that for an appropriate parameter \( \mu \), the minimizer will exhibit similar behavior of (8).

As observed in [23, 9], the nuclear norm objective can be omitted provided that positive semidefiniteness is enforced: In absence of any measurement noise, there is typically only one positive semidefinite solution; likewise in noisy scenarios, an accurate reconstruction can be obtained by choosing the positive semidefinite matrix that best fits the data in \( \ell_q \)-norm such that

\[
\Delta_{q,\eta}^{\geq 0}(b) := \arg \min_{Z \geq 0} \|\mathcal{A}Z - b\|_q.
\]

If the matrix of interest \( X \) is positive semidefinite, arguably a situation of particular interest in many applications, this provides an affirmative answer to \( Q_3 \), as confirmed by the recovery guarantees of [9] (for the case \( q = 1 \)) and [39]. However, the positive semidefiniteness of \( X \) is crucial for \( \Delta_{q,\eta}^{\geq 0} \) and its analysis in [9, 39] does extend directly beyond that case.

In fact, we are not aware of any approaches to \( Q_4 \) nor any alternative approaches to \( Q_3 \) that would cover general Hermitian matrices. While arguably this scenario has not found many applications yet, we still believe that it is an important case for a thorough understanding of low-rank matrix recovery problems. We expect that its analysis can help to pave the way for an understanding of other structured measurement scenarios without positive semidefiniteness such as randomized blind deconvolution [2].
1.3 Our Contribution and Outlook

In this article, we provide answers for questions Q3 and Q4 for random rank-one measurements and general Hermitian matrices \(X\), thus including the case of Hermitian matrices that are not positive semidefinite.

In Theorem 5, we show that with high probability the rank-one measurement operator \(\mathcal{A} : \mathcal{H}_a \to \mathbb{R}^n\) defined by independent Gaussian vectors \(a_i\) fulfills, under appropriate conditions, the nuclear norm quotient property [12, 49]. Our proof technique for Theorem 5 is entirely different from techniques used to establish the various results for the quotient property in compressed sensing and low-rank matrix recovery mentioned in Section 1.1, and might be of independent interest. As our techniques are tailored to a low-rank recovery problem from structured measurements, we expect that they will prove useful to study other structured measurement scenarios such as randomized blind deconvolution.

In analogy to sparse recovery, our findings for rank-one measurements (see Theorem 6) entail recovery guarantees for the noise-blind nuclear norm minimizer \(\Delta\) and its inequality-constrained variant \(\Delta_{\eta,\theta}\) with incompatible choice of parameter \(\eta\). These results answer Q3 and Q4 affirmatively.

The rest of the paper is structured as following. The next section establishes basic notation and definitions used throughout the paper. In Section 1.5, we discuss other possible choices of reconstruction maps beyond \(\Delta\) and results regarding their performance provided in [39]. After presenting our results in Section 2, we perform numerical experiments in Section 3 to shed light on the actual difference in the performance of various reconstruction methods for noisy measurements. Finally, we detail the proofs of our results in Section 4.

1.4 Preliminaries and Notation

We now set up some notation that will be used throughout the paper. We denote the Frobenius (Hilbert-Schmidt) product \(\langle \cdot, \cdot \rangle_F\) of two matrices \(B, D \in \mathbb{C}^{n_1 \times n_2}\) by \(\langle B, D \rangle_F = \text{tr}(BD^*)\). For any matrix \(X \in \mathbb{C}^{n \times n}\) we use the following notation for its singular value decomposition (SVD)

\[
X = \sum_{j=1}^{n} \sigma_j(X) u_j v_j^*,
\]

where \(U\) and \(V\) are \(n \times n\) unitary matrices and \(\Sigma \in \mathbb{C}^{n_1 \times n_2}\) is diagonal with entries non-negative entries \(\sigma_j(X)\), the singular values of \(X\) ordered in decreasing order. The rank of matrix \(X\) is given by the number of positive singular values.

For \(1 \leq p \leq \infty\), the Schatten-\(p\) norm of the matrix \(X \in \mathbb{C}^{n \times n}\) is the \(\ell_p\)-norm of its singular values, that is

\[
\|X\|_{S_p} := \|\sigma_1(X), \ldots, \sigma_n(X)\|_{\ell_p},
\]

We will always write \(\|\cdot\|_p = \|\cdot\|_{S_p}\) for matrices and \(\|\cdot\|_p = \|\cdot\|_{\ell_p}\) for vectors.

The dual norm \(\|\cdot\|_*\), associated with \(\|\cdot\|\) is given by \(\|v\|_* = \sup_{\|u\|=1} \langle v, u \rangle\). For \(\ell_p\)- and \(S_p\)-norms, the associated dual norms are \(\ell_{p^*}\) and \(S_{p^*}\) respectively, where \(p^*\) is a Hölder dual of \(p\), that is \(1/p + 1/p^* = 1\). We recall that Hölder’s inequality holds for Schatten-\(p\) norms [5, p. 92, 95] such that

\[
\langle X, Z \rangle_F \leq \|X\|_p \|Z\|_{p^*}, \quad \text{for all } Z, X \in \mathbb{C}^{n \times n}\text{ and } 1 \leq p \leq \infty.
\]

The best rank-\(r\) approximation \(X_r\) of \(X\) and its complement \(X_r^c\) are defined as

\[
X_r := \sum_{j=1}^{r} \sigma_j(X) u_j v_j^*, \quad \text{and} \quad X_r^c := X - X_r = \sum_{j=r+1}^{n} \sigma_j(X) u_j v_j^*,
\]
and $X_r$ is a matrix which minimizes the projection error on the manifold of rank $r$ matrices, that is
\[
\|X_r\|_1 = \|X - X_r\|_1 = \min_{Z \in \mathbb{C}^{n \times n}, \text{rank}(Z) = r} \|X - Z\|_1.
\] (11)

The measurement operator $\mathcal{A}$ is a linear operator mapping Hermitian matrices $\mathcal{H}_n$ to real vectors $\mathbb{R}^m$. The kernel (null space) and the range of $\mathcal{A}$ are denoted by
\[
\ker \mathcal{A} := \{Z \in \mathcal{H}_n : \mathcal{A}(Z) = 0\},
\]
\[
\text{Ran} \mathcal{A} := \{v \in \mathbb{R}^m : \text{there exists } Z \in \mathcal{H}_n \text{ such that } \mathcal{A}(Z) = v\},
\]
respectively.

1.5 Technical foundations

In this section, we review previous results about recovery guarantees for convex recovery methods for low-rank recovery, with a particular focus on those suitable for rank-one measurement operators as defined in (5).

An important tool in the performance analysis of recovery methods $\Delta$ and $\Delta_{q,\eta}$ as introduced in (7) and (8) in Section 1.2 is the robust rank null space property [39], which is similar in its core to the robust null space property that is used to characterize bounds for $\ell_1$-type optimization methods for the sparse vector recovery problem [31].

Definition 1 ([39], Definition 3.1). For $p \geq 1$, the measurement operator $\mathcal{A} : \mathcal{H}_n \rightarrow \mathbb{R}^m$ satisfies the $S_p$-robust rank null space property with respect to norm $\|\cdot\|$ on $\mathbb{R}^m$ of order $r$ with constants $0 < \rho < 1$ and $\tau > 0$ if for all $X \in \mathcal{H}_n$, the inequality
\[
\|X_r\|_p \leq \frac{\rho}{r^{1-\frac{1}{p}}} \|X_r^c\|_1 + \tau \|\mathcal{A}X\|
\]
holds.

The following theorem shows that the $S_p$-robust rank null space property is sufficient to establish the recovery guarantees for nuclear norm minimization problems such as $\Delta$ from (7) and $\Delta_{q,\eta}$ from (8), both in the noiseless case and in the presence of noise.

Proposition 2 ([39, version of Theorem 3.1]). For $q \geq 1$ and $1 \leq p \leq 2$, let the measurement operator $\mathcal{A} : \mathcal{H}_n \rightarrow \mathbb{R}^m$ satisfy the $S_2$-robust rank null space property with respect to norm $\|\cdot\|_q$ of order $r$ with constants $0 < \rho < 1$ and $\tau > 0$. Then, for any $X \in \mathcal{H}_n$ and any noise $w \in \mathbb{R}^m$, $\|w\|_q \leq \eta$, the inequalities
\[
\|X - \Delta (\mathcal{A}X)\|_p \leq \frac{2(1 + \rho)^2}{(1 - \rho)r^{1-1/p}} \|X_r^c\|_1
\]
and
\[
\|X - \Delta_{q,\eta} (\mathcal{A}X + w)\|_p \leq \frac{2(1 + \rho)^2}{(1 - \rho)r^{1-1/p}} \|X_r^c\|_1 + \frac{2\tau(3 + \rho)}{1 - \rho} r^{1/p-1/2} \eta
\] (12)
hold.

Similar results involving weaker versions of the null space property were established for the recovery of the real symmetric positive semidefinite matrices [18], recovery of general [49] and Hermitian matrices [28] with Pauli measurements and phase retrieval [15].

Similarly to Proposition 2, our noise-blind recovery guarantees also rely on the $S_2$-robust rank null space property. The next theorem analyzes this property for rank-one Gaussian measurements and will hence be a crucial ingredient of our proof.
Theorem 3 ([39, Section 6]). Let $A : \mathcal{H}_n \to \mathbb{R}^m$ be a measurement operator defined by $m$ rank-one measurements of the form (5) such that the entries of the vectors $a_j$ defining the measurement matrices $A_j = a_ja_j^*$ are i.i.d. standard complex Gaussian random variables. If

$$m \geq c_1 \rho^{-2} r n,$$

then the scaled measurement operator $\frac{1}{\sqrt{m}}A$ possesses the $S_2$-robust rank null space property with respect to the norm $\|\cdot\|_2$ on $\mathbb{R}^m$ of order $r$ with constants $0 < \rho < 1$ and $\tau > 0$ with probability at least $1 - e^{-r^2 m}$, where $c_1, \gamma_1 > 0$ are absolute constants.

Another important component of the noise-blind recovery will be the following property.

Definition 4 ($S_q$-quotient property [12, 49]). Given $q \geq 1$, a measurement operator $A : \mathcal{H}_n \to \mathbb{R}^m$ is said to possesses the $S_q$-quotient property with constant $d$ and rank $r$, relative to norm $\|\cdot\|$ on $\mathbb{R}^m$ if for all $\omega \in \mathbb{R}^m$ there exists $U \in \mathcal{H}_n$ such that

$$AU = \omega \text{ and } \|U\|_q \leq d r^{-\frac{1}{2}} \|\omega\|.$$

Beyond an algebraic relationship on the elements of the quotient space $S_q/k\ker A$, the $S_q$-quotient property has also a geometric interpretation: It is easy to check that Definition 4 is equivalent to the measurement operator $A$ fulfilling

$$A(B_{S_q}^{\times n}) \supset \frac{1}{d} B_2^m,$$

where $B_{S_q}^{\times n} = \{X \in \mathcal{H}_n : \|X\|_{S_q} \leq 1\}$ and $B_2^m = \{y \in \mathbb{R}^m : \|y\|_2 \leq 1\}$ are the nuclear norm unit ball and $\ell_2$-unit ball, respectively. In this sense, the quotient property means that the an $\ell_2$-ball of radius $1/d$ is contained in the image of the $S_q$-unit ball with respect to the measurement operator. In other words, the quotient property provides a lower bound on the inradius of the image of the $S_q$-unit ball.

The $S_q$-quotient property has been used in the literature on low-rank matrix recovery before: The $S_1$-quotient property coincides with the Nuclear Norm Quotient property (NNQ) of [12], which has been used to analyze the performance of the matrix Lasso $\mathcal{L}_q^{\text{Lasso}}$ and another related reconstruction map, the Matrix Dantzig selector [12, (I.3)], in the presence of the Gaussian noise with known variance. In that paper, the $S_1$-quotient property was shown for random measurement operators (5) that are defined by independent measurement matrices $A_j$ with i.i.d. Gaussian or subgaussian entries (which each are of full rank with high probability) if $m \leq c n^2 / \log(m/n)$, where $c$ is a constant, $m$ is the number of measurements and $n$ the dimension of the domain of the measurement operator $A$. For an analysis of the same algorithms, [49] showed the $S_1$-quotient property for Pauli measurements, which are of importance in quantum state tomography. To the best of our knowledge, the results on the $S_1$-quotient property of [12, 49] are the only ones available in the literature.

2 Main results

This section presents the main results of this paper. First, we present Theorem 5, which shows that the $S_1$-quotient property holds for measurement operators defined from Gaussian rank-one measurements. Theorem 6 establishes then that noise-blind recovery is indeed possible from Gaussian rank-one measurements.

For the latter result, it is necessary to combine the $S_1$-quotient property with a $S_2$-robust rank null space property of Definition 1. In Theorem 7, we provide an extensive statement characterizing the interplay of these two properties, noise-aware and noise-blind guarantees for convex reconstruction maps for low-rank matrix recovery.
We recall the measurement model of (5): Let $\mathcal{A}: \mathcal{H}_n \rightarrow \mathbb{R}^m$ be an operator such that
\[
\mathcal{A}(X)_j = \langle a_j a_j^*, X \rangle
\] (14)
for all $j = 1, \ldots, m$. With this, we state our main theorem about the $S_1$-quotient property.

**Theorem 5** ($S_1$-quotient property for Gaussian rank-one measurements). Let $\mathcal{A} : \mathcal{H}_n \rightarrow \mathbb{R}^m$ be a measurement operator defined by $m$ rank-one measurements of the form (14). If the entries of the vectors $a_j$ are i.i.d. standard complex Gaussian random variables and if
\[
m \leq \left( \frac{n}{c_2 \log(545m)} \right)^{8/7} \text{ and } m \text{ is sufficiently large,}
\]
for a constant $c_2 > 0$, then there exists $\gamma_2 > 0$, such that the scaled measurement operator $\frac{1}{\sqrt{m}} \mathcal{A}$ possesses the $S_1$-quotient property with constant $d = \frac{128\sqrt{2}}{\kappa} \sqrt{\kappa m/n}$ and rank $\kappa m/n$ relative to the norm $\|\cdot\|_2$ on $\mathbb{R}^m$ for all $\kappa > 0$ with probability at least $1 - 14e^{-\gamma_2 m^{7/8}}$.

Compared to the related result involving dense, full-rank Gaussian measurement operators [12], Theorem 5 contains a dimension dependence of the factor $d$ as it scales with $\sqrt{m/n}$. However, when used in the analysis of noise-blind reconstruction maps, this will only lead to a slightly worse order in front of the noise term $\|w\|_2$ in the theoretical guarantees, as can be seen below in Theorem 7.

With the $S_1$-quotient property being established, we can go ahead and extend the scope of Proposition 2 to a noise-blind setting. With the following theorem, we show that equality-constrained nuclear norm minimization $\Delta$, which does not require any estimate of the magnitude of a noise vector $w$ and therefore is noise-blind, exhibits similar noise-robustness properties as $\Delta_{d,\eta}$ with properly tuned parameters $\eta$ for the recovery of low-rank matrices.

**Theorem 6** (Noise-blind recovery guarantees for Gaussian rank-one measurements). Let $1 \leq p \leq 2$ and $0 < \rho < 1$. Let $\mathcal{A} : \mathcal{H}_n \rightarrow \mathbb{R}^m$ be a measurement operator defined by $m$ rank-one measurements of the form (14) such that the entries of vectors $a_j$ are i.i.d. standard complex Gaussian random variables. If
\[
m \leq \left( \frac{n}{c_1 c_2 \log(545m)} \right)^{8/7}, \text{ } m \text{ sufficiently large and } r \leq \frac{\rho^2 m}{c_1 n} = r_*
\]
for absolute constants $c_1, c_2 > 0$, then there exists constant $\gamma$ such that with probability at least $1 - 15 \exp \{ -\gamma m^{7/8} \}$, for all $X \in \mathcal{H}_n$ and for all $w \in \mathbb{R}^m$ the error bound
\[
\|X - \Delta (\mathcal{A}X + w)\|_p \leq \frac{D_1}{r^{1-p}} \|X^c\|_1 + \frac{D_4 r^{1/2}}{\sqrt{m}} \|w\|_2
\]
holds, where $D_1$ and $D_4$ are positive constants depending only on $\rho$, and $\Delta$ refers to the solution of the equality nuclear norm under constrained minimizer (7).

Furthermore, a minimizer $\Delta_{d,\eta}$ of inequality-constrained nuclear norm minimization (8) with noise estimate $\eta$ then fulfills
\[
\|X - \Delta_{d,\eta} (\mathcal{A}X + w)\|_p \leq \frac{D_1}{r^{1-p}} \|X^c\|_1 + \frac{D_5 r^{1/2}}{\sqrt{m}} \max \{\|w\|_2, \eta\}
\]
for all $X \in \mathcal{H}_n$ and all $w \in \mathbb{R}^m$, where $D_5$ is a positive constant depending on $\rho$. 

Besides providing a thorough theoretical understanding of the noise-blind reconstruction map \( \Delta \), Theorem 6 also provides improved guarantees for inequality-constrained nuclear norm minimization \( \Delta_{q,n} \) that go beyond the ones of Proposition 2, very much in the spirit of [6]. In particular, unlike inequality (12) of Proposition 2, which only applies if \( \|w\|_q \leq \eta \), we have obtained a guarantee that depends on \( \max(\|w\|_2, \eta) \), suggesting a good performance of \( \Delta_{q,n} \) in the case of underestimated noise level, too.

As discussed in Section 1.2, a noise-blind guarantee similar to Theorem 6 has been shown for the recovery of positive semidefinite low-rank matrices from rank-one measurements via (10) [9, 39]. However, the scaling of the guarantees of [9, 39] is slightly better with respect to the rank \( r_* \) in the second summand of the error bound, as their results have a factor \( r_*^{1/p-1/2} \) instead of \( r_*^{1/p} \). On the other hand, their method and theoretical analysis is explicitly designed for positive semidefinite matrices and cannot generalize beyond this case.

The proof of Theorem 6 relies on a relation between the quotient property and noise-blind recovery, which is independent of the specific measurement scenario. The next statement summarizes this relation, which is very much analogous to the results in [65] (see also [31, Chapter 11]) and, for underestimated noise levels, in [6] that have been obtained in the context of sparse recovery.

**Theorem 7** (Low-rank matrix recovery analogue of [31, Theorem 11.12] and [6, Theorem 4]). Let \( 1 \leq p \leq 2 \). Assume that a measurement operator \( A : \mathcal{H}_n \to \mathbb{R}^m \) satisfies:
- the \( S_2 \)-robust rank null space property of order \( r_* \) and constants \( 0 < \rho < 1 \) and \( \tau > 0 \) relative to norm \( \| \cdot \| \), and
- the \( S_1 \)-quotient property with constant \( d \) and rank \( r_* \) relative to the norm \( \| \cdot \| \).

Let \( \eta > 0 \). If a \( \eta \)-dependent recovery map \( R_\eta : \mathbb{R}^m \to \mathcal{H}_n \) possesses robust low-rank matrix recovery guarantees, that is for all \( X \in \mathcal{H}_n \) and \( w \in \mathbb{R}^m \) such that \( \|w\| \leq \eta \) inequality

\[
\|X - R_\eta (AX + w)\|_p \leq \frac{2(1 + \rho)^2}{(1 - \rho)^{1/p - 1/2} r_*^{1/p}} \|X^*_r\|_1 + \frac{2 \tau (3 + \rho)}{1 - \rho} r_*^{1/p - 1/2} \eta
\]

holds, then for \( r \leq r_* \) and for all \( w \in \mathbb{R}^m \) it holds that

\[
\|X - R_\eta (AX + w)\|_p \leq \frac{D_1}{r^{1/p}} \|X^*_r\|_1 + (D_2d + D_3)r_*^{\frac{1}{2} - \frac{1}{p}} \max(\|w\|, \eta),
\]

where \( D_1, D_2 > 0 \) are constants depending on \( \rho \) and \( D_3 > 0 \) is a constant depending on \( \rho \) and \( \tau \).

We note that Theorem 7 does not apply to positive semidefinite-based reconstruction maps. The reason for this is the fact that the psd cone is a cone, but not a linear space and the quotient property will not hold. For instance, if the noise vector \( w \) has negative entries, the corresponding matrix \( U \) in the quotient property is Hermitian, but not a positive semidefinite and, hence, infeasible for the solvers.

### 3 Numerical Experiments

In this section, we explore the validity of the noise-blind recovery guarantees presented in Theorem 6 for low-rank matrix recovery problems with noisy rank-one measurements. We also compare reconstruction errors achieved by noise-blind reconstruction programs to those achieved by noise-aware methods such as inequality-constrained nuclear norm minimization (8).

We conduct the experiments for simple, small-sized random problem instances where the dimension \( n \) of the matrix to be recovered and the number of measurements \( m \) relate
in a way that would allow for exact recovery in the noiseless case. In all experiments, we construct the random measurement operator \( \mathcal{A} \) as in (14) by drawing i.i.d. complex Gaussian rank-one measurements.

As reconstruction methods, we use many of the optimization problems that have been discussed, in particular equality-constrained nuclear norm minimization \( \Delta \) (NucNorm), which is the method defined by (7), inequality-constrained nuclear norm minimization \( \Delta_{2,\eta} \) with parameter \( \eta \) based on an \( \ell_2 \)-error (8) (also called nuclear norm denoising, NucNorm\( \Delta \)), and the MatrixLasso \( \Delta^{Lasso}_{2,\mu} \) with parameter data fit parameter \( \mu \), cf. (9).

When the matrix to be recovered is positive semidefinite, we also compare these methods with approaches that make explicit use of this property, such as inequality-constrained nuclear norm minimization \( \Delta_{2,\eta}^{\geq 0} \) with parameter \( \eta \), analogue of \( \Delta_{2,\eta} \) constrained to the psd cone (phase lift denoising, PhaseLiftDN) and \( \ell_2 \)- and \( \ell_1 \)-minimization of the residual \( \mathcal{A}(X) - b \) on the cone of positive semidefinite matrices, that is, the solution \( \Delta^{\geq 0}_q \) of (10) for \( q = 2 \) and \( q = 1 \), respectively (PosDef-\( \ell_2 \)-min and PosDef-\( \ell_1 \)-min).

The experiments were conducted on a Linux node with Intel Xeon E5-2690 v3 CPU with 28 cores and 64 GB RAM, using MATLAB R2019a. All optimization problem were modeled using the CVX package \cite{grant2014cvx} and solved by SDPT3 \cite{szekely2001sdpt3}.

### 3.1 Robust Recovery of Rank-One Matrices

In our first set of experiments, we study the noise robustness of different methods for the task of recovering rank-one matrices \( X_0 = x_0x_0^* \) from rank-one measurements perturbed by different types of noise. This setting corresponds to (noisy) phase retrieval \cite{candes2009phase, candes2015phase, candes2014exact, candes2015phase}, as explained above in (6).

In particular, we sample vectors \( x_0 \in \mathbb{C}^n \) randomly with respect to the Haar measure on the complex unit sphere \( S^{n-1} = \{ x \in \mathbb{C}^n : \|x\|_2 = 1 \} \) to define \( (n \times n) \)-ground truth matrices \( X_0 = x_0x_0^* \). Independently from \( x_0 \) and the complex Gaussian vectors \( a_j \) defining \( \mathcal{A} \), we sample a random real vector \( w \in \mathbb{R}^m \) from the sphere \( \eta S^{m-1} = \{ w \in \mathbb{R}^m : \|w\|_2 = \eta \} \) with radius \( \eta = 0.01 \) and

\[
b := |a_j^*x_0|^2 + w = \mathcal{A}(x_0x_0^*) + w,
\]

a measurement vector \( b \) that is perturbed by spherical noise \( w \) such that \( \|w\|_2 = \eta = 0.01 \).

For \( n = 50 \) and a range of parameters \( m \) between \( m = 50 \) and \( m = 500 \), we compare the reconstructions \( \hat{X} \) of the recovery algorithms NucNorm, NucNorm\( \Delta \), PhaseLiftDN, PosDef-\( \ell_2 \)-min and PosDef-\( \ell_1 \)-min, which are provided with \( b \) as an input, with \( X_0 \) and measure the relative Frobenius error \( \|\hat{X} - X_0\|_F / \|X_0\|_F \). For NucNorm\( \Delta \) and PhaseLiftDN, we provide the oracle noise level estimate \( \eta \) as an input parameter.

In Figure 1, the resulting recovery errors are reported, averaged across 100 independent realizations of experimental setup.

We observe that for the algorithms PhaseLiftDN, PosDef-\( \ell_2 \)-min and PosDef-\( \ell_1 \)-min, which all optimize on the cone of positive semidefinite matrices, the relative error falls below \( 10^{-2} \) if the number of measurements surpasses a number between \( m = 200 \) and \( m = 250 \). On the other hand, NucNorm, NucNorm\( \Delta \), which do not use the positive definiteness, require at least \( m = 300 \) measurements to pass the threshold of a relative Frobenius error of \( 10^{-2} \). This shows that for few measurements, including positive definiteness as a constraint helps to identify the desired matrix. As can we see in the right column of Figure 1, the behavior of NucNorm\( \Delta \) starts to follow closely the one of PhaseLiftDN for \( m \geq 350 \). These two methods are fully noise-aware and use oracle knowledge of the \( \ell_2 \)-norm of the noise \( \eta \) as an input parameter. We observe that NucNorm, which is a noise-blind method, consistently exhibits a lower error than the noise-aware ones in the stable region between \( m = 350 \) and \( m = 500 \) (for example, it is 24% lower for \( m = 500 \) with \( 5.18 \times 10^{-4} \) to \( 6.80 \times 10^{-4} \), respectively).
This validates somewhat Theorem 6, confirming that NucNorm returns estimates that are proportional to the noise magnitude $\|w\|_2$ for a reasonable set range of parameters $m$ and $n$. Furthermore, we observe that PosDef-$\ell_2$-min and PosDef-$\ell_1$-min, which are also noise-blind methods, consistently return the reconstructions with the smallest error compared to the ones based on the nuclear norm. The experiments suggest that for phase retrieval, at least for the considered noise model, incorporating the positive definiteness constraint indeed is beneficial if used in the formulations PosDef-$\ell_2$-min and PosDef-$\ell_1$-min.

Next, we repeat the same experiment for the methods NucNorm as well as NucNormDN (NNDN) $\Delta_{2,\eta}$ and MatrixLasso (MLasso) $\Delta_{2,\mu}$ with different choices of the noise balancing parameters $\eta$ and $\mu$, respectively, concentrating on a higher range of measurements with $m$ between $m = 300$ and $m = 2500$. For $m = 2500$, given that we consider square matrices of with size $n = 50$, we would be able to solve the reconstruction problem exactly in the absence of noise by simple linear algebra. We choose the parameter $\eta$ of NucNormDN in the range of $\eta \in \{0.1, 0.5, 1, 2\}$ and the parameter $\mu$ of MatrixLasso in the range of $\mu \in \{0.025, 0.1, 0.25, 0.5\}$. We do not report results of this experiment for a number of measurements $m < 300$, as the results are all very similar in that range. We illustrate the results of this experiment in Figure 2.

Our first observation is that the the choice of $\eta$ and $\mu$ is especially sensitive to overestimates: We see in the first column of Figure 2 that for a moderate number of measurements $m < 1000$, the smallest reconstruction errors are obtained by the noise-blind method NucNorm (which corresponds to NucNormDN with $\eta = 0$) and by the choices of $\eta = 0.1\|w\|_2$.

![Graph](image-url)

**Figure 1:** Comparison of reconstruction algorithms for phase retrieval with spherical noise: Recovery of normalized rank-1 matrices $X_0 = x_0x_0^* \in \mathbb{C}^{m \times n}$, $n = 50$, from noisy rank-one measurements $b = A(X_0) + w \in \mathbb{R}^m$ perturbed by spherical noise $w$ such that $\|w\|_2 = 0.01$. $x$-axis: Number of measurements $m$, $y$-axis: Relative Frobenius error $\|X - X_0\|_F / \|X_0\|_F$ of reconstruction $\hat{X}$, averaged across 100 experiments.

Left column: Logarithmic scaling of y-axis for $m \in \{50, \ldots, 500\}$, right column: Linear scaling of y-axis for $m \in \{250, \ldots, 500\}$. 
for the considered parameter choices of \( \eta \). In the right column of Figure 2, we see that MatrixLasso for different parameters \( \eta \), right column: MatrixLasso (MLasso) \( \Delta_{2,\eta}^{\text{Lasso}} \) for different parameters \( \mu \).

Increasing the number of measurements \( m \) reduces the relative Frobenius reconstruction error consistently for all methods as long as \( m < 1000 \).

For very large numbers of measurements \( 1500 \leq m \leq 2500 \), however, we observe an additional phenomenon: The performance of NucNorm and NucNormDN with \( \eta = 0.1 \|\mu\|_2 \) actually deteriorates as \( m \) grows in that range. For MatrixLasso, this is not observed, at least not for the considered parameter choices of \( \mu \geq 0.025 \).

One explanation of this observation is that if \( m = 2500 \) or close to that, the linear system \( \mathcal{A}X = b \) has only a unique solution (or a very low-dimensional set of solutions), so that the noise-blind method NucNorm has only a unique feasible point (or a very low-dimensional feasible set). If \( b \) is quite noisy, it is reasonable to expect that the solution of NucNorm is farther away from the ground truth rank-one matrix \( X_0 \) than methods that do not use a strict equality constraint, but allow for a trade-off between data-fit and complexity measure, such as NucNormDN or MatrixLasso for larger noise parameters \( \eta \) and \( \mu \).

These observations can be also interpreted in view of the \( S_1 \)-quotient property. Evidently, for large \( m \) and severely underestimated noise level, a reconstruction error that is proportional to the \( \ell_2 \)-norm of the noise \( \|\mu\|_2 \), as in the statement of Theorem 6, is not possible in general. We recall that the proof of Theorem 6 is based on the \( S_1 \)-quotient property. Thus, it is clear that some sort of upper bound on the number of measurements \( m \) with respect to the dimension \( n \) in the assumption of Theorem 6 is to be expected and not just an artifact of our proof.
3.2 Robust Recovery of Hermitian Rank-Two Matrices

We recall that in the experiment of Figure 1, it was observed that methods incorporating the positive definiteness constraint into their optimization such as PosDef-$\ell_2$-min and PosDef-$\ell_1$-min resulted in the lowest reconstruction errors for the recovery of rank-one matrices.

In this subsection, we consider the recovery of Hermitian rank-2 matrices $X_0 = v_1v_1^* + (-0.5)v_2v_2^*$ with orthonormal vectors $v_1, v_2 \in \mathbb{R}^n$ that are drawn uniformly from the Stiefel manifold. For this case, the only available reconstruction methods (among the ones considered in this paper) are NucNorm, NucNormDN and the MatrixLasso as $X_0$ is not positive semidefinite. Independently from $v_1$ and $v_2$ and the complex Gaussian vectors $a_j$ defining $\mathcal{A}$, we sample a random real vector $w \in \mathbb{R}^m$ from the sphere $\eta S^{m-1} = \{w \in \mathbb{R}^m : \|w\|_2 = \eta\}$ with radius $\eta = 0.01$ and define the measurement vector

$$b := \mathcal{A}(X_0) + w,$$

that is perturbed by spherical noise $w$ such that $\|w\|_2 = \eta = 0.01$.

For $n = 25$ and a range of parameters $m$ between $m = r n = 50$ and $m = n^2 = 625$, we compare the reconstructions $\hat{X}$ of the recovery algorithms NucNorm, as well as NucNormDN (NNDN) $\Delta_{2,0}$ and MatrixLasso (MLasso) $\Delta_{2,\mu}$, which are provided with $b$ as an input, with $X_0$ and measure the relative Frobenius error $\|\hat{X} - X_0\|_F / \|X_0\|_F$. For NucNormDN and MatrixLasso, we provide different noise level parameters $\eta$ and $\mu$ as an input parameter.

In Figure 3 we report the resulting recovery errors for $\eta \in \{0.05\|w\|_2, 1\|w\|_2, 2\|w\|_2\}$ and $\eta \in \{0.025, 0.25\}$, averaged across 100 independent realizations of experimental setup. We observe that the relative Frobenius error of the reconstruction falls below $10^{-2}$ between $m = 200$ and $m = 250$ for all considered methods. This is interesting since compared to the experiments in Section 3.1, the dimension was halved from $n = 50$ to $n = 25$, but the rank was doubled from $r = 1$ to $r = 2$, and these methods needed more than $m = 300$ measurements to obtain an error below $10^{-2}$. We note that among the considered methods,
NucNormDN for $\eta = 0.5\|w\|_2$ and MatrixLasso for $\mu = 0.025$ result in the lowest errors for all considered $m$, with $\eta = 0.5\|w\|_2$ being a choice that underestimates the noise level by 50%. NucNorm does almost equally well for a moderate number of measurements until $m \approx 350$, after which its performance deteriorates from around $3.8 \cdot 10^{-4}$ to $1.5 \cdot 10^{-2}$ at $m = 625$, when the system becomes square.

Finally, in Figure 4, we illustrate the results of the same experiment for more parameters $\eta$ and $\mu$ when restricted to $m \in \{225, \ldots, 625\}$ – such a measurement complexity would result in exact recovery via NucNormDN in the case of noiseless measurements $w = 0$. We observe also here that an overestimate of $\eta$ and $\mu$ (e.g., by 100% with the choice of $\eta = 2\|w\|_2$ or with $\mu = 0.5$ for MatrixLasso) has more negative consequences than underestimating their “oracle” choice. Choosing $\eta$ as small as $\eta = 0.1\|w\|_2$ results in a qualitatively very similar behavior as NucNorm with a performance deterioration for large $m$.

As a summary, we note these experiments show that a noise-blind recovery of low-rank indefinite matrices is indeed possible via convex formulations such as NucNorm if the measurement matrices are random and rank-one. This is compatible with the theory of Theorems 5 to 7 that are based on the $S_2$-robust null space property and the $S_1$-quotient property of the measurement operator $\mathcal{A}$. We also note that noise-blind recovery works particularly well if the number of measurements $m$ is only moderate such that $m$ is closer to $m \approx Crn$ than to $m \approx Cn^2$.

4 Proofs

In this section, we detail the proofs of our main results Theorem 5 about the $S_1$-quotient property of random measurement operators with Gaussian rank-one matrices and Theorem 6 about noise-blind recovery guarantees for such measurements. The proof of a general
result for arbitrary measurement operators, Theorem 7 concludes this section.

4.1 Proof of Theorem 5

4.1.1 Proof concept and structure

The goal of this section is to establish that the scaled measurement operator $\frac{1}{\sqrt{m}} A$ possesses the $S_1$-quotient property with constant $\frac{128 \sqrt{2}}{k} \sqrt{km/n}$ and rank $\kappa m/n$ relative to the norm $\|\cdot\|_2$ on $\mathbb{R}^m$ for all $\kappa > 0$ with high probability. We show this property by establishing an equivalent form, as given by the following proposition, which is exactly in line with the analogous result for sparse recovery (see, e.g., [31, Lemma 11.17]). For completeness, we provide a proof in the appendix.

**Proposition 8.** For $q \geq 1$, a measurement operator $\mathcal{A} : \mathcal{H}_n \to \mathbb{R}^m$ possess the $S_q$-quotient property with constant $d$ and rank $\gamma$, relative to norm $\|\cdot\|$ if and only if

$$\|\mathcal{A}^* w\|_q \leq d \gamma^{-\frac{1}{2}} \|\mathcal{A}^* w\|_q, \text{ for all } w \in \mathbb{R}^m,$$

where $\mathcal{A}^*$ is the adjoint of the measurement operator, $\|\cdot\|_q$ is a dual norm associated with $\|\cdot\|$ and $q^*$ is a Hölder dual of $q$.

Recalling that the dual of $\|\cdot\|_2$ is $\|\cdot\|_2$ itself and the Hölder dual of $1$ is $\infty$, we observe that after normalization of $\mathcal{A}$, Inequality (17) reads as

$$\left\| \frac{1}{\sqrt{m}} \mathcal{A}^* w \right\|_\infty \geq \frac{n}{128 \sqrt{2m}}, \text{ for all } w \in \mathbb{R}^m \text{ satisfying } \|w\|_2 = 1. \quad (18)$$

We will establish this inequality via a covering argument (see Section 4.1.5), for which we need

$$\left\| \frac{1}{\sqrt{m}} \mathcal{A}^* w \right\|_\infty \geq \frac{n}{64 \sqrt{2m}}, \quad (19)$$

to hold with high probability for any fixed vector $w$ with $\|w\|_2 = 1$.

The left hand side can be made explicit via the following well-known formula for the dual of $\mathcal{A}$ (see, e.g., [10, Lemma 3.1]). Again we include a proof in the appendix for completeness.

**Lemma 9.** The adjoint operator $\mathcal{A}^* : \mathbb{R}^m \to \mathcal{H}_n$ is given by

$$\mathcal{A}^* w := \sum_{k=1}^m w_k a_k a_k^*.$$

With this lemma and using that $\mathcal{A}^* w$ is Hermitian, we obtain the following estimate for the left hand side of (19).

$$\|\mathcal{A}^* w\|_\infty = \max_{\nu \in \mathbb{C}^n, \|\nu\|_2 = 1} |\nu^*(\mathcal{A}^* w)\nu| = \max_{\nu \in \mathbb{C}^n, \|\nu\|_2 = 1} \left| \sum_{k=1}^m w_k |a_k^* \nu|^2 \right| \quad (20)$$

Depending on the properties of the vector $w$, the remainder of the proof is split into three separate cases, each requiring a different approach.

We will first consider vectors whose entries sum to a number significantly different from zero. As this leads to a non-zero expectation of the sum in (20), the lower bound can be established via a concentration argument.
Secondly, we will consider vectors with some large entries, as formalized by a lower bound on the supremum norm. In this case, we can bound the right hand side of (20) by choosing \( \nu \) to be the normalized measurement vector \( a_j \) corresponding to the largest entry.

In the last case of a vector with entries averaging to a small number, but without large entries, we select a subset of entries with sufficiently large magnitudes and construct a suitable \( \nu \) from the associated measurement vectors.

The common ingredient in all three cases is Bernstein’s inequality for subexponential random variables. Recall that a random variable \( X \) is subexponential if \( \sup_{p \geq 1} P^{-1}(\mathbb{E}|X|^p)^{1/p} \) is finite. For more details about subexponential random variables we refer reader to Section 2.7 of [64].

**Proposition 10** (Bernstein’s inequality, version of [64, Theorem 2.8.2]). Let \( K > 0 \) and let \( \xi_1, \ldots, \xi_N \) be independent subexponential random variables with \( \|\xi_i\|_\psi_1 \leq K \) for all \( i \in [N] \). Then, for every \( w \in \mathbb{R}^N \) and every \( t > 0 \) it holds that

\[
\mathbb{P}\left( \left| \sum_{j=1}^{N} w_j (\xi_j - \mathbb{E}\xi_j) \right| \geq t \right) \leq 2 \exp \left\{ -c \min \left\{ \frac{t^2}{K^2 \|w\|_2^2}, \frac{t}{K \|w\|_\infty} \right\} \right\},
\]

as well as

\[
\mathbb{P}\left( \sum_{j=1}^{N} w_j \xi_j \leq \sum_{j=1}^{N} w_j \mathbb{E}\xi_j - t \right) \leq \exp \left\{ -c \min \left\{ \frac{t^2}{K^2 \|w\|_2^2}, \frac{t}{K \|w\|_\infty} \right\} \right\}.
\]

It will be applied to random variables drawn from a \(|\mathcal{C}/N(0,1)|^2\) distribution, which are subexponential. Indeed, by definition, standard complex Gaussian random variable \( \xi \) is defined as

\[
\xi = (\alpha + i\beta)/\sqrt{2},
\]

where \( \alpha \) and \( \beta \) are independent standard Gaussian random variables. Then, \( \alpha^2 + \beta^2 \) follows a chi-squared distribution with 2 degrees of freedom, which coincides with the exponential distribution with parameter 1/2 and hence \( |\xi|^2 = (\alpha^2 + \beta^2)/2 \) is subexponential with norm \( K = 1 \). It has expectation \( 1 \) and the sum of the expectations in Bernstein’s inequality becomes

\[
\sum_{j=1}^{N} w_j \mathbb{E}\xi_j = \sum_{j=1}^{N} w_j.
\]

We note that in further statements we also denote constants by \( c, C, \tilde{C} \), but their values may differ; even within a proof or a chain of inequalities.

### 4.1.2 Non-centered vectors

The first case only considers vectors \( w \) with mean value far from zero. In this case, the following holds.

**Theorem 11.** Let \( w \in \mathbb{R}^m \), \( \|w\|_2 = 1 \) and assume that \( w \) satisfies \( \sum_{k=1}^{m} w_k \geq \frac{n}{32\sqrt{2}m} \). Suppose that the number of measurements \( m \) satisfies \( m \leq n^2 \). Then, inequality (10) holds on a random event \( E_{1,1}(w) \) (depending on \( w \)), which occurs with a probability of at least \( 1 - 2 \exp(-cn^2/\sqrt{m}) \).
Thus, it suffices to establish a lower bound for a maximum of
\[ \ell_{\max} \] occurring in the bilinear representation of \( \|A^*w \|_\infty \) to standard basis vectors. More precisely,
\[
\|A^*w\|_\infty = \max_{v \in \mathbb{C}^N, \|v\|_2 = 1} |v^*(A^*w)v| \geq \max_{\ell \in [n]} |e_\ell^*(A^*w)e_\ell| = \max_{\ell \in [n]} \left| \sum_{k=1}^m w_k |(a_k)_\ell|^2 \right| =: \max_{\ell \in [n]} |S_{\ell}^1(w)|. 
\]

Thus, it suffices to establish a lower bound for a maximum of \( |S_{\ell}^1(w)| \). The first step in this direction is to fix index \( \ell \) and establish the lower bound for a single \( |S_{\ell}^1(w)| \). In order to do so, we introduce a random event
\[
E_{S_{\ell}^1}(w) := \left\{ |S_{\ell}^1(w) - \mathbb{E}[S_{\ell}^1(w)]| < \frac{n}{64 \sqrt{2m}} \right\}.
\]

Under event \( E_{S_{\ell}^1}(w) \), by reverse triangle inequality, it holds that
\[
|S_{\ell}^1(w) - |\mathbb{E}[S_{\ell}^1(w)]|| \leq |S_{\ell}^1(w) - \mathbb{E}[S_{\ell}^1(w)]| < \frac{n}{64 \sqrt{2m}}, 
\]
and consequently
\[
|S_{\ell}^1(w)| > |\mathbb{E}[S_{\ell}^1(w)]| - \frac{n}{64 \sqrt{2m}}.
\]

The expectation of \( S_{\ell}^1(w) \) is given by \( \mathbb{E}[S_{\ell}^1(w)] = \sum_{k=1}^m w_k \), and thus, using the assumption on \( w \), we obtain
\[
|S_{\ell}^1(w)| > |\mathbb{E}[S_{\ell}^1(w)]| - \frac{n}{64 \sqrt{2m}} = \left| \sum_{k=1}^m w_k \right| - \frac{n}{64 \sqrt{2m}} \geq \frac{n}{32 \sqrt{2m}} - \frac{n}{64 \sqrt{2m}} = \frac{n}{64 \sqrt{2m}}.
\]

The tail probability of the event \( E_{S_{\ell}^1}(w) \) can be bounded via Proposition 10. We recall that \( (a_k)_\ell \) are i.i.d standard complex Gaussian random variables, and hence the probability of the complement of \( E_{S_{\ell}^1}(w) \) is bounded from above as
\[
\mathbb{P}(E_{S_{\ell}^1}(w))^C \leq 2 \exp \left\{ -c \min \left\{ \frac{n^2}{64^2 \cdot 2m \cdot \|w\|_2^2}, \frac{n}{64 \sqrt{2m} \|w\|_\infty} \right\} \right\}.
\]

Since the number of measurements satisfies \( m \leq n^2 \) and condition \( \|w\|_2 = 1 \) implies \( \|w\|_\infty \leq 1 \), this bound simplifies to
\[
\mathbb{P}(E_{S_{\ell}^1}(w))^C \leq 2 \exp \left\{ -cn/\sqrt{m} \right\}.
\]

This establishes the desired result for a single \( \ell \in [n] \). Our next step is to establish a similar upper bound for the random event
\[
E_{1,1}(w) := \bigcup_{\ell \in [n]} E_{S_{\ell}^1}(w) \subseteq \left\{ \max_{\ell \in [n]} |S_{\ell}^1(w)| > \frac{n}{64 \sqrt{2m}} \right\}.
\]

In order to extend it for \( E_{1,1}(w) \) without losses in probability, we observe that \( S_{\ell}^1(w), \ldots, S_{\ell}^1(w) \) are i.i.d random variables as a consequence of the fact that the entries of the measurement vectors \( (a_k)_\ell, k \in [m], \ell \in [n] \) are independent and the vector are independent as well.
This implies that random events \( E^C_{S\ell} (w) \) are independent and, thus, using the De Morgan’s law, the probability of \( E_1 (w) \) is bounded from below as

\[
\Pr (E_1 (w)) = 1 - \Pr \left( E^C_{1,1} (w) \right) = 1 - \Pr \left( \bigcap_{\ell \in [n]} E^C_{S\ell} (w) \right) = 1 - \prod_{\ell \in [n]} \Pr \left( E^C_{S\ell} (w) \right) \geq 1 - 2 \exp \left\{ -cn^2 / \sqrt{m} \right\},
\]

which concludes the proof of Inequality (19) in the first case. \( \square \)

### 4.1.3 Spiky vectors

The second case considers those of the remaining vectors \( w \) which have at least one entry with large magnitude.

**Theorem 12.** Let \( w \in \mathbb{R}^m, \|w\|_2 = 1 \) and assume that \( w \) satisfies \( \|w\|_\infty \geq m^{-1/4} \). Suppose that the number of measurements \( m \) satisfies

\[
m \leq \min \{ (n/16)^{4/3}, (n/8)^{8/7} \} \text{ and } m \text{ is sufficiently large.}
\]

Then, there exists a random event \( E_{2,1} (w) \) depending on \( w \) and random event \( E_{2,2} \) independent of \( w \) with tail probabilities

\[
\Pr \left( E^C_{2,1} (w) \right) \leq 2 \exp \left\{ -\tilde{C} n^{1/8} \right\} \text{ and } \Pr \left( E^C_{2,2} \right) \leq 2m \exp \left\{ -cn \right\} + 4m^{3/4 + 1} \exp \left\{ -C m^{7/8} \right\}
\]

such that on \( E_{2,1} (w) \cap E_{2,2} \), Inequality (19) holds.

**Proof.** Let \( w_j \) be the entry of \( w \) with the largest magnitude so that \( |w_j| = \|w\|_\infty \geq m^{-1/4} \). Then, in Equality (20), we select single \( v = a_j / \|a_j\|_2 \) and apply the reverse triangle inequality to obtain the lower bound

\[
\|A^*w\|_\infty \geq \sum_{k=1}^{m} w_k \left| a_k^* a_j \right| / \|a_j\|_2^2 = \left| w_j \right| \left| a_j \right|_2^2 + \sum_{k=1, k \neq j}^{m} w_k \left| a_k^* a_j \right| / \|a_j\|_2^2 \geq \left| w_j \right| \left| a_j \right|_2^2 - \sum_{k=1, k \neq j}^{m} w_k \left| a_k^* a_j \right| / \|a_j\|_2^2.
\]

Next step is to further split the sum in two by separating the high and low magnitude entries. Let \( J \) be an index set containing high magnitude entries, so that

\[
J := J(w) = \{ k \in [m] \mid |w_k| > m^{-3/8} \}.
\]

The cardinality of \( J \) is bounded by \( \lfloor m^{3/4} \rfloor \) since

\[
1 = \|w\|_2^2 = \sum_{k=1}^{m} |w_k|^2 \geq \sum_{k \in J} |w_k|^2 > \sum_{k \in J} m^{-3/4} = m^{-3/4} |J|.
\]

Then, we split the sum such that

\[
\sum_{k=1, k \neq j}^{m} w_k \left| a_k^* a_j \right| / \|a_j\|_2^2 \leq \sum_{k \in J, k \neq j}^{m} w_k \left| a_k^* a_j \right| / \|a_j\|_2^2 + \sum_{k \in [m] \setminus J}^{m} w_k \left| a_k^* a_j \right| / \|a_j\|_2^2 = \sum_{k \in J}^{m} w_k \left| a_k^* a_j \right| / \|a_j\|_2^2 + |S^2 (w)|.
\]
The first sum is further bounded from above as
\[
\left| \sum_{k \in J, k \neq j} w_k a_k^* a_j^T \|a_j\|_2^2 \right| \leq \sum_{k \in J, k \neq j} |w_k| |a_k^* a_j^T| \|a_j\|_2^2 \leq |w_j| \sum_{k \in J, k \neq j} a_k^* a_j^T \|a_j\|_2^2 := |w_j| S^3(J, j),
\]
with \(S^3(J, j)\) depending only on index set \(J\) and \(j\). Combining these bounds, we obtain
\[
\|A^* w\|_\infty \geq |w_j| \left( |\|a_j\|_2^2 - S^3(J, j) | - |S^2(w) | \right).
\]
Further, we proceed with separate bounds for each of the obtained terms. For each \(j \in \{m\}\), we can expand \(\|a_j\|_2^2\) as a sum of independent \(|CN(0, 1)|^2\) random variables with expectation 1, that is
\[
\|a_j\|_2^2 = \sum_{\ell=1}^n |(a_j)_\ell|^2
\]
Consider now the random events
\[
E_{a_j} := \{ |\|a_j\|_2^2 - n| < n/2 \},
\]
and
\[
E_{\|\cdot\|} := \left\{ \max_{j \in \{m\}} |\|a_j\|_2^2 < \frac{3n}{2} \text{ and } \min_{j \in \{m\}} |\|a_j\|_2^2 > \frac{n}{2} \right\}.
\]
On \(E_{a_j}\), it holds that \(n/2 \leq |\|a_j\|_2^2 \leq 3n/2\) and observe that \(E_{\|\cdot\|} = \cap_{j=1}^m E_{a_j}\). By Proposition 10, the probability of \(E_{a_j}^C\) is bounded from above by
\[
\mathbb{P}(E_{a_j}^C) \leq 2 \exp \left\{ -c \min \left\{ \frac{n^2}{4n^2}, \frac{n}{2} \right\} \right\} = 2 \exp \{ -cn \}.
\]
Then, using union bound and De Morgan’s law, we obtain
\[
\mathbb{P}(E_{\|\cdot\|}) = 1 - \mathbb{P}(E_{\|\cdot\|})^{\cap} \geq 1 - \mathbb{P} \left( \bigcup_{j=1}^m E_{a_j}^C \right) \geq 1 - \sum_{j=1}^m \mathbb{P}(E_{a_j}^C) \geq 1 - 2m \exp \{ -cn \}.
\]
We finally note that on \(E_{\|\cdot\|}\), it holds that
\[
\|a_j\|_2^2 \geq \min_{j \in \{m\}} \|a_j\|_2^2 > \frac{n}{2}.
\]
For upper bound on \(S^3(J, j)\), we follow similar steps. When \(a_j\) is fixed, the \(a_k \|a_j\|_2^2, k \in J, k \neq j\) are independent random variables distributed as \(CN(0, 1)\) distribution, since they are projections of complex Gaussian random vectors \([19]\). Hence, \(S^3(J, j)\) is a sum of independent \(|CN(0, 1)|^2\) random variables. Consider a random event
\[
E_{S^3,J,j} := \{|S^3(J, j) - \mathbb{E}S^3(J, j)| < m^{7/8}\}.
\]
The expectation \(\mathbb{E}S^3(J, j)\) is a number of summands in \(S^3\), that is \(|J| - 1\). Therefore, on the event \(E_{S^3,J,j}\), it holds that
\[
S^3(J, j) < |J| - 1 + m^{7/8} \leq 2m^{7/8}.
\]
The probability of its complement can be again bounded by Proposition 10. More precisely,

\[ \Pr \left( E^C_{S^3, J, j} \mid a_j \right) \leq 2 \exp \left\{ -C \min \left\{ \frac{m^{7/4}}{4(|J| - 1)}, \frac{m^{7/8}}{2} \right\} \right\} \]

\[ \leq 2 \exp \left\{ -C \min \left\{ \frac{m^{7/4}}{4m^{3/4}}, \frac{m^{7/8}}{2} \right\} \right\} = 2 \exp \left\{ -Cm^{7/8} \right\}. \]

Integrating out \( a_j \) leads to the bound for the unconditional probability

\[ \Pr \left( E^C_{S^3, J, j} \right) = \int_{a_j} \Pr \left( E^C_{S^3, J, j} \mid a_j \right) d\Pr(a_j) \leq 2 \exp \left\{ -Cm^{7/8} \right\}. \]

The exponent of the obtained tail probability has order \( m^{7/8} \) which is less than \( m \) (dimension of \( w \)). It makes it impossible to apply covering argument (for details see Section 4.1.5). However, the random event \( E_{S^3, J, j} \) depends only on the choice of the index set \( J \) and the index \( j \). Therefore, we can consider all possible selections of \( J \) and \( j \) and resulting events \( E_{S^3, J, j} \). Define a random event

\[ E_{S^3} := \bigcap_{J \subset [m]} \bigcap_{|J| \leq m^{3/4}} \bigcup_{j \in J} E_{S^3, J, j}. \]

Again, by union bound and De Morgan’s law, the probability of \( E_{S^3} \) is bounded from below as

\[ \Pr \left( E_{S^3} \right) = 1 - \Pr \left( \bigcup_{J \subset [m]} \bigcup_{|J| \leq m^{3/4}} \bigcup_{j \in J} E_{S^3, J, j} \right) \geq 1 - \sum_{J \subset [m]} \sum_{|J| \leq m^{3/4}} \Pr(E^C_{S^3, J, j}). \]

The total number of all non-empty subsets of \([m]\) with cardinality up to \([m^{3/4}]\) is given by

\[ \sum_{r=1}^{m^{3/4}} \binom{m}{r} \leq \sum_{r=1}^{m^{3/4}} \frac{m^r}{r!} \leq \sum_{r=1}^{m^{3/4}} m^r \leq m \frac{m^{m^{3/4}}}{m - 1} \leq 2m^{3/4}. \]

Thus, returning to probability we obtain

\[ \Pr \left( E_{S^3} \right) \geq 1 - 2m^{3/4} |J| \cdot 2 \exp \left\{ -Cm^{7/8} \right\} \geq 1 - 4m^{m^{3/4} + 1} \exp \left\{ -Cm^{7/8} \right\}, \]

so the order in the exponent

\[ -Cm^{7/8} + (m^{3/4} + 1) \log m \]

is negative for sufficiently large \( m \). The proof for the last part yet again follows the same logic. For fixed \( a_j \), sum \( S^2(w) \) is a weighted sum of independent \( |CN(0, 1)|^2 \) random variables. Its expectation is the sum of weights, that is

\[ \mathbb{E}S^2(w) = \sum_{k \in [m] \setminus J} w_k \quad \text{and} \quad ||\mathbb{E}S^2(w)|| \leq \sum_{k \in [m] \setminus J} |w_k| \leq ||w||_1 \leq \sqrt{m}. \]

Consider a random event

\[ E_{2,1}(w) := \{|S^2(w) - \mathbb{E}S^2(w)| < n/16m^{1/4}\}. \]
Under $E_{2,1}(w)$, by triangle inequality and assumptions on $m$, it holds that

$$|S^2(w)| \leq |S^2(w) - \mathbb{E}S^2(w)| + |\mathbb{E}S^2(w)| < \frac{n}{16m^{1/4}} + \sqrt{m} \leq \frac{n}{8m^{1/4}}$$  \hfill (24)

In order to apply Proposition 10 consider a vector $\tilde{w}$ defined as

$$\tilde{w}_k = \begin{cases} w_k, & k \in [m] \setminus J, \\ 0, & k \in J. \end{cases}$$

Then, by definition of $J$, $\tilde{w}$ satisfies

$$\|\tilde{w}\|_2 \leq 1 \text{ and } \|\tilde{w}\|_\infty \leq m^{-3/8}.$$  

Hence, the conditional probability of the complement of $E_{2,1}(w)$ is bounded from above as

$$\mathbb{P}\left( E_{2,1}^c(w) \mid a_j \right) \leq 2 \exp \left\{ -\tilde{C} \min \left\{ \frac{n^2}{128m^{1/2} \|\tilde{w}\|_2^2}, \frac{n}{16m^{1/4} \|\tilde{w}\|_\infty} \right\} \right\}$$

$$\leq 2 \exp \left\{ -\tilde{C} \min \left\{ \frac{n^2}{128m^{1/2}}, \frac{n}{16m^{1/4}m^{-3/8}} \right\} \right\}$$

$$\leq 2 \exp \left\{ -\tilde{C} n m^{1/8} \min \left\{ \frac{n}{m^{5/8}}, 1 \right\} \right\} \leq 2 \exp \left\{ -\tilde{C} n m^{1/8} \right\},$$

where in the last inequality we used that $n \geq 8m^{7/8} \geq 8m^{5/8}$. Integrating out $a_j$ grants us

$$\mathbb{P}\left( E_{2,1}^c(w) \right) = \int_{a_j} \mathbb{P}\left( E_{2,1}^c(w) \mid a_j \right) d\mathbb{P}(a_j) \leq 2 \exp \left\{ -\tilde{C} n m^{1/8} \right\}.$$  

Finally, we define a random event $E_{2,2} := E_{\|\cdot\\|} \cap E_{S^3}$ with tail probability

$$\mathbb{P}\left( E_{2,2}^c \right) = \mathbb{P}\left( E_{\|\cdot\\|}^c \cup E_{S^3}^c \right) \leq 2m \exp \left\{ -cn \right\} + 4m^{3/4+1} \exp \left\{ -Cm^{7/8} \right\}.$$  

Then, under $E_{2,1}(w) \cap E_{2,2}$ all established bounds (22), (23), (24) hold and we can return to the Inequality (21). Hence, it holds that

$$\|A^*w\|_\infty \geq |w_j| \left( \|a_j\|_2^2 - S^2(J, j) \right) - |S^2(w)| > |w_j| \left( \frac{n}{2} - 2m^{7/8} \right) - \frac{n}{8m^{1/4}}$$

$$\geq |w_j| \frac{n}{4} - \frac{n}{8m^{1/4}} \geq \frac{n}{4m^{1/4}} - \frac{n}{8m^{1/4}} = \frac{n}{8m^{1/4}},$$

where we used condition on $m$ in the second inequality and the choice of $w_j$ in the last inequality. Finally, note that

$$\|A^*w\|_\infty > \frac{n}{8m^{1/4}} \geq \frac{n}{64 \sqrt{2m}},$$

which concludes the proof. \hfill $\square$
4.1.4 Flat vectors

The last case is when the mean of \( \omega \) is not big enough, so we cannot proceed and in Section 4.1.2 and at the same time there is no significantly big entries to compensate the rest as in Section 4.1.3. Therefore, the idea is to do something in between these two proof approaches, separate several relatively big entries which will have bigger impact than the rest. Our main result in this section is the following.

**Theorem 13.** Let \( \omega \in \mathbb{R}^m, \| \omega \|_2 = 1 \) and assume that \( \omega \) satisfies

\[
\left| \sum_{k=1}^{m} w_k \right| < \frac{n}{32 \sqrt{2m}} \quad \text{and} \quad \| \omega \|_\infty < m^{-1/4}.
\]

Suppose that the number of measurements \( m \) satisfies

\[ m \leq (n/16)^{4/3} \quad \text{and} \quad m \text{ is sufficiently large}. \]

Then, there exists a random event \( E_{3,1}(\omega) \) depending on \( \omega \) and random event \( E_{3,2} \) independent of \( \omega \) with tail probabilities

\[
\mathbb{P} \left( E_{3,1}(\omega) \right) \leq \exp \left\{ -\tilde{C} nm^{1/4} \right\} \quad \text{and} \quad \mathbb{P} \left( E_{3,2} \right) \leq 4m^{2m/8} \exp \left\{ -Cn \right\}
\]

such that on \( E_{3,1}(\omega) \cap E_{3,2} \), Inequality (19) holds.

**Proof.** As in the previous two cases, we start with the representation (20). Similarly to the spiky case, we want to select vector providing high magnitudes \( \omega_j \). Since, \( \| \omega \|_\infty < m^{-1/4} \) its effect is not strong enough to compensate the rest of the entries with sufficiently high probability. However, it is possible to apply independence argument similar to the one used in the proof of the first case. Let us first introduce the set from which we will select \( \omega_j \)'s by defining the index set \( I \) as

\[ I = I(\omega) := \left\{ k \in [m] \mid |w_k| > 1/\sqrt{2m} \right\}. \]

Using the bound on infinity norm, we obtain the following bound

\[ 1 = \sum_{k \in [m]} |w_k|^2 = \sum_{k \in I} |w_k|^2 + \sum_{k \in [m] \setminus I} |w_k|^2 \leq |I| \frac{1}{\sqrt{m}} + m \frac{1}{2m} = \frac{1}{2} + \frac{|I|}{\sqrt{m}}. \]

Hence, the cardinality of \( I \) is bounded from below as \( |I| \geq \sqrt{m}/2 \). Since entries of \( \omega \) are real, they either satisfy \( \omega_j > 0 \) or \( \omega_j \leq 0 \). Consider \( I_+ \) and \( I_- \), subsets of \( I \) with all positive and negative entries, so that

\[ I_+ = I_+(\omega) := \{ j \in I, \omega_j > 0 \} \quad \text{and} \quad I_- = I_-(\omega) := \{ j \in I, \omega_j < 0 \}. \]

Since \( |I_+| + |I_-| = |I| \), one of the sets has at least half of the entries of \( I \). Without loss of generality, let \( |I_+| \geq \lceil \sqrt{m}/2 \rceil \geq \lceil \sqrt{m}/4 \rceil \). Otherwise, for the rest of the proof consider \( \nu = -\omega \) satisfying \( \| \mathcal{A}^* \nu \|_\infty = \| \mathcal{A}^* \omega \|_\infty \) with \( |I_+(\nu)| \geq |I_-(\nu)| \). Finally, we select an subset \( L \) of \( I_+ \) with cardinality \( |L| = \lceil \sqrt{2m}/16 \rceil \leq \lceil \sqrt{m}/4 \rceil \) with indices sorted in increasing order.

Now, we introduce a lower bound for representation (20) as

\[
\| \mathcal{A}^* \nu \|_\infty = \max_{\nu \in \mathbb{C}^n, \| \nu \|_2 = 1} \left| \sum_{k=1}^{m} w_k |a_k^* \nu|^2 \right| \geq \max_{j \in L} \left| \sum_{k=1}^{m} w_k |a_k^* \tilde{a}_{kj}|^2 \right|
\]
where $GS := \{\tilde{a}_j\}_{j \in L}$ is the Gram-Schmidt orthogonalization of the vectors $\{a_j\}_{j \in L}$ according to the order in $L$. With notation

$$S_j^4(w) := \sum_{k \in [m] \setminus \ell} w_k |a_k^* \tilde{a}_j|^2,$$

the bound can be further elaborated as

$$\|A^* w\|_\infty \geq \max_{j \in L} \sum_{k=1}^m w_k |a_k^* \tilde{a}_j|^2 = \max_{j \in L} \left[ \sum_{k \in [m] \setminus \ell} w_k |a_k^* \tilde{a}_j|^2 + \sum_{k \in \ell} w_k |a_k^* \tilde{a}_j|^2 \right]$$

$$\geq \max_{j \in L} S_j^4(w) + \min_{j \in L} \sum_{k \in \ell} w_k |a_k^* \tilde{a}_j|^2 \geq \max_{j \in L} S_j^4(w) + \min_{j \in L} w_j |a_j^* \tilde{a}_j|^2$$

$$\geq \max_{j \in L} S_j^4(w) + \frac{1}{\sqrt{2m}} \min_{j \in L} |a_j^* \tilde{a}_j|^2,$$

where we used properties of $L$ in the last two inequalities.

We note the first term can be treated similarly to the $\max_{j \in [m]} S_j^4(w)$ in the proof of Theorem 11. Let vectors in $GS$ to be fixed. Then, consider two sums $S_j^4$ and $S_\ell^4$ for $j, \ell \in L, j \neq \ell$. The sums are independent of each other, since $a_k^* \tilde{a}_j$ is independent of $a_r^* \tilde{a}_\ell$ for $k, r \in [m] \setminus L, k \neq r$ and $a_k^* \tilde{a}_j$ is independent of $a_k^* \tilde{a}_\ell$ as a projections on orthogonal directions for all $k \in [m] \setminus L$ [19]. The summands within the single sum are independent due to the independence of the $a_k$’s and follow a $|CN(0, 1)|^2$ distribution. Hence, we introduce a random event

$$E_{S_j^4}(w) := \left\{ S_j^4(w) > \sum_{k \in [m] \setminus \ell} w_k - \frac{n}{16 \sqrt{2m}} \right\}.$$

The tail probability of $E_{S_j^4}(w)$ bounded from above by Proposition 10 and assumptions on $w$ and $m$ as

$$\mathbb{P}\left( E_{S_j^4}(w) \mid GS \right) \leq \exp\left\{ -\tilde{c} \min \left\{ \frac{n^2}{512m \|w\|_2^2}, \frac{n}{16 \sqrt{2m} \|w\|_\infty} \right\} \right\}$$

$$\leq \exp\left\{ -\tilde{c} \min \left\{ \frac{n^2}{512m^2}, \frac{n}{16 \sqrt{2m}} \right\} \right\}$$

$$\leq \exp\left\{ -\tilde{c} \min \left\{ \frac{n}{m^{1/4}}, \frac{n}{16m^{3/4}} \right\} \right\} \leq \exp\left\{ -\tilde{c} \frac{n}{m^{1/4}} \right\}.$$

By integrating out random variables $GS$, we obtain bound for the unconditional probability

$$\mathbb{P}\left( E_{S_j^4}(w) \right) = \int_{\Omega_{GS}} \mathbb{P}\left( E_{S_j^4}(w) \mid GS \right) d\mu_{GS} \leq \exp\left\{ -\tilde{c} \frac{n}{m^{1/4}} \right\},$$

where $(\Omega_{GS}, \mu_{GS})$ denotes probability space generated by $GS$. Finally, we define a random event

$$E_{3,1}(w) := \left\{ \max_{j \in \Omega_{GS}} S_j^4(w) > \sum_{k \in [m] \setminus \ell} w_k - \frac{n}{16 \sqrt{2m}} \right\} = \bigcup_{j \in \ell} E_{S_j^4}(w)$$

(26)
with tail probability

$$\mathbb{P} \left( E_{3,1}^C (w) \right) = \mathbb{P} \left( \bigcap_{j \in \mathcal{L}} E_{S_j}^C (w) \right) = \prod_{j \in \mathcal{L}} \mathbb{P} \left( E_{S_j}^C (w) \right) \leq \left( \exp \left\{ - \frac{\tilde{C}n}{m^{1/4}} \right\} \right)^{|\mathcal{L}|} \leq \exp \left\{ -\tilde{C}nm^{1/4} \right\} .$$

Turning to the second term in Inequality (25), \( \min_{j \in \ell} |a_j^* \tilde{a}_j| \), we recall that \( \{ \tilde{a}_k \}_{k \in \ell} \) is an orthonormal system obtained by Gram-Schmidt orthogonalization. Hence, \( |a_j^* \tilde{a}_j|^2 \) can be expressed by the Pythagorean theorem as

$$|a_j^* \tilde{a}_j|^2 = \|a_j\|^2 - \sum_{k \in \ell, k \neq j} |a_j^* \tilde{a}_k|^2 = \|a_j\|^2 - \sum_{k \in \ell, k < j} |a_j^* \tilde{a}_k|^2 =: \|a_j\|^2 - S_j^5 (L),$$

where in the second equality we used that \( a_j \in \text{span} \{ \tilde{a}_1, \ldots, \tilde{a}_j \} \) by construction. In the proof of Theorem 12, we showed that for a random event

$$E_{a_j} := \{|\|a_j\|^2 - n| < n/2\} \text{ it holds that } \mathbb{P} \left( E_{a_j}^C \right) \leq 2 \exp \{|-cn\}.\right.$$ 

The sum \( S_j^5 (L) \) is again a sum of independent \( \mathcal{CN}(0,1)^2 \) distributed random variables when \( S_j := \{ \tilde{a}_1, \ldots, \tilde{a}_{j-1} \} \) are fixed. Hence, a random event

$$E_{S_j}^5 (L) := \{|S_j^5 (L) - ES_j^5 (L)| < n/4\}$$

has tail probability

$$\mathbb{P} \left( E_{S_j}^C (L) \mid S_j \right) \leq 2 \exp \left\{ -C \min \left\{ \frac{n^2}{16|L|}, \frac{n}{4} \right\} \right\} \leq 2 \exp \left\{ -C \min \left\{ \frac{n^2}{2 \sqrt{2m}}, \frac{n}{4} \right\} \right\} \leq 2 \exp \left\{ -Cn \min \left\{ \frac{n}{\sqrt{m}}, 1 \right\} \right\} \leq 2 \exp \{|-Cn\},$$

where in the last inequality we used that \( n \geq 16m^{3/4} \geq 16\sqrt{m} \). Again, by integrating out \( S_j \), unconditional tail probability is

$$\mathbb{P} \left( E_{S_j}^C (L) \right) \leq 2 \exp \{|-Cn\}$$

Under \( E_{a_j} \cap E_{S_j}^5 (L) \) it holds that

$$|a_j^* \tilde{a}_j|^2 = \|a_j\|^2 - S_j^5 (L) > \frac{n}{2} - \frac{n}{4} - \mathbb{E} S_j^5 (L) = \frac{n}{4} - |L| \geq \frac{n}{4} - \frac{\sqrt{m}/4}{2} \geq \frac{n}{4} - \frac{\sqrt{m}^2}{16} \geq \frac{n}{4} - \frac{n^{2/3} - 22}{16} \geq \frac{n}{4} - \frac{n}{16^{1/2}} = \frac{n}{8}.\$$

Then, a random event

$$E_{a_j} (L) := \{|a_j^* \tilde{a}_j|^2 > \frac{n}{8}\} \supseteq E_{a_j} \cap E_{S_j}^5 (L)$$

has tail probability bounded from above by a union bound as

$$\mathbb{P} \left( E_{a_j}^C (L) \right) \leq \mathbb{P} \left( E_{a_j}^C \cup E_{S_j}^C (L) \right) \leq \mathbb{P} \left( E_{a_j}^C \right) + \mathbb{P} \left( E_{S_j}^C (L) \right) \leq 4 \exp \{|-Cn\}.\right.$$
Consequently, a random event
\[
E_{\text{min}}(L) := \left\{ \min_{j \in L} |\alpha_j^* \tilde{a}_j|^2 > \frac{n}{8} \right\} = \bigcap_{j \in L} E_{\tilde{a}_j}(L) \tag{27}
\]
has tail probability bounded from above by a union bound as
\[
\mathbb{P} \left( E_{\text{min}}^c(L) \right) = \mathbb{P} \left( \bigcup_{j \in L} E_{\tilde{a}_j}^c(L) \right) \leq \sum_{j \in L} \mathbb{P} \left( E_{\tilde{a}_j}^c \right) \leq 4|L| \exp \{-Cn\} .
\]

Since order in the exponent is \( n \leq m \), it is insufficient to apply the covering argument. However, \( \min_{j \in L} |\alpha_j^* \tilde{a}_j|^2 \) is independent of entries of \( w \) and only depends on a choice of an index set \( L \). The number of all possible choices of the subset \( L \) is given by
\[
\left( \frac{m}{|L|} \right) \leq \frac{m^{1/|L|}}{(|L|)!} \leq \frac{m^{\sqrt{2m}/8}}{(|L|)!} .
\]

Thus, a random event
\[
E_{3,2} := \bigcap_{L \subset [m], |L| = \sqrt{2m}/16} E_{\tilde{a}_j}(L)
\]
has tail probability
\[
\mathbb{P} \left( E_{3,2}^c \right) = \mathbb{P} \left( \bigcup_{L \subset [m], |L| = \sqrt{2m}/16} E_{\text{min}}^c(L) \right) \leq \sum_{L \subset [m], |L| = \sqrt{2m}/16} \mathbb{P} \left( E_{\text{min}}^c(L) \right) \leq 4 \left( \frac{|L|m^{\sqrt{2m}/8}}{(|L|)!} \right) \exp \{-Cn\} \leq 4m^{\sqrt{2m}/8} \exp \{-Cn\}
\]
The order of the exponent can be bounded from above as
\[
-Cn + \frac{\sqrt{2m}}{8} \log m \leq -16Cm^{3/4} + \frac{\sqrt{2m}}{8} \log m < 0
\]
for sufficiently large \( m \).

Last step is to observe that due to inequalities \( m^{3/4} \geq n/16 \) and \( \|w\|_{\infty} \leq m^{-1/4} \) it holds that
\[
\sum_{k \in L} w_k \leq |L|m^{-1/4} = \left[ \frac{\sqrt{2m}}{16} \right] m^{-1/4} \leq \frac{\sqrt{2m}}{8m^{1/4}} = \frac{\sqrt{2m^{1/4}}}{8} = \frac{2m^{3/4}}{8\sqrt{2m}} \leq \frac{n}{64\sqrt{2m}} . \tag{28}
\]

Returning to the Inequality (25), under \( E_{3,1}(w) \cap E_{3,2} \) in the view of Inequalities (26), (27), (28) and assumption on \( w \), we obtain
\[
\|A^* w\|_{\infty} \geq \max_{j \in L} \xi_j^c(w) + \frac{1}{\sqrt{2m}} \min_{j \in L} |\alpha_j^* \tilde{a}_j|^2 > \sum_{k \in [m] \setminus L} w_k - \frac{n}{16\sqrt{2m}} + \frac{n}{8\sqrt{2m}}
\]
\[
= \sum_{k=1}^{m} w_k - \sum_{k \in L} w_k + \frac{n}{16\sqrt{2m}} > \frac{n}{32\sqrt{2m}} - \frac{n}{64\sqrt{2m}} + \frac{n}{16\sqrt{2m}} = \frac{n}{64\sqrt{2m}},
\]
which concludes the proof. \( \square \)
4.1.5 Combining the results

All three cases can be united as a single statement.

**Corollary 14.** Let \( w \in \mathbb{R}^m \), \( \|w\|_2 = 1 \). Suppose that the number of measurements \( m \) satisfies

\[
m \leq \min \{ (n/16)^{4/3}, (n/8)^{8/7}, n^2 \}
\]

and \( m \) is sufficiently large.

Then, there exists a random event \( E_1(w) \) depending on \( w \) and random event \( E_2 \) independent of \( w \) with tail probabilities

\[
\mathbb{P} \left( E_1^c(w) \right) \leq 2 \exp \left\{ -\bar{C}nm^{1/8} \right\},
\]

and

\[
\mathbb{P} \left( E_2^c \right) \leq 8m^{\sqrt{2m}/8} \exp \{ -cn \} + 4m^{3/4+1} \exp \left\{ -Cm^{7/8} \right\}
\]

such that on \( E_1(w) \cap E_2 \), Inequality (19) holds.

**Proof.** The proof of the corollary is obtained by the comparison of the tail of the corresponding events in Theorems 11 to 13. The random event \( E_1(w) \) is either \( E_{1,1}(w) \), \( E_{2,1}(w) \) or \( E_{3,1}(w) \) depending on \( w \). Let \( \bar{C} \) be the smallest of the constants in the tail probabilities of these random events. Due to the condition \( 16m^{3/4} \leq n \), the following line of inequalities holds

\[
nm^{1/8} \leq nm^{1/4} = nm^{3/4}/\sqrt{m} \leq n^2/(16\sqrt{m}) \leq n^2/\sqrt{m},
\]

and hence tail probability of the event \( E_1(w) \) is bounded from above as

\[
\mathbb{P} \left( E_1^c(w) \right) \leq \max \left\{ \mathbb{P} \left( E_{1,1}^c(w) \right), \mathbb{P} \left( E_{2,1}^c(w) \right), \mathbb{P} \left( E_{3,1}^c(w) \right) \right\} \leq 2 \exp \left\{ -\bar{C}nm^{1/8} \right\}.
\]

We set random event \( E_2 \) as an intersection of events \( E_{2,2} \) and \( E_{3,2} \) and its tail probability is bounded from above by De Morgan’s law and union bound, that is

\[
\mathbb{P} \left( E_2^c \right) = \mathbb{P} \left( E_{2,2}^c \cup E_{3,2}^c \right) \leq \mathbb{P} \left( E_{2,2}^c \right) + \mathbb{P} \left( E_{3,2}^c \right)
\]

\[
\leq 2 \exp \{ -cn \} + 4m^{3/4+1} \exp \left\{ -Cm^{7/8} \right\} + 4m^{\sqrt{2m}/8} \exp \{ -cn \}
\]

\[
\leq 8m^{\sqrt{2m}/8} \exp \{ -cn \} + 4m^{3/4+1} \exp \left\{ -Cm^{7/8} \right\},
\]

where constant \( c \) is chosen to be smaller of two constants in the corresponding exponentials. \( \square \)

Finally, we extend our results for a single vector \( w \) to a uniform bound, which hold for all \( w \) simultaneously and grant us the \( S_1 \)-quotient property of the scaled measurement operator \( \frac{1}{\sqrt{m}}A \).

**Proof of Theorem 5.** In order to extend the obtained results for all \( w, \|w\|_2 = 1 \), we will consider an \( \delta \)-covering (net) \( C \) of the unit ball \( B := \{ w \in \mathbb{R}^m, \|w\|_2 = 1 \} \), that is for each \( w \in B \) exist vector \( \bar{w} \) in \( C \) such that

\[
\|w - \bar{w}\|_2 \leq \delta.
\]

The radius \( \delta \) is chosen to be \( \frac{1}{192\sqrt{2m}} \). The covering \( C \) is a finite subset of \( B \) and its cardinality is bounded from above by Corollary 4.2.13 in [64] as

\[
|C| \leq (1 + 2/\delta)^m = (1 + 384\sqrt{2m})^m \leq (1 + 544.1m)^m \leq (545m)^m.
\]
First, we note that under conditions $c_2 m^{7/8} \log 545m \leq n$ and $m$ being sufficiently large all inequalities $m \leq (n/8)^{8/7}$, $m \leq (n/16)^{3/3}$ and $m < n^2$ are satisfied and thus Corollary 14 can be applied for $\tilde{\omega} \in C$. Further, Inequality (19) can be extended to hold for all $\tilde{\omega} \in C$ simultaneously. That is the tail probability of the random event $E_C$ defined as

$$E_C := \left\{ \frac{1}{\sqrt{m}} \mathcal{A}^* \tilde{\omega} \right\}_\infty \geq \frac{n}{64\sqrt{2m}}$$

for all $\tilde{\omega} \in C$ is bounded from above by De Morgan's laws and union bound

$$\mathbb{P} \left( E_C^c \right) = \mathbb{P} \left( \bigcup_{\tilde{\omega} \in C} E_1(\tilde{\omega})^c \cup E_2^c \right) \leq \sum_{\tilde{\omega} \in C} \mathbb{P} \left( E_1(\tilde{\omega})^c \right) + \mathbb{P} \left( E_2^c \right)$$

$$\leq 2(545m)^m \exp \left\{ -\bar{c} nm^{1/8} \right\} + \mathbb{P} \left( E_2^c \right)$$

$$\leq 2(545m)^m \exp \left\{ -\bar{c} nm^{1/8} \right\} + 8m^{2m/8} \exp \{-cn\} + 4m^{m/4+1} \exp \{-Cm^{7/8}\}.$$  

Note that if $c_2$ is selected to satisfy $c_2 > \bar{C}^{-1}$, then there exists $\gamma_2 > 0$ such that $c_2 = (1 + \gamma_2)\bar{C}^{-1}$ and due to condition $c_2 m^{7/8} \log 545m \leq n$ the first term transforms as

$$\exp \left\{ m \log(545m) - \bar{C} nm^{1/8} \right\} \leq \exp \left\{ -\gamma_2 m \log(545m) \right\} \leq \exp \left\{ -\gamma_2 m^{7/8} \right\}.$$  

Similarly, the second and the third terms are bounded from above as

$$\exp \left\{ (\sqrt{2m/8}) \log m - cn \right\} \leq \exp \left\{ (\sqrt{2m/8}) \log m - c_2 m^{7/8} \log 545m \right\} \leq \exp \left\{ -\gamma_2 m^{7/8} \right\} ,$$

and

$$\exp \left\{ m^{3/4} \log m + \log m - Cm^{7/8} \right\} \leq \exp \left\{ -\gamma_2 m^{7/8} \right\} ,$$

respectively, for a sufficiently large $m$. Thus, the tail probability of $E_C$ is bounded from above by

$$\mathbb{P} \left( E_C^c \right) \leq 14 \exp \left\{ -\gamma_2 m^{7/8} \right\} .$$  

Final step is to use the properties of the covering under event $E_C$ to derive the uniform bound. Let $\omega \in B$ be arbitrary vector. Then, there exists $\tilde{\omega} \in C$ such that $\|\omega - \tilde{\omega}\|_2 \leq \delta$ and we bound the spectral norm $\left\| \frac{1}{\sqrt{m}} \mathcal{A}^* \omega \right\|_\infty$ as

$$\left\| \frac{1}{\sqrt{m}} \mathcal{A}^* \omega \right\|_\infty = \left\| \frac{1}{\sqrt{m}} \mathcal{A}^* (\omega - \tilde{\omega} + \tilde{\omega}) \right\|_\infty \geq \left\| \frac{1}{\sqrt{m}} \mathcal{A}^* \tilde{\omega} \right\|_\infty - \left\| \frac{1}{\sqrt{m}} \mathcal{A}^* (\omega - \tilde{\omega}) \right\|_\infty.$$  

Under $E_C$ it holds that $\left\| \frac{1}{\sqrt{m}} \mathcal{A}^* \tilde{\omega} \right\|_\infty \geq \frac{n}{64\sqrt{2m}}$ and hence what remains is an upper bound for the second term. Using Lemma 9, we bound it as

$$\left\| \frac{1}{\sqrt{m}} \mathcal{A}^* (\omega - \tilde{\omega}) \right\|_\infty = \left\| \frac{1}{\sqrt{m}} \sum_{k=1}^n (\omega - \tilde{\omega})_k a_k a_k^* \right\|_\infty \leq \frac{1}{\sqrt{m}} \sum_{k=1}^n |(\omega - \tilde{\omega})_k| \|a_k a_k^*\|_\infty$$

$$\leq \frac{1}{\sqrt{m}} \sum_{k=1}^n |(\omega - \tilde{\omega})_k| \|a_k\|_2^2 \leq \frac{1}{\sqrt{m}} \|\omega - \tilde{\omega}\|_1 \max_{k \in [n]} \|a_k\|_2^2$$

$$\leq \|\omega - \tilde{\omega}\|_2 \frac{3n}{2} \leq \frac{3n\delta}{2} = \frac{n}{128\sqrt{2m}},$$
where we used that by construction
\[ E_C \subseteq E_2 \subseteq E_{2,2} \subseteq E_{\|u\|} = \left\{ \max_{j \in [m]} \|a_j\|_2 < \frac{3n}{2} \text{ and } \min_{j \in [m]} \|a_j\|_2 > \frac{n}{2} \right\}. \]

Thus, we obtain that under \( E_C \) for all \( w \) such that \( \|w\|_2 = 1 \) it holds that
\[ \left\| \frac{1}{\sqrt{m}} A^* w \right\|_\infty \geq \left\| \frac{1}{\sqrt{m}} A^* \tilde{w} \right\|_\infty - \left\| \frac{1}{\sqrt{m}} A^*(w - \tilde{w}) \right\|_\infty \geq \frac{n}{64\sqrt{2}m} - \frac{n}{128\sqrt{2}m} = \frac{n}{128\sqrt{2}m}. \]

The final term can be split as
\[ \frac{n}{128\sqrt{2}m} = \left( \frac{\kappa}{128\sqrt{2}} \sqrt{\frac{n}{\kappa m}} \right) \sqrt{\frac{n}{\kappa m}}. \]

Then, by Proposition 8, the scaled measurement operator \( \frac{1}{\sqrt{m}} A \) possess the \( S_1 \)-quotient property with constant \( \frac{128\sqrt{2}}{\kappa} \sqrt{\kappa m/n} \) and rank \( \kappa m/n \) relative to the norm \( \|\cdot\|_2 \). \( \Box \)

### 4.2 Proof of Theorem 6

Theorem 6 is a consequence of applying Theorem 7.

**Proof of Theorem 6.** This proof is essentially joins the results. By Theorem 3 and the choice of rank \( r_* = c^{-1}_1 \rho^2 m/n \), the scaled measurement operator \( \frac{1}{\sqrt{m}} A \) satisfies the \( S_2 \)-rank robust null space property of order \( r_* \), with constants \( 0 < \rho < 1 \) and \( \tau > 0 \) relative to the \( \|\cdot\|_2 \) on \( \mathbb{R}^m \) with probability at least \( 1 - e^{-y_1 m} \). By Theorem 5 with \( \kappa = c_1^2 \rho^2, \frac{1}{\sqrt{m}} A \) posses the \( S_1 \)-quotient property with constant \( \frac{128\sqrt{2}}{\rho^2} \sqrt{\tau} \) and rank \( r_* \) relative to the norm \( \|\cdot\|_2 \) on \( \mathbb{R}^m \) with probability at least \( 1 - 14e^{-y_2 m^{7/8}} \). Thus, by union bound, it satisfies both properties simultaneously with probability at least
\[ 1 - e^{-y_1 m} - 14e^{-y_2 m^{7/8}} \geq 1 - e^{-y m} - 14e^{-y m^{7/8}} = 1 - 15e^{-y m^{7/8}}, \]

with \( \gamma := \min\{y_1, y_2\} \). By Proposition 2 the reconstruction program \( \Delta \) admits the error bound (7) in the noiseless case \( (\eta = 0) \). Hence, by Theorem 7 we obtain
\[ \left\| X - \Delta \left( \frac{1}{\sqrt{m}} AX + \frac{1}{\sqrt{m}} w \right) \right\|_p \leq \frac{D_1}{r_1^{-1}} \left\| X_\epsilon \right\|_1 + (D_2 \frac{128\sqrt{2}c_1}{\rho^2} \sqrt{r_*} + D_3) r_*^{\frac{1}{2} - \frac{1}{2}} \|w\|_2. \]

The last term can be bounded from above as
\[ (D_2 \frac{128\sqrt{2}c_1}{\rho^2} \sqrt{r_*} + D_3) r_*^{\frac{1}{2} - \frac{1}{2}} \leq (D_2 \frac{128\sqrt{2}c_1}{\rho^2} \sqrt{r_*} + D_3 \sqrt{r_*}) r_*^{\frac{1}{2} - \frac{1}{2}} =: D_4 r_*^{\frac{1}{2}}, \]

since \( r_* \geq 1 \). Note that constraint of reconstruction program \( \Delta \) rescales error in the following way
\[ \left\| \frac{1}{\sqrt{m}} AZ - \frac{1}{\sqrt{m}} AX + \frac{1}{\sqrt{m}} w \right\| = 0 \text{ is equivalent to } \|AZ - AX + w\| = 0, \]

and therefore
\[ \left\| X - \Delta (AX + w) \right\|_p \leq \frac{D_1}{r_1^{-1}} \left\| X_\epsilon \right\|_1 + \frac{D_4}{r_1} \left\| w \right\|_2. \]

This concludes the proof for \( \Delta \). The proof for \( \Delta_{2,\eta} \) is analogous. \( \Box \)
4.3 Proof of Theorem 7

The proof of the Theorem 7 closely follows the steps of its sparse recovery analogue Theorem 11.12 in [31].

**Proof of Theorem 7.** Our goal is to bound \( \|X - \mathcal{R}_\eta (\mathcal{A}X + w)\|_p \) from above in terms of \( \|X_c^\ell\| \) and \( \|w\| \) for all \( X \in \mathcal{H}_\eta \) and \( w \in \mathbb{R}^m \). In the first case, when \( \|w\| \leq \eta \), by assumption we have

\[
\|X - \mathcal{R}_\eta (\mathcal{A}X + w)\|_p \leq \frac{D_1}{r_*^{1-1/p}} \|X_c^\ell\|_1 + \frac{2\tau (3 + \rho)}{1 - \rho} r_*^{1/p-1/2} \max\{\|w\|, \eta\},
\]

with \( D_1 := \frac{2(1+\rho)^2}{(1-\rho)} \).

In the second case, let \( \|w\| > \eta \) and consider decomposition

\[
w = \frac{\eta}{\|w\|} w + \left(1 - \frac{\eta}{\|w\|}\right) w =: w_\eta + v.
\]

Note that \( \|w_\eta\| = \eta \) and \( \|v\| = \|w\| - \eta \). Furthermore, by the \( S_1 \)-quotient property there exists matrix \( U \) such that \( \mathcal{A}U = v \). Thus, by triangle inequality, it holds that

\[
\|X - \mathcal{R}_\eta (\mathcal{A}X + w)\|_p = \|X - \mathcal{R}_\eta (\mathcal{A}X + w_\eta + \mathcal{A}U) + U - U\|_p \\
\leq \|X + U - \mathcal{R}_\eta (\mathcal{A}(X + U)) + w_\eta\|_p + \|U\|_p.
\]

For matrix \( X + U \) and noise \( w_\eta \) the first case applies and one obtains the bound

\[
\|X - U + \mathcal{R}_\eta (\mathcal{A}(X + U) + w_\eta)\|_p \leq \frac{D_1}{r_*^{1-1/p}} \|(X + U)^\ell\|_1 + \frac{2\tau (3 + \rho)}{1 - \rho} r_*^{1/p-1/2} \eta.
\]

Further the first norm bounded using the best rank approximation properties (11) of \( (X + U)^\ell\) and triangle inequality, that is

\[
\|(X + U)_{r_*}^\ell\|_1 = \min_{Z \in \mathbb{C}^{n \times n}} \|X + U - Z\|_1 \leq \|U\|_1 + \min_{Z \in \mathbb{C}^{n \times n}} \|X - Z\|_1 = \|U\|_1 + \|X_{r_*}^\ell\|_1.
\]

Consequently, we obtain

\[
\|X - \mathcal{R}_\eta (\mathcal{A}X + w)\|_p \leq \frac{D_1}{r_*^{1-1/p}} \|X_{r_*}^\ell\|_1 + \frac{D_1}{r_*^{1-1/p}} \|U\|_1 + \|U\|_p + \frac{2\tau (3 + \rho)}{1 - \rho} r_*^{1/p-1/2} \eta. \tag{29}
\]

The first term is already a finalized. The second term is bounded by the \( S_1 \)-quotient property as

\[
\frac{D_1}{r_*^{1-1/p}} \|U\|_1 \leq D_1 \|r_*^{1/p-1/2} \|v\|,
\]

and hence, only the third term remains. By Lemma 3.1 in [39], it holds that

\[
\|U_{r_*}^\ell\|_p \leq \|U\|_1.
\]

On the other hand, by Hölder’s inequality and \( S_2 \)-robust rank null space property, it follows that

\[
\|U_{r_*}^\ell\|_p \leq \frac{r_*^{1/p-1/2}}{r_*^{1-1/p}} \|U_{r_*}^\ell\|_2 \leq \frac{\rho r_*^{1/p-1/2}}{r_*^{1-1/p}} \|U_{r_*}^\ell\|_1 + r_*^{1/p-1/2} \tau \|\mathcal{A}U\| \\
\leq \frac{\rho}{r_*^{1-1/p}} \|U\|_1 + r_*^{1/p-1/2} \tau \|v\|.
\]
Thus, by triangle inequality and the $S_1$-quotient property we obtain
\[
\|U\|_p \leq \|U_r\|_p + \|U^c\|_p \leq \frac{\|U\|_1}{r_s^{1-1/p}} + \frac{\rho}{r_s^{1-1/p}} \|U\|_1 + r_s^{1/p-1/2} \|v\| \\
\leq \frac{(1 + \rho)d\sqrt{r_s}}{r_s^{1-1/p}} \|v\| + r_s^{1/p-1/2} \|v\| = (1 + \rho)d + \tau r_s^{1/p-1/2} \|v\|.
\]

Next, we combine established bounds for the second and third terms with the inequality (29), so it holds that
\[
\|X - \mathcal{R}_\eta(AX + w)\|_p \leq \frac{D_1}{r_s^{1-1/p}} \|X^c\|_1 + (D_2d + \tau)d_s^{1/p-1/2} \|v\| + \frac{2\tau(3 + \rho)}{1 - \rho} r_s^{1/p-1/2} \eta,
\]
where $D_2 = D_1 + (1 + \rho)$. Recall that $\|v\| = \|\|w\| - \eta$. Then, we can unite coefficients by $\eta$ as $d_1 := \frac{2\tau(3 + \rho)}{1 - \rho} - (D_2d + \tau)$. If $d_1 \leq 0$ then the $\eta$-term can be neglected. Otherwise we bound $\eta \leq \|w\| = \max\{\|w\|, \eta\}$ and obtain
\[
\|X - \mathcal{R}_\eta(AX + w)\|_p \leq \frac{D_1}{r_s^{1-1/p}} \|X^c\|_1 + d_2 r_s^{1/p-1/2} \max\{\|w\|, \eta\},
\]
where $d_2 := (D_2d + \tau) + \max\{d_1, 0\}$. Finally, we join two cases and select coefficient by $\max\{\|w\|, \eta\}$ as maximum of $d_2$ and $\frac{2\tau(3 + \rho)}{1 - \rho}$. Depending on the sign of $d_1$, $d_2$ is either $D_2d + \tau$ or $\frac{2\tau(3 + \rho)}{1 - \rho}$. Hence, it holds that
\[
\max\left\{d_2, \frac{2\tau(3 + \rho)}{1 - \rho}\right\} \leq \max\left\{D_2d + \tau, \frac{2\tau(3 + \rho)}{1 - \rho}\right\} \leq D_2d + \tau + \frac{2\tau(3 + \rho)}{1 - \rho} =: D_2d + D_3,
\]
and desired inequality
\[
\|X - \mathcal{R}_\eta(AX + w)\|_p \leq \frac{D_1}{r_s^{1-1/p}} \|X^c\|_1 + (D_2d + D_3)d_s^{1/p-1/2} \max\{\|w\|, \eta\}
\]
holds. At last we note that for all $r \leq r_s$, it holds that $1/r_s \leq 1/r$ and $\|X^c\|_1 \leq \|X^c\|_1$, and thus
\[
\|X - \mathcal{R}_\eta(AX + w)\|_p \leq \frac{D_1}{r_s^{1-1/p}} \|X^c\|_1 + (D_2d + D_3)d_s^{1/p-1/2} \max\{\|w\|, \eta\},
\]
which concludes the proof. 

\section{Conclusion and Future work}

In this paper, we established the $S_1$-quotient property for Gaussian rank-one measurements, allowing us to derive noise-blind guarantees for the recovery low-rank matrices from such measurements. We expect that the proof technique we used can help with the development of similar results for other applications with more structure such as randomized blind deconvolution. An additional difficulty of this scenario is that even for noise-aware nuclear norm minimization recovery guarantees analogous to those discussed in Section 1.5 are only possible with additional dimensional scaling factors [42].

For follow-up work, it will be of interest to study to which extent noise-blind recovery guarantees from rank-one measurements via a square-root Lasso as studied in [32] for random noise can also be established under a robust null space property in analogy to the results of [54] that have been achieved for sparse recovery.
Another interesting line of research concerns the refinement of our proofs to get an improvement on the maximum number of measurements admissible in Theorem 5. Our numerical experiments in Section 3 and contributions regarding the quotient property for a different measurement scenario [12] suggest that the optimal bound of a form comparable to $m \leq cn^2/\log^d(m/n)$, potentially leaving room for further improvement of Theorem 5.

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A Appendix

A.1 Proof of Proposition 8

Let $\mathcal{A}$ satisfy the $S_q$-quotient property with constant $d$ and rank $r_*$ relative to $\|\cdot\|$. For $w \in \mathbb{R}^m$ by definition of dual norm we have that

$$\|w\|_* = \sup_{\|v\|=1} \langle v, w \rangle.$$ 

Since $\mathbb{R}^m$ is finite-dimensional space and the unit ball is a compact set, there exists element $y \in \mathbb{R}^m$, $\|y\| = 1$ such that supremum is attained $\|w\|_* = y^*w$. By definition of the quotient property, $y = AU$ for some $U \in \mathcal{H}_n$ with

$$\|U\|_q \leq dr_1^{1/q-1/2} \|y\| = dr_1^{1/q-1/2}.$$ 

Next, we apply definition of the adjoint operator and Hölder’s inequality for Schatten norms and obtain

$$\|w\|_* = \langle y, w \rangle = \langle AU, w \rangle = \langle U, A^*w \rangle_F \leq \|U\|_q \|A^*w\|_{q^*} \leq dr_1^{1/q-1/2} \|A^*w\|_{q^*},$$

where $q^*$ is Hölder dual of $q$ and equals to $q/(q-1)$. It finishes the proof in one direction.

To show that reverse statement is true, let us assume that inequality (17) holds. We start with case $q > 1$. Recall that $q^* = q/(q-1)$. If $w = 0$, then $U$ from the definition of the $S_q$-quotient property can be set as zero matrix. Let $w \in \mathbb{R}^m \setminus \{0\}$ and set $U \in \mathcal{H}_n$ as

$$U = \Delta_q(w) := \arg \min_{Z \in \mathcal{H}_n} \{ \|Z\|_q \text{ s.t. } AZ = w \}.$$ 

The existence of the feasible point can be justified with proof by contradiction.

Suppose that there exists $v \in \mathbb{R}^m$ such that for all $Z \in \mathcal{H}_n$ it holds that $AZ \neq v$. This means that the measurement operator $\mathcal{A} : \mathcal{H}_n \to \mathbb{R}^m$ is not surjective, i.e., $\dim(\text{Ran } \mathcal{A}) < m$ and therefore $\dim((\text{Ran } \mathcal{A})^\perp) > 0$. Consider now the adjoint operator $A^* : \mathbb{R}^m \to \mathcal{H}_n$ of $\mathcal{A}$. Since $\mathcal{A}$ is linear operator between finite-dimensional spaces, it holds that $(\text{Ran } \mathcal{A})^\perp = \ker(\mathcal{A}^*)$ and this means that

$$\dim(\ker(\mathcal{A}^*)) \geq 1$$

and, therefore, there exists $\beta \in \mathbb{R}^m$, $\beta \neq 0$ such that $\mathcal{A}^*\beta = 0$. So, $\mathcal{A}^*\beta$ is a zero matrix. Using Inequality (17) we obtain a contradiction

$$0 < \|\beta\|_* \leq dr_1^{1/q-1/2} \|A^*\beta\|_{q^*} = dr_1^{1/q-1/2} \|0\|_{q^*} = 0.$$
Thus, for all \( \omega \in \mathbb{R}^m \) optimization problem \( \Delta_q(\omega) \) is feasible and \( U = \Delta_q(\omega) \) is well-defined.

Fix \( V \in \ker \mathcal{A} \). Define \( \tau := \tau e^{i\theta}, \) where \( \theta \in [0, 2\pi) \) and \( \tau > 0 \) small enough to have \( U + \tau V \neq 0 \). We can present \( U + \tau V \) using the singular value decomposition.

\[
U + \tau V = \sum_{j=1}^{n} \sigma_j(U + \tau V)u_jv_j^*,
\]

Using the same vectors \( u_j \) and \( v_j, j \in [n], \) we define

\[
W^\tau := \frac{1}{\|U + \tau V\|_q^{-q}} \sum_{j=1}^{n} \sigma_j(U + \tau V)^{q-1} u_jv_j^*,
\]

Note that

\[
\|W^\tau\|_{q^*} = \left( \frac{1}{\|U + \tau V\|_q^{-q}} \sum_{j=1}^{n} (\sigma_j(U + \tau V))^{q-1} \right)^{1/q^*} = \left( \frac{1}{\|U + \tau V\|_q^{-q}} \sum_{j=1}^{n} (\sigma_j(U + \tau V)^{q}) \right)^{q/q^*} = 1.
\]

Using that both \( u_j \) and \( v_j, j \in [n] \) form a basis in \( \mathbb{C}^n \) we obtain

\[
\langle W^\tau, U + \tau V \rangle_F = \frac{1}{\|U + \tau V\|_q^{-q}} \sum_{j=1}^{n} (\sigma_j(U + \tau V))^{q-1} \langle u_jv_j^*, U + \tau V \rangle_F
\]

\[
= \frac{1}{\|U + \tau V\|_q^{-q}} \sum_{j=1}^{n} (\sigma_j(U + \tau V))^{q-1} \sum_{k=1}^{n} \sigma_k(U + \tau V) \langle u_jv_j^*, u_kv_k^* \rangle_F
\]

\[
= \frac{1}{\|U + \tau V\|_q^{-q}} \sum_{j=1}^{n} \sum_{k=1}^{n} (\sigma_j(U + \tau V))^{q-1} \sigma_k(U + \tau V) \langle u_jv_j^*, u_kv_k^* \rangle_F
\]

\[
= \frac{1}{\|U + \tau V\|_q^{-q}} \sum_{j=1}^{n} \sum_{k=1}^{n} (\sigma_j(U + \tau V))^q \langle u_j, u_k \rangle \langle v_k, v_j \rangle
\]

\[
= \frac{1}{\|U + \tau V\|_q^{-q}} \sum_{j=1}^{n} \sigma_j(U + \tau V)^q = \frac{\|U + \tau V\|_q^q}{\|U + \tau V\|_q^{q-1}} = \|U + \tau V\|_q.
\]

Out next goal is to take a limit \( \tau \to 0 \), but first we need to justify its existence. By Weil’s Theorem [5, Theorem III.2.1], we have that for all \( j \in [m] \)

\[
0 \leq |\sigma_j(U + \tau V) - \sigma_j(U)| \leq \sigma_j(\tau V) = \|\tau V\|_{\infty} = |\tau| \|V\|_{\infty} \to 0, \text{ as } \tau \to 0.
\]

Therefore, \( \sigma_j(U + \tau V) \) converges to \( \sigma_j(U) \), for \( \tau \to 0 \) and, consequently,

\[
W^\tau \to \frac{1}{\|U\|_q^{-q}} \sum_{j=1}^{n} \sigma_j(U)^{q-1}u_jv_j^* := W, \quad \tau \to 0.
\]

Firstly, due to assumption \( q > 1 \), \( W \) is independent of \( V \). Secondly, it preserves properties of \( W^\tau \), namely

\[
\|W\|_{q^*} = 1 \text{ and } \langle W, U \rangle_F = \|U\|_q.
\]
Recall that $U$ is solution of $Δ_q(w)$ and $U + τV$ is a feasible point since $V ∈ \ker A$ and hence $∥U∥_q ≤ ∥U + τV∥_q$. Together with Hölder’s inequality it yields that

$$\text{Re}(W^T, U)_F ≤ |⟨W^T, U⟩_F| ≤ ∥U∥_q ∥W^T∥_{q'} = ∥U∥_q ≤ ∥U + τV∥_q = \text{Re}(W^T, U + τV)_F.$$  

Consequently, it holds that

$$0 ≤ \frac{1}{τ} \text{Re}(W^T, τV)_F = \frac{1}{τ} \text{Re}(W^T, e^{iθ}V)_F = \text{Re}(e^{-iθ} ⟨W^T, V⟩)_F.$$  

Taking limit $t → 0$ implies $τ → 0$ and we obtain that for all $θ ∈ [0, 2π]$ inequality

$$\text{Re}(e^{-iθ} ⟨W, V⟩)_F ≥ 0$$  

is true. It is only possible if $⟨W, V⟩_F = 0$. Again, recall that $V$ was an arbitrary matrix from $\ker A$ and we proved that $⟨W, V⟩_F = 0$. It implies that $W ∈ (\ker A)^⊥ = \text{Ran } A^*$. Thus, there exists $y ∈ \mathbb{R}^m$ so that $W = A^*y$. By Inequality (17) and Equalities (30), it holds that

$$∥y∥_* ≤ d_r^{1/q - 1/2} ∥A^*y∥_{q'} = d_r^{1/q - 1/2} ∥W∥_{q'} = d_r^{1/q - 1/2}.$$  

Therefore, by Hölder’s inequality and Equalities (30) we obtain

$$∥U∥_q = ⟨W, U⟩ = ⟨A^*y, U⟩ = ⟨y, A^*U⟩ = ⟨y, w⟩ ≤ ∥y∥_* ∥w∥ ≤ d_r^{1/q - 1/2} ∥w∥,$$

what establishes the quotient property for $q > 1$. For the case $q = 1$, consider a sequence $(q_j)_{j ∈ \mathbb{N}}$ such that $q_j > 1$ for all $j ∈ \mathbb{N}$ and $q_j → 1$ for $j → ∞$. We first note that

for all $1 ≤ \tilde{q} ≤ \tilde{p} ≤ ∞$, for all $Z ∈ \mathcal{H}_n$ it holds that $∥Z∥_{\tilde{q}} ≤ ∥Z∥_{\tilde{p}}.$  \hspace{1cm} (31)

Combining Inequality (17) with (31) results in

$$∥w∥_* ≤ d_r^{1/2} ∥A^*w∥_∞ = (d_r^{1-1/q_j}) r_*^{1/q_j - 1/2} ∥A^*w∥_∞ ≤ (d_r^{1-1/q_j}) r_*^{1/q_j - 1/2} ∥A^*w∥_{q_j}.$$  

Since for all $j ∈ \mathbb{N}$ it holds that $q_j > 1$, above inequality is equivalent to the $S_{q_j}$-quotient property of $A$ and hence there exist matrix $U_j$ such that

$$A^*U_j = w$$

and $∥U_j∥_{q_j} ≤ (d_r^{1-1/q_j}) r_*^{1/q_j - 1/2} ∥w∥ = d_r^{1/2} ∥w∥.$

By (31), sequence $\{U_j\}_{j ∈ \mathbb{N}}$ is bounded in the $S_{∞}$-Schatten norm, that is

$$∥U_j∥_∞ ≤ ∥U_j∥_{q_j} ≤ d_r^{1/2} ∥w∥,$$  

for all $j ∈ \mathbb{N}$.

Space $\mathcal{H}_n$ is a finite-dimensional space and thus, there exists a convergent in $S_{∞}$-norm subsequence $(U_{j_k})_{k ∈ \mathbb{N}}$, such that $U_{j_k} → U$ when $k → ∞$. Consequently, we obtain that $A^*U = w$. Moreover, since all norms in finite-dimensional spaces are equivalent $U_{j_k}$ converges to $U$ in $S_1$ norm as well.

Last step is to prove that $∥U∥_1 = \lim_{k → ∞} ∥U_{j(k)}∥_{q_j(k)}$. Using continuity of the $S_p$ norms in parameter $p$, that is $∥U∥_1 = \lim_{k → ∞} ∥U∥_{q_j(k)}$, it follows that

$$∥U∥_1 = \lim_{k → ∞} ∥U∥_{q_j(k)} ≤ \lim_{k → ∞} ∥U - U_{j(k)}∥_{q_j(k)} + \lim_{k → ∞} ∥U_{j(k)}∥_{q_j(k)}$$

$$≤ \lim_{k → ∞} ∥U - U_{j(k)}∥_1 + \lim_{k → ∞} ∥U_{j(k)}∥_{q_j(k)} = \lim_{k → ∞} ∥U_{j(k)}∥_{q_j(k)}.$$
where we used triangle inequality and (31). On the other hand,
\[ \lim_{k \to \infty} \| U_j(k) \|_{q_j(k)} \leq \lim_{k \to \infty} \| U_j(k) \|_1 \leq \lim_{k \to \infty} \| U_j(k) - U \|_1 + \lim_{k \to \infty} \| U \|_1 = \| U \|_1. \]
Thus,
\[ \| U \|_1 = \lim_{k \to \infty} \| U_j(k) \|_{q_j(k)} \leq d r_*^{1/2} \| w \|, \]
which concludes the proof.

A.2 Proof of Lemma 9

Let \( \{e^{(1)}, \ldots, e^{(m)}\} \) be a standard basis vectors in \( \mathbb{R}^m \). By the definition of adjoint linear operators in Hilbert spaces, for all \( U \in \mathcal{H}_n \) and for all \( w \in \mathbb{R}^m \) it holds that
\[ \langle \mathcal{A}U, w \rangle = \langle U, \mathcal{A}^* w \rangle_F. \]
Applying this equality for basis vector \( e^{(k)}, k \in [m] \), we obtain
\[ \left\langle U, \mathcal{A}^* e^{(j)} \right\rangle_F = \left\langle \mathcal{A}U, e^{(j)} \right\rangle = \sum_{j=1}^{m} \text{tr}(U(a_j a_j^*)^*)(e^{(k)}) = \text{tr}(U(a_k a_k^*)^*) = \left\langle U, a_k a_k^* \right\rangle_F. \]
Since it holds for all \( U \in \mathcal{H}_n \), we have that the action of \( \mathcal{A}^* \) on basis vectors is \( \mathcal{A}^* e^{(k)} = a_k a_k^* \).

Therefore, by the linearity of \( \mathcal{A}^* \) we derive that
\[ \mathcal{A}^* w = \mathcal{A}^* \left( \sum_{k=1}^{m} w_k e^{(k)} \right) = \sum_{k=1}^{m} w_k \mathcal{A}^* e^{(k)} = \sum_{k=1}^{m} w_k a_k a_k^*, \]
for all \( w \in \mathbb{R}^m \), which concludes the proof.

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