Abstract

We prove lower bounds for several (dynamic) data structure problems conditioned on the well known conjecture that \textsc{3SUM} cannot be solved in $O(n^{2-\Omega(1)})$ time. This continues a line of work that was initiated by Pătraşcu [STOC 2010] and strengthened recently by Abboud and Vassilevska-Williams [FOCS 2014]. The problems we consider are from several subfields of algorithms, including text indexing, dynamic and fault tolerant graph problems, and distance oracles. In particular we prove polynomial lower bounds for the data structure version of the following problems: Dictionary Matching with Gaps, Document Retrieval problems with more than one pattern or an excluded pattern, Maximum Cardinality Matching in bipartite graphs (improving known lower bounds), $d$-failure Connectivity Oracles, Preprocessing for Induced Subgraphs, and Distance Oracles for Colors.

Our lower bounds are based on several reductions from \textsc{3SUM} to a special set intersection problem introduced by Pătraşcu, which we call Pătraşcu’s Problem. In particular, we provide a new reduction from \textsc{3SUM} to Pătraşcu’s Problem which allows us to obtain stronger conditional lower bounds for (some) problems that have already been shown to be \textsc{3SUM} hard, and for several of the problems examined here. Our other lower bounds are based on reductions from the \textsc{Convolution3SUM} problem, which was introduced by Pătraşcu. We also prove that up to a logarithmic factor, the \textsc{Convolution3SUM} problem is equivalent to \textsc{3SUM} when the inputs are integers. A previous reduction of Pătraşcu shows that a subquadratic algorithm for \textsc{Convolution3SUM} implies a similarly subquadratic \textsc{3SUM} algorithm, but not that the two problems are asymptotically equivalent or nearly equivalent.

1 Introduction

When designing data structures and algorithms for (dynamic) data, a time bound that is considered “good” for a query or update is typically constant or poly-logarithmic. Generally speaking, polynomial time per operation is not a desired outcome. Nevertheless, some data structure problems seem to be too difficult to be solved with fast preprocessing while supporting queries and updates in sub-polynomial time. A common method for giving evidence of such hardness for a data
structure problem is by reducing some other problem that is conjectured to be hard to the data structure problem. In particular, the notorious 3SUM problem \cite{GO95} problem, where we are given a set of \( n \) real numbers and want to determine if any three sum up to zero, is conjectured to be be unsolvable in subquadratic expected time. This is known as the 3SUM Conjecture. Gajentaan and Overmars \cite{GO95} were the first to conjecture that 3SUM is unsolvable in sub-quadratic time leading to several conditional lower bounds lower in computational geometry \cite{GO95} \cite{BHP01} \cite{SEO03}. Recently, Gronlund and Pettie \cite{JP14} showed that 3SUM can be solved in \( O(n^2 \frac{\log \log n}{\log n}) \) expected time, thereby refuting the conjecture made by Gajentaan and Overmars. But it is still conjectured that 3SUM is unsolvable in \( O(n^{2-\Omega(1)}) \) expected time.

Pătraşcu \cite{P10} used the 3SUM conjecture to prove some conditional polynomial lower bounds for various algorithmic challenges. To obtain these lower bounds, Pătraşcu \cite{P10} showed a connection between 3SUM and a simplified version of it called Convolution3SUM. Both problems deal with numerical inputs. For clarity we differentiate between the real and integer versions of each problem. Define \( [N] = \{0, 1, \ldots, N-1\} \) to be the first \( N \) natural numbers.

**3SUM:** Given a set \( A \subset \mathbb{R} \), report any triple \((a, b, c)\) \( \in A^3 \) of distinct numbers such that \( a + b = c \).

**Integer3SUM:** The same as 3SUM, except that \( A \subset [u] \subset \mathbb{Z} \) (\( u \) us a parameter of the problem).

**Convolution3SUM:** Given a vector \( A \in \mathbb{R}^n \), report any pair \((i, j)\) \( \in [n]^2 \) for which \( A(i) + A(j) = A(i+j) \).

**IntegerConv3SUM:** The same as Convolution3SUM, except that \( A \subset [u]^n \subset \mathbb{Z}^n \).

We emphasize that the 3SUM conjecture is conjectured to be true even for the relaxed Integer3SUM. Baran, Demaine, and Pătraşcu proved in \cite{BDP08} that Integer3SUM can be solved in \( O(n^2 \frac{\log \log n}{\log n}) \) expected time. Pătraşcu \cite{P10} defined the Convolution3SUM problem and used it as a stepping stone in a series of reductions from Integer3SUM to triangle enumeration, set intersection, and various dynamic graph problems. It is straightforward to reduce Convolution3SUM and IntegerConv3SUM to 3SUM and Integer3SUM, respectively, but there are no obvious reductions in the reverse direction. Pătraşcu’s \cite{P10} randomized reduction from Integer3SUM to IntegerConv3SUM is summarized with the following Theorem.

**Theorem 1.1.** (Pătraşcu \cite{P10}) Define \( T_{13S}(n) \) and \( T_{1IC3S}(n) \) to be the randomized (Las Vegas) complexities of Integer3SUM and IntegerConv3SUM on instances of size \( n \). For any parameter \( k \), \( T_{13S}(n) = O(n^2/k + (k^3 + k^2 \log n) \cdot T_{1IC3S}(n/k)) \).

In the last year there has been a surge of research on lower bounds conditioned on the Integer3SUM conjecture. Abboud, Vassilevska-Williams, and Weimann \cite{AWW14} proved that the Local Alignment problem, which is of great importance in computational biology, cannot be solved in truly sub-quadratic time unless the Integer3SUM conjecture is false. Amir, Chan, Lewenstein, and Lewenstein \cite{ACL14} proved that the Jumbled Indexing problem on a text with \( n \) integers (from a large enough alphabet) either the preprocessing time needs to be truly quadratic or the query time needs to be truly linear unless the Integer3SUM conjecture is false. Very recently, Abboud and Vassilevska-Williams \cite{AW14} showed several lower bounds conditioned on popular conjectures of hardness. In particular, they showed, conditioned on the Integer3SUM conjecture, that data structure versions of \((s, t)\)-teachability, Strong Connectivity, Subgraph Connectivity, Bipartite Perfect Matching, and variations of a problem known as Pagh’s problem all require non-trivial polynomial preprocessing/query/update times.
1.1 Our Results

In this paper, we continue this line of work by proving lower bounds conditioned on the \textit{Integer3SUM} conjecture for several (dynamic) data structure problems in several sub-fields of algorithms, including text indexing, dynamic and fault tolerant graph problems, and computational geometry. The lower bounds follow by reductions from \textit{IntegerConv3SUM}.

**Dictionary Matching with Gaps.** In the famous \textit{Dictionary Matching problem} we wish to preprocess a dictionary which is a set of patterns $D = \{P_1, \ldots, P_k\}$, over an alphabet $\Sigma$, such that given a text $T$ we can report all of the occurrences of all of the patterns from $D$ in $T$. In the \textit{Dictionary Matching with Gaps problem} the goal is the same as in the dictionary problem but now the patterns in the dictionary are over alphabet $\Sigma \cup \{\ast\}$ where $\ast \notin \Sigma$ can match any substring. This problem is of great interest in, for example, virus detection algorithms which use signatures to detect viruses in streams of data. In particular, SNORT (a comprehensive list of rules/patterns used for virus detection) contains many patterns with gaps.

The Aho-Corasick automata [AC75] is typically used to solve the \textit{Dictionary Matching} problem. However, it does not seem fit for solving the gap version. Haapasalo, Silvasti, Sippu, and Soisalon-Soininen in [HSSSS11] showed a solution for variable length gaps where the runtime depends on parameters of the dictionary that capture the combinatorial difficulty of the problem. Unfortunately, no significant progress has been obtained on this problem although many researchers in the pattern matching community and the industry have worked on it. We provide some evidence to this hardness by showing a polynomial lower bound conditioned on the \textit{IntegerConv3SUM} conjecture.

**Theorem 1.2.** Suppose there is a \textit{Dictionary Matching with Gaps} algorithm for a dictionary $D$ where $d = |D|$ with amortized expected preprocessing time $t_p(d)$ and amortized expected query time $t_q(d) = N \cdot t_{q1}(d) + op \cdot t_{q2}(d)$ where $N$ is the size of the query and $op$ is the size of the output. If the complexity of \textit{Integer3SUM} is expected $\Omega(n^2/f(n))$ then for any constants $0 < \epsilon < 1$ and $\delta > 0$:

$$t_p(d) + d^{3+\epsilon+\delta} \cdot t_{q1}(d) + d^{2+\delta} \cdot t_{q2}(d) = \Omega(d^{2/(1+\epsilon)} / f(d^{1/(1+\epsilon)}))$$

We prove Theorem 1.2 in Section 3. Notice that the strongest lower bounds are obtained by making $\epsilon$ and $\delta$ as small as possible. In particular, either $t_p(d) = \Omega(d^{2-\omega(1)})$, $t_{q1} = \Omega(d^{1/2-\omega(1)})$, or $t_{q2} = \Omega(d^{1/(1+\epsilon)}-\omega(1))$.

**Two Patterns Document Retrieval.** In the \textit{Document Retrieval problem} [Mut02] we are interested in preprocessing a collection of documents $X = \{D_1, \ldots, D_k\}$ where $x = \sum_{D \in X} |D|$, so that given a pattern $P$ we can quickly report all of the documents that contain $P$. Typically, we are interested in run time that depends on the number of documents that contain $P$ and not in the total number of occurrences of $P$ in the entire collection of documents. In the \textit{Two Patterns Document Retrieval problem} we are given two patterns during query time, say $P_1$ and $P_2$, and wish to report all of the documents that contain both $P_1$ and $P_2$.

All known solutions for the Two Patterns Document Retrieval problem with non trivial preprocessing use at least $\Omega(\sqrt{x})$ time per query [Mut02, CP10, HSTV10, HSTV12]. In a recent paper, Larsen, Munro, Nielsen, and Thankachan [LMNT14] show lower bounds for the Two Patterns Document Retrieval problem conditioned on the hardness of boolean matrix multiplication. We provide some additional evidence of hardness conditioned on the \textit{Integer3SUM} conjecture:
Theorem 1.3. Suppose there is a Two Patterns Document Retrieval algorithm for a collection of documents $X$ where $x = \sum_{D \in X} |D|$, with amortized expected preprocessing time $t_p(x)$ and amortized expected query time $t_q(x, |P_1|, |P_2|)$. If the complexity of IntegerConv3SUM is $\Omega(n^2/f(n))$ then for any constant $0 < \epsilon < 1/2$:

$$t_p(x) + x^{1+2\epsilon/3} \cdot t_q(x, |P_1|, |P_2|) = \Omega(x^{4/3}/f(x^{2/3})).$$

This holds even for $|P_1| = |P_2| = 1$.

We prove Theorem 1.3 in Appendix A. Notice that the strongest lower bounds are obtained by making $\epsilon$ as large as possible. In particular, either $t_p(x) = \Omega(x^{4/3-o(1)})$ or $t_q(x, |P_1|, |P_2|) = \Omega(x^{1/3-o(1)})$.

Forbidden Patterns Document Retrieval. In the Forbidden Pattern Document Retrieval problem [FGK+12], we are also interested in preprocessing the collection of documents but this time given a query $P^+$ and $P^-$ we are interested in reporting all of the documents that contain $P^+$ and do not contain $P^-$. 

All known solutions for the Forbidden Pattern Document Retrieval problem with non trivial preprocessing use at least $\Omega(\sqrt{x})$ time per query [FGK+12] [HSTV12]. In a recent paper, Larsen, Munro, Nielsen, and Thankachan [LMNT14] show lower bounds for the Forbidden Pattern Document Retrieval problem conditioned on the hardness of boolean matrix multiplication. We provide some additional evidence of hardness conditioned on the IntegerConv3SUM conjecture. Notice that here we opted to phrase the lower bound without the use of the function $f$ in order to simplify the exposition:

Theorem 1.4. Suppose there is a Forbidden Pattern Document Retrieval algorithm for a collection of documents $X$ where $x = \sum_{D \in X} |D|$, with amortized expected preprocessing time $t_p(x)$ and amortized expected query time $t_q(x, |P^+|, |P^-|)$. If the complexity of IntegerConv3SUM is $\Omega(n^{2-o(1)})$ then for any constant $0 < \epsilon < 1/2$:

$$t_p(x) + x^{\frac{5+\epsilon}{2}} \cdot t_q(x, |P^+|, |P^-|) = \Omega(x^{\frac{5+\epsilon}{2}} - o(1)).$$

This holds even for $|P^+| = |P^-| = 1$.

We prove Theorem 1.4 in Appendix A. Notice that for $\epsilon = 1/4$ we have that either $t_p(x) = \Omega(x^{8/7-o(1)})$ or $t_q(x, |P^-|, |P^-|) = \Omega(x^{1/7-o(1)})$.

Induced Subgraph. In the Induced Subgraph problem we wish to preprocess a (directed) graph $G = (V, E)$ so that given a query list of vertices $(v_1, \ldots, v_k)$ we can output all edges $(v_j, v_i) \in E$ such that $j < i$. We show that the Induced Subgraph problem has an interesting connection with the Dictionary with Gaps, thereby obtaining the following:

Theorem 1.5. Suppose there is an Induced Subgraph algorithm for a graph $G = (V, E)$ where $M = |E|$ with amortized expected preprocessing time $t_p(M)$ and amortized expected query time $t_q(M) = k \cdot t_q(M) + op \cdot t_q(M)$ where $k$ the query size. If the complexity of IntegerSUM is expected $\Omega(n^2/f(n))$ then for any constants $0 < \epsilon < 1$ and $\delta > 0$:

$$t_p(M) + M^{\frac{3+\epsilon+\delta}{2+\epsilon+\delta}} \cdot t_q(M) + M^{\frac{2+\delta}{1+\epsilon}} \cdot t_q(M) = \Omega(M^{2/(1+\epsilon)} / f(M^{1/(1+\epsilon)})).$$
We prove Theorem 1.5 in Section 3. Notice that the strongest lower bounds are obtained by making $\epsilon$ and $\delta$ as small as possible. In particular, either $t_p(M) = \Omega(M^{2-\omega(1)})$, $t_q1 = \Omega(M^{1/2-\omega(1)})$, or $t_q2 = \Omega(M^{\frac{\delta}{1+\epsilon}-\omega(1)})$.

**Maximum Cardinality Matching.** In the Dynamic Maximum Cardinality Matching problem we are interested in maintaining a dynamic graph $G = (V,E)$ while supporting maximum cardinality matching (MCM) queries returning the current size of the MCM. When both insertions and deletions are supported we say that $G$ is fully dynamic, while if only insertions are supported we say that $G$ is incremental. The trivial algorithm for updating an MCM takes $O(m)$ time by finding an augmenting path. Sankowski [San07] gave a fully dynamic algorithm with an amortized time bound of $O(n^{1.495})$ based on fast matrix multiplication. For the bipartite vertex-addition model where vertices on one side of the graph arrive online with all of their edges, B. Bosek, Leniowski, Sankowski and Zych [BLSZ14] recently showed how to maintain a maximum matching where the total cost for all insertions is $O(\sqrt{nm})$ time.

Independently from this paper, Abboud and Vassilevska-Williams [AW114] showed that for a fully dynamic graph for $1/6 \leq \alpha \leq 1/3$, either the preprocessing time is $\Omega(m^{4/3-\omega(1)})$, the amortized update time is $\Omega(m^{\alpha-\omega(1)})$, or the amortized query time is $\Omega(m^{2/3-\alpha-\omega(1)})$. In our setting we consider an initially empty graph $G$ and so the amortized lower bounds for updates and queries are relevant; a nonempty graph can be simulated by inserting all of the edges through updates. For the incremental case, Abboud and Vassilevska-Williams [AW114] show that the same lower bounds hold if the time for each operation is in the worst-case and not amortized. Abboud and Vassilevska-Williams [AW114] also mention the difficulty in obtaining an amortized lower bound for the incremental case using their approach, as they simulate deletions by rolling the structure back after each insertion to its state prior to the insertion. The worst-case lower bounds of Abboud and Vassilevska-Williams [AW114] can be stated in terms of $\hat{n}$ which is the number of vertices when an operation takes place, as opposed to the parameter $m$, and the update and query lower bound meet at $\Omega(\hat{n}^{1/2-\omega(1)})$ (we prove this independently as well). We show in Section 4 that if we allow the graph to grow with each query, it is straightforward to obtain an amortized expected $\Omega(\hat{n}^{1/3-\omega(1)})$ time lower bound.

We focus on proving an improved amortized lower bound for incremental MCM in terms of $\hat{n}$. We do this by combining two ideas. Sometimes we roll back an insertion while at other times we let the graph grow, but we always maintain the invariant that the graph has a perfect matching. It is also important to note that Abboud and Vassilevska-Williams [AW114] prove the hardness of MCM by a chain of reductions from Integer3SUM to MCM. However, our lower bound methods do not seem to work with all of those reductions. So to overcome this we need a reduction directly to MCM. Nevertheless, the general framework of our reduction shares the same flavors of the ideas used in [AW114]. Using the combination of techniques we prove the following in Section 4.

**Theorem 1.6.** Suppose there is an Incremental MCM algorithm for an incremental graph $G = (V,E)$. If the complexity of IntegerConv3SUM is $\Omega(n^{2-\omega(1)})$ then the amortized expected time of either insertion or query is $\Omega(\hat{n}^{0.366-\omega(1)})$ where $\hat{n}$ is the size of the graph during the operation.

**d-failure Connectivity Oracles.** In the d-failure Connectivity Oracles problem we wish to preprocess an undirected graph $G = (V,E)$ and some integer parameter $d > 0$ in order to support: (1) updates to $G$ in which a vertex is deleted, and (2) connectivity queries in which given two vertices $s$ and $t$ we wish to determine if they are in the same connected component of $G$. 

In Duan and Pettie’s [DP10] $d$-failure connectivity structure the preprocessing and deletion
times are $O(n^{1/c} \cdot \text{poly}(\log(n)))$ and $O(d^{2c+4} \cdot \text{poly}(\log(n)))$, where $c \geq 1$ is a parameter. The query
time is $O(d)$, independent of $c$. In Section 5 we prove the following:

**Theorem 1.7.** Suppose there is a $d$-failure connectivity structure for $M$-edge, $N$-vertex graphs with
amortized expected preprocessing time $t_p(M,N,d)$, amortized expected deletion time $t_d(M,N,d)$, and
amortized expected query time $t_q(M,N,d)$. If the complexity of IntegerConv3SUM is $\Omega(n^2/f(n))$
then

$$t_p(M,N,d) + (n^3/\sqrt{d}) \cdot t_d(M,N,d) + (n^2/\sqrt{d}) \cdot t_q(M,N,d) = \Omega(n^2/f(n) \log n),$$

where $M = O(n^3/2)$, $N = O(n^3/\sqrt{d})$, and $d$ is arbitrary.

This lower bound shows that with similar preprocessing and deletion times, the time to answer
a connectivity query must be $\Omega(\sqrt{d}/f(n)) = \Omega(\sqrt{d}/f((N\sqrt{d})^{2/3}) \log n)$. In particular, if $f$ is
polylogarithmic, then the query time must be $\tilde{O}(\sqrt{d})$. The connectivity oracles of [PT07, DP10, KKM13] answer queries in $O(\log\log n)$ time after $d$ edge deletions, independent of $d$. Our lower bound precludes such a $\tilde{O}(1)$ query time for $d$ vertex deletions.

**Distance Oracles for Colors.** Let $S$ be a set of points in some metric with distance $d(\cdot, \cdot)$, where
each point $p \in S$ has some associated colors $C(p) \subset [\ell]$. For $c \in [\ell]$ we denote by $P(c)$ the set
of points from $S$ with color $c$. We generalize $d$ so that the distance between a point $p$ and a color $c$ is
denoted by $d(p,c) = \min_{q \in P(c)} \{d(p,q)\}$. In the (Approximate) Distance Oracles for Vertex-Labeled
Graphs problem [HLWY11, Che12] we are interested in preprocessing $S$ so that given a query of a
point $q$ and a color $c$ we can return $d(q,c)$ (or some approximation). We further generalize $d$ so that
the distance between two colors $c$ and $c'$ is denoted by $d(c,c') = \min_{p \in P(c)} \{d(p,c')\}$. In the
Distance Oracle for Colors problem we are interested in preprocessing $S$ so that given two query
colors $c$ and $c'$ we can return $d(c,c')$. In the Approximate Distance Oracle for Colors problem we
are interested in preprocessing $S$ and some constant $\alpha > 1$ so that given two query colors $c$ and $c'$
we can return some value $\hat{d}$ such that $d(c,c') \leq \hat{d} \leq \alpha d(c,c')$.

For a text $T$ and a pattern $P$ let $L(P,T)$ be the set of locations in which $P$ occurs in $T$. A
special case of the Distance Oracle for Colors problem is the Snippets problem in which one is given
a text $T$ of length $N$ to preprocess so that given pattern queries $P_1, P_2, \ldots, P_k$ one can quickly
compute $\min_{1 \leq i \leq N} \{\max_{1 \leq j \leq k} \{\min_{o \in L(P_i,T)} |o - j|\}\}$. In words, we are interested in the location in
the text which minimizes the distance from that location to any query pattern. This problem is
of interest for search engines where one is interested in demonstrating the relevance of documents
or webpages to the queries patterns. A common method of demonstrating such relevance is by
providing a so called snippet of the document in which the queried patterns appear close to each
other.

We show evidence of the hardness of the Distance Oracle for Colors problem and the Approxi-
mate Distance Oracle for Colors problem by focusing on the 1-D case (the snippets). The following
Theorem is proven in Section 6.

**Theorem 1.8.** Suppose there is a 1-D Distance Oracle for Colors with constant stretch $\alpha \geq 1$
algorithm for an array of size $N$ with amortized expected preprocessing time $t_p(N)$ and amortized
expected query time $t_q(N)$. If the complexity of Integer3SUM is $\Omega(n^2/f(n))$ then for any constant
$0 < \epsilon < 1/2$

$$t_p(N) + N^{1+\epsilon} \cdot t_q(N) = \Omega(N^{4/3}/f(N^{2/3})),$$
The implication of this theorem is that, assuming the Integer3SUM conjecture, any data structure for solving the (Approximate) Distance Oracle for Colors problem must use either at least \( \Omega(N^{4/3-o(1)}) \) preprocessing time, or at least \( \Omega(N^{1/3-\Omega(1)}) \) query time, even in expectation.

1.2 A Stronger Reduction From 3SUM

Pătraşcu’s Problem. Pătraşcu in [P10] considered the following problem, which we call Pătraşcu’s problem. Given a set \( C \) of elements and two families \( A \) and \( B \) of subsets of \( C \), we are interested in listing the intersections of \( t \) pairs of subsets \( (S, S') \in A \times B \). In particular, Pătraşcu proved the following:

**Theorem 1.9 (Pătraşcu [P10]).** For any constant \( 0 < \epsilon < 1/2 \) and \( R = n^{1/2+\epsilon} \), let \( \mathcal{A} \) be an algorithm for Pătraşcu’s problem where \( |C| = n \), \( |A| = |B| = \sqrt{n}R \), each set in \( A \cup B \) has at most \( 3n/R \) elements from \( C \), each element in \( C \) appears in \( \sqrt{n} \) sets from \( A \) and \( \sqrt{n} \) sets from \( B \), there are \( t = \Theta(nR) \) pairs of subsets whose disjointness needs to be determined, and the total size of the intersections of these \( t \) pairs is at most \( O(n^2/R) \). If \( \mathcal{A} \) runs in expected \( O(n^2/f(n)) \) time, then IntegerConv3SUM can be solved in expected \( O(n^2/f(n)) \) time.

The following theorem is a direct consequence of Theorem 1.9, and is hinted in Appendix of [P10]. For completeness’s sake we provide a proof here.

**Theorem 1.10 (Pătraşcu [P10]).** For any constant \( 0 < \epsilon < 1/2 \) and \( R = n^{1/2+\epsilon} \), let \( \mathcal{A} \) be an algorithm for Pătraşcu’s problem where \( |C| = \Theta((n/R)^2) \), \( |A| = |B| = \sqrt{n}R \), each set in \( A \cup B \) has at most \( 3n/R \) elements from \( C \), there are \( t = \Theta(nR) \) pairs of subsets whose disjointness needs to be determined, and the total size of the intersections of these \( t \) pairs is at most \( O(n^2/R) \). If \( \mathcal{A} \) runs in expected \( O(n^2/f(n)) \) time, then IntegerConv3SUM can be solved in expected \( O(n^2 \log n/f(n)) \) time.

**Proof.** Let \( A, B \) and \( C \) be the sets from Theorem 1.9. Each set in \( A \cup B \) has size at most \( 3n/R \), so we can afford to hash the universe \( C \) down to one of size \( K = 2(3n/R)^2 \). Choose \( 2 \log n \) hash functions \( h_1, \ldots, h_{2 \log n} : C \to [K] \) independently from a universal hash family \( \mathcal{H} \). For any \( a \in A, b \in B \), if \( a \cap b \neq \emptyset \) then \( h_i(a) \cap h_i(b) \neq \emptyset \) as well, and if \( a \cap b = \emptyset \) then \( h_i(a) \cap h_i(b) = \emptyset \) with probability at least 1/2. We only look for an element in \( a \cap b \) if \( h_i(a) \cap h_i(b) \neq \emptyset \) for all \( i \leq 2 \log n \). Non-intersecting sets will appear to intersect with probability at most \( 1/n^2 \), so the cost of checking these false positives will be negligible.

**Improved Reductions.** Some of the hardness results conditioned on the 3SUM conjecture suffer from the limitation of \( R \gg n^{0.5} \). In particular, it implies that the size of the instance of Pătraşcu’s Problem must be fairly large. We overcome this obstacle by allowing \( R \) to be \( n^\epsilon \) for any constant \( 0 < \epsilon < 1 \) through two new reductions. The first reduction is summarized by the following theorem.

**Theorem 1.11.** For any constant \( 0 < \epsilon < 1 \) and \( R = n^\epsilon \), let \( \mathcal{A} \) be an algorithm for Pătraşcu’s problem where \( |C| = \Theta((n/R)^2) \), \( |A| = |B| = n \log n \), each set in \( A \cup B \) has at most \( 3n/R \) elements from \( C \), and there are \( t = \Theta(nR \log n) \) pairs of subsets whose disjointness needs to be determined. If \( \mathcal{A} \) runs in expected \( O(n^2/f(n)) \) time, then Integer3SUM can be solved in expected \( O(n^2/f(n)) \) time.

We also consider the reporting version of Pătraşcu’s problem where for each of the \( t \) set intersections we are interested in reporting the intersection instead of just deciding if two sets are disjoint or not. Of course, the reporting version is harder than the decision version. We prove the following theorem.
Theorem 1.12. For any constants \(0 < \epsilon < 1\) and \(R = n^\epsilon\) and \(\delta > 0\), let \(\mathcal{A}\) be an algorithm for the reporting version of Pătrașcu’s problem where \(|C| = \Theta((n^{1+\delta}/R))\), \(|A| = |B| = \sqrt{R}n^{1+\delta}\), each set in \(A \cup B\) has at most \(3n/R\) elements from \(C\), there are \(t = \Theta(nR)\) pairs of subsets whose intersections need to be reported, and the total size of the intersections of these \(t\) pairs is expected to be \(O(n^{2-\delta})\). If \(\mathcal{A}\) runs in expected \(O(n^2/f(n))\) time, then \(\text{Integer3SUM}\) can be solved in expected \(O(n^2/f(n))\) time.

We prove Theorem 1.11 and Theorem 1.12 in Section 2.

1.3 Logarithmic equivalence between \(\text{Integer3SUM}\) and \(\text{IntegerConv3SUM}\). Theorem 1.1 implies that \(\text{Integer3SUM}\) and \(\text{IntegerConv3SUM}\) would be subquadratic in qualitatively similar ways, for example, both \(\Theta(n^2/\text{polylog}(n))\) or both \(\Theta(n^{2-o(1)})\) or both \(\Theta(n^{2-O(1)})\). However, the gap between the two could be quite large. For example, if \(\text{IntegerConv3SUM}\) can be solved (optimally) in \(O(n)\) time, Theorem 1.1 only implies an \(O(n^{5/3})\) \(\text{Integer3SUM}\) algorithm.

Remark 1.1. Pătrașcu [P10] actually claimed that \(T_{3\text{SUM}}(n) = O(n^2/k + k^3, T_{1\text{CS3}}(n/k))\) but there is a minor error in his proof arising from the fact that the Convolution3SUM instances may contain witnesses that do not correspond to witnesses for the original 3SUM problem. (There may be triples of the form \((a, a, 2a)\) with repeated occurrences of some number.) The methods used in Appendix B.2 can be used to fix his proof at the cost of adding \(O(k^2 \log n)\) calls to an \(\text{IntegerConv3SUM}\) algorithm.

We prove (in Appendix B) that up to a logarithmic factor, the \(\text{Integer3SUM}\) and \(\text{IntegerConv3SUM}\) problems are equivalent, at least for the class of randomized algorithms.

Theorem 1.13. Define \(T_{3\text{SUM}}(n)\) and \(T_{1\text{CS3}}(n)\) to be the randomized (Las Vegas) complexities of \(\text{Integer3SUM}\) and \(\text{IntegerConv3SUM}\) on instances of size \(n\). Then \(T_{3\text{SUM}}(n) = O(\log n \cdot T_{1\text{CS3}}(n))\).

Our reduction is very similar to Pătrașcu’s. It applies an almost linear hash function \(h : [u] \to [m]\) of Dietzfelbinger, Hagerup, Katajainen, and Penttonen [DHKP97] to the elements of the \(\text{Integer3SUM}\) instance \(A\), thereby mapping them to buckets \(B(0), \ldots, B(m-1)\). If the hash function were perfectly linear (modulo \(m\)) and injective we would have \(a + b = c\) if and only if \(B(h(a)) + B(h(b)) = B(h(c)) = B(h(a) + h(b))\). Roughly speaking, we would have an instance \(B\) of \(\text{IntegerConv3SUM}\) whose witnesses are in one-to-one correspondence with the witnesses of the \(\text{Integer3SUM}\) instance \(A\).

One needs to overcome three technical difficulties to make this approach work. We could have false negatives due to the fact that \(h(c) \equiv h(a) + h(b) \pmod{m}\) but \(h(c) \neq h(a) + h(b)\). Pătrașcu solves this problem by concatenating two identical copies of \(B\), thereby catching these witnesses. We could also have false negatives due to \(h\) being only almost linear, say \(h(c) \equiv h(a) + h(b) \pm 1 \pmod{m}\). These witnesses are also fairly easy to catch by creating a few more \(\text{IntegerConv3SUM}\) instances with elements shifted appropriately. The most significant difficulty is that \(h\) is not injective: the loads of the buckets could vary widely. A lemma of Baran, Demaine, and Pătrașcu [BDP08] states that if \(m = n/k\), the expected number of elements in buckets with load greater than \(3k\) is at most \(n/k\). Pătrașcu [P10] searches for witnesses involving elements in overloaded buckets using the standard 3SUM algorithm, in time \(O(n^2/k)\). The witnesses involving elements in lightly loaded buckets are found by generating \(O(k^3 + k^2 \log n)\) instances of \(\text{IntegerConv3SUM}\) on vectors with length \(O(n/k)\).

In Pătrașcu’s reduction one chooses \(k\) based on the complexity of \(\text{IntegerConv3SUM}\) in order to balance the two costs: \(O(n^2/k)\) and \(O(k^3 + k^2 \log n) \cdot T_{1\text{CS3}}(n/k)\). The reduction becomes progressively less efficient as \(k\) grows. In our reduction we do away with the \(O(n^2/k)\) cost for dealing with
elements in overloaded buckets and manage to create only $O(\log n)$ instances of \texttt{IntegerConv3SUM}, independent of its algorithmic complexity.

2 An Improved Reduction

\textbf{Definition 2.1. (Universality and Linearity)} Let $\mathcal{H}$ be a family of hash functions from $[u] \rightarrow [m]$.

1. $\mathcal{H}$ is called $c$-universal if for any distinct $x, x' \in [u],$

   \[ \Pr_{h \in \mathcal{H}}(h(x) = h(x')) \leq \frac{c}{m}. \]

2. $\mathcal{H}$ is called almost linear if for any $h \in \mathcal{H}$ and any $x, x' \in [u],$

   \[ h(x) + h(x') \in h(x + x') + \{-1, 0\} \pmod{m}. \]

In our application any $O(1)$-universal almost linear hash function suffices.

\textbf{Theorem 2.1. (Dietzfelbinger, Hagerup, Katajainen, and Penttonen 1997 \cite{DHKP97})} Let $u$ and $m$ be powers of two, with $m < u$. The family $\mathcal{H}_{u, m}$ is 2-universal and almost linear, where

$$\mathcal{H}_{u, m} = \{h_a : [u] \rightarrow [m] \mid a \in [u] \text{ is an odd integer}\}$$

and $h_a(x) = (ax \mod u) \div (u/m)$.

Because the modular arithmetic and division are by powers of two, the hash functions of Theorem 2.1 are very easy to implement using standard multiplication and shifts. If $u = 2^w$, where $w$ is the number of bits per word, and $m = 2^s$, the function is written in C as $(a \times x) >> (w-s)$. Dietzfelbinger et al. \cite{DHKP97} proved that it is 2-universal. It is clearly almost linear.

2.1 Proof of Theorem 1.11

\textit{Proof.} We will reduce an instance of Integer3SUM to an instance of Pătrașcu’s problem. To simplify the exposition we assume that we are looking for three elements $x, y, z$ from the input such that $x - y = z$, and that $\frac{2n}{R}$ is an integer.

We begin by picking a 2-universal and almost linear hash function $h_1 : U \rightarrow [R]$, and creating $R$ buckets $B_1, \ldots, B_R$ such that $B_i = \{x : h_1(x) = i\}$. We assume that each bucket contains at most $\frac{3n}{R}$ elements. If this is not the case then we can deal with the overflow just like we do in the proof of Theorem 1.13 and so details are omitted.

We now pick a 2-universal and almost linear hash function $h_2 : U \rightarrow [(\frac{2n}{R})^2]$. For each bucket $B_i$ we create $\frac{14n}{R}$ copies as follows: for each $0 \leq j \leq \frac{2n}{R}$ let $B_{i,j}^2 = \{h_2(x) + j \cdot \frac{2n}{R} \pmod{(\frac{2n}{R})^2} \mid x \in B_i\}$ and $B_{i,j}^3 = \{h_2(x) - j \pmod{(\frac{2n}{R})^2} \mid x \in B_i\}$. Next, we set $z \in A$ and we want to find if there exist $x$ and $y$ in $A$ such that $x - y = z$. To do this we utilize the almost linearity of $h_1$ and $h_2$. Without loss of generality we assume that the hash functions are truly linear, and note that the almost linearity will induce a multiplicative 2 to the cost from each hash function (for a total multiplicative overhead of 4). This implies that $h_1(x) - h_1(y) = h_1(z) \pmod{R}$ and $h_2(x) - h_2(y) = h_2(z) \pmod{(\frac{2n}{R})^2}$. Thus, if $x \in B_i$ then $y$ must be in $B_{i-h_1(z) \pmod{R}}$. To this end, for each $i \in [R]$ we intersect
$B_i$ with $B_{i-h_1(z) \pmod R}$ shifted by $z$ in order to find possible candidates for $x$ and $y$. Denote by $h_2^\uparrow(z) = \lceil \frac{h_2(z)}{R} \rceil$ and $h_2^\downarrow(z) = h_2(z) \pmod {7n/R}$. Being that we are interested in reducing the universe size, the intersection between $B_i$ and $B_{i-h_1(z) \pmod R} + z$ is computed by utilizing the linearity of $h_2$ and intersecting $B_{i-h_1(z) \pmod R}^\uparrow \cap B_{i-h_1(z) \pmod R}^\downarrow$ to see if it is empty. If the intersection is empty, then there is no $x \in B_i$. If the intersection is not empty, then it is likely that $x \in B_i$ but we may have false positives.

Using the fact that $h_2$ is 2-universal and the union bound, it is straightforward to show that the probability that $B_i \cap B_{i-h_1(z) \pmod R} + z = \emptyset$ but $B_{i-h_1(z) \pmod R}^\uparrow \cap B_{i-h_1(z) \pmod R}^\downarrow \neq \emptyset$ is at most a small enough constant. To reduce this probability to be polynomially small, we repeat the intersection process $O(\log n)$ times, each time with a new almost linear $h_2$ function. If the sets intersect under all mappings of choices of $h_2$ then we can spend $O(\frac{n}{R}) = o(n)$ time to find $x$ and $y$. The expected contribution to the runtime from false positives is $o(1)$ time in expectation. We emphasize that $O(\frac{n}{R}) = o(n)$ time is also spent if we there exists a witness for $\text{Integer3SUM}$, however, this is negligible as $\Omega(n)$ time is clearly a lower bound for solving $\text{Integer3SUM}$.

To summarize, for each bucket $B_i$ we create a total of $O(n \log n)$ copies ($\frac{4n}{R}$ copies for each of the $R$ buckets for each of the $O(\log n)$ hash functions). These copies can be partitioned into two families $A$ and $B$ where all of the copies with the $\uparrow$ belong to $A$ and all of the copies with the $\downarrow$ belong to $B$. Thus, we never ask an intersection query between two sets where both are either in $A$ or in $B$. The universe $C$ of the elements in the sets is of size $O((\frac{n}{R})^2)$. The number of set intersection queries to be answered is $O(nR \log n)$. 

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reporting the intersection of \( B^\uparrow_{i,h_2(\mod R),h_2(\mod R)} \cap B^\downarrow_{i-h_1(\mod R),h_2(\mod R)} \). If the intersection is empty, then there is no \( x \in B_i \). If the intersection is not empty, then it is possible that \( x \in B_i \) but we most likely need to deal with false positives.

To verify the false positives, for each value \( k \in B^\uparrow_{i,h_2(\mod R),h_2(\mod R)} \cap B^\downarrow_{i-h_1(\mod R),h_2(\mod R)} \), the expected number of elements \( x \in B_i \) for which \( h_2(x) + h_2^2(z) \cdot \sqrt{n^{1+\delta}/R} = k \) is \( O(1) \) and the expected number of elements \( y \in B_{i-h_1(\mod R)} \) for which \( h_2(y) - h_2^2(z)(\mod \sqrt{n^{1+\delta}/R}) = k \) is \( O(1) \). So the time to verify the candidates is \( O(1) \) expected time per element in the intersection. The expected number of false positives in a single intersection is \( O(\sqrt{n_1^3+\delta}/R) \). Since we are performing \( O(nR) \) intersections, the number of false positives is \( O(n^2-\delta) \).

To summarize, we create a total of \( O(\sqrt{Rn^{1+\delta}}) \) copies (\( \sqrt{n^{1+\delta}/R} \) copies for each of the \( R \) buckets). These copies can be partitioned into two families \( A \) and \( B \) where all of the copies with the \( \uparrow \) belong to \( A \) and all of the copies with the \( \downarrow \) belong to \( B \). Thus, we never ask an intersection query between two sets where both are either in \( A \) or in \( B \). The universe \( C \) of the elements in the sets is of size \( O(nR) \). The number of set intersection queries to be answered is \( O(nR) \). The sum of the sizes of the set intersections will be an expected \( O(n^2+\delta) \).

\[ \square \]

3 The Dictionary Matching With Gaps Problem - Theorem 1.2

Denote \( d = |D| \) and \( N = |T| \). We will separate the query time into two components \( t_q(d) = N \cdot t_1(d) + \text{op} \cdot t_2(d) \) where \( \text{op} \) is the size of output. This will allow us to investigate the dependence of the cost of the query process that does not depend on the query or output size.

We reduce the reporting version of Pătraşcu’s problem to the dictionary matching with gaps problem as follows. Create an alphabet \( \Sigma = \{A \cup B\} \) so there is one unique character for each set. The Dictionary \( D \) is composed of \( d = O(nR) = n^{1+\epsilon} \) patterns, one for each set intersection query we are interested in. Specifically, for each set intersection query between \( a \in A \) and \( b \in B \) we create a pattern "\( a \ast b \)". We preprocess \( D \) to support dictionary matching with gaps queries.

For an element \( c \in C \) we create a text \( T_c \) which contains all of the characters in \( \Sigma \) that correspond to sets in \( A \cup B \) that contain \( c \). The first characters correspond to sets from \( A \) while the last characters correspond to sets from \( B \). Notice that \( \sum_{c \in C} |T_c| = O(n\sqrt{Rn^{1+\delta}}) = O(\sqrt{n^{3-\epsilon+\delta}}) \).

We then input \( T_c \) to the algorithm for solving dictionary matching with gaps. Each pattern reported corresponds to a non-empty set intersection that contains \( c \). We can now repeat this process for each of the \( O(\sqrt{n_1^3+\delta}/R) \) elements in \( C \) so that each pattern that is reported corresponds to an element in the output of a non empty set intersection query, and each pattern that is never reported corresponds to an empty set intersection.

The total cost for solving Pătraşcu’s problem is therefore \( t_p(n^{1+\epsilon}) \) time for the preprocessing of the dictionary, and another \( O(\sum_{c \in C} (|T_c| \cdot t_1(n^{1+\epsilon}) + \text{output}_c \cdot t_2_2(n^{1+\epsilon})) \) time for processing the elements in \( C \) where \( \text{output}_c \) is the number of patterns listed as part of processing \( c \). Notice that \( \sum_{c \in C} \text{output}_c \) is exactly the summation of the sizes of all intersections which is at most \( O(n^2-\delta) \). Therefore, the total cost is \( O(t_p(n^{1+\epsilon}) + \sqrt{n^{3-\epsilon+\delta}} \cdot t_1(n^{1+\epsilon}) + n^{2-\delta} \cdot t_2(n^{1+\epsilon})) \) time.

Notice that \( n = d^{1/(1+\epsilon)} \). Thus, if \( t_p(d) = d^{2(1/(1+\epsilon))} \cdot t_1(d) + d^{1/(1+\epsilon)} \cdot t_2(d) = o(d^{2(1/(1+\epsilon))}/f(d^{1/(1+\epsilon)})) \) then we obtain a contradiction as the time needed to solve IntegerConv3SUM is:
if two sets \( A \) and \( B \)

As pointed out by Abboud and Vassilevska-Williams [AW114], it is enough to be able to determine that correspond to sets in \( V \) the dictionary with gaps problem we create a list of vertices which contains all of the vertices in \( V \) gaps problem since we only used patterns that have a gap surrounded by two characters. To see the subgraph problem follows directly from the lower bound proof for the dictionary matching with

The Induced Subgraph problem- Theorem 1.5

The lower bound proof for the induced graph problem follows directly from the lower bound proof for the dictionary matching with

Theorem 1.5

The Induced Subgraph problem- Theorem 1.5

The lower bound proof for the induced subgraph problem follows directly from the lower bound proof for the dictionary matching with gaps problem since we only used patterns that have a gap surrounded by two characters. To see this, let \( V = \Sigma \) and for each pattern "a * b" create an edge \((a, b)\). Instead of a query text \( T_c \) in the dictionary with gaps problem we create a list of vertices which contains all of the vertices in \( V \) that correspond to sets in \( A \cup B \) that contain \( c \). The rest of the proof is straightforward.

4 Maximum Cardinality Matching - Theorem 1.6

As pointed out by Abboud and Vassilevska-Williams [AW114], it is enough to be able to determine if two sets \( a \in A \) and \( b \in B \) intersect in our reductions, since there are at most \( O(n^2) \) such pairs that intersect, and listing the intersections of each such pair can be done trivially in \( O(n^2/f(n)) \) time, for a total of \( O(n^3) = O(n^{2-2\epsilon}) \). We make use of Theorem 1.9.

4.1 Fully Dynamic Maximum Cardinality Matching

We consider the task of maintaining the cardinality of the maximum matching in a fully dynamic graph \( G(V, E) \) where edges may be added or removed from \( G \). To simplify the exposition, we assume that a query is performed after each update.

Consider the following instance of the fully dynamic MCM problem which is created from an instance of Pătraşcu’s problem. For each \( c \in C \) we create two vertices \( c_A \) and \( c_B \) with an edge between them. We say that \( c_A \) and \( c_B \) are copies of \( c \). For each \( a \in A \) we create two vertices \( a' \) and \( a'' \) with an edge between them, and for each \( c \in a \) there is an edge between \( a' \) and \( c_A \). We say that \( a' \) and \( a'' \) are copies of \( a \). For each \( b \in B \) we create two vertices \( b' \) and \( b'' \) with an edge between them, and for each \( c \in b \) there is an edge between \( b' \) and \( c_B \). We say that \( b' \) and \( b'' \) are copies of \( b \).

The initialization of this graph is implemented by inserting all of the edges one at a time using the fully dynamic MCM algorithm. This initial graph has \( n^{1+\epsilon} \) vertices and \( \theta(n^{1.5}) \) edges. We also add 2 additional vertices, \( x \) and \( y \), that will be used during the implementation of the set intersections. Before we continue in the description of the reduction, it is important to notice that our initial graph (without the extra 2 vertices) has a unique perfect matching where we match each pair of vertices that are copies of the same entity. This perfect matching is of course also a MCM.

As mentioned, we make use of 2 dummy vertices \( x, y \). To implement a set intersection query we first add an edge from \( x \) to \( a'' \) and an edge from \( y \) to \( b'' \). See Figure 1(A). There are two cases to consider. In the first case, there exists an element \( c \in a \cap b \). In this case before the addition of the new edges our graph has a path from \( a'' \) to \( b'' \) which is \((a'', a', c_A, c_B, b', b'')\). This path has 5 edges with 3 of them being in the perfect matching that we assume exists by induction. Adding the edges \((x, a'')\) and \((y, b'')\) creates an augmenting path, and so the MCM has increased due to the insertion of the two edges. On the other hand, if the intersection is empty then there is no

\[
O(t_p(n^{1+\epsilon}) + \sqrt{n^{3-\epsilon}} \cdot t_{q_1}(n^{1+\epsilon}) + n^{2-\delta} \cdot t_{q_2}(n^{1+\epsilon})) = O(t_p(d) + d^{\frac{3-\epsilon+\delta}{1+\epsilon}} \cdot t_{q_1}(d) + d^{\frac{2-\delta}{1+\epsilon}} \cdot t_{q_2}(d)) = o(d^{2/(1+\epsilon)} / f(d^{1/(1+\epsilon)})) = o(n^2/f(n)).
\]
Figure 1: An illustration of the graph before the set intersection query \( a \cap b = \emptyset \). There is a unique perfect matching before the query: matched edges are drawn thick and unmatched ones thin. Dashed edges are inserted in the course of the query. (A) For the worst case bound we insert edges \((x, a''), (y, b'')\), check if the size of the MCM has increased (implying \( a \cap b \neq \emptyset \)), then delete them. (B) For the amortized bound we insert new vertices \( x_{a,b}, x'_{a,b}, y_{a,b}, y'_{a,b} \) insert edges \((x_{a,b}, a''), (y_{a,b}, b'')\), then check whether the size of the MCM has increased, then insert edges \((x'_{a,b}, x_{a,b}), (y'_{a,b}, y_{a,b})\). Depending on the actual time of these operations, we either do nothing or roll back all edge and vertex insertions.

augmenting path in the graph. This is because paths from \( x \) to \( y \) must have two consecutive edges that are unmatched.

Thus, the MCM increases due to the insertion of the two edges if and only if the two sets intersect. In order to facilitate additional set intersection queries we then delete the two edges from the graph, thereby reverting back to the original graph with the original perfect matching and the original size of the MCM.

The total number of insertions and deletions performed is \( n^{1.5+\epsilon} \). If \( t_u(\hat{n}, m) \) is the amortized expected update time (either insertion or deletion), then by the Integer3SUM conjecture and by Theorem 1.9 we have that \( t_u(n^{1.5}, n^{1+\epsilon}) = \Omega(n^{2-o(1)}) \). Given that the Theorem 1.9 holds for any constant \( 0 < \epsilon < 1/2 \) we have \( t_u(\hat{n}, m) = \Omega(m^{1/3-o(1)}) \) and \( t_u(\hat{n}, m) = \Omega(\hat{n}^{1/2-o(1)}) \).

4.2 Incremental version

We now consider the task of maintaining the MCM of an incremental graph \( G(V, E) \) where edges and vertices may be added to \( G \). Again, to simplify the exposition, we assume that a query is performed after each update. The difficulty compared to the fully dynamic case is that we can no longer remove edges from the graph in order to facilitate additional set intersection queries. We discuss two techniques for overcoming this difficulty, and then combine the two techniques to obtain our strongest results.

Rollback. The first technique we discuss is the rollback technique which takes advantage of the fact that in the fully dynamic reduction we only delete edges right after they were inserted. So an edge deletion can be simulated by tracking all of the changes made due to the previous insertion, and then rolling back the data structures used to their previous state prior to the insertion of the edge. The time of the deletion process is the same as the time of the insertion process. We do however need to use extra space for tracking the changes made during an insertion, but the amount of space is sub-linear as we are interested in times that are sub-linear per update.
The downside of this approach is that the lower bound now only holds in the worst-case, and not for an amortized analysis. We conclude that the rollback technique immediately implies a $t_u(\hat{n}, m) = \Omega(n^{1/2-o(1)})$ worst-case expected time lower bound from the reduction of the fully dynamic case.

Creating Perfect Matchings. The second technique we consider comes from the following observation. Consider a set intersection query that we compute using the framework of the fully dynamic case. We know that the MCM increases if and only if the sets intersect. See Figure 1(B). After we decide if the sets intersect or not, we add edges $(x_{a,b}, a'')$ and $(y_{b,a}, b'')$ to the graph and just like in the fully dynamic case we know that the MCM increases if and only if the sets $a$ and $b$ intersect. See Figure 1(B). After we decide if the sets intersect or not, we add edges $(x_{a,b}, x'_{a,b})$ and $(y_{b,a}, y'_{b,a})$. These two additional edges guarantee that the resulting graph has a perfect matching. This perfect matching is comprised of the perfect matching of the graph prior to the insertion of the 4 edges together with edges $(x_{a,b}, x'_{a,b})$ and $(y_{b,a}, y'_{b,a})$. The MCM has also increased by 2 after the insertion of the 4 edges, regardless of whether the sets intersect or not.

The downside of this method is that the number of vertices increases as we perform more and more set intersection queries. In particular, being that we perform $\theta(n^{1.5+\epsilon})$ intersection queries, the resulting graph has $N = \theta(n^{1.5+\epsilon})$ vertices and $M = \theta(n^{1.5+\epsilon})$ edges. So, we have that the amortized expected time per insertion is $t_u(\hat{n}, m) = \Omega(n^{1/2-o(1)}) = \Omega(N^{1/3-o(1)})$. Thus the exponent in the dependency of the lower-bound on the number of vertices in the graph goes down from $1/2$ to $1/3$.

Combining the Two. To obtain our higher amortized lower bound we combine the two approaches of the rollbacks and creating perfect matchings. In particular, we will always begin by adding four edges as before, thereby creating a perfect matching. If the time of the insertion of the four edges is low (to be defined soon) then we will rollback the insertion. Otherwise, we leave the four edges and continue to the next set intersection query. Intuitively, our goal with this combined method is to guarantee that the graph does not grow by too much, while maintaining the amortized cost (to obtain a higher lower-bound).

Before delving into the detailed proof of the lower bound, recall that when we are interested in the amortized cost of an operation, we are actually interested in the cost of a sequence of operations. The so called amortized cost of each operation $\sigma$ can be viewed as a function $f(\sigma)$ that assigns each operation a value (which may depend on various parameters), so that the time of a sequence $S = (\sigma_1, \cdots, \sigma_k)$ of $k$ operations is always at most $\sum_{i=1}^k f(\sigma_i)$. The lower bound that we prove is that if $n_i$ is the number of vertices in $G$ during the $i^{th}$ operation, then $\sum_{i=1}^k f(\sigma_i) \geq \Omega(\sum_{i=1}^k n_i^{(\sqrt{3}-1)/2}) \geq \Omega(\sum_{i=1}^k n_i^{0.366})$. In other words, there is no algorithm for incremental MCM in which the amortized cost of each insertion is $O(\hat{n}^{0.366-O(1)})$ where $\hat{n}$ is the size of the graph during the insertion.
Now to the lower bound. Assume for contradiction that the amortized cost of each insertion is \( \hat{n}^\alpha \) (ignoring constant coefficients) for some constant \( \alpha > 0 \), where \( \hat{n} \) is the size of the graph during that insertion. Recall that we want to prove a lower bound on \( \alpha \). Our threshold for deciding when to rollback edge insertions is \( 4\hat{n}^\alpha \). If the time of an insertion is less than \( 4\hat{n}^\alpha \) then we rollback; otherwise we do not. For the time, notice that set intersection queries for which we perform a rollback will end up costing \( O(\hat{n}^\alpha) \leq O(N^\alpha) \) time each. The cost of the edge insertions due to set intersection queries for which we do not perform rollbacks, together with the edge insertions performed to setup the graph each cost an amortized \( O(\hat{n}^\alpha) \).

The next step will be to express \( N \) as a function of \( n \), by taking advantage of the fact that after the setup of the graph, every four edge insertions implementing a set intersection query that remain in the graph must have time that is expensive in a worst-case sense. Notice that after the graph setup there is \( O(n^{1.5}n^\alpha) \) credit for performing expensive insertions later, as the graph has \( m = \theta(n^{1.5}) \) edges at the start. Each expensive set intersection query uses up at least \( \Omega(\hat{n}^\alpha) \) of that credit, and so the total credit used during all of the expensive insertions is at least \( \Omega(\sum_{i=1}^{N}(n + i)^\alpha) = \Omega(N^{1+\alpha}) \). Being that we can never be in credit debt, we have that

\[
O(n^{1.5}n^\alpha) \geq \Omega(N^{1+\alpha}) \quad \text{and} \quad N \leq O(n^{\frac{1.5+\alpha}{1+\alpha}}).
\]

The number of cheaper insertions that we rolled back is \( O(nR) = O(n^{1.5+\varepsilon}) \). Each one of these costs at most \( N^\alpha \). So the total time of the entire sequence of operations which solves \texttt{IntegerConv3SUM} is \( O(n^{1.5+\varepsilon}N^\alpha + N^{1+\alpha}) \leq O(n^{1.5+\varepsilon + \Omega(\frac{1.5+\alpha}{1+\alpha})}) \). Given the \texttt{Integer3SUM} conjecture this runtime cannot be \( O(n^{2-\Omega(1)}) \) and so we must have that \( 2 < 1.5 + \frac{(1.5+\alpha)\alpha}{1+\alpha} \), or \( \alpha^2 + \alpha - 1/2 > 0 \). Solving for \( \alpha \) we have \( \alpha > \frac{\sqrt{3} - 1}{2} > 0.366 \).

5 \hspace{1cm} \textbf{\textit{d}-failure Connectivity - Theorem 1.7}

We reduce Pătraşcu’s problem to the \textit{d}-failure connectivity as follows. We make use of Theorem 1.10 and set \( d = (3n/R)^2 = |C| \). This entails an extra log factor. Construct a tripartite graph \( G = (V,E) \) on vertices \( V = A \cup B \cup C \) and edges

\[
E = \{(a,c) \mid c \in a\} \cup \{(b,c) \mid c \in b\}
\]

We now need to answer \( nR = O(n^2/\sqrt{d}) \) set intersection queries using a black-box data structure for \textit{d}-failure connectivity on \( G \). For each \( a \in A \) separately we perform up to \( d \) deletions and then answer all set intersection queries involving \( a \) using connectivity queries. To do this we delete all vertices in \( C \) that correspond to elements not in \( a \) and let \( G[a] \) be the resulting graph. Notice that in \( G[a] \), \( a \) is only connected to sets in \( A \cup B \) that intersect \( a \). We can therefore answer any set intersection query “\( a \cap b = \emptyset \)” by asking one connectivity query in \( G[a] \).

Observe that \( G \) is an \( M \)-edge, \( N \)-vertex graph where \( N = |A| + |B| + |C| = O(R\sqrt{n}) = O(n^{3/2}/\sqrt{d}) \) and \( M = O(n^{3/2}) \). Thus, if \( t_p(M,N,d) + (n^{3/2}/\sqrt{d})t_d(M,N,d) + (n^{3/2}/\sqrt{d})t_q(M,N,d) = o(n^2/f(n) \log n) \) then we obtain a contradiction as the time needed to solve \texttt{Integer3SUM} is \( o(n^2/f(n)) \).

6 \hspace{1cm} \textbf{Distance Oracles for Colors - Theorem 1.8}

We reduce Pătraşcu’s problem to the Colored Distance problem as follows. Let \( A \cup B = \{S_1, \cdots, S_{2\sqrt{\pi R}}\} \). For each \( S_i \) we define a unique color \( c_i \). The following array \( X \) is constructed from \( C = \{e_1, \cdots, e_n\} \) by assigning an interval \( I_i = [f_i, \ell_i] \) in \( X \) to each \( e_i \in C \) such that no two intervals overlap. Every
interval $I_i$ contains a list of all of the colors of subsets in $A \cup B$ that contain $e_i$. This implies that $|I_i| = \ell_i - f_i + 1 = O(\sqrt{n})$. Furthermore, for each $e_i$ and $e_{i+1}$ we separate $I_i$ from $I_{i+1}$ with a dummy color $d$ listed $\beta \sqrt{n}$ times at locations $[\ell_i + 1, f_{i+1} - 1]$, for a carefully chosen constant $\beta$. The only requirement we have on $\beta$ is that for any element $e \in C$ the number of subsets in $A \cup B$ that contain $e$ is strictly less than $\beta \sqrt{n}$. The size of $X$ is clearly $N = O(n^{3/2})$.

We can now simulate an intersection query on subsets $(S_i, S_j) \in A \times B$ by performing a colored distance query on colors $c_i$ and $c_j$. By our choice of $\beta$, the two points returned from the query are at distance less than $\beta \sqrt{n}$ if and only if there is an element in $C$ that is contained in both $S_i$ and $S_j$. The number of queries that need to be performed is $t = O(nR) = O(N^{1+\epsilon})$. Thus, if $t_p(N) + O(N^{1+\epsilon}) \cdot t_q(N) = o(N^{4/3}/f(N^{2/3}))$ then we obtain a contradiction as the time needed to solve $\text{Integer3SUM}$ is:

$$O(n^{3/2} + t_p(N) + nR \cdot t_q(N)) = O(N + t_p(N) + N^{1+\epsilon} \cdot t_q(N))$$

$$= o(N^{4/3}/f(N^{2/3}))$$

$$= o((n^{3/2})^{4/3}/f(n))$$

$$= o(n^2/f(n)).$$

Finally, notice that the lower bound also holds for the approximate case, as for any constant $\alpha$ the reduction can overcome the $\alpha$ approximation by separating intervals using $\alpha \beta \sqrt{n}$ listings of $d$.

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A Document Retrieval Problems with Multiple Patterns - Theorems 1.3 and 1.4

It is straightforward to see that the document retrieval problem, where each query is comprised of two patterns and the goal is to list all documents that contain both patterns, solves the set intersection problem. In particular, this can be obtained by creating an alphabet $\Sigma$ which corresponds to all of the sets in $A \cup B$. Then we can create a document for each $c \in C$ which has the corresponding character for each set that contains $c$. So $x = 2n^{1.5}$. A set intersection query between $a \in A$ and $b \in B$ can be answered by finding all documents that contain both $a$ and $b$. These documents correspond exactly to the elements in $a \cap b$. There are $n^{1.5+\epsilon} = x^{1+2\epsilon/3}$ queries that need to be computed. Thus we have that $t_p(x) + x^{1+2\epsilon/3} \cdot t_q(x, |P_1|, |P_2|) = \Omega(x^{4/3}/f(x^{2/3})).$

For the Forbidden Pattern problem, where we are interested in all documents that contain one pattern but do not contain the second one, we prove a lower bound as follows. $\Sigma$ is the same as above, but now for each $c$ we create a document that contains all of the characters corresponding to sets from $A$ that contain $c$ and sets from $B$ that do not contain $c$. Here we make use of Theorem 1.10 and so there is an extra log $n$ factor that we can ignore by expressing the runtime of IntegerConv3SUM as $\Omega(n^{2-o(1)})$. So $x = n^{1.5} + n^{2.5}/R = \Theta(n^{2-\epsilon})$. To answer an intersection query $a \cap b$ we search for all documents that contain $a$ and do not contain $b$. These documents correspond exactly to the elements in $a \cap b$. There are $n^{1.5+\epsilon} = x^{1+2\epsilon/3}$ queries that need to be computed. Thus we have that $t_p(x) + x^{1+2\epsilon/3} \cdot t_q(x) = \Omega(x^{2\epsilon/3-\Omega(1)}).

B Proof of Theorem 1.13

B.1 Hashing and Coding Preliminaries

The reduction in the next section makes use of any constant rate, constant relative distance binary code. The expander codes of Sipser and Spielman [SS96] are sufficient for our application.

Theorem B.1. (See Sipser and Spielman [SS96]) There is a constant $\epsilon > 0$ such that for any sufficiently large $\delta > \delta(\epsilon)$, there is a binary code $C : \{0, 1\}^N \rightarrow \{0, 1\}^{\delta N}$ such that for any $x,y \in \{0, 1\}^N$, the Hamming distance between $C(x)$ and $C(y)$ is at least $\epsilon \cdot \delta N$. Moreover, $C(x)$ can be computed in $O(\delta N)$ time.

B.2 The Reduction

Let $[n]\{0\} = [2^n]\{0\}$ be the universe. It is convenient to assume that 0 is excluded from $A$, but this is without loss of generality since all witnesses involving 0 can be enumerated in $O(n \log n)$ time by sorting $A$. Choose $L$ hash functions $(h_i)_{i \in [L]}$ independently from $\mathcal{H}_{a,m}$, where $m = 2^{[\log n]}$ is the least power of two larger than $n$. Ideally a hash function will map $A$ injectively into the buckets $[m]$, or at least put a constant load on each bucket, but this cannot be guaranteed. Some buckets will be overloaded and the items in them discarded.

Definition B.1. (Overloaded Buckets, Discarded Elements) For each $i \in [L]$ and $j \in [m]$ define

$$\text{BUCKET}_i(j) = \{x \in A \mid h_i(x) = j\}$$
to be the set of elements hashed by \( h_i \) to the \( j \)th bucket. The truncation of this bucket is defined as

\[
\text{BUCKET}^*_i(j) = \begin{cases} 
\text{BUCKET}_i(j) & \text{if } |\text{BUCKET}_i(j)| \leq T, \text{ and} \\
\emptyset & \text{otherwise},
\end{cases}
\]

where \( T = O(1) \) is a constant threshold to be determined. If \( \text{BUCKET}^*_i(j) = \emptyset \) we say that the elements of \( \text{BUCKET}_i(j) \) were discarded by \( h_i \). An element is called bad if it is discarded by a \( 4/T \)-fraction of the hash functions.

**Lemma B.2.** The probability that an element is bad is at most \( \exp \left(-\frac{2L}{3T}\right) \).

**Proof.** Since each \( h_i \) is 2-universal, the expected number of other elements in \( x \)'s bucket is, by linearity of expectation, at most \( 2(n - 1)/m < 2 \). By Markov’s inequality the probability that \( x \) is discarded by \( h_i \) is less than \( 2/T \). Let \( X \) be the number of hash functions that discard \( x \), so \( \mathbb{E}(X) < 2L/T \). By definition \( x \) is bad if \( X > 4L/T > 2 \cdot \mathbb{E}(X) \). Since the hash functions were chosen independently, by a Chernoff bound, \( \Pr(x \text{ is bad}) < \exp \left(-\frac{3L}{3T}\right) \).

We will set \( T = O(1) \) and \( L = \Theta(\log n) \) to be sufficiently large so that the probability that no elements are bad is \( 1 - 1/poly(n) \). We proceed under the assumption that there are no bad elements.

**Lemma B.3.** Suppose there are no bad elements with respect to \( (h_i)_{i \in [L]} \). For any three \( a, b, c \in A \), there are more than \( (1 - \frac{L}{4})L \) indices \( i \in [L] \) such that \( h_i \) discards none of \( \{a, b, c\} \).

**Proof.** Each of \( a, b, c \) is discarded by less than \( 4L/T \) hash functions, so none are discarded by at least \( L - 12L/T \) hash functions.

Let \( \delta > 1, \epsilon > 0 \) be the parameters of Theorem B.1 where \( N = \lfloor \log n \rfloor \) and \( L = \delta N \). We assign each \( x \in A \) an \( L \)-bit codeword \( C_x \) such that any two \( C_x, C_y \) disagree in at least \( \epsilon L \) positions.

We will make \( 8TL \) calls to an IntegerConv3SUM algorithm on vectors \( \{A_x\}_{x \in [L]\times \{-1,0\}\times\{0,1\}\times[2T]} \), each of length \( 14m = O(n) \). For reasons that will become clear we index the calls by tuples \( \ell = (i, \alpha, \beta, \gamma) \in [L] \times [-1,0] \times \{0,1\} \times [2T] \). The first coordinate \( i \) of \( \ell \) identifies the hash function. The second coordinate \( \alpha \) indicates that we are looking for witnesses \( a, b, a + b \in A \) for which \( h_i(a) + h_i(b) = h_i(a + b) + \alpha \pmod{m} \). A natural way to define \( A_\ell \) creates multiple copies of elements but can lead to a situation where there are false positives: we may have \( A_\ell(p) + A_\ell(q) = A_\ell(p + q) \) and yet this is not a witness for the original 3SUM instance because \( A_\ell(p) = A_\ell(q) \). In each call to IntegerConv3SUM we look for witnesses where each element can play the role of either “\( p \)” or “\( q \)” in the example above, but not both; all elements will be eligible to play the role of “\( p + q \).” The parity of \( C_x(i) \text{ XOR } \beta \) tells us which roles \( x \) is allowed to play, where \( \beta \) is the third coordinate of \( \ell \). The fourth coordinate \( \gamma \) of \( \ell \) effects a cyclic shift of the order of elements within a bucket.

Each vector \( A_\ell \) is partitioned into \( 2m \) contiguous blocks, each of length \( 7T \). Many of the locations of \( A_\ell \) are filled with a dummy value \( \infty \), which is some sufficiently large number that cannot be part of any witness, say \( 2 \max(A) + 1 \). The elements of the \( j \)th bucket each appear three times in \( A_\ell \), twice in the first half and once in the second.\footnote{This minor bug appears in Pătraşcu’s reduction from Integer3SUM to IntegerConv3SUM.}
Figure 2: Block $j$ in $A_\ell$ occupies positions $j(7T)$ through $(j + 1)(7T) - 1$. In the first half of $A_\ell$, a block is partitioned into five intervals. The first interval covers positions 0 through $T - 1$ and is always filled with a dummy value $\infty$. The second and third intervals run, respectively, from positions $T$ through $2T - 1$ and positions $2T$ through $3T - 1$. They contain those elements $x \in \text{bucket}_i^*(j - \alpha)$ for which $C_x(i) \text{ XOR } \beta$ is, respectively, 0 and 1. The fourth interval runs from positions $3T$ through $5T - 1$ and contains all members of $\text{bucket}_i^*(j - \alpha)$, cyclically shifted by $\gamma$. The last interval, from positions $5T$ through $7T - 1$, is always filled with dummies. The composition of a block $j$ in the second half of $A_\ell$ is similar, except that the second and third intervals (positions $T$ through $3T - 1$) contain only dummies, and the fourth interval contains all members of $\text{bucket}_i^*(((j - \alpha) \mod m) )$.

Order the elements of $\text{bucket}_i^*(j)$ arbitrarily as $(x(i, j, k))_{k \in [T]}$, where $x(i, j, k)$ does not exist if $k \geq |\text{bucket}_i^*(j)|$. Define the vector $A_{(i, \alpha, \beta, \gamma)}$ as follows.

$$A_{(i, \alpha, \beta, \gamma)}(j(7T) + t) = \begin{cases} 
  x(i, j, k) & \text{when } t = T + k, k \in [T], \text{ and } C_x(i,j,k)(i) \text{ XOR } \beta = 0, \\
  x(i, j, k) & \text{when } t = 2T + k, k \in [T], \text{ and } C_x(i,j,k)(i) \text{ XOR } \beta = 1, \\
  x(i, (j - \alpha) \mod m, k) & \text{when } t = 3T + ((k + \gamma) \mod 2T) \text{ and } k \in [T], \\
  \infty & \text{in all other cases.}
\end{cases}$$

The last case applies when $j, k$, or $t$ is out of range or if the given element, say $x(i, j, k)$, does not exist because $|\text{bucket}_i^*(j)| \leq k$. See Figure 2.

**Lemma B.4. (No False Negatives)** Suppose $a, b, a + b \in A$ is a witness to the 3SUM instance $A$. For some $\ell = (i, \alpha, \beta, \gamma)$, this is also a witness in the Convolution3SUM instance $A_\ell$.

**Proof.** Set the threshold $T = 12/\epsilon = O(1)$. By Lemma B.3 there are more than $L(1 - 12/T) = L(1 - \epsilon)$ indices $i \in [L]$ such that none of $\{a, b, a + b\}$ are discarded by $h_i$. Moreover, by the properties of the error correcting code (Theorem B.1) there are at least $\epsilon L$ indices $i$ for which $C_a(i) \neq C_b(i)$, which implies that both criteria are satisfied for at least one $i$. Fix any such $i$.

Let $j_a = h_i(a), j_b = h_i(b)$, and $j_{a+b} = h_i(a + b)$ be the bucket indices of $a, b,$ and $a + b$. Let $k_a, k_b, k_{a+b}$ be their positions in those buckets, that is, $a = x(i, j_a, k_a)$ and $b = x(i, j_b, k_b)$, and $a + b = x(i, j_{a+b}, k_{a+b})$. Without loss of generality $j_a \leq j_b$. Let $\beta = C_a(i)$, so $C_a(i) \text{ XOR } \beta = 0$ and $C_b(i) \text{ XOR } \beta = 1$. Let $\alpha \in \{-1, 0\}$ be such that $h_i(a) + h_i(b) \equiv h_i(a + b) + \alpha (\mod m)$.

In the vector $A_{(i, \alpha, \beta, \gamma)}$,

- $a$ is at position $j_a(7T) + T + k_a$, because $C_a(i) \text{ XOR } \beta = 0$,
- $b$ is at position $j_b(7T) + 2T + k_b$, because $C_b(i) \text{ XOR } \beta = 1$,
- and since $j_{a+b} \equiv j_a + j_b - \alpha (\mod m), a + b$ is at position $(j_a + j_b)(7T) + 3T + ((k_{a+b} + \gamma) \mod 2T)$.

Thus, for $\gamma = (k_a + k_b - k_{a+b}) \mod 2T$, the triple $(a, b, a + b)$ forms a witness for the Convolution3SUM vector $A_\ell$. 

Lemma B.5. (No False Positives) If \((a, b, a+b)\) is a witness in some \texttt{Convolution3SUM} instance \(A_{\ell}\), it is also a witness in the original \texttt{3SUM} instance \(A\).

Proof. None of \(\{a, b, a+b\}\) can be the dummy \(\infty\) in \(A_{\ell}\), so they must all be members of \(A\). The only way it cannot be an witness for \texttt{3SUM} is if \(b = a\), that is, \((a, a, 2a)\) is not a triple of distinct numbers. If \(a\) is not discarded, it appears at exactly three positions in \(A_{\ell}\). Regardless of the bit \(C_a(i)\), \(a\) appears at both

\[
A_{\ell}((ja + \alpha)(7T) + 3T + ((ka + \gamma) \mod 2T))
\]

and

\[
A_{\ell}((m + ja + \alpha)(7T) + 3T + ((ka + \gamma) \mod 2T)),
\]

for some \(ka \in [T]\) and \(\gamma \in [2T]\).

Depending on the parity of \(C_x(i) \oplus \beta\), \(a\) also appears at either

\[
A_{\ell}(ja(7T) + T + ka)
\]

or

\[
A_{\ell}(ja(7T) + 2T + ka).
\]

For \((a, a, 2a)\) to be a \texttt{Convolution3SUM} witness we would need \(2a\) to appear either at

\[
A_{\ell}((2ja + \alpha)(7T) + 4T + ka + ((ka + \gamma) \mod 2T))
\]

or

\[
A_{\ell}((2ja + \alpha)(7T) + 5T + ka + ((ka + \gamma) \mod 2T)).
\]

However, in both of those positions \(A_{\ell}\) is \(\infty\) by definition. See Figure \[2\] \(\square\)

B.3 Conclusions

We have shown that the randomized (Las Vegas) complexities of \texttt{Integer3SUM} and \texttt{IntegerConv3SUM} are equivalent up to a logarithmic factor. Since hashing plays such an essential role in the reduction, it would be surprising if our construction could be efficiently derandomized, or if it could be generalized to show that \texttt{3SUM} and \texttt{Convolution3SUM} (over the reals) are essentially equivalent.

The \(O(\log n)\)-factor gap in Theorem \[1.13\] stems from our solution to two technical difficulties, (i) ensuring that all triples appear in lightly loaded buckets with respect to a large fraction of the hash functions, and (ii) ensuring that no non-\texttt{3SUM} witnesses \((a, a, 2a)\) occur as witnesses in any \texttt{Convolution3SUM} instance. We leave it as an open problem to show that \texttt{Integer3SUM} and \texttt{IntegerConv3SUM} are asymptotically equivalent, without the \(O(\log n)\)-factor gap.