Pion Light-Cone Wave Functions and Light-Front Quark Model

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Abstract

We discuss a relation between the light-front quark model and QCD. We argue that this model can be used for an evaluation of the light-cone wave functions for moderate values of \(u\), but that it is inapplicable for this purpose in the region near the end points \(u = (0,1)\). We find additional support for a recent analysis in which it was claimed that the twist-two pion wave function attains its asymptotic form. The asymptotic twist-four two-particle wave function is also in good agreement with the light-front quark model.

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1 Introduction

The light-front quark model (LFQM) [1], is based on the algebra of the generators of the Lorentz-group in light-front dynamics [2]. The specific dynamical input is made through a parametrized form for the quark wave function. With specific choices of the parameters, the model describes numerous hadron properties, including form factors for $Q^2 \sim 1 \text{ GeV}^2$ (see for example [3]). Thus, this model establishes a phenomenological link between hadron properties and the wave function of the quark constituents that has been successful in many instances.

It is desirable to investigate a deeper theoretical connection between the LFQM wave function and QCD, not only to establish more firmly the dynamical basis of the LFQM but also with an eye to improve the general understanding and interconnections among various approaches. In our work, we derive a connection between the LFQM and QCD light-cone wave functions by equating the matrix element $\langle 0|\bar{d}(0)\gamma_\mu\gamma_5u(x_{1})|\pi^+(P)\rangle$ in the two approaches. This leads to an expression for the light-cone QCD wave function in terms of the LFQM wave function.

Some relevant questions were considered in [4]. Particularly, in [4] it was noted that the asymptotic behavior $u \to (0, 1)$ of the QCD light-cone wave function can not be reproduced within a constituent quark model (CQM) with equal-time wave functions [5] $\psi_{CM}(q^2) \sim \exp(-q^2)$, which can be represented in the form of the light-cone (LC) wave function by identification (see [4]):

$$q^2 \leftrightarrow \frac{k_1^2 + m^2}{4u(1 - u)} - m^2, \quad \psi_{CQM}(q^2) \leftrightarrow \psi_{LC} \left( \frac{k_1^2 + m^2}{4u(1 - u)} - m^2 \right), \quad (1)$$

where $m \simeq 300 \text{MeV}$ is the constituent mass.

The same conclusion is reached in the case of the LFQM [7]. Therefore, we can not expect to obtain a good description of the QCD light-cone wave function in terms of the quark model near the end points $u = (0, 1)$. In the next section we discuss the reason for this disagreement. However, in most cases hadronic matrix elements calculated in the quark model are saturated at moderate $u$, and the contribution of the region near the end points is small. In this case, one can expect that the LFQM can describe the light-cone wave functions for moderate $u$. 

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As mentioned above, the LFQM is able to describe hadron matrix elements even for large values of momentum transfer: $Q^2 \sim 1GeV^2$. This means that the LFQM gives a good description of hadron matrix elements at small distances $x^2 \ll \Lambda_{QCD}^{-2}$. Therefore, it is reasonable to use such a model for the evaluation of matrix elements of operators defined at a scale $\mu \sim 1GeV \gg \Lambda_{QCD}$. Below, we will assume that the matrix elements (which depend on the normalization point) are defined at the scale $\mu \sim 1GeV$.

In this paper we consider the transition amplitude of a pion to the vacuum by a nonlocal gauge invariant operator of the axial current. This amplitude can be expressed in terms of QCD light-cone wave functions [8, 9, 10]. Assuming that the LFQM can give a good description for this type of amplitude (except the region near the end points $u = (0, 1)$, as discussed above) we express the QCD light-cone wave functions of a pion as an integral over the LFQM wave function. Using the published parametrizations [11] for the latter, we show that the twist-2 pion wave function is very close to its asymptotic form. This observation confirms the result obtained in [12] based on constraints for the light-cone wave function determined from QCD sum rules [13] for $g_{\pi NN}$ coupling constant [14, 15] and from the QCD sum rule for the pion structure function [16, 17]. The constraint obtained in [18] also indicates that the the light-cone wave function is close to its asymptotic form.

In the LFQM, we also evaluate the two-particle twist-4 light-cone pion wave function. We find the asymptotic form for the twist-4 light-cone wave function to be in good agreement with the LFQM description.

2 Pion Wave Function

In our work, we derive a connection between LFQM and QCD light-cone wave functions by equating the gauge invariant matrix element

\[
< 0|\bar{d}(0)\gamma_\mu U[0,x]\gamma_5 u(x)|\pi^+(P) >
\]  

(2)

in the two approaches, where $U[0,x] = P e^{ig\int_x^0 A_\mu(y)dy_\mu}$. Note that in LFQM considered here the gauge fields are not taken into account. Nevertheless we can assume that in the LFQM the quark wave functions correspond to the fixed-point gauge $x_\mu A_\mu(x) = 0$. Then $U[0,x] = 1$ and the matrix element (2) can be evaluated in terms of the constituent quark wave functions.
In QCD, the twist-2 and twist-4 two-particle wave functions of the pion are defined by the following matrix element:

\[
<0|\bar{d}(0)\gamma_{\mu}\gamma_5u(x)|\pi^+(P)>
\]

\[=
if_\pi P_\mu \int_0^1 e^{-iu(Px)}[\varphi_{\pi}(u) + x^2g_1(u) + x^2G_2(u) + O(x^4)]du
\]

\[+
 f_\pi x_\mu \int_0^1 e^{-iu(Px)}g_2(u)du + O(x^2)
\]  

(3)

where \(\varphi_{\pi}(u)\) is the twist-2 wave function, where \(g_1(u)\) and \(g_2(u)\) are the twist-4 pion wave functions, where and \(G_2(u) = -\int_0^u g_2(u)du\).

It is useful to first examine the situation where the quark field \(u(x_1)\) is placed at the origin, in which case

\[<0|\bar{d}(0)\gamma_{\mu}\gamma_5u(x_1 = 0)|\pi^+(P)>| = iP_\mu f_\pi,
\]

(4)

where

\[
\bar{d}(x_2 = 0) = \int e^{ip_2\cdot x_2} \mid_{x_2=0} \bar{d}(p_2) \frac{d^3p_2}{(2\pi)^3}
\]

(5)

and

\[
u(x_1 = 0) = \int e^{-ip_1\cdot x_1} \mid_{x_1=0} v(p_1) \frac{d^3p_1}{(2\pi)^3}.
\]

(6)

Following analysis of Ref.[11], one can determine from Eq.(4) the relationship of the pion decay constant to the LFQM wave function. Substituting Eqs.(5,6) into Eq.(4), introducing the LFQM pion wave function \(\psi\) defined in terms of the S-wave orbital wave function \(\phi(p)\),

\[
\phi(p) = \sqrt{\frac{(2\pi)^3}{N_c}} \frac{1}{\pi^{3/4}\beta^{3/2}} e^{-\frac{p^2}{2\beta^2}},
\]

(7)

using the analysis of Ref.[11], and changing variables to the total and relative momenta, one finds that \((N_c = 3)\)

\[
f_\pi = \frac{\sqrt{3}}{\pi^{5/4}\beta^{3/2}} \int_0^\infty \frac{p^2e^{-\frac{p^2}{2\beta^2}}dp}{(p^2 + m^2)^{3/4}},
\]

(8)
where \( m \) is the constituent mass of a \( u \) or \( d \) quark. In Ref.\([11]\) it was suggested to use the following set of parameters: \( \beta = 0.3194 \text{GeV} \) and \( m = 0.25 \text{GeV} \) \([11]\); this gives \( f_\pi = 0.130 \text{GeV} \).

To draw a correspondence with Eq.(3), let us consider the case \( x_1 = x \) in Eq.(4). It is then clear that the \( u \)-quark wave function gets a phase factor \( e^{i p_1 x} \). In terms of light-cone variables,

\[
p_1 = (\zeta P^+, p_1^- , \vec{p}_\perp) \quad P^\pm = P^0 \pm P^3 \quad p_1^\pm = p_1^0 \pm p_1^3 ,
\]

the phase factor has the following form:

\[
e^{i p_1 x} = e^{-i \zeta P^+ x^- + i \vec{p}_\perp \vec{x}_\perp} ,
\]

where by definition \( p_1^+ = \zeta P^+ \). In Eq.(10) we have dropped the \( p^- \) term, consistent with the fact that in the LFQM the \( p^- \) dependence is usually not considered \([1]\). This is justified as long as \( \zeta \neq 0 \) since, in the infinite momentum frame \( (P^+ \to \infty) \), the dependence on \( p^- \)

\[
p_1^- = \frac{\vec{p}_\perp^2 + m^2}{\zeta P^+} ,
\]

is suppressed. However, the suppression is absent in the limit when \( \zeta \to 0 \). In the region of small \( \zeta \) it is incorrect to neglect the \( p^- \) dependence, and one therefore cannot apply the LFQM for calculations of matrix elements here. It is clear that one draws the same conclusion for matrix elements where the dominant contribution comes from the region \( \zeta \simeq 1 \). It is nevertheless possible to make a comparison to the light-front wave functions at \( x^+ = 0 \), being mindful of the concerns in the region of the end points, \( \zeta \simeq (0,1) \). These remarks are useful for understanding of limits of LFQM applicability.

Then, from Eqs.(3,10) one can find that

\[
< 0|\bar{d}(0)\gamma_\lambda \gamma_5 u(x)|\pi(P)> = iP_\perp \frac{\sqrt{3}}{\pi^{5/2}} \frac{m}{\beta^{3/2}} \frac{1}{4\pi} \int \frac{e^{\frac{-x^2}{2\beta^2}} d^3p}{(p^2 + m^2)^{3/4}} e^{-i p_1 x} ,
\]
where we work in the frame of reference in which \( P_\perp = 0 \). The integral over \( p_\perp^+ \) can be rewritten in the form of the integral over the parameter \( \zeta \), and the Jacobian of the transformation is
\[
J = \frac{\sqrt{p_\perp^2 + m^2}}{[4\zeta (1 - \zeta)]^{3/2}}. \tag{13}
\]

As a result of this transformation, Eq.(12) becomes:
\[
iP_\perp \pi \sqrt{3} m \frac{P_\perp}{\pi^{5/4} \beta^{3/2}} \int_0^1 e^{-i\zeta p_\perp^+ x^-} d\zeta \int d^2p_\perp e^{-\frac{1}{2\beta^2}[p_\perp^2 + m^2]} e^{ip_\perp \cdot x_\perp} \\
= i f_\pi P_\perp \int_0^1 e^{-iu(p_\perp^+ x^-)}[\varphi_\pi(u) - x_\perp^2 g_1(u) - x_\perp^2 G_2(u)] du. \tag{14}
\]

Here we use the fact that \( x^2 = -x_\perp^2 \) when \( x^+ = 0 \). From Eq.(14) it follows that \( \zeta = u \).

Using Eq.(14) and expanding this expression up to the terms \( x_\perp^2 \) we obtain the following formulae for the twist-2 and twist-4 light-cone wave functions:
\[
f_\pi \varphi_\pi(u) = \frac{\sqrt{3}m}{\pi^{5/4} \beta^{3/2} [4u(1-u)]^{3/4}} \int_0^\infty dp_\perp^2 e^{-\frac{1}{2\beta^2}[p_\perp^2 + m^2]} \frac{e^{ip_\perp \cdot x_\perp}}{(p_\perp^2 + m^2)^{1/4}} \tag{15}
\]
and
\[
f_\pi [g_1(u) + G_2(u)] = \frac{\sqrt{6}m}{4 \pi^{5/4} \beta^{3/2} [4u(1-u)]^{3/4}} \int dp_\perp^2 e^{-\frac{1}{2\beta^2}[p_\perp^2 + m^2]} \frac{p_\perp^2}{(p_\perp^2 + m^2)^{1/4}} \tag{16}
\]

where the additional factor of \( 1/4 \) in Eq.(16) arises from the expansion of the exponential to second order and subsequent integration over the square of the angle between \( p_\perp \) and \( x_\perp \).

It was noted above that we cannot apply the LFQM to determine matrix elements having their dominant contribution occurring near the end points \( \zeta = (0, 1) \). Therefore, since \( \zeta = u \), we cannot expect a good description of the QCD light-cone wave functions in terms of the LFQM in the region near the end points \( u = (0, 1) \).
3 Numerical Results

The integrals in Eqs. (15,16) may be expressed in terms of the incomplete Gamma function \[ \Gamma(a, x) = \int_x^\infty \frac{t^{a-1}}{e^{t}} \, dt \]. The result is

\[ f_\pi \varphi_\pi(u) = \frac{2^{1/4} \sqrt{6}}{\pi^{3/4}} \frac{e^{m^2/2 \beta^2}}{m^2} \frac{\Gamma\left(\frac{3}{4}, z_0\right)}{z_0} \] (17)

and

\[ f_\pi [g_1(u) + G_2(u)] = \frac{2^{1/4} \sqrt{6}}{4\pi^{3/4}} \frac{e^{m^2/2 \beta^2}}{m^3} \left[ \frac{1}{z_0} \Gamma\left(\frac{7}{4}, z_0\right) - \Gamma\left(\frac{3}{4}, z_0\right) \right], \] (18)

where \( z_0 = \frac{m^2}{8 \beta^2 u(1-u)} \).

We compare our results with asymptotic QCD light-cone wave functions:

\[ \varphi_\pi(u) = 6u \bar{u} \]
\[ g_1(u) = \frac{5}{2} \delta^2 u^2 \bar{u}^2 \]
\[ g_2(u) = \frac{10}{3} \delta^2 \bar{u}u(u - \bar{u}) \]
\[ G_2(u) = \frac{5}{3} \delta^2 \bar{u}^2 u^2, \] (19)

where \( \bar{u} = 1 - u \), and with constraints on the light-cone wave function obtained from the QCD light-cone sum rule for the \( g_{\pi NN} \) coupling constant \[13\] and from the light-cone QCD sum rule for the pion structure function \[14\]. The QCD sum rule estimate \[20\] for \( \delta^2 \) yields \( \delta^2 = 0.2 \text{ GeV}^2 \). Comparing Eq. (19) to Eqs. (17,18), we find the results shown in Fig.1 and Fig.2, respectively.

We see that the agreement for the twist-2 wave function is about 10\% and for the twist-4 wave function is 12 \% at the peaks, \( u = 0.5 \). The LFQM predicts a broader distribution and is consistent with results of Ref. \[12\].

4 Discussion and Conclusion

We have obtained a connection between the LFQM wave function of the pion and the corresponding two-particle wave functions in QCD by identifying
matrix elements of quark fields, \( \langle 0|\bar{d}(0)\gamma_\mu\gamma_5u(x_1)|\pi^+(P)\rangle \). This connection has permitted us to check conclusions obtained from the QCD sum rule analysis for the pion against findings in the LFQM. We also found a simple reason to be wary of the comparison between the LFQM predictions and the QCD light-cone wave functions in the region where \( u = (0,1) \): one of the main LFQM results, namely the absence of a sensitivity to the \( p^- \) component of the quark momentum, does not permit us to take seriously the comparisons near the end points \( u = (0,1) \). It was shown that the LFQM description of QCD light-cone wave functions indicates not only that the twist-2 pion light-cone wave function is close to its asymptotic form but also that the form of the twist-4 light-cone wave function \( g_1(u) + G_2(u) \) is not far from asymptotic.

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Figure Captions

Figure 1. Comparison between the asymptotic twist-2 light-cone wave function \( \varphi_\pi(u) \) (solid curve) and the result of the LFQM (dashed curve), given by Eq.(17). The points at \( u = 0.3 \) and \( u = 0.5 \) are constraints determined in Refs.[16] and [15], respectively.

Figure 2. Comparison between the asymptotic twist-4 light-cone wave function \( g_1(u) + G_2(u) \) (solid curve) and the result of the LFQM (dashed curve), given by Eq.(18).
Figure 1
Figure 2