EVERY EYELET ENABLES AN ESCAPE

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Abstract. The aim of this paper is to establish a relation between the tangent cone of the medial axis of $X$ at given point $a \in \mathbb{R}^n$ and the medial axis to the set of points in $X$ realising the $d(a, X)$. As a consequence we obtain the lower bound for the dimension of the medial axis of $X$ in terms of the dimension of the medial axis of $m(a)$.

1. Introduction

As medial axes are closely related to the conflict sets, one can believe most of the theorems about conflict sets should have their counterparts in medial axis theory. Unfortunately medial axis is infamous for its unstability, thus the proofs are seldom transferable between these two objects. The aim of this paper is to prove the medial axis analog of the [2] Theorem 2.2. Since the proof presented by Birbrair and Siersma heavily depends on the monotonicity of Conflict Sets - the phenomena that has no counterpart in medial axis setting - we are forced to develop a completely new approach to the problem based on the analysis of the graph of the distance function.

Throughout the paper definable means definable in some polynomially bounded o-minimal structure expanding the field of reals $\mathbb{R}$, $\mathbb{B}(a, r)$ denotes closed ball with center at $a$ and radius $r$, and $\mathbb{S}(a, r)$ denotes its boundary - a n-1 sphere of radius $r$ centered at $a$.

For a closed subset $X$ of $\mathbb{R}^n$ endowed with euclidean metric, we define the distance of a point $a \in \mathbb{R}^n$ from $X$ by

$$d(a, X) := \inf \{d(a, x) | x \in X\},$$

which allows us to define the set of closest points of $X$ to $a$ by

$$m(a) := \{x \in X | d(a, X) = \|a - x\|\}.$$

The main object discussed in this paper is the Medial axis of $X$, that is the set of points of $\mathbb{R}^n$ admitting more than one closest point to the set $X$ i.e.

$$M_X := \{a \in \mathbb{R}^n | \#m(a) > 1\}$$

In the next section we will also extensively use the graph of the function $d : \mathbb{R}^n \ni a \to d(a, X) \in \mathbb{R}_+$ denoted from now on by $\Gamma$. 


2. Tangent cone of the medial axis

Let’s begin by reminding, that we have the explicit formula for the directional derivative of the distance function

\[ D_v d(0, X) = \inf \{- < v, \frac{y}{\|y\|} >, y \in m(0)\} \]

Assuming that \( m(0) \) is the subset of the unit sphere, we can use the polarization identity to express \( D_v d(0, X) \) in the form

\[ \frac{1}{2} \inf \{d(v, y)^2 - \|v\|^2 - 1, y \in m(0)\} \]

Since the infimum is attained at \( y \in m(0) \) which is closest to \( v \) in fact the formula is

\[ \frac{1}{2}(d(v, m(0))^2 - \|v\|^2 - 1) \]

As for the graph \( \Gamma \) of the function \( d \) we have the following.

**Proposition 2.1.** The set \( \Gamma \) has following properties:

1. For any \((a, d(a))\) \(\in\) \(\Gamma\),
   \[\{ (x, y) \in \mathbb{R}^m \times \mathbb{R} \mid \|y - d(a)\| > \| x - a \| \} \cap \Gamma = \emptyset \]
2. For any \(a \in \mathbb{R}^m\) and \(x \in m(a)\), \(\{(x, 0), (a, d(a))\} \subset \Gamma\)
3. For any \((x, y) \in \mathbb{R}^m \times \mathbb{R}\), \((x, y) \in M_\Gamma \iff x \in M_X\) provided \(y < d(x)\) (in other words: the medial axis to the epigraph of \(d\) is equal \(M_X \times \mathbb{R} \cap \{(x, y) \mid y < d(x)\}\)).

**Proof.** (1) is a consequence of the Lipschitz condition for the distance function. (2) comes from \(m(t x + (1 - t)a) = x\) for \(t \in [0, 1], x \in m(a)\) and \(d(t x + (1 - t)a, x) = d((1 - t)a, (1 - t)a) = (1 - t)d(a, x)\). (3) can be proved by observing that (1) and (2) gives us:

\[\{ (x, y) \in \mathbb{R}^m \times \mathbb{R} \mid \|y - d(a)\| \geq \| x - a \| \} \cap \Gamma = \bigcup_{x \in m(a)} [(a, d(a)), (x, 0)]\]

for every \((a, d(a))\) \(\in\) \(\Gamma\). Together with Proposition 4.1 from [1] we obtain that the axis of the cone \(\{y \leq -\|x\|\}\) translated by \((a, d(a))\) is a subset of \(M_\Gamma\) if and only if \(a\) belongs to \(M_X\).

\[\square\]

With the properties of \(\Gamma\) at hand we are ready to prove

**Theorem 2.2.** For any closed definable \(X \subset \mathbb{R}^m\) assuming \(0 \in M_X\) we have \(M_{m(0)} \subset C_0(M_X)\).

**Proof.** Without loss of generality we can assume that \(d(0) = 1\). As was shown in [1] during the proof of Theorem 4.6 \(M_{C(0,1)\Gamma} \subset C(0,1)M_\Gamma\). In order to prove the theorem we will establish the relation between those sets and \(M_{m(0)}\) and \(C_0(M_X)\).

Let’s begin with \(C_{(0,1)M_\Gamma}\). Since \((M_\Gamma - (0, 1)) \cap \{y \leq \|x\|\} = M_X \times \mathbb{R} \cap \{y \leq \|x\|\}\) the tangent cones of \(M_\Gamma - (0, 1)\) and \(M_X \times \mathbb{R}\) must coincide in the cone \(\{y \leq -2\|x\|\}\). Because \(\mathbb{R}\) is a cone, we obtain in
the end the equity of $C_{(0,1)}M_{\Gamma}$ and $C_0M_{\mathcal{X}} \times \mathbb{R}$ in the aforementioned cone.

As it comes to $M_{C_{(0,1)}\Gamma}$ we will study first the set $C_{(0,1)}\Gamma$. The explicit formula for the directional derivative $D_xd(0,X)$ allows us to express $C_{(0,1)}\Gamma$ as a graph of the function

$$x \rightarrow D_xd(0,X) = \frac{1}{2}(|x|^2 + 1 - d(x, m(0))^2).$$

Consider for a moment the graph $\Gamma_1$ as in the case for $\Gamma$ namely $x \rightarrow D_xd(0,X)$. The medial axis of the epigraph $d(x, m(0))$ after the translation by $(0, -1)$ have to coincide with $M_{C_{(0,1)}\Gamma}$, thus their intersections with the cone $\{y \leq -2|x|\}$ are also equal.

We have obtained

$$C_0M_{\mathcal{X}} \times \mathbb{R} \cap \{y \leq -2|x|\} = C_{(0,1)}M_{\Gamma} \cap \{y \leq -2|x|\}$$

and

$$M_{m(0)} \times \mathbb{R} \cap \{y \leq -2|x|\} = M_{C_{(0,1)}\Gamma} \cap \{|y| \leq -2||x||\}.$$

As the following example shows we cannot expect the equality between $M_{n}M_{X}$ and $M_{m(a)}$ in general.

**Example 2.3.** Let $X = \{(x, y, z) \in \mathbb{R}^3 | z^2 = 1, (x + y)(x - y) = 0\}$, then we have

- $M_{X} = \{ (x, y, z) \in \mathbb{R}^3 | xy = 0, z \neq 0 \} \cup \{ (x, y, z) \in \mathbb{R}^3 | z = 0 \}$
- $m(0) = \{(0, 0, 1), (0, 0, -1)\}$
- $M_{m(0)} = \{(x, y, z) \in \mathbb{R}^3 | z = 0 \}$

It is easy to check that indeed $M_{m(0)}$ is a proper subset of $C_0M_{\mathcal{X}}$.

It is also possible to state the condition under which the sets become equal.

**Corollary 2.4.** Assume that $0 \in M_{X}$ for a closed definable $X \subset \mathbb{R}^n$. If there exist a neighbourhood of the origin $U$ and $r > 0$ such that for any $a \in U$ there is diam $m(a) > r$ then $C_0M_{X} = M_{m(0)}$.

**Proof.** Theorem 2.2 gives us one of the inclusions, to prove the other start by taking $v \in C_0M_{X}$.

By the definition we can find subsequences $\{a_n\}$ in $M_{X}$ and $\{\lambda_n\}$ in $\mathbb{R}^+$ such that $a_n \rightarrow 0, \lambda_n \rightarrow 0, a_n/\lambda_n \rightarrow v$.

Take any convergent sequence of elements $x_n \in m(a_n) \rightarrow x \in m(0)$, we will show that $x \in m_{m(0)}(v)$. Since the additional assumption on the diameters of $m(a_n)$ ensures diam lim sup $m(a_n) \geq r > 0$ that will mean $v \in M_{m(0)}$. 
Consider $B(a_n, d(a_n)) \cap S(0, d(0))$. For $\|a_n\| < d(0)$ it is an open ball $B_n$ in $S(0, d(0))$ centered at $a_n/\|a_n\|$. Moreover $B(a_n, d(a_n)) \cap X = \emptyset$ also ensures $B_n \cap m(0) = \emptyset$. Since $x_n \to x$, the balls $B_n$ also converge to a certain ball $B$ centered at $v$ with $x$ on its border. Of course $B \cap m(0) = \emptyset$ which proves that $x$ is the closest point to $v$ in $m(0)$.

Theorem 2.2 yields two immediate corollaries more.

**Corollary 2.5.** In the considered situation $\dim_a M_X \geq \dim M_{m(a)}$.

**Proof.** From the Theorem 2.2 $M_{m(a)} \subset C_a(M_X)$, thus $\dim C_a(M_X) \geq \dim M_{m(a)}$. Since $M_X$ is definable $\dim M_X \geq \dim C_a(M_X)$ and the assertion follows.

**Corollary 2.6.** Point $a \in M_X$ is isolated in $M_X$ if and only if $m(a)$ is a full sphere.

**Proof.** Sufficiency of the condition is obvious. For the proof of the necessity suppose that $m(a)$ is not a full sphere. It is easy to observe (for example using the compactness of the sphere and the continuity of the distance function), that its medial axis is a cone of dimension $\dim M_{m(a)} > 0$. From Corollary 2.5 we derive that $\dim_a M_X > 0$.

In other words every hole in $m(a)$ enables the escape of $M_X$ in its direction.

**References**

[1] Lev Birbrair and Maciej Denkowski. Medial axis and singularieties. *J. Geom. Anal.*, 27(3):2339–2380, 2017.

[2] Lev Birbrair and Dirk Siersma. Metric properties of conflict set. *Houston Mathematical Journal*, 35(1):73–80, 2009.