Reversibility revival in non-Markovian quantum evolutions

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The relation between the thermodynamic entropy production and non-Markovian evolutions is matter of current research. Here, we study the behavior of the stochastic entropy production in open quantum systems undergoing unitary non-Markovian dynamics. In particular, we show that in some specific time intervals both the average entropy production and the variance can decrease, provided that the quantum dynamics fails to be P-divisible. This may be interpreted as a transient tendency towards reversibility, described as a delta peaked distribution of entropy production around zero. Finally, we also provide analytical bounds on the parameters in the generator giving rise to the quantum system dynamics, so as to ensure reversibility revival of the corresponding non-Markovian evolution.

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In out-of-equilibrium settings the entropy production is a fundamental thermodynamic quantity allowing to measure the degree of irreversibility of the dynamical evolution of physical systems. Classically, in the framework of stochastic thermodynamics [1], this irreversible contribution lies in the ratio (different from one) between the probability to observe a specific trajectory of the system and the probability to observe its time-reversed partner. The discrepancy between them is a consequence of an irreversible loss of the system internal energy, usually in the form of heat [2, 3].

A similar framework of stochastic thermodynamics has been developed for quantum systems too [4–12], where the trajectory has to be mapped into the sequence of outcomes of measurements performed on the system. This scheme, where just two measurements are taken into account, has been successfully used to prove a number of fluctuation relations that hold far from equilibrium and allow for the derivation of all the statistical moments of thermodynamic quantities [13–21]. In such a framework, relevant information about the dynamics of closed and open quantum systems can be extracted by looking at the probability distribution of the quantum entropy production [8, 9]. In this respect, a dynamics is said (fully) reversible if it is unitary and the effect due to possible measurements is negligible. When the dynamics is non-unitary and there are memory noise effects, one usually deals with non-Markovian quantum dynamics, which has become a topic of extensive research in the last decades [22–24]. Apart from the theoretical foundational interest, this is also due to a number of experimental platforms where non-Markovianity turns out to be necessary to fully catch the relevant physics [25–28].

Here, we address the relation between the non-Markovianity of the evolution of a quantum system and its thermodynamic reversibility, as described by the first two moments of the entropy production distribution. First results on this topic have started to appear very recently [29–34], only considering the average entropy production rate. However, it is reasonable to expect that the non-Markovian character of the quantum dynamics may display some relevant features on the whole distribution of the entropy production, and not only on the 1st moment. In this respect, we show that it is possible to have time intervals where both the 1-st and 2-nd cumulant of the quantum entropy distribution are decreasing, if the dynamics display a strong form of non-Markovianity known as essential non-Markovianity [35]. The importance of our result lies in the following consideration: Despite essential non-Markovian evolutions allow for the existence of a time interval in which the average entropy production rate is negative [30], this does not necessarily imply a mitigation of irreversibility in general, due to huge fluctuations of the entropy production. In other terms, although the mean value of the entropy distribution could decrease in a given time interval, the variance may not have the same behaviour. What we show is that, already at the level of qubits, it is possible to have a transient reduction of both the average entropy production and the variance, thus signalling a tendency towards reversibility.

Essential Non-Markovianity.— Many different approaches to quantum non-Markovianity can be found in the literature [22–24]. In the following, we adopt the point of view first presented in Ref. [36], associating the concept of quantum Markovianity to the divisibility of the dynamical evolution. In particular, given a quantum evolution described by a one-parameter family of completely positive (CP) and trace preserving (TP) maps \( \Lambda_t \), one says that the dynamics is CP-divisible if for any \( t, s \) such that \( t \geq s \geq 0 \) one has \( \Lambda_t = V_{t,s} \Lambda_s \), with \( V_{t,s} \) CP map. A dynamics is Markovian if it is CP-divisible, otherwise is non-Markovian otherwise. Actually, one can go a step further and distinguish between...
different degrees of non-Markovianity, as suggested in [35], depending on whether the intertwining map \( V_{t,s} \) is \( k \)-positive, namely whether the map \( V_{t,s} \otimes \text{id}_k \) is positive (\( \text{id}_k \) is the identity map on \( \mathbb{C}^k \)) or not. Given a Hilbert space of dimension \( n \), dynamical maps corresponding to \( n \)-positive \( V_{t,s} \) are CP-divisible, those corresponding to \( V_{t,s} \) that are only 1-positive are called P-divisible, while if a dynamics is not even P-divisible we call it as essentially non-Markovian. Resorting to a recently proved inequality [37], essential non-Markovianity is necessary condition to find negative entropy production rates [30], even though it may not be sufficient [32]. In this paper we consider unital dynamics, namely those evolutions that preserve the identity operator, fixed point of the map. Among them we focus on Pauli channels, whose Markovianity degree has been studied in detail in these recent papers [38–41].

Non-Markovian Pauli channels.– In the following, we consider a two-level quantum system described by a density matrix \( \rho_t \) evolving in time through a unital dynamics. Notice that any unital qubit dynamics can be always described by a random unitary map [42, 43] (this is not true in higher dimension) that corresponds to the family of Pauli channels. They are defined through the following Kraus representation:

\[
\Lambda_t(\rho) = \sum_{\alpha=0}^{3} p_\alpha(t) \sigma_\alpha \rho \sigma_\alpha, \tag{1}
\]

where \( \{\sigma_\alpha\}^3_{\alpha=1} = \{1, \sigma_x, \sigma_y, \sigma_z\} \) is the set of Pauli matrices plus the identity, while the coefficients \( p_\alpha \) obey the relation \( \sum_\alpha p_\alpha(t) = 1, \forall t \) (trace preservation). The initial condition \( \Lambda_0 = \text{id} \) implies that \( p_0(0) = 1 \) and \( p_\alpha(0) = 0 \) for \( \alpha \neq 0 \). The map \( \Lambda_t(\rho) \) is CP if \( p_\alpha(t) \geq 0, \forall t, \alpha \).

The conditions for CP-divisibility of the system dynamics are usually provided by introducing the generator \( \mathcal{L}_t \) associated to the quantum map (1). The generator satisfies the differential equation \( \partial_t \Lambda_t = \mathcal{L}_t \Lambda_t \). Thus, under the hypothesis that the inverse \( \Lambda_t^{-1} \) exists, the generator is defined as \( \mathcal{L}_t = \partial_t \Lambda_t \circ \Lambda_t^{-1} \) [38] with \( \circ \) denoting the composition of quantum maps. In this respect, one finds that the invertibility of the quantum map is ensured if \( p_1 + p_2, p_2 + p_3 \) and \( p_1 + p_3 \) are different from 1/2 for any \( t \). Moreover, being \( p_1(0) = p_2(0) = p_3(0) = 0 \), the invertibility of the map together with the continuity in time of the functions \( p_\alpha \) implies \( p_1 + p_2 < 1/2, p_2 + p_3 < 1/2 \) and \( p_1 + p_3 < 1/2, \forall t \). This working hypothesis will be assumed in the following.

As discussed in the Supplemental Material (SM), the Pauli channels generator can be written in the following general form [35, 38, 39]

\[
\mathcal{L}_t(\rho) = \sum_{\alpha=1}^{3} \gamma_\alpha(t) \left( \sigma_\alpha \rho \sigma_\alpha - \rho \right), \tag{2}
\]

where \( \gamma_\alpha \) are the so-called Lindblad coefficients. Therefore, the dynamics originated from \( \Lambda_t(\rho) \) is CP-divisible if and only if \( \gamma_\alpha \geq 0, \forall t, \alpha \), while necessary and sufficient condition for P-divisibility is \( \gamma_\alpha(t) + \gamma_\beta(t) \geq 0 \) with \( \alpha, \beta = 1, 2, 3 \) and \( \alpha \neq \beta \). Finally, one also finds that the relations linking together all the Kraus and Lindblad coefficients of the map are given by the following equation, with \( \alpha, \beta = 1, 2, 3 \) and \( \alpha \neq \beta \) (see also SM):

\[
\exp \left(-2 \int_0^t \gamma_\alpha(s) + \gamma_\beta(s) \, ds \right) = 1 - 2[p_\alpha(t) + p_\beta(t)]. \tag{3}
\]

Stochastic quantum entropy production.– The distribution of the quantum entropy production, originated by a generic quantum dynamics, can be obtained by realizing two distinct experimental procedures, i.e., a forward and a backward protocol that are appropriately chosen [8, 9, 11, 12]. Both protocols are interspersed by the application of two projective measurements at the initial and final time instants, according to the the well-known two-point measurement (TPM) scheme [14]. The two measurements are defined as projections on the eigenstates of the arbitrary observables \( O_{\text{in}} \) and \( O_{\text{fin}} \). By using the spectral decomposition theorem, the observables \( O_{\text{in}} \) and \( O_{\text{fin}} \) can be generally written as \( O_{\text{in}} = \sum_k a_k^{\text{in}} \Pi_k^{\text{in}} \) and \( O_{\text{fin}} = \sum_m a_m^{\text{fin}} \Pi_m^{\text{fin}} \), where \( \{\Pi\} \) is the set of projectors associated to the set of observable eigenvalues \{\( a \)\} (measurement outcomes). The stochastic quantum entropy production \( \Delta \sigma \) is then defined as [8]:

\[
\Delta \sigma(a^{\text{fin}}_m, a^{\text{in}}_k) \equiv \ln \frac{p_F(a^{\text{fin}}_m, a^{\text{in}}_k)}{p_B(a^{\text{fin}}_m, a^{\text{in}}_k)}, \tag{4}
\]

where \( p_F(a^{\text{fin}}_m, a^{\text{in}}_k) \) and \( p_B(a^{\text{fin}}_m, a^{\text{in}}_k) \) are the joint probabilities to simultaneously measure the outcomes \{\( a \)\} in a single realization of the forward and backward processes, respectively [44]. In Eq. (4) \( a^{\text{ref}}_m \) is obtained from the state after the 1st measurement of the backward process, which is generally named as reference state. Explicitly, the joint probabilities read \( p_F(a^{\text{fin}}_m, a^{\text{in}}_k) = \text{Tr}[\Pi_m^{\text{fin}} \Lambda^{F}_{\text{fin}}(\Pi_k^{\text{in}})]p(a^{\text{in}}_k) \) and \( p_B(a^{\text{fin}}_m, a^{\text{in}}_k) = \text{Tr}[\Pi_m^{\text{fin}} \Lambda^{B}_{\text{fin}}(\Pi_k^{\text{in}})]p(a^{\text{ref}}_m) \). If the CPTP map \( \Lambda_t \) governing the system dynamics is unital, then it is customary to consider the backward dynamics \( \Lambda^B_t \) as the dual of the forward one (because it is itself a proper quantum dynamics). As a result, the stochastic quantum entropy production \( \Delta \sigma \) becomes \( \Delta \sigma(a^{\text{fin}}_m, a^{\text{in}}_k) = \ln \frac{p(a^{\text{fin}}_m)}{p(a^{\text{in}}_k)} \), with \( p(a^{\text{ref}}_m) \) denoting the probability to get the measurement outcome \( a^{\text{ref}}_m \).

It is reasonable to choose the reference state equal to the final density operator after the 2nd measurement of the forward process. This means that for our purposes the stochastic quantum entropy production is equal to:

\[
\Delta \sigma(a^{\text{fin}}_m, a^{\text{in}}_k) = \ln p(a^{\text{in}}_k) - \ln p(a^{\text{fin}}_m), \tag{5}
\]

where \( p(a^{\text{fin}}_m) \) denotes the probability to measure the \( m \)-th outcome at the final time instant \( t_{\text{fin}} \).

The statistics of the stochastic quantum entropy production \( \Delta \sigma \) can be computed by evaluating the corresponding probability distribution \( \text{Prob} (\Delta \sigma) \). Each time we repeat the TPM scheme, one has a different realization for \( \Delta \sigma \) within a set of discrete values, whereby \( \text{Prob}(\Delta \sigma) \) is fully determined by the knowledge of the measurement outcomes and the respective probabilities:

\[
\text{Prob}(\Delta \sigma) = \sum_{k,m} \delta \left[ \Delta \sigma - \Delta \sigma(a^{\text{fin}}_m, a^{\text{in}}_k) \right] p(a^{\text{in}}_k, a^{\text{fin}}_m), \tag{6}
\]
where \( \delta[.] \) denotes the Kronecker delta and
\( p(a^n_k, a^m_n) = Tr[\Pi^m_n \Lambda_{t_{fin}}(\Pi^m_n)] p(a^n_k) \) with
\( p(a^n_k) = Tr[\rho_0 \Pi^m_k] \).

All the statistical moments of \( \Delta \sigma \) can be obtained by using
the characteristic function \( G_{\Delta \sigma}(u) \equiv \int \text{Prob}(\Delta \sigma) e^{iu\Delta \sigma} d\Delta \sigma \)
with \( u \in \mathbb{C} \). As formally shown in the SM, there exists a
\textit{closed-form expression} for each quantum entropy statistical
moment, provided that a TPM scheme is used to derive the
entropy fluctuations. As a consequence, one can determine the
1st and 2nd moments of \( \Delta \sigma \). The former equals to
\[
\langle \Delta \sigma \rangle = -Tr[\ln \rho_t \Lambda_{t_{fin}}(\rho_{t_{fin}})] + Tr[\rho_{t_{fin}} \ln \rho_{t_{fin}}] \\
= \Delta S + S(\rho_{t_{fin}}|\rho_t) \tag{7}
\]
with \( S(\rho|\sigma) \) denoting the quantum relative entropy of \( \rho \)
with respect to \( \sigma \) and \( \Delta S \equiv S(\rho_{t_{fin}}) - S(\rho_{t_{ini}}) \) the
difference of the von-Neumann entropies of \( \rho_{t_{fin}} \) and \( \rho_{t_{ini}} \). In Eq. (7),
\( \rho_{t_{fin}} = \sum_k p(a^n_k) \Pi^m_k \) and \( \rho_{t} = \sum_m p(a^m_m) \Pi^m_m \) are, respectively, the
ensemble average of the quantum system after the 1st and 2nd
measurements of the TPM scheme, while \( \rho_{t_{fin}} = \Lambda_t(\rho_{t_{ini}}) \) is the
density operator before the 2nd projective measurement.
Instead, the 2nd statistical moment of \( \Delta \sigma \) is given by the
following relation:
\[
\langle \Delta \sigma \rangle^2 = Tr[(\ln \rho_t)^2 \Lambda_{t_{fin}}(\rho_{t_{fin}})] \nonumber \\
- 2Tr[\ln \rho_t \Lambda_{t_{fin}}(\rho_{t_{fin}} \ln \rho_{t_{fin}})] + Tr[\rho_{t_{fin}}(\ln \rho_{t_{fin}})^2]. \tag{8}
\]
In this manuscript, we will mostly focus on the variance
that is related to the 2nd moment as usual, i.e.,
\[
\text{Var}(\Delta \sigma) \equiv \langle \Delta \sigma^2 \rangle - \langle \Delta \sigma \rangle^2. \tag{9}
\]

**Pauli channels and stochastic quantum entropy.**— We apply
the formalism of the stochastic thermodynamics to the
Pauli channel model. Without loss of generality, we take the
observable \( \mathcal{O} \), associated to both the quantum projective
measurements of the TPM, equal to the Pauli operator \( \sigma_z \).
As a result, the projectors \( \Pi^m \) and \( \Pi^m \) are described by the pure
states \( |l\rangle\langle l| \) with \( l \in \{0, 1\} \), whereby \( |0\rangle\langle 0| = (1 + \sigma_z)/2 \) and
\( |1\rangle\langle 1| = (1 - \sigma_z)/2 \). By initializing the system in the state
\( \rho_0 \), the 1st measurement of the TPM scheme makes the
quantum system collapse in one of the eigenstates of \( \sigma_z \). Thus,
the ensemble average of the system after such a measurement
is given by the mixed state \( \rho_{t_{ini}} \) with diagonal elements
\( 1/\delta_{t} \) and \( 1-1/\delta_{t} \), where \( \delta_{t} \equiv 2\theta_{10}^{(1)} - 1 \) and \( \theta_{10}^{(1)} \equiv |1\rangle\langle 0| \) .
Since \( \Lambda_t(|0\rangle\langle 0|) = \frac{|1\rangle + 2(1/2 - p_1 - p_2)\sigma_z)/2}{2} \), also \( \rho_{t_{fin}} = \Lambda_t(\rho_{t_{ini}}) \)
\begin{align*}
\text{is a mixed state} & , \text{with} \frac{1+\delta_{t}}{2} \text{and} \frac{1-\delta_{t}}{2} \text{being its diagonal elements. Here,} \\
\delta_{t} & \equiv \frac{1}{2} \theta_{10}, \quad \text{where} \\
0 & \leq \delta_{t} \leq 1 - 2|p_1(t) + p_2(t)| = e^{-2f_t}|\gamma_1(s) + \gamma_2(s)| ds \leq 1. \tag{10}
\end{align*}

Since \( \sigma_z \), \( \rho_{t_{ini}} \), one has \( \rho_t = \rho_{t_{ini}} \). Hence, simplifying
the expressions of \( \langle \Delta \sigma \rangle \) and \( \langle \Delta \sigma^2 \rangle \), we obtain
\( \langle \Delta \sigma \rangle = \Delta S \),
while the 2nd moment of \( \Delta \sigma \) in case of initial pure state with
\( \rho_0 = 1 \), reads \( \langle \Delta \sigma^2 \rangle = Tr[(\ln \rho_{t_{fin}})^2 \Lambda_t(\rho_{t_{fin}})] \).

**Revival of thermodynamic reversibility.**— In this paragraph,
we present our results relating non-Markovianity and
stochastic entropy production. They are based on the
computation of the time-derivative of the average \( \langle \Delta \sigma \rangle \) and the
variance \( \text{Var}(\Delta \sigma) \). Depending on the final time \( t \) of the TPM
scheme, one has a different probability distribution for the
entropy production, with an average value that typically (always
if the dynamics is P-divisible) increases with time and a non-
monotonic behaviour on the variance. Below we are going to
show that it is possible to have a time interval in which
the average decreases and the width of the distribution
decreases as well, thus signaling a reversibility revival.

For simplicity, let us assume \( \delta_0 = 1 \). The dynamics origin-
ated by a Pauli channel is unital. However, for a non-P-di-
visible unital quantum map the time-derivative of \( \langle \Delta \sigma \rangle \) is not
necessarily positive. Indeed, \( \partial_t \langle \Delta \sigma \rangle \) explicitly reads
\[
\partial_t \langle \Delta \sigma \rangle = \ln \left( \frac{1 + \lambda_t}{1 - \lambda_t} \right) \langle \gamma_1(t) + \gamma_2(t) \rangle \lambda_t \tag{11}
\]
so that it is negative whenever the sum \( \gamma_1(t) + \gamma_2(t) \) becomes
negative, namely when the dynamics fails to be P-divisible.
Then, in order to see some effects due to non-Markovianity
on system reversibility, we look at the time-derivative of the
variance \( \text{Var}(\Delta \sigma) \) and study its sign. The latter is explicitly
given by
\[
\partial_t \text{Var}(\Delta \sigma) = \delta_t \ln \left( \frac{1 + \lambda_t}{1 - \lambda_t} \right) \left[ \langle \Delta \sigma \rangle + 1 + \frac{1}{2} \ln \left( \frac{1 - \lambda_t^2}{4} \right) \right] \tag{12}
\]
As shown in the SM, the sign of \( \partial_t \text{Var}(\Delta \sigma) \) can be
determined by looking at the sign of the function
\[
\chi_t \equiv [\gamma_1(t) + \gamma_2(t)] \tag{13}
\]
with \( \delta_t \equiv \frac{\lambda_t}{2} \ln \left( \frac{1 + \lambda_t}{1 - \lambda_t} \right) - 1 \). Therefore, it is convenient to
study \( f_t \) and its time-derivative. On one hand, the function \( f_t \)
is known to be always greater or equal than \( -1 \) and is such that
\( \lim_{t \to \infty} f_t = +\infty \) and \( \lim_{t \to -\infty} f_t = -1 \). On the other
hand, by computing \( \partial_t f_t \), one observes that \( f_t \) is increasing or
decreasing depending of \( \delta_t \). In particular, it is increasing in
the region where \( \gamma_1(t) + \gamma_2(t) \) is negative, which means when
P-divisibility is broken. Therefore, assuming \( \gamma_1 + \gamma_2 \leq 0 \)
in a single interval \( [t_1, t_2] \), the function \( f_t \) is decreasing up to
\( t_1 \), then increases from \( t_1 \) to \( t_2 \) and finally decreases for
\( t > t_2 \). As a result, one can have three different cases for the
sign of \( f_t \):
\[
(\text{I}) \quad f_{t_1} \geq 0 \text{ and } f_{t_2} \geq 0; \quad (\text{II}) \quad f_{t_1} < 0 \text{ and } f_{t_2} \geq 0; \quad (\text{III}) \quad f_{t_1} < 0 \text{ and } f_{t_2} < 0. \tag{14}
\]

In Fig. 1 we report a sketch of the possible behaviour of \( f_t \)
as a function of time in the three different cases (I), (II) and
(III), corresponding to three different situations for the sign of
\( \partial_t \text{Var}(\Delta \sigma) \) in the interval \( [t_1, t_2] \), where the time-derivative
of the average is also negative. In particular, one has:
\[
(\text{I}) \quad \partial_t \text{Var}(\Delta \sigma) \leq 0 \text{ in } [t_1, t_2] \\
(\text{II}) \quad \partial_t \text{Var}(\Delta \sigma) \geq 0 \text{ in } [t_3, t_2] \text{ with } t_1 < t_3 < t_2 \\
(\text{III}) \quad \partial_t \text{Var}(\Delta \sigma) \geq 0 \text{ in } [t_1, t_2]. \tag{15}
\]
Therefore, in cases (I) and (II) there is a time interval in which the system tends to be more reversible, in the sense that both the average and the variance of Prob(Δσ) are reducing, so that the distribution becomes sharper. As a matter of fact, the reversibility of a quantum system dynamics is associated to a shrinking of the quantum entropy distribution Prob(Δσ) up to approach a Kronecker delta δ[Δσ]. So, the decreasing of ∂tProb(Δσ) and ∂tVar(Δσ) in a given time interval can be considered as a witness of a reversibility revival of the system dynamics, induced by the presence of non-Markovian effects.

Now, we provide analytical bounds on the Kraus coefficients pα(t) of the Pauli channel that are sufficient to achieve reversibility revival. Above, provided that the dynamics of the system is not P-divisible, we have shown that the tendency of Var(Δσ) to decrease just depends on the sign of ft. By introducing φt ≡ ∫0t[γ1(s) + γ2(s)]ds, the inequality ft ≥ 0 can be recast in the relation

\[ e^{-2φt} \ln \left( \frac{1 + e^{-2φt}}{1 - e^{-2φt}} \right) \geq 2 . \]  

(16)

The function \( x \ln \left( \frac{1 + e^{-2x}}{1 - e^{-2x}} \right) - 2 \), with 0 ≤ x < 1, has an unique zero at \( x^* \approx 0.8336 \) and is positive for \( x \geq x^* \). This implies that the inequality (16) is verified for \( x \geq x^* \), i.e.,

\[ 0 \leq φt ≤ φ^* \equiv -\frac{1}{2} \ln(x^*) \approx 0.091 \]  

(17)

for all \( t > 0 \). Eq. (17) clearly shows that the reversibility revival can be found only in a quite small range of dynamical parameters. This means that essential non-Markovianity has to be usually associated to irreversibility, except some narrow regimes whereby a transient tendency to reversibility could be observed.

Analytical example.— Here, we present an example of legitimate (namely completely positive and trace preserving) unital dynamics for a qubit such that the evolution is not P-divisible in a single time interval \([t_1, t_2]\). As discussed before, one has to satisfy the following constraints:

(i) \( \gamma(t) = \gamma_1(t) + \gamma_2(t) \leq 0 \) in \([t_1, t_2]\), (no P-divisibility)

(ii) \( φt \geq 0 \) for any \( t \geq 0 \), (CP dynamics).

This in turn implies that \( λ_t = e^{-2φt} \leq 1 \). We assume the following explicit form for the function \( γ(t) \)

\[ γ(t) = γ - e^{-αt} \left( 1 - e^{-αt} \right) , \]  

(18)

where \( α, γ > 0 \) are two positive parameters. As a consequence the function \( φ_1 \) reads \( φ_1 = γt - \left(1 - e^{-αt} \right)^2 \). Then, the sign of \( γ(t) \) can be easily studied. In particular, one finds that two zeros exist at times \( t_1 \) and \( t_2 \) corresponding to

\[ t_1 = -\frac{1}{α} \ln \left( \frac{1 + \sqrt{1 - 4γ}}{2} \right) , \]  

\[ t_2 = -\frac{1}{α} \ln \left( \frac{1 - \sqrt{1 - 4γ}}{2} \right) , \]  

(19)

provided that \( γ < \frac{1}{4} \). Moreover, it turns out that \( γ(t) \) is negative between \( t_1 \) and \( t_2 \) and positive otherwise, thus satisfying condition (i). Instead, condition (ii) corresponds to the requirement \( γ ≥ \frac{(1 - e^{-αt})^2}{2αt} \), \( \forall t > 0 \). Therefore, one has to impose a lower bound \( τ \) to \( γ \), which is given by

\[ τ = \max_{t > 0} \frac{(1 - e^{-αt})^2}{2αt} . \]  

(20)

It can be easily found that the maximum of the function is implicitly defined by the relation \( e^{−2αt} = 1 + 2τ_max \), with \( τ_max \equiv αt_max \). Numerically, one obtains a value \( τ_max \approx 1.25 \) that in turn implies \( γ ≤ 2τ_max e^{-2αt} \approx 0.2 \) for any value of \( α \). As a result, conditions (i) and (ii) bound the parameter \( γ \) to be

\[ τ < γ < \frac{1}{4} \]  

(21)

because \( τ \approx 0.2 \) is a nontrivial lower bound smaller than 1/4. As shown in Fig. 2, one can span the three different regimes
corresponding to a vanishing function \( f_t \).

Conclusions.— We have further investigated on the relations between entropy production and non-Markovianity using the formalism of stochastic thermodynamics. We have shown that it is possible to have legitimate non-Markovian dynamics, namely 1-parameter families of completely positive and trace preserving maps, that allow for both the average entropy production and its variance to be transiently decreasing. This can happen when the dynamics is not P-divisible. Being a dynamics reversible if the distribution of the entropy production is a Kronecker delta, we interpret our finding as a transition in qubit dynamics, for which we provide analytical bounds in dependence of the parameter \( \alpha \). In particular, according to Eq. (17), these regimes are obtained by comparing the values \( \phi_{t_1}(\alpha) \) and \( \phi_{t_2}(\alpha) \) with \( \phi_0 = -\frac{1}{2} \ln(x^*) \approx 0.091 \), which is the value corresponding to a vanishing function \( f_t \).

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If we choose the measurement outcomes \( \{ a \} \) to be equal to the energies \( \{ E \} \) of the system at \( t_0 \) and \( t_{\text{fin}} \), then one can evaluate the energy entropy of the open quantum system [8].
A. Brief overview on Pauli channels

Pauli channels are quantum dynamical maps of the form

\[ \Lambda_i(g) = \sum_{\alpha=0}^{3} p_{\alpha}(t) \sigma_\alpha \otimes \sigma_\alpha, \]  

(S1)

where \( \{\sigma_\alpha\}_{0}^{3} = \{1, \sigma_x, \sigma_y, \sigma_z\} \) is the set of Pauli matrices plus the identity, the coefficients \( p_{\alpha} \) obey the relation \( \sum_{\alpha} p_{\alpha}(t) = 1, \forall t \) (trace preservation), and the initial condition \( \Lambda_0 = \text{id} \) enforces \( p_0(0) = 1 \). Each map \( \Lambda_i \) is completely positive if \( p_{\alpha}(t) \geq 0 \forall \alpha, t \geq 0 \). One can easily check that the Pauli matrices are the eigen-operators of the linear map \( \Lambda_i \) and, in particular, one has

\[ \begin{align*}
\Lambda_i(1) &= 1, \\
\Lambda_i(\sigma_1) &= (1 - 2p_2(t) - 2p_3(t))\sigma_1, \\
\Lambda_i(\sigma_2) &= (1 - 2p_1(t) - 2p_3(t))\sigma_2, \\
\Lambda_i(\sigma_3) &= (1 - 2p_1(t) - 2p_2(t))\sigma_3.
\end{align*} \]  

(S2)-(S5)

The map is invertible provided that \( p_1(t) + p_2(t) \neq 1/2, p_1(t) + p_3(t) \neq 1/2, p_2(t) + p_3(t) \neq 1/2 \) at any time \( t > 0 \). Since initially the coefficients \( p_i \) with \( i \in \{1, 2, 3\} \) are vanishing (because \( p_0(0) = 1 \)) we can enforce continuity of the functions \( p_i(t) \) and invertibility of the map \( \Lambda_i \) at any time if the constraints \( p_1(t) + p_2(t) < 1/2, p_1(t) + p_3(t) < 1/2, p_2(t) + p_3(t) < 1/2 \) are satisfied. For an invertible dynamics the time-dependent generator turns out to be \( \mathcal{L}_i = \partial_t \Lambda_i \circ \Lambda_i^{-1} \). By comparing the following ansatz for the generator

\[ \mathcal{L}_i(g) = \sum_{i=1}^{3} \gamma_i(t) (\sigma_i \otimes \sigma_i - g), \]  

(S6)

with the expression derived computing \( \partial_t \Lambda_i \) and \( \Lambda_i^{-1} \) one obtains the following relation between the functions \( \gamma_i(t) \) and the functions \( p_i(t) \)

\[ \gamma_i(t) + \gamma_j(t) = \frac{p_i(t) + p_j(t)}{1 - 2\partial_t(p_i(t) + p_j(t))}, \]  

(S7)

for any pair \( i, j \) with \( i \neq j \). The previous differential equations can be easily integrated by recognizing that

\[ \frac{p_i(t) + p_j(t)}{1 - 2\partial_t(p_i(t) + p_j(t))} = -\frac{1}{2} \partial_t \ln \left[ 1 - 2(p_i(t) + p_j(t)) \right], \]  

(S8)

so that finally one has

\[ p_i(t) + p_j(t) = \frac{1 - e^{-2 \int_0^t (\gamma_i + \gamma_j) ds}}{2}. \]  

(S9)

Therefore, we have two equivalent characterizations of the dynamics, one based on the Lindblad coefficients \( \gamma_i \) and the other one based on the Kraus parameters \( p_i \) and we know how to connect the two. This is important because the conditions for complete positivity (CP) are easily given for the \( p_i \) while conditions for CP-divisibility and P-divisibility are given on the \( \gamma_i \). These conditions are reported in the main text, as first derived in Phys. Rev. A 91 (1), 012104 (Ref. [39] of the main text).

B. Closed-form expression of quantum entropy statistical moments

The statistical moments of a random variable \( X \) with probability distribution \( \text{Prob}(X) \) can be generally computed by introducing the characteristic function

\[ G_X(u) = \int \text{Prob}(X)e^{iuX}dX \]  

(S10)
associated to $\text{Prob}(X)$, with $u$ complex number. In this regard, it holds that $\langle X^k \rangle = (-i)^k \partial^k_u G_X(u)\big|_{u=0}$, namely the $k$th statistical moment of $X$ is proportional to the $k$th derivative of $G_X(u)$ with respect to $u$ and evaluated at $u = 0$. This property can be thus applied to the computation of the statistical moments of $\Delta \sigma$, so that we are allowed to write

$$\langle \Delta \sigma^k \rangle = (-i)^k \partial^k_u G_{\Delta \sigma}(u)\big|_{u=0}. \quad (S11)$$

Provided that a TPM scheme is used to derive the fluctuations of entropy, we here show that there exists a closed-form expression for each quantum entropy statistical moment. In particular, the $k$-th statistical moment $\langle \Delta \sigma^k \rangle$, with $k \geq 1$ ($k$ arbitrary integer), is equal to

$$\langle \Delta \sigma^k \rangle = \sum_{n=0}^{l} (-1)^{l-n}\binom{l}{n} \text{Tr} \left[\left(\ln \varrho_t\right)^{l-n} A_{\text{fin}} \left((\ln \varrho_{\text{fin}})^n \varrho_{\text{fin}}\right)\right], \quad (S12)$$

where $\varrho_{\text{fin}} \equiv \sum_k p(a_{m}^{\text{fin}}) \Pi_k^{\text{fin}}$ and $\varrho_{\text{fin}} \equiv \sum_m p(a_{m}^{\text{fin}}) \Pi_m^{\text{fin}}$ are, respectively, the ensemble average of the quantum system after the 1st and 2nd measurement of the TPM scheme. The validity of (S12) can be easily shown starting from the definition of the stochastic variable $\Delta \sigma$, as given by Eq. (5) and its probability distribution (6) in the main text. Indeed, one gets

$$\langle \Delta \sigma^k \rangle = \sum_{m,k}^{} p(a_{m}^{\text{fin}},a_{k}^{\text{fin}}) \left[\Delta \sigma(a_{m}^{\text{fin}},a_{k}^{\text{fin}})\right] = \sum_{m,k}^{} \text{Tr} \left[\Pi_m^{\text{fin}} \Lambda_{\text{fin}} \Pi_k^{\text{fin}}\right] p(a_m^{\text{fin}}) \left[\ln p(a_m^{\text{fin}}) - \ln p(a_k^{\text{fin}})\right] =$$

$$= \sum_{n=0}^{l} (-1)^{l-n}\binom{l}{n} \text{Tr} \left[\sum_m^{} \Pi_m^{\text{fin}} \left(\ln p(a_m^{\text{fin}})\right)^{l-n} A_{\text{fin}} \left((\ln p(a_{m}^{\text{fin}}))^{n}\right)\right] =$$

$$= \sum_{n=0}^{l} (-1)^{l-n}\binom{l}{n} \text{Tr} \left[\left(\ln \varrho_t\right)^{l-n} A_{\text{fin}} \left((\ln \varrho_{\text{fin}})^n \varrho_{\text{fin}}\right)\right]. \quad (S13)$$

C. Reversibility revival: Details of the mathematical derivation

Here, we present further details on the mathematical derivation of our result relating non-Markovianity and stochastic entropy production of a qubit whose evolution is described by Pauli channels. In particular, we will compute the time-derivative of the first two statistical moments $\langle \Delta \sigma \rangle$ and $\langle \Delta \sigma^2 \rangle$, and then evaluate their sign. As first step, we start from the computation of the time-derivative of $\langle \Delta \sigma \rangle$. The latter, for the specific analyzed class of models, equals to

$$\Delta S \equiv S(\varrho_{\text{fin}}) - S(\varrho_{\text{in}}) = \frac{1 + \zeta_0}{2} \ln \left(\frac{1 + \zeta_0}{2}\right) + \frac{1 - \zeta_0}{2} \ln \left(\frac{1 - \zeta_0}{2}\right) - \frac{1 + \zeta_t}{2} \ln \left(\frac{1 + \zeta_t}{2}\right) - \frac{1 - \zeta_t}{2} \ln \left(\frac{1 - \zeta_t}{2}\right), \quad (S14)$$

where $\zeta_0 \equiv 2 \rho_0^{(11)} - 1$ and $\zeta_t \equiv \lambda_t \zeta_0$, with $\rho_0^{(11)}$ and $\lambda_t$ ($\lambda_t \in [0,1]$) defined as in the main text. Thus, by assuming without loss of generality that $\zeta_0 = 1$, $\partial_t \langle \Delta \sigma \rangle$ explicitly reads

$$\partial_t \langle \Delta \sigma \rangle = \ln \left(\frac{1 + \lambda_t}{1 - \lambda_t}\right) [\gamma_1(t) + \gamma_2(t)] \lambda_t. \quad (S15)$$

This implies that $\partial_t \langle \Delta \sigma \rangle$ is negative whenever the sum $\gamma_1(t) + \gamma_2(t)$ becomes negative, namely when the dynamics fails to be P-divisible.

As second step, in order to determine the existence of reversibility revival within the quantum system dynamics due to non-Markovian memory effects, we need to look also at the time-derivative of $\langle \Delta \sigma^2 \rangle = \text{Tr} \left[\left(\ln \varrho_{\text{fin}}\right)^2 A_{t}(\varrho_{\text{fin}})\right]$. By substituting the expressions of $\varrho_{\text{fin}}$ and $\varrho_{\text{fin}}$ depending on $\zeta_t$, one finds that

$$\partial_t \langle \Delta \sigma^2 \rangle = \dot{\zeta}_t \ln \left(\frac{1 + \lambda_t}{1 - \lambda_t}\right) \left[1 + \frac{1}{2} \ln \left(\frac{1 - \lambda_t^2}{4}\right)\right] \quad (S16)$$

for the time-derivative of the 2nd statistical moment and

$$\partial_t \text{Var}(\Delta \sigma) = \dot{\zeta}_t \ln \left(\frac{1 + \lambda_t}{1 - \lambda_t}\right) \left[\langle \Delta \sigma \rangle + 1 + \frac{1}{2} \ln \left(\frac{1 - \lambda_t^2}{4}\right)\right] \quad (S17)$$
for the stochastic entropy variance, where
\[ \dot{z}_t = -2 \left[ \gamma_1(t) + \gamma_2(t) \right] \lambda_t. \]  \hspace{1cm} (S18)

Let us study the sign of \( \partial_t \text{Var}(\Delta \sigma) \). In Eq. (S17), the factorized logarithm term is always positive and the term \( \dot{z}_t \) can be easily controlled, being its sign related to the P-divisibility of the dynamics. Instead, the term in square brackets has a nontrivial sign behaviour. This term can be rewritten as
\[ 1 - \frac{\lambda_t}{2} \ln \left( \frac{1 + \lambda_t}{1 - \lambda_t} \right), \]  \hspace{1cm} (S19)
so that we only need to look at the sign of
\[ \chi_t = \left[ \gamma_1(t) + \gamma_2(t) \right] \left[ \frac{\lambda_t}{2} \ln \left( \frac{1 + \lambda_t}{1 - \lambda_t} \right) - 1 \right] = \left[ \gamma_1(t) + \gamma_2(t) \right] f_t, \]  \hspace{1cm} (S20)
where
\[ f_t = \frac{\lambda_t}{2} \ln \left( \frac{1 + \lambda_t}{1 - \lambda_t} \right) - 1. \]  \hspace{1cm} (S21)

Therefore, it is convenient to study \( f_t \) and its time-derivative, i.e.
\[ \frac{df_t}{dt} = \dot{\lambda}_t \left[ \frac{1}{2} \ln \left( \frac{1 + \lambda_t}{1 - \lambda_t} \right) + \frac{\lambda_t}{1 - \lambda_t^2} \right]. \]  \hspace{1cm} (S22)

In Eq. (S22) the term in the brackets is always positive so that \( f_t \) is increasing or decreasing depending of \( \dot{\lambda}_t \). For the complete analysis of the sign of \( f_t \) and thus of \( \partial_t \text{Var}(\Delta \sigma) \), the reader can just refer to the main text.