Fermionic vacuum polarization induced by a non-Abelian vortex

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Abstract

In this paper, we analyze the fermionic condensate (FC) and the vacuum expectation value (VEV) of the energy-momentum tensor associated with an isospin-1/2 charged massive fermionic field induced by the presence of a SU(2) vortex, taking into account the effect of the conical geometry produced by this object. We consider the vortex as an idealized topological defect, i.e., very thin, straight and carrying a magnetic flux running along its core. Besides the direct coupling of the fermionic field with the iso-vector gauge field, we also admit the coupling with the scalar sector of the non-Abelian vortex system, expressed as a vector in the three-dimensional isospace. Due to this interaction, the FC is expressed as the sum of two contributions associated with the two different effective masses for the ±1/2 fermionic components of the isospin operator, τ/2. The VEV of the energy-tensor also presents a similar structure. The vacuum energy density is equal to the radial and axial stresses. As to the azimuthal one, it is expressed in terms of the radial derivative of energy-density. Regarding to the magnetic flux, both, the FC and the VEV of the energy-momentum tensor, can be positive or negative. Another interesting consequence of the interaction with the bosonic sector, the FC and VEV of the energy-momentum tensor, present different intensity for different values of the ratio between the scalar coupling constant and the mass of the fermionic field. This is a new feature that the system presents.

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1 Introduction

According to the Big Bang theory, the Universe has been undergone a series of phase transitions during its expansion, characterized by spontaneous symmetry breaking [1]. As consequence of these transitions, topological defects such as domain walls, monopoles, vortices, among others have been formed [2]. One of the first theoretical model analyzing the formation of topological objects, has been proposed by Nielsen and Olesen [3] many years ago. They pointed out that, in analogy with Landau-Ginzburg theory of superconductivity, the Lagrangian comprised by vector and Higgs fields, that undergo spontaneously U(1) gauge symmetry breaking, presents string-like solutions, named by vortex. The influence of this object on the geometry of the spacetime has been analyzed in [4,5], by coupling the energy-momentum tensor associated with the Nielsen and Olesen vortex system with the Einstein equations. The authors showed by numerical analysis...
that, asymptotically, the spacetime around the string is a Minkowski one minus a wedge. The core of the vortex has a nonzero thickness and a magnetic flux running along it. Two years later, Linet [6] was able to obtain exact solutions for the complete system, considering specific values for the parameters. Linet showed that the structure of the two-geometry orthogonal to the string is a conical one, with the conicity parameter being expressed in terms of the energy per unit length of the vortex. In addition to the Abelian vortex, in [3] has also been analyzed the formation of a $SU(2)$ vortex, where it has been proved that it is necessary the presence of two bosonic iso-vectors coupled to the gauge field.

More recently, the analysis of the geometry of the spacetime due to the presence of the $SU(2)$ non-Abelian Higgs vortex, has been developed in [7]. There it was shown that, similarly what happens in the Abelian system, static and cylindrically solution of the system is present, and asymptotically the space is flat with a deficit planar angle. In fact for the non-Abelian vortex, the deficit planar angle in the two-geometry orthogonal to the string is larger than the Abelian case for the same energy scale where the gauge symmetry is spontaneously broken.

In this way, the vortexes solutions of the Nielsen and Olesen model, can also be considered as good candidates to represent a cosmic string.

The geometry produced by an idealized cosmic string, i.e., infinitely long, straight with zero transverse size, has a conical topology with a planar angle deficit in the two-geometry orthogonal to the string proportional to its linear mass density. The vacuum polarization effects in quantum field theory induced by this object have been considered in a large number of papers. Specifically the analysis for the VEV of the energy-momentum tensor, has been developed for scalar, fermionic and electromagnetic fields [8]-[22]. For charged fields, considering the presence of a magnetic flux running along the cosmic strings, there appear additional contributions to the corresponding vacuum polarization effects [23]-[27].

The analysis of the dynamics of fermionic fields interacting with vortices in Abelian and $SU(2)$ non-Abelian gauge theories under relativistic quantum mechanics viewpoint, has been developed in [28]; moreover in a $(1 + 2)$-dimensions, considering quantum fluctuations around the non-Abelian vortex fields, the fermionic condensate, induced quantum numbers and Wilson loops, have been investigate in [29], [30] and [31], respectively.

As far as we know, the analysis of fermionic vacuum polarization has been developed considering Abelian vortex only; therefore, the main objective of this paper is to investigate this effects associated with a isospin-$1/2$ charged massive fermionic field, induced by the presence of a $SU(2)$ vortex, considering the effect of the conical geometry produced by this object, and also taking into account the coupling of the fermionic field with the iso-scalar sector of the system. Specifically we want to calculate the fermionic condensate (FC), $\langle \bar{\Psi} \Psi \rangle$, and the vacuum expectation value (VEV) of the corresponding energy-momentum tensor, $\langle T_{\mu}^{\nu} \rangle$. In order to develop these calculations, we will assume the idealized case, where the size of the vortex is considered zero.

The paper is organized as follows. In Sec. 2 we present the system that we want to investigate. First we introduce the $SU(2)$ non-Abelian vortex system, and the influence of this system on the geometry of the space-time. It is also presented the Lagrangian density associated with the isospin$-1/2$ fermionic field coupled to a non-Abelian gauge fields, and the iso-vector bosonic sector. Also we provide the complete set of normalized positive- and negative-energy fermionic wave-functions. By using the mode-summation procedure, we furnish in Sec. 3 the main steps to calculate the fermionic condensate, FC. It is presented some analytical results for the FC for some limiting values of the parameters of the system, and relevant plots displaying the behavior of FC. The calculation of the VEV of the energy-momentum tensor is developed in Sec. 4 by using the mode-summation procedure; there we also provide some asymptotic expressions for the energy-density for specific values of the parameters, and also relevant plots. The properties satisfied by this tensor are analyzed in Sec. 5. Finally in Sec. 6 we summarize the most relevant
results of this paper, and present a brief comments about them. Throughout the paper we use the units with $G = \hbar = c = 1$

2 The model

2.1 SU(2) vortex

The SU(2) vortex configuration can be obtained by the matter Lagrangian density below:

$$\mathcal{L}_m = -\frac{1}{4} F_{\mu\nu}^a F^{\mu\nu a} + \frac{1}{2} (D_\mu \varphi^a)^2 + \frac{1}{2} (D_\mu \chi^a)^2 - V(\varphi^a, \chi^a), \quad a = 1, 2, 3,$$

being the field strength tensor given by,

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g \epsilon^{abc} A_\mu^b A_\nu^c,$$

expressed in terms of the SU(2) gauge potential $A_\mu^b$, with $g$ being the gauge coupling constant.

The covariant derivatives of the isovectors Higgs fields are given by

$$D_\mu \varphi^a = \partial_\mu \varphi^a + g \epsilon^{abc} A_\mu^b \varphi^c,$$

$$D_\mu \chi^a = \partial_\mu \chi^a + g \epsilon^{abc} A_\mu^b \chi^c.$$

The latin indexes denote the internal gauge groups ($a, b = 1, 2, 3$). The analysis of the influence of this system on the geometry of the spacetime was developed in [7] considering a non-negative interaction potential, $V(\varphi^a, \chi^a)$, that admits spontaneous symmetry breaking. Moreover, the following configurations for the iso-scalar fields have been taking into account,

$$\varphi(r, \phi) = f(r) \begin{pmatrix} \cos(\phi) \\ \sin(\phi) \\ 0 \end{pmatrix},$$

$$\chi(r, \phi) = h(r) \begin{pmatrix} -\sin(\phi) \\ \cos(\phi) \\ 0 \end{pmatrix} \quad (2.5)$$

and for the iso-vector field,

$$A_t^a = A_r^a = A_z^a = 0, \quad \text{and} \quad A_\phi^a = -\frac{A(r)}{g} \delta_{a,3}, \quad a = 1, 2, 3.$$

We can see that both iso-vectors bosonic fields satisfy the orthogonality condition, $\varphi^a \chi^a = 0$.

Considering cylindrically symmetric line element invariant under boosts along $z-$direction below,

$$ds^2 = N^2(r)dt^2 - dr^2 - L^2(r)d\phi^2 - N^2(r)dz^2,$$

in [7], was shown that for points outside the vortex, the metric tensor functions, $N(r)$, $L(r)$ and the gauge function $A(r)$, satisfy the asymptotic behavior below,

$$N \to 1, \quad L \to \beta r, \quad A \to 1.$$

being $\beta$ a constant smaller than unity, providing a planar angle deficit, $\delta = 2\pi(1 - \beta)$, on the two-geometry orthogonal to the vortex.

The idealized vortex configuration corresponds to the situation where all the fields and metric tensor can be represented by their corresponding vacuum values. This approximation is valid in the analysis of phenomena where the energies considered are smaller than the energy scale where the gauge symmetry is broken.
2.2 Fermionic field

Now let us consider the charged massive isospin $-1/2$ fermionic field coupling to isovectors, $\varphi^a$, $\chi^a$ and $A_\mu^a$ in curved spacetime. The corresponding Lagrangian density is,

$$\mathcal{L} = \bar{\psi} \left[ i \gamma^\mu \left( \nabla_\mu + i e A_\mu^a \tau^a / 2 \right) - m \right] \psi + \bar{\psi} \tau^a \psi (g_1 \varphi^a + g_2 \chi^a),$$

(2.10)

where $\tau^a$ are the Pauli matrices acting on the isospin indices, and $\gamma^\mu$ the Dirac matrices in curved spacetime. In (2.10), $e$, $g_1$ and $g_2$ correspond to the coupling constants between the fermionic field to the gauge and scalars iso-vectors, $A_\mu^a$, $\varphi^a$ and $\chi^a$, respectively.

The fermionic covariant derivative reads,

$$\nabla_\mu = \partial_\mu + \Gamma_\mu,$$

(2.11)

being $\Gamma_\mu$ the spin connection. Both matrices are given in terms of the flat spacetime Dirac matrices, $\gamma^{(a)}$, by the relations,

$$\gamma^\mu = e^{\mu}_{(a)} \gamma^{(a)}, \quad \Gamma_\mu = \frac{1}{4} \gamma^{(a)} \gamma^{(b)} e^{(a)}_{\nu} e^{(b)}_{\nu} \gamma_\mu.$$

(2.12)

In (2.12), $e^{\mu}_{(a)}$ represents the tetrad basis satisfying the relation $e^{\mu}_{(a)} e^{\nu}_{(b)} \eta^{ab} = g_{\mu\nu}$, with $\eta^{ab}$ being the Minkowski spacetime metric tensor.

The analysis that we want to perform here is the calculations of the fermionic vacuum fluctuations induced by the presence of the vortex configuration. To make this analysis simpler we will assume the idealized configuration for the vortex. In this way we consider that the geometry of the spacetime can be written, by using cylindrical coordinates, through the line element below:

$$ds^2 = dt^2 - dr^2 - r^2 d\phi^2 - dz^2,$$

(2.13)

where the coordinates take values in the ranges $r \geq 0$, $\phi \in [0, 2\pi/q]$, $-\infty < (t, z) < +\infty$. In the metric tensor above, we have redefined the azimuthal angular coordinate incorporating the parameter $\beta$ specified in (2.9); so the parameter $q = 1/\beta \geq 1$. In the geometry described by (2.13) the gamma matrices can be taken in the form [33]-[34]:

$$\gamma^0 = \gamma^{(0)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^l = \begin{pmatrix} 0 & \sigma^l \\ -\sigma^l & 0 \end{pmatrix},$$

(2.14)

where for the $2 \times 2$ Pauli matrices $\sigma^l$ in this conical geometry, with $l = (r, \phi, z)$, reads,

$$\sigma^r = \begin{pmatrix} 0 & e^{-iq\phi} \\ e^{iq\phi} & 0 \end{pmatrix}, \quad \sigma^\phi = -\frac{i}{r} \begin{pmatrix} 0 & e^{-iq\phi} \\ -e^{iq\phi} & 0 \end{pmatrix}, \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(2.15)

It is easy to check that with this choice the matrices (2.14) obey the Clifford algebra with the metric tensor from (2.13). For the spin connection and the combination appearing in the Dirac equation we find

$$\Gamma_\mu = \frac{1 - q}{2} \gamma^{(1)} \gamma^{(2)} \delta^{(\phi)}_\mu, \quad \gamma^\mu \Gamma_\mu = \frac{1 - q}{2r} \gamma^r.$$

(2.16)

As to the non-Abelian gauge field we assume

$$A_\mu^a = (0, 0, A_\phi^a, 0).$$

(2.17)

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1 In principle there is no need that the electric charge of a fermion coupling to the gauge field of vortex be quantized [32]. So in this paper we will assume that $g$, in general, is different from $e$. 
The only non-vanishing component $A^3_\phi$ is related to an infinitesimally thin magnetic flux, $\Phi$, running along the core of the vortex by $A^3_\phi = -q\Phi/(2\pi) = -1/g$.

In the present paper we are interested in the calculation of the fermionic condensate, and the VEV of the energy-momentum tensor induced by the non-Abelian vortex. For the evaluation of these observables, a complete set of fermionic mode-functions is needed. In order to develop this analysis, we want to exhibit below, the notation adopted along this paper:

$$\tilde{\gamma}^\mu \equiv I_{(2)} \otimes \gamma^\mu = \begin{pmatrix} \gamma^\mu \\ 0 \\ 0 \end{pmatrix}, \quad (2.18)$$

$$\tilde{\tau}^a \equiv \tau^a \otimes I_{(4)}, \quad (2.19)$$

being $I_{(2)}$ and $I_{(4)}$ the $2 \times 2$ and $4 \times 4$ identities matrices.

The effective $8 \times 8$ matrix Hamiltonian operator associated with this system, is:

$$\hat{H} = -i\tilde{\alpha}^l(\nabla_l + ieA^a_l\tilde{\tau}^a/2) + \beta m - \beta(g_1\varphi + g_2\chi), \quad (2.20)$$

with

$$\varphi = \varphi^a\tilde{\tau}^a, \chi = \chi^a\tilde{\tau}^a \text{ and } \tilde{\alpha}^l = \tilde{\beta}\tilde{\gamma}^l. \quad (2.21)$$

Due to the non-Abelian feature of the vortex solution, the system is invariant under space plus isospin rotations, consequently the conserved angular momentum is,

$$\hat{J} = \frac{1}{i} \frac{\partial}{\partial \phi} + q\frac{\tilde{\Sigma}(z)}{2} + q\frac{\tilde{\tau}(3)}, \quad (2.22)$$

being $\tilde{\Sigma}(3) = \text{diag}(\sigma_3, \sigma_3, \sigma_3, \sigma_3)$, with $\sigma_3$ being the Pauli matrix.

Using explicit matrix notation, (2.22) reads,

$$\hat{J} = -iI_{(8)}\partial_\phi + q\text{diag}(1, 0, 1, 0, -1, 0, -1) . \quad (2.23)$$

Having this conserved operator, we now look for solutions of the Dirac equation eigenfunctions of it:

$$\hat{J}\psi = qj\psi, \quad (2.24)$$

with eigenvalues $j = 0, \pm 1, \pm 2, \ldots$.

Although in the last subsection we have briefly introduced the $SU(2)$ vortex system and also mentioned the influence of the system on the spacetime, in the analysis that we are going to develop in this paper, we will adopt another direction in the isospace for $\chi(r, \phi)$, given below,

$$\chi(r) = h(r) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad (2.25)$$

that preserves the condition $\varphi^a\chi^a = 0$. This is very convenient for our purpose\footnote{In fact this choice was presented in [35] analyzing the electrically charged non-Abelian vortex in $(1 + 2)$–dimensions in the presence of Chern-Simons term.}. Moreover, we will adopt in (2.20) the coupling constant $g_1 = 0$. In this case the interaction term between the fermionic field with the iso-scalar takes a diagonal form, and we can express the full Dirac equation,

$$\hat{H}\psi = i\partial_t\psi. \quad (2.26)$$
Writing

\[ \hat{H} = \begin{pmatrix} H_{(+)} & 0 \\ 0 & H_{(-)} \end{pmatrix}, \quad \psi = \begin{pmatrix} \psi_{(+)} \\ \psi_{(-)} \end{pmatrix}, \] (2.27)

two independents $4 \times 4$ matrices Dirac equations,

\[ H_{(\pm)}\psi_{(\pm)} = i\partial_t \psi_{(\pm)}, \] (2.28)

with

\[ H_{(\pm)} = -i\alpha^I(\partial_t \pm ieA_t/2) - i\left(\frac{1-q}{2r}\right)\sigma^r + \beta(m \pm g_Y). \] (2.29)

In above equation $g_Y = g_2 h_0$, being $h_0$ the asymptotic value for the function $h(r)$, and $A_t = (0, A^3_{\phi}, 0)$. In (2.27) the positive/negative signal in the iso-spinor refers to its projections, $\pm 1/2$, along $I_3 = r^3/2$ operator. In this present analysis it is considered only the asymptotic values of the profiles for the gauge and the scalar fields. So in this way we set $A^3_{\phi} = -q\Phi/(2\pi)$ and $h(r) = h_0$ in the Hamiltonian above. The physical reason for this procedure is because we are considering the fermions far from the vortex center, i.e., its corresponding energy is much smaller than the energy scale where the gauge symmetry is broken. Of course a more precise analysis should take into account the inner structure of the metric tensor, and also the gauge and scalar fields. In this way we can say that this paper is the first step in the direction to investigate the quantum system, fermions and non-Abelian vortex system, in a more complete treatment.

In order to solve the above system, we will adopt the following procedure to find positive energy solutions for the system:

\[ H_{(\pm)}\psi_{(\pm)}(\vec{r}, t) = i\partial_t \psi_{(\pm)}(\vec{r}, t), \quad \text{with } \psi_{(\pm)}(\vec{r}, t) \equiv e^{-iE^{(\pm)}t}\psi_{(\pm)}(\vec{r}), \] (2.30)

where due to the presence of coupling constant, $g_Y$, two effective masses terms take place, $m^{(\pm)} = m \pm g_Y$. Due to the presence of these effective masses and because we want to specify both components of the iso-spinor by the same set of quantum numbers, we assumed different energies for each component of the fermionic field.

### 2.2.1 Wave-function $\psi_{(+)}$

For positive projection four-component spinor $\psi_{(+)}$, and decomposing it into upper and lower components, denoted by $\psi_\uparrow$ and $\psi_\downarrow$, respectively, we find the equations

\[
\begin{align*}
\left( \sigma^I(\partial_t + ieA_I/2) + \frac{1-q}{2r} \sigma^r \right) \psi_\uparrow - i \left( E^{(+)} + m^{(+)} \right) \psi_\downarrow &= 0, \\
\left( \sigma^I(\partial_t + ie2A_I/2) + \frac{1-q}{2r} \sigma^r \right) \psi_\downarrow - i \left( E^{(+)} - m^{(+)} \right) \psi_\uparrow &= 0,
\end{align*}
\] (2.31)

with $m^{(+)} = m + g_Y$. Substituting the function $\psi_\downarrow$ from the first equation into the second one, we obtain the second order differential equation for the spinor $\psi_\uparrow$:

\[
\left[ \partial^2_r + \frac{1}{r} \partial_r + \frac{1}{r^2} \left( \partial_\phi + ieA_\phi^3/2 - i\frac{1-q}{2} \sigma^z \right)^2 + \partial^2_z + (E^{(+)} - m^{(+)} - m^{(\downarrow)})^2 \right] \psi_\uparrow = 0.
\] (2.32)

---

4 A similar procedure where only asymptotic values for the bosonic fields were considered was in the analysis of the catalysis of the proton decay by magnetic monopole.

5 Due to the presence of $g_Y$, the wave-function (2.27) is not an eigenstate of the full Dirac Hamiltonian. Both components, iso-spin up and down, present different energies. A similar situation also happens due to the Zeeman effect in the analysis of electron spin precession in an homogeneous magnetic field.
The same equation is obtained for the spinor \( \psi_\downarrow \).

In order to look for solution for (2.32), we use the ansatz below, compatible with the cylindrical symmetry of the physical system:

\[
\psi_\uparrow = e^{i(qj_\phi + kz)} \left( \frac{C_1 R_1(r)e^{-iq_\phi}}{C_2 R_2(r)} \right),
\]

with \( C_1 \) and \( C_2 \) being two arbitrary constants. Substituting this function into (2.32), we can see that the solutions of the equations for the radial functions regular at \( r = 0 \), are expressed in terms of the Bessel function of the first kind [40],

\[
R_l(r) = J_{|\nu|}(\lambda r), \text{ for } l = 1, 2,
\]

with corresponding order,

\[
\nu_1 = q(j + \alpha - 1/2) - 1/2, \quad \nu_2 = q(j + \alpha - 1/2) + 1/2.
\]

In the above expression we have introduced the notation,

\[
e^{A_3/2} = 2q_\alpha.
\]

We can see that \(|\nu_2| = |\nu_1| + \epsilon_{\nu_1}\), where \( \epsilon_{\nu_1} \) is equal to +1 for \( \nu_1 \geq 0 \) and −1 for \( \nu_1 < 0 \).

Having obtained \( \psi_\uparrow \), we can find \( \psi_\downarrow \), by using (2.31). The two-component spinor has the form,

\[
\psi_\downarrow = e^{i(qj_\phi + kz)} \left( \frac{B_1 R_1(r)e^{-iq_\phi}}{B_2 R_2(r)} \right).
\]

The coefficients \( B_1 \) and \( B_2 \) are related to \( C_1 \) and \( C_2 \) through (2.33), by,

\[
B_1 = \frac{1}{E^{(+)} + m^+} \left( C_1 k - iC_2 \epsilon_{\nu_1} \lambda \right),
\]

\[
B_2 = \frac{1}{E^{(+)} + m^+} \left( iC_1 \lambda \epsilon_{\nu_1} - C_2 k \right).
\]

The fermionic wave-function presents two independent coefficients. The normalization condition provides an extra restriction, consequently one of the coefficient remains arbitrary. In order to determine this coefficient some additional condition should be imposed on the coefficients. The necessity for this imposition is related to the fact that the quantum numbers \((\lambda, k, j)\) do not specify the fermionic wave-function uniquely and some additional quantum number is required. In order to specify the second constant we impose the condition

\[
C_1/B_1 = -C_2/B_2.
\]

By taking into account (2.38), we can write,

\[
C_2 = sC_1, \quad B_1 = -sB_2 = \frac{k - is\epsilon_{\nu_1} \lambda}{E^{(+)} + m^{(+)}}, \quad s = \pm 1.
\]

With the condition (2.39), the fermionic mode functions are specified by the set of quantum numbers \( \sigma = (\lambda, k, j, s) \). Of course, instead of (2.39) we could use another condition. The only restriction is that the resulting fermionic mode functions should form a complete set. For example, the condition similar to (2.39) with the opposite sign of the right-hand side gives another set of fermionic mode functions. The vacuum state is defined by the time dependence of the mode functions and it does not depend on the particular choice of condition required for
the coefficients. Consequently, different conditions would result the same fermionic condensate
and the VEV of the energy-momentum tensor.

The negative-energy solution associated with this Hamiltonian, can also be constructed fol-
lowing analogous procedure. This state is also characterized by the same set of quantum number,
\( \sigma \).

On the basis of all these considerations, the positive- and negative-energy fermionic wave
functions can be writing in a compact notation below:

\[
\psi_{\sigma(\pm)}(x) = C_{\sigma(\pm)} e^{\pm i E(\pm)t + i(kz + qj \phi)} \begin{pmatrix}
J_{\tilde{\beta}_j}(\lambda r)e^{-iq\phi} \\
\pm k-i\tilde{\epsilon}_j\lambda J_{\tilde{\beta}_j}(\lambda r)
\end{pmatrix},
\]  

(2.41)

with

\[
\tilde{\beta}_j = q|j + \alpha - 1/2| - \epsilon_j/2,
\]  

(2.42)

being \( \epsilon_j = \text{sgn}(j + \alpha - 1/2) \), and \( C_{\sigma(\pm)} \) a normalization constant.

The energy it is expressed in terms of \( \lambda \) and \( k \) by the relation,

\[
E(\pm) = \sqrt{\lambda^2 + k^2 + (m(\pm))^2}.
\]  

(2.43)

2.2.2 Wave-function \( \psi_{\ell(-)} \)

To obtain the four-component spinor \( \psi_{\ell(-)} \), we have to adopt a similar procedure as before. Only small modifications on the main differential equations are required. These modifications
are related to the sign of the charge, \( e \to -e \), the definition of the effective mass, now being
\( m_{\ell(-)} = m - gY \), and the energy,

\[
E(-) = \sqrt{\lambda^2 + k^2 + (m(-))^2}.
\]  

(2.44)

The positive and negative-energy fermionic wave functions are expressed as,

\[
\psi_{\sigma(\ell(-))}(x) = C_{\sigma(\ell(-))} e^{\mp i E(-)t + \pm (kz + qj \phi)} \begin{pmatrix}
J_{\tilde{\beta}_j}(\lambda r)
\end{pmatrix},
\]  

(2.45)

with

\[
\tilde{\beta}_j = q|j - \alpha + 1/2| - \tilde{\epsilon}_j/2,
\]  

(2.46)

being \( \tilde{\epsilon}_j = \text{sgn}(j - \alpha + 1/2) \), and \( C_{\sigma(\ell(-))} \) a normalization constant.

2.2.3 Normalization constant

The normalization constants in (2.41) and (2.45) can be determined from the orthonormalization condition

\[
\int d^3x \sqrt{\gamma} (\psi_{\sigma(\ell(\pm))})^\dagger \psi_{\sigma'(\ell(\pm))} = \delta_{\sigma\sigma'},
\]  

(2.47)

where we assume,

\[
\psi_{\sigma(\ell(\pm))} = \begin{pmatrix}
\psi_{\sigma(\ell(\pm))} \\
\psi_{\sigma(\ell(-))}
\end{pmatrix},
\]  

(2.48)
a 8–component spinor. In (2.47), γ represents the determinant of the spatial metric tensor. The delta symbol on the right-hand side is understood as the Dirac delta function for continuous quantum numbers, (λ, k), the Kronecker delta for discrete ones, (j, s).

Considering first positive-energy solution, we get the following result,

\[ |C^{(+)}_{\sigma}|^2 \cdot \frac{16\pi^2 E^{(+)}}{q\lambda(E^{(+)} + m^{+})} + |C^{(-)}_{\sigma}|^2 \cdot \frac{16\pi^2 E^{(-)}}{q\lambda(E^{(-)} + m^{-})} = 1. \]  

(2.49)

Here we adopt the specific choice below for each constant, which contains information specifically of the corresponding wave-function:

\[ |C^{(+)}_{\sigma}|^2 = \frac{q\lambda(E^{(\pm)} + m^{(\pm)})}{32\pi^2 E^{(\pm)}}. \]  

(2.50)

Adopting similar procedure, the normalization constant associated with negative-energy fermionic mode, is

\[ |C^{(-)}_{\sigma}|^2 = \frac{q\lambda(E^{(\pm)} - m^{(\pm)})}{32\pi^2 E^{(\pm)}}. \]  

(2.51)

3 Fermionic condensate

The phenomenon of Fermi condensation is important in both quantum field theories and in condensed matter physics. It plays a relevant role in the investigation of superconductivity and phase transitions in models of dynamical mass generation and symmetry breaking. Its characteristic feature is the appearance of non vanishing fermion condensate (FC) defined as the expectation value \( \langle 0 | \bar{\psi} \psi | 0 \rangle \equiv \langle \bar{\psi} \psi \rangle \), where \( | 0 \rangle \) is the vacuum state and \( \bar{\psi} = \psi^\dagger \gamma^{(0)} \) is the Dirac adjoint. Various mechanisms for the formation of the FC have been considered in the literature. They include different kinds of interactions of fermion fields, in particular, the Nambu-Jona-Lasinio-type models with self-interacting fields \[38, 39\]. An interesting line of investigation of the Fermi condensation is the dependence of the FC on the local topology of the background spacetime and interaction with gauge fields. Here in this section we are interested to evaluate the FC induced by the presence of non-Abelian vortex.

Expanding the field operator in terms of the complete set of positive- and negative-energy solutions, and using the anticommutation relations for the creation and annihilation operators, the FC can be evaluated by the following mode sum formula:

\[ \langle \bar{\psi} \psi \rangle = -\frac{1}{2} \sum_\sigma \sum_{\chi = -1}^{+1} \chi \bar{\psi}_\sigma^{(\chi)} \psi^{(\chi)} \cdot \]  

(3.1)

In the above expression, \( \psi^{(\chi)}_\sigma \), for \( \chi = +1, -1 \), represents positive and negative energy solutions, given in (2.41) and (2.45). The summation goes over the complete set of quantum numbers,

\[ \sum_\sigma = \sum_j \int_0^\infty d\lambda \int_{-\infty}^\infty dk \sum_{s = \pm 1}. \]  

(3.2)

In (3.2) we use the notation \( \sum_j = \sum_{j = 0, \pm 1, \pm 2, \ldots} \). In addition we assume that the parameter \( \alpha \) in (2.36) is in the range,

\[ \alpha \in [-1/2, 1/2]. \]  

(3.3)

In fact if we have written \( \alpha \) as an integer number plus a fractional part, \( \alpha = n + \epsilon \), \( n \) can be absorbed in a redefinition of the quantum number \( j \), present in the summation over this
number, and all the physical result will depend only on the fractional part. This corresponds to the Aharanov-Bohm effect.

By using the expression (2.48), we can see that,
\[ \tilde{\psi}^{(x)}_{\sigma} \bar{\psi}^{(x)}_{\sigma} = \psi^{(x)}_{\sigma} \frac{1}{2} \gamma^{(0)} \psi^{(x)}_{\sigma} = \psi^{(x)}_{\sigma(+) \gamma^{(0)}} \psi^{(x)}_{\sigma(-)} + \psi^{(x)}_{\sigma(-) \gamma^{(0)}} \psi^{(x)}_{\sigma(+)} \],
(3.4)
with \( \chi = \pm \).

Substituting the fermionic modes, (2.41) and (2.45) into the above equation, we can observe that the terms with \( s = 1 \) and \( s = -1 \) provide the same contributions. So after some intermediate steps, we obtain,
\[ \langle \psi \psi \rangle = - \frac{q}{2 \pi^2} \sum_{\sigma} \left[ \frac{\lambda_m^{(+)}(E(+))}{E(+) \left( J_{\beta_j}^2(\lambda r) + J_{\beta_j+\epsilon_j}^2(\lambda r) \right)} + \frac{\lambda_m^{(-)}(E(-))}{E(-) \left( J_{\beta_j}^2(\lambda r) + J_{\beta_j+\tilde{\epsilon}_j}^2(\lambda r) \right)} \right]. \]
(3.5)
In the above expression the notation \( \sum'_{\sigma} \) means that the summation over \( s \) has already been developed.

In the second summation on the right hand side of (3.5), we can change \( j \rightarrow -j \); doing this we obtain
\[ \tilde{\beta}_j \rightarrow \beta_j + \epsilon_j, \tilde{\beta}_j + \tilde{\epsilon}_j \rightarrow \beta_j. \]
(3.6)
So we get,
\[ \langle \tilde{\psi} \tilde{\psi} \rangle = - \frac{q}{4 \pi^2} \sum_{j} \int_{0}^{\infty} d\lambda \int_{0}^{\infty} d\lambda' \left( J_{\beta_j}^2(\lambda r) + J_{\beta_j+\epsilon_j}^2(\lambda r) \right) \left( \frac{m^{(+)}}{E(+) + m^{(-)}} \right). \]
(3.7)
In order to obtain a more workable expression, we use the identity below,
\[ \frac{1}{\sqrt{k^2 + \lambda^2 + (m')^2}} = \frac{2}{\sqrt{\pi}} \int_{0}^{\infty} ds \ e^{-r^2 + \lambda^2 + (m')^2}s^2, \]
(3.8)
with \( l = +, - \). Substituting this identity into (3.7), the integral over variable \( k \) is easily performed. As to the integral over \( \lambda \), we use the integral involving the square of the Bessel function from [40]:
\[ \int_{0}^{\infty} d\lambda \ e^{-\lambda^2} J_{\beta_j}^2(\lambda r) = \frac{e^{-r^2/(2s^2)}}{2s^2} I_{\mu}(r^2/(2s^2)) \]
(3.9)
being \( I_{\mu}(z) \) the modified Bessel function. As a result, the FC is presented in the form
\[ \langle \tilde{\psi} \tilde{\psi} \rangle = - \frac{q}{(2\pi r)^2} \left[ \frac{m^{(+)}(E(+) + m^{(-)}}{2} \int_{0}^{\infty} dy \ e^{-y-(m^{(+)}+r^2/2y)} \mathcal{F}(q, \alpha, y) \right] + \frac{m^{(-)}(E(-))}{2} \int_{0}^{\infty} dy \ e^{-y-(m^{(-)}+r^2/2y)} \mathcal{F}(q, \alpha, y) \]
(3.10)
where we have introduced the function
\[ \mathcal{F}(q, \alpha, y) = \mathcal{I}(q, \alpha, y) + \mathcal{I}(q, -\alpha, y), \]
(3.11)
with \( \mathcal{I}(q, \alpha, y) = \sum_{j} I_{\beta_j}(y) \) and \( \mathcal{I}(q, -\alpha, y) = \sum_{j} I_{\beta_j+\epsilon_j}(y) \).
An integral representation for the function $I(q, \alpha, y)$, that allows us to extract the divergent part in the FC, has been derived in [41]. Here we use that representation for the function (3.11)  

$$J(q, \alpha, y) = \frac{2}{q} e^y + \frac{4}{\pi} \int_{0}^{\infty} dx \frac{h(q, \alpha, x) \sinh x}{\cosh(2qx) - \cos(q\pi)} e^{-y\cosh(2x)} + \frac{4}{q} \sum_{k=1}^{p} (-1)^k \cos(\pi k/q) \cos(2\pi k\alpha) e^y \cos(2\pi k/q) ,$$  

(3.12)

where $p$ is an integer defined by $2p \leq q < 2p + 2$ and for $1 \leq q < 2$ the last term on the right-hand side is absent. The function in the integrand of (3.12) is given by the expression

$$h(q, \alpha, x) = \sum_{\eta=\pm} \cos[q\pi(1/2 + \eta\alpha)] \sinh[(1-2\eta\alpha)qx] .$$  

(3.13)

Note that $J(q, \alpha, y)$ is an even function of $\alpha$.

We can observe that the first term on the right-hand side of (3.12) provides a contribution independent of $\alpha$ and $q$ in (3.10), that corresponds to the FC in the absence of the vortex. This term is divergent. In order to obtain a finite and well defined FC, we have to adopt some renormalization procedure. Because we are interested here to calculate the contribution to the FC induced by the presence of the vortex, the renormalization for $\langle \bar{\psi}\psi \rangle$ reduces to subtract the exponential term in (3.12). The other terms provide contributions to the FC due to the magnetic flux and conical topology. These terms are finite and do not require any renormalization procedure. Substituting these terms into (3.10), we get,

$$\langle \bar{\psi}\psi \rangle_{\text{ren}} = -\frac{2}{(2\pi)^2} \sum_{l=+,\alpha} m(l) \left[ \sum_{k=1}^{p} (-1)^k \cos(\pi k/q) \cos(2\pi k\alpha) \int_{0}^{\infty} dy e^{-2y\sin^2(\pi k/q) - (m(l)r)^2/(2y)} \right]$$

$$+ \frac{q}{\pi} \int_{0}^{\infty} dx \frac{h(q, \alpha, x) \sinh x}{\cosh(2qx) - \cos(q\pi)} \int_{0}^{\infty} dy e^{-2y\cosh^2(x) - (m(l)r)^2/(2y)} ,$$  

(3.14)

Using the integral representation below for the Macdonald function [40],

$$\int_{0}^{\infty} dy e^{-\gamma y - \beta/(4y)} = \sqrt{\frac{\beta}{\gamma}} K_1(\sqrt{\beta\gamma}) ,$$  

(3.15)

the final result for the renormalized FC can be written as:

$$\langle \bar{\psi}\psi \rangle_{\text{ren}} = -\frac{1}{\pi^2} \sum_{l=+,\alpha} (m(l))^3 \left[ \sum_{k=1}^{p} (-1)^k \cos(\pi k/q) \cos(2\pi k\alpha) f_1(2|m(l)|r s_k) \right]$$

$$+ \frac{q}{\pi} \int_{0}^{\infty} dx \frac{h(q, \alpha, x) \sinh x}{\cosh(2qx) - \cos(q\pi)} f_1(2|m(l)|r \cosh x) ,$$  

(3.16)

where we have adopted the notations

$$f_\nu(x) = K_\nu(x)/x^\nu, \ s_k = \sin(\pi k/q) .$$  

(3.17)

Because (3.16) is an odd function of $m(\pm)$, the FC vanishes for massless field.

---

6 In fact the summation over the quantum number $j$ was developed assuming that this quantum number is semi-integer; however, in this analysis, $j$ is a integer number. Redefining in (2.42), $j - 1/2 = j'$, the resulting expression coincides with the one given in [41], so we can proceed the summation over $j'$. 
We can express Eq. (3.16) in terms of the parameter \( \delta = g_Y/m \), defined as the ratio between the coupling constant, \( g_Y = g_2\hbar_0 \), defined below (2.29) and the mass. It reads,

\[
\langle \bar{\psi} \psi \rangle^\text{ren} = -\frac{m^2}{2\pi r} \left\{ (1 + \delta)^2 \left[ \sum_{k=1}^{p} (-1)^k \cot(\pi k/q) \cos(2\pi k\alpha) K_1(2|1 + \delta|mrs_k) \right] + \frac{q}{\pi} \int_0^\infty dx \frac{h(q, \alpha, x) \tanh x}{\cosh(2qx) - \cos(q\pi)} K_1(2|1 + \delta|m c\cosh x) \right\}. (3.18)
\]

Moreover, at large distance from the string, i.e., for \( mr >> 1/|\delta \pm 1| \) and for \( q \geq 2 \), the dominant contribution in the (3.18) comes from the term \( k = 1 \). So, for the leading term we find

\[
\langle \bar{\psi} \psi \rangle^\text{ren} \approx \frac{1}{\sqrt{2}} \left( \frac{m}{2\pi r} \right)^{3/2} \cos(\pi/q) \cos(2\pi\alpha) \sin(3/2(\pi/q)) \left[ (1 + \delta)^{3/2} e^{-2|1 + \delta|mrs\sin(\pi/q)} + (\delta \rightarrow -\delta) \right]. (3.20)
\]

For \( 1 \leq q < 2 \) the sum over \( k \) does not exist and the integral terms are suppressed by the factor \( e^{-2|1 + \delta|m r} \).

We also can see that for \( \delta = 0 \), (3.18) reproduces previous result \( [34] \); in addition, admiring \( \alpha = 0 \), we get the expression given in \( [42] \). For fixed \( mr \), in the limit \( \delta \rightarrow \pm 1 \), the leading order in (3.18) is,

\[
\langle \bar{\psi} \psi \rangle^\text{ren} \approx -\frac{2m^2}{\pi^2 r} \left[ \sum_{k=1}^{p} (-1)^k \cot(\pi k/q) \cos(2\pi k\alpha) K_1(4mrs_k) \right] + \frac{q}{\pi} \int_0^\infty dx \frac{h(q, \alpha, x) \tanh x}{\cosh(2qx) - \cos(q\pi)} K_1(4m c\cosh x) \right\}. (3.21)
\]

In Fig. 1 we exhibit the behavior of the FC as function of \( \delta \) for \( \alpha = 1/2 \) (left plot) and \( \alpha = 0 \) (right plot), considering different values of \( q \). Because (3.18) is a even function of \( \delta \) we assume only this parameter in the interval \([0, 1]\). Moreover, we take \( mr = 1 \). As we can see the module of the intensity of the FC increases with \( q \) for a given value of \( \delta \).

Figure 2 presents the dependence of the FC as function of \( \alpha \) for different values of \( \delta \), cosdering \( mr = 1 \) and \( q = 3.5 \). By this plot we see that the intensity of the FC is larger for \( \delta = 0 \) and \( \alpha = 0 \), i.e., in absence of interaction between the fermionic field with the scalar one, \( \chi \), and the magnetic flux. The physical explanation for this result resides in the fact that for massless field (3.18) vanishes. So, for a given \( \alpha \), considering \( \delta \neq 0 \) there is an imbalance between the contributions of both effective masses, decreasing the magnitude of FC.

### 4 Energy-momentum tensor

The analysis of VEV of the energy-momentum tensor is one of the most relevant physical quantity associated with the quantum vacuum. Besides to describing the distribution of the energy density and vacuum stresses around the non-Abelian vortex, it appears as a source of gravity.
Figure 1: The quantity $2\pi^2\langle \bar{\psi}\psi\rangle^\text{ren}/m^3$, is exhibited as function of $\delta$ for $\alpha = 1/2$ (left plot) and $\alpha = 0$ (right plot) for $q = 2.5, 2.7, 2.9, 3.1$. In both plots we consider $\delta$ in the interval $[0, 1]$ and assume $mr = 1.0$.

Figure 2: The FC, $2\pi^2\langle \bar{\psi}\psi\rangle^\text{ren}/m^3$, is exhibited as function of $\alpha$ for $\delta = 0.0, 0.3, 0.6, 0.9$. In this plot was considered $mr = 1$ and $q = 3.5$.

in the semiclassical Einstein equations and provides the backreaction of quantum effects on the gravitational field.

So, motivated by these facts, in this section we want to calculate the VEV of the energy-momentum tensor. For a charged field coupled with an electromagnetic field and in curved space, this quantity can be calculated using the expression below, by expanding the fermionic operator in terms of the complete set of positive and negative normalized wave-functions:

$$\langle 0|T_{\mu\nu}|0\rangle = \langle T_{\mu\nu}\rangle = -\frac{i}{4} \sum_\sigma \sum_{\chi=+,-} \chi \left[ \bar{\psi}_\sigma(\chi) \tilde{\psi}_\sigma(\chi) \tilde{D}_\nu \psi_\sigma(\chi) - (D_\mu \tilde{\psi}_\sigma(\chi)) \psi_\sigma(\chi) \right] ,$$  \hspace{1cm} (4.1)

where $D_\mu \tilde{\psi} = \partial_\mu \tilde{\psi} - ieA_\mu \tilde{\psi} - \tilde{\psi} \Gamma_\mu$, and the brackets in the index expression means the symmetrization over the enclosed indexes.

### 4.1 Energy density

Let us first consider the energy density, $\langle T^0_0\rangle$. By taking into account that $A_0 = \Gamma_0 = 0$, we have,

$$\bar{\psi}_\sigma(\chi) \gamma_0 D_0 \psi_\sigma(\chi) = \psi_\sigma(\chi) \uparrow \partial_0 \psi_\sigma(\chi) + \psi_\sigma(\chi) \uparrow \partial_0 \psi_\sigma(\chi) .$$  \hspace{1cm} (4.2)

Now writing in a compact form $\partial_0 \psi_\sigma(\pm) = -i\chi E(\pm) \psi_\sigma(\pm)$, and taking explicitly the fermionic wave-functions, (2.41) and (2.45), with the corresponding normalization constant, (2.50) and
we can see that the contributions of the terms \( s = 1 \) and \( s = -1 \) are the same. After some intermediate steps, we get:

\[
\langle T_0^0 \rangle = - \frac{q}{4\pi^2} \sum_{l=+,-} \sum_j \int_0^\infty d\lambda \int_0^\infty d\epsilon \left[ J_{\beta_j}^2(\lambda r) + J_{\beta_j+\epsilon}^2(\lambda r) \right],
\]

where we have changed \( j \to -j \) in the indexes of Bessel functions associated with the projections \(-1/2\) of the iso-spinor.

In order to present the above result in a more suitable expression for the renormalization, we will use the identity below,

\[
\sqrt{k^2 + \lambda^2 + (m^{(l)})^2} = -\frac{2}{\sqrt{\pi}} \int_0^\infty ds \partial_s e^{-(k^2 + \lambda^2 + (m^{(l)})^2)s^2}.
\]

(4.4)

Substituting (4.4) into (4.3), the integration over the variable \( k \) can be developed trivially. As to the integral over \( \lambda \) we use Eq. (3.9). So after some intermediate steps, we can write,

\[
\langle T_0^0 \rangle = - \frac{q}{4\pi^2 r^4} \sum_{l=+,-} \int_0^\infty dy \ y^{1/2} \partial_y [y^{3/2} e^{-y - (m^{(l)})^2} \epsilon/(2y) \mathcal{J}(q, \alpha, y)],
\]

(4.5)

where \( \mathcal{J}(q, \alpha, y) \) is explicitly given in (3.12). The exponential term in that expression, \( \frac{2}{q} e^y \), provides a divergent result, and corresponds to the contribution associated with the Minkowski background in absence of magnetic flux. To obtain a finite and well defined value for the energy-density we should adopt some renormalization procedure. As we did in the FC analysis we will discard the exponential term, that is equivalent to the subtraction of the Minkowskian counterterms. In this way, we may now integrate by parts to evaluate \( \langle T_0^0 \rangle_{\text{ren}} \):

\[
\langle T_0^0 \rangle_{\text{ren}} = - \frac{1}{2\pi^2 r^4} \sum_{l=+,-} \int_0^\infty dy \ y^{1/2} \partial_y [y^{3/2} e^{-y - (m^{(l)})^2} \epsilon/(2y) \mathcal{J}(q, \alpha, y)] + \frac{q}{\pi} \int_0^\infty dx \ \frac{h(q, \alpha, x) \sinh x}{\cosh(2q x) - \cos(q \pi)} e^{-y \cosh(2x)}.
\]

(4.6)

After some minor simplifications, and using the integral representation for the the Macdonald function, \( K_2(z) \), below [40],

\[
K_2(z) = \frac{2}{z^2} \int_0^\infty dt e^{-t - z^2/(4t)},
\]

(4.7)

we get:

\[
\langle T_0^0 \rangle_{\text{ren}} = \frac{1}{\pi^2} \sum_{l=+,-} (m^{(l)})^4 \left[ \sum_{k=1}^p (-1)^k \cos(\pi k/q) \cos(2\pi k \alpha) f_2(2|m^{(l)}|rs_k) \right. \\
+ \left. \frac{q}{\pi} \int_0^\infty dx \ \frac{h(q, \alpha, x) \sinh x}{\cosh(2q x) - \cos(q \pi)} f_2(2|m^{(l)}|r \cosh x) \right]
\]

(4.8)

with the notation (3.17). For massless field, the above expression reduces to,

\[
\langle T_0^0 \rangle_{\text{ren}} = \frac{2}{\pi^2} (g_Y)^4 \left[ \sum_{k=1}^p (-1)^k \cos(\pi k/q) \cos(2\pi k \alpha) f_2(2|g_Y|rs_k) \right. \\
+ \left. \frac{q}{\pi} \int_0^\infty dx \ \frac{h(q, \alpha, x) \sinh x}{\cosh(2q x) - \cos(q \pi)} f_2(2|g_Y|r \cosh x) \right].
\]

(4.9)
Finally considering the limit $r |y_r| \to 0$, we obtain,
\[
\langle T_{0}^{\gamma}\rangle_{\text{ren}} = \frac{1}{4\pi^{2}r^{4}} \left[ \sum_{k=1}^{p} (-1)^{k} \frac{\cos(\pi k/q)}{\sin^{4}(\pi k/q)} \cos(2\pi k \alpha) \right] + \frac{q}{\pi} \int_{0}^{\infty} dx \frac{h(q, \alpha, x)}{\cosh(2qx) - \cos(q\pi) \cosh^{4}(x)} .
\] (4.10)

### 4.2 Radial stress

Our objective in this subsection is to evaluate the radial stress, $\langle T_{r}^{r}\rangle$. In order to do that, we take in the covariant derivative of the fermionic field $A_{r} = \Gamma_{r} = 0$. In this way, we can write,
\[
\langle T_{r}^{r}\rangle = -\frac{i}{4} \sum_{\sigma} \sum_{\lambda^{\pm}} \chi \left[ \bar{\psi}^{(\lambda)}_{\sigma} \gamma_{r} (\partial_{r} \psi^{(\lambda)}_{\sigma}) - (\partial_{r} \bar{\psi}^{(\lambda)}_{\sigma}) \gamma_{r} \psi^{(\lambda)}_{\sigma} \right] .
\] (4.11)

Taking the definition (2.18) for the Dirac gamma matrices, the mode functions from (2.41) and (2.45) with their respective normalizations constants, into the above expression, after some intermediate steps, we arrive at,
\[
\langle T_{r}^{r}\rangle = -\frac{q}{16\pi^{2}} \sum_{\sigma} \sum_{l=\pm} \epsilon_{l} \lambda^{3} \left[ J_{\beta_{l}}^{r}(\lambda r) J_{\beta_{l} + \epsilon_{l}}^{r}(\lambda r) - J_{\beta_{l}}^{r}(\lambda r) J_{\beta_{l} + \epsilon_{l}}^{r}(\lambda r) \right] \frac{1}{E(l)} ,
\] (4.12)

where the primes means derivative with respect to the argument of the function.

Using the recurrence relation involving the Bessel functions \[43\] and develop the summation over $s$, (4.12) can be written as,
\[
\langle T_{r}^{r}\rangle = \frac{q}{8\pi^{2}} \sum_{l=\pm} \int_{-\infty}^{\infty} dk \int_{0}^{\infty} d\lambda \lambda^{3} S(\lambda r) \frac{1}{\sqrt{k^{2} + \lambda^{2} + (m(l))^{2}}} ,
\] (4.13)

where
\[
S(x) = \sum_{j} \left[ J_{\beta_{j}}^{2}(x) + J_{\beta_{j} + \epsilon_{j}}^{2}(x) - \frac{2\beta_{j} + \epsilon_{j}}{x} J_{\beta_{j}}(x) J_{\beta_{j} + \epsilon_{j}}(x) \right] .
\] (4.14)

Using the identity [3.8] we can develop the integration over $k$ trivially. It remains to evaluate the integral over $\lambda$, as shown below:
\[
\langle T_{r}^{r}\rangle_{s} = \frac{q}{(2\pi)^{2}} \sum_{l=\pm} \int_{-\infty}^{\infty} ds e^{-(m(l))^{2}s^{2}} \int_{0}^{\infty} d\lambda \lambda^{3} e^{-\lambda^{2}s^{2}} S(\lambda r) .
\] (4.15)

Now we have to proceed the integration,
\[
\int_{0}^{\infty} d\lambda \lambda^{3} e^{-\lambda^{2}s^{2}} S(\lambda r) = \sum_{j} \int_{0}^{\infty} d\lambda \lambda^{3} e^{-\lambda^{2}s^{2}} \left[ J_{\beta_{j}}^{2}(\lambda r) + J_{\beta_{j} + \epsilon_{j}}^{2}(\lambda r) \right] - \sum_{j} \frac{2\beta_{j} + \epsilon_{j}}{r} \int_{0}^{\infty} d\lambda \lambda^{2} e^{-\lambda^{2}s^{2}} J_{\beta_{j}}(\lambda r) J_{\beta_{j} + \epsilon_{j}}(\lambda r) .
\] (4.16)

As to the first two integrals we can use the previous result, Eq. (3.9), by applying the derivative $-\partial_{\lambda^{2}}$. However to integrate the third term in $S(\lambda r)$, we have to adopt a more subtle method. First we observe that we can write,
\[
J_{\beta_{j}}(\lambda r) J_{\beta_{j} + \epsilon_{j}}(\lambda r) = \frac{1}{2\lambda} (\epsilon_{j} \partial_{r} + 2\beta_{j}/r) J_{\beta_{j}}^{2}(\lambda r) .
\] (4.17)
With this result in hand, we can use again (3.9). So, after some intermediate steps, we arrive to,
\[
\int_0^\infty d\lambda \lambda^2 e^{-\lambda^2 s^2} J_{\beta_j}(\lambda r)J_{\beta_j+\epsilon_j}(\lambda r) = \frac{r \epsilon_j e^{-y}}{4\pi^4} \left[ I_{\beta_j}(y) - I_{\beta_j+\epsilon_j}(y) \right],
\]  
with \( y = r^2/(2s^2) \). To obtain a more workable result, in order to use the identity (3.12), we use the relation
\[
(1 + 2\epsilon_j \beta_j) [I_{\beta_j}(y) - I_{\beta_j+\epsilon_j}(y)] = 2 (y \partial_y - y + 1/2) [I_{\beta_j}(y) + I_{\beta_j+\epsilon_j}(y)].
\]  
After combine all the above results, we get:
\[
\langle T_r^r \rangle = \frac{q}{8\pi^2 r^4} \sum_{l=+,-} \int_0^\infty dy y e^{-y-(m(l))^2r^2/(2y)} f(q, \alpha, y).
\]  
As we have already mentioned, the exponential term in (3.12) corresponds to contribution associated with the Minkowski spacetime in absence of magnetic flux. The renormalized expression for the radial stress, \( \langle T_r^r \rangle^{\text{ren}} \), can be obtained in a manifest form by subtracting this exponential term. Doing this we can observe that the result obtained coincides with (4.5) after we have developed the integration by part. So we can conclude that,
\[
\langle T_r^r \rangle^{\text{ren}} = \langle T_0^r \rangle^{\text{ren}}.
\]  
\[ 4.3 \text{ Azimuthal stress} \]

In this subsection we will evaluation of the VEV for the azimuthal stress, \( \langle T_{\phi}^\phi \rangle \). In order to do that we have to take into account, \( A_\phi = 2q\alpha/e \) and
\[
\tilde{\Gamma}_\phi = \frac{1-q}{2} \tilde{\gamma}^{(1)} \tilde{\gamma}^{(2)} = -\frac{i}{2} (1-q) \tilde{\Sigma}^{(3)}.
\]  
So this component reads,
\[
\langle T_{\phi\phi} \rangle = -\frac{i}{4} \sum_{\sigma} \sum_{\chi=+,-} \left[ (\tilde{\psi}_\sigma^{(1)} \tilde{\gamma}_\phi \psi^{(1)}_\sigma) - (D_\phi \tilde{\psi}_\sigma^{(1)}) \tilde{\gamma}_\phi \psi^{(1)}_\sigma \right].
\]  
In the development of the term inside the bracket, it is convenient to express the angular derivative in terms of the total angular momentum operator: \( \partial_\phi = iJ - i\frac{q}{2} (\tilde{\Sigma}^{(3)} + \tilde{\gamma}_3) \). Also, we can observe that the anticommutator, \( \{ \tilde{\gamma}_\phi, \tilde{\Sigma}^{(3)} \} \), which appears in this development, vanishes. So after some steps, we get:
\[
\langle T_{\phi\phi} \rangle = \frac{q}{2} \sum_{\sigma} \sum_{\chi} \left[ (j - 1/2) \tilde{\psi}_\sigma^{(1)} \gamma_\phi \psi^{(1)}_\sigma \right] + (j - \alpha + 1/2) \tilde{\psi}_\sigma^{(1)} \gamma_\phi \psi^{(1)}_\sigma \right].
\]  
Now substituting the explicit expressions for positive- and negative-energy fermionic wavefunctions, Eq.s (2.41) and (2.45), the Dirac matrix \( \gamma^{\phi} \) and developing the summation over \( s \) and \( \chi \), we obtain
\[
\langle T_{\phi}^\phi \rangle = \frac{q^2}{2\pi^2 r} \sum_{j} \int_0^\infty d\lambda \lambda^2 \int_0^\infty dk \left[ (j + \alpha - 1/2) \epsilon_j J_{\beta_j}(\lambda r) J_{\beta_j+\epsilon_j}(\lambda r) \frac{1}{E^{(+)}} + (j - \alpha + 1/2) \epsilon_j J_{\beta_j}(\lambda r) J_{\beta_j+\epsilon_j}(\lambda r) \frac{1}{E^{(-)}} \right].
\]  
(4.25)
Finally changing \( j \to -j \) in the second term inside the bracket, we get a more convenient expression to work:

\[
\langle T_\phi^\phi \rangle = \frac{q^2}{2\pi^2 \tau} \sum_j \epsilon_j (j + \alpha - 1/2) \int_0^\infty dk \int_0^\infty d\lambda \lambda^2 J_{\beta_j}(\lambda r)J_{\beta_j + \epsilon_j}(\lambda r) \sum_{l=+,-} \frac{1}{E(l)} \cdot (4.26)
\]

To continue our development we will use (3.8). Doing this the integral over \( k \) is trivially performed. As to the integral over \( \lambda \) we use (4.18), so we get,

\[
\langle T_\phi^\phi \rangle = \frac{q}{8\pi^2} \sum_{l=+,-} \int_0^\infty \frac{ds}{s^5} e^{-(m(l))^2 s^2 - s^2/(2s^2)} \sum_j q(j + \alpha - 1/2) \times [I_{\beta_j}(y) - I_{\beta_j + \epsilon_j}(y)]_{y=r^2/(2s^2)} \cdot (4.27)
\]

Using the fact that \( q(j + \alpha - 1/2) = \epsilon_j \beta_j + 1/2 \) and the identity (4.19), we can present the above expression by,

\[
\langle T_\phi^\phi \rangle = \frac{q}{8\pi^2} \sum_{l=+,-} \int_0^\infty \frac{ds}{s^5} e^{-(m(l))^2 s^2} (y\partial_y + 1/2) e^{-y} \mathcal{J}(q, \alpha, y)_{y=r^2/(2s^2)} \cdot (4.28)
\]

Moreover, using the identity below into (4.28),

\[
(y\partial_y + 1/2) e^{-y} \mathcal{J}(q, \alpha, y)_{y=r^2/(2s^2)} = 1/2 (r\partial_r + 1) e^{-r^2/(2s^2)} \mathcal{J}(q, \alpha, r^2/(2s^2)) \quad (4.29)
\]

and changing the variable of integration by, \( y = r^2/(2s^2) \), we can easily shown that,

\[
\langle T_\phi^\phi \rangle^{\text{ren}} = (r\partial_r + 1) \langle T_0^\phi \rangle^{\text{ren}} \cdot (4.30)
\]

An explicit expression for \( \langle T_\phi^\phi \rangle^{\text{ren}} \) can be derived by applying the radial differential operator on the energy density, (4.8), and using the identity,

\[
\partial_x (xf_2(x)) = -f_1(x) - 3f_2(x) \cdot (4.31)
\]

### 4.4 Axial stress

In the calculation of the axial stress, we have to consider \( A_z = \Gamma_z = 0 \). So, for this case we have

\[
\mathcal{D}_z \psi^{(x)}(x) = \partial_x \psi^{(x)}(x) = ik \psi^{(x)}(x) .
\]

Consequently we have,

\[
\langle T_z^z \rangle = \frac{1}{2} \sum_\sigma \sum_\chi \chi k \left[ \bar{\psi}_{\sigma(\pm)}^{(x)} \gamma^z \psi_{\sigma(\pm)}^{(x)} + \bar{\psi}_{\sigma(\pm)}^{(x)} \gamma^z \psi_{\sigma(\pm)}^{(x)} \right] \cdot (4.32)
\]

Substituting the expressions for positive- and negative-energy mode functions, Eq.s (2.41) and (2.45), into the above expression, develop the summations over \( s \) and \( \chi \), and conveniently changing the quantum number \( j \to -j \) in the second contribution inside the bracket, we get

\[
\langle T_z^z \rangle = \frac{q}{4\pi^2} \sum_{l=+,-} \int_0^\infty d\lambda \lambda \int_0^\infty dk k^2 \sum_j [J_{\beta_j}(\lambda r) + J_{\beta_j + \epsilon_j}(\lambda r)] \frac{1}{E(l)} \cdot (4.33)
\]

Using the identity (3.8), and the previous integral over the square of Bessel functions, Eq. (3.9), one obtains,

\[
\langle T_z^z \rangle = \frac{q}{16\pi^2} \sum_{l=+,-} \int_0^\infty \frac{ds}{s^5} e^{-(r^2/(2s^2) + (m(l))^2 s^2)} \mathcal{J}(q, \alpha, r^2/(2s^2)) \cdot (4.34)
\]
where we have used the general expression for the summation over the modified Bessel function, (3.11).

Finally changing the variable of integration, \( y = r^2/(2s^2) \), we can see that,

\[
\langle T_z \rangle_{\text{ren}} = \langle T_0 \rangle_{\text{ren}} .
\] (4.35)

The above result is consequence of the invariance of the system with respect to a boost along the \( z \) coordinate.

## 5 Properties of the VEV of the energy-momentum tensor

Here we want to investigate the most important properties obeyed by the VEV of the energy-momentum tensor. They are the trace relation, that that establishes the relation between its trace and the femionic condensate, and its conservation condition, necessary to be considered as the backreaction of the quantum fermionic field on the Einstein equation.

For the system under consideration, we can express,

\[
\langle T_\mu^\nu \rangle_{\text{ren}} = \langle T_\mu^\nu \rangle_{\text{ren}}^{(+)} + \langle T_\mu^\nu \rangle_{\text{ren}}^{(-)} ,
\] (5.1)

where the positive/negative signal corresponds to the signal in the effective mass \( m^{(\pm)} = m \pm g_Y \).

(See Eq.s (4.8), (4.21), (4.30), (4.31) and (4.35)). On the other hand the FC has a similar structure. So we can write,

\[
\langle \bar{\psi} \psi \rangle_{\text{ren}} = \langle \bar{\psi} \psi \rangle_{\text{ren}}^{(+)} + \langle \bar{\psi} \psi \rangle_{\text{ren}}^{(-)} ,
\] (5.2)

as shown in (3.16). By using the explicit expressions for the components of the VEV of the energy-momentum tensor and FC, we can verify,

\[
\langle T_\mu^\nu \rangle_{\text{ren}}^{(\pm)} = m^{(\pm)} \langle \bar{\psi} \psi \rangle_{\text{ren}}^{(\pm)} .
\] (5.3)

As to the conservation condition, \( \nabla_\mu \langle T_\mu^\nu \rangle = 0 \), for the problem under considerations, it reduces to a single expression,

\[
\partial_r (r \langle T_r^r \rangle) = \langle T_\phi^\phi \rangle .
\] (5.4)

According to the relations (4.21) and (4.30), we conclude that the VEV of the energy-momentum tensor is conserved.

Another point that we want to explore is the behavior of the energy-density with the variable \( \delta \). To obtain this information let us express Eq. (4.8) as,

\[
\langle T_0 \rangle_{\text{ren}} = \frac{m^2}{4\pi^2 r^2} \left\{ (1 + \delta)^2 \left[ \sum_{k=1}^{p} (-1)^k \frac{\cos(\pi k/q)}{\sin^2(\pi k/q)} \cos(2\pi k\alpha) K_2(2m|1 + \delta|rs_k) + \right. \right.
\]

\[
\left. + \left. \frac{q}{\pi} \int_0^{\infty} dx \frac{h(q , \alpha , x)}{\cosh(2qx) - \cos(q\pi) \cosh^2(x)} \sinh(x) K_2(2m|1 + \delta|r \cosh x) \right] \right. \}
\]

\[
+ \left. (\delta \to -\delta) \right\} . \] (5.5)

As in the FC case for \( \delta = 0 \), the above expression reproduces previous result given in [34].
Moreover, in the limit $\delta \to \pm 1$, we obtain

$$\langle T_0^0 \rangle_{\text{ren}} = \frac{1}{\pi^2 r^2} \left\{ m^2 \sum_{k=1}^p (-1)^k \frac{\cos(\pi k/q)}{\sin^2(\pi k/q)} \cos(2\pi k\alpha) K_2(4mrk) \right. + \frac{q}{\pi} \int_0^\infty dx \frac{h(q, \alpha, x)}{\cosh(2qx) - \cos(q\pi) \cosh^2(x)} K_2(4mr \cosh x) \left. \right\} .$$

So for $mr >> 1$ the above expression is dominated by the second contribution presenting a $1/r^4$ decay.

At large distance from the string and for $q \geq 2$, the dominant contribution in the (5.5) comes from the term $k = 1$. Under this condition the dominant contribution is given by,

$$\langle T_0^0 \rangle_{\text{ren}} \approx \frac{m^4}{8\pi^3/2} \cos(\pi/q) \cos(2\pi\alpha) \left[ (1 + \delta)^{3/2} e^{-2m|1+\delta|r \sin(\pi/q)} + (1 - \delta)^{3/2} e^{-2m|1-\delta|r \sin(\pi/q)} \right] .$$

For $1 \leq q < 2$ the sum over $k$ in (5.5) is absent and the integral terms are suppressed by the factor $e^{-2|1\pm\delta|m}$.

In Fig. 3 we exhibit the behavior of the energy-density as function of $\delta$ for $\alpha = 1/2$ (left plot) and $\alpha = 0$ (right plot), considering different values of $q$. In these plots we take $mr = 1$. Moreover, we can observe that the intensities of the module of the energy-densities increase with $q$.

Figure 3: The quantity $4\pi^2 \langle T_0^0 \rangle_{\text{ren}}/m^4$, is exhibited as function of $\delta$ for $\alpha = 1/2$ (left plot) and $\alpha = 0$ (right plot) for $q = 2.5, 2.7, 2.9, 3.1$. In both plots we consider $\delta$ in the interval $[0, 1]$ and assume $mr = 1.0$.

Figure 4 presents the dependence of the energy-density as function of $\alpha$ for different values of $\delta$, considering $mr = 1$ and $q = 1.5$. By this graph we can see that the energy density can be positive or negative, depending on the value of $\alpha$, for every value of $\delta$ considered. By this plot we can see that the energy-density, $\langle T_0^0 \rangle_{\text{ren}}$, can assume positive or negative values. For vanishing magnetic flux, $\alpha = 0$, this quantity is always negative, by turning on the magnetic flux, the corresponding Aharonov-Bohm effect provides a growth in the intensity of this observable.
Figure 4: The energy-density, $4\pi^2(T_0^{\text{ren}}/m^4)$, is exhibited as function of $\alpha$ for $\delta = 0.0, 0.3, 0.6, 0.9$. In this plot was considered $mr = 1$ and $q = 1.5$.

6 Conclusion

The analysis of fermionic and bosonic vacuum polarizations induced by an Abelian vortex configuration taking into account the conical geometry of the spacetime produced by this topological defect, have been developed by many researchers, great part of them cited in this paper; however, for the non-Abelian vortex configuration, as fas as we know, this analysis has been missed. Here in this paper we try to fill that gap. In fact the analysis developed in this paper is for the $SU(2)$ vortex field. For this specific configuration, only iso-vectors field can be considered. In this way we consider the system of isospin−1/2 fermionic field coupled with iso-scalars and iso-vector gauge fields. Because the complete analysis of this system is almost impossible to be developed, we assume two important simplifications: i) that the vortex is an idealized topological defect, i.e., very thin and straight. So we can discard the inner structure of the iso-vectors, gauge and scalars field, and ii) the second is to consider the coupling of the fermion with iso-scalar $\chi^a$, given in (2.25). So, the $8 \times 8$ matrix Hamiltonian could be written in a diagonal form, Eq. (2.27). As consequence the 8−component isospin−1/2 fermionic field effectively behaves as two-independent 4−component ones. Each of them presenting different energy as exhibit in (2.30) and different normalizations constants. The corresponding fermionic modes, $\psi^{(\pm)}_{\sigma(+)}$ and $\psi^{(\pm)}_{\sigma(+)}$, are given in (2.41) and (2.45), expressed in terms of Bessel functions with orders (2.42) and (2.46), respectively. 

Having obtained the complete set of normalized isospin−1/2 fermionic modes, we proceed to the calculations of the FC and the VEV of the energy-momentum tensor. Due to the presence of the coupling with the iso-scalar field, there appears different corrections in the mass term for the fermionic Hamiltonian, $H_{(+)}$ and $H_{(-)}$, given in (2.29). In the calculation of the FC by equation (3.5), we changed in the summation over the eigenvalues of the total angular momentum $j$ by $-j$ in the second contribution, providing the expression (3.7). The next steps was to obtain the renormalized FC. After some intermediate steps, our final result was given by Eq. (3.16). Defining by $\delta$ the ratio $g_Y/m$, the renormalized FC could be expressed by (3.18). This FC presents two distinct contributions: one due to the conical topology of the background spacetime, and the other due to the interaction of the fermionic field with the magnetic flux running along the vortex. From our result we can observe that for massless field the FC vanishes. Some asymptotic behavior of this quantity were developed: for large distance from the string; it decays as $e^{-2|1\pm\delta|mr\sin(\pi/q)}$ for $q \geq 2$, and as $e^{-2|1\pm\delta|mr}$ for $1 \leq q < 2$. Near the string the FC behaves as $1/r^2$, as exhibited in (3.19). In Eq. (3.21), we provide the asymptotic behavior for fixed distance and $\delta \to \pm 1$. Because the analysis of the behavior of the FC with the parameter $\delta$, cannot be observed in terms of analytical functions, we only can answer this question numerically. In Fig. 1 we exhibit the behavior of the FC as function of $\delta$ in the interval $[0, 1]$, for $\alpha = 1/2$ (left plot),
and $\alpha = 0$ (right plot), for different values of $q$. By these plots we conclude that the module of the intensity increases with $q$ for a given $\delta$. On the other hand, for a given $q$ the modulus of the intensity is higher for $\delta = 0$, that corresponding the absence of coupling between the fermionic field with the bosonic sector of the system. In Fig. 2 we exhibit the behavior of the FC with $\alpha$ for different values of $\delta$ keeping $q = 3.5$ and $mr = 1$. We note that FC is higher for $\delta = \alpha = 0$. In fact the interaction of the fermionic field with the magnetic flux, provides a decreases in the FC; moreover, for a given $\alpha$, the coupling with the iso-scalar field contributes to the imbalance between the contributions of both effective masses, decreasing, by its turn, the magnitude of FC. In addition we can see that the FC can be positive or negative depending the value of $\alpha$, and it seems that this changing in the signal occurs for all $\delta$ at the same value.

Another important characteristic of the fermionic vacuum is the VEV of the energy momentum tensor. In section 4 we have presented the mains procedures concerning with all calculations. Specifically the obtainment of the renormalized energy-density required many steps given in detail in the corresponding subsection; one of them was in the changing of the quantum number $j \to -j$ in the summation over the contribution associated with the mode $\psi^{(\pm)}_{\sigma(\delta)}$. As to the other components, they could be expressed in terms of $\langle T^0_0 \rangle^{\text{ren}}$. The renormalized expression for energy-density was presented in (4.8). The expression for $\langle T^0_0 \rangle^{\text{ren}}$, considering massless field, but $gy \neq 0$ is given in (4.9). Finally considering in the latter the limit $|gy|r \to 0$ the energy density behaves as $1/r^4$, as shown in (4.10). The two more important properties satisfied by the VEV of the energy-momentum tensor, have been presented in section 5. They are: trace relation and the conservation condition. In addition some important analytical result were obtained for some limiting values of the physical parameter of the theory. For large distance from the string, and for $q \geq 2$, the energy-density decays as $e^{-2mr/\sin(\pi/q)}$, and decays with $e^{-2|\pm mr}$ for $1 \leq q < 2$. Similarly what we have pointed out in the last section, the knowledge of the dependence of the energy-density with the parameters $\delta$ and $q$, can only be provided numerically. With this objective, in Fig 3 we exhibit the behavior of $\langle T^0_0 \rangle^{\text{ren}}$ as function of $\delta$ defined in the interval $[0, 1]$ for $\alpha = 1/2$ (left panel) and $\alpha = 0$ (right panel), considering different values of $q$. The Fig. 4 exhibits its behavior as function of $\alpha$, taking $mr$ and $q = 1.5$, and varying $\delta$. We note that the energy-density can be positive or negative depending on the value of $\alpha$ for each value of $\delta$.

At this point we have to say that we have calculated 1-loop energy-momentum tensor generated by the fermionic field. Another source of contributions are due to the bosonic fields. Because we are considering that the energy scale where the gauge symmetry is broken is much larger than the energy associated with the fermionic field, it is expected that the contributions of the Higgs and vector fields are suppressed by some negative power of their corresponding masses. However considering only the gauge and matter fields such contributions are important since they allow to obtain analytic results for the vacuum energy [46, 47]. In addition, the analysis of stability of the $SU(2)$ Higgs-gauge system taking into account quantum effects has been developed in [48]. Finally a review about the computation of 1-loop order corrections to the classical masses of some topological object, specifically the planar Abelian-Higgs model, with respective Feynmann rules, have been provided in [49].

Finally, as our last comments, we would like to say that the singular behavior of the FC and the energy-density near the vortex, is consequence of the idealized model adopted for this topological defect. In this way, we considered for the vortex and iso-scalar fields their vacuum values. Unfortunately there is no analytical function capable to provide the behavior associated with the non-Abelian vortex field for all values of radial distance $r$, only numerical representation can do that. Approximated model may be adopted, like in the case of Abelian vortex [44, 45]. This procedure deserves to be developed in a separate work, as the natural continuation. In this sense we can say that the present calculation can be considered as the first step in the analysis of fermionic vacuum polarization induced by the presence of non-Abelian vortex. Still
considering the idealized model for the non-Abelian vortex, a possible future calculation could be the obtainment of the induced fermionic current.

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