MATRIX UNITS AND GENERIC DEGREES FOR THE ARIKI–KOIKE ALGEBRAS

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ABSTRACT. We compute the generic degrees of the Ariki–Koike algebras by first constructing a basis of matrix units in the semisimple case. As a consequence, we also obtain an explicit isomorphism from any semisimple Ariki–Koike algebra to the group algebra of the corresponding complex reflection group.

1. Introduction

The cyclotomic Hecke algebras were introduced by Ariki and Koike \cite{ARK} and Broué and Malle \cite{BM}. It is conjectured \cite{BM} that these algebras play a rôle in the representation theory of reductive groups similar to (but more complicated than) that played by the Iwahori–Hecke algebras (see, for example, \cite{J}). In particular, it should be possible to use these algebras to compute the degrees (and more generally characters) of certain representations of reductive groups; more precisely, we can attach a polynomial to each irreducible representation \( \mathcal{H} \), called its generic degree, and appropriate specialisations of this polynomial should compute the dimensions of the corresponding irreducible representations of the finite groups of Lie type.

The purpose of this paper is to compute the generic degrees for the cyclotomic Hecke algebras of type \( G(r, 1, n) \); these polynomials have also been computed by Geck, Iancu and Malle \cite{GIM}. Further results of Malle \cite{Malle} and Malle and the author \cite{MM} give the generic degrees for all of the cyclotomic Hecke algebras corresponding to imprimitive complex reflection groups. Malle \cite{Malle} has recently computed the generic degrees for cyclotomic algebras for the primitive complex reflection groups (modulo the assumption that the corresponding cyclotomic Hecke algebras are symmetric), so this completes the calculation of the generic degrees of the cyclotomic Hecke algebras associated with complex reflection groups.

Two important special cases of the Ariki–Koike algebras are the Iwahori–Hecke algebras of types \( A \) and \( B \); the generic degrees of these algebras are well–known and were first computed by Hoefsmit \cite{Hof}. Later Murphy \cite{Murphy} gave an easier derivation of Hoefsmit’s formulae for the generic degrees of the Iwahori–Hecke algebras of type \( A \) using different, but related, techniques.

This article is largely inspired by Murphy’s paper \cite{Murphy}; however, with hindsight we are able to take quite a few shortcuts. Along the way we give a new and quite elegant treatment of the representation theory of the semisimple Ariki–Koike algebras. In particular, we explicitly construct the primitive idempotents and the matrix units in the Wedderburn decomposition of \( \mathcal{H} \). One of the nice features of our approach is that we use the modular theory (more accurately, the cellular
2. The Ariki–Koike algebras

Let \( R \) be a commutative domain with 1 and fix elements \( q, Q_1, \ldots, Q_r \) in \( R \) with \( q \) invertible. Let \( \mathfrak{q} = (q; Q_1, \ldots, Q_r) \). The Ariki–Koike algebra \( \mathcal{H} = \mathcal{H}_\mathfrak{q}(n) \) is the unital associative algebra with generators \( T_0, T_1, \ldots, T_{n-1} \) and relations

\[
(T_0 - Q_1) \cdots (T_0 - Q_r) = 0, \\
T_0 T_1 T_0 T_1 = T_1 T_0 T_1 T_0, \\
(T_i + q)(T_i - 1) = 0, \quad \text{for } 1 \leq i \leq n-1, \\
T_{i+1} T_i T_{i+1} = T_i T_{i+1} T_i, \quad \text{for } 1 \leq i \leq n-2, \\
T_i T_j = T_j T_i, \quad \text{for } 0 \leq i < j - 1 \leq 2.
\]

The Ariki–Koike algebra is a deformation of the group algebra of \( \text{Sym}(n) \). It follows from (2.2) that \( \mathcal{H} \) is isomorphic to the Iwahori–Hecke algebra of \( \mathfrak{S}_n \) of degree \( n \).

Define elements \( L_m = q^{1-m} T_{m-1} \cdots T_1 T_0 T_1 \cdots T_{m-1} \) for \( m = 1, 2, \ldots, n \); these are analogues of the \( q \)-Murphy operators of the Iwahori–Hecke algebras of type A \( \mathcal{A}_{10} \). An easy calculation using the relations in \( \mathcal{H} \) (see \( \mathcal{A}_{10} \)) shows that we have the following results.

2.1 Suppose that \( 1 \leq i \leq n-1 \) and \( 1 \leq m \leq n \). Then

(i) \( L_i \) and \( L_m \) commute.
(ii) \( T_i \) and \( L_m \) commute if \( i \neq m - 1, m \).
(iii) \( T_i \) commutes with \( L_i L_{i+1} \) and \( L_i + L_{i+1} \).
(iv) If \( a \in R \) and \( i \neq m \) then \( T_i \) commutes with \( (L_1 - a)(L_2 - a) \cdots (L_m - a) \).

Using the elements \( T_w \) and \( L_m \) defined above, Ariki and Koike proved the following result which gives a basis for \( \mathcal{H} \).

2.2 (Ariki–Koike \( \mathcal{A}_{10} \)) The algebra \( \mathcal{H} \) is free as an \( R \)-module with basis

\[
\{ L_{c_1}^1 L_{c_2}^2 \cdots L_{c_n}^n T_w \mid w \in \mathfrak{S}_n \text{ and } 0 \leq c_m \leq r - 1 \text{ for } m = 1, 2, \ldots, n \}.
\]

In particular, \( \mathcal{H} \) is free of rank \( r^n n! \).

We call this basis the Ariki–Koike basis of \( \mathcal{H} \). Let \( \mathcal{H}(\mathfrak{S}_n) \) be the subalgebra of \( \mathcal{H} \) generated by \( T_1, \ldots, T_{n-1} \). It follows from (2.2) that \( \mathcal{H}(\mathfrak{S}_n) \) is isomorphic to the Iwahori–Hecke algebra of \( \mathfrak{S}_n \) and is free as an \( R \)-module with basis \( \{ T_w \mid w \in \mathfrak{S}_n \} \).

Let \( \tau : \mathcal{H} \rightarrow R \) be the \( R \)-linear map determined by

\[
\tau(L_{c_1}^1 L_{c_2}^2 \cdots L_{c_n}^n T_w) = \begin{cases} 
1, & \text{if } c_1 = \cdots = c_n = 0 \text{ and } w = 1, \\
0, & \text{otherwise},
\end{cases}
\]

where \( w \in \mathfrak{S}_n \) and \( 0 \leq c_i < r \) for \( i = 1, \ldots, n \).

The function \( \tau \) was introduced by Bremke and Malle \( \mathcal{A}_{10} \) who showed that \( \tau \) is a trace form and that \( \tau \) is essentially independent on the choice of basis of \( \mathcal{H} \). It is not obvious from the definition above that \( \tau \) coincides with the form introduced by Bremke and Malle; however, this was proved by Malle and the author in \( \mathcal{A}_{21} \) where we also showed that \( \tau \) is non–degenerate whenever \( Q_1, \ldots, Q_r \) are invertible in \( R \).
For future reference we note the following two important properties of \( \tau \); the first is Bremke and Malle’s result that \( \tau \) is a trace form and the second follows easily from the definition and well–known properties of the trace form \( \tau \) in the case \( r = 1 \) (see, for example, [22, Prop. 1.16]).

2.3 (i) Suppose \( h_1, h_2 \in \mathcal{H} \). Then \( \tau(h_1 h_2) = (h_2 h_1) \).
(ii) Suppose that \( x, y \in \mathfrak{S}_n \) and that \( 0 \leq c_i < r \) for \( i = 1, \ldots, n \). Then 

\[
\tau(L_1^{c_1} L_2^{c_2} \cdots L_n^{c_n} T_x T_y) = \\
\begin{cases} 
q^{\ell(x)}, & \text{if } c_1 = \cdots = c_n = 0 \text{ and } x = y^{-1}, \\
0, & \text{otherwise}.
\end{cases}
\]

In this paper we will mainly be concerned with the semisimple Ariki–Koike algebras; these were classified by Ariki [1] who showed that when \( R \) is a field \( \mathcal{H} \) is semisimple if and only if

\[
P_{\mathcal{H}}(q) = \prod_{i=1}^{n}(1+q+\cdots+q^{i-1}) \cdot \prod_{1 \leq i<j \leq r-n<d<n}(q^d Q_i - Q_j)
\]

is a non–zero element of \( R \). For most of what we do it will be enough to assume that \( R \) is a ring in which \( P_{\mathcal{H}}(q) \) is invertible.

A multipartition of \( n \) is an ordered \( r \)-tuple \( \lambda = (\lambda^{(1)}, \ldots, \lambda^{(r)}) \) of partitions \( \lambda^{(s)} \) such that \( n = \sum_{s=1}^{r} |\lambda^{(s)}| \); we write \( \lambda \vdash n \). In the semisimple case Ariki and Koike [4] constructed an irreducible \( \mathcal{H} \)-module \( S^\lambda \), called a Specht module, for each multipartition \( \lambda \) of \( n \). Further, they showed that \( \{ S^\lambda \mid \lambda \vdash n \} \) is a complete set of pairwise non–isomorphic irreducible \( \mathcal{H} \)-modules. Let \( \chi^\lambda \) be the character of \( S^\lambda \).

Assume that \( \mathcal{H} \) is semisimple. Then \( \tau \) can be written as a linear combination of the irreducible characters of \( \mathcal{H} \) because \( \tau \) is a trace form.

2.5. Definition. Suppose that \( R \) is a field and that \( P_{\mathcal{H}}(q) \neq 0 \). The Schur elements of \( \mathcal{H} \) are the elements \( s_\lambda(q) \in R \) such that

\[
\tau = \sum_\lambda \frac{1}{s_\lambda(q)} \chi^\lambda,
\]

where \( \lambda \) runs over the multipartitions of \( n \).

The rational functions \( \frac{1}{s_\lambda(q)} \) are also called the weights of \( \mathcal{H} \).

The generic degrees of \( W_{r,n} \) are certain “spetsial” specializations of the rational functions \( s_\eta(q)/s_\lambda(q) \), where \( \eta = ((n), (0), \ldots, (0)) \); so \( s_\eta(q) \) is the Schur element corresponding to the trivial representation of \( \mathcal{H} \) (\( s_\eta(q) = s_\eta(q) \)) is computed in Example 2.3. The spetsial specializations of the rational functions \( s_\eta(q)/s_\lambda(q) \) are polynomials in \( q \) with rational coefficients; moreover, for these specializations \( s_\eta(q) \) is equal to the Poincaré polynomial of the coinvariant algebra of the reflection representation of \( W_{r,n} \). These results are due to Malle and can be found in [19][20].

In the special case when \( r = 1, 2 \) the group \( W_{r,n} \) is a Weyl group (rather than just a complex reflection group), and here the generic degrees were first computed by Hoefsmit [16]; they can be found, for example, in [8].

One of the motivations for writing this paper was to compute the Schur elements and hence the generic degrees. Ostensibly the Schur elements depend in a non–uniform way upon the choice of \( q, Q_1, \ldots, Q_r \); however, we shall see that in fact they can be expressed as rational functions in \( q, Q_1, \ldots, Q_r \) which depend only on \( \lambda \). The expression we obtain is a generalization of the hook length formula of Frame,
Now, for all \( i \), that \( \chi^\lambda(1) = |W_{r,n}|/s_\lambda(1; \zeta^*) \). As the referee remarked, it is worth noting that in general \( P_{\mathcal{H}}(1; \zeta^*) \) is only a scalar multiple of \( |W_{r,n}| = r^n n! \) since \( \prod_{1 \leq i < j \leq n} (\zeta^i - \zeta^j) = (-1)^{\frac{n(n-1)}{2}} e^{\frac{2\pi i}{n}} \) by [13, 2.22].

We will compute the Schur elements of \( \mathcal{H} \) by explicitly constructing a set of primitive idempotents in \( \mathcal{H} \) and then applying the following Lemma (which is really a well-known fact about symmetric algebras).

**2.6. Lemma.** Assume that \( R \) is a field and that \( \mathcal{H} \) is semisimple. Let \( \lambda \) be a multipartition of \( n \) and suppose that \( e_\lambda \) is a primitive idempotent in \( \mathcal{H} \) such that \( S^\lambda \cong e_\lambda \mathcal{H} \). Then \( s_\lambda(q) = \frac{1}{\tau(\lambda)} \) for any \( q \in \mathbb{C} \).

**Proof.** Suppose first that \( R \) is a field of characteristic zero. Let \( E_\lambda \) be the primitive central idempotent corresponding to the irreducible module \( S^\lambda \). By definition \( \tau = \sum_\mu s_\mu(q) \chi^\mu \), where \( \mu \) runs over the multipartitions of \( n \); therefore,

\[
\tau(E_\lambda) = \frac{1}{s_\lambda(q)} \chi^\lambda(E_\lambda) = \frac{1}{s_\lambda(q)} \chi^\lambda(1) = \sum_\mu s_\mu(q) \chi^\mu(E_\lambda) = \frac{1}{s_\lambda(q)} \chi^\lambda(1).
\]

Now \( E_\lambda = e_1 + \cdots + e_N \), where \( e_1, \ldots, e_N \) are primitive idempotents with \( e_i \mathcal{H} \cong S^\lambda \), for all \( i \), and \( N = \dim S^\lambda = \chi^\lambda(1) \). The idempotents \( e_1, \ldots, e_N \) belong to the same Wedderburn component of \( \mathcal{H} \), so there exist invertible elements \( u_i \in \mathcal{H} \) such that \( e_i = u_i e_i u_i^{-1} \) for all \( i \). Consequently, \( \tau(e_i) = \tau(u_i e_i u_i^{-1}) = \tau(e_1) \) since \( \tau \) is a trace form; hence, \( \tau(E_\lambda) = \chi^\lambda(1) \tau(e_1) \). Without loss, \( e_1 = e_\lambda \) so the Lemma follows.

The case where \( R \) is a field of positive characteristic now follows by a specialization argument (using, for example, Theorem 3.14) which we leave to the reader. \( \square \)

**2.7. Example.** Fix \( t \) with \( 1 \leq t \leq r \) and let \( \eta_t = (\eta_t^{(1)}, \ldots, \eta_t^{(r)}) \) be the multipartition of \( n \) with \( \eta_t^{(r)} = (n) \) if \( s = t \) and \( \eta_t^{(s)} = (0) \) otherwise. We will compute the Schur elements \( s_{\eta_t}(q) \). Let \( x_n = \sum_{w \in S_n} T_w \) and \( u_{\eta_t} = \prod_{s \neq t} \prod_{k=t}^n (L_k - Q_s) \), in the product, \( 1 \leq s \leq r \) and set \( m_{\eta_t} = u_{\eta_t} x_n = x_n u_{\eta_t} \) (cf. (3.1)). It follows from (2.1) that \( u_{\eta_t} \) is central in \( \mathcal{H} \). Further, the relations imply that \( T_0 u_{\eta_t} = Q_t u_{\eta_t} \) and \( T_w x_{\eta_t} = q^{l(w)} x_{\eta_t} \) for \( w \in S_n \); it follows that \( L_k m_{\eta_t} = q^{k-1} Q_t m_{\eta_t} \) for \( k = 1, \ldots, n \). Therefore, the module \( m_{\eta_t} \mathcal{H} \) is one dimensional and, in particular, irreducible; in fact, \( S^m \cong m_{\eta_t} \mathcal{H} \cong R m_{\eta_t} \) (for example, use (3.2)). Moreover, by what we have said

\[
m_{\eta_t}^2 = [n]_q^r \prod_{s \neq t}^{n} \prod_{k=1}^{r} (q^{k-1} Q_t - Q_s) \cdot m_{\eta_t},
\]

where \( [n]_q = \prod_{s=1}^{n} (1 + q + \cdots + q^{s-1}) \); so \( m_{\eta_t} \) is a scalar multiple of the primitive idempotent which generates \( S^m \). Hence, by the Lemma,

\[
s_{\eta_t}(q) = (\tau(m_{\eta_t}))^{-1} [n]_q^r \prod_{s \neq t}^{n-1} \prod_{k=0}^{r-1} (q^k Q_t - Q_s)
\]

\[
= (-1)^{n(r-1)} [n]_q^r \prod_{s \neq t}^{n-1} Q_s^n \cdot \prod_{s \neq t}^{n-1} (q^k Q_t - Q_s).
\]
Similar arguments give the Schur elements for the multipartition which is conjugate to \( \eta_t \); alternatively, they are given by Corollary 4.3 and the calculation above.

There is an action of \( S_r \) on the set of multipartitions of \( n \) (by permuting components) and also on the rational functions in \( Q_1, \ldots, Q_r \) (by permuting parameters). When \( \mathcal{H} \) is semisimple the Specht modules are determined up to isomorphism by the action of \( L_1, \ldots, L_n \); as the relation \( \prod_{s=1}^r (T_0 - Q_s) = 0 \) is invariant under the \( S_r \)-action it follows that \( s_{\nu, \lambda}(q) = v \cdot s_{\lambda}(q) \) for all multipartitions \( \lambda \) and all \( v \in S_r \); this is also clear from Theorem 3.14(i). In the case where \( \lambda = \eta_t \) this symmetry is evident in the formulae above.

3. AN ORTHOGONAL BASIS FOR \( \mathcal{H} \)

If \( R \) is a field and \( P_\mathcal{H}(q) \) is non–zero then \( \mathcal{H} \) is a split semisimple algebra; hence, \( \mathcal{H} \) has a basis which corresponds to the matrix units in its Wedderburn decomposition. In this section we explicitly construct a Wedderburn basis for \( \mathcal{H} \). We begin by recalling the standard basis of \( \mathcal{H} \) from [12].

A multipartition of \( n \) (with \( r \) components) is an \( r \)-tuple \( \lambda = (\lambda^{(1)}, \ldots, \lambda^{(r)}) \) of partitions such that \( |\lambda^{(1)}| + \cdots + |\lambda^{(r)}| = n \). Let \( \Lambda^+ \) be the set of multipartitions of \( n \); then \( \Lambda^+ \) becomes a poset under dominance where \( \lambda \geq \mu \) if for all \( 1 \leq s \leq r \) and all \( i \geq 1 \) we have

\[
\sum_{i=1}^{s-1} |\lambda^{(t)}| + \sum_{j=1}^{i} \lambda_j^{(s)} \geq \sum_{t=1}^{s-1} |\mu^{(t)}| + \sum_{j=1}^{i} \mu_j^{(s)}.
\]

We also write \( \lambda \triangleright \mu \) if \( \lambda \geq \mu \) and \( \lambda \neq \mu \).

The diagram of a multipartition \( \lambda \) is the set

\[
[\lambda] = \{ (i, j, c) \mid 1 \leq j \leq \lambda_i^{(c)} \text{ and } 1 \leq c \leq r \}.
\]

We will think of \( [\lambda] \) as being the \( r \)-tuple of diagrams of the partitions \( \lambda^{(c)} \), for \( 1 \leq c \leq r \). A \( \lambda \)-tableau is a bijection \( : [\lambda] \to \{1, 2, \ldots, n\} \). If \( t \) is a \( \lambda \)-tableau write \( \text{Shape}(t) = \lambda \). As with diagrams, we will think of a tableau \( t \) as an \( r \)-tuple of tableaux \( t = (t^{(1)}, \ldots, t^{(r)}) \), where \( t^{(c)} \) is a \( \lambda^{(c)} \)-tableau. The tableaux \( t^{(c)} \) are called the components of \( t \). A tableau is standard if in each component the entries increase along the rows and down the columns; let \( \text{Std}(\lambda) \) be the set of standard \( \lambda \)-tableaux.

We identify a tableau \( t \) with an \( r \)-tuple of labelled diagrams; for example

\[
\left( \begin{array}{ccc}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9
\end{array} \right) \quad \text{and} \quad \left( \begin{array}{ccc}
1 & 3 & 6 \\
2 & 4 & 9 \\
7 & 8 & 5
\end{array} \right)
\]

are two \( ((3, 1), (2, 1), (1^2)) \)-tableaux. Both of these tableaux are standard.

Given a multipartition \( \lambda \) let \( t^\lambda \) be the \( \lambda \)-tableau with the numbers \( 1, 2, \ldots, n \) entered in order first along the rows of \( t^{(1)} \) and then the rows of \( t^{(2)} \) and so on. For example, the first of the \( ((3, 1), (2, 1), (1^2)) \)-tableaux above is \( t^{((3, 1), (2, 1), (1^2))} \).

The symmetric group \( S_n \) acts from the right on the set of \( \lambda \)-tableaux; let \( S_\lambda = S_{\lambda^{(1)}} \times \cdots \times S_{\lambda^{(r)}} \) be the row stabilizer of \( t^\lambda \). For any \( \lambda \)-tableau \( t \) let \( d(t) \) be the unique element of \( S_\lambda \) such that \( t = t^\lambda d(t) \).

Let \( * \) be the \( R \)-linear antiautomorphism of \( \mathcal{H} \) determined by \( T_i^* = T_{i} \) for all \( i \) with \( 0 \leq i \leq n - 1 \). Then \( T_{w}^* = T_{w^{-1}} \) and \( L_{k}^* = L_{k} \) for all \( w \in S_n \) and for \( k = 1, 2, \ldots, n \).
We can now define the cellular basis of $\mathcal{H}$ constructed in [12]. Fix a multipartition $\lambda$ and let $a = (a_1, \ldots, a_r)$, where $a_s = |\lambda^{(1)}| + \cdots + |\lambda^{(s-1)}|$ for $1 \leq s \leq r$. Define $m_\lambda = x_\lambda u_\lambda^\dagger$, where

$$x_\lambda = \sum_{w \in \mathcal{D}_\lambda} T_w \quad \text{and} \quad u_\lambda^\dagger = \prod_{s=2}^r \prod_{k=1}^r (L_k - Q_s).$$

It follows from (2.1) that $x_\lambda$ and $u_\lambda^\dagger$ commute. Finally, given standard $\lambda$-tableaux $s$ and $t$ let $m_{st} = T_{d(t)} m_\lambda T_{d(s)}$.

Whenever we write $m_{st}$ in what follows $s$ and $t$ will be standard tableaux of the same shape (and similarly, for $f_{st}$ etc.).

3.1 (Dipper–James–Mathas [12, Theorem 3.26]) The Ariki–Koike algebra $\mathcal{H}$ is free as an $R$–module with cellular basis \{ $m_{st} \mid s, t \in \text{Std}(\lambda)$ for some $\lambda \in \Lambda^+$ \}.

One consequence of this result is that the $R$–module $\mathcal{H}^{\lambda}$ which has as basis the set of $m_{st}$ where Shape($u$) = Shape($v$) $\triangleright$ $\lambda$ is a two–sided ideal of $\mathcal{H}$. It follows from (3.1) that $m_{st} \mathcal{H} \subseteq \mathcal{H}^{\lambda}$ whenever $\mu \triangleright \lambda$. The Specht module $S^\lambda$ is the right $\mathcal{H}$–module $m_{st} \mathcal{H} / (m_{st} \mathcal{H} \cap \mathcal{H}^{\lambda})$, a submodule of $\mathcal{H} / \mathcal{H}^{\lambda}$. Thus, $S^\lambda$ is free as an $R$–module with basis \{ $m_t \mid t \in \text{Std}(\lambda)$ \}, where $m_t = m_{tt} + \mathcal{H}^{\lambda}$. Further, there is a natural associative bilinear form $\langle \ , \ \rangle$ on $S^\lambda$ which is determined by

$$\langle m_s, m_t \rangle m_\lambda = m_{st} m_\lambda \mod \mathcal{H}^{\lambda}.$$ 

Before we can begin we need some further notation and a result from [17]. If $t$ is any tableau and $k \geq 0$ is an integer let $t[k]$ be the subtableau of $t$ which contains the integers $1, 2, \ldots, k$. Observe that $t$ is standard if and only if Shape($t[k]$) is a multipartition for all $k$. We extend the dominance order to the set of standard tableaux by defining $s \triangleright t$ if Shape($s[k]$) $\triangleright$ Shape($t[k]$) for $k = 1, 2, \ldots, n$; again, we write $s \triangleright t$ if $s \triangleright t$ and $s \neq t$.

Write $\text{res}_t(k) = q^{|t|-|k|} Q_c$ if $k$ appears in row $i$ and column $j$ of $t$; then $\text{res}_t(k)$ is the residue of $k$ in $t$. We can now state the result we need from [17].

3.2 [17, Prop. 3.7] Let $s$ and $t$ be standard $\lambda$–tableaux and suppose that $k$ is an integer with $1 \leq k \leq n$. Then there exist $a_{tv} \in R$ such that

$$m_{st} L_k = \text{res}_t(k) m_{st} + \sum_{v \in \text{Std}(\lambda) \cup \triangleright t} a_{tv} m_{sv} \mod \mathcal{H}^{\lambda}.$$ 

From our current point of view the importance of this result derives from the observation that $m_{st} (L_k - \text{res}_s(k)) / (\text{res}_t(k) - \text{res}_s(k)) = m_{st}$ plus a linear combination of more dominant terms, providing that $\text{res}_u(k) \neq \text{res}_s(k)$; this motivates the next definition.

Let $\mathcal{R}(k) = \{ q^d Q_s \mid 1 \leq s \leq r, |d| < k \text{ and } d \neq 0 \text{ if } r = 1 \text{ and } k = 2, 3 \}$; be the set of possible residues $\text{res}_t(k)$ as $t$ runs over the standard tableaux.

3.3. Definition (cf. [17, Defn. 3.11]) Suppose that $s$ and $t$ are standard $\lambda$–tableaux.

(i) Let $F_t = \prod_{k=1}^n \prod_{c \in \mathcal{R}(k)}^{c \neq \text{res}_t(k)} \frac{L_k - c}{\text{res}_t(k) - c}$.

(ii) Let $f_{st} = F_s m_{st} F_t$.

There are some remarks worth making about the definition of $F_t$. First, we do not need to specify an order for the product in the definition of $F_t$ since the $L_k$
generate a commutative subalgebra of $\mathcal{H}$. Secondly, the definition of $F_1$ is very conservative in the sense that many of the factors of $F_1$ can be omitted without changing the element $f_{st}$. Finally, historically this construction has been used to produce an orthogonal basis for the Specht modules; we are going to modify this procedure to give an orthogonal basis for the whole of $\mathcal{H}$.

Let $s$ and $t$ be two standard tableaux, not necessarily of the same shape. The proof of next result rests upon the easy fact [17, Lemma 3.12] that, because $P_{\mathcal{H}}(q)$ is invertible, $s = t$ if and only if $\text{res}_s(k) = \text{res}_t(k)$ for $k = 1, \ldots, n$.

3.4. Proposition. Suppose that $P_{\mathcal{H}}$ is invertible in $R$ and that $s$ and $t$ are standard $\lambda$-tableau and that $k$ is an integer with $1 \leq k \leq n$. Then

(i) $f_{st} = m_{st} + \sum_{(u,v)\triangleright(s,t)} a_{uv}m_{uv}$ for some $a_{uv} \in R$;

(ii) $f_{st}L_k = \text{res}_t(k)f_{st}$;

(iii) $f_{st}F_u = \delta_{tu}f_{st}$; and,

(iv) $F_u f_{st} = \delta_{su}f_{st}$.

Proof. Given (3.2), this is a variation on a well known argument; see, for example, [22, Prop. 3.35]. The proof in [22] can be copied out verbatim except that $N$ should be replaced by $\sum_{k=1}^n |R(k)|$.

In particular, part (i) together with (3.1) shows that

$$\{ f_{st} \mid s, t \in \text{Std}(\lambda) \text{ for some } \lambda \vdash n \}$$

is a basis of $\mathcal{H}$; shortly we will see that it is an orthogonal basis of $\mathcal{H}$ with respect to the trace form $\tau$. As a first step we describe the action of $\mathcal{H}(S_n)$ on this basis; note that the action of the $L_k$ (and, in particular, $T_0 = L_1$), on this basis is given by Proposition 3.3(ii).

3.5. Proposition. Suppose that $t = s(i, i + 1)$ where $s$ and $u$ are standard $\lambda$-tableaux and $i$ is an integer with $1 \leq i < n$. If $t$ is standard then

$$f_{us}T_i = \begin{cases} (q - 1)\frac{\text{res}_s(i)}{\text{res}_s(i) - \text{res}_u(i)} f_{us} + f_{ut}, & \text{if } s \triangleright t, \\ (q - 1)\frac{\text{res}_s(i)}{\text{res}_s(i) - \text{res}_u(i)} f_{us} + \frac{(q \text{res}_s(i) - \text{res}_s(i))(\text{res}_s(i) - q \text{res}_s(i))}{(\text{res}_s(i) - \text{res}_u(i))^2} f_{ut}, & \text{if } t \triangleright s. \end{cases}$$

If $t$ is not standard then

$$f_{us}T_i = \begin{cases} q f_{us}, & \text{if } i \text{ and } i + 1 \text{ are in the same row of } s, \\ -f_{us}, & \text{if } i \text{ and } i + 1 \text{ are in the same column of } s. \end{cases}$$

Proof. By the above remarks, $\{ f_{us} \}$ is a basis of $\mathcal{H}$, so $f_{us}T_i = \sum_{a,b} r_{ab}f_{ab}$ for some $r_{ab} \in R$. By Proposition 3.4(iv), $F_u f_{ab} = \delta_{au}f_{ab}$. Therefore, multiplying the equation for $f_{us}T_i$ on the left by $F_u$ shows that $r_{ab} = 0$ whenever $a \neq u$; in particular, $r_{ab} = 0$ if $\text{Shape}(b) \neq \lambda$. Hence, $f_{us}T_i = \sum_{a,b} a_{ab}f_{ab}$, for some $a_{ab} \in R$, where $b$ runs over the set of standard $\lambda$-tableaux. The argument of [22, Theorem 3.36] can now be repeated, essentially word for word, to complete the proof.

For each standard $\lambda$-tableau $s$ let $f_s = f_s^{1s} + \mathcal{H}^\lambda$. Then $\{ f_s \mid s \in \text{Std}(\lambda) \}$ is a basis of the Specht module $S^\lambda$ by Proposition 3.3(i). Note that $f_{1s} = m_{1s}$; further, by Proposition 3.3, if $s$ and $t$ are standard $\lambda$-tableaux with $s = t(i, i + 1) \triangleright t$ then $f_t = f_s(T_t - \alpha)$, where $\alpha = (q - 1)\text{res}_s(i)/(\text{res}_t(i) - \text{res}_s(i))$; hence, by [17, Proof]...
and let $s$ be a $\lambda$–tableau. Then for each integer $i$ there is a unique node $x \in [\lambda]$ such that $s(x) = i$, $1 \leq i \leq n$. Let $A_s(i)$ be the set of addable nodes for $\text{Shape}(s,i)$ which are strictly less than $x$ (with respect to $\prec$); similarly, let $R_s(i)$ be the set of removable nodes strictly less than $x$ for the multipartition $\text{Shape}(s,i-1)$. Essentially as in [17, Defn. 3.15] let

$$\gamma_s = q^{\ell(d(s)) + \alpha(\lambda)} \prod_{i=1}^{n} \prod_{y \in A_s(i)} \frac{(\text{res}_s(i) - \text{res}(y))}{(\text{res}_s(i) - \text{res}(y))}$$

where $\alpha(\lambda) = \frac{1}{2} \sum_{s=1}^{r} \sum_{i \geq 1} (\lambda_i^{(s)} - 1)\lambda_i^{(s)}$.

If $k$ is an integer let $[k]_q = \frac{q^k - 1}{q - 1}$ if $q \neq 1$; more generally, set $[k]_q = 1 + q + \ldots + q^{k-1}$ if $k \geq 0$ and $[k]_q = -q^k - [k]_q$ if $k < 0$. When $k \geq 0$ we also set $[k]_q^! = [1]_q[2]_q \cdots [k]_q$.

Finally, if $\lambda$ is a multipartition let $[\lambda]_q^! = \prod_{s=1}^{r} \prod_{i \geq 1} [\lambda_i^{(s)}]_q^!$.

3.7 (James–Mathas [17 (3.17)–(3.19)]) Suppose that $P_{w}(q)$ is invertible in $R$ and let $t \in \text{Std}(\lambda)$. Then $\gamma_t$ is uniquely determined by the two conditions

(i) $\gamma_t = [\lambda]_q^! \prod_{1 \leq s < t \leq r, (i,j) \in [\lambda^{(s)}]} (q^{res_s(i) - res_s(i)} - Q_s - Q_t)$; and,

(ii) if $s = t(i, i + 1) \triangleright t$ then $\gamma_t = \frac{(q^{res_s(i) - res_s(i)}(res_s(i) - q \cdot res_s(i)))}{(res_s(i) - res_s(i))^2} \gamma_s$.

Further, $(f_s, f_t) = \delta_{st}\gamma_t$ for all $s, t \in \text{Std}(\lambda)$.

3.8. Remarks. (i) In [17] the rational functions $\gamma_t$ were important only up to a power of $q$ (which is a unit in $R$). In this paper we need to know the inner products $\langle f_s, f_t \rangle$ exactly; for this reason our definition of $\gamma_s$ differs from that in [17] by the factor $q^{\ell(d(s)) + \alpha(\lambda)}$. The argument in [17] computes $\langle f_s, f_t \rangle$ using only (3.7)(i) and (3.7)(ii); the inner product $\langle f_\lambda, f_\lambda \rangle$ is easily seen to be given by the formula on the right hand side of (3.7)(i).

(ii) A factor of $q$ was omitted from the formula for $\gamma_t$ in [17, Lemma 3.17]); this is corrected in (3.7)(ii) — note that $\ell(d(1)) = \ell(d(\lambda)) + 1$.

(iii) The definition of $\gamma_s$ above is simpler than that given in [17] because $s$ is a standard tableau (rather than the more general semistandard tableau which were considered in [17]).

Let $(.,.)$ be the inner product on $\mathcal{H}$ given by $(h_1, h_2) = \tau(h_1 h_2^\ast)$, for $h_1, h_2 \in \mathcal{H}$. Then $(.,.)$ is a symmetric associative bilinear form on $\mathcal{H}$.

3.9. Theorem. Suppose that $P_{w}(q)$ is invertible in $R$. Then

$$\{ f_{st} | s, t \in \text{Std}(\lambda) \text{ for some } \lambda \in \Lambda^+ \}$$

is a orthogonal basis of $\mathcal{H}$ with respect to the trace form $\tau$. In addition, if $s, t, u$ and $v$ are standard tableaux then $f_{st} f_{uv} = \delta_{ut} \gamma_t f_{uv}$.
Proof. By $\{3,1\}$ the set $\{m_\lambda\}$ is a basis of $\mathcal{H}$; therefore, $\{f_{st}\}$ is also a basis of $\mathcal{H}$ by Proposition $\{3,1\}$). Next we prove that $f_{st}f_{uw} = \delta_{ut}\gamma_t f_{sv}$. First, if $u \neq t$ then $f_{st}f_{uw} = f_{st}F_u n_{uw} F_v = 0$ by Proposition $\{3,4\}(iii)$. Now consider $f_{st}f_{uw}$; since $\{f_{ab}\}$ is a basis we can write $f_{st}f_{uw} = \sum_{a,b} r_{ab}f_{ab}$ for some $r_{ab} \in R$. Applying Proposition $\{3,4\}(iii)$ and its left handed analogue shows that

$$f_{st}f_{uw} = F_s f_{st} F_u F_v = \sum_{a,b} r_{ab} f_{st} F_a F_b F_v = r_{sv} f_{sv}.$$

Now, for any $s,v \in \text{Std}(\lambda)$ the definition of the inner product on the Specht module gives

$$(f_t,f_t) f_{sv} \equiv f_{st}f_{tv} \mod \mathcal{H}^\lambda.$$ 

Therefore, $r_{sv} = (f_t,f_t) = \gamma_t$ by $\{3,7\}$; so $r_{sv}$ depends only on $t$ and $f_{st}f_{tv} = \gamma_t f_{sv}$ as claimed.

Finally, it remains to show that the basis $\{f_{st}\}$ is orthogonal with respect to the bilinear form $(\ ,\ )$. First, $(f_{st},f_{uw}) = \tau(f_{st}f_{uw}) = \tau(f_{st}f_{uv}) = \delta_{uv}\gamma_t \tau(f_{su})$; in particular, $(f_{st},f_{uw}) = 0$ if $t \neq u$. On the other hand, $\tau$ is a trace form so $(f_{st},f_{uw}) = \tau(f_{st}f_{uw}) = \delta_{uv}\gamma_s \tau(f_{st})$. Therefore, $(f_{st},f_{uw}) = \delta_{uv}\delta_v\gamma_s \tau(f_{st})$. Consequently, $\{f_{st}\}$ is an orthogonal basis of $\mathcal{H}$ (and $\tau(f_{st})$ is non–zero for all $t$).

The basis $\{f_{st}\}$ is cellular (with respect to the involution $*$); but this is not surprising as is was constructed from a cellular basis.

3.10. Remark. In $[21]$ it was shown that if $\mathcal{H}$ is defined over a ring $R$ in which the parameters $q, Q, 1, \ldots, Q_r$ are invertible then $\mathcal{H}$ is a symmetric algebra with respect to the trace form $\tau$; however, this was proved indirectly without constructing a pair of dual bases. The Theorem gives a self–dual basis of the semisimple Ariki–Koike algebras; no such basis is known in general.

As a first consequence, Theorem $\{3,9\}$ identifies a submodule of $\mathcal{H}$ which is isomorphic to the Specht module $S^\lambda$.

3.11. Corollary. Suppose that $P_{\mathcal{H}}(q)$ is invertible in $R$ and let $s$ and $t$ be standard $\lambda$–tableaux. Then $S^\lambda \cong f_{st}\mathcal{H} = \sum_{\nu \in \text{Std}(\lambda)} R f_{sv}$.

Proof. By the Theorem, $f_{st}\mathcal{H}$ has as basis the set $\{ f_{sv} \mid v \in \text{Std}(\lambda) \}$. The isomorphism is given by the linear map $S^\lambda \rightarrow f_{st}\mathcal{H}$ determined by $f_v \mapsto f_{sv}$ for all tableaux $v \in \text{Std}(\lambda)$. 

Set $\tilde{f}_{st} = \gamma_t^{-1} f_{st}$. Then $\tilde{f}_{st} f_{uv} = \delta_{tu} \tilde{f}_{sv}$ and $\{ \tilde{f}_{st} \}$ is a basis of $\mathcal{H}$. Hence, we have the following.

3.12. Corollary. Suppose that $P_{\mathcal{H}}(q)$ is invertible in $R$. Then

$\{ \tilde{f}_{st} \mid s, t \in \text{Std}(\lambda) \text{ for some } \lambda \in \Lambda^+ \}$

is a basis of matrix units in $\mathcal{H}$.

The last result yields an explicit isomorphism from $\mathcal{H}$ to the group ring of $W_{r,n}$ when $P_{\mathcal{H}}(q)$ is invertible. Assume that $R$ contains a primitive $r$th root of unity $\zeta$; then, $\mathcal{H} \cong RW_{r,n}$ when $q = 1$ and $Q_s = \zeta^s$ for $s = 1, 2, \ldots, r$. Write $f_{st}^1$ for the element of $RW_{r,n}$ corresponding to $\tilde{f}_{st} \in \mathcal{H}$ under the canonical isomorphism $\mathcal{H} \rightarrow RW_{r,n}$. 

3.13. Corollary. Assume that $R$ contains a primitive $r$th root of unity and that $P_{\mathcal{H}}(q)$ is invertible in $R$. Then $\mathcal{H} \cong RW_{r,n}$ via the $R$-algebra homomorphism determined by $f_{st} \mapsto f_{st}^1$.

By parts (i) and (iii) of Theorem 3.14 below, $T_i = \sum f_{ui}T_i$, for $0 \leq i < n$; so, in principle, we can determine the image of the generators of $\mathcal{H}$ under this isomorphism.

Lusztig [18] has shown that there exists a homomorphism $\Phi$ from the Hecke algebra $\mathcal{H}(W)$ of any finite Weyl group $W$ to the group ring $RW$ and he shows that $\Phi$ induces an isomorphism when $\mathcal{H}(W)$ is semisimple. Our map is not an analogue of Lusztig’s isomorphism; rather it is an explicit realization of the Tits deformation theorem in this setting. It would be good to find a generalization of Lusztig’s isomorphism theorem for the Ariki–Koike algebras.

We next construct the primitive (central) idempotents in $\mathcal{H}$.

3.14. Theorem. Suppose that $R$ is a field and that $P_{\mathcal{H}}(q) \neq 0$.

(i) Let $t$ be a standard $\lambda$-tableau. Then $F_t = \frac{1}{\gamma_t}f_{tt}$ and $F_t$ is a primitive idempotent with $S^\lambda \cong F_t\mathcal{H}$.

(ii) For any multipartition $\lambda$ let $F_\lambda = \sum_{t \in \text{Std}(\lambda)} F_t$. Then $F_\lambda$ is a primitive central idempotent.

(iii) $\{ F_\lambda \mid \lambda \vdash n \}$ is a complete set of primitive central idempotents; in particular,

$$
1 = \sum_{\lambda \vdash n} F_\lambda = \sum_{\text{t standard}} F_t.
$$

Proof. We may write $F_t = \sum_{u,v} a_{uv}f_{uv}$ for some $a_{uv} \in R$ by Theorem 3.3. By Proposition 3.4(iii) and Theorem 3.9, $f_{st} = f_{st}F_t = \sum_{u,v} a_{uv}f_{st}f_{uv} = \sum_{u,v} \gamma_t a_{uv}f_{uv}$; equating coefficients on both sides shows that $a_{uv} = 0$ if $v \neq t$ and that $1 = a_{tt}\gamma_t$. Since $F_t^2 = F_t$ we also have that $a_{tt} = 0$ if $v \neq t$. Hence, $F_t = \frac{1}{\gamma_t}f_{tt}$ as claimed. By Theorem 3.9, $\frac{1}{\gamma_t}f_{tt}$ is idempotent; further, it is primitive because $S^\lambda$ is irreducible and $S^\lambda \cong \frac{1}{\gamma_t}f_{tt}\mathcal{H}$ by Corollary 3.11. Hence, (i) is proved.

Parts (ii) and (iii) now follow because $\mathcal{H} = \bigoplus_{\lambda \vdash n} \bigoplus_{t \in \text{Std}(\lambda)} F_t\mathcal{H}$ is a decomposition of $\mathcal{H}$ into a direct sum of simple modules with each simple module $F_t\mathcal{H} \cong S^\lambda$ appearing with multiplicity equal to its dimension (the sum is direct because $\{ f_{tv} \}$ is a basis of $F_t\mathcal{H}$ by Theorem 3.9 and the set of all $f_{uv}$ is a basis of $\mathcal{H}$).

3.15. Corollary. Let $t$ be a standard tableau and let $k$ be an integer with $1 \leq k \leq n$. Then $F_t L_k = \text{res}_t(k) F_t$ and $f_{st} L_k = \text{res}_t(k) f_{st}$.

Proof. Since $F_t = \frac{1}{\gamma_t}f_{tt}$ by part (i), the formula for $F_t L_k$ follows from Proposition 3.5(ii); this also implies the second statement because $f_{st} L_k = f_{st}F_t L_k$.

In particular, the Corollary describes the action of $T_0 = L_1$ on the orthogonal basis $\{ f_{st} \}$ of $\mathcal{H}$. Proposition 3.5 gives the action of the remaining generators $T_1, \ldots, T_{n-1}$ of $\mathcal{H}$ on the basis $\{ f_{st} \}$.

3.16. Corollary. Suppose that $1 \leq k \leq n$. Then $\prod_{c \in R(k)}(L_k - c) = 0$ and this is the minimum polynomial for $L_k$ acting on $\mathcal{H}$.
Proof. By Theorem 3.14
\[
\prod_{c \in \mathcal{R}(k)} (L_k - c) = 1 \cdot \prod_{c \in \mathcal{R}(k)} (L_k - c) = \sum_{t \text{ standard}} F_t \prod_{c \in \mathcal{R}(k)} (L_k - c) = 0,
\]
since \(F_t(L_k - \text{res}_t(k)) = 0\) by Corollary 3.15. Moreover, if we remove any factor \((L_k - c)\) from the product \(\prod_{c \in \mathcal{R}(k)} (L_k - c)\) then what remains is non-zero because it divides \(F_t\) for some standard tableau \(t\). Hence, \(\prod_{c \in \mathcal{R}(k)} (L_k - c)\) is the minimum polynomial of \(L_k\).

\[\text{3.17. Corollary. Suppose that } 1 \leq k \leq n. \text{ Then } L_k = \sum_{i} \text{res}_i(k)F_i, \text{ where the sum is over the set of all standard tableaux (of arbitrary shape).}
\]

Proof. Combining part (iii) of Theorem 3.14 with Corollary 3.15 shows that
\[
L_k = 1 \cdot L_k = \sum_{t \text{ standard}} F_t L_k = \sum_{t \text{ standard}} \text{res}_t(k)F_t
\]
as required.

We close this section with a description of the centre of \(\mathcal{H}\) in the semisimple case; this result is due to Ariki and Koike [3, Theorem 3.20].

\[\text{3.18. Theorem (Ariki–Koike). Suppose that } R \text{ is a field and that } P_{\mathcal{H}}(q) \neq 0.
\]
(i) The centre of \(\mathcal{H}\) is the set of symmetric polynomials in \(L_1, L_2, \ldots, L_n\).
(ii) Let \(L\) be the subalgebra of \(\mathcal{H}\) generated by \(L_1, L_2, \ldots, L_n\). Then \(L\) is a maximal abelian subalgebra of \(\mathcal{H}\).

Proof. First consider (ii). By definition \(L\) contains each of the primitive idempotents \(F_t\) for an arbitrary standard tableau \(t\). On the other hand, by Corollary 3.17, \(L\) is contained in the subalgebra of \(\mathcal{H}\) generated by the primitive idempotents \(F_t\); hence, \(L\) is the subalgebra of \(\mathcal{H}\) generated by the idempotents \(F_t\). As the primitive idempotents generate a maximal abelian subalgebra of \(\mathcal{H}\) the result follows.

Now consider (i). By (2.7) every symmetric polynomial in \(L_1, \ldots, L_n\) belongs to the centre of \(\mathcal{H}\). Conversely, the centre of \(\mathcal{H}\) has as basis the set of idempotents \(F_\lambda\), as \(\lambda\) runs over the multipartitions of \(n\). So to prove (i) it is enough to show that each \(F_\lambda\) is symmetric in \(L_1, \ldots, L_n\).

Let \(\mathcal{R} = \bigcup_{k=1}^{n} \mathcal{R}(k)\) be the set of all possible residues for \(\mathcal{H}\). By Corollary 3.15 if \(t\) is a \(\lambda\)-tableau and \(1 \leq m \leq n\) then \(F_t(L_k - c) = (\text{res}_t(k) - c)F_t\); therefore, multiplying \(F_t\) by the appropriate extra factors we see that
\[
F_t \overset{\text{def}}{=} \prod_{k=1}^{n} \prod_{c \in \mathcal{R}(k) \setminus \{\text{res}_t(k)\}} \frac{L_k - c}{\text{res}_t(k) - c} = \prod_{k=1}^{n} \prod_{c \in \mathcal{R} \setminus \{\text{res}_t(k)\}} \frac{L_k - c}{\text{res}_t(k) - c}.
\]
Notice that the denominator is equal to \(f_\lambda = \prod_{k=1}^{n} \prod_{c \in \mathcal{R} \setminus \{\text{res}_\lambda(k)\}} (\text{res}_\lambda(k) - c)\) and \(f_\lambda\) depends only on \(\lambda\) and not directly on \(t\).

Next, following Murphy [25] say that a \(\lambda\)-tableau is regular if its entries increase from left to right along the nodes in \(\lambda\) of constant residue. Now, because \(P_{\mathcal{H}}(q) \neq 0\) two nodes in \(\lambda\) have the same residue if and only if they lie on the same diagonal \(\{i + d, j + d, s\} \in [\lambda^{(s)}] \mid d \geq 0\}\) of \(\lambda\); thus, a tableau is regular if and only if its entries increase from left to right along each diagonal. (So, for example, every
standard tableau is regular, but not conversely.) Extending the formula above for \( F_t \), for each regular tableau \( t \) define

\[
F_t = \prod_{k=1}^{n} \frac{L_k - c}{\text{res}_t(k) - c} = \frac{1}{f_t} \prod_{k=1}^{n} \prod_{c \in \mathcal{R} \setminus \{ \text{res}_t(k) \}} (L_k - c).
\]

Observe that if \( c \in \mathcal{R} \) then \( L_k - c \) is a factor of \( F_t \) if and only if \( c \neq \text{res}_t(k) \); therefore, \( F_t \) determines \( \text{res}_t(k) \), for \( k = 1, \ldots, n \). As remarked above, the residues on the different diagonals of \([\lambda]\) are distinct; consequently, a regular tableau \( t \) is uniquely determined by the sequence of residues \( (\text{res}_t(1), \ldots, \text{res}_t(n)) \) and hence by the 'polynomial' \( F_t \). It follows that if we permute \( L_1, \ldots, L_n \) then \( F_t \) is mapped to \( F_{s^t} \) where \( s \) is the regular tableau determined by the corresponding permutation of the residue sequence of \( t \). Again, \( s \) is necessarily a \( \lambda \)-tableau because the shape of \( s \) is determined by the lengths of its diagonals which, in turn, are determined by the multiplicity of each residue in \( s \) (or \( t \)).

Finally, notice that a regular tableau \( t \) is not standard if and only if \( \text{res}_t(k) \notin \mathcal{R}(k) \) for some \( k \). Therefore, if \( t \) is not standard then \( \prod_{c \in \mathcal{R}(k)} (L_k - c) \) is a factor of \( F_t \) and, consequently, \( F_t = 0 \) by Corollary 3.16; hence,

\[
F_{\lambda} = \sum_{t \text{ a standard } \lambda \text{-tableau}} F_t = \sum_{t \text{ a regular } \lambda \text{-tableau}} F_t.
\]

By the last paragraph the right hand side is a symmetric polynomial in \( L_1, \ldots, L_n \) so the theorem follows.

For the Iwahori–Hecke algebras of type \( A \) (that is, when \( r = 1 \)), it is conjectured that the centre of \( \mathcal{H} \) is always the set of symmetric polynomials in \( L_1, \ldots, L_n \), even when \( \mathcal{H} \) is not semisimple; see [10, 23]. When \( r > 1 \) and \( \mathcal{H} \) is not semisimple there are cases where the centre of \( \mathcal{H} \) is larger than the set of symmetric polynomials in \( L_1, \ldots, L_n \); for an example see [3, p. 792].

4. ANOTHER CONSTRUCTION OF THE SPECHT MODULES

In this section we give another two constructions of the Specht modules. The first is via a second cellular basis of \( \mathcal{H} \) which is, in a certain sense, dual to the basis described in the previous section. The second construction combines these two approaches to produce submodules of \( \mathcal{H} \) which are isomorphic to the Specht modules \( S^\lambda \). Some of these results can be found in the work of Du and Rui [13]. In the next section we will use these results to compute the Schur elements of \( \mathcal{H} \).

Let \( Z = \mathbb{Z}[\hat{q}, \hat{q}^{-1}, \hat{Q}_1, \ldots, \hat{Q}_r] \), where \( \hat{q}, \hat{Q}_1, \ldots, \hat{Q}_r \) are indeterminates over \( \mathbb{Z} \), and let \( \mathcal{H}_Z \) be the Ariki–Koike algebra with parameters \( \hat{q}, \hat{Q}_1, \ldots, \hat{Q}_r \). Consider the ring \( R \) as a \( Z \)-module by letting \( \hat{q} \) act on \( R \) as multiplication by \( q \) and \( \hat{Q}_s \) by multiplication by \( Q_s \), for \( 1 \leq s \leq r \). Then \( \mathcal{H} \cong \mathcal{H}_Z \otimes_Z R \), since \( \mathcal{H} \) is free as an \( R \)-module; we say that \( \mathcal{H} \) is a specialization of \( \mathcal{H}_Z \) and call the map which sends \( h \in \mathcal{H}_Z \) to \( h \otimes 1 \in \mathcal{H} \) the specialization homomorphism.

Let \( ' : Z \to Z \) be the \( Z \)-linear map given by \( \hat{q} \mapsto \hat{q}^{-1} \) and \( \hat{Q}_s \mapsto \hat{T}_{r-s+1} \) for \( 1 \leq s \leq r \). Define \( T_0 = T_0 \) and \( T_i = -\hat{q}^{-1}T_i \) for \( 1 \leq i < n \); using the relations of \( \mathcal{H}_Z \) it is easy to verify that \( ' \) now extends to a \( Z \)-linear ring involution \( ' : \mathcal{H}_Z \to \mathcal{H}_Z \) of \( \mathcal{H}_Z \). Hereafter, we drop the distinction between \( \hat{q} \) and \( q \), and \( \hat{Q}_s \) and \( Q_s \).

Suppose that \( h \in \mathcal{H} \). Then there exists a (not necessarily unique) \( h_Z \in \mathcal{H}_Z \) such that \( h = h_Z \otimes 1 \) under specialization; we sometimes abuse notation and write
4.3 Proposition. Let for certain rings \( R \) integer with \( \lambda \) \( L \) and \( L_i = L_i \) for all \( w \in \mathfrak{S}_n \) and \( 1 \leq i \leq n \).

If \( \lambda \) is a multipartition of \( n \) let \( y_\lambda = \sum_{w \in \mathfrak{S}_n} (-q)^{-\ell(w)} T_w; \) then \( y_\lambda = x'_\lambda \). Similarly, we define \( n_\lambda = y_\lambda u_\lambda = m'_\lambda \) where

\[
u_\lambda = \prod_{s=2}^{r} L_k - Q_{r-s+1} = \prod_{s=1}^{r-1} L_k - Q_s.
\]

Observe that \( u_\lambda = (u_\lambda^+) \); here, as usual, \( a_\lambda = |\lambda^{(1)}| + \cdots + |\lambda^{(s-1)}| \) for all \( s \). For standard tableaux \( s, t \in \text{Std}(\lambda) \) set \( n_{st} = T_{d(s)}^d T_{d(t)}^d \); then \( m'_{st} \in \mathcal{H}^Z \) is mapped to \( n_{st} \) under specialization. Hence, from [3.1] we obtain the following.

4.1 \( [13, (2.7)] \) The Ariki–Koike algebra \( \mathcal{H} \) is free as an \( R \)-module with cellular basis \( \{ n_{st} \mid \sigma, t \in \text{Std}(\lambda) \text{ for some } \lambda \in \Lambda^+ \} \).

Let \( \lambda \) be a multipartition of \( n \). Then \( (\mathcal{H}^\lambda)' \) is a two–sided ideal of \( \mathcal{H} \) which is free as an \( R \)-module with basis \( \{ n_{uv} \mid u, v \in \text{Std}(\mu) \text{ for some } \mu \triangleright \lambda \} \). Let \( \mathcal{S}^\lambda \) be the Specht module (or cell module) corresponding to \( \lambda \) determined by the basis \( \{ n_{st} \}; \) then \( \mathcal{S}^\lambda \cong n_{\lambda, \mathcal{H}}/(n_{\lambda, \mathcal{H}} \cap (\mathcal{H}^\lambda)' \) and \( \mathcal{S}^\lambda \) is free as an \( R \)-module with basis \( \{ n_{s} \mid t \in \text{Std}(\lambda) \} \), where \( n_{s} = n_{s, \lambda} + (\mathcal{H}^\lambda)' \) for all \( t \in \text{Std}(\lambda) \).

In order to compare the two modules \( S^\lambda \) and \( \mathcal{S}^\lambda \) we need to introduce some more notation. Given a partition \( \sigma \) let \( \sigma' = (\sigma_1^*, \sigma_2^*, \ldots) \) be the partition which is conjugate to \( \sigma \); thus, \( \sigma_i^* \) is the number of nodes in column \( i \) of the diagram of \( \sigma \). If \( \lambda = (\lambda^{(1)}, \ldots, \lambda^{(r)}) \) is a multipartition then the conjugate \( \lambda' = (\lambda^{(1)}, \ldots, \lambda^{(r)}) \) of \( \lambda' \) is the multipartition with \( \lambda^{(s)} = (\lambda^{(r-s+1)})' \) for \( 1 \leq s \leq r \).

Now suppose that \( t = (t^{(1)}, \ldots, t^{(r)}) \) is a standard \( \lambda \)-tableau. Then the conjugate of \( t \) is the standard \( \lambda' \)-tableau \( t' = (t^{(1)}, \ldots, t^{(r)}) \) where \( t^{(s)} \) is the tableau obtained by interchanging the rows and columns of \( t^{(s)} \).

With these definitions in place, we see that the following holds in \( Z \).

4.2 Let \( t \) be a standard \( \lambda \)-tableau. Then \( \text{res}_t^* (k) \) is \( \text{res}_t (k) \) in \( Z \), for \( 1 \leq k \leq n \).

The expression \( \text{res}_t^* (k) \) is always well–defined; whereas \( \text{res}_t (k) \) is ambiguous for certain rings \( R \). As a first consequence we determine how the \( L_k \) act on the \( n_{st} \)-basis of \( \mathcal{H} \).

4.3. Proposition. Let \( s \) and \( t \) be standard \( \lambda \)-tableaux and suppose that \( k \) is an integer with \( 1 \leq m \leq n \). Then there exist \( a_\sigma \in \mathbb{R} \) such that

\[
n_{st} L_k = \text{res}_t (k) n_{st} + \sum_{s < \sigma \in \text{Std}(\lambda)} a_\sigma n_{st} \text{ mod } (\mathcal{H}^\lambda)'.
\]

Proof. First assume that \( R = Z \). Then \( ' \) is a \( Z \)-linear ring involution on \( \mathcal{H}^Z \) and \( L_k = L_k \); therefore, by [3.2],

\[
n_{st} L_k = (m_{st} L_k)' = \left( \text{res}_t (k) m_{st} + \sum_{s < \sigma \in \text{Std}(\lambda)} a_\sigma m_{st} \text{ mod } \mathcal{H}^\lambda \right)'.
\]

Using [12] this proves the Proposition for \( \mathcal{H}^Z \). The general case now follows by specialization since \( \mathcal{H} \cong \mathcal{H}^Z \otimes Z \).
Next consider the orthogonal basis \( \{ f_{st} \} \) of \( \mathcal{H} \) in the case where \( P_{\mathcal{H}}(q) \) is invertible. Let \( \mathcal{Z}_P \) be the localization of \( \mathcal{Z} \) at \( P_{\mathcal{H}}(q) \) and let \( \mathcal{H}_{\mathcal{Z}_P} \) be the corresponding Ariki–Koike algebra. The involution \( \cdot \) extends to \( \mathcal{H}_{\mathcal{Z}_P} \) and \( \mathcal{H} \) is a specialization of \( \mathcal{H}_{\mathcal{Z}_P} \) whenever \( P_{\mathcal{H}}(q) \) is invertible in \( R \). (Note that \( Q_1, \ldots, Q_r \) are indeterminates in \( \mathcal{Z}_P \).)

In general, \( f_{st} \not\in \mathcal{H}_{\mathcal{Z}_P} \); however, if \( t \neq u \) then \( \text{res}_u(k) - \text{res}_u(k) \) is a factor of \( P_{\mathcal{H}}(q) \) for all \( k \), so \( f_{st} \in \mathcal{H}_{\mathcal{Z}_P} \) and we can speak of the elements \( F_t \) and \( f_{st} \) in \( \mathcal{H}_{\mathcal{Z}_P} \). More generally, whenever \( P_{\mathcal{H}}(q) \) is invertible in \( R \) we have an element \( f_{st} \in \mathcal{H} \) via specialization because \( \mathcal{H} \cong \mathcal{H}_{\mathcal{Z}_P} \otimes_{\mathcal{Z}_P} R \).

4.4. Proposition. Suppose that \( t \) is a standard tableau. Then \( F_t = F_{\lambda} \) in \( \mathcal{H}_{\mathcal{Z}_P} \).

Proof. Applying the definitions together with \([4.2]\) gives

\[
F_t = \prod_{k=1}^{n} \prod_{c \in R(k) \cap \text{res}_u(k)} \left( \frac{L_k - c}{\text{res}_u(k) - c} \right) \quad \text{and} \quad F_{\lambda} = \prod_{k=1}^{n} \prod_{c \in R(k) \cap \text{res}_u(k)} \left( \frac{L_k - c'}{\text{res}_u(k) - c} \right)
\]

the last equality following because \( R(k) \) is invariant under \( \cdot \).

By Lemma \([2.6]\) and Theorem \([3.14]\) the Schur elements are given by \( s_\lambda(q) = \tau(F_{\lambda})^{-1} \); consequently, the Schur elements have the following “palindromy” property.

4.5. Corollary. Suppose that \( \lambda \) is a multipartition of \( n \). Then \( s_{\lambda'}(q) = (s_\lambda(q))' \).

Returning to the general case, let \( g_{st} = F_s m_{st} F_{sv} \); then \( g_{st} = F_s m_{st} F_s = f_{st} \) (in \( \mathcal{H}_{\mathcal{Z}_P} \)). Applying \( \cdot \) to \( \{ f_{st} \} \) and using Theorem \([3.8]\) (and a specialization argument) shows that \( \{ g_{st} \mid s, t \in \text{Std}(\lambda) \} \) is a basis of \( \mathcal{H} \). Consequently, as in Corollary \([3.11]\), \( \tilde{S}^\lambda \cong g_{st} \mathcal{H} \) for any standard \( \lambda \)-tableaux \( s, t \in \text{Std}(\lambda) \).

4.6. Remark. By the Proposition and Theorem \([3.14][i]\),

\[
g_{st} = f_{st}' = (\gamma_s F_t)' = \gamma_s' F_t' = \frac{\gamma_s'}{\gamma_t'} F_{t'}.
\]

More generally, we can write \( g_{st} = \sum_{u, v} a_{uv} f_{uv} \) for some \( a_{uv} \in R \). By Proposition \([3.4]\) and Proposition \([4.4]\), \( F_{t'} g_{st} F_{t'} = g_{st} \); so it follows that \( a_{uv} = 0 \) unless \( u = s \) and \( v = t' \). Therefore, \( g_{st} = a_{st} f_{sv} \) for some \( a_{st} \in R \). Applying the \( \ast \)-involution shows that \( a_{st} = a_{ts} \). Finally, by looking at the product \( g_{st} g_{ta} \) we see that \( a_{ta}^2 = \gamma'_{st} / \gamma_{st} \). \( A \) priori, there is no reason why the square root of this element should belong to \( R \); nor do I see a way to determine the sign of \( a_{st} \).

Combining Proposition \([4.4]\) with Corollary \([3.11]\) and the corresponding result for the \( g \)-basis shows that \( S^\lambda \cong f_{st} \mathcal{H} = g_{t'} \mathcal{H} \cong S^{\lambda'} \), for any \( t \in \text{Std}(\lambda) \). Hence, we have the following.

4.7. Corollary. Suppose that \( P_{\mathcal{H}}(q) \) is invertible in \( R \). Then \( \tilde{S}^\lambda \cong S^{\lambda'} \).

When \( R \) is field the assumption that \( P_{\mathcal{H}}(q) \) is invertible is equivalent to \( \mathcal{H} \) being semisimple. This assumption is necessary because, in general, \( S^{\lambda'} \) and \( \tilde{S}^\lambda \) are not isomorphic; rather, \( S^{\lambda'} \) is isomorphic to the dual of \( \tilde{S}^\lambda \) \([24]\). In the semisimple
Therefore, interchanging the roles of \( \lambda \) that is to say that if \( \text{Proposition 3.4(i)} \), b and then specializing) we see that there exist n of first along the rows of t
\[ \text{algebra } H(4 \text{ degenerate bilinear form). Accordingly, we call the module } \hat{S}^\lambda \text{ a dual Specht module.} \]

Here is another useful application of Proposition 4.4.

4.8. Corollary. Suppose that s and t are standard \( \lambda \)-tableaux and that u and v are standard \( \mu \)-tableaux where \( \lambda \) and \( \mu \) are multipartitions of n. Then \( f_{st} g_{uv} = 0 \) if \( t \neq u' \).

Proof. Applying the definitions, \( f_{st} g_{uv} = F_s m_{st} F_t m_{uv} F_v' \); however, \( F_u' = F_v' \) by Proposition 4.4 so \( F_t F_u' = \delta_{u'} F_t \) by Theorem 3.14(i), giving the result. (By Remark 4.6, \( f_{st} g_{uv} \) is a scalar multiple of \( f_{uv'} \).)

The Specht modules \( S^\lambda \) and the dual Specht modules \( \hat{S}^\lambda \) are both constructed as quotient modules using the bases \( \{ m_{st} \} \) and \( \{ n_{st} \} \) respectively (see Corollary 3.11). In the cases where \( P_{H^0}(q) \) is invertible in \( R \) we have also constructed these modules as submodules of \( H \). Next we produce submodules of an arbitrary Hecke algebra \( H \) which are isomorphic to the Specht modules; these results will also play a rôle in computing the Schur elements in the next section.

Recall that \( t^\lambda \) is the \( \lambda \)-tableau which has the numbers 1, 2, \ldots, n entered in order first along the rows of \( t^{(1)} \lambda \) and then the rows of \( t^{(2)} \lambda \) and so on. Let \( t_\lambda = (t^\lambda)^{'} \); that is to say that \( t_\lambda \) is the \( \lambda \)-tableau with the numbers 1, 2, \ldots, n entered in order first down the columns of \( t^{(r)} \lambda \) and then the columns of \( t^{(r-1)} \lambda \) etcetera. Observe that if \( t \) is a standard \( \lambda \)-tableau then \( t^\lambda \supseteq t \supseteq t_\lambda \).

4.9. Proposition. Suppose that \( P_{H^0}(q) \) is invertible and let \( \lambda \) be a multipartition of n. Then \( m_\lambda H n_\lambda = R f_{t^\lambda t_\lambda} \).

Proof. By Proposition 3.4(i), \( m_\lambda = f_{t^{\lambda^{'} t'}} + \sum_{u, b \supseteq t^{(1)} \lambda} a_{ub} f_{uv} \) for some \( a_{ub} \in R \). Therefore, interchanging the roles of \( \lambda \) and \( \lambda' \) and applying the involution \( ^{'} \) (in \( Z_p \) and then specializing) we see that there exist \( b_{ab} \in R \) such that

\[ n_\lambda = g_{t^{\lambda^{'} t'}} + \sum_{a, b \supseteq t^{(1)} \lambda} b_{ab} g_{ab} = \gamma_{t_\lambda} f_{t_\lambda t_\lambda} + \sum_{t_\lambda \supseteq a', b'} b_{ab} g_{ab} \]

where for the second equality we have used Remark 1.6 (note that \( (t^{(1)} \lambda)^{'} = t_\lambda \) and the observation that \( a, b \supseteq t^{(1)} \lambda' \) if and only if \( t_\lambda \supseteq a', b' \)). Now \( m_\lambda H n_\lambda \) is spanned by the elements \( m_\lambda f_{st} n_\lambda \), where \( s \) and \( t \) range over all pairs of standard tableaux of the same shape. Now, (4.10) and Corollary 4.8 imply that

\[ m_\lambda f_{st} n_\lambda = \left( f_{t^{(1)} \lambda} + \sum_{u, b \supseteq t^{(1)} \lambda} a_{ub} f_{uv} \right) f_{st} (\gamma_{t_\lambda} f_{t_\lambda t_\lambda} + \sum_{t_\lambda \supseteq a', b'} b_{ab} g_{ab}) \]

\[ = \gamma_{t_\lambda} f_{t^{(1)} \lambda} f_{st} f_{t_\lambda t_\lambda} + \sum_{t_\lambda \supseteq a', b'} b_{ab} f_{uv} f_{uv} f_{st} g_{ab}, \]

\[ = \gamma_{t_\lambda} f_{t^{(1)} \lambda} f_{st} f_{t_\lambda t_\lambda} \]

\[ = \left\{ \begin{array}{ll} \gamma_{t_\lambda} f_{t^{(1)} \lambda}, & \text{if } s = t^{(1)} \lambda \text{ and } t = t_\lambda, \\ 0, & \text{otherwise}, \end{array} \right. \]

with the last equality following from Theorem 3.9. Therefore, \( m_\lambda H n_\lambda = R f_{t^{(1)} \lambda} \) as required.
Let $w_\lambda = d(t_\lambda)$; thus, $w_\lambda$ is the unique element of $\mathfrak{S}_n$ such that $t_\lambda = t^d w_\lambda$.

4.11. Example. Let $\lambda = ((2, 1^2), (2, 1), (2))$. Then

$$t^\lambda = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 & 8 & 9 \end{pmatrix}$$
and $t_\lambda = \begin{pmatrix} 6 & 9 \\ 7 & 8 \\ 3 & 5 & 1 & 2 \end{pmatrix}$

and $w_\lambda = (1, 6, 5, 3, 7, 4, 8)(2, 9)$.

In order to compute the Schur elements we will need to know quite a few properties of the permutations $w_\lambda$; these permutations enter the story through the following definition and Corollary.

4.12. Definition. Suppose that $\lambda$ is a multipartition of $n$. Let $z_\lambda = m_\lambda T_{w_\lambda} n_\lambda$.

The element $z_\lambda$ and the following result are crucial to our computation of the Schur elements.

4.13. Corollary. Suppose that $P_{\mathcal{H}}(q)$ is invertible in $R$. Then $z_\lambda = \gamma^t_\lambda f_t^\lambda t_\lambda$. In particular, $m_\lambda \mathcal{H} n_{\lambda'} = R z_\lambda$.

Proof. Consulting the definitions, $z_\lambda = m_\lambda T_{w_\lambda} n_{\lambda'} = m_\lambda t_\lambda n_{\lambda'}$. Furthermore, by Proposition 3.3(i), there exist $c_{uv} \in R$ such that $m_\lambda t_\lambda = f_t^\lambda t_\lambda + \sum c_{uv} f_{uw}$ where the sum is over the pairs $(u, v)$ of standard tableaux which strictly dominate $t_\lambda$. Therefore, by (1.10),

$$z_\lambda = \left( f_t^\lambda t_\lambda + \sum_{u, v \triangleright t_\lambda} c_{uv} f_{uv} \right) \left( \frac{\gamma^t_{\lambda'}}{\gamma^\lambda_{t_\lambda}} f_{t_\lambda t_\lambda} + \sum_{t_\lambda \triangleright a', b'} b_{a'b} \right).$$

By Theorem 3.9, $f_t^\lambda f_{t_\lambda t_\lambda} = \gamma_{t_\lambda} f_{t_\lambda t_\lambda}$ and this is the only non-zero term in this product by Corollary 1.8. Hence, $z_\lambda = \gamma^t_{t_\lambda} f_{t_\lambda t_\lambda}$ as required. \square

Now, $z_\lambda$ is an element of $\mathcal{H}_Z$, so $\gamma^t_{t_\lambda} f_{t_\lambda t_\lambda} \in \mathcal{H}_Z$. By definition, $z_\lambda \mathcal{H}$ is a submodule of $m_\lambda \mathcal{H}$ and a quotient module of $n_{\lambda'} \mathcal{H}$. Over an arbitrary ring $R$, Du and Rui [3, Remark 2.5] showed that $S^\lambda \cong z_\lambda^* \mathcal{H}$ and $S^\lambda \mathcal{H}$ is a module. The isomorphisms being given by the natural quotient maps $m_\lambda \mathcal{H} \to z_\lambda^* \mathcal{H}$ and $n_{\lambda'} \mathcal{H} \to z_\lambda \mathcal{H}$. Note that $S^\lambda \cong S^\lambda \mathcal{H}$ when $\mathcal{H}$ is semisimple by Corollary 5.11.

5. The Schur elements

Using the results of the previous sections we are now ready to compute the Schur elements of $\mathcal{H}$. By Lemma 2.6, $s_\lambda(q) = \tau(F_\lambda)$, so it is enough to calculate $\tau(F_\lambda)$. Our basic strategy, which is inspired by Murphy [25], is to write $F_\lambda$ as a product of two terms and, in effect, to evaluate $\tau$ on each of these factors separately.

5.1. Proposition. Suppose that $t$ is a standard $\lambda$-tableau. Then there exist elements $\Phi_t$ and $\Psi_t$ in $\mathcal{H}(\mathfrak{S}_n)$ such that

(i) $\Psi_t F_t = F_t \Phi_t$;
(ii) $\Phi_t = T_{d(t)} + \sum_{w < d(t)} p_{tw} T_w$ for some $p_{tw} \in R$; and,
(iii) $\gamma^t_{\lambda} \Phi_t \Psi_t^* = \gamma^t_{t_\lambda}$. 

Then there exists an integer $i$, with $1 \leq i < n$, such that $s = (i, i + 1) \triangleright t$. Let 
$\alpha = \frac{(q-1) \text{res}_s(i)}{\text{res}_t(i) - \text{res}_s(i)}$ and $\beta = \frac{(q-1) \text{res}_s(i)}{\text{res}_t(i) - \text{res}_s(i)}$.
Then $f_{st}(T_i - \alpha) = f_{st}$ by Proposition 3.3.
Similarly, by the left hand analogue of Proposition 3.5 (interchanging the roles of $s$ and $t$), together with (3.7)(ii), $(T_i - \beta)f_{tt} = (\gamma_t/\gamma_s)f_{st}$. Therefore,
\[(T_i - \beta)f_t = \frac{1}{\gamma_t}(T_i - \beta)f_{tt} = \frac{1}{\gamma_s}f_{st} = \frac{1}{\gamma_s}f_{st}(T_i - \alpha) = F_s(T_i - \alpha).
\]
By induction, there exist elements $\Phi_s$ and $\Psi_s$ which satisfy properties (i)–(iii). Define $\Psi_t = \Psi_s(T_i - \beta)$ and $\Phi_t = \Phi_s(T_i - \alpha)$; then, by induction and the last equation,
\[\Psi_t F_t = \Psi_s(T_i - \beta) F_t = \Psi_s F_s(T_i - \alpha) = F_{ts} \Phi_s(T_i - \alpha) = F_{ts} \Phi_t.
\]
Hence, (i) holds. Next, again by induction we have
\[\Psi_t = \Psi_s(T_i - \beta) = \left(T_{d(\delta)} + \sum_{v < d(\delta)} p_{sv} T_v\right)(T_i - \beta) = T_{d(t)} + \sum_{w < d(t)} p_{tw} T_w,
\]
by standard properties of the Bruhat order since $d(t) = d(\delta)(i, i + 1) > d(\delta)$. This proves (ii). Finally, using induction once more (and a quick calculation for the second equality),
\[\gamma_t \Phi_t \Psi_t^* = \gamma_t \Phi_s(T_i - \alpha)(T_i - \beta) \Psi_s^* = (q \text{res}_s(i) - \text{res}_t(i))(\text{res}_s(i) - q \text{res}_t(i)) \frac{(\text{res}_s(i) - \text{res}_t(i))^2}{(\text{res}_s(i) - \text{res}_t(i))^2} \gamma_s = \gamma_t,
\]
the last equality coming from (3.7)(ii). This proves (iii) and so completes the proof.

We are not claiming that the elements $\Phi_t$ and $\Psi_t$ are uniquely determined by the
conditions of the Proposition; ostensibly, these elements depend upon the choice of
reduced expression for $d(t)$. In what follows we only need to know that elements
with these properties exist.

5.2. Corollary. Suppose that $t$ is a standard $\lambda$–tableau. Then
(i) $F_{\lambda} = \frac{\gamma_t}{\gamma_t} \Phi_t \Psi_t^* F_t \Psi_t^*$; and,
(ii) $F_t = \frac{\gamma_t}{\gamma_t} \Phi_t^* F_{\lambda} \Psi_t$.

Proof. Using parts (iii) and (i) of the Proposition, respectively, shows that
\[F_{\lambda} = \frac{\gamma_t}{\gamma_t} F_{\lambda} \Phi_t \Psi_t^* = \frac{\gamma_t}{\gamma_t} \Psi_t F_t \Psi_t^*;
\]
this proves (i). Part (ii) follows from (i) by ‘conjugating’ (i) by $\Phi_t$.

The main reason why we are interested in $\Psi_t$ and $\Phi_t$ is the following.

5.3. Proposition. Suppose that $s$ and $t$ are standard $\lambda$–tableaux. Then
\[f_{st} = \Phi_t^* F_{\lambda} \Psi_t.
\]
Proof. By the definition of $f_{st}$ and Proposition 5.4(ii) we have
\[ f_{st} = F_s \Psi^*_m \Psi_t F_t = F_s \Psi^*_m \Psi_t F_t - \sum_{(v,w) < (d(s),d(t))} p_{uw} F_s T_v^* m \lambda T_w F_t. \]

Now if $(v,w) < (d(s),d(t))$ then $T_v^* m \lambda T_w$ belongs to the span of the $m_{ub}$ where $(u,v) \triangleright (s,t)$. Therefore, by Proposition 5.4(i), $T_v^* m \lambda T_w$ belongs to the span of the $m_{uw}$ where either $u$ and $v$ are standard $\lambda$–tableaux and $(u,v) \triangleright (s,t)$, or $\text{Shape}(u) = \text{Shape}(v) \triangleright \lambda$; consequently, $F_s T_v^* m \lambda T_w F_t = 0$ by Proposition 5.4(iii).

Hence, by Theorem 5.1(i) and Proposition 5.1(i),
\[ f_{st} = F_s \Psi^*_m \Psi_t F_t = F_s \Psi^*_m \lambda \Psi_t F_t = F_s \Psi^*_m \lambda \Phi_t \]
as required. \hfill \square

Applying this result to $f_{t^\lambda t_\lambda}$ shows that $f_{t^\lambda t_\lambda} = \Phi^*_\lambda \Psi^*_t \Phi_t \Phi_{t^\lambda}$, the last equality following because $\Phi_{t^\lambda} = 1$. Using Proposition 5.4(iii) to multiply this equation on the right by $\Psi^*_t$, and recalling Corollary 4.13, now yields the following.

5.4. Corollary. Suppose that $\lambda$ is a multipartition of $n$. Then
\[ F_{t^\lambda} = \frac{1}{\gamma_{t^\lambda}} f_{t^\lambda t_\lambda} \Psi^*_t = \frac{1}{\gamma_{t^\lambda} \gamma_{t^\lambda^r}} z_\lambda \Psi^*_t. \]

By Lemma 2.6 in order to compute the Schur elements it suffices to calculate $\tau(F_{t^\lambda})$ for each $\lambda$; so we are reduced to finding $\tau(z_\lambda \Psi^*_t)$. To do this we rewrite $z_\lambda$ with respect to the Arikî–Koike basis. In types $A$ and $B$ (that is, $r = 1$ or $r = 2$) this is reasonably straightforward; in general we have to work much harder.

Until further notice, fix a multipartition $\lambda$ and let $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(r)})$ be the multipartition with $\lambda^{(s)} = (|\lambda^{(s)}|)$ for $1 \leq s \leq r$. Then $w_{\lambda} = d(t^\lambda)$ is the permutation
\[ (1 \cdots a_2 | a_2 + 1 \cdots a_3 | \cdots | a_r + 1 \cdots n | b_1 + 1 \cdots b_2 | b_2 + 1 \cdots b_1 | \cdots | 1 \cdots b_{r-1}), \]
where $a_s = \sum_{t=1}^{s-1} |\lambda^{(t)}|$ and $b_s = \sum_{t=s+1}^{r} |\lambda^{(t)}|$ for $1 \leq s \leq r$.

Now, $t^\lambda w_{\lambda}$ is a standard tableau; therefore, $w_{\lambda}$ is a distinguished right coset representative for $S_{\lambda}$ in $S_n$ (for example, by 22 Prop. 3.3). Equivalently, $w_{\lambda}$ is a distinguished left coset representative for $S_{\lambda^r}$ in $S_n$ since $S_{\lambda} w_{\lambda} = w_{\lambda} S_{\lambda^r}$. Consequently, there exists $w_{\lambda^r} \in S_{\lambda^r}$ such that $w_{\lambda} = w_{\lambda^r} w_{\lambda^r} \lambda$ and $\ell(w_{\lambda}) = \ell(w_{\lambda^r}) + \ell(w_{\lambda^r} \lambda)$. To proceed we need to factorize $w_{\lambda}$.

Suppose that $a$ and $b$ are non–negative integers let $w_{a,b} = w_{((a),(b))}$; so, $t_{(a),(b)} = t_{((a),(b))} w_{a,b}$. More concretely, $w_{a,0} = 1 = w_{0,b}$ and if $a > 0$ and $b > 0$ then
\[ w_{a,b} = \left( \begin{array}{cccc} 1 & 2 & \cdots & a \\ b+1 & b+2 & \cdots & a+b \end{array} \right). \]
If $i \leq j$ we also set $s_{i,j} = s_i s_{i+1} \ldots s_j$ and $s_{j,i} = 1; \text{so } s_{i,j} = (j+1,i,\ldots,i)$. For convenience let $T_{i,j} = T_s^{s_{i,j}}$. It is not hard to see that $w_{a,b} = (s_{a+b-1,1})^b$.

The permutations $w_{a,b}$ were studied by Dipper and James [11]; in particular, they observed that $w_{a,b} = s_{a,a+b-1} w_{a-1,b}$, with the lengths adding, which implies the following. (Note that our $s_{i,j}$ is Dipper and James’ $s_{i,j-1}$ when $i < j$; also if $i \geq j$ then Dipper and James set $s_{i,j} = (s_{j,i-1})^{-1}$ whereas we have $s_{i,j} = 1$.) We let $S_{(b,a)} = S_b \times S_a \hookrightarrow S_n$ (natural embedding).
5.5. Lemma. Suppose that $a$ and $b$ are positive integers with $a + b \leq n$. Then $w_{a,b} = s_{a,a+b-1} \cdots s_{1,b}$ and $\ell(w_{a,b}) = \ell(s_{a,a+b-1}) + \cdots + \ell(s_{1,b})$. Moreover, $w_{a,b}$ is a distinguished left coset representative for $\text{S}_{(b,a)}$ in $\text{S}_n$; that is, $\ell(w_{a,b}v) = \ell(w_{a,b}) + \ell(v)$ for all $v \in \text{S}_{(b,a)}$.

For $s = 1, \ldots, r$ define $w_{s,b_s} = w_{n_s,b_s}$, where $n_s = |\lambda(s)|$ and $b_s = \sum_{t=s+1}^{r} |\lambda(t)|$; in particular, $b_0 = n$ and $w_{\lambda,1} = 1$. Now $n_s + b_s = b_{s-1}$ so $w_{\lambda,s} \in \text{S}_{b_{s-1}}$ for each $s$; therefore, $w_{\lambda,1} \cdots w_{\lambda,r-1}$ is an element of $\text{S}_{b_{r-1}}$ and, consequently,

$$
\ell(w_{\lambda,1} \cdots w_{\lambda,r-1}) = \ell(w_{\lambda,1}) + \ell(w_{\lambda,2} \cdots w_{\lambda,r-1}) = \cdots = \ell(w_{\lambda,1}) + \cdots + \ell(w_{\lambda,r-1})
$$

by Lemma 5.5. Noting that $(k)w_{k} = (k)w_{\lambda,1} \cdots w_{\lambda,s}$ for $k = 1, 2, \ldots, a_{s+1}$ we have shown the following.

5.6 Let $\lambda$ be a partition of $n$. Then $w_\lambda = w_{\lambda,1}w_{\lambda,\lambda} = w_{\lambda,1} \cdots w_{\lambda,r-1}w_{\lambda,\lambda}$. Moreover,

$$
\ell(w_\lambda) = \ell(w_{\lambda,1}) + \ell(w_{\lambda,\lambda}) = \ell(w_{\lambda,1}) + \cdots + \ell(w_{\lambda,r-1}) + \ell(w_{\lambda,\lambda}).
$$

The point of the lengths adding is that $T_{w_\lambda} = T_{w_{\lambda,1}} \cdots T_{w_{\lambda,r-1}}T_{w_{\lambda,\lambda}}$. We will use this below without further comment.

Although we won’t need it notice that $\ell(w_{a,n-a}) = a(n-a)$ by Lemma 5.5 (or directly); so (5.4) implies that $\ell(w_\lambda) = \sum_{1 \leq s < t \leq r} |\lambda(s)||\lambda(t)|$.

5.7. Example. Let $\lambda = ((2, 1^2), (2, 1), (2))$. Then $\bar{\lambda} = ((4), (3), (2))$ and

$$
w_\lambda = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 6 & 9 & 7 & 8 \\ 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 \\ 8 & 9 \end{pmatrix} = (1, 6, 5, 3, 7, 4, 8)(2, 9),
$$

$$
w_{\bar{\lambda}} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 6 & 7 & 8 & 9 \\ 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 \\ 8 & 9 \end{pmatrix} = (1, 6, 4, 9, 2, 7, 5, 3, 8);
$$

therefore, $w_{\lambda,\bar{\lambda}} = (7, 9, 8)(4, 5)$. Further, $b = (5, 2, 0)$ so, using Lemma 5.5,

$$
w_{\lambda,1} = w_{4,5} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 6 & 7 & 8 & 9 \\ 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 \\ 8 & 9 \end{pmatrix} = (1, 6, 2, 7, 3, 8, 4, 9, 5) = s_{4,8}s_{3,7}s_{2,6}s_{1,5},
$$

$$
w_{\lambda,2} = w_{3,2} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 5 & 1 \\ 2 & 3 & 4 & 5 \\ 8 & 9 \end{pmatrix} = (1, 3, 5, 2, 4) = s_{3,4}s_{2,3}s_{1,2},
$$

and $w_{\lambda,3} = 1$. It is evident that $w_\lambda = w_{\lambda,1}w_{\lambda,2}w_{\lambda,3}$.

Given $i \leq j$ and $Q \in R$ let $L_{i,j}(Q) = (L_i - Q) \cdots (L_j - Q)$; if $i > j$ set $L_{i,j}(Q) = 1$. In particular, $w_{\lambda}^{-1} = L_{1,b_1}(Q_1) \cdots L_{1,b_{r-1}}(Q_{r-1})$ where $b_s = |\lambda(s+1)| + \cdots + |\lambda(r)|$, as above.

As a final piece of notation, given $0 \leq l \leq m < n$ let $H_{k,m}$ be the $R$-submodule of $\mathcal{H}$ spanned by the elements $\{ L_{1}^{l_1} \cdots L_{k}^{l_k} T_w \mid 0 \leq c_i < r \text{ and } w \in \mathcal{S}_m \}$. In general $H_{k,m}$ is neither a subalgebra nor a submodule of $\mathcal{H}$; however, $H_{k,m}$ is a right $\mathcal{H}(\mathcal{S}_m)$–module.

One of the difficulties in working with $\mathcal{H}$ is that, in general, the powers of the $L_k$ are not necessarily linear combinations of products of the $L_k^{s}$ for $1 \leq i \leq k$ and $1 \leq s < r$; however, it is always true that $L_k^{s} \in H_{k,m}$ for all $s \geq 0$ provided that $m \geq k$ (since in this case $T_0, \ldots, T_{k-1} \subseteq H_{k,m}$ by (5.2)).

5.8. Lemma. Suppose that $h \in H_{k-1,k}$ for some $k$ with $1 \leq k \leq n$. 
(i) If \( 1 \leq i < n \) then
\[
T_i h = \begin{cases} 
H_{k-1,k}, & \text{if } 1 \leq i < k-1, \\
H_{k,k}, & \text{if } i = k-1, \\
H_{k-1,i+1}, & \text{if } k \leq i < n.
\end{cases}
\]

(ii) If \( 1 \leq s < r \) then \( h L_k^s = \sum_{t=0}^{s} L_k^t h_t \) for some \( h_t \in H_{k-1,k} \).

**Proof.** By definition \( h \) is a linear combination of terms of the form \( L_{k-1}^{c_i} \ldots L_1^{c_1} T_w \), for some \( 0 \leq e_i < r \) and \( w \in \mathcal{S}_k \). By (2.1)(ii), if \( i \neq m-1, m \) then \( T_i L_m = L_m T_i \); whereas \( T_m L_m = L_{m+1} (T_m - q + 1) \) and \( T_{m-1} L_m = L_{m-1} T_{m-1} + (q-1) L_m \).

Combining these facts proves (i).

For part (ii), observe that if \( w \in \mathcal{S}_k \) then \( T_w L_k = L_k T_w \) unless \( s_{k-1} \) appears in a reduced expression of \( w \); however, \( T_{k-1} L_k = L_{k-1} T_{k-1} + (q-1) L_k \). Part (ii) now follows by induction on \( s \) using the remarks of the last paragraph.

**5.9. Lemma.** Fix \( Q \in R \) and let \( i, j, c \) and \( d \) be positive integers with \( i \leq c \leq d \leq j < n \). Then
\[
T_{i,j} L_{c,d} (Q) = \left\{ L_{c+1,d+1}(Q) + \sum_{k=c+1}^{d+1} L_k h_k \right\} T_{i,j}
\]
for some elements \( h_k \in H_{k-1,k} \) for \( c+1 \leq k \leq d+1 \).

**Proof.** If \( d = j \) then by convention \( T_{d+1,j} = 1 \). Therefore, by (2.1)(iv) we have
\[
T_{i,j} L_{c,d} (Q) = T_{i,c-1} T_{c,d} (L_c - Q) \ldots (L_d - Q) T_{d+1,j} = T_{i,c-1} T_{c} (L_c - Q) \ldots T_{d} (L_d - Q) T_{d+1,j}.
\]
Now, \( T_k L_k = q L_{k+1} T_k^{-1} = L_{k+1} T_k - (q-1) L_{k+1} \), so
\[
T_{i,j} L_{c,d} (Q) = T_{i,c-1} \prod_{k=c+1}^{d+1} \left\{ (L_k - Q) T_{k-1} - (q-1) L_k \right\} \cdot T_{d+1,j},
\]
where we read the terms from left to right with increasing values of \( k \) — the order of the factors is important here. Now \( T_{k-1} \) commutes with \( L_m \) if \( k < m \); therefore, each \( T_{k-1} \) in this product commutes with all of \( L_m \)'s which appear to its right. A straightforward induction on \( (d-c) \) using (2.1) shows that there exist \( h'_k \in H_{k-1,k} \) such that
\[
T_{i,j} L_{c,d} (Q) = T_{i,c-1} \left\{ L_{c+1,d+1}(Q) T_{c,d} + \sum_{k=c+1}^{d+1} L_k h'_k T_{k,d} \right\} T_{d+1,j}.
\]
(Recall that our convention is that \( T_{d+1,d} = 1 \).) If \( c+1 \leq k \leq d+1 \) then by (2.1)
\[
T_{i,c-1} L_k h'_k T_{k,d} T_{d+1,j} = L_k T_{i,c-1} h'_k T_{k,j} = L_k T_{i,c-1} h'_k T_{i-1,k-1} T_{i,j}.
\]
Also, using (2.1)(ii), \( T_{i,c-1} L_{c+1,d+1} (Q) = L_{c+1,d+1}(Q) T_{i,c-1} \). Therefore,
\[
T_{i,j} L_{c,d} (Q) = L_{c+1,d+1}(Q) T_{i,j} + \sum_{k=c+1}^{d+1} L_k h_k T_{i,j},
\]
where \( h_k = T_{i,c-1} h'_k T_{i,k-1}^{-1} \in H_{k-1,k} \) by Lemma 5.8, for \( c+1 \leq k \leq d+1 \).
This brings us to the key technical lemma.

5.10. Lemma. Fix $Q \in R$ and let $a$, $b$, $c$ and $d$ be non-negative integers such that $a + b \leq n$ and $1 \leq c \leq d \leq b$. Then there exist elements $h_k \in H_{n-1,k}$ such that

$$T_{w_{a,b}}L_{c,d}(Q) = \left\{ L_{a+c,a+d}(Q) + \sum_{k=a+c}^{a+d} L_k h_k \right\} T_{w_{a,b}}.$$

Proof. We argue by induction on $a$. If $a = 0$ then $w_{a,b} = 1$ and there is nothing to prove; so suppose that $a \geq 1$. Then $w_{a,b} = s_{a,a+b-1}w_{a-1,b}$ by Lemma 5.3. Therefore, by induction and Lemma 5.4, respectively, there exist elements $h_k', h_k'' \in H_{n-1,k}$ such that

$$T_{w_{a,b}}L_{c,d}(Q) = T_{a,a+b-1}L_{w_{a-1,b}}L_{c,d}(Q)$$

$$= T_{a,a+b-1}L_{a+c-1,a+d-1}(Q) + \sum_{k=a+c-1}^{a+d-1} L_k h_k' T_{w_{a-1,b}}$$

$$= \left\{ L_{a+c,a+d}(Q) + \sum_{k=a+c}^{a+d} L_k h_k'' \right\} T_{a,a+b-1}T_{w_{a-1,b}}$$

$$+ \sum_{k=a+c}^{a+d} T_{a,a+b-1}L_k h_k' T_{w_{a-1,b}}.$$

If $a + c - 1 \leq k \leq a + d - 1$ then by (2.1)

$$T_{a,a+b-1}L_k h_k' T_{w_{a-1,b}} = T_{a,k-1}T_k L_k T_{k+1,a+b-1}h_k' T_{w_{a-1,b}}$$

$$= q T_{a,k-1}L_{k+1} T_{k+1,a+b-1} h_k' T_{w_{a-1,b}}$$

$$= q L_{k+1} T_{a,k-1} T_{k+1,a+b-1} h_k' T_{w_{a-1,b}}$$

$$= q T_{a,k-1} T_{k}^{-1} h_k' T_{a,k-1} T_{a,a+b-1} T_{w_{a-1,b}}$$

$$= L_k T_{k-1} h_k' T_{a,k-1} T_{w_{a-1,b}},$$

where the third equality follows by (2.1) because $h_k' \in H_{k-1,k}$. By Lemma 5.4, the element $T_{a,k-1}(T_k - q + 1) h_k' T_{a,k}^{-1}$ belongs to $H_{k,k+1}$; so, combining these equations proves the Lemma.

Lemma 5.10 is enough to compute the Schur elements when $r = 2$; in order to cover the general case we delicately apply Lemma 5.10 several times.

5.11. Corollary. Suppose that $a$, $b$, $c$ and $s$ are non-negative integers with $a + b \leq n$, $1 \leq c \leq b$ and $1 \leq s < r$. Then $T_{w_{a,b}}L^s_c = \sum_{t-1}^s L_{a+c} h_t T_{w_{a,b}}$ some $h_t \in H_{a+c-1,a+c}$.

Proof. If $s = 1$ this is the special case of Lemma 5.10 corresponding to the choices $d = c$ and $Q = 0$; in particular, this implies that the exponent $t$ is always at least 1. If $s > 1$ then the result follows by induction using Lemma 5.8 (ii).

For each integer $k$ let $c(k) = s$ if $k$ appears in component $s$ of $\lambda$. Observe that for all $k$ we have $c(k) = \min\{ 1 \leq s \leq r \mid k \leq |\lambda^{(1)}| + \cdots + |\lambda^{(s)}| \}$. For the next Lemma recall that $a_s = |\lambda^{(1)}| + \cdots + |\lambda^{(s-1)}|$ and $b_s = |\lambda^{(s+1)}| + \cdots + |\lambda^{(r)}|$ for $1 \leq s \leq r$. 


5.12. Lemma. Suppose that $\lambda$ is a multipartition of $n$. Then

$$T_{w_\lambda} u_{\lambda'} = L_{\alpha_2+1,n}(Q_1) \ldots L_{\alpha_r+1,n}(Q_{r-1}) T_{w_\lambda} + \sum_{k=1}^n c_{\lambda}(k) \frac{1}{L_k h_k}$$

for some $h_k \in H_{k-1,n}$.

Proof. First note that if $r = 1$ then $u_{\lambda'} = 1$ and there is nothing to prove; so we assume that $r > 1$. Next, $T_{w_\lambda} u_{\lambda'} = T_{w_\lambda} T_{w_\lambda} u_{\lambda'} = T_{w_\lambda} u_{\lambda'} T_{w_\lambda}$ by part (iv) of (2.1) since $w_{\lambda'/\lambda} \in \mathcal{S}_\lambda$. Consequently, it is enough to consider $T_{w_\lambda} u_{\lambda'}$; equivalently, we may assume that $\lambda = \lambda'$ (for clarity we will continue to write $\lambda$).

By (2.1) (iv),

$$T_{w_\lambda} u_{\lambda'} = T_{w_{\lambda,1}} \ldots T_{w_{\lambda,r-1}} L_{1,b_1}(Q_1) \ldots L_{1,b_{r-1}}(Q_{r-1})$$

$$= T_{w_{\lambda,1}} L_{1,b_1}(Q_1) \ldots T_{w_{\lambda,r-1}} L_{1,b_{r-1}}(Q_{r-1}).$$

We also let $n_s = |\lambda^{(s)}|$. Suppose that $1 \leq s \leq r - 1$. For $k = 1, \ldots, b_{s-1}$ set

$$c_{\lambda,s}(k) = \min \{ 0 \leq t \leq r - s | k \leq |\lambda^{(s)}| + \cdots + |\lambda^{(s+t)}| \}$$

and let $H^{(s)}$ be the $R$-submodule of $\mathcal{H}$ spanned by elements of the form $L_k h_k$ where $n_s < k \leq b_{s-1}$, $1 \leq c \leq c_{\lambda,s}(k)$ and $h_k \in H_{k-1,b_{s-1}}$. We claim that

$$T_{w_{\lambda,s}} L_{1,b_s}(Q_s) \ldots T_{w_{\lambda,r-1}} L_{1,b_{r-1}}(Q_{r-1}) \equiv P^{(s)} \pmod{H^{(s)}},$$

where $P^{(s)} = \prod_{t=s}^{r-1} L_{n_s+\cdots+n_t+1,b_s-1}(Q_t) T_{w_{\lambda,s}} \ldots T_{w_{\lambda,r-1}}$. Taking $s = 1$ will prove the Lemma because $b_0 = n$ and $c_{\lambda}(k) = c_{\lambda,1}(k) + 1$ by the remarks before the Lemma. Note that $P^{(s)}$ has degree $c_{\lambda,s}(k)$ when we consider it as a polynomial in $L_k$.

To prove the claim we argue by downwards induction on $s$. If $s \geq c_{\lambda}(n)$ then $b_s = 0$ so that $T_{w_{\lambda,s}} = L_{1,b_s}(Q_s) = P^{(s)} = 1$ and there is nothing to prove. Suppose then that $s < c_{\lambda}(n)$. Then, by induction,

$$T_{w_{\lambda,s}} L_{1,b_s}(Q_s) \ldots T_{w_{\lambda,r-1}} L_{1,b_{r-1}}(Q_{r-1}) = T_{w_{\lambda,s}} L_{1,b_s}(Q_s) \left\{ P^{(s+1)} + h^{(s+1)} \right\}$$

for some $h^{(s+1)} \in H^{(s+1)}$. Now $b_s > 0$ since $s < c_{\lambda}(n)$, so $b_{s-1} = b_s + n_s \geq b_s \geq 1$. Hence, by Lemma 5.10,

$$T_{w_{\lambda,s}} L_{1,b_s}(Q_s) \left\{ P^{(s+1)} + h^{(s+1)} \right\} = T_{w_{n_s+\cdots+n_t+1,b_s-1}} L_{1,b_s}(Q_s) \left\{ P^{(s+1)} + h^{(s+1)} \right\}$$

$$= \left\{ L_{n_s+\cdots+n_t+1}(Q_s) T_{w_{n_s+\cdots+n_t+1,b_s-1}} + \sum_{k=n_s+1}^{b_{s-1}} L_k h_k T_{w_{n_s+\cdots+n_t+1,b_s-1}} \right\} \left\{ P^{(s+1)} + h^{(s+1)} \right\},$$

for some $h_k \in H_{k-1,k}$. We will move $T_{w_{\lambda,s}}$ past $P^{(s+1)}$ and $h^{(s+1)}$ using Lemma 5.10.

Viewing $P^{(s+1)}$ as a polynomial in the $L_m$'s, if $n_{s+1} < m \leq b_s$ then the degree of $L_m$ in $P^{(s+1)}$ is at most $c_{\lambda,s+1}(m)$; further, $L_m$ does not appear in $P^{(s+1)}$ if either $m \leq n_{s+1}$ or $m > b_s$. Let $p^{(s+1)} = \prod_{t=s+1}^{r-1} L_{n_{s+1}+\cdots+n_{t-1}+1,b_t}(Q_t)$. Then by
Lemma 5.10 and Corollary 5.11 there exist $h_{k,1}, h_{ck,2} \in H_{k-1,k}$ such that

$$T_{w_{n,s}, b_s} P^{(s+1)} = \left\{ L_{n_s+n_{s+1}+1, b_{s-1}} (Q_{s+1}) + \sum_{k=n_s+n_{s+1}+1}^{b_{s-1}} L_k h_{k,1} \right\} T_{w_{n,s}, b_s} P^{(s+2)}$$

$$= \left\{ L_{n_s+n_{s+1}+1, b_{s-1}} (Q_{s+1}) L_{n_s+n_{s+1}+n_{s+2}+1, b_{s-1}} (Q_{s+2}) + \sum_{k=n_s+n_{s+1}+1}^{b_{s-1}} \sum_{c=1}^{b_{s-1}} c_2(k) L_k h_{ck,2} \right\} T_{w_{n,s}, b_s} P^{(s+3)},$$

where $c_2(k) = 1$ if $n_s + n_{s+1} < k \leq n_s + n_{s+1} + n_{s+2}$ and $c_2(k) = 2$ if $n_s + n_{s+1} + n_{s+2} < k \leq b_{s-1}$. For the second equality we have used Lemma 5.10, Lemma 5.8 and Corollary 5.11. Looking at the definition of $c_{\lambda,s}(m)$, if $n_{s+1} < m \leq b_s$, then $c_{\lambda,s+1}(m) = c_{\lambda,s}(m + n_k) - 1$; therefore, continuing in this way we see that

$$T_{w_{n,s}, b_s} P^{(s+1)} = \prod_{t=s+1}^{r-1} L_{n_s+n_{s+1}+\cdots+n_t+1, b_{t-1}} (Q_t) \cdot T_{w_{n,s}, b_s} \pmod{H^{(s)}}.$$

In obtaining this equation notice that if $m > k$ then $L_k h_k L_m h_k = L_m (L_k h_k h_m)$ and $L_k h_k h_m \in H_{m-1,m}$, for any $h_k \in H_{k-1,k}$ and $h_m \in H_{m-1,m}$; similarly, if $m < k$ then $h_k L_m h_m \in H_{k-1,k}$ by Lemma 5.8. It follows that if $L_k h$ appears in this expansion, for some $h \in H_{k-1,k}$, then $n_s + n_{s+1} < k \leq b_{s-1}$ and so $1 \leq c \leq c_{\lambda,s+1}(k - n_s) = c_{\lambda,s}(k) - 1$ by the remarks above; hence, $L_k h_k L_m h_m \in H^{(s)}$ for all $k$ and $m$.

Now consider a term from the inductive step of the form $L_k h_k T_{w_{n,s}, b_s} P^{(s+1)}$, where $n_s < k \leq b_{s-1}$ and $h_k \in H_{k-1,k}$. What we have just shown combined with Lemma 5.8(ii) shows that $L_k h_k T_{w_{n,s}, b_s} P^{(s+1)}$ is equal to a linear combination of terms of the form $L_k h_m$, where $n_s < m \leq b_{s-1}$, $1 \leq c \leq c_{\lambda,s}(m)$ and $h_m \in H_{m-1,m}$. Moreover, $c \leq c_{\lambda,s}(m)$ with equality only if $k = m > n_s + n_{s+1}$ and $c = 1 = c_{\lambda,s}(m)$ if $n_s < m \leq n_s + n_{s+1}$. Hence, $L_k h_k T_{w_{n,s}, b_s} P^{(s+1)} \in H^{(s)}$.

Therefore, combining the last two paragraphs we have shown that

$$T_{w_{n,s}, b_s} L_1 b_s (Q_s) P^{(s+1)} = P^{(s)} \pmod{H^{(s)}}.$$

By similar arguments, the terms $L_{n_s+n_{s+1}+1, b_{s-1}} (Q_s) T_{w_{n,s}, b_s} h^{(s+1)}$ and $L_k h_k T_{w_{n,s}, b_s} h^{(s+1)}$ from the inductive step also belong to $H^{(s)}$. We leave the details to the reader. \(\square\)

We are now basically done. The next result essentially computes $\tau(z \Psi^*_{\lambda})$; we record it separately because it is also the key to showing that $F^\lambda$ is a self–dual $\mathcal{H}$–module and that $\tilde{S}^\lambda$ is isomorphic to the dual of $S^\lambda$; see 24.

5.13. Proposition. Suppose that $\lambda$ is a multipartition of $n$. Then

$$\tau(z \Psi^*_{\lambda}) = (-1)^{n(r-1)} q^{w_3} \prod_{s=1}^{r} Q_n^{-1} L^{n-\lambda(s)}.$$
Proof. Let \( \tilde{u}_\lambda \), \( L_{a+1,n}(Q_1) \ldots L_{a+1,n}(Q_{r-1}) \). Using (2.3)(i) and Lemma 5.12 shows that

\[
\tau(z_\lambda T_{w_\lambda}^*) = \tau(m_\lambda T_{w_\lambda} u_\lambda T_{w_\lambda}^*) = \tau(x_\lambda u_\lambda^+ T_{w_\lambda} u_\lambda^- y_\lambda T_{w_\lambda}^*) \\
= \tau(u_\lambda^+ T_{w_\lambda} u_\lambda^- y_\lambda T_{w_\lambda}^* x_\lambda) \\
= \tau(u_\lambda^+ \tilde{u}_\lambda T_{w_\lambda} y_\lambda T_{w_\lambda}^* x_\lambda) + \tau(u_\lambda^+ h_\lambda y_\lambda T_{w_\lambda}^* x_\lambda),
\]

where \( h = \sum_{k=1}^n \sum_{s=1}^{c_s(k)-1} L_s^k h_{kc} \) for some \( h_{kc} \in H_{k-1,n} \). Consider \( u_\lambda^+ \) as a polynomial in \( L_k \). Then the degree of \( L_k \) in \( u_\lambda^+ \) is \( r - c_\lambda(k) \). Therefore, if \( h_{kc} \neq 0 \) then \( u_\lambda^+ L_k^k h_{kc} \) is a polynomial in \( L_k \) which is left divisible by \( L_k \) and has degree at most \( r - 1 \). If \( m > k \) then \( L_m^k \) appears in \( u_\lambda^+ L_k^k h_{kc} \) with exponent at most \( r - c_\lambda(m) < r \). (If \( m < k \) then \( L_m^k \) can appear in \( u_\lambda^+ L_k^k h_{kc} \) for \( d \geq r \); however, this does not matter because such terms can be written as a linear combination of Ariki–Koike basis elements in \( H_{k-1,n} \).) Therefore, \( \tau(u_\lambda^+ L_K^k h_{kc} y_\lambda T_{w_\lambda}^* x_\lambda) = 0 \) by (2.3)(ii) and, consequently, \( \tau(u_\lambda^+ h_\lambda y_\lambda T_{w_\lambda}^* x_\lambda) = 0 \); hence,

\[
\tau(z_\lambda T_{w_\lambda}^*) = \tau(u_\lambda^+ \tilde{u}_\lambda^- T_{w_\lambda} y_\lambda T_{w_\lambda}^* x_\lambda).
\]

Considered as polynomials in \( L_k \), \( u_\lambda^+ \) has degree \( r - c_\lambda(k) \) and \( \tilde{u}_\lambda^- \) has degree \( c_\lambda(k) - 1 \); consequently, each \( L_k \) has degree \( r - 1 \) in \( u_\lambda^+ \tilde{u}_\lambda^- \). Therefore, by (2.3) again,

\[
\tau(z_\lambda T_{w_\lambda}^*) = \tau(u_\lambda^+ \tilde{u}_\lambda^-) \cdot \tau(T_{w_\lambda} y_\lambda T_{w_\lambda}^* x_\lambda) \\
= \prod_{s=2}^r (-Q_s)^{a_s} \prod_{s=1}^{r-1} (-Q_s)^{b_s} \cdot \tau(T_{w_\lambda} y_\lambda T_{w_\lambda}^* x_\lambda) \\
= (-1)^{n(r-1)} \prod_{s=1}^r Q_s^{n - |\lambda(s)|} \cdot \tau(T_{w_\lambda} y_\lambda T_{w_\lambda}^* x_\lambda),
\]

since \( a_s + b_s = n - |\lambda(s)| \) for all \( s \) and \( a_1 = 0 = b_1 \). To complete the proof recall that \( \mathcal{G}_\lambda \cap w_\lambda \mathcal{G}_{\lambda'} = \{1\} \) and that \( w_\lambda \) is a distinguished \( (\mathcal{G}_\lambda, \mathcal{G}_{\lambda'}) \)-double coset representative (consider the tableaux \( t_\lambda^\lambda \) and \( t_\lambda \), so

\[
T_{w_\lambda} y_\lambda T_{w_\lambda}^* x_\lambda = \sum_{u \in \mathcal{G}_{\lambda'}} (-1)^{\ell(u)} T_{w_\lambda} T_u T_{w_\lambda}^* T_v = \sum_{u \in \mathcal{G}_{\lambda'}} (-1)^{\ell(u)} T_{w_\lambda} T_{uw_\lambda^{-1}} T_v.
\]

Therefore, \( \tau(T_{w_\lambda} y_\lambda T_{w_\lambda}^* x_\lambda) = q^{\ell(w_\lambda)} \) by (2.3)(ii) (corresponding to \( u = v = 1 \)), so the Proposition follows.

We can now give our first formula for the Schur elements \( s_\lambda(q) \) of \( \mathcal{H} \).

5.14. Corollary. Suppose that \( \lambda \) is a multipartition of \( n \). Then

\[
s_\lambda(q) = (-1)^{n(r-1)} q^{-\ell(w_\lambda)} \prod_{s=1}^r Q_s^{[\lambda|[s]|} - n.
\]

Proof. By Lemma 5.4, \( s_\lambda(q) = 1/\tau(F_{t_\lambda}) \) and \( F_{t_\lambda} = \frac{1}{\gamma_{t_\lambda} \gamma_{t_\lambda'}^*} z_{t_\lambda'} \Psi_{t_\lambda}^* \) by Corollary 5.2, therefore, \( s_\lambda(q) = \gamma_{t_\lambda} \gamma_{t_\lambda'}^*/\tau(z_{t_\lambda'} \Psi_{t_\lambda}^*) \). Now, \( \tau \) is a trace form by (2.3)(i), so

\[
\tau(z_{t_\lambda'} \Psi_{t_\lambda}^*) = \tau(x_{t_\lambda} u_{t_\lambda}^+ T_{w_\lambda} u_{t_\lambda}^- y_{t_\lambda} x_{t_\lambda}^*). 
\]
It is well-known and easy to check that the permutation \( w_\lambda \) has the “trivial intersection property”; that is, \( \mathfrak{S}_\lambda \cap \mathfrak{S}_\mu \neq \{1\} \) if and only if \( \mathfrak{S}_\lambda \mathfrak{S}_\mu = \mathfrak{S}_\lambda w_\lambda \mathfrak{S}_\mu \). Therefore, \( y_\lambda T^*_w x_\lambda \neq 0 \) if and only if \( w \in \mathfrak{S}_\lambda w_\lambda \mathfrak{S}_\mu \) (see, for example, [9, (4.9)]).

Now, \( \Psi^*_t \lambda = T^*_w z_\lambda + \sum_{w < w'} \mathfrak{S} z_\lambda T^*_w \) by Proposition 5.1(ii); so \( y_\lambda \Psi^*_t \lambda x_\lambda = y_\lambda T^*_w x_\lambda \) since \( w_\lambda \) is the unique element of minimal length in \( \mathfrak{S}_\lambda w_\lambda \mathfrak{S}_\mu \). Therefore,

\[
\tau(z_\lambda \Psi^*_t \lambda) = \tau(u_\lambda T^*_w u_\lambda^\gamma y_\lambda T^*_w x_\lambda) = \tau(x_\lambda u_\lambda T^*_w u_\lambda x_\lambda y_\lambda T^*_w) = \tau(z_\lambda T^*_w x_\lambda).
\]

The result now follows from Proposition 5.13.

A closed formula for \( \gamma'_t \lambda \), is given by \((3.7)(i)\); therefore, in order to find an explicit formula for \( s_\lambda(q) \) we need only compute \( \gamma'_t \lambda \). Although the formula below looks formidable, its proof follows readily enough from the definition of \( \gamma'_t \lambda \).

Recall that the \( ij \)th hook in the diagram \( |\lambda(t)\rangle \) is the collection of nodes to the right of and below the node \( (i, j, s) \), including the node \( (i, j, s) \) itself. The \( ij \)th hook length \( h_{ij}^{\lambda} = \lambda^s(s) + \lambda^s(j) - i - j + 1 \) is the number of nodes in the \( ij \)th hook and the leg length \( l_{ij}^{\lambda} = \lambda^s(j) - j + 1 \), is the number of nodes in the “leg” of this hook. Observe that if \( (a, b, c) \) and \( (i, j, c) \) are two removable nodes in \( |\lambda(t)\rangle \) with \( a \leq i \) and \( j \leq b \) then \( h_{ij}^{\lambda} = b - a - j + i + 1 \).

5.15. Lemma. Suppose that \( \lambda \) is a multipartition of \( n \). Then

\[
\gamma_t \lambda = q^{\ell(w_\lambda)} \prod_{(i, j, s) \in |\lambda(t)\rangle} \frac{[h_{ij}^{\lambda(s)}]}{[q]^{[h_{ij}^{\lambda(s)}]}} q^{\prod_{t = s + 1}^{r-1} (q^{j-i} Q_{s} - q^{j-i} Q_{t})} \prod_{k=1}^{\lambda^{(i)}} \frac{[q^{j-i} Q_{s} - q^{k-1} \lambda^{(i)}]}{[q^{j-i} Q_{s} - q^{k} \lambda^{(i)}]}.
\]

(Note that \( \lambda^{(i)} \) is the length of the \( k \)th column of \( |\lambda(t)\rangle \).)

Proof. We argue by induction on \( n \). If \( n = 0 \), both sides are 1 and there is nothing to prove (by convention, empty products are 1). Suppose that \( n > 0 \). Let \( \mu = \text{Shape}(t_\lambda(n-1)) \); then \( \mu \) is a multipartition of \( n - 1 \). Recall that \( \alpha(\lambda) = \frac{1}{2} \sum_{s=1}^{r} \sum_{i \geq 1} (\lambda^s(i) - 1) \lambda^s(i) \). Applying the definitions (see \((3.4)\)),

\[
\frac{\gamma_t \lambda}{\gamma_t \mu} = q^{\ell(w_\lambda) + \alpha(\lambda) - \ell(w_\mu) - \alpha(\mu)} \prod_{x \in \mathfrak{S} \lambda \lambda(n)} \frac{\text{res}_t \lambda(n) - \text{res}(x)}{\prod_{y \in \mathfrak{S} \lambda \lambda(n)} \text{res}_t \lambda(n) - \text{res}(y)}.
\]

Assume that \( n \) appears in row \( a \) and column \( b \) of \( t_\lambda^{(i)} \). First consider the contribution that the addable and removable nodes in \( |\lambda(t)\rangle \) make to \( \gamma_t \lambda \). Looking at the definitions above \((3.6)\), these nodes occur in pairs \((x, y)\) where \( y < (a, b, c) \) is a removable node in row \( i \) and \( x \) is an addable node in row \( i + 1 \) for some \( i \geq a \). If \( x \) is in column \( d \) of \( |\lambda(t)\rangle \) and \( y \) is in column \( d' \) then \( d \leq d' < b \) and

\[
\frac{\text{res}_t \lambda(n) - \text{res}_t(x)}{\text{res}_t \lambda(n) - \text{res}_t(y)} = \frac{q^{b-a} Q_e - q^{d-(i+1)} Q_e}{q^{b-a} Q_e - q^{d-i} Q_e} = \prod_{j=d}^{d'} q^{b-a} - q^{j-1}.
\]

\[
= \prod_{j=d}^{d'} q^{j-i-1} (q^{b-a-j+i+1} - 1) = \prod_{j=d}^{d'} q^{j-1} (q^{h_{ij}^{\lambda(c)}} - 1) = \prod_{j=d}^{d'} q^{j-1} [h_{ij}^{\lambda(c)} q].
\]

\[
= \prod_{j=d}^{d'} q^{-1} [h_{ij}^{\lambda(c)} q].
\]
Therefore,
\[
\prod_{(i,j,c) \in \mathcal{A}_t(n)} \left( \frac{\text{res}_{t+1}(n) - \text{res}_{t+1}(i,j,c)}{\text{res}_{t+1}(n) - \text{res}_{t+1}(i,j,c)} \right) = \prod_{j=1}^{b-1} q^{-1} [h_{a_j}^{(c)}]_q = q^{b-1} \prod_{j=1}^{b-1} [h_{a_j}^{(c)}]_q.
\]

Note that \( \alpha(\lambda) = \alpha(\mu) + b - 1 \); hence, by induction, this accounts for the factor \( q^{\ell(w_s)} \) in Lemma 5.15.

Now \( \gamma_{\ell_s} \) is known by induction and it contains as a factor the left hand term in the product below. Further, \( \ell_{ij}^{(c)} = \ell_{ij}^{(c)} \) and \( h_{ij}^{(c)} = h_{ij}^{(c)} \) if \( (i,j) \neq (a,c) \), and \( \ell_{aj}^{(c)} = \ell_{aj}^{(c)} \) for \( 1 \leq j < b \), so
\[
\left( \prod_{(i,j,c) \in [\lambda]} [h_{ij}^{(c)}]_q \right) \left( \prod_{j=1}^{b-1} [h_{a_j}^{(c)}]_q \right) = \left( \prod_{(i,j,s) \in [\lambda]} [\ell_{ij}^{(c)}]_q \right) \left( \prod_{j=1}^{b-1} [\ell_{a_j}^{(c)}]_q \right) = \prod_{(i,j,s) \in [\lambda]} [h_{ij}^{(c)}]_q,
\]

since \( [h_{ab}^{(c)}]_q = 1 = [\ell_{ab}^{(c)}]_q \). This accounts for the left hand factor in the expression for \( \gamma_{\ell_s} \) given in the statement of the Lemma.

Finally, consider the nodes in \( \mathcal{A}_t(n) \) and \( \mathcal{R}_t(n) \) which are in component \( t \) for some \( t > c \) (there are no such nodes for \( t < c \)). Again, almost all of the addable and removable nodes in component \( t \) occur in pairs placed in consecutive rows; however, this time there is also an additional addable node at the end of the first row of \( \lambda^{(t)} \).

As above, it is easier to insert extra factors which cancel out and so take a product over all of the columns of \( \lambda^{(t)} \). An argument similar to that above shows that the nodes in \( \mathcal{A}_t(n) \) and \( \mathcal{R}_t(n) \) which do not belong to component \( c \) contribute the factor
\[
\prod_{t=c+1}^{r} \left( q^{b-a}Q_c - q^{\lambda^{(t)}_c}Q_t \right) \prod_{k=1}^{\lambda^{(t)}_c} \left( q^{b-a}Q_c - q^{b-1-\lambda^{(t)}_k}Q_t \right) \left( q^{b-a}Q_c - q^{k-\lambda^{(t)}_k}Q_t \right)
\]
to \( \gamma_{\ell_s} \). Using induction to combine the formulae above proves the Lemma.

We can now give a closed formula for the Schur elements.

5.16. Corollary. Suppose that \( \lambda \) is a multipartition of \( n \) and for \( 1 \leq s < t \leq r \) let
\[
X_{st}^\lambda = \prod_{(i,j) \in [\lambda^{(t)}]} (q^{j-i}Q_t - Q_s) \cdot \prod_{(i,j) \in [\lambda^{(r)}]} (q^{j-i}Q_s - q^{\lambda^{(t)}_i}Q_t) \prod_{k=1}^{\lambda^{(t)}_c} \frac{q^{b-a}Q_c - q^{b-1-\lambda^{(t)}_k}Q_t}{q^{b-a}Q_c - q^{k-\lambda^{(t)}_k}Q_t}.
\]

Then
\[
s_\lambda(q) = (-1)^{n(r-1)}(Q_1 \cdots Q_r)^{-n} q^{-\alpha(\lambda)} \prod_{s=1}^{r} \prod_{(i,j) = [\lambda^{(s)}]} Q_s [h_{ij}^{(s)}]_q \cdot \prod_{1 \leq s < t \leq r} X_{st}^\lambda.
\]

Proof. Looking at the definitions, \( [k]_q^{(r)} = 1 + q^{-1} + \cdots + q^{-k} = q^{-k}[k]_q \); therefore, \( ([k]_q^{(r)})' = q^{-\frac{1}{2}(k-1)}[k]_q^{(r)} \); consequently, \( ([\lambda^{(s)}]_q)^{(r)} = q^{-\alpha(\lambda)}[\lambda^{(s)}]_q^{(r)} \). Next observe that \( [\lambda^{(s)}]_q = \prod_{(i,j,s) \in [\lambda]} [\ell_{ij}^{(s)}]_q \). Further, \( (i,j) \in [\lambda^{(s)}] \) if and only if \( (j,i) \in [\lambda^{(r-s+1)}] \).
Therefore, applying \( \gamma ' \) to Corollary 3.7 (i) (and swapping the roles of \( s \) and \( t \) in the right hand factor),

\[
\gamma '_{\lambda'} = q^{-\alpha(\lambda')} \left( \prod_{(i,j) \in [\lambda]} \left[ \gamma_{ij}^{(s)} \right] q \right) \left( \prod_{1 \leq s < t \leq r} \prod_{(i,j) \in [\lambda^{(s)}]} (q^{i-j}Q_t - Q_s) \right).
\]

By Corollary 5.14, \( s_\lambda(q) = (-1)^{n(r-1)}q^{-\ell(\lambda)} \gamma_{1t} \gamma '_{1t} \prod_{s=1}^r Q_s^{\lambda^{(s)}-\eta} \); so the result now follows by Lemma 5.15.

5.17. Example. It is straightforward to check that Corollary 5.16 gives the same rational functions for the Schur elements \( s_q(\mathbf{q}) \) as were obtained in Example 2.7.

It is not at all obvious that the formula in Corollary 5.16 for the Schur elements agrees with that conjectured by Malle, so we now show that this is the case. This takes quite a lot of work; however, it also results in a more symmetrical formula for \( s_\lambda(\mathbf{q}) \). To do this we need to rewrite the Schur elements as functions of beta numbers (or first column hook lengths).

Define the length of a partition \( \sigma \) to be the smallest integer \( \ell(\sigma) \) such that \( \sigma_i = 0 \) for all \( i > \ell(\sigma) \). The length of a multipartition \( \lambda \) is \( \ell(\lambda) = \max \{ \ell(\lambda^{(s)}) \mid 1 \leq s \leq r \} \).

Fix an integer \( L \) such that \( L \geq \ell(\lambda) \). The \( L \)-beta numbers for \( \lambda^{(s)} \) are the integers \( \beta_i^{(s)} = \lambda_i^{(s)} + L - i \) for \( i = 1, \ldots, L \); note that \( \beta_1^{(s)} > \cdots > \beta_L^{(s)} \geq 0 \). Malle calls the \( r \times L \) matrix \( B = (\beta_i^{(s)})_{s,i} \) the \( L \)-symbol of \( \lambda \). Actually, this is not quite one of Malle's symbols; it is what Broué and Kim [4] call \( B \) an ordinary symbol. We also let \( B_s = \{ \beta_1^{(s)}, \ldots, \beta_L^{(s)} \} \) for \( s = 1, \ldots, r \).

If we change \( L \) to \( L + 1 \) the beta set \( B_s \) is shifted to \( \{ \beta_1^{(s)} + 1, \ldots, \beta_L^{(s)} + 1, 0 \} \).

A function of beta numbers is invariant under beta shifts if it is unchanged by such transformations; equivalently, the function is independent of \( L \) provided that \( L \) is large enough. For example, the formula for \( s_\lambda(\mathbf{q}) \) below is invariant under beta shifts since \( s_\lambda(\mathbf{q}) \) does not depend on \( L \).

5.18. Theorem. Suppose that \( \lambda \) is a multipartition of \( n \) with \( L \)-symbol \( B = (\beta_i^{(s)})_{s,i} \) such that \( L \geq \ell(\lambda) \). Then

\[
s_\lambda(\mathbf{q}) = (-1)^{a_{rL}} q^{b_{rL}} \frac{\prod_{1 \leq s \leq r} (Q_s - Q_1)^L \cdot \prod_{1 \leq s \leq r} \prod_{a \in B_s, 1 \leq k \leq a_s} (q^k Q_s - Q_t)}{(q-1)^n (Q_1 \ldots Q_r)^n \prod_{1 \leq s \leq r} \prod_{(a_s, a_t) \in B_s \times B_t, a_s > a_t, if s \neq t} (q^{a_s} Q_s - q^{a_t} Q_t)},
\]

where \( a_{rL} = n(r-1) + \binom{r}{2} \binom{L}{2} \) and \( b_{rL} = \frac{r(L-1)(2L^2-r-3)}{12} \).

Proof. Adopting the notation of Corollary 5.16,

\[
s_\lambda(\mathbf{q}) = (-1)^{n(r-1)} (Q_1 \ldots Q_r)^{-n} q^{-\alpha(\lambda)} \prod_{s=1}^r \prod_{(i,j) \in [\lambda^{(s)}]} Q_s h_{ij}^{\lambda^{(s)}} q_i \prod_{1 \leq s \leq t \leq r} X_s X_t.
\]

We consider each of these factors separately.
First, let $\sigma$ be a partition of $m$ with beta numbers $(\beta_1, \ldots, \beta_L)$, where $L \geq \ell(\sigma)$. Then

$$\prod_{(i,j) \in [\sigma]} [h^*_{ij}]_q = (q - 1)^{-m} \prod_{(i,j) \in [\sigma]} (q^{h^*_{ij}} - 1) = (q - 1)^{-m} \prod_{1 \leq i < j \leq L} (q^{\beta_i - \beta_j} - 1).$$

This is quite well-known and is easily proved by first observing that the right hand side is invariant under beta shifts and then by arguing by induction on the number of columns.

Ignoring the leading term $(q - 1)^m$ on the right hand side, the number of factors in the numerator is $m + \binom{L}{2}$, whereas the number of factors in the denominator is $\binom{L}{2}$. Consequently, multiplying the left hand side by $Q^m$, say, is the same as multiplying each factor, top and bottom, on the right hand side by $Q$. Therefore, returning to the multipartition $\lambda$,

$$q^{-\alpha(\lambda)} \prod_{s=1}^{r} \prod_{(i,j) \in [\lambda^{(s)}]} Q_s^*[a^*_{ij}]_q = q^{N-\alpha(\lambda)}(q-1)^{-n} \prod_{s=1}^{r} \prod_{1 \leq i < j \leq L} \frac{\prod_{i=1}^{L} \beta_i}{\prod_{1 \leq i < j \leq L} (q^{\beta_i - \beta_j} - 1)}.$$

where $N = \sum_{s=1}^{r} \sum_{j=1}^{L} (j-1)\beta_j^{(s)}$. It is well-known (in the right circles) and easy enough to check that $\alpha(\lambda') = \sum_{s=1}^{r} \sum_{j=1}^{L} (j-1)\lambda_j^{(s)}$; therefore, $N - \alpha(\lambda') = r\binom{L}{2}$.

Fix $s$ and $t$ with $1 \leq s < t \leq r$; there are $\binom{r}{2}$ such choices. A quick calculation shows that $b_{rL} = r\binom{r}{2} + \binom{r}{2} \sigma(L)$, where $\sigma(L) = (2L - 1)L(L - 1)/6 = \sum_{j=1}^{L-1} j^2$. Therefore, in order to complete the proof it is enough to show that $X^\lambda_{st} = Y^\lambda_{st}$ where

$$Y^\lambda_{st} = (-1)^{\binom{L}{2}} q^{\sigma(L)} \prod_{\alpha_s \in B_s} \prod_{1 \leq k \leq \alpha_s} (q^{k^s Q_s - Q_t}) \prod_{\alpha_t \in B_t} \prod_{1 \leq k \leq \alpha_t} (q^{k^t Q_t - Q_s}) \prod_{(\alpha_s, \alpha_t) \in B_s \times B_t} (q^{\alpha_s Q_s - Q^{\alpha_t} Q_t}).$$

It is not hard to see that $Y^\lambda_{st}$ is invariant under beta shifts so we may change $L$ arbitrarily, provided that $L \geq \max \{\ell(\lambda^{(s)}), \ell(\lambda^{(t)})\}$. We prove our claim that $X^\lambda_{st} = Y^\lambda_{st}$ by induction in three incremental steps. We start the induction by observing that both products are equal to 1 when $\lambda^{(s)} = \lambda^{(t)} = (0)$ — to see this it is easiest to take $L = 0$.

Next consider the case where $\lambda^{(t)} = (0)$. Since $Y^\lambda_{st}$ is invariant under beta shifts we may assume that $\ell(\lambda^{(s)}) = L$. Assume by way of induction that we have proved the claim for $(\lambda^{(s)}_1, \ldots, \lambda^{(s)}_{L-1})$. Adding a non-empty $L^{th}$ row to $\lambda^{(s)}$ changes $X^\lambda_{st}$
by the factor
\[
\prod_{k=1-L}^{L} (q^k Q_s - Q_t) = \prod_{k=1}^{L-1} q^k (Q_s - q^{k} Q_t) \cdot (Q_s - Q_t) \prod_{k=1}^{L} (q^k Q_s - Q_t)
\]

\[
= (-1)^{L-1} \frac{\sum_{1 \leq k \leq L-1} q^{L-k} \prod_{1 \leq k \leq \lambda^{(s)}_L} (q^k Q_s - Q_t)}{\prod_{\alpha \in B_t} (q^\beta \prod_{s=1}^{L} Q_s - q^{\alpha} Q_t)}
\]

the last equality following because \( \beta^{(s)}_L = \lambda^{(s)}_L \) and \( \beta^{(t)}_i = L - i \) for \( i = 1, \ldots, L \).

Notice that the \((L-1)\)-beta numbers for \((\lambda^{(s)}_1, \ldots, \lambda^{(s)}_{L-1})\) all increase by 1 when we add the extra row \( \lambda^{(s)}_L \) to \( \lambda^{(s)} \). Let \( B'_t \) be the set of \((L-1)\)-beta numbers for \((\lambda^{(s)}_1, \ldots, \lambda^{(s)}_{L-1})\) and let \( B'_s \) be the set of \((L-1)\)-beta numbers for \( \lambda^{(t)} \). Then

\[
\prod_{(\alpha, \alpha') \in B_s \times B_t} (q^{\alpha} Q_s - Q^{\alpha'} Q_t) = \prod_{1 \leq i < L} \prod_{0 \leq k < L} (q^{\beta^{(s)}_i} Q_s - q^{\beta^{(t)}_k} Q_t)
\]

This equation allows us to rewrite the denominator of the preceding equation and so see that the change in \( X_{st}^\lambda \) is the same as the change in \( Y_{st}^\lambda \) (in particular, the change of the scalar is \((-1)^{L-1} q^{L-1} \lambda^{(s)}_L \lambda^{(t)}_L \) in both cases). This proves our claim when \( \lambda^{(t)} = (0) \).

The next step is to fix \( \lambda^{(s)} \) and assume that \( \lambda^{(t)} = (a) \) for some \( a \geq 0 \). If \( \lambda^{(s)} = (0) \) in this case then it is straightforward to check the claim (or to modify the argument below), so assume that \( \lambda^{(s)} \neq (0) \) and let \( L = \ell(\lambda^{(s)}) \geq 1 \geq \ell(\lambda^{(t)}) \).

The case \( a = 0 \) we already understand. Next, changing \( \lambda^{(t)} \) from \((a-1)\) to \( (a) \) changes \( X_{st}^\lambda \) by the factor

\[
(q^{a-1} Q_t - Q_s) \prod_{(i,j) \in \lambda^{(s)}} \frac{(q^{a-1} Q_t - q^a Q_s)}{(q^{a-1} Q_t - q^a Q_s - q^{a-1} Q_t)} \cdot \frac{(q^{a-1} Q_t - q^{a-2} Q_s)}{(q^{a-1} Q_t - q^{a-1} Q_s - q^{a-1} Q_t)}
\]

\[
= (q^{a-1} Q_t - Q_s) \prod_{(i,j) \in \lambda^{(s)}} \frac{(q^{a-1} Q_t - q^i Q_s)}{(q^{a-1} Q_t - q^{a-1} Q_s - q^{a-1} Q_t)} \cdot \frac{(q^{a-1} Q_t - q^{a-2} Q_s)}{(q^{a-1} Q_t - q^{a-1} Q_s - q^{a-1} Q_t)}
\]

\[
= (q^{a-1} Q_t - q^{L} Q_s) \prod_{i=1}^{L} \frac{(q^{a-1} Q_t - q^{a-1} Q_s)}{(q^{a-1} Q_t - q^{a-1} Q_s - q^{a-1} Q_t)} \cdot \frac{(q^{a-1} Q_t - q^{a-1} Q_s)}{(q^{a-1} Q_t - q^{a-1} Q_s - q^{a-1} Q_t)}
\]

\[
= (q^{a} Q_t - q^{L} Q_s) \prod_{i=1}^{L} \frac{(q^{a} Q_s - q^{a} Q_t)}{(q^{a} Q_s - q^{a} Q_t - q^{a} Q_t)}
\]

\[
= (q^{a} Q_t - Q_s) \prod_{i=1}^{L} \frac{(q^{a} Q_s - q^{a} Q_t)}{(q^{a} Q_s - q^{a} Q_t - q^{a} Q_t)}
\]
This last product is exactly the change in $Y^\lambda_{st}$ so we now know that $X^\lambda_{st} = Y^\lambda_{st}$ when $\lambda^{(t)} = (a)$.

Finally, suppose that $\lambda^{(t)}$ has more than one row. For convenience, let $l = \ell(\lambda^{(s)})$, $m = \ell(\lambda^{(t)}) > 1$ and $b = \lambda^{(t)}_m$ and assume that $L \geq l, m$. When we add row $m$ to $\lambda^{(t)}$, $X^\lambda_{st}$ changes by the factor

$$
\prod_{j=1}^b (q^{j-m}Q_t - Q_s) \cdot \prod_{(i,j) \in [\lambda^{(s)}]} \prod_{k=1}^b \left( (q^{j-i}Q_s - q^{k-m-1}Q_t) \left( (q^{j-i}Q_s - q^{k-m+1}Q_t) \right) \right) \left( (q^{j-i}Q_s - q^{k-m}Q_t) \right) \left( (q^{j-i}Q_s - q^{k-m}Q_t) \right)
$$

$$
= \prod_{j=1}^b (q^{j-m}Q_t - Q_s) \cdot \prod_{(i,j) \in [\lambda^{(s)}]} \prod_{k=1}^b \left( (q^{j-i+1}Q_s - q^{k-m}Q_t) \left( (q^{j-i}Q_s - q^{k-m}Q_t) \right) \right) \left( (q^{j-i}Q_s - q^{k-m}Q_t) \right) \left( (q^{j-i}Q_s - q^{k-m}Q_t) \right)
$$

$$
= \prod_{j=1}^b (q^{j-m}Q_t - Q_s) \cdot \prod_{(i,j) \in [\lambda^{(s)}]} \prod_{k=1}^b \left( q^{j(i-1)}Q_s - q^{k-m-1}Q_t \right) \left( (q^{j(i-1)}Q_s - q^{k-m}Q_t) \right) \left( (q^{j(i-1)}Q_s - q^{k-m}Q_t) \right)
$$

$$
= \prod_{j=1}^b (q^{j-m}Q_t - Q_s) \cdot \prod_{(i,j) \in [\lambda^{(s)}]} \prod_{k=1}^b \left( q^{j(i-1)}Q_s - q^{k-m}Q_t \left( q^{j(i-1)}Q_s - q^{k-m}Q_t \right) \right) \left( (q^{j(i-1)}Q_s - q^{k-m}Q_t) \right) \left( (q^{j(i-1)}Q_s - q^{k-m}Q_t) \right)
$$

$$
= \prod_{j=1}^b (q^{j-m}Q_t - Q_s) \cdot \prod_{(i,j) \in [\lambda^{(s)}]} \prod_{k=1}^b \left( q^{j(i-1)}Q_s - q^{k-m}Q_t \right) \left( (q^{j(i-1)}Q_s - q^{k-m}Q_t) \right) \left( (q^{j(i-1)}Q_s - q^{k-m}Q_t) \right)
$$

$$
= \prod_{j=1}^b (q^{j-m}Q_t - Q_s) \cdot \prod_{(i,j) \in [\lambda^{(s)}]} \prod_{k=1}^b \left( q^{j(i-1)}Q_s - q^{k-m}Q_t \right) \left( (q^{j(i-1)}Q_s - q^{k-m}Q_t) \right) \left( (q^{j(i-1)}Q_s - q^{k-m}Q_t) \right)
$$

where the last line follows by a short calculation since $\beta^{(t)}_m = b + L - m$ (and $l \leq L$). As $\beta^{(t)}_m$ is the only beta number that has changed, this factor is precisely the change in $Y^\lambda_{st}$ when an extra row is added to $\lambda^{(t)}$.

We have now shown that $X^\lambda_{st} = Y^\lambda_{st}$ in all cases, so the theorem is proved. \hfill \Box

Finally, comparing Theorem 3.18 with [8, Prop 3.17] we see that our formula for the Schur elements agrees with Malle’s — as it must because Malle’s conjecture has already been proved by Geck, Iancu and Malle [13]. Actually, there is still a small amount of work to be done in reconciling the two formulas because Bröché and Kim [8] write the exponent of $q$ as a sum of binomial coefficients; however, their expression simplifies to give $b_{rL}$.

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