1. Introduction

Suppose \( \{Z_i\}_{i \geq 0} \) is a supercritical Galton-Watson process with \( Z_0 = 1 \) (the root), branching random variable \( Z_1 \) (size of the first generation), and progeny mean \( \mu := E(Z_1) \in (1, \infty) \). It is well-known that \( Z_n/\mu^n \) is a martingale sequence that converges almost surely to a non-negative random variable \( W \). We assume that the branching random variable satisfies the Kesten-Stigum condition

\[
E(Z_1 \log^+ Z_1) < \infty. \tag{1.1}
\]

We shall condition on the survival of this Galton-Watson process and (1.1) ensures that the limiting random variable \( W \) is almost surely positive; see Kesten and Stigum (1966). This rooted infinite Galton-Watson tree will be denoted in this article by \( T = (V, E) \), where the collection of all vertices is denoted by \( V \), the collection of all edges is denoted by \( E \) and the root is denoted by \( o \). Note that every vertex \( v \) is connected to the root \( o \) by
a unique geodesic path which will be denoted by $I_v$ and the length of this path will be denoted by $|v|$.

We define a branching random walk with balanced regularly varying step size as follows. After obtaining the entire infinite tree $T$, we assign independent and identically distributed random variables $\{X_e : e \in E\}$ (that are also independent of the Galton-Watson process $\{Z_i\}_{i \geq 0}$) on the edges satisfying the regular variation condition

$$P(|X_\varepsilon| > x) = x^{-\alpha}L(x),$$

(1.2)

where $\alpha > 0$ and $L(x)$ is a slowly varying function (i.e., for all $x > 0$, $L(tx)/L(t) \to 1$ as $t \to \infty$), and the tail balance condition

$$\frac{P(X_\varepsilon > x)}{P(|X_\varepsilon| > x)} \to p \quad \text{and} \quad \frac{P(X_\varepsilon < -x)}{P(|X_\varepsilon| > x)} \to q$$

(1.3)

as $x \to \infty$ for some $p, q \geq 0$ with $p + q = 1$. For an encyclopaedic treatment of regularly varying and slowly varying functions, see Bingham et al. (1987).

To each vertex $v$, we assign displacement labels $S_v$, which is the sum of all edge random variables on the geodesic path from the root $o$ to the vertex $v$, i.e.,

$$S_v = \sum_{e \in I_v} X_e.$$  

(1.4)

The collection of displacement random variables $\{S_v : |v| = n\}$ forms the $n^{th}$ generation of our branching random walk.

Branching random walk has been of interest starting from the classical works of Hammersley (1974), Kingman (1975), Biggins (1976, 1977b,a). Recently, extremes of branching random walk has gained much prominence due to its connection to tree indexed random walk and Gaussian free field; see Bramson and Zeitouni (2012), Hu and Shi (2009); Addario-Berry and Reed (2009), Aïdékon (2013), Madaule (2011), Biskup and Louidor (2013, 2014), Bramson et al. (2013). See also Bramson (1978, 1983), Lalley and Sellke (1987), Arguin et al. (2011, 2012, 2013), Aïdékon et al. (2013) for related results on extremes of branching Brownian motion.

Heavy tailed edge random variables were introduced in branching random walks by Durrett (1979, 1983); see also Kyprianou (1999), Gantert (2000), and the recent works of Lalley and Shao (2013) and Béard and Maillard (2014). It was shown in Durrett (1983) that when the step sizes have regularly varying tails, then the maximum displacement grows exponentially and converges (after scaling) to a $W$-mixture of Fréchet random variables. This limiting behaviour is very different from the ones obtained by Biggins (1976) and Bramson (1978) in the light tailed case.

It was predicted in Brunet and Derrida (2011) that the limits of point processes of properly normalized displacements of branching random walk and branching Brownian motion should be decorated Poisson point processes. This conjecture was proved to be true for branching Brownian motion by
Arguin et al. (2012, 2013) and Aïdékon et al. (2013), and for branching random walks with step sizes having finite exponential moments by Madaule (2011) relying on a work of Maillard (2013).

A natural question arising out of the works of Durrett (1983) and Madaule (2011) is the following: where do the point processes based on the scaled displacements converge in the regularly varying case? The main aim of this article is to show the convergence of this point process sequence and also explicitly identify the limit as a Cox cluster process. We establish that the prediction of Brunet and Derrida (2011) on this limit remains true for branching random walk with regularly varying step size even though the finiteness of exponential moments fails to hold. In order to overcome this obstacle, we use a twofold truncation technique based on multivariate extreme value theory.

We also discuss the superposability properties of our limiting point process in parallel to the recent works of Maillard (2013) and Subag and Zeitouni (2014) and confirm the validity of a related prediction of Brunet and Derrida (2011) in our setup. As a consequence of our main result, we give explicit formulae for the asymptotic distributions of the properly scaled order and gap statistics from which various problems mentioned in Brunet and Derrida (2011) can be investigated. In particular, we recover a slightly improved version of Theorem 1 of Durrett (1983) in our framework.

This paper is organized as follows. Section 2 contains the statements of the main result (Theorem 2.1) and its consequences (Theorems 2.3 and 2.5). Since the proof of Theorem 2.1 is long and notationally complicated, we first give a detailed outline of the main steps based on four lemmas in Section 3. These lemmas, and Theorems 2.3 and 2.5 are finally proved in Section 4.

2. The Results

We consider point processes as a random elements in the space $\mathcal{M}$ of all Radon point measures on a locally compact and separable metric space $\mathcal{E}$. Here $\mathcal{M}$ is endowed with the vague convergence (denoted by "$\overset{\text{v}}{\longrightarrow}$"), which is metrizable by the metric $\rho(\mu, \nu) = \sum_{i=1}^{\infty} 2^{-i} \min(|\mu(h_i) - \nu(h_i)|, 1)$, where $\{h_i\}_{i \geq 1}$ is a suitably chosen subset (consisting only of Lipschitz functions) of the collection $C^+_c(\mathcal{E})$ of all non-negative continuous real-valued functions on $\mathcal{E}$ with compact support. $(\mathcal{M}, \rho)$ is a complete and separable metric space. Therefore the standard theory of weak convergence is readily available for point processes and can be characterized by the pointwise convergence of corresponding Laplace functionals on $C^+_c(\mathcal{E})$ (see Proposition 3.19 in Resnick (1987)). For further details on point processes, see Kallenberg (1986), Resnick (1987), Embrechts et al. (1997) and Resnick (2007).

Because of (1.2) and (1.3), we can choose scaling constants $b_n$ such that (see, e.g., Resnick (1987), Davis and Resnick (1985), Davis and Hsing (1995))

$$\mu^n P(b_n^{-1} X_e \in \cdot) \overset{\text{v}}{\longrightarrow} \nu_0$$

(2.1)
on $[-\infty, \infty] \setminus \{0\}$, where
\[
\nu_\alpha(dx) = \alpha px^{-\alpha-1}1_{(0,\infty)}(x)dx + \alpha q(-x)^{-\alpha-1}1_{(-\infty,0)}(x)dx.
\] (2.2)

Note that one can write $b_n = \mu^{n/\alpha}L_0(\mu^n)$ for some slowly varying function $L_0$. In this paper, conditioned on the survival of the tree, we investigate the asymptotic behaviour of the sequence of point processes defined by
\[
N_n = \sum_{|v|=n} \delta_{b_n^{-1}S_v}, \quad n \geq 1,
\] (2.3)
where $S_v$ is as in (1.4).

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be the probability space where all the random variables are defined and let $\mathbf{P}^*$ denote the probability obtained by conditioning $\mathbf{P}$ on the non-extinction of the underlying Galton-Watson tree. We shall denote by $\mathbf{E}$ and $\mathbf{E}^*$, the expectation operators with respect to $\mathbf{P}$ and $\mathbf{P}^*$, respectively. We introduce two sequences of random variables $\{T_l\}_{l \geq 1}$ and $\{j_l\}_{l \geq 1}$ as follows. Suppose $\{T_l\}_{l \geq 1}$ is a sequence of independent and identically distributed positive integer valued random variables with probability mass function
\[
\gamma(y) := \mathbf{P}(T_1 = y) = \frac{1}{r} \sum_{i=0}^{\infty} \frac{1}{\mu^i} \mathbf{P}(Z_i = y), \quad y \in \mathbb{N},
\] (2.4)
where $r = \sum_{i=0}^{\infty} \frac{1}{\mu^i} P(Z_i > 0)$. Let $\{j_l\}_{l \geq 1}$ be a sequence of random variables such that $\sum_{i=1}^{\infty} \delta_{j_i} \sim \text{PRM}(\nu_\alpha)$. We also assume that the sequences $\{T_l\}_{l \geq 1}$ and $\{j_l\}_{l \geq 1}$ are independent of each other and are both independent of the martingale limit $W$.

Our main result says that the limiting point process is a Cox cluster process in which a typical Cox point $(rW)^{1/\alpha}j_l$ appears with random multiplicity $T_l$. The clusters appear here due to the strong dependence structure of the displacement random variables $\{S_v : |v| = n\}$. The randomness in the intensity measure arises from the martingale limit $W$ in contrast to the light tailed case, where similar randomness arise due to the appearance of derivative martingale limit (see, e.g., Theorem 1.1 in Aïdékon (2013)). Note also that a $W$-mixture was already present in Theorem 1 of Durrett (1983); see Remark 2.6 below.

Theorem 2.1. With the assumptions (1.1), (1.2), (1.3) and $\{b_n\}$ as in (2.1), under $\mathbf{P}^*$, the sequence of point processes defined in (2.3) converges weakly in the space $\mathcal{M}$ of all Radon point measures on $[-\infty, \infty] \setminus \{0\}$ to a Cox cluster process with representation
\[
N_* \overset{\mathcal{L}}{=} \sum_{l=1}^{\infty} T_l \delta_{(rW)^{1/\alpha}j_l}
\] (2.5)
and Laplace functional given by
\[
\Psi_{N_*}(g) = \mathbf{E}^* \left(e^{-N_*(g)}\right)
\]
\[
E^* \left[ \exp \left\{ -W \int_{|x|>0} \sum_{i=0}^{\infty} \frac{1}{\mu^i} \P(Z_i > 0) E(1 - e^{-\tilde{Z}_i g(x)}) \nu_\alpha(dx) \right\} \right] \quad (2.6)
\]
for all \( g \in C^\infty_c([-\infty, \infty] \setminus \{0\}) \). Here \( \tilde{Z}_i \) denotes the random variable \( Z_i \) conditioned to be positive.

2.1. Scale-decorated Poisson point processes. For any point process \( \P \) and any \( a > 0 \), we denote by \( s_a \P \) the point process obtained by multiplying the atoms of \( \P \) by \( a \). The following is an analogue of Definition 1 in Subag and Zeitouni (2014) suitable for our framework.

**Definition 2.2.** A point process \( N \) is called a randomly scaled scale-decorated Poisson point process (SSDPPP) with intensity measure \( \nu \), scale-decoration \( \P \) and random scale \( \Theta \) if \( N \overset{d}{=} s_\Theta \sum_{i=1}^{\infty} s_\lambda \P_i \), where \( \Lambda = \sum_{i=1}^{\infty} \delta_\lambda_i \sim PRM(\nu) \) on the space \((0, \infty)\) and \( \P_i \), \( i \geq 1 \) are independent copies of the point process \( \P \) and are independent of \( \Lambda \), and \( \Theta \) is a positive random variable independent of \( \Lambda \) and \( \{\P_i\}_{i \geq 1} \). We shall denote this by \( N \sim SSDPPP(\nu, \P, \Theta) \). If \( \Theta \equiv 1 \), we call \( N \) a scale-decorated Poisson point process (SDPPP) and denote it by \( N \sim SDPPP(\nu, \P) \).

Our next result establishes that the limiting point process \( (2.5) \) admits an SSDPPP representation and confirms that a prediction of Brunet and Derrida (2011) remains valid in our setup. Moreover, the scale-Laplace functional of \( N \) (i.e., the left hand side of \( (2.7) \) below) can always be expressed as a multiplicative convolution of an \( \alpha \)-Fréchet distribution with some measure. This is a scale-analogue of a property investigated in Subag and Zeitouni (2014) (see property (SUS) therein).

**Theorem 2.3.** Under the assumptions of Theorem 2.1, the limiting point process \( N_\ast \sim SSDPPP(\nu^\ast_\alpha, T_\delta, (rW)^{1/\alpha}) \), where \( \varepsilon \) is a \pm 1-valued random variable with \( \P(\varepsilon = 1) = p \), \( \nu^\ast_\alpha(dx) = \alpha x^{-\alpha - 1} dx \) is a measure on \((0, \infty)\) and \( T \) is a positive integer valued random variable (independent of \( \varepsilon \)) with probability mass function \( (2.3) \). Furthermore, for all \( g \in C^\infty_c([\varepsilon, \infty] \setminus \{0\}) \), \( N_\ast \) satisfies
\[
E^* \left[ \exp \left\{ - \int g(x/y) N_\ast(dx) \right\} \right] = E^* \left( \Phi_\alpha(c_g y W^{-1/\beta}) \right), \quad y > 0, \quad (2.7)
\]
where \( \Phi_\alpha \) denotes the distribution function of an \( \alpha \)-Fréchet random variable, i.e. \( \Phi_\alpha(x) = \exp\{-x^{-\alpha}\} \), \( x > 0 \), and \( c_g \) is a positive constant that depends on \( g \) but not on \( y \).

The scale-decorating in the SSDPPP representation of \( N_\ast \) is the point process consisting of \( T \) many repetitions of the random point \( \varepsilon \). This is due to the fact that very few (more precisely, a \( W \)-mixture of Poisson many) edge random variables survive the scaling by \( b_n \) and the surviving ones come with random cluster-sizes that are independent copies of \( T \). The presence of \( \varepsilon \) in the scale-decoration can be justified by the fact that the surviving
edge random variables are positive and negative with probabilities \( p \) and \( q \), respectively (see (1.3)).

**Remark 2.4** (Superposability). Let \( N_s^{(i)} = \sum_{l=1}^{\infty} T^{(i)}_l \delta^{(i)}_{(rW_1)^{1/\alpha} d^{(i)}} \), \( i = 1, 2 \) be two independent copies of (2.5). Then using Laplace functionals, it can easily be verified that for two positive constants \( a_1 \) and \( a_2 \), \( s_{a_1} N_s^{(1)} + s_{a_2} N_s^{(2)} \sim SDPPP(\nu_\alpha^+, T\delta_\varepsilon, (r(a_1^2 W_1 + a_2^2 W_2)^{1/\alpha})) \). In particular, when the underlying Galton-Watson tree is a \( d \)-regular tree (i.e., \( Z_1 \equiv d \geq 2 \)), then the limiting point process is the Poisson cluster process \( SDPPP(\nu_\alpha^+, d^\varepsilon \delta_\varepsilon) \), where \( G \) follows a Geometric(1/d) distribution (independently of \( \varepsilon \)) with probability mass function \( P(G = k) = (1 - 1/d)^k d^{-1}, k \geq 0 \), and \( N_s, d \) satisfies the superposability property described as follows.

If \( N_{s,d}^{(i)}, i = 1, 2 \) are two independent copies of \( N_s, d \), then \( s_{a_1} N_{s,d}^{(1)} + s_{a_2} N_{s,d}^{(2)} \subseteq N_{s,d} \) for any two positive constants \( a_1, a_2 \) such that \( a_1^2 + a_2^2 = 1 \). For a similar statement in case of exp-1-stable point processes, see Brunet and Derrida (2011) and Maillard (2013).

### 2.2. Order and gap statistics

The point process convergence in Theorem 2.1 helps us to derive some properties of the order and gap statistics; see also Ramola et al. (2014) for related works on branching Brownian motion.

Let \( M_n^{(k)} \) denote the \( k^{th} \) upper order statistic coming from the \( n^{th} \) generation, \( G_n^{(k)} = M_n^{(k)} - M_n^{(k+1)} \) be the \( k^{th} \) gap statistic and \( M'_n := \min_{|\ell|=n} S_\ell \) be the minima. In order to study the asymptotic properties of these statistics, we need a few more notations as described below. We denote by \( \pi \) a partition of an integer \( l \) of the form \( l = i_1 y_1 + i_2 y_2 + \cdots + i_m y_{|\pi|} \), where each \( i_j \) repeats \( y_j \) many times in the partition, and \( i_1 < i_2 < \cdots < i_{|\pi|} \). Here \( | \cdot | \) denotes the number of distinct elements in a partition. Let \( \Pi_l \) be the set of all such partitions of the integer \( l \).

**Theorem 2.5.** With the assumptions of Theorem 2.1 and \( \{b_n\} \) as in (2.1), the following asymptotic properties hold.

(a) (Minima) For all \( x > 0 \),

\[
\lim_{n \to \infty} \mathbf{P}^* \left( M'_n > -b_n x \right) = \mathbf{E}^* \left( \exp \left\{ -rWqx^{-\alpha} \right\} \right).
\]

(b) \( (k^{th} \) upper order statistic) For all \( x > 0 \),

\[
\lim_{n \to \infty} \mathbf{P}^* \left( M_n^{(k)} \leq b_n x \right) = \mathbf{E}^* \left( \exp \left\{ -rWpx^{-\alpha} \right\} \right) \tag{2.8}
\]

\[
+ \sum_{l=1}^{k-1} \sum_{\pi \in \Pi_l} \mathbf{E}^* \left[ \prod_{j=1}^{|\pi|} \left( rWpx^{-\alpha} \gamma(i_j) \right)^{y_j} \exp \left\{ -rWpx^{-\alpha} \gamma(i_j) \right\} \right].
\]
(c) (Joint distribution of \(k^{th}\) and \((k+1)^{th}\) upper order statistics) For all \((u,v)\) such that \(0 < u < v\),

\[
\lim_{n \to \infty} P^* (M_{n}^{(k+1)} \leq b_n u, M_{n}^{(k)} \leq b_n v) = E^* \left( \xi_{0,(u,\infty)}(W) \right) + \sum_{j=1}^{k} E^* \left( \xi_{0,(v,\infty)}(W) \xi_{j,(u,v)}(W) \right) + \sum_{l=1}^{k-1} \sum_{j=0}^{k-l} E^* \left( \xi_{l,(v,\infty)}(W) \xi_{j,(u,v)}(W) \right),
\]

where for all \(l \geq 0\) and for all \(A \subset [-\infty, \infty] \setminus \{0\}\) such that \(\nu_\alpha(A) < \infty\),

\[
\xi_{l,A}(W) := \begin{cases} 
0 & \text{if } l = 0, \\
\sum_{\pi \in \Pi} \prod_{j=1}^{l} \left( r W \nu_\alpha(A) \gamma(i_j) \right)^{y_j} \frac{1}{y_j!} e^{-r W \nu_\alpha(A) \gamma(i_j)} & \text{if } l \geq 1.
\end{cases}
\]

(d) \((k^{th}\) gap statistic\) Let \(L : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+\) be the map \(L(u,v) = v - u\). Then \(P^* (b_n^{-1} G_{n}^{(k)} \in \cdot) \to \zeta_k \circ L^{-1}\) where \(\zeta_k\) is a probability measure on \(\mathbb{R}^+ \times \mathbb{R}^+\) with joint cumulative distribution function \((2.9)\).

The second term in \((2.8)\) and the third term in \((2.9)\) are both interpreted as zero when \(k = 1\).

**Remark 2.6 (Maxima).** Note that putting \(k = 1\) in \((2.8)\), we recover Theorem 1 of Durrett (1983) in our framework. In fact, from the proof (see Section 4 below), it transpires that as long as the right tail of the point process sequence \(N_n\) converges weakly to that of \(N_s\), this convergence will hold. Therefore the condition \(\log(-x) P(X_e \leq x) \to 0\) as \(x \to -\infty\) of the aforementioned paper becomes redundant leading to a slight improvement of Theorem 1 therein under Kesten-Stigum condition on the progeny distribution of the underlying branching process.

By Theorem 8.2 (Page 15) of Harris (1963) (see also Athreya and Ney (2004), Theorem 2, Page 29), the limiting distribution function of the scaled maxima of the \(n^{th}\) generation can be written as

\[
\lim_{n \to \infty} P^* \left( M_{n}^{(1)} \leq b_n x \right) = \phi(r p x^{-\alpha}), \quad x > 0,
\]

where \(\phi\) is the unique (up to a scale-change) completely monotone function on \(\mathbb{R}^+\) satisfying

\[
\phi(z) = f(\phi(z/\mu))
\]

with \(f\) being the probability generating function of the branching random variable \(Z_1\).

**Example 2.1 (Maxima for geometric branching).** Suppose that the offspring distribution of the underlying branching process is geometric with parameter \(b \in (0, 1)\) and probability mass function \(P(Z_1 = k) = b(1-b)^{k-1}, k \geq 1\). It is easy to check that the completely monotone function \(\phi(u) = \frac{1}{1+du}, u > 0\) satisfies the functional equation \((2.11)\) for any scaling constant \(d > 0\). Therefore using \((2.10)\) and the fact that \(E(W) = 1\) (a consequence
of Kesten-Stigum condition (1.1); see Kesten and Stigum (1966)), it follows that \( d = 1 \) and
\[
\lim_{n \to \infty} P(M_n \leq b_n x) = 1 - \frac{1 - b}{1 - b + px^{-\alpha}}, \quad x > 0.
\]

3. Outline of Proof of Theorem 2.1

In this section, we outline the main result’s proof, which is based on a twofold truncation technique using extreme value theory. We attain this via four lemmas, whose proofs will be given in the next section. For ease of presentation, we shall use Ulam-Harris labeling system described recursively as follows. The \( i \)th descendant of the root \( o \) is denoted by \( i \) and \( j \)th descendant of an \((n-1)\)th generation vertex \((i_1, \ldots, i_{n-1})\) is denoted by \((i_1, \ldots, i_{n-1}, j)\). We abuse the notation and denote an edge joining an \((n-1)\)th generation vertex and an \(n\)th generation vertex using the same label as the latter vertex. Such an edge is assumed to belong to the \(n\)th generation. Let \(D_n\) denote the vertices (and hence edges because of the abuse of notation) in the \(n\)th generation and \(C_n = \bigcup_{i=1}^n D_i\) denote the vertices (as well as edges) up to the \(n\)th generation of the underlying Galton-Watson tree. With these notations, we describe below the main steps of the proof of Theorem 2.1.

3.1. One large jump. Following Durrett (1983), it is easy to see that with very high probability, exactly one edge random variable \(X_e\) in \(N_n = \sum_{|v|=n} \delta_{e(V)} b_n^{-1} x_e\) will be large enough to survive the scaling by \(b_n\). Hence we can expect that the asymptotic behavior of \(N_n\) will be same as that of
\[
\tilde{N}_n = \sum_{|v|=n} \sum_{e \in I_v} \delta_{b_n^{-1} x_e}.
\]

(3.1)

More precisely, we shall establish the following lemma.

**Lemma 3.1.** Under the assumptions of Theorem 2.1, for every \(\epsilon > 0\),
\[
\lim_{n \to \infty} \sup P^* \left( \rho \left( N_n, \tilde{N}_n \right) > \epsilon \right) = 0,
\]

(3.2)

where \(\rho\) is the vague metric introduced in Section 2.

This lemma formalizes the well-known principle of one large jump (see, e.g., Steps 3 and 4 in Section 2 of Durrett (1983)) at the level of point processes and it can be shown by molding the proof of Theorem 3.1 in Resnick and Samorodnitsky (2004). Because of Lemma 3.1, it is enough to investigate the weak convergence of (3.1), which is much easier compared to that of (2.3).

3.2. Cutting the tree. The first truncation is a standard one that has been used in branching random walks. First fix a positive integer \(K\). Taking \(n > K\), look at the tree \(T\) up to the \(n\)th generation and cut it at \((n-K)\)th generation keeping last \(K\) generations alive; see Figure 1. This means that after cutting the tree, we will be left with a forest containing \(K\) generations.
of $|D_n-K|$ many independent (under $P$) Galton-Watson trees with roots being the vertices at the $(n-K)^{th}$ generation of the original tree $T$ and the same offspring distribution as before. We label the new sub-trees in this forest as $\{T_j\}_{j=1}^{D_n-K}$.

![Figure 1](image)

Figure 1: Cutting the Galton-Watson tree ($n = 3, K = 1$) at generation 2.

Each vertex $v$ in the $n^{th}$ generation of the original tree $T$ belongs to the $K^{th}$ generation of some sub-tree $T_j$ and we denote by $I^K_v$ the unique geodesic path from the root of $T_j$ to the vertex $v$. We introduce another point process generated by the i.i.d. heavy-tailed random variables attached to the edges of the forest as follows:

$$\tilde{N}^{(K)}_n := \sum_{|v|=n} \sum_{e \in I^K_v} \delta_{b_n^{-1}X_e},$$

(3.3)

where $|v|$ denotes the generation of $v$ in the original tree $T$. The following lemma asserts that as long as $K$ is large, (3.3) is a good approximation of (3.1).

Lemma 3.2. Under the assumptions of Theorem 2.1, for every $\epsilon > 0$,

$$\lim_{K \to \infty} \limsup_{n \to \infty} P^* \left( \rho \left( \tilde{N}_n, \tilde{N}^{(K)}_n \right) > \epsilon \right) = 0. \quad (3.4)$$

In light of the above lemma, it is enough to find the weak limit of (3.3) as $n \to \infty$ keeping $K$ fixed, and then letting $K \to \infty$. This can be achieved with the help of another truncation as mentioned below.

3.3. Pruning the forest. This is the second truncation step, which is also quite standard in branching process theory. Fix an integer $K > 0$ and for each edge $e$ in the forest $\bigcup_{j=1}^{D_n-K} T_j$, define $A_e$ to be the number of descendants of $e$ at $n^{th}$ generation of $T$. Fix another integer $B > 1$ large
enough so that \( \mu_B := \mathbb{E}(Z_1^{(B)}) > 1 \), where \( Z_1^{(B)} := Z_1 \mathbb{1}(Z_1 \leq B) + B \mathbb{1}(Z_1 > B) \). We modify the forest according to the pruning algorithm mentioned below (see also Figure 2).

P1. Start with the sub-tree \( T_1 \) and look at its root.

P2. If the root has more than \( B \) many children (edges), then keep the first \( B \) many edges according to our labeling, and delete the others and their descendants. If the number of children (edges) of the root is less than or equal to \( B \), then do nothing.

P3. Now we can have at most \( B \) many vertices in the first generation of the sub-tree \( T_1 \). Repeat Step P2 for children (edges) of each of these vertices. Continue with this algorithm up to the children (edges) of the \((K - 1)^{th}\) generation vertices (of the sub-tree \( T_1 \)).

P4. Repeat Steps P2 and P3 for the other sub-trees \( T_2, \ldots, T_{|D_n - K|} \).

Note that under \( \mathbf{P} \), these \(|D_n - K|\) many pruned sub-trees are independent copies of a Galton-Watson tree (up to the \( K^{th} \) generation) with a bounded branching random variable \( Z_1^{(B)} \). For each \( j \), we denote by \( T_j^{(B)} \) the pruned version of \( T_j \). For each edge \( e \) in \( \bigcup_{j=1}^{|D_n - K|} T_j^{(B)} \), we define \( A_e^{(B)} \) to be the number of descendants of \( e \) in the \( K^{th} \) generation of the corresponding pruned sub-tree. Observe that for every vertex \( e \) at the \( i^{th} \) generation of any sub-tree \( T_j^{(B)} \), \( A_e^{(B)} \) is equal to distribution to \( Z_{K-i}^{(B)} \), where \( \{Z_i^{(B)}\}_{i \geq 0} \) denotes a branching process with \( Z_0^{(B)} \equiv 1 \) and branching random variable \( Z_1^{(B)} \). For each \( i = 1, 2, \ldots, K \), we denote by \( D_{n-K+i}^{(B)} \) the union of all \( i^{th} \) generation vertices (as well as edges) from the pruned sub-trees \( T_j^{(B)} \), \( j = 1, 2, \ldots, |D_n - K| \). We introduce another point process as follows.

\[
\tilde{N}_n^{(K,B)} := \sum_{v \in D_{(n)}} \sum_{e \in I_v} \delta_{b_n^{-1}X_e}.
\]  

The point processes \( \tilde{N}_n^{(K)} \) and \( \tilde{N}_n^{(K,B)} \) are not simple point processes since both of them have alternative representations as given below.

\[
\tilde{N}_n^{(K)} = \sum_{i=0}^{K-1} \sum_{e \in D_{n-i}} A_e \delta_{b_n^{-1}X_e}
\]  

Figure 2: Pruning of the forest obtained in Figure 1 with \( B = 2 \).
and
\[ \tilde{N}_n^{(K,B)} = \sum_{i=0}^{K-1} \sum_{e \in \mathcal{D}_n^{(B)}} A_e^{(B)} \delta_{b_n^{-1}X_e}. \]  

The next lemma justifies the second truncation step and reduces our work to computation of weak limit of (3.7) obtained by letting \( n \to \infty \), and then \( B \to \infty \), and finally \( K \to \infty \).

**Lemma 3.3.** Under the assumptions of Theorem 2.1, for each fixed positive integer \( K > 1 \) and for all \( \epsilon > 0 \),
\[ \lim_{B \to \infty} \limsup_{n \to \infty} P^* \left( \rho \left( \tilde{N}_n^{(K)}, \tilde{N}_n^{(K,B)} \right) > \epsilon \right) = 0. \] (3.8)

The asymptotics of (3.7) is nontrivial and is the key step of the proof of Theorem 2.1. This is based on a regularization of the pruned sub-trees \( T_j^{(B)}, j = 1, 2, \ldots, \left| D_{n-K} \right| \) followed by use of machineries from multivariate extreme value theory.

### 3.4. Regularization and multivariate extremes

In order to investigate the weak convergence of (3.7), we need to modify the pruned sub-trees \( T_j^{(B)}, j = 1, 2, \ldots, \left| D_{n-K} \right| \) to a bunch of \( B \)-regular trees using the following regularization algorithm.

**R1.** Fix the sub-tree \( T_1^{(B)} \) and look at its root.
**R2.** The root can have now at most \( B \) many children (edges). If it has exactly \( B \) many children, then do nothing. Otherwise if it has \( m < B \) many children, then add \( B - m \) new children (edges) to the root.
**R3.** To each newly added edge \( e \), assign two random variables as follows:
(a) an edge random variable \( X_e \) satisfying (1.2) and (1.3) and independent of all other edge random variables, and (b) a degenerate random variable \( A_e^{(B)} \equiv 0 \).
**R4.** For each of the \( B \) many vertices in the first generation, repeat whatever has been done to the root in Steps R2 and R3. Continue this up to the \( (K-1)^{th} \) generation vertices (including the new ones).
**R5.** Repeat Steps R2 - R4 for the remaining trees \( T_j^{(B)}, 2 \leq j \leq \left| D_{n-K} \right| \).

See Figure 3 below for the regularized versions of the sub-trees obtained in Figure 2. The newly added edges are the dashed ones.

![Figure 3](image-url)
For each $1 \leq j \leq |D_{n-K}|$, $1 \leq i \leq K$ and $1 \leq l \leq B^i$, the triplet $(j, i, l)$ will indicate the $l^{th}$ edge (in our original labelling) at the $j^{th}$ generation of the $j^{th}$ modified (and hence $B$-regular) sub-tree. The corresponding $A_n^{(B)}$ and $X_n$ will be denoted by $A_{i,l}^{(j,B)}$ and $X_{i,l}^{(j)}$, respectively. It is now easy to see from (3.7) that
\[
\tilde{N}_n^{(K,B)} = \sum_{j=1}^{|D_{n-K}|} \sum_{i=1}^K \sum_{l=1}^{B^i} A_{i,l}^{(j,B)} \delta_{b_n^{-1}X_{i,l}^{(j)}}.
\] (3.9)

To investigate the asymptotics of (3.9), we use a marking technique based on multivariate extreme value theory; see Section 4 below.

The random coefficient $A_{i,l}^{(j,B)}$ denotes the number of repetitions of the edge random variable $X_{i,l}^{(j)}$ and if it does not correspond to a newly added edge, then it has the same distribution as that of $Z_{K-i}$. Roughly speaking, properly scaled $X_{i,l}^{(j)}$s give rise to the Cox points $j_l$ and slightly adjusted $A_{i,l}^{(j,B)}$s are responsible for the appearance of $T_l$ in (2.5). This is confirmed by the following lemma, which should be regarded as the key step of proving Theorem 2.1.

**Lemma 3.4.** Under the conditions of Theorem 2.1, the following weak convergence results hold in the space $M$ of all Radon point measures on $[-\infty, \infty] \setminus \{0\}$ under the measure $P^*$.

(a) For each positive integer $K$ and each integer $B > 1$ with $\mu_B > 1$, there exists a point process $N_s^{(K,B)}$ such that $\tilde{N}_n^{(K,B)} \Rightarrow N_s^{(K,B)}$ as $n \to \infty$.

(b) For each positive integer $K$, there exists a point process $N_s^{(K)}$ such that $N_s^{(K,B)} \Rightarrow N_s^{(K)}$ as $B \to \infty$.

(c) As $K \to \infty$, $N_s^{(K,B)} \Rightarrow N_s^{(K)}$.

For detailed descriptions of the point processes $N_s^{(K,B)}$ and $N_s^{(K)}$, see Section 4 below.

3.5. **Proof of Theorem 2.1** It is easy to check that (2.5) is $P^*$-almost surely Radon and hence is a random element of $M$ with $E = [-\infty, \infty] \setminus \{0\}$. To compute the Laplace functional of $N_s$, take any $g \in C^+_\alpha([-\infty, \infty] \setminus \{0\})$ and observe that
\[
E^*\left(e^{-N_s(g)}\right) = E^*\left[ E^*\left(e^{-N_s(g)}|W\right)\right] = E^*\left[ E^*\left(e^{-\bar{N}(f)}|W\right)\right],
\] (3.10)
where $f$ is the function $f(t,x) = tg(x)$ defined on $N \times (\mathbb{R} \setminus \{0\})$ and $N$ is the Cox process $N = \sum_{l=1}^{\infty} \delta_{(T_l,(rW)^{1/\alpha}j_l)}$. Using Propositions 3.6 and 3.8 in Resnick [1987], we get
\[
E^*\left(e^{-\bar{N}(f)}|W\right) = \exp\left\{ -rW \int_{|x|>0} E(1 - e^{-f(T_l,x)}) \nu_\alpha(dx) \right\},
\]
which can be shown to be equal to the random quantity inside the expectation in (2.6). Therefore, the second part of Theorem 2.1 follows from (3.10).

Using Lemmas 3.2, 3.3 and 3.4 and applying twice a standard converging together argument (see, e.g., Theorem 3.5 in Resnick (2007)), it follows that under $P^*$,

$$\tilde{N}_n \Rightarrow N_\ast \text{ as } n \rightarrow \infty,$$

from which the weak convergence in Theorem 2.1 follows by a simple application of Theorem 3.4 of Resnick (2007) combined with Lemma 3.1 above.

4. Rest of the Proofs

Throughout this section $P^*_T$ will denote the probability obtained by conditioning $P^*$ on the whole Galton-Watson tree $T$ and $E$ will denote the space $[-\infty, \infty] \setminus \{0\}$. Also we will use the notation $S$ to denote the event that the Galton-Watson tree survives.

4.1. Proof of Lemma 3.2. Let $g \in C_c^+(E)$ with support($g$) $\subseteq \{x : |x| > \delta\}$, for some $\delta > 0$. By definition of the vague convergence, it is enough to show that for all $\epsilon > 0$,

$$\lim_{K \rightarrow \infty} \limsup_{n \rightarrow \infty} P^*_T(\tilde{N}_n(g) - \tilde{N}_n^K(g) > \epsilon) = 0. \quad (4.1)$$

Define $B_{n,K}$ to be the event that all the random variables in the collection $\{X_e : e \in C_{n-K}\}$ are less than $b_n \delta/2$ in modulus. We claim that $\lim_{K \rightarrow \infty} \limsup_{n \rightarrow \infty} P^*_T(B_{n,K}) = 0$, which will follow provided we show that

$$\lim_{K \rightarrow \infty} \limsup_{n \rightarrow \infty} P^*_T(B_{n,K}^c) = 0 \text{ for } P^*\text{-almost all } T. \quad (4.2)$$

To this end, note that conditioned on the tree $T$, $\sum_{e \in C_{n-K}} \delta_{b_n^{-1}|X_e|}(\theta, \infty)$ follows a Binomial($|C_{n-K}|, P(|X_e| > b_n \theta)$) distribution, and for each $K \geq 1$, the following $P^*$-almost sure convergence holds:

$$|C_{n-K}| P(|X_e| > b_n \theta) = \frac{|C_{n-K}| \mu^{n-K}}{\mu^{n-K}} \left[1 - \frac{1}{\mu^K} \right]^{b_n \theta - \alpha} =: \lambda(\theta, K).$$

Therefore, for all $K \geq 1$, $\sum_{e \in C_{n-K}} \delta_{b_n^{-1}|X_e|}(\theta, \infty) \Rightarrow \mathcal{P} \sim \text{Poisson}(\lambda(\theta, K))$ as $n \rightarrow \infty$ under $P^*_T$ for $P^*$-almost all $T$. Because of Kesten-Stigum condition (1.1), $\lambda(\theta, K) \rightarrow 0$ $P^*$-almost surely as $K \rightarrow \infty$. In particular, we get $\lim_{K \rightarrow \infty} P^*_T(B_{n,K}^c) = P^*_T(\mathcal{P} > 1)$, which tends to zero as $K \rightarrow \infty$ for $P^*$-almost all $T$ and hence (4.2) holds. To finish the proof from here, observe that $P^*_T(\tilde{N}_n(g) - \tilde{N}_n^K(g) > \epsilon, B_{n,K}) \equiv 0$ since the support of $g$ is contained in $\{x : |x| > \delta\}$. Hence (4.1) follows immediately from (4.2).
4.2. Proof of Lemma 3.3. Let $g \in C_c^+(\mathbb{E})$ be as in proof of Lemma 3.2 with support $(g) \subseteq \{x : |x| > \delta\}$ and $\|g\|_\infty := \sup_{x \in \mathbb{E}} |g(x)| < \infty$. To show (3.8), it is enough to show that for each $g \in C_c^+(\mathbb{E})$ and $\epsilon > 0$,

$$\lim_{B \to \infty} \limsup_{n \to \infty} P^* \left[ \left| \hat{N}_n^K(g) - \hat{N}_n^{(K,B)}(g) \right| > \epsilon \right] = 0.$$

Noting that the points from the point process $\hat{N}_n^{(K,B)}$ are contained in the point process $N_n^K$, and using (3.6) and (3.7), we have $|\hat{N}_n^K(g) - \hat{N}_n^{(K,B)}(g)| = \sum_{i=0}^{K-1} (S_{i,n,B}^{(1)} + S_{i,n,B}^{(2)})$, where $S_{i,n,B}^{(1)} = \sum_{e \in D_{n-i}} (A_e - A_e^{(B)}) g(b_n^{-1}X_e)$ and $S_{i,n,B}^{(2)} = \sum_{e \in D_{n-i}\setminus D^{(B)}_{n-i}} A_e^{(B)} g(b_n^{-1}X_e)$. Since $P^*(\cdot) \leq (P(S))^{-1} P(\cdot)$, it is enough to show that for each $i$, both $S_{i,n,B}^{(1)}$ and $S_{i,n,B}^{(2)}$ are negligible under $P$.

To this end, fix $0 \leq i \leq K - 1$ and $\eta > 0$. Using Markov’s inequality, Wald’s identity and the bound $|g| \leq \|g\|_\infty 1_{[-\infty, -\delta] \cup [\delta, \infty]}$, we get

$$P \left( S_{i,n,B}^{(1)} > \eta \right) \leq \frac{1}{\eta} E(Z_{n-i}) E(A_e - A_e^{(B)}) E(g(b_n^{-1}X_e))$$

$$\leq \frac{1}{\eta \mu^i} \|g\|_\infty E(A_e - A_e^{(B)}) \mu^n P(|X_e| > b_n \delta),$$

from which first letting $n \to \infty$ based on (2.1) and then letting $B \to \infty$, it follows that $\lim_{B \to \infty} \limsup_{n \to \infty} P \left( S_{i,n,B}^{(1)} > \eta \right) = 0$. We can deal with $S_{i,n,B}^{(2)}$ in a similar fashion and obtain

$$P \left( S_{i,n,B}^{(2)} > \eta \right) \leq \frac{\|g\|_\infty}{\eta} E(A_e^{(B)}) P(|X_e| > b_n \delta) E(|D_{n-i} - D^{(B)}_{n-i}|)$$

$$\leq \frac{\|g\|_\infty}{\eta} E(A_e) \mu^n P(|X| > b_n \delta) \left( \frac{1}{\mu^i} - \frac{\mu^i}{\mu_B^i} \mu^{-K} \right).$$

Therefore, $\lim_{B \to \infty} \limsup_{n \to \infty} P \left( S_{i,n,B}^{(2)} > \eta \right) = 0$. This suffices.

4.3. Proof of Lemma 3.4. This is the our key lemma and we shall use the convergence of Laplace functional to investigate the asymptotic behaviour of the sequence of point processes $\hat{N}_n^{(K,B)}$ defined in (3.9). This will be carried out using a marking technique based on multivariate extreme value theory; see, e.g., Resnick (2007).

To prove (a), fix a positive integer $K$ and an integer $B > 1$ such that $\mu_B > 1$ and introduce a bunch of random vectors as follows. For each $j = 1, 2, \ldots, |D_{n-K}|$, let $X_j^{(j)}$ denote the random vector components as the edge random variables attached to the regularized version of the $j$th sub-tree $T_j^{(B)}$, i.e.,

$$\tilde{X}^{(j)} := \left( X_{1,1}^{(j)}, \ldots, X_{1,B}^{(j)}, X_{2,1}^{(j)}, \ldots, X_{2,B^2}^{(j)}, \ldots, X_{K,1}^{(j)}, \ldots, X_{K,B^K}^{(j)} \right).$$
Let $S_p := \{0, 1, \ldots, p\}$ for every $p \in \mathbb{N}$. For each $j = 1, 2, \ldots, |D_{n-K}|$, define

$$\tilde{A}^{(j)} = \left(A_{1,1}^{(j)}, \ldots, A_{1,B}^{(j)}, A_{2,1}^{(j)}, \ldots, A_{2,B^2}^{(j)}, \ldots, A_{K,1}^{(j)}, \ldots, A_{K,B^K}^{(j)}\right)$$

to be an $\tilde{S}^{(B)}$-valued random vector, where

$$\tilde{S}^{(B)} = S_{B^{K-1}} \times \ldots \times S_{B^{K-2}} \times S_{B^{K-2}} \times \ldots \times S_{B^1} \times \ldots \times S_{B^1}$$

with law $G$.

Using (2.1) and independence of the components of $\tilde{X}^{(j)}$ for each $j$, we get that as $n \to \infty$,

$$Z_{n-K} \mathbb{P} \left(b_n^{-1} \tilde{X}^{(j)} \in \cdot \right) \xrightarrow{v} \frac{1}{\mu^K} W \tau := \frac{1}{\mu^K} W \sum_{i=1}^{K} \sum_{l=1}^{B^i} \tau_{i,l}$$

$\mathbb{P}$-almost surely, where

$$\tau_{i,l} := \delta_0 \times \ldots \times \delta_0 \times \nu_{\delta_0} \times \delta_0 \times \ldots \times \delta_0$$

is a measure on $\tilde{R}^{(B)} := [-\infty, \infty]^{B+B^2+\ldots+B^K} \setminus \{0\}$ that concentrates on the $(i,l)^{th}$ axis. Using the independence of $\tilde{X}^{(j)}$ and $\tilde{A}^{(j)}$ for each $j$, we obtain

$$Z_{n-K} \mathbb{P} \left((b_n^{-1} \tilde{X}^{(j)}, \tilde{A}^{(j)}) \in \cdot \right) \xrightarrow{v} \frac{1}{\mu^K} W \tau \otimes G$$

(4.3) $\mathbb{P}$-almost surely as $n \to \infty$.

Note that $\tilde{A}^{(j)}$'s are independent under $\mathbb{P}$ but not under $\mathbb{P}^*$. However, the relation $d \mathbb{P}^* = (\mathbb{P}(S))^{-1} \mathbb{1}_S \, d \mathbb{P}$ turns out to be extremely useful in computation of the Laplace functional of the marked point process

$$\mathcal{P}_n := \sum_{j=1}^{Z_{n-K}} \delta_{(b_n^{-1} \tilde{X}^{(j)}, \tilde{A}^{(j)})}$$

under $\mathbb{P}^*$. More precisely, for any $\psi \in C_+ \left(\tilde{R}^{(B)} \times \tilde{S}^{(B)}\right)$,

$$
\mathbb{E}^* \left[ \exp \left\{ - \sum_{i=1}^{Z_{n-K}} \psi \left(b_n^{-1} \tilde{X}^{(j)}, \tilde{A}^{(j)}\right) \right\} \right] = \frac{1}{\mathbb{P}(S)} \int \exp \left\{ - \sum_{i=1}^{Z_{n-K}} \psi \left(b_n^{-1} \tilde{X}^{(j)}, \tilde{A}^{(j)}\right) \right\} \mathbb{1}_S \, d \mathbb{P},
$$

(4.4)
whose asymptotics will be investigated below and will form a major building block of our proof. To this end, we introduce an event $S_{n-K}$ which is empty if $Z_{n-K} = 0$, and on $(Z_{n-K} > 0)$, it is the event that there is at least one infinite tree rooted at the $(n - K)^{th}$ generation of the underlying Galton-Watson tree. Using $1_S = 1_{(Z_{n-K}>0)} 1_{S_{n-K}}$, (4.4) equals

$$= \frac{1}{P(S)} \int \exp \left\{ - \sum_{i=1}^{Z_{n-K}} \psi(\tilde{X}^{(j)}, \tilde{A}^{(j)}) \right\} \, 1_{(Z_{n-K}>0)} \, dP \quad (4.5)$$

$$- \frac{1}{P(S)} \int \exp \left\{ - \sum_{i=1}^{Z_{n-K}} \psi(\tilde{X}^{(j)}, \tilde{A}^{(j)}) \right\} \, 1_{(Z_{n-K}>0)} 1_{S_{n-K}^c} \, dP. \quad (4.6)$$

We shall compute the asymptotics of the first term above and then show that the second term converges to zero. Denoting by $F_n$ the $\sigma$-field generated by the first $n$ generations of the underlying Galton-Watson tree, we rewrite (4.5) as

$$\frac{1}{P(S)} E \left[ 1_{(Z_{n-K}>0)} E \left( \exp \left\{ - \sum_{i=1}^{Z_{n-K}} \psi(\tilde{X}^{(j)}, \tilde{A}^{(j)}) \right\} \right| F_{n-K} \right].$$

Using Theorem 5.3 of Resnick [2007], (4.3) above, and the independence of the random vectors $\{\tilde{A}^{(j)}\}$ under $P$, we have

$$E \left( \exp \left\{ - \sum_{i=1}^{Z_{n-K}} \psi(\tilde{X}^{(j)}, \tilde{A}^{(j)}) \right\} \right| F_{n-K} \right)$$

$$\rightarrow \exp \left\{ - \frac{1}{\mu_K} W \int_{\tilde{R}(B)} \sum_{\tilde{a} \in \tilde{S}(B)} (1 - e^{\psi(\tilde{x}, \tilde{a})}) G(\tilde{a}) \tau(\tilde{d}\tilde{x}) \right\}$$

$P$ almost surely as $n \rightarrow \infty$. We also know that $1_{(Z_{n-K}>0)} \rightarrow 1_S$ almost surely under $P$ as $n \rightarrow \infty$. Hence dominated convergence theorem yields that (4.5) converges to

$$E^* \left[ \exp \left\{ - \frac{1}{\mu_K} W \int_{\tilde{R}(B)} \sum_{\tilde{a} \in \tilde{S}(B)} (1 - e^{\psi(\tilde{x}, \tilde{a})}) G(\tilde{a}) \tau(\tilde{d}\tilde{x}) \right\} \right].$$

On the other hand, (4.6) is bounded by

$$\frac{1}{P(S)} E(1_{(Z_{n-K}>0)} 1_{S_{n-K}^c}) = \frac{1}{P(S)} E \left( 1_{(Z_{n-K}>0)} p_{eZ_{n-K}} \right), \quad (4.7)$$

where $p_e = P(S^c)$ denotes the probability of extinction of the underlying Galton-Watson tree. Since our tree is supercritical, $p_e < 1$ and $P$-almost surely, $Z_{n-K} \rightarrow \infty$ as $n \rightarrow \infty$ on the event $S$. Using the $P$-almost sure convergence $1_{(Z_{n-K}>0)} \rightarrow 1_S$ once more, it is easy to see that (4.7) tends to
zero as $n \to \infty$. Hence we have shown that for all $\psi \in C_{c}^{+}(\tilde{R}(B) \times \tilde{S}(B))$,

$$
E^* \left[ \exp \left\{ - \sum_{i=1}^{Z_{n,K}} \psi \left( b_{n}^{-1} \tilde{X}^{(j)}, \tilde{A}^{(j)} \right) \right\} \right]
\to E^* \left[ \exp \left\{ - \frac{1}{\mu K} W \int_{\tilde{R}(B)} \sum_{\tilde{a} \in \tilde{S}(B)} (1 - e^{\psi(\tilde{x}, \tilde{a})}) G(\tilde{a}) \tau(d\tilde{x}) \right\} \right]. \quad (4.8)
$$

The next step of the proof is to use the above convergence of Laplace functionals to establish the same for the point process $\tilde{N}_{n}^{(K,B)}$. To achieve this goal, take a function $g \in C_{c}^{+}([-\infty, \infty]\setminus\{0\})$ and observe that $\tilde{\psi}(\tilde{x}, \tilde{a}) := \sum_{i=1}^{K} \sum_{l=1}^{B} a_{i,l} g(x_{i,l}) \in C_{c}^{+}(\tilde{R}(B) \times \tilde{S}(B))$. Therefore, as a special case of (4.8), we get

$$
E^* \left( \exp \left\{ - \tilde{N}_{n}^{(K,B)}(g) \right\} \right) = E^* \left( \exp \left\{ - \sum_{j=1}^{Z_{n,K}} \tilde{\psi} \left( b_{n}^{-1} \tilde{X}^{(j)}, \tilde{A}^{(j)} \right) \right\} \right)
\to E^* \left[ \exp \left\{ - W \frac{1}{\mu K} \int_{\tilde{R}(B)} \sum_{\tilde{a} \in \tilde{S}(B)} (1 - e^{-\sum_{i=1}^{K} \sum_{l=1}^{B} a_{i,l} g(x_{i,l})}) G(\tilde{a}) \sum_{i=1}^{K} \sum_{l=1}^{B} \tau_{i,l}(d\tilde{x}) \right\} \right]
= E^* \left[ \exp \left\{ - W \frac{1}{\mu K} \int_{|x| > 0} \sum_{i=1}^{K} \left[ \sum_{\tilde{a} \in \tilde{S}(B)} (1 - e^{-a_{i,l} g(x)}) G(\tilde{a}) \right] \nu_{\alpha}(dx) \right\} \right]. \quad (4.9)
$$

The last step uses that any $a_{i,l} = 0$ does not have any contribution to the integral appearing at the exponent in (4.9). Note that keeping track of a positive $a_{i,l}$ amounts to doing the same for $A^{(j,B)}_{i,l} \equiv Z_{K-i}^{(B)}$ and hence is related to the random variable $\tilde{Z}_{K-i}^{(B)}$ obtained by conditioning $Z_{K-i}^{(B)}$ to stay positive. Taking an independent copy $\{Y_{i}^{(B)}\}_{i \geq 1}$ of the sequence $\{Z_{i}^{(B)}\}_{i \geq 1}$, we get that for fixed $i$ and $x$,

$$
\sum_{\tilde{a} \in \tilde{S}(B)} \sum_{a_{i,l} \neq 0} (1 - e^{-a_{i,l} g(x)}) G(\tilde{a})
= E \left( \sum_{j=1}^{Y_{i}^{(B)}} (1 - e^{-\tilde{Z}_{K-i}^{(B)} g(x)}) \right) P(\tilde{Z}_{K-i}^{(B)} > 0)
= \mu_{B} E \left( 1 - e^{-\tilde{Z}_{K-i}^{(B)} g(x)} \right) P(\tilde{Z}_{K-i}^{(B)} > 0). \quad (4.10)
$$
This, combined with (4.9), yields
\[
\lim_{n \to \infty} E^* \left( \exp \left\{ -\tilde{N}_n^{(K,B)}(g) \right\} \right) = E^* \left[ \exp \left\{ -W \frac{\mu_B^K}{\mu^K} \int_{|x| > 0} \sum_{i=0}^{K-1} \frac{1}{\mu_B^i} P(Z_i^{(B)} > 0) \right. \right.
\times E \left( 1 - e^{-\tilde{Z}_i^{(B)} g(x)} \right) \nu_\alpha(dx) \left. \right] ,
\]
which can easily be shown (using an argument similar to the one used in Subsection 3.5) to be the Laplace functional of the point process
\[
N_*^{(K,B)} := \sum_{l=1}^{\infty} T_l^{(K,B)} \delta_{(r_{K,B} \left( \frac{\mu^K W}{\mu_B^K} \right)^{1/\alpha} j_l) ,}
\]
where \( r_{K,B} = \sum_{i=0}^{K-1} \frac{1}{\mu_B^i} P(Z_i^{(B)} > 0) \) and \( \{T_l^{(K,B)}\}_{l \geq 1} \) is a sequence of i.i.d. random variables (independent of \( \{j_l\}_{l \geq 1} \) and \( W \)) with probability mass function
\[
P(T_1 = y) = \frac{1}{r_{K,B}} \sum_{i=0}^{K-1} \frac{1}{\mu_B^i} P(Z_i^{(B)} = y) , \quad y \in \mathbb{N}.
\]
Thus (a) follows using Theorem 5.2 in Resnick (2007).

To establish (b), fix a positive integer \( K \) and observe that applying dominated convergence theorem as \( B \to \infty \), the Laplace functional of \( \tilde{N}_n^{(K,B)} \) can be shown to converge to that of
\[
N_*^{(K)} := \sum_{l=1}^{\infty} T_l^{(K)} \delta_{(r_K \left( \frac{\mu W}{\mu^K} \right)^{1/\alpha} j_l) ,}
\]
where \( r_K = \sum_{i=0}^{K-1} \frac{1}{\mu_i} P(Z_i > 0) \) and \( \{T_l^{(K)}\}_{l \geq 1} \) is a sequence of i.i.d. random variables (independent of \( \{j_l\}_{l \geq 1} \) and \( W \)) with probability mass function
\[
P(T_1 = y) = \frac{1}{r_K} \sum_{i=0}^{K-1} \frac{1}{\mu_i} P(Z_i = y) , \quad y \in \mathbb{N}.
\]
In a similar fashion, (c) can be shown. This completes the proof of Lemma 3.4.

4.4. Proof of Lemma 3.1 To show (3.2), it is enough to take a Lipschitz function \( g \in C^+_c(\Xi) \) (with Lipschitz constant \( \|g\| \) and support(\( g \)) \( \subseteq \{x : |x| > \delta\} \) for some \( \delta > 0 \) and show that for every \( \epsilon > 0 \),
\[
\lim_{n \to \infty} P^* \left( |N_n(g) - \tilde{N}_n(g)| > \epsilon \right) = 0.
\]
This will be attained by slightly revamping the proof of the convergence in (3.14) of Resnick and Samorodnitsky (2004). Some of the estimates used therein will not work for us mainly because we are dealing with general regularly varying random variables as opposed to stable ones with an inbuilt
Poissonian structure. This hurdle will be overcome by use of Potter’s bound
and a mild modification of the event $AMO(\theta)$ defined in page 201 of the
aforementioned reference.

For every $\theta > 0$, let $A_n(\theta)$ denote the event that for all $v \in D_n$, at most
one of the random variables in the collection $\{X_e : e \in I_v\}$ is bigger than $b_n\theta/n$ in absolute value. We claim that $\lim_{n \to \infty} P^*(A_n(\theta)^c) = 0$. As in the
proof of Lemma 3.2, this follows easily if we can establish that

$$\lim_{n \to \infty} P^*_T(A_n(\theta)^c) = 0$$

for $P^*$-almost all $T$. (4.12)

To this end, observe that conditioned on the tree $T$, $\sum_{e \in I_v} \delta_{|X_e|}(b_n^{\theta/n}, \infty) \sim \text{Binomial}(n, P(n|X_e| > b_n\theta))$ for each $v \in D_n$. Hence using Potter’s
bound (see, e.g., Proposition 0.8(ii) in Resnick (1987)), (1.1), (2.1), and
the fact that $P(U_n \geq 2) = O(n^2 p_n^2)$ for any $U_n \sim \text{Binomial}(n, p_n)$ with $p_n = o(1/n)$, we get

$$P^*_T(A_n(\theta)^c) \leq \sum_{|v| = n} P^*_T\left(\sum_{e \in I_v} \delta_{|X_e|}(b_n^{\theta/n}, \infty) \geq 2\right)$$

$$\leq (\text{const}) |D_n| n^2 \left( P(n|X_e| > b_n\theta) \right)^2 \to 0$$

$P^*$-almost surely as $n \to \infty$.

In light of (4.12), to prove (4.11), it is enough to show that

$$\lim_{n \to \infty} P^*\left(|N_n(g) - \tilde{N}_n(g)| > \epsilon, A_n(\theta)\right) = 0.$$  (4.13)

This can be achieved by following verbatim the proof of the convergence in (3.14) of Resnick and Samorodnitsky (2004). More precisely, choosing $\theta < \frac{\delta}{2}$ and defining $T_v$ to be the the largest (in absolute value) summand in $\sum_{e \in I_v} X_e$ on the set $A_n(\theta)$, it can be shown that

$$P^*_T\left(|N_n(g) - \tilde{N}_n(g)| > \epsilon, A_n(\theta)\right)$$

$$\leq P^*_T\left(\sum_{|v| = n} |g(b_n^{-1}T_v) - g(b_n^{-1}S_v)| > \epsilon, \max_{|v| = n} T_v > b_n\theta/n, A_n(\theta)\right)$$

$$\leq P^*_T\left(||g||_\infty \theta \tilde{N}_n\left(\frac{\delta}{2}, \infty\right) > \epsilon\right),$$

from which (4.13) follows by first taking expectation with respect to the tree,
then taking limit as $n \to \infty$ relying on (3.11), and finally letting $\theta \to 0$.

**Remark 4.1.** The proof of Lemma 3.1 uses (3.11), which is a consequence of
Lemmas 3.2, 3.3 and 3.4; see Subsection 3.5. However, this is not a problem
because the latter lemmas are proved without using Lemma 3.1.
4.5. **Proof of Theorem 2.3.** Take two independent sequences of random variables \( \{ \varepsilon_i \}_{i \geq 1} \) and \( \{ \lambda_i \}_{i \geq 1} \) such that \( \sum_{i=1}^{\infty} \delta_{\varepsilon_i, \lambda_i} \sim PRM(\nu_\alpha) \) and \( \varepsilon_1, \varepsilon_2, \ldots \) are i.i.d. random variables with same distribution as that of \( \varepsilon \). Straightforward applications of Propositions 5.2 and 5.3 of Resnick (2007) yield that \( \sum_{i=1}^{\infty} \delta_{\varepsilon_i, \lambda_i} \sim PRM(\nu_\alpha) \), which, together with (2.5), gives the SSDPPP representation of \( N_* \).

To show the second part of this theorem, we follow the computation of Laplace functional in Subsection 3.5 along with the scaling property of \( \nu_\alpha \) and express the left hand side of (2.7) as

\[
E^* \left( \exp \left\{ -W \int |x|>0 \sum_{i=0}^{\infty} \frac{1}{\mu^i} P(Z_i > 0) E \left( 1 - e^{-\tilde{Z}_i f(y^{-1}x)} \right) \nu_\alpha(dx) \right\} \right) = E^* \left( \exp \left\{ -y^{-\alpha}W \int |x|>0 \sum_{i=0}^{\infty} \frac{1}{\mu^i} P(Z_i > 0) E \left( 1 - e^{-\tilde{Z}_i g(x)} \right) \nu_\alpha(dx) \right\} \right),
\]

which equals the right hand side with

\[
c_g = \left( \int |x|>0 \sum_{i=0}^{\infty} \frac{1}{\mu^i} P(Z_i > 0) E \left( 1 - e^{-\tilde{Z}_i g(x)} \right) \nu_\alpha(dx) \right)^{-1/\alpha} > 0.
\]

4.6. **Proof of Theorem 2.5.** Using Theorem 3.1 and Theorem 3.2 of Resnick (2007) and Theorem 2.1 above, it transpires that \( N_n([-\infty,-x]) \) converges weakly to \( N_*([-\infty,-x]) \) under \( P^* \). Therefore, for each \( x > 0 \),

\[
P^*(M_n' > -b_n x) = P^*(N_n([-\infty,-x]) = 0) \rightarrow P^*(N_*([-\infty,-x]) = 0),
\]

from which (a) follows because \( T_i > 0 \) for all \( l \geq 1 \) and this implies

\[
P^*(N_*([-\infty,-x]) = 0 | W) = P^* \left( \sum_{l=1}^{\infty} \delta_{(rW)^{1/\alpha} j_l}([-\infty,-x]) = 0 | W \right)
\]

\[
= \exp \left\{ -rW qx^{-\alpha} \right\}.
\]

The \( k = 1 \) case of (b) follows similarly from the weak convergence of \( N_n((x,\infty]) \) to \( N_*((x,\infty]) \) under \( P^* \). For \( k \geq 2 \), using the same weak convergence, we get

\[
\lim_{n \to \infty} P^*(M_n^{(k)} \leq b_n x) = \lim_{n \to \infty} P^*(N_n((x,\infty]) \leq k-1)
\]

\[
= E^* \left( \exp \left\{ -rW px^{-\alpha} \right\} \right) + \sum_{l=1}^{k-1} P^*(N_*((x,\infty]) = l). \tag{4.14}
\]

We need to show that the second term of (4.14) is same as that of (2.8). To this end, considering the marked point process \( N = \sum_{l=1}^{\infty} \delta_{(rW)^{1/\alpha} j_l} \sim PRM(rW(\gamma \otimes \nu_\alpha)) \) conditioned on \( W \), and analyzing exactly how each event
\((N_n((x, \infty)) = l)\) can occur, the second term in (4.14) becomes
\[
\sum_{l=1}^{k-1} \sum_{\pi \in \Pi_l} P^* \left( \bigcap_{j=1}^{\vert \pi \vert} \{ N(\{i_j\} \times (x, \infty)) = y_j \} \right)
\]
\[
= \sum_{l=1}^{k-1} \sum_{\pi \in \Pi_l} E^* \left[ \prod_{j=1}^{\vert \pi \vert} \left( (rW_{px}^{-\alpha} \gamma(i_j))^{y_j} \frac{1}{y_j!} \exp \left\{ -rW_{px}^{-\alpha} \gamma(i_j) \right\} \right) \right].
\]

This establishes (b).

In order to verify (c), we need a similar (but slightly tedious) calculation as in the proof of (b) based on the following observation: for \(0 < u < v\),
\[
P^* (M_n^{k+1} \leq b_n u, M_n^k \leq b_n v) = P^* (N_n((u, \infty]) = 0) + P^* (N_n((v, \infty]) = 0, 1 \leq N_n((u, v]) \leq k)
\]
\[
+ P^* (1 \leq N_n((v, \infty]) \leq k-1, N_n((u, \infty]) \leq k).
\]

Finally, (d) follows from (c) using continuous mapping theorem (see, e.g., Theorem 3.1 in Resnick (2007)).

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