Is Modified Newtonian Dynamics a fractional theory?

Andrea Giusti

1Department of Physics & Astronomy, Bishop’s University, 2600 College Street, Sherbrooke Québec, Canada J1M 1Z7

I provide a derivation of Milgrom’s Modified Newtonian Dynamics from a fractional version of Newton’s theory based on the fractional Poisson equation. I employ the properties of the fractional Laplacian to investigate the features of the fundamental solution of the proposed model. Taking advantage of the Tully-Fisher relation I then connect the fundamental length scale ℓ, emerging from this modification of Newton’s gravity, with the critical acceleration a0 of MOND. Finally, implications for galaxy rotation curves of a variable-order version of the model are discussed.

General Relativity (GR) and the Standard Model of particle physics have proven to be invaluable tools for our current understanding of nature. Yet we can only account for less then 5% of the content of the universe, while the rest remains mostly uncharted territory that we dub as dark components. Over the decades, several observations [1] [2] have proven that the present universe is expanding at an accelerating pace. To explain this effect within GR one assumes the existence of a mysterious dark energy component pervading the universe and accounting for around 70% of its energy content [3] [4]. However, since dark energy has to be included ad hoc in Einstein’s theory to reproduce the observed accelerated expansion, it is worth mentioning that some substantial effort has been devoted to the study of large-scale modifications of gravity aimed at reproducing this effect while dispensing of the notion of dark energy (see e.g. [5] [11]). Yet, all of the proposal aimed at reproducing the accelerated expansion of the universe still do not provide an accurate explanation of many astronomical observations. For instance, if one assumes spherical symmetry then a test mass laying on a stable Keplerian orbit around a galaxy should experience a rotational velocity \( v^2(r) \sim G_N m(r)/r \), with \( m(r) \) the total mass within the orbit. Observations, however, show that \( v(r) \) flattens as we move away from the galaxy center (see e.g. [12] [15]). This effect is typically accounted for by assuming the existence of an exotic form of matter, called dark matter, such that \( m(r) \sim r \) even well outside the core of the galaxy. From the several observations of galaxy rotation curves, as well as from measures of the mass of several galaxy clusters, this dark component of the universe has some pretty peculiar features, namely it does not interact with electromagnetic radiation and it has an (almost) imperceptible pressure. Over the years there have been several proposals (see e.g. [16] [22]) concerning the physical nature and origin of dark matter, ranging from primordial black holes to new physics beyond the standard model, though it does not seem that we are getting any closer to a definite answer to this conundrum. An alternative to the ad hoc addition of dark matter consists in Milgrom’s Modified Newtonian Dynamics (MOND) [23] [26], according to which Newtonian gravity is modified when the acceleration of a test mass falls below a certain threshold \( a_0 \), whose value is empirically determined. In detail, considering a test particle on a Keplerian orbit around a core of mass \( M \), then for \( a \gg a_0 \) the acceleration follows the standard Newtonian theory yielding \( a \approx G_N M/r^2 \), whilst when \( a \ll a_0 \) it gets modified according to \( a^2/a_0 \approx G_N M/r^2 \). In other words, it is possible to dispense of the notion dark matter provided that one assumes that Newton’s gravity is modified at large scales, specifically leading to a transition between a short-scale \( a(r) \sim 1/r^2 \) behavior and \( a(r) \sim 1/r \) at galactic scales.

Fractional calculus [27] [29] is the collection of tools that allows one to extend the classical theory of calculus to the case of fractional powers of the standard integrals and derivatives. This theory then turns out to be related to the theory of weakly singular Volterra-type integro-differential operators [30] [31] and naturally leads to the notions of memory and non-locality. One of the most important results of this general approach consists in a mathematically sound definition of the so-called fractional Laplacian (see e.g. [32] [33] and references therein).

Here, I derive the empirical asymptotic behaviors of MOND form the fundamental solution of the fractional Poisson equation

\[
(-\triangle)^s \Phi(x) = -4 \pi G_N \ell^2 \rho(x),
\]

where \( \triangle \) denotes the standard Laplacian, \( 1 < s \leq 3/2 \), \( \Phi(x) \) the modified gravitational potential, \( \rho(x) = M \delta^{(3)}(x) \) the matter density distribution, and \( \ell \) a constant with the dimension of a length. Then, since MOND is an inherently scale dependent effect, I discuss how this picture can be framed within the theory of variable-order fractional operators (see e.g. [34]) with \( s \) becoming a scale-dependent quantity \( s = s(x) \).

The canonical way of approaching the problem of the mathematical definition of the fractional Laplacian is to start from its Fourier transform. Let \( f(x) \) be a function of the Schwartz class on \( \mathbb{R}^3 \), then we define the Fourier transform of \( f(x) \) as

\[
\hat{f}(k) \equiv \mathcal{F}[f(x); k] = \int_{\mathbb{R}^3} e^{-ik \cdot x} f(x) \, dx,
\]

\[1\] Also known as the space of rapidly decreasing functions.
with \( \cdot \) denoting the Euclidean scalar product. This implies that the Fourier transform of the Laplacian simply yields

\[
\mathcal{F}[(-\Delta) f(x) : k] = |k|^2 \hat{f}(k),
\]
with \(|k|^2 \equiv k : k\). Thus, a natural requirement for the fractional generalization of \(-\Delta\) is to preserve this nice feature, i.e.

\[
\mathcal{F}[(-\Delta)^s f(x) : k] = |k|^{2s} \hat{f}(k).
\]

Note that the choice of \(-\Delta\) over \(\Delta\) is particularly important since the first yields a positive-definite operator, allowing one to take advantage of the method of semigroups, see e.g. [32, 33, 35] and references therein. Specifically, it is not hard to see that (see e.g. [35]) for any \(\lambda \geq 0\) one has

\[
\lambda^s = \frac{1}{\Gamma(-s)} \int_{0}^{\infty} (e^{-t\lambda} - 1) t^{-s-1} dt,
\]
with \(0 < s < 1\) and \(\Gamma(z)\) denoting Euler’s Gamma function, and

\[
\lambda^{-s} = \frac{1}{\Gamma(s)} \int_{0}^{\infty} e^{-t\lambda} t^{-s} dt,
\]
for any \(s > 0\) and \(\lambda > 0\).

Since [3] is not particularly helpful when trying to solve (1) for \(s > 1\), one has to relay on [6]. Indeed, considering (1) in the Fourier domain one has

\[
\hat{\Phi}(k) = -4\pi G_N \ell^{2-2s} M |k|^{-2s},
\]

since, again, I am considering a configuration of the system with \(\rho(x) = M \delta^{(3)}(x)\). This expression suggests the general restriction \(0 < s < 3/2\), since otherwise \(|k|^{-2s}\) is not a tempered distribution [35]. Eq. (7), brought back to the space domain, implies that

\[
\Phi(x) = -4\pi \ell^{2-2s} G_N M (-\Delta)^{-s} \delta^{(3)}(x),
\]

with \((-\Delta)^{-s}\) denoting the inverse fractional Laplacian that, taking advantage of (6) can be expressed as [32, 33]

\[
(-\Delta)^{-s} f(x) = \frac{1}{\Gamma(s)} \int_{0}^{\infty} dt \ t^{s-1} e^{t \Delta} f(x),
\]
for any \(s > 0\). If \(0 < s < 3/2\), it was shown that (see e.g. [32])

\[
(-\Delta)^{-s} \delta^{(3)}(x) = \frac{1}{4\pi^{3/2} \Gamma(s)} \frac{1}{|x|^{3-2s}},
\]

thus leading to a potential

\[
\Phi_s(x) = -\frac{\Gamma\left(\frac{3}{2} - s\right)}{4^{s-1} \sqrt{\pi} \Gamma(s)} \left(\frac{\ell}{|x|}\right)^{2-2s} \frac{G_N M}{|x|}.
\]

This expression clearly shows that for \(s \to 1\) one easily recovers the Newtonian potential.

Now, considering the limit for \(s \to (3/2)^+\) one is required to extend the previous argument taking into account the regularity problem that comes with this limit [30]. Going back to (7) and setting \(s = 3/2\) one finds

\[
\hat{\Phi}(k) = -4\pi G_N \ell^{-1} M |k|^{-3},
\]
that implies

\[
\Phi_{3/2}(x) = \frac{4\pi G_N M}{\ell} \int_{\mathbb{R}^3} \frac{d^3 k}{(2\pi)^3} \frac{e^{i k \cdot x}}{|k|^3}
\]

To compute this inverse Fourier transform one can start-off by denoting

\[
\eta(k) = \mathcal{F}[\log (|x|/\ell) : k] = \int_{\mathbb{R}^3} d^3 x \ e^{i k \cdot x} \log (|x|/\ell),
\]
and by recalling that

\[
\triangle \log (|x|/\ell) = \frac{1}{|x|^2}.
\]

On the one hand, taking the Fourier transform the letter one finds

\[
\mathcal{F}\left[\triangle \log (|x|/\ell) : k\right] = \int_{\mathbb{R}^3} d^3 x \ \frac{e^{-i k \cdot x}}{|x|^2}
\]

where with P.V. it is understood that we are taking the principal value of the Dirichlet integral. On the other hand, taking advantage of the properties of the Fourier transform one finds

\[
\mathcal{F}\left[\triangle \log (|x|/\ell) : k\right] = -|k|^2 \eta(k).
\]

Then, putting together (16) and (17) one concludes that

\[
\eta(k) = \mathcal{F}\left[\log (|x|/\ell) : k\right]_{\text{P.V.}} = -\frac{2\pi^2}{|k|^3}.
\]

Backtracking to (13) and exploiting the last result one finds that

\[
\Phi_{3/2}(x)_{\text{P.V.}} = \frac{2 G_N M}{\pi \ell} \log (|x|/\ell).
\]

We can therefore summarize these computations as follows

\[
\Phi_s(x) = \begin{cases} 
-\frac{\Gamma\left(\frac{3}{2} - s\right)}{4^{s-1} \sqrt{\pi} \Gamma(s)} \left(\frac{\ell}{r}\right)^{2-2s} \frac{G_N M}{r}, \\
\frac{2 G_N M}{\ell} \log (r/\ell), \quad \text{for } 0 < s < 3/2,
\end{cases}
\]

\[2 \text{ Rigorously speaking, it means that the solution of } 4 \text{ corresponds to its fundamental solution.}\]
where the case $s = 3/2$ is understood in the regularized sense discussed above.

This discussion suggests that a modified Newtonian theory based on the fractional Poisson equation \(1\) allows one to naturally derive the transition from Newton’s gravity to MOND’s large-scale behavior. These kind of transitions are actually a rather common feature of fractional models, see e.g. [42, 43]. Additionally, this approach would suggest an inherent non-local nature of MOND. Note that, differently from Milgrom’s approach [23, 26], here we have introduced a constant $\ell$ with the dimension of a length, rather than $a_0$, to maintain the correct dimensions in \(1\). But again, from $a = -\nabla \Phi_s$ we recover $a(r) \sim 1/r^2$ and $a(r) \sim 1/r$ for $s = 1$ and $s = 3/2$, respectively. Besides, from $s = 3/2$ one finds that

$$a(r) = \frac{2 G_N M}{\pi \ell r},$$

that, coupled to the condition $a = v^2/r$, yields

$$v^2 = \frac{2 G_N M}{\pi \ell},$$

which leads to the empirical Tully–Fisher relation [40]

$$v^4 = G_N M a_0,$$

provided that

$$\ell = \frac{2}{\pi} \sqrt{\frac{G_N M}{a_0}},$$

thus fully reconnecting the proposed fractional model with MOND’s results and phenomenology.

In order to properly equip the theory with an explicit scale-dependent behavior of $\Phi_s(r)$ one would need to replace $s$ in \(1\) with $s(x) = s(r/\ell)$. In other words, one should convert the fractional Poisson equation of order $s$ into a variable-order fractional differential equation. This is a subject which has been largely studied (see e.g. [41] for a review) over the years paying particular attention for 1–dimensional and $(1 + 1)$–dimensional problems. However, it is rather clear that such differential equations can only be treated numerically, whilst closed-form solutions are fairly rare and hard to find. The situation clearly worsens, both from the analytical and numerical standpoints, when one considers the case of the variable-order Laplacian. Indeed, this topic still represents a rather uncharted territory, even though a few works have started tackling this problem in the last couple of years [42, 43].

It is then interesting to study the behavior of the rotational velocity predicted by the proposed model as a function of $s$. Recalling that

$$a(r) = \frac{v(r)^2}{r} = |\nabla \Phi_s(r)|,$$

and that

$$|\nabla \Phi_s(r)| = \begin{cases} 
\frac{4^{2-s} \Gamma \left( \frac{5}{2} - s \right)}{\sqrt{\pi} \Gamma(s)} \left( \frac{\ell}{r} \right)^{2-2s} \frac{G_N M}{r^2}, & \text{for } 0 < s < 3/2, \\
\frac{2 G_N M}{\pi \ell r}, & \text{for } s = 3/2,
\end{cases}$$

with $\ell$ as in [23], one finds

$$v(r) = \begin{cases} 
\frac{2^{2-s} \sqrt{\Gamma \left( \frac{5}{2} - s \right)} \Gamma(s)}{\sqrt{\pi} \Gamma(s)} \left( \frac{\ell}{r} \right)^{1-s} \frac{G_N M}{r}, & \text{for } 0 < s < 3/2, \\
\frac{2 G_N M}{\pi \ell}, & \text{for } s = 3/2,
\end{cases}$$

The velocity profiles are then plotted in Fig. 1 depicting the transition from Newton’s gravity to MOND.

To sum up, fractional calculus has proven to be a valuable tool for studying several physical problems. Its role in fundamental physics has however been largely ignored so far, even though the last few years have finally seen the emergence of some studies pointing out its potential relevance in quantum field theory and gravity [44–47]. In this regard, the fractional Laplacian seems to play an important role in light of its connection to the heat kernel and the Euclidean pictures of non-local quantum field theories. Here, I have derived MOND’s asymptotic behaviors for the gravitational potential (and, as a consequence, for the acceleration) as the result of a fractional Newtonian theory of gravity. This model is based on the fractional Poisson equation \(1\), where the standard Laplacian $-\triangle$ is replaced by its fractional power $(-\triangle)^s$. Taking advantage of the method of semigrups

![Image of rotational velocity as a function of radius](image-url)
and of the property [4] I have discussed the fundamental solution of [1] in light of [22,33]. The result is that [1] reduces to Newton’s theory for \( s = 1 \), whereas it reproduces MOND’s large-scale behavior for \( s = 3/2 \). This transition is, however, all but trivial since \(|k|^{-3}\) clearly does not belong to the class of tempered distributions and therefore \( \Phi_{3/2}(r) \) is obtained from a regularization of the inverse Fourier transform of \(|k|^{-3}\). Then, comparing \( a(r) = |\nabla \Phi|_a \) with MOND’s expression for the Tully-Fisher relation one can identify the relation between \( \ell \) and \( a_0 \), i.e. [23], fully reconciling the two approaches. The fact that MOND predicts \( a \sim 1/r \) for \( a \ll a_0 \) implies that \( s = 3/2 \) for \( r \gg \ell \), thus identifying a scale at which these deviations should emerge from the non-locality of the theory. One can then infer that the proper way to fully describe MOND as a fractional Newtonian theory requires to treat [1] as a variable-order fractional differential equation, with \( s = s(r/\ell) \). Then MOND’s conditions \( a_0 \gg a \) and \( a_0 \ll a \), that allows one to recover the flattening of the tangential velocity as one moves away from the galaxy center and the Newtonian force respectively, are recast as \( s(r/\ell) = 3/2 \) for \( r \gg \ell \) and \( s(r/\ell) = 1 \) for \( r \ll \ell \). Hence, we can treat MOND as a fractional model with a variable-order spanning \( 1 \leq s(r/\ell) \leq 3/2 \), taking proper care to the upper extreme. This nice interpretation of the model comes at a price, indeed the variable-order nature of the theory makes its analytical treatment rather subtle, if not impervious. A full numerical treatment of the variable-order counterpart of [1] is needed in order to constrain the functional form of \( s(r/\ell) \). Note that this variability could, in principle, help explaining the fact that rotational velocities as functions of the distance from the galaxy center are not exactly flat and display some variability. Finally, the scale dependence of the fractional order \( s(r/\ell) \) is consistent with the corpuscular interpretation of MOND [19,20], which predicts that \( a_0 \sim cH \), with \( H \) denoting the Hubble parameter and \( c \) being the speed of light. This is actually consistent with many observational evidences according to which \( a_0 \sim cH_0 \) [23]. As a consequence, from [23] one finds that \( \ell \sim \sqrt{G_N M/cH_0} \), providing an estimate for the critical scale for these fractional effects.

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