Bell as the Copernicus of Probability

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January 13, 2015

Abstract

Our aim is to emphasize the role of mathematical models in physics, especially models of geometry and probability. We briefly compare developments of geometry and probability by pointing to similarities and differences: from Euclid to Lobachevsky and from Kolmogorov to Bell. In probability Bell played the same role as Lobachevsky in geometry. In fact, violation of Bell’s inequality implies the impossibility to apply the classical probability model of Kolmogorov (1933) to quantum phenomena. Thus quantum probabilistic model (based on Born’s rule) is an example of non-Kolmogorovian model of probability, similarly to the Lobachevskian model – the first example of non-Euclidean model of geometry. We also discuss coupling of the classical probabilistic model with classical (Boolean) logic. The Kolmogorov model of probability is based on the set-theoretic presentation of the Boolean logic. In this framework violation of Bell’s inequality implies the impossibility to use the Boolean structure of events for quantum phenomena; instead of it, events have to be represented by linear subspaces. This is the “probability model” interpretation of violation of Bell’s inequality. We also criticize the standard interpretation – an attempt to add to rigorous mathematical probability models additional elements such as (non)locality and (un)realism. Finally, we compare embeddings of non-Euclidean geometries into the Euclidean space with embeddings of the non-Kolmogorovian probabilities (in particular, quantum probability) into the Kolmogorov probability space. As an example, we consider the CHSH-test.
1 Introduction

The argument which will be presented in this paper, namely, that violations of Bell type inequalities [1], [2] signal us that the classical model of probability [3] (Kolmogorov, 1933) is inapplicable to quantum phenomena, was already discussed in very detail by many authors, see, e.g., [4]–[53]. Here I try to formulate this argument so clearly as possible. I also argue that any attempt to assign to these violations some additional value e.g., to philosophize about (non)locality and (un)reality, is not consistent with the mathematical model approach to description of natural (and mental) phenomena (see [54] for a discussion).

The modern physics is based on creation of mathematical models describing natural phenomena. We emphasize that a model has not be identified with reality. Each model has boundaries of its application. Thus the evolution of physics can be considered as a chain of creation of mathematical models, finding boundaries of their applicability, and creation of new models. Personally I think that any attempt to proceed without keeping to this mathematical modeling ideology leads to philosophizing which only shadows the structure of theory (although it may be generates the feeling of understanding).

From this viewpoint the Bell argument [1], [2] led to the recognition that the classical Kolmogorov model of probability [3] which served so well for classical statistical physics has to be rejected and one has to use the quantum model of probability. And (!) nothing more.

Now we move to the issue of similarities and differences in the evolutions of geometry and probability and their impact to physics. We recall that Clifford called Lobachevsky the Copernicus of Geometry due to the revolutionary character of his work. Lobachevsky “discovered” the first non-Euclidean model which nowadays is known as the Lobachevskian or hyperbolic geometry [55]. We also point out that he discussed possible experiments in astrophysics to check whether the geometry of Universe is Euclidean or Lobachevskian.\footnote{In 1826 he noted that it is possible to find experimentally the deviation from $\pi$ of the sum of the angles of cosmic triangles of great size; in a later work he moved to the opposite scale and suggested that his geometry might find application in the intimate sphere of molecular attractions, see, e.g., [56].}

I would call Bell the Copernicus of Probability. In fact, violation of Bell’s inequality implies the impossibility to apply the classical probability model of Kolmogorov (1933) to quantum phenomena. Thus the quantum probability model (based on Born’s rule) gives us an example of non-Kolmogorovian model of probability, similarly to the Lobachevskian model (an example of non-Euclidean model). How-
ever, there are not only similarities, but also differences. Physicists sufficiently quickly understood that a variety of models of geometry can be fruitfully explored. For example, the Lobachevskian geometry is used in special relativity. And the combination of the efforts of mathematicians (first of all, Riemann and Hilbert) and physicists (first of all, Einstein) led to the wide use of non-Euclidean geometries in general relativity. At the same time physicists have no (or very little) interest to the interpretation of quantum mechanics (QM) as a theory based on a special non-Kolmogorovian model of probability.

We shall also discuss coupling of the classical probabilistic model with classical (Boolean) logic. The Kolmogorov model of probability is based on the set-theoretic presentation of the Boolean logic, the crucial point is that events are represented by subsets of a “set of elementary events” Ω and operations for events are classical Boolean operations. In this framework violations of Bell-type inequalities can be interpreted simply as a signal that one cannot use this set-theoretic representation of events for quantum phenomena, instead of it, one has to represent events (detections for quantum systems) by linear subspaces (orthogonal projectors) in a complex Hilbert space.

All previous attempts, see especially [33], [35], to communicate with the physical community about Kolmogorovness/non-Kolmogorovness issue of Bell’s argument were not successful. One of the roots of

\footnote{One can also say “Boolean/non-Boolean”. It is a good place to recall that J. Boole by himself analyzed a possibility that probabilities for some group of random observations cannot be embedded in the Boolean algebra \cite{57}. He even investigated the standard Bell’s situation. There are given three dichotomous random observables. It is possible to perform their pairwise measurements and the corresponding joint probability distributions are given. The natural question arises: Is it always possible to define the joint probability distribution for this triple of observables? Boole presented the inequality which is nowadays (2023) not satisfied even for very simple inputs.}
such miscommunication is that majority of physicists are poorly educated in probability theory (it seems that students in physics do not have a mathematically rigorous course in probability theory, neither in Europe nor in USA). Therefore we start the paper with a brief presentation of the modern axiomatic approach to probability theory [3].

Since classical probability is fundamentally based on classical logic, rejection of the Kolmogorov model of probability automatically implies rejection of the Boolean model of logic. Thus Bell’s “no-go theorem” tells us that for analysis of results of quantum observations it is impossible to use the Boolean laws. In particular, the conjunction-disjunction distributivity is violated. There is again nothing mystical. The Boolean logic is just one special model of thought [57]. It was proposed by mathematician and it spread widely, because of its clearness and matching intuition. However, in cognitive science and psychology it was found that humans often make decisions by violating laws of Boolean logic [59] (and in particular Bell’s type inequalities [60]). Hence, there is nothing surprising that some natural phenomena can be described by novel (non-Boolean) models of logic. Such formal logic was created by von Neumann [61], see also von Neumann and Birkhoff [62]. From this viewpoint, Bell’s “no-go theorem” supports (indirectly via coupling of probability with logic) the interpretation of quantum theory as a departure from classical to special nonclassical (quantum) logic. However, inter-relation of classical and nonclassical logics is not the main topic of this paper; we shall concentrate on probability. For reader’s convenience, we presents basics of quantum logic in the appendix.

The final theme of this paper is about a possibility to “embed” a nonclassical probabilistic model, e.g., quantum probability, into the classical (Kolmogorov) probability model. From the viewpoint of the presented analogy geometry-probability, it would not be surprising if a non-Kolmogorovian model were embedded in some (may be nonunique) way into the Kolmogorovian model. We know that non-Euclidean geometries can be modeled by using surfaces in the Euclidean space. As was shown in [12], [45] even in this aspect probability is similar to geometry: it is really possible to embed, e.g., quantum probability into the classical probability model. The embedding proposed in these papers is based on treatment of quantum probabilities (correlations) as conditional classical probabilities (correlations). In

\[ \text{known as Bell’s inequality as a necessary condition for the existence of this probability.} \]
\[ \text{Later this problem, of the existence of the joint probability distribution, was studied in} \]
\[ \text{very details by Vorob'jev [58], for any group of observables taking the finite number of values.} \]
particular, the EPR-Bohm correlations used in Bell’s “no-go theorem” can be treated in this way. In this paper we present the construction proposed in [12], [45] in the compact form.

It may be useful to remark that after the appearance of the arXiv-preprint [63] I received a few critical comments on the title of this paper, this issue is discussed in section 9.4.

2 Methodology of classical probability theory

The reader may be surprised that by discussing Bell’s type inequalities we do not say practically any word about hidden variables, see only section 9.3. One of the main reasons for this is that classical probability theory does not admire at all hidden parameters, may be opposite to classical statistical physics. For example, how does classical probability theory describe the random experiment on coin tossing? Does it present the space of hidden parameters leading to various results of coin tossing? Not at all! It uses so to say the operational description based solely on the “results of measurements”, head (H) or tail (T). So, the space of “elementary events” for the random experiment with n-times tossing consists of vectors $\omega = (x_1, \ldots, x_n)$, where $x_j = H, T$. We can say that classical probability theory does not like counterfactuals.

As another example, consider the classical Brownian motion. To construct the classical probability space, we do not try to present the space of hidden parameters generating the trajectories of Brownian particles. Such trajectories by themselves are used as elementary events. This is again a kind of the operational description. It involves as much as possible information which can be gained from measurements. Classical probabilist would never discuss results which would appear if the experimental context were changed. Andrei Kolmogorov emphasized that each experimental context induces its own probability space [3], see also [35], [41], [64]. Although Kolmogorov had never discussed explicitly incompatibility of experimental contexts, it seems that such a possibility was completely clear for him [3].

One may be curious whether in classical probability the probabilistic results obtained for incompatible experimental contexts (in other words described by different probability spaces) are used at all. My position was that from the classical probability viewpoint to combine incompatible probabilistic data is meaningless; in particular, to put correlations collected in incompatible experiments in a single inequality. However, at least for the first part of the previous statement my position was weaken after an exciting recent debate with Richard Gill who pointed that in classical statistics data collected for incompatible experimental contexts are used to compare outputs for different contexts, see section 9.1.
We can say that in classical probability theory the problem of hidden variables does not arise, because the operational description based on the measured quantities is simpler and this is the main reason for its use. So, one simply is not interested whether hidden variables exist or not, since there is a possibility to use simple and self-consistent operational description. This position is surprisingly close to Bohr’s position \[65\]. It seems that Bohr would not demonstrate any interest to Bell’s inequality, since Bohr was completely fine with the operational representation of quantum probabilities.

Nevertheless, if one likes it is possible to treat the elementary events of the classical probability model for violation of CHSH-inequality as hidden variables. However, such “hidden variables” cannot be represented solely by using the degrees of freedom of quantum systems, e.g., pairs of entangled photons. They include additional parameters determined by the experimental context, cf. \[\], namely, parameters of random generators for selections of experimental settings; in this case the pairs of angles for polarization beam splitters. Such hidden variables are in some sense nonlocal, since one random generator is located in one lab and another in another lab. However, this is simply completely classical nonlocality of the experimental context which has nothing to do with so-called “quantum nonlocality.” At the same time quantum observables are represented by local classical random variables, each of them depends only on functioning of the random selection generator located in the lab, where this variable is measured.

We also remark that the presented methodology of treatment of random experiments is heavily explored in one special interpretation of quantum mechanics, namely, the consistent histories interpretation, see especially the monograph of Bob Griffiths \[66\]. Our explanation of violation of Bell’s inequality is similar to the one used in the consistent histories approach to quantum mechanics \[66\]: the existence of inconsistent families of histories implies non-Kolmogorovness. However, the adherences of the consistent histories approach are fine with the recognition of non-Kolmogorovness, they did not constructed embeddings of inconsistent histories in one “large Kolmogorov probability space.”

3 Boolean logic and Kolmogorovian probability

In 19th century George Boole wrote the book \[57\], in which he formalized the semantics of classical logic, he also formulated the laws of for details. (Of course, one has to have in mind that statistics and probability may differ essentially in their methodologies for treating of outputs of random experiments.)
probability based on this logic.

One of the most important futures of Boolean logic is that it serves as the basis of the modern probability theory \[3\]: the representation of events by sets, subsets of some set \(\Omega\), so called sample space, or space of elementary events. The system of sets representing events, say \(\mathcal{F}\), matches with the operations of Boolean logics; \(\mathcal{F}\) is so-called \(\sigma\)-algebra of sets. It is closed with respect to the (Boolean) operations of (countable) union, intersection, and complement (or in logical terms “and”, “or”, “no”). Thus the first lesson for a physics student is that by applying any theorem of probability theory, e.g., the law of large numbers, one has to be aware that Boolean logic is in the use. The set-theoretic model of probability was presented by Andrei Nikolaevich Kolmogorov in 1933 \[3\]; it is based on the following two natural (from the Boolean viewpoint) axioms:

- AK1) events are represented as elements of a \(\sigma\)-algebra and operations for events are described by Boolean logic;
- AK2) probability is represented as a probabilistic measure.

We remind that a probabilistic measure \(p\) is a (countably) additive function on a \(\sigma\)-algebra \(\mathcal{F}\):

\[
p(\bigcup_{j=1}^{\infty} A_j) = \sum_{j=1}^{\infty} p(A_j) \quad \text{for} \quad A_j \in \mathcal{F}, A_i \cap A_j = \emptyset, i \neq j,
\]

which is valued in \([0, 1]\) and normalized by \(1, p(\Omega) = 1\).

The triple \(\mathcal{P} = (\Omega, \mathcal{F}, p)\) is called (Kolmogorov) probability space.

We also remind the definition of a random variable as a measurable function, \(a: \Omega \rightarrow \mathbb{R}\). In classical probability theory random variables represent observables.

Here measurability has the following meaning. The set of real numbers \(\mathbb{R}\) is endowed with the Borel \(\sigma\)-algebra \(\mathcal{B}\): the minimal \(\sigma\)-algebra containing all open and closed intervals. Then for any \(A \in \mathcal{B}\) its inverse image \(a^{-1}(A) \in \mathcal{F}\). This gives the possibility to define on \(\mathcal{B}\) the probability distribution of a random variable, \(p_a(A) = p(a^{-1}(A))\).

The mathematical expectation (average) of a random variable \(\xi\) is defined as its integral:

\[
E\xi = \int_{\Omega} \xi(\omega) dp(\omega).
\]

Finally, we point to an exceptional role which is played by conditional probability in the Kolmogorov model. This sort of probabilities is not derived in any way from “usual probability”; conditional probability is per definition given by the Bayes formula:

\[
P(B|C) = \frac{P(B \cap C)}{P(C)}, P(C) > 0.
\]
By Kolmogorov’s interpretation it is the probability of an event $B$ to occur under the condition that an event $C$ has occurred. One can immediately see that this formula is one of strongest exhibitions of the Boolean structure of the model; one cannot even assign conditional probability to an event without using the Boolean operation of intersection.

Thus the second lesson for a physics student is that probability is an axiomatic theory, as, e.g., geometry.

4 CHSH-inequality as a theorem of Kolmogorov probability theory

Let $\mathcal{P} = (\Omega, \mathcal{F}, p)$ be a Kolmogorov probability space. For two random variables $A$ and $B$, we set

$$< A, B > = E(AB) = \int_{\Omega} A(\omega)B(\omega)dp(\omega).$$

**Theorem 1.** (CHSH-inequality) Let $A^{(i)}, B^{(j)}, i, j = 1, 2$, be random variables with values in $[-1,1]$. Then the corresponding combination of correlation

$$S = < A^{(1)}, B^{(1)} > + < A^{(1)}, B^{(2)} > + < A^{(2)}, B^{(1)} > - < A^{(2)}, B^{(2)} >, \tag{1}$$

satisfies the CHSH-inequality:

$$|S| \leq 2. \tag{2}$$

**Proof.** It is easy to show that for any quadruple of random variables valued in $[-1,1]$ the following inequality holds:

$$2 \leq A^{(1)}(\omega)B^{(1)}(\omega) + A^{(1)}(\omega)B^{(2)}(\omega) + A^{(2)}(\omega)B^{(1)}(\omega) - A^{(2)}(\omega)B^{(2)}(\omega) \leq 2.$$

By integrating this inequality with respect to the probability measure $p$ we obtain (2).

5 From Euclid to Lobachevsky and from Kolmogorov to Bell

To understand better the axiomatic nature of the modern set-theoretic model of probability, it is useful to make comparison with another
axiomatic theory - geometry. We can learn a lot from history of development of geometry. Of course, the biggest name in geometry is Euclid. His axiomatics of geometry was considered as the only possible during about two thousand years. It became so common that people started to identify Euclidean model of geometry with physical space. In particular, Immanuel Kant presented deep philosophic arguments \cite{67} that physical space is Euclidean. The Euclidean dogma was rejected as the result of internal mathematical activity, the study of a possibility of derivation of one of axioms from others. This axiom was the famous fifth postulate: through a point not on a line, there is precisely one line parallel to the given one. Nikolay Ivanovich Lobachevsky was the first who demonstrated that this postulate can be replaced with one of its negations. This led him to a new geometric axiomatics, the model which nowadays is known as Lobachevsky geometry (or hyperbolic geometry). Thus the Euclidean geometry started to be treated as just one of possible models of geometry. And this discovery revolutionized first mathematics (with contributions of Gauss, Bolyai, and especially Riemann) and then physics (Minkowski, Einstein, Hilbert).

This geometry lesson tells us that there is no reason to expect that the Kolmogorovian model is the only possible axiomatic model of probability. One can expect that by playing with the Kolmogorovian axioms as Lobachevsky played with the Euclidean ones mathematicians could create non-Kolmogorovian models of probability which may be useful for various applications, in particular in physics. However, in the case of probability the historical pathway of development of geometry was not repeated. Mathematicians did not have two thousand years to rethink the Kolmogorovian axiomatics...

6 Non-Kolmogorovian nature of quantum probability; no-go theorems

New physics, QM, intervened brutally in the mathematical kingdom. The probabilistic structure of QM did not match with classical probability theory based on the set-theoretic approach of Kolmogorov. At the first stage of development of QM this mismatching was not so visible. The first clink came in the form of Born’s rule:

\[ p(x) = |\psi(x)|^2; \quad (1) \]

where \( \psi(x) \) is the wave function and \( p(x) \) is the probability to detect a particle at the point \( x \). Here not the probability, but the wave function
is primary. What is encoded in this presence of complex amplitudes behind probabilities obtained in quantum measurements?

The first who paid attention to peculiarity of the probabilistic structure of QM comparing with the probabilistic structure of classical statistical mechanics was John von Neumann [61]. In particular, he generalized Born’s rule to quantum observables represented by Hermitian operators; for observable represented by an operator with purely discrete spectrum, the probability to obtain the value \( \lambda \) as the result of measurement is given as

\[
p(\lambda) = \| P_\lambda \psi \|^2, \tag{2}
\]

where \( P_\lambda \) is the projector corresponding to the eigenvalue \( \lambda \). (Here \( A = \sum_\lambda \lambda P_\lambda \).)

In his seminal book [61] von Neumann pointed out that, opposite to classical statistical mechanics where randomness of results of measurements is a consequence of variability of physical parameters such as, e.g., the position and momentum of a classical particle, in QM the assumption about the existence of such parameters (for a moment may be still hidden and unapproachable by existing measurement devices) leads to contradiction. This statement presented in [61] is known as von Neumann no-go theorem, theorem about impossibility to go beyond the description of quantum phenomena based on quantum states: it is impossible to construct a theoretical model providing a finer description of those phenomena than given by QM. Thus von Neumann was sure that it is impossible to construct a classical probability measure on the space of some hidden variables which would reproduce probabilities obtained in quantum measurements.

Later this statement was confirmed by other no-go theorems, e.g., of Bell [2].

Bell’s “no-go theorem” says that Bell type inequalities, e.g., the CHSH-inequality [4], which are derived in Kolmogorovian model of probability are violated for correlations calculated in the quantum probability model.

Thus we conclude that the former model of probability has to be rejected as inapplicable to these correlations and that the latter model (which is non-Kolmogorovian) has to be in the use. From the viewpoint of the scientific methodology based on the transition from one

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5This theorem was criticized for unphysical assumptions used by von Neumann to approach his no-go conclusion; especially strong critique was from the side of John Bell [2], the author of another famous no-go theorem; calmer critical arguments were presented by Leslie Ballentine [68]. (We also remark that, although in the modern literature the von Neumann statement is called “theorem”, in the German edition he called it “ansatz”.)
If one try to go beyond this methodology and to interpret the violation of Bell’s inequality not simply as generating the aforementioned transition from one model of probability to another, but to attract additional elements (which do not have formal mathematical meaning) such as (non)locality or (un)realism, then he starts a random walk in jungles of arguments and counter-arguments. And this happened in debates on Bell’s argument.

7 Kolmogorovization of quantum probability

7.1 Embedding: Geometry-probability analogy

As we pointed out in the introduction, the geometry-probability analogy stimulates us to construct embeddings of the quantum model of probability into the classical probability model. On the other hand, two models of probability seems to be so different that such embeddings are impossible. In the classical model events are represented by sets and operations for events are based on classical logic, in the quantum model events are represented by subspaces and operations for them violate the laws of classical logic, they are based on quantum logic of von Neumann-Birkhoff, see the appendix. In the classical model probability is given by a measure on a $\sigma$-algebra of events and in the quantum model it is given by Born’s rule, by the operator trace. However, again the geometry-probability analogy tells us that such differences may be not a problem.

We know that a non-Euclidean geometry having very exotic features (from the Euclidean viewpoint) can be modeled by using surfaces in the Euclidean space. Let us turn again to the Lobachevskian geometry; consider, for example, the so called hyperbolic plane. There are various models of this plane based on the Euclidean plane. Consider, for example, the Poincaré disc model. It is based the interior of a circle (in the Euclidean plane) and lines are represented by arcs of circles that are orthogonal to the boundary circle, plus diameters of the boundary circle. We stress that, although in this representation the hyperbolic plane is given by a domain of the Euclidean plane, the correspondence between the basic entities of the models is nontrivial. Lines of the hyperbolic geometry are not at all straight lines of the Euclidean geometry. Therefore if we hope to embed the quantum model of probability into the classical model of probability, we have
to be prepared that the quantum probability will be represented in some nontrivial way.

As was pointed out in the introduction, besides geometry-probability similarities, there are also differences. Before to proceed to embedding of quantum probability into classical probability, we discuss one specialty of the Kolmogorov model.

### 7.2 Kolmogorov model: Coupling with experiment

The quantum model describes *all possible quantum measurements*. We can always select a Hilbert state space such that all possible quantum observables are represented by Hermitian operators acting in this space (more generally by POVMs). This representation of quantum observables, all in one Hilbert space, is definitely a mathematical idealization. Here even incompatible observables peacefully coexist, although the joint probability distributions are well defined only for groups of *compatible observables*.

Surprisingly the classical model is closer to real experimental situation. Here we do not try to construct some huge Kolmogorov space, i.e., a triple $\mathcal{P} = (\Omega, \mathcal{F}, p)$, such that all possible observables would be represented by random variables on this space, measurable functions $a : \Omega \to \mathbb{R}$. As was emphasized by Kolmogorov in section 2 of his monograph [3], each experimental context $C$ is represented by its own probability space $\mathcal{P}_C = (\Omega_C, \mathcal{F}_C, p_C)$. (This message was practically washed out from modern textbooks in probability.) Of course, sometimes one needs to operate with data corresponding to a few experimental contexts $C_1, ..., C_N$. They are represented by the probability spaces $\mathcal{P}_{C_j} = (\Omega_{C_j}, \mathcal{F}_{C_j}, p_{C_j})$. How can one construct the “unifying probability space”? The procedure is known as randomization. First of all one have to decide how often each experimental context will be in the use and to assign probabilities $q_1, ..., q_N$ to contexts-realizations. This is an important step. One has to take into account the randomness of selection of experimental contexts. This randomness is real and it cannot be ignored. Then by selecting the probability spaces $\mathcal{P}_{C_j}$ with the probabilities $q_j$ a “big probability space” is constructed.

This fundamental coupling of the classical probability model to experiment has to be taken into account if one wants to represent quantum probability in a classical probability space. In the light of the above discussion it is natural to construct the classical probabilistic

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6Although typically the opposite is claimed, namely, that the quantum model is an operational model designed just to describe results of measurements, cf. also with the operational derivations of QM in the spirit of D’ Ariano et al. [69]
representation for any finite group of quantum measurements. We shall consider the most interesting example, the test to violate the CHSH-inequality (2). (Since here we have experimental verification of violation under the assumption of fair sampling.)

8 Classical probability model for quantum correlations violating the CHSH inequality

Now we present embedding of the probabilities (and correlations) for joint measurements of polarizations for pairs of photons given by QM and violating the CHSH-inequality (2) into a Kolmogorov probability space. This construction was used in [12], [45]. Here we present it in a clearer way.

To verify an inequality of this type, one should put in it statistical data collected for four pairs of settings of polarization beam splitters (PBSs):

\[ \theta_{11} = (\theta_1, \theta'_1), \theta_{12} = (\theta_1, \theta'_2), \theta_{21} = (\theta_2, \theta'_1), \theta_{22} = (\theta_2, \theta'_2). \]

Here \( \theta = \theta_1, \theta_2 \) and \( \theta' = \theta'_1, \theta'_2 \) are selections of angles for orientations of respective PBSs. The selection of the angle \( \theta_i \) determines the corresponding polarization observable, \( a_{\theta_i}(\omega) = \pm 1 \). There are two detectors coupled to the PBS with the fixed \( \theta \)-orientation: “up-polarization” detector and “down-polarization” detector. A click of the up-detector assigns to the random variable \( a_{\theta}(\omega) \) the value +1 and a click of the down-detector assigns to it the value -1. In the same way selection of the angle \( \theta' \) determines the corresponding polarization observable, \( b_{\theta'}(\omega) = \pm 1 \). Thus, in fact, the CHSH-test consists of four different experiments corresponding to settings \( \theta_{ij} \). Our aim is to unify these four experiments into a single experiment with random selection of experimental settings. In principle, such unification is used in modern tests of the CHSH-inequality in which settings are selected with the aid of random generators.

For the illustrative purpose, it is more useful to map this experiment with random switching of orientations of two fixed PBSs onto the experiment in which all settings are unified at the “hardware level”, i.e., the experiment with 4 PBSs oriented with the angles \( \theta_1, \theta_2 \) and \( \theta'_1, \theta'_2 \) and each PBS is equipped with its own two detectors, so there are totally 8 detectors.

Such an experimental scheme was used in the pioneer experiment of A. Aspect [70] with one difference: he used single channel PBSs.
We, finally present the corresponding citation of Aspect [71], see also [70], section “Difficulties of an ideal experiment”:

“We have done a step towards such an ideal experiment by using the modified scheme shown on Figure 15. In that scheme, each (single-channel) polarizer is replaced by a setup involving a switching device followed by two polarizers in two different orientations: $a$ and $a'$ on side I, $b$ and $b'$ on side II. The optical switch $C1$ is able to rapidly redirect the incident light either to the polarizer in orientation $a$, or to the polarizer in orientation $a'$. This setup is thus equivalent to a variable polarizer switched between the two orientations $a$ and $a'$. A similar set up is implemented on the other side, and is equivalent to a polarizer switched between the two orientations $b$ and $b'$. In our experiment, the distance $L$ between the two switches was 13 m, and $L/c$ has a value of 43 ns. The switching of the light was effected by home built devices, based on the acousto-optical interaction of the light with an ultrasonic standing wave in water. The incidence angle (Bragg angle) and the acoustic power, were adjusted for a complete switching between the 0th and 1st order of diffraction.”

Figure 15 can be found in [71], p.26. The only difference of our scheme that each of four PBSs has two output channels.
8.1 Experiment taking into account random choice of settings

a). There is a source of entangled photons.

b). There are 4 PBSs and corresponding pairs of detectors for each PBS, totally 8 detectors. PBSs are labeled as \( \ell = 1, 2 \) (at the left-hand side, LHS) and \( \ell = 1, 2 \) (at the right-hand side, RHS).

c). Directly after source there are 2 distribution devices, one at LHS and one at RHS. At each instance of time, \( t = 0, \tau, 2\tau, \ldots \) each device opens the port to only one (of two) optical fibers going to the corresponding two PBSs. For simplicity, we suppose that each pair of ports \( (i,j), (1,1), (1,2), (2,1), (2,2), \) can be opened with equal probabilities \( 7 \):

\[
P(i,j) = 1/4.
\]

Now we introduce the observables measured in this experiment. They are modifications of the polarization observables \( a_{\theta, i}, i = 1, 2 \), and \( b_{\theta, j}, j = 1, 2 \). We start with the “LHS-observables”:

1) \( A^{(i)} = \pm 1, i = \ell = 1, 2 \) if the corresponding (up or down) detector is coupled to \( \ell \)th PBS (at LHS) fires and the \( i \)-th channel is open;

2) \( A^{(i)} = 0 \) if the \( i \)-th channel (at LHS) is blocked.

In the same way we define the “RHS-observables” \( B^{(j)} = 0, \pm 1 \), corresponding to PBSs \( j = \ell = 1, 2 \).

Thus unification of 4 incompatible experiments of the CHSH-test into a single experiment modifies the range of values of polarization observables for each of 4 experiments; the new value, zero, is added to reflect the random choice of experimental settings. We emphasize that this value has no relation to the efficiency of detectors. In this model we assume that detectors have 100% efficiency. The observables take the value zero when the optical fibers going to the corresponding PBSs are blocked.

8.2 Kolmogorov space for incompatible observables

Our aim is to construct a proper Kolmogorov probability space for the experiment which was described in the previous section. In fact,
we shall present a general construction for combining probabilities produced by a few experiments which can be incompatible.

In the CHSH-test we operate with probabilities $p_{ij}(\epsilon, \epsilon')$, $\epsilon, \epsilon' = \pm 1$, – to get the results $a_{\theta_i} = \epsilon, b_{\theta'_j} = \epsilon'$ in the experiment with the fixed pair of orientations $(\theta_i, \theta'_j)$. From QM we know that, for the singlet state, these EPR-Bohm probabilities are given by expressions:

$$ p_{ij}(\epsilon, \epsilon') = \frac{1}{2} \cos^2 \frac{\theta_i - \theta'_j}{2}, p_{ij}(\epsilon, -\epsilon') = \frac{1}{2} \sin^2 \frac{\theta_i - \theta'_j}{2}. $$

However, this special form of probabilities is not important for us. Our construction of unifying Kolmogorov probability space works well for any collection of probabilities $p_{ij} > 0$:

$$ \sum_{\epsilon, \epsilon'} p_{ij}(\epsilon, \epsilon') = 1. $$

Thus we proceed in this general situation, (2). Hence, we consider the experiment described in section 8.1 and producing some collection of probabilities $(p_{ij})$. The special choice of the probabilities (1) will be used only to concrete considerations and to violate the CHSH-inequality, see section 8.5.

Let us now consider the set of points Ω (the space of “elementary events” in Kolmogorov’s terminology):

$$ \Omega = \{(\epsilon_1, 0, \epsilon'_1, 0), (\epsilon_1, 0, 0, \epsilon'_2), (0, \epsilon_2, \epsilon'_1, 0), (0, \epsilon_2, 0, \epsilon'_2)\}, $$

where $\epsilon = \pm 1, \epsilon' = \pm 1$. These points correspond to the following events: e.g., $(\epsilon_1, 0, \epsilon'_1, 0)$ means: at LHS the PBS with $i = 1$ is coupled and the PBS with $i = 2$ is uncoupled and the same situation is at RHS: the PBS with $j = 1$ is coupled and the PBS with $j = 2$ is uncoupled; the result of measurement at LHS after passing the PBS with $i = 1$ is given by $\epsilon_1$ and at RHS by $\epsilon'_1$.

We define the following probability measure on $\Omega$:

$$ P(\epsilon_1, 0, \epsilon'_1, 0) = \frac{1}{4} p_{11}(\epsilon_1, \epsilon'_1), P(\epsilon_1, 0, 0, \epsilon'_2) = \frac{1}{4} p_{12}(\epsilon_1, \epsilon'_2) $$

$$ P(0, \epsilon_2, \epsilon'_1, 0) = \frac{1}{4} p_{21}(\epsilon_2, \epsilon'_1), P(0, \epsilon_2, 0, \epsilon'_2) = \frac{1}{4} p_{22}(\epsilon_2, \epsilon'_2). $$

To match completely with Kolmogorov’s terminology, we have to select a $\sigma$-algebra $\mathcal{F}$ of subsets of $\Omega$ representing events and define the probability measure on $\mathcal{F}$. However, in the case of a finite $\Omega = \{\omega_1, ..., \omega_k\}$ the system of events $\mathcal{F}$ is always chosen as consisting of all subsets of $\Omega$. To define a probability measure on such $\mathcal{F}$, it is sufficient to define it for one-point sets, $(\omega_m) \rightarrow P(\omega_m), \sum_m P(\omega_m) = 1$, and to extend it by additivity: for any subset $O$ of $\Omega, P(O) = \sum_{\omega_m \in O} P(\omega_m)$.
We now define random variables \( A^{(i)}(\omega), B^{(j)}(\omega) : \)

\[
\begin{align*}
A^{(1)}(\epsilon_1, 0, \epsilon'_1, 0) &= A^{(1)}(\epsilon_1, 0, \epsilon'_2, 0) = \epsilon_1, \\
A^{(2)}(0, \epsilon_2, 0, \epsilon'_2) &= A^{(2)}(0, \epsilon_2, 0, \epsilon'_1) = \epsilon_2; \\
B^{(1)}(\epsilon_1, 0, \epsilon'_1, 0) &= B^{(1)}(0, \epsilon_2, 0, \epsilon'_2) = \epsilon'_1, \\
B^{(2)}(0, \epsilon_2, 0, \epsilon'_2) &= B^{(2)}(0, \epsilon_2, 0, \epsilon'_1) = \epsilon'_2.
\end{align*}
\]

and we put these variables equal to zero in other points.

**Remark.** (Locality of the model) We remark that the values of the LHS-variables \( A^{(1)}(\omega), A^{(2)}(\omega) \) depend only on the first two coordinates of \( \omega \) and the values of the RHS-variables \( B^{(1)}(\omega), B^{(2)}(\omega) \) depend only on the last two coordinates of \( \omega \). Thus the value of \( A^{(i)}(\omega) \) does not depend on the values of \( B^{(j)}(\omega) \). They neither depend on the action of the RHS distribution device; for the LHS-variable it is not important which port is open or closed by the RHS distribution device. The RHS-variables \( B^{(j)}(\omega) \) behave in the same way. Thus the random variables under consideration are determined locally.

We find two dimensional probabilities

\[
\begin{align*}
P(\omega \in \Omega : A^{(1)}(\omega) = \epsilon_1, B^{(1)}(\omega) = \epsilon'_1) &= P(\epsilon_1, 0, \epsilon'_1, 0) = \frac{1}{4} p_{11}(\epsilon_1, \epsilon'_1), \\
P(\omega \in \Omega : A^{(2)}(\omega) = \epsilon_2, B^{(2)}(\omega) = \epsilon'_2) &= P(0, \epsilon_2, 0, \epsilon'_2) = \frac{1}{4} p_{22}(\epsilon_2, \epsilon'_2).
\end{align*}
\]

We also consider the random variables which are responsible for selections of pairs of ports to PBSs. For the device at LHS:

\[
\eta_L(\epsilon_1, 0, \epsilon'_1, 0) = \eta_L(\epsilon_1, 0, \epsilon'_1, 0) = 1, \eta_L(0, \epsilon_2, 0, \epsilon'_2) = \eta_L(0, \epsilon_2, 0, \epsilon'_2) = 2.
\]

For the device at RHS:

\[
\eta_R(\epsilon_1, 0, \epsilon'_1, 0) = \eta_R(0, \epsilon_2, 0, \epsilon'_2) = 1, \eta_R(0, \epsilon_2, 0, \epsilon'_2) = \eta_R(0, \epsilon_2, 0, \epsilon'_2) = 2.
\]

### 8.3 Validity of CHSH-inequality for correlations taking into account randomness of selection of experimental settings

Consider the correlations with respect to the probability \( P \) (which takes into account randomness of selections of experimental settings),

\[
< A^{(i)}, B^{(j)} > = \int_{\Omega} A^{(i)}(\omega) B^{(j)}(\omega) dP(\omega).
\]

These are classical correlations for random variables taking values in \([-1,1]\) and for them the CHSH-inequality \(|S| \leq 2\), see \([2]\), holds (Theorem 1). Here \( S \) is defined by \([1]\).
We remark that
\[< A^{(i)} , B^{(j)} > = \frac{1}{4} \sum_{\epsilon_1, \epsilon'_1 = \pm 1} A^{(i)}(\epsilon_1, 0, \epsilon'_1, 0) B^{(j)}(\epsilon_1, 0, \epsilon'_1, 0) p_{11}(\epsilon_1, \epsilon'_1) \]
\[+ \sum_{\epsilon_2, \epsilon'_2 = \pm 1} A^{(i)}(0, \epsilon_2, \epsilon'_2, 0) B^{(j)}(0, \epsilon_2, \epsilon'_2, 0) p_{21}(\epsilon_2, \epsilon'_1) \]
\[+ \sum_{\epsilon_2, \epsilon'_2 = \pm 1} A^{(i)}(0, \epsilon_2, 0, \epsilon'_2) B^{(j)}(0, \epsilon_2, 0, \epsilon'_2) p_{22}(\epsilon_2, \epsilon'_2) \] .

Let us fix some setting, e.g., \(i = 1, j = 1\). Then taking into account the definitions of \(A^{(1)}\) and \(B^{(1)}\) we found that in the above expression only the first summand is nonzero. Thus
\[< A^{(1)} , B^{(1)} > = \frac{1}{4} \sum_{\epsilon_1, \epsilon'_1 = \pm 1} \epsilon_1 \epsilon'_1 p_{11}(\epsilon_1, \epsilon'_1) .\]

Thus classical correlations (taking into account randomness of setting selections) coincide (up to the factor 1/4) with the correlations corresponding to the probability measures \((p_{ij})\) :
\[C_{ij} \equiv \sum_{\epsilon, \epsilon'_j = \pm 1} \epsilon_i \epsilon'_j p_{ij}(\epsilon_i, \epsilon'_j), \]
namely,
\[C_{ij} = 4 < A^{(i)} , B^{(j)} > . \]

In particular, we can select the probabilities \((p_{ij})\) as the EPR-Bohm probabilities for the singlet state, see [1], then \(C_{ij}\) are the EPR-Bohm correlations used to violate the CHSH-inequality.

We stress that the correlations \(C_{ij}\) are larger than classical ones. In the general case, see footnote [9],
\[C_{ij} = \frac{1}{P(i,j)} < A^{(i)} , B^{(j)} > . \]

Thus randomization washes out a part of correlation. However, this washing effect is the probabilistic reality: in any CHSH-test experimentalists have to determine selection of experimental settings.

\[\text{[9]}\] If one ignores this fact, then his/her description of the CHSH-test would not match the real experimental situation; an important part of randomness involved in the experiment would be missed. The specialty of this sort of randomness is that it is not present in
8.4 Quantum correlations as conditional classical correlations

In classical probability theory one uses the notion of *conditional expectation* of a random variable (under the condition that some event occurred). This notion is based on Bayes’ formula defining conditional probability, see [1]. We shall use this notion to define conditional correlation – under the condition that the fixed pair \((i, j)\) of experimental settings is selected.

Let \((\Omega, \mathcal{F}, P)\) be an arbitrary probability space and let \(\Omega_0 \subset \Omega, \Omega_0 \in \mathcal{F}, P(\Omega_0) \neq 0\). We also consider an arbitrary random variable \(\xi : \Omega \to \mathbb{R}\). Then the conditional mathematical expectation (average) of the random variable \(\xi\), conditioned to the event \(\Omega_0\), is defined as follows:

\[
E(\xi|\Omega_0) = \int_{\Omega} \xi(\omega) dP_{\Omega_0}(\omega),
\]

where the conditional probability \(P_{\Omega_0}\) is defined by Bayes’ formula (1):

\[
P_{\Omega_0}(U) \equiv P(U|\Omega_0) = \frac{P(U \cap \Omega_0)}{P(\Omega_0)}.
\]

Let us come back to our unifying probability space. Take in the above definition \(\Omega_0 \equiv \Omega_{ij} = \{\omega \in \Omega : \eta_L(\omega) = i, \eta_R(\omega) = j\}\).

We remark that \(P(\Omega_{ij}) = 1/4\). The latter implies that

\[
E(A^{(i)}B^{(j)}|\eta_L = i, \eta_R = j) = \int_{\Omega} A^{(i)}(\omega)B^{(j)}(\omega) dP_{\Omega_{ij}}(\omega) \quad (9)
\]

\[
= 4 \int_{\Omega_{ij}} A^{(i)}(\omega)B^{(j)}(\omega) dP(\omega) = 4 \int_{\Omega} A^{(i)}(\omega)B^{(j)}(\omega) dP(\omega) = 4 < A^{(i)}, B^{(j)}> 
\]

(since the product \(A^{(i)}(\omega)B^{(j)}(\omega)\) is nonzero only on the set \(\Omega_{ij}\)). Hence, we have:

\[
E(A^{(i)}B^{(j)}|\eta_L = i, \eta_R = j) \equiv 4 < A^{(i)}, B^{(j)}> . \quad (10)
\]

By comparing (7) and (10) we obtain that the correlations \(C_{ij}\) which correspond to the collection of probabilities \((p_{ij})\) coincide with the classical conditional correlations, condition with respect to the choice of experimental setting.

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Thus, we can identify the correlations $C_{ij}$ obtained with the aid of probabilities $(p_{ij})$ with the corresponding conditional correlation for the unifying Kolmogorov space:

$$C_{ij} = E(A^{(i)}B^{(j)}|\eta_L = i, \eta_R = j).$$

(11)

We were able to embed the correlations $C_{ij}$ collected (separately) for in general incompatible experimental settings into the classical probability space. We remark that in principle by themselves these correlations can have the non-Kolmogorovian structure. It can happen that there is no a single classical probability space in which $C_{ij}$ can be considered as unconditional correlations.

In particular, we can select the probabilities $(p_{ij})$ as the EPR-Bohm probabilities for the singlet state, see (1). Then we obtain the representation of the corresponding quantum correlations as classical conditional correlations.

Hence, the quantum correlations are present in our classical probability model for the CHSH-test, but they are not simply correlations: they are conditional conditional; cf. with the embedding of the hyperbolic plane (the Poincaré model) into the Euclidean place, the lines of the hyperbolic plane are not simply Euclidean lines.

### 8.5 Violation of the CHSH-inequality for classical conditional correlations

As we have seen, the CHSH-inequality is satisfied for classical (uncoditional) correlations $[5].$

Is there any reason to expect that it is also satisfied for classical conditional correlations?

The answer is no. In the classical probability model Theorem 1 can be proven only for unconditional correlations. Thus in principle conditional correlations can violate the CHSH-inequality. And this is not surprising from the Kolmogorovian viewpoint. Set

$$S_C = E(A^{(1)}B^{(1)}|\eta_L = 1, \eta_R = 1) + E(A^{(1)}B^{(2)}|\eta_L = 1, \eta_R = 2) + E(A^{(2)}B^{(1)}|\eta_L = 2, \eta_R = 1) - E(A^{(2)}B^{(2)}|\eta_L = 2, \eta_R = 2)$$

(12)

(here “C” is the abbreviation for conditional”). By using the equality $[10]$ we obtain that

$$S_C = 4S.$$  

(13)

Since by general Theorem 1 the quantity $S$ is majorated by 2, the quantity $S_C$ is majorated by 8. But the upper bound 8 is really too
rough and, to obtain a better upper bound, we have to proceed more carefully.

**Theorem 2.** ("Strong CHSH-inequality") Let $A^{(i)}, B^{(j)}, i, j = 1, 2,$ be random variables defined as (3), (4). Then the corresponding combination of correlation $S$, see (1), satisfies the stronger version of the CHSH-inequality:

\[ |S| \leq 1. \quad (14) \]

**Proof.** Consider classical correlation $< A^{(i)}, B^{(j)} > = \int_{\Omega} A^{(i)}(\omega)B^{(j)}(\omega)dP(\omega)$.

We state again that the product $A^{(i)}(\omega)B^{(j)}(\omega)$ is nonzero only on the set $\Omega_{ij}$ and $P(\Omega_{ij}) = 1/4$. We also use the condition that all random variables are valued in [-1,1]. Hence, in fact,

\[ | < A^{(i)}, B^{(j)} > | = \left| \int_{\Omega_{ij}} A^{(i)}(\omega)B^{(j)}(\omega)dP(\omega) \right| \leq P(\Omega_{ij}) = 1/4. \]

Thus each correlation in the combination $S$ of four correlations is bounded by 1/4. Hence, the inequality (14) holds.

We remark that, for this concrete Kolmogorov space, the CHSH-inequality is “trivialized”; in particular, the signs of correlations in $S$ do not play any role. We remark that this theorem is valid for any collection of probabilities $(p_{ij})$, see (2).

**Corollary 1.** (CHSH-inequality for conditional correlations.) Let $A^{(i)}, B^{(j)}, i, j = 1, 2,$ be random variables defined as (3), (4). Then the corresponding combination of conditional correlation $S_C$ satisfies the inequality:

\[ |S_C| \leq 4. \quad (15) \]

Take now probabilities given by (1), the EPR-Bohm probabilities for the singlet state. For them $S_C = 2\sqrt{2}$ and the inequality (15) is not violated.

# Discussion

## 9.1 On the use of data from incompatible experimental contexts in statistics

Our previous analysis of the CHSH-experiment showed that one has to be very careful by operating with statistical data collected in incompatible “sub-experiments” of some “compound experiment.” However, we do not claim that one cannot use such incompatible data to derive some statistical conclusions. As was pointed by Richard Gill (email...
exchange) data collected for incompatible experimental contexts is widely used in statistics, especially in medicine:

“For instance we compare the health of non-smokers who are passively exposed to cigarette smoke (living with a smoking partner), with those who don’t. The conclusion is that a lifetime of exposure roughly doubles your risk of lung cancer and other smoking related negative outcomes: doubled from something tiny, to something also tiny.”

Two experimental contexts, $C_1$: “living with a smoking partner”, and $C_2$: “not”, are incompatible, we are not able to collect statistics by using the same person in both contexts. Nevertheless, we compare the probability distributions $p(\cdot|C_i), i = 1, 2$, to come to conclusions. Is such a practice acceptable from our viewpoint? Definitely yes. We can do everything with these contextual probabilities, but we should not forget that they are contextual, i.e., they are not so to say “absolute Kolmogorovian probabilities.”

In the same way in the CHSH-experiment, we can compare the probabilities $p_{ij}(\epsilon, \epsilon')$ and say that, for one pair of angles (one experimental context) the probability of, e.g., the $(+,+)$-configuration is larger than for another pair (another context).

Even in medical experiments we can apply the construction of the underlying Kolmogorov space which was constructed for the CHSH-data. Only in this space we can work with probabilities by using the laws of classical probability theory, but absolute probabilities are less than contextual-conditional, this is the result of taking into account randomness of selections of the contexts $C_1$ and $C_2$. This should be remembered. And it seems that statisticians remembered this difference between “absolute” and conditional probabilities. (May be I wrong, but then those who treated $p(\cdot|C_i), i = 1, 2$, as “absolute probabilities” might (but need not) get the problems which are similar to the problems generated by violation of Bell’s inequality for quantum systems.)

We also point out that, although in medical statistics the data from incompatible experimental contexts is widely used (as Richard Gill stressed), in another domain (where statistical methods also play the crucial role), namely, cognitive psychology and related studies in game theory and economics, the similar use of such data is considered as leading to ambiguous conclusions and paradoxes [59], [82]-[84], see [59] for contextual probabilistic analysis of the situation. This situation is enlighten well in games of the Prisoner’s Dilemma type. There are the following incompatible contexts: $C_+ (C_-)$: one prisoner knows that another will cooperate (not) with her and $C$: she has no information about the plans of another prisoner [59]. In cognitive
psychology and behavioral economics there was collected a plenty of statistical data for such games in different setups. As was pointed out, for psychologists comparison of such data led to paradoxes. We remark that non-Kolmogorovness of data collected for the contexts $C_+, C_-, C$ is demonstrated via violations of the formula of total probability. This formula, as well as Bell’s inequality, can be considered as a test of non-Kolmogorovness. I proposed to use this test in cognitive science, psychology, and economics in [85]. The first experimental study was devoted to recognition of ambiguous figures, see Conte at al. [87]. Later Jerome Bussemeyer coupled violation of the formula of total probability with data collected in games of the Prisoner’s Dilemma type and with violation of the Savage Sure Thing Principle [86]. The latter axiomatizes the rationality of players, in particular, of agents acting at the market. Generally in cognitive psychology non-Kolmogorovness was coupled to irrationality of players or more generally the presence of some bias. We also remark that in cognitive psychology Bell’s type inequalities are also used as statistical tests of non-Kolmogorovness, see Conte et al. [88], Asano et al. [60].

9.2 Consistent histories

The consistent histories interpretation of quantum mechanics provides a similar viewpoint on violation of Bell’s inequality [66].

This approach (opposite to the majority of other approaches) is heavily based on the use of the Kolmogorov axiomatics of probability theory in quantum physics. The adherents of consistent histories proceed at the mathematical level of rigorousness, they “even” define explicitly the spaces of elementary events serving different quantum experiments [66] (which is very unusual for other approaches).

If a set of histories is consistent, then it is possible to introduce a probability measure on it, i.e., to use the Kolmogorov probability model. The crucial point is that in quantum physics there exist inconsistent families of histories, so quantum measurements cannot be described by a single probability space; quantum probability theory is non-Kolmogorovian. The violation of Bell’s inequality is explained in the same way as in the present paper as a consequence of the non-Kolmogorovian structure of experimental data collected for different experimental contexts. In particular, the nonlocality issue is irrelevant (as well as in the present paper). However, the consistent histories approach stops at this point, i.e., recognition of non-Kolmogorovness. I proceeded further and showed that, in fact, quantum probabilities can be considered as the classical conditional probabilities and that the Kolmogorov model covers even the quantum probabilities.
Finally, we remark that in the consistent histories approach there are no counterfactuals as well as in our approach.

### 9.3 Hidden variables

Our model of embedding of the quantum probabilities in the Kolmogorov model can be considered as an extension of the space of hidden variables to include parameters generating selections of experimental settings. Such a hidden variable depends on the parameters for the selections of angles at both labs. One can say that a hidden variable is nonlocal (although observed quantities are local). However, this nonlocal structure of a hidden variable reflects the nonlocal setup of the experiment, and nothing else.

### 9.4 On the contribution of Bell to foundations of probability

It is clear that not everybody would agree to call Bell as the *Copernicus of Probability*. In particular, after the publication of the preprint [63], I got a few comments such as, e.g.,

> “Regarding the title of your work, “Bell as the Copernicus of Probability”, I think you may be overdoing things. Whereas I very much admire Bell’s work and think he made some very important contributions to quantum foundations, he did not solve its fundamental measurement problem (as is evident from his “Against Measurement”), and so far as I can tell he had not the slightest idea of how to apply standard (Kolmogorov) probability, or some modification of it, to resolve the quantum mysteries. He did not provide the inspiration for my CH ideas, and I think that is also true for my colleagues who were in on its development. He did apply ideas of probability to draw conclusions about (classical) hidden variables, but that seems to rather different from Copernicus’ putting the sun at the center of the solar system, an idea that did work (even if Copernicus’ formulation of it left something to be desired.) I am not sure you want to call Bell the Ptolemy of Probability, though that might be more accurate. While I think it appropriate to honor great men for their achievements, we don’t do so by an excessive hero worship.”

And such a position is understandable, because Bell really did not proceed by using the standard for modern probability theory formalism of the Kolmogorov probability space and he did not point explicitly to violation of Kolmogorovness by data collected in experimental
tests of this inequality. However, the impact of his work to the probabilistic foundations of quantum mechanics and the foundations of theory of probability in general was really great! Without Bell’s works [1], [2] the development of quantum probability would be concentrated on merely mathematical issues without the direct connection with experiment. The key point is that in Bell’s framework the assumption on the existence of a single probability distribution serving to four incompatible measurements and the possibility of its experimental testing enlightened (at least for experts in foundations of quantum mechanics and probability theory) the role of (non-)Kolmogorovness. Of course, the main problem was (as I pointed out in few my recent publications, e.g., [64]) that Bell as the majority of physicists did not get a proper education in probability theory. Therefore it was difficult for him to formulate explicitly the thesis about the non-Kolmogorovian structure of his argument. However, he prepared the soil and the harvest is really good, see, e.g., [33], [35] for reviews. Therefore I compared Bell with Copernicus, although I agree that comparisons with other heroes of science might be more appropriate.

Finally, I remark that in general Bell’s works played the crucial role in stimulation of the presently growing interest to quantum foundations, e.g., [90]. Older experts in quantum foundations told me that in 1960th quantum foundations were not considered as a serious field of research in physics; they were treated merely as philosophical games. As Alain Aspect told me, at his PhD defense he was asked whether he had some job opportunities, and the members of jury were happy that he already got the offer for the permanent position. It was completely clear that with such a topic as the (first!) experimental violation of Bell’s inequality he would never find job.

10 Conclusion

The analogy between the evolutions of geometry and probability and their impacts to physics support the scientific methodology based on construction of mathematical models of physical phenomena. By this methodology the development of physics can be treated as transition from one mathematical model to another. We compare the contribution of Bell to probability with the contribution of Lobachevsky to geometry. Violations of Bell type inequalities imply rejection of the classical model of probability (Kolmogorov, 1933); quantum probability based on Born’s rule is an example of non-Kolmogorovian probability. We propose to use this purely probabilistic interpretation of violations of Bell type inequalities. Such an interpretation is very natural from the viewpoint of analogy with geometry – the use of non-
Euclidean models play the fundamental role in special and general relativity. We also discuss the logical aspects of “Bell’s revolution” in foundations of probability as leading to rejection of Boolean logic and the applications of quantum logic. Again by exploring the analogy between geometry and probability we come to the problem of embedding of a non-Kolmogorov probability (e.g., quantum probability) into the Kolmogorov model. We illustrate this problem by the CHSH-test (and probabilities generated by it) as an example. We show that it is possible to embed the CHSH-probabilities and correlations (which have the intrinsic non-Kolmogorovian structure) into the Kolmogorov model. Our embedding is based on accounting randomness of selection of the experimental settings in the CHSH-test. In such an approach the quantum correlations are interpreted as the classical conditional correlations with respect to selection of the fixed experimental settings. Even in the classical (Kolmogorovian) probability approach such conditional correlations can violate the CHSH-inequality. Thus we explain the violation of the CHSH-inequality by quantum correlations without appealing to nonlocality or rejecting realism.

As in geometry, the same quantum probabilistic structure can be embedded into the Kolmogorov model in various ways. In this paper we discussed only one possibility based on the treatment of quantum probabilities for the CHSH-experiment as conditional probabilities. And we used a special sort of conditioning: on the selection of the angles of PBSs. Other sorts of conditioning can serve for the same purpose. For example, in [2] quantum correlations violating the CHSH-inequality were obtained as the result of conditioning on the simultaneous exceeding of the detection threshold by the two (of the four) components of a classical random field passing two (two channel) PBSs. Conditioning on the simultaneous (more generally coincidence window) detection was used in [3]–[5], but for particle-like classical systems (see [6] for a review on the computer simulation approach to violations of Bell’s inequality; see also [7] for another approach to computer simulation for violation of Bell’s inequality).

We remark that in this paper the present experimental status of violations of Bell’s inequality was not discussed at all. The main reason for this is that our argument does not depend on complete justification of experimental violations of Bell’s inequality, i.e., on closing of various loophole (and first of all the locality and detection efficiency loopholes).

We point out that in [28], [29] Khrennikov and Volovich formulated an analog of the Heisenberg uncertainty principle for these loopholes (or more precisely the Bohr complementarity principle): they cannot be closed jointly in a single experimental test. (Here the “detection
efficiency” is treated as the efficiency of the complete experimental scheme.) The argument used by the authors of [28, 29] were of the heuristic nature. The recent studies on the asymptotic of the space-time dependence of the EPR-Bohm-Bell correlations [78]–[80] support at least implicitly the Khrennikov-Volovich uncertainty principle for Bell’s test. However, as was emphasized, from the viewpoint of the probabilistic opposition all these problems with closing experimental loopholes are not important. Our position would not change even if the final loophole free experimental test demonstrating violation of Bell’s inequality were performed.

Bell’s inequality is just one of various tests of non-Kolmogorovness of quantum probabilities (if they are not treated as conditional probabilities). For example, R. Feynman considered the two slit experiment as a test of nonclassicality of quantum probability [81]. He did not know about the modern mathematical model of probability, Kolmogorov, 1933. Therefore he spoke about Laplacian classical probability. Feynman’s argument in the modern probabilistic fashion was presented, e.g., in [91] – [93], [41].

We recall that violations of Bell’s inequality can be modeled not only with the aid of quantum probability, but even with the aid of other nonclassical probabilistic models. For example, negative probabilities were widely explored, see [41] for a detailed review, see also [95] for a recent study. In [22, 24] I used so called $p$-adic probabilities [41, 94, 96]. (In principle, such models are not less exotic than the quantum probabilistic model based on representations of probabilities with the aid of Born’s rule. The latter became commonly accepted through extensive applications.) Roughly speaking this is merely the matter of test whether to explore the complex probability amplitudes and Born’s rule, or signed, or even $p$-adic valued probabilities for the mathematical representations of quantum correlations. However, it seems that from the formal representational viewpoint the complex Hilbert space representation is the most convenient, in particular because it is linear.

One can also proceed another way around: to represent the classical probabilities in the quantum-like manner, with the aid of complex probability amplitudes or in the abstract framework with the aid of vectors from complex Hilbert space. This is so-called inverse Born’s problem. It was studied in very detail in [35], see also [97, 98].
Appendix: Representation of events by subspaces, quantum logic

Bell’s “theorem” is a consequence of the mathematical structure of QM. While classical probability theory is based on the set-theoretical description, QM is founded on the premise that events are associated with subspaces (or orthogonal projectors on these subspaces) of a vector space, complex Hilbert space. The adoption of subspaces as the basis for predicting events also entails a new logic, the logic of subspaces (projectors) which relaxes some of the axioms of classical Boolean logic (e.g., commutativity and distributivity).

First time this viewpoint that QM is based on a new type of logic, quantum logic, was expressed in the book of von Neumann [61], where he treated projectors corresponding to the eigenvalues of quantum observables (represented by Hermitian operators) as propositions, see also [62]. The explicit formulation of logic of QM as a special quantum logic is based on the lattice of all orthogonal projectors. For reader’s convenience, below we present the mathematical structure of quantum logic.

10.0.1 Logical operations on for projectors

For an orthogonal projector $P$, we set $H_P = P(H)$, its image, and vice versa, for subspace $L$ of $H$, the corresponding orthogonal projector is denoted by the symbol $P_L$.

The set of orthogonal projectors is a lattice with the order structure: $P \leq Q$ iff $H_P \subset H_Q$ or equivalently, for any $\psi \in H$, $\langle \psi | P \psi \rangle \leq \langle \psi | Q \psi \rangle$.

We recall that the lattice of projectors is endowed with operations “and” ($\wedge$) and “or” ($\vee$). For two projectors $P_1, P_2$, the projector $R = P_1 \wedge P_2$ is defined as the projector onto the subspace $H_R = H_{P_1} \cap H_{P_2}$ and the projector $S = P_1 \vee P_2$ is defined as the projector onto the subspace $H_R$ defined as the minimal linear subspace containing the set-theoretic union $H_{P_1} \cup H_{P_2}$ of subspaces $H_{P_1}, H_{P_2}$: this is the space of all linear combinations of vectors belonging these subspaces. The operation of negation is defined as the orthogonal complement.

In the language of subspaces the operation “and” coincides with the usual set-theoretic intersection, but the operations “or” and “not” are nontrivial deformations of the corresponding set-theoretic operations. It is natural to expect that such deformations can induce deviations from classical Boolean logic.

Consider the following simple example. Let $H$ be two dimensional Hilbert space with the orthonormal basis $(e_1, e_2)$ and let $v = (e_1 + e_2)$. 

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Then $P_v \wedge P_{e_1} = 0$ and $P_v \wedge P_{e_2} = 0$, but $P_v \wedge (P_{e_1} \vee P_{e_2}) = P_v$. Hence, for quantum events, in general the distributivity law is violated:

$$P \wedge (P_1 \vee P_2) \neq (P \wedge P_1) \vee (P \wedge P_2) \quad (1)$$

As can be seen from our example, even mutual orthogonality of the events $P_1$ and $P_2$ does not help to save the Boolean laws.

We remark that for commuting projectors quantum logical operations have the Boolean structure. Thus noncommutativity can be considered as the algebraic representation of nonclassicality of quantum logic. In particular, for a single observable with purely discrete spectrum $A = \sum \lambda P_{\lambda}$, the projectors corresponding to different eigenvalues are orthogonal and, hence, they commute. Therefore deviations from classical logic and probability can be found only through analysis of results of a few incompatible measurements.

At first sight the representation of events by projectors/linear subspaces might look as exotic. However, this is simply a prejudice of the common use of the set-theoretic representation of events in the modern classical probability theory. The tradition to represent events by subsets was firmly established by A. N. Kolmogorov only in 1933. We remark that before him the basic classical probabilistic models were not of the set-theoretic nature. For example, the main competitor of the Kolmogorov model, the von Mises frequency model, was based on the notion of a collective, see [4] for formulation of QM on the basis of the von Mises model.

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