Parallel Worldline Numerics: Magnetic Flux Tubes in a Dense Lattice

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If the superconducting nuclear material of a neutron star contains magnetic flux tubes, the magnetic field is likely to vary rapidly on the scales where QED effects are important. In this paper, we construct a cylindrically symmetric toy model of a flux tube lattice in which the non-local influence of QED on neighbouring flux tubes is taken into account. We compute the effective action densities using the worldline numerics technique. The numerics predict a greater effective energy density in the region of the flux tube, but a smaller energy density in the regions between the flux tubes compared to a locally-constant-field approximation. We also compute the interaction energy between a flux tube and its neighbours as the lattice spacing is reduced from infinity. Because our flux tubes exhibit compact support, this energy is entirely non-local and predicted to be zero in local approximations such as the derivative expansion. This Casimir-Polder energy can take positive or negative values depending on the magnetic field profile and the specific definition of the interaction energy.

I. INTRODUCTION

In this article, we will present calculations of the QED effective action in cylindrically symmetric, extended tubes of magnetic flux using the worldline numerics method. These configurations may be called flux tubes, strings, or vortices, depending on the context. Flux tubes are of interest in astrophysics because they describe magnetic structures near stars and planets, cosmic strings [1], and vortices in the superconducting core of neutron stars [2, 3]. Outside of astrophysics, magnetic vortex systems are at the forefront of research in condensed matter physics for the role they play in superconducting systems and in QCD research for their relation to center vortices, a gluonic configuration analogous to magnetic vortices which is believed to be important to quark confinement [4, 5]. Currently, we are most interested in the roles played by magnetic flux tubes in neutron star cores.

Our motivation for discussing flux tubes comes from the fact that superconductivity is predicted in the nuclear matter of neutron stars and that some superconducting materials produce a lattice of flux tubes when placed in an external magnetic field. Superconductivity is a macroscopic quantum state of a fluid of fermions that, most notably, allows for the resistanceless conduction of charge. In 1933, Meissner and Ochsenfeld observed that magnetic fields are repelled from superconducting materials [6]. In 1935, F. and H. London described the Meissner effect in terms of a minimization of the free energy of the superconducting current [7]. Then, in 1957, by studying the superconducting electromagnetic equations of motion in cylindrical coordinates, Abrikosov predicted the possible existence of line defects in superconductors which can carry quantized magnetic flux through the superconducting material [8].

A more complete microscopic description of superconducting materials is given by BCS (Bardeen, Cooper, and Schrieffer) theory [9]. However, detailed discussions are outside the scope of this paper. Instead, we will simply outline the main features of superconductivity that motivate the study of flux tubes in neutron stars. Interested readers may pursue more thorough reviews of superconductivity and superfluidity in neutron stars [2, 3].

In section II of this article, we will briefly review the physics of magnetic flux tubes and of nuclear superconductivity in neutron stars to provide context and motivation. We present our models for solitary flux tubes and dense flux tube lattices in section III as well as the details of the calculations. Our calculations use the worldline numerics method which is reviewed in detail in a separate article [10]. The results of our calculations for scalar and spinor QED are presented in section IV. Our results suggest a small but possibly influential Casimir interaction between flux tubes in a dense lattice that may cause the flux tubes to form bunches. This result and other implications of our calculations are discussed in section V.

II. THEORETICAL BACKGROUND

A. Superconductivity

To see how superconductivity is possible, consider fermions at zero temperature with chemical potential, $\mu$.
The free energy of this system is
\[ \Omega = E - \mu N. \] (1)

For non-interacting fermions, we could add fermions at the Fermi surface without changing the free energy, since the first and second terms would change by the same amount. If, instead, the fermions are interacting, the binding energy between the fermions means that the free energy can be reduced by adding fermions. If there is an attractive interaction between the fermions, then there is a new ground state of the fermion fluid where pairs of fermions are created at the Fermi surface. These pairs of fermions are called Cooper pairs and (since a pair of fermions can be viewed as a boson) they form a Bose condensate.

The particles in a Cooper pair do not form a bound state, but they are interacting through attractive forces, so there is an energy \( \Delta \) associated with separating them. The consequence of this is that there is an energy gap in the dispersion relation for the Cooper pair:
\[ \epsilon_k = \sqrt{(\sqrt{k^2 + m^2} - \mu)^2 + \Delta^2}. \] (2)

This energy gap means that a finite amount of energy is required to excite a single electron state, even near the Fermi surface. An amazing consequence of this is that particles flowing through the Bose condensate cannot scatter inelastically from the phonons because fermions cannot be excited at low energies. For neutral fermions, such as neutrons, this can give rise to superfluidity, characterized by frictionless flows. If the fermions forming the Cooper pairs are charge carriers such as protons or electrons, this effect gives rise to superconductivity, characterized by the resistanceless conduction of charge.

### B. Energy Interpretation of the QED Effective Action

The effective action can be thought of as the additive inverse of the energy. Because this idea is important for understanding the free energy associated with a magnetic flux tube, we will make the relationships between the Hamiltonian expectation value, i.e. the energy, \( E[J^\mu, \bar{\eta}, \eta] \), and the effective action, \( \Gamma[A^\mu_\alpha, \bar{\psi}^0, \psi^0] \) clear and explicit. This section follows section 16.3 of Weinberg [11].

To study the vacuum energy of a static electromagnetic field, we would like to minimize the expectation value of the Hamiltonian,
\[ \langle H \rangle_\Omega = \langle \Omega | H | \Omega \rangle, \] (3)
under the constraint that the vacuum expectation value of the quantum gauge field is the classical field,
\[ \langle \Omega | A_\mu(x) | \Omega \rangle = A^\mu_\mu(x). \] (4)

We assume that the classical fields, \( \bar{\psi}^0 \) and \( \psi^0 \), vanish. Using the method of Lagrange multipliers, the function which we would like to minimize is:
\[ \langle H \rangle_\Omega - \alpha(\Omega) + \int d^3 x \beta^\mu(x) A^\mu_\alpha(x) | \Omega \rangle. \] (5)

So, we find that we must satisfy
\[ H(\Omega) = \alpha(\Omega) + \int d^3 x \beta^\mu(x) A^\mu_\alpha(x) | \Omega \rangle. \] (6)

This can be done if the Lagrange multipliers \( \alpha \) and \( \beta^\mu(x) \) are functionals of \( A^\mu_\alpha(x) \).

Let \( |\Psi\rangle_{J^\mu} \) be normalized eigenvectors of the Hamiltonian in the presence of external source \( J^\mu \). The energy eigenvalue equation is
\[ \left[ H - \int d^3 x \left( J^\mu A^\mu_\alpha + \bar{\eta} \bar{\psi}^0 + \bar{\psi} \eta \right) \right] |\Psi\rangle_{J^\mu} = \frac{E[J^\mu, \bar{\eta}, \eta]}{\mathcal{T}} |\Psi\rangle_{J^\mu}, \] (7)

where \( \mathcal{T} \) is the time extent of the functional integration. If we turn on the external source \( J^\mu \) adiabatically to put the vacuum into the energy eigenstate \( |\Psi\rangle_{J^\mu=0} \), then equation (6) is satisfied by taking
\[ |\Omega\rangle = |\Psi\rangle_{J^\mu=0}, \] (8)
\[ \alpha = E[J^\mu=0, \bar{\eta}=0, \eta=0], \] (9)
and
\[ \beta^\mu(x) = J^\mu_\alpha(x). \] (10)

Under these substitutions, equation (6) provides an expression for the vacuum expectation value for the Hamiltonian in the external field in terms of the external fields and currents.
\[ \langle H \rangle_{A^\mu_\alpha} = \frac{E[J^\mu_\alpha=0, \bar{\eta}=0, \eta=0]}{\mathcal{T}} + \int d^3 x J^\mu_\alpha A^\mu_\alpha \] (11)

The above equation is related simply to the definition of the effective action as a Legendre transformation of the energy
\[ \Gamma[A^\mu_\alpha, \bar{\psi}^0, \psi^0] = -E[J^\mu, \bar{\eta}, \eta] - \int d^3 y (J^\mu(y) A^\mu_\alpha(y) + \bar{\eta}(y) \psi^0(y) + \bar{\psi}^0(y) \eta(y)). \] (12)

So, we arrive at the relationship between the expectation value of the Hamiltonian in the external field and the effective action.
\[ \langle H \rangle_{A^\mu_\alpha} = -\frac{1}{\mathcal{T}} \Gamma[A^\mu_\alpha] \] (13)

Because the energy is extensive, the most interesting quantity for magnetic flux tube configurations is the total energy per unit length.
\[ \langle H \rangle_{A^\mu_\alpha} = \frac{\Gamma[A^\mu_\alpha]}{L_z \mathcal{T}} \] (14)

Thus, we have explicitly confirmed the expectation that the negative effective action is related to the total vacuum energy of a specific field configuration.
C. Flux Tube Free Energy

Many of the basic properties of flux tubes in superconductors can be understood by examining the free energy in Ginzburg-Landau theory. The Ginzburg-Landau free energy is given by

\[ E = \int d^2 x \left\{ \frac{1}{2m} \left| \nabla - i q A(x) \right| \psi(x)^2 \right. \]
\[ \left. - \mu \left| \psi(x) \right|^2 + \frac{a}{2} \left| \psi(x) \right|^4 + \frac{1}{8\pi} (\nabla \times A(x))^2 \right\}, \quad (16) \]

where \( \psi(x) \) is a complex order field such that the density of superconducting fermions is proportional to \( |\psi|^2 \), \( \mu \) is the chemical potential and \( a \) is related to the scattering length. The first term represents the dynamical energy of fluctuations of the order field. The next two terms represent the potential energy of the order field. The final term is the familiar energy of the classical magnetic field.

Incorporating the 1-loop effects from QED, the classical term must be replaced with the energy of the field, \( E_{\text{1-loop}} = -\frac{\Gamma[A_i^0]}{L^2} \).

\[ E = \int d^2 x \left\{ \frac{1}{2m} \left| \nabla - i q A(x) \right| \psi(x)^2 \right. \]
\[ \left. - \mu \left| \psi(x) \right|^2 + \frac{a}{2} \left| \psi(x) \right|^4 \right\} - \frac{\Gamma[A_i^0]}{L^2}. \quad (17) \]

This addition can be viewed as a correction to the classical magnetic field energy in the Ginzburg-Landau free energy. Many important behaviours of superconductors are determined by minimizing the free energy.

The Ginzburg-Landau free energy plays an important role in determining the difference between type-I and type-II superconductors. For example, the free energy of a pair of flux tubes will contain terms characterizing the energy of each flux tube, but also cross terms arising from the interaction between these flux tubes. The sign of this interaction energy determines if the flux tubes will repel each other and form a lattice, or attract each other and collapse. In the classical case, when the fields are separated spatially, there is no interaction energy arising from the last term in equation (16). However, when the 1-loop QED effects are taken into account, the non-local effects arising due to a nearby flux tube make a contribution to the free energy [12].

D. Meissner Effect

The free energy in a superconductor is minimized if the magnetic field obeys its equation of motion, the London equation,

\[ \nabla \times \nabla \times B(x) = -\lambda_L^2 B(x), \quad (18) \]

where \( \lambda_L \) is called the London penetration depth, which is defined as

\[ \lambda_L = \sqrt{\frac{2m_f c^2}{4\pi(2q_f)^2 n_0}} \quad (19) \]

where \( m_f \) and \( q_f \) are the fermion mass and charge, respectively, and \( n_0 \) is the fermion density.

The solution to this equation implies that magnetic fields are repelled exponentially from the surface of the superconductor.

\[ B(x) = e^{-\frac{x}{\lambda_L}} \quad (20) \]

Superconducting regions of a material must repel magnetic fields according to this Meissner effect. However, there can be a large free energy cost to expelling field lines around a superconducting region. For some materials, it is therefore energetically favourable to allow magnetic field lines to penetrate the material through a non-superconducting core. When this occurs, tubes of magnetic flux are permitted in the material.

E. Interaction Energy Between Vortices

The interaction energy between two vortices, and therefore the force between them, can be found by examining the free energy of a two vortex system and subtracting off the energy expected from each individual flux tube [13]. The interaction energy is then given by the remaining cross terms

\[ E_{\text{int}} = \int d^2 x \left( J^\mu_i A^2_\mu - \rho_1 \psi_2 \right) \quad (21) \]

where \( \psi_i \) and \( A^i_\mu \) are the fields induced by sources \( \rho_i \) and \( J^\mu_i \) [14].

In a superconductor, these two terms tend to oppose one another. Two neighbouring vortices have charged fermion currents in opposite directions and are repelled from each other. However, there is also an attractive force due to a free energy associated with defects in the superconductor which can be reduced if two vortices are combined [13]. Therefore, the vortices can be either attractive or repulsive depending on the relative sizes of these terms.

This leads to two different behaviours for superconductors in the presence of magnetic fields. The behaviour is naively determined by the Ginzburg-Landau parameter,

\[ \kappa = \frac{\lambda_L}{\xi}, \quad (22) \]

The coherence length, \( \xi \), is the length scale associated with density variations of superconducting fermions. This length scale is given in terms of \( a \) in the Ginzburg-Landau theory (see equation (16)), or in terms of the
energy gap, $\Delta$, and the velocity of Cooper pairs, $v_f$ in BCS theory.

$$\xi = \sqrt{\frac{\hbar^2}{4m_f|a|}} \quad (23)$$

$$= \frac{2\hbar v_f}{\pi \Delta} \quad (24)$$

When $\kappa > \frac{1}{\sqrt{2}}$, the vortices will repel, and when $\kappa < \frac{1}{\sqrt{2}}$, the vortices will attract. Of course, this analysis has assumed that the free energy has been completely accounted for by the standard analysis. If there are additional considerations in the interaction energy, they will influence the properties of the superconducting material. Previously, the effects of an asymmetry in the scattering length between neutron and proton Cooper pairs [15], and the effects of currents transported along vortices [16] have been suggested as mechanisms for making a type-II superconductor behave like a type-I superconductor. Friction between superconducting and normal fluid domains in a type-I proton superfluid causes a dissipation which is consistent with precession [17]. Thus, a major motivation for studying the QED contribution to the free energy of flux tubes, is to learn how this contribution may affect their interaction energies and the rotational dynamics of neutron stars.

If $\kappa$ is large, meaning the vortices repel, the superconductor can be in a so-called mixed state where the vortices will form a triangular Abrikosov lattice with each vortex carrying a single quanta of flux [8]. The superconductors where this happens allow partial penetration of the magnetic field and are called type-II superconductors. However, this mixed state only occurs for magnetic fields strong enough to form vortices, but not so strong that they destroy superconductivity. If $\kappa$ is small, the vortices will attract each other and coalesce. When this happens, the superconductor does not allow magnetic flux to penetrate the material and is called a type-I superconductor.

In this section, we have argued that without the electromagnetic forces, the vortices in a superconductor will experience an attractive force. Naively, this suggests that superfluids should be type-I since they lack a coupling to the gauge fields. This contrasts with the experimental observation that superfluids are type-II (the evidence for this is reviewed in section II F). The subtlety here is that the nature of the self-interaction of the order field is different in the absence of gauge fields, and is always repulsive for superfluids. We can understand a superconductor as the $q_e \to 0$ limit of a superfluid. In this limit, the London penetration depth, equation (19), becomes infinite and so does the Ginzburg-Landau parameter, equation (22).

$$\lim_{q_e \to 0} \kappa = \lim_{q_e \to 0} \frac{\lambda_L}{\xi} = \lim_{q_e \to 0} \frac{1}{q_e} = \infty \quad (25)$$

Therefore, we predict type-II behaviour from superfluids, consistent with experimental results. The self-interaction of the order field is repulsive in the absence of a gauge field.

F. Nuclear Superconductivity in Neutron Stars

In the dense nuclear matter of a neutron star, it may be possible to have neutron superfluidity, proton superconductivity, and even quark colour superconductivity [2, 3]. The first prediction of neutron-star superfluidity dates back to Migdal in 1959 [18]. The arguments which make superfluidity seem likely are based on the temperatures of neutron stars. A short time after their creation, neutron stars are very cold compared to nuclear energy scales. The temperature in the interior may be a few hundred keV. Studies of nuclear matter show that the transition temperature is $T_c \approx 500$ keV [19]. So, it is expected that the nuclear matter in a neutron star forms condensates of Cooper pairs.

Because of this, some fraction of neutrons in the inner crust of a neutron star are expected to be superfluid. These neutrons make up about a percent of the moment of inertia of the star and are weakly coupled to the nuclear crystal lattice which makes up the remainder of the inner crust. If the neutron vortices in the superfluid component move at nearly the same speed as the nuclear lattice, the vortices can become pinned to the lattice so that the superfluid shares an angular velocity with the crust. This pinning between the fluid and solid crust has observable impacts on the rotational dynamics of the neutron star.
neutron star, for the same reasons that hard-boiling an egg (pinning the yolk to the shell) produces an observable difference in the way it spins.

This picture of a neutron superfluid co-rotating with a solid crust has been used to interpret several types of pulsar timing anomalies. Pulsars are nearly perfect clocks, although they gradually spin-down as they radiate energy. Occasionally, though, pulsars demonstrate deviations from their expected regularity. A glitch is an abrupt increase in the rotation and spin-down rate of a pulsar, followed by a slow relaxation to pre-glitch values over weeks or years. This behaviour is consistent with the neutron superfluid suddenly becoming unpinned from the crust and then dynamically relaxing due to its weak coupling to the crust until it is pinned once again [20].

The neutron star in Cassiopeia A has been observed to be rapidly cooling [21]. The surface temperature has decreased by about 4% over 10 years. This observation is also strong evidence of superfluidity and superconductivity in neutron stars [22, 23]. The observed cooling is too fast to be explained by the observed x-ray emissions and standard neutrino cooling. However, the cooling is readily explained by the emission of neutrinos during the formation of neutron Cooper pairs. Based on such a model, the superfluid transition temperature of neutron star matter is \( \sim 10^9 \) K or \( \sim 90 \) keV.

Further hints regarding superfluidity in neutron stars come from long-term periodic variability in pulsar timing data. For example, variations in PSR B1828-11 were initially interpreted as free precession (or wobble) of the star [24]. If neutron stars can precess, observations could strongly constrain the ratio of the moments of inertia of the crust and the superfluid neutrons. Moreover, the existence of flux tubes (i.e. type-II superconductivity) in the crust is generally incompatible with the slow, large amplitude precession suggested by PSR B1828-11 [24]. The neutron vortices would have to pass through the flux tubes, which should cause a large dissipation of energy and a dampening of the precession which is not observed [25]. However, recent arguments suggest that the timing variability data is not well explained by free precession and that it more likely suggests that the star is switching between two magnetospheric states [26]. Nevertheless, other authors suggest not being premature in throwing out the precession hypothesis without further observations [27].

G. Magnetic Flux Tubes in Neutron Stars

If a magnetic field is able to penetrate the proton superfluid on a microscopic level, it must do so by forming a triangular Abrikosov lattice with a single quanta of flux in each flux tube. So, the density of flux tubes is given simply by the average field strength. If the distance between flux tubes is \( l_f \), the flux in a circular region within \( l_f/2 \) of a flux tube is given by

\[
F = \frac{2\pi F}{e} = 2\pi \int_0^{l_f/2} B \rho d\rho
\]

where we have introduced a dimensionless measure of flux \( F = \frac{\phi}{2\pi F} \). So, the distance between flux tubes is

\[
l_f = \sqrt{\frac{8\pi F}{\kappa eB}}.
\]

If the magnetic field is the quantum critical field strength, \( B_k = \frac{\mu_0^2}{e^2} = 4.4 \times 10^{13} \) Gauss, then the flux tubes are separated by a few Compton electron wavelengths. This is particularly interesting since this is the distance scale associated with non-locality in QED.

The size of a flux tube profile in laboratory superconductors is on the nanometer or micron scale [28]. Because the flux is fixed, the size of the tube profile determines the strength of the magnetic field within the tube. For laboratory superconductors the field strengths are small compared to the quantum critical field, and the field is slowly varying on the scale of the Compton wavelength. In this case, the quantum corrections to the free energy are known to be much smaller than the classical contribution (see section [11]). The size scale for the flux tubes in a superconductor is determined by the London penetration depth. In a neutron star, this quantity is estimated to be a small fraction of a Compton wavelength, much smaller than in laboratory superconductors [25, 29]. In this case, the magnetic field strength at the centre of the tube exceeds the quantum critical field strength and the field varies rapidly, rendering the derivative expansion description of the effective action unreliable.

The Ginzburg-Landau parameter, equation [22], is the ratio of the proton coherence length, \( \xi_p \sim 30 \) fm, and the London penetration depth of a proton superconductor, \( \lambda_p \sim 80 \) fm [25].

\[
\kappa = \frac{\lambda_p}{\xi_p} \sim 2
\]

We therefore expect that the proton Cooper pairs most likely form a type-II superconductor [2]. However, it is possible that physics beyond what is taken into account in the standard picture affects the free energy of a magnetic flux tube. In that case, the interaction between two flux tubes may indeed be attractive in which case the neutron star would be a type-I superconductor.

H. QED Effective Actions of Flux Tubes

Vortices of magnetic flux have very important impacts on the quantum mechanics of electrons. In particular, the phase of the electron’s wavefunction is not unique in such a magnetic field. This is demonstrated by the Aharonov-Bohm effect [30, 31]. The first calculations of the fermion effective energies of these configurations were
for infinitely thin Aharonov-Bohm flux strings \[32\, 33\]. Calculations for thin strings were also performed for cosmic string magnetic configurations \[34\]. For these infinitely-thin string magnetic fields, the energy density is singular for small radii. So, it is not possible to define a total energy per unit length. Another approach was to compute the effective action for a finite-radius flux tube where the magnetic flux was confined entirely to the radius of the tube \[35\]. This approach results in infinite classical energy densities.

Physical flux tube configurations would have a finite radius. The earliest paper to deal with finite radius flux tubes in QED considered the effective action of a step-function profiled flux tube using the Jost function of the related scattering problem \[36\]. One of the conclusions from this research was that the quantum correction to the classical energy was relatively small for any value of the flux tube size, for the entire range of applicability of QED. The techniques from this study were soon generalized to other field profiles including, a delta-function cylindrical shell magnetic field \[37\], and more realistic flux tube configurations such as the Gaussian \[38\] and the Nielsen-Olesen vortex \[39\]. Flux tube vacuum energies were also analyzed extensively using a spectral method \[10\, 12\].

The effective actions of flux tubes have been previously analyzed using worldline numerics \[12\]. This research investigated isolated flux tubes, but also made use of the loop cloud method’s applicability to situations of low symmetry to investigate pairs of interacting vortices. One conclusion from that investigation was that the fermionic effects resulted in an attractive force between vortices with parallel orientations, and a repulsive force between vortices with anti-parallel orientations. Due to the similarity in scope and technique, the latter mentioned research is the closest to the research presented in this paper.

### III. CALCULATIONS

In this section, we will further explore the nature of this phenomenon in QED using a highly parallel implementation of the worldline numerics technique implemented on a heterogeneous CPU and GPU architecture. Specifically, we explore cylindrically symmetric magnetic field profiles for isolated flux tubes and periodic profiles designed to model properties of a triangular lattice. For these calculations, the classical magnetic field configurations are a chosen input to the algorithms and the physical processes that may have created the field configurations do not factor in to the calculations. The worldline numerics algorithm cannot be applied to spinor QED calculations in our model lattice because of the well-known fermion problem of worldline numerics \[13\, 14\]. However, the problem does not affect the scalar QED (Sc-QED) calculations. Therefore, we explore the quantum-corrected energies of isolated flux tubes for both scalar and spinor electrons and use this comparison to speculate about the relationship of our cylindrical lattice model and the spinor QED energies of an Abrikosov lattice of flux tubes that may be found in neutron stars.

#### A. Cylindrical Magnetic Fields

We consider our flux tubes to have cylindrical symmetry so that the field points along the \(\hat{z}\)-direction with a profile that depends on the radial coordinate, \(\rho\). In this case, we have \(\mathbf{B} = B(\rho)\hat{z}\). We can find the magnetic field in terms of the vector potential from \(\mathbf{B} = \nabla \times \mathbf{A}\) and the gauge choice that \(A_0 = A_\phi = A_z = 0\). Then,

\[
B(\rho) = \frac{A_\phi(\rho)}{\rho} + \frac{dA_\phi(\rho)}{d\rho}.
\]

The vector potential, \(A_\phi(\rho)\), can be characterized by a profile function, \(f_\lambda\), and a dimensionless flux parameter, \(\mathcal{F} = \frac{\Phi}{2\pi F}\):

\[
A_\phi(\rho) = \frac{\mathcal{F}}{e\rho} f_\lambda(\rho).
\]

If we choose \(f_\lambda(\rho = 0) = 0\), then the total flux within in a radius of \(L_\rho\) is

\[
\Phi = \frac{2\pi}{e}\mathcal{F} f_\lambda(L_\rho).
\]

In terms of the profile function, the magnetic field is written

\[
B_z(\rho) = \frac{\mathcal{F}}{e\rho} \frac{df_\lambda(\rho)}{d\rho} = \frac{2\mathcal{F}}{e} \frac{df_\lambda(\rho^2)}{d(\rho^2)}.
\]

#### B. Isolated Flux Tubes

To explore isolated magnetic flux tubes, we consider the following profile function:

\[
f_\lambda(\rho^2) = \frac{\rho^2}{\rho^2 + \lambda^2}.
\]

From equation \(32\), this give a magnetic field with a profile

\[
B_z(\rho^2) = \frac{2\mathcal{F}}{e} \frac{\lambda^2}{(\rho^2 + \lambda^2)^2}.
\]

This profile is a smooth flux tube representation that can be evaluated quickly. Moreover, flux tubes with this profile were studied previously using worldline numerics \[13\, 15\].
to compute the scalar part of the Wilson loop. In order to keep each tube as a distinct entity which stays within its assigned region, we assign a smooth function with compact support to represent each tube. This is most easily done with the bump function, \( \Psi(x) \), defined as

\[
\Psi(x) = \begin{cases} 
  e^{-1/(1-x^2)} & \text{for } |x| < 1 \\
  0 & \text{otherwise}
\end{cases}
\]

This function can be viewed as a rescaled Gaussian.

We start by defining the magnetic field outside of the central flux tube. Here, the magnetic field is a constant background field, with the flux ring contributing a bump of width \( \lambda \). The height of the bump must go to zero as the width of the flux tube approaches the distance between flux tubes, and should become infinite as the flux tube width goes to zero:

\[
B_z(\rho > a/2) = B_{bg} + A \left( \frac{a - \lambda}{\lambda} \right) [\Psi(2(\rho - na)/\lambda) - B] .
\]

with \( n \equiv \left[ \frac{\rho + \lambda/2}{a} \right] \).

If we require 6 units of flux in the first outer ring, 12 in the second, and so on (see figure 2), the size of the background field is fixed to \( B_{bg} = \frac{2}{eaq_1} \). The total flux contribution due to the \( \lambda \)-dependent terms must be zero:

\[
\int_{(n-1/2)a}^{(n+1/2)a} \rho A \left( \frac{a - \lambda}{\lambda} \right) [\Psi(2(\rho - na)/\lambda) - B] d\rho = 0
\]

\[
\frac{\lambda}{2} \int_{-1}^{1} \left( \frac{\lambda}{2} n + a \right) \Psi(x) dx - B a^2 n = 0.
\]

This fixes the value of the constant \( B \) to

\[
B = \frac{q_1 \lambda}{2a} .
\]

The numerical constant \( q_1 \) is defined by

\[
q_1 = \int_{-1}^{1} \Psi(x) dx \approx 0.443991.
\]

For a given bump amplitude, \( A \), the magnetic field will become negative if \( \lambda \) becomes small enough. Therefore, we replace \( A \) with its maximum value for which the field is positive if \( \lambda > \lambda_{\text{min}} \) for some choice of minimum flux tube size:

\[
A = \frac{12F}{eaq_1(a - \lambda_{\text{min}})} .
\]

The choice of \( \lambda_{\text{min}} \) sets the tube width at which the field between the flux tubes vanishes. If \( \lambda < \lambda_{\text{min}} \), the magnetic field between the flux tubes will point in the \( -z \)-direction.

Because we are trying to fit a hexagonal peg into a round hole, we must treat the central flux tube differently. For example, the average field inside the central

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FIG. 2: In a type-II superconductor, there are neighbouring flux tubes arranged in a hexagonal Abrikosov lattice which have a non-local impact on the effective action of the central flux tube (left). In our model, we account for the contributions from these neighbouring flux tubes in cylindrical symmetry by including surrounding rings of flux (right).

C. Cylindrical Model of a Flux Tube Lattice

In a neutron star, we do not have isolated flux tubes. The tubes are likely arranged in a dense lattice with the spacing between tubes on the order of the Compton wavelength, with the size of a flux tube a few percent of the Compton wavelength. Specifically, the maximum size of a flux tube is on the order of the coherence length of the superconductor, which for neutron stars has been estimated to be \( \xi \approx 30 \text{ fm} \) \([25]\). This situation can be directly computed in the worldline numerics technique. However, this requires us to integrate over two spatial dimensions instead of one. Moreover, it requires the use of more loops to more precisely probe the spatial configurations of the magnetic field. Despite these problems, it is very interesting to consider a dense flux tube lattice. Unlike the isolated flux tube, the wide-tube limit of the configuration doesn’t have zero field, but an average, uniform background field. If this background field is the size of the critical field, there are interesting quantum effects even in the wide-tube limit.

In this section, we build a cylindrically symmetric toy model of a hexagonal flux tube lattice. We focus on one central flux tube and treat the surrounding six flux tubes as a continuous ring with six units of flux at a distance \( a \) from the central tube. The next ring will contain twelve units of flux at a distance of \( 2a \), etc (see figure 2). Because of this condition, the average strength of the field is fixed, and the field becomes uniform in the wide tube limit instead of going to zero. For small values of \( \lambda \), we will have non-local contributions from the surrounding rings in addition to the local contributions from the central flux tube.

It is difficult to construct a model of this scenario if the flux tubes bleed into one another as they are placed close together. For example, with Gaussian flux tubes or flux tubes with the profile used in the previous section, it is difficult to increase the width of the flux tubes while accounting for the magnetic flux that bleeds out of their regions. Moreover, it is difficult to integrate these schemes to find the profile function \( f_x(\rho) \) which is needed
region for a unit of flux, is different than the average field in the exterior region. Therefore, even when \( \lambda \to a \), the field cannot be quite uniform. We consider the field in the central region to be a constant field with a bump centered at \( \rho = 0 \):

\[
B_z(\rho < \frac{a}{2}) = A_0 \Psi(2\rho/\lambda) + B_0. \tag{42}
\]

The constant term, \( B_0 \), is fixed by requiring continuity with the exterior field at \( \rho = a/2 \):

\[
B_0 = \frac{6 \mathcal{F}}{ea^2} \left(1 - \frac{a - \lambda}{a - \lambda_{\text{min}}} \right). \tag{43}
\]

The bump amplitude, \( A_0 \), is determined by fixing the flux in the central region,

\[
\int_0^{a/2} \rho \left[ A_0 \Psi(2\rho/\lambda) + B_0 \right] d\rho = \frac{\mathcal{F}}{e}; \tag{44}
\]

\[
A_0 \left(\frac{\lambda}{2}\right)^2 \int_0^1 x\Psi(x)dx + \frac{B_0}{2} \left(\frac{a}{2}\right)^2 = \frac{\mathcal{F}}{e} \tag{45}
\]

\[
A_0 = \frac{4 \mathcal{F}}{\lambda^2 e q_2} \left(1 - \frac{3}{4} \left(1 - \frac{a - \lambda}{a - \lambda_{\text{min}}} \right) \right), \tag{46}
\]

where the numerical constant, \( q_2 \), is defined by

\[
q_2 = \int_0^1 x\Psi(x)dx \approx 0.0742478. \tag{47}
\]

Finally, collecting together the important expressions, the cylindrically symmetric flux tube lattice model is

\[
B_z(\rho < \frac{a}{2}) = \frac{4 \mathcal{F}}{\lambda^2 e q_2} \left(1 - \frac{3}{4} \left(\lambda - \lambda_{\text{min}} \right) \right) \Psi(2\rho/\lambda) \\
+ \frac{6 \mathcal{F}}{ea^2} \left(\frac{\lambda - \lambda_{\text{min}}}{a - \lambda_{\text{min}}} \right) + \\
\frac{12 \mathcal{F}}{q_1 e a \lambda} \left(\frac{a - \lambda}{a - \lambda_{\text{min}}} \right) \Psi(2(\rho - na)/\lambda). \tag{48}
\]

The magnetic field profile defined by these equations is shown in figures 3 and 4. The current density required to created fields with this profile is shown in figure 5.

D. The Classical Action

The classical action, \( \Gamma^0 \), is infinite for this configuration. To obtain a finite result, we must look at the action per unit length in the \( z \)-direction, per unit time, and per flux tube region (i.e. for \( \rho < a/2 \)). The action for such a region in cylindrical coordinates is given by

\[
\frac{\Gamma^0}{TL_z} = -\pi \int_0^{a/2} \rho B_z(\rho)^2 d\rho. \tag{50}
\]

After substituting the value of equation (48), the magnetic field in the interior region:

\[
\frac{\Gamma^0}{TL_z} = -\pi \int_0^{a/2} \rho \left[ \frac{4 \mathcal{F}}{\lambda^2 e q_2} \left(1 - \frac{3}{4} \left(\lambda - \lambda_{\text{min}} \right) \right) \Psi(2\rho/\lambda) \\
+ \frac{6 \mathcal{F}}{ea^2} \left(\frac{\lambda - \lambda_{\text{min}}}{a - \lambda_{\text{min}}} \right) \right]^2 d\rho. \tag{51}
\]
After some algebra, we are left with an expression for the classical action,

\[
\frac{\Gamma_0}{TL_z} = \pi \int_{0}^{\alpha/2} \rho B_\perp(\rho)^2 d\rho \\
= -\frac{\pi F^2}{\varepsilon^2 a^2} \left[ 4 \frac{a^2 q_3}{\lambda^2 q_3^2} + \frac{\lambda - \lambda_{\min}}{a - \lambda_{\min}} \right] \left[ 9 \frac{a^2 q_3}{4 \lambda^2 q_3^2} - \frac{9}{2} \frac{\lambda - \lambda_{\min}}{a - \lambda_{\min}} - 6 \frac{a^2 q_3}{\lambda^2 q_3^2} + 12 \right]
\]

where \( q_3 \) is another numerical constant related to integrating the bump function:

\[
q_3 = \int_{0}^{1} x (\Psi(x))^2 dx = \int_{0}^{1} x e^{-\frac{x^2}{2}} dx \approx 0.0187671.
\]

E. Integrating to Find the Potential Function

To compute the Wilson loops, it is generally required to use the vector potential which describes the magnetic field. For us, this means that we must find \( f_\lambda(\rho) \) for our magnetic field model. This could always be done numerically, but can be computationally costly since it is evaluated by every discrete point of every worldline in the ensemble. For computations on the CUDA device, an increase in the complexity of the kernel often means that less memory resources are available per processing thread, limiting the number of threads that can be computed concurrently. It is therefore preferable to find an analytic expression for this function. From equation (52), this function is related to the integral of the magnetic field with respect to \( \rho^2 \). For the inner region, we have

\[
f_\lambda(\rho < a/2) = \frac{e}{2F} \left[ \frac{4F}{\lambda^2 e q_2} \left( 1 - \frac{3}{4} \frac{\lambda - \lambda_{\min}}{a - \lambda_{\min}} \right) \right] \int_{0}^{\rho^2} \Psi \left( \frac{2\rho'}{\lambda} \right) d\rho'^2 + \frac{6F}{e a^2} \left( \frac{\lambda - \lambda_{\min}}{a - \lambda_{\min}} \right) \rho^2.
\]

The integral over the bump function can be computed in terms of the exponential integral \( E_i(x) = \int_{-\infty}^{x} \frac{e^{t}}{t} dt \):

\[
\int_{0}^{\rho^2} \Psi \left( \frac{2\rho'}{\lambda} \right) d\rho'^2 = \left( \lambda - \frac{2}{3} \right) 2q_2 + \left( \frac{4\rho^2}{\lambda^2} - 1 \right) e^{-\frac{1}{4\rho^2}} E_i \left( \frac{1}{1 - \frac{4\rho^2}{\lambda^2}} \right)
\]

for \( \rho < \lambda/2 \) and

\[
\int_{0}^{\rho^2} \Psi \left( \frac{2\rho'}{\lambda} \right) d\rho'^2 = \frac{q_2 \lambda}{2}
\]

for \( \rho \geq \lambda/2 \). Our expression for the profile function in the inner region is

\[
f_\lambda(\rho < a/2) = \left( 1 - \frac{3}{4} \frac{\lambda - \lambda_{\min}}{a - \lambda_{\min}} \right) \Phi(2\rho/\lambda) + 3\rho^2 \frac{\lambda - \lambda_{\min}}{a - \lambda_{\min}}.
\]

with

\[
\Phi(x) = \left\{ \begin{array}{ll}
1 + \frac{1}{2q_2} (x^2 - 1) e^{-\frac{1}{2q_2}} - \frac{1}{2q_2} E_i \left( \frac{1}{1 - \frac{1}{2q_2}} \right) & \text{for } x < 1 \\
1 & \text{for } x \geq 1.
\end{array} \right.
\]

The exterior integral is a bit more challenging, but we can make significant progress and obtain an approximate expression. The first term is a constant given by the value of the profile function at \( \rho = a/2 \). This value is given by the flux in the central flux tube, which we have already chosen to be 1,

\[
f_\lambda(\rho > a/2) = 1 + \frac{e}{F} \int_{a/2}^{\rho} B(p'/a/2) dp'.
\]

We may put the magnetic field, equation (48), into this expression to get

\[
f_\lambda(\rho > a/2) = 1 + \frac{3}{4} \left( \frac{4\rho^2}{a^2} - 1 \right) \left( \frac{\lambda - \lambda_{\min}}{a - \lambda_{\min}} \right) + \frac{12}{q_1 a \lambda} \left( \frac{a - \lambda}{a - \lambda_{\min}} \right) \int_{a/2}^{\rho} \Psi \left( \frac{2(p - na)}{\lambda} \right) dp'.
\]

The remaining integral is over every bump between \( p' = a/2 \) and \( p' = \rho \). We express the result as a term which accounts for each completely integrated bump, and an
integral over the partial bump if $\rho$ is within a bump:

$$f_{\lambda}(\rho > a/2) = 1 + \frac{3}{4} \left( \frac{4\rho^2}{a^2} - 1 \right) \left( \frac{\lambda - \lambda_{\min}}{a - \lambda_{\min}} \right) + 3n(n - 1) \left( \frac{a - \lambda}{a - \lambda_{\min}} \right) + \frac{3\lambda}{q_1a} \left( \frac{a - \lambda}{a - \lambda_{\min}} \right)^2 \chi(2(\rho - na)/\lambda),$$

where

$$\chi(x_0) = \begin{cases} 0 & \text{for } x_0 \leq -1 \\ \int_{-1}^{x_0} xe^{-\frac{1}{1-x^2}} dx & \text{for } |x_0| < 1 \\ \int_{-1}^{x_0} xe^{-\frac{1}{1-x^2}} dx + \frac{2na}{\lambda} \int_{-1}^{x_0} e^{-\frac{1}{1-x^2}} dx & \text{for } x_0 \geq 1 \end{cases}$$

(63)

One of the integrals in $\chi(x_0)$ can be expressed in terms of the exponential integral:

$$\int_{-1}^{x_0} xe^{-\frac{1}{1-x^2}} dx = \frac{1}{2} \left( x_0^2 - 1 \right) e^{-\frac{1}{1-x_0^2}} - E_1 \left( -\frac{1}{1-x_0^2} \right).$$

(64)

The remaining integral cannot be simplified analytically. To use this integral in our numerical model, it must be computed for each discrete point on each loop for each $\rho_{cm}$ and $T$ value. Therefore, it is worthwhile to consider an approximate expression which models the integral, and can be computed faster than performing a numerical integral each time. To find this approximation, we computed the numerical result at 300 values between $x_0 = -1.2$ and $x_0 = 1.2$. The data was then input into Eureqa Formulize, a symbolic regression program which uses genetic algorithms to find analytic representations of arbitrary data [46]. A similar technique has been used to produce approximate analytic solutions of ODEs [47].

The result is a model of the numerical data points with a maximum error of 0.0001 on the range $|x_0| < 1.0$:

$$\int_{-1}^{x_0} e^{-\frac{1}{1-x^2}} dx \approx \frac{0.444}{1 + e^{-3.31g(x_0)}}$$

(65)

where

$$g(x_0) = \frac{5.25x_0^3 - 3.31x_0^2 \sin(x_0) \cos(-0.907x_0^2 - 1.29x_0^8)}{\cos(x_0)}$$

(66)

This function evaluates ten times faster than the numerical integral evaluated at the same level of precision with the GSL Gaussian quadrature library functions and with fewer memory registers. Using this approximation introduces a systematic uncertainty which is small compared to that associated with the discretization of the loop integrals, and considerably smaller than the statistical error bars. Using these expressions for the integrals, we can express $f_{\lambda}(\rho)$ in any region in terms of exponential integrals, exponential, and trigonometric functions with suitable precision. Furthermore, for computation we may express the exponential integral as a continued fraction.

The profile function, $f_{\lambda}(\rho)$, is plotted in figure 6.

FIG. 6: The function $f_{\lambda}(\rho)$ for the above described magnetic field model. The flux conditions require the function to pass through the black dots. A quadratic function corresponds to a uniform field while a staircase function corresponds to delta-function flux tubes. The parameter $\lambda$ smoothly makes the transition between these two extremes. Each of these functions corresponds to a magnetic field profile in figure 3.

IV. RESULTS

A. Comparing Scalar and Fermionic Effective Actions

Because of the fermion problem of worldline numerics [43, 44], the 1-loop effective action for the cylindrical flux tube lattice model could not be computed for the case of spinor QED. The fermion problem does not affect the scalar case. So, we are confined to analyzing this model for ScQED.

For isolated flux tubes, the decay of the magnetic field for large distances protects the calculations from the fermion problem. Therefore, the effective action can be computed for both scalar and spinor QED. In figure 7, we plot the ratio of the spinor to scalar 1-loop correction term for identical magnetic fields, along with the prediction of the LCF approximation for large values of $\lambda$. The LCF approximation in ScQED is given by

$$\Gamma_{\text{scal}}^{(1)} = -\frac{1}{2\pi} \int_0^\infty dT \int_0^\infty \rho_{\text{cm}} d\rho_{\text{cm}} \frac{e^{-m^2T}}{T^3} \left\{ \frac{eB(\rho_{\text{cm}})T}{\sinh(eB(\rho_{\text{cm}})T)} - 1 + \frac{1}{6}(eB(\rho_{\text{cm}})T)^2 \right\}$$

(67)
This can be compared to the spinor QED approximation,

\[
\Gamma_{\text{term}}^{(1)} = \frac{1}{4\pi} \int_{0}^{\infty} dT \int_{0}^{\infty} \rho_{\text{cm}} d\rho_{\text{cm}} \frac{e^{-m^2T}}{T^3} \times \left\{ eB(\rho_{\text{cm}}) T \coth( eB(\rho_{\text{cm}}) T) - 1 - \frac{1}{3}(eB(\rho_{\text{cm}}) T)^2 \right\}. \tag{68}
\]

There are two important notes to make about figure 7. Firstly, the LCF approximation is only a good approximation for \( \lambda \gg \lambda_e \), and isn’t accurate when pushed near its formal validity limits \([13]\). The second note is that the statistics of the points computed with worldline numerics are strongly correlated. So, we conclude that the ScQED 1-loop correction is larger than the QED correction for large \( \lambda \), and this appears to be reversed for small \( \lambda \). However, the large worldline numerics error bars and the invalidity of the LCF approximation near \( \lambda = \lambda_e \) prevent us from seeing how this transition happens. Nevertheless, the main conclusion from this figure is that the scalar 1-loop correction reflects the behaviour of the full QED 1-loop correction to within a factor of about 2 over a wide range of flux tube widths.

Besides using a finite field profile, the fermion problem can also be circumvented by increasing the electron mass. The square of the electron mass sets the scale for the exponential suppression of the large proper time Wilson loops that contribute to the fermion problem. However, if we increase the fermion mass, we are reducing the Compton wavelength of our theory so that the flux tube lattice is no longer dense in terms of the modified Compton wavelength. It is the Compton wavelength of the theory that determines what is meant by ‘dense’. We therefore cannot avoid the fermion problem for dense lattice models by changing the electron mass.

Based on the results presented in figure 7, we conclude that the coupling between the electron’s spin and the magnetic field do not have a dramatic effect on the vacuum energy for isolated flux tubes. Therefore, we expect that ScQED provides a good model of the underlying vacuum physics near these flux tubes, at least at the level our toy model flux tube lattice.

### B. Flux Tube Lattice

The worldline numerics technique computes an effective action density which is then integrated to obtain the effective action. This quantity differs from the Lagrangian in that it is not determined by local operators, but encodes information about the field everywhere through the worldline loops. Like the classical action, the 1-loop term of the effective action per unit length is infinite for a flux tube lattice because the field extends infinitely far. For this reason, we define the effective action to be the action density integrated over the region of a central flux tube \((0 < \rho < a/2)\):

\[
\frac{\Gamma}{T L_z} = -\pi \int_{0}^{a/2} \rho B_\perp(\rho)^2 d\rho - \frac{1}{2\pi} \int_{0}^{a/2} \rho_{\text{cm}} d\rho_{\text{cm}} \int_{0}^{\infty} dT \frac{e^{-m^2T}}{T^3} \times \left\{ (W)_{\rho_{\text{cm}}} - 1 + \frac{1}{6}(eB(\rho_{\text{cm}}) T)^2 \right\}. \tag{69}
\]

The 1-loop term of the effective action density is plotted in figure 8 for the cylindrical flux tube lattice model. The most pronounced feature of this density is that there is a negative contribution from the regions where the field is strong. This contribution has the same sign as the classical term. Therefore, quantum correction tends to reinforce the classical action. A less pronounced feature is that there is a positive contribution arising from the \( \rho_{\text{cm}} > \lambda/2 \) region, in between the lumps of magnetic field which represent the flux tubes. In this region, the local magnetic field is positive, but small.

To interpret this feature, we consider the relative contributions between the Wilson loops and the counter term. These terms are shown in figure 9 for the constant field case. For all values of proper time, \( T \), the counter terms dominate, giving an overall negative sign. In order for the action density to be positive, there must be a greater contribution from the Wilson loop average than from the counter term, since this term tends to give a positive contribution to the action. In our flux tube model, this seems to occur in the regions between the flux tubes. In these regions, the local contribution from the counter term is relatively small because the field is
small. However, the contribution from the Wilson loop average is large because the loop cloud is exploring the nearby regions where the field is much larger. The effect is largest where the field is small, but becomes large in a nearby region. We therefore interpret the positive contributions to the 1-loop correction from these regions as a non-local effect. A similar example of such an effect from fields which vary on scales of the Compton wavelength has been observed previously using the worldline numerics technique [48].

In figure [10] we plot the magnitude of the 1-loop ScQED term of the effective action as a function of the flux tube width. As the flux tubes become smaller, there is an amplification of the 1-loop term, just as there is for the classical action. Similarly, for more closely spaced flux tubes, $a$ is smaller, and the 1-loop term increases in magnitude. The ratio of the 1-loop term to the classical term is plotted in figure [13]. The quantum contribution is greatest for closely spaced, narrow flux tubes, but does not appear to become a significant fraction of the total action.
We observe that the LCF approximation is surprisingly good despite the fact that the magnetic field is varying rapidly on the Compton wavelength scale of the electron. We plot the residuals showing the deviations between the worldline numerics results and the LCF approximation in figure [11]. To understand this, recall the discussion surrounding figure [9]. The Wilson loop term is sensitive to the average magnetic field through the loop ensemble, \( \langle B \rangle_{c} \). In contrast, the counter term is sensitive to the magnetic field at the center of mass of the loop, \( B_{cm} \). Since these terms carry opposite signs, we can understand the difference from the constant field approximation in terms of a competition between these terms. When \( B_{cm} < \langle B \rangle_{c} \), such as when the center of mass is in a local minimum of the field, there is a reduction of the energy relative to the locally constant field case, with a possibility of the quantum term of the energy density becoming negative. However, when \( B_{cm} > \langle B \rangle_{c} \), such as in a local maximum of the field, there is an amplification of the energy relative to the constant field case. We can put a bound on the difference between the mean field through a loop and the field at the center of mass for small loops (i.e. small \( T \)),

\[
|\langle B \rangle - B_{cm}| \lesssim |B''(\rho_{0})|T
\]

where \(|B''(\rho_{0})| \geq B''(\rho)\) for all \( \rho \) in the loop. This expression is proved the same way as determining the error in numerical integration using the midpoint rectangle rule.

If the field varies rapidly about some mean value on the Compton wavelength scale, the various contributions from local minima and local maxima are averaged out and the mean field approximation provided by the LCF method becomes appropriate. A similar argument applies in the fermion case, where the important quantity is the mean magnetic field along the circumference of the loop. This quantity is also well served by a mean-field approximation when integrating over rapidly varying fields.

Another interesting feature of figure [11] is that the LCF approximation appears to describe narrower flux tubes better than wider ones, even when the spacing between the flux tubes is held constant. This effect is likely a result of the compact support given to the flux tube profiles. For narrow tubes, we are guaranteed to have many more center of mass points outside the flux tube than inside, giving a smaller energy contribution than for isolated flux tubes without compact support where the distinction between inside and outside is not as abrupt. This also explains why narrow, closely spaced tubes produce a lower energy than is predicted by the LCF approximation.

This argument does not apply to the smooth isolated flux tubes given by equation [34]. For these flux tubes, the only region where there is a large discrepancy between \( \langle B \rangle_{c} \) and \( B_{cm} \) is near the center of the flux tube. This is a global maximum of the field, and the only maximum of \(|B''(\rho)|\). There are no regions where the average field in the loop ensemble is much stronger than the center of mass magnetic field. So, we expect an amplification of the energy near the flux tube relative to the constant field case. In the flux tubes with compact support, however, there is such a region just outside the flux tube. We can understand the surprisingly close agreement of these results to the LCF approximation in our model in terms of competition between these regions of local minima and maxima of the field (see figure [12]).

Finally, we find that the quantum term remains small compared to the classical action for the range of parameters investigated. This is shown in figure [13] where we plot the ratio of the ScQED term of the action to the classical action. The relative smallness of this correction is consistent with the predictions from homogeneous fields and the derivative expansion, as well as with previous studies on flux tube configurations [30].

C. Interaction Energies

Using this model, we may also investigate the energy associated with interactions between the flux tubes. Since the flux tubes in our model exhibit compact support, the interaction energy is entirely due to a nonlocal interaction between nearby flux tubes. Thus, it contrasts with previous research which has investigated the interaction energies between flux tubes which have overlapping fields [12]. In this case, there is a classical interaction energy (\( \propto B_{1}B_{2} \)) as well as a quantum correction (\( \propto (B_{1} + B_{2})^{4} - B_{1}^{4} - B_{2}^{4} \) in the weak-field limit). Even when these field overlap interactions are not present, there are also nonlocal energies in the vicinity of a flux
tube due to the presence of other flux tubes. For example, the energy from nearby flux tubes can interact with a flux tube through the quantum diffusion of the magnetic field. Because of this phenomenon, we expect an interaction energy in the region of the central flux tubes due to the proximity of neighboring flux tubes, even though no changes are made to the field profile or its derivatives in the region of interest. Since this interaction represents a force due to quantum fluctuations under the influence of external conditions, it is an example of a Casimir force. The Casimir force between two infinitely thin flux tubes in ScQED has previously been found to be attractive [49]. Our model can shed light specifically on this interaction, which is not predicted by local approximations such as the derivative expansion.

Consider a central flux tube with a width $\lambda = 0.5\lambda_e$. When $\lambda_{\text{min}} = \lambda$, the magnetic field outside of the flux tube, $B_{\text{bg}}$, is zero. Then, if the distance between flux tubes, $a$, is set very large, the energy density will be localized to the central flux tube and there will be no non-local interaction energy due to neighbouring flux tubes. We define the interaction energy, $E_{\text{int}}$, as the difference in energy within a distance $a/2$ of the central flux tube between a configuration with a given value of $a$ and a configuration with $a = \infty$. In practice, we use $a = 10,000\lambda_e$ as a suitable stand-in for $a = \infty$:

$$\frac{E_{\text{int}}}{L_z} = \frac{-\Gamma_{\text{scal}}(a)}{L_z T} + \frac{\Gamma_{\text{scal}}(a = 1 \times 10^4\lambda_e)}{L_z T}. \quad (71)$$

With this definition, the interaction energy is the energy associated with lowering the distance between flux rings from infinity. This the the analogue in our model of reducing the lattice spacing of the flux tubes.

One complication of this definition is that there is no clear distinction between energy density which 'belongs' to the central flux tube and energy density which 'belongs' to the neighbouring flux tubes. We continue to use our convention that the total energy for the central flux tube is determined by the integral over the non-local action density in a region within a radius of $a/2$ of the flux tube. As $a$ is taken smaller and smaller, some energy from nearby flux tubes is included within this region, but also, some energy associated with the central flux tube is diffused out of the region. This ambiguity is unavoidable within this model. We can’t numerically compute the energy over all of space and subtract off different contributions, because these energies are infinite.

The interaction energy is plotted in figure 14. In this plot, the error bars are 1-sigma error bars that account for the correlations in the means computed for each group of worldlines,

$$\sigma_{E_{\text{int}}} = \sqrt{\sigma_{E_a}^2 + \sigma_{E_{a=10000}}^2} - 2\text{Cov}(E_a, E_{a=10000}), \quad (72)$$

where Cov$(a, b)$ is the covariance between random variables $a$ and $b$. Recall from figure 8 that there is a positive contribution to the effective action, and therefore a negative contribution to the energy from the region between flux tubes. As we reduce $a$, bringing the flux tubes closer together, two considerations become important. Firstly, we are increasing the average field strength meaning there tends to be more flux through the worldline loops which tends to give a negative contribution to the interaction energy. Secondly, we are reducing the spatial volume over which we integrate the energy since we only integrate $\rho$ from 0 to $a/2$. This effect makes a positive contribution to the interaction energy since we include less and less of the region of negative energy density in our integral.
In figure 14, there appears to be a landscape with both positive and negative interaction energies at different values of $a$. These appear to be consistent with the interplay between positive and negative contributions described in the previous paragraph. This is consistent with the usual expectation of attractive Casimir forces 49. The dominant contribution for the positive energy values is caused by less of the negative energy region contributing as the distance assigned to the flux tube is reduced. However, the point at $a/\lambda_c = 2$ is negative ($-0.3 \pm 0.1\text{MeV}/\lambda_c$) indicating that an attractive interaction from nearby flux tubes is dominant.

At $a/\lambda_c = 0.5$, the flux tubes are positioned right next to one another, and the negative contribution from the increase in the mean field appears to be slightly larger than the positive contribution from the loss of the region of negative energy density from the integral. Beyond this, the flux tubes would overlap each other, which approximately corresponds to the critical background field which destroys superconductivity. Based on the above explanation, it appears that the non-local interaction energy between magnetic fields has a strong dependence on the specific profile of the classical magnetic field that was used. This makes it difficult to predict if it will result in attractive or repulsive forces in a more realistic model of a flux tube lattice.

The energy density of a critical strength magnetic field is about $17\text{GeV}/\lambda_c^2$, so the energy associated with this interaction is relatively small. However, there are no other interactions which affect the energy of moving the flux tubes closer together when they are separated by many coherence lengths. Here, the characteristic distance associated with the interaction, $\lambda_c$, is considerably larger than coherence length or London penetration depth, so the interactions between flux tubes through the order field are heavily suppressed.

V. CONCLUSION

Computing the quantum effective action for magnetic flux tube configurations is a problem that has generated considerable interest and has been explored through a variety of approaches 12 32 42 (see section II H). Partly, this is because it is a relatively simple problem for analyzing non-homogeneous generalizations of the Heisenberg-Euler action and for exploring limitations of techniques such as the derivative expansion. But, this is also a physically important problem because tubes of magnetic flux are very important for the quantum mechanics of electrons due to the Aharonov-Bohm effect, and they appear in a variety of interesting physical scenarios such as stellar astrophysics, cosmic strings, in superconductor vortices, and quark confinement 5.

In the present context, we are concerned with the role that magnetic flux tubes play in the superconducting nuclear material of compact stars. In this scenario, the QED effects are particularly interesting because the magnetic flux tubes, if they exist, are confined to tubes which may be only a few percent of the Compton wavelength, $\lambda_c$, in radius. Specifically, the flux must be confined to within the London penetration depth of the superconducting material, which for neutron stars has been estimated to be $80\text{ fm} = 0.032\lambda_c$ 25. Moreover, the flux tube density is expected to be proportional to the average magnetic field. For a background field near the quantum critical strength, $B_k$, such as in a neutron star, the distance between flux tubes is comparable to a Compton wavelength. This Compton wavelength scale is also the scale at which the non-locality of QED becomes important and at which powerful local techniques like the derivative expansion are no longer appropriate for computing the effective action.

The free energy associated with these flux tubes is a factor in determining whether the nuclear material of a neutron star is a type-I or type-II superconductor. The free energy of a flux tube is determined by looking at the energies associated with the magnetic field, with the creation of a non-superconducting region in the superconductor, and with interactions between the flux tubes. Flux tubes can only form if it is energetically favourable to do so compared to expelling the field due to the Meissner effect. For a lattice of flux tubes, there is also an energy contribution from the presence of neighbouring flux tubes because of the non-local nature of quantum field theory.

The energy of two flux tubes has been previously computed using worldline numerics methods and for flux tubes with aligned fields, the energy is larger than twice the energy of a single flux tube when the flux tubes are closely spaced 12. This result implies that there is a repulsive interaction between the flux tubes due to QED effects, strengthening the likelihood of the type-II scenario in neutron stars. This interaction energy increases as the flux tubes are placed closer together, and can have a similar magnitude as the QED correction to the energy when the flux tubes are closely spaced.

We have developed a cylindrically symmetric magnetic field model which reproduces some of the features of a flux tube lattice: for a given central flux tube, there are nearby regions of large magnetic field which interact non-locally, and the large flux tube size limit goes to a large uniform magnetic field instead of to zero field. We have investigated the 1-loop effects from ScQED in this model using the worldline numerics technique for various combinations of flux tube size, $\lambda$, and flux tube spacing, $a$.

In contrast to isolated flux tubes, we find that there are some regions where the worldline numerics results are greater than the LCF approximation and other regions where they are less than the LCF approximation. This can be understood by thinking of the difference from the LCF approximation as a competition between the local counter term and the Wilson loop averages. For magnetic fields which vary on the Compton wavelength scale about some mean field strength, the LCF approximation provides a poor approximation of the energy density, but
may provide a good approximation to the total energy density of the field due to it being a good mean field theory approximation to the energy density. The appropriateness of the LCF approximation in this case can be understood as an approximate balance between regions where the field is a local maximum and the magnitude of the quantum correction to the action density is larger than in the constant field case, and regions where the field is a local minimum and the quantum corrections predict a smaller action density than the constant field case. This washing out of the field structure due to non-local effects has also been observed in worldline numerics studies of the vacuum polarization tensor [48].

There is a force between nearby magnetic flux tubes due to the quantum diffusion of the energy density. This interaction is non-local and is not predicted by the local derivative expansion. It is an example of a Casimir force (i.e. a force resulting from quantum vacuum fluctuations) and it is computed in a very similar way as the Casimir force between conducting bodies in the worldline numerics technique [50]. The size of the energy densities involved in this force are small even compared to the 1-loop corrections to the energy densities, which are in turn small compared to the classical magnetic energy density.

Although this interaction energy is small, the interactions between flux tubes in a neutron star due to the order field of the superconductor are suppressed because the distance between the tubes is considerably larger than the coherence length and London penetration depth. Therefore, this force is possibly important for the behaviour of flux tubes in neutron star crusts and interiors. For example, in our lattice model, this force could contribute to a bunching of the worldlines, producing regions where flux tubes are separated by $\sim 2 \lambda_c$, and other regions which have no flux tubes. Consequently, this force may have important implications for neutron star physics. However, investigating these implications is outside the scope of this paper.

The nature of this interaction energy is expected to depend on the model of the magnetic field profile for the reasons discussed in section [14]. It is therefore reasonable that forces of either sign, attractive or repulsive, may be possible depending on the particular landscape formed by the magnetic field and the particular definition used of the interaction energy. In a superconductor, the profiles of the magnetic flux tubes are determined by the minimization of the free energy for the interacting system formed by the magnetic and order parameter fields. Therefore, investigating this phenomenon using more realistic models is an interesting direction for future research. In particular, it would be interesting to determine if certain conditions allowed for a non-local interaction between magnetic flux tubes to be experimentally observable.

These conclusions are directly applicable to ScQED. However, we have also investigated the relationship between spinor and scalar QED for isolated flux tubes where the worldline numerics technique can be applied to both cases. We find that both theories have the same qualitative behaviour, and agree within a factor of order unity quantitatively. The arguments and explanations given for the ScQED results have strong parallels in the spinor QED case. The spinor case can also be understood in terms of a competition between the Wilson loop averages and the local counter term. We therefore speculate that the results from this paper will hold in the spinor case, at least qualitatively. However, addressing the fermion problem so that the spinor case can be studied explicitly for flux tube lattices would be valuable progress in this area of research.

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