Neutral perfect fluids of Majumdar-type
in general relativity

Victor Varela
Centro de Física Teórica y Computacional,
Escuela de Física, Facultad de Ciencias,
Universidad Central de Venezuela,
AP 47270, Caracas 1041-A, Venezuela
E-mail: vvarela@fisica.ciens.ucv.ve

Abstract

We consider the extension of the Majumdar-type class of static solutions for the Einstein-Maxwell equations, proposed by Ida to include charged perfect fluid sources. We impose the equation of state $\rho + 3p = 0$ and discuss spherically symmetric solutions for the linear potential equation satisfied by the metric. In this particular case the fluid charge density vanishes and we locate the arising neutral perfect fluid in the intermediate region defined by two thin shells with respective charges $Q$ and $-Q$. With its innermost flat and external (Schwarzschild) asymptotically flat spacetime regions, the resultant condenser-like geometries resemble solutions discussed by Cohen and Cohen in a different context. We explore this relationship and point out an exotic gravitational property of our neutral perfect fluid. We mention possible continuations of this study to embrace non-spherically symmetric situations and higher dimensional spacetimes.

Our previous contribution [1] was devoted to the study of static charged dust solutions of the Einstein-Maxwell equations. The derivation of the solutions was based on three main assumptions: (i) the spacetime metric has the conformastatic form

$$ds^2 = -V^2 dt^2 + \frac{1}{V^2} h_{ij} dx^i dx^j,$$

where Latin indices indicate $1, 2, 3$; (ii) the background three-dimensional metric $h_{ij}$ is Euclidean; (iii) $h_{ij}$ and $V$ depend only on the space-like coordinates $x^1, x^2, x^3$. As a consequence, the corresponding static solutions of the Einstein-Maxwell-charged dust equations satisfied a linear relation between $V$ and the Coulombian potential $A_0$. We will refer to solutions of the field equations with such a linear relationship as being of Majumdar-type. See Guilfoyle’s paper [2] for a discussion of these geometries within the framework of the Weyl class of solutions [3]. More recent papers by Vogt and Letelier [4], Kleber et al. [5], Ivanov [6], and Wickramasuriya [7] deal with interesting physical and mathematical aspects of Majumdar-type solutions.
When the static charged dust source is Majumdar-type, the field equations imply that the charge density $\sigma$ and the energy density $\rho$ satisfy the relation $\sigma = \pm \rho$ and the metric function $\lambda = \frac{1}{V}$ is a solution of the potential equation
\[ \nabla^2 (h) \lambda + 4\pi \rho \lambda^3 = 0, \] (2)
where $\nabla^2 (h)$ denotes the Laplacian operator associated to the background space with flat metric $h_{ij}$. Gürses [8] discussed internal solutions without space-like Killing vectors when Eq. (2) is linear.

The extension of the Majumdar-type class of solutions to include static charged perfect fluid sources is easily achieved using a modified version of the charged dust procedure outlined in [1]. In this case the conformastatic metric (1) is assumed again, but $h_{ij}$ is allowed to have non-vanishing Riemannian curvature.

In the static perfect fluid case with metric (1) and four-velocity $u^\mu = \frac{1}{V} \delta^\mu_0$ (we use Greek indices for the range $0, 1, 2, 3$), the components of the matter energy-momentum tensor
\[ M_{\mu\nu} = \rho u_\mu u_\nu + p (g_{\mu\nu} + u_\mu u_\nu) \] (3)
are given by
\[ M_{00} = \rho V^2, \quad M_{0i} = 0, \quad M_{ij} = p \frac{h_{ij}}{V^2}. \] (4)

As in the charged dust case, the electrostatic forms of the four-potential $A_\mu$ and the four-current $J^\mu$ are given by
\[ A_\mu = A_0 (x^i) \delta^0_\mu, \] (5)
\[ J^\mu = \frac{\sigma (x^i)}{V} \delta^\mu_0. \] (6)

The definitions of the electromagnetic tensor
\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \] (7)
and the Maxwell energy-momentum tensor
\[ E_{\mu\nu} = \frac{1}{4\pi} \left( F_{\mu\alpha} F^\alpha_{\nu} - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \right) \] (8)
lead to the components
\[ E_{00} = \frac{1}{8\pi} V^2 h^{ij} A_0 \partial_i A_0 \partial_j A_0, \] (9)
\[ E_{0i} = 0, \] (10)
\[ E_{ij} = \frac{1}{4\pi} \left( - \frac{1}{V^2} \partial_i A_0 \partial_j A_0 + \frac{1}{2} \frac{1}{V^2} h_{ij} h^{kl} \partial_k A_0 \partial_l A_0 \right), \] (11)
where $h^{ij}$ is the inverse of $h_{ij}$.
Using Eq. (1) we determine the components of the Einstein tensor, given by

\[ G_{00} = -3V^2 h^{ij} \partial_i V \partial_j V + 2V^3 \nabla^2_{(h)} V + \frac{1}{2} V^4 R_{(h)}, \]

\[ G_{0i} = 0, \]

\[ G_{ij} = R_{(h)ij} - \frac{2}{V^2} \partial_i V \partial_j V + h_{ij} \left( \frac{1}{V^2} h^{mk} \partial_m V \partial_k V - \frac{1}{2} R_{(h)} \right), \]

where \( R_{(h)} \) and \( R_{(h)ij} \) respectively denote the Ricci scalar and the Ricci tensor constructed with \( h_{ij} \).

When we combine the above results with the Einstein equations

\[ G_{\mu\nu} = 8\pi (M_{\mu\nu} + E_{\mu\nu}) \]

the following non-trivial equations are obtained:

\[ \frac{1}{2} V^4 R_{(h)} - 3V^2 h^{ij} \partial_i V \partial_j V + 2V^3 \nabla^2_{(h)} V = V^2 h^{ij} \partial_i A_0 \partial_j A_0 + 8\pi \rho V^2, \]

\[ V^2 G_{(h)ij} - 2\partial_i V \partial_j V + h_{ij} h^{kl} \partial_k V \partial_l V = -2\partial_i A_0 \partial_j A_0 + h_{ij} h^{kl} \partial_k A_0 \partial_l A_0 + 8\pi p h_{ij}, \]

where \( G_{(h)ij} \) is the Einstein tensor constructed with \( h_{ij} \).

A great simplification of these equations is achieved if we assume (i) that the three-dimensional background space with metric \( h_{ij} \) is maximally symmetric, and (ii) that \( h_{ij}, p, \) and \( V \) are related by

\[ G_{(h)ij} = 8\pi p \frac{h_{ij}}{V^2}. \]

As a consequence, the extended Majumdar-type solutions satisfy

\[ R_{(h)} = \text{constant}, \]

and the pressure is given by

\[ p = -\frac{R_{(h)} V^2}{48\pi}. \]

Combining Eqs. (17) and (18), and using \( h^i_i = 3 \) we obtain

\[ h^{kl} \partial_k V \partial_l V = h^{kl} \partial_k A_0 \partial_l A_0. \]

Plugging this result and Eq. (18) into Eq.(17) we find that

\[ \partial_i V \partial_j V = \partial_i A_0 \partial_j A_0, \]

which integrates up to

\[ V = \kappa A_0 + C, \quad \kappa^2 = 1, \]

for some constant \( C \). Using Eqs. (16), (20), (23), and the identity

\[ \nabla^2_{(h)} V = \frac{2}{V} h^{ij} \partial_i V \partial_j V - V^2 \nabla^2_{(h)} \left( \frac{1}{V} \right), \]

3
we get the extended potential equation

\[ \nabla^2_{(h)} \lambda + 4\pi (\rho + 3p) \lambda^3 = 0. \]  

(25)

We can use Eq. (23) to eliminate \( A_0 \) in the non-trivial Maxwell equation

\[ \frac{1}{\sqrt{h}} \partial_j \left( \sqrt{h} h^{jk} \frac{\partial A_0}{\sqrt{V^2}} \right) = \frac{4\pi \sigma}{V^3}, \]  

(26)

where \( h \) is the determinant of \( h_{ij} \). The resultant expression is

\[ \nabla^2_{(h)} \lambda + \frac{4\pi \sigma}{\kappa} \lambda^3 = 0. \]  

(27)

Comparing Eqs. (25) and (27) we conclude that \( \sigma, \rho, \) and \( p \) are related by

\[ \sigma = \kappa (\rho + 3p). \]  

(28)

Equation (28) characterizes the class of static charged perfect fluid solutions of Majumdar-type. It was obtained by Ida [9] using a different formalism. Guilfoyle [2] studied properties of this class of solutions. Cho et al. [10] and Yazadjiev [11] have generalized Eqs. (25) and (28) with the inclusion of a dilaton field.

We have seen that extended Majumdar-type solutions must satisfy Eq. (20). Hence non-zero values for constant \( R_h \) cause metric singularities at spacetime points where \( p = 0 \). This fact complicates the construction of bounded, perfect fluid sources surrounded by ordinary vacuum, characterized by \( \rho = p = 0 \). However we easily conceive composite sources where these geometries describe only internal regions, so that metric singularities can be avoided. In this work we shall consider composite models that include Majumdar-type perfect fluids in the case of spherical symmetry. (Composite charged dust spheres have been considered in references [1] and [5].)

Equations (25) and (28) relate perfect fluid properties, charge distribution and analytic structure of the solutions. For example, if \( \rho + p = 0 \) then \( M_{\mu\nu} \) is isotropic and \( \lambda \) is an eigenfunction of \( \nabla^2_{(h)} \) with eigenvalue \( \frac{R_h}{6} \). We are particularly interested in perfect fluids with equation of state

\[ \rho + 3p = 0. \]  

(29)

Majumdar-type sources with this property are necessarily neutral as a consequence of Eq. (28). In this case \( \lambda \) is a solution of the Laplace equation in a background space of constant curvature.

Using spherical coordinates Eq. (1) can be written as

\[ ds^2 = -V^2 dt^2 + \frac{1}{V^2} \frac{1}{(1 + K r^2)^2} \left( dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right), \]  

(30)

where \( K \) is a constant. Also Eq. (20) takes the form

\[ p = -\frac{K V^2}{8\pi}, \]  

(31)

\[ p \]
so that $\rho$ and $K$ always have the same sign.

The dominant energy condition implies that

$$\dot{\rho} \geq 0, \quad -\dot{\rho} \leq \dot{p}_i \leq \dot{\rho}, \quad (32)$$

where $\dot{\rho}$ and $\dot{p}_i$, $i = 1, 2, 3$, are the components of the covariant, total energy-momentum tensor in an orthonormal base. The spherically symmetric expressions for these components have the form

$$\dot{\rho} = \frac{1}{8\pi} \left(1 + \frac{Kr^2}{4}\right)^2 \left(\frac{\partial A_0}{\partial r}\right)^2 + \rho, \quad (33)$$

$$\dot{p}_1 = -\frac{1}{8\pi} \left(1 + \frac{Kr^2}{4}\right)^2 \left(\frac{\partial A_0}{\partial r}\right)^2 + p, \quad (34)$$

$$\dot{p}_2 = \frac{1}{8\pi} \left(1 + \frac{Kr^2}{4}\right)^2 \left(\frac{\partial A_0}{\partial r}\right)^2 + p, \quad (35)$$

$$\dot{p}_3 = \frac{1}{8\pi} \left(1 + \frac{Kr^2}{4}\right)^2 \left(\frac{\partial A_0}{\partial r}\right)^2 + p. \quad (36)$$

Using Eqs. (32)-(36) we conclude that the dominant energy condition is equivalent to

$$\rho \geq 0, \quad -\rho \leq p \leq \rho. \quad (37)$$

We observe that every spherically symmetric Majumdar-type solution describing matter with equation of state (29) will be compatible with Eq. (37) whenever $p < 0$, or equivalently $K > 0$.

In the spherically symmetric case $\lambda = \lambda(r)$, and the general solution of the Laplace equation

$$\nabla^2_{(\text{tr})} \lambda = 0 \quad (38)$$

is given by

$$\lambda = a + b \left(\frac{1}{r} - \frac{Kr}{4}\right), \quad (39)$$

where $a$ and $b$ are real integration constants.

Projecting the electromagnetic tensor onto the unit vector $n = \left(1 + \frac{Kr^2}{4}\right) V \frac{\partial}{\partial r}$ and the four-velocity $u = \frac{1}{V} \frac{\partial}{\partial t}$, and using the linear relationship between $A_0$ and $V$ we obtain the scalar

$$E = F_{\mu\nu} n^\mu u^\nu = \kappa \frac{\partial V}{\partial r} \left(1 + \frac{Kr^2}{4}\right). \quad (40)$$

Additionally, we use Eq. (39) to derive the expression

$$E = \kappa b \frac{V^2}{r^2} \left(1 + \frac{Kr^2}{4}\right)^2. \quad (41)$$
The invariant area of the 2-sphere \( t = \text{constant}, r = \text{constant} \) with unit normal \( n \) is given by
\[
S = \frac{4\pi r^2}{V^2 \left(1 + \frac{K r^2}{4}\right)^2}.
\] (42)

Combining Eqs. (41) and (42) we obtain
\[
ES = 4\pi \kappa b.
\] (43)

In view of the Gauss theorem [12], we derive the expression
\[
b = \kappa Q,
\] (44)
where \( Q \) is the total charge enclosed by the 2-sphere.

We devote the rest of this work to the construction of bounded sources for asymptotically flat geometries using the neutral perfect fluid solution (39). We restrict our attention to \( C^0 \) junctions with both the internal and external metrics represented in isotropic coordinates.

First we consider the \( C^0 \) junction of the internal solution given by Eqs. (30) and (39) to the Schwarzschild solution
\[
ds^2 = -\left(1 - \frac{M}{2r}\right)^2 dt^2 + \left(1 + \frac{M}{2r}\right)^4 \left(dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2\right)
\] (45)
at \( r = r_2 > 0 \). The continuity of the metric coefficients and Eq. (44) imply expressions
\[
K = \frac{M^2}{r_2^4 \left(1 - \frac{M^2}{4r_2^2}\right)}
\] (46)
and
\[
a = \frac{\left(1 + \frac{M}{2r_2}\right)^2 - \kappa Q}{r_2^2 \left(1 - \frac{M^2}{4r_2^2}\right)}.
\] (47)

In view of Eq. (46), \( M < 2r_2 \Rightarrow K > 0 \). In this case Eq. (31) implies \( p < 0 \), and \( \rho \) is positive as a consequence of Eq. (29). We conclude that the dominant energy condition is satisfied by the neutral perfect fluid whenever the Schwarzschild horizon is excluded from the external solution.

The external geometry is determined by the Schwarzschild metric (45), and we expect the hyper-surface \( r = r_2 \) to define a thin shell with charge \(-Q\), so that the composite source has zero net charge. Indeed, the singular character of this hyper-surface shows up when we calculate the integrated stresses
\[
\Sigma_{\alpha\beta} = \lim_{\epsilon \to 0} \int_{r_2 - \epsilon}^{r_2 + \epsilon} T_{\alpha\beta}^\gamma d\hat{r},
\] (48)
where $T^\alpha_\beta$ and $\dd r$ denote the components of the total (mixed) energy-momentum tensor and the infinitesimal, invariant (spatial) distance, respectively. In order to evaluate these integrals we use the Einstein equations, assume the general form of the invariant element in isotropic coordinates

$$ds^2 = -e^{-2\gamma(r)}dt^2 + e^{2\alpha(r)} \left( dr^2 + r^2 d\theta^2 + r^2 \sin \theta^2 d\phi^2 \right),$$

(49)

use the expression $\dd r = e^{\alpha(r)}dr$, and note that only terms in the mixed Einstein tensor with second derivatives of $\gamma(r)$ and $\alpha(r)$ can contribute [13]. Using the internal metric (30) combined with Eqs. (39), (44), (46), (47) for $r < r_2$, the external metric (45) for $r > r_2$, and the corresponding formulae derived in [13] we finally obtain

$$\Sigma_0^\alpha = \frac{8\pi r_3 Q + 4M^3 r_2 + M^4 - 8r_3^2 M}{2\pi r_2 (4r_2^2 + 4r_2 M + M^2)^2},$$

(50)

$$\Sigma_1^1 = 0,$$

(51)

$$\Sigma_2^2 = \Sigma_3^3 = \frac{(8r_2^2 - M^2) M^2}{4\pi r_2 (4r_2^2 + 4r_2 M + M^2)(4r_2^2 - M^2)}.$$

(52)

To complete the construction of the bounded source with $C^0$ metric and zero net charge, we impose the junction of the internal solution (39) to the flat metric

$$ds^2 = -dt^2 + dr^2 + r^2 \left( d\theta^2 + \sin \theta^2 d\phi^2 \right)$$

(53)

at $r = r_1$, with $0 < r_1 < r_2$. We expect the hyper-surface $r = r_1$ to define a thin shell with charge $Q$. In fact, if we evaluate the integrated stresses (48) at $r = r_1$ then we obtain more complicated expressions that impose finite values for $\Sigma_0$, $\Sigma_2$, and $\Sigma_3$ on this hyper-surface.

We summarize the construction details of the source as follows.

The source is composed by (i) the innermost flat region, which is empty in the ordinary sense ($\rho = p = 0$) and is defined for $0 \leq r < r_1$; (ii) the intermediate region $r_1 < r < r_2$, which contains a neutral perfect fluid with equation of state $\rho + 3p = 0$ and $\rho > 0$; (iii) the external region $r > r_2$, with $\rho = p = 0$ and asymptotically flat (Schwarzschild) metric. Hyper-surfaces $r = r_1$ and $r = r_2$ constitute thin shells with charges $Q$ and $-Q$, respectively. The Maxwell tensor vanishes only in the innermost and external regions.

Our composite source resembles the self-gravitating electrical condenser presented by Cohen and Cohen [14]. Two important differences are (i) the use of curvature coordinates by these authors; (ii) the imposition of ordinary vacuum ($\rho = p = 0$) in the intermediate region of their condenser, where only the electric field contributes to the total energy-momentum tensor.

Combining Eqs. (39), (42), (44), (46), (47), and defining $\mu = \frac{M}{2r_2}$, $\beta = \frac{Q}{r_2}$, $\alpha = \frac{r_2}{r_1}$, we derive the following expression for the invariant area

$$\tilde{S} = \frac{\left(1+\mu^2\right)^2 - \kappa \beta \left(1-2\mu^2\right)}{1-\mu^2} x + \kappa \alpha \beta \left(1 - \frac{\mu^2 x^2}{\alpha^2 \left(1-\mu^2\right)}\right)^2 \left(1 + \frac{\mu^2 x^2}{\alpha^2 \left(1-\mu^2\right)}\right)^2,$$

(54)
where $\bar{S} = \frac{\hat{S}}{4\pi r_1^2}$ and $x = \frac{r}{r_1}$. Certainly, $\bar{S}$ has a complicated dependence on parameters $\alpha$, $\beta$, $\kappa$ and $\mu$.

Distinct behaviors of $\bar{S}$ arise when we examine few simple examples. If we choose $\alpha = 2$, $\beta = 1$, $\kappa = 1$ and $\mu = 0.1, 0.2, 0.3, 0.4, 0.5$, then we obtain monotonously increasing $\bar{S}$ functions for $x \in [1, 2]$. Nevertheless $\bar{S}$ has local maxima when $\mu = 0.6, 0.7, 0.8$, and the choice $\mu = 0.9$ leads to a monotonously decreasing $\bar{S}$ in the same interval. If $\kappa = -1$ and the values of $\alpha$ and $\beta$ remain the same, then we obtain monotonously increasing $\bar{S}$ functions for $x \in [1, 2]$ corresponding to the choices $\mu = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8$. In this case $\bar{S}$ has a local maximum when $\mu = 0.9$.

On the other hand, the model defined by $\alpha = 10$, $\beta = 1$, $\kappa = -1$, $\mu = 0.5$ develops a singularity at $x \approx 2.663$.

If we represent the metric of the intermediate region in curvature coordinates $t, R, \theta, \phi$, then we conclude that $S = 4\pi R^2$. Therefore solutions with monotonously increasing invariant area in the interval $x \in [1, \alpha]$ have the simplest interpretation in terms of the radial marker $R$, and are closely related to the Cohen and Cohen solution.

We have filled the intermediate region of our composite source with an electric field and a neutral perfect fluid. In contrast, the intermediate region of the Cohen and Cohen condenser contains only an electric field. Surprisingly, this difference diminishes when we have a closer look at the gravitational properties of perfect fluids with equation of state $\rho + 3p = 0$.

As discussed by Ponce de León [15, 16], the use of curvature coordinates entails an expression for the gravitational mass $M_G(R)$ inside a sphere of radius $R$ (the Tolman-Wittaker formula). (Gron [17] considered the discontinuities in $M_G(R)$ caused by thin shell singularities.) The crucial point is that $\rho + 3p$ determines the contribution of the matter to $M_G(R)$ in the case of isotropic pressures. The analysis of these authors suggest that our neutral perfect fluid has no direct effect on local gravitational interactions. (Matter with the odd equation of state $\rho + 3p = 0$ was considered by Gott and Rees in the context of cosmic strings [18]. Wesson related $\rho + 3p = 0$ to the existence of zero-point fields [19]. Dadhich and Narayan [20] regarded various topological defects in connection with this equation of state.)

Equations (31) and (46) show the effect of mass $M$ on pressure $p$, and we presume that our fluid affects the contribution of the charged thin shells to the gravitational mass of the system (via boundary conditions), in analogy with the induction of surface charge on the plates of a condenser by a neutral, polarized dielectric substance. Systematic use of curvature coordinates should lead to an expression for $M$ as a function of $Q$ and the condenser’s capacity, modified by the exotic matter presence in the intermediate region (see Eq. (30) in [14]).

We finally remark that both the neutral fluid condition and the simplification that leads to the Laplace equation for $\lambda$ are consequences of $\rho + 3p = 0$ even in the absence of space-like Killing vectors. Notably, Eqs. (21) and (24) in [10] suggest that the equation of state $\rho + \frac{N}{N-2}p = 0$ plays an analogous role in $N + 1$ dimensional spacetimes.
Acknowledgements

The author thanks his colleagues P. J. Arias, A. Bellorín, and L. Leal for timely computational support.

References

[1] Varela, V. (2003). Gen. Rel. Grav. 35, 1815.
[2] Guilfoyle, B.S. (1999). Gen. Rel. Grav. 31, 1645.
[3] Weyl, H. (1917). Annalen. Phys. 54, 117.
[4] Vogt, D. and Letelier, P.S. (2004). Class. Quant. Grav. 21, 3369.
[5] Kleber, A., Lemos, J.P.S., and Zanchin, V.T. Preprint gr-qc/0406053.
[6] Ivanov, B.V. Preprint gr-qc/0407048.
[7] Wickramasuriya, S.B.P. Preprint gr-qc/0410003.
[8] Gürses, M. (1998). Phys. Rev. D 58, 044001.
[9] Ida, D. (2000). Prog. Theor. Phys. 103, 573.
[10] Cho, Y., Degura, Y., and Shiraishi, K. (2000). Phys. Rev. D 62, 084038.
[11] Yazadjiev, S.S. Preprint gr-qc/0411132.
[12] Wald, R.M. (1984). General Relativity (University of Chicago Press, Chicago, U.S.A.).
[13] Lightman, A.P., Press, W.H., Price, R.H., and Teukolsky, S.A. (1975). Problem Book in Relativity and Gravitation (Princeton University Press, Princeton, U.S.A.).
[14] Cohen, J.M., and Cohen, M.D. (1969). Nuovo Cim. 60, 241.
[15] Ponce de León, J. (1993). Gen. Rel. Grav. 25, 1123.
[16] Ponce de León, J. (2004). Gen. Rel. Grav. 36, 1453.
[17] Grøn, Ø. (1985). Phys. Rev. D 31, 2129.
[18] Gott, J.R. and Rees, M.J. (1987). Mon. Not. Roy. Astron. Soc. 227, 453.
[19] Wesson, P.S. (1991). Astrophys. J. 378, 466.
[20] Dadhich, N. and Narayan, K. (1998). Gen. Rel. Grav. 30, 1133.