THE ENDOSCOPIC FUNDAMENTAL LEMMA FOR UNITARY
FRIEDBERG-JACQUET PERIODS

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Abstract. We prove the endoscopic fundamental lemma for the Lie algebra of the
symmetric space $U(2n)/U(n) \times U(n)$, where $U(n)$ denotes a unitary group of rank $n$.
This is the first major step in the stabilization of the relative trace formula associated
to the $U(n) \times U(n)$-periods of automorphic forms on $U(2n)$.

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1. Introduction

In this paper, we prove the endoscopic fundamental lemma for the Lie algebra of
the symmetric space $U(2n)/U(n) \times U(n)$, stated below as Theorem 1.2. Conjectured in
[Les19a], this is the first example of such a fundamental lemma and is the first major step
in the stabilization of the relative trace formula associated to the $U(n) \times U(n)$-periods
of automorphic forms on $U(2n)$. Let us now explain the context and motivation.

1.1. Global motivation. Let $E/F$ be a quadratic extension of number fields, $\mathbb{A}_E$ and
$\mathbb{A}_F$ the associated rings of adeles. Let $W_1$ and $W_2$ be two $n$ dimensional Hermitian
spaces over $E$. The direct sum $W_1 \oplus W_2$ is also a Hermitian space and we have the
embedding of unitary groups

$$U(W_1) \times U(W_2) \hookrightarrow U(W_1 \oplus W_2)$$.
Let \( \pi \) be an irreducible cuspidal automorphic representation of \( U(W_1 \oplus W_2)(\mathbb{A}_F) \). Then \( \pi \) is said to be distinguished by the subgroup \( U(W_1) \times U(W_2) \) if the period integral

\[
\int_{[U(W_1) \times U(W_2)]} \varphi(h)\,dh
\]

is not equal to zero for some vector \( \varphi \) in the \( \pi \)-isotypic subspace of automorphic forms on \( U(W_1 \oplus W_2)(\mathbb{A}_F) \). Here, \([H] = H(F) \backslash H(\mathbb{A}_F)\) for any \( F \)-group \( H \). These periods are a unitary version of the “linear periods” first studied by Friedberg and Jacquet [FJ93], who showed that a cuspidal automorphic representation \( \Pi \) of \( GL_n(\mathbb{A}_F) \) is distinguished by \( GL_n(\mathbb{A}_F) \times GL_n(\mathbb{A}_F) \) if and only if the central \( L \)-value \( L(\frac{1}{2}, \Pi) \) is non-zero and the exterior square \( L \)-function \( L(s, \Pi, \wedge^2) \) has a pole at \( s = 1 \). While the literature has stuck with the name linear periods for integrals against the subgroup

\[
GL_n(\mathbb{A}_F) \times GL_n(\mathbb{A}_F) \hookrightarrow GL_{2n}(\mathbb{A}_F),
\]

the name “unitary linear periods” for the integrals (1) is clearly problematic. As a result, we refer to these periods as unitary Friedberg-Jacquet periods.

Recently, these periods have appeared in the literature in several ways (for example, [IP18], [PWZ19], and indirectly in [LZ19]) with several interesting applications to arithmetic and relative functoriality. As a simple example, we have the following conjecture, which is a special case of conjectures of Getz and Wambach [GW14].

**Conjecture 1.1.** Let \( U(W_1 \oplus W_2)(\mathbb{A}_E) \) be quasi-split and let \( \pi \) be a generic cuspidal automorphic representation. Let \( \Pi = BC(\pi) \) be the base change of \( \Pi \) to \( GL_{2n}(\mathbb{A}_E) \). The following are equivalent:

1. The exterior square \( L \)-function \( L(s, \Pi, \wedge^2) \) has a pole at \( s = 1 \) and the central \( L \)-value \( L(\frac{1}{2}, \Pi) \) is non-zero,
2. There is a cuspidal automorphic representation \( \pi' \) in the same Arthur packet of \( \pi \) that is distinguished by \( U(W_1)(\mathbb{A}_F) \times U(W_2)(\mathbb{A}_F) \).

Theorem 1.5 of [PWZ19] proves one direction of this conjecture under the assumption that \( \pi \) is discrete series at a split place of \( F \). In [Les19], we outlined a comparison of relative trace formulas conjectured by Wei Zhang (first suggested in a less precise form in [GW14]) which would prove this conjecture if established.

The crucial observation is that, unlike other relative trace formulas in the literature, the relative trace formula associated to the unitary Friedberg-Jacquet periods on \( U(W_1 \oplus W_2)(\mathbb{A}_F) \) is not stable: when we consider the action of \( U(W_1) \times U(W_2) \) on the symmetric variety \( U(W_1 \oplus W_2)/U(W_1) \times U(W_2) \), invariant polynomials distinguish only geometric orbits. Appropriately, stability issues also arise in the local spectral theory of these periods [BPW]. We therefore must “stabilize” the geometric side of the relative trace formula if we hope to prove global results like Conjecture 1.1.

### 1.2. Local theory of endoscopy and the main result

Now suppose that \( E/F \) is a quadratic extension of non-archimedean local fields of characteristic zero and set \( W = W_1 \oplus W_2 \). In [Les19], we initiated a program to stabilize the relative trace formula associated to the symmetric pair \( (U(W), U(W_1) \times U(W_2)) \) by developing the local theory of endoscopy for the “Lie algebra” of the symmetric space

\[
Q = U(W)/U(W_1) \times U(W_2).
\]

Current work-in-process of the author should ultimately reduce the full stabilization of the elliptic part of the relative trace formula to this infinitesimal case. Let us recall the basic notions; Section 2.3 reviews this theory in more detail.

The \( 2n \) dimensional Hermitian space \( W = W_1 \oplus W_2 \) is naturally equipped with an involutive linear map: \( \sigma(w_1 + w_2) = w_1 - w_2 \) for \( w_i \in W_i \). This induces an involution
on the unitary group $U(W)$ with the fixed-point subgroup $U(W)^\sigma = U(W_1) \times U(W_2)$. Letting $\mathfrak{u}(W)$ denote the Lie algebra of $U(W)$, the differential of $\sigma$ induces a $\mathbb{Z}/2\mathbb{Z}$-grading
\[
\mathfrak{u}(W) = \mathfrak{u}(W)_0 \oplus \mathfrak{u}(W)_1,
\]
where $\mathfrak{u}(W)_1$ is the $(-1)^i$-eigenspace of $\sigma$. Then $\mathfrak{u}(W)_1$ is the “Lie algebra” of the symmetric variety $Q$ and the subgroup $U(W_1) \times U(W_2)$ acts on $\mathfrak{u}(W)_1$ via restriction of the adjoint action.

Section 2.3 reviews the notions of relative endoscopic data, endoscopic symmetric spaces, orbital integrals, and transfer. We postpone the details until then and content ourselves with the following special case: suppose that the extension $E/F$ is unramified and that $W_1 = W_2 = V_n$ is a split Hermitian space, so that there is a lattice $\Lambda_n \subset V_n$ that is self-dual with respect to the Hermitian form. There is a natural identification in this case
\[
\mathfrak{u}(W)_1 = \text{End}(V_n),
\]
where the $U(V_n)_1 \times U(V_n)$-action is given by pre- and post-composition. An elliptic endoscopic datum $\xi_{a,b}$ determines integers $a, b$ such that $n = a + b$. To such a datum, we associate the endoscopic symmetric space
\[
\text{End}(V_a) \oplus \text{End}(V_b),
\]
where $V_a$ denotes a split Hermitian space of dimension $a$ and similarly with $V_b$. Let $\Lambda_a \subset V_a$ and $\Lambda_b \subset V_b$ be self-dual lattices.

For a regular semi-simple element $\delta \in \text{End}(V_n)$, the endoscopic datum determines a character $\kappa$, with respect to which we define the relative $\kappa$-orbital integral
\[
\text{RO}^\kappa(f, \delta) = \sum_{\delta' \sim_{\kappa} \delta} \kappa(\delta') \text{RO}(f, \delta'),
\]
where $\delta'$ runs over rational $U(V_n) \times U(V_n)$-orbits that lie in the same stable orbit of $\delta$. We show that there is a good notion of the matching of regular semi-simple elements
\[
\delta \in \text{End}(V_n)^{rss} \text{ and } (\delta_a, \delta_b) \in (\text{End}(V_a) \oplus \text{End}(V_b))^{rss},
\]
and transfer factors
\[
\Delta_{rel} : (\text{End}(V_a) \oplus \text{End}(V_b))^{rss} \times \text{End}(V_n)^{rss} \to \mathbb{C}.
\]
With these definitions, we say that
\[
f \in C_c^\infty(\text{End}(V_a)) \text{ and } f_{a,b} \in C_c^\infty(\text{End}(V_a) \oplus \text{End}(V_b))
\]
are smooth transfers (or match) if
\[
\text{SRO}(f_{a,b}, (\delta_a, \delta_b)) = \Delta_{rel}((\delta_a, \delta_b), \delta) \text{RO}^\kappa(f, \delta)
\]
whenever $(\delta_a, \delta_b)$ and $\delta$ match. Here $\text{SRO} = \text{RO}^\kappa$ when $\kappa = 1$ is the trivial character. Our main result establishes following explicit smooth transfer.

**Theorem 1.2.** Let $\text{End}(\Lambda_n) \subset \text{End}(V_n)$ be the compact-open subring of endomorphisms of the lattice $\Lambda_n$, and let $\text{End}(\Lambda_a) \oplus \text{End}(\Lambda_b)$ be the analogous subring of $\text{End}(V_a) \oplus \text{End}(V_b)$.

The characteristic functions $1_{\text{End}(\Lambda_n)}$ and $1_{\text{End}(\Lambda_a) \otimes 1_{\text{End}(\Lambda_b)}}$ are smooth transfers of each other.

This is the endoscopic fundamental lemma referred to in the title and was conjectured in [Les19a] where we proved the special case $n = 2$ and $a = b = 1$ via explicit computation.
Remark 1.3. In current work-in-progress, we expect to show that the entire stabilization of the elliptic part of the relative trace formula follows from this result. This is entirely analogous to the Arthur-Selberg trace formula: work of Waldpurger [Wal95, Wal97] and Hales [Hal95] reduced both the smooth transfer and fundamental lemma for the entire Hecke algebra to the fundamental lemma for the Lie algebra. This final statement was further reduced to the case of positive characteristic local fields in [Wal06]. Famously, Ngô utilized the geometry of the Hitchin fibration to prove this last form in [Ngo10].

Our proof is firmly planted in characteristic-zero harmonic analysis. Drawing from several recent developments in a novel way, we show that this result follows from a new fundamental lemma for an entire Hecke algebra in the context of the Lie algebra version of Jacquet-Rallis transfer from [Zha14]. We then introduce a new comparison of relative trace formulas to prove this fundamental lemma via spectral techniques.

1.3. Outline of the proof. The first part of our proof follows a series of reductions, each one replacing an explicit statement of matching of orbital integrals for another. In each of these reductions, the varieties and groups involved in the orbital integrals changes: the argument deals with no less than 6 different types of orbital integral! The goal is to obtain a statement to which global methods may be applied; this is the case for Theorem 1.6 below.

We outline these reductions in Figure 1.3 below, which indicates the relevant sections for each component of the argument. Beginning in the lower left-hand corner, we are in the context for Theorem 1.2. We recall the contraction map $r_n : \text{End}(V_n) \rightarrow \text{Herm}(V_n)$ introduced in [Les19a], where

$$\text{Herm}(V_n) = \{ y \in \text{End}(V_n) : \langle yv, w \rangle = \langle v, yw \rangle \text{ for any } v, w \in V_n \}$$

is the twisted Lie algebra for the quasi-split unitary group $U(V_n)$. The terminology “twisted” Lie algebra refers to the fact that

$$\text{Lie}(U(V_n)) = \text{Herm}(V_n) \cdot \varepsilon,$$

where $\varepsilon \in E = F(\varepsilon)$ is a generator such that $\varpi = -\varepsilon$. In Section 3, we consider the Hermitian symmetric space

$$X_n = \left( \text{Res}_{E/F} \text{GL}_n / U(V_n) \right)(F) = \{ y \in \text{Herm}(V_n) : \det(x) \neq 0 \}.$$ 

The contraction map translates Theorem 1.2 into a matching of orbital integrals for non-standard test functions on $X_n$ that are not compactly supported. These functions possess additional symmetries due to the properties of the endomorphism ring $\text{End}(\Lambda_n)$, allowing us to study them in terms of the spherical Hecke algebra of the symmetric space $\mathcal{H}_{K_n,E}(X_n)$ (see Section 3.3 for details). Here $K_{n,E} = \text{GL}_n(\mathcal{O}_E)$ is a maximal compact subgroup and

$$\mathcal{H}_{K_n,E}(X_n) := C^\infty_c(X_n)^{K_{n,E}}.$$ 

A theorem of Hironaka [Hir99] shows that this ring is a free $\mathcal{H}_{K_{n,E}}(\text{GL}_n(E))$-module of rank $2^n$; in particular, there is a distinguished rank 1 sub-module given by the embedding (see Section 3.3 for the notation)

$$-\ast 1_0 : \mathcal{H}_{K_{n,E}}(\text{GL}_n(E)) \rightarrow \mathcal{H}_{K_{n,E}}(X_n).$$

Extension-by-zero gives an embedding of $\mathcal{H}_{K_{n,E}}(X_n) \hookrightarrow C^\infty_c(\text{Herm}(V_n))$. Our first reduction shows that Theorem 1.2 follows from the following result.

Proposition 1.4. There is a morphism of Hecke algebras

$$\xi_{(a,b)} : \mathcal{H}_{K_{n,E}}(\text{GL}_n(E)) \rightarrow \mathcal{H}_{K_{a,E}}(\text{GL}_a(E)) \otimes \mathcal{H}_{K_{b,E}}(\text{GL}_b(E)).$$
such that for any $\varphi \in \mathcal{H}_{K_{n,E}}(\text{GL}_n(E))$, the functions
\[ \varphi \ast 1_0 \text{ and } \xi_{(a,b)}(\varphi) \ast 1_0 \]
are smooth transfers with respect to endoscopic transfer for the twisted Lie algebra. Here, $\xi_{(a,b)}(\varphi) \ast 1_0$ denotes the image of $\xi_{(a,b)}(\varphi)$ in $\mathcal{H}_{K_{n,E}}(X_a) \otimes \mathcal{H}_{K_{n,E}}(X_b)$ under the analogous embedding.

This result gives new explicit endoscopic transfers of test functions on the twisted Lie algebra, generalizing the fundamental lemma of Laumon and Ngô [LN08].

To prove Proposition 1.4, we utilize a recent alternative proof of the existence of smooth transfer for the twisted Lie algebra due to Xiao [Xia18]. This argument is indicated by the rectangle in the lower right of Figure 1.3. The arrows denote the following relationships:

- $\text{ev}_0$: this arrow indicates the evaluation-at-0 map $\text{ev}_0(F)(-)=F(-,0)$;
- $\text{JR}$: this arrow indicates the Jacquet-Rallis transfer between the spaces
  \[ \mathcal{H}_{\text{erm}}(V_n) \times V_n \text{ and } \mathfrak{gl}_n(F) \times F^n \times F_n, \]
  where $F_n = (F^n)^*$ is the space of $1 \times n$ row vectors;
- $\text{PD}$: this arrow indicates Lie-algebraic parabolic descent of relative orbital integrals.

Roughly speaking, the matching of orbital integrals comprising the endoscopic transfer between $\mathcal{H}_{\text{erm}}(V_n)$ and $\mathcal{H}_{\text{erm}}(V_a) \oplus \mathcal{H}_{\text{erm}}(V_b)$ may be obtained from parabolic descent of orbital integrals from $\mathfrak{gl}_n(F) \times F^n \times F_n$ to the Levi factor $\prod_{i=a,b} \mathfrak{gl}_i(F) \times F^i \times F_i$ by applying the Jacquet-Rallis transfer of [Zha14] and then taking a limit to certain non-regular orbits. We outline this argument in greater detail in Section 4.3. The new tool for this proof is Xiao’s analysis of certain generalized nilpotent orbital integrals in the context of the Jacquet-Rallis transfer. We review these notions in Sections 4.1 and 4.2.

The upshot is that Proposition 1.4 follows if we can construct sufficiently many explicit pairs of functions that a smooth transfers of each other with respect to the Jacquet-Rallis transfer. To this end, we prove the following new fundamental lemma in the context of the Jacquet-Rallis transfer.

**Proposition 1.5.** Let $\Lambda_n \subset V_n$ be our self-dual lattice and set $\mathcal{L}_n = O_{F}^{1} \times O_{F_n}$. Let $BC : \mathcal{H}_{K_{n,E}}(\text{GL}_n(E)) \rightarrow \mathcal{H}_{K_{n,F}}(\text{GL}_n(F))$ be the base change homomorphism of Hecke algebras. Then for any $\varphi \in \mathcal{H}_{K_{n,E}}(\text{GL}_n(E))$, the functions
\[ \{(\varphi \ast 1_0) \otimes 1_{\Lambda_n}, 0\} \text{ and } BC(\varphi) \otimes 1_{\mathcal{L}_n} \]
are smooth transfers of each other with respect to the matching (14). This implies Proposition 1.4.

This generalizes the Jacquet-Rallis fundamental lemma of Yun [Yun11]. To prove this proposition, we need to remove the characteristic functions $1_{\Lambda_n}$ and $1_{\mathcal{L}_n}$ in the above comparison so that we can apply global tools. Strikingly, the recently-explicated Weil representation ([BP19]; see also [Zha19]) of $\text{SL}_2(F)$ on the function spaces
\[ C^\infty_c(\mathcal{H}_{\text{erm}}(V) \times V) \text{ and } C^\infty_c(\mathfrak{gl}_n(F) \times F^n \times F_n) \]
is exactly what we need. We recall the details of these representations in Section 5.

Beuzart-Plessis recently used this structure to give a new proof of the Jacquet-Rallis fundamental lemma for any residual characteristic. We carry out a similar computation to reduce Proposition 1.5 to the final form of the fundamental lemma.
Figure 1. Various spaces and the relations between their orbital integrals
Theorem 1.6. Consider the Jacquet-Rallis transfer between the spaces
\[ C_\infty^c(\text{Herm}(V)) \text{ and } C_\infty^c(\text{gl}_n(F)); \]
see Section 4.1 for details. Then for any \( \varphi \in \mathcal{H}_{K_n,E}(\text{GL}_n(E)) \), the functions
\[ \{(\varphi \ast 1_0),0\} \text{ and } BC(\varphi) \]
are transfers of each other with respect to the matching (18). This implies Proposition 1.5.

This theorem is our final reduction of Theorem 1.2. Its proof is global, relying on a new comparison of trace formulas. We refer to these trace formulas as the \textit{twisted Jacquet-Rallis relative trace formulas} as they arise by “switching the roles” of the unitary group \( U(V_n) \) and the linear group \( \text{GL}_n(F) \) in the original Jacquet-Rallis comparison. This switching is explained in terms of orbits at the beginning of Part 2, and we refer the curious reader there. While several spectral consequences of this comparison are known by work of Feigon, Lapid, and Offen [FLO12] and Jacquet [Jac10] on unitary periods of cusp forms, the resulting geometric comparison is exactly what is needed to translate Theorem 1.6 into a spectral problem, despite ostensibly being a statement in the \textit{Lie algebra version} of Jacquet-Rallis transfer with no clear spectral content.

This argument is the content of Part 2, which we have written to be essentially self-contained. To avoid making this introduction overlong, we refer the reader to the beginning of Part 2 for more details as the ideas and techniques used are rather different. We simply remark that the final piece is Theorem 11.1, which establishes the fundamental lemma for the Hecke algebra for our comparison. This is the \( BC \) arrow in Figure 1.3, indicating that base change is the functoriality underlying this comparison.

Below we introduce notations and conventions which are in force throughout both Part 1 and Part 2. We caution the reader that notations adopted within the two parts differ from one another in certain important aspects; we indicate these changes at the start of the second part.

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1.5. Preliminaries.

1.5.1. Invariant theory. For any field \( F \) and any non-singular algebraic variety \( Y \) over \( F \) with \( G \) an algebraic group over \( F \) acting algebraically on \( Y \), we set \( Y^{rss} \) to be the invariant-theoretic regular semi-simple locus. That is, \( x \in Y^{rss} := Y^{rss}(F) \) if and only if its \( G(F) \)-orbit is of maximal possible dimension and is closed as a subset of \( Y \).

For \( x, x' \in Y^{rss} \), we say that \( x' \) is in the \textit{rational} \( G \)-\textit{orbit} of \( x \) if there exists \( g \in G(F) \) such that
\[ g \cdot x = x'. \]

Fixing an algebraic closure \( \overline{F} \), \( x \) and \( x' \) are in the same \textit{stable orbit} if \( g \cdot x = x' \) for some \( g \in G(\overline{F}) \) and such that the cocycle
\[ (\sigma \mapsto g^{-1}g^\sigma) \in Z^1(F,G) \]
lies in \( Z^1(F, G^0_x) \), where \( G^0_x \subset G_x \) is the connected component of the identity of the stabilizer of \( x \) in \( G \). A standard computation (see \cite[Lemma 2.1.5]{Kal11}) shows that the set \( \mathcal{O}_{sl}(x) \) of rational orbits in the stable orbit of \( x \) are in natural bijection with

\[
\mathcal{D}(G^0_x/F) := \ker(H^1(F, G^0_x) \to H^1(F, G)).
\]

Here we ignore the dependence on \( G \) in the notation. When \( G^0_x \) is abelian (as will be the case for us), \( \mathcal{O}_{sl}(x) \) is a torsor for the group \( \mathcal{D}(G^0_x/F) \).

1.5.2. Local fields. When \( F \) is a non-archimedean field, we set \( |\cdot|_F \) to be the normalized valuation so that if \( \varpi \) is a uniformizer, then

\[
|\varpi|_F = \#(\mathcal{O}_F/p) =: q
\]

is the cardinality of the residue field. Here \( p \) denotes the unique maximal ideal of \( \mathcal{O}_F \).

For any quadratic étale algebra \( E/F \) of local fields, we set \( \eta := \eta_{E/F} : F^\times \to \mathbb{C}^\times \) for the character associated to the extension by local class field theory.

1.5.3. Groups and Hermitian spaces. For a field \( F \) and for \( n \geq 1 \), we consider the algebraic group \( \text{GL}_n \) of invertible \( n \times n \) matrices. Suppose that \( E/F \) is a quadratic étale algebra and consider the restriction of scalars \( \text{Res}_{E/F}(\text{GL}_n) \). For any \( F \)-algebra \( R \) and \( g \in \text{Res}_{E/F}(\text{GL}_n)(R) \), we set

\[
g \mapsto \overline{g}
\]

to be the Galois involution associated to the extension \( E/F \); we also denote this involution by \( \sigma \). We denote by \( T_n \subset \text{GL}_n \) the diagonal maximal split torus, \( B_n = T_n N_n \) the Borel subgroup of upper triangular matrices with unipotent radical \( N_n \). Set

\[
X_n := X_n(F) = \{ x \in \text{GL}_n(E) : \overline{x} = x \}.
\]

Note that \( \text{GL}_n(E) \) acts on \( X_n \) via

\[
g * x = gx^t \overline{g}, \quad x \in X_n, \ g \in \text{GL}_n(E),
\]

where \( {}^t \) denotes the transpose. We let \( \mathcal{V}_n \) be a fixed set of orbit representatives. For any \( x \in X_n \), set \( \langle \cdot, \cdot \rangle_x \) to be the Hermitian form on \( E^m \) associated to \( x \). Denote by \( V_x \) the associated Hermitian space and \( U(V_x) \) the corresponding unitary group. Note that if \( g * x = x' \) then

\[
V_x \xrightarrow{\overline{g}} V_{x'}
\]

is an isomorphism of Hermitian spaces. Thus, \( \mathcal{V}_n \) gives a set of representatives \( \{ V_x : x \in \mathcal{V}_n \} \) of the equivalence classes of Hermitian vector space of dimension \( n \) over \( E \). When convenient, we will abuse notation and identify this set with \( \mathcal{V}_n \). If we are working with a fixed but arbitrary Hermitian space, we often drop the subscript. For any Hermitian space, we set

\[
U(V) = U(V)(F).
\]

1.5.4. Measures and centralizers. Suppose now that \( E/F \) is an extension of local fields and fix an additive character \( \psi : F \to \mathbb{C}^\times \). By composing with the trace \( \text{Tr}_{E/F} \), we also obtain an additive character for \( E \). We fix here our choice of Haar measures on the groups involved, choosing to follow \cite{FLO12} closely. This is primarily to aid in Part 2 of the paper; the main point for Part 1 is that our choices are normalized to give the appropriate maximal compact subgroup volume 1 in the unramified setting.

For any non-singular algebraic variety \( Y \) over \( F \) of dimension \( d \) and gauge form \( \omega_Y \), the Tamagawa measure \( dy_{\text{Tam}} \) of \( Y = Y(F) \) is defined by transferring the standard Haar measure on \( F^d \) to \( Y \) by \( \omega_Y \).
For the varieties we consider, we set our measure to be of the form $dy = c(\psi)^d/2 \lambda y dy_{Tam}$, where

$$c(\psi) = \begin{cases} q^m & : F \text{ non-archimedean and } \text{cond}(\psi) = \infty^m O_F, \\ |a|_F & : F \text{ archimedean and } \psi(x) = e^{2\pi i T_E/F(ax)}.
\end{cases}$$

For the other terms, we impose the choice that for any $Y$, $\omega_{Res/F} Y = p^*(\omega_Y)$, where $p^*$ is given in [Wei82, pg. 22]. We now fix $\omega_Y$:  

- For $Y = GL_n$, we take $\omega_{GL_n} = \prod_{i,j} \frac{dg}{det(g)^m}$ and take $\lambda_{GL_n} = \prod_{i=1}^n L(i,1_{F^\times})$, where for any character $\chi : F^\times \to \mathbb{C}^\times$, $L(s,\chi)$ is the local Tate $L$-factor. We also set $\lambda_{Res/F(GL_n)} = \prod_{i=1}^n L(i,1_{E^\times})$.
- For $Y = N$ for any unipotent subgroup of $GL_n$, we set $\omega_N = \prod_i dx_i$, where the product ranges over the non-constant coordinate functions on $N$. We set $\lambda_N = 1$.
- For $Y = X_n$, set $\omega_{X_n} = \prod_{i,j} \frac{dx_i dx_j}{det(x)^n}$, and take $\lambda_{X_n} = \prod_{i=1}^n L(i,\eta^{i+1})$, where $\eta = \eta_{E/F}$ is the quadratic character associated to $E/F$.
- For $Y = U(V)$, we take $\omega_{U(V)}$ to be compatible with $\omega_{Res/E(F,GL_n)}$ and $\omega_{X_n}$.

Finally, we take $\lambda_{U(V)} = \prod_{i=1}^n L(i,\eta^i)$. In particular, the isomorphism

$$X_n \cong \bigcup_{\chi \in V_n} GL_n(E)/U(V_{\chi})$$

is compatible with these measures.

When $F$ is $p$-adic and $\psi$ of conductor $O_F$, our choice of measure gives $K_n := GL_n(O_F)$ volume 1. When $E/F$ is also unramified, the same holds for the maximal compact subgroups $K_{n,E} = GL_n(O_E) \subset GL_n(E)$ as well as $X_n(O_E) := GL_n(O_E) * I_n$.

Finally, we consider the measures on regular semi-simple centralizers. Fix a Hermitian form $x$ and consider $U(V) = U(V_x)$. We will be interested in the twisted Lie algebra

$\mathcal{Herm}(V) = \{ \delta \in \text{End}(V) : \langle \delta v, u \rangle = \langle v, \delta u \rangle \}$.

The group $U(V)$ acts on this space by the adjoint action, and an element $\delta$ is regular semi-simple if its centralizer is a maximal torus $T_{\delta} \subset U(V)$. To construct $T_{\delta}$ note that there is a natural decomposition

$$F[\delta] := F[X]/(\text{char}_F(X)) = \prod_{i=1}^m F_i,$$

where $F_i/F$ is a field extension and $\text{char}_F(X)$ denotes the characteristic polynomial of $\delta$. Setting $E_i = E \otimes F_i$, we have

$$E[\delta] = \prod_i E_i = \prod_{i \in S_1} E_i \times \prod_{i \in S_2} F_i \oplus F_i,$$

where $S_1 = \{ i : F_i \not\subset E \}$.

**Lemma 1.7.** The centralizer $T_{\delta}$ of $\delta$ in $U(W)$ sits in a short exact sequence

$$1 \longrightarrow Z_{U(W)}(F) \longrightarrow T_{\delta} \longrightarrow E[\delta]^\times / F[\delta]^\times \longrightarrow 1.$$

Moreover, $H^1(F, T_{\delta}) = \prod_{S_1} \mathbb{Z}/2\mathbb{Z}$, and

$$D(T_{\delta}/F) = \ker \left( H^1(F, T_{\delta}) \to H^1(F, U(V)) \right) = \ker \left( \prod_{S_1} \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \right).$$

Set $T_{S_1} \cong Z_{U(V)}(F) \prod_{i \in S_1} E_i^\times / F_i^\times$ for the unique maximal compact subgroup of $T_{\delta}$. We choose the measure $dt$ on $T_{\delta}$ giving this subgroup volume 1.
Part 1. Endoscopic theory and reduction

In this first part, we recall the basic theory of endoscopy for the infinitesimal symmetric space from [Les19a]. We then state our main result in Theorem 2.11. In Section 3, we show that the main theorem follows from a fundamental lemma for an entire Hecke algebra on the symmetric space $X_n$. In Section 4, we use recent results relating endoscopic transfer for unitary Lie algebras and Jacquet-Rallis transfer to translate the problem into a statement about Jacquet-Rallis transfer. Finally, we use the Weil representation on certain spaces of orbital integrals to reduce the statement to its final form in Theorem 5.3. The proof of this final reformulation is the content of Part 2.

For the entirety of this part, $F$ is a non-archimedean local field and $E/F$ is a quadratic étale $F$-algebra. For the identity form $I_n \in X_n$, we set $V_n := V_{I_n}$ and note that when $E/F$ is unramified, then $V_n$ is a split Hermitian space and $U(V_n)$ is the quasi-split unitary group.

2. The relative endoscopic fundamental lemma

In this section, we recall the basics of the theory of endoscopy for the infinitesimal symmetric space; our reference is [Les19a]. We then state our main result in Theorem 2.11.

2.1. The Lie algebra of the symmetric space. Let $(W_1, \langle \cdot, \cdot \rangle_1)$, $(W_2, \langle \cdot, \cdot \rangle_2)$ be two Hermitian spaces of dimension $n$ over $E$. Setting $W = W_1 \oplus W_2$, we consider the Lie algebra $u(W)$ of the rank $2n$ unitary group $U(W)$. The differential of the involution $\sigma$ acts on $u(W)$ by the same action and induces a $\mathbb{Z}/2\mathbb{Z}$-grading

$$u(W) = u(W)_0 \oplus u(W)_1,$$

where $u(W)_i$ is the $(-1)^i$-eigenspace of the map $\sigma$. We recall the basic properties of this setting, referring the interested reader to [Les19a, Section 3] for proofs.

Lemma 2.1. We have natural identifications

$$u(W)_0 = u(W_1) \oplus u(W_2), \text{ and } u(W)_1 = \text{Hom}_E(W_2, W_1).$$

Here $U(W_1) \times U(W_2)$ acts on $u(W)_1$ by the restriction of the adjoint action. In terms of $W_1$ and $W_2$, the action is given by $(g, h) \cdot \varphi = g \circ \varphi \circ h^{-1}$. \hfill \Box

In particular, any element $\delta \in u(W)_1$ may be uniquely written

$$\delta = \delta(X) = \begin{pmatrix} X & X \tau \\ -X \tau & X \end{pmatrix},$$

where $X \in \text{Hom}_E(W_2, W_1)$ and where for any $w_1 \in W_i$

$$\langle X w_2, w_1 \rangle_1 = \langle w_2, X \tau w_1 \rangle_2.$$  

For any such $\delta$, we denote by

$$H_\delta = \{(h, g) \in U(W_1) \times U(W_2) : h^{-1} x g = x \} \subset U(W_1) \times U(W_2)$$

the stabilizer of $\delta$.

Set $u(W)_1^{rss}$ to be the regular semi-simple locus with respect to this action of $U(W_1) \times U(W_2)$ on $u(W)_1$. That is, $\delta \in u(W)_1^{rss}$ if and only if its orbit under $U(W_1) \times U(W_2)$ is closed and of maximal dimension. In our present case, we have

$$u(W)_1^{rss} := u(W)_1 \cap u(W)^{rss},$$

where $u(W)^{rss}$ is the classical regular semi-simple locus of the Lie algebra. This is due to the fact that the symmetric pair $(U(W), U(W_1) \times U(W_2)$ is quasi-split. See [Les19b, Sec. 2] for more details on quasi-split symmetric spaces. In particular, if $\delta \in u(W)_1^{rss}$, then $H_\delta$ is a torus of rank $n$. 


Let $$u(W)_1^{iso} \cong Iso_F(W_2, W_1)$$ be the open subvariety of elements $$\delta(X)$$ where $$X : W_2 \to W_1$$ is a linear isomorphism. There are natural contraction maps $$r_i : u(W)_1 \to \Herm(V)$$ given by

$$r_i(\delta(X)) = \begin{cases} XX^\tau : & i = 1 \\ X^\tau X : & i = 2. \end{cases}$$ (2)

**Proposition 2.2.** Set $$r := r_1$$. Then $$r : u(W)_1 \to \Herm(W_1)$$ is equivariant with respect to the $$U(W_1)$$ action on $$u(W)_1$$ and the adjoint action on $$\Herm(W_1)$$. Moreover, the pair $$(\Herm(W_1), r)$$ is a categorical quotient for the $$U(W_2)$$-action on $$u(W)_1$$. □

The next lemma shows that the contraction map preserves centralizers over the non-singular locus.

**Lemma 2.3.** The restriction of $$r$$ to $$u(W)_1^{iso}$$ is a $$U(W_2)$$-torsor. Moreover, for $$\delta \in u(W)_1^{iso}$$, we have an isomorphism

$$\phi_\delta : H_\delta \sim \rightarrow T_r(\delta)$$

given by $$(h_1, h_2) \mapsto h_1$$. Moreover, $$\phi_\delta$$ induces an isomorphism between

$$\mathcal{D}(H_\delta/F) \sim \rightarrow \mathcal{D}(T_r(\delta)/F)$$ (3)

where

$$\mathcal{D}(H_\delta/F) = \ker (H^1(F, H_\delta) \to H^1(F, U(W_1) \times U(W_2)))$$

and

$$\mathcal{D}(T_r(\delta)/F) = \ker (H^1(F, T_r(\delta)) \to H^1(F, U(W_1)))$$.

The isomorphism (3) implies that there is a bijection of rational orbits $$\mathcal{O}_u(\delta)$$ of $$U(W_1) \times U(W_2)$$ inside the stable orbit of $$\delta$$ and rational conjugacy classes of $$\Herm(W_1)$$ inside a stable class $$r(\delta)$$. □

**Definition 2.4.** For $$f \in C_\infty^r(u(W)_1)$$, and $$\delta \in u(W)_1$$ a semi-simple element, we define the relative orbital integral of $$f$$ by

$$\text{RO}(f, \delta) = \int_{H_\delta \setminus U(W_1) \times U(W_2)} f(h_1^{-1} \delta h_2) \frac{dh_1 dh_2}{dt_\delta},$$

where $$dh_i$$ and $$dt_\delta$$ are Haar measures on $$U(W_i)$$ and $$H_\delta$$, respectively.

Our primary tool for studying relative orbital integrals is to relate them via the contraction map to orbital integrals of *non-standard test functions* on the twisted Lie algebra $$\Herm(W_1)$$. The next lemma explains why this is effective for regular semi-simple orbits.

**Lemma 2.5.** There is an inclusion $$u(W)_1^{rss} \subset u(W)_1^{iso}$$.

**Proof.** Since the claim is geometric, we may pass to the algebraic closure $$F = \bar{F}$$ and assume that $$u(W) \cong \mathfrak{gl}_n(F)$$. This translates the $$U(W_1) \times U(W_2)$$-action to the action of $$\text{GL}_n(F) \times \text{GL}_n(F)$$ on $$\mathfrak{gl}_n(F) \times \mathfrak{gl}_n(F)$$ given by

$$(g, h) \cdot (X, Y) = (gXh^{-1}, hYg^{-1}).$$

The categorical quotient of this action is given by taking the coefficients of the characteristic polynomial $$\pi(X, Y)(t) = \det(tI - XY)$$ [Les19a, Lemma 3.3].
Recalling that the infinitesimal symmetric space $\mathfrak{gl}_n(F) \times \mathfrak{gl}_n(F)$ is quasi-split, the element $(X,Y)$ is regular semi-simple in $\mathfrak{gl}_n(F) \times \mathfrak{gl}_n(F)$ if and only if the element

$$Z = \begin{pmatrix} X & \ast \\ \ast & \ast \end{pmatrix} \in \mathfrak{gl}_{2n}(F)$$

is regular semi-simple in the classical sense. Letting $\chi_Z(t) = \det(tI - Z)$ denote the characteristic polynomial, $Z$ is regular semi-simple if and only if $\chi_Z$ has distinct roots. Now a simple exercise in linear algebra shows that

$$\chi_Z(t) = \pi(X,Y)(t^2).$$

Thus, $Z \in \mathfrak{gl}_{2n}(F)^{rss}$ is possible only if 0 is not a root of $\pi(X,Y)$, implying the lemma.

This inclusion allows us to express relative orbital integrals at regular semi-simple points in terms of classical orbital integrals.

**Lemma 2.6.** For $f \in C_c^\infty(\mathfrak{u}(W)_1)$ and for $x \in \mathfrak{u}(W)_1$ regular semi-simple, define

$$r_1(f)(r(x)) = \int_{U(W_2)} f(xu)du.$$

We have the equality

$$\text{RO}(f,x) = \int_{T_r(x) \backslash U(W_1)} r_1(f)(g^{-1}r(x)g)dg =: \text{Orb}(r_1(f),r(x)).$$

**Proof.** If $x$ is a regular semi-simple element, then everything is absolutely convergent. By Lemma 2.5, we know that $x \in \mathfrak{u}(W)^{rss}$, so that Lemma 2.3 implies that

$$\text{RO}(f,x) = \int_{T_r(x) \backslash U(W_1)} r_1(f)(g^{-1}r(x)g)dg.$$

\[ \square \]

### 2.2. Endoscopy for the twisted Lie algebra

Lemmas 2.3 and 2.6 allow us to utilize the contraction map to define endoscopic symmetric spaces for $\mathfrak{u}(W)_1$ and the associated transfer factors in terms of those for the twisted Lie algebra $\mathfrak{Herm}(W)_1$. We briefly recall the necessary facts from this theory. We refer the reader to [Xia18] for proofs of these facts.

#### 2.2.1. Matching

An elliptic endoscopic datum for $\mathfrak{Herm}(W_1)$ is the same as a datum for the group $U(W_1)$, namely a triple $(U(V_a) \times U(V_b), s, \eta)$ where $a + b = n$, with $s \in \hat{U}(W_1)$ a semi-simple element of the Langlands dual group of $U(W_1)$, and an embedding

$$\eta: \hat{U}(V_a) \times \hat{U}(V_b) \hookrightarrow \hat{U}(W_1)$$

identifying $\hat{U}(V_a) \times \hat{U}(V_b)$ with the neutral component of the centralizer of $s$.

Fixing such a datum, we consider the endoscopic Lie algebra $\mathfrak{Herm}(V_a) \oplus \mathfrak{Herm}(V_b)$. Let $y \in \mathfrak{Herm}(W_1)$ and $(y_a,y_b) \in \mathfrak{Herm}(V_a) \oplus \mathfrak{Herm}(V_b)$ be regular semi-simple. While the general notion of matching orbits is involved, this situation has the following simple characterization: if we identify the underlying vector spaces (but not necessarily the Hermitian structures)

$$W_1 \cong E^n \cong V_a \oplus V_b,$$

we have embeddings

$$\mathfrak{Herm}(W_1), \mathfrak{Herm}(V_a) \oplus \mathfrak{Herm}(V_b) \hookrightarrow \mathfrak{gl}_n(E).$$

Then $y$ and $(y_a,y_b)$ match in the endoscopic sense if they are $\text{GL}_n(E)$-conjugate in $\mathfrak{gl}_n(E)$. This is well defined since the above embeddings are determined up to $\text{GL}_n(E)$-conjugacy.
2.2.2. Orbital integrals. To an elliptic endoscopic datum \((U(V_a) \times U(V_b), s, \eta)\) and regular semi-simple element \(y \in \text{Herm}(W_1)\), there is a natural character (see [Rog90, Chapt. 3], for example)

\[
\kappa : \mathcal{D}(T_y/F) \rightarrow \mathbb{C}^\times.
\]

Recalling that the set of rational conjugacy classes \(\mathcal{O}_d(y)\) in the stable conjugacy class of \(y\) form a \(\mathcal{D}(T_y/F)\)-torsor, we have a map

\[
\text{inv}(y,-) : \mathcal{O}_d(\delta) \xrightarrow{\sim} \mathcal{D}(T_\delta/F)
\]

trivializing the torsor by fixing the base point \(y\). We then form the \(\kappa\)-orbital integral

\[
\text{Orb}^\kappa(f, y) = \sum_{y' \sim^\kappa y} \kappa(\text{inv}(y, y')) \text{Orb}(f, y')
\]

where \(f \in C^\infty_c(\text{Herm}(W_1))\). When \(\kappa = 1\) is trivial, write \(\text{SO} = \text{Orb}^\kappa\).

In our case, the character \(\kappa\) is easy to describe. For matching elements \(y\) and \((y_a, y_b),\)

\[
H^1(F, T_y) = \prod_{S_1} \mathbb{Z}/2\mathbb{Z} = \prod_{S_1(a)} \mathbb{Z}/2\mathbb{Z} \times \prod_{S_1(b)} \mathbb{Z}/2\mathbb{Z} = H^1(F, T_{y_a} \times T_{y_b}),
\]

where the notation indicates which elements of \(S_1\) arise from the torus \(T_{y_a}\) or \(T_{y_b}\).

**Lemma 2.7.** Consider the character \(\tilde{\kappa} : H^1(F, T_y) \rightarrow \mathbb{C}^\times\) such that on each \(\mathbb{Z}/2\mathbb{Z}\) factor arising from \(S_1(a)\), \(\tilde{\kappa}\) is the trivial map, while it is the unique nontrivial map on each \(\mathbb{Z}/2\mathbb{Z}\)-factor arising from \(S_1(b)\). Then

\[
\kappa = \tilde{\kappa}|_{\mathcal{D}(T_y/F)}.
\]

2.2.3. Smooth transfer. The final notion is the transfer factor of Langlands-Shelstad and Kottwitz. This is a function

\[
\Delta : [\text{Herm}(V_a) \oplus \text{Herm}(V_b)]^{rss} \times \text{Herm}(W_1)^{rss} \rightarrow \mathbb{C}.
\]

The general definition is subtle, though see [Les19a, Section 2] (following [LN08]) for formulas in this special setting. The two important properties are

1. \(\Delta((y_a, y_b), y) = 0\) if \(y\) does not match \((y_a, y_b),\)
2. if \(y\) is stably conjugate to \(y'\), then

\[
\Delta((y_a, y_b), y) \text{Orb}^\kappa(f, y) = \Delta((y_a, y_b), y') \text{Orb}^\kappa(f, y').
\]

A pair of functions

\[
f \in C^\infty_c(\text{Herm}(W_1)) \quad \text{and} \quad f_{a,b} \in C^\infty_c(\text{Herm}(V_a) \oplus \text{Herm}(V_b))
\]

are said to be smooth transfers (or matching functions) if the following conditions are satisfied:

1. for any matching elements regular semi-simple elements \(y\) and \((y_a, y_b),\)

\[
\text{SO}(f_{a,b}, (y_a, y_b)) = \Delta((y_a, y_b), y) \text{Orb}^\kappa(f, y);
\]

2. if there does not exist \(y\) matching \((y_a, y_b),\) then

\[
\text{SO}(f_{a,b}, (y_a, y_b)) = 0.
\]

The following theorem was first shown by combining [LN08], [Wal06], and [Wal97]; we will outline an alternative proof due to [Xia18] in Section 4.

**Theorem 2.8.** For any \(f \in C^\infty_c(\text{Herm}(W_1))\), there exists a smooth transfer \(f_{a,b} \in C^\infty_c(\text{Herm}(V_a) \oplus \text{Herm}(V_b)).\)
2.3. Relative endoscopy for \((U(W), U(W_1) \times U(W_2))\). Recall that \(V_n\) denotes our fixed set of representatives of the \(\text{GL}_n(E)\)-orbits on \(X_n\). Since we only consider the non-archimedean setting, \(|V_n| = 2\) for any \(n\); we always assume that \(I_n \in V_n\).

In [Les19a], we defined a relative elliptic endoscopic datum of \(u(W)_1\) to be a quintuple

\[
\xi = (U(V_a) \times U(V_b), s, \eta, \alpha, \beta),
\]

where \((U(V_a) \times U(V_b), s, \eta)\) is an elliptic endoscopic datum for \(U(W_1)\) and \(\alpha \in V_a\) and \(\beta \in V_b\) are Hermitian forms on \(E^a\) and \(E^b\) respectively. We denote \(V_a = (E^a, \alpha)\) and \(V_\beta = (E^b, \beta)\). For such a datum, we consider the Lie algebras

\[
u(V_a \oplus V_\alpha) \text{ and } u(V_b \oplus V_\beta),
\]

and associated symmetric pairs

\[(U(V_a) \times U(V_a), u(V_a \oplus V_a))_1 \text{ and } (U(V_b) \times U(V_b), u(V_b \oplus V_\beta))_1 .
\]

We call the direct sum of these symmetric pairs an endoscopic symmetric pair associated to \(\xi\). This space comes equipped with the contraction map

\[r_{\alpha, \beta}: u(V_a \oplus V_\alpha) \oplus u(V_b \oplus V_\beta)_1 \rightarrow \text{Herm}(V_a) \oplus \text{Herm}(V_b)
\]

\[(\delta_a, \delta_b) \mapsto (r(\delta_a), r(\delta_b)) \]

We say that a regular semi-simple element \(\delta \in u(W)^{rss}_1\) matches the pair

\[(\delta_a, \delta_b) \in [u(V_a \oplus V_\alpha)_1 \oplus u(V_b \oplus V_\beta)_1]^{rss}
\]

if \(r(\delta) \in \text{Herm}(W_1)\) and \(r_{\alpha, \beta}(\delta_a, \delta_b) \in \text{Herm}(V_a) \oplus \text{Herm}(V_b)\) match in the endoscopic sense. When such an \(\delta\) exists, we say that \((\delta_a, \delta_b)\) is \(u(W)_1\)-regular semi-simple.

For matching elements \((\delta_a, \delta_b)\) and \(\delta\), we define the transfer factor

\[\Delta_{rel}(\delta_a, \delta_b, \delta) := \Delta(r_{\alpha, \beta}(\delta_a, \delta_b), r(\delta))\]

where the right-hand side is the Langlands-Shelstad-Kottwitz transfer factor the the twisted Lie algebra.

2.3.1. Smooth transfer. Fix \(\delta \in u(W)^{rss}_1\) and let \(\xi\) be a relative endoscopic datum. Combining Lemma 2.3 with the construction of Section 2.2.2 gives a character

\[\kappa: D(H_\delta/F) \rightarrow \mathbb{C}^\times,
\]

with which we define the associated relative \(\kappa\)-orbital integral to be

\[\text{RO}^{\kappa}(f, \delta) := \sum_{\delta' \sim_{\kappa} \delta} \kappa(\text{inv}(\delta, \delta')) \text{RO}(f, \delta'),
\]

where \(\delta'\) runs over the set of rational orbits in \(u(W)_1\) in the stable orbit of \(\delta\) and

\[\text{inv}(\delta, \delta_a) := \text{inv}(r(\delta), r(\delta_a)).
\]

Here, \(\text{inv}(r(\delta), -)\) is the invariant map (4). When \(\kappa = 1\), this is called the stable relative orbital integral and denoted by \(\text{SRO} = \text{RO}^1\).

**Definition 2.9.** We say that \(f \in C^\infty_c(u(W)_1)\) and \(f_{\alpha, \beta} \in C^\infty_c(u(V_a \oplus V_\alpha)_1 \oplus u(V_b \oplus V_\beta)_1)\) match (or are smooth transfers) if the following conditions are satisfied:

1. For any matching orbits \(\delta \in u(W)^{rss}_1\) and \((\delta_a, \delta_b) \in [u(V_a \oplus V_\alpha)_1 \oplus u(V_b \oplus V_\beta)_1]^{rss}\), we have an identify

\[\text{SRO}(f_{\alpha, \beta}, (\delta_a, \delta_b)) = \Delta_{rel}(\delta_a, \delta_b, \delta) \text{RO}^{\kappa}(f, \delta).
\]

2. If there does not exist \(\delta\) matching \((\delta_a, \delta_b)\), then \(\text{SRO}(f_{\alpha, \beta}, (\delta_a, \delta_b)) = 0\).

We conjectured that smooth transfers always exist in [Les19a, Conjecture 4.7], and showed that transfers exist for many test functions.
Remark 2.10. Recall that $u(W)_1$ has two natural contraction maps (2). For the reader concerned with canonicity, we remark that it is straightforward to show using the properties of the endoscopic transfer, Jacquet-Rallis transfer, and the Langlands-Shelstad-Kottwitz transfer factors that these definitions are independent of our choice of contraction $r = r_1$.

2.4. The endoscopic fundamental lemma. We now assume that $E/F$ is an unramified extension of non-archimedean local fields of characteristic zero. Suppose that $V_n = W_1 = W_2$ is split, and let $\Lambda_n \subset V_n$ be a self-dual lattice. In this case, $u(W)_1 = \text{Hom}_E(V_n, V_n) = \text{End}(V_n)$ and the ring of endomorphisms $\text{End}(\Lambda_n) \subset \text{End}(V_n)$ of the lattice $\Lambda_n$ is a compact open subset. Let $1_{\text{End}(\Lambda_n)}$ denote the indicator function for this subset. This also induces a hyperspecial maximal compact subgroup $U(\Lambda_n) \subset U(V_n)$.

Now suppose that $(U(V_a) \times U(V_b), s, \eta, \alpha, \beta)$ is an elliptic relative endoscopic datum. Under our assumptions, we have $V_n \cong V_a \oplus V_b$ and we fix an isomorphism by imposing $\Lambda_n = \Lambda_a \oplus \Lambda_b$ for fixed self-dual lattices $\Lambda_a \subset V_a$ and $\Lambda_b \subset V_b$; this choice is determined up to $U(\Lambda_n) \times U(\Lambda_n)$-conjugation. Our measures conventions in Section 1.5.4 ensure that the given hyperspecial maximal subgroups of $U(V_n) \times U(V_n)$ and $(U(V_a) \times U(V_a)) \times (U(V_b) \times U(V_b))$

each have volume 1.

The following was conjectured in [Les19a], and is the main result of this paper.

Theorem 2.11. (Relative fundamental lemma) If $(\alpha, \beta) = (I_a, I_b)$, the functions $1_{\text{End}(\Lambda_a)}$ and $1_{\text{End}(\Lambda_a)} \otimes 1_{\text{End}(\Lambda_a)}$ match. Otherwise, $1_{\text{End}(\Lambda_n)}$ matches 0.

The proof of this statement follows a series of reductions, each of which changes the orbital integrals involved and the comparison needed. These reductions take up the rest of Part 1, culminating in Theorem 5.3.

3. A relative fundamental lemma for the Hecke algebra

As an elementary first step, we use Lemmas 2.3 and 2.6 to reduce the comparison of relative orbital integrals in Theorem 2.11 to a countable number of explicit endoscopic transfers for the twisted Lie algebra $\text{Herm}(V_n)$.

A useful observation is that this reduction is possible for any $x \in u(W)_1^{rss}$ since Lemma 2.5 implies that $x \in u(W)_1^{rss}$.

This motivates us to study orbital integrals of special functions on the Hermitian symmetric space $X_n$, which we may view as elements of $C_0^\infty(\text{Herm}(V_n))$ via extension-by-zero.

To this end, we first define a morphism of certain Hecke algebras below. We then relate this to the spherical Hecke algebra of the symmetric variety $X_n$ studied by Hironaka [Hir99]. We then reduce Theorem 2.11 to a countable family of explicit transfers in this context, which we show follow from a relative fundamental lemma for an entire Hecke algebra.

3.1. Endoscopy for $X_n$. Set $X_n^{rss} = X_n \cap \text{Herm}(V_n)^{rss}$; this agrees with the invariant-theoretic notion of regular semi-simple locus of $X_n$ as a $U(V_n)$-variety since the open embedding $X_n \hookrightarrow \text{Herm}(V_n)$.
is equivariant.

Fix an elliptic endoscopic datum \((U(V_a) \times U(V_b), s, \eta)\) for \(\text{Herm}(V_a)\) and let \(y \in X^{rss}_n\). Note that any element \((y_a, y_b) \in \text{Herm}(V_a) \times \text{Herm}(V_b)\) matching \(y\) necessarily lies in \(X_a \times X_b\). Thus, it is reasonable to view \(X_a \times X_b\) as an endoscopic symmetric space for \(X_n\). In this way, an elliptic endoscopic datum of the symmetric space \(X_n\) is just an elliptic endoscopic datum \((U(V_a) \times U(V_b), s, \eta)\) for \(\text{Herm}(V_n)\).

### 3.2. A morphism of Hecke algebras.

Assume for the remainder of the section that \(E/F\) is unramified. We now construct the map of Hecke algebras which arises in the fundamental lemma of Hecke algebras stated below. Let \(\mathcal{H}_{K_n,E}(\text{GL}_n(E))\) denote the spherical Hecke algebra of \(\text{GL}_n(E)\). Recall that the Satake isomorphism gives an isomorphism

\[
\text{Sat} : \mathcal{H}_{K_n,E}(\text{GL}_n(E)) \cong \mathbb{C}[T_n]^{S_n},
\]

where \(T_n \subset \text{GL}_n(\mathbb{C})\) is the diagonal split torus in the dual group of \(\text{GL}_n(E)\). Note that \(\mathbb{C}[T_n] \cong \mathbb{C}[Z_1^\pm, \ldots, Z_n^\pm]\), where we normalize the monomials \(\{Z_i\}_{i=1}^n\) as

\[
Z_i \begin{pmatrix} t_1 \\ \vdots \\ t_n \end{pmatrix} = (-1)^i q^{n+1-2i} \prod_{j \geq n+1-i} t_j,
\]

where \(q = \#(\mathcal{O}_F/\mathfrak{p}_F)\) is the cardinality of the residue field of \(F\) (not \(E\)).

**Remark 3.1.** This normalization is chosen to match that of [Off09, Section 6] as it facilitates comparison with the work of Hironaka.

Suppose now that \(n = a + b\). Let \(P_{(a,b)} = M_{(a,b)}N_{(a,b)} \subset \text{GL}_n\) be the standard parabolic subgroup of \(\text{GL}_n\) such that \(M_{(a,b)} \cong \text{GL}_a \times \text{GL}_b\). On the dual group side, consider the embedding

\[
\text{GL}_a(\mathbb{C}) \times \text{GL}_b(\mathbb{C}) \hookrightarrow \text{GL}_n(\mathbb{C})
\]

\[
(m_1, m_2) \mapsto \left( \begin{array}{cc} \mu_b(\varpi)m_1 \\ \mu_a(\varpi)m_2 \end{array} \right),
\]

where \(\mu_s(t) = |t|_{E}^{s/2}\) for any \(t \in E^\times\) and \(s \in \mathbb{C}\). If \(\pi_1 \boxtimes \pi_2\) is a smooth irreducible representation of \(M_{(a,b)}(E)\), this map of dual groups corresponds to the parabolic induction

\[
\pi_1 \boxtimes \pi_2 \mapsto \text{Ind}_{P_{(a,b)}(E)}^{\text{GL}_n(E)}((\pi_1 \mu_b \circ \det) \boxtimes (\pi_2 \mu_a \circ \det))
\]

where \(\text{Ind}_{P_{(a,b)}(E)}^{\text{GL}_n(E)}\) is normalized induction. Restricting to unramified representations, this induces a map on Hecke algebras

\[
\xi_{(a,b)} : \mathcal{H}_{K_n,E}(\text{GL}_n(E)) \rightarrow \mathcal{H}_{K_{a,E}}(\text{GL}_a(E)) \otimes \mathcal{H}_{K_{b,E}}(\text{GL}_b(E)).
\]

The following lemma makes this map explicit.

**Lemma 3.2.** Define the parabolic descent \(f^{P_{(a,b)}} \in C_c^\infty(M_{(a,b)}(E))\) to be

\[
f^{P_{(a,b)}}(m_1, m_2) = \delta_{P_{(a,b)}}^{1/2}(m_1, m_2) \int_{N_{(a,b)}(E)} \int_{K'} f \left( k \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} nk^{-1} \right) dk dn,
\]

where the measures are normalized so that the volumes of the integral points is 1 and \(\delta_{P_{(a,b)}}\) is the modular character of \(P_{(a,b)}(E)\).

Then the morphism \(\xi_{(a,b)}\) of spherical Hecke algebras is given as follows: let \(f \in \mathcal{H}_{K_E}(\text{GL}_n(E))\)

\[
\xi_{(a,b)}(f)(m_1, m_2) = \mu_b(\det(m_1))\mu_a(\det(m_2))f^{P_{(a,b)}}(m_1, m_2).
\]

Here, \(\mu_s : E^\times \rightarrow \mathbb{C}^\times\) for \(s \in \mathbb{C}\) is the unramified character \(\mu_s(t) = |t|_E^{s/2}\).
Proof. This expression is a direct consequence of the Satake isomorphism (see [M11], for example).

Using the Satake isomorphism, this morphism fits into the diagram

$$
\begin{array}{ccc}
\mathcal{H}_{K_n,E}(\text{GL}_n(E)) & \xrightarrow{\text{Sat}} & \mathbb{C}[^T_n]^{S_n} \\
\downarrow \xi_{(a,b)} & & \downarrow \hat{\xi}_{(a,b)} \\
\mathcal{H}_{K_n,E}(\text{GL}_a(E)) \otimes \mathcal{H}_{K_n,E}(\text{GL}_b(E)) & \xrightarrow{\text{Sat}} & \mathbb{C}[^T_a]^{S_a} \otimes \mathbb{C}[^T_b]^{S_b},
\end{array}
$$

where if \( \mathbb{C}[^T_a]^{S_a} \otimes \mathbb{C}[^T_b]^{S_b} \cong \mathbb{C}[X_1, \ldots, X_a]^{S_a} \otimes \mathbb{C}[Y_1, \ldots, Y_b]^{S_b} \) and the variables \( \{X_i\} \) and \( \{Y_j\} \) are normalized analogously to (7), the map \( \xi_{(a,b)} \) is given by

$$
\hat{\xi}_{(a,b)} : Z_i \mapsto \begin{cases} 
q^b X_i & : i \leq a \\
q^a Y_i & : i \geq a + 1,
\end{cases}
$$

where again \( q \) is the cardinality of the residue field of \( F \).

3.3. **The spherical Hecke algebra for \( X_n \).** The group \( \text{GL}_n(E) \) acts on \( X_n \) via twisted conjugation: for any \( g \in \text{GL}_n(E) \) and \( y \in X_n \)

$$
g * y = g y^g. 
$$

It follows from [Jac62] that the \( K_{n,E} \)-orbits on \( X_n \) are

$$
X_n = \bigsqcup_{\lambda \in P_n} K_{n,E} * \omega^\lambda, 
$$

where

$$
\omega^\lambda = \begin{pmatrix}
\omega^{\lambda_1} \\
\vdots \\
\omega^{\lambda_n}
\end{pmatrix},
$$

and

$$
P_n = \{ \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{Z}^n : \lambda_1 \geq \cdots \geq \lambda_n \geq 0 \}.
$$

The \( \text{GL}_n(E) \)-action on \( X_n \) induces an action of \( \text{GL}_n(E) \) on \( C^\infty_c(X_n) \) given by

$$
g * f(y) = f(g^{-1} * y), 
$$

for any \( f \in C^\infty_c(X_n) \), \( g \in \text{GL}_n(E) \) and \( y \in X_n \).

Set \( \mathcal{H}_{K_n,E}(X_n) = C^\infty_c(X_n)^{K_{n,E}} \) to be the vector space of \( K_{n,E} \)-invariant functions. This is known as the spherical Hecke algebra of the symmetric space \( X_n \). Set \( 1_\lambda \) to be the indicator function of the orbit \( K_{n,E} * \omega^\lambda \). The above orbit decomposition implies that \( \{1_\lambda\}_{\lambda \in P_n} \) is a \( \mathbb{C} \)-basis for \( \mathcal{H}_{K_n,E}(X_n) \). Note that with this notation

$$
1_0 = 1_{X_n(\mathcal{O}_E)}. 
$$

The spherical Hecke algebra \( \mathcal{H}_{K_n,E}(\text{GL}_n(E)) \) acts on this space by

$$
f * \phi(y) = \int_G f(g^{-1})\phi(g * y)dg.
$$

The induced \( \mathcal{H}_{K_n,E}(\text{GL}_n(E)) \)-module structure of \( \mathcal{H}_{K_n,E}(X_n) \) is well understood thanks to the work of Hironaka.

**Proposition 3.3.** [Hir99, Theorem 2] As an \( \mathcal{H}_{K_n,E}(\text{GL}_n(E)) \)-module, the spherical Hecke algebra \( \mathcal{H}_{K_n,E}(X_n) \) is free of rank \( 2^n \).
In particular, we have a distinguished rank 1 sub-$\mathcal{H}_{K_{n,E}}(\text{GL}_n(E))$-module given by the embedding

$$- \ast 1_0 : \mathcal{H}_{K_{n,E}}(\text{GL}_n(E)) \hookrightarrow \mathcal{H}_{K_{n,E}}(X_n)$$

Suppose now that $(U(V_a) \times U(V_b), s, \eta)$ is an elliptic endoscopic datum of $X_n$. By a slight abuse of notation, we also denote the map

$$\mathcal{H}_{K_{a,E}}(\text{GL}_a(E)) \otimes \mathcal{H}_{K_{b,E}}(\text{GL}_b(E)) \rightarrow \mathcal{H}_{K_{a,E}}(X_a) \otimes \mathcal{H}_{K_{b,E}}(X_b)$$

$$f_a \otimes f_b \mapsto (f_a \ast 1_0) \otimes (f_b \ast 1_0)$$

by $- \ast 1_0$. Much of this paper consists of the proof of the following Theorem.

**Theorem 3.4.** For any $\varphi \in \mathcal{H}_{K_{n,E}}(\text{GL}_n(E))$, the functions $\varphi \ast 1_0$ and $\xi_{(a,b)}(\varphi) \ast 1_0$ are smooth transfers of each other in the sense of Theorem 2.8.

**Remark 3.5.** In proving Proposition 3.3, Hironaka introduces the spherical Fourier transform which she uses to give an isomorphism of $\mathcal{H}_{K_{n,E}}(\text{GL}_n(E))$-modules,

$$H : \mathcal{H}_{K_{n,E}}(X_n) \xrightarrow{\sim} \mathcal{H}_{K_{n}}(\text{GL}_n(F)),$$

where the module structure on the right being induced by the (injective) base change homomorphism

$$BC : \mathcal{H}_{K_{n,E}}(\text{GL}_n(E)) \rightarrow \mathcal{H}_{K_{n}}(\text{GL}_n(F)),$$

The algebra structure on $\mathcal{H}_{K_{n,E}}(X_n)$ is given via transfer of the algebra structure of $\mathcal{H}_{K_{n}}(\text{GL}_n(F))$ via $H$. In particular, we have a commutative diagram of $\mathcal{H}_{K_{n,E}}(\text{GL}_n(E))$-modules,

$$\mathcal{H}_{K_{n,E}}(\text{GL}_n(E)) \xrightarrow{- \ast 1_0} \mathcal{H}_{K_{n,E}}(X_n) \xrightarrow{H} \mathcal{H}_{K_{n}}(\text{GL}_n(F)).$$

**Remark 3.6.** It is tempting to extend the statement of Theorem 3.4 to the entire Hecke algebra $\mathcal{H}_{K_{n,E}}(X_n)$. Indeed, using the spherical Fourier transform of Hironaka, we may extend the morphism $\xi_{(a,b)}$ to a module homomorphism

$$\xi_{(a,b)} : \mathcal{H}_{K_{n,E}}(X_n) \rightarrow \mathcal{H}_{K_{n,E}}(X_a) \otimes \mathcal{H}_{K_{b,E}}(X_b),$$

and conjecture that for any $\varphi \in \mathcal{H}_{K_{n,E}}(X_n)$, $\xi_{(a,b)}(\varphi)$ is a smooth transfer in the sense of Theorem 2.8. This should play the role of the full fundamental lemma for the relative trace formula for the Galois pair $(\text{GL}_n(E), U(V_n))$.

To make this precise, we would need to deal with several complications not germane to our current discussion. For example, preliminary calculations suggest augmenting the Langlands-Shelstad-Kottwitz transfer factors in a precise way for such a generalization to hold. We plan to return to this in a future paper.

### 3.4. The initial reduction.

We now show that Theorem 3.4 implies the relative endoscopic fundamental lemma. Let $1_{\text{End}(\Lambda_n)}$ and $1_{\text{End}(\Lambda_0)} \otimes 1_{\text{End}(\Lambda_b)}$ be as in the statement of Theorem 2.11.

Set $\Phi^n := r_1 1_{\text{End}(\Lambda_n)}$. This gives a non-standard test function on $X_n$ that is not compactly supported. Nevertheless, it is almost-compactly supported in that if we consider the decomposition of $X_n$ into disjoint closed (in the Hausdorff topology) subsets

$$X_n = \bigsqcup_{d \in \mathbb{Z}} X_{n,d}, \quad X_{n,d} = \{h \in X_n : |\det(h)|_F = q^{-d}\},$$

and set $\Phi^n_d = \Phi^n \cdot 1_{X_{n,d}}$, then $\Phi^n_d \in C_c^\infty(X_n)$ for all $d \in \mathbb{Z}$.
Lemma 3.7. We have $\Phi_d^n \equiv 0$ if $d$ is odd or $d < 0$. If $d \geq 0$, there exist integers $a_\lambda \in \mathbb{Z}_{\geq 0}$ such that

$$\Phi_{2d}^n = \sum_{\lambda \in P_{n,2d}} a_\lambda 1_\lambda,$$

where

$$P_{n,2d} = \left\{ \lambda = (\lambda_1, \ldots, \lambda_n) \in P_n : \sum_i \lambda_i = 2d \right\}.$$

Proof. Since $\text{supp}(\Phi^n) \subset r(\text{End}(\Lambda_n))$, if $x \in \text{supp}(\Phi^n)$, then $\det(x) \in Nm_{E/F}(\mathcal{O}_E) \setminus \{0\}$. Our assumption that $E/F$ is unramified now implies the vanishing statement. Now for any $g \in K_{n,E}$,

$$\Phi^n(gr(x)^7) = \int_{U(V_n)} 1_{\text{End}(\Lambda_n)}(gxh)dh = \int_{U(V_n)} 1_{\text{End}(\Lambda_n)}(xh)dh = \Phi^n(r(x)).$$

Thus, $\Phi^n$ is constant on $K_{n,E}$-orbits of $X_n$. The lemma now follows from the $K_{n,E}$-orbit decomposition (9).

A corollary of this Lemma and Lemma 2.6 is the following restatement of Theorem 2.11.

Corollary 3.8. Theorem 2.11 holds if and only if for every $d \in \mathbb{Z}_{\geq 0}$, the functions

$$\Phi_{2d}^n \text{ and } \sum_{d_a+d_b=d} \Phi_{2d_a}^a \otimes \Phi_{2d_b}^b$$

match in the sense of Theorem 2.8.

Proof. This follows in a straightforward fashion from our previous discussion and Lemma 2.6. \qed

To relate this corollary to Theorem 3.4, we record the following elementary lemma.

Lemma 3.9. For $\phi \in H_{K_{n,E}}(\text{GL}_n(E))$, one has

$$r_1(\phi) = \phi \ast 1_0.$$

Proof. First we prove the claim for $\phi = 1_{K_{n,E}}$. In this special case, it is immediate that $1_{K_{n,E}} \ast 1_0 = 1_0$. On the other hand,

$$r_1(1_{K_{n,E}})(XX^\tau) = \int_{U(V_n)} 1_{K_{n,E}}(Xu)du.$$

The right-hand side is only non-zero if there exists $u \in U(V_n)$ such that $Xu \in K_{n,E}$. This implies that the left-hand side is non-zero only if $XX^\tau \in K_{n,E} \ast I_n$. Since $r_1(1_{K_{n,E}}) \in H_{K_{n,E}}(X_n)$, we must have $r_1(1_{K_{n,E}}) = c1_0$ for some constant $c \in \mathbb{C}$. Since our measure conventions give $U(\Lambda_n) = U(V_n) \cap K_{n,E}$ volume 1, we check that

$$c = r_1(1_{K_{n,E}})(1) = \int_{U(V_n)} 1_{K_{n,E}}(u)du = \text{vol}(U(\Lambda_n)) = 1,$$

proving the claim for $\phi = 1_{K_{n,E}}$. \qed
In general, if \( \phi \in \mathcal{H}_{n,E}(\text{GL}_n(E)) \), then for any other \( \phi_1 \in \mathcal{H}_{n,E}(\text{GL}_n(E)) \),

\[
\phi \ast r_1(\phi_1)(XX^\tau) = \int_{\text{GL}_n(E)} \phi(g^{-1})r_1(\phi_1)(gXX^\tau)f \, dg
\]

\[
= \int_{\text{GL}_n(E)} \int_{U(V_\chi)} \phi(g^{-1})\phi_1(\chi)du \, dg
\]

\[
= \int_{U(V_\chi)} (\phi \ast \phi_1)(\chi)du = r_1(\phi \ast \phi_1)(XX^\tau),
\]

where \( \phi \ast \phi_1 \) denotes the convolution product.

Setting \( \phi_1 = 1_{K_n,E} \), and using the \( \mathcal{H}_{n,E}(\text{GL}_n(E)) \)-module structure, we find that

\[
r_1(\phi) = r_1(\phi \ast 1_{K_n,E}) = \phi \ast r_1(1_{K_n,E}) = \phi \ast (1_{K_n,E} \ast 1_0) = (\phi \ast 1_{K_n,E}) \ast 1_0 = \phi \ast 1_0
\]

\(\square\)

**Proposition 3.10.** Theorem 3.4 implies Theorem 2.11.

**Proof.** Set \( 1_d = \sum_{\lambda \in P_{n,d}} 1_{\lambda} \). Combining the definition of \( \Phi^n_{2d} \) with Lemma 3.9,

\( \Phi^n_{2d} = r_1(1_d) = 1_d \ast 1_0. \)

If we assume the statement of Theorem 3.4, Corollary 3.8 implies that it suffices to show that

\[
\xi_{(a,b)}(1_d) = \sum_{d_a + d_b = d} 1_{d_a} \otimes 1_{d_b}.
\]

(10)

We claim this follows if for all non-negative integers \( n \) and \( d \), we can show that

\[
\text{Sat}(1_d)(Z_i) = \sum_{d_a + d_b = d} Z^{-m},
\]

(11)

where \( m \in \mathbb{Z}_{d,+} \) as in (11) and we set

\[
\sum_{d_a + d_b = d} Z^{-m} = \prod_{i} Z^{-m_i}.
\]

On the other hand, (11) implies that for each \( d_a + d_b = d \),

\[
\text{Sat}(1_{d_a} \otimes 1_{d_b})(X_i, Y_j) = q^{d_a(a-1) + d_b(b-1)} \sum_{a \in \mathbb{Z}_{n_a,+}} \sum_{b \in \mathbb{Z}_{n_b,+}} X^{-\sigma_a(\alpha)}Y^{-\sigma_b(\beta)},
\]

which may be expressed as

\[
q^{d_a(a-1) + d_b(b-1)} \sum_{a \in \mathbb{Z}_{n_a,+}} \sum_{b \in \mathbb{Z}_{n_b,+}} X^{-a}Y^{-b}.
\]

There is a bijection

\[
\bigoplus_{d_a + d_b = d} \mathbb{Z}_{d_a,+} \otimes \mathbb{Z}_{d_b,+} \xrightarrow{\sim} \mathbb{Z}_{d,+}
\]

\[ (a, b) \mapsto a \cup b, \]

where \( \cup \) denotes concatenation. If \( m = a \cup b \) for \( (a, b) \in \mathbb{Z}_{n_a,+} \otimes \mathbb{Z}_{n_b,+} \), then

\[
\xi_{(a,b)}(Z^{-m}) = q^{-d_a}q^{-d_b}X^{-a}Y^{-b}.
\]
In particular, the power of $q$ in front of the $X^{-a}Y^{-b}$ term of $\hat{\xi}_{(a,b)}(Sat(1_d))$ is
$$q^{d(n-1)}q^{-da}q^{-db} = q^{da(a-1)}q^{db(b-1)}.$$ Therefore, the equality (11) implies that
$$\hat{\xi}_{(a,b)}(Sat(1_d))(X_i, Y_j) = \sum_{n_a + n_b = n} q^{da(a-1)+db(b-1)} \sum_{a \in \mathbb{Z}_{n_a, +}} \sum_{b \in \mathbb{Z}_{n_b, +}} X^{-a}Y^{-b}.$$

By the commutativity of (8), this is equivalent to (10).

Using the Satake transform (see [Off09, Section 6], for example), we compute that
$$Sat(1_d)(Z_i) = \sum_{\lambda \in \mathbb{Z}^n} q^{\langle \lambda, 2\rho \rangle} Z^{-\lambda} \int_{N_n(E)} 1_d(u \varpi^\lambda) du,$$
where $N_n(E)$ is the $E$-points of the unipotent radical of the Borel subgroup $B_n(E) \subset GL_n(E)$ of upper triangular matrices. A standard computation shows that
$$\int_{N_n(E)} 1_d(u \varpi^\lambda) du = \begin{cases} q^{\sum_i (i-1)\lambda_i} : \lambda \in P_{n,d} \\ 0 : \text{otherwise} \end{cases}.$$ Therefore, since $\langle \lambda, 2\rho \rangle + \sum_i (i-1)\lambda_i = 2d(n-1)/2 = d(n-1)$, we get
$$Sat(1_d) = q^{d(n-1)} \sum_{\lambda \in P_{n,d}} \sum_{\sigma \in S_n} Z^{-\sigma(\lambda)}. \quad \square$$

4. Nilpotent orbital integrals and the second reduction

In this section, we reduce Theorem 3.4 to a statement of explicit transfers in the context of the Lie algebra version of the Jacquet-Rallis transfer. This relies on recent results of Xiao relating endoscopic transfer for the twisted Lie algebra to germ expansion of orbital integrals in the context of the Jacquet-Rallis transfer. We recall the fundamental notions and results in the next section, review the main result of Xiao in Section 4.2, and execute the reduction in Section 4.3.

4.1. Jacquet-Rallis transfer and fundamental lemma. Let $E/F$ be a quadratic extension of non-archimedean local fields and let $V$ be an arbitrary $n$-dimensional Hermitian space. The linear side of the Lie algebra version of the Jacquet-Rallis comparison is
$$\mathfrak{gl}_n(F) \times F^n \times F_n,$$
where $F_n = (F^n)^*$ is the vector space of $1 \times n$ row vectors. We consider the diagonal action of $GL_n(F)$ on this space. The unitary side of Jacquet-Rallis transfer considers the space
$$Herm(V) \times V,$$
with the diagonal action of $U(V)$. [191x766]

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In particular, the power of $q$ in front of the $X^{-a}Y^{-b}$ term of $\hat{\xi}_{(a,b)}(Sat(1_d))$ is
$$q^{d(n-1)}q^{-da}q^{-db} = q^{da(a-1)}q^{db(b-1)}.$$ Therefore, the equality (11) implies that
$$\hat{\xi}_{(a,b)}(Sat(1_d))(X_i, Y_j) = \sum_{n_a + n_b = n} q^{da(a-1)+db(b-1)} \sum_{a \in \mathbb{Z}_{n_a, +}} \sum_{b \in \mathbb{Z}_{n_b, +}} X^{-a}Y^{-b}.$$

By the commutativity of (8), this is equivalent to (10).

Using the Satake transform (see [Off09, Section 6], for example), we compute that
$$Sat(1_d)(Z_i) = \sum_{\lambda \in \mathbb{Z}^n} q^{\langle \lambda, 2\rho \rangle} Z^{-\lambda} \int_{N_n(E)} 1_d(u \varpi^\lambda) du,$$
where $N_n(E)$ is the $E$-points of the unipotent radical of the Borel subgroup $B_n(E) \subset GL_n(E)$ of upper triangular matrices. A standard computation shows that
$$\int_{N_n(E)} 1_d(u \varpi^\lambda) du = \begin{cases} q^{\sum_i (i-1)\lambda_i} : \lambda \in P_{n,d} \\ 0 : \text{otherwise} \end{cases}.$$ Therefore, since $\langle \lambda, 2\rho \rangle + \sum_i (i-1)\lambda_i = 2d(n-1)/2 = d(n-1)$, we get
$$Sat(1_d) = q^{d(n-1)} \sum_{\lambda \in P_{n,d}} \sum_{\sigma \in S_n} Z^{-\sigma(\lambda)}. \quad \square$$

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4.1. Jacquet-Rallis transfer and fundamental lemma. Let $E/F$ be a quadratic extension of non-archimedean local fields and let $V$ be an arbitrary $n$-dimensional Hermitian space. The linear side of the Lie algebra version of the Jacquet-Rallis comparison is
$$\mathfrak{gl}_n(F) \times F^n \times F_n,$$
where $F_n = (F^n)^*$ is the vector space of $1 \times n$ row vectors. We consider the diagonal action of $GL_n(F)$ on this space. The unitary side of Jacquet-Rallis transfer considers the space
$$Herm(V) \times V,$$
with the diagonal action of $U(V).$
4.1.1. Linear side. We define the invariants of \((x, v, v^*) \in \mathfrak{gl}_n(F) \times F^n \times F_n\) to be 
\[
\chi(x, v, v^*) = (a, b) \in F^n \times F_n
\]
with 
\[
a_i = \text{coefficient of } t^i \text{ in } \det(I - x), \quad \text{and } b_i = v^*(x^i v).
\]
An element \((x, v, v^*)\) is regular semi-simple if and only if 
\[
\det \left( \left( (v^*, x^{i+j} v) \right)_{i,j} \right) \neq 0.
\]
Moreover, the stabilizer of a regular semi-simple element is trivial and two regular semi-simple elements have the same invariants if and only if they are in the same \(\text{GL}_n(F)\)-orbit \([\text{RS07}].\)

For \(f \in C^\infty_c(\mathfrak{gl}_n(F) \times F^n \times F_n),\) we consider the orbital integrals 
\[
\text{Orb}_{\text{GL}_n(F), \eta}(f, (x, v, v^*)) = \int_{\text{GL}_n(F)} f(\text{Ad}(g)x, gv, v^* g^{-1}) \eta(g) dg,
\]
where \(\eta = \eta_{E/F}\) is the quadratic character associated to \(E/F\). This gives a \((\text{GL}_n(F), \eta)\)-invariant distribution. To compare with unitary orbital integrals, we multiply by the transfer factor \(\omega\) introduced in [Zha14, Section 3]. This function is defined for any regular semi-simple \((x, v, v^*)\) as 
\[
\omega(x, v, v^*) = \eta \left( \det[v|xv| \ldots |x^{n-1}v]\right),
\]
where \([v|xv| \ldots |x^{n-1}v]\) denotes the \(n \times n\) matrix with columns \(x^i v\) for \(i = 0, 1, \ldots, n-1\).

4.1.2. Unitary side. We similarly associate to an element \((y, w) \in \text{Herm}(V) \times V\) the invariants \(\chi_V(y, w) = (a, b),\) where 
\[
a_i = \text{coefficient of } t^i \text{ in } \det(I - y), \quad \text{and } b_i = \langle w, x^i w \rangle_V.
\]
It is clear that these values lie in \(F\). For \(f \in C^\infty_c(\text{Herm}(V) \times V),\) we consider the orbital integrals 
\[
\text{Orb}_{U(V)}(f, (y, w)) = \int_{U(V)} f(\text{Ad}(g)y, gw) dg.
\]
As in the linear case, the stabilizer of a regular semi-simple element is trivial and two regular semi-simple elements have the same invariants if and only if they are in the same \(U(V)\)-orbit.

4.1.3. Transfer. Two regular semi-simple elements \((x, v, v^*) \in \mathfrak{gl}_n(F) \times F^n \times F_n\) and 
\((y, w) \in \text{Herm}(V) \times V\) are said to match if their invariants agree. It is helpful to view this matching invariant theoretically.

Suppose that \(\mathcal{A}\) is the categorical quotient \(\mathfrak{gl}_n \times \mathbb{G}_a^n \times (\mathbb{G}_a^n)^*\) by \(\text{GL}_n\). There is a natural isomorphism of affine varieties (see [Cha19, Proposition 2.2.2.1])
\[
\mathcal{A} = \mathfrak{gl}_n \times \mathbb{G}_a^n \times (\mathbb{G}_a^n)^*/\text{GL}_n \cong \text{Herm}(V) \times V/ U(V) \cong \mathbb{A}^{2n},
\]
where here \(\mathbb{A}\) denotes the affine line over \(F\).

The image of the regular locus is an open subvariety \(\mathcal{A}^{rss} \subset \mathcal{A}\), and for any \(a \in \mathcal{A}^{rss}(F)\), the inverse image of \(a\) in \(\mathfrak{gl}_n(F) \times F^n \times F_n\) consists of a single \(\text{GL}_n(F)\)-orbit. On the other hand, the preimage of \(a\) in \(\text{Herm}(V) \times V\) is either empty or a single \(U(V)\)-orbit. On \(F\)-points, this gives a bijection of regular semi-simple orbits [RS07]:
\[
[\text{GL}_n(F) \setminus \mathfrak{gl}_n(F) \times F^n \times F_n]^{rss} \sim \bigsqcup_{V \in \mathcal{V}_n} [U(V) \setminus \text{Herm}(V) \times V]^{rss}.
\]
Here \(V \in \mathcal{V}_n\) runs through our representatives of the isomorphism classes of non-degenerate Hermitian spaces of dimension \(n\).
We say that functions \( f \in \mathcal{C}_c(\mathfrak{gl}_n(F) \times F^n \times F_n) \) and \( \{ f_V \} \) with \( f_V \in \mathcal{C}_c(\mathcal{H}(V) \times V) \) are said to be Jacquet-Rallis transfers if for any matching regular semi-simple elements \((x, v, v^*)\) and \((y, w)\), the following identify holds
\[
\omega(x, v, v^*) \text{Orb}^{\mathcal{GL}_n(F), \eta}_n(f, (x, v, v^*)) = \text{Orb}^U(V)(f_V, (y, w)). \tag{14}
\]
The existence of smooth transfer follows from Theorem 4.2 below.

4.1.4. A variant. In Section 5 below, we will need to also consider a slight variant of the preceding set-up, which is the version of the Jacquet-Rallis transfer for the Lie algebra considered by [Zha14].

To this end, note that there is a natural embedding of \( \mathcal{GL}_n(F) \)-modules
\[
\mathfrak{gl}_n(F) \times F^n \times F_n \hookrightarrow \mathfrak{gl}_{n+1}(F)
\]
\[
(x, v, v^*) \mapsto \begin{pmatrix} x & v \\ v^* & 0 \end{pmatrix},
\]
where \( \mathcal{GL}_n(F) \) acts on \( \mathfrak{gl}_{n+1}(F) \) via the adjoint action as a subgroup of \( \mathcal{GL}_{n+1}(F) \). In particular, we have an isomorphism of \( \mathcal{GL}_n(F) \)-representations
\[
\mathcal{C}_c(\mathfrak{gl}_{n+1}(F)) \xrightarrow{\sim} \mathcal{C}_c(\mathfrak{gl}_n(F) \times F^n \times F_n) \otimes \mathcal{C}_c(F). \tag{15}
\]
Similarly, for an \( n \)-dimensional Hermitian space \( V \) there is a natural embedding of \( U(V) \)-modules
\[
\mathcal{H}(V) \times V \hookrightarrow \mathcal{H}(V \oplus E e_0)
\]
\[
(y, w) \mapsto \begin{pmatrix} x & w \\ \langle w, - \rangle_V & 0 \end{pmatrix},
\]
where we impose that \( \langle e_0, e_0 \rangle = 1 \) and that the sum is direct. As in the linear case, this induces an isomorphism of \( U(V) \)-representations
\[
\mathcal{C}_c(\mathcal{H}(V \oplus E e_0)) \xrightarrow{\sim} \mathcal{C}_c(\mathcal{H}(V) \times V) \otimes \mathcal{C}_c(F). \tag{16}
\]
Noting that the spaces on the right-hand sides of (15) and (16) are related by the matching of orbital integrals (14), we extend the notion of matching functions to one between \( \mathcal{C}_c(\mathfrak{gl}_n(F)) \) and \( \mathcal{C}_c(\mathcal{H}(V_n)) \) compatible with these isomorphisms.

More specifically, we say that
\[
X = \begin{pmatrix} x & v \\ v^* & d \end{pmatrix} \in \mathfrak{gl}_{n+1}(F)
\]
and
\[
Y = \begin{pmatrix} y & w \\ \langle w, - \rangle_V & \lambda \end{pmatrix} \in \mathcal{H}(V \oplus E e_0)
\]
match (resp. are regular semi-simple) if \((x, v, v^*)\) matches \((y, w)\) and \(d = \lambda\) (resp. if \((x, v, v^*)\) and \((y, w)\) are regular semi-simple in the sense of Section 4.1).

For \( f \in \mathcal{C}_c(\mathfrak{gl}_{n+1}(F)) \), we consider the orbital integrals
\[
\text{Orb}^{\mathcal{GL}_n(F), \eta}_n(f, X) = \int_{\mathcal{GL}_n(F)} f(\text{Ad}(g)X)\eta(g)dg, \quad \text{for } X \in \mathfrak{gl}_{n+1}(F)^{rss}
\]
For any \( X \in \mathfrak{gl}_{n+1}(F)^{rss} \), we define the transfer factor \( \omega: \mathfrak{gl}_{n+1}(F)^{rss} \to \mathbb{C} \) to be
\[
\omega(X) = \eta(y \text{det}[e_n+1, X e_{n+1}, \ldots X^n e_{n+1}]) \tag{17}
\]
where
\[
e_{n+1} = t[0, \ldots, 0, 1] \in F^{n+1}.
For $f_V \in C_c^\infty(\Herm(V \oplus Ee_0))$, we consider the orbital integrals

$$\text{Orb}^{U(V)}(f_V, Y) = \int_{U(V)} f_V(\text{Ad}(h)Y)dh, \quad \text{for } Y \in \Herm(V \oplus Ee_0)^{rss}.$$  

The functions $f$ and $\{f_V\}_{V \in V_n}$ are said to be Jacquet-Rallis transfers if for any regular semi-simple $X \in \mathfrak{gl}_{n+1}(F)$ and $Y \in \Herm(V \oplus Ee_0)$ that match, we have

$$\omega(X)\text{Orb}^{\text{GL}_n(F)}(f, X) = \text{Orb}^{U(V)}(f_V, Y). \quad (18)$$

**Remark 4.1.** There are now two notions of “Jacquet-Rallis transfer.” These are on different spaces, so it will always be clear in context which comparison is meant. Nevertheless, to ensure that this does not cause confusion, we will refer to Jacquet-Rallis transfer in the sense of (14) or (18) to specify which is intended.

We now state the two main results in this theory: the existence of Jacquet-Rallis transfers and the fundamental lemma for the Lie algebra. We note that the results of Part 1 do not rely on either of these results, though both are crucial to Part 2.

**Theorem 4.2.** [Zha14] For any $f \in C_c^\infty(\mathfrak{gl}_{n+1}(F))$, there exists a transfer $\{f_V\}_{V \in V_n}$. Conversely, for any $\{f_V\}_V$, there exists a transfer $f$.

Assume now that $E/F$ is an unramified extension of $p$-adic fields and assume $V = V_n$ is our fixed split Hermitian form.

**Theorem 4.3.** [Yun11] The functions $1_{\mathfrak{gl}_{n+1}(O_F)}$ and $\{1_{\Herm(V_n \oplus Ee_0)(O_F)}, 0\}$ are Jacquet-Rallis transfers.

**Remark 4.4.** This theorem was first proved by Yun for characteristic $p$ local fields when $p > n+1$ and transferred to characteristic zero by Gordan in [Yun11], provided the residual characteristic sufficiently high. Recently, Beuzart-Plessis gave a remarkable proof of this statement in characteristic zero with arbitrary residual characteristic [BP19].

### 4.2. Nilpotent orbital integrals

We recall some results from [Xia18] that we use in the sequel. Roughly speaking, one may recover $\kappa$-orbital integrals on the twisted Lie algebra $\Herm(V)$ as limits of the orbital integrals discussed in the previous sections by studying certain singular orbits in the context of Jacquet-Rallis transfer.

Recall from Section 1.5.4 that for a regular semi-simple element $\delta \in \Herm(V)$, there is a decomposition

$$F[\delta] := F[X]/(\text{char}_{\delta}(X)) = \prod_{i=1}^{m} F_i,$$

where $F_i/F$ is a field extension and $\text{char}_{\delta}(X)$ denotes the characteristic polynomial of $\delta$. Setting $S_1 = \{ i : F_i \not\subseteq E \}$, we have $H^1(F, T_\delta) = \prod_{S_1} \mathbb{Z}/2\mathbb{Z}$. Recall that the subgroup $D(T_\delta/F) \subset H^1(F, T_\delta)$ parametrizes rational conjugacy classes in the stable conjugacy class $O_{ad}(\delta)$. Let $S$ be the union of these conjugacy classes and the set of conjugacy classes $O \subset \Herm(V')$ that are Jacquet-Langlands transfers of $\delta$, where $V'$ represents the other isomorphism class of $n$ dimensional Hermitian space over $F$.

**Proposition 4.5.** [Xia18, Proposition 3.8] There is a natural $H^1(F, T_\delta)$-torsor structure on $S$ extending the above classical $D(T_\delta/F)$-torsor structure. In particular, there is a natural bijection between $S$ and $\prod_{S_1} \mathbb{Z}/2\mathbb{Z}$.

Now fix a regular semi-simple element $(x, v, v^*) \in \mathfrak{gl}_n(F) \times F^m \times F_n$, and let $F[x] = \prod_{i=1}^{m} F_i$, $\{1, \ldots, m\} = S_1 \sqcup S_2$ be as above. Then $T_x = F[x]^\times$ is the centralizer of $x$ in $\text{GL}_n(F)$, and we define $T_1 = \prod_{i \in S_1} F_i^\times$. 


The action of $x$ on $F^n$ induces the decomposition

$$F^n = \bigoplus_{i=1}^{m} M_i,$$

where $M_i = F_i \cdot F^n$. We similarly have $F_n = \bigoplus_{i=1}^{m} M_i^*$. With these decompositions, write $v = (v_1, \ldots, v_m)$ and $v^* = (v_1^*, \ldots, v_m^*)$. The assumption that $(x, v, v^*)$ is regular semi-simple implies that $v_i \neq 0$ and $v_i^* \neq 0$ for all $i$.

For any subset $\Sigma \subset S_1$, let $v_{\Sigma}$ denote the vector where

$$v_{\Sigma,i} = \begin{cases} v_i & : i \in \Sigma \\ 0 & : i \in (S_1 \setminus \Sigma) \cup S_2 \end{cases},$$

and likewise for $v_{\Sigma}^*$.

For any $f \in C_c^\infty(\mathfrak{gl}_n(F) \times F^n \times F_n)$ and $\Sigma \subset S_1$, define the generalized nilpotent orbital integral

$$\text{Orb}^{\mathfrak{gl}_n(F), \eta}(f, (x, v_{\Sigma}, v_{S_1 \setminus \Sigma})) = \int_{\text{GL}_n(F)/T} \left( \int_{T_1} f(\text{Ad}(g)x, gtv_{\Sigma}, v_{S_1 \setminus \Sigma}t^{-1}g^{-1}) \prod_{i \in \Sigma} |t_i|^s \prod_{i \in S_1 \setminus \Sigma} |t_i|^{-s} \eta(t) \eta(g) dt \right)_{s=0} dg,$$

where $t_i \in F^\times$ and the integral is understood in terms of a natural meromorphic continuation. The point is that the subsets of $S_1$ are in canonical bijection with certain non-regular orbits in $\mathfrak{gl}_n(F) \times F^n \times F_n$, and these nilpotent orbital integrals are natural $(\text{GL}_n(F), \eta)$-invariant distributions supported on these orbits.

On the other hand, for any $\zeta \in F[x]^\times$, there exists a unique space $\text{Herm}(V_\zeta) \times V_\zeta$ containing a regular semi-simple orbit matching $(x, v, v^*\zeta)$; denote by $(\delta_\zeta, w_\zeta)$ a representative of this orbit. Set $(\delta, w) := (\delta_1, w_1)$.

**Lemma 4.6.** [Xia18, Lemma 4.6] The map $\zeta \mapsto \delta_\zeta$ defines a bijection of $H^1(F, T_\delta)$-torsors

$$F[x]^\times/\text{Nm}_{E[x]/F}(E[x]^\times) \xrightarrow{\sim} S \cong H^1(F, T_\delta).$$

Associate to the subset $\Sigma \subset S_1$ the element of $\Sigma = (\Sigma_i) \in \prod_{i \in S_1} \mathbb{Z}/2\mathbb{Z}$ by setting $\Sigma_i = 1$ if $i \notin \Sigma$ and 0 otherwise. Then for any $(\delta, w) \in \text{Herm}(V) \times V$ matching $(x, v, v^*)$, we obtain a character

$$\kappa_\Sigma : H^1(F, T_\delta) \to \mathbb{C}^\times,$$

$$\delta_\zeta \mapsto (-1)^{\Sigma(\zeta)},$$

where $\Sigma(\zeta) = \#\{i \notin \Sigma : \zeta_i \notin \text{Nm}_{E_i/F}(E_i)\}$. It is evident that all characters $\kappa \in H^1(F, T_\delta)^*$ arise in this fashion for some $\Sigma \subset S_1$. This motivates the following germ expansion, relating these generalized nilpotent orbital integrals and $\kappa$-orbital integrals on the twisted Lie algebra $\text{Herm}(V)$.

**Theorem 4.7.** [Xia18, Theorem 4.7] Suppose that $f$ and $\{f_V\}_V$ are smooth transfers with respect to the Jacquet-Rallis transfer of (14). Then for any regular semi-simple $(x, v, v^*) \in \mathfrak{gl}_n(F) \times F^n \times F_n$, we have the equality

$$\omega(x, v, v^*) \text{Orb}^{\mathfrak{gl}_n(F), \eta}(f, (x, v_{\Sigma}, v_{S_1 \setminus \Sigma})) = \sum_{\zeta \in S} \kappa_\Sigma(\zeta) \int_{U(V_\zeta)/T_{\delta_\zeta}} f_{V_\zeta}(\text{Ad}(g)\delta_\zeta, 0) dg,$$

where $\omega$ is the transfer factor in (12) and $(\delta_\zeta, w_\zeta) \in \text{Herm}(V_\zeta) \times V_\zeta$. 
Note that the right-hand side is essentially a $\kappa$-orbital integral of the function $f_V(-,0) \in C_c^\infty(\text{Herm}(V))$. One caveat is that this sum is over $S$, rather than conjugacy classes in a single stable orbit of $\text{Herm}(V)$. In particular, to recover a $\kappa$-orbital integral on $V$, we must apply the Jacquet-Langlands transfer to the function $f_V$.

4.3. The second reduction. Returning to the context of Theorem 3.4, we assume that $E/F$ is unramified and recall the diagram

$$
\begin{array}{c}
\text{H}_K, E(GL_n(E)) \\
\text{H}_K, E(X_n) \\
\end{array}
\xleftarrow{-1_0} H \xrightarrow{BC} \text{H}_K, (GL_n(F))
$$

where $H$ denotes the isomorphism given by Hironaka. Fix the self-dual lattice $\Lambda_n = \mathcal{O}_E^n \subset V_n$ and the lattice $L_n = \mathcal{O}_E \times \mathcal{O}_{F_n} \subset F^n \times F_n$. Let $1_{\Lambda_n}$ and $1_{L_n}$ be the indicator functions. Extension-by-zero gives an embedding

$$
\mathcal{H}_{K, E}(X_n) \hookrightarrow C_c^\infty(\text{Herm}(V_n));
$$

composing this with tensor multiplication by $1_{\Lambda_n}$ gives

$$
\mathcal{H}_{K, E}(X_n) \hookrightarrow C_c^\infty(\text{Herm}(V_n) \times V_n)
$$

$$
f \mapsto f1_{\Lambda_n} := f \otimes 1_{\Lambda_n}.
$$

We similarly embed $\mathcal{H}_{K, E}(GL_n(F))$ in $C_c^\infty(\mathfrak{gl}_n(F) \times F^n \times F_n)$ using $1_{L_n}$. These latter two spaces are related by the Jacquet-Rallis transfer in the sense of (14).

**Proposition 4.8.** Suppose that for any $\varphi \in \mathcal{H}_{K, E}(GL_n(E))$, the functions

$$
\{(\varphi \ast 1_0)1_{\Lambda_n}, 0\}
$$

and $BC(\varphi)1_{L_n}$ are Jacquet-Rallis transfers in the sense of (14). Then Theorem 3.4 follows.

**Proof.** Fix an elliptic endoscopic datum $(U(V_a) \times U(V_b), s, \eta)$ for $X_n$, matching regular semi-simple elements $y \in X_n$ and $(y_a, y_b) \in X_a \times X_b$, and let $\kappa : \mathcal{D}(T_y/F) \to C^\infty$ be the associated character. We recall the construction of endoscopic transfer for the twisted Lie algebra $\text{Herm}(V_n)$ from [Xin18]. Consider the diagram

$$
\begin{array}{c}
\text{Herm}(V_a) \\
\text{Herm}(V_b) \\
\end{array}
\xleftarrow{ev_0} \text{Herm}(V_n) \times V_n \xleftarrow{JR} \mathfrak{gl}_n(F) \times F^n \times F_n
$$

$$
\begin{array}{c}
\text{Herm}(V_a) \oplus \text{Herm}(V_b) \\
\end{array}
\xleftarrow{ev_0} \prod_{i=a,b} \text{Herm}(V_i) \times V_i \xleftarrow{JR} \prod_{i=a,b} \mathfrak{gl}_i(F) \times F^i \times F_i.
$$

Here, the arrows indicate relations between certain orbital integrals as follows:

- $ev_0$: this arrow indicates the map $ev_0(F)(-) = F(-,0)$;
- $JR$: this arrow indicates the Jacquet-Rallis transfer;
- $PD$: this arrow indicates parabolic descent of relative orbital integrals.

Fixing $f \in C_c^\infty(\text{Herm}(V_n))$, we will construct an endoscopic transfer $f_{a,b}$ of $f$. Choose $F \in C_c^\infty(\text{Herm}(V_n) \times V_n)$ such that $ev_0(F) = f$ and let $\phi \in C_c^\infty(\mathfrak{gl}_n(F) \times F^n \times F_n)$ be a Jacquet-Rallis transfer of $\{F, 0\}$.

We now describe the parabolic descent that arises in the above diagram.

**Definition 4.9.** Let $P_{(a,b)} = M_{(a,b)}N_{(a,b)}$ be the standard maximal parabolic subgroup of $\text{GL}_n(F)$ with Levi factor $M_{(a,b)} \cong \text{GL}_a \times \text{GL}_b$, unipotent radical $N_{(a,b)}$, and set $p_{(a,b)} =$
Lie($P_{a,b}(F)$). For a function $\phi \in C^\infty_c(\mathfrak{gl}_n(F) \times F^n \times F_n)$, denote by $\phi^p := \phi^p_{a,b}$ the following **Lie-algebraic parabolic descent** of $\phi$ to $\prod_{i=a,b} \mathfrak{gl}_i(F) \times F^i \times F_i$:

$$
\phi^p((m_1, m_2), v, v^*) = \int_{n(a,b)} \int_{K_n} \phi \left( k \begin{pmatrix} m_1 & n \\ m_2 & 0 \end{pmatrix} k^{-1}, kv, v^* k^{-1} \right) dk dn,
$$

where $n(a,b) = \text{Lie}(N_{a,b})$.

Returning to the argument, denote by $\tilde{f}$ the parabolic descent of $\phi$ to $\prod_{i=a,b} \mathfrak{gl}_i(F) \times F^i \times F_i$. We now use the Jacquet-Rallis transfer on both lower rank spaces to obtain four functions

$$
F_{a,b}^\alpha, \beta \in C^\infty_c(\text{Herm}(V_\alpha) \times V_\alpha \times \text{Herm}(V_\beta) \times V_\beta),
$$

where $\alpha \in V_a$ and $\beta \in V_b$. Set

$$
f_{a,b}^\alpha, \beta = ev_0(F_{a,b}^\alpha, \beta) \in C^\infty_c(\text{Herm}(V_\alpha) \times \text{Herm}(V_\beta)).
$$

Finally, if $f_{a,b}^{\tilde{\alpha}, \tilde{\beta}}$ denotes the Jacquet-Langlands transfer to $\text{Herm}(V_a) \oplus \text{Herm}(V_b)$ then define

$$
f_{a,b} = \sum_{a, \beta} (-1)^k(\alpha, \beta) f_{a,b}^{(\alpha, \beta)},
$$

where $k(\alpha, \beta)$ is the number of the forms $\{\alpha, \beta\}$ that are split. Theorem 6.1 of [Xia18] asserts that $f_{a,b}^{(a,b)}$ is an endoscopic transfer of $f$.

Now let $\varphi \in \mathcal{H}_{K_n,E}(\text{GL}_n(E))$. To prove the Proposition, we apply this approach to the matching functions $\{(\varphi \ast 1_0)_{1_{L_n}}, 0\}$ and $BC(\varphi)1_{L_n}$. Set $F = \varphi 1_{L_n}$ so that $ev_0(F) = \varphi$.

Fix now an auxiliary element $w \in V_n$ so that $(y, w)$ is regular semi-simple in $\text{Herm}(V_n) \times V_n$ and let $(x, v, v^*)$ be a matching element in $\mathfrak{gl}_n(F) \times F^n \times F_n$. The assumption that $\{(\varphi \ast 1_0)_{1_{L_n}}, 0\}$ and $BC(\varphi)1_{L_n}$ match and Theorem 4.7 implies that there exists a subset $\Sigma \subset S_1$ such that

$$
\omega(x, v, v^*) \text{Orb}^{\mathcal{GL}_n(F), n}(BC(\varphi)1_{L_n}, (x, v, v^*) \Sigma) = \text{Orb}^c(\varphi \ast 1_0, y).
$$

We similarly fix auxiliary vectors $w_a \in V_a$ and $w_b \in V_b$ such that $(y_a, w_a)$ is regular semi-simple in $\text{Herm}(V_a) \times V_a$ and similarly for $(y_b, w_b)$. Using our assumption again, we know that the functions

$$
\{(\xi_{a,b}(\varphi) \ast 1_0)_{1_{L_n} \times X_b}, 0, 0, 0\} \text{ and } BC(\xi_{a,b}(\varphi))1_{L_n} \times X_b
$$

match with respect to Jacquet-Rallis transfer (14). In particular, we have no need to appeal to Jacquet-Langlands transfer in this case.

We may assume that $\xi_{a,b}(\varphi) = \varphi_a \otimes \varphi_b$, so that $BC(\xi_{a,b}(\varphi)) = BC(\varphi_a) \otimes BC(\varphi_b)$. Applying Theorem 4.7 for $V_n$, for any regular semi-simple $(x_a, v_a, v_a^*)$ matching $(y_a, w_a) \in X_a \times V_a$, we have

$$
\omega(x_a, v_a, v_a^*) \text{Orb}^{\mathcal{GL}_n(F), n}(BC(\varphi)_a)1_{L_a} \times X_b, (x_a, v_a, \Sigma_a, v_a^*, \Sigma_a) = \text{SO}(\varphi_a \ast 1_0, y_a),
$$

where the subset $\Sigma_a \subset S_1(a)$ arises from

$$
\Sigma = (\Sigma_a, \Sigma_b) \subset S_1(a) \times S_1(b) = S,
$$

with $S_1(a)$ and $S_1(b)$ as in (5). A similar identity holds for $V_b$.

Applying the argument outlined above, it remains to verify that

$$
BC(\xi_{a,b}(\varphi))1_{L_n} \times X_b = (BC(\varphi)1_{L_n})^p.
$$

Using Lemma 3.2, we check that

$$
BC(\xi_{a,b}(\varphi)) = \xi_{a,b}(BC(\varphi)),
$$
where for any $f' \in C_c^\infty(\GL_n(F))$
\[
\xi'_{(a,b)}(f)(m_1, m_2) = \mu'_b(\det(m_1))\mu'_a(\det(m_2))(f')P_{(a,b)}(m_1, m_2),
\]
where $\mu'_s(t) = |t|_F^{s/2}$ for any $s \in \mathbb{C}$ and the parabolic descent $(f')P_{(a,b)}$ is defined using the modular character of $P_{(a,b)}(F)$. Therefore, it suffices to show that for any $\phi \in \mathcal{H}_{K_n}(\GL_n(F))$
\[
\xi'_{(a,b)}(\phi)1_{\mathcal{L}_a \times \mathcal{L}_b} = (\phi1_{\mathcal{L}_n})^p
\]
as functions on $M_{(a,b)} \times (F^a \oplus F^b) \times (F_a \oplus F_b)$.

Similarly, for $(y, w) \in \mathcal{L}_n \times \mathcal{L}_n^\times = (L_a \times L_b) \oplus (L_a^\times \times L_b^\times)$. In particular, $(\phi1_{\mathcal{L}_n})^p((m_1, m_2), v, v^*)$ equals $1_{\mathcal{L}_a \times \mathcal{L}_b}(v, v^*)$ times
\[
\int_{n(a, b)} \int_{K'} \phi \left(k \begin{pmatrix} m_1 & n \\ m_2 & 1 \end{pmatrix} \right) dk dn
\]
\[
= \int_{n(a, b)} \int_{K'} \phi \left(k \begin{pmatrix} m_1 & n \\ m_2 & 1 \end{pmatrix} \right) \left( 1 \begin{pmatrix} m_1^{-1}n \\ 1 \end{pmatrix} \right) k^{-1} dk dn
\]
\[
= |\det(m_1)|_{F_a}^p \int_{n(a, b)} \int_{K'} \phi \left(k \begin{pmatrix} m_1 & n \\ m_2 & 1 \end{pmatrix} \right) \left( 1 \begin{pmatrix} n \\ 1 \end{pmatrix} \right) k^{-1} dk dn
\]
\[
= \mu'_b(\det(m_1))\mu'_a(\det(m_2))\phi^{\mathcal{P}_{(a,b)}}(m_1, m_2),
\]
where we have used the formula
\[
\delta_{(a,b)}(m_1, m_2) = |\det(m_1)|_{F_a}^p |\det(m_2)|_{F_b}^{-a}.
\]
By Lemma 3.2 and the explanation above, this last expression is precisely $\xi'_{(a,b)}(\phi)1_{\mathcal{L}_a \times \mathcal{L}_b}$, completing the proof. \hfill \Box

5. The Weil representation and the third reduction

We now wish to “peel off” the indicator functions $1_{\mathcal{L}_a}$ and $1_{\mathcal{L}_n}$ from the conjectured transfer for the Hecke algebra. This requires the full power of the Weil representation on the spaces $C_c^\infty(\Herm(W) \times W)$ and $C_c^\infty(\gl_n(F) \times F^n \times F_n^\times)$ studied in [BP19]. We recall this representation now.

5.1. The Weil representation. Now fix an additive character $\psi : F \to \mathbb{C}^\times$ of conductor $O_F$. Let $V$ be an $n$-dimensional Hermitian space. For an element $(x, v, v^*) \in \gl_n(F) \times F^n \times F_n^\times$, we set
\[
q(x, v, v^*) = v^*(v) \in F.
\]
Similarly, for $(y, w) \in \Herm(V) \times V$ we set $q(y, w) = \langle w, w \rangle_V$.

Recall the partial Fourier transforms $\mathcal{F}$ on $C_c^\infty(\gl_n(F) \times F^n \times F_n)$ and $C_c^\infty(\Herm(V) \times V)$: for $f \in C_c^\infty(\gl_n(F) \times F^n \times F_n)$, we set
\[
\mathcal{F}(f)(x, v, v^*) = \int_{F^n \times F_n} f(x, w, w^*)\psi(w^*(v) + v^*(w))dw dw^*.
\]
Similarly, for $f \in C_c^\infty(\Herm(V) \times V)$ we set
\[
\mathcal{F}(f)(y, w) = \int_V f(x, u)\psi(Nm_E/F(\langle u, w \rangle))du.
\]

These transforms induce a Weil representation of $\SL_2(F)$ on these function spaces in the standard way. Indeed, since $\SL_2(F)$ is generated by the elements $\begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$ and...
More precisely, recall from Section 4.1.3 that \( A \) denotes the categorical quotient \( \mathfrak{gl}_n \times G^*_a \times (G^*_a)^*/GL_n \). The image of the regular locus is an open sub-variety \( A^{rss} \subset A \).

We denote the canonical quotient maps by

\[
p_{GL} : \mathfrak{gl}_n(F) \times F^n \times F_n \to A(F)
\]

and

\[
p_V : \text{Herm}(V) \times V \to A(F).
\]

For any \( a \in A^{rss}(F) \) and functions \( f' \in C_c(\mathfrak{gl}_n(F) \times F^n \times F_n) \) and \( f \in C_c(\text{Herm}(V) \times V) \), set

\[
O(a, f) = \begin{cases} 
\text{Orb}_{U(W)}^{U(W)}(f, Y_a) & : p^{-1}_W(a) \neq \emptyset \text{ and } Y_a = (y, w) \in p^{-1}_W(a), \\
0 & : \text{otherwise},
\end{cases}
\]

and

\[
O(a, f') = \omega(X_a) \text{Orb}_{GL_n(F) \times GL_n(F)}^{GL_n(F)}(f', X_a) \text{ for any } X_a = (x, v, v^*) \in p^{-1}_{GL}(a).
\]

With this notation, \( f' \) and \( f \) are transfers in the sense of (14) if and only if

\[
O(a, f') = O(a, f)
\]

as functions on \( A^{rss}(F) \). With this in mind, let

\[
\text{Orb}(\mathfrak{gl}_n(F) \times F^n \times F_n) = \{ a \to O(a, f') : f \in C_c(\mathfrak{gl}_n(F) \times F^n \times F_n) \}
\]

and let

\[
\text{Orb}(\text{Herm}(V) \times V) = \{ a \to O(a, f) : f \in C_c(\text{Herm}(V) \times V) \}.
\]

There are natural Weil representations on \( \text{Orb}(\mathfrak{gl}_n(F) \times F^n \times F_n) \) and \( \text{Orb}(\text{Herm}(V) \times V) \): as before, we need only describe the action of a unipotent element and the Weyl element. For and \( t \in F \) and any \( \Phi \in \text{Orb}(\mathfrak{gl}_n(F) \times F^n \times F_n) \), set

\[
W \left( \begin{array}{cc} 1 & t \\ 1 & 1 \end{array} \right) \Phi(a) = \psi(tq(a))\Phi(a),
\]

where \( q(a) = q(x, v, v^*) \) for any \( (x, v, v^*) \in p^{-1}_{GL}(a) \). Realizing \( \Phi = O(-, f) \) for some \( f \in C_c(\mathfrak{gl}_n(F) \times F^n \times F_n) \), then set

\[
W \left( \begin{array}{cc} 1 & t \\ -1 & 1 \end{array} \right) O(a, f) = O(a, \mathcal{F}(f)).
\]

The formulas for the unitary case are identical.

The compatibility of Jacquet-Rallis transfer and Fourier transforms [Zha14, Theorem 4.17] allows us to conclude the following result.
Proposition 5.1. [BP19, Proposition 1] The Weil representations on
\[ C_c^\infty(g_n(F) \times F^n \times F_n) \text{ and } C_c^\infty(Herm(V) \times V) \]
descend to the Weil representations on
\[ \text{Orb}(g_n(F) \times F^n \times F_n) \text{ and } \text{Orb}(Herm(V) \times V). \]
Moreover, these latter representations coincide on the intersection.

5.2. The third reduction. We utilize these Weil representations to affect our final reduction. For this, we need to consider both forms of the Jacquet-Rallis transfer discussed in Section 4.1.

Proposition 5.2. Suppose that for any \( \varphi \in \mathcal{H}_{K_n,E}(GL_n(E)) \), the functions
\[ \{ \varphi \ast 1_0, 0 \} \text{ and } BC(\varphi) \]
are Jacquet-Rallis transfers in the sense of (18). Then the functions \( \{ (\varphi \ast 1_0)1_{\Lambda_n}, 0 \} \)
and \( BC(\varphi)1_{L_n} \) are transfers in the sense of Theorem 4.2.

Proof. The argument is similar to the proof of the Jacquet-Rallis fundamental lemma in [BP19]. Fix \( \varphi \in \mathcal{H}_{K_n,E}(GL_n(E)) \) and consider
\[ \Phi_\varphi(a) := O(a, (\varphi \ast 1_0)1_{\Lambda_n}) - O(a, BC(\varphi)1_{L_n}) \]
as a function on \( A^{rss}(F) \). We claim that the assumption that \( \{ \varphi \ast 1_0, 0 \} \) and \( BC(\varphi) \) are transfers forces \( \Phi_\varphi \equiv 0 \); it is clear that this implies the proposition. Since \( \Phi_\varphi \) is locally constant, it suffices to show \( \Phi_\varphi(a) = 0 \) for \( a \) in the open dense set where \( q(a) \neq 0 \), where
\[ q(a) = q(x, v, v^*) \]
for any \( (x, v, v^*) \in p_{GL}^{-1}(a) \).

Note that it is immediate that \( \Phi_\varphi(a) = 0 \) if \( |q(a)|_F > 1 \) as the indicator functions are supported away from such orbits. We now assume that \( |q(a)|_F = 1 \). Supposing that \( (x, v, v^*) \in p_{GL}^{-1}(a) \) and \( (y, w) \in p_{V_n}^{-1}(a) \), we see
\[ q(a) = q(x, v, v^*) = q(y, w) \in O_F^{\times}. \]
Since \( q(y, w) \in Nm_{E/F}(E) \), there is an \( \nu \in O_F^{\times} \) such that \( q(a) = Nm_{E/F}(\nu) \). Setting \( e_n = t[0, \ldots, 0, 1] \in O_F^{\times} \), we are free to conjugate \( (x, v, v^*) \in p_{GL}^{-1}(a) \) and \( (y, w) \in p_{V_n}^{-1}(a) \) and assume that \( w = \nu e_n \) and \( (v, v^*) = (Nm_{E/F}(\nu) e_n, t e_n) \).

By the definition of \( O(a, -) \), we have
\[ O(a, (\varphi \ast 1_0)1_{\Lambda_n}) = \int_{U(V_n)} (\varphi \ast 1_0)(Ad(h^{-1})y)1_{\Lambda_n}(h^{-1} e_n)dh. \]

For \( 1_{\Lambda_n}(h^{-1} e_n) \neq 0 \), we must have \( h^{-1} e_n \in O_E^{\times} \). Since the stabilizer of \( e_n \) in \( U(V_n) \) is \( U(V_n-1) \), it follows that this integral is supported on \( U(\Lambda_n)U(V_n-1) \). Since the function \( \varphi \ast 1_0 \in \mathcal{H}_{K_n,E}(X_n) \) is invariant under the action of \( U(\Lambda_n) \), our choice of Haar measure implies that
\[ O(a, (\varphi \ast 1_0)1_{\Lambda_n}) = \text{Orb}^{U(V_n-1)}((\varphi \ast 1_0), y). \]
A similar argument shows that \( |q(a)|_F = 1 \) implies that
\[ O(a, BC(\varphi)1_{L_n}) = \omega(x) \text{Orb}^{GL_{n-1}(F), \eta}(BC(\varphi), x). \]
Thus, our assumption implies that
\[ \Phi_\varphi(a) = 0 \text{ whenever } |q(a)|_F \geq 1. \] (19)

To complete the proof, we make use of the Weil representation. We first note since we assumed that \( \psi \) is unramified, we have
\[ \mathcal{F}(\phi 1_{L_n}) = \phi 1_{L_n}, \]
for any $\phi \in C_c^\infty(\mathfrak{gl}_n(F))$; a similar statement holds for any $\phi' \in C_c^\infty(\text{Herm}(V_n))$. Considering the Weil representation on

$$\text{Orb}(\mathfrak{gl}_n(F) \times F^n \times F_n) \cap \text{Orb}(\text{Herm}(V_n) \times V_n),$$

Proposition 5.1 now implies that

$$W\begin{pmatrix} 1 & t \\ -1 & 1 \end{pmatrix}\Phi_{\varphi} = F(\Phi_{\varphi}) = \Phi_{\varphi}.$$ 

Moreover, (19) implies that for any $t \in p_F^{-1}$,

$$W\begin{pmatrix} 1 & t \\ 1 & 1 \end{pmatrix}\Phi_{\varphi} = \psi(tq(a))\Phi_{\varphi} = \Phi_{\varphi}.$$ 

Since $\text{SL}_2(F)$ is generated by $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & t \\ 1 & 1 \end{pmatrix}$ for $t \in p_F^{-1}$, it follows that

$$W(g)\Phi_{\varphi} = \Phi_{\varphi}$$

for all $g \in \text{SL}_2(F)$.

Now for any $a \in A^{rs}(F)$ with $q(a) \neq 0$, there exists a $t \in F$ such that $\psi(tq(a)) \neq 1$. But then

$$\psi(tq(a))\Phi_{\varphi}(a) = W\begin{pmatrix} 1 & t \\ 1 & 1 \end{pmatrix}\Phi_{\varphi}(a) = \Phi_{\varphi}(a),$$

showing that $\Phi_{\varphi}(a) = 0$. This proves the proposition. $\square$

We now arrive at the final reduction of Theorem 3.4.

**Theorem 5.3.** For any $\varphi \in \mathcal{H}_{K_n,E}(\text{GL}_n(E))$, and for any $X \in \text{GL}_n(F)^{rs}$, we have

$$\omega(X)\text{Orb}^{\text{GL}_{n-1}(F) \cdot \eta(\text{BC}(\varphi), X)} = \begin{cases} \text{Orb}^{U_\eta(V_n-1)}(\varphi * 1_0, Y) & : X \leftrightarrow Y \in X_n^{rs}, \\ 0 & : \text{otherwise}. \end{cases}$$

We prove this in Section 11.1 by spectral techniques. Note that combining Propositions 4.8 and 5.2 with this theorem completes the proof of Theorem 3.4. By Proposition 3.10, we conclude Theorem 2.11.

**Part 2. Spectral transfer and a comparison of relative trace formulas**

In this part, we prove Theorem 5.3. Our approach is a comparison of relative trace formulas we refer to as the twisted Jacquet-Rallis trace formula. This name indicates both a strong analogy with the Jacquet-Rallis case, as well as our dependence on the Jacquet-Rallis transfer and fundamental lemma for the Lie algebra in Theorems 4.2 and 4.3 to obtain the needed geometric comparison.

Let $E/F$ denote a quadratic extension of number fields. Heuristically, the comparison of Jacquet-Rallis may be stated in terms of the matching of orbits

$$\text{GL}_n(E) \backslash \text{GL}_n(E) \times \text{GL}_{n+1}(E) / \text{GL}_n(F) \times \text{GL}_{n+1}(F)$$

with

$$\bigcup_{V \in \mathcal{V}_n} U(V) \backslash U(V) \times U(V \oplus Ee_0)/U(V),$$

where $\mathcal{V}_n$ runs over a set of representatives of the isomorphism classes of $n$ dimensional Hermitian spaces, and the Hermitian form on $V \oplus Ee_0$ is determined by that of $V$. 

The first observation is that the matching of orbital integrals in Theorem 5.3 may be studied globally by *switching the roles* of the rational linear group and the unitary group in the Jacquet-Rallis case. This leads to a matching of orbits

\[
\bigcup_{V \in \mathcal{V}_n} \bigcup_{W \in \mathcal{V}_{n+1}} GL_n(E) \backslash GL_n(E) \times GL_{n+1}(E)/U(V) \times U(W)
\]

where we impose no assumptions on \(V\) and \(W\), with

\[
GL_n(F) \backslash GL_n(F) \times GL_{n+1}(F)/GL_n(F).
\]

We show below that this matching of orbits suggests a comparison of relative trace formulas, the geometric side of which may be calibrated to study the comparison in Theorem 5.3; see Section 11. This allows us to translate the problem into one of *spectral transfer of spherical characters*, a classical method for proving fundamental lemmas for Hecke algebras in the context of the Arthur-Selberg trace formula; see, for example, [Clo90], [Hal95], and [Luo18].

We next observe that the spectral results of Feigon, Lapid, and Offen [FLO12] and Jacquet [Jac10] on unitary periods of cuspidal automorphic forms are precisely what we need to make the spectral comparison manageable. We review the necessary results in Section 8.3. In particular, we have the factorization (37). This is the crucial input, allowing for access to local spherical characters by the relative trace formulas. In fact, the spectral results of *loc. cit.* are so complete that our comparison does not appear to reveal any new information about unitary periods. On the other hand, the global theory of our comparison does not rely on any previous work on these periods. With refined results about non-vanishing of central values of Rankin-Selberg \(L\)-functions of the form of [Li09], our comparison would give a new proof of several of the main results of these works. See Remark 9.8 below.

In the next section, we establish our notational conventions for this part, highlighting important changes from the notation in Part 1. Section 7 covers the local geometric comparison of orbital integrals, proving existence of smooth transfer and the fundamental lemma for the unit element by reducing our comparison to the Lie algebra version of Jacquet-Rallis transfer as in Section 4.1. Next, we review the global and local theory of the invariant distributions we use to build the spherical characters in Section 8, and compare the relative trace formulas in Section 9. The main result of these sections is the transfer of global spherical characters in Theorem 9.7. We then prove a weak transfer of local spherical characters in Section 10.

Finally, we use these results to prove the fundamental lemma for this comparison in Section 11; this is Theorem 11.1. The point is to reduce the local equality of orbital integrals to a statement about global transfer of spherical characters by first globalizing the orbital integrals and then using the comparison of relative trace formulas. The results of Sections 8 and 10 then reduce this problem to a local spectral identity at a single finite place, which we verify directly. Theorem 5.3 is then readily deduced from Theorem 11.1, completing the proof of Theorem 2.11.

*Remark 5.4.* For the reader who is inclined to believe that most of the analytic properties of the Jacquet-Rallis relative trace formula comparison are enjoyed by our set up, we recommend skipping Section 9 except for the statement of Theorem 9.7 as it mirrors [Zha14, Section 2] closely. Some additional care is needed to isolate the comparison for a single pair of Hermitian spaces, but this is not difficult.

*Remark 5.5.* In the final application, we work with globally quasi-split unitary group to prove Theorem 5.3. Despite this, we develop the comparison in general as restricting to the quasi-split case does not simplify the arguments, and in some instances would overly complicate the notation. The general comparison may also be of independent interest.
6. Preliminaries

In this section, we fix our conventions regarding groups, Hermitian spaces, and measures. In order for this part to be self-contained, we only continue to hold to those conventions established in Section 1.5 and do not refer to Part 1 for notation. This allows for additional flexibility, despite a good amount of notation being consistent across both parts. For example, we scrub our notations for orbital integrals, transfer factors, etc. unless making explicit reference to a formula.

6.1. Involutions. For a field $F$, recall the element

$$w_n := \begin{pmatrix} & & & 1 \\ & & \ddots & \\ & -1 & & \\ (-1)^{n-1} & & & \end{pmatrix} \in \text{GL}_n(F).$$

For any $F$-algebra $R$ and $g \in \text{GL}_n(R)$, we define

$$g^\theta = w_n g^{-1} w_n.$$

Now suppose that $E/F$ is a quadratic étale algebra and consider the restriction of scalars $\text{Res}_{E/F}(\text{GL}_n)$. Then for any $F$-algebra $R$ and $g \in \text{Res}_{E/F}(\text{GL}_n)(R)$, we set

$$g^\sigma = \overline{g}$$

to be the Galois involution associated to the extension $E/F$.

**Important notational difference:** In this part, we set $V_n = V_{w_n}$. Thus, the fixed point subgroup $\text{Res}_{E/F}(\text{GL}_n)_{w_n} = U(V_n)$ is a quasi-split unitary group over $F$. We make this choice as it will be convenient to have a form that is split both globally and locally.

6.2. Groups and Hermitian spaces. Let $F$ be a field and fix $E/F$ a quadratic étale algebra. Let $F^n$ be a fixed $n$ dimensional vector space, $F^{n+1} = F^d \oplus F e_0$ with a fixed vector $e_0$. This gives rise to an embedding of $\text{GL}_n(E)$ as the subgroup preserving this decomposition:

$$g \mapsto \begin{pmatrix} g \\ 1 \end{pmatrix}.$$ 

With this, set $G = \text{GL}_n \times \text{GL}_{n+1}$ and $H \cong \text{GL}_n \subset G$, where $H$ is embedded diagonally:

$$g \mapsto \left( g, \begin{pmatrix} g \\ 1 \end{pmatrix} \right).$$

Now consider the product

$$X_n \times X_{n+1}$$

parametrizing pairs of Hermitian vector spaces of dimension $n$ and $n + 1$. A point $(x, y) \in X_n \times X_{n+1}$ determines the unitary groups

$$U(V_x) \hookrightarrow \text{Res}_{E/F}(\text{GL}_n)$$

and

$$U(V_y) \hookrightarrow \text{Res}_{E/F}(\text{GL}_{n+1}).$$

We set $G' = \text{Res}_{E/F}(\text{GL}_n) \times \text{Res}_{E/F}(\text{GL}_{n+1})$ and $H' \cong \text{Res}_{E/F}(\text{GL}_n)$ embedded diagonally as above. For any $(x, y) \in X_n \times X_{n+1}$, set $H'_{x,y} = U(V_x) \times U(V_y)$. Note that

$$H'_{w_n, w_{n+1}} = U(V_n) \times U(V_{n+1})$$

is a product of quasi-split unitary groups.
6.3. Representations and Whittaker models. Suppose that $F$ is a local field and let $\mathcal{GR}_n(F)$ be the set of equivalence classes of generic representation of $GL_n(F)$. For any additive character $\psi : F \to \mathbb{C}^\times$, we denote by $\psi_0$ the generic character of $N_n(F)$

$$\psi_0(u) = \psi \left( \sum_i u_{i,i+1} \right).$$

For a quadratic étale algebra $E/F$, we set $\psi_0^{\prime} = \psi \circ \text{Tr}_{E/F}$ for the induced additive character and $\psi_0^{\prime}$ the generic character of $N_n(E)$. Set $\mathcal{GR}_n(E)$ for the set of equivalence classes of generic representations of $GL_n(E)$ that are isomorphic to their Galois twists. It follows from [AC89] that such representations arise as the base change of a representation $\pi \in \mathcal{GR}_n(F)$ on $GL_n(F)$; we write $\Pi = BC(\pi)$ to denote this relationship. It follows from [FLO12, Theorem 0.2.1] that $\Pi$ has non-trivial invariant $U(V_x)$-invariant functionals for any $x \in X_n$.

For any $\pi \in \mathcal{GR}_n(F)$ we denote by $\pi^{\vee}$ the abstract contragredient representation. Set $W(\pi) := W^{\psi}(\pi)$ to be the Whittaker model of $\pi$ with respect to the generic character $\psi_0$. The action is given by

$$W(g, \pi)W(h) = W(hg), \quad g, h \in GL_n(F), \ W \in W(\pi).$$

Then we obtain an isomorphism

$$\hat{\cdot} : W(\pi)^0 \longrightarrow W^{\psi^{-1}}(\pi^{\vee}),$$

given by $\hat{W}(g) = W(g^0)$.

7. Orbital integrals and transfer

We begin by describing the regular semi-simple orbits and the matching of orbits between our two models. We then describe the local orbital integrals and describe the necessary transfer of test functions and fundamental lemma needed for our global applications.

7.1. Matching and transfer. Let $F$ be a field and let $E/F$ be a quadratic étale algebra over $F$.

7.1.1. Linear side. Recall $G = GL_n \times GL_{n+1}$ and $H = GL_n$ regarded as a subgroup of $G$ via the diagonal embedding. We define the regular semi-simple locus $G(F)^{rss}$ to be the set of points $\gamma$ such that the double coset $H(F)\gamma H(F) \subset G(F)$ is closed and of maximal possible dimension.

Lemma 7.1. Let $GL_{n+1}(F)^{rss}$ to denote the locus of elements $g$ such that, under the adjoint action of $GL_n(F)$, the orbit of $g$ is closed and of maximal dimension. Then there is a natural homeomorphism

$$H(F)\backslash G(F)^{rss}/H(F) \cong GL_{n+1}(F)^{rss}/GL_n(F).$$

Proof. This follows from considering the natural map

$$GL_n(F) \backslash GL_n(F) \times GL_{n+1}(F) \longrightarrow GL_{n+1}(F) \quad GL_n(F)(h, g) \quad \mapsto \quad h^{-1} g.$$
7.1.2. Twisted side. Recall that $V_n$ denotes our set of $GL_n(E)$-orbit representatives for $X_n$. Thus,

$$\{V_x : x \in V_n\}$$

is a fixed set of representatives for the isomorphism classes of $n$-dimensional Hermitian spaces over $E$. In this part, we always require that $w_n \in V_n$ in keeping with our choice of split Hermitian space. Denoting the $GL_n(E)$-orbit of $x \in V_n$ by $X^x_n$, there is a decomposition

$$X_n = \bigsqcup_{x \in V_n} X^x_n. \quad (20)$$

For any $x \in X_n$, set

$$y(x) = \begin{pmatrix} x & \vline & 1 \end{pmatrix} \in X_{n+1}. $$

Then $V_{y(x)} = V_x \oplus E e_0$ where the sum is orthogonal and $(e_0, e_0)_{y(x)} = 1$. With this construction, there is a natural embedding of unitary groups

$$U(V_x) \to U(V_{y(x)}).$$

Note that if $V_x$ is split, then so is $V_{y(x)}$, albeit with a Hermitian form conjugate to $w_{n+1}$.

For any $y \in V_{n+1}$, denote by

$$X_y := X_{n+1} \cdot y \subset \text{Herm}(V_y)$$

the set of invertible elements in the twisted Lie algebra $\text{Herm}(V_y)$.

For any pair $(x, y) \in X_n \times X_{n+1}$, consider the subgroups $H'_{x,y} = U(V_x) \times U(V_y) \subset G'(F)$ and $H' = \text{Res}_{E/F} GL_n$ embedded diagonally. Set $G'(F)^{rss}$ to be the set of points $\delta$ such that the double coset $H'(F)\delta H_{x,y}(F) \subset G'(F)$ is closed and of maximal possible dimension.

We have a similar reduction of the regular orbits in this case.

**Lemma 7.2.** For any $x \in X_n$, define $X_{y(x)}^{rss}$ to be the set of elements $z$ such that, under the adjoint action of $U(V_x)$, the orbit of $z$ is closed and of maximal dimension. Then there is a natural bijection

$$\bigsqcup_{y \in V_{n+1}} H'(F) \backslash G'(F)^{rss} / H'_{x,y}(F) \simeq X_{y(x)}^{rss} / U(V_x).$$

**Proof.** In view of the decomposition (20), this follows by considering the map

$$\bigsqcup_{y \in V_{n+1}} H'(F) \backslash G'(F)/\{1\} \times U(V_y) \to X_y(x)$$

$$H'(F)(g_1, g_2) \mapsto (g_1^{-1} g_2) y(g_1^{-1} g_2) y(x).$$

\[\square\]

**Proposition 7.3.** There is a natural matching of regular semi-simple orbits, giving a bijection:

$$H(F) \backslash G(F)^{rss} / H(F) \sim \bigsqcup_{x \in V_n} \bigsqcup_{y \in V_{n+1}} H'(F) \backslash G'(F)^{rss} / H'_{x,y}(F).$$

**Proof.** By Lemmas 7.1 and 7.2, the claim reduces to the claim that there is a natural matching

$$\text{GL}_{n+1}(F)^{rss} \sim \bigsqcup_x X_{y(x)}^{rss}.$$

This is precisely the setting of the Lie algebra version of Jacquet-Rallis matching of orbits described in Section 5.2. We need only check that this matching respects restriction to the invertible elements of both sides. This may be checked directly via the explicit
invariant polynomials reviewed below, but is more readily seen from noting that regular semi-simple elements \( g \in \mathfrak{gl}_{n+1}(F)^{rss} \) and \( x \in \mathcal{H}erm(V_{g(x)})^{rss} \) match if and only if, viewed naturally as elements of \( \mathfrak{gl}_{n+1}(E) \), they are conjugate by \( \text{GL}_n(E) \).

We say that two regular semi-simple elements \( \gamma \) and \( \delta \) match with respect to \( (x, y) \) and write \( \gamma \xrightarrow{x,y} \delta \) if the orbits

\[
[\gamma] \in H(F)\backslash G(F)^{rss}/H(F) \quad \text{and} \quad [\delta] \in H'(F)\backslash G'(F)^{rss}/H'_{x,y}(F)
\]

match.

7.1.3. Invariant polynomials. We recall the invariant polynomials used in [Zha14] as it will aid certain arguments in Section 11. Let \( X \in \mathfrak{gl}_{n+1}(F) \) which we can express uniquely as

\[
X = \begin{pmatrix} A & b \\ c & d \end{pmatrix}, \quad A \in \mathfrak{gl}_n(F), \ b \in F^n, \ c \in F_n, \ \text{and} \ d \in F.
\]

Then we define the invariant map \( \pi : \mathfrak{gl}_{n+1}(F) \to \mathfrak{h}(F)^{2n+1} \) by

\[
c(X) = (a_1(X), \ldots, a_n(X), b_0(X), \ldots, b_{n-1}(X), d) = (c_i(X))_{i=1}^{2n+1}
\]

where

\[
a_i(X) = \text{Tr}(\wedge^i A), \quad \text{and} \quad b_j(X) = c \cdot A^j \cdot b.
\]

These polynomials are similarly defined for \( Y \in \mathcal{H}erm(V_{g(x)}) \) and two regular semi-simple elements \( X \in \mathfrak{gl}_{n+1}(F)^{rss} \) and \( Y \in \mathcal{H}erm(V_{g(x)})^{rss} \) match if and only if they have the same invariants [RS07].

By a slight abuse of notation, we define the invariant polynomials \( c_i : G(F) \to F \) for \( i = 1, \ldots, 2n+1 \) by setting

\[
c_i(\gamma) := c_i(\gamma_1^{-1}\gamma_2),
\]

where \( \gamma = (\gamma_1, \gamma_2) \in G(F) \).

Similarly, for any pair \( (x, y) \in X_n \times X_{n+1} \), we define the invariant polynomials \( c_i^{x,y} : G'(F) \to F \) for \( i = 1, \ldots, 2n+1 \) by setting

\[
c_i^{x,y}(\delta) := c_i(\pi_{x,y}(\delta))
\]

where \( \delta = (\delta_1, \delta_2) \in G'(F) \) and

\[
\pi_{x,y}(\delta) := (\delta_1^{-1}\delta_2) g(\delta_1^{-1}\delta_2) y(x) \in X_{g(x)}.
\]

7.2. Orbital integrals. Assume now that \( F \) is a local field, and let \( E/F \) be a quadratic étale algebra.

7.2.1. Linear Side. Let \( f \in C_c^\infty(G(F)) \). We define the relative orbital integrals of interest

\[
\text{Orb}^\gamma(f, \gamma) := \int_H \int_H f(h_1^{-1}(\gamma_1, \gamma_2)h_2) \eta(h_2) dh_1 dh_2,
\]

where \( \gamma = (\gamma_1, \gamma_2) \in G(F)^{rss} \) is a regular semi-simple element. This assumption implies that the centralizer of \( \gamma \) is trivial and that the orbit of \( \gamma \) is closed, so the integral is well defined. Consider the function \( \tilde{f} \in C_c^\infty(\text{GL}_{n+1}(F)) \) defined as

\[
\tilde{f}(g) := \int_H f(h^{-1}(1, g)) dh.
\]

We note that the map

\[
C_c^\infty(G) \to C_c^\infty(\text{GL}_{n+1}(F))
\]

\[
f \mapsto \tilde{f}
\]
is surjective. Since the integrals are absolutely convergent, a simple rearrangement gives
\[ \text{Orb}^\eta(f, \gamma) = \text{Orb}^{\GL_n(F), \eta}(\tilde{f}, \gamma_1^{-1}\gamma_2) := \int_{\GL_n(F)} \tilde{f}(h^{-1}\gamma_1^{-1}\gamma_2 h)\eta(h)dh. \] (23)

Note that this orbital integral is of the type arising on the linear side of the Jacquet-Rallis transfer in the sense of (18).

The transfer factor in this case is built out of the transfer factor (17) for the Lie algebra version of Jacquet-Rallis transfer. For an element \( X \in \mathfrak{gl}_{n+1}(F)^{rss} \), set
\[
\omega(X) = \eta(\det([e_{n+1} | X e_{n+1} | \ldots | X^n e_{n+1}]))
\]
where
\[
e_{n+1} = t[0, \ldots, 0, 1] \in F^{n+1}.
\]

**Definition 7.4.** We define the transfer factor \( \Omega : G(F)^{rss} \to \mathbb{C} \) by
\[
\Omega(\gamma_1, \gamma_2) := \omega(\gamma_1^{-1}\gamma_2).
\]

**7.2.2. Twisted side.** For any pair \((x, y) \in X_n \times X_{n+1}\), we define the orbital integral
\[
\text{Orb}(f', \delta) := \int_{H'} \int_{H'_{x,y}} f'(h_1^{-1}(\delta_1, \delta_2)h_2)dh_1dh_2,
\]
where \( f' \in C^\infty(G') \) and \( \delta = (\delta_1, \delta_2) \in G'(F)^{rss} \). Similarly to the previous case, we first define \( \tilde{f}' : X^y_{y(x)} \to \mathbb{C} \) by
\[
\tilde{f}'(gy) = \int_{H'} \int_{U(V_y)_{y}} f'(h^{-1}(1, gu))dhdu.
\]
We see that
\[
\text{Orb}(f', \delta) = \text{Orb}^{U(V_y)}(\tilde{f}', \pi_{x,y}(\delta)) := \int_{U(V_y)} \tilde{f}'(h^{-1}\pi_{x,y}(\delta)h)dh.
\] (24)

We similarly note that this orbital integral is of the type arising on the unitary side of the Jacquet-Rallis transfer.

**7.2.3. Taking care of the center.** We need to take the action on the center into account. Fixing \((x, y) \in X_n \times X_{n+1}\) and setting \( Z_{G'} \subset G' \) denote the center, consider the \( Z_{G'(F)}H'(F) \times H'_{x,y}(F)\)-action on \( G'(F) \).

Following the reductions above, it suffices to consider the \( Z_{G'(F)} \times U(V_x)\)-action on \( X_{y(x)} \). Here \( U(V_x) \) acts via conjugation, while the center acts by
\[
(z_1, z_2) \cdot s = (z_1^{-1}z_2)s(z_1^{-1}z_2).
\]
Set
\[
Z_0 = \{(z_1, z_2) \in Z_{G'(F)} \times U(V_x) : z_1 \in Z_{U(V_y)}(F), z_2 \in Z_{U(V_y)}(F) \}
\cong Z_{U(V_y)}(F) \times Z_{U(V_y)}(F).
\]

It is simple to check that \( Z_0 \) acts trivially under the above action. For any \( s \in X_{y(x)} \), we may write
\[
s = \begin{pmatrix} A & b \\ \langle b, - \rangle_x & d \end{pmatrix},
\]
where \( A \in \mathbb{Herm}(V_x), b \in E^n, \) and \( d \in F \). It is easy to check that
\[
\text{Tr}(A) \text{ and } d
\]
are scaled by a non-zero norm class under the $Z_{G'}(F) \times U(V_x)$-action, so that their norm classes are invariant. In analogy to [Zha14], we call $s \in X_{g(x)}$ $Z$-regular semi-simple if it is regular semi-simple in $X_{g(x)}$ and if

$$\text{Tr}(A), d \in F^\times.$$ 

This gives a Zariski open dense subset of $X_{g(x)}$.

**Lemma 7.5.** If $s$ is $Z$-regular semi-simple, then its centralizer under the $Z_{G'}(F) \times U(V_x)$-action is $Z_0$ and its orbit is closed. In particular, a $Z$-regular semi simple element is $Z_{G'} \times U(V_x)$-regular semi-simple.

**Proof.** If $(z, h) \circ s = s$, then since $\text{Tr}(A)$ and $d$ are invertible, we may augment $(z, h)$ by an element of $Z_0$ to assume that $z = (1, 1)$. But now $h$ lies in the centralizer of $s$ under the adjoint action of $U(V_x)$. This is trivial since $s$ is regular semi-simple, proving the first claim.

We now note that when $\text{Tr}(A), d \in E^\times$, the rational functions

$$\frac{\text{Tr}(\wedge^i A)}{\text{Tr}(A)^i} \text{ and } \frac{\langle b, A^2 b \rangle_y}{\text{Tr}(A)^{i+1} d}$$

for $1 \leq i \leq n$ and $0 \leq j \leq n - 1$ are invariant under $Z_{G'}(F) \times U(V_x)$. We claim that two $Z$-regular semi-simple elements $s_1$ and $s_2$ are in the same $Z_{G'}(F) \times U(V_x)$-orbit if and only if they have the same values under the invariants (25) and

$$\text{Tr}(A_1) \equiv \text{Tr}(A_2) \mod \text{Nm}(E^\times) \text{ and } d_1 \equiv d_2 \mod \text{Nm}(E^\times),$$

where $A_i$ and $d_i$ are as above. Indeed, sufficiency is immediate. To prove necessity, suppose that they have the same invariants and norm classes. By augmenting $s_2$ to $z \circ s_2$ for an appropriate central element $z \in Z_{G'}(F)$, we may assume that $\text{Tr}(A_1) = \text{Tr}(A_2)$ and $d_1 = d_2$.

Considering the invariants above, this implies that $\text{Tr}(\wedge^i A_1) = \text{Tr}(\wedge^i A_2)$ for each $i$ and $\langle b_1, A_1^2 b_1 \rangle_x = \langle b_2, A_2^2 b_2 \rangle_x$ for all $j$. These are precisely the invariants noted in (22), so it follows from our assumption that $s_1$ and $s_2$ are regular semi-simple that they lie in the same $U(V_x)$-orbit. As in [Zha14], this implies that the $Z_{G'}(F) \times U(V_x)$-orbit of $s_1$ is closed. \hfill \Box

We say that $\delta \in G(F)$ is $Z$-regular semi-simple if $\pi_{x,y}(\delta)$ is. In this case, for any central character $\omega' : Z_{G}(F) \to \mathbb{C}^\times$ and any $f' \in C_c^\infty(G(F))$ we set

$$\text{Orb}_{\omega'}(f', \delta) := \int_{H(F)} \int_{Z_{G'}(F) \backslash H(F)} f(h^{-1} \delta) \omega'(zh) dzh dh.$$ 

The integration is absolutely convergent by the closed orbit assertion of the lemma.

For linear case, if $Z_{G}(F) \subset G(F)$ is the center, we consider the action of $Z_{G}(F)H(F) \times H(F)$ on $G(F)$. As before this reduces to considering the $Z_{G}(F) \times \text{GL}_n(F)$-action on $\text{GL}_{n+1}(F)$, and we say that $g \in \text{GL}_{n+1}(F)$ is $Z$-regular semi-simple if it is regular semi-simple under the $\text{GL}_n(F)$-action and $\text{Tr}(A), d \neq 0$ where

$$g = \begin{pmatrix} A & b \\ c & d \end{pmatrix} \text{ where } A \in \text{gl}_n(F), b, c \in F^n, \text{ and } d \in F.$$

A similar but easier argument now shows that a $Z$-regular semi-simple element of $\text{GL}_{n+1}(F)$ has trivial centralizer under $Z_{G}(F) \times \text{GL}_n(F)$ and has a closed orbit. We say $\gamma = (\gamma_1, \gamma_2)$ is $Z$-regular semi-simple if $\gamma_1 \gamma_2$ is. In this case, for any central character $\omega : Z_{G}(F) \to \mathbb{C}^\times$ and $f \in C_c^\infty(G(F))$ then set

$$\text{Orb}^0(\omega, \gamma) := \int_{H(F)} \int_{H(F)} \int_{Z_{G}(F)} f(h^{-1}(\gamma_1, \gamma_2) z) \omega(z) dz dh_1 dh_2.$$
The integration is absolutely convergent.

7.3. Smooth transfer. We say that functions \( f \in C_c^\infty(G(F)) \) and \( \{f'_{x,y}\}_{x,y} \) with \( f'_{x,y} \in C_c^\infty(G'(F)) \) and \((x, y) \in V_n \times V_{n+1}\) match if for any matching regular orbits \( \gamma \xrightarrow{\mathcal{R}} \delta \), the following identify holds

\[
\Omega(\gamma) \text{Orb}^\eta(f, \gamma) = \text{Orb}(f'_{x,y}, \delta).
\]  

(26)

When \( E = F \times F \), the transfer of functions may be made explicit. Here, \( V_n \) and \( V_{n+1}\) are both singletons and \( \eta \) is trivial. For \( k = n, n+1 \), we may choose isomorphisms \( GL_k(E) \cong GL_k(F) \times GL_k(F) \) such that the unitary groups \( U(V_x) \cong GL_n(F) \hookrightarrow GL_n(E) \) and \( U(V_y) \cong GL_{n+1}(F) \hookrightarrow GL_{n+1}(E) \) are sent to

\[
U(V_x) \cong \{(g, g^\theta) \in GL_n(F) \times GL_n(F) : g \in GL_n(F)\}
\]

and

\[
U(V_y) \cong \{(g, g^\theta) \in GL_{n+1}(F) \times GL_{n+1}(F) : g \in GL_n(F)\},
\]

where we recall that for \( g \in GL_k(E) \), \( g^\theta = w_k^1g^{-1}w_k \). The proof of the next proposition is a simple computation, which we omit.

**Proposition 7.6.** When \( E = F \times F \) as above, the functions \( f_1 \otimes f_2 \in C_c^\infty(G(F) \times G(F)) \) and \( f_1 \ast f_2^{\theta_1} \in C_c^\infty(G(F)) \) are smooth transfers. Here \( \ast \) denotes convolution and 

\[
f^{\theta_1}(g) = f(g^{-\theta}).
\]

Assume now that \( F \) is non-archimedean and that \( E/F \) is a quadratic field extension. The existence of smooth transfer now follows from the existence of smooth transfer for the Jacquet-Rallis transfer.

**Theorem 7.7.** Assume that \( E/F \) is a quadratic extension of non-archimedean fields. For any \( f \in C_c^\infty(GL_{n+1}(F)) \), there exists a transfer \( \{f'_{x,y} \}_{x,y} \). Conversely, for any \( \{f'_{x,y} \}_{x,y} \), there exists a transfer \( f \).

**Proof.** This follows directly from Theorem 4.2 by the reductions (23) and (24) in the previous section. Indeed, the identity of orbital integrals (26) may be reduced to

\[
\omega(\gamma_{1}^{-1}\gamma_2) \text{Orb}^{GL_n(F), \eta}(f, \gamma_{1}^{-1}\gamma_2) = \text{Orb}^{U(W)}(f'_{x,y}, \pi_{V_y(x)}(W)(\delta)).
\]

This is precisely the context of Theorem 4.2. \( \square \)

Now fix a single pair \((x, y) \in X_n \times X_{n+1}\), and consider a function \( f'_{x,y} \in C_c^\infty(G'(F)) \). The above theorem tells us that there exists \( f \in C_c^\infty(G(F)) \) such that

\[
\Omega(\gamma) \text{Orb}^\eta(f, \gamma) = \begin{cases} 
\text{Orb}(f'_{x,y}, \delta) & : \gamma \xrightarrow{x,y} \delta \in G'(F)_{\text{rss}}, \\
0 & : \text{otherwise}.
\end{cases}
\]  

(27)

Consider the closed and open subset \( G[x, y] = G_F[x, y] \subset G(F) \) such that

\[
G[x, y] = \{(g_1, g_2) \in GL_n(F) \times GL_{n+1}(F) : \eta(g_1) = \eta(x), \eta(g_2) = \eta(y)\}.
\]

**Lemma 7.8.** Assume that either \( E/F \) is split or that \( F \) is non-archimedean. For \( f'_{x,y} \) and \( f \) as above, we may assume that \( \text{supp}(f) \subset G[x, y] \).

**Proof.** When \( E/F \) is split, \( G[x, y] = G(F) \) so that the statement is vacuous. We now assume that \( F \) is non-archimedean. In this case, there are four possible pairs of Hermitian spaces. We index them as follows:

\[
\{(i, j) : (i, j) \in \mathbb{F}_2^2 \text{ such that } \eta(x_i) = (-1)^i, \eta(y_j) = (-1)^j\}.
\]
There is then a decomposition of $G(F)$ into open and closed subsets.

$$G(F) = \bigcup_{(i,j) \in \mathbb{F}_2^2} G[x_i, y_j].$$

Similarly, we may decompose $\text{GL}_{n+1}(F) = G_0 \sqcup G_1$ where

$$G_i = \{ g \in \text{GL}_{n+1}(F) : \eta(g) = (-1)^i \}.$$ 

Recall that the map

$$p : G(F) \to \text{GL}_{n+1}(F)$$

$$(g_1, g_2) \mapsto g_1^{-1} g_2$$

is a submersion. Since

$$p(G[x_0, y_0]) = p(G[x_1, y_1]) = G_0 \quad \text{and} \quad p(G[x_0, y_1]) = p(G[x_1, y_0]) = G_1,$$

the disjoint unions above implies that the restrictions

$$G[x_0, y_0] \xrightarrow{p} G_0, \quad G[x_1, y_1] \xrightarrow{p} G_0 \quad \text{and} \quad G[x_0, y_1] \xrightarrow{p} G_1, \quad G[x_1, y_0] \xrightarrow{p} G_1$$

are all submersions. Therefore, the map

$$C^\infty_c(G[x_i, y_j]) \longrightarrow C^\infty_c(G_{i+j})$$

$$f \mapsto \tilde{f},$$

is surjective, for each $(i, j) \in \mathbb{F}_2^2$ where the sum $i+j$ is considered modulo 2. To conclude the lemma, note that (27) is equivalent to

$$\omega(g) \, \text{Orb}_{\text{GL}_n(F), \eta}^{\text{GL}_n(F), \eta}(\tilde{f}, g) = \begin{cases} \text{Orb}_{U(V_x)}^{U(V_x)}(\tilde{f}_{x,y}, h) : g \leftarrow h \in X_{y(rss)}^{y(x)}; \\ 0 : \text{otherwise}. \end{cases}$$

(29)

By construction, the support of $\tilde{f}_{x,y}$ lies in

$$X_{y(x)}^{y} = \{ h \in X_{y(x)} : \eta(h) = \eta(x)\eta(y) = (-1)^{k(x,y)} \},$$

where $k(x,y) \in \{0,1\}$. Thus, we may replace $\tilde{f}$ by $\tilde{f} \cdot 1_{G_{k(x,y)}}$ without affecting the matching (29). The surjectivity of (28) now implies that we are free to choose $f$ so that $\text{supp}(f) \subset G[x,y]$, proving the lemma. \hfill \Box

7.4. The fundamental lemma. Now assume that $E/F$ is an unramified quadratic extension of non-archimedean local fields.

Set $K = G(\mathcal{O}_F)$ and $1_K$ the corresponding characteristic function. Also, set $K' = G'(\mathcal{O}_F)$ and let $1_{K'}$ denote the characteristic function.

**Theorem 7.9.** The functions $1_K$ and $\{ f'_{x,y} \}$ are transfers where

$$f'_{x,y} = \begin{cases} 1_{K'} : (x, y) = (w_n, w_{n+1}), \\ 0 : \text{otherwise}. \end{cases}$$

**Proof.** This follows from the previous reductions and Theorem 4.3 by restricting to the integral locus such that $|\det(X)|_F = 1$. As previously stated, this was recently reproved in characteristic zero with no assumption on the residue characteristic in [BP19]. We therefore do not need to make any assumptions on the residue characteristic. \hfill \Box
8. Factorization of Certain Global Distributions

We recall the definitions of certain global and local distributions that arise in the spectral decomposition of the relative trace formulas. In this section, $F$ is a number field and $E/F$ a quadratic extension. We set $\mathbb{A}_F$ for the adele ring of $F$, and $\mathbb{A}_E$ for that of $E$. As always, we consider the diagonal embedding $\text{GL}_n(F) \hookrightarrow \text{GL}_n(\mathbb{A}_F)$. For any $n$, we set $A_n \subset \text{GL}_n(\mathbb{A}_F)$ to be the connected component of the identity in the maximal $\mathbb{Q}$-split torus in the center of $\text{GL}_n(F_\infty) = \prod_{v \mid \infty} \text{GL}_n(F_v)$. We set

$$[\text{GL}_n] := A_n \text{GL}_n(F) / \text{GL}_n(\mathbb{A}_F).$$

We adopt similar notations for other algebraic groups. Finally, we set $\eta := \eta_{\mathbb{A}_E/\mathbb{A}_F}$ to be the idele class character associated to the quadratic extension.

8.1. Peterson inner product. Suppose $\pi$ is a cuspidal automorphic representation of $\text{GL}_n(\mathbb{A}_F)$, and let $\tilde{\pi} \cong \pi^\vee$ denote the contragredient representation of $\pi$ realized on the space of functions $\{ \phi^\theta : \phi \in \pi \}$, where $g^\theta = w_0^i g^{-1} w_0$. Consider the inner product

$$(\phi, \tilde{\phi}) = \int_{A_n \text{GL}_n(F) / \text{GL}_n(\mathbb{A}_F)} \phi(g) \tilde{\phi}(g) dg;$$

this is a $\text{GL}_n(\mathbb{A}_F)$-invariant inner product on $\pi$. Denote by $W^\phi$ the $\psi$-th Fourier coefficient of a cusp form $\phi$:

$$W^\phi(g) = \int_{[N_n]} \phi(n g) \psi_0^{-1}(n) dn,$$

where $\psi_0$ is our generic character of the unipotent radical $N_n(\mathbb{A}_F)$.

Suppose now that $S$ is a finite set of places, containing the archimedean ones, such that $\pi_v$ is unramified and $\psi_{0,v}$ has conductor $\mathfrak{o}$ for $v \not\in S$. Let $\phi \in \pi$ be factorizable, write $W^\phi(g) = \prod_v W_v^\phi(g_v)$, where $W_v \in W^\psi_v(\pi_v)$. Similarly, let $\tilde{\phi} \in \tilde{\pi}$ be factorizable and set $W^\tilde{\phi}(g) = \prod_v \tilde{W}_v(g_v)$, where $\tilde{W}_v \in W^{\psi_v^{-1}}(\tilde{\pi}_v) = W^{\psi_v^{-1}}(\pi_v^\vee)$. We may assume that for all $v \not\in S$, $W_v$ and $\tilde{W}_v$ are spherical and normalized so that $W_v(e) = \tilde{W}_v(e) = 1$.

We recall the canonical inner product

$$[\cdot, \cdot]_{\pi_v} : W^{\psi_v}(\pi_v) \otimes W^{\psi_v^{-1}}(\pi_v) \rightarrow \mathbb{C}.$$

It is defined by considering the integral

$$I_s(W_v, \tilde{W}_v^\prime) = L(n, 1_{F_v^\circ}) \int_{\mathbb{P}_n(F_v) \backslash \mathbb{P}_n(F_v)} W_v(h) \tilde{W}_v^\prime(h) \det(h)^{\sigma} f_d dh,$$

where $W_v, \tilde{W}_v^\prime \in W^{\psi_v}(\pi_v)$ and $P_v \cong \text{GL}_{n-1} \times G_v^{n-1}$ is the mirabolic subgroup of $\text{GL}_n$. The integral converges for $\text{Re}(s) \gg 0$, and has meromorphic continuation. It is known for any local field of characteristic zero (see [FLO12, Appendix A] and the references therein) that this continuation is holomorphic at $s = 0$ and gives a non-degenerate $\text{GL}_n(F_v)$-invariant pairing. We set

$$[W_v, \tilde{W}_v^\prime]_{\pi_v} := I_0(W_v, \tilde{W}_v^\prime).$$

Moreover, when $\pi_v$ is unramified and $W_v$ is the spherical vector normalized so that $W_v(e) = 1$, then

$$[W_v, \tilde{W}_v^\prime]_{\pi_v} = L(1, \pi_v \times \pi_v^\vee),$$

where $L(s, \pi_v \times \pi_v^\vee)$ denotes the local Rankin-Selberg $L$-factor.

Proposition 8.1. [FLO12, Section 10.3] Assume that $\phi \in \pi$ is factorizable as above. There is a corresponding factorization

$$(\phi, \tilde{\phi}) = \text{Res}_{s=1} L(s, \pi \times \pi^\vee) \prod_v [W_v, \tilde{W}_v^\prime]_{\pi_v}^2,$$
where
\[
[W_v, \hat{W}_v]_{\pi_v}^2 = \frac{[W_v, W_v]_{\pi_v}}{L(1, \pi_v \times \pi_v^\pm)}.
\]

8.2. Rankin-Selberg period. The results of this section are found in [JPSS83]. Let \( \Pi = \Pi_n \otimes \Pi_{n+1} \) be a generic cuspidal automorphic representation of \( G(\mathbb{A}_F) \), where \( \Pi_n \) is a generic cuspidal automorphic representation of \( GL_i(\mathbb{A}_F) \). The global Rankin-Selberg period is given by
\[
\lambda_\Pi(\phi) = \int_{[GL_n]} \phi_n(h)\phi_{n+1} \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} dh,
\]
where \( \phi_n \in \Pi_n \). Now let \( \mathcal{W}^{-1}(\Pi_n) \) and \( \mathcal{W}(\Pi_{n+1}) \) be the corresponding Whittaker models. The local Rankin-Selberg period is defined in terms of the local Whittaker model \( \mathcal{W}(\Pi_v) := \mathcal{W}^{-1}(\Pi_{n,v}) \otimes \mathcal{W}^\psi(\Pi_{n+1,v}) \) by
\[
\lambda_{\Pi_v}(s, W_v) = \int_{N_n(F) \backslash GL_n(F)} W_v(h) |\det(h)|^s dh, \quad s \in \mathbb{C}, \ W_v \in \mathcal{W}(\Pi_v).
\]
We also consider the normalized period by introducing the local Rankin-Selberg \( L \)-function \( L(s, \Pi) \):
\[
\lambda^\sharp_{\Pi_v}(s, W_v) = \frac{\lambda_{\Pi_v}(s, W_v)}{L(s + \frac{1}{2}, \Pi_v)}.
\]
For any generic \( \Pi_v \), the integral \( \lambda_{\Pi_v}(s, \cdot) \) is absolutely convergent when \( \text{Re}(s) \) is large and extends meromorphically to \( \mathbb{C} \). Moreover, the normalized integral \( \lambda^\sharp_{\Pi_v}(s, \cdot) \) is entire in \( s \in \mathbb{C} \), and we set
\[
\lambda^\sharp_{\Pi_v}(W_v) = \lambda^\sharp_{\Pi_v}(0, W_v);
\]
this gives a non-zero element of the one-dimensional space \( \text{Hom}_{H(F_v)}(\Pi_v, \mathbb{C}) \) for any generic \( \Pi \).

Remark 8.2. When \( \Pi_v \) is tempered, the integral is absolutely convergent for \( \text{Re}(s) > -\frac{1}{2} \), so that there is no need to analytically continue the integral to \( s = 0 \).

With our measure conventions, when \( W_v \) is the normalized spherical vector and \( \psi_v \) has conductor \( \mathcal{O}_F \),
\[
\lambda_{\Pi_v}(s, W_v) = L(s + \frac{1}{2}, \Pi_v),
\]
where \( L(s, \Pi_v) = L(s, \Pi_{n,v} \times \Pi_{n+1,v}) \) is the local Rankin-Selberg \( L \)-factor. This implies that
\[
\lambda^\sharp_{\Pi_v}(W_v) = 1. \tag{32}
\]

Proposition 8.3. We have the following decomposition when \( \Pi \) is unitary, and \( \phi = \phi_n \otimes \phi_{n+1} \in \Pi \) is factorizable:
\[
\lambda_{\Pi}(\phi) = L\left(\frac{1}{2}, \Pi_n \times \Pi_{n+1}\right) \prod_v \lambda^\sharp_{\Pi_v}(W_v), \tag{33}
\]
where \( W^\Pi(g : \phi) = \prod_v W_v(g_v) \).

We will also need the twisted version \( \lambda^\eta \) of this period, were \( \eta : F^\times \rightarrow \mathbb{C}^\times \) is a quadratic unitary character. This distribution is given, both globally and locally, by setting
\[
\lambda^\eta_{\Pi_n \otimes \Pi_{n+1}} = \lambda_{\Pi_n \otimes \Pi_{n+1}} \lambda^\eta_{\Pi_n},
\]
and similarly for the normalized distribution.
Corollary 8.4. We have the following decomposition when $\Pi$ is unitary, and $\phi = \phi_n \boxtimes \phi_{n+1} \in \Pi$ is factorizable:

$$\lambda^n_\Pi(\phi) = L\left(\frac{1}{2} \Pi_n \times \Pi_{n+1} \cdot \eta\right) \prod_v \lambda^n_{\Pi_v}(W_v), \quad (34)$$

where $W^\Pi(\phi) = \prod_v W_v(g_v)$.

8.3. Unitary periods. Let $\Pi$ be a cuspidal automorphic representation of $\operatorname{GL}_n(\mathbb{A}_E)$. For any Hermitian form $x \in X_n$, we have the associated unitary group $U(V_x) \subset \operatorname{Res}_{E/F} \operatorname{GL}_n$. For $\phi \in \Pi$, we define the unitary period $\mathcal{P}_x(\phi)$ by the (convergent) integral

$$\mathcal{P}_x(\phi) = \int_{[U(V_x)]} \phi(h) dh.$$  

Then $\Pi$ is said to be distinguished by $U(V_x)$ if there exists a vector $\psi$ such that $\mathcal{P}_x(\phi) \neq 0$. If $V_x = V_n$ is split, then a theorem of Jacquet [Jac10] states that if there is a cuspidal automorphic representation $\pi$ of $\operatorname{GL}_n(\mathbb{A}_F)$ such that $\Pi = B\pi(\pi)$, then $\Pi$ is distinguished by $U(V_n)$. In general, Corollary 10.3 of [FLO12] gives a vast generalization to other forms.

To define the local unitary periods, we need a bit more terminology. Fix a place $v$ of $F$ and consider the quadratic étale extension $E_v/F_v$.

Let $\Pi_v \in \operatorname{Temp}(\operatorname{GL}_n(E_v))$ and denote by $\mathcal{E}(X_n, \mathcal{W}^\psi_v(\Pi_v)^\times)$ the set of all maps

$$\alpha : X_n(F_v) \times \mathcal{W}^\psi_v(\Pi_v) \to \mathbb{C},$$

which are continuous and $\operatorname{GL}_n(E)$-invariant with respect to the diagonal action. Note that we have an isomorphism

$$\mathcal{E}(X_n, \mathcal{W}^\psi_v(\Pi_v)^\times) \sim \bigoplus_{x \in V_v} \operatorname{Hom}_{U(V_v)}(\mathcal{W}^\psi_v(\Pi_v), \mathbb{C})$$

$$\alpha \mapsto (\alpha(x, \cdot))_{x \in V_v}.$$  

Now for any such $\alpha$, we consider the twisted Bessel character $J^\alpha_{\Pi_v} : C^\infty_c(X_n(F_v)) \to \mathbb{C}$ given by

$$J^\alpha_{\Pi_v}(f') = \langle f' \cdot \alpha, \lambda^\vee_1 \rangle,$$

where $f' \cdot \alpha$ is the smooth functional

$$W \mapsto \int_{X_n(F_v)} f'(x) \alpha(x, W) dx,$$

which we identify with an element of $\mathcal{W}^{\psi_v^{-1}(\Pi_v)^\vee}$ via the pairing $[\cdot, \cdot]_{\Pi}$, and $\lambda^\vee_1$ denotes the functional $W \mapsto W(1)$. Similarly, for $\pi_v \in \operatorname{Temp}(\operatorname{GL}_n(F_v))$, we define the Bessel character $I_{\pi_v} : C^\infty_c(\operatorname{GL}_n(F_v)) \to \mathbb{C}$ by

$$I_{\pi_v}(f) = \langle f \cdot \lambda_{w_n}, \lambda^\vee_1 \rangle,$$

where $f = \lambda_{w_n}$ denotes the smooth functional

$$W \mapsto \int_{\operatorname{GL}_n(F_v)} f(g) W(w_n g) dg,$$

which we again identify with an element of $\mathcal{W}^{\psi_v^{-1}(\pi_v)^\vee}$ via the pairing $[\cdot, \cdot]_{\pi_v}$, and $\lambda^\vee_1$ denotes the functional $W \mapsto W(1)$.

One of the main results of [FLO12] is the following theorem.
Theorem 8.5. For every $\pi_v \in \text{Temp}(\text{GL}_n(F_v))$, there exists a unique
$$\alpha_{\pi_v} \in \mathcal{E}(X_n, \mathcal{W}_\psi^v(BC(\pi_v))^*)$$
such that the identity
$$J_{BC(\pi_v)}^\alpha(f) = I_{\pi_v}^a(f)$$
holds for all pairs of test functions $(f, f')$ which are transfers with respect to Jacquet-Ye transfer as defined in [FLO12, Section 3].

We refer to the functionals $\alpha_{\pi_v}$ as FLO functionals. When $E_v \cong F_v \times F_v$ is split, so that $BC(\pi_v) \cong \pi_v \otimes \pi_v$, these functionals are very simple [FLO12, Corollary 7.2]:
$$\alpha_{\pi_v}^w(h, h)(W' \otimes W'') = \left[\mathcal{W}(h, \pi_v)W', \mathcal{W}(w_0, \pi_v)\mathcal{W}'''\right]_{\pi_v}$$
for any $h \in \text{GL}_n(F_v)$ and $W', W'' \in \mathcal{W}_\psi^v(\pi_v)$.

Lemma 8.6. [FLO12, Lemma 3.9] Assume that $E_v$ is non-archimedean of odd residue characteristic. Further assume that $E_v/F_v$ is an unramified extension and that $\psi_v'$ has conductor $O_{E_v}$. Let $\Pi_v = BC(\pi_v)$ be unramified and let $W_0 \in \mathcal{W}_\psi^v(\Pi_v)$ denote the normalized spherical vector. Then for any $x \in X_n(O_{E_v}) = \text{GL}_n(O_{E_v}) \ast w_n$
$$\alpha_{\pi_v}^x(W_0) = L(1, \pi_v \times \pi_v^\vee \cdot \eta_v).$$

We define for $W \in \mathcal{W}_\psi^v(\Pi_v)$
$$\alpha_{\pi_v}^{x, \delta}(W) = \frac{\alpha_{\pi_v}^x(W)}{L(1, \pi_v \times \pi_v^\vee \cdot \eta_v)}.
$$
Returning to our extension of number fields $E/F$, we have the following factorization of unitary periods of cusp forms.

Proposition 8.7. [FLO12, Theorem 10.1] Let $\pi$ be an irreducible cuspidal automorphic representation, and let $\Pi = BC(\pi)$. Then for any $x \in X_n$, we have
$$\mathcal{P}_x(\hat{\phi}) = 2L(1, \pi \times \pi^\vee \cdot \eta) \prod_v \alpha_{\pi_v}^{x, \delta}(W_v),$$
where $W_{\Pi}(g : \phi) = \prod_v W_v(g_v)$.

8.4. Global spherical characters. Assume that $\pi = \pi_n \boxtimes \pi_{n+1}$ is a cuspidal automorphic representation of $G(\mathbb{A}_F)$.

Definition 8.8. We define the global spherical character $I_\pi$ as the following distribution: for $f \in C_c^\infty(G(\mathbb{A}_F))$, we set
$$I_\pi(f) = \sum_\phi \lambda_\pi(\pi(f)\phi)\lambda_{\pi^\vee}(\hat{\phi}) \langle \phi, \hat{\phi} \rangle,$$
where the sum runs over an orthogonal basis for $\Pi$ and where $\hat{\phi}(g) = \phi(g^0)$ is a vector in the contragredient representation $\Pi^\vee$.

Now let $\Pi = \Pi_n \boxtimes \Pi_{n+1}$ be an irreducible cuspidal automorphic representation of $G'(\mathbb{A}_F)$. For any $(x, y) \in V_n \times V_{n+1}$, we set $\mathcal{P}_{x, y} = \mathcal{P}_x \otimes \mathcal{P}_y$ to be the product of unitary periods.

Definition 8.9. We define the global twisted spherical character $J_{\Pi}^{x, y}$ as the following distribution: for $f' \in C_c^\infty(G'(\mathbb{A}_F))$, we set
$$J_{\Pi}^{x, y}(f') = \sum_\phi \lambda_{\Pi}(\Pi(f')\phi)\mathcal{P}_{x, y}(\hat{\phi}) \langle \phi, \hat{\phi} \rangle,$$
where the sum runs over an orthogonal basis for $\Pi$ and where $\hat{\phi}(g) = \phi(g^0)$. 


Note that \( J_{\Pi}^{x,y} \equiv 0 \) unless \( \Pi = \Pi_n \boxtimes \Pi_{n+1} \) is \( H_{x,y}' \)-distinguished. When this is the case, a theorem of Jacquet [Jac05] implies that there must exist a cuspidal representation \( \pi = \pi_n \boxtimes \pi_{n+1} \) of \( G(\mathbb{A}_F) = \mathrm{GL}_n(\mathbb{A}_F) \times \mathrm{GL}_{n+1}(\mathbb{A}_F) \) such that
\[
\Pi_i = BC(\pi_i), \quad i = n, n + 1.
\]
Since \( \Pi_i \) is cuspidal, we know \( \pi_i \not\cong \pi_i \cdot \eta \). Therefore, the Rankin-Selberg \( L \)-function \( L(s, \pi \times \pi^\vee \cdot \eta) \) is holomorphic at \( s = 1 \). Using the relation
\[
L(s, \Pi \times \Pi^\vee) = L(s, \pi \times \pi^\vee)L(s, \pi \times \pi^\vee \cdot \eta),
\]
we see that
\[
\frac{\mathrm{Res}_{s=1}(L(s, \Pi \times \Pi^\vee))}{L(1, \pi \times \pi^\vee \cdot \eta)} = \frac{\mathrm{Res}_{s=1}(L(s, \pi \times \pi^\vee))}{\neq 0}.
\]

8.5. Local spherical characters and factorization. Denote by \( \mathcal{W}(\Pi_v) \) the Whittaker model \( \mathcal{W}^\psi_{v^-1}(\Pi_{n,v}) \otimes \mathcal{W}^\psi_{v^+}(\Pi_{n+1,v}) \). Also denote by \( \alpha_{x,y} = \alpha_{n,x,v} \otimes \alpha_{n+1,y,v} \) the product of FLO functionals.

**Definition 8.10.**  
1. We define the normalized local twisted spherical character \( J_{\Pi_v}^{x,y} \) associated to a unitary generic representation \( \Pi_v \) of \( G'(F_v) \) and \( (x, y) \in \mathcal{V}_n \times \mathcal{V}_{n+1} \) as follows: for \( f'_v \in C_c^{\infty}(G'(F_v)) \)

\[
J_{\Pi_v}^{x,y}(f'_v) = \sum_{W_v} \frac{\lambda_{\Pi_v}^L(\Pi_v(f'_v)W_v)\alpha_{V_v}(W_v)}{|W_v|_{\Pi_v}},
\]

where the sum ranges over an orthogonal basis for \( \mathcal{W}(\Pi_v) \). We denote by \( J_{\Pi}^{x,y} \) the distribution defined using the unnormalized local periods.

2. We similarly define the normalized local spherical character \( I_{\pi_v}^x \) for any unitary generic representation \( \pi_v \) of \( G(F_v) \); for \( f_v \in C_c^{\infty}(G(F_v)) \)

\[
I_{\pi_v}^x(f_v) = \sum_{W_v} \frac{\lambda_{\pi_v}^L(\pi_v(f_v)W_v)\lambda_{\pi_v}^{y,v}(W_v)}{|W_v|_{\pi_v}},
\]

where the sum ranges over an orthogonal basis for \( \mathcal{W}(\pi_v) \). We denote by \( I_{\pi}^x \) the distribution defined using the unnormalized local periods.

We now state the factorization results.

**Proposition 8.11.** Consider an irreducible cuspidal automorphic representation \( \Pi = \otimes_v \Pi_v \) and fix a pair \((x, y)\). If \( J_{\Pi}^{x,y} \) is not identically zero, let \( \pi = \pi_n \boxtimes \pi_{n+1} \) be an irreducible cuspidal automorphic representation of \( G(\mathbb{A}_F) \) such that \( \Pi = BC(\pi) \).

We have the product decomposition: for any factorizable \( f' = \prod_v f'_v \in C_c^{\infty}(G'(\mathbb{A}_F)) \),
\[
J_{\Pi}^{x,y}(f') = 4\frac{L \left( \frac{1}{2}, \Pi_n \times \Pi_{n+1} \right)}{\mathrm{Res}_{s=1}(L(s, \pi \times \pi^\vee))} \prod_v J_{\Pi_v}^{x,y}(f'_v). \tag{42}
\]

**Proof.** The product decomposition follows immediately from (31), (33), (37), and (41). \( \square \)

This implies that the global twisted spherical character \( J_{\Pi}^{x,y} \) is non-vanishing if and only if

1. the global \( L \)-factor \( L \left( \frac{1}{2}, \Pi_n \times \Pi_{n+1} \right) \) is non-vanishing, and
2. the local FLO functionals \( \alpha_{x,y} = \alpha_{n,x,v} \otimes \alpha_{n+1,y,v} \) are non-vanishing for every place \( v \) of \( F \).
This is due to the non-vanishing of the local Rankin-Selberg periods for any generic representation $\Pi_v$ [JPSS83].

In the linear case, we have a similar factorization.

**Proposition 8.12.** Consider an irreducible cuspidal automorphic representation $\pi = \otimes'_v \pi_v$. For any factorizable $f = \prod_v f_v \in C_c^\infty(G(\mathbb{A}_F))$, the spherical character $I_\pi$ associated to $\pi$ factorizes as

$$I_\pi(f) = \frac{L\left(\frac{1}{2}, \Pi_n \times \Pi_{n+1}\right)}{\text{Res}_{s=1}(L(s, \pi \times \pi_v))} \prod_v I_{\pi_v}(f_v),$$

(43)

where $\Pi_n \boxtimes \Pi_{n+1} = BC(\pi_n) \boxtimes BC(\pi_{n+1})$.

**Proof.** This similarly follows from (31), (33), and (40). \qed

In particular, for any cuspidal automorphic representation $\pi_n \boxtimes \pi_{n+1}$, the global spherical character is non-vanishing if and only if the central $L$-value $L\left(\frac{1}{2}, \Pi_n \times \Pi_{n+1}\right)$ is non-vanishing.

### 9. Comparison of relative trace formulas

The main result of this section is Theorem 9.7, showing the following spectral transfer of global spherical characters. Assume that $E/F$ is a quadratic extension of number fields that is split at all archimedean places of $F$. Let $\Pi = \Pi_n \boxtimes \Pi_{n+1}$ be a cuspidal automorphic representation of $G'(\mathbb{A}_F)$ satisfying certain local assumptions and such that $\Pi \cong \Pi''$, and for any pair of Hermitian forms $(x,y) \in X_n \times X_{n+1}$, there exist “nice” matching test functions $f' \in C_c^\infty(G'(\mathbb{A}_F))$ and $f \in C_c^\infty(G(\mathbb{A}_F))$ such that

$$J^{x,y}_{\Pi}(f') = \sum_{\pi \in B(\Pi)} I_\pi(f).$$

where $B(\Pi)$ is the (finite) set of cuspidal automorphic representations $\pi$ of $G(\mathbb{A}_F)$ such that $\Pi = BC(\pi)$.

We will prove this and more via a comparison of (simple forms of) two relative trace formulas, which we now introduce. Much of this section mirrors [Zha14, Section 2] closely.

**9.1. The linear side.** Suppose that $f \in C_c^\infty(G(\mathbb{A}_F))$ and consider the automorphic kernel

$$K_f(x,y) = \sum_{\gamma \in G(F)} f(x^{-1} \gamma y).$$

We consider the distribution on $C_c^\infty(G(\mathbb{A}_F))$

$$I(f) = \int_{[H]} \int_{[H]} K_f(h_1, h_2) \eta(h_2) dh_1 dh_2.$$

We also consider versions of this distribution $I_\omega$, where $\omega$ is a central character for $G(\mathbb{A}_F)$ by replacing $K_f$ with

$$K_{f,\omega}(x,y) = \int_{[Z_G(\mathbb{A}_F)]} \sum_{\gamma \in G(F)} f(x^{-1} \gamma y) \omega(\gamma) d\gamma z$$

Note that the intersection of $Z_G$ and $H$ is trivial.

This integral does not converge in general. Following [Zha14], we introduce the space of *nice* test functions.

**Definition 9.1.** We say $f = \prod_v f_v \in C_c^\infty(G(\mathbb{A}_F))$ is a *nice test function* with respect to the central character $\omega = \prod_v \omega_v$ if
(1) For at least one place $v_1$, the function $f_{v_1}$ is essentially a matrix coefficient of a supercuspidal representation with respect to $\omega_{v_1}$: this means that

$$f_{v_1, \omega_{v_1}}(g) = \int_{Z_G(F_{v_1})} f_{v_1}(gz)\omega_{v_1}(z)dz$$

is a matrix coefficient of a supercuspidal representation of $G(F_{v_1})$.

(2) For at least one split place $v_2 \neq v_1$, the test function $f_{v_2}$ is supported on the $Z$-regular locus of $G(F_{v_2})$. This place is not required to be non-archimedean.

**Lemma 9.2.** Let $\omega$ be a unitary character of $Z_G(F)\backslash Z_G(\mathbb{A}_F)$. Suppose that $f = \prod_v f_v$ is nice with respect to $\omega$. Then

1. As a function on $H(\mathbb{A}_F) \times H(\mathbb{A}_F)$, $K_f(x,y)$ is compactly supported modulo $H(F) \times H(F)$. In particular, $I(f)$ converges absolutely.
2. As a function on $H(\mathbb{A}_F) \times H(\mathbb{A}_F)$, $K_{f,\omega}(x,y)$ is compactly supported modulo $H(F) \times H(F)Z_G(\mathbb{A}_F)$. In particular, $I_{\omega}(f)$ converges absolutely.

**Proof.** The argument is verbatim as in the case of the Jacquet-Rallis relative trace formula [Zha14, Lemma 2.2].

This implies that when $f$ is nice, we have the decomposition into a finite sum of integrals.

$$I(f) = \sum_{\gamma} \text{Orb}^0(f, \gamma),$$

where the sum is over regular semi-simple $\gamma \in H(F)\backslash G(F)/H(F)$ and

$$\text{Orb}^0(f, \gamma) := \int_{H(\mathbb{A}_F)} \int_{H(\mathbb{A}_F)} f(h_1^{-1}\gamma h_2)\eta(h_2)dh_2dh_1 = \prod_v \text{Orb}^{0_v}(f_v, \gamma_v).$$

We have a similar decomposition for $I_{\omega}$:

$$I_{\omega}(f) = \sum_{\gamma} \text{Orb}^0_{\omega}(f, \gamma),$$

where the sum is over $Z$-regular semi-simple $\gamma \in H(F)\backslash G(F)/Z_G(F)H(F)$ and where

$$\text{Orb}^0_{\omega}(f, \gamma) := \int_{H(\mathbb{A}_F)} \int_{H(\mathbb{A}_F)} \int_{Z_G(\mathbb{A}_F)} f(h_1^{-1}\gamma zh_2)\eta(h_2)\omega(z)dzdh_2dh_1 = \prod_v \text{Orb}^{0_v}_{\omega}(f_v, \gamma_v).$$

If $\pi = \pi_\pi \boxtimes \pi_{n+1}$ is a cuspidal automorphic representation of $G(\mathbb{A}_F)$, recall the definition (38) of the spherical character $I_{\pi}$.

**Proposition 9.3.** Let $\omega$ be a unitary central character. If $f$ is nice with respect to $\omega$, then we have the equality

$$\sum_{\gamma} \text{Orb}^0(f, \gamma) = \sum_{\pi} I_{\pi}(f), \quad (44)$$

where the first sum is over $Z$-regular semi-simple orbits

$$\gamma \in H(F)\backslash G(F)^{rss}/Z_G(F)H(F),$$

and where the second sum runs over irreducible cuspidal automorphic representations of $G(\mathbb{A}_F)$ with central character $\omega$.

**Proof.** The proof of this is standard so we omit the details. See [Zha14, Theorem 2.3] for an analogous argument and [GH19, Theorem 18.2.2] for a general treatment including absolute convergence of both sides. \qed
9.2. The twisted side. This case is entirely analogous to the previous case, so we just state the results. Fix a pair of Hermitian forms \((x, y) \in X_n \times X_{n+1}\). Let \(G' = \text{GL}_n(F) \times \text{GL}_{n+1}(F) = \text{Res}_{F/F}(G)\) and consider the two subgroups \(H' = \text{GL}_n(F)\) embedded diagonally and \(H'_{x,y} = \text{U}(V_x) \times \text{U}(V_y)\). For \(f' \in C_c^\infty(G'(\mathbb{A}_F))\), we form the analogous kernel \(K_{f'}\) and consider the distribution

\[
J^{x,y}(f') = \int_{H'(F)} \int_{H'_{x,y}(F)} K_{f'}(h_1, h_2) dh_1 dh_2.
\]

We also consider versions of this distribution \(J^{x,y}_w\), where \(\omega'\) is a central character for \(Z_{G'}(\mathbb{A}_F)\) that is trivial on \(Z_{G'}(\mathbb{A}_F) \cap H'_{x,y}\) by replacing \(K_{f'}\) with

\[
K_{f', \omega'}(x, y) = \int_{Z_{G'}(\mathbb{A}_F)} \sum_{\gamma \in G'(F)} f'(x^{-1} \gamma y) \omega'(z) dz.
\]

Note that such a central character is the base change of a central character \(\omega\) of \(G(\mathbb{A}_F)\). That is, it is of the form \(\omega' = \omega \circ N_{\mathbb{A}_F/F'}\).

As in the linear case, we introduce the space of nice test functions. We say \(f' = \prod_v f'_v \in C_c^\infty(G'(\mathbb{A}_F))\) is nice with respect to the central character \(\omega'\) if

1. For at least one place \(v_1\), the function \(f'_{v_1}\) is essentially a matrix coefficient of a supercuspidal representation with respect to \(\omega'_{v_1}\): this means that

\[
f'_{v_1, \omega'_{v_1}}(g) = \int_{Z_{G'}(F_{v_1})} f'_{v_1}(gz) \omega'_{v_1}(z) dz
\]

is a matrix coefficient of a supercuspidal representation of \(G'(F_{v_1})\).

2. For at least one split place \(v_2 \neq v_1\), the test function \(f'_{v_2}\) is supported on the \(Z\)-regular locus of \(G'(F_{v_2})\). This place is not required to be non-archimedean.

If \(\Pi = \prod_{n} \Pi_{n+1}\) is a cuspidal automorphic representation of \(G'(\mathbb{A}_F)\), recall the definition (39) of the twisted spherical character \(J_{\Pi}^{x,y}\).

**Proposition 9.4.** Let \(\omega'\) be a unitary central character. If \(f\) is nice with respect to \(\omega'\), then we have the equality

\[
\sum_{\delta} \text{Orb}_{\omega'}(f', \delta) = \sum_{\Pi} J_{\Pi}^{x,y}(f'),
\]

where the first sum is over \(Z\)-regular semi-simple orbits

\[
\delta \in H'(F) \backslash G'(F)/\text{Res}_{F/F}(G)(F) H'_{x,y}(F),
\]

and where the second sum runs over irreducible cuspidal automorphic representations of \(G'(\mathbb{A}_F)\) with central character \(\omega'\).

9.3. Global matching of test functions. Suppose now that \(f = \prod_v f_v \in C_c^\infty(G(\mathbb{A}_F))\) and \(\{f_{x,y}^{(x, y)}(x, y) = \prod_v f_{x,y,v} \in C_c^\infty(G'(\mathbb{A}_F))\}\) where \(f_{x,y}^{(x, y)} = 0\) for all but finitely many \((x, y)\). Suppose further that for each \(v\), the local functions \(f_v\) and \(\{f_{x,y,v}\}\) are smooth transfers of each other.

If we consider global \(Z\)-regular semi-simple classes \(\gamma \in G(F)\) and \(\delta \in G'(F)\) that match with respect to the pair \((x, y)\), it follows that for each place \(v\)

\[
\Omega_v(\gamma) \text{Orb}^{\gamma}_{\omega}'(f_v, \gamma) = \text{Orb}(f'_{x,y,v}, \delta).
\]

Noting that the transfer factor satisfies the global product formula

\[
\prod_v \Omega_v(\gamma) = 1 \text{ whenever } \gamma \in G(F),
\]
this implies the comparison of global orbital integrals
\[ \operatorname{Orb}^0(f, \gamma) = \prod_v \Omega_v(\gamma) \operatorname{Orb}^0(f_v, \gamma) = \prod_v \operatorname{Orb}(f'_{x,y,v}, \delta) = \operatorname{Orb}(f'_{x,y}, \delta). \]

Using the $Z$-regular semi-simple assumption, we further obtain the matching of orbital integrals with central character
\[ \operatorname{Orb}^0(f, \gamma) = \operatorname{Orb}_{\omega'}(f'_{x,y}, \delta), \quad (46) \]
where $\omega = \prod_v \omega_v : Z_G(F) \backslash G(\mathbb{A}_F) \to \mathbb{C}$ is a unitary central character of $G(\mathbb{A}_F)$ and $\omega' = \omega \circ \text{Nm}_{\mathbb{A}_F/\mathbb{A}_F}^x$.

9.4. Comparison. To obtain the necessary refined comparison, we make use of the automorphic-Cebotarev-density theorem of Ramakrishnan.

**Theorem 9.5.** [Ram18] Let $E/F$ be a quadratic extension of global fields. Two cuspidal automorphic representations $\Pi_1$ and $\Pi_2$ of $\text{Res}_{E/F}(\text{GL}_n)(\mathbb{A}_F)$ are isomorphic if and only if $\Pi_{1,v} \cong \Pi_{2,v}$ for almost all places $v$ of $F$ that are split in $E/F$.

We have the following comparison of trace formulas.

**Proposition 9.6.** Fix a character $\omega$ of $Z_G(\mathbb{A}_F)$ and let $\omega'$ be its base change. Fix a split place $v_0$ and a supercuspidal representation $\pi_{v_0}$ of $G(F_{v_0})$ Suppose that
- $f \in C_c^\infty(G(\mathbb{A}_F))$ and $\{f'_{x,y}\}_{(x,y) \in \mathbb{V}_n \times \mathbb{V}_{n+1}}$ are nice test functions and are smooth transfers of each other;
- let $\Pi_{v_0}$ be the base change of $\pi_{v_0}$. We assume that $f_{v_0}$ is essentially a matrix coefficient of $\Pi_{v_0}$ and that $f'_{x,y,v_0}$ is related to $f_{v_0}$ as in Proposition 7.6. Then $f'_{x,y,v_0}$ is essentially a matrix coefficient of $\pi_{v_0}$.

Now fix a representation $\otimes_v \pi_{v_0}$, where the product is over almost all split places $v$ and with each $\pi_{v_0}$ an irreducible unramified representation. Then we have
\[ \sum_{x,y} J^{x,y}_{\Pi}(f_{x,y}) = \sum_{\pi \in \mathcal{B}(\Pi)} I_\pi(f), \]
where the sum on the left runs over all $(x, y) \in \mathbb{V}_n \times \mathbb{V}_{n+1}$ while the sum on the right runs over all automorphic representations $\pi$ of $G(\mathbb{A}_F)$ such that
- $\pi_v \cong \pi_{v_0}$ for almost all split $v$,
- $\pi_{v_0}$ is our fixed supercuspidal representation,

and where $\Pi = BC(\pi)$ is the base change of any $\pi$ appearing in the sum.

**Proof.** We may assume that all test functions are factorizable. Let $S$ be a finite set of places such that
- all Hermitian spaces $V_x$ and $V_y$ with $f'_{x,y} \neq 0$ are unramified outside $S$, and
- for any $v \notin S$, $f_v$ and $f'_{x,y,v}$ are units of the spherical Hecke algebras. These match by the fundamental lemma (Theorem 7.9).

Consider now the spherical Hecke algebra $\mathcal{H}_{K_s}(G(\mathbb{A}_F^S))$, where $\mathbb{A}_F^S = \prod_{v \notin S} F_v$ and $K_s = \prod_{v \notin S} K_v$, as well as $\mathcal{H}_{K_s}(G(\mathbb{A}_F^S))$, where $K_s = \prod_{v \notin S} K_v$.

Given $f_s \in \mathcal{H}_{K_s}(G(\mathbb{A}_F^S))$ and $f_{x,y} \in \mathcal{H}_{K_s}(G(\mathbb{A}_F^S))$ such that for each non-split place $v \notin S$, $f_v$ and $f_{x,y,v}$ are the units of the algebra, the matching of global orbital integrals (46) implies the identity
\[ \sum_{x,y} J^{x,y}(f_{x,y,s} \otimes f^S_{x,y}) = I(f_s \otimes f^S). \]
Applying the simple relative trace formulas (44) and (45), we obtain the spectral identity
\[ \sum_{x,y} \sum_{\Pi} J_{\Pi}^{x,y}(f_{x,y,S} \otimes f_{x,y}^S) = \sum_{\pi} I_{\pi}(f_S \otimes f^S), \]
where \( \Pi \) and \( \pi \) runs over cuspidal automorphic representations with the prescribed super-cuspidal component at \( v_0 \). For the unramified representations \( \Pi^S \) (resp. \( \pi^S \)), let \( \lambda_{\Pi^S} \) (resp. \( \lambda_{\pi^S} \)) be the associated linear functionals of \( \mathcal{H}_{K,S}(G'(\mathbb{A}_F^{S})) \) (resp. \( \mathcal{H}_{K,S}(G(\mathbb{A}_F^{S})) \)). Then we observe that
\[ I_{\pi}(f_S \otimes f^S) = \lambda_{\pi^S}(f^S)I_{\Pi}(f_S \otimes 1_{K^S}), \]
and
\[ J_{\Pi}^{x,y}(f_{x,y,S} \otimes f_{x,y}^S) = \lambda_{\Pi^S}(f_{x,y}^S)J_{\Pi}^{x,y}(f_{x,y,S} \otimes 1_{K^S}). \]
Noting that we are only allowing non-identity elements of the Hecke algebras, we may view the above two equations as identities of linear functionals on the Hecke algebra \( \mathcal{H}_{K,S,\text{split}}(G'(\mathbb{A}_F^{S,\text{split}})) \), where the superscript \( \text{split} \) indicates that we only take the product over the split places outside of \( S \). By the infinite linear independence of Hecke characters (see [Bad18, Appendix] for a short proof), for any fixed \( \otimes_v \pi_v^0 \) we obtain the sum
\[ \sum_{x,y} \sum_{\Pi} J_{\Pi}^{x,y}(f_{x,y}) = \sum_{\pi \in \mathcal{B}} I_{\pi}(f), \]
where \( \mathcal{B} \) is the set of cuspidal automorphic representations satisfying the two bullet points in the statement of the proposition, and where
\[ \Pi \in \{ \Pi : \text{for almost all split primes, } \Pi_v = BC(\pi_v) \text{ for some } \pi \in \mathcal{B} \}. \]

Applying Theorem 9.5, we see that there is only one representation appearing on the left-hand side. Furthermore, this implies that \( \mathcal{B} = \mathcal{B}(\Pi) \).

We now fix a pair \((x, y) \in X_n \times X_{n+1}\) and obtain a comparison of spherical characters which is compatible with the factorizations in Propositions 8.11 and 8.12.

**Theorem 9.7.** Suppose that \( E/F \) is a quadratic extension of number fields such that every archimedean place \( v|\infty \) of \( F \) splits in \( E \). Fix \((x, y) \in X_n \times X_{n+1}\). Let \( \Pi \) be a cuspidal automorphic representation of \( G' \) such that

1. \( \Pi \cong \Pi'^{\sigma} \), and
2. there is a split place \( v_0 \) at which \( \Pi_{v_0} \) is supercuspidal.

Consider a nice factorizable function \( f' \in C_c^\infty(G'(\mathbb{A}_F)) \) and let \( f \in C_c^\infty(G(\mathbb{A}_F)) \) be a nice factorizable matching function. Then we have the identity
\[ J_{\Pi}^{x,y}(f') = \sum_{\pi \in \mathcal{B}(\Pi)} I_{\pi}(f). \tag{47} \]
In fact, fix \( \pi \in \mathcal{B}(\Pi) \). If \( f' = \prod_v f'_v \) and \( f = \prod_v f_v \) where the pairs \((f'_v, f_v)\) match for each place \( v \), then \( f \) may be chosen so that
\[ \prod_v J_{\Pi_v}^{x,y}(f'_v) = \prod_v I_{\pi_v}^2(f_v). \tag{48} \]

**Remark 9.8.** We note that the proof of (47) does not rely on [FLO12]. Indeed, we expect that this theorem, along with generalizations of [Zha14, Theorem 1.2] about the existence of twists \( \sigma \otimes \pi \) of \( \pi \) by cuspidal representations of \( GL_n(\mathbb{A}_F) \) such that
\[ L \left( \frac{1}{2}, \sigma \times \pi \right) \neq 0 \]
(see [Li09] for the $n = 2$ case, for example) may be used to give a new proof of several results such as the characterization of the image of the base change map and the factorization of unitary periods for cuspidal representations.

**Proof.** That such a transfer $f \in C_c^\infty(G(\mathcal{A}_F))$ exists follows from Proposition 7.6, Theorem 7.7, and the properties of the $Z$-regular semi-simple loci. We now apply the previous proposition to the unramified representation $\otimes_v \pi_v^0$ where $v$ runs over those split places over which II is unramified and $\pi_v^0$ is determined by

$$II_v = BC(\pi_v^0) \cong \pi_v^0 \otimes \pi_v^0.$$  

This gives (47).

Now fix $\pi \in B(II)$. Let $\eta_{i,j} : G(\mathcal{A}_F) \to \mathbb{C}^\times$ be the characters

$$\eta_{i,j}(g_1, g_2) = \eta(g_1)^i \eta(g_2)^j.$$

Note that

$$B(II) = \{ \pi \cdot \eta_{i,j} : (i, j) \in F_2^2 \},$$

where if $\pi = \pi_n \boxtimes \pi_{n+1}$, then

$$\pi \cdot \eta_{i,j} := \pi_n \cdot (\eta^i \circ \det) \boxtimes \pi_{n+1} \cdot (\eta^j \circ \det).$$

By Lemma 7.8 and our assumptions on the global extension $E/F$, we may assume that for each place $v$ of $F$,

$$\text{supp}(f_v) \subset G_v[x, y] := G_{F_v}[x, y].$$

Since the two Hermitian forms $x$ and $y$ are global, this implies that $f = \eta_{i,j} \cdot f$ for any $(i, j) \in F_2^2$.

Considering the local distribution $I_{\pi_v}$, we have

$$I_{\pi_v \eta_{i,j}}(f_v) = I_{\pi_v}(f_v \cdot \eta_{i,j}).$$

Combining this with the product formula (43) implies that

$$\sum_{(i,j) \in F_2^2} I_{\pi \eta_{i,j}}(f) = \sum_{(i,j) \in F_2^2} I_{\pi}(f \cdot \eta_{i,j}) = 4I_{\pi}(f).$$

Thus, the global matching of spherical characters becomes

$$J_{II}^{\#}(f') = 4I_{\pi}(f)$$

whenever $f'$ and $f$ are matching functions as in the proposition. Combining this with the factorizations (42) and (43) gives (48).

$\square$

10. Weak transfer of local spherical characters

In this section, we show that Theorem 9.7 implies a weak form of the local transfer of spherical characters for matching test functions. Here “weak form” means that our results only apply to certain representations $\pi$. This is sufficient for our final (geometric) goal.

10.1. Split places and non-vanishing under regular support conditions. In the global comparison, we imposed certain support conditions at a single place $v$ of our number field in order to affect a simple trace formula. As we are only making the regular semi-simple support assumption at split places, the local distributions are precisely the ones discussed in [Zha14, Appendix A]. This allows for the following non-vanishing result.

**Lemma 10.1.** Assume that $F$ is a local field. Suppose that $\pi$ is a supercuspidal representation of $G(F)$ with central character $\omega$. Then there exists a matrix coefficient $\Phi$ of $\pi$, and a test function $f \in C_c^\infty(G(F))$ such that
\( \bullet f_\omega(g) = \int_{Z_G(F)} f(gz)\omega^{-1}(z)dz = \Phi(g) \) for all \( g \in G(F) \), and  
\( \bullet \) there exists a \( Z \)-regular semi-simple element \( \gamma \) such that \( \text{Orb}_\omega(f, \gamma) \neq 0 \).

**Proof.** The first requirement is follows from the fact that map 
\[ C_c^\infty(G(F)) \to C_c^\infty(Z_G(F) \backslash G(F), \omega) \]
\[ f \mapsto f_\omega \]
is surjective. Since \( \pi \) is supercuspidal, any matrix coefficient \( \Phi \) lies in \( C_c^\infty(Z_G(F) \backslash G(F), \omega) \), so there exists an \( f \) satisfying \( f_\omega = \Phi \).

Recall now that for any generic representation \( \pi \), the spherical character \( I_\pi \) is a non-zero distribution [JPSS83]. For simplicity, we work instead with the unnormalized distribution \( I_\pi \). Since the pair \( (G, H) \) is a strongly-tempered spherical pair, a theorem of Sakellaridis and Venkatesh [SV17, Section 6] tells us that there exists a vector \( W_0 \in \mathcal{W}(\pi) \) such that the local Rankin-Selberg period \( \lambda_\pi \) may be expressed as
\[ \lambda_\pi(W) = \int_{H(F)} |\mathcal{W}(\pi, h)W, W_0| \pi dh. \]

With this, define the matrix coefficient
\[ \Phi_0(g) = [\mathcal{W}(\pi, g)W_0, W_0]|_\pi. \]

Ichino and Zhang show in [Zha14, Appendix A] that \( \Phi_0 \) satisfies the properties that the integral
\[ \text{Orb}(\Phi, \gamma) = \int_H \int_H \Phi(h_1^{-1}\gamma h_2)dh_1dh_2 \]
is convergent on a subset \( G(F)_{\text{con}} \subset G(F) \) the compliment of which has measure zero. Moreover, this orbital integral is non-zero on a subset of positive measure. In particular, since \( G(F)^{Z-rss} \) is Zariski open and dense, there exists an element \( \gamma \in G(F)^{Z-rss} \) such that
\[ \text{Orb}(\Phi, \gamma) \neq 0. \] (49)

This follows from the following lemma and Theorem A.2 of [Zha14], which states that there is a function \( f' \in C_c^\infty(G(F)^{Z-rss}) \) such that \( I_\pi(f') \neq 0 \).

**Lemma 10.2.** [Zha14, Lemma A.5] The function \( g \mapsto \text{Orb}(\Phi_0, g) \) on \( G(F) \) is locally \( L^1 \) and for any \( f \in C_c^\infty(G(F)) \), we have
\[ I_\pi(f) = \int_{G(F)} f(g) \text{Orb}(\Phi_0, g)dg. \]

In particular, if \( f' \in C_c^\infty(G(F)^{Z-rss}) \) such that \( I_\pi(f') \neq 0 \), the lemma implies that we cannot have \( \text{Orb}(\Phi_0, -)|_{G(F)^{Z-rss}} \equiv 0 \). We remark that the theorem of loc. cit. is stated for the regular semi-simple locus. The proof, however, works for any open dense subset of \( G(F) \).

Now take \( f \in C_c^\infty(G(F)) \) such that \( f_\omega = \Phi_0 \) and let \( \gamma \in G(F)^{Z-rss} \) be an element satisfying (49). Since \( \gamma \) is semi-simple and \( f \) has compact support, the orbital integral is absolutely convergent and we may rearrange the integration to find
\[ \text{Orb}_\omega(f, \gamma) = \int_{Z_G(F)} \text{Orb}(f, \gamma z)\omega^{-1}(z)dz \]
\[ = \int_H \int_H \int_{Z_G(F)} f(h_1^{-1}\gamma z h_2)\omega^{-1}(z)dzdh_1dh_2 \]
\[ = \text{Orb}(\Phi_0, \gamma) \neq 0. \]

\( \square \)
In particular, this ensures that there always exists a test function \( f \) satisfying this assumption with \( I_{\hat{\Pi}}^1(f) \neq 0 \). To ensure that we have a similar non-vanishing statement for \( J_{\Pi}^{x,y,2} \) under this support assumption, we give a direct local transfer of spherical characters in the split case. For simplicity, we work with the unnormalized distributions \( J_{\Pi}^{x,y} \) and \( I_{\pi} \).

We continue to assume that \( F \) is local and now assume that \( E = F \times F \). As before, we choose isomorphisms \( GL_k(E) \cong GL_k(F) \times GL_k(F) \) such that our unitary groups are given by

\[
U(V_n) \cong \{(g, g^\theta) \in GL_n(F) \times GL_n(F) : g \in GL_n(F)\}
\]

and

\[
U(V_{n+1}) \cong \{(g, g^\theta) \in GL_{n+1}(F) \times GL_{n+1}(F) : g \in GL_{n+1}(F)\}.
\]

Set \( J_{\Pi} := J_{\Pi}^{w_0,w_{n+1}} \).

**Proposition 10.3.** Consider matching smooth functions \( f_1 \otimes f_2 \in C_c^\infty(G(F) \times G(F)) \) and \( f = f_1 \ast f_2^{\psi} \in C_c^\infty(G(F)) \). Then for any irreducible representation \( \pi \) of \( G(F) \),

\[
J_{BC(\pi)}(f_1 \otimes f_2) = I_{\pi}(f).
\]

**Proof.** Identifying \( \Pi = BC(\pi) = \pi \boxtimes \pi \), (35) implies that for any \( W', W'' \in W(\pi) \)

\[
\alpha_{(h,i,h)}(W' \otimes W'') = \left[ W(h, \pi)W', W(w_0, \hat{\pi})W'' \right]_\pi.
\]

For the purposes of computing \( J_{\Pi} \) we note that for \( \hat{W}', \hat{W}'' \in W(\hat{\pi}) \)

\[
\alpha_{(w_0,w_0)}(\hat{W}' \otimes \hat{W}'') = \left[ W(w_0, \hat{\pi})\hat{W}', W(w_0, \pi)\hat{W}' \right]_{\hat{\pi}} = \left[ W', \hat{W}' \right]_{\hat{\pi}}.
\]

Thus, we have

\[
J_{BC(\pi)}(f_1 \otimes f_2) = \sum_{W' \boxtimes W''} \lambda_{\pi}(\pi(f_1)W') \lambda_{\pi}(\pi(f_2)W'') \alpha_{(w_0,w_0)}(\hat{W}' \otimes \hat{W}'') \frac{\left[ W', W'' \right]_\pi}{\left[ W', W'' \right]_{\hat{\pi}}},
\]

where we use (50) and the fact that we are summing over an orthogonal basis to reduce the sum to a single basis element \( W' \in W^\psi(\pi) \). We now claim that

\[
\lambda_{\pi}(\pi(f_2)W') = \lambda_{\hat{\pi}}(\hat{\pi}(f_2^\psi)\hat{W}').
\]

This follows from the fact that

\[
\hat{\pi}(f^\theta)\hat{W}(h) = \int_{G(F)} f^\theta(g)\hat{W}(hg)dg = \int_{G(F)} f(g^\theta)\hat{W}(h^\theta g^\theta)dg = \pi(f)W(h^\theta),
\]

and that the change of variables \( h \mapsto h^\theta \) is unimodular. Applying this, we obtain

\[
\sum_{W'} \frac{\lambda_{\pi}(\pi(f_1)W') \lambda_{\hat{\pi}}(\hat{\pi}(f_2^\psi)\hat{W}')}{\left[ W', W' \right]_\pi} = \sum_{W'} \frac{\lambda_{\pi}(\pi(f_1 \ast f_2^{\psi})W') \lambda_{\hat{\pi}}(\hat{W}')}{\left[ W', W' \right]_{\hat{\pi}}} = I_{\pi}(f).
\]

\( \square \)
10.2. **Unramified case.** We now consider the case that \( E/F \) is an unramified extension of non-archimedean local fields. Let \( \mathcal{H}_{K'}(G'(F)) \) denote the spherical Hecke algebra for \( G'(F) \) and let \( \mathcal{H}_K(G(F)) \) the corresponding algebra for \( G(F) \). We have the morphism

\[
BC : \mathcal{H}_{K'}(G'(F)) \to \mathcal{H}_K(G(F)),
\]
defined by \( \text{Sat}(BC(\varphi))(\pi) = \text{Sat}(f)(BC(\pi)) \) and is injective.

**Lemma 10.4.** Let \( f' \in \mathcal{H}_{K'}(G'(F)) \) and let \( (x, y) \in G'(F) \ast (w_n, w_{n+1}) \), where \( * \) denotes the action on \( X_n \times X_{n+1} \). For any representation \( \pi \) of \( G(F) \), we have

\[
J_{BC(\pi)}(f') = I_{BC(\pi)}^2(BC(f')).
\]

**Proof.** When \( \pi \) is not unramified, both sides are zero so we can assume \( \pi \) is unramified. Using (30), (36), and (32), the left hand side is equal to

\[
J_{BC(\pi)}(f') = \text{Sat}(f'(BC(\pi))) \frac{\lambda_{BC(\pi)}^2(W_0^{BC(\pi)})}{[W_0^{BC(\pi)}, W_0^{BC(\pi)}]_2^{BC(\pi)}} = \text{Sat}(f'(BC(\pi))),
\]
where \( W_0^{BC(\pi)} \) is the normalized spherical Whittaker function for \( BC(\pi) \). A similar argument shows

\[
I_{BC(\pi)}^2(BC(f')) = \text{Sat}(BC(f'))(\pi) \frac{\lambda_{BC(\pi)}^2(W_0^\pi W_0^\pi \pi)}{[W_0^\pi, W_0^\pi]_2^\pi} = \text{Sat}(BC(f'))(\pi),
\]
where \( W_0^\pi \) is the normalized spherical Whittaker function for \( \pi \). The result follows from the definition of the base change homomorphism \( BC \). \( \square \)

10.3. **Weak transfer of spherical characters.** For non-split places more generally, the global Theorem 9.7 implies the following weak local spectral transfer of spherical characters.

**Proposition 10.5.** Assume now that \( E/F \) is a quadratic extension of number fields such that every archimedean place \( v' \) of \( F \) splits in \( E \). Let \( \Pi = BC(\pi) \) be an irreducible cuspidal automorphic representation of \( G(\mathbb{A}_E) \) and \( (x, y) \in X_n \times X_{n+1} \) such that there exists a nice test function \( f' \) such that

\[
J_{\Pi}(f') \neq 0.
\]

Then for any non-split place \( v_0 \) of \( F \), there exists a non-zero constant \( C(\Pi_{v_0}, x, y, \pi) \in \mathbb{C}^\times \) depending only on \( (x, y) \) and \( \Pi_{v_0} \) such that for any pair of matching functions \( f'_{v_0} \in C_c^\infty(G'(F_{v_0})) \) and \( f_{v_0} \in C_c^\infty(G(F_{v_0})) \) we have

\[
J_{\Pi_{v_0}}^{x,y}(f'_{v_0}) = C(\Pi_{v_0}, x, y) I_{\Pi_{v_0}}^2(f_{v_0}).
\]

**Proof.** Let \( \mathbb{A}_{v_0}^E \) denote the adeles away from the place \( v_0 \) and let \( f^{v_0} = \prod_{v \neq v_0} f'_v \in C_c^\infty(G'(k_{v_0})) \) be a factorizable test function. Using the factorization (42) we have the equality

\[
J_{\Pi_{v_0}}^{x,y}(f'_{v_0} \otimes f^{v_0}) = C J_{\Pi_{v_0}}^{x,y}(f'_{v_0}).
\]

Since \( J_{\Pi_{v_0}}^{x,y} \neq 0 \), we may choose \( f^{v_0} \) so that \( C \neq 0 \). Moreover, since the distribution is non-vanishing for nice test functions, we know that there is a finite split place \( v_1 \) (necessarily distinct from \( v_0 \)) such that \( \Pi_{v_1} \) is supercuspidal. We may assume that \( f'_{v_1} \in C_c^\infty(G'(F_{v_1})) \) is essentially a matrix coefficient of \( \Pi_{v_1} \). Additionally, we may impose that there exists a second split place \( v_2 \) such that the local test function \( f'_{v_2} \) is supported in the \( Z \)-regular semi-simple locus. In particular, we know that

- the test function \( f'_{v_0} \otimes f^{v_0} \) is nice, and
- \( C = 4 \prod_{v \neq v_0} J_{\Pi_{v_0}}^{x,y}(f'_{v_0}) \neq 0. \)
Now by Theorem 9.7, there exists a factorizable test function $f^\nu_0 = \prod_{v \neq v_0} f_v \in C^\infty_c(G(F_v))$ such that for any function $f_{\nu_0}$ matching $f^\nu_0$, the test function $f = f_{\nu_0} \otimes f^\nu_0$ is nice and that

$$J^x,y_\Pi(f') = 4I_x(f).$$

Since we chose $f^\nu_0$ such that $C \neq 0$, the factorization (43) implies that there is a non-zero constant $C'$ such that

$$CJ^x,y_\Pi(f^\nu_0) = J^x,y_\Pi(f') = 4I_x(f) = C'\hat{I}^2_{\pi_{\nu_0}}(f_{\nu_0}).$$

Since the initial test function $f^\nu_0$ was arbitrary, the constant

$$C(\Pi_{\nu_0}, x, y) = C^{-1}C' \neq 0$$

is independent of the matching test functions $f_{\nu_0}'$ and $f_{\nu_0}$, finishing the proof. \qed

Combining this with our unramified computation, we have the following corollary for unramified representations.

**Corollary 10.6.** Let all notations be as in the previous proposition. If $\Pi_{\nu_0}$ is unramified and $(x, y) \in G'(F) \ast (w_n, w_{n+1})$, then $C(\Pi_{\nu_0}, x, y) = 1$.

**Proof.** By the proposition, for any pair of matching functions $f^\nu_{\nu_0} \in C^\infty_c(G'(F_{\nu_0}))$ and $f_{\nu_0} \in C^\infty_c(G(F_{\nu_0}))$ we have

$$J^x,y_\Pi(f^\nu_{\nu_0}) = C(\Pi_{\nu_0}, x, y)\hat{I}^2_{\pi_{\nu_0}}(f_{\nu_0}),$$

for some $C(\Pi_{\nu_0}, x, y) \in C^\times$.

Assume now that $\Pi_{\nu_0}$ is unramified, and recall that the fundamental lemma (Theorem 7.9) states that we can take $f^\nu_{\nu_0} = 1_{K_{\nu_0}'}$ and $f_{\nu_0} = 1_{K_{\nu_0}}$. As these functions lie in the spherical Hecke algebras of the two groups and $BC(f^\nu_{\nu_0}) = f_{\nu_0}$, Lemma 10.4 implies that

$$J^x,y_\Pi(f^\nu_{\nu_0}) = \hat{I}^2_{\pi_{\nu_0}}(f_{\nu_0}) = 1 \neq 0.$$

It follows that $C(\Pi_{\nu_0}, x, y) = 1$. \qed

11. A base change fundamental lemma and the proof of Theorem 2.11

The following application of our local and global spectral results will suffice to prove Theorem 5.3.

**Theorem 11.1.** Let $E/F$ be an unramified extension of $p$-adic local fields. For any $\varphi \in \mathcal{H}_{K'}(G'(F))$, the function $BC(\varphi) \in \mathcal{H}_K(G(F))$ matches the functions $\{\phi_{x,y}(x,y)\in \mathcal{V}_n \times \mathcal{V}_{n+1} \}$ where

$$\phi_{x,y} = \begin{cases} \varphi : (x,y) = (w_n,w_{n+1}), \\ 0 : \text{otherwise.} \end{cases}$$

The idea of the proof is to reduce this statement to the spectral transfer in Lemma 10.4. To do this, we globalize the problem by carefully picking a global test function and then use the comparison of relative trace formulas to reduce the statement to a global transfer of spherical characters. This requires care in order to ensure non-vanishing of the global orbital integral, which takes up most of the argument. We then apply results of Sections 8 and 10 to show that this follows from Lemma 10.4 to finish the proof.

**Proof.** Let $\varphi \in \mathcal{H}_{K'}(G'(F))$ and let $\beta \in G'(F)^{rss}$. Our goal is to show that if $\alpha \xrightarrow{w_n,w_{n+1}} \beta$, then

$$\text{Orb}(\varphi, \beta) = \Omega(\alpha)\text{Orb}^\eta(BC(\varphi), \alpha).$$

(51)
Note that we automatically have supp$(BC(\varphi)) \subset G_{w_n^1}^1[w_n, w_{n+1}]$. In particular, if $\alpha \xleftarrow{x,y} \beta$ for some $(x,y)$ not in the $G'_0(F)$-orbit of $(w_n, w_{n+1})$, then

$$\text{Orb}^\eta(BC(\varphi), \alpha) = 0,$$

giving the vanishing statement of the theorem.

We claim that it suffices to prove (51) for $Z$-regular semi-simple classes. Indeed, the equality reduces via (24) and (23) to

$$\text{Orb}^{U(V_n)}(\tilde{\varphi}, \pi_{w_n, w_{n+1}}(\beta)) = \omega(\alpha_1^{-1} \alpha_2) \text{Orb}^{GL_n(F), \eta}(BC(\varphi), \alpha_1^{-1} \alpha_2),$$

where $\alpha = (\alpha_1, \alpha_2)$. The orbital integrals here are precisely those arising in Jacquet-Rallis transfer (18), implying that they are locally constant on the regular semi-simple locus [Zha14, Proposition 3.13].

Stated in terms of the categorical quotient

$$\mathcal{A}' = GL_{n+1}///GL_n \cong X_{n+1}///U(V_n) \cong A^{2n+1},$$

where the identification with $A^{2n+1}$ is given by the invariant maps $c$ and $c^\pi_{w_n, w_{n+1}}$ (21), we may view

$$\Phi_\varphi(z) = \text{Orb}^{U(V_n)}(\tilde{\varphi}, \pi_{w_n, w_{n+1}}(\beta)) - \omega(\alpha_1^{-1} \alpha_2) \text{Orb}^{GL_{n+1}(F), \eta}(BC(\varphi), \alpha_1^{-1} \alpha_2)$$

as a smooth function on the regular semi-simple locus $\mathcal{A}'^{rss}(F)$, where $c(\alpha) = c^\pi_{w_n, w_{n+1}}(\beta) = z$. Arguing as in the proof of Proposition 5.2, it suffices to show that $\Phi_\varphi \equiv 0$ on the open dense set

$$\{(a_1, \ldots, a_{2n+1}) \in \mathcal{A}'^{rss}(F) : a_1 \neq 0 \text{ and } a_{2n+1} \neq 0\}.$$ 

This is precisely the $Z$-regular semi-simple locus (see Section 7.2.3), proving the claim. Therefore, we assume that $\alpha \xleftarrow{w_n, w_{n+1}} \beta$ are matching $Z$-regular semi-simple elements.

We now globalize our quadratic extension. That is, we let $E/F$ be a quadratic extension of number fields such that every archimedean place of $E$ splits in $E$ and there exists a place $v_0$ of $E$ such that $E_{v_0}/F_{v_0} \cong E/F$. We also set aside two distinct split places $v_{cusp}$ and $v_{reg}$, and a third auxiliary finite place $v_{max}$.

We now construct nice global matching test functions $f'$ and $f$ such that the distributions

$$J^\pi_{w_n, w_{n+1}}(f') \text{ and } I(f)$$

have particularly simple geometric expansions. Let $\pi_{cusp}$ be a supercuspidal automorphic representation of $G(F_{v_{cusp}})$. By Lemma 10.1, we may find a test function $f_{cusp}$ which is essentially a matrix coefficient of $\pi_{cusp}$ and such that there exists a $Z$-regular semi-simple element $\alpha_{cusp}$ such that

$$\text{Orb}^\eta(f_{cusp}, \alpha_{cusp}) \neq 0.$$ 

Let $f'_{cusp}$ be an essential matrix coefficient of $BC(\pi_{cusp})$ matching $f_{cusp}$. It is clear that such an $f'_{cusp}$ exists. Then we know

$$\text{Orb}(f'_{cusp}, \beta_{cusp}) = \text{Orb}^\eta(f_{cusp}, \alpha_{cusp}) \neq 0,$$

where $\beta_{cusp} \leftrightarrow \alpha_{cusp}$.

Similarly for the place $v_{reg}$, we let $\pi_{reg}$ be a supercuspidal representation of $G(F_{v_{reg}})$, Again applying Lemma 10.1, there exists an $f_{reg}$ and a $Z$-regular semi-simple element $\alpha_{reg}$ such that

1. $\text{Orb}^\eta(f_{reg}, \alpha_{reg}) \neq 0$, and
2. supp$(f_{reg}) \subset G(F_{v_{reg}})^{rss}$. 
Indeed, if we take $\tilde{f}_{v_{\text{reg}}}$ to be essentially a matrix coefficient and $\alpha_{\text{reg}}$ as in the previous case, let $Z \subset G(F_{v_{\text{reg}}})_{\text{rss}}$ be a subset that is closed in $G(F_{v_{\text{reg}}})$ such that

$$Z_G(F_{v_{\text{reg}}})H(F_{v_{\text{reg}}})\alpha_{\text{reg}}H(F_{v_{\text{reg}}}) \subset Z.$$  

This is possible since $G(F_{v_{\text{reg}}})$ is Hausdorff and the orbit of $\alpha_{\text{reg}}$ is closed by assumption. Setting $f_{v_{\text{reg}}} = 1_Z \cdot \tilde{f}_{v_{\text{reg}}}$ gives the appropriate function. We now similarly obtain $f'_{v_{\text{reg}}}$ and $\beta_{\text{reg}}$ matching $f_{v_{\text{reg}}}$ and $\alpha_{\text{reg}}$ so that

$$\text{Orb}(f'_{v_{\text{reg}}}, \beta_{\text{reg}}) = \text{Orb}^\eta(f_{v_{\text{reg}}}, \alpha_{\text{reg}}) \neq 0,$$

To study orbital integrals at $\alpha$ and $\beta$ by global means, we first approximate these points by global elements. Indeed, since the diagonal embeddings

$$G(F) \hookrightarrow G(F_{v_{\text{cusp}}}) \times G(F_{v_{\text{reg}}}),$$

and

$$G'(F) \hookrightarrow G'(F_{v_{\text{cusp}}}) \times G'(F_{v_{\text{reg}}})$$

are dense, local constancy of the orbital integrals imply that we may find matching global $Z$-regular semi-simple elements $a \in G(F)^{\text{rss}}$ and $b \in G'(F)^{\text{rss}}$ such that

$$\text{Orb}(f'_{v_{\text{cusp}}}, b) = \text{Orb}(f'_{v_{\text{cusp}}}, \beta_{\text{cusp}}) = \text{Orb}^\eta(f_{v_{\text{cusp}}}, \alpha_{\text{cusp}}) = \text{Orb}^\eta(f_{v_{\text{cusp}}}, a) \neq 0,$$

$$\text{Orb}(f'_{v_{\text{reg}}}, b) = \text{Orb}(f'_{v_{\text{reg}}}, \beta_{\text{reg}}) = \text{Orb}^\eta(f_{v_{\text{reg}}}, \alpha_{\text{reg}}) = \text{Orb}^\eta(f_{v_{\text{cusp}}}, a) \neq 0,$$

and

$$\Omega(a) \text{Orb}^\eta(BC(\varphi), a) = \Omega(\alpha) \text{Orb}^\eta(BC(\varphi), \alpha).$$

In particular, to prove (51), it suffices to prove the equality with $a$ and $b$. For this we may utilize the comparison of relative trace formulas of Section 9.

Now let $S$ be a finite set of places of $F$ containing all infinite places and the places $v_0$, $v_{\text{cusp}}$, and $v_{\text{reg}}$ such that for each $v \notin S$, $a \in K_v$ and $b \in K'_v$. For every $v \in S \setminus \{v_{\text{cusp}}, v_{\text{reg}}\}$, select matching $f'_v$ and $f_v$ such that

$$\text{Orb}(f'_v, b) = \Omega(a) \text{Orb}^\eta(f_v, a) \neq 0.$$

For each place $v \in S$, let $C_v$ be a compact set containing the support of $f_v$ and assume that $C_v$ is large enough to contain the support of $BC(\varphi)$; set

$$C = \prod_{v \in S} C_v \times \prod_{v \notin S} K_v \subset G(\mathbb{A}_F).$$

For all places $v \notin S \setminus \{v_{\text{aux}}\}$, we take $f_{v} = 1_{K_v}$ and $f'_{v} = 1_{K'_v}$ to be the unit spherical functions. In particular, the fundamental lemma Theorem 4.3 implies that

$$\text{Orb}(f'_v, b) = \Omega(a) \text{Orb}^\eta(f_v, a) \neq 0$$

for all $v \notin S \setminus \{v_{\text{aux}}\}$. For $v_{\text{aux}}$, recall that the matching of orbits may be characterized by the invariant polynomials (21). With this in mind, we set $f_{v} = 1_{G_{v_{\text{aux}}}[l]} \cdot 1_{K_{v_{\text{aux}}}}$ and $f'_{v_{\text{aux}}} = 1_{G'_{v_{\text{aux}}}[l]} \cdot 1_{K'_{v_{\text{aux}}}}$, where

$$G_{v_{\text{aux}}}[l] = \{g \in G(F_{v_{\text{aux}}}) : c_l(g) \in c_l(a) + p^l, \text{ for all } i = 1, \ldots, 2n + 1\}$$

and

$$G'_{v_{\text{aux}}}[l] = \{g \in G'(F_{v_{\text{aux}}}) : c'_l(g) \in c'_l(a) + p^l, \text{ for all } i = 1, \ldots, 2n + 1\}.$$

As the polynomials $c_i$ are the invariant polynomials of $H'(F_{v_{\text{aux}}}) \times H'(F_{v_{\text{aux}}})$ acting on $G(F_{v_{\text{aux}}})$ and $c'_i_{\text{aux}}[l]$ are the invariant polynomials of $H'(F_{v_{\text{aux}}}) \times H'_{v_{\text{aux}}}[l]$, we see that $f_{v_{\text{aux}}}$ and $f'_{v_{\text{aux}}}$ match for any choice of $l \in \mathbb{Z}_{\geq 0}$ and that

$$\text{Orb}(f'_{v_{\text{aux}}}, b) = \Omega(a) \text{Orb}^\eta(f_{v_{\text{aux}}}, a) \neq 0.$$
Now set \( f = \prod_v f_v \) and \( f' = \prod_v f'_v \). By linearity, we may assume without loss of generality that
\[
\text{supp}(f) \subset G_\mathbb{A}[w_n, w_{n+1}] := \prod_v G_v[w_n, w_{n+1}]
\]
Our choices ensure that \( f \) and \( f' \) match and that
\[
\text{Orb}(f', b) = \text{Orb}(f, a) \neq 0.
\]

We claim that we may choose \( l \) large enough at \( v_{\text{aux}} \) so that this is the only non-vanishing global orbital integral for \( f \). Our assumption on the support of \( f_{v_{\text{reg}}} \) already reduces this to \( \mathbb{Z} \)-regular semi-simple classes.

For each \( i \), the image of \( C \subset G(\mathbb{A}_F) \) under \( c_i \) gives a compact subset \( c_i(C) \subset \mathbb{A}_F \). Since \( F \subset \mathbb{A}_F \) is discrete, the intersection \( c_i(C) \cap F \) is finite for each \( i = 1, \ldots, 2n + 1 \). We may now choose \( l \) so large that if \( a' \in G(F)^{\text{rss}} \) was not in the same orbit as \( a \), then \( c_i(a') \notin c_i(C) \cap F \) for some \( i \). Since \( \text{supp}(f) \subset C \), this implies that \( \text{Orb}^0(f, a') = 0 \).

By our choices of local test functions at \( v_{\text{cusp}} \) and \( v_{\text{reg}} \), \( f \) and \( f' \) are nice matching functions such that
\[
I(f) = \text{Orb}^0(f, a) = \text{Orb}(f', b) = J^{w_n, w_{n+1}}(f') \neq 0.
\]

Now set \( \hat{f} = BC(\varphi) \otimes \prod_{v \neq v_0} f_v \) and \( \hat{f}' = \varphi \otimes \prod_{v \neq v_0} f'_v \). We note that we still have \( \text{supp}(\hat{f}) \subset G_\mathbb{A}[w_n, w_{n+1}] \) since \( \text{supp}(BC(\varphi)) \subset G_{v_0}[w_n, w_{n+1}] \). By our assumptions at the other places, \( \hat{f} \) and \( \hat{f}' \) are still nice and
\[
\text{Orb}^0(\hat{f}, a') = \text{Orb}(\hat{f}', b') = 0
\]
for any global match regular semi-simple elements \( a' \overset{w_n, w_{n+1}}{\leftrightarrow} b' \) lying in distinct orbits from \( a \). This implies that
\[
I(\hat{f}) = \text{Orb}^0(\hat{f}, a) \quad \text{and} \quad J^{w_n, w_{n+1}}(\hat{f}') = \text{Orb}(\hat{f}', b).
\]
To finish the proof, it suffices show that
\[
I(\hat{f}) = J^{w_n, w_{n+1}}(\hat{f}').
\]

For this, consider the spectral expansions
\[
I(\hat{f}) = \sum_{\pi} I_{\pi}(\hat{f}) \quad \text{and} \quad J^{w_n, w_{n+1}}(\hat{f}') = \sum_{\Pi} J^{w_n, w_{n+1}}_{\Pi}(\hat{f}').
\]

Note that if \( \pi \cong \pi \cdot \eta_{i, j} \) for any non-trivial \( (i, j) \in \mathbb{F}^2_2 \), then \( BC(\pi) \) is not cuspidal. Then for any \( f^{\gamma_\Pi} \) matching \( \hat{f} \),
\[
J^{w_n, w_{n+1}}_{BC(\pi)}(f^{\gamma_\Pi}) = 0.
\]

By Theorem 9.7, this forces \( I_{\pi}(\hat{f}) = 0 \). Thus, we may assume that \( \pi \not\cong \pi \cdot \eta_{i, j} \) for any non-trivial \( (i, j) \in \mathbb{F}^2_2 \). In particular, it suffices to show that
\[
J^{w_n, w_{n+1}}_{\Pi}(\hat{f}') = \sum_{\pi \in B(\Pi)} I_{\pi}(\hat{f})
\]
for all cuspidal automorphic representations \( \Pi \) such that \( \Pi \cong \Pi^\pi \). Since \( \text{supp}(\hat{f}) \subset G_\mathbb{A}[w_n, w_{n+1}] \), the argument in the proof of Theorem 9.7 implies that this reduces to showing that for any such \( \pi \),
\[
J^{w_n, w_{n+1}, z}_{\Pi \otimes v_0}(\varphi) \prod_{v \neq v_0} J^{w_n, w_{n+1}, z}_{\Pi_v}(f'_v) = I_{\pi v_0}^2(BC(\varphi)) \prod_{v \neq v_0} I_{\pi_v}^2(f_v).
\]

Note that if \( \pi \) is not unramified at \( v_0 \), then both sides are zero. We thus assume that \( \pi v_0 \) is unramified.
Theorem 9.7 tells us that there exists a test function \( f^0 \in C_c^\infty(G(\mathbb{A}_F)) \) matching \( \hat{f}' \) such that for all such cuspidal representations
\[
J^{w_n,w_n+1}_{\Pi_v}(\varphi) \prod_{v \neq v_0} J^{w_n,w_n+1}_{\Pi_v}(f^0_v) = I^{\mathcal{L}}_{\pi_{v_0}}(f^0_v) \prod_{v \neq v_0} I^{\mathcal{L}}_{\pi_v}(f^0_v).
\]
We may assume that \( f^0_v = f_v \) for all \( v \neq v_0 \). Corollary 10.6 now tells us that
\[
J^{w_n,w_n+1}_{\Pi_v}(\varphi) = I^{\mathcal{L}}_{\pi_{v}}(f^0_v),
\]
so that
\[
J^{w_n,w_n+1}_{\Pi_v}(\varphi) \prod_{v \neq v_0} J^{w_n,w_n+1}_{\Pi_v}(f^0_v) = J^{w_n,w_n+1}_{\Pi_v}(\varphi) \prod_{v \neq v_0} I^{\mathcal{L}}_{\pi_v}(f_v).
\]
Lemma 10.4 now states that
\[
J^{w_n,w_n+1}_{\Pi_v}(\varphi) = I^{\mathcal{L}}_{\pi_{v}}(BC(\varphi)),
\]
allowing us to conclude (52). Since this holds for all \( \Pi \), we obtain \( I(\hat{f}) = J^{w_n,w_n+1}(\hat{f}') \) and the theorem follows.

11.1. Proof of Theorem 2.11. We continue to assume that \( E/F \) is an unramified extension of \( p \)-adic local fields. In Part 1, we reduced Theorem 2.11 to Theorem 5.3 which states that for any \( \varphi \in \mathcal{H}_{K_n,E}(GL_n(E)) \), and for any \( Y \in GL_n(F)^{rss} \), we have
\[
\omega(X) \text{Orb}^{GL_n-1(F),\eta}(BC(\varphi), X) = \begin{cases} 
\text{Orb}^{GL_n(V_n-1)}(\varphi \ast 1_0, Y) & : X \leftrightarrow Y \in X_n^{rss}, \\
0 & : \text{otherwise}.
\end{cases}
\]
We deduce this from Theorem 11.1. We remark that while we proved Theorem 11.1 with respect to the split Hermitian form \( w_n \), it is easy to see that the result holds with respect to the identity form \( I_n \) since the two Hermitian spaces are isomorphic under the assumption that \( E/F \) is unramified.

Considering the contraction map
\[
C_c^\infty(GL_n^{-1}(E) \times GL_n(E)) \rightarrow C_c^\infty(GL_n(E))
\]
\[
f \mapsto \hat{f},
\]
there exists a natural lift
\[
\Phi \in \mathcal{H}_{K_{n-1,E} \times K_n,E}(GL_n^{-1}(E) \times GL_n(E))
\]
such that \( \tilde{\Phi} = \varphi \). Indeed, the function \( \Phi = 1_{K_{n-1,E}} \otimes \varphi \) works.

Recall the commutative diagram
\[
\mathcal{H}_{K_n,E}(GL_n(E)) \xleftarrow{\ast 1_0} \mathcal{H}_{K_n,E}(X_n) \xrightarrow{H} \mathcal{H}_{K_n}(GL_n(F)),
\]
where \( \ast 1_0 \) indicates convolution with the unit element and \( H \) denotes the \( \mathcal{H}_{K_n,E}(GL_n(E)) \)-module isomorphism of Hironaka. As we are multiplying both sides by the unit of the appropriate Hecke algebra, a simple computation and the commutativity of the above diagram imply that
\[
BC(\Phi) = BC(\tilde{\Phi}) = H(\varphi \ast 1_0).
\]
Now Theorem 11.1 implies that \( \{ \Phi, 0 \} \) and \( BC(\Phi) \) are transfers of one another.
To make this useful, we first lift $X \in \text{GL}_n(F)^{rss}$ to a regular semi-simple $\gamma = (\gamma_1, \gamma_2) \in [\text{GL}_{n-1}(F) \times \text{GL}_n(F)]^{rss}$ and lift $Y$ to $\delta = (\delta_1, \delta_2) \in [\text{GL}_{n-1}(E) \times \text{GL}_n(E)]^{rss}$. By the relations of orbital integrals and transfer factors in (24) and (23), we conclude
\[
\text{Orb}^U(V_{n-1})(\varphi \ast 1_0, Y) = \text{Orb}^{\Phi}(\Phi, \delta) = \Omega(\gamma) \text{Orb}^\eta(BC(\Phi), \gamma) \quad \text{(Theorem 11.1)}
\]
\[
= \omega(X) \text{Orb}^{\eta(BC(\Phi))}(BC(\varphi), X).
\]
Additionally, the vanishing component of Theorem 11.1 gives the correct vanishing of orbital integrals for $BC(\varphi)$, completing the proof of Theorem 5.3. □

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