The exact solutions of differential equation with delay

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The exact solutions of the first order differential equation with delay are derived. The equation has been introduced as a model of traffic flow. The solution describes the traveling cluster of jam, which is characterized by Jacobi’s elliptic function. We also obtain the family of solutions of such type of equations.
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In this paper, we investigate following differential-difference equation,
\[
\frac{d}{dt} x_n(t + \tau) = f(x_{n+1}(t) - x_n(t)),
\]
where \(\tau\) is a real positive constant called “delay”. The index \(n\) takes integer. The set of equations of this type has been introduced for a car-following model of traffic flow \[1\] \[2\]. In that case \(x_n\) is the position of the \(n\)th car. Equation \[1\] has been popular in many physical phenomena of relaxation towards an optimal equilibrium state, such as relaxation effect in gases, chemical reactions and synchronization problem.

As a model for traffic flow, \(f(x) = \tanh(x)\) is the reasonable choice \[3\] \[4\], which was first introduced in the model \[3\].

\[
\tau \frac{d^2}{dt^2} x_n(t) = f(x_{n+1}(t) - x_n(t)) - \frac{d}{dt} x_n(t).
\]

This model may have some relation to Eq.\[4\] for relatively small \(\tau\) \[2\]. Many studies have been made to the model of Eq.\[4\] with \(f(x) = \tanh(x)\), which has the stable traveling cluster solution as traffic jam \[3\]. The traveling cluster solution is characterized as a soliton of modified Korteweg-de Vries (MKdV) equation in the vicinity of the critical point \[3\]. One interesting question is whether Eq.\[4\] has such traveling cluster solution or not. Our simulation result suggest the existence of such solution, which describes a traveling cluster moving backward with the velocity \(v = 1/(2\tau)\) \[4\]. Recently, we have an information that Igarashi, Itoh and Nakashishi have found the exact solution of Eq.\[4\], which characterized by the theta function \[3\]. In this paper we also derive a series of analytic solutions of traveling cluster in the context of our work, and confirm their stability by the numerical simulations.

It is convenient to introduce new variable \(h_n(t) = x_{n+1}(t) - x_n(t)\) and rewrite Eq.\[1\],

\[
\frac{d}{dt} h_n(t + \tau) = f(h_{n+1}(t)) - f(h_n(t)).
\]

We start at the linear theory assuming that the amplitude \(h_n(t)\) is infinitesimal and \(f(h_n) = f(0) + f'(0)h_n\). Equation \[3\] becomes

\[
\frac{d}{dt} h_n(t + \tau) = h_{n+1}(t) - h_n(t).
\]

Here we set \(f'(0) = 1\) without loss of generality. We investigate the solution of the form

\[
h_n(t) = \exp(i\alpha n + i\omega t)
\]

for real \(\alpha\) and \(\omega\). Inserting Eq. \[5\] to Eq. \[3\] we obtain

\[
\sin\alpha/2 = \frac{1}{2\tau},
\]

and

\[
\alpha = 2\omega\tau.
\]

Equation \[3\] has a solution if \(2\tau\) is larger than 1, which means \(\tau = 1/2\) is critical. If \(\tau\) is a little bit larger than \(1/2\), Eq. \[3\] has only two solutions \(\alpha = \pm \lambda\). Then, we obtain the solution of Eq. \[3\] as

\[
h_n(t) = \exp \pm i\lambda(n + \frac{t}{2\tau}).
\]

This represents a traveling wave solution with the velocity \(1/(2\tau)\) in the space of index \(n\), which is treated as a
continuous variable. The wave moves backward against the numbering direction, which appears as the traveling wave in the real space moving backward in the flow of \( x_n \). The above analysis is first given by Whitham \[3\].

Now let us investigate the exact traveling wave solution of Eq. (3). We treat the index \( n \) as a continuous variable, and change the notation \( h_n(t) \) to \( h(n,t) \). We introduce new variables on the moving flame of traveling wave as \( u = n + vt \), where \( v \) is the velocity of the traveling wave. We search the solution which does not change its form on this flame. We define the amplitude of traveling wave

\[
H(u) = H(n + vt) \equiv h(n, t). \tag{9}
\]

Eq. (3) for the amplitude \( H(u) \) is expressed as

\[
v \frac{d}{du} H(u + v \tau) = f(H(u + 1)) - f(H(u)). \tag{10}
\]

Replacing \( u \) by \( u - 1/2 \), we get more symmetric form

\[
v \frac{d}{du} H(u + \sigma) = f(H(u + \frac{1}{2})) - f(H(u - \frac{1}{2})), \tag{11}
\]

where

\[
\sigma = v \tau - \frac{1}{2}. \tag{12}
\]

We investigate the solution under the condition \( \sigma = 0 \). This means that the traveling wave of such solutions propagates backward with just the same velocity as the linear theory,

\[
v = \frac{1}{2 \tau}. \tag{13}
\]

Now we present the definite form of exact solutions giving concrete examples of \( f(x) \). First we take \( f(x) = \tanh(x) \), which is a suitable choice for a model of traffic flow. Introducing a new amplitude

\[
G = f(H), \tag{14}
\]

Eq. (11) with \( \sigma = 0 \) is rewritten as

\[
v \frac{dG(u)/du}{1 - G(u)^2} = G(u + \frac{1}{2}) - G(u - \frac{1}{2}). \tag{15}
\]

We can easily find a solution of Eq. (15) in the form

\[
G(u) = \beta \ sn(\alpha u, k), \tag{16}
\]

where \( \text{sn} \) is Jacobi’s elliptic function with modulus \( k \). The parameter \( \alpha \) is determined by

\[
\frac{\text{sn}(\alpha/2, k)}{\alpha/2} = \frac{1}{2 \tau}. \tag{17}
\]

and \( \beta \) is given by

\[
\beta = \pm k \frac{\alpha}{4 \tau}. \tag{18}
\]

Eq. (17) has a real solution only if \( 1/(2 \tau) < 1 \). In the case of \( k = 0 \), Eq. (17) reduces to the result of linear theory, Eq. (4). The modulus \( k \) is a free parameter of the solution, which indicates the existence of many solutions for the same traveling velocity \( v = 1/(2 \tau) \). The relation between the modulus and solutions is discussed later. Next we take \( f(x) = \tan(x) \). In this case Eq. (15) is replaced by

\[
v \frac{dG(u)/du}{1 + G(u)^2} = G(u + \frac{1}{2}) - G(u - \frac{1}{2}). \tag{19}
\]

It is also easy to see that Eq. (19) has a solution

\[
G(u) = \pm k \frac{\alpha}{4 \tau} \text{cn}(\alpha u, k), \tag{20}
\]

where \( \text{cn} \) is another Jacobi’s elliptic function. The parameter \( \alpha \) is determined by

\[
\frac{\text{sd}(\alpha/2, k)}{\alpha/2} = \frac{1}{2 \tau}, \tag{21}
\]

which also reduces to the result of linear theory, if we take \( k = 0 \).

The above two solutions suggest us to expect the existence of another solutions using elliptic functions. Actually, we have found the family of solutions of Eqs. (15) or (19). The solutions of this family are denoted in the form

\[
G(u) = \pm C_k \frac{\alpha}{4 \tau} g(\alpha u, k), \tag{22}
\]

where \( \alpha \) is determined by

\[
\frac{A(\alpha/2, k)}{\alpha/2} = \frac{1}{2 \tau}. \tag{23}
\]

In the above, \( g \) and \( A \) are appropriate elliptic functions. The solution is given by each set of \( (g,A,C_k) \) in Tables I and II.

| \( g \) | \( \text{sn} \) | \( \text{ns} \) | \( \text{sc} \) | \( \text{cs} \) | \( \text{ds} \) | \( \text{cd} \) | \( \text{dc} \) | \( \text{nc} \) |
|---|---|---|---|---|---|---|---|---|
| 1 | \( \sqrt{1 - k^2} \) | 1 | \( k \) | \( \sqrt{1 - k^2} \) |

| \( A \) | \( \text{sn} \) | \( \text{sd} \) | \( \text{sc} \) | \( \text{dn} \) | \( \text{sd} \) |
|---|---|---|---|---|---|
| 1 | \( \sqrt{1 - k^2} \) | \( k \) | \( \sqrt{1 - k^2} \) |

Thus, all Jacobi’s elliptic functions are the solutions for either Eq. (15) or (14). This result is well understood from the following fact. Jacobi’s elliptic functions \( g(\alpha u, k) \) are connected each other by Jacobi’s imaginary transformation: \( \alpha \rightarrow i \alpha \), the imaginary transformation of modulus: \( k \rightarrow ik \) and the translation of \( u \):
$u \rightarrow u + K(k)/\alpha$, where $K(k)$ is a quarter of the period of elliptic functions. These transformations preserve the form of Eq. (13) or (19), or exchange each other. Each pair of two elliptic functions connected by the translation gives an equivalent solution. The relation of solutions is shown in Fig.1.

![Diagram of elliptic functions]

FIG. 1. The relation of elliptic solutions for Eq. (13) denoted by − sign and Eq. (19) denoted by + sign. They are connected by transformations $k \rightarrow ik$, $\alpha \rightarrow i\alpha$ and $u \rightarrow u + K(k)/\alpha$, drawn by thin lines, dashed lines and thick lines, respectively. They form a double hexagonal structure. Each pair of two elliptic functions connected with a thick line gives an equivalent solution. The functions in the left and right sides divided by the vertical center line are inverse with each other.

The explicit form for the solution of Eq.(3) is given by $H = f^{-1}(G)$. In this step some solutions for $G$ needs appropriate interpretation for the realistic meaning. The solution $G \sim cs$ for $G = \tanh(H)$ is the case, which has the region of $u$ where $|G| > 1$. On the other hand, the solution $G \sim sn$ has a realistic meaning in the whole range of $u$.

Equation (13) or (19) includes the solutions for another choice of $f(x)$ besides $\tanh(x)$ and $\tan(x)$. Actually, $f(x) = \coth(x)$ leads to Eq. (13), and $f(x) = \cot(x)$ leads to Eq. (19) with $v \rightarrow -v$. Jacobi’s elliptic functions also provide the solutions for these systems. This result presents some interesting picture. Let us compare the systems controlled by $f(x) = \tanh(x)$ and $\coth(x)$. At first sight, the behavior of these two systems seems different from each other. We note sn and ns are the solutions for both systems, which obey Eq. (13). The solution $H \sim \text{arccoth}(ns)$ in the system of $f(x) = \coth(x)$ has essentially the same form as the solution $H \sim \text{arctanh}(sn)$ in the system of $f(x) = \tanh(x)$. As the result, these two systems may have the same global phenomenon in spite of the different type of local interaction. This analysis is not simply applicable to the solution nc in the system of $f(x) = \coth(x)$, because $cn$ is the solution of the system of $f(x) = \tan(x)$, which obeys the other Eq. (19).

We discuss the meaning of the modulus in our solution. The modulus $k$ determines the period of elliptic functions, which is related to the number of traveling cluster and the boundary condition. For example, $k = 1$ gives kink like solution corresponding to the boundary condition $G(-\infty) = -G(\infty)$,

$$G(z) = \tanh(\alpha/2) \tanh(\alpha u)$$  \hspace{1cm} (24)

with $\alpha$ satisfying

$$\frac{\tanh(\alpha/2)}{\alpha/2} = \frac{1}{2\tau}.$$  \hspace{1cm} (25)

We perform the simulation to check the stability of our analytic solution. Fig.2 shows an elliptic solution for $f(x) = \tanh(x)$ given by $sn$

$$H(u, k) = \text{arc} \tanh(\frac{\alpha}{4\tau} \text{sn}(\alpha u, k)),$$  \hspace{1cm} (26)

with $\tau = 0.501$, $k = 0.9965$ and $\alpha = 0.15522$ together with the result of simulation for Eq. (1) performed in the periodic boundary condition with a suitable initial condition.

![Simulation result of $h_n$](image)

FIG. 2. Simulation result of $h_n$ in the system of $f(x) = \tanh(x)$ is shown by diamonds together with the analytic solution given by $sn$. The curve of analytic solution is shifted by $u \rightarrow u + \text{constant}$.

We are convinced the set of Eqs. (13) and (19) has much more solutions constructed with elliptic functions. As a fact, the solution found by Igarashi, Itoh and Nakamishi is rewritten in our formulation as

$$G(u) = \beta \frac{\text{sn}(\alpha + a)) + \text{sn}(\alpha(u - a))}{\text{sn}(\alpha u)} + \gamma,$$  \hspace{1cm} (27)
where $a$ is a free parameter. Actually this satisfies Eq.(15). Their solution includes Eq.(16) as a special case if we take $\alpha a = K(k)/2$. The parameter $\alpha$ is determined by

$$\frac{\alpha \operatorname{cn}(\alpha a) \operatorname{dn}(\alpha a)}{4\tau} \frac{\operatorname{sn}(\alpha a)}{\operatorname{x}_+ + \operatorname{x}_-} \operatorname{x}_+ - \operatorname{x}_- = 0, \tag{28}$$

where

$$\operatorname{x}_+ = 1 - \frac{\operatorname{sn}^2(\alpha a)}{\operatorname{sn}^2(\alpha (a \pm \frac{1}{2}))). \tag{29}$$

Then $\beta$ and $\gamma$ are given by

$$\beta = \pm \frac{\alpha}{4\tau \operatorname{sn}(\alpha a)}, \tag{30}$$

and

$$\gamma = \pm \frac{\operatorname{x}_+ + \operatorname{x}_-}{\operatorname{x}_+ - \operatorname{x}_-}. \tag{31}$$

This solution represents the asymmetry of the widths of upper and lower plateaus for traveling cluster in contrast of the solution given by a single $\operatorname{sn}$ shown in Fig.1. Some combinations of elliptic functions in the numerator and denominator of Eq.(27) are solutions, those are constructed by the three transformations represented in Fig.1. This type of solutions has asymmetry in contrast of the solutions given by a single elliptic function in Tables I and II. Probably, the set of Eqs. (15) and (19) has some algebraic structure in the space of solutions constructed with elliptic functions.

We remark all these solutions have the common velocity $v = 1/(2\tau)$. Numerical simulation shows that the traveling cluster solutions of Eq.(1) preserve their velocity as $v = 1/(2\tau)$ in the deformation of $f(x)$ beyond $\tanh(x)$. This fact suggests $v$ is some invariant quantity of the structure in the set of the solutions.

The set of differential-difference equations for $G$: Eqs. (15) and (19) offers rich contents of the system with traveling cluster solutions which characterized by elliptic functions. Eqs. (15) and (19) are related to some soliton systems. Our equations can be derived as the traveling wave equations for such soliton systems. Eqs. (15) and (19) correspond to one of the evolution equations discussed by Ablowitz and Ladik [9]. The soliton systems related to Eq.(19) were widely discussed in the self dual network equations of nonlinear inductors and capacitors by Wadati [10], Hirota and Satsuma [11]. Wadati showed the corresponding soliton system to Eq.(19) was derived from Lotka-Volterra system by Bäcklund transformation [10]. These soliton systems reduce to MKdV equation. Our system of difference equation with delay may be related to soliton systems.

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