1. Introduction

A major implication of confinement in QCD is to generate an ordered structure for the ground state. As a consequence, at low energies, the non-perturbative QCD dynamics can be described by an effective theory (called ChPT) in this approach, an expansion in either two or three of the light quark masses is performed and it is important to verify its accuracy. This is now becoming possible as many observables have been computed at NNLO and impressive progress has also been achieved on the experimental side. A summary of the most interesting results concerning the SU(2) expansion are presented by Irinel Caprini at this conference. The SU(3) chiral expansion is of considerable practical interest as it embodies the whole field of kaon physics. The difference between the SU(2) and the SU(3) expansions is not simply that the expansion parameter is larger for SU(3). It reflects also a difference between the respective chiral vacuums, which is an OZI suppressed dynamical effect. This topic is discussed in sec. 2 below. The computation of radiative corrections is important for precision tests and for properly dealing with isospin breaking and this has motivated an extension of ChPT. Some aspects and applications of this extension are presented in sec.3.

2. SU(3) chiral expansion and the OZI rule:

According to the OZI rule (or, equivalently, the large $N_c$ expansion) the role of the light quarks in the dynamics of chiral symmetry breaking in QCD is suppressed as compared to that of the gluons. Consequently, if one sets $N_f^0$ quark masses equal to zero, the value of the chiral condensate $\Sigma$ should be essentially independent of $N_f^0$. That this is unlikely to be true is suggested by the Banks-Casher formula which relates $\Sigma$ to the gluon averaged density of small eigenvalues of the Dirac operator.
The dependence upon $N_f$ shows up in the integration measure through a factor $[\det(iD)]^{N_f}$. Obviously, this tends to suppress the weight of gluon configurations which generate a high density of small eigenvalues. In fact, as we have heard from Th. Appelquist’s talk, a critical value for $N_f$ can be argued to exist $N_f^{\text{crit}} \simeq 6$ above which spontaneous breaking of chiral symmetry will no longer occur. One therefore expects the $SU(3)$ condensate to be reduced

$$\langle \bar{u}u \rangle_{N_f=3} < \langle \bar{u}u \rangle_{N_f=2}$$

(1)

What is the size of this reduction and what are the physical implications? The OZI rule starts to manifests itself in the $SU(3)$ chiral Lagrangian at order $p^4$ and implies that the coupling constants $L_4$, $L_6$ and the combination $L_2 - 2L_1$ are suppressed. How can one determine these couplings? It was first noted in ref. 6, 7 that $L_2 - 2L_1$ can be determined from experimental $Kl_4$ decay data. Both $L_4$ and $L_6$ can be determined from form-factors associated with the isospin zero scalar currents $\bar{s}(x)s(x)$ and $\bar{u}(x)u(x) + \bar{d}(x)d(x)$. These scalar form-factors are not directly accessible to experiment. However, it was shown in ref. 8 that they can be determined in the low-energy region $E \lesssim 1$ GeV from $\pi\pi \rightarrow \pi\pi$ and $\pi\pi \rightarrow KK$ amplitudes, based on analyticity properties and a few plausible assumptions/approximations (notably a simplified treatment of unitarity in the high energy region and a minimal implementation of the Brodsky-Lepage asymptotic scaling behaviour). This construction provides a way of fixing the Kaplan-Manohar ambiguity. Table 1 shows the results for $L_4$ and $L_6$ obtained in this manner in ref. 11 and more recently in ref. 12 who computed the scalar form factors up to $O(p^6)$. Significant violation of the OZI rule is observed. The value of $L_6$ corresponds to a reduction of the $SU(3)$ condensate by approximately a factor of two.

| Evaluation                        | $10^4 L_4^2(m_\rho)$ | $10^4 L_6^2(m_\rho)$ |
|-----------------------------------|-----------------------|-----------------------|
| OZI rule [6, 7]                   | $-0.3 \pm 0.3$        | $-0.2 \pm 0.5$        |
| Scalar form-factors, sum-rule [11] | $+0.3 \pm 0.2$        | $+0.3 \pm 0.3$        |
| Scalar form-factors, $O(p^6)$ [12]| $+0.4 \pm 0.2$        | $+0.1 \pm 0.3$        |

A sensitive probe of the $SU(3)$ chiral expansion is the $\pi K$ scattering process. Based on the $O(p^4)$ calculation of ref. 13 it was pointed out [14] that $L_2 - 2L_1$ as well as $L_4$ can be determined from $\pi K \rightarrow \pi K$ and the crossing symmetric amplitude $\pi\pi \rightarrow KK$. On the experimental side, data of good accuracy exists in the medium energy region $1 \lesssim E \lesssim 2.5$ GeV for both $\pi K$ and $\pi\pi \rightarrow K\bar{K}$. The general properties of analyticity, crossing and elastic unitarity allow one to extrapolate these experimental results down to low energies. More precisely, as first shown by Steiner and by Roy [15], a set of integral equations can be written down for the $l = 0$ and $l = 1$ partial-waves. A set of equations of this type was recently derived and analyzed using as input, for the first time, the available high-statistics experimental data [16].
Table 2. \( L_i \) couplings (\( \times 10^3 \)) from \( \pi K \) and from \( Kl_4 \)

|   | \( L_2 \) | \( L_3 \) | \( L_2 - 2L_4 \) | \( L_4 \) | \( \times 10^3 \) |
|---|----------|----------|-----------------|-------|----------|
| \( \pi K \) | 1.3 ± 0.1 | -4.5 ± 0.1 | -0.8 ± 0.2 | 0.5 ± 0.4 | \( L_4 \) |
| \( Kl_4 \) | 1.5 ± 0.2 | -3.2 ± 0.8 | +0.6 ± 0.5 | - | \( L_2 - 2L_4 \) |

Table 2 shows the results for the \( O(p^4) \) coupling constants \( L_1, L_2, L_3, L_4 \) determined from the \( \pi K \) amplitude as obtained from the Roy-Steiner solution. For the coupling \( L_4 \), one observes good agreement with the determination from the scalar form-factors. The table also shows the determination (from ref.17) which uses data on \( K_{l_4} \) decays. The results for \( L_1 \) and \( L_3 \) are in reasonably good agreement. The only discrepancy concerns the OZI suppressed combination \( L_2 - 2L_1 \). This suggests that the \( SU(3) \) chiral expansion is functioning properly and that including the \( O(p^6) \) corrections should resolve the discrepancies.

The computation of the \( \pi K \) amplitude at order \( p^6 \) has recently been completed.18 The expression involves 28 couplings \( C_i \) from the \( O(p^6) \) Lagrangian.19 In order to make predictions, one may estimate the relevant \( C_i \)'s from resonance saturation models. This approach has proved reasonably successful for the \( O(p^4) \) couplings.20 Clearly, the task is harder in the case of the \( C_i \)'s because many more terms in the resonance Lagrangians must be considered. The numerical predictions presented in ref.18 are based on a rather minimal resonance model. Their results concerning the S-wave scattering lengths agree rather well with those generated by solving the Roy-Steiner equations (see Table 3 below). However, a more detailed comparison of the chiral and the dispersive amplitudes shows several discrepancies and, sometimes, unphysically large \( O(p^6) \) contributions. Efforts to refine the estimates of the \( C_i \)'s are now needed in order to improve the predictivity of the \( SU(3) \) expansion and our understanding of its workings.

### 3. Electroweak extensions

The ChPT formalism is well adapted to the computation of radiative corrections. The extension, which consists in treating the photon as a dynamical field was first developed by Urech21. A further extension has been performed, allowing the light leptons to be treated dynamically as well22 (which is necessary for computing radiative corrections in semi-leptonic processes). A natural chiral counting for the charge is to set \( O(e) \sim O(p) \) and, for a lepton field \( l \), to set \( O(l) = O(p^{1/2}) \). The resulting setup represents the low-energy effective theory of the full standard model since it includes all of the light particles.
The extended chiral Lagrangian involves a number of new coupling constants: one coupling $C$ at order $e^2$, then 13 couplings $K_i$ at order $e^2p^2$ and 7 couplings $X_i$ at order $e^2l^2p^2$. A basic property of the chiral couplings in this sector is that they can be expressed as sum rules involving a QCD Green’s function and the photon propagator. This was noted long ago for the $\pi^+ - \pi^0$ mass difference. Systematic generalizations have been derived for the couplings $K_i$ which involve 2, 3 and 4-point QCD Green’s functions. These sum rules can be used to estimate the coupling constants based on light resonance models. It is important to constrain the models such as to reproduce the proper QCD short distance behaviour of the relevant Green’s functions. Several sum rules actually diverge in four dimensions. The divergences are canceled by direct contributions from QED+QCD counterterms. As a consequence, the 4 couplings $K_9, K_{10}, K_{11}$ and $K_{12}$ depend not only on the chiral scale $\mu$ but also on the short distance renormalization scale $\mu_0$.

3.1. “Strong” quark mass

As an application, let us consider the definition of a “strong” quark mass $\bar{m}_f$, i.e. a quantity which runs according to QCD only and not QCD+QED as the physical mass $m_f$ does. This issue was recently discussed and illustrated in several models. ChPT provides a very simple answer to this question

$$\bar{m}_f(\mu, \mu_0) = m_f(\mu_0)(1 + 4e^2Q_f^2[K'_9(\mu, \mu_0) + K'_{10}(\mu, \mu_0)]) + O(e^4)$$

The strong mass $\bar{m}_f$ depends on two scales in accordance with ref. In practice, the quark mass ratios that can be extracted from low energy data using ChPT concern precisely these “strong” masses since all the ratios are QCD renormalization group invariants (up to $O(e^4)$). The physical mass ratios can then be determined knowing the value of $K_9 + K_{10}$. This combination satisfies the following sum rule,

$$K'_9 + K'_{10} = \frac{1}{8F_0^2(m_K^2 - m_\pi^2)} \int d^4x \langle 0|T[V^{ud}_\mu(x)V^{\bar{d}u}_\nu(x) - V^{us}_\mu(x)V^{\bar{s}u}_\nu(x)]|0\rangle D^{\mu\nu}_\gamma(x)\right.$$  

$$+ \frac{3C}{64\pi^2 F_0^4} \left[ \frac{m_\pi^2 \log \frac{m_K^2}{m_\pi^2}}{m_\pi^2 - m_K^2} + \log \frac{m_K^2}{\mu^2} \right] + (Z_2 - Z_2) + O(m_s)$$

where $Z_2$ and $Z_s$ are QED renormalization parameters. The integrand can be rewritten in terms of spectral functions which, in principle, can be obtained from data on $\tau$ decays into hadrons with $S = 0$ and $S = 1$. At present, the $\tau$ decay data with $S = 1$ has not enough statistics for a fully quantitative evaluation but this will become possible with the advent of $\tau$-charm factories.

3.2. $X_i$ sum rules

One might anticipate that the chiral couplings $X_i$, which appear at $O(p^4)$ in the chiral Lagrangian with dynamical leptons and photons should also satisfy sum rules. These have been investigated recently. A basic ingredient is the calculation.
at order one loop in the standard model, of the semi-leptonic decay amplitude 
\[ l(p) \rightarrow \bar{u}(q) + d(q') + \nu(p'). \] 
The matching to the analogous calculation performed with ChPT must be done in two steps. One first considers a four-fermion Fermi type effective theory \( L_{\text{Fermi}} \) obtained by integrating out the heavy bosons in the SM. This effective theory is valid in a range of energies \( E \ll M_W \) and \( E \gtrsim 2 \text{ GeV} \) such that it makes sense to treat the quarks and gluons perturbatively. Matching the calculation at order one loop in this theory and in the SM determines a set of four counterterms to be added to \( L_{\text{Fermi}} \). From a technical point of view it is wise to rely on Pauli-Villars rather than dimensional regularization which avoids all problems with \( \gamma^5 \). In a second step one performs a matching with ChPT. The objects that one matches are spurion-Green’s functions which are obtained by performing functional differentiations of the generating functional with respect to the charge spurions. This method generates in a straightforward way the sum rules for the \( X_i \) parameters. As an illustration the expression for the combination \( X_6 - 4K_{12} \), which corresponds to Sirlin’s logarithmically enhanced universal factor \( S_{EW} \) is:

\[
X_6(\mu) - 4K_{12}(\mu) \simeq \frac{1}{32\pi^2} \int_0^{M_Z^2} dx \left[ \Gamma_{VV}(-x) + \Gamma_{AA}(-x) \right] + \frac{1}{16\pi^2} \left[ -6 \log \frac{M_Z}{\mu} + \frac{5}{2} \right]
\]

where \( \Gamma_{VV} \), \( \Gamma_{AA} \) are the form-factors associated with the matrix elements \( \langle 0 | V_{\mu} V_{\nu} | \pi \rangle \) and \( \langle 0 | A_{\mu} A_{\nu} | \pi \rangle \). The sum rule allows to estimate the contributions from the resonance region and the perturbative \( \alpha_s \) one in addition to the large logarithm.

Several new measurements of \( K_0 \rightarrow \pi l^- \nu \) decays\(^{29}\) as well as \( K^+ \rightarrow \pi^0 l^+ \nu \) decays\(^{30}\) have been performed with the aim of refining the determination of \( V_{us} \).

The compatibility of the two sets of results was questioned\(^{31}\). As a measure of this, let us consider the ratio of the \( f_+ \) form factors which, in ChPT, reads

\[
r_{+0} \equiv \frac{f_{+}^{K^+ \pi^0}(0)}{f_{+}^{K^0 \pi^+}(0)} = 1 + \frac{3}{4} \frac{1}{R} + O(p^4, e^2 p^2), \quad R = \frac{2m_s - m_u - m_d}{2(m_d - m_u)}
\]

Complete expressions for the \( O(p^4) \) contributions (including electromagnetic ones) can be found in ref.\(^{32}\) The sum rules allow one to estimate all the \( K_i \) and \( X_i \) coupling constants involved. The quark mass combination \( R \) can be determined from the \( K^+ - K^0 \) mass difference as a function of the mass ratio \( r = 2m_s/(m_u + m_d) \) (for which we will use \( r = 27.1 \) as obtained by MILC\(^{33}\)). Again here, the electromagnetic contributions beyond the leading order result (given by Dashen’s theorem\(^{34}\)) can be determined from the sum rules. Numerical results are collected in table 4. The table shows that the isospin breaking ratio \( R \) is likely to be significantly different from its leading order determination. An important test of \( SU(3) \) ChPT will be to verify the compatibility of the value of \( R \) with the \( \eta \rightarrow 3\pi \) process\(^{35}\).

Table 4. Isospin breaking \( K_{13} \) form-factor ratio and mass ratio

|         | Dashen | EM sum rules | MILC\(^{33}\) | experiment |
|---------|--------|--------------|--------------|------------|
| \( R \) | 41.5   | 30.5         | 33.3         | –          |
| \( r_{+0} \) | 1.020  | 1.030        | –            | 1.040 ± 0.010 |
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