A Generalization of the Lifting Lemma for Logic Programming

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Abstract. Since the seminal work of J. A. Robinson on resolution, many lifting lemmas for simplifying proofs of completeness of resolution have been proposed in the literature. In the logic programming framework, they may also help to detect some infinite derivations while proving goals under the SLD-resolution. In this paper, we first generalize a version of the lifting lemma, by extending the relation “is more general than” so that it takes into account only some arguments of the atoms. The other arguments, which we call neutral arguments, are disregarded. Then we propose two syntactic conditions of increasing power for identifying neutral arguments from mere inspection of the text of a logic program.

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1 Introduction

Since the seminal work of J. A. Robinson on resolution [16], many lifting lemmas have been proposed in the literature, see for instance [6] p. 84, [6] p. 848, [11] p. 47, [1] p. 60 or where a stronger version than that of [1] is given. Lifting results are used as a tool to simplify proofs of completeness of resolution.

In this paper, we are interested in generalizing the lifting lemma presented in [1] where it is called the Lifting Theorem. Given a logic program \( P \), our aim is the further design of a mechanism that generates at compile-time classes of queries that have an infinite SLD-derivation w.r.t. \( P \). Towards this end, we propose a criterion in the form of a sufficient condition that enables us to identify infinite derivations.

Notice that such a condition can be easily designed from the Lifting Theorem that is proposed by Apt in [1], as we explain now. Assuming that the reader is familiar with the standard notations about logic programming reviewed in the next section, let us first recall the notion of a lift. An SLD-derivation \( \xi' \) is a lift of another SLD-derivation \( \xi \) if there exists a prefix \( D \) of \( \xi' \) that is more general than \( \xi \), i.e., \( D \) is of the same length than \( \xi \) and in \( D \) and \( \xi \) the same clauses are used in the same order, atoms in the same positions are selected at each step and for each \( i \), the \( i \)-th query of \( D \) is more general than that of \( \xi \). The following result holds.

**Theorem 1 (Lifting [1]).** Let \( P \) be a logic program, \( Q \) be a query and \( \eta \) be a substitution. For every SLD-derivation \( \xi \) of \( P \cup \{Q\eta\} \), there exists an SLD-derivation of \( P \cup \{Q\} \) which is a lift of \( \xi \).

This theorem provides a sufficient condition that can be used to identify infinite SLD-derivations.

**Corollary 1.** Let \( P \) be a logic program and \( A \) be an atom. Suppose that there exists a sequence of SLD-derivation steps from \( A \) to a query \( Q \) using the clauses of \( P \) and that \( Q \) contains an atom \( B \) that is more general than \( A \). Then there exists an infinite SLD-derivation of \( P \cup \{A\} \).

**Proof.** Let \( A \xrightarrow{P} Q \) denote that there exists a sequence of SLD-derivation steps from \( A \) to \( Q \) using the clauses of \( P \). As \( B \) is more general than \( A \), by the Lifting Theorem we have \( B \xrightarrow{P} Q_1 \) where \( Q_1 \) is a query that is more general than \( Q \). So there exists \( B_1 \) in \( Q_1 \) that is more general than \( B \). Iterating this process, we construct an infinite sequence of queries \( Q_1, Q_2, \ldots \) and an infinite sequence of atoms \( B_1, B_2, \ldots \) such that for \( i \geq 1 \), \( B_i \) is in \( Q_i \) and \( B_i \xrightarrow{P} Q_{i+1} \).

Nevertheless, such a condition is rather weak because it fails at identifying some simple loops. This is illustrated by the following example.

**Example 1.** Let \( c \) be the clause \( p(x, y) \leftarrow p(f(x), z) \). Then from the head of \( c \) we get an SLD-derivation step \( p(x, y) \xrightarrow{c} p(f(x_1), z_1) \). Since the atom \( p(f(x_1), z_1) \) is not more general than \( p(x, y) \), Corollary [1] cannot be used. Moreover,

\[
p(f(x_1), z_1) \xrightarrow{c} p(f(f(x_2)), z_2) \xrightarrow{c} p(f(f(f(x_3))), z_3) \ldots
\]
As the first argument of $p$ grows from step to step, we will never be able to use Corollary 1 to show that there exists an infinite SLD-derivation of $\{c\} \cup \{p(x, y)\}$.

In this article, we extend the relation “is more general than” so that it takes into account only some arguments of the atoms, the others (which are called neutral arguments) being, roughly, disregarded. We show that the Lifting Theorem 1, and hence Corollary 1, can be extended to this new relation. Neutral arguments correspond to the following intuition. Suppose we have an SLD-derivation $\xi$ of a query $Q$ w.r.t. a logic program $P$. Let $Q'$ be the query obtained by replacing the neutral arguments of the atoms of $Q$ by any term. Then, there exists an SLD-derivation $\xi'$ of $Q'$ w.r.t. $P$ such that $\xi'$ is a lift of $\xi$ up to the neutral arguments.

Example 2. Consider Example 1 again. For any derivation step $p(s_1, s_2) \Rightarrow_c p(s_3, s_4)$ if we replace $s_1$ by any term $t_1$ then there exists a derivation step $p(t_1, s_2) \Rightarrow_c p(t_3, t_4)$. Moreover, notice that $s_4$ and $t_4$ are variables, so $p(t_3, t_4)$ is more general than $p(s_3, s_4)$ up to the first argument of $p$. Consequently, by the intuition described above, the first argument of $p$ is neutral for derivation w.r.t. $c$. Finally, as $p(x, y) \Rightarrow_c p(f(x_1), z_1)$ and $p(f(x_1), z_1)$ is more general than $p(x, y)$ up to the first argument which is neutral, by the extended version of Corollary 1 there exists an infinite SLD-derivation of the query $p(x, y)$ w.r.t. $c$.

Last but not least, we offer two syntactic conditions of increasing power for easily identifying neutral arguments from mere inspection of the text of a logic program.

The paper is organized as follows. In Section 2 we review basic concepts concerning logic programming and introduce some notations. Then in Section 3 we give a generic presentation of what we mean for an argument to be neutral. In Section 4 we propose some particular concrete means for detecting neutral arguments. In Section 5 we apply our results to generate some queries that have an infinite SLD-derivation w.r.t. a given logic program. Finally, in Section 6 we discuss related works.

2 Preliminaries

We try to strictly adhere to the notations, definitions, and results presented in [1].

$\mathbb{N}$ denotes the set of non-negative integers and for any $n \in \mathbb{N}$, $[1, n]$ denotes the set $\{1, \ldots, n\}$. If $n = 0$ then $[1, n] = \emptyset$.

From now on, we fix a language $\mathcal{L}$ of programs. We assume that $\mathcal{L}$ contains an infinite number of constant symbols. The set of relation symbols of $\mathcal{L}$ is $\Pi$, and we assume that each relation symbol $p$ has an unique arity, denoted as $\text{arity}(p)$. $\text{TU}_\mathcal{L}$ (resp. $\text{TB}_\mathcal{L}$) denotes the set of all (ground and non ground) terms of $\mathcal{L}$ (resp. atoms of $\mathcal{L}$). A query is a finite sequence of atoms $A_1, \ldots, A_n$ (where $n \geq 0$). Queries are denoted by $Q$, $Q'$, $\ldots$ or by bold upper-case letters $\mathbf{A}$, $\mathbf{B}$, $\ldots$. 

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Let $t$ be a term. Then $\text{Var}(t)$ denotes the set of variables occurring in $t$. This notation is extended to atoms, queries and clauses. Let $\theta := \{x_1/t_1, \ldots, x_n/t_n\}$ be a substitution. We denote by $\text{Dom}(\theta)$ the set of variables $\{x_1, \ldots, x_n\}$, and by $\text{Ran}(\theta)$ the set of variables appearing in $t_1, \ldots, t_n$. We define $\text{Var}(\theta) = \text{Dom}(\theta) \cup \text{Ran}(\theta)$. Given a set of variables $V$, $\theta|_V$ denotes the substitution obtained from $\theta$ by restricting its domain to $V$.

Let $t$ be a term and $\theta$ be a substitution. Then, the term $t\theta$ is called an instance of $t$. If $\theta$ is a renaming (i.e. a substitution that is a 1-1 and onto mapping from its domain to itself), then $t\theta$ is called a variant of $t$. Finally, $t$ is called more general than $t'$ if $t'$ is an instance of $t$.

A logic program is a finite set of definite clauses. In program examples, we use the ISO-Prolog syntax. Let $P$ be a logic program. Then $\Pi_P$ denotes the set of relation symbols appearing in $P$. Consider a non-empty query $A, B, C$ and a clause $c$. Let $H \leftarrow B$ be a variant of $c$ variable disjoint with $A, B, C$ and assume that $B$ and $H$ unify. Let $\theta$ be an mgu of $B$ and $H$. Then $A, B, C \theta \Rightarrow c(A, B, C)\theta$ is an SLD-derivation step with $H \leftarrow B$ as its input clause and $B$ as its selected atom. If the substitution $\theta$ or the clause $c$ is irrelevant, we drop a reference to it.

Let $Q_0$ be a query. A maximal sequence $Q_0 \xrightarrow{a_1}_{c_1} Q_1 \xrightarrow{a_2}_{c_2} \cdots$ of SLD-derivation steps is called an SLD-derivation of $P \cup \{Q_0\}$ if $c_1, c_2, \ldots$ are clauses of $P$ and if the standardization apart condition holds, i.e.: each input clause used is variable disjoint from the initial query $Q_0$ and from the mgu’s and input clauses used at earlier steps. A finite SLD-derivation may end up either with the empty query (then it is a successful SLD-derivation) or with a non-empty query (then it is a failed SLD-derivation). We say $Q_0$ loops with respect to $P$ if there exists an infinite SLD-derivation of $P \cup \{Q_0\}$.

Finally, we write $Q \xrightarrow{P} Q'$ (resp. $Q \xrightarrow{P} Q'$) if there exists a finite prefix (resp. a finite non-empty prefix) ending at $Q'$ of an SLD-derivation of $P \cup \{Q\}$.

Let $E$ and $F$ be two sets. Then, $f : E \rightarrow F$ denotes that $f$ is a function from $E$ to $F$ and $f : E \rightarrow F$ denotes that $f$ is a mapping from $E$ to $F$. The domain of a function $f$ from $E$ to $F$ is denoted by $\text{Dom}(f)$ and is defined as: $\text{Dom}(f) = \{x \mid x \in E, f(x) \text{ exists}\}$. Thus, if $f$ is a mapping from $E$ to $F$, then $\text{Dom}(f) = E$.

### 3 Neutral Arguments for SLD-Derivation

The basic idea in the work we present here relies on some arguments in clauses which we can be disregarded when unfolding a query. For instance, the second argument of the non-unit clause of the program

```
append([],Ys,Ys).
append([X|Xs],Ys,[X|Zs]) :- append(Xs,Ys,Zs).
```
is such a candidate. Notice that a very common programming technique
called *accumulator passing* (see for instance e.g. [15], p. 21–25), always
produces such patterns.

**Example 3.** A classical example of the accumulator passing technique is
the following program **REVERSE**.

\[
\begin{align*}
\text{reverse}(L,R) & :\rightarrow \text{rev}(L,\emptyset,R). \\
\text{rev}(\emptyset,R,R). & \\
\text{rev}(X\{Xs\},R0,R) & \rightarrow \text{rev}(Xs,\{X|R0\},R).
\end{align*}
\]

Concerning termination, we may ignore the third and the second argu-
ment of the recursive clause of **rev** while unfolding a query with this
clause. Only the first argument can stop the unfolding. □

But we can be more precise. Instead of only identifying arguments that
can be totally disregarded as in the above examples, we can try to identify
arguments that can be replaced, when unfolding a query, by any terms
for which a given condition holds. Consider for instance the program

\[
p(x,f(z)) \rightarrow q(x,y), p(y,f(z)).
\]

If we unfold a query \(p(t_1,t_2)\) with this program, then if we replace the
second argument of \(p\) by any instance \(t_3\) of \(f(z)\), we can still unfold
\(p(t_1,t_3)\).

In the sequel of this section, we give a technical tool to describe specific
arguments inside a program and present an extension of the relation
"is more general than". Then we formalize the concept of derivation
neutrality and we propose an extended version of the Lifting Theorem [1].

### 3.1 Filters

Let us first introduce the notion of a **filter**. We use filters in order to
distinguish atoms some arguments of which satisfy a given condition.
A condition upon atom arguments, *i.e.* terms, can be defined as a mapping
in the following way.

**Definition 1 (Term-Condition).** A term-condition is a mapping from
\(TU_L\) to \{true,false\}.

**Example 4.** The following mappings are term-conditions.

\[
\begin{align*}
\text{f}_{\text{true}} : TU_L & \rightarrow \{\text{true, false}\} \\
& \quad t \rightarrow \text{true} \\
\text{f}_1 : TU_L & \rightarrow \{\text{true, false}\} \\
& \quad t \rightarrow \text{true} \text{ iff } t \text{ is an instance of } [x|y] \\
\text{f}_2 : TU_L & \rightarrow \{\text{true, false}\} \\
& \quad t \rightarrow \text{true} \text{ iff } t \text{ unifies with } h(a,x)
\end{align*}
\]
Now we can precise what we exactly mean by a filter.

**Definition 2 (Filter).** A filter, denoted by $\Delta$, is a mapping from $\Pi$ such that: for each $p \in \Pi$, $\Delta(p)$ is a function from $[1, \text{arity}(p)]$ to the set of term-conditions.

**Example 5 (Example continued).** Let $\Pi := \{p\}$ with $p$ a relation symbol whose arity equals 3. Then, $\Delta := \{p \mapsto \{1 \mapsto \text{true}, 2 \mapsto f_1\}\}$ is a filter.

Notice that, given a filter $\Delta$, the relation “is more general than” can be extended in the following way: an atom $A := p(\cdots)$ is $\Delta$-more general than $B := p(\cdots)$ if the “is more general than” requirement holds for those arguments of $A$ whose position is not in the domain of $\Delta(p)$ while the other arguments satisfy their associated term-condition.

**Definition 3 ($\Delta$-More General).** Let $\Delta$ be a filter, $A$ and $B$ be two atoms and $Q := A_1, \ldots, A_n$ and $Q' := B_1, \ldots, B_m$ be two queries.

- Let $\eta$ be a substitution. Then $A$ is $\Delta$-more general than $B$ for $\eta$ if:

\[
\begin{align*}
A &= p(s_1, \ldots, s_n) \\
B &= p(t_1, \ldots, t_n) \\
\forall i \in [1, n] \setminus \text{Dom}(\Delta(p)), & t_i = s, \eta \\
\forall i \in \text{Dom}(\Delta(p)), & \Delta(p)(i)(s_i) = \text{true}.
\end{align*}
\]

- $A$ is $\Delta$-more general than $B$ if there exists a substitution $\eta$ s.t. $A$ is $\Delta$-more general than $B$ for $\eta$.

- Let $\eta$ be a substitution. Then $Q$ is $\Delta$-more general than $Q'$ for $\eta$ if:

\[
\begin{align*}
\{ n = m & \text{ and } \\
\forall i \in [1, n], & A_i \text{ is } \Delta \text{-more general than } B_i \text{ for } \eta.
\end{align*}
\]

- $Q$ is $\Delta$-more general than $Q'$ if there exists a substitution $\eta$ s.t. $Q$ is $\Delta$-more general than $Q'$ for $\eta$.

**Example 6.** Let $\Pi := \{p\}$ and $A := p(b, x, h(a, x)), B := p(a, [a][b], x), C := p(a, [a][b], h(y, b))$.

- Consider the filter $\Delta$ defined in Example 5. Then, $A$ is not $\Delta$-more general than $B$ and $C$ because, for instance, its second argument $x$ is not an instance of $[x][y]$ as required by $f_1$. On the other hand, $B$ is $\Delta$-more general than $A$ for the substitution $\{x/h(a, x)\}$ and $B$ is $\Delta$-more general than $C$ for the substitution $\{x/h(y, b)\}$. Finally, $C$ is not $\Delta$-more general than $A$ because $h(y, b)$ is not more general than $h(a, x)$ and $C$ is not $\Delta$-more general than $B$ because $h(y, b)$ is not more general than $x$.

- Consider the term-conditions defined in Example 5. Let $\Delta' := \{p \mapsto \{1 \mapsto \text{true}, 2 \mapsto f_1, 3 \mapsto f_2\}\}$. Then, $A$ is not $\Delta'$-more general than $B$ and $C$ for the same reason as above. On the other hand, $B$ is $\Delta'$-more general than $A$ and $C$ for any substitution and $C$ is $\Delta'$-more general than $A$ and $B$ for any substitution.

The following proposition states an intuitive result:

**Proposition 1.** Let $\Delta$ be a filter and $Q$ and $Q'$ be two queries. Then $Q$ is $\Delta$-more general than $Q'$ if and only if there exists a substitution $\eta$ such that $\text{Var}(\eta) \subseteq \text{Var}(Q, Q')$ and $Q$ is $\Delta$-more general than $Q'$ for $\eta$.

**Proof.** The proof of this proposition is given in Appendix 1. □
3.2 Derivation Neutral Filters: Operational Definition

Before we give a precise definition of the kind of filters we are interested in, we review the notion of a lift. The definition we propose below is the same as that of \cite{1} p. 57 up to the arguments whose position is distinguished by a filter.

Definition 4 (\(\Delta\)-Lift). Let \(\Delta\) be a filter. Consider a sequence of SLD-derivation steps

\[ \xi := Q_0 \Rightarrow_{c_1} Q_1 \cdots Q_n \Rightarrow_{c_{n+1}} Q_{n+1} \cdots \]

We say that the sequence of SLD-derivation steps

\[ \xi' := Q'_0 \Rightarrow_{c_1} Q'_1 \cdots Q'_n \Rightarrow_{c_{n+1}} Q'_{n+1} \cdots \]

is a \(\Delta\)-lift of \(\xi\) if

- \(\xi\) is of the same or smaller length than \(\xi'\),
- for each \(Q_i\) in \(\xi\), \(Q'_i\) is \(\Delta\)-more general than \(Q_i\),
- for each \(Q_i\) in \(\xi\), in \(Q_i\) and \(Q'_i\) atoms in the same positions are selected.

In the sequel of this paper, we focus on “derivation neutral” filters. The name “derivation neutral” stems from the fact that in any derivation of a query \(Q\), the arguments of \(Q\) whose position is distinguished by such a filter can be safely replaced by any terms satisfying the associated term-condition. Such a replacement does not modify the derivation process.

Definition 5 (Derivation Neutral). Let \(\Delta\) be a filter and \(c\) be a clause. We say that \(\Delta\) is DN for \(c\) if

- for each SLD-derivation step \(Q \Rightarrow_{c} Q_1\),
- for each query \(Q'\) that is \(\Delta\)-more general than \(Q\), there exists a query \(Q'_1\) such that \(Q \Rightarrow_{c} Q'_1\) and \(Q' \Rightarrow_{c} Q'_1\) is a \(\Delta\)-lift of \(Q \Rightarrow_{c} Q_1\).

This definition is extended to finite sets of clauses: \(\Delta\) is DN for a logic program \(P\) if it is DN for each clause of \(P\).

Example 7. The following examples illustrate the above definitions.

- Consider the following program APPEND:

\[
\begin{align*}
\text{append}([],Xs,Xs). & \quad \% C1 \\
\text{append}([X|Xs],Ys,[X|Zs]) & \leftarrow \text{append}(Xs,Ys,Zs). \quad \% C2
\end{align*}
\]

Consider the term-condition \(f_{true}\) defined in Example \(4\), the set of relation symbols \(\Pi := \{\text{append}\}\) and the filter \(\Delta := (\text{append} \mapsto (2 \mapsto f_{true}))\). Then, \(\Delta\) is DN for \(C2\). However, \(\Delta\) is not DN for \(\text{APPEND}\) because it is not DN for \(C1\).

- Consider the following program MERGE:

\[
\begin{align*}
\text{merge}([X|Xs],[Y|Ys],[X|Zs]) & \leftarrow \text{merge}(Xs,Ys,Zs). 
\end{align*}
\]

Consider the term-condition \(f_1\) defined in Example \(4\), the set of relation symbols \(\Pi := \{\text{merge}\}\) and the filter \(\Delta := (\text{merge} \mapsto (2 \mapsto f_1))\). Then, \(\Delta\) is DN for \(\text{MERGE}\). \(\square\)
Repeatedly using this definition, we get an extended version of the Lifting Theorem.\footnote{\textsuperscript{2}}

**Theorem 2 (\(\Delta\)-Lifting).** Let \(P\) be a logic program and \(\Delta\) be a DN filter for \(P\). Let \(\xi\) be an SLD-derivation of \(P \cup \{Q_0\}\) and \(Q'_0\) be a query that is \(\Delta\)-more general than \(Q_0\).

Then there exists an SLD-derivation of \(P \cup \{Q'_0\}\) that is a \(\Delta\)-lift of \(\xi\).

**Proof.** If \(\xi\) is infinite, we can construct a required SLD-derivation \(\xi'\) whose length is infinite by repeatedly using Definition \(\footnote{\textsuperscript{2}}\). Now if \(\xi\) is of finite length, then

- either \(\xi\) is successful: in this case, we can construct a required SLD-derivation \(\xi'\) of the same length than \(\xi\) by repeatedly using Definition \(\footnote{\textsuperscript{2}}\),

- either \(\xi\) fails. Suppose that \(\xi := Q_0 \Rightarrow Q_1 \cdots Q_n \Rightarrow Q_{n+1}\). Then, by repeatedly using Definition \(\footnote{\textsuperscript{2}}\), we can construct a finite sequence of SLD-derivation steps

\[
Q'_0 \Rightarrow Q'_1 \cdots Q'_n \Rightarrow Q'_{n+1}
\]

that is a \(\Delta\)-lift of \(\xi\). But as \(Q'_{n+1}\) is \(\Delta\)-more general than \(Q_{n+1}\), this sequence may not be a maximal one, i.e. from \(Q'_{n+1}\) and \(P\), we may derive \(Q'_{n+2}\) and so forth. Hence, in this case, there may exist an SLD-derivation of \(P \cup \{Q'_0\}\) that is a \(\Delta\)-lift of \(\xi\) but of longer length than \(\xi\). \(\square\)

### 3.3 Model Theoretic Results Induced by DN Filters

Lifting lemmas are used in the literature to prove completeness of SLD-resolution. Now that we have established an extended Lifting Theorem, it may be worth to investigate its consequences from the model theoretic point of view.

Let \(E\) be an atom or a query and \(\Delta\) be a filter. Then one may “expand” the atoms occurring in \(E\) by replacing every argument whose position is distinguished by \(\Delta\) by any term that satisfies the associated term-condition.

**Definition 6 (Expansion of atoms and queries by a filter).** Let \(\Delta\) be a filter.

- Let \(A\) be an atom. The expansion of \(A\) w.r.t. \(\Delta\), denoted \([A]^\Delta\), is the set defined as

\[
[A]^\Delta := \{ B \in TB \mid B \text{ is } \Delta\text{-more general than } A \text{ for } \epsilon\}
\]

where \(\epsilon\) denotes the empty substitution.

- Let \(Q := A_1, \ldots, A_n\) be a query. The expansion of \(Q\) w.r.t. \(\Delta\), denoted \([Q]^\Delta\), is the set defined as:

\[
[Q]^\Delta := \{(B_1, \ldots, B_n) \mid B_1 \in [A_1]^\Delta, \ldots, B_n \in [A_n]^\Delta\}.
\]
Term interpretations in the context of logic programming were introduced in [6] and further investigated in [7] and [12]. A term interpretation for \( \mathcal{L} \) is identified with a (possibly empty) subset of the term base \( \mathcal{T} \mathcal{B}_\mathcal{L} \). So, as for atoms, a term interpretation can be expanded by a set of positions.

**Definition 7 (Expansion of term interpretations by a filter).** Let \( \Delta \) be a filter and \( I \) be a term interpretation for \( \mathcal{L} \). Then \( [I]^\Delta \) is the term interpretation for \( \mathcal{L} \) defined as:

\[
[I]^\Delta := \bigcup_{A \in I} [A]^\Delta.
\]

For any logic program \( P \), we denote by \( C(P) \) its least term model.

**Theorem 3.** Let \( P \) be a logic program and \( \Delta \) be a DN filter for \( P \). Then \( [C(P)]^\Delta = C(P) \).

**Proof.** The inclusion \( C(P) \subseteq [C(P)]^\Delta \) is trivial so let us concentrate on the other one i.e. \( [C(P)]^\Delta \subseteq C(P) \). Let \( A' \in [C(P)]^\Delta \). Then there exists \( A \in C(P) \) such that \( A' \in [A]^\Delta \). A well known result states:

\[
C(P) = \{ B \in \mathcal{T} \mathcal{B}_\mathcal{L} \mid \text{there exists a successful SLD-derivation of } B \} \quad (1)
\]

Consequently, there exists a successful SLD-derivation of \( A \). Therefore, by the \( \Delta \)-Lifting Theorem 2, there exists a successful SLD-derivation of \( A' \). So by \( (1) \), \( A' \in C(P) \). \( \Box \)

### 4 Some Particular DN Filters

In this section, we consider some instances of the definitions of Section 3.

#### 4.1 DN Sets of Positions

The first instance we consider corresponds to filters whose associated term-conditions are all equal to \( f_{\text{true}} \) (see Example 4.) Within such a context, as the term-conditions are fixed, each filter \( \Delta \) is uniquely determined by the domains of the functions \( \Delta(p) \) for \( p \in \Pi \). Hence the following definition.

**Definition 8 (Set of Positions).** A set of positions, denoted by \( \tau \), is a mapping from \( \Pi \) to \( 2^{\mathbb{N}} \) such that: for each \( p \in \Pi \), \( \tau(p) \) is a subset of \( [1, \text{arity}(p)] \).

**Example 8.** Let \( \Pi := \{ \text{append, append3} \} \) with \( \text{arity(append)} = 3 \) and \( \text{arity(append3)} = 4 \). Then

\[
\tau := \langle \text{append} \mapsto \{2\}, \text{append3} \mapsto \{2, 3\} \rangle
\]

is a set of positions. \( \Box \)

Not surprisingly, the filter that is generated by a set of positions is defined as follows.
Definition 9 (Associated Filter). Let $\tau$ be a set of positions and $f_{\text{true}}$ be the term-condition defined in Example 4. The filter $\Delta[\tau]$ defined as:

for each $p \in \Pi$, $\Delta[\tau](p)$ is the mapping from $\tau(p)$ to $\{f_{\text{true}}\}$

is called the filter associated to $\tau$.

Example 9 (Example 8 continued). The filter associated to $\tau$ is

$$\Delta[\tau] := \langle \text{append} \mapsto \{2 \mapsto f_{\text{true}}\}, \text{append3} \mapsto \{2 \mapsto f_{\text{true}}, 3 \mapsto f_{\text{true}}\} \rangle.$$ 

Now we define a particular kind of sets of positions. These are named after “DN” because, as stated by Theorem 4 below, they generate DN filters.

Definition 10 (DN Set of Positions). Let $\tau$ be a set of positions. We say that $\tau$ is DN for a clause $p(s_1, \ldots, s_n) \leftarrow B$ if:

$$\forall i \in \tau(p), \left\{ \begin{array}{l}
    s_i \text{ is a variable} \\
    s_i \text{ occurs only once in } p(s_1, \ldots, s_n) \\
    \text{for each } q(t_1, \ldots, t_m) \in B: \\
    \forall j \in [1, m], s_i \in \text{Var}(t_j) \Rightarrow j \in \tau(q). 
\end{array} \right.$$ 

A set of positions is DN for a program $P$ if it is DN for each clause of $P$.

The intuition of Definition 10 is the following. If for instance we have a clause $c := p(x, y, f(z)) \leftarrow p(g(y, z), x, z)$ then in the first two positions of $p$ we can put any terms and get a derivation step w.r.t. $c$ because the first two arguments of the head of $c$ are variables that appear exactly once in the head. Moreover, $x$ and $y$ of the head reappear in the body but again only in the first two positions of $p$. So, if we have a derivation step $p(s_1, s_2, s_3) \Rightarrow c p(t_1, t_2, t_3)$, we can replace $s_1$ and $s_2$ by any terms $s'_1$ and $s'_2$ and get another derivation step $p(s'_1, s'_2, s_3) \Rightarrow c p(t'_1, t'_2, t'_3)$ where $t'_3$ is the same as $t_3$ up to variable names.

Example 10 (Example 8 continued). $\tau$ is DN for the following program:

$$\text{append}(\text{[X|Xs]}, Ys, \text{[X|Zs]}) \leftarrow \text{append}(Xs, Ys, Zs).$$

$$\text{append3}(Xs, Ys, Zs, Ts) \leftarrow \text{append}(Xs, Ys, Us), \text{append}(Us, Zs, Ts).$$

DN sets of positions generate DN filters.

Theorem 4. Let $\tau$ be a DN set of positions for a logic program $P$. Then $\Delta[\tau]$ is DN for $P$.

Proof. See Lemma 4 and Theorem 5 further. $\square$

Notice that the set of DN sets of positions of a logic program is not empty as stated by the following proposition.

Proposition 2. Let $P$ be a logic program. Then $\tau_0 := \langle p \mapsto \emptyset | p \in \Pi \rangle$ is DN for $P$.

Proof. By Definition 3

Moreover, an atom $A$ is $\Delta[\tau_0]$-more general than an atom $B$ iff $A$ is more general than $B$. So, in the context of the filter $\Delta[\tau_0]$, the $\Delta$-Lifting Theorem 2 is the same as the Lifting Theorem 1.
4.2 DN Sets of Positions with Associated Terms

Now we consider another instance of the definitions of Section 3. As we will see, it is more general than the previous one. It corresponds to filters whose associated term-conditions have all the form “is an instance of $t$” where $t$ is a term that uniquely determines the term-condition. Hence the following definition.

**Definition 11 (Sets of Positions with Associated Terms).** A set of positions with associated terms, denoted by $\tau^+$, is a mapping from $\Pi$ such that: for each $p \in \Pi$, $\tau^+(p)$ is a function from $[1, \text{arity}(p)]$ to $\text{TU}_\land$.

**Example 11.** Let $\Pi := \{p, q\}$ where the arity of $p$ and $q$ is 2. Then,

$$\tau^+ := \langle p \mapsto \langle 2 \mapsto x \rangle, q \mapsto \langle 2 \mapsto g(x) \rangle \rangle$$

is a set of positions with associated terms.

The filter that is generated by a set of positions with associated terms is defined as follows.

**Definition 12 (Associated Filter).** Let $\tau^+$ be a set of positions with associated terms. The filter associated to $\tau^+$, denoted by $\Delta[\tau^+]$, is defined as: for each $p \in \Pi$, $\Delta[\tau^+](p)$ is the mapping

$$\text{Dom}(\tau^+(p)) \rightarrow \text{The set of term-conditions}$$

$$i \mapsto \left\{ \begin{array}{l}
\text{TU}_\land \mapsto \{\text{true, false}\} \\
\text{true} \iff t \text{ is an instance of } \tau^+(p)(i)
\end{array} \right.$$ 

**Example 12 (Example 11 continued).** The filter associated to $\tau^+$ is

$$\Delta[\tau^+] := \langle p \mapsto \langle 2 \mapsto f_1 \rangle, q \mapsto \langle 2 \mapsto f_2 \rangle \rangle$$

where

$$f_1 : \text{TU}_\land \mapsto \{\text{true, false}\}$$

$$t \mapsto \text{true} \iff t \text{ is an instance of } x$$

$$f_2 : \text{TU}_\land \mapsto \{\text{true, false}\}$$

$$t \mapsto \text{true} \iff t \text{ is an instance of } g(x)$$

As for sets of positions, we define a special kind of sets of positions with associated terms.

**Definition 13 (DN Sets of Positions with Associated Terms).** Let $\tau^+$ be a set of positions with associated terms. We say that $\tau^+$ is DN for a clause $p(s_1, \ldots, s_n) \leftarrow B$ if these conditions hold:

1. (DN1) $\forall i \in \text{Dom}(\tau^+(p)), \forall j \in [1, n] \setminus \{i\} : \text{Var}(s_i) \cap \text{Var}(s_j) = \emptyset$,
2. (DN2) $\forall (i \mapsto u_i) \in \tau^+(p) : s_i \text{ is more general than } u_i$,
3. (DN3) $\forall q \in \text{Dom}(\tau^+(p)), \forall q(t_1, \ldots, t_m) \in B, \forall j \notin \text{Dom}(\tau^+(q)) : \text{Var}(s_i) \cap \text{Var}(t_j) = \emptyset$,
4. (DN4) $\forall q(t_1, \ldots, t_m) \in B, \forall (j \mapsto u_j) \in \tau^+(q) : t_j \text{ is an instance of } u_j$. 

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A set of positions with associated terms is DN for a program $P$ if it is DN for each clause of $P$.

This definition says that any $s_i$ where $i$ is in the domain of $\tau^+(p)$ (i.e. position $i$ is distinguished by $\tau^+$): (DN1) does not share its variables with the other arguments of the head, (DN2) is more general than the term $u_i$ that $i$ is mapped to by $\tau^+(p)$, (DN3) distributes its variables to some arguments $t_j$ of some atoms $q(t_1, \ldots, t_m)$ in $B$ such that $j$ is in the domain of $\tau^+(q)$ (i.e. position $j$ is distinguished by $\tau^+$). Moreover, (DN4) says that any argument $t_j$, where $j$ is distinguished by $\tau^+$, of any atom $q(t_1, \ldots, t_m)$ in $B$ is such that $t_j$ is an instance of the term $u_j$.

Example 13 (Example 11 continued). $\tau^+$ is DN for the following program:

\[
\begin{align*}
p(f(x), y) & : - q(x, g(x)), p(x, g(y)) \\
q(a, g(x)) & : - q(a, g(b))
\end{align*}
\]

The preceding notion is closed under renaming:

**Proposition 3.** Let $c$ be a clause and $\tau^+$ be a set of positions with associated terms that is DN for $c$. Then $\tau^+$ is DN for every variant of $c$.

**Proof.** The proof of this proposition is given in Appendix 1. □

Notice that a set of positions is a particular set of positions with associated terms in the following sense.

**Proposition 4.** Let $\tau$ be a set of positions and $x$ be any variable. Let $\tau^+$ be the set of positions with associated terms defined as: for each $p \in \Pi$, $\tau^+(p) := (\tau(p) \ni \{x\})$. Then, the following holds.

1. An atom $A$ is $\Delta[\tau]$-more general than an atom $B$ iff $A$ is $\Delta[\tau^+]$-more general than $B$.
2. For any clause $c$, $\tau$ is DN for $c$ iff $\tau^+$ is DN for $c$.

**Proof.** The proof follows from these remarks.

- Item 1 is a direct consequence of the definition of “$\Delta$-more general” (see Definition 3) and the definition of the filter associated to a set of positions (see Definition 10) and to a set of positions with associated terms (see Definition 13).
- Item 2 is a direct consequence of the definition of DN sets of positions (see Definition 14), and DN sets of positions with associated terms (see Definition 15). □

Finally, the sets of positions with associated terms of Definition 13 were named after “DN” because of the following result.

**Theorem 5.** Let $P$ be a logic program and $\tau^+$ be a set of positions with associated terms that is DN for $P$. Then $\Delta[\tau^+]$ is DN for $P$.

**Proof.** The proof of this theorem is given in Appendix 2. □
For each predicate symbol \( p \), let \( \phi_p \) be the function from \([1, \text{arity}(p)]\) to \( \text{TU}_C \) whose domain is empty. As in the case of sets of positions, the set of DN sets of positions with associated terms of a logic program is not empty as stated by the following proposition.

**Proposition 5.** Let \( P \) be a logic program. Then \( \tau_0^+ := \{ p \mapsto \phi_p \mid p \in \Pi \} \) is DN for \( P \).

**Proof.** By Definition \( \Xi \). \( \square \)

Moreover, an atom \( A \) is \( \Delta[\tau_0^+]\)-more general than an atom \( B \) iff \( A \) is more general than \( B \). So, in the context of the filter \( \Delta[\tau_0^+] \), the \( \Delta \)-Lifting Theorem \( \Xi \) is the same as the Lifting Theorem \( \mathbb{I} \).

## 5 Examples

In this section, we focus on left derivations i.e. we only consider the leftmost selection rule: \( Q \Rightarrow Q' \) is a left derivation step if it is an SLD-derivation step whose selected atom is the first atom of \( Q \) from the left. We say that a query \( Q \) left loops w.r.t. a program \( P \) if there exists an infinite left derivation of \( P \cup \{ Q \} \). Notice that the \( \Delta \)-Lifting Theorem \( \Xi \) provides a sufficient condition to identify left loops as it generates the following corollary.

**Corollary 2 (from Theorem \( \Xi \)).** Let \( P \) be a logic program and \( \Delta \) be a DN filter for \( P \). If \( A \xrightarrow{P} B_1, B_1 \xrightarrow{P} B_2, B_2 \) and \( B_2 \) is \( \Delta \)-more general than \( B_1 \) then \( P \cup \{ A \} \) left loops.

**Proof.** The proof is similar to that of Corollary \( \mathbb{I} \).

This section presents some examples of atomic queries that have an infinite left derivation w.r.t. a given logic program. We use Corollary \( \Xi \) with DN sets of positions and DN sets of positions with associated terms. In each case, \( A \), it is not possible to conclude using the classical Lifting Theorem \( \mathbb{I} \). The last example exhibits a case where we are not able conclude.

**Example 14.** Let \( \Pi := \{ p \} \) with \( p \) a relation symbol whose arity equals 2 and let

\[
 c := p(f(x), y) \leftarrow p(x, g(y)).
\]

Then

\[
 \tau := \{ p \mapsto \{ \} \}
\]

is a DN set of positions for \( c \). The filter associated to \( \tau \) (see Definition \( \mathbb{I} \)) is

\[
 \Delta[\tau] := \{ p \mapsto \{ \} \mapsto f_{true} \}.
\]

Notice that, by Theorem \( \mathbb{I} \),

\[
 \Delta[\tau] \text{ is DN for } \{ c \}.
\]

Moreover, from the head \( p(f(x), y) \) of \( c \) we get

\[
 p(f(x), y) \xrightarrow{c} p(x', g(y')).
\]
Applying Corollary 2 as \( p(x', g(y')) \) is \( \Delta[\tau] \)-more general than \( p(f(x), y) \), we get that
\[
p(f(x), y) \text{ left loops w.r.t. } \{c\}.
\]

We point out that we do not get this result from the classical Lifting Theorem \( \tilde{1} \) as \( p(x', g(y')) \) is not more general than \( p(f(x), y) \).

By the \( \Delta \)-Lifting Theorem \( \tilde{2} \), we can also conclude that each query that is \( \Delta[\tau] \)-more general than \( p(f(x), y) \) also left loops w.r.t. \( \{c\} \). This means that each query of form \( p(t_1, t_2) \) where \( t_1 \) is a term that is more general than \( f(x) \) and \( t_2 \) is any term (because \( \tau(p) = \{2\} \)) left loops w.r.t. \( \{c\} \).

Example 15. Let \( \Pi := \{p\} \) with \( p \) a relation symbol whose arity equals 2 and let
\[
\begin{align*}
c := p(f(x), g(y)) &\leftarrow p(x, g(h(y))) \end{align*}
\]
Notice that from the head \( p(f(x), g(y)) \) of \( c \) we get
\[
p(f(x), g(y)) \Rightarrow p(x', g(h(y))) \Rightarrow c.
\]

The only DN set of positions for \( c \) is \( \tau_0 := \{p \mapsto 2\} \) because each argument of the head of \( c \) is not a variable (see Definition \( \tilde{1} \)). Hence, as \( p(x', g(h(y'))) \) is not \( \Delta[\tau_0] \)-more general than \( p(f(x), g(y)) \), we can not conclude using \( \tau_0 \) that \( p(f(x), g(y)) \) left loops w.r.t. \( \{c\} \). However,
\[
\tau^+ := \{p \mapsto \langle 2 \mapsto g(y) \rangle\}
\]
is a set of positions with associated terms that is DN for \( \{c\} \). Hence, by Theorem \( \tilde{2} \), the associated filter \( \Delta[\tau^+] \) (see Definition \( \tilde{2} \)) is DN for \( \{c\} \). As \( p(x', g(h(y'))) \) is \( \Delta[\tau^+] \)-more general than \( p(f(x), g(y)) \), by Corollary \( \tilde{2} \) we get that
\[
p(f(x), g(y)) \text{ left loops w.r.t. } \{c\}.
\]

By the \( \Delta \)-Lifting Theorem \( \tilde{2} \), we can also conclude that each query that is \( \Delta[\tau^+] \)-more general than \( p(f(x), g(y)) \) also left loops w.r.t. \( \{c\} \). This means that each query of form \( p(t_1, t_2) \) where \( t_1 \) is a term that is more general than \( f(x) \) and \( t_2 \) is any instance of \( g(y) \) (because \( \tau^+(p) = \{2 \mapsto g(y)\} \)) left loops w.r.t. \( \{c\} \).

Example 16. Let \( \Pi := \{p, q\} \) with \( p \) and \( q \) two relation symbols whose arity equals 2. Let
\[
\begin{align*}
c_1 := p(f(x), y) &\leftarrow q(x, g(x)), p(x, g(y)) \\
c_2 := q(a, g(x)) &\leftarrow q(a, g(b))
\end{align*}
\]

Then
\[
\tau^+ := \{p \mapsto \langle 2 \mapsto y \rangle, q \mapsto \langle 2 \mapsto g(x) \rangle\}
\]
is a set of positions with associated terms that is DN for \( \{c_1, c_2\} \). Hence, by Theorem \( \tilde{2} \), the associated filter \( \Delta[\tau^+] \) (see Definition \( \tilde{2} \)) is DN for \( \{c_1, c_2\} \). Moreover, from the head \( q(a, g(x)) \) of \( c_2 \) we get
\[
q(a, g(x)) \Rightarrow q(a, g(b)) \Rightarrow c_2.
\]
Applying Corollary 3 as \( q(a, g(b)) \) is \( \Delta[\tau^\perp] \)-more general than \( q(a, g(x)) \), we get that

\[
q(a, g(x)) \text{ left loops w.r.t. } c_2.
\]

By the \( \Delta \)-Lifting Theorem \( \overline{3} \) each query of form \( q(t_1, t_2) \) where \( t_1 \) is a term that is more general than \( a \) and \( t_2 \) is any instance of \( g(x) \) (because \( \tau^\perp(q) = \langle x \mapsto g(x) \rangle \) left loops w.r.t. \( c_2 \)). Notice that from the head \( p(f(x), y) \) of \( c_1 \) we get

\[
p(f(x), y) \Rightarrow q(x', g(x')), p(x', g(y')) \text{.}
\]

As \( q(x', g(x')) \) is such that \( x' \) is more general than \( a \) and \( g(x') \) is an instance of \( g(x) \), we get that \( q(x', g(x')) \) left loops w.r.t. \( c_2 \). Consequently, \( p(f(x), y) \) left loops w.r.t. \( \{c_1, c_2\} \). So, again by the \( \Delta \)-Lifting Theorem \( \overline{3} \) each query of form \( p(t_1, t_2) \), where \( t_1 \) is a term that is more general than \( f(x) \) and \( t_2 \) is any instance of \( y \) (because \( \tau^\perp(p) = \langle 2 \mapsto y \rangle \) left loops w.r.t. \( \{c_1, c_2\} \).

**Example 17.** Let \( \Pi := \{p\} \) with \( p \) a relation symbol whose arity equals 2 and let

\[
c := p(x, x) \leftarrow p(f(x), f(x)) \text{.}
\]

As the arguments of the head of \( c \) have one common variable \( x \), the only set of positions with associated terms that is DN for \( c \) in \( \tau_0^\perp \) such that the domain of \( \tau_0^\perp(p) \) is empty (see (DN1) in Definition \( \overline{23} \)). Notice that from the head \( p(x, x) \) of \( c \) we get

\[
p(x, x) \Rightarrow p(f(x_1), f(x_1)) \cdots \\
\cdots \Rightarrow p(f^n(x_n), f^n(x_n)) \Rightarrow p(f^{n+1}(x_{n+1}), f^{n+1}(x_{n+1})) \cdots
\]

As the arguments of \( p \) grow from step to step, there cannot be any query in the derivation that is \( \Delta[\tau_0^\perp] \)-more general than one of its ancestors. Consequently, we can not conclude that \( p(x, x) \) left loops w.r.t \( c \). □

## 6 Related Works

Some extensions of the Lifting Theorem with respect to infinite derivations are presented in \( \overline{15} \), where the authors study numerous properties of finite failure. The non-ground finite failure set of a logic program is defined as the set of possibly non-ground atoms which admit a fair finitely failed SLD-tree w.r.t. the program. This denotation is shown correct in the following sense. If two programs have the same non-ground finite failure set, then any ground or non-ground goal which finitely fails w.r.t. one program also finitely fails w.r.t. the other. Such a property is false when we consider the standard ground finite failure set. The proof of correctness of the non-ground finite failure semantics relies on the following result. First, a derivation is called non-perpetual if it is a fair infinite derivation and there exists a finite depth from which unfolding does not instantiate the original goal any more. Then the authors define the definite answer goal of a non-perpetual derivation as the maximal instantiation of the original goal. A crucial lemma states that any instance...
of the definite answer goal admits a similar non-perpetual derivation. Compared to our work, note that we do need fairness as an hypothesis for the $\Delta$-Lifting Theorem \[\text{(2)}\]. On the other hand, investigating the relationships between non-ground arguments of definite answer and neutral arguments is an interesting problem.

Loop checking in logic programming is also a subject related to our work. In this area, \[\text{(3)}\] sets up some solid foundations. A loop check is a device to prune derivations when it seems appropriate. A loop checker is defined as sound if no solution is lost. It is complete if all infinite derivations are pruned. A complete loop check may also prune finite derivations. The authors shows that even for function-free programs (also known as Datalog programs), sound and complete loop checks are out of reach. Completeness is shown only for some restricted classes of function-free programs.

We now review loop checking in more details. To our best knowledge, among all existing loop checking mechanisms only OS-check \[\text{(17,18)}\], EVA-check \[\text{(19)}\] and VAF-check \[\text{(20)}\] are suitable for logic programs with function symbols. They rely on a structural characteristic of infinite SLD-derivations, namely, the growth of the size of some generated subgoals. This is what the following theorem states.

**Theorem 6.** Consider an infinite SLD-derivation $\xi$ where the leftmost selection rule is used. Then there are infinitely many queries $Q_{i_1}, Q_{i_2}, \ldots$ (with $i_1 < i_2 < \ldots$) in $\xi$ such that for any $j \geq 1$, the selected atom $A_{i_j}$ of $Q_{i_j}$ is an ancestor of the selected atom $A_{i_{j+1}}$ of $Q_{i_{j+1}}$ and $\text{size}(A_{i_{j+1}}) \geq \text{size}(A_{i_j})$.

Here, \text{size} is a given function that maps an atom to its size which is defined in terms of the number of symbols appearing in the atom. As this theorem does not provide any sufficient condition to detect infinite SLD-derivations, the three loop checking mechanisms mentioned above may detect finite derivations as infinite. However, these mechanisms are complete w.r.t. the leftmost selection rule i.e. they detect all infinite loops when the leftmost selection rule is used.

OS-check (for OverSize loop check) was first introduced by Shalin \[\text{(17,18)}\] and was then formalized by Bol \[\text{(3)}\]. It is based on a function \text{size} that can have one of the three following definitions: for any atoms $A$ and $B$, either $\text{size}(A) = \text{size}(B)$, either $\text{size}(A)$ (resp. $\text{size}(B)$) is the count of symbols appearing in $A$ (resp. $B$), either $\text{size}(A) \leq \text{size}(B)$ if for each $i$, the count of symbols of the $i$-th argument of $A$ is smaller than or equal to that of the $i$-th argument of $B$. OS-check says that an SLD-derivation may be infinite if it generates an atomic subgoal $A$ that is oversized, i.e. that has ancestor subgoals which have the same predicate symbol as $A$ and whose size is smaller than or equal to that of $A$.

EVA-check (for Extented Variant Atoms loop check) was introduced by Shen \[\text{(19)}\]. It is based on the notion of generalized variants. EVA-check says that an SLD-derivation may be infinite if it generates an atomic subgoal $A$ that is a generalized variant of some of its ancestor $A'$, i.e. $A$ is a variant of $A'$ except for some arguments whose size increases from $A'$ to $A$ via a set of recursive clauses. Here the size function that is
used applies to predicate arguments, i.e., terms, and it is fixed: it is defined as the count of symbols that appear in the terms. EVA-check is more reliable than OS-check because it is less likely to mis-identify infinite loops. This is mainly due to the fact that, unlike OS-check, EVA-check refers to the informative internal structure of subgoals.

VAF-check (for Variant Atoms loop check for logic programs with Functions) was proposed by Shen et al. [20]. It is based on the notion of expanded variants. VAF-check says that an SLD-derivation may be infinite if it generates an atomic subgoal $A$ that is an expanded variant of some of its ancestor $A'$, i.e., $A$ is a variant of $A'$ except for some arguments $t_{i_1}, \ldots, t_{i_n}$ such that: $t_{i_1}$ grows from $A'$ to $A$ into a function containing $t_{i_1}, \ldots, t_{i_n}$ grows from $A'$ to $A$ into a function containing $t_{i_n}$. VAF-check is as reliable as and more efficient than EVA-check [20].

The main difference with our work is that we want to pinpoint some infinite derivations, based on syntactical properties of the program. We are not interested in completeness nor in soundness. Notice, however, that using the $\Delta$-Lifting Theorem as a loop checker leads to a device that is neither complete (see Example 17) nor sound since the Lifting Theorem is a particular case of the $\Delta$-Lifting Theorem.

7 Conclusion

We have presented a generalization of the lifting lemma for logic programming, which allows to disregard some arguments, termed neutral arguments, while checking for subsumption. We have investigated the model theoretic consequence of this generalization and have proposed two syntactic criteria for statically identifying neutral arguments.

A first application of this work has already been presented in [14], in the area of termination analysis of logic programs. We combine cTI [13], a termination inference tool with a non-termination inference analyzer whose correctness relies on our generalized lifting lemma. The resulting combined analysis may sometimes characterize the termination behavior of some concrete logic program w.r.t. to the left selection rule and the language we use to describe classes of queries.

Finally, this paper leaves numerous questions open. For instance, it might be interesting to try to generalize this approach to constraint logic programming [14]. Can we obtain higher level proofs compared to those we give? Can we propose more abstract or semantically-based criteria for identifying neutral arguments?

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8 Appendix 1

8.1 Proof of Proposition I

As By definition.

As Q is $\Delta$-more general than $Q'$, there exists a substitution $\sigma$ such that $Q$ is $\Delta$-more general than $Q'$ for $\sigma$. Notice that $Q$ is also $\Delta$-more general than $Q'$ for the substitution obtained by restricting the domain of $\sigma$ to the variables appearing the positions of $Q$ not distinguished by $\Delta$. More precisely, let

$T := \{ t \in T U_{\mathcal{L}} \mid \exists p(t_1, \ldots, t_n) \in Q, \exists i \in [1,n] \setminus Dom(\Delta(p)), t = t_i \}$

and

$\eta := \sigma | Var(T)$.

Then, $Dom(\eta) \subseteq Var(T)$ i.e.

$$Dom(\eta) \subseteq Var(Q)$$

(2)

and $Q$ is $\Delta$-more general than $Q'$ for $\eta$.

Now, let $x \in Dom(\eta)$. Then, as $Dom(\eta) \subseteq Var(T)$, there exists $A := p(t_1, \ldots, t_n) \in Q$ and $i \in [1,n] \setminus Dom(\Delta(p))$ such that $x \in Var(t_i)$.

As $Q$ is $\Delta$-more general than $Q'$ for $\eta$, there exists $A' := p(t'_1, \ldots, t'_n)$ in $Q'$ such that $A$ is $\Delta$-more general than $A'$ for $\eta$. But, as $i \in [1,n] \setminus Dom(\Delta(p))$, we have $t'_i = t_i \eta$. So, as $x \in Var(t_i)$, $xy$ is a subterm of $t'_i$. Consequently, $Var(x\eta) \subseteq Var(t'_i)$, so $Var(x\eta) \subseteq Var(Q')$.

So, we have proved that for each $x \in Dom(\eta)$, $Var(x\eta) \subseteq Var(Q')$, i.e. we have proved that

$$Ran(\eta) \subseteq Var(Q').$$

(3)

Finally, (2) and (3) imply that $Dom(\eta) \cup Ran(\eta) \subseteq Var(Q, Q')$ i.e. that

$$Var(\eta) \subseteq Var(Q, Q').$$

$\square$

8.2 Proof of Proposition II

Let $c := p(s_1, \ldots, s_n) \leftarrow B$ and $c' := p(s'_1, \ldots, s'_n) \leftarrow B'$ be a variant of $c$. Then, there exists a renaming $\gamma$ such that $c' = c\gamma$.

(DN1) Let $i \in Dom(\tau^+(p))$. Suppose that there exists $j \neq i$ such that $Var(s'_j) \cap Var(s'_j) \neq \emptyset$ and let us derive a contradiction.

Let $x' \in Var(s'_j) \cap Var(s'_j)$. As $s'_j = s_j \gamma$, there exists $x \in Var(s_j)$ such that $x' = x\gamma$.

For such an $x$, as $j \neq i$ and as $Var(s_j) \cap Var(s_j) = \emptyset$ (because $\tau^+$ is DN for $c$), we have $x \notin Var(s_i)$. So, as $\gamma$ is a 1-1 and onto mapping
Because if $x\gamma \notin \text{Var}(s_i\gamma)$, then either $x \in \text{Var}(s_i)$, or $x\gamma \notin \text{Var}(s_i)$ and $(x\gamma)\gamma = x\gamma$. The former case is impossible because we said that $x \notin \text{Var}(s_i)$. The latter case is impossible too because $(x\gamma)\gamma = x\gamma$ implies that $x\gamma \notin \text{Dom}(\gamma)$ i.e. $x \notin \text{Dom}(\gamma)$ (because $\gamma$ is a 1-1 and onto mapping from its domain to itself); so, $x = x\gamma$ i.e., as $x\gamma \in \text{Var}(s_i)$, $x \notin \text{Var}(s_i)$. 

Consequently, $\text{Var}(s'_i) \cap \text{Var}(t'_i) = \emptyset$.

\textbf{(DN2)} Let $(i \mapsto u_i) \in \tau^+(p)$. As $s_i$ is more general than $u_i$ (because $\tau^+$ is DN for $c$) and as $s'_i$ is a variant of $s_i$, $s'_i$ is more general than $u_i$.

\textbf{(DN3)} Let $i \in \text{Dom}(\tau^+(p))$. Suppose that there exists $q(t'_{1,\ldots,t'}_{m}) \in B'$ and $j \notin \text{Dom}(\tau^+(q))$ such that $\text{Var}(s'_i) \cap \text{Var}(t'_i) \neq \emptyset$. Let us derive a contradiction. Let $x' \notin \text{Var}(s'_i) \cap \text{Var}(t'_i)$. As $B' = B\gamma$, there exists $q(t_{1,\ldots,t_m}) \in B$ such that $q(t'_{1,\ldots,t'_{m}}) = q(t_{1,\ldots,t_m})\gamma$, i.e. $t'_j = t_j\gamma$. So, as $x' \notin \text{Var}(t'_i)$, there exists $x \in \text{Var}(t_i)$ such that $x' = x\gamma$. For such an $x$, as the elements of $\text{Var}(s_i)$ only occur in those $t_k$ such that $k \in \text{Dom}(\tau^+(q))$ (because $\tau^+$ is DN for $c$) and as $x \in \text{Var}(t_i)$ with $j \notin \text{Dom}(\tau^+(q))$, we have $x \notin \text{Var}(s_i)$. So, as $\gamma$ is a 1-1 and onto mapping from its domain to itself, we have $x\gamma \notin \text{Var}(s_i\gamma)$ (see footnote 3), i.e. $x' \neq \text{Var}(s'_i)$. Contradiction! Therefore, for each $q(t'_{1,\ldots,t'}_{m}) \in B'$ and $j \notin \text{Dom}(\tau^+(q))$, we have $\text{Var}(s'_i) \cap \text{Var}(t'_i) = \emptyset$.

\textbf{(DN4)} Let $q(t'_{1,\ldots,t'}_{m}) \in B'$ and $(j \mapsto u_j) \in \tau^+(q)$. As $t_j$ is an instance of $u_j$ (because $\tau^+$ is DN for $c$) and as $t'_j$ is a variant of $t_j$, $t'_j$ is an instance of $u_j$. Finally, we have established that $\tau^+$ is DN for $c'$. 

\section{Appendix 2: DN Sets of Positions with Associated Terms Generate DN Filters}

In this section, we give a proof of Theorem 4.

\subsection{Context}
All the results of this section are parametric to the following context:
- $\tau^+$ denotes a set of positions with associated terms that is DN for a program $P$.
- $Q \Rightarrow Q_1$ is an SLD-derivation step where
  - $c \in P$,
  - $Q := A, p(t_1,\ldots,t_n)$, $C$ where $p(t_1,\ldots,t_n)$ is the selected atom,
  - $c_1 := p(s_1,\ldots,s_n) \leftarrow B$ is the input clause used,
- $Q' := A', p(t'_{1,\ldots,t'_{m}}), C'$ is $\Delta[\tau^+]$-more general than $Q$ i.e., by Proposition 3 there exists a substitution $\eta$ such that $\text{Var}(\eta) \subseteq \text{Var}(Q,Q')$ and $Q'$ is $\Delta[\tau^+]$-more general than $Q$ for $\eta$. Moreover, the position of $p(t'_1,\ldots,t'_m)$ in $Q'$ is the same as that of $p(t_1,\ldots,t_n)$ in $Q$.

Because if $x\gamma \in \text{Var}(s_i\gamma)$, then either $x \in \text{Var}(s_i)$, either $x\gamma \notin \text{Var}(s_i)$ and $(x\gamma)\gamma = x\gamma$. The former case is impossible because we said that $x \notin \text{Var}(s_i)$. The latter case is impossible too because $(x\gamma)\gamma = x\gamma$ implies that $x\gamma \notin \text{Dom}(\gamma)$ i.e. $x \notin \text{Dom}(\gamma)$ (because $\gamma$ is a 1-1 and onto mapping from its domain to itself); so, $x = x\gamma$ i.e., as $x\gamma \in \text{Var}(s_i)$, $x \in \text{Var}(s_i)$. 

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9.2 Technical Definitions and Lemmas

**Definition 14 (Technical Definition).** Let $c'_1 := p(s'_1, \ldots, s'_n) \leftarrow B'$ be a clause such that
- $\text{Var}(c'_1) \cap \text{Var}(Q, Q') = \emptyset$ and
- $c_1 = c'_1 \gamma$ for some renaming $\gamma$ satisfying $\text{Var}(\gamma) \subseteq \text{Var}(c_1, c'_1)$.

As $c'_1$ is a variant of $c_1$ and $c_1$ is a variant of $c$, then $c'_1$ is a variant of $c$. Moreover, as $\tau^+$ is DN for $c$, by Proposition 11, $\tau^+$ is DN for $c'_1$.

So, by (DN2) in Definition 13, for each $(i \mapsto u_i) \in \tau^+(p)$ there exists a substitution $\delta_i$ such that $u_i = s'_i \delta_i$.

Moreover, as $p(t'_1, \ldots, t'_n)$ is $\Delta[\tau^+]$-more general than $p(t_1, \ldots, t_n)$, for each $(i \mapsto u_i) \in \tau^+(p)$, $t'_i$ is an instance of $u_i$. So, there exists a substitution $\delta'_i$ such that $t'_i = u_i \delta'_i$.

For each $i \in \text{Dom}(\tau^+(p))$, we set
- $\sigma_i \overset{\text{def}}{=} (\delta, \delta'_i)|\text{Var}(s'_i)$.

Moreover, we set:
- $\sigma \overset{\text{def}}{=} \bigcup_{i \in \text{Dom}(\tau^+(p))} \sigma_i$.

**Lemma 1.** The set $\sigma$ of Definition 14 is a well-defined substitution.

**Proof.** Notice that, as $\tau^+$ is DN for $c'_1$, by (DN1) in Definition 13 we have
- $\forall i \in \text{Dom}(\tau^+(p)), \forall j \in [1, n] \setminus \{i\}, \text{Var}(s'_i) \cap \text{Var}(s'_j) = \emptyset$.

Consequently,
- $\forall i, j \in \text{Dom}(\tau^+(p)), i \neq j \Rightarrow \text{Dom}(\sigma_i) \cap \text{Dom}(\sigma_j) = \emptyset$.

Moreover, for each $i \in \text{Dom}(\tau^+(p))$, $\sigma_i$ is a well-defined substitution. So, $\sigma$ is a well-defined substitution.

**Lemma 2 (Technical Lemma).** Let $c'_1 := p(s'_1, \ldots, s'_n) \leftarrow B'$ be a clause such that
- $\text{Var}(c'_1) \cap \text{Var}(Q, Q') = \emptyset$ and
- $c_1 = c'_1 \gamma$ for some renaming $\gamma$ satisfying $\text{Var}(\gamma) \subseteq \text{Var}(c_1, c'_1)$.

Let $\sigma$ be the substitution of Definition 14. Then, the substitution $\sigma \eta \gamma \theta$ is a unifier of $p(t'_1, \ldots, t'_n)$ and $p(s'_1, \ldots, s'_n)$.

**Proof.** The result follows from the following facts.
- For each $(i \mapsto u_i) \in \tau^+(p)$, we have:
  $s'_i \sigma = s'_i \sigma_i = s'_i \delta_i \delta'_i = (s'_i \delta_i) \delta'_i = u_i \delta'_i = t'_i$

  and $t'_i \sigma = t'_i$ because $\text{Dom}(\sigma) \subseteq \text{Var}(c'_1)$ and $\text{Var}(Q') \cap \text{Var}(c'_1) = \emptyset$. So, $s'_i \sigma = t'_i \sigma$ and $s'_i \sigma \eta \gamma \theta = t'_i \sigma \eta \gamma \theta$. 

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Moreover, as $c_i$ for $i \in [1, n] \setminus Dom(\tau^+(p))$, we have:

$$s'_i \eta \gamma \theta = (s'_i \eta) \gamma \theta = s'_i \gamma \theta = (s'_i \gamma) \theta = s_i \theta$$

and

$$t'_i \eta \gamma \theta = (t'_i \eta) \gamma \theta = t_i \gamma \theta = (t_i \gamma) \theta = t_i \theta$$

and $s_i \theta = t_i \theta$ because $\theta$ is a unifier of $p(s_1, \ldots, s_n)$ and $p(t_1, \ldots, t_n)$ (because $Q = \theta \cdot Q_1$ with $c_1$ as input clause used). So,

$$s'_i \eta \gamma \theta = t'_i \eta \gamma \theta \quad (4)$$

For each $i \in [1, n] \setminus Dom(\tau^+(p))$, we also have:

- $s'_i \gamma = s'_i$ because $\operatorname{Dom}(\sigma) \subseteq \operatorname{Var}\{s'_j \mid j \in Dom(\tau^+(p))\}$ and, by (DN1) in Definition 13, $\operatorname{Var}(s_i') = \emptyset$;
- $t'_i \gamma = t'_i$ because $\operatorname{Dom}(\sigma) \subseteq \operatorname{Var}(c'_1)$ and $\operatorname{Var}(Q') \cap \operatorname{Var}(c'_1) = \emptyset$.

Therefore, because of (4), $s'_i \sigma \eta \gamma \theta = t'_i \sigma \eta \gamma \theta$. □

### 9.3 $\Delta$-Propagation

Now we extend the following Propagation Lemma that is proved by Apt in [1] p. 54–56.

#### Lemma 3 (Propagation). Let $G$, $G_1$, $G'$ and $G'_1$ be queries such that $G \Rightarrow c_1$ and $G' \Rightarrow c'_1$ and:

- $G$ is an instance of $G'$
- in $G$ and $G'$ atoms in the same positions are selected.

Then, $G_1$ is an instance of $G'_1$.

First we establish the following result.

#### Lemma 4. Suppose that there exists an SLD-derivation step of form $Q' \Rightarrow Q'_1$ where $p(t'_1, \ldots, t'_n)$ is the selected atom and the input clause is $c'_1$ such that $\operatorname{Var}(Q) \cap \operatorname{Var}(c'_1) = \emptyset$. Then, $Q'_1$ is $\Delta|\tau^+|$-more general than $Q_1$.

**Proof.** Notice that we have

$$\operatorname{Var}(Q) \cap \operatorname{Var}(c_1) = \operatorname{Var}(Q, Q') \cap \operatorname{Var}(c'_1) = \emptyset.$$ 

Moreover, as $c_1$ is a variant of $c'_1$, there exists a renaming $\gamma$ such that

$$\operatorname{Var}(\gamma) \subseteq \operatorname{Var}(c_1, c'_1) \quad \text{and} \quad c_1 = c'_1 \gamma.$$ 

Let $c'_1 := p(s'_1, \ldots, s'_n) \leftarrow B'$. Then,

$$Q_1 = (A, B, C) \theta \quad \text{and} \quad Q'_1 = (A', B', C') \theta'.$$

$\tau^+$ is DN for $c$ and $c'_1$ is a variant of $c$. So, by Proposition 3, $\tau^+$ is DN for $c'_1$. Let $\sigma$ be the substitution of Definition 14.
Let \( A' \) := \( q(v'_1, \ldots, v'_m) \in A' \). As \( A' \) is \( \Delta[\tau^+] \)-more general than \( A \) for \( \eta, A' \) and \( A \) have the same length. Moreover, if \( k \) denotes the position of \( A' \) in \( A' \), then the \( k \)th atom of \( A \) has form \( q(v_1, \ldots, v_m) \).

- As \( A' \) is \( \Delta[\tau^+] \)-more general than \( A \) for \( \eta \), for each \( j \mapsto q(j) \in \tau^+(q) \), \( v'_j \) is an instance of \( u_j \).
- For each \( j \in [1, m] \setminus \text{Dom}(\tau^+(q)) \) we have:
  \[
  v'_j = \sigma = \eta = \eta
  \]
  because \( \text{Dom}(\sigma) = \text{Var}((s'_i \mid i \in \text{Dom}(\tau^+(p)))) \subseteq \text{Var}(c'_i) \) and \( \text{Var}(c'_i) \cap \text{Var}(Q') = \emptyset \). Moreover,
  \[
  v_j \eta = \eta \gamma \theta = \gamma \theta
  \]
  because \( A' \) is \( \Delta[\tau^+] \)-more general than \( A \) for \( \eta \). Finally, \( v_j \gamma \theta = v_j \theta \)
  because \( \text{Var}(\gamma) \subseteq \text{Var}(c_1, c'_1) \) and \( \text{Var}(c_1, c'_1) \cap \text{Var}(Q) = \emptyset \).

Consequently, we have proved that \( q(v'_1, \ldots, v'_m) \) is \( \Delta[\tau^+] \)-more general than \( q(v_1, \ldots, v_m) \) for \( \sigma \eta \gamma \theta \).

As \( q(v'_1, \ldots, v'_m) \) denotes any atom of \( A', \) we have proved that \( A' \) is \( \Delta[\tau^+] \)-more general than \( A \) for \( \sigma \eta \gamma \theta \). (5)

- Let \( B' := q(v'_1, \ldots, v'_m) \in B' \). As \( B = B' \gamma \), then \( B' \) and \( B \) have the same length. Moreover, if \( k \) denotes the position of \( A' \) in \( B' \), then the \( k \)th atom of \( B \) has form \( q(v_1, \ldots, v_m) \).
- For each \( j \mapsto q(j) \in \tau^+(q) \), \( v'_j \) is an instance of \( u_j \) (because \( \tau^+ \) is DN for \( c'_i \) and (DN4) in Definition [4].
- For each \( j \in [1, m] \setminus \text{Dom}(\tau^+(q)) \) we have:
  \[
  v'_j = \sigma = \eta = \eta
  \]
  because, by (DN3) in Definition [4],
  \[
  \text{Var}(v'_j) \cap \text{Var}(\{s'_i \mid i \in \text{Dom}(\tau^+(p))) = \emptyset
  \]
  with \( \text{Dom}(\sigma) = \text{Var}(\{s'_i \mid i \in \text{Dom}(\tau^+(p))) \). Moreover,
  \[
  v_j \eta \gamma \theta = \eta \gamma \theta
  \]
  because \( \text{Var}(\eta) \subseteq \text{Var}(Q, Q') \) and \( \text{Var}(c'_i) \cap \text{Var}(Q, Q') = \emptyset \). Finally, \( v_j \gamma \theta = v_j \theta \)
  because \( B = B' \gamma \).

Consequently, we have proved that \( q(v'_1, \ldots, v'_m) \) is \( \Delta[\tau^+] \)-more general than \( q(v_1, \ldots, v_m) \) for \( \sigma \eta \gamma \theta \).

As \( q(v'_1, \ldots, v'_m) \) denotes any atom of \( B' \), we have established that \( B' \) is \( \Delta[\tau^+] \)-more general than \( B \) for \( \sigma \eta \gamma \theta \). (6)
– Finally, by the same reasoning as for $A$ and $A'$ above, we show that

$$C' \triangleleft \Delta[\tau^+]\text{-more general than } C\theta \text{ for } \sigma\eta\gamma\theta. \quad (7)$$

So, we conclude from (3), (4) and (5) that $(A', B', C')$ is $\Delta[\tau^+]\text{-more general than } (A, B, C)\theta$ i.e. that

$$(A', B', C') \triangleleft \Delta[\tau^+]\text{-more general than } Q_1 \text{ for } \sigma\eta\gamma\theta. \quad (8)$$

But, by the Technical Lemma 2, $\sigma\eta\gamma\theta$ is a unifier of $p(s_1', \ldots, s_n')$ and $p(t_1', \ldots, t_n')$. As $\theta'$ is an mgu of $p(s_1', \ldots, s_n')$ and $p(t_1', \ldots, t_n')$ (because $Q' \xrightarrow{\theta'} c$), there exists $\delta$ such that $\sigma\eta\gamma\theta = \theta'\delta$.

Therefore, we conclude from (5) that $(A', B', C')$ is $\Delta[\tau^+]\text{-more general than } Q_1$ for $\theta'\delta$. But this result implies that $(A', B', C')\theta'$ is $\Delta[\tau^+]\text{-more general than } Q_1$ for $\delta$ i.e. that $Q_1'$ is $\Delta[\tau^+]\text{-more general than } Q_1$ for $\delta$.

Finally, we have proved that $Q_1'$ is $\Delta[\tau^+]\text{-more general than } Q_1$. □

Using the Propagation Lemma 3, the preceding result can be extended as follows.

**Proposition 6 ($\Delta$-Propagation).** If there exists an SLD-derivation step $Q' \xrightarrow{\theta'} c Q_1'$ where $p(t_1', \ldots, t_n')$ is the selected atom then $Q_1'$ is $\Delta[\tau^+]\text{-more general than } Q_1$.

**Proof.** Let $c_1'$ be the input clause used in $Q' \xrightarrow{\theta'} c Q_1'$. Take a variant $Q''$ of $Q$ such that

$$\text{Var}(Q'') \cap \text{Var}(c_1') = \emptyset$$

and a variant $c''_1$ of $c$ such that

$$\text{Var}(c''_1) \cap \text{Var}(Q'') = \emptyset.$$ 

Then, the SLD-resolvent $Q''_1$ of $Q''$ and $c$ exists with the input clause $c''_1$ and with the atom selected in the same position as in $Q$. So, for some $\theta''$, we have $Q'' \xrightarrow{\theta''} c Q''_1$ with input clause $c''_1$. Consequently, we have:

$$Q \xrightarrow{\theta} c Q_1 \quad \text{ and } \quad Q' \xrightarrow{\theta'} c Q'_1.$$ 

$Q$ and $Q''$ are instances of each other because $Q''$ is a variant of $Q$. So, by the Proposition Lemma 3 used twice, $Q''_1$ is an instance of $Q_1$ and $Q_1'$ is an instance of $Q''_1$. So,

$$Q'_1 \text{ is a variant of } Q_1. \quad (9)$$

But we also have

$$Q'' \xrightarrow{\theta''} c Q''_1 \quad \text{ and } \quad Q' \xrightarrow{\theta'} c Q'_1$$

with input clauses $c''_1$ and $c'_1$, with $Q'$ that is $\Delta[\tau^+]\text{-more general than } Q''$ (because $Q''$ is a variant of $Q$ and $Q'$ is $\Delta[\tau^+]\text{-more general than } Q$) and $\text{Var}(Q'') \cap \text{Var}(c'_1) = \emptyset$. So, by Lemma 3,

$$Q'_1 \text{ is } \Delta[\tau^+]\text{-more general than } Q''_1. \quad (10)$$

Finally, from (3) and (10) we have: $Q'_1 \text{ is } \Delta[\tau^+]\text{-more general than } Q_1$. □

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9.4 Epilogue

Theorem 5 is a direct consequence of the following result.

**Proposition 7 (One Step $\Delta$-Lifting).** Let $c'$ be a variant of $c$ variable disjoint with $Q'$. Then, for some $\theta'$ and $Q'_1$,
- $Q' \overset{\theta'}{\Rightarrow} Q'_1$ where $c'$ is the input clause used,
- $Q' \overset{\theta'}{\Rightarrow} Q'_1$ is a $\Delta[\tau^+]$-lift of $Q \overset{\theta}{\Rightarrow} Q_1$.

**Proof.** Let $c'_1 := p(s'_1, \ldots, s'_n) \leftarrow B'$ be a variant of $c_1$. Then there exists a renaming $\gamma$ such that $\text{Var}(\gamma) \subseteq \text{Var}(c_1, c'_1)$ and $c_1 = c'_1 \gamma$. Suppose also that $\text{Var}(c'_1) \cap \text{Var}(Q, Q') = \emptyset$.

By the Technical Lemma, $p(s'_1, \ldots, s'_n)$ and $p(t'_1, \ldots, t'_n)$ unify. Moreover, as $\text{Var}(c'_1) \cap \text{Var}(Q') = \emptyset$, $p(s'_1, \ldots, s'_n)$ and $p(t'_1, \ldots, t'_n)$ are variable disjoint. Notice that the following claim holds.

**Claim.** Suppose that the atoms $A$ and $H$ are variable disjoint and unify. Then, $A$ also unifies with any variant $H'$ of $H$ variable disjoint with $A$.

**Proof.** For some $\gamma$ such that $\text{Dom}(\gamma) \subseteq \text{Var}(H')$, we have $H = H' \gamma$. Let $\theta$ be a unifier of $A$ and $H$. Then, $A \gamma \theta = A \theta = H \theta = H' \gamma \theta$, so $A$ and $H'$ unify.

Consequently, as $c'$ is a variant of $c'_1$ and $c'$ is variable disjoint with $Q'$, $p(t'_1, \ldots, t'_n)$ and the head of $c'$ unify. As they also are variable disjoint, we have

$$Q' \overset{\theta'}{\Rightarrow} Q'_1$$

for some $\theta'$ and $Q'_1$ where $p(t'_1, \ldots, t'_n)$ is the selected atom and $c'$ is the input clause used. Moreover, by the $\Delta$-Propagation Proposition, $Q'_1$ is $\Delta[\tau^+]$-more general than $Q_1$. \qed