1 Introduction

One of the central themes of Riemannian geometry is the study of how local properties (curvature) of a Riemannian manifold affect its global (topological or metric) properties. The most famous example of this is the classical Gauss-Bonnet Theorem. Many results relating local and global properties are based on the injectivity radius estimates. One example of this sort is Klingenberg’s injectivity radius estimate for the quarter-pinched \(^1\) (compact simply connected Riemannian) manifolds as the main part of the proof of the Sphere Theorem ([9], also chapter 13 of [3]).

Looking at the Klingenberg’s injectivity radius estimate for the quarter pinched manifolds, one would like to get an injectivity radius estimate for \(\delta\)-pinched compact simply connected Riemannian manifolds with any \(\delta \in (0, 1]\).

Actually, the problem only exists for odd dimensional manifolds since, two years before proving the injectivity radius estimate for the quarter pinched manifolds, Klingenberg showed, in [8], that, for any compact simply connected even dimensional manifold \(M\) with positive sectional curvature \(K_M\), the injectivity radius \(i(M)\) satisfies

\[
i(M) \geq \frac{\pi}{\sqrt{\max K_M}}.
\]

The first instinct is to try to get an estimate depending only on \(\delta\) and the dimension. This turns out to be impossible. In fact, Aloff-Wallach spaces provide a counterexample to such an estimate.

In [10], Klingenberg and Sakai conjectured that, if one fixed a compact simply connected differential manifold \(M\) and then considered all possible \(\delta\)-pinched Riemannian structures on \(M\), then one should be able to find a uniform lower bound for the injectivity radii of the obtained Riemannian manifolds. In the positively pinched case, finding a lower bound on the injectivity radius is the same as finding a lower bound on the volume. Therefore, the conjecture can be reformulated as: “A sequence of \(\delta\)-pinched Riemannian structures on a given compact simply connected differential manifold can not collapse,” where

\(^1\)A manifold is called \(\delta\)-pinched if its sectional curvatures lie between two positive constants whose ratio is bounded by \(\delta\).
“collapse” means “volume goes to zero.” In this form, the problem asks for application of methods of Gromov-Hausdorff convergence. This approach (and in particular usage of the N-structures introduced in [5]) brought significant success in proving the conjecture under different special assumptions. In particular, in [11], it is proven that Klingenberg-Sakai conjecture holds if, instead of considering all possible metrics, one considers only metrics with bounded distance function, and [6] contains a proof of the conjecture for the manifolds satisfying special topological condition, namely that the second Betti number is zero. As far as we know, the conjecture in its general form is still open.

In this paper, we are going to focus on a particular example of δ-pinched manifolds, which may be interesting in its own right. The topic of our study - Aloff-Wallach spaces, which were first introduced in [1], are the quotients of SU(3) by various images of S^1. In [7], Huang showed that there is an infinite family of uniformly pinched simply connected topologically distinct Aloff-Wallach spaces and then used Cheeger’s Finiteness Theorem [4] to conclude that this family does not have a common lower injectivity bound.

The main results of this paper are two-sided volume estimates for all Aloff-Wallach spaces [Theorem 2.1] and sharp (sectional) curvature estimates for the Aloff-Wallach spaces from the family mentioned in the last paragraph [Theorem 2.2]. The estimation of the volumes uses generalized Euler angles on SU(3), and the sectional curvature bounds are obtained using modified curvature operators and the computational procedures given by Püttmann in [12]. As an application of these results, we obtain injectivity radii estimates [Corollary 2.3], which, in particular, give a different proof of the Huang’s result.

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2 Description of the Spaces and Statement of the Results

For each pair of integers p and q, we define the subgroup T(p, q) of SU(3) by

\[ T(p, q) = \left\{ \begin{pmatrix} e^{2\pi i p \theta} & 0 & 0 \\ 0 & e^{2\pi i q \theta} & 0 \\ 0 & 0 & e^{-2\pi i (p+q) \theta} \end{pmatrix} \mid \theta \in \mathbb{R} \right\}. \]  

If at least one of the numbers p and q is not zero, the subgroup T(p, q) is nontrivial, and the factor space

\[ W(p, q) = SU(3)/T(p, q) \]

is called an Aloff-Wallach space.

It is shown in [1] that, if neither of p, q, and p + q is zero, W(p, q) can be equipped with a positively curved metric. The positively curved metric on this
space is obtained by deforming the metric induced by the Killing form. The standard Killing metric $k$ on $SU(3)$ is given by the formula

$$k(X, Y) = \frac{1}{2} \text{Tr}(XY^*)$$

for $X, Y \in T_I(SU(3))$ and then extended by left invariance. This metric induces a $SU(3)$-invariant metric on $W(p, q)$ in the following way. We decompose $T_I(SU(3)) = \mathfrak{su}(3)$ as

$$\mathfrak{su}(3) = \mathfrak{T} \oplus \mathfrak{T}^\perp,$$

where $\mathfrak{T}$ is the Lie algebra of $T(p, q)$ and $\mathfrak{T}^\perp$ is its orthogonal complement in $\mathfrak{su}(3)$ with respect to $k$. Let

$$\pi : SU(3) \to W(p, q)(= SU(3)/T(p, q))$$

be the canonical projection. The differential of the canonical projection at the identity, $d\pi_I$, gives an isomorphism of $\mathfrak{T}^\perp = \mathfrak{su}(3)/\mathfrak{T}$ and $T_{T(p, q)}(W(p, q))$. Therefore, we shall have a scalar product on $T_{T(p, q)}(W(p, q))$ once we have a scalar product on $\mathfrak{T}^\perp$. In order to be able to extend this product by left invariance and obtain a $SU(3)$-invariant metric on $W(p, q)$, the scalar product must be $Ad_{T(p, q)}$-invariant. One obvious way to get such a scalar product on $\mathfrak{T}^\perp$ is to restrict the scalar product given by $k$ from $\mathfrak{su}(3)$ to $\mathfrak{T}^\perp \subset \mathfrak{su}(3)$. More generally, supposing that there is an orthogonal, with respect to $k$, $Ad_{T(p, q)}$-invariant decomposition $\mathfrak{T}^\perp = V_1 \oplus V_2$, we can deform $k$ to obtain a new $Ad_{T(p, q)}$-invariant scalar product $\tilde{k}$ on $\mathfrak{T}^\perp$ as follows

$$\tilde{k}(X, Y) = a_1 k(X_1, Y_1) + a_2 k(X_2, Y_2),$$

where $a_1$ and $a_2$ are positive constants, $X_1$ and $X_2$ are the projections of $X$ on $V_1$ and $V_2$, and analogously for $Y$.

The construction of the aforementioned metric on $W(p, q)$ is based on a particular choice of subspaces $V_1$ and $V_2$ and constants $a_1$ and $a_2$. First, we choose $V_1$ and $V_2$. The choice is made in the following way to ensure that the decomposition satisfies certain conditions, called “condition II” in $\Pi$, which guarantee that $\tilde{k}$ (with appropriately chosen constants $a_1$ and $a_2$) will induce an $SU(3)$-invariant positively curved metric on $W(p, q)$. More precisely, it is shown in $\Pi$ that, if the $Ad_{T(p, q)}$-invariant orthogonal decomposition $\mathfrak{T}^\perp = V_1 \oplus V_2$ satisfy:

1. $[V_1, V_2] \subset V_2$,
2. $[V_1, V_1] \subset \mathfrak{T} \oplus V_1$,
3. $[V_2, V_2] \subset \mathfrak{T} \oplus V_1$,
4. for any pair of linearly independent vectors $x = x_1 + x_2$ and $y = y_1 + y_2$, with $x_i, y_i \in V_i$, $[x, y] = 0$ implies $[x_1, y_1] \neq 0$,
then the metric $\tilde{k}(X,Y) = a_1 k(X_1,Y_1) + a_2 k(X_2,Y_2)$ as above has positive curvature for $a_2 = 1$ and any $a_1 \in (0,1)$. We shall refer to the list above as condition II.

In order to choose $V_1$ and $V_2$, we start with the subgroup $U$ of $SU(3)$ given by

$$U = \left\{ \begin{pmatrix} g & 0 \\ 0 & (\det g)^{-1} \end{pmatrix} \middle| g \in U(2) \right\}.$$  

Note that this subgroup contains $T(p,q)$. The Lie algebra $u$ of $U$ is given by

$$u = \left\{ \begin{pmatrix} u & 0 \\ 0 & -\text{Tr} u \end{pmatrix} \middle| u \in u(2) \right\}.$$  

Let us point out that $\mathfrak{T} \subset u$, which follows from $T(p,q) \subset U$, or could be seen directly from the fact that $T = \{ \begin{pmatrix} 2\pi ip\theta & 0 \\ 0 & 2\pi iq\theta \\ 0 & 0 \\ 0 & 0 & 2\pi i(p+q)\theta \end{pmatrix} \middle| \theta \in \mathbb{R} \}$.  

We form the decomposition of $\mathfrak{T}^\perp$ by taking

$$V_1 = \mathfrak{T}^\perp \cap u,$$
$$V_2 = u^\perp,$$  \hspace{1cm} (2)

where $u^\perp$ is the orthogonal complement of $u$ with respect to the Killing form $k$. The fact that $\mathfrak{T}^\perp = V_1 \oplus V_2$ follows from the fact that $\mathfrak{T} \subset u$. A series of matrix computations shows that $V_1$ and $V_2$ given by (2) are $Ad_T(p,q)$-invariant, and, if $pq > 0$, the decomposition $\mathfrak{T}^\perp = V_1 \oplus V_2$ (with $V_1$ and $V_2$ given by (2)) satisfy condition II.

We complete the construction of our particular version of the positively curved metric on $W(p,q)$ by picking $a_1 = 1/2$ and $a_2 = 1$, which makes

$$\tilde{k}(X,Y) = \frac{1}{2} k(X_1,Y_1) + k(X_2,Y_2).$$

Since, for $pq > 0$, the decomposition $\mathfrak{T}^\perp = V_1 \oplus V_2$, with $V_1$ and $V_2$ given by (2), satisfies condition II, Theorem 2.4 of [1] says that $SU(3)$-invariant metric induced on $W(p,q)$ [with $pq > 0$] by $\tilde{k}$ is positively curved; using this, Theorem 3.2 of [1] shows that the result holds as long as neither of $p$, $q$, or $p+q$ is zero.

In [7], Huang proved that the curvature of $W(p,q)$ (with this metric) depends only on the ratio $p/q$ and established that the curvature of $W(1,1)$ is pinched between $2/37$ and $29/8$. Using this, he showed that the Aloff-Wallach spaces $W(i,i+1)$, with $i$ sufficiently big, are uniformly pinched, simply connected, and topologically distinct, and, therefore, do not have a common lower injectivity radius.

Let us now formulate our results precisely.
Theorem 2.1 (Volumes). If $\text{Vol}(W(p,q))$ denotes the volume of the Aloff-Wallach space $W(p,q)$ with respect to the metric that we have just chosen, then

$$\frac{\sqrt{3}\pi^4 \gcd(p,q)}{32\sqrt{p^2 + q^2 + pq}} \leq \text{Vol}(W(p,q)) \leq \frac{\sqrt{3}\pi^4 \gcd(p,q)}{2\sqrt{p^2 + q^2 + pq}}.$$

Theorem 2.2 (Curvatures). For any positive integer $n$, the sectional curvature of the Aloff-Wallach space $W(n,n+1)$, satisfies the sharp inequality

$$c(n) \leq K(W(n,n+1)) \leq C(n),$$

where

$$c(n) = \frac{17 + 63n + 63n^2}{16 + 48n + 48n^2}$$

$$- \frac{1}{16} \left[ \left( \frac{7 + 33n + 33n^2}{(1 + 3n + 3n^2)^2} + 4 \left( \frac{9(1 + 2n)}{\sqrt{3 + 9n + 9n^2}} \right) \right) \right.$$

$$+ \left\{ \left( 32 + 552n + 3132n^2 + 8037n^3 + 9648n^4 + 4401n^5 \right) \sqrt{3 + 9n + 9n^2} \right.$$

$$- \sqrt{3n(16 + 60n + 57n^2)(-56 - 555n - 1935n^2 - 1620n^3 + 7173n^4}$$

$$+ 22788n^5 + 26649n^6 + 11907n^7)^{1/2} \right\} \left[ \{ 64 + 672n + 2916n^2 + 6624n^3 + 8181n^4 + 4995n^5 + 999n^6 \} \right]^{1/2}$$

and

$$C(n) = 4 - \frac{9n^2}{8(1 + 3n + 3n^2)},$$

which implies non-sharp inequality

$$\frac{1}{25} \leq K(W(n,n+1)) \leq 4.$$

For any Riemannian manifold $M$, let $i(M)$ denote its injectivity radius.

Corollary 2.3 (Injectivity Radii). The injectivity radii of the various Aloff-Wallach spaces satisfy the following inequalities:

1. $i(W(1,1)) \geq 4.65 \cdot 10^{-5};$

2. $i(W(n,n+1)) \geq \frac{3\sqrt{3}\pi(c(n))^3}{32\sqrt{3n^2 + 3n + 1}}$

where $c(n)$ is the functions from Theorem 2.2.

3. $i(W(p,q)) \leq \pi \left[ \frac{3\sqrt{3} \gcd(p,q)}{2\sqrt{p^2 + q^2 + pq}} \right]^{1/7}.$

Theorem 2.1 is established in section 3, Theorem 2.2 in section 4, and Corollary 2.3 in section 5.
3 Volume of $W(p, q)$

3.1 Preliminary considerations

In order to estimate the volume of $W(p, q)$ and prove Theorem 2.1, we are going to use the following result: If $\pi: (G, g) \to (M, f)$ is a Riemannian submersion, then

$$\text{Vol}(G, g) = \int_M \text{Vol} \left( \pi^{-1}(x) \right) \sqrt{\det f(x)} \, dx,$$

which is given as Corollary II.5.7 in [13]. If in addition $G$ is a Lie group and $M$ is its homogeneous space, say $M = G/H$, then points of $M$ are left cosets: for any $x \in M$ there exists $g \in G$ such that $x = [gH]$, and $\pi^{-1}(x) = \pi^{-1}([gH]) = gH$. If further the metric on $G$ is left-invariant, all $gH$ mentioned in the previous sentence are isometric, and, in particular, their volumes are equal. Thus, if $\pi: G \to G/H$ is the canonical projection,

$$\text{Vol}(G, g) = \int_{G/H} \text{Vol}(H) \sqrt{\det f(x)} \, dx = \text{Vol}(H) \cdot \text{Vol}(G/H, f). \quad (3)$$

In order to apply this formula to $(G, H) = (SU(3), T(p, q))$, we need to pick a metric on $SU(3)$ such that the canonical projection $\pi: SU(3) \to W(p, q)$ is a Riemannian submersion. Such a metric is induced by the scalar product

$$w: \mathfrak{su}(3) \times \mathfrak{su}(3) \to \mathbb{R}$$

defined by

$$w(X, Y) = k(X_\bar{\tau}, Y_\bar{\tau}) + \frac{1}{2} k(X_1, Y_1) + k(X_2, Y_2),$$

where $X_\bar{\tau} \in \mathfrak{t}$, $X_1 \in V_1$, $X_2 \in V_2$, and analogously for $Y$. The actual metric, which we are also going to call $w$, is given by extending this scalar product by left invariance.

The fact that $\pi: (SU(3), w) \to (W(p, q), \bar{k})$ is a Riemannian submersion at $I \in SU(3)$ follows from the definitions of $w$ and $\bar{k}$: With the identification of $T_{T(p, q)}(W(p, q))$ and $\mathfrak{t}_\perp$ that was made in order to construct the metric on $W(p, q)$, $d\pi_I: (\mathfrak{su}(3), w) \to (T_{T(p, q)}(W(p, q)), \bar{k})$ is an orthogonal projection. Since the metrics on $SU(3)$ and $W(p, q)$ are left-invariant, this implies that $\pi$ is a Riemannian submersion everywhere.

The application of (3) to $\pi: (SU(3), w) \to (W(p, q), \bar{k})$ yields

$$\text{Vol}(W(p, q), \bar{k}) = \text{Vol}(SU(3), w) / \text{Vol}(T(p, q)). \quad (4)$$

Now, our goal is to estimate the volume of $SU(3)$ (in metric $w$) and compute the length of $T(p, q)$. In view of (4), the tangent vector $t = t(\theta)$ of $T(p, q)$ is given by

$$t = \begin{pmatrix} 2\pi ip e^{2\pi ip\theta} & 0 & 0 \\ 0 & 2\pi iq e^{2\pi iq\theta} & 0 \\ 0 & 0 & -2\pi i(p + q) e^{-2\pi i(p+q)\theta} \end{pmatrix}. $$
Therefore,
\[ \text{Vol}(T(p, q)) = \int_0^{1/\gcd(p, q)} \sqrt{k(t, t)} \, dt = \frac{2\pi}{\gcd(p, q)} \sqrt{p^2 + q^2 + pq}. \]  

3.2 Volume of $SU(3)$

3.2.1 Euler angle parametrization

In order to compute the volume of $SU(3)$, we are going to introduce the generalized Euler angles. Before we start describing the parametrization of $SU(3)$, let us recall the Euler angles on $SU(2)$. In the case of $SU(2)$, one uses the Pauli matrices $\sigma_i$ given by

\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]

to write a generic $s \in SU(2)$ as

\[ s = s(\phi, \theta, \psi) = e^{i\phi \sigma_3} e^{i\theta \sigma_2} e^{i\psi \sigma_3}, \]

thus parameterizing $SU(2)$ (outside a set of measure zero) by $\phi$, $\theta$, and $\psi$. Let us recall how the coordinate ranges of this parametrization are found.

It follows directly from the definition of $SU(2)$ that

\[ SU(2) = \left\{ \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix} : \begin{array}{l} \{a, b \in \mathbb{C} & |a|^2 + |b|^2 = 1 \} \end{array} \right\}. \]

On the other hand, since

\[ e^{i\theta \sigma_2} = \begin{pmatrix} \cos \frac{\theta}{2} & \sin \frac{\theta}{2} \\ -\sin \frac{\theta}{2} & \cos \frac{\theta}{2} \end{pmatrix}, \]

\[ s(\phi, \theta, \psi) = e^{i\phi \sigma_3} e^{i\theta \sigma_2} e^{i\psi \sigma_3} \]

\[ = \begin{pmatrix} e^{i\phi} \cos \frac{\theta}{2} e^{i\psi} \sin \frac{\theta}{2} & e^{i\phi} \sin \frac{\theta}{2} e^{i\psi} \cos \frac{\theta}{2} \\ -e^{-i\phi} \cos \frac{\theta}{2} \sin \frac{\theta}{2} \end{pmatrix}. \]

Therefore, in order to find the ranges of the Euler angles, we need to find three intervals $I_\phi$, $I_\theta$, and $I_\psi$ so that, outside a set of measure 0, there is a diffeomorphism

\[ f : \{(a, b) \in \mathbb{C}^2 : |a|^2 + |b|^2 = 1 \} \rightarrow I_\phi \times I_\theta \times I_\psi, \]

with the property that, if $f(a, b) = (\phi, \theta, \psi)$, then

\[ a = e^{i\phi} \cos \frac{\theta}{2}, \]

\[ b = e^{i\phi} \sin \frac{\theta}{2}. \]
Writing \((a, b)\) in the form \((|a|e^{i\alpha}, |b|e^{i\beta})\), with \(\alpha, \beta \in [0, 2\pi)\), we define \(f : (a, b) \mapsto (f_\phi(a, b), f_\theta(a, b), f_\psi(a, b))\) by \(f_\theta(a, b) = 2 \arccos(|a|)\) and

\[
(f_\phi(a, b), f_\psi(a, b)) = \begin{cases} 
(\alpha + \beta, \alpha - \beta) & \text{if } \alpha \geq \beta, \\
(\alpha + \beta - 2\pi, \alpha - \beta + 2\pi) & \text{if } \alpha < \beta \text{ and } \alpha + \beta \geq 2\pi, \\
(\alpha + \beta + 2\pi, \alpha - \beta + 2\pi) & \text{if } \alpha < \beta \text{ and } \alpha + \beta < 2\pi,
\end{cases}
\]

which gives

\[I_\phi \times I_\theta \times I_\psi = [0, 4\pi) \times [0, \pi) \times [0, 2\pi).\]

In the case of \(SU(3)\), Gell-Mann matrices \(\lambda_i\) are used in place of Pauli matrices. The Gell-Mann matrices that are used in the parametrization are

\[
\lambda_2 = \begin{pmatrix} 0 & -i & 0 \\
i & 0 & 0 \\
0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 0 \end{pmatrix},
\]

\[
\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\
0 & 0 & 0 \\
0 & 0 & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -2 \end{pmatrix}.
\]

We claim that any \(g \in SU(3)\) can be written as

\[g = g(\phi, \theta, \psi, \xi, \alpha, \beta, \gamma, \tau) = s(\phi, \theta, \psi)e^{i\lambda_5 \xi}s(\alpha, \beta, \gamma)e^{i\lambda_8 \tau},\]

where

\[s(x, y, z) = e^{\frac{x}{2} \lambda_3}e^{\frac{y}{2} \lambda_2}e^{\frac{z}{2} \lambda_3}\]

is the Euler-angle parametrization of \(SU(2) \subset SU(3)\). The coordinates \(\phi, \theta, \psi, \xi, \alpha, \beta, \gamma\) and, \(\tau\) as above are called generalized Euler angles.

Direct computation, which can be done painlessly with the help of Mathematica, shows that, for any choice of parameters,

\[g(\phi, \theta, \psi, \xi, \alpha, \beta, \gamma, \tau) \in SU(3).\]

To find the ranges of the coordinates we look at the matrix elements of

\[g = s(\phi, \theta, \psi)e^{i\lambda_5 \xi}s(\alpha, \beta, \gamma)e^{i\lambda_8 \tau},\]

and, by considerations similar to the ones used in the \(SU(2)\) case, we establish that

\[\beta, \theta, \xi \in [0, \pi); \quad \alpha, \phi \in [0, 4\pi); \quad \gamma, \psi \in [0, 2\pi).\]

### 3.2.2 Estimation

In computation of the volume, we are going to use the coordinate vector fields, which are obtained by differentiating \(g = g(\phi, \theta, \psi, \xi, \alpha, \beta, \gamma, \tau)\), for example \(\partial_\phi = \frac{\partial}{\partial \phi}\). Using these as a basis, we, theoretically, could compute the determinant of the metric \(w\) and then integrate the square root of this determinant to
get the volume. In reality, however, this computation is too complicated even for Mathematica to handle. Therefore, we shall settle for the estimate of the volume near the volume in the Killing metric, whose volume element turns out to be given by a nice formula. Recall that we decomposed the Lie algebra of $SU(3)$ as

$$su(3) = T \oplus V_1 \oplus V_2,$$

and defined the Wallach scalar product $w$ on $su(3)$ by

$$w(X, Y) = k(X_T, Y_T) + \frac{1}{2}k(X_1, Y_1) + k(X_2, Y_2),$$

where $k$ is the killing form on $SU(3)$ and the subscripts denote the projections on the corresponding subspaces. Note that $w$ can be easily estimated in terms of $k$:

$$\frac{1}{2}k(X, X) \leq w(X, X) \leq k(X, X),$$

and, hence,

$$\frac{1}{16}Vol(SU(3), k) \leq Vol(SU(3), w) \leq Vol(SU(3), k). \quad (6)$$

Therefore, once we know the volume of $SU(3)$ in the Killing metric, we’ll have a two-sided estimate on the volume in the Wallach metric, which is our main goal.

Using the strategy described in the beginning of the last paragraph, we compute, with the help of Mathematica, that the volume element of the Killing metric, $k$, at a generic point is given by the formula

$$dV = \frac{\sqrt{3}}{512} \sin \beta \sin \theta \sin \xi \sin^2 \frac{\xi}{2}.$$

Integrating this formula over the ranges of the generalized Euler angles, we get

$$Vol(SU(3), k) = \sqrt{3}\pi^5.$$

Inserting this into (6), we get the following two-sided estimate for the volume of $SU(3)$ with the Aloff-Wallach metric:

$$\frac{1}{16}\pi^5 \leq Vol(SU(3), w) \leq \sqrt{3}\pi^5.$$

Combining this with (4) and (5), we get the promised two-sided estimate for the volume of the Aloff-Wallach spaces:

$$\frac{\sqrt{3}\pi^4 \gcd(p, q)}{32\sqrt{p^2 + q^2 + pq}} \leq Vol(W(p, q)) \leq \frac{\sqrt{3}\pi^4 \gcd(p, q)}{2\sqrt{p^2 + q^2 + pq}}, \quad (7)$$

completing the proof of Theorem 2.1.
4 The Pinching of $W(n,n+1)$

4.1 General Remarks about Curvatures

Our estimation of the curvature (and computation of the pinching) is based on the procedure given in [12]. The central tool of the procedure is modified curvature operators. Let us describe the relevance of the modified curvature operators to the estimation of the sectional curvature.

For any Riemannian manifold $(M, g)$ and the corresponding Levi-Civita connection $\nabla$, one defines the Riemann curvature tensor $Rm : \Gamma(TM) \times \Gamma(TM) \times \Gamma(TM) \times \Gamma(TM) \to (R)$ by the formula

$$Rm(X, Y, Z, W) = g(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, W).$$

It can be shown that $Rm$ can be used to define a symmetric bilinear form on the bundle of bivectors, $\widehat{Rm} : \Lambda_2 TM \otimes \Lambda_2 TM \to \mathbb{R}$, by

$$\widehat{Rm}(X \wedge Y, W \wedge Z) = Rm(X, Y, Z, W).$$

The self-adjoint linear operator associated to this symmetric bilinear form is called the curvature operator and will be denoted by $\mathcal{R}$. In other words, $\mathcal{R}$ is defined by the equality

$$\hat{g}(\mathcal{R}(X \wedge Y), W \wedge Z) = \widehat{Rm}(X \wedge Y, W \wedge Z),$$

where $\hat{g}$ is the metric induced by $g$ on $\Lambda_2 TM$.

The sectional curvature $K$ of a given Riemannian manifold $(M, g)$ is a function that associates to any pair of linearly independent vectors $\{X, Y\} \subset T_p M$ (for some $p \in M$) a number

$$K(X, Y) = \frac{\widehat{Rm}(X \wedge Y, X \wedge Y)}{\hat{g}(X \wedge Y, X \wedge Y)}.$$

The value of the sectional curvature depends only on the 2-plane spanned by $X$ and $Y$ (which makes it possible to write $K(X \wedge Y)$ in place of $K(X, Y)$). Therefore, in order to estimate the sectional curvature, it is enough to look at its values on the orthonormal pairs of vectors. For an orthonormal pair of vectors $\{X, Y\}$,

$$K(X, Y) = \widehat{Rm}(X \wedge Y, X \wedge Y) = \hat{g}(\mathcal{R}(X \wedge Y), X \wedge Y),$$

which lies between the smallest and the biggest eigenvalues of $\mathcal{R}$. Thus, one way to estimate the sectional curvature is to compute the eigenvalues of the curvature operator. Unfortunately, this estimate is not optimal because an eigenvector of $\mathcal{R}$ might happen to be a bivector that cannot be written as a wedge of two tangent vectors. In particular, the smallest eigenvalue of the curvature operator on $W(1, 1)$ is negative.
4.2 Modified Curvature Operators

The shortcomings of the curvature operator method of estimating the sectional curvature described at the end of the last subsection can be overcome if one considers modified curvature operators in place of the curvature operator. The construction of the modified curvature operators, is based on the function $i : \Lambda^4 TM \rightarrow S^2(\Lambda^2 TM)$ that assigns to each 4-form $\Omega$ a symmetric bilinear form (on the space of bivectors) $i(\Omega)$ defined by $i(\Omega)(\alpha_1, \alpha_2) = \Omega(\alpha_1 \wedge \alpha_2)$. Now, for each $\Omega \in \Lambda^4 TM$, we define modified Riemann curvature tensor by $Rm_{\Omega} = Rm + i(\Omega)$. The self-adjoint linear operator associated to the symmetric bilinear form $Rm_{\Omega}$ is called a modified curvature operator and is denoted by $R_{\Omega}$.

Since $Rm_{\Omega}(X \wedge Y, X \wedge Y) = Rm(X \wedge Y, X \wedge Y)$ for all $\Omega \in \Lambda^4 TM$, the sectional curvature is controlled by the eigenvalues of the modified curvature operators:

$$\lambda_{\min}(\mathcal{R}_{\Omega_1}) \leq K \leq \lambda_{\max}(\mathcal{R}_{\Omega_2}), \quad (8)$$

where $\Omega_1$ and $\Omega_2$ are any two 4-forms.

Let us describe how this inequality can be used to estimate the sectional curvature of the Aloff-Wallach spaces. The strategy is to estimate the curvature on certain subspaces of $\Lambda^2(T_T(p,q)W(p,q))$ and then to show that the bounds are stricter than the bounds given by the corresponding eigenvalues of some modified curvature operators. To formulate this more precisely, we introduce

$$\hat{\lambda} := \inf \{K(\omega) | \omega \in G \cap E_1\},$$

$$\bar{\lambda} := \inf \{K(\omega) | \omega \in G \cap E_2\},$$

$$\Lambda_j := \inf \{K(\omega) | \omega \in G \cap F_i\},$$

where $j \in \{0, 1, 2\}$, $G$ is the Grassmannian of oriented 2-planes, and $E_i$ and $F_i$ are subspaces of $\Lambda^2(T_T(p,q)W(p,q))$. It is established in [12] that $\min\{\hat{\lambda}, \bar{\lambda}\}$ is weakly smaller than the minimal eigenvalue of certain modified curvature operator. Using inequality (8), this implies that $\min\{\hat{\lambda}, \bar{\lambda}\}$ bounds the sectional curvature from below, and, since it is clear from the definition that $\hat{\lambda}$ and $\bar{\lambda}$ are weakly larger than the minimum of the (unrestricted) sectional curvature, $K_{\min}$, we get $\min\{\hat{\lambda}, \bar{\lambda}\} = K_{\min}$. Similar reasoning works for $\Lambda_j$ and $K_{\max}$.

Sections 5 of [12] gives concrete recipes for computing $\hat{\lambda}$, $\bar{\lambda}$, and $\Lambda_j$. In his paper, Püttmann uses these recipes to determine the optimal pinching among certain class of metrics on the Aloff-Wallach spaces. We shall employ these procedures to determine the minimum and the maximum of the sectional curvatures on the Aloff-Wallach spaces with the metric $\tilde{k}$, which we defined in section 2.

In order to proceed to the computations, we need to introduce some notation, which we are going to take from [12], but adapt to our case. In proposition 4.10 (of [12]), Püttmann introduces quantities $a_j, b_j, c_j, d_j$, and $\xi_j$, with $j \in \{0, 1, 2\}$ (and shows that they are the matrix elements of the curvature operators restricted to various subspaces of $\Lambda^2 TM$). Here are the definitions of these...
Quantities for an Aloff-Wallach space $W(p, q)$ adapted to our choice of metric:

\[
\begin{align*}
  a_0 &= 8 - \frac{9(p+q)^2}{2(p^2+pq+q^2)}, & a_1 &= 4 - \frac{9p^2}{8(p^2+pq+q^2)}, & a_2 &= 4 - \frac{9q^2}{8(p^2+pq+q^2)}; \\
  b_0 &= -2 - \frac{9pq}{8(p^2+pq+q^2)}, & b_1 &= -\frac{10p^2+pq+q^2}{4(p^2+pq+q^2)}, & b_2 &= -\frac{p^2+pq+10q^2}{4(p^2+pq+q^2)}; \\
  c_0 &= \frac{3(p+q)^2}{2(p^2+pq+q^2)}, & c_1 &= \frac{3p^2}{8(p^2+pq+q^2)}, & c_2 &= \frac{3q^2}{8(p^2+pq+q^2)}; \\
  d_0 &= \frac{5}{8}, & d_1 &= \frac{1}{8}, & d_2 &= \frac{1}{8}; \\
  \xi_0 &= -\frac{3\sqrt{3}(p+q)}{8\sqrt{p^2+pq+q^2}}, & \xi_1 &= \frac{\sqrt{3}(2p+q)}{8\sqrt{p^2+pq+q^2}}, & \xi_2 &= \frac{\sqrt{3}(p+q)}{8\sqrt{p^2+pq+q^2}}.
\end{align*}
\]

Here are more details on these quantities: It turns out that it is enough to consider invariants modified curvature operators, which are the operators $\mathcal{R}_\Omega$ with $T^2$-invariant $\Omega$, where $T^2$ is the extension of $T^2 \subset SU(3)$ by the complex conjugation on $SU(3)$. To decompose these operators, their domain $\Lambda_2(T_{T(p,q)}W(p,q)) = \Lambda_2\mathbb{T}^4$, is decomposed into the sum of $T^2$-invariant subspaces by first identifying $\mathbb{T}^4$ with $\mathbb{R} \oplus \mathbb{C}^3$, and then decomposing the corresponding space of bivectors as

\[\Lambda_2(\mathbb{T}^4) = \mathbb{R}^3 \oplus (\oplus_{j=0}^2 V_j) \oplus (\oplus_{j=0}^2 (\mathbb{C}_j^a \oplus \mathbb{C}_j^b)),\]

where the only non-obvious terms $V_j$ are copies of $\mathbb{C}$. Then it is shown that $T^2$-invariant 4-forms on $\mathbb{T}^4$ are parameterized by four real numbers. Calling these numbers $\eta_0, \eta_1, \eta_2$, and $\xi$ and writing $\mathcal{R}(\eta, \xi)$ for $\mathcal{R}_{\Omega}(\eta_0, \eta_1, \eta_2, \xi)$, one gets the following decomposition for the invariants modified curvature operator:

\[
\mathcal{R}(\eta, \xi)|_{V_j} = \eta_j, \\
\mathcal{R}(\eta, \xi)|_{\mathbb{R}^3} = \begin{pmatrix}
  a_0 & b_2 - \eta_2 & b_1 - \eta_1 \\
  b_2 - \eta_2 & a_1 & b_0 - \eta_0 \\
  b_1 - \eta_1 & b_0 - \eta_0 & a_2
\end{pmatrix}, \\
\mathcal{R}(\eta, \xi)|_{\mathbb{C}_j^a \oplus \mathbb{C}_j^b} = \begin{pmatrix}
  c_j & \sqrt{2}(\xi_j - \xi) \\
  \sqrt{2}(\xi_j - \xi) & 2d_j - \eta_j
\end{pmatrix}.
\]

We can also mention that

\[
E_1 = (\oplus_{j=0}^2 V_j) \oplus (\oplus_{j=0}^2 (\mathbb{C}_j^a \oplus \mathbb{C}_j^b)), \\
E_2 = \mathbb{R}^3 \oplus (\oplus_{j=0}^2 V_j), \\
F_j = \mathbb{R}^3 \oplus \mathbb{C}_j^a \oplus \mathbb{C}_j^b.
\]
4.3 Minimal Curvature

As we explained in the previous subsection, in order to find the minimum of the sectional curvature, we need to compute \( \hat{\lambda} \) and \( \hat{\lambda} \). Let us describe how these two numbers are computed in \([12]\). In order to compute \( \hat{\lambda} \), one considers three functions \( \lambda_j(x) \) (where \( j \in \{ 0, 1, 2 \} \)), each of which is the smallest root of the corresponding polynomial

\[
P_x(\lambda) = \det(\mathfrak{R}(\lambda, \xi)|_{\mathfrak{R}^n} - \lambda I),
\]

where \( \mathfrak{R}(\lambda, \xi) \) is the modification with \( (\eta_0, \eta_1, \eta_2) = (\lambda, \lambda, \lambda) \). In terms of these functions, \( \hat{\lambda} \) can be computed as \( \hat{\lambda} = \max_x \min_j \{ \lambda_j(x) \} \). It follows from the decomposition of the modified curvature operators shown in the previous subsection that

\[
\lambda_j(x) = c_j + d_j \sqrt{\left(\frac{c_j - d_j}{2}\right)^2 + (\xi_j - x)^2}.
\]

Since \( y = \lambda_j(x) \) are lower branches of hyperbolas with maxima at \( x = \xi_j \), \( \hat{\lambda} \) is achieved either at \( \xi_j \) or at an intersection of two curves between their maxima. We shall now compute \( \hat{\lambda} \) for \( W(n, n + 1) \) (with \( n \) a positive integer). First let us look at the maxima of \( \lambda_j \), that is \( \lambda_j(\xi_j) \). In the following table, we write \( \xi_j(n) \) for \( \xi_j(p, q) \) with \( p = n \) and \( q = n + 1 \) and \( \lambda_j(\xi_j(n)) \). We shall now compute \( \hat{\lambda} \) for \( W(n, n + 1) \) (with \( n \) a positive integer). First let us look at the maxima of \( \lambda_j \), that is \( \lambda_j(\xi_j) \). In the following table, we write \( \xi_j(n) \) for \( \xi_j(p, q) \) with \( p = n \) and \( q = n + 1 \) and \( \lambda_j(\xi_j(n)) \).

| \( \xi_0(n) \) | \( \xi_1(n) \) | \( \xi_2(n) \) |
|-------------|-------------|-------------|
| \( \frac{9(1 + 2n)}{8\sqrt{3} + 9n + 9n^2} \) | \( \frac{3(1 + 3n)}{8\sqrt{3} + 9n + 9n^2} \) | \( \frac{3(2 + 3n)}{8\sqrt{3} + 9n + 9n^2} \) |

\[
\lambda_0(\xi_0) = \frac{5}{8}, \quad \lambda_1(\xi_1) = \frac{3n^2}{8 + 24n + 24n^2}, \quad \lambda_2(\xi_2) = \frac{1}{8}.
\]

The numbers from the second row can not be \( \hat{\lambda} \) since, for any \( j \in \{ 0, 1, 2 \} \), there exists \( i \in \{ 0, 1, 2 \} \), with \( i \neq j \), such that \( \lambda_j(\xi_j) > \lambda_i(\xi_i) \). Namely,

\[
\lambda_1(\xi_0) = \frac{1 + 3n + 6n^2 - \sqrt{193 + 1446n + 4149n^2 + 5508n^3 + 2916n^4}}{16 + 48n + 48n^2} < 0,
\]

which implies \( \lambda_1(\xi_0) < \lambda_0(\xi_0) \) and \( \lambda_0(\xi_0) \neq \hat{\lambda} \);

\[
\lambda_0(\xi_1) = \frac{17 + 63n + 63n^2 - \sqrt{241 + 1902n + 5691n^2 + 7686n^3 + 4005n^4}}{16 + 48n + 48n^2}
\]

\[
< \frac{3n^2}{8 + 24n + 24n^2} = \lambda_1(\xi_1)
\]

for any \( n \geq 1 \) which implies \( \lambda_1(\xi_1) \neq \hat{\lambda} \) by the same logic as above; and, finally,

\[
\lambda_0(\xi_1) = \frac{17 + 63n + 63n^2 - \sqrt{349 + 2442n + 6663n^2 + 8334n^3 + 4005n^4}}{16 + 48n + 48n^2} < 0
\]

shows \( \lambda_2(\xi_2) \neq \hat{\lambda} \).
The other candidates for $\hat{\lambda}$ are the intersections, so let us consider these. Setting $\lambda_0(x) = \lambda_1(x)$, we get two roots:

$$x_1^{01} = \left[ (32 + 552n + 3132n^2 + 8037n^3 + 9648n^4 + 4401n^5)\sqrt[3]{3 + 9n + 9n^2} 
- (16 + 60n + 57n^2)\sqrt{3n(-56 - 555n - 1935n^2 - 1620n^3) + 7173n^4 + 22788n^5 + 26649n^6 + 11907n^7} \right] / [8(64 + 672n + 2916n^2 + 6624n^3 + 8181n^4 + 4995n^5 + 999n^6)],$$

$$x_2^{01} = \left[ (32 + 552n + 3132n^2 + 8037n^3 + 9648n^4 + 4401n^5)\sqrt[3]{3 + 9n + 9n^2} 
+ (16 + 60n + 57n^2)\sqrt{3n(-56 - 555n - 1935n^2 - 1620n^3) + 7173n^4 + 22788n^5 + 26649n^6 + 11907n^7} \right] / [8(64 + 672n + 2916n^2 + 6624n^3 + 8181n^4 + 4995n^5 + 999n^6)].$$

The second root lies outside the interval $[\xi_0, \xi_1]$ and, therefore, could not lead to $\hat{\lambda}$. More precisely, for any $n$, $x_2^{01}(n) > \xi_1(n)$, as one can see by checking the inequality for $n = 1$ and then checking that the derivative of $x_2^{01}(n) - \xi_1(n)$ (with respect to $n$) is positive. In order for $\lambda_0(x_1)$ to be a valid candidate for $\hat{\lambda}$ we shall need to check that $\lambda_0(x_1^{01}) = \lambda_1(x_1^{01}) < \lambda_2(x_2^{01})$ or rule out all other candidates for $\hat{\lambda}$. We choose the second route and proceed to analyze the other intersections of $\lambda_1$.

The solutions of the equation $\lambda_1(x) = \lambda_2(x)$ are

$$x_1^{12} = \frac{3 + 6n}{8\sqrt{3 + 9n + 9n^2}},$$

$$x_2^{12} = \frac{9 + 18n}{8\sqrt{3 + 9n + 9n^2}}.$$

This time both roots lie outside the interval $[\xi_1, \xi_2]$. More precisely, $x_1^{12} < \xi_1$ and $x_2^{12} > \xi_2$, which is clear once one recalls the formulae for $\xi_j(n)$. Thus, this intersection does not produce any candidates for $\hat{\lambda}$.

Turning to the last intersection, we find that $\lambda_2(x) = \lambda_0(x)$ for

$$x_1^{20} = \left[ (178 + 1812n + 7101n^2 + 13455n^3 + 12357n^4 + 4401n^5)\sqrt[3]{3 + 9n + 9n^2} 
- (13 + 54n + 57n^2)\sqrt{3(689 + 7847n + 39387n^2 + 113562n^3 + 205110n^4 + 236718n^5 + 169641n^6 + 68607n^7 + 11907n^8)} \right] / [8(-131 - 969n - 2835n^2 - 3870n^3 - 1809n^4 + 999n^5 + 999n^6)],$$

$$x_2^{20} = \left[ (178 + 1812n + 7101n^2 + 13455n^3 + 12357n^4 + 4401n^5)\sqrt[3]{3 + 9n + 9n^2} 
+ (13 + 54n + 57n^2)\sqrt{3(689 + 7847n + 39387n^2 + 113562n^3 + 205110n^4 + 236718n^5 + 169641n^6 + 68607n^7 + 11907n^8)} \right] / [8(-131 - 969n - 2835n^2 - 3870n^3 - 1809n^4 + 999n^5 + 999n^6)].$$

A computation shows that the second root lies outside the interval $[\xi_0, \xi_2]$ and, thus, irrelevant to the computation of $\hat{\lambda}$.
Therefore, (9) implies that
\[ x_1^{12} < x_1^{20} < x_2^{12}. \]  

Now, for any \( x \in (x_1^{12}, x_2^{12}) \), \( \lambda_1(x) < \lambda_2(x) \) since \( \max_x \{ \lambda_1(x) \} = \lambda_1(\xi_1) < \lambda_2(\xi_2) = \max_x \{ \lambda_2(x) \} \), and \( x_1^{12} \) and \( x_2^{12} \) (the ordinates of the intersections of \( \lambda_1 \) and \( \lambda_2 \)) lie to the left and to the right of the interval \( [\xi_1, \xi_2] \) respectively. Therefore, (9) implies that \( \lambda_2(x_1^{20}) > \lambda_1(x_1^{20}) \), which means that this intersection of \( \lambda_2 \) and \( \lambda_0 \) lies above \( \lambda_1 \) and (its ordinate) can not be \( \lambda \).

Therefore, the only remaining candidate for \( \lambda \), which is \( \lambda_0(x_1^{01}) \), is \( \hat{\lambda} \):

\[
\hat{\lambda} = \lambda_0(x_1^{01}) = \frac{17 + 63n + 63n^2}{16 + 48n + 48n^2} - \frac{1}{16} \left( \frac{(7 + 33n + 33n^2)^2}{(1 + 3n + 3n^2)^2} + 4 \left( \frac{9(1 + 2n)}{\sqrt{3 + 9n + 9n^2}} + \frac{\{(32 + 552n + 3132n^2 + 8037n^3 + 9648n^4 + 4401n^5)\sqrt{3 + 9n + 9n^2} - \sqrt{3n(16 + 60n + 57n^2)(-56 - 555n - 1935n^2 - 1620n^3 + 7173n^4 + 22788n^5 + 26649n^6 + 11907n^7)^{1/2}}}{\{(64 + 672n + 2916n^2 + 6624n^3 + 8181n^4 + 4995n^5 + 999n^6)^2 \}} \right) \right)^{1/2}
\]

We make two remarks about \( \hat{\lambda} \). First, \( \hat{\lambda}(n) \) is monotonously increasing (as one can see by checking the positivity of \( \lambda(n+1) - \lambda(n) \)), which implies that \( \hat{\lambda}(n) \geq \hat{\lambda}(1) > 1/25 \), and, second, \( \lim_{n \to \infty} \hat{\lambda}(n) = 2/37 \), which is the minimal curvature of \( W(p,q) \) with \( p/q = 1 \).

Let us now describe how to compute \( \hat{\lambda} \) - the other quantity needed to estimate \( K_{\min}(W(p,q)) \). The number \( \hat{\lambda} \) is computed through the following auxiliary quantities:

\[
A = \begin{pmatrix}
 a_0 & b_2 & b_1 \\
 b_2 & a_1 & b_0 \\
 b_1 & b_0 & a_2
\end{pmatrix} = \begin{pmatrix}
 8 - \frac{9(p+q)^2}{4(p^2+pq+q^2)} & \frac{p^2+pq+10q^2}{4(p^2+pq+q^2)} & \frac{-10p^2+pq+9q^2}{8(p^2+pq+q^2)} \\
 -\frac{p^2+pq+10q^2}{4(p^2+pq+q^2)} & 4 - \frac{9p^2}{8(p^2+pq+q^2)} & \frac{-2p^2+pq+9q^2}{8(p^2+pq+q^2)} \\
 -\frac{10p^2+pq+9q^2}{4(p^2+pq+q^2)} & \frac{-2p^2+pq+9q^2}{8(p^2+pq+q^2)} & 4 - \frac{9p^2}{8(p^2+pq+q^2)}
\end{pmatrix};
\]
\[
D_1 = a_1a_2 - b_0^2 - a_1b_1 - a_2b_2 + b_0b_1 + b_2b_0
= \frac{3(113p^2 + 86pq + 113q^2)}{16(p^2 + pq + q^2)},
\]
\[
D_2 = a_2a_0 - b_1^2 - a_2b_2 - a_0b_0 + b_1b_2 + b_0b_1
= \frac{9(38p^2 + 13pq + 45q^2)}{16(p^2 + pq + q^2)},
\]
\[
D_3 = a_0a_1 - b_2^2 - a_0b_0 - a_1b_1 + b_0b_1 + b_1b_2
= \frac{9(106p^4 + 141p^3q + 192p^2q^2 + 77pq^3 + 60q^4)}{32(p^2 + pq + q^2)}.
\]

It can be shown that
\[
\bar{\lambda} = \min \left\{ x^T A x | x_0, x_1, x_2 \geq 0, x_0 + x_1 + x_2 = 1 \right\},
\]
where
\[
A = R(0, 0) |_{\mathbb{R}^3} = \begin{pmatrix} a_0 & b_2 & b_1 \\ b_2 & a_1 & b_0 \\ b_1 & b_0 & a_2 \end{pmatrix}.
\]

Using this characterization of \( \bar{\lambda} \) one sees that
\[
\bar{\lambda} = \frac{\det A}{D_1 + D_2 + D_3},
\]
under certain conditions on the elements of \( A \) and \( D_i \).

According to proposition 5.11 of [12], if \( \sum_{j=0}^2 (a_j - b_j) \geq 0 \) and \( D_j > 0 \) for all \( j \in \{0, 1, 2\} \),
\[
\bar{\lambda} = \frac{\det A}{D_1 + D_2 + D_3}.
\]

For any \( (p, q) \in \mathbb{Z}^+ \times \mathbb{Z}^+ \), \( \sum_{j=0}^2 (a_j - b_j) = \frac{121p^2 + 85pq + 121q^2}{8(p^2 + pq + q^2)} > 0 \) and all \( D_j \) are positive (as evident from the above formulae). Therefore, in this case,
\[
\bar{\lambda} = \frac{\det A}{D_1 + D_2 + D_3} = \frac{(p^2 + pq + q^2)(59p^2 - 22pq + 59q^2)}{772p^4 + 1127p^3q + 1776p^2q^2 + 977pq^3 + 676q^4}.
\]

In particular, if \( (p, q) = (n, n+1) \) (where \( n \) is a positive integer),
\[
\bar{\lambda} = \frac{(1 + 3n + 3n^2)(59 + 96n + 96n^2)}{676 + 3681n + 8763n^2 + 10314n^3 + 5328n^4}.
\]

We remark that the function \( \bar{\lambda}(n) \) is decreasing (as one can see by checking that \( \bar{\lambda}' \) is negative), and \( \lim_{n \to \infty} \bar{\lambda} = \frac{326}{1105} = \frac{2}{3} \), which is the minimal curvature of \( W(1, 1) \). Figure 1 compares \( \bar{\lambda} \) and \( \bar{\lambda} \) and shows that they both approach
the minimal curvature of $W(1,1)$, but from different sides. Summarizing this subsection, we can say that the minimal curvature of the Aloff-Wallach space $W(n,n+1)$ is given by the formula

$$K_{min}(W(n,n+1)) = \frac{17 + 63n + 63n^2}{16 + 48n + 48n^2} - \frac{1}{16} \left[ \frac{(7 + 33n + 33n^2)^2}{(1 + 3n + 3n^2)^2} + 4 \left( \frac{9(1 + 2n)}{\sqrt{3 + 9n + 9n^2}} \right) + \left\{ (32 + 552n + 3132n^2 + 8037n^3 + 9648n^4 + 4401n^5)\sqrt{3 + 9n + 9n^2} \right. \\
- \sqrt{3n(16 + 60n + 57n^2)(-56 - 555n - 1935n^2 - 1620n^3 + 7173n^4} \\
+ 22788n^5 + 26649n^6 + 11907n^7)^{1/2} \right\} \left\{ (64 + 672n + 2916n^2 + 6624n^3 + 8181n^4 + 4995n^5 + 999n^6) \right\}^{2/3} \right]^{1/2} < \frac{1}{25},$$

which proves the left parts of the inequalities from Theorem 2.2.

### 4.4 Maximal Curvature

Let us now describe the computation of the maximum of the sectional curvature. To do this, we need to compute numbers $\Lambda_j$ (with $j \in \{0,1,2\}$). In fact, we shall show that $K_{max} = \Lambda_0$.

According to lemma 5.13 of [13], if $a_1 > 2d_0 - b_0$, $\Lambda_0 = \max\{a_0, a_1, a_2, c_0\}$. Using the formulae above we compute (for $W(n,n+1)$):

$$a_1 = 4 - \frac{9n^2}{8(1 + 3n + 3n^2)} > \frac{26 + 87n + 87n^2}{8(1 + 3n + 3n^2)} = 2d_0 - b_0,$$
which allows us to apply lemma 5.13 (of [12]) and get

\[ \Lambda_0 = \max\{a_0, a_1, a_2, c_0\} \]
\[ = \max\left\{2 - \frac{3}{2(1 + 3n + 3n^2)}, 4 - \frac{9n^2}{8(1 + 3n + 3n^2)}, 4 - \frac{9(1 + n)^2}{8(1 + 3n + 3n^2)}, \frac{3(1 + 2n)^2}{2(1 + 3n + 3n^2)}\right\} \]
\[ = 4 - \frac{9n^2}{8(1 + 3n + 3n^2)}. \]

Lemma 5.14 of [12] says that, if \(a_1 > 2d_0 - b_0\) (which we already verified) and

\[ \Lambda_0 \geq \max\left\{b_0, b_1, b_2, c_0, \lambda_{\text{max}}\left(\sqrt{2}(\xi_1 - \xi_0) \quad 2d_1 - b_1\right), \lambda_{\text{max}}\left(\sqrt{2}(\xi_2 - \xi_0) \quad 2d_2 - b_2\right)\right\}, \]

(where \(\lambda_{\text{max}}\) denotes the largest eigenvalue of the matrix written next to it), \(K_{\text{max}} = \Lambda_0\). Since we already know that \(\Lambda_0 = a_1 \geq c_0\), and all \(b_j\), namely

\[ b_0 = -2 - \frac{9n(n + 1)}{8(1 + 3n + 3n^2)}, \quad b_1 = -1 + 3n + 12n^2 - \frac{1 + 3n + 12n^2}{4(1 + 3n + 3n^2)}, \quad b_2 = -10 + 21n + 12n^2 - \frac{10 + 21n + 12n^2}{4(1 + 3n + 3n^2)}, \]

are negative, we only need to check the eigenvalues. Introducing the notation

\[ \nu_1 = \lambda_{\text{max}}\left(\sqrt{2}(\xi_1 - \xi_0) \quad 2d_1 - b_1\right), \]
\[ \nu_2 = \lambda_{\text{max}}\left(\sqrt{2}(\xi_2 - \xi_0) \quad 2d_2 - b_2\right), \]

we compute

\[ \nu_1 = \frac{4 + 12n + 33n^2 + \sqrt{400 + 2976n + 8640n^2 + 11664n^3 + 6561n^4}}{16(1 + 3n + 3n^2)}, \]
\[ \nu_2 = \frac{25 + 54n + 33n^2 + \sqrt{961 + 5556n + 13014n^2 + 14580n^3 + 6561n^4}}{16(1 + 3n + 3n^2)}. \]

Evidently \(\nu_2 > \nu_1\); therefore, once we show that \(\Lambda_0 = a_1 > \nu_2\), we shall conclude that \(K_{\text{max}} = \Lambda_0\). Subtracting \(\nu_2\) from \(a_1\), we get

\[ a_1 - \nu_2 = \frac{39 + 138n + 141n^2 - \sqrt{961 + 5556n + 13014n^2 + 14580n^3 + 6561n^4}}{16(1 + 3n + 3n^2)}. \]

A computation shows that this fraction is always positive. Thus, the maximum curvature of the Aloff-Wallach space \(W(n, n + 1)\) is given by the formula

\[ K_{\text{max}}(W(n, n + 1)) = 4 - \frac{9n^2}{8(1 + 3n + 3n^2)}. \]
Figure 2: $K_{max}(W(n,n+1))$ approaches $K_{max}(W(1,1))$ as $\frac{n}{n+1}$ tends to one.

which proves the right part of the first inequality from Theorem 2.2. We note that

$$\lim_{n \to \infty} K_{max}(W(n,n+1)) = 4 - \frac{9}{24} = \frac{29}{8} = K_{max}(W(1,1)).$$

Figure 2 shows $K_{max}(W(n,n+1))$ together with the asymptote.

5 Application: Injectivity Radius Estimates

5.1 Estimating Injectivity Radius from Below

Cheeger’s injectivity radius estimate, first obtained in [4], gives a lower bound on the injectivity radius in terms of dimension, a lower bound on the volume, a two-sided bound on the curvature, and an upper bound on the diameter. We shall use an improved version of this estimate, which is given in [13] as Theorem IV.3.9(2). For any compact Riemannian manifold $M$, let $i(M)$ denote the injectivity radius of $M$, $K(M)$ - the set of the sectional curvatures of $M$, and $d(M)$ - the diameter of $M$. Also let $s_\delta$ be the solution of $f'' + \delta f = 0$ with the initial conditions $f(0) = 0$ and $f'(0) = 1$. The above mentioned theorem from [13] says that, if $K(M) \subset [\delta, \Delta]$, 

$$i(M) \geq \min \left\{ \frac{\pi}{\sqrt{\Delta}} \cdot \frac{Vol(M)}{Vol(S^n)} \cdot \left[ s_\delta \left( \min \left\{ d(M), \frac{\pi}{2\sqrt{\delta}} \right\} \right) \right]^{1-n} \right\},$$

where $n$ is the dimension of $M$. For $\delta > 0$, which the case for $M = W(p,q)$, $s_\delta(t) = \frac{\sin(\sqrt{\delta}t)}{\sqrt{\delta}}$. Since $\sin(t) \leq 1$, $s_\delta(t) \leq \frac{1}{\sqrt{\delta}}$, which implies that $[s_\delta(t)]^{1-n} \geq [\sqrt{\delta}]^{n-1}$ for any $n > 1$. Putting this into the injectivity radius estimate, we see that, if $K(M) \subset [\delta, \Delta]$ and $\delta > 0$,

$$i(M) \geq \min \left\{ \frac{\pi}{\sqrt{\Delta}} \cdot \frac{Vol(M)}{Vol(S^n)} \frac{1}{\sqrt{\delta}^{n-1/2}} \right\}$$

\[10\]
Since
\[ \dim(W(p, q)) = \dim(SU(3)) - \dim(T(p, q)) = 8 - 1 = 7, \]
when we apply (10) to \( W(p, q) \), we shall need to put
\[ \text{Vol}(S^7) = \frac{2\pi^{(7+1)/2}}{\Gamma((7+1)/2)} = \frac{\pi^4}{3} \]
in place of \( \text{Vol}(S^n) \) and \( \delta^3 \) in place of \( \delta^{(n-1)/2} \). Applying (10) to \( M = W(p, q) \) and using the left part of (7), we get
\[ i(W(p, q)) \geq \min \left\{ \frac{\pi}{\sqrt{\Delta}} \cdot \frac{\text{Vol}(W(p, q))}{\text{Vol}(S^7)}, \frac{3\sqrt{3}\pi \gcd(p, q)\delta^3}{32\sqrt{p^2 + q^2 + pq}} \right\}. \]
Applying this inequality to \( W(n, n+1) \) (where \( n \) is a positive integer) and using the curvature estimates derived in section 4, we get part 2 of Corollary 2.3.

Applying the inequality to \( W(1, 1) \), where \( \delta = 2/37 \) and \( \Delta = 29/8 \), we see that
\[ i(W(1, 1)) \geq \min \left\{ \frac{\pi}{\sqrt{\Delta}} \cdot \frac{3\sqrt{3}\pi (2/37)^3}{32\sqrt{1^2 + 1^2 + 1 \cdot 1}} = \frac{3\pi}{4 \cdot 37^3} \geq 4.65 \cdot 10^{-5}, \right\}
which proves item 2 of Corollary 2.3.

5.2 Estimating Injectivity Radius from Above

In [2], Berger proved that
\[ \text{Vol}(M^m, g) \geq \left[ \frac{i(M^m, g)}{\pi} \right]^m \cdot \text{Vol}(S^m). \] (11)
This result is known as the Berger isoembolic inequality and its expository account can be found, for example, in [13] as Theorem VI.2.1. Since we have an upper bound on the volumes of the Aloff-Wallach spaces [7], we can apply (11) to \( M^m = W(p, q) \) and get upper bounds on their injectivity radii. Since (as we saw in the previous subsection) \( \dim(W(p, q)) = 7 \) and \( \text{Vol}(S^7) = \pi^4/3 \), combining the right part of (7) with (11), we get
\[ i(W(p, q)) \leq \left[ \frac{3\sqrt{3}\gcd(p, q)}{2\sqrt{p^2 + q^2 + pq}} \right]^{1/7} \cdot \pi, \]
and Corollary 2.3 is proven. In particular, we see that, if at least one of the indices \( p \) and \( q \) goes to infinity, the injectivity radius of \( W(p, q) \) tends to zero.

Applying this estimate to the family of spaces considered by Huang in [7], which are \( W(i, i+1) \) with \( i \) sufficiently large, we see that the family contains spaces with arbitrary small injectivity radii, and, thus, we conclude that there cannot be a common lower injectivity radius for this family, getting an alternative prove of Huang’s result mentioned in the introduction.
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