The $\varepsilon$-factorised differential equation for the four-loop equal-mass banana graph

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The four-loop equal-mass banana integral is the simplest Feynman integral whose geometry is related to a Calabi–Yau three-fold. We show that the differential equation for this Feynman integral can be cast into an $\varepsilon$-factorised form. This allows us to obtain the solution to any desired order in the dimensional regularisation parameter $\varepsilon$. Our calculation also shows that the four-loop banana integral is only minimally more complicated than the corresponding Feynman integrals at two or three loops.

INTRODUCTION

An $\varepsilon$-factorised differential equation for a system of Feynman integrals is the essential tool to solve the system to all orders in the dimensional regularisation parameter $\varepsilon$ $[4]$. This has been applied successfully to many Feynman integrals evaluating to multiple polylogarithms, to several elliptic Feynman integrals $[2, 5]$ and quite recently to the three-loop equal-mass banana integral $[6]$. The geometry of the latter system is related to a Calabi–Yau two-fold. The relevant Picard–Fuchs operator (a third-order differential operator) has the special property that it is the symmetric square of a second-order differential operator related to an elliptic curve $[7, 8]$. This property can be used to treat the three-loop equal-mass banana graph with methods similar to the elliptic case $[6, 9–12]$. Unfortunately the property of being a symmetric product does not carry over to the higher loop banana integrals and less is known for the higher $l$-loop banana graphs $[13, 15]$.

The first genuinely generic Calabi–Yau Feynman integral is the four-loop equal-mass banana integral. Its geometry is given by a Calabi–Yau three-fold and the relevant Picard–Fuchs operator is no longer a symmetric product. We are interested in the question, if an $\varepsilon$-factorised differential equation exists and if yes, what are the differential one-forms appearing in the differential equation. In this letter we show that an $\varepsilon$-factorised differential equation exists, give the differential one-forms and show how to solve the four-loop equal-mass banana integral to all orders in the dimensional regularisation parameter $\varepsilon$.

We need two essential ingredients: The first one is a change of variables. In the two-loop and three-loop case an important step is the change of variables from the dimensionless ratio $m^2/(-p^2)$ to the modular parameter $\tau$, defined as the ratio of the two periods of the elliptic curve. In the four-loop case there is no elliptic curve, but the above change of variables can also be viewed as the mirror map $[19, 21]$ for a family of Calabi–Yau manifolds and this generalises to the four-loop case.

The second ingredient is the following: The $\varepsilon$-factorised form is achieved by redefining the master integrals in $D = 2 - 2\varepsilon$ space-time dimensions at $l$-loops is given by

$$I_l = \frac{\varepsilon_i}{\omega_1^i} I_1^{(1)} \cdots I_{l-1}^{(1)},$$

where $\omega_1$ is a specific solution of the homogeneous differential equation. With $q = \exp(2\pi i\tau)$ and $\theta = q^{1/2}$ the Picard–Fuchs operators for $I_2$ at two and three loops are

$$\theta^2 \text{ and } \theta^3.$$ (2)

At four loops we find

$$\theta^2 \frac{1}{K} \theta^2.$$ (3)

The appearance of the new function $K$ is related to the fact that the Picard–Fuchs operator at four loops is not a symmetric product. It also implies that at four loops the first term of $I_2$ in the $\varepsilon$-expansion is not an Eichler integral. In the literature on Calabi–Yau manifolds the function $K$ is known as the “Yukawa coupling” $[22, 29]$.

NOTATION

We are interested in the integrals

$$I_{\nu_1\nu_2\nu_3\nu_4\nu_5} = e^{\Delta \varepsilon \tau} \left( m^2 \right)^{\nu_1 - 2D} \int \left( \prod_{a=1}^{5} \frac{d^D k_a}{(2\pi)^D} \right).$$

1 In particle physics a Yukawa coupling describes the coupling of the Higgs boson to a fermion. This notion has been transferred to superstring models, compactified on Calabi–Yau three-folds. From there it diffused into the mathematical literature, where it is used in the context of Calabi–Yau operators. In this letter we are back to particle physics, but the term “Yukawa coupling” does not refer to the original particle physics meaning. In order to avoid confusion, we will not use any further the term “Yukawa coupling”.

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The four-loop banana graph.

\[ i\pi \frac{\partial}{\partial D} \left( p - \sum_{b=1}^{5} k_b \prod_{c=1}^{5} \left( 1 - k_c^2 + m_c^2 \right)^{\nu_c} \right) , \]  

where \( D \) denotes the number of space-time dimensions, \( \varepsilon \) the dimensional regularisation parameter and \( \gamma_E \) the Euler-Mascheroni constant. We consider these integrals in \( D = 2 - 2\varepsilon \) space-time dimensions. The corresponding Feynman graph is shown in fig. 1. It is convenient to introduce the dimensionless variable

\[ y = -\frac{m^2}{p^2}. \]

It is well-known that this family of Feynman integrals has five master integrals. A possible choice for a basis of master integrals is

\[ I_{11110}, I_{11111}, I_{11112}, I_{11113}, I_{11114}. \]

The Feynman integral \( I_{11110} \) is a product of four one-loop tadpole integrals and rather simple. We set

\[ I_1 = \varepsilon^4 I_{11110} = [\varepsilon^{\gamma_E} \Gamma (1 + \varepsilon)]^4. \]

The integral \( I_1 \) has uniform weight.

Let us now look at \( J_{11111} \). Useful information can be extracted from the Picard–Fuchs operator \([30, 33]\). For \( I_{11111} \) we have the inhomogeneous fourth-order differential equation

\[ L_4 I_{11111} = \frac{120\varepsilon^4}{y^4 (1 + y) (1 + 9y) (1 + 25y)} I_{11110}. \]

The dependence on \( \varepsilon \) of the fourth-order differential operator \( L_4 \) is polynomial. The \( \varepsilon \)-independent part of \( L_4 \), which we denote by \( L_4^{(0)} \), is of particular importance:

\[ L_4^{(0)} I_{11111} = \int_0^1 \frac{d^4}{dy^4} + 2 \left( \frac{1}{y} + \frac{1}{1 + y} + \frac{9}{1 + 9y} + \frac{25}{1 + 25y} \right) \frac{d^3}{dy^3} \]
\[ + \frac{(1 + 98y + 1839y^2 + 3150y^3)}{y^2 (1 + y) (1 + 9y) (1 + 25y)} \frac{d^2}{dy^2} \]
\[ - \frac{(1 + 15y) (1 - 15y - 60y^2)}{y^3 (1 + y) (1 + 9y) (1 + 25y)} \frac{d}{dy} \]
\[ + \frac{1 + 5y}{y^4 (1 + y) (1 + 9y) (1 + 25y)}. \]

This operator appears as entry 34 in the table of ref. \([24]\). The associated Calabi–Yau manifold has been studied in ref. \([34]\).

The indicial equation for the operator \( L_4^{(0)} \) at the point \( y = 0 \) is \( (\rho - 1)^4 = 0 \), showing that \( y = 0 \) is a point of maximal unipotent monodromy.

### Master Integrals

The construction of the master integrals, which put the differential equation into an \( \varepsilon \)-factorised form, follows with one exception the pattern found at two-loops \([2]\) and three-loops \([6]\). Let

\[ \omega_1 (y), J (y), K (y), \]
\[ F_{32} (y), F_{42} (y), F_{43} (y), F_{52} (y), F_{53} (y), F_{54} (y) \]

be nine a priori unknown functions of \( y \). They are however independent of \( \varepsilon \). We start from the following ansatz for the master integrals

\[ I_1 = \frac{\varepsilon^4}{\omega_1} I_{111110}, \]
\[ I_2 = \frac{\varepsilon^4}{\omega_1} I_{111111}, \]
\[ I_3 = \frac{J}{\varepsilon} \frac{d}{dy} I_2 + F_{32} I_2, \]
\[ I_4 = \frac{J}{\varepsilon} \frac{d}{dy} I_3 + F_{42} I_2 + F_{43} I_3, \]
\[ I_5 = \frac{J}{\varepsilon} \frac{d}{dy} I_4 + F_{52} I_2 + F_{53} I_3 + F_{54} I_4. \]

The new ingredient at four loops is the appearance of the function \( K \) in the definition of \( I_4 \). This ansatz leads to the differential equation

\[ dI = AI, \]

where \( A \) is a \( 5 \times 5 \)-matrix. The entries in the first four rows and the entry \( A_{5,1} \) are already proportional to \( \varepsilon \). The remaining entries in the fifth row may be written as

\[ A_{5,k} = \sum_{j=k-5}^{1} A_{5,k} \varepsilon^j, \quad k \in \{2, 3, 4, 5\}, \]

where \( A_{5,k} \) is independent of \( \varepsilon \). We now require that the \( A_{5,k}^{(j)} \) with \( j < 1 \) vanish. This gives ten equations for nine unknown functions. These equations can be satisfied as follows: Let \( \omega_1 \sim \omega_4 \) be four independent solutions of \( L_4^{(0)} \omega = 0 \). From Frobenius it follows that we may write them as

\[ \omega_k = \frac{1}{(2\pi i)^{k-1}} \sum_{j=0}^{k-1} \ln^j \frac{y}{j!} \sum_{n=0}^{\infty} a_{k-1-j,n} y^{n+1}. \]

As normalisation we choose \( a_{0,0} = 1 \). We identify the holomorphic solution \( \omega_1 \) with the one appearing in eq. (10) and eq. (11). The first few terms read

\[ \omega_1 = y (1 - 5y + 45y^2 - 545y^3) + O (y^5). \]
We note that the expansion coefficients \( a_{0,n} \) of \( \omega_1 \) are related to the Bessel function \( I_0 \) by
\[
[I_0 (2\sqrt{-y})] \sum_{n=0}^{\infty} \frac{a_{0,n}}{(n!)^2} y^n.
\]
(16)

We set
\[
\tau = \frac{\omega_2}{\omega_1}, \quad q = e^{2\pi i \tau}
\]
(17)
and
\[
J = \frac{1}{2\pi i} \frac{dy}{d\tau}
\]
(18)

Eq. (17) defines a change of variables from \( y \) to \( \tau \) (or \( q \)). This map is also called the mirror map. We have
\[
y = q + 8q^2 + 36q^3 + 168q^4 + O(q^5).
\]
(19)
As \( y \) goes to zero, \( \tau \) approaches \( i\infty \) and \( q \) goes to zero. The function \( K \) is given by
\[
K = \frac{f^3}{\omega_1^3} y (1+y) (1+9y) (1+25y).
\]
(20)

\( K \) can also be expressed as
\[
K = \frac{d^2 \omega_3}{d\tau^2} \omega_1 = \frac{1}{\tau} \frac{d^2 \omega_4}{d\tau^2} \omega_1.
\]
(21)

We define iterated integrals by
\[
I(f_1,\ldots,f_n;\tau) = 
\lim_{q_0 \to 0} R \left[ \int_0^q \frac{dq_1}{q_1} \ldots \int_0^{q_{n-1}} \frac{dq_n}{q_n} f_1(\tau_1) \ldots f_n(\tau_n) \right],
\]
(22)
where \( R \) is the operator which removes all \( \ln(q_0) \)-terms, corresponding to the standard “trailing zero” or “tangential base point” regularisation. As the last argument of all iterated integrals will always be \( \tau \), we simply write \( I(f_1,\ldots,f_n) \) instead of \( I(f_1,\ldots,f_n;\tau) \). The functions \( F_{\tau j} \) are given by
\[
F_{32} = -f_{2,a} - f_{2,b},
\]
(23)
\[
F_{43} = -\frac{1}{K} (f_{2,a} - f_{2,b}), \quad F_{42} = -\frac{1}{K} f_{4,a},
\]
\[
F_{54} = -f_{2,a} + f_{2,b}, \quad F_{53} = -f_{4,b}, \quad F_{52} = -f_6,
\]
where
\[
f_{2,a} = J \left( \frac{5}{2y} - \frac{1}{1+y} - \frac{9}{1+9y} - \frac{25}{1+25y} \right),
\]
(24)
\[
f_{2,b} = I(K,h_6),
\]
\[
f_{4,a} = -[I(K,h_6)]^2 - 2KI(1,K,h_6,h_6), \quad -K I(1,h_{8,b}),
\]
\[
f_{4,b} = 4I(1,K,h_6,h_6) + 2I(1,h_{8,b}) + h_4,
\]
\[
f_5 = 120 \frac{J\omega}{y^2},
\]
\[
f_6 = f_{4,b} I(K,h_6) - 4h_6,
\]
\[
f_8 = f_{4,b} [I(K,h_6)]^2 - 8h_6 I(K,h_6) + K [I(1,h_{8,b}) + 2I(1,K,h_6,h_6)]^2 + h_{8,a}.
\]
The helper functions \( h_4, h_6, h_{8,a} \) and \( h_{8,b} \) are defined in the appendix. This completes the definition of the master integrals and it can be verified that with this definition the terms \( A_{n,j} \) with \( j < 1 \) vanish in eq. (13). Hence, the differential equation is in \( \varepsilon \)-factorised form
\[
dI = 2\pi i \varepsilon A_{\tau} d\tau,
\]
(25)
with
\[
A_{\tau} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & f_{2,a} + f_{2,b} & 1 & 0 & 0 \\
0 & f_{4,a} & f_{2,a} - f_{2,b} & 0 & 0 \\
f_{5} & f_{8} & f_{4,b} & f_{2,a} - f_{2,b} & 1 \\
f_{5} & f_{8} & f_{4,b} & f_{4,a} & f_{2,a} + f_{2,b}
\end{pmatrix}.
\]
(26)

This is a differential equation with an alphabet consisting of nine letters
\[
\mathcal{A} = \{1,K,f_{2,a},f_{2,b},f_{4,a},f_{4,b},f_{5},f_{6},f_{8}\}.
\]
(27)

We observe the symmetry
\[
A_{i,j} = A_{7-j,7-i} \quad \text{for } i,j \geq 2.
\]
(28)

We further observe that the \( q \)-expansions of all entries of the matrix \( A_{\tau} \) have integer coefficients (apart from a possible rational prefactor):
\[
K = 1 - q + 17q^2 - 253q^3 + 3345q^4 + O(q^5),
\]
\[
f_{2,a} = \frac{5}{2} - 15q + 167q^2 - 2787q^3 + 40631q^4 + O(q^5),
\]
\[
f_{2,b} = 7q - 135q^2 + 2275q^3 - 34759q^4 + O(q^5),
\]
\[
f_{4,a} = \frac{1}{4} (147q - 3267q^2 + 59943q^3 - 1017027q^4)
\]
\[
+ O(q^5),
\]
\[
f_{4,b} = \frac{5}{2} - 52q + 1460q^2 - 33316q^3 + 652212q^4
\]
\[
+ O(q^5),
\]
\[
f_5 = 120 (1 + 3q - 27q^2 + 147q^3 - 1467q^4) + O(q^5),
\]
\[
f_6 = -\frac{1}{2} (21q - 2805q^2 + 10877q^3 - 2772213q^4)
\]
\[
+ O(q^5),
\]
\[
f_8 = -\frac{1}{16} (9 - 855q + 7623q^2 + 606789q^3
\]
\[
- 31766841q^4) + O(q^5).
\]
(29)
RESULTS

We write
\[ I_k = \sum_{j=0}^{\infty} I_k^{(j)} \varepsilon^j \]  
(30)
for the \( \varepsilon \)-expansion of the master integral \( I_k \).

With the differential equation (25) at hand we only need the boundary values as additional input. We choose \( y = 0 \) as boundary point. The boundary values are easily obtained with the help of the Mellin–Barnes technique.

The calculation follows the lines of the corresponding calculation at three loops [6, 11]. We need the constant term and all logarithms \( \ln(y) \). We obtain for the boundary value of \( I_2 \)
\[ I_2|_{y=0} = 5 \varepsilon^4 \varepsilon^3 \sum_{j=0}^{4} \binom{4}{j} (-1)^j y^j \]  
(31)
\[ \frac{\Gamma(1+\varepsilon)\Gamma(1-\varepsilon)\Gamma(1+j\varepsilon)}{\Gamma(1+(j+1)\varepsilon)}. \]

Note that eq. (31) determines the boundary values of all master integrals.

With the boundary values we may now obtain analytical results for all master integrals. As an example we focus on \( I_2 \). With
\[ f_2^+ = f_{2,a} + f_{2,b}, \quad f_2^- = f_{2,a} - f_{2,b} \]  
(32)
the analytic result for the integral \( I_2 \) up to \( \mathcal{O}(\varepsilon^5) \) is given by
\[ I_2 = \varepsilon^4 [I(1, K, 1, f_3) - 80 \zeta_3 \ln q] + \varepsilon^5 \left\{ I(1, K, 1, f_2^+, f_3) + I(1, K, f_2^-, 1, f_3) + I(f_2^+, 1, K, 1, f_3) - 120 \zeta_2 I(1, K, 1) - 40 \zeta_3 [13 I(1, K) + 2 I(f_2^-) \right. \]
\[ + 2 I(f_2^+, 1)] - 120 \zeta_4 \ln q + 80 \zeta_2 \zeta_3 - 600 \zeta_5 \} + \mathcal{O}(\varepsilon^6). \]

From eq. (29) we obtain the \( q \)-expansion of all iterated integrals and hence the \( q \)-expansion of the master integrals. For example, the first few terms of \( I_2^{(4)} \) read with \( L = \ln q \)
\[ I_2^{(4)} = 5 L^4 - 80 \zeta_3 L + 60 L (4 - L) \]  
(34)
\[ + 15 \left( 6 - 34 L + 17 L^2 \right) q^2 \]
\[ - \frac{10}{9} \left( 123 - 2024 L + 1518 L^2 \right) q^3 \]
\[ - \frac{25}{8} \left( 851 + 4014 L - 4014 L^2 \right) q^4 + \mathcal{O}(q^5). \]

In addition we must compute the value of \( q \) from a given value \( y \). In the region \( |y| < 1/25 \) we may use the series for \( \omega_1 \) and \( \omega_2 \) given in eq. (14) to obtain from a given value of \( y \) the corresponding value of \( q \). Note that this is different from the situation at two and three loops, where we may use complete elliptic integrals to obtain the value of \( q \) from \( y \).

In the plots we will use the kinematical variable \( x = -1/y = p^2/m^2 \) instead of \( y \). The condition \( |y| < 1/25 \) translates to \( |x| > 25 \). This is the region where we may evaluate the integral numerically. The correct branch is selected by giving \( x \) an infinitesimal positive imaginary part according to Feynman’s \( i\delta \)-prescription. Fig. 2 and fig. 3 show the numerical results for the \( \varepsilon^4 \)-term \( I_2^{(4)} \) and the \( \varepsilon^5 \)-term \( I_2^{(5)} \) of \( I_2 \). We also plotted the results from the program pySecDec [38]. We observe excellent agreement.

Numerical evaluations around the singular points \( x = 0, x = 1, x = 9 \) and \( x = 25 \) will be discussed in a longer publication.
CONCLUSIONS

In this letter we showed that the differential equation for the four-loop equal-mass banana graph can be cast into an $\varepsilon$-factorised form, which allows for a systematic solution to any desired order in the dimensional regularisation parameter.

We used the mirror map to define new variables $\tau$ and $q$. The relevant Picard–Fuchs operator is no longer a symmetric product. We showed that this is not an essential complication, it merely introduces the function $K$.

Our calculation shows that the four-loop equal-mass banana integral is only minimally more complicated than the corresponding Feynman integrals at two or three loops. This is good news, as it opens a path towards Feynman integrals related to genuinely generic Calabi–Yau manifolds.

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The helper functions

In this appendix we give the explicit expressions for the helper functions $h_4$, $h_6$, $h_{8,a}$ and $h_{8,b}$:

\begin{align*}
\displaystyle h_4 &= \frac{5 + 14y - 569y^2 - 12044y^3 + 77427y^4 - 302850y^5 + 50625y^6}{2y(1+y)(1+9y)(1+25y)}\omega_1^2 J, \\
\displaystyle h_6 &= \left[\frac{7}{y} + 192y \left(\frac{1}{(1+y)^2} + \frac{64}{(1+9y)^2} - \frac{192}{(1+25y)^2}\right)\right]\omega_1^2, \\
\displaystyle h_{8,a} &= -\left((1+33y^2-225y^3)(3-25y-187y^2+225y^3)\right) \\
&\quad \frac{25045562 + 569y^2 - 12044y^3 + 77427y^4 - 302850y^5 + 50625y^6}{16y^3(1+y)^4(1+9y)^3(1+25y)^3}\omega_1^2 J, \\
\displaystyle h_{8,b} &= \left[\frac{h_4 J d^2\omega_1}{\omega_1^2 d y^2} - \left(\frac{d}{d y} \frac{h_4 J}{\omega_1^2}\right)\left(\frac{d\omega_1}{d y}\right) - \frac{X}{4y^3(1+y)^3(1+9y)^3(1+25y)^3}\omega_1\right]\omega_1 J, \\
\displaystyle X &= 10 + 953y + 36365y^2 + 624143y^3 + 5506553y^4 + 25045562y^5 - 43328414y^6 - 1595702658y^7 - 3663006612y^8 \\
&\quad + 4836275325y^9 + 29858270625y^{10} + 9146671875y^{11} + 2562890625y^{12}. \\
\end{align*}

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