The position-momentum symmetry principle

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It is shown that the Fourier transformation that relates position and momentum representations of quantum mechanics can be understood as a consequence of a symmetry principle that establishes the equivalence of being and becoming in the description of reality. There are however other transformation compatible with the same principle that could lead to different formalisms of quantum mechanics.

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I. INTRODUCTION

In the attempt to develop an understanding of quantum mechanics it is convenient to treat the simplest possible quantum system consisting in a structureless free particle moving in one dimension. This system has only two relevant observables: position and momentum. Their relation in classical mechanics has several degrees of abstraction: from the simplest definition of momentum as $p = mv$ to the relevant conjugate variables in the Legendre transformation of the Lagrangian in order to obtain the Hamiltonian. In any case, their conceptual meaning is related to the space-time location of matter and its movement or evolution: being and becoming.

In quantum mechanics these observables are less well understood. Not only they have the usual indetermination, characteristic of all quantum observables, but in addition they acquire a mysterious correlation that is essentially formalized in

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Heisenberg’s uncertainty principle: we have gotten used to it, but we don’t really understand it.

There are several features of the formalism of quantum mechanics concerning the observables of position and momentum that are closely related and are just different manifestations of some profound relation between these two fundamental observables. These are:

- The commutator between position and momentum is \([X, P] = i\) (we assume units with \(\hbar = 1\)).

- The momentum operator is the generator of translations because, for any position dependent operator \(F(X)\), we have \([F, P] = \frac{i dF}{dx}\) and therefore \(U^a F(X) U_a = F(X + a)\), where \(U_a = \exp(-iaP)\). In a similar way, position is the generator of impulsions.

- The internal product between the basis elements corresponding to position and momentum operators, \(\{\varphi_x\}\) and \(\{\phi_p\}\), is \(\langle \varphi_x, \phi_p \rangle = \frac{1}{\sqrt{2\pi}} \exp(ixp)\). This implies that position and momentum representations are related by the Fourier transformation. The fact that the bases are unbiased, that is, \(|\langle \varphi_x, \phi_p \rangle|\) is independent of \(x\) and \(p\), is a manifestation of the physical independence of position and momentum observables.

The three statements above are equivalent since from any of them one can prove the others and they contain all the essential relations between position and momentum. Therefore if we manage to develop an understanding or an interpretation of any of them we gain a deep insight in this central point of quantum mechanics. In this note we attempt to give a physical meaning to the third feature: we will find out why position and momentum representations are related by the Fourier transformation. For this we postulate a symmetry principle that establishes the equivalence of the being and becoming of a physical system. In particular, we propose that the roles of position and momentum can be interchanged or permuted.

**II. PHILOSOPHICAL ROOTS**

Being and becoming were the main issues in the opposing ontologies of Heraclitus and Parmenides in the fifth century before the Christian Era. For Heraclitus,
nothing in the world is constant; “everything flows” and the unique reality is the permanent change. Parmenides, on the contrary, claimed that all change is impossible because nothing can stop being what it is in order to start being what is not. For him all changes are a dilution of our senses: existence is timeless and unchanging. Accordingly, his disciple Zeno devised paradoxes tending to show that all motion is impossible. A somewhat conciliating view was adopted by Leucippus (that with his pupil Democritus developed the atomistic hypothesis) assuming that atoms are permanent but can change in their movement and combinations.

Apparently, in the antique Greek philosophy it was important to establish some sort of priority between the different views of reality as permanent and unchanging or as in a continuous change. For Heraclitus becoming has priority, Parmenides denies every change and Leucippus favours the being of atoms but does not deny their change. Today, the question of priority between being and becoming is not a problem that requires a solution: reality may well be a non contradicting, complementary, combination of being and becoming from which we perceive different perspectives, sometimes exhibiting the being and other times the becoming. Each perspective brings a complete, but not unique, view of reality and both perspectives are equivalent in the sense that from the being we can derive its becoming and vice-versa. This is precisely what we do in elementary calculus: from a function we can derive its changes (derivatives) and from the changes we can obtain the function (integration).

In this work we postulate a symmetry principle based in the equivalence of being and becoming: these are two different, but complementary, perspectives to approach reality. In the case of classical mechanics, this principle is formalized in a well known canonical transformation and therefore it does not bring anything really new, but in quantum mechanics it provides a novel understanding or explanation for the relation between position and momentum that is at the basis of all essential features of the theory.

III. BEING AND BECOMING IN MECHANICS

We can identify the two different philosophical considerations of reality –being and becoming– with two different perspectives in the analysis of a physical system: the space-time or the energy-momentum point of view. The being-becoming symme-
try principle means for mechanics the equivalence of description through space-time
or energy-momentum.

For the case of the simplest physical system consisting in a moving structureless
particle in one dimension, the description of the system by means of its time de-
pending coordinate \( x \) is equivalent to the corresponding description by means of its
momentum \( p \). The symmetry principle, in this simplest case, establishes then that
the roles of position and momentum can be interchanged.

In classical mechanics the symmetry principle is not very surprising. It is, in
fact, well known that there is a transformation \( x \rightarrow x' = p \) and \( p \rightarrow p' = -x \) that
leaves the Poisson brackets invariant. This canonical transformation is a special
case of a family of transformations that amount to a rotation of the phase space by
an arbitrary angle \( x \rightarrow x' = x \cos \theta + p \sin \theta \) and \( p \rightarrow p' = -x \sin \theta + p \cos \theta \). For
\( \theta = \frac{\pi}{2} \) we obtain the transformation mentioned before and for \( \theta = \pi \) we get the
parity transformation \( x \rightarrow -x, \quad p \rightarrow -p \).

We will now see that in quantum mechanics the symmetry principle has an in-
teresting consequence. The quantum mechanical predictions for position and mo-
mentum are given by the distribution of their eigenvalues \( \rho(x) \) and \( \varpi(p) \). These
are non-negative and normalized, in the sense that their integration is unity. They
are measured in an experiment by the frequency of appearance of each eigenvalue,
however, in all rigour, they are not probability distributions and a proper name for
them could be the existential weight for position and momentum [2]. Unfortunately,
the misnomer probability distribution is irreversibly installed in quantum mechanics.

One striking feature of quantum mechanics is that the complete knowledge of
both existential weights \( \rho(x) \) and \( \varpi(p) \) is not sufficient for an unambiguous de-
termination of the state of the system, although position and momentum are the
unique relevant observables. Therefore these distributions do not provide a complete
description of the system. This fact, known as the Pauli problem [3], has triggered
much activity in trying to establish sufficient conditions for state determination; a
problem still unsolved [4–9]. In order to fix the state of the system we need, besides
the distributions \( \rho(x) \) or \( \varpi(p) \), something more.

From the position point of view, besides the information contained in the exis-
tential weight \( \rho(x) \) we define another function \( \alpha(x) \) that somehow encodes all the
missing information about all other observables of the system (functions of position
and momentum). Furthermore, we can combine both functions in a single complex function $f(x)$ that is a candidate for the state of the system because it contains information on all relevant observables. A representation for the state of the system is then

$$f(x) = \sqrt{\rho(x)} \exp(i\alpha(x)). \quad (1)$$

In the same way, but from the momentum perspective, we obtain another representation for the state

$$g(p) = \sqrt{\omega(p)} \exp(i\beta(p)). \quad (2)$$

The squared roots in these definitions were introduced because it is convenient to consider these two normalized complex functions as members of the Hilbert space $L^2$ with unit norm: $\|f\|^2 = \langle f, f \rangle = 1$ and $\|g\|^2 = \langle g, g \rangle = 1$.

Both functions contain all relevant information about the system and therefore they are redundant. There must exist then an invertible operator $\mathcal{F}$ that relates them:

$$g = \mathcal{F} f. \quad (3)$$

We will now prove that the being-becoming symmetry principle implies some conditions on $\mathcal{F}$ that are satisfied by the Fourier transformation.

Let us assume then a state of the system given, redundantly, by $f(x)$ and $g(p)$. The symmetry principle under the transformation $x \rightarrow x' = p$ and $p \rightarrow p' = -x$ means that another possible state of the system is given by $f'(x) = g(x)$ and $g'(p) = f(-p) = \mathcal{P} f(p)$, where we have applied the parity operator $\mathcal{P}$ to perform the change of sign. As a consequence of this symmetry principle, $f$ and $g$ as well as $f'$ and $g'$ are related by the same operator $\mathcal{F}$. Therefore we have $g' = \mathcal{F} f' = \mathcal{F} g = \mathcal{F} \mathcal{F} f$.

Replacing $g' = \mathcal{P} f$ and considering that $f$ is arbitrary we conclude that

$$\mathcal{F}^2 = \mathcal{P}. \quad (4)$$

Now, since $\mathcal{P}$ is an involution, $\mathcal{P}^2 = 1$, the transformation $\mathcal{F}$ has period 4:

$$\mathcal{F}^4 = 1, \quad (5)$$

its eigenvalues are $\{1, -i, -1, i\}$ and the inverse is $\mathcal{F}^{-1} = \mathcal{F}^3$.

It is easy to check that the Fourier transformation

$$(\mathcal{F} f)(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \exp(-ipx)f(x). \quad (6)$$
satisfies all these requirements and therefore we can take the Fourier transformation for the operator \( \mathcal{F} \) that relates the position and momentum perspectives of the physical system. It is interesting to notice however that the Fourier transformation is not the unique transformation that satisfies the requirements of the proposed symmetry principle. In the appendix it is shown that there are in fact infinite many transformations \( \mathcal{K} \), different from the Fourier transformation \( \mathcal{F} \), having the same properties \( \mathcal{K}^2 = \mathcal{P} \) and \( \mathcal{K}^4 = 1 \).

IV. FINAL COMMENTS

We have seen that the symmetry principle that establishes an equivalence of position and momentum imposes some requirements on the formalism that are satisfied by the Fourier transformation between the position and momentum pictures or perspectives of the quantum system. In this sense, the Fourier transformation in quantum mechanics is related to the philosophical equivalence of being and becoming.

However, we have also seen that the Fourier transformation is not the unique one to have these properties and a natural question is if one can develop a different formalism of quantum mechanics based on some other transformation different from the Fourier one. In this case many questions arise: what predictions do these other theories make? are they the same as the conventional quantum mechanics? do all these other theories have some form of an uncertainty principle? what are the commutation relations of position and momentum for these other theories? etc.

In a different approach, one can try to find an additional physical principle that excludes all other transformation leaving the Fourier transformation as the unique one acceptable for quantum mechanics.

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V. APPENDIX

We prove here that there are infinite many transformations \( \mathcal{K} \), different from the Fourier transformation \( \mathcal{F} \), with the property \( \mathcal{K}^2 = \mathcal{P} \) and \( \mathcal{K}^4 = 1 \).

Let \( \{ \psi_n \} \ n = 0, 1, 2, \ldots \) be the orthonormal basis in \( \mathcal{L}_2 \) built with the Hermite
Functions
\[ \psi_n(x) = \exp\left(-\frac{x^2}{2}\right)H_n(x), \quad H_n(x) = (-1)^n \exp(x^2) \left(\frac{d}{dx}\right)^n \exp(-x^2). \quad (7) \]

These are the eigenvectors of the Fourier transformation operator:
\[ \mathfrak{F} \psi_n = (-i)^n \psi_n. \quad (8) \]

Let us consider a decomposition of \( L^2 \) in the invariant subspaces corresponding to the four degenerate eigenvalues \( \{1, -i, -1, i\} \), \( \mathcal{L}_2 = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \) where the subspaces \( \mathcal{H}_k \) \( k = 0, 1, 2, 3 \) are spanned by the subbases \( \{\psi_{4r+k}\} \) \( r = 0, 1, 2, \ldots \)

Notice that all functions in \( \mathcal{H}_0 \) and \( \mathcal{H}_2 \) are even whereas those in \( \mathcal{H}_1 \) and \( \mathcal{H}_3 \) are odd under the parity transformation \( \mathcal{P} \). Let \( P_k \) be the projectors in the subspaces \( \mathcal{H}_k \). With this, the Fourier operator has a spectral decomposition \( \mathfrak{F} = P_0 - iP_1 - P_2 + iP_3 \) and one can easily check that it has the properties \( \mathfrak{F}^2 = \mathcal{P} \) and \( \mathfrak{F}^4 = 1 \).

Now let us build another decomposition of the even subspace \( \mathcal{H}_0 \oplus \mathcal{H}_2 \) by choosing an arrangement of the basis elements different from \( \{\psi_{4r}\} \) \( r = 0, 1, 2, \ldots \) of \( \mathcal{H}_0 \) and \( \{\psi_{4r+2}\} \) \( r = 0, 1, 2, \ldots \) of \( \mathcal{H}_2 \). For instance, we can take randomly two disjoint sets of indices from \( \{0, 2, 4, 6, 8, 10, 12, \ldots \} \) in order to build the two sub bases for the decomposition \( \tilde{\mathcal{H}}_0 \oplus \tilde{\mathcal{H}}_2 \). There are infinite ways to chose the decomposition \( \tilde{\mathcal{H}}_0 \oplus \tilde{\mathcal{H}}_2 \) and similarly we make a different decomposition for the odd space \( \mathcal{H}_1 \oplus \mathcal{H}_3 \) resulting in \( \tilde{\mathcal{H}}_1 \oplus \tilde{\mathcal{H}}_3 \).

Let \( \tilde{P}_k \) be the projectors in the subspaces \( \tilde{\mathcal{H}}_k \). With this, we define an operator \( \mathfrak{R} = \tilde{P}_0 - i\tilde{P}_1 - \tilde{P}_2 + i\tilde{P}_3 \) that is clearly different from \( \tilde{\mathfrak{F}} \) because it has different invariant subspaces, and one can easily check that it has the properties \( \mathfrak{R}^2 = \mathcal{P} \) and \( \mathfrak{R}^4 = 1 \).

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