Reinforcement of a plate weakened by multiple holes with several patches for different types of plate-patch attachment

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Abstract

The most general situation of the reinforcement of a plate with multiple holes by several patches is considered. There is no restriction on the number and the location of the patches. Two types of patch attachment are considered: only along the boundary of the patch or both along the boundary of the patch and the boundaries of the holes which this patch covers. The unattached boundaries of the holes may be loaded with given in-plane stresses. The mechanical problem is reduced to a system of singular integral equations which can be further reduced to a system of Fredholm equations. A new numerical procedure for the solution of the system of singular integral equations is proposed in this paper. It is demonstrated on numerical examples that this procedure has advantages in the case of multiple patches and holes and allows achievement of better numerical convergence with less computational effort.

Keywords

Reinforcement, patch repair, complex potentials, singular integral equations.

1. Introduction

Patch repair is one of the most common techniques used to strengthen thin constructions containing imperfections such as cracks and holes. There is a significant amount of scientific literature dedicated to the topic with different types of plate-patch attachment being considered. In particular, the plate and the patches can be attached to each other using adhesive applied along parts of their surfaces (two-dimensional) or along some contours (one-dimensional), or only at certain points by using rivets. The detailed review of the literature on the bonded (two-dimensional) repair of the cracked plates can be seen in [1–3] and many others. Reinforcement of plates with holes is studied less extensively. Engels et al. [4] have considered a bonded repair of an anisotropic plate with a hole. Some problems on the reinforcement of a cracked plate with circular holes have been studied numerically in [5]. The patch repair of a single hole in a thin plate reinforced by a single patch is considered for two different types of plate-patch attachment in [6, 7]. To the author’s knowledge, a patch repair of multiple holes in a plate by multiple patches of arbitrary shapes has not been the subject of many studies, with an exception of [8].

In this paper, a reinforcement of a plate with multiple holes with several patches is considered for two types of plate-patch attachment: only along the boundary of the patch or along both the boundary of the patch and the boundaries of the holes covered by the patch. It is assumed that the patch is loaded with in-plane stresses applied to the free unattached boundaries of the holes and at the infinity point of the plate. The location, the
quantity and the shape of the patches can be arbitrary with the only condition that the boundaries of the patches and the holes are smooth and do not touch each other or intersect.

The complex analysis methods are used to solve the problem. First, Muskhelishvili’s formulas [9] are used to describe the stresses and the displacements in the plate and the patches through the complex potentials in the plate and the patches. The singular integral representations of the complex potentials are then used to reformulate the problem as a system of singular integral equations. These representations were first proposed by Savruk [10] to study the problems for cracked plates. The uniqueness of the solution of the resulting system of singular integral equations can be proved in a manner similar to [6–8].

An alternative technique for a numerical solution of the derived system of singular integral equations is proposed here. The method of mechanical quadratures applied for the solution of similar systems in [6–8] is difficult to implement for a large number of contours due to decreasing precision. The idea behind the method discussed here is in the approximation of the unknown functions by trigonometric polynomials. Substituting these approximations into the system of singular integral equations reduces this system to a system of linear algebraic equations for the coefficients of the trigonometric polynomials. Numerical examples show that even systems of relatively small order are sufficient to achieve good accuracy in the results. By considering numerical examples it can be shown that this method demonstrates a noticeably better convergence, larger independence from the symmetry of the construction and, hence, can be applied to a wider range of practical problems.

2. Boundary conditions

Consider an infinite thin elastic plate $S$ weakened by several holes with the boundaries $L_j, j = 1, \ldots, n + r$. Thin elastic patches $S_k, k = 1, \ldots, n + m$, with the boundaries $\Gamma_k$ are used to reinforce the plate $S$ (Figure 1). It is assumed that the patch $S_k, k = 1, \ldots, n$, covers the hole $L_k$ and is attached to the plate $S$ both along the boundary of the hole $L_k$ and the boundary of the patch $\Gamma_k$. Other patches $S_k, k = n + 1, \ldots, n + m$, are attached to the plate only along their boundaries $\Gamma_k$ and can be located anywhere in the plate $S$, in particular, they may cover some of the holes $L_j, j = n + 1, \ldots, n + r$. Given in-plane stresses $p_j(t)$ act on the free boundaries of the holes $L_j, j = n + 1, \ldots, n + r$. It is assumed that the lines $L_j$ and $\Gamma_k$ are smooth Lyapunov curves and do not touch or intersect each other.

The plate and the patches are homogeneous and isotropic. Their thicknesses, shear moduli and Poisson ratios are given by $h, \mu, \nu$ and $h_k, \mu_k, \nu_k, k = 1, \ldots, m + n$, correspondingly. The principal in-plane stresses $\sigma_1^{\infty}$ and $\sigma_2^{\infty}$ are applied at infinity of the plate and act in the directions constituting the angles $\alpha$ and $\alpha + \pi/2$ with the positive direction of the real axis.

Assume that the patches and the plate are joined perfectly along the junction lines. This means that the displacements are equal both in the patch and the plate from both sides of the junction lines and the stress equilibrium conditions are satisfied on the junction lines. These conditions together with the given stresses on
the free boundaries of the holes lead to the following set of the boundary conditions:

\[ (u + iv)^+(t) = (u + iv)^-\; q_k(t), \quad t \in L_k, \quad k = 1, \ldots, n, \]  
\[ h(\sigma_n + i\tau_n)^+(t) + h_k(\sigma_n + i\tau_n)^-\; q_k(t) = h_k(\sigma_n + i\tau_n)^-\; q(t), \]  
\[ t \in L_k, \quad k = 1, \ldots, n, \]

\[ (\sigma_n + i\tau_n)^+(t) = p(t), \quad t \in L_k, \quad k = n + 1, \ldots, n + r, \]  
\[ (u + iv)^+(t) = (u + iv)^-\; \tau(t), \quad t \in L_k, \quad k = 1, \ldots, n + m, \]

\[ h(\sigma_n + i\tau_n)^+(t) + h_k(\sigma_n + i\tau_n)^-\; \tau(t) = h(\sigma_n + i\tau_n)^-\; \tau(t), \]  
\[ t \in L_k, \quad k = 1, \ldots, n + m, \]

where \((u + iv)(t)\) is the vector of displacements at a point \(t\) of the plate \(S\) or the patch \(S_k\), and \(\sigma_n\) and \(\tau_n\) are the tensile and the shear components of the stress vector acting on the tangent line to the curves \(L_k\) or \(\Gamma_k\). Parameters without a subscript are related to the plate \(S\); parameters with a subscript ‘\(k\)’ are related to the patch \(S_k\). Here and henceforth, superscripts ‘+’ and ‘−’ denote the limit values of the stresses, the displacements and other parameters from the left-hand and the right-hand sides correspondingly of the lines \(L_k\) or \(\Gamma_k\). The direction of the curve \(L_k\) is chosen to be clockwise, and of the curve \(\Gamma_k\), counterclockwise (Figure 1).

### 3. Integral representations of complex potentials

The stresses and the derivatives of the displacements in the plate \(S\) or in the patch \(S_k, k = 1, \ldots, m + n\), on any contour \(L'\) lying in \(S\) or in \(S_k\) correspondingly can be obtained from Muskhelishvili’s complex potentials \(\Phi(z), \Psi(z)\) by the following formulas [9]

\[
(\sigma_n + i\tau_n)(t) = \Phi(t) + \overline{\Phi(t)} + \frac{dt}{dt}(i\Phi'(t) + \Psi(t)),
\]

\[
2\mu \frac{d}{dt}(u + iv)(t) = \kappa \Phi(t) - \overline{\Phi(t)} - \frac{dt}{dt}(i\Phi'(t) + \Psi(t)), \quad t \in L',
\]

where the overbar denotes complex conjugation and \(\overline{dt}/dt = e^{-2i\theta(t)}, \) where \(\theta(t)\) is the angle which the tangent vector to the curve \(L'\) makes with the positive direction of the real axis. For the patch \(S_k\), all the parameters and the functions in these formulas should be taken with the subscript ‘\(k\)’. In the case of plane stress take \(\kappa = (3 - v)/(1 + v)\) and \(\kappa_k = (3 - v)/(1 + v_k), \) \(k = 1, \ldots, m + n\).

We will derive the integral representations of the complex potentials in the spirit of [10]. The main idea is to divide the ‘plate-patches’ system into separate components and introduce the unknown functions describing the jumps of the stresses or the derivatives of the displacements on each of the lines \(L_j\) and \(\Gamma_k\). These unknown functions may be chosen in such a manner that some of conditions (2.1)–(2.5) are satisfied automatically.

First, consider the infinite plate \(S\) with the holes \(L_j\) which is subjected to the given in-plane stresses at infinity and on the lines \(L_j, j = n + 1, \ldots, n + r\). The stresses in the plate \(S\) have a jump discontinuity on the curve \(\gamma = \bigcup_{k=1}^{n+r} \Gamma_k\) lying in the interior of the plate, and the displacements are continuous on these curves:

\[
q(t) = \frac{(\sigma_n + i\tau_n)^+(t) - (\sigma_n + i\tau_n)^-\; \tau(t)}{2}, \quad t \in \gamma.
\]

Formally extend the plate \(S\) to the full complex plane so that the stresses are continuous through the curve \(L = \bigcup_{j=1}^{n+r} L_j\) and the derivatives of the displacements have a jump discontinuity along this line:

\[
g(t) = \frac{2\mu}{i(\kappa + 1)} \frac{d}{dt} \left( (u + iv)^+(t) - (u + iv)^-\; \tau(t) \right), \quad t \in L.
\]

Then the complex potentials \(\Phi(z)\) and \(\Psi(z)\) can be taken in the form [10]

\[
\Phi(z) = \Gamma - \sum_{k=n+1}^{n+r} \frac{Q_k}{z - z_k} + \frac{1}{2\pi} \int_L \frac{g(t)\; dt}{t - z} + \frac{(\kappa + 1)^{-1}}{\pi i} \int_\gamma \frac{q(t)\; dt}{t - z},
\]
Therefore, the complex potentials 

$$\Psi(z) = \Gamma' + \sum_{k=n+1}^{n+r} \frac{\kappa}{z - z_k} + \frac{1}{2\pi} \int_{L_k} \left( \frac{g(t) - \bar{g}(t)}{t - z} \right) \frac{dt}{(t - z)^2} \quad (3.4)$$

$$+ \frac{(\kappa + 1)^{-1}}{\pi i} \int_{\gamma} \left( \frac{\kappa g(t) - \bar{g}(t)}{t - z} \right) \frac{dt}{(t - z)^2}, \quad z \in S,$$

$$\Gamma = (\sigma_1^\infty + \sigma_2^\infty)/4, \quad \Gamma' = (\sigma_2^\infty - \sigma_1^\infty)e^{2\pi i}/2,$$

$$Q_k = \frac{X_k + iY_k}{2\pi(1 + \kappa)}, \quad X_k + iY_k = -i \int_{L_k} p_k(t) dt,$$

where $z_k, k = n + 1, \ldots, n + r$, is the arbitrarily fixed point inside of the contour $L_k$, and $g'(t)$ and $q(t)$ are the unknown functions which can be represented through the jump of the vector of displacements $u + iv$ on the line $L$ and the jump of the vector of stresses $\sigma_n + i\tau_n$ on the line $\Gamma$ by formulas (3.2), (3.3).

According to condition (2.5) and formula (3.2) we have

$$(\sigma_n + i\tau_n)^+(t) = -2d_k^{-1}q(t), \quad t \in \Gamma_k, \quad d_k = h_k/h.$$ 

Hence, the stressed state of the ‘plate-patches’ system is described by the complex potentials (3.4), (3.7) which contain unknown functions $g'(t), t \in L_j, j = 1, \ldots, n + r; q(t), t \in L_j, j = 1, \ldots, n; g'(t), t \in \Gamma_k, k = 1, \ldots, n + m$. We will look for these functions in the class of functions satisfying the Hölder condition on the corresponding curves $L_j$ and $L_k$. This choice guarantees the existence of all the principal and limit values of the integrals of the Cauchy type in formulas (3.4), (3.7).
4. The system of singular integral equations

To find the unknown functions \( g'(t), q_k(t), q(t), g_j(t) \), we have conditions (2.1)–(2.4) and (3.6). Recall that condition (2.5) has been used to derive representations (3.7) and thus is satisfied automatically. Observe also that condition (3.6) appears from the way we extend each patch \( S_k \) to the full complex plane (that is, it guarantees that the displacements and the stresses outside of the patch \( S_k \) are equal to zero).

From formulas (3.1) and representations (3.4), (3.7), satisfying conditions (2.1)–(2.4) and (3.6), we obtain the system of singular integral equations on the closed curves \( L_j, j = 1, \ldots, n + r \), and \( \Gamma_k, k = 1, \ldots, n + m \), with the unknown functions \( g'(t), q_k(t), q(t), g_j(t) \):

\[
\frac{\mu_k}{\mu} \left[ \kappa \Gamma - \tilde{\Gamma} - \tilde{\Gamma}' \frac{dt}{dt} - \sum_{j=n+1}^{n+r} \left( \frac{\kappa Q_j}{\tau - z_j} - \frac{\tilde{Q}_j}{\tau - \bar{z}_j} + \frac{\kappa Q_j}{\bar{\tau} - z_j} + \frac{\tilde{Q}_j}{\bar{\tau} - \bar{z}_j} \right) \right]
+ \frac{i(\kappa + 1)}{2} g'(t) + \frac{1}{2\pi} \int_{L} \left( \kappa \frac{dt}{\tau - t} - \frac{1}{\bar{\tau} - t} \frac{dt}{dt} \right) g'(\tau) d\tau
+ \frac{1}{2\pi} \int_{L} \left( -\frac{1}{\bar{\tau} - t} + \frac{\tau - t}{(\bar{\tau} - t)^2} \frac{dt}{dt} \right) g'(\tau) d\tau
+ \frac{1}{\pi i} \int_{\gamma} \left( \frac{\kappa}{\tau - t} + \frac{\kappa}{\bar{\tau} - t} \right) q(\tau) d\tau
+ \frac{1}{\pi i} \int_{\gamma} \left( \frac{1}{\tau - t} - \frac{\tau - t}{(\tau - t)^2} \frac{dt}{dt} \right) q(\tau) d\tau
+ \frac{1}{\pi i} \int_{L_k} \left( \frac{\kappa_k}{\tau - t} + \frac{\kappa_k}{\bar{\tau} - t} \right) q_k(\tau) d\tau
+ \frac{1}{\pi i} \int_{L_k} \left( \frac{1}{\tau - t} - \frac{\tau - t}{(\tau - t)^2} \frac{dt}{dt} \right) q_k(\tau) d\tau
+ \frac{1}{2\pi} \int_{\Gamma_k} \left( \frac{\kappa_k}{\tau - t} - \frac{1}{\bar{\tau} - t} \frac{dt}{dt} \right) g_k'(\tau) d\tau
+ \frac{1}{2\pi} \int_{\Gamma_k} \left( -\frac{1}{\bar{\tau} - t} + \frac{\tau - t}{(\bar{\tau} - t)^2} \frac{dt}{dt} \right) g_k'(\tau) d\tau
+ \frac{id_k^{-1}}{\pi (\kappa_k + 1)} \int_{\Gamma_k} \left( \frac{\kappa_k}{\tau - t} + \frac{\kappa_k}{\bar{\tau} - t} \right) q(\tau) d\tau
+ \frac{id_k^{-1}}{\pi (\kappa_k + 1)} \int_{\Gamma_k} \left( \frac{1}{\tau - t} - \frac{\tau - t}{(\tau - t)^2} \frac{dt}{dt} \right) q(\tau) d\tau
\]
\[
2 \delta_k q_k(t) + \frac{1}{2\pi} \int_{L} \left( \frac{\tau - t}{\tau - t} + \frac{1}{\bar{\tau} - t} \frac{dt}{dt} \right) g'(\tau) d\tau
+ \frac{1}{2\pi} \int_{L} \left( \frac{1}{\tau - t} - \frac{\tau - t}{(\tau - t)^2} \frac{dt}{dt} \right) g'(\tau) d\tau
+ \frac{1}{\pi i} \int_{\gamma} \left( \frac{1}{\tau - t} - \frac{\kappa}{\bar{\tau} - t} \right) q(\tau) d\tau
\]
\begin{align*}
\frac{\kappa + 1}{\pi i} \int_Y \left( -\frac{1}{\bar{z} - \bar{t}} + \frac{\tau - t}{(\bar{z} - \bar{t})^2} \right) \overline{q(\tau)} \, d\tau \\
- 2 \operatorname{Re} \Gamma - \frac{\partial \overline{Q_j}}{\partial t} + \sum_{j=n+1}^{n+r} \left( 2 \operatorname{Re} \left( \frac{Q_j}{i - \bar{z}_j} \right) - \frac{\partial \overline{Q_j}}{\partial t} \left( \frac{\kappa Q_j}{i - \bar{z}_j} + \frac{i \overline{Q_j}}{(i - \bar{z}_j)^2} \right) \right),
\end{align*}
\quad t \in L_k, \quad k = 1, \ldots, n;
\begin{align*}
\frac{1}{\pi} \frac{|dt|}{dt} \int_{\Gamma_k} g'(\tau) \, d\tau + \frac{1}{2\pi} \int_L \left( \frac{1}{\tau - t} + \frac{1}{\bar{\tau} - \bar{t}} \right) g'(\tau) \, d\tau \\
+ \frac{1}{2\pi} \int_L \left( \frac{1}{\bar{\tau} - \bar{t}} - \frac{\kappa \overline{d_t}}{(\bar{\tau} - \bar{t})^2} \right) g'(\tau) \, d\tau \\
+ \frac{(\kappa + 1)^{-1}}{\pi i} \int_Y \left( \frac{1}{\tau - t} + \frac{\kappa \overline{d_t}}{(\tau - t)^2} \right) q(\tau) \, d\tau \\
+ \frac{(\kappa + 1)^{-1}}{\pi i} \int_Y \left( \frac{1}{\bar{\tau} - \bar{t}} + \frac{\kappa \overline{d_t}}{(\bar{\tau} - \bar{t})^2} \right) q(\tau) \, d\tau \\
- \frac{\mu_k}{\mu} \left[ \kappa \Gamma - \bar{\Gamma} - \bar{\Gamma} \frac{\partial \overline{Q_j}}{\partial t} - \sum_{j=n+1}^{n+r} \left( \frac{\kappa Q_j}{t - \bar{z}_j} - \frac{\overline{Q_j}}{i - \bar{z}_j} + \frac{\partial \overline{Q_j}}{\partial t} \left( \frac{\kappa Q_j}{i - \bar{z}_j} + \frac{i \overline{Q_j}}{(i - \bar{z}_j)^2} \right) \right) \\
+ \frac{1}{\pi} \frac{|dt|}{dt} \int_{\Gamma_k} q(\tau) \, d\tau + \frac{1}{2\pi} \int_L \left( \frac{\kappa}{\tau - t} - \frac{1}{\bar{\tau} - \bar{t}} \right) g'(\tau) \, d\tau \\
+ \frac{1}{2\pi} \int_L \left( \frac{1}{\bar{\tau} - \bar{t}} + \frac{\tau - t}{(\bar{\tau} - \bar{t})^2} \right) g'(\tau) \, d\tau \\
+ \frac{(\kappa + 1)^{-1}}{\pi i} \int_Y \left( \frac{1}{\tau - t} - \frac{\tau - t}{(\tau - t)^2} \right) q(\tau) \, d\tau \\
+ \frac{(\kappa + 1)^{-1}}{\pi i} \int_Y \left( \frac{1}{\bar{\tau} - \bar{t}} - \frac{\tau - t}{(\bar{\tau} - \bar{t})^2} \right) q(\tau) \, d\tau \right] \\
= \frac{i(\kappa_k + 1)}{2} g_k'(t) + \frac{(\kappa_k + 1)^{-1}}{\pi i} \int_{L_k} \left( \frac{\kappa_k}{\tau - t} + \frac{\kappa_k \overline{d_t}}{i - \bar{t}} \right) q_k(\tau) \, d\tau \\
+ \frac{(\kappa_k + 1)^{-1}}{\pi i} \int_{L_k} \left( \frac{1}{\tau - t} - \frac{\tau - t}{(\tau - t)^2} \right) q_k(\tau) \, d\tau \\
+ \frac{1}{2\pi} \int_{\Gamma_k} \left( \frac{\kappa_k}{\tau - t} + \frac{1}{\bar{\tau} - \bar{t}} \right) g_k'(\tau) \, d\tau \\
+ \frac{1}{2\pi} \int_{\Gamma_k} \left( \frac{1}{\bar{\tau} - \bar{t}} + \frac{\tau - t}{(\bar{\tau} - \bar{t})^2} \right) g_k'(\tau) \, d\tau.
\end{align*}
where \(q_k(t) = 0, k = n + 1, \ldots, n + m\), and \(d_k = h_k/h, k = 1, \ldots, n + m\).

Observe that the additional terms
\[
\int \left( \frac{1}{\tau - t} - \frac{\tau - t}{(\tau - t)^2} \right) q(\tau) d\tau = i(\kappa_k + 1)g^*_k(t),
\]
where \(t \in \Gamma_k, \ k = 1, \ldots, n + m\).

and
\[
\int q(\tau) d\tau = 0, \ k = 1, \ldots, n + m,
\]

have been included in equations (4.3) and (4.4) correspondingly. Adding these terms provides that the following conditions are satisfied:
\[
\int_{L_k} g'(\tau) d\tau = 0, \ k = n + 1, \ldots, n + r,
\]
and
\[
\int_{\Gamma_k} q(\tau) d\tau = 0, \ k = 1, \ldots, n + m,
\]

and does not change conditions (2.1)–(2.4) and (3.6). To show this fact it is sufficient to integrate equations (4.1)–(4.4) over the corresponding contours \(L_k\) and \(\Gamma_k\). After that, substituting conditions (4.5), (4.6) back into the system (4.1)–(4.4) reduces the system to the original boundary conditions (2.1)–(2.4) and (3.6).

From the physical viewpoint, condition (4.5) means that the displacements are single-valued in the plate along the contours \(L_k, k = n + 1, \ldots, n + r\), while condition (4.6) provides that the resultant force applied to the patch \(S_k\) is equal to zero [6, 8].

It can be shown that the system (4.1)–(4.4) has a unique solution. To achieve this goal the system (4.1)–(4.4) can be rewritten in the following form:
\[
\frac{1}{\pi} \left( \frac{\mu_k d_k (\kappa + 1)}{2 \mu} + \frac{2 \kappa_k}{\kappa_k + 1} \right) \int_{L_k} q_k(\tau) d\tau = \frac{\mu_k (\kappa - 1)}{2 \pi \mu} \int_{L_k} g'(\tau) d\tau,
\]
\[
d_k q_k(t) + \int_{L_k} g'(\tau) d\tau = K_{2k}(g', \bar{g}', q, \bar{q}, q_k, \bar{q}_k, g_k, \bar{g}_k) + f_{1k}(t), \ t \in L_k, \ k = 1, \ldots, n;
\]
\[
\frac{1}{\pi} \int_{L_k} g'(\tau) d\tau = K_{3k}(g', \bar{g}', q, \bar{q}) + f_{3k}(t), \ t \in L_k, \ k = n + 1, \ldots, n + r;
\]
\[
\frac{2}{\pi i} \left( \frac{\mu_k \kappa}{\mu (\kappa + 1)} + \frac{\kappa_k d_k^{-1}}{\kappa_k + 1} \right) \int_{\Gamma_k} q(\tau) d\tau = \frac{\kappa_k - 1}{2 \pi} \int_{\Gamma_k} g'_k(\tau) d\tau,
\]
\[
\frac{1}{\pi i} \left( \frac{\mu_k}{\mu (\kappa + 1)} + \frac{\kappa_k d_k^{-1}}{\kappa_k + 1} \right) \int_{\Gamma_k} q(\tau) d\tau = \frac{2 \mu_k \kappa}{\pi i \mu (\kappa + 1)} \int_{\Gamma_k} g'_k(\tau) d\tau + f_{4k}(t), \ t \in \Gamma_k, \ k = 1, 2, \ldots, n + m;
\]
\[
= K_{4k}(g', \bar{g}', q, \bar{q}) + f_{4k}(t), \ t \in \Gamma_k, \ k = 1, 2, \ldots, n + m,
\]

where \(K_{1k}, K_{2k}, \ldots, K_{5k}\), are Fredholm operators of the first kind, and \(f_{1k}(t), f_{2k}(t), \ldots, f_{5k}(t)\) are given functions.

Since the singular parts of equations (4.7) have indices equal to zero, it can be seen that the system can be equivalently reduced to the system of Fredholm equations [11]. From here, using the uniqueness of the solution of the mechanical problem which has been proved for one hole and one patch in [6], it can be seen that the solution of the system (4.1)–(4.4) exists and is unique.
5. Numerical procedure

In this section we present an alternative technique for solving the system of singular integral equations (4.1)–(4.4). The method of mechanical quadratures presented to study the problems for cracked plates in [10] and adopted for plates with holes reinforced by patches in [6–8] provides sufficient accuracy with a reasonable computational time for the systems of similar type in the cases where the number of contours \( L_j \) and \( \Gamma_k \) is small. However, the results become unsatisfactory in the sense of both accuracy and computational time if a larger number of contours is considered. Another observed disadvantage of the method of mechanical quadratures is that the accuracy of the results obtained using this method tends to deteriorate significantly with the loss of the symmetry in the construction.

To overcome these disadvantages the following approach is proposed and shown to be computationally effective. First, parametrize each curve \( L_j \) and \( \Gamma_k \) with a parameter \( \theta \in [0, 2\pi] \). Let \( \tau_j = \tau_j(\theta) \) be a parametrization of the curves \( L_j \), and \( \xi_k = \xi_k(\theta) \) be a parametrization of the curves \( \Gamma_k \). Next, approximate the unknown functions \( g'(t) \), \( q_k(t) \), \( t \in L_j \), and \( q(t) \), \( g'_k(t), t \in \Gamma_k \), by the partial sums of the complex Fourier series:

\[
g'(t) = \sum_{m=-N}^{N} g_{0m} e^{im\theta}, \quad t \in L_j, \quad j = 1, \ldots, n + r;
\]

\[
q_j(t) = \sum_{m=-N}^{N} q_{jm} e^{im\theta}, \quad t \in L_j, \quad j = 1, \ldots, n;
\]

\[
q(t) = \sum_{m=-N}^{N} q_{m} e^{im\theta}, \quad t \in \Gamma_j, \quad j = 1, \ldots, n + m;
\]

\[
g'_k(t) = \sum_{m=-N}^{N} g'_{km} e^{im\theta}, \quad t \in \Gamma_j, \quad j = 1, \ldots, n + m,
\]

where \( g_{0m}, q_{jm}, q_{0m}, g'_{km} \) are the unknown complex coefficients of the partial sums (5.1) which need to be found.

Substituting formulas (5.1) into the system (4.1)–(4.4) leads to the following equations:

\[
\sum_{k=1}^{n+r} \left( \sum_{m=-N}^{N} g_{0m} A_{m}^{k} + \sum_{m=-N}^{N} \overline{g_{0m}} B_{m}^{k} \right) + \sum_{k=1}^{n+m} \left( \sum_{m=-N}^{N} q_{0m} A_{m}^{k} \overline{f_{j}} + \sum_{m=-N}^{N} \overline{q_{0m}} B_{m}^{k} \overline{f_{j}} \right) + \sum_{k=1}^{n} \left( \sum_{m=-N}^{N} q_{m} A_{m}^{k+2n+m+r} + \sum_{m=-N}^{N} \overline{q_{m}} B_{m}^{k+2n+m+r} \right) + \sum_{k=1}^{n+m} \left( \sum_{m=-N}^{N} g'_{m} A_{m}^{k+3n+m+r} + \sum_{m=-N}^{N} \overline{g'_{m}} B_{m}^{k+3n+m+r} \right) = f_{j}(\theta),
\]

\( \theta \in [0, 2\pi], \quad j = 1, \ldots, 4n + 2m + r, \)

where \( N \) is the degree of the trigonometric polynomials (5.1), and the functions \( A_{m}^{k} (\theta), B_{m}^{k} (\theta) \) and \( f_{j}(\theta) \) can be obtained from the system (4.1)–(4.4). The integrals in these functions can be computed numerically by using any appropriate method.

Equations (5.2), in general, cannot be satisfied for all the values of the parameter \( \theta \) due to the approximation of the unknown functions by the truncated Fourier series (5.1). Instead, we assume that equations (5.2) are satisfied only in the finite number of points \( \theta_{s} = \pi (2s - 1)/(2N + 1), \quad s = 1, \ldots, 2N + 1 \). This transforms (5.2) into the system of linear algebraic equations with real coefficients with \( 2(2N+1)(4n+2m+r) \) unknowns. Thus,
the approximate solution of the system of singular integral equations (4.1)–(4.4) can be obtained by solving the system of linear algebraic equations.

While the precise investigation of the convergence of the solution of the resulting system of linear algebraic equations to the solution of the system of singular integral equations (4.1)–(4.4) is outside of the scope of this paper, the numerical results obtained by using this method and also comparison with the numerical results obtained by the method of mechanical quadratures in [8] suggests that even a relatively small number of terms \( N \) (\( N \leq 20 \) in most cases) is sufficient to obtain a satisfactory convergence, thus reducing the problem to the solution of a relatively small system of linear algebraic equations.

For instance, consider a plate with two holes \( L_1 \) and \( L_2 \) in the shape of the squares with rounded corners given by the equations \( \tau_{1,2}(\theta) = 0.45(e^{i\theta} + e^{-3i\theta}/9) \mp 1 \), which is reinforced by two patches in the shape of the squares with rounded corners with the boundaries \( \Gamma_1 \) and \( \Gamma_2 \) given by the equations \( \xi_{1,2}(\theta) = 0.7(e^{i\theta} + e^{-3i\theta}/14) \mp 1 \) (Figure 2). Assume that the patches are attached to the plate along both contours \( L_1, \Gamma_1 \) and \( L_2, \Gamma_2 \) correspondingly. The plate is loaded at infinity by the tensile stress \( \sigma_\infty \) acting in the direction constituting the angle \( \alpha = \pi/4 \) with the positive direction of the \( x \)-axis, and \( \sigma_\infty^2 = 0 \). The mechanical parameters of the plate and of the patches are \( \mu = 60, \nu = 0.4 \) and \( \mu_1 = \mu_2 = 40, \nu_1 = \nu_2 = 0.3 \) correspondingly. The thicknesses of the patches and of the plate are equal: \( h = h_1 = h_2 \). Figure 3 shows the approximations of the real (solid line) and imaginary (dashed line) parts of the unknown function \( g'_1(\tau_1(\theta)) \) plotted for different numbers \( N \) of terms of the trigonometric polynomials (5.1): \( N = 10, N = 15 \) and \( N = 20 \). It can be seen that a good approximation is already obtained for \( N = 15 \). Similar results have been observed in other considered examples. The comparison of the results with the method of mechanical quadratures [8] is presented in Figure 4. The computation is made for the same mechanical and geometric parameters of the plate and the patches as in Figure 2, and \( \sigma_\infty = 1, \sigma_\infty^2 = 0, \alpha = 0 \). The patches are joined with the plate only along their boundaries \( \Gamma_1 \) and \( \Gamma_2 \). The solid-diamond line corresponds to the results obtained by the method of trigonometric polynomials with \( N = 20 \) presented.
Figure 4. The comparison of the approximation of the function $\text{Im} g'_1(\tau_1(\theta))$ computed by the method of trigonometric polynomials and the method of mechanical quadratures [8].

Figure 5. The stresses on the lines $L_1$ and $\Gamma_1$ in the case of two square holes and two square patches.

here, and the dash-dot line corresponds to the method of mechanical quadratures with 100 points on each of the lines [8]. It can be seen from Figure 4 that the results are in good agreement with each other. At the same time it needs to be stressed that the method of mechanical quadratures tends to lose its efficiency for a larger number of contours and that is why alternative numerical techniques should be used.

Figure 5 shows the graphs of the tensile and the shear stresses $\sigma_n$ and $\tau_n$ on the boundaries of the left hole $L_1$ and the left patch $\Gamma_1$ for the construction shown in Figure 2. The plate is loaded at infinity by the tensile stress $\sigma_{\infty}^1 = 1$ acting in the direction constituting the angle $\alpha = \pi/4$ with the positive direction of the real axis, and $\sigma_{\infty}^2 = 0$. The mechanical parameters of the plate and the patches are the same as in the previous example. Here and further, on the hole boundary $L_j$, the graphs given by the solid line represent the stresses in the plate, the graphs given by the dashed line represent the stresses in the patch from the outside of the hole boundary $L_j$, and the graphs given by the dash-dot line represent the stresses in the patch from the inside of the hole boundary $L_j$. Similarly, on the patch boundary $\Gamma_j$, the graphs given by the solid line represent the stresses in the plate from
the inside of $\Gamma_j$, the graphs given by the dashed line show the stresses in the plate from the outside of $\Gamma_j$, and finally, the graphs given by the dash-dot line represent stresses in the patch.

The graphs of the dependence of the maximums of the stresses $\sigma_n$ and $\tau_n$ on the lines $L_1$ and $\Gamma_1$ on the patch size characterized by the parameter $r$ are shown in Figure 6. The parameter $r$ here is the radius of the circle circumscribed around each square patch. The proportions of the patch stay the same as in the previous example and only the size of the patch changes. The graphs on the lines $L_2$ and $\Gamma_2$ are the same due to the symmetry of the construction. It can be seen from the graphs that while the maximums of the stresses on the hole boundary $L_1$ decrease as the size of the patch increases, the maximums on the patch boundary $\Gamma_1$ increase with $r$.

Consider the same construction as in Figure 2 where the left patch $S_1$ is attached to the plate $S$ along both lines $L_1$ and $\Gamma_1$ while the right patch $S_2$ is attached only along its boundary $\Gamma_2$. All other mechanical and geometrical parameters are the same as in the previous example. The plate $S$ is loaded at infinity with the tensile stress $\sigma_1^\infty = 1$ acting in the direction constituting the angle $\alpha$ with the positive direction of the real axis. The dependence of the maximums of the stresses $\sigma_n$ and $\tau_n$ on the line $L_1$ on the direction of the loading given by the angle $\alpha$ is shown in Figure 7. Observe that the stresses $\sigma_n$ and $\tau_n$ are equal to zero on the line $L_2$. The graphs of the stresses on the lines $\Gamma_1$ and $\Gamma_2$ are shown in Figure 8. It can be seen that the direction of loading makes a significant impact on the maximums of the stresses and, hence, on the possibility of the failure of the construction. Thus, each potential repair should be made with consideration for the direction of the loading which is applied to the construction.
Figure 8. The dependence of the maximal stresses on the lines $\Gamma_1$ and $\Gamma_2$ on the angle $\alpha$.

Figure 9. A plate with three circular holes reinforced by three square patches.

Figure 9 shows a plate with three circular holes each of radius $R = 0.5$ and reinforced with three square patches $S_1, S_2, S_3$ with rounded corners. The centers of the holes and the corresponding patches are located at the points $z_1 = -1 - i$, $z_2 = 1 - i$ and $z_3 = -1 + i$. The boundaries of the patches are given by the equations $\xi_j(\theta) = 0.675 \cdot e^{i(b - \pi/4)}(e^{i\theta} + e^{-3i\theta}/9) + z_j$, $j = 1, 2, 3$. The patches are attached to the plate both along their boundaries and the boundaries of the holes they cover. The angle $\beta$ here represents the orientation of the patches (Figure 9). The plate is loaded at infinity with a normal stress $\sigma_{1}^\infty = 1$ acting in the direction of the real axis ($\alpha = 0$). The mechanical parameters of the plate and of the patches are $\mu = 60$, $\nu = 0.4$ and $\mu_1 = \mu_2 = \mu_3 = 40$, $v_1 = v_2 = v_3 = 0.3$, $h = h_1 = h_2 = h_3$ correspondingly.
The graphs of the dependence of the maximal stresses $\sigma_n$ and $\tau_n$ on the lines $L_1, L_2$ and $L_3$ on the orientation of the patches $\beta$ are shown in Figure 10. The corresponding graphs on the lines $\Gamma_1, \Gamma_2$ and $\Gamma_3$ are shown in Figure 11. It can be seen that the maximums of the stresses in the plate and patches depend strongly on the orientation of the patches with respect to the loading applied. The minimal stresses are achieved at $\beta \approx 0^\circ$, and the maximal stresses are achieved at $\beta \approx 60^\circ$. There is a noticeable difference in the magnitude of the stresses acting in the construction depending on the orientation of the patches, which means that an improper repair can increase the stresses even compared to the unreinforced case and can potentially lead to failure of the construction. Thus each potential repair should be carefully evaluated in order to reduce the maximal stresses in the construction.
The numerical method proposed here for the solution of the systems of singular integral equations of the type (4.1)–(4.4) can be easily extended to more general constructions. It also has been observed to provide more stable numerical results for a wider range of geometries of the holes and patches, compared to the method of mechanical quadratures [6–8], to be less dependent on the symmetry of the construction and to be more computationally efficient.

6. Conclusions
In this paper we extended a recently suggested technique for solving the problem of an infinite elastic plate containing several holes of arbitrary shapes reinforced by multiple patches of arbitrary shapes to the case where each patch can be attached to the plate by one of two methods: either only along the boundary of the patch or both along the boundary of the patch and the boundaries of the holes covered by the patch. The problem is reduced to a system of singular integral equations. A new and more efficient numerical approach is proposed for numerical solution of this system. The numerical examples for some particular cases of the plate and patch geometries are given. Obvious future developments of the method are the implementation of finite boundaries of the plate and inhomogeneously imperfect interfaces, and also the optimization of the patch repair in accordance with a given plate geometry and an applied loading.

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Conflict of interest
None declared.

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