Sauter-Schwinger effect for colliding laser pulses

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Via a combination of analytical and numerical methods, we study electron-positron pair creation by the electromagnetic field $A(t, r) = [f(t - x) + f(t + x)]e_y$ of two colliding laser pulses. Employing a generalized WKB approach, we find that the pair creation rate along the symmetry plane $x = 0$ (where one would expect the maximum contribution) displays the same exponential dependence as for a purely time-dependent electric field $A(t) = 2f(ct)e_y$. The pre-factor in front of this exponential does also contain corrections due to focusing or de-focusing effects induced by the spatially inhomogeneous magnetic field. We compare our analytical results to numerical simulations using the Dirac-Heisenberg-Wigner method and find good agreement.

Introduction As one of the most striking and fundamental predictions of quantum electrodynamics (QED), the vacuum should become unstable in the presence of strong electric fields, leading to the spontaneous creation of electron-positron pairs (“matter from light”) [1] [2]. For a constant electric field $E$, the pair-creation probability $P$ displays an exponential dependence ($\hbar = c = 1$)

$$ P \sim \exp \left\{ -\pi \frac{m^2}{qE} \right\} = \exp \left\{ -\pi \frac{E_s}{E} \right\}, $$

(1)

with the electron mass $m$ and elementary charge $q$, which can be combined to yield the Schwinger critical field $E_s = m^2/q \approx 1.3 \times 10^{18}$ V/m. The above functional dependence does not admit a Taylor expansion in $q$ which indicates that this Sauter-Schwinger effect [1] is a non-perturbative phenomenon [3] [4]. As a result, the corresponding calculations can be quite non-trivial and our knowledge beyond the case of constant fields is very limited [5] [6]. For slowly varying fields, we may apply the locally constant field approximation by evaluating Eq. (1), together with its generalization to additional magnetic fields, at each space-time point [2]. However, this approximation has a limited range of applicability and does not capture many important effects, such as the dynamically assisted Sauter-Schwinger effect [7] [11].

From a fundamental point of view as well as in anticipation of experimental initiatives aiming at ultra-high field strengths [12], it is important to better understand the Sauter-Schwinger effect for non-constant fields [13] [14]. While there has been progress regarding fields which depend on one coordinate (e.g., space $x$ or time $t$ [15] [16], or a light-cone variable $t - x$ [17] [18]), our understanding of more complex field dependences, e.g., the interplay between spatial and temporal variations, is still in its infancy [2] [20]. Furthermore, going from simple models towards realistic field configurations requires the consideration of transversal fields which are vacuum solutions of the Maxwell equations. In the following, we venture a step into this direction by employing a combination of analytical and numerical methods.

The model In order to treat a potentially realistic yet simple field configuration, we consider the head-on collision of two equal plane-wave laser pulses, see also [21]

$$ A(t, r) = [f(t - x) + f(t + x)]e_y. $$

(2)

For asymmetric collision scenarios, see, e.g., [22] [25]. This vector potential (2) is an even function of $x$, i.e., $A_y(t, x) = A_y(t, -x)$ such that $\partial_x A_y(t, x) = 0$. Thus, along the symmetry plane $x = 0$, the electric field components $E_y$ add up while the magnetic fields $B_z$ of the two pulses cancel each other. As a result, one would expect the maximum contribution to pair creation there.

In the following, we assume that the typical frequency scale $\omega$ describing the rate of change of the function $f(t)$ is sub-critical, i.e., much smaller than the electron mass $\omega \ll m$. The characteristic electric field strength $E$ should also be sub-critical $E \ll E_s$ and the Keldysh parameter (or inverse laser parameter $1/a_0$) [26] [28]

$$ \gamma = \frac{m \omega}{qE} = \frac{1}{a_0}, $$

(3)

should be roughly of order unity such that $qE = \mathcal{O}(\omega m)$.

WKB approach In this limit, where the electron mass $m$ is the largest scale, we may employ semi-classical methods such as world-line instantons [1] [4] discussed in Section A of the Supplemental Material [32] or the WKB approach [33] [34] used here. For simplicity and because spin effects are not expected to play a major role here, we start from the Klein-Fock-Gordon equation

$$ [\partial_\mu + iqA_\mu)(\partial^\mu + iqA^\mu) - m^2] \phi = 0. $$

(4)

Via the standard WKB ansatz [35]

$$ \phi(t, x, y, z) = \alpha(t, x)e^{iS(t, x, y, z)}, $$

(5)

we split $\phi$ into a slowly varying amplitude $\alpha$ and a rapidly oscillating phase $e^{iS}$. More precisely, $\partial_\mu S$ and $qA_\mu$ are large quantities of the order of the electron mass $\mathcal{O}(m)$ while $\partial_\mu \alpha = \mathcal{O}(\omega)$ is much smaller. Inserting this ansatz [5] into Eq. (4), the leading order $\mathcal{O}(m^2)$ yields

$$ \left. \left( \partial^2_{xx} - \frac{1}{c^2} \partial^2_t - \frac{q^2}{c^2} \right) \right|_{x=0} \alpha(t, x) = 0. $$

(6)

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the eikonal equation \((\partial_\mu S + qA_\mu)(\partial^\mu S + qA^\mu) = m^2\). In view of the translational invariance in \(y\) and \(z\), we make the separation ansatz \(S(t, x, y, z) = k_y y + k_z z \pm s(t, x)\), where \(s(t, x)\) is determined by the first-order equation
\[
\partial_t s = \sqrt{m^2 + (\partial_x s)^2 + (k_y + qA_y)^2 + k_z^2}.
\] (6)

We expect the maximum contribution to pair creation along the symmetry plane \(x = 0\) where the electric field assumes its maximum, i.e., from those wave packets staying close to \(x = 0\) throughout the evolution, which implies zero momentum in \(x\)-direction \(\partial_x s|_{x=0} = 0\) \[30\]. Thus (and since \(A_y\) is an even function of \(x\)), we take \(s(t, x)\) to be an even function of \(x\) for simplicity. After a Taylor expansion around \(x = 0\)
\[
s(t, x) = s_0(t) + \frac{x^2}{2} s_2(t) + \mathcal{O}(x^4),
\] (7)

we find that the zeroth order \(s_0(t)\), i.e., the eikonal along \(x = 0\) is given by
\[
\partial_t s_0 = \sqrt{m^2 + [k_y + qA_y(t, x = 0)]^2 + k_z^2},
\] (8)
in complete analogy to a purely time-dependent field. 

**Focusing and de-focusing effects**  As the next step, let us study the impact of the curvature \(s_2(t)\) in Eq. \((7)\). Having determined the phase function \(S\) by the leading-order \(\mathcal{O}(m^2)\) contribution to Eq. \((6)\), the sub-leading order \(\mathcal{O}(m\omega)\) determines the evolution of \(\alpha\) via
\[
(\partial^\mu s)\partial_\mu \alpha = -\frac{\alpha}{2} \Box s,
\] (9)

where the higher-order term \(\Box \alpha = \mathcal{O}(\omega^2)\) has been neglected. Along the symmetry plane \(x = 0\) where \(s_2 = 0\), the spatial derivative \(\partial_x \alpha\) drops out and thus the left-hand side of Eq. \((9)\) is again the same as in a purely time-dependent field.

The right-hand side of Eq. \((9)\), on the other hand, contains the additional term \(\partial_\mu \xi(x)\partial_\mu s|_{x=0} = s_2\). This curvature contribution can be obtained by inserting Eq. \((7)\) into Eq. \((6)\) followed by a Taylor expansion
\[
\partial_t s_2 = \frac{s_2^2 + [k_y + qA_y]q\partial_\mu A_\mu|_{x=0}}{\sqrt{m^2 + [k_y + qA_y]^2 + k_z^2}}.
\] (10)

In analogy to Eq. \((9)\), we obtain a closed first-order differential equation for \(s_2(t)\). In contrast to Eq. \((6)\), however, this is a non-linear equation which can display (blow-up) singularities. Similar to caustics, they do not imply singularities of the solutions \(\phi\) to the original (linear) Klein-Fock-Gordon equation \((1)\), but indicate a break-down of the WKB ansatz \((5)\), as also discussed in \[39\]. Fortunately, for a large class of parameters including the cases of interest here, such singularities do not occur – see also Section F in the Supplemental Material \[32\].

In order to provide an intuitive interpretation of the above equation \((10)\), we note that \(k_y + qA_y\) is the mechanical momentum in \(y\)-direction, proportional to the velocity \(v_y\). As \(\partial_y A_y\) is the magnetic field \(B_z\), the numerator in Eq. \((10)\) yields, apart from the non-linearity \(s_2^2\), the divergence \(\partial_y F_y\) of the Lorentz force. Thus the curvature \(s_2\) is associated to the focusing or de-focusing effect of the inhomogeneous magnetic field \(B_z\).

**Particle creation**  The simple WKB ansatz \((5)\) is not well suited for studying pair creation because this phenomenon is associated with a mixing of positive and negative frequency solutions, which is not captured by the ansatz \((5)\) for slowly varying \(\alpha\). Thus, we adapt a generalized WKB ansatz, see also \[27, 30, 39\].

To this end, we define the phase-space pseudo-vector \(\varphi = (\phi, \dot{\phi})^T\) which allows us to cast the original second-order equation \((4)\) into a first-order form
\[
\partial_t \varphi = \left[ \begin{array}{c} 0 \\ \partial_x^2 - \mu^2 \end{array} \right] \cdot \varphi,
\] (11)

where \(\sigma_{\pm}\) are the Pauli ladder matrices and \(\mu(t, x)\) denotes the effective mass \(\mu^2 = m^2 + (k_y + qA_y)^2 + k_z^2\).

In order to include pair creation, we generalize the original WKB ansatz \((5)\) via
\[
\varphi = \alpha u_{\pm} e^{i\chi} + \beta u_{\mp} e^{-i\chi},
\] (12)

where \(\alpha(t, x)\) and \(\beta(t, x)\) are the Bogoliubov coefficients, which are assumed to be slowly varying. The basis vectors \(u_{\pm}(t, x)\) are eigenvectors of the matrix
\[
[\sigma_{+} - \sigma_{-} ((\partial_x s)^2 + \mu^2)] \cdot u_{\pm} = \pm i\chi u_{\pm},
\] (13)

with eigenvalues \(\pm i\chi\) where \(\chi(t, x) = \partial_t s(t, x)\) is given by Eq. \((6)\). Thus, after inserting the generalized ansatz \((12)\) into Eq. \((11)\), the leading order again corresponds to the eikonal equation \((6)\).

For simplicity, we use the (non-normalized) eigenvectors \(u_{\pm} = (1, \pm i\chi)^T\) in the following. Since \(A_y\) and \(s\) are even functions of \((x, y, z)\), we can view \(u_{\pm}\) as eigenvectors of the matrix \(\alpha \in \mathbb{C}\) and \(\beta \in \mathbb{C}\) do not vanish along the symmetry plane \(x = 0\). Furthermore, although the second \(x\)-derivatives of \(\alpha, \beta, u_{\pm}\) vanish along the symmetry plane \(x = 0\), they scale with \(\mathcal{O}(\omega^2)\). Thus, they are neglected within the next-to-leading order \(\mathcal{O}(m\omega)\) of the WKB approach, which yields (along the symmetry plane \(x = 0\))
\[
\begin{align*}
(\dot{\alpha} u_{+} + \alpha \dot{u}_{+} - i\alpha \sigma_{-} \cdot u_{+} \partial_\mu^2 s) e^{i\chi} + \\
(\dot{\beta} u_{-} + \beta \dot{u}_{-} + \beta \sigma_{-} \cdot u_{-} \partial_\mu^2 s) e^{-i\chi} &= 0.
\end{align*}
\] (14)

Note that \(\partial_\mu^2 s = \mathcal{O}(m\omega)\) is kept, in complete analogy to Eq. \((9)\). Projection with \(u_{\pm}^* = (\pm i\chi, 1)^T\) gives
\[
\begin{align*}
2\chi \dot{\alpha} + \alpha \Box s &= \beta (\Box s) e^{-2i\chi}, \\
2\chi \dot{\beta} + \beta \Box s &= \alpha (\Box s) e^{2i\chi}.
\end{align*}
\] (15)
For the spatially homogeneous limit where $\partial^2_t s = 0$, we recover the well-known evolution equations for a purely time-dependent field as $\square s \to \dot{s}$. For our colliding-pulse scenario [2], these two evolution equations [15] for $\alpha$ and $\beta$ along the $x = 0$ plane contain the same exponents $e^{2i\omega t}$ as in the case of a purely time-dependent field, the only difference are the pre-factors $\square s$ which now contain the additional $\partial^2_s$ term. The $\dot{s}$ contribution $\partial^2_t s = \partial_t \chi = q A_y (k_y + q A_y) / \chi$ already present in a purely time-dependent scenario contains the electric field $E_y$ while the additional $\partial^2_s$ contribution stems from the inhomogeneities of the magnetic field $B_z$ and describes the focusing or defocusing effects, see the discussion below.

As in the purely time-dependent scenario, we may combine the two linear evolution equations (15) for the Bogoliubov coefficients into a single Riccati equation $R = \square s (e^{2i\omega t} - R^2 e^{-2i\omega t})/(2\chi)$ for their ratio $R = \beta/\alpha$.

**Numerical simulations** Let us compare our analytical findings with numerical simulations. Numerical approaches to the Sauter-Schwinger effect include direct integrations of the Klein-Fock-Gordon or Dirac equations (see, e.g., [8–16]), quantum Monte-Carlo methods (see, e.g., [22]), or numerical world-line instanton solvers (see, e.g., [48–49]). Each of these methods has advantages and drawbacks, but calculating an exponentially small pair-creation probability $P$ in a complex higher-dimensional field configuration $A(t, r)$ is always challenging.

In order to reduce the computational complexity as much as possible, we consider the Dirac equation in 2+1 dimensions, where we can use two-component spinors, but still incorporate a transversal field (2). Employing the Dirac-Heisenberg-Wigner formalism, the problem is mapped onto a set of first-order transport equations involving bi-linear expectation values, see Section B in the Supplemental Material [32].

We consider the following field profile in Eq. (2)

$$f(t) = \frac{E t}{2} \exp \left\{ -\omega^2 t^2 \right\}, \quad (16)$$

which displays the maximum electric field $E$ at $t = 0$ and $x = 0$. Since the vector potential vanishes asymptotically $f(t \to \pm \infty) = 0$ the wavenumber $k_y$ coincides with the mechanical momentum at those times. This simplifies the numerical analysis and will be relevant for the pair-creation spectra discussed in Section D in the Supplemental Material [32].

**Numerical results** In the following, we set the field parameter $E$ in Eq. (16) to $E = E_S/3$, i.e., the peak field strength is one third of the Schwinger critical field. In this case, we are already in the sub-critical regime where the pair-creation probability $P$ is exponentially suppressed as in Eq. (1), but the numbers are not too small for a reliable numerical computation.

The computed mean particle numbers are plotted in Fig. 1. The locally constant field approximation just reflects the trivial space-time volume scaling with $1/\omega^2$. As expected, the results of the Dirac-Heisenberg-Wigner formalism converge to that approximation for small $\omega$, i.e., small Keldysh parameters [3], but start to show significant deviations for Keldysh parameters of order unity, which is the regime we are interested in.

Motivated by the above findings based on the WKB approach, we also compared those results with the spatially homogeneous field approximation: To this end, we calculated the pair-creation probability $P$ for a purely time-dependent scenario $A(t) = 2 f(ct)e_y$ corresponding to the field at the symmetry plane $x = 0$ [22–24]. While this is expected to yield the correct pair-creation exponent, this scenario grossly overestimates the pre-factor because particles are now created in the whole spatial volume. For the colliding pulses [2], however, pair creation predominantly occurs in the vicinity of the symmetry plane $x = 0$ where the electric field assumes its maximum. In order to correct this over-estimation, we introduce a pre-factor accounting for the finite extent (in $x$-direction) of the effective pair-creation volume [20]. As a natural and minimal assumption, we take this pre-factor to be proportional to $1/\omega$, i.e., the pulse width, where the proportionality constant is fixed by demanding convergence to the locally constant field approximation at small $\omega$, see also [55–59].

As we may observe in Fig. 1, this spatially homogeneous field approximation still over-estimates the pair-creation probability a bit, but provides a much better description than the locally constant field approximation. Even for frequencies of the order of the electron mass, it reproduces the qualitative behavior of the full Dirac-Heisenberg-Wigner results, such as the peak of the particle number at $\omega = \mathcal{O}(m)$. The quantitative disagreement regarding the height and location of the peaks can presumably be explained by a threshold effect marking the transition from the non-perturbative to the perturbative
regime at large \( \omega \) (where the WKB approach is expected to break down), see Section E in the Supplemental Material \[32\].

**Focusing/de-focusing corrections** The spatially homogeneous field approximation explained above does only take into account the \( \bar{s} \) term in Eqs. \[15\], i.e., the electric field \( E_y \). In order to include the effects of the magnetic field \( B_z \), one should replace \( \bar{s} \rightarrow \Box \bar{s} \), cf. Eqs. \[15\], which also contains \( \partial_x^2 \bar{s} \), i.e., the curvature \( s_2 \) in Eq. \[10\]. The effect of this replacement can be studied by numerically solving the set of ordinary differential equations \[8\], \[10\] and \[15\] for the profile \[16\]. As example parameters, we choose \( E = E_S/3 \) as before and \( \omega = m/3 \), i.e., \( \gamma = 1 \).

As shown in Section C of the Supplemental Material \[32\], the behavior of \( \bar{s}_0(t) \) and \( s_2(t) \) strongly depends on the momentum \( k_y \). For \( k_y = \pm m \), for example, the curvature \( s_2(t) \) is quite close to \( \bar{s}_0(t) \) thus almost canceling each other in the pre-factor \( \Box \bar{s} \). For \( k_y = 0 \), this is not the case as the curvature \( s_2(t) \) varies more slowly with time than \( \bar{s}_0(t) \).

We find that including the curvature term \( s_2 \) reduces the pair-creation probability, e.g., roughly by a factor of two for the case \( k_y = 0 \) (which yields the dominant contribution), see the Supplemental Material \[32\]. Thus, including the focusing/de-focusing effects corrects the over-estimate of the spatially homogeneous field approximation and brings the estimated pair-creation probability almost on top of the value obtained by the Dirac-Heisenberg-Wigner approach. However, more systematic investigations are needed to assess the overall accuracy of this approach.

**Conclusions** As a prototypical example for a space-time dependent and transversal field configuration (as a vacuum solution to the Maxwell equations), we consider the head-on collision of two plane-wave laser pulses. Via the WKB approach, we study electron-positron pair creation in this background for sub-critical fields \( E \ll E_S \) and Keldysh parameters of order unity. Along the symmetry (i.e., collision) plane, where we expect that dominant contribution, we find that the pair-creation exponent is the same as for a purely time-dependent electric field, only the pre-factor \( \bar{s} \rightarrow \Box \bar{s} \) does also include the impact of the magnetic field, leading to focusing/de-focusing effects.

This approximate mapping to a purely time-dependent electric field allows us to employ the spatially homogeneous field approximation, which we compare to numerical simulations using the Dirac-Heisenberg-Wigner approach. We find that the spatially homogeneous field approximation over-estimates the pair-creation probability slightly, but provides a much better description than the locally constant field approximation, see Fig. 1. It even reproduces qualitative features of the pair-creation spectra, see Section D in the Supplemental Material \[32\].

Going beyond the spatially homogeneous field approximation, we may also study the impact of the magnetic field, leading to focusing/de-focusing effects. Along the symmetry plane, this amounts to replacing \( \bar{s} \) by \( \Box \bar{s} \) in the evolution equations for the Bogoliubov coefficients, which also contains the curvature term \( \partial_x^2 \bar{s} \). For the cases we studied, we found that this replacement tends to lower the pair-creation probability, which brings it closer to the results of the Dirac-Heisenberg-Wigner approach.

However, one might also imagine other scenarios. Note that \( \bar{s} \) is a local function of \( A_y \) and \( A_y \), while the curvature \( \partial_x^2 \bar{s} \) is non-local, i.e., depends on whole history of the evolution. This could be exploited in pulse-shape optimization schemes aimed at increasing the pair-creation probability. As an intuitive picture, if the initial wave-packet of the fermionic quantum vacuum fluctuations is focused onto the symmetry plane, where the electric field assumes its maximum, it can react to this strong field (i.e., produce particles) much better than a wave-packet which is more de-localized.

**Experimental scenarios** Finally, let us discuss potential experimental tests of our results. Ultra-strong optical laser foci have very small Keldysh parameters \( \gamma \ll 1 \) and should thus be treatable via the locally constant field approximation. X-ray free electron lasers (XFEL), on the other hand, have much larger \( \gamma \) and could require going beyond that approximation. Unfortunately, however, present-day facilities do not reach the necessary field strengths \( E \) yet. An interesting idea to achieve this goal is high-harmonic focusing (see, e.g., \[60–62\]) which typically also corresponds to non-negligible \( \gamma \).

As a completely different scenario for generating ultra-strong fields, collisions of heavy nuclei have been studied theoretically and experimentally, see, e.g., \[63–65\]. Considering ultra-peripheral “collisions” at relativistic velocities \( v \) along the trajectories \( r(t) = \pm(vt, b/2, 0)^T \) with impact parameters \( b \), the superpositions of the boosted Coulomb fields of the two nuclei can be approximated by Eq. \[2\] at sufficiently large distances \( |r| \gg b \), say, of the order of the Compton length \[69,71\]. The associated field strengths may reach or even exceed the Schwinger critical field \( E_S \) \[74,75\] and the Keldysh parameters \( \gamma \) will also be non-negligible (especially for ultra-relativistic \( v \)). Of course, the field strengths and their spatial and temporal gradients will be even larger at smaller distances \( |r| \sim b \), such that the total electron-positron yield will also contain contributions from this region. Nevertheless, this again shows the importance of understanding the impact of space-time dependence on pair creation, i.e., to go beyond the locally constant field approximation.

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Sauter-Schwinger laser assisted Pair creation electric QED Ionization Electron pair im strong QED shape pair a Dirac's elektrischen Locally WKB Electromagnetic Exact the atoms structures Verhalten for Finite invariance Elektrons in quantum in processes, Phys. Rev. A atoms the a of Pulse time-dependent Schwinger a of Vacuum Pair in Plane Improved local-constant-field to laser the in vacuum and Catalysis lightfront, Phys. Rev. and by WKB (Keldysh In of nonlinear Schwinger gauge in approach Complex Fermion pair instantons Ultrarelativistic pulses, Phys. Rev. D shell on mechanism, Phys. Rev. Lett. a distribution Diracs, Z. Phys. for of production of intense time-dependent in vacuum fields, Phys. Rev. D fields, Phys. Rev. D monochromatic field, Phys. Rev. D periodic colliding Pair approximation by HIBEF, ELI, CILEX, CoReLS, ShenGuang-II as well as SLAC; cf. the websites http://www.hibe.eu/, https://www.eli-beams.eu/, https://eli-laser.eu/, http://cilexaclay.fr/, http://corels.ibs.re.kr/, http://issf.cam.cn/en/, https://www6.slac.stanford.edu/

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A. World-line instanton technique

It might be illuminating to study the colliding-pulse scenario in Eq. (2) via the world-line instanton technique. This method is another semi-classical approach and allows us to estimate the pair-creation probability $P$ via
\[ P \sim \exp \{ -A_{\text{inst}} \} , \quad (S1) \]
where $A_{\text{inst}}$ is the action of the associated world-line instanton, i.e., a closed loop $x^\mu(\tau)$ in Euclidean space-time as a solution of the Euclidean semi-classical or classical equations of motion
\[ m \frac{d^2 x^\mu}{d\tau^2} = q F^\mu_{\nu} \frac{dx^\nu}{d\tau} , \quad (S2) \]
parametrized by the “proper” time $\tau$.

As explained after Eq. (2), the magnetic field vanishes in the $x = 0$ plane and thus the Euclidean field strength tensor $F^\mu_{\nu}(x = 0)$ does only contain the electric field $E_\nu$. As a result, the instanton moves in imaginary time and $y$-direction, but stays on the $x = 0$ plane, which yields the same exponent $A_{\text{inst}}$ as for a purely time-dependent field $A(t) = 2f(ct)e_y$, see also [S3] [S4].

Within the world-line instanton technique, the pre-factor in front of the exponential (S1) can be obtained (at least in principle) by considering perturbations around the instanton trajectory, but such a calculation can be quite non-trivial for genuinely space-time-dependent fields [S5] [S6].

B. Dirac-Heisenberg-Wigner approach

Let us provide a brief outline of the Dirac-Heisenberg-Wigner approach [S7]. We start from the Lagrangian
\[ \mathcal{L} = \frac{1}{2} (i \bar{\Psi} \gamma^\mu D_\mu \Psi - i \bar{\Psi} D_\mu \gamma^\mu \Psi) - m \bar{\Psi} \Psi \quad (S3) \]
with the Dirac matrices $\gamma^\mu$ and the covariant derivatives $D_\mu = \partial_\mu + iq A_\mu$ and $D_\mu^\dagger = \partial_\mu - iq A_\mu$. As usual in the Furry picture, the Dirac spinors $\hat{\Psi}$ and $\Psi$ are treated as dynamical quantum fields whereas the electromagnetic vector potential $A_\mu$ is an external field, i.e., a $c$-number. As is well known, by doing so we neglect higher-order interactions, e.g., radiative emission, backreaction or electron-electron coupling [S8] [S9].

Considering bilinear forms of the Dirac spinors $\hat{\Psi}$ and $\Psi$ at different space-time points $x_1^\mu$ and $x_2^\mu$, we may transform them to center-of-mass $v^\mu = (x_1^\mu + x_2^\mu)/2$ and relative coordinates $s^\mu = x_1^\mu - x_2^\mu$ which yields the generalized density operator
\[ \hat{C}_{\alpha\beta}(r,s) = \mathcal{U}_A(r,s) \left[ \hat{\Psi}_\beta(r-s/2), \bar{\Psi}_\alpha(r+s/2) \right] , \quad (S4) \]
where the Wilson line factor
\[ \mathcal{U}_A(r,s) = \exp \left( iq \int_{-1/2}^{1/2} d\xi \ s^\mu A_\mu(r + \xi s) \right) \quad (S5) \]
is implemented to ensure gauge-invariance.

The Fourier transform of this quantity (S4) with respect to the relative coordinate $s^\mu = x_1^\mu - x_2^\mu$ yields the covariant Wigner operator [S7] in 2 + 1 dimensions
\[ \hat{W}_{\alpha\beta}(r,p) = \frac{1}{2} \int d^3 s \ e^{ip_\mu s^\mu} \hat{C}_{\alpha\beta}(r,s) , \quad (S6) \]
where $p_\mu$ can be identified as the kinetic or mechanical momentum because the vector potential $A_\mu$ is already contained in the Wilson line factor (S5). Thus $\hat{W}_{\alpha\beta}$ represents a kinetic quantity defined in the particles’ coordinate-momentum phase-space.

The time evolution of the Wigner operator is determined by the Dirac equation. In order to formulate transport equations, we take its expectation value in the initial vacuum state $|0\rangle$ which yields the Wigner function
\[ \langle \hat{W}(r,p) \rangle = \langle 0 | \hat{W}(r,p) | 0 \rangle , \quad (S7) \]
where we omitted the indices for the sake of simplicity.

We expand the Wigner function in Dirac bilinears in an irreducible representation

$$\mathcal{W}(\mathbf{r}, p) = \frac{1}{2} (\mathbb{1} \mathcal{S} + \gamma^\mu \mathcal{W}_\mu),$$

(S8)

where the gamma matrices are (in 2+1 dimensions) given in terms of the Pauli matrices

$$\gamma^0 = \sigma_3, \quad \gamma^1 = i \sigma_1, \quad \gamma^2 = -i \sigma_2.$$

(S9)

**Transport equations** Projection on equal-time

$$\mathcal{W}(t, \mathbf{x}, p) = \int \frac{dp}{2\pi} \mathcal{W}(\mathbf{r}, p),$$

(S10)

with \( \mathbf{r} = (t, \mathbf{x}) \) gives rise to a closed set of differential equations describing the time evolution of particle distributions, namely mass \( s \), charge \( v_0 \) and current \( \mathbf{v} \) density [S7] [S14] [S15]. For a potential of the form (2), within QED\(_{2+1}\) and one set of 2-spinors we obtain on the basis of \( p_x = k_x, \ p_y = k_y + q \int d\xi A_y (x + \xi s_x, t) \) [S10] [S18]

$$\partial_t s + 2k_x v_2 - 2\Pi v_1 = 0,$$

(S11)

$$\partial_t v_0 + D v_2 + \partial_2 v_1 = 0,$$

(S12)

$$\partial_t v_1 + \partial_2 v_0 + 2\Pi s = +2m v_2,$$

(S13)

$$\partial_t v_2 + D v_0 - 2k_x s = -2m v_1,$$

(S14)

with pseudo-differential operators [S19]

$$D = +iq \mathcal{F}_{k_x}^{-1} \left[ A_y \left( x + \frac{g_x}{2}, t \right) - A_y \left( x - \frac{g_x}{2}, t \right) \right] \mathcal{F}_{k_x},$$

(S15)

$$\Pi = k_y + \frac{q}{2} \mathcal{F}_{k_x}^{-1} \left[ A_y \left( x + \frac{g_x}{2}, t \right) + A_y \left( x - \frac{g_x}{2}, t \right) \right] \mathcal{F}_{k_x}.$$

(S16)

Here, the Fourier operators \( \mathcal{F}_{k_x} \) transform from canonical momentum space \( k_x \) to relative coordinate space \( s_x \). As a matter of fact, Eqs. (S11)-(S14) describe a partial differential equation in \( t, x \) and \( k_x \) with the (conserved) canonical momentum \( k_y \) serving as an external parameter.

In order to determine the pair production rate from an initial vacuum state, we employ initial conditions of the form

$$s_{\text{vac}}(k) = -\frac{m}{\sqrt{m^2 + k^2}}, \quad v_{\text{vac}}^i(k) = -\frac{k^i}{\sqrt{m^2 + k^2}}, \quad v_0 = 0.$$

(S17)

Evaluating the transport equations on the basis of these initial conditions, we obtain the particle distribution function at asymptotic times \( (t \to \pm \infty) \)

$$n(x, k) = \frac{m(s - s_{\text{vac}}) + k \cdot (v - v_{\text{vac}})}{2\sqrt{m^2 + k^2}}.$$

(S18)

The total particle number is given by

$$N = \int \frac{d^2 k \, d^2 x}{(2\pi)^2} n(x, k).$$

(S19)

**Spatially homogeneous approximation** In the spatially homogeneous field approximation we calculate a fully time-dependent, but spatially localized pair production rate \( (x = 0) \) on the basis of Eqs. (S11)-(S14), see also [S20].

At \( x = 0 \) the vector potential takes on the form \( \mathbf{A}(t) = 2f(ct)\mathbf{e}_y \), see Eq. (10). Consequently, the differential operators (S15)-(S16) simplify to ordinary factors, \( D = 0 \) and \( \Pi = qA_y \). Furthermore, within an entirely localized description of particle creation, particle propagation can be neglected thus derivatives with respect to spatial coordinates vanish.

As a result, the corresponding equations of motion take on the much simpler form

$$\partial_t s^{\text{SHA}} + 2k_x v_2^{\text{SHA}} - 2(k_y + qA_y(t)) v_1^{\text{SHA}} = 0,$$

(S20)

$$\partial_t v_1^{\text{SHA}} + 2(k_y + qA_y(t)) s^{\text{SHA}} = +2m v_2^{\text{SHA}},$$

(S21)

$$\partial_t v_2^{\text{SHA}} - 2k_x s^{\text{SHA}} = -2m v_1^{\text{SHA}},$$

(S22)

where we used the superscript SHA (spatially homogeneous approximation) to indicate that these components do not depend on \( x \). Initial conditions remain unchanged, see Eq (S17). Equations (S20)-(S22) can also be derived employing a quantum kinetic approach [S22] [S24].
FIG. S1. Left panel: Time evolution of \( \dot{s}_0 \) (blue line) in Eq. (8) and the curvature \( s^2 \) in Eq. (10) in the linear (green diamonds) and non-linear (orange circles) form, respectively. Data are obtained for the profile with \( E = E_S / 3 \), \( \omega = m / 3 \) and \( k_y = -m \) (top), \( k_y = 0 \) (middle), \( k_y = m \) (bottom). In the right panel, we display a set of characteristic curves for the same parameters in order to visualize the focusing/de-focusing effects – where one may observe a strong correlation to the behavior of \( s^2 \), as expected from the discussion after Eq. (10).

**Locally constant field approximation**  Within the locally constant field approximation we assume instantaneous particle creation. While this approximation fails to capture time-memory effects, e.g., photon absorptive processes, it is expected to provide correct particle numbers if spatial and temporal variations in the employed fields are sufficiently smooth.

For a field of the form \( \mathbf{A}(t, \mathbf{r}) \) we model the source term heuristically after the Sauter-Schwinger effect in constant fields in 2 + 1 dimensions

\[
S = \left| q \right|^{3/2} \frac{a(t, x)^{3/2}}{(2\pi)^2} \exp \left( -\frac{\pi m^2}{\left| q \right| a(t, x)} \right), \tag{S23}
\]

with \( a(t, x) = \sqrt{|\mathcal{F}(t, x)| - \mathcal{F}(t, x)} \) and \( \mathcal{F}(t, x) = -1/2 \left( E(t, x)^2 - B(t, x)^2 \right) \). In order to obtain the total particle number integrations over all coordinates are in order, \( N_{\text{LCFA}} = \int dt \int dx S(t, x) \).

**C. Curvature contribution**

Solving Eq. (10) numerically, we plot the evolution of the curvature \( s_2 \) (starting at zero) in comparison to \( \dot{s}_0 \) for \( E = E_S / 3 \), \( \omega = m / 3 \) and the values \( k_y = 0 \) and \( k_y = \pm m \) in Fig. S1.

As our first observation, we find no blow-up singularities for these parameters. Furthermore, we compare the full solution of the non-linear differential equation (10) with its linearized approximation obtained by neglecting the nonlinearity \( s_2^2 \) in Eq. (10). We find reasonably good agreement in the time interval relevant for pair creation. Note that the potential blow-up singularities are caused by the nonlinearity \( s_2^2 \) in Eq. (10) and thus never occur in the linearized solution.
FIG. S2. Time evolution of $\ddot{s}_0$ (solid blue line) in Eq. (8) and the curvature $s_2$ (dashed orange line) in Eq. (10) as in Fig. S1 (top); the quantity $|R|^2$ associated with the rate of particle creation, obtained in the spatially homogeneous field approximation (solid blue line), i.e., just from $\ddot{s}_0$, in comparison to the full solution (dashed orange line) obtained by replacing $\ddot{s}_0 \rightarrow \Box s = \ddot{s}_0 - s_2$ taking into account the focusing/de-focusing corrections (bottom). Data are obtained for the profile (16) with $E = E_S/3$, $\omega = m/3$ and $k_y = 0$.

For $k_y = \pm m$, we see that the curvature $s_2(t)$ lies very close to $\ddot{s}_0(t)$. As a result, this curvature contribution (stemming from the magnetic field) almost completely cancels the term $\ddot{s}_0(t)$ in the pre-factor $\Box s$ of the evolution equations (15) for the Bogoliubov coefficients, leading to a significantly reduced pair creation. This cancellation does not occur for $k_y = 0$, but the pair creation probability is still reduced by roughly a factor of two, see Fig. S2 and Section G below.

Whether the differences between the full Dirac-Heisenberg-Wigner data and the results of the spatially homogeneous field approximation observed in Fig. S3 below, most notably the “shoulders” of the curve with $\omega = m/3$ starting around $|k_y| \approx m/2$, can be traced back (at least partially) to this cancellation for large $|k_y|$ should be investigated in future studies.

For the sake of completeness, we also examine the characteristic curves for a better visualization of focusing and de-focusing effects created by the inhomogeneity in the field. To this end, we employ the method of characteristics in order to turn Eq. (6) into a family of first-order, ordinary differential equations

$$\begin{align*}
\dot{x}_0(\tau) &= \frac{2}{m} p_0(\tau), \\
\dot{x}_1(\tau) &= -\frac{2}{m} p_1(\tau), \\
p_0(\tau) &= \frac{2}{m} (k_y + qA_y) (qE_y), \\
p_1(\tau) &= \frac{2}{m} (k_y + qA_y) (qB_z),
\end{align*}$$
(S24)

with $p_0 = \partial_t s$ and $p_1 = \partial_x s$, respectively. The parameter $\tau$ specifies the location on a distinctive characteristic curve determined by a particular initial condition. A set of solutions obtained by varying the initial starting position $x_{1,\text{init}}$ is displayed in Fig. S1. It is apparent that there is a strong correlation between $s_2$ being positive/negative and the curves coming closer or being dispersed, respectively.
D. Particle spectra

While the world-line instanton technique – at least in its simplest form discussed above – only yields the total pair creation probability \( S_1 \), the WKB and Dirac-Heisenberg-Wigner approaches allow direct access to the particle spectra. Since \( k_x \) is not a conserved quantity in our case (2), we focus on the \( k_y \)-dependence. Figure S3 displays exemplary spectra for the frequency values \( \omega = m/3 \), \( \omega = m/4 \) and \( \omega = m/5 \), where we have used the same scenario as in Fig. 1 i.e., the profile (16) with \( E = E_S/3 \). Shown are the results of the Dirac-Heisenberg-Wigner approach in comparison to the spatially homogeneous field approximation (after re-scaling), which are basically the spectra produced by a purely time-dependent field.

We find that the spectra obtained by the two approaches match quite well, especially for small \( k_y \) (consistent with the considerations in the previous Section), but there are also distinctive differences. Most notably, the spatially homogeneous field approximation predicts two “shoulders” starting around \( k_y \approx \pm m/2 \) in the spectrum for \( \omega = m/3 \), which are not reproduced by the Dirac-Heisenberg-Wigner approach. These differences might be explainable by the additional curvature contributions \( s_2 \) discussed in the previous Section, but further investigations are necessary to settle this point.

As another interesting feature, both approaches agree on a plateau or even dip at small \( k_y \) in the spectrum for \( \omega = m/4 \). This is quite remarkable since it seems to contradict the standard expectation that the spectrum should have its maximum at \( k_y = 0 \). Assuming that the particles are predominantly created at \( t = 0 \) (i.e., the maximum of the electric field) and with vanishing initial velocity \( v_y(t = 0) = 0 \), one would indeed expect the spectrum to have its maximum at \( k_y = 0 \) in view of \( A_y(t = 0) = A_y(t \to \pm \infty) = 0 \). Investigating the reasons for this apparent contribution (e.g., interference phenomena, see also [S27–S29]) should be the subject of further studies.

For the sake of completeness, we illustrate the two-dimensional spectra (displaying the dependence on \( k_x \) and \( k_y \)) obtained through solving the transport equations (S11)-(S14) for a variety of parameters in Fig. S5. Note that integrating these spectra over \( k_x \) yields the blue curves in Fig. S3 while another integration over \( k_y \) then gives the total particle number, i.e., the orange curve in Fig. 1.

E. Threshold effects

As a working hypothesis, we interpret the peaks observed in Fig. 1 as threshold effects marking the transition from the non-perturbative to the perturbative regime. In order to test this hypothesis, we study the dependence of these peaks on the maximum electric field strength \( E \) in Fig. S4.

To lowest (non-vanishing) order perturbation theory, one would expect an \( E^4 \)-scaling of the mean particle number in the collision scenario (2) where two photons collide to form an electron-positron pair. Such an \( E^4 \)-scaling is indeed consistent with the dependence of the peak heights as well as the behavior of the curves for larger frequencies in Fig. S4(right). For smaller frequencies, however, the curves display deviations from this scaling, indicating the failure of lowest-order perturbation theory.

For purely time-dependent fields, one would expect an \( E^2 \)-scaling instead, as already the first-order amplitude in

![FIG. S3. Plot of the pair-creation spectra obtained by the spatially homogeneous field approximation for the profile (16) with \( E = E_S/3 \) and \( \omega = m/3 \) (green circles), \( \omega = m/4 \) (orange diamonds), and \( \omega = m/5 \) (red squares). The data have been rescaled by a constant factor in order to compensate the overestimation mentioned above. For comparison, the blue curves correspond to the results obtained through the Dirac-Heisenberg-Wigner approach.](image_url)
FIG. S4. Mean number of created particles as a function of collision. Thus, for moderate values of \(\omega\) we again find a negative value of \(s_2\). Apart from the total derivative in the last term, we find that the non-linearity \(\gamma\) can account for pair creation. Again, this scaling is consistent with the peak heights as well as the behavior for larger frequencies in Fig. S4 (left), while the dependence at smaller frequencies deviates.

These perturbative arguments can also help to understand the different locations of the peaks. While the collision of two photons with frequency \(\Omega \geq m\) may create \(e^+e^-\)-pairs in the collision scenario, a purely time-dependent field must contain frequency components with \(\Omega \geq 2m\) in order to obtain a non-zero first-order amplitude in \(E\). Consistently, the peak of the collision scenario occurs at lower frequencies than that for purely time-dependent fields.

Note that making these points more precise is complicated by the fact that the Fourier transform of the field profile (16) yields a rather broad frequency spectrum which has its maximum at a frequency of \(\Omega = 2\omega\). Elucidating this issue should be the subject of further studies.

\[s_2 = \frac{s_{22}^2 + [k_y + qA_y]q\partial_2^2 A_y}{\sqrt{m^2 + [k_y + qA_y]^2 + k_z^2}} \approx \frac{q^2A_y\dot{A}_y}{m} \bigg|_{x=0} = -\frac{q^2(\dot{A}_y)^2}{m} \bigg|_{x=0} + \frac{d}{dt} \left(\frac{q^2A_y\dot{A}_y}{m}\right) \bigg|_{x=0}. \tag{S25}\]

Apart from the total derivative in the last term, we find that \(\dot{s}_2\) is negative. Thus, \(s_2(t)\) may oscillate during the collision of the two pulses, but assumes a negative value afterwards. The subsequent evolution is then governed by the non-linearity \(\dot{s}_2 = s_2^2/m\) which implies that \(s_2(t)\) is slowly approaching the \(t\)-axis from below as \(1/t\). Blow-up singularities would occur if \(s_2\) was positive after the collision and could be visualized as self-focusing, but they are absent in this case and the de-focusing effects dominate.

Actually, as we have observed in Fig. S1, the linearization (i.e., neglect of \(s_2^2\)) provides a fairly good approximation even for moderate values of \(k_y = O(m)\) and for Keldysh parameters \(\gamma\) of order unity. Thus, if we rewrite \(\dot{s}_2\) as

\[\dot{s}_2 = -\frac{q^2A_y\dot{A}_y}{m} \left(\frac{3m^2 + q^2[1 + qA_y]^2 + k_z^2}{m^2 + [k_y + qA_y]^2 + k_z^2}\right)^{3/2} \bigg|_{x=0} + \frac{d}{dt} \left(\frac{qA_y[k_y + qA_y]}{\sqrt{m^2 + [k_y + qA_y]^2 + k_z^2}}\right) \bigg|_{x=0} + O(s_2^2), \tag{S26}\]

we again find a negative value of \(s_2\) after the collision, provided that the non-linearity \(s_2^2\) can be neglected during the collision. Thus, for moderate values of \(\gamma\) and \(k_y\) (note that \(k_z\) can simply be absorbed into \(m\)), we basically get the same picture as above, i.e., an approximately linear evolution of \(s_2(t)\) during the collision resulting in a negative value of \(s_2\) after the collision, which then implies a slow \(1/t\)-decay of \(|s_2|\) governed by the non-linear evolution.

Consistent with these analytic approximations, we only found blow-up singularities in our numerical simulation for very large \(k_y\) and/or for extremely long pulses (which can be treated via the locally constant field approximation).
G. Bogoliubov coefficients

The Bogoliubov coefficients $\alpha(t, x = 0)$ and $\beta(t, x = 0)$ along the symmetry plane can either be obtained by solving Eqs. (15) directly or from the Riccati equation $R = \Box s(e^{2i\alpha} - R^2 e^{-2i\alpha})/(2\chi)$ together with their normalization. For our choice of the eigenvectors $u_\pm = (1, \pm i\chi)^T$, the normalization of the Bogoliubov coefficients along the symmetry plane can be derived from Eqs. (15) and reads

$$|\alpha(t, x = 0)|^2 - |\beta(t, x = 0)|^2 = \exp \left\{ -\frac{1}{2} \int_{-\infty}^{t} dt' \frac{\Box s}{\chi} |v', x = 0| \right\}. \tag{S27}$$

For purely time-dependent fields $\Box s = \dot{s}$, the integrand $\dot{s}/\chi = \ddot{s}$ in Eq. (S27) is a total derivative and thus we recover the well-known $1/\sqrt{\chi}$ normalization. Including the spatial curvature $s_2$ also incorporates focusing/de-focusing effects. Actually, inserting the dependence $s_2(t) \propto 1/t$ for late times (as explained above), the right-hand side of Eq. (S27) behaves as $1/\sqrt{t}$ which corresponds to the spread of the wave packets. Obviously, this does not imply that the number of created particles decreases – their number should be constant after the collision is over – it just means that they do not stay at $x = 0$ but eventually move away. In order to factor out this trivial spreading effect, we consider the ratio $R = \beta/\alpha$ where the overall normalization (S27) cancels.

More specifically, starting with a normalized wave-packet of purely positive frequency (corresponding to the $\alpha$ coefficient), the probability of particle creation is determined by the norm of the final negative-frequency part of the wave-packet (corresponding to the $\beta$ coefficient). Obviously, the trivial spreading of the wave-packet does not change this norm. Thus, in order to determine the pair-creation probability, we introduce the normalized Bogoliubov coefficients $\tilde{\alpha} = \alpha/N$ and $\tilde{\beta} = \beta/N$ where $N$ is given by Eq. (S27) via $|\alpha|^2 - |\beta|^2 = N^2$. These normalized Bogoliubov coefficients satisfy the usual unitarity relation $|\tilde{\alpha}|^2 - |\tilde{\beta}|^2 = 1$ and have the same ratio $R = \tilde{\beta}/\tilde{\alpha} = \beta/\alpha$. The pair-creation probability is then given by $|\tilde{\beta}|^2$ which can be obtained from $R$ via

$$|\tilde{\beta}|^2 = \frac{|R|^2}{1 - |R|^2}. \tag{S28}$$

For small $|R|^2 \ll 1$ as in Fig. S2 (implying $|\tilde{\alpha}|^2 \gg |\tilde{\beta}|^2$) this simplifies to $|\tilde{\beta}|^2 \approx |R|^2$.

H. Numerical Simulation

The transport equations (S11)-(S14) have been solved according to the blueprint presented in [S10]. Thus, the time evolution is performed on the basis of a high-order Dormand-Prince Runge-Kutta integrator with adaptive time-stepping. This includes an artificial super-exponential adiabatic turn on/off of the field in order to avoid high-frequency components spoiling the simulation corresponding to a computational initial time $t_i = 40/m$ even for profiles where $\omega \sim O(m)$. Derivative operators (S15)-(S16) are evaluated employing discrete (inverse) Fourier transforms [31].

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Note that the spatially homogeneous field approximation applied here is not synonymous with the locally homogeneous approximation used in, for example, [S21]. In the latter, the field $A(t, x)$ at each space point $x$ is mapped onto a purely time-dependent field and the resulting pair creation density is then integrated over all positions $x$ (in analogy to the locally constant field approximation, just with $x$ instead of $x$ and $t$). Thus, this approximation scheme completely neglects the impact of the magnetic field component [S16]. As a result, it would yield a finite particle creation rate even for a single propagating pulse $A(t, r) = A_0(t, x)e_y = f(t - x)e_y$, which is not correct.

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FIG. S5. Particle momentum spectra obtained from solving the Dirac-Heisenberg-Wigner transport equations for a field profile of two identical, colliding laser pulses \(2\). The field strength is given by \(E = E_S/3\), the different pulse frequencies are displayed as insets. Through integration over the momentum space variables \(k_x\) and \(k_y\) the mean number of created particles is obtained, cf. the orange circles in Fig. 1.