Simplicial Vertices in Graphs with no Induced Four-Edge Path or Four-Edge Antipath, and the $H_6$-Conjecture

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Abstract: Let $\mathcal{G}$ be the class of all graphs with no induced four-edge path or four-edge antipath. Hayward and Nastos [6] conjectured that every prime graph in $\mathcal{G}$ not isomorphic to the cycle of length five is either a split graph or contains a certain useful arrangement of simplicial and antisimplicial vertices. In this article, we give a counterexample to their conjecture, and prove a slightly weaker version. Additionally, applying a result of the first author and Seymour [1] we give a short proof of Fouquet’s result [3] on the structure of the subclass of bull-free graphs contained in $\mathcal{G}$. © 2013 Wiley Periodicals, Inc. J. Graph Theory 76: 249–261, 2014

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1. INTRODUCTION

All graphs in this article are finite and simple. Let $G$ be a graph. The complement $\overline{G}$ of $G$ is the graph with vertex set $V(G)$, such that two vertices are adjacent in $G$ if and only if they are nonadjacent in $\overline{G}$. For a subset $X$ of $V(G)$, we denote by $G[X]$ the subgraph of $G$ induced by $X$, that is, the subgraph of $G$ with vertex set $X$ such that two vertices are adjacent in $G[X]$ if and only if they are adjacent in $G$. If $H$ be a graph. If $G$ has no induced subgraph isomorphic to $H$, then we say that $G$ is $H$-free. If $G$ is not $H$-free, $G$ contains $H$, and a copy of $H$ in $G$ is an induced subgraph of $G$ isomorphic to $H$. For a family $\mathcal{F}$ of graphs, we say that $G$ is $\mathcal{F}$-free if $G$ is $F$-free for every $F \in \mathcal{F}$.

We denote by $P_{n+1}$ the path with $n + 1$ vertices and $n$ edges, that is, the graph with distinct vertices $\{p_0, \ldots, p_n\}$ such that $p_i$ is adjacent to $p_j$ if and only if $|i - j| = 1$. For a graph $H$, and a subset $X$ of $V(G)$, if $G[X]$ is a copy of $H$ in $G$, then we say that $X$ is an $H$. By convention, when explicitly describing a path we will list the vertices in order. In this article, we are interested in understanding the class of $\{P_5, \overline{P}_5\}$-free graphs.

Let $A$ and $B$ be disjoint subsets of $V(G)$. For a vertex $b \in V(G) \setminus A$, we say that $b$ is complete to $A$ if $b$ is adjacent to every vertex of $A$, and that $b$ is anticomplete to $A$ if $b$ is nonadjacent to every vertex of $A$. If every vertex of $A$ is complete to $B$, we say that $A$ is complete to $B$, and if every vertex of $A$ is anticomplete to $B$, we say that $A$ is anticomplete to $B$. If $b \in V(G) \setminus A$ is neither complete nor anticomplete to $A$, we say that $b$ is mixed on $A$. A homogeneous set in a graph $G$ is a subset $X$ of $V(G)$ with $1 \leq |X| \leq |V(G)|$ such that no vertex of $V(G) \setminus X$ is mixed on $X$. We say that a graph is prime if it has at least four vertices, and no homogeneous set.

Let us now define the substitution operation. Given graphs $H_1$ and $H_2$, on disjoint vertex sets, each with at least two vertices, and $v \in V(H_1)$, we say that $H$ is obtained from $H_1$ by substituting $H_2$ for $v$, or obtained from $H_1$ and $H_2$ by substitution (when the details are not important) if:

- $V(H) = (V(H_1) \cup V(H_2)) \setminus \{v\}$,
- $H[V(H_2)] = H_2$,
- $H[V(H_1)] \setminus \{v\} = H_1[V(H_1)] \setminus \{v\}$, and
- $u \in V(H_1)$ is adjacent in $H$ to $w \in V(H_2)$ if and only if $w$ is adjacent to $v$ in $H_1$.

Thus, a graph $G$ is obtained from smaller graphs by substitution if and only if $G$ is not prime. Since $P_3$ and $\overline{P}_3$ are both prime, it follows that if $H_1$ and $H_2$ are $\{P_3, \overline{P}_3\}$-free graphs, then any graph obtained from $H_1$ and $H_2$ by substitution is $\{P_3, \overline{P}_3\}$-free. Hence, in this article, we are interested in understanding the class of prime $\{P_3, \overline{P}_3\}$-free graphs.

Let $C_n$ denote the cycle of length $n$, that is, the graph with distinct vertices $\{c_1, \ldots, c_n\}$ such that $c_i$ is adjacent to $c_j$ if and only if $|i - j| = 1$ or $n - 1$. A theorem of Fouquet [3] tells us that:

**Lemma 1.1.** Any $\{P_3, \overline{P}_3\}$-free graph that contains $C_5$ is either isomorphic to $C_5$ or has a homogeneous set.

That is, $C_5$ is the unique prime $\{P_3, \overline{P}_3\}$-free graph that contains $C_5$, and so we concern ourselves with prime $\{P_3, \overline{P}_3, C_5\}$-free graphs, the main subject of this article.

Let $G$ be a graph. A clique in $G$ is a set of vertices all pairwise adjacent. A stable set in $G$ is a set of vertices all pairwise nonadjacent. The neighborhood of a vertex $v \in V(G)$ is the set of all vertices adjacent to $v$, and is denoted $N(v)$. A vertex $v$ is simplicial if $N(v)$
is a clique. A vertex \( v \) is *antisimplicial* if \( V(G) \setminus N(v) \) is a stable set, that is, if and only if \( v \) is a simplicial vertex in the complement.

In [6] Hayward and Nastos proved:

**Lemma 1.2.** If \( G \) is a prime \( \{P_5, \overline{P_5}, C_5\} \)-free graph, then there exists a copy of \( P_4 \) in \( G \) whose vertices of degree one are simplicial, and whose vertices of degree two are antisimplicial.

A graph \( G \) is a *split graph* if there is a partition \( V(G) = A \cup B \) such that \( A \) is a stable set and \( B \) is a clique. Földes and Hammer [2] showed:

**Lemma 1.3.** A graph \( G \) is a split graph if and only if \( G \) is a \( \{C_4, \overline{C_4}, C_5\} \)-free graph.

Drawn in Figure 1 with its complement, \( H_6 \) is the graph with vertex set \( \{v_1, v_2, v_3, v_4, v_5, v_6\} \) and edge set \( \{v_1v_2, v_2v_3, v_3v_4, v_2v_5, v_3v_6, v_5v_6\} \).

Hayward and Nastos conjectured the following:

**Theorem 1.4 (The \( H_6 \)-Conjecture).** If \( G \) is a prime \( \{P_5, \overline{P_5}, C_5\} \)-free graph that is not split, then there exists a copy of \( H_6 \) in \( G \) or \( \overline{G} \) whose two vertices of degree one are simplicial, and whose two vertices of degree three are antisimplicial.

First, in Figure 2 we provide a counterexample to 1.4. On the other hand, we prove the following slightly weaker version:

**Lemma 1.5.** If \( G \) is a prime \( \{P_5, \overline{P_5}, C_5\} \)-free graph that is not split, then there exists a copy of \( H_6 \) in \( G \) or \( \overline{G} \) whose two vertices of degree one are simplicial, and at least one of whose vertices of degree three is antisimplicial.
We say that a graph $G$ admits a 1-join, if $V(G)$ can be partitioned into four nonempty pairwise disjoint sets $(A, B, C, D)$, where $A$ is anticomplete to $C \cup D$, and $B$ is complete to $C$ and anticomplete to $D$. In trying to use 1.5 to improve upon 1.1 we conjectured the following:

**Theorem 1.6.** If $G$ is a $\{P_5, \overline{P_5}\}$-free graph, then either

- $G$ is isomorphic to $C_5$, or
- $G$ is a split graph, or
- $G$ has a homogeneous set, or
- $G$ or $\overline{G}$ admits a 1-join.

However, 1.6 does not hold, and we give a counterexample in Figure 3.

The **bull** is a graph with vertex set $\{x_1, x_2, x_3, y, z\}$ and edge set $\{x_1x_2, x_2x_3, x_1x_3, x_1y, x_2z\}$. Lastly, applying a result of the first author and Seymour [1] we give a short proof of 1.1, and Fouquet’s result [3] on the structure of $\{P_5, \overline{P_5}, \text{bull}\}$-free graphs.

This article is organized as follows. Section 2 contains results about the existence of simplicial and antisimplicial vertices in $\{P_5, \overline{P_5}\}$-free graphs. In Section 3, we give a counterexample to the $H_6$-conjecture 1.4, and prove 1.5, a slightly weaker version of the conjecture. We also give a simpler proof of 1.2, and provide a counterexample to 1.6. Finally, in Section 4 we give a new proof of 1.1, and a structure theorem for $\{P_5, \overline{P_5}, \text{bull}\}$-free graphs.

### 2. SIMPLICIAL AND ANTISIMPLICIAL VERTICES

In this section, we prove the following result:

**Lemma 2.1.** All prime $\{P_5, \overline{P_5}, C_5\}$-free graphs have both a simplicial vertex, and an antisimplicial vertex.

Along the way we establish 2.9, a result that is helpful in finding simplicial and antisimplicial vertices in prime $\{P_5, \overline{P_5}\}$-free graphs.
Let \( G \) be a graph. We say \( G \) is connected if \( V(G) \) cannot be partitioned into two disjoint sets anticomplete to each other. If \( \overline{G} \) is connected we say that \( G \) is anticonnected. Let \( X \subseteq Y \subseteq V(G) \). We say \( X \) is a connected subset of \( Y \) if \( G[X] \) is connected, and that \( X \) is an anticonnected subset of \( Y \) if \( G[X] \) is anticonnected. A component of \( X \) is a maximal connected subset of \( X \), and an anticomponent of \( X \) is a maximal anticonnected subset of \( X \).

First, we make the following three easy observations:

**Lemma 2.2.** If \( G \) is a prime graph, then \( G \) is connected and anticonnected.

**Proof.** Passing to the complement if necessary, we may suppose \( G \) is not connected. Since \( G \) has at least four vertices, there exists a component \( C \) of \( V(G) \) such that \( |V(G) \setminus C| \geq 2 \). However, then \( V(G) \setminus C \) is a homogeneous set, a contradiction. This proves 2.2. \( \blacksquare \)

We say a vertex \( v \in V(G) \setminus X \) is mixed on an edge of \( X \), if there exist adjacent \( x, y \in X \) such that \( v \) is mixed on \( \{x, y\} \). Similarly, a vertex \( v \in V(G) \setminus X \) is mixed on a nonedge of \( X \), if there exist nonadjacent \( x, y \in X \) such that \( v \) is mixed on \( \{x, y\} \).

**Lemma 2.3.** Let \( G \) be a graph, \( X \subseteq V(G) \), and suppose \( v \in V(G) \setminus X \) is mixed on \( X \).

1. If \( X \) is a connected subset of \( V(G) \), then \( v \) is mixed on an edge of \( X \).
2. If \( X \) is an anticonnected subset of \( V(G) \), then \( v \) is mixed on a nonedge of \( X \).

**Proof.** Suppose \( X \) is a connected subset of \( V(G) \). Since \( v \) is mixed on \( X \), both \( X \cap N(v) \) and \( X \setminus N(v) \) are nonempty. As \( G[X] \) is connected, there exists an edge given by \( x \in X \cap N(v) \) and \( y \in X \setminus N(v) \), and \( v \) is mixed on \( \{x, y\} \). This proves 2.3.1. Passing to the complement, we get 2.3.2. \( \blacksquare \)

**Lemma 2.4.** Let \( G \) be a graph, \( X_1, X_2 \subseteq V(G) \) with \( X_1 \cap X_2 = \emptyset \), and \( v \in V(G) \setminus (X_1 \cup X_2) \).

1. If \( G \) is \( P_5 \)-free, and \( X_1, X_2 \) are connected subsets of \( V(G) \) anticomplete to each other, then \( v \) is not mixed on both \( X_1 \) and \( X_2 \).
2. If \( G \) is \( \overline{P}_5 \)-free, and \( X_1, X_2 \) are anticonnected subsets of \( V(G) \) complete to each other, then \( v \) is not mixed on both \( X_1 \) and \( X_2 \).

**Proof.** Suppose \( G \) is \( P_5 \)-free, \( X_1, X_2 \) are disjoint connected subsets of \( V(G) \) anticomplete to each other, and \( v \) is mixed on both \( X_1 \) and \( X_2 \). By 2.3.1, \( v \) is mixed on an edge of \( X_1 \), given by say \( x_1, y_1 \in X_1 \) with \( v \) adjacent to \( x_1 \) and nonadjacent to \( y_1 \), and an edge of \( X_2 \), given by say \( x_2, y_2 \in X_2 \) with \( v \) adjacent to \( x_2 \) and nonadjacent to \( y_2 \). However, then \( \{y_1, x_1, v, x_2, y_2\} \) is a \( P_5 \), a contradiction. This proves 2.4.1. Passing to the complement, we get 2.4.2. \( \blacksquare \)

As a consequence of 2.3 and 2.4 we obtain the following two useful results:

**Lemma 2.5.** Let \( u \) and \( v \) be nonadjacent vertices in a \( \overline{P}_5 \)-free graph \( G \), and let \( A \) be an anticonnected subset of \( N(u) \cap N(v) \). Then no vertex \( w \in V(G) \setminus (A \cup \{u, v\}) \) can be mixed on both \( A \) and \( \{u, v\} \).

**Proof.** Since \( A \) and \( \{u, v\} \) are disjoint anticonnected subsets of \( V(G) \) complete to each other, 2.5 follows from 2.4.2. \( \blacksquare \)
Lemma 2.6. Let $u, v,$ and $w$ be three pairwise nonadjacent vertices in a $\{P_5, \overline{P}_5\}$-free graph $G$ such that $w$ is mixed on an anticonnected subset $A$ of $N(u) \cap N(v)$. Then no vertex $z \in N(w) \setminus (A \cup \{u, v\})$ can be mixed on $\{u, v\}$.

Proof. Suppose there exists a vertex $z \in N(w) \setminus (A \cup \{u, v\})$ that is mixed on $\{u, v\}$, with say $z$ adjacent to $v$ and nonadjacent to $u$. Since $w$ is mixed on $A$, by 2.3.2, it follows that $w$ is mixed on a nonedge of $A$, given by say $x, y \in A$ with $w$ adjacent to $x$ and nonadjacent to $y$. By 2.5, $z$ is not mixed on $A$. However, if $z$ is anticomplete to $A$, then $\{y, u, x, w, z\}$ is a $P_5$, and if $z$ is complete to $A$, then $\{x, y, w, u, z\}$ is a $\overline{P}_5$, in both cases a contradiction. This proves 2.6. \hfill \blacksquare

Now, we can start to prove 2.1.

Lemma 2.7. Let $G$ be a prime $\{P_5, \overline{P}_5, C_5\}$-free graph. Then $G$ has an antisimplicial vertex, or admits a 1-join.

Proof. Suppose $G$ does not admit a 1-join. Let $W$ be a maximal subset of vertices that has a partition $A_1 \cup \ldots \cup A_k$ with $k \geq 2$ such that:

- $A_1, \ldots, A_k$ are all anticonnected subsets of $V(G)$, and
- $A_1, \ldots, A_k$ are pairwise complete to each other.

(1) $V(G) \setminus W$ is nonempty.

By 2.2, $G$ is anticonnected, which implies that $V(G) \setminus W$ is nonempty. This proves (1).

(2) Every $v \in V(G) \setminus W$ is either anticomplete to or mixed on $A_i$ for each $i \in \{1, \ldots, k\}$.

Suppose $v \in V(G) \setminus W$ is complete to some $A_i$. Take $B$ to be the union of all the $A_j$ to which $v$ is complete. However, since $\{v\} \cup W \setminus B$ is anticonnected and complete to $B$, it follows that $W' = B \cup (\{v\} \cup W \setminus B)$ contradicts the maximality of $W$. This proves (2).

(3) If for some $i \in \{1, \ldots, k\}$, $v \in V(G) \setminus W$ is mixed on $A_i$, then $v$ is anticomplete to $W \setminus A_i$.

By 2.4.2, any $v \in V(G) \setminus W$ is mixed on at most one $A_i$, and so together with (2) this proves (3).

(4) Every vertex in $V(G) \setminus W$ is mixed on exactly one $A_i$ for some $i \in \{1, \ldots, k\}$.

Suppose not. Let $X \subseteq V(G) \setminus W$ be the set of vertices anticomplete to $W$, which is nonempty by (2) and (3). By 2.2, $G$ is connected, and so there exists an edge given by $v \in X$ and $u \in V(G) \setminus (X \cup W)$. By (2), $u$ is mixed on some $A_i$, and so, by 2.3.2, $u$ is mixed on a nonedge of $A_i$, given by say $x_i, y_i \in A_i$ with $u$ adjacent to $x_i$ and nonadjacent to $y_i$. However, by (3), $u$ is anticomplete to $W \setminus A_i$, and so for $j \neq i$ and a vertex $z \in A_j$ we get that $\{v, u, x_i, z, y_i\}$ is a $P_5$, a contradiction. This proves (4).

And so, by (3) and (4), we can partition $V(G) = A_1 \cup \ldots \cup A_k \cup B_1 \cup \ldots \cup B_k$, where each $B_i$ is the set of vertices mixed on $A_i$ and anticomplete to $(A_1 \cup \ldots \cup A_k) \setminus A_i$.

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(5) \( B_1, \ldots, B_k \) are pairwise anticomplete.

Suppose for \( i \neq j, b_i \in B_i \) is adjacent to \( b_j \in B_j \). By 2.3.2, \( b_i \) is mixed on a nonedge of \( A_i \), given by say \( x_i, y_i \in A_i \) with \( b_i \) adjacent to \( x_i \) and nonadjacent to \( y_i \). As \( b_j \) is mixed on \( A_j \), there exists \( x_j \in A_j \) nonadjacent to \( b_j \), however then \( \{b_j, b_i, x_i, x_j, y_i\} \) is a \( P_5 \), a contradiction. This proves (5).

(6) Exactly one \( B_i \) is nonempty.

By (1) and (4), at least one \( B_i \) is nonempty. Suppose for \( i \neq j, B_i \) and \( B_j \) are both non-empty. Then, by (5), \( A = B_i, B = A_i, C = (A_1 \cup \ldots \cup A_k) \setminus A_i, \) and \( D = (B_1 \cup \ldots \cup B_k) \setminus B_i \) is a 1-join, a contradiction. This proves (6).

Hence, by (6), we may assume \( B_1 \) is nonempty while \( B_2, \ldots, B_k \) are all empty.

(7) \( k = 2 \) and \( |A_2| = 1 \).

Since \( A_2 \cup \ldots \cup A_k \) is not a homogeneous set, (6) implies that \( k = 2 \) and \( |A_2| = 1 \).

This proves (7).

Let \( a \) be the vertex in \( A_2 \).

(8) \( B_1 \) is a stable set.

Suppose not. Then there exists a component \( B \) of \( B_1 \) with \( |B| > 1 \). Since \( a \) is anticomplete to \( B_1 \), and \( B \) is a component of \( B_1 \), as \( G \) is prime, it follows that there exist \( a_1 \in A_1 \) that is mixed on \( B \). Thus, by 2.3.1, \( a_1 \) is mixed on an edge of \( B \), given by say \( b, b' \in B \) with \( a_1 \) adjacent to \( b \) and nonadjacent to \( b' \). Next, partition \( A_1 = C \cup D \) with \( C = A_1 \cap (N(b) \setminus N(b')) \) and \( D = A_1 \setminus C \), where both \( C \) and \( D \) are nonempty, as \( a_1 \in C \) and \( b' \) is mixed on \( A_1 \). Since \( A_1 \) is anticonnected there exists a nonedge given by \( c \in C \) and \( d \in D \). However, since \( d \in D \), it follows that \( \{d, a, c, b, b'\} \) is either a \( P_5 \), \( \overline{P_5} \), or \( C_5 \), a contradiction. This proves (8).

Thus, by (8), \( a \) is an antisimplicial vertex. This proves 2.7.

Next, we observe:

**Lemma 2.8.** Let \( u \) and \( v \) be nonadjacent vertices in a prime \( \overline{P_5} \)-free graph \( G \). Then either

- \( \mathcal{N}(u) \cap \mathcal{N}(v) \) is a clique, or
- there exists a vertex \( w \in V(G) \setminus (\mathcal{N}(u) \cup \mathcal{N}(v) \cup \{u, v\}) \) that is mixed on an anticonnected subset of \( \mathcal{N}(u) \cap \mathcal{N}(v) \).

**Proof.** Suppose \( \mathcal{N}(u) \cap \mathcal{N}(v) \) is not a clique. Then there exists an anticomponent \( A \) of \( \mathcal{N}(u) \cap \mathcal{N}(v) \) with \( |A| > 1 \). Since \( \{u, v\} \) is complete to \( \mathcal{N}(u) \cap \mathcal{N}(v) \), and \( A \) is an anticomponent of \( \mathcal{N}(u) \cap \mathcal{N}(v) \), as \( G \) is prime, it follows that there exists \( w \in V(G) \setminus ((\mathcal{N}(u) \cap \mathcal{N}(v)) \cup \{u, v\}) \) that is mixed on \( A \). Thus, by 2.5, \( w \) is not mixed on \( \{u, v\} \), and so \( w \) is anticomplete to \( \{u, v\} \). This proves 2.8.

A useful consequence of 2.8 is the following:

**Lemma 2.9.** Let \( v \) be a vertex in a prime \( \{P_5, \overline{P_5}\} \)-free graph \( G \).

1. If \( v \) is ant simplicial, and we choose \( u \) nonadjacent to \( v \) such that \( |\mathcal{N}(u) \cap \mathcal{N}(v)| \) is minimum, then \( u \) is a simplicial vertex.
2. If \( v \) is simplicial, and we choose \( u \) adjacent to \( v \) such that \( |\mathcal{N}(u) \cup \mathcal{N}(v)| \) is maximum, then \( u \) is an antisimplicial vertex.
Proof. Suppose \( v \) is antisimplicial, we choose \( u \) nonadjacent to \( v \) such that \( |N(u) \cap N(v)| \) is minimum, and \( u \) is not simplicial. Since \( v \) is antisimplicial, it follows that \( N(u) \setminus N(v) \) is empty, and thus, as \( u \) is not simplicial, \( N(u) \cap N(v) \) is not a clique. Hence, by 2.8, there exists some \( w \), nonadjacent to both \( u \) and \( v \), which is mixed on an anticonnected subset of \( N(u) \cap N(v) \). However, then, by our choice of \( u \), there exists a vertex \( z \in N(v) \setminus N(u) \) adjacent to \( w \), contradicting 2.6. This proves 2.9.1. Passing to the complement, we get 2.9.2. 

Lemma 2.10. Let \( G \) be a prime \( \{P_5, P_5, C_5\}\)-free graph. Then \( G \) has a simplicial vertex, or an antisimplicial vertex.

Proof. Suppose \( G \) does not have an antisimplicial vertex. Then, by 2.7, it admits a 1-join \( (A, B, C, D) \).

(1) \( A \) and \( D \) are stable sets. 
By symmetry, it suffices to argue that \( A \) is a stable set. Suppose not. Then there exists a component \( A' \) of \( A \) with \( |A'| > 1 \). Since \( C \cup D \) is anticomplete to \( A \), and \( A' \) is a component of \( A \), as \( G \) is prime, it follows that there exists \( b \in B \) that is mixed on \( A' \). Thus, by 2.3.1, \( b \) is mixed on an edge of \( A' \), given by say \( a, a' \in A' \) with \( b \) adjacent to \( a' \) and nonadjacent to \( a \). By 2.2, \( G \) is connected, and so there exists an edge given by \( c \in C \) and \( d \in D \). However, then \( \{a, a', b, c, d\} \) is a \( P_5 \), a contradiction. This proves (1).

Next, fix some \( c \in C \), and choose a vertex \( a \in A \) such that \( |N(a) \cap N(c)| \) is minimum.

(2) \( a \) is a simplicial vertex.
Suppose not. Then, by (1), \( N(a) \cap N(c) = N(a) \subseteq B \) is not a clique, and so, by 2.8, there exists \( w \), nonadjacent to both \( a \) and \( c \), which is mixed on an anticonnected subset of \( N(a) \cap N(c) \). Since \( B \) is complete to \( C \) and anticomplete to \( D \), it follows that \( w \) belongs to \( A \). However, then, by our choice of \( a \), there exists a vertex \( z \in N(c) \setminus N(a) \) adjacent to \( w \), contradicting 2.6. This proves (2).

This completes the proof of 2.10. 

Putting things together we can now prove 2.1.

Proof of 2.1. By 2.10, passing to the complement if necessary, there exists an antisimplicial vertex \( a \). And so, by 2.9.1, if we choose \( s \) nonadjacent to \( a \) such that \( |N(a) \cap N(s)| \) is minimum, then \( s \) is simplicial. This proves 2.1. 

3. THE \( H_6 \)-CONJECTURE

In this section, we give a counterexample to the \( H_6 \)-conjecture 1.4, and prove 1.5, a slightly weaker version of the conjecture. We also give a proof of 1.2, and provide a counterexample to 1.6.

We begin by establishing some properties of prime graphs. Recall the following theorem of Seinsche [7]:

Lemma 3.1. If \( G \) is a \( P_4 \)-free graph with at least two vertices, then \( G \) is either not connected or not anticonnected.
Lemma 3.2. Every prime graph contains $P_4$.

Next, as first shown by Hoàng and Khouzam [4], we observe that:

Lemma 3.3. Let $G$ be a prime graph.

1. A vertex $v \in V(G)$ is simplicial if and only if $v$ is a degree one vertex in every copy of $P_4$ in $G$ containing it.
2. A vertex $v \in V(G)$ is antisimplicial if and only if $v$ is a degree two vertex in every copy of $P_4$ in $G$ containing it.

Proof. Both forward implications are clear. To prove the converse of 3.3.1, suppose there exists a vertex $v$ that is not simplicial and yet is a degree one vertex in every copy of $P_4$ in $G$ containing it. Then there exists an anticomponent $A$ of $N(v)$ with $|A| > 1$. Since $v$ is complete to $A$, and $A$ is a anticomponent of $N(v)$, as $G$ is prime, it follows that there exists $u \in V(G) \setminus (N(v) \cup \{v\})$ that is mixed on $A$. Thus, by 2.3.2, $u$ is mixed on a nonedge of $A$, given by say $x, y \in A$ with $u$ adjacent to $x$ and nonadjacent to $y$. However, then $\{y, v, x, u\}$ is a $P_4$ with $v$ having degree two, a contradiction. This proves 3.3.1. Passing to the complement, we get 3.3.2.

Finally, we observe that:

Lemma 3.4. Let $G$ be a prime graph.

1. The set of antisimplicial vertices in $G$ is a clique.
2. The set of simplicial vertices in $G$ is a stable set.

Proof. Suppose there exist nonadjacent antisimplicial vertices $a, a' \in V(G)$. Since $a$ is antisimplicial, it follows that $N(a') \setminus N(a)$ is empty. Similarly, $N(a) \setminus N(a')$ is also empty. However, this implies that $\{a, a'\}$ is a homogeneous set in $G$, a contradiction. This proves 3.4.1. Passing to the complement, we get 3.4.2.

Lemma 3.5. Let $G$ be a prime $\{P_5, \overline{P_5}, C_5\}$-free graph. Let $A$ be the set of antisimplicial vertices in $G$, and let $S$ be the set of simplicial vertices in $G$. Then $G[A \cup S]$ is a split graph which is both connected and anticonnected.

Proof. 3.4 implies that $G[A \cup S]$ is a split graph, where $A$ is a clique and $S$ is a stable set. By 2.9.1, every vertex in $A$ has a nonneighbor in $S$, and, by 2.9.2, every vertex in $S$ has a neighbor in $A$. Thus, $G[A \cup S]$ is both connected and anticonnected. This proves 3.5.

We are finally ready to give a proof of 1.2, first shown in [6] by Hayward and Nastos.

Lemma 3.6. If $G$ is a prime $\{P_5, \overline{P_5}, C_5\}$-free graph, then there exists a copy of $P_4$ in $G$ whose vertices of degree one are simplicial, and whose vertices of degree two are antisimplicial.

Proof. Let $A$ be the set of antisimplicial vertices in $G$, and let $S$ be the set of simplicial vertices in $G$. By 2.1, both $A$ and $S$ are nonempty. Hence, $G[A \cup S]$ is a graph with at least two vertices, which, by 3.5, is both connected and anticonnected, and so, by
3.1, it follows that \( G[A \cup S] \) contains \( P_4 \). Since 3.4 implies that \( A \) is a clique and \( S \) is a stable set, it follows that every copy of \( P_4 \) in \( G[A \cup S] \) is of the desired form. This proves 3.6.

Next, we turn our attention to the \( H_6 \)-conjecture. A result of Hoàng and Reed [5] implies the following:

**Theorem 3.7.** If \( G \) is a prime \( \{P_5, \overline{P}_5, C_5\} \)-free graph that is not split, then \( G \) or \( \overline{G} \) contains \( H_6 \).

In hopes of saying more along these lines, motivated by 3.6 and 3.7, Hayward and Nastos posed 1.4, which we restate:

**Theorem 3.8 (The \( H_6 \)-Conjecture).** If \( G \) is a prime \( \{P_5, \overline{P}_5, C_5\} \)-free graph that is not split, then there exists a copy of \( H_6 \) in \( G \) or \( \overline{G} \) whose two vertices of degree one are simplicial, and whose two vertices of degree three are antisimplicial.

In Figure 2, we give a counterexample to 3.8. The graph \( G \) in Figure 2 contains \( C_4 \), and so, by 1.3, is not split. The mapping \( \phi : V(G) \to V(\overline{G}) \)

\[
\phi := \begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\
3 & 1 & 4 & 2 & 7 & 5 & 8 & 6 & 11 & 9 & 12 & 10
\end{pmatrix}
\]

is an isomorphism between \( G \) and \( \overline{G} \). Thus, as \( G \) is self-complementary, it suffices to check that \( G \) is \( P_5 \)-free, which is straightforward, as is verifying that \( G \) is prime, and we leave the details to the reader. The set of simplicial vertices in \( G \) is \( \{1, 4\} \), and the set of antisimplicial vertices in \( G \) is \( \{2, 3\} \). However, no copy of \( C_4 \) in \( G \) contains \( \{2, 3\} \), and so there does not exist a copy of \( H_6 \) of the desired form.

However, all is not lost as we can prove 1.5, a slightly weaker version of the \( H_6 \)-conjecture, which we restate:

**Lemma 3.9.** If \( G \) is a prime \( \{P_5, \overline{P}_5, C_5\} \)-free graph that is not split, then there exists a copy of \( H_6 \) in \( G \) or \( \overline{G} \) whose two vertices of degree one are simplicial, and at least one of whose vertices of degree three is antisimplicial.

**Proof.** By 3.6, there exist simplicial vertices \( s, s' \), and antisimplicial vertices \( a, a' \) such that \( \{s, a, a', s'\} \) is a \( P_5 \) in \( G \). Now, choose maximal subsets \( A \) of antisimplicial vertices in \( G \), and \( S \) of simplicial vertices in \( G \) such that \( a, a' \in A \), \( s, s' \in S \), every vertex in \( A \) has a neighbor in \( S \), and every vertex in \( S \) has a nonneighbor in \( A \).

1. **Any graph containing a vertex that is both simplicial and antisimplicial is split.**
   By definition, if a vertex \( v \in V(G) \) is both simplicial and antisimplicial, then \( N(v) \) is a clique and \( V(G) \setminus N(v) \) is a stable set. This proves (1).

2. **There exists no vertex \( v \in V(G) \setminus (A \cup S) \) adjacent to a vertex \( u \in S \) and nonadjacent to a vertex \( w \in A \).**
   Suppose not. If \( u \) is adjacent to \( w \), then \( N(u) \) is not a clique, and if \( u \) is nonadjacent to \( w \), then \( V(G) \setminus N(w) \) is not a stable set, in both cases a contradiction. This proves (2).

By (1) and (2), we can partition \( V(G) = A \cup S \cup B \cup C \cup D \), where \( B \) is the set of vertices complete to \( A \) and anticomplete to \( S \), \( C \) is the set of vertices complete to \( A \) with a neighbor.
in $S$, and $D$ is the set of vertices anticomplete to $S$ with a nonneighbor in $A$. Recall 3.4 implies that $A$ is a clique and $S$ is a stable set.

(3) No vertex of $C \cup D$ is simplicial or antisimplicial.

Consider a vertex $c \in C$. Then there exists $s_c \in S$ adjacent to $c$. Hence, $c$ is not antisimplicial, as otherwise we could add $c$ to $A$ contrary to maximality. By construction, $s_c$ has a nonneighbor $a_c \in A$. Since $c$ is complete to the $A$, it follows that $N(c)$ is not a clique, and thus $c$ is not simplicial. Hence, $C$ contains no simplicial or antisimplicial vertices. Passing to the complement, we get that no vertex in $D$ is simplicial or antisimplicial. This proves (3).

(4) We may assume that $C$ is a clique, and $D$ is a stable set.

By symmetry, it is enough to argue that if $D$ is not a stable set, then the theorem holds. Suppose we have an edge given by $x, y \in D$. By definition, any antisimplicial vertex is adjacent to at least one of $x$ and $y$. And so, as $x$ and $y$ both have nonneighbors in $A$, there exists $a_x, a_y \in A$ such that $a_x$ is adjacent to $x$ and nonadjacent to $y$, and $a_y$ is adjacent to $y$ and nonadjacent to $x$. Since $S$ is anticomplete to $D$, it follows that $a_x$ and $a_y$ do not have a common neighbor $s'' \in S$, as otherwise \{a_x, y, s'', x, a_y\} is a $P_4$. By construction, every vertex in $A$ has a neighbor in $S$, and so there exists $s_x \in S$ adjacent to $a_x$ and nonadjacent to $a_y$, and $s_y \in S$ adjacent to $a_y$ and nonadjacent to $a_x$. However, then \{s_x, a_x, a_y, s_y, x, y\} is a copy of $H_6$ in $G$ of the desired form. Passing to the complement, we may also assume that $C$ is a clique. This proves (4).

(5) For all $d \in D$ and $u \in A$, $N(d) \subseteq N(u) \cup \{u\}$.

By (4), $A \cup C$ is a clique and $D \cup S$ is a stable set. Thus, for any $d \in D$, it follows that $N(d) \subseteq A \cup B \cup C$. Since $A$ is complete to $B$, it follows that any $a \in A$ is complete to $(A \setminus \{a\}) \cup B \cup C$. This proves (5).

(6) We may assume both $C$ and $D$ are empty.

By symmetry, it is enough to argue that if $D$ is nonempty, then the theorem holds. Suppose $D$ is nonempty, and choose $d \in D$ with $|N(d)|$ minimum. Then there exists $a_d \in A$ nonadjacent to $d$. By (3) and (5), $N(a_d) \cap N(d) = N(d)$ is not a clique, and so, by 2.8, there exists a vertex $w$, nonadjacent to both $a_d$ and $d$, which is mixed on an anticonnected subset of $N(d)$. Since $a_d$ is complete to $(A \setminus \{a_d\}) \cup B \cup C$, it follows that $w \in D \cup S$. If $w \in D$, then, by our choice of $d$, there exists $z \in N(w) \setminus N(d)$ which, by (5), is adjacent to $a_d$, contradicting 2.6. Hence, $w \in S$. Since $w$ is mixed on an anticonnected subset of $N(d)$, by 2.3.2, $w$ is mixed on a nonedge of $N(d)$, given by say $x, y \in N(d)$ with $w$ adjacent to $x$ and nonadjacent to $y$. Since $A \cup C$ is a clique, and $B$ is complete to $A$ and anticomplete to $S$, it follows that $x \in C$ and $y \in B$. By construction, every vertex in $A$ has a neighbor in $S$, and so there exists $s_d \in S$ adjacent to $a_d$. Since $a_d$ is mixed on \{a_d, d\} and nonadjacent to $y$, 2.5 implies that $s_d$ is anticomplete to $\{x, y\}$. However, then \{s_d, a_d, x, w, y, d\} is a copy of $H_6$ in $G$ of the desired form. Passing to the complement, we may also assume that $C$ is empty. This proves (6).

By (6), since $G$ is prime, it follows that $|B| \leq 1$, implying that $G$ is a split graph, a contradiction. This proves 3.9.}

Another conjecture that seemed plausible for a while is 1.6, which we restate:
Theorem 3.10. If \( G \) is a \([P_5, \overline{P_5}]\)-free graph, then either

- \( G \) is isomorphic to \( C_5 \), or
- \( G \) is a split graph, or
- \( G \) has a homogeneous set, or
- \( G \) or \( \overline{G} \) admits a 1-join.

However, with Paul Seymour we found the counterexample in Figure 3. The graph in Figure 3 contains \( C_4 \) and \( \overline{C_4} \), and so, by 1.3, is not split; we leave the rest of the details to the reader.

4. \([P_5, \overline{P_5}, \text{bull}]\)-FREE GRAPHS

In this section, we give a short proof of 1.1, and of Fouquet’s result 4.4 on the structure of \([P_5, \overline{P_5}, \text{bull}]\)-free graphs. The following is joint work with Max Ehramm.

Let \( O_k \) be the bipartite graph on \( 2^k \) vertices with bipartition \((\{a_1, \ldots, a_k\}, \{b_1, \ldots, b_k\})\) in which \( a_i \) is adjacent to \( b_j \) if and only if \( i + j \geq k + 1 \). If a graph \( G \) is isomorphic to \( O_k \) for some \( k \), then we call \( G \) a half graph. Note that by construction half graphs are prime.

In [1] the first author and Seymour proved:

Lemma 4.1. Let \( G \) be a graph, and let \( H \) be a proper induced subgraph of \( G \). Assume that both \( G \) and \( H \) are prime, and that both \( G \) and \( \overline{G} \) are not half graphs. Then there exists an induced subgraph \( H' \) of \( G \), isomorphic to \( H \), and a vertex \( v \in V(G) \setminus V(H') \), such that \( G[V(H') \cup \{v\}] \) is prime.

Next, we give a proof of Fouquet’s result 1.1, which we restate:

Lemma 4.2. If \( G \) is a prime \([P_5, \overline{P_5}]\)-free graph that contains \( C_5 \), then \( G \) is isomorphic to \( C_5 \).

Proof. Suppose not. By 3.2, \( G \) contains \( P_4 \), which is isomorphic to \( O_2 \). Since \( G \) and \( \overline{G} \) are not half graphs, it follows that \( P_4 \) is a proper induced subgraph of \( G \). As \( P_4 \) is prime, by 4.1, there exists a subgraph \( H \) induced by \( \{v_1, v_2, v_3, v_4\} \) isomorphic to \( P_4 \), and a vertex \( v \in V(G) \setminus V(H) \) such that the subgraph of \( G \) induced by \( V(H) \cup \{v\} \) is prime. This proves 4.2.

Thus, to understand prime \([P_5, \overline{P_5}, \text{bull}]\)-free graphs it is enough to study prime \([P_5, \overline{P_5}, C_5], \text{bull}\)-free graphs.

Lemma 4.3. If \( G \) is a prime \([P_5, \overline{P_5}, C_5, \text{bull}]\)-free graph, then either \( G \) or \( \overline{G} \) is a half graph.

Proof. Suppose not. By 3.2, \( G \) contains \( P_4 \), which is isomorphic to \( O_2 \). Since \( G \) and \( \overline{G} \) are not half graphs, it follows that \( P_4 \) is a proper induced subgraph of \( G \). As \( P_4 \) is prime, by 4.1, there exists a subgraph \( H \) induced by \( \{v_1, v_2, v_3, v_4\} \) isomorphic to \( P_4 \), and a vertex \( v \in V(G) \setminus V(H) \) such that the subgraph of \( G \) induced by \( V(H) \cup \{v\} \) is prime.
Considering the complement, we may assume \( v \) is adjacent to at most two vertices in \( H \). To avoid a homogeneous set in \( G[V(H) \cup \{v\}] \), by symmetry, the only possibilities are for \( N(v) = \{v_1\} \), in which case \( \{v, v_1, v_2, v_3, v_4\} \) is a \( P_5 \), for \( N(v) = \{v_1, v_4\} \), in which case \( \{v, v_1, v_2, v_3, v_4\} \) is a bull, in all cases a contradiction. This proves 4.3.

Putting things together we obtain Fouquet’s original structural result [3]:

**Theorem 4.4.** If \( G \) is a \( \{P_5, \overline{P_5}, \text{bull}\} \)-free graph, then either

- \( |V(G)| \leq 2 \), or
- \( G \) is isomorphic to \( C_5 \), or
- \( G \) has a homogeneous set, or
- \( G \) or \( \overline{G} \) is a half graph.

**Proof.** As all graphs on three vertices have a homogeneous set, 4.4 immediately follows from 4.2 and 4.3.

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