Introduction—Two paths are known whereby local Hamiltonians in lattice models can stabilize complex spin order—meaning both that the spin configurations are complex in space, and that the phase diagram contains a zoo of different phases. One well-known path to such complexity is state frustration—meaning the ground states are massively degenerate. Any small perturbation, such as disorder [1], dipolar interactions [2], or simply the intrinsic quantum or thermal fluctuations [3–5], then suffice to select a particular state as the unique ground state.

A second path to complexity is through frustrated interactions, i.e. there are multiple kinds of Heisenberg spin couplings that cannot be satisfied simultaneously. Complexity may be realized with as few as two isotropic neighbor distances, but only when the spin sites form a non-Bravais lattice, such as Kagome or Pyrochlore lattices: rigorously, on Bravais lattices with isotropic couplings—at most simple coplanar spin spirals are realized [6].

The noncoplanar complex spin states are particularly intriguing for their unusual rigid-body-like order parameters. They are also motivated experimentally as they realize an Anomalous Hall effect due to Berry phases [7–10], and theoretically since if such a phase loses long range order at sufficiently small spin-length, it is expected to become a chiral spin liquid, induced without any spin-orbit effects [12].

Even more complex behavior is possible when the frustrated spin-spin interactions decay slowly with distance. That is easily realized by coupling local moments to a band of fermions, which mediate oscillating couplings between the local Heisenberg moments—the so called Kondo Lattice Model (KLM) [9–11, 13].

In this letter, we construct the full phase diagram for a Kagome lattice coupled to fermions, as a function of the filling and the coupling strength. We used recipes to identity and classify states laid out previously [14], which we have further extended here. The phase diagram so obtained includes many competing orders, with incommensurate wave vectors and non-coplanar spin arrangements. The complexity of the phase diagram motivates both a further theoretical examination of Kondo Lattice models and the experimental detection of novel spin orders found in this study.

Kondo Lattice Model to RKKY—We adopt the KLM Hamiltonian given by

$$\mathcal{H}_{KLM} = -t \sum_{\langle ij \rangle, \alpha, \beta} \hat{c}_{i\alpha}^\dagger c_{j\beta}^\dagger e_{ij}^{(\beta)\sigma} - J_K \sum_{i=1, N} S_{i(\alpha)} \cdot S_{i(\alpha)} \quad (1)$$

The first term is nearest-neighbor hopping with amplitude $t$ of a single band of noninteracting electrons, with creation operator $c_{i\alpha}^\dagger$ at unit cell $i$ and sublattice $\alpha$ [23]. The second term is the Kondo coupling, with $s_{i(\alpha)}$ being the electron spin and $S_{i(\alpha)}$ being classical Heisenberg spins representing the local moments. We seek to find the background-state configuration of the local moments $\{S_{i(\alpha)}^{opt}\}$, for every fermion filling.

Previous methods for finding $\{S_{i(\alpha)}^{opt}\}$ were either variational MC, with costly fermionic diagonalization at each MC step (this has been recently overcome by an efficient algorithm [17, 18] allowing exploration of large system sizes) or else a variational optimization of $\langle \mathcal{H}_{KLM} \rangle$ using a trial basis [10],
limited to commensurate orders with small unit cells. We instead developed an approach valid in the limit $J_K/t \ll 1$, which allows us to access large system sizes with complex orders. The effective Hamiltonian is the lowest-order ($O(J_K/t)^2$) perturbation expansion in the Kondo coupling. This is an exchange interaction between pairs of spins, the familiar RKKY interaction:

$$\mathcal{H}_{\text{RKKY}} = \frac{1}{2} \sum_{\langle i,j \rangle} J_{ij} S_i \cdot S_j$$  \hspace{1cm} (2)

To compute the couplings (at $T = 0$), we take a grid in reciprocal space; the corresponding lattice sizes were up to $N = 3 \times 36^2$. We first locate the Fermi surface corresponding to the chosen filling, then numerically evaluate the usual analytic formula for $J_{\langle i,j \rangle}$ from second-order perturbation theory (see Supplementary Information I) [24]. The result is an interaction $J_{\langle i,j \rangle}(q)$ [25] that Fourier transformed to give the direct-space RKKY couplings $J_{\langle i,j \rangle}(\beta)$.

The first five couplings are plotted in Figure 2-B, as a function of filling. Negative (positive) couplings correspond to ferromagnetic (antiferromagnetic) interactions. Observe that the farther the interaction, the more oscillations are seen; and there are some fillings at which both $J_1$ and $J_2$ are dominated by more distant couplings. It should also be noted that (at $T = 0$) the magnitude of RKKY couplings decays in two dimensions (as $1/(\mathcal{R})^2$), a long-range coupling, so it may be dangerous to truncate couplings to the first few neighbors [19].

Keeping all possible $J_{\langle i,j \rangle}(\beta)$ on a finite lattice, we analyze spin orders that minimize the RKKY interaction in two steps. First, using Luttinger-Tisza (L.T.) and second by direct Monte Carlo (MC) minimization.

Orders from Luttinger-Tisza analysis—An informed guess about the wave vector composition of states $\{S_{i(\alpha)}\}$ minimizing Eq.(2) can be made using Luttinger-Tisza (L.T.) analysis [20]. The L.T. framework gives a star of $C_6$ symmetry related L.T. wave vectors $\{Q_{L,T}\}$ which have the most dominant (negative) contribution to $J_{\langle i,j \rangle}(q)$. Each such contribution, coming from a wave vector in the zone, is obtained by diagonalizing $J_{\alpha \beta}(q)$:

$$J_{\alpha \beta}(q) u^{\nu}(q) = \lambda^{\nu}(q) u^{\nu}(q)$$  \hspace{1cm} (3)

where the $\nu$ index labels the three bands, $\lambda^{\nu}(q)$ is the eigenvalue corresponding to the mode $u^{\nu}(q)$ and band $\nu$.

The wave vector that goes along with the smallest eigenvalue of $J_{\alpha \beta}(q)$ is identified to be L.T. ordering vector $\{Q_{L,T}\}$. In the RKKY limit of Eq.(1), the set of ordering wave vectors are determined, not surprisingly, by the F.S. geometry.

The F.S., as plotted in Fig. 2(A) evolves as follows with filling $n$: the essentially circular F.S. at $n \lesssim 0.05$ evolves in the range $0.1 < n < 0.25$ to lie along straight lines connecting the zone $M$ points. Beyond $n = 0.25$ the F.S. shrinks again to a set of circles around the zone’s $K$ point, which it reaches at filling $n = 1/3$. Beyond $n = 1/3$, the F.S. enters the Kagome $\nu = 2$ band.

As the F.S. evolves, so do the nesting vectors (see Supplementary Information V), and hence so does the optimal ordering wave vector $Q_{L,T}(n)$ as shown in Fig. 2(c). In the limit $n \to 0$, all RKKY interactions are ferromagnetic, i.e. the order is at the $\Gamma$ point. For $n \lesssim 0.05$, some distant interactions become antiferromagnetic: we conjecture the L.T. wave vector steadily evolves towards the zone $M$ point, so the spin states are ferromagnetic with some incommensurate twist added [26]. The $Q_{L,T}(n)$ vector hits the zone edge at $n = 0.105$, and continues to move (or, equivalently, to move back) smoothly along the same mirror line. Throughout this trajectory, the $Q_{L,T}(n)$ vector arises from connections between points on the F.S.—an observation also noted in a recent study of KLM on Bravais lattices [21]. As $n \to 0.25$, the dominant F.S. wavevector approaches the zone-boundary where the F.S. kisses itself, and $Q_{L,T}(n)$ approaches the $\Gamma$ point again. However, exactly at $n = 1/4$, and probably for a small interval on either side of it, a different nesting vector beats out the one we described: this comes from a conventional nesting between flat segments of F.S. and includes a brief detour of $Q_{L,T}(n)$ to the $M$ point—a first order transition. For $1/4 < n < 1/3$, the $Q_{L,T}(n)$ resumes moving smoothly, but switches trajectory to the other kind of mirror symmetry, reaching the $K$ point at $n = 1/3$.

For $n > 1/3$, the first band is filled and the F.S. falls in the second band. Due to a symmetry relating these two bands, the evolution from $n = 1/3$ to $n = 2/3$, as shown in Fig. 2-C, is the same as the first band, so that $n$ and $2/3 - n$ always have the same $Q_{L,T}$.
The trajectory of $Q_{L,T}$ also guides the construction of candidate ground states of KLM at special fillings (Van Hove) where $J_{\alpha\beta}(q)$ is singular[27]. Two examples are the non-coplanar cuboc1[15] and cuboc2[16](Fig.1 (A)-(C)) orders with the 12 spin directions (in the magnetic unit cell) aligned with the corners of a cuboctahedron. Both these states are composed (see Supplementary Information III) of wave vectors connecting parallel parts of the F.S. (shown in Fig.2-(A) and labeled $Q_{3\mu}$). The stability of these orders, against a host of other states, will be tested in the variational phase diagram in a later section.

Symmetry broken orders—The L.T. wavevector is insufficient to define a spin order, because the L.T. star usually includes several symmetry-related wavevectors, and there are many different ways these could be distributed on the three sublattices of the Kagome lattice. We use both zero and finite temperature MC to get access to the rich family of spin orders $\{\text{S}^{\mu\nu}\}$ resulting from different ways of combining both the dominant L.T. wave vectors and the subdominant modes (required for spin normalization). The optimal spin configurations from MC is our starting point for the classification of symmetry broken orders of the RKKY Hamiltonian (2).

Let us imagine Fourier transforming the actual ground state $\{\text{S}^{\mu\nu}\}$ from MC. The first step is to identify the dominant mode (which, virtually always, is closely similar to the dominant L.T. mode). This is done by defining $\eta_{\alpha}(q)$:

$$\eta_{\alpha}(q) = \frac{|S^{opt}_{\alpha}(q)|^2}{\sum_{q\in \text{1st B.Z.}} |S^{opt}_{\alpha}(q)|^2} \quad (4)$$

which maps out the Fourier weights of different wave vectors $q$ in the optimal spin configuration. The combined set of wave vectors with the highest $\eta_{\alpha}(q)$ across all sublattices is defined as the star of dominant modes. We zero out the wave vectors except those from that dominant star, and reassemble the configuration in real space: this state will be called the “purified” configuration. This is the key starting point, for both our nomenclature to classify orders, and also in the practical processing of configurations from simulation results.

The second part of our label is to specify the way in which different ordering vectors go with the three Kagomé sublattices. The $n$ different ordering wavevectors in use (from that same star) are labeled $a, b, ...$. (As illustrated in Fig. 2(c), it turns out the L.T. vectors and correspondingly the dominant modes, always fall on mirror axes of the zone, which implies the star has at most 3 inequivalent wave vectors hence $m \leq 3$.) Then the label, e.g. $(a, b, c)$, has three entries referring to sublattices 1, 2, and 3 respectively, each listing which wave vectors are used on that sublattice [28] (For short, it is conventional to use the term “$nQ$” order, distinguishing the number of wave vectors but not their relation to the sublattices.)

As the filling varies, the L.T. wavevectors and the ordering wavevectors found from simulations vary continuously. We classify these phases as being in the same phase family if they share the same broken symmetries and evolve smoothly with $n$.

This classification is done by assigning a numeric subscript to dominant modes on a sublattice based on the symmetry of $\eta_{\alpha}(q)$ across $\alpha$ [29](see Supplementary Information II). A phase label $(a_1,b_1,c_1)$ indicates that the order is of type $3Q$ and the dominant modes on the three sublattices have the same Fourier weight. For convenience, we skip writing numerical indices for phases which have no symmetry in the distribution of $\eta_{\alpha}(q)$. All spin orders recovered in the RKKY limit are classified in to eight competing phases shown in Fig.4. Most of these phases consist of non-coplanar incommensurate spins: such states dominate the RKKY phase diagram for small $J_K$. We now turn to discuss two of the simplest orders.

![Figure 3: Two simple Incommensurate Orders](image)

(a1) (a2) (a3) (a4)

(a1) (a2) (a3) (a4)

(b1) (b2) (b3) (b4)

**Figure 3:** Two simple Incommensurate Orders. (a1)-(a3) Common origin plot [14] of spins on each of the three sublattices for a $(a_1, a_2, a_3)$ incommensurate coplanar spiral on $N = 3 \times 36^2$ found at $n = 0.325$, a(4): The "purified" state (see text) is a perfect coplanar spiral (b1)-(b3) Common origin plot for a $(a_1,b_1,c_1)$ 3Q order on $N = 3 \times 36^2$ found at $n = 0.488$, b(4) The "purified" state (see Supplementary Information III)

The first of these simple orders, shown in Fig.3-(a1)-(a4), is an $(a_1,a_2,a_3)$ incommensurate twist of the nine sublattice $\sqrt{3} \times \sqrt{3}$ order Fig.1-(D) with ordering wave vector at the zone corner. Although the optimal spin order is non-coplanar Fig.3-(a1)-(a3), the "purified" version in Fig.3-(a4) is an incommensurate coplanar spiral with wave vector $Q^{opt} = 2\pi(7/24,0)$ lying close to $K$. The second, more complicated spin order found at $n = 0.488$, is a representative spin set from the $(a_1,b_1,c_1)$ phase and is shown in Fig.3(b1)-(b4).

Spins on each sublattice, in the "purified" configuration in Fig.3-(b4), lie in a plane which is orthogonal to the planes of the other two sublattices.

Variational phase diagram—To test the stability of non-coplanar incommensurate orders at finite Kondo coupling strengths, we numerically diagonalize Eq. (1) to map out the KLM phase diagram. At a given value of $(n, J_K)$, the KLM ground state is approximated to be the lowest energy state from within a trial data base of exact ground states of the RKKY Hamiltonian. This data base also includes the incommensurate orders in Fig.1.

Fig.4 highlights the existence of non-coplanar orders even for large $J_K/l$. The interplay between different orders in Fig.4 is interpreted by dividing the phase diagram, across all fillings, into small, intermediate and high $J_K/l$ regime.

The small $J_K/l \ll 1$ regime, same as the RKKY limit, is dominated by non-coplanar orders. Most of these orders are incommensurate (as in Fig.3) and typically fan out from filling...
intervals where they were exact ground states of Eq. (2). On entry in to the intermediate \( J_K \) regime (typically around one) these orders lose out to commensurate states.

Large regions of the intermediate \( J_K / t \) regime of the phase diagram also stabilize non-coplanar orders. Of these, the non-coplanar \( \text{cuboc}1 \) (found at \( n = 5/12 \) [30], Supplementary InformationV) and \( \text{cuboc}2 \) (at \( n = 2/3 \)) orders are commensurate, while the orders making up the \( (a_1, b_1, c_1) \) phase shown in red in Fig.4 are incommensurate. Commensurate orders evolve smoothly from the low to the intermediate regime, the incommensurate orders are reentrant and cannot be continuously evolved (without crossing another state) from the small \( J_K \) regime.

Finally, the large \( J_K \) regime (also the Double Exchange limit of (1)) favors the trivial ferromagnetic order.

**Conclusion**— In this paper, we have shown that, on a Kagomé lattice, a Kondo-lattice coupling local moments and electrons with the simplest possible electron band structure, yields a host of generically non-coplanar and incommensurate spin states. We computed the RKKY spin-spin couplings induced by fermions, and verified that the “Luttinger-Tisza” framework (dominant eigenmodes of the interaction matrix) is a reliable guide to the orders we found by Monte Carlo simulations. This framework explains the overall evolution of the spin order with filling in terms of Fermi surface nesting (Fig.2). There are additional dimensions to spin orders than the L-T vectors, in particular the possibility of non-coplanar states based on multiple ordering vectors, examples of which are shown in Fig.3.

Finally, we used a different, “variational” method to obtain the approximate phase diagram for larger Kondo couplings (Fig. 4). We discovered an intermediate regime (before the ferromagnetic state sweeps the phase diagram in the Double Exchange limit): commensurate states are more prominent here, as expected, but (empirically) we found “reentrant” incommensurate orders also appear at different fillings than they were found in the limit of small \( J_K \). Such orders were neglected in previous studies[10] of KLM phase diagram. The intermediate regime could be more systematically studied using improved MC algorithm[17] or with an effective spin Hamiltonian including multi-spin interactions[11].

Non-coplanar orders appear rarely in Kagomé systems. Our results suggest that, in Kondo-coupled systems without spin-orbit effects, non-coplanar orders are abundant. Experimental signatures of these orders will be a non-zero Anomalous Hall response [7–10] and Bragg peaks at incommensurate positions in the zone.

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A recent work[22] on the same model with similar results was brought to our attention after completion of this draft.

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At this notation is incomplete, in that it does not specify whether /index. Using Eq. (5) we first write down the free fermion part /for small \( K_f \), this cannot be proven numerically: in a finite interval /the Kondo part of (1) can be expressed in real space as /Domenge et al. Phys. Rev. B 72, 024433 (2005) /Y. Kato, I. Martin and C. D. Batista, Phys. Rev. Lett. 105, 266405 (2010) /T. Yavors’kii, M. J. P. Gingras and M. Enjalran, J. Phys. Condens. Matter 19, 145274 (2007) /K. Barros, J. Venderbose, G.-W. Chern, and C. D. Batista (unpublished) /Realistic Kagome materials have more complex band structures /If, for that filling, some reciprocal vectors must be partially occupied or ambiguous, we use averaging and symmetrization procedures, interpolating between fillings and ensuring the result always has the full symmetry of the lattice /This is simply the generalization of the single particle electron Lindhard susceptibility for non-Bravais lattices /This cannot be proven numerically: in a finite interval \( n = [0, 0.05] \) the \( \alpha \) limit of the magnetic field vanishes. /Due to divergence in the single particle D.O.S., \( J_{\alpha \beta} (q) \) has a logarithmic divergence \( \sim V \log |q - Q_M| \) at Van Hove fillings /This notation is incomplete, in that it does not specify whether the coefficients of the different wavevectors are equal or unequal, nor does it specify particular phase relationships between the amplitudes of the same wavevector appearing in different sublattices. Hence, two qualitatively distinct phases may have the same label,.) /We can also look at the angles between the different sublattices. For simplicity of classification and due to a limited resolution in fillings, we choose \( \eta_{\alpha \beta} (q) \) as the sole criterion /At \( n = 1/4 \) for small \( K_f \) in Fig. 4 the coplanar \( q = 0 \) order is stable instead of cuboc1. This is a finite size effect, cuboc1 wins for larger lattice sizes

**I. SUPPLEMENTARY INFORMATION S1**

We will derive the RKKY couplings via two different methods in this section and compare results obtained from each method. The first way to get \( J_{\alpha \beta} (q) \) is in Fourier space using second order perturbation theory. We define the following electron operators in momentum space:

\[
c_{i \alpha} = \frac{1}{\sqrt{N/3}} \sum_{\ell} \epsilon_{i \ell} e^{i \mathbf{k} \cdot \mathbf{R}_{i}} \tag{5}
\]

where \( N/3 \) is the number of unit cells and \( \alpha \) is the sublattice index. Using Eq. (5) we first write down the fre fermion part of the KLM Hamiltonian in the basis \( |k\alpha \rangle \) as follows,

\[
\begin{pmatrix}
0 & -2t \cos(\mathbf{k} \cdot \mathbf{a}_{12}) & -2t \cos(\mathbf{k} \cdot \mathbf{a}_{13}) \\
-2t \cos(\mathbf{k} \cdot \mathbf{a}_{12}) & 0 & -2t \cos(\mathbf{k} \cdot \mathbf{a}_{23}) \\
-2t \cos(\mathbf{k} \cdot \mathbf{a}_{13}) & -2t \cos(\mathbf{k} \cdot \mathbf{a}_{23}) & 0
\end{pmatrix}
\tag{6}
\]

where \( \mathbf{a}_{\alpha \beta} = \mathbf{a}_{\alpha} - \mathbf{a}_{\beta} \), \( \mathbf{a}_1 = (0, 0) \), \( \mathbf{a}_2 = (1, 0) \) and \( \mathbf{a}_3 = (1/2, \sqrt{3}/2) \). Both \( \mathbf{a}_2(3) \) are one half of lattice vectors that define the Kagome unit cell. Diagonalizing (6) yields three bands \( \nu \) with single particle energies \( \epsilon_{\nu} \), eigenvectors \( \mathbf{c}_\nu \) and a new set of operators \( \mathbf{c}_\nu^\dagger = \sum \mathbf{c}_{\nu, \alpha} e^{i \mathbf{k} \cdot \mathbf{R}_{\alpha}} \). We can now look at the Kondo perturbation to the free fermion dispersion.

The Kondo part of (1) can be expressed in real space as

\[
\mathcal{H}_{\text{Kondo}} = -\frac{J_K}{2} \sum_i \sum_{\alpha} (S_{\iota(\alpha)}^{\dagger} S_{\iota(\alpha)}^{\dagger} S_{\iota(\alpha)}^{\dagger} S_{\iota(\alpha)}^{\dagger} + S_{\iota(\alpha)}^{\dagger} S_{\iota(\alpha)}^{\dagger} S_{\iota(\alpha)}^{\dagger} S_{\iota(\alpha)}^{\dagger} - S_{\iota(\alpha)}^{\dagger} S_{\iota(\alpha)}^{\dagger} S_{\iota(\alpha)}^{\dagger} S_{\iota(\alpha)}^{\dagger}) \tag{7}
\]

Second order perturbation theory will carry two copies of (7), each of them sandwiched between pairs of electronic states inside \( |K_{\text{in}}\rangle \) and outside \( K_{\text{out}} \) the F.S. given by

\[
E_2(n) = J_K^2 \sum_{\mathbf{k}_{\text{in}}, \mathbf{k}_{\text{out}}, \nu, \nu'} \left| \frac{\langle K_{\text{in}}^\nu | \mathcal{H}_{\text{Kondo}} | K_{\text{out}}^{\nu'} \rangle^2}{(\epsilon_{K_{\text{in}}} - \epsilon_{K_{\text{out}}})^2} \right| \tag{8}
\]

Insertion of (7) in to (8) produces 16 terms, out of which only 4 are non zero due to spin rotational invariance. Each of these four terms have the same contribution and \( E_2(n) \), expressed in Fourier space, then becomes

\[
E_2(n) = \sum_{q \in \text{first B.Z.}} J_{\alpha \beta}(q) \mathbf{S}_\alpha(q) \cdot \mathbf{S}_\beta(-q) \tag{9}
\]

where \( J_{\alpha \beta}(q) \) is a \( 3 \times 3 \) matrix in the sublattice basis and is given by

\[
J_{\alpha \beta}(q) = -\frac{J_K^2}{2} \sum_{\mathbf{k}_{\text{in}}, \nu, \nu'} u^{\nu}_{K_{\text{in}}, \alpha} u^{\nu}_\mathbf{k_{\text{in}}+q, \alpha} u^{\nu'}_{\mathbf{k}_{\text{in}}+q, \beta} u^{\nu'}_\mathbf{k_{\text{in}}, \beta} e^{-iq \cdot \mathbf{a}_{\alpha \beta}} \tag{10}
\]

where \( u^{\nu}_{K_{\text{in}}, \alpha} \) is an amplitude for destroying an electron with wave vector \( q \) in band \( \nu \) and the summation is restricted to states \( K_{\text{in}} \) below the F.S. Note that in going from (8) to (10) we have switched dummy momentum indices from \( \mathbf{k}_{\text{in}}, \mathbf{k}_{\text{out}} \) to \( \mathbf{k}_{\text{in}}, q \). Use of zone symmetries \( C_q \) and mirror reflections requires us to compute \( J_{\alpha \beta}(q) \) for only 1/12 of the zone. The real space couplings are obtained by inverse Fourier transform of (10)

\[
J_{\iota(\alpha)j(\beta)} = \sum_q J_{\alpha \beta}(q) e^{-iq \cdot \mathbf{R}_{ij}} \tag{11}
\]

There are two limitations of computing the set of \( \{ J_{\iota(\alpha)j(\beta)} \} \) via the method above. Firstly, symmetries in the zone at any filling lead to degeneracies in the single particle energies (typically six leading to twelve missing electronic states) requiring us to ”hop” over fillings so as to avoid zero energy denominators in (10). The second limitation is a more severe form of the first constraint, where, for a window of fillings near Van Hove points, computation of (10) is restricted by large parallel parts of the F.S. with degenerate energies. The first limitation is resolved by averaging (10) over two non-local boundary phases which break the six fold symmetry to
two (associated with spins). To circumvent the second restriction we propose an alternative methodology for computations of \(\{ J_{(\alpha)}(\beta) \} \) as follows.

The second method for extracting the set of couplings is an approach in real space, cruder in spirit, but works as well as the explicit calculation in Fourier space. Using exact numerical diagonalization, we evaluate the single particle energies of fermions in (1) in the background of a set of random spin configurations of size \( N_s \) for a given \( J_K \). These single particle energies are summed up to the F.S. to find the total energy \( E_{ED}(n) \) as a function of filling. Each element from the set of \( \{ E_{ED}(n) \} \) is fit to the following functional form at different \( J_K \), for all fillings:

\[
E_{ED}(n) = E_{0}^{fit}(n) + (J_K/t)^2 E_{2}^{fit}(n) + (J_K/t)^4 E_{4}^{fit}(n) + \ldots \tag{12}
\]

with fit parameters \( \{ E_{0}^{fit}, E_{2}^{fit}, E_{4}^{fit} \} \). Once we recover the set of \( \{ E_{ED}(n) \} \) for all spin configurations in the data base, we fit it to the following functional form

\[
E_{2}^{fit}(n) = \varepsilon_{0}(n) + J_{1}^{fit}(n) \sum_{\langle ij \rangle} S_{i} \cdot S_{j} + J_{2}^{fit}(n) \sum_{\langle \langle ij \rangle \rangle} S_{i} \cdot S_{j} \tag{13}
\]

by minimizing the norm of the following matrix equation

\[
\min_{J_{1}^{fit}(n), J_{2}^{fit}(n), \ldots} |M(n) \cdot \text{x}(n) - \text{b}(n)| \tag{14}
\]

with \( \{ J_{1}^{fit}(n), J_{2}^{fit}(n), \ldots \} \) as fit parameters. \( M \) is a \( N_s \times (n_f + 1) \), where \( n_f \) is the number of couplings to be fit along with an additional constant \( \varepsilon_{0}(n) \) in (13). The matrix \( M \) contains the classical energies corresponding to the couplings \( J_{1,2, \ldots}(n) \) for each random spin configuration, arranged along the rows. \( \text{x} \) is a vector of length \( n_f + 1 \) given by \( \text{x}^T = (\varepsilon_{0}(n), J_{1}(n), J_{2}(n), \ldots) \) and the vector \( \text{b} \) contains the extracted \( E_{2}^{fit}(n) \) from (12) for each of the \( N_s \) spin configurations.

A comparison of the Fourier and real space methods is shown for the first four RKKY couplings in Fig.5 as a function of filling \( n \). Both methods have maximum susceptibility to finite size effects at small fillings \( n \leq 0.1 \) and to Ferromagnetic background orders where electrons have longer mean free paths comparable to system sizes. We next investigate, in more detail, the various spin orders making up the phase diagram in Fig.4.

II. SUPPLEMENTARY INFORMATION S2

Spin orders with the same broken symmetries in Fourier space are classified as a single phase. The phase diagram in Fig.4 shows eight such phases, each shown in a different color. The most prominent of these phases is also listed in Table 1. In this section, we take representative spin configurations from the most dominant phases in Fig.4 and by looking at their Fourier space composition, illustrate why they form part of a smoothly connected second order phase. This section will also helps to establish the nomenclature for the different phases.

We begin with the kind of phases where the dominant modes on each sublattice are made out of two of the six fold Luttinger-Tisza star of wave-vectors. The first kind of order has different weights of the modes on the three sublattices– indicated as a \( \{ab, bc, ca\} \) phase. The second phase is more symmetric as can be seen in Table II and is labeled \( (a_1 b_2, c_3; a_2, b_1 c_2) \) indicating two distinct \( \eta_{\alpha}(q) \) magnitudes across all \( \alpha \). A wave vector \( a \) uses one of the two \( \eta_{\alpha} \) values denoted by subscript (1) on sublattice one and the second Fourier weight (2) on sublattice two. The same goes for the other wave-vectors \( b, c \) highlighting the symmetry of the state.

The next kind of phase is where each sublattice is made out of its own independent single dominant Luttinger-Tisza wave vector. There are again two types of such phases shown in Fig.11. One in which the Fourier weights of each of the three wave-vectors \( a, b, c \) is different on each sublattice. This phase is labeled as \( \{a, b, c\} \). The second type of phase is a highly symmetric version of the former type. Each sublattice has the same weight of the dominant Fourier mode and is labeled

| Filling range | Phase Label | non-coplanar? | CO/ICO | Ex. |
|--------------|-------------|--------------|--------|-----|
| (0, 0.03)    | \( \{a, a, a\} \) | no           | CO     | FM  |
| (0.115, 0.146) | \( \{ab, bc, ca\} \) | yes         | ICO    | Fig.7 |
| (0.226, 0.25)  | \( \{a, b, c\} \) | yes         | ICO    | Fig.11 |
| (0.492, 0.551) | \( \{a_1, a_1, a_1\} \) | yes | ICO | Fig.3a |
| (0.362, 0.485)  | \( \{a_1, b_1, c_1\} \) | yes | ICO | Fig.3b |
Table II: Dominant wave vectors and their Fourier weights on the sublattices. Left to right: Filling at which the spin order originates in the RKKY limit, dominant wave vectors in the first B.Z., weights of the dominant wave vectors on each of the three sublattices. Fillings 0.115 and 0.146 correspond to $(ab, bc, ca)$ phase, while the more symmetric spin order at 0.181 is a spin set from the $(a_1b_2, c_1a_2, b_1c_2)$ phase. Spin configurations are shown in Fig. 7

(a₁, b₁, c₁). The Fourier components and weights of the spin orders from these two types of phases are enlisted in Table IV.

The next category of phases are the 1Q type orders, where all sublattices are made out of the same dominant wave-vector. These again fall in to a symmetric $-(a_1, a_1, a_1)$ and an asymmetric version $(a, a, a)$. Fourier weights of the dominant mode of spin orders from these two types of phases is shown in Table III.

Table III: Dominant wave vectors and their Fourier weights on the sublattices. Left to right: Filling at which the spin order originates in the RKKY limit, dominant wave vector in the first B.Z., weight of the dominant wave vectors on each of the three sublattices. Spin order at filling 0.321 corresponds to $(a, a, a)$ phase. The more symmetric spin orders found at 0.325 (coplanar spiral Fig.3) and $1/3(\sqrt{3} \times \sqrt{3}$ order Fig.1) form part of the $(a_1, a_1, a_1)$ phase.

Depending on the location of $q$ in the zone, the spiral can further be classified as commensurate - if $q$ lies at a special symmetry point in the zone) or incommensurate - if $q$ lies at an arbitrary wave vector. Examples of special commensurate coplanar spirals are the two well known $q = 0$ and the $\sqrt{3} \times \sqrt{3}$ order with ordering wave vectors $q = \Gamma$ and $q = K$ lying at the zone center and zone corner, respectively. For both orders, $\varphi_{a} = \pi/(a-1)/3$ which makes an angle of $2\pi/3$ between spins, locally, on every triangle.

An incommensurate order, on the other hand, has a special direction in real space defined by $q$ and a set of points $\{R_{i(a)}\}$, such that $q \cdot R_{i(a)} = 2\pi m(\sqrt{N/3})^{-1}$ for an integer $m$. As we move along $\{R_{i(a)}\}$, we trace out $(\sqrt{N/3})^{-1}$ equally spaced coplanar directions in spin space as shown in Fig.3(a1)-(a3). An incommensurate coplanar spiral might locally have angles close to $2\pi/3$ as in the twisted $\sqrt{3} \times \sqrt{3}$ order -Fig.3 and Table III. As discussed in the previous section, the two kinds of spiral orders can belong to an $(a, a, a)$ phase or a more symmetric $(a_1, a_1, a_1)$ phase. States from these phases are shown in Fig.6 below.

### III. SUPPLEMENTARY INFORMATION S3

A parametrization of the different spin orders $\{S_{\mu \nu}^{\text{opt}}\}$ making up the phases in Fig. 4 is provided in this section. There are two steps for parametrizing a spin state. In the first step, we construct a purified version $\{S_{\mu \nu}^{\text{opt}}\}$ of the optimal spin order obtained from MC by filtering out the dominant Fourier modes on each sublattice. In the second step, we try fitting simple functional forms to $\{S_{\mu \nu}^{\text{opt}}\}$. Since, most of the states are combinations of simple coplanar incommensurate spirals, we begin by parametrizing coplanar spirals.

A coplanar spiral is parametrized by its ordering wave vector $q$ and two phases dependent on the locking between the sublattices. For two orthonormal unit vectors $e_{1, 2}$, a coplanar spiral can be parametrized as

$$S_{\mu}(q) = \text{Re}[e^{i(q \cdot R_{i} + \varphi_{a})}(e_{1} - ie_{2})]$$  \hspace{1cm} (15)
| $n$  | $q_1(2\pi)$ | $q_2(2\pi)$ | $q_3(2\pi)$ | $S_1(q_1)$ | $S_2(q_2)$ | $S_3(q_3)$ | $\eta_1(q_1)$ | $\eta_2(q_2)$ | $\eta_3(q_3)$ |
|------|-------------|-------------|-------------|------------|------------|------------|--------------|--------------|--------------|
| 0.226 | (0.25, 0.12) | (−0.25, 0.12) | (0.02, −0.27) | 0.29e$^{-1.38}$ | 0.38e$^{-1.48}$ | 0.46e$^{-0.18}$ | 0.26e$^{-1.82}$ | 0.38e$^{-1.61}$ | 0.46e$^{0.95}$ |
| 0.341 | (0.12, −0.24) | (0.12, 0.24) | (−0.27, 0.01) | 0.2e$^{-12.1}$ | 0.41e$^{-11.36}$ | 0.2e$^{-11.87}$ | 0.49e$^{-11.54}$ | 0.18e$^{-12.5}$ | 0.14e$^{-10.97}$ |
| 0.355 | (0.1, −0.2) | (0.1, 0.2) | (−0.23, 0.01) | 0.18e$^{-12.04}$ | 0.46e$^{-11.76}$ | 0.22e$^{-12.28}$ | 0.42e$^{-10.21}$ | 0.19e$^{-12.6}$ | 0.42e$^{-11.7}$ |
| 0.492 | (0.1, 0.13) | (0.1, −0.13) | (0.06, 0.16) | 0.27e$^{-0.94}$ | 0.43e$^{1.48}$ | 0.47e$^{-0.26}$ | 0.45e$^{-13.02}$ | 0.28e$^{3}$ | 0.43e$^{1.84}$ |
| 0.527 | (0.23, 0.16) | (0.23, −0.15) | (0.02, 0.27) | 0.47e$^{+1.88}$ | 0.48e$^{-1.3}$ | 0.2e$^{+1.23}$ | 0.45e$^{-12.7}$ | 0.17e$^{-12.55}$ | 0.49e$^{-0.34}$ |
| 0.551 | (0.21, 0.1) | (−0.2, 0.1) | (0.02, −0.22) | 0.01e$^{-12.12}$ | 0.49e$^{+1.15}$ | 0.47e$^{+7.7}$ | 0.45e$^{-12.66}$ | 0.49e$^{-13.35}$ | 0.19e$^{+3.10}$ |
| 0.28 | (0.125, 0.12) | (−0.17, 0.04) | (0.04, −0.17) | 0.25e$^{-11.69}$ | 0.49e$^{+1.02}$ | 0.43e$^{-11.69}$ | 0.41e$^{+0.43}$ | 0.44e$^{-11.47}$ | 0.36e$^{-0.69}$ |
| 0.362 | (0.1, −0.15) | (0.08, 0.16) | (0.18, 0.01) | 0.43e$^{-10.16}$ | 0.4e$^{-11.51}$ | 0.22e$^{-11.06}$ | 0.36e$^{-12.31}$ | 0.29e$^{-11.76}$ | 0.42e$^{-2.6}$ |
| 0.394 | (0.06, −0.08) | (0.04, 0.09) | (0.1, 0.01) | 0.47e$^{+2.78}$ | 0.47e$^{-2.34}$ | 0.47e$^{-2.7}$ | 0.38e$^{+0.74}$ | 0.47e$^{-2.44}$ | 0.29e$^{+1.1}$ |
| 0.485 | (0.1, 0.08) | (−0.12, 0.04) | (0.02, −0.13) | 0.44e$^{-11.71}$ | 0.46e$^{+0.24}$ | 0.22e$^{+2.38}$ | 0.33e$^{+1.45}$ | 0.44e$^{+2.54}$ | 0.4e$^{-12.5}$ |
| 0.579 | (0.25, 0.1) | (0.22, −0.15) | (0.02, 0.27) | 0.44e$^{-12.12}$ | 0.28e$^{-11.76}$ | 0.47e$^{-11.36}$ | 0.44e$^{-11.88}$ | 0.41e$^{-0.91}$ | 0.42e$^{0.21}$ |
| 0.596 | (0.23, 0.15) | (−0.25, 0.12) | (0.02, 0.27) | 0.48e$^{-12.78}$ | 0.19e$^{-12.6}$ | 0.47e$^{+1.55}$ | 0.48e$^{-12.53}$ | 0.47e$^{+2.08}$ | 0.21e$^{-0.44}$ |
| 0.624 | (0.17, 0.12) | (−0.18, 0.08) | (0.02, −0.2) | 0.48e$^{-11.46}$ | 0.32e$^{-0.1}$ | 0.38e$^{-12.87}$ | 0.35e$^{-10.31}$ | 0.37e$^{-12.2}$ | 0.47e$^{-12.21}$ |

Table IV: Dominant wave-vectors and their Fourier weights on the sublattices for representative spin orders from the phases $-(a, b, c)$ and $(a_1, b_1, c_1)$. Corresponding spin configurations are shown in Fig. 11

We now consider the more complicated 3Q (ab, bc, ca) orders in the phase diagram in Fig. 4. The simplest of these orders are the two commensurate cuboc$215$ and cuboc$216$ orders shown in Fig. 1 and found at fillings $n = 5/12$ and $n = 2/3$. Each of these orders are made from the three ordering vectors $Q_{1,2,3} \in M$ belonging to the zone mid points and leading to a twelve site magnetic unit cell. Each sublattice $\alpha$ uses only two of the three vectors leading to the label 3Q (ab, bc, ca). Cuboc$1$ is parametrized as:

$$S_{i(1)} = \frac{1}{\sqrt{2}} [\cos(Q_2 \cdot R_1)e_2 + \cos(Q_3 \cdot R_1)e_3]$$
$$S_{i(2)} = \frac{1}{\sqrt{2}} [\cos(Q_1 \cdot R_1)e_1 - \cos(Q_3 \cdot R_1)e_3]$$
$$S_{i(3)} = -\frac{1}{\sqrt{2}} [\cos(Q_1 \cdot R_1)e_1 + \cos(Q_2 \cdot R_1)e_2]$$

(16)
Table V: Ordering wave vectors and spin F.T. for reconstructing order at \( n = 0.311 \) occurring in the phase diagram in Fig. 4. From left to right: order number corresponding to labeling in 4. Ordering wave vector \( \mathbf{q} \) for the \( 1\mathbf{Q}(a, a, a) \) state, spin F.T. at each wave vector. The spin order Fig. 6 has an additional significant contribution (\( \sim 30\% \)) on sublattice \( \alpha = 1 \) from an additional wave vector \( \mathbf{q}_2 = 2\pi(0.12, 0.19) \) with Fourier weight \( S_{\alpha=1}(\mathbf{q}_2) = (0.25e^{i1.90}, 0.1e^{i1.06}, 0.3e^{-i1.19}) \).

| \( \mathbf{q}(2\pi) \) | \( S_1(\mathbf{q}) \) | \( S_2(\mathbf{q}) \) | \( S_3(\mathbf{q}) \) |
|------------------|------------------|------------------|------------------|
| \( 0.12 \) \<br\> \( -0.19 \) | \( 0.32e^{-i2.9} \)<br\> \( 0.4e^{i1.55} \)<br\> \( 0.26e^{i2.8} \) | \( 0.37e^{i0.9} \)<br\> \( 0.45e^{-i0.8} \)<br\> \( 0.3e^{-i0.4} \) | \( 0.36e^{-i0.5} \)<br\> \( 0.43e^{-i2.2} \)<br\> \( 0.29e^{-i1} \) |

Table VI: Ordering wave vectors and spin F.T. for reconstructing orders found at \( n = 0.321 \) and \( n = 0.325 \). From left to right: Filling at which order was found, ordering wave vector \( \mathbf{q} \) for the \( 1\mathbf{Q}(a, a, a) \) state, sublattice phases \( \varphi_{2,3} \) (see (15)) and \( \text{Norm}(S^\text{recons}_i) \).

| Filling(\( n \)) | \( q(2\pi) \) | \( \varphi_2(2\pi) \) | \( \varphi_3(2\pi) \) | \( \text{Norm}(S^\text{recons}_i) \) |
|-----------------|-----------------|-----------------|-----------------|-----------------|
| \( 0.321 \) | \( (0.27, 0.04) \) | 0.27 | 2/3 | \( (0.95, 0.97) \) |
| \( 0.325 \) | \( (0.14, 0.29) \) | 0.65 | 0.27 | \( (0.99, 0.99) \) |

where \( \mathbf{Q}_2 = 2\pi(1/4, -1/(4\sqrt{3})) \), \( \mathbf{Q}_3 = \mathcal{R}_{\pi/3}\mathbf{Q}_2 \) and \( \mathbf{Q}_1 = \mathcal{R}_{2\pi/3}\mathbf{Q}_2 \). \( \mathcal{R}_\theta \) is the \( 2 \times 2 \) rotation matrix. The state does not elicit an Anomalous Hall response due to the coplanarity of spins on every triangle.

Near neighbor spins in the Cuboc2 state make an angle of \( \pi/3 \), while the next nearest neighbor spins have an angle of \( 2\pi/3 \) between them. The state is thus favored in the presence of a ferromagnetic \( J_1 \) and an AFM \( J_2 \) interaction. The non-coplanarity of spins within each triangle in cuboc2 leads to a non-vanishing value of the scalar spin chirality \( \chi = \mathbf{S}_1 \cdot \mathbf{S}_2 \times \mathbf{S}_3 \). \( \chi \) is \( \pm (1/\sqrt{2}) \) on all the up (down) triangles. The equal and opposite fluxes leads to zero overall flux and no Anomalous Hall response. The state has a \( Z_2 \) symmetric partner and at zero temperature, one of the two states is spontaneously selected, breaking \( Z_2 \) symmetry. The spin order is parametrized as follows:

\[
\begin{align*}
S_{i(1)} &= \frac{1}{\sqrt{2}} [\cos(\mathbf{Q}_2 \cdot \mathbf{R}_i)\mathbf{e}_2 + \cos(\mathbf{Q}_3 \cdot \mathbf{R}_i)\mathbf{e}_3] \\
S_{i(2)} &= \frac{1}{\sqrt{2}} [\cos(\mathbf{Q}_1 \cdot \mathbf{R}_i)\mathbf{e}_1 + \cos(\mathbf{Q}_3 \cdot \mathbf{R}_i)\mathbf{e}_3] \\
S_{i(3)} &= \frac{1}{\sqrt{2}} [\cos(\mathbf{Q}_1 \cdot \mathbf{R}_i)\mathbf{e}_1 + \cos(\mathbf{Q}_2 \cdot \mathbf{R}_i)\mathbf{e}_2]
\end{align*}
\]

(17)

Parametrization of other constituent states of the \((ab, bc, ca)\) type phase, such as the spin configurations shown in Fig 7 is done using the Fourier transform of the spin orders \( \{S_\alpha(\mathbf{q}_i)\} \) from MC minimization. The "purified" order is obtained by simply inverse F.T. the vectors \( \{S_\alpha(\mathbf{q}_i)\} \), given in Table VIII, using Eq. 18.

\[
S^\text{recons}_i(\alpha) = N_i \sum_{\mathbf{q} \in \{\mathbf{q}_0\}} S_\alpha e^{i\mathbf{q} \cdot \mathbf{R}_i}
\]

(18)

The ordering wave vectors and spins in Fourier space for the three orders in Fig. 7 is given in Table VIII.

We next turn to spin orders from the \((a, b, c)\) phase. For all orders in this phase, spins on the three sublattices are defined by mutually exclusive wave vectors tracing out coplanar spirals. For most spin orders, the three sublattice dependent planes, are mutually orthogonal. Spins are parametrized as:

\[
\begin{align*}
S^\text{recons}_i(\alpha=1) &= \text{Re}[e^{i(\mathbf{q}_i \cdot \mathbf{R}_i + \varphi_1)}(\mathbf{e}_1 - i\mathbf{e}_2)] \\
S^\text{recons}_i(\alpha=2) &= \text{Re}[e^{i(\mathbf{q}_i \cdot \mathbf{R}_i + \varphi_2)}(\mathbf{e}_2 - i\mathbf{e}_3)] \\
S^\text{recons}_i(\alpha=3) &= \text{Re}[e^{i(\mathbf{q}_i \cdot \mathbf{R}_i + \varphi_3)}(\mathbf{e}_1 - i\mathbf{e}_3)]
\end{align*}
\]

(19)

where \( \mathbf{e}_{1,2,3} \) form a triad of orthonormal vectors (see Table VII). For a few spin orders at \( n = 0.226, 0.228 \) and at \( n = 0.527 \) from within this phase, spins on two of the three sublattices lie in the same plane perpendicular to the plane in which spins on the third sublattice lie.

IV. SUPPLEMENTARY INFORMATION S4

Here we show that the Cuboc1 state is also selected from the family of degenerate states of the nearest neighbor HAF on Kagome by turning on the KLM Hamiltonian. To this end, we add a strong nearest neighbor antiferromagnetic interaction of strength \( J_{\perp} \) to the KLM Hamiltonian and using exact diagonalization explore the stability of states within the variational approach outlined before.
Table VII: Set of parameters for constructing the purified spin configurations from the \((a, b, c)\) and \((a_1, b_1, c_1)\) phases according to the ansatz outlined in 19 for spin configurations shown in Fig. 11

Fig. 9 shows the phase diagram in the presence of a strong \(J_{ce} = 10t\) which suppresses all the incommensurate orders and selects within the manifold of the three 120 degree states the \(\mathbf{q} = 0, \sqrt{3} \times \sqrt{3}\) and the \(\text{cuboc1}\) order. As can be seen, large parts of the phase diagram are dominated by the \(\text{cuboc1}\) state.

Figure 8: Variational phase diagram in the presence of a nearest neighbor antiferromagnetic exchange interaction \(J_{ce} = 10t\). (A): \(\mathbf{q} = 0\), (B): \(\sqrt{3} \times \sqrt{3}\) (C): Common origin plot of spins for orders in (A),(B)

\[
\begin{array}{cccccc}
\# & q_1(2\pi) & q_2(2\pi) & q_3(2\pi) & \phi_1(2\pi) & \phi_2(2\pi) \\
(6) & 0.125 & -0.17 & 0.05 & -0.17 & -0.91 & -1.36 & -0.86 \\
(12) & 0.1 & 0.08 & 0.19 & 0.01 & -2.04 & -0.58 & 0.99 \\
(13) & 0.06 & -0.04 & 0.1 & 0.01 & 0.39 & 1.9 & 1 \\
(14) & 0.1 & 0.08 & -0.12 & -0.05 & 0.02 & 0.91 & 2.26 & 1.1 \\
(16) & 0.22 & 0.22 & 0.02 & 0.27 & -0.12 & -1.53 & -3.02 \\
(17) & 0.21 & -0.2 & 0.02 & -0.23 & -0.48 & 1.79 & -0.97 \\
(18) & 0.25 & 0.23 & -0.02 & -0.28 & 1.8 & 0.64 & -0.37 \\
(19) & 0.22 & -0.25 & 0.02 & 0.27 & 0.97 & -0.35 & 0.96 \\
\end{array}
\]

Figure 9: Eigenvalues \(\lambda^\nu(\mathbf{q})\) of the \(J_{\alpha\beta}(\mathbf{q})\) in the lowest band \(\nu = 1\) (see Eq.3) in the Kagome first B.Z. (a) 0.333 (b)0.343 (c) 0.352 (d) 0.354 (e) 0.364 (f) 0.372 (g) 0.375 (h) 0.381 (i) 0.41. Data for \(N = 3 \times 36^2\) lattice and color scheme is red(high), blue (low)

Figure 10: Common origin plot of spins and the Fourier weights for a spin configuration found at \(n = 0.375\) on a \(N = 3 \times 36^2\) lattice. (A)-(C) common origin plots showing \((a_1, b_1, c_1)\) phase. (D)-(F) Fourier profile \(\eta_o(\mathbf{q})\) (see Eq.4) for \(\alpha = 1, 2, 3\)

V. SUPPLEMENTARY INFORMATION S5

The evolution of the Luttinger-Tisza matrix in the first B.Z. and the Fourier weights of a spin configuration from the highly symmetrical \((a_1, b_1, c_1)\) phase is shown in Figs. 10 and 11 respectively. An experimental signature of incommensurate orders will be Neutron Scattering Bragg peaks at incommensurate positions in the zone as shown in Fig. 11 (D)-(F).
Figure 11: Common origin plots of the spin configurations from the \((a, b, c)\) and the \((a_1, b_1, c_1)\) phases shown in Fig. 4. (A)-(C) label common origin plots for spins on each of the three sublattices and (D) shows the spins in the "purified" configuration. (5):0.226, (6):0.28, (12):0.362, (13): 0.394, (14):0.485, (16):0.527, (17):0.551, (18):0.579, (19):0.596, (20):0.624.

| #  | \(\mathbf{q}_1\) (2\(\pi\)) | \(\mathbf{q}_2\) (2\(\pi\)) | \(\mathbf{q}_3\) (2\(\pi\)) | \(S_1(\mathbf{q}_1)\) | \(S_1(\mathbf{q}_2)\) | \(S_1(\mathbf{q}_3)\) | \(S_2(\mathbf{q}_1)\) | \(S_2(\mathbf{q}_2)\) | \(S_2(\mathbf{q}_3)\) | \(S_3(\mathbf{q}_1)\) | \(S_3(\mathbf{q}_2)\) | \(S_3(\mathbf{q}_3)\) |
|----|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| (1) | 0.19          | -0.13         | 0.02           | -0.22         | 0.21           | 0.09           | 0.32\(e^{1.4}\) | 0.36\(e^{-0.8}\) | 0.09\(e^{-12.8}\) | 0.37\(e^{11.2}\) | 0.34\(e^{-10.4}\) | 0.15\(e^{13}\) |
| (2) | 0.15          | -0.1          | 0.02           | -0.18         | 0.17           | 0.07           | 0.29\(e^{1.1}\) | 0.32\(e^{1.9}\)  | 0.19\(e^{12}\)  | 0.33\(e^{7}\)   | 0.16\(e^{-12.4}\) | 0.39\(e^{1.9}\) |
| (3) | 0.02          | -0.13         | -0.12          | 0.05          | 0.1            | 0.08           | 0.06\(e^{-0.6}\) | 0.29\(e^{1.2}\)  | 0.06\(e^{-10.5}\) | 0.36\(e^{-12.5}\) | 0.37\(e^{-10.6}\) | 0.29\(e^{-12.9}\) |

Table VIII: Ordering wave vectors and spin F.T. for reconstructing orders (1) at \(n = 0.115\), (2) at \(n = 0.146\) and (3) at \(n = 0.181\) occurring in the phase diagram in 4. From left to right: order number corresponding to labeling in 4. Ordering wave vectors \(\mathbf{q}_{1,2,3}\) for the \(3\mathbf{Q}(ab, bc, ca)\) state, spin F.T. at each wave vector. An approximate and un-normalized spin order can be constructed using the information provided above using the recipe provided in text (see Eq.18)