Estimation in nonparametric regression model with additive and multiplicative noise via Laguerre series

Rida Benhaddou
Department of Mathematics, Ohio University, Athens, OH, USA

ABSTRACT
We look into the nonparametric regression estimation with additive and multiplicative noise and construct adaptive thresholding estimators based on Laguerre series. The proposed approach achieves asymptotically near-optimal convergence rates when the unknown function belongs to Laguerre–Sobolev space. We consider the problem under two noise structures; (1) i.i.d. Gaussian errors and (2) long-memory Gaussian errors. In the i.i.d. case, our convergence rates are similar to those found in the literature. In the long-memory case, the convergence rates depend on the long-memory parameters only when long-memory is strong enough in either noise source, otherwise, the rates are identical to those under i.i.d. noise.

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1. Introduction
Consider a nonparametric regression model with both multiplicative and additive noise

\[ y_i = f(t_i)e_i + \sigma z_i, \quad i = 1, 2, \cdots, N, \]  

where \( e_i \) and \( z_i \) are zero-mean (1) independent and identically distributed Gaussian random variables with variance equal to 1 and (2) \( e_i \) and \( z_i \) are Gaussian with long-memory structure, and \( \sigma \) is a known positive constant. The function \( f(t) \) is the unknown response, it is real-valued and is defined on the interval \([0, b]\) with \( b > 0 \) a fixed real number. In addition, \( t_i, i = 1, 2, \cdots, N, \) are independent and identically distributed random variables, drawn from a known probability density function \( g \) with support \([0, b]\).

It is assumed that the quantities \( t_i, e_i \) and \( z_i \) are independent from one another for any \( i \in \{1, 2, \cdots, N\} \). The goal is to estimate \( h(t) = f^2(t) \) based on data points \((t_1, y_1), (t_2, y_2), \cdots, (t_N, y_N)\).

This problem, under various settings, has been studied considerably by means of a number of nonparametric methods, including kernel smoothing, splines and wavelets, and the list of articles includes, in chronological order, Hardle and Tsybakov (1997), Brown and Levine (2007), Cai and Wang (2008), Kulik and Wichelhaus (2011) and Chichignoud (2012). Most recently, Chesneau et al. (2020) considered the problem in a multivariate setting and proposed a wavelet thresholding approach to solve it. This problem has a great deal of applications, for instance, in Global Positioning System (GPS) signal propagation modeling where there is empirical evidence that in heavy multi-
path urban areas, the GPS signal encounters both additive and multiplicative noise (see Huang, Pi, and Progri (2013)), or in finance where one is interested in estimating the variance from the returns of an asset and the interested reader may refer to Chesneau et al. (2020) for more. Almost all of these articles assume that the error terms are white noise processes or i.i.d. noise. However, empirical evidence has shown that, even at large lags, the correlation structure in the errors may take the power-like form. This phenomenon is referred to as long-memory (LM) or long-range dependence (LRD).

Long-memory has been investigated quite considerably in many nonparametric estimation problems, including regression and deconvolution and the list includes Wang (1996, 1997), Comte, Dedecker, and Taupin (2008), Kulik and Raimondo (2009), Kulik and Wichelhaus (2011), Wishart (2013), Benhaddou et al. (2014), Benhaddou (2016), Benhaddou (2018a, 2018b) and Benhaddou and Liu (2019).

The application of Laguerre series to nonparametric estimation has become popular as of late and the list includes the application to density estimation in Comte, Dedecker, and Taupin (2008) and Comte and Genon-Catalot (2015), the estimation of linear functionals of a density function in Mabon (2016) and Laplace deconvolution in Vareschi (2015), Comte et al. (2017) and Benhaddou, Pensky, and Rajapakshage (2019).

The objective of the paper is to solve the nonparametric regression model with both multiplicative and additive i.i.d., and long-memory Gaussian noise via Laguerre hard-thresholding when the design points are random and follow known probability density function $g$. We derive lower bounds for the $L^2$-risk when $h = f^2$ belongs to some Laguerre-Sobolev ball of radius $A > 0$, and then construct an adaptive Laguerre-thresholding estimator for $h = f^2$. In addition, we show that the proposed estimator attains asymptotically near-optimal convergence rates. Furthermore, we demonstrate that long-memory has a detrimental effect on the convergence rates only when it is strong enough in either noise source. In which case, the convergence rates depend on the smoothness of the unknown function $h = f^2$ and the long-memory parameter associated with the stronger dependence between the two noise sources. It turns out that the present rates are identical to those in Chesneau et al. (2020) with $d = 1$ for the i.i.d. case and with $\alpha_1 = \alpha_2 = 1$ for the long-memory case. Similarly, our rates are comparable to those in Brown and Levine (2007) and Cai and Wang (2008) in their treatment of the regression variance estimation when the unknown mean function is smooth enough.

2. Estimation algorithm

For the rest of the paper, let $||h||$ denote the $L^2([0, \infty))$-norm of the function $h$. Given a matrix $A$, let $A^T$ be its transpose, $\lambda_{\text{max}}(A)$ be its largest eigenvalue in magnitude, $||A||_F = \sqrt{\text{Tr}(A^T A)}$ and $||A||_p = \lambda_{\text{max}}(A^T A)$ be, respectively, its Frobenius and the spectral norms. In addition, let $(a \vee b) = \max(a, b)$ and $(a \wedge b) = \min(a, b)$. Consider the orthonormal basis that consists of the system of Laguerre functions

$$\varphi_k(t) = e^{-t^2/2}L_k(t), \quad k = 0, 1, \cdots,$$

(2)

where $L_k(t)$ are Laguerre polynomials (see, e.g., Gradstein and Ryzhik 1980, Section 8.97). Since the functions $\varphi_k(t), k = 0, 1, \cdots$, form an orthonormal basis on $[0, \infty)$, the function $h(.) = f^2(.)$ can be expanded over this basis as follows
\[ h(t) \equiv f^2(t) = \sum_{k=0}^{\infty} \theta_k \varphi_k(t), \quad (3) \]

where \( \theta_k = \int_0^b h(t) \varphi_k(t) dt = \int_0^b f^2(t) \varphi_k(t) dt \). Under i.i.d noise case, similar to Chesneau et al. (2020), an estimator for \( \theta_k \) is given by

\[
\hat{\theta}_k = \frac{1}{N} \sum_{i=1}^{N} \left[ y_i^2 \frac{\varphi_k(t_i)}{g(t_i)} - \sigma^2 \int_0^b \varphi_k(t) dt \right] \mathbb{I}(\Omega_k(i)),
\]

where \( \Omega_k(i) = \left\{ i : \left| y_i^2 \frac{\varphi_k(t_i)}{g(t_i)} - \sigma^2 \int_0^b \varphi_k(t) dt \right| \leq \frac{\sqrt{N}}{\ln(N)} \right\} \). Similarly, under long-memory noise case, an unbiased estimator for \( \theta_k \) is given by

\[
\hat{\theta}_k = \frac{1}{N} \sum_{i=1}^{N} \left[ y_i^2 \frac{\varphi_k(t_i)}{g(t_i)} - \sigma^2 \int_0^b \varphi_k(t) dt \right].
\]

Then, consider the hard-thresholding estimator for \( h = f^2 \)

\[
\hat{h}_M(t) = \sum_{i=0}^{M-1} \hat{\theta}_i \mathbb{I}(|\hat{\theta}_i| > \lambda_i) \varphi_i(t),
\]

where the quantities \( M \) and \( \lambda_i \) will be determined under the two different setups in the proceeding sections.

Next is the list of conditions that will be utilized in the derivation of the theoretical results.

**Assumption A.1.** \( f \in L^2[0,b) \) is bounded above, that is, there exists positive constant \( M_2 < \infty \) such that \( f(t) \leq M_2 \), for all \( t \in [0,b) \).

**Assumption A.2.** The probability density function \( g \) is uniformly bounded, that is, on \( [0,b) \) there exist positive constants \( m_1 \) and \( m_2 \), with \( 0 < m_1 \leq m_2 < \infty \), such that \( m_1 \leq g(t) \leq m_2 \).

**Remark 1.** Assumption A.2. is valid for instance when \( g \) is the uniform distribution. In such case \( m_1 = m_2 = 1/b \). If \( g \) is not bounded below, such as in the case of beta distribution of the form \( g(x) = \beta x^{b-1} \) and \( \beta > 1 \) with \( b = 1 \), a variation of the present procedure will be needed and this would be another direction for future research. The idea is to consider the generalized Laguerre function basis instead, which is defined by

\[
\varphi_k^{(a)}(x) = \left[ \frac{k!}{\Gamma(k+a+1)} \right]^{1/2} e^{-x/2} x^{a/2} L_k^{(a)}(x), \quad k = 0, 1, \ldots,
\]

where \( L_k^{(a)}(t) \) are generalized Laguerre polynomials with parameter \( a \), \( a > 0 \) (see, e.g., Gradshtein and Ryzhik 1980, Section 8.97), and select the parameter \( a \) which achieves the smallest variance for estimators (4) and (5) according to \( g \) at hand.

**Assumption A.3.** The function \( h(t) = f^2(t) \) belongs to a Laguerre-Sobolev space. In particular, Laguerre coefficients of \( h, \theta_i \) satisfy
We are now in the position to fill in the details of the estimator and find the minimax lower bound for the quadratic risk and compare it to asymptotic upper bound for the mean squared error of our estimator.

Remark 2. Functional spaces of Assumption A.3, have been introduced in Bongioanni and Torrea (2009) to study Laguerre operators, and the connection with Laguerre coefficients was established in Comte and Genon-Catalot (2015).

3. Asymptotic minimax and adaptivity: the i.i.d. case

We define the minimax \( L^2 \)-risk over a set \( \Theta \) as

\[
R(\Theta) = \inf_{\tilde{h}} \sup_{h \in \Theta} \mathbb{E} ||\tilde{h} - h||^2,
\]

where the infimum is taken over all possible estimators \( \tilde{h} \) of \( h \).

Theorem 1. Let Assumptions A.1–A.3 hold. Then, as \( N \to \infty \),

\[
R(\mathcal{B}^s(A)) \leq CA^2 \left[ \frac{1}{A^2 N} \right]^\frac{s}{2}.
\]

Lemma 1. Let conditions A.1 and A.2 hold and let \( \hat{\theta}_l \) be defined in (4). Then, for \( l = 1, 2, \ldots, M - 1 \), as \( N \to \infty \), one has

\[
\mathbb{E} |\hat{\theta}_l - \theta_l|^2 = O\left( \frac{1}{N} \right).
\]

Based on Lemma 1, choose the thresholds \( \lambda_l \) such that

\[
\lambda_l = \gamma \sqrt{\frac{\ln(N)}{N}}.
\]

In addition choose the truncation level \( M \) as

\[
M = N.
\]

Lemma 2. Let conditions A.1 and A.2 hold and let \( \hat{\theta}_l \) be defined in (4). Then, for \( l = 1, 2, \ldots, M - 1 \), if \( \rho \gamma > 1 \), as \( N \to \infty \), one has

\[
\text{Pr} \left( |\hat{\theta}_l - \theta_l| > \rho \lambda_l \right) = O \left( \left[ \frac{1}{N} \right]^\tau \right),
\]

where \( \tau \) is a positive parameter that is large enough and \( \rho \) is such that \( 0 < \rho < 1 \).

Theorem 2. Let \( s \geq 1/2 \) and let \( \tilde{h}_M(t) \) be the Laguerre estimator defined in (6) with \( M \) given in (12) and \( \lambda_l \) given in (11). Suppose Assumptions A.1–A.3 hold. Then, if \( \tau \) is
large enough, as $N \to \infty$, one has

$$
\sup_{h \in B^r(A)} \mathbb{E}[\|\hat{h}_M - h\|^2] \leq CA^2 \left[ \frac{\ln(N)}{A^2N} \right]^{\frac{2s}{2s+1}} \cdot (14)
$$

Remark 3. Theorems 1 and 2 imply that, for the $L^2$-risk, the estimator (6) with $\lambda_1$ given by (11) and $M$ chosen according to (12) is adaptive and asymptotically near-optimal, within a logarithmic factor of $N$, over all Laguerre-Sobolev balls $B^r(A)$.

Remark 4. The convergence rates match those in Chesneau et al. (2020), with $d = 1$ in their treatment of the problem using wavelets when the function under consideration belongs to a certain Besov ball.

Remark 5. Our rates are comparable to those in Brown and Levine (2007) and Cai and Wang (2008) in their treatment of the regression variance estimation when the unknown mean function is smooth enough.

Remark 6. Notice that $r$ in equation (1) may not be known in advance but it can be estimated from the data. In addition, GPS signal detection application, the size of $r$ will dictate whether the multiplicative noise will be considered or ignored in the analysis (see Huang, Pi, and Progri 2013).

4. Asymptotic minimax and adaptivity: the long-memory case

Let $\epsilon_N$ be the random vector with elements $\epsilon_i$, $i = 1, 2, \cdots, N$, and covariance matrix $\Sigma_1 = \text{Cov}(\epsilon_N) = \mathbb{E}[\epsilon_N \epsilon_N^T]$, and let $z_N$ be the random vector with elements $z_i$, $i = 1, 2, \cdots, N$ and covariance matrix $\Sigma_2 = \text{Cov}(z_N) = \mathbb{E}[z_N z_N^T]$.

Assumption A.4. The vectors $\epsilon_N$ and $z_N$ allow the decomposition

$$
\epsilon_N = A_1 \eta_N^{(1)}, \quad z_N = A_2 \eta_N^{(2)}, \quad (15)
$$

where $\eta_i^{(j)}$, $j = 1, 2$, is a random vector with zero-mean independent Gaussian $\eta_i^{(j)}$ having equal variance, $i = 1, 2, \cdots, N$ and $A_j$ is some nonrandom matrix. $\{\epsilon_i\}_{i \geq 1}$ and $\{z_i\}_{i \geq 1}$ are zero-mean, stationary Gaussian sequences with auto-covariance functions $\gamma_1(h) = \mathbb{E}[\epsilon_i \epsilon_{i+h}]$ and $\gamma_2(h) = \mathbb{E}[z_i z_{i+h}]$, satisfying

$$
\gamma_1(h) \asymp h^{-x_1}, \quad \gamma_2(h) \asymp h^{-x_2}. \quad (16)
$$

Assumption A.5. There exist constants $c_i^{(j)}$, $j = 1, 2$, $(0 < c_i^{(j)} \leq c_2^{(j)} < \infty)$, independent of $N$, such that

$$
c_i^{(j)} n^{1-x_j} \leq \lambda_{\min}(\Sigma_j) \leq \lambda_{\max}(\Sigma_j) \leq c_2^{(j)} n^{1-x_j}, \quad 0 < x_j \leq 1, \quad (17)
$$

where $x_j$, $j = 1, 2$, are the long-memory parameters associated with the matrices $\Sigma_j$, respectively.
**Theorem 3.** Let Assumptions A.1–A.3 and A.5 hold. Then, provided that \( f \) is bounded away from zero, as \( N \to \infty \), one has

\[
R(B^*(A)) \geq CA^2 \begin{cases} \left[ \frac{1}{A(N)} \right]^{\frac{1}{\alpha+1}}, & \text{if } (\alpha_1 \land \alpha_2) \geq 1/2, \\ \left[ \frac{1}{A^2(N^{\alpha_1} \vee N^{\alpha_2})} \right]^{\frac{1}{\alpha+1}} & \text{if otherwise.} \end{cases}
\]

(18)

**Lemma 3.** Let conditions A.1, A.2, A.4 and A.5 hold and let \( \hat{\theta}_l \) be defined in (5). Then, for \( l = 1, 2, \ldots, M - 1 \), as \( N \to \infty \), one has

\[
\mathbb{E} |\hat{\theta}_l - \theta_l|^2 = \begin{cases} O\left( \frac{1}{N} \right) & \text{if } (\alpha_1 \land \alpha_2) \geq 1/2, \\ O\left( \frac{1}{N^{2\alpha_1} \vee N^{2\alpha_2}} \right) & \text{if otherwise.} \end{cases}
\]

(19)

Based on Lemma 3, choose the thresholds \( \lambda_l \) such that

\[
\lambda_l = \begin{cases} \gamma \sqrt{\frac{\ln (N)}{N}}, & \text{if } (\alpha_1 \land \alpha_2) \geq 1/2, \\ \gamma_1 \frac{\ln (N)}{N^{\alpha_1}} \sqrt{\gamma_2 \frac{\ln (N)}{N^{\alpha_2}}} & \text{if otherwise.} \end{cases}
\]

(20)

In addition, choose the maximal level \( M \)

\[
M = \begin{cases} N, & \text{if } (\alpha_1 \land \alpha_2) \geq 1/2, \\ N^{2\alpha_1} \land N^{2\alpha_2} & \text{if otherwise.} \end{cases}
\]

(21)

**Lemma 4.** Let conditions A.1, A.2, A.4 and A.5 hold and let \( \hat{\theta}_l \) be defined in (5). Then, for \( l = 1, 2, \ldots, M - 1 \), as \( N \to \infty \), one has

\[
\text{Pr}
\left( |\hat{\theta}_l - \theta_l| > \rho \lambda_l \right) = O\left( \left[ \frac{1}{N} \right]^\tau \right),
\]

(22)

where \( \tau \) is a positive parameter that is large enough and \( \rho \) is such that \( 0 < \rho < 1 \).

**Theorem 4.** Let \( s \geq 1/2 \) and let \( \tilde{h}_M(t) \) be the Laguerre estimator defined in (6) with \( M \) given in (21) and \( \lambda_l \) given in (20). Suppose assumptions A.1–A.5 hold. Then, if \( \tau \) is large enough, as \( N \to \infty \), one has

\[
\sup_{h \in B^*(A)} \mathbb{E} |\tilde{h}_M - h|^2 \leq CA^2 \begin{cases} \left[ \frac{\ln (N)}{A^2} \right]^{\frac{1}{\alpha+1}}, & \text{if } (\alpha_1 \land \alpha_2) \geq 1/2, \\ \left[ \frac{\ln^2 (N)}{A^2(N^{\alpha_1} \vee N^{\alpha_2})^{\frac{1}{2\alpha+1}}} \right] ^{\frac{1}{\alpha+1}} & \text{if otherwise.} \end{cases}
\]

(23)

Remark 7. Notice that when both long-memory parameters are large enough, in particular when \( (\alpha_1 \land \alpha_2) \geq 1/2 \), the convergence rates are identical to those under i.i.d.
errors. In such case they match those in Chesneau et al. (2020) directly if, in their notation, \( d = 1 \).

**Remark 8.** When the long-memory is strong, which corresponds to relatively low \( \alpha_j \), the convergence rates depend on the smoothness of the unknown function \( h = f^2 \) and the long-memory parameter associated with the stronger dependence, \( (\alpha_1 \wedge \alpha_2) \), between the two noise sources. These rates are completely new and provide an extension of the problem in a different direction.

**Remark 9.** Note that the i.i.d. case can also be handled using estimators (5) along with the choice of thresholds (11) and truncation level \( M \) based on (12) and achieve the same convergence rates.

5. Proofs

In order to prove Theorem 1, we use the following lemma

**Lemma 5.** (Lemma A.1 of Bunea, Tsybakov, and Wegkamp (2007)) Let \( \Theta \) be a set of functions of cardinality \( \text{card}(\Theta) \geq 2 \) such that

\[
\begin{align*}
(\text{i}) & \quad \|f - g\|_p^p \geq 4\delta^p, \text{ for } f, g \in \Theta, f \neq g, \\
(\text{ii}) & \quad \text{the Kullback divergences } K(P_f, P_g) \text{ between the measures } P_f \text{ and } P_g \text{ satisfy the inequality } K(P_f, P_g) \leq \log(\text{card}(\Theta))/16, \text{ for } f, g \in \Theta.
\end{align*}
\]

Then, for some absolute positive constant \( C_1 \), one has

\[
\inf_{f_n} \sup_{f \in \Theta} \mathbb{E}_f \|f_n - f\|_p^p \geq C_1\delta^p,
\]

where \( \inf_{f_n} \) denotes the infimum over all estimators.

**Proof of Theorem 1 and Theorem 3.** Let \( \omega \) be the vector with components \( \omega_l \in \{-1, 1\}, l = 0, 1, \ldots, L - 1 \), and denote the set of all possible values of \( \omega \) by \( \Omega \). Let \( h_L \) be the functions of the form

\[
h_L(t) = \rho_L \sum_{l=0}^{L-1} \omega_l \varphi_l(t), \quad \omega_l \in \{-1, 1\}.
\]

(24)

Observe that \( \omega \) has \( L \) components and therefore \( \Omega \) will have cardinality \( \text{card}(\Omega) = 2^L \).

By (8), it is easy to verify that \( h_L(t) \in B^s(A) \) with the choice \( \rho_L^2 = CA^2L^{-2s+1} \). Take \( \tilde{h}_L \) of the form of (24), but with \( \tilde{\omega}_l \in \{-1, 1\} \), then applying Varshamov-Gilbert Lemma (Tsybakov 2008, 104), the \( L^2 \)-norm of the difference is

\[
\|h_L(t) - \tilde{h}_L(t)\|^2 \geq \frac{L\rho_L^2}{8}.
\]

(25)

To prove Theorem 1, define the quantities \( h_i = h_L(t_i) + f_L(t_i)\nu_i + \sigma_i\mu_i, i = 1, 2, \ldots, N \), where \( \{\nu_i\}_{i=1}^{N} \) and \( \{\mu_i\}_{i=1}^{N} \) are i.i.d. \( N(0, 1) \) sequences that are independent of each other. Let \( P_{h_L} \) be the probability law of the process \( h_i \) under the hypothesis \( h_L \) defined
in (24). Then, by Assumptions A.2 and A.3, the Kullback divergence can be written as

\[
K(P_{h_L}, P_{h_L^l}) \leq N \rho_L^2 \sum_{l=0}^{L-1} |\tilde{w}_l - \omega_l|^2 \frac{\int_0^b \varphi_l^2(t)g(t)dt}{2(\int_0^b f_L^2(t)g(t)dt + \sigma_l^2)}
\]

(26)

\[
\leq N \rho_L^2 \frac{2L}{\sigma_l^2} \max_{l=1}^{L-1} |\varphi_l(t)|^2 = C N^2 L^{-(2s+1)} L.
\]

Now, to apply Lemma 5, choose

\[
A^2 L^{-(2s+1)} LN \leq \pi_0 L.
\]

(27)

Hence, the proof is complete by taking

\[
L = C[A^2 N]^\frac{1}{2s+1}.
\]

(28)

To prove Theorem 3, define the vectors \(h_N\) whose elements are quantities \(h_i = h_L(t_i) + f_L(t_i)\nu_i + \sigma_i \mu_i, i = 1, 2, \ldots, N\), such that \(h_N \sim N(h_L, \Sigma_1 + \sigma_i^2 \Sigma_2)\). Here, \({\nu_i}_{i \geq 1}\) and \({\mu_i}_{i \geq 1}\) are zero-mean, stationary Gaussian sequences with auto-covariance functions \(\gamma_1(h) = \mathbb{E}[\nu_i \nu_{i+k}]\) and \(\gamma_2(h) = \mathbb{E}[\mu_i \mu_{i+k}],\) satisfying

\[
\gamma_1(h) \sim h^{-2s_1}, \quad \gamma_2(h) \sim h^{-2s_2}.
\]

(29)

Notice that under (29), (1) \(\lambda_{\min}(\Sigma_i) \approx 1, \ i = 1, 2,\) if \((s_1 + s_2) \geq 1/2,\) and (2) \(\lambda_{\min}(\Sigma_1) \approx k_1 N^{1-2s_1}\) and \(\lambda_{\min}(\Sigma_2) \approx k_1 N^{1-2s_2},\) otherwise, provided that the function \(f\) is bounded away from zero. Let \(P_{h_L}\) be the probability law of the process \(h_i\) under the hypothesis \(h_L\) defined in (24). We consider two cases; case when \(\nu_i = 0\) and case when \(\sigma_i = 0.\)

**Case when \(\nu_i = 0\) and \(s_2 \leq 1/2.\)** Then, by Assumptions A.2 and A.3, the Kullback divergence can be written as

\[
K(P_{h_L}, P_{h_L^l}) \leq N \rho_L^2 \sum_{l=0}^{L-1} |\tilde{w}_l - \omega_l|^2 \frac{\int_0^b \varphi_l^2(t)g(t)dt}{2(\int_0^b f_L^2(t)g(t)dt + \sigma_l^2)}
\]

(30)

\[
\leq N \rho_L^2 \frac{2L}{k_2 \sigma_l^2 N^{1-2s_2}} \max_{l=1}^{L-1} |\varphi_l(t)|^2 = C N^{2s_2} A^2 L^{-(2s+1)} L.
\]

Now, to apply Lemma 5, choose

\[
A^2 L^{-(2s+1)} LN^{2s_2} \leq \pi_0 L,
\]

(31)

which gives

\[
L = C[A^2 N^{2s_2}]^{\frac{1}{2s+1}}.
\]

(32)

Case when \(\sigma_i = 0\) and \(s_1 \leq 1/2.\) Then, the Kullback divergence gives
\[
K(P_{h_1}, P_{h_z}) \leq \frac{N\rho^2}{2} \lambda_{\max} \left[ \left( \Sigma_1 \right)^{-1} \right] \sum_{l=0}^{L-1} |\omega_l - \omega_0|^2 \int_0^b \varphi_l^2(t) g(t) dt \\
\leq \frac{N\rho^2}{2} \left[ \left( \lambda_{\min}(\Sigma_1) \right)^{-1} \right] \sum_{l=0}^{L-1} |\omega_l - \omega_0|^2 \int_0^b \varphi_l^2(t) g(t) dt \\
\leq \frac{N\rho^2}{k_1N^{1-2x_1}} 2L \max_{l \leq L-1} |\varphi_l(t)|^2 = CN^{2x_2}A^2L^{-(2x+1)}L.
\]

Now, to apply Lemma 5, choose
\[
A^2L^{-(2x+1)}LN^{2x_2} \leq \pi_0 L,
\]
which gives
\[
L = C[A^2N^{2x_2}]^{\frac{1}{x_1}}.
\]
Notice that cases Case when \(z_1 = 0\) and \(x_1 > 1/2\) and Case when \(\nu_i = 0\) and \(x_2 > 1/2\) the matrices \(\Sigma_1\) and \(\Sigma_2\) have finite eigenvalues (they do not depend on \(N\)) can be dealt with in a similar fashion as to the i.i.d case, so we skip them. To complete the proof, keep in mind that the highest of the lower bounds corresponds to
\[
\tilde{L} = \min\left\{ [A^2N^{2x_2}]^{\frac{1}{x_1}}, [A^2N^{2x_2}]^{\frac{1}{x_2}}, [A^2N]^{\frac{1}{x}} \right\}.
\]

**Proof of Lemma 1.** Notice that with \(\hat{\theta}_l\) defined in (4), one has
\[
\hat{\theta}_l - \theta_l = \frac{1}{N} \sum_{i=1}^N \left[ \eta_i \mathbb{I}(\Omega_l(i)) - \mathbb{E}[\eta_i \mathbb{I}(\Omega_l(i))] \right] - \mathbb{E}[\eta_i \mathbb{I}(\Omega_l(i))],
\]
where the quantities \(\eta_i = \left[ y_l^2 \frac{\varphi_l(t_i)}{g(t_i)} - \sigma^2 \int_0^b \varphi_l(t) dt \right]\). Define the quantities \(\Delta_i = [\eta_i \mathbb{I}(\Omega_l(i)) - \mathbb{E}[\eta_i \mathbb{I}(\Omega_l(i))]\) and notice they are independent zero-mean random variables with variance
\[
\mathbb{E}[\Delta_i]^2 \leq 2\mathbb{E} \left[ y_l^4 \frac{\varphi_l^2(t_i)}{g^2(t_i)} \right] + 8\sigma^4
\]
\[
= 2 \left[ \mathbb{E} \left[ y_l^4 \right] \int f^4(t) \frac{\varphi_l^2(t)}{g^2(t)} dt + 6\sigma^2 \int f^2(t) \frac{\varphi_l^2(t)}{g(t)} dt + \sigma^4 \mathbb{E} \left[ z_l^4 \right] \int \varphi_l^2(t) g(t) dt \right] + 8\sigma^4 = \sigma_0^4.
\]
In addition, by Cauchy–Schwarz inequality and the Gaussian tail probability inequality, we want to show that
\[
\left( \mathbb{E}[\eta_i \mathbb{I}(\Omega_l^c(i))] \right)^2 \leq \mathbb{E}[\eta_i^2^2] \Pr(\Omega_l^c(i)) = o(N^{-1}).
\]
Bear in mind that conditional on the distribution \(g\), the quantities \(y_i = \omega f(t_i) + \sigma z_i\) are \(N(0, f^2(t_i) + \sigma^2)\). Therefore, by Assumptions A.1 and A.2 and Equation (2.5) of Muckenhoupt (1970), we obtain
\[ \Pr(\Omega^c_i(i)) \leq \Pr\left( \frac{|\varphi_k(t_i)|y_i^2}{m_1(f^2(t_i) + \sigma^2)} \geq \frac{\sigma^2 \int_0^b \varphi_1(t)dt + \sqrt{N/\ln(N)}}{f^2(t_i) + \sigma^2} \right) \]

\[ \leq \Pr\left( \frac{C_0y_i^2}{m_1(f^2(t_i) + \sigma^2)} \geq \frac{\sigma^2 \int_0^b \varphi_1(t)dt + \sqrt{N/\ln(N)}}{f^2(t_i) + \sigma^2} \right) \]

\[ = \Pr\left( \frac{y_i}{\sqrt{(f^2(t_i) + \sigma^2)}} \geq \sqrt{\frac{m_1 \sigma^2 \int_0^b \varphi_1(t)dt + \sqrt{N/\ln(N)}}{C_0 f^2(t_i) + \sigma^2}} \right) \]

\[ \leq C_2 \exp \left\{ -\frac{m_1 \sqrt{N/\ln(N)}}{2C_0 M_2^2 + \sigma^2} \right\}. \]

To complete the proof, apply the expectation to the square of (37) and use results (38) and (39).

**Proof of Lemma 2.** In order to prove (13), we make use of Bernstein inequality.

**Lemma 6.** (Bernstein Inequality). Let \( Y_i, i = 1, 2, \cdots, N, \) be independent and identically distributed random variables with mean zero and finite variance \( \sigma^2, \) with \( ||Y_i|| \leq ||Y||_\infty < \infty. \) Then,

\[ \Pr\left( N^{-1} \sum_{i=1}^N Y_i > z \right) \leq 2 \exp \left\{ -\frac{Nz^2}{2(\sigma^2 + ||Y||_\infty z/3)} \right\}. \]  

(40)

Recall the notation used in the proof of **Lemma 1.** Thus, since \( \text{Var}(\Delta_i) \leq \sigma_0^2 < \infty, \) and \( ||\Delta_i|| \leq c_0 \sqrt{N/\ln(N)}, \) for \( \gamma < 1, \) Bernstein inequality gives

\[ \Pr\left( |\hat{\theta} - \theta| > \rho \lambda_i \right) \leq \Pr\left( N^{-1} \sum_{i=1}^N \Delta_i + \mathbb{E}[\eta_i(\Omega^c_i(i))] > \rho \lambda_i \right) \]

\[ \leq \Pr\left( N^{-1} \sum_{i=1}^N \Delta_i > (\gamma \rho - 1)\sqrt{\ln(N)} \right) \]

\[ \leq 2 \exp \left\{ -\frac{2(\gamma \rho - 1)^2 \ln(N)}{2\sigma^2 + c_0(\gamma \rho - 1)/3} \right\} \]

\[ = 2N^{-\frac{(\gamma \rho - 1)^2}{2(\sigma_0^2 + c_0(\gamma \rho - 1)/3)}}. \]

**Proof of Theorem 2.** Denote

\[ \chi_N = \frac{\ln(N)}{A^2N}, \quad M_o = [\chi_N]^{-\frac{1}{2}}, \]

and note that with the choice of \( M \) and \( \lambda_i \) given by (12) and (11), respectively, the estimation error can be decomposed as \( \mathbb{E}[|h_M - h|^2] \leq \mathbb{E}_1 + \mathbb{E}_2 + \mathbb{E}_3 + \mathbb{E}_4 + \mathbb{E}_5, \) where
\[ E_1 = \sum_{l=1}^{M-1} \mathbb{E} \left[ \left| \hat{\theta}_l - \theta_l \right|^2 \mathbb{I} \left( \left| \hat{\theta}_l - \theta_l \right| > \frac{1}{2} \lambda_l \right) \right], \] (42)

\[ E_2 = \sum_{l=1}^{M-1} \mathbb{E} \left[ \left| \hat{\theta}_l - \theta_l \right|^2 \mathbb{I} \left( |\theta_l| > \frac{1}{2} \lambda_l \right) \right], \] (43)

\[ E_3 = \sum_{l=1}^{M-1} \theta_l^2 \text{Pr} \left( \left| \hat{\theta}_l - \theta_l \right| > \frac{1}{2} \lambda_l \right), \] (44)

\[ E_4 = \sum_{l=1}^{M-1} \theta_l^2 \mathbb{I} \left( |\theta_l| < \frac{3}{2} \lambda_l \right), \] (45)

\[ E_5 = \sum_{l=M}^{\infty} \theta_l^2. \] (46)

Then, by (8) and (12), (46) becomes

\[ E_5 = O \left( \sum_{l=M}^{\infty} A^2(l \mathbb{N})^{-2s} \right) = O \left( A^2[\chi_n]^{2s} \right) = O \left( A^2[\chi_n]^{\frac{2s}{2s+1}} \right). \] (47)

Now, combining \( E_1 \) and \( E_3 \), and applying Cauchy-Schwarz inequality, the moments property of the Gaussian, Lemma 1 with the choice \( \tau > 2 \), (8) and (12), yields

\[ E_1 + E_3 = O \left( \frac{M}{N} \frac{1}{N}^{\tau/2} + A^2 M \left[ \frac{1}{N} \right]^{\tau} \right) = O \left( \frac{1}{N} \right). \] (48)

Now, combining \( E_2 \) and \( E_4 \) and using condition (8) yields

\[ \Delta = E_2 + E_4 = O \left( \sum_{l=1}^{M-1} \min \left\{ \theta_l^2, A^2[\chi_n] \right\} \right). \] (49)

Finally, \( \Delta \) can be decomposed into the following components

\[ \Delta_1 = O \left( \sum_{l=M_0}^{M-1} A^2(l \mathbb{N})^{-2s} \right) = O \left( A^2[\chi_n]^{\frac{2s}{2s+1}} \right), \] (50)

\[ \Delta_2 = O \left( \sum_{l=1}^{M_0-1} A^2[\chi_n] \right) = O \left( M_0 A^2[\chi_n] \right) = O \left( A^2[\chi_n]^{\frac{2s}{2s+1}} \right). \] (51)

Hence, combining (47), (48), (50) and (51) completes the proof.

Proof of Lemma 3. Notice that with \( \hat{\theta}_l \) defined in (5), and using property (16) the properties \( \mathbb{E}[z_l^4] = 3 \) and \( \mathbb{E}[z_l^2 z_{j}^2] = \mathbb{E}[z_l^2] + 2\mathbb{E}^2[z_l z_j] \), and the Assumptions A.1, A.2 for \( m_1 \geq 1 \), as \( N \to \infty \), one has
\[
\mathbb{E}[(\hat{\theta}_1 - \theta)]^2 = \mathbb{E}\left[\frac{1}{N} \sum_{i=1}^{N} \left(\epsilon_i^2 f^2(t_i) + 2\sigma_i z_i f(t_i) + \sigma_i^2 z_i^2\right) \frac{\varphi_i(t_i)}{g(t_i)}\right]^2 - \sigma^4 \left(\int_0^b \varphi_i(t) dt\right)^2 - \theta_1^2 - 2\sigma^2 \theta_1 \int_0^b \varphi_i(t) dt
\]
\[
= \frac{3}{N} \left[\int_0^b f^4(t) \varphi^2(t)/g(t) dt + \sigma^2 \left(\int_0^b f^2(t) \varphi^2(t)/g(t) dt + \sigma^4 \int_0^b \varphi_i^2(t)/g(t) dt\right)\right]
\]
\[
+ \frac{1}{N^2} \sum_{i\neq j} \left[\theta_i^2 \mathbb{E}[\epsilon_i^2 \epsilon_j^2] + 2\sigma^2 \theta_i \int_0^b \varphi_i(t) dt + 4c_1 c_2 \sigma^2 |i-j|^{-2\zeta_1} \left(\int_0^b f(t) \varphi_i(t) dt\right)^2\right]
\]
\[
+ \frac{1}{N^2} \sum_{i\neq j} \left[\sigma^4 \mathbb{E}[\epsilon_i^2 \epsilon_j^2] \left(\int_0^b \varphi_i(t) dt\right)^2 - 2\sigma^2 \theta_i \int_0^b \varphi_i(t) dt - \sigma^4 \left(\int_0^b \varphi_i(t) dt\right)^2 - \theta_i^2\right]
\]
\[
\leq \frac{3}{N} [M_2^2 + \sigma^2]^2 + \frac{2}{N^2} \sum_{i\neq j} \left[M_2^2 c_1 |i-j|^{-2\zeta_1} + \sigma^2 c_2 |i-j|^{-2\zeta_1}\right]^2.
\]

(52)

Notice that in the last line, if both \(\zeta_1\) and \(\zeta_2\) are greater than 1/2, the first term with dominate, otherwise, the variance will be bounded by the larger of \(N^{-2\zeta_j}, j = 1, 2\).

**Proof of Lemma 4.** Below, we use a combination of Lemma 2 in Benhaddou (2018a), which is an adaptation of Hanson-Wright inequality to matrices, and large deviation result that was developed in Comte (2001) and further improved in Gendre (2014) which states that for any \(x > 0\), if \(\xi_n\) is a zero mean Gaussian vector with independent elements, and \(Q\) is nonnegative definite matrix, then

\[
\Pr\left(|\xi_n^T Q \xi_n > \sigma^2 \left[\sqrt{(tr(Q))} + \sqrt{x \rho_{max}^2(Q)}\right]^2\right) \leq e^{-x}.
\]

(53)

Let \(F\) and \(\Phi\) be the \(N\)-dimensional diagonal matrices whose diagonal elements are \(f(t_1), f(t_2), \ldots, f(t_N)\), and \(\varphi_1(t_1)/g(t_1), \varphi_1(t_2)/g(t_2), \ldots, \varphi_1(t_N)/g(t_N)\), respectively. Then,

\[
\Pr\left(|\hat{\theta}_1 - \theta_1| > \rho \hat{\lambda}_1\right) \leq \Pr\left(2\mathbf{e}_N^T \Phi F \mathbf{e}_N - N\theta_1 > \frac{\rho \gamma \ln(N)}{2 N^{2\zeta_1}}\right)
\]
\[
+ \Pr\left(2\mathbf{z}_N^T \Phi \mathbf{z}_N - N \int \varphi_1(t) dt > \frac{\rho \gamma \ln(N)}{2 \sigma^2 N^{2\zeta_1}}\right)
\]
\[
= P_1 + P_2.
\]

(54)

For the first term and if \(\theta_1 > 0\), we apply (53). Therefore, with (15) and (17) as \(N \to \infty\), one has

\[
\text{Tr}(A_i^T F \Phi F A_i) = \text{Tr}(F \Phi F A_i A_i^T) = \text{Tr}(F \Phi F \Sigma_1) = \sum_{i=1}^{N} f^2(t_i) \varphi_1(t_i)/g(t_i) \leq \frac{N}{b m_1} \theta_1,
\]

(55)

and
\[
\rho_{\max}^2(A_1^T F \Phi F A_1) = \rho_{\max}^2(F \Phi F \Sigma_1) \leq \rho_{\max}^2(F \Phi F)^2 \max_{i=1}^{m_1} |\varphi_i(t)| c_2^{(1)} N^{1-2\gamma}
\]

\[
= \frac{M_2^2}{m_1} \pi_0 c_2^{(1)} N^{1-2\gamma}
\]

Therefore, if \(bm_1 \geq 4 \text{Var}(\eta_i^{(1)})\),

\[
\Pr \left( \left\| 2\varepsilon_N^T F \Phi F e_N - N \theta_1 \right\| > \frac{\rho \gamma \ln (N)}{2 N^{2\gamma -1}} \right) \leq \Pr \left( \varepsilon_N^T F \Phi F e_N > \frac{bm_1}{4} \left[ \sqrt{\frac{N \theta_1}{bm_1}} + \sqrt{\frac{\rho \gamma \ln (N)}{2bm_1 N^{2\gamma -1}}} \right]^2 \right).
\]

(53) is applied then by taking \(x = \frac{\rho \gamma \ln (N)}{\pi_0 b^2 M_2^2}\). Now, if \(\theta_1 < 0\), then we apply Hanson–Wright inequality from Rudelson and Vershynin (2013) to

\[
\Pr \left( \left\| 2\varepsilon_N^T F \Phi F e_N - N \theta_1 \right\| > \frac{\rho \gamma \ln (N)}{2 N^{2\gamma -1}} \right) \leq \Pr \left( \left\| \varepsilon_N^T F \Phi F e_N - N \theta_1 \right\| > \frac{\rho \gamma \ln (N)}{4 N^{2\gamma -1}} \right),
\]

with matrix \(B = A_1^T F \Phi F A_1\) having Frobenius norm

\[
||A_1^T F \Phi F A_1||_F \leq \text{Tr}^2(A_1^T F \Phi F A_1) = \text{Tr}^2(F \Phi F \Sigma_1) \leq N \frac{M_1^4}{b^2 m_1^2}, \text{ as } N \to \infty.
\]

Hence, applying Hanson–Wright inequality yields

\[
\Pr \left( \left\| \varepsilon_N^T F \Phi F e_N - N \theta_1 \right\| > \frac{\rho \gamma \ln (N)}{4 N^{2\gamma -1}} \right) \leq \exp \left\{ - \frac{c_0 m_1^2 \rho \gamma \ln (N)}{4k_2^2 \pi_0 c_2^{(1)}} \right\}. \tag{57}
\]

In a similar fashion, one can evaluate \(P_2\) taking into consideration whether \(\int_0^b \varphi_1(t) dt\) is positive, in which case we use (53), or negative in which case we apply Hanson-Wright inequality.

\[\square\]

**Proof of Theorem 4.** The proof is similar to that of Theorem 2, so we skip it.

\[\square\]

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