Exact solution for many-body Hamiltonian of interacting particles with linear spectrum

M. V. Entin and L. Braginsky

Rzhanov Institute of Semiconductor Physics, Siberian Branch of the Russian Academy of Sciences
Novosibirsk 630090, Russia and
Novosibirsk State University - Novosibirsk 630090, Russia

received 29 September 2017; accepted in final form 27 November 2017
published online 29 December 2017

PACS 73.20.-r – Electron states at surfaces and interfaces
PACS 73.20.At – Surface states, band structure, electron density of states

Abstract – The exact solution of the Schrödinger equation for the one-dimensional system of interacting particles with linear dispersion law in an arbitrary external field is found. The solution is reduced to two groups of particles moving with constant velocities in the opposite directions with a fixed distance between the particles in each group. The problem is applied to the edge states of the 2D topological insulator.

Introduction. – One-dimensional electron systems with linear dispersion are the topical issue now: the edge states [1–3] of the 2D topological insulator [4], graphene strips, carbon nanotubes [5–7], etc. The specificity of these systems is the presence of the double degenerate state at the zero longitudinal momentum that has a linear splitting at a finite momentum.

The 1D systems with near-linear spectrum are the subject of study in the theory of bosonization and Luttinger liquid [8]. This approach considers the electron Fermi liquid of strongly interacting electrons assuming their energy spectrum to be approximately linear near the Fermi energy. The bosonization procedure separates the long- ($q \ll p_F$) and short-range ($q \approx 2p_F$) interactions considering them in different ways. In general, this approach is approximative and applicable near the Fermi points only.

Note that the curvature of the energy spectrum caused by the $k$-$p$ expansion violates the linearity. For example, such non-linear corrections to the energy spectrum exist in graphene where they determine the unconventional character of the e-e scattering [9] and e-h coupling to the excitons [10]. The linear spectrum of the 2D TI edge states is the main reason of disappearance of the electron correlation energy [11].

In [12] we have studied the linearity of the energy spectrum in the edge states of the 2D topological insulator. We concluded that in two models of the edge states, [13] and [1–3], the linearity is absolute, while in other cases the non-linear corrections are extremely weak. This pushes forward the problem of the many-electron states in the system with the linear single-electron energy spectrum.

The purpose of the present paper is the general consideration of the one-dimensional system with the many-body Hamiltonian

$$H = \sum_i \left( \frac{i}{\hbar} \sigma_i p_i + U(x_i) \right) + \sum_{i<j} V(x_i - x_j). \quad (1)$$

Here $U(x)$ is an external field, and $V(x_i - x_j)$ is the interaction between the particles, $[x_i, x_j] = 0$, $[p_i, p_j] = 0$, $[p_i, x_j] = -ih$. For certainty we consider $\sigma_i = \pm$ (or equivalently, $\sigma_i = \uparrow, \downarrow$) as a spin quantum number. Below we shall set $\hbar = 1$. The coordinates of spin-up and spin-down electrons are denoted as $x_i$ and $y_i$, respectively. The Hamiltonian equation (1) is valid for the edge states of electrons in 2D topological insulators [12].

The linearity of the energy spectrum is the most important for our consideration. We obtain an exact solution of this quantum problem. Unlike the case of the Luttinger liquid, we do not need low temperatures and the vicinity of the energy to the Fermi level. Owing to the exact linearity of the spectrum our results are valid for all energies.

Exact solution of the many-body Schrödinger equation. – In the absence of interaction the direction of motion coincides with the spin sign, so that the spin-up and spin-down electrons are right-movers and left-movers, respectively. The exact solution of the Schrödinger equation for a single particle with energy $E = pv$ is
\[\psi(x) = \exp(ipx - i \int dx U(x)/v).\] This is the wave function of constant density \(|\psi|^2\). The result essentially differs from that for a particle with a quadratic kinetic energy.

Consider now two free (\(U(x) = 0\)) particles with positive spins. The Hamiltonian
\[H = v \rho_1 + v \rho_2 + V(x_1 - x_2)\]
commutes with \(x_1 - x_2\) conserving, therefore, the distance between the particles. Then the wave function can be chosen as the eigenfunction of \(x_1 - x_2\) and the total momentum \(P = p_{1,1} + p_{2,1}\): \(\psi = A \exp(iP(x_1 + x_2)/2)\delta(x_1 - x_2 - a)\), where \(a\) is the eigenvalue of \(x_1 - x_2\). The corresponding self-energy is
\[E = P v + V(a).\]
The normalizing coefficient \(A\) should be chosen to exclude the divergence of the integral \(\int dx \delta^2(x)\). This divergence inevitably appears when one uses the self-functions of the coordinate operator. It can be formally fixed by the choice of \(A^2 = 1/\delta(0)\).

For the case of \(n\) right-movers we write
\[\psi(x_1, \ldots, x_n) = A^n \exp(iPX)(\Pi_k \delta(x_{k+1} - x_k - a_k)), \quad (2)\]
where \(X = \sum x_i/n, \; P = \sum_i p_{i,1}\) is the total momentum. The self-energy is
\[E = P v + \sum_{i<j} V(a_{ij}), \quad a_{ij} = \sum_{k=1}^{j-1} a_k. \quad (3)\]
To take an external, e.g., impurity, potential \(U(x_i)\) into consideration, we multiply the wave function, eq. (2), by the factor
\[u = \exp(-i \int dx \sum_j U(x_1 + a_j)).\]

The coordinate part of the same-spin wave function should be antisymmetrized. To do that, the Slater determinant has to be constructed.

Consider now two particles of the opposite spins with the Hamiltonian
\[H = v \rho_1 - v \rho_1 + V(x - y).\]
The wave function can be chosen as the eigenfunction of \(x + y\) with self-value \(2c\):
\[\psi = \exp\left(iP'(x - y)/2 - i \int dx V(2b - 2x)/v\right) \times A \delta(x + y - 2c). \quad (4)\]
Here \(P' = p_1 - p_2\), the corresponding self-energy is \(E = P' v\).

Finally, consider a general case of \(n\) spins-up and \(m\) spins-down. The Hamiltonian commutes with the central point between each left-mover and right-mover \((x_1 + y_1)/2\).

The sufficient condition is the fixation of one of these variables, e.g., \(x_1 + y_1 = 2c\). Hence, the state with the quantum numbers \(P, c, \{a_k\}, \{b_k\}\) is
\[\psi(x_1, \ldots, x_n; y_1, \ldots, y_m) = \Phi \exp(iP') A \delta(y_1 + x_1 - 2c) \times \prod_{k=1}^{n-1} A \delta(x_{k+1} - x_k - a_k) \times \prod_{k=1}^{m-1} A \delta(y_{k+1} - y_k - b_k), \quad (5)\]
where \(\zeta = (x_1 - y_1)/2, \; m > 1, \; n > 1\). If \(m = 1\) or \(n = 1\), the corresponding product in eq. (5) should be replaced by unity. Substituting into the Schrödinger equation, we find the proportionality factor
\[\Phi = \exp\left(-i \int dx V(x_1) + U(x_1)\right), \quad (6)\]

The corresponding energy is
\[E = P v + \bar{V}, \quad \bar{V} = \sum_{i>j} V(a_{ij}) + \sum_{i>j} V(b_{ij}). \quad (8)\]

The case of an arbitrary number of identical electrons with different spins should be considered using the permutation symmetry. The ground state of the system corresponds to the equal numbers of up and down spins, so that the total spin is zero. Note that in this state the average velocity vanishes.

Neglecting the particle exchange, each of the subsystems of the right- and left-movers represents a solid superparticle, inside which the distances between the electrons are fixed. The right- and left-moving superparticles, (RMS) and (LMF), obey the linear dispersion. The Hamiltonian of the RMS and LMS interaction depends on the distance between the electrons belonging to the different superparticles. Thus, RMS and LMS can be considered as two opposite moving superparticles and the two-particle wave function can be used to describe their relative motion. This is an explanation of eqs. (5) and (6).

**Cyclic boundary conditions.** – In the previous consideration we assumed that the coordinates change in the infinite domain \(-\infty < x_i < \infty\). In this case the energy is expressed via the total and relative momenta of the left- and right-movers (8) and the interaction energies inside
the groups. It is important to point out that the interactions between the carriers from different groups as well as the carriers with the impurities do not contribute to the total energy.

Actually, this is the consequence of the problem formulation. We have considered an infinite system with a finite number of interacting electrons. In this case electrons with different spins being separated at infinite distance can be characterized by their momenta at the infinity. In a dense system, however, this is not the case. Consider now a cyclic system of the length $L$ assuming $x_i + L = x_i$. Suppose the potentials $V(x)$ and $U(x)$ to be the periodic functions of $L$. For this reason we replace $x$ by the distance between points on the circle $x \equiv (L/2\pi) \sin(2\pi x/L)$. In particular, the $\delta$-functions in the previous expressions have to be replaced by their periodic generalizations $\delta(x) \rightarrow (2\pi/L)\delta(\sin(2\pi x/L))$.

The cyclic boundary condition reads $\psi(x_1, \ldots, x_i + L, \ldots) = \psi(x_1, \ldots, x_i, \ldots)$. Consider first the two-particle problem with opposite spins. In this case the quantization rule, eq. (3), is

$$E = \frac{2\pi vN}{L} + \frac{1}{L} \int_0^L (V(x) + 2U(x))dx,$$

where $N$ is an integer. The second term here is the average interaction, which has to be added to the total energy. The generalization of the quantization rule to many particles is

$$E = 2\pi vN(n - m)/L + \sum_{\sigma = \pm, i,j} V(s_{\sigma,i,j}) + nm \int_0^L V(x)dx + (n + m) \int_0^L U(x)dx.$$  \hspace{1cm} (9)

Here $s_{\sigma,i,j}$ is the distance between the $i$-th and $j$-th particles of the same spin. In accordance with eq. (9), the total energy incorporates the intra-group interaction and averages of the external field and inter-group interaction. The physical meaning can be simply understood: electrons with the same spins conserve the distance between each other (and, consequently, the sum of the potentials in eq. (9)) and move through the impurity lattice and the electrons with opposite spins, averaging the interaction with them (the second line in eq. (9)).

It is clear from eq. (9) that the electron density is not affected by an external potential. This explains the absence of the correlation energy [11] for the system with the linear spectrum.

Note that $P$, $a_k$, $b_k$, and $c$ compose the full set of the numbers describing the system state. The variable $P$ is a quantum number. It is quantized in a closed edge in the same way as the non-interacting particles momenta. The distances between the same-spin particles $a_k$, $b_k$ and the quantity $c$ are the classical variables. This can be seen from the Hamiltonian which corresponds to the limit $h \rightarrow 0$. The set of classical variables has arbitrary values; they have infinite masses and are resting. To some extent, this situation recalls the molecular systems where the electron coordinates are quantum quantities, while the ion coordinates are classical. It is known that the molecular system can be considered via the molecular terms: the electronic levels are determined at fixed arbitrary positions of the ions, while the motion of the latter is considered classically where the terms play the role of the interaction potential. (This description is limited by the crossing of the molecular terms.) The relative momentum $P$ is a global variable. Thus, the quantities $a_k$, $b_k$ and $c$ obey the Boltzmann statistics. This explains how to make the average of the observable quantities.

### Scattering of interacting electrons at a magnetic impurity.

The exact solution permits one to include perturbingly other interaction mechanisms that can affect the responses, for example, an interaction with magnetic impurities or the spin-orbit interaction with phonons. Here, as an example, we consider backscattering of the electrons by a magnetic impurity. The backscattering is forbidden without such an interaction violating the time-reversibility. The e-e interaction essentially modifies the magnetic impurity scattering. The transition between the states occurs at the terms crossing points (when the total energies of two states are equal at coinciding electron positions).

Consider an impurity whose spin $\mathbf{S}$ interacts with the electron spin $\mathbf{\sigma_i}/2$. The Hamiltonian of the spin-spin interaction is

$$H_{ss} = U_0 \sum_n \delta(x_n)(\mathbf{S}\mathbf{\sigma_n}).$$  \hspace{1cm} (10)

For the two-electron system we find the matrix elements

$$M = U_0/2e^{(P' + P)^2}\delta(a - 2b)$$

and the transition rate $T = U_0^2/4\delta(E - (P' - P)v - \delta V)$. Here $\delta V = V(a) - \int_0^L V(x)dx$ is a correction to the total energy due to interaction. In the general case, the matrix element between the wave functions (5) is equal to $M = U_0/2\sum_{ij} e^{(P + P')^2}\delta(a_j - 2b)$ and $T = NU_0^2/4\delta(E - (P' - P)v - \delta V)$, where $\delta V = \sum_{ij} V_{ij} - (n - m + 1) \int_0^L V(x)dx$ includes the correction to the interaction energy $V_{ij}$ after the transition of the electron from the left- to the right-moving ensemble or vice versa.

Now let us study the many-body problem. Consider the transition between the states $|1\rangle = |P, c, \{a_1, \ldots, a_{n-1}\}, \{b_1, \ldots, b_{m-1}\}\rangle$ and $|2\rangle = |P', c', \{a_2, \ldots, a_{n-1}\}, \{b_1, \ldots, b_{m-1}\}\rangle$. The backscattering rate at fixed quantum numbers is

$$2\pi|\langle 1|H_{ss}|2\rangle|^2 \times \delta(vP - vP' + \sum_{j=1}^{n-1} V(a_j) - \sum_{j=1}^{m-1} V(b_j)) =$$

$$2\pi|U_0|^2 \delta(vP - vP' + \sum_{j=1}^{n-1} V(a_j) - \sum_{j=1}^{m-1} V(b_j)) \times \sum_{j,k} \delta(a_j + 2c)\delta(b_k + 2c').$$  \hspace{1cm} (11)
The physical meaning of eq. (11) is simple. An electron changing its spin simultaneously changes its interaction energy with all electrons with the same spins to the interaction energy with the opposite-spin electrons. This difference of potential energies is transmitted to the difference of the kinetic energies establishing the thermal equilibrium between the kinetic and potential energies.

Classical variables obey the Boltzmann statistics in the thermal equilibrium. Let us now average the energy delta function over this distribution:

\[ R = \frac{\int \prod d\alpha d\beta \exp(-\beta V) \delta(\ldots)}{\int \prod d\alpha d\beta \exp(-\beta V)}, \tag{12} \]

where \( \beta \) is the inverse temperature. In the nearest-neighbor approximation we obtain

\[
R = \int_{-\infty}^{\infty} \frac{dt}{2\pi} e^{itv(P-P')} \exp \left\{ \frac{l}{L} \int \frac{1}{\beta} \left[ \frac{m \left( e^{(it-\beta)V(a)} - 1 \right)}{L} \right] \right\} \exp \left\{ \frac{n + m}{L} \right\}.
\]

Equation (12) gives a symmetric dependence of the transition probability on \( P - P' \), which is determined by the function \( V(a) \). A careful examination goes beyond the scope of the paper.

Discussion. – Let us discuss some properties of the edge states that can be observed in experiments. The many-electron edge states have an extremely unusual organization. They consist of two groups of electrons moving with the same velocity in the opposite directions with the stable distances in each group. At the same time they do not perceive the elements of other group. What physical consequences follow from this?

Consider a hypothetic way to observe the moving electron lattices by diffraction of an external wave (visual light, slow electrons, x-ray, neutron etc.). After elastic scattering at the moving 1D solid lattice (this is the case of the strong e-e interaction) the wave with the wave vector \( \mathbf{k} \) obtains an additional wave vector \( \mathbf{q} \). This produces the change of the wave frequency by \( \delta \omega = \pm q_x v \) \( (q_x \) is the wave vector component along the edge). The sign of \( \delta \omega \) is arbitrary. On the contrary, in the absence of e-e interaction the Fermi distribution allows only negative \( \delta \omega \). This permits distinguishing the situations experimentally. Note that rigid connection of the frequency change with the direction of the wave scattering should be the evidence of the absence of the velocity dispersion.

The value of \( \delta \omega \) is relatively small even for the visual light wave. However, \( \delta \omega \) can be observed by heterodyning of incident and scattered waves. Disorder in the electrons position randomises the \( q_x \) value. However, if the electrons are ordered into the Vigner lattice, the \( q_x \) value has to be quantized.

In conclusion, we have found the exact solution of the interacting many-particle 1D system with linear single-particle spectrum. The Hamiltonian includes an external potential (e.g., impurities) as well. The Schrödinger equation solution is reduced to the separation of the system to the groups of right- and left-moving carriers with the constant velocity. The interactions between groups and with impurities are reflected in the phase factor in the wave function. The relative coordinates in each group turn out to be classical conserving variables, while the relative momentum of all carriers is a global quantum number. The collective self-energy consists of the linear-in-momentum kinetic part and the potential energy of interaction at a fixed interparticle distance inside the groups. In the framework of the Hamiltonian equation (1) the backscattering is absent. The formal solution is applicable to the edge states of the 2D topological insulator. In a separate paper [12] we have found that the edge states have either an exact linear single-electron spectrum in most models of the 2D topological insulator or this spectrum is numerically linear. Hence, the results of the present paper directly pertain to these edge states.

Unlike the Luttinger liquid, our solution is not linked to the Fermi level of the non-interacting system. It is also valid in a strongly non-equilibrium situation.

The conservation of the distances between the same-spin electrons makes relaxation of such system to the equilibrium impossible, unless some additional terms are taken into account.

In accordance with the obtained equations, the implementation of the e-e and electron-impurities interactions has no effect on the velocity matrix elements. This means that the conductivity of the system is also not changed and stays infinite for the system of interacting electrons.

The exact solutions permitted one to include perturbingly the interaction with the magnetic impurities that was not included in the Hamiltonian (1). It was found that the e-e interaction essentially affects the backscattering. Note that other mechanisms, like the spin-orbit interaction with phonons, can be studied in the same way.

One other remark concerns the possibility of the generalization of the Hamiltonian (1). It obviously can be generalized to

\[
H = \sum_i (v_\sigma p_i + U_i(x_i) + \sigma_i U^{(1)}_i(x_i)) + \sum_{i<k} (V_{ij}(x_i - x_j) + \sigma_i \sigma_j V^{(1)}_{ij}(x_i - x_j)), \tag{13}
\]

with similar consequences.
Exact solution for many-body Hamiltonian of interacting particles with linear spectrum

***

This research was supported by RFBR grant No. 17-02-00837. The authors thank A. V. Chaplik for stimulating discussions.

REFERENCES

[1] Qi Xiao-Liang and Zhang Shou-Cheng, Rev. Mod. Phys., 83 (2011) 1057.
[2] Zhou Bin, Lu Hai-Zhou, Chu Rui-Lin, Shen Shun-Qing and Niu Qian, Phys. Rev. Lett., 101 (2008) 246807.
[3] König M., Buhmann H., Molenkamp Laurens W., Hughes T., Liu Chao-Xing, Qi Xiao-Liang and Zhang Shou-Cheng, J. Phys. Soc. Jpn., 77 (2008) 031007.
[4] Bernevig B. A., Hughes T. L. and Zhang Shou-Cheng, Science, 314 (2006) 1757.
[5] Miserev D. S. and Entin M. V., Zh. Eksp. Teor. Fiz., 142 (2012) 784; J. Exp. Theor. Phys., 115 (2012) 694.
[6] Miserev D. S. and Entin M. V., J. Exp. Theor. Phys. Lett., 99 (2014) 410.
[7] Miserev D. S., J. Exp. Theor. Phys., 149 (2016) 1223.
[8] Tsvelik Alexei M., Quantum Field Theory in Condensed Matter Physics, 2nd edition (Cambridge University Press, Cambridge, UK) 2003.
[9] Golub L. E., Tarasenko A., Entin M. V. and Magarill L. I., Phys. Rev. B, 84 (2011) 195408.
[10] Mahmoodian M. M. and Entin M. V., EPL, 102 (2013) 37012.
[11] Entin M. V. and Braginsky L., Phys. Rev. B, 96 (2017) 115403.
[12] Entin M. V., Mahmoodian M. M. and Magarill L. I., EPL, 118 (2017) 57002.
[13] Volkov B. A. and Pankratov O. A., Pis’ma Zh. Eksp. Teor. Fiz., 42 (1985) 145; JETP Lett., 42 (1985) 178.