AN ANALOGUE OF THE LÉVY-CRAMÉR THEOREM FOR RAYLEIGH DISTRIBUTIONS

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Abstract. In the present paper we prove that every $k$-dimensional Cartesian product of Kingman convolutions can be embedded into a $k$-dimensional symmetric convolution ($k=1, 2, \ldots$) and obtain an analogue of the Cramér-Lévy theorem for multi-dimensional Rayleigh distributions.

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1. Introduction, Notations and Preliminaries

In probability theory and statistics, the Rayleigh distribution is a continuous probability distribution which is widely used to model events that occur in different fields such as medicine, social and natural sciences. A multivariate Rayleigh distribution is the probability distribution of a vector of norms of random Gaussian vectors. The purpose of this paper, is to introduce and study the fractional indexes multivariate Rayleigh distributions via the Cartesian product of Kingman convolutions and, in particular, to prove an analogue of the Lévy-Cramér theorem for multivariate Rayleigh distributions.

Let $\mathcal{P} := \mathcal{P}(\mathbb{R}^+)$ denote the set of all probability measures (p.m.’s) on the positive half-line $\mathbb{R}^+$. Put, for each continuous bounded function $f$ on $\mathbb{R}^+$,

$$(1) \quad \int_0^\infty f(x)\mu *_1,\delta \nu(dx) = \frac{\Gamma(s + 1)}{\sqrt{\pi} \Gamma(s + \frac{1}{2})} \int_0^\infty \int_0^\infty \int_{-1}^1 f((x^2 + 2uxy + y^2)^{1/2}/(1 - u^2))^{s-1/2} \mu(dx)\nu(dy)du,$$

where $\mu$ and $\nu \in \mathcal{P}$ and $\delta = 2(s+1) \geq 1$ (cf. Kingman [7] and Urbanik [17]). The convolution algebra $(\mathcal{P}, *_{1,\delta})$ is the most important example of Urbanik convolution algebras (cf Urbanik [17]). In language of the Urbanik convolution algebras, the characteristic measure, say $\sigma_s$, of the Kingman convolution has the Rayleigh density

$$(2) \quad d\sigma_s(y) = \frac{2(s+1)^{s+1}}{\Gamma(s+1)} y^{2s+1} \exp\left(-\left(s+\frac{1}{2}\right)y^2\right)dy$$

with the characteristic exponent $\kappa = 2$ and the kernel $\Lambda_s$

$$(3) \quad \Lambda_s(x) = \Gamma(s+1)J_s(x)/(1/2x)^s,$$

where $J_s(x)$ denotes the Bessel function of the first kind,

$$(4) \quad J_s(x) := \sum_{k=0}^\infty \frac{(-1)^k (x/2)^{\nu+2k}}{k! \Gamma(\nu+k+1)}.$$

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It is known (cf. Kingman [7, Theorem 1]), that the kernel $\Lambda_s$ itself is an ordinary characteristic function (ch.f.) of a symmetric p.m., say $F_s$, defined on the interval $[-1,1]$. Thus, if $\theta_s$ denotes a random variable (r.v.) with distribution $F_s$ then for each $t \in \mathbb{R}^+$,

$$
\Lambda_s(t) = E \exp(it\theta_s) = \int_{-1}^{1} \cos(tx)dF_s(x).
$$

Suppose that $X$ is a nonnegative r.v. with distribution $\mu \in \mathcal{P}$ and $X$ is independent of $\theta_s$. The radial characteristic function (rad.ch.f.) of $\mu$, denoted by $\hat{\mu}(t)$, is defined by

$$
\hat{\mu}(t) = E \exp(itX\theta_s) = \int_{0}^{\infty} \Lambda_s(tx)\mu(dx),
$$

for every $t \in \mathbb{R}^+$. The characteristic measure of the Kingman convolution $*_{1,\delta}$, denoted by $\sigma_s$, has the Maxwell density function

$$
d\sigma_s(x) = \frac{2(s+1)^{s+1}}{\Gamma(s+1)} x^{2s+1} \exp\{- (s+1)x^2\}, \quad 0 < x < \infty.
$$

and the rad.ch.f.

$$
\hat{\sigma}_s(t) = \exp\{-t^2/4(s+1)\}.
$$

Let $\hat{\mathcal{P}} := \hat{\mathcal{P}}(\mathbb{R}^+)$ denote the class of symmetric p.m.’s on $\mathbb{R}^+$. Putting, for every $G \in \mathcal{P}$,

$$
F_s(G) = \int_{0}^{\infty} F_{cs}G(dc),
$$

we get a continuous homeomorphism from the Kingman convolution algebra $(\mathcal{P}, *_{1,\delta})$ onto the ordinary convolution algebra $(\hat{\mathcal{P}}, *)$ such that

$$
F_s\{G_1 *_{1,\delta} G_2\} = (F_sG_1) * (F_sG_2) \quad G_1, G_2 \in \mathcal{P}
$$

$$
F_s\sigma_s = N(0, 2s + 1)
$$

which shows that every Kingman convolution can be embeded into the ordinary convolution $*$. 

2. Cartesian product of Kingman convolutions

Denote by $\mathbb{R}^{+k}, k = 1, 2, ...$ the k-dimensional nonnegative cone of $\mathbb{R}^k$ and $\mathcal{P}(\mathbb{R}^{+k})$ the class of all p.m.’s on $\mathbb{R}^{+k}$ equipped with the weak convergence. In the sequel, we will denote the multidimensional vectors and random vectors (r.vec.’s) and their distributions by bold face letters.

For each point $z$ of any set $A$ let $\delta_z$ denote the Dirac measure (the unit mass) at the point $z$. In particular, if $x = (x_1,x_2,\ldots,x_k) \in \mathbb{R}^{k+}$, then

$$
\delta_X = \delta_{x_1} \times \delta_{x_2} \times \ldots \times \delta_{x_k}, \quad (k \text{ times}),
$$

where the sign ” $\times$ ” denotes the Cartesian product of measures. We put, for $x = (x_1,\ldots,x_k)$ and $y = (y_1,y_2,\ldots,y_k) \in \mathbb{R}^{+k},$

$$
\delta_X \bigcirc_{s,k} \delta_Y = \{\delta_{x_1} o_{s} \delta_{y_1}\} \times \{\delta_{x_2} o_{s} \delta_{y_2}\} \times \ldots \times \{\delta_{x_k} o_{s} \delta_{y_k}\}, \quad (k \text{ times}),
$$
Then, the following formula is equivalent to (15) (cf. [13])

\[ G_1 \circ_{s,k} G_2 = \int_{\mathbb{R}^k} \delta_x \circ_{s,k} \delta_y G_1(dx)G_2(dy) \]

which means that for each continuous bounded function \( \phi \) defined on \( \mathbb{R}^k \)

\[ \int_{\mathbb{R}^k} \phi(z) G_1 \circ_{s,k} G_2(dz) = \int_{\mathbb{R}^k} \{ \int_{\mathbb{R}^k} \phi(z) \delta_x \circ_{s,k} \delta_y (dz) \} G_1(dx)G_2(dy). \]

In the sequel, the binary operation \( \circ_{s,k} \) will be called the \( k \)-times Cartesian product of Kingman convolutions and the pair \( (\mathcal{P}(\mathbb{R}^k), \circ_{s,k}) \) will be called the \( k \)-dimensional Kingman convolution algebra. It is easy to show that the binary operation \( \circ_{s,k} \) is continuous in the weak topology which together with (11) and (15) implies the following theorem.

**Theorem 1.** The pair \( (\mathcal{P}(\mathbb{R}^k), \circ_{s,k}) \) is a commutative topological semigroup with \( \delta_0 \) as the unit element. Moreover, the operation \( \circ_{s,k} \) is distributive w.r.t. convex combinations of p.m.’s in \( \mathcal{P}(\mathbb{R}^k) \).

For every \( G \in \mathcal{P}(\mathbb{R}^k) \) the \( k \)-dimensional rad.ch.f. \( \hat{G}(t) = (t_1, t_2, \cdots, t_k) \in \mathbb{R}^k \), is defined by

\[ \hat{G}(t) = \int_{\mathbb{R}^k} \prod_{j=1}^k \Lambda_s(t_j x_j) G(dx), \]

where \( x = (x_1, x_2, \cdots, x_k) \in \mathbb{R}^k \). Let \( \Theta_s = \{ \theta_{s,1}, \theta_{s,2}, \cdots, \theta_{s,k} \} \), where \( \theta_{s,j} \) are independent r.v.’s with the same distribution \( F_s \). Next, assume that \( X = \{ X_1, X_2, \ldots, X_k \} \)

is a k-dimensional r.vec. with distribution \( G \) and \( X \) is independent of \( \Theta_s \). We put

\[ [\Theta_s, X] = \{ \theta_{s,1} X_1, \theta_{s,2} X_2, \ldots, \theta_{s,k} X_k \}. \]

Then, the following formula is equivalent to (15) (cf. 13)

\[ \hat{G}(t) = E e^{i \langle t, [\Theta_s, X] \rangle}, \quad t \in \mathbb{R}^k. \]

The Reader is referred to Corollary 2.1, Theorems 2.3 & 2.4 [13] for the principal properties of the above rad.ch.f. Given \( s \geq -1/2 \) define a map \( F_{s,k} : \mathcal{P}(\mathbb{R}^k) \rightarrow \mathcal{P}(\mathbb{R}^k) \) by

\[ F_{s,k}(G) = \int_{\mathbb{R}^k} \left( T_{c_1} F_s \right) \times \left( T_{c_2} F_s \right) \times \cdots \times \left( T_{c_k} F_s \right) G(dc), \]

here and in the sequel, for a distribution \( G \) of a r.vec. \( X \) and a real number \( c \) we denote by \( T_c G \) the distribution of \( cX \). Let us denote by \( \mathcal{P}_{s,k}(\mathbb{R}^k) \) the sub-class of all p.m.’s defined by the right-hand side of (15). By virtue of (15) it is easy to prove the following theorem.

**Theorem 2.** The set \( \mathcal{P}_{s,k}(\mathbb{R}^k) \) is closed w.r.t. the weak convergence and the ordinary convolution * and the following equation holds

\[ \hat{G}(t) = \mathcal{F}\left(F_{s,k}(G)\right)(t) \quad t \in \mathbb{R}^k \]
where \( F(K) \) denotes the ordinary characteristic function (Fourier transform) of a p.m. \( K \). Therefore, for any \( G_1 \) and \( G_2 \in \mathbb{R}^{+k} \)

\[(20) \quad F_{s,k}(G_1) \ast F_{s,k}(G_2) = F_{s,k}(G_1 \circ_{s,k} G_2)\]

and the map \( F_{s,k} \) commutes with convex combinations of distributions and scale changes \( T_{c}, c > 0 \). Moreover,

\[(21) \quad F_{s,k}(\Sigma_{s,k}) = N(0, 2(s+1)I)\]

where \( \Sigma_{s,k} \) denotes the \( k \)-dimensional Rayleigh distribution and \( N(0, 2(s+1)I) \) is the symmetric normal distribution on \( \mathbb{R}^{k} \) with variance operator \( R = 2(s+1)I, I \) being the identity operator. Consequently, Every Kingman convolution algebra \( (\mathcal{P}(\mathbb{R}^{+k}), \circ_{s,k}) \) is embedded in the ordinary convolution algebra \( (\mathcal{P}_{s,k}(\mathbb{R}^{+k}), \ast ) \) and the map \( F_{s,k} \) stands for a homeomorphism.

Proof. First we prove the equation (19) by taking the Fourier transform of the right-hand side of (18). We have, for \( t \in \mathbb{R}^{k} \),

\[(22) \quad \mathcal{F}(F_{s,k}(G))(t) = \int_{\mathbb{R}^{k}} \prod_{j=1}^{k} \cos(t_{j}x_{j})H_{s,k}(G)dx = \int_{\mathbb{R}^{k}} \int_{\mathbb{R}^{+k}} \prod_{j=1}^{k} \cos(t_{j}x_{j})(T_{c}F_{s}(dx)G(dc)) = \int_{\mathbb{R}^{+k}} \prod_{j=1}^{k} \Lambda_{s}(t_{j}c_{j})G(dc) = \hat{G}(t)\]

which implies that the set set \( \mathfrak{P}_{s,k}(\mathbb{R}^{+k}) \) is closed w.r.t. the weak convergence and the ordinary convolution \( \ast \) and, moreover, the equations (20) and (21) hold. \( \square \)

Definition 1. A p.m. \( F \in \mathcal{P}(\mathbb{R}^{+k}) \) is called \( \circ_{s,k} \)-infinitely divisible (\( \circ_{s,k} \)-ID), if for every \( m=1, 2, \ldots \) there exists \( F_{m} \in \mathcal{P}(\mathbb{R}^{+k}) \) such that

\[(23) \quad F = F_{m} \circ_{s,k} F_{m} \circ_{s,k} \ldots \circ_{s,k} F_{m} \quad (m \text{ times}).\]

Definition 2. \( F \) is called stable, if for any positive numbers \( a \) and \( b \) there exists a positive number \( c \) such that

\[(24) \quad T_{a}F \circ_{s,k} T_{b}F = T_{c}F\]

By virtue of Theorem 2 it follows that the following theorem holds.

Theorem 3. A p.m. \( G \) is \( \circ_{s,k} \)-ID, resp. stable if and only if \( H_{s,k}(G) \) is ID, resp. stable, in the usual sense.

The following lemma will be used in the representation of \( \circ_{s,k} - \text{ID}, k \geq 2 \).

Lemma 1. (i) For every \( t \geq 0 \)

\[(25) \quad \lim_{x \to 0} \frac{1 - \Lambda_{s}(tx)}{x^{2}} = \lim_{x \to 0} \frac{1 - E_{e^{it\theta}}}{x^{2}} = \frac{t^{2}}{2}.\]

(ii) For any \( x = (x_{0}, x_{1}, \cdots, x_{k}) \) and \( t = (t_{0}, t_{1}, \cdots, t_{k}) \in \mathbb{R}^{k+1}, k = 1, 2, \ldots \)

\[(26) \quad \lim_{\rho \to 0} \frac{1 - \prod_{r=0}^{k} \Lambda_{s}(t_{r}x_{r})}{\rho^{2}} = \frac{1}{2} \sum_{r=0}^{k} \lambda_{s}(Arg(x))t_{r}^{2} \]
where $\rho = \|x\|$, $\text{Arg}(x) = \frac{x}{\|x\|}$, and $\lambda_r(\text{Arg}(x)), r = 0, 1, \ldots, k$ are given by

$$
\lambda_r(\text{Arg}(x)) = \begin{cases} 
\cos \phi & r = 0, \\
\sin \phi \sin \phi_1 \cdots \sin \phi_{r-1} \cos \phi_r & 1 \leq r \leq k-2, \\
\sin \phi \sin \phi_1 \cdots \sin \phi_{k-2} \sin \psi & r = k-1, \\
\sin \phi \sin \phi_1 \cdots \sin \phi_{k-2} \sin \psi & r = k,
\end{cases}
$$

where $0 \leq \psi, \phi, \phi_r \leq \pi/2$, $r = 1, 2, \ldots, k-2$ are angles of $x$ appearing its polar form.

The following theorem gives a representation of rad.ch.f.'s of $\sigma_{s,k}$-ID distributions (see [13], Theorem 2.6 for the proof).

**Theorem 4.** A p.m. $\mu \in \text{ID}(\sigma_{s,k})$ if and only if there exist a $\sigma$-finite measure $M$ (a Lévy's measure) on $\mathbb{R}^+ \times \cdots \times \mathbb{R}^+$ with the property that $M(0) = 0$, $M$ is finite outside every neighborhood of $0$ and

$$
\int_{\mathbb{R}^+ \times \cdots \times \mathbb{R}^+} \frac{\|x\|^2}{1 + \|x\|^2} M(dx) < \infty
$$

and for each $t = (t_1, \ldots, t_k) \in \mathbb{R}^+$

$$
- \log \hat{\mu}(t) = \int_{\mathbb{R}^+ \times \cdots \times \mathbb{R}^+} \left\{ 1 - \prod_{j=1}^k \Lambda_s(t_jx_j) \right\} \frac{1 + \|x\|^2}{\|x\|^2} M(dx),
$$

where, at the origin $0$, the integrand on the right-hand side of (29) is assumed to be

$$
\Sigma_{j=1}^k \lambda_j^2 t_j^2 = \lim_{\|x\| \to 0} \left\{ 1 - \prod_{j=1}^k \Lambda_s(t_jx_j) \right\} \frac{1 + \|x\|^2}{\|x\|^2}
$$

for nonnegative $\lambda_j, j = 1, 2, \ldots, k$ given by equations (27) in Lemma 1. In particular, if $M = 0$, then $\mu$ becomes a Rayleighian p.m. on $\mathbb{R}^+$ as defined as follows:

**Definition 3.** A $k$-dimensional distribution, say $\Sigma_{s,k}$, is called a Rayleigh distribution, if

$$
\Sigma_{s,k} = \sigma_s \times \sigma_s \times \cdots \times \sigma_s \quad (k \text{ times}).
$$
Further, a distribution $F \in \mathcal{P}(\mathbb{R}^+)$ is called a Rayleighian distribution if there exist nonnegative numbers $\lambda_r, r = 1, 2 \cdots k$ such that

$$F = \{T_{\lambda_1} \sigma_s\} \times \{T_{\lambda_2} \sigma_s\} \times \cdots \times \{T_{\lambda_k} \sigma_s\}.$$  

(33)

It is evident that every Rayleigh distribution is a Rayleighian distribution. Moreover, every Rayleighian distribution is $\bigcirc_{s,k} - ID$. By virtue of (7) and (32) it follows that the k-dimensional Rayleigh density is given by

$$g(x) = \prod_{j=1}^{k} \frac{2^{k(s+1)}(s+1)^k}{\Gamma^k(s+1)} x_j^{2s+1} \exp\{- (s+1)\|x\|^2\},$$

where $x = (x_1, x_2, \ldots, x_k) \in \mathbb{R}^+$ and the corresponding rad.ch.f. is given by

$$\hat{\Sigma}_{s,k}(t) = \exp(-|t|^2/(4(s+1)), \ t \in \mathbb{R}^+.$$  

Finally, the rad.ch.f. of a Rayleighian distribution $F$ on $\mathbb{R}^+$ is given by

$$\hat{F}(t) = \exp\left(-\frac{1}{2} \sum_{j=1}^{k} \lambda_j^2 t_j^2\right)$$

where $\lambda_j, j = 1, 2, \ldots, k$ are some nonnegative numbers.

3. An analogue of the Lévy-Cramér Theorem in multi-dimensional Kingman convolution algebras

We say that a distribution $F$ on $\mathbb{R}^k$ has dimension $m$, $1 \leq m \leq k$, if $m$ is the dimension of the smallest hyper-plane which contains the support of $F$. The following theorem can be regarded as a version of the Lévy-Cramér Theorem for multi-dimensional Kingman convolution. The case $k=1$ was proved by Urbanik ([18]).

**Theorem 5.** Suppose that $G_i \in \mathcal{P}(\mathbb{R}^+), i = 1, 2$ and

$$\Sigma_{s,k} = G_1 \bigcirc_{s,k} G_2.$$  

(37)

Then, $G_i, i = 1, 2$ are both Rayleighian distributions fulfilling the condition that there exist nonnegative numbers $\lambda_{i,r}, i = 1, 2$ and $r = 1, 2, \ldots, k$ such that for each $i=1, 2$ the number of non-zero coefficients $\lambda_{i,r}$’s among $\lambda_{i,1}, \lambda_{i,2}, \ldots, \lambda_{i,k}$ are equal to the dimension of $G_i$, respectively. Moreover,

$$\lambda_{i,1}^2 + \lambda_{i,2}^2 + \cdots + \lambda_{i,k}^2 = 1, \ r = 1, 2, \ldots, k$$

(38)

and

$$G_i = T_{\lambda_{i,1}} \sigma_s \times T_{\lambda_{i,2}} \sigma_s \times \cdots \times T_{\lambda_{i,k}} \sigma_s$$

(39)

Proof. Suppose that the equation (37) holds. Using the map $F_{s,k}$ we have

$$F_{s,k}(\Sigma_{s,k}) = F_{s,k}(G_1) \ast F_{s,k}(G_2)$$

where, by virtue of (21), implies that

$$N(0, 2(s+1)I) = F_{s,k}(G_1) \ast F_{s,k}(G_2).$$

By the well-known Lévy-Cramér Theorem on $\mathbb{R}^k$ (cf. Linnik and Ostrovskii [9]), that they are both symmetric Gaussian distributions on $\mathbb{R}^k$. Consequently, they must be of the form (39) and the coefficients $\lambda_{i,r}$’s satisfy the above stated conditions. \qed
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