Unification of Newtonian Physics with Einstein Relativity Theory by using Generalized Metrics of Complex Spacetime and application to the Motions of Planets and Stars, eliminating Dark Matter

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Keywords: 5th Euclidean postulate, complex spacetime, dark matter, Einstein Relativity Theory, electromagnetic tensor, Euclidean closed linear transformations, Euclidean complex relativistic mechanics, Euclidean metric, four-momentum, four-velocity, Galaxies: kinematics and dynamics of, Galilean transformation, general relativity, geodesics, gravitation, human senses, isometry, linear spacetime transformation, Lomonosov-Lavoisier Law, Lorentz boost, Lorentz matrix, Lorentz metric, Lorentz transformation, Maxwell equations, Minkowski space, Modified Newtonian Dynamics (MOND), Newtonian Physics, precession of Mercury’s perihelion, proper time, real spacetime, relativistic Doppler shift, Schwarzschild metric, Solar System: kinematics and dynamics of, special relativity, universal rotation curve, Vossos matrix, Vossos transformation.

PACS: 02.10.Ud, 02.40.Dr, 03.30.+p, 03.50.De, 04.20.-q, 04.50.Kd, 04.80.Cc, 95.35.+d, 96.12.De, 96.15.De, 98.20.+d

Abstract. In this paper we unify Theories of Physics (TPs) with real or infinite Universal Speed \((c)\), such as Einstein Relativity Theory (ERT) and Newtonian Physics (NPs). Generalized Special Relativity (SR) relates the frames of Relativistic Inertial observers (RIOs) where the spacetime has the metric \(g=\text{diag}(g_00, g_{11}, g_{11}, g_{11})\). The parameter \(ξ=\sqrt{−g_{11}/g_{00}}\) is contained in the matrix \(Λ(I)\) of the Euclidean Closed Linear Transformation of complex spacetime (ECLSTT). Besides the elements of \(Λ(I)\) are complex numbers. So, the corresponding spacetime is necessarily complex and there exists real \(c_1=c/ξ\). In addition, the complex Cartesian Coordinates (CCs) of the theory, may be turned to the corresponding real CCs, in order to be perceived by human senses. The new real transformation is not closed (the corresponding real matrices \(A_{\text{IR}}\) do not form a group) and the successive real transformations produce Generalized Thomas Rotation. The specific value \(ξ=1\) gives Vossos transformation (VT) endowed with Lorentz metric (for \(g_{11}=1\)) of complex spacetime and invariant speed of light in vacuum \((c_1=c)\), which produce the Lorentzian version of Euclidean Complex Relativistic Mechanics (ECRMs). The corresponding real matrix \((A_{\text{IR}})\) is the matrix of Lorentz...

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Boost ($\Lambda_{IR}=\Lambda_{L}$). The specific value $\zeta_{I} \rightarrow 0$ gives Galilean Transformation (GT) with invariant time, in which any other ECLSTT is reduced, if one RIO has small velocity wrt another RIO. Thus we unify TPs such as NPs and ERT, keeping the formalism of ERT. The generalized definition of Proper Time ($\tau$) gives us the possibility to compute four-velocity, four-momentum, Relativistic Doppler Shift etc, building the whole structure of the Generalized SR and General Relativity (GR). For instance the Generalized Relativistic Energy of Rest Mass ($m$) is $E_{\text{rest}}=mc^2/\zeta_{I}^2$. In case of NPs, the annihilation energy becomes infinite. Thus the Lomonosov-Lavoisier Law becomes clear theorem of NPs. In addition, the case of observers with variable metric of spacetime leads to GR. Thus we produce the 1<sup>st</sup> Generalized Schwarzschild metric (1GSM) and 2<sup>nd</sup> Generalized Schwarzschild metric (2GSM), which are in accordance with Einstein field equations for any TPs. In case of 1GSM, we compute the corresponding Lagrangian, geodesics, equations of motion, precession of planets’ orbits etc, resulting formulas which are referred to any TPs. We then combine the theoretical results to the experimental data of our Solar system, producing a set of valid values of $\zeta_{I}$. In case of 2GSM, the combination of its Galilean version with Modified Newtonian Dynamics (MOND), leads to MOND relativization. After, we pass to RIOs with ordinary flat spacetime (Minkowski space), extending MOND methods to ERT. We use Simple and Standard Interpolating Function ($\mu$) to the Lorentzian-Einsteinian version of 2GSM for the explanation of Rotation Curves in Galaxies as well as the Solar system, eliminating Dark Matter. Generally, this approach, in non rotating black hole, planetary and star system scale, coincides to the original Schwarzschild metric, while in galactic scale, it gives MONDian results. In universal scale, the gravitational field strength becomes negative, producing slight antigravity.

1. Introduction

The main communication of Einstein Relativity Theory (ERT) with Newtonian Physics (NPs), is the low velocity limit and the weak-field approximation. In this paper, we present a generalized Relativity Theory (RT), which contains ERT and NPs, keeping the formalism of ERT. Thus the differences between these two Theories of Physics (TPs) are limited to their different value of metric coefficients of spacetime of the corresponding Relativistic Inertial observers (RIOs).

This generalized RT combined with Quantum Mechanics (QMs), leads to the generalized Relativistic Quantum Mechanics (RQMs). For instance, Galilean Transformation (GT) endowed with the corresponding metric produces Newtonian Physics (NPs), which is associated with the Newtonian QMs, producing Schrödinger Equation. Thus, many low velocities phenomena, like the atomic spectra without fine structure, are explained. On the other hand, Lorentz Transformation (LT) endowed with the corresponding metric produces ERT, which is associated with the Einsteinian RQMs, producing Klein-Gordon Equation. Thus, many high velocities phenomena and the fine structure of atomic spectra are explained [1,2].

Special Relativity (SR) relates the frames of RIOs, through Linear Spacetime Transformations (LSTTs) of linear spacetime. The main approach of Einsteinian SR uses real spacetime endowed with Lorentz Metric and the frames of two RIOs with parallel spatial axes are always related using Lorentz Boost (LB). But it is well-known that LB is not closed transformation. On the contrary, Lorentz Transformation (LT) (combination of Euclidean Rotation with LB) is closed [3] (pp. 35, 41, eq 1.104). Thus, if three observers $O$, $O'$ and $O''$ are related, where the frame of observer $O'$ has parallel axes not only to the axes of observer $O$, but also to the observer $O''$ axes, then the axes of observers $O$ and $O''$ are not parallel. Thus the transitive attribute in parallelism (which is equivalent to the 5<sup>th</sup> Euclidean postulate) is cancelled. This option leads to successful results, such as Thomas Precession which explains the fine structure of atomic spectra. But this happens only if we take successive observers $O'$ and $O''$ with the Thomas order [2]. If we reverse the order of this sequence, it yields a result with 200% relative error.

In the referred paper [4], we have proved that there exist Euclidean Closed Linear Transformations of Complex Spacetime endowed with the Corresponding Metrics (ECLSTTs) which correlate frames
with the corresponding spatial axes being parallel. Thus, the transitive attribute in parallelism (or equivalently the 5th Euclidean postulate) is valid. Moreover, if we demand the transformation respecting spacetime isometry a Generalized SR is produced. These isometric ECLSTTs of complex spacetime (with common solution the GT), can be applied not only to the SR problems, but also to the General Relativity (GR). This is achieved, because the production of the corresponding matrices has become without adapting one specific metric. In addition, any complex Cartesian Coordinates (CCs) of the theory may be turned to the corresponding real CCs, in order to be perceived by human senses. In this way, the axes rotation that happens in case that real CCs are used (when more than two observers are related) is only an equivalent phenomenon.

In this paper, we demand real spacetime for the unmoved observer (O) where time and space coefficients of metric have different sign. Then, using isometric ECLSTTs for moving observers, we prove that the four-vectors’ zeroth component (such as energy) is real, in contrast to the spatial components that are complex, but their norm is real. It is also proved that moving (wrt O) human O’ meters length, according to a Generalized Real Boost (GRB) which describes at the same time, not only the LB of Einsteinian SR, but also the GT of NPs. In addition, we find Generalized Rotation Matrix Complex-Real \( R_{CR} \) that turns natural sizes’ complex components to real. The application of this theory to the Mechanics, Electromagnetism and Gravitation, gives results that describe at the same time, not only these extracted by ERT, but also, those extracted by NPs.

The case of observers with variable metric of spacetime, leads to the corresponding GR. For being this clear, we produce the 1st Generalized Schwarzschild metric (1GSM) and 2nd Generalized Schwarzschild metric (2GSM), which are in accordance with any SR based on isotropic Generalized metrics \((g_i)\) and Einstein field equations. In case of 1GSM, we compute the corresponding Lagrangian, geodesics, equations of motion, precession of planets’ orbits etc, resulting formulas which are referred to any TPs. We then combine the theoretical results to the experimental data of our Solar system, producing a set of valid values of \( \xi \). In case of 2GSM, the combination of its Newtonian version with Modified Newtonian Dynamics (MOND), leads to MOND relativization. After, we pass to RIOs with ordinary flat spacetime with Lorentz metric (Minkowski space), extending MOND methods to ERT. We use Simple and Standard Interpolating Function (\( \mu \)) to the Lorentzian version of 2GSM, for the explanation of the Rotation Curves in Galaxies as well as the Solar system, eliminating Dark Matter. Generally, this approach, in non rotating black hole, planetary and star system scale, coincides to the original Schwarzschild metric, while in galactic scale, it gives MONDian results. In universal scale, the gravitational field strength becomes negative producing slight antigravity.

The used theory is open to any other modification as well as the Generalization can be extended to other metrics of spacetime such as Kerr metric, producing Generalized field strength of rotating mass systems. Besides, the theory gives us the possibility of using mathematical objects (such as \( \mu \)) which have been produced by one specific TPs (such as NPs) to be suitably transformed and then effectively be used in another TPs (such as ERT).

2. Isometric Euclidean Closed Linear Transformations of Complex Spacetime endowed with the Corresponding Metrics

In this paper, the metric coefficients of time and space have different signs. We remind the well-known theorem that “any space of constant curvature is conformally flat; that is, has a metric that can be expressed as a multiple of the Euclidean metric” [5] (p.364). Moreover 3D-space is isotropic, in case of Isometric ECLSTTs [4]. So for RIOs, the representation of the non-degenerate inner product in holonomic basis \( \{e_o, e_x, e_y, e_z\} \) of ‘flat’ spacetime is the real matrix

\[
g_1 = \text{diag}(g_{100}, g_{111}, g_{122}, g_{133}) = g_{111} \text{diag} \left( -\frac{1}{\xi_i^2}, 1, 1, 1 \right) = g_{100} \text{diag} \left( 1, -\xi_i^2, -\xi_i^2, -\xi_i^2 \right)
\]

where

\[
\xi_i = \sqrt{-\frac{g_{111}}{g_{100}}}
\]
The index \( I \) remind us that we are referred to the spacetime of the RIOs of each specific Theory of Physics (TPs). The corresponding Special Relativity (SR) has real Universal Speed \((c)\):

\[
c_I = \frac{1}{\xi_I} c
\]

and the transformation of a contravariant four-vector is

\[
d\textbf{X}' = \Lambda_{(\xi_I, \beta)} \textbf{dX}
\]

where

\[
\Lambda_{(\xi_I, \beta)} = \gamma_{(\xi_I, \beta)} \begin{bmatrix}
1 & -\xi_I^2 \beta_x & -\xi_I^2 \beta_y & -\xi_I^2 \beta_z \\
-\beta_x & 1 & i \xi_I \beta_z & -i \xi_I \beta_y \\
-\beta_y & -i \xi_I \beta_z & 1 & i \xi_I \beta_x \\
-\beta_z & i \xi_I \beta_y & -i \xi_I \beta_x & 1
\end{bmatrix}
\]

\[
\gamma_{(\xi_I, \beta)} = \frac{1}{\sqrt{1 - \beta^2}}
\]

For simplicity reasons, wherever it is written \( i \), it is meant \( \pm i \). Besides the norm of the infinitesimal position four-vector for RIOs is the following invariant quantity

\[
d S^2 = d X^r \sqrt{g} \, d X = g_{111} c^2 \, d t^2 + g_{110} \, d x^2 = g_{111} \left( -\frac{1}{\xi_I^2} c^2 \, d t^2 + d \tilde{x}^2 \right) = g_{100} \left( c^2 \, d t^2 - \xi_I^2 \, d \tilde{x}^2 \right)
\]

and the typical matrix along x-axis is

\[
A_{typ} = \gamma_{(\xi_I, \beta)} \begin{bmatrix}
1 & -\xi_I^2 \beta \\
\beta & 1 \end{bmatrix}
\]

The specific value \( \xi_I = 0 \) \((g_{111} \rightarrow 0, \, g_{100} \neq 0)\) gives GT with Infinite Universal Speed \((c_I \rightarrow +\infty)\) and the corresponding metric of the spacetime (let us call Galilean metric)

\[
g_{11} = \lim_{\xi_I \rightarrow -0} \text{diag} \{g_{100}, \, g_{111}, \, g_{111}, \, g_{111}\} = \lim_{\xi_I \rightarrow -0} \text{diag} \{g_{100}, \, 0, \, 0, \, 0\}
\]

The corresponding spacetime (let us call Galilean spacetime) has infinite curvature \((K \rightarrow +\infty)\) in any orientation \(k e_x + \lambda e_y + \mu e_z\) of the 3D space. This is the reason that time is absolute for any type of observers as well as the Universal speed is infinite.

The specific value \( \xi_I = 1 \) \((g_{111} = -g_{100})\) gives Vossos Transformation (VT) with \(c_I = c\) (the universal speed is the speed of light in vacuum) and the corresponding metric of spacetime (let us call Vossos metric)

\[
g_8 = g_{111} \text{diag} \{-1, \, 1, \, 1, \, 1\} = g_{111} \eta
\]

which for \( g_{111} = 1 \) becomes the Lorentz metric \((\eta)\). Thus, we have the Lorentzian case of Euclidean Complex Relativistic Mechanics (ECRMs) \([6]\), which is associated with Einstein Relativity Theory (ERT). Besides \( \Lambda_{(1, \beta)} = \Lambda_{(\eta, \beta)} \) is called Vossos matrix.

### 3. Measure of Time, Length, Velocity, Momentum, Energy and Doppler Effect

As matrix \( A_I \) contains some elements which are imaginary numbers, we conclude that the spacetime of one moving observer is complex. Thus, we put an index \( C \) to the complex natural sizes and the real natural sizes have no index. We now make the option that observer \( O \) measures real spacetime where
time and space coefficients of metric have different sign. This simplifies the problem, because the following are proven:
i) Time is real for every observer.
ii) The unmoved observer measures real velocity.
iii) Every observer measures square of velocity which is real and positive.
iv) Every observer measures Generalized Lorentz γ-factor which is real and positive, for particles with velocity \( \upsilon < c_1 \).
v) The unmoved observer measures natural sizes that can be suitably determined to be real.

Transformation (4) may be written as

\[
X'_C = \Lambda_{0\beta_1} X
\]  

Using vectors, the isometric ECLSTT becomes

\[
c t' = \gamma_{(\xi_1, \beta)} (c t - \xi_1^2 \beta \cdot \bar{x}) ; \quad \bar{x}'_C = \gamma_{(\xi_1, \beta)} (\bar{x} - \beta c t) - i \xi_1 \gamma_{(\xi_1, \beta)} \beta \times \bar{x}
\]  

\[
c d t' = \gamma_{(\xi_1, \beta)} (c d t - \xi_1^2 \beta \cdot d \bar{x}) ; \quad d \bar{x}'_C = \gamma_{(\xi_1, \beta)} (d \bar{x} - \beta c d t) - i \xi_1 \gamma_{(\xi_1, \beta)} \beta \times d \bar{x}
\]

3.1. Properties of matrix \( \Lambda_I \) – Inverse isometric ECLSTT

The matrix \( \Lambda_I \) has the following properties:

\[
\Lambda_{(0)} = I
\]  

\[
\Lambda_{I_{-\beta}} = \Lambda_{(I \beta)}
\]  

\[
det \Lambda_{(I \beta)} = 1
\]

3.2. Time – Proper Time

Let have a particle P, moving with velocity \( \tilde{\upsilon}_P \) wrt observer O (\( \tilde{\upsilon}_P ' \) wrt observer O’) in spacetime. The contravariant and covariant infinitesimal four-vector correspondingly are:

\[
dx'_C = \begin{bmatrix} c d t' \\ d \bar{x}'_C \end{bmatrix} ; \quad dx'_{C_\mu} = dx'_{C^*} \cdot g_{\mu \nu} = g_{111} \left[ -\frac{1}{\xi_1^2} c^2 d t'^2 + d \bar{x}'_C ^T \right] = g_{100} \left[ c^2 d t'^2 - \xi_1^2 d \bar{x}'_C ^T \right]
\]  

So

\[
dS'^2 = dx'_{C_\mu} \cdot dx'_{C^*} = g_{111} \left[ -\frac{1}{\xi_1^2} c^2 d t'^2 + d \bar{x}'_C ^T \right] = g_{100} \left[ c^2 d t'^2 - \xi_1^2 d \bar{x}'_C ^T \right]
\]

which respects the isometry of spacetime: \( dS'^2 = dS^2 \).

The Proper Time (PT) maintains the same definition as ERT

\[
d \tau = \frac{1}{c} \sqrt{dS'^2}
\]  

and this is the relation to the time

\[
\tau' = \frac{d \tau'}{d \tau} = \gamma_{(\xi_1, \beta)}
\]

For GT where \( \xi_1 = 0 \), it emerges \( d \xi = d \tau = d t \) (NPs).

In case of VT with \( \xi_1 = 1 \), we have the same result as ERT

\[
\tau' = \frac{d \tau'}{d \tau} = \gamma_{(1, \beta)}
\]

3.3. Generalized Real Boost and Generalized Rotation Matrix \( R_{CR} \)

Human perceives real spacetime, through senses. In contrast, if \( g_{100} g_{111} < 0 \), then (according to the ECLSTT) for a moving observer, time is real, but 3D-space is complex. This conflict is solved as following:

Let have a rod O’\( \Sigma \), perpendicular to x-axis moving along it, with velocity (\( \beta c, 0, 0 \)) wrt Oxyz (Figure 1). Observer O projects point \( \Sigma \) (edge or the rod), perpendicularly to \( y' \) and \( z' \)-axis. Observer O’
considers as Real CCs the lengths of the resultant rods Ο′Α και Ο′Β. In addition, observer Ο perceives the following spacetime events: Ο′(ct , ξt , 0, 0) ; Σ(ct , ξt , 0, 0). A(ct , ξt , 0, 0) ; B(ct , ξt , 0, 0). For observer Ο′, the same events through typical isometric ECLSTT, are described using complex CCs:

\[\Sigma(ct', 0, \gamma_{(\xi,t)} y + i\xi_0 \beta \gamma_{(\xi,t)} z, \gamma_{(\xi,t)} z - i\xi_0 \beta \gamma_{(\xi,t)} y) ;\]

\[A(ct', 0, \gamma_{(\xi,t)} y, -i\xi_0 \beta \gamma_{(\xi,t)} y) ;\]

\[B(ct', 0, i\xi_0 \beta \gamma_{(\xi,t)} z, \gamma_{(\xi,t)} y) .\]

For observer Ο, there are the following four-vectors:

\[\text{O}'\tilde{A}(0, 0, y, 0) ; \text{O}'\tilde{B}(0, 0, 0, z) .\]

The same four-vectors for observer Ο′ are complex

\[\left(\text{O}'\tilde{A}\right)^{0, 0, \gamma_{(\xi,t)} y, -i\xi_0 \beta \gamma_{(\xi,t)} y) ;\]

\[\left(\text{O}'\tilde{B}\right)^{0, 0, i\xi_0 \beta \gamma_{(\xi,t)} z, \gamma_{(\xi,t)} y} .\]

So for observer Ο′, the Real CCs of point Σ are:

\[y' = \left|\left(\text{O}'\tilde{A}\right)^{0, 0, \gamma_{(\xi,t)} y, -i\xi_0 \beta \gamma_{(\xi,t)} y) ;\right| = \left|\left(\text{O}'\tilde{B}\right)^{0, 0, i\xi_0 \beta \gamma_{(\xi,t)} z, \gamma_{(\xi,t)} y} \right| = \sqrt{-\gamma_{(\xi,t)}^2 \beta^2 y^2 + \gamma_{(\xi,t)}^2 z^2} = |z| = z .\]

We observe that using Real CCs, the transformation from observer Ο to observer Ο′, happens according to the typical Generalized Real Boost (GRB)

\[X' = \Lambda_{IRyp(\beta)} X\]

using the Typical matrix of GRB

\[\Lambda_{IRyp(\beta)} = \begin{bmatrix} \gamma_{(\xi,t)} & -\xi_0 \beta \gamma_{(\xi,t)} & 0 & 0 \\ -\xi_0 \beta \gamma_{(\xi,t)} & \gamma_{(\xi,t)} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} .\]

We then obtain the General matrix of GRB

\[\Lambda_{IRyp(\beta)} = \begin{bmatrix} \gamma_{(\xi,t)} & -\xi_0 \beta \gamma_{(\xi,t)}^2 & 0 & 0 \\ -\xi_0 \beta \gamma_{(\xi,t)}^2 & \gamma_{(\xi,t)} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} .\]

In case that \(\xi_0 = 0\), it emerges the GT.

If \(\xi_0 = 1\), we have the original typical proper Lorentz Boost (LB) (see e.g. [3] p. 21, eq. 1.38)

\[X' = \Lambda_{LYp(\beta)} X\]

using the typical proper Lorentz matrix

\[\Lambda_{LYp(\beta)} = \begin{bmatrix} \gamma_{(\beta)} & -\gamma_{(\beta)} & 0 & 0 \\ -\gamma_{(\beta)} & \gamma_{(\beta)} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} .\]

The corresponding general proper Lorentz Boost (LB) (see e.g [3] p. 24, eq. 1.47) is

\[X' = \Lambda_{Ly(\beta)} X\]

using the general proper Lorentz matrix

\[\Lambda_{Ly(\beta)} = \begin{bmatrix} \gamma_{(\beta)} & -\gamma_{(\beta)} & 0 & 0 \\ -\gamma_{(\beta)} & \gamma_{(\beta)} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} .\]
Here, we can make one of the following options:

i) We demand the GRB always correlating any RIOs with parallel spatial axes. This destroys the 5\textsuperscript{th} Euclidean postulate. So we adapt a Generalized Hyperbolic Geometry. This is the option made by the most of scientists working on ERT with Lorentz Group and Lorentz Transformation.

ii) We demand the GRB correlating only the unmoved observer O with any RIO with parallel spatial axes. This respects the 5\textsuperscript{th} Euclidean postulate in complex 3D-space. So we adapt a Generalized Euclidean Geometry. The equations (20-22) proves that the isometric ECLSTT produce a group of elements $g=\left(A_1, b\right)$ with operation

$$g_1 \ast g_2 = (A_1 A_2, A_1 b_1 + b_2)$$

(30)

where $b^\mu_c$ is $\mu$-CC measured by O′, if observer O measures $x^\nu = 0$ and $b^\mu_c$ is $\mu$-CC measured by O′′, if observer O′ measures $x^\nu_c = 0$ [7] (pp. 35-37).

In this paper we correlate only two RIOs. So the results are in accordance with the both of options.

The transformation which turns one complex four-vector $X_c′$ to the corresponding real four-vector $X^r$ is

$$X^r = R_{CR(\beta)} X_c′$$

(31)

where

$$R_{CR(\beta)} = A_{IR(\beta)} A_{IR(\beta)}$$

(32)

is the Spacetime Imaginary Rotation Matrix Complex-Real. It is proven that

$$\tilde{R}_{CR(\beta)} = \begin{bmatrix} 1 & 0 \\ 0 & R_{CR(\beta)} \end{bmatrix}$$

(33)

where

$$R_{CR(\beta)} = \gamma(\xi, \beta) I_3 - i \xi \gamma(\xi, \beta) A_{\gamma(\beta)} + \frac{1 - \gamma(\beta)}{\beta^T \beta} \beta \beta^T$$

(34)

is the Spatial Imaginary Rotation Matrix Complex-Real with the following properties

$$\det R_{CR(\beta)} = 1 \ ; \ R_{CR(\beta)}^T = R_{CR(\beta)}$$

(35)

So we also have

$$\det \tilde{R}_{CR(\beta)} = 1 \ ; \ R_{CR(\beta)}^T = \tilde{R}_{CR(\beta)}$$

(36)

For typical transformation along x-axis, it is

$$R_{CR_{\gamma x}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \gamma(\xi, \beta) & -i \xi \gamma(\xi, \beta) \beta \\ 0 & i \xi \gamma(\xi, \beta) \beta & \gamma(\xi, \beta) \end{bmatrix}$$

(37)

The following relation expresses the transformation of a complex vector to real

$$x^r = R_{CR(\beta)} x_c′$$

(38)

In case that $\xi=0$, there is no need for the above transformation, because $R_{CR(\beta)} = I_3$.

In case that $\xi=1$, the transformation (38) becomes

$$x^r = R_{BL(\beta)} x_c′$$

(39)

where

$$R_{BL(\beta)} = A_{1\gamma(\beta)} A_{BL(\beta)}$$

(40)

is the Spacetime Imaginary Rotation Matrix Vossos-Lorentz

$$\tilde{R}_{BL(\beta)} = \begin{bmatrix} 1 & 0 \\ 0 & \gamma \gamma(\beta) I + \frac{1 - \gamma}{\beta^T \beta} \beta \beta^T \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & R_{BL(\beta)} \end{bmatrix}$$

(41)
and

\[ R_{BL(\beta)} = \gamma I - i \gamma A_{(\beta)} + \frac{1 - \gamma}{\beta^2} \beta \beta^T \]  

(42)

is the spatial imaginary rotation matrix Vossos-Lorentz with the following properties

\[ \det R_{BL(\beta)} = 1 ; \quad R_{BL(\beta)}^T R_{BL(-\beta)} \]

(43)

So we also have

\[ \det \tilde{R}_{BL(\beta)} = 1 ; \quad \tilde{R}_{BL(-\beta)}^T \tilde{R}_{BL(\beta)} \]

(44)

For typical transformation, it is

\[ R_{BLtyp} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \gamma & -i \gamma \beta & 0 \\ 0 & i \gamma \beta & \gamma & 0 \end{bmatrix} \]

(45)

The following relation expresses the transformation of a complex vector to real

\[ \tilde{x} = R_{BL(\beta)} x_C \]

(46)

3.4. Complex Four-velocity

Supposing observer O’ moving with velocity \( \beta c \) wrt O and one moving particle \( P \), we define as complex contravariant four-velocity measured by O’:

\[ u^\mu_c = \frac{d x^\mu_c}{d\tau} \]

(47)

with \( \mu = 0, 1, 2, 3 \). Thus the contravariant and covariant four-velocity are correspondingly:

\[ U_C^\mu = \gamma' \xi_{(\xi,\beta)} \begin{bmatrix} c \\ \xi_c C \end{bmatrix} ; \quad U_C^\mu = U_C^\mu g_{00} = g_{11} \gamma' \xi_{(\xi,\beta)} \left( -\frac{1}{\xi_1} c \xi_1 \xi_c C \right) = g_{100} \gamma' \xi_{(\xi,\beta)} \left[ c \xi_1^2 \xi_c C^T \right] \]

(48)

and we have the property

\[ u_c^\mu \cdot u_c^\mu = g_{00} c^2 \]

(49)

The relation between complex velocity and complex four-velocity:

\[ \frac{\xi_c C}{c} = \frac{u_c^0}{u_c^0} \]

(50)

has the same formula as ERT. The transformation of complex four-velocity at the frame O’ to the corresponding four-velocity at the frame O’’ (which has complex velocity \( \beta’ c \) wrt O’’) is

\[ U' = \Lambda_{\beta(\beta')} U' \]

(51)

The cases of \( \xi_1 = 0 \) or \( \xi_1 = 1 \) \((g_{100} = g_{111} = -1)\) give correspondingly:

Relation between complex velocity and complex four-velocity:

\[ U' = \gamma' \xi_{(\xi,\beta)} \begin{bmatrix} c \\ \xi_c C \end{bmatrix} ; \quad U_C^\mu = \gamma' \xi_{(\xi,\beta)} \begin{bmatrix} c \\ \xi_c C \end{bmatrix} \]

(52)

The transformation of complex four-velocity at the frame O’ to the corresponding four-velocity at the frame O’’ (which has complex velocity \( \beta’ c \) wrt O’’) is

\[ U' = \Lambda_{\beta(\beta')} U' ; \quad U' = \Lambda_{\beta(\beta')} U' \]

(53)

The Lorentz metric \((g_{100} = -g_{111} = -1)\) gives

\[ u_c^\mu \cdot u_c^\mu = -c^2 \]

(54)

3.5. Complex Velocity Transformation

The complex velocity at the frame O (not especially the unmoved observer) is transformed to the corresponding complex velocity at the frame O’ moving with velocity \( \beta c \) relative to O, as following:
The cases of $\xi_I=0$ or $\xi_I=1$ ($g_{00}=-g_{11}$) give Galilean transformation of velocities or Vossos transformation of complex velocities correspondingly:

\[
\tilde{v} = \tilde{v} - \tilde{\beta}c ; \quad \tilde{v}' = \tilde{v} - \tilde{\beta}c - i\beta \times \tilde{v} + \frac{1}{c} \beta \cdot \tilde{v}
\]

### 3.6. Transformation of Complex Velocity to Real Velocity - Transformation of Real Velocities

If we derivative equation (31) wrt Proper Time, we obtain

\[
U' = R_{CR(\beta)} U'C
\]

This emerges that

\[
\gamma_{(\xi,\tilde{v})} = \gamma_{(\xi,\tilde{v}')} ; \quad |\tilde{v}'| = |\tilde{v}'|
\]

and

\[
\tilde{v}' = R_{CR(\beta)} \tilde{v}'
\]

The above equation and (34) give the general transformation of real velocities

\[
\tilde{v} = \frac{c}{\gamma_{(\xi,\tilde{v})} (c-\xi^2 \beta \cdot \tilde{v})} \left[ \tilde{v} - \gamma_{(\xi,\tilde{v})} - \frac{1}{\beta} \gamma_{(\xi,\tilde{v})} (\beta \cdot \tilde{v}) \right]
\]

So, the typical transformation of real velocities is

\[
v'_x = \frac{v_x - \beta c}{c-\xi^2 \beta v_x} c ; \quad v'_y = \frac{v_y}{\gamma_{(\xi,\tilde{v})} (c-\xi^2 \beta v_y)} c ; \quad v'_z = \frac{v_z}{\gamma_{(\xi,\tilde{v})} (c-\xi^2 \beta v_z)} c
\]

In addition, the Generalized Lorentz $\gamma$-factor of a particle $P$, moving with velocity $\tilde{\beta}_p$ wrt O (not especially the unmoved observer) is transformed to the frame $O'$ (moving with velocity $\tilde{\beta}$ wrt O), according to the formula

\[
\gamma'_{(\tilde{\beta},\tilde{\beta}_p)} = \gamma_{(\xi,\tilde{v})} \cdot \gamma'_{(\xi,\tilde{v})} \left( 1 - \xi^2 \tilde{\beta} \cdot \tilde{\beta}_p \right)
\]

The case of $\xi_I=1$ ($g_{00}=-g_{11}$) gives

\[
U' = R_{BL(\beta)} U'C
\]

\[
\gamma_{(v)} = \gamma_{(\tilde{v})} \gamma'_{(\xi,\tilde{v})} \left( |\tilde{v}'| = |\tilde{v}'| \right)
\]

\[
\tilde{v}' = R_{BL(\beta)} \tilde{v}'
\]

\[
\tilde{v}' = \frac{c}{\gamma_{(\tilde{v})} (c-\beta \tilde{v})} \left[ \tilde{v} - \gamma \beta c + \frac{1}{\beta^2} (\beta \cdot \tilde{v}) \right]
\]

The typical transformation of real velocities is

\[
v'_x = \frac{v_x - \beta c}{c-\beta v_x} c ; \quad v'_y = \frac{v_y}{\gamma (c-\beta v_y)} c ; \quad v'_z = \frac{v_z}{\gamma (c-\beta v_z)} c
\]

We observe that we have obtained the formula of Lorentz velocities transformation (see e.g. [3] p. 161, eq. 6.14). Besides (62) becomes

\[
\gamma'_{(\tilde{\beta},\tilde{\beta}_p)} = \gamma'_{(\tilde{\beta})} \gamma_{(\tilde{\beta},\tilde{\beta}_p)} (1 - \tilde{\beta} \cdot \tilde{\beta}_p)
\]

which the same formula as ERT (see e.g. [3] p. 162, eq. 6.15).

### 3.7. Four-momentum – Relativistic Force - Relativistic Energy – Universal speed particle

Supposing observer $O'$ moving with velocity $\tilde{\beta}c$ wrt O and one Particle ($P$) with real mass $m$ moving...
with complex velocity \( \vec{v}_C = \vec{\beta}_C c \) wrt O', we define as complex four-momentum measured by O':

\[
P_C^\mu = m u_C^\mu ; \quad P_C^\mu = m u_C^\mu
\]

(69)

with \( \mu = 0, 1, 2, 3 \). Thus the contravariant and covariant four-momentum are correspondingly:

\[
P_C^\mu = m \gamma'_{(\xi, \beta_C)} \left[ \frac{c}{\tilde{u}^\mu_C} \right] ; \quad P_C^\mu = g_{111} m \gamma'_{(\xi, \beta_C)} \left[ -\frac{1}{\xi_1^2 c} \tilde{u}^\mu_C \right] = g_{100} m \gamma'_{(\xi, \beta_C)} \left[ c - \frac{\xi_1^2 \tilde{u}^\mu_C}{c} \right]
\]

(70)

which have the same properties as ERT (see e.g. [3] pp. 267-71). The spatial part of the contravariant four-momentum is defined as the complex Generalized relativistic momentum of one particle

\[
\vec{p}_C = m \gamma'_{(\xi, \beta_C)} \tilde{u}_C = m c \gamma'_{(\xi, \beta_C)} \vec{\beta}_C
\]

(71)

The definition of the complex Generalized relativistic force, is the same as ERT (see e.g. [3] p. 325, eq. 11.1)

\[
F_c^{\mu} = \frac{d P_C^\mu}{dt'}
\]

(72)

So, the work of the Generalized relativistic force which accelerates one particle from initial velocity \( v' = 0 \) to final velocity \( \vec{u}'_C \) (in other words the Generalized relativistic kinetic energy) is calculated:

\[
K' = W' = \int_{t_0}^{t'} \! d x' . F_c^{\mu} = \int_{t_0}^{t'} \! \frac{d P_C^\mu}{dt'} \left| \gamma'_{(\xi, \beta_C)} - \frac{1}{\xi_1^2} \right| m c^2
\]

We also define the Generalized relativistic energy

\[
E' = \frac{\gamma'_{(\xi, \beta_C)}}{\xi_1^2} m c^2
\]

(74)

and the Generalized energy of Rest mass

\[
E_{\text{rest}} = \frac{1}{\xi_1^2} m c^2
\]

(75)

The zeroth coordinate of the contravariant four-momentum is relevant to the Generalized relativistic energy of one particle:

\[
p^{0} = \xi_1^2 \frac{E'}{c}
\]

(76)

Hence the contravariant and covariant four-momentum can be written as

\[
P_C^\mu = \left[ \xi_1^2 \frac{E'}{c} ; \vec{P}_C \right] ; \quad P_C^\mu = g_{111} \left[ -\frac{E'}{c} \vec{P}_C \right] = g_{100} \left[ \xi_1^2 \frac{E'}{c} - \xi_1^2 \vec{P}_C \right]
\]

(77)

Combining the above equations with (49), we obtain the property

\[
E'^2 = \frac{1}{\xi_1^4} m^2 c^4 + \frac{1}{\xi_1^2} \left| \vec{P} \right|^2 c^2
\]

(78)

Below we shall produce Generalized Maxwell equations which are covariant under isometric ECLSTT and GRB. The corresponding electromagnetic waves are propagating in vacuum with velocity equal to the universal speed \( (c) \). For any monochromatic electromagnetic wave with frequency \( f' \) wrt O', we define a corresponding particle without mass \( (m = 0) \) moving also with the universal speed \( (c) \) and total Relativistic energy \( E' = h f' \), where h is Planck constant. Thus the above equation gives

\[
\left| \vec{P} \right| = \xi_1 \frac{E'}{c}
\]

(79)

The ECLSTT of one four-momentum at the frame O' to that of O', which has complex velocity \( \vec{\beta}_C c \), wrt O':
\( P^\mu_c = \Lambda_{\mathcal{I} (\rho')} P^\mu_c \)  

and the complex relativistic momentum of one particle is transformed to real, by using 
\( \vec{p}^i_r = R_{CR (\rho')} \vec{p}^i_c \)  

Besides the transformation of real contravariant four-momentum is 
\( P' = \Lambda_{1R (\rho')} P \)  

In case that \( \xi_i \rightarrow 0 \) and particle with finite velocity, the above equations become

\[
P^\mu = m \left[ \begin{array}{c} \frac{c}{\gamma} \\ \nu' \end{array} \right]; \quad P'_\mu = g_{100} m \left[ c \quad \vec{\nu}^T \right]
\]

\textit{(Newtonian momentum)}

\[
\vec{p}_N = m \vec{\nu} = m c \vec{\beta}_p
\]

\textit{(Newton's 2nd Law)}

\[
F'_\mu = m \frac{d \vec{p}'}{d t'} = m \frac{d \vec{\nu}'}{d t'} = ma'^i
\]

\textit{(Newtonian kinetic energy)}

\[
K'_N = \frac{1}{2} [\vec{\beta}_p]^2 m c^2 = \frac{1}{2} m [\vec{\nu}]^2
\]

\textit{(Newtonian Relativistic energy)}

\[
E'_N = \lim_{\zeta_i \rightarrow 0} \gamma' \left( \xi_i (\beta'_p) \right) m c^2 = +\infty
\]

\textit{(Newtonian energy of Rest mass)}

\[
E_{\text{restN}} = \lim_{\zeta_i \rightarrow 0} \frac{1}{\zeta_i} m c^2 = +\infty
\]

Thus the annihilation energy of particle at rest \((m \neq 0)\) becomes infinite and \textit{Lomonosov-Lavoisier Law} of 'total mass conservation' becomes clear theorem of NPs. For a \textit{particle without mass} \((m = 0)\), the rest energy is finite. The \textit{GT} of one four-momentum at the frame \(\Omega'\) to that at \(\Omega''\) (which has velocity \(\beta' c\) wrt \(\Omega'\)) is

\[
P' = \Lambda_{1R (\rho')} P'
\]

In case that \(\xi_i = 1\) and particle with sub-luminous velocity, the Generalized equations become

\[
P'^\mu_c = m \gamma' (\beta'_p) \left[ \begin{array}{c} \frac{c}{\gamma'} \\ \nu' \end{array} \right]; \quad P'_\mu_c = g_{111} m \gamma' (\beta'_p) - c \quad \vec{\nu}^T
\]

\textit{(Einsteinian relativistic momentum)}

\[
\vec{p}'_c = m \gamma' (\beta'_p) \vec{p}'_c = m c \gamma' (\beta'_p) \vec{\beta}_p
\]

\textit{(Vassos Complex relativistic force)}

\[
F'_\mu = m \frac{d \vec{p}'_c}{d t'} = m \frac{d \gamma' (\beta'_p)}{d t'} \vec{\nu}'_c + m \gamma' (\beta'_p) \frac{d \vec{\nu}'_c}{d t'}
\]

\textit{(Einsteinian relativistic kinetic energy)}

\[
K'_E = \left[ \gamma' (\beta'_p) - 1 \right] m c^2
\]

\textit{(Einsteinian relativistic energy)}

\[
E'_E = \gamma' (\beta'_p) m c^2
\]

\textit{(Einsteinian energy of Rest mass)}

\[
P_{\text{restE}}^0 = E_{\text{E}}^0 = \frac{E_{\text{E}}}{c}
\]

\[
P'_\mu_c = \left[ \begin{array}{c} E'_c \\ \vec{p}'_c \end{array} \right]; \quad P'_\mu_c = \left[ - \frac{E'}{c} \quad \vec{p}'_c \right]
\]

\textit{(Einsteinian energy of Rest mass)}

\[
E_{\text{E}}^2 = m^2 c^4 + \left| \vec{p}'_E \right|^2 c^2
\]

For a \textit{photon} \((m = 0, E' = h f')\), we also have

\[
\left| \vec{p}'_E \right| = \frac{E'}{c}
\]
The transformation of one four-momentum at the frame \( O' \) to that of \( O'\) (which has complex velocity \( \beta' C \), wrt \( O' \)) is

\[
P'_C = \Lambda_{B(\beta')} P_C
\]  

(100)

The complex relativistic momentum of one particle is transformed to real, by using

\[
\vec{p}'_E = R_{B(\beta')} \vec{p}'_C
\]

(101)

Besides the transformation of real contravariant four-momentum becomes

\[
P' = \Lambda_{\xi(\beta')} P
\]

(102)

Thus we obtain the original Einsteinian relativistic force (see e.g. [3] p. 329, eq. 11.17)

\[
F'_E = \frac{d}{d\tau'} m \gamma'(\beta') \nabla_p'^{\tau'} + m \gamma'(\beta') \frac{d \nabla_p'^{\tau'}}{d\tau'}
\]

(103)

### 3.8. Relativistic Doppler Shift (General Case) by using isometric ECLSTT

Let have an unmoved source \( O \) which emits monochromatic Generalized electromagnetic beams having frequency \( f \) or equivalently the corresponding particles without mass \((m = 0)\) and total Relativistic energy \( E' = h f' \). Supposing the frame \( O'xyz \) moving with velocity \( \vec{v} = (\beta c, 0, 0) \) wrt \( O \), we will find the frequency \( f' \) that is measured by an observer which is placed on a point \( A \) on \( y' \)-axis.

At the moment of measurement, the angle \( (OO', OA) = \phi \) (Figure 2). Then it is

\[
P'_C = \Lambda_{\xi p} P
\]

(104)

Using equations (9), (77) and (79), we obtain the formula of the Generalized Relativistic Doppler Shift

\[
f' = f' (1 - \xi_1 \beta \cos \phi)
\]

(105)

The typical case of \( \phi = 0 \) gives

\[
f' = f \frac{1 - \xi_1 \beta}{1 + \xi_1 \beta}
\]

(106)

In addition, the complex relativistic momentum is transformed to real, by using equations (37) and (81). Thus we obtain

\[
\vec{p}' = \begin{bmatrix}
\frac{h f}{c} \gamma'(\xi, \beta') (-\xi_1^2 \beta + \xi_1 \cos \phi) \\
\xi_1 \frac{h f}{c} \sin \phi \\
0
\end{bmatrix}
\]

(107)

The same result would emerge, if we used (82) directly.

The cases where \( \xi_1 \to 0 \) or \( \xi_1 = 1 \), give correspondingly

\[
f'_N = f ; \quad f'_E = f' \gamma (1 - \beta \cos \phi)
\]

(108)

\[
\vec{p}'_N = 0 ; \quad \vec{p}'_E = \begin{bmatrix}
\frac{h f}{c} \gamma (-\beta + \cos \phi) \\
\frac{h f}{c} \sin \phi \\
0
\end{bmatrix}
\]

(109)

The typical case of \( \phi = 0 \) gives
\[ f'_\beta = f \cdot \left( 1 - \beta^2 \right)^{-1/2} \]

The first equations are Newtonian and the second equations are the corresponding of ERT [7] (pp. 80-81).

### 4. Electromagnetism

We apply the isometric ECLSTT to electromagnetism. This demands the production of Generalized Maxwell Equations (GMEs) which are covariant under isometric ECLSTT:

\[ \nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \]  
\[ \nabla \times \vec{B} - \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{J} \]  
\[ \nabla \cdot \vec{B} = 0 \]  
\[ \nabla \times \vec{E} + \frac{\varepsilon_0}{c^2} \frac{\partial \vec{B}}{\partial t} = 0 \]

We observe that only Faraday’s Law needs generalization. The contravariant and covariant Generalized Electromagnetic Tensors (GETs) are respectively expressed by antisymmetric matrices

\[ F^{\mu\nu} = \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & cB_z & -cB_y \\ -E_y & -cB_z & 0 & cB_x \\ -E_z & cB_y & -cB_x & 0 \end{bmatrix} \quad \text{and} \quad \tilde{F}^{\mu\nu} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & \varepsilon_0 c B_z & -\varepsilon_0 c B_y \\ E_y & -\varepsilon_0 c B_z & 0 & \varepsilon_0 c B_x \\ E_z & \varepsilon_0 c B_y & -\varepsilon_0 c B_x & 0 \end{bmatrix} \]

The contravariant GET is a second order tensor, so it is transformed by using isometric ECLSTT as

\[ F'_C = \Lambda_{\nu(\rho)} F^{\mu\nu} \Lambda_{\mu(\nu)}^T \]

The corresponding real transformation is

\[ F'_C = \Lambda_{\nu(\rho)} F^{\mu\nu} \Lambda_{\mu(\nu)}^T \]

Moreover, the GMEs in tensor form are the same as ERT

\[ \frac{\partial F^{\mu\nu}}{\partial x^\sigma} = \sqrt{\frac{\varepsilon_0}{\varepsilon_0}} j^\mu \quad \varepsilon^{\rho\delta\beta\gamma} \frac{\partial F_{\beta\gamma}}{\partial x^\rho} = 0 \quad c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} \]

where \( \varepsilon^{\rho\delta\beta\gamma} \) is the totally antisymmetric tensor, \( j^\mu \) is the conduction four-current, \( \mu_0 \) is the magnetic permeability and \( \varepsilon_0 \) is the dielectric constant of vacuum (see e.g. [7] p. 64).

The GMEs give generalized electromagnetic waves in vacuum

\[ \nabla^2 \vec{E} + \frac{\varepsilon_0}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} = 0 \quad \nabla^2 \vec{B} - \frac{\varepsilon_0}{c^2} \frac{\partial^2 \vec{B}}{\partial t^2} = 0 \quad \vec{E} = \frac{\varepsilon_0}{c} \vec{B} \]

which are propagating with velocity equal to the universal speed \((c_0 = c)\) in \( \xi \). In this way every classic physical law keeps its form as well as a Generalized RQMs can be produced.

In case that \( \xi = 0 \) (Galilean metric) we have the Galilean form of Maxwell equations

\[ \nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \quad \nabla \times \vec{B} - \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{J} \quad \nabla \cdot \vec{B} = 0 \quad \nabla \times \vec{E} = 0 \]

which are covariant under GT. The Galilean version of contravariant GET is transformed by using GT as

\[ F'_C = \Lambda_{\nu(\rho)} F^{\mu\nu} \Lambda_{\mu(\nu)}^T \]

This emerges \( \tilde{E}' = \tilde{E} \). Besides the corresponding electromagnetic waves are reduced to electric waves, propagating in vacuum with infinite velocity \((c_1 = +\infty)\).

In case that \( \xi \rightarrow 1 \) (Vossos metric) we obtain the original Maxwell equations (see e.g. [3] p. 412)
\[ \nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} ; \quad \nabla \times \vec{B} - \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{J} ; \quad \nabla \cdot \vec{B} = 0 ; \quad \nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \] (122)

which are covariant under VT and LT. We also obtain the original contravariant and covariant electromagnetic tensors (see e.g [7] pp. 62-63), expressed by the anti-symmetric matrix

\[ F^\mu{}^\nu = \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & cB_z & -cB_y \\ -E_y & -cB_z & 0 & cB_x \\ -E_z & cB_y & -cB_x & 0 \end{bmatrix} ; \quad F_{\mu\nu} = g^{\alpha\beta} \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & cB_z & -cB_y \\ E_y & -cB_z & 0 & cB_x \\ E_z & cB_y & -cB_x & 0 \end{bmatrix} \] (123)

The original contravariant electromagnetic tensor is transformed by using VT and LT correspondingly

\[ F_{C'} = \Lambda_{B(\beta)} F^{\alpha\nu} \Lambda_{B(\beta)}^T ; \quad F' = \Lambda_{L(\beta)} F^{\mu\nu} \Lambda_{L(\beta)}^T \] (124)

Besides we obtain the usual electromagnetic waves in vacuum

\[ \nabla^2 \vec{E} - \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} = 0 ; \quad \nabla^2 \vec{B} - \frac{1}{c^2} \frac{\partial^2 \vec{B}}{\partial t^2} = 0 : \quad |\vec{B}| = \frac{1}{c}|\vec{E}| \] (125)

which are propagating in vacuum with velocity \( c = c \) (see e.g. [3] p. 421).

5. Generalized Schwarzschild metric and Combination with MOND

We define a new relativistic potential \( \Phi \) around a center of gravity with mass \( M \) (let us call Modified Generalized Schwarzschild Relativistic potential [8] p.13) as

\[ \Phi = h(r) \frac{c^2}{2 \xi} \ln \left( 1 - f(r) \frac{r_s}{r} \right) \] (126)

where

\[ r_s = \frac{2G M}{c^2} \] (127)

is Schwarzschild radius and \( h(r) ; f(r) \) are unspecified functions. The corresponding metric may be obtain, by using Einstein field equations in vacuum [5] (pp. 303, 396):

\[ R_{\mu\nu} - \frac{1}{2} Rg_{\mu\nu} + Ag_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} ; \quad A=0 ; \quad T_{\mu\nu}=0 ; \quad R = g^{\alpha\nu}R_{\mu\nu} \] (128)

which are reduced to the single tensor equation

\[ R_{\mu\nu} = 0 ; \quad R = 0 \] (129)

in accordance with any TPs.

5.1. 1st Generalized Schwarzschild metric, Relativistic potential, Field strength, Lagrangian, geodesics, equations of motion and precession of planets' orbits

In case that \( h(r)=1 ; f(r)=1, (126) \) gives the 1st Generalized Schwarzschild Relativistic potential (1GSRP) [8] (p.11):

\[ \Phi = \frac{c^2}{2 \xi} \ln \left( 1 - \frac{r_s}{r} \right) = \frac{c^2}{2} \frac{r_s}{r} + ... = -\frac{G M}{r} + ... \] (130)

Besides (129) emerges the 1st Generalized Schwarzschild metric (1GSM):

\[ ds^2 = g^{10} \left( 1 - \frac{r_s}{r} \right) c^2 dt^2 + \frac{g_{11}}{1 - \frac{r_s}{r}} dr^2 + g_{11} r^2 d\theta^2 + g_{11} r^2 \sin^2 \theta d\phi^2 \] (131)

The field strength (g) is radial:

\[ \vec{g} = \sqrt{g_{11}} \frac{d\Phi}{dr} \hat{r} = \sqrt{g_{11}} \frac{d\Phi}{dl} \hat{r} \] (132)
where $l$ is the radial ruler distance. So

$$\ddot{r} = -\frac{GM}{r^3} \left(1 - \frac{\xi^2}{r} \right)^{-\frac{3}{2}} \hat{r}$$  \quad (133)

The usual definition of Lagrangian of gravitational system $(M, m)$ \[5\] (p.205) is

$$L = mx^\mu g_{\mu\nu} \dot{x}^\nu$$  \quad (134)

For orbit on the ‘plane’ $\theta=\pi/2$, we obtain the 1$^{\text{st}}$ Generalized Schwarzschild Lagrangian (1GSL):

$$L = mg_{100} \left[1 - \frac{\xi^2}{\xi_1} \frac{r_s}{r} \right] c^2 i^2 + \frac{mg_{111}}{1 - \frac{\xi^2}{\xi_1} \frac{r_s}{r}} \dot{r}^2 + mg_{111} r^2 \dot{\phi}^2 \quad ; \quad \frac{d}{dr}$$  \quad (135)

or equivalently

$$L = mg_{100} \left[1 - \frac{\xi^2}{\xi_1} \frac{r_s}{r} \right] c^2 i^2 - \frac{\xi^2}{\xi_1} \frac{r_s}{r} \dot{r}^2 - \frac{\xi^2}{\xi_1} \frac{r_s}{r} \dot{\phi}^2 \quad ; \quad \frac{d}{dr}$$  \quad (136)

The well-known Euler-Lagrange equations

$$\frac{d}{d\tau} \left( \frac{\partial L}{\partial \dot{\chi^\mu}} \right) - \frac{\partial L}{\partial \chi^\mu} = 0 \quad ; \quad \mu=0, 1, 3$$  \quad (137)

give us

$$E = \left(1 - \frac{\xi^2}{\xi_1} \frac{r_s}{r} \right) \frac{mc^2}{\xi^2 i^2} \quad ; \quad \frac{d}{d\tau}$$  \quad (138)

$$\left[ \frac{2\dot{i}}{1 - \frac{\xi^2}{\xi_1} \frac{r_s}{r}} \right] - \frac{r_s}{r} c^2 i^2 + \frac{\partial}{\partial r} \left[ \frac{1}{1 - \frac{\xi^2}{\xi_1} \frac{r_s}{r}} \right] \dot{r}^2 + 2r \dot{\phi}^2 \right] = 0 \quad ; \quad \frac{d}{d\tau}$$  \quad (139)

$$J = mr^2 \dot{\phi} \quad ; \quad \frac{d}{d\tau}$$  \quad (140)

where $E$ is the total energy and $J$ is the total angular momentum of the system (the integrals of motion). The solutions of the above equations of motion satisfies the condition

$$L = mg_{100} c^2$$  \quad (141)

So, they can also be used for the practical determination of geodesics \[5\] (p. 205).

Now, we study the motion of particle $P$ around the center of gravity. The case of circular motion is obtained by putting $r=R=\text{constant}$ to (139). This gives Uniform Circular Motion (UCM) with the same angular velocity for any TPs

$$\omega = \frac{d\phi}{dt} = \sqrt{\frac{GM}{R^3}} \quad ; \quad g = \frac{\nu^2}{R} = \omega^2 R$$  \quad (142)

The orbit of non-circular motion comes with similar way to the original Schwarzschild space \[5\] (pp. 238-45). Thus the exact differential equation of motion is

$$\frac{d^2 u}{d\phi^2} - u = GM \frac{h}{h^2} + 3 e^2 \frac{GM}{c^2} \frac{u^2}{h^2} \quad ; \quad u = \frac{1}{r} \quad ; \quad h = r^2 \dot{\phi} \quad ; \quad \frac{d}{d\tau}$$  \quad (143)

where $h=J/m$ is the angular momentum per mass unit.

In case of small velocities relative to $c_1 (\nu<<\xi\zeta)$, we replace the solution of the simplified differential equation

$$\frac{d^2 u}{d\phi^2} + u = GM \frac{h}{h^2} \quad ; \quad u = GM \frac{h^2}{h^2} (1+\cos\phi) \quad ; \quad GM \frac{h^2}{h^2} = \frac{1}{a(1-e^2)}$$  \quad (144)
to the last term of the exact differential equation of motion (\(e\) is the eccentricity of the conic section, \(\alpha\) is the semimajor axis in case of ellipse). Thus we have the approximate differential equation of motion (which also validates UCM):

\[
\frac{d^2 u}{d \phi^2} + u = \frac{GM}{h^2} + \frac{3\xi^2 G^3 M^3}{c^4 h^2} (1 + e \cos \phi)^2 ; \quad u = \frac{1}{r} ; \quad h = r^2 \hat{\phi} ; \quad \dot{\phi} = \frac{d}{dt} \tag{145}
\]

with exact and approximate solution, correspondingly

\[
u = \frac{GM}{h^2} \left(1 + e \cos \phi + \frac{3\xi^2 G^2 M^2}{c^2 h^2} e \phi \sin \phi \right) ; \quad u = \frac{1}{r} ; \quad h = r^2 \hat{\phi} ; \quad \dot{\phi} = \frac{d}{dt} \tag{146}

\[
u \approx \frac{GM}{h^2} \left(1 + e \cos \left(1 - \frac{3\xi^2 G^2 M^2}{c^2 h^2} \phi \right) \right) ; \quad 0 < \frac{6\pi^2 G^2 M^2}{c^2 h^2} << 1 \tag{147}
\]

Hence the orbit can be regarded as an ellipse that rotates (‘precesses’) about one of its foci by an amount

\[
\Delta = \frac{2\pi}{1 - \frac{3\xi^2 G^2 M^2}{c^2 h^2}} - 2\pi \approx \frac{6\pi^2 G^2 M^2}{c^2 h^2} = \frac{6\pi^2 G M}{a(1 - e^2)c^2} ; \quad h = r^2 \hat{\phi} ; \quad \dot{\phi} = \frac{d\phi}{dt} 	ag{148}
\]

per revolution.

Accordingly to the mainstream approach in textbooks, the further study is based on the superposition principle. This emerges relation of time to the proper time. Replacing this to (138), they obtain the final formula of the total relativistic energy. Finally, the generalized potential energy is calculated, by reducing the kinetic energy (which is considered equal to this of SR) from the total relativistic energy. But Schwarzschild metric is a static and stationary metric of non-rotating mass. So, there is no gravitomagnetism and we expect that the gravitational force is independent from the velocity of the particle. Thus we adapt the following approach which gives simple central potential which describes Gravitoelectric Effect (GE).

The isometry of spacetime relieves us the relation of time to the proper time:

\[
dS^2 = g_{10} c^2 dr^2 = g_{101} \left(1 - \frac{\xi^2 r_k}{r} \right) c^2 dt^2 + \frac{g_{111}}{1 - \frac{\xi^2 r_k}{r}} dr^2 + g_{111} r^2 d\theta^2 + g_{111} r^2 \sin^2 \theta d\phi^2 ; \quad \theta = \frac{\pi}{2} \tag{149}
\]

or

\[
d\tau^2 = \left(1 - \frac{\xi^2 r_k}{r} \right) dt^2 - \frac{\xi^2}{1 - \frac{\xi^2 r_k}{r}} \beta_{\rho \rho}^2 dt^2 - \xi^2 \beta_{\rho \phi}^2 dt^2 ; \quad \theta = \frac{\pi}{2} \tag{150}
\]

This gives

\[
i = \frac{dt}{d\tau} = \left[1 - \xi^2 \left(\frac{r_k}{r} + \frac{1}{1 - \frac{\xi^2 r_k}{r}} \beta_{\rho \rho}^2 + \beta_{\rho \phi}^2 \right) \right]^{\frac{1}{2}} \geq 1 ; \quad \theta = \frac{\pi}{2} \tag{151}
\]

The same result emerges from the combination of (136) with (141). Besides, for a particle or planet at the perihelion or aphelion of an ellipse or in UCM, the above equation becomes

\[
i = \frac{dt}{d\tau} = \left[1 - \xi^2 \left(\frac{r_k}{r} + \beta_{\rho \phi}^2 \right) \right]^{\frac{1}{2}} \geq 1 ; \quad \theta = \frac{\pi}{2} \tag{152}
\]

Replacing (151) to (138), we obtain the final formula of the total relativistic energy
\[ E = \sqrt{\frac{1 - \frac{\xi^2}{s_1^2}}{1 - \frac{s_1^2}{s_2^2}}} \frac{m c^2}{\xi^2 s_1^2} \geq 0 \quad \text{; } \theta = \frac{\pi}{2} \]  

(153)

We observe the different contribution of the radial and orbital velocity to the total energy! Now we demand zero kinetic energy \((K=0)\), in case that the particle is static \(\left(\hat{\beta}_r = 0\right)\). Then \(E_{(\beta_r=0)} = E_{\text{rest}} + U\), where \(U\) is the potential energy. Replacing (75) and (153) to the above equation, we have

\[ U = \left( \sqrt{1 - \frac{s_1^2}{s_2^2}} - 1 \right) \frac{m c^2}{\xi^2 s_1^2} \leq 0 \]  

(154)

\[ V = \left( \sqrt{1 - \frac{s_1^2}{s_2^2}} \frac{r}{r} - 1 \right) \frac{c^2}{\xi^2 s_1^2} \leq 0 \]  

(155)

where \(V\) is the 1st Generalized Schwarzschild potential (1GSP). This is a central potential:

\[ \ddot{g} = -\frac{dV}{dr} = -\frac{GM}{r^2} \left( 1 - \frac{s_1^2}{s_2^2} \frac{\dot{r}}{r} \right)^{\frac{1}{2}} \]  

(156)

We observe that the result is the same as (133). The generalized Relativistic Kinetic energy is defined as \(K_g = E - E_{\text{rest}}\). So

\[ K_g = \sqrt{\frac{1 - \frac{s_1^2}{s_2^2}}{1 - \frac{s_1^2}{s_2^2}}} \frac{m c^2}{\xi^2 s_1^2} \geq 0 \quad \text{; } \theta = \frac{\pi}{2} \]  

(157)

We observe that if \(r \to +\infty\), the above equation becomes the corresponding of Generalized SR: (73). Moreover it is \(K_g U \geq 0\). Finally the Relativistic mechanic energy \(E_m = E - E_{\text{rest}} = K_g + U\) is

\[ E_m = \sqrt{\frac{1 - \frac{s_1^2}{s_2^2}}{1 - \frac{s_1^2}{s_2^2}}} - \frac{1}{\sqrt{1 - \frac{s_1^2}{s_2^2}}} \frac{m c^2}{\xi^2 s_1^2} \geq 0 \quad \text{; } \theta = \frac{\pi}{2} \]  

(158)

Alternatively, we may define a new 1st Generalized Schwarzschild Lagrangian (n1GSL):

\[ L_1 = \frac{1}{2 \xi^2 r} \left( \frac{L}{-g_{100}} + m c^2 \right) \]  

(159)

If we replace (141) to the above equation, then it emerges that \(L_1 = 0\) except for the case of \(\xi_1 = 0\), where it is proven that \(L_{1N} = 0\). \(K_{N} U_{N} \geq 0\). Besides the combination of (159) with (136) gives

\[ L_1 = \frac{1}{2 \xi^2 r} \left( -m \left( 1 - \frac{s_1^2}{s_2^2} \right) \dot{c} \dot{r}^2 + \frac{s_1^2 m}{1 - \frac{s_1^2}{s_2^2}} \dot{\phi}^2 + m c^2 \right) \quad ; \quad \frac{d}{d \tau} \quad ; \quad \theta = \frac{\pi}{2} \]  

(160)
or equivalently

\[
L_1 = \frac{m}{2} \left( \frac{c^2}{\epsilon_1^2} - \frac{1}{\epsilon_1^2} - \frac{\epsilon_1^2}{r^2} \right) \left( c^2 i^2 + \frac{1}{1 - \epsilon_1^2} \frac{r^2 + r^2 \phi^2}{r} \right) ; \quad \frac{d}{dr} ; \quad \theta = \frac{\pi}{2}
\]  

(161)

In case that \( \epsilon_1 \rightarrow 0^+ \) (Galilean metric), (151) gives \( i = 1 \). Thus we obtain the Newtonian results:

\[
\Phi_N = \lim_{\epsilon_1 \rightarrow 0} \Phi = \frac{c^2}{2} \lim_{\epsilon_1 \rightarrow 0} \left[ \frac{1}{\epsilon_1^2} \ln \left( 1 - \frac{\epsilon_1^2}{\ell_0} \right) \right] = \frac{c^2}{4} \lim_{\epsilon_1 \rightarrow 0} \left[ \frac{1}{\epsilon_1^2} \left( 1 - \frac{\epsilon_1^2}{\ell_0} \right) \right] = \frac{-2}{2} \left( \frac{r \ell_0}{r} \right) = -\frac{G M}{r}
\]

\[
dS_N^2 = g_{100} c^2 d^2 r^2 + 2 r d r \left. d r^2 + r^2 d \theta^2 + 2 r^2 \sin^2 \theta d \phi^2 \right) = \frac{G M}{r^2} - \frac{2}{r}
\]

\[
L_N = mg_{100} c^2 ; \quad E_N = +\infty ; \quad \dot{r} + \frac{G M}{r^2} - \frac{r}{r^2} = 0 ; \quad J_N = mr^2 \phi ; \quad \dot{\theta} = \frac{d}{dt} ; \quad \theta = \frac{\pi}{2}
\]

The Newtonian differential equation of motion and the corresponding solution are

\[
\frac{d^2 u_N}{d \phi^2} + u_N = \frac{G M}{h_N} ; \quad u_N = \frac{G M}{h_N^2} (1 + \epsilon_N \cos \phi) ; \quad u = \frac{1}{r} ; \quad h_N = r^2 \phi ; \quad \dot{\epsilon} = \frac{d}{dt} ; \quad \theta = \frac{\pi}{2}
\]

(162)

\[
e_N = \sqrt{1 + 2E_{MN}^2} ; \quad E_{MN} = -\frac{G M}{2d_N}
\]

(163)

where \( \alpha_N \) is the semimajor axis of Newtonian ellipse which do not rotate \( (A_N = 0) \). Besides

\[
U_N = -\frac{G M}{r} ; \quad V_N = -\frac{G M}{r} ; \quad K_N = \frac{1}{2} \beta_0^2 m c^2 = \frac{1}{2} m |p|^2 ; \quad E_{MN} = \frac{1}{2} m |p|^2 - \frac{G M}{r}
\]

(164)

Finally, the Galilean version of n1GSL is

\[
L_N = \frac{m}{2} \left( \frac{\ell_0^2}{r} + \dot{r}^2 + r^2 \phi^2 \right) = \frac{G M}{r} + \frac{1}{2} m |p|^2 = K_N - V_N ; \quad \dot{\epsilon} = \frac{d}{dt} ; \quad \theta = \frac{\pi}{2}
\]

(165)

which is the original Newtonian Lagrangian!

In case that \( \epsilon_1 = 1 \) (GSL = g_{100}0) (Vassos metric), it emerges the well-known results of the original Schwarzschild metric in ERT (see e.g. [5] pp. 228-45):

\[
\Phi_E = \frac{c^2}{2} \ln \left( 1 - \frac{\ell_0}{r} \right)
\]

(170)

\[
dS_E^2 = g_{100} \left[ \left( 1 - \frac{\ell_0}{r} \right) c^2 d^2 r^2 - \frac{1}{1 - \ell_0} d r^2 - r^2 d \theta^2 - r^2 \sin^2 \theta d \phi^2 \right]
\]

(171)

\[
\bar{g}_E = -\frac{G M}{r^2} \left( \frac{1}{1 - \ell_0} \right) \frac{1}{2} \frac{1}{r}
\]

(172)

\[
L_E = mg_{100} \left[ \left( 1 - \frac{\ell_0}{r} \right) c^2 i^2 - \frac{1}{1 - \ell_0} \dot{r}^2 - r^2 \phi^2 \right] ; \quad E_E = \left( 1 - \frac{\ell_0}{r} \right) m c^2 ; \quad \dot{\epsilon} = \frac{d}{dt} \frac{1}{2} \theta = \frac{\pi}{2}
\]

(173)
Finally, the ERT\ approximate differential equation of motion (which also validates UCM) is:

\[
\frac{d^2 u}{d \phi^2} + u = \frac{GM}{h_E^4} + 3 \frac{G^2 M^2}{c^2 h_E^4} \left(1 + e \cos \phi \right)^2 ; \ u = \frac{1}{r} ; \ h_E = r^2 \phi ; \ \cdot = \frac{d}{d \tau_E}
\]

with exact and approximate solution, correspondingly

\[
u = \frac{GM}{h_E^2} \left(1 + e \cos \phi + 3 \frac{G^2 M^2}{c^2 h_E^4} e \sin \phi \right) ; \ \frac{GM}{h_E} \approx \frac{1}{a_E \left(1 - e^2\right)}
\]

\[
u \approx \frac{GM}{h_E^2} \left(1 + e \cos \left(1 - 3 \frac{G^2 M^2}{c^2 h_E^2} \phi \right) \right) ; \ \frac{6 \pi G^2 M^2}{c^2 h_E^2} < < 1
\]

Hence the ERT orbit can be regarded as an Einsteinian ellipse (with \(a_E\) semimajor axis) which rotates (*precesses*) about one of its foci by an amount

\[
\Delta E = \frac{2 \pi}{1 - \frac{3 G^2 M^2}{c^2 h_E^2}} - 2 \pi \approx \frac{6 \pi G^2 M^2}{c^2 h_E^2} = \frac{6 \pi GM}{a_E \left(1 - e^2\right) \beta^2} ; \ h_E = r^2 \phi ; \ \frac{d \phi}{d \tau_E} = \frac{d \phi}{d t} i_E
\]

per revolution. Accordingly to our non-mainstream approach

\[
i_E = \frac{d \tau}{d r} = \left[1 - \left(\frac{\beta_{y^2}}{r} + \frac{1}{\beta_{y^2}} \right) \left(1 - \frac{\beta_{y^2} + \beta_{y^2}}{\beta_{y^2}} \right) \right]^{-\frac{1}{2}} ; \ E_E = \frac{1 - \frac{\beta_{y^2}}{r}}{\left[1 - \frac{\beta_{y^2}}{1 - \frac{\beta_{y^2}}{r}} \right]^{\frac{1}{2}}} \sqrt{m c^2 \geq 0}
\]

\[
U_E = \left(1 - \frac{\beta_{y^2}}{r} - 1 \right) m c^2 \leq 0 ; \ V_E = \left(1 - \frac{\beta_{y^2}}{r} - 1 \right) c^2 \leq 0 ; \ K_{EE} = \left[1 - \frac{\beta_{y^2}}{r} + \frac{1}{\beta_{y^2}} \left(1 - \frac{\beta_{y^2} + \beta_{y^2}}{\beta_{y^2}} \right) \right]^{-\frac{1}{2}} \sqrt{m c^2 \geq 0}
\]

\[
E_{mE} = \left[1 - \left(\frac{\beta_{y^2}}{r} + \frac{1}{1 - \frac{\beta_{y^2}}{r}} \beta_{y^2} + \beta_{y^2} \right) \right]^{-\frac{1}{2}} \sqrt{m c^2 ; \ \theta = \frac{\pi}{2}}
\]

Finally, the Lorentzian-Einsteinian version of n1GSL is
\[ L_E = \frac{m}{2} \left( c^2 \left( 1 - \frac{\xi_0}{r} \right) c^2 t^2 + \frac{1}{1 - \frac{\xi_0}{r}} r^2 \phi^2 \right) \ ; \ \phi = \frac{d}{dE} \ ; \ \theta = \frac{\pi}{2} \] (183)

5.2. 2nd Generalized Schwarzschild metric, Relativistic potential & Field strength and Combination with MOND

In case that only \( f(r) = 1 \), (126) gives the 2nd Generalized Schwarzschild Relativistic potential (2GSRP) [8] (p.13):

\[ \Phi = \frac{c^2}{2 \xi_1^2} \ln \left( 1 - \frac{\xi_1^2 \xi_0}{r} \right) \] (184)

We demand the 2GSRP having value equal to the corresponding of the 1G SRP at the distance of the 1st Generalized Schwarzschild radius \( (r_{SI} = \xi_1 \xi_2) \). Thus, we have the condition

\[ h_{(0)} = h_{(\xi_1 \xi_2)} = 1 \] (185)

Besides (129) emerges the 2nd Generalized Schwarzschild Metric (2GSM) [8] (p.15)

\[ dS^2 = g_{110} \left( 1 - \frac{\xi_1^2 \xi_0}{r} \right)^{h_{(0)}} \ c^2 dt^2 - \frac{g_{111}}{\left( 1 - \frac{\xi_1^2 \xi_0}{r} \right)^{h_{(0)}}} dr^2 + g_{111} r^2 \ d \theta^2 + g_{111} r^2 \sin^2 \theta \ d \phi^2 \] (186)

So, we obtain the corresponding radial 2nd Generalized Schwarzschild Field Strength (2GSFS)

\[ g = \frac{GM}{r^2} \left( 1 - \frac{\xi_1^2 \xi_0}{r} \right)^{\frac{h_{(0)}}{2}} + c^2 \frac{dh}{dr} \left( 1 - \frac{\xi_1^2 \xi_0}{r} \right)^{\frac{h_{(0)}}{2}} \ln \left( 1 - \frac{\xi_1^2 \xi_0}{r} \right) \] (187)

where the positive value of field strength means gravity, while negative value means antigravity.

Modified Newtonian Dynamics (MOND) is using suitable Interpolating Function \( (\mu) \) in Milgrom’s Law [9]. In case of a spherical or cylindrical distribution of mass, the Modified Newtonian field strength is

\[ g = \frac{GM}{r^2} \frac{1}{\mu} \] (188)

Two common choices are the Simple and Standard interpolating function, correspondingly

\[ \mu = \frac{1}{1 + \frac{a_0}{g}} = \frac{2}{1 + \left( \frac{r}{r_0} \right)^2} \ ; \ \mu = \frac{1}{1 + \left( \frac{a_0}{g} \right)^2} = \frac{\sqrt{2}}{1 + \left( \frac{1}{4} \left( \frac{r}{r_0} \right)^2 \right)} \ ; \ r_0 = \frac{GM}{\sqrt{4a_0}} \] (189)

where \( r_0 \) is called Milgrom radius [8] (p. 3) and \( a_0 \approx 1.2 \times 10^{-10} ms^{-2} \) [9] (p. 1) is a new fundamental physical constant. Both the Interpolating functions give the same velocity at infinite distance from the center of gravity

\[ \upsilon = \sqrt{GMa_0} \] (190)

The combination of Galilean version of 2GSFS (187) with MOND (188) emerges

\[ \frac{1}{\mu_{(r)}} = h_{(r)} - \frac{d h}{dr} \] (191)

or equivalently

\[ h_{(r)} = -r \int \frac{dr}{r^2 \mu_{(r)}} = -r \int \frac{1}{\frac{r}{r_0}} \frac{d}{d \left( \frac{r}{r_0} \right)} \left( \frac{r}{r_0} \right) = -r \int \frac{1}{r_0} \] (192)
The integrals of Simple and Standard interpolating functions are correspondingly
\[
I_{\text{simp}} = \int \frac{1}{2x^{3}}(1 + \sqrt{1 + x^{2}}) \, dx = \frac{1}{2} \left( -\frac{1}{x} - \frac{\sqrt{1 + x^{2}}}{x} + \text{ArcSinh} \, x \right) = \frac{1}{2} \left( -\frac{1}{x} \sqrt{1 + x^{2}} + \ln(x + \sqrt{1 + x^{2}}) \right)
\]
(193)
\[
I_{\text{stand}} = \int \frac{1}{\sqrt{x^{2}}} \left[ \frac{\sqrt{1 + x^{4}}}{4} \right]^{\frac{1}{2}} \, dx = \frac{1}{2} \left[ \frac{\sqrt{1 + x^{4}}}{4} + 1 \right]^{\frac{1}{2}} \, \text{ArcSinh} \, \frac{\sqrt{1 + x^{4}}}{4} + 2 \left( 1 - \frac{\sqrt{1 + x^{4}}}{4} \right)^{\frac{1}{2}} \left[ \frac{1}{4} \int \frac{1}{4} \frac{5}{4} \frac{1}{2} \left( 1 + \frac{\sqrt{1 + x^{4}}}{4} \right) \right]
\]
(194)

The last solution contains Gauss hypergeometric function. We then extend the above Newtonian solutions to other TPs, by according the integration constant with condition (185). Thus we have
\[
C_{\text{simp}} = \frac{-1}{2r_{0}^{2} \tilde{\xi}} \left( -1 + \ln \left[ \frac{\tilde{\xi}^{2} \tilde{r}_{0}}{\tilde{r}_{0}} \right] - \frac{\tilde{\xi}^{2} \tilde{r}_{0}}{\tilde{r}_{0}} \text{ArcSinh} \left( \frac{\tilde{\xi}^{2} \tilde{r}_{0}}{\tilde{r}_{0}} \right) \right)
\]
(195)
\[
C_{\text{stand}} = \frac{-r_{0}}{\tilde{\xi}^{2} \tilde{r}_{0}} \left( -1 + \ln \left[ \frac{\tilde{\xi}^{2} \tilde{r}_{0}}{\tilde{r}_{0}} \right] - \frac{\tilde{\xi}^{2} \tilde{r}_{0}}{\tilde{r}_{0}} \text{ArcSinh} \left( \frac{\tilde{\xi}^{2} \tilde{r}_{0}}{\tilde{r}_{0}} \right) \right)
\]
(196)

In case of ERT (\(\tilde{\xi}=1\)), the Combination of 2GSM with MOND Simple interpolating function gives
\[
h_{\text{simp}} = \frac{1}{2} \frac{r}{\tilde{r}_{0}} + 1 + \frac{r}{\tilde{r}_{0}} \left[ -\frac{r}{\tilde{r}_{0}} \right] \frac{\tilde{r}_{0}}{\tilde{r}_{0}} \text{ArcSinh} \left( \frac{\tilde{r}_{0}}{\tilde{r}_{0}} \right)
\]
(197)
while the MOND Standard interpolating function gives more complicated solution:
\[
h_{\text{stand}} = \frac{r}{\tilde{r}_{0}} - \frac{r}{\tilde{r}_{0}} \left[ -\frac{r}{\tilde{r}_{0}} \right] \text{ArcSinh} \left( \frac{\tilde{r}_{0}}{\tilde{r}_{0}} \right)
\]
(198)

6. Experimental Validation

6.1. The Precession of Mercury’s Perihelion according to the 1st Generalized Schwarzschild metric 1GSM contains only one free parameter: \(\tilde{\xi}\). Besides the precession of Mercury’s perihelion was one of the first confirmations of ERT. Thus they can be combined in order to find a set of valid values of \(\tilde{\xi}\) in our Solar system. The GE precession of Mercury’s perihelion is stated as [10] (p.1):
\[
\Delta = \frac{2\pi(2 - \beta + 2\gamma)GM}{a(1 - e^{2})c^{2}}
\]
(199)
per revolution, where $\beta$ is a measure of the nonlinearity of superposition for gravity and $\gamma$ is a measure of the curvature of space due to unit rest mass. This combined to (148) gives

$$\xi_i = \frac{2 - \beta + 2\gamma}{3}; \quad \bar{\xi}_i = \sqrt{\frac{3(1 - \beta) + 2(\gamma - 1)}{3}}$$  \hspace{1cm} (200)

We replace $(1-\beta)=(2.7\pm3.9) \times 10^{-5}$ from the processing MESENGER ranging data (Nordtvedt effect) and $(\gamma-1)=(2.1\pm2.3) \times 10^{-5}$ from the Cassini mission (Shapiro delay experiment) [10]. Thus we obtain

$$\bar{\xi}_i = 1.0000115(75)$$  \hspace{1cm} (201)

Another way of finding a set of valid values of $\xi_i$ in our Solar system is the direct replacement of all the experimental data to (148): $\Delta=42.9799\pm0.0009\,\text{cy}^{-1}$ [10] (p.6), $G=6.67428(67)\times10^{-11}\,\text{m}\,\text{kg}^{-1}\text{s}^{-2}$ and $c=299792458\,\text{ms}^{-1}$ (exact) [11] (pp 1-14-2), $M=1,988,500\times10^{24}\,\text{kg}$ [12], $\alpha=0.38709893\,\text{AU}$, $e=0.20563069$ and $T=87.968\,\text{days}$ [13]. Thus we also have

$$\bar{\xi}_i = 0.99999754(105)$$  \hspace{1cm} (202)

Finally, we obtain the mean value

$$\bar{\xi}_i = 0.9999934(91); \quad \bar{\xi}_i \in (0.9999843, 1.0000025) \quad \text{(confidence level 68%)}$$  \hspace{1cm} (203)

This represents a slight curvature of the spacetime even for RIOs:

$$\Delta\bar{\xi}_i = \frac{\bar{\xi}_i - 1}{1} = -0.000066(91)\%$$  \hspace{1cm} (204)

6.2. The Combination of Lorentzian version of 2nd Generalized Schwarzschild metric with MOND Simple & Standard Interpolating Function in Galaxy NGC 3198 & the Solar System

The effect of 2GSM modification was examined at large mass and size systems, such as Galaxy NGC 3198 (data from [14] (p. 2), results from [8] (pp. 21-24)) and medium mass and size systems, such as our Solar System (data from [11] (p. 14-3), results from [8] (pp. 24-27)). The corresponding Rotation Curves are shown in Figures 3 & 4 and the Plots of function $h(r)$ are shown in Figures 5 & 6, respectively.

**Figure 3.** Universal Rotation Curves in Galaxy NGC 3198. Rotational Velocities [experimental ($V_{\text{exp}}$), calculated by Schwarzschild or Newtonian field strength ($V_d$) and the Combination of Lorentzian version of 2nd Generalized Schwarzschild metric with MOND Simple or Standard Interpolating Function ($V_{\text{simp}}, V_{\text{stand}}$)] wrt the distance ($r$) from the center of Galaxy NGC 3198.
We observe that in scale of black hole, planetary and star system, the Combination of Lorentzian version of 2GSM metric with MOND coincides to the original Schwarzschild metric ($h \approx 1$), while in galactic scale, it gives MONDian results. In universal scale, the gravitational field strength (187) becomes negative producing slight antigravity.
Conclusion

Einstein Relativity Theory (ERT) with Relativistic Inertial Observers (RIOs) in Minkowski Space was generalized to a Relativity Theory (RT) where the RIOs have metric $g_{I}=\text{diag}(g_{00}, g_{11}, g_{11}, g_{11})$. Thus Newtonian Physics (NPs) has Galilean metric $g_{I}=\text{diag}(g_{00}, 0, 0, 0)$, keeping the formalism of ERT. Besides infinite number of Theories of Physics (TPs) was produced where the critical parameter is $\xi_{I}=\sqrt{-\frac{g_{11}}{g_{00}}}$ and the universal speed is $c_{I}=c/\xi_{I}$. Maxwell equations were also generalized in order to have electromagnetic waves which are propagating with $c_{I}$. Thus a Generalized Quantum Mechanics can also be developed. The generalization is extended to General Relativity, by producing the 1st Generalized Schwarzschild metric (1GSM) and 2nd Generalized Schwarzschild metric (2GSM), which are in accordance with Einstein field equations for any TPs. In case of 1GSM, the corresponding Lagrangian, geodesics, equations of motion, precession of planets’ orbits etc was found, resulting formulas which are referred to any TPs. The theoretical results were combined with experimental data of our Solar system, producing a set of valid values of $\xi_{I}$. In case of 2GSM, the combination of its Galilean version of 2GSM with Modified Newtonian Dynamics (MOND), leads to MOND relativization. Moreover, we extended MOND methods to ERT. The combination of Simple or Standard Interpolating Function ($\mu$) with the Lorentzian version of 2GSM, leads to the explanation of Rotation Curves in Galaxies as well as the Solar system, eliminating Dark Matter. Generally, this approach, in non rotating black hole, planetary and star system scale, coincides to the original Schwarzschild metric, while in galactic scale, it gives MONDian results. In universal scale, it emerges slight antigravity, improving the aspects for elimination of dark energy, too.

Acknowledgments

The authors would like to thank Nikolaos Veronikiatis (Physicist MSc, Department of Physics, School of Applied Mathematics and Physical Science, National Technical University of Athens) for the calculation (194).
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