Choice of MUBs Affects Measurement Outcome Secrecy

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Mutually unbiased bases (MUBs) are a crucial ingredient for many protocols in quantum information processing. Measurements performed in these bases are unbiased to the maximally possible extent, which is used to prove randomness or secrecy of measurement results. In this work we show that certain properties of sets of MUBs crucially depend on their specific choice — if measurements are chosen in a coherent way, the secrecy of the result can be completely lost for specific sets of MUB measurements, while partially retained for others. This fact has a large impact on broad spectrum of applications.

Introduction – One of the defining features of quantum mechanics is the impossibility to simultaneously measure a certain set of physical quantities. This fact led to the definition of the famous Heisenberg uncertainty principle [1] or understanding of the quantum model of hydrogen atom [2]. If a simultaneous measurement of two quantities is not possible, or, in other words, if a measurement of one quantity influences the expectation of the other measurement, we call these two measurements incompatible. In this context a very natural question arises – how much incompatible a pair of measurements can be? The answer to this question is simple – as incompatible as possible: for any system, even as simple as a qubit, one can find a pair of measurements where irrespective of the starting state of the system, after performing one of the measurements the result of the other one is completely random.

A straightforward generalization is on hand – can one form a larger set of measurements that are pairwise fully incompatible? Here again one can answer affirmatively – for each system one can find at least three such measurements and the size of the set depends on the dimension of the system.

In order to tackle with these questions more formally, the notion of mutually unbiased bases (MUBs) [3–6] was introduced. Two $d$-dimensional bases $\{|\psi_i\rangle\}_{i=0,...,d-1}$ and $\{|\phi_j\rangle\}_{j=0,...,d-1}$ corresponding to two full projective measurements are mutually unbiased, when

$$\forall i, j : |\langle \psi_i | \phi_j \rangle| = \frac{1}{\sqrt{d}}. \tag{1}$$

Due to their properties, mutually unbiased bases have become an important cornerstone of contemporary quantum information processing [7]. They are being used for quantum tomography [4, 6], uncertainty relations [5, 8, 9], quantum key distribution [10–13], quantum error correction [14], as well as for witnessing entanglement [15–21], design of Bell inequalities [22, 23] and more general forms of quantum correlations [24–26].

The natural question of the number of unbiased bases in a given dimension $d$ turned out to be unexpectedly complicated. While the answer is rather simple for qubits – there are three pairwise mutually unbiased bases, defined as eigenvectors of Pauli $\sigma_x$, $\sigma_y$, $\sigma_z$ operators up to unitary equivalencies, in general, the construction of MUBs is a very difficult task. It is known that the number of MUBs cannot be larger than $d + 1$ for any dimension and the constructions of $d + 1$ MUBs are known for $d = p^r$, where $p$ is a prime. However, for non-prime-power $d$ only the trivial tensor product construction is known. Most notably, for $d = 6$ the tensor product construction using three qubit and three qutrit MUBs results in three MUBs, which is conjectured to be optimal [27, 28].

Fortunately, in many applications one needs to use only $k \leq d+1$ MUB measurements. Clearly, there are different ways to pick the subset of $k$ out of all MUBs. In fact, it is known that different sets of MUBs are not necessarily equivalent under different mathematical operations, such as global unitary operations, changing individual vector phases, relabelling of outcomes, relabelling of moments or introducing complex conjugation [29]. This mathematical inequivalence is however irrelevant in many practical applications where just satisfying the defining property (1) is required for the task.

More interestingly, it was recently shown that different subsets of MUBs of can be inequivalent operationally as well. For example, MUBs turn out to be an optimal strategy in a communication task called quantum random access coding (QRAC). Generally, in this task one user is trying to encode $n$ messages into a qudit with $n > d$ and the goal of the second party is to access a randomly chosen $i$-th message. In [30] it was shown that in certain variant of QRAC, different subsets of $k$ out of $d + 1$ MUBs lead to different strategies with different average success rates. More recently, it was shown that different subsets of $k$ out of $d + 1$ MUBs behave differently under a measure called incompatibility robustness [31]. Last but not least, very specific MUBs are required to obtain Bell inequalities [22], which are maximally violated by maximally entangled states and MUBs.

The full definition of a measurement consist of specifying the basis as a set of states and labeling these states. Two measurements consisting of the same set of states
are in principle different, even if they measure the same property and their results can be classically transformed at any later stage (in the same way as two rulers, one measuring in cm and the other one in mm are different, but pretty similar). From the experimental and operational point of view it makes sense to distinguish between different measurements that only differ in labeling (we call this a classical difference) and two measurements that differ in the states per se (quantum difference). One can then naturally ask, to what extent the properties of MUBs do change if one only makes a classical change in them. In other words, do the properties of the subsets change by simple re-labeling of their vectors? In this work, we affirmatively answer this question by introducing a quantum information task called guessing game. There a subset of $d$ out of $d+1$ MUBs is used to hide and guess information between two parties. We show that in spite of the fact that just one out of the full set of $d+1$ MUBs is removed from the game, the achievable results of the parties critically depend on this choice. Even more interestingly, for a suitable chosen subset of $d$ out of $d+1$ MUBs, one can achieve the full spectrum of results (from perfect guessing to maximal hiding) just by relabeling the measurement outcomes. In other words, for a given set of measurement bases, the results of the game can provide maximal secrecy or no secrecy based on labeling of them.

Results – The incompatibility of measurements can be demonstrated and examined with the help of a very simple quantum game, studied in [32, 33]. Here Alice realizes one of $m$ possible measurements on a $d$-dimensional system and records the result of this measurement. The task of Bob is to guess this result using the following strategy: first, he prepares the state for Alice to be measured and second, he receives information about which measurement was performed (see the next section for the full definition of the guessing game).

If the game is described by classical physics, a pure state has a determined outcome for all possible measurements. Therefore, trivially Bob can prepare a state in such a form that irrespective on the measurement performed by Alice, he will be able to guess the outcome with certainty. This is true due to the fact that in classical physics incompatible measurements do not exist.

One can make the scenario partially quantum, by making Bob’s probe state as well as the measurements quantum, but the information about the measurement chosen by Alice is classical – we call this a classical coin scenario. This is the the traditional way to demonstrate incompatibility of quantum measurements – for compatible measurements Bob still can guess with certainty, but with increasing incompatibility of the measurements the uncertainty of his guess increases.

In a fully quantum scenario – called quantum coin scenario – depicted in Figure (1), both the probe state and the information about the measurement chosen are quantum. Here Alice realizes the chosen measurement by first applying a coherently controlled unitary, followed by a measurement in a standard basis. Bob receives the control state and can use it to determine Alice’s outcome.

The authors of [32] have thoroughly analyzed the guessing game for two specific measurements ($m = 2$) with carefully chosen MUBs. They have shown that for qubits ($d = 2$), in the quantum coin scenario Bob can guess Alice’s outcome with certainty. In contrast, this was not the case for higher dimensions. This is however very natural due to the fact that in case of two measurements the control state is always a two dimensional state and it is impossible to use it to determine a higher dimensional outcome. The authors have also shown the monotonous dependence of the guessing probability on the coherence of the control qubit (i.e. they considered also scenario with a partially quantum coin) – the more coherent the control, the higher the achievable guessing probability.

In [33] we have further analyzed the guessing game with the quantum coin and we have shown that for qubits, with any number of measurements (independent on their level of compatibility) it is always possible for Bob to obtain the result of Alice with probability 1. In contrast, for higher dimensions this is not the case, so even if Bob receives a large enough control state, he will not be able to guess the result perfectly for a specific set of MUBs chosen by Alice.

Here we analyze the problem further. We fix the number of measurements to $m = d$, which will make the size of the measurement outcomes alphabet equal to the dimension of the control state available to Bob. We show that in a quantum coin scenario with a specific set of MUBs, i.e. fixed measurement bases of Alice’s measurement, Bob will either be able to guess the measurement outcomes perfectly for any dimension, or be able to guess with a probability strictly decreasing with the dimension.
His guessing probability depends on the labeling Alice chooses. This shows that different sets of $d$ out of $d+1$ MUB bases, which however only differ in a classical sense (i.e. by relabeling), exhibit very different operational properties. With differently chosen set of MUBs, corresponding to different quantum measurements, we numerically (and analytically for $d=3$ and $d=5$) obtain a smaller spread of probabilities – Bob can neither achieve perfect guessing, nor can Alice achieve such a good hiding by changing the labeling only. This shows that there exist sets of quantumly different MUBs that lead to operationally different properties even under relabeling.

**Guessing game** – Here we give a formal definition of the guessing game and define a set of $d$ out of $d+1$ MUB measurements which allows Bob to construct a perfect guessing strategy. In the guessing game, Alice receives an initial state $\rho_B$ of dimension $d$ prepared by Bob. She performs a coherently controlled unitary transformation $CU$ defined by the set of $\{|U_i\rangle\}_{i=0}^{d-1}$ controlled by the “coin” state $\rho_C$. In the classical coin scenario the pure state $\rho_C = |+\rangle\langle+|$ is used, where $|+\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i\rangle$, while in the classical coin scenario a fully mixed state $\rho_C = \frac{1}{d}$ is used. After the transformation, Alice measures the state $\rho_B$ in the computational basis and sends the control state $\rho_C$ to Bob, who also performs a general measurement defined by POVM elements $\{M_{ai}\}_{a=0}^{d-1}$ to obtain his guess $b$. Bob wins if the results coincide, otherwise Alice is the winner.

The average guessing probability of Bob is defined as:

$$P_g = \sum_{a=0}^{d-1} \text{Tr}_{AB} \left[ (\rho_B \otimes \rho_C) CU \left( |a\rangle \langle a| \otimes M_a \right) CU^\dagger \right].$$

(2)

Although there are multiple constructions of MUBs for prime dimensions, to demonstrate our result we will use a construction of Wooters and Fields (WF) [6]:

$$U_\text{WF}^a = \frac{1}{\sqrt{d}} \sum_{i,j=0}^{d-1} \omega^{ai^2 + ji^3} |i\rangle \langle j|.$$  

(3)

In prime dimension $d$, this construction defines $d$ different bases and can be supplemented by the computational basis to for the full set of $d+1$ MUBs. There are $d+1$ different ways to select the set of $d$ bases. Additionally, for each set of $d$ bases we will consider relabeling of the vectors which allows to construct additional sets of $d$ measurements used in the guessing game.

**Classical coin scenario** – In the case of a classical coin state, we have that $\rho_C = \frac{1}{d}$. Clearly, this is equivalent to Alice choosing the measurement uniformly at random and Bob then receiving the information about which measurement was chosen. Based on this information he has to guess the result obtained by Alice. While for qubits the optimal strategy for Bob is straightforward and easy to understand (he prepares a coherent superposition of two basis states of the two possible measurements of Alice) and yields the guessing probability of $\frac{1}{2} \left( 1 + \frac{1}{\sqrt{2}} \right)$, for the higher dimensional variant of the game the situation is much more complicated. In Appendix A.4 we derive an upper bound in the form $\frac{1}{d} \left( 1 + \frac{d-1}{\sqrt{d}} \right)$ valid for any set of MUBs (this includes relabelling, since it does not influence the Bob’s guessing probability in the classical coin scenario), which converges to 0 for high $d$. Furthermore, for the set of MUBs defined in (3) up to $d = 7$ we also obtain exact values of the guessing probability for the classical coin case by exhaustive search. For higher $d$ we provide numerical estimates that show that the bound obtained is not tight. In spite of this, it is more than clear that without coherent information, with increasing $d$, Bob can only obtain negligible information about the result obtained by Alice irrespective on which set of MUBs she uses.

**Quantum coin scenario** – The situation is dramatically different for the quantum coin scenario, in which $\rho_C = |+\rangle\langle+|$. First we show that for a specific selection of MUBs chosen by Alice, it is possible for Bob to obtain the result of Alice with certainty. To achieve this, Alice needs to select both the proper $d$ MUB bases, (quantum setting) and label the individual measurement basis vectors in a suitable way as well (classical setting). For example if Alice chooses $d$ WF bases, Bob can never achieve perfect guessing, as we have shown in [33].

A stark contrast can be shown by redefining the set of MUBs as

$$U_\text{DPP}^a = \frac{1}{\sqrt{d}} \sum_{i,j=0}^{d-1} \omega^{ai^2 + ji^3} |i\rangle \langle j|,$$  

(4)

which consists of simple relabeling of the vectors of the bases of the WF construction: $U_\text{DPP}^a |j\rangle = U_\text{WF}^j |j - a^2\rangle$.

Let us define Bob’s (pure) probe state $|\psi_B\rangle$ and measurements $\{M_a\}_{a=0}^{d-1}$ as:

$$|\psi_B\rangle = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} \omega^{k^3} |k\rangle,$$

$$M_a = \frac{\langle \phi_a | \phi_a \rangle}{\langle \phi_a | \phi_a \rangle},$$

$$|\phi_a\rangle := \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i\rangle U_\text{DPP}^i |\psi_B\rangle |a\rangle,$$

(5)

where $|\phi_a\rangle$ are unnormalized pure states. In Appendix (A.2) we show that $\{M_a\}_{a=0}^{d-1}$ form a projective measurement. Subsequently we show that such measurement allows Bob to guess perfectly Alice’s measurement outcomes if used in conjunction with the probe state $|\psi_B\rangle$. The construction and proof are slightly different specifically for $d = 3$, as shown in (A.3).

Interestingly, if one of the WF bases is exchanged for the computational basis (which corresponds to a quantum difference), there is no way for Bob to achieve perfect guessing for any labeling of the individual measurements. In other words, if the computational basis is included in the set of MUBs used, we have strong numerical evidence that Alice can retain some secrecy towards Bob.
irrespective of the labeling used; for dimensions 3 and 5 this can be shown by exhaustive search over all the possible relabellings, for higher dimensions we performed a randomized search (see Appendix A.V for details).

Optimal hiding in the quantum coin case – We have shown that if Bob can influence the choice of MUBs used by Alice, he can perfectly guess her outcome. It is thus very natural to ask the complementary question – if Alice can retain full control about her measurements, what is the maximum Bob can learn about her outcome? And how does this maximum depend on the quantum setting of her measurements and actual labeling?

To answer this question fully, one would have to search through all possible MUBs including their labeling and find optimal values. To keep the task tractable, first we have focused on the standard WF set of MUBs plus the computation basis (leading to \( d + 1 \) possibilities) plus possible relabellings expressed via permutation matrices \( P_\pi \), which relabel the computational basis states and leave the MUB property intact:

\[
| \langle i | U_a | j \rangle | = \frac{1}{\sqrt{d}} = | \langle i | P_\pi U_a | j \rangle | .
\]

Due to the intractably large number of combinations, for dimensions higher than 5 we first restricted ourselves to cyclic permutation matrices. On top of it, we have also tested randomly a large set of non-cyclic permutation matrices (see Appendix A.V for details).

For a fixed set of MUBs, we cast the problem as a see-saw SDP [34](A.I), which allows us to obtain a lower-bound on the maximum of \( P_g \). We have randomized the initial point and repeated the optimization to obtain the lower bounds as depicted in Figure (2). As a last step, to look a bit behind the strict limit of WF construction and its relabelling, we applied the see-saw algorithm to unitaries close to the MUBs in the space of unitary matrices. In all cases we obtained values higher than the WF construction; this shows that the found values constitute (at least) a local minimum in the space of unitary matrices, while the search over permutation matrices suggests that they constitute a global minimum over the space of MUB unitary matrices as well.

While the obtained minima decrease with the dimension, they stay far above the upper bounds of the classical coin scenario. Thus it is clear that irrespective of the selection of measurements by Alice, obtaining coherent information about her measurement allows Bob to take a more accurate guess. At the same time the maximal and minimal guessing probability in the case of the quantum coin changes with the choice of both measurement bases and their labelling, making it important for Alice to carefully choose the MUBs used in the guessing game.

An analysis of the actual MUBs that leads to the obtained minimal guessing probability sheds some light on the problem. Surprisingly, it turned out that the minimal guessing probabilities are obtained for the standard WF construction of MUBs \( \{U_a^{WF}\}_{a=0}^{d-1} \). So in the case when Alice can make her choice of the measurements, including the labeling, it is best for her to select the standard construction to minimize the knowledge of Bob. At the same time we could see that the perfect guessing by Bob was achieved for the DPP construction (4), which only differs from the WF construction by relabelling – i.e. boundary values are achieved for MUBs that differ only by labeling.

On the contrary, if the computational basis is included into the system by exchanging it with any of the WF bases, we have strong numerical evidence that neither Bob can perfectly guess the outcome, nor Alice can hide it as well as in the WF case. This suggests that the set of \( d \) WF constructed bases including its relabelling is structurally different than any set where the computational basis is used with \( d – 1 \) WF bases. It is worth mentioning that this fact is not connected to the computational basis itself. One can find sets of MUBs containing computation basis that exhibit the same properties as the WF or DPP set respectively, but the remaining bases are not given by the WF construction.

Discussion – In our work we have shown on a simple quantum mechanical game that different choices of mutually unbiased bases have dramatic effects on experimentally achievable results. Interestingly, for any prime dimension \( d \) one can choose a set of \( d \) MUBs that provide the possibility of perfect guessing by Bob of the result obtained by Alice in the quantum coin scenario. At the same time, with a set of MUBs that differs only by relabeling of the individual vectors, Alice can obtain the maximum hiding of her result the game allows.

This result is very striking on its own, as it shows a very interesting and deep structure of the seemingly simple construction of MUBs. Even though all of the bases look...
very similar in its mathematical form, the subtle phase interdependences allow for some of the subsets to deliver truly different results than others.

More than that, the result is interesting from a practical viewpoint as well. While it might be considered as very artificial to introduce a quantum control of the measurement chosen by Alice, this is in fact the way how control works for instance on the IBM quantum computer, where no classical control is available [35]. In the future design of quantum security elements it is more than possible that due to technological reasons, quantum controls will be a standard procedure. In such a case, it will be very important to carefully consider the design of the quantum part so that the selected MUBs are not only secure as designed, but are (reasonably) secure even in the case of coherent control and possible relabeling.

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APPENDIX

A.I. Optimization Algorithm

Given a MUB construction encoded by the unitaries \( \{U_a\}_{a=0}^{d-1} \), we want to estimate the associated optimal strategy that Bob can use to guess Alice’s outcomes in the quantum coin scenario. The optimal strategy would be the result of the following optimization:

\[
P^\text{max}_g = \max_{\rho_B, \{M_a\}_{a=0}^{d-1}} \sum_{a=0}^{d-1} \text{Tr}_{AB} \left[ (\rho_B \otimes \rho_C)CU (|a\rangle \langle a| \otimes M_a) CU^\dagger \right] \\
\text{s.t.} \quad \rho_B \geq 0 \quad \text{Tr} \rho_B = 1 \quad M_a \geq 0 \quad \forall a \in \{0, \ldots, d-1\} \quad \sum_{a=0}^{d-1} M_a = 1,
\]

where the optimization variables are Bob’s probe state \( \rho_B \) and Bob’s POVM elements \( M_a \) corresponding to the outcome \( a \). Also recall that \( \rho_C \) is the control state representing the choice of measurements, and \( CU \) is a controlled unitary used to implement Alice’s measurement settings coherently. The target function of this optimization problem is non-linear, therefore it cannot be solved directly by Semi-Definite Programming (SDP). We therefore cast it as two SDPs, which we run alternatively. In the first SDP we optimize over \( \rho_B \) while \( \{M_a\}_{a=0}^{d-1} \) are constant:

\[
\text{given } \rho_B \quad \{M_a\}_{a=0}^{d-1} = \arg \max \frac{1}{d} \sum_{i,j,a=0}^{d-1} \langle i | M_a | j \rangle \langle a | U_j^\dagger \rho_B U_i | a \rangle \\
\text{s.t. } M_a \geq 0 \quad \forall a \in \{0, \ldots, d-1\} \quad \sum_{a=0}^{d-1} M_a = 1
\]

and in the second one we optimise over \( \rho_B \) while \( \{M_a\}_{a=0}^{d-1} \) are constant:

\[
\text{given } \{M_a\}_{a=0}^{d-1} \quad \rho_B = \arg \max \frac{1}{d} \sum_{i,j,a=0}^{d-1} \langle i | M_a | j \rangle \langle a | U_j^\dagger \rho_B U_i | a \rangle \\
\text{s.t. } \rho_B \geq 0 \quad \text{Tr} \rho_B = 1,
\]

where we simplified the notation with

\[
\sum_{a=0}^{d-1} \text{Tr}_{AB} \left[ (\rho_B \otimes \rho_C)CU (|a\rangle \langle a| \otimes M_a) CU^\dagger \right] = \frac{1}{d} \sum_{i,j,a=0}^{d-1} \langle i | M_a | j \rangle \langle a | U_j^\dagger \rho_B U_i | a \rangle ,
\]

for \( CU = \sum_{i=0}^{d-1} U_i^\dagger \otimes |i\rangle \langle i|, \quad \rho_C = |+\rangle \langle +|, \quad |+\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i\rangle \).

The two SDPs are each guaranteed to converge, the see-saw, however must stop at a ‘convergence parameter’ \( \varepsilon \) that we set to be \( 10^{-6} \); explicitly, the see-saw algorithm is the following:

**Algorithm 1 See-saw**

1: **Initialization:** Generate a random density matrix \( \rho_0 \), distributed according to the Hilbert-Schmidt measure. Set \( P_W = 0 \).
2: **POVM optimization:** Given \( \rho_0 \), solve the SDP with \( \{M_a\}_{a=0}^{d-1} \) as variable, and find the solution \( \{M_a^*\}_{a=0}^{d-1} \).
3: **State optimization:** Given \( \{M_a^*\}_{a=0}^{d-1} \) from step 2, solve the SDP with \( \rho_B \) as variable, and find the solution \( \rho_B^* \) and \( P_W^* \).
4: **Convergence check:**
   - If \( P_W^* - P_W > \varepsilon \), then set \( \rho_0 = \rho_B^* \) and \( P_W = P_W^* \). Repeat from step 2.
   - If \( P_W^* - P_W < \varepsilon \), stop the algorithm. The complete solution is given by \( P_W^*, \rho_B^*, \{M_a^*\}_{a=0}^{d-1} \).

The algorithm is then applied to a large number of initial random points \( \rho_0 \). We observed that for \( \varepsilon \) small enough it yields always the same result \( P_W^* \), suggesting that the see-saw algorithm lower bounds tightly the solution of (A.I.1).
A.II. Optimal strategy

In (5) we considered Alice’s MUB measurements defined as $U_a = \frac{1}{\sqrt{d}} \sum_{i,j=0}^{d-1} \omega^{ai^2 + ij - a^2} |i⟩⟨j|$. Bob’s optimal strategy in this case is:

$$|\psi_B⟩ = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} \omega^{3d^2 - k^2} |k⟩,$$

$$M_a = \frac{|φ_a⟩⟨φ_a|}{⟨φ_a|φ_a⟩},$$

$$|φ_a⟩ = \frac{1}{\sqrt{d}} \sum_{a=0}^{d-1} ⟨j| U_a^† |ψ_B⟩ |a⟩,$$

where $|ψ_B⟩$ is Bob’s (pure) probe state and $\{M_a\}_{a=0}^{d-1}$ are POVM elements of the measurement he uses on the probe state $ρ_C = |+⟩⟨+|$ to guess Alice’s outcome. Note that states $|φ_a⟩$ are not normalized.

Here we show that $\{M_a\}_{a=0}^{d-1}$ is indeed a valid POVM, i.e. $M_a ≥ 0 \ \forall a$ and $\sum_{a=0}^{d-1} M_a = 1$. Positivity is guaranteed by definition. To prove summation to identity we notice that $M_a$ are projectors and span the Hilbert space of Bob if $\{ |φ_j⟩\}_{j=0}^{d-1}$ form an orthonormal basis. Normalization is guaranteed by definition, so it remains to prove orthogonality:

$$⟨φ_i|φ_j⟩ = \frac{1}{d^3} \sum_{a=0}^{d-1} ⟨j| U_a^† |ψ_B⟩ ⟨ψ_B| U_a |i⟩ =$$

$$= \frac{1}{d^3} \sum_{a,k,l=0}^{d-1} \omega^{-(ak^2 + jk - a^2)} \omega^{3d^2 - k^2} \omega^{-3d^2 - l^2} \omega^{al^2 + il - a^2 l}$$

$$= \frac{1}{d^3} \sum_{a,k,l=0}^{d-1} \omega^{-ak^2 - jk + a^2 k + 3d^2 - k^2 - 3d^2 - l^2 + al^2 + il - a^2 l}.$$

In what follows, we will show that for $d > 3$ ($d = 3$ and $d = 2$ are treated separately) the above expression can be simplified using quadratic Gauss sums. In order to do so, we will manipulate the exponents of $ω$. The key idea is to realize that since $ω^d = 1$, we can work with its exponent modulo $d$. Additionally, we introduce a substitution:

$$m = l + k \quad (mod \ d) \quad and \quad n = l - k \quad (mod \ d),$$

and two constants

$$α = 3^{d-2} \equiv 3^{-1} \quad (mod \ d) \quad and \quad β = 2^{d-2} \equiv 2^{-1} \quad (mod \ d).$$

From these definitions it follows that

$$l \equiv β(m + n) \quad (mod \ d), \quad 3α \equiv 1 \quad (mod \ d),$$

$$k \equiv β(m - n) \quad (mod \ d), \quad 2β \equiv 1 \quad (mod \ d),$$

$$l^2 - k^2 \equiv mn \quad (mod \ d), \quad il - jk \equiv βm(i - j) + βn(i + j) \quad (mod \ d),$$

$$l^3 - k^3 \equiv β^2 n(3m^2 + n^2) \quad (mod \ d).$$

We will also use the quadratic Gauss sum:

$$\sum_{a=0}^{d-1} \omega^{a^2 m} = \begin{cases} (\frac{m}{d}) \sqrt{d} & \text{if } m \neq 0 \ (mod \ d) \\ \frac{d}{d} & \text{if } m \equiv 0 \ (mod \ d) \end{cases},$$

where $(\frac{m}{d})$ is the Legendre symbol:

$$(\frac{m}{d}) = \begin{cases} 1 & \text{if } \exists n : m \equiv n^2 \ (mod \ d) \\ -1 & \text{if } \nexists n : m \equiv n^2 \ (mod \ d) \end{cases}, \quad \text{and} \quad ε_d = \begin{cases} 1 & \text{if } d \equiv 1 \ (mod \ 4) \\ i & \text{if } d \equiv 3 \ (mod \ 4) \end{cases}.$$
After the substitution, the expression reads:

\[
\langle \phi_i | \phi_j \rangle = \frac{1}{d^3} \sum_{a,m,n=0}^{d-1} \omega^{amn-a^2n-a\beta^2n^3-\beta^2m^2n+\beta m(i-j)+\beta n(i+j)} \\
= \frac{1}{d^3} \sum_{m,n=0}^{d-1} \omega^{a\beta^2n^3-\beta^2m^2n+\beta m(i-j)+\beta n(i+j)} \sum_{a=0}^{d-1} \omega^{amn-a^2n}.
\]

The sum over \(a\) is a quadratic Gauss sum:

\[
\sum_{a=0}^{d-1} \omega^{-a^2n+amn} = \sum_{a=0}^{d-1} \omega^{-n(a-\beta m)^2} \omega^{\beta^2m^2n} \\
= \omega^{\beta^2m^2n} \sum_{a=0}^{d-1} \omega^{-a^2n} \\
= \begin{cases} \\
\omega^{\beta^2m^2n} \left( \frac{-n}{d} \right) \varepsilon_d \sqrt{d} & \text{if } n \not\equiv 0 \pmod{d} \\
n \varepsilon_d \sum_{n=1}^{\frac{d}{2}} \left( \frac{n}{d} \right) \omega^{12(d-2)n^3-nj} \varepsilon_d \sqrt{d} \varepsilon_d \sqrt{d} & \text{if } n \equiv 0 \pmod{d} \\
\end{cases},
\]

where the second equality follows from the fact that \((a-\beta m)^2\) iterates over the same values \(\pmod{d}\) as \(a^2\). Substituting this expression in the previous one, we obtain:

\[
\langle \phi_i | \phi_j \rangle = \frac{1}{d^3} \sum_{m=0}^{d-1} \omega^{\beta m(i-j)} \sum_{n=1}^{d-1} \varepsilon_d \sqrt{d} \left( \frac{-n}{d} \right) \omega^{-a\beta^2n^3+\beta n(i+j)} + d \\
= \frac{\delta_{ij}}{d} \varepsilon_d \sum_{n=1}^{\frac{d}{2}} \left( \frac{n}{d} \right) \omega^{12(d-2)n^3-nj} + 1,
\]

which shows that they are orthogonal as requested. We then show that this construction gives a guessing probability \(P_g = 1\):

\[
P_g = \sum_{k=0}^{d-1} \text{Tr}_{AB} \left[ CU \uparrow (\rho_B \otimes \rho_C) \right] C U \left( \langle k | \langle k \otimes M_k \rangle \right) \\
= \sum_{k=0}^{d-1} \text{Tr}_{AB} \left( \sum_{a=0}^{d-1} U_a \uparrow \otimes |a\rangle \langle a| \right) \left( \langle \psi_B | \psi_B \rangle \otimes |1\rangle \langle 1| \right) \sum_{b=0}^{d-1} U_b \otimes |b\rangle \langle b| \\
= \text{Tr}_{B} \sum_{k=0}^{d-1} \left( \frac{1}{\sqrt{d}} \sum_{a=0}^{d-1} \langle k | U_a \uparrow | \psi_B \rangle |a\rangle \right) \left( \frac{1}{\sqrt{d}} \sum_{b=0}^{d-1} \langle \psi_B | U_b | k \rangle |b\rangle \right) |\phi_k\rangle \langle \phi_k| \\
= \text{Tr}_{B} \sum_{k=0}^{d-1} \langle \phi_k| \langle \phi_k| |\phi_k\rangle \langle \phi_k| \\
= \sum_k \langle \phi_k| \langle \phi_k| \\
= \frac{1}{d} \sum_{a,k=0}^{d-1} \langle b| a\rangle \langle k | U_a \uparrow | \psi_B \rangle \langle \psi_B | U_b | k \rangle \\
= \frac{1}{d} \sum_{a=0}^{d-1} \langle \psi_B | U_a \uparrow \langle k | U_a \uparrow | \psi_B \rangle \langle \psi_B | U_b | k \rangle \\
= \frac{1}{d} \sum_{a=0}^{d-1} \langle \psi_B | U_a \uparrow | \psi_B \rangle = 1.
\]

**A.III. Optimal strategy in the quantum coin scenario for \(d = 3\)**

In **A.II** we showed that Bob can guess with probability one for \(d > 3\). For the case \(d = 2\) the optimal strategy can be found in [32], For \(d = 3\), the proof needs to be adapted due to the fact that a multiplicative inverse \(\pmod{3}\) of 3 does not exist; we then use \(\omega = e^{\frac{2\pi i}{3}}\) for Alice’s MUB construction \(U_a = \frac{1}{\sqrt{d}} \sum_{i,j=0}^{d-1} \omega^{ai^2+i+aj} |i\rangle |j\rangle\) and with
\( \omega_9 = e^{2\pi i} \) we define Bob’s strategy as:

\[
M_a = \frac{|\phi_a\rangle\langle \phi_a|}{\langle \phi_a|\phi_a \rangle},
\]

\[
|\phi_a\rangle = \frac{1}{\sqrt{d}} \sum_{a=0}^{2} \langle j | U_a \rangle |\psi_B \rangle |a\rangle,
\]

\[
|\psi_B\rangle = \frac{1}{\sqrt{d}} \sum_{k=0}^{2} \omega_9^k |k\rangle,
\]

The proof follows exactly the same steps of A.II.1, with all the substitutions remaining valid, with the exception of

\[
\omega_9^{k^3-\ell^3} = \omega_9^{-\beta^2 n(3m^2+n^2)} = \omega_9^{-\beta^2 m^2 \omega_9^{-\beta^2 n^3}} = \omega_9^{-\beta^2 m^2 \omega_9^{-7 n^3}},
\]

where we made use of the fact that \( \beta = 5 \) is the multiplicative inverse of 2 (mod 3) and (mod 9). We then get

\[
\langle \phi_i | \phi_j \rangle = \frac{\delta_{ij}}{3} \left[ \frac{\xi_3}{\sqrt{3}} \sum_{n=1}^{2} \left( \frac{n}{d} \right) \omega_9^{n^3-3nj} + 1 \right],
\]

concluding the proof.

**A.IV. Classical coin**

In the classical case, the control state can only contain the information about the basis Alice measures in. Bob’s optimal measurement is therefore a simple projection onto the computational basis, which reveals Alice’s measurement basis \( i \), followed by a map \( \tilde{n}(i) \) that associates to each basis \( i \) the most probable outcome of Alice for that basis. Note that this also means that the maximum guessing probability in the classical scenario does not depend on the labelling of the outcomes, since the labelling does not change the probability of the most probable outcome. Formally:

\[
\tilde{n}(i) := \arg\max_{j \in \{0,\ldots,d-1\}} P_A(j | U_i),
\]

\[
P_A(j | U_i) = \text{Tr} \left( \rho_B U_i |j\rangle \langle j | U_i^\dagger \right),
\]

\[
M_i = \sum_{j=0}^{d-1} \langle i | \tilde{n}(i) \rangle.
\]

With these definitions we can state the problem as follows:

\[
P_g^c := \max_{\rho_B, \{M_k\}_{k=0}^{d-1}} \text{Tr} \left( \rho_B \otimes \frac{1}{d^2} \right) \text{CU} \left( \langle k | \otimes M_k \right) \text{CU}^\dagger \right] =
\]

\[
= \max_{\rho_B} \max_{n_0, n_1, \ldots, n_d} \frac{1}{d} \text{Tr} \left( \sum_{j=0}^{d-1} \rho_B U_j |n_j\rangle \langle n_j | U_j^\dagger \right) =
\]

\[
= \frac{1}{d} \max_{\rho_B} \text{Tr} \left( \sum_{j=0}^{d-1} \rho_B U_j |\tilde{n}(j)\rangle \langle \tilde{n}(j) | U_j^\dagger \right) =
\]

\[
= \frac{1}{d} \lambda_{\text{max}} \left[ \sum_{j=0}^{d-1} U_j |\tilde{n}(j)\rangle \langle \tilde{n}(j) | U_j^\dagger \right].
\]
where $\lambda_{\text{max}}[T]$ is the largest eigenvalue of a matrix $T$. For small dimensions, the maximum probability can be found by evaluating all possible mappings $\tilde{n}(j)$ (there are $d^d$ of them). This however quickly becomes infeasible, therefore we look for an upper bound:

$$P_g^c = \frac{1}{d} \left( 1 + \lambda_{\text{max}} \left[ \sum_{j=0}^{d-1} U_j |\tilde{n}(j)\rangle\langle \tilde{n}(j)| U_j^\dagger - 1 \right] \right);$$

to simplify the notation we define

$$T_j := U_j |\tilde{n}(j)\rangle\langle \tilde{n}(j)| U_j^\dagger - \frac{I}{d},$$

$$T := \sum_{j=0}^{d-1} T_j,$$

which satisfy the following properties:

$$\text{Tr}(T_j) = 0 \quad \forall j \in \{0, \ldots, d - 1\},$$
$$\text{Tr}(T_i^\dagger T_j) = 0 \quad \forall i \neq j \in \{0, \ldots, d - 1\},$$
$$\text{Tr}(T_j^2) = \frac{d - 1}{d} \quad \forall j \in \{0, \ldots, d - 1\},$$
$$\text{Tr}(T^2) = \text{Tr}\left( \sum_{i,j=0}^{d-1} T_i^\dagger T_j \right) = \sum_{i=0}^{d-1} \text{Tr}\left( T_i^\dagger T_i \right) + \sum_{i,j=0}^{d-1} \text{Tr}\left( T_i^\dagger T_j \right) = d - 1.$$

The guessing probability with a classical coin can be then expressed as

$$P_g^c : = \frac{1}{d} \left( 1 + \lambda_{\text{max}}[T] \right).$$

Since $T$ is trace-less and Hermitian, its largest eigenvalue is positive. We then use the following inequality:

$$\text{Tr}(T^2) = \lambda_{\text{max}}^2[T] \text{Tr}\left( \frac{T^2}{\lambda_{\text{max}}^2[T]} \right) \geq \lambda_{\text{max}}^2[T] \left( 1 + \frac{1}{d - 1} \right) = \lambda_{\text{max}}^2[T] \frac{d}{d - 1};$$

where we denoted the space of Hermitian matrices of order $d - 1$ by $M_{d-1}$. Substituting the trace of $T^2$ we get the desired upper bound:

$$\lambda_{\text{max}}[T] \leq \frac{d - 1}{\sqrt{d}}, \quad (A.IV.1)$$
$$P_g^c \leq \frac{1}{d} \left( 1 + \frac{d - 1}{\sqrt{d}} \right). \quad (A.IV.2)$$

**A.V. Numeric search**

a. Classical coin

When considering a classical coin, the optimal strategy is given by searching over all possible maps $\tilde{n} : Z_d \to Z_d$ and taking the largest eigenvalue of the matrix $T = \sum_{j=0}^{d-1} U_j |\tilde{n}(j)\rangle\langle \tilde{n}(j)| U_j^\dagger - 1$. There are $d^d$ such mappings, and we
could perform this extensive search for \( d = 2, 3, 5, 7 \), obtaining exact bounds for these dimensions. In other dimensions lower bounds were obtained by applying the see-saw algorithm (A.1) with \( \rho_C = 1/d \) and randomized initial points. This algorithm tends to get stuck in local maxima; however in the dimensions in which we could perform the extensive search we observed that the see-saw algorithm returned the maximum value more often then by a random sampling of \( \tilde{n} \) in the space of maps \( \mathbb{Z}_d \rightarrow \mathbb{Z}_d \).

b. Quantum coin

For the quantum coin, for the convergence parameter \( \varepsilon \) small enough (10\(^{-6}\)) we didn’t observe convergences to local maxima different from the global maximum. Differently from the classical case, the choice of unitaries changes the value of the maximum. We then search for the smallest such value among all possible unitary constructions. The space over which we search is given by choosing \( d \) unitaries out of the \( d + 1 \) available from the WF construction, and by relabeling, i.e. applying a permutation matrix to each unitary. For \( d = 3, 5 \) we searched over all possible permutations, for \( d = 7 \) we only considered cyclic permutations, while for higher dimension we randomly sampled over the space of permutation matrices. Each search is performed for all \( d + 1 \) choices of \( d \) unitaries. We observed that the WF unitaries give the lowest value when the excluded unitary is the identity.