Chiral Transition and Baryon-number Susceptibility \(^a\)

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We discuss the baryon-number susceptibility \(\chi_B\) and related topics which include the density fluctuations around the critical point of the chiral transition at finite temperature \(T\) and baryon density \(\rho_B\). Phenomenological implications of the density fluctuations near the first-order chiral transition at finite density are also discussed.

1 Introduction

When exploring a phase transition in any physical system, the study of fluctuations of physical quantities, especially ones related to the order parameter is as important as that of the phase diagram for the system in equilibrium. The fluctuations of observables are related with dynamical phenomena including the transport properties of the system. The chirally restored and deconfined phase is expected to be created dynamically in the intermediate stage of the ultra-relativistic heavy ion collisions and in the early universe. Therefore the study of the fluctuations has a great relevance to phenomenology.

In the present report, we discuss the baryon-number susceptibility \(\chi_B\) \(^1\) and related topics which include the density fluctuations around the critical point of the chiral transition at finite temperature \(T\) and baryon density \(\rho_B\).

2 QCD phase diagram in \((T, \rho_B)\)-plane and the vector coupling

The lattice simulations of QCD \(^4\) suggest that the order and even the existence of the phase transition(s) at finite temperature \(T\) are largely dependent on the number of the active flavors when the physical current quark masses are used: For \(m_u \sim m_d \sim 10\text{MeV} < 100\text{MeV} \lesssim m_s\), the phase transition may be weak 1st order or 2nd order or not exist. The lattice QCD is, unfortunately, still not matured enough to predict a definite thing about the phase transition at finite baryon density \(\rho_B\) (or chemical potential \(\mu\)).

Low-energy effective models \(^3\) and the chiral random-matrix theory \(^7\) have given suggestive pictures of the phase diagram of QCD in the \(T-\mu\) (or \(T-\rho_B\)) plane. For example, the NJL model \(^8\) well describes the gross features of the \(T\)

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dependence of the quark condensates of the lightest three quarks as given by
the lattice QCD, and predict that the chiral transition at \( \mu \neq 0 \) is of rather
strong first order at low temperatures \( T \), say, lower than 50 MeV, provided that
the vector coupling between the quarks as given by \( g_v/2 \cdot (\overline{q} \gamma_{\mu} q)^2 \) is absent \(^4\).
\( ^4 \) As a matter of fact, the strength and even the existence of the 1st order
transition are strongly dependent on the strength of the vector coupling \( g_v \)
\( ^5 \); the vector term prevents a high-density state.

The reason why the vector coupling weakens the phase transition and
postpones the chiral restoration is understood as follows. Thermodynamics
tells us that when two phases I and II are coexistent, their temperatures \( T_{I,II} \),
pressures \( P_{I,II} \) and the chemical potentials \( \mu_{I,II} \) are the same:

\[
T_I = T_{II}, \quad P_I = P_{II}, \quad \mu_I = \mu_{II}.
\] (2.1)

If the phase I (II) is the chirally broken (chirally restored) phase, the last
equality further tells us something because \( \mu_{I,II} \) at vanishing temperature are
given by \( \mu_i = \sqrt{M^2 + p_{F_i}^2} \), \( (i = I, II) \), where \( p_{F_i} \) is the Fermi momentum of
the \( i \)-th phase, and \( M \) and \( m \) are the constituent quark mass and the current
quark mass that vanishes in the chiral limit. One readily sees that \( p_{F_I} < p_{F_{II}} \),
accordingly, \( \rho_{B_I} < \rho_{B_{II}} \), i.e., the chirally restored phase is in higher density
than the coexistent broken phase. The vector coupling above gives rise to
a repulsion proportional to the density squared i.e., \( g_v \rho_B^2/2 \) which is bigger
in the restored phase than in the broken phase. Thus the vector coupling
weakens and/or postpone the phase transition of the chiral restoration at low
temperatures.

Since the vector coupling contribute to the energy repulsively, it also sup-
press the density fluctuations, or the baryon-number susceptibility, which is
the main subject of the present report.

3  Baryon-number susceptibility

The baryon-number susceptibility \( \chi_B \) is the measure of the response of the
baryon number density \( \rho_B = \sum_{i=1}^{N_f} \rho_i \) to infinitesimal changes in the quark
chemical potentials \( \mu_i \): \( ^6 \)

\[
\chi_B(T, \mu) = \langle \sum_{i=1}^{N_f} \frac{\partial}{\partial \mu_i} \sum_{i=1}^{N_f} \rho_i \rangle = \langle \langle N_B^2 \rangle \rangle / VT, \quad (3.1)
\]

\(^c\) The first order chiral transition in the density direction at low temperatures is also obtained
in the chiral random-matrix theory. \( ^7 \)
where \( N_B \) is the baryon-number operator given by \( N_B \equiv \sum_{i=1}^{N_f} N_i \), with

\[
\rho_i = \text{Tr} N_i \exp[-\beta (H - \sum_{i=u,d} \mu_i N_i)] / V \equiv \langle \langle N_i \rangle \rangle / V
\]  

(3.2)

the \( i \)-th quark-number density, \( V \) the volume of the system and \( \beta = 1/T \).

In the following, we shall confine ourselves to the \( SU_f(N_f) \)-symmetric case; \( \mu_u = \mu_d = \mu_s = \ldots = \mu \).

The baryon-number susceptibility at \( \rho_B \neq 0 \) is related with the (isothermal) compressibility of the system

\[
\kappa_T \equiv -N_B^{-1} \langle \partial V / \partial \mu \rangle_{T,N_B} = \frac{1}{\beta^2},
\]

which tells that if \( \chi_B \) is large and so is the density fluctuation, the system is easy to compress.

Another physical meaning of \( \chi_B \) is that it is the density-density correlation which is nothing but the 0-0 component of the vector-vector correlations or fluctuations.

\[
\chi_B(T, \mu_q) = \beta \int dx S_{00}(0, x),
\]  

(3.3)

where \( S_{\mu\nu}(t, x) = \langle \langle j_\mu(t, x) j_\nu(0, 0) \rangle \rangle \), with \( j_\mu(t, x) = \bar{q}(t, x) \gamma_\mu q(t, x) \) being the current operator. Using the fluctuation-dissipation theorem, one has

\[
\chi_B(T, \mu_q) = -\lim_{k \to 0} L(0, k),
\]  

(3.4)

where \( L(\omega, k) \) is the longitudinal component of the retarded Green’s function or the response function in the vector channel;

\[
R_{\mu\nu}(\omega, k) = \text{F.T.}(-i\theta(t) \langle \langle [j_\mu(t, x), j_\nu(0, 0)] \rangle \rangle).
\]

The lattice simulations of QCD\textsuperscript{[11]} show that \( \chi_B \) at \( \mu = 0 \) is suppressed in the low temperature phase, increases with \( T \) sharply around the critical point of the chiral transition and takes almost the free quark gas value then saturates. This behavior may be understood intuitively and roughly as follows:\textsuperscript{[4]}

In the confined phase at low \( T \), the density fluctuation picks up the Boltzmann factor \( e^{-M_N/T} \) with \( M_N \) being the nucleon mass, which is much smaller than the factor \( e^{-M_q/T} \) with \( M_q \) being the current quark mass (constituent quark mass), which factor will be picked up in the deconfined and chirally restored (chirally broken) phase.

Our point here is however that the nature of the chiral transition and also presence (or absence) of the vector coupling affect the baryon-number susceptibility, especially when \( \rho_B \neq 0 \), for which the lattice data is not available so far.
3.1 Free quark gas

It is instructive to examine $\chi_B(T, \mu)$ in the simple free-quark gas model:

$$\rho_B = 2N_f N_c \int \frac{d\mathbf{p}}{(2\pi)^3} (n(T, \mu) - \bar{n}(T, \mu))$$

(3.5)

where $n(T, \mu) = 1/[\exp \beta(E_p - \mu) + 1]$ and $\bar{n}(T, \mu) = 1/[\exp \beta(E_p + \mu) + 1]$ with $E_p = \sqrt{M^2 + p^2}$, and $N_c = 3$ is the number of the colors. Then one readily obtains

$$\chi_B(T, \mu) = 2N_f N_c \beta \int \frac{d\mathbf{p}}{(2\pi)^3} \left\{ n(1 - n) + \bar{n}(1 - \bar{n}) \right\} \equiv \chi_B^{(0)}(T, \mu),$$

(3.6)

which is reduced to

$$\chi_B^{(0)}(T, 0) \equiv \chi_B^{(0)}(T) = 4N_f N_c \beta \int \frac{d\mathbf{p}}{(2\pi)^3} \frac{\exp(E_p/T)}{[\exp(E_p/T) + 1]^2},$$

(3.7)

at $\mu = 0$.

If $M(T)$ is decreased as in the chiral restoration, $\chi_B$ increases and reaches $N_f T^2$ at $M(T) = 0$: The enhancement is, however, found to be modest and not so large as obtained in the lattice simulations.

3.2 Model calculation

To demonstrate the relevance of the nature of the chiral transition and the presence of the vector coupling to $\chi_B$, we perform a calculation with an effective model, which is given by adding the vector-coupling terms to the Nambu–Jona-Lasinio model:

$$\mathcal{L} = \bar{q}(i\gamma \cdot \partial - m)q + \sum_{a=0}^{N_f^2 - 1} \frac{g_s}{2} [\bar{q} \lambda_a q]^2 + (\bar{q} i \lambda_a \gamma q)]$$

$$- \frac{g_V}{2} \sum_{a=0}^{N_f^2 - 1} [\bar{q} \lambda_a \gamma q]^2 + (\bar{q} \lambda_a \gamma_5 q)]$$

(3.8)

The realistic value of $g_V$ used in the literature is roughly in the range of $g_V \Lambda^2 = 5 \sim 9$.

In the self-consistent mean field approximation, the constituent mass (dynamical mass) $M_i(T, \mu_i)$, the quark condensates $\langle \langle q_i q_i \rangle \rangle$ and the quark density
ρ\textsubscript{i} are all coupled with the vector coupling g\textsubscript{V} and determined by the following equations;

\[ M\textsubscript{i} = m\textsubscript{i} - 2g\textsubscript{V} \langle \langle \bar{q}\textsubscript{i}q\textsubscript{i} \rangle \rangle, \quad (3.9) \]

\[ \langle \langle \bar{q}\textsubscript{i}q\textsubscript{i} \rangle \rangle = -2Nc \int \frac{dp}{(2\pi)^3} \{ 1 - n\textsubscript{i}(T, \tilde{\mu}) - \bar{n}\textsubscript{i}(T, \tilde{\mu}) \}, \quad (3.10) \]

\[ \rho\textsubscript{i} = 2Nc \int \frac{dp}{(2\pi)^3} (n\textsubscript{i}(T, \tilde{\mu}) - \bar{n}\textsubscript{i}(T, \tilde{\mu})), \quad (3.11) \]

where it is to be noted that the shifted chemical potential \( \tilde{\mu} = \mu - 2g\textsubscript{V} \rho\textsubscript{i} \) enters the distribution functions instead of the naive one, \( \mu \).

Simply differentiating these equations with respect to \( \mu \), one obtains \( \chi_B(T, \mu) \).

It is noteworthy that when \( \mu \neq 0 \) there arises a coupling between \( \chi_B \) and the scalar-density susceptibility \( \chi_s \) owing to the non-vanishing “vector-scalar susceptibility” \( \chi_{V S} \). They are defined by

\[ \chi_s = -\frac{d\langle \langle \bar{q}q \rangle \rangle}{dm} = \beta \int dx (\langle \langle \bar{q}(0, x)q(0, x)\bar{q}(0, 0)q(0, 0) \rangle \rangle), \quad (3.12) \]

\[ \chi_{V S} = \frac{\partial \langle \langle \bar{q}q \rangle \rangle}{\partial \mu_B} = \beta \int dx (\langle \langle \bar{q}(0, x)\gamma_0 q(0, x)\bar{q}(0, 0)q(0, 0) \rangle \rangle), \quad (3.13) \]

respectively. \( \chi_s \) represents the fluctuation of the order parameter of the chiral transition, and is related with the sigma meson propagator. The differentiation leads to the coupled equation

\[
\begin{pmatrix}
1 + 2g_V \chi_B^{(0)} \\
4g_s g_V \chi_s^{(0)}
\end{pmatrix}
\begin{pmatrix}
-\chi_s^{(0)} \\
1 - 2g_s \chi_s^{(0)}
\end{pmatrix}
= \begin{pmatrix}
\chi_B^{(0)} \\
-2g_s \chi_{V S}^{(0)}
\end{pmatrix}
\]

\[ \quad (3.14) \]

where \( \chi_B^{(0)}(T, \mu_q) \) is the zero-th order baryon-number susceptibility given before,

\[ \chi_{V S}^{(0)} = -2Nc \sum_{i=1}^{N_f} \int \frac{dp}{(2\pi)^3} \left\{ n_i(1 - n_i) - \bar{n}_i(1 - \bar{n}_i) \right\} \]

\[ \quad (3.15) \]

the zero-th order vector-scalar one and

\[ \chi_s^{(0)} = 2Nc \sum_{i=1}^{N_f} \int \frac{dp}{(2\pi)^3} \beta M_i^2 E_i^3 \left\{ n_i(1 - n_i) + \bar{n}_i(1 - \bar{n}_i) \right\} \]

\[ + \beta \frac{M_i^2}{E_i} \left\{ n_i(1 - n_i) + \bar{n}_i(1 - \bar{n}_i) \right\} \]

\[ \quad (3.16) \]
the scalar-density susceptibility in the zero-th order. One should note that when \( \mu = 0 \), then \( \chi^{(0)}_{V,S} \) vanishes. One can readily obtain \( \chi_q \) and \( \chi_s \) in terms of \( \chi^{(0)}_{V,S} \), so we do not write them down to save the space.

### 3.3 (A) Finite density case \( g_V = 0 \)

To see how the nature of the chiral transition can affect the behavior of \( \chi_B \), let us first take a simple case where \( g_V \) is negligible.

The essential point lies in the fact that the distribution function \( n(T, \mu) = \left[ e^{\beta(E_p - \mu)} + 1 \right]^{-1} \) depends on \( \mu \) not only explicitly but also implicitly through the dynamical mass \( M(T, \mu) \) in \( E_p = \sqrt{M^2 + p^2} \); hence

\[
T \frac{\partial n}{\partial \mu} = n(1 - n) - \frac{M}{E_p} \frac{\partial M}{\partial \mu} n(1 - n). \tag{3.17}
\]

Thus

\[
\chi_B(T, \mu) = 2N_c \sum_{i=1}^{N_f} \beta \int \frac{d\mathbf{p}}{(2\pi)^3} \left\{ n_i(1 - n_i) + \bar{n}_i(1 - \bar{n}_i) \right\} - \frac{M_i}{E_{ip}} \left( n_i(1 - n_i) - \bar{n}_i(1 - \bar{n}_i) \right). \tag{3.18}
\]

The notable point is the presence of the derivative \( \frac{\partial M}{\partial \mu} \). The constituent mass \( M \) varies with the quark condensate \( \langle \langle \bar{q}q \rangle \rangle \) by Eq. (3.9), the order parameter of the chiral transition. We saw in §2 that the chiral transition at low temperatures is likely to be of first order in the chemical potential direction. It means that the derivative \( \frac{\partial M}{\partial \mu} \) diverges at the critical point at low temperatures, hence so does the susceptibility \( \chi_B \).

### 3.4 (B) Zero-density case with \( g_V \neq 0 \)

Putting \( \mu = 0 \) into the expressions one gets

\[
\chi_B = \frac{\chi^{(0)}_B(T)}{1 + 2g_v \chi^{(0)}_B(T)}, \tag{3.19}
\]

where \( \chi^{(0)}_B(T) \) is the susceptibility for the free-quark gas. The denominator of \( \chi_B \) is essentially the inverse of the propagator of the vector meson in the ring approximation at the vanishing four momenta. The above expression shows that \( \chi_B \) is suppressed by the vector coupling (\( g_v > 0 \)). This is reasonable.
at least for a system with a finite $\mu$; because the system becomes hard to compress when the vector coupling is present, the number fluctuations will be suppressed. Recall also that $\chi_B$ is proportional to the compressibility $\kappa_T$.

The comparison with the lattice data shows that the vector coupling is rather small in the high temperature phase. It is interesting that this suppression of the vector coupling at high temperatures is consistent with the observation that the screening masses of the vector modes obtained in the lattice simulations almost coincides with $2\pi T$, the lowest screening mass of the $q\bar{q}$ system in the chiral limit; a similar result is also obtained in the instanton approach.

4 Estimate of $g_V$ and $g_S$ in the lattice QCD

Boyd et al. once extracted the effective coupling constants $g_V$ and $g_S$ from the lattice data using the expressions given in the NJL model as given above;

$$\chi_B = \chi_B^{(0)}/(1 + g_V \chi_B^{(0)})$$

and

$$\chi_S = \chi_S^{(0)}/(1 - g_S \chi_S^{(0)})$$.

They concluded that when $T \sim T_C$, $g_S$ is much bigger than $g_V$; $g_S \approx 4g_V$. This result is consistent with our analysis and the behavior of the screening masses.

5 Implications to phenomenology

Kumagai, Miyamura and Sugitate discussed the implications of the baryon number susceptibility and the strangeness susceptibility to the observables in the relativistic heavy-ion collisions. They argued that in the stopping region where the chemical potential is large, large fluctuations of the baryon and the strangeness numbers may be a signature of the chiral transition.

Here we wish to also indicate that the large number fluctuations cause those in the scalar channel (the sigma meson channel) at finite density.

The observability of the possible large density fluctuations caused as a critical phenomenon is discussed by several authors.

6 Summary and concluding remarks

We have examined the baryon-number susceptibility $\chi_B$ as an observable which reflects the confinement-deconfinement and the chiral phase transitions in hot and/or dense hadronic matter.

The suppression of $\chi_B$ at low temperatures and steep rise around the critical temperature as shown in the lattice QCD may be roughly attributed...
to the confinement-deconfinement transition. Nevertheless, we have shown that such a behavior of $\chi_B$ is also affected by the chiral transition.

Since $\chi_B$ is a measure of the rate of the density fluctuation in the system, the chiral transition at finite chemical potential especially leads to an interesting phenomenological consequence to $\chi_B$. When the vector coupling is small, the chiral transition at low temperatures is of first order in the density direction, which implies a divergent behavior of $\chi_B$, accordingly a huge density fluctuations. We have emphasized that such a large enhancement of the fluctuation can be also expected for the scalar density fluctuations due to the scalar-vector mixing at finite density. Such a large enhancement may leads to an enhancement of the sigma-meson production $^7$. The above phenomena all have relevance to experiments to be done in RHIC and LHC.

We have indicated that the nature of the chiral transition as to the first order or not etc is sensitively dependent on the strength of the vector coupling. An analysis of the lattice data suggests that the vector coupling is small in comparison with the scalar coupling at high temperature.

The susceptibility $\chi_B$ is nothing but the generalized susceptibility $\chi(\omega, k)$ at $\omega = k = 0$. One should examine $\chi(\omega, k)$ in the whole region of $\omega$ and $k$ to get more information about the vector correlations and the density fluctuations.

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$^a$ As for the significance of the sigma meson for the chiral transition at finite $T$ and/or $\rho_B$, see $^4$. 

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