On the Smarr formula
for rotating dyonic black holes

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Abstract

We revisit the derivation by Tomimatsu of the generalized Komar integrals giving the mass and angular momentum of rotating Einstein-Maxwell black holes. We show that, contrary to Tomimatsu’s claim, the usual Smarr formula relating the horizon mass and angular momentum still holds in the presence of both electric and magnetic charges. The simplest case is that of dyonic Kerr-Newman black holes, for which we recover the modified Smarr formula relating the asymptotic mass and angular momentum, the difference between asymptotic and horizon masses being equal to the sum of the two Dirac string masses. Our results apply in particular to the case of dyonic dihole solutions which have been investigated recently.

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1 Introduction

The Smarr formula [1] relating the mass, angular momentum, entropy and electric charge of black holes was originally designed for the electrically charged Kerr-Newman solution. Its possible generalizations were extensively analyzed by Carter [2] on the basis of Komar integrals [3]. More general solutions describing axisymmetric configurations of multiple rotating black holes (possibly joined by strings) endowed with electric and also magnetic charges were discussed recently. These solutions usually have a simple description in terms of Ernst potentials, while the metric and the electromagnetic potentials are rather complicated. For such situations Tomimatsu designed his original formulas [4, 5] expressing black hole parameters in terms of both the metric variables and the Ernst potentials taken on the symmetry axis. These formulas suggested in 1984 were successfully applied for multiple electrically charged rotating black holes [6].

On the other hand, when the Tomimatsu formulas (completed by an analogous expression for magnetic charge) were applied to multi-dyons [5, 7, 8, 9, 10], it was observed that the resulting values for the black hole parameters failed to obey the standard Smarr relation, but obeyed a generalized Smarr relation with both electric and magnetic contributions. However, the derivation by Tomimatsu [5] gives little details of the underlying calculations, so to clarify the situation a new derivation is necessary. Here such a derivation is presented, showing that in the original Tomimatsu formulas an important term is missing. Correcting the Tomimatsu formulas, we obtain a new version which, when applied to dyons, leads to the standard Smarr relation between the local horizon mass, angular momentum and electric charge.

In passing we establish the crucial role played by the Dirac strings associated with magnetic monopoles in the mass and angular momentum balance equations. We show that for the Kerr-Newman solution with both electric and magnetic charges the Dirac strings are endowed with non-zero generalized Komar masses which should be taken into account in the Smarr formula for the total mass. We also find that the symmetric choice of gauge for the vector potential (with both North and South pole Dirac strings present with equal weights) for dyons is essential to achieve the total angular momentum balance of the configuration.
2 Generalized Komar mass and angular momentum

We first review the generalized Komar formulas [2] giving the masses and angular momenta of extended sources of Einstein-Maxwell fields. The Komar mass and angular momentum for an asymptotically flat, stationary, axisymmetric configuration are given by the integrals over a spacelike 2-surface at infinity [3,4]:

\[ M = \frac{1}{4\pi} \oint_{\Sigma_{\infty}} D^{\nu} k^{\mu} d\Sigma_{\mu\nu}, \quad (2.1) \]
\[ J = -\frac{1}{8\pi} \oint_{\Sigma_{\infty}} D^{\nu} m^{\mu} d\Sigma_{\mu\nu} \quad (2.2) \]

where \( k^{\mu} = \delta^{\mu}_{t} \) and \( m^{\mu} = \delta^{\mu}_{\phi} \) are the Killing vectors associated with time translations and rotations around the z-axis.

Because the integrand \( D^{\nu} k^{\mu} \) is antisymmetric, one can apply the Ostrogradsky theorem to transform

\[ M = \sum_{n} \frac{1}{4\pi} \oint_{\Sigma_{n}} D^{\nu} k^{\mu} d\Sigma_{\mu\nu} + \frac{1}{4\pi} \int_{D_{n}} D^{\nu} k^{\mu} dS_{\mu}, \quad (2.3) \]

where \( \Sigma_{n} \) are the spacelike surfaces bounding the various sources, and the second integral is over the bulk. Using again the fact that \( k \) is a Killing vector and the Einstein equations, we obtain

\[ D_{\nu} D^{\nu} k^{\mu} = -[D_{\nu}, D^{\mu}] k^{\nu} = -R^{\mu}_{\nu} k^{\nu} = -8\pi T^{\mu}_{\nu} k^{\nu}. \quad (2.4) \]

Here \( T^{\mu}_{\nu} \) is the electromagnetic energy-momentum tensor

\[ T^{\mu}_{\nu} = \frac{1}{4\pi} \left[ F^{\mu\rho} F^{\nu}_{\rho} - \frac{1}{4} \delta^{\mu}_{\nu} F^{\rho\sigma} F_{\rho\sigma} \right], \quad (2.5) \]

with \( F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} \). It follows that the bulk contribution to the Komar mass [2,3], which after Tomimatsu [5] we will call \( M^{E} \), may be transformed to

\[ M^{E} = \frac{1}{4\pi} \int_{D_{n}} D^{\nu} k^{\mu} dS_{\mu} = -2 \int_{D_{n}} T^{t}_{t} \sqrt{|g|} d^{3}x \]

\[ = \frac{1}{4\pi} \int D_{\nu} D^{\nu} k^{\mu} dS_{\mu} = -2 \int T^{t}_{t} \sqrt{|g|} d^{3}x \]

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We use the metric signature (−+++ + ) and the convention \( d\Sigma_{\mu\nu} = 1/2\sqrt{|g|} \epsilon_{\mu\nu\lambda\tau} dx^{\lambda} dx^{\tau} \) with \( \epsilon_{t\phi z} = 1 \) in Weyl coordinates. We will label \( t, \phi \) by an index \( a \), and the remaining coordinates \( \rho, z \) by \( i, j \). In Sect. 4 we will also use prolate spheroidal coordinates \( x, y \) instead of \( \rho, z \). The two-dimensional Levi-Civita symbol \( \epsilon_{ij} \) is defined with \( \epsilon_{t\rho} = 1 \) and \( \epsilon_{\rho z} = 1 \) respectively.
\[
\begin{align*}
\mathcal{L}_{\text{Harm}} &= -\frac{1}{4\pi} \int \left( F_{it} F^{it} - F_{i\varphi} F^{i\varphi} \right) \sqrt{|g|} d^3 x \\
&= -\frac{1}{4\pi} \int \partial_i \left[ \sqrt{|g|} \left( A_t F^{it} - A_\varphi F^{i\varphi} \right) \right] d^3 x \\
&= \sum_n \frac{1}{4\pi} \oint_{\Sigma_n} \left( A_t F^{it} - A_\varphi F^{i\varphi} \right) d\Sigma_i, \quad (2.6)
\end{align*}
\]

where we have used the Maxwell equations in the bulk outside the sources, and again the Ostrogradsky theorem. Note that we have implicitly assumed in the last step that

\[
\oint_{\infty} \left( A_t F^{it} - A_\varphi F^{i\varphi} \right) d\Sigma_i = 0. \quad (2.7)
\]

Were this condition not satisfied, the final result (2.6) should also include a surface integral at infinity.

Returning to (2.1), we can write the total mass as the sum of the masses of the individual sources

\[
M = \sum_n M_n, \quad (2.8)
\]

with

\[
M_n = \frac{1}{8\pi} \oint_{\Sigma_n} \left[ g^{ij} g^{\alpha\beta} \partial_j g_{\alpha\beta} + 2(A_t F^{it} - A_\varphi F^{i\varphi}) \right] d\Sigma_i. \quad (2.9)
\]

where the first term may be viewed as the gravitational contribution to the source mass, and the second term as the electromagnetic contribution.

Similarly, (2.2) can be transformed to

\[
J = -\sum_n \frac{1}{8\pi} \oint_{\Sigma_n} D^\mu m^\mu d\Sigma_{\mu\nu} + \int T^{\mu\nu} m^\nu dS_\mu, \quad (2.10)
\]

and the second, bulk contribution \( J^E \) can be further transformed to

\[
J^E = \frac{1}{4\pi} \int F_{i\varphi} F^{i\varphi} \sqrt{|g|} d^3 x \\
= \frac{1}{4\pi} \int \partial_i \left( \sqrt{|g|} A_\varphi F^{i\varphi} \right) d^3 x \\
= -\sum_n \frac{1}{4\pi} \oint_{\Sigma_n} A_\varphi F^{i\varphi} d\Sigma_i, \quad (2.11)
\]

under the assumption

\[
\oint_{\infty} A_\varphi F^{i\varphi} d\Sigma_i = 0. \quad (2.12)
\]
The total angular momentum then decomposes as

\[ J = \sum_n J_n, \quad J_n = -\frac{1}{16\pi} \oint_{\Sigma_n} \left[ g^{ij} g^{\alpha\beta} \partial_j g_{\phi a} + 4A_{\phi} F^{\alpha\beta} \right] d\Sigma_i. \quad (2.13) \]

These formulas give the masses and angular momenta as fluxes through surfaces, and thus necessitate the knowledge of the gravitational and electromagnetic potentials off these surfaces. In the case where \( \Sigma_n \) is the horizon of a rotating black hole, Tomimatsu \[5\] derived formulas which have the advantage of involving only potentials on-shell, that is on the horizon.

### 3 Correcting the Tomimatsu formulas

Now we revisit the derivation of the mixed formulas for the mass and angular momentum of rotating black holes given by Tomimatsu in \[5\]. We call these “mixed” because they involve both physical metric and Maxwell field components, and Ernst potentials. We will spell out a number of details which were omitted in the rather elliptic derivation of \[5\], with the conclusion that the Tomimatsu formula for the black hole mass should be corrected. The use of an incorrect formula has led Tomimatsu himself, and other authors \[7\]-\[10\] to the incorrect conclusion that the usual Smarr mass formula should be modified when magnetic charges are present.

Before proceeding we need to recall the definition of the complex Ernst potentials, which will be used in the following. In Weyl coordinates, these are defined by

\[ \mathcal{E} = F - \overline{\psi}\psi + i\chi, \quad \psi = v + iu, \quad (3.1) \]

where the electric and magnetic scalar potentials \( v \) and \( u \) are such that

\[ v = A_t, \quad \partial_i u = \rho^{-1} \epsilon_{ij} \partial_j A_{\phi} \quad (3.2) \]

with \( x^1 = \rho, \ x^2 = z \), and the twist potential \( \chi \) is defined by

\[ \partial_i \chi = -F^2 \rho^{-1} \epsilon_{ij} \partial_j \omega + 2(v \partial_i v - u \partial_i u). \quad (3.3) \]

#### 3.1 Mass

First consider the generalized Komar mass formula \[2.9\], where \( \Sigma_n \) is the horizon \( H \) of a rotating black hole, and the spacetime metric is written in the standard Weyl form

\[ ds^2 = -F(dt - \omega d\phi)^2 + F^{-1}[e^{2k}(d\rho^2 + dz^2) + \rho^2 d\phi^2]. \quad (3.4) \]
The horizon corresponds to \( N^2 \equiv \rho^2 / g_{\varphi \varphi} = 0 \) with

\[
g_{\varphi \varphi} = F^{-1} \rho^2 - F \omega^2 > 0,
\]

so that \( H \) is a cylindrical surface \( \rho = 0 \), \( t = \) constant, on which \( \sqrt{|g|} g^{\rho \rho} = \rho = 0 \), and \( \omega \) takes a constant value \( \omega_H = \Omega_H^{-1} \), with \( \Omega_H \) the horizon angular velocity. The horizon mass \( (2.9) \) decomposes as \( M_H = M^G_H + M^E_H \), with the gravitational contribution

\[
M^G_H = \frac{1}{8\pi} \int_H \sqrt{|g|} g^{\rho \rho} g^{tn} \partial_j g_{tn} dz d\varphi
\]

\[
= \frac{1}{8\pi} \int_H \left[ \rho F^{-1} \partial_n F + \rho^{-1} F^2 \omega \partial_n \omega \right] dz d\varphi
\]

\[
= \frac{1}{8\pi} \int_H \omega [\partial_z \chi + 2(v \partial_z u - u \partial_z v)] dz d\varphi,
\]

where we have discarded the first term of the integrand in the second line, which (assuming \( F_H \neq 0 \)) vanishes on the horizon \( \rho = 0 \), and used the definition \( (3.3) \) of the Ernst twist potential \( \chi = \text{Im} E \) to express the second term in terms of Ernst potentials. The electromagnetic contribution is

\[
M^E_H = \frac{1}{4\pi} \int_H \sqrt{|g|} \left( A_t F^t - A_\varphi F^{\rho \varphi} \right) dz d\varphi
\]

\[
= \frac{1}{4\pi} \int_H \sqrt{|g|} (\omega v - A_\varphi) F^{\rho \varphi} dz d\varphi,
\]

on account of \( \sqrt{|g|} (F^t - \omega F^{\rho \varphi}) = -\sqrt{|g|} g^{\rho \rho} F^{-1} F_{\rho \varphi} = 0 \) on the horizon. Using the electromagnetic duality equation \( (3.2) \) this can be transformed to

\[
M^E_H = -\frac{1}{4\pi} \int_H (\omega v - A_\varphi) \partial_z u dz d\varphi
\]

\[
= -\frac{1}{4\pi} \int_H [\omega (v \partial_z u - u \partial_z v) - \partial_z (u A_\varphi)] dz d\varphi
\]

where we have used the constancy over the horizon of

\[
A_\varphi + \omega v = -\omega_H \Phi_H,
\]

with

\[
-\Phi_H = A_t + \Omega_H A_\varphi
\]

the horizon electric potential in the horizon co-rotating frame \( [2] \). Adding \( (3.6) \) and \( (3.7) \), we obtain

\[
M_H = \frac{1}{8\pi} \int_H [\omega \partial_z \text{Im} E + 2 \partial_z (A_\varphi \text{Im} \psi)] dz d\varphi,
\]

which differs from Tomimatsu’s \( [5] \) Eq. (52) by the presence of the second term. We will discuss the consequences of this difference shortly.
3.2 Angular momentum

Similarly, the first, gravitational contribution to the horizon angular momentum is

\[ J^G_H = -\frac{1}{16\pi} \int_H \sqrt{|g|} g^{\rho\rho} g^{\alpha\alpha} \partial_\rho g_{\rho\alpha} dzd\varphi \]
\[ = -\frac{1}{16\pi} \int_H \left[ 2\omega(1 - \rho F^{-1} \partial_\rho F) - \rho^{-1}(\rho^2 + F^2 \omega^2) \partial_\rho \omega \right] dzd\varphi \]
\[ = -\frac{1}{16\pi} \int_H \left\{ 2\omega - \omega^2 [\partial_z \chi + 2(\nu \partial_z u - u \partial_z v)] \right\} dzd\varphi. \quad (3.12) \]

The electromagnetic contribution is

\[ J^E_H = -\frac{1}{4\pi} \int_H \sqrt{|g|} A_\varphi F^{\varphi t} dzd\varphi \]
\[ = \frac{1}{4\pi} \int_H \omega A_\varphi \partial_z u dzd\varphi \]
\[ = \frac{1}{8\pi} \int_H \omega [(A_\varphi + \omega v) \partial_z u - \omega v \partial_z u + u(\omega \partial_z v + \partial_z A_\varphi) + A_\varphi \partial_z u] dzd\varphi \]
\[ = \frac{1}{8\pi} \int_H \omega [(A_\varphi + \omega v) \partial_z u - \omega(v \partial_z u - u \partial_z v) + \partial_z(A_\varphi u)] dzd\varphi, \quad (3.13) \]

where we have used the constancy of \( A_\varphi + \omega v \) over the horizon in the third line. Adding these two contributions together, we obtain

\[ J_H = \frac{1}{16\pi} \int_H \omega \left[ -2 + \omega \partial_z \text{Im} \chi + 2\partial_z(A_\varphi \text{Im} \psi) - 2\omega \Phi_H \partial_z \text{Im} \psi \right] dzd\varphi, \quad (3.14) \]
in agreement with Eqs. (54)-(55) of [5].

3.3 The horizon Smarr formula

To be self-contained, we first recall briefly the derivation of the Tomimatsu formula giving the horizon electric charge [5]. In Weyl coordinates, this is defined by the flux

\[ Q_H = \frac{1}{4\pi} \int_H \sqrt{|g|} F^{\varphi t} dzd\varphi. \quad (3.15) \]

The electric field is related to the Ernst potentials by

\[ F^{ti} = \frac{g^{ij}}{g_{tt}} F_{tj} - \frac{g_{\varphi t}}{g_{tt}} F_{\varphi i} = e^{-2k} \left[ \partial_i v + \frac{F\omega}{\rho} \epsilon_{ij} \partial_j u \right], \quad (3.16) \]
leading on the horizon $\rho = 0$ to

$$Q_H = \frac{1}{4\pi} \int_H \omega \partial_z \text{Im} \psi \, dz d\phi.$$  \hfill (3.17)

Comparing (3.11) and (3.14), using the definitions of the Hawking temperature $T_H = \kappa / 2\pi$ and entropy $S = A_H / 4$, with the surface gravity $\kappa = e^{-k} / |\omega|$ and the horizon area $A_H = \int_H e^k|\omega| \, dz d\phi$ (so that $T_H S = \kappa A_H / 8\pi = \sigma / 2$), and the value (3.17) of the horizon electric charge, we recover the usual Smarr mass formula

$$M_H = 2\Omega_H J_H + 2T_H S + \Phi_H Q_H.$$ \hfill (3.18)

We have thus shown that this formula, which involves only electric charges, is valid on the horizons for any asymptotically flat configuration, including horizons carrying also magnetic charges.

As mentioned above, Tomimatsu obtained in [5] an expression $M'_H$ for the horizon mass which does not contain the second term in (3.11). Consequently he obtained for his $M'_H$ a modified Smarr formula (Eq. (57) of [5]) which contains an additional term $M_A S$, equal to the opposite of the “lost” second term of (3.11), so as to account for the difference $M'_H - M_H$. This extra term can contribute to the horizon mass in the case of magnetically charged black holes, and the consequences of this were explored in [7]-[10] in the case of dyonic diholes, pairs of rotating black holes carrying opposite electrical and magnetic monopole charges (so that the global system has only dipole electromagnetic moments). However, as we have shown, the usual Smarr relation holds in this case for the horizon observables, provided they are correctly computed. It is true that the two black holes are connected by a string with conical singularity and carrying magnetic flux (Dirac string), but the contributions of this string to the total mass and angular momentum should be included as third separate contributions $M_{\text{string}}$ and $J_{\text{string}}$ (integrals on cylinders centered on the strings) to the sums (2.8) and (2.13), without effect on the horizon observables.

The situation is different in the case of the first and third examples discussed in [10] (the dyonic Kerr-Newman solution, and a system of two counter-rotating black holes with the same electric and magnetic charges). In this case, not only will some Dirac strings necessarily extend to infinity, but also the net magnetic monopole potential

$$A_\varphi \sim -P(\cos \theta + C)$$ \hfill (3.19)

(where $P$ is the net magnetic charge, and $C$ a constant governing the strength of the Dirac strings) will go to constant values at infinity for all
\[\theta\], so that the sums \((2.8)\) and \((2.13)\) could include, besides the horizon and string contributions, additional contributions from a surface at infinity. We revisit the case of the dyonic Kerr-Newman black hole in the next section.

4 The case of the dyonic Kerr-Newman black hole

The metric and electromagnetic fields of this solution are given by \([10]\):

\[
F = \frac{f}{\Sigma}, \quad e^{2k} = \frac{f}{\sigma^2(x^2 - y^2)},
\]

\[
f = \sigma^2(x^2 - 1) - a^2(1 - y^2), \quad \Sigma = (\sigma x + M)^2 + a^2 y^2,
\]

\[
\omega = -a(1 - y^2)\frac{2M(\sigma x + M) - Q^2 - P^2}{f},
\]

\[
A_t = -\frac{Q(\sigma x + M) + aPy}{\Sigma}, \quad A_\varphi = -Py - C - aA_t(1 - y^2), \quad (4.1)
\]

where the prolate spheroidal coordinates \(x \geq 1, y \in [-1, +1]\) are related to the Weyl coordinates by

\[
\rho = \sigma\left(\frac{x^2 - 1}{2(1 - y^2)}\right)^{1/2}, \quad z = \sigma xy,
\]

\[\sigma\] being related to the mass \(M\), electric charge \(Q\), magnetic charge \(P\) and rotation parameter \(a\) by \(\sigma^2 = M^2 - Q^2 - P^2 - a^2\). The corresponding Ernst potentials are

\[
\mathcal{E} = \frac{\sigma x - M + iay}{\sigma x + M + iay}, \quad \psi = \frac{-Q + iP}{\sigma x + M + iay} \quad (4.2)
\]

(our \(\mathcal{E}\) is the complex conjugate of that of \([10]\), and our \(\psi\) is minus the complex conjugate of the \(\Phi\) of \([10]\)), and their imaginary parts are

\[
\chi = \text{Im}\mathcal{E} = \frac{2aMy}{\Sigma}, \quad u = \text{Im}\psi = \frac{P(\sigma x + M) + aQy}{\Sigma}. \quad (4.3)
\]

First we observe that \(A_tF^t\) and \(A_\varphi F^\varphi\varphi\) (with \(r \sim \sigma x\)) fall off at infinity more quickly than \(1/r^2\), so that the condition \((2.7)\) is satisfied. However the situation is different for the angular momentum. Namely, in transforming the Komar angular momentum \((2.2)\) into the sum \((2.13)\) we have dropped an integral over a large sphere at spatial infinity parameterized by the angles \(\theta\) (with \(y = \cos \theta\)) and \(\varphi\),

\[
\frac{1}{4\pi} \int_{\infty} A_\varphi F^t r^2 \sin \theta d\theta d\varphi = \frac{PQ}{2} \int_{-1}^{+1} (y + C) dy = CPQ. \quad (4.4)
\]

So \((2.13)\) is valid for dyons provided the constant \(C\) (\(-b_0\) in \([10]\)) is set to zero. Let us emphasize that this conclusion depends only on the asymptotic
behaviour, and so obviously extends to the case of axisymmetric dyonic multi-black hole configurations, such as that discussed in Sect. IV of [10]. For such configurations with net total electric and magnetic charges \(Q\) and \(P\) both different from zero, horizon and string angular momenta can consistently be defined only in the gauge where the vector magnetic potential is asymptotically \(A = -P \cos \theta d \varphi\), so that that the asymptotic angular momentum (2.2) can be transformed into the sum (2.13).

The horizon is \(x = 1\). On the horizon we find from (4.11) \(\omega_H = \Omega_H^{-1}\), with the horizon angular velocity

\[
\Omega_H = \frac{a}{(M + \sigma)^2 + a^2} = \frac{a}{\Sigma_0}.
\]

We use this to evaluate the integrals (3.17), (3.11), (3.14). The evaluation of the horizon electric charge (3.17) leads naturally to

\[
Q_H = \frac{\omega_H}{2} \int_{-1}^{+1} \partial_y u_H dy = \frac{\omega_H}{2} \left[ u(1, 1) - u(1, -1) \right] = Q.
\]  

The evaluation of the first term \(1/8\pi \int_H \omega \partial_y u dy d\varphi\) of the horizon mass results similarly, as in [10], in the value \(M\). Concerning the second term, we note that \(A_\varphi = -Py\) on the horizon, so that only the part of \(u\) which is even in \(y\) will contribute, leading to

\[
\frac{1}{8\pi} \int_H 2 \partial_y (A_\varphi u) dy d\varphi = -\frac{P^2(M + \sigma)}{\Sigma_0},
\]

and to the total horizon mass (defined à la Tomimatsu)

\[
M_H = M - \frac{P^2(M + \sigma)}{\Sigma_0}.
\]

To compute the horizon angular momentum (3.14), we need to know the horizon potential \(\Phi_H\). From (3.10) we obtain [10]

\[
\Phi_H = \frac{Q(M + \sigma)}{\Sigma_0}.
\]

We can then use (3.18) to obtain

\[
J_H = \frac{\omega_H}{2} [M_H - \sigma - \Phi_H Q_H]
\]

\[
= \frac{\omega_H}{2} \left[ M - \sigma - \frac{(P^2 + Q^2)(M + \sigma)}{\Sigma_0} \right] = Ma = J.
\]
Comparing (4.8) and (4.10), we see that while the (local) horizon mass and angular momentum satisfy the usual Smarr formula (3.18), the corresponding (global) quantities evaluated at infinity satisfy a modified Smarr formula \[11, 10, 12\] where the electric and magnetic charges stand on equal footings:

\[ M = 2\Omega H J + 2T_H S + \Phi H Q + \tilde{\Phi}_H P, \]  

(4.11)

with the horizon magnetic potential \( \tilde{\Phi}_H = P(M + \sigma)/\Sigma_0 \).

A deeper issue is to understand the reason for the difference between the horizon mass \( M_H \) and the global mass \( M \). This can only be due to the contribution of the two Dirac strings \( y = 1 \) and \( y = -1 \) with \( x > 1 \) to the sum (2.8). Consider the integral (2.9) on a surface \( \Sigma_{S\pm} \) of equation \( y = \pm(1 - \varepsilon) \), where the constant \( \varepsilon \) shall be taken to zero. Using

\[ dp^2 + dz^2 = \sigma^2(x^2 - y^2) \left[ \frac{dx^2}{x^2 - 1} + \frac{dy^2}{1 - y^2} \right], \]

(4.12)

the flux (2.9) through this surface is

\[ M_{S\pm} = \pm \frac{1}{8\pi} \int_{\Sigma_{S\pm}} \sqrt{|g|} g^{\mu
u} \left[ g^{ta} \partial_y g_{ta} + 2(g^{ta} A_t \partial_y A_a - g^{\rho a} A_\rho \partial_y A_a) \right] dx d\varphi \]

(4.13)

(\( a = t, \varphi \)), with \( \sqrt{|g|} g^{\mu
u} = 1 - y^2 \). It follows that in the limit \( \varepsilon \to 0 \) \( (y \to \pm 1) \), only the terms inside the square brackets with a pole in \( (1 - y^2) \) will contribute to the integral. The covariant metric tensor and electromagnetic vector components remain finite in this limit, as well as the contravariant metric components \( g^{\alpha\beta} \), with the exception of \( g^{\rho\varphi} \sim F/\sigma^2(x^2 - 1)(1 - y^2) \). So (4.13) reduces to

\[ M_{S\pm} = \pm \frac{1}{4\pi \sigma} \int_{y=\pm 1} \frac{F}{x^2 - 1} A_\varphi \partial_y A_\varphi dx d\varphi \]

\[ = \pm \frac{\sigma}{2} \int_1^\infty \frac{Py(P - 2aA_t y)}{\Sigma} dx \]

\[ = \frac{P}{2} \int_{a+M}^\infty \frac{P(\xi^2 - a^2) \pm 2aQ\xi}{(\xi^2 + a^2)^2} d\xi \]

\[ = \frac{P[M + \sigma \pm aQ]}{2[(M + \sigma)^2 + a^2]}, \]

(4.14)

where we have put \( \xi = \sigma x + M \). The sum of the two string masses

\[ M_{S+} + M_{S-} = \frac{P^2(M + \sigma)}{\Sigma_0}, \]

(4.15)
leads to
\[ M_H + M_{S_+} + M_{S_-} = M, \] as required. A similar computation for the Dirac string angular momenta yields consistently \( J_{S_\pm} = 0 \), as the contravariant \( g^{\phi \phi} \) does not occur in (2.13).

5 Conclusion

Thus we solved the dilemma “Tomimatsu vs Smarr” in favor of the latter and presented a corrected version of the Tomimatsu formulas which reproduce the standard Smarr relation for the horizon mass of dyons without an additional term associated with the magnetic charge. Our modification should settle some problems which have been raised in recent discussions of multi-dyonic black hole solutions of the Einstein-Maxwell equations. We should mention that it has no effect in the case of the recently presented quasiregular two-NUTty dyonic black hole solution [13], for which the term omitted by Tomimatsu vanishes.

We have also discovered that Dirac strings give a non-zero contribution to the total mass of the Kerr-Newman dyonic black hole, so that Dirac strings are “heavy”. Taking them into account, one finds that the Smarr relation for the total mass includes the magnetic term. One unexpected feature concerns the gauge choice for the vector potential of the magnetic charge. Usually it is believed that by adding a constant term to \( A_\phi \) one can eliminate either the North pole or the South pole Dirac strings without affecting physics. This is not valid if gravity is taken into account: only the symmetric gauge choice with both strings present with equal weights leads to correct balance of mass and angular momentum. This emphasizes the fact that adding a constant term to \( A_\phi \) is a “large” gauge transformation affecting the physical properties of the solution.

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References

[1] L. Smarr, Phys. Rev. Lett. 30, 71 (1973).

[2] B. Carter, “Black Hole Equilibrium States, Part 2: General Theory of Stationary Black Hole States”, in Black Holes (Les Houches 1972) ed B and C. DeWitt, (Gordon Breach, New York 1973)) 125-214.

[3] A. Komar, Phys. Rev. 113, 934 (1959).

[4] A. Tomimatsu, Progr. Theor. Phys. 70, 385 (1983).

[5] A. Tomimatsu, Progr. Theor. Phys. 72, 73 (1984).

[6] V. S. Manko, R. I. Rabadán and J. Sanabria-Gómez, Phys. Rev. D 89, 064049 (2014) [arXiv:1311.2326 [gr-qc]].

[7] I. Cabrera-Munguia, C. Lämmerzahl, L.A. López and A. Macías, Phys. Rev. D 88, 084062 (2013) [arXiv:1309.2556 [gr-qc]].

[8] I. Cabrera-Munguia, C. Lämmerzahl, L.A. López and A. Macías, Phys. Rev. D 90, 024013 (2014) [arXiv:1405.2629 [gr-qc]].

[9] I. Cabrera-Munguia, C. Lämmerzahl and A. Macías, Phys. Lett. B 743, 357 (2015).

[10] V. S. Manko and H. García-Compeán, “Remarks on Smarr’s mass formula in the presence of both electric and magnetic charges” [arXiv:1506.03870 [gr-qc]].

[11] M. Heusler, Phys. Rev. D 56, 961 (1997) [arXiv:gr-qc/9703015].

[12] M. Cárdenas, O. Fuentealba and J. Matulich, JHEP 1605, 001 (2016) [arXiv:1603.03760 [hep-th]].

[13] G. Clément and D. Gal’tsov, Phys. Lett. B 771, 457 (2017) [arXiv:1705.08017 [gr-qc]].