CURVATURE PROPERTIES OF MELVIN MAGNETIC METRIC

ABSOS ALI SHAIKH\textsuperscript{1}, AKRAM ALI\textsuperscript{2}, ALI H. ALKHALDI\textsuperscript{2} AND DHYANESH CHAKRABORTY\textsuperscript{1}

Abstract. This paper aims to investigate the curvature restricted geometric properties admitted by Melvin magnetic spacetime metric, a warped product metric with 1-dimensional fibre. For this, we have considered a Melvin type static, cylindrically symmetric spacetime metric in Weyl form and it is found that such metric, in general, is generalized Roter type, \textit{Ein}(3) and has pseudosymmetric Weyl conformal tensor satisfying the pseudosymmetric type condition $R \cdot R - Q(S, R) = L'Q(g, C)$. The condition for which it satisfies the Roter type condition has been obtained. It is interesting to note that Melvin magnetic metric is pseudosymmetric and pseudosymmetric due to conformal tensor. Moreover such metric is 2-quasi-Einstein, its Ricci tensor is Reimann compatible and Weyl conformal 2-forms are recurrent. The Maxwell tensor is also pseudosymmetric type.

1. Introduction

We consider $M$ as a connected and smooth manifold of dimension $n(\geq 3)$, on which a semi-Reimannian metric $g$ with signature $(\sigma, n - \sigma)$, $0 \leq \sigma \leq n$, is endowed. Then $(M, g)$ is called an $n$-dimensional Reimannian (resp, Lorentzian) manifold if $\sigma = 0$ or $n$ (resp, $\sigma = 1$ or $n - 1$). The Levi Civita connection, the Riemann curvature tensor, the Ricci tensor and the scalar curvature of $M$ are respectively denoted by $\nabla$, $R$, $S$ and $\kappa$. We mention that a spacetime is a 4-dimensional connected Lorentzian manifold.

A stationary, cylindrically symmetric electrovac solution to Einstein-Maxwell system of equations in general relativity was obtained by Melvin \cite{36} in 1964 and such solution is often referred as ‘Melvin Magnetic Universe’. Physically, this spacetime describes a bundle of parallel magnetic lines of force which under their mutual gravitational attraction remain in equilibrium. It is interesting to note that such spacetime is not asymptotically flat and has no singularity but is geodesically complete. About an axis of symmetry the spacetime is invariant under rotation and translation, and also invariant under reflection in planes containing that axis or perpendicular to it. Moreover it admits four Killing vectors and Weyl conformal tensor is Petrov type D.

Melvin himself in \cite{37} proved his spacetime to be stable against small radial perturbations. Thorne \cite{73} further showed the stability of such spacetime against any large perturbations and

\textsuperscript{0}\textsuperscript{0} Corresponding author.

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also suggested that this spacetime has a great value in the study of extragalactic sources of radio waves. In this paper we are really tempted to investigate the geometric structures of such magnetic spacetime.

The geometry of Melvin magnetic spacetime in cylindrical symmetry is described by the metric

$$ds^2 = U_B^2(-dt^2 + dr^2 + dz^2) + \frac{r^2}{U_B^2}d\phi^2$$

(1.1)

where $U_B = 1 + \frac{B_0 r^2}{4}$, $B_0$ is the magnetic field at $r = 0$ axis. The Maxwell field is given by

$$F = \frac{B_0 r^2}{U_B^2}d\phi \wedge dr^2.$$  

(1.2)

The coordinates $t$, $r$, $z$ and $\phi$ are respectively dimensionless time, radial, longitudinal and azimuthal angle. It is easy to see that for vanishing magnetic field, $U_B = 1$ and the metric (1.1) reduces to the Minkowski metric in cylindrical coordinates. It may be noted that the metric (1.1) can be written as a warped product metric

$$ds^2 = d\bar{s}^2 + F^2d\tilde{s}^2$$

where $d\bar{s}^2 = (1 + \frac{B_0 r^2}{4})^2(-dt^2 + dr^2 + dz^2)$, $d\tilde{s}^2 = d\phi^2$ and the warping function $F(r) = r/(1 + \frac{B_0 r^2}{4})$. In the Weyl form an electrovac or vacuum static, cylindrically symmetric metric can be written as

$$ds^2 = -e^{2\psi}dt^2 + e^{-2\psi}[e^{2\lambda}(dr^2 + dz^2) + r^2 d\phi^2]$$

(1.3)

where $\psi$ and $\lambda$ are no-where vanishing smooth functions of $r$ and $z$. We see that for $\psi = 2\lambda = \ln U_B$, the metric (1.1) is obtained from the metric (1.3). Thus we can see Melvin type metric in Weyl form as

$$ds^2 = e^{2f(r)}(-dt^2 + dr^2 + dz^2) + r^2 e^{-2f(r)}d\phi^2.$$  

(1.4)

In the literature of differential geometry, the notion of pseudosymmetry generalizes the notions of (local) symmetry ($\nabla R = 0$) and semisymmetry ($R \cdot R = 0$) introduced by Cartan [4, 5]. Such a notion of pseudosymmetry has a great importance in the study of general relativity and cosmology due to its applications. Precisely, many spacetimes are pseudosymmetric. The extension of this notion to other curvature tensors are called pseudosymmetric type curvature conditions. By curvature restricted geometric structures of a manifold mean the structures arose due to restriction of covariant derivative(s) on various curvature tensors of that manifold. Again, the study of local symmetry has been extended to various concepts
such as recurrency \[39, 40, 41\], generalized recurrency \[43, 49, 60, 61, 62, 63, 64\], curvature 2-forms of recurrency \[3, 30\], weakly symmetry \[72\], Chaki pseudosymmetry \[6\] etc. our aim in this paper is to investigate such type of geometry by means of restriction on several curvature tensors of Melvin magnetic metric (1.1). We mention that Robertson-Walker spacetimes \[2, 8, 24\], the warped products of 1-dimensional base and 3-dimensional fibre, are pseudosymmetric. Also Schwarzschild spacetime \[29\], Robinson-Trautman spacetime \[46\] etc. are models of pseudosymmetric warped product spacetimes of 2-dimensional base and 2-dimensional fibre. The curvature properties of an warped product metric have also been investigated in \[55\] Thus, it is worthy to investigate the geometric structures of a warped product spacetime like Melvin of 3-dimensional base and 1-dimensional fibre.

The planning of this paper is as follows: Section 2 is devoted to the preliminaries of various curvature tensors and curvature restricted geometric structures. In section 3 we obtain the geometric structures of Melvin type metric (1.4) and also obtain the conditions for which it is conformally flat, pseudosymmetric and Roter type manifold. The curvature properties of Melvin magnetic spacetime have been determined in section 4 and it is shown that such metric is pseudosymmetric and has pseudosymmetric Maxwell tensor. Also the tensor $C \cdot R - R \cdot C$ is linearly dependent with \(Q(S, C)\) i.e., a generalized Einstein metric condition is satisfied on this spacetime. Moreover, it is 2-quasi-Einstein and Roter type. Finally we add some conclusion in section 5.

2. Preliminaries

In this section we will review the notations and definitions of various curvature tensors and some useful curvature restricted geometric structures. For this the endomorphisms $U \wedge_E V$, $\mathcal{R}$, $\mathcal{C}$, $\mathcal{P}$, $\mathcal{W}$ and $\mathcal{K}$ on $M$ are respectively defined as \[22, 51\]

\[
(U \wedge_E V)Z = E(V, Z)U - E(U, Z)V,
\]

\[
\mathcal{R}(U, V) = [\nabla_U, \nabla_V] - \nabla_{[U, V]},
\]

\[
\mathcal{C} = \mathcal{R} - \frac{1}{(n-2)}(\wedge_g \mathcal{J} + \mathcal{J} \wedge_g - \frac{\kappa}{n-1} \wedge_g),
\]

\[
\mathcal{P} = \mathcal{R} - \frac{1}{(n-1)} \wedge_S,
\]

\[
\mathcal{W} = \mathcal{R} - \frac{\kappa}{n(n-1)} \wedge_g \quad \text{and}
\]

\[
\mathcal{K} = \mathcal{R} - \frac{1}{(n-2)}(\wedge_g \mathcal{J} + \mathcal{J} \wedge_g)
\]
where $E$ is a symmetric $(0, 2)$-tensor, $\mathcal{J}$ is the Ricci operator defined by $S(X, Y) = g(X, \mathcal{J}Y)$ and throughout this paper we consider $X, Y, Z, Z_i, U, V \in \chi(M)$, the Lie algebra of all smooth vector fields on $M$. The Kulkarni-Nomizu product $E \wedge F$ for two symmetric $(0, 2)$ tensors $E$ and $F$ is defined by \cite{14, 15, 16, 21, 26, 53}

$$(E \wedge F)(X, Y, U, V) = E(X, V)F(Y, U) - E(X, U)F(Y, V) + E(Y, U)F(X, V) - E(Y, V)F(X, U).$$

Now we define the $(0, 4)$-tensor $H$ corresponding to an endomorphism $\mathcal{H}(Z_1, Z_2)$ on $M$ as

$$(2.1) \quad H(Z_1, Z_2, Z_3, Z_4) = g(\mathcal{H}(Z_1, Z_2)Z_3, Z_4)$$

Replacing $\mathcal{H}$ in above by $\mathcal{C}$ (resp., $\mathcal{P}$, $\mathcal{W}$, $\mathcal{K}$ and $Z_1 \wedge g Z_2$) one can easily obtain the conformal curvature tensor $C$ (resp., the projective curvature tensor $P$, the concircular curvature tensor $W$, the conharmonic curvature tensor $K$ and the Gaussian curvature tensor $G$). Let $(M, g)$ be covered with a system of coordinate charts $(\mathcal{U}, x^\alpha)$. Then the local components of the $(0, 4)$-tensors $E \wedge F$, $R, C, P, W$ and $K$ are written as

$$(E \wedge F)_{abcd} = E_{ad}F_{bc} - E_{ac}F_{bd} + E_{be}F_{ad} - E_{bd}F_{ac},$$

$$R_{abcd} = g_{aa}(\partial_d \Gamma_{bc}^a - \partial_c \Gamma_{bd}^a + \Gamma_{bc}^r \Gamma_{rd}^a - \Gamma_{bd}^r \Gamma_{rc}^a),$$

$$C_{abcd} = R_{abcd} - \frac{1}{(n - 2)}(g \wedge S)_{abcd} + \frac{\kappa}{(n - 1)(n - 2)}G_{abcd},$$

$$P_{abcd} = R_{abcd} - \frac{1}{(n - 1)}(g_{ad}S_{bc} - g_{bd}S_{ac}),$$

$$W_{abcd} = R_{abcd} - \frac{\kappa}{n(n - 1)}G_{abcd} \quad \text{and}$$

$$K_{abcd} = R_{abcd} - \frac{1}{(n - 2)}(g \wedge S)_{abcd}$$

where $\Gamma_{bc}^a$ are Christoffel symbols of 2nd kind, $\partial_\alpha = \frac{\partial}{\partial x^\alpha}$ and $G_{abcd} = \frac{1}{2}(g \wedge g)_{abcd}$. Let $T$ be a $(0, k)$-tensor, $k \geq 1$, and the operation of the endomorphisms $\mathcal{A}(X, Y)$ and $X \wedge_B Y$ on it give rise two $(0, k + 2)$-tensors $A \cdot T$ and $Q(B, T)$ \cite{14, 16, 47, 49, 71} respectively given as

$$(A \cdot T)(Z_1, Z_2, \ldots, Z_k; U, V) = (\mathcal{A}(U, V) \cdot T)(Z_1, Z_2, \ldots, Z_k)$$

$$= -T(\mathcal{A}(U, V)Z_1, Z_2, \ldots, Z_k) - \cdots - T(Z_1, Z_2, \ldots, \mathcal{A}(U, V)Z_k) \quad \text{and}$$
Theorem \[\dim \bar{\text{dim}}\] Remark 2.1. for some function examples of such warped products.

A \[\text{metric}\] and pseudosymmetric (resp., Ricci, conformally, concircularly and conharmonically obtain semisymmetric (resp., Ricci, conformally, concircularly and conharmonically semisymmetric) manifolds respectively. Also replacing \(A\) by \(C\), \(W\) and \(K\) we obtain the corresponding pseudosymmetric type curvature conditions.

Definition 2.1. If on \((M, g)\) the relation \(A \cdot T = 0\) holds then it is said to be \(T\)-semisymmetric due to \(A\) and said to be \(T\)-pseudosymmetric due to \(A\) if \(A \cdot T = \mathcal{L}_T Q(g, T)\) holds on \(M\), where \(\mathcal{L}_T\) is smooth function on \(\{x \in M : Q(g, T)_x \neq 0\}\).

If we replace \(A\) by \(R\) and \(T\) by \(R\) (resp., \(S\), \(C\), \(W\) and \(K\)) in the above definition then we obtain semisymmetric (resp., Ricci, conformally, concircularly and conharmonically semisymmetric) and pseudosymmetric (resp., Ricci, conformally, concircularly and conharmonically pseudosymmetric) manifolds respectively. Also replacing \(A\) by \(C\), \(W\) and \(K\) we obtain the corresponding pseudosymmetric type curvature conditions.

Remark 2.1. 1. In Theorem 4.1 of \([9]\) it was proved that every warped product \(\bar{M} \times_f \bar{N}\) with \(\dim \bar{M} = 1\) and \(\dim \bar{N} = 3\) satisfies

\[
(A \cdot T)_{a_1 a_2 \ldots a_k \alpha \beta} = -g^{rs}[A_{\alpha \beta a_1} T_{a_2 \ldots a_k} + \cdots + A_{\alpha \beta a_k} T_{a_1 a_2 \ldots r}],
\]

\[
Q(B, T)_{a_1 a_2 \ldots a_k \alpha \beta} = B_{\beta a_1} T_{a_2 \ldots a_k} + \cdots + B_{\beta a_k} T_{a_1 a_2 \ldots \alpha} - B_{\alpha a_1} T_{\beta a_2 \ldots a_k} - \cdots - B_{\alpha a_k} T_{a_1 a_2 \ldots \beta}.
\]

\[
(C \cdot C = \mathcal{L}_C' Q(g, C)
\]

for some function \(\mathcal{L}_C'\) and Robertson-Walker spacetime \([2, 8, 24]\) admits this property.

2. In Theorem 2 of \([11]\) it was given that every warped product \(\bar{M} \times_f \bar{N}\) with \(\dim \bar{M} = 2\) and \(\dim \bar{N} = 2\) satisfies

\[
R \cdot R - Q(S, R) = \mathcal{L}' Q(g, C)
\]

for some function \(\mathcal{L}'\) and Schwarzschild spacetime \([29]\), Robinson-Trautman spacetime \([46]\) are examples of such warped products.

The manifold \(M\) is called a \(k\)-quasi-Einstein manifold if the rank of \((S - \alpha g)\) is \(k\), \(0 \leq k \leq (n - 1)\), for some scalar \(\alpha\). For \(k = 1\) (resp., \(k = 0\)) the manifold is called quasi-Einstein (resp.,
Einstein) and the Ricci tensor locally takes the form \[57, 58, 59\]
\[
S_{ab} = \alpha g_{ab} + \beta \Pi_a \otimes \Pi_b
\]
(resp., \(S_{ab} = \alpha g_{ab}\))
where \(\alpha, \beta\) are scalars and \(\Pi_a\) are the components of the covector \(\Pi\). Again on the class of non-quasi-Einstein manifolds, the notion of pseudo quasi-Einstein manifolds was introduced in \[42\] by decomposing the Ricci tensor
\[
S_{ab} = \alpha g_{ab} + \beta \Pi_a \otimes \Pi_b + \gamma F_{ab}
\]
where \(\gamma\) is also a scalar and \(F_{ab}\) are components of a trace free \((0,2)\)-tensor \(F\) such that \(F(Y, U) = 0\), \(U\) is the associated vector field of \(\Pi\). In particular, for \(F_{ab} = \Pi_a \otimes \delta_b + \delta_a \otimes \Pi_b\), \(\delta_a\) are the components of the covector \(\delta\), it is generalized quasi-Einstein due to Chaki \[7\]. We note that Siklos spacetime \[45\] is quasi-Einstein and Som-Raychaudhuri spacetime \[50\] is 2-quasi-Einstein as well as generalized quasi-Einstein. For curvature properties of Vaidya metric, we refer the reader to see \[54\].

**Definition 2.2.** \([3, 51]\) If on a semi-Reimannian manifold \(M\) the relation
\[
n_0 g + n_1 S + n_2 S^2 + n_3 S^3 + n_4 S^4 = 0, \ n_4 \neq 0
\]
holds where \(S^i(Z_1, Z_2) = g(Z_1, J^j Z_2)\) and \(n_0, n_1, n_j \in C^\infty(M)\), the ring of all smooth functions on \(M\) \((2 \leq j \leq 4)\) then it is called \(\text{Ein}(4)\) manifold. For \(n_4 = 0\) (resp., \(n_3 = n_4 = 0\)) it is \(\text{Ein}(3)\) (resp., \(\text{Ein}(2)\)) manifold.

We mention that Siklos spacetime \[45\] is \(\text{Ein}(2)\) while Lifshitz spacetime \[65\] is \(\text{Ein}(3)\).

**Definition 2.3.** \([17, 18, 20, 47, 50, 51, 66]\) If the tensor \(R\) of \(M\) can be expressed as
\[
R = e_1 S \land S + e_2 S \land S^2 + e_3 S^2 \land S^2 + e_4 g \land g + e_5 g \land S + e_6 g \land S^2
\]
for some \(e_i \in C^\infty(M)\), \(1 \leq i \leq 6\), then it is called generalized Roter type manifold. For \(e_2 = e_3 = e_6 = 0\) it is Roter type manifold \([13, 14, 19, 23, 27]\).

**Definition 2.4.** \([22, 28, 44]\) If on a semi-Riemannian manifold \(M\) the Ricci tensor satisfies the condition
\[
\sum_{Z_1, Z_2, Z_3} \nabla_{Z_1} S(Z_2, Z_3) = 0
\]
(resp., \(\nabla_{Z_1} S(Z_2, Z_3) - \nabla_{Z_2} S(Z_1, Z_3) = 0\))
where \(\sum\) denotes the cyclic sum over \(Z_1, Z_2, Z_3\), then it is said to be of cyclic parallel Ricci tensor (resp., Codazzi type Ricci tensor).
Definition 2.5. ([48] [51] [56]) A weakly symmetric manifold is defined by the equation

$$\nabla_\alpha R_{abcd} = \Pi_\alpha R_{abcd} + \Phi_\alpha R_{a,b,c,d} + \Psi_\alpha R_{aabcd} + \Psi_d R_{abca}$$

where $\Pi, \Phi, \Phi, \Psi$ and $\Psi$ are covectors on $M$ with local components $\Pi_a, \Phi_a, \Phi_a, \Psi_a, \Psi_a$ respectively. In particular, if $\frac{1}{2} \Pi = \Phi = \Phi = \Psi = \Psi = 0$ (resp., $\Phi = \Phi = \Psi = \Psi = 0$), then the manifold reduces to Chaki pseudosymmetric manifold [6] (resp., recurrent manifold [39] [40] [75]).

Definition 2.6. ([31] [32]) Let $H$ be a $(0,4)$-tensor and $E$ be a symmetric $(0,2)$-tensor corresponding to the endomorphism $\mathcal{E}$ on $M$. Then the tensor $A$ on $M$ is said to be $H$-compatible if

$$\sum_{Z_1,Z_2,Z_3} H(\mathcal{E} Z_1, X, Z_2, Z_3) = 0$$

holds. Again an 1-form $\Phi$ is said to be $H$-compatible if $\Phi \otimes \Phi$ is $H$-compatible.

Replacing $H$ by $R, W, K, C$ and $P$ we can define, respectively, Reimann compatibility, concircular compatibility, conharmonic compatibility, conformal compatibility and projective compatibility.

Definition 2.7. Let $H$ be a $(0,4)$-tensor and $E$ be a symmetric $(0,2)$-tensor corresponding to the endomorphisms $\mathcal{E}$ on $M$. Then the corresponding curvature 2-forms $\Omega_{(H)}^m$ ([33] [30]) are recurrent if ([33] [34] [35])

$$\sum_{Z_1,Z_2,Z_3} (\nabla Z_1 H)(Z_2, Z_3, X, Y) = \sum_{Z_1,Z_2,Z_3} \Pi(Z_1) H(Z_2, Z_3, X, Y)$$

and the 1-forms $\Lambda_{[E]}$ ([67]) are recurrent if

$$(\nabla Z_1 E)(Z_2, X) - (\nabla Z_2 E)(Z_1, X) = \Pi(Z_1) E(Z_2, X) - \Pi(Z_2) E(Z_1, X)$$

for some covector $\Pi$.

Definition 2.8. ([38] [74]) Let $L(M)$ be the vector space formed by all covectors $\theta$ on $M$ satisfying

$$\sum_{Z_1,Z_2,Z_3} \theta(Z_1) H(Z_2, Z_3, X, Y) = 0$$

where $H$ is a $(0,4)$ tensor. Then $M$ is said to be a $H$-space by Venzi if $\dim L(M) \geq 1$.

3. Geometric structures of Melvin type metric

Let $\tilde{M} = \{(t,r,z) : r > 0\}$ be an open connected non-empty subset of $\mathbb{R}^3$ and on $\tilde{M}$ we define the metric $-\tilde{g}_{11} = \tilde{g}_{22} = \tilde{g}_{33} = e^{2f(r)}$. Then the Melvin type spacetime in Weyl form is a warped product $M = \tilde{M} \times_F \tilde{N}$ of 3-dimensional base $(\tilde{M}, \tilde{g})$ and 1-dimensional fibre
\( (\tilde{N}, \tilde{g}) = (S^1(1), dg^2) \) with the warping function \( F(r) = re^{-f(r)} \).

The non-zero values of the local components \( \bar{R}_{abcd}, \bar{S}_{ab}, \bar{K}_{abcd} \) and \( \bar{\kappa} \) (upto symmetry) are

\[
\begin{align*}
\bar{R}_{1212} &= e^{2f} f'' = -\bar{R}_{2323}, \quad \bar{R}_{1313} = e^{2f} f'^2; \quad \bar{S}_{11} = -(f'^2 + f'') = -\bar{S}_{33}, \\
\bar{S}_{22} &= 2f''; \quad \bar{\kappa} = 2e^{-2f}(f'^2 + 2f''); \\
\bar{K}_{1212} &= -e^{2f}(f'^2 + 2f'') = \bar{K}_{1313} = -\bar{K}_{2323}.
\end{align*}
\tag{3.1}
\]

For every 3-dimensional manifold the conformal curvature tensor vanishes and hence \( \bar{C} = 0 \) of \( (\bar{M}, \bar{g}) \). The values of \( \bar{K}_{abcd, a}, (\bar{R} \cdot \bar{R})_{abcd\beta} \) and \( Q(\bar{g}, \bar{R})_{abcd\beta} \) (upto symmetry) are

\[
\begin{align*}
\bar{K}_{1212, 2} &= 2e^{2f}(f'^3 + f'f'' - f''') = \bar{K}_{1313, 2} = -\bar{K}_{2323, 2}; \\
(\bar{R} \cdot \bar{R})_{123312} &= -e^{2f}f''(f'^2 - f'') = -(\bar{R} \cdot \bar{R})_{123312}, \\
Q(\bar{g}, \bar{R})_{123312} &= -e^{4f}(f'^2 - f'') = -Q(\bar{g}, \bar{R})_{123312}.
\end{align*}
\tag{3.2}
\]

From (3.1) and (3.2) we get the following geometric properties for \( (\bar{M}, \bar{g}) \):

**Theorem 3.1.** The 3-dimensional base \( (\bar{M}, \bar{g}) \) admits the following geometric structures:

1. \( \bar{R} \cdot \bar{R} = \mathcal{L}_R Q(\bar{g}, \bar{R}) \) for \( \mathcal{L}_R = e^{-2f}f'' \) i.e., pseudosymmetric,
2. Rank of \( (\bar{S} - \alpha \bar{g}) = 1 \) for \( \alpha = e^{-2f}(f'^2 + f'') \) i.e., it is quasi-Einstein,
3. satisfies the Ein(2) condition \( S^2 + \mu_1 S + \mu_2 g = 0 \) for
   \[
   \mu_1 = -e^{-2f}(f'^2 + 3f f''), \quad \mu_2 = 2e^{-4f}f''(f'^2 + f''),
   \]
4. if \( (f'^2 + 2f'') \neq 0 \) then the conharmonic tensor is recurrent with 1-form of recurrency
   \[
   \Pi_1 = 0, \quad \Pi_2 = \frac{2(f'' - f'f'' - f'^3)}{(f'^2 + 2f''), \quad \Pi_3 = 0}
   \]
5. if \( (f'^2 + f'') \neq 0 \) then the Ricci 1-form is recurrent with 1-form of recurrency
   \[
   \Pi_1 = 0, \quad \Pi_2 = -\frac{(f'' - f'f'' - f'^3)}{(f'^2 + f''), \quad \Pi_3 = 0}.
   \]

The non-zero values (upto symmetry) of \( R_{abcd}, S_{ab}, \kappa \) and \( C_{abcd} \) of the metric (1.4) are as follows:

\[
\begin{align*}
R_{1313} &= e^{2f} f'^2, \quad R_{1212} = e^{2f} f'' = -R_{2323}, \quad R_{1414} = e^{-2f}f'(1 - r f') = -R_{4343}, \\
R_{2424} &= e^{-2f}f'(3f' - 2r f'^2 + r f''), \quad S_{11} = -\frac{f' + f''}{r} = -S_{33}, \\
S_{22} &= -\frac{3f' - 2r f'^2 - r f''}{r}, \quad S_{44} = e^{4f} f(r' + f''), \quad \kappa = \frac{2e^{-2f}}{r}(r f'' + r f'^2 - f'); \\
C_{1212} &= \frac{e^{2f}}{3r}(r f'' - 2r f'^2 + 2f') = -2C_{1313} = \frac{e^{4f}}{r^2} C_{1414} = -C_{2323} = \frac{2e^{-4f}}{r^2} C_{2424} = -\frac{e^{-4f}}{r^2} C_{3434}.
\end{align*}
\]

**Lemma 3.1.** The Weyl conformal tensor of Melvin type metric (1.4) vanishes if and only if \( (r f'' - 2r f'^2 + 2f') = 0 \).

**Proof.** From the values of the non-zero components \( C_{ijkl} \) the proof is obvious. \( \square \)
Example 3.1. We see that \( f(r) = c_1 + \frac{1}{2} \ln \left( \frac{r}{c_1r+2} \right) \) is the general solution of \( r f'' - 2rf'^2 + f' = 0 \), where \( c_1 \) and \( c_2 \) are arbitrary constants. In particular, we take \( c_1 = 0 \) and \( c_2 = 1 \) such that \( f(r) = \frac{1}{2} \ln \frac{r}{r+2} \). Then the metric \((\text{3.1})\) with such \( f(r) \) takes the form
\[
ds^2 = \frac{r}{(r+2)} (-dt^2 + dr^2 + d\phi^2) + r(r+2)d\phi^2
\]
of which conformal curvature tensor vanishes.

From above Lemma we get
\[
U_C = \{ x \in M : (C)_x \neq 0 \} = \{ (t, r, z, \phi) : (rf'' - 2rf'^2 + 2f') \neq 0 \}.
\]
The non-zero values (upto symmetry) of \((R \cdot R)_{abcd} \) and \(Q(g, R)_{abcd} \) are given by
\[
\begin{align*}
(R \cdot R)_{1234} &= -e^{-2f} f''(f'^2 - f'') = -(R \cdot R)_{121324}, \\
(R \cdot R)_{1421} &= -e^{-2f} f''(4f' - 3rf'^2 + rf'') = (R \cdot R)_{234324}, \\
(R \cdot R)_{1344} &= -e^{-2f} f'^2(1 - rf')(1 - 2rf') = -(R \cdot R)_{131434}, \\
(R \cdot R)_{1244} &= -e^{-2f} f'(1 - rf')(3f' - 2rf'^2 + 2rf'') = (R \cdot R)_{232434}, \\
(R \cdot R)_{1244} &= -e^{-2f} (3f' - 2rf'^2 + rf'')(f' - rf'^2 - rf'') = (R \cdot R)_{23424};
\end{align*}
\]
\[
Q(g, R)_{1324} = -e^{4f} (f'^2 - f'') = -Q(g, R)_{121324},
\]
\[
Q(g, R)_{1424} = -r(4f' - 3rf'^2 + rf'') = Q(g, R)_{142412} = Q(g, R)_{134414} = -r f'(1 - 2rf') = Q(g, R)_{131434}.
\]
\[
Q(g, R)_{1244} = r(3f' - 2rf'^2 + 2rf'') = Q(g, R)_{232434},
\]
\[
Q(g, R)_{1244} = r(f' - rf'^2 - rf'') = Q(g, R)_{23424}. 
\]

From the above components we see that metric \((\text{3.1})\) is not pseudosymmetric but it is pseudosymmetric under certain condition.

Lemma 3.2. The metric \((\text{1.4})\) satisfies the condition \( R \cdot R = \mathcal{L}_R Q(g, R) \) with \( \mathcal{L}_R = \frac{e^{-2f}}{r}(f' - rf'^2) \) if \( (rf'' + rf'^2 - f') = 0 \).

Proof. We consider the endomorphism \( \mathcal{D}(X, Y) = \mathcal{R}(X, Y) - \mathcal{L}'_R X \wedge Y, \mathcal{L}'_R \in C^\infty(M) \).

Then by operating this endomorphism on the tensor \( R \), we obtain the (0,6)-tensor \((DR)(U_1, U_2, U_3, U_4, X, Y)\) as
\[
(D \cdot R)(U_1, U_2, U_3, U_4, X, Y) = (\mathcal{D}(X, Y) \cdot R)(U_1, U_2, U_3, U_4) = (R \cdot R)(U_1, U_2, U_3, U_4, X, Y) - \mathcal{L}'_R Q(g, R)(U_1, U_2, U_3, U_4, X, Y).
\]

Using the values of the components \( R \cdot R_{ijklmn} \), \( Q(g, R)_{ijklmn} \), and putting \( \mathcal{L}'_R = \mathcal{L}_R \) in above, we see that the non-zero components (upto symmetry) of the tensor \( D \cdot R \) are \((DR)_{123124}, (DR)_{124412}, (DR)_{121323}, (DR)_{234324}, \) and \((DR)_{232434}\). Now, if \( (rf'' + rf'^2 - f') = 0 \) then \( (DR)_{ijklmn} = 0 \) for \( 1 \leq i, j, k, l, m, n \leq 4 \). Hence \( R \cdot R = \mathcal{L}_R Q(g, R) \). This completes the proof. \(\square\)
Example 3.2. It is easy to see that \( f(r) = c_1 + \ln(r^2 + 2c_2) \) is the general solution of \( r f'' + r f' - f'^2 = 0 \), where \( c_1 \) and \( c_2 \) are arbitrary constants. In particular, we take \( c_1 = 0 \) and \( c_2 = \frac{1}{2} \) such that \( f(r) = \ln(1 + r^2) \). Then the metric \((1.4)\) with such \( f(r) \) takes the form
\[
ds^2 = \frac{1}{(1 + r^2)^2} \left(-dt^2 + dr^2 + dz^2\right) + r^2(1 + r^2)^2d\phi^2
\]
which is pseudosymmetric.

Again the non-zero components \( C_{abcd, \alpha}, (C \cdot C)_{abcd\beta}, Q(g, C)_{abcd\beta} \) and \( Q(S, R)_{abcd\beta} \) (upto symmetry) are given by
\[
C_{1313,2} = \frac{2e^{rf}}{3r^2}(2r f'(2f' - 2rf'^2 + 3rf'') + (2f' - 2rf'' - 3rf'^2)) = -2C_{1212,2}
\]
\[
C_{1213,3} = \frac{2e^{rf}}{3r}(2f' - 2rf'^2 + rf'') = C_{1321,3} = -\frac{e^{rf}}{3r}C_{1424,1} = \frac{e^{rf}}{3r}C_{2434,3};
\]
\[
r^2 e^{-4f}(C \cdot C)_{123212} = \frac{e^{-4f}}{3}(2f' - 2rf'^2 + rf'')^2 = -(C \cdot C)_{124212} = (C \cdot C)_{122414}
\]
\[
= -(C \cdot C)_{13414} = -r^2 e^{-4f}(C \cdot C)_{121323} = -(C \cdot C)_{243423} = (C \cdot C)_{131434} = (C \cdot C)_{234234};
\]
\[
r^2 e^{-4f}Q(g, C)_{123212} = -Q(g, C)_{142412} = r(2f' - 2rf'^2 + rf'') = -r^2 e^{-4f}Q(g, C)_{121323}
\]
\[
= Q(g, C)_{13414} = -Q(g, C)_{122414} = -Q(g, C)_{131434} = Q(g, C)_{243423} = Q(g, C)_{234234};
\]
\[
Q(S, R)_{121323} = \frac{3e^{-2f}}{r}(f'^3(3 - 2rf') + f''(f' - r f' + rf'')) = -Q(S, R)_{121323},
\]
\[
Q(S, R)_{142412} = -e^{-2f} f'^3(3 - 2rf') + f''(5f' - 3rf'^2 + rf'') = Q(S, R)_{243423},
\]
\[
Q(S, R)_{133414} = e^{-2f} f' f'' = -Q(S, R)_{131434},
\]
\[
Q(S, R)_{122414} = -e^{-2f} f'(3 - 2rf')(f' + rf'') = Q(S, R)_{232434},
\]
\[
Q(S, R)_{121424} = -e^{-2f}(3f' - 2rf'^2 + rf'')(f' - rf'^2 - rf'') = Q(S, R)_{233424}.
\]
The values of the above components help us to obtain the following relations on \( U_C \):
\[
C \cdot C = \mathcal{L}_C Q(g, C) \quad \text{for} \quad \mathcal{L}_C = \frac{e^{-2f}}{3r}(2f' - 2rf'^2 + rf''),
\]
\[
R \cdot R - Q(S, R) = \mathcal{L} Q(g, C) \quad \text{for} \quad \mathcal{L} = -\frac{e^{-2f} f''}{3r\mathcal{L}_C}(3f'^3 - 2rf'^3 + f'').
\]
The relation \((3.3)\) shows that the metric \((1.4)\) has pseudosymmetric Weyl tensor. Throughout this paper we consider the smooth functions \( \mathcal{L}_R, \mathcal{L}_C \) and \( \mathcal{L} \) as defined respectively in Lemma 3.2 relation \((3.3)\) and \((3.4)\). By a straightforward calculation the local components of the tensors \( g \wedge g, g \wedge S, g \wedge S^2, S \wedge S \) and \( S \wedge S^2 \) can easily be computed. Now these tensors help us to decompose the tensor \( R \) as
\[
R = \mathcal{L}_{12} g \wedge S + \mathcal{L}_{13} g \wedge S^2 + \mathcal{L}_{22} S \wedge S + \mathcal{L}_{23} S \wedge S^2
\]
where \( L_{12} = \frac{r e^{2f} L_{R} + f'}{2(r e^{2f} L_{R} + f')} + \frac{f'}{2L_e}, \ L_{13} = -\frac{e^{2f} r^2 f''}{4(r e^{2f} L_{R} + f') L_e^2}, \ L_{22} = (L_{13} L_{e} e^{-2f} - \frac{r^2 e^{2f}}{2L_e^2} L_{R}), \ L_{23} = 2r e^{2f} L_{13} L_{R} \) and \( L_e = (f' + r f'') \). Now contracting the relation (3.3) we get

(3.6) \[ a_{11} g + a_{22} S + a_{33} S^2 + a_{44} S^3 = 0 \]

where \( a_{11} = -(2L_{R} + \frac{e^{2f}}{r} L_{e} a_{22}), \ a_{22} = -\frac{e^{2f} L_{R}}{r e^{2f} L_{R} + f'}, \ a_{33} = -\frac{e^{2f}}{L_{e}} (a_{22} + \frac{2 r e^{2f}}{L_{R}} L_{R}), \ a_{44} = -\frac{r^2}{L_e} e^{2f} a_{22} \)

and \( L_e \) is defined above. Further from the non-zero components \( C_{abcd} \) and \( C_{abcd, \alpha} \) on \( U_C \) we obtain the curvature 2-forms of recurrence for conformal curvature tensor with 1-form of recurrence as

\[ \Pi_1 = 0, \ \Pi_2 = -\frac{3 r e^{2f}}{L_C} (2 f' - 2 r f'^2 + 2 r^2 f'^3 - 2 r f'' + 3 r^2 f' f'' - r^2 f'''), \ \Pi_3 = 0, \ \Pi_4 = 0. \]

From relation (3.3)–(3.6) we can state the following:

**Theorem 3.2.** The metric (1.4) admits the following geometric structures on \( U_C \):

(i) it has pseudosymmetric Weyl conformal tensor,

(ii) it satisfies the pseudosymmetric type condition

\[ R \cdot R - Q(S, R) = -e^{-2f} f' \left( \frac{3 f'^2 - 2 r f'^3 + f''}{2 f' - 2 r f'^2 + f''} \right) Q(g, C), \]

(iii) it is generalized Roter type manifold,

(iv) it is Ein(3) manifold,

(v) its conformal 2-forms are recurrent.

**Lemma 3.3.** The metric (1.4) satisfies the Roter type condition on \( U_C \) if \( r f'' + r f' - f'^2 = 0 \).

**Proof.** Remark 2.2 of [25] states that a manifold \((M, g), n \geq 4\), is Roter type on \( \{ x \in M : (C)_x \neq 0 \} \) if it is pseudosymmetric i.e., \( R \cdot R = L'_R Q(g, R) \) and satisfies the pseudosymmetric type condition \( R \cdot R - Q(S, R) = L'_R Q(g, C) \), where \( L'_R \) and \( L'_R \) are some functions on \( \{ x \in M : (C)_x \neq 0 \} \).

Hence, Lemma 3.2 and the relation (3.3) imply that the metric (1.4) satisfies the Roter type condition on \( U_C \) if \( r f'' + r f' - f'^2 = 0 \).

**Remark 3.1.** If \( r f'' + r f' - f'^2 = 0 \) then by Lemma 3.3 and from Remark 2.2 of [25], we can write the tensor \( R \) on \( U_C \) for the metric (1.4) as

(3.7) \[ R = \mu S \wedge S - 2 \mu (L_R - L) g \wedge S + (\mu (L_R - L)^2 - \frac{L'}{4}) g \wedge g \]

where \( \mu \) is some function on \( U_C \) and is given by \( \mu = -\frac{1}{4(L_R - L)} \).
Corollary 3.1. Thus from Theorem 6.7 of [15] and by the Lemma 3.3 we obtain the condition \( rf'' + rf'^2 - f' = 0 \) as the sufficient condition of the metric (1.4) to realise the following geometric structures:

1. \( R \cdot R = \mathcal{L}_R Q(g, R), \quad \mathcal{L}_R = \frac{e^{-rf}}{r}(f' - rf^2), \)
2. \( C \cdot R = \mathcal{L}_R Q(g, R), \)
3. \( S^2 + \lambda g = 0, \quad \lambda = -3\mathcal{L}_R(\mathcal{L}_R - \mathcal{L}), \)
4. \( Q(S, C) = C \cdot R - R \cdot C. \)

4. Melvin magnetic metric admitting geometric structures

It is easy to see that \( f(r) = \ln(1 + \frac{B_0 r^2}{4}) \) satisfies the equation \( rf'' + rf'^2 - f' = 0 \). Hence from Lemma 3.2, Lemma 3.3 and Corollary 3.1 we can state the following:

Theorem 4.1. The Melvin magnetic metric (1.1) admits the following properties:

(i) scalar curvature \( \kappa = 0 \) and hence \( C = K \) and \( R = W \),

(ii) it is pseudosymmetric and pseudosymmetric due to conformal curvature tensor i.e., satisfies the relations

\[
R \cdot R = \mathcal{L}_1 Q(g, R), \quad C \cdot R = \mathcal{L}_1 Q(g, R) \quad \text{where} \quad \mathcal{L}_1 = \frac{32B_0^2(4 - B_0^2r^2)}{(4 + B_0^2r^2)^4},
\]

(iii) it also satisfies the following pseudosymmetric type curvature conditions

\[
\begin{align*}
(a) \quad R \cdot R - Q(S, R) &= \mathcal{L}_2 Q(g, C), \quad \mathcal{L}_2 = -\frac{32B_0^2(16 + 24B_0^2r^2 - 3B_0^4r^4)}{3(4 - B_0^2r^2)(4 + B_0^2r^2)^4}, \\
(b) \quad Q(S, C) &= C \cdot R - R \cdot C, \\
(c) \quad C \cdot R - R \cdot C &= \mathcal{L}_3 Q(g, R) + \mathcal{L}_4 Q(S, R), \quad \mathcal{L}_3 = -\frac{2048B_0^6(4 - B_0^2r^2)}{\omega(4 + B_0^2r^2)^4}, \\
& \quad \mathcal{L}_4 = 1 + \frac{64}{\omega}, \quad \omega = -16 - 24B_0^2r^2 + 3B_0^4r^4,
\end{align*}
\]

(iv) curvature 2-forms for \( C \) or \( K \) are recurrent with associated 1-form of recurrency

\[
\Pi = \left\{ 0, 0, -\frac{16B_0^2r}{(4 - B_0^2r^2)(4 + B_0^2r^2)^4}, 0 \right\},
\]

(v) it is Roter type spacetime i.e., satisfies \( R = N_1 S \wedge S + N_2 g \wedge S + N_3 g \wedge g \) where

\[
\begin{align*}
N_1 &= -3(4 - B_0^2r^2)(4 + B_0^2r^2)^4 \quad \text{where} \quad N_1 = -\frac{3(4 - B_0^2r^2)(4 + B_0^2r^2)^4}{8192B_0^8}, \\
N_2 &= \frac{1}{2}, \quad N_3 = -\frac{8B_0^6(4 - B_0^2r^2)}{(4 + B_0^2r^2)^4}
\end{align*}
\]
(vi) 2-quasi Einstein for $\alpha = \frac{256B_{0}^{2}}{(4+B_{0}^{2}r^{2})^{4}}$.

(vii) generalized quasi Einstein due to Chaki for

$$\alpha = -\frac{256B_{0}^{2}}{(4+B_{0}^{2}r^{2})^{4}}, \quad \beta = -1, \quad \gamma = 1, \quad \Pi = \left\{ -\frac{4\sqrt{2}B_{0}}{(4+B_{0}^{2}r^{2})^{2}}, 0, \frac{4\sqrt{2}B_{0}}{\sqrt{(16+8B_{0}^{2}r^{2}+B_{0}^{4}r^{4})}}, 0 \right\},$$

$$\delta = \left\{ 0, 0, \frac{4\sqrt{2}B_{0}}{(4+B_{0}^{2}r^{2})^{2}}, 0 \right\}, \quad \text{with} \quad \|\Pi\| = 0 \quad \text{and} \quad \|\delta\| = \frac{512B_{0}^{2}}{(4+B_{0}^{2}r^{2})^{4}},$$

(viii) Ein(2) spacetime, since $S^{2} + \lambda g = 0$ for $\lambda = -\frac{65536B_{0}^{2}}{(4+B_{0}^{2}r^{2})^{8}}$.

(ix) Ricci tensor is Reimann compatible only.

Remark 4.1. It is worthy to note the following facts about the Melvin magnetic spacetime:

1. Melvin magnetic spacetime is a pseudosymmetric warped product of 3-dimensional pseudosymmetric base and 1-dimensional fibre with warping function $F(r) = r/U_{B}$, where $U_{B} = 1 + \frac{B_{0}r^{2}}{4}$.

2. Melvin magnetic spacetime, a warped product of 3-dimensional conharmonically recurrent base and 1-dimensional fibre, is not conharmonically recurrent but its curvature 2-forms for conharmonic tensor are recurrent.

3. Melvin magnetic spacetime is a 2-quasi-Einstein warped product spacetime of 3-dimensional quasi-Einstein base and 1-dimensional fibre.

4. Referred to the Remark 2.1, the Melvin magnetic spacetime is a warped product spacetime of 3-dimensional base and 1-dimensional fibre satisfying both the relations (2.2) and (2.3).

The non zero components of Maxwell tensor from (1.2) are given by $F_{24} = \frac{8B_{0}r}{(4+B_{0}^{2}r^{2})^{2}} = -F_{42}$.

The non-zero components of the tensors $R \cdot F$ and $Q(g, F)$ are

$$(R \cdot F)_{1412} = -(R \cdot F)_{1214} = -\frac{16B_{0}^{3}r(4-B_{0}^{2}r^{2})}{(4+B_{0}^{2}r^{2})} = R \cdot F_{3423} = -R \cdot F_{2334};$$

$$Q(g, F)_{1214} = -Q(g, F)_{1412} = \frac{B_{0}r}{2} = Q(g, F)_{2334} = -Q(g, F)_{3423}.$$

From above we have the following:

Theorem 4.2. $R \cdot F = \mathcal{L}_{F} Q(g, F)$ where $\mathcal{L}_{F} = \frac{32B_{0}^{2}(4-B_{0}^{2}r^{2})}{(4+B_{0}^{2}r^{2})^{4}}$ i.e., Maxwell tensor of Melvin spacetime is pseudosymmetric type.

5. Conclusion

We have studied the curvature restricted geometric properties of Melvin magnetic spacetime, a warped product with 1-dimensional fibre, and found that it is non-semisymmetric pseudosymmetric warped product with 3-dimensional pseudosymmetric base. Also it is non-quasi-Einstein.
2-quasi-Einstein warped product spacetime with 3-dimensional quasi-Einstein base. Moreover its Maxwell tensor is pseudosymmetric type. Hence, Melvin magnetic spacetime is evidently a model of 4-dimensional pseudosymmetric and 2-quasi-Einstein warped product manifolds with 1-dimensional fibre.

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References

[1] Adamów, A. and Deszcz, R., On totally umbilical submanifolds of some class of Riemannian manifolds, Demonstratio Math., 16 (1983), 39–59.
[2] Arslan, K., Deszcz, R., Ezentaş, R., Hotloš, M. and Murathan, C., On generalized Robertson-Walker spacetimes satisfying some curvature condition, Turk. J. Math., 38(2) (2014), 353–373.
[3] Besse, A.L., Einstein Manifolds, Springer-Verlag, Berlin-New York, 1987.
[4] Cartan, É., Sur une classe remarquable de désaces de Riemann, I, Bull. de la Soc. Math. de France, 54 (1926), 214–216.
[5] Cartan, É., Leçons sur la Géométrie des Espaces de Riemann, 2nd edn. (Gauthier Villars, Paris, 1946).
[6] Chaki, M. C., On pseudosymmetric manifolds, An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.), 33 (1987), 53–58.
[7] Chaki, M. C., On generalized quasi-Einstein manifolds, Publ. Math. Debrecen, 58 (2001), 683–691.
[8] Defever, F., Deszcz, R., Hotloš, M., Kucharaki, M. and Sentürk, Z., Generalisations of Robertson-Walker spaces, Ann. Univ. Sci. Budapest, Eötvös Sect. Math., 44 (2001), 13–24.
[9] Defever, F., Deszcz. R. and Prvanović, M., On warped product manifolds satisfying some curvature conditions of pseudosymmetric type, Bull. Greek Math. Soc. 36 (1994), 43–67.
[10] Defever, F., Deszcz, R., Verstraelen, L. and Vrancken, L., On pseudosymmetric spacetimes, J. Math. Phys., 35(11) (1994), 5908–5921.
[11] Deszcz, R., On four-dimensional Reimannian warped product manifolds satisfying certain pseudo-symmetric curvature conditions, Colloq. Math., 62 (1991), 103–120
[12] Deszcz, R., On pseudosymmetric spaces, Bull. Belg. Math. Soc., Series A, 44 (1992), 1–34.
[13] Deszcz, R., On Roter type manifolds, 5-th Conference on Geometry and Topology of Manifolds April 27 - May 3, 2003, Krynica, Poland.
[14] Deszcz, R. and Głogowska, M., Some examples of nonsemisymmetric Ricci-semisymmetric hypersurfaces, Colloq. Math., 94 (2002), 87–101.
[15] Deszcz, R., Głogowska, M, Hotloš, M and Sawicz, K., A Survey on Generalized Einstein Metric Conditions, Advances in Lorentzian Geometry, Proceedings of the Lorentzian Geometry Conference in Berlin, AMS/IP Studies in Advanced Mathematics, 49, S.-T. Yau (series ed.), M. Plaue, A.D. Rendall and M. Scherfner (eds.), 2011, 27–46.
[16] Deszcz, R., Głogowska, M., Hotloš, M. and Sentürk, Z., On certain quasi-Einstein semi-symmetric hypersurfaces, Ann. Univ. Sci. Budapest Eötvös Sect. Math., 41 (1998), 151–164.
[17] Deszcz, R., Głogowska, M., Jelowicki, L., Petrović-Torgašev, M. and Zafindratafa, G., On Riemann and Weyl compatible tensors, Publ. Inst. Math. (Beograd) (N.S.), 94(108) (2013), 111–124.

[18] Deszcz, R., Głogowska, M., Jelowicki, J. and Zafindratafa, Z., Curvature properties of some class of warped product manifolds, Int. J. Geom. Meth. Modern Phys., 13 (2016), 1550135 (36 pages).

[19] Deszcz, R., Głogowska, M., Petrović-Torgašev, M. and Verstraelen, L., On the Roter type of Chen ideal submanifolds, Results Math., 59 (2011), 401–413.

[20] Deszcz, R., Głogowska, M., Petrović-Torgašev, M. and Verstraelen, L., Curvature properties of some class of minimal hypersurfaces in Euclidean spaces, Filomat, 29 (2015), 479–492.

[21] Deszcz, R. and Hotloš, M., n hypersurfaces with type number two in spaces of constant curvature, Ann. Univ. Sci. Budapest Eötvös Sect. Math., 46 (2003), 19–34.

[22] Deszcz, R., Hotloš, M., Jelowicki, J., Kundu, H. and Shaikh, A.A., Curvature properties of Gõdel metric, Int J. Geom. Methods Mod. Phy., 11 (2014), Article ID: 1450025, 20 pp and Erratum: Curvature properties of Gõdel metric, Int. J. Geom. Methods Mod. Phys., 16 (2019), Article ID: 1992002, 4 pp.

[23] Deszcz, R., Kowalczyk, D., On some class of pseudosymmetric warped products, Colloq. Math. 97 (2003), 7–22.

[24] Deszcz, R. and Kucharski, M., On curvature properties of certain generalized Robertson-Walker spacetimes, Tsukuba J. Math., 23(1) (1999), 113–130.

[25] Deszcz, R., Plaue, M., Scherfner, M.: On Roter type warped products with 1-dimensional fibres. J. Geom. Phys. 69 (2013), 1–11.

[26] Głogowska, M., Semi-Riemannian manifolds whose Weyl tensor is a Kulkarni-Nomizu square, Publ. Inst. Math. (Beograd) (N.S.), 72(86) (2002), 95–106.

[27] Głogowska, M., emphOn Roter type manifolds, in: Pure and Applied Differential Geometry- PADGE 2007, Shaker Verlag, Aachen, 2007, 114–122.

[28] Gray, A., Einstein-like manifolds which are not Einstein, Geom. Dedicata, 7 (1978), 259–280.

[29] Griffiths, J. B. and Podolský, J., Exact space-times in Einsteins general relativity, Cambridge University Press, 2009.

[30] Lovelock, D. and Rund, H., Tensors, differential forms and variational principles, Courier Dover Publications, 1989..

[31] Mantica, C.A. and Molinari, L.G., Extended Derdzinski-Shen theorem for curvature tensors, Colloq. Math., 128 (2012), 16.

[32] Mantica, C.A. and Molinari, L.G., Riemann compatible tensors, Colloq. Math., 128 (2012), 197–210.

[33] Mantica, C. A. and Suh, Y. J., The closedness of some generalized curvature 2-forms on a Riemannian manifold I, Publ. Math. Debrecen, 81/3-4 (2012), 313–326.

[34] Mantica, C. A. and Suh, Y. J., The closedness of some generalized curvature 2-forms on a Riemannian manifold II, Publ. Math. Debrecen, 82/1 (2013), 163–182.

[35] Mantica, C. A. and Suh, Y. J., Recurrent conformal 2-forms on pseudo-Riemannian manifolds, Int. J. Geom. Meth. Mod. Phy., 11(6) (2014), 1450056 (29 pages).

[36] Melvin, M. A., Pure magnetic and electric geons, Phys. Lett., 8 (1964), 65–68

[37] Melvin, M. A., Dynamics of Cylindrical Electromagnetic Universe, Phys. Lett., 139 (1965), B225–B243.

[38] Prvanović, M., On weakly symmetric Riemannian manifolds, Publ. Math. Debrecen, 46(1-2) (1995), 19–25.

[39] Ruse, H. S., On simply harmonic spaces, J. London Math. Soc., 21 (1946), 243–247.

[40] Ruse, H. S., On simply harmonic kappa spaces of four dimensions, Proc. London Math. Soc., 50 (1949), 317–329.
[41] Ruse, H. S., *Three dimensional spaces of recurrent curvature*, Proc. London Math. Soc., **50** (1949), 438–446.

[42] Shaikh, A. A., *On pseudo quasi-Einstein manifolds*, Period. Math. Hungarica, **59(2)** (2009), 119–146.

[43] Shaikh, A. A., Al-Solamy, F. R. and Roy, I., *On the existence of a new class of semi-Riemannian manifolds*, Math. Sci. (Springer), **7:46** (2013), 1–13.

[44] Shaikh, A.A. and Binh T.Q., *On some class of Riemannian manifolds*, Bull. Transilvania Univ. **15(50)** (2008), 351–362.

[45] Shaikh, A. A., Das, L. and Chakraborty, D., *Curvature properties of Siklos metric*, to appear in Diff. Goem.-Dyn. Syst.

[46] Shaikh, A. A., Ali, M. and Ahsan, Z., *Curvature properties of Robinson-Trautman metric*, J. Geom., **109(38)** (2018), [http://dx.doi.org/10.1007/s000022-018-0443-1](http://dx.doi.org/10.1007/s000022-018-0443-1).

[47] Shaikh, A.A., Deszcz, R., Hotloś, M., Jelówicki, J. and Kundu, H., *On pseudosymmetric manifolds*, Publ. Math. Debrecen, **86(3-4)** (2015), 433–456.

[48] Shaikh, A. A., Kundu, H., *On weakly symmetric and weakly Ricci symmetric warped product manifolds*, Publ. Math. Debrecen **81(34)** (2012), 487–505.

[49] Shaikh, A. A. and Kundu, H., *On equivalency of various geometric structures*, J. Geom., **105** (2014), 139–165.

[50] Shaikh, A. A. and Kundu, H., *On warped product generalized Roter type manifolds*, Balkan J. Geom. Appl., **21(2)** (2016), 82–95.

[51] Shaikh, A. A. and Kundu, H., *On generalized Roter type manifolds*, Kragujevac J. Math 43(3) (2019), Pages 471–493.

[52] Shaikh, A. A. and Kundu, H., *On some curvature restricted geometric structures for projective curvature tensor*, Int. J. Geom. Meth. Modern Phys., **15(9)** (2018), 18501157 (38 pages).

[53] Shaikh, A. A. and Kundu, H., *On warped product manifolds satisfying some pseudosymmetric type conditions*, Differ. Geom. Dyn. Syst., **19** (2017), 119–135.

[54] Shaikh, A.A., Kundu, H. and Sen, J., *Curvature properties of Vaidya metric*, Indian J. Math., **61(1)**, 2019, 41–59

[55] Shaikh, A. A. and Matsuyama, Y., *Curvature properties of some 4-dimensional semi-Riemannian metrics*, Kragujevac Journal of Mathematics, **41(2)** (2017), 259-278.

[56] Shaikh, A.A. and Jana S.K., *On weakly cyclic Ricci symmetric manifolds*, Ann. Pol. Math., **89(3)** (2006), 139–146.

[57] Shaikh, A.A. and Hui, S.K., *On decomposable quasi-Einstein spaces*, Math. Reports, **13(63)** (2011), 89–94.

[58] Shaikh, A.A., Hui, S. K. and Yoon, W. Dae, *On quasi Einstein Spacetimes*, Tsukuba J. Math., **33(2)** (2009), 305–326.

[59] Shaikh, A.A., Kim, Y.H. and Hui, S.K., *On Lorentzian quasi-Einstein manifolds*, J. Korean Math. Soc., **48** (2011), 669–689, and Erratum to: On Lorentzian quasi-Einstein manifolds, J. Korean Math. Soc. **48** (2011), 1327–1328.

[60] Shaikh, A. A. and Patra, A., *On a generalized class of recurrent manifolds*, Arch. Math. (BRNO), **46** (2010), 71–78.

[61] Shaikh, A.A. and Roy, I., *On quasi generalized recurrent manifolds*, Math. Pannon., **21(2)** (2010), 251–263.

[62] Shaikh, A. A. and Roy, I., *On weakly generalized recurrent manifolds*, Ann. Univ. Sci. Budapest, Eötvös Sect. Math., **54** (2011) 35–45.

[63] Shaikh, A. A., Roy, I. and Kundu, H., *On some generalized recurrent manifolds*, Bull. Iranian Math. Soc., **43(5)** (2017).
Shaikh, A. A., Roy, I. and Kundu, H., *On the existence of a generalized class of recurrent manifolds*, An. Ştiinţ. Univ. Al. I. Cuza Iaşi Mat. (N. S.), LXIV(2) (2018), 233–251.

Shaikh, A. A., Srivastava, S. K. and Chakraborty, D., *Curvature properties of anisotropic scale invariant metrics*, Int. J. Geom. Meth. Mod. Phys., 16 (2019), 195086 (17 pages).

Sawicz, K., *Curvature properties of some class of hypersurfaces in Euclidean spaces*, Publ. Inst. Math. (Beograd) (N.S.) 98(112) (2015), 167–177.

Suh, Y. J., Kwon, J-H. and Pyo, Y. S., *On semi-Riemannian manifolds satisfying the second Bianchi identity*, J. Korean Math. Soc., 40(1) (2003), 129–167.

Szabó, Z. I., *Structure theorems on Riemannian spaces satisfying $R(X, Y ) \cdot R = 0$, I. The local version*, J. Diff. Geom., 17 (1982), 531–582.

Szabó, Z. I., *Classification and construction of complete hypersurfaces satisfying $R(X, Y ) \cdot R = 0$*, Acta Sci. Math., 47 (1984), 321–348.

Szabó, Z. I., *Structure theorems on Riemannian spaces satisfying $R(X, Y ) \cdot R = 0$, II. The global version*, Geom. Dedicata, 19 (1985), 65–108.

Tachibana, S., *A Theorem on Riemannian manifolds of positive curvature operator*, Proc. Japan Acad., 50 (1974), 301–302.

Támassy, L. and Binh, T.Q., *On weakly symmetric and weakly projective symmetric Riemannian manifolds*, Coll. Math. Soc. J. Bolyai, 50 (1989), 663–670.

Thorne, K. S., *Absolute Stability of Melvin’s Magnetic Universe*, Phys. Rev., 139 (1965), B244–B254.

Venzi, P., *Una generalizzazione degli spazi ricorrenti*, Revue Roumaine de Math. Pure at appl., 30 (1985), 295–305.

Walker, A.G., *On Ruses spaces of recurrent curvature*, Proc. London Math. Soc., 52 (1950), 36–64.

Shaikh, A.A. and Kundu, H., *On weakly symmetric and weakly Ricci symmetric warped product manifolds*, Publ. Math. Debrecen, 48(3-4) (2012), 487–505.

1 DEPARTMENT OF MATHEMATICS, UNIVERSITY OF BURDWAN, GOLAPBAG, BURDWAN-713104, WEST BENGAL, INDIA

E-mail address: aask2003@yahoo.co.in, aashaikh@mathburuniv.ac.in

E-mail address: dhyanesh2011@gmail.com

2 DEPARTMENT OF MATHEMATICS, COLLEGE OF SCIENCE, KING KHALID UNIVERSITY, 9004 ABHA, SAUDI ARABIA

E-mail address: akramali133@gmail.com ; akali@kku.edu.sa

E-mail address: ahalkhaldi@kku.edu.sa