The stochastic limit in the analysis of the open BCS model

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Abstract

In this paper we show how the perturbative procedure known as stochastic limit may be useful in the analysis of the Open BCS model discussed by Buffet and Martin as a spin system interacting with a fermionic reservoir. In particular we show how the same values of the critical temperature and of the order parameters can be found with a significantly simpler approach.

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\(^1\)This paper is dedicated with all my love to my sweet father
I Introduction

In this paper we analyze the Open BCS model as given in [1, 2] using the techniques of the stochastic limit approach (SLA), which is described in many details in the monograph [3]. Instead of considering a fermionic reservoir, as the authors do in [1, 2] (following the original suggestion contained in [4] which allow to avoid dealing with unbounded operators), we will consider here a bosonic thermal bath. This choice is made to try to stay closer to the real physical world, where the reservoir is bosonic. This means that some of our formulas are only formal, but they can be made rigorous with just a little effort, using, for instance, the same framework for unbounded operators developed in [5] and references therein. We will comment again on this aspect of our model in the next section.

The main outcome of this paper is that the same values of the critical temperature and of the order parameters can be found using the SLA, in a significantly simpler way, as we will show in Section III. This simplification allows us to focus our attention on some aspects of the model which could appear not so clearly using the standard technique. This is what has been already observed in other physical applications: for instance, in [6], we used the SLA to explore in details some relations between different models of matter interacting with the radiation, as the Hepp-Lieb and the Alli-Sewell models. Also, in [7], the SLA was used in connection with the fractional quantum Hall effect, giving some interesting results. Other applications are contained in [3] and references therein.

The paper is organized as follows:

in the next section we introduce the model and compute its generator using the SLA together with a semiclassical approximation, already introduced in [2], useful to obtain the free evolution of the matter operators;

in Section III we write the equations of motion for some macroscopic variables of the matter and we recover the same results as in [1];

our conclusions are contained in Section IV, while the Appendix is devoted to review some facts concerning the SLA, useful to keep the paper self contained.

II The Physicals Model and its stochastic limit

Our model consists of two main ingredients, the system, which is described by spin variables, and the reservoir, which is given in terms of bosonic operators. It is contained in a box of
volume $V = L^3$, with $N$ lattice sites. We define, following [1, 2]

$$H_N^{(sys)} = \tilde{\epsilon} \sum_{j=1}^{N} \sigma_j^0 - \frac{g}{N} \sum_{i,j=1}^{N} \sigma_i^+ \sigma_j^-,$$  

(2.1)

where the indexes $i, j$ represent the discrete values of the momentum that an electron in a fixed volume can have, $\sigma_j^+$ creates a Cooper pair with given momentum while $\sigma_j^-$ annihilates the same pair, $\tilde{\epsilon}$ is the energy of a single electron and $-g < 0$ is the interaction close to the Fermi surface. As we can see, only the $\pm$ component of the spin, that is the $x, y$ components, have a mean field interaction, while the $z$ component interacts with a constant external magnetic field. The algebra of the Pauli matrices is given by

$$[\sigma_i^+ \sigma_j^-, \sigma_k^0] = \delta_{ij} \sigma_k^0, \quad [\sigma_i^\pm, \sigma_j^0] = \mp 2 \delta_{ij} \sigma_i^\pm.$$  

(2.2)

We will use the following realization of these matrices:

$$\sigma_i^0 \equiv \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$  

If we now define the following operators,

$$S_\alpha^N = \frac{1}{N} \sum_{i=1}^{N} \sigma_i^\alpha, \quad R_N = S_N^+ S_N^- = R_N^\dagger,$$  

(2.3)

$H_N^{(sys)}$ can be simply written as $H_N^{(sys)} = N(\tilde{\epsilon} S_0^N - g R_N)$ and it is easy to check that the following commutation rules hold:

$$[S_0^N, R_N] = [H_N^{(sys)}, R_N] = [H_N^{(sys)}, S_0^N] = 0,$$

for any given $N > 0$. It is also worth noticing that the intensive operators $S_\alpha^N$ are all bounded by 1 in the operator norm, and that the commutators $[S_\alpha^N, \sigma_j^\beta]$ go to zero in norm as $1/N$ when $N \to \infty$, for all $j, \alpha$ and $\beta$.

As we have already mentioned in the Introduction, we consider here a realistic bosonic reservoir, so that some of the following formulas must be understood to be formal. However, using for instance the same algebraic framework discussed in [5] for some different spin-bosons models, or replacing the bosonic operators with their smeared versions, everything can be made rigorous. We avoid here this useless complication, since it would make all the treatment much more complicated, hiding in this way our main results.
Our construction of the reservoir follows the same steps given in [2], but for the commutation rules. We introduce here as many bosonic modes $a_{\vec{p},j}$ as lattice sites are present in $V$. This means that $j = 1, 2, ..., N$. $\vec{p}$ is the value of the momentum of the $j$-th boson which, if we impose periodic boundary condition on the wave functions, has necessarily the form $\vec{p} = \frac{2\pi}{L} \vec{n}$, where $\vec{n} = (n_1, n_2, n_3)$ with $n_j \in \mathbb{Z}$. These operators satisfy the following CCR,

$$[a_{\vec{p},i}, a_{\vec{q},j}] = [a_{\vec{p},i}^\dagger, a_{\vec{q},j}^\dagger] = 0, \quad [a_{\vec{p},i}, a_{\vec{q},j}^\dagger] = \delta_{ij} \delta_{\vec{p} \vec{q}}$$

(2.4)

and their free dynamics is given by

$$H_N^{(\text{res})} = \sum_{j=1}^{N} \sum_{\vec{p} \in \Lambda_N} \epsilon_{\vec{p}} a_{\vec{p},j}^\dagger a_{\vec{p},j}. \quad (2.5)$$

Here $\Lambda_N$ is the set of values which $\vec{p}$ may take, according to the previous remark: $\Lambda_N = \{\vec{p} = \frac{2\pi}{L} \vec{n}, \vec{n} \in \mathbb{Z}^3\}$. It is useful to stress that the energy of the different bosons is clearly independent of the lattice site: $\epsilon_{\vec{p}} = \frac{p^2}{2m} = \frac{4\pi^2(n_1^2 + n_2^2 + n_3^2)}{2mL^2}$.

The form of the interaction between reservoir and system is assumed to be of the following form:

$$H_N^{(I)} = \sum_{j=1}^{N} (\sigma_j^+ a_j(f) + \text{h.c.}), \quad (2.6)$$

where we have introduced $a_j(f) = \sum_{\vec{p} \in \Lambda_N} a_{\vec{p},j} f(\vec{p})$, $f$ being a given test function which will be asked to satisfy some extra conditions, see equation (2.24) below and the related discussion. We would like to stress that, in order to keep the notation reasonably simple, we will not use the tensor product symbol along this paper whenever the meaning of the symbols is clear.

The finite volume open system is now described by the following hamiltonian,

$$H_N = H_N^{(0)} + \lambda H_N^{(I)}$$

where $H_N^{(0)} = H_N^{(\text{sys})} + H_N^{(\text{res})}$

(2.7)

and $\lambda$ is the coupling constant.

The first step in the SLA is the computation of the free evolution of the interaction hamiltonian:

$$H_N^{(I)}(t) = e^{iH_N^{(I)}t} H_N^{(I)} e^{-iH_N^{(I)}t} = \sum_{j=1}^{N} (e^{iH_N^{(\text{sys})}t} \sigma_j^+ e^{-iH_N^{(\text{sys})}t} e^{iH_N^{(\text{res})}t} a_j(f) e^{-iH_N^{(\text{res})}t} + \text{h.c.}). \quad (2.8)$$

The computation of the part of the reservoir is trivial and produces

$$e^{iH_N^{(\text{res})}t} a_j(f) e^{-iH_N^{(\text{res})}t} = a_j(f e^{-it\epsilon}),$$
where 
\[ a_j (f e^{-it\varepsilon}) = \sum_{\bar{p} \in \Lambda_N} a_{\bar{p},j} f(\bar{p}) e^{-it\varepsilon}. \]
This is an easy consequence of the CCR (2.4). The free evolution of the spin operators is more difficult and its expression can be found in [1, 2], for instance. Here it is shown how to obtain the time evolution in a semiclassical approximation, i.e., when the free time evolution of the intensive operators \( S_N^\alpha \) is replaced by its limit (taken in the strong topology restricted to a certain family of relevant vectors, [8]).

The differential equations of motion for the spin variables are

\[
\begin{align*}
\frac{d\sigma_j^+(t)}{dt} &= 2i\tilde{e}\sigma_j^+(t) + igS_j^0(t)\sigma_j^0(t) \\
\frac{d\sigma_j^0(t)}{dt} &= 2ig(\sigma_j^+(t)S_j^- - \sigma_j^-(t)S_j^+),
\end{align*}
\]

where we have called, with a little abuse of language which is quite useful to maintain the notation simple, \( \sigma_j^\alpha(t) = e^{iH_j^{(sys)}t}\sigma_j^\alpha e^{-iH_j^{(sys)}t} \). In fact, to be more precise, instead of \( \sigma_j^\alpha(t) \), we should write \( \sigma_j^\alpha,\text{free}(t) \), to stress the fact that \( e^{iH_j^{(sys)}t}\sigma_j^\alpha e^{-iH_j^{(sys)}t} \) only produces the free evolution of \( \sigma_j^\alpha \), i.e., the evolution without any reservoir, and for \( N \) fixed. Moreover, in (2.9) we have introduced \( S_N^\alpha(t) = e^{iH_j^{(sys)}t}S_N^\alpha e^{-iH_j^{(sys)}t} = \frac{1}{N}\sum_{j=1}^N e^{iH_j^{(sys)}t}\sigma_j^\alpha e^{-iH_j^{(sys)}t} = \frac{1}{N}\sum_{j=1}^N \sigma_j^\alpha(t) \).

Let us now call \( S^\alpha = F - \text{strong lim}_{N \to \infty} S_N^\alpha \). The proof of the existence of this limit (together with all its powers) may be found in [8] and references therein. We can now take the sum over \( j = 1,2,...,N \) of (both sides of) the equations in (2.9), divide the result by \( N \), and consider the \( F - \text{strong lim}_{N \to \infty} \) of the equations obtained in this way. We find that \( \dot{S}^0(t) = 0 \) and \( \dot{S}^+(t) = i(2\tilde{e} + gS^0(t))S^+(t) \). These equations can be easily solved: \( S^0(t) = S^0 = (S^0)^\dagger \) and \( S^+(t) = S^+ e^{i(2\tilde{e} + gS^0)t} \). Of course \( S^-(t) = (S^+(t))^\dagger \). The system (2.9) gives now, if we replace \( S_N^\alpha(t) \) with its \( F - \text{strong} \) limit \( S^\alpha(t) \),

\[
\begin{align*}
\frac{d\sigma_j^+(t)}{dt} &= 2i\tilde{e}\sigma_j^+(t) + igS^+(t)\sigma_j^0(t) \\
\frac{d\sigma_j^0(t)}{dt} &= 2ig(\sigma_j^+(t)S^- - \sigma_j^-(t)S^+).
\end{align*}
\]

This system is called the semiclassical approximation of (2.9), and it can be explicitly solved using, for instance, the Laplace transform technique. The computation is rather long and we omit here all the details, which can be found in [1, 2]. Also, since only \( \sigma_j^+(t) \) appear in (2.8), together with its hermitian conjugate, we give here only the result we need. We have

\[
\sigma_j^+(t) = e^{it\nu}\rho^-_0 + e^{i(\nu+\omega)t}\rho^- + e^{i(\nu-\omega)t}\rho^+_0,
\]

where we have defined the following operators

\[
\begin{align*}
\rho^-_0 &= \frac{\tilde{g}S^-}{\omega^-} \\
\rho^+_0 &= \frac{gS^0}{\omega^+} \\
\rho^- &= \frac{gS^0}{\omega^-} \\
\rho^+_0 &= \frac{2S^-\sigma_j^+ + S^0\sigma_j^0 + 2S^+\sigma_j^-}{2} \\
\rho^-_0 &= \frac{gS^-\omega^+ + gS^0}{\omega^+} \\
\rho^- &= \frac{gS^-\omega^- - gS^0}{\omega^-} - \frac{\omega^+gS^0}{2} \sigma_j^0 + \frac{\omega^-gS^0}{2} \sigma_j^0 + gS^+ \sigma_j^-. \quad \text{(2.12)}
\end{align*}
\]
and the following quantities
\[ \omega = g \sqrt{S^0 + 4S^+S^-}, \quad \nu = 2\bar{\epsilon} + gS^0. \]  
(2.13)

Defining further
\[ \nu_\alpha(\vec{p}) = \nu - \epsilon_\vec{p} + \alpha\omega, \]  
(2.14)

where \( \alpha \) takes the values 0, + and −, the operator \( H_N^{(f)}(t) \) in (2.8) becomes
\[ H_N^{(f)}(t) = \sum_{j=1}^{N} \sum_{\alpha=0,\pm} \left( \rho^j_\alpha a_j(f e^{i\nu_\alpha}) + h.c. \right). \]  
(2.15)

**Remark:** It may be worth remarking that we would have obtained exactly this free time evolution even for a fermionic reservoir, since CCR and CAR produce the same free time evolution for both the annihilation and the creation operators. From this point of view, the difference between a fermionic and a bosonic thermal bath appears really only a minor aspect of the model.

The next step in the SLA consists in computing the following quantity
\[ I_\lambda(t) = \left( -\frac{i}{\lambda} \right)^2 \int_0^t dt_1 \int_0^{t_1} dt_2 \omega_{tot} \left( H_N^{(f)}(\frac{t_1}{\lambda^2})H_N^{(f)}(\frac{t_2}{\lambda^2}) \right), \]  
(2.16)

and its limit for \( \lambda \) going to zero. Here the state \( \omega_{tot} \) is the following product state \( \omega_{tot} = \omega_{sys} \omega_{\beta} \), where \( \omega_{sys} \) is a state of the system, while \( \omega_{\beta} \) is a state of the reservoir, which we will take to be a KMS state corresponding to an inverse temperature \( \beta = \frac{1}{kT} \). It is convenient here to use the so-called **canonical representation of thermal states**, [3], which is sketched in the Appendix. Then we introduce two sets of mutually commuting bosonic operators \( \{ c^{(\gamma)}_{\vec{p},j} \} , \gamma = a, b \), as follows:
\[ a_{\vec{p},j} = \sqrt{m(\vec{p})} c^{(a)}_{\vec{p},j} + \sqrt{n(\vec{p})} c^{(b)\dagger}_{\vec{p},j}, \]  
(2.17)

where
\[ m(\vec{p}) = \omega_\beta(a_{\vec{p},j}a_{\vec{p},j}^\dagger) = \frac{1}{1 - e^{-\beta \epsilon_\vec{p}}}, \quad n(\vec{p}) = \omega_\beta(a_{\vec{p},j}^\dagger a_{\vec{p},j}) = \frac{e^{-\beta \epsilon_\vec{p}}}{1 - e^{-\beta \epsilon_\vec{p}}}. \]  
(2.18)

The operators \( c^{(a)}_{\vec{p},j} \) satisfy the following commutation rules
\[ [c^{(a)}_{\vec{p},j}, c^{(\gamma)\dagger}_{\vec{q},k}] = \delta_{jk} \delta_{\vec{p},\vec{q}} \delta_{\alpha \gamma}, \]  
(2.19)

while all the other commutators are trivial. Furthermore, we introduce the vacuum of the operators \( c^{(a)}_{\vec{p},j}, \Phi_0 \):
\[ c^{(a)}_{\vec{p},j} \Phi_0 = 0, \quad \forall \vec{p} \in \Lambda_N, \quad j = 1,...N, \quad \alpha = a, b. \]  
(2.20)
Finally, if we define \( f_m(\vec{p}) = \sqrt{m(\vec{p})} f(\vec{p}) \) and \( f_n(\vec{p}) = \sqrt{n(\vec{p})} f(\vec{p}) \), we get

\[
a_j(f e^{it\nu_\alpha}) = c_j^{(a)}(f_m e^{it\nu_\alpha}) + c_j^{(b)\dagger}(f_n e^{it\nu_\alpha}),
\]

(2.21)

where we have used the usual following notation \( c_j^{(a)}(g) = \sum_{\vec{p}\in\Lambda_N} c_{\vec{p},j}^{(a)} g(\vec{p}) \) and \( c_j^{(b)\dagger}(g) = \sum_{\vec{p}\in\Lambda_N} c_{\vec{p},j}^{(b)\dagger} g(\vec{p})^2 \). Therefore we have

\[
H_N^{(I)}(t) = \sum_{j=1}^N \sum_{\alpha=0,\pm} \left\{ \rho^{\dagger}_\alpha \left( c_j^{(a)}(f_m e^{it\nu_\alpha}) + c_j^{(b)\dagger}(f_n e^{it\nu_\alpha}) \right) + h.c \right\},
\]

(2.22)

and the KMS state \( \omega_\beta \) can be represented as the following vector state, as in a GNS-like representation:

\[
\omega_\beta(X_r) = \langle \Phi_0, X_r \Phi_0 \rangle,
\]

(2.23)

for any observable of the reservoir, \( X_r \), since \( \omega_\beta \) is a gaussian state, [3]. This fact, together with (2.20) and with the commutation rules (2.19), simplifies the computation of the two point function \( \omega_{tot} \left( H_N^{(I)}(t) H_N^{(I)}(t') \right) \), which, after some algebraic computations, produces

\[
\omega_{tot} \left( H_N^{(I)}(t_1) H_N^{(I)}(t_2) \right) = \sum_{j=1}^N \sum_{\alpha,\beta=0,\pm} \sum_{\vec{p}\in\Lambda_N} \left\{ \omega_{sys}(\rho^{\dagger}_\alpha \rho^{\dagger}_\beta) | f_m(\vec{p}) |^2 e^{i \frac{4}{\lambda^2} \nu_\alpha(\vec{p})} e^{-i \frac{4}{\lambda^2} \nu_\beta(\vec{p})} + \omega_{sys}(\rho^{\dagger}_\alpha \rho^{\dagger}_\beta) | f_n(\vec{p}) |^2 e^{-i \frac{4}{\lambda^2} \nu_\alpha(\vec{p})} e^{i \frac{4}{\lambda^2} \nu_\beta(\vec{p})} \right\},
\]

Since we are interested to the limit \( \lambda \to 0 \) of \( I_\lambda(t) \) we need to impose some conditions on the test function \( f(\vec{p}) \), [3]. In particular, we will require that the following integral exists finite:

\[
\int_{-\infty}^0 d\tau \sum_{\vec{p}\in\Lambda_N} | f_r(\vec{p}) |^2 e^{\pm i \tau \nu_\alpha(\vec{p})} < \infty,
\]

(2.24)

where \( f_r(\vec{p}) \) is \( f_m(\vec{p}) \) or \( f_n(\vec{p}) \) and \( \nu_\alpha(\vec{p}) \) is given in (2.14). Under this assumption we find that

\[
I(t) = \lim_{\lambda \to 0} I_\lambda(t) = -t \sum_{j=1}^N \sum_{\alpha=0,\pm} \left\{ \omega_{sys}(\rho^{\dagger}_\alpha \rho^{\dagger}_\alpha) \Gamma^{(a)}(\alpha) + \omega_{sys}(\rho^{\dagger}_\alpha \rho^{\dagger}_\alpha) \Gamma^{(b)}(\alpha) \right\},
\]

(2.25)

where the two complex quantities

\[
\Gamma^{(a)}(\alpha) = \int_{-\infty}^0 d\tau \sum_{\vec{p}\in\Lambda_N} | f_m(\vec{p}) |^2 e^{-i \tau \nu_\alpha(\vec{p})}, \quad \Gamma^{(b)}(\alpha) = \int_{-\infty}^0 d\tau \sum_{\vec{p}\in\Lambda_N} | f_n(\vec{p}) |^2 e^{i \tau \nu_\alpha(\vec{p})}
\]

(2.26)

\[\text{It may be worth noticing that both } c_j^{(\gamma)(f)} \text{ and } c_j^{(\gamma)(f)\dagger} \text{ are linear in their argument } f.\]
both exist because of the assumption (2.24).

To this same result we could also arrive starting with the following stochastic limit hamiltonian

\[
H_N^{(sl)}(t) = \sum_{j=1}^{N} \sum_{\alpha=0,\pm} \left\{ \rho^j_\alpha \left( c^{(a)}_{\alpha_j}(t) + c^{(b)}_{\alpha_j}(t) \right) + h.c. \right\},
\]

where the operators \(c^{(a)}_{\alpha_j}(t)\) are assumed to satisfy the following commutation rule,

\[
[c^{(a)}_{\alpha_j}(t), c^{(b)}_{\beta_k}(t')] = \delta_{jk} \delta_{\alpha\beta} \delta(t - t') \Gamma^{(\gamma)}_\alpha,
\]

for \(t > t'\). (2.28)

We mean that, as it is easily checked, the following quantity

\[
\Omega = \omega_{sys} \Omega = \omega_{sys} \langle \Psi_0, \Psi_0 \rangle, \quad \text{where } \Psi_0 \text{ is the vacuum of the operators } c^{(a)}_{\alpha_j}(t): c^{(a)}_{\alpha_j}(t)\Psi_0 = 0 \text{ for all } \alpha, j, \gamma \text{ and } t, [3].
\]

Following the SLA, [3], we now use \(H_N^{(sl)}(t)\) to compute the generator of the theory. In fact, this is the main reason why this operator is introduced in the game. Let \(X\) be an observable of the system and \(\mathbb{1}_r\) the identity of the reservoir. Its time evolution (after the stochastic limit is taken) is \(j_t(X \otimes \mathbb{1}_r) = U_t^\dagger (X \otimes \mathbb{1}_r)U_t\), where \(U_t\) is the wave operator satisfying the following differential equation \(\partial_t U_t = -i H_N^{(sl)}(t)U_t\), whose adjoint is \(\partial_t U_t^\dagger = i U_t^\dagger H_N^{(sl)}(t)\).

Then we find

\[
\partial_t j_t(X \otimes \mathbb{1}_r) = i U_t^\dagger [H_N^{(sl)}(t), X \otimes \mathbb{1}_r]U_t =
\]

\[
i U_t^\dagger \sum_{j=1}^{N} \sum_{\alpha=0,\pm} \left\{ \left[ \rho^j_\alpha, X \right] (c^{(a)}_{\alpha_j}(t) + c^{(b)}_{\alpha_j}(t)) + \left[ \rho^j_\alpha, X \right] (c^{(a)}_{\alpha_j}(t) + c^{(b)}_{\alpha_j}(t)) \right\} U_t
\]

Next we have to normal order the formula above, i.e. to move to the right all the annihilation operators \(c^{(a)}_{\alpha_j}(t)\) and to the left the creation operators \(c^{(a)}_{\alpha_j}(t)\). To achieve this result we need to compute first the commutator \(c^{(a)}_{\alpha_j}(t), U_t\), and this can be done easily by means of the time consecutive principle, [3], and of the commutation rules (2.28):

\[
[c^{(a)}_{\alpha_j}(t), U_t] = -i \int_0^t [c^{(a)}_{\alpha_j}(t), H_N^{(sl)}(t')] U_{t'} dt' = -i \int_0^t \left( \rho^j_\alpha \Gamma^{(\gamma)}_\alpha \delta(t - t') \right) U_{t'} dt' = -i \rho^j_\alpha \Gamma^{(\gamma)}_\alpha U_t.
\]

(2.29)

Similarly we get

\[
[c^{(b)}_{\alpha_j}(t), U_t] = -i \rho^j_\alpha \Gamma^{(\gamma)}_\alpha U_t,
\]

(2.30)

and, taking the adjoint of these two equations,

\[
[U_t^\dagger, c^{(a)}_{\alpha_j}(t)] = i U_t^\dagger \rho^j_\alpha \Gamma^{(\gamma)}_\alpha \quad \text{and} \quad [U_t^\dagger, c^{(b)}_{\alpha_j}(t)] = i U_t^\dagger \rho^j_\alpha \Gamma^{(\gamma)}_\alpha.
\]
Going back to $\partial_t j_t(X \otimes I_r)$ we find that

$$\partial_t j_t(X \otimes I_r) = \sum_{j=1}^N \sum_{\alpha=0}^{\pm} \left\{ \left( i U_t^{\dagger} \rho^j_{\alpha} \Gamma^{(a)}_{\alpha} + c_{\alpha j}^{(a)}(t) U_t^{\dagger} \right) \left[ \rho^j_{\alpha}, X \right] U_t^{\dagger} + \left( i U_t^{\dagger} \rho^j_{\alpha} \Gamma^{(a)}_{\alpha} + c_{\alpha j}^{(a)}(t) U_t^{\dagger} \right) \left[ \rho^j_{\alpha}, X \right] \rho^j_{\alpha}^{\dagger} \right\}$$

which has to be computed on the state $\Omega_{\text{tot}}$. Therefore, since the generator $L$ satisfies the equality $\Omega_{\text{tot}}(\partial_t j_t(X \otimes I_r)) = \Omega_{\text{tot}}(j_t(L(X)))$, we get

$$L(X) = \sum_{j=1}^N \sum_{\alpha=0}^{\pm} \left\{ \left[ \rho^j_{\alpha}, X \right] \rho^j_{\alpha} \Gamma^{(a)}_{\alpha} + \left[ \rho^j_{\alpha}, X \right] \rho^j_{\alpha} \Gamma^{(b)}_{\alpha} - \rho^j_{\alpha} \rho^j_{\alpha}^{\dagger} \left[ \rho^j_{\alpha}, X \right] \Gamma^{(a)}_{\alpha} - \rho^j_{\alpha} \rho^j_{\alpha}^{\dagger} \left[ \rho^j_{\alpha}, X \right] \Gamma^{(b)}_{\alpha} \right\} \quad (2.31)$$

This expression can be made simpler if the observable $X$ is self-adjoint ($X = X^\dagger$). In this case we have

$$L(X) = L_1(X) + L_2(X), \quad (2.32)$$

where

$$L_1(X) = \sum_{j=1}^N \sum_{\alpha=0}^{\pm} \left\{ \left[ \rho^j_{\alpha}, X \right] \rho^j_{\alpha} \Gamma^{(a)}_{\alpha} + h.c. \right\}, \quad L_2(X) = \sum_{j=1}^N \sum_{\alpha=0}^{\pm} \left\{ \rho^j_{\alpha} \rho^j_{\alpha}^{\dagger} \Gamma^{(b)}_{\alpha} + h.c. \right\}. \quad (2.33)$$

This formula will be the starting point for the analysis in the next section.

**Remark:** Before going on, it may be interesting to stress that, when compared with the standard perturbative approach for the master equation for open quantum systems, [2], the perturbative approach based on the SLA appears quite friendly. For instance, the so called time consecutive principle and the new Hilbert space with ground vector $\Psi_0$ arising after the limit $\lambda \to 0$ is taken, are typical tools of the SLA and they are essential to make many computations almost trivial.

### III The phase transition

As discussed in [1, 2], $S^0_N$ and $R_N$ are the relevant variables whose dynamics must be considered to analyze the phase structure of the model. These intensive operators are both self-adjoint, so that we can use equations (2.32) and (2.33) instead of (2.31). As a matter of fact, in both [1] and [2] these equations of motion are considered only as an intermediate step to compute the
equation for $\Delta_N = \frac{1}{2}R_N^{1/2}$, which is called the gap operator. We will see in a while that the same conclusions as in [1, 2] can be obtained without introducing $\Delta_N$ but working directly with $R_N$ and $S_N^0$.

As a first step we compute $L(S_N^0) = L_1(S_N^0) + L_2(S_N^0)$. We have, using (2.2), (2.3) and (2.33)

$$L_1(S_N^0) = \frac{1}{N} \sum_{j=1}^N L_1(\sigma_j^0) = \frac{1}{N} \sum_{j=1}^N \sum_{\alpha=0,\pm} \left\{ |\rho^j_\alpha, \sigma_j^0| \rho^j_\alpha \, \Gamma^{(a)}_\alpha + h.c. \right\}$$

which can be written as

$$L_1(S_N^0) = \sum_{\alpha=0,\pm} \left\{ (b_\alpha + S_N^+ + b_\alpha - S_N^- + b_{\alpha 0} S_N^0 + b_{\alpha 1} \mathbb{1}_N) \Gamma^{(a)}_\alpha + h.c. \right\}, \quad (3.1)$$

where $\mathbb{1}_N = \frac{1}{N} \sum_{j=1}^N \mathbb{1}_j$ and the various coefficients $\{b_{\alpha \gamma}\}$ have been introduced here only to stress the fact that $L_1(S_N^0)$ is linear in the intensive operators. As we have already mentioned before, the limit of the right hand side of the formula exists in the strong topology restricted to a certain family $\mathcal{F}$ of states, since all the operators $S_N^0$ converge in this topology. Therefore also the limit of the left hand side does exist in the same topology. After some non trivial algebra we find

$$L_1(S^0) := \mathcal{F} - \text{strong} \lim_{N \to \infty} L_1(S_N^0) = - \frac{8g^4 S^0 (S^+ S^-)^2}{\omega^3} \left\{ \Re \Gamma_+^{(a)} \frac{\omega - g}{(\omega + g S^0)^2} + \Re \Gamma_-^{(a)} \frac{\omega + g}{(\omega - g S^0)^2} \right\}, \quad (3.2)$$

where $\Re \Gamma^{(a)}_\pm$ indicates the real part of $\Gamma^{(a)}_\pm$.

The computation of $L_2(S^0) := \mathcal{F} - \text{strong} \lim_{N \to \infty} L_2(S_N^0)$ follows essentially the same steps and produces

$$L_2(S^0) = - \frac{8g^4 S^0 (S^+ S^-)^2}{\omega^3} \left\{ \Re \Gamma_+^{(b)} \frac{\omega + g}{(\omega + g S^0)^2} + \Re \Gamma_-^{(b)} \frac{\omega - g}{(\omega - g S^0)^2} \right\}, \quad (3.3)$$

so that the final result is

$$L(S^0) = - \frac{8g^4 S^0 (S^+ S^-)^2}{\omega^3} h(S^0, S^+ S^-). \quad (3.4)$$

Here we have introduced, for brevity, the function

$$h(S^0, S^+ S^-) = \Re \Gamma_+^{(a)} \frac{\omega - g}{(\omega + g S^0)^2} + \Re \Gamma_-^{(a)} \frac{\omega + g}{(\omega - g S^0)^2} + \Re \Gamma_+^{(b)} \frac{\omega + g}{(\omega + g S^0)^2} + \Re \Gamma_-^{(b)} \frac{\omega - g}{(\omega - g S^0)^2}, \quad (3.5)$$

and we have made explicit the fact that $h$ depends on $S^+ S^- = \mathcal{F} - \text{strong} \lim_{N \to \infty} S_N^+ S_N^-$ via the pulsation $\omega$, see (2.13). It is interesting to observe that the same function $h(S^0, S^+ S^-)$ appears
in the computation of $L(S^+S^-) := \mathcal{F} - \text{strong lim}_{N \to \infty} L(S^+_NS^-_N)$. Again, since $(S^+_NS^-_N)^\dagger = S^+_N S^-_N$, we can use formulas (2.32) and (2.33). Here the computations are significantly harder, but no difficulty of principle arises. As a technical tool it is convenient to use the fact that, in the limit $N \to \infty$, all the intensive operators commute with all the local operators of the system, $\text{lim}_{N \to \infty} [S^\alpha_N, \sigma^\beta_j] = 0$, for all $\alpha, \beta$ and $j$. Therefore we get

$$L(S^+S^-) = -\frac{16g^4(S^+S^-)^3}{\omega^3}h(S^0, S^+S^-). \quad (3.6)$$

The phase structure of the model is now given by the right-hand sides of equations (3.4) and (3.6), see [1, 2], and, in particular, from the zeros of the functions

$$f_1(x, y) = -\frac{8g^4xy^2}{\omega^3}h(x, y), \quad f_2(x, y) = -\frac{16g^4y^3}{\omega^3}h(x, y), \quad (3.7)$$

where we have introduced, to simplify the notation, $x = S^0$ and $y = S^+S^-$, so that $\omega = g\sqrt{x^2 + 4y}$ and $\nu = 2\varepsilon + gx$. In particular, the existence of a superconducting phase corresponds to the existence of a non trivial zero of $f_1$ and $f_2$, [1, 2]. Due to the definition of $f_1$ and $f_2$ it is clear that any $(x_o, y_o)$, with $x_o \neq 0$ and $y_o \neq 0$, is such that $f_1(x_o, y_o) = f_2(x_o, y_o) = 0$ if and only if it is a zero of the function $h$: $h(x_o, y_o) = 0$. In order to find such a solution, it is first necessary to obtain an explicit expression for the coefficients $\mathcal{R} \Gamma^{(\gamma)}_{\pm}$. This is easily done using the definitions in (2.26), since we get

$$\mathcal{R} \Gamma^{(a)}_{\pm} = \frac{1}{2} \int_{-\infty}^\infty \sum_{\vec{p} \in \Lambda_N} |f_m(\vec{p})|^2 e^{-i\tau \nu_{\pm}(\vec{p})} d\tau = \pi \sum_{\vec{p} \in \Lambda_N} |f_m(\vec{p})|^2 \delta(\nu_{\pm}(\vec{p})), \quad (3.8)$$

and

$$\mathcal{R} \Gamma^{(b)}_{\pm} = \pi \sum_{\vec{p} \in \Lambda_N} |f_n(\vec{p})|^2 \delta(\nu_{\pm}(\vec{p})). \quad (3.9)$$

It is now almost straightforward to recover the results of [1, 2]. Following Buffet and Martin’s original idea, we look for solutions corresponding to $\nu = 0$. This means that, because of (2.13), the value of $x = S^0$ is fixed: $x = -2\varepsilon/g$. Moreover, with this choice, $\nu_{+}(\vec{p}) = \omega - \epsilon_{\vec{p}}$, which is zero if and only if $\omega = \epsilon_{\vec{p}}$. Also, we have $\nu_{-}(\vec{p}) = -\omega - \epsilon_{\vec{p}}$, which is never zero. For these reasons we deduce that $\mathcal{R} \Gamma^{(\gamma)}_{-} = 0$, $\gamma = a, b$, while the sums in (3.8) and (3.9) for $\mathcal{R} \Gamma^{(\gamma)}_{+}$ are restricted to the smaller set, $\mathcal{E}_N \subset \Lambda_N$, of those values of $\vec{p}$ such that, if $\vec{q} \in \mathcal{E}_N$ then $\epsilon_{\vec{q}} = \omega$. Therefore, recalling the expression of $m(\vec{p})$ and $n(\vec{p})$ in (2.18), we find

$$\mathcal{R} \Gamma^{(a)}_{+} = \pi \frac{e^{\beta \omega}}{e^{\beta \omega} - 1} \sum_{\vec{p} \in \mathcal{E}_N} |f(\vec{p})|^2, \quad \mathcal{R} \Gamma^{(b)}_{+} = \pi \frac{1}{e^{\beta \omega} - 1} \sum_{\vec{p} \in \mathcal{E}_N} |f(\vec{p})|^2. \quad (3.10)$$
From definition (3.5), therefore, we get the following equation

$$\pi e^{\beta \omega} \sum_{\vec{p} \in E} \left| f(\vec{p}) \right|^2 \frac{\omega - g}{(\omega + gx)^2} + \pi \frac{1}{e^{\beta \omega} - 1} \sum_{\vec{p} \in E} \left| f(\vec{p}) \right|^2 \frac{\omega + g}{(\omega + gx)^2} = 0,$$

or

$$e^{\beta \omega} = \frac{g + \omega}{g - \omega}. \tag{3.11}$$

This equation is the crucial one, which replaces the equation obtained in [1, 2], \(g \tanh \left( \frac{\beta \omega}{2} \right) = \omega\).

We conclude that:

1) first of all, introducing a new variable \(\xi = \frac{\omega}{g}\), equation (3.11) has a non-trivial solution if and only if the function \(g(\xi) = e^{\beta g \xi} - \frac{1 + \xi}{1 - \xi}\) has a zero \(\xi \neq 0\). It is clear that such a solution does exist only if the first derivative of \(g(\xi)\), computed in \(\xi = 0\) is positive, i.e. when \(\beta g - 2 > 0\). This is because \(g(0) = 0\) and \(\lim_{\xi \to -1} g(\xi) = -\infty\). We recover therefore the first result of [1, 2], since this inequality implies the existence of a critical temperature, \(T_c := \frac{2\pi}{\beta g}\), coinciding with that found by Martin and Buffet, such that, when \(T < T_c\), a \(\tilde{\xi} \neq 0\) does necessarily exist such that \(g(\tilde{\xi}) = 0\), and the system is in a superconducting phase.

2) it is also possible to find the value of \(y = S^+S^-\) directly from equation (3.11). However, in order to recover the same value of the gap operator known in literature, we prefer to play a little bit with equation (3.11) in the following way:

$$g \tanh \left( \frac{\beta \omega}{2} \right) = \frac{g e^{\beta g} - e^{-\beta g}}{e^{\beta g} + e^{-\beta g}} = g e^{\beta g} - 1 \left( \frac{g + \omega}{g - \omega} + 1 \right) = \omega.$$ 

This chain of equalities shows once again how our equation (3.11) returns the same equation obtained in [1, 2] with completely different techniques.

Vice versa, it is also straightforward to check that equation \(g \tanh \left( \frac{\beta \omega}{2} \right) = \omega\) implies equation (3.11):

$$e^{\beta \omega} = \frac{\tanh \left( \frac{\beta \omega}{2} \right) + 1}{\tanh \left( \frac{\beta \omega}{2} \right) - 1} = \frac{\omega}{1 - \omega} + 1 = \frac{g + \omega}{g - \omega},$$

and this concludes the proof of the equivalence of our approach with that of Buffet and Martin.

**IV Conclusions and comments**

We have shown how the SLA can be successfully used to analyze the phase structure of low temperature superconductivity analyzing a strong coupling BCS model, considered as an open system interacting with a bosonic thermal bath.
The procedure, which makes use of the canonical representation of thermal states, is rather direct and is technically much simpler than the one used in the original paper, [1]. Among the other simplifications, for instance, a single equation $h(x, y) = 0$ must be solved instead of the system $f_1(x, y) = f_2(x, y) = 0$, which is the highly transcendental system which must be solved in [1].

For this reason we believe that it may be worth considering other models, still unsolved, with the simplifying strategy provided by the SLA, since new insights may eventually come out. For instance, one could first replace the bosonic reservoir with a reservoir made of quons, [9], in the attempt of getting a different free time evolution for the creation and annihilation quon operators. Following our analysis, and in particular definition (3.5) of the function $h$, this is in fact the easiest way to get an higher value of the critical temperature ($T_c > \frac{g}{2\kappa}$). Another possibility to achieve the same result is to consider a second reservoir interacting with the first one: again, in this way the free time evolution of the bosonic operators will be different from the one considered here, $a_j(f e^{-i\omega t})$. These models will be considered in a forthcoming paper, [10].

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Appendix: Few results on the stochastic limit

In this Appendix we will briefly summarize some of the basic facts and properties concerning the SLA which are used all throughout the paper. We refer to [3] and references therein for more details.

Given an open system $S + R$ we write its Hamiltonian $H$ as the sum of two contributions, the free part $H_0$ and the interaction $\lambda H_I$. Here $\lambda$ is a coupling constant, $H_0$ contains the free evolution of both the system $S$ and the reservoir $R$, while $H_I$ contains the interaction between $S$ and $R$. Working in the interaction picture, we define $H_I(t) = e^{iH_0t}H_I e^{-iH_0t}$ and the so-called wave operator $U_\lambda(t)$ which is the solution of the following differential equation

$$\partial_t U_\lambda(t) = -i\lambda H_I(t) U_\lambda(t),$$

(A.1)

with the initial condition $U_\lambda(0) = 1$. Using the van-Hove rescaling $t \rightarrow \frac{t}{\lambda^2}$, see [2, 3] for instance, we can rewrite the same equation in a form which is more convenient for our perturbative approach, that is

$$\partial_t U_\lambda\left(\frac{t}{\lambda^2}\right) = -\frac{i}{\lambda} H_I\left(\frac{t}{\lambda^2}\right) U_\lambda\left(\frac{t}{\lambda^2}\right),$$

(A.2)

with the same initial condition as before. Its integral counterpart is

$$U_\lambda\left(\frac{t}{\lambda^2}\right) = 1 - \frac{i}{\lambda} \int_0^t H_I\left(\frac{t'}{\lambda^2}\right) U_\lambda\left(\frac{t'}{\lambda^2}\right) dt',$$

(A.3)

which is the starting point for a perturbative expansion, which works in the following way.

Suppose, to begin with, that we are interested in the zero temperature situation. Then let $\varphi_0$ be the ground vector of the reservoir and $\xi$ a generic vector of the system. Now we put $\varphi_0(\xi) = \varphi_0 \otimes \xi$. We want to compute the limit, for $\lambda$ going to 0, of the first non-trivial order of the mean value of the perturbative expansion of $U_\lambda(t/\lambda^2)$ above in $\varphi_0(\xi)$, that is the limit of

$$I_\lambda(t) = \left(\frac{-i}{\lambda}\right)^2 \int_0^t dt_1 \int_0^{t_1} dt_2 < H_I\left(\frac{t_1}{\lambda^2}\right) H_I\left(\frac{t_2}{\lambda^2}\right) >_{\varphi_0(\xi)},$$

(A.4)

for $\lambda \rightarrow 0$. Under some regularity conditions on the functions which are used to smear out the (typically) bosonic fields of the reservoir, this limit is shown to exist for many relevant physical models, see [3], and [7, 6] for few recent applications to quantum many body theory. It is at this stage that all the complex quantities like the $\Gamma^{(\gamma)}$'s we have introduced in the main body of this paper appear. We define $I(t) = \lim_{\lambda \rightarrow 0} I_\lambda(t)$. In the same sense of the convergence of the (rescaled) wave operator $U_\lambda(t/\lambda^2)$ (the convergence in the sense of correlators), it is possible to check that also the (rescaled) reservoir operators converge and define new operators which
do not satisfy canonical commutation relations but a modified version of these. For instance, in Section II this procedure has produced the operators $c^{(\gamma)}_{\alpha j}$ starting from $c^{(\gamma)}_{\vec{\rho},j}$. Moreover, these limiting operators depend explicitly on time and they live in a Hilbert space which is different from the original one. In particular, they annihilate a vacuum vector, $\eta_0$, which is no longer the original one, $\varphi_0$. This is what happens, for instance, if $\varphi_0$ depends on $\lambda$, $\varphi_0 \rightarrow \varphi^{(\lambda)}_0$, and considering $\eta_0$ as the following limit: $\eta_0 = \lim_{\lambda \rightarrow 0} \varphi^{(\lambda)}_0$.

It is not difficult to deduce the form of a time dependent self-adjoint operator $H_I^{(sd)}(t)$, which depends on the system operators and on the limiting operators of the reservoir, such that the first non trivial order of the mean value of the expansion of $U_t = I - i \int_0^t H_I^{(sd)}(t')U_{t'}dt'$ on the state $\eta_0^{(\xi)} = \eta_0 \otimes \xi$ coincides with $I(t)$. The operator $U_t$ defined by this integral equation is called again the wave operator.

The form of the generator follows now from an operation of normal ordering. More in details, we start defining the flux of an observable $\tilde{X} = X \otimes I_r$, where $I_r$ is the identity of the reservoir and $X$ is an observable of the system, as $j_t(X) = U_t^{\dagger} \tilde{X} U_t$. Then, using the equation of motion for $U_t$ and $U_t^{\dagger}$, we find that $\partial_t j_t(X) = i U_t^{\dagger} [H_I^{(sd)}(t), \tilde{X}] U_t$. In order to compute the mean value of this equation on the state $\eta_0^{(\xi)}$, so to get rid of the reservoir operators, it is convenient to compute first the commutation relations between $U_t$ and the limiting operators of the reservoir. At this stage the so called time consecutive principle is used in a very heavy way to simplify the computation. This principle, which has been checked for many classes of physical models, [3], states that, if $\beta(t)$ is any of these limiting operators of the reservoir, then

$$[\beta(t), U_{t'}] = 0, \text{ for all } t > t'.$$

(A.5)

Using this principle and recalling that $\eta_0$ is annihilated by the limiting annihilation operators of the reservoir, it is now a simple exercise to compute $< \partial_t j_t(X) >_{\eta_0^{(\xi)}}$ and, by means of the equation $< \partial_t j_t(X) >_{\eta_0^{(\xi)}} = < j_t(L(X)) >_{\eta_0^{(\xi)}}$, to identify the form of the generator of the physical system.

Let us now consider the case in which $T > 0$. In this case the state of the reservoir is no longer given by $\varphi_0$. It is now convenient to use the so-called canonical representation of thermal states, [3]. Using the same notation of Section 2, any annihilator operator $a_{\vec{p},j}$ can be written as the following linear combination

$$a_{\vec{p},j} = \sqrt{m(\vec{p})} c_{\vec{p},j}^{(a)} + \sqrt{n(\vec{p})} c_{\vec{p},j}^{(b),\dagger},$$

(A.6)
where $m(\vec{p})$ and $n(\vec{p})$ are the following two-points functions,

$$
m(\vec{p}) = \omega_\beta(a_{\vec{p},j}a_{\vec{p},j}^\dagger) = \frac{1}{1 - e^{-\beta \vec{p}}} , \quad n(\vec{p}) = \omega_\beta(a_{\vec{p},j}^\dagger a_{\vec{p},j}) = \frac{e^{-\beta \vec{p}}}{1 - e^{-\beta \vec{p}}} ,
$$

(A.7)

for our bosonic reservoir, if $\omega_\beta$ is a KMS state corresponding to an inverse temperature $\beta$. The operators $c_{\vec{p},j}^{(\alpha)}$ are assumed to satisfy the following commutation rules

$$
[c_{\vec{p},j}^{(\alpha)}, c_{\vec{q},k}^{(\gamma)}] = \delta_{j,k} \delta_{\vec{p},\vec{q}} \delta_{\alpha \gamma} ,
$$

(A.8)

while all the other commutators are trivial. Let moreover $\Phi_0$ be the vacuum of the operators $c_{\vec{p},j}^{(\alpha)}$:

$$
c_{\vec{p},j}^{(\alpha)} \Phi_0 = 0 , \quad \forall \vec{p}, j, \alpha .
$$

Then it is immediate to check that the results in (A.7) for the KMS state can be found, using these new variables, representing $\omega_\beta$ as the following vector state $\omega_\beta(\cdot) = \langle \Phi_0, \cdot \Phi_0 \rangle$. With this GNS-like representation it is trivial to check that both the CCR and the two-point functions are easily recovered. This representation is also called in [3] the Fock-anti Fock representation because of the different sign in the free time evolution of the annihilation operators $c_{\vec{p},j}^{(a)}$ and $c_{\vec{p},j}^{(b)}$. Once this representation is introduced, all the same steps as for the situation with $T = 0$ can be repeated, and the expression for the generator can be deduced using exactly the same strategy.
References

[1] E. Buffet, P.A. Martin, *Dynamics of the Open BCS Model*, J. Stat. Phys., **18**, No. 6, 585-632, (1978)

[2] P.A. Martin, *Modèles en Mécanique Statistique des Processus Irréversibles*, Lecture Notes in Physics, 103, Springer-Verlag, Berlin, (1979)

[3] L. Accardi, Y.G. Lu, I. Volovich, *Quantum Theory and its Stochastic Limit*, Springer (2002)

[4] K. Hepp and E. H. Lieb, *Phase transitions in reservoir-driven open systes with applications to lasers and superconductors*, Helv. Phys. Acta **46**, (1973), 573.

[5] F. Bagarello, *Applications of Topological *-Algebras of Unbounded Operators*, J. Math. Phys., **39**, 2730-2747, (1998)

[6] F. Bagarello, *Relations between the Hepp-Lieb and the Alli-Sewell laser models*, Ann. H. Poincarè, **3**, 983-1002, (2002)

[7] L. Accardi, F. Bagarello, *The stochastic limit of the Fröhlich Hamiltonian: relations with the quantum Hall effect*, Int. Jour. Theor. Phys., in press

[8] F. Bagarello, G. Morchio, *Dynamics of mean field spin models from basic results in abstract differential equations*, J. Stat. Phys. **66**, 849-866 (1992)

[9] R.N. Mohapatra, *Infinite statistics and a possible small violation of the Pauli principle*, Phys. Lett. B, **242**, 407-411, (1990); D.I. Fivel, *Interpolation between Fermi and Bose statistics using generalized commutators*, Phys. Rev. Lett., **65**, 3361-3364, (1990); O.W. Greenberg, *Particles with small violations of Fermi or Bose statistics*, Phys. Rev. D, **43**, 4111-4120, (1991)

[10] F. Bagarello, *The role of a double reservoir in the open BCS model*, in preparation