Codes defined by forms of degree 2 on quadric and non-degenerate hermitian varieties in $\mathbb{P}^4(\mathbb{F}_q)$

Frédéric A. B. Edoukou

CNRS, Institut de Mathématiques de Luminy
Luminy case 907 - 13288 Marseille Cedex 9 - France
E.mail : edoukou@iml.univ-mrs.fr

Abstract

We study the functional codes of second order defined by G. Lachaud on $X \subset \mathbb{P}^4(\mathbb{F}_q)$ a quadric of rank($X$)=3,4,5 or a non-degenerate hermitian variety. We give some bounds for the number of points of quadratic sections of $X$, which are the best possible and show that codes defined on non-degenerate quadrics are better than those defined on degenerate quadrics. We also show the geometric structure of the minimum weight codewords and estimate the second weight of these codes. For $X$ a non-degenerate hermitian variety, we list the first five weights and the corresponding codewords. The paper ends with two conjectures. One on the minimum distance for the functional codes of order $h$ on $X \subset \mathbb{P}^4(\mathbb{F}_q)$ a non-singular hermitian variety. The second conjecture on the distribution of the code-words of the first five weights of the functional codes of second order on $X \subset \mathbb{P}^N(\mathbb{F}_q)$ the non-singular hermitian variety.

Keywords: functional codes, hermitian surface, hermitian variety, hermitian curve, quadric, weight.

Mathematics Subject Classification: 05B25, 11T71, 14J29

1 Introduction

We study the codes $C_2(X)$ defined on the quadric $X$ and on the non-singular hermitian variety $X : x_0^{t+1} + x_1^{t+1} + x_2^{t+1} + x_3^{t+1} + x_4^{t+1} = 0$ in $\mathbb{P}^4(\mathbb{F}_q)$ with
$q = t^2$ ($t$ is a prime power).
When $X$ is a quadric, we will consider the cases where rank$(X) = 3, 4, 5$, since otherwise, the computation of the minimum distance of the codes $C_2(X)$ is reduced to the study of plane or hyperplane sections of quadrics which is well known.
The case where $X$ is a non-degenerate quadric in $\mathbb{P}^4(\mathbb{F}_q)$ (i.e. rank$(X) = 5$, $X$ is a parabolic quadric), a bound for the minimum distance is given by D. B. Leep and L. M. Schueller [11, p.172]:

$$\# X_{Z(\mathbb{Q})}(\mathbb{F}_q) \leq 3q^2 + q + 1.$$  

It is not optimal. When $X$ is a degenerate quadric of rank 3 or 4, the hypothesis of D. B. Leep and L. M. Schueller is too much restrictive and we cannot find the minimal distance. In this case, however, we have a bound given by G. Lachaud [10, proposition 2.3 ]:

$$\# X_{Z(\mathbb{Q})}(\mathbb{F}_q) \leq 4(q^2 + q + 1)$$

which is not the best possible.
In the case where $X$ is a non-singular hermitian variety, the result of Lachaud gives an upper bound [14, p.208],

$$\# X_{Z(\mathbb{Q})}(\mathbb{F}_q) \leq 2(t + 1)(q^2 + q + 1)$$

which is also not optimal.
In this paper we find some optimal bounds excepting the case where $X$ is a degenerate quadric with rank$(X) = 4$ and projective index $g(X)=1$.
The paper is organized as follows. First of all we recall some generalities on quadrics and hermitian varieties. Secondly by using the projective classification of quadrics in [9, p.4] and their geometric structure, we find some interesting bounds on the number of rational points on the intersection of two quadrics. Thus, we show that for $X$ a non-degenerate quadric:

$$\# X_{Z(\mathbb{Q})}(\mathbb{F}_q) \leq 2q^2 + 3q + 1$$

is the best possible. We also show that for $X$ degenerate and rank $(X) = 3$, we get

$$\# X_{Z(\mathbb{Q})}(\mathbb{F}_q) \leq 4q^2 + q + 1,$$

and this bound is optimal. Identically, for rank$(X) = 4$ and $g(X)=2$, we have the optimal bound:

$$\# X_{Z(\mathbb{Q})}(\mathbb{F}_q) \leq 4q^2 + 1.$$
For $\mathcal{X}$ of rank($\mathcal{X}$) = 4 and $g(\mathcal{X})=1$, we get the bound

$$\#\mathcal{X}_{Z(\mathcal{Q})}(\mathbb{F}_q) \leq 3q^2 + q + 1.$$ 

Next, we express the exact parameters of the codes $C_2(\mathcal{X})$, the geometric structure of the minimum weight codewords and show that the performances of the codes $C_2(\mathcal{X})$ defined on the non-degenerate quadrics are better than the ones defined on the degenerate quadrics.

In the section 5, by using again the projective classification of quadrics in [9, p.4], those of hermitian surfaces in [8, p.112], some properties of hermitian varieties, we find the first five best upper bounds:

$$\#\mathcal{X}_{Z(\mathcal{Q})}(\mathbb{F}_q) = 2t^5 + t^3 + 2t^2 + 1, 2t^5 + t^2 + 1,$$

$$2t^5 + t^2 + 1, 2t^5 - t^3 + 2t^2 + 1,$$

on the number of rational points on the intersection of the non-singular hermitian variety $\mathcal{X}$ and any quadric $\mathcal{Q}$. In section 6, when $\mathcal{X}$ is a non-singular hermitian variety, we determine exactly the minimum distance of the code $C_2(\mathcal{X})$ and the minimum weight codewords. We also list the second, third, fourth, fifth weight, and describe the corresponding codewords reaching these weights.

The paper ends with two conjectures: the first, on the minimum distance for the code $C_h(\mathcal{X})$ where $\mathcal{X} \subset \mathbb{P}^4(\mathbb{F}_q)$ is the non-singular hermitian variety and the second, on the distribution of the codewords of the first five weights of $C_2(\mathcal{X})$ when $\mathcal{X} \subset \mathbb{P}^N(\mathbb{F}_q)$ is the non-singular hermitian variety.

2 Generalities

We denote by $\mathbb{F}_q$ the field with $q$ elements. Let $V = \mathbb{A}^{n+1}(\mathbb{F}_q)$ be the affine space of dimension $n + 1$ over $\mathbb{F}_q$ and $\mathbb{P}^n(\mathbb{F}_q) = \Pi_n$ the corresponding projective space. Then

$$\pi_n = \#\mathbb{P}^n(\mathbb{F}_q) = \#\Pi_n = q^n + q^{n-1} + \ldots + 1.$$ 

We use the term forms of degree two to describe homogeneous polynomials $f$ of degree two, and $\mathcal{Q} = Z(f)$ (the zeros of $f$ in the projective space $\mathbb{P}^n(\mathbb{F}_q)$) is a quadric.

Let $\mathcal{Q}$ be a quadric. The rank of $\mathcal{Q}$, denoted $r(\mathcal{Q})$, is the smallest number of indeterminates appearing in $f$ under any change of coordinate system. The quadric $\mathcal{Q}$ is said to be degenerate if $r(\mathcal{Q}) < n + 1$; otherwise it is non-degenerate. For $\mathcal{Q}$ a degenerate quadric and $r(\mathcal{Q})=r$, $\mathcal{Q}$ is a cone $\Pi_{n-r} \mathcal{Q}_{r-1}$
with vertex $\Pi_{n-r}$ (the set of singular points of $Q$) and base $Q_{r-1}$ in a subspace $\Pi_{r-1}$ skew to $\Pi_{n-r}$. For degenerate hermitian varieties we have an analogous decomposition as for quadrics.

**Definition 2.1** For $Q = \Pi_{n-r}Q_{r-1}$ a degenerate quadric with $r(Q) = r$, $Q_{r-1}$ is called the non-degenerate quadric associated to $Q$. The degenerate quadric $Q$ will said to be of type hyperbolic (resp. elliptic, parabolic) if its associated non-degenerate quadric is of the same type.

**Definition 2.2** [11, p.158] Let $Q_1$ and $Q_2$ be two quadrics. The order $w(Q_1, Q_2)$ of the pair $(Q_1, Q_2)$, is the minimum number of variables, after invertible linear change of variables, necessary to write $Q_1$ and $Q_2$.

**Definition 2.3** For any variety $V$, the maximum dimension $g(V)$ of linear subspaces lying on $V$, is called the projective index of $V$.

First of all, let us recall the classification of quadrics in $\mathbb{P}^4(F_q)$. It can be found in the book of J. W. P. Hirschfeld [9, p.4].

```
| r(Q) | Description            | |Q|   | g(Q) |
|------|------------------------|---|-----|------|
| 1    | repeated hyperplane $\Pi_3P_0$ | $q^3 + q^2 + q + 1$ | 3   |
| 2    | pair of distinct hyperplanes $\Pi_2H_1$ | $2q^3 + q^2 + q + 1$ | 3   |
| 2    | the plane $\Pi_2E_1$ | $q^2 + q + 1$ | 2   |
| 3    | the cone $\Pi_1P_2$ | $(q + 1)(q^2 + 1)$ | 2   |
| 4    | the cone $\Pi_0H_3(R, R')$ | $q(q + 1)^2 + 1$ | 2   |
| 4    | the cone $\Pi_0E_3$ | $q(q^2 + 1) + 1$ | 2   |
| 5    | the parabolic quadric $\mathcal{P}_4$ | $(q + 1)(q^2 + 1)$ | 1   |
```

Table 1: Quadrics in $\mathbb{P}^4(F_q)$

From the work of Ray-Chaudhuri [13, pp.132-136] we deduce that a non-degenerate quadric $\mathcal{P}_4$ in $\mathbb{P}^4(F_q)$ contains exactly $\alpha_q = \pi_3$ lines and there are exactly $q + 1$ lines contained in $\mathcal{P}_4$ through a given point. A regulus defined in the work of Hirschfeld [8, p.4], is the set of transversals of three skew lines in $\mathbb{P}^3(F_q)$; it consists of $q + 1$ skew lines. Thus, a hyperbolic quadric $Q$ in $\mathbb{P}^3(F_q)$ is a pair of complementary reguli (each one of the two reguli generates the whole hyperbolic quadric). It is denoted by $Q = \mathcal{H}_3(R, R')$ where $R$ and $R'$ are the two reguli.

Let $\mathcal{F}_h$ be the vector space of forms of degree $h$ in $V = \mathbb{A}^{n+1}(F_q)$, $X \subset \mathbb{P}^n(F_q)$ a variety and $|X|$ the number of rational points of $X$ over $F_q$. We denote by
$W_i$ the set of points with homogeneous coordinates $(x_0 : \ldots : x_n) \in \mathbb{P}^n(\mathbb{F}_q)$ such that $x_j = 0$ for $j < i$ and $x_i \neq 0$. The family $\{W_i\}_{0 \leq i \leq n}$ is a partition of $\mathbb{P}^n(\mathbb{F}_q)$. The code $C_h(X)$ is the image of the linear map $c : \mathcal{F}_h \rightarrow \mathcal{F}_q^{[X]}$, defined by $c(f) = (c_x(f))_{x \in X}$, where $c_x(f) = f(x_0, \ldots, x_n)/x_i^h$ with $x = (x_0 : \ldots : x_n) \in W_i$. The length of $C_h(X)$ is equal to $\#X(\mathbb{F}_q)$. The dimension of $C_h(X)$ is equal to $\dim \mathcal{F}_h - \dim \ker c$. Therefore, when $c$ is injective we get:

$$\dim C_h(X) = \left(\frac{n + h}{h}\right).$$  \hspace{1cm} (1)

The minimum distance of $C_h(X)$ is equal to the minimum over all $f$ of $\#X(\mathbb{F}_q) - \#X_Z(f)(\mathbb{F}_q)$.

3 Intersection of two quadrics

In this section, we will estimate the number of points in the intersection of two quadrics $X$ and $Q$ in $\mathbb{P}^4(\mathbb{F}_q)$. Some bounds in $\mathbb{P}^4(\mathbb{F}_q)$ have been given by Y. Aubry [1], G. Lachaud [10], D. B. Leep and L. M. Schueller [11]. These bounds are not the best possible even in the case of $\mathbb{P}^4(\mathbb{F}_q)$. Here $X$ denotes a quadric of rank $(X) = 3, 4, 5$ in $\mathbb{P}^4(\mathbb{F}_q)$.

Let us recall the result of D. B. Leep and L. M. Schueller [11, p.172] which gives an upper bound for the number of intersection of a pair of quadrics.

**Theorem 3.1** Let $Q_1$ and $Q_2$ be two quadrics in $\mathbb{P}^n(\mathbb{F}_q)$. Suppose that $w(Q_1, Q_2) = n + 1$. If $n + 1 \geq 5$ and $n + 1$ odd, then

$$|Q_1 \cap Q_2| \leq (2q^{n-1} - q^{n-2} + q^{\frac{n-5}{2}} - q^\frac{n}{2} - 1)/(q - 1).$$

From this theorem, we deduce the following result.

**Corollary 3.2** Let $Q_1$ and $Q_2$ be two quadrics in $\mathbb{P}^4(\mathbb{F}_q)$ with $w(Q_1, Q_2) = 5$. Then,

$$|Q_1 \cap Q_2| \leq 3q^2 + q + 1.$$

In the theorem above the hypothesis on the order of the two quadrics $Q_1$ and $Q_2$ is too restrictive and does not work in general case. Let us state another property, free from this condition.

**Lemma 3.3** Let $Q_1$ and $Q_2$ be two quadrics in $\mathbb{P}^n(\mathbb{F}_q)$ and $l$ an integer such that $1 \leq l \leq n - 1$. Suppose that $w(Q_1, Q_2) = n - l + 1$ (i.e. there exists a linear transformation such that $Q_1$ and $Q_2$ are defined with the indeterminates $x_0, x_1, \ldots, x_{n-l}$) and
|Q_1 \cap Q_2 \cap \mathbb{E}_{n-l}(F_q)| \leq m \text{ where } \mathbb{E}_{n-l}(F_q) = \{x \in \mathbb{P}^n(F_q) \mid x_{n-l+1} = \ldots = x_n = 0\}. \text{ Then}
|Q_1 \cap Q_2| \leq mq^l + \pi_{l-1}.

This bound is optimal as soon as m is optimal in \mathbb{E}_{n-l}(F_q).

**Proof** If w(Q_1, Q_2) = n - l + 1, then there exists a linear transformation such that Q_1 and Q_2 are defined with the indeterminates x_0, x_1, \ldots, x_{n-l}. Let m_0 \leq m such that |Q_1 \cap Q_2 \cap \mathbb{E}_{n-l}(F_q)| = m_0. From the fact that Q_1 \cap Q_2 has exactly m_0 projective zeros in \mathbb{E}_{n-l}(F_q), we deduce that it has exactly m(q-1)+1 affine zeros in \bar{E}(F_q) (the corresponding affine space of dimension n - l + 1).

Let \hat{y} = (y_0, y_1, \ldots, y_{n-l}) be a zero of Q_1 \cap Q_2 in \bar{E}(F_q). We get that y = (y_0, y_1, \ldots, y_{n-l-1}, x_{n-l+1}, \ldots, x_n) is a zero of Q_1 \cap Q_2 in \mathbb{A}^{n+1}(F_q) for any (x_{n-l+1}, \ldots, x_n) \in \mathbb{F}_q. And reciprocally every zero of Q_1 \cap Q_2 in \mathbb{A}^{n+1}(F_q) comes from a zero of Q_1 \cap Q_2 in \bar{E}(F_q). Therefore, we deduce that Q_1 \cap Q_2 contains exactly \{m_0(q-1)+1\}q^l affine zeros in \mathbb{A}^{n+1}(F_q). Thus, in \mathbb{P}^n(F_q), we get |Q_1 \cap Q_2| = (\{m_0(q-1)+1\}q^l - 1)/(q-1) = m_0q^l + \pi_{l-1}.

**Lemma 3.4** For Q_1 and Q_2 two quadrics in \mathbb{P}^n(F_q), we get
w(Q_1, Q_2) \geq \sup\{r(Q_1), r(Q_2)\}.

**Proof** We have r(Q_i) \leq w(Q_1, Q_2) for i = 1, 2.

### 3.1 Plane section of the quadric \(X'\): \(g(X) = 2\)

Here we study the section of the quadric \(X'\) by a quadric \(Q\) with \(g(X) = 2\). We have three cases: \(Q = \Pi_2 E_1\), \(Q = \Pi_1 P_2\) and \(Q = \Pi_0 H_3(R, R')\). Here \(Q = \Pi_1 P_2\) and \(Q = \Pi_0 H_3(R, R')\) are respectively a set of \(q + 1\) planes through a common line or a common point.

#### 3.1.1 \(Q\) is a quadric with \(r(Q) = 2\): \(Q = \Pi_2 E_1\)

Here \(Q\) consists of one plane of \(\mathbb{P}^4(F_q)\). If this plane is not contained in \(X'\), then \(Q_1 \cap X'\) is a plane quadric. In the book of J. W. P. Hirschfeld [7, p.156], we have the following classification of plane quadrics.

Therefore we deduce that if \(Q\) is a plane of \(\mathbb{P}^4(F_q)\):
- If it is contained in \(X'\), we get |\(X' \cap Q| = q^2 + q + 1.
- Otherwise, we get from the table above that |\(X \cap Q| \leq 2q + 1.
Table 2: Plane quadrics in $\mathbb{P}^2(\mathbb{F}_q)$

| r($Q'$) | Description | $|Q'|$ | g($Q'$) |
|---------|-------------|-------|---------|
| 1       | repeated line $\Pi_1P_0$ | $q + 1$ | 1       |
| 2       | pair of lines $\Pi_0H_1$  | $2q + 1$ | 1       |
| 2       | point $\Pi_0E_1$          | 1      | 0       |
| 3       | parabolic $P_2$           | $q + 1$ | 0       |

3.1.2 $Q$ is a quadric with $r(Q)=3,4$: $Q = \Pi_1P_2$ and $Q = \Pi_0H_3$

For $X$ a non-degenerate quadric, there is no plane in $X$. Thus each of the $q+1$ planes of $Q$ intersects $X$ in at most $2q+1$ points. Therefore, we deduce that $|X \cap Q| \leq 2q^2 + 3q + 1$.

For $X$ degenerate of $r(X) = 3,4$, let us consider the order $w(X, Q)$ of the pair $\{X, Q\}$. We have three possibilities $w(X, Q) = 3, 4, 5$:

(i) If $w(X, Q) = 5$, from the corollary 3.2 we get $|X \cap Q| \leq 3q^2 + q + 1$.

(ii) If $w(X, Q) = 4$, then there exists a linear transformation such that $X$ and $Q$ are defined with the indeterminates $x_0, x_1, x_2, x_3$.

Suppose rank($X$) = 3. From [5, IV-D-3,4], we get $|X \cap Q \cap E_3(\mathbb{F}_q)| = 4q + 1$ or $|X \cap Q \cap E_3(\mathbb{F}_q)| \leq 3q$. From lemma 3.3 we deduce that either $|X \cap Q| = 4q^2 + q + 1$ in the case where $X$ and $Q$ have exactly four common planes through a line or $|X \cap Q| \leq 3q^2 + 1$ otherwise.

If rank($X$) = 4 and $g(X) = 1$, from [5, IV-D-2] we get $|X \cap Q \cap E_3(\mathbb{F}_q)| \leq 2(q + 1)$. Therefore lemma 3.3 gives $|X \cap Q| \leq 2q^2 + 2q + 1$.

If rank($X$) = 4 and $g(X) = 2$, [5, IV-D-3 and IV-E-1] give $|X \cap Q \cap E_3(\mathbb{F}_q)| = 4q$ or $|X \cap Q \cap E_3(\mathbb{F}_q)| \leq 3q + 1$. In fact, in the case $|X \cap Q \cap E_3(\mathbb{F}_q)| = 4q$, $X \cap Q \cap E_3(\mathbb{F}_q)$ is exactly a set of four lines with two lines in each regulus of $H_3$. Therefore, from lemma 3.3 and the geometric structure of $X$ and $Q$, we get $|X \cap Q| = 4q^2 + 1$ when $X$ and $Q$ have the same vertex $\Pi_0$ and contain four common planes with the following configuration:

the first two planes $\Pi_2^{(1)}$ and $\Pi_2^{(2)}$ meet at the point $\Pi_0$, and the two others $\Pi_2^{(3)}$ and $\Pi_2^{(4)}$ also meet at the point $\Pi_0$,

the plane $\Pi_2^{(1)}$ meets $\Pi_2^{(3)}$ and $\Pi_2^{(4)}$ respectively at two distinct lines $D_{1,3}$ and $D_{1,4}$,

the plane $\Pi_2^{(2)}$ meets $\Pi_2^{(3)}$ and $\Pi_2^{(4)}$ respectively at two distinct lines $D_{2,3}$ and $D_{2,4}$,
each one of the four lines $D_{1,3}, D_{1,4}, D_{2,3}$ and $D_{2,4}$ passes through the point $\Pi_0$). For example, for the two quadrics defined by $f_X = x_0x_1 + x_2x_3$ and $f_Q = x_3x_0 + x_1x_2$ we have $|X \cap Q| = 4q^2 + 1$. Otherwise, we get $|X \cap Q| \leq 3q^2 + q + 1$.

(iii) If $w(X, Q) = 3$, necessary from lemma 3.4, we get $r(X) = r(Q) = 3$. Here $Q \cap E_2(F_q)$ and $X \cap E_2(F_q)$ are two irreducible curves (conics). Therefore from the theorem of Bézout (the fact that two plane conics have exactly four common points or less than three points) and lemma 3.3 we deduce that $|X \cap Q| = 4q^2 + q + 1$, or otherwise $|X \cap Q| \leq 3q^2 + q + 1$.

3.2 Hyperplane section of the quadric $X$: $g (Q) = 3$

This paragraph deals with the section of $X$ by $Q$ in the case $g(Q) = 3$ i.e. $Q$ contains a hyperplane. We have two cases: $r(Q) = 1$ (i.e. $Q$ is a repeated hyperplane), or $r(Q) = 2$ with $Q$ a pair of hyperplanes.

3.2.1 $Q$ is a quadric with $r(Q) = 1$: $Q = \Pi_3P_0$

Here $Q$ is a repeated hyperplane $H$. Let us recall three general results on hyperplane section of quadrics in $\mathbb{P}^n(F_q)$.

**Lemma 3.5** Swinnerton-Dyer [18, p.264], Let $\tilde{X} \subset \mathbb{P}^n(F_q)$ be a degenerate quadric of rank $r$ and $H$ an hyperplane. Then $\tilde{X} \cap H$ is a quadric in $\mathbb{P}^{n-1}(F_q)$ of rank $r, r-1$, or $r-2$.

**Lemma 3.6** Primrose [12, pp.299-300], Let $\tilde{X}$ be a non-degenerate quadric in $\mathbb{P}^n(F_q)$ and $H$ a hyperplane. If $H$ is tangent to $\tilde{X}$, then $\tilde{X} \cap H$ is a degenerate quadric of rank $n-1$ in $\mathbb{P}^{n-1}(F_q)$. If $H$ is not tangent to $\tilde{X}$, then $\tilde{X} \cap H$ is a non-degenerate quadric in $\mathbb{P}^{n-1}(F_q)$.

Let $H \subset \mathbb{P}^4(F_q)$ be a hyperplane. If $X$ is a non-degenerate quadric, from lemma 3.6 we get

\[
\#X_H(F_q) = \begin{cases}
(q + 1)^2, q^2 + 1 & \text{if } H \text{ is not tangent to } X, \\
q^2 + q + 1 & \text{if } H \text{ is tangent to } X.
\end{cases}
\]

If $X$ is a degenerate quadric, from lemma 3.5 we get

\[
\#X_H(F_q) \leq 2q^2 + q + 1.
\]

The third important result on hyperplane section of a quadric is:
**Lemma 3.7** J. Wolfmann [19, pp.191-192], Let \( \tilde{X} \) be a non-degenerate quadric in \( \mathbb{P}^n(\mathbb{F}_q) \) and \( H \) a hyperplane. If \( H \) is tangent to \( \tilde{X} \), then \( \tilde{X} \cap H \) is of the same type as \( \tilde{X} \) (in the sense of the definition 2.1).

### 3.2.2 \( Q \) is a quadric with \( r(Q)=2 \) and \( Q \) is a pair of hyperplanes

Let \( H_1 \) and \( H_2 \) be the two distinct hyperplanes generating \( Q \). We have \( Q = H_1 \cup H_2 \). Let \( P = H_1 \cap H_2 \) be the plane of intersection of the two hyperplanes. We have

\[
|Q \cap X| = |H_1 \cap X| + |H_2 \cap X| - |P \cap X|.
\]

Let \( \tilde{X}_1 = H_1 \cap X \), and \( \tilde{X}_2 = H_2 \cap X \). We have

\[
P \cap X = P \cap \tilde{X}_1 = P \cap \tilde{X}_2.
\]

#### 3.2.2.1 If \( X \) is non-degenerate (i.e. parabolic)

(i) In the case where each hyperplane is tangent to \( X \), we know that \( \tilde{X}_1 \) and \( \tilde{X}_2 \) are quadric cones of rank 3. We get that \( |\tilde{X}_1| = |\tilde{X}_2| = q^2 + q + 1 \). From lemma 3.5 and table 2, we deduce that \( |P \cap \tilde{X}_1| \geq 1 \). Therefore, from relation (3), we deduce that \( |X \cap Q| \leq 2q^2 + 2q + 1 \).

(ii) In the case where one hyperplane \( H_1 \) is tangent to \( X \), and the second hyperplane \( H_1 \) is non-tangent to \( X \), \( \tilde{X}_1 \) is a non-degenerate quadric surface in \( \mathbb{P}^3(\mathbb{F}_q) \): elliptic or hyperbolic.
   - If \( \tilde{X}_1 \) is an elliptic quadric, from lemmas 3.6, 3.7 and table 2, we get that \( P \cap \tilde{X}_1 \) is either a plane conic (parabolic), or a single point according to \( P \) being non-tangent or tangent to \( \tilde{X}_1 \). Therefore, from relations (4) and (3), we deduce that \( |X \cap Q| \leq 2q^2 + q + 1 \).
   - If \( \tilde{X}_1 \) is an hyperbolic quadric, from lemmas 3.6, 3.7 and table 2, we get that \( P \cap \tilde{X}_1 \) is either a plane conic (parabolic), or a pair of two distinct lines according to \( P \) being non-tangent or tangent to \( \tilde{X}_1 \). Therefore, from (4) and (3), we deduce that \( |X \cap Q| \leq 2q^2 + q + 1 \).

(iii) In the case where each hyperplane is non-tangent to \( X \), we know that \( \tilde{X}_1 \) and \( \tilde{X}_2 \) are non-singular quadric surfaces: elliptic or hyperbolic. They can be of the same type or of different types.
   - If one of the two quadrics is elliptic, from lemma 3.6 and table 2, we deduce that \( |P \cap \tilde{X}_1| \geq 1 \). Therefore from (3) and (⋆) we get that \( |X \cap Q| \leq 2q^2 + 2q + 1 \).
   - If the two quadrics are hyperbolic, from lemmas 3.6, 3.7 and table 2,
we deduce that $|P \cap \hat{X}_1| \geq q + 1$. Therefore, from (11) and (3), we get $|X \cap Q| \leq 2q^2 + 3q + 1$. And this upper bound is reached when $P$ is non-tangent to $\hat{X}_1$ and $\hat{X}_2$. For example, for the two quadrics defined by $f_X = x_0x_1 + x_2x_3 + x_4^2$ and $f_Q = (x_0 + x_1)(x_2 + x_3)$ with $\text{car}(F_q) \neq 2$.

### 3.2.2.2 If $X$ is degenerate (i.e. rank $(X)$=3,4)

Here we have $w(X, Q) = 3, 4, 5$:

(i) If $w(X, Q) = 5$, from corollary 3.2 we get $|X \cap Q| \leq 3q^2 + q + 1$.

(ii) If $w(X, Q) = 4$. Suppose rank$(X)$=3. From [5, IV-C] we get $|X \cap Q \cap \mathbb{E}_3(F_q)| = 4q + 1$ or $|X \cap Q \cap \mathbb{E}_3(F_q)| \leq 3q + 1$. From lemma 3.3 we deduce that either $|X \cap Q| = 4q^2 + q + 1$ in the case where $Q$ is union of two hyperplanes (non-tangent) each through a pair of planes of $X$ and the plane of intersection of the two hyperplanes intersecting $X$ in a line or $|X \cap Q| \leq 3q^2 + q + 1$ otherwise.

If rank $(X)$=4 and $g(X) = 1$ from [5, IV-B] we get $|X \cap Q \cap \mathbb{E}_3(F_q)| \leq 2(q + 1)$. Therefore lemma 3.3 gives $|X \cap Q| \leq 2q^2 + 2q + 1$.

If rank $(X)$=4 and $g(X) = 2$, the result of [5, IV-B] gives $|X \cap Q \cap \mathbb{E}_3(F_q)| = 4q$ or $|X \cap Q \cap \mathbb{E}_3(F_q)| \leq 3q + 1$. Therefore from lemma 3.3 we get either $|X \cap Q| = 4q^2 + 1$ when each hyperplane is tangent to $X$ with the plane of intersection meeting $X$ at two lines or $|X \cap Q| \leq 3q^2 + q + 1$ otherwise.

(iii) If $w(X, Q) = 3$, necessary from lemma 3.4 we get $r(X) = 3$. Here, the quadric $Q$ describes a pair of lines in $\mathbb{E}_2(F_q)$. The number of points in the intersection of two secant lines with a conic (non-singular plane quadric) is exactly four or less than three. From table 2 and lemma 3.3 we get that either $|X \cap Q| = 4q^2 + q + 1$ or $|X \cap Q| \leq 3q^2 + q + 1$.

#### 3.3 Line section of the quadric $X$: $g(Q)$=1

In this section we estimate the number of points in the intersection of $X$ and a quadric $Q$ with $g(Q) = 1$. In this case $Q$ is the degenerate quadric $\Pi_0\mathbb{E}_3$ or the non-degenerate quadric (i.e. parabolic) $\mathcal{P}_4$.

#### 3.3.1 $Q$ is a quadric with $r(Q)$=4: $Q = \Pi_0\mathbb{E}_3$

(i) For $X$ non-degenerate quadric, we get two possibilities:

- If there is a line of $X \cap Q$ through the vertex $\Pi_0$ of the cone $Q = \Pi_0\mathbb{E}_3$, it is obvious that this vertex is a point of $X$. Therefore there are
at most \( q + 1 \) lines of the cone \( Q \) through \( \Pi_0 \) contained in \( X \); the other lines of \( Q \) meet \( X \) at most two points each. Thus, we get

\[ |X \cap Q| \leq 2q^2 + 2q + 1. \]

If there is no line of \( X \cap Q \) through the vertex of the cone \( Q = \Pi_0 \mathcal{E}_3 \), each line of \( Q \) intersecting \( X \) in at most two points, we deduce that

\[ |X \cap Q| \leq 2(q^2 + 1). \]

(ii) For \( X \) degenerate quadric (i.e. \( r(X) = 3, 4 \)).

From lemma \( \text{[3.4]} \) we have \( w(X, Q) = 4, 5 \). The case \( w(X, Q) = 5 \) follows from corollary \( \text{[3.2]} \). Let us consider now that \( w(X, Q) = 4 \).

If \( r(X) = 3, 4 \) and \( g(X) = 2 \), or \( r(X) = 4 \) and \( g(X) = 1 \), from \([5, \text{IV-D-2, IV-E-2}] \) we get

\[ |X \cap Q \cap \mathbb{P}_3(F_q)| \leq 2(q + 1). \]

Thus, we deduce from lemma \( \text{[3.3]} \) that

\[ |X \cap Q| \leq 2q^2 + 2q + 1. \]

Remark 3.8

Let \( Q \) be a non-degenerate quadric (i.e. \( Q = \mathcal{P}_4 \)) and \( r(X) = 3, 4 \).

The cases \( r(X) = 3, 4 \) and \( g(X) = 2 \), correspond to the first part of 3.1.2. with \( Q \) at the place of \( X \). In the same way \( r(X) = 4 \) and \( g(X) = 1 \), corresponds to 3.3.1.(i).

3.3.2 Intersection of two non-degenerate quadrics

Here we study the number of points in the intersection of two non-degenerate quadrics \( X \) and \( Q \).

Proposition 3.9

Let \( X \) and \( Q \) be two non-degenerate quadrics in \( \mathbb{P}^4(F_q) \). If there is no line in \( X \cap Q \), then

\[ |X \cap Q| \leq 2(q^2 + 1). \]

Proof

We use the fact that \( \alpha_q \) is the number of lines of \( X \), each one of them meeting \( Q \) in at most two points. Moreover there pass exactly \( q + 1 \) lines through each point contained in \( X \).

Now we will study the section of two non-degenerate quadrics containing a common line. Let us consider a line \( D \) contained in \( X \cap Q \) and \( P \) a plane through \( D \). By the principle of duality in the projective space \([15, \text{pp.49-51}] \), or \([7, \text{p.33, theorem 3.1 p.85}] \), we deduce that there are exactly \( q + 1 \) hyperplanes \( (H_i) \) through \( P \). These \( q + 1 \) hyperplanes \( (H_i) \) generate \( \mathbb{P}^4(F_q) \).

For \( i = 1, ..., q + 1 \), we denote \( \hat{X}_i = H_i \cap X \) and \( \hat{Q}_i = H_i \cap Q \). Thus, we get:

\[ |X \cap Q| \leq |\hat{X}_1 \cap \hat{Q}_1| + \sum_{i=2}^{q+1} |(\hat{X}_i \cap \hat{Q}_i) - D|. \]

(5)
From lemma 3.6, we deduce that $\hat{X}_i$ and $\hat{Q}_i$ are quadrics of rank 3 or 4 in $\mathbb{P}^3(F_q)$. Since they contain the line $(D)$, they can not be elliptic. They are either hyperbolic or cone quadrics (of rank 3). Thus one has to study the three types of intersection in $\mathbb{P}^3(F_q)$ given by the table 3.

| Types | $\hat{X}_i \cap \hat{Q}_i$ |
|-------|-----------------------------|
| 1     | (hyperbolic quadric) $\cap$ (quadric cone) |
| 2     | (quadric cone) $\cap$ (quadric cone) |
| 3     | (hyperbolic quadric) $\cap$ (hyperbolic quadric) |

Table 3: Intersection of $\hat{X}_i \cap \hat{Q}_i$ in $\mathbb{P}^3(F_q)$

**Lemma 3.10** Let $\mathcal{X}$ and $\mathcal{Q}$ be two non-degenerate quadrics in $\mathbb{P}^n(F_q)$, and $\mathcal{K}$ a linear space of codimension 2. Then there exists at most two hyperplanes $H_i$, $i = 1, 2$ through $\mathcal{K}$ such that $\hat{X}_i = \hat{Q}_i$.

**Proof** Let

$$Q = \sum_{0 \leq i \leq j \leq n} a_{ij}x_ix_j \text{ and } \mathcal{X} = \sum_{0 \leq i \leq j \leq n} a'_{ij}x_ix_j.$$ 

Without loss of generality, we can choose a system of coordinates, such that $H_1 = \{x_n = 0\}$, $H_2 = \{x_{n-1} = 0\}$. For $H_1 \cap \mathcal{X} = H_1 \cap \mathcal{Q}$, we get that $a_{ij} = a'_{ij}$ except maybe for $(i, n)$ with $i = 0, ..., n$. For $H_2 \cap \mathcal{X} = H_2 \cap \mathcal{Q}$, we get that $a_{ij} = a'_{ij}$ except maybe for $(i, n-1)$ with $i = 0, ..., n-1$ and $(n-1, n)$. Therefore, we deduce that $a_{ij} = a'_{ij}$ except for $(i, j) = (n-1, n)$; and there exists $(\alpha, \beta) \in F_q^2 (\alpha \neq \beta)$ such that:

$$\begin{cases} 
Q = Q_0(x_0, x_1, ..., x_{n-1}, x_n) + \alpha x_{n-1}x_n \\
\mathcal{X} = Q_0(x_0, x_1, ..., x_{n-1}, x_n) + \beta x_{n-1}x_n.
\end{cases}$$

Let us suppose that there is a third hyperplane $H_3$ through $\mathcal{K}$ such that $H_3 \cap \mathcal{X} = H_3 \cap \mathcal{Q}$. We can suppose that $H_3 = \{ax_{n-1} + bx_n = 0\}$ and $b \neq 0$, therefore $x_n = -\frac{a}{b}x_{n-1}$. From the above system and the fact that $H_3 \cap \mathcal{X} = H_3 \cap \mathcal{Q}$ we deduce that $-\frac{a}{b}x_{n-1}^2 = -\frac{\beta}{b}x_{n-1}^2$. Since $\alpha \neq \beta$, we deduce that $a = 0$, which leads to $H_3 = H_2$. 

12
Proposition 3.11 Let $\mathcal{X}$ and $\mathcal{Q}$ be two non-degenerate quadrics in $\mathbb{P}^4(\mathbb{F}_q)$ containing a line ($\mathcal{D}$). There is a plane ($\mathcal{P}$) containing ($\mathcal{D}$) such that $\hat{\mathcal{X}}_i \neq \hat{\mathcal{Q}}_i$ for $i = 1, ..., q + 1$.

Proof We know from lemma 3.10 that there exists at most two hyperplanes $\mathcal{H}_i$, $i = 1, 2$ such that $\hat{\mathcal{X}}_i = \hat{\mathcal{Q}}_i$. There are also at most $2(q + 1)$ planes through ($\mathcal{D}$) contained in $H_1 \cup H_2$. Since there are exactly $q^2 + q + 1$ planes through ($\mathcal{D}$), we conclude that there are at least $q^2 - q - 1 > 0$ possibilities of choice for the plane ($\mathcal{P}$) such that $\hat{\mathcal{X}}_i \neq \hat{\mathcal{Q}}_i$ for $i = 1, ..., q + 1$.

Lemma 3.12 Let $\mathcal{D}_1$, $\mathcal{D}_2$ be two secant lines of the non-degenerate quadric $\mathcal{X} \subset \mathbb{P}^4(\mathbb{F}_q)$ and $\mathcal{P} = \langle \mathcal{D}_1, \mathcal{D}_2 \rangle$ the plane defined by these two lines. Let $(\mathcal{H}_i)_{i=1,...,q+1}$ be the $q+1$ hyperplanes containing $\mathcal{P}$. If there exists a hyperplane $\mathcal{H}_1$ tangent to $\mathcal{X}$, then it is unique (i.e. the remaining $q$ hyperplanes are non-tangent to $\mathcal{X}$).

Proof Let $H_1$ containing $\mathcal{P}$, be a hyperplane which is tangent to $\mathcal{X}$ at a point $P_1$. Then from lemma 3.6 $\hat{\mathcal{X}}_1$ is a quadric cone of rank 3 in $\mathbb{P}^3(\mathbb{F}_q)$: $\hat{\mathcal{X}}_1$ is a set of $q + 1$ lines through $P_1$ ($\mathcal{D}_1$ and $\mathcal{D}_2$ are two of them). Let $H_2$ containing $\mathcal{P}$ be another hyperplane which is tangent to $\mathcal{X}$ at a point $P_2$. Then $\hat{\mathcal{X}}_2$ is a set of $q + 1$ lines through $P_2$ and ($\mathcal{D}_1$ and $\mathcal{D}_2$ are two of them). The lines ($\mathcal{D}_1$) and ($\mathcal{D}_2$) intersect at $P_1$ and $P_2$, therefore we deduce that $P_1 = P_2$. Thus, there exist a unique hyperplane $\mathcal{H}_1$ tangent to $\mathcal{X}$ at the point $P = \mathcal{D}_1 \cap \mathcal{D}_2$.

Remark 3.13 We also know from [5, §IV-D-4 and IV-E-1] that if the intersection of two quadrics cone (of rank 3) or two hyperbolic quadrics in $\mathbb{P}^3(\mathbb{F}_q)$ contains three lines, it contains exactly four common lines.

From the results of [5, §IV] and table 3 above, we deduce the following table 4.

Let us explain the table 4. For the type 1, the intersection of a hyperbolic quadric and a quadric cone, contains at most two lines; therefore $3q$

| Types | 4 lines | 2 lines | 1 line |
|-------|---------|---------|--------|
| 1     | $3q$    | $2q + 1$|
| 2     | $4q+1$  | $3q$    | $2q + 1$|
| 3     | $4q$    | $3q + 1$| $2(q + 1)$|

Table 4: Number of points and lines in $\hat{\mathcal{X}}_i \cap \hat{\mathcal{Q}}_i$
and $2q + 1$ are respectively the maximum number of this intersection when containing (exactly or) at most two lines or exactly one line. From the above remark, the types 2 and 3 are describe as before.

Now an estimation on the number of points in the intersection of the two non-degenerate quadrics $X$ and $Q$ containing a common line is reduced to the two following simple cases.

**If $X \cap Q$ contains exactly one common line:** Each $\hat{X}_i \cap \hat{Q}_i$ contains exactly one line, and from the table 4, we get for $i = 1, \ldots, q + 1$ $|\hat{X}_i \cap \hat{Q}_i| \leq 2(q + 1)$. Therefore from relation (5), we deduce that $|X \cap Q| \leq q^2 + 3q + 2$.

**If $X \cap Q$ contains at least two common lines:** We distinguish two following cases:

1. **In the case where $X \cap Q$ contains only skew lines:** from table 4, we get for $i = 1, \ldots, q + 1$ $|\hat{X}_i \cap \hat{Q}_i| \leq 2q + 1$ for the types 1 and 2. Indeed, if $\hat{X}_i \cap \hat{Q}_i$ contains more than one line, from the fact that one of the two quadrics $\hat{X}_i$ or $\hat{Q}_i$ is a quadric cone, they are secant. For type 3, if $\hat{X}_i \cap \hat{Q}_i$ contains exactly four common lines, then it contains necessarily two secant lines. This is a contradiction. From remark 3.13, $\hat{X}_i \cap \hat{Q}_i$ can only contain at most two lines. Thus, from table 4, $3q + 1$ is an upper bound for the number of points in $\hat{X}_i \cap \hat{Q}_i$. The lines of $\hat{X}_i \cap \hat{Q}_i$ are skew; so they belong to the same regulus. From [5, IV-E-1] we get $|\hat{X}_i \cap \hat{Q}_i| \leq 2(q + 1)$. Finally, from relation (5) we deduce that $|X \cap Q| \leq q^2 + 3q + 2$.

2. **In the case where there exist some secant lines in $X \cap Q$, let** $\{D_1, D_2\}$ **denote a pair of secant lines and** $P$ **be the plane generated by this pair of secants. Let**

$$Q = \sum_{0 \leq i \leq j \leq 4} a_{ij} x_i x_j \text{ and } X = \sum_{0 \leq i \leq j \leq 4} a'_{ij} x_i x_j.$$ 

(i) If there are exactly two hyperplanes $H_i, i = 1, 2$ such that $\hat{X}_i = \hat{Q}_i$, then we can choose a system of coordinates, such that $H_1 = \{x_4 = 0\}$ and $H_2 = \{x_3 = 0\}$. From the proof of lemma 3.10 there exists $(\alpha, \beta) \in \mathbb{F}^2_q (\alpha \neq \beta)$ such that:

$$\begin{cases} Q = Q_0(x_0, x_1, x_2, x_3, x_4) + \alpha x_3 x_4 \\ X = Q_0(x_0, x_1, x_2, x_3, x_4) + \beta x_3 x_4. \end{cases}$$
Thus, we have $\mathcal{X} \cap Q = (H_1 \cap \mathcal{X}) \cup (H_2 \cap \mathcal{X})$ and from the relation (3) we get that $|\mathcal{X} \cap Q| \leq 2q^2 + 2q + 1$.

(ii) If there is exactly one hyperplane $H_1$ such that $\hat{X}_1 = \hat{Q}_1$, then by the same reasoning as above, we can choose two distinct linear forms $h_1(x_0, x_1, x_2, x_3, x_4)$ and $h_2(x_0, x_1, x_2, x_3, x_4)$ such that:

$$\begin{cases}
Q = \mathcal{X}_0(x_0, x_1, x_2, x_3) + x_4 h_1(x_1, x_2, x_3, x_4) \\
\mathcal{X} = \mathcal{X}_0(x_0, x_1, x_2, x_3) + x_4 h_2(x_1, x_2, x_3, x_4).
\end{cases}$$

We have $\mathcal{X} \cap Q = (H_1 \cap \mathcal{X}) \cup (H_2 \cap \mathcal{X} \cap Q)$ where $H_2$ is the hyperplane defined by the linear form $h_1 - h_2$. We also have $|H_2 \cap \mathcal{X} \cap Q| \leq 4q + 1$ from table 4 and $|H_1 \cap \mathcal{X}| \leq q^2 + 2q + 1$. Therefore we get that $|\mathcal{X} \cap Q| \leq q^2 + 6q + 2$.

(iii) If for $i = 1, \ldots, q + 1$ we get $\hat{X}_i \neq \hat{Q}_i$, one has two possibilities: If the $q + 1$ hyperplanes $H_i$ are all non-tangent to $\mathcal{X}$, $\hat{X}_i$ are non-degenerate quadrics and only the types 1 and 3 of the table 3 can appear. From table 4 and remark 3.13 we deduce that $|\hat{X}_i \cap \hat{Q}_i| \leq 4q$. If there exists a hyperplane $H_i$ tangent to $\mathcal{X}$, then from lemma 3.12 it is unique. Let $H_1$ be this tangent hyperplane, $\hat{X}_1$ is a quadric cone and $\hat{X}_1 \cap \hat{Q}_1$ is of type 1 or 2. Therefore we get $|\hat{X}_1 \cap \hat{Q}_1| \leq 4q + 1$. Finally from relation

$$|\mathcal{X} \cap Q| \leq |\hat{X}_1 \cap \hat{Q}_1| + \sum_{i=2}^{q+1} (|\hat{X}_i \cap \hat{Q}_i| - (D_1 \cup D_2)|) \quad (6)$$

we deduce that, $|\mathcal{X} \cap Q| \leq 2q^2 + 3q + 1$.

4 The parameters of the code $C_2(\mathcal{X})$ defined on the quadric $\mathcal{X}$

When $\mathcal{X} = Z(f') \subset \mathbb{P}^n(\mathbb{F}_q)$ is a quadric, the linear map $c : \mathcal{F}_2 \rightarrow \mathbb{F}_q^{\mathcal{X}}$ is not injective because $\ker c = \{ \lambda f' | \lambda \in \mathbb{F}_q \}$. Therefore we deduce that:

$$\dim C_2(\mathcal{X}) = \left( \frac{n + 2}{2} \right) - 1 = \frac{n(n + 3)}{2}$$

From the results of section 3, we deduce the followings results.
Theorem 4.1 Let $Q$ be a quadric in $\mathbb{P}^4(\mathbb{F}_q)$ and $X$ the non-degenerate (parabolic) quadric in $\mathbb{P}^4(\mathbb{F}_q)$. We get
\[
\#X_Z(Q)(\mathbb{F}_q) \leq 2q^2 + 3q + 1
\]
and this bound is the best possible.
The code $C_2(X)$ defined on the parabolic quadric $X$ is a $[n, k, d]_q$-code where $n = (q + 1)(q^2 + 1)$, $k = 14$, $d = q^3 - q^2 - 2q$.

Theorem 4.2 The minimum weight codewords of the code $C_2(X)$ correspond to:
– either quadrics which are union of two (non-tangent) hyperplanes each intersecting $X$ at a hyperbolic quadric such that the plane of intersection of the two hyperplanes intersects $X$ at a plane conic.
– or quadrics with $r(Q) = 3$ (i.e. $q + 1$ planes through a line) and each plane containing exactly two lines of $X$.
– or non-degenerate (i.e. parabolic) quadrics containing two secant lines of $X$ defining a plane contained in the $q + 1$ hyperplanes which has one tangent hyperplane, the $q$ other non-tangent to $X$, and with maximal hyperplane section.

Theorem 4.3 Let $Q$ be a quadric in $\mathbb{P}^4(\mathbb{F}_q)$ and $X$ a degenerate quadric in $\mathbb{P}^4(\mathbb{F}_q)$. We get :
– If $X$ is degenerate with $r(X) = 3$, then
\[
\#X_Z(Q)(\mathbb{F}_q) = 4q^2 + q + 1 \quad \text{or} \quad \#X_Z(Q)(\mathbb{F}_q) \leq 3q^2 + q + 1.
\]
– If $X$ is degenerate with $r(X) = 4$ and $g(X) = 2$, then
\[
\#X_Z(Q)(\mathbb{F}_q) = 4q^2 + 1 \quad \text{or} \quad \#X_Z(Q)(\mathbb{F}_q) \leq 3q^2 + q + 1.
\]

Theorem 4.4 The code $C_2(X)$ defined on the degenerate quadric $X$ in $\text{PG}(4,q)$ is a $[n, k, d]_q$-code where:
– If $X$ is degenerate with $r(X) = 3$, then $n = q^3 + q^2 + q + 1$, $k = 14$, $d = q^3 - 3q^2$.
– If $X$ is degenerate with $r(X) = 4$ and $g(X) = 2$, then $n = q^3 + 2q^2 + q + 1$, $k = 14$, $d = q^3 - 2q^2 + q$.

Theorem 4.5 The minimum weight codewords of the code $C_2(X)$ correspond:
For $X$ degenerate with $r(X) = 3$ to:
– quadrics which are union of two hyperplanes (non-tangent) each through a...
pair of planes and the plane of intersection of the two hyperplanes intersecting \( X \) in a line.

–quadratics with \( r(Q) = 3 \) and a plane containing exactly four lines of \( X \).

For \( X \) degenerate quadric with \( r(X) = 4 \) and \( g(X) = 2 \):

–quadratics which are union of two tangent hyperplanes to \( X \) and the plane of intersection of the two hyperplanes meeting \( X \) at two secant lines.

–quadratics with \( r(Q) = 4 \) and \( g(Q) = 2 \) containing exactly four common planes of \( X \) with the following configuration: (the two first planes \( \Pi_1^{(1)} \) and \( \Pi_2^{(2)} \) meet at the point \( \Pi_0 \), and the two others planes \( \Pi_2^{(3)} \) and \( \Pi_2^{(4)} \) meet at the point \( \Pi_0 \).

the plane \( \Pi_2^{(1)} \) meets \( \Pi_2^{(3)} \) and \( \Pi_2^{(4)} \) respectively at two distinct lines \( D_{1,3} \) and \( D_{1,4} \).

the plane \( \Pi_2^{(2)} \) meets \( \Pi_2^{(3)} \) and \( \Pi_2^{(4)} \) respectively at two distinct lines \( D_{2,3} \) and \( D_{2,4} \).

each one of the four lines \( D_{1,3}, D_{1,4}, D_{2,3} \) and \( D_{2,4} \) pass through the point \( \Pi_0 \).

**Theorem 4.6** Let \( Q \) be a quadric in \( \mathbb{P}^4(\mathbb{F}_q) \) and \( X \) a degenerate quadric in \( \mathbb{P}^4(\mathbb{F}_q) \) of rank \( (X) = 4 \) with \( g(X) = 1 \). We get :

\[
\#X_{Z(Q)}(\mathbb{F}_q) \leq 3q^2 + q + 1.
\]

The code \( C_2(X) \) defined on the degenerate quadric \( X \) of \( \text{rank}(X) = 4 \) with \( g(X) = 1 \) is a \([n, k, d]_q\)-code where \( n = q^3 + q + 1 \), \( k = 14 \), \( d \geq q^3 - 3q^2 \).

**Remark 4.7** From the study of the parameters of these codes, we can assert that the performances of the codes \( C_2(X) \) defined on the non-degenerate quadrics are better than the ones defined on the degenerate quadrics.

### 5 Quadratic section of the non-degenerate hermitian variety

In this section \( \mathbb{F}_q \) denotes the field with \( q \) elements, where \( q = t^2 \) and \( X \) denotes the non-degenerate (i.e. non-singular) hermitian variety of \( \mathbb{P}^4(\mathbb{F}_q) \) of equation \( X : x_0^{t+1} + x_1^{t+1} + x_2^{t+1} + x_3^{t+1} + x_4^{t+1} = 0 \).

In [3, p.1175] Bose and Chakravarti proved the following result:

**Theorem 5.1** Let \( \tilde{X} \subset \mathbb{P}^n(\mathbb{F}_q) \) be a non-degenerate hermitian variety. Then,

\[
\#\tilde{X}(\mathbb{F}_q) = \Phi(n, t^2) = [t^{n+1} - (1)^{n+1}][t^n - (1)^n]/(t^2 - 1) \quad (7)
\]
For $\tilde{X} \subset \mathbb{P}^n(F_q)$ a degenerate hermitian variety of rank $r < n + 1$, we have:

$$\#\tilde{X}(F_q) = (t^2 - 1)\pi_{n-r}\Phi(r-1,t^2) + \pi_{n-r} + \Phi(r-1,t^2),$$

where $\Phi(n,t^2)$ is given by [7].

From theorem 5.1 we get that

$$\#X(F_q) = (t^2 + 1)(t^5 + 1).$$

It has been shown by Bose et al. [3, p.1176] that $g(X) = 1$ (i.e. $X$ contains lines and does not contain a plane). In the work of Hirschfeld [9, p.60] we get that there are $t^3 + 1$ lines contained in $X$ through each point of $X$.

We will study here the section of $X$ by any quadric $Q$ of $\mathbb{P}^4(F_q)$. An approach to this problem has also been considered by F. Rodier in [14, pp.207-208]. He used the result of G. Lachaud [10, proposition 2.3] which gives an upper bound for the number of rational points of an algebraic set of a given degree. He found a bound which is not optimal.

We will use a different method depending mainly on the geometric structure of the quadric $Q$ and the hermitian variety $X$.

Let us recall the classification of hermitian varieties in $\mathbb{P}^3(F_q)$, table.5, which can be found in the work of J.W.P. Hirschfeld [9, p.60].

| $r(V)$ | Description | $|V|\quad g(V)$ |
|--------|-------------|---------------|
| 1      | repeated plane $\Pi_2\mathcal{U}_0$ | $t^4 + t^2 + 1$ | 2 |
| 2      | $t+1$ collinear planes $\Pi_1\mathcal{U}_1$ | $t^5 + t^4 + t^2 + 1$ | 2 |
| 3      | a cone $\Pi_0\mathcal{U}_2$ | $t^5 + t^2 + 1$ | 1 |
| 4      | non-singular hermitian surface $\mathcal{U}_3$ | $t^5 + t^3 + t^2 + 1$ | 1 |

Table 5: Hermitian surfaces in $\mathbb{P}^3(F_q)$

5.1 Plane section of the non-degenerate hermitian variety $X$: $g(Q)=2$

We are interested in the case where $g(Q)=2$ for the quadric $Q$. These are the cases where $Q = \Pi_2\mathcal{E}_1$, $Q = \Pi_1\mathcal{P}_2$ or $Q = \Pi_0\mathcal{H}_3(\mathcal{R},\mathcal{R}')$. In the book of J. W. P. Hirschfeld [7, p.160], we have also the classification of plane hermitian curves.
\[
\begin{array}{|c|c|c|c|}
\hline
r(\mathcal{V}) & \text{Description} & |\mathcal{V}| & g(\mathcal{V}) \\
\hline
1 & \text{repeated line } \Pi_1 U_0 & t^2 + 1 & 1 \\
2 & \text{cone } \Pi_0 U_1 & t^4 + t^2 + 1 & 1 \\
3 & \text{non-singular hermitian curve } U_2 & t^4 + 1 & 0 \\
\hline
\end{array}
\]

Table 6: Plane hermitian curves

From the table above, for the section of the non-singular hermitian variety \( \mathcal{X} \) by a plane \( Q \) we get \( |\mathcal{X} \cap Q| \leq t^3 + t^2 + 1 \). In the case \( r(Q) = 3 \) or \( r(Q) = 4 \), \( (Q \) is a union of \( q + 1 \) planes) we get \( |\mathcal{X} \cap Q| \leq t^5 + t^4 + t^3 + 2t^2 + 1 \).

### 5.2 Hyperplane section of the non-degenerate hermitian variety \( \mathcal{X} \): \( g(Q) = 3 \)

In the case \( g(Q) = 3 \), i.e. when \( Q \) contains a hyperplane, we have two possibilities: \( r(Q) = 1 \) (\( Q \) is a hyperplane), or \( r(Q) = 2 \) and \( Q \) is a pair of distinct hyperplanes.

#### 5.2.1 \( Q \) is a quadric with \( r(Q) = 1 \): \( Q = \Pi_3 P_0 \)

Here \( Q \) is a repeated hyperplane \( H \). Let us recall two general results of Bose and Chakravarti on hyperplane section of a non-degenerate hermitian variety.

**Theorem 5.2** [3, p.1173] Let \( \tilde{\mathcal{X}} \) be a non-degenerate hermitian variety in \( \mathbb{P}^n(\mathbb{F}_q) \) and \( H \) a hyperplane. If \( H \) is tangent to \( \tilde{\mathcal{X}} \) at \( P \), then \( \tilde{\mathcal{X}} \cap H \) is a degenerate hermitian variety of rank \( n-1 \) in \( \mathbb{P}^{n-1}(\mathbb{F}_q) \). The singular space of \( \tilde{\mathcal{X}} \cap H \) consists of the single point \( P \).

**Theorem 5.3** [4, p.272] Let \( \tilde{\mathcal{X}} \) be a non-degenerate hermitian variety in \( \mathbb{P}^n(\mathbb{F}_q) \) and \( H \) a hyperplane. If \( H \) is not tangent to \( \tilde{\mathcal{X}} \), then \( \tilde{\mathcal{X}} \cap H \) is a non-degenerate variety in \( \mathbb{P}^{n-1}(\mathbb{F}_q) \).

From theorems 5.1, 5.2 and 5.3 we deduce the following result.

**Theorem 5.4** Let \( H \subset \mathbb{P}^4(\mathbb{F}_q) \) be a hyperplane

\[
\#\mathcal{X}_H(\mathbb{F}_q) = \begin{cases} 
  t^5 + t^3 + t^2 + 1 & \text{if } H \text{ is not tangent to } \mathcal{X}, \\
  t^5 + t^2 + 1 & \text{if } H \text{ is tangent to } \mathcal{X}.
\end{cases}
\]
If \( H \) is tangent to \( X \), then \( X \cap H \) is a singular hermitian surface of rank 3 in \( \mathbb{P}^3(\mathbb{F}_q) \). More precisely, \( X \cap H \) is a set of \( t^3 + 1 \) lines passing through a common point \( P \). If \( H \) is non-tangent to \( X \), then \( X \cap H \) is a non-singular hermitian surface in \( \mathbb{P}^3(\mathbb{F}_q) \); thus we get \( \#X_{H}(\mathbb{F}_q) = t^5 + t^3 + t^2 + 1 \).

5.2.2 \( Q \) is a quadric with \( r(Q) = 2 \) and \( Q \) is a pair of hyperplanes

Here we will also use two important results of Bose and Chakravarti on degenerate hermitian varieties in \( \mathbb{P}^n(\mathbb{F}_q) \).

**Theorem 5.5** [3, p.1171] Let \( \tilde{X} \) be a degenerate hermitian variety of rank \( r < n + 1 \) in \( \mathbb{P}^n(\mathbb{F}_q) \) and \( \Pi_{r-1} \) a linear projective space of dimension \( r - 1 \) disjoint from the singular space \( \Pi_{n-r} \) of \( \tilde{X} \). Then \( \Pi_{r-1} \cap \tilde{X} \) is a non-degenerate hermitian variety in \( \Pi_{r-1} \).

**Theorem 5.6** [3, p.1171] Let \( \tilde{X} \subset \mathbb{P}^n(\mathbb{F}_q) \) be a degenerate hermitian variety of rank \( r < n + 1 \). If \( P \) is any point belonging to the singular space of \( \tilde{X} \) and \( D \) is an arbitrary point of \( \tilde{X} \), then any point of the line \( (PD) \) belongs to \( \tilde{X} \).

Now let \( H_1 \) and \( H_2 \) be two distinct hyperplanes generating \( Q \). We have \( Q = H_1 \cup H_2 \). Let \( \mathcal{P} = H_1 \cap H_2 \) be the plane of intersection of the two hyperplanes. Let \( \hat{X}_1 = H_1 \cap X \), and \( \hat{X}_2 = H_2 \cap X \).

(i) In the case where each hyperplane is tangent to \( X \), we know that \( \hat{X}_1 \) and \( \hat{X}_2 \) as singular hermitian surfaces are sets of \( t^3 + 1 \) lines passing respectively through the point \( P_1 \) and \( P_2 \) (\( P_1 \neq P_2 \)). From theorem 5.2, we deduce that \( P_1 \) and \( P_2 \) are the singular spaces of \( \hat{X}_1 \) and \( \hat{X}_2 \).

- If the plane \( \mathcal{P} \) does not pass through at least one of the two points \( P_1 \) and \( P_2 \), without loss of generality we can suppose that \( \mathcal{P} \) does not pass through \( P_1 \). Since \( P_1 \) is the singular space of \( \hat{X}_1 \), from theorem 5.6, we deduce that \( \mathcal{P} \cap X_1 \) is a non-singular curve in \( \mathbb{P}^2(\mathbb{F}_q) \). Therefore, we get \( |\mathcal{P} \cap \hat{X}_1| = t^3 + 1 \), which with the relations (3) and (4) give \( |X \cap Q| = 2t^5 - t^3 + 2t^2 + 1 \).

- If the plane \( \mathcal{P} \) passes through \( P_1 \) and \( P_2 \), then it is obvious that \( P_1 \) and \( P_2 \) belong to \( \mathcal{P} \cap \hat{X}_1 \) and \( \mathcal{P} \cap \hat{X}_2 \) respectively. From the relation (4), we deduce that \( P_2 \in \mathcal{P} \cap \hat{X}_1 \). Therefore \( P_2 \in \hat{X}_1 \). Since \( P_1 \) is the singular space of \( \hat{X}_1 \), from theorem 5.6, we deduce that the line \( (P_1P_2) \) is contained in \( \hat{X}_1 \). The line \( (P_1P_2) \) is also contained in \( \mathcal{P} \). Therefore \( (P_1P_2) \) is contained in \( \mathcal{P} \cap \hat{X}_1 \) and from table 6, we get that \( \mathcal{P} \cap \hat{X}_1 \) is a degenerate hermitian curve in \( \mathbb{P}^2(\mathbb{F}_q) \). Here \( \mathcal{P} \cap \hat{X}_1 \) cannot be a set of \( t + 1 \) lines through a common point, because it would imply
\( P_1 = P_2. \) Thus, \( \mathcal{P} \cap \hat{X}_1 \) is the line defined by the two points \( P_1 \) and \( P_2. \) Therefore, from the relations (3) and (4) we get \(|X \cap Q| = 2t^5 + t^4 + 1.\)

(ii) In the case where one hyperplane \((H_2)\) is tangent to \(X\) and the second hyperplane \((H_1)\) is non-tangent to \(X, \hat{X}_2\) and \(\hat{X}_1\) are singular and non-singular hermitian surfaces respectively. From theorems 5.2, 5.5 and relation (4) we deduce that the following two conditions are equivalent:

- the plane \(\mathcal{P}\) is non-tangent to \(\hat{X}_1\)
- the plane \(\mathcal{P}\) is disjoint from the singular space \(\{P_2\}\) of \(\hat{X}_2\)

Therefore, from the theorems above and relation (3) we get either \(|X \cap Q| = 2t^5 + 2t^2 + 1\) when \(\mathcal{P}\) is non-tangent to \(\hat{X}_1\) or \(|X \cap Q| = 2t^5 + t^2 + 1\) when \(\mathcal{P}\) is tangent to \(\hat{X}_1\).

(iii) In the case where both hyperplanes are non-tangent to \(X\), we know that \(\hat{X}_1\) and \(\hat{X}_2\) are both non-singular hermitian surfaces. From the relation (4) and the fact that \(\mathcal{P} \cap \hat{X}_1\) (resp. \(\mathcal{P} \cap \hat{X}_2\)) is singular if and only if \(\mathcal{P}\) is tangent to \(\hat{X}_1\) (resp. \(\hat{X}_2\)), we deduce that \(\mathcal{P}\) is either tangent to \(\hat{X}_1\) and \(\hat{X}_2\), or non-tangent to \(\hat{X}_1\) and \(\hat{X}_2\).

- If the plane \(\mathcal{P}\) is tangent to \(\hat{X}_1\) (i.e. it is also tangent to \(\hat{X}_2\)), then from theorem 5.2, \(\mathcal{P} \cap \hat{X}_1\) is a singular hermitian curve of rank 2 in \(\mathbb{P}^2(F_q)\). Therefore, we get \(|\mathcal{P} \cap \hat{X}_1| = t^3 + t^2 + 1\), which with the relation (3) give \(|X \cap Q| = 2t^5 + t^3 + t^2 + 1\)

- If the plane \(\mathcal{P}\) is non-tangent to \(\hat{X}_1\) (i.e. it is also non-tangent to \(\hat{X}_2\)), \(\mathcal{P} \cap \hat{X}_1\) is a non-singular hermitian curve in \(\mathbb{P}^2(F_q)\). Therefore, we get \(|\mathcal{P} \cap \hat{X}_1| = t^3 + 1\), which with the relation (3) give \(|X \cap Q| = 2t^5 + t^3 + 2t^2 + 1\).

### 5.3 Line section of the non-degenerate hermitian variety \(X\):

\(g(Q)=1\)

We will now study the section of the non-degenerate hermitian variety \(X\) by \(Q\) with \(g(Q)=1\). In this case \(Q\) is the non-degenerate quadric (parabolic) or the degenerate quadric \(\Pi_0E_3\).

For the section of \(X\) by \(Q\) we need to distinguish two cases.

#### 5.3.1 \(Q \cap X\) does not contain any line

When \(Q\) is degenerate (i.e. \(Q = \Pi_0E_3\)) and \(Q \cap X\) does not contain any line, every line of \(Q\) intersects \(X\) in at most \(t + 1\) points. From the fact that \(Q\) consists of \(q^2 + 1\) lines through the point \(\Pi_0\), we deduce that \(|Q \cap X| \leq t^5 + t^4 + t + 1\).
When $Q$ is non-degenerate, it contains exactly $\alpha_q$ lines and there are $q + 1$ lines of $Q$ through any point of $Q$. Therefore, we deduce that $|Q \cap X| \leq \alpha_q(t + 1)/(q + 1)$ i.e. $|Q \cap X| \leq t^5 + t^4 + t + 1$.

5.3.2 $Q \cap X$ contains a line

First, we will study the simple case of the section of $X$ by the degenerate quadric. Next, we will treat the more technical case, section of $X$ by the non-degenerate quadric.

5.3.2.1 When $Q$ is degenerate: We get two possibilities:
- If there is no line of $X \cap Q$ through the vertex $\Pi_0$ of the cone $\Pi_0 \in \mathbb{F}_q$. As in 5.3.1 we deduce that $|X \cap Q| \leq t^5 + t^4 + t + 1$.
- If there is a line of $X \cap Q$ through the vertex $\Pi_0$ of the cone, it is obvious that $\Pi_0$ is a point of $X$. Therefore, there are at most $t^3 + 1$ lines of $Q$ contained in $X$. Each one of the other $t^4 - t^3$ lines of $Q$ intersects $X$ in at most $t + 1$ points. Thus, we deduce that $|Q \cap X| \leq 2t^5 - t^4 + t^2 + 1$.

5.3.2.2 When $Q$ is non-degenerate: We will use the same technique as in §3.3.2 (intersection of two non-degenerate quadrics). Let us consider a line $D$ contained in $Q \cap X$, $P$ a plane through $D$ and $(H_i)$ the $q + 1$ hyperplanes passing through $P$ which generate $\mathbb{F}_q^4$. As in §3.3.2, for $i = 1, \ldots, q + 1$, $Q_i = H_i \cap Q$ are either hyperbolic quadrics or cone quadrics in $\mathbb{F}_q^3$. From theorems 5.2, 5.3 we get that for $i = 1, \ldots, q + 1$, $X_i = H_i \cap X$ are hermitian surfaces of rank 3 or 4 (non-singular hermitian surfaces) in $\mathbb{F}_q^3$.

Thus, one has to study the four types of intersection in $\mathbb{F}_q^3$ given by the table 7.

| Type | $Q_i \cap X_i$ |
|------|----------------|
| 1    | (hyperbolic quadric) $\cap$ (non-sing herm surf) |
| 2    | (quadric cone) $\cap$ (non-sing herm surf) |
| 3    | (hyperbolic quadric) $\cap$ (sing herm surface) |
| 4    | (quadric cone) $\cap$ (sing hermitian surface) |

Table 7: Intersection of $Q_i \cap X_i$ in $\mathbb{F}_q^3$

From the paragraphs 4.1 and 4.2 of [6], we can be more precise about
\# \hat{X}_{iZ(\hat{Q}_i)}(\mathbb{F}_q) \) for the types 1 and 2 of table 7 (section of the non-singular hermitian surface \( \hat{X}_i \) by a cone or hyperbolic quadric \( \hat{Q}_i \)). Thus, we have the following table 8.

| \( r(\hat{Q}_i) \) | Type  | \( L(\hat{X}_i \cap \hat{Q}_i) \) | \# \( \hat{X}_{iZ(\hat{Q}_i)}(\mathbb{F}_q) \) |
|-------------------|-------|-----------------------------|----------------------------------|
| \( 3 \) (cone)   | 2     | 2                          | \( t^3 + 2t^2 - t + 1 \)        |
|                   | 1     | 1                          | \( t^4 + t^2 + 1 \)             |
| \( 4 \) (hyperbolic) \( \mathcal{H}_3(\mathcal{R}_i, \mathcal{R}'_i) \) | 2     | 3                          | \( 2t^5 + t^2 + 1 \)            |
|                   | 1     | 2                          | \( \leq t^3 + 3t^2 - t + 1 \)   |
|                   |       | 1                          | \( \leq t^3 + 2t^2 + 1 \)       |

Table 8: Number of points and lines in \( \hat{Q}_i \cap \hat{X}_i \)

In table 8, \( \hat{X}_i \) and \( L(\hat{X}_i \cap \hat{Q}_i) \) denote respectively the non-degenerate hermitian surface and the number of lines contained in \( \hat{X}_i \cap \hat{Q}_i \). In the case where \( \hat{Q}_i \) is a hyperbolic, we consider \( L(\hat{X}_i \cap \mathcal{R}_i) \) (where \( \mathcal{R}_i \) is a regulus of the hyperbolic quadric \( \hat{Q}_i \)) instead of \( L(\hat{X}_i \cap \hat{Q}_i) \).

From the tables above, we deduce that an estimation on the number of points in the intersection of \( \mathcal{X} \cap \mathcal{Q} \) where \( \mathcal{X} \) and \( \mathcal{Q} \) are respectively a non-degenerate hermitian variety and a non-degenerate quadric variety with a common line, is resolved by considering the two following simple cases.

**(a) If \( \mathcal{Q} \cap \mathcal{X} \) contains exactly one common line:** From table 8, we get \( |\hat{Q}_i \cap \hat{X}_i| \leq t^3 + 2t^2 + 1 \) for the types 1 and 2 of table 7.

Likewise, for the type 3, if \( \hat{Q}_i \) and \( \hat{X}_i \) contain exactly the only common line \( \mathcal{D} \), a fortiori there is a regulus \( \mathcal{R}_i \) of the hyperbolic quadric \( \hat{Q}_i \) containing \( \mathcal{D} \). Each one of the \( q \) other lines of \( \mathcal{R}_i \) intersects \( \hat{X}_i \) in at most \( t + 1 \) points. We deduce that \( |\hat{Q}_i \cap \hat{X}_i| \leq t^3 + 2t^2 + 1 \).

For the type 4, the quadric cone \( \hat{Q}_i \) consists of \( q + 1 \) lines through a common point. The fact that there is only one line of \( \hat{Q}_i \) in \( \hat{X}_i \), implies that each one of the \( q \) other lines of \( \hat{Q}_i \) intersects \( \hat{X}_i \) in at most \( t + 1 \) points (the vertex of the cone is one of them). We deduce that \( |\hat{Q}_i \cap \hat{X}_i| \leq t^3 + t^2 + 1 \). Finally, when \( \mathcal{Q} \cap \mathcal{X} \) contains exactly one common line, we get:

\[
\text{for}\quad i = 1, 2, ..., q + 1 \quad |\hat{Q}_i \cap \hat{X}_i| \leq t^3 + 2t^2 + 1.
\] (8)

In this way, by using the relations (5) and (8) we conclude that \( |\mathcal{Q} \cap \mathcal{X}| \leq t^5 + t^4 + t^3 + 2t^2 + 1 \).
(b) If $Q \cap \mathcal{X}$ contains at least two common lines: We distinguish two cases: $Q \cap \mathcal{X}$ containing only skew lines or some secant lines. We will use an important property on the intersection of a hyperbolic quadric and a non-singular hermitian surface in $\mathbb{P}^3(F_q)$.

Lemma 5.7 [8, pp.123-124] Let $\hat{Q}_i = \mathcal{H}(R_i, R'_i)$ and $\hat{X}_i$ denote respectively the hyperbolic quadric and the non-singular hermitian surface in $\mathbb{P}^3(F_q)$. If $\hat{Q}_i = \mathcal{H}(R_i, R'_i)$ has three skew lines on $\hat{X}_i$, then $\hat{X}_i \cap \hat{Q}_i$ consists of $2(t+1)$ lines of $R_0 \cup R'_{0}$ where $R_0 \subset R_i$, $R'_0 \subset R'_i$ and $|R'_0| = |R_0| = t + 1$.

Remark 5.8 Lemma 5.7 says that, if a hyperbolic quadric contains three skew lines on the non-singular hermitian surface $\hat{X}_i$, then it contains exactly $2(t+1)$ lines of the surface $\hat{X}_i$, and $t + 1$ lines in each of the two reguli.

We will also use the following result.

Lemma 5.9 Let $D_1, D_2$ be two secant lines of the non-degenerate hermitian variety $\mathcal{X} \subset \mathbb{P}^4(F_q)$ and $P = \langle D_1, D_2 \rangle$ the plane defined by these two lines. Let $(H_i)_{i=1,...,q+1}$ be the $q + 1$ hyperplanes containing $P$. If there exists a hyperplane $H_1$ tangent to $\mathcal{X}$, then it is unique (i.e. the remaining $q$ hyperplanes are non-tangent to $\mathcal{X}$).

The proof of this lemma is analogous to the one of lemma 3.12.

(i) If $Q \cap \mathcal{X}$ contains only skew lines.

For the types 2 and 4 of table 7, since the quadric cone consists of $q + 1$ lines through a common point, we deduce that $\hat{Q}_i \cap \hat{X}_i$ contains exactly one common line. And therefore from table 8, we get that $|\hat{Q}_i \cap \hat{X}_i| \leq t^3 + t^2 + 1$ for the type 2. For the type 4, we also have $|\hat{Q}_i \cap \hat{X}_i| \leq t^3 + t^2 + 1$ as in the subparagraph (a) above.

For the type 3, since the singular hermitian surface $\hat{X}_i$ is a set of $t^3 + 1$ lines through a common point, we also deduce that $\hat{Q}_i \cap \hat{X}_i$ contains exactly one common line. Thus, as in subparagraph (a), we get $|\hat{Q}_i \cap \hat{X}_i| \leq t^3 + 2t^2 + 1$.

For the type 1, from remark 5.8, we deduce that $\hat{Q}_i$ as a hyperbolic quadric in $\mathbb{P}^3(F_q)$, cannot contain three skew lines of $\hat{X}_i$. Otherwise $\hat{Q}_i$ would contain two secant lines of $\hat{X}_i$. Thus, under the condition of the non-existence of secant lines in $Q \cap \mathcal{X}$, any regulus of $\hat{Q}_i$ contains at most two skew lines of $\hat{X}_i$ and therefore from the table 8, we get $|\hat{Q}_i \cap \hat{X}_i| \leq t^3 + 3t^2 - t + 1$.

Finally, when $Q \cap \mathcal{X}$ contains only skew lines, we get:

$$|\hat{Q}_i \cap \hat{X}_i| \leq t^3 + 3t^2 - t + 1.$$  (9)
In this way, by using the relations (5) and (9) we conclude that $|Q \cap \mathcal{X}| \leq t^5 + 2t^4 - 3t^2 + 1$.

(ii) If $Q \cap \mathcal{X}$ contains some secant lines, let $(D_1)$ and $(D_2)$ be two of them (which are not skew) and $P$ the plane generated by them. One has two cases:

- The $q + 1$ hyperplanes $H_i$ are all non-tangent to $\mathcal{X}$: in this case $\hat{X}_i$ is a non-singular hermitian surface. From the four types of table 7, only the first two types appear in this case. From the table 8, we deduce that for $i = 1, 2, \ldots, q + 1$, $|\hat{Q}_i \cap \hat{X}_i| \leq 2t^3 + t^2 + 1$. (10)

Thus, from (6) and (10) we get $|Q \cap \mathcal{X}| \leq 2t^5 - t^4 + 2t^3 + t^2 + 1$.

- If the $q + 1$ hyperplanes are not all non-tangent to $\mathcal{X}$, from lemma 5.9, there is a unique tangent hyperplane to $\mathcal{X}$, say $H_1$. Now, $\hat{Q}_1 \cap \hat{X}_1$ can be of the types 3 and 4.

For the type 3, $\hat{Q}_1$ and $\hat{X}_1$ contain exactly two common lines. Each one of the remaining $t^2 - 1$ lines of $\hat{X}_1$ intersects $\hat{Q}_1$ at most two points (the vertex of $\hat{X}_1$ is one of them). We deduce that $|\hat{Q}_1 \cap \hat{X}_1| \leq t^2 + 2t^2$.

Thus, when there is a unique hyperplane tangent to $\mathcal{X}$, from (6) we get that:

$$|\mathcal{X} \cap Q| \leq \begin{cases} 2t^5 + t^2 + 1 & \text{if } t = 2, \\ 2t^5 - t^4 + 3t^3 - t + 1 & \text{if } t \geq 3. \end{cases}$$

6 The parameters of the code $C_2(\mathcal{X})$ defined on the non-degenerate hermitian variety $\mathcal{X}$

From the results of section 5, we deduce the following results.

**Theorem 6.1** Let $Q$ be a quadric in $\mathbb{P}^4(\mathbb{F}_q)$ and $\mathcal{X}$ the non-degenerate hermitian variety $\mathcal{X}: x_0^{t+1} + x_1^{t+1} + x_2^{t+1} + x_3^{t+1} + x_4^{t+1} = 0$, we get

$$\# \mathcal{X}_Z(Q)(\mathbb{F}_q) = 2t^5 + t^3 + 2t^2 + 1, \ 2t^5 + t^3 + t^2 + 1$$
\[
\#\mathcal{X}_Z(\mathbb{F}_q) = 2t^5 + 2t^2 + 1, \quad 2t^5 + t^2 + 1,
\]

and for \( t \neq 2,3 \)
\[
\#\mathcal{X}_Z(\mathbb{F}_q) = \begin{cases} 
2t^5 - t^3 + 2t^2 + 1 \\
\text{or} \\
\leq 2t^5 - t^4 + 3t^3 - t + 1.
\end{cases}
\]

**Theorem 6.2** (Serre-Sørensen) ([16, p.351], [17, chp.2, pp.7-10]) Let \( f(x_0, ..., x_m) \) be a homogeneous polynomial in \( m+1 \) variables with coefficients in \( \mathbb{F}_q \) and degree \( h \leq q \). Then the number of zeros of \( f \) in \( \mathbb{P}^m(\mathbb{F}_q) \) satisfies:
\[
\#Z(f)(\mathbb{F}_q) \leq hq^{m-1} + \pi_m - 2.
\]
This upper bound is attained when \( Z(f) \) is a union of \( h \) hyperplanes passing through a common linear space of codimension 2.

**Theorem 6.3** The code \( C_2(\mathcal{X}) \) defined on the hermitian variety \( \mathcal{X} : x_0^{t+1} + x_1^{t+1} + x_2^{t+1} + x_3^{t+1} + x_4^{t+1} = 0 \) is a \([n,k,d]_q\) code where \( n = (t^2 + 1)(t^5 + 1) \), \( k = 15 \), \( d = t^7 - t^5 - t^3 - t^2 \).

**Proof:** Here we need to prove that the linear map \( c : \mathbb{F}_2 \rightarrow \mathbb{F}_q^{|\mathcal{X}|} \) is injective. In general, when \( f \) is an homogeneous polynomial of degree \( h \) in the affine space \( \mathbb{A}^5(\mathbb{F}_q) \), by the Serre-Sørensen’s bound on hypersurfaces, we deduce that \( \#Z(f)(\mathbb{F}_q) \leq h6t^6 + t^4 + t^2 + 1 \). From the fact that \( \#\mathcal{X}(\mathbb{F}_q) = t^7 + t^5 + t^2 + 1 \) we deduce that for \( h \leq t \), the map \( c \) is injective. From the relation (1), we get that \( k = 15 \). From theorem 6.1 and the definition of \( d \) at the end of section 2, we deduce that \( d = \#\mathcal{X}(\mathbb{F}_q) - (2t^5 + t^2 + 2t^2 + 1) = t^7 - t^5 - t^3 - t^2 \).

Observe that theorem 6.3 gives the exact parameters (i.e. the minimum distance) of the codes \( C_2(\mathcal{X}) \) which cannot be found by F. Rodier in [14, pp.207-208].

**Theorem 6.4** The minimum weight codewords correspond to quadrics which are pair of hyperplanes non-tangent to \( \mathcal{X} \) such that the plane of intersection of the two hyperplanes intersects \( \mathcal{X} \) at a non-singular hermitian plane curve.

**Theorem 6.5** The second weight of the code \( C_2(\mathcal{X}) \) is \( t^7 - t^5 - t^3 \). The codewords of second weight correspond to quadrics which are pair of hyperplanes non-tangent to \( \mathcal{X} \) and the plane of intersection of the two hyperplanes intersecting \( \mathcal{X} \) at a singular hermitian plane curve of rank 2 (i.e. a set of \( t + 1 \) lines through a common point).
Theorem 6.6 The third weight of the code $C_2(\mathcal{X})$ is $t^7 - t^5 - t^2$.

The codewords of third weight correspond to quadrics which are pair of hyperplanes, one tangent to $\mathcal{X}$, the second non-tangent to $\mathcal{X}$ and such that the plane of intersection of the two hyperplanes intersects $\mathcal{X}$ at a non-singular hermitian plane curve.

Theorem 6.7 The fourth weight of the code $C_2(\mathcal{X})$ is $t^7 - t^5$.

The codewords of fourth weight correspond to:

– quadrics which are pair of hyperplanes, one tangent to $\mathcal{X}$, the second non-tangent to $\mathcal{X}$ and the plane of intersection of the two hyperplanes is tangent at the non-singular hermitian surface obtained from the second hyperplane.

– quadrics which are pair of tangent hyperplanes and the plane of intersection of the two hyperplanes intersecting $\mathcal{X}$ at a single line.

Theorem 6.8 The fifth weight (for $t \neq 2, 3$) of the code $C_2(\mathcal{X})$ is $t^7 - t^5 + t^3 - t^2$.

The codewords of fifth weight (for $t \neq 2, 3$) correspond to quadrics which are pair of tangent hyperplanes and the plane of intersection of the two hyperplanes intersecting $\mathcal{X}$ at a non-singular hermitian plane curve.

7 Conjectures

7.1 Conjecture on the minimum distance of the code $C_h(\mathcal{X})$ in $\mathbb{P}^4(\mathbb{F}_q)$

The author has tried to generalize the study of the code $C_2(\mathcal{X})$ to $C_h(\mathcal{X})$ where $\mathcal{X}$ is the non-degenerate hermitian variety defined by $x_0^{t+1} + x_1^{t+1} + x_2^{t+1} + x_3^{t+1} + x_4^{t+1} = 0$ in $\mathbb{P}^4(\mathbb{F}_q)$ ($q = t^2$) and conjecture that:

Conjecture 1

For $h \leq t$  \#$\mathcal{X}_{Z(f)}(\mathbb{F}_q) \leq h(t^5 + t^2) + t^3 + 1$.

This upper bound is the best possible. The minimum weight codewords correspond to hypersurfaces which have the following configuration:

– hypersurfaces reaching the Serre-Sørensen’s upper bound for hypersurfaces (i.e. union of $h$ hyperplanes $H_i$ passing through a common plane $P$)

– each hyperplane $H_i$ is non-tangent to $\mathcal{X}$

– and the plane $P$ of intersection of the $h$ hyperplanes intersecting $\mathcal{X}$ at a non-singular hermitian plane curve.
Unfortunately no proof has been found yet. In general, it seems that when we use this strategy of code-construction, the combinatorical complexity of finding the minimum distance increases drastically with the degree $h$.

Remark 7.1 The preceding conjecture is true for $h = 1$ (theorem 5.4) and for $h = 2$ (theorems 6.1, 6.4).

7.2 Conjecture on the first five weights of the code $C_2(\mathcal{X})$ in $\mathbb{P}^n(\mathbb{F}_q)$

The author has also tried to generalize the study of the code $C_2(\mathcal{X})$ to $\mathcal{X}$ the non-degenerate hermitian variety defined by $x_0^{t+1} + x_1^{t+1} + \ldots + x_N^{t+1} = 0$ in $\mathbb{P}^N(\mathbb{F}_q)$ ($q = t^2$) and conjecture that:

Conjecture 2

1 If $w_i$ ($1 \leq i \leq 5$) are the first five weights of the code $C_2(\mathcal{X})$, then there exist degenerate quadrics $Q$ reaching the Serre-Sørensen’s upper bound for hypersurfaces (i.e. $Q$ is a pair of distinct hyperplanes $Q = H_1 \cup H_2$), giving codewords of weight $w_i$.

2 The minimum weight (i.e. $w_1$) codewords are only given by degenerate quadrics which are pair of distinct hyperplanes ($Q = H_1 \cup H_2$) such that $(H_1 \cap H_2) \cap \mathcal{X}$ is a non-singular hermitian variety in $\mathbb{P}^{N-2}(\mathbb{F}_q)$ and:
   - If $N$ is even, the two hyperplanes $H_1$ and $H_2$ are non-tangent to $\mathcal{X}$
   - If $N$ is odd, the two hyperplanes $H_1$ and $H_2$ are tangent to $\mathcal{X}$.

3 For $i > 5$, there is no quadric which is a pair of distinct hyperplanes, giving codewords of weight $w_i$.

Unfortunately no proof has been found yet.

Remark 7.2 The second part of conjecture 2 is true for $N = 3$ (see [6, §4.1-4.2]). The conjecture is also true for $N = 4$: theorem 6.1, theorems 6.3 to 6.8 give the result.

Acknowledgment The author thanks Prof. F. Rodier. His remarks and patience encouraged him to work on the problem.

References
[1] Y. Aubry, Reed-Muller codes associated to projective algebraic varieties. In “Algebraic Geometry and Coding Theory”. (Luminy, France, June 17-21, 1991). Lecture Notes in
Math., Vol. 1518, Springer-Verlag, Berlin, (1992), 4-17.

[2] M. Boguslavsky, On the number of solutions of polynomial systems. Finite Fields and Their Applications 3 (1997), 287-299.

[3] R. C. Bose and I. M. Chakravarti, Hermitian varieties in finite projective space $PG(N, q)$. Canadian J. of Math. 18 (1966), 1161-1182.

[4] I. M. Chakravarti, Some properties and applications of Hermitian varieties in finite projective space $PG(N, q^2)$ in the construction of strongly regular graphs (two-class association schemes) and block designs, Journal of Comb. Theory, Series B, 11(3) (1971), 268-283.

[5] F. A. B. Edoukou, Codes defined by forms of degree 2 on quadric surfaces. arXiv:math. AG/0511679 v 1 28 Nov 2005. 5 pp., Submitted to IEEE. Trans. Inform. Theory (2005).

[6] F. A. B. Edoukou, The weight distribution of the functional codes defined by forms of degree 2 on hermitian surfaces. arXiv:math. AG/0512476 v 1 20 Dec 2005. 13 pp., Submitted to Finite Fields and Their Applications (2005).

[7] J. W. P. Hirschfeld, Projective Geometries Over Finite Fields (Second Edition) Clarendon Press. Oxford 1998.

[8] J. W. P. Hirschfeld, Finite projective spaces of three dimensions, Clarendon press. Oxford 1985.

[9] J. W. P. Hirschfeld, General Galois Geometries, Clarendon press. Oxford 1991.

[10] G. Lachaud, Number of points of plane sections and linear codes defined on algebraic varieties; in ” Arithmetic, Geometry, and Coding Theory ”. (Luminy, France, 1993), Walter de Gruyter, Berlin-New York, (1996), 77-104.

[11] D. B. Leep and L. M. Schueller, Zeros of a pair of quadric forms defined over finite field. Finite Fields and Their Applications 5 (1999), 157-176.

[12] E. J. F. Primrose, Quadrics in finite geometries, Proc. Camb. Phil. Soc., 47 (1951), 299-304.

[13] D. K. Ray-Chaudhuri, Some results on quadrics in finite projective geometries based on Galois fields, Can. J. Math. Vol. 14 (1962), 129-138.

[14] F. Rodier, Codes from flag varieties over a finite field, Journal of Pure and Applied Algebra 178 (2003), 203-214.

[15] P. Samuel, Géométrie projective, Presses Universitaires de France, 1986.

[16] J. P. Serre, Lettre à M. Tsfasman, In ”Journées Arithmétiques de Luminy (1989)”, Astérisque 198-199-200 (1991), 3511-353.

[17] A. B. Sørensen, Rational points on hypersurfaces, Reed-Muller codes and algebraic-geometric codes. Ph. D. Thesis, Aarhus, Denmark, 1991.

[18] H. P. F. Swinnerton-Dyer, Rational zeros of two quadratics forms, Acta Arith. 9 (1964), 261-270.

[19] J. Wolfmann, Codes projectifs à deux ou trois poids associés aux hyperquadriques d’une géométrie finie, Discrete Mathematics 13 (1975), 185-211.