Ground state cooling of a radio-frequency LC circuit in an optoelectromechanical system

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We present a complete theory for laser cooling of a macroscopic radio-frequency LC electrical circuit by means of an optoelectromechanical system, consisting of an optical cavity dispersively coupled to a nanomechanical oscillator, which is in turn capacitively coupled to the LC circuit of interest. We determine the optimal parameter regime where the LC resonator can be cooled down to its quantum ground state, which requires a large optomechanical cooperativity, and a larger electromechanical cooperativity. Moreover, comparable optomechanical and electromechanical coupling rates are preferable for reaching the quantum ground state.

I. INTRODUCTION

Over the past decade, the experimental realization of quantum states of macroscopic objects has made significant progress in the fields of opto- and electromechanics. These include mechanical ground state cooling [1–5], mechanical squeezing [6, 7], entanglement between mechanical, microwave and optical modes [8–12]. Also facilitated by this progress, hybrid quantum systems [13] provide interesting opportunities and a variety of novel platforms for new technological applications. In particular, optoelectromechanical devices has received significant attention, especially in transcoding radio frequency and microwave signals to the optical domain [14–29].

However, most of optoelectromechanical systems are using a GHz microwave resonator. Here, we consider a more macroscopic low-frequency LC resonator, and show that in such a tripartite system it is possible to cool a (sub-)MHz LC resonator into its quantum ground state. Realizing ground state cooling of a large-sized LC circuit is of significance to the study of macroscopic quantum phenomena. Laser cooling of an LC circuit via the intermediate coupling to a mechanical resonator has been first proposed in Ref. [14]. Here we extend that analysis, focusing on the possibility to reach ground state cooling of a macroscopic LC resonator, providing an alternative route to what has been recently achieved with a superconducting qubit in Ref. [30]. In this paper, we provide a detailed analysis of the system, by first determining its optimal working point, and then analysing its stationary state, focusing onto the parameter regime in which the radiofrequency (rf) LC resonant circuit can be cooled down to its quantum ground state. We show that ground state cooling of such a macroscopic system is possible if a large optomechanical cooperativity, and even larger electromechanical cooperativity characterize the system.

The remainder of the paper is organized as follows. In Sec. II, we introduce our tripartite optoelectromechanical system and provide its Hamiltonian and the corresponding Langevin equations. In Sec. III we determine the working point of the system and derive the linearized equations for the system quantum fluctuations. In Sec. IV we show how to exactly solve these linearized equations and determine the steady state of the system, while in Sec. V we provide an approximate analytical theory for the steady state occupancy of the rf resonator. In Sec. VI we describe the results and determine the optimal parameter regime for laser cooling the LC circuit to its quantum ground state. Then, in Sec. VII we discuss in detail the challenges one has to face for an unambiguous detection of the stationary state of rf resonator, while Sec. VIII is for concluding remarks.
II. THE SYSTEM

We consider a generic hybrid optomechanical system, which consists of an optical cavity, a nanomechanical oscillator, and a radiofrequency (rf) resonant circuit. Different kind of systems and configurations have been already proposed and characterized experimentally [14–28] and the treatment presented here can be applied to all the cases in which the electromechanical coupling is capacitive, and the optomechanical coupling is dispersive. Nonetheless, in order to be more specific, we will refer to the configuration in which the optomechanical system is the membrane-in-the-middle (MIM) one [31–35], i.e., a driven optical Fabry-Perot cavity with a thin semitransparent membrane inside. The membrane is metalized [14, 18, 19, 22, 36] and capacitively coupled via an electrode to an LC resonant circuit formed by a coil and additional capacitors, see Fig. 1. The Hamiltonian of the system can be written in general as the sum of an optical, mechanical and electrical term,

\[ H = H_{\text{opt}} + H_{\text{mech}} + H_{\text{LC}}, \]

where

\[ H_{\text{opt}} = \hbar \omega(x) a^\dagger a + i \hbar E \left( a^\dagger e^{-i\omega t} - a e^{i\omega t} \right), \]
\[ H_{\text{mech}} = \frac{p^2}{2m} + \frac{m \omega_m^2 x^2}{2}, \]
\[ H_{\text{LC}} = \frac{\phi^2}{2L} + \frac{q^2}{2C(x)} - qV. \]

In the optical contribution we consider a specific cavity mode, with photon annihilation (creation) operator \( a \) (\( a^\dagger \)), with the usual bosonic commutation relations \([a, a^\dagger] = 1\), which is driven by a laser of frequency \( \omega_L \) and input power \( P \). Consequently, the driving rate can be written as \( E = \sqrt{2\kappa_0} P / \hbar \omega_L \), with \( \kappa_0 \) the cavity amplitude decay rate through the input port. The mechanical Hamiltonian corresponds to a resonator with mass \( m \), displacement operator \( x \) and conjugated momentum \( p \), with commutation rule \([x, p] = i \hbar \), which is associated to a given vibrational mode of the metalized membrane with bare frequency \( \omega_0 \). The dispersive optomechanical coupling arises due to the dependence of the cavity mode frequency \( \omega(x) \) upon the membrane displacement \( x \), as discussed in Refs. [31–35].

The electrical contribution \( H_{\text{LC}} \) refers to the rf resonator, which we will describe here as a lumped-element series RLC circuit (see Fig. 1), whose dynamical variables are given by the concatenated flux \( \phi \) and the total capacitor charge \( q \), with the canonical commutation rules \([q, \phi] = i \hbar \). We have also included a driving term associated with the possibility to control the circuit via a voltage bias \( V \). The electromechanical coupling is capacitive and it arises from the displacement dependence of the effective circuit capacitance \( C(x) \). In the case of the chosen optoelectromechanical setup based on a metalized membrane, such as those of Refs. [14, 18, 19, 22], one can write

\[ C(x) = C_0 + C_m(x) = C_0 + \frac{\varepsilon_0 A_{\text{eff}}}{h_0 + x}. \]
$a_{in}$ and $a_{ex}$, which are uncorrelated and whose only nonzero correlation is $\langle \delta a(t) \delta a^\dagger(t') \rangle = \delta(t - t')$, $j = in, ex$.

We have included two zero-mean noise terms in the equations. $F(t)$ is the Langevin force operator which accounts for the Brownian motion of the mechanical oscillator, whose symmetrized correlation function is in general equal to [37, 38]

$$\frac{1}{2} \langle F(t) F(t') + F(t') F(t) \rangle = m \gamma_m \int \frac{d\omega}{2\pi} \cos \omega(t-t') \hbar \omega \coth \left( \frac{\hbar \omega}{2k_B T} \right),$$

which, in the case of a large mechanical quality factor $Q_m = \omega_0/\gamma_m \gg 1$ valid here, can be approximated with the Markovian expression [38],

$$\langle F(t) F(t') + F(t') F(t) \rangle / 2 = m \gamma_m \hbar \omega_0 (2 \hbar \omega_0 + 1) \delta(t-t'),$$

where $\hbar \omega_0 = [\omega^{\text{zpf}} / \kappa_T - 1]^{-1}$ is the equilibrium mean thermal phonon number, with $k_T$ the Boltzmann constant and $T$ the environmental temperature. We have also rewritten the external bias voltage as $V(t) = V_{DC} + \delta V(t)$, i.e., the sum of a DC bias and the Johnson-Nyquist voltage noise $\delta V$ with autocorrelation function [37, 39],

$$\frac{1}{2} \langle \delta V(t) \delta V(t') + \delta V(t') \delta V(t) \rangle = R \int \frac{d\omega}{2\pi} \cos \omega(t-t') \hbar \omega \coth \left( \frac{\hbar \omega}{2k_B T} \right),$$

which again, in the case of a large LC quality factor can be approximated with the Markovian expression $\langle \delta V(t) \delta V(t') + \delta V(t') \delta V(t) \rangle / 2 = R \hbar \omega_{LC} (2 \hbar \omega_{LC} + 1) \delta(t-t')$, where $\hbar \omega_{LC} = \left[ \omega^{\text{zpf}} / \kappa_{T_{LC}} - 1 \right]^{-1}$ is the mean thermal rf photon number. We have assumed in general $T_{LC} \neq T$ because the rf circuit tends to pick up ambient noise and the effective rf noise temperature can be larger than room temperature.

### III. WORKING POINT AND LINEARIZED DYNAMICS OF THE QUANTUM FLUCTUATIONS

In order to look for the possibility to reach the quantum regime for the macroscopic rf resonator, we have to evaluate the stationary quantum fluctuations around the classical steady state of the system, which is obtained by replacing all the operators in the Heisenberg-Langevin equations Eqs. (8)-(12) with the corresponding $c$-numbers, neglecting all noise terms, and setting all the derivatives to zero. In this way one defines the working point of the system, which is determined by the two external driving, i.e., the laser driving rate $E$ and the DC bias voltage $V_{DC}$. If stability conditions are satisfied (see Appendix), the steady state is characterized by the cavity mode in a coherent state with amplitude $\alpha_s$, the membrane in an equilibrium position displaced by $x_s$, the rf circuit with no current, and the capacitor with a stationary charge $q_s$. Using the fact that $p_s = \phi_s = 0$, one can express the working point parameters in terms of $x_s$, i.e.,

$$a_s = \frac{E}{\kappa + i \Delta},$$

$$q_s = C(x_s) V_{DC},$$

where $\Delta = \omega(x_s) - \omega_L$ is the effective cavity mode detuning, and it is the parameter which is actually fixed in an experiment by the cavity locking system. The static membrane displacement $x_s$ is the solution of the equilibrium condition for the three forces applied to the membrane, i.e., the membrane elastic force, the electrostatic force and the radiation pressure force,

$$m \omega_s^2 x_s = -\frac{e_0 A_{eff} V_{DC}^2}{2 (\hbar_0 + x_s)^2} - \hbar \frac{\partial \omega}{\partial x_s} (x_s) n_{cav},$$

where $n_{cav} = |\alpha_s|^2 = E^2 / (\kappa^2 + \Delta^2)$ is the intracavity photon number.

The quantum fluctuations dynamics is obtained by linearizing the exact Heisenberg-Langevin equations Eqs. (8)-(12) around the chosen working point, i.e., by keeping only first-order terms in such fluctuations. It is convenient to express these equations in terms of dimensionless quantities, scaled by the corresponding quantum zero-point fluctuation units, i.e.,

$$x \rightarrow x_s + x_{qpl} \delta x = x_s + \frac{\hbar}{m \omega_0} \delta x,$$

$$p \rightarrow p_s + p_{qpl} \delta p = \sqrt{\hbar m \omega_0} \delta p,$$

$$q \rightarrow q_s + q_{\omega_{LC}} \delta q = q_s + \frac{\hbar}{\omega_{LC}} \delta q,$$

$$\phi \rightarrow \phi_s + \phi_{\omega_{LC}} \delta \phi = \sqrt{\hbar \omega_{LC}} \delta \phi,$$

so that the commutation rules are rewritten as $[\delta x, \delta p] = [\delta q, \delta \phi] = i$. By introducing also the two intracavity quadrature fluctuations

$$\delta X = \frac{\delta a e^{i\theta} + \delta a^\dagger e^{-i\theta}}{\sqrt{2}},$$

$$\delta Y = \frac{\delta a e^{i\theta} - \delta a^\dagger e^{-i\theta}}{i \sqrt{2}},$$

where $\tan \theta = \Delta / \kappa$, after straightforward steps, one gets the following linearized Heisenberg-Langevin equations

$$\delta x = \omega_0 \delta p,$$

$$\delta \dot{p} = -\frac{\omega_x^2}{\omega_0} \delta x - \gamma_a \delta p + G \delta X - g \delta q + \xi,$$

$$\delta q = \omega_{LC} \delta \phi,$$

$$\delta \dot{\phi} = -\omega_{LC} \delta q - \gamma_{LC} \delta \phi - g \delta x + \delta V,$$

$$\delta X = \Delta \delta Y - \kappa \delta X + \sqrt{2} \kappa X_{vac},$$

$$\delta Y = -\Delta \delta X - \kappa \delta Y + G \delta x + \sqrt{2} \kappa Y_{vac}.$$

We have introduced the two relevant coupling rates, the optomechanical coupling rate

$$G = -x_{qpl} \frac{\partial \omega (x_s)}{\partial x} \sqrt{2 n_{cav}},$$

and the electromechanical coupling rate

$$g = \frac{e_0 A_{eff} V_{DC}}{C(x_s) (\hbar_0 + x_s)^2 \sqrt{\hbar m \omega_{LC} \omega_0}}.$$

on the treatment of Ref. [34], the contribution of the radiation force, as we show in Appendix B, where we provide the expression versus the electrode distance \( h \).

We also notice that the bare mechanical frequency \( \omega_0 \) is modified when the cavity is driven and the DC voltage bias is applied, acquiring the new value \( \omega_m \) given by the expression

\[
\omega_m^2 = \omega_0^2 + \frac{\hbar}{m} \frac{\partial^2 \omega(x_i)}{\partial x_i^2} n_{\text{vac}} - \frac{V_{\text{DC}}^2 \epsilon_0 A_{\text{eff}}}{m(h_0 + x_i)^3}.
\]

We recall that the system is stable provided that \( \omega_m^2 > 0 \) and the latter expression shows that there is a maximum value for \( V_{\text{DC}} \), the pull-in voltage, beyond which the effective mechanical frequency \( \omega_m \) becomes imaginary and the membrane is pulled onto the other electrode of the capacitor (see Appendix A). We also notice that for physically interesting parameter regimes, the shift \( x_i \) may be not negligible with respect to \( h_0 \) and tends to \(-h_0/3\) when approaching the pull-in voltage (see Appendix A). As a consequence, due to Eq. (17) and Eq. (30), the coupling \( g \) has a nonlinear dependence upon \( V_{\text{DC}} \), and it never surpasses a maximum value when \( V_{\text{DC}} \) approaches its maximum value \( V_{\text{pull}} \). This is explicitly shown in Fig. 2, where the electromechanical coupling \( g \) is shown versus the electrode distance \( h_0 \) and \( V_{\text{DC}} \).

In principle \( x_i \) is determined by the equilibrium between the mechanical stress, the electrostatic force and the radiation force, as we show in Appendix B, where we provide the explicit expressions for the membrane-in-the-middle case, based on the treatment of Ref. [34], the contribution of the radiation force on \( x_i \) is negligible.

Finally we have also introduced rescaled noise operators: i) the mechanical thermal noise term \( \dot{\xi}(t) = F(t)/\kappa_{\text{eff}} \), with symmetrized autocorrelation function (in the high \( Q_{\text{m}} \) Markovian limit)

\[
\frac{1}{2} \langle \xi(t)\xi(t') + \xi(t')\xi(t) \rangle = \gamma_m(2\hbar_m + 1)\delta(t-t');
\]

ii) the rescaled Nyquist noise operator on the rf circuit \( \delta \dot{V} = \delta \dot{V}(t)/\phi_{\text{rf}} \), with symmetrized autocorrelation function (in the high \( Q_{\text{LC}} \) Markovian limit)

\[
\frac{1}{2} \langle \delta \dot{V}(t)\delta \dot{V}(t') + \delta \dot{V}(t')\delta \dot{V}(t) \rangle = \gamma_{\text{LC}}(2\hbar_{\text{LC}} + 1)\delta(t-t');
\]

iii) the two vacuum optical noises

\[
X_{\text{vac}} = \frac{1}{\sqrt{2\kappa}} \left[ \sqrt{\kappa_{\text{in}}} (a^{\dagger} e^{i\theta} + a e^{-i\theta}) + \sqrt{\kappa_{\text{ex}}} (a^{\dagger} e^{i\theta} + a e^{-i\theta}) \right],
\]

\[
Y_{\text{vac}} = \frac{-i}{\sqrt{2\kappa}} \left[ \sqrt{\kappa_{\text{in}}} (a^{\dagger} e^{i\theta} - a e^{-i\theta}) + \sqrt{\kappa_{\text{ex}}} (a^{\dagger} e^{i\theta} - a e^{-i\theta}) \right],
\]

which are uncorrelated and possess the same autocorrelation function

\[
\frac{1}{2} \langle X_{\text{vac}}(t)X_{\text{vac}}(t') + X_{\text{vac}}(t')X_{\text{vac}}(t) \rangle = \frac{1}{2} \delta(t-t'),
\]

\[
\frac{1}{2} \langle Y_{\text{vac}}(t)Y_{\text{vac}}(t') + Y_{\text{vac}}(t')Y_{\text{vac}}(t) \rangle = \frac{1}{2} \delta(t-t').
\]

IV. DETERMINATION OF THE STEADY STATE

The linearized Heisenberg-Langevin Equations in (24)-(29) can be rewritten in the following compact matrix form

\[
u(t) = Au(t) + n(t),
\]

where \( u(t) = [\delta x(t), \delta p(t), \delta q(t), \delta \phi(t), \delta X(t), \delta Y(t)]^T \) is the vector of fluctuation (\( [\cdot]^T \) denotes the transposition operator), \( n(t) = [0, \xi(t), 0, \delta \dot{V}(t), \sqrt{2\kappa} X_{\text{vac}}(t), \sqrt{2\kappa} Y_{\text{vac}}(t)]^T \) the corresponding vector of noises, and \( A \) the matrix

\[
A = \begin{pmatrix}
0 & \omega_0 & 0 & 0 & 0 & 0 \\
-\frac{\omega_m}{\kappa_{\text{m}}} & -\gamma_m & -g & 0 & G & 0 \\
0 & 0 & 0 & \omega_{\text{LC}} & 0 & 0 \\
-g & 0 & -\omega_{\text{LC}} & -\gamma_{\text{LC}} & 0 & 0 \\
0 & 0 & 0 & 0 & -\kappa & \Delta \\
G & 0 & 0 & 0 & -\Delta & -\kappa
\end{pmatrix}.
\]

The formal solution of Eq. (38) is \( u(t) = M(t)u(0) + \int_0^t ds M(s)n(t-s) \), where \( M(t) = \exp[At] \). The system is stable and reaches its steady state for \( t \to \infty \) when all the eigenvalues of \( A \) have negative real parts so that \( M(\infty) = 0 \). Notice that when \( \Delta > 0 \), (laser red-detuned with respect to the cavity) which is relevant here because we want to laser cool the rf resonator via the intermediate interaction with the mechanical resonator, the instability threshold corresponds to the onset of optical bistability [40]. This is achieved only for very large values of the optomechanical coupling \( G \) which are not relevant here, corresponding physically to a situation where the renormalized mechanical frequency associated with the optical spring effect [41] becomes equal to zero.
In the linearized regime, the steady state of the tripartite optomelectromechanical system can be fully characterized because the noise terms are zero-mean quantum Gaussian noises, and as a consequence, the steady state of the system is a zero-mean tripartite Gaussian state, fully determined by its $6 \times 6$ correlation matrix (CM) $V_{ij} = \left\langle \left( u_i(\omega) u_j(\omega) + u_i(\omega) u_j(\omega) \right) \right\rangle / 2$.

Starting from Eq. (38), this steady state CM can be determined in two equivalent ways. Using the Fourier transforms $\tilde{u}_i(\omega)$ of $u_i(t)$, one has

$$V_{ij}(\omega) \equiv \int \frac{d\omega d\omega'}{(2\pi)^2} e^{-i(\omega + \omega')\omega'} \frac{1}{2} \left\langle \tilde{u}_i(\omega) \tilde{u}_j(\omega') + \tilde{u}_j(\omega') \tilde{u}_i(\omega) \right\rangle. \quad (40)$$

Then, by Fourier transforming Eq. (38) and the correlation functions of the noises in the Markovian limit, Eqs. (33), (34) and (37), one gets

$$\left\langle \tilde{u}_i(\omega) \tilde{u}_j(\omega') + \tilde{u}_j(\omega') \tilde{u}_i(\omega) \right\rangle \equiv \left[ \tilde{M}(\omega) D \tilde{M}(\omega')^T \right] \delta(\omega + \omega'), \quad (41)$$

where we have defined the $6 \times 6$ matrix

$$\tilde{M}(\omega) = (i\omega + A)^{-1}, \quad (42)$$

and the diagonal diffusion matrix

$$D = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \gamma_m(2\bar{n}_m + 1) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma_{LC}(2\bar{n}_{LC} + 1) & 0 & 0 \\ 0 & 0 & 0 & 0 & \kappa & 0 \\ 0 & 0 & 0 & 0 & 0 & \kappa \end{pmatrix}. \quad (43)$$

The $\delta(\omega + \omega')$ factor is a consequence of the stationarity of the noises, and inserting Eq. (41) into Eq. (40), one gets the following expression for the stationary correlation matrix

$$V^\infty = \int_{-\infty}^{\infty} d\omega \tilde{M}(\omega) D \tilde{M}(\omega)^T, \quad (44)$$

which can be equivalently rewritten as an integral in the time domain as

$$V^\infty = \int_0^\infty dt M(t) D(t) M(t)^T. \quad (45)$$

When the stability conditions are satisfied [$M(\infty) = 0$], Eq. (45) is equivalent to the following Lyapunov equation for the steady-state CM,

$$AV^\infty + V^\infty A^T = -D, \quad (46)$$

which is a linear equation for $V^\infty$ and can be analytically solved, but the general exact expression is too cumbersome and will not be reported here. The numerical analysis and the plots of Sec. VI are obtained from the numerical solution of Eq. (46).

In this paper we are interested only in the stationary state of the rf resonator and in its stationary energy in particular, which is equal to

$$U_{LC} = \frac{\hbar \omega_{LC}}{2} \left[ \langle \delta q^2 \rangle + \langle \delta \phi^2 \rangle \right] = \frac{\hbar \omega_{LC}}{2} \left( V_{33}^\infty + V_{44}^\infty \right) \quad (47)$$

$$\equiv \hbar \omega_{LC} \left( \tilde{n}_{eff} + \frac{1}{2} \right),$$

where $\tilde{n}_{eff}$ is the effective mean occupation number of the LC oscillator.

V. APPROXIMATE EXPRESSION FOR THE RF RESONATOR OCCUPANCY

One can adapt standard resolved sideband cooling theory of optomechanical systems for the derivation of an approximate expression of the stationary occupancy of the rf resonator. In the case of a standard optomechanical system, the stationary occupancy of the mechanical resonator, far from the strong coupling regime, can be very well approximated as [42–44]:

$$\tilde{n}_{eff} = \gamma_m \tilde{n}_m + \Gamma_m \tilde{n}_e + A_+, \quad (48)$$

where $\tilde{n}_e \approx 0$ is the mean excitation of the optical reservoir at zero temperature, $\Gamma_m = A_+ - A_-$ is the net laser cooling rate, and

$$A_+ = \frac{G^2 k_2}{\kappa^2 + (\Lambda \pm \omega_m)^2}, \quad (49)$$

are the scattering rates into the Stokes ($A_+$) and anti-Stokes ($A_-$) sidebands, corresponding respectively to the absorption or emission of a mechanical vibrational quantum. Eq. (49) implies that the net laser cooling rate is

$$\Gamma_m = A_- - A_+ > 0. \quad (50)$$

Eq. (48) can be seen as the result of the balance between the two energy exchange processes involving the mechanical resonator: i) the one with rate $\gamma_m$ with its thermal reservoir with $\tilde{n}_m$ mean excitations; ii) the other one with rate $\Gamma_m$ with the effective optical reservoir at zero temperature ($\tilde{n}_e = 0$) represented by the driven and decaying cavity, and which is responsible for cooling. The scattering rate $A_+$ is responsible for the quantum back-action limit associated with the quantum fluctuations of the radiation pressure force.

In the optoelectromechanical system under study, the rf resonator we are interested in is directly coupled to the mechanical resonator, which is in turn coupled to the driven optical cavity. In the proposal of Ref. [14] one can laser cool the rf resonator by driving on the red sideband of the optical cavity as in the usual optomechanical sideband cooling, and then exploiting the resonant electromechanical interaction in order to extend cooling to the rf circuit.

An equivalent description of the desired cooling process is that the rf resonator is cooled because its energy exchange processes at a rate $\gamma_{LC}$ with its thermal reservoir described by $\tilde{n}_{LC}$ mean excitation, dominated by the exchange processes with the much colder “polariton” reservoir represented by the
mechanical resonator hybridized with the optical cavity excitation in the regime of efficient sideband cooling. This latter effective reservoir is characterised by an effective decay rate \( \gamma_m = \gamma_m + \Gamma_m \), a nonzero mean number of excitations \( \bar{n}_m \) [see Eq. (48)], and the LC resonator will scatter polaritons into the corresponding Stokes and anti-Stokes sidebands with rates that are respectively given by

\[
A_{\text{eff}}^LC = \frac{g^2 \gamma_m}{(\gamma_m)^2 + 4(\omega_m \pm \omega_{LC})^2}.
\]

An intuitive explanation of the present expression is the fact that, when comparing with the optomechanical case of Eq. (49), the rate \( \gamma_m / 2 \) plays the role of the cavity decay rate \( \kappa \), and the electromechanical coupling \( g \) plays the role of the optomechanical coupling \( G \). As a consequence one has an effective polariton cooling rate

\[
\Gamma_{LC} = A_{\text{eff}}^LC - A_{\text{eff}}^LC > 0.
\]

One can then apply the same arguments used for deriving Eq. (48) to the present situation, and arrive at the following expression for the rf resonator occupancy

\[
\bar{n}_{LC}^{\text{eff}} = \frac{\gamma_{LC}\bar{n}_{LC} + \Gamma_{LC}^{\text{eff}}\bar{n}_{LC} + A_{\text{eff}}^LC}{\gamma_{LC} + \Gamma_{LC}^{\text{eff}}}.
\]

This is the desired approximation we were looking for. It works in the optimal regime for optimal sideband cooling, that is, \( \Delta \sim \omega_m \sim \omega_{LC} / \kappa > G \) as well as \( \gamma_m / 2 \sim G^2 / 4\kappa > g \). From Eq. (53) one can see that the rf resonator cannot be cooled more than the mechanical resonator and that therefore at best one can achieve \( \bar{n}_{LC}^{\text{eff}} \approx \bar{n}_m \). The latter condition is achieved when \( \Gamma_{LC} \sim A_{\text{eff}}^LC \gg A_{\text{eff}}^LC, \gamma_{LC} \), which is obtained at resonance \( \Delta = \omega_m = \omega_{LC} \gg \gamma_m \sim G^2 / 2\kappa \), when \( 2g^2 \kappa / \kappa \gg \gamma_{LC} \). Defining the two relevant cooperativities, the optomechanical cooperativity \( C_{om} = G^2 / 2\gamma_m \) and the electromechanical cooperativity \( C_{em} = g^2 / \gamma_{LC} \gamma_m \), the necessary condition to achieve simultaneous ground state cooling, \( \bar{n}_{LC}^{\text{eff}} \approx \bar{n}_m < 1 \), can be written as

\[
C_{em} \gg C_{om} \gg 1.
\]

This latter condition for the cooperatives can be satisfied only for an LC circuit with a large enough value of its quality factor, so that \( C_{LC} \ll g, \kappa \), because the electromechanical coupling \( g \) cannot be too large with respect to \( G^2 / \kappa \) for the validity of the above expressions. Nonetheless, the results of Sec. VI based on the exact numerical solution of the Lyapunov equation of Eq. (46) show that cooling of the rf resonator close to the quantum regime is possible also when the above assumptions are not fully satisfied and Eq. (53) is not too accurate.

VI. RESULTS FOR THE COOLING OF THE LC RESONATOR

Let us now determine the optimal parameter conditions under which one can cool a macroscopic LC circuit down to its quantum ground state. We show the main results in Fig. 3 and Fig. 4, where we apply the exact treatment of Eq. (47), and we compare it with the approximate expression of Eq. (53) (see Fig. 4). Since the stationary photon occupation number of the LC resonator reached during the cooling process, \( \bar{n}_{LC}^{\text{eff}} \) depends on the choice of the parameters, i.e., temperature, the LC quality factor \( Q_{LC} \) and so on, it is useful to introduce the resonator cooling efficiency \( \eta_{LC} \), aiming to easily compare results corresponding to different parameter regimes.

\[
\eta_{LC} = \frac{\bar{n}_{LC}^{\text{eff}}}{\bar{n}_{LC}^{\text{eff}}},
\]

that is, the ratio between the LC photon occupation number without cooling (no electromechanical interaction \( g = 0 \)) and the LC occupancy with cooling on, \( \bar{n}_{LC} \). For the chosen model of a DC-biased metalized membrane, the two conditions are equivalent to \( V_{DC} = 0 \) and \( V_{DC} \neq 0 \).

In Fig. 3(a) and Fig. 3(c) we display the solution of Eq. (47) in terms of the cooling efficiency as a function of the scaled electromechanical coupling \( g / \kappa \) and of the scaled optomechanical coupling \( G / \kappa \), for two different choices of the LC resonator quality factor and of the environment temperature, \( Q_{LC} = 10^6 \), \( T = 300 \) mK in Fig. 3(a),(b), and \( Q_{LC} = 4 \times 10^4 \), \( T = 10 \) mK, in Fig. 3(c),(d). All the other parameters are identical for each subgraph of Fig. 3, and correspond to typical values for a metalized membrane-in-the-middle configuration [19, 22], that is, laser optical wavelength \( \lambda = 1064 \) nm, membrane effective mass \( m = 0.7 \times 10^{-10} \) kg, membrane intensity reflectivity \( R = 0.4 \), mechanical resonance frequency \( \omega_0 = 2\pi \times 1 \) MHz, mechanical quality factor \( Q_m = 10^6 \), optical cavity length \( L_c = 8 \times 10^{-3} \) m, optical cavity finesse \( F = 5 \times 10^5 \), yielding a total optical cavity amplitude decay rate \( \kappa = 2\pi \times 374.74 \) kHz. We have also chosen \( \kappa_m = 0.4\kappa \), laser driving on the red mechanical sideband, that is, \( \Delta = \omega_0 \), the LC circuit resonant with the uncoupled mechanical resonator, \( \omega_{LC} = \omega_0 \), an equivalent circuit inducance \( L = 1 \) mH, and a membrane capacitor with an effective area \( A_{eff} = 1.1 \times 10^{-4} \) m\(^2\) and distance between the electrodes \( h_0 = 2 \) \( \mu \)m. In Fig. 3(b) and Fig. 3(d) we show instead the LC resonator occupation number \( \bar{n}_{LC}^{\text{eff}} \), as a function of the scaled electromechanical coupling \( g / \kappa \) for a fixed value of \( G \), corresponding to the red lines in Fig. 3(a) and Fig. 3(c), respectively. It is evident and expected that the maximum of the cooling efficiency strongly depends on the quality factor of the LC circuit, \( \eta_{LC}^{\text{max}} \sim 40000 \) in Fig. 3(a), and \( \eta_{LC}^{\text{max}} \sim 300 \) in Fig. 3(c), but the two graphs show that a region of maximal cooling is always present, and it is characterised by the condition that \( g \) and \( G \) are approximately of the same order.

The relevant result of our analysis is that experimentally reasonable parameter regions exist where it is possible to reach the quantum regime with an LC resonator occupation number below 1, as shown in Fig. 3(b) and Fig. 3(d). In both cases the optimal value of electromechanical coupling \( g \sim 0.12\kappa \), corresponding to \( g \sim 2\pi \times 44.968 \) kHz for both figures, is reachable with a value of \( V_{DC} \) far enough from the pull-in voltage (see Fig. 2). What is instead more demanding is to achieve a large enough value of the LC quality factor \( Q_{LC} \), and in order to reach the quantum regime one has to
compare it by considering low enough values of the environmental temperature $T_{\text{e}}$. In fact, in Fig. 3(a),(b) the LC resonator is cooled to the quantum regime $n_{\text{LC}}^{\text{e}} \ll 1$ starting from a cryogenic temperature of $T = 0.3$ K, but assuming an hard to reach, high quality LC resonator, $Q_{\text{LC}} = 10^8$. Instead, in Fig. 4(c),(d), we choose a state-of-the-art LC quality factor $Q_{\text{LC}} = 10^5$, but one has to decrease the environmental temperature to the ultra-cryogenic value $T_{\text{e}} = 10 \text{ mK}$ in order to cool down to an occupation number, $n_{\text{LC}}^{\text{e}} \lesssim 0.1$. This is promising because all the other parameters involved (optical cavity, circuitry, electromechanical device) are experimentally achievable even if demanding. In Fig. 3, at fixed $G$, the cooling efficiency (the LC resonator occupation number) shows always a maximum (minimum) as a function of $g$, and it decreases (increases) for increasing values of $g$. As discussed in Sec. V, for too large electromechanical couplings, the rf energy is dissipated through the mechanical resonator and the optical cavity in a less efficient way. Moreover, for increasing $g$, the mechanical resonator moves out of resonance with the LC resonator due to the electromechanical interaction [see Eq. (32)], and the cooling mechanism is less effective. We also recall here that in the present model, the coupling $g$ cannot be arbitrarily increased because, for any choice of the electromechanical parameters, i.e., the capacitor electrode distance $h_0$ and the capacitor effective area $A_{\text{eff}}$, $g$ is always upper limited by the maximum $V_{\text{DC}}$ that can be applied before the pull-in effect.

In Fig. 4 we compare the exact numerical result with the approximate analytical theory developed in Sec. V for different parameter choices. We show the LC resonator occupation number as a function of $g$ for the numerical solution (continuous lines) and for the approximate analytical theory of Eq. (53) (dashed lines) at different values of the environmental temperature and of the rf resonator quality factor $Q_{\text{LC}}$. In all situations, the approximate theory follows the numerical solution for relatively low values of the electromechanical coupling $g$, up to the value corresponding to the minimum occupancy. For larger $g$, the prediction of Eq. (53) rapidly diverges from the numerical solution, which is somehow expected because the approximated theory is valid as long as $g$ is not larger than the effective optomechanical decay rate $G^2/2\kappa$. Nonetheless, the approximate theory provides a very good estimate of the achievable cooling limit as well as of the $g$-interval where the minimum rf-photon occupancy could be achieved.

VII. DETECTION OF THE RF RESONATOR STEADY STATE

The effective mean photon number of the rf circuit at the steady state can be measured following two ways: i) measuring directly the rf voltage signal between two points of the circuit; ii) measuring the optical output of the cavity and trying to get information about the rf circuit state from it. In both cases these measurements are carried out in the frequency domain and therefore here we will focus on the solution of the Fourier transform of the Heisenberg-Langevin equations, Eq. (38). This solution has been already given in compact form in Sec. IV, but it will be convenient to re-express it in
more physical terms using effective quadrature susceptibilities.

Separating equally the two quadrature equations for each mode in the equations from Eq. (24) to Eq. (29), we get

\[ \chi_{\omega}^{(-1)}(\omega) \delta \chi(\omega) = \frac{\kappa - i \omega}{\Delta} \chi_{\omega}(\omega) + \chi_{\omega}(\omega), \]  
\[ \chi_{\omega}^{(-1)}(\omega) \delta \chi(\omega) = \frac{\kappa - i \omega}{\Delta} \chi_{\omega}(\omega) + \chi_{\omega}(\omega), \]  
\[ \chi_{\omega}^{(-1)}(\omega) \delta \chi(\omega) = \frac{\kappa - i \omega}{\Delta} \chi_{\omega}(\omega) + \chi_{\omega}(\omega). \]

where \( \chi_{\omega}(\omega), \chi_{\omega}(\omega), \) and \( \chi_{\omega}^{(-1)}(\omega) \) are the natural susceptibilities of the cavity, mechanical, and electrical modes, respectively, given by

\[ \chi_{\omega}(\omega) = \frac{\Delta}{2} \left( \frac{k - i \omega}{\Delta} \chi_{\omega}(\omega) + \chi_{\omega}(\omega) \right), \]
\[ \chi_{\omega}(\omega) = \frac{\Delta}{2} \left( \frac{k - i \omega}{\Delta} \chi_{\omega}(\omega) + \chi_{\omega}(\omega) \right), \]
\[ \chi_{\omega}^{(-1)}(\omega) = \frac{\Delta}{2} \left( \frac{k - i \omega}{\Delta} \chi_{\omega}(\omega) + \chi_{\omega}(\omega) \right). \]

The mutual interactions among the three modes lead to the modification of their natural susceptibilities. Inserting \( \delta \chi(\omega) \) and \( \delta \chi(\omega) \) in Eq. (57) into the equation \( \delta \chi(\omega) \), we obtain

\[ \chi_{\omega}^{(-1)}(\omega) \delta \chi(\omega) = \chi_{\omega}(\omega) G \sqrt{2 h} \left( \frac{k - i \omega}{\Delta} \chi_{\omega}(\omega) + \chi_{\omega}(\omega) \right) \]
\[ + \frac{\chi_{\omega}(\omega) G \sqrt{2 h}}{\Delta} \chi_{\omega}(\omega) + \frac{\chi_{\omega}(\omega) G \sqrt{2 h}}{\Delta} \chi_{\omega}(\omega), \]

where \( \chi_{\omega}(\omega) \) is the effective mechanical susceptibility, defined by

\[ \chi_{\omega}^{(-1)}(\omega) = \chi_{\omega}(\omega) - \frac{g^2 \chi_{\omega}^{(-1)}(\omega)}{\omega}, \]

with

\[ \chi_{\omega}(\omega) = \chi_{\omega}(\omega) - \frac{g^2 \chi_{\omega}(\omega)}{\omega}. \]

where \( \chi_{\omega}(\omega) \) is the effective mechanical susceptibility in the presence of only the optomechanical interaction. Eq. (60) together with \( \delta \chi(\omega) = -i \omega(\omega) \delta \chi(\omega) \) provides the mechanical response of the system to external perturbations.

Following the same approach, for the electrical mode we obtain

\[ \chi_{\omega}^{(-1)}(\omega) \delta \chi(\omega) = \delta \chi(\omega) - \chi_{\omega}(\omega) G \sqrt{2 h} \left( \frac{k - i \omega}{\Delta} \chi_{\omega}(\omega) + \chi_{\omega}(\omega) \right) \]
\[ + \frac{\chi_{\omega}(\omega) G \sqrt{2 h}}{\Delta} \chi_{\omega}(\omega) + \frac{\chi_{\omega}(\omega) G \sqrt{2 h}}{\Delta} \chi_{\omega}(\omega), \]

where \( \chi_{\omega}(\omega) \) is the effective rf circuit susceptibility, given by

\[ \chi_{\omega}^{(-1)}(\omega) = \chi_{\omega}(\omega) - \frac{g^2 \chi_{\omega}(\omega)}{\omega}. \]

In the same way Eq. (63) together with \( \delta \chi(\omega) = -i \omega(\omega) \delta \chi(\omega) \) provides the rf response of the system to external perturbations.

Eq. (47) can be rewritten as

\[ \tilde{R}_{\omega}^{\text{eff}}(\omega) = \frac{\langle \delta q^2 \rangle + \langle \delta q^2 \rangle - 1}{2}, \]

that is, the effective stationary rf photon number can be expressed in terms of the dimensionless charge and flux variances. In turn, using Eq. (44), these variances are given by the integral over the corresponding noise spectra

\[ \langle \delta q^2 \rangle = \int_{-\infty}^{\infty} d\omega \left[ M(\omega) D M(\omega) \right], \]
\[ \langle \delta q^2 \rangle = \int_{-\infty}^{\infty} d\omega \left[ M(\omega) D M(\omega) \right], \]

Therefore laser cooling of the rf resonator can be experimentally verified by measuring the charge noise spectrum \( \delta q(\omega) \), which can be explicitly written in terms of the effective susceptibilities defined above as

\[ S_{\delta q}(\omega) = \left| \chi_{\omega}^{(-1)}(\omega) \right|^2 \left[ \left| \chi_{\omega}(\omega) \right|^2 \left( S_{\text{rf}} + S_{\gamma} \right) + S_{\gamma} \right]. \]

where \( S_{\text{rf}} \) is the radiation pressure noise spectral contribution

\[ S_{\delta q}(\omega) = G^2 \kappa \left( \Omega^2 + \kappa^2 + \omega^2 \right)^2 + 4 \kappa^2 \omega^2, \]

and \( S_{\gamma} \) and \( S_{\gamma} \) are, respectively, the Brownian force noise and the voltage noise spectra, which are constant, white, contributions due to the Markovian approximation made on the Brownian and Johnson-Nyquist noise.
that, as suggested by Eq. (68), in all the typical physical conditions, the charge noise spectrum is characterized by a single Lorentzian-like peak essentially determined by the rf effective susceptibility \( \chi_{ee}^{\text{eff}} \) of Eq. (64). In fact, one can write with a very good approximation the effective rf susceptibility as a standard susceptibility with modified effective frequency \( \omega_{\text{eff}}^{\text{LC}} \) and damping \( \gamma_{\text{eff}}^{\text{LC}} \) [44, 45],

\[
|\chi_{\text{eff}}^{\text{LC}}(\omega)|^2 \approx \frac{\omega_{\text{eff}}^{\text{LC}}}{(\omega_{\text{eff}}^{\text{LC}})^2 - \omega^2} + (\omega \gamma_{\text{eff}}^{\text{LC}})^2, \tag{74}
\]

where

\[
\omega_{\text{eff}}^{\text{LC}} \approx \sqrt{\omega_{\text{LC}}^2 + \frac{g^2\kappa^2}{G^2}} \approx \omega_{\text{LC}}, \tag{75}
\]

under typical experimental conditions, and

\[
\gamma_{\text{eff}}^{\text{LC}} \approx \gamma_{\text{LC}} + \Gamma_{\text{LC}}, \tag{76}
\]

in agreement with the analysis of Sec. V. Therefore the charge noise spectrum \( S_{\delta q}(\omega) \) is peaked at \( \omega = \omega_{\text{eff}}^{\text{LC}} \approx \omega_{\text{LC}} \), and, using Eqs. (68)-(71), one can write its maximum value with very good approximation as

\[
S_{\delta q}^{\text{peak}} \approx S_{\delta q}(\omega_{\text{LC}}) = \frac{1}{4\chi_{\text{LC}}^{\text{eff}}} |\gamma_{\text{LC}}(2\bar{n}_{\text{LC}} + 1) + \frac{g^2\kappa^2}{(\omega_{\text{LC}} - \omega_{\text{LC}}^2 + (\omega_{\text{LC}})\gamma_{\text{LC}}) \omega_{\text{LC}}} + \frac{g^2(2\omega_{\text{LC}}^2 + \omega_{\text{LC}}^2)}{4\chi_{\text{LC}}^{\text{eff}} \omega_{\text{LC}}^2} \bigg| \gamma_{\text{LC}} \left(2\bar{n}_{\text{LC}} + 1\right) + \frac{g^2(2\omega_{\text{LC}}^2 + \omega_{\text{LC}}^2)}{4\chi_{\text{LC}}^{\text{eff}} \omega_{\text{LC}}^2} \bigg|, \tag{77}
\]

where we have approximated also the effective optomechanical susceptibility in the Lorentzian-like form [44, 45]

\[
|\chi_{\text{me}}(\omega)|^2 \approx \frac{\omega_{\text{me}}^2}{(\omega_{\text{me}} - \omega_{\text{LC}}^2 + (\omega_{\text{LC}})\gamma_{\text{LC}})(\omega_{\text{me}})^2 + (\omega \gamma_{\text{me}})^2}. \tag{78}
\]

Due to the peaked structure of \( S_{\delta q}(\omega) \), one has \( \omega^3 \approx \omega_{\text{LC}}^3 \) in Eq. (73), and therefore the calibration factor for the two voltage noise measurements is practically the same, implying that the condition for a faithful, direct, spectral measurement of the LC resonator photon occupancy reads

\[
S_{\delta q}^{\text{LC}} = \frac{\delta q^3}{C_{\text{LC}}^2} S_{\delta q}^{\text{peak}}. \tag{79}
\]

We also notice that, again due to the peaked form of \( S_{\delta q}(\omega) \), one has \( \langle \delta q^2 \rangle = \langle \delta q^3 \rangle \) [see Eqs. (66)-(67)] and therefore

\[
\delta q_{\text{eff}}^3 \approx \langle \delta q^3 \rangle - \frac{1}{2}. \tag{80}
\]

If we consider experimentally achievable parameters, enabling to approach the quantum regime for the rf circuit, \( \delta q_{\text{eff}} \approx 1 \), one sees that the condition of Eq. (79) is hard to satisfy, because its right hand side is of the order of \( 10^{-20} \text{ V}^2/\text{Hz} \). In fact, in this regime the charge noise spectrum peak is flattened and broadened because \( \gamma_{\text{LC}} \) becomes larger and larger. Under these conditions the resonance peak falls below the background noise level and a direct measurement of the rf photon occupancy is no longer possible. In this case however one can design an approximate indirect experimental detection of the rf circuit cooling process. In fact, by driving the rf circuit with a tunable AC voltage \( V_{\text{AC}}(\omega) \), much larger than Brownian, Johnson-Nyquist and radiation pressure noises, but small enough not to modify the working point of the system, one has from Eq. (63),

\[
\delta q(\omega) \approx \chi_{\text{eff}}^{\text{LC}}(\omega) V_{\text{AC}}(\omega), \tag{81}
\]

that is, one directly measures the effective susceptibilities of the LC circuit, and in particular its FWHM \( \gamma_{\text{eff}}^{\text{LC}} = \gamma_{\text{LC}} + \Gamma_{\text{LC}} \) [see Eq. (74)]. However, such a measurement provides also an indirect measurement of the rf photon occupancy in a large parameter regime, i.e., when the Johnson-Nyquist spectral contribution dominates over the mechanical and radiation pressure ones in the charge noise spectrum of Eq. (68). In fact, in this regime, one has simply [see also Eq. (53)]

\[
\delta q_{\text{eff}}^3 \approx \frac{\gamma_{\text{eff}}^{\text{LC}}}{\gamma_{\text{LC}}}, \tag{82}
\]

that is, the temperature of the rf circuit is scaled down by the ratio \( \gamma_{\text{LC}}/\gamma_{\text{eff}}^{\text{LC}} \).

An alternative way to probe the system properties is to detect the output of the optical cavity. However, any optical cavity mode interacts directly only with the mechanical resonator, and therefore it detects the dynamics of the rf circuit only indirectly, via its effects on the mechanical motion. As it is customary in cavity optomechanics [41], the resulting optical output spectra allows a good measurement of the effective mechanical occupancy, from which however it is hard to extract direct information about the steady state of the rf circuit.

VIII. CONCLUSIONS

We have investigated a tripartite optoelectromechanical system formed by an optical cavity, a mechanical oscillator, and an LC circuit, focusing on the possibility to achieve ground state cooling of the LC circuit. We have derived the optimal conditions to achieve such a regime, which requires a large optomechanical cooperativity, and an even larger electromechanical cooperativity. Under these conditions, the LC resonator can be cooled close to its quantum ground state, as confirmed by the exact numerical results in the linearized regime around the optimal working point of the circuit. Reaching the ground state of a macroscopic LC circuit can be viewed as a further, striking manifestation of macroscopic quantum state, and thus is useful for the study of macroscopic quantum phenomena. Furthermore, as shown in Ref. [46], macroscopic entanglement between the LC and mechanical resonators is possible when both resonators are cooled to their quantum ground state. Finally, manipulating rf resonant circuits at the quantum level will be also a promising starting point for the quantum-limited detection of weak rf signals.
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Appendix A: Pull-in voltage

As discussed in the main text in Sec. III, soon after Eq. (32), we cannot apply a too large value of the DC voltage bias $V_{DC}$ due to the pull-in effect of the electrode in front of the metalized membrane, softening the intrinsic spring constant of the membrane mechanical mode. The quantity $\omega_m^2$ of Eq. (32) must be always positive, and using Eq. (17), one can rewrite the stability condition of Eq. (32) as

$$\frac{m \omega_m^2 (h_0 + 3x_s) + \hbar n_{cav} [2\omega'(x_s) + \omega''(x_s)(h_0 + x_s)]}{h_0 + x_s} > 0,$$

where $\omega'(x_s)$ and $\omega''(x_s)$ denote respectively the first and second order derivatives of the cavity frequencies with respect to $x$. The denominator $h_0 + x_s$ is always positive because it is just the distance between the two electrodes of the effective plane-parallel capacitor modeling the membrane capacitor, so that the stability condition is equivalent to impose the positivity of the above numerator. However, it is possible to verify that the static radiation pressure frequency shift proportional to $n_{cav}$ is always negligible with respect to that of electrostatic origin under typical experimental values, and therefore one gets the very simple stability condition

$$x_s > -\frac{h_0}{3}. \quad \text{(A2)}$$

Using Eq. (17) without the negligible radiation pressure term, the critical point $x_s = -h_0/3$ can be re-expressed as a condition for the maximum applicable voltage, which is given by

$$V_{pull} = \frac{8m \omega_m^2 h_0^3}{27 \hbar \omega_c A_{eff}}, \quad \text{(A3)}$$

which can be rewritten as a condition on the maximum electrical field within the membrane capacitor

$$\left(\frac{V_{DC}}{h_0}\right)_{\text{max}} = \frac{8m \omega_m^2}{27 \hbar \omega_c (0)}. \quad \text{(A4)}$$

Appendix B: Explicit expressions in the case of a membrane-in-the-middle setup

We have not specified in the text the explicit form of the function $\omega(x)$, which is responsible for the radiation pressure coupling between the optical mode and the mechanical resonator. In fact, the results shown in the main text can be applied to a generic geometry of the optoelectromechanical setup. However, here we provide more details for the membrane-in-the-middle case, based on the treatment of Ref. [34]. One can always express the frequency of a chosen cavity mode in the presence of a semi-transparent membrane with intensity reflectivity $R$, placed at the static position $z_0$ along the cavity axis, as

$$\omega(x) = \omega_c + \frac{L_c}{c} \arcsin \left( \sqrt{R} \cos [2(k(z_0 + x))] \right), \quad \text{(B1)}$$

where $L_c$ is the cavity length, $k = \omega_c/c$ is the wave vector associated with the chosen cavity mode, and $\Theta$ is the overlap parameter, $0 \leq \Theta \leq 1$, quantifying the transverse overlap between the chosen optical and membrane vibrational modes.

The first order derivative determines the optomechanical coupling according to Eq. (30), and it is given by

$$\frac{\partial \omega}{\partial x} = -\Theta \frac{2 \omega_c}{L_c} \cos[2(k(z_0 + x))] \sqrt{R} \cos [2(k(z_0 + x))], \quad \text{(B2)}$$

The second order derivative instead enters into the expression for the renormalized mechanical frequency of Eq. (32) and it is given by

$$\frac{\partial^2 \omega}{\partial x^2} = -\Theta \frac{4 \omega_c}{c L_c} \sqrt{R} \cos [2(k(z_0 + x))] \frac{1 - 2R \cos^2 [2(k(z_0 + x))]}{[1 - R \cos^2 [2(k(z_0 + x))]]^{3/2}}, \quad \text{(B3)}$$

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