The Relative Value of the $\delta$—Symmetric 
Strangle under the Black Scholes Model

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Abstract

In this article we propose and calculate, under the Black Scholes option pricing model, a measure of the relative value of a delta-Symmetric Strangle. The proposed measure accounts for the price of the strangle, relative to the (present value of the) spread between the two strikes, all expressed, after a natural re-parameterization, in terms of delta and a volatility parameter. We show the startling main result that under the standard BS option pricing model, this measure of relative value is a function of delta only and is independent of the time to expiry, the price of the underlying security or the prevailing volatility used in the pricing model. In fact, the simple and intuitively appealing expression for this measure allows us to study the strangle's exit strategy and the corresponding optimal choice as values of delta.

Keywords: Call-put parity, option pricing, the Black-Merton-Scholes model, European options

1 Introduction

Options, as asset’s derivatives, are the primary tools available to the market participants for hedging their portfolio from directional risk and/or volatility risk. The so-called option’s delta, which typically denoted as $\delta$ or $\Delta$, measures the ‘sensitivity’ of the option’s price to changes in the price of the
underlying security, is the primary parameter one considers when using an option to mitigate directional risk. The option’s delta is seen as the hedging ratio and is often also used (near expiration) by market participants as a surrogate to the probability that the option will expire in the money. For standard option pricing models (i.e., the Black Scholes model, abbreviated here as the BS model, see below), these probabilities are readily available for direct calculations under the prevailing log-normality assumption of the asset’s returns. Roughly speaking, a trader that sells (or buys) a call option at a strike located one standard deviation above the current asset’s price, ends up with a 16-delta call option (i.e. \( \delta = 0.16 \)). We denote the corresponding strike for this 16-delta call contract as \( k_{0.16}^+ \). Similarly, a trader that sells (or buys) a put option at a strike located one standard deviation below the current asset’s price, ends up with a 16-delta put contract option (i.e. with \( \delta = -0.16 \)). We denote the corresponding strike for this 16-delta put contract option as \( k_{0.16}^- \). Thus, the corresponding strangle, which is obtained by selling a (negative) 16-delta put option and a (positive) 16-delta call option, is a delta-neutral strategy that is associated, very roughly, with a 0.68 probability for the asset’s price to remain between the two strikes, \( k_{0.16}^- \) and \( k_{0.16}^+ \) by expiration, all as resulting from the governing normal distribution assumption. We refer to such a strangle as a 16-delta Symmetric Strangle, only to indicate the common (absolute) delta value of its put and call components.

In a similar fashion we use the term a \( \delta \)-Symmetric Strangle to indicate the strangle obtained, for some fixed \( \delta \in (0, 0.5) \), from buying (or selling) a \( \delta \)-units put and call option contracts at the corresponding strikes \( k_{\delta}^- \) and \( k_{\delta}^+ \), respectively. Such a symmetric strangle would be a delta-neutral strategy offering zero directional risk but potentially useful for mitigating volatility risk. We further denote by \( \Pi_{\delta} \) the price of (or the credit received from) such \( \delta \)-Symmetric Strangle. In this paper, we study, for a a given \( \delta \), the value of this \( \delta \)-Symmetric Strangle relative to the width of the corresponding spread \( (k_{\delta}^+ - k_{\delta}^-) \), adjusted for its present value (PV). More precisely, for a given \( \delta \in (0, 0.5) \), we define the relative value of the corresponding \( \delta \)-Symmetric Strangle as

\[
\mathcal{R}_\delta := \frac{\Pi_{\delta}}{PV(k_{\delta}^+ - k_{\delta}^-)}.
\] (1)

In Section 2, we show the startling result that under the standard BS option pricing model, this measure of relative value, \( \mathcal{R}_\delta \), is a function of \( \delta \)
only and is independent of the time to expiry, the price of the underlying security or the prevailing volatility used in the pricing model. In fact, as we will see in Theorem 1 below, \( R_\delta \) takes the simple form

\[
R_\delta = -\frac{\phi(z_\delta)}{z_\delta} - \delta, \tag{2}
\]

where \( \phi(\cdot) \) is the standard normal density, \( \phi(u) = \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} \), and \( z_\delta \equiv \Phi^{-1}(\delta) \), is usual \( \delta^{th} \) percentile of the standard normal distribution, with \( \Phi(z) = \int_{-\infty}^{z} \phi(u) du \). We point out that since \( \delta < 0.5 \), we have \( z_\delta < 0 \) in expression (2) of \( R_\delta \).

As an illustration, by utilizing (2) one quickly finds that the 16-delta Symmetric Strangle has a relative value of \( R_{0.16} = 0.08467 \) and that the 30-delta Symmetric Strangle has a relative value of \( R_{0.30} = 0.36 \). That is to say that irrespective of the security’s price, or time to expiry, and irrespective of the prevailing volatility, one would expect the price of the 30-delta Symmetric Strangle to be 36% of the width of the spread between the corresponding strikes. Specifically, for a given \( \delta = 0.3 \) the corresponding strangle’s price is expected to be \( \Pi_{0.3} = 0.36 \times PV(k_{0.3}^+ - k_{0.3}^-) \). These straightforward calculations of the relative value measure, \( R_\delta \), of the \( \delta \)-Symmetric Strangle as is given in (2) allow us to address, in Section 3, the strangle’s exit strategy and the corresponding optimal choice of \( \delta \) for it. Section 4 provides some technical details and perspectives along with concluding remarks.

## 2 Pricing the \( \delta \)-unit option contract

One of the most widely celebrated option pricing model for equities (and beyond) is that of Black and Scholes (1973). Their pricing model is derived under some simple assumptions concerning the distribution of the asset’s returns, coupled with presumptive continuous hedging, zero dividend, risk-free interest rate, \( r \), and no cost of carry or transactions fees. In its standard form, the BS model evaluates, for a risky asset with a current market price \( \mu \), the price of an European call option contract at a strike \( k \) and \( t \) days to expiration as:

\[
c_\mu(k) = \mu \times \Phi(d_1(k)) - k \cdot e^{-rt} \times \Phi(d_2(k)). \tag{3}
\]
Here, using the standard notation,
\[
d_1(k) = \frac{\log(\frac{k}{K}) + (r + \frac{\sigma^2}{2})t}{\sigma \sqrt{t}} \quad \text{and} \quad d_2(k) = d_1(k) - \sigma \sqrt{t},
\]
where \(\sigma\) denotes the standard deviation of the daily asset’s returns, and \(\Phi(\cdot)\) is the standard normal c.d.f defined above. The model for the corresponding price of a put option contract, \(p_\mu(k)\), may be obtain from expression (3) of \(c_\mu(k)\), by exploiting the so-called put-call parity which is expressed by the equation
\[
\mu - c_\mu(k) = k \cdot e^{-rt} - p_\mu(k),
\]
see for example Peskir and Shiryaev (2002) or Knoll (1997) for details as well as [22] below. This parity implies that the price of the corresponding put option contract is,
\[
p_\mu(k) = k \cdot e^{-rt} \times \Phi(-d_2(k)) - \mu \times (1 - \Phi(d_1(k)))
\]
There is substantial body of literature dealing with the BS option pricing model in (3)-(4), its refinements, its extensions and the so-called Igreek's (i.e. various partial derivatives of various orders, representing the model’s ”sensitivities” to changes in its parameters). The interested reader is referred to standard textbooks such as Wilmott, Howison, Dewynne (1995), Hull (2005) or Iacus (2011).

As we already mentioned in the Introduction, here we focus our attention on the option’s delta, which we denote by \(\Delta\) as a function with a corresponding value of \(\delta \in (0, 1)\). More specifically, while suppressing (for sake of simplicity for now) from the notation \(\mu, r, t\) and \(\sigma^2\), we define for the call and the put contracts options their respective \(\Delta\) functions as,
\[
\Delta_c(k) := \frac{\partial c_\mu(k)}{\partial \mu} \quad \text{and} \quad \Delta_p(k) := \frac{\partial p_\mu(k)}{\partial \mu}.
\]
It follows immediately from the put-call parity equation in (5) that
\[
\Delta_p(k) = -(1 - \Delta_c(k)).
\]
It is well known (see also Lemma 2 and [25] below and the subsequent discussion there) that for the BS pricing model in (3),
\[
\Delta_c(k) = \Phi(d_1(k)),
\]

where \(d_1(k)\) is given in (4), and hence,

\[
\Delta_p(k) = -(1 - \Phi(d_1(k))) \equiv -\Phi(-d_1(k)). \tag{8}
\]

For its supreme importance to portfolio hedging, the investor/trader often needs to buy (or sell) an option at a strike, \(k\), which is associated with a specified and desired value \(\delta\) of the option’s \(\Delta\). For any given \(\delta \in (0, 1)\), we let \(k_\delta^+\) denote the (unique) solution of the equation \(\Delta_c(k_\delta^+) = \delta\), or equivalently by (7), the solution of

\[
\Phi(d_1(k_\delta^+)) = \delta. \tag{9}
\]

Accordingly, it follows immediately that \(k_\delta^+\) satisfies the equation

\[
d_1(k_\delta^+) = \Phi^{-1}(\delta) \equiv z_\delta, \tag{10}
\]

and hence, by utilizing (4) in (10) leads to the solution as

\[
k_\delta^+ = \mu \cdot e^{-z_\delta \nu + \nu^2/2 + rt}, \tag{11}
\]

where we have substituted \(\nu \equiv \sigma^2 t\) throughout. It should be clear from (11) that if \(\delta < 0.5\), one has \(z_\delta < 0\) and therefore \(k_\delta^+ > \mu\), so that the corresponding call option is said to be ‘out of the money’ (OTM). Also, note that it follows from (4) and (10) that

\[
d_2(k_\delta^+) = d_1(k_\delta^+) - \sigma \sqrt{t} \equiv z_\delta - \nu,
\]

so that

\[
\Phi(d_2(k_\delta^+)) = \Phi(z_\delta - \nu) \tag{12}
\]

in (3). Indeed, with the re-parameterization by \((\delta, \nu)\) (with \(\nu \equiv \sigma \sqrt{t}\)), of the BS option pricing model in (3), we may re-express, upon using the matching expressions (9)-(12) in equation (3), the \textit{current} price of a \(\delta\)--unit call option in a much simpler form as

\[
c(\delta, \nu) \equiv c_\mu(k_\delta^+) = \mu \times \left[ \delta - e^{-z_\delta \nu + \nu^2/2} \times \Phi(z_\delta - \nu) \right], \tag{13}
\]

for any \(\delta \in (0, 1)\) and with \(\nu > 0\).

**Remark 1:** Note in passing that in practice, the option’s \(\delta\) is often used as a crude approximation to the probability the option will end in the money, \(Pr(\text{ITM})\), which is equals, by (12), (13) and (23) below, to \(\Phi(d_2(k_\delta^+)) \equiv \)
\( \Phi(z - \nu). \) However, since \( \nu \equiv \sigma \sqrt{t} > 0, \) it immediately follows that \( \Phi(z - \nu) \leq \Phi(z) \) \( \equiv \delta. \) Hence, for any \( \delta \in (0,1) \) and \( \nu > 0, \) Pr(ITM) \( \leq \delta \) and only near expiration, as \( \lim_{t \to 0} \Pr(IMT) = \delta, \) it holds.

Similarly to (13), we calculate the current price of the \( \delta \)-unit put contract option by using the put-call parity equation in (5), and by noting that by (8)-(9) the corresponding \( k^- \) strike for the put contract is the same as the strike \( k^+_{1-\delta} \) of the \( (1-\delta) \)-unit call option contract, so that \( k^-_\delta \equiv k^+_{1-\delta}. \) Accordingly, by (5) we have

\[
p_{\mu}(k^-_\delta) = k^+_{1-\delta} \cdot e^{-rt} + c_{\mu}(k^+_{1-\delta}) - \mu
\]

\[
= -\delta \mu + k^+_{1-\delta} \cdot e^{-rt} \times (1 - \Phi(z_{1-\delta} - \nu)),
\]

which since \( z_{\delta} \equiv -z_{1-\delta}, \) we have as in (11),

\[
k^-_\delta = \mu \cdot e^{z_{\delta} + \nu^2/2} \cdot e^{-rt}.
\]

Hence, we obtain that under the \((\delta, \nu)\) re-parameterization, the expression for the current price of the \( \delta \)-unit put option is,

\[
p_{\mu}(\delta, \nu) \equiv p_{\mu}(k^-_\delta) = -\mu \times \left[ \delta - e^{z_{\delta} + \nu^2/2} \times \Phi(z_{\delta} + \nu) \right],
\]

(15)

**Remark 2:** The two strikes, \( k^+_{\delta} \) and \( k^-_{\delta}, (\equiv k^+_{1-\delta}), \) of the \( \delta \)-Symmetric Strangle need not be symmetrical with respect of the current price \( \mu \) of the underlying security. It is well-known that the occasional asymmetry of these equal \( \delta \)-units strikes is a fixture of the skew in the volatility surface that is affecting the option pricing model, see for example Doran and Krieger (2010).

Consider now a trader that simultaneously sells (say), at some desired level of \( \delta < 0.5, \) the \( \delta \)-unit put and the \( \delta \)-unit call contracts so as to form the \( \delta \)-Symmetric Strangle strategy. The total credit received from selling this strangle is therefore, \( \Pi_{\delta} = c_{\mu}(k^+_{\delta}) + p_{\mu}(k^-_{\delta}). \) As a measure for assessing the 'worthiness' of this strangle, we consider the value of the credit received, \( \Pi_{\delta}, \) relative to the present value of the spread between the strikes, namely, \( PV(k^+_{\delta} - k^-_{\delta}) = (k^+_{\delta} - k^-_{\delta}) \times e^{-rt}. \) We express this relative value measure in
as
\[ R(\delta, \nu) = \frac{\Pi_{\delta}}{PV(k^+_\delta - k^-_\delta)} \]
\[ = \frac{c_\mu(\delta, \nu) + p_\mu(\delta, \nu)}{(k^+_\delta - k^-_\delta)} \times e^{-rt}. \]

Note that by its definition, \( R(\delta, \nu) \geq 0 \) for all \( \delta \in (0, 0.5) \) and \( \nu > 0 \). By utilizing expressions (11), (13), (14) and (15) in \( R(\delta, \nu) \) and simplifying the resulting terms, we obtain, for each \( \delta < 0.5 \) and \( \nu > 0 \), the expression,
\[ R(\delta, \nu) = \frac{e^{z\delta \nu} \cdot \Phi(z_{\delta} + \nu) - e^{-z\delta \nu} \cdot \Phi(z_{\delta} - \nu)}{e^{-z\delta \nu} - e^{z\delta \nu}}, \]
for the relative value of the \( \delta\text{-Symmetric Strangle} \). We point out that by construction, \( R(\delta, \nu) \) in (16), is independent of \( \mu \), the current price of the underlying security as well as the risk-free interest rate, \( r \), and that its values for any \( (\delta, \nu) \) are easy and straightforward to calculate.

Figure 1: The Relative Value function \( R(\delta, \nu) \) of the \( \delta\text{-Symmetric Strangle} \) for \( \delta < 0.5 \) and \( \nu \in (0, 1) \).

Figure 1 below provides the graph of \( R(\delta, \nu) \) for various values of \((\delta, \nu)\), with \( 0 < \delta < 0.5 \) and \( 0 < \nu < 1 \), where \( \nu = \sigma\sqrt{t} \) representing realistic
values for $t$ (the time in days to expiry) and the model’s daily (implied or historical) volatility, $\sigma$. The startling revelation from Figure 1 is that for each fixed $0 < \delta < 0.5$, $R(\delta, \nu)$ is a constant independent of $\nu$ ($\nu \in (0, 1)$), and hence independent of $t$ and $\sigma$ in their appropriate practical domains.

**Theorem 1** Under the BS model and irrespective of the current asset’s price, $\mu$, the current risk-free interest rate, $r$, and irrespective of the time to expiry, $t$, and the presumed volatility (either implied or historical), the relative value of the OTM $\delta$-Symmetric Strangle with $\delta \in (0, 0.5)$, depends only on $\delta$ and is given by, $R(\delta, \nu) \equiv R_\delta > 0$, where

$$R_\delta := \lim_{\nu \to 0} R(\delta, \nu) = -\frac{\phi(z_\delta)}{z_\delta} - \delta.$$  

(17)

Moreover, $\lim_{\delta \to 0} R_\delta = 0$, and

$$R'_\delta := \frac{d}{d\delta} R_\delta = \frac{1}{z_\delta^2} > 0,$$

(18)

for all $\delta \in (0, 0.5)$.

**Proof.** The results stated in (17) follow immediately by a straightforward application of L’Hopital’s Rule to the numerator and denominator that comprise expression (16) of $R(\delta, \nu)$ and noting that it trivially also independent of $\mu$ and $r$ by construction. By another direct application L’Hopital’s Rule to the quotient $\phi(z_\delta)/z_\delta$ along with the facts that $\phi'(z_\delta) = -z_\delta \phi(z_\delta) z'_\delta$ and $z'_\delta = 1/\phi(z_\delta)$ leads to the second result as well as to the result stated in (18).

The results of Theorem 1 provide in (18) a unified benchmark for assessing the value, in relative terms, of a $\delta$-Symmetric Strangle under the BS option pricing model in (3) and (6), as applicable to any security (i.e. independent of the current underlying security price $\mu$), to any expiry (independent of $t$), and under any presumed volatility (independent of $\sigma$). In fact, if $\hat{R}_\delta$ denote the market value of a $\delta$-Symmetric Strangle (i.e. the market version of (18)), then, this strangle would be deemed 'well-priced' as long as $\hat{R}_\delta \geq R_\delta$.

In Figure 2 below, we graph the values of this function, $R_\delta$ (in (2)) for all $0 < \delta < 0.5$.

**Remark 3:** The results stated in Theorem 1 and their derivations are valid in the BS 'world', in which the distribution of the asset’s returns assumed
to have a constant variability throughout and do not take into account the volatility 'skew' or 'smile' that is often being observed by the traders across the options' grid (see also Remark 2).

Example 1: As an illustration of it’s usage, consider the market option pricing of the SPY as of March 5, 2020. We find that the 21-delta symmetric strangle with expiration of March 25, 2020 ($\mu = 300.75, t = 20, 42\%$ (average) implied volatility, so that $\sigma = IV/\sqrt{365} = 0.0219838$) trades for $6.58$ with the corresponding (asymmetric) strikes of $274$ and $319$ for the sold put and call respectively. This results with $\hat{R}_{0.21} = 6.58/(319 - 274) = 0.1462222$ whereas, by using (2), we have $R_{0.21} = 0.147383$ (marked in red in Figure 2). The reader is invited to check the validity of Theorem 1 results with any other traded security options at any expiration.

![Figure 2](image)

Figure 2: The relative value $R_{\delta}$ as a function of $\delta$. Marked in red is the market (current, as of March 5th 2020) relative value, $\hat{R}_{0.21} = 0.1462222$, of a 21-delta symmetric strangle in the SPY.

3 Strategizing

One of the appealing aspects of a $\delta$—Symmetric Strangle is that from the outset, it is a delta-neutral strategy with zero directional risk, initially. Moreover a trader that sells such a strangle, for some fixed $\delta < 0.05$, at the matching
two strikes \( k^-_\delta \) and \( k^+_\delta \), benefit from a well defined probability of success, that may be calculated under the \emph{current} distribution of the asset’s returns implied by BS option pricing model in (3) and (6). Specifically, for a given value \( \delta < 0.5 \) and \( \nu > 0 \), the \emph{initial} probability that the underlying security price would remain, at expiration, between \( k^-_\delta \), \( \equiv k^-_{1-\delta} \) and \( k^+_\delta \) is simply (see Remarks 1 and 2 and (25)),

\[
\alpha \equiv \Phi(-z_\delta - \nu) - \Phi(z_\delta - \nu).
\] (19)

Hence, the expected reward for a trader that sells the strangle for \( \Pi_\delta = c_\mu(k^+_\delta) + p_\mu(k^-_\delta) \) (as credit) and plans to exit and buy it back for a fraction \( \lambda \in (0, 1] \) of the credit received is

\[
E_\lambda(\delta) := \alpha \Pi_\delta - (1 - \alpha) \lambda \Pi_\delta.
\]

In relative terms, this expected reward, relative to the present value of the spread between the strikes, becomes

\[
\mathcal{E}_\lambda(\delta) := \frac{E_\lambda(\delta)}{PV(k^+_\delta - k^-_\delta)} = \alpha \mathcal{R}_\delta - (1 - \alpha) \lambda \mathcal{R}_\delta,
\] (20)

where \( \mathcal{R}_\delta \) is given in (2). As was mentioned in the Introduction and pointed out in Remark 1, for small values of \( \nu \) (i.e. near expiration) we may approximate the ‘success’ probability in (19) as \( \alpha \approx (1 - 2\delta) \). Accordingly, for any given fractional loss \( \lambda \in (0, 1] \) the expected relative reward in (20), under this approximation would be,

\[
\mathcal{E}'_\lambda(\delta) = (1 - 2\delta(1 + \lambda)) \times \mathcal{R}_\delta.
\] (21)

Observe that \( \mathcal{E}_\lambda(\delta) \geq 0 \) as long as \( \delta \leq 1/2(1 + \lambda) \) and that, upon using (17) and (18), the equation

\[
E'_\lambda(\delta) \equiv -2(1 + \lambda)\mathcal{R}_\delta + (1 - 2\delta(1 + \lambda))\mathcal{R}'_\delta = 0,
\]

has a unique root, \( \delta^* \), at which point \( \mathcal{E}_\lambda(\delta) \) attains its maximal value. That is, for a given fractional lose \( \lambda \in (0, 1] \), this root \( \delta^* = h(\lambda) \), must ’solves’ the equation

\[
\delta(1 - z^2_\delta) - z_\delta \phi(z_\delta) = \frac{1}{2(1 + \lambda)},
\]

at which point \( \mathcal{E}^*_\lambda := \mathcal{E}_\lambda(\delta^*) \geq \mathcal{E}_\lambda(\delta) \).
Figure 3: The expected relative reward function, $E_\lambda(\delta)$ as a function of $\delta$ with $\lambda = 0.5$. The maximal value is achieved at $\delta^* = 0.2336$ at which point, $E_\lambda(\delta^*) = 0.05615$.

In Figure 3 below we present the graph of the relative reward function, $E_\lambda(\delta)$, for a trader who sells a $\delta-$Symmetric Strangle, and wishes, as a matter of strategy, to exit it upon a loss of 50% of the credit received. This case corresponds to $\lambda = 0.5$ and results with an optimal choice for $\delta$ of $\delta^* = 0.2336$ for this strategy to yield a maximal expected relative reward of $E_{0.5}^* = 0.05615$.

In Table 1 below, we provide the 'optimal' values for $\delta$ as were calculated for various choices of $\lambda$, along with the corresponding values of the maximal expected (relative) reward $E_\lambda^*$, and the matching initial probability of "success" of this $\delta^*-$Symmetric Strangle strategy. As can be seen from Table 1, the selling 'standard' 16-delta symmetric strangle with its 0.68 'success' probability should be coupled with an exit strategy limits losses at 100% of the credit received.
| \( \lambda \) | \( \delta^* \) | \( \xi^* \) | \( \alpha(\delta^*) \) |
|-----|-----|-----|-----|
| 0.25 | 0.300 | 0.091 | 0.400 |
| 0.40 | 0.256 | 0.067 | 0.489 |
| 0.50 | 0.234 | 0.056 | 0.533 |
| 0.60 | 0.216 | 0.048 | 0.567 |
| 0.75 | 0.194 | 0.040 | 0.611 |
| 1.00 | 0.164 | 0.031 | 0.670 |

Table 1: The optimal choice for \( \delta \) for the \( \delta \)-Symmetric Strangle strategy, calculated for ‘exits’ with the various fractional loss \( \lambda \).

4 Some technical perspectives

In this section we provide some basic technical perspectives of the distributional form of the option price model in (3). In particular, as it relates to the model’s \( \Delta \)-function and the calculations of \( Pr(ITM) \).

Let \( X \) a positive random variable with a c.d.f. \( F(\cdot) \) and p.d.f. \( f(\cdot) \), having a mean \( \mu \) and a variance \( \sigma^2 < \infty \). To emphasize the dependency on \( \mu \), as a parameter, we write \( E_{\mu}(\cdot) \) for the expectation under \( F_{\mu} \equiv F \) (\( f_{\mu} \equiv f \)).

For any \( x \in \mathbb{R} \), let \( x^- = \max(-x, 0) \) and \( x^+ = \max(x, 0) \) and recall the decompositions, \( x = x^+ - x^- \) and \( |x| = x^+ + x^- \). Accordingly, for each \( s \geq 0 \), we define:

\[
  c_{\mu}(s) \equiv E_{\mu}[(X - s)^+], \text{ and } \quad p_{\mu}(s) \equiv E_{\mu}[(X - s)^-].
\]

It follows immediately that \( E_{\mu}(|X - s|) = c_{\mu}(s) + p_{\mu}(s) \) and that \( E_{\mu}(X - s) = c_{\mu}(s) - p_{\mu}(s) \), which in the more familiar form

\[
  p_{\mu}(s) - c_{\mu}(s) = s - \mu, \quad (22)
\]

represents, as in (5), the put-call parity (as for an option pricing model with no dividends, no cost of carry and with \( r = 0 \)). Along with the availability of (22), we will focus our attention here on \( c_{\mu}(\cdot) \) only, as the treatment of \( p_{\mu}(\cdot) \) would be similar. Note that by its definition, we may write

\[
  c_{\mu}(s) = \int_{s}^{\infty} (x - s) f_{\mu}(x) \, dx = \int_{s}^{\infty} x f_{\mu}(x) \, dx - s (1 - F_{\mu}(s)), \quad (23)
\]

or alternatively,

\[
  c_{\mu}(s) \equiv \int_{s}^{\infty} (1 - F_{\mu}(x)) \, dx. \quad (24)
\]
Hence, it follows immediately from (xx) that for each \( s > 0 \),

\[
c'_\mu(s) := \frac{\partial c_\mu(s)}{\partial s} = -(1 - F_\mu(s)).
\]

The Lemma given below provides the implied re-scaling property of \( c_\mu \) (an similarly of \( p_\mu \)) under an assumed scale parameterization of \( F_\mu \).

**Lemma 2** If for a given \( \mu > 0 \), \( F_\mu(s) \equiv F_1(s/\mu) \), for some c.d.f. \( F_1 \) whose mean is 1, then for \( s > 0 \), \( c_\mu(s) = \mu c_1(k/\mu) \) where \( c_1 \) is as defined in (24), but with respect to \( F_1 \), \( c_1(s) = \int_{s/\mu}^\infty (1 - F_1(u))du \).

**Proof.** This 'standardization' result follows from observing that

\[
c_\mu(s) = \int_{s/\mu}^\infty (1 - F_\mu(x)) \, dx \equiv \int_{s/\mu}^\infty (1 - F_1(x/\mu)) \, dx
\]

\[
= \mu \int_{s/\mu}^\infty (1 - F_1(u)) \, du = \mu c_1(s/\mu).
\]

Now, with \( c_\mu(s) \) as given in (23) with \( s > 0 \), define \( \Delta_\mu(s) := \partial c_\mu(s)/\partial \mu \), for the \( \Delta \) function under \( F_\mu(\cdot) \). Then, it follows directly from Lemma 2 that

\[
\Delta_\mu(s) = \frac{\partial}{\partial \mu} [\mu c_1(s/\mu)] = c_1(s/\mu) - \frac{k}{\mu} c'_1(s/\mu)
\]

\[
= 1/\mu [c_\mu(s) + sk(1 - F_\mu(s))]
\]

\[
= 1/\mu \int_s^\infty x f_\mu(x) dx,
\]

since \( c'_1(s/\mu) \equiv c'_\mu(s) = -(1 - F_\mu(s)) \) and \( c_\mu(s) = \int_{s/\mu}^\infty x f_\mu(x) \, dx - s(1 - F_\mu(s)) \), by (23). Accordingly, we may rewrite the expression for \( c_\mu(s) \) in (24) as

\[
c_\mu(s) \equiv \mu \times \Delta_\mu(s) - s \times (1 - F_\mu(s)), \tag{25}
\]

where

\[
\Delta_\mu(s) = \frac{1}{\mu} \int_s^\infty x f_\mu(x) dx \equiv \int_{s/\mu}^\infty uf_1(u) du := \Delta_1(s/\mu) < 1.
\]

Upon comparing the expression for \( c_\mu(s) \) as in (25) to the expression given by (3), with \( s = k \cdot e^{-rt} \), it becomes evident that for the BS option pricing model...
(with its assumed log-normal distribution of the returns), \( \Delta_\mu(s) \equiv \Phi(d_1(s)) \) and \( Pr(ITM) := (1 - F_\mu(s)) \equiv \Phi(d_2(s)) \), there. This point is demonstrated further by the following Example.

**Example 2:** Suppose that the random variable \( X \) has the 'standard' log-normal distribution having mean \( E(X) = 1 \) and variance \( V(X) = e^{\nu^2} - 1 \), for some \( \nu > 0 \), so that \( W = \log(X) \sim N(-\nu^2/2, \nu^2) \). Accordingly, the p.d.f of \( X \) is given by;

\[
f_1(x) = \frac{1}{x\nu} \times \phi\left(\frac{\log(x) + \nu^2/2}{\nu}\right), \quad x > 0.
\]

Therefor, the c.d.f of \( X \) is therefore

\[
F_1(s) = P_1(X \leq s) = \int_0^s f_1(x)dx = \Phi\left(\frac{\log(s) + \nu^2/2}{\nu}\right), \quad \forall s > 0,
\]

and trivially,

\[
1 - F_1(s) = P_1(X > s) = 1 - \Phi\left(\frac{\log(s) + \nu^2/2}{\nu}\right).
\]

Next, in order to calculate \( c_1(s) \) as given in (xx), we need to calculate \( \Delta_1(s) \) (as in (xx)), which upon using the relation \( X \equiv e^W \), where \( W \sim N(-\nu^2/2, \nu^2) \), becomes

\[
\Delta_1(s) = \int_s^\infty xf_1(x)dx = \int_{\log(s)}^\infty e^w \phi\left(\frac{w + \nu^2/2}{\nu}\right) \frac{dw}{\nu} = \int_{\log(s)}^\infty \phi\left(\frac{w - \nu^2/2}{\nu}\right) \frac{dw}{\nu} = 1 - \Phi\left(\frac{\log(s) - \nu^2/2}{\nu}\right).
\]

Hence, for the 'standardized' model we have

\[
c_1(s) = \Delta_1(s) - s(1 - F_1(s)) \equiv \\
1 - \Phi\left(\frac{\log(s) - \nu^2/2}{\nu}\right) - s \left[1 - \Phi\left(\frac{\log(s) + \nu^2/2}{\nu}\right)\right]
\]

It is straightforward to verify that if \( Y \equiv \mu X \) for some \( \mu > 0 \), then the p.d.f. of \( Y \) is the 'scaled' version of \( f_1 \), namely,

\[
f_\mu(y) = \frac{1}{\mu} \times f_1(y/\mu) \equiv \frac{1}{y\nu} \times \phi\left(\frac{\log(y/\mu) + \nu^2/2}{\nu}\right).
\]
Accordingly, by Lemma 2, \( c_\mu(s) \equiv \mu \times c_1(s/\mu) \) and we therefore immediately obtain the following expression for \( c_\mu(s) \) as,

\[
c_\mu(s) = \mu \times \left[ \Delta_1(s/\mu) - \frac{s}{\mu} \times (1 - F_1(s/\mu)) \right]
= \mu \times \left[ 1 - \Phi \left( \frac{\log(s/\mu) - \nu^2/2}{\nu} \right) \right] - s \times \left[ 1 - \Phi \left( \frac{\log(s/\mu) + \nu^2/2}{\nu} \right) \right]
= \mu \times \Phi \left( \frac{\log(s/\mu) + \nu^2/2}{\nu} \right) - s \times \Phi \left( \frac{\log(s/\mu) - \nu^2/2}{\nu} \right),
\]

where the last equality utilized the symmetry of the normal distribution. This is just expression (xx), for Black-Scholes option pricing model (though with \( r = 0 \), there). To account for the effects of the risk free interest rate \( r \) and of the dividend rate \( q \) on the call option price, one would substitute in \( c_\mu(s) \) as in (xx), the expressions \( \mu e^{-qt} \) and \( se^{-rt} \) for \( \mu \) and \( s \) respectively, and \( \nu = \sigma \sqrt{t} \), to obtain the ‘usual’ expression for the price of the call option under the Black-Scholes, Morton model,

\[
c_\mu(s) = \mu e^{-qt} \times \Phi \left( \frac{\log(s/\mu) + (r - q + \sigma^2/2)t}{\sigma \sqrt{t}} \right)
- se^{-rt} \times \Phi \left( \frac{\log(s/\mu) + (r - q - \sigma^2/2)t}{\sigma \sqrt{t}} \right).
\]

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