Looking for new solutions to the hierarchy problem

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Abstract

While in first and second quantization the fundamental operators are respectively coordinates and fields (functions), an extension of quantum field theory can be achieved if the usual pair of conjugate momenta is represented by functionals. After a brief introduction on the hierarchy problem, we show how the ordinary quantum field theory can arise from a specific limit of this extension. We will also show how this extension can offer new solutions to the hierarchy problem. The peculiarity that makes this scenario appealing (as possible solution of the hierarchy problem), is the absence of new light particles at (or close to) the electroweak scale. This is in much better agreement with the experimental observation, since (until now) all searches for new physics signals have confirmed the remarkable success of the Standard Model (contrary to common expectations).

1 Introduction

The incredible experimental success of the renormalizable Standard Model, well beyond any theoretical expectation, puts severe constraints to any attempt to solve the hierarchy problem introducing new particles or states at the weak scale. All experiments confirm the Standard Model. It is true that all these experiments could have missed new light particles signals for some fortuitous coincidence or for some theoretical and still unknown reason, but we should not neglect the impressive convergence of all experiments towards the same conclusion: the spectrum and the properties of the low energy theory coincide with that of the renormalizable Standard Model. We stress the word renormalizable, because it is ruled out not only the real production of such new light particles but also their virtual exchange, that could appear under the form of new effective non renormalizable operators. Limits to these operators can be found in the literature. They arise from precision electroweak data, from FCNC tests, but also from proton stability and neutrino masses. Here we argue that this experimental scenario discloses a clue that can help us to guess how to extend the Standard Model.

1.1 The problem

With the (renormalizable) Standard Model Hamiltonian, we are able to compute the scattering matrix for processes that involve different type of $|\text{in}\rangle$ and $|\text{out}\rangle$ states. All experimental tests prove the amazing accuracy of such predictions. In particular, experiments rule out the possibility to modify the renormalizable Standard Model Hamiltonian either adding new light particles or introducing new higher dimensional operators with a characteristic scale $\Lambda$ (according
to the operator choice, Λ can range from the TeV scale, up to the Grand Unification Scale).
On the other hand the same Hamiltonian is not able to give a realistic estimate of the effective potential, explaining the origin of the Hierarchy between the weak scale and the unification scale. The mass parameter in the Higgs potential is unstable under radiative corrections and it depends quadratically on the cut-off Λ, the scale of the new physics. Theories that introduces this cut-off close to the weak scale directly to the Hamiltonian, do not explain why present experiments have not yet found evidence for departures from the Standard Model. Theorists acknowledge the severe constraints to a relatively light new physics scale, and cope with them, assuming new symmetries that could reduce the impact of such light scale to the experimental tests, and getting used to the idea that a certain amount of fine tuning must be accepted. This scenario becomes less puzzling, if we look closer at the physical quantities that we are comparing. From one side we have very accurate predictions of the scattering matrix elements, between $|in \rangle$ and $|out \rangle$ states with the average $<in, out| φ |in, out >$ very close to $v$, the electroweak vev. These matrix elements, and the time evolution of these states are very well described by the Standard Model. On the other hand, we know that the same Standard Model hamiltonian is unable to evaluate the energy of states with a very different vev $<ψ|φ|ψ > = V >> v$ (i.e. the SM does not give an acceptable effective potential). However this set of states is experimentally very different from the previous one, since a state with $<ψ|φ|ψ > = V$ cannot be reproduced in a laboratory. To our opinion these considerations raise a doubt: can we extrapolate ordinary quantum field theory to include the time evolution of these states, with very different vev, and very far from our experience? In the following we will try to see if we can give a theoretical basis to this question.

1.2 A theoretical clue to solve the Hierarchy problem: the mathematical example of the non linear Schrödinger equation

It is useful to exemplify the problem as follows: we show that a non linear Schrödinger equation could explain an apparent paradox in a similar situation but in the simpler context of first quantization. To be concrete we take an explicit mathematical example. Consider the following equation

$$i \frac{\partial}{\partial t} \psi(x,t) = - \frac{\partial^2}{\partial x^2} \psi(x,t) + x^2 \cdot V \left( \int \psi^*(y,t) y^2 \psi(y,t) dy \right) \psi(x,t).$$  \hspace{1cm} (1)

Where $V(x)$ is an analytic real function with $V > 0$, and $V(x)$ going exponentially to zero when $x \to \infty$. The solutions of this equation conserve the probability, and for any $t > 0$ we have (if $ψ$ is normalized, at $t = 0$)

$$\int \psi^*(x,t) \psi(x,t) dx = 1$$  \hspace{1cm} (2)

This is simple to check once we observe that $\int \psi^*(y,t) y^2 \psi(y,t) dy$, and its time derivatives, are real for any $t$. Then

$$\frac{\partial}{\partial t} \int \psi^*(x,t) \psi(x,t) dx = \int \psi^*(x,t) \frac{\partial}{\partial t} \psi(x,t) dx + \int \psi(x,t) \frac{\partial}{\partial t} \psi^*(x,t) dx = 0$$  \hspace{1cm} (3)

where we have used [1]. One can also verify that the energy

$$E = - \int \psi^*(x,t) \frac{\partial^2}{\partial x^2} \psi(x,t) \ dz + F \left( \int \psi^*(y,t) y^2 \psi(y,t) dy \right)$$  \hspace{1cm} (4)

$^1$φ is the Higgs field.
with $F' = V$ is conserved. One can study the class of solutions of this equation that satisfy
\[
\int \psi^*(y,t) \psi(y,t) dy << V(0)/V'(0).
\]
In this approximation the (1) is an ordinary Schrödinger Equation with the potential of a harmonic oscillator (and with bounded states) provided that we replace $V(x) \to V(0)$. On the other hand, since $V(x)$ exponentially goes to zero when $x \to \infty$, we can also have solutions that are similar to a free gaussian wave packet, centered in $x >> 1$ and going to infinity when $t$ goes to infinity. This situation is similar to the puzzle that we have discussed in the previous section, where the experimental scenario apparently required two distinct Hamiltonians with two different class of solutions: one with small vev and another one with very large vev. Clearly a unique linear equation cannot describe the evolution of the physical system in both physical regions, and only a non linear equation can correctly describe the time evolution in all regions. In the following we would like to see if this naive idea can be exploited to solve the hierarchy problem (taking into account all necessary modifications required by the different physical context).

## 2 Second Quantization: Definitions

To start, we remind the formalism and the basics of the second quantization of a scalar field. The classical action for a free scalar field is
\[
\mathcal{A} = \int \frac{1}{2} (-\phi \partial^2 \phi - m^2 \phi^2) d^4 x
\]
from which we get the Hamiltonian
\[
\mathcal{H} = \int \frac{1}{2} (\pi^2(x) - \phi(x) \nabla^2 \phi(x) + m^2 \phi^2(x)) d^3 x.
\]

The rule of second quantization imposes replacing $\phi$ and $\pi$ with field operators $\hat{\phi}$ and $\hat{\pi}$ satisfying the equal time commutation relations below
\[
[\hat{\pi}(x), \hat{\phi}(y)] = [\hat{\phi}(y), \hat{\phi}(x)] = i\hbar \delta^3(x-y).
\]

We can give an explicit representation of the commutation (7) in the space of functionals $S[\phi]$ with $\phi$ being a real function $\phi(x)$ of the three dimensional space. The product of two states $\langle \psi_1 | \psi_2 \rangle$ can be (formally) defined in terms of a path integral (i.e. a functional integration) as follows:
\[
\langle \psi_1 | \psi_2 \rangle = \int \mathcal{D}\phi S_1[\phi] S_2[\phi].
\]

With these definitions, it is easy to guess the action of $\hat{\pi}(x)$ and $\hat{\phi}(y)$ onto a state $|S \rangle$. Namely
\[
\hat{\phi}(y) \ |S \rangle \Rightarrow \phi(y) S[\phi]
\]

\(^2\)For convention, $S[\phi]$ (with square brackets) denotes a functional of the real function $\phi(x)$, while $S(\phi)$ denotes an ordinary function $S$ of the real number $\phi$.

\(^3\)Note the analogy with first quantization where the product of $\langle \psi_1 | \psi_2 \rangle = \int \psi_1^*(x) \psi_2(x) dx$. The functional integration is defined as usual \(^3\): an infinite constant factor is always understood to get a well defined and finite result. We will come back to this infinite constant after.
and
\[
\hat{\pi}(y) |\psi\rangle \Rightarrow i \frac{\delta}{\delta \phi(y)} S[\phi]
\]  
(10)

where \(i\hbar \delta/\delta \phi(y)\) is a functional derivative. One can check that
\[
[i\hbar \frac{\delta}{\delta \phi(y)}, \phi(x)] = i\hbar \delta^3(x - y)
\]
(11)
as required by the (7). With the replacement (9,10), the Hamiltonian (6) in this functional representation becomes
\[
H = \int \frac{1}{2} (\hbar^2 \frac{\delta^2}{\delta \phi(x)^2} - \phi \nabla^2 \phi + m^2 \phi^2) d^3x.
\]
(12)

This functional operator leads to the Schrödinger functional equation
\[
\int \frac{1}{2} (\hbar^2 \frac{\delta^2 S[\phi]}{\delta \phi(x)^2} - (\phi \nabla^2 \phi) S[\phi] + m^2 \phi^2 S[\phi]) d^3x = E S[\phi]
\]
(13)

where \(S\) is a generic eigenfunctional of \(H\) with eigenvalue \(E\). The state with minimal \(E\) can be written\footnote{The eigenvalue \(E\) is divergent, but this divergence can be removed by a proper subtraction procedure.}
\[
S_0[\phi] = N \exp(-\frac{1}{2} \int d^3x \phi(x) \sqrt{-\nabla^2 + m^2 \phi(x)})
\]
(14)

with a (divergent) normalization constant \(N\) that sets
\[
\int \mathcal{D}\phi S_0^\dagger[\phi] S_0[\phi] = 1.
\]
(15)

The above integration \(\mathcal{D}\phi\) is understood in the functional sense [3] as in common path integrals. One can recognize that \(S_0[\phi]\), as given in (14), is similar to the wave function of the common harmonic oscillator, apart from the obvious change of functions with functionals. Starting from the vacuum \(S_0\) we can build other states, for instance we can create a new state through the action of the operator \(\hat{\phi}\) at the position \(z\), \(\hat{\phi}(z) |0\rangle\) and the corresponding functional becomes
\[
S_z[\phi] = N \phi(z) \exp(-\int d^3x \phi(x) \sqrt{-\nabla^2 + m^2 \phi(x)})
\]
(16)

We can also compute the product
\[
<0|\phi^\dagger(z) \phi(y) |0> = \int \mathcal{D}\phi S_z^\dagger[\phi] S_y[\phi] = N^2 \int \mathcal{D}\phi \phi(x) \phi(y) \exp(-\int d^3x \phi(x) \sqrt{-\nabla^2 + m^2 \phi(x)})
\]
(17)

This is easily evaluated once we realize that this is simply the definition of the two point Green function of three dimensional (euclidean) scalar theory with inverse propagator \(\sqrt{-\nabla^2 + m^2}\). This yields
\[
\frac{1}{2\sqrt{-\nabla^2 + m^2}} \delta^3(z - y) = \int \frac{d^3p}{2(2\pi)^3} \frac{e^{ip(z-y)}}{\sqrt{p^2 + m^2}}
\]
(18)
defined once we have chosen a vacuum functional (14), that sets the normalization \( N \) (15), once for all.

We stress that until now we have not introduced any new physics or new physical concept. We have only introduced a rather unusual formalism to define second quantization. Our aim, here, is to put in better evidence the parallel between the well known Schrödinger equation in first quantization and its (less popular) analogue in second quantization (13). This will become useful after, because it will make more transparent some essential clues of the hierarchy problem that will lead us to naturally extend quantum field theory.

3 Quantization of free functional fields

In what follows we will only introduce some basic concepts concerning the quantization of functional fields. We do not have the intention to be comprehensive, and sometimes we will be not rigorous. Even if the way to proceed is not unique, some important conclusions concerning the hierarchy problem can be achieved without entering into difficult theoretical details. In the discussion below, we will often put in evidence the parallel with first and second quantization, this will help us to guess how some results of these well known theories could be generalized into some sort of third quantization. Namely a theory where the role of fields \( \psi(x) \) is now played by functionals \( S[\phi] \). Note that eq. (13) can come from the action \( A \) below, where the integral \( \mathcal{D}\phi \) is taken in the functional sense

\[
\mathcal{A} = \int \mathcal{D}\phi \, S^{\dagger}[\phi, t] \left( i \frac{\partial}{\partial t} + \frac{1}{2} \left( -\hbar^2 \frac{\delta^2}{\delta\phi(x)^2} - (\phi \nabla^2 \phi) + m^2 \phi^2 \right) \right) S[\phi, t] \, d^3x \, dt + \text{h.c.} \quad (19)
\]

In fact, if we formally define a functional derivative, such that

\[
\frac{\delta}{\delta S[\phi']} S[\phi] = \prod_x \delta (\phi(X) - \phi'(X)) \equiv \delta^\infty[\phi - \phi']
\]

and

\[
\frac{\delta}{\delta S[\phi']} \int \mathcal{D}\phi \, S[\phi] G[\phi] = G[\phi'],
\]

the equation \( \delta \mathcal{A}/\delta S^{\dagger}[\phi] = 0 \) is equivalent to the (13). Let us try to quantize the action \( \mathcal{A} \). As usual the momentum conjugate of \( S[\phi] \) is obtained taking the derivative of \( \mathcal{A} \) with respect \( \dot{S}[\phi] \). Thus \( S[\phi] \) and \( i \, S^{\dagger}[\phi] \) represent a couple of conjugate momenta, that for the rule of quantization must obey the equal time anti-commutation prescription (if we consider fermion-like functional operators)

\[
\{ i S^{\dagger}[\phi, t], S[\phi', t] \} = \prod_{i=1,N} \delta^k (\phi(X_i) - \phi'(X_i)) \equiv \delta^\infty[\phi - \phi']
\]

As expected, the space-time Dirac delta is replaced by a functional Dirac delta. The Hamiltonian for these free functional fields \( S \) is given by

\[
\mathcal{H} = \int \mathcal{D}\phi \, S^{\dagger}[\phi] \, H \, S[\phi]
\]

where \( H \) is a functional operator given by (12), in the representation (11).
We can also define the operator
\[ \mathcal{N} = \int \mathcal{D}\phi \ S^\dagger[\phi] \ S[\phi] \] (24)
that commutes with the hamiltonian \( \mathcal{H} \). From the commutation relation \( [\mathcal{N}, S[\phi]] = S[\phi] \), we recognize that \( S^\dagger \) and \( S \) are creation and annihilation \emph{functional} operators. The eigenstates of
the hamiltonian (23) can be found, following the same arguments used in second quantization, and exploiting the commutativity of the Hamiltonian with the number of fields operator \( \mathcal{N} \).

Namely, we start from the vacuum \( |0\rangle \), defined by
\[ S|0\rangle = 0 \] (25)
from which we get
\[ \mathcal{H}|0\rangle = 0 \] (26)
The vacuum is an eigenstate of the Hamiltonian with zero energy. Then we proceed building all the other states. We can apply iteratively and several times the creation \emph{functional} operator. For example, the one \emph{functional} creation operator \( S^\dagger \) can creates one field states with a generic wave functional \( F \), as follows
\[ |n_1\rangle = \int \mathcal{D}\phi F_n[\phi] S^\dagger[\phi]|0\rangle \] (27)
with
\[ \langle n_1|n_1\rangle = \int \mathcal{D}\phi F^*_n[\phi] F_n[\phi] = 1 \] (28)
The low index 1 in the label \( n_1 \), means that the state (27) is an eigenstate of \( \mathcal{N} \), with eigenvalue 1. It is easy to verify that \( \mathcal{H}|n_1\rangle = E_n \ |n_1\rangle \) implies the Schrödinger equation (13) (with \( S \) replaced by \( F \)). For certain choices of the Hamiltonian \( H \), the state with minimum energy could be of the type (27), and we can label it \( |0_1\rangle \). States with two functional fields \( |n_2\rangle \) would lead to two identical (and decoupled) equations for two different functionals \( F_1[\phi_1] \) and \( F_2[\phi_2] \): they would describe two parallel worlds that do not talk to each other, as long as we restrict to the free Hamiltonian (24).

A (Higgs) field that in second quantization is represented by a scalar field operator \( \hat{\phi} \) (see section 2), in the context of a third quantization can be replaced by a proper combination of the functional operators \( S^\dagger[\phi] \) and \( S[\phi] \). Namely,
\[ \hat{\phi}(x) \Rightarrow \hat{\Phi}(x) \equiv \int \mathcal{D}\phi \ S^\dagger[\phi] \phi(x) S[\phi] \] (29)
\[ \hat{\pi}(x) \Rightarrow \hat{\Pi}(x) \equiv i \int \mathcal{D}\phi \ S^\dagger[\phi] \frac{\delta S[\phi]}{\delta \phi(x)} \] (30)
One can verify that the commutation relations (11) are satisfied (using (22)). The operators \( \hat{\Phi} \) and \( \hat{\Pi} \) are common field operators (i.e. not functional operators), and until now the (29) can be considered an alternative representation of the algebra (11).

\(^5\)The Hamiltonian \( H \) includes interaction in the common second quantization sense, but it is free in the third quantization sense. This will become more clear in the next section, when we will add a third quantization interaction.

\(^6\)This means that if we restrict to a theory described by the Hamiltonian (23) (bilinear in the functional operators \( S \)), and with physical observables defined by \( \Pi \) and \( \Phi \), we have a theory that is completely equivalent to a quantum field theory. Departures from quantum field theory will become manifest when we will add to (23) an interaction with the insertion of several functional operators \( S \).
In the Heisenberg picture the functional field operator $S[\phi]$ becomes time dependent, and satisfies the quantum mechanical equation

$$i \frac{d S[\phi, t]}{dt} = [\mathcal{H}, S[\phi, t]]$$

(31)

from which we get

$$i \frac{d \hat{\Phi}(x, t)}{dt} = [\mathcal{H}, \hat{\Phi}(x, t)] = \hat{\Pi}(x, t)$$

(32)

$$\frac{d^2 \hat{\Phi}(x, t)}{dt^2} = -i \frac{d \hat{\Pi}(x, t)}{dt} = - [\mathcal{H}, \hat{\Pi}(x, t)] = (\nabla^2 - m^2) \hat{\Phi}(x, t)$$

(33)

In particular we can find that the ordinary two point Green function of the scalar field, applying the definition (29),

$$G(x, t; y, t') = \langle 0_1 | T \{ \Phi(x, t) \Phi(y, t') \} | 0_1 \rangle$$

(34)

where $T$ is the time ordered product. $|0_1 \rangle$ is the state with minimum energy, with $\mathcal{N} | n_1 \rangle = | n_1 \rangle$, and $F_0$ is equal to (14), namely

$$|0_1 \rangle = \int D\phi F_0[\phi] S^\dagger[\phi] |0\rangle$$

(35)

Then applying twice (32), we can check the equation

$$(\partial^2 + m^2) G(x, t; y, t') = \delta^3(x - y) \delta(t - t')$$

(36)

This confirms that the state $|0_1 \rangle$, in third quantization, describes a physical system that is equivalent to the one described by a scalar quantum field theory with Green functions given by the rules of second quantization. The concept of Green functions can be generalized to third quantization, provided that we replace functions with functionals. For example, in complete analogy with the second quantization, we can define a two-point Green Functional

$$G[\phi, t; \phi', t'] = \langle 0 | T \{ S[\phi', t'] S^\dagger[\phi, t] \} | 0 \rangle$$

(37)

One can verify that it corresponds to the inverse functional operator appearing in (19), once we define a proper time ordering prescription to get rid of the pole singularities (the $i\varepsilon$ Feynman prescription).

### 3.1 The Hierarchy problem: Interacting functional fields

We have seen in the first section that the hierarchy puzzle could be solved if we add a non-linear term to equation (13). One could introduce it by hand modifying (13), as in a non-linear Schrödinger equation, and studying the phenomenological consequences. However here we prefer to stick to quantum theory. In fact, we notice that adding a non-linear interaction reminds us the embedding of first quantization into second quantization, i.e. the possibility of interaction changing the number of particles, that in the second quantization language corresponds to operators involving more than two fields. In other words the hierarchy puzzle can be seen as an hint to move forward with some sort of third quantization beyond the second one.
Unfortunately, until now, we have not many clues to guess which type of interaction to add. We have not yet any principle as powerful as that of gauge theories leading to unambiguous forms for the interactions. Nevertheless, note that (22) implies that the mass dimension of the functional field \( S[\phi] \) is not well defined since the functional delta has the same dimensions of the product of an infinite number of dirac delta. But we also know that

\[
\int \mathcal{D}\phi \{ iS^\dagger[\phi, t], S[\phi', t] \} = \int \mathcal{D}\phi \ \delta^\infty[\phi - \phi'] = 1,
\]

thus the dimension of the integral \( \int \mathcal{D}\phi \) is the inverse of a pair of \( S \). If we want to add a pair of functional fields \( S^\dagger S \) we are obliged to add an integral \( \int \mathcal{D}\phi \), if we want to keep the action with the right mass dimensions. For instance \( \int \mathcal{D}\phi S^\dagger[\phi]S[\phi] = \mathcal{N} \) is a dimensionless operator. In the following, we restrict ourselves to Hamiltonians that commutes with \( \mathcal{N} \), thus the number of fields is conserved. The phenomenological consequences of additional insertions to the Hamiltonian of different powers of \( \mathcal{N}, \mathcal{N}^2 \), etc. are rather trivial. Instead, let us consider the operator\(^7\) \( \mathcal{O}(x) = \int \mathcal{D}\phi S^\dagger[\phi]\phi^2(x)S[\phi] \). \( \mathcal{O} \) commutes with \( \mathcal{N} \), it has well defined mass dimensions, and can be used to build Hamiltonians with much more interesting phenomenology. As example, we will consider the Hamiltonian (23) with an additional interaction of the form

\[
\Delta \mathcal{H} = \int d^3x \sum_n c_n \mathcal{O}(x)^n = \int d^3x \ V(\mathcal{O}(x))
\]

where \( V \) is an arbitrary function, defined by the expansion above with coefficients \( c_n \). This interaction is local in the space-time dimensions. It is understood that some renormalization prescription for the \( c_n \) are considered; for our purpose we can also look at the (39) as an effective hamiltonian, arising from a more fundamental theory involving different functional operators in addition to \( S \).

### 3.1.1 The two point Green functional in the non perturbative vacuum of the Theory

The insertion of (39), leads to a new action \( \mathcal{A} \), given by

\[
\mathcal{A} = \int \mathcal{D}\phi \ dt \ S^\dagger[\phi, t] \left( i \frac{\partial}{\partial t} + H \right) S[\phi, t] + \int dt d^3x V \left( \int \mathcal{D}\phi' S^\dagger[\phi', t] \phi^2(x) S[\phi', t] \right)
\]

The Green functional (37) at the zeroth order approximation, as anticipated in the previous section, is given by

\[
G_0[\phi, t; \phi', t'] = i \left( i \frac{\partial}{\partial t} + H + i \varepsilon \right)^{-1} [\phi, t; \phi', t']
\]

However the exact two point Green functional is affected by the interaction \( V \) that can change the vacuum of the theory (in the following we have in mind an analogous example in second

\(^7\)Even if not explicitly stated, we assume that the field \( \phi \) can carry internal indices of some internal symmetry. This justifies why we have inserted the square \( \phi^2 \) (to build an invariant operator), instead of \( \phi \). Also \( S \) can carry an index of an internal symmetry, but to simplify the notation we remove this index.
quantization, the Nambu-Jona-Lasinio model). Naively, taking the derivative $\delta A/\delta S = 0$, we deduce the equation

$$i \frac{\partial}{\partial t} S[\phi, t] = H S[\phi, t] + V' (\langle 0|O|0 \rangle) \int d^3 x \phi^2(x) S[\phi, t]$$

(42)

where, in the mean field approximation, we have replaced the interacting term with its vacuum expectation value $\langle 0|V'(O)|0 \rangle \simeq V'(v^2)$. In this approximation, we see that we still have a linear Schrödinger equation (42), as in second quantization (13). The only effect of the interaction $V$, is to modify the bare mass of the field $\phi$. In fact, a mass $V'(v^2)$ adds directly to the Hamiltonian $H$, in (42), and thus modifies also the Green functional (41). Apart from this renormalization of the bare mass, the physics described by the approximate equation (42), is identical to an ordinary quantum field theory.

If we wish to compute the exact Green functional, we have to compute all loop contributions induced by the interaction $V$. These can be formally taken into account in a non-perturbative approach for composite operators (see chapter 8 in [4]): the exact Green functions arise from the functional minimization of an effective action $\Gamma[G]$, a functional of the Green functional $G$. $\Gamma$ is the sum of all two-particle irreducible vacuum loop diagrams. We will not repeat here the arguments leading to the equations below instead we assume a straightforward generalization of the discussion in [4]. We replace Green functions with Green functionals, and integrations over space coordinates with functional integrations over the function $\phi(x)$. As a result, the effective action for a composite functional operator $S^\dagger[\phi, t] S[\phi', t']$ is a functional $\Gamma[G]$ of the Green functional (37), that can be written (compare with eq. 8.47 of [4])

$$\Gamma[G] = -i Tr (\log(G^{-1})) - i Tr (G_0^{-1} G) + \Gamma_2[G]$$

(43)

where $\Gamma_2$ is given by all two-loop (and higher) two-particle irreducible vacuum graphs. In the following (see Fig. 1), we will consider only vacuum loop diagrams, at first order in $V$, i.e. with only one space-time point (represented in Figure 1, by the big grey point). $V$ contains several powers of the operator $O(x)$; each $O(x)$ contains a functional integration with respect a distinct function $\phi$. This explains why we have also depicted few small dark points, each one representing a distinct functional point $\phi, \phi', \phi''$, etc. For example, the top diagram on the left comes from the interaction term $O^2$. $O^2$ two functional integrations with respect the two functions $\phi, \phi'$ (depicted by the two distinct black points). In the same diagram there are two lines (i.e. two Green functionals or propagators), attached at the same black point. That is to say each Green functional is evaluated at the same functional point $(G[\phi, \phi])$.

From the (43), and reminding the definition (20) of functional derivative, we can write the equation

$$\frac{\delta \Gamma[G]}{\delta G} = i G^{-1} - i G_0^{-1} + \frac{\delta \Gamma_2[G]}{\delta G}.$$  

(44)

This yields the exact functional $G$ that we want to compute.

We can distinguish two type of contributions, when we take the derivative of the functional, $\delta \Gamma_2/\delta G$. The first is obtained when the derivative $\delta /\delta G[\phi, \phi']$ acts on the Green functional $G[\phi'', t; \phi''', t]$, computed at the same function $\phi''$, (and same time $t$). For example, those coming

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8 Also because the generalization of path integrals into an equivalent mathematical object in third quantization is involved and unclear.

9 We assume $\langle 0|S(\phi)|0 \rangle = 0$. 

9
Figure 1: Two particle irreducible Feynman diagrams. The small black points stand for distinct points in the functional space \( \phi, \phi', \ldots \). The lighter and bigger points stand for a point in the ordinary space-time. Diagrams on the left involve only Green functionals at the same functional point \( \phi \), i.e. \( G[\phi, t; \phi, t] \). Instead on the right they only involve \( G[\phi, t; \phi', t] \) with \( \phi \neq \phi' \).

From the diagrams on the left of Figure 1. After the integration over \( \phi'' \), these yield a term proportional to a functional Dirac delta \( \delta^\infty[\phi - \phi'] \). A second one, arises when the derivative acts on a Green functional \( G[\phi''', t; \phi'', t] \) with different functions \( \phi''' \) and \( \phi'' \) (but same time \( t \)): the full derivative takes the form

\[
\frac{\delta \Gamma_2[G]}{\delta G[\phi, t; \phi', t']} = A(v^2) \int d^3x \phi^2[x] \delta^\infty(\phi - \phi') +
\]

\[
+ B(v^2) \int d^3x \phi^2[x] G[\phi, t; \phi', t] \phi'^2[x] + \ldots
\]

Where the dots \( \ldots \) stand for all other possible combinations of the Green functionals \( G \) arising from the same interaction \( V \). For brevity, we do not list them all, since they are unnecessary for the discussion below. The functions \( A(v^2) \) and \( B(v^2) \) have appeared as a result of all integrations of the functionals \( G \) over \( \phi \). Their explicit dependance on the vacuum expectation value \( v = \langle 0 | \hat{\phi}(x) | 0 \rangle \), can be computed only if a specific interaction \( V \) is chosen, and after the evaluation of all functional integrals. Among these, we note for instance

\[
\int \mathcal{D}\phi G[\phi, t; \phi, t] \phi[x]^2 \equiv \left\langle 0 \left| \hat{\phi}^2(x) \right| 0 \right\rangle = v^2
\]

In this step we have assumed\(^{10}\) the boundary condition at \( t' = t \)

\[
G[\phi, t; \phi', t] = F_0(\phi) F_0^*(\phi').
\]

\(^{10}\)The Green function is obtained as the inverse of the kinetic operator of the classical action (41). The choice (47) corresponds to a particular \( i\varepsilon \) prescription for the pole singularities in (41). These prescriptions set the boundary conditions for \( G \) at \( t' = t \).
Where $F_0$ is of the form (44). With this prescription, we get a vacuum with one (and only one) higgs field (see eq. (35) and eq. (37)). As in ordinary quantum field theory, where the evaluation of the composite operator $\hat{\phi}^2(x)$ needs a suitable renormalization prescription, we understand that suitable counterterms have been added to ensure a finite result in the integration (46). Note that $v$ is the true vev of the theory, i.e. all radiative corrections are taken into account, since $G$ is the full Green Functional of the theory. The full equation (44) for the Green functional $G[\phi, t; \phi', t']$ can be written

$$G[\phi, t; \phi', t']^{-1} = i \frac{\partial}{\partial t} + H + i \varepsilon + A[v^2] \delta(t-t') \int d^3x \phi^2[x] \delta^{\infty}(\phi - \phi') +$$

$$+ B[v^2] \delta(t-t') \int d^3x \phi^2[x] G[\phi, t; \phi', t] \phi'^2[x] + ...$$

We can distinguish two parts in the right-hand side: the one proportional to $A[v^2] \delta^{\infty}(\phi - \phi')$ is similar to a mass term in $H$ (see eq. (13)) and can be reabsorbed into a redefinition of the bare mass of the field $m_0^2 \phi^2 \rightarrow m_1^2 \phi^2 = (m_0^2 + A[v^2]) \phi^2$. One can recognize that in the mean field approximation (eq. (12)) $A[v^2] = V'[v^2]$ (and $B = 0$). The last term is not proportional to $\delta^{\infty}(\phi - \phi')$; for the moment, as a first approximation, we assume $B[v^2]$ to be very small and negligible. The equations (48) and (46) are a system of coupled equations, from which we would like to determine $v$ and $G$. Let us forget for a moment (46). We look for solutions $G$ of (48), treating $v$ as a free parameter. We recognize that (48), (with $B = 0$) is the same equation that we would get in absence of the interaction $V$ (see the section before), apart from a redefinition of the bare mass

$$m_0^2 \phi^2 \rightarrow m_1^2 \phi^2 = (m_0^2 + A[v^2]) \phi^2.$$  

(49)

This implies that we can compute $G$ following the more familiar procedure of second quantization. We remember that $G$ is defined in (14). In particular we can compute the vacuum (16) of this modified Hamiltonian as follows. We first calculate the full effective potential of an ordinary quantum field theory with Hamiltonian $H$, but with modified bare masses (49). The minimization of this non-perturbative effective potential would give

$$v^2 = V^2 \left( m_1^2, g_1^2, ... \right) = \mu^2 \left( m_1^2, g_1^2, ... \right) / \lambda \left( m_1^2, g_1^2 \right) + ...$$  

(50)

$v$ comes from a function $V$ of the bare parameters $m_1^2, g_1^2, ...$ at the large scale $\Lambda$. The function $V$ can be computed in the case of a perturbative quantum field theory; exploiting the renormalization group equations (RGE) one usually finds $v$ as a simple function of the common Higgs potential parameters $\mu$ and $\lambda$. In any case if $m_1$ and the other mass parameters at the high energy scale are of the order of the large scale $\Lambda$ also $v$ from eq. (50) is of order $\Lambda$. This is the origin of the Hierarchy problem: if any of the mass parameters of the theory entering the $V$ is large, then also $v$ is large. Only with a fine tuning of those large masses $m_1$ etc. one can get a small vev $v$. Now let us see how this paradox can be solved in the context of third quantization. In this case $m_1$, the bare mass at the scale $\Lambda$, is not a free parameter but it is itself a function of the vev $v$ through equation (16) and (49). Using these equations the (50) becomes

$$v^2 = V^2 \left( (m_0^2 + A[v^2]), g_1^2, ... \right)$$  

(51)

\footnote{We remind that we look for space-time translation invariant solutions, such that $G[\phi, t; \phi', t'] \rightarrow G[\phi, t; \phi', t']$ when $\phi(x) \rightarrow \phi(x + x_0)$, $t \rightarrow t + t_0$, $t' \rightarrow t' + t_0$ and $\phi'(x) \rightarrow \phi'(x + x_0)$ then the integral $\int D\phi$ in eq. (13) yields a constant $v$ independent of $x$.}

\footnote{We stress that this redefinition occurs at the level of the bare mass, before any integration over loop momenta and before the action of the renormalization group equations.}
It is understood that the above equation is exact, i.e. it includes all quantum corrections: the function $V$ includes the exact (all loops) dependance from the parameters $m_1^2, g_1^2, \ldots$ given at the very high energy. It is clear that the equation (41) can have solutions with $v << \Lambda$: even if the function $V^2$ is rather simple (e.g. polynomial). Mathemical examples could be easily built if $A$ contains logarithms or exponentials of the vev $v$. The hierarchy arises exclusively from the interactions at the third quantization level, that in the (51) are represented by the function $A$. This interplay between second and third quantization is parallel to a similar connection between first and second quantization. An interesting example, is provided by the hydrogen atom. In this specific physical system, one finds the minimum of an Hamiltonian (in first quantization), with the fine structure constant $\alpha$, that can be calculated in second quantization. Namely, firstly we solve the Schrödinger equation (first quantization), and we find the electron wave function $\psi$. $\psi$ contains the characteristic scale $p$ of the virtual momentum exchanged between the proton and the electron. This $p$ is clearly a function of the input parameter $\alpha$. But this input parameter $\alpha$ is in turn a function of $p$, since the fine structure constant is obtained (in second quantization) from the $\beta$ function and the RGE, evaluated at the characteristic scale $p$. The interplay between first and second quantization can be summarized as follows: $p$ is obtained from $\alpha$ applying the common procedure of first quantization. In turn $\alpha$ is a function of the characteristic scale $p$, and can be computed using second quantization and the RGE equations. This connection leads to a non linear equation that is absolutely equivalent to (51): in fact second quantization (and the minimization of the full effective potential) is used to derive the characteristic scale $v$ as a function $V$ of the bare parameters $m_1^2, g_1^2$, while third quantization (through the interaction $V$) is necessary to calculate $m_1^2$ (the bare mass) as a function of $v$. This leads to (51).

Even if the phenomenological consequences of a third quantization need an appropriate study of well defined and realistic models, we can anticipate some generic possible consequences implied by this scenario. These could lead to new and exotic experimental signatures. One of these can be inferred by the additional terms proportional to $B$, that appeared in (48) (and that we have neglected until now). If we consider non trivial vacua, then the functional $G$ solving (48) could be different from that one of an ordinary quantum field theory. Let us consider the modification to the solution $G$, induced by the additional term $B$. If $G_0$ indicates the Green functional at $B = 0$, then eq. (48) gives us the first order perturbative correction

$$G_1^{-1}[\phi, t; \phi', t'] = G_0^{-1}[\phi, t; \phi', t'] + B(v^2) \delta(t - t') \int d^3x \phi^2(x) \phi'^2(x) G_0[\phi, t; \phi', t].$$

(52)

Plugging the (47) in (52), we get the Green functional $G_1$ describing the time evolution of the quantum state (the wave functional) from the initial time $t'$ to the final time $t$. To the ordinary Hamiltonian $H$, we have to add an operator of the form

$$\Delta H = B(v^2) \int d^3x \left( \hat{\phi}^2(x)|0 > < 0|\hat{\phi}^2(x) \right).$$

(53)

The insertion of the vacuum projector $|0 > < 0|$ is rather unusual. However note, that if $G_0^{-1}$ is the renormalizable hamiltonian of the Standard Model, it includes the Standard Model symmetries, like the baryon and the lepton number. Thus the vacuum $F_0$ (14), (obtained from $G_0^{-1}$, eq. (11)) is invariant under these symmetries. Thus also the new interaction (53) looks symmetric. All accidental symmetries of the renormalizable standard Model are preserved. At first glance, we can also anticipate few possible experimental signatures. Let us write down the Green function involved in the scattering of two Higgses into two Higgses $h \ h \rightarrow h \ h$. For simplicity we consider the case where $H$ in (11) contains only the kinetic term, thus the only interaction in this process,
comes from $\Delta H$ \cite{53}, and it appears only at the first order in $B(v^2)$. The four point Green function is

$$g(x_1, x_2, x_3, x_4) = <0|\phi(x_1, t_1)\phi(x_2, t_2)\phi(x_3, t_3)\phi(x_4, t_4)|0>$$

(54)

with the $t_{1,2} \rightarrow +\infty$ in the far future and $t_{3,4} \rightarrow -\infty$ in the far past. At the first non trivial order we get

$$g(x_1, x_2, x_3, x_4) =$$

\begin{align*}
&\int_{-\infty}^{+\infty} dt dt' dt'' dt''' \\
&\phi(x_1, t_1)\phi(x_2, t_2)\Delta H(t')\phi(x_3, t_3)\phi(x_4, t_4)|0> =
\end{align*}

(55)

\begin{align*}
&= B(v^2) \int_{-\infty}^{+\infty} d^3 x dt dt' dt'' dt''' \\
&\phi(x_1, t_1)\phi(x_2, t_2)\hat{\phi}^2(x, t)|0> <0|\hat{\phi}^2(x, t)\phi(x_3, t_3)\phi(x_4, t_4)|0> 
\end{align*}

(56)

(57)

where $\Delta H$ is inserted in the middle, given that $t_{1,2} >> t >> t_{3,4}$, and we have to respect the time ordering prescription. $hh \rightarrow hh$ is mediated by $\Delta H$ as if it were an ordinary $\hat{\phi}^4$ operator. On the other hand, the same effective $\hat{\phi}^4$ interaction, added to the original interaction, would also modify a process like $h \rightarrow hh hh$; but this process is not affected by $\Delta H$, as it can be easily checked, because we know that $<0|\hat{\phi}^2(x, t)\phi(x_4, t_4)|0> = 0$. In other words, in such a scenario, an anomalous self-interaction of the Higgs boson could be proven by the existence of two discrepant precision measurements\cite{13} of the same effective coupling constant but in two different physical processes.

4 Conclusions

The unexpected experimental success of the renormalizable Standard Model puts severe constraints to any attempt to solve the Hierarchy problem by adding new particles in the low energy spectrum of the theory. Their real (or virtual) production is ruled out, at least in the region around the electroweak scale. We have seen that the main difficulty is due to the theoretical assumption that the time evolution operator of quantum states with very different vacuum expectation values of $\phi$ are described by the same linear operator, the Hamiltonian $H$. While the Standard Model Hamiltonian, seems to be very accurate to describe particle physics phenomenology (in experimental tests concerning only quantum states very close to the electroweak vacuum), it seems to be unable to estimate the energy of vacua with very different expectation values of the field operator $\phi$. This paradox becomes more affordable if we accept the possibility that the time evolution of the quantum states is described by a non-linear Schrödinger equation. Motivated by this example, we have explored how such a modification can be achieved still in the context of quantum mechanics, but with an embedding of second quantization into some sort of third quantization. We already know that first quantization arises as an approximation of second quantization: namely, if we neglect the interaction between particles and we consider only two-point Green functions. In fact, two-point Green functions satisfy the Schrödinger equation of first quantization. Similarly, we can imagine that second quantization (i.e. quantum field theory) arises as an approximation of third quantization, where the role of points and fields is now played, respectively, by fields and functional fields. As for first quantization, second quantization is obtained in the limit when only the two-point Green functional is relevant. In this limit, the Hamiltonian of quantum field theory is simply the inverse of the full two-point Green functional of the third quantization theory. Thus,

\footnote{This discrepancy would inevitably prove the failure of ordinary quantum field theory in favor of an extension with a new type (third quantized) interactions.}
to derive the Hamiltonian of second quantization from the full third quantization theory, we have used the formalism of the effective action of composite operators. We have found that the interaction of third quantization $A$, can significantly change the equation (51) without adding any new particles in the low energy spectrum. This minimal impact in the low energy spectrum is, to our opinion, in better agreement with the experimental evidence. Even if we have not yet shown that the idea suggested in this paper can lead to a realistic and well defined theory, we have anticipated few possible experimental signature, from a simple and naive analysis of our mathematical example above. Namely, a new type of operators could appear in the effective Hamiltonian of second quantization. Beside the common self interaction of the operators $\phi^4(x)$, one could also have some exotic operators of the form $\phi^2(x)|0><0|\phi^2(x)$ with the insertion of the vacuum projector. This interaction would lead to some anomalous (and non-linear) interactions in scattering processes involving the Higgs particle.

References

[1] G.’t Hooft and M. Veltman, Nucl. Phys. B44 (1972)189.

[2] S. Weinberg, Phys. Rev. Lett. 19, 1264 (1967). A. Salam, in Elementary Particle Physics, ed. N. Svartholm (Almqvist and Wiksells, Stockholm,1968):p.367; S.L. Glashow, J. Iliopoulos, and L. Maiani, Phys. Rev. D2,1285 (1970).

[3] C.Itzykson and J.Zuber, Quantum Field Theory, McGraw-Hill Book Co., NewYork, 1980.

[4] V. A. Miransky, Dinamical Symmetry Breaking in Quantum Field Theories, World Scientific Publishing, Singapore, 1993.