Generic tripartite Bell nonlocality sudden death under local phase noise

Kevin Ann$^a$ and Gregg Jaeger$^b$

$^a$Department of Physics, Boston University
590 Commonwealth Avenue, Boston, MA 02215

$^b$Quantum Imaging Lab, Department of Electrical and Computer Engineering, and Division of Natural Sciences, Boston University, Boston, MA 02215

Abstract

We definitively show, using an explicit and broadly applicable model, that local phase noise that is capable of eliminating state coherence only in the infinite-time limit is capable of eliminating nonlocality in finite time in three two-level systems prepared in the Bell-nonlocal tripartite states of the generic entanglement class.

Key words: Bell-type inequality, Bell nonlocality, Entanglement sudden death, multi-local dephasing

PACS: 03.65.Ta, 03.65.Ud, 03.67.-a

1 Introduction

It has recently been demonstrated that when external noise acts on bipartite states of compound quantum systems, a sudden total loss of entanglement can occur in finite time in a context where there is persistence of some quantum coherence for all finite times, an effect known as Entanglement Sudden Death (ESD) [1,2,3,4,5,6,7,8]. An explicit local hidden-variables model for entangled mixed states of three two-level systems has also recently been found [9], illustrating the distinction between entanglement and nonlocality first made by Werner [10]. The demonstration of ESD under noise in the case of multipartite states has been difficult because defining practical multipartite entanglement measures for the mixed states inevitably produced by such noise is highly...
nontrivial. Despite this difficulty, an phenomenon analogous to ESD can more rigorously be studied in multipartite systems, namely, the effect of Bell Non-locality Sudden Death (BNSD). The loss of nonlocal properties due to effects that are entirely local is the most significant element of this, particularly in the case of multiple subsystems where nonlocal behavior is not “encoded” in local states, as it can be in the case of pure bipartite two-level states. For example, the state entropy for the subsystems of a pair of two-level systems determines the global properties of entanglement and nonlocality in the joint bipartite pure states \( \text{cf.} \ [11] \), whereas for multipartite states, such a simple relationship no longer holds.

This effect was recently indicated by the demonstration \[8\] that a tripartite system prepared in the W state initially violating the Mermin-Ardehali-Belinskii-Klyshko (MABK) inequality \[12,13,14\] fails to violate it at a later finite time in a local phase noise environment. The W class is a set of zero measure compared to the class of generic entangled pure states of three two-level systems \[15\]. Here, a far stronger and more general result is obtained, namely, a definitive demonstration that the death of Bell nonlocality occurs suddenly in finite time in any system prepared in any one of the members of the generic class of tripartite-entangled pure states and subject to local phase noise alone, a result that requires the examination not only of the MABK inequality but of the full representative subset of the entire 256-element set of WWZB Bell-type inequalities \[16,17\] and the Svetlichny inequality for three two-level systems \[18\]. This result is demonstrated using an explicit and broadly applicable model which includes explicit time-dependence.

These results fall within the context of other recent results regarding decoherence of multipartite nonlocal quantum states. For example, Sen(De), Sen, Wieśniak, Kaszlikowski, and Żukowski \[19\] performed an analysis focused on nonlocality rather than entanglement; they considered the persistence of Bell-type nonlocality in multipartite GHZ and W states under multilocal phase noise and found that the nonlocality properties of W-type states were more robust against multilocal phase noise than those of the GHZ class. Our results reinforce this latter observation by showing that, in the case of \( n = 3 \) with an explicit and physically motivated noise model, not only are the generalized GHZ state not robust, but they exhibit nonlocality sudden death.

In particular, we study the relatively small but illuminating case of triples of two-level systems in detail and demonstrate, for the first time in a situation where \( n \geq 3 \), that sudden death of multi-partite nonlocality occurs in a system for a range of state preparations due to such local phase noise alone. Moreover, we show sudden death of two distinct types of nonlocal correlation: tripartite correlations associated with the inequality of Svetlichny \[18\] and nonlocal correlations associated with the Werner and Wolf \[16\] and Żukowski and Brukner \[17\] inequalities, which subsume the MABK form.
We thereby extend the study of Bell nonlocality sudden death in several ways. First, because previous sudden death results for tripartite states considered only correlations addressed by the MABK inequality, which is the representative of only one of the five distinct types of inequality of the full set of WWZB Bell-type inequalities [8], those preliminary results concerned the sudden death of only one species of Bell-nonlocal correlations, whereas we here show the sudden failure to violate the entire 256-element set of WWZB Bell-type inequalities under local phase noise. That is, the sudden death of all species of Bell-nonlocal correlation in the presence of local dephasing noise alone is proven. Second, we demonstrate Bell nonlocality sudden death as captured by the Svetlichny inequality for initially genuinely tripartite-entangled pure states of the generic class (GHZ-class) [15]. Thus, we show that Bell nonlocality sudden death occurs in this class of states in two distinct senses: there is the sudden loss of genuinely tripartite Bell nonlocality and of subsystem bipartite Bell nonlocality. Finally, we explicitly confirm that nonlocality death in the even-odd bipartite state-split of the system of three two-level systems occurs in precisely the same manner and timescale as that of genuinely tripartite Bell nonlocality death.

The simple, pervasive character of the local phase noise considered here is noteworthy. Local phase noise appears in a broad range of physical situations and is of great concern, for example, in attempts to distribute quantum states, even in a very simple environment. That such a simple form of noise is unavoidable and can lead to the loss of Bell nonlocality for tripartite states is of great significance for entanglement distribution and quantum computing [20], where entangled states of multiple two-level system appear in algorithms offering exponential speedups over classical computing and such states are used as encoding states [21].

2 BELL-TYPE NONLOCALITY IN VARIOUS CONTEXTS

Svetlichny’s Bell-type inequality [18] distinguishes genuinely three-subsystem nonlocal correlations A-B-C of a system ABC composed of two-level subsystems A, B, and C, from those that can be described by a hybrid local-nonlocal model for a 1-2 subsystem A-BC (or B-AC or C-AB) bipartite split and furthermore, from “convex sums” of such hybrid local-nonlocal models. In contrast, the Mermin-Ardehali-Belinskii-Klyshko (MABK) Bell-type inequality for three-component systems [12,13,14], which has often been used in studies of nonlocality and was recently used to explore a precondition for Bell nonlocality sudden death [8], is incapable of addressing effects involving the element of genuinely tripartite Bell-nonlocal correlation or loss thereof. Let us write Svetlichny’s inequality as
\[ |S| \equiv |E(ABC) + E(AB'C') + E(A'BC) + E(A'B'C') - E(A'B'C') - E(A'B'C') - E(A'B'C') - E(AB'C')| \leq 4, \]

where \( E(\cdots) \) denotes the expectation value of the measured outcomes in state-components \( A, B, \) and \( C \), for example, a component of spin, primes denoting alternative directions of measurement. When \( |S| > 4 \), one has genuine tripartite Bell-nonlocal correlations, rather than simply bipartite correlations between subsystems within a tripartite system \([22]\); as expected, the maximum quantum value of \( \max(|S|) = 4\sqrt{2} \), compared to the algebraic maximum of 8, is attained only when the system is prepared in the maximally entangled \((cf. [23])\) GHZ state, \( |\text{GHZ} \rangle = \frac{1}{\sqrt{2}}(|000 \rangle + |111 \rangle) \), the representative of one of the two entanglement classes of tripartite pure states, the generic class \([15]\). 

We refer to the following four distinct notions of Bell nonlocality in this paper.

i. **Generic Bell nonlocality** - The most general class of tripartite Bell nonlocality, for which Bell nonlocality of any type is present within the tripartite system. This class contains states in which Bell-locality and nonlocality may both be present in subsystems or genuinely tripartite Bell nonlocality may be present in the tripartite state. All generic Bell nonlocality no longer exists when our state is describable using a local classical model, occurring when all of the WWZB inequalities are satisfied.

ii. **Genuinely tripartite Bell nonlocality** - Exists when the Svetlichny inequality for a tripartite state is violated, that is, when a hybrid local-nonlocal model cannot be used to describe the state.

iii. **Subsystem bipartite Bell nonlocality** - Refers to the nonlocality existing in a bipartite two-level system within a larger tripartite system, for example, the subset of bipartite two-level systems \( AB, BC, \) or \( AC \) within a tripartite system \( ABC \). Implicit in this definition is that each subsystem is nonlocally separated from the other subsystems. This sort of nonlocality occurs when a single tripartite WWZB inequality is violated.

iv. **Nonlocality of the even-odd bipartite state split** - Bell nonlocality for a bipartite two-level partition of the tripartite state. Two of those four dimensions are within the Hilbert space of one two-level system and the other two dimensions are those of the remaining two-level system space. We can have, for example, two of four dimensions within the Hilbert space of subsystem \( A \) and the remaining two dimensions within the joint Hilbert space of \( B \) and \( C \). Regardless of how the two-level system pair split is made, one can analyze the corresponding Bell nonlocality properties using the CHSH inequality. The development of this type of Bell nonlocality, its significance, and its relation to the previous notions of Bell nonlocality is discussed in Sec. 4.

Despite the differences between the Svetlichny and the MABK inequalities,
they can be related mathematically. Consider the following two pertinent instances of the MABK inequality.

\[ |\mathcal{M}| = |E(ABC') + E(AB'C) + E(A'BC) - E(A'B'C')| \leq 2 \tag{2} \]
\[ |\mathcal{M}'| = |E(ABC) - E(AB'C) - E(A'BC) - E(A'B'C)| \leq 2 \tag{3} \]

where \(\mathcal{M}\) and \(\mathcal{M}'\) are Bell-type operators, with differing arguments all of which appear in the single instance of the Svetlichny inequality above. Either \(|\mathcal{M}| > 2\) or \(|\mathcal{M}'| > 2\) indicates the presence of Bell-nonlocal correlation via the MABK inequality, although this does not indicate genuine tripartite Bell-nonlocal correlation; tripartite Bell nonlocality is not guaranteed even when \(\max(|\mathcal{M}|) = \max(|\mathcal{M}'|) = 4\), because these values can be achieved by convex combinations of bipartite correlations alone. The left-hand-side of the Svetlichny inequality for genuine tripartite correlations is rather

\[ |S| = |\mathcal{M} + \mathcal{M}'| \leq |\mathcal{M}| + |\mathcal{M}'| . \tag{4} \]

For the state \(|W\rangle = 1/\sqrt{3}(|100\rangle + |010\rangle + |001\rangle|\), which is the representative of the other class of tripartite entangled states than that represented by \(|\text{GHZ}\rangle\), the maximum value attainable for the left-hand-side of the Svetlichny inequality is \(\max(|S|_W) = 4.354 > 4\), which occurs when \(\max(|\mathcal{M}|_W) = \max(|\mathcal{M}'|_W) = 2.177\), which is inferior to the maximum quantum mechanical violation attained for the GHZ-state, even though in this case the nonlocal correlations take the form of convex combinations of bipartite Bell-nonlocal correlations. Thus, the greatest possible extent of destruction of tripartite Bell nonlocality can be greater for states in the GHZ class.

The necessary and sufficient condition for the behavior of a system of three two-level subsystems to be describable by a fully Bell-local hidden-variables model, however, is provided jointly by the WWZB set of inequalities: all elements of the entire 256-element set of WWZB Bell-type inequalities \([16,17]\) must be satisfied for Bell locality and the violation of even a single member of the set of WWZB inequalities is sufficient for Bell nonlocality. Therefore, in order to demonstrate the death of all Bell nonlocality in such a system due to some physical influence, it is necessary for all members of this set of inequalities to become satisfied after at least one of them is not at some previous time, in addition to the demonstration of the same for similar obeysance and violation of the Svetlichny inequality. In Section IV below, this is shown to occur for states of the generic pure state entanglement class \(|\Psi_3\rangle\), which is represented by the GHZ state. The WWZB inequalities are discussed in the next section, in its subsection B, after the pertinent noise model, states and notation is introduced in its subsection A.
Let us take the system of three two-level systems under study to be prepared in the generic pure entanglement-class state \([24]\),

\[
|\Psi_3\rangle = \bar{a}_0|000\rangle + \bar{a}_4|100\rangle + \bar{a}_5|101\rangle + \bar{a}_6|110\rangle + \bar{a}_7|111\rangle
\]  

in \(\mathcal{H}_{ABC} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C\), where \(\bar{a}_i \in \mathcal{C}\) and \(\sum_i |\bar{a}_i|^2 = 1\), that is,

\[
\rho(0) = \begin{pmatrix}
|\bar{a}_0|^2 & 0 & 0 & \bar{a}_0\bar{a}_4^* & \bar{a}_0\bar{a}_5^* & \bar{a}_0\bar{a}_6^* & \bar{a}_0\bar{a}_7^* \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\bar{a}_4\bar{a}_0^* & 0 & 0 & |\bar{a}_4|^2 & \bar{a}_4\bar{a}_5^* & \bar{a}_4\bar{a}_6^* & \bar{a}_4\bar{a}_7^* \\
\bar{a}_5\bar{a}_0^* & 0 & 0 & \bar{a}_5\bar{a}_4^* & |\bar{a}_5|^2 & \bar{a}_5\bar{a}_6^* & \bar{a}_5\bar{a}_7^* \\
\bar{a}_6\bar{a}_0^* & 0 & 0 & \bar{a}_6\bar{a}_4^* & \bar{a}_6\bar{a}_5^* & |\bar{a}_6|^2 & \bar{a}_6\bar{a}_7^* \\
\bar{a}_7\bar{a}_0^* & 0 & 0 & \bar{a}_7\bar{a}_4^* & \bar{a}_7\bar{a}_5^* & \bar{a}_7\bar{a}_6^* & |\bar{a}_7|^2
\end{pmatrix}.
\]  

The tripartite generic state is analyzed because of its relations to the other pure tripartite classes: GHZ, W, biseparable (B), and separable (S). The generic state may be locally transformed with some finite probability into the GHZ class of states, which in turn may be converted stochastically by means positive-operators-valued measures (POVMs) into any of the other classes described by the following ordered relation \([15]\): \(S \subset B \subset W \subset GHZ\). The analysis of the generic tripartite state completes and extends the analysis of \([8]\), where the phenomenon of Bell nonlocality sudden death was shown to exist in the W class. A feature that distinguishes the GHZ state from the W state is that the former is genuinely entangled at the tripartite level as opposed to the latter, which may be described by a convex sum of bipartite entangled states and is of measure zero.

The following results for the generic class of tripartite state apply immediately to the GHZ state itself, due to the measurement operators we have used that are composed of the tensored products of the Pauli matrices: \(\sigma_X\) and \(\sigma_Y\). Furthermore, due to the fact that multi-local operations only cannot change the nonlocality properties of state, it is noteworthy that it suffices to use a specific GHZ state as representative of the GHZ class. We have not assigned specific values to the coefficients in order to get the most general expressions.
for demonstration and clarity. However, we do assign them in specific instances to demonstrate, for example, the coefficients $\bar{a}_0 = \bar{a}_7 = 1/\sqrt{2}$ correspond to maximum violation of the Svetlichny inequality and the longest timescale in which genuinely tripartite nonlocality is lost.

Let the components of ABC be noninteracting and subject only to local external phase noise. The time-evolved state of an open quantum system under such external noise, written in the operator-sum representation, is

$$\rho(t) = \mathcal{E}(\rho(0)) = \sum_{\mu} D_{\mu}(t) \rho(0) D_{\mu}^\dagger(t), \quad (7)$$

where the $\{D_{\mu}(t)\}$, with the index $\mu$ running over all elements of the chosen operator-sum decomposition, satisfy the completeness condition that guarantees that the evolution be trace-preserving [25]. For a collection of local noise sub-environments, noise operates locally on individual subsystems, that is, the $D_{\mu}(t)$ are of the form $G_k(t)F_j(t)E_i(t)$. Hence,

$$\rho(t) = \mathcal{E}(\rho(0)) = \sum_{i=1}^2 \sum_{j=1}^2 \sum_{k=1}^2 G_k(t) F_j(t) E_i(t) \rho(0) E_i^\dagger(t) F_j^\dagger(t) G_k^\dagger(t) \quad (8)$$

In particular, let this local noise to be the basis-dependent pure phase noise for which

$$E_1(t) = \text{diag}(1, \gamma_A(t)) \otimes I_2 \otimes I_2, \quad E_2(t) = \text{diag}(0, \omega_A(t)) \otimes I_2 \otimes I_2, \quad (9)$$

$$F_1(t) = I_2 \otimes \text{diag}(1, \gamma_B(t)) \otimes I_2, \quad F_2(t) = I_2 \otimes \text{diag}(0, \omega_B(t)) \otimes I_2, \quad (10)$$

$$G_1(t) = I_2 \otimes I_2 \otimes \text{diag}(1, \gamma_C(t)), \quad G_2(t) = I_2 \otimes I_2 \otimes \text{diag}(0, \omega_C(t)), \quad (11)$$

$$\gamma_A(t) = \gamma_B(t) = \gamma_C(t) = \gamma(t) = e^{-\Gamma t}, \omega_A(t) = \omega_B(t) = \omega_C(t) = \omega(t) = \sqrt{1 - \gamma^2(t)} = \sqrt{1 - e^{-2\Gamma t}}, \Gamma$$ being the parameter describing the rate of local asymptotic dephasing taken to be that induced by all three sub-environments in their local subsystems: The $\{E_i(t)\}$, $\{F_j(t)\}$, and $\{G_k(t)\}$ dephase the local state of each two-level subsystem individually at the same rate, $\Gamma$. For clarity, the time-dependence of $\gamma(t)$’s are implicitly written from here on, particularly when displaying full density matrices. This local phase noise appears in a broad range of physical situations and is of concern, for example, in attempts to distribute entanglement. That such a simple form of noise is unavoidable is of great significance for entanglement distribution and quantum computing.

In the multi-local noise environment described above, for the composite system initially prepared at $t = 0$ in $\rho(0) = |\Psi_3\rangle\langle\Psi_3|$, the solution of Eq. (8) at later time $t$ is
Nonetheless, as we now demonstrate, the tripartite Bell-nonlocality
γ

\(\gamma\)

exponential decay with one of the rates \(\Gamma\), \(2\Gamma\), or \(3\Gamma\). The full triple two-level system
state, therefore, fully decoheres only in the infinite-time limit, because the off-diagonal dephasing factors \(\gamma_A\), \(\gamma_B\), and \(\gamma_C\) only asymptotically approach zero. Nonetheless, as we now demonstrate, the tripartite Bell-\textit{nonlocality} of these states is entirely lost in a specific and finite time-scale.

The measurement operators \(M_K\) and \(M'_K\) of Eqs. (2)-(4) in the Bell-type inequalities for \(n = 3\) correspond to measurements on each of the subsystems \(K\) (A, B, or C), with the primed and unprimed terms denoting two different measurement directions for the corresponding party. Defining \(M_A \equiv \sigma_y\) and \(M'_A \equiv \sigma_x\), the measurement operator acting upon each successive subsystem is defined with respect to the first by a rotation by \(\theta_K\):

\[
\begin{pmatrix}
M_K \\
M'_K
\end{pmatrix} = R(\theta_K)
\begin{pmatrix}
M_A \\
M'_A
\end{pmatrix},
\text{ where } R(\theta_K) =
\begin{pmatrix}
\cos \theta_K & -\sin \theta_K \\
\sin \theta_K & \cos \theta_K
\end{pmatrix}.
\tag{13}
\]

There are two such rotation angles \(\theta_B\) and \(\theta_C\) \((K = B, C)\); the corresponding measurement operators for two-level systems A, B, and C are

\begin{align}
M_A &= \sigma_y \otimes I_2 \otimes I_2, \quad \tag{14} \\
M'_A &= \sigma_x \otimes I_2 \otimes I_2, \quad \tag{15} \\
M_B &= I_2 \otimes [\cos (\theta_B) \sigma_y - \sin (\theta_B) \sigma_x] \otimes I_2, \quad \tag{16} \\
M'_B &= I_2 \otimes [\sin (\theta_B) \sigma_y + \cos (\theta_B) \sigma_x] \otimes I_2, \quad \tag{17} \\
M_C &= I_2 \otimes I_2 \otimes [\cos (\theta_C) \sigma_y - \sin (\theta_C) \sigma_x], \quad \tag{18} \\
M'_C &= I_2 \otimes I_2 \otimes [\sin (\theta_C) \sigma_y + \cos (\theta_C) \sigma_x]. \quad \tag{19}
\end{align}
3.1 Svetlichny Inequality

The Svetlichny operator appearing in Eq. (1) is, in terms of the measurement operators introduced above,

\[ S = M_A M_B M_C + M_A M_B' M_C + M_A M_B M_C' + M_A' M_B M_C 
- M_A' M_B' M_C - M_A' M_B M_C' - M_A M_B' M_C' - M_A M_B M_C'. \] (20)

Recall that if \( |\langle S\rangle_{\rho(t)}| = \text{tr}[S\rho(t)] > 4\), the state \( \rho(t) \) is genuinely tripartite Bell nonlocal. In order to demonstrate tripartite Bell nonlocality sudden death in \( \rho \) due to the effect of external noise, we must show that both \( |\langle S\rangle_{\rho(0)}| > 4 \) and \( |\langle S\rangle_{\rho(t)}| \leq 4 \) for some finite \( t > 0 \) under it. We now show that this indeed occurs for a system composed of three two-level subsystems prepared in generic state \( |\Psi_3\rangle \) under local phase noise described by the model of the previous section. Considering the complex coefficients \( \bar{a}_0 \) and \( \bar{a}_7 \) in polar forms \( \bar{a}_0 = |a_0|e^{i\phi(\bar{a}_0)} \) and \( \bar{a}_7 = |a_7|e^{i\phi(\bar{a}_7)} \), let us write the relative phase angle between the amplitudes as \( \alpha = \phi(\bar{a}_0) - \phi(\bar{a}_7) \) and \( \theta_{BCa} = \theta_B + \theta_C + \alpha \). Therefore,

\[
|\langle S\rangle_{\rho(t)}| = \text{tr}[S\rho(t)] \\
= \text{tr}[(M_A M_B M_C + M_A M_B M_C') + (M_A M_B M_C + M_A M_B M_C')] \\
- M_A M_B M_C' - M_A M_B' M_C - M_A M_B M_C' - M_A M_B M_C'.
\] (21)

The Svetlichny inequality is violated whenever \( |\langle S\rangle_{\rho(t)}| > 4 \), which is seen to occur for any state \( |\Psi_3\rangle \) for which \( |\bar{a}_0||\bar{a}_7| > 1/(2\sqrt{2}) \), the maximal violation for each state occurring at \( \theta_{BCa} = -\pi/4, 3\pi/4 \) and \( t = 0 \), at which time \( \gamma_A = \gamma_B = \gamma_C = 1 \). The maximum quantum mechanically allowed value, \( |\langle S\rangle_{\rho(0)}| = 4\sqrt{2} \), is attained by elements of the generic class \( |\Psi_3\rangle \) for which \( |\bar{a}_0| = |\bar{a}_7| = 1/\sqrt{2} \), for example, the standard GHZ state with \( \theta_{BC} = -\pi/4 \) at \( t = 0 \). Furthermore, recalling that \( \gamma_A = \gamma_B = \gamma_C = e^{-\Gamma t} \) and assuming a natural local decoherence rate of \( \Gamma = 1 \), one sees that the maximum value of the left-hand-side of the Svetlichny inequality for these initially tripartite Bell-nonlocal states evolves according to \( |\langle S\rangle_{\rho(t)}| = 8\sqrt{2}|\bar{a}_0||\bar{a}_7|e^{-3\Gamma t} \), and so approaches the critical value \( |\langle S\rangle_{\rho(t^*_3)}| = 4 \) in the finite timescale

\[
t^*_3 = \frac{\ln(2\sqrt{2}|\bar{a}_0||\bar{a}_7|)}{3\Gamma}.
\] (22)

Thus, for example, when the system is initially prepared in the standard GHZ
state, we find $t_3^* = \ln(\sqrt{2})/3\Gamma$. For any initial preparation of the pure generic class, genuine tripartite Bell nonlocality is lost from $t_3^*$ onward.

Before proceeding, one should note that there exist “two” Svetlichny inequalities. The second Svetlichny inequality, denoted by $S'$, is given by

$$|S'| \equiv |E(ABC) - E(ABC') - E(AB'C) - E(A'BC) + E(A'B'C') - E(A'B'C) - E(A'BC') - E(AB'C')| \leq 4.$$  

(23)

In particular, note that a minus sign appears in front of $E(ABC')$. (Also note that a typographical error was made in front of that term in Eq. 6 of the published version of Svetlichny’s original paper of 1987 [18], which was pointed out in footnote 9 of a later paper [26].) In the current analysis, $S' = -S$ upon the substitution $\theta_{BC} \rightarrow -\theta_{BC}$; because only the maximum magnitude of the Svetlichny expression is relevant in this analysis, one gets similar results for $S'$, so that here it is only necessary to refer to $S$.

3.2 WWZB Inequality

Werner and Wolf [16] and Zukowski and Brukner [17] have derived a set of $2^n$ Bell-type inequalities the conjunction of the truth values of which is a necessary and sufficient condition for a system composed of $n$ two-level subsystems to be describable by a fully local hidden-variables model. For $n = 3$, there are 256 of these inequalities, which fall into five classes with elements forming subsets related by symmetries under (1) changing the labels of the measured observables at each site, (2) changing the names of the measurement outcomes, or (3) permuting subsystems. The behavior of a single element of each class is identical to that of all members of that class, as explicitly shown in the appendix of [16]. As a result, one need consider only one inequality from each of the five distinct classes, for example, those with left-hand-sides with Bell-type operators of the forms

(P1) $B_{P1} = 2M_AM_BM_C$,

(P2) $B_{P2} = \frac{1}{2}(-M_AM_BM_C + M_AM_BM'_CM + M_AM'_BM_C + M_AM'_BM'_C$ $+ M'_AM_BM_C + M'_AM_BM'_C + M'_AM'_BM_C + M'_AM'_BM'_C)$,

(P3) $B_{P3} = [M_A(M_B + M'_B) + M'_A(M_B - M'_B)]M_C$,

(P4) $B_{P4} = M_AM_B(M_C + M'_C) - M'_AM'_B(M_C - M'_C)$,

(P5) $B_{P5} = M_AM_BM'_C + M_AM'_BM_C + M'_AM_BM_C - M'_AM'_BM'_C$,  

(24)

which we consider here. For the entire class of local hidden-variables models, the corresponding Bell-type inequalities are $|\langle B_{PI} \rangle_\rho| \leq 2$ (for $I = 1, 2, 3, 4,$...).
5). In order to show definitively that Bell nonlocality sudden death occurs in a system at \( t^* \) for a class of state preparations, one must demonstrate both that (i) these system states are initially incapable of description by a local hidden-variables model at \( t = 0 \), that is, that at least one of the PI > 2 (for \( I = 1, 2, 3, 4, 5 \)), and (ii) they are describable by a hidden-variables model at some later time \( t^* < \infty \), that is, \( |\langle B_{PI}\rangle_\rho| \leq 2 \) for all \( I = 1, 2, 3, 4, 5 \) at \( t^* \).

We first show (i), in particular, that at time \( t = 0 \) the inequality of form P5 is violated by the same range of generic entanglement class pure states as considered above, and therefore that the system is not describable by an entirely local hidden-variables model—as opposed to local-nonlocal hybrid model, as pertained in Subsection IIA. The expectation value of the \( B_{P5} \) operator for the state under the influence of multi-local noise on the composite system of three two-level systems ABC initially prepared in the GHZ-class pure state is

\[
\langle B_{P5}\rangle_\rho(t) = \text{tr} [B_{P5}\rho(t)] = \text{tr} \left[ (M_A M_B M_C + M_A M'_B M_C + M'_A M_B M_C - M'_A M'_B M'_C) \rho(t) \right] \\
= 4\gamma_A \gamma_B \gamma_C |(\bar{a}_0 \bar{a}_7^* + \bar{a}_7 \bar{a}_0^*) \cos(\theta_B + \theta_C) - i(\bar{a}_0 \bar{a}_7^* - \bar{a}_7 \bar{a}_0^*) \sin(\theta_B + \theta_C)| \\
= 8\gamma_A \gamma_B \gamma_C |\bar{a}_0| |\bar{a}_7| \sin(\theta_{BCa}) . \tag{25}
\]

Taking \( \gamma_A = \gamma_B = \gamma_C = e^{-\Gamma t} \) as before, the left-hand-side of this form of inequality evolves as \( |\langle B_{P5}\rangle_\rho(t)| = 8|\bar{a}_0||\bar{a}_7|e^{-3\Gamma t} \) approaching the critical value \( |\langle B_{P5}\rangle_\rho(t^*)| = 2 \) from above on a timescale \( t^* = \ln(4|\bar{a}_0||\bar{a}_7|)/3\Gamma \). \tag{26}

For example, the maximum value \( |\langle B_{P5}\rangle_\rho(t)| = 4 \) for initial Bell nonlocality occurs at \( t = 0 \) (when \( \gamma_A = \gamma_B = \gamma_C = 1 \), for \( |\bar{a}_0| = |\bar{a}_7| = 1/\sqrt{2} \), that is, in the standard GHZ state (for which \( \alpha = 0 \) and when the trigonometric term takes its maximum value, \( \sin(\theta_{BCa}) = 1 \), that is, when \( \theta_{BC} = \pi/2 \); the critical time is then \( t^* = \ln(2)/3\Gamma < \infty \).

We now show (ii), that is, that all the remaining inequalities, given this set of initial state preparations, are later satisfied in the timescale \( t^* \), so that the condition for a local hidden-variables model to suffice to explain the resulting correlations is satisfied in it. This occurs when the absolute value of the left-hand-side of the following expressions are less than or equal to the value two. Let us evaluate the operator expectation values \( \langle B_{PI}\rangle_\rho(t) \), for each remaining inequality for \( I = 1, 2, 3, 4 \) in turn.
Finally, for the remaining form, P4, one finds

\[ \langle B_{P4}\rangle_{\rho(t)} = \text{tr} \left[ B_{P4}\rho(t) \right] \]

\[ = \text{tr} \left[ (2M_A M_B M_C)\rho(t) \right] \]

\[ = 2\gamma_A \gamma_B \gamma_C \left( |\langle B_0\rangle| + 2|\langle B_0\rangle| \sin(\theta_B + \theta_C) + 2|\langle B_0\rangle| \cos(\theta_B + \theta_C) \right) \]

\[ = 4\gamma_A \gamma_B \gamma_C |\bar{a}_0||\bar{a}_7| \sin(\theta_{BCa}). \]  \hspace{1cm} (27)

One immediately sees that \( \max(\langle B_{P4}\rangle_{\rho(t)} \rangle = 4|\bar{a}_0||\bar{a}_7|e^{-3\Gamma t} \leq 2 \) for all times \( t > 0 \) and over the full range of values of \( \theta_{BCa} \), because for the states of interest \( 1/2 \geq |\bar{a}_0||\bar{a}_7| > 1/4 \), where the upper bound 1/2 represents a maximally entangled state and the lower bound 1/4 represents a maximally mixed state.

For the inequality of form P2, one finds

\[ \langle B_{P2}\rangle_{\rho(t)} = \text{tr} \left[ B_{P2}\rho(t) \right] \]

\[ = \text{tr} \left[ \left( \frac{1}{2} - M_A M_B M_C + M_A M_B M_C + M_A M_B M_C + M_A M_B M_C \right) \rho(t) \right] \]

\[ = - (1 + i) \gamma_A \gamma_B \gamma_C \left( |\langle 2 + i|\bar{a}_7\bar{a}_0^* - |\langle 1 + 2i |\bar{a}_0\bar{a}_7^* \cos(\theta_B + \theta_C) \right) \]

\[ + (1 + 2i) |\bar{a}_7\bar{a}_0^* + |\langle 2 - i |\bar{a}_0\bar{a}_7^* \sin(\theta_B + \theta_C) \right) \]

\[ = 2\gamma_A \gamma_B \gamma_C |\bar{a}_0||\bar{a}_7| \sin(\theta_{BCa} + \cos(\theta_{BCa})). \]  \hspace{1cm} (28)

One sees that for \( t \geq t^* \), \( \langle B_{P2}\rangle_{\rho(t)} < 2 \) for all choices of \( \theta_{BCa} \): in that range the maximum with respect to \( \theta_{BCa} \) of \( |\langle B_{P2}\rangle_{\rho(t^*)} | = 8|\bar{a}_0||\bar{a}_7|e^{-3\Gamma t^*} < 2 \), because the trigonometric factor is strictly bounded by 4.

For the inequality of form P3, one finds

\[ \langle B_{P3}\rangle_{\rho(t)} = \text{tr} \left[ B_{P3}\rho(t) \right] \]

\[ = \text{tr} \left[ \left( M_A M_B M_C + M_A M_B M_C + M_A M_B M_C - M_A M_B M_C \right) \rho(t) \right] \]

\[ = 2\gamma_A \gamma_B \gamma_C (1 + i) \left( |\langle i|\bar{a}_7\bar{a}_0^* - |\langle 1 - 2i |\bar{a}_0\bar{a}_7^* \cos(\theta_B + \theta_C) \right) \]

\[ + (1 + 2i) |\bar{a}_7\bar{a}_0^* - |\langle 2 - i |\bar{a}_0\bar{a}_7^* \sin(\theta_B + \theta_C) \right) \]

\[ = 4\gamma_A \gamma_B \gamma_C |\bar{a}_0||\bar{a}_7| [[\cos(\theta_{BCa}) - \sin(\theta_{BCa})]]. \]  \hspace{1cm} (29)

The maximum with respect to \( \theta_{BCa} \) is \( |\langle B_{P3}\rangle_{\rho(0)} | = 4\sqrt{2}|\bar{a}_0||\bar{a}_7| \), which occurs for \( \theta_{BCa} = -\pi/4, 3\pi/4 \). At \( t^* \), one has \( |\langle B_{P3}\rangle_{\rho(t^*)} | = 4\sqrt{2}|\bar{a}_0||\bar{a}_7|e^{-3\Gamma t^*} = 2\sqrt{2}|\bar{a}_0||\bar{a}_7| \leq 2 \), and similarly for all later times for these optimal angles.

Finally, for the remaining form, P4, one finds
\[ \langle B_{P4} \rangle_{\rho(t)} = \text{tr} [ B_{P4} \rho(t) ] \]
\[ = \text{tr} \left[ \left( M_A M_B M_C + M_A M_B M'_C + M'_A M_B M'_C - M_A M'_B M_C \right) \rho(t) \right] \]
\[ = 2 \left( \bar{a}_0 \bar{a}_7 + \bar{a}_7 \bar{a}_0^* \right) \sin(\theta_B + \theta_C) + i \left( \bar{a}_7 \bar{a}_0^* - \bar{a}_0 \bar{a}_7^* \right) \cos(\theta_B + \theta_C) \gamma_A \gamma_B \gamma_C \]
\[ = 4 \gamma_A \gamma_B \gamma_C |\bar{a}_0||\bar{a}_7| \sin(\theta_{BC\alpha}) \].

(30)

One sees immediately, as in case P1, that
\[ | \langle B_{P4} \rangle_{\rho(t)} | = 4 |\bar{a}_0||\bar{a}_7|e^{-3\Gamma t} \leq 2, \]
for all times \( t \) and for all values of \( \theta_{BC\alpha} \).

Thus, in the timescale \( t^* = \ln(4 |\bar{a}_0||\bar{a}_7|)/3\Gamma \), all the WWZB inequalities are satisfied for all measurement angles and all initially Bell-nonlocal generic pure-state entanglement class preparations \( |\Psi_3 \rangle \). Therefore, the composite quantum system has entirely and irreversibly lost its Bell nonlocality in finite time under the influence only of local phase noise.

4 BELL NONLOCALITY SUDDEN DEATH IN THE BIPARTITIONS

The destruction of genuine tripartite Bell nonlocality in finite time for states of three two-level systems was demonstrated in Section IIIA above using the Svetlichny inequality. In a three-component system, the loss of genuine tripartite Bell nonlocality entails the loss in the same system considered as composed of two subsystems of bipartite Bell nonlocality, one subsystem (e.g., A) being one of the two-level systems alone and the other being the subsystem constituted by the remaining pair of two-level systems (e.g., BC). We now verify that this is indeed the case, by considering the remaining two systems as a single unit in a bipartition of the system. In particular, we show that bipartite Bell nonlocality sudden death occurs, by using the CHSH inequality, in exactly the same time scale found when using the Svetlichny inequality.

Without loss of generality, because in our model local phase noise affects each subsystem in exactly the same way, we take the solo two-level system to be subsystem A and the remaining subsystems, B and C, to jointly form subsystem BC with states lying in a four-dimensional Hilbert space \( \mathcal{H}_{BC} = \mathcal{H}_B \otimes \mathcal{H}_C \). The maximally Bell-nonlocal state in this bipartite splitting of the system corresponds to the GHZ state, as can be seen by noting that with \( |0\rangle \equiv |00\rangle = (1, 0, 0, 0)^T \in \mathcal{H}_{BC} \) and \( |1\rangle \equiv |11\rangle = (0, 0, 0, 1)^T \in \mathcal{H}_{BC} \), respectively, the GHZ state is formally similar to the Bell state \( |\Phi^+\rangle \), in that \( |\text{GHZ}\rangle = 1/\sqrt{2}(|00\rangle + |11\rangle) \). This decomposition is of the Schmidt form, which naturally exposes nonlocal correlations, and shows how one can construct the CHSH spin-measurement operators in the two-dimensional subspace of \( \mathcal{H}_{BC} \), in terms of which the measurement outcomes on the quantum states are written when evaluating the inequality. In particular, writing \( \tau = |\bar{1}\rangle\langle \bar{0}| = \)
the left-hand-side of the CHSH inequality is maximized when $\tau = \tau_1 + \tau_2$, $\tau_2 = i\tau - i\tau_1$, $\tau_3 = \tau_1 - \tau_2$, and $\tau_4 = \tau_1 + \tau_2$, where $\tau_1 = \sigma_y \otimes \sigma_x$ and $\tau_2 = \sigma_x \otimes \sigma_y$. One sees that $\tau_1$ and $\tau_2$ act analogously on $|0\rangle$ and $|\bar{1}\rangle$ as $\sigma_x$ and $\sigma_y$ act on the natural basis states of two-dimensional Hilbert space: $\tau_1|0\rangle = \tau_1|1\rangle = |\bar{1}\rangle$, $\tau_2|0\rangle = \tau_2|1\rangle = |\bar{0}\rangle$, $\tau_2|0\rangle = i|1\rangle = i|\bar{1}\rangle$, and $\tau_2|1\rangle = -i|0\rangle = -i|\bar{0}\rangle$, as required.

The measurement generators appearing in the Bell operator of the CHSH inequality, therefore, for the first subsystem are the usual ones and, for the larger, second subsystem are

$$
\begin{pmatrix}
\bar{M}_{BC} \\
\bar{M}'_{BC}
\end{pmatrix}
= R(\theta_{BC}) \begin{pmatrix} M_A \\ M'_A \end{pmatrix},
$$

with

$$
R(\theta_{BC}) = \begin{pmatrix} \cos \theta_{BC} - \sin \theta_{BC} \\ \sin \theta_{BC} \cos \theta_{BC} \end{pmatrix},
$$

that is,

$$
\bar{M}_A = \sigma_y \otimes \mathbf{I}_4, \tag{32}
$$

$$
\bar{M}'_A = \sigma_x \otimes \mathbf{I}_4, \tag{33}
$$

and

$$
\begin{aligned}
\bar{M}_{BC} &= \mathbf{I}_2 \otimes [\cos(\theta_{BC}) \tau_2 - \sin(\theta_{BC}) \tau_1] , \\
\bar{M}'_{BC} &= \mathbf{I}_2 \otimes [\sin(\theta_{BC}) \tau_2 + \cos(\theta_{BC}) \tau_1] .
\end{aligned} \tag{34,35}
$$

In terms of these measurement operators, the appropriate Bell-CHSH operator is then

$$
\mathcal{B}_{CHSH} = \bar{M}_A \bar{M}_{BC} + \bar{M}_A \bar{M}'_{BC} + \bar{M}'_A \bar{M}_{BC} - \bar{M}'_A \bar{M}'_{BC} . \tag{36}
$$

Writing $\tilde{\theta}_{BCa} = \theta_{BC} + \alpha$, the Bell-operator expectation value for state $\rho(t)$ is

$$
\langle \mathcal{B}_{CHSH} \rangle_{\rho(t)} = \text{tr} \left[ \mathcal{B}_{CHSH} \rho(t) \right]
= \text{tr} \left[ (\bar{M}_A \bar{M}_{BC} + \bar{M}_A \bar{M}'_{BC} + \bar{M}'_A \bar{M}_{BC} - \bar{M}'_A \bar{M}'_{BC}) \rho(t) \right]
= -2(2 + 2i)\gamma_A \gamma_B \gamma_C \left[ (a_7a^*_0 - ia_0a^*_7) \cos(\theta_{BC}) + (-i\bar{a}_7\bar{a}^*_0 + \bar{a}_0\bar{a}^*_7) \cos(\theta_{BC}) \right]
= 4\gamma_A \gamma_B \gamma_C \left[ |\bar{a}_0||\bar{a}_7| \cos(\tilde{\theta}_{BCa}) - \sin(\tilde{\theta}_{BCa}) \right] . \tag{37}
$$

Recall that $| \langle \mathcal{B}_{CHSH} \rangle_{\rho(t)} | \leq 2$ holds for all local hidden-variables models and that $2\sqrt{2}$ is the Tsirelson bound [27,28], the maximum violation attainable by quantum mechanical states. Whenever $| \langle \mathcal{B}_{CHSH} \rangle_{\rho(t)} | > 2$ the system in $\rho$ exhibits Bell nonlocality. Before local phase decoherence begins at $t = 0$, the left-hand-side of the CHSH inequality is maximized when $| \cos(\tilde{\theta}_{BCa}) - \sin(\tilde{\theta}_{BCa}) | = \sqrt{2}$, that is, when $\tilde{\theta}_{BCa} = -\pi/4, 3\pi/4$, and $|\bar{a}_0| = |\bar{a}_7| = 1/\sqrt{2}$, so
that $|\langle B_{\text{CHSH}} \rangle_{\rho(t)}| = 2\sqrt{2}$. After the local dephasing noise has begun acting, one finds that $\langle B_{\text{CHSH}} \rangle_{\rho(t)} = 2$ in the timescale

$$t^*_2 = \frac{\ln(2\sqrt{2}|a_0||\bar{a}_7|)}{3\Gamma}. \quad (38)$$

The extent of inequality violation is thus seen to evolve in time in exactly the same manner as the violation of Svetlichny inequality. In particular, one sees that Bell nonlocality sudden death occurs in precisely the same timescale in these alternative perspectives on the same process, that is, $t^*_2 = t^*_3$, as it should.

5 Conclusion

We have shown that local phase noise that is capable of eliminating all state coherence only in the infinite-time limit is nonetheless capable of eliminating Bell nonlocality in finite time, for three-component systems prepared in the generic entanglement class of tripartite states for all preparations in which they are initially Bell nonlocal. It is noteworthy that the noise acting on the initially entangled states is merely local, whereas the central characteristic of entanglement and Bell-type inequality violation is nonlocality.

This Bell nonlocality sudden death was examined in both of its aspects. One is the certain sudden death of all Bell-nonlocal correlations irreducible to convex sums of internal bipartite correlations in such states, exhibited by the sudden failure to violate Svetlichny’s inequality. The other is the certain sudden death of Bell-nonlocal correlations reducible to such convex combinations of bipartite correlations, in that the three subsystems suddenly become jointly describable by a fully local hidden-variables model, as exhibited by their suddenly obeying the entire set of Werner–Wolf–Żukowski–Brukner inequalities. The results were also shown to accord with the behavior of correlations under bi-partitioning of the system. The loss of nonlocal properties due to effects that are entirely local is the most significant element of this, particularly in the case of multiple subsystems where nonlocal behavior is not “encoded” in local states, as it can sometimes be in the case of bipartite two-level states, for example via state entropy in the pure case.

Acknowledgement: We thank Michael P. Seevinck for pointing out the need for a more detailed analysis than previously carried out (in Ref. 8) in order definitively to demonstrate BNSD.
References

[1] L. Diósì, in F. Benatti and R. Floreanini (eds.), *Irreversible quantum dynamics*, (Springer; Berlin, 2003), pp. 157-163.

[2] P. J. Dodd and J. J. Halliwell, Phys. Rev. A 69, 052105 (2004).

[3] T. Yu and J.H. Eberly, Phys. Rev. Lett. 93, 140404 (2004).

[4] T. Yu and J.H. Eberly, Phys. Rev. Lett. 97, 140403 (2006).

[5] T. Yu and J.H. Eberly, Opt. Commun. 264, 393 (2006).

[6] T. Yu and J.H. Eberly, Science 316, 555 (2007).

[7] K. Ann and G. Jaeger, Phys. Rev. A 76, 044101 (2007).

[8] G. Jaeger and K. Ann, Phys. Lett. A 372, 2212 (2008).

[9] G. Tóth and A. Acín, Phys. Rev. A 74, 030306 (R) (2006).

[10] R. F. Werner, Phys. Rev. A 40, 4277 (1989).

[11] C. H. Bennett, S. Popescu, D. Rohrlich, J. A. Smolin, and A. V. Thapliyal, Phys. Rev. A 63, 012307 (2000).

[12] N. D. Mermin, Phys. Rev. Lett. 65, 1838 (1990).

[13] M. Ardehali, Phys. Rev. A 46, 5375 (1992).

[14] A. V. Belinskii and D. N. Klyshko, Sov. Phys. Usp. 36, 653 (1993).

[15] W. Dür, G. Vidal, and J. I. Cirac, Phys. Rev. A 62, 062314 (2000).

[16] R. F. Werner and M. M. Wolf, Phys. Rev. A 64, 032112 (2001).

[17] M. Žukowski and Č. Brukner, Phys. Rev. Lett. 88, 210401 (2002).

[18] G. Svetlichny, Phys. Rev. D 35, 3066 (1987).

[19] A. Sen(De), U. Sen, M. Wieśniak, D. Kaszlikowski, M. Żukowski, Phys. Rev. A 68, 062306 (2003).

[20] W. G. Unruh, Phys. Rev. A 51, 992 (1995).

[21] A. Steane, Phys. Rev. Lett. 77, 793 (1996).

[22] J. L. Cereceda, Phys. Rev. A 66, 024102 (2002).

[23] V. Coffman, J. Kundu, W. K. Wootters, Phys. Rev. A 61, 052306 (2000).

[24] A. Acín, A. Andrianov, L. Costa, E. Jané, J. I. Latorre, and R. Tarrach, Phys. Rev. Lett. 85, 1560 (2000).

[25] K. Kraus, *States, effects and operations*, (Springer-Verlag; Berlin, 1983)
[26] M. Seevinck and G. Svetlichny, Phys. Rev. Lett. 89, 060401 (2002).

[27] B. S. Tsirel’son, Lett. Math. Phys. 4, 93 (1980)

[28] B. S. Tsirel’son, J. of Sov. Math. 36, 557 (1987)