Constraints On The BPS Spectrum Of $\mathcal{N} = 2$, $D=4$ Theories With A-D-E Flavor Symmetry

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Abstract

BPS states of $\mathcal{N} = 2, D = 4$ Super Yang-Mills theories with $ADE$ flavor symmetry arise as junctions joining a D3-brane to a set of 7-branes defining the enhanced flavor algebra. We show that the familiar BPS spectrum of SU(2) theories with $N_f \leq 4$ is simply given by the set of junctions whose self-intersection is bounded below as required by supersymmetry. This constraint, together with the relations between junction and weight lattices, is used to establish the appearance of arbitrarily large flavor representations for the case of $D_{n \geq 5}$ and $E_n$ symmetries. Such representations are required by consistency with decoupling down to smaller flavor symmetries.
1 Introduction

It has become quite clear that in IIB superstring theory $(p, q)$ strings \[1\] and string junctions \[2, 3\] are actually on the same footing. More precisely, states defined by a fixed set of charges on a moduli space of backgrounds are sometimes realized as strings, but more generally, as junctions. This viewpoint has been obtained by examination of backgrounds with 7-branes \[4, 5, 6\], and with 7-branes together with 3-branes \[7, 8\]. We focus here on the case of a single 3-brane on the background of some 7-branes defining an ADE type configuration.

The theory on the 3-brane is a four dimensional $N = 2$ Yang-Mills theory and when the 3-brane sits away from the 7-branes, the gauge symmetry of the 7-branes appears as a flavor symmetry for the 3-brane. BPS states of the 3-brane field theory are now recognized as junctions joining the 3-brane and (some of) the 7-branes.

In a recent interesting work Mikhailov, Nekrasov and Sethi \[9\] discussed a constraint that selected the junctions giving the BPS spectrum for the SU(2) theory with no matter. This constraint is based on a subtle argument comparing pairs of junctions. Motivated by this work, we show here that a very simple constraint also gives the well-known BPS spectrum, not only for the pure SU(2) theory discussed in \[9\] but also for the cases when we have $N_f \leq 4$ flavors. This constraint requires any BPS junction $J$ with asymptotic charges $(p, q)$ to have self-intersection

$$(J, J) - GCD(p, q) \geq -2,$$ (1.1)

and it arises as follows. By construction \[9, 10\], $(J, J) = \#(J \cdot J)$ where $J$ is the two-dimensional cycle associated to $J$ in the F/M theory picture and $\#$ denotes intersection number. In order for $J$ to be BPS, $J$ must be holomorphic, and then $\#(J \cdot J) = 2g - 2 + b$, where $g$ is the genus of the curve and $b$ is the number of boundaries. We show that $b$ can be identified with $GCD(p, q)$, the greatest common divisor of the charges, and since $g \geq 0$, (1.1) follows. We then explore this constraint on the BPS spectrum for the general ADE case.

For the cases of $D_{n \geq 5}$ and $E_6, E_7$ and $E_8$ flavor groups our discussion is based on the recently-found relation between the junction lattice and the corresponding Lie-algebra weight vector lattice \[10\]. The self-intersection of a junction has two contributions, a negative-definite one given by minus the length squared of the corresponding weight vector, and a positive-definite one from a quadratic form on the asymptotic $(p, q)$ charges of the junction. These charges are seen on the D3-brane as the electric and magnetic charges of the BPS state. In contrast to the case of $D_{n \leq 4}$ where only vectors, spinors and singlets appear, here the constraint permits arbitrarily large representations for sufficiently large $(p, q)$ charges. Encouraged by the precise agreement between our predictions and the known spectra, one can speculate that all states allowed by (1.1) are present in the spectra of the $D_{n \geq 5}$, $E_6, E_7$ and $E_8$ theories as well. In support of this, we show that consistency with smaller algebras after a brane decoupling requires some of the large representations to belong to the spectrum. However other representations decouple completely and so their presence in the BPS spectrum cannot be confirmed by these consistency arguments. Finally we examine the remnant of the $SL(2, \mathbb{Z})$ duality group that acts on the BPS spectra of the various theories.
2 Selection rule based on Self-Intersection

\((p, q)\) strings in Type IIB string theory compactified on a manifold \(B\) are holomorphic curves of F/M theory compactified on a four real dimensional elliptically fibered manifold \(X\) with base \(B\). These curves are formed by taking a geodesic on \(B\) and a cycle of the elliptic fiber above the geodesic. \([p, q]\) 7-branes on the base \(B\) correspond to singular fibers of \(X\). A D3-brane lifts to a regular elliptic fiber \(F_0\) above its position on \(B\).

In ref.\([11]\), \(N=2\) \(SU(2)\) Seiberg-Witten theory was interpreted as the worldvolume theory of a D3-brane in the presence of mutually non-local 7-branes. Charged states in the D3-brane theory are junctions with legs on 7-branes and ending on the D3-brane with non-vanishing asymptotic charge. In the F/M theory picture BPS states of charge \((p, q)\) correspond to holomorphic curves with a \((p, q)\) cycle of \(F_0\) as the boundary.

The manifold \(X\) is hyper-Kähler of vanishing first Chern class and therefore has a two-sphere worth of complex structures \([12]\). The elliptic fiber \(F_0\) corresponding to the D3-brane is holomorphic in one of the complex structures. The BPS states are curves holomorphic in a complex structure orthogonal to that of the elliptic fiber. Thus the space of allowed complex structures for the curves corresponding to BPS states is a circle.

The self-intersection number of a smooth holomorphic curve \(J\) of genus \(g\) with \(b\) boundary components in a complex surface \(X\) is equal to the degree of the normal bundle of \(J\) in \(X\) \([13, 12]\), therefore

\[
\#(J \cdot J) = \text{deg} N_{J/X} = \int_J c_1(N_{J/X}) = -\chi(J) = 2g - 2 + b. \tag{2.1}
\]

since the first Chern class of \(X\) is zero. In the case of a single D3-brane the boundary of \(J\) is a \((p, q)\) cycle of the elliptic fiber \(F_0\). If \(p\) and \(q\) are not relatively prime, the greatest common divisor \(\gcd(p, q) \geq 1\) gives the number \(b\) of boundary components (at the D3-brane), this is necessary for \(J\) to be smooth. Consider now the junction \(J\) associated to \(J\), which by construction \([9, 10]\) satisfies \((J, J) = \#(J \cdot J)\). Since the genus \(g\) is nonnegative, it follows from (2.1) that

\[
(J, J) - \gcd(p, q) \geq -2. \tag{2.2}
\]

This constraint will be the primary tool in our analysis.

The selection rule of \([8]\) was based on the fact that submanifolds holomorphic in the same complex structure have positive intersection number. By intersecting the junction shown in Fig. 1(a) with an \((r, 0)\) string starting at the 7-brane, it was shown that the junction may only be BPS if \(-r^2 + mr \geq 0\). We recover this as follows. Imagining that the strings end on 3-branes (to make a well-defined junction) we have at least two boundary components, and therefore \((J, J) \geq 0\). On the other hand, from the rules of \([11]\) we find \((J, J) = -r^2 + rm\), thus reproducing the claimed result.
The fact that holomorphic curves intersect positively was also used to select the BPS spectrum of $\mathcal{N} = 2$ $SU(2)$ pure SW \cite{9}. Cycles corresponding to BPS states of different charges ending on the same D3-brane are holomorphic in different complex structures (Fig. 1(b)). This is the case because, having different charges, the associated junctions must depart the 3-brane in different directions, and thus hit the curve of marginal stability at different points. This requires identical prongs departing the 7-branes to do so at different angles, a signal of different complex structures. The cycles can be made holomorphic in the same complex structure but in this case they have boundaries on different elliptic fibers of $X$ (Fig. 1(c)). These two curves can be considered BPS states of two different D3-brane worldvolume theories. The spectrum does not change as long as a D3-brane is outside the curve of marginal stability, therefore if there is a BPS state of charge $(p, q)$ in one theory it must also exist in the other. It was shown that under the assumption that a state with magnetic charge one exists, the only states which have positive intersection number with this state are the ones with magnetic charge $\pm 1$ or $0$. It is known from semi-classical analysis that in the weak coupling regime the only BPS states are the dyons with magnetic charge $\pm 1$ and the W-boson. Thus this argument gives exactly the known spectrum if the initial assumption about the existence of a BPS state of magnetic charge one is correct.

A direct argument based on self-intersection number does not require comparing states of different theories. Consider the state of charge $(Q_B + Q_C, Q_C - Q_B)$ in the $\mathcal{N} = 2$ $SU(2)$ SW theory and represented by a junction $J$, with $Q_B$ legs on the B brane and $Q_C$ legs on the C brane. In the notation of \cite{10}

$$J = Q_B \mathbf{b} + Q_C \mathbf{c}. \quad (2.3)$$

Using the rules given in \cite{11} we can calculate the self-intersection number.

$$\#(J \cdot J) = (J, J) = -(Q_B - Q_C)^2 \geq -1, \quad (2.4)$$
implying that the magnetic charge \((Q_C - Q_B)\) of a BPS state is either 0 or \(\pm 1\), thus recovering the familiar result.

## 3 Recovering the familiar BPS spectra

This section is devoted to testing the selection rule proposed above by applying it to brane configurations of familiar field theories with known spectrum. The examples will be the well known Seiberg-Witten theories \([14, 15]\) with \(N_f = 0, 1, 2, 3, 4\) flavors. The pure \(\mathcal{N} = 2\) SYM theory is realized on a D3-brane in the vicinity of two mutually nonlocal 7-branes with charges \([1, 1]\) and \([-1, 1]\) (B- and C-brane) which stand for the two strong-coupling singularities on the “u-plane”. The theories with quarks are obtained by adding up to four \([1, 0]\) 7-branes (A-branes). These conventions agree with \([5]\) and differ by an overall SL(2,Z) transformation by \((\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix})\) from the one used in \([15]\).

A state in the D3-brane field theory is represented by a junction which ends on the 3-brane and on some or all of the 7-branes and is characterized by the invariant charges \((Q_C, Q_B, Q_1^A \ldots Q_{N_f}^A)\) of each brane. The self-intersection number and the central charges of such a state are determined as \([10]\):

\[
\begin{align*}
(J, J) &= -(Q_C - Q_B)^2 + (Q_C - Q_B)(Q_1^A + \ldots + Q_{N_f}^A) - ((Q_1^A)^2 + \ldots + (Q_{N_f}^A)^2) \\
(p, q) &= (Q_C + Q_B + Q_1^A + \ldots + Q_{N_f}^A, Q_C - Q_B).
\end{align*}
\]

(3.1) (3.2)

It is straightforward to find all the solutions to (2.2) in terms of the invariant charges. The complete list of states for the theories with \(N_f = 1 \ldots 4\) is provided in the following tables.

In the last columns we list the representations of the flavor symmetry algebra, \(so(2N_f)\).

| \((Q_C, Q_B)\) \(N_f = 0\) | \((p, q)\) |
|---|---|
| \((1, 1)\) | \((2, 0)\) |
| \((n+1, n)\) | \((2n+1, 1)\) |

| \((Q_C, Q_B, Q_A)\) \(N_f = 1\) | \((p, q)\) | \(so(2)\) |
|---|---|---|
| \((1, 1, 0)\) | \((2, 0)\) | \(0\) |
| \((0, 0, 1)\) | \((1, 0)\) | \(1\) |
| \((1, 1, -1)\) | \((1, 0)\) | \(-1\) |
| \((n+1, n, 0)\) | \((2n+1, 1)\) | \(1/2\) |
| \((n, n-1, 1)\) | \((2n, 1)\) | \(-1/2\) |

| \((Q_C, Q_B, Q_1^A, Q_{N_f}^A)\) \(N_f = 2\) | \((p, q)\) | \(so(4)\) |
|---|---|---|
| \((1, 1, 0, 0)\) | \((2, 0)\) | \((1, 1)\) |
| \((0, 0, 1, 0)\), \((0, 0, 0, 1)\) | \((1, 0)\) | \((2, 2)\) |
| \((1, 1, -1, 0)\), \((1, 1, 0, -1)\) | \((1, 0)\) | \((2, 2)\) |
| \((n+1, n, 0, 0)\), \((n, n-1, 1, 1)\) | \((2n+1, 1)\) | \((2, 1)\) |
| \((n, n-1, 1, 0), (n, n-1, 0, 1)\) | \((2n, 1)\) | \((1, 2)\) |
For the $N_f < 4$ tables $n$ is an arbitrary integer, and there is an identical set of states (not listed) with all invariant charges reversed and all representations conjugated. For $N_f = 4$ in the first row, $n$ and $m$ are coprime while in the rest of the table $p$ and $q$ must be coprime; “...” stands for obvious permutations of the A-brane invariant charges. The $N_f = 4$ results for $n, m = 0$ were anticipated by [16].

Instead of using the invariant charges, we may also solve directly for the allowed representations using the results of [16]. Consider the case of $so(8)$ and note that the four conjugacy classes are identified by $(p, q) \bmod 2$. After translating the invariant charges to weight vectors we obtain $(J, J) = -\vec{\lambda} \cdot \vec{\lambda}$. Then (3.1) becomes:

$$\vec{\lambda} \cdot \vec{\lambda} \leq 2 - \text{GCD}(p, q). \quad (3.3)$$

This implies that the singlet representation survives with $\text{GCD}(p, q) = 2$ and the 8’s with $\text{GCD}(p, q) = 1$. This is in accord with the fact that there are two null junctions (having zero intersection with any other junction) of charges $(2, 0)$ and $(0, 2)$ in the $so(8)$ lattice which, when added to another junction, do not change its weight vector. Therefore if a representation is allowed, it is present with every $(p, q)$ compatible with its conjugacy class.

Notice that the listings give exactly the weak-coupling spectrum of the Seiberg-Witten theories with $N_f \leq 4$. We emphasize that the above table is simply the full set of solutions to (2.2), no extra assumption about the field theories was made. Thus we find that the

| $(Q_C, Q_B, Q_A^1, Q_A^2, Q_A^3)$ | $N_f = 3$ | $(p, q)$ | $so(6)$ |
|-----------------------------------|-----------|---------|---------|
| $(1, 1, 0, 0, 0)$ | $(2, 0)$ | 1 |
| $(0, 0, 1, 0, 0)$, $(0, 0, 0, 1, 0)$, $(0, 0, 0, 0, 1)$ | $(1, 0)$ | 6 |
| $(1, 1, -1, 0, 0)$, $(1, 1, 0, -1, 0)$, $(1, 1, 0, 0, -1)$ | $(1, 0)$ | 6 |
| $(n+1, n, 0, 0, 0)$, $(n, n-1, 1, 1, 0)$, $(n, n-1, 0, 1, 0)$, $(n, n-1, 0, 1, 0)$, $(n-1, 0, 1, 0)$, $(n, n-1, 0, 0, 1)$ | $(2n+1, 1)$ | 4 |
| $(n-2, 1, 1, 1)$ | $(2n+1, 2)$ | 1 |

| $(Q_C, Q_B, Q_A^1, Q_A^2, Q_A^3, Q_A^4)$ | $N_f = 4$ | $(p, q)$ | $so(8)$ |
|------------------------------------------|-----------|---------|---------|
| $(n-m, n-3m, m, m, m)$ | $(2n, 2m)$ | 1 |
| $(n-m, n-3m, m, m, m, m+1)$,  ... | $(2n+1, 2m)$ | 8_v |
| $(n-m+1, n-3m+1, m, m, m, m-1)$,  ... | $(2n+1, 2m)$ | 8_s |
| $(n-m, n-3m-1, m, m, m, m+1)$,  ... | $(2n, 2m+1)$ | 8_s |
| $(n-m+1, n-3m+1, m, m, m, m)$, $(n-m, n-3m+1, m, m, m)$ | $(2n+1, 2m+1)$ | 8_c |
supersymmetric string states which are allowed by our simple selection rule are in one-to-one correspondence with the states of the spectrum of both the \( N_f = 0 \ldots 3 \) \([13, 17]\) and the \( N_f = 4 \) \([15, 18]\) models. This result tempts us to leave the area of models with known spectrum and propose \((2.2)\) as a tool to investigate those field theories which we have little information about. This is what we shall do in the next section.

4 Larger flavor symmetries and decoupling

Configurations of parallel seven-branes can produce any Lie algebra of \( ADE\) type. We consider here the worldvolume theory on a three-brane in the presence of \( D_{n \geq 5} \) and \( E_6, E_7, E_8\) backgrounds, and constrain the BPS spectra of these theories using the self-intersection criterion. In terms of its associated Lie algebra weight vector \( \vec{\lambda} \) and charges \((p, q)\), the self-intersection of a junction \( \mathbf{J} \) can be written \([10]\)

\[
(J, J) = -\vec{\lambda} \cdot \vec{\lambda} + f(p, q) \geq -2 + \gcd(p, q).
\] (4.1)

where \( f(p, q) \) is a quadratic form of definite sign given for the various algebras in Table 1.

| Algebra | \( f(p, q) \) | \( \mu^2(p, q) \) |
|---------|-------------|----------------|
| \( A_n \) | \(-\frac{1}{n} p^2\) | — |
| \( D_n \) | \( \frac{1}{2} q^2(n-4) \) | — |
| \( E_6 \) | \( \frac{1}{2} p^2 - pq + q^2 \) | \( p^2 - pq + q^2 \) |
| \( E_7 \) | \( \frac{1}{2} p^2 - 2pq + \frac{5}{3} q^2 \) | \( p^2 + q^2 \) |
| \( E_8 \) | \( p^2 - 5pq + 7q^2 \) | \( p^2 - pq + q^2 \) |

Table 1: The quadratic forms \( f(p, q) \) and rescaled mass per unit length squared \( \mu^2(p, q) \) for the \( D_n, n \geq 5 \) and \( E_6, E_7, E_8\) algebras.

For the \( A_n \) singularities obtained by collapsing \( n + 1 \) A-branes the only consistent junctions fall into fundamentals, antifundamentals and singlets of \( A_n \). We will not consider this case any further. For \( D_{n \geq 5} \) and \( E_6, E_7, E_8\), \( f(p, q) \) is positive-definite instead of negative-definite or vanishing. Arbitrarily large representations, having arbitrarily large weight vectors, can then be associated to junctions satisfying \((4.1)\) if sufficiently large \((p, q)\) values are chosen. As we will show, consistency with brane decoupling transitions actually requires arbitrarily large representations. Brane decoupling has been considered in other contexts in \([16]\). Also indicated in the table is \( \mu^2 \equiv |p - q \tau|^2 \), the rescaled mass per unit length squared of a junction \([21]\), evaluated for constant \( \tau \) values \([22]\). Even though arbitrarily large representations appear, only a finite number of representations yield states with mass-squared less than or equal to any fixed value.
Thus, as the brane is moved away, the invariant charges, the asymptotic charges (invariant charge associated to the brane; all others become infinitely massive and decouple.

As a 7-brane is moved to infinity, the only junctions that survive are those having zero invariant charge associated to the brane; all others become infinitely massive and decouple. Thus, as the brane is moved away, the invariant charges, the asymptotic charges \((p, q)\), and the self-intersection number of the surviving junctions do not change. Junctions satisfying the self-intersection constraint before the removal of the brane continue to do so afterwards.

The decoupling of a single brane induces the removal of a simple root \(\vec{\alpha}_i\) with dual weight \(\vec{\omega}^i\). The rank decreases by one, and one can define the \(u(1)\) generator \(H^\ast \propto \vec{\omega}^i \cdot \vec{H}\), where \(\vec{H}\) are the Cartan generators of the parent algebra. \(H^\ast\) commutes with the full subalgebra, and its eigenvalue on weight vectors is denoted by \(Q^\ast\). For example, for \(so(10) \rightarrow so(8)\), the brane \(a_1\) is decoupled, removing the simple root \(\vec{\alpha}_1\). Fixing the normalization, we act on a weight vector \(\lambda\) to find

\[
Q^\ast = H^\ast(\vec{\lambda}) = 2\vec{\omega}^1 \cdot \vec{\lambda} = 2(A^{1i}\vec{\alpha}_i) \cdot (a_j\vec{\omega}^j) = 2A^{1i}a_i \\
= 2a_1 + 2a_2 + 2a_3 + a_4 + a_5 = 2Q^1_A + Q^B - Q^C \\
= 2Q^1_A - q ,
\]

where \(A^{ij}\) is the inverse Cartan matrix of \(so(10)\) and use was made of the relation between Dynkin labels and invariant charges (III, eqn. (6.27)). Thus an \(so(10)\) junction survives to \(so(8)\) if

\[
Q^1_A = \frac{1}{2}(Q^\ast + q) = 0 \quad \rightarrow \quad Q^\ast = -q .
\]

Since \(H^\ast\) commutes with \(so(8)\), all states in a given \(so(8)\) representation have the same value of \(Q^\ast\). Depending on the \(q\) value of the original \(so(10)\) representation, the \(so(8)\) representation will either decouple or survive as a whole. The analysis of brane removal for other symmetries follows along the above lines. The results are summarized in Table 2, where we show the brane that decouples, the simple root that is removed, the coefficients of the Dynkin labels in the expression for \(Q^\ast\), and the invariant charge on the brane to be removed. Note that this charge is written entirely in terms of the charges \((p, q)\) and \(Q^\ast\).
We now consider in further detail the case of $so(10) \rightarrow so(8)$. We will show that to get the complete spectrum for $so(8)$ with all possible $(p, q)$ charges, we need arbitrarily large representations in $so(10)$. Indeed, consider the decompositions

$$10 \rightarrow (8_v)_0 + 1_2 + 1_{-2},$$

$$16 \rightarrow (8_c)_1 + (8_s)_{-1},$$

$$16 \rightarrow (8_s)_1 + (8_c)_{-1},$$

where the subscript is $Q^*$. Since by (4.3) only junctions with charge $q = -Q^*$ survive in $so(8)$, these $so(10)$ representations only produce an $8_v$ with $q = 0$ and $8_s, 8_c$ with $q = \pm 1$. $8$’s with larger $q^2$ are embedded in other $so(10)$ representations.

We now show that an $8$ or $1$ of $so(8)$ with fixed $(p, q)$ charges arises from a unique representation of $so(10)$. Let $R$ denote an $so(10)$ representation that contains an $8_v$. This will be the case if $R$ contains the weight $\vec{\lambda}_k = (k, 1, 0, 0, 0)$ for some integer $k$. It follows from (1.2) that $\vec{\lambda}_k$ has $Q^* = 2k + 2$. Moreover, $\vec{\lambda}_k \cdot \vec{\lambda}_k = k^2 + 2k + 2$, and since the $(p, q)$ charges are coprime (4.1) becomes

$$- \vec{\lambda} \cdot \vec{\lambda} + \frac{1}{2} q^2 = (J, J) \geq -1 \quad \rightarrow \quad q^2 \geq (2k + 2)^2 .$$

For $\vec{\lambda}_k$, as well as the rest of the states giving the $8_v$, to survive decoupling we must have $q = -Q^* = -(2k + 2)$. This fixes $k$ in terms of $q$, and as a consequence $\vec{\lambda}_k$ is also fixed. Note that the $8_v$ junctions saturate the self-intersection bound. Therefore, $\vec{\lambda}_k$ must be one of the longest weights in $R$; any longer weight would violate (1.3). There is a unique representation which contains a given weight and none longer, and hence $R$ is unique. Since the longest weights in a representation occur with multiplicity one, the $8_v$ occurs in $R$ only once. Analogous arguments apply for $8_s, 8_c$ and $1$; they are also embedded uniquely in $so(10)$ representations. We indicate the $so(10)$ representation that contains each $so(8)$ representation for given values of $(p, q)$ in Table 3.

Thus we have shown that when an additional brane is brought in from infinity to an $so(8)$ configuration, the $8$ and $1$ representations of various $q$ charges transform in arbitrarily large
Table 4: \( so(8) \) representations of given \((p, q)\) charges embedded in successively larger groups.

| \((p, q)\) | \( so(8) \) | \( so(10) \) | \( E_6 \) | \( E_7 \) | \( E_8 \) |
|---|---|---|---|---|---|
| (1, 0) | \( 8_v \) | 10 | 27 | 56 | 248 |
| (0, 1) | \( 8_s \) | 16 | 78 | 912 | 147250 |
| (1, 1) | \( 8_e \) | 16 | 27 | 133 | 3875 |
| (2, 0) | 1 | 1 | 27 | 133 | 3875 |
| (1, 2) | \( 8_v \) | 45 | 351 | 27664 | 6899079264 |

Table 5: Junctions with various \((p, q)\) charges and representations realizing the smallest values of the quadratic form \( f(p, q) \) for \( E_6 \).

| \( f(p, q) \) | Possible \( \pm (p, q) \) | Representations |
|---|---|---|
| 1/3 | (1, 0), (1, 1), (2, 1) | \( 27, 27 \) |
| 1 | (0, 1), (3, 2), (3, 1) | \( 1^\dagger, 78 \) |
| 4/3 | (2, 0), (2, 2), (4, 2) | \( 27, 27 \) |
| 7/3 | (1, 2), (1, −1), (4, 3), (4, 1), (5, 3), (5, 2) | \( 27^\dagger, 27^\dagger, 351, 351 \) |
| 3 | (3, 0), (3, 3), (6, 3) | \( 1^\dagger, 78^\dagger, 650 \) |

representations of \( so(10) \); thus consistency requires all these states to be in the \( so(10) \) spectrum. Note however that there are junctions in \( so(10) \) representations which decouple completely for any values of \( q \); these are the ones which do not include \( 8 \) or \( 1 \) representations in their decomposition. For example, the \( 126 \) of \( so(10) \) has highest weight \( \bar{\lambda}_0 = (0, 0, 0, 0, 2) \) and decomposes as

\[
126 \rightarrow (56_v)_0 + (35_c)_2 + (35_s)_{-2}.
\]  

(4.6)

However, \( \bar{\lambda}_0 \cdot \bar{\lambda}_0 = 5 \), and so by (1.3) \( q^2 \geq 16 \); there is no acceptable value of \( q \) for which \( q = -Q^* \). Such representations are permitted by self-intersection for appropriate values of \( q \), but consistency with \( so(8) \) makes no statement about their presence in the \( so(10) \) spectrum.

In Table 4 we show how \( so(8) \) representations with specific \((p, q)\) charges are embedded in successively larger groups. Table 4 presents junctions with various \((p, q)\) charges and representations for the smallest possible values of the quadratic form \( f(p, q) \) for \( E_6 \). Conjugacy requires that the \( 27 \) and \( 351 \) representations occur only for \( p = 1 \) (mod 3), the \( 27 \) and \( 351 \) occur only for \( p = 2 \) (mod 3), and the \( 1, 78 \) and \( 650 \) occur for \( p = 0 \) (mod 3). The presence of the representations marked with a dagger is not required by consistency with decoupling.
5 Duality constraints

The theory with $so(8)$ flavor symmetry has an $SL(2, \mathbb{Z})$ duality acting on the BPS charges, which induces an action on the representations via the $S_3$ permutation group implementing $so(8)$ triality \[3\]. Given a representation $R$ with highest weight vector $(a_1, a_2, a_3, a_4)$ an element of $g \in SL(2, \mathbb{Z})$ will map it to a representation where the labels $a_1, a_3$ and $a_4$ have been permuted according to the element of $S_3$ associated to $g$ by the homomorphism $h : SL(2, \mathbb{Z}) \to S_3$. More concretely, the spectrum of the theory, defined by the BPS charges and representations $\sum_i \{(p_i, q_i); R_i\}$, is invariant under the action of a group with elements of the form $(g, h(g))$, where $g \in SL(2, \mathbb{Z})$ acts on the $(p, q)$ charges, and $h(g) \in S_3$ acts on the representation $R_i$. This claim is readily verified by examination of the table corresponding to $N_f = 4$ in section 3.

We will now develop corresponding results for $so(10), E_6, E_7$ and $E_8$. While in the $so(8)$ case the duality conjecture is well supported by additional evidence, in the other cases such evidence is not available. Our analysis is therefore in essence a proposal for the duality symmetry of the spectrum of these unfamiliar theories. This proposed symmetry of the spectrum implies nontrivial constraints on representations and multiplicities, especially for the representations that are not required by the decoupling argument discussed in the previous section. The analysis has one new element: we claim that in each case the relevant $SL(2, \mathbb{Z})$ transformations must preserve the quadratic form $f(p, q)$. (For $so(8)$ the quadratic form vanishes \[10\], and therefore the full $SL(2, \mathbb{Z})$ is relevant.) The symmetry transformation must relate representations of the same size with weight vectors of equal lengths. Thus if the quadratic form $f(p, q)$ is not left invariant, one could find that $SL(2, \mathbb{Z})$ action maps junctions allowed by self-intersection to forbidden junctions. \footnote{One could, in principle, allow a larger subgroup of $SL(2, \mathbb{Z})$ and demand that representations that can become forbidden junctions be eliminated. It seems, however, that such a constraint would actually eliminate representations that we know must be present.} We will find that in general the group $SL(2, \mathbb{Z})$ is broken down to a subgroup $M$. We will then find a homomorphism $h : M \to A$ to the outer automorphisms of the particular algebra. The symmetry group of the spectrum will be generated by elements of the form $(m, h(m))$ with $m \in M$ and $h(m) \in A$.

For the case of $so(10)$ the group $\mathbb{Z}_2$ of automorphisms of this algebra is generated by $\sigma : a_4 \leftrightarrow a_5$. Conjugacy classes of representations are given by $C = 2a_1 + 2a_3 + a_4 - a_5 \pmod{4}$. One can readily see that $\sigma$ exchanges representations with $C = 1$ and $C = 3$, while leaving invariant those with $C = 0, 2$. The conjugacy class is correlated with asymptotic charges as $C = 2p - q \pmod{4}$ \[10\]. For $so(10)$ the group $SL(2, \mathbb{Z})$ is broken down to the subgroup $M$ of transformations preserving the quadratic form $q^2$. The elements of $M$ are given by

\[ M_{\pm}^n = \begin{pmatrix} \pm 1 & n \\ 0 & \pm 1 \end{pmatrix}, \quad n \in \mathbb{Z} . \tag{5.1} \]

By checking the action of the matrices on $(p, q)$ and thus on $C$, one readily verifies that the ho-
momorphism \( M \to \mathbb{Z}_2 \) is defined by \( \{ M^{\text{odd}}, M^{\text{even}} \} \to \sigma \), while all others map to the identity. The group of symmetries is therefore given by \( \{ (M^{\text{even}}, e), (M^{\text{odd}}, e), (M^{\text{odd}}, \sigma), (M^{\text{even}}, \sigma) \} \). This is an infinite abelian group. It is simple to verify that the necessary spectrum of \( so(10) \) listed in Table 3 is consistent with this group of symmetries. The same is true for the \( so(6) \) case (\( N_f = 3 \)) discussed in section 3.

For the case of \( E_6 \) the group \( \mathbb{Z}_2 \) of automorphisms is generated by the element \( \sigma : (a_1, a_2) \leftrightarrow (a_4, a_5) \). Conjugacy classes of representations are given by \( C = a_1 - a_2 + a_4 - a_5 \pmod{3} \), and \( \sigma \) exchanges representations with \( C = 1 \) and \( C = 2 \), while leaving invariant ones with \( C = 0 \). We also have \( C = p \pmod{3} \) \([10]\). In this case the group \( SL(2, \mathbb{Z}) \) is broken down to the subgroup \( M(6) \) of transformations preserving the quadratic form given in Table 4. The elements of \( M(6) \) are given by

\[
M^0_\pm(6) = \begin{pmatrix} \pm1 & 0 \\ 0 & \pm1 \end{pmatrix}, \quad M^1_\pm(6) = \begin{pmatrix} \pm1 & \mp3 \\ \pm1 & \mp2 \end{pmatrix}, \quad M^2_\pm(6) = \begin{pmatrix} \mp2 & \pm3 \\ \mp1 & \pm1 \end{pmatrix}. \tag{5.2}
\]

By checking the action of the matrices on \((p, q)\) and thus on \( C \), one readily verifies that the homomorphism \( M \to \mathbb{Z}_2 \) is defined by \( M^i_\pm(6) \to e, M^i_\pm(6) \to \sigma \) for \( i = 1, 2, 3 \). The group of symmetries is therefore given by \( \{(M^i_\pm, e), (M^i_\pm(6), \sigma)\} \). This is isomorphic to \( \mathbb{Z}_2 \times \mathbb{Z}_3 \).

For the case of \( E_7 \) the group of automorphisms is trivial. Conjugacy classes of representations are given by \( C = a_3 + a_4 + a_6 + a_7 \pmod{2} \), and \( C = p + q \pmod{2} \) \([10]\). In this case the group \( SL(2, \mathbb{Z}) \) is broken down to the subgroup \( M(7) \) of transformations preserving the quadratic form given in the table. The elements of \( M(7) \) are given by

\[
M^0_\pm(7) = \begin{pmatrix} \pm1 & 0 \\ 0 & \pm1 \end{pmatrix}, \quad M^1_\pm(7) = \begin{pmatrix} \pm2 & \mp5 \\ \pm1 & \mp2 \end{pmatrix}. \tag{5.3}
\]

Since there is no algebra automorphism, we expect that the above matrices act on \((p, q)\) leaving \( C \) invariant. This is readily verified to be the case. The group of symmetries is simply \( \{(M^0_\pm(7), e)\} \). This is isomorphic to \( \mathbb{Z}_4 \).

For the case of \( E_8 \) the group of automorphisms is trivial and there only a single conjugacy class. In this case the group \( SL(2, \mathbb{Z}) \) is broken down to the subgroup \( M(8) \) of transformations preserving the relevant quadratic form. The elements of \( M(8) \) are given by

\[
M^0_\pm(8) = \begin{pmatrix} \pm1 & 0 \\ 0 & \pm1 \end{pmatrix}, \quad M^1_\pm(8) = \begin{pmatrix} \pm2 & \mp7 \\ \pm1 & \mp3 \end{pmatrix}, \quad M^2_\pm(8) = \begin{pmatrix} \mp3 & \pm7 \\ \pm1 & \mp2 \end{pmatrix}. \tag{5.4}
\]

Since there is no algebra automorphism, nor conjugacy classes, the action of \( M(8) \) on the spectrum must leave the \( E_8 \) representations invariant. The group of symmetries is simply \( \{(M^0_\pm(8), e)\} \). This is isomorphic to \( \mathbb{Z}_2 \times \mathbb{Z}_3 \).

6 Conclusions and open questions

We have considered configurations of 7-branes with \( D_n \) and \( E_n \) symmetry giving rise, on a 3-brane probe, to four-dimensional \( \mathcal{N} = 2 \) theories with global \( D_n \) and \( E_n \) symmetry.
respectively. The theories with global exceptional symmetry, in particular, are only defined for strong coupling; they are believed to be non-Lagrangian interacting fixed-point theories \[ 19, 20 \], and little is known about them.

We have shown that the self-intersection constraint selects the junctions giving the well-known spectrum for the familiar $\mathcal{N} = 2$ SU(2) SYM theories with $N_f = 0, \ldots, 4$. This striking result led us to investigate the $D_n \geq 5$ and $E_n$ theories as well. This constraint, together with the results of \[ 19 \], allowed us to derive some new facts about their BPS spectra. In fact, we suspect that the junctions allowed by the self-intersection constraint are all BPS and all appear in the spectrum. More work will be necessary to be sure about this.

We have exhibited a major change in the nature of the BPS spectrum when we go from the familiar theories to the case of $D_n \geq 5$ and $E_n$ flavor symmetries. In the latter cases arbitrarily large representations are required, while in the former only a few representations of the flavor group appear. While for the familiar theories all BPS states arise from junctions of self-intersection minus one or zero, in the less familiar theories all self-intersection numbers are realized, and in general the junctions correspond to curves of higher genus. We have also carried out a preliminary investigation of duality constraints on the BPS spectrum. These constraints relate representations and their multiplicities for different values of the asymptotic charges.

While we believe to have made some concrete progress in elucidating the BPS spectrum of the mysterious theories, much remains to be investigated. The multiplicities of representations not constrained by decoupling are not known. The representations of supersymmetry associated to general BPS states are also unknown. These are questions that are related to the quantization of zero modes of general junctions, and have been addressed for particular situations in \[ 23, 24 \]. A rich spectrum of states is suggested also by the authors of \[ 25 \] in 6D theories. A complete description of the BPS spectrum of four-dimensional theories with $ADE$ flavor symmetries appears to be within reach.

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