SOME MINIMUM NETWORKS FOR FOUR POINTS IN THE
THREE DIMENSIONAL EUCLIDEAN SPACE

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Abstract. We construct a minimum tree for some boundary symmetric tetrahedra \( \mathbb{R}^3 \), which has two nodes (interior points) with equal weights (positive numbers) having the property that the common perpendicular of some two opposite edges passes through their midpoints. We prove that the length of this minimum tree may have length less than the length of the full Steiner tree for the same boundary symmetric tetrahedra.

1. Introduction

In 1951, Courant and Robbins introduced the Steiner problem (\([3, pp.360]\)) in \( \mathbb{R}^2 \):

**Problem 1.**\([3, pp.360]\) Given \( n \) points \( A_1, \ldots, A_n \) to find a connected system of straight line segments of shortest total length such that any two of the given points can be joined by a polygon consisting of segments of the system.

The solution of the unweighted Steiner problem is called a Steiner tree (\([4, 2]\)). A characterization of the solutions of the unweighted Steiner problem in \( \mathbb{R}^3 \) is given by the following theorem:

**Theorem 1.**\([1, pp. 328, 4]\) Any solution of the unweighted Steiner problem is a tree (a Steiner tree) with at most \( n - 2 \) Fermat-Torricelli points, where each Fermat-Torricelli point has degree three and the angles formed between any two edges incident with a Fermat-Torricelli point are equal (120\(^\circ\)). The \( n - 2 \) Fermat-Torricelli points are vertices of the polygonal tree which do not belong to \( \{A_1, \ldots, A_n\} \).

In 2011, we characterize a weighted Steiner minimal tree for convex quadrilaterals on the K-plane (Two dimensional sphere with radius \( R = \sqrt{-K} \), \( K > 0 \), Hyperbolic plane with constant Gaussian curvature \( K < 0 \), Euclidean plane) in [7, Theorem 2.1, pp. 140-149].

**Theorem 2.**\([7, Theorem 2.1, p. 140]\) A weighted (full) Steiner minimal tree of \( A_1A_2A_3A_4 \) consists of two (weighted) Fermat-Torricelli points \( A_0, A'_0 \) which are located at the interior convex domain with corresponding weights \( B_0 = B'_0 = B_5 \) and minimizes the objective function:

\[
B_1a_1 + B_2a_2 + B_3a_3 + B_4a_4 + B_5d = \text{minimum},
\]

such that:

\[
|B_i - B_j| < B_k < B_i + B_j \quad (1.2)
\]
and

$$|B_i - B_m| < B_n < B_i + B_m$$  \hspace{1cm} (1.3)$$

for \(i, j, k \in \{1, 4, 5\}, \ l, m, n \in \{2, 3, 5\}\) and \(i \neq j \neq k, \ l \neq m \neq n\).

By setting \(B_4 = 0\), we obtain a weighted Fermat-Torricelli tree which coincides with the weighted Fermat-Torricelli tree w.r.t to the triangle \(\triangle A_1A_2A_3\).

In this paper, we construct a minimum tree for some boundary symmetric tetrahedra in \(\mathbb{R}^3\), whose common perpendicular of some two non-neighbouring edges pass from their midpoints. By performing a rotation by a specific angle (twist angle) w.r.t to the line which pass from their midpoints the problem of finding a minimum network for these boundary symmetric tetrahedra is transformed to the equivalent problem of finding a minimum network for isosceles trapezium.

Thus, we consider the problem:

Find a minimum network which has two interior points with corresponding weights (positive real numbers) which depend on the angle \(\theta\) formed by the two diagonals of the isosceles trapezium (Problem 2).

We shall solve Problem 2 by constructing two points which lie on the midperpendicular, which are the intersections of the distances of each vertex with the diagonals and we prove that the corresponding weights are \(w(\theta) = \sin \frac{\theta}{2}\) (Solution of Problem 2).

Furthermore, we prove that the length of this minimum tree is less than the corresponding length of the Steiner tree if \(0^\circ < \theta < 60^\circ\) (Theorem 3).

By applying Theorem 3, we derive that the length of the construction tree may be less than the length of the Steiner tree for a rectangle (Corollary 1). Finally, by taking into account Corollary 2 and remark 3, we show that the length of the construction tree is greater than the length of the corresponding Steiner tree for the square.

2. The Steiner problem for some boundary symmetric tetrahedra in the three-dimensional Euclidean Space.

We shall introduce the Steiner problem for some boundary symmetric tetrahedra in \(\mathbb{R}^3\). These boundary symmetric tetrahedra are tetrahedra whose common perpendicular of some two non-neighbouring edges pass from their midpoints.

Let \(A_1A_2A_3A_4\) be a tetrahedron in \(\mathbb{R}^3\), such that \(d\) is the length of the common perpendicular of the edges \(A_1A_2\) and \(A_4A_3\) (euclidean distance) which pass from the midpoints \(M_{12}\) and \(M_{34}\) of \(A_1A_2\) and \(A_4A_3\), respectively.

We denote by \(F_{12}, \ F_{34}\) two points at the interior of \(A_1A_2A_3A_4\) in \(\mathbb{R}^3\) with corresponding positive numbers (weights) \(w_{12}\) and \(w_{34}\), respectively, by \(a_{12}\) the Euclidean distance of the line segment \(A_iF_{12}\), by \(a_{13}\) the Euclidean distance of the line segment \(A_iF_{34}\), by \(a_{ij}\) the Euclidean distance of the line segment \(A_iA_j\), for \(i, j = 1, 2, 3, 4\) and by \(d_{12,34}\) the Euclidean distance of the line segment \(F_{12}F_{34}\).

The twist angle is referred as the angle between the planes formed by \(\triangle A_1A_2A_{12}\) and \(\triangle A_4A_{34}\), at the edge \(F_{12}F_{34}\).

The twist angle \(\varphi\) for this particular tetrahedron \(A_1A_2A_3A_4\) is given by:

$$\varphi = \arccos \left( \frac{a_{12}a_{34}}{a_{12}a_{34}} \right)$$  \hspace{1cm} (2.1)$$

By rotating \(A_1A_2\) w.r.t \(M_{12}\) by an angle \(\varphi\) we derive an isosceles trapezium \(A_1'A_2'A_3A_4\).
We denote by $F$ the intersection point of the two equal diagonals $A'_1A_3$ and $A'_2A_4$ and by $\theta$ the angle $\angle A'_1FA'_2 = \angle A_4FA_3$.
Assume that $d > \max\{a_{12}, a_{34}\}$.

**Problem 2.** Find $F_{12}$ and $F_{34}$ with corresponding weights (positive real numbers) $w_{12}$ and $w_{34}$, such that $w_{12} = w_{34} = w(\theta) > 0$

and

$$f(a_{12}, a_{212}, a_{3,34}, a_{4,34}, \theta, d) = a_{1,12} + a_{2,12} + a_{3,34} + a_{4,34} + w(\theta)d_{12,34} \to \min$$  \hspace{1cm} (2.2)

**Solution of Problem 2.** Without loss of generality, we assume that:

$M_{43} = \{0, 0, 0\}$, $M_{34}M_{12}$ lie on the $z$ axis, $M_{12} = \{0, 0, 0\}$, $A_1 = \{-x_1, -y_1, z_1\}$, $A_2 = \{x_1, y_1, z_1\}$, $A_4 = \{-x_4, 0, z_1\}$, $A_3 = \{x_4, 0, z_1\}$.

The angle $\phi$ is given by:

$$\phi = \arccos\left(\frac{2x_4, 0, 0}{4x_4\sqrt{x_1^2 + y_1^2}}\right),$$  \hspace{1cm} (2.3)

or

$$\phi = \arccos\left(\frac{x_1}{\sqrt{x_1^2 + y_1^2}}\right).$$  \hspace{1cm} (2.4)

By rotating by $\phi$ counterclockwise $A_1A_2$ w.r. to $M_{12}$, we derive an isosceles trapezium $A'_1A'_2A_4A_3$. We get: $x'_1 = -\frac{x_1}{\cos\phi}$, $y'_1 = 0$, $z'_1 = z_1$, $x'_2 = \frac{x_1}{\cos\phi}$, $y'_2 = 0$, $z'_2 = z_1$.

From $\triangle A'_1FM_{12}$ and $\triangle A_3FM_{12}$, we derive that:

$$\tan \theta = \frac{A_3M_{34}}{d - FM_{12}},$$  \hspace{1cm} (2.5)

and

$$\tan \theta = \frac{A_1M_{12}}{FM_{12}}.$$  \hspace{1cm} (2.6)

By subtracting (2.5) from (2.6), we obtain:

$$FM_{12} = \frac{d}{A_1M_{12}} + 1$$  \hspace{1cm} (2.7)

or

$$FM_{12} = \frac{d}{a_{12}} + 1.$$  \hspace{1cm} (2.8)

By replacing (2.8) in (2.6), we get:

$$\tan \theta = \frac{a_{12}}{\frac{a_{12}}{2d} + 1}.$$  \hspace{1cm} (2.9)

The intersection of the two heights of $\triangle A_3FA_4$ w.r. to the sides $FA_3$ and $FA_4$ is the point $F_{34}$.

The intersection of the two heights of $\triangle A'_1FA'_2$ w.r. to the sides $FA'_1$ and $FA'_2$ is the point $F_{12}$. The points $F_{12}$, $F_{34}$ belong to $M_{12}M_{34}$.

Thus, we get
Figure 1.
\[ \angle A_3 F_{34} A_4 = \angle A'_1 F_{12} A'_2 = 180^\circ - \theta. \]

By applying Theorem 2, for \( A'_1 A'_2 A_4 A_3, B_1 = B_2 = B_3 = B_4 = 1, \) we derive that \( B_0 = B_0' = w(\theta). \)

Taking into account that \( F_{12} \) is the weighted Fermat-Torricelli point of \( \triangle A'_1 F_{34} A'_2 \)
and \( F_{34} \) is the weighted Fermat-Torricelli point of \( \triangle A_4 F_{12} A_3, \) we derive that:

\[ \frac{1}{\sin(90^\circ + \frac{\theta}{2})} = \frac{w(\theta)}{\sin(180^\circ - \theta)}. \]

or

\[ w(\theta) = 2 \sin \frac{\theta}{2}. \]

Proposition 1. The length of the minimum construction tree of \( A'_1 A'_2 A_4 A_3 \) having two weighted Fermat-Torricelli points \( F_{12} \) and \( F_{34} \) with corresponding equal weights \( w(\theta) = 2 \sin \frac{\theta}{2} \) is given by

\[ l_{\min \text{ST}}(A'_1 A'_2 A_4 A_3) = 2A'_1 O_{12} + 2A_3 O_{34} + O_{12} O_{34} \]

and

\[ l_{\min \text{T}}(A'_1 A'_2 A_4 A_3) = 2A'_1 F_{12} + 2A_3 F_{34} + w(\theta)F_{12} F_{34}. \]

Proof. From \( \triangle A_3 F_{34} M_{34} \), and \( \triangle A'_1 F_{12} M_{12}, \) we derive:

\[ A_3 F_{34} = \frac{a_{34}}{2 \cos \frac{\theta}{2}}, \]

\[ A'_1 F_{12} = \frac{a_{12}}{2 \cos \frac{\theta}{2}}, \]

\[ F_{34} M_{34} = \frac{a_{34}}{2} \tan \frac{\theta}{2}, \]

and

\[ F_{12} M_{12} = \frac{a_{12}}{2} \tan \frac{\theta}{2}. \]

Taking into account that

\[ F_{12} F_{34} = d - F_{12} M_{12} - F_{34} M_{34} \]

and by replacing (2.18) and (2.19) in (2.20), we get:

\[ F_{12} F_{34} = d - \frac{(a_{12} + a_{34})}{2} \tan \frac{\theta}{2}. \]
\[ d = M_{12}M_{34} = (a_{12} + a_{34}) \frac{1}{2 \tan \frac{\theta}{2}} \]  
(2.22)

and by replacing (2.21), (2.22), (2.16) and (2.17) in (2.14) we obtain (2.15).

\[ \square \]

**Proposition 2.** The length of the full (equally weighted) Steiner tree of \( A_1'A_2'A_4A_3 \) is given by

\[ l_{\min ST}(A_1'A_2'A_4A_3) = (a_{34} + a_{12})\left(\frac{\sqrt{3}}{2} + \frac{\cos \frac{\theta}{2}}{2 \sin \frac{\theta}{2}}\right) \]  
(2.23)

**Proof.** From \( \triangle A_3O_{34}M_{34} \), and \( \triangle A_1'O_{12}M_{12} \), we derive:

\[ A_3O_{34} = \frac{a_{34}}{2 \cos 30^\circ}, \]  
(2.24)

\[ A_1'O_{12} = \frac{a_{12}}{2 \cos 30^\circ}, \]  
(2.25)

\[ O_{34}M_{34} = \frac{a_{34}}{2} \tan 30^\circ, \]  
(2.26)

and

\[ O_{12}M_{12} = \frac{a_{12}}{2} \tan 30^\circ. \]  
(2.27)

Taking into account that

\[ O_{12}O_{34} = d - O_{12}M_{12} - O_{34}M_{34} \]  
(2.28)

and by replacing (2.26) and (2.27) in (2.28), we get:

\[ O_{12}O_{34} = d - \frac{(a_{12} + a_{34})}{2} \tan 30^\circ. \]  
(2.29)

By replacing (2.29), (2.22), (2.24) and (2.25) in (2.28) we obtain (2.23).

\[ \square \]

We consider a class of isosceles trapezium \( A_1'A_2'A_4A_3 \), such that \( a_{12}, a_{34} \) are constant positive real numbers and \( \theta = \angle A_1'F A_2' \), \( d = M_{12}M_{34} \) are variables. The class of isosceles trapezium are the isosceles trapezium which are formed by a parallel translation of \( a_{12} \) or \( a_{34} \) w.r. to \( M_{12}M_{34} \).

**Theorem 3.** If \( 0 < \theta < 60^\circ \), then

\[ l_{\min T}(A_1'A_2'A_4A_3) < l_{\min ST}(A_1'A_2'A_4A_3), \]

and If \( 60^\circ < \theta < 90^\circ \), then

\[ l_{\min T}(A_1'A_2'A_4A_3) > l_{\min ST}(A_1'A_2'A_4A_3). \]

**Proof.** We set

\[ g(\theta) = l_{\min ST}(A_1'A_2'A_4A_3) - l_{\min T}(A_1'A_2'A_4A_3) \]  
(2.30)

or

\[ g(\theta) = (a_{12} + a_{12})\left(\frac{\sqrt{3}}{2} + \frac{\cos \frac{\theta}{2}}{2 \sin \frac{\theta}{2}} - 2 \cos \frac{\theta}{2}\right). \]  
(2.31)

By replacing the trigonometric transformations
\[
\tan \frac{\theta}{2} = \frac{2 \tan^2 \frac{\theta}{4}}{1 - \tan^2 \frac{\theta}{4}} 
\]  
(2.32)

and

\[
\cos \frac{\theta}{2} = \frac{1 - \tan^2 \frac{\theta}{4}}{1 + \tan^2 \frac{\theta}{4}} 
\]  
(2.33)

in (2.31) and by setting

\[ t = \tan^2 \frac{\theta}{4}, \]

we obtain a polynomial of fourth order w.r. to \( t \):

\[
-t^4 + (2\sqrt{3} + 8)t^3 + (2\sqrt{3} - 8)t + 1 = 0,
\]  
(2.34)

There is only one real solution \( t = 0.26794919243112275 \), which gives \( \theta = 60^\circ \in (0^\circ, 90^\circ) \). The other real solutions give values for \( \theta \notin (0^\circ, 90^\circ) \). The function attains a global minimum at \( \theta \approx 78.09^\circ \) which corresponds to the Fermat condition of the first derivative \( \sin^3 \frac{\theta}{2} = \frac{1}{4} \).

Thus, we derive that \( g(\theta) \) is decreasing for \( \theta \in (0^\circ, 78.09^\circ) \) increasing for \( \theta \in (78.09^\circ, 90^\circ) \) and positive for \( \theta \in (0^\circ, 60^\circ) \) (Fig 2).

□

**Corollary 1.** If \( a_{12} = a_{34} \) and \( \theta < 90^\circ \), then

\[
l_{\text{min}T}(A_1'A_2A_4A_3) = 4a_{12} \cos \frac{\theta}{2}.
\]  
(2.35)

and

\[
l_{\text{min}ST}(A_1'A_2A_4A_3) = (a_{12})\sqrt{3} + \frac{\cos \frac{\theta}{2}}{2\sin \frac{\theta}{2}}
\]  
(2.36)
Proof. For $a_{12} = a_{34}$, $A'_1A'_2A_4A_3$ is a rectangle.
By replacing $a_{12} = a_{34}$ in (2.15) and (2.23), we obtain (2.35) and (2.36).
\[ \square \]

**Remark 1.** By replacing $\cos \frac{\theta}{2} = \frac{d}{\sqrt{d^2 + a_{12}^2}}$ in (2.36) we get
\[ l_{\min T}(A'_1A'_2A_4A_3) = 4a_{12} \frac{d}{\sqrt{d^2 + a_{12}^2}}. \]  
(2.37)

**Corollary 2.** If $a_{12} = a_{34}$ and $\theta = 90^\circ$, then
\[ l_{\min T}(A'_1A'_2A_4A_3) = 2a_{12}\sqrt{2}, \]  
(2.38)
\[ l_{\min ST}(A'_1A'_2A_4A_3) = a_{12}(\sqrt{3} + 1) \]  
(2.39)

and
\[ l_{\min T}(A'_1A'_2A_4A_3) > l_{\min ST}(A'_1A'_2A_4A_3). \]

Proof. For $a_{12} = a_{34}$ and $\theta = 90^\circ$, $A'_1A'_2A_4A_3$ is a square.
By replacing $a_{12} = a_{34}$ and $\theta = 90^\circ$ in (2.15) and (2.23), we obtain (2.38) and (2.39).
The inequality
\[ \sqrt{3} + 1 < 2\sqrt{2} \]  
yields
\[ l_{\min T}(A'_1A'_2A_4A_3) > l_{\min ST}(A'_1A'_2A_4A_3). \]  
\[ \square \]

**Remark 2.** Taking into account Corollary 2, the length of the minimum tree $T$ for the square $A'_1A'_2A_4A_3$ is the sum of the two equal diagonals. The intersection $F$ of the two diagonals is the Fermat-Torricelli point of the square $A'_1A'_2A_4A_3$.

**Remark 3.** The length of the minimum tree $T$ for the rectangle $A'_1A'_2A_4A_3$ is given by:
\[ l_{\min T}(A'_1A'_2A_4A_3) = 4 \frac{a_{12}}{\cos \frac{\theta}{2}} + 2 \cos \frac{\theta}{2} \left( d - \frac{a_{12}^2}{d} \right). \]

For $a_{12} = d$, $A'_1A'_2A_4A_3$ is a square and the second term vanishes which corresponds to the weight $w(\theta)$. It is important to note that for this reason the length of the Steiner minimum tree of the square is less than the length of the minimum construction tree $T$ for the same square (Fermat-Torricelli tree).

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