SIEGEL METRIC AND CURVATURE OF THE MODULI
SPACE OF CURVES

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ABSTRACT. We study the curvature of the moduli space \( M_g \) of curves of

\text{genus} \( g \) with the Siegel metric induced by the period map \( j : M_g \rightarrow A_g \).

We give an explicit formula for the holomorphic sectional curvature of

\( M_g \) along a Schiffer variation \( \xi_P \), for \( P \) a point on the curve \( X \), in terms

of the holomorphic sectional curvature of \( A_g \) and the second Gaussian

map. Finally we extend the Kähler form of the Siegel metric as a closed

current on \( \overline{M}_g \) and we determine its cohomology class as a multiple of

\( \lambda \).

1. INTRODUCTION

During the last thirty years, some natural metrics on the moduli space

genus \( g \) curves \( M_g \) have been extensively studied. Many of these met-

rics come from metrics on the Teichmüller space of which the moduli space

of curves is the quotient by the mapping class group. One of these is the

Weil-Petersson metric \( \omega_{WP} \). It was introduced by Weil and is known to be

Kähler, to have non-positive curvature operator and negative Ricci curva-

ture, and to be geodesically convex. S. A. Wolpert showed that both its

holomorphic sectional curvature and Ricci curvature have negative (genus

dependent) upper bounds (but no lower bounds do exist). Moreover it is not

complete (23). The other canonical metrics, namely the Teichmüller met-

ric (or the Kobayashi metric), the Caratheodory metric, the Kähler-Einstein

metric, the induced Bergman metric, the McMullen metric, are complete.

Recently Liu, Sun and Yau (10, 11) showed their equivalence on \( M_g \) and

the equivalence with the Ricci metric and the perturbed Ricci metric intro-

duced by them. The Kähler form of the WP metric has been extended by

Masur (12) as a closed current on the Deligne-Mumford compactification

\( \overline{M}_g \) of \( M_g \). Wolpert (22) determined its cohomology class in terms of the

first Chern class of the Hodge bundle \( \lambda \) and the classes of the boundary.

Let \( A_g \) be the moduli space of principally polarized abelian varieties of

dimension \( g \) and let \( j : M_g \rightarrow A_g \) be the period map sending a curve to

its jacobian. It is an interesting and classical problem to understand the

geometry of the image of \( M_g \) in \( A_g \).

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On $A_g$ there is a natural metric coming from the unique $Sp(2g, \mathbb{R})$ invariant metric on the Siegel space $H_g \simeq Sp(2g, \mathbb{R})/U(g)$ of which $A_g$ is the quotient by $Sp(2g, \mathbb{Z})$. The purpose of this paper is to study the metric on $M_g$ induced by this metric through the period map, which we call the Siegel metric. In [4] an explicit expression for the second fundamental form of the immersion $j$ is given and it is proven that the second fundamental form lifts the second Gaussian map $\mu_2 : I_2(K_X) \to H^0(X, 4K_X)$, as stated in an unpublished paper of Green-Griffiths (cf. [7]).

Here we use it to compute the curvature of the Siegel metric. In particular we give an explicit formula for the holomorphic sectional curvature of $M_g$ along the a Schiffer variation $\xi_P$, for $P$ a point on the curve $X$, in terms of the holomorphic sectional curvature of $A_g$ and the second Gaussian map $\mu_2 : I_2(K_X) \to H^0(X, 4K_X)$.

Finally we give some properties of the holomorphic sectional curvature of $M_g$, using results of [3]. In particular along a Schiffer variation $\xi_P$ the holomorphic sectional curvature $H(\xi_P)$ of $M_g$ is strictly smaller than the holomorphic sectional curvature of $A_g$ unless $P$ is either a Weierstrass point of a hyperelliptic curve or a ramification point of the $g_3^1$ on a trigonal curve. In these last cases $H(\xi_P) = -1$.

Furthermore we study the asymptotic behaviour of the Kähler form of the Siegel metric on $M_g$ showing that it extends as a closed current to $\overline{M_g}$, hence it defines a cohomology class in $H^2(\overline{M_g}, \mathbb{C})$ which we compute to be $\pi \lambda$.

In all what we have stated, we have considered $M_g$ as an orbifold. In the paper we make all computations using the covering of $M_g$ given by the moduli space of curves with level $n \geq 3$ structures $M_g^{(n)}$ and the moduli space $A_g^{(n)}$ of principally polarized abelian varieties with level $n$ structures.

In fact $M_g^{(n)}$ and $A_g^{(n)}$ are smooth and by Local Torelli theorem proven in [18] we know that the period map $j^{(n)} : M_g^{(n)} \to A_g^{(n)}$ is a two to one immersion outside the hyperelliptic locus and it is an injective immersion if we restrict to the hyperelliptic locus.

The paper is organized as follows: in Section 2 we define the Siegel metric and compute it on the tangent directions given by the Schiffer variations (Lemma 2.2). In Section 3 we give the expression of the curvature of the Siegel metric on $A_g^{(n)}$ restricted to the Schiffer variations. Then, we show that the second fundamental form of the immersion of $M_g^{(n)}$ in $A_g^{(n)}$ is non zero at any non hyperelliptic curve and we exhibit a formula for the curvature of $M_g^{(n)}$ (Thm 3.7). Finally we write the holomorphic sectional curvature of $M_g^{(n)}$ along a Schiffer variation $\xi_P$, using the second Gaussian map. In Section 4 we give some applications of results of [3] to the holomorphic sectional curvature of $M_g^{(n)}$. In Section 5 we study in particular the hyperelliptic locus $HE_g$ and we show that the second fundamental form of $HE_g$ in $A_g^{(n)}$ is non zero at any point. In Section 6 we extend the Kähler form of the Siegel
metric as a closed current on $\overline{M}_g$ and we determine its cohomology class \[6.1\].

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2. The Siegel metric

We introduce some notations. Let $M_g$, resp. $M_g^{(n)}$ be the moduli space of smooth genus $g$ curves, resp. of smooth genus $g$ curves with a fixed $n$-level structure. Denote by $T_g$ the Teichmüller space and by $\Gamma_g$ the mapping class group acting on $T_g$ with quotient $M_g$. Let $K(n) := ker(\Gamma_g \rightarrow Sp(2g, \mathbb{Z}/n\mathbb{Z}))$ and recall that $M_g^{(n)}$ is the quotient of $T_g$ by the action of $K(n)$. Moreover, let $K := ker(\Gamma_g \rightarrow Sp(2g, \mathbb{Z}))$ be the Torelli group and define the Torelli space $Tor_g$ as the quotient of $T_g$ by the action of the Torelli group.

Let $A_g$, resp. $A_g^{(n)}$ be the moduli space of $g$-dimensional principally polarized Abelian varieties, resp. of $g$-dimensional principally polarized Abelian varieties with a $n$-level structure. Denote by $H_g := \{ Z \in M(g, \mathbb{C}) \mid Z = t Z, Im Z > 0\}$ the Siegel space so that $A_g$ is the quotient of $H_g$ by the action of $Sp(2g, \mathbb{Z})$ and $A_g^{(n)}$ is the quotient of $H_g$ by $ker(Sp(2g, \mathbb{Z}) \rightarrow Sp(2g, \mathbb{Z}/n\mathbb{Z}))$. Denote by $j^{Tor}$, $j$ and $j^{(n)}$ the period maps which send a curve to its jacobian. We have the following diagram

$$
\begin{array}{ccc}
T_g & \downarrow & \\
\downarrow & \downarrow & \downarrow \\
Tor_g & j^{Tor} & H_g \\
\downarrow & \downarrow & \downarrow \\
M_g^{(n)} & j^{(n)} & A_g^{(n)} \\
\downarrow & \downarrow & \downarrow \\
M_g & j & A_g \\
\end{array}
$$

The Torelli theorem states that $j$ is injective, while $j^{Tor}$ and $j^{(n)}$ are two to one on the image and ramified over the hyperelliptic locus. In fact multiplication by $-1$ in $H^1(X, \mathbb{Z}) = H^1(JX, \mathbb{Z})$, where $JX$ is the Jacobian of the curve $X$, is induced by an automorphism of abelian varieties but not by an automorphism of non hyperelliptic curves. Local Torelli Theorem says that outside the hyperelliptic locus and restricted to the hyperelliptic locus the period map is an immersion (cf. [15]).

From now on we shall work on $M_g^{(n)}$ and $A_g^{(n)}$, with $n \geq 3$, since they are smooth, everything works in the same way on $M_g$ and $A_g$ but in the orbifold context.

We will now define the Siegel metric.

The Siegel space $H_g$ is a homogeneous space and it can be seen as the quotient $Sp(2g, \mathbb{R})/U(g)$. We call the unique (up to scalar) invariant metric the Siegel metric.
Let $F$ be the homogeneous vector bundle on $H_g$ associated to the standard $g$-dimensional representation of $U(g, \mathbb{C})$. The Hodge metric $h$ on $F$ is the only (up to multiplication by scalars) invariant metric on the homogeneous bundle $F$. Moreover through the identification

$$\Omega^1_{H_g} \simeq S^2 F$$

the Hodge metric on $F$ defines the Siegel metric on $H_g$.

The Siegel metric on $H_g$ defines a metric on $A_{g}^{(n)}$ and $A_g$ and, through the period map, an induced metric on $M_{g}^{(n)}$ and $M_g$ outside the hyperelliptic locus, and on the hyperelliptic locus itself. We call all these metrics the Siegel metrics.

These metrics can be described in terms of polarized variation of Hodge structures. More precisely, on $A_{g}^{(n)}$ we have the universal family $\phi : \mathcal{A} \to A_{g}^{(n)}$, and the polarized variation of Hodge structures associated to the local system $R^1 \phi_* \mathbb{Z}$. The associated Hodge bundle $F^1$ can be identified with $\phi_* (\Omega^1_{\mathcal{A}/A_{g}^{(n)}})$, where $\Omega^1_{\mathcal{A}/A_{g}^{(n)}}$ is the sheaf of relative holomorphic one forms. The polarization induces a Hermitian metric on $R^1 \phi_* \mathbb{C}$ and on $F^1$, which we call the Hodge metric. In fact the pullback of $F^1$ on $H_g$ is the bundle $F$ and the pullback of the metric is the Hodge metric on $F$. Hence the Siegel metric is induced by the Hodge metric through the identification $S^2 F^1 \cong \Omega^1_{A_{g}^{(n)}}$.

On $M_{g}^{(n)}$ we have the universal family $\psi : \mathcal{C} \to M_{g}^{(n)}$ with induced relative dualizing sheaf $K_{\mathcal{C}/M_{g}^{(n)}}$. The local system $R^1 \psi_* \mathbb{Z}$ coincides with the pullback of $R^1 \phi_* \mathbb{Z}$ through the period map: at a point $[X] \in M_{g}^{(n)}$, we have $H^1(X, \mathbb{Z}) \cong H^1(JX, \mathbb{Z})$. The non-degenerate Hermitian product on $H^1(X, \mathbb{C})$, defined by the polarization is the following: for any $[\eta], [\xi] \in H^1(X, \mathbb{C})$, we have

$$\langle [\eta], [\xi] \rangle = i \int_X \eta \wedge \bar{\xi}.$$ 

The Hodge bundle can be identified with $\psi_* (K_{\mathcal{C}/M_{g}^{(n)}})$, and the corresponding Hodge metric yields a metric on $S^2 F^1 \cong j^{(n)*} \Omega^1_{A_{g}^{(n)}}$, hence on $j^{(n)*} \mathcal{T}_{A_{g}^{(n)}}$, and by restriction the Siegel metric on $T_{M_{g}^{(n)}}$.

We finally observe that for the sake of simplicity we defined the Siegel metric on the fine moduli space $M_{g}^{(n)}$, but we also have a Siegel metric on $M_g$ viewed as an orbifold.

### 2.1. An explicit formula

We shall now give an explicit formula for the Siegel metric on $M_{g}^{(n)}$ at a point $[X] \in M_{g}^{(n)}$ in terms of the basis of $H^1(T_X)$ given by Schiffer variations $\xi_P$, for a set of $3g - 3$ general points on $X$.

Now, we briefly recall the definition of $\xi_P$. Consider the exact sequence

$$0 \to T_X \to T_X(P) \to T_X(P)_{|P} \to 0.$$
Notice that $H^0(T_X(P)|_P) \cong \mathbb{C}$. If we denote the coboundary map by $\delta : H^0(T_X(P)|_P) \rightarrow H^1(T_X)$, we have $\text{dim}(\text{Im}(\delta)) = 1$. Any non-zero element $\xi_P$ in $\text{Im}(\delta)$ is called a Schiffer variation. Let us choose a local coordinate $z$ in a neighborhood of $P$. Under the Dolbeault isomorphism $H^1(T_X) \cong H^{0,1}(T_X)$, it is represented by the form

$$\theta_P = \frac{1}{z} \bar{\partial} b_P \otimes \frac{\partial}{\partial z},$$

where $b_P$ is a bump function around $P$. Notice that if we choose $b_P$ to be one in a neighborhood of $P$ for this choice of local coordinate $z$, $\xi_P$ depends only on the choice of $z$. In what follows, we need to express $\xi_P$ in terms of a basis of $S^2(H^0(K_X)^*)$, through the inclusion of $H^1(T_X)$ in $S^2(H^0(K_X)^*)$.

Fix an orthonormal basis $\{\omega_i\}_{i=1, \ldots, g}$ of $H^0(K_X)$. Choose a local coordinate $z$ around a point $P \in X$ and write $\omega_j = f_j(z) \, dz$. Since $H^0(K_X) \cong H^{1,0}(X)$, the set $\{\bar{\omega}_i\}_{i=1, \ldots, g}$ is a basis of $H^{0,1}(X)$. This set can be viewed as the dual basis of $\{\omega_i\}$, where the non-degenerate pairing is given by

$$\bar{\omega}_i(\omega_j) = i \int_X \omega_i \wedge \bar{\omega}_j = \langle \omega_i, \omega_j \rangle = \delta_{ij}.$$

Observe that we have

$$\langle \bar{\omega}_i, \bar{\omega}_j \rangle = i \int_X \bar{\omega}_i \wedge \omega_j = -i \int_X \omega_j \wedge \bar{\omega}_i = -\delta_{ij}. \quad (1)$$

**Lemma 2.1.** For a choice of a local coordinate $z$ at $P$, we have

$$\langle \xi_P(\omega_i), \bar{\omega}_j \rangle = -2\pi f_i(P) f_j(P), \quad (2)$$

Hence in $S^2(H^0(K_X)^*) \cong S^2(H^{0,1}(X))$ it holds

$$\xi_P = 2\pi \sum_{i,j} f_i(P) f_j(P) (\bar{\omega}_i \otimes \bar{\omega}_j). \quad (3)$$

**Proof.** Since $\xi_P = \sum_{i=1}^g \xi_P(\omega_i) \otimes \bar{\omega}_i$ and

$$\xi_P(\omega_i) = \sum_j -\langle \xi_P(\omega_i), \bar{\omega}_j \rangle \bar{\omega}_j, \quad (3)$$

we get

$$\xi_P = \sum_{i,j} -\langle \xi_P(\omega_i), \bar{\omega}_j \rangle (\bar{\omega}_j \otimes \bar{\omega}_i). \quad (4)$$

By definition of $\xi_P$, the element $\xi_P(\omega_i) \in H^{0,1}(X)$ is represented by the $(0,1)$-form

$$\left(\frac{1}{z} \bar{\partial} b_P \otimes \frac{\partial}{\partial z}\right) |(f_i(z) \, dz) = \frac{1}{z} \bar{\partial} b_P f_i(z).$$

Let $C$ be a small circle around $P$ such that $b_P \equiv 1$ on $C$. Then the lemma follows by Stokes and Cauchy Theorems:

$$\langle \xi_P(\omega_i), \bar{\omega}_j \rangle = i \int_X \frac{\bar{\partial} b_P}{z-z(P)} f_i(z) \wedge f_j(z) \, dz = i \int_X \bar{\partial} \left( \frac{b_P f_i(z) f_j(z)}{z-z(P)} \right) \wedge dz =$$
\[ i \int_X d \left( \frac{b_P f_i(z) f_j(z)}{z - z(P)} \right) dz = i \int_C \frac{f_i(z) f_j(z)}{z - z(P)} dz = -2\pi f_i(P) f_j(P) \]

Lemma 2.2. The scalar product of the two Schiffer variations \( \xi_P, \xi_{P'} \) has the following form:

\[ \langle \xi_P, \xi_{P'} \rangle = 8\pi^2 (\alpha_{P,P'})^2, \]

where

\[ \alpha_{P,P'} = \sum_i f_i(P) f_i(P'). \]

Proof. Recall that on \( S^2 H^{0,1} \) the scalar product is:

\[ \langle a \otimes b, c \otimes d \rangle = \langle a,c \rangle \langle b,d \rangle + \langle a,d \rangle \langle b,c \rangle, \]

induced by the scalar product \( \langle a \otimes b, c \otimes d \rangle = 2\langle a,c \rangle \langle b,d \rangle \) on \( H^{0,1} \otimes H^{0,1} \) via the inclusion of \( S^2 H^{0,1} \hookrightarrow H^{0,1} \otimes H^{0,1}, \ i(a \otimes b) = \frac{1}{2} (a \otimes b + b \otimes a) \).

So, by (2.1) one immediately computes

\[ \langle \xi_P, \xi_{P'} \rangle = 8\pi^2 \sum_{i,j} f_i(P) f_i(P') f_j(P) f_j(P') = 8\pi^2 (\alpha_{P,P'})^2. \]

\[ \square \]

3. Curvature

We would like now to give a formula for the curvature of the Siegel metric on \( M_g^{(n)} \). We will do the computation on the tangent vectors given by the \( \xi_P \)'s. These depend on the choice of the local coordinates, but by linearity one can immediately derive the formulas at the tangent vectors \( \xi_P |_{\xi_P} = \frac{\xi_P}{|\xi_P|} \), which are intrinsic.

Recall that outside the hyperelliptic locus we have the sequence of tangent bundles:

\[ 0 \to T_{M_g^{(n)}} \to j^{(n)}* T_{A_g^{(n)}} \xrightarrow{\pi} N \to 0, \]

whose dual, under the identifications \( j^{(n)}* \Omega^1_{A_g^{(n)}} \cong S^2(\psi_* K_{C|\mathcal{M}_g^{(n)}}), \Omega^1_{M_g^{(n)}} \cong \psi_*(K^2_{C|\mathcal{M}_g^{(n)}}), \) is

\[ 0 \to I_2 \to S^2(\psi_* K_{C|\mathcal{M}_g^{(n)}}) \xrightarrow{m} \psi_*(K^2_{C|\mathcal{M}_g^{(n)}}) \to 0, \]

where \( I_2 : = N^* \) and \( m \) is the multiplication map.

The Hermitian connection of the variation of Hodge structures \( \mathcal{R}^1 \psi_* \mathbb{C}, \) the Gauss-Manin connection, defines a Hermitian connection on \( \mathcal{F}^1 = \psi_* K_{C|\mathcal{M}_g^{(n)}}, \) thus on \( \mathcal{F}^1^* \), as well as \( S^2\mathcal{F}^1 \) and \( S^2\mathcal{F}^1^* \cong j^{(n)}* T_{A_g^{(n)}}, \) which we denote by \( \nabla \).

The exact sequence (5) defines a second fundamental form,
\[ \sigma \in \text{Hom}(\mathcal{T}_{M_g(n)}^1, N \otimes \Omega^1_{M_g(n)}) \], \quad \sigma : s \mapsto \pi(\nabla(s)). \]

Similarly the exact sequence \( [3] \) defines the second fundamental form \( \rho \in \text{Hom} \left( \mathcal{T}_2, \psi^*(K_{C|M_g(n)}^2) \otimes \Omega^1_{M_g(n)} \right) \).

The curvature form \( R \) of \( \mathcal{T}_{M_g(n)}^1 \) is computed in terms of the curvature form \( \tilde{R} \) of \( j^{(n)*}(\mathcal{T}_{M_g(n)}^1) \) and the second fundamental form \( \sigma \). Namely, we have

\[ \langle R(s), t \rangle = \langle \tilde{R}(s), t \rangle - \langle \sigma(s), \sigma(t) \rangle, \]

where \( s, t \) are local sections of \( \mathcal{T}_{M_g(n)}^1 \).

At the point \([X] \in M_g(n)\), we need to compute

\[ \langle R(\xi_P), \xi_{P'} \rangle(\xi_R, \xi_T) = \langle \tilde{R}(\xi_P), \xi_{P'} \rangle(\xi_R, \xi_T) \]

\[ \langle \sigma(\xi_P), \sigma(\xi_{P'}) \rangle(\xi_R, \xi_T). \]

Let us now determine \( \langle \tilde{R}(\xi_P), \xi_{P'} \rangle(\xi_R, \xi_T) \) in terms of the curvature form of the Hodge bundle.

Consider the exact sequence

\[ 0 \to \mathcal{F}^1 \to \mathcal{R}^1 \psi_* \mathbb{C} \otimes C^\infty_{\mathcal{M}_g(n)} \to (\mathcal{R}^1 \psi_* \mathbb{C} \otimes C^\infty_{\mathcal{M}_g(n)})/\mathcal{F}^1 \to 0 \]

At \([X] \in M_g(n)\) we have

\[ 0 \to H^{1,0}(X) \to H^1(X, \mathbb{C}) \to H^{0,1}(X) \to 0. \]

**Lemma 3.1.** The curvature form of the Hodge bundle is given by

\[ \langle R_{\mathcal{F}^1}(\omega_j), \omega_l \rangle(\xi_R, \xi_T) = 4\pi^2 \alpha_{R,T} f_j(R) f_l(T). \]

**Proof.** Since the Gauss-Manin connection on \( \mathcal{R}^1 \psi_* \mathbb{C} \otimes C^\infty_{\mathcal{M}_g(n)} \) is flat, the following holds:

\[ \langle R_{\mathcal{F}^1}(\omega_j), \omega_l \rangle = -\langle \epsilon(\omega_j), \epsilon(\omega_l) \rangle, \]

where \( \epsilon \in \text{Hom}(H^{1,0}(X), H^{0,1}(X) \otimes H^0(2K_X)) \) is the second fundamental form of \( [10] \) at the point \([X]\). We can also view \( \epsilon \) as an element in \( \text{Hom}(H^{1,0}(X) \otimes H^1(T_X), H^{0,1}(X)) \), and by a result of Griffiths (cf. e.g. [7] p.32), we have \( \epsilon(\omega_i \otimes \zeta) = \zeta(\omega_i) \). Hence, we can write \( \epsilon(\omega_j) = \sum_{P,S} (\xi_P(\omega_j) \otimes \xi_P^*) \). Therefore, we have

\[ \langle \epsilon(\omega_j), \epsilon(\omega_l) \rangle = \sum_{P,S} \langle \xi_P(\omega_j), \xi_S(\omega_l) \rangle (\xi_P^* \otimes \xi_S^*). \]

This implies

\[ \langle \epsilon(\omega_j), \epsilon(\omega_l) \rangle (\xi_R, \xi_T) = \langle \xi_R(\omega_j), \xi_T(\omega_l) \rangle. \]
Proposition 3.2. The curvature $\tilde{R}$ of $j^{(n)^*}(T_{\mathcal{A}_g^{(n)}}) = S^2(F^1^*)$ is given by

$$\langle \tilde{R}(\xi_P), \xi_{P'} \rangle(\xi_S, \xi_T) = -64\pi^4 \alpha_{S,T} \sum_{i,j,k,l} f_i(P)f_j(P)f_k(P')f_l(P')f_l(S)f_l(T) = -64\pi^4 \alpha_{S,T} \alpha_{P,T} \alpha_{P'} \alpha_{S,P'}.$$

Proof. To begin with, by Lemma 3.1 we have

$$\langle \tilde{R}(\xi_P), \xi_{P'} \rangle = 4\pi^2 \sum_{i,j,k,l} f_i(P)f_j(P)f_k(P')f_l(P')f_l(S)f_l(T).$$

By standard facts on complex bundles [9], we have

$$\langle \tilde{R}(\xi_P), \xi_{P'} \rangle = 4\pi^2 \sum_{i,j,k,l} f_i(P)f_j(P)f_k(P')f_l(P')f_l(S)f_l(T).$$

Now, we observe that

$$\langle R_{\mathcal{F}^1,}\alpha_{\{P \cap T\}}\rangle = \langle R_{\mathcal{F}^1,}(\xi_P), \xi_{P'} \rangle.$$

In fact, set $R_{\mathcal{F}^1,}(\xi_P) = \sum a_{ij} \xi_i \xi_j$, where $a_{ij} \in \Omega^{1,1}_{M_{g}(n)}$. By duality, we have

$$R_{\mathcal{F}^1,}(\xi_P) = -\sum a_{ij} \xi_i \xi_j. \text{ Hence } \langle R_{\mathcal{F}^1,}(\xi_P), \xi_{P'} \rangle = -a_{ij} = \langle R_{\mathcal{F}^1,}(\xi_P), \xi_{P'} \rangle.$$

By (11) and Lemma 3.1 we deduce

$$\langle \tilde{R}(\xi_P), \xi_{P'} \rangle(\xi_S, \xi_T) = -16\pi^4 \alpha_{S,T} \sum_{i,j,k,l} f_i(P)f_j(P)f_k(P')f_l(P')f_l(S)f_l(T).$$

Therefore, we obtain

$$\langle \tilde{R}(\xi_P), \xi_{P'} \rangle = -64\pi^4 \alpha_{S,T} \alpha_{P,T} \alpha_{P'} \alpha_{S,P'}.$$

In order to apply (17), we still need to compute

$$\langle \sigma(\xi_P), \sigma(\xi_{P'}) \rangle(\xi_S, \xi_T).$$
Recall that the exact sequence \( [5] \) of which \( \sigma \) is the second fundamental form, at \([X] \in M_g^{(n)} \) is
\[
0 \to H^1(T_X) \to S^2(H^0(K_X))^* \to I_2(X)^* \to 0,
\]
thus \( \sigma \) yields a homomorphism
\[
\sigma : H^1(T_X) \to Hom(I_2(K_X), H^0(2K_X)).
\]
Analogously, at \([X] \in M_g^{(n)} \) the exact sequence \( [6] \) is:
\[
0 \to I_2(K_X) \to S^2(H^0(K_X)) \xrightarrow{m} H^0(2K_X) \to 0,
\]
hence the second fundamental form \( \rho \) gives a homomorphism
\[
\rho : I_2(K_X) \to Hom(H^1(T_X), H^0(2K_X))
\]
and for every \( v \in H^1(T_X) \), and for every \( Q \in I_2(X) \), we have
\[
\rho(Q)(v) = \sigma(v)(Q) = \rho(Q)(v).
\]

We recall now some results of \([4] \) on the second fundamental form \( \rho \). In particular we want to use Thm. 2.1 and Lemma 3.2 of \([4] \), (cf. also \([19] \) (4.8)). Let us fix a point \( P \in X \), where \([X] \in M_g^{(n)} \). We have the inclusion \( H^0(K_X(2P)) \hookrightarrow H^1(X - \{P\}, \mathbb{C}) \cong H^1(X, \mathbb{C}) \). By Riemann Roch and Hodge decomposition we immediately see that \( \text{dim}(H^0(K_X(2P)) \cap H^{0,1}(X)) = 1 \), so we define \( \eta_P \in H^0(K_X(2P)) \cap H^{0,1}(X) \) as the only generator of \( H^0(K_X(2P)) \cap H^{0,1}(X) \) having in a neighborhood of \( P \) the following local expression:
\[
\eta_P = \left(-\frac{1}{(z - z(P))^2} + g(z)\right)dz,
\]
with \( g(z) \) holomorphic.

**Lemma 3.3.** (cf. \([4] \) (Thm 2.1), (Lemma 3.2)) Let \( Q \in I_2(K_X) \), \( Q = \sum_{i,j} a_{ij} \omega_i \otimes \omega_j \), then
\[
\rho(Q)(\xi_P) = -\eta_P \sum_{i,j} a_{ij} f_i(P) \omega_j \in H^0(2K_X).
\]

**Corollary 3.4.** If \( X \) is any non hyperelliptic curve \( \rho \) is injective, and \( \sigma \) is non zero. In particular at any point \([X] \in M_g^{(n)} \) outside the hyperelliptic locus the curvature \( R \) of \( T_{M_g^{(n)}} \) and the curvature \( \tilde{R} \) of \( j(n)^* (T_{A_g^{(n)}}) \) are different.

**Proof.** By lemma \( [3, 3] \), for any \( Q \in I_2 \), \( Q = \sum_{i,j} a_{ij} \omega_i \otimes \omega_j \), \( \rho(Q)(\xi_P) = 0 \) implies \( \sum_{i,j} a_{ij} f_i(P) \omega_j = 0 \), hence \( \forall j, \sum_i a_{ij} f_i(P) = 0 \). Then \( Q \in \ker(\rho) \) if and only if \( \sum_i a_{ij} f_i(P) = 0 \) \( \forall j, \forall P \in X, \) so \( \sum_i a_{ij} \omega_i = 0 \), which implies \( Q = 0 \).

Since \( \sigma(\xi_P)^*(Q) = \rho(Q)(\xi_P) \) and \( \rho \) is injective, there must exist a point \( P \in X \) such that \( \sigma(\xi_P) \neq 0 \).

Now we compute \( \xi_S(\rho(Q)(\xi_P)) \), where \( P \) and \( S \) are two points in \( X \).
Let $z$ be a local coordinate in a neighborhood of $S$, and consider a local expression of $\rho(Q)(\xi_P) \in H^0(2K_X)$,
$$
\rho(Q)(\xi_P) = \Psi^Q_P(z) dz^2.
$$

**Lemma 3.5.** Let $Q \in I_2(K_X)$, then
$$
\xi_S(\rho_Q(\xi_P)) = 2\pi i \Psi^Q_P(S).
$$

**Proof.** Recall that $\xi_S$ is represented by a form
$$
\theta_S = \frac{1}{z} \partial b_S \otimes \frac{\partial}{\partial z},
$$
where $z$ is a local coordinate in a neighborhood of $S$ and $b_S$ is a bump function around $S$ which is equal to one in a neighborhood of $S$.

Let $C$ be a small circle around $S$ such that $b_S \equiv 1$ on $C$. We have
$$
\xi_S(\rho(Q)(\xi_P)) = \int_X \theta_S(\rho(Q)(\xi_P)) = \int_X \mathcal{J} \left( \frac{b_S \Psi^Q_P(z)}{z - z(S)} \right) \wedge dz = \int_C \frac{\Psi^Q_P(z)}{z - z(S)} dz = 2\pi i \Psi^Q_P(S).
$$

We want now to compute $\Psi^Q_P(S)$.

If $P \neq S$ the form $\eta_P$ has the following local expression in a neighborhood of $S$:
$$
\eta_P(z) = G_P(z) dz,
$$
where $G_P(z)$ is holomorphic, so
$$
\Psi^Q_P(z) = -G_P(z) \left( \sum_{i,j} a_{ij} f_i(P) f_j(z) \right).
$$

If $P = S$, the local expression of $\eta_P$ in a neighborhood of $P$ is
$$
\eta_P = \left( -\frac{1}{(z - z(P))^2} + g(z) \right) dz,
$$
and we have (cf. also [4], Thm.3.1)
$$
\Psi^Q_P(z) = - \left( -\frac{1}{(z - z(P))^2} + g(z) \right) \left( \sum_{i,j} a_{ij} f_i(P) f_j(z) \right) = 
\sum_{i,j} a_{ij} f_i(P) \left( f_j(P) + f_j'(P)(z - z(P)) + \frac{1}{2} f_j''(P)(z - z(P))^2 + h.o.t. \right)
\frac{1}{(z - z(P))^2} = 
\frac{1}{2} \sum_{i,j} a_{ij} f_i(P) f_j''(P) + O(1),
$$
since $Q \in I_2(K_X)$, so $\sum_{i,j} a_{ij} f_i(P) f_j(P) = 0$, and $\sum_{i,j} a_{ij} f_i(P) f'_j(P) = 0$. Thus we have
\begin{equation}
\Psi^Q_P(P) = \frac{1}{2} \sum_{i,j} a_{ij} f_i(P) f'_j(P) = \frac{1}{2} (\mu_2(Q))(P),
\end{equation}
where $\mu_2(Q)$ is the second Gaussian map of $X$ in $Q$. For the definition of the second Gaussian map see Section 4.

**Proposition 3.6.** Let $\xi_P$ be a Schiffer variation, and let $\{Q_i\}$ be an orthonormal basis of $I_2(K_X)$, denote by $\Psi^Q_P := \Psi^Q_{PQ}$. Then the following holds:
\begin{equation}
\langle \sigma(\xi_P), \sigma(\xi_{P'}) \rangle(\xi_S, \xi_T) = 4\pi^2 \sum_i \Psi^Q_P(S) \Psi^Q_{P'}(T).
\end{equation}

**Proof.** Fix an orthonormal basis $\{Q_i\}$ of $I_2(K_X) \subset S^2(H^0(K_X))$. Let $\{Q^*_i\}$ be the dual basis of $I^*_2(K_X)^*$. By (13), $\sigma(\xi_P) \in I_2 \otimes H^0(2K_X)$; hence
\begin{equation}
\sigma(\xi_P) = \sum_i \sigma(\xi_P)(Q_i) \otimes Q^*_i.
\end{equation}

$\sigma(\xi_P)(Q_i) = \rho(Q_i)(\xi_P) =: \rho_{Q_i}(\xi_P) \in H^0(2K_X)$, so
\begin{equation}
\sigma(\xi_P) = \sum_i \rho_{Q_i}(\xi_P) \otimes Q^*_i.
\end{equation}

On the other hand, a basis of $H^0(2K_X)$ is given by the set $\{\xi^*_S\}$, where $S$ runs in a set of $3g - 3$ general points of $X$. This implies that
\begin{equation}
\rho_{Q_i}(\xi_P) = \sum_S \xi_S(\rho_{Q_i}(\xi_P)) \xi^*_S.
\end{equation}
Therefore, the following holds:
\begin{equation}
\langle \sigma(\xi_P), \sigma(\xi_{P'}) \rangle(\xi_S, \xi_T) =
\end{equation}
\begin{equation}
\sum_i \sum_{V, V'} \langle \xi_V(\rho_{Q_i}(\xi_P)) \xi^*_V, \xi_{V'}(\rho_{Q_i}(\xi_{P'})) \xi^*_{V'} \rangle(\xi_S, \xi_T) =
\end{equation}
\begin{equation}
\sum_i \xi_S(\rho_{Q_i}(\xi_P)) \xi_T(\rho_{Q_i}(\xi_{P'})).
\end{equation}

Using lemma (3.3) we get
\begin{equation}
\langle \sigma(\xi_P), \sigma(\xi_{P'}) \rangle(\xi_S, \xi_T) = \sum_i \xi_S(\rho_{Q_i}(\xi_P)) \xi_T(\rho_{Q_i}(\xi_{P'})) =
\end{equation}
\begin{equation}
= 4\pi^2 \sum_i \Psi^Q_P(S) \Psi^Q_{P'}(T).
\end{equation}

From Proposition 3.2 and Proposition 3.6 we obtain a closed expression for the curvature form of $T_{M^g_9(n)}$ at $[X] \in M^g_9(n)$. More precisely, the following holds.
Theorem 3.7.
\[ \langle R(\xi_P), \xi_{P'} \rangle(\xi_S, \xi_T) = -64\pi^4 \alpha_{S,T} \alpha_{P,T} \alpha_{P',P} \alpha_{S,P'} - 4\pi^2 \sum_i \Psi_i^P(S) \Psi_i^{P'}(T). \]

Corollary 3.8. The holomorphic sectional curvature of \( T_{M_g(\mathfrak{n})} \) at \( [X] \in M_g(\mathfrak{n}) \) computed at the tangent vector \( \xi_P \) is given by
\[
H(\xi_P) = \frac{1}{\langle \xi_P, \xi_P \rangle \langle \xi_P, \xi_P \rangle} \langle R(\xi_P), \xi_P \rangle(\xi_P, \xi_P) = -1 - \frac{1}{64\pi^2(\alpha_{P,P})^4} \sum_i |\mu_2(Q_i)(P)|^2.
\]

Proof. The proof immediately follows from (3.7), (15) and (2.2).

By corollary (3.8) we see that the holomorphic sectional curvature of \( A_g(\mathfrak{n}) \) calculated along the tangent directions at \( [X] \in M_g(\mathfrak{n}) \) given by the Schiffer variations \( \xi_P \) is equal to \(-1\), for all \( P \in X \).

We shall now give another proof of this. We recall that the image of the sectional curvature of \( H_\mathfrak{g} \) is the segment \([-1, -\frac{1}{g}]\) and that the tangent directions \( V \) such that \( H(V) = -1 \) correspond to the symmetric matrices of rank 1.

Let us now see as usual an element \( \xi \in H^1(T_X) \) as a symmetric homomorphism \( H^0(K_X) \to H^0(K_X)^* \) through the exact sequence (12). Then the above observation shows that \( H(\xi) = -1 \) if and only if \( \xi \) has rank one. We therefore recall the characterisation of the elements \( \xi \in H^1(T_X) \) such that \( \xi \) has rank 1. Moreover observe that the Schiffer variations are the points of the bicanonical curve \( \phi_2 K(X) \subset \mathbb{P} H^1(X, T_X) \). Then the statement follows as a corollary by the following result of Griffiths and by the theorem of Enriques-Babbage and Petri.

Define \( \mathcal{X} \subset \mathbb{P} H^1(X, T_X) \),
\[ \mathcal{X} = \{ \xi \in \mathbb{P} H^1(X, T_X) \mid \text{rank}(\xi) \leq 1 \}. \]

Theorem 3.9. Assume that \( g \geq 3 \) and \( X \) is not hyperelliptic. Consider the image of the bicanonical map \( \phi_2 K(X) \subset \mathbb{P} H^1(X, T_X) \). Then \( \phi_2 K(X) \subset \mathcal{X} \) with equality holding if and only if the canonical curve \( \phi_K(X) \) is cut out by quadrics.

Corollary 3.10. Assume that \( g \geq 3 \) and \( X \) is not hyperelliptic. Then \( \phi_2 K(X) \subset \mathcal{X} \) with equality holding if and only if the canonical curve \( \phi_K(X) \) is not trigonal, and it is not isomorphic to a plane quintic.
4. Second Gaussian map and holomorphic sectional curvature

We first recall the definition of the Gaussian maps (cf. [21]). Let $X$ be a smooth projective curve, $S := X \times X$, $\Delta \subset S$ be the diagonal. Let $L$ be a line bundle on $X$ and $L_S := p_1^*(L) \otimes p_2^*(L)$, where $p_i : S \to X$ are the natural projections. Consider the restriction map

$$\tilde{\mu}_{n,L} : H^0(S, L_S(-n\Delta)) \to H^0(\Delta, L_S(-n\Delta|\Delta)).$$

Notice that since $O(\Delta)|\Delta \cong T_X$, we have

$$H^0(\Delta, L_S(-n\Delta|\Delta)) \cong H^0(X, 2L \otimes nK_X).$$

In the case $L = K_X$, $I_2(K_X) \subset H^0(S, K_S(-2\Delta))$, so we can define the second Gaussian map

$$\mu_2 : I_2(K_X) \to H^0(X, 4K_X),$$

as the restriction $\tilde{\mu}_{2,K|I_2(K_X)}$.

As above we fix a basis $\{\omega_i\}$ of $H^0(K_X)$. In local coordinates $\omega_i = f_i(z)dz$. Let $Q \in I_2(K_X)$, $Q = \sum a_{ij}\omega_i \otimes \omega_j$, recall that $\sum_{i,j} a_{ij}f_i f_j \equiv 0$, and since $a_{ij}$ are symmetric, we also have $\sum_{i,j} a_{ij}f_i' f_j' \equiv 0$. The local expression of $\mu_2(Q)$ is

$$\mu_2(Q) = \sum_{i,j} a_{ij}f_i'' f_j'(dz)^4 = -\sum_{i,j} a_{ij}f_i' f_j'(dz)^4. \quad (17)$$

We recall the following results of [3].

**Theorem 4.1.** ([3] Lem.4.1, Thm.4.3) For any trigonal non hyperelliptic curve $X$ of genus $g \geq 4$, the image of $\mu_2$ is contained in $H^0(4K_X - (q_1 + \ldots + q_{2g+4}))$, where $q_1 + \ldots + q_{2g+4}$ is the ramification divisor of the $g_3^1$.

If $g \geq 8$, the rank of $\mu_2$ is $4g - 18$.

We also recall

**Theorem 4.2.** ([3] Thm.6.1) Assume that $X$ is smooth curve of genus $g \geq 5$, which is non-hyperelliptic and non-trigonal. Then for any $P \in X$ there exists a quadric $Q \in I_2$ such that $\mu_2(Q)(P) \neq 0$. Equivalently $\text{Im}(\mu_2) \cap H^0(4K_X - P) \neq \text{Im}(\mu_2)$, $\forall P \in X$.

Assume $[X] \in M_g(n)$, with $g \geq 4$, $X$ non hyperelliptic. Then corollary [3,8] allows us to define a function $F : X \to \mathbb{R}$, given by the holomorphic sectional curvature evaluated along the tangent vectors given by the Schiffer variations:

$$F(P) = H(\xi_P) = -1 - \frac{1}{64\pi^2(\alpha_{P,P})^4} \sum_i |\mu_2(Q_i)(P)|^2 \leq -1,$$

where $\{Q_i\}$ is an orthonormal basis of $I_2(K_X)$.
Proposition 4.3. If \( g = 4 \), the set of points \( P \in X \) such that \( F(P) = -1 \) is finite, which implies that \( F \) is non constant.

If \( g \geq 5 \), \( X \) not hyperelliptic, nor trigonal, then \( F(P) < -1 \) for all \( P \in X \).

If \( X \) is a trigonal curve of genus \( \geq 4 \), \( F(P) = H(\xi_P) = -1 \) for every \( P \in X \) which is a ramification point of the \( g^1_3 \).

Proof. Assume \( X \) has genus 4, then the dimension of \( I_2 \) is one and \( I_2 \) can be generated by a quadric \( Q \) of rank 4 which has norm 1. So \( \forall P \in X, F(P) = -1 - \frac{1}{6\pi \alpha(P,P)}(\mu_2(Q)(P))^2 \), hence there is a finite number of points \( P \) such that \( \mu_2(Q)(P) = 0 \), so in these points we have \( F(P) = -1 \), while \( F(P) < -1 \) elsewhere.

As regards the second statement, we observe that \( F(P) = -1 \) if and only if \( \mu_2(Q_i)(P) = 0 \) for all \( i \), where \( \{Q_i\} \) is an orthonormal basis of \( I_2 \). But then we must have \( \mu_2(Q)(P) = 0 \) for all \( Q \in I_2 \). So the proof follows by Theorem (4.2).

The last statement follows from (4.1).

Remark 4.4. The previous statements imply that for any curve \( X \in M_g^{(n)} \), not hyperelliptic, nor trigonal, for every point \( P \in X \) the holomorphic sectional curvature of \( M_g^{(n)} \), at \( X \) along the tangent directions given by \( \xi_P \) is strictly smaller than the holomorphic sectional curvature of \( A_g^{(n)} \). Hence the Schiffer variations are never tangent directions of totally geodesic submanifolds of \( A_g^{(n)} \).

On the other hand, in the trigonal case, along the Schiffer variations at the ramification points of the \( g^1_3 \), (which are a basis of the tangent space to the trigonal locus) the holomorphic sectional curvature of \( M_g^{(n)} \), coincides with the holomorphic sectional curvature of \( A_g^{(n)} \).

5. The hyperelliptic locus

We will now study the hyperelliptic locus \( HE_g \subset M_g^{(n)} \). Recall that by local Torelli, the restriction of the period map to \( HE_g \) is an injective immersion (cf. [18]). Therefore we have the exact sequence

\[
0 \rightarrow T_{HE_g} \rightarrow T_{A_g^{(n)}}|_{HE_g} \rightarrow N_{HE_g|A_g^{(n)}} \rightarrow 0,
\]

and we denote by

\[
\sigma_{HE} : T_{HE_g} \rightarrow \text{Hom}(T_{HE_g}, N_{HE_g|A_g^{(n)}})
\]

the associated second fundamental form and by \( \rho_{HE} \) the second fundamental form of the dual exact sequence. At the point \( [X] \in HE_g \) the dual exact sequence is

\[
0 \rightarrow I_2 \rightarrow S^2(H^0(K_X)) \rightarrow H^0(2K_X)^+ \rightarrow 0,
\]

where \( H^0(2K_X)^+ \) is the invariant part of \( H^0(2K_X) \) under the hyperelliptic involution and \( I_2 \) is the vector space of the quadrics containing the rational
normal curve, so that

\[ \rho_{HE} : I_2 \to \text{Hom}(T_{HE_g([X])}, H^0(2K_X)^+) \].

We recall that the set of Schiffer variations at the Weierstrass points \( P_i \) generates \( T_{HE_g([X])} \).

**Proposition 5.1.** If \( X \) is hyperelliptic, \( \rho_{HE} \) is injective and thus \( \sigma_{HE} \) is non zero. This implies that the curvature \( R_{HE} \) of \( T_{HE_g} \) is different from the curvature \( \tilde{R} \) of \( T_{A^g([X])} \) at any point \([X] \in HE_g\).

**Proof.** With the same proof of Thm 2.1, Lemma 3.2 of [4] one can show that

\[ \rho_{HE}(Q)(\xi_P) = -\eta_P \sum_{i,j} a_{ij} f_i(P) \omega_j \in H^0(2K_X)^+ \]

if \( P \) is a Weierstrass point of \( X \) and \( Q = \sum_{i,j} a_{ij} \omega_i \otimes \omega_j \in I_2 \).

So \( \rho_{HE}(Q)(\xi_P) = 0 \) implies \( \sum_{i,j} a_{ij} f_i(P) \omega_j = 0 \), hence \( \forall j, \sum_{i,j} a_{ij} f_i(P) = 0 \). Then \( Q \in \ker(\rho_{HE}) \) if and only if \( \sum_{i,j} a_{ij} f_i(P) = 0 \) for every Weierstrass point \( P \in X \). Since there are \( 2g + 2 \) Weierstrass points, this implies that \( \sum_{i,j} a_{ij} \omega_i = 0 \), hence \( Q = 0 \).

Since \( \sigma_{HE}(\xi_P)(Q) = \rho_{HE}(Q)(\xi_P) \) and \( \rho_{HE} \) is injective, there must exist a Weierstrass point \( P \in X \) such that \( \sigma_{HE}(\xi_P) \neq 0 \).

We also observe that with the same proof as in Lemma (3.5) and formula (15) one shows that

\[ \xi_P(\rho_{HE}(Q)(\xi_P)) = \mu_2(Q)(P) \]

at a Weierstrass point \( P \in X \).

Let us denote by \( H_{HE} \) the holomorphic sectional curvature of \( T_{HE_g} \), if \( [X] \in HE_g \) and \( P \in X \) is a Weierstrass point, we have the same expression for \( H_{HE}(\xi_P) \) as in (3.8), namely

\[ H_{HE}(\xi_P) = -1 - \frac{1}{64\pi^2(\alpha_{P,P})^4} \sum_i |\mu_2(Q_i)(P)|^2 \]

where \( \{Q_i\} \) is an orthonormal basis of \( I_2 \).

We recall now a result on the second Gaussian map proven in [3].

**Proposition 5.2.** ([3] Lem.4.1, Prop.4.2) Let \( X \) be a hyperelliptic curve of genus \( g \geq 3 \). Then the rank of \( \mu_2 \) is \( 2g - 5 \) and its image is contained in \( H^0(4K_X - (q_1 + \ldots + q_{2g+2})) \), where \( \{q_1, \ldots, q_{2g+2}\} \) are the Weierstrass points.

**Corollary 5.3.** Let \([X] \in HE_g\), then \( H_{HE}(\xi_P) = -1 \), for any Weierstrass point \( P \in X \).

**Proof.** The proof immediately follows from (18) and from (5.2). \( \square \)
6. The class of the Siegel metric

Let \( \overline{M}_g \) be the Deligne-Mumford compactification of \( M_g \). In [15] it is shown that the Hodge bundle extends to \( \overline{M}_g \) and its \( g \)-th exterior power is ample on \( M_g \).

We denote by \( \lambda \) both the first Chern class of the extension of the Hodge bundle on \( \overline{M}_g \) and on \( M_g \). We will prove that the Kähler form of the Siegel metric on \( \overline{M}_g \) extends as a closed current to \( \overline{M}_g \), hence it defines a cohomology class in \( H^2(\overline{M}_g, \mathbb{C}) \) which is a multiple of \( \lambda \).

**Theorem 6.1.** The Kähler form \( \omega \) of the Siegel metric on \( M_g \) extends as a closed current to \( \overline{M}_g \). Its class \([\omega] \in H^2(\overline{M}_g, \mathbb{C})\) satisfies \([\omega] = \pi \lambda\).

**Proof.** On \( H_g \) the Hodge metric is the only (up to multiplication by scalars) invariant metric on the homogeneous bundle \( F \).

Therefore we have an invariant metric on the line bundle \( \Lambda^g F \) and thus its curvature is an invariant \((1,1)\) form \( \beta \) on \( H_g \).

On the other hand, the Siegel metric is the invariant metric obtained by the metric on \( S^2 F^* \) induced by the Hodge metric and we denote by \( \tilde{\omega} \) its Kähler form.

Since both \( \beta \) and \( \tilde{\omega} \) are invariant \((1,1)\) forms and we are on the irreducible symmetric domain \( H_g \), there exists a constant \( c \) such that \( \tilde{\omega} = c \beta \). This relation still holds on the corresponding forms on \( A_g(n) \) which we denote in the same way.

In [1] a compactification \( A_g(n) \) is constructed and it has the property that it is nonsingular and that \( D_\infty := A_g(n) - A_g^{(n)} \) is a divisor with normal crossings.

In [14] it is shown that the Hodge bundle \( F^1 \) on \( A_g(n) \) extends as a bundle on \( A_g^{(n)} \), such that the Hodge metric has only logarithmic singularities at \( D_\infty \).

Moreover in [14] (see also [5]), it is also proven that the extension of the second symmetric power is isomorphic to the sheaf of differential forms with logarithmic poles at \( D_\infty \):

\[
S^2(F^1) \cong \Omega^1_{A_g^{(n)}}[D_\infty].
\]

Furthermore Mumford proves in ([14] Thm.(3.1), Thm.(1.4)) that the extension of the Hodge metric has “good” singularities and that this implies that its first Chern class yields a closed current on \( A_g^{(n)} \) and thus a cohomology class \( \tilde{\lambda} \in H^2(A_g^{(n)}, \mathbb{C}) \).
Therefore, since on $A_g^{(n)}$ our Kähler form $\tilde{\omega} = c\beta$, then also $\tilde{\omega}$ can be extended as a closed $(1,1)$ current on $A_g^{(n)}$, which we still call $\tilde{\omega}$ and its cohomology class $[\tilde{\omega}] \in H^2(A_g^{(n)}, \mathbb{C})$ is given by $[\tilde{\omega}] = c\lambda$.

In ([16] (18.9), see also [17]) it is shown that the period map $j^{(n)} : M_g^{(n)} \to A_g^{(n)}$ extends to a period map $\overline{j} : M_g^{(n)} \to A_g^{(n)}$ so we can consider the pull-back $\overline{j}^*([\tilde{\omega}]) \in H^2(M_g^{(n)}, \mathbb{C})$. Moreover, since the image of $\overline{j}$ is not contained in the locus where the current is singular, the pullback $\omega := \overline{j}^* (\tilde{\omega})$ is a well defined closed current and $[\omega] = \overline{j} ([\tilde{\omega}])$ (cf. [13]). Moreover it gives a closed current on $\overline{M}_g$ still denoted by $\omega$. Observe that $\overline{j}^*(\lambda) = \lambda$ so $[\omega] = c\lambda$ in $H^2(M_g^{(n)}, \mathbb{C})$, hence in $H^2(\overline{M}_g, \mathbb{C})$.

In order to compute the constant $c$, we use the cycles introduced by Wolpert in ([22]). In our case, since $[\omega]$ is a multiple of $\lambda$, it is sufficient to compute the value of $[\omega]$ on the 1-dimensional family given by a varying 1-pointed elliptic curve attached to a fixed $g−1$ curve with 1 marked point $\mathcal{E}_i$ of ([22] (2.2)). More precisely, let us denote by $H := \{ z \in \mathbb{C} \mid Im(z) > 0\}$, by $\Gamma := SL(2, \mathbb{Z})$, and by

$$\Gamma_l = \left\{ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma \mid \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \equiv \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) \mod l \right\}.$$ 

Since the $g−1$ curve in $\mathcal{E}_i$ is constant we identify $\mathcal{E}_i$ with the curve $H/\Gamma_l = A_1^{(l)}$. We have then to compute

$$\int_{H/\Gamma_l} \omega = [\Gamma : \Gamma_l] \int_{H/\Gamma} \omega.$$ 

Set $E_z := \mathbb{C}/(\mathbb{Z} \oplus z\mathbb{Z})$, where $z \in H/\Gamma$, let $\xi_z$ be a holomorphic coordinate on $E_z$, so that $H^{1,0}(E_z) = \langle d\xi_z \rangle$. Then a cotangent direction to the curve $\mathcal{E}_i$ can be identified with $d\xi_z \otimes d\xi_z$ and we have: $\langle d\xi_z \otimes d\xi_z, d\xi_z \otimes d\xi_z \rangle = 2\langle d\xi_z, d\xi_z \rangle^2,$

$$\langle d\xi_z, d\xi_z \rangle = i \int_{E_z} d\xi_z \wedge \overline{d\xi_z} = 2Im(z).$$ 

Then

$$\langle [\omega], \mathcal{E}_i \rangle = \int_{H/\Gamma_l} \omega = i[\Gamma : \Gamma_l] \int_{D} \frac{1}{2s(Im(z))^2} (dz \wedge \overline{dz}) = [\Gamma : \Gamma_l] \frac{\pi}{12},$$ 

where $D$ is the fundamental domain of the action of $\Gamma$ on $H$ and the last equality is a standard integral computation.

Since one has $\langle \lambda, \overline{\mathcal{E}_i} \rangle = \frac{\pi}{12}$, we have

$$\frac{\pi}{12} = \langle \xi, \mathcal{E}_i \rangle = \langle c\lambda, \overline{\mathcal{E}_i} \rangle = c \frac{1}{12},$$ 

we obtain $c = \pi$, so finally $[\omega] = \pi\lambda$. 

$\Box$
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