Subtractive Renormalization of Strong-Coupling Quenched QED in Four Dimensions

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We study renormalized quenched strong-coupling QED in four dimensions in arbitrary covariant gauge, in the Dyson-Schwinger equation formalism. Above the chiral critical coupling, we show that there is no finite chiral limit. This behaviour is found to be independent of the detailed choice of proper vertex, provided that the vertex is consistent with the Ward-Takahashi identity and multiplicative renormalizability. The finite solutions previously reported lie in an unphysical regime of the theory with multiple solutions and oscillating mass functions. This study is consistent with the assertion that strong coupling QED$_4$ does not have a continuum limit in the conventional sense.

1 Introduction

The abelian nature of quantum electrodynamics (QED) in many ways makes it a much simpler system to study than a nonabelian theory such as quantum chromodynamics (QCD). For this reason it has been the subject of many nonperturbative studies, which have as their long-term goal a detailed understanding of nonperturbative QCD. On the other hand, strong-coupling QED$_4$ is widely anticipated to behave unconventionally in the continuum limit and for this reason is a theory of considerable interest in its own right.

In previous work we introduced a numerical renormalization procedure for the Dyson-Schwinger equations, and applied it to QED$_4$ with a quenched photon propagator, under momentum cutoff regularization. The central result of those works was to demonstrate that the numerical renormalization procedure works extremely well and allows the continuum limit (cutoff $\Lambda \to \infty$) to be taken numerically, while giving rise to stable finite solutions for the renormalized fermion propagator.

Here we describe studies of the chiral limit in renormalized quenched strong-coupling QED$_4$ using a photon-fermion vertex that satisfies the Ward-
Takahashi Identity (WTI) and makes the fermion DSE multiplicatively renormalizable. We find that for couplings above the chiral critical coupling, keeping the bare mass $m_0(\Lambda) \equiv 0$ as the cutoff is relaxed results in either a dynamical mass function which is identically zero, or a dynamical mass function which diverges with the cutoff. The previously reported finite solutions showed damped oscillations in the dynamical mass functions at large $p^2$, which suggested that they were unphysical. Further, we show here that for a given supercritical coupling and the same bare mass $m_0(\Lambda)$, it is possible to have multiple solutions corresponding to different renormalized masses $m(\mu)$. We conclude that above the chiral phase transition, quenched strong-coupling QED in four dimensions does not have a chiral limit in the conventional sense.

2 Formalism

We write the fermion propagator as
\[ S(p) = \frac{Z(p^2)}{\not{p} - M(p^2)} = \frac{1}{A(p^2)} \frac{\not{p} - B(p^2)}{\not{p} - M(p^2)}, \]
where we refer to $A(p^2) \equiv 1/Z(p^2)$ as the finite momentum-dependent fermion renormalization and where $M(p^2) \equiv B(p^2)/A(p^2)$ is the fermion mass function. Dynamical chiral symmetry breaking (DCSB) occurs when the fermion propagator develops a nonzero scalar self-energy in the absence of an explicit chiral symmetry breaking (ECSB) fermion mass. We will refer to coupling constants strong enough to induce DCSB as supercritical and those weaker as subcritical.

The DSE for the renormalized fermion propagator, in an arbitrary covariant gauge, is
\[ S^{-1}(p) = Z_2(\mu, \Lambda)[\not{p} - m_0(\Lambda)] - iZ_3(\mu, \Lambda)e^2 \int_{\Lambda} \frac{d^4k}{(2\pi)^4} \gamma^\mu S(k)\Gamma^\nu(k, p)D_{\mu\nu}(q); \]
here $q = k - p$ is the photon momentum, $\mu$ is the renormalization point, and $\Lambda$ is a regularizing parameter (taken here to be an ultraviolet momentum cutoff), and $m_0(\Lambda)$ is the regularization-parameter dependent bare mass. We work in the quenched approximation, i.e., the photon propagator is bare, so that for the coupling strength and gauge parameter we have $\alpha \equiv e^2/4\pi = \alpha_0 \equiv e_0^2/4\pi$ and $\xi = \xi_0$.

Among other general requirements, the vertex Ansatz is constrained by the Ward-Takahashi Identity (WTI), which is necessary for gauge invariance and guarantees the equality of the propagator and vertex renormalization
constants, \( Z_2 \equiv Z_1 \). Ball and Chiu\(^\square\) have described the most general fermion-photon vertex that satisfies the WTI; it consists of a longitudinally-constrained (i.e., “Ball-Chiu”) part \( \Gamma^\mu_{BC} \); which is a minimal solution of the WTI with no artificial kinematic singularities, and a set of eight vectors \( T^\mu_i(k, p) \), which span the transverse subspace and are also kinematically regular as \( k \to p \) (improved versions of the \( T^\mu_i \) can be found in Ref.\(^9\)). A general vertex is then written as \( \Gamma^\mu(k, p) = \Gamma^\mu_{BC}(k, p) + \sum_{i=1}^8 \tau_i(k^2, p^2, q^2) T^\mu_i(k, p) \), where the \( \tau_i \) are functions that must be chosen to ensure the correct discrete symmetry properties, perturbative limits, etc. For our numerical examples we have used the Curtis-Pennington vertex\(^10\), however our results and conclusions regarding the chiral limit will not depend on the specific vertex \textit{Ansatz} as long as it respects multiplicative renormalizability.

The renormalization procedure is entirely analogous to one-loop perturbative renormalization: One first determines a finite, regularized self-energy, \( \Sigma'(\mu, \Lambda; p) \), depending on both the regularization parameter \( \Lambda \) and the renormalization point \( \mu \). The DSE for the renormalized fermion propagator can be written in terms of \( \Sigma'(\mu, \Lambda; p) \) as

\[
S^{-1}(p) = Z_2(\mu, \Lambda)[\not{p} - m_0(\Lambda)] - \Sigma'(\mu, \Lambda; p) = \not{p} - m(\mu) - \tilde{\Sigma}(\mu; p) = A(p^2) \not{p} - B(p^2) ,
\]

where \( \tilde{\Sigma}(\mu; p) \) denotes the renormalized self-energy. The self-energies are decomposed into Dirac and scalar parts, \( \Sigma'(\mu, \Lambda; p) = \Sigma'_d(\mu, \Lambda; p^2) \not{p} + \Sigma'_s(\mu, \Lambda; p^2) \), (and similarly for \( \tilde{\Sigma}(\mu, p) \)). By imposing the renormalization boundary condition,

\[
S^{-1}(p)\bigg|_{p^2 = \mu^2} = \not{p} - m(\mu) ,
\]

one finds relations for the scalar and Dirac self-energies, propagator renormalization constant \( Z_2(\mu, \Lambda) \), and bare mass. A mass renormalization constant can also be defined as \( Z_m(\mu, \Lambda) = m_0(\Lambda)/m(\mu) \), i.e., as the ratio of the bare to renormalized mass.

We approach the chiral limit by setting the bare mass to zero and removing the regularization, i.e., maintaining \( m_0(\Lambda) = 0 \) while taking the limit \( \Lambda \to \infty \); this is the same definition used in nonperturbative studies of QCD\(^1\). Obviously, any limiting procedure where we take \( m_0(\Lambda) \to 0 \) sufficiently rapidly as \( \Lambda \to \infty \) will also lead to the chiral limit.

The behavior of the solutions under a renormalization point transformation is governed by a “renormalization group flow”: under any transformation \( \mu \to \mu' \), we must have for all \( p^2 \)

\[
M(\mu'; p^2) = M(\mu; p^2) \equiv M(p^2) ,
\]
\[
\frac{A(\mu'; p^2)}{A(\mu; p^2)} = \frac{Z_2(\mu', \Lambda)}{Z_2(\mu, \Lambda)} = A(\mu'; \mu^2) = \frac{1}{A(\mu; \mu'^2)}, \quad (5)
\]
so that, for the fermion propagator, \(S(\mu'; p)/S(\mu; p) = Z_2(\mu, \Lambda)/Z_2(\mu', \Lambda)\) in the usual way. We have verified Eq. (5) to within 1 part in 10\(^4\) for our calculations.

The DSE is solved in Euclidean momentum-space, after separation into Dirac-odd and -even parts describing the finite renormalization \(A(p^2)\) and the scalar self-energy \(B(p^2)\). We use a "gauge-covariance-improved" momentum cutoff scheme, which is described in detail in Appendix A of Ref. 5. Choosing the renormalized mass, \(m(\mu)\), and then solving for the bare mass, \(m_0(\Lambda)\), leads to well-behaved finite solutions for \(A(p^2)\) and \(M(p^2)\), which do not vary as we take the continuum limit (\(\Lambda \to \infty\)).

\section{Results and Conclusions}

Figure 1: The behaviour of the finite renormalization \(A(p^2)\) and the mass function \(M(p^2)\) as a function of the ultraviolet cut-off \(\Lambda\) for \(m_0(\Lambda) = 0\). These solutions were for renormalization point \(\mu^2 = 10^4\), coupling \(\alpha = 1.15\), and gauge parameter \(\xi = 0.25\). As \(\Lambda \to \infty\) we find \(A(p^2) \to 1\) for all \(p^2\) and \(M(p^2)\) diverges proportionally to \(\Lambda\). Conversely, for subcritical coupling (i.e., \(\alpha < \alpha_c\)), the chiral limit does exist and we find simply that \(A(\mu; p^2) = (p^2/\mu^2)^{-\alpha \xi/4\pi}\). It seems clear from these numerical studies that above critical...
coupling there is no finite chiral limit in the continuum quenched theory in any covariant gauge, even in the presence of the renormalization procedure.

While these conclusions were based on a numerical study with a specific choice of vertex, one can easily construct a general argument which applies irrespective of this choice: Consider any vertex $\Gamma$ which satisfies the WTI and which leads to multiplicative renormalizability. It automatically follows that $M(p^2)$ and $A(p^2)/Z_2(\mu, \Lambda)$ are renormalization point independent, as in Eq. (6). We can then define dimensionless quantities by appropriately scaling with $\Lambda$, i.e., $\hat{\mu} \equiv \mu/\Lambda$, $\hat{p}^2 \equiv p^2/\Lambda^2$, $\hat{M}(\hat{p}^2) \equiv M(p^2)/\Lambda$, $\hat{A}(\hat{p}^2) \equiv A(\mu; p^2)/Z_2(\mu, \Lambda)$, and $\hat{m}_0 \equiv m_0(\Lambda)/\Lambda$. Note that the renormalization condition $A(\mu; \mu^2) = 1$ automatically determines $Z_2(\mu, \Lambda)$ for a given solution for fixed $\Lambda$. The dimensionless functions $\hat{M}(\hat{p}^2)$ and $\hat{A}(\hat{p}^2)$ can only depend on dimensionless parameters, i.e., $\alpha$, $\xi$, $\hat{\mu}$, and $\hat{m}_0$. Furthermore, since for any fixed $\Lambda$ they are independent of $\mu$ (recall that we are working only in the quenched approximation), then it follows that they must in turn be independent of $\hat{\mu}$. Now for any finite $\Lambda$ and the choice $m_0(\Lambda) = 0$ we have $\hat{m}_0 = 0$. Hence solving for any $\mu$ and $\Lambda$ with $m_0(\Lambda) = 0$ allows us to form $\hat{M}(\hat{p}^2)$ and $\hat{A}(\hat{p}^2)$, from which we can read off the solutions for $A(p^2)$ and $M(p^2)$ for any other $\mu$ and $\Lambda$ with vanishing bare mass using the above rules. We see then that the resulting $M(p^2)$ must diverge with $\Lambda$. So if $M(p^2) \neq 0$, it must diverge with $\Lambda$ as was found numerically. To summarize, above critical coupling any vertex which satisfies the WTI and leads to multiplicative renormalizability will lead to a diverging mass function in the chiral limit for quenched QED in four dimensions.

It remains to discuss the meaning of the well-behaved finite solutions. Our conventional thinking about the chiral limit would imply that since ECSB should only increase the mass function above that found in its absence, then any solution above critical coupling which also has ECSB must also correspond to a divergent mass in the continuum limit. Thus the finite solutions do not correspond to the chiral limit nor to any conventional concept of ECSB. Further, the finite solutions show damped oscillations in the dynamical mass function, periodic in $\ln(p^2)$, and a corresponding oscillatory behavior for the bare mass. Thus it is possible for a given set of parameters $\alpha$, $m_0(\Lambda)$, $\mu$, $\xi$, and $\Lambda$ to admit more than one finite solution. We explicitly show this in Figs. 2 and 3, where for a given set $\alpha$, $m_0(\Lambda)$, $\mu$, $\xi$, and $\Lambda$ there are distinct solutions. As $\Lambda \to \infty$ the number of simultaneous solutions becomes infinite. It seems likely that this behavior for the finite solutions is independent of the detailed vertex choice and furthermore that the same behavior can be induced in QCD by forcing the renormalized mass into an unphysical regime (by choosing $m(\mu) \equiv M(\mu^2)$ below the value corresponding to the chiral limit). It also seems likely that unquenching the theory will not remove this rather undesirable behaviour in
Figure 2: The relationship between the bare mass ($m_0(\Lambda)$) and the renormalized mass ($m(\mu)$) for the renormalized finite solutions. These result from solving for a given $m(\mu)$ and extracting $m_0(\Lambda)$. The parameters for these solutions were the renormalization point $\mu^2 = 10^4$, $\alpha = 1.25$, gauge parameter $\xi = 0.25$ and $\Lambda^2 = 10^{14}$. The dashed horizontal line shows, e.g., that for $m_0(\Lambda) = 0.1$ there are three solutions. (The dashed vertical lines are merely to guide the eye on this back-to-back log scale.)

QED$_4$, since the running coupling increases with scale rather than decreasing as in QCD. These latter conjectures are the subject of current investigation$^{12}$. 

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Figure 3: The three renormalized finite solutions corresponding to the same bare mass, i.e., $m_0(\Lambda) = 0.1$, as indicated in Fig. 2. The other parameters for these solutions were renormalization point $\mu^2 = 10^4$, coupling $\alpha = 1.25$, gauge parameter $\xi = 0.25$ and $\Lambda^2 = 10^{14}$. (The short-long dashed vertical lines connecting the upper and lower curves are merely to guide the eye on this back-to-back log scale.)

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