ASYMPTOTICS OF THE STRESS CONCENTRATION IN HIGH-CONTRAST ELASTIC COMPOSITES

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Abstract. A long-standing area of materials science research has been the study of electrostatic, magnetic, and elastic fields in composite with densely packed inclusions whose material properties differ from that of the background. For a general elliptic system, when the coefficients are piecewise Hölder continuous and uniformly bounded, an $\varepsilon$-independent bound of the gradient was obtained by Li and Nirenberg [40], where $\varepsilon$ represents the distance between the interfacial surfaces. However, in high-contrast composites, when $\varepsilon$ tends to zero, the stress always concentrates in the narrow regions. As a contrast to the uniform boundedness result of Li and Nirenberg, in order to investigate the role of $\varepsilon$ played in such kind of concentration phenomenon, in this paper we establish the blow-up asymptotic expressions of the gradients of solutions to the Lamé system with partially infinite coefficients in dimensions two and three. We discover the relationship between the blow-up rate of the stress and the relative convexity of adjacent surfaces, and find a family of blow-up factor matrices with respect to the boundary data. Therefore, this work completely solves the Babuška problem on blow-up analysis of stress concentration in high-contrast composite media. Moreover, as a byproduct, we establish an extended Flaherty-Keller formula on the effective elastic property of a periodic composite with densely packed fibers, which is related to the “Vigdergauz microstructure” in the shape optimization of fibers.

1. Introduction

In this paper we are concerned with the blow-up behavior of the gradients of solutions to a class of elliptic systems, stimulated by the study of composite media with closely spaced interfacial boundaries. It is a long-standing area of material science research to study the high concentration of electrostatics, magnetic, and elastic fields in high-contrast composites with densely packed inclusions since the time of Maxwell and Reyleigh. This requires an understanding of micro-structural effects, especially from the distances (say, $\varepsilon$) between inclusions, because when the inclusions are close to touching, the charge density becomes nearly singular. To evaluate the electrostatic fields (where the potential function is scalar-valued), the potential theory, Fourier analysis, and numerical method have been fully developed. While, for the elastic field (where the deformation displacement is vector-valued), in order to predict damage initiation and growth in carbon-fiber epoxy composites at the fiber scale level, Babuška, et al. [8] assumed the systems of linear elasticity

$$\mathcal{L}_{\lambda,\mu} u = \mu \Delta u + (\lambda + \mu) \nabla (\nabla \cdot u)$$
in unidirectional composites to numerically analyze the residual stresses and stresses due to mechanical loads, where \( u = (u^1, u^2, u^3)^T \) expresses the displacement. Obviously, this multiscale problem need more rigorous mathematical treatment and numerical analysis to control the errors of the analysis. On the other hand, we emphasize that there is a significant difficulty in applying the method developed for scalar equations to systems of equations. For instance, the maximum principle does not hold for the Lamé system. Due to these difficulties on PDE theory and numerical analysis as well as the importance in practical applications, it arouses great interest of many applied mathematicians and engineers. In the last last two decades, there has been an extensive study on the gradient estimates of solutions to elliptic equations and systems with discontinuous coefficients, to show whether the stress remains bounded or blows up when inclusions touch or nearly touch.

Bonnetier and Vogelius considered the elliptic equation with piecewise constant coefficients in dimension two

\[
\nabla (a_k(x) \nabla u) = 0 \quad \text{in } D,
\]

where the scalar \( u \) is the out of plane displacement, \( D \) represents the cross-section of a fiber-reinforced composite taken perpendicular to the fibers, containing a finite number of inhomogeneities, which are very closely spaced and may possibly touch. The coefficients \( 0 < a_k(x) < \infty \) take two different constant values, after rescaling,

- \( a_k(x) = k \) for \( x \) inside the cross-sections of the fibers,
- \( a_k(x) = 1 \) elsewhere in \( D \).

Despite the discontinuity of the coefficient along the interfaces, they proved that any variational solution \( u \) is in \( W^{1,\infty} \), which actually improves a classical regularity result due to De Giorgi and Nash, which asserts that \( H^1 \) solution is in some Hölder class. A general result was established by Li and Vogelius for a class of divergence form elliptic equations with piecewise Hölder continuous coefficients. They obtained a uniform bound of \( |\nabla u| \) regardless of \( \varepsilon \) in any dimension \( d \geq 2 \). Li and Nirenberg extended the results in to general elliptic systems including systems of elasticity. This, in particular, answered in the affirmative the question that is naturally led to by the above mentioned numerical indication in for the boundedness of the stress as \( \varepsilon \) tends to zero. Dong and Xu further showed that a \( W^{1,1} \) weak solution is Lipschitz and piecewise \( C^1 \). Recently, Dong and Li used an image charge method to construct a Green’s function for two adjacent circular inclusions and obtained more interesting higher-order derivative estimates for non-homogeneous equations making clear their specific dependence on \( k \) and \( \varepsilon \) exactly. But for more general elliptic equations and systems, and more general shape of inclusions, it is still an open problem to estimate higher-order derivatives in any dimension. We draw the attention of readers to the open problem on page 894 of.

As mentioned above, the concentration of the stresses is greatly influenced by the thickness of the ligament between inclusions. To figure out the influence from this thickness \( \varepsilon \), one assumes that the material parameters of the inclusions degenerate to infinity. However, this makes the situation become quite different. As a matter of fact, in 1960’s, in the context of electrostatics, Keller computed the effective electrical conductivity for a composite medium consisting of a dense cubic array of identical perfectly conducting spheres (that is, \( k \) degenerates
to $\infty$) imbedded in a matrix medium and first discovered that it becomes infinite when sphere inclusions touch each other. Keller found that this singularity is not contained in the expressions given by Maxwell, and by Meredith and Tobias [51]. See also Budiansky and Carrier [16], and Markenscoff [47]. Rigorous proofs were later carried out by Ammari et al. [6, 7] for the case of circular inclusions by using layer potential method, together with the maximum principle. Since then, there is a long list of literature in this direction of research, for example, see [1,3,12–14,19,24–26,29,34,36–38,43,45,46,54,55] and the references therein. It is proved that the blow-up rate of $|\nabla u|$ is $\varepsilon^{-1/2}$ in dimension two and $|\varepsilon \log \varepsilon|^{-1}$ in dimension three. From the perspective of practical application in engineering and the requirement of numerical algorithm design, it is more interesting and important to characterize the singular behavior of $\nabla u$, see [2,30,31,39,41,44].

In the context of linear elasticity, for Lamé system with partially infinite coefficients, by building an iteration technique with respect to the energy, the first author and his collaborators overcame the difficulty caused by the lack of maximum principle, obtained the upper and lower pointwise bounds of $|\nabla u|$, and showed that $|\nabla u|$ may blow up on the shortest line between two adjacent inclusions, see [9–11, 35]. By using the polynomial function $x_d = |x'|^m$, $m \geq 2$, as a local expression of inclusion’s boundary to measure its order of convexity, Li and Hou [27] revealed the relationship between the blow-up rate of $|\nabla u|$ and the convexity order $m$. However, under the same logic as in the electrostatics problem, what one cares more about in practical applications is how to obtain an asymptotic formula to characterize the singular behaviour of $\nabla u$ in the whole narrow region between two adjacent inclusions. The main contribution of the paper is that we completely solve this problem in two physically relevant dimensions $d = 2$ and $3$, and for all $m \geq 2$. For $d \geq 4$, the result is similar. Our asymptotic expressions of the gradients of solutions not only show the optimality of the blow-up rates, which depend only on the dimension $d$ and the convexity order $m$ of the inclusions, but also provide a family of blow-up factor matrices, which are linear functionals of boundary value data, determining whether or not blow-up occurs. Notice that when $m > 2$, the curvature of the inclusions vanishes at the two nearly touch points, so in general we can not use a spherical inclusion to approximate an $m$-convex inclusion.

The asymptotic formulas obtained above clearly reflect the local property of $\nabla u$. Beyond this, they can further influence the global property of a composite. For the effective elastic moduli of a composite, Flaherty and Keller [22] obtained an asymptotic formula for a retangular array of cylinders ($m = 2$) in the nearly touching, when the cylinders are hard inclusions and showed their validity numerically. As an application of the above local asymptotic formulas, we give an extended Flaherty-Keller formula for $m$-convex inclusions, which is also related to the “Vigdergauz microstructure” [53], having a large volume fraction in the theory of structure optimization, see Grabovsky and Kohn [23].

To end this introduction, we make some comments on the corresponding numerical problem. Accurate numerical computation of the gradient in the present of closely spaced inclusions is also a well-known challenging problem in computational mathematics and sciences. Here it should be noted that Lord Rayleigh, in his classic paper [52], use Fourier approach to determine the effective conductivity of a composite material consisting of a periodic array of disks in a uniform background. In the case that inclusions are reasonably well separated or have conductivities close
to that of the background, Rayleigh’s method gives excellent result. Unfortunately, if the inclusions are close to touching and their conductivities differ greatly from that of the background, the charge density becomes nearly singular and the number of computation degrees of freedom required extremely large. Recently, a hybrid numerical method was developed by Cheng and Greengard [18], and Cheng [17]. Related works can be referred to Kang, Lim, and Yun [30], and McPhedran, Poladian, and Milton [50]. For high-contrast elastic composite, a serious difficulty arises in applying the methods for scalar equations to systems of equations. We expect our asymptotic formulas of $\nabla u$ in the narrow regions, the most difficult areas to deal with, can open up a way to do some computation for inclusions of arbitrary shape.

This paper consists of eight sections including introduction. In Section 2, we first fix our domain and formulate the problem with partially infinite coefficients, and then introduce a family of vector-valued auxiliary functions with several preliminary estimates including the main ingredient Proposition 2.3 for the asymptotics of $a_{\alpha\alpha}^{\alpha \alpha}$, $\alpha = 1, \cdots, d$. In Section 3, a family of the blow-up factors is defined. Then our main results are stated. Theorem 3.2 and Theorem 3.4 are for 2-convex inclusions in 2D and 3D, respectively, Theorem 3.6 and Theorem 3.9 are for $m$-convex inclusions. Finally we give an example to show the dependence on the precise geometry feature of $D_1$ and $D_2$. In Section 4, we prove the two important ingredients Propositions 2.3 and 3.1 where two improved estimates, Theorem 2.2 and Theorem 2.6 are used. The proofs of Theorems 3.2 and 3.4 are given in Section 5. We prove Theorems 3.6 and 3.9 in Section 6. Finally, by applying the local asymptotic formulas established in the previous sections, we give an extended Flaherty-Keller formula in Section 7. The proof of Theorem 2.2 and Theorem 2.6 is given in Appendix.

2. Problem formulation and auxiliary functions

In this section we first fix our notations and formulate the problems, then identify the key difficulties and present the strategy to solve them, and finally introduce our auxiliary functions, involving the parameters of Lamé system, and give some preliminary results.

2.1. Problem formulation. Because the aim of this paper is to study the asymptotic behavior of $\nabla u$ in the narrow region between two adjacent inclusions, we may without loss of generality restrict our attention to a situation with only two adjacent inclusions. The basic notations used in this paper follow from [10].

We use $x = (x', x_d)$ to denote a point in $\mathbb{R}^d$, where $x' = (x_1, \cdots, x_{d-1})$ and $d = 2, 3$. Let $D$ be a bounded open set in $\mathbb{R}^d$ with $C^2$ boundary. $D_1$ and $D_2$ are two disjoint convex open subsets in $D$ with $C^{2, \gamma}$ ($0 < \gamma < 1$) boundaries, $\varepsilon$-apart, and far away from $\partial D$. That is,

$$D_1, D_2 \subset D, \quad \varepsilon := \text{dist}(D_1, D_2) > 0, \quad \text{dist}(D_1 \cup D_2, \partial D) > \kappa_0 > 0,$$

where $\kappa_0$ is a constant independent of $\varepsilon$. We also assume that the $C^{2, \gamma}$ norms of $\partial D_1$, $\partial D_2$, and $\partial D$ are bounded by some positive constant independent of $\varepsilon$. Set

$$\Omega := D \setminus (D_1 \cup D_2).$$

Assume that $\Omega$ and $D_1 \cup D_2$ are occupied, respectively, by two different isotropic and homogeneous materials with different Lamé constants $(\lambda, \mu)$ and $(\lambda_1, \mu_1)$. Then
the elasticity tensors for the background and the inclusion can be written, respectively, as $C^0$ and $C^1$, with

$$C^0_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}),$$

and

$$C^1_{ijkl} = \lambda_1 \delta_{ij} \delta_{kl} + \mu_1 (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}),$$

where $i, j, k, l = 1, 2, \ldots, d$ and $\delta_{ij}$ is the kronecker symbol: $\delta_{ij} = 0$ for $i \neq j$, $\delta_{ij} = 1$ for $i = j$. Let $u = (u^1, u^2, \ldots, u^d)^T : D \to \mathbb{R}^d$ denote the displacement field. For a given vector-valued function $\varphi = (\varphi^1, \varphi^2, \ldots, \varphi^d)^T$, we consider the following Dirichlet problem for the Lamé system with piecewise constant coefficients:

$$\begin{align*}
\nabla \cdot ((\chi_D^0 + \chi_{D_1 \cup D_2}^1) e(u)) &= 0, & \text{in } D, \\
u = \varphi, & \text{on } \partial D,
\end{align*}$$

(2.1)

where $\chi_D$ is the characteristic function of $\Omega \subset \mathbb{R}^d$, and

$$e(u) = \frac{1}{2} (\nabla u + (\nabla u)^T)$$

is the strain tensor.

Assume that the standard ellipticity condition holds for (2.1), that is,

$$\mu > 0, \quad d\lambda + 2\mu > 0, \quad \mu_1 > 0, \quad d\lambda_1 + 2\mu_1 > 0.$$ 

For $\varphi \in C^{1,\gamma}(\partial D; \mathbb{R}^d)$, it is well known that there exists a unique solution $u \in H^1(D; \mathbb{R}^d)$ to the Dirichlet problem (2.1), which is also the minimizer of the energy functional

$$J_1[u] := \frac{1}{2} \int_{\Omega} ((\chi_D^0 + \chi_{D_1 \cup D_2}^1) e(u), e(u)) \, dx$$

on

$$H_0^1(D; \mathbb{R}^d) := \{ u \in H^1(D; \mathbb{R}^d) \mid u - \varphi \in H_0^1(D; \mathbb{R}^d) \}.$$

As mentioned previously, Li and Nirenberg [40] proved that $\nabla u$ is uniformly bounded with respect to $\varepsilon$. But, in high-contrast composite media, the concentration of $\nabla u$ is a very usual phenomenon when the distance $\varepsilon$ is sufficiently small. In order to investigate the role of $\varepsilon$ in such concentration phenomenon, let us assume that the Lamé constant in $D_1 \cup D_2$ degenerates to infinite and consider this extreme case. To this end, we first introduce the linear space of rigid displacement in $\mathbb{R}^d$:

$$\Psi := \{ \psi \in C^1(\mathbb{R}^d, \mathbb{R}^d) \mid \nabla \psi + (\nabla \psi)^T = 0 \},$$

with a basis $\{ \psi_\alpha \mid \alpha = 1, 2, \ldots, \frac{d(d+1)}{2} \}$, namely,

$$\{ e_i, x_j e_k - x_k e_j \mid 1 \leq i \leq d, 1 \leq j < k \leq d \},$$

where $e_1, \ldots, e_d$ denote the standard basis of $\mathbb{R}^d$. For fixed $\lambda$ and $\mu$, denoting $u_{\lambda_1, \mu_1}$ as the solution of (2.1), then we have [10]

$$u_{\lambda_1, \mu_1} \to u \quad \text{in } H^1(D; \mathbb{R}^d), \quad \text{as } \min\{\mu_1, d\lambda_1 + 2\mu_1\} \to \infty,$$
where $u$ is the unique $H^1(D; \mathbb{R}^d)$ solution of

$$
\begin{align*}
\mathcal{L}_{\lambda, \mu} u &:= \nabla \cdot (\mathbb{C}^{0}(e(u))) = 0, & \text{in } \Omega, \\
u_i^+ &= u_i^- , & \text{on } \partial D_i, i = 1, 2, \\
e(u) &= 0, & \text{in } D_i, i = 1, 2, \\
\int_{\partial D_i} \frac{\partial u}{\partial \nu} + \psi \alpha &= 0, & i = 1, 2, \alpha = 1, 2, \ldots, \frac{d(d+1)}{2}, \\
u &= \varphi, & \text{on } \partial D,
\end{align*}
$$

(2.2)

where

$$
\frac{\partial u}{\partial \nu} := (\mathbb{C}^{0}(e(u)))\vec{n} = \lambda(\nabla \cdot u)\vec{n} + \mu(\nabla u + (\nabla u)^T)\vec{n},
$$

and $\vec{n}$ is the unit outer normal of $D_i$, $i = 1, 2$. Here and throughout this paper the subscript $\pm$ indicates the limit from outside and inside the domain, respectively. The existence, uniqueness and regularity of weak solutions to (2.2) can be referred to the Appendix of [10]. We note that it suffices to consider the problem (2.2) with $\varphi \in C^0(\partial D; \mathbb{R}^d)$ replaced by $\varphi \in C^{1, \gamma}(\partial D; \mathbb{R}^d)$. Indeed, it follows from the maximum principle [48] that $\|u\|_{L^\infty(D)} \leq C\|\varphi\|_{C^{1, \gamma}(\partial D)}$. Taking a slightly small domain $\hat{D} \subset \subset D$, then in view of the interior derivative estimates for Lamé system, we find that $\hat{\varphi} := u|_{\partial \hat{D}}$ satisfies

$$
\|\hat{\varphi}\|_{C^{1, \gamma}(\partial \hat{D})} \leq C\|u\|_{L^\infty(D)} \leq C\|\varphi\|_{C^{1, \gamma}(\partial D)}.
$$

Without loss of generality, we assume that $\|\varphi\|_{C^{1, \gamma}(\partial D)} = 1$ by considering $u/\|\varphi\|_{C^{1, \gamma}(\partial D)}$ if $\|\varphi\|_{C^{1, \gamma}(\partial D)} > 0$. If $\varphi|_{\partial D} = 0$, then $u \equiv 0$.

2.2. Main difficulties and the strategy. We first point out that problem (2.2) has free boundary value feature. Although $e(u) = 0$ implies $u$ in $\hat{D}_i$ is linear combination of $\psi_\alpha$,

$$
u = \sum_{\alpha=1}^{\frac{d(d+1)}{2}} C_i^\alpha \psi_\alpha \quad \text{in } \hat{D}_i,
$$

these $d(d+1)$ constants $C_i^\alpha$ are free, which will be uniquely determined by $u$. We would like to emphasize that this is exactly the biggest difference with the conductivity model [12], where only two free constants need us to handle in any dimension. It is the increase of the number of free constants that makes elastic problem quite difficult to deal with. Therefore, how to determine such many constants is one of main difficulties we need to solve.

The strategy is as follows. First, by continuity of $u$ across $\partial D_i$, we can decompose the solution of (2.2), as in [10].

$$
u(x) = \sum_{i=1}^{2} \sum_{\alpha=1}^{\frac{d(d+1)}{2}} C_i^\alpha v_i^\alpha(x) + v_0(x), \quad x \in \Omega, \quad (2.3)
$$

where $v_i^\alpha, v_0 \in C^2(\Omega; \mathbb{R}^d)$, respectively, satisfying

$$
\begin{align*}
\mathcal{L}_{\lambda, \mu} v_i^\alpha &= 0, & \text{in } \Omega, \\
v_i^\alpha &= \psi_\alpha, & \text{on } \partial D_i, i = 1, 2, \alpha = 1, \ldots, d(d+1)/2, \\
v_i^\alpha &= 0, & \text{on } \partial D_j \cup \partial D, j \neq i,
\end{align*}
$$

(2.4)
By integration by parts, and
\[
\begin{aligned}
L_{\lambda, \mu} v_0 = 0, & \quad \text{in } \Omega, \\
v_0 = 0, & \quad \text{on } \partial D_1 \cup \partial D_2, \\
v_0 = \varphi, & \quad \text{on } \partial D.
\end{aligned}
\]

So
\[
\nabla u(x) = \sum_{i=1}^{2} \sum_{\alpha=1}^{d(d+1)/2} C_i^\alpha \nabla v_i^\alpha (x) + \nabla v_0 (x), \quad x \in \Omega.
\]

To investigate the asymptotic behavior of \( \nabla u \), we need both the asymptotic formulas of \( \nabla v_i^\alpha \) and the exact value of \( C_i^\alpha \). To solve \( C_i^\alpha \), from the fourth line in (2.2) and the decomposition (2.3), we have the following linear system of these free constants \( C_i^\alpha \),

\[
\sum_{i=1}^{2} \sum_{\alpha=1}^{d(d+1)/2} C_i^\alpha \int_{\partial D_j} \frac{\partial v_i^\alpha}{\partial \nu} \cdot \psi_j + \int_{\partial D_j} \frac{\partial v_0}{\partial \nu} \cdot \psi_j = 0,
\]

where \( j = 1, 2, \beta = 1, \ldots, \frac{d(d+1)}{2} \). But these coefficients are all boundary integrals. By integration by parts,

\[
a_{ij}^{\alpha \beta} := - \int_{\partial D_j} \frac{\partial v_i^\alpha}{\partial \nu} \cdot \psi_j = \int_{\Omega} \left( C_i^\alpha e(v_i^\alpha), e(v_j^\beta) \right) \ dx.
\]

Therefore, in order to solve \( C_i^\alpha \) from (2.6), we have to calculate the energy integral on the right hand side of (2.7). This in turn needs exact asymptotics of \( \nabla v_i^\alpha \). In fact, even if we can have the asymptotic formulas of \( \nabla v_i^\alpha \), it is still hard to solve every \( C_i^\alpha \). To this end, the following theorem can make it a little bit better to handle.

Because \( (v_1^\alpha + v_2^\beta) \) and \( v_0 \) have no displacement difference on \( \partial D_1 \) and \( \partial D_2 \), we have

**Theorem 2.1**. Let \( v_i^\alpha \) and \( v_0 \) be defined by (2.4) and (2.5), respectively, \( i = 1, 2 \). Then we have

\[
\| \nabla (v_1^\alpha + v_2^\beta) \|_{L^\infty(\Omega)} \leq C, \quad \alpha = 1, \ldots, d, \quad \text{and} \quad \| \nabla v_0 \|_{L^\infty(\Omega)} \leq C.
\]

If we knew that \( C_2^\alpha \), \( \alpha = 1, \ldots, d \), are bounded, then combining this with Theorem 2.1 we have the gradient of

\[
u_b := \sum_{\alpha=1}^{d} C_2^\alpha (v_1^\alpha + v_2^\beta) + v_0
\]

is bounded. Thus, \( \nabla u_b \) is a “good” term which has no singularity in the narrow region. We write

\[
\nabla u = \sum_{\alpha=1}^{d} (C_1^\alpha - C_2^\alpha) \nabla v_1^\alpha + \sum_{i=1}^{2} \sum_{\alpha=d+1}^{d(d+1)/2} C_i^\alpha \nabla v_i^\alpha + \nabla u_b, \quad \text{in } \Omega,
\]

because for \( \alpha = d+1, \ldots, \frac{d(d+1)}{2} \), \( \nabla v_i^\alpha \) are also “not too bad” near the origin; see Theorem 2.6.

Thus, we reduce the establishment of the asymptotics of \( \nabla u \) to that of the asymptotics of \( \nabla v_i^\alpha \), \( \alpha = 1, \ldots, d \), \( i = 1, 2 \), and to solving \( C_1^\alpha - C_2^\alpha, \alpha = 1, \ldots, d \). These are two main difficulties that we need to solve in this paper. For the former, we separate all singular terms of \( \nabla v_i^\alpha \), up to constant terms, by using a family
of improved auxiliary functions, which depend on the parameters of Lamé system and the geometry informations of $\partial D_1$ and $\partial D_2$. This essentially improves the results in [10][11], where only estimates of $|\nabla v^\alpha_i|$ are obtained. For the latter, we need to characterize the coefficients $\alpha^{ij}_\beta$ to solve the big linear system generated by (2.9), and we also establish their asymptotics of $C_1^\alpha - C_2^\alpha$. Here one crucial step is to derive the asymptotic expression of $a^\alpha_{11}$, which is of independent interest, see Proposition 2.3 and Remark 2.4 below. To state it precisely, we further fix our results in [10, 11], where only estimates of $|\nabla \partial D|$ and the geometry informations of improved auxiliary functions, which depend on the parameters of Lamé system.

### 2.3. Further assumptions on inclusions and the construction of auxiliary functions.

Recalling the assumptions about $D_1$ and $D_2$, there exist two points $P_1 \in \partial D_1$ and $P_2 \in \partial D_2$, respectively, such that

$$\text{dist}(P_1, P_2) = \text{dist}(\partial D_1, \partial D_2) = \varepsilon.$$  

By a translation and rotation of coordinates, if necessary, we suppose that $P_1 = (0', \varepsilon) \in \partial D_1$, $P_2 = (0', 0) \in \partial D_2$.

Now we further assume that there exits a constant $R$, independent of $\varepsilon$, such that the portions of $\partial D_1$ and $\partial D_2$ near $P_1$ and $P_2$, respectively, can be represented by

$$x_d = \varepsilon + h_1(x') \quad \text{and} \quad x_d = h_2(x'), \quad \text{for } |x'| < 2R.$$  

Suppose that the convexity of $\partial D_1$ and $\partial D_2$ is of order $m \geq 2$ ($m \in \mathbb{N}^+$) near the origin,

$$h_1(x') = \kappa_1 |x'|^m + O(|x'|^{m+1}), \quad h_2(x') = -\kappa_2 |x'|^m + O(|x'|^{m+1}), \quad \text{for } |x'| < 2R,$$

and

$$\|h_1\|_{C^{2, \gamma}(B_{2R}^c)} + \|h_2\|_{C^{2, \gamma}(B_{2R}^c)} \leq C,$$

where $\kappa_1$, $\kappa_2$, and $C$ are constants independent of $\varepsilon$. We call these inclusions $m$-convex inclusions. For simplicity, we assume that $\kappa_1 = \kappa_2 = \frac{\kappa}{2}$. For $0 < r \leq 2R$, set the narrow region between $\partial D_1$ and $\partial D_2$ as

$$\Omega_r := \{(x', x_d) \in \mathbb{R}^d \mid h_2(x_d) < x_d < \varepsilon + h_1(x'), \ |x'| < r\}.$$  

By the standard theory for elliptic systems, we have

$$\|\nabla v^\alpha_i(x)\|_{L^\infty(\Omega_r \setminus \Omega_R)} \leq C, \quad \alpha = 1, \cdots, \frac{d(d+1)}{2}, \ i = 1, 2.$$  

(2.12)

Therefore, in the following we only need to deal with the problems in $\Omega_R$. To this end, we denote

$$\delta(x') := \varepsilon + h_1(x') - h_2(x'),$$

and introduce a scalar auxiliary function $\bar{u} \in C^2(\mathbb{R}^d)$ such that

$$\bar{u}(x) = \frac{x_d - h_2(x')}{\delta(x')}, \quad \text{in } \Omega_{2R},$$

(2.13)

$\bar{u} = 1$ on $\partial D_1$, $\bar{u} = 0$ on $\partial D_2 \cup \partial D$, and $\|\bar{u}\|_{C^2(\Omega_1 \setminus \Omega_R)} \leq C$. Next we use the function $\bar{u}$ to generate a family of vector-valued auxiliary functions $u^\alpha_1, \alpha = 1, \cdots, d$.

For $d = 2$, recalling that

$$\psi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \psi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$
we define $u^\alpha_1 \in C^2(\Omega)$ such that $u^\alpha_1 = v^\alpha_1$ on $\partial \Omega$, and, in $\Omega_{2R}$

$$u_1^\alpha := \ddot{u}_1^\alpha + \dddot{u}_1^\alpha := \ddot{u}_1^\alpha + \frac{\lambda + \mu}{\lambda + 2\mu} f(\ddot{u}) \delta'(x_1) \psi_2,$$

$$u_2^\alpha := \ddot{u}_2^\alpha + \dddot{u}_2^\alpha := \ddot{u}_2^\alpha + \frac{\lambda + \mu}{\mu} f(\ddot{u}) \delta'(x_1) \psi_1,$$

(2.14)

where $f(\ddot{u}) := \frac{1}{2}(\dddot{u} - \frac{1}{2})^2 - \frac{1}{8} f(\ddot{u}) = 0$ on $(\partial D_1 \cup \partial D_2) \cap \{ |x_1| < R \}$, and $\|u^\alpha_1\|_{C^2(\Omega \setminus \Omega_R)} \leq C$.

For $d = 3$, noting that

$$\psi_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \psi_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \text{and} \quad \psi_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

we define $u^\alpha_3 \in C^2(\Omega)$ such that, in $\Omega_{2R}$

$$u_1^\alpha := \ddot{u}_1^\alpha + \dddot{u}_1^\alpha := \ddot{u}_1^\alpha + \frac{\lambda + \mu}{\lambda + 2\mu} f(\ddot{u}) \partial_{x_1} \delta \psi_3, \quad \alpha = 1, 2,$$

$$u_3^\alpha := \ddot{u}_3^\alpha + \dddot{u}_3^\alpha := \ddot{u}_3^\alpha + \frac{\lambda + \mu}{\mu} f(\ddot{u}) (\partial_{x_1} \delta \psi_1 + \partial_{x_2} \delta \psi_2).$$

(2.15)

Notice that the terms $\dddot{u}_1^\alpha$ depend on the Lamé parameters $\lambda$ and $\mu$.

2.4. Asymptotic expression of $\nabla v^\alpha_3$, $\alpha = 1, \cdots, d$. We now use these auxiliary functions $u^\alpha_3$ to obtain the asymptotics of $\nabla v^\alpha_3$ in the narrow region $\Omega_R$, $\alpha = 1, \cdots, d$.

**Theorem 2.2.** Let $v^\alpha_3 \in H^1(\Omega; \mathbb{R}^d)$ be the weak solutions of (2.4). Then for sufficiently small $0 < \varepsilon < 1/2$, we have

$$\nabla v^\alpha_3(x) = \nabla u_3^\alpha + O(1), \quad \alpha = 1, \cdots, d, \quad x \in \Omega_R.$$

(2.16)

Here we would like to emphasize the importance of the introduction of $\dddot{u}_1^\alpha$. Although $\dddot{u}_1^\alpha$ can be used to obtain the upper bound estimates of $|\nabla v^\alpha_3|$ by $|\nabla (v^\alpha_3 - \dddot{u}_1^\alpha)| \leq \frac{C_2}{\sqrt{\varepsilon}}$ derived in [27], Corollary 5.2], it as well shows to us it is possible that there remains more other singular terms in $\nabla (v^\alpha_3 - \dddot{u}_1^\alpha)$. As it turns out, the appearance of $\nabla \dddot{u}_1^\alpha$ can find all the singular terms of $\nabla v^\alpha_3$ and make $|\nabla (v^\alpha_3 - \dddot{u}_1^\alpha)|$ be bounded. The proof is left in Appendix for readers’ convenience.

The asymptotic expression (2.16) is an essential improvement of the estimate $|\nabla (v^\alpha_3 - \dddot{u}_1^\alpha)| \leq \frac{C_2}{\sqrt{\varepsilon}}$. It allows us to obtain the asymptotics of $a^\alpha_{11}$, $\alpha = 1, \cdots, d$, defined by (2.17). This kind of formulas is one of our main ingredients to prove our main results. We here give the results for $m = 2$. For the general cases $m \geq 3$, see Section 4 below.

**Proposition 2.3.** [The asymptotics of $a^\alpha_{11}$] Under the assumptions (2.10) and (2.11) with $m = 2$, we have, for sufficiently small $0 < \varepsilon < 1/2$,

(i) for $d = 2$, there exist constants $C_{21}^1$ and $C_{22}^2$, independent of $\varepsilon$, such that

$$a_{11}^1 = \frac{\pi \mu}{\sqrt{\varepsilon}} + C_{21}^1 + O(\varepsilon^{1/8}) \quad \text{and} \quad a_{11}^{22} = \frac{\pi (\lambda + 2\mu)}{\sqrt{\varepsilon}} + C_{22}^2 + O(\varepsilon^{1/8});$$


(ii) for $d = 3$, there exist constants $C_3^{\alpha}$ and $C_3^{\alpha}$, independent of $\varepsilon$, such that for $\alpha = 1, 2$,

$$a_{11}^{\alpha} = \frac{\pi \mu}{\kappa} \log \varepsilon + C_3^{\alpha} + O(\varepsilon^{1/8}) \quad \text{and} \quad a_{11}^{33} = \frac{\pi (\lambda + 2\mu)}{\kappa} \log \varepsilon + C_3^{33} + O(\varepsilon^{1/8}).$$

**Remark 2.4.** Here we would point out that if we only use $\bar{u}_1^\beta$ as the auxiliary function like in [10], it is also not possible to obtain these asymptotic formula for $a_{11}^{\alpha}$. The details can be found in the proof of Proposition 2.3 see Section 4 below. So the introduction of $\bar{u}_1^\beta$ is an essential improvement. Its advantage will be also shown in the calculation of other integrals in later sections, for instance, to calculate the global effective elastic property of a periodic composite containing $m$-convex inclusions. This relates the “Vigdergauz microstructure” in the shape optimization of fibers. For more details, see Section 7.

**Remark 2.5.** As we know, in electrostatics, the condenser capacity of $\partial D_1$ relative to $\partial (D \setminus D_2)$ is given by

$$\text{Cap}(D_1) := - \int_{\partial D_1} \frac{\partial v_1}{\partial v} = \int_{\Omega} |\nabla v_1|^2 \, dx,$$

where $v_1 \in C^2(\Omega)$ satisfies

$$\begin{cases}
\Delta v_1 = 0, & \text{in } \Omega, \\
v_1 = 1, & \text{on } \partial D_1, \\
v_1 = 0, & \text{on } \partial D_2 \cup \partial D.
\end{cases}$$

Henceforth, in this sense $a_{11}^{\alpha}$ is an “elasticity capacity” of $\partial D_1$ relative to $\partial (D \setminus D_2)$.

For $\alpha = d + 1, \ldots, d(d + 1)/2$, we use the auxiliary functions as in [10][11],

$$u_1^\alpha := \bar{u}_1^\psi.$$  \hfill (2.17)

**Theorem 2.6.** ([10][11]) Let $v_1^\alpha \in H^1(\Omega; \mathbb{R}^d)$ be the weak solutions of (2.4). Then for sufficiently small $0 < \varepsilon < 1/2$, we have

$$\nabla v_1^\alpha(x) = \nabla u_1^\alpha + O(1), \quad \alpha = d + 1, \ldots, d(d + 1)/2, \quad x \in \Omega_R.$$

We remark that Theorems 2.2 and 2.6 also hold true for $v_2^\alpha$ by replacing $\bar{u}$ with $u$, where $u \in C^2(\mathbb{R}^d)$ is a scalar function satisfying $u = 1$ on $\partial D_2$, $u = 0$ on $\partial D_1 \cup \partial D$, $u = 1 - \bar{u}$ in $\Omega_R$, and $\|u\|_{C^2(\Omega \setminus \Omega_R)} \leq C$. In this case we denote the auxiliary functions by $u_2^\alpha$.

Throughout the paper, unless otherwise stated, we use $C$ to denote some positive constant, whose values may vary from line to line, depending only on $d, \kappa_0, R$, and an upper bound of the $C^2$ norms of $\partial D_1, \partial D_2$, and $\partial D$, but not on $\varepsilon$. We call a constant having such dependence a universal constant.

### 3. Main results

In this section, we state our main theorems. First, in Subsection 3.1 for the 2-convex inclusion case we introduce a blow-up factor matrix, which is a linear functional of boundary data $\varphi$ and then give the asymptotic formulas of $\nabla v$ in Theorem 3.2 for 2D and Theorem 3.4 for 3D. The results for the generalized $m$-convex inclusion cases are presented in Subsection 3.2. We find that the blow-up rates depend on the dimension $d$ and the convexity order $m$, and the blow-up points vary with the increase of $m$. Finally, in Subsection 3.3 we give an example that
where more details, see the proof of Theorem 3.2 below.

1. For the 2-convex inclusions. As mentioned in Subsection 2.2 we know $|\nabla u_0|$ is a bounded quantity, because $u_0$ has no displacement difference on $\partial D_1$ and $\partial D_2$. In order to derive the asymptotics of the solution gradient with respect to the sufficiently small parameter $\varepsilon > 0$, we consider the case when two inclusions touch. Let $u_0^\varepsilon$ verify

$$
\begin{align*}
\mathcal{L}_{\lambda,\mu} u_0^\varepsilon &= 0, & \text{in } \Omega^*, \\
u_b^\varepsilon &= \sum_{\alpha=1}^d C^\alpha \psi_\alpha, & \text{on } \partial D_1^* \cup \partial D_2^*, \\
u_b^\varepsilon &= \varphi(x), & \text{on } \partial D,
\end{align*}
$$

(3.1)

where $D_1^* := \{ x \in \mathbb{R}^d | x + P_1 \in D_1 \}$, $D_2^* := D_2$, and $\Omega^* := D \setminus D_1^* \cup D_2^*$, and the constants $C^\alpha$, $\alpha = 1, \ldots, d$, are uniquely determined by minimizing the energy

$$
\int_{\Omega^*} \left( \mathcal{C}(v), e(v) \right) dx
$$

in an admissible function space

$$
\mathcal{A}^* := \{ v \in H^1(D; \mathbb{R}^d) \mid e(v) = 0 \text{ in } D_1^* \cup D_2^*, \text{ and } v = \varphi \text{ on } \partial D \}.
$$

Denote

$$
\rho_d(\varepsilon) = \begin{cases} \varepsilon, & d = 2, \\
\log \varepsilon, & d = 3, \end{cases}
$$

and for $i = 1, 2$ and $\beta = 1, \ldots, d(d + 1)/2$,

$$
b_i^\beta[\varphi] := \int_{\partial D_i} \frac{\partial u_0^\varepsilon}{\partial \nu} \cdot \psi_\beta \quad \text{and} \quad b_i^{*\beta}[\varphi] := \int_{\partial D_i^*} \frac{\partial u_i^\varepsilon}{\partial \nu} \cdot \psi_\beta.
$$

(3.2)

The following proposition shows that $b_i^{*\beta}[\varphi]$ are exactly the limits of $b_i^\beta[\varphi]$.

**Proposition 3.1.** For sufficiently small $\varepsilon > 0$, we have

$$
b_i^\beta[\varphi] - b_i^{*\beta}[\varphi] = O(\rho_d(\varepsilon)), \quad \beta = 1, \ldots, d,
$$

(3.3)

and

$$
b_i^\beta[\varphi] - b_i^{*\beta}[\varphi] = O(\rho_d(\varepsilon)), \quad i = 1, 2, \quad \beta = d + 1, \ldots, d(d + 1)/2.
$$

(3.4)

The proof will be given in Section 4. Due to Proposition 3.1 we here define the blow-up factors for $d = 2, 3$ by

$$
B^\alpha_d[\varphi] := b_1^{*\alpha} + \sum_{i=1}^2 (-1)^{d+\beta+i-1} \sum_{\beta=d+1}^{d(d+1)/2} A_i^{\alpha}\beta b_i^{*\beta}, \quad \alpha = 1, \ldots, d,
$$

(3.5)

where $A_i^{\alpha}\beta$ are some constants independent of $\varepsilon$, see Lemma 5.3 below. We would like to mention that $B^\alpha_d[\varphi]$ will determine whether or not the blow-up occurs. For more details, see the proof of Theorem 3.2 below.
We shall use $O(1)$ to denote those quantities satisfying $|O(1)| \leq C$, for some constant independent of $\varepsilon$. We assume that for some $\delta_0 > 0$,

$$\delta_0 \leq \mu, d\lambda + 2\mu \leq \frac{1}{\delta_0}.$$ 

The first asymptotic expression of $\nabla u$ in dimension two for $m = 2$ follows.

**Theorem 3.2.** Let $D, D_1, D_2 \subset \mathbb{R}^2$ be defined as above and satisfy (2.10) and (2.11) with $m = 2$. Assume $u \in H^1(D; \mathbb{R}^2) \cap C^1(\Omega; \mathbb{R}^2)$ is the solution to (2.2). Then we have, for sufficiently small $0 < \varepsilon < 1/2$ and for $x \in \Omega_R$,

$$
\nabla u(x) = \frac{\sqrt{\kappa \varepsilon}}{\pi} \left( \frac{B_2^1[\varphi]}{\mu} - \frac{B_3^2[\varphi]}{\lambda + 2\mu} \nabla u_1^1 \right) (1 + O(\sqrt{\varepsilon} \log \varepsilon))
+ O(1)\nabla (u_1^1 + u_2^1) + O(1)\|\varphi\|_{C^0(\partial D)},
$$

(3.6)

where $u_1^1$ are specific functions, constructed in (2.14), $\alpha = 1, 2$, and $u_3^1$ are defined by (2.17), $i = 1, 2$.

**Remark 3.3.** Recalling the definition of $u_1^1$, (2.14), a direct calculation yields

$$
\nabla u_1^1(x_1, x_2) = \frac{1}{\delta(x_1)}E_{12} + O(1), \quad x \in \Omega_x,
$$

where "$E_{\alpha\beta}$" denotes the basic matrix with only one non-zero entry 1 in the $\alpha^{th}$ row and the $\beta^{th}$ column. So in a neighbourhood of $x_1 = 0$, say $x \in \Omega_x$, we find that (3.6) becomes

$$
\nabla u(x, x_2) = \frac{\sqrt{\kappa \varepsilon}}{\pi} \frac{1}{\delta(x_1)} \mathbb{B}_2[\varphi] (1 + O(\sqrt{\varepsilon} \log \varepsilon)) + O(1)\|\varphi\|_{C^0(\partial D)},
$$

(3.7)

where

$$
\mathbb{B}_2[\varphi] := \frac{1}{\mu} B_2^1[\varphi] E_{12} + \frac{1}{\lambda + 2\mu} B_3^2[\varphi] E_{22}.
$$

For this moment, to get the exact asymptotic expression of $\nabla u$ near the origin, we only need to evaluate these boundary integrals $b_{ij}^{(\nu)}$ in $\mathbb{B}_2[\varphi]$ by numerical method. We would like to point out that they no longer depend on $\varepsilon$ and there is no singularity near the origin. It is completely a computation problem. We leave it to interested readers.

**Theorem 3.4.** Let $D, D_1, D_2 \subset \mathbb{R}^3$ be defined as above and satisfy (2.10) and (2.11) with $m = 2$. Assume $u \in H^1(D; \mathbb{R}^3) \cap C^1(\Omega; \mathbb{R}^3)$ is the solution to (2.2). Then for sufficiently small $0 < \varepsilon < 1/2$, and for $x \in \Omega_R$, we have

$$
\nabla u = \frac{\kappa}{|\log \varepsilon|} \frac{1}{\delta(x')} \left( \frac{1}{\mu} \sum_{\alpha=1}^{2} B_2^\alpha[\varphi] \nabla u_1^\alpha + \frac{1}{\lambda + 2\mu} B_3^2[\varphi] \nabla u_1^3 \right) (1 + O(|\log \varepsilon|^{-1}))
+ O(1)\nabla (u_1^1 + u_2^1) + O(1)\|\varphi\|_{C^0(\partial D)},
$$

(3.8)

where $u_1^1$ are specific functions, constructed in (2.15) and (2.17).

**Remark 3.5.** (1) Similarly as in Remark 3.3, for $x \in \Omega_{\sqrt{\varepsilon}}$, we have from (3.8) that

$$
\nabla u(x', x_3) = \frac{\kappa}{|\log \varepsilon|} \frac{1}{\delta(x')} \mathbb{B}_3[\varphi] (1 + O(|\log \varepsilon|^{-1})) + O(1)\|\varphi\|_{C^0(\partial D)},
$$

(3.9)
where
\[
\mathcal{B}_3[\varphi] := \frac{1}{\mu} (B_1^3[\varphi]E_{13} + B_3^3[\varphi]E_{23}) + \frac{1}{\lambda + 2\mu} B_3^3[\varphi]E_{33}.
\] (3.10)

(2) It is worth mentioning that the asymptotic formulas (3.7) and (3.9) immediately imply that the blow-up rates of \( |\nabla u|, \varepsilon^{-1/2} \) for \( d = 2 \), and \((\varepsilon \log \varepsilon)^{-1}\) for \( d = 3 \), obtained in \([10, 11, 35]\), are optimal.

3.2. For the \( m \)-convex inclusions, \( m \geq 3 \). Let \( u^* \) be the solution of
\[
\begin{cases}
\mathcal{L}_{\lambda, \mu} u^* = 0, & \text{in } \Omega^*, \\
u^* = \sum_{\alpha=1}^{d(d+1)/2} C_{\alpha}^\ast \psi_\alpha, & \text{on } \partial D_1^* \cup \partial D_2^*, \\
\int_{\partial D_1^*} \frac{\partial u^*}{\partial \nu} |_{\beta} \cdot \psi_\beta + \int_{\partial D_2^*} \frac{\partial u^*}{\partial \nu} |_{\beta} \cdot \psi_\beta = 0, & \beta = 1, \ldots, d(d+1)/2, \\
u^* = \varphi, & \text{on } \partial D.
\end{cases}
\] (3.11)

Notice that here \( u^* \) is a little different in the third line of (3.11) with that of \( u^*_m \) in (3.1). Moreover, we emphasize that we use the third line of (3.11), where we assume that total flux of \( u^* \) along the boundaries of both two inclusions is zero, to make a distinction with the forth line of (2.2). The kind of limit function for conductivity problem was also used in \([24, 26, 29]\). We define another class of blow-up factors as follows:
\[
B_{\alpha}^\ast [\varphi] := \int_{\partial D_1^*} \frac{\partial u^*}{\partial \nu} |_{\alpha} \cdot \psi_\alpha, \quad \alpha = 1, \ldots, d(d+1)/2.
\] (3.12)

In the following, we shall reveal the role of the order of the relatively convexity between \( \partial D_1 \) and \( \partial D_2 \), \( m \), playing in the asymptotics of \( \nabla u \). Define
\[
Q_{d,m} = 2 \int_0^\infty \frac{t^{d-2}}{1 + tm} dt, \quad \tilde{Q}_{d,m} = 2 \int_0^\infty \frac{t^d}{1 + tm} dt;
\]
\[
\rho_{m,2}(\varepsilon) = \begin{cases} \frac{|\log \varepsilon|^{-1}}{\varepsilon}, & m = 3, \\
\varepsilon^{1/4}, & m = 4, \text{ and } E(\kappa, \varepsilon, m) = \begin{cases} \frac{3m}{\varepsilon^{3/4} |\log \varepsilon|^{-2}}, & m = 3, \\
\frac{2\varepsilon^{1/4} |\log \varepsilon|^{-1}}{Q_{2, m}}, & m \geq 4.
\end{cases}
\end{cases}
\]

Then we have

**Theorem 3.6.** Let \( D, D_1, D_2 \subset \mathbb{R}^2 \) be defined as above and satisfy (2.10) and (2.11) with \( m \geq 3 \). Let \( u \in H^1(D; \mathbb{R}^2) \cap C^1(\overline{D}; \mathbb{R}^2) \) be the solution to (2.2). Then for sufficiently small \( 0 < \varepsilon < 1/2 \) and \( x \in \Omega_R \),
\[
\nabla u(x) = \frac{\kappa^{1/m} \varepsilon^{1-1/m}}{Q_{2, m}} \left( B_1^1[\varphi] \nabla u_1^* + \frac{B_1^3[\varphi]}{\lambda + 2\mu} \nabla u_3^* \right) (1 + O(\rho_{m,2}(\varepsilon)))
\]
\[
+ E(\kappa, \varepsilon, m) B_3^3[\varphi] \lambda + 2\mu \nabla u_3^* (1 + O(\rho_{m,2}(\varepsilon))) + O(1)\|\varphi\|_{C^0(\partial D)},
\]
where \( u_\alpha^* \) are defined by (2.14), \( \alpha = 1, 2 \), and \( u_3^* \) is defined by (2.17).

**Remark 3.7.** We would like to remark that the pointwise upper bound estimates of \( |\nabla u| \) in \([27]\) imply that when \( m \geq d + 1 \), the maximum of the upper bounds obtain at \( x \in \Omega_R \) and \(|x'| = c\varepsilon^{1/m}\) for some positive constant \( c > 0 \). Therefore, for \( x \in \Omega_{1/(m(m-1))} \), recalling the definition of \( u_1^* \), we have
\[
\nabla u(x_1, x_2) = \frac{\kappa^{1/m} \varepsilon^{1-1/m}}{Q_{2, m}} \cdot \frac{1}{\delta(x_1)} B_2^1[\varphi] (1 + O(\rho_{m,2}(\varepsilon))) + E(\kappa, \varepsilon, m) \frac{x_1}{\delta(x_1)} B_{2,11}[\varphi]
\]
\begin{equation}
\cdot (1 + O(\rho_{m,2}(\varepsilon))) + O(1)\|\varphi\|_{C^0(\partial D)},
\end{equation}

where
\[ B_{2,1}[\varphi] := \frac{1}{\mu} B_1^* [\varphi] E_{12} + \frac{1}{\lambda + 2\mu} B_2^* [\varphi] E_{22}, \quad B_{2,II}[\varphi] := \frac{1}{\lambda + 2\mu} B_2^* [\varphi] E_{22}. \]

**Remark 3.8.** If \( B_{2,1}[\varphi] = 0 \) for some \( \varphi \), then the concentration mechanism of the stress is determined by \( B_{2,II}[\varphi] \). Thus, Theorem 3.6 combining with the upper bounds in [27] implies that the blow-up occurs at the segments \( S := \{(\bar{x}_1, \bar{x}_2) \in \Omega_R \mid |\bar{x}_1| = \varepsilon^{1/m}\} \), with blow-up rate \( \frac{1}{\log \varepsilon} \) when \( m = 3 \) and \( \varepsilon^{1/m} \) when \( m \geq 4 \). Consequently, the gradient will not blow up any more and \( S \) will be more and more away from the shortest segment \( \overline{P_1P_2} \) as \( m \) goes to infinity. From (2.10) with \( R < 1 \), we can see that when \( m \to \infty \), the boundaries of \( \partial D_1 \) and \( \partial D_2 \) parallel. However, in fact, it was showed in [27] that there is no blow-up in this situation. The result of Theorem 3.6 describes this diffuse process of the stress concentration phenomenon when \( m \) changes from \( 2 \) to infinity.

Finally, when \( d = 3 \), we define
\[ \rho_{m,3}(\varepsilon) = \begin{cases} \left| \log \varepsilon \right|^{-1}, & m = 4, \\ \varepsilon^{1-4/m}, & 5 \leq m \leq 7, \\ 0, & m \geq 8; \end{cases} \quad \text{and} \quad F(\kappa, \varepsilon, m) = \begin{cases} \frac{2\kappa}{\pi \log \varepsilon}, & m = 4, \\ \frac{2}{\pi Q_{3,m}} \kappa^{2/m} \varepsilon^{-4/m}, & m \geq 5. \end{cases} \]

Then

**Theorem 3.9.** Let \( D, D_1, D_2 \subset \mathbb{R}^3 \) be defined as above and satisfy (2.10) and (2.11) with \( m \geq 3 \). Let \( u \in H^1(D;\mathbb{R}^3) \cap C^1(\Omega;\mathbb{R}^3) \) be the solution to (2.2).

(i) When \( m = 3 \), for sufficiently small \( 0 < \varepsilon < 1/2 \), \( x \in \Omega_R \), we have
\begin{equation}
\nabla u(x) = \frac{\kappa^{2/3}}{\pi} \cdot \frac{\varepsilon^{1/3}}{Q_{3,3}} \left( \frac{1}{\mu} \sum_{\alpha=1}^2 B_3^\alpha [\varphi] \nabla u_\alpha^3 + \frac{1}{\lambda + 2\mu} B_3^2 [\varphi] \nabla u_3^3 \right) (1 + O(\varepsilon^{1/3})) \\
+ O(1) \sum_{\alpha=4}^6 \nabla (u_\alpha^3 + u_\alpha^2) + O(1)\|\varphi\|_{C^0(\partial D)},
\end{equation}

where \( u_\alpha^3 \) are specific functions, constructed in (2.15) and (2.17).

(ii) When \( m \geq 4 \), for sufficiently small \( 0 < \varepsilon < 1/2 \) and \( x \in \Omega_R \),
\begin{equation}
\nabla u(x) = \frac{\kappa^{2/m} \varepsilon^{1-2/m}}{\pi Q_{3,m}} \left( \sum_{\alpha=1}^2 \frac{1}{\mu} B_3^\alpha [\varphi] \nabla u_\alpha^3 + \frac{1}{\lambda + 2\mu} B_3^2 [\varphi] \nabla u_3^3 \right) (1 + O(\rho_{m,3}(\varepsilon))) \\
+ F(\kappa, \varepsilon, m) \left( \frac{1}{\mu} B_3^2 [\varphi] \nabla u_3^4 + \sum_{\alpha=5}^6 \frac{1}{\lambda + 2\mu} B_3^\alpha [\varphi] \nabla u_\alpha^4 \right) (1 + O(\rho_{m,3}(\varepsilon))) \\
+ O(1)\|\varphi\|_{C^0(\partial D)},
\end{equation}

where \( u_\alpha^4 \) are specific functions, constructed in (2.15) and (2.17).

**Remark 3.10.** The above results, together with Theorem 3.6, imply that the blow-up rate of \( |\nabla u| \) depends on the space dimension \( d \), the convexity order \( m \), and the first term’s coefficient \( \kappa \). For more generalized cases, we refer readers to the following Example 3.11. Our method can also be applied to study the case in dimensions \( d \geq 4 \), which is left to the interested readers.
3.3. An example. Finally, we give an example in dimension three to show the dependence of the constants in above asymptotics upon the mean curvature of the inclusions more precisely.

Example 3.11. For \( m \geq 2 \), we denote the top and bottom boundaries of \( \Omega_R \) as
\[
\Gamma^+_R = \left\{ x \in \mathbb{R}^3 \mid x_3 = \varepsilon + \frac{\kappa}{2} |x_1|^m + \frac{\kappa'}{2} |x_2|^m \right\}
\]
and
\[
\Gamma^-_R = \left\{ x \in \mathbb{R}^3 \mid x_3 = -\frac{\kappa}{2} |x_1|^m - \frac{\kappa'}{2} |x_2|^m \right\},
\]
where \( \kappa \) and \( \kappa' \) are two positive constants, may different. For \( \theta \in [0, 2\pi] \), denote
\[
E_m(\theta) := \sin^{2/m-1} \theta \cos^{2/m+1} \theta + \cos^{2/m-1} \theta \sin^{2/m+1} \theta,
\]
\[
F_m(\theta) := \left( \frac{\cos^2 \theta}{\kappa} \right)^{2/m} + \left( \frac{\sin^2 \theta}{\kappa'} \right)^{2/m},
\]
and
\[
G_m := \int_0^{2\pi} E_m(\theta) \, d\theta, \quad \tilde{G}_m := \int_0^{2\pi} E_m(\theta) F_m(\theta) \, d\theta.
\]
Then the results in Theorems 3.4 and 3.9 hold true, except that
(i) if \( m = 2 \), \( \kappa \) in (3.8) is replaced by \( \sqrt{\kappa \kappa'} \), the square root of the relative Gauss curvature of \( \partial D_1 \) and \( \partial D_2 \); if \( m = 3 \), \( \frac{\kappa^{2/3}}{\pi} \) in (3.13) becomes \( \frac{3(\kappa \kappa')^{1/3}}{2G_3} \);
(ii) if \( m \geq 4 \), the terms \( \frac{\kappa^{2/m}}{2 \pi} \) and \( \frac{\kappa^{1/m}}{\pi} \) in (3.14) become \( \frac{2(\kappa \kappa')^{1/m}}{G_m} \) and \( \frac{m(\kappa \kappa')^{2/m}}{\tilde{G}_m} \), respectively.

4. Proof of Proposition 2.3 and Proposition 3.1

This section is devoted to proving Proposition 2.3 and Proposition 3.1 which are two main ingredients to establish our asymptotic formulas of \( \nabla u \). We first need some preliminary estimates on \( v_1^\alpha \), \( \alpha = 1, \cdots, d(d+1)/2 \).

4.1. Auxiliary estimates for \( v_1^\alpha \). Suppose \( v_1^\alpha \) satisfies
\[
\begin{align*}
\mathcal{L}_{\lambda, \mu} v_1^\alpha &= 0, \quad \text{in } \Omega^*, \\
\psi_\alpha &= \psi_\alpha, \quad \text{on } \partial D_1^* \setminus \{0\}, \quad \alpha = 1, \cdots, d(d+1)/2, \\
v_1^\alpha &= 0, \quad \text{on } \partial D_2^* \cup \partial D,
\end{align*}
\]
We will prove that \( v_1^\alpha \) converge to \( v_1^{*\alpha} \), for each \( \alpha \), with proper convergence rates. Define
\[
V := D \setminus D_1 \cup D_1^* \cup D_2^*.
\]
see Figure [II] and
\[
C_r := \left\{ x \in \mathbb{R}^d \mid |x'| < r, \ 2 \min_{|x'|=r} h_2(x') \leq x_d \leq \varepsilon + 2 \max_{|x'|=r} h_1(x') \right\}, \quad r < R.
\]
Then we have
Lemma 4.1. Let $v_1^\alpha$ and $v_1^{*\alpha}$ satisfy (2.4) and (4.1), respectively. Then we have
\[ |(v_1^\alpha - v_1^{*\alpha})(x)| \leq C\varepsilon^{\frac{\alpha}{2}}, \quad \alpha = 1, \ldots, d, \quad x \in V \setminus \mathcal{C}_{\frac{1}{2}}, \tag{4.2} \]
and
\[ |(v_1^\alpha - v_1^{*\alpha})(x)| \leq C\varepsilon^{\frac{\alpha}{2}}, \quad \alpha = d + 1, \ldots, d(d+1)/2, \quad x \in V \setminus \mathcal{C}_{\frac{1}{4}}, \tag{4.3} \]
where $C$ is a universal constant.

Proof. For $\varepsilon = 0$, we define similarly the auxiliary functions, $\tilde{u}$ and $u_1^{*\alpha}$, as limits of $\bar{u}$ and $u_1^\alpha$, where $\bar{u} = 1$ on $\partial D^*$, $\bar{u} = 0$ on $\partial D^*_2 \cup \partial D$ and
\[ \bar{u}^* = \frac{x_d - h_2(x')}{h_1(x') - h_2(x')}, \quad \text{in } \Omega^*_R, \quad \|\bar{u}^*\|_{C^2(\Omega^* \setminus \Omega^*_R)} \leq C, \]
where $\Omega^*_r := \{(x', x_d) \in \mathbb{R}^d \mid h_2(x') < x_d < h_1(x'), \quad |x'| < r\}$, $r < R$. A direct computation yields
\[ |\partial_{x_d} \tilde{u}(x)| \leq \frac{C|x'|}{\varepsilon + |x'|^2}, \quad \partial_{x_d} \tilde{u}(x) = \frac{1}{\delta(x')}, \quad x \in \Omega^*_R, \tag{4.4} \]
and
\[ |\partial_{x_d} \tilde{u}^*(x)| \leq \frac{C}{|x'|}, \quad \frac{1}{C|x'|^2} \leq \partial_{x_d} \tilde{u}^*(x) \leq \frac{C}{|x'|^2}, \quad x \in \Omega^*_R. \tag{4.5} \]

Recalling the construction of $u_1^\alpha$ in (2.14), (2.15), and (2.17), we can construct $u_1^{*\alpha}$ in the same way.

Case 1. $\alpha = 1, \ldots, d$. By using (2.14), (2.15), (4.4) and (4.5), we have for $x \in \Omega^*_R$,
\[ |\partial_{x_d}(u_1^\alpha - u_1^{*\alpha})| \leq |\partial_{x_d}(\tilde{u}_1^\alpha - \bar{u}_1^{*\alpha})| + |\partial_{x_d}(\bar{u}_1^\alpha - \tilde{u}_1^{*\alpha})| \]
\[ C\varepsilon \frac{1}{|x'|^2(\varepsilon + |x'|^2)} + C\varepsilon \frac{1}{|x'|^3} \leq C\varepsilon \left(1 + \frac{\varepsilon}{|x'|} \right). \quad (4.6) \]

Notice that \( v_1^\alpha - v_1^{\alpha_0} \) satisfies
\[
\begin{align*}
\mathcal{L}_{\lambda, \mu}(v_1^\alpha - v_1^{\alpha_0}) &= 0, \quad \text{in } V, \\
v_1^\alpha - v_1^{\alpha_0} &= \psi_\alpha - v_1^{\alpha_0}, \quad \text{on } \partial D_1 \setminus D_1^*, \\
v_1^\alpha - v_1^{\alpha_0} &= -v_1^{\alpha}, \quad \text{on } \partial D_2 \setminus D_2^*, \\
v_1^\alpha - v_1^{\alpha_0} &= v_1^{\alpha} - \psi_\alpha, \quad \text{on } \partial D_1^* \setminus (D_1 \cup \{0\}), \\
v_1^\alpha - v_1^{\alpha_0} &= v_1^{\alpha}, \quad \text{on } \partial D_2^* \setminus D_2, \\
v_1^\alpha - v_1^{\alpha_0} &= 0, \quad \text{on } \partial D.
\end{align*}
\]

By \( \text{[2.12]} \), we have
\[ |\partial_{x_d} v_1^{\alpha_0}| \leq C, \quad \text{in } \Omega^* \setminus \Omega^*_R. \quad (4.7) \]

For \( x \in \partial D_1 \setminus D_1^* \subset \Omega^* \setminus \Omega^*_R \) (see Figure 1), by using mean value theorem and \( (4.7) \),
\[ |(v_1^\alpha - v_1^{\alpha_0})(x', x_d)| = |(\psi_\alpha - v_1^{\alpha_0})(x', x_d)| \]
\[ = |v_1^{\alpha_0}(x', x_d - \varepsilon) - v_1^{\alpha_0}(x', x_d)| \leq C\varepsilon. \quad (4.8) \]

For \( x \in \partial D_1^* \setminus (D_1 \cup C_{\varepsilon^*}) \), by mean value theorem again and Theorem \( \text{[2.2]} \) we have
\[ |(v_1^\alpha - v_1^{\alpha_0})(x', x_d)| = |(v_1^{\alpha_0} - \psi_\alpha)(x', x_d)| \leq \frac{C\varepsilon}{\varepsilon + |x'|^2} \leq C\varepsilon^{1-2\theta}, \quad (4.9) \]
where \( 0 < \theta < 1 \) is some constant to be determined later. Similarly, for \( x \in \partial D_2 \setminus D_2^* \), we have
\[ |(v_1^\alpha - v_1^{\alpha_0})(x', x_d)| \leq C\varepsilon, \quad (4.10) \]
and for \( x \in \partial D_2^* \setminus (D_2 \cup C_{\varepsilon^*}) \), we have
\[ |(v_1^\alpha - v_1^{\alpha_0})(x', x_d)| \leq C\varepsilon^{1-2\theta}. \quad (4.11) \]

For \( x \in \Omega^*_R \) with \( |x'| = \varepsilon^\theta \), it follows from Theorem \( \text{[2.2]} \) and \( (4.6) \) that
\[ |\partial_{x_d}(v_1^\alpha - v_1^{\alpha_0})(x', x_d)| \]
\[ = |\partial_{x_d}(u_1^{\alpha_0} - u_1^\alpha) + \partial_{x_d}(u_1^{\alpha_0} - v_1^{\alpha_0})| \leq C \left(1 + \frac{\varepsilon}{|x'|^2(\varepsilon + |x'|^2)} \right) \leq C \left(1 + \frac{1}{\varepsilon^{4\theta-1}} \right). \quad (4.12) \]

Thus, for \( x \in \Omega^*_R \) with \( |x'| = \varepsilon^\theta \), by using the triangle inequality, \( (4.11) \), the mean value theorem, and \( (4.12) \), we have
\[ |(v_1^\alpha - v_1^{\alpha_0})(x', x_d)| \leq |(v_1^\alpha - v_1^{\alpha_0})(x', x_d) - (v_1^\alpha - v_1^{\alpha_0})(x', h_2(x'))| + C\varepsilon^{1-2\theta} \]
\[ \leq |\partial_{x_d}(v_1^\alpha - v_1^{\alpha_0})| \cdot |h_1(x') - h_2(x')| \leq C \left(\varepsilon^{2\theta} + \varepsilon^{1-2\theta} \right). \quad (4.13) \]

By taking \( 2\theta = 1 - 2\theta \), we get \( \theta = \frac{1}{2} \). Substituting it into \( (4.9) \), \( (4.11) \), and \( (4.13) \), and using \( (4.8),(4.10) \), and \( v_1^\alpha - v_1^{\alpha_0} = 0 \) on \( \partial D \), we obtain
\[ |v_1^\alpha - v_1^{\alpha_0}| \leq C\varepsilon^{\frac{1}{2}}, \quad \text{on } \partial(V \setminus C_{\varepsilon^\frac{1}{2}}). \quad (4.14) \]
Applying the maximum principle for Lamé systems in $V \setminus C_{\varepsilon}^\pm$, we get
\[ |(v_1^\alpha - v_1^{\ast\alpha})(x)| \leq C\varepsilon^{1/2}, \quad \text{in } V \setminus C_{\varepsilon}^\pm. \]

**Case 2.** $\alpha = d + 1, \ldots, d(d + 1)/2$. By using (2.17), (4.4), and (4.5), we obtain for $x \in \Omega_R^*$,
\[ |\partial_{x_d}(u_1^\alpha - u_1^{\ast\alpha})| \leq |\partial_{x_d}(\bar{u} - \bar{u}^\ast)\psi_\alpha| + |(\bar{u} - \bar{u}^\ast)\partial_{x_d}\psi_\alpha| \leq C \left(1 + \frac{\varepsilon}{|x'|(|\varepsilon + |x'|^2)}\right). \quad (4.15) \]

By using Theorem 2.6, we find that (4.9) and (4.11) become
\[ |(v_1^\alpha - v_1^{\ast\alpha})(x', x_d)| \leq C\varepsilon^{1-\theta}, \quad x \in \left(\partial D_1^* \setminus (D_1 \cup C_\varepsilon^\ast)\right) \cup \left(\partial D_2^* \setminus (D_2 \cup C_\varepsilon^\ast)\right) \cup \left(\partial D_3^* \setminus (D_3 \cup C_\varepsilon^\ast)\right), \]
where $0 < \theta < 1$ is some constant to be fixed later. For $x \in \Omega_R^*$ with $|x'| = \varepsilon^\theta$, (4.13) becomes
\[ |(v_1^\alpha - v_1^{\ast\alpha})(x', x_d)| \leq C \left(\varepsilon^{2\theta} + \varepsilon^{1-\theta}\right). \]

By taking $2\theta = 1 - \theta$, we get $\theta = \frac{1}{3}$. We henceforth obtain
\[ |v_1^\alpha - v_1^{\ast\alpha}| \leq C\varepsilon^{2/3}, \quad \text{on } (V \setminus C_{\varepsilon}^\pm). \]
The proof of Lemma 4.1 is finished. \qed

4.2. **Proof of Proposition 2.3** \textit{(The asymptotics of $a_1^{\alpha \alpha}$)} We will use Theorems 2.2 and 2.6 and Lemma 4.1 to prove Proposition 2.3. Let us first prove the case in dimension two.

**Proof of Proposition 2.3** for $d = 2$. (1) First consider $a_1^{11}$. We divide it into three parts:
\[ a_1^{11} = \int_\Omega \left(C^0 e(v_1^1), e(v_1^1)\right) dx \]
\[ = \int_{\Omega \setminus \Omega_R} \left(C^0 e(v_1^1), e(v_1^1)\right) dx + \int_{\Omega_R \setminus \Omega_{1/\varepsilon}} \left(C^0 e(v_1^1), e(v_1^1)\right) dx \]
\[ + \int_{\Omega_{1/\varepsilon}} \left(C^0 e(v_1^1), e(v_1^1)\right) dx =: I_1 + I_2 + I_3. \quad (4.16) \]

In the follow we estimate $I_1$, $I_2$, and $I_3$ one by one.

**Step 1. Claim:** There exists a constant $M_1^*$ independent of $\varepsilon$, such that
\[ I_1 = M_1^* + O(\varepsilon^{1/4}). \quad (4.17) \]

Notice that
\[ \mathcal{L}_{\lambda, \mu}(v_1^1 - v_1^{\ast 1}) = 0, \quad x \in D \setminus D_1 \cup D_2 \cup D_3^* \cup \Omega_R, \]
and
\[ 0 \leq |v_1^1|, |v_1^{\ast 1}| \leq 1, \quad x \in D \setminus D_1 \cup D_2 \cup D_3^* \cup \Omega_R. \]

Then
\[ \partial D_1, \partial D_2, \partial D_3^*, \partial D \text{ and } \partial D \text{ are } C^{2,\alpha}, \]
we have
\[ |\nabla^2(v_1^1 - v_1^{\ast 1})| \leq |\nabla^2 v_1^1| + |\nabla^2 v_1^{\ast 1}| \leq C \quad \text{in } D \setminus (D_1 \cup D_3^* \cup D_2 \cup \Omega_R). \quad (4.18) \]

Moreover, we obtain from (4.2) that
\[ ||v_1^1 - v_1^{\ast 1}||_{L^\infty(D \setminus (D_1 \cup D_3^*) \cup D_2 \cup \Omega_{1/\varepsilon})} \leq C\varepsilon^{1/2}. \quad (4.19) \]
By using the interpolation inequality, (4.18), and (4.19), we obtain
\[ |\nabla (v^*_1 - v^*_1)| \leq C \varepsilon^{1/2(1 - \frac{1}{2})} = C \varepsilon^{1/4} \quad \text{in } D \setminus (D_1 \cup D_1^* \cup D_2 \cup \Omega_R). \] (4.20)

Denote
\[ M^*_I := \int_{\Omega \setminus \Omega_R^*} (C^0 e(v^*_1), e(v^*_1)) \, dx. \]

Then
\[ I_1 - M^*_I = \int_{\Omega \setminus (D_1 \cup \Omega_R)} \left( (C^0 e(v^*_1), e(v^*_1)) - (C^0 e(v^*_1), e(v^*_1)) \right) \, dx 
+ \int_{D_1^* \setminus (D_1 \cup \Omega_R)} (C^0 e(v^*_1), e(v^*_1)) \, dx 
- \int_{D_1 \setminus D_1^*} (C^0 e(v^*_1), e(v^*_1)) \, dx. \]

It follows from \(|D_1^* \setminus (D_1 \cup \Omega_R)| \leq C \varepsilon, |D_1 \setminus D_1^*| \leq C \varepsilon, \) and the boundedness of \(|\nabla v_1^*| \) and \(|\nabla v_1^*| \) in \(D_1^* \setminus (D_1 \cup \Omega_R)\) and \(D_1 \setminus D_1^*\), respectively, that
\[ \left| \int_{D_1^* \setminus (D_1 \cup \Omega_R)} (C^0 e(v^*_1), e(v^*_1)) \, dx - \int_{D_1 \setminus D_1^*} (C^0 e(v^*_1), e(v^*_1)) \, dx \right| \leq C \varepsilon. \] (4.21)

So by using (4.20) and (4.21), we have
\[ I_1 - M^*_I = \int_{\Omega \setminus (D_1 \cup \Omega_R)} (C^0 e(v^*_1 - v^*_1), e(v^*_1 - v^*_1)) \, dx 
+ 2 \int_{\Omega \setminus (D_1 \cup \Omega_R)} (C^0 e(v^*_1 - v^*_1), e(v^*_1 - v^*_1)) \, dx + O(\varepsilon) 
= O(\varepsilon^{1/4}). \]

We henceforth get (4.17).

**Step 2.** Proof of
\[ I_2 = I_2' + O(\varepsilon^{1/8}), \] (4.22)

where
\[ I_2' = \int_{\Omega_R \setminus \Omega^*_r} (C^0 e(v^*_1), e(v^*_1)) \, dx. \]

We further divide \( I_2 - I_2' \) into three terms:
\[ I_2 - I_2' = \int_{(\Omega_R \setminus \Omega^*_r) \setminus (\Omega_R^* \setminus \Omega^*_r)} (C^0 e(v^*_1), e(v^*_1)) \, dx 
+ \int_{\Omega_R^* \setminus \Omega^*_r} (C^0 e(v^*_1 - v^*_1), e(v^*_1 - v^*_1)) \, dx 
=: I_{2,1} + I_{2,2} + I_{2,3}. \] (4.23)

For \( \varepsilon^{1/8} \leq |z_1| \leq R \), we rescale \( \Omega_{|z_1| + |z_2|} \setminus \Omega_{|z_1|} \) into a nearly cube \( Q_1 \) in unit size, and \( \Omega^*_{|z_1| + |z_2|} \setminus \Omega^*_{|z_1|} \) into \( Q^*_1 \) by using the following change of variables:
\[ \begin{aligned}
& x_1 - z_1 = \frac{|z_1|^2 y_1}, \\
& x_2 = \frac{|z_1|^2 y_2}.
\end{aligned} \]

After rescaling, let
\[ V^*_1 = v^*_1(z_1 + z_1^2 y_1, |z_1|^2 y_2), \quad \text{in } Q_1, \quad \text{and } \quad V^*_1 = v^*_1(z_1 + z_1^2 y_1, |z_1|^2 y_2), \quad \text{in } Q^*_1. \]
By the same reason that leads to (4.18), we get
\[ |\nabla^2 v^1_1| \leq C \quad \text{in} \quad Q_1, \quad \text{and} \quad |\nabla^2 v^{s_1}| \leq C \quad \text{in} \quad Q^*_1.\]

Using the interpolation inequality, we obtain
\[ |\nabla (v^1_1 - v^{s_1})| \leq C\varepsilon^{1/4}. \]

Rescaling back to \( v^1_1 - v^{s_1}, \) we get
\[ |\nabla (v^1_1 - v^{s_1})| \leq C\varepsilon^{1/4}|x_1|^{-2} \quad \text{in} \quad \Omega^*_R \setminus \Omega^*_{s_1/8}. \tag{4.24} \]

Similarly, we have
\[ |\nabla v^1_1| \leq C|x_1|^{-2} \quad \text{in} \quad \Omega^*_R \setminus \Omega^*_{s_1/8}, \tag{4.25} \]

and
\[ |\nabla v^{s_1}| \leq C|x_1|^{-2} \quad \text{in} \quad \Omega^*_R \setminus \Omega^*_{s_1/8}. \tag{4.26} \]

Now, by (4.25) and \(|\Omega_R \setminus \Omega^*_{s_1/8}\) \setminus \(|\Omega^*_R \setminus \Omega^*_{s_1/8}| \leq C\varepsilon, we have
\[ |I_{2,1}| \leq C\varepsilon \int_{\varepsilon^{1/8} < |x_1| \leq R} \frac{dx_1}{|x_1|^2} \leq C\varepsilon^{5/8}. \]

Also, by using (4.24) and (4.26), we obtain
\[ |I_{2,2}| \leq C\varepsilon^{1/4} \int_{\varepsilon^{1/8} < |x_1| \leq R} \frac{dx_1}{|x_1|^2} \leq C\varepsilon^{3/8}, \]

and by (4.24), we get
\[ |I_{2,3}| \leq C\varepsilon^{1/2} \int_{\varepsilon^{1/8} < |x_1| \leq R} \frac{dx_1}{|x_1|^2} \leq C\varepsilon^{3/8}. \]

Substituting the estimates above into (4.23), we obtain (4.22).

**Step 3.** We next further approximate \( I'_2 \) by some specific functions. Note that
\[
I'_2 = \int_{\Omega^*_R \setminus \Omega^*_{s_1/8}} (C^0 e(u^s_1), e(u^{s_1}_1)) \, dx + 2 \int_{\Omega^*_R \setminus \Omega^*_{s_1/8}} (C^0 e(u^{s_1}_1), e(v^{s_1}_1 - u^{s_1}_1)) \, dx + \int_{\Omega^*_R \setminus \Omega^*_{s_1/8}} (C^0 e(u^{s_1}_1 - u^{s_1}_1), e(v^{s_1}_1 - u^{s_1}_1)) \, dx.
\]

By using Theorem 2.2 for the second term, we have
\[ 2 \left| \int_{\Omega^*_{s_1/8}} (C^0 e(u^{s_1}_1), e(v^{s_1}_1 - u^{s_1}_1)) \, dx \right| \leq C\varepsilon^{1/8}, \]

and for the third term,
\[ \left| \int_{\Omega^*_{s_1/8}} (C^0 e(v^{s_1}_1 - u^{s_1}_1), e(v^{s_1}_1 - u^{s_1}_1)) \, dx \right| \leq C\varepsilon^{3/8}. \]

Hence,
\[
I'_2 = \int_{\Omega^*_R \setminus \Omega^*_{s_1/8}} (C^0 e(u^{s_1}_1), e(u^{s_1}_1)) \, dx + M^*_2 + O(\varepsilon^{1/8}), \tag{4.27}
\]

where
\[ M^*_2 := 2 \int_{\Omega^*_R} (C^0 e(u^{s_1}_1), e(v^{s_1}_1 - u^{s_1}_1)) \, dx + \int_{\Omega^*_R} (C^0 e(v^{s_1}_1 - u^{s_1}_1), e(v^{s_1}_1 - u^{s_1}_1)) \, dx. \]
is a constant independent of \( \varepsilon \). Coming back to (4.16), and using (4.17), (4.22), and (4.27), so far we obtain

\[
a_{11}^1 = I_3 + \int_{\Omega_*^{1/s}} (C^0 e(u_1^1), e(u_1^{1*})) \, dx + M_1^* + M_2^* + O\left(\varepsilon^{1/8}\right). \tag{4.28}
\]

**Step 4.** Now we are in a position to complete the rest of the proof by direct computations. First, similar to (4.27), we obtain

\[
I_3 = \int_{\Omega_*^{1/s}} (C^0 e(v_1^1), e(v_1^1)) \, dx
\]

\[
= \int_{\Omega_*^{1/s}} (C^0 e(u_1^1), e(u_1^1)) \, dx + 2 \int_{\Omega_*^{1/s}} (C^0 e(u_1^1), e(v_1^1 - u_1^1)) \, dx
\]

\[
+ \int_{\Omega_*^{1/s}} (C^0 e(v_1^1 - u_1^1), e(v_1^1 - u_1^1)) \, dx
\]

\[
= \int_{\Omega_*^{1/s}} (C^0 e(u_1^1), e(u_1^1)) \, dx + O(\varepsilon^{1/8}). \tag{4.29}
\]

Second, by a direct computation, we obtain in \( \Omega_R \),

\[
|\partial_{x_1} (\tilde{u}_1^1)|, |\partial_{x_2} (\tilde{u}_1^1)| \leq \frac{C|x_1|}{\delta(x_1)}, \quad |\partial_{x_1} (\tilde{u}_1^1)| \leq C, \quad \partial_{x_2} (\tilde{u}_1^1) = \frac{1}{\delta(x_1)}. \tag{4.30}
\]

For any \( u = (u_1^1, u_2^1)^T \), recalling the definition of \( C^0 \), a direct calculation yields

\[
(C^0 e(u), e(u))
\]

\[
= \lambda (\partial_{x_1} u_1^1 + \partial_{x_2} u_2^1)^2 + \mu (2(\partial_{x_1} u_1^1)^2 + (\partial_{x_2} u_1^1 + \partial_{x_1} u_2^1)^2 + 2(\partial_{x_2} u_2^1)^2). \tag{4.31}
\]

Substituting \( u_1^1 \) and \( u_1^{1*} \) into (4.31) and using (4.30), we have

\[
\int_{\Omega_*^{1/s}} (C^0 e(u_1^1), e(u_1^1)) \, dx + \int_{\Omega_*^{1/s}} (C^0 e(u_1^{1*}), e(u_1^{1*})) \, dx
\]

\[
= \mu \int_{|x_1| \leq \varepsilon^{1/8}} \frac{dx_1}{\varepsilon + \kappa |x_1|^2 + o(|x_1|^2)} + \mu \int_{\varepsilon^{1/8} < |x_1| \leq R} \frac{dx_1}{\kappa |x_1|^2 + o(|x_1|^2)} + O(\varepsilon^{1/8}).
\]

Notice that for the first term,

\[
\int_{|x_1| \leq \varepsilon^{1/8}} \frac{dx_1}{\varepsilon + \kappa |x_1|^2 + o(|x_1|^2)} = \int_{|x_1| \leq \varepsilon^{1/8}} \frac{dx_1}{\varepsilon + \kappa |x_1|^2} + O(\varepsilon^{1/8}),
\]

and for the second term,

\[
\int_{\varepsilon^{1/8} < |x_1| \leq R} \frac{dx_1}{\kappa |x_1|^2 + o(|x_1|^2)} = \int_{\varepsilon^{1/8} < |x_1| \leq R} \frac{dx_1}{\kappa |x_1|^2} + O(\varepsilon^{1/8}).
\]

Let two right hand sides subtract,

\[
\int_{\varepsilon^{1/8} < |x_1| \leq R} \left( \frac{1}{\kappa |x_1|^2} - \frac{1}{\varepsilon + \kappa |x_1|^2} \right) \, dx_1 = O(\varepsilon^{5/8}).
\]

Then we obtain

\[
\int_{\Omega_*^{1/s}} (C^0 e(u_1^1), e(u_1^1)) \, dx + \int_{\Omega_*^{1/s}} (C^0 e(u_1^{1*}), e(u_1^{1*})) \, dx
\]

\[
= \frac{\pi \mu}{\sqrt{\kappa \varepsilon}} - \frac{2\mu}{\kappa R} + O(\varepsilon^{1/8}). \tag{4.32}
\]
Substituting (4.32) into (4.28) and using (4.29), we have
\[ a_{11}^{11} = \frac{\pi \mu}{\sqrt{\kappa \varepsilon}} + M_1^* + M_2^* - \frac{2\mu}{\kappa R} + O(\varepsilon^{1/8}) = \frac{\pi \mu}{\sqrt{\kappa \varepsilon}} + C_2^1 + O(\varepsilon^{1/8}), \] (4.33)
where \( C_2^1 := M_1^* + M_2^* - \frac{2\mu}{\kappa R} \) is a constant independent of \( \varepsilon \). Furthermore, it is also independent of \( R \). In fact, if there exist \( C_2^1(R) \) and \( C_2^1(\tilde{R}) \) such that (4.33) holds true, then for small enough \( \varepsilon > 0 \), we have
\[ C_2^1(R) - C_2^1(\tilde{R}) = O(\varepsilon^{1/8}). \]
Thus, \( C_2^1(R) = C_2^1(\tilde{R}) \).

(2) For \( a_{11}^{21} \), we apply the auxiliary function defined by (4.14). The rest of the proof is the same as in (1), except that we substitute \( u_1^2 \) and \( u_1^3 \) into (4.31). A direct calculation gives
\[ |\partial_{x_1}(\tilde{u}_1^2)|, |\partial_{x_2}(\tilde{u}_1^2)| \leq \frac{C|x_1|}{\delta(x_1)}, \quad |\partial_{x_1}(\tilde{u}_1^3)| \leq C, \quad |\partial_{x_2}(\tilde{u}_1^2)|^2 = \frac{1}{\delta(x_1)}. \] (4.34)
By using (4.34) and the same argument as that in (4.32), we obtain
\[ a_{11}^{22} = \frac{\pi(\lambda + 2\mu)}{\sqrt{\kappa \varepsilon}} + C_2^2 + O(\varepsilon^{1/8}), \]
where \( C_2^2 \) is a constant independent of \( \varepsilon \) and \( R \). So we complete the proof of Proposition 2.3 in dimension two. \( \square \)

**Proof of Proposition 2.3 for \( d = 3 \).** The proof is similar to the above until (4.31). For \( u = (u^1, u^2, u^3)^T \), (4.31) becomes
\[
(C^0 e(u), e(u)) = \lambda \left( \partial_{x_1} u^1 + \partial_{x_2} u^2 + \partial_{x_3} u^3 \right)^2 + \mu \left( 2(\partial_{x_1} u^1)^2 + 2(\partial_{x_2} u^2)^2 + 2(\partial_{x_3} u^3)^2 \right) + (\partial_{x_2} u^1 + \partial_{x_1} u^2)^2 + (\partial_{x_2} u^1 + \partial_{x_1} u^3)^2 + (\partial_{x_3} u^3 + \partial_{x_3} u^2)^2. \] (4.35)
Substituting the specific function \( u_1^1 \), defined by (2.15), into (4.35) and recalling that in \( \Omega_R \),
\[ |\partial_{x_k} u_1^1| \leq \frac{C|x'|}{\delta(x')} \quad \text{and} \quad \partial_{x_k} u_1^1 = \frac{1}{\delta(x')}, \quad k = 1, 2, \]
we find that (4.33) becomes
\[ a_{11}^{11} = \mu \int_{|x'| \leq \varepsilon^{1/8}} \frac{dx'}{\varepsilon + \kappa |x'|^2 + o(|x'|^2)} + \mu \int_{\varepsilon^{1/8} < |x'| \leq R} \frac{dx'}{\kappa |x'|^2 + o(|x'|^2)} + O(\varepsilon^{1/8}) \]
\[ = \mu \int_{|x'| \leq R} \frac{dx'}{\varepsilon + \kappa |x'|^2} + O(\varepsilon^{1/8}) \]
\[ = \frac{\pi \mu}{\kappa} \log \varepsilon + C_3^1 + O(\varepsilon^{1/8}), \]
where \( C_3^1 \) is a constant independent of \( \varepsilon \) and \( R \). Similarly,
\[ a_{11}^{22} = \frac{\pi \mu}{\kappa} \log \varepsilon + C_3^2 + O(\varepsilon^{1/8}), \quad \text{and} \quad a_{11}^{33} = \frac{\pi(\lambda + 2\mu)}{\kappa} \log \varepsilon + C_3^3 + O(\varepsilon^{1/8}), \]
where \( C_3^2 \) and \( C_3^3 \) are constants independent of \( \varepsilon \) and \( R \). Hence, Proposition 2.3 in dimension three is proved. \( \square \)
4.3. Proof of Proposition 3.1. In this section, we aim to prove Proposition 3.1. Let $v_0^*$ satisfy
\[
\begin{aligned}
&\left\{\begin{array}{ll}
\mathcal{L}_{\lambda,\mu} v_0^* = 0, & \text{in } \Omega^*, \\
v_0^* = 0, & \text{on } \partial D_1^* \cup \partial D_2^*, \\
v_0^* = \varphi, & \text{on } \partial D,
\end{array}\right.
\end{aligned}
\tag{4.36}
\]
and
\[
v^\alpha := v_1^\alpha + v_2^\alpha, \quad v^\alpha := v_1^\alpha + v_2^\alpha
\]
satisfy
\[
\begin{aligned}
&\left\{\begin{array}{ll}
\mathcal{L}_{\lambda,\mu} v^\alpha = 0, & \text{in } \Omega, \\
v^\alpha = \psi_\alpha, & \text{on } \partial D \cup \partial D_2, \\
v^\alpha = 0, & \text{on } \partial D,
\end{array}\right.
\end{aligned}
\tag{4.37}
\]
respectively. Recalling (2.8), (3.1), and the definitions of $b_1^{\beta \varphi}$ and $b_1^{*\beta \varphi}$, (3.2), we have
\[
b_1^{\beta \varphi} - b_1^{*\beta \varphi} = \int_{\partial D_1} \frac{\partial u_0}{\partial \nu} \cdot \psi_\beta - \int_{\partial D_1^*} \frac{\partial u_0^*}{\partial \nu} \cdot \psi_\beta
\]
\[
= \int_{\partial D} \frac{\partial v_0}{\partial \nu} \cdot \psi_\beta - \int_{\partial D_1} \frac{\partial v_0^*}{\partial \nu} \cdot \psi_\beta + \sum_{\alpha=1}^d C_2^\alpha \int_{\partial D_1} \frac{\partial v_\alpha}{\partial \nu} \cdot \psi_\beta
\]
\[
- \int_{\partial D_1} \frac{\partial v_\alpha}{\partial \nu} \cdot \psi_\beta + \sum_{\alpha=1}^d \left( C_2^\alpha - C_2^* \right) \int_{\partial D_1^*} \frac{\partial v_\alpha}{\partial \nu} \cdot \psi_\beta.
\tag{4.38}
\]
From (35 Lemma 3.2), we have
\[
|C_2^\alpha - C_2^*| \leq C_{\rho_d}(\varepsilon), \quad \alpha = 1, \ldots, d.
\tag{4.39}
\]
So the proof of Proposition 3.1 is reduced to the estimates of
\[
\int_{\partial D_1} \frac{\partial v_0}{\partial \nu} \cdot \psi_\beta - \int_{\partial D_1^*} \frac{\partial v_0^*}{\partial \nu} \cdot \psi_\beta \quad \text{and} \quad \int_{\partial D_1} \frac{\partial v_\alpha}{\partial \nu} \cdot \psi_\beta - \int_{\partial D_1^*} \frac{\partial v_\alpha}{\partial \nu} \cdot \psi_\beta.
\]
In order to (4.3), we firstly need the following

**Lemma 4.2.** Let $v_0$ and $v_0^*$ be defined in (2.5) and (4.36), respectively. Then for $\beta = 1, \ldots, d$, we have
\[
\left| \int_{\partial D_1} \frac{\partial v_0}{\partial \nu} \cdot \psi_\beta - \int_{\partial D_1^*} \frac{\partial v_0^*}{\partial \nu} \cdot \psi_\beta \right| \leq C \varepsilon^{\frac{1}{2}} \| \varphi \|_{L^1(\partial D)}.
\]

**Proof.** It follows from (2.5), (4.36), and the integration by parts that
\[
\int_{\partial D_1} \frac{\partial v_0}{\partial \nu} \cdot \psi_\beta = \int_{\partial D_1} \frac{\partial v_0}{\partial \nu} \cdot \psi_\beta = - \int_{\partial D} \frac{\partial v_1^\beta}{\partial \nu} \cdot \varphi,
\]
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\[ \int_{\partial D_1} \frac{\partial v_0^\alpha}{\partial \nu} \cdot \psi_\beta = - \int_{\partial D} \frac{\partial v_1^\beta}{\partial \nu} \cdot \varphi. \]

Thus, we have

\[ \int_{\partial D_1} \frac{\partial v_0}{\partial \nu} \cdot \psi_\beta - \int_{\partial D_1^*} \frac{\partial v_0^*}{\partial \nu} \cdot \psi_\beta = - \int_{\partial D} \frac{\partial (v_1^\beta - v_1^{*\beta})}{\partial \nu} \cdot \varphi. \] (4.40)

By using (4.2) and the standard boundary gradient estimates for Lamé system (see [4]), we have

\[ |\nabla (v_1^\beta - v_1^{*\beta})(x)| \leq C \varepsilon^\frac{1}{2}, \quad \text{on } \partial D. \] (4.41)

Substituting it into (4.40), we finish the proof of Lemma 4.2. \[ \square \]

Secondly, similar to Lemma 4.2, we can get

**Lemma 4.3.** Let \( v^\alpha \) and \( v^{*\alpha} \) be defined by (4.37), respectively, \( \alpha = 1, \ldots, d \). Then

\[ \left| \int_{\partial D_1} \frac{\partial v^\alpha}{\partial \nu} \cdot \psi_\beta - \int_{\partial D_1^*} \frac{\partial v^{*\alpha}}{\partial \nu} \cdot \psi_\beta \right| \leq C \varepsilon^\frac{1}{2} |\partial D|, \quad \beta = 1, \ldots, d. \]

_Proof._ For \( \alpha = 1, \ldots, d \), it follows from (4.37) that

\[
\begin{cases}
L_{\lambda, \mu}(v^\alpha - \psi_\alpha) = 0, & \text{in } \Omega, \\
v^\alpha - \psi_\alpha = 0, & \text{on } \partial D_1 \cup \partial D_2, \\
v^\alpha - \psi_\alpha = -\psi_\alpha, & \text{on } \partial D.
\end{cases}
\]

By using the integration by parts, we have for \( \alpha = 1, \ldots, d \),

\[
\int_{\partial D_1} \frac{\partial v^\alpha}{\partial \nu} \cdot \psi_\beta = \int_{\partial D_1} \frac{\partial (v^\alpha - \psi_\alpha)}{\partial \nu} \cdot \psi_\beta = \int_{\partial D} \frac{\partial v_1^\beta}{\partial \nu} \cdot \psi_\alpha.
\]

Similarly,

\[
\int_{\partial D_1^*} \frac{\partial v^{*\alpha}}{\partial \nu} \cdot \psi_\beta = \int_{\partial D} \frac{\partial v_1^{*\beta}}{\partial \nu} \cdot \psi_\alpha.
\]

Hence,

\[
\int_{\partial D_1} \frac{\partial v^\alpha}{\partial \nu} \cdot \psi_\beta - \int_{\partial D_1^*} \frac{\partial v^{*\alpha}}{\partial \nu} \cdot \psi_\beta = \int_{\partial D} \frac{\partial (v_1^\beta - v_1^{*\beta})}{\partial \nu} \cdot \psi_\alpha.
\]

Thus, by using (4.41), we have

\[
\left| \int_{\partial D_1} \frac{\partial v^\alpha}{\partial \nu} \cdot \psi_\beta - \int_{\partial D_1^*} \frac{\partial v^{*\alpha}}{\partial \nu} \cdot \psi_\beta \right| \leq C \varepsilon^\frac{1}{2} |\partial D|.
\]

The proof of Lemma 4.3 is finished. \[ \square \]

Finally, we are ready to complete the proof of (3.3).

**Proof of (3.3).** By using (4.38), (4.39), Lemmas 4.2 and 4.3, we have

\[
|b_1^\beta [\varphi] - b_1^{*\beta} [\varphi]| \leq C \varepsilon^\frac{1}{2} \| \varphi \|_{L^1(\partial D)} + \sum_{\alpha=1}^d |C_{2}^\alpha| \left| \int_{\partial D_1} \frac{\partial v^\alpha}{\partial \nu} \right| \cdot \psi_\beta - \int_{\partial D_1^*} \frac{\partial v^{*\alpha}}{\partial \nu} \cdot \psi_\beta \biggr|
\]
\[\sum_{\alpha=1}^{d} |C_{2}^{\alpha} - C_{1}^{\alpha}| \left| \int_{\partial D_1} \frac{\partial v^*_{\alpha}}{\partial\nu} \right| + |\psi_\beta| \leq C \rho_{d}(\varepsilon) \left( |\partial D| + \|\varphi\|_{C^0(\partial D)} \right), \tag{4.42}\]

here we used the fact that \(|\nabla v^*_{\alpha}| \leq C\) in \(\Omega^*\) since the displacement takes the same value on the boundaries of both inclusions, see Theorem 2.1. We thus prove (3.3). \(\square\)

To prove (3.4), we need the following

**Lemma 4.4.** Let \(v_0\) and \(v^*_0\) be defined in (2.5) and (4.36), respectively. Then for \(i = 1, 2, \beta = d + 1, \cdots, d(d + 1)/2,\)

\[\left| \int_{\partial D} \frac{\partial v_0}{\partial\nu} \right| + |\psi_\beta| - \left| \int_{\partial D^*} \frac{\partial v^*_0}{\partial\nu} \right| + |\psi_\beta| \leq C \varepsilon^{\frac{d}{2}} \|\varphi\|_{L^1(\partial D)}. \tag{4.43}\]

**Proof.** As in (4.40), we have

\[\left| \int_{\partial D} \frac{\partial v_0}{\partial\nu} \right| + |\psi_\beta| - \left| \int_{\partial D^*} \frac{\partial v^*_0}{\partial\nu} \right| + |\psi_\beta| = - \left| \int_{\partial D} \frac{\partial (v^\beta_i - v^*_{i})}{\partial\nu} \right| + |\varphi|. \tag{4.43}\]

By using (4.3) and the standard boundary gradient estimates for Lamé system (see [4]),

\[|\nabla (v^\beta_i - v^*_{i})(x)| \leq C \varepsilon^{\frac{d}{2}}, \text{ on } \partial D. \tag{4.44}\]

This is also holds for \(|\nabla (v^\beta_i - v^*_{i})(x)|\) on \(\partial D\). Coming back to (4.43),

\[\left| \int_{\partial D} \frac{\partial v_0}{\partial\nu} \right| + |\psi_\beta| - \left| \int_{\partial D^*} \frac{\partial v^*_0}{\partial\nu} \right| + |\psi_\beta| \leq C \varepsilon^{\frac{d}{2}} \|\varphi\|_{L^1(\partial D)}. \tag{4.45}\]

Lemma 4.4 is proved. \(\square\)

Similarly, we have

**Lemma 4.5.** Let \(v^\alpha\) and \(v^*\alpha\) be defined in (4.37), \(\alpha = 1, \cdots, d\). Then for \(\beta = d, \cdots, d(d + 1)/2,\)

\[\left| \int_{\partial D} \frac{\partial v^\alpha}{\partial\nu} \right| + |\psi_\beta| - \left| \int_{\partial D^*} \frac{\partial v^*\alpha}{\partial\nu} \right| + |\psi_\beta| \leq C \varepsilon^{\frac{d}{2}} |\partial D|. \tag{5.1}\]

Hence, substituting Lemmas 4.4 and 4.5 into (4.38), using (4.39) and the argument in the proof of (4.42), we obtain (3.4). Therefore, Proposition 3.1 is proved.

### 5. The Proof of Theorem 3.2 and Theorem 3.4

In this section, we prove our main results, Theorem 3.2 and Theorem 3.4. Recalling the forth line of (2.2) and (2.9), we obtain

\[
\begin{cases}
\sum_{\alpha=1}^{d} (C_{1}^{\alpha} - C_{2}^{\alpha})a_{i1}^{\alpha\beta} + \sum_{i=1}^{2} \sum_{\alpha=d+1}^{d(d+1)/2} C_{1}^{\alpha} a_{i1}^{\alpha\beta} = b_{1}^{\beta}, \\
\sum_{\alpha=1}^{d} (C_{1}^{\alpha} - C_{2}^{\alpha})a_{i2}^{\alpha\beta} + \sum_{i=1}^{2} \sum_{\alpha=d+1}^{d(d+1)/2} C_{1}^{\alpha} a_{i2}^{\alpha\beta} = b_{2}^{\beta},
\end{cases}
\]
where \( a_{ij}^{\alpha \beta} \) is defined by (2.7), and \( b_i^\beta \) is defined by (3.2). In the following, we want to solve \( C_1^0 - C_2^0 \) and derive their asymptotic formula, \( \alpha = 1, \ldots, d \). Let us first consider \( d = 2 \).

### 5.1. Proof of Theorem 3.2

We denote (5.1) in block matrix as follows:

\[
AX := \begin{pmatrix}
    a_{11}^{11} & a_{12}^{11} & a_{13}^{11} & a_{12}^{12} \\
    a_{21}^{11} & a_{22}^{11} & a_{23}^{11} & a_{22}^{12} \\
    a_{31}^{11} & a_{32}^{11} & a_{33}^{11} & a_{32}^{12} \\
    a_{31}^{21} & a_{32}^{21} & a_{33}^{21} & a_{32}^{22}
\end{pmatrix}
\begin{pmatrix}
    C_1^1 - C_2^1 \\
    C_1^2 - C_2^2 \\
    C_1^3 \\
    C_2^3
\end{pmatrix}
= \begin{pmatrix}
    b_1^1 \\
    b_2^1 \\
    b_1^3 \\
    b_2^3
\end{pmatrix}.
\]

To solve \( X \), except the asymptotics of \( a_{11}^{\alpha \beta} \), established in Proposition 2.3, we also need the following two lemmas, one from [27] and the other from [35].

**Lemma 5.1.** [27, Lemma 5.3] For \( d = 2, 3 \), we have

\[
|a_{11}^{\alpha \beta}| = |a_{11}^{\beta \alpha}| \leq C \rho_{2(d-1),m}(\varepsilon), \quad \alpha, \beta = 1, \ldots, d, \alpha \neq \beta.
\]

\[
|a_{11}^{\beta \alpha}| = |a_{11}^{\alpha \beta}| \leq C \rho_{2d,m}(\varepsilon), \quad \alpha = 1, \ldots, d, \beta = d + 1, \ldots, \frac{d(d+1)}{2}.
\]

\[
|a_{11}^{\alpha \beta}| = |a_{11}^{\beta \alpha}| \leq C \rho_{2(d+1),m}(\varepsilon), \quad \alpha, \beta = d + 1, \ldots, \frac{d(d+1)}{2}, \alpha \neq \beta.
\]

where

\[
\rho_{k,m}(\varepsilon) = \begin{cases}
    1, & m < k, \\
    |\log \varepsilon|, & m = k, \\
    \varepsilon^{-k-m}, & m > k;
\end{cases}
\]

**Lemma 5.2.** [35, Lemma 4.4] Let \( a_{ij}^{\alpha \beta} \) be defined as in (2.7). Then we have

\[
a_{11}^{33}a_{22}^{33} - a_{12}^{33}a_{21}^{33} > \frac{1}{C} \quad \text{for some constant } C \text{ independent of } \varepsilon.
\]

By Cramer’s rule, we have

\[
C_1^1 - C_2^1 = \frac{\det A_1}{\det A},
\]

where

\[
A_1 := \begin{pmatrix}
    b_1^{11} & a_{12}^{11} & a_{13}^{11} & a_{12}^{12} \\
    b_1^{21} & a_{22}^{11} & a_{23}^{11} & a_{22}^{12} \\
    b_1^{31} & a_{32}^{11} & a_{33}^{11} & a_{32}^{12} \\
    b_1^{32} & a_{32}^{21} & a_{33}^{21} & a_{32}^{22}
\end{pmatrix}, \quad A_2 := \begin{pmatrix}
    a_{11}^{11} & b_1^{11} & a_{13}^{11} & a_{12}^{12} \\
    a_{11}^{21} & b_1^{21} & a_{23}^{11} & a_{12}^{12} \\
    a_{11}^{31} & b_1^{31} & a_{33}^{11} & a_{12}^{12} \\
    a_{21}^{32} & b_2^{32} & a_{33}^{21} & a_{32}^{22}
\end{pmatrix}.
\]

By Lemma 5.1, we can see that

\[
\det A_1 = a_{11}^{22}(b_1^{11}A_1(34;34) - b_3^{32}A_1(14;34) + b_2^{32}A_1(13;34)) + O(\|\log \varepsilon\|),
\]

where \( A_1(kl;mn) \) denotes the determinant of submatrix of the matrix \( A_1 \), by choosing its row \( k, l \) and column \( m, n \), \( k, l, m, n = 1, 2, 3, 4 \).
By using Proposition 2.3 and Lemma 5.1, we have
\[ \det A = a_{11}^{11}a_{12}^{22}A_1(34; 34) + \varepsilon^{-1/2}O(1). \]
Substituting (5.4) and (5.5) into (5.3) and using (5.2), we obtain
\[ C_1 - C_2 = \frac{1}{a_{11}^{11} + O(1)} \left( b_1 - b_1^3 A_1(14; 34) + b_2^3 A_1(13; 34) \right) + O(\varepsilon |\log \varepsilon|). \]
Similarly,
\[ C_1^2 - C_2^2 = \frac{1}{a_{22}^{11} + O(1)} \left( b_2 - b_1^2 A_2(24; 34) + b_2^3 A_2(23; 34) \right) + O(\varepsilon |\log \varepsilon|), \]
where \( A_2(kl; mn) \) denotes the determinant of the submatrix of \( A_2 \), by choosing its row \( k, l \) and column \( m, n, k, l, m, n = 1, 2, 3, 4 \). From Proposition 2.3, we have
the asymptotic formula of \( a_{11}^{11} \). On the other hand, we have had the convergence
property for \( b_1^\beta \) and \( b_3^\beta \) in Proposition 3.1, \( \beta = 1, 2 \) and \( i = 1, 2 \).

Denote \( \varepsilon \)-independent constants
\[ A_{11}^3 := \frac{a_{11}^{13} + a_{22}^{12} - a_{12}^{13} + a_{22}^{12}}{a_{11}^{13} a_{22}^{12} - a_{12}^{13} a_{22}^{12}}, \quad A_{21}^1 := \frac{a_{11}^{13} - a_{12}^{13} + a_{22}^{13}}{a_{11}^{13} a_{22}^{12} - a_{12}^{13} a_{22}^{12}}, \]
and
\[ A_{12}^3 := \frac{a_{11}^{23} + a_{22}^{12} - a_{12}^{23} + a_{22}^{12}}{a_{11}^{23} a_{22}^{12} - a_{12}^{23} a_{22}^{12}}, \quad A_{22}^1 := \frac{a_{11}^{23} - a_{12}^{23} + a_{22}^{23}}{a_{11}^{23} a_{22}^{12} - a_{12}^{23} a_{22}^{12}}, \]
where
\[ a_{ij}^{\alpha \beta} := - \int_{\partial D_j^i} \frac{\partial v_{\alpha}^{i}}{\partial \nu} \cdot \psi_{\beta} \]
and \( v_{\alpha}^{i} \) are defined by (4.1). Then we have

**Lemma 5.3.** As \( \varepsilon \to 0 \),
\[ A_1(14; 34) \xrightarrow{A_{11}^3} A_1(13; 34) \xrightarrow{A_{21}^1} A_1(34; 34) \]
and
\[ A_2(24; 34) \xrightarrow{A_{12}^3} A_2(23; 34) \xrightarrow{A_{22}^1} A_1(34; 34). \]

**Proof.** It follows from the maximum principle for Lamé systems (see [48]) that
\( \|v_\beta^\alpha\|_{L^\infty(O)} \) is bounded by a constant independent of \( \varepsilon \). On the other hand, by
using the definition of \( a_{ij}^{\alpha \beta} \), (2.7), we obtain
\[ a_{ij}^{\alpha \beta} - a_{ij}^{\alpha \beta} = - \int_{\partial D_j} \frac{\partial v_{\alpha}^{i}}{\partial \nu} \cdot \psi_{\beta} + \int_{\partial D_j^i} \frac{\partial v_{\alpha}^{i}}{\partial \nu} \cdot \psi_{\beta} = - \int_{\partial D_j} \frac{\partial v_{\alpha}^{i} - v_{\alpha}^{i}}{\partial \nu} \cdot \psi_{\beta} + \int_{\partial D_j^i} \frac{\partial v_{\alpha}^{i} - v_{\alpha}^{i}}{\partial \nu} \cdot \psi_{\beta} + \int_{\partial D_j^i \setminus \partial D_j} \frac{\partial v_{\alpha}^{i}}{\partial \nu} \cdot v_{\beta}^i. \]
As \( \varepsilon \) tends to zero, the uniform boundedness of \( v_{\beta}^i \) implies that the third term in (5.6) goes to zero. For the second term, by using a similar argument that led to Lemma 4.1, we find that it also vanishes. Finally, for the first term in (5.6), similar to (4.44), using the standard boundary gradient estimates for Lamé system (see [3]), we can conclude that it is convergent to zero. Thus, as \( \varepsilon \to 0 \), we get
\(a_{ij} \to a_{ij}^{**}\). Hence, \(A_{11}(14;34), A_{11}(34;34), A_{21}(24;34), A_{21}(23;34)\) are convergent to \(A_{11}^{**}, A_{21}^{**}, A_{12}^{**}, \) and \(A_{22}^{**}\), respectively, as \(\varepsilon \to 0\).

We now present the proof of Theorem 3.2.

**Proof of Theorem 3.2.** For any \(x \in \Omega_R\), by using (2.9), Theorems 2.1, 2.2 and 2.6

\[
\nabla u = \sum_{\alpha=1}^{2} (C_1^\alpha - C_2^\alpha) \nabla u^\alpha + O(1)\nabla (u_1^3 + u_2^3) + O(1). \tag{5.7}
\]

Recalling the asymptotics of \(a_{11}^{\beta\beta}\), Proposition 2.3 for 2-convex inclusions case, we have

\[
C_1^1 - C_2^1 = \frac{\sqrt{K\varepsilon}}{\pi\mu} \left( b_1^1 - b_2^1 A_{11}(14;34) + b_2^1 A_{11}(34;34) \right) + O(\varepsilon |\log \varepsilon|), \tag{5.8}
\]

and

\[
C_1^2 - C_2^2 = \frac{\sqrt{K\varepsilon}}{\pi\mu} \left( b_1^2 - b_2^2 A_{21}(24;34) + b_2^2 A_{21}(34;34) \right) + O(\varepsilon |\log \varepsilon|). \tag{5.9}
\]

Denote

\[
B_2^1[\varphi] := b_1^* - A_{11}^* b_1^3 + A_{21}^* b_2^3.
\]

Then coming back to (5.8), using Proposition 3.1 and Lemma 5.3, we have

\[
C_1^1 - C_2^1 = \frac{\sqrt{K\varepsilon}}{\pi\mu} \left( b_1^1 - A_{11}^* b_1^3 + A_{21}^* b_2^3 + O(\varepsilon^{1/2}) \right) + O(\varepsilon |\log \varepsilon|)
\]

\[
= \frac{\sqrt{K\varepsilon}}{\pi\mu} \left( B_2^1[\varphi] + O(\varepsilon^{1/2}) \right) + O(\varepsilon |\log \varepsilon|)
\]

\[
= \frac{\sqrt{K\varepsilon}}{\pi\mu} B_2^1[\varphi](1 + O(\sqrt{\varepsilon} \log \varepsilon)).
\]

Similarly,

\[
C_1^2 - C_2^2 = \frac{\sqrt{K\varepsilon}}{\pi(\lambda + 2\mu)} B_2^2[\varphi](1 + O(\sqrt{\varepsilon} \log \varepsilon)),
\]

where

\[
B_2^2[\varphi] := b_2^* - A_{12}^* b_1^3 + A_{22}^* b_2^3.
\]

Substituting the above estimates into (5.7), we have for \(x \in \Omega_R\),

\[
\nabla u = \frac{\sqrt{K\varepsilon}}{\pi\mu} B_2^1[\varphi] \nabla u^1_1 (1 + O(\sqrt{\varepsilon} \log \varepsilon)) + \frac{\sqrt{K\varepsilon}}{\pi(\lambda + 2\mu)} B_2^2[\varphi] \nabla u^2_1 (1 + O(\sqrt{\varepsilon} \log \varepsilon)) + O(1) \nabla (u_1^3 + u_2^3) + O(1)
\]

\[
= \frac{\sqrt{K\varepsilon}}{\pi} \left( \frac{B_2^1[\varphi]}{\mu} \nabla u^1_1 + \frac{B_2^2[\varphi]}{\lambda + 2\mu} \nabla u^2_1 \right) (1 + O(\sqrt{\varepsilon} \log \varepsilon)) + O(1) \nabla (u_1^3 + u_2^3) + O(1).
\]

The proof of Theorem 3.2 is finished. \qed
5.2. **Proof of Theorem 3.4** In dimension three, in order to solve $C_1^\alpha - C_2^\alpha$ from (5.1), we choose the equations with $\beta = 1, 2, \cdots, 6$ for $j = 1$ and $\beta = 4, 5, 6$ for $j = 2$, and denote them in block matrix

$$AX := \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix},$$

where

$$A_{11} := \begin{pmatrix} a_{11}^{\alpha \beta} \\ a_{21}^{\alpha \beta} \end{pmatrix}_{\alpha, \beta = 1, 2, 3}, \quad A_{12} := \begin{pmatrix} a_{11}^{\alpha \beta} \\ a_{12}^{\alpha \beta} \end{pmatrix}_{\alpha = 1, 2, 3; \beta = 4, 5, 6},$$

$$A_{21} := \begin{pmatrix} a_{21}^{\alpha \beta} \\ a_{22}^{\alpha \beta} \end{pmatrix}_{\alpha = 4, 5, 6; \beta = 1, 2, 3}, \quad A_{22} := \begin{pmatrix} a_{21}^{\alpha \beta} \\ a_{22}^{\alpha \beta} \end{pmatrix}_{\alpha, \beta = 4, 5, 6}.$$

$$X_1 = \left(C_1^1 - C_2^1, C_1^2 - C_2^2, C_1^3 - C_2^3 \right)^T, \quad X_2 = \left(C_1^4, C_1^5, C_1^6, C_2^4, C_2^5, C_2^6 \right)^T,$$

and

$$B_1 = \left(b_1^1, b_1^2, b_1^3 \right)^T, \quad B_2 = \left(b_2^1, b_2^2, b_2^3 \right)^T.$$

Before giving the expressions of $C_1^\alpha - C_2^\alpha$ by using (5.10) and Cramer’s rule, we first denote for $\alpha = 1, 2, 3$, $\beta_0 = 4, 5, 6$, and $j = 1, 2$, we get

$$B_{\beta_0}^{\alpha_j} := \det \begin{pmatrix} a_{11}^{\alpha \beta} & a_{12}^{\alpha \beta} \\ a_{21}^{\alpha \beta} & a_{22}^{\alpha \beta} \end{pmatrix}_{\alpha, \beta = 1, 2, 3},$$

here, $\alpha_1 \neq \beta_0$ if $j = 1$; $\alpha_2 \neq \beta_0$ if $j = 2$. Then similarly as (5.8), we get

$$C_1^1 - C_2^1 = \frac{a_{11}^{22}a_{11}^{33}}{\det A} (b_1^1 \det A_{22} - b_1^4 B_{11}^{11} + \cdots + b_2^6 B_{12}^{12}) + O(|\log \varepsilon|^{-1})$$

$$= \frac{\kappa}{\pi \mu} \cdot \frac{1}{|\log \varepsilon|} \left(b_1^1 - b_1^4 \frac{B_{11}^{11}}{\det A_{22}} + \cdots + b_2^6 \frac{B_{12}^{12}}{\det A_{22}} \right) + O(|\log \varepsilon|^{-1});$$

$$C_1^2 - C_2^2 = \frac{\kappa}{\pi \mu} \cdot \frac{1}{|\log \varepsilon|} \left(b_1^2 + b_4^1 \frac{B_{21}^{21}}{\det A_{22}} - \cdots - b_2^6 \frac{B_{22}^{62}}{\det A_{22}} \right) + O(|\log \varepsilon|^{-1});$$

$$C_1^3 - C_2^3 = \frac{\kappa}{\pi(\lambda + 2\mu)} \cdot \frac{1}{|\log \varepsilon|} \left(b_1^3 - b_1^4 \frac{B_{31}^{41}}{\det A_{22}} + \cdots + b_2^6 \frac{B_{32}^{62}}{\det A_{22}} \right) + O(|\log \varepsilon|^{-1}).$$

(5.11)

We now prove Theorem 3.4

**Proof of Theorem 3.4** For any $x \in \Omega_R$, by using (2.9), Theorems 2.2 and 2.6, we have

$$\nabla u = \sum_{\alpha=1}^{3} (C_1^\alpha - C_2^\alpha) \nabla u^\alpha + O(1) \sum_{\alpha=4}^{6} \nabla (u_1^\alpha + u_2^\alpha) + O(1).$$

(5.12)

Similar to (5.2), we have

$$\det A_{22} > \frac{1}{C} \text{ for some constant independent of } \varepsilon.$$
It follows from (5.11), (5.13), and Propositions 3.1 that
\[ C_1^1 - C_2^1 = \frac{\kappa}{\pi \mu} \cdot \frac{1}{\log \varepsilon} B_3^1[\varphi] \left( 1 + O\left( \log \varepsilon^{-1} \right) \right), \]
where
\[ B_3^1[\varphi] := b_1^* - b_4^* B_{11}^4 + \cdots + b_6^* B_{12}^4, \]
\[ B_{11}^4, \ldots, B_{12}^4 \]
are the limits of \[ \frac{B_{11}^4}{\det \mathcal{A}_{22}}, \ldots, \frac{B_{12}^4}{\det \mathcal{A}_{22}} \]
as \( \varepsilon \to 0 \), respectively. Similarly, we obtain
\[ C_1^2 - C_2^2 = \frac{\kappa}{\pi \mu} \cdot \frac{1}{\log \varepsilon} B_3^2[\varphi] \left( 1 + O\left( \log \varepsilon^{-1} \right) \right), \]
and
\[ C_1^3 - C_2^3 = \frac{\kappa}{\pi (\lambda + 2\mu)} \cdot \frac{1}{\log \varepsilon} B_3^3[\varphi] \left( 1 + O\left( \log \varepsilon^{-1} \right) \right), \]
where
\[ B_3^\beta[\varphi] := b_1^\beta - b_4^\beta B_{\beta 1}^4 + \cdots + b_6^\beta B_{\beta 2}^4, \]
\[ B_{\beta 1}^4, \ldots, B_{\beta 2}^4 \]
are the limits of \[ \frac{B_{\beta 1}^4}{\det \mathcal{A}_{22}}, \ldots, \frac{B_{\beta 2}^4}{\det \mathcal{A}_{22}} \]
as \( \varepsilon \to 0 \), respectively, \( \beta = 2, 3 \). Coming back to (5.12), for sufficiently small \( 0 < \varepsilon < 1/2 \), and \( x \in \Omega_R \), we have
\[ \nabla u = \frac{\kappa}{\pi} \cdot \frac{1}{\log \varepsilon} \left( \frac{1}{\mu} \sum_{\alpha=1}^{d+2} C_\alpha^\beta (\varphi) \nabla v_\alpha^\beta + \frac{1}{\lambda + 2\mu} \sum_{\alpha=1}^{d+2} C_\alpha^3 (\varphi) \nabla u_3^\beta \right) \left( 1 + O\left( \log \varepsilon^{-1} \right) \right) + O(1) \sum_{\alpha=4}^{6} \nabla (u_\alpha^1 + u_\alpha^2) + O(1). \]
The proof of Theorem 3.4 is completed. \( \square \)

6. The proof of Theorem 3.6 and Theorem 3.9

6.1. The proof of Theorem 3.6 We rewrite (2.3) as
\[
\nabla u = \sum_{\alpha=1}^{d(d+1)/2} (C_1^\alpha - C_2^\alpha) \nabla v_\alpha^\alpha + \sum_{\alpha=1}^{d(d+1)/2} C_2^\alpha \nabla(u_\alpha^\alpha + v_\alpha^\alpha) + \nabla v_0 \\
= \sum_{\alpha=1}^{d(d+1)/2} (C_1^\alpha - C_2^\alpha) \nabla v_\alpha^\alpha + \nabla u^b, \quad x \in \Omega, \tag{6.1}
\]
where \( u^b := \sum_{\alpha=1}^{d(d+1)/2} C_2^\alpha v_\alpha^\alpha + v_0 \), \( v_\alpha^\alpha := v_\alpha^\alpha + v_\alpha^\alpha \). From (2.4), we get
\[
\begin{cases}
L_{\lambda, \mu} v_\alpha^\alpha = 0, & x \in \Omega, \\
v_\alpha^\alpha = \psi_\alpha, & x \in \partial D_1 \cap \partial D_2, \\
v_\alpha^\alpha = 0, & x \in \partial D. \tag{6.2}
\end{cases}
\]
Hence, we have
\[ \| \nabla v_\alpha^\alpha \|_{L^\infty(\Omega)} \leq C. \]
The case \( \alpha = 1, 2 \) follows from Lemma 2.1. For \( \alpha = 3 \), we can use the result in [28, Theorem 1.1] by taking \( \varphi = \psi_3 \approx (\varepsilon - h_1(x_1), x_1) \) and \( \psi \approx (-h_2(x_1), x_1) \) there. Meanwhile,
\[ \| \nabla v_0 \|_{L^\infty(\Omega)} \leq C. \]
Therefore,
\[
\|\nabla u^b\|_{L^\infty(\Omega)} \leq C. \tag{6.3}
\]
By using \(2.9\) and the forth line of \(2.2\), we have
\[
\sum_{\alpha=1}^{d(d+1)/2} (C_1^\alpha - C_2^\alpha) a_{11}^{\alpha \beta} = \int_{\partial D_1} \frac{\partial u^b}{\partial \nu} \bigg|_t \cdot \psi_\beta. \tag{6.4}
\]
Denote
\[
B_\beta[\varphi] = \int_{\partial D_1} \frac{\partial u^b}{\partial \nu} \bigg|_t \cdot \psi_\beta, \quad \beta = 1, \ldots, d(d+1)/2. \tag{6.5}
\]
From \(6.3\), we have
\[
|B_\beta| \leq C, \quad x \in \Omega.
\]
By \(6.4\) and Cramer’s rule,
\[
C_1^1 - C_2^1 = \frac{1}{\det a_{11}} \left( B_1[\varphi] \begin{bmatrix} a_{11}^{11} & a_{11}^{12} & a_{11}^{13} \\ a_{11}^{21} & a_{11}^{22} & a_{11}^{23} \\ a_{11}^{31} & a_{11}^{32} & a_{11}^{33} \end{bmatrix} - B_2[\varphi] \begin{bmatrix} a_{11}^{11} & a_{11}^{12} & a_{11}^{13} \\ a_{11}^{21} & a_{11}^{22} & a_{11}^{23} \\ a_{11}^{31} & a_{11}^{32} & a_{11}^{33} \end{bmatrix} \right),
\]
\[
C_1^2 - C_2^2 = \frac{1}{\det a_{11}} \left( -B_1[\varphi] \begin{bmatrix} a_{11}^{11} & a_{11}^{12} & a_{11}^{13} \\ a_{11}^{21} & a_{11}^{22} & a_{11}^{23} \\ a_{11}^{31} & a_{11}^{32} & a_{11}^{33} \end{bmatrix} + B_2[\varphi] \begin{bmatrix} a_{11}^{11} & a_{11}^{12} & a_{11}^{13} \\ a_{11}^{21} & a_{11}^{22} & a_{11}^{23} \\ a_{11}^{31} & a_{11}^{32} & a_{11}^{33} \end{bmatrix} \right),
\]
\[
C_1^3 - C_2^3 = \frac{1}{\det a_{11}} \left( B_1[\varphi] \begin{bmatrix} a_{11}^{11} & a_{11}^{12} & a_{11}^{13} \\ a_{11}^{21} & a_{11}^{22} & a_{11}^{23} \\ a_{11}^{31} & a_{11}^{32} & a_{11}^{33} \end{bmatrix} - B_2[\varphi] \begin{bmatrix} a_{11}^{11} & a_{11}^{12} & a_{11}^{13} \\ a_{11}^{21} & a_{11}^{22} & a_{11}^{23} \\ a_{11}^{31} & a_{11}^{32} & a_{11}^{33} \end{bmatrix} \right),
\]
where \(a_{ij}\) is a \(3 \times 3\) matrix, \((a_{ij}^{\alpha \beta})\), \(i, j = 1, 2, \alpha, \beta = 1, 2, 3\). By Lemma \(5.1\) for \(\alpha = 1, 2\),
\[
C_1^\alpha - C_2^\alpha = \begin{cases} \frac{B_{1,\beta}[\varphi]}{a_{11}^{\alpha \beta}} + O(\varepsilon^{2/3} \log \varepsilon^{-1}), & m = 3, \\ \frac{B_{1,\beta}[\varphi]}{a_{11}^{\alpha \beta}} + O(\varepsilon |\log \varepsilon|), & m = 4, \\ \frac{B_{1,\beta}[\varphi]}{a_{11}^{\alpha \beta}} + O(\varepsilon^{3m/4}), & m \geq 5; \end{cases}
\]
and
\[
C_1^3 - C_2^3 = \begin{cases} \frac{B_{1,\beta}[\varphi]}{a_{11}^{\alpha \beta}} + O(\varepsilon^{2/3} \log \varepsilon^{-1}), & m = 3, \\ \frac{B_{1,\beta}[\varphi]}{a_{11}^{\alpha \beta}} + O(\varepsilon |\log \varepsilon|), & m = 4, \\ \frac{B_{1,\beta}[\varphi]}{a_{11}^{\alpha \beta}} + O(\varepsilon^{3m/4}), & m \geq 5. \end{cases}
\]
By using a similar argument that led to Propositions \(2.3\) and \(3.1\), we obtain

**Proposition 6.1.** Under the above assumptions, for sufficiently small \(\varepsilon > 0\),
\[
a_{11}^{11} = \frac{\mu Q_{2,m}}{\kappa^{1/m} \varepsilon^{1-1/m}} + O(1), \quad a_{11}^{22} = \frac{(\lambda + 2\mu) Q_{2,m}}{\kappa^{1/m} \varepsilon^{1-1/m}} + O(1), \quad m \geq 3,
\]
and
\[
a_{11}^{33} = \begin{cases} \frac{2(\lambda + 2\mu)}{3\varepsilon^{3/m}} |\log \varepsilon| + O(1), & m = 3, \\ \frac{(\lambda + 2\mu) \bar{Q}_{2,m}}{3\varepsilon^{3/m} \varepsilon^{1-1/m} + O(1), & m \geq 4,
\end{cases}
\]
where
\[
Q_{2,m} = 2 \int_0^\infty \frac{1}{1 + t^m} \, dt, \quad \bar{Q}_{2,m} = 2 \int_0^\infty \frac{t^2}{1 + t^m} \, dt.
\]
Proposition 6.2. Under the above assumptions, we have for small enough $\varepsilon > 0$, if $\alpha = 1, 2$,
\[
\mathcal{B}_\alpha[\varphi] - \mathcal{B}_\alpha^*[\varphi] = \begin{cases} 
O(\vert \log \varepsilon \vert^{-1}), & m = 3, \\
O(\varepsilon^{1/4}), & m = 4, \\
O(\varepsilon^{1/3}), & m \geq 5,
\end{cases} \quad \mathcal{B}_3[\varphi] - \mathcal{B}_3^*[\varphi] = \begin{cases} 
O(\vert \log \varepsilon \vert^{-1}), & m = 3, \\
O(\varepsilon^{1+\frac{\alpha}{c}}), & m = 4, 5, \\
O(\varepsilon^{\frac{m+2}{3m}}), & m \geq 6,
\end{cases}
\]
where $\mathcal{B}_\alpha^*[\varphi]$ is defined by (3.12), $\alpha = 1, 2, 3$.

We are ready to finish the proof of Theorem 3.6.

Completion of the proof of Theorem 3.6. For any $x \in \Omega_R$, by using (6.1), (6.3), and Theorem 2.2 we have
\[
\nabla u = \sum_{\alpha=1}^{3} (C_1^\alpha - C_2^\alpha) \nabla u_1^\alpha + O(1), \quad m \geq 3. \tag{6.9}
\]
For any $x \in \Omega_R$, by (6.7), (6.8), Propositions 6.1 and 6.2, if $m = 3$,
\[
C_1^1 - C_2^1 = \frac{\mathcal{B}_1^1[\varphi] + O(\vert \log \varepsilon \vert^{-1})}{\kappa^{1/3} 2^{2/3}} + O(1) \\
= \frac{\kappa^{1/3} 2^{2/3}}{\mu \kappa^{1/3} 2^{2/3}} \mathcal{B}_1^1[\varphi] \left(1 + O(\vert \log \varepsilon \vert^{-1})\right),
\]
\[
C_1^2 - C_2^2 = \frac{\kappa^{1/3} 2^{2/3}}{\lambda + 2\mu} \mathcal{B}_2^2[\varphi] \left(1 + O(\vert \log \varepsilon \vert^{-1})\right),
\]
\[
C_1^3 - C_2^3 = \frac{\kappa^{1/3} 2^{2/3}}{\lambda + 2\mu} \mathcal{B}_3^2[\varphi] \left(1 + O(\vert \log \varepsilon \vert^{-1})\right).
\]
Hence, coming back to (6.9), for $x \in \Omega_R$,
\[
\nabla u = \frac{\kappa^{1/3} 2^{2/3}}{Q_{2,3}} \left( \frac{\mathcal{B}_1^1[\varphi]}{\mu} \nabla u_1^1 + \frac{\mathcal{B}_2^2[\varphi]}{\lambda + 2\mu} \nabla u_2^2 \right) \left(1 + O(\vert \log \varepsilon \vert^{-1})\right) \\
+ \frac{3\kappa}{2(\lambda + 2\mu)\vert \log \varepsilon \vert} \mathcal{B}_3^1[\varphi] \nabla u_3^1 \left(1 + O(\vert \log \varepsilon \vert^{-1})\right) + O(1).
\]
The cases $m \geq 4$ follow from the same argument as above, we omit them here. This completes the proof of Theorem 3.6. \hfill \Box

6.2. The proof of Theorem 3.9

(i) The proof of the case $m = 3$ is very similar to that in the proof of Theorem 3.2 when $d = 3$ and $m = 2$. We omit it here.

(ii) For $m \geq 4$, (6.1) becomes
\[
\nabla u = \sum_{\alpha=1}^{6} (C_1^\alpha - C_2^\alpha) \nabla u_1^\alpha + \nabla u_b, \quad x \in \Omega. \tag{6.10}
\]
Recalling Theorems 2.2 and 2.6, one can see that we only need to obtain the asymptotics of $C_1^\alpha - C_2^\alpha$, $\alpha = 1, \cdots, 6$.
From (6.4), we have
\[
\begin{pmatrix}
  a_{11}^{11} & a_{11}^{12} & \cdots & a_{11}^{15} & a_{11}^{16} \\
  a_{11}^{21} & a_{11}^{22} & \cdots & a_{11}^{25} & a_{11}^{26} \\
  \vdots & \vdots & \cdots & \vdots & \vdots \\
  a_{11}^{51} & a_{11}^{52} & \cdots & a_{11}^{55} & a_{11}^{56} \\
  a_{11}^{61} & a_{11}^{62} & \cdots & a_{11}^{65} & a_{11}^{66}
\end{pmatrix}
\begin{pmatrix}
  C_1^1 - C_2^1 \\
  C_1^2 - C_2^2 \\
  \vdots \\
  C_1^5 - C_2^5 \\
  C_1^6 - C_2^6
\end{pmatrix}
= \begin{pmatrix}
  B_1[\varphi] \\
  B_2[\varphi] \\
  \vdots \\
  B_5[\varphi] \\
  B_6[\varphi]
\end{pmatrix}.
\]

Then by Cramer’s rule, we have for \( \alpha = 1, \cdots, 6 \),
\[
C_1^\alpha - C_2^\alpha = \frac{(-1)^{\alpha+1}}{\det a_{11}} \left( B_1[\varphi]B_1^\alpha - B_2[\varphi]B_2^\alpha - \cdots - B_6[\varphi]B_6^\alpha \right),
\]
where \((a_{ij})\) is a \(6 \times 6\) matrix, \(B_\beta^\alpha\) denotes \(B_\beta[\varphi]\)’s cofactor. Then we get for \( \alpha = 1, 2, 3, 4, 5, 6 \),
\[
C_1^\alpha - C_2^\alpha = \frac{B_\alpha[\varphi]}{a_{11}^{\alpha \alpha}} + O(\varepsilon^{m+2}), \quad m \geq 4.
\]
(6.11)

Therefore, we need to estimate \(B_\alpha[\varphi]\) and \(a_{11}^{\alpha \alpha}\).

**Proposition 6.3.** Under the above assumptions, we have for sufficiently small \( \varepsilon > 0 \), if \( \alpha = 1, 2, 3 \),
\[
B_\alpha[\varphi] - B_\alpha'[\varphi] = \begin{cases} 
O(|\log \varepsilon|^{-1}), & m = 4, \\
O(\varepsilon^{1/5}), & m = 5, \\
O(\varepsilon^{1/3}), & m \geq 6,
\end{cases}
\]
and \( \alpha = 4, 5, 6 \),
\[
B_\alpha[\varphi] - B_\alpha'[\varphi] = \begin{cases} 
O(|\log \varepsilon|^{-1}), & m = 4, \\
O(\varepsilon^{-1/4}), & m = 5, 6, \\
O(\varepsilon^{m+2}), & m \geq 7,
\end{cases}
\]
where \(B_\alpha'[\varphi]\) is defined by (3.12), \( \alpha = 1, 2, 3, 4, 5, 6 \).

**Proposition 6.4.** Under the above assumptions, for sufficiently small \( \varepsilon > 0 \) and \( m \geq 4 \),
\[
a_{11}^{\alpha \alpha} = \frac{\pi \mu Q_{3,m}}{k^{2/m} \varepsilon^{1-2/m}} + O(1), \quad \alpha = 1, 2, \quad a_{11}^{33} = \frac{\pi (\lambda + 2\mu) Q_{3,m}}{k^{2/m} \varepsilon^{1-2/m}} + O(1),
\]
\[
a_{11}^{44} = \begin{cases}
\frac{\pi \mu}{k^m} \log \varepsilon + O(1), & m = 4, \\
\frac{\pi \mu Q_{3,m}}{k^{2/m} \varepsilon^{1-4/m}} + O(1), & m \geq 5,
\end{cases}
\]
and for \( \alpha = 5, 6 \),
\[
a_{11}^{\alpha \alpha} = \begin{cases}
\frac{\pi (\lambda + 2\mu)}{k^m} \log \varepsilon + O(1), & m = 4, \\
\frac{\pi (\lambda + 2\mu) Q_{3,m}}{k^{2/m} \varepsilon^{1-4/m}} + O(1), & m \geq 5,
\end{cases}
\]
where
\[
Q_{3,m} = 2 \int_0^\infty \frac{t^3}{1 + t^m} \, dt, \quad \bar{Q}_{3,m} = 2 \int_0^\infty \frac{t^3}{1 + t^m} \, dt.
\]
Finally, we close this section by giving the proof of Theorem 3.9 (ii).

**Proof of Theorem 3.9 (ii).** For any \( x \in \Omega_R \), by \( \text{[6.11]} \), Lemma 5.1, Propostions 6.3 and 6.4, we give the proof only for \( m = 4 \). The other cases are similar. When \( m = 4 \),

\[
C_1^\alpha - C_2^\alpha = \frac{\sqrt{\kappa \varepsilon}}{\pi \mu Q_{3,4}} B^*_\alpha[\varphi](1 + O(\| \log \varepsilon \|^{-1})), \quad \alpha = 1, 2,
\]

\[
C_3^\alpha - C_2^\alpha = \frac{\sqrt{\kappa \varepsilon}}{\pi (\lambda + 2\mu) Q_{3,4}} B^*_\alpha[\varphi](1 + O(\| \log \varepsilon \|^{-1})),
\]

\[
C_4^\alpha - C_2^\alpha = \frac{2\kappa}{\pi |\log \varepsilon|} B^*_\alpha[\varphi](1 + O(\| \log \varepsilon \|^{-1})),
\]

and for \( \alpha = 5, 6 \),

\[
C_1^\alpha - C_2^\alpha = \frac{4\kappa}{\pi (\lambda + 2\mu) |\log \varepsilon|} B^*_\alpha[\varphi](1 + O(\| \log \varepsilon \|^{-1})).
\]

Substituting the above terms into \( \text{[6.10]} \), using Theorems 2.2 and 2.6, for \( x \in \Omega_R \),

\[
\nabla u = \sum_{\alpha=1}^{6} (C_1^\alpha - C_2^\alpha) \nabla u_1^\alpha + O(1)
\]

\[
= \frac{\sqrt{\kappa \varepsilon}}{\pi Q_{3,4}} \left( \sum_{\alpha=1}^{2} \frac{1}{\mu} B^*_\alpha[\varphi] \nabla u_1^\alpha + \frac{1}{\lambda + 2\mu} B^*_3[\varphi] \nabla u_1^3 \right) (1 + O(\| \log \varepsilon \|^{-1}))
\]

\[
+ \frac{2\kappa}{\pi |\log \varepsilon|} \left( \frac{1}{\mu} B^*_1[\varphi] \nabla u_1^4 + \sum_{\alpha=5}^{6} \frac{1}{\lambda + 2\mu} B^*_\alpha[\varphi] \nabla u_1^\alpha \right) (1 + O(\| \log \varepsilon \|^{-1})) + O(1).
\]

The proof is finished. \( \square \)

7. Application: An Extended Flaherty-Keller Formula

As an application of the asymptotic expressions in Propositions 2.3 and 6.1, we prove an extended Flaherty-Keller formula on the effective elastic property of a periodic composite with densely packed inclusions. We are going to follow the setting in \( \text{[22,32,36]} \) other than the symmetry conditions. Specifically we denote the period cell by \( Y := (-L_1, L_1) \times (-L_2, L_2) \), where \( L_1 \) and \( L_2 \) are two positive numbers. Let \( D \subset Y \) be a \( m \)-convex domain containing the origin with \( C^2 \) boundary. Assume that \( D \) is close to the horizontal boundary of \( Y \) and away from the vertical boundary. Let \( \varepsilon/2 \) be the distance between \( D \) and the horizontal boundary of \( Y \), so that \( \varepsilon \) becomes the distance between two adjacent inclusions, see Figure 2.

As in \( \text{[36]} \), after translation, we denote

\[
Y_1 := Y + (0, L_2) = (-L_1, L_1) \times (0, 2L_2),
\]

\[
D_1 := D + (0, 2L_2 + \varepsilon/2), \quad D_2 := D + (0, \varepsilon/2), \quad Y' := Y_1 \setminus \overline{D_1 \cup D_2},
\]

and set

\[
\Gamma_+ := (\partial D_1 \cup \{x_2 = 2L_2 + \varepsilon/2\}) \cap \partial Y', \quad \Gamma_- := (\partial D_2 \cup \{x_2 = \varepsilon/2\}) \cap \partial Y'.
\]
Then we obtain the effective shear modulus $\mu_m^*$ and the effective extensional modulus $E_m^*$ defined by

$$\mu_m^* = \frac{L_2^2}{L_1} \epsilon_1, \quad E_m^* = \frac{E}{\lambda + 2\mu} \frac{L_2^2}{L_1} \epsilon_1,$$

where

$$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}$$

and

$$\epsilon_1^\alpha = \int_{Y'} (C_0 e^{(v_1^\alpha)}) e^{(v_1^\alpha)} dx, \quad \alpha = 1, 2,$$

here $v_1^\alpha \in H^1(Y')$ is the solution to

$$\begin{cases}
\mathcal{L}_{\lambda, \mu} v_1^\alpha := \nabla \cdot (C_0 e^{(v_1^\alpha)}) = 0, & \text{in } Y', \\
v_1^\alpha = \psi_\alpha, & \text{on } \Gamma_+, \\
v_1^\alpha = 0, & \text{on } \Gamma_-, \\
\frac{\partial v_1^\alpha}{\partial \nu} = 0, & \text{on } x_1 = \pm L_1.
\end{cases}$$

Note that the definition of $\epsilon_1^\alpha$ is similar to that of $a_1^\alpha$ in (2.7). Then by using Proposition 2.3 and Proposition 6.1, we have

**Theorem 7.1.** Given $m \geq 2$. As $\varepsilon \to 0$,

$$\mu_m^* = \mu \frac{L_2}{L_1} \frac{Q_{2,m}}{\kappa^{1/m} \varepsilon^{1-1/m}} + O(1), \quad \text{and} \quad E_m^* = E \frac{L_2}{L_1} \frac{Q_{2,m}}{\kappa^{1/m} \varepsilon^{1-1/m}} + O(1),$$

where $\kappa$ is the curvature of $\partial D$ near the origin, and

$$Q_{2,m} = 2 \int_0^\infty \frac{1}{1 + t^m} \, dt.$$
Clearly, by a direct calculation, we find that Theorem 7.1 for \( m = 2 \) is actually the result in [32]. Furthermore, we would like to remark that compared with [32], our method can do not need to assume that \( D_1 \) and \( D_2 \) are symmetric.

8. Appendix: the proof of Theorem 2.2 and Theorem 2.6

We here give the proof of Theorem 2.2 and Theorem 2.6. The key point is that \( |\mathcal{L}_{\lambda, \mu} u_1^n| \) is improved to be controled by \( \frac{C}{\delta(x)} \). This is due to the introduction of \( \tilde{u}_1^n \). Then we adapt the iteration technique developed in [10] to capture all singular terms of \( \nabla u_1^n \) and to obtain the asymptotic formulas.

Proof of Theorems 2.2 and 2.6

Step 1. Claim:

\[
|\mathcal{L}_{\lambda, \mu} u_1^n| \leq C \left\{ \frac{1}{\delta(x)}, \frac{|x|^m-2}{\delta(x)} \right\}, \quad m = 2, \quad \alpha = 1, \cdots, d; \\
|\mathcal{L}_{\lambda, \mu} u_1^n| \leq \frac{C}{\delta(x)}, \quad \alpha = d + 1, \cdots, \frac{d(d+1)}{2}.
\]

We will prove the claim in the light of the following two cases.

Case 1. \( d = 2 \). A direct calculation yields in \( \Omega_R \),

\[
\left| \partial_{x_1 x_1} (\tilde{u}_1) \right|, \left| \partial_{x_1 x_2} (\tilde{u}_1) \right| \leq C \frac{|x_1|^{m-2}}{\delta(x_1)}, \quad \left| \partial_{x_1 x_1} (\tilde{u}_1) \right| \leq C \left\{ \frac{|x_1|}{\delta(x_1)}, m = 2, \quad \alpha = 1, \cdots, d; \right\}, \quad m = 3.
\]

and

\[
\partial_{x_1 x_2} (\tilde{u}_1) = \frac{-\partial_{x_1} (h_1 - h_2)}{\delta^2(x_1)}, \quad \partial_{x_1 x_1} (\tilde{u}_1) = \frac{\lambda + \mu}{\lambda + 2\mu} \frac{\partial_{x_1} (h_1 - h_2)}{\delta^2(x_1)}.
\]

By using (8.2), we have

\[
\left| (\mathcal{L}_{\lambda, \mu} u_1^n)^1 \right| = \lambda \left( \partial_{x_1 x_1} (u_1^n)^1 + \partial_{x_1 x_2} (u_1^n)^1 \right) + \mu \left( 2\partial_{x_1 x_1} (u_1^n)^1 + \partial_{x_2 x_2} (u_1^n)^1 + \partial_{x_1 x_2} (u_1^n)^1 \right) \\
= \mu \Delta (u_1^n)^1 + (\lambda + \mu) \left( \partial_{x_1 x_1} (u_1^n)^1 + \partial_{x_2 x_2} (u_1^n)^1 \right) \\
= \mu \Delta (\tilde{u}_1^n)^1 + (\lambda + \mu) \left( \partial_{x_1 x_1} (\tilde{u}_1^n)^1 + \partial_{x_2 x_2} (\tilde{u}_1^n)^1 \right) \leq C \frac{|x_1|^{m-2}}{\delta(x_1)}.
\]

By using (8.3), we get

\[
(\lambda + \mu) \partial_{x_1 x_2} (\tilde{u}_1^n)^1 + (\lambda + 2\mu) \partial_{x_2 x_2} (\tilde{u}_1^n)^2 = 0,
\]

which means that the “bad” terms in (8.3) are eliminated. Combining this and (8.2), we obtain

\[
\left| (\mathcal{L}_{\lambda, \mu} u_1^n)^2 \right| = \left| (\lambda + \mu) \left( \partial_{x_1 x_2} (u_1^n)^1 + \partial_{x_2 x_2} (u_1^n)^2 \right) + \mu \Delta u_1^n \right| \\
= \left| \mu \partial_{x_1 x_1} (\tilde{u}_1^n)^2 \right| \leq C \left\{ \frac{|x_1|}{\delta(x_1)}, m = 2, \quad \alpha = 1, \cdots, d; \right\}, \quad m \geq 3.
\]
By (8.7), we obtain
\[ |\mathcal{L}_{\lambda, \mu} u_1^1| \leq C \left( \frac{1}{\delta(x_1)} \right)^{m-2} \left( 1 + \frac{r}{|x_1|} \right), \quad m = 2, \]
Similarly, we have
\[ |\mathcal{L}_{\lambda, \mu} u_1^2| \leq C \left( \frac{1}{\delta(x_1)} \right)^{\frac{m-2}{m-3}} \left( 1 + \frac{r}{|x_1|} \right), \quad m \geq 3. \]
Furthermore, we have
\[ |\partial_{x_1 x_1} u_1^3| \leq \frac{C |x_1|^{m-1}}{\delta(x_1)}, \quad |\partial_{x_1 x_2} u_1^3|, |\partial_{x_2 x_2} u_1^3| \leq \frac{C}{\delta(x_1)}. \]
Then we obtain
\[ |\mathcal{L}_{\lambda, \mu} u_1^3| \leq \frac{C}{\delta(x_1)}. \]

**Case 2.** \( d = 3 \). We have in \( \Omega_R \),
\[ |\partial_{x_1 x_1}(\tilde{u}_1^1)|, |\partial_{x_2 x_2}(\tilde{u}_1^1)|, |\partial_{x_1 x_2}(\tilde{u}_1^1)|, |\partial_{x_1 x_3}(\tilde{u}_1^1)|, |\partial_{x_2 x_3}(\tilde{u}_1^1)| \leq \frac{C|x'|^{m-2}}{\delta(x')}, \]
\[ |\partial_{x_1 x_1}(\tilde{u}_1^1)|, |\partial_{x_2 x_2}(\tilde{u}_1^1)| \leq C \left( \frac{|x'|}{|x'|^{m-3}} \right), \quad m = 2, \]
and \( m \geq 3; \) (8.6)
and
\[ \partial_{x_1 x_1}(\tilde{u}_1^1) = -\frac{\partial_{x_1}(h_1 - h_2)}{\delta(x')^2}, \quad \partial_{x_1 x_3}(\tilde{u}_1^1) = \frac{\lambda + \mu}{\lambda + 2\mu} \frac{\partial_{x_1}(h_1 - h_2)}{\delta(x')^2}. \] (8.7)
By (8.6), we have
\[ |(\mathcal{L}_{\lambda, \mu} u_1^1)| = |\lambda(\partial_{x_1 x_1}(u_1^1) + \partial_{x_3 x_3}(u_1^1)) + \mu(2\partial_{x_1 x_1}(u_1^1) + \partial_{x_2 x_2}(u_1^1) + \partial_{x_1 x_3}(u_1^1))| \leq \frac{C|x'|^{m-2}}{\delta(x')}, \] (8.8)
and
\[ |(\mathcal{L}_{\lambda, \mu} u_1^2)| = |\lambda(\partial_{x_1 x_1}(u_1^1) + \partial_{x_3 x_3}(u_1^1)) + \mu(\partial_{x_1 x_1}(u_1^1) + \partial_{x_2 x_2}(u_1^1) + \partial_{x_1 x_3}(u_1^1))| \leq \frac{C|x'|^{m-2}}{\delta(x')}. \] (8.9)
By (8.7), we obtain
\[ (\lambda + \mu)\partial_{x_3 x_1}(\tilde{u}_1^1) + (\lambda + 2\mu)\partial_{x_3 x_3}(\tilde{u}_1^1) = 0. \] (8.10)
Then (8.6) and (8.10) imply that
\[ |(\mathcal{L}_{\lambda, \mu} u_1^1)| = |\lambda(\partial_{x_1 x_1}(u_1^1) + \partial_{x_3 x_3}(u_1^1)) + \mu(\partial_{x_1 x_1}(u_1^1) + \partial_{x_2 x_2}(u_1^1) + \partial_{x_1 x_3}(u_1^1)) + 2\partial_{x_3 x_3}(u_1^1)|
Therefore, \( \text{the boundedness of the global energy of } \nabla(v_1^\alpha - \tilde{u}_1^\alpha) \) is established.
Step 3. Proof of
\[
\int_{\Omega_s(z')} |\nabla w_1^\alpha|^2 \, dx \leq C \delta^d(z') \begin{cases} 
\delta^{2(1-\frac{2}{m})}(z'), & \alpha = 1, \ldots, d, \\
\delta(1), & \alpha = (d+1), \ldots, d(d+1)/2.
\end{cases}
\]  
(8.12)

where
\[
\Omega_s(z') := \{(x', x_d) \in \mathbb{R}^d \mid h_2(x') < x_d < \varepsilon + h_1(x'), |x' - z'| < r\}, \quad s < R,
\]
and \(w_1^\alpha = v_1^\alpha - u_1^\alpha\), \(\alpha = 1, \ldots, d(d+1)/2\), satisfying
\[
\begin{cases}
\mathcal{L}_{\lambda, \mu} w_1^\alpha = -\mathcal{L}_{\lambda, \mu} u_1^\alpha, & \text{in } \Omega, \\
w = 0, & \text{on } \partial \Omega.
\end{cases}
\]  
(8.13)

We will use the iteration scheme developed in [10] to prove (8.12). For \(0 < t < s < R\), let \(\eta\) be a smooth cutoff function satisfying \(\eta(x') = 1\) if \(|x' - z'| < t\), \(\eta(x') = 0\) if \(|x' - z'| > s\), \(0 \leq \eta(x') \leq 1\) if \(t \leq |x' - z'| \leq s\), and \(|\nabla x' \cdot \eta(x')| \leq \frac{2}{s-t}\).

Multiplying the equation in (8.13) by \(w_1^\alpha \eta^2\) and applying integration by parts, Hölder’s inequality, and Cauchy inequality, we get
\[
\int_{\Omega_s(z')} |\nabla w_1^\alpha|^2 \, dx \leq \frac{C}{(s-t)^2} \int_{\Omega_s(z')} |w_1^\alpha|^2 \, dx + C(s-t)^2 \int_{\Omega_s(z')} |\mathcal{L}_{\lambda, \mu} u_1^\alpha|^2 \, dx.
\]  
(8.14)

On one hand, we obtain from Hölder’s inequality that
\[
\int_{\Omega_s(z')} |w_1^\alpha|^2 \, dx = \int_{\Omega_s(z')} \left| \int_{h_2(x')}^{x_d} \partial_{x_d} w_1^\alpha(x', \xi) \, d\xi \right|^2 \, dx \leq C \delta^2(z') \int_{\Omega_s(z')} |\nabla w_1^\alpha|^2 \, dx.
\]  
(8.15)

On the other hand, we estimate the second term on the right hand side of (8.14) according to the following two cases.

**Case 1.** \(|z'| \leq \varepsilon^{1/m}\). By using (8.1), we have for \(0 < s < \varepsilon^{1/m}\),
\[
\int_{\Omega_s(z')} |\mathcal{L}_{\lambda, \mu} u_1^\alpha|^2 \, dx \leq Cs^{d-1} \begin{cases} 
\varepsilon^{\frac{2(m-2)}{m}-1}, & \alpha = 1, \ldots, d, \\
\varepsilon^{-1}, & \alpha = (d+1), \ldots, d(d+1)/2.
\end{cases}
\]  
(8.16)

This is an improvement of [10] (3.32),(3.35)]. Denote
\[
F(t) := \int_{\Omega_t(z')} |\nabla w_1^\alpha|^2.
\]

Then substituting (8.15) and (8.16) into (8.14), we have
\[
F(t) \leq \left( \frac{C_1 \varepsilon}{s-t} \right)^2 F(s) + C(s-t)^2 s^{d-1} \begin{cases} 
\varepsilon^{\frac{2(m-2)}{m}-1}, & \alpha = 1, \ldots, d, \\
\varepsilon^{-1}, & \alpha = (d+1), \ldots, d(d+1)/2.
\end{cases}
\]  
(8.17)

where \(c_1\) is a universal constant.

Let \(k = \left\lfloor \frac{1}{3c_1 \varepsilon^{1/m}} \right\rfloor\) and \(t_i = 2c_1 \varepsilon i, i = 1, \ldots, k\). Then by (8.17) with \(s = t_{i+1}\) and \(t = t_i\), we have
\[
F(t_i) \leq \frac{1}{4} F(t_{i+1}) + C(i+1)^{d-1} \varepsilon^d \begin{cases} 
\varepsilon^{\frac{2(m-2)}{m}-1}, & \alpha = 1, \ldots, d, \\
1, & \alpha = (d+1), \ldots, d(d+1)/2.
\end{cases}
\]
After $k$ iterations, using the global boundedness of $\nabla w_1^0$, we have
\[
F(t_1) \leq Cc^d \begin{cases} 
\varepsilon^{2(m-2) \over m}, & \alpha = 1, \ldots, d, \\
1, & \alpha = (d+1), \ldots, d(d+1)/2.
\end{cases}
\]

**Case 2.** $\varepsilon^{1/m} < |z'| < R$. For $0 < s < \frac{2}{3}|z'|$, \(8.16\) becomes
\[
\int_{\Omega_s(z')} |\mathcal{L}_{\lambda,\mu} w_1^0|^2 \, dx \leq C s^{d-1} \begin{cases} 
|z'|^{m-4}, & \alpha = 1, \ldots, d, \\
|z'|^{-m}, & \alpha = (d+1), \ldots, d(d+1)/2.
\end{cases}
\]

\(8.17\) becomes
\[
F(t) \leq \left( c_2 |z'|^{m \over s-t} \right)^2 F(s) + C(s-t)^2 s^{d-1} \begin{cases} 
|z'|^{m-4}, & \alpha = 1, \ldots, d, \\
|z'|^{-m}, & \alpha = (d+1), \ldots, d(d+1)/2,
\end{cases}
\]
where $c_2$ is another universal constant. Let $k = \left[ \frac{1}{4c_2|z'|} \right]$ and $t_i = 2c_2 i |z'|^m, i = 1, \ldots, k$. Then by \(8.17\) with $s = t_{i+1}$ and $t = t_i$, we have
\[
F(t_i) \leq \frac{1}{4} F(t_{i+1}) + C(i+1)^{d-1} |z'|^{md} \begin{cases} 
|z'|^{2(m-2)}, & \alpha = 1, \ldots, d, \\
1, & \alpha = (d+1), \ldots, d(d+1)/2.
\end{cases}
\]

After $k$ iterations, using the global boundedness of $\nabla w_1^0$, we have
\[
F(t_k) \leq C |z'|^{md} \begin{cases} 
|z'|^{2(m-2)}, & \alpha = 1, \ldots, d, \\
1, & \alpha = (d+1), \ldots, d(d+1)/2.
\end{cases}
\]

So \(8.12\) is proved.

**Step 4.** Scaling and $L^\infty$-estimates. It follows from \([10, (3.40)]\) that
\[
\|\nabla w_1^0\|_{L^\infty(\Omega_{4\delta}(z'))} \leq C \left( \delta^{d-\frac{d}{2}} \|\nabla w_1^0\|_{L^2(\Omega_{\delta}(z'))} + \delta^2 |z'| \|\mathcal{L}_{\lambda,\mu} w_1^0\|_{L^\infty(\Omega_{\delta}(z'))} \right).
\]

By using \(8.12\) and \(8.1\), we obtain
\[
|\nabla w_1^0(z', x_d)| \leq C, \quad h_2(z') < x_d < \varepsilon + h_1(z').
\]

Theorems \(2.2\) and \(2.6\) are proved.

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