Random data Cauchy problem for the wave equation on compact manifold

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Abstract. Inspired by the work of Burq and Tzvetkov (Invent. math. 173(2008), 449-475.), firstly, we construct the local strong solution to the cubic nonlinear wave equation with random data for a large set of initial data in $H^s(M)$ with $s \geq \frac{5}{14}$, where $M$ is a three dimensional compact manifold with boundary, moreover, our result improves the result of Theorem 2 in (Invent. math. 173(2008), 449-475.); secondly, we construct the local strong solution to the quintic nonlinear wave equation with random data for a large set of initial data in $H^s(M)$ with $s \geq \frac{1}{6}$, where $M$ is a two dimensional compact boundaryless manifold; finally, we construct the local strong solution to the quintic nonlinear wave equation with random data for a large set of initial data in $H^s(M)$ with $s \geq \frac{23}{90}$, where $M$ is a two dimensional compact manifold with boundary.

1. Introduction

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In this paper, we investigate the Cauchy problem for
\[ u_{tt} - \Delta u + u^3 = 0, \quad (1.1) \]
with real initial data \( f = (f_1, f_2) \in \mathcal{H}^s(M) = H^s(M) \times H^{s-1}(M) \), where \( M \) is a three dimensional compact manifold with boundary. We also investigate the Cauchy problem for
\[ u_{tt} - \Delta u + u^5 = 0, \quad (1.3) \]
with real initial data \( f = (f_1, f_2) \in \mathcal{H}^s(M) = H^s(M) \times H^{s-1}(M) \), where \( M \) is a two dimensional compact manifold.

Rauch [26] established the global regularity of (1.3)-(1.4) in three dimension space with small initial energies. Struwe [32] obtained a unique global radially symmetric solution for any radially symmetric initial data \( f_1 \in C^3(\mathbb{R}^3), f_2 \in C^2(\mathbb{R}^3) \). Some people studied the global well-posedness, scattering and global space-time bounds [1, 2, 13–18, 20, 21, 24, 27, 28, 33]. Mockenhaupt et al. [19] studied local smoothing of Fourier integral operators and Carleson-Sjölin estimates of the wave equation. Smith and Sogge [29] proved that some Strichartz estimates for (1.3)-(1.4) hold on \( n \)-dimensional Riemannian manifolds with smooth, strictly geodesically concave boundaries and \( n \geq 2 \). Christ et al. [12] proved that the solution map of (1.3)-(1.4) fails to be continuous at zero in the \( H^s(\mathbb{R}^3) \times H^{s-1}(\mathbb{R}^3) \)-topology for \( 0 < s < 1 \). Christ et al. [12] also proved that the solution map of (1.3)-(1.4) fails to be continuous at zero in the \( H^s(\mathbb{R}^2) \times H^{s-1}(\mathbb{R}^2) \)-topology for \( 0 < s < \frac{1}{2} \). Burq et al. [7] studied global existence for energy critical waves in 3-D domains.

Burq and Tzvetkov [8] investigated the invariant measure for a three dimensional wave equation
\[ w_{tt} - \Delta w + |w|^\alpha w = 0, (w, w_t)|_{t=0} = (g_1, g_2), \quad \alpha < 2 \quad (1.5) \]
with Dirichlet boundary condition and random initial data. Burq and Tzvetkov [9] studied the local theory of the Cauchy problem for (1.1)-(1.2) with random data in supercritical case on the three compact manifold. More precisely, they constructed the
local strong solution for a large set of initial data in $H^s(M)$ with $s \geq \frac{1}{4}$, where $M$ is a three dimensional boundaryless compact manifold and constructed the local strong solution for a large set of initial data in $H^s(M)$ with $s \geq \frac{8}{27}$, where $M$ is a three dimensional compact manifold with boundary; they also established the ill-posedness in $H^s(M)$ with $s < \frac{1}{2}$ in the sense that the flow map on $H^s(M)$ with $s < \frac{1}{2}$ is discontinuous at zero. Burq and Tzvetkov [10] obtained the global existence result of the Cauchy problem for a supercritical wave equation. Bourgain and Bulut [4–6] studied Gibbs measure evolution in radial nonlinear wave on a three dimensional ball. Burq and Tzvetkov [11] established the probabilistic well-posedness for (1.1) in $H^s(M)$, $0 < s < \frac{1}{2}$ with a suitable randomization on the three dimensional torus. Lührmann and Mendelson [22] established an almost sure global existence result of the defocusing nonlinear wave equation of power-type on $R^3$ with respect to a suitable randomization of the initial data. Recently, Pocovnicu [25] studied the almost surely global well-posedness for the energy-critical defocusing nonlinear wave equation on $R^d, d = 4$ and $5$ with random data. Very recently, Oh and Pocovnicu [23] studied the probabilistic global well-posedness of (1.1) on $R^3$.

In this paper, inspired by [3, 9, 31], firstly, for a large set of initial data in $H^s(M)$ with $s \geq \frac{5}{14}$, we construct the local strong solution to the random data Cauchy problem for (1.1), where $M$ is a three dimensional compact manifold with boundary; secondly, for a large set of initial data in $H^s(M)$ with $s \geq \frac{1}{6}$, we construct the local strong solution to the random data Cauchy problem for (1.3), where $M$ is a two dimensional compact boundaryless manifold; finally, for a large set of initial data in $H^s(M)$ with $s \geq \frac{23}{90}$, we construct the local strong solution to the random data Cauchy problem for (1.3), where $M$ is a two dimensional compact manifold with boundary.

We give some notations and some definitions before presenting the main results. We assume that $\Delta$ is the Laplace-Beltrami operator on compact manifold and $\Delta_D$ is the Laplace-Beltrami operator associated to the Dirichlet boundary condition and $\Delta_N$ is the Laplace-Beltrami operator associated to the Neumann boundary condition. $(\Omega, \mathcal{F}, P)$ is a probability space. We define

$$
\|v(t, x)\|_{L^q_t([0,T])L^r_x(M)} = \left( \int_0^T \left( \int_M |u|^r dx \right)^{\frac{q}{r}} dt \right)^{\frac{1}{q}}
$$
and

\[\|u(\omega, t, x)\|_{L^p_t(L^q_x([0,T) \times M))} = \left( \int_\Omega \left( \int_0^T \left( \int_M |u|^p \, dx \right)^{\frac{q}{p}} \, dt \right)^{\frac{p}{q}} \, dP(\omega) \right)^{\frac{1}{p}}.\]

**Definition 1.1.** Assume that \((e_n) \in C^\infty(M) (n = 1, 2, \cdots)\) is an orthonormal basis of \(L^2(M)\) and \((h_n(\omega), l_n(\omega))_{n=1}^\infty\) is a sequence of independent, 0 mean, real random variables on a probability space \((\Omega, \mathcal{F}, P)\) such that

\[\exists C > 0, \forall n \geq 1, \int_\Omega (|h_n(\omega)|^6 + |l_n(\omega)|^6) \, dP(\omega) < C. \tag{1.6}\]

Let \(f = (f_1, f_2)\), where

\[f_1 = \sum_{n=1}^\infty \alpha_n e_n(x), f_2(x) = \sum_{n=1}^\infty \beta_n e_n(x), \alpha_n, \beta_n \in \mathbb{R}\]

and the map

\[\omega \mapsto f^\omega = \left( f_1^\omega(x) = \sum_{n=1}^\infty h_n(\omega) \alpha_n e_n(x), f_2(x) = \sum_{n=1}^\infty l_n(\omega) \beta_n e_n(x) \right) \tag{1.7}\]

is equipped with the Borel sigma algebra from \((\Omega, \mathcal{F})\). From (1.7), we know that the map \(\omega \mapsto f^\omega\) is measurable and \(f^\omega \in L^2(\Omega; \mathcal{H}^s(M))\). Hence, this defines a \(\mathcal{H}^s(M)\) valued random variable, which is the random function related to \(f\).

**Definition 1.2.** Assume that \(M\) is a smooth compact manifold and \((e_n) \in C^\infty(M) (n = 1, 2, \cdots)\) is an orthonormal basis of \(L^2(M)\) and \(-\Delta e_n = \lambda^2_n e_n\). Let

\[H^s(M) = \left\{ h \in H^s(M), h = \sum_{n=1}^\infty \gamma_n e_n(x), \|h\|_{H^s(M)}^2 = \sum_{n=1}^\infty (1 + \lambda^2_n)^{2s} |\gamma_n|^2 < \infty \right\}.\]

Define \(\mathcal{H}^s(M) = H^s(M) \times H^{s-1}(M)\).

**Definition 1.3.** Assume that \(M\) is a smooth manifold with boundary, compact closure and \((e_n) \in C^\infty(M) (n = 1, 2, \cdots)\) is an orthonormal basis of \(L^2(M)\) and \(-\Delta_D e_n = \lambda^2_n e_n\) with \(e_n(x) \mid_{\partial M} = 0\). Let

\[H^s_D(M) = \left\{ h = \sum_{n=1}^\infty \gamma_n e_n(x), \|h\|_{H^s_D(M)}^2 = \sum_{n=1}^\infty (1 + \lambda^2_n)^{2s} |\gamma_n|^2 < \infty \right\}.\]

Define \(\mathcal{H}^s_D(M) = H^s_D(M) \times H^{s-1}_D(M)\).
Definition 1.4. Assume that $M$ is a smooth manifold with boundary, compact closure and $(e_n) \in C^\infty(M)(n = 1, 2, \cdots)$ is an orthonormal basis of $L^2(M)$ and $-\Delta_N e_n = \lambda_n^2 e_n$ with $N_x \nabla_x e_n(x) = 0$, where $x \in \partial M$ and $N_x$ is a unit field with respect to the metric. Let

$$H^s_N(M) = \left\{ h = \sum_{n=1}^{\infty} \gamma_n e_n(x), \|h\|_{H^s_N(M)}^2 = \sum_{n=1}^{\infty} (1 + \lambda_n^2)^s |\gamma_n|^2 < \infty \right\}.$$ 

Define $\mathcal{H}^s_M = H^s_N(M) \times H^{s-1}_N(M)$.

The main result of this paper are as follows.

Theorem 1.1. Let (1.6) be valid and $M$ be a three dimensional manifold with boundary and $s \geq \frac{5}{14}$ and $f = (f_1, f_2) \in \mathcal{H}^s_D(M)$ and $f^\omega \in L^2(\Omega; \mathcal{H}^s_D(M)$ be defined by the randomization (1.7). For a.s. $\omega \in \Omega$, there exist $T_\omega > 0$ and a unique solution to (1.1) with $u \mid_{R \times \partial M} = 0$ and the initial data $f^\omega$ in a space continuously embedded in

$$X_\omega = \left( \cos(t \sqrt{-\Delta_D}) f_1^\omega + \frac{\sin(t \sqrt{-\Delta_D})}{\sqrt{-\Delta_D}} f_2^\omega \right) + C([-T_\omega, T_\omega]; H^2_D(M)).$$

More precisely, for $0 < T \leq 1$, there exists $C > 0, \delta > 0$, an event $\Omega_T$ satisfying

$$P(\Omega_T) \geq 1 - CT^\frac{2s}{s+1}$$

such that for every $\omega \in \Omega_T$ there exists a unique solution of (1.1) with data $f^\omega$ in a space continuously embedded in $C(\lbrack 0, T \rbrack; H^s(M))$. Moreover, when $h_n, g_n$ are standard real Gaussian or Bernoulli variables, we have

$$P(\Omega_T) \geq 1 - C \exp \left( cT^{-\frac{s}{2}} \right).$$

Remark 1: In Theorem 1.1, if $\Delta_D, \mathcal{H}^s_D(M)$ and Dirichlet boundary condition $u \mid_{R \times \partial M} = 0$ are replaced by $\Delta_N, \mathcal{H}^s_N(M)$ and Neumann boundary condition $N_x \nabla_x u(x) \mid_{R \times \partial M} = 0$, respectively, the conclusion is still valid. $\mathcal{H}^\frac{1}{2}(M)$ is the critical space of (1.1)-(1.2). In Theorem of [9], the authors have constructed the local strong solution to the cubic nonlinear wave equation with random data for a large set of initial data in $H^s(M)$ with $s \geq \frac{8}{21}$. Thus, our result improves the result of [9].

Theorem 1.2. Let (1.6) be valid and $M$ be a two dimensional boundaryless manifold and $s \geq \frac{1}{6}$ and $f = (f_1, f_2) \in \mathcal{H}^s(M)$ and $f^\omega \in L^2(\Omega; \mathcal{H}^s(M)$ be defined by the
randomization (1.7). For a.s. $\omega \in \Omega$, there exist $T_\omega > 0$, $\sigma \geq \frac{1}{2}$ and a unique solution to (1.3) with initial data $f^\omega$ in a space continuously embedded in

$$X_\omega = \left( \cos(t\sqrt{-\Delta}) f^\omega_1 + \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}} f^\omega_2 \right) + C([-T_\omega, T_\omega]; H^{\sigma}(M)).$$

More precisely, for $0 < T \leq 1$, there exists $C > 0, \delta_1 > 0$, an event $\Omega_T$ satisfying

$$P(\Omega_T) \geq 1 - CT^{1+\delta_1}$$

(1.10)

such that for every $\omega \in \Omega_T$ there exists a unique solution of (1.1) with data $f^\omega$ in a space continuously embedded in $C([0, T]; H^{\sigma}(M))$. Moreover, when $h_n, g_n$ are standard real Gaussian or Bernoulli variables, we have

$$P(\Omega_T) \geq 1 - C \exp \left( cT^{-\delta_1} \right).$$

(1.11)

**Theorem 1.3.** Let (1.6) be valid and $M$ be a three dimensional manifold with boundary and $s \geq \frac{23}{90}$ and $f = (f_1, f_2) \in \mathcal{H}_D^s(M)$ and $f^\omega \in L^2(\Omega; \mathcal{H}_D^s(M))$ be defined by the randomization (1.7). For a.s. $\omega \in \Omega$, there exist $T_\omega > 0$ and a unique solution to (1.3) with $u \mid_{R_t \times \partial M} = 0$ and the initial data $f^\omega$ in a space continuously embedded in

$$X_\omega = \left( \cos(t\sqrt{-D}) f^\omega_1 + \frac{\sin(t\sqrt{-D})}{\sqrt{-D}} f^\omega_2 \right) + C([-T_\omega, T_\omega]; H_D^{\frac{7}{12}}(M)).$$

More precisely, for $0 < T \leq 1$, there exists $C > 0, \delta_2 > 0$, an event $\Omega_T$ satisfying

$$P(\Omega_T) \geq 1 - CT^{1+\delta_2}$$

(1.12)

such that for every $\omega \in \Omega_T$ there exists a unique solution of (1.1) with data $f^\omega$ in a space continuously embedded in $C([0, T]; H^{\sigma}(M))$. Moreover, when $h_n, g_n$ are standard real Gaussian or Bernoulli variables, we have

$$P(\Omega_T) \geq 1 - C \exp \left( cT^{-\delta_2} \right).$$

(1.13)

**Remark 2:** In Theorem 1.3, if $\Delta_D, \mathcal{H}_D^s(M)$ and Dirichlet boundary condition $u \mid_{R_t \times \partial M} = 0$ are replaced by $\Delta_N, \mathcal{H}_N^s(M)$ and Neumann boundary condition $N_x \nabla_x u(x) \mid_{R_t \times \partial M} = 0$, respectively, the conclusion is still valid.

The rest of the paper is arranged as follows. In Section 2, we give Strichartz estimates and $L^p(p = 5, 6)$ norm of eigenfunction associated to $-\Delta$ on compact manifold. In
Section 3, we give some properties of two random series. In Section 4, we give averaging effects. In Section 5, we prove the Theorem 1.1. In Section 6, we prove the Theorem 1.2. In Section 7, we prove the Theorem 1.3.

2. Strichartz estimates and $L^p$ norm of eigenfunction associated to $-\Delta$ on compact manifolds

In this section, we give some Strichartz estimates and $L^p$ norm of eigenfunction associated to Laplace-Beltrami operator on two and three compact manifolds, which play a crucial role in establishing Lemmas 4.1-4.6.

Definition 2.1. Let $0 \leq s < 1$ and $M$ be a three dimensional compact manifold with boundary. A couple of real numbers $(p, q)$ is called $s$-admissible provided that $p, q, s$ satisfy

$$\frac{1}{p} + \frac{3}{q} = \frac{3}{2} - s$$

and $p \geq \frac{7}{2s}$ if $s \leq \frac{7}{10}$; $p = 5$ if $s \geq \frac{7}{10}$. For $T > 0$, $0 \leq s < 1$, we define $X^s_T$ space and $Y^s_T$ space as follows.

$$X^s_T = C^0([0, T]; H^s(M)) \bigcap_{(p, q)\text{-admissible}} L^p((0, T); L^q(M)),$$

$$Y^s_T = L^1([0, T]; H^{-s}(M)) + \bigcap_{(p, q)\text{-admissible}} L^p((0, T); L^q(M)),$$

where $\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1$. Obviously, $Y^s_T$ is the dual space of $X^s_T$.

Definition 2.1 can be found in Definition 6.3 of [9].

Inspired by (1.1)-(1.3) of [3], we give the definition 2.2.

Definition 2.2. Let $0 \leq s < 1$ and $M$ be a two dimensional compact boundaryless manifold. A couple of real numbers $(p, q)$ is called $s$-admissible provided that $p, q, s$ satisfy

$$\frac{1}{p} + \frac{2}{q} = 1 - s$$

and $\frac{3}{s} \leq p \leq \infty$. For $T > 0$, $0 \leq s \leq 1$, we define $X^s_T$ space and $Y^s_T$ space as follows.

$$X^s_T = C^0([0, T]; H^s(M)) \bigcap_{(p, q)\text{-admissible}} L^p((0, T); L^q(M)),$$

$$Y^s_T = L^1([0, T]; H^{-s}(M)) + \bigcap_{(p, q)\text{-admissible}} L^p((0, T); L^q(M)),$$

where $\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1$. Obviously, $Y^s_T$ is the dual space of $X^s_T$. 


Inspired by (1.1)-(1.2), (1.4) of [3], we give the definition 2.3.

**Definition 2.3.** Let \(0 \leq s < 1\) and \(M\) be a two dimensional compact manifold with boundary. A couple of real numbers \((p, q)\) is called \(s\)-admissible provided that \(p, q, s\) satisfy

\[ \frac{1}{p} + \frac{2}{q} = 1 - s \]

and \(p \geq \frac{5}{s}\) if \(s \leq \frac{5}{8}\); \(p = 8\) if \(s \geq \frac{5}{8}\). For \(T > 0, 0 \leq s < 1\), we define \(X^s_T\) space and \(Y^s_T\) space as follows.

\[
X^s_T = C^0([0, T]; H^s(M)) \bigcap_{(p, q)\text{ s-admissible}} L^p((0, T); L^q(M)),
\]

\[
Y^s_T = L^1([0, T]; H^{-s}(M)) +_{(p, q)\text{ s-admissible}} L^{p'}((0, T); L^{q'}(M))
\]

where \(\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1\). Obviously, \(Y^s_T\) is the dual space of \(X^s_T\).

**Lemma 2.1.** Let \((p, q)\) be an \(s\)-admissible couple of Definition 2.1 and \(M\) be a three dimensional compact manifold with boundary. For \(0 \leq s < 1\), there exists \(C > 0\) such that

\[
\left\| \cos(t \sqrt{-\Delta})(f_1) \right\|_{X^s_T} + \left\| \sin(t \sqrt{-\Delta}) \frac{f_2}{\sqrt{-\Delta}} \right\|_{X^s_T} \leq C\|f\|_{X^s(M)},
\]

\[
\left\| \int_0^t \sin((t - \tau) \sqrt{-\Delta}) \frac{g(\tau)}{\sqrt{-\Delta}} d\tau \right\|_{X^s_T} \leq C\|g\|_{Y^{1-s}_T}
\]

for all \(T \in (0, 1]\) and \(g \in \mathcal{H}^s(M)\).

Lemma 2.1 can be found in [9].

**Lemma 2.2.** Let \((p, q)\) be an \(s\)-admissible couple of Definition 2.2 and \(M\) be a two dimensional compact boundaryless manifold. For \(0 \leq s < 1\), there exists \(C > 0\) such that

\[
\left\| e^{\pm it \sqrt{-\Delta}}(g) \right\|_{L^p((0, T); L^q(M))} \leq C\|g\|_{H^s(M)}
\]

for all \(T \in (0, 1]\) and \(g \in H^s(M)\).

For Lemma 2.2, we refer the readers to [3].
Lemma 2.3. Let $(p, q)$ be an $s$-admissible couple of Definition 2.2 and $M$ be a two dimensional compact boundaryless manifold. For $0 \leq s < 1$, there exists $C > 0$ such that

\[
\left\| \cos(t \sqrt{-\Delta})(f_1) \right\|_{X_T^s} + \left\| \sin(t \sqrt{-\Delta}) \right\|_{X_T^s} \leq C \|f\|_{\mathcal{H}^s(M)},
\]

\[
\left\| \int_0^t \frac{\sin((t - \tau) \sqrt{-\Delta})}{\sqrt{-\Delta}} g(\tau) d\tau \right\|_{X_T^s} \leq C \|g\|_{Y_{T}^{1-s}},
\]

for all $T \in (0, 1]$ and $g \in \mathcal{H}^s(M)$.

Combining Lemma 2.2 with the Corollary 4.3 of [8], we have Lemma 2.3.

Lemma 2.4. Let $(p, q)$ be an $s$-admissible couple of Definition 2.3 and $M$ be a two dimensional compact manifold with boundary. For $0 \leq s < 1$, there exists $C > 0$ such that

\[
\left\| e^{\pm it \sqrt{-\Delta}}(f) \right\|_{L^p((0,T):L^q(M))} \leq C \|f\|_{H^s(M)}
\]

for all $T \in (0, 1]$ and $f \in H^s(M)$.

For Lemma 2.4, we refer the readers to Theorem 1.1 of [3].

Lemma 2.5. Let $(p, q)$ be an $s$-admissible couple of Definition 2.3 and $M$ be a two dimensional compact manifold with boundary. For $0 \leq s < 1$, there exists $C > 0$ such that

\[
\left\| \cos(t \sqrt{-\Delta})(f_1) \right\|_{X_T^s} + \left\| \sin(t \sqrt{-\Delta}) \right\|_{X_T^s} \leq C \|f\|_{\mathcal{H}^s(M)},
\]

\[
\left\| \int_0^t \frac{\sin((t - \tau) \sqrt{-\Delta})}{\sqrt{-\Delta}} g(\tau) d\tau \right\|_{X_T^s} \leq C \|g\|_{Y_{T}^{1-s}},
\]

for all $T \in (0, 1]$ and $g \in \mathcal{H}^s(M)$.

Combining Lemma 2.4 with the Corollary of [8], we have Lemma 2.5.

Lemma 2.6. Let $M$ be a three dimensional compact manifold with boundary and $(e_n)_{n=1}^{\infty}$ be an $L^2$-normalized basis consisting in eigenfunctions of the Laplace-Beltrami operator with Dirichlet (resp. Neumann) boundary conditions, associated to eigenvalues $\lambda_n^2$. Then, there exists $C > 0$ such that

\[
\|e_n\|_{L^p(M)} \leq C(1 + \lambda_n^2)^{\frac{s}{4}}.
\]
For the proof of Lemma 2.6, we refer the readers to Theorem 2 of [30].

**Lemma 2.7.** Let $M$ be a two dimensional compact boundaryless manifold and $(e_n)_{n=1}^\infty$ be an $L^2$-normalized basis consisting in eigenfunctions of the Laplace-Beltrami operator, associated to eigenvalues $\lambda_n^2$. Then, there exists $C > 0$ such that

$$\|e_n\|_{L^6(M)} \leq C(1 + \lambda_n^2)^{\frac{1}{12}}.$$  

For the proof of Lemma 2.7, we refer the readers to Theorem 2.1 of [31].

**Lemma 2.8.** Let $M$ be a two dimensional compact manifold with boundary and $(e_n)_{n=1}^\infty$ be an $L^2$-normalized basis consisting in eigenfunctions of the Laplace-Beltrami operator with Dirichlet (resp. Neumann) boundary conditions, associated to eigenvalues $\lambda_n^2$. Then, there exists $C > 0$ such that

$$\|e_n\|_{L^6(M)} \leq C(1 + \lambda_n^2)^{\frac{1}{5}}.$$  

For the proof of Lemma 2.8, we refer the readers to Theorem 1.1 of [30].

3. Properties of two random series

In this section, we present $L^p$ properties of two random series which play a crucial role in proving Lemmas 4.1-4.6.

**Lemma 3.1.** Let $(l_n(\omega))_{n=1}^\infty$ be a sequence of independent, 0-mean value, complex random variables satisfying

$$\exists C > 0, \forall n \geq 1, \left| \int_{\mathbb{R}} |l_n(\omega)|^{2k} dp(\omega) \right| \leq C.$$  

Then, we have that

$$\forall 2 \leq p \leq 2k, \exists C > 0, \forall (c_n)_{n \in N^*} \in l^2(N^*, \mathbb{C}), \left\| \sum_{n=1}^\infty c_n l_n \right\|_{L^p(\Omega)} \leq C \left( \sum_{n=1}^\infty |c_n|^2 \right)^{\frac{1}{p}}.$$  

Lemma 3.1 can be found in Lemma 4.2 of [9].
Lemma 3.2. Let \((l_n(\omega))_{n=1}^\infty\) be a sequence of real, 0-mean, independent random variables with associated sequence of distributions \((\mu_n)_{n=1}^\infty\). Suppose that \(\mu_n\) satisfy

\[\exists C > 0: \forall \gamma \in \mathbb{R}, \forall n \geq 1, \left| \int_{\mathbb{R}} e^{\gamma x} d\mu_n(x) \right| \leq e^{C\gamma^2}. \tag{3.1}\]

Then, there exists \(\alpha > 0\) such that for every \(\lambda > 0\), every sequence \((c_n)_{n=1}^\infty \in l^2\) of real numbers,

\[P \left( \omega: \left| \sum_{n=1}^\infty c_n l_n(\omega) \right| \right) \leq 2e^{\frac{-\alpha \lambda^2}{\sum_{n=1}^\infty c_n^2}}. \]

Consequently, there exists \(C > 0\) such that

\[\left\| \sum_{n=1}^\infty c_n l_n(\omega) \right\|_{L^p(\Omega)} \leq \sqrt{p} \left( \sum_{n=1}^\infty c_n^2 \right)^{\frac{1}{2}} \]

for every \(p \geq 2\) and every \((c_n)_{n=1}^\infty \in l^2\).

Lemma 3.2 can be found in Lemma 3.1 of [9].

4. Averaging effects

In this section, motivated by Propositions 4.1, 4.4, 6.4 of [9], we use Lemmas 2.6-2.8, 3.1, 3.2 to establish some mixed norm estimates about \(u^\omega_f(x, t)\) defined below.

Lemma 4.1. Let \(s \in \mathbb{R}, 1 < p \leq 5\) and \(0 < T \leq 1\) and \(f = (f_1, f_2) \in \mathcal{H}^s(M)\). Under the assumptions of Theorem 1, we have that

\[\left\| (-\Delta + 1)^{\frac{s}{2} - \frac{1}{2}} u^\omega_f \right\|_{L^5_t(\Omega)L^5_x([0,T])L^5_x(M)} \leq C T^{\frac{1}{p}} \| f \|_{\mathcal{H}^s(M)}, \tag{4.1}\]

where \(u^\omega_f(x, t) = \cos(t\sqrt{-\Delta}) f_1^\omega + \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}} f_2^\omega\). In particular, for \(s \in \mathbb{R}\), the following inequality is valid

\[P( E_{\lambda,T,f} ) \leq C T^{\frac{1}{p}} \lambda^{-\frac{5}{2}} \| f \|_{\mathcal{H}^s(M)}^5, \tag{4.2}\]

where \(E_{\lambda,T,f} = \left\{ \omega \in \Omega: \left\| (-\Delta + 1)^{\frac{s}{2} - \frac{1}{2}} u^\omega_f \right\|_{L^5_t([0,T])L^5_x(M)} \geq \lambda \right\} \).
Proof. By using Lemma 3.1 and Minkowski inequality and Lemma 2.6, we have that

\[
\left\| \cos(t\sqrt{-\Delta})f_1^w \right\|_{L^2_\omega(\Omega)L^p_t([0,T])L^2_x(M)} \\
\leq C \left\| \left( \sum_{\lambda_n=1}^\infty \left| \cos(t\lambda_n)\alpha_n e_n(x) \right|^2 \right)^{\frac{1}{2}} \right\|_{L^p_t([0,T])L^2_x(M)} \\
= \left\| \left( \sum_{n=1}^\infty |\alpha_n e_n(x)|^2 \right)^{\frac{1}{2}} \right\|_{L^p_t([0,T])L^2_x(M)} \\
\leq CT \frac{1}{p} \left[ \sum_{n=1}^\infty |\alpha_n|^2 \right]^{\frac{1}{2}} \\
\leq CT \frac{1}{p} \left[ \sum_{n=1}^\infty |\alpha_n|^2 \right]^{\frac{1}{2}} \\
\leq CT \frac{1}{p} \left[ \sum_{n=1}^\infty |\alpha_n|^2 \right]^{\frac{1}{2}} = CT \frac{1}{p} \left\| f_1 \right\|_{H^{\frac{p}{2}}(M)}. \tag{4.3}
\]

From (4.3), for \( s \in \mathbb{R} \), we have that

\[
\left\| (-\Delta + 1)^{\frac{s}{2}} \cos(t\sqrt{-\Delta})f_1^w \right\|_{L^2_\omega(\Omega)L^p_t([0,T])L^2_x(M)} \leq CT \frac{s}{2} \left\| f_1 \right\|_{H^{s}(M)}. \tag{4.4}
\]

From the property of \( \lambda_n^2(1 \leq n \leq \infty, n \in \mathbb{N}) \), we know that there exists \( k \in \mathbb{N}^+ \) such that

\[
\lambda_n^2 \leq 1, (1 \leq n \leq k, n \in \mathbb{N}); \lambda_n^2 \geq 1, (n \geq k + 1, n \in \mathbb{N}). \tag{4.5}
\]

From (4.5), for \( 0 \leq t < 1 \) and \( n \in \mathbb{N} \), we have that

\[
\left| \frac{\text{sint} \lambda_n}{\lambda_n} \right| = |t| \left| \frac{\text{sint} \lambda_n}{t \lambda_n} \right| \leq t \leq 1, (1 \leq n \leq k); \left| \frac{\text{sint} \lambda_n}{\lambda_n} \right| \leq |\lambda_n|^{-1}, (n \geq k + 1). \tag{4.6}
\]

By using Lemma 3.1 and Minkowski inequality and Lemma 2.6 as well as (4.6), we have
Remark 3: Our result improves the result of Proposition 6.4 of [9].
Lemma 4.3. Let $s \in \mathbb{R}$, $1 < q \leq 5$, $p \geq 5$ and $0 < T \leq 1$ and $f = (f_1, f_2) \in \mathcal{H}^s(M)$. Under the assumptions of Theorem 1, if (3.1) is valid, then we have that

$$\left\| (-\Delta + 1)^{\frac{q}{2} - \frac{1}{2}} u_j^x \right\|_{L^p(0,T; L^q(M))} \leq CT^{\frac{1}{q}} \sqrt{p} \| f \|_{\mathcal{H}^s(M)},$$

(4.10)

where $u_j^x(x,t) = \cos(t\sqrt{-\Delta}) f_1^x + \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}} f_2^x$. In particular, for $s \in \mathbb{R}$, the following inequality is valid

$$P(E_{\lambda,T,f}) \leq C \exp \left( -c \frac{\lambda^2}{\| f \|_{\mathcal{H}^s(M)}^2} \right),$$

where $E_{\lambda,T,f} = \left\{ \omega \in \Omega : \left\| (-\Delta + 1)^{\frac{q}{2} - \frac{1}{2}} u_j^x \right\|_{L^p(0,T; L^q(M))} \geq \lambda \right\}$.

Combining Lemmas 3.2, 4.1 with the method of Proposition 4.4 of [9], we derive that Lemma 4.2 is valid.

Lemma 4.3. Let $s \in \mathbb{R}$, $1 < q \leq 6$, $0 < T \leq 1$ and $f = (f_1, f_2) \in \mathcal{H}^s(M)$. Under the assumptions of Theorem 2, we have that

$$\left\| (-\Delta + 1)^{\frac{q}{2} - \frac{1}{2}} u_j^x \right\|_{L^p(0,T; L^q(M))} \leq CT^{\frac{1}{q}} \| f \|_{\mathcal{H}^s(M)},$$

(4.11)

where $u_j^x(x,t) = \cos(t\sqrt{-\Delta}) f_1^x + \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}} f_2^x$. In particular, for $s \in \mathbb{R}$, the following inequality is valid

$$P(E_{\lambda,T,f}) \leq C T^6 \lambda^{-6} \| f \|_{\mathcal{H}^s(M)}^6,$$

(4.12)

where $E_{\lambda,T,f} = \left\{ \omega \in \Omega : \left\| (-\Delta + 1)^{\frac{q}{2} - \frac{1}{2}} u_j^x \right\|_{L^p(0,T; L^q(M))} \geq \lambda \right\}$.

Lemma 4.3 can be proved similarly to Lemma 4.1 with the aid of Lemma 2.7.

Lemma 4.4. Let $s \in \mathbb{R}$, $1 < q \leq 6$, $p \geq 5$, and $0 < T \leq 1$ and $f = (f_1, f_2) \in \mathcal{H}^s(M)$. Under the assumptions of Theorem 2, if (3.1) is valid, then we have that

$$\left\| (-\Delta + 1)^{\frac{q}{2} - \frac{1}{2}} u_j^x \right\|_{L^p(0,T; L^q(M))} \leq CT^{\frac{1}{q}} \| f \|_{\mathcal{H}^s(M)},$$

(4.13)

where $u_j^x(x,t) = \cos(t\sqrt{-\Delta}) f_1^x + \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}} f_2^x$. In particular, for $s \in \mathbb{R}$, the following inequality is valid

$$P(E_{\lambda,T,f}) \leq C \exp \left( -c \frac{\lambda^2}{\| f \|_{\mathcal{H}^s(M)}^2} \right),$$

where $E_{\lambda,T,f} = \left\{ \omega \in \Omega : \left\| (-\Delta + 1)^{\frac{q}{2} - \frac{1}{2}} u_j^x \right\|_{L^p(0,T; L^q(M))} \geq \lambda \right\}$.
Lemma 4.4 can be proved similarly to Lemma 4.2 with the aid of Lemma 4.3.

**Lemma 4.5.** Let \( s \in \mathbb{R}, \, 1 < q \leq 6, \, 0 < T \leq 1 \) and \( f = (f_1, f_2) \in \mathcal{H}^s(M) \). Under the assumptions of Theorem 3, we have that

\[
\left\| (-\Delta + 1)^{\frac{3}{2} - \frac{1}{q}} u_\omega^f \right\|_{L^p(\Omega) L^q([0,T]) L^6_M} \leq C T^{\frac{1}{q}} \| f \|_{\mathcal{H}^s(M)},
\]

where \( u_\omega^f(x,t) = \cos(t\sqrt{-\Delta}) f_1^\omega + \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}} f_2^\omega \). In particular, for \( s \in \mathbb{R} \), the following inequality is valid

\[
P(E_{\lambda,T,f}) \leq C T^6 \lambda^{-6} \| f \|^6_{\mathcal{H}^s(M)},
\]

where \( E_{\lambda,T,f} = \left\{ \omega \in \Omega : \left\| (-\Delta + 1)^{\frac{3}{2} - \frac{1}{q}} u_\omega^f \right\|_{L^p(\Omega) L^q([0,T]) L^6_M} \geq \lambda \right\} \).

Lemma 4.5 can be proved similarly to Lemma 4.1 with the aid of Lemma 2.8.

**Lemma 4.6.** Let \( s \in \mathbb{R}, \, 1 < q \leq 6 \) and \( p \geq 6, \, 0 < T \leq 1 \), \( f = (f_1, f_2) \in \mathcal{H}^s(M) \). Under the assumptions of Theorem 3, if (3.1) is valid, then we have that

\[
\left\| (-\Delta + 1)^{\frac{3}{2} - \frac{1}{q}} u_\omega^f \right\|_{L^p(\Omega) L^q([0,T]) L^6_M} \leq C T^{\frac{1}{q}} \| f \|_{\mathcal{H}^s(M)},
\]

where \( u_\omega^f(x,t) = \cos(t\sqrt{-\Delta}) f_1^\omega + \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}} f_2^\omega \). In particular, for \( s \in \mathbb{R} \), the following inequality is valid

\[
P(E_{\lambda,T,f}) \leq C \exp \left( -c \frac{\lambda^2}{\| f \|^2_{\mathcal{H}^s(M)}} \right),
\]

where \( E_{\lambda,T,f} = \left\{ \omega \in \Omega : \left\| (-\Delta + 1)^{\frac{3}{2} - \frac{1}{q}} u_\omega^f \right\|_{L^p(\Omega) L^q([0,T]) L^6_M} \geq \lambda \right\} \).

Lemma 4.6 can be proved similarly to Lemma 4.2.

**5. Proof of Theorem 1.1**

In this section, following the method of [9], we use Lemmas 4.1, 4.2 and contraction map theorem to prove Theorem 1.1. We give Lemma 5.1 before proving Theorem 1.1.

**Lemma 5.1.** Let \( s \geq \frac{5}{14} \) and \( 0 < T \leq 1 \). Then, we have that

\[
K_\omega^f(v) = -\int_0^t \frac{\sin((t - \tau)\sqrt{-\Delta})}{\sqrt{-\Delta}} ((u_\omega^f + v)^3)(\cdot, \tau) \, d\tau
\]
and \((v, v_t)|_{t=0} = (0, 0)\). Then, there exists \(C > 0\) such that for every \(f \in \mathcal{H}^s(M)\) and \(\omega \in E^c_{\lambda, f}\), the map \(K^\omega_f\) satisfies

\[
\|K^\omega_f(u)\|_{X^T_f}^2 \leq C \left( \lambda^3 + T^{1/2} \|v\|_{X^T_f}^3 \right), \tag{5.1}
\]

\[
\|K^\omega_f(v) - K^\omega_f(w)\|_{X^T_f}^2 \leq C T^{1/2} \left[ \lambda^2 + \|v\|_{X^T_f}^2 + \|w\|_{X^T_f}^2 \right]. \tag{5.2}
\]

**Proof.** Since \(\left(\frac{21}{4}, \frac{14}{3}\right)\) is \(\frac{2}{3}\)-admissible, we have that

\[
\|g\|_{L^\infty([0,T];H^{\frac{2}{3}}(M))} + \|g\|_{L^2_t([0,T];L^{\frac{14}{3}}(M))} \leq C \|g\|_{X^T_f}. \tag{5.3}
\]

By using (5.3) and Lemma 2.1, we have that

\[
\|K^\omega_f(u)\|_{X^T_f}^2 \leq C \left\| \int_0^t \frac{\sin((t - \tau)\sqrt{-\Delta})}{\sqrt{-\Delta}} \left( (u^\omega_f + v)^3 \right) (\cdot, \tau) d\tau \right\|_{X^T_f}^2
\]

\[
\leq C \left\| (u^\omega_f + v)^3 \right\|_{Y^T_f}^2 \leq C \left\| (u^\omega_f + v)^3 \right\|_{L^2_t([0,T];L^{\frac{14}{3}}(M))}^2
\]

\[
\leq C \left( \left\| u^\omega_f \right\|_{L^{63}_t([0,T];L^{\frac{14}{3}}(M))}^{63} + \left\| v \right\|_{L^{63}_t([0,T];L^{\frac{14}{3}}(M))}^{63} \right).
\tag{5.4}
\]

By using the Hölder inequality, from (5.3), we have that

\[
\|v\|_{L^{63}_t([0,T];L^{\frac{14}{3}}(M))} \leq CT^{1/2} \left\|v\right\|_{L^2_t([0,T];L^{\frac{14}{3}}(M))} \leq C T^{1/2} \|v\|_{X^T_f}. \tag{5.5}
\]

When \(s \geq 1\), since \(5(s - \frac{2}{3}) \geq 3\), we have that

\[
W^{s - \frac{2}{3}, 5}(M) \hookrightarrow L^{\frac{14}{3}}(M). \tag{5.6}
\]

By using (5.6), for \(\omega \in E^c_{\lambda,T,f}\), we have that

\[
\left\| u^\omega_f \right\|_{L^{63}_t([0,T];L^{\frac{14}{3}}(M))} \leq C \left\| u^\omega_f \right\|_{L^{63}_t([0,T];W^{s - \frac{2}{3}, 5}(M))} \leq \lambda. \tag{5.7}
\]

When \(s < 1\), by using the Sobolev embedding Theorem, we have

\[
W^{s - \frac{2}{3}, 5}(M) \hookrightarrow L^{q_1}(M), \quad \frac{1}{q_1} = \frac{1}{5} - \frac{s - \frac{2}{3}}{3} = \frac{1 - s}{3}. \tag{5.8}
\]

By using (5.8), for \(\omega \in E^c_{\lambda,T,f}\), we have that

\[
\left\| u^\omega_f \right\|_{L^{63}_t([0,T];L^{\frac{14}{3}}(M))} \leq C \left\| u^\omega_f \right\|_{L^{63}_t([0,T];W^{s - \frac{2}{3}, 5}(M))} \leq C \lambda. \tag{5.9}
\]

Since \(\frac{14}{3} \leq \frac{3}{1 - s}\) which is equivalent to \(s \geq \frac{5}{11}\), from (5.9), we have that

\[
\left\| u^\omega_f \right\|_{L^{63}_t([0,T];L^{\frac{14}{3}}(M))} \leq \left\| u^\omega_f \right\|_{L^{63}_t([0,T];L^{\frac{14}{3}}(M))} \leq C \left\| u^\omega_f \right\|_{L^{63}_t([0,T];W^{s - \frac{2}{3}, 5}(M))} \leq C \lambda. \tag{5.10}
\]
Inserting (5.5), (5.7), (5.9)-(5.10) into (5.4) yields that
\[ \| K_f^\omega (u) \|_{X_T^+} \leq C \left( \lambda^3 + T^{\frac{1}{3}} \| v \|_{X_T^+}^3 \right). \] (5.11)

By using a proof similar to (5.11), we have that
\[ \| K_f^\omega (v) - K_f^\omega (w) \|_{X_T^+} \leq CT^{\frac{1}{3}} \| v - w \|_{X_T^+} \left[ \lambda^2 + T^{\frac{2}{3}} \| v \|_{X_T^+}^2 + T^{\frac{2}{3}} \| w \|_{X_T^+}^2 \right]. \] (5.12)

This completes the proof of Lemma 5.1.

Now we prove Theorem 1.1. Fix \( 0 < T \leq 1 \). Let
\[ B(0, 2C\lambda^3) = \left\{ u | u \in X_T^+ : \| u \|_{X_T^+} \leq 2C\lambda^3 \right\} \] (5.13)
and
\[ 4CT^{\frac{1}{3}} \lambda^3 \leq \lambda. \] (5.14)

From (5.11)-(5.14), we have that
\[ \| K_f^\omega (u) \|_{X_T^+} \leq C \left( \lambda^3 + T^{\frac{1}{3}} \| v \|_{X_T^+}^3 \right) \leq C \left( \lambda^3 + T^{\frac{1}{3}} (2C\lambda^3)^3 \right) \leq 2C\lambda^3, \] (5.15)
\[ \| K_f^\omega (v) - K_f^\omega (w) \|_{X_T^+} \leq CT^{\frac{1}{3}} \| v - w \|_{X_T^+} \left[ \lambda^2 + T^{\frac{2}{3}} \| v \|_{X_T^+}^2 + T^{\frac{2}{3}} \| w \|_{X_T^+}^2 \right], \]
\[ \leq \frac{1}{2} \| v - w \|_{X_T^+}. \] (5.16)

Thus, \( K_f^\omega \) is a contraction map on the ball \( B(0, 2C\lambda^3) \). We define
\[ \Omega_T = E_{\lambda, T, f}^c, \sum_n = \bigcup_{n \in \mathbb{N}^*} \Omega_n. \] (5.17)

Combining (4.2) with (5.17), we have that
\[ P(\Omega_T) \geq 1 - CT^{\frac{1}{2}}, P \left( \sum_n \right) = 1. \] (5.18)

When \( h_n, g_n \) are standard real Gaussian or Bernoulli variables, (3.1) is valid. In this case, by using a proof similar to case (1.6), we have that \( K_f^\omega \) is a contraction map on the ball \( B(0, 2C\lambda^3) \). We define
\[ \Omega_T = E_{\lambda, T, f}^c, \sum_n = \bigcup_{n \in \mathbb{N}^*} \Omega_n. \] (5.19)
Combining (5.19) with Lemma 4.2, we have that

\[
P(\Omega_T) \geq 1 - C \exp \left( -c T^{-\frac{1}{9}} \right), \quad P \left( \sum \right) = 1.
\]

This completes the proof of Theorem 1.1.

6. Proof of Theorem 1.2

In this section, following the method of [9], we use Lemmas 4.3, 4.4 and contraction map theorem to prove Theorem 1.2. We present Lemmas 6.1, 6.2 before proving Theorem 1.2.

Lemma 6.1. Let \( s = \frac{1}{6} \) and \( 0 < T \leq 1 \). Then, we have that

\[
K_\omega^\omega(v) = - \int_0^t \frac{\sin((t-\tau)\sqrt{-\Delta})}{\sqrt{-\Delta}} \left( (u_\omega^\omega + v)^5 \right) (\cdot, \tau) d\tau
\]

and \( (v, v_t)|_{t=0} = (0, 0) \). Then, there exists \( C > 0 \) such that for every \( f \in \mathcal{C}^s(M) \) and \( \omega \in E_\lambda,f \), the map \( K_\omega^\omega \) satisfies

\[
\| K_\omega^\omega(u) \|_{\mathcal{L}^6((0,T) \times M)} \leq C \left( \lambda^5 + \| v \|_{\mathcal{L}^5([0,T] \times M)}^5 \right), \tag{6.1}
\]

\[
\| K_\omega^\omega(v) - K_\omega^\omega(w) \|_{\mathcal{L}^6((0,T) \times M)} \leq C \| v - w \|_{\mathcal{L}^5([0,T] \times M)} \left[ \lambda^4 + \| v \|_{\mathcal{L}^4([0,T] \times M)}^4 + \| w \|_{\mathcal{L}^4([0,T] \times M)}^4 \right]. \tag{6.2}
\]

Proof. For \( \omega \in E_\lambda,T,f \), we have that

\[
\left\| u_\omega^\omega \right\|_{\mathcal{L}^6((0,T) \times M)} \leq \lambda, \tag{6.3}
\]

from (6.3) and Lemma 2.3, we have that

\[
\left\| K_\omega^\omega(v) \right\|_{\mathcal{L}^6([0,T] \times M)} \leq C \left\| (u_\omega^\omega + v)^5 \right\|_{\mathcal{L}^6([0,T] \times M)} \leq C \left[ \left\| u_\omega^\omega \right\|_{\mathcal{L}^5([0,T] \times M)}^5 + \| v \|_{\mathcal{L}^5([0,T] \times M)}^5 \right]. \tag{6.4}
\]

Since (6.6) is \( \frac{1}{2} \)-admissible, combining (6.3) with (6.4), we have that

\[
\left\| K_\omega^\omega(v) \right\|_{\mathcal{L}^6([0,T] \times M)} \leq C \left\| (u_\omega^\omega + v)^5 \right\|_{\mathcal{L}^6([0,T] \times M)} \leq C \left( \lambda^5 + \| v \|_{\mathcal{L}^5([0,T] \times M)}^5 \right). \tag{6.5}
\]

By using a proof similar to (6.5), we have that

\[
\left\| K_\omega^\omega(v) - K_\omega^\omega(w) \right\|_{\mathcal{L}^6([0,T] \times M)} \leq C \| v - w \|_{\mathcal{L}^5([0,T] \times M)} \left[ \lambda^4 + \| v \|_{\mathcal{L}^4([0,T] \times M)}^4 + \| w \|_{\mathcal{L}^4([0,T] \times M)}^4 \right]. \tag{6.6}
\]

This completes the proof of Lemma 6.1.
Lemma 6.2. Let \( s \geq \frac{1}{6} + \frac{4\epsilon}{15} \), \( 0 < \epsilon \ll \frac{1}{2} \) and \( 0 < T \leq 1 \). Then, we have that

\[
K^\omega_f(v) = -\int_0^t \frac{\sin((t - \tau)\sqrt{-\Delta})}{\sqrt{-\Delta}} \left( (u^\omega_f + v)^5 \right) (\cdot, \tau) d\tau
\]

and \( (v, v_0)_{L^2} = (0, 0) \). Then, there exists \( C > 0 \) such that for every \( f \in \mathcal{H}^s(M) \) and \( \omega \in E^c_{\lambda, T, f} \), the map \( K^\omega_f \) satisfies

\[
\|K^\omega_f(u)\|_{X^{\frac{1}{2} + \epsilon}_T} \leq C \left( \lambda^5 + T^{\frac{204}{175}} \|v\|_{L^{\frac{5}{4}}(M)}^5 \right), \quad (6.7)
\]

\[
\|K^\omega_f(v) - K^\omega_f(w)\|_{X^{\frac{1}{2} + \epsilon}_T} \leq CT^{\frac{23}{175}} \left[ \lambda^4 + \|v_1\|_{L^{\frac{4}{3}}(M)}^4 + \|w\|_{L^{\frac{4}{3}}(M)}^4 \right]. \quad (6.8)
\]

Proof. Since \((\frac{30}{5-2\alpha}, \frac{30}{5-4\alpha})\) is \((\frac{1}{2} + \epsilon)\)-admissible, we have that

\[
\|g\|_{L^\infty([0,T];H^{\frac{1}{2} + \epsilon}(M))} + \|g\|_{L^\infty([0,T];L^{\frac{30}{22}}(M))} \leq C\|g\|_{X^{\frac{1}{2} + \epsilon}_T}. \quad (6.9)
\]

By using (6.9), we have that

\[
\|K^\omega_f(u)\|_{X^{\frac{1}{2} + \epsilon}_T} \leq C \left\| \int_0^t \frac{\sin((t - \tau)\sqrt{-\Delta})}{\sqrt{-\Delta}} \left( (u^\omega_f + v)^5 \right) (\cdot, \tau) d\tau \right\|_{X^{\frac{1}{2} + \epsilon}_T}
\]

\[
\leq C \left\| (u^\omega_f + v)^5 \right\|_{Y^{\frac{1}{2} - \epsilon}_T} \leq C \left\| (u^\omega_f + v)^5 \right\|_{L^{\frac{6}{4}}(M)} \leq C \left( \|u^\omega_f\|_{L^{\frac{30}{22}}(M)}^5 + \|v\|_{L^{\frac{30}{22}}(M)}^5 \right). \quad (6.10)
\]

By using the Hölder inequality, from (6.9), we have that

\[
\|v\|_{L^{\frac{30}{22}}(M)} \leq CT^{\frac{23}{175}} \|v\|_{L^{\frac{30}{22}}(M)} \leq CT^{\frac{23}{175}} \|v\|_{X^{\frac{1}{2} + \epsilon}_T}. \quad (6.11)
\]

When \( s \geq \frac{1}{2} \), since \( 6(s - \frac{1}{6}) \geq 2 \), we have that

\[
W^{s - \frac{1}{6}, 6}(M) \hookrightarrow L^{\frac{30}{22}}(M). \quad (6.12)
\]

By using (6.12), for \( \omega \in E^c_{\lambda, T, f} \), we have that

\[
\|u^\omega_f\|_{L^{\frac{30}{22}}(M)} \leq C \|u^\omega_f\|_{L^{\frac{30}{22}}(W^{s - \frac{1}{6}, 6}(M))} \leq \lambda. \quad (6.13)
\]

When \( s < \frac{1}{2} \), by using the Sobolev embedding Theorem, we have

\[
W^{s - \frac{1}{6}, 6}(M) \hookrightarrow L^{q_1}(M), \quad \frac{1}{q_1} = \frac{1}{6} - s - \frac{1}{6} = \frac{1 - 2s}{4}. \quad (6.14)
\]

By using (6.14), for \( \omega \in E^c_{\lambda, T, f} \), we have that

\[
\|u^\omega_f\|_{L^{\frac{30}{22}}(M)} \leq C \|u^\omega_f\|_{L^{\frac{30}{22}}(W^{s - \frac{1}{6}, 6}(M))} \leq C\lambda. \quad (6.15)
\]
Since \( \frac{30}{5 - 4\epsilon} \leq \frac{2}{1 - 2s} \), which is equivalent to \( s \geq \frac{5 + 8\epsilon}{30} \), from (6.15), we have that
\[
\| u_f \|_{L_t^{\frac{30}{5 - 4\epsilon}} ([0,T];L_x^{\frac{30}{5 - 4\epsilon}} (M))} \leq C \| u_f \|_{L_t^{\frac{30}{5 - 4\epsilon}} ([0,T];L_x^{\frac{22}{5 - 4\epsilon}} (M))} \leq C . \tag{6.16}
\]
Inserting (6.11), (6.13), (6.15)-(6.16) into (6.10) yields that
\[
\| K_f^\omega (u) \|_{L_t^{\frac{2}{1 + s}} (\mathbb{R}^+, L_{\mathcal{X}})} \leq C \left( \lambda^5 + T^{\frac{22}{5 - 4\epsilon}} \| v \|_{L_{\mathcal{X}}}^5 \right). \tag{6.17}
\]
By using a proof similar to (6.17), we have that
\[
\| K_f^\omega (v) - K_f^\omega (w) \|_{L_t^{\frac{2}{1 + s}} (\mathbb{R}^+, L_{\mathcal{X}})} \leq C T^{\frac{22}{5 - 4\epsilon}} \| v - w \|_{L_t^{\frac{2}{1 + s}} (\mathbb{R}^+, L_{\mathcal{X}})} \left( \lambda^4 + \| v \|_{L_t^{\frac{2}{1 + s}} (\mathbb{R}^+, L_{\mathcal{X}})}^4 + \| w \|_{L_t^{\frac{2}{1 + s}} (\mathbb{R}^+, L_{\mathcal{X}})}^4 \right). \tag{6.18}
\]
This completes the proof of Lemma 6.2.

Now we prove Theorem 1.2. Fix \( 0 < T \leq 1 \). Firstly, we consider \( s = \frac{1}{6} \). Let
\[
B(0, 2C\lambda^5) = \left\{ u | u \in X_{T}^{\frac{1}{2}} : \| u \|_{X_{T}^{\frac{1}{2}}} \leq 2C\lambda^5 \right\} . \tag{6.19}
\]
and
\[
4C\lambda^5 \leq \lambda . \tag{6.20}
\]
From (6.1)-(6.2), we have that
\[
\| K_f^\omega (u) \|_{X_{T}^{\frac{1}{2}}} \leq C \left( \lambda^5 + \| v \|_{X_{T}^{\frac{1}{2}}}^5 \right) \leq C \left( \lambda^5 + (2C\lambda^5)^5 \right) \leq 2C\lambda^5 , \tag{6.21}
\]
\[
\| K_f^\omega (v) - K_f^\omega (w) \|_{X_{T}^{\frac{1}{2}}} \leq C \| v - w \|_{X_{T}^{\frac{1}{2}}} \left( \lambda^4 + \| v \|_{X_{T}^{\frac{1}{2}}}^4 + \| w \|_{X_{T}^{\frac{1}{2}}}^4 \right) \leq \frac{1}{2} \| v - w \|_{X_{T}^{\frac{1}{2}}} . \tag{6.22}
\]
Thus, \( K_f^\omega \) is a contraction map on the ball \( B(0, 2C\lambda^5) \). We define
\[
\Omega_T = E_{\lambda,T,f}^c, \sum_n = \bigcup_{n \in N^*} \Omega_n . \tag{6.23}
\]
Combining (4.2) with (6.23), we have that
\[
P(\Omega_T) = 1 - CT, P \left( \sum_n \right) = 1 . \tag{6.24}
\]
We consider case \( s > \frac{1}{6} \). Fix \( 0 < T \leq 1 \). Let
\[
B(0, 2C\lambda^3) = \left\{ u | u \in X_{T}^{\frac{1}{2} + \epsilon} : \| u \|_{X_{T}^{\frac{1}{2} + \epsilon}} \leq 2C\lambda^5 \right\} . \tag{6.25}
\]
and

\[ 2CT^{\frac{22}{15}} \lambda^5 \leq \lambda. \quad (6.26) \]

From (6.17)-(6.18), we have that

\[
\|K^\omega_f(u)\|_{X_T^{1+s}} \leq C \left( \lambda^5 + T^{\frac{22}{15}} \|v\|_{X_T^{1+s}}^5 \right) \leq C \left( \lambda^5 + T^{\frac{22}{15}} (2C\lambda^5)^5 \right) \leq 2C\lambda^5, \quad (6.27)
\]

Thus, \(K^\omega_f\) is a contraction map on the ball \(B(0, 2C\lambda^5)\). We define

\[ \Omega_T = E_{\lambda,T,f}^c, \bigcup_{n \in \mathbb{N}^*} \Omega_n \quad (6.29) \]

Combining (4.2) with (6.29), we have that

\[ P(\Omega_T) \geq 1 - CT^{1+\frac{38}{15}}, P\left(\sum\right) = 1. \quad (6.30) \]

When \(h_n, g_n\) are standard real Gaussian or Bernoulli variables, (3.1) is valid. In this case, by using a proof similar to case (1.6), we have that \(K^\omega_f\) is a contraction map on the ball \(B(0, 2C\lambda^5)\). We define

\[ \Omega_T = E_{\lambda,T,f}^c, \bigcup_{n \in \mathbb{N}^*} \Omega_n \cdot \quad (6.31) \]

Combining (6.31) with Lemma 4.4, we have that

\[ P(\Omega_T) \geq 1 - C\exp \left( -cT^{\frac{14}{15}} \right), P\left(\sum\right) = 1. \]

This completes the proof of Theorem 1.2.

\section{Proof of Theorem 1.3}

In this section, following the method of [9], we use Lemmas 4.5, 4.6 and contraction map theorem to prove Theorem 1.3. We give Lemma 7.1 before proving Theorem 1.3.

\textbf{Lemma 7.1.} Let \( s \geq \frac{23}{90} \) and \( 0 < T \leq 1 \). Then, we have that

\[
K^\omega_f(v) = -\int_0^t \frac{\sin((t-\tau)\sqrt{-\Delta})}{\sqrt{-\Delta}} (u_f^\omega + v)^5 (\cdot, \tau) d\tau
\]
and \((v, v_t)\)|_{t=0} = (0, 0). Then, there exists \(C > 0\) such that for every \(f \in \mathcal{H}^s(M)\) and \(\omega \in E^c_{\lambda, T, f}\), the map \(K^{\omega}_f\) satisfies
\[
\|K^{\omega}_f(u)\|_{X^\omega_T(M)} \leq C \left( \lambda^5 + T^{\frac{4}{5}} \|v\|_{X^\omega_T(M)}^5 \right), \quad (7.1)
\]
\[
\|K^{\omega}_f(v) - K^{\omega}_f(w)\|_{X^\omega_T(M)} \leq C T^{\frac{1}{2}} \|v - w\|_{X^\omega_T(M)} \left[ \lambda^4 + T^{\frac{1}{5}} \|v\|_{X^\omega_T(M)}^4 + T^{\frac{1}{5}} \|w\|_{X^\omega_T(M)}^4 \right]. \quad (7.2)
\]

Proof. Since \((\frac{60}{7}, \frac{20}{3})\) is \(\frac{7}{12}\)-admissible, we have that
\[
\|g\|_{L^\infty([0,T]; H^\omega_T(M))} + \|g\|_{L^{60}_{xT}([0,T]; L^{20}_{xT}(M))} \leq C \|g\|_{X^\omega_T(M)}. \quad (7.3)
\]
By using (7.3) and Lemma 2.5, we have that
\[
\|K^{\omega}_f(u)\|_{X^\omega_T(M)} \leq C \left\| \int_0^t \frac{\sin((t - \tau)\sqrt{-\Delta})}{\sqrt{-\Delta}} (\|u^{\omega}_f + v\|)^5 \right\|_{X^\omega_T(M)} \leq C \left\| (u^{\omega}_f + v)^5 \right\|_{L^{60}_{xT}([0,T]; L^{20}_{xT}(M))} \leq C \left( \|u^{\omega}_f\|_{L^{60}_{xT}([0,T]; L^{20}_{xT}(M))} + \|v\|_{L^{60}_{xT}([0,T]; L^{20}_{xT}(M))}^5 \right). \quad (7.4)
\]
By using the Hölder inequality, from (7.3), we have that
\[
\|v\|_{L^{60}_{xT}([0,T]; L^{20}_{xT}(M))} \leq C T^{\frac{1}{2}} \|v\|_{L^{60}_{xT}([0,T]; L^{20}_{xT}(M))} \leq C T^{\frac{1}{2}} \|v\|_{X^\omega_T(M)}. \quad (7.5)
\]
When \(s \geq \frac{5}{9}\), since \(6(s - \frac{2}{3}) \geq 2\), we have that
\[
W^{s - \frac{2}{3}, 6}(M) \hookrightarrow L^{14}_{xT}(M). \quad (7.6)
\]
By using (7.6), for \(\omega \in E^c_{\lambda, T, f}\), we have that
\[
\|u^{\omega}_f\|_{L^{60}_{xT}([0,T]; L^{20}_{xT}(M))} \leq C \|u^{\omega}_f\|_{L^{60}_{xT}([0,T]; W^{s - \frac{2}{3}, 6}(M))} \leq \lambda. \quad (7.7)
\]
When \(s < \frac{5}{9}\), by using the Sobolev embedding Theorem, we have that
\[
W^{s - \frac{2}{3}, 6}(M) \hookrightarrow L^{q_1}(M), \quad \frac{1}{q_1} = \frac{1}{6} - \frac{s - \frac{2}{3}}{2} = \frac{1 - s}{3}. \quad (7.8)
\]
By using (7.8), for \(\omega \in E^c_{\lambda, T, f}\), we have that
\[
\|u^{\omega}_f\|_{L^{60}_{xT}([0,T]; L^{q_1}_{xT}(M))} \leq C \|u^{\omega}_f\|_{L^{60}_{xT}([0,T]; W^{s - \frac{2}{3}, 6}(M))} \leq C \lambda. \quad (7.9)
\]
Since \(\frac{20}{3} \leq \frac{18}{5 - 9\delta}\) which is equivalent to \(s \geq \frac{23}{90}\), from (7.9), we have that
\[
\|u^{\omega}_f\|_{L^{60}_{xT}([0,T]; L^{14}_{xT}(M))} \leq \|u^{\omega}_f\|_{L^{60}_{xT}([0,T]; L^{q_1}_{xT}(M))} \leq C \|u^{\omega}_f\|_{L^{60}_{xT}([0,T]; W^{s - \frac{2}{3}, 6}(M))} \leq C \lambda. \quad (7.10)
\]
Inserting (7.5), (7.7), (7.9)-(7.10) into (7.4) yields that

\[ \|K_T^\omega(u)\|_{X^{T,5}_T} \leq C \left( \lambda^5 + T^{\frac{1}{2}} \|v\|_{X^{T,5}_T}^5 \right). \quad (7.11) \]

By using a proof similar to (7.11), we have that

\[ \|K_T^\omega(v) - K_T^\omega(w)\|_{X^{T,5}_T} \leq CT^{\frac{1}{2}} \|v - w\|_{X^{T,5}_T} \left[ \lambda^4 + \|v\|_{X^{T,5}_T}^4 + \|w\|_{X^{T,5}_T}^4 \right]. \quad (7.12) \]

This completes the proof of Lemma 7.1.

Now we prove Theorem 1.3.

By using Lemmas 7.1, 4.5, 4.6 and a proof similar to Theorem 1.1, we can obtain Theorem 1.3.

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