FOURIER ALGEBRAS OF PARABOLIC SUBGROUPS

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Abstract. We study the following question: Given a locally compact group when does its Fourier algebra coincide with the subalgebra of the Fourier-Stieltjes algebra consisting of functions vanishing at infinity? We provide sufficient conditions for this to be the case.

As an application we show that when \( P \) is the minimal parabolic subgroup in one of the classical simple Lie groups of real rank one or the exceptional such group, then the Fourier algebra of \( P \) coincides with the subalgebra of the Fourier-Stieltjes algebra of \( P \) consisting of functions vanishing at infinity. In particular, the regular representation of \( P \) decomposes as a direct sum of irreducible representations although \( P \) is not compact.

We also show that \( P \) contains a non-compact closed normal subgroup with the relative Howe-Moore property.

1. Introduction

If \( G \) is a locally compact abelian group with dual group \( \hat{G} \), then the Fourier transform on \( \hat{G} \) maps the group algebra \( L^1(\hat{G}) \) injectively onto a subset \( A(G) \) of the continuous functions on \( G \). Also, the Fourier-Stieltjes transform on \( \hat{G} \) maps the measure algebra \( M(\hat{G}) \) injectively onto a subset \( B(G) \) of the continuous functions on \( G \). Using the usual identification \( L^1(\hat{G}) \subseteq M(\hat{G}) \) we see that \( A(G) \subseteq B(G) \). Every function in \( B(G) \) is bounded, and every function in \( A(G) \) vanishes at infinity. In the very special case when \( \hat{G} = \mathbb{R}^n \), the fact that functions in \( A(G) \) vanish at infinity is the Riemann-Lebesgue lemma.

In the paper [11], Eymard introduced the algebras \( A(G) \) and \( B(G) \) in the setting where \( G \) is no longer assumed to be abelian. Let \( G \) be a locally compact group. The Fourier-Stieltjes algebra \( B(G) \) is defined as the linear span of the continuous positive definite functions on \( G \). There is a natural identification of \( B(G) \) with the Banach space dual of the full group \( C^* \)-algebra \( C^*(G) \), and under this identification \( B(G) \) inherits a norm with which it is a Banach space. The Fourier algebra \( A(G) \) is the closed subspace in \( B(G) \) generated by the compactly supported functions in \( B(G) \). Other descriptions of \( A(G) \) and \( B(G) \) are available (see Section 2). The Fourier and Fourier-Stieltjes algebras play an important role in non-commutative harmonic analysis.

Date: January 13, 2014.

Supported by ERC Advanced Grant no. OAFPG 247321 and the Danish National Research Foundation through the Centre for Symmetry and Deformation (DNRF92).
For general locally compact groups it is still true that $A(G) \subseteq C_0(G)$ just as in the abelian case, and it is natural to ask whether every function in $B(G)$ which vanishes at infinity belongs to $A(G)$.

**Question 1.** Let $G$ be a locally compact group. Does the equality
\[
A(G) = B(G) \cap C_0(G)
\]
(1.1)
hold?

Of course, if $G$ is compact then $B(G) = A(G)$, and (1.1) obviously holds. But for non-compact groups the question is more delicate.

In 1916, Menchoff [25] proved the existence of a singular probability measure $\mu$ on the circle such that its Fourier-Stieltjes transform $\hat{\mu}$ satisfies $\hat{\mu}(n) \to 0$ as $|n| \to \infty$. In other words, $\hat{\mu} \in B(\mathbb{Z}) \cap C_0(\mathbb{Z})$, but $\hat{\mu} \notin A(\mathbb{Z})$, and thus the answer to Question 1 is negative when $G$ is the group $\mathbb{Z}$ of integers. In 1966, Hewitt and Zuckerman [14] proved that for any abelian locally compact group $G$ the answer to Question 1 is always negative, unless $G$ is compact. In 1983 it was shown that for any discrete group $G$ one has $A(G) \neq B(G) \cap C_0(G)$, unless $G$ is finite (see [29, p. 190] and [5]).

The first non-compact example of a group satisfying (1.1) was given by Khalil in [18] and is the (non-unimodular) $ax + b$ group consisting of affine transformations $x \mapsto ax+b$ of the real line, where $a > 0$ and $b \in \mathbb{R}$. We remark that the $ax+b$ group is isomorphic to the minimal parabolic subgroup in the simple Lie group $\text{PSL}_2(\mathbb{R})$ of real rank one.

It is proved in [12],[5] that if (1.1) holds for some second countable, locally compact group $G$, then the regular representation of $G$ is completely reducible, i.e., a direct sum of irreducible representations. For a while, this was thought to be a characterization of groups satisfying (1.1), but this was shown not to be the case (see [4] or [24]). However, it follows from the fact that second countable, locally compact groups satisfying (1.1) have completely reducible regular representations combined with [22, Theorem 3.1] that (1.1) fails for second countable, locally compact IN-groups, unless they are compact. Recall that an IN-group is a group which has a compact neighborhood of the identity which is invariant under all inner automorphisms. In particular, abelian, discrete and compact groups are all IN-groups.

It follows from Baggett’s work [3] that if $G$ is a locally compact, second countable group which is also connected, unimodular and has a completely reducible regular representation, then $G$ is compact (see [30, Theorem 3]). In particular, Question 1 has a negative answer for locally compact second countable connected unimodular groups which are non-compact. This gives an abundance of examples of groups where Question 1 has a negative answer. An example given in [31] of a unimodular group satisfying (1.1) shows that the assumption about connectedness cannot be removed from the previous statement, and of course the assumption about unimodularity cannot be removed as the $ax+b$ group shows.

It should be apparent from the above that there are plenty of examples of groups for which Question 1 has a negative answer. In this paper we provide new examples of groups answering Question 1 in the affirmative. Our main source of examples is
formed by the minimal parabolic subgroups in connected simple Lie groups of real rank one. But first we give a more straightforward example which is a subgroup of $\text{SL}_3(\mathbb{R})$. The method of proof in this example can be seen as an easy version of what follows after. We prove the following.

**Theorem 2.** For the group

$$P = \left\{ \begin{pmatrix} \lambda & a & c \\ 0 & \lambda^{-1} & b \\ 0 & 0 & 1 \end{pmatrix} \bigg| a, b, c \in \mathbb{R}, \lambda > 0 \right\}$$

(1.2)

we have $A(P) = B(P) \cap C_0(P)$.

If we think of $\text{SL}_2(\mathbb{R}) \ltimes \mathbb{R}^2$ as a subgroup of $\text{SL}_3(\mathbb{R})$ in the following way

$$\begin{pmatrix} \text{SL}_2(\mathbb{R}) \\ 0 & 1 \end{pmatrix},$$

then we can think of $P$ as a subgroup of $\text{SL}_2(\mathbb{R}) \ltimes \mathbb{R}^2$. This viewpoint will be relevant in a forthcoming paper [14] by the author and U. Haagerup.

Apart from the group in (1.2), our examples of groups satisfying (1.1) arise in the following way. Let $n \geq 2$, let $G$ be one of the classical simple Lie groups $SO_0(n, 1)$, $SU(n, 1)$, $\text{Sp}(n, 1)$ or the exceptional group $F_4(-20)$, and let $G = KAN$ be the Iwasawa decomposition. If $M$ is the centralizer of $A$ in $K$, then $P = MAN$ is the minimal parabolic subgroup of $G$. We refer to Section 6 for explicit descriptions of the groups $G$, $K$, $A$, $N$ and $M$. We prove the following theorem concerning the Fourier algebra of the minimal parabolic subgroup.

**Theorem 3.** Let $P$ be the minimal parabolic subgroup in one of the simple Lie groups $SO_0(n, 1)$, $SU(n, 1)$, $\text{Sp}(n, 1)$ or $F_4(-20)$. Then $A(P) = B(P) \cap C_0(P)$.

In order to prove Theorem 2 and Theorem 3 we develop a general strategy for providing examples of groups that answer Question 1 affirmatively. The strategy is based on (1) determining all irreducible representations of the group, (2) determining the irreducible subrepresentations of the regular representation and (3) disintegration theory. An often useful tool for (1) is the Mackey Machine (see [13, Chapter 6]).

Our strategy for proving Theorem 2 and Theorem 3 is contained in the following theorem.

**Theorem 4.** Let $G$ be a second countable, locally compact group satisfying the following two conditions.

1. $G$ is type I.
2. There is a non-compact, closed subgroup $H$ of $G$ such that every irreducible unitary representation of $G$ is either trivial on $H$ or is a subrepresentation of the left regular representation $\lambda_G$.

Then

$$A(G) = B(G) \cap C_0(G).$$
In particular, the left regular representation $\lambda_G$ is completely reducible.

It was pointed out to the author by T. de Laat that with the assumptions of Theorem 4 one can deduce that $(G, H)$ has the relative Howe-Moore property (defined in [6]). In fact, condition (1) can be dropped, and condition (2) still implies that $(G, H)$ has the relative Howe-Moore property as is immediately seen from [6, Proposition 2.3] and the well-known fact that the regular representation is a $C_0$-representation.

Since we prove Theorem 2 and Theorem 3 by verifying the conditions in Theorem 4 for the groups in question, we obtain the following corollary.

**Corollary 5.** Let $P$ be the group in (1.2) or the minimal parabolic subgroup in one of the simple Lie groups $\text{SO}_0(n,1)$, $\text{SU}(n,1)$, $\text{Sp}(n,1)$ or $\text{F}_4(-20)$. Then there is a normal, non-compact closed subgroup $H$ in $P$ such that $(P, H)$ has the relative Howe-Moore property.

The non-compact subgroup $H$ can be described explicitly. For a more precise statement see Corollary 28 below. We refer to [6] for a treatment of the relative Howe-Moore property.

In order to verify the two conditions in Theorem 4 for the minimal parabolic subgroups $P$, we rely primarily on earlier work of J.A. Wolf. In [32], the irreducible representations of some parabolic subgroups are determined by employing the Mackey Machine, and the approach of [32] carries over to our situation almost without changes. Using [19] we can easily determine the irreducible subrepresentations of the regular representation of $P$.

The paper is organized as follows. In Section 2 we describe the basic properties of the Fourier and Fourier-Stieltjes algebra, and Section 3 contains the proof of Theorem 4. Section 4 contains a few results to be used later when we verify condition (2) of Theorem 4 for the groups in question. In Section 5 we prove Theorem 2. This includes determining all irreducible unitary representations of the group (1.2), determining the Plancherel measure for the group and finally verifying conditions (1) and (2) of Theorem 4 for the group.

In Section 6 we turn to the minimal parabolic subgroups $P$ in the simple Lie groups of real rank one that we will be working with. We give an explicit description of the groups as matrix groups (at least in the classical cases). In Section 7 we describe the irreducible representations of the minimal parabolic subgroups, and then, in Section 8 we verify the two conditions in Theorem 4 for the minimal parabolic subgroups. Theorem 3 then follows immediately.

Section 9 contains the proof of Corollary 5 concerning the Howe-Moore property, and Section 10 contains some concluding remarks.
This section contains a brief description of the Fourier and Fourier-Stieltjes algebra of a locally compact group introduced by Eymard in [11]. We refer to the original paper [11] for more details. Let $G$ be a locally compact group equipped with a left Haar measure. By a representation of $G$ we always mean a continuous unitary representation of $G$ on some Hilbert space (except for the vector and spin representations in Section 6.2). If $\pi$ is a unitary representation of $G$ on a Hilbert space $\mathcal{H}$, and $x, y \in \mathcal{H}$, then the continuous complex function

$$\varphi(g) = \langle \pi(g)x, y \rangle, \quad (g \in G)$$

is a matrix coefficient of $\pi$. The Fourier-Stieltjes algebra of $G$ is denoted $B(G)$ and consists of the complex linear span of continuous positive definite functions on $G$. It coincides with the set of all matrix coefficients of representations of $G$,

$$B(G) = \{ \langle \pi(\cdot)x, y \rangle \mid (\pi, \mathcal{H}) \text{ is a representation of } G \text{ and } x, y \in \mathcal{H} \}.$$ 

Since the pointwise product of two positive definite functions is again positive definite, $B(G)$ is an algebra under pointwise multiplication. Given $\varphi \in B(G)$, the map

$$f \mapsto \langle f, \varphi \rangle = \int_G f(x)\varphi(x) \, dx$$

is a linear functional on $L^1(G)$ which is bounded, when $L^1(G)$ is equipped with the universal $C^*$-norm. Hence $\varphi$ defines a functional on $C^*\bigl(G,\mathcal{H}\bigr)$, the full group $C^*$-algebra of $G$, and this gives the identification of $B(G)$ with $C^*\bigl(G,\mathcal{H}\bigr)$ as vector spaces. The Fourier-Stieltjes algebra inherits the norm

$$\|\varphi\| = \sup\{ |\langle f, \varphi \rangle| \mid f \in L^1(G), \|f\|_{C^*\bigl(G,\mathcal{H}\bigr)} \leq 1 \}$$

of $C^*\bigl(G,\mathcal{H}\bigr)$ from this identification. With this norm $B(G)$ is a unital Banach algebra.

Given $\varphi \in B(G)$, a representation $(\pi, \mathcal{H})$ and $x, y \in \mathcal{H}$ such that $\varphi(g) = \langle \pi(g)x, y \rangle$ we have

$$\|\varphi\| \leq \|x\|\|y\|,$$

and conversely, it is always possible to find $(\pi, \mathcal{H})$ and $x, y \in \mathcal{H}$ such that $\varphi(g) = \langle \pi(g)x, y \rangle$ and $\|\varphi\| = \|x\|\|y\|$.

The Fourier algebra of $G$ is denoted $A(G)$ and is the closure of the set of compactly supported functions in $B(G)$, and $A(G)$ is in fact an ideal. The Fourier algebra coincides with the set of all matrix coefficients of the left regular representation of $G$,

$$A(G) = \{ \langle \lambda(\cdot)x, y \rangle \mid x, y \in L^2(G) \},$$

and given any $\varphi \in A(G)$, there are $x, y \in L^2(G)$ such that $\varphi(g) = \langle \lambda(g)x, y \rangle$ and $\|\varphi\| = \|x\|\|y\|$. This can be rephrased as follows. Given $\varphi \in A(G)$, there are $f, h \in L^2(G)$ such that $\varphi = f \ast \hat{h}$ and $\|\varphi\| = \|f\|\|h\|$, where $\hat{h}(g) = h(g^{-1})$. This is often written as

$$A(G) = L^2(G) \ast L^2(G).$$

It is known that $\|\varphi\|_\infty \leq \|\varphi\|$ for any $\varphi \in B(G)$, and hence $A(G) \subseteq C_0(G)$.

Although we will not study von Neumann algebras in this paper, we note that $A(G)$ may be identified with the predual of the group von Neumann algebra $L(G)$.
of \( G \). When \( G \) is abelian, the Fourier transform provides an isometric isomorphism between \( L^1(\hat{G}) \) and \( \text{A}(G) \), and in this way \( \text{A}(G) \) is identified isometrically with the predual of group von Neumann algebra \( L(G) \simeq L^\infty(\hat{G}) \). In the non-abelian case it is still true that \( \text{A}(G) \) identifies isometrically with the predual of the group von Neumann algebra via the duality

\[
\langle T, \varphi \rangle = \langle T f, h \rangle,
\]

where \( T \in L(G) \) and \( \varphi = \tilde{h} \ast \tilde{f} \) for some \( f, h \in L^2(G) \).

### 3. Proof of Theorem 4

In this section we prove Theorem 4, which is the basis of proving Theorems 2 and 5. We first prove that the conditions in Theorem 4 ensure that the regular representation is completely reducible.

**Lemma 6.** Let \( G \) be a locally compact group. Any unitary representation of \( G \) on a separable Hilbert space has at most countably many inequivalent (with respect to unitary equivalence) irreducible subrepresentations.

**Proof.** Let \( \pi \) be a unitary representation of \( G \). The subrepresentations of \( \pi \) are in correspondence with the projections in the commutant \( \pi(G)' \), equivalent subrepresentations correspond to projections that are equivalent in \( \pi(G)' \) (in the sense of Murray-von Neumann), and the irreducible subrepresentations correspond to minimal projections in \( \pi(G)' \). It is therefore enough to show that a von Neumann algebra on a separable Hilbert space has at most countably many inequivalent minimal projections. Let \( M \) be such a von Neumann algebra.

Recall that two minimal projections are inequivalent if and only if their central supports are orthogonal (see [16, Proposition 6.1.8]). Let \( (p_i)_{i \in I} \) be a family of inequivalent minimal projections, and let \( c_i \) be the central support of \( p_i \). Then \( (c_i)_{i \in I} \) is a family of orthogonal projections. By separability of the Hilbert space, \( I \) must be countable. Hence there are at most countably many inequivalent minimal projections in \( M \). \( \square \)

**Corollary 7.** Let \( G \) be a locally compact, second countable group. Then the left regular representation of \( G \) has at most countably many inequivalent irreducible subrepresentations.

**Proof.** The left regular representation represents \( G \) on the Hilbert space \( L^2(G) \), which is separable, since \( G \) is second countable. The statement now follows. \( \square \)

We recall that a unitary representation is of type I, if the image of the representation generates a type I von Neumann algebra. A locally compact group is said to be of type I, if all its unitary representations are of type I (see [10, Chapter 13]). Disintegration theory works especially well in the setting of type I groups. We refer to [13, Chapter 7] for more on type I groups and disintegration theory. Several equivalent characterizations of type I groups can also be found in [10, Chapter 9], but let us just mention one characterization here. The unitary equivalence classes of irreducible representations form a set \( \hat{G} \) called the unitary dual of \( G \). The dual
\( \hat{G} \) is equipped with the Mackey Borel structure, and \( G \) is of type I if and only if \( \hat{G} \) is a standard Borel space. When \( G \) is abelian, the unitary dual coincides with the usual dual group.

**Proposition 8.** Let \( G \) be a second countable, locally compact group satisfying the following two conditions.

1. \( G \) is type I.
2. There is a non-compact, closed subgroup \( H \) of \( G \) such that every irreducible unitary representation of \( G \) is either trivial on \( H \) or is a subrepresentation of the left regular representation \( \lambda_G \).

Then the left regular representation \( \lambda_G \) is completely reducible.

**Proof.** For each \( p \in \hat{G} \), we let \( \pi_p \) denote a representative of the class \( p \), and we assume that the choice of representative is made in a measurable way (\cite{13}, Lemma 7.39). We write the left regular representation as a direct integral of irreducibles,

\[
\lambda_G = \int_{\hat{G}} n_p \pi_p \, d\mu(p),
\]

where \( \mu \) is a Borel measure on \( \hat{G} \) and \( n_p \in \{0, 1, 2, \ldots, \infty\} \) (see \cite{13} Theorem 7.40).

Let \( A = \{ [p] \in \hat{G} \mid p(h) = 1 \text{ for all } h \in H \} \) and let \( B = \hat{G} \setminus A \). It is not hard to check that \( A \subseteq \hat{G} \) is a Borel set.

We note that if \( \pi_p \in B \), then \( \pi_p \) is a subrepresentation of \( \lambda_G \). By the previous corollary, \( B \) is countable. Since \( \lambda_G \) has no subrepresentation which is trivial on a non-compact subgroup, we must have \( \mu(A) = 0 \). Then

\[
\lambda_G = \int_B n_p \pi_p \, d\mu(p),
\]

and since \( B \) is countable, \( \lambda_G \) is a direct sum of irreducibles. \( \square \)

**Lemma 9.** Let \( G \) be a locally compact, second countable group with left regular representation \( \lambda \) and a closed subgroup \( H \) such that

1. \( G \) is type I;
2. Every irreducible unitary representation of \( G \) is either trivial on \( H \) or is a subrepresentation of \( \lambda \);
3. \( \lambda \) is completely reducible.

Then every unitary representation \( \pi \) of \( G \) is a sum \( \sigma_1 \oplus \sigma_2 \), where \( \sigma_1 \) is trivial on \( H \) and \( \sigma_2 \leq \lambda^\infty \). Here \( \lambda^\infty \) denotes a countably infinite direct sum \( \lambda \oplus \lambda \oplus \cdots \) of the representation \( \lambda \).

**Proof.** For each \( p \in \hat{G} \), we let \( \pi_p \) denote a representative of the class \( p \), and we assume that the choice of representative is made in a measurable way (\cite{13}, Lemma 7.39). There is a decomposition

\[
\lambda \simeq \bigoplus_{p \in C} m_p \pi_p.
\]
for some countable $C \subseteq \widehat{G}$ and suitable multiplicities $m_p \in \{1, 2, \ldots, \infty\}$. We may write $\pi$ is a direct integral of irreducibles,

$$\pi = \int_G \oplus n_p \pi_p \, d\mu(p),$$

where $\mu$ is a Borel measure on $\widehat{G}$ and $n_p \in \{0, 1, 2, \ldots, \infty\}$ (see [13, Theorem 7.40]). Let $A = \{[p] \in \widehat{G} \mid p(h) = 1 \text{ for all } h \in H\}$ and let $B = \widehat{G} \setminus A$. Then $A \subseteq \widehat{G}$ is a Borel set. It follows from our assumptions that $B \subseteq C$. If

$$\sigma_1 = \int_A \oplus n_p \pi_p \, d\mu(p), \quad \sigma_2 = \int_B \oplus n_p \pi_p \, d\mu(p),$$

then we see that

$$\pi = \sigma_1 \oplus \sigma_2,$$

where $\sigma_1(g) = 1$ for every $g \in H$. Also,

$$\sigma_2 \leq \bigoplus_{p \in B} n_p \pi_p \leq \bigoplus_{p \in C} (n_p m_p) \lambda \leq \lambda^\infty.$$

\[\square\]

**Lemma 10.** Let $G$ be a locally compact group with left regular representation $\lambda$ and a closed, non-compact subgroup $H$. Suppose every unitary representation $\pi$ of $G$ is a sum $\sigma_1 \oplus \sigma_2$, where $\sigma_1$ is trivial on $H$ and $\sigma_2 \leq \lambda^\infty$. Then $A(G) = B(G) \cap C_0(G)$.

**Proof.** The inclusion $A(G) \subseteq B(G) \cap C_0(G)$ holds for any locally compact group $G$. Suppose $\varphi \in B(G) \cap C_0(G)$. Then there is a continuous, unitary representation $\pi$ of $G$ on some Hilbert space $\mathcal{H}$ and vectors $x, y \in \mathcal{H}$ such that

$$\varphi(g) = \langle \pi(g)x, y \rangle \quad \text{for all } g \in G.$$ 

By assumption we may split $\pi = \sigma_1 \oplus \sigma_2$. Accordingly, we split $\varphi = \varphi_1 + \varphi_2$, where $\varphi_1$ is a coefficient of $\sigma_1$ etc. We will show that $\varphi_1 = 0$ and $\varphi_2 \in A(G)$, which will complete the proof.

Since $\sigma_2 \leq \lambda^\infty$, we see that $\varphi_2$ is of the form

$$\varphi_2(g) = \sum_{i=1}^\infty \langle \lambda(g)x_i, y_i \rangle$$

for some $x_i, y_i \in L^2(G)$ with $\sum_i \|x_i\|^2 < \infty$ and $\sum_i \|y_i\|^2 < \infty$. Each of the maps

$$g \mapsto \langle \lambda(g)x_i, y_i \rangle$$

is in $A(G)$ with norm at most $\|x_i\|\|y_i\|$. Since $A(G)$ is a Banach space and $\sum_i \|x_i\|\|y_i\| < \infty$, we deduce that $\varphi_2 \in A(G)$, and in particular $\varphi_2 \in C_0(G)$. It then follows that $\varphi_1 \in C_0(G)$. Since $\sigma_1$ is trivial on $H$, we see that $\varphi_1$ is constant on $H$ cosets. Since $H$ is non-compact, we deduce that $\varphi_1 = 0$. Then $\varphi = \varphi_2 \in A(G)$. This proves $B(G) \cap C_0(G) = A(G)$. \[\square\]

Theorem [3] is an easy consequence of the previous statements.
Proof of Theorem 4. We assume that $G$ is a locally compact, second countable group satisfying the two conditions in the statement of the theorem. It follows from Proposition 8 that $\lambda_G$ is completely reducible. So by Lemma 9, every unitary representation $\pi$ of $G$ is a sum $\sigma_1 \oplus \sigma_2$, where $\sigma_1$ is trivial on $H$ and $\sigma_2 \leq \lambda^\infty$. From Lemma 10 we conclude that $A(G) = B(G) \cap C_0(G)$. □

4. Invariant measures on homogeneous spaces

To describe the irreducible representations of the groups $P$ in question, we rely on a general method known to the common man as the Mackey Machine. Essential in the Mackey Machine is the notion of induced representations. For a general introduction to the theory of induced representations we refer to [13, Chapter 6] which also contains a description of (a simple version of) the Mackey Machine. The general results about the Mackey Machine can be found in the original paper [21].

The construction of an induced representation from a closed subgroup $H$ to a group $G$ is more easily described when the homogeneous space $G/H$ admits an invariant measure for the $G$-action given by left translation. Regarding homogeneous spaces and invariant measures we record the following easy (and well-known) facts.

Proposition 11. Consider topological groups $G$, $N$, $H$, $K$, $A$, $B$ and topological spaces $X$ and $Y$.

1. Suppose $G$ is the semi-direct product $G = N \rtimes H$, where $N$ is normal in $G$. If $K \leq H$ is a closed subgroup of $H$, then there is a canonical isomorphism

$$NH/NK \simeq H/K$$

as $G$-spaces. Here the $G$-action on $H/K$ is the $H$-action, and $N$ acts trivially on $H/K$.

2. Suppose $G = N \times H$, and $A \leq N$, $B \leq H$ are closed subgroups. Then there is a canonical isomorphism

$$(N \times H)/(A \times B) \simeq N/A \times H/B$$

as $G$-spaces, where the $G$-action on $N/A \times H/B$ is the product action of $N \times H$.

3. Suppose $G \acts X$ and $H \acts Y$ have invariant, $\sigma$-finite Borel measures. Then the product $G \acts X \times Y$ has an invariant, $\sigma$-finite Borel measure.

4. Suppose $G$ is compact (or just amenable) and $X$ is compact. Then any action $G \acts X$ has an invariant probability measure.

Proof.

1. The map $[nh]_{NK} \mapsto [h]_K$ is a well-defined, equivariant homeomorphism.

2. The map $[(n, h)]_{A \times B} \mapsto ([n]_A, [h]_B)$ is a well-defined, equivariant homeomorphism.

3. Take the product measure on $X \times Y$ of the invariant measures on $X$ and $Y$.

4. This is Proposition 5.4 in [27].
The following lemma will be relevant in Section 5 and Section 8 when we verify condition (2) of Theorem 4 for the minimal parabolic groups $P$.

**Lemma 12.** Let $G$ be a locally compact group with closed subgroups $N \subseteq H \subseteq G$, and suppose $N \triangleleft G$. If $\sigma$ is a unitary representation of $H$ which is trivial on $N$, and if $G/H$ admits a $G$-invariant measure, then the induced representation $\text{Ind}_H^G \sigma$ is also trivial on $N$.

**Proof.** Let $\mathcal{H}$ denote the Hilbert space of $\sigma$, and let $q : G \to G/H$ be the quotient map. The induced representation $\pi = \text{Ind}_H^G \sigma$ acts on a completion of the space $\mathcal{F}_0 = \left\{ f \in C(G,\mathcal{H}) \mid f(gh) = \sigma(h^{-1})f(g) \text{ for } g \in G, h \in H \right\}$.

Since $G/H$ admits an invariant measure, the action on $\mathcal{F}_0$ is simply given by left translation, $(\pi(x)f)(g) = f(x^{-1}g)$. With $f \in \mathcal{F}_0$, $g \in G$ and $n \in N$, we compute $(\pi(n)f)(g) = f(n^{-1}g) = f(g(g^{-1}n^{-1}g)) = \sigma(g^{-1}ng)f(g) = f(g)$, since $g^{-1}ng \in N$. It follows that $\pi(n) = 1$. □

**5. The First Example**

In this section we prove Theorem 2. Let $P$ be the group defined in (1.2). In the following proposition we describe the unitary dual of $P$, i.e. the equivalence classes of the irreducible representations of $P$. To do so we apply the Mackey Machine, which works particularly well in our case, where $P$ decomposes as a semidirect product $N_0 \rtimes P_0$ with $N_0$ abelian. For an account on the Mackey Machine we refer to Chapter 6 in [13].

Consider the following closed subgroups of $P$.

$$P_0 = \left\{ \begin{pmatrix} \lambda & a & 0 \\ 0 & \lambda^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid a \in \mathbb{R}, \lambda > 0 \right\} \quad (5.1)$$

$$P_1 = \left\{ \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid a \in \mathbb{R} \right\} \quad (5.2)$$

$$N_0 = \left\{ \begin{pmatrix} 1 & 0 & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \mid b, c \in \mathbb{R} \right\} \quad (5.3)$$

$$N_1 = \left\{ \begin{pmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid c \in \mathbb{R} \right\} \quad (5.4)$$
Observe that $P = N_0 \rtimes P_0$. We note that $P_0$ is isomorphic to the $ax + b$ group, i.e. the group of affine transformations $x \mapsto ax + b$ of the real line, where $a > 0$ and $b \in \mathbb{R}$. The dual of the $ax + b$ group is well-known (see for instance [13, Section 6.7]). The dual of $N_0 \simeq \mathbb{R}^2$ is $\tilde{N}_0 \simeq \mathbb{R}^2$ which we as usual identify with $\mathbb{R}^2$.

**Proposition 13.** Let $\pi$ be an irreducible representation of $P$. Then $\pi$ is equivalent to one of the following representations (and these are all inequivalent).

1. $\pi_1 = \text{Ind}_{N_0}^P(\nu)$, where $\nu \in \tilde{N}_0$ is $\nu = (1, 0)$.
2. $\pi_2 = \text{Ind}_{N_0}^P(\nu)$, where $\nu \in \tilde{N}_0$ is $\nu = (-1, 0)$.
3. $\pi_{3, \rho} = \text{Ind}_{N_0 P_1}^P(\nu \rho)$, where $\nu \in \tilde{N}_0$ is $\nu = (0, 1)$ and $\rho$ is a character in $\tilde{P}_1 \simeq \mathbb{R}$.
4. $\pi_{4, \rho} = \text{Ind}_{N_0 P_1}^P(\nu \rho)$, where $\nu \in \tilde{N}_0$ is $\nu = (0, -1)$ and $\rho$ is a character in $\tilde{P}_1 \simeq \mathbb{R}$.
5. $\pi_{5, \sigma} = \sigma \circ q$, where $\sigma \in \tilde{P}_0$ and $q : P \rightarrow P_0$ is the quotient map.

**Proof.** We follow the strategy of the Mackey Machine as described in Theorem 6.42 in [13], which gives a complete description of the unitary dual of $P$. We think of $P = N_0 \rtimes P_0$ as a subgroup of $\mathbb{R}^2 \rtimes \text{SL}_2(\mathbb{R})$, where $\text{SL}_2(\mathbb{R})$ acts on $\mathbb{R}^2$ by matrix multiplication. The action $P_0 \cap N_0$ is then simply matrix multiplication, and the dual action $P_0 \cap \tilde{N}_0$ is given by $(p, \nu)(n) = \nu(p^{-1}n)$ for $p \in P_0$, $\nu \in \tilde{N}_0$ and $n \in N_0$. Under the usual identification $\tilde{N}_0 \simeq \mathbb{R}^2$ we see that $p \in P_0$ acts on $\mathbb{R}^2$ by matrix multiplication by the transpose of the inverse of $p$. Thus, if $p$ has the form in (5.1), then the action of $p$ on $\mathbb{R}^2$ is

$$
\begin{pmatrix}
  s \\
  t
\end{pmatrix}
\rightarrow
\begin{pmatrix}
  \lambda^{-1} & 0 \\
  -a & \lambda
\end{pmatrix}
\begin{pmatrix}
  s \\
  t
\end{pmatrix}.
$$

There are five orbits in $\tilde{N}_0$ under this action, which give the five alternatives in the proposition. The orbits are

- $\mathcal{O}_1 = \{(s, t) \mid s > 0\},$
- $\mathcal{O}_2 = \{(s, t) \mid s < 0\},$
- $\mathcal{O}_3 = \{(0, t) \mid t > 0\},$
- $\mathcal{O}_4 = \{(0, t) \mid t < 0\},$
- $\mathcal{O}_5 = \{(0, 0)\}.$

Since there are only finitely many orbits, the action of $P_0$ on $\tilde{N}_0$ is regular. As representatives of the orbits we choose the points

$$
(1, 0) \in \mathcal{O}_1, \quad (-1, 0) \in \mathcal{O}_2, \quad (0, 1) \in \mathcal{O}_3, \quad (0, -1) \in \mathcal{O}_4, \quad (0, 0) \in \mathcal{O}_5.
$$

Case 1: $\nu = (1, 0)$. In this case the stabilizer subgroup of $\nu$ inside $P_0$ is trivial, and hence we obtain the representation $\pi = \text{Ind}_{N_0}^P(\nu)$.

Case 2: $\nu = (-1, 0)$. This is similar to case 1.

Case 3: $\nu = (0, 1)$. The stabilizer subgroup of $\nu$ inside $P_0$ is $P_1$, and hence we obtain $\pi = \text{Ind}_{N_0 P_1}^P(\nu \rho)$, where $\rho \in \tilde{P}_1$. Here the representation $\nu \rho$ on $N_0 P_1$ is
given by
\[(\nu \rho)(nh) = \nu(n)\rho(h), \quad \text{for all } n \in N_0, \ h \in P_1.\]

Case 4: \(\nu = (0, -1).\) This is similar to case 3.

Case 5: \(\nu = (0, 0).\) In this case the stabilizer subgroup of \(\nu\) inside \(P_0\) is everything. It follows that \(\pi\) is a representation which satisfies \(\pi(n) = \langle n, \nu \rangle\) for every \(n \in N_0.\) In other words, \(\pi\) is trivial on \(N_0\) and factors to an irreducible representation \(\sigma\) of \(P_0.\) That is, \(\pi = \sigma \circ q.\) \(\square\)

The Plancherel measure of a group describes how the left regular representation decomposes as a direct integral of irreducible representations. For example, the Plancherel measure of a locally compact abelian group is simply the Haar measure on the dual group. This is seen using the Fourier transform. The following proposition determines the Plancherel measure of \(P\) and shows in particular that the measure is purely atomic. Hence the left regular representation of \(P\) is completely reducible.

**Proposition 14.** The left regular representation \(\lambda_P\) of \(P\) is (equivalent to) the countably infinite direct sum of \(\pi_1 \oplus \pi_2,\) where \(\pi_1\) and \(\pi_2\) are as in Proposition 13.

**Proof.** Again it is useful to view \(P\) as the semidirect product \(P = N_0 \rtimes P_0.\) We follow the approach described in [4, Section 1]. Their results are stated for the right regular representation, but everything works mutatis mutandis for the left regular. As before, the dual group \(\hat{N}_0\) is identified with \(\mathbb{R}^2,\) and the Plancherel measure on \(\hat{N}_0\) is simply Lebesgue measure. The orbits under the dual action \(P_0 \curvearrowright \hat{N}_0\) which have positive Lebesgue measure are \(O_1\) and \(O_2,\) and their complement in \(\hat{N}_0\) is a null set (the \(y\)-axis). The stabilizer subgroups inside \(P_0\) of the points \((1, 0)\) and \((-1, 0)\) are trivial, so in particular these stabilizer subgroups have completely reducible regular representations. Thus criteria (a) and (b) of [4] are satisfied, and it follows from their calculation on page 595 that
\[\lambda_P = \bigoplus_{n=1}^{\infty} (\pi_1 \oplus \pi_2).\] \(\square\)

**Lemma 15.** Consider the groups \(P = N_0 \rtimes P_0\) and \(N_1 \subseteq P.\) If \(\pi\) is an irreducible unitary representation of \(P,\) then one (and only one) of the following holds.

1. \(\pi(g) = 1\) for every \(g\) in the subgroup \(N_1,\)
2. \(\pi\) is a subrepresentation of \(\lambda_P.\)

**Proof.** We divide the proof into the cases according to the description in Proposition 13.

If \(\pi = \pi_1\) or \(\pi = \pi_2,\) then it follows from Proposition 14 that \(\pi \leq \lambda_P.\)

Suppose now \(\pi = \pi_{3, \rho},\) where \(\rho \in \hat{P}_1.\) If we let \(\nu = (0, 1) \in \hat{N}_0,\) then we see that \(N_1 = \ker \nu.\) Hence the representation \(\nu \rho\) of \(N_0 P_1\) is trivial on \(N_1\) which is normal in \(P.\) Since \(N_0 P_1\) is a normal subgroup of \(P,\) the homogeneous space
$P/(N_0P_1)$ has a $P$-invariant measure, Haar measure. It follows from Lemma 12 that $\pi = \text{Ind}_{N_0P_1}^P (\nu \rho)$ is trivial on $N_1$.

The case $\pi = \pi_{4, \rho}$ is similar to the previous case. We simply note that $\ker \bar{\nu} = N_1$, where $\bar{\nu} = (0, -1)$.

In the case $\pi = \pi_{5, \sigma}$, it is clear that $\pi(g) = 1$ for every $g \in N_0$, and hence in particular for every $g \in N_1$.

\textbf{Lemma 16.} The group $P$ is of type I.

\textit{Proof.} The group $P$ is a connected, real algebraic group, and such groups are of type I according to [8, Theorem 1].

\hspace{1cm} \Box

We collect the previous results of this section in the following proposition, which together with Theorem 4 immediately implies Theorem 2.

\textbf{Proposition 17.} Let $P$ and $N_1$ be the groups in (1.2) and (5.4). The following holds.

1. $P$ is of type I.
2. Every irreducible unitary representation of $P$ is either trivial on the non-compact closed subgroup $N_1$ or is a subrepresentation of $\lambda_P$.

6. Simple Lie groups of real rank one

Let $G$ be a connected simple Lie group with finite center and of real rank one. Let $G = KAN$ be an Iwasawa decomposition of $G$. Then $K$ is a maximal compact subgroup, $A$ is abelian of dimension 1, and $N$ is nilpotent. Let $M$ be the centralizer of $A$ in $K$, and let $P = MAN$ be the minimal parabolic subgroup of $G$.

It is known that $G$ is locally isomorphic to one of the classical groups $\text{SO}_0(n, 1)$, $\text{SU}(n, 1)$, $\text{Sp}(n, 1)$ or the exceptional group $F_4(-20)$ (see for instance the list on p. 426 in [20]), and we now describe these groups in more detail, including explicit descriptions of the Iwasawa subgroups and the minimal parabolic subgroup $P$.

6.1. \textbf{The classical cases.} Let $F$ be one of the three finite-dimensional division algebras over the reals, the real field $\mathbb{R}$, the complex field $\mathbb{C}$ or the quaternion division ring $\mathbb{H}$. In the exceptional case treated later, $F$ will be the non-associative real algebra $\mathbb{O}$ of octonions, also known as the Cayley algebra. We let $\text{Re} F$ and $\text{Im} F$ denote the real and imaginary part of $F$, so that $F = \text{Re} F + \text{Im} F$, and in the standard notation

$$\text{Im} \mathbb{R} = 0, \quad \text{Im} \mathbb{C} = \mathbb{R}i, \quad \text{Im} \mathbb{H} = \mathbb{R}i + \mathbb{R}j + \mathbb{R}k.$$ We use the notation $F'$ to denote the unit sphere in $F$, 

$$F' = \{ x \in F \mid \| x \| = 1 \}.$$
and $F^* = F \setminus \{0\}$.

Let $F_{p,q}$ denote the real vector space $F^{p+q}$ equipped with the hermitian form

$$\langle x, y \rangle = \sum_{i=1}^{p} x_i \bar{y}_i - \sum_{i=p+1}^{p+q} x_i \bar{y}_i.$$  

We also think of $F_{p,q}$ as a right $F$-module. Of course, $F_n = F^n_0$. We write $w^t$ for the row vector which is the transpose of a column vector $w \in F^n$. Also, $w^* = \bar{w}^t$ and $|w|^2 = w^*w = \langle w, w \rangle$, when $w \in F^n$.

Let $U(p, q, F)$ denote the unitary group of $F_{p,q}$, i.e. the square matrices over $F$ of size $p+q$ that preserve the hermitian form. If $F$ is $\mathbb{R}$ or $\mathbb{C}$, we will be concerned with the unitaries of determinant 1. We write $SU(p, q, F)$ for this group. It is customary to write $U(p, q, \mathbb{R}) = O(p, q)$, $SU(p, q, \mathbb{R}) = SO(p, q)$, $SU(p, q, \mathbb{C}) = SU(p, q)$, $U(p, q, \mathbb{H}) = Sp(p, q)$.

The groups $SU(p, q)$ and $Sp(p, q)$ are connected, and $Sp(p, q)$ is even simply connected (see [20, Section I.17]).

We let $U_0(p, q, F)$ denote the connected component of $U(p, q, F)$. Note that $U_0(n, \mathbb{R}) = SO(n)$, $U_0(n, \mathbb{C}) = U(n)$, $U_0(n, \mathbb{H}) = Sp(n)$, and in particular $U_0(1, \mathbb{R}) = \{1\}$. We remark that $U_0(n, F)$ acts transitively on the unit sphere in $F^n$ except for the case $n = 1$ and $F = \mathbb{R}$.

The following is taken from [23]. Let $G$ be one of $SO_0(n, 1)$, $SU(n, 1)$, $Sp(n, 1)$. Then the subgroups related to the Iwasawa decomposition of $G$ are the following.

\[
K = \begin{pmatrix} k & 0 \\ 0 & \beta \end{pmatrix}, \quad \beta \in U_0(1, F), \quad \beta \det k = 1 \text{ if } F \neq \mathbb{H} \quad (6.1)
\]

\[
M = \begin{pmatrix} \beta & 0 & 0 \\ 0 & u & 0 \\ 0 & 0 & \beta \end{pmatrix}, \quad \beta^2 \det u = 1 \text{ if } F \neq \mathbb{H} \quad (6.2)
\]

\[
A = \begin{pmatrix} \cosh t & 0 & \sinh t \\ 0 & I & 0 \\ \sinh t & 0 & \cosh t \end{pmatrix}, \quad t \in \mathbb{R}
\]

\[
N = \begin{pmatrix} 1 + z - \frac{1}{2} |w|^2 & w^* & -z + \frac{1}{2} |w|^2 \\ w & I & w \\ z - \frac{1}{2} |w|^2 & w^* & 1 - z + \frac{1}{2} |w|^2 \end{pmatrix}, \quad w \in F^{n-1}, \quad z \in \text{Im } F \quad (6.3)
\]

The subgroups $M$ and $A$ of $P$ commute. The group $N$ is normal in $P$, and $P$ is the semi-direct product of $MA$ and $N$. To describe the action of $M$ and $A$ on $N$, \[\text{...}\]
it will be easier to work with a group isomorphic to $P$ (but no longer a subgroup of $G$) obtained by conjugating $P$ by the orthogonal matrix

\[
\begin{pmatrix}
\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\
0 & I & 0 \\
-\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}
\end{pmatrix}.
\]

Then $A$ and $N$ become, with $\alpha = e^t$,

\[
A = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & \alpha^{-1} \end{pmatrix}, \quad \alpha > 0 \quad (6.4)
\]

\[
N = \begin{pmatrix} 1 & w^t & z + \frac{1}{2}|w|^2 \\ 0 & I & \bar{w} \\ 0 & 0 & 1 \end{pmatrix}, \quad w \in \mathbb{F}^{n-1}, \quad z \in \text{Im } \mathbb{F} \quad (6.5)
\]

while $M$ remains the same. We have chosen to rescale the parameter $z$ in (6.5) by a factor of two compared with (6.3) and replace $w$ by its conjugate $\bar{w}$, so that the group law in $N$ matches the one from [32]. We think of the group $N$ as $\mathbb{F}^{n-1} \times \text{Im } \mathbb{F}$ with group structure

\[
(w_1, z_1)(w_2, z_2) = (w_1 + w_2, z_1 + z_2 + \text{Im}(w_1, w_2))
\]

and write $(w, z)$ for the matrix in (6.5). The action $MA \lhd N$ is given by

\[
\alpha.(w, z) = (\alpha w, \alpha^2 z) \quad (6.6)
\]

and

\[
\text{diag}(\beta, u, \beta).(w, z) = (uw\beta^{-1}, \beta z\beta^{-1}). \quad (6.7)
\]

Note that the three subsets $\{0\} \times \text{Im } \mathbb{F}$, $\{0\} \times \text{Im } \mathbb{F}^*$ and $\mathbb{F}^{n-1} \times \{0\}$ are invariant under the action $MA$. If $\mathbb{F} = \mathbb{R}$, then $N$ is abelian, and otherwise the center of $N$ is

\[
Z(N) = \{(0, z) \mid z \in \text{Im } \mathbb{F}\}.
\]

### 6.2. The exceptional case.

The exceptional group $F_4(-20)$ has a realization as automorphisms of a Jordan algebra. A detailed treatment of the group $F_4(-20)$ can be found in [28] including a description of the Iwasawa decomposition $F_4(-20) = KAN$ (see [28, §5 Théorème 1]). Here we only describe the components $M$, $A$ and $N$ of the minimal parabolic subgroup $P = MAN$ and not the group $F_4(-20)$ itself. The group $P$ is best described using the octonion non-associative division algebra $\mathcal{O}$. For a detailed description of the octonions we refer to [28, §1]. Another reference is [11 2].

We recall that $\mathcal{O}$ is an 8-dimensional real vector space, and thus we usually identify $\mathcal{O}$ with $\mathbb{R}^8$. We use the notation $\bar{y}$ for the conjugate of $y \in \mathcal{O}$, and we let $(x, y) = x\bar{y}$. The real bilinear form $(x|y) = \text{Re}(x, y)$ corresponds to the usual inner product on $\mathbb{R}^8$. The imaginary octonions $\text{Im } \mathcal{O}$ form a subspace identified with $\mathbb{R}^7$.

The group $N$ is $\mathcal{O} \times \text{Im } \mathcal{O}$ with group product

\[
(w_1, z_1)(w_2, z_2) = (w_1 + w_2, z_1 + z_2 + \text{Im}(w_1, w_2)).
\]
The center of $N$ is $Z(N) = \{(0, z) \mid z \in \text{Im } \mathcal{O}\}$, and the quotient $N/Z(N)$ is then isomorphic to $(\mathcal{O}, +)$. The group $N$ is connected and nilpotent.

The group $A$ is $\mathbb{R}_+$, and the action $A \curvearrowright N$ is given by

$$\alpha.(w, z) = (\alpha w, \alpha^2 z), \quad \alpha \in \mathbb{R}_+.$$ 

The group $M$ is the spin group Spin(7), which is the (2-sheeted) universal cover of SO(7). In order to describe the action $M \curvearrowright N$, we need to consider two orthogonal representations of Spin(7), the spin representation $\sigma : M \to \text{SO}(8)$ and the vector representation $\nu : M \to \text{SO}(7)$. Then the action $M \curvearrowright N$ is then given as

$$u.(w, z) = (\sigma(u)w, \nu(u)z), \quad u \in \text{Spin}(7).$$ 

If we identify Im $\mathcal{O}$ with $\mathbb{R}^7$ in the usual way, then SO(7) acts on Im $\mathcal{O}$ by matrix multiplication. The vector representation $\nu$ is simply the covering homomorphism $\nu : \text{Spin}(7) \to \text{SO}(7)$. Under the identification of Im $\mathcal{O}$ with $\mathbb{R}^7$, the purely imaginary unit octonions are identified with the unit sphere $S^6$. Since SO(7) acts transitively on $S^6$, it follows that $MA$ acts transitively on Im $\mathcal{O}^*$.

The spin representation $\sigma : \text{Spin}(7) \to \text{SO}(8)$ gives a transitive action of Spin(7) on $S^7$ (see [28, §4 Lemme 1]).

The actions of $M$ and $A$ on $N$ commute and thus give an action $M \times A \curvearrowright N$. The group $P$ is the semidirect product $P = MA \ltimes N$.

Note that the three subsets $\{0\} \times \text{Im } \mathcal{O}$, $\{0\} \times \text{Im } \mathcal{O}^*$ and $\mathcal{O} \times \{0\}$ of $N$ are invariant under the action $MA$.

7. The irreducible representations of parabolic subgroups

In this section we describe the unitary dual of the minimal parabolic subgroups $P$ from the previous section. The result is contained in Theorem [18] and Theorem [20]. We also prove that $P$ and $N$ are of type I.

7.1. The classical cases. Let $G$ be one of the classical groups $\text{SO}_0(n, 1)$, $\text{SU}(n, 1)$, $\text{Sp}(n, 1)$, and let $P = MAN$ be the minimal parabolic subgroup of $G$. To describe the irreducible representations of $P$ we rely on the work of [32], in which groups very similar to our $P$ are considered as well as many other groups. In fact, they consider the group $\tilde{M}AN$, where $\tilde{M}$ is

$$\tilde{M} = \left( \begin{array}{ccc} \beta & 0 & 0 \\ 0 & u & 0 \\ 0 & 0 & \beta \end{array} \right), \quad u \in \text{U}(n-1, \mathbb{F}), \ \beta \in \text{U}(1, \mathbb{F})$$

Their conclusion about the irreducible representations is contained in [32, Proposition 7.8]. Actually, if $\mathbb{F} = \mathbb{H}$, which is the case we are most interested in because of future applications [14], then $\tilde{M} = M$, and Theorem [18] is a special case of [32, Proposition 7.8].
The discussion below is based on Section 4 and 7 from [32] to which we refer for proofs and more details. The arguments carry over without any challenges to our situation. The representations of $P$ fall into three series.

1) The subgroup $N$ is normal in $P$, and $P/N \simeq MA$. We let $q : P \to P/N$ denote the quotient map. Of course, any irreducible representation $\sigma$ of $P/N$ gives rise to the irreducible representation $\sigma \circ q$ of $P$, and these are precisely the irreducibles of $P$ that annihilate $N$.

2) Next we describe the irreducibles of $P$ arising from characters on $N$. Let $v \in \mathbb{F}^{n-1}$ be non-zero, and define the character $\chi_v$ on $N$ by

$$\chi_v(w, z) = \exp(i \text{Re}(w, v)).$$

The group $MA$ acts on $N$ by conjugation, and this induces a dual action of $MA$ on $\hat{N}$. Let $L_v$ be the stabilizer of $\chi_v$ in $MA$ under this action. Then $\chi_v$ extends to a character of $N \rtimes L_v$ by the formula

$$\chi_v(w, z, g) = \chi_v(w, z) = \exp(i \text{Re}(w, v)), \quad (w, z, g) \in \mathbb{F}^{n-1} \times \text{Im } \mathbb{F} \times L_v.$$

Let $\gamma$ be an irreducible representation of $L_v$. Extend $\gamma$ to be the irreducible representation of $N \rtimes L_v$ defined by letting $\gamma$ be trivial on $N$. Form the tensor product representation $\chi_v \otimes \gamma$ and induce this representation from $N \rtimes L_v$ to $P$ to get a representation $\pi_{2,v,\gamma}$ of $P$,

$$\pi_{2,v,\gamma} = \text{Ind}_{N L_v}^P(\chi_v \otimes \gamma).$$

Before we move on to the last series in $\widehat{P}$, we describe the action of $MA$ on the non-trivial characters on $N$ in more detail. The action is given by (6.6) and (6.7),

$$(u, \beta, \alpha) \cdot \chi_v = \chi_{v'} \quad \text{where } v' = u^{-1} \alpha^{-1} v \beta. \quad (7.1)$$

We see that unless $G = \text{SO}_0(2, 1)$, the action of $MA$ on $\mathbb{F}^{n-1} \setminus \{0\}$ is transitive, and if $G = \text{SO}_0(2, 1)$, the action of $MA$ on $\mathbb{R}^*$ has two orbits $\mathbb{R}_+$ and $\mathbb{R}_-$. A set of representatives for the orbits $MA \cap \mathbb{F}^{n-1} \setminus \{0\}$ is then

$$S_2 = \{-1,1\} \quad \text{if } G = \text{SO}_0(2, 1) \quad \text{and} \quad S_2 = \{1\} \quad \text{if } G \neq \text{SO}_0(2, 1)$$

The stabilizer $L_v$ is $L_v = \{(u, \beta) \in M \mid uv = v \beta\}$, and we note that $L_v \subseteq M$.

3) Finally, we consider representations that do not come from characters on $N$. This happens only when $F \neq \mathbb{R}$. Let $m \in \text{Im } \mathbb{F}^*$, and define $\lambda : \text{Im } \mathbb{F} \to \mathbb{R}$ by $\lambda(z) = -\text{Re}(m \bar{z})$. Then $\lambda$ is a non-trivial $\mathbb{R}$-linear map. It is known that there exists an infinite dimensional irreducible representation $\eta_m$ of $N$, uniquely determined be the property

$$\eta_m(w, z) = e^{i\lambda(z)} \eta_m(w, 0), \quad (w, z) \in \mathbb{F}^{n-1} \times \text{Im } \mathbb{F}.$$ 

Moreover, $\eta_m$ is uniquely determined within unitary equivalence by the central character $\lambda$ (see [32] Lemma 4.4)). The group $MA$ acts on the classes of representations $\eta_m$. Let $L_m$ denote the stabilizer in $MA$ of the class $[\eta_m]$,

$$L_m = \{g \in MA \mid g \eta_m \simeq \eta_m\}.$$ 

Then $\eta_m$ extends to a representation of $N \rtimes L_m$ as discussed in [32] Section 7], and the extension is of course still irreducible.
Let \( \gamma \) be an irreducible representation of \( L_m \). Extend \( \gamma \) to be the irreducible representation of \( N \rtimes L_m \) defined by letting \( \gamma \) be trivial on \( N \). Form the tensor product representation \( \eta_m \otimes \gamma \) and induce this representation to get a representation \( \pi_{3,m,\gamma} \) of \( P \),

\[
\pi_{3,m,\gamma} = \text{Ind}_{N L_m}^P (\eta_m \otimes \gamma).
\]

We now describe the action of \( MA \) on the infinite dimensional representations of \( N \) in more detail. Since \( \eta_m \) is uniquely determined within unitary equivalence by \( \lambda \) (or equivalently by \( m \)), the action is best described by the action \( MA \rtimes \text{Im} F^* \) given by (6.6) and (6.7),

\[
(u, \beta, \alpha) \eta_m = \eta_{m'} \quad \text{where} \quad m' = \beta \alpha^{-2} m \beta^{-1}.
\]

If \( F = \mathbb{C} \), there are two orbits under this action, \( i \mathbb{R}^+ \) and \( i \mathbb{R}^- \), and if \( F = \mathbb{H} \), there is only one orbit \( \text{Im} F^* \). As a set of representatives for the orbits we choose

\[
S_3 = \{-i, i\} \quad \text{if} \quad F = \mathbb{C} \quad \text{and} \quad S_3 = \{i\} \quad \text{if} \quad F = \mathbb{H}.
\]

The stabilizer of \( m \in \{-i, i\} \) is

\[
L_m = \{(u, \beta) \in M \mid \beta \in \mathbb{R} + i \mathbb{R}\}
\]

We note that the stabilizer \( L_m \subseteq M \).

The three constructions given above exhaust the unitary dual of \( P \).

**Theorem 18.** Let \( G \) be one of the classical groups \( \text{SO}_0(n, 1) \), \( \text{SU}(n, 1) \), \( \text{Sp}(n, 1) \), let \( F \) be the corresponding division algebra \( (\mathbb{R}, \mathbb{C} \text{ or } \mathbb{H}) \), and let \( P = MAN \) be the minimal parabolic subgroup of \( G \). The irreducible representations of \( P \) fall into three series as follows.

1. The series

   \[
   \pi_{1,\sigma} = \sigma \circ q,
   \]
   where \( q : P \to P/N \) is the quotient map, and \( \sigma \in \widehat{P/N} = \widehat{MA} \). The classes are parametrized by \( \sigma \in \widehat{P/N} \).

2. The series

   \[
   \pi_{2,v,\gamma} = \text{Ind}_{N L_v}^P (\chi_v \otimes \gamma),
   \]
   where \( v \in F^{n-1} \) is non-zero and \( \gamma \in \widehat{L_v} \). The classes are parametrized by \( v \in S_2 \) and \( \gamma \in \widehat{L_v} \).

3. The series (only when \( F \neq \mathbb{R} \))

   \[
   \pi_{3,m,\gamma} = \text{Ind}_{N L_m}^P (\eta_m \otimes \gamma)
   \]
   where \( m \in \text{Im} F^* \) and \( \gamma \in \widehat{L_m} \). The classes are parametrized by \( m \in S_3 \) and \( \gamma \in \widehat{L_m} \).

**Lemma 19.** Consider the minimal parabolic subgroup \( P = MAN \) in one of the classical groups \( \text{SO}_0(n, 1) \), \( \text{SU}(n, 1) \), \( \text{Sp}(n, 1) \). Then \( P \) and \( N \) are of type I.

**Proof.** It is known that connected nilpotent Lie groups are of type I (see [21 Corollaire 4]), and it follows that \( N \) is of type I.

Theorem 9.3 in [21] provides a way of establishing that \( P \) is type I. First of all, \( \widehat{N} \) is a standard Borel space, because \( N \) is of type I. The action \( MA \rtimes \widehat{N} \) has only
According to [21, Theorem 9.3] we may now conclude that $P$ is of type I. Indeed, if $\pi$ is the trivial character on $N$, then $L_\pi = MA$ which is a direct product of the compact group $M$ and the abelian group $A$. Hence the stabilizer $MA$ is of type I. If $\pi$ is not the trivial character, then $\pi = \eta_m$ or $\pi = \chi_v$, where $m \in \mathbb{F}^\ast$ or $v \in \text{Im} \mathbb{F}^\ast$, and we already saw that $L_m$ and $L_v$ are closed subgroups of $M$ and hence compact. In particular the stabilizers are of type I.

According to [21, Theorem 9.3] we may now conclude that $P$ is of type I. \hfill \Box

7.2. The exceptional case. Let $P = MAN$ be the minimal parabolic subgroup of $F_{4(-20)}$. We will now describe the irreducible representations of $P$. Again, this is based on [32]. They consider the group $\tilde{MAN}$, where $\tilde{M} = \text{Spin}(7) \times \{\pm 1\}$. The complete description of the unitary dual of $\tilde{MAN}$ can be found in (8.12) and (8.15) in [32]. The discussion below is based on Section 8 in [32] to which we refer for proofs and more details. The representations fall into two series.

(1) Irreducible representations of $N$ that annihilate the center $Z(N) = \text{Im} \mathcal{O}$ are characters of the form $\chi_v$ for some $v \in \mathcal{O}$, where $\chi_v$ is given by

$$\chi_v(w, z) = e^{i \text{Re}(w, v)} = e^{i |w| v}$$

The group $MA$ acts on $N$, and this induces a dual action of $MA$ on $\tilde{N}$.

Let $L_v$ be the stabilizer of $\chi_v$ in $MA$. Then $\chi_v$ extends to a character of $N \rtimes L_v$ by the formula

$$\chi_v(w, z, g) = \chi_v(w, z) = e^{i |w| v}, \quad (w, z, g) \in \mathcal{O} \times \text{Im} \mathcal{O} \times L_v.$$ 

Let $\gamma$ be an irreducible representation of $L_v$. Extend $\gamma$ to be the irreducible representation of $N \rtimes L_v$ defined by letting $\gamma$ be trivial on $N$. Form the tensor product representation $\chi_v \otimes \gamma$ and induce this representation from $N \rtimes L_v$ to $P$ to get a representation $\pi_{1,v,\gamma}$ of $P$,

$$\pi_{1,v,\gamma} = \text{Ind}_{N L_v}^P (\chi_v \otimes \gamma).$$

This representation $\pi_{1,v,\gamma}$ is a representation in the first series.

From the definition of the action $MA \curvearrowright N$ we see that

$$(u, \alpha).\chi_v(w, z) = e^{i(\alpha \sigma(u))^{-1} |w| v} = e^{i (u | \alpha \sigma(u) v) = \chi_{\alpha \sigma(u) v}(w, z).}$$

Since $M$ acts transitively on $S^7 \subseteq \mathcal{O}$, we see that $MA$ acts transitively $\mathcal{O}^\ast$ and thus on the characters $\{\chi_v\}_{v \in \mathcal{O}^\ast}$.

If $v = 0$, the stabilizer $L_v$ is of course all of $MA$. Otherwise the stabilizer $L_v$ is

$$L_v = \{(u, \alpha) \in MA | \sigma(u) \alpha v = v\}.$$ 

Since $\sigma(u)$ preserves the norm of elements in $\mathcal{O}$, we see that if $(u, \alpha) \in L_v$, then $\alpha = 1$. Hence $L_v \subseteq M$. 

Let $m \in \text{Im } \mathbb{O}^*$ be non-zero, and define $\lambda : \text{Im } \mathbb{O} \to \mathbb{R}$ by $\lambda(z) = -\text{Re}(mz)$. Then $\lambda$ is a non-trivial $\mathbb{R}$-linear map which is uniquely determined by $m$. Irreducible representations of $N$ that do not annihilate the center are infinite dimensional and of the form $\eta_m$ for some $m \in \text{Im } \mathbb{O}^*$, where $\eta_m$ is uniquely determined by the property

$$\eta_m(z, z) = e^{i\lambda(z)}\eta_m(z, 0), \quad (z, z) \in N.$$ 

Moreover, the equivalence class of $\eta_m$ is uniquely determined by the central character $\lambda$ and hence by $m$. Since the action of $MA$ on $\text{Im } \mathbb{O}^*$ is transitive, $MA$ acts transitively on the set $\{\eta_m\}_{m \in \text{Im } \mathbb{O}^*}$.

Let $L_m$ denote the stabilizer in $MA$ of the class of $\eta_m$. Then

$$L_m = \{u \in M \mid \nu(u)m = m\}.$$ 

It follows from [32, Lemma 8.14] that $\eta_m$ extends to a representation of $N \rtimes L_m$. Let $\gamma$ be an irreducible representation of $L_m$, and extend $\gamma$ to $N \rtimes L_m$ by letting $\gamma$ be trivial on $N$. Form the tensor product representation $\eta_m \otimes \gamma$ and induce this representation to get a representation $\pi_{2,m,\gamma}$ of $P$,

$$\pi_{2,m,\gamma} = \text{Ind}_{N L_m}^P (\eta_m \otimes \gamma).$$

**Theorem 20.** Let $P = MAN$ be the minimal parabolic subgroup of $F_4(-20)$ and let $\pi$ be an irreducible representation of $P$. Then $\pi$ is unitarily equivalent to one of the following.

1. $\pi_{1,v,\gamma} = \text{Ind}_{N L_v}^P (\chi_v \otimes \gamma)$ for some $v \in \mathbb{O}$ and $\gamma \in \hat{L}_v$.
2. $\pi_{2,m,\gamma} = \text{Ind}_{N L_m}^P (\eta_m \otimes \gamma)$ for some $m \in \text{Im } \mathbb{O}^*$ and $\gamma \in \hat{L}_m$.

**Lemma 21.** Consider the minimal parabolic subgroup $P = MAN$ of $F_4(-20)$. Then $P$ and $N$ are of type I.

**Proof.** Since $N$ is a connected nilpotent Lie group, $N$ is of type I (see [9, Corollaire 4]).

Theorem 9.3 in [21] provides a way of establishing that $P$ is type I. First of all, $\hat{N}$ is a standard Borel space, because $N$ is of type I. The action $MA \curvearrowright \hat{N}$ has only three orbits,

$$\mathcal{O}_1 = \{\chi_0\}, \quad \mathcal{O}_2 = \{\chi_v\}_{v \in \mathbb{O}^*}, \quad \mathcal{O}_3 = \{\eta_m\}_{m \in \text{Im } \mathbb{O}^*},$$

where $\chi_0$ is the trivial representation. Then, clearly, there is a Borel set in $\hat{N}$ which meets each orbit exactly once. By [21, Theorem 9.2] the action $MA \curvearrowright \hat{N}$ is regular.

We now verify that when $\pi \in \hat{N}$, the stabilizer $L_\pi = \{g \in MA \mid g \pi \simeq \pi\}$ is of type I. Indeed, if $\pi = \eta_m$ or $\pi = \chi_v$, where $m \in \mathbb{O}^*$ or $v \in \text{Im } \mathbb{O}^*$, then we already saw that $L_m$ and $L_v$ are compact and in particular of type I. If $\pi = \chi_0$, then the stabilizer is $MA$ which is a direct product of the compact group $M$ and the abelian group $A$. Hence the stabilizer $MA$ is of type I.

According to [21, Theorem 9.3] we may now conclude that $P$ is of type I. \qed
8. The Fourier algebra of $P$

In this section we verify the last condition in Theorem 4 for the minimal parabolic subgroups $P$. The result is contained in Proposition 27.

Recall that part of the Peter-Weyl Theorem asserts that the left regular representation of a compact group is completely reducible, and every irreducible representation of the compact group occurs as a direct summand (see [13, Theorem 5.12]).

We now set out to determine which irreducible representations of $P$ that occur as subrepresentations of the left regular representation. For this we will rely on Corollary 11.1 in [19]. In order to apply the corollary we first need to verify the assumptions I-IV from [19]. For this it will suffice to observe that $N$ and $P$ are of type I (see Lemma 19 and Lemma 21), and all stabilizers $L_v$ and $L_m$ are closed and contained in $M$, so they are compact and in particular of type I.

In the case of the representation $\pi_{1, v, \gamma}$, Corollary 11.1 in [19] then applies to show that $\pi_{1, v, \gamma}$ is a subrepresentation of the left regular representation of $P$ if and only if $\gamma$ is a subrepresentation of the left regular representation of the stabilizer group $L_v$ and the orbit of $\chi_v$ inside $\hat{N}$ has positive Plancherel measure. Similar conclusions hold in the other cases.

8.1. The classical cases. We first consider the case where $F = \mathbb{R}$.

**Lemma 22.** Let $G$ be the group $\text{SO}_0(n, 1)$, and let $P = MAN$ be the minimal parabolic subgroup of $G$. Any irreducible unitary representation $\pi$ of $P$ is either trivial on the non-compact subgroup $N$ or is a subrepresentation of $\lambda_P$.

**Proof.** We divide the proof into the cases according to the description in Theorem 18.

In the case $\pi = \pi_{1, \sigma}$, it is clear that $\pi(g) = 1$ for every $g \in N$.

Consider now a representation $\pi = \pi_{2, v, \gamma}$ where $v$ is non-zero. Since $L_v$ is compact, $\gamma \in \hat{L}_v$ is a subrepresentation of the regular representation of $L_v$. If $n \neq 2$, then the action of $MA$ on the non-zero characters of $N$ is transitive. In particular, the orbit has positive Plancherel measure in $\hat{N}$. If $n = 2$, then the orbit of $\chi_v$ is either $\mathbb{R}_+$ or $\mathbb{H}$ inside $\hat{N} \simeq \mathbb{R}$, and both of these sets have positive measure. By Corollary 11.1 in [19] we conclude that $\pi$ is a subrepresentation of $\lambda_P$.

The case $\pi = \pi_{3, m, \gamma}$ does not occur, when $F = \mathbb{R}$. 

From Proposition 8 we can now conclude that the left regular representation of $P$ is completely reducible. From the proof of Lemma 22 we then obtain the following.

**Corollary 23.** Let $G$ be the group $\text{SO}_0(n, 1)$, and let $P = MAN$ be the minimal parabolic subgroup of $G$. The left regular representation of $P$ is completely reducible with the representations $\pi_{2, v, \gamma}$ as its subrepresentations. Here $v \in S_2$ and $\gamma \in \hat{L}_v$.

When $F$ equals $\mathbb{C}$ or $\mathbb{H}$ we have the following.
Lemma 24. Let $G$ be one of the groups $SU(n, 1)$, $Sp(n, 1)$, and let $P = MAN$ be the minimal parabolic subgroup of $G$. Any irreducible unitary representation $\pi$ of $P$ is either trivial on the non-compact subgroup $Z(N)$ or is a subrepresentation of $\lambda_P$.

Proof. We divide the proof into the cases according to the description in Theorem 18.

In the case $\pi = \pi_{1, \sigma}$, it is clear that $\pi(g) = 1$ for every $g \in N$, and hence in particular for every $g \in Z(N)$.

Suppose now $\pi = \pi_{2, v, \gamma}$. Since $\chi_v$ is trivial on $Z(N)$ and $\gamma$ is trivial on $N$, it follows that $\chi_v \otimes \gamma$ is trivial on $Z(N)$. Since $Z(N) \triangleleft P$, it now follows from Lemma 12 that $\pi$ is trivial on $Z(N)$, once we show that the homogeneous space $P/N L_v$ admits a $P$-invariant measure. Using Proposition 11 we find

$$P/N L_v \simeq M A/L_v \simeq M/L_v \times A.$$ 

The left translation action $A \curvearrowleft A$ has the Haar measure as an invariant measure. Since $M$ is compact, the action $M \curvearrowleft M/L_v$ has an invariant measure. It follows that $P \curvearrowleft N L_v$ has an invariant measure, and then by Lemma 12 the representation $\pi$ is trivial on $Z(N)$.

Consider now a representation $\pi = \pi_{3, m, \gamma}$. Since $L_m$ is compact, $\gamma \in \hat{L}_m$ is a subrepresentation of the regular representation of $L_m$. It remains to show that the orbit of $\eta_m$ in $\hat{N}$ has positive Plancherel measure.

If $F = \mathbb{H}$, then the third series of Theorem 18 forms a single orbit, which must then have positive Plancherel measure, because all other irreducible representations of $N$ are trivial on $Z(N)$ and hence must form a null set for the Plancherel measure.

If $F = \mathbb{C}$, then the action of $MA$ on the representations $\{\eta_m \in \hat{N} \mid m \in \text{Im } F^*\}$ has two orbits, so the simple argument for $\mathbb{H}$ does not apply. Luckily, the Plancherel measure of $N$ is well-known. In fact, $N$ is the Heisenberg group of dimension $2n - 1$, and the Plancherel measure for the Heisenberg group can be found on p. 241 in [13]. We see that the measure of the orbit of $\eta_i$ is

$$\mu_{N}(P, \eta_i) = \int_0^\infty |m|^{n-1} \, dm.$$ 

Hence the orbit of $\eta_i$ has positive, in fact infinite, measure. Similarly, the orbit of $\eta_{-i}$ has positive measure. By Corollary 11.1 in [19] we conclude that $\pi$ is a subrepresentation of $\lambda_P$.  

\[ \square \]

From Proposition 8 we can now conclude that the left regular representation of $P$ is completely reducible. From the proof of Lemma 24 we then obtain the following.

Corollary 25. Let $G$ be one of the classical groups $SU(n, 1)$, $Sp(n, 1)$, and let $P = MAN$ be the minimal parabolic subgroup of $G$. The left regular representation of $P$ is completely reducible with the representations $\pi_{3, m, \gamma}$ as its subrepresentations. Here $m \in S_3$ and $\gamma \in \hat{L}_m$. 

From Proposition 8 we can now conclude that the left regular representation of $P$ is completely reducible. From the proof of Lemma 24 we then obtain the following.
8.2. The exceptional case.

**Lemma 26.** Let $P = MAN$ be the minimal parabolic subgroup of $F_4(-20)$. Any irreducible unitary representation $\pi$ of $P$ is either trivial on the non-compact subgroup $Z(N) \simeq \text{ImO}$ or is a subrepresentation of $\lambda_P$.

**Proof.** Recall that any irreducible representation of $P$ is given as in Theorem 20. We will show that representations $\pi_{1,v,\gamma}$ are trivial on $Z(N)$ and that representations $\pi_{2,m,\gamma}$ are subrepresentations of $\lambda_P$.

Suppose first $\pi = \pi_{1,v,\gamma}$. The proof from Lemma 24 carries over verbatim and shows that $\pi$ is trivial on $Z(N)$.

Consider now a representation $\pi = \pi_{2,m,\gamma}$. Since $L_m$ is compact, $\gamma \in \hat{L}_m$ is a subrepresentation of the regular representation of $L_m$. It remains to show that the orbit of $\eta_m$ in $\hat{N}$ has positive Plancherel measure.

Clearly, the characters $\{\chi_v\}_{v \in \Omega}$ form a null set for the Plancherel measure on $\hat{N}$, because they are all trivial on the non-compact subgroup $Z(N)$. As mentioned in the proof of Lemma 21 the complement of $\{\chi_v\}_{v \in \Omega}$ forms a single orbit in $\hat{N}$ under the action of $MA$. Thus, the orbit of $\eta_m$ must have positive Plancherel measure.

By Corollary 11.1 in [19] we conclude that $\pi$ is a subrepresentation of $\lambda_P$. □

8.3. Conclusion. The following proposition sums up the necessary results from this and last section so that we may apply Theorem 4.

**Proposition 27.** Let $n \geq 2$, and let $G$ be one of the simple Lie groups $\text{SO}_0(n,1)$, $\text{SU}(n,1)$, $\text{Sp}(n,1)$ or $F_4(-20)$. Let $P = MAN$ be the minimal parabolic subgroup in $G$. The following holds.

1. $P$ is type I.
2. There is a non-compact, closed subgroup $H$ of $P$ such that every irreducible unitary representation of $P$ is either trivial on $H$ or is a subrepresentation of the regular representation $\lambda_P$.

In fact, if $G = \text{SO}_0(n,1)$, then one can take $H = N$, and otherwise one can take $H = Z(N)$.

From Proposition 27 and Theorem 4 we immediately obtain Theorem 3.

9. The relative Howe-Moore property

In this section we prove Corollary 5 concerning the relative Howe-Moore property. We recall from [6] that if $H$ is a closed subgroup of a locally compact group $G$, then the pair $(G,H)$ has the relative Howe-Moore property, if every representation $\pi$ of $G$ either has $H$-invariant vectors, or the restriction $\pi|_H$ is a $C_0$-representation, i.e. all coefficients of $\pi|_H$ vanish at infinity. Using a direct integral argument, it is proved in [4] Proposition 2.3] that it is sufficient to consider only irreducible representations of $G$. 
From the results in the previous sections we easily obtain the following, which obviously implies Corollary 5.

**Corollary 28.** If $P$ is the group in (1.2) and $N_1$ is the group in (5.4), then $N_1$ is a normal, non-compact closed subgroup of $P$, and $(P, N_1)$ has the relative Howe-Moore property.

Let $n \geq 2$. If $P = MAN$ the minimal parabolic subgroup in the simple Lie group $SO_0(n, 1)$, then $N$ is a normal, non-compact closed subgroup of $P$, and $(P, N)$ has the relative Howe-Moore property.

If $P = MAN$ the minimal parabolic subgroup in one of the simple Lie groups $SU(n, 1)$, $Sp(n, 1)$ or $F_4(-20)$, then $Z(N)$ is a normal, non-compact closed subgroup of $P$, and $(P, Z(N))$ has the relative Howe-Moore property.

**Proof.** Apply Proposition 17 or Proposition 27, respectively. Since any subrepresentation of the left regular representation $\lambda_P$ is a $C_0$-representation, we immediately obtain the result. $\square$

10. **Concluding remarks**

Theorem 3 shows that Question 1 has a positive answer for the minimal parabolic subgroups $P = MAN$ in the groups $SO_0(n, 1)$, $SU(n, 1)$, $Sp(n, 1)$ and $F_4(-20)$. One could ask if the same is true for the smaller groups $MN$, $AN$ or $N$. We will now discuss these cases. Recall from the introduction that a non-compact second countable connected unimodular groups never satisfy (1.1).

Let $G$ be one of the groups classical groups $SO_0(n, 1)$, $SU(n, 1)$, $Sp(n, 1)$ with $n \geq 2$ or the exceptional group $F_4(-20)$. Let $F$ be the corresponding division algebra, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$. We start by discussing the groups $N$. Since $N$ is nilpotent, $N$ is unimodular. Indeed, a locally compact group $G$ is unimodular if and only if $G/Z$ is unimodular, where $Z$ is the center of $G$ (see [26, p. 92]). Induction on the length of an upper central series then shows that all locally compact nilpotent groups are unimodular. Since $N$ is also connected and second countable, it follows that

$$A(N) \neq B(N) \cap C_0(N).$$

Next we discuss the groups $MN$. Since $MN$ is a semi-direct product of the unimodular group $N$ by the compact group $M$, we will argue that $MN$ itself is unimodular. Indeed, this follows directly from [26 Proposition 23] but we also include another argument here. If we use $\Delta_G$ to denote the modular function of a locally compact group $G$, then since $N$ is normal in $MN$, we see that the quotient space $MN/N$ has an invariant measure, Haar measure on $M$, and using [13, Theorem 2.49] we see that $\Delta_{MN}/N = \Delta_N = 1$. Also, since $M$ is compact, $\Delta_{MN}|M = 1$ by [13 Proposition 2.27]. Since $M$ and $N$ generate $MN$, it follows that $\Delta_{MN} = 1$. So $MN$ is connected and unimodular, and hence

$$A(MN) \neq B(MN) \cap C_0(MN).$$
Alternatively, one could show that all orbits in $\hat{N}$ under the action of $M$ have zero Plancherel measure. This type of argument will be used below for the groups $AN$.

For the groups $SO_0(n, 1)$, $Sp(n, 1)$ and $F_4(-20)$ it will usually also be the case that Question 1 has a negative answer for the groups $AN$. However, there is one exception. If $G = SO_0(2, 1)$, then $M$ is trivial and $P$ coincides with $AN$. Hence it follows from Theorem 3 that Question 1 has an affirmative answer for the group $AN$. In this special case let us remark that in fact $AN$ is isomorphic to the $ax + b$ group, and the result that $A(AN) = B(AN) \cap C_0(AN)$ is actually the original result of Khalil from [18].

The unimodularity argument used for the groups $N$ and $MN$ cannot be replicated for $AN$, since these groups are not unimodular (see [17, (1.14)]). As mentioned in the introduction, a group satisfying (1.1) has a completely reducible left regular representation, and in particular the left regular representation has irreducible subrepresentations. Then by [19, Corollary 11.1] at least one of the orbits of the action $A \rtimes \hat{N}$ must have positive Plancherel measure. To show that $A(AN) \neq B(AN) \cap C_0(AN)$ it therefore suffices to show that any orbit of $A \rtimes \hat{N}$ has zero Plancherel measure.

It this point we split the argument in cases. Consider first the case when $F = \mathbb{R}$ and $n \geq 3$. Then $N \simeq \mathbb{R}^{n-1}$, and the Plancherel measure on $\hat{N} \simeq \mathbb{R}^{n-1}$ is the Lebesgue measure. Since $A$ acts on $\hat{N}$ by dilation, every orbit except $\{0\}$ is a half-line. Since $n \geq 3$ every half-line in $\mathbb{R}^{n-1}$ has vanishing Lebesgue measure, and hence every orbit in $\hat{N}$ has vanishing Plancherel measure.

In the other cases $F$ is $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$. For convenience, when $F = \mathbb{O}$, we set $n = 2$. As mentioned earlier, the dual $\hat{N}$ then consists of the characters $\{\chi_v\}_{v \in \mathbb{F}^n}$ and the infinite dimensional representations $\hat{N}_r = \{\eta_m\}_{m \in \text{Im} \mathbb{F}^*}$.

Fortunately, the Plancherel measure for $N$ is known. It is described in [7, Section 3]. Since the characters are trivial on the center $\text{Im} \mathbb{F}$ of $N$ which is non-compact, the characters form a null set for the Plancherel measure. Let $k$ be the dimension of $\text{Im} \mathbb{F}$ as a real vector space so that $k$ is either 1, 3 or 7. If we identify $\hat{N}_r$ with $\text{Im} \mathbb{F}^*$ which in turn is identified with the punctured Euclidean space $\mathbb{R}^k \setminus \{0\}$, then it follows from [7, p. 524] that the Plancherel measure on $\hat{N}_r$ is absolutely continuous (has density) with respect to the Lebesgue measure.

Since $A$ acts on $\hat{N}_r$ by dilation, every orbit in $\hat{N}_r$ is a half-line. Every half-line has vanishing Lebesgue measure, unless $k = 1$, and hence every orbit in $\hat{N}_r$ has vanishing Plancherel measure, except when $F = \mathbb{C}$. Combined with the fact that the characters have vanishing Plancherel measure, we conclude that every orbit in $\hat{N}$ has vanishing Plancherel measure. As pointed out, the argument breaks down when $F = \mathbb{C}$.

We collect the discussion above in the following proposition.

**Proposition 29.** Let $G$ be one of the simple Lie groups $SO_0(n, 1)$ ($n \geq 3$), $Sp(n, 1)$ ($n \geq 2$) or $F_4(-20)$. Let $G = KAN$ be the Iwasawa decomposition of $G$. Then if $H$
is either $N$, $MN$ or $AN$, then

$$A(H) \neq B(H) \cap C_0(H).$$

Finally, we consider the group $AN$ in $G = SU(n,1)$.

**Proposition 30.** Let $G$ be the simple Lie group $SU(n,1)$ ($n \geq 2$) with Iwasawa decomposition $G = KAN$. Then

$$A(AN) = B(AN) \cap C_0(AN).$$

**Proof.** We will verify the conditions of Theorem 4 for the group $AN$.

First we verify that $AN$ is a group of type I. We mimic the proof of Lemma 19. Recall that $N$ is of type I, and hence $\hat{N}$ is a standard Borel space. Using the notation from Section 7, we identify $\hat{N}$ with the union of the characters $\{\chi_v\}_{v \in \mathbb{C}^{n-1}}$ and the infinite dimensional representations $\{\eta_m\}_{m \in i\mathbb{R}^*}$. The action $A \acts \hat{N}$ is described by (7.1) and (7.2), and it is easy to read off the orbits of the action.

The characters in $\hat{N}$, which we think of simply as $\mathbb{C}^{n-1}$, form an invariant subset whose orbits consist of the origin $\{0\}$ and half-lines originating at the origin. The infinite dimensional representations in $\hat{N}$, which we think of simply as $i\mathbb{R}^*$ also form an invariant subset which has two orbits, $i\mathbb{R}^+$ and $i\mathbb{R}^-$. If $S$ denotes the unit sphere in $\mathbb{C}^{n-1} \simeq \mathbb{R}^{2n-2}$, then $R = \{0\} \cup S \cup \{i, -i\}$ is a set of representatives for the orbits of $A \acts \hat{N}$. We claim that $R$ is a Borel subset of $\hat{N}$. To see this, it suffices to prove that $S$ is a Borel subset, since points are always Borel subsets in a standard Borel space.

The group $N$ is the Heisenberg group of dimension $2n - 1$, and the Fell topology on $\hat{N}$ is well-known (see e.g. [13, Chapter 7]). The characters $\{\chi_v\}_{v \in \mathbb{C}^{n-1}}$ form a closed subset in $\hat{N}$, and on the set of characters the Fell topology coincides with the Euclidean topology (on $\mathbb{C}^{n-1}$). In particular $S$ is closed in the Fell topology. By [13, Theorem 7.6], the Mackey Borel structure on $\hat{N}$ is induced by the Fell topology, since $N$ is of type I. It follows that $S$ is a Borel set.

We may now conclude from [21, Theorem 9.2] that the action $A \acts \hat{N}$ is **regular**, that is, $N$ is **regularly embedded** in $AN$.

Next we verify that if $\pi \in \hat{N}$, then the stabilizer $L_\pi = \{\alpha \in A \mid \alpha \pi \simeq \pi\}$ is of type I. Indeed, if $\pi$ is the trivial character on $N$, then $L_\pi = A$ which is abelian group. Hence the stabilizer $A$ is of type I. If $\pi$ is not the trivial character, then the stabilizer $L_\pi$ is trivial. So all stabilizers are of type I. According to [21, Theorem 9.3] we may now conclude that $AN$ is of type I.

The unitary dual of $AN$ is described in [32, Proposition 7.6]. The irreducible representations of $AN$ fall into three series as follows (retaining earlier notation).

1. The series $\pi_{1,\sigma} = \sigma \circ q$, where $q : AN \to A$ is the quotient map, and $\sigma \in \hat{A}$. 


(2) The series
\[ \pi_{2,v} = \text{Ind}^A_N (\chi_v), \]
where \( v \in \mathbb{C}^{n-1} \) is non-zero. The classes are parametrized by the orbits of \( A \ltimes \mathbb{C}^{n-1} \setminus \{0\} \).

(3) The series
\[ \pi_{3,m} = \text{Ind}^A_N (\eta_m) \]
where \( m \in i\mathbb{R}^* \). The classes are parametrized by \( m \in \{i, -i\} \).

We claim that \( \pi_{1,\sigma} \) and \( \pi_{2,v} \) are trivial on the center \( Z(N) \) of \( N \), and that \( \pi_{3,m} \) is a subrepresentation of the regular representation.

Clearly, \( \pi_{1,\sigma} \) annihilates \( N \) and in particular \( Z(N) \). Consider now a representation \( \pi_{2,v} = \text{Ind}^A_N (\chi_v) \), where \( v \in \mathbb{C}^{n-1} \) is non-zero. The character \( \chi_v \in \hat{N} \) is trivial on \( Z(N) \). Both \( A \) and \( N \) normalize \( Z(N) \), so \( Z(N) \) is normal in \( AN \). The representation \( \pi_{2,v} \) is induced from \( N \) to \( AN \), and when the quotient space \( AN/N \) is identified with \( A \) in the natural way, it is obvious that \( AN/N \) carries an invariant measure for the \( AN \)-action, namely the Haar measure on \( A \). From Lemma 12 it now follows that \( \pi_{2,v} \) is trivial on \( Z(N) \).

Finally, consider a representation \( \pi_{3,m} = \text{Ind}^A_N (\eta_m) \) where \( m \in i\mathbb{R}^* \). We will show that the orbit of \( \eta_m \) in \( \hat{N} \) has positive Plancherel measure, and then it follows from [19, Corollary 11.1] that \( \text{Ind}^A_N (\eta_m) \) is a subrepresentation of the left regular representation of \( AN \).

As mentioned before, the action of \( A \) on the representations \( \{\eta_m \in \hat{N} \mid m \in i\mathbb{R}^*\} \) has two orbits, \( i\mathbb{R}_+ \) and \( i\mathbb{R}_- \). The Plancherel measure of \( N \) is known and can be found on p. 241 in [13]. We see that the measure of the orbit of \( \eta_i \) is
\[ \mu_N(A.\eta_i) = \int_0^\infty |m|^{n-1} \, dm. \]
Hence the orbit of \( \eta_i \) has positive, in fact infinite, measure. Similarly, the orbit of \( \eta_{-i} \) has positive measure. By [19, Corollary 11.1] we conclude that \( \pi_{3,m} \) is a subrepresentation of the left regular representation of \( AN \).

The conditions of Theorem 4 have now been verified for the group \( AN \), and our proof is complete. \( \square \)

References

[1] John C. Baez. The octonions. Bull. Amer. Math. Soc. (N.S.), 39(2):145–205, 2002.
[2] John C. Baez. Errata for: “The octonions” [Bull. Amer. Math. Soc. (N.S.) 39 (2002), no. 2, 145–205; mr1886087]. Bull. Amer. Math. Soc. (N.S.), 42(2):213 (electronic), 2005.
[3] L. Baggett. Unimodularity and atomic Plancherel measure. Math. Ann., 266(4):513–518, 1984.
[4] Larry Baggett and Keith Taylor. Groups with completely reducible regular representation. Proc. Amer. Math. Soc., 72(3):593–600, 1978.
[5] Larry Baggett and Keith Taylor. A sufficient condition for the complete reducibility of the regular representation. J. Funct. Anal., 34(2):250–265, 1979.
[6] Raf Cluckers, Yves Cornulier, Nicolas Louvet, Romain Tessera, and Alain Valette. The Howe-Moore property for real and p-adic groups. Math. Scand., 109(2):201–224, 2011.
[7] Michael Cowling and Uffe Haagerup. Completely bounded multipliers of the Fourier algebra of a simple Lie group of real rank one. *Invent. Math.*, 96(3):507–549, 1989.

[8] J. Dixmier. Sur les représentations unitaires des groupes de Lie algébriques. *Ann. Inst. Fourier, Grenoble*, 7:315–328, 1957.

[9] Jacques Dixmier. Sur les représentations unitaires des groupes de Lie nilpotents. V. *Bull. Soc. Math. France*, 87:65–79, 1959.

[10] Jacques Dixmier. *C*-algebras. North-Holland Publishing Co., Amsterdam, 1977. Translated from the French by Francis Jellett, North-Holland Mathematical Library, Vol. 15.

[11] Pierre Eymard. L’algèbre de Fourier d’un groupe localement compact. *Bull. Soc. Math. France*, 92:181–236, 1964.

[12] Alessandro Figà-Talamanca. Positive definite functions which vanish at infinity. *Pacific J. Math.*, 69(2):355–363, 1977.

[13] Gerald B. Folland. *A course in abstract harmonic analysis*. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1995.

[14] Uffe Haagerup and Søren Knudby. The weak Haagerup property: Examples. In preparation, 2013.

[15] Edwin Hewitt and Herbert S. Zuckerman. Singular measures with absolutely continuous convolution squares. *Proc. Cambridge Philos. Soc.*, 62:399–420, 1966.

[16] Richard V. Kadison and John R. Ringrose. *Fundamentals of the theory of operator algebras. Vol. II*, volume 16 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1997. Advanced theory, Corrected reprint of the 1986 original.

[17] Frederick W. Keene, Ronald L. Lipsman, and Joseph A. Wolf. The Plancherel formula for parabolic subgroups. *Israel J. Math.*, 28(1-2):68–90, 1977.

[18] Idiss Khalil. Sur l’analyse harmonique du groupe affine de la droite. *Studia Math.*, 51:139–167, 1974.

[19] Adam Kleppner and Ronald L. Lipsman. The Plancherel formula for group extensions. I, II. *Ann. Sci. École Norm. Sup.* (4), 5:459–516; ibid. (4) 6 (1973), 103–132, 1972.

[20] Anthony W. Knapp. *Lie groups beyond an introduction*, volume 140 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, second edition, 2002.

[21] George W. Mackey. Unitary representations of group extensions. I. *Acta Math.*, 99:265–311, 1958.

[22] Peter F. Mah and Tianxuan Miao. Extreme points of the unit ball of the Fourier-Stieltjes algebra. *Proc. Amer. Math. Soc.*, 128(4):1097–1103, 2000.

[23] Robert Paul Martin. On the decomposition of tensor products of principal series representations for real-rank one semisimple groups. *Trans. Amer. Math. Soc.*, 201:177–211, 1975.

[24] Giancarlo Mauceri. Square integrable representations and the Fourier algebra of a unimodular group. *Pacific J. Math.*, 73(1):143–154, 1977.

[25] D. Menchoff. Sur L’unicité du Développement Trigonométrique. *C. R. Acad. Sci. Paris*, 163:433–436, 1916.

[26] Leopoldo Nachbin. *The Haar integral*. D. Van Nostrand Co., Inc., Princeton, N.J.-Toronto-London, 1965.

[27] Jean-Paul Pier. *Amenable locally compact groups*. Pure and Applied Mathematics (New York). John Wiley & Sons Inc., New York, 1984. A Wiley-Interscience Publication.

[28] Reiji Takahashi. Quelques résultats sur l’analyse harmonique dans l’espace symétrique non compact de rang 1 du type exceptionnel. In *Analyse harmonique sur les groupes de Lie (Sém., Nancy-Strasbourg 1976–1978), II*, volume 739 of *Lecture Notes in Math.*, pages 511–567. Springer, Berlin, 1979.

[29] Keith F. Taylor. Geometry of the Fourier algebras and locally compact groups with atomic unitary representations. *Math. Ann.*, 262(2):183–190, 1983.

[30] Keith F. Taylor. Groups with atomic regular representation. In *Representations, wavelets, and frames*, Appl. Numer. Harmon. Anal., pages 33–45. Birkhäuser Boston, Boston, MA, 2008.

[31] Martin E. Walter. On a theorem of Figà-Talamanca. *Proc. Amer. Math. Soc.*, 60:72–74 (1977), 1976.

[32] Joseph A. Wolf. Representations of certain semidirect product groups. *J. Functional Analysis*, 19(4):339–372, 1975.
FOURIER ALGEBRAS OF PARABOLIC SUBGROUPS

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