A NEW TAKE ON SPHERICAL, WHITTAKER AND
BESSEL FUNCTIONS

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0. Introduction

This paper grew out of the lectures given by the first author at Harvard in February and March, 2009. A draft of the lecture notes was prepared by the second author, and then expanded and brought to the final form by the first author.

0.1. Objectives and main results. The general aims of this paper are as follows.

1) The theory of DAHA at arbitrary levels \( l \), which, technically, means that any \( l \)-powers of the Gaussian are to be considered (not only \( l = 0, 1 \) as in [Ch1]).

2) The affine Satake isomorphism and affine Hall functions via DAHA; the latter attract growing attention of the specialists, though not much is known so far on these functions.

3) Establishing connections with the theory of Kac-Moody characters, treated as the \( t \to \infty \) limit of the affine Hall functions.

4) The theory of coinvariants of DAHA, their relations to the symmetric forms on DAHA of higher levels and to the Looijenga functions.

5) Revisiting the classical \( p \)-adic theory of the Satake-Macdonald, Matsumoto and Whittaker functions via DAHA.

6) The study of the new spinor Dunkl operators for the \( q \)-Toda operators and \( q \)-Whittaker functions including the related theory of the nil-DAHA.

7) Developing the technique of \( W \)-spinors in the differential setting; applications to the Bessel functions and the AKZ\( \leftrightarrow \)QMBP isomorphism theorem.

8) Last but not the least, an outline of the DAHA approach to the quantum Langlands program via the Verlinde algebras and \( q \)-Whittaker functions.

0.1.1. Affine Satake isomorphisms. Among the main topics we consider, is the \textit{DAHA-Satake map}, the infinite symmetrizer on its affine Hecke subalgebra, and its relation to the \textit{affine Satake map} (and related constructions) defined by the formulas used in [Ka, FGT, BK]. The latter map is directly connected with the theory of Jackson integration developed in [Ch3, Ch4, Sto], which provide exact formulas at levels 0, 1; see also [FGT], Section 12.7 “Lattice-hypergeometric sums”. Interestingly, the DAHA-Satake map and the affine Satake map have different convergence ranges for higher levels. The latter one is well defined for any nonzero \( t \), the former only as \( \Re k < -1/h \) for \( t = q^k \) and the Coxeter number \( h \). When both converge, they are proportional to each other.
The affine Satake summation becomes essentially the \textit{Weyl-Kac character formula} in the limit \( t \to \infty \); the DAHA-Satake map is related to the \textit{Demazure characters} (in the form with all \( Y \) on the left from Theorem 2.8). The \( t \)-counterparts of the Kac-Moody string functions (and related matters) are not discussed in this paper; see [FGT, Vi]. What seems promising to me is the study of the monodromy of the affine Hall functions (generalizing the classical theorem due to Kac and Peterson); I hope to consider this problem in other works.

Concerning the algebraic theory of DAHA, the Satake map and affine Hall functions are closely related to the \textit{DAHA coinvariants}, which, in their turn, are directly connected with the symmetric bilinear forms on DAHA of levels \( l \geq 0 \). The bilinear forms of level 0 and 1 are exactly the key inner products from [Ch1] and other author’s works. More generally, the space of DAHA coinvariants is isomorphic to the Looijenga space for level \( l > 0 \). These and other links discussed in this paper obviously indicate that theory of higher level affine Hall functions is very fruitful.

0.1.2. \textbf{Whittaker functions}. The second important group of results is the theory of the Dunkl operators for the \( q \)-Toda operators and \( q \)-Whittaker functions. It requires the \textit{technique of spinors}. The construction of the \textit{Dunkl-spinor operators} can be presented as an isomorphism between the standard polynomial representation of the \textit{nil-DAHA} and the spinor-polynomial representation of its dual. The reproducing kernel of this isomorphism is the \textit{spinor nonsymmetric Whittaker function}, which was mentioned in [Ch8] as a possible major continuations of the theory of \( q \)-Whittaker functions. We note that the difference (relativistic) Toda chain in the case of \( A_n \) (in the classical and quantum variants) is essentially due to Ruijsenaars; see [Rui] for a review.

In this paper, the formula for the \textit{nonsymmetric} Whittaker function is discussed in the \textit{A}_1-case only; its extension to arbitrary (reduced) root systems is relatively straightforward. See [Ch8] for the theory of \textit{global symmetric} \( q \)-Whittaker functions, which is expected to be related to the theory of affine flag varieties and the Givental-Lee theory. They may have other applications too; see [GLO]. Technically, the introduction of \textit{nonsymmetric} Whittaker functions is an important step for using the DAHA methods at full potential.

It is important that the same limit \( t \to \infty \) serves the \( q \)-Whittaker functions and the passage to the Kac-Moody theory. However, this limit must be calibrated in a very special way in the Whittaker case following the construction from [Et] (extended recently in [Ch8]). As a
matter of fact, obtaining the Kac-Moody characters is not immediate from DAHA too; the affine Satake isomorphism is needed here. The \( q \)-Hermite polynomials emerge in the limit \( t \to \infty \) for both, the \( q \)-Whittaker and Kac-Moody theories. They play an important role in our analysis. The resulting connection between the Kac-Moody theory and the \( q \)-Whittaker theory is expected to be related to the geometric quantum Langlands program.

0.1.3. The setting of the paper. I mainly use the standard affine root systems in contrast to the twisted affine root systems considered in [Ch1] and almost all my other papers on DAHA. The standard (untwisted) “affinization” is (presumably) exactly the one compatible with the quantum Langlands duality. For instance, the untwisted affine exponents from [Ch6], describing the reducibility of the polynomial representation, obey the quantum Langlands-type duality for the modular transformation \( q \mapsto \hat{q} \). However, this kind of duality does not hold in the twisted case (at least, I do not know how to formulate it). On the other hand, the twisted affinization has obvious advantages (versus the standard setting) for the theory of Gaussians. It is parallel to the advantages of the twisted case for the level 1 character formulas in the Kac-Moody theory.

Due to the standard (untwisted) setting, I need to state some of the results of this paper, especially where the Gaussians are involved, only for the simply-laced root systems. It includes the level one formulas for the affine Hall polynomials. I am going to consider the corresponding twisted case in other publications; then the root system can be really arbitrary (reduced) and this restriction can be removed. Using \( t \) in this paper is relaxed as well; we simply treat it as a single parameter. Generally, \( t \) (or \( k \)) are supposed to depend on the length of the corresponding root.

I present several constructions mainly in the \( A_1 \)-case, where practically everything can be calculated explicitly. However there are almost no \( A_1 \)-specific results and methods in this paper. I use the \( A_1 \)-setting to simplify definitions and justifications, but the major results of this paper can be transferred to arbitrary (at least, reduced) root systems. Concerning \( A_1 \), only the “differential” part of the paper is somewhat exceptional. Not all results obtained in the sections devoted to the “tilde”-Bessel functions, nonsymmetric solutions of the equation for the symmetric Bessel functions, are known at the moment for general root systems.

0.2. Dunkl operators via DAHA. Trying to put this paper into perspective, let me outline the (current) status of the DAHA theory
from the viewpoint of the constructions of the Dunkl operators we have at our disposal now. The families of the Dunkl operators are, essentially, in one-to-one correspondence with the constructions of “polynomial representations”. The latter are induced ones from the affine Hecke subalgebra of DAHA, their variants and degenerations. Not all of them are exactly induced and not all are really polynomial; Fock representations may be a better name.

Such approach to reviewing the theory of DAHA is of course simplified, but maybe not too much. For instance, if the polynomial representation is known and studied well, then we know a lot about the corresponding DAHA. It gives the PBW theorem, the zeros of the corresponding Bernstein-Sato polynomial, the definition of the localization functor, the construction of the corresponding spherical function and more.

0.2.1. Families of Dunkl operators. I will stick to the crystallographic case; there are important developments for other groups generated by complex and symplectic reflections. With this reservation, the list of major families of Dunkl operators and corresponding polynomial representations seems as follows.

a) The rational-differential operators due to Charles Dunkl; the rational DAHA is self-dual and its theory (including the polynomial representation) is the most developed now.

b) Differential-trigonometric and difference-rational polynomial representations of the degenerate DAHA; they are connected by the generalized Harish-Chandra transform.

c) The Macdonald theory and the $q,t$–DAHA, corresponding to the difference-trigonometric polynomial representation and the corresponding Dunkl operators; it is self-dual as in the rational case.

d) Differential-elliptic representation of the degenerate DAHA and the difference-elliptic representation of the $q,t$–DAHA [Ch9, Ch10]; their dual counterparts are not studied so far.

e) The specializations of the representations from (b) in the theory of Yang-type systems of spin-particles. The references are [Ug] and recent [EOS]; the same degenerate DAHA is used in this theory.

From this (limited) viewpoint, the theory of the nonsymmetric $q$–Whittaker functions adds new spinor family of Dunkl operators for the nil-DAHA and in some other cases. Developing the $q$–Whittaker theory is important because of its various known and expected applications. The list of such applications seems significantly greater than that for the difference spherical functions; the coefficients of the $q$–Whittaker functions are $q$–integers!
The technique of \( W \)-spinors is an important tool in the DAHA theory. The spinors address the problem that the Dunkl operators are not local; they become local in the space of spinors. This technique incorporates in the DAHA theory all solution (not only \( W \)-invariant) of the QMBP (the Heckman-Opdam eigenvalue problem). The spinor representations (generally) do not coincide with the DAHA-modules induced from representations of AHA or from its maximal commutative subalgebras. However, there are connections, especially, at the analytic level. Analytically, the spinors are designed to include all, not necessarily symmetric, solutions of the differential and difference equations for spherical, Whittaker and Bessel functions and their \( q \)-generalizations.

As for the affine Hall functions, another major direction of this paper, they will certainly refresh our interest in (d), the theory of differential- elliptic and difference- elliptic polynomial-type representations. These families were introduced in [Ch9] and [Ch10], but there is no reasonably complete theory of these representations so far. We mention that there are other “elliptic” theories (which we will not review here).

0.3. The technique of spinors. As far as I know, this technique was used explicitly for the first time in [Ch11], when proving the so-called Matsuo- Cherednik isomorphism theorem. This theorem establishes an equivalence of the affine Knizhnik-Zamolodchikov equation, AKZ, in the modules of the degenerate Hecke algebra induced from (dominant) characters and the corresponding Heckman-Opdam system (QMBP). See Chapter 1 of [Ch1] and Section 4.6 below.

0.3.1. Connections to AKZ. The Matsuo proof from paper [Mats] was a direct algebraic one. I used the Grothendieck-type notion of the monodromy without a fixed point, which made the proof short and entirely conceptual. Let me note that [Ch11] was written for the vector-valued solutions and included the rational QMBP as well. Then I extended this equivalence to the difference and elliptic cases. In the difference theory, it gives an embedding rather than an isomorphism of the spaces of solutions. In the elliptic case, the critical level condition must be imposed: \( l = -kh \) for the Coxeter number \( h \), \( t = q^k \).

Using the technique of spinor systematically (see Section 4.6) makes my proof from [Ch11] entirely algebraic and establishes its direct connection with the proof suggested (several years later) in [O4]; compare Lemma 3.2 there with Theorem 4.17 below. Generally speaking, there is nothing new about the definition of \( W \)-spinors. They are simply sets of functions \( \{f_w\} \) numbered by the elements from the Weyl group \( W \) with the action of \( W \) on the indices. The principle spinors are in
the form \( \{ w^{-1}(f), w \in W \} \) for a global function \( f \); generally, \( f_w \) are absolutely independent functions. For instance, the real spinors are functions on the disjoint union of all Weyl chambers, collected (using \( W \)) in the fundamental Weyl chamber. The Dunkl operators can be naturally extended to spinor space; we note that they are different from those of induced type defined in [C12] (and for various degenerations).

Let me also mention that the spinor representation may be related to the (differential) theory of the nonsymmetric Fourier transform from [O4]. The technique of spinors, generally, results in solutions the symmetric differential or difference equations, including non-\( W \)-invariant ones. However, the technique of spinors (combined with the explicit calculation of the AKZ-monodromy) was used in [Ch11] to obtain the nonsymmetric spherical function, called the \( G \)-function in [O4].

0.3.2. Isomorphism theorems. The isomorphism theorem from [Ch11] (see also [Ch1], Chapter 1) is as follows.

**Theorem 0.1 (AKZ \( \rightarrow \) Dunkl \( \rightarrow \) QMBP).** Given an arbitrary weight \( \lambda \), the space of AKZ-solutions in the induced module \( I_\lambda \) of the (degenerate) affine Hecke algebra can be identified with the \( \lambda \)-eigenspace of Dunkl operators in the corresponding DAHA spinor representation. Then the latter eigenspace can be mapped to the space of all, not necessarily symmetric, solutions of the corresponding QMBP (the Heckman-Opdam system). For generic \( \lambda \), this map is an isomorphism (an embedding in the difference setting).

The spinors needed here are complex, defined in the domain \( U = \{ z \} \) such that \( \Im(z) \) belongs to the corresponding fundamental Weyl chamber. They can be interpreted as functions in the disjoint union \( \bigcup_{w \in W} w(U) \); then the principle spinors are global analytic functions. Only functions in \( U \) emerge in the spinor theory of the Dunkl-type eigenvalue problem, including the integration theory and related inner products.

0.3.3. The localization functor. This construction is connected with the localization functor, one of the most powerful tools in the theory of DAHA. See [GGOR] and [VV]. A link to the AKZ is essentially as follows.

The localization construction assigns a local system to a module of DAHA (from a proper category); the case of induced representations is related to AKZ. In [Ch11] and further papers, the starting point was the AKZ with the values in an arbitrary finite dimensional module \( V \) of AHA (or degenerate AHA). Then it can be interpreted as a DAHA-module in the space of \( V \)-valued analytic functions via the
spinor Dunkl operators. The action of DAHA is a combination of the action of the resulting spinor Dunkl-type operators and multiplications by functions.

The action of the spinor Dunkl operators can be associated with the monodromy of AKZ. The monodromy cocycle on $W$ I used can be expressed in terms of the (usual) monodromy homomorphism of the braid group; see [Ch1], Chapter 1. It gives a link to the localization functor.

However, the construction $\text{AKZ} \to \text{Dunkl} \to \text{QMBP}$ was aimed at applications to the corresponding eigenvalue problems and was done only within the class of induced modules. In the theory of the localization functor, the projective modules are of key importance.

0.3.4. The Whittaker limit. Solving QMBP in the class of all functions, not only $W$–invariant, has interesting algebraic and analytic aspects. We will not try to review them here. From the DAHA viewpoint, the definition of the spinor representation requires using meromorphic functions; Laurent polynomials are, generally, not enough. However, sometimes analytic functions are sufficient. For instance, it occurs under the nonsymmetric Whittaker limit, when the DAHA is degenerated to the corresponding nil-DAHA.

The nonsymmetric Whittaker limit is a spinor variant of the limiting procedure from the difference QMBP to the difference Toda chain mainly due to Ruijsenaars; see [Rui, Et]. It must be significantly modified in the nonsymmetric case using the spinor setting and eventually leads to the spinor polynomial representation, an irreducible modules of $\text{nil-DAHA}$ of a new kind. To be exact, the Whittaker limit naturally results in this representation multiplied by the Gaussian. Its Fourier-dual equals the Gaussian times the standard polynomial representation of the nil-DAHA. The latter module is induced from the trivial character of the nil-AHA. The map intertwining these two representations is given in terms of the nonsymmetric spinor global $q$–Whittaker function. The construction is a general one, but we stick to the $A_1$–case in this work.

I think that these and other examples provide a solid motivation for the spinors. Hopefully, this technique can be useful for the following.

0.4. On Langlands’ program. Although not much is known so far concerning the relations of DAHA to the geometric quantum Langlands program, I think, it makes sense to touch this topic upon in this paper. I will not try to review here the applications (known and expected) of the symmetric global $q$–Whittaker functions, including the Shintani
-Casselman-Shalika formula, the relations to the Givental-Lee theory and possible applications in physics. See [Ch8] and [GLO] for a discussion. Generally, the (coefficients of) $q$–Whittaker functions are expected to contain a lot of information about the quantum $K$–theory and $IC$–theory of affine flag varieties. The Givental-Lee theory deals with the quantum $K$–theory of the flag variety.

I am very thankful to David Kazhdan, Dennis Gaitsgory, Roman Bezrukavnikov and Alexander Braverman, who introduced me to the quantum Langlands program and neighboring topics. The following is (partially) based on our conversations.

0.4.1. **Key connection.** The relations of LP to DAHA I mean are expected upon taking $K_0$; then the monoidal, rigid, modular categories become respectively commutative rings with inner products and with a projective action of $PGL_2(\mathbb{Z})$. Generally, the number of simple objects must be finite for the latter action. As it was pointed out in Section “Abstract Verlinde Algebras” from [Ch1], such rings (even if some of these structures are missing) form a very rigid class. For instance, one can formally prove counterparts of the Macdonald conjectures in the abstract Verlinde-type setting; cf. [Ch6]. It is unlikely that there are many commutative rings with such rich structures. The major candidates are quotients of $(W$–invariants of) the polynomial and various similar representations of DAHA, including infinite dimensional ones. The expected connections to the Langlands program are grouped around the following conjecture.

**Key Conjecture 0.2.** The commutative algebra $K_0(\text{Rep}_q G)$ for the category $\text{Rep}_q G$ of finite dimensional representations of Lusztig’s quantum group can be canonically identified with the algebra of $W$–invariants of the polynomial representation of DAHA as $t = q$ defined for the corresponding root system. For generic $q$, the simple objects correspond to the “characters” (eigenfunctions of the $Y$–operators) and the fusion procedure becomes the multiplication.

When $q = e^{2\pi i/N}$, the Verlinde algebra, becomes canonically isomorphic to the $W$–invariant part of the corresponding perfect representation from [Ch1] in the special case $k = 1$ ($t = q$). The projective DAHA-action of the $PGL(2, \mathbb{Z})$ there leads to the Verlinde $T, S$–operators.

The Verlinde algebra was originally defined in terms of the integrable Kac-Moody modules with the fusion. It is isomorphic to the quotient of $K_0(\text{Rep}_q G)$ by the modules of zero $q$–dimension, the so-called reduced category, due to Kazhdan -Lusztig [KL] and Finkelberg (at roots of unity).
Generally, a lack of information about $K_0(\text{Rep}_q G)$ makes Conjecture 0.2 very preliminary at the moment; the DAHA–side of it is better understood.

Extrapolating the relation between the perfect modules for $t = q$ of DAHA and the Verlinde algebras, I expect that the so-called parallelogram quotient of $K_0(\text{Rep}_q G)$ corresponds to the algebra of $W$–invariants of the DAHA parallelogram module. The latter is

$$V^{-2} = \mathbb{C}[X, X^{-1}]/(X^{2N} + X^{-2N} - 2) = \mathbb{C}[X, X^{-1}]/(X^N - X^{-N})^2$$

for $A_1$ (it is of dimension $4N$) in the notation from [Ch1] Section 2.9.3.

0.4.2. The rank one case. Continuing, the perfect quotient of $V^{-2}$ for integral $0 \leq k < N/2$ is $V_{2N-4k}$ (of dimension $2N - 4k$), where $q = \exp(2\pi i/N)$. The kernel, $V_{2N+4k}$, of the natural map $V^{-2} \to V_{2N-4k}$ is an also and irreducible DAHA-module with the projective $PGL(2, \mathbb{Z})$–action. However, the action of $X$ is non-semisimple in $V_{2N+4k}$ for $k > 0$, namely, there will be $4k$ Jordan 2–dimensional blocks (with pairwise different eigenvalues) and $2N - 4k$ simple eigenvectors. Due to the projective $PGL(2, \mathbb{Z})$–action, this answer remains the same for $Y$.

However, the Jordan decompositions of $X$ and $Y$ in the whole $V^{-2}$ are different. Namely, $Y$ has $4k$ Jordan 2–blocks and the rest of it is semisimple (although of multiplicity 2). The decomposition of $X$ in $V^{-2}$ consists of the 2–blocks only.

Upon taking the $W$–invariants, $\dim_{\mathbb{C}} (V^{-2})^W = 2N$, $\dim_{\mathbb{C}} V_{2N-4k}^W = N - 2k + 1$ and $\dim_{\mathbb{C}} V_{2N+4k}^W = N + 2k - 1$. The latter two modules are projective $PGL(2, \mathbb{Z})$–invariant.

When $k = 1$,

$$V_{2N-4} = \mathbb{C}[X, X^{-1}]/(F) \quad \text{for} \quad F = \frac{X^{2N} - 1}{(X^2 - 1)(X^2 - q)}.$$ 

for instance, $F = X^2 - q^2$ for (the minimal possible) $N = 3$. We note that the symmetric kernel-module $V_{2N+4}^W$ becomes semisimple under the action of $X + X^{-1}$ or $Y + Y^{-1}$ in this case. It is due to the fact that the $(Y + Y^{-1})$–eigenvectors in $\mathbb{C}[X + X^{-1}]$ do not depend on $q$ and are proportional to the $SL(2)$–Schur functions, which is a special feature of the case $k = 1$.

It is not impossible that the irreducible constituents of the parallelogram DAHA modules for (sufficiently small) integral $k$ always are projective $PGL(2, \mathbb{Z})$–modules, but this is known only for $A_1$. The parallelogram module, as the whole, has no natural (projective) $PGL(2, \mathbb{Z})$–structure. Not much is known about the action of $PGL(2, \mathbb{Z})$ on the sub-quotients of the category $\text{Rep}_q G$ at roots of unity.
0.4.3. **Discussion.** I would like to mention that Tipunin and others successfully calculated certain generalized Verlinde algebras of non-semisimple type using the *logarithmic conformal theory*; see, e.g., [MT]. They obtained exactly the ones described in [Ch1]. Let me also note that the limit of the minimal models as $c \to 1$ is important in physics applications; the corresponding infinite dimensional Verlinde-type algebra may be the polynomial DAHA representation itself.

For verifying the conjecture, the key fact seems that DAHA are closely related (almost at level of definitions) to the $K$–theory of affine flag varieties. Also, the affine Knizhnik-Zamolodchikov equations and the so-called $r$–matrix KZ (see [Ch1]) can be employed. These KZ are directly connected with the coinvariants and the $\tau$–function for the factorizable Kac-Moody algebras associated with $r$–matrices (introduced in author’s works).

Let me note that the true power of DAHA is the nonsymmetric theory, although we do not know much about the geometric meaning of this theory. There are two major facts. The nonsymmetric Macdonald polynomials are connected with the Matsumoto spherical functions and also with the Demazure characters, but these examples are degenerate. Generally, taking the $W$–invariants in DAHA-modules is necessary to relate them to the Lie-Kac-Moody groups and algebras. The technique of spinors, which establishes a connection of DAHA to non-$W$–invariant sections of local systems like QMBP (the Heckman-Opdam system), could be a bridge from the non-symmetric theory to geometry.

It is worth mentioning that the specialization $t = q$ used in Conjecture 0.2 does not seem the only one related to $\text{Rep}_q G$. Interesting finite-dimensional DAHA-modules can be obtained when

a) $t = q^{s/d}$ for singular rational numbers $s/d < 0$ (and any unimodular $q$!),

b) $t$ is an arbitrary complex number but $q$ is a root of unity (a variant of the case of the parallelogram),

c) and when $q$ is a root of unity and $t \to 0$ or $t \to \infty$,

although not all structures expected in the conjecture are present in these cases. The limits from (c) are very interesting because of possible (no direct confirmations so far) relations to the following.

0.4.4. **The category Whit.** Assuming that the DAHA polynomial representation can be used as a substitute for $K_0(\text{Rep}_q G)$ (at least apart from the roots of unity), the limits $t \to 0$ and $t \to \infty$ could be a step towards the DAHA-interpretation of $K_0$ of

$$\text{Whit}^c(Gr_G) = D \text{ mod }^c (G((z))/G_0\text{)}^{N((z))}.$$
It is defined as the category of $N((z))$–equivariant $c$–twisted $D$–modules on the affine flag variety $Gr_G$ defined as the group $G((z))$ of (formal) meromorphic loops divided by the group $G_0 = G[[z]]$ of all holomorphic ones. Here $N \subset G$ is the standard unipotent subgroup and we need to choose an unramified character on $N((z))$ to define the equivariant modules. This category was proven by Gaitsgory in [Ga] (under some technical restrictions) to be equivalent to $\text{Rep}_q LG$ for $q = \exp(\pi c)$, where $LG$ is the Langlands dual (it was conjectured by Lurie).

The relation of $\text{Whit}^c(Gr_G)$ to the $W$–algebras (and their Verlinde algebras) is an indication that DAHA modules can be used for describing $K_0(\text{Whit})$. The spinor representation of the nil-DAHA from Theorems 3.9 and 3.10, called the hat-representation, may be a candidate (upon the symmetrization). It is dual to the bar-representation, the limit of the standard polynomial representation of DAHA as $t \to 0$. We cannot provide any evidence at the moment, but the $q$–Toda eigenvalue problem could be a reasonable candidate for the Hitchin system in the $q$–setting of the Gaitsgory theorem.

Another source of inspiration could be Theorem 3 from [BF]. In its $K$–theoretic variant (still a conjecture), it looks similar to the Fourier duality we establish between too different definitions of the nil-DAHA from Theorem 3.9. The spinor $q$–Whittaker function from (3.49), which connects the hat-representation and the bar-representation, may be connected to [BF] too. We expect relations of the translation functor to the intertwiners acting on the spinor $q$–Whittaker function. The mod $p$ methods (already used for DAHA) could be also a powerful tool, including the modular translation functor due to Bezrukavnikov.

We note that DAHA can be applied to interpret certain structures beyond $K_0$. Generally, changing the asymptotic sectors of the Knizhnik-Zamolodchikov equations gives the associators due to Drinfeld. In certain special cases, the associators are expected to be related to different choices of maximal commutative subalgebras, in AHA and DAHA. If it can be done, then the resulting $q, t$–pentagon-type relation may be connected with [FG] in the limit $t \to 1$. It is worth mentioning that the well-known pentagon relation for the quantum dilogarithm (the $q$–Gamma function) does not play any significant role in the DAHA theory so far.

0.5. Acknowledgements.

0.5.1. My Harvard lectures. The paper is based on the series of lectures delivered by the first author at Harvard (Spring 2009); he is responsible for the scientific contents of this paper.
It was a somewhat unusual series, directly based on the research I performed at Harvard during my visit, a sort of reporting the latest research activities on weekly basis. The output of this venture appeared better than the lecturer expected (hopefully, the listeners too). I am very thankful for the invitation and this unique opportunity; the listeners greatly stimulated me to do my best. I am grateful to Xiaoguang Ma for preparing the initial TeX files of the lectures and for his various suggestions concerning improving the quality of the text.

Let me mention that extensive using examples and brief expositions of the classical topics are an organic part of the design of this work. However, the focus is on general approaches and new results.

0.5.2. Special thanks. My special thanks go to Dennis Gaitsgory and Pavel Etingof for participating in these lectures, various discussions and help with the direction and contents of the lectures.

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Quite a few topics were stimulated by my talks to physicists; special thanks to Anton Gerasimov who introduced me to the brave new world of $q$–Whittaker functions.

I am thankful for the invitations and grateful to many people, mathematicians and physicists, I talked to on these and related matters at Harvard, MIT, RIMS and many other places.

—Ivan Cherednik

1. P-adic theory revisited

There is a lot of research in the area of affine Hecke algebras, AHA. The classical $p$–adic spherical functions were subject to various generalizations. For instance, Ian Macdonald obtained them as a limit of the symmetric Macdonald polynomials in his very first works on these
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polynomials. See Section 2.11 from Chapter 2 in [Ch1] (and references therein) and [O3] concerning the relations of the nonsymmetric Macdonald polynomials to the Matsumoto spherical functions. The DAHA methods help in clarifying algebraic aspects of their theory. See also [Ion2, O1].

The purpose of this section, is to continue revisiting the \( p \)-adic theory from the viewpoint of DAHA aiming at establishing connections with the so-called “double arithmetic”, for instance, with the affine Hall functions.

1.1. Affine Weyl group.

1.1.1. Root systems. Concerning the classical theory of root systems and Weyl groups, the standard references are [B, Hu]; if the latter sources are not sufficient, then see [Ch1].

In this paper, \( R = \{ \alpha \} \subset \mathbb{R}^n \) is a simple reduced root system with respect to a nondegenerate symmetric bilinear form \((,\) on \( \mathbb{R}^n \). Let \( \{ \alpha_i \}_{i=1}^n \subset R \) be the set of simple roots and \( R_+ \) (or \( R_- \)) be the set of positive (or negative) roots. The coroots are denoted by \( \alpha^\vee = 2\alpha/(\alpha, \alpha) \); \( W \) is the Weyl group generated by \( s_\alpha \).

Let \( Q^\vee = \bigoplus_{i=1}^n \mathbb{Z}\alpha_i^\vee \) be the coroot lattice and \( P^\vee = \bigoplus_{i=1}^n \mathbb{Z}\omega_i^\vee \) the coweight lattice, where \( \omega_i^\vee \)'s are the fundamental coweights, i.e., \( (\omega_i^\vee, \alpha_j) = \delta_{ij} \). Replacing \( \mathbb{Z} \) by \( \mathbb{Z}_+ = \mathbb{Z}_{\geq 0} \), we obtain \( Q^\vee_+ \) and \( P^\vee_+ \).

Finally, \( \theta \) is the maximal positive root, the bilinear form is normalized by the condition \( (\theta, \theta) = 2 \) and \( \rho \overset{\text{def}}{=} \frac{1}{2} \sum_{\alpha \in R_+} \alpha \). Due to this normalization,

\[
Q^\vee \subset P^\vee \subset P \cup Q
\]

Concerning excluding the case \( BC \) in this paper, there is an important \( C^\vee C \)-direction; this root system is in many ways more symmetric than \( BC \). Almost all results in the theory of DAHA and related Macdonald polynomials for reduced root systems were transferred to the case of \( C^\vee C \) and, correspondingly, to the case of Koornwinder polynomials. A unification of this theory with the one for reduced root systems in one exposition seems not very reasonable.

1.1.2. Affine root systems. The vectors \( \alpha = [\alpha, j] \in \mathbb{R}^n \times \mathbb{R} \) for \( \alpha \in R \), \( j \in \mathbb{Z} \) form the standard affine root system \( \tilde{R} \). The set of positive affine roots is \( \tilde{R}_+ = \{ [\alpha, j] \mid j \in \mathbb{Z}_{\geq 0} \} \cup \{ [\alpha, 0] \mid \alpha \in R_+ \} \). Define \( \alpha_0 = [-\theta, 1] \), where \( \theta \) is the maximal positive root in \( R \). We will identify \( \alpha \in \tilde{R} \)
with \( \tilde{\alpha} = [\alpha, 0] \in \tilde{R} \). The affine simple roots \( \{\alpha_i, 0 \leq i \leq n\} \) form the extended (also called affine) Dynkin diagram \( \text{Dyn}^{\text{aff}} \supset \text{Dyn} = \{\alpha_i, 1 \leq i \leq n\} \).

For an arbitrary affine root \( \tilde{\alpha} = [\alpha, j] \) and \( \tilde{z} = [z, \zeta] \in \mathbb{R}^{n+1} \), the corresponding reflection is defined as follows:

\[
    s_{\tilde{\alpha}}(\tilde{z}) = \tilde{z} - 2\frac{\langle z, \alpha \rangle}{\langle \alpha, \alpha \rangle} \tilde{\alpha} = \tilde{z} - (z, \alpha^\vee) \tilde{\alpha}.
\]

We set \( s_i = s_{\alpha_i} \) for \( i = 0, \ldots, n \). The affine Weyl group \( \tilde{W} \) is generated by \( \{s_{\tilde{\alpha}} | \tilde{\alpha} \in \tilde{R}_+\} \); \( \{s_i\} \) for \( i \geq 0 \) are sufficient.

**Theorem 1.1.** We have an isomorphism:

\[
    \tilde{W} \cong W \ltimes Q^\vee,
\]

where the translation \( \alpha^\vee \in Q^\vee \) is naturally identified with the composition \( s_{[-\alpha, 1]}s_\alpha \in \tilde{W} \). In terms of the action in \( \mathbb{R}^{n+1} \ni \tilde{z} \), one has:

\[
    b(\tilde{z}) = [z, \zeta - (b, z)] \text{ for } \tilde{z} = [z, \zeta], \ b \in Q^\vee.
\]

Define the extended affine Weyl group to be \( \tilde{W} = W \ltimes P^\vee \) acting on \( \mathbb{R}^{n+1} \) via the last formula from the theorem with \( b \in P^\vee \). Then \( \tilde{W} \subset \tilde{W} \).

Moreover, we have the following theorem. Let \( \text{Aut} = \text{Aut}(\text{Dyn}^{\text{aff}}) \), \( O \overset{\text{def}}{=} \{r\} \) for \( \text{Aut}(\alpha_0) = \{\alpha_r\} \).

**Theorem 1.2.** (i) The group \( \tilde{W} \) is a normal subgroup of \( \tilde{W} \) and \( \tilde{W}/\tilde{W} = P^\vee/Q^\vee \). The latter group can be identified with the group \( \Pi = \{\pi_r\} \) of the elements of \( \tilde{W} \) permuting simple affine roots under their action in \( \mathbb{R}^{n+1} \). It is a normal commutative subgroup of \( \text{Aut} \); the quotient \( \text{Aut}/\Pi \) is isomorphic to the group \( A_0 = \text{Aut}(\text{Dyn}) \) of the automorphisms preserving \( \alpha_0 \).

(ii) The indices \( r \in O^* \overset{\text{def}}{=} O \setminus \{0\} \) are exactly those for the minuscule coweights \( \omega_r^\vee \) satisfying the inequalities \( (\alpha, \omega_r^\vee) \leq 1 \) for all \( \alpha \in R_+ \). The elements \( \pi_r \in \Pi \) are uniquely determined by the relations \( \pi_r(\alpha_0) = \alpha_r \) \( (\pi_0 = \text{id}) \). An arbitrary element \( \tilde{w} \in \tilde{W} \) can be uniquely represented as \( \tilde{w} = \pi_r \tilde{w} \) for \( \tilde{w} \in \tilde{W} \).

It is not difficult to calculate \( \pi_r \) explicitly (see [Ch1]):

\[
    \pi_r = \omega_r^\vee u_r^{-1} \text{ for minuscule } \omega_r^\vee \in P_+^\vee \subset \tilde{W}, \ u_r = w_0w_0^{(r)},
\]

where \( w_0^{(r)} \) is the element of maximal length in the centralizer of \( \omega_r^\vee \) in \( W \) for \( r \in O^* \), \( w_0 \) is the element of maximal length in \( W \). Equivalently, \( u_r \) is of minimal possible length such that \( u_r(\omega_r) \in P_- = -P_+ \). Note that \( \pi_r s_i \pi_r^{-1} = s_j \) if \( \pi_r(\alpha_i) = \alpha_j, 0 \leq i \leq n \).
1.1.3. The length function. For any element \( \hat{w} \in \hat{W} \), it can be written as \( \hat{w} = \pi_r \tilde{w} \) for \( \pi_r \in \Pi \) and \( \tilde{w} \in \tilde{W} \). The length \( l(\hat{w}) \) is defined to be the length of the reduced decomposition \( \tilde{w} = s_{i_1} \cdots s_{i_l} \) (i.e., with minimal possible \( l \)) in terms of the simple reflections \( s_i \). Thus, by definition, \( l(\pi_r) = 0 \).

This is the standard group-theoretical definition, but there are two other (equivalent) definitions of the length for the crystallographic groups, combinatorial and geometric. Namely, the length \( l(\hat{w}) \) is the cardinality \( |\tilde{R}_+ \cap \tilde{w}^{-1}(\tilde{R}_-)\| \) and can be also interpreted as the “distance” from the standard affine Weyl chamber to its image under \( w \). Both definitions readily give that \( l(\pi_r) = 0 \); indeed, \( \pi_r \) sends positive roots \( \tilde{\alpha} \) to positive ones and (therefore) leaves the standard affine Weyl chamber invariant.

Either the combinatorial or the geometric definition can be used to check that \( l(w(b)) = 2(\rho, b) \) for \( b \in P_+^\vee \) and for an arbitrary \( w \in W \).

All three approaches to the length-function are important in the combinatorial theory of affine Weyl groups, which is far from being simple and complete.

1.1.4. Twisted affinization. There is another affine extension \( R^\nu \) of \( R \), convenient in quite a few constructions (especially, when the DAHA Fourier transform and the Gaussians are studied). It is the main one in [Ch1] and quite a few first author’s papers. It is defined for the maximal short root \( \vartheta \) instead of the maximal root \( \theta \). Accordingly, \( (\alpha, \alpha) = 2 \) for short roots, and affine roots are introduced as \( \tilde{\alpha} = [\alpha, \nu_\alpha] \) for \( \nu_\alpha = \frac{(\alpha, \alpha)}{2} (= 1, 2, 3) \). Adding \( \alpha_0 = [-\vartheta, 1] \) for such \( \vartheta \), the resulting diagram is the extended Dynkin diagram \( (Dyn^\vee)^{aff} \) for \( R^\vee \) where all the arrows are reversed. On can simply set \( \tilde{R}^\nu \overset{\text{def}}{=} ((R^\vee)^{aff})^\vee \), where the form in \( R^\vee \) is normalized by the (usual) condition \( (\alpha^\vee, \alpha^\vee) = 2 \) for long \( \alpha^\vee \), so \( \vartheta \) becomes the maximal root in \( R^\vee \). The second check is applied to the affine roots. The formula \( s_{[-\alpha, \nu_\alpha]} s_{-\alpha} = \alpha \) naturally results in the definition of affine Weyl groups with unchecked \( Q, P \):

\[
\text{for } R^\nu : \quad \hat{W} \cong W \ltimes Q, \quad \hat{W} \cong W \ltimes P.
\]

In the \( p \)-adic theory, the corresponding Chevalley group is a form of the split one.

The appearance of \( Q, P \) in \( \hat{W}, \tilde{W} \) leads to the invariance of the corresponding DAHA with respect to the Fourier transform and other basic automorphisms. It is the main reason why the book [Ch1] is mainly written in this, “self-dual”, setting. Due to the special choice of the
normalization, $Q \subset Q'$ in this case. The term “twisted” matches similar name in the Kac-Moody theory.

1.2. AHA and spherical functions.

1.2.1. Affine Hecke algebras. The affine Hecke algebra $\mathcal{H}$ is generated by $T_0, T_1, \ldots, T_n$ and the group $\Pi = \{\pi_r\}$ with the relations:

$$T_i T_j T_i \ldots = T_j T_i T_j \ldots,$$

\(m_{ij}\) times \(m_{ij}\) times (1.2)

$$(T_i - t^{-1/2})(T_i + t^{-1/2}) = 0,$$

$$(T_0 - t^{-1/2})(T_0 + t^{-1/2}) = 0.$$ 

To be precise, $\pi_r(i)$ must be understood here as the suffix of $\pi_r(\alpha_i)$; $m_{ij}$ is the number of edges between vertex $i$ and vertex $j$ in the affine Dynkin diagram $\text{Dyn}^{\text{aff}}$, $t$ is a formal parameter (later, mainly a nonzero number).

Comment. The above definition gives the affine Hecke algebra with equal parameters. More systematically, we can introduce a family of formal parameters $\{t_\alpha\}$ depending only on $|\alpha|$, setting $t_i = t_{\alpha_i}$ for $0 \leq i \leq n$. Replacing relations (1.2) by the relations $(T_i - t_i^{-1/2})(T_i + t_i^{-1/2}) = 0$, we come to the definition of the affine Hecke algebra standard in their (modern) geometric and/or algebraic theory; it is called the case of unequal parameters.

The formulas below can be readily adjusted to this setting, namely, $t_i$ must be used for $T_i$ and the subscript $\alpha$ must be added to $t$ in the formulas involving $Y_{\alpha^\vee}$. In the DAHA theory, the same must be done for $X_\alpha$; also, $t = q^k$ reads as $t_\alpha = q^{k_\alpha}$. If $\tilde{R}^\vee$ is used instead of $R$, with $Y_\alpha$ instead of $Y_{\alpha^\vee}$, then $q$ must be also replaced by $q_\alpha = q^{\rho_\alpha}$ in the formulas; accordingly, $t_\alpha = q^{k_\alpha}$.

For any element $\hat{w} \in \hat{W}$, define $T_{\hat{w}} = \pi_r T_i \cdots T_{i_1}$, where $\hat{w} = \pi_r s_{i_1} \cdots s_{i_1}$ is a reduced representation of $\hat{w}$. The definition of $T_{\hat{w}}$ does not depend on the choice of the reduced decomposition.

Setting $Y_b = T_b$ for $b \in P_+^\vee \subset \hat{W}$, one has $Y_b Y_c = Y_c Y_b$ for such (dominant) $b, c$; use that $l(b) = 2(\rho, b)$ for dominant $b$. For any $a \in P^\vee$, we set $Y_a \overset{\text{def}}{=} Y_b Y_c^{-1}$ for $a = b - c$ with some $b, c \in P_+^\vee$; the commutativity guarantees that $Y_a$ depends only on $a$. This definition is due to Bernstein, Zelevinsky and Lusztig, see, e.g., [L].

Let $\mathcal{Y} \overset{\text{def}}{=} \mathbb{C}[Y_{\omega_i^\vee}] \subset \mathcal{H}$. Then

$$\mathcal{H} = \langle \mathcal{Y}, T_1, \ldots, T_n \rangle.$$ 

Indeed, $T_0 = Y_\theta T_{s_\theta}$ and $\pi_r = Y_{\omega_r^\vee} T_{u_r}^{-1}$ (see (1.1)).
**Theorem 1.3.** (i) An arbitrary element $H \in \mathcal{H}$ can be uniquely represented as $H = \sum c_{b,i} Y_b T_i$ for $b \in P^\vee$, $1 \leq i \leq n$ (called the PBW Theorem).

(ii) The subalgebra $\mathcal{Y}^W$ of $W$–invariant $Y$–polynomials is the center of $\mathcal{H}$ (the Bernstein Lemma); see Lemma 1.6.

1.2.2. Matsumoto functions. Let $\mathcal{H} = \mathcal{H}_{\text{nonaff}}$ be the Hecke algebra associated with the nonaffine root system $R$, i.e., generated by $T_i$ with $1 \leq i \leq n$ and without adding the group $\Pi$. We can define the $t$–symmetrizer by the formula:

$$P_+ = \frac{\sum_{w \in W} t^{l(w)/2} T_w}{\sum_{w \in W} t^{l(w)}} \in \mathcal{H};$$

indeed, using (1.3) below,

$$\frac{(1 + t^{1/2} T_i) P_+}{1 + t} = P_+, \ 1 \leq i \leq n.$$

The following renormalization $\delta_{\hat{w}} = t^{-l(\hat{w})/2} T_{\hat{w}}$ of $T_{\hat{w}}$ (any $\hat{w} \in \hat{W}$) is convenient to establish the connection with the $p$-adic theory. We have

$$T_i \delta_{\hat{w}} = \begin{cases} 
 t^{1/2} \delta_{s_i \hat{w}}, & \text{if } l(s_i w) = l(w) + 1; \\
 t^{-1/2} \delta_{s_i \hat{w}} + (t^{1/2} - t^{-1/2}) \delta_{\hat{w}}, & \text{otherwise.}
\end{cases}$$

Now let $\Delta = \bigoplus_{\hat{w} \in \hat{W}} \mathbb{C} \delta_{\hat{w}}$ be the (left) regular representation of $\mathcal{H}$. Its spherical submodule is defined as follows:

$$\Delta^\sharp = \Delta \mathcal{P}_+ \cong \mathcal{Y} \mathcal{P}_+.$$  

The identification with the Laurent $Y$–polynomials is based on the PBW property from Theorem 1.3.

From now on, $\mathcal{Y}$ will be understood as $\Delta^\sharp$, i.e., $1 \in \mathcal{Y}$ is actually $\mathcal{P}_+$. By $\delta_{\hat{w}}^\sharp$, we denote the image of $\delta_{\hat{w}}$ in $\Delta^\sharp$, namely, $\delta_{\hat{w}}^\sharp = \delta_{\hat{w}} \mathcal{P}_+$.

The Matsumoto functions [Mat], also called nonsymmetric $p$-adic spherical functions, are defined (in this approach) to be

$$\varepsilon_b = \delta_b^\sharp, \ \forall b \in P^\vee,$$

i.e., we simply restrict $\delta^\sharp$ to $P^\vee$ here. From this definition, $\varepsilon_b = t^{-(b,\rho)} Y_b$ for any $b \in P^\vee$. The fundamental problem is calculating $\varepsilon_b$ for any $b \in P^\vee$.

1.2.3. The rank one case. In the $A_1$ case, we can set $\omega = \omega_1^\vee = \omega^\vee$; then $\alpha = \alpha_1 = 2\omega$ and $\rho = \omega$. The extended affine Weyl group $\hat{W}$ is generated by $\pi = \pi_1$ and the reflection $s = s_\alpha$. As an element of $\hat{W}$, $\omega = \pi s$. Let $T = T_1 \in \mathcal{H}$, then $Y = Y_\omega = \pi T$. 

The affine Hecke algebra can be written as $H = \langle Y, T \rangle$ with only one relation involving $Y$: $T^{-1}YT^{-1} = Y^{-1}$. It readily gives that $\pi^2 = 1$ for $\pi$ introduced as $YT^{-1}$.

The symmetrizer is
\[ \mathcal{P}_+ = \frac{1 + t^{1/2}T}{1 + t}. \]

For any $m \in \mathbb{Z}$, let $\delta_m = \delta_{m\omega}$ and $\varepsilon_m = \delta_{m\omega}^\# = t^{-m/2}T_{m\omega}\mathcal{P}_+$. Then we have for $m \geq 0$,
\begin{align*}
T\varepsilon_m &= t^{1/2}\varepsilon_m, \\
T\varepsilon_{-m} &= t^{-1/2}\varepsilon_{-m} + (t^{1/2} - t^{-1/2})\varepsilon_m.
\end{align*}

Similarly, for $m \geq 0$,
\begin{align*}
T^{-1}\varepsilon_{-m} &= t^{-1/2}\varepsilon_{-m}, \\
T^{-1}\varepsilon_m &= (T - (t^{1/2} - t^{-1/2}))\varepsilon_m = t^{1/2}\varepsilon_{-m} - (t^{1/2} - t^{-1/2})\varepsilon_m.
\end{align*}

**Lemma 1.4.** For any $m \in \mathbb{Z}$, $\pi\varepsilon_m = \varepsilon_{1-m}$.

**Proof.** Since $\pi^2 = 1$, it suffices to calculate $\pi\varepsilon_{-m}$ for $m \leq 0$. Using that $Y\varepsilon_m = t^{1/2}\varepsilon_{m+1}$ (it results from the definition of $\varepsilon$ for such $m$),
\[ \pi\varepsilon_{-m} = YT^{-1}\varepsilon_{-m} = t^{-1/2}Y\varepsilon_m = \varepsilon_{1-m}. \]

Let us apply the lemma to write down the action of $Y^\pm$ on $\varepsilon_m, \varepsilon_{-m}$ for $m \geq 0$:
\begin{align*}
Y\varepsilon_m &= t^{1/2}\varepsilon_{m+1}, \\
Y\varepsilon_{-m} &= t^{-1/2}\varepsilon_{m+1} + (t^{1/2} - t^{-1/2})\varepsilon_{m+1}, \\
Y^{-1}\varepsilon_{m+1} &= t^{-1/2}\varepsilon_m, \\
Y^{-1}\varepsilon_{-m} &= t^{1/2}\varepsilon_{m-1} - (t^{1/2} - t^{-1/2})\varepsilon_{m+1}.
\end{align*}

The last two formulas do not overlap as $m \geq 0$; the pairs of formulas for the action of $T, T^{-1}$ and $Y$ intersect (and coincide) at $m = 0$. The formulas for the action of $Y, Y^{-1}$ are called *nonsymmetric Pieri rules*; they are obviously sufficient to calculate the $\varepsilon$–functions (it holds in any ranks).

Generally, the *technique of intertwiners* is more efficient for calculating the $\varepsilon$–polynomials and their generalizations than direct using the Pieri formulas (see, e.g., [Ch1]). In this example, formula (1.4) is...
We are now ready to introduce the (exact) algebraic counterparts of \( p \)-adic spherical functions:

\[
\varphi_m \overset{\text{def}}{=} \frac{1 + t^{1/2} T^m}{1 + t^{-m}}, \quad m \geq 0.
\]

Using the formulas (1.6), (1.8) and the commutativity of \( Y + Y^{-1} \) with \( T \) (check it directly or see below), we establish the symmetric Pieri rules:

\[
(Y + Y^{-1}) \varphi_m = t^{1/2} \varphi_{m+1} + t^{-1/2} \varphi_{m-1} \quad \text{as } m > 0,
\]

(1.11) \[
(Y + Y^{-1}) \varphi_0 = (t^{1/2} + t^{-1/2}) \varphi_1.
\]

Note that the latter relation follows from the former if one formally imposes the periodicity condition \( \varphi_{-1} = \varphi_1 \), but of course we can not use it for the justification. By construction, \( \varphi_0 = 1 \); all other functions can be calculated using the Pieri rules. All \( \varphi_i \)'s are invariant under \( s : Y \mapsto Y^{-1} \) due to the commutativity \([Y + Y^{-1}, T] = 0\).

The first three \( \varphi_m \)'s are as follows:

\[
\varphi_0 = 1, \quad \varphi_1 = \frac{Y + Y^{-1}}{t^{1/2} + t^{-1/2}}, \quad \varphi_2 = \frac{(Y + Y^{-1})^2}{1 + t} - t^{-1}.
\]

For the system \( A_1 \), the symmetric Pieri rules look simpler than their \( \varepsilon \)-counterparts, but it is exactly the other way round in higher ranks. Generally, there are no good formulas for the action of \( W \)-orbit sums in the form \( \sum_w Y_{w(b)} \) on the spherical functions (see (1.11)) except for the minuscule \( b = \omega_r^\vee \) and \( b = \theta \). Theoretically, the Pieri formulas are sufficient to calculate all \( \varphi \)-polynomials, but it can be used practically mainly for \( A_n \) and in some cases of small ranks. The nonsymmetric formulas of type (1.6–1.9) exist (and are reasonably convenient to deal with) for arbitrary root systems.

1.3. Spherical functions as Hall polynomials.

1.3.1. Macdonald's formula. In general (for any \( R \) above), we can define the spherical function as follows:

\[
\varphi_b \overset{\text{def}}{=} \mathcal{P}_+ \varepsilon_b = t^{-(\rho, b)} \mathcal{P}_+ Y_b \mathcal{P}_+ \in \mathcal{Y}, \quad b \in P^\vee_+.
\]
They are $W$–invariant $Y$–polynomials for the natural action $w(Y_b)$ (the Bernstein Lemma). Their $(p$-adic) theory was developed by Satake, Macdonald and others; we will mainly call them the Macdonald spherical functions. He established the following fundamental fact.

**Theorem 1.5.** Let $P(t)$ be the Poincaré polynomial, namely, $P(t) = \sum_{w \in W} t^{l(w)}$. Then

\[
\varphi_b(Y) = \frac{t^{-(\rho, b)} P(t-1)}{1} \sum_{w \in W} Y_w(b) \prod_{\alpha \in R_+} \frac{1 - t^{-1}Y^{-1}_{w(\alpha^\vee)}}{1 - Y^{-1}_{w(\alpha^\vee)}}.
\]

The summation in the right-hand side is proportional to the Hall-Littlewood polynomial associated with $b \in P_+$. It results in a Laurent $Y$–polynomial, indeed, which readily follows from the fact that all antisymmetric polynomials in $Y$ are divisible by the discriminant, the common denominator in the right-hand side. The proof of this theorem will be given in the next section.

In the case of $A_1$, we obtain

\[
\varphi_m = \frac{t^{-m/2}}{1 + t^{-1}} \left( \frac{Y^m - Y^{-1}Y^{-m-2} + t^{-1}Y^{-m}}{1 - Y^{-2}} \right)
\]

\[
= \frac{t^{-m/2}}{1 + t^{-1}} \left( \frac{Y^{m+1} - Y^{-m-1} - t^{-1}(Y^{m-1} - Y^{1-m})}{Y - Y^{-1}} \right),
\]

which matches our calculations above based directly on the Pieri rules. Compare with the “non-symmetric” formulas (1.10). Macdonald established his formula by calculating the Satake $p$–adic integral representing the spherical function (see below).

One can try to use Pieri rules to justify the theorem, but, as we noted above, sufficiently explicit formulas exist only for $A_n$ and in some cases of small ranks. There is another, much more direct approach (any root systems), which can be generalized to the DAHA theory. We will follow it after the following remarks, clarifying the origins of the Pieri rules (to be continued in the section on the classical $p$–adic theory).

**1.3.2. Comments on Pieri rules.** Formulas (1.11) match the classical arithmetical definition of the (one-dimensional) Hecke operator. Let $t$ be the cardinality of the residue field of a $p$–adic field $K$ ($t = p$ for $\mathbb{Q}_p$). The Bruhat-Tits building of type $A_1$ is a tree with $t + 1$ edges from each vertex; the vertices $\{v\}$ correspond to the maximal parahoric subgroups of $G = PGL_2(K)$, which are (all) conjugated to $U = PGL_2(\mathcal{O}) \subset G = PGL_2(K)$ for the ring of integers $\mathcal{O} \subset K$. 

Two vertices are connected by an edge if their intersection is an Iwahori subgroup, i.e., is conjugated to $B = \{ g \in U \mid g_{21} \in p \}$ for the maximal ideal $p \subset O$. The group $G$ naturally acts on this tree by conjugation. Identifying the vertices with the cosets of $G/U$, the action of $G$ becomes left regular; we will use this interpretation too.

Let $d(v)$ be the distance (in the tree) of the vertex $v$ from the origin $o$, which corresponds to $U$. The functions $f(m)$ on this tree depending only on the distance $m = d(v) \geq 0$ are exactly the functions on $G//U = U \backslash G/U$. The figure is as follows ($t = p = 3$):

The classical Hecke operator is the (radial) Laplace operator $\Delta$ on this tree, the averaging over the neighbors. Explicitly,

$$\Delta f(m) = \frac{tf(m+1) + f(m-1)}{t+1} \quad \text{for} \quad m > 0, \quad \Delta f(0) = \frac{f(1)}{t+1}.$$

Thus (1.11) is exactly the eigenvalue problem for $t^{-1/2} \Delta$ with the eigenvalue $Y + Y^{-1}$, where $Y$ is treated as a free parameter.

For arbitrary Chevalley groups, a combinatorial definition of the Laplace-type operator and its higher analogs in terms of the Bruhat-Tits buildings becomes involved. The case of $A_n$ was considered by Drinfeld.

The Bruhat-Tits building is equally useful in the theory of Whittaker functions; see Section 1.4.4 below for more detail. There is a unique infinite path from the origin such that the elements of the unipotent subgroup $N \subset G$ preserve its direction to infinity; only the direction, any finite number of vertices can be ignored. Let us extend this path to a road, infinite in both directions. Then any vertex can be mapped onto this road (identified with $N\backslash G/U$) using $N$; its image is unique. The Whittaker function can be interpreted as a function on this road nonzero only on the original path.

1.3.3. The major limits. Let us switch from the normalization we used (compatible with the $p$–adic Hecke operators), to the one more convenient algebraically. Namely, we set $\tilde{\varphi}_m \overset{\text{def}}{=} t^{m/2} \varphi_m$, which readily simplifies the (symmetric) Pieri rules:

$$(Y + Y^{-1}) \tilde{\varphi}_m = \tilde{\varphi}_{m+1} + \tilde{\varphi}_{m-1}.$$
This recurrence has the following elementary solutions for \( m \geq 0 \).

1) The monomial symmetric functions (slightly renormalized):

\[
\mathcal{M}_m = \frac{(Y^m + Y^{-m})}{2}.
\]

2) The Schur functions \( \chi_m \):

\[
\chi_m = \frac{Y^{m+1} - Y^{-m-1}}{Y - Y^{-1}}.
\]

3) The renormalized Macdonald spherical functions:

\[
\tilde{\varphi}_m = \frac{1}{1 + t^{-1}} \cdot \frac{Y^{m+1} - Y^{-m-1} - t^{-1}(Y^{m-1} - Y^{1-m})}{Y - Y^{-1}}.
\]

All three sequences begin with 1 at \( m = 0 \). They are different due to the boundary conditions; extending them from \( m = 0 \) to \( m = -1 \):

1) \( \mathcal{M}_{-1} = \mathcal{M}_1 \), 2) \( \chi_{-1} = 0 \), 3) \( \tilde{\varphi}_{-1} = \tilde{\varphi}_1 t^{-1} \).

The first two cases are limits of the third one:

\[
-\chi_{m-2} \xrightarrow{t \to 0} \tilde{\varphi}_m \xrightarrow{t \to \infty} \chi_m \\
\xrightarrow{t \to 1} \mathcal{M}_m
\]

The limit \( t \to \infty \) is actually the degeneration of the Macdonald spherical functions to the Whittaker functions (see Section 1.4.4), although this limiting procedure makes little sense if we interpret \( t \) as \( p \).

1.3.4. The nonsymmetric case. The Matsumoto spherical functions are left Iwahori–invariant (not bi-\( U \)–invariant), so they can be naturally identified with the functions depending on the distances from the origin \( o \) in the following two subtrees formed by:

(+) the paths from \( o \) through the \( p \) non-affine neighbors of \( o \),

(−) the paths from \( o \) through the one affine neighbor \( \hat{o} \) of \( o \).

The defining property of \( B \subset U \) is that its action preserves the edge between \( o \) and \( \hat{o} \). Now we measure the distance using negative numbers in the second subtree. Accordingly, the functions on \( B \setminus G/U \) become \( f(m) \) for \( m \in \mathbb{Z} \), where \( m = d'(v) \in \mathbb{Z} \) for the new distance (can be negative).

Check that \( d'(v) \) is the only invariant of the vertex under the action of the Iwahori subgroup as an exercise and interpret combinatorially formulas (1.6, 1.7) in terms of \( m = d'(v) \).

Considering the limits, let us switch in (1.10) to \( \tilde{\varepsilon}_m = t^{|m|/2} \varepsilon_m \). Then

\[
(1.14) \quad \tilde{\varepsilon}_m = t^{m/2} \varepsilon_m = Y^m, \quad \tilde{\varepsilon}_{-m} = Y^{-m} + (1 - t^{-1}) \frac{Y^{-m} - Y^m}{Y^2 - 1},
\]
where $m \geq 0$. There is no dependence on $t$ for non-negative indices (so the corresponding limits are obvious). The graph of the limits for $-m (m > 0)$ is as follows:

\[
\begin{array}{c}
\text{\Huge$\infty$} \\
\text{\Huge$t^{-1}$} \\
\text{\Huge$t^{-\infty}$} \\
\text{\Huge$\tilde{\varepsilon}$} \\
\text{\Huge$m$} \\
\text{\Huge$\chi_{m}$} \\
\text{\Huge$Y^{-m}$}
\end{array}
\]

1.3.5. Proof of Macdonald’s formula. Recall that the affine Hecke algebra $\mathcal{H}$ in the $T$-$Y$–presentation is generated by the elements $T_1, \ldots, T_n$ and $Y_b$ for $b \in P^\vee$. The defining relations between $T_i$’s and $Y_b$’s are:

\begin{align}
T_i Y_b &= Y_b T_i, \quad \text{if } (b, \alpha_i) = 0, \ i > 0, \\
T_i^{-1} Y_b T_i^{-1} &= Y_b Y_{\alpha_i}^{-1}, \quad \text{if } (b, \alpha_i) = 1.
\end{align}

The relation to the original definition is as follows:

\begin{align}
T_0 &= Y_\theta T_{s_\theta}^{-1}, \\
\pi_r &= Y_{\omega^\vee_r} T_{u_r}^{-1},
\end{align}

where $u_r$ are from (1.1).

These relations are actually the relations of the orbifold braid group of $C^*/W$. It is straightforward to extend them to any $b \in P^\vee$ using the quadratic relations. The formulas are due to Lusztig (see, e.g., [L]):

\begin{align}
T_i Y_b - Y_{s_i(b)} T_i &= (t^{1/2} - t^{-1/2}) \frac{Y_{s_i(b)} - Y_b}{Y_{\alpha_i}^{-1} - 1}, \ i > 0.
\end{align}

Lemma 1.6. The center of the affine Hecke algebra is

\[Z(\mathcal{H}) = \mathcal{B}^W = \mathbb{C}[Y_b]^W.\]

Proof. By regarding both side of (1.17) as operators on $\mathcal{B} \ni f(Y)$, we have

\begin{align}
T_i f &= t^{1/2} s_i f + (t^{1/2} - t^{-1/2}) \frac{s_i f - f}{Y_{\alpha_i}^{-1} - 1}.
\end{align}

Thus $T_i f = t^{1/2} f$ for all $i > 0$ are equivalent to the relations $s_i(f) - f = 0$ for all $i > 0$, which means that $f \in \mathcal{B}^W$. \qed

Theorem 1.7 (Operator Macdonald’s Formula). Let

\[\tilde{M} \overset{\text{def}}{=} \prod_{\alpha \in R^+} \frac{1 - t^{-1} Y_{\alpha}^{-1}}{1 - Y_{\alpha}^{-1}}.\]

Then we have the following operator identity in $\mathcal{B}$:

\begin{align}
P(t^{-1}) \mathcal{P}_+ &= (\sum_{w \in W} w) \circ \tilde{M},
\end{align}
or, equivalently,

\begin{equation}
\sum_{w \in W} T_w^{-1} t^{-l(w)/2} = (\sum_{w \in W} w) \circ \tilde{M}.
\end{equation}

**Proof.** The equivalence of identities (1.19) and (1.20) is due to

\[
\mathcal{P} + = \frac{\sum_{w \in W} t^{l(w)/2} T_w}{\sum_{w \in W} t^{l(w)}} = \frac{\sum_{w \in W} t^{-l(w)/2} T_w^{-1}}{\sum_{w \in W} t^{-l(w)}}.
\]

Indeed, both operators are divisible by \(1 + t^{1/2} T_i\) on the right and on the left for any \(i > 0\) and act identically on \(1 \in \mathcal{Y}\).

Following [Ch5] (upon the affine degeneration), let us introduce the following involution:

\begin{equation}
\iota : Y_b \mapsto Y_b, \quad t^{1/2} \mapsto -t^{-1/2}, \quad s_i \mapsto -s_i.
\end{equation}

Applying it to the operator from (1.18)

\[ T_i = t^{1/2} s_i + \frac{t^{1/2} - t^{-1/2}}{Y_{\alpha_i}^{-1} - 1} (s_i - 1), \]

one readily obtains:

\[ T_i^\iota = t^{-1/2} s_i - \frac{t^{1/2} - t^{-1/2}}{Y_{\alpha_i}^{-1} - 1} (s_i + 1). \]

To match [Ch5] and the general theory of DAHA, let us begin with

\[ M \overset{\text{def}}{=} \prod_{\alpha \in R_+} \frac{1 - Y_{\alpha^\vee}}{1 - tY_{\alpha^\vee}}. \]

It is the \(q \to 0\) limit of the “truncated theta-function” \(\mu\) playing the key role in the theory of polynomial representation of DAHA. It is equivalent \((\Leftrightarrow)\) to \(\tilde{M}\) in the following natural sense: they coincide up to a \(W\)-invariant factor. Indeed:

\[ \tilde{M} \Leftrightarrow \tilde{M}' \overset{\text{def}}{=} \prod_{\alpha \in R_+} \frac{1 - Y_{\alpha^\vee}}{1 - t^{-1} Y_{\alpha^\vee}} \Rightarrow M. \]

**Lemma 1.8.** One has: \(MT_i M^{-1} = T_i^\iota\) for \(i = 1, \ldots, n.\) \(\square\)

**Lemma 1.9.** For \(i \geq 1,\)

\[ T_i + t^{-1/2} = (s_i + 1) \cdot F_i \text{ for a rational function } F_i(Y), \]

\[ T_i + t^{-1/2} = G_i \cdot (s_i + 1) \text{ for a rational function } G_i(Y). \] \(\square\)

Now, returning to the proof of the theorem, \(\mathcal{P} + \circ \tilde{M}^{-1} \Rightarrow \mathcal{P} + \circ M^{-1},\) and these operators are divisible by \((1 + t^{1/2} T_i)\) on the left and on the operator \((1 + t^{1/2} T_i^\iota)\) on the right. The divisibility on the left is straight
from the divisibility of $\mathcal{P}_+$. With the divisibility on the right, we need Lemma 1.8.

Using Lemma 1.9, we obtain that $\mathcal{P}_+ \circ \widetilde{M}^{-1}$ is divisible on the right and on the left, both, by $(s_i + 1)$. Thus it commutes with the operators of multiplication by functions from $\mathcal{P}^W$ and must be in the form $G(Y) \circ \sum_{w \in W} w$ for a $W$–invariant (rational) function $G(Y)$. However, $G = P(t^{-1})^{-1}$ due to $\sum_{w \in W} w(\widetilde{M}) = P(t^{-1})$ from [Hu], formula (35), Section 3.20. Note that the latter formula is an immediate corollary of the divisibility of any $W$–invariant antisymmetric Laurent polynomial by the discriminant; see [B]. □

The operator Macdonald formula is actually from [Ma5], formula (5.5.14). His proof is similar to what we did. We deduced it from [Ch5]; Macdonald checks the divisibility of the operator $(\sum_{w \in W} w) \circ \widetilde{M}$ by $1 + t^{1/2}T_i$ on the left and on the right directly. Then he equates the leading terms in (1.19), the coefficients of the longest element $w_0 \in W$. The last step cannot be used in the DAHA theory (the longest element does not exist in $\hat{W}$). We think that the interpretation of $M$ and (later) $\mu$ from [Ch5] as an intertwiner between the symmetric and antisymmetric polynomial representations clarifies well which property of $M$ is needed in this calculation. The claim of Lemma 1.8 is of more fundamental nature, similar to the well-known interpretations of $M, \mu$ as “measures” in the theory of Hall and Macdonald polynomials.

1.4. Satake-Macdonald theory.

1.4.1. Chevalley groups. Let $K$ be a $p$–adic field and $\mathcal{O} \subset K$ the valuation ring in $K$ with the (unique) prime ideal $(\varpi)$ for the uniformizing element $\varpi$. We set $t = |k|$, where $k$ is the residue field $\mathcal{O}/(\varpi)$.

For an irreducible reduced root system $R$ as above and the coweight lattice $P^\vee$, the Lie algebra $\mathfrak{g}_K$ is defined as the $\mathfrak{g} \otimes K$ for the Lie algebra $\mathfrak{g}$ defined over $\mathbb{Z}$ as the span of $\{x_\alpha, h_b\}$ for $\alpha \in R, b \in P^\vee$ subject to the relations:

$$
[h_a, h_b] = 0, \quad [h_b, x_\alpha] = (b, \alpha)x_\alpha, \quad [x_\alpha, x_{-\alpha}] = h_{\alpha^\vee},
$$

$$
[x_\alpha, x_\beta] = N_{\alpha, \beta}x_{\alpha + \beta} \quad \text{if} \quad \alpha + \beta \in R, \quad \text{otherwise} \quad 0.
$$

Accordingly, $\mathfrak{g}_\mathcal{O} = \mathfrak{g} \otimes \mathcal{O}$. The integers $N_{\alpha, \beta}$ can be chosen here uniquely up to the signs; we will omit their discussion.

The unipotent groups $X_\alpha$ are defined for $\alpha \in R$ as “exponents” of $Kx_\alpha$; $H$ is the $K$–torus corresponding to $P^\vee$. By construction, these groups act on $\mathfrak{g}_K$. We will also need the group lattice formed by the
elements \( \varpi^b \in H \) for \( b \in P^\vee \) defined as follows:
\[
\varpi^b(x_\alpha) = \varpi^{(b,\alpha)}x_\alpha, \forall \alpha \in R.
\]

Finally, the (split) Chevalley group \( G \) is the span of \( X_\alpha \) for all \( \alpha \in R \) and \( H \). The standard unipotent subgroup \( N \) is the group span of \( X_\alpha \) for \( \alpha \in R_+ \). The maximal parahoric subgroup \( U \) is the centralizer of \( g_\mathcal{O} \) in \( G \). Note that \( P^\vee \) is used here; if it is replaced by \( Q^\vee \), then the corresponding group is the group of \( K \)-points of the connected simply connected split algebraic group associated with \( R \).

We have the Cartan decomposition of \( G \):
\[
G = U H U = \bigcup_{b \in P^\vee} U \varpi^b U, \tag{1.22}
\]
and the Iwasawa decomposition:
\[
G = U H N = \bigcup_{b \in P^\vee} U \varpi^b N; \tag{1.23}
\]
the unions are disjoint.

As an exercise, introduce the Chevalley group corresponding to the twisted affinization \( \check{\mathcal{R}}^\nu \) of \( R \) considered in Section 1.1.4. Using algebraic groups, it will be a group of \( K \)-points of a non-split group over \( K \), which splits over certain ramified extension of \( K \).

1.4.2. The Satake integral. Let \( L(G, U) \) be the space of complex valued functions \( f \) on \( G \), compactly supported, satisfying the bi-\( U \)-invariance condition:
\[
f(u_1 x u_2) = f(x) \text{ for all } x \in G, \text{ and any } u_1, u_2 \in U.
\]

It is a ring; the product of two functions \( f, g \in L(G, U) \) is defined by the convolution:
\[
f * g(x) = \int_G f(xy^{-1})g(y)dy,
\]
where \( dy \) is the Haar measure on \( G \) normalized by \( \int_U dy = 1 \). Moreover, it is a commutative ring (use the \(~-1\)–automorphism of \( R \) and \( R^\vee \) extended to \( G \)).

The zonal spherical function on \( G \) relative to \( U \) are continuous bi-\( U \)-invariant complex-valued function \( \Phi \) on \( G \) satisfying the following condition:
\[
\Phi * f = c_f \Phi \text{ for any } f \in L(G, U) \tag{1.24}
\]
and for constants \( c_f \) depending on \( f \). In other words, \( \Phi \) is a common eigenfunction of all the convolution operators with the elements \( f \in \)
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\[ L(G, U); \] then \( c_f \) are the corresponding eigenvalues. The normalization is \( \Phi(1) = 1 \).

Satake (following Harish-Chandra) found that an arbitrary zonal spherical function can be uniformly described in terms of the vector \( \lambda \in \mathbb{C} \otimes_{\mathbb{Z}} R \cong \mathbb{C}^n \). Using the Iwasawa decomposition (1.23), let us define the projection map onto \( P^\vee \):

\[ \text{pr} : G \to P^\vee, \quad x \in U \, \mathbb{w}^b N \mapsto b. \]

Using this map, the zonal spherical functions are given as follows:

\[ \Phi_{\lambda}(x) = \int_U t^{(\text{pr}(x^{-1})u, \rho - \lambda)} \, du \]

for the Haar measure restricted to \( U \).

Macdonald calculated this integral in [Ma1] using the combinatorics of \( U \). It was not too simple; see his Madras lectures [Ma2] (the lectures also include the relations to the real theory, the positivity matters and other issues). It suffices to evaluate \( \Phi_{\lambda} \) at \( \mathbb{w}^b \). His formula reads as:

\[ \Phi_{\lambda}(\mathbb{w}^b) = \frac{1}{P(t^{-1})} \sum_{w \in W} t^{(b, w(\lambda) - \rho)} \prod_{\alpha \in R_+} \frac{1 - t^{-1}(\alpha, w(\lambda))}{1 - t^{-1}(\alpha, w(\lambda))}. \]

Connecting the \( p \)-adic theory and the algebraic one can be now achieved by replacing \( Y_b \) by \( t^{(b, \lambda)} \), namely,

\[ \varphi_b(Y) = \Phi_{\lambda}(\mathbb{w}^b) \left[ t^{(b, \lambda)} \mapsto Y_b \right]. \]

Recall that in (1.12),

\[ \varphi_b(Y) = \frac{t^{-(\rho, b)}}{P(t^{-1})} \sum_{w \in W} Y_w(b) \prod_{\alpha \in R_+} \frac{1 - t^{-1}Y_{w(\alpha^\vee)}}{1 - Y_{w(\alpha^\vee)}}. \]

1.4.3. The universality principle. As a matter of fact, the approach via the Matsumoto spherical functions establishes the required bridge between the algebraic theory above and the \( p \)-adic theory and proves (1.27) (without taking a single \( p \)-adic integral).

The coincidence of these two theories, algebraic and \( p \)-adic, can be also seen in a more direct way by observing that the defining relations from (1.24) are nothing but the Pieri rules in the algebraic theory. However it is with the reservation that the Pieri rules are, generally, not explicit.

One can also use the following universality principle.

We need formula (1.24) only to ensure that there exists a family of pairwise commutative difference operators in terms of \( b \); they are convolutions with different \( f \in L(G, U) \). It is not necessary to know exactly how the convolution is defined; it can be of any origin, say,
from the geometric theories. Provided the existence of such operators (differential or difference) and certain natural symmetries, such family is essentially unique. This claim can be made rigorous if more information on the structure of difference or differential operators under consideration is available.

Informally, we have very few such families (subject to certain symmetries and boundary conditions). Cf. the discussion in Section 1.3.3. So far the major known examples come from the theory of Macdonald polynomials and DAHA, their counterparts, generalizations and degenerations. In physics, the universality of the quantum many body problem reflects the same phenomenon.

Thus, one can expect a priori (or even conclude rigorously) that \( p \)-adic spherical functions are proper specializations of the Macdonald polynomials. In our case, the specialization of the general \( q,t \)-theory is by letting \( q \to 0 \) under minor renormalization. The relation of the symmetric and nonsymmetric Macdonald polynomials to spherical function introduced algebraically (as we did) is straightforward.

1.4.4. Whittaker functions. The universality principle above works well for the Whittaker functions. We introduce them mainly following [CS] with some simplifications; see also [Shi] for the \( GL_n \)-case. The notation is from Section 1.4.1.

The unramified \( p \)-adic Whittaker function \( \mathcal{W} \) is defined for a character \( \psi \) that is the product of the \((K\text{-additive})\) characters \( \psi_i : K \to K/\mathcal{O} \to \mathbb{C}^* \) (\( i = 1, \ldots, n \)); each \( \psi_i \) must be nontrivial on \( \varpi^{-1}\mathcal{O}/\mathcal{O} \). It can be naturally extended to a character of the group \( N \) (vanishing on \( X_\alpha \) for non-simple roots \( \alpha > 0 \)).

For an algebra homomorphism \( \chi : L(G, U) \to \mathbb{C} \), there is a unique function \( \mathcal{W}_\chi \) on \( G \) such that \( \mathcal{W}_\chi(1) = 1 \),

\[
\mathcal{W}_\chi(nu) = \psi(n) \mathcal{W}_\chi(g) \quad \text{for} \quad n \in N, u \in U, g \in G,
\]

\[
\mathcal{W}_\chi * f = \chi(f) \mathcal{W} \quad \text{for any} \quad f \in L(G, U).
\]

As with the spherical function \( \Phi \), it suffices to know the values \( \mathcal{W}_\chi(\varpi^b) \) for \( b \in P^\vee \). However, the difference is dramatic; \( \mathcal{W}_\chi(\varpi^b) \) is not a \( \mathcal{W} \)-invariant function of \( b \). Moreover, \( \mathcal{W}_\chi(\varpi^b) = 0 \) unless \( b \in P^\vee_\chi \) (anti-dominant in Lemma 5.1 from [CS]).

This vanishing property and the universality principle are actually sufficient to conclude/expect that, up to a certain renormalization, \( \mathcal{W}_\chi(\varpi^b) \) does not depend on \( t \) (a surprising fact!) and that it is a classical finite-dimensional character of the Langlands dual group of \( G \). The corresponding dominant weight is \( b \) and \( \chi \) is treated as the argument. See Theorem 5.4 from [CS] and [Shi] for the precise statements.
The fact that $\mathcal{W}_\chi(\varpi^b)$ vanishes for $b \not\in P^+_r$ is the key here. It provides the boundary condition sufficient to identify the Whittaker functions with the characters (without calculations). Cf. Section 1.3.3, case (2). A counterpart of this property in the theory of real and complex Whittaker functions is a certain decay condition. See [Ch8] for the corresponding fact for the $q$–Whittaker functions.

Let us demonstrate the mechanism of this vanishing condition in the case of $GL_2(K)$. Using the first relation from the definition of $\mathcal{W} = \mathcal{W}_\chi$, $\psi(\varpi^{-1})\mathcal{W}(\begin{pmatrix} \varpi^n & 0 \\ 0 & \varpi^{n+1} \end{pmatrix}) = \mathcal{W}(\begin{pmatrix} 1 & \varpi^{-1} \\ 0 & 1 \end{pmatrix}) (\begin{pmatrix} \varpi^n & 0 \\ 0 & \varpi^{n+1} \end{pmatrix})$

$= \mathcal{W}(\begin{pmatrix} \varpi^n & \varpi^n \\ 0 & \varpi^{n+1} \end{pmatrix}) = \mathcal{W}(\begin{pmatrix} \varpi^n & \varpi^n \\ 0 & \varpi^{n+1} \end{pmatrix}) (\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix})$

$= \mathcal{W}(\begin{pmatrix} \varpi^n & 0 \\ 0 & \varpi^{n+1} \end{pmatrix}) = 0$ due to $\psi(\varpi^{-1}) \neq 1$.

At level of formulas, $\mathcal{W}_\chi(\varpi^b)$ is a limit $t \to \infty$ of (1.27); see Section (1.3.3) in the $A_1$–case. However, sending $t$, the cardinality of the residue filed $k$, to $\infty$ makes absolutely no sense in the $p$–adic theory. We need to go to the $q,t$–setting, establish the $q,t$–generalization of the Shintani -Casselman -Shalika formulas for the global $q,t$–spherical functions and then consider the Whittaker limit; see paper [Ch8].

2. Double affine generalizations

2.1. Double affine Hecke algebra. We continue to use the notations from Section 1.1. Let $\widehat{P} = \{\hat{a} = [a, j] \mid a \in P, j \in \mathbb{Z}\} \subset \mathbb{R}^n \times \mathbb{R}$ be the affine weight lattice. Correspondingly, let $X_{[a,j]} \overset{\text{def}}{=} X_a q^j$ for pairwise commutative $X_a \ (X_{a+b} = X_a X_b)$ and a parameter $q$ (later, a nonzero number). Setting $X_j = X_\omega j$ for $j = 1, \ldots, n$ (they are algebraically independent),

$X_a = \prod_{j=1}^{n} X_{l_j}^j$, where $l_j = (a, \alpha_j^\vee)$ due to $a = \sum_{j=1}^{n} l_j \omega_j$.

Recall the definition of the action of the extended affine Weyl group $\widehat{W} = W \ltimes P^\vee$ in $\mathbb{R}^{n+1}$:

$b[z, \xi] = [z, \xi - (b, z)] \ (b \in P^\vee), \ w[z, \xi] = [w(z), \xi] \ (w \in W)$.

Accordingly, we set $\hat{w}(X_{a}) \overset{\text{def}}{=} X_{\hat{w}(a)}$.

It is compatible with the standard affine action of $\widehat{W} \ni \hat{w}$ in $\mathbb{R}^n \ni x$ via the translations. This action is defined as $wb(x) = w(x + b)$ for
$w \in W, b \in P^\vee$; it reads as $\hat{w}(f)(x) = f(\hat{w}^{-1}(x))$ in the space of functions of $x$ (notice the sign). It will be convenient to use $x$ instead of $X$ in many formulas:

$$\hat{w}(X_a) = q^{(w^{-1}x-b,a)} = q^{(x,w(a)-(b,a))} = X_{[w(a),-(b,a)]} = X_{\hat{w}(a)}. \tag{2.1}$$

The double affine Hecke algebra (DAHA), denoted by $\mathcal{H}$, is defined over the ring of constants $\mathbb{C}_{q,t}[q^{\pm 1/m}, t^{\pm 1/2}]$ for $m \in \mathbb{Z}_+$. In this paper, we will mainly consider it over the field $\mathbb{C}_{q,t}$. This algebra is generated by the affine Hecke algebra $\mathcal{H} = \langle T_i, i = 0, \ldots, n, \Pi \rangle$ defined above and pairwise commutative elements $\{X_a, a \in P\}$ subject to the following cross-relations:

$$T_iX_aT_i = X_aX_{\alpha_i}^{-1} \text{ if } (a, \alpha_i) = 1, \tag{2.2}$$
$$T_iX_a = X_aT_i \text{ if } (a, \alpha_i) = 0,$$
$$\pi_rX_b\pi_r^{-1} = X_{\pi_r(b)},$$

where, recall, $r \in O$ for the orbit $O$ of $\alpha_0$ in $Dyn^{aff}$. See (1.2).

Recall that $Y_b$ for $b \in P^\vee$ from (1.15) satisfy the dual cross-relations:

$$T_iY_bT_i = Y_bY_{\alpha_i}^{-1}, \text{ if } (b, \alpha_i) = 1, \tag{2.2}$$
$$T_iY_b = Y_bT_i, \text{ if } (b, \alpha_i) = 0.$$

Using these elements, $\mathcal{H} = \langle X_a (a \in P), Y_b (b \in P^\vee), T_1, \ldots, T_n \rangle$.

2.1.1. PBW Theorem. An important fact is the PBW Theorem (actually, there are 6 of them depending on the ordering of $X, T, Y$):

**Theorem 2.1** (PBW for DAHA). Every element in $\mathcal{H}$ can be uniquely written in the form

$$\sum_{a, w, b} C_{a, w, b} X_aT_wY_b \text{ for } C_{a, w, b} \in \mathbb{C}_{q,t}, \ a \in P, \ w \in W, \ b \in P^\vee. \tag{2.3}$$

The theorem readily results in the definition of the polynomial representation of $\mathcal{H}$ in $\mathcal{X} \overset{\text{def}}{=} \mathbb{C}_{q,t}[X_b]$, where, recall, $\mathbb{C}_{q,t}$ is the field of rational functions in terms of $q^{1/m}, t^{1/2}$ (actually the ring $\mathbb{Z}[q^{\pm 1/m}, t^{\pm 1/2}]$ is sufficient). As a linear space, $\mathcal{X}$ is generated by $\{X_b | b \in P\}$. Using Theorem 2.1, we can identify $\mathcal{X}$ with the induced representation $\text{Ind}_{\mathcal{H}} \mathbb{C}_+$, where $\mathbb{C}_+$ is the one-dimensional module of $\mathcal{H}$ such that $T_{\hat{w}} \mapsto t^{\langle \hat{w} \rangle}/2$. 

\[ \square \]
The generators $X_b$ act by multiplication; $T_i(i \geq 0)$ and $\pi_r(r \in O^*)$ act as follows:

$$\pi_r \mapsto \pi_r, \quad T_i \mapsto t^{1/2}s_i + \frac{t^{1/2} - t^{-1/2}}{X_{\alpha_i} - 1}(s_i - 1).$$

Here, for instance, $s_0(X_b) = X_bX_{\theta}^{-\langle b,\theta \rangle}q^{\langle b,\theta \rangle}$.

**Comment.** If one begins with formulas (2.4), then the DAHA relations for these operators are not difficult to check directly. This approach gives the PBW Theorem for $H\mathcal{H}$ (the polynomial representation is faithful if $q$ is not a root of unity). In the affine case, the deduction of the PBW Theorem from the (non-affine) formulas (2.4), checked directly, is actually due to Lusztig (in one of his first papers on AHA). Kato interpreted these formulas as those in $\text{Ind}_{\mathcal{H}}^{\mathcal{H}\mathcal{H}}\mathbb{C}_+^+$ for nonaffine $\mathcal{H}$ and the plus-representation $\mathbb{C}_+$ (but then you need to use the PBW Theorem).

In the DAHA case, the best way to obtain the PBW Theorem is by constructing $X$ and checking that it is faithful. Formally, there is no problem to order $X,Y,T$ as in (2.3) using the DAHA relations, but the uniqueness of such expansions must be proved; the polynomial representation provides the simplest way.

### 2.1.2. The $\mu$-functions.

The major functions in the theory of $\mathcal{X}$ are:

$$\mu(X;q,t) = \prod_{\beta > 0} \frac{1 - X_{\alpha}}{1 - tX_{\beta}}, \quad \tilde{\mu}(X;q,t) = \prod_{\beta > 0} \frac{1 - t^{-1}X_{\alpha}}{1 - X_{\beta}}. \quad (2.5)$$

Following Section 1.1.3, given $\hat{w} \in \hat{W}$, the subset

$$\Lambda(\hat{w}) \overset{\text{def}}{=} \tilde{R}_+ \cap \hat{w}^{-1}(\tilde{R}_-) = \{\tilde{\alpha} > 0 \mid \hat{w}(\tilde{\alpha}) < 0\} = \{-\hat{w}(\tilde{\alpha}) > 0 \mid \tilde{\alpha} > 0\} \quad (2.6)$$

consists of $l(\hat{w})$ positive roots. The following are the key relations for the functions $\mu, \tilde{\mu}$:

$$\frac{\hat{w}^{-1}(\mu)}{\mu} = \frac{\hat{w}^{-1}(\tilde{\mu})}{\tilde{\mu}} = \prod_{\alpha \in \Lambda(\hat{w})} \frac{1 - t^{-1}X_{\alpha}^{-1}}{1 - X_{\alpha}} \cdot \frac{1 - X_{\alpha}}{1 - t^{-1}X_{\alpha}}. \quad (2.7)$$

We see that $\mu/\tilde{\mu}$ is (formally) a $\hat{W}$-invariant function. Note that both functions, $\mu$ and $\tilde{\mu}$, are invariant under the action of $\Pi = \{\pi_r, r \in O\}$.
We will need the formula for the constant term $c_t(t)$ of $\mu$ (the coefficient of $X^0$):

\begin{equation}
  c_t(t) = \prod_{\alpha \in R_+} \prod_{i=1}^\infty \frac{(1 - t^{(\alpha,\rho)})q^t}{(1 - t^{(\alpha,\rho)+1}q^t)(1 - t^{(\alpha,\rho)-1}q^t)}.
\end{equation}

It will be treated as an element in $\mathbb{C}[t][[q]]$; we will use it mainly for $t^{-1}$ instead of $t$.

2.2. Affine symmetrizers.

2.2.1. The hat-symmetrizers. Let us define the infinite counterpart of the $P$–symmetrizer as follows:

\begin{equation}
  \hat{\mathcal{P}}_+ = \sum_{\hat{\omega} \in \hat{W}} t^{-l(\hat{\omega})/2} T^{-1}_{\hat{\omega}} / \hat{P}(t^{-1}) \quad \text{for} \quad \hat{P}(t^{-1}) = \sum_{\hat{\omega} \in \hat{W}} t^{-l(\hat{\omega})},
\end{equation}

the latter is the affine Poincaré series evaluated at $t^{-1}$. Ignoring the exact projection normalization, we set $\hat{\mathcal{P}}_+ \overset{\text{def}}{=} \sum_{\hat{\omega} \in \hat{W}} t^{-l(\hat{\omega})/2} T^{-1}_{\hat{\omega}}$. Similarly,

\begin{equation}
  \hat{\mathcal{S}}_+ \overset{\text{def}}{=} \sum_{\hat{\omega} \in \hat{W}} \hat{\omega}.
\end{equation}

Following Theorem 1.7, let us establish that $\hat{\mathcal{P}}_+$ and $\hat{\mathcal{S}}_+ \circ \hat{\mu}$ are proportional to each other provided the existence. All constructions below can be extended to the anti-symmetrizers (generally, we need an arbitrary starting character of the affine Hecke algebra in DAHA). However, we will stick to the “plus-case”.

Let us move all $\hat{\omega} \in \hat{W}$ in the series for $\hat{\mathcal{S}}_+ \circ \hat{\mu}$ to the right and expand the coefficients in terms of $X_{\alpha_i}$ for $i = 0, \ldots, n$. Note that such expansions will contain only non-negative powers of $q$. Similarly, let us use replace $t^{-l(\hat{\omega})/2} T^{-1}_{\hat{\omega}}$ in $\hat{\mathcal{P}}_+$ by the corresponding formulas for their action in the polynomial representation. Then we move all $\hat{\omega}$ to the right. The resulting coefficients will be infinite sums of the formal series in terms of $X_{\alpha_i}$, $i \geq 0$.

2.2.2. The kernel and the image.

**Theorem 2.2.** (i) The coefficients of $\hat{\omega}$ in the above representations of $\hat{\mathcal{S}}_+ \circ \hat{\mu}$ and $\hat{\mathcal{P}}_+$ will contain only non-positive powers of $t$. They are well defined as formal series in terms of $X_{\alpha_i}$ for $i \geq 0$ and $t^{-1}$. Moreover, provided that $|q| < 1$ and $|t| > 1$, the coefficients of individual $X_a$ ($a \in Q \subset P$) will converge as series in terms of $q, t^{-1}$.  

(ii) Letting $A = \hat{P}'_+$ or $A = \hat{P}'_+ \circ \tilde{\mu}$, the following annihilation properties hold:

\[
(\hat{\omega} - 1)A = 0 = (T_{\hat{\omega}} - t^{\frac{1}{2}})A
\]

(2.10)

Here $\{T_{\hat{\omega}}\}$ from $\hat{P}'_+$ are expressed via $\{\hat{\omega}\}$ using (2.4). Then expressions from (2.10) must be transformed in the same way as above with all elements $\hat{\omega}$ moved to the right, treating the resulting coefficients of $\hat{\omega}$ as series from $\mathbb{Z}[[t^{-1/2}, X_{\alpha_i}, i \geq 0]]$.

**Proof.** To check (i) for $\hat{P}'_+ \circ \tilde{\mu}$, let us divide it by $\tilde{\mu}$ on the left.

Then, using (2.7),

\[
\tilde{\mu}^{-1} \circ \hat{P}'_+ \circ \tilde{\mu} = \sum_{\hat{\omega} \in \hat{\omega}} \prod_{\tilde{\alpha} \in \Lambda(\hat{\omega})} \frac{t^{-1} - X_{\tilde{\alpha}}}{1 - t^{-1}X_{\tilde{\alpha}}} \circ \hat{\omega}^{-1},
\]

which can be readily expanded in terms of $t^{-1}$. Multiplying (2.11) by the expansion of $\tilde{\mu}$ as in (i), we obtain the required.

As for $t^{-\frac{1}{2}}T_{\hat{\omega}}^{-1}$, only the non-negative powers of $t^{-1}$ appear in the expression for $t^{-\frac{1}{2}}T_i^{-1}$ from (2.4). Indeed:

\[
t^{-\frac{1}{2}}T_i^{-1} = t^{-\frac{1}{2}}(t^{-\frac{1}{2}}s_i + \frac{t^{-\frac{1}{2}} - t^{\frac{1}{2}}}{X_{\alpha_i}^{-1} - 1}(s_i - 1))
\]

\[
= t^{-1}s_i + \frac{(t^{-1} - 1)X_{\alpha_i}(s_i - 1)}{1 - X_{\alpha_i}}.
\]

The $\hat{\omega}$–coefficients of resulting expansions (upon all transformation) are well defined as elements from $\mathbb{Z}[[t^{-1/2}, X_{\alpha_i}, i \geq 0]]$. We will omit the justification. The operators $\hat{P}'_+ \circ \tilde{\mu}$ and $\hat{P}'_+$ will be mainly needed in concrete analytic spaces and these coefficients will be treated as (meromorphic) functions.

Let $\iota$ be the involution (not an anti-involution) of $\mathcal{H}$ (or in a proper localization of $\mathcal{H}$) given by

\[
\iota : s_i \mapsto -s_i \ (i \geq 0), \ \pi_r \mapsto \pi_r, \ X_a \mapsto X_a, \ q \mapsto q, \ t^{1/2} \mapsto -t^{-1/2}.
\]

Then we have the following two lemmas extending the non-affine claims used for verifying the Macdonald formula.

**Lemma 2.3.** One has $\mu T_i \mu^{-1} = T_i$, for $i = 0, \ldots, n$. □

**Lemma 2.4.** For $i \geq 0$,

\[
t^{1/2}T_i + 1 = (s_i + 1) \cdot F_i \text{ for a rational function } F_i,
\]

\[
t^{1/2}T_i + 1 = G_i \cdot (s_i + 1) \text{ for a rational function } G_i.
\]

They are sufficient to establish (ii).
2.2.3. **Employing the E-polynomials.** From now on, we will frequently represent \( t \) in the form \( t = q^k \). The Macdonald polynomials \( E_a, a \in P \), can be introduced by the relations:

\[
Y_b^{-1}(E_a) = q^{(b,a_\sharp)}E_a, \quad b \in P^\vee, \quad a_\sharp = a - ku_a^{-1}(\rho),
\]

which fix them uniquely up to proportionality for generic \( k \). Here \( u_a \) is the element of minimal possible length in \( W \) such that \( u_a(a) \in P_- \); we set:

\[
a_\pm \overset{\text{def}}{=} u_a(a) \in P_- , \quad \pi_a \overset{\text{def}}{=} au_a^{-1}.
\]

Note that \( 0_\sharp = -k\rho \); more generally, \( u_a = \text{id} \) for \( a \in P_- \) and \( Y_b^{-1}(E_a) = q^{(b,a-k\rho)}E_a \) in this case for any \( b \in P^\vee \). Concerning \( \pi_a \), the following is the key property: \( l(\pi_aw) = l(\pi_a) + l(w) \) for an arbitrary \( w \in W \).

The standard normalization condition is \( E_a = X_a + (\text{lower terms}) \); together with (2.12), it fixes the \( E \)-polynomials uniquely. See books [Ma4, Ch1].

**Comment.** These polynomials were introduced by Heckman and Opdam in the differential setting, then by Macdonald for \( t = q^k \) for integers \( k \) and then in [Ch2] in complete generality (in the reduced case). They are orthogonal with respect to the \( \mu \)-measure (see below).

We will not discuss here the \( C^\vee C \)-theory. See [OS] for some historic remarks and references. The symmetric Macdonald polynomials for the classical root systems were defined (and used) for the first time by Kevin Kadell.

Among quite a few properties of the \( E \)-polynomials let us mention the nonsymmetric Macdonald conjectures, namely, the norm-formula, the duality-evaluation formula and the Pieri rules. The modern approach is entirely conceptual (see [Ch1] and [Ch6]). We deduce these properties from the self-duality of DAHA, practically, without calculations.

In a sense, the duality claim is the first in the chain of the properties mentioned above and the constant term formula is the last. To be exact, the Pieri rules do not belong to the list of Macdonald’s conjectures, but they are the key to connect the duality with the evaluation and norm formulas (and in other applications too). We note that their proof was given in [Ch2] via the reduction to the roots of unity, as well as in the symmetric setting.

The symmetric (usual) Macdonald conjectures can be deduced from the nonsymmetric ones or can be obtained directly from the DAHA theory upon the symmetrization. There is one feature of the nonsymmetric theory which has no symmetric counterpart, the technique of
intertwiners. It simplifies dealing with the $E$–polynomials significantly vs. the theory of the $P$–polynomials.

We note that [Ch1] and other works of the first author are mainly written for the twisted affinization $\tilde{R}'$ (in the reduced case). A natural notation is $H(\tilde{R}; \tilde{R}')$, which means that the $X$–generators and $Y$–generators are labeled by same lattice $P$. Then the $H$ from this paper must be denoted $H(\tilde{R}; \tilde{R}')$.

The technique of intertwiners can be transferred to $H(\tilde{R}; \tilde{R}')$ (the algebra of this paper). The norm and evaluation formulas for $\tilde{R}'$ hold for $H(\tilde{R}; \tilde{R}')$ upon natural modifications at level of the resulting formulas. For instance, the evaluation formula for $E_a(t^{-\rho'})$ can be obtained from the one from [Ch1] or from the Main Theorem of [Ch2] (formula (5.4)) by the following transformations:

a) adding check to $\rho$,  
b) replacing $q^\alpha$ by $q$ and  
c) setting $t_\alpha = q^{k\alpha}$.

Explicitly, for $b \in P$,

$$E_b(t^{-\rho'}) = t^{(\rho'^{\vee}, b)} \prod_{[\alpha, j] \in \Lambda'(\pi_b)} \left( \frac{1 - q^j t_{1+\rho'^{\vee}, \alpha}}{1 - q^j t_{\rho'^{\vee}, \alpha}} \right), \tag{2.14}$$

$$\Lambda'(\pi_b) = \{ [\alpha, j] | [-\alpha, \nu_{\alpha} j] \in \Lambda(\pi_b) \} \text{ for } \pi_b \overset{\text{def}}{=} bu_b^{-1},$$

where we use the elements $u_b, \pi_b$ from (2.12),(2.13). The same transformation must be done with the norm-formula (5.5) from [Ch2].

Comment. We note that DAHA of untwisted type $H(\tilde{R}; \tilde{R}')$ are expected to satisfy the quantum Langlands duality (see [Ch6]). Trying to help the readers interested in this setting, let us discuss briefly the changes with the key DAHA–automorphisms from [Ch1] needed in the untwisted case. The $\sigma$ from [Ch1] (coinciding with $\omega^{-1}$ from [Ch2]) maps now from $H(\tilde{R}; \tilde{R}')$ to $H(\tilde{R}', \tilde{R})$. The automorphism $\tau_+$ acts in the former DAHA, $\tau_-$ in the latter one. One has:

$$\sigma \tau_+^{-1} = \tau_- \sigma, \; \sigma \tau_+ = \tau_- \sigma.$$  

See (3.2) below (for $A_1$). There are unsettled questions with the difference Mehta-Macdonald formulas from [Ch4] in the untwisted case; they will be partially addressed when discussing the affine Hall functions of level one.

2.2.4. The convergence at level zero. Let us begin with the remark that the formula from [Ma3] we used above was interpreted in [Ch3] as the Jackson integration variant of the constant term conjecture. It was extended there to arbitrary $E$–polynomials as a norm-formula in the Jackson setting. The relation to the approach we use here is direct;
the definition of Jackson integral of $f(X)$ from [Ch3] is nothing but the specialization
\[ \hat{\mathcal{P}}'_+((\bar{\mu}f(X))[X \mapsto \xi]) \text{ for } \xi \in \mathbb{C}^n; \]
the vector $\xi$ (arbitrary) is called the origin. The following theorem is a particular case of the Jackson norm-formulas from [Ch3].

**Theorem 2.5.** For $|q| < 1$, $t = q^k$ and $a \in P$ such that $E_{a'}$ are well defined for all $a' \in W(a)$, the sums $\hat{\mathcal{P}}'_+((\bar{\mu}E_{a'})$ absolutely converge if and only if $\Re(2k\rho + a_+, \omega_i) < 0$ for all $i = 1, \ldots, n$. Here \( \{a_-\} = W(a) \cap P_-, a_+ = w_0(a_-) \) for the element $w_0$ of maximal length in $W$, $\Re$ denotes the real part. Under this condition, $\hat{\mathcal{P}}'_+((\bar{\mu}E_{a'}) = 0$ for all $a'$ as $a \neq 0$.

E.g., the (absolute) convergence range for $a = \rho = \alpha_1 + \alpha_2$ in the case of $A_2$ is $\{\Re k > -1/2\}$; it is $\{\Re k > -1/3\}$ for $a = \omega_1 = \omega_1^\vee = \frac{2\alpha_1 + \alpha_2}{3}$.

We continue to assume that $k$ is generic (we will need it to employ the $E$–polynomials). Considering generic $k$ is essentially sufficient for the convergence matters. Indeed, if we know the inequalities for $\Re k$ providing the convergence in a given finite dimensional subspace of $\mathcal{X}$ for all but finitely many special values (satisfying these inequalities), then the convergence will hold automatically for these special values. It can be better at such values, but no worse than at generic $k$ in this domain, which is sufficient in what will follow.

**Theorem 2.6.** The sum $\hat{\mathcal{P}}'_+(E_{a'}) = \sum_{\hat{\omega} \in \hat{W}} t^{-i(\hat{\omega})/2}T_{\hat{\omega}}^{-1}(E_{a'})$ absolutely converges for any $a' \in W(a)$ if and only the following its sub-sum converges absolutely: $\sum_{b \in P'_k} t^{-(\rho,b)}Y_b^{-1}(E_{a_-})$. Using (2.12), it readily results in the same condition as from the previous theorem, namely, $\Re(2k\rho + a_+, \omega_i) < 0$ for all $i = 1, \ldots, n$. Provided the convergence,

\[ (2.15) \quad \hat{\mathcal{P}}'_+ = ct(t^{-1})\hat{\mathcal{P}}'_+ \circ \bar{\mu} \quad \text{acting in } \mathcal{X}, \]

where $ct(t^{-1})$ is the constant term of $\mu(X; q, t^{-1})$:

\[ ct(t^{-1}) = \prod_{\alpha \in \mathbb{R}^+} \prod_{i=1}^\infty \frac{(1 - t^{-(\alpha, \rho')})q^i}{(1 - t^{-(\alpha, \rho') - 1}q^i)(1 - t^{-(\alpha, \rho') + 1}q^i)} \in \mathbb{C}[t^{-1}][[q]]. \]

**Proof.** Let us begin with establishing the proportionality from (2.15) assuming the convergence. Copying the affine case, $\hat{\mathcal{P}}'_+ \circ \bar{\mu}^{-1}$ is divisible by $(t^{1/2}T_i + 1)$ on the left and by $(i^{1/2}T_i + 1)$ on the right. Hence it is divisible by $(s_i + 1)$ on the left and on the right. Thus, it suffices
to justify that this operator must be in the form
\[ \tilde{P}^t \circ \tilde{\mu}^{-1} = G(X) \cdot \tilde{P}^t = G(X) \sum_{\tilde{w} \in \tilde{W}} \tilde{w} \]
for a certain \( \tilde{W} \)-invariant function \( G(X) \). Using [Ma3], \( G = ct(t^{-1}) \).

Another, more direct, justification is by establishing that
\[ \tilde{P}^t'(E_a) = 0 \]
for any \( a \in P \); cf. Theorem 2.5.

Of course, the operator \( \tilde{S}^t + \tilde{\mu}^{-1} \) diverges in (the whole) \( X \), so we must apply this argument as follows. Given \( N \in C_{<N} \), formulas (2.10) guarantee that the images and the kernels of \( \tilde{P}^t \) and \( \tilde{S}^t + \tilde{\mu} \) coincide upon acting in the linear spaces \( V_N = \oplus_{\rho, a^+ < N} C_{X_a} \), provided that \( \Re k \) is sufficiently large negative (depending on \( N \)). Thus these operators are proportional in every \( V_N \) and the coefficient of proportionality (a constant) does not depend on \( N \).

The convergence analysis for \( \tilde{P}^t \) is somewhat different from that for \( \tilde{S}^t \). First, it suffices to assume that \( a \in P_{-} \), using the standard relations between the polynomials \( E_{a^+} \) for \( a^+ \) from in the same \( \tilde{W} \)-orbit \( W(a) \). Second, we observe that the combination of \( a \in P_{-} \) with \( Y_{-1}^b \) for \( b \in P_{+}^\vee \) is the worst possible as far as the convergence is concerned. Thus, we need to analyze
\[ \sum_{b \in P_{+}^\vee} t^{-(\rho, b)} Y_{-1}^b(E_a) = \sum_{b \in P_+} q^{(b, a^+ - 2k\rho)} E_a; \]
it converges absolutely if and only if \( \Re(2k\rho + a^+) \in \mathbb{R}_{>0}Q_{+} \). A formalization of this argument is based on the following theorem.

2.2.5. The \( Y \)-formulas for \( \tilde{P} \)-hat. Recall that \( \tilde{P}^t \) is the \( t^{-1} \)-symmetrizer without the exact projector normalization, i.e., without the division by \( \tilde{P}(t^{-1}) \). By \( P(t) \), we denote the non-affine Poincaré polynomial. For a subset \( I \subset \{1, 2, \ldots, n\} \), the Poincaré polynomial of the root subsystem \( R_I \subset R \) generated by the simple roots \( \{\alpha_i \mid i \in I\} \) will be denoted by \( P_I(t) \). It is 1 if \( I = \emptyset \).

Theorem 2.7. The symmetrizer \( \tilde{P}^t \) can be presented as the following summation over all subsets \( I \subset \{1, 2, \ldots, n\} \) including the empty set and \( I = \{1, \ldots, n\} \),
\[ (2.16) \quad \tilde{P}^t_+ = P(t^{-1}) \sum_I P_I(t) \prod_{i \in I} \frac{t^{-(\omega_i, \rho)} Y_{-1}^{\omega_i}}{1 - t^{-(\omega_i, \rho)} Y_{-1}^{\omega_i}} \mathcal{P}_+, \]
where the rational expressions in the products are supposed to be expanded in terms of \( t^{-1} \).
Proof. We employ the key property of the elements ρb from (2.13), namely, the equality l(πbw) = l(πb) + l(w) for any w ∈ W. Since πb = bn−1, one has πbw = u−1b−1w. The element u = un can be arbitrary such that its length is minimal possible for a given b = u−1(b−), i.e., minimal in the coset Z(b−)u for the centralizer Z(b−) of b− in W. It results in (2.16).

Note that formula (2.16) gives a rational expression for the affine Poincaré series \( \hat{P}(t^{-1}) = \hat{\mathcal{P}}'_+(1) \). Provided that \( \hat{P}(t^{-1}) \neq 0 \), the theorem gives a universal map onto the space of \( \hat{Y} \)-spherical vectors

\[
\{ v \mid T_{\hat{w}}(v) = t^{l(\hat{w})/2}v \text{ for } \hat{w} \in \hat{W} \},
\]

which is applicable to \( \mathcal{H} \)-modules that are unions of finite-dimensional \( \hat{Y} \)-invariant subspaces. There is a natural generalization of Theorem 3.4 to arbitrary one-dimensional characters of \( \mathcal{H}_Y \); the case of the affine antisy-mmetrizer, corresponding to \{T_{\hat{w}} \mapsto (-t^{-1/2})^{l(\hat{w})}\}, is important.

The right-hand side of formula (2.16) is a rational function and, generally, does not require using \( t^{-1} \)-expansions. However, one must ensure that the denominators in (2.16) are nonzero. For instance, this formula can be used in the (whole) polynomial representation \( \mathcal{X} \) for \( A_1 \) for any \( q, t \) unless \( t^2 \in q^{1-Z_+} \) and for \( A_2 \) unless \( t^6 \in q^{1-Z_+} \) or \( t^3 \in q^{1+Z_+} \). It is under the assumption that \( q \) is not a root of unity and \( \hat{P}(t^{-1}) \neq 0 \). At roots of unity, it can be applied only in certain quotients of \( \mathcal{X} \).

See Theorem 3.4 for this formula in the case of \( A_1 \). For \( A_2 \), it reads as follows:

\[
\hat{\mathcal{P}}' = P(t)P(t^{-1})\mathcal{P} + \left( \frac{t^{-2}Y_{\omega_1+\omega_2}^{-1}}{(1-t^{-1}Y_{\omega_1}^{-1})(1-t^{-1}Y_{\omega_2}^{-1})} \right) \mathcal{P} + \frac{1}{1+t} \left( \frac{t^{-1}Y_{\omega_1}^{-1}}{1-t^{-1}Y_{\omega_1}^{-1}} + \frac{t^{-1}Y_{\omega_2}^{-1}}{1-t^{-1}Y_{\omega_2}^{-1}} \right) \mathcal{P} + \frac{1}{(1+t)(1+t+t^2)} \hat{\mathcal{P}}'.
\]

Here \( \rho = \alpha_1 + \alpha_2 \) and \( (\rho, \omega_i) = 1 \) for \( i = 1, 2 \); \( P(t) = (1+t)(1+t+t^2) \). Applying this formula to \( 1 \in \mathcal{X} \) and using that \( t^{-1}Y_{\omega_i}^{-1}(1) = t^{-2} \), one can readily calculate the resulting series. It is the \( t^{-1} \)-expansion of \( \hat{P}(t^{-1}) = 3(1-t^{-3})/(1-t^{-1})^3 \).

We note that the expression from the theorem treated as an element in the localization of affine Hecke subalgebra \( \mathcal{H}_Y = \langle T_{\hat{w}}, \hat{w} \in \hat{W} \rangle \) is zero. It follows from the fact that no affine symmetrizers exist in \( \mathcal{H}_Y \) or its localizations unless completions are allowed. Similarly, formula (2.16) becomes zero when applied in \( \mathcal{H} \)-modules that are unions of finite-dimensional \( \mathcal{H}_Y \)-modules containing no \( \hat{Y} \)-spherical vectors. It results in the following a presentation of \( \hat{\mathcal{P}}' \). See Section 3.3.2 and
Theorem 3.5 for explicit rank one calculations. We assume that the affine Poincaré series $\hat{P}(t^{-1})$ is invertible.

**Theorem 2.8.** Given a set of representatives $b = \{b_1, \ldots, b^p\} \subset P^\vee_+$ for the group $\Pi = P^\vee/Q^\vee$ (of cardinality $p$), let

\begin{equation}
\tilde{\Sigma}_\hat{P} = \prod_{\alpha \in R_+} \frac{(1 - tY_{\alpha}^{-1})}{(1 - Y_{\alpha}^{-1})} \sum_{j=1}^{p} \frac{t^{-(b_j, \rho)}Y_{b_j}}{\prod_{i=1}^{n}(1 - tY_{\alpha_i}^{-1})},
\end{equation}

\begin{equation}
\Sigma_\hat{P} = \prod_{\alpha \neq \alpha > 0} \frac{(1 - t^{1-(\alpha, \rho)})}{(1 - \alpha, \rho)} \sum_{j=1}^{p} t^{-(b_j, \rho)}Y_{b_j}.
\end{equation}

In any representations of $\mathcal{H}_Y$ where $\hat{P}_+'$ is well defined (or coefficient-wise provided their existence),

\begin{equation}
\hat{P}_+ = \lim_{b \to \infty} \tilde{\Sigma}_\hat{P} = \lim_{b \to \infty} \Sigma_\hat{P} \text{ for } \hat{P}_+ = P(t^{-1}),
\end{equation}

assuming that $(b_j, \alpha_i) \to \infty$ for all $1 \leq j \leq p, i > 0$, and that

\begin{equation}
\lim_{b \to \infty} t^{-(b_j, \rho)}Y_{a(b)} \hat{P}_+ = \left\{ \begin{array}{ll}
\text{exists for all } w \in W & \\
\text{equals zero for } w \neq \text{id} & 
\end{array} \right\}.
\end{equation}

As an application, one obtains formula (5.9) from [Ma3]:

\begin{equation}
\hat{P}(t) = \frac{|\Pi|}{(1 - t)^n} \prod_{i=1}^{n} \frac{1 - t^{d_i}}{1 - t^{d_i - 1}}, \text{ where } P(t) = \prod_{i=1}^{n} \frac{(1 - t^{d_i})}{(1 - t)^n}.
\end{equation}

**Sketch of the proof.** Using (2.20), it is not too difficult to see that $\Sigma_\hat{P}$ from (2.19) exists and gives the affine symmetrizer up to proportionality, i.e., satisfies the standard properties upon the multiplication by $T_{\vec{w}}$ ($\vec{w} \in \hat{W}$) on the right and on the left. Cf. Theorem 3.5 below (for $A_1$). It is obvious for $T_{\vec{w}} = Y_a(a \in P_+)$; apply it, the $t^{-(a, \rho)}Y_a$–invariance, when managing $T_i(i > 0)$. The calculation of the coefficient of proportionality is straightforward (see below).

As an application, the $\Sigma_\hat{P}$ and $\Sigma_\hat{P}$ coincide in the limit provided the convergence of the latter expression. Indeed, the multiplication/division by $(1 - CY_{\alpha})/(1 - Ct^{(\alpha, \rho)})$ will not change $\Sigma_\hat{b}$ in the limit for a (sufficiently general) constant $C$.

The $\tilde{\Sigma}$–formula can be obtained directly from (2.16); let us outline the procedure. We introduce the truncation $\hat{\Upsilon}_b$ of the $Y$–expression between the two $\hat{P}$ in formula (2.16). Upon the $Y^{-1}$–expansion, only the monomials $Y_a^{-1}$ will subject to $b^j - Q_+ \ni a \in P_+$ for $b^j = a \mod Q$.
will be kept. Let \( b^j = \sum_{i=1}^{n} r^j_i \alpha_i^\vee \); recall that \((b^j, \alpha_i) \to \infty \). The sum

\[
(2.22) \quad \Upsilon_b^i = P(t) \sum_{j=1}^{p} \prod_{i=1}^{n} \frac{(1 - t^{-1} r^j_i (\alpha_i^\vee, \rho) Y^{-r^j_i - 1})}{(1 - t^{-\alpha_i^\vee, \rho} Y^{-1})}
\]

contains all such \( Y_a^{-1} \), but there will be (many) non-dominant terms (with \( a \not\in P_+^{\vee} \)).

We are going to use formulas (1.19) and (1.20):

\[
(2.23) \quad P(t^{-1}) \mathcal{P}_+ = (\sum_{w \in W} w) \circ \tilde{M} \quad \text{for} \quad \tilde{M} \overset{\text{def}}{=} \prod_{\alpha \in R_+} \frac{1 - t^{-1} Y_{\alpha^\vee}^{-1}}{1 - Y_{\alpha^\vee}}.
\]

Due to these formulas combined with the vanishing property from (2.20), the contribution of \( Y_a^{-1} \) in (2.22) with \((a, \rho) \ll (b^j, \rho)\) for \( a \in b^j - Q_+ \) tend to zero in the limit. For instance, the non-dominant terms can be disregarded in \( \Upsilon_b^i \). Similarly, it suffices to consider only \( I = \emptyset \) in Theorem 2.7 upon applying the operator from (2.23) in the limit. Recall that the terms \( Y_a^{-1} \) for the weights \( a \) linearly generated by \( \omega_i^\vee \not\in I \) are treated differently for different \( I \) in this theorem.

Moreover, the numerator in formula (2.22) can be actually reduced to

\[
(1 + (-1)^n \prod_{i=1}^{n} t^{-1-\alpha_i^\vee, \rho} Y_{\alpha_i^\vee}^{-r^j_i - 1}).
\]

Furthermore, as we noticed above,

\[
\mathcal{P}_+ \Upsilon_{\infty}^i \mathcal{P}_+ = 0 \quad \text{for} \quad \Upsilon_{\infty}^i \quad \text{obtained by deleting the terms with} \quad r^j_i
\]

from \( \Upsilon_b^i \) (and treated as a rational \( Y \)-function). It can be readily seen directly from (2.23); use the divisibility of anti-invariant Laurent polynomials by the discriminant.

We conclude that \( \Upsilon_b^i \) can be replaced by

\[
\Upsilon_b^i = P(t) (-1)^n \sum_{j=1}^{p} t^{-(b^j, \rho)} Y_{b^j}^{-1} \prod_{i=1}^{n} \frac{t^{-\alpha_i^\vee, \rho} Y_{\alpha_i^\vee}^{-1}}{(1 - t^{-1} Y_{\alpha_i^\vee})},
\]

and

\[
\tilde{P}_+^i = \lim_{b \to \infty} \sum_{w \in W} w(\Upsilon_b^i M) \mathcal{P}_+.
\]

Once again using the vanishing condition from (2.20), we see that only \( w = w_0 \) really contributes to \( \tilde{P}_+^i \) in the limit. Changing \( b \mapsto -w_0(b) \) in the resulting expression, \( \tilde{P}_+^i \) becomes the limit of

\[
(-1)^n \sum_{j=1}^{p} \prod_{i=1}^{n} \frac{t^{-(b^j + \rho^\vee, \rho)} Y_{b^j + \rho^\vee}}{(1 - t^{-(\alpha_i^\vee, \rho)} Y_{\alpha_i^\vee})} \prod_{\alpha \in R_+} \frac{(1 - t^{-1} Y_{\alpha^\vee})}{(1 - Y_{\alpha^\vee})} P(t) \mathcal{P}_+,
\]

where we use \( \rho^\vee = \sum_{i=1}^{n} \omega_i^\vee \). Rewriting it in terms of \( Y_{\alpha^\vee}^{-1} \), we come to the required relation from (2.19).
Applying (2.19) to 1, one arrives at (2.21). Indeed, $Y_a$ become $t^{(a,\rho)}$ upon this evaluation and $\mathcal{P}(1) = 1$. This formula is due to Matsumoto and Macdonald; see formula (5.9) from [Ma3].

**Comment.** Formula (2.17) for $\sum_b$ coincides with formula (3.22) for $\sum_M$ below in the case of $A_1$. One needs to set there $b^1 = M\omega$, $b^2 = (M - 1)\omega$ for $\omega = \omega_1$. Generally, $b^1$ and $b^2$ can be taken arbitrary (approaching infinity). Note that the expansions of $Y_{b_j}$ in terms of $\hat{w}$ (placed on the right) are disjoint in the following sense. The corresponding sets $\{\hat{w}\}$ do not intersect because the images of $b_j$ in $\Pi$ are different for different $j$. □

**Comment.** In the Kac-Moody limit $t \to \infty$, combining formula (2.18) with the proportionality claim from (2.15), we obtain the presentation of Kac-Moody characters as limits of the (affine) Demazure characters. The latter are directly related to the operators $T_{\hat{w}}^\infty = \lim_{t \to \infty} t^{-l(\hat{w})/2} T_{\hat{w}}$. More precisely, the corresponding Demazure characters are proportional to $q^{-l^2/2} T_{\hat{w}}^\infty (X_{-a} q^{l^2/2})$. Here $a$ are affine $l$–dominant weights, i.e., $a \in P_+$ and $(a, \theta) \leq l$. For $\hat{w} = b \in P_+$ as $b \to \infty$, they approach $\prod_{i=1}^{\infty} \frac{1}{(1-q^a)^{X_\alpha}} \chi_{\alpha}^{(b)}$, see (2.32) below.

Here it is not necessary to stick to the affine dominant weights $a$ of level $l$. One can define the Kac-Moody characters formally for arbitrary $a \in P$ using the Kac-Weyl formula.

For arbitrary $t$, (2.18) can be interpreted as a presentation of the corresponding affine Hall functions from (2.25) as the (inductive) limit of the Demazure $t$–characters. The latter are defined formally in terms of $t^{-l(\hat{w})/2} T_{\hat{w}}$. □

2.2.6. **Coefficient-wise proportionality.** Theorem 2.6 is actually sufficient to claim the coefficient-wise proportionality in (2.15) as formal series. We will outline here an analytic approach; see Theorem 2.17 for the proof based on the DAHA theory.

The following theorem states that the coefficients of the operator $\mathcal{P}'_+$ are meromorphic functions (which is obvious for $\mathcal{P}'_+ \circ \tilde{\mu}$).

**Theorem 2.9.** Let us assume that $|t| > q^{1/h}$ for the Coxeter number $h = (\theta, \rho) + 1$. Expanding $\mathcal{P}'_+ = \sum_{\hat{w} \in \hat{W}} F_{\hat{w}}(X) \hat{w}$, the coefficients $F_{\hat{w}}$ converge absolutely to functions of $X$ analytic when $0 \neq X_\alpha \notin q\mathbb{Z}$ for every $\alpha \in R$. Moreover, $F_{\hat{w}}$ coincide with the corresponding coefficients of $ct(t^{-1}) \mathcal{P}'_+ \circ \tilde{\mu}$; for instance, $F_{id} = ct(t^{-1}) \tilde{\mu}$. □

The proof of the existence of $\{F_{\hat{w}}\}$ is based on the estimates for the coefficients of operators $Y_b$; see Theorem 3.6 below for the case of $A_1$. Theorem 2.17 contains the justification of the proportionality claim.
(based on the representations of $\mathcal{H}$ in the space of delta-functions). Let us outline an approach to the coefficient-wise proportionality utilizing the following analytic modification of Theorem 2.6.

When dealing with the affine symmetrizers analytically, it is convenient to replace $\mathcal{P}$ by the union of Paley–Wiener-type spaces $\mathcal{PW}_M(\mathcal{U})$ of analytic functions in a given $\hat{\mathcal{W}}$–invariant domain $\mathbb{R}^n \subset \mathcal{U} \subset \mathbb{C}^n$. Here $M \in \mathbb{Z}_+$ and the growth condition is as follows:

$$f(x) \in \mathcal{PW}_M(\mathcal{U}) \Rightarrow bw f(x) < C_x(M) q^{-M(b+\rho)}, \quad b \in P^\nu, \; w \in W,$$

for a constant $C_x(M)$ continuously depending on $x \in \mathcal{U}$. For $M = 0$, this space includes $1$ and all $\hat{\mathcal{W}}$–invariant functions analytic in $\mathcal{U}$, for instance, the images of $\hat{\mathcal{P}}_+$ and $\hat{\mathcal{P}}_+ \circ \hat{\mu}$. These two operators act in $\mathcal{PW}_M(\mathcal{U})$ for sufficiently large negative $\Re k$, depending on $M$, and for sufficiently small $\mathcal{U}$ containing $\mathbb{R}^n$.

Provided the convergence, the kernels and images of these operators in $\mathcal{H}$–invariant subspaces of $\cup_{M \geq 0} \mathcal{PW}_M(\mathcal{U})$ coincide and

$$\hat{\mathcal{P}}'_+ = ct(t^{-1})_+ \hat{\mathcal{P}}_+ \circ \hat{\mu} \quad \text{provided the convergence}.$$

The coincidence of the kernels and the images is controlled by Theorem 2.2 (in an analytic variant), which implies the proportionality.

To “extract analytically” and then equate the coefficients of the operators under consideration, we need certain delta-type functions in the space $\mathcal{PW}_0(\mathcal{U})$ for a sufficiently small neighborhood $\mathcal{U}$ of $0 \in \mathbb{R}^n$. Let $\mathcal{A} = \sum_{\hat{w} \in \hat{\mathcal{W}}} F_{\hat{w}}(X) \hat{w}$; this operator is assumed convergent with the coefficients analytic in $\mathcal{U}$ subject to the conditions from (2.10). It suffices to find $\tilde{F}_b \overset{\text{def}}{=} \sum_{w \in W} F_{bw}(X) b \in P^\nu$; expand $\mathcal{A}$ in terms of $bT_w$ for $\hat{w} = bw$ to see it (use that $q, t$ are generic).

Let us begin with recovering the value of the coefficient $\tilde{F}_0$ at $x = 0$. Recall the notation: $X = q^x$, $x_\alpha = (\alpha, x)$. The following probe function from $\mathcal{PW}_0(\mathcal{U})$ can be used:

$$\zeta_N(x) = -\prod_{\alpha \in R_+} \frac{(\exp(N\pi i x_\alpha) - \exp(-N\pi i x_\alpha))^2}{(\exp(N\pi x_\alpha) - \exp(-N\pi x_\alpha))^2},$$

where $N \in \mathbb{N}$, $\nu^2 = -1$. It is of order $1 + O(|x|^2/N)$ near $x = 0$ and of order $O\left(\frac{|x - b|^2 \exp(-CN)}{N}\right)$ for $x \approx b \in P^\nu \setminus 0$ for some constant $C > 0$. Obviously, $\mathcal{A}(\zeta_N)(x = 0) = \tilde{F}_0(x = 0)$, and we recover the value of $\tilde{F}$ at $x = 0$.

Using the function $\sum_w \zeta_N(w(x) - x_0)$ in the same manner we can find the values $\tilde{F}_0(x = x_0)$ for any given $x_0$ in a sufficiently small neighborhood of $x = 0$. It gives the function $\tilde{F}_0$ in $\mathcal{U}$ pointwise in
terms of the action of $A$ in $\mathcal{PW}_0(U)$. Alternatively, recovering $\tilde{F}_0(x)$ for small $x$ can be achieved by tending $N$ to $\infty$ (we will omit the detail).

The same approach can be used for recovering any $\tilde{F}_b$, upon applying the translations by $b \in P^\vee$ to the argument $x$ in the probe function (which fix its numerator).

This is of course based on the existence of $A$ applied to $\zeta_N$ in a neighborhood of $x = 0$. The numerator of $\zeta_N$ is a pseudo-constant, an $\hat{W}$–invariant function. Thus, the rate of convergence depends only on the denominator and the convergence of the operators $\hat{P}^I_+$ and $\hat{P}^I_+ \circ \tilde{\mu}$ applied to $\zeta_N$ is no worse than that for constants (or pseudo-constants). Actually, it is better; it holds for small positive $\Re k$ too (presumably, the inequality $\Re k < 1/h$ is sufficient).

As a matter of fact, we need to know the convergence only for large negative $\Re k$ (for recovering the coefficients), a weaker fact. Indeed, the coefficients of $\hat{P}^I_+$ and $\hat{P}^I_+ \circ \tilde{\mu}$ are meromorphic functions in $k$ (provided the convergence). Thus, if the proportionality of these operators is known for $\Re k \ll 0$, then it holds coefficient-wise. We need Theorem 2.15, extended analytically to the functions (like) $\zeta_N$. The proportionality factor must be $ct(t^{-1})$ automatically. Theorem 2.17 below is an algebraic variant of this approach.

### 2.3. Affine Hall functions.

#### 2.3.1. Main definition. The above considerations were the 0–level case of the general theory of affine Hall functions of arbitrary levels, the subject of this section. We continue to assume that $|q| < 1$.

Expressing $X_a = q^{x_a} = q^{(x,a)}$, let us introduce the $l$–Gaussian as $q^{lx^2/2}$ for $x^2 \overset{\text{def}}{=} \sum_{i=1}^n x_i x_\alpha^{(\vee)}$. In the case of $A_2$, for example, we have $\alpha_1 = \alpha_1^{(\vee)} = 2\omega_1 - \omega_2$, $\alpha_2 = \alpha_2^{(\vee)} = 2\omega_2 - \omega_1$ and

$$\frac{x^2}{2} = \frac{x_1(2x_1 - x_2)}{2} + \frac{x_2(2x_2 - x_1)}{2} = x_1^2 - x_1x_2 + x_2^2.$$ 

One readily checks that

$$\hat{w}(q^{lx^2/2}) = q^{lb^2/2} X_{lb}^{-1} q^{lx^2/2} \text{ for } \hat{w} = bw, b \in P^\vee, w \in W.$$ 

As a matter of fact, these formulas are the defining relations of the Gaussian in what will follow.

To simplify the notations, we set

$$\hat{\mathcal{I}} \overset{\text{def}}{=} \hat{P}^I_+ \circ \tilde{\mu},$$

where $\mathcal{I}$ (fancy $I$) stays for “integration”.
The Hall functions of level \( l > 0 \) are defined as
\[
H_a^{(l)} \overset{\text{def}}{=} \mathcal{J}(X_a q^{lx^2/2}), \quad a \in P, \quad \mathcal{H}_l \overset{\text{def}}{=} \mathcal{J}(\mathcal{X} q^{lx^2/2}).
\] (2.25)

Thanks to the presence of the Gaussian, the absolute convergence is granted for any \( t \) (including \( t = 0 \)). Moreover, \( q^{-lx^2/2} H_a^{(l)} \) are absolute convergent Laurent series in terms of \( X_b \) \((b \in P)\) for all \( x \) and \( t \); we continue to assume that \( |q| < 1 \). Actually, the absolute convergence holds here for any \( l \in \mathbb{C} \) such that \( \Re l > 0 \), but then we will not be able to represent \( q^{-lx^2/2} H_a^{(l)} \) as Laurent series. The singularities in \( x \) can appear too for non-integral \( l \) at non-real poles of \( \tilde{\mu}(q^x) \), i.e., in the set
\[
\{ x \mid (x, \alpha) + j \in 2\pi \log(q) i \{ P \setminus \emptyset \}, \quad [\alpha, j] \in \tilde{R}_+ \},
\] (2.26)
where \( i \) is the imaginary unit. There will be no singularities in a sufficiently small neighborhood of \( \mathcal{C} \subset \mathcal{M} \).

Note that for any \( \tilde{W} \)-invariant function \( f \), called a pseudo-constant,
\[
\tilde{P}'_+(f) = \tilde{P}(t^{-1}) f = c t(t^{-1}) \mathcal{J}(f),
\] (2.27)
where we need to assume that \( \Re k < 0 \) to ensure the convergence. Here \( \tilde{P}(t) \) is the affine Poincaré series.

The coefficient of proportionality is the same as in (2.15), because the action of our operators on any invariant \( f \) is exactly the same as on \( 1 \in \mathcal{X} \). For instance, we can use (2.27) applied to functions from \( \mathcal{H}_l \) assuming that \( \Re k < 0 \).

**Comment.** If the proportionality from (2.15) holds for all \( k \), then we come to the conclusion that \( H_a^{(l)} \) must vanishes identically for all \( a \in P \) at the poles of \( c t(t^{-1}) \). For instance, \( \mathcal{H}_l = \{0\} \) as \( t = q^{1/h} \) for the Coxeter number \( h \). Indeed, the proportionality always holds when both operators are well defined. However, generally, the operator \( \tilde{P}'_+ \) converges only for \( \Re k < 1/h \) unless \( l = 1 \). Thus, the vanishing test at \( 1/h \) (it would be an indication of the proportionality) fails for \( l > 1 \).

The following holds at \( t = q^{1/h} \) and at other zeros of \( H_a^{(l=1)} \) from part (ii) of the next Theorem 2.10 (in the simply-laced case). We claim that for any (integral) \( l > 0 \), the space \( \mathcal{H}_l \) is always smaller than the corresponding Looijenga space (see below) at such \( k \). The justification of this and similar facts is based on diminishing the level due to formula (2.27).

Numerical calculations of the space \( \mathcal{H}_l = \mathcal{J}(\mathcal{X}) \) for \( A_1, A_2, B_2 \) show that this space is really nonzero at \( t = q^{1/h} \), i.e., that, generally, \( \tilde{P}'_+ \) cannot be continued analytically to all positive \( \Re k > 1/h \). The latter inequality seems sharp for \( l > 1 \), namely, the convergence of \( \tilde{P}'_+ \) and (its corollary) the vanishing property \( \mathcal{H}_l(k = 1/h) = \{0\} \) are not
expected to hold for \( l = 1 \pm \varepsilon \) for arbitrarily small \( \varepsilon > 0 \). Only integral \( l \) are considered in this paper, but the definition of the corresponding spaces for any complex \( l \) is straightforward. \( \square \)

2.3.2. **Discussion, some references.** The formula for the affine Satake-type operator \( \hat{\mathcal{I}}' \circ \hat{\mu} \) was known to quite a few specialists. The theory of the operator \( \hat{\mathcal{P}}_+ \) from this paper seems new. Its definition is “affine”, however it becomes really fruitful only when using in DAHA modules.

Both operators, \( \hat{\mathcal{I}} = \hat{\mathcal{I}}' \circ \hat{\mu} \) and \( \hat{\mathcal{P}}_+ \), can be used as affine Satake maps. Since they are proportional whenever the operator \( \hat{\mathcal{P}}_+ \) exists (see Theorem 2.10).

The convergence of the \( \hat{\mathcal{I}} \) for \( l > 0 \) is better and simpler to manage than that of \( \hat{\mathcal{P}}_+ \). However, \( \hat{\mathcal{P}}_+ \) is an exact DAHA-version of the classical Satake isomorphism in the AHA theory and, importantly, does not require any a priori knowledge of the \( \mu \)–function. It provided an alternative way to supply the polynomial representation with an inner product. The following is a certain clarification of its general role in the theory.

Under the limit \( t \to \infty \), the operator \( \hat{\mathcal{I}} \) is directly connected with the Weyl-Kac formula for the Kac-Moody characters; the functions \( ct(t^{-1})H^{(l)}_b \) tend to the corresponding characters for affine dominant weights \( b \). In the approach based on \( \hat{\mathcal{P}}_+ \), Theorem 2.8 generalizes the presentation of the corresponding Kac-Moody character as an inductive limit of the Demazure characters. Thus the proportionality claim can be viewed as an abstract (operator) \( t \)–variant of the link between the Demazure characters and the Kac-Moody characters.

The definition of the Hall functions we use is not new; equivalent definitions of the affine Hall-Littlewood functions were suggested by several authors (not always published); see [Ka, FGT, BK]. Let us also mention earlier Garland’s results.

Independently, the affine Hall functions of level one were calculated in [Ch4] in the context of Jackson integrals (see also [Sto]). Paper [FGT] contains an important interpretation of the affine Hall functions via the Dolbeault cohomology of the affine Grassmannian and related flag varieties. The appearance of the \( ct(t^{-1}) \) in the formulas is interpreted there as the “failure of the Hodge decomposition”. See also Section 12.7 there concerning the level one formulas.

2.3.3. **The proportionality.** Let us begin with the level one case. Then we have a reasonably complete theory from [Ch4] (see also [Ch1]) and
paper [Sto] devoted to the \( C^\vee C \)-case. Let us mention paper [Vi], where the level one case is addressed in the simply-laced case. Theorem 2 there is a special case of Theorem 7.1 from [Ch4] (for simply-laced root systems). The relation of Theorem 2 to the difference Mehta-Macdonald formulas from [Ch4] in the compact case is discussed in [Vi]. The compact case is that based on the constant term inner product (more generally, on the imaginary integration). However, it is the non-compact case, namely the Jackson integration formula from [Ch4] (not mentioned in [Vi]), that is directly connected with the affine Hall functions of level one.

Works [Ch4, Ch1] were written in the self-dual setting, i.e., for the twisted affine root system \( \tilde{R}^\vee \), where the same lattice \( P \) is used in \( \tilde{W} \) and for \( X_a \) (and \( E_a \)). Accordingly, the operator \( T_0 \) changes to the one with \( \alpha_0 = [-\vartheta, 1] \) for the maximal short root \( \vartheta \). Restricting ourselves with the simply-laced case, the results from [Ch4] on the Mehta-Macdonald formulas in the context of Jackson integration can be formulated as follows. Recall, that \( \alpha^\vee = \alpha, \omega^\vee_i = \omega_i \) in this case due to the normalization \( (\alpha, \alpha) = 2 \) for \( \alpha \in R \).

**Theorem 2.10.** Let \( R \) be a simply-laced root system. We set \( \gamma(x) = \frac{|W|^{-1}}{n_{\tilde{W}}} \sum_{w \in \tilde{W}} w(q^{\vartheta/2}) = q^{x/2} \sum_{b \in P} X_b q^{\vartheta/2} \) for the order \( |W| \) of the non-affine Weyl group \( W \). Let \( X_\beta(q^\gamma) = q^{(\beta, a)} \), \( \hat{P}(t^{-1}) \) is from (2.21). The level will be \( l = 1 \).

(i) The series \( \hat{\mathcal{P}}^+_+ \) considered as an operator in \( \mathcal{X} q^{x^2/2} \) converges element-wise for all \( t \in C^* \). The coincidence relation

\[
\hat{\mathcal{P}} = \hat{\mathcal{P}}^+_+ \circ \hat{\mu} = \text{ct}(t^{-1})^{-1} \hat{\mathcal{P}}^+_+
\]

holds for any \( t \neq 0 \) as well; cf. (2.15).

(ii) Assuming that \( E_a \) is well defined,

\[
(2.28) \quad \hat{\mathcal{P}}^+(\hat{\mu} E_a q^{x^2/2}) = \frac{\hat{P}(t^{-1})}{\text{ct}(t^{-1})} \hat{\mathcal{P}}^+(E_a q^{x^2/2})
\]

\[
= E_a(q^{-k\vartheta}) q^{-a^2/2 - k(a_+, \rho)} \prod_{\alpha \in R_+} \prod_{j=0}^\infty \frac{1 - t^{-1-(\rho, \alpha)} q^j}{1 - t^{-(\rho, \alpha)} q^j} \cdot \gamma(x).
\]

(iii) If \( t \) is not a root of unity, then the linear map \( \hat{\mathcal{P}}^+_+ \) is identically zero in \( \mathcal{X} q^{x^2/2} \) if and only if \( t^{m_j} = q^j \) for \( j \in \mathbb{N} \) (for instance, for \( t = q \)). Here \( \{m_1, m_2, \ldots, m_n\} \) are the exponents of \( R \); \( m_i = d_i - 1 \) for the degrees \( \{d_i\} \). The map \( \hat{\mathcal{P}} \) is identically zero on \( \mathcal{X} q^{x^2/2} \) if and only if \( t^{d_j} = q^j \) for \( j \in \mathbb{N} \) and \( j/d_i \not\in \mathbb{N} \) (for instance, it vanishes identically at \( t = q^{1/h} \), where \( h = (\theta, \rho) + 1 \) is the Coxeter number).
Sketch of the proof. We will omit the justification of the convergence of \( \widehat{P}_+ \) for all \( k \in \mathbb{C} \). The proportionality relation formally follows from this claim. Paper [Ch4] contains formula (2.28). To check (iii), use the explicit formula for \( \text{ct}(t^{-1}) \) and the fact that all \( E_a \) are well defined with nonzero \( E_a(q^{-k\varrho}) \) for positive \( \Re k \).

Comment. The levels 0 and 1 are exceptional from the viewpoint of convergence. For \( l = 0 \), the convergence of both, \( \hat{I} \) and \( \hat{P}_+ \), is (naturally) significantly worse than that in the presence of the Gaussian. For \( l = 1 \), \( \hat{P}_+ \) converges much better than for (integral) \( l > 1 \) due to the fact that its image is one-dimensional. Recall that \( \hat{I} \) always converges for \( l > 0 \), but this is not the case with \( \hat{P}_+ \). The Gaussian helps to improve the convergence of the latter operator (say, it adds the points \( |t| = 1 \) to the convergence range), but not too much.

\[ \text{Comment.} \] Let us mention the symmetrizer \( \sum b_w t(l\omega) T_{b_w} / 2 \), with \( t, T \) instead of \( t^{-1}, T^{-1} \). Its convergence range in the space \( \mathcal{X} q^{-lx^2/2} \) is the opposite of the range for \( \hat{P}_+ \) acting in \( \mathcal{X} q^{lx^2/2} \), i.e., with \( \Re k \mapsto -\Re k \).

This symmetrizer corresponds to the theory of imaginary integration. Applying it to \( \mathcal{X} q^{lx^2/2} \) with positive \( \Re l \) is possible provided that \( \Re k > 0 \), but the result will be zero identically.
2.3.4. The Looijenga spaces. For positive integral levels \( l > 0 \), let us introduce the Looijenga space

\[
\mathcal{L}_l = \{ \sum_{\tilde{w} \in \tilde{W}} \tilde{w}(X_0 q^{l x^2/2}), \ a \in P \}.
\]

It can be identified with the space \( \text{Funct}(P/IP)^W \) formed by the \( \Pi W \)-invariant functions on the set \( P/IP^\vee \). Recall that \( P^\vee \subset P \) due to the normalization \( (\theta, \theta) = 2 \). The action of \( W \) is natural. The action of the group \( \Pi = \{ \pi_r = \omega_r u_r^{-1} \mid r \in O \} \) is as follows.

Let us identify the space \( \text{Funct}(P/IP)^W \) with the space \( \text{Funct}(C_l) \) defined for the set \( C_l \) as \( \{ b \in P_+ \mid (b, \theta) \leq l \} \). Then \( \mathcal{L}_l \) becomes isomorphic to \( \text{Funct}(C_l) \Pi \) for the action of \( \Pi \) on the set \( C_l \) through its affine action on the closed fundamental affine Weyl chamber \( \{ x \in \mathbb{R}_+ \cdot P_+ \mid (x, \theta) \leq 1 \} \) “multiplied” by \( l \). The latter is the affine action of the group \( \{(l \omega_r) u_r \mid r \in O\} \); it permutes the points of the set \( C_l \).

For instance, in the case of \( A_2 \), the permutation induced by \( \pi_1 \in \Pi \) on \( C_2 \) reads as follows:

\[
\mathcal{C}_2 = \{ 0, \omega_1, \omega_2, \omega_1 + \omega_2, 2\omega_1, 2\omega_2 \}
\]

\[
\pi_1(C_2) = \{ 2\omega_1, \omega_1 + \omega_2, \omega_1, \omega_2, 2\omega_2, 0 \}.
\]

Thus \( \dim \mathcal{L}_2 = 6/|\Pi| = 2 \) in this example. Only the sets \( C_{3p} \) contain a (unique) \( \Pi \)-invariant point, which is \( p(\omega_1 + \omega_2) \). The general dimension formula for \( A_2 \ (l > 0) \) is

\[
\dim \mathcal{L}_l = \frac{(l+2)(l+1)}{2} + \delta_l /3 \text{ for } \delta_3 = 2, \delta_{3p\pm 1} = 0.
\]

For \( A_1 \), \( \dim \mathcal{L}_l = 1 + \lfloor l/2 \rfloor \), where \( \lfloor \cdot \rfloor \) is the integer part. Indeed, \( \pi_1 \) transposes 0 and \( lw_1 \) in this case and has a fixed point if and only if \( l \) is even.

**Theorem 2.12.** The space \( \mathcal{H}_l = \tilde{\mathcal{F}}(\mathcal{X} q^{lx^2/2}) \), belongs to \( \mathcal{L}_l \). For generic \( k \), for instance, provided that \( \Re k < 0 \), this space coincides with \( \mathcal{L}_l \).

**Proof.** The surjectivity of the map \( \tilde{\mathcal{F}} : \mathcal{X} q^{lx^2/2} \to \mathcal{L}_l \) for generic \( k \) is straightforward; adding \( \tilde{\mu} \) does not change the image. One can also use that this map is zero on \( \mathcal{F}(\mathcal{X}) q^{lx^2/2} \) (see below) and apply Theorem 2.13. \( \square \)

Note that the group of the automorphisms of the non-affine Dynkin diagram acts in \( \text{Funct}(G)^\Pi \). This action commutes with the action of this groups on \( \mathcal{X} \) under the map \( \tilde{\mathcal{F}} \), since the Gaussian is invariant.
with respect to such automorphisms. The main example is the formula
\[(H^l_a)^\varsigma = H^l_{\varsigma(a)} \quad \text{for} \quad \varsigma(a) = -w_0(a), \ X_\varsigma = X_{\varsigma(a)}.
\]

2.4. DAHA coinvariants.

2.4.1. Polynomial coinvariants. We will introduce the coinvariants only for the polynomial representation. The space of coinvariants of level \(l\) is \(X/J_l(X)\) for the subspace
\[J_l(X) = \langle q^{-lx^2/2} T_\hat{\bar{w}} q^{lx^2/2}(X_a) - t^{l(\hat{\bar{w}})/2} X_a \mid \hat{\bar{w}} \in \hat{W}, a \in P \rangle \subset X.\]
Actually, taking only finitely many \(X_a\) is sufficient in this definition (and all \(\hat{\bar{w}}\)). For instance, it suffices to make \(a = 0\) if the quotient is one-dimensional (say, when \(l = 1\) in the simply-laced case).

By construction, \(J_l(X)q^{lx^2/2}\) belongs to the kernel of the map \(\hat{J}\). Denoting the map \(\mathcal{H} \ni A \mapsto q^{x^2/2} A q^{-x^2/2}\) by \(\tau\) (it is an automorphism of \(\mathcal{H}\)), \(J_l(X) = \tau^{-1}(J_0(X))\).

We claim that the dimension of \(X/J_l(X)\) always coincides with that of the Looijenga space (defined above). The dimension of the space of coinvariants can be calculated without any reference to the Looijenga space.

**Theorem 2.13.** For any \(q, t \in \mathbb{C}^*\) and \(l > 0\),
\[\dim_\mathbb{C}(X/J_l(X)) = \dim_\mathbb{C}(\text{Funct}(C_l)^\Pi).\]

**A sketch of the proof.** We use the PBW Theorem to establish the inequality
\[(2.30) \quad \dim_\mathbb{C}(X/J_l(X)) \leq \dim_\mathbb{C}(\text{Funct}(C_l)^\Pi).\]
Let \(k \to 0\) (\(t = q^k \to 1\)). Then \(T_\hat{\bar{w}} \to \hat{\bar{w}}\) and \(\mathcal{H}(t = 1)\) becomes the classical Weyl algebra generated by \(X_a\) and \(Y_\delta\) extended by \(W\). The dimension can be readily calculated at \(k = 0\); it equals \(\dim_\mathbb{C}(\text{Funct}(\{b \in P_+, (b, \theta) \leq l\})^\Pi)\). Due to (2.30), it must remain the same for all \(q, t\). \(\square\)

2.4.2. The B-case. Avoiding the non-simply-laced root systems in Theorem 2.10 is not only a technicality. The dimension of \(L_1\) is greater than one if \(P \neq P^\vee\), so it is not true (generally) that all level one Hall functions are proportional to \(\gamma(x)\), as stated in this theorem. However for \(B_n\), there is the following possibility to make the image really one-dimensional (for \(l = 1\)).

We use that \(Q = P^\vee\) in this case and consider \(\mathcal{X}' = \mathbb{C}_q,t[X_a, a \in Q]\) instead of the complete polynomial representation \(\mathcal{X}\). The space \(\mathcal{X}'\) is a module over the little DAHA (in the terminology from [Ch1]),
which is generated by $X$ and the same $\{T_\tilde{w}, \tilde{w} \in \tilde{W}\}$; all the considerations above hold under this restriction. The corresponding level one Looijenga space will be isomorphic to $\text{Funct} \left( \frac{Q}{lQ} \right)^W$, i.e., will be of dimension one as $l = 1$. The formula $(2.28)$ holds if $\rho$ is replaced by $\rho^v$ and $a \in Q$.

Generally, if there is any DAHA-submodule $X'$, then, automatically,

$$\tilde{\mathcal{F}}(X' \ q^{l2^2/2}) \subset \left\{ \sum_{\tilde{w} \in \tilde{W}} \tilde{w}(G(X) q^{l2^2/2}), G(X) \in X' \right\} \text{ for any } l > 0.$$  

2.4.3. One-dimensional coinvariants. Let us consider the (simplest) cases when the space of coinvariants is one-dimensional.

Theorem 2.14. In the level zero case, provided that the space of $Y$–eigenvectors with the eigenvalue $t^{\rho}$ (i.e., containing $E_0 = 1$) is one dimensional in $X$, $\dim_C (X / J(X)) = 1$ and $\oplus q^{\lambda} \neq t^{\rho} \in X_\lambda = J(X)$, where $X_\lambda = \{ f \in X \mid (Y_a - q^{(\lambda,a)})^N(f) = 0 \}$ for sufficiently large $N$; we identify $q^{\lambda}$ if they give coinciding $Y$–eigenvalues. This dimension is one for $l = 1$ in the simply-laced case too. Moreover, $q, t$ can be arbitrary nonzero in this case.

Proof. Let us assume that the nonsymmetric Macdonald polynomials $E_a$ are well defined; they form a basis for $X$. Recall that the action of $Y_b$ is given by $Y_b^{-1}(E_a) = q^{(a+b)}E_a$ for $a \in P, b \in P^\vee$. So for any $a \in P$ such that $q^{(a+b)} \neq q^{(\rho,b)}$, we have $E_a \in J(X)$. Then $E_0 = 1$ is of multiplicity one in $X$ and $\dim_C (X / J(X)) = 1$. Generally, we need to use the generalized $Y$–eigenvectors here.

In the case $l = 1$, the proof is very different. The PBW Theorem for DAHA is used and the properties of its basic automorphisms. See Lemma 3.3 below for the case of $A_1$.  

2.5. The Kac-Moody limit. The limiting case $t \to \infty$ ($k \to -\infty$) is important. Then the Hall function $\tilde{H}_a^{(l)}$ for a weight $a \in P_+$ subject to $(a, \theta) \leq l$ becomes proportional to the character of the corresponding integrable Kac-Moody module. The level $l \in \mathbb{N}$ equals the action of the central element $c$ in the standard normalization; we consider the split case. It results directly from $(2.7)$. Notice that we use the extended affine Weyl group $\tilde{W}$ with $P^\vee$ instead of $Q^\vee$ (standard in the Kac-Moody theory) and that, in our approach, no inequalities for weights $a \in P$ are imposed. The Hall functions were defined for any $a$; their interpretation as characters of integrable modules of level $l$ in the limit requires $a \in P_+$ and the inequality $(a, \theta) \leq l$. The relation of the affine
Hall functions to the Kac-Moody characters is, essentially, known; see, e.g., [Vi]. However, we prefer to state it explicitly in this paper.

2.5.1. Explicit formulas. From (2.5) and (2.8),

\[ \tilde{\mu}(t \to \infty) = \prod_{\alpha > 0} \frac{1}{1 - X_\alpha}, \quad \lim_{t \to \infty} ct(t^{-1}) = \prod_{i=1}^{\infty} \frac{1}{(1 - q^i)^n}. \]

Also, \( \tilde{\mathcal{P}}(t^{-1}) \to |\Pi| \) as \( t \to \infty \) for \( \Pi = P^\vee/Q^\vee \). Setting

\[ \hat{\chi}_a(\lambda) \overset{\text{def}}{=} q^{-l_2} \lim_{t \to \infty} \tilde{H}(l_2) \]

\[ \frac{1}{\prod_{\alpha \in eR^+} (1 - X_\alpha)}. \]

Here the summation is over all \( b \in P^\vee, w \in W \) and we set (symbolically) \( \tilde{\rho} = \frac{1}{2} \sum_{\alpha} \tilde{\alpha} \) (as for the Kac-Moody algebras). What we really need is the relation:

\[ \sum_{\alpha \in \Lambda(\tilde{w}^{-1})} \tilde{\alpha} = \tilde{\rho} - \tilde{w}(\tilde{\rho}) \]

for the sets \( \Lambda(\tilde{w}) \) defined in (2.6); notice \( \tilde{w}^{-1} \) here. Using the level zero and level one formulas for \( \mathcal{P}_+ \circ \tilde{\mu} \),

\[ \prod_{\tilde{\alpha} \in R^+_+} (1 - X_{\tilde{\alpha}}) = \prod_{\tilde{w} = bw, \tilde{w}(\tilde{\rho}) = \tilde{\rho}, w \in W} (1 - q^{l_2}) \]

\[ \prod_{\tilde{\alpha} \in R^+_+} X_{\tilde{\alpha}} q^{l_2}. \]

Formula (2.34) is stated here in the simply-laced case as in (2.28). One can readily adjust it to the setting of [Ch1], i.e., to the case of twisted \( \tilde{R}^\vee \)–affinization (any nonaffine \( R \) can be used).

These two formulas are the denominator identity and the level one formula due to Kac. See Theorem 10.4, Lemma 12.7 and (12.13.6) from [Kac]. We conclude that \( ct(t^{-1}) \hat{\chi}_a(t) \) is the character of the corresponding Kac-Moody integrable module of level \( l \) provided that \( a \in P_+ \) and \( (a, \theta) \leq l \).
Let us provide the first few terms of the numerators of these formulas in the case of $A_1$:

$$
\sum_{\tilde{\omega}=bw} (-1)^{l(\tilde{\omega})} X_{\tilde{\omega}^\vee} q^{lb^2/2} \mod (q^2)
$$

$$
= \begin{cases} 
1 - X^2 + q^{1/4}(X^{-1} - X^3 - qX^{-3} + qX^5) & \text{for } l = 1, \\
2(1 - X^2 + qX^4 - qX^2) & \text{for } l = 0,
\end{cases}
$$

where $X = X_{\omega_1}$. Compare with the left-hand side of (2.33) and (2.34) multiplied by the corresponding denominators (here the calculations are direct).

In our approach, there are no clear reasons to stick here to affine $l$–dominant weights, i.e. to $a \in P_+$ subject to $(a, \theta) \leq l$. Apart from the weights of integrable modules, i.e., for arbitrary $a \in P$, the following level one formulas in terms of the polynomials $\tilde{E}_a \overset{\text{def}}{=} E_a(t \to \infty)$ are worth mentioning:

$$
\mathcal{H}_+^{\prime}(\tilde{\mu}(t \to \infty) \tilde{E}_a q^{x^2/2}) = \begin{cases} 
q^{-a^2/2} t_\gamma(x), & \text{if } a \in P_, \\
0, & \text{otherwise.}
\end{cases}
$$

We use formula (2.28). The polynomials $\tilde{E}_a$ are closely connected with the $q$–Hermite polynomials $E_a(t \to 0)$ studied in [Ch8] (and playing the key role in the theory of $q$–Whittaker functions); see (3.31) below.

**Comment.** Let us consider briefly the limit $t \to 0$ ($k \to \infty$). Then the series $\tilde{\mu}^{-1} \circ \mathcal{H}_+^{\prime} \circ \tilde{\mu}$ can be interpreted via the Kac-Moody characters too. Due to (2.7),

$$
q^{-1/2} \lim_{t \to 0} \mathcal{H}_a^{(l)} = \sum_{\tilde{\omega}=bw} (-1)^{l(\tilde{\omega})} X_{\tilde{\omega}^\vee} q^{lb^2/2} \prod_{\tilde{\alpha} \in \tilde{R}_+} (1 - X_{\tilde{\alpha}}).
$$

2.5.2. **Match at level one.** We note that (12.13.6) from Kac’ book is stated in the simply-laced case, which matches the setting we use for formulas (2.28) and (2.34). Calculating the level one characters in the cases $B_n, F_4, G_2$ is due to Kac and Peterson. As for the $k$–case, we explained in Section 2.4.2 how to proceed in the $B$–case for the lattice $Q^\vee$. The cases $F_4$ and $G_2$ with $k$ seem doable too. The most difficult case in the theory of level one Kac-Moody characters is $C_n$ (managed by Kac and Wakimoto); the problem with $C_n$ seems exactly parallel to that in the $k$–theory. The paper [Sto] devoted to the $C^\vee C$ may contain the methods and results sufficient to deal with the $C$–case in the DAHA setting.

The above discussion and considerations of this section are in the untwisted case. The formulas for the twisted Kac-Moody characters are
known for any root systems. The twisted KM–characters correspond (with some reservations) to our using $\tilde{R}$, the twisted affinization from Section 1.1.4. Similarly to the Kac-Moody theory, the level one formulas with $k$ are obtained (uniformly) for any reduced root systems in [Ch4].

It is worth mentioning that the classification of the Kac-Moody algebras is not the same as for DAHA (which continues the classical classification of symmetric spaces). However, when they intersect, it seems that there is almost exact match between the problems arising in the theory of the Kac-Moody characters and those for the affine Hall functions (with $k$). At least, it is so in the level one case. It is not very surprising because both constructions have outputs in the same Looijenga space.

We do not discuss the $k$–string functions in this work, where not much is known for levels $l > 1$; see [Vi]. In the level one case, the affine Demazure characters are directly connected with the nonsymmetric $q$–Hermite polynomials $E_a(t \to 0)$ (see above). They become $W$–invariant for $a \in P_-$ and their coefficients in this case are given in terms of the $q$–Kostka numbers (see [San],[Ion1]).

We are grateful to Victor Kac who helped us to establish the correspondence between the two theories, the classical KM theory and the one for arbitrary $k$. We thank Boris Feigin for a helpful discussion.

2.6. Shapovalov forms. We will begin with a very general approach to constructing inner products (in functional analysis, known as GNS construction). Let $\mathcal{F}$ be a cyclic $\mathcal{H}_1$–module, i.e., $\mathcal{F} = \mathcal{H}_1(vac)$ for some $vac \in \mathcal{F}$. Actually $\mathcal{F}$ can be absolutely arbitrary in the following (formal) considerations, but we prefer to restrict ourselves with cyclic modules here. We assume that $\mathcal{H}_1$ and $\mathcal{F}$ are defined over a field $\tilde{\mathbb{C}}$. It can be $\mathbb{C}_{q,t}$, the definition field for the polynomial representation of $\mathcal{H}_1$, or its extension by the parameters of $\mathcal{F}$ (treated as independent variables). If $q,t$ and the parameters of $\mathcal{F}$ are considered as nonzero complex numbers, then $\tilde{\mathbb{C}} = \mathbb{C}$.

2.6.1. Symmetric $J$-coinvariants. We set $\mathcal{J} = \{A \in \mathcal{H}_1 | A(vac) = 0\}$ (a left ideal). Then $\mathcal{F} \cong \mathcal{H}_1/\mathcal{J}$. **Any** form on $\mathcal{F}$ which is symmetric and $\mathcal{H}_1$–invariant with respect to a given anti-involution $*$ of $\mathcal{H}_1$ can be obtained as follows.

We begin with an anti-involution $*$ on $\mathcal{H}_1$; automatically, $*^2 = 1$ because the form is symmetric. Let $\varphi : \mathcal{H}_1 \to \tilde{\mathbb{C}}$ be a functional on $\mathcal{H}_1$ such that $\varphi(A^*) = \varphi(A)$ and $\varphi(\mathcal{J}) = 0$. Automatically, we have that $\varphi(\mathcal{J}^*) = 0$ ($\mathcal{J}^*$ is a right ideal in $\mathcal{H}_1$). Since $\varphi(\mathcal{J}) = 0$, we know that
it comes from a functional \( g' : \mathcal{F} \to \mathbb{C} \). The anti-involution \( \star \) can be naturally defined on the elements \( f \in \mathcal{F} \); we lift them to \( \bar{f} \in \mathcal{H} \) and set \( f^\star = f^*(\text{vac}) \).

The form on \( \mathcal{F} \) is introduced as follows:

\[
\langle f, g \rangle \overset{\text{def}}{=} \tilde{\varrho}(\bar{f}^\star \bar{g}) = g'(f^\star g), \quad f, g \in \mathcal{F}.
\]

This form \( \langle , \rangle \) is obviously symmetric and \( \star \)-invariant:

\[
\langle A(f), g \rangle = \langle f, A^*(g) \rangle, \quad f, g \in \mathcal{F}, A \in \mathcal{H}.
\]

Vice versa, let us introduce the space

\[
\mathcal{H}/(\mathcal{J} + \mathcal{J}^\star) = \mathcal{F}/\mathcal{J}^*(\mathcal{F}).
\]

It has a natural action of \( \star \) and is a direct sum of \( \pm 1 \)-eigenspaces, as well as its dual \( \text{Hom}_{\mathbb{C}}(\mathcal{F}/\mathcal{J}^*(\mathcal{F}), \mathbb{C}) \). The subspace of \( \star \)-invariants of the latter space is called the co-space of \( \star \)-symmetric \( \mathcal{J} \)-coinvariants.

We assume that \( 1^\star = 1 \) and, correspondingly, \( \text{vac}^\star = \text{vac} \).

The \( \pm 1 \)-eigenvectors of \( \star \) from \( \text{Hom}_{\mathbb{C}}(\mathcal{F}/\mathcal{J}^*(\mathcal{F}), \mathbb{C}) \) lead to either \( \star \)-invariant forms or to \( \star \)-anti-invariant ones respectively. In the examples we consider, the action of \( \star \) is trivial in the whole space from \((2.36)\) and its dual, but, generally, the minus-sign (equally interesting) may occur.

Let us discuss basic examples.

### 2.6.2. Shapovalov pairs

We call the form \( \langle , \rangle \) a Shapovalov form if

\[
\dim_{\mathbb{C}}(\mathcal{H}/(\mathcal{J} + \mathcal{J}^\star)) = 1 = \dim_{\mathbb{C}}(\mathcal{F}/\mathcal{J}^*(\mathcal{F})),
\]

and therefore it is a unique symmetric \( \star \)-invariant form in \( \mathcal{F} \) up to proportionality. Accordingly, \( \{\mathcal{J}, \star\} \) is called a Shapovalov pair.

This terminology may be somewhat misleading; the anti-involutions we are going to consider, generally, have little to do with “changing the signs” of all roots in the Lie theory. The connection with the theory of Heisenberg and Weyl algebras is more direct. Nevertheless, we think that the name we use explains our approach well (at least to specialists in the Lie theory).

Given a Shapovalov pair \( \{\mathcal{J}, \star\} \), calculating \( \langle f, g \rangle \) for any \( f, g \) is a pure algebraic problem similar to the calculations based on the PBW Theorem. For instance, \( \langle f, g \rangle \) always depends rationally on the parameters \( t, q \) of \( \mathcal{H} \) for such anti-involutions. It is a valuable feature, since the forms given by integrals (or similar) are almost always well defined only for \( t, q \) satisfying certain inequalities. Their meromorphic continuation to other values can be very involved; compare with the Bernstein-Sato theory.
Comment. We follow here unpublished notes by the first author devoted to the Arthur-Heckman-Opdam formulas [HO2] in the theory of the spectral decomposition of AHA (due to Lusztig and many others). The DAHA version of this decomposition is finished (by now) only for $A_n$ (unpublished). The best reference we can give so far is [Ch7]. In this theory, the Shapovalov form appears as a result of analytic continuation of the inner product in the polynomial representation multiplied by the Gaussian to all $t$. This inner product is defined via the integral over $\mathbb{C}_n$ subject to the constraint $\Re k > 0$. Its continuation to negative $\Re k$ appeared a generalization of the Arthur-Heckman-Opdam method [HO2]. However the result of this analytic continuation is known a priori and is analytic for all $t$ due to the Shapovalov property; see Theorem 2.15 below.

The case of the standard anti-involution $\star$ of the polynomial representation, sending $t, q, X_a, Y_b, T_i$ to their inverses, was treated using the approach from this section in [Ch1], Proposition 3.3.2. It gave a complete description of all forms on $\mathcal{X}$ associated with $\star$ due to their alternative definition as symmetric coinvariants. The Shapovalov property ensures the uniqueness of such form up to proportionality and results in its $q, t$–rationality theorem. Similar approach was applied to $\phi$ in [Ch1] and to the forms based on the $q$–Gaussians. See the cases (1,2,3) below.

2.6.3. $Y$-induced modules. Let us discuss the Shapovalov forms for the $Y$–induced modules $F = I_\lambda$, where $\lambda \in \tilde{C}^n$. By definition, $I_\lambda$ is a free $\mathcal{H}$–module over $\tilde{C}$ generated by $\text{vac}$ with the defining relations $Y_b(\text{vac}) = q^{(\lambda,b)} \text{vac}$. It belongs to the category $\mathcal{O}$ with respect to the action of $Y$–elements, i.e., can be represented as a direct sum of the finite-dimensional spaces of generalized $Y$–eigenvectors. For the sake of definiteness, let us assume that $T_i^* = T_i$ for $i = 1, \ldots, n$. The following conditions for $\rho$ are obvious.

$$\rho(Y_a^* T_w Y_b) = q^{(\lambda,a+b)} \rho(T_w), \quad \rho(T_w) = \rho(T_{w^{-1}}) \quad \text{for} \quad w \in W. \quad (2.37)$$

The latter relation simply means that $\rho$ is a trace functional on the non-affine Hecke algebra $\mathcal{H}$.

Generally speaking, there can be other conditions for $\rho$ beyond (2.37). We call the anti-involution $\star$ of strong Shapovalov type with respect to $\mathcal{Y}$ if $\mathcal{H}$ satisfies the PBW condition for $\mathcal{Y}$, $\mathcal{H}$ and $\mathcal{Y}^\star$ replacing $\mathcal{X}^\star$. Namely, if an arbitrary $A \in \mathcal{H}$ can be uniquely represented as $c_{aw} Y_a^* T_w Y_b$ for $a, b \in P^\vee$ and $w \in W$. Then the conditions from (2.37) determine $\rho$ completely. We see that the simply-laced root systems are, generally, needed here, unless the self-dual setting (with $\tilde{R}^\vee$)
is used, as in [Ch1]. Note, that the definition of strong Shapovalov anti-involutions depends only on $\star$, not on the module $I_\lambda$.

A simple but important observation is that if $Y = Y$, i.e., $Y$ is a normal subalgebra with respect to $\star$, then the Shapovalov condition holds for $I_\lambda$ if the generalized $Y$–eigenspace containing $\text{vac}$ is one-dimensional in $I_\lambda$. Indeed, then the linear span of the spaces $(Y_a - q^{(a,\lambda)})I_\lambda \subset \text{Ker}(\rho)$ is of codimension one in $I_\lambda$. Here $\star$ can be arbitrary, provided that $Y$ is normal; of course it is not of strong Shapovalov type.

There are only few strong Shapovalov anti-involutions in the DAHA theory, essentially, the examples (1) and (3) considered below. However, they play a very significant role. In all known cases, the corresponding PBW property holds for any (nonzero) $q,t$. The following rationality theorem clarifies the importance of the Shapovalov property and its strong variant. We follow Proposition 3.3.2 from [Ch1]. The algebra $\mathcal{H}$, the representation $\mathcal{F}$ and the functional $\rho$ are defined over the same field $\tilde{\mathbb{C}}$, for instance, the field of rationals $\mathbb{C}(q^{1/m}, t^{1/2})$ can be taken for the polynomial representation.

**Theorem 2.15.** (i) A form $\langle \cdot, \cdot \rangle$ on $\mathcal{F}$ corresponding to a Shapovalov pair $\{J, \star\}$ is a unique symmetric $\star$–invariant form in $\mathcal{F}$ up to proportionality; let us normalize it by the condition $\langle 1, 1 \rangle = 1$. Then given $f, g \in \mathcal{F}$, their inner product $\langle f, g \rangle$ belongs to the field $\tilde{\mathbb{C}}$.

(ii) Assuming that $\star$ satisfies the strong Shapovalov property for any nonzero $q,t$, let $f,g$ be taken from $\mathcal{H}_{\text{int}}(\text{vac})$ for

$$H_{\text{int}} = \mathbb{C}[q^{\pm 1/m}, t^{\pm 1}] [X_a, Y_b, T_w] \subset \mathcal{H},$$

where the ring of coefficients is the smallest $\mathbb{C}$–algebra necessary for the defining DAHA relations. Then the inner product $\langle f, g \rangle$ is well defined for any nonzero $q,t$. In other words, if the PBW property holds for $\mathcal{Y}$, $\mathcal{H}$ and $\mathcal{Y}^*$, then the corresponding form is regular in terms of $q^{\pm 1/m}, t^{\pm 1}$.

2.6.4. The polynomial case. Let us discuss the Shapovalov condition for an arbitrary anti-involution $\star$ fixing $T_i$ for $i > 0$ coupled with the polynomial representation $\mathcal{X}$. This representation is a quotient of $I_\lambda$ for $\lambda = k\rho$; the vacuum element (the cyclic generator) becomes $1 \in \mathcal{X}$. One has:

$$\mathcal{H}/(\mathcal{H}J + J^*\mathcal{H}) \cong \mathcal{X}/J^*(\mathcal{X})$$

for the left ideal $J$ linearly generated by the spaces $\mathcal{H}(T_w^* - t^{(w)/2})$. For instance, $\rho(Y^*_a T_w Y_b) = t^{(\rho,a+b) + l(w)/2}$. 

Chapter 3 of book [Ch1] is actually the theory of the following three anti-involutions and the corresponding symmetric forms:

\begin{align}
(1) & \quad \varphi : X_a \leftrightarrow Y_a^{-1}, T_w \mapsto T_{w^{-1}}, \\
(2.39) & \quad \Diamond : X_a \mapsto T_{w_0}^{-1}X_{-w_0(a)}T_{w_0}, Y_b \mapsto Y_b, T_w \mapsto T_{w^{-1}}, \\
(3) & \quad \Diamond_1 = q^{-x^2/2} \circ \Diamond \circ q^{x^2/2} : Y_a \mapsto q^{-x^2/2}Y_aq^{x^2/2}.
\end{align}

We assume that $R$ is simply-laced in (1) (it can be arbitrary if $\tilde{R}^\nu$ is considered as in [Ch1]). Let us provide more detail.

(1) This anti-involution controls the duality and evaluation conjectures and is related to the Fourier transform. The Shapovalov property for $\varphi$ is exactly the PBW Theorem (any $q,t$). The corresponding form is well defined for any $q,t$ and the study of its radical is an important tool in the theory of the polynomial representation of DAHA.

(2) This one is about the inner product in $\mathcal{H}$; $\Diamond$ is of Shapovalov type only for generic $k$ (and there is no direct relation to the PBW Theorem); so it is not strong. The corresponding form is the key in the DAHA harmonic analysis, including the Plancherel formula for $\mathcal{H}$ and its Fourier image, the representation of $\mathcal{H}$ in delta-functions.

(3) The third one controls the difference Mehta-Macdonald formulas and is used to prove that the Fourier transform of the DAHA module $\mathcal{X}q^{-x^2/2}$ is $\mathcal{X}q^{x^2/2}$. The strong Shapovalov property holds here, so the form is well defined for any $q,t$. The radical of the corresponding pairing is closely related to that from (1) (they coincide in the rational theory).

2.7. Further examples.

2.7.1. Level zero forms. Let us consider the case $l = 0$ via the affine symmetrizer $\widehat{P}_+$. In this case, the $P$–hat symmetrizer is more convenient than $\widehat{\mathcal{I}}$, which we mainly use in this part of the paper. One can follow the definition from (2.9) or use the rational formula of Theorem 2.7, which gives a $t$–meromorphic continuation of this operator acting in $\mathcal{X}$.

Recall that $\widehat{\mathcal{P}}_+(f) = \mathcal{P}_+(f)/\mathcal{P}(t^{-1})$. See (2.9); $\mathcal{P}(t)$ is the affine Poincaré series. We continue using the notation $J \subset \mathcal{H}$ for the ideal such that $\mathcal{X} = \mathcal{H}/J$; it is the linear span of subspaces

$\mathcal{H}(T_\tilde{w} - t^{l(\tilde{w})/2})$ for $\tilde{w} \in \hat{W}$.

For the anti-involution $\Diamond$ from (2.39), let the functional be

$q_+ : \mathcal{H} \to \mathbb{C}_{q,t}$ sending $A \mapsto \mathcal{P}_+A(1)$. 

It satisfies the \(\diamond\)–invariance property: \(\varphi_+ (J^\diamond + J) = 0\). Indeed, in terms of \(X\ni f\) and \(\varphi'_+\):

\[
\varphi'_+(f) = \hat{P}_+(f), \quad \varphi'_+((T^\omega_0 - t^{l(\bar{w})/2}) f) = 0,
\]

since \(\diamond\) preserves \(H = \langle T^\omega_0 \rangle\). Thus, \(\varphi_+\) can be used to construct a symmetric form on \(X\) corresponding to the anti-involution \(\diamond\).

This argument is of course formal; one needs to address the existence of \(\hat{P}_+(f)\). Theorem 2.7 can be applied instead of the definition of \(\hat{P}_+(f)\) if there are no \(Y_{\omega_i}\)–eigenvectors in \(X\) with the eigenvalue \(t^{-(\rho, \omega_i)}\) for \(i = 1, 2, \ldots, n\).

Note that the \(Y\)–eigenvalue of \(1 \in X\) is \(t^\rho\), so the rational formula for \(\hat{P}_+(f)\) can not be used in (the whole) \(X\) if \(q\) is a root of unity even if \(t\) is sufficiently general. For generic \(q\), the parameter \(t\) can be a \(N\)-th root of unity for sufficiently large \(N\); the zeros of \(\hat{P}(t^{-1})\) must be certainly excluded.

Under these conditions, \(\hat{P}_+\) is a universal \(\diamond\)–coinvariant, which leads to the following construction. Recall that \(\varsigma(a) = -w_0(a), \ X_\varsigma = X_{\varsigma(a)}\); see (2.29).

**Theorem 2.16.** (i) Let us assume that \(X\) posses a nonzero symmetric form \(\langle f, g \rangle\) with the anti-involution \(\diamond\) normalized by \(\langle 1, 1 \rangle = 1\). Given any \(f, g \in X\), \(\langle f, g \rangle\) is a rational function in terms of \(q, t\). Provided that \(\Re k\) is sufficiently large negative (depending on \(f, g\)),

\[
\langle f, g \rangle = t^{-l(w_0)/2} \hat{P}_+ (fT_{w_0}(g^*))
\]

(2.40)

(ii) Let \(\hat{P}(t^{-1}) \neq 0\) for the affine Poincaré series expressed as in (2.21), \(F\) be a \(\mathcal{H}\)–quotient of \(X\) such that it has no \(Y_{\omega_i}\)–eigenvectors with the eigenvalue \(t^{-(\rho, \omega_i)}\). for any \(i = 1, 2, \ldots, n\). Using the rational presentation for \(\hat{P}_+\) from Theorem 2.7, formula (2.40) supplies \(F\) with a bilinear symmetric form associated to the anti-involution \(\diamond\) and satisfying \(\langle 1, 1 \rangle = 1\).

\(\square\)

Compare with Proposition 3.3.2 from [Ch1] and with Theorem 2.15 above.

**2.7.2. X-induced modules.** A modification of formula (2.40) can be used in \(X\)–induced \(\mathcal{H}\)–modules. They are defined as universal \(\mathcal{H}\)–modules \(\mathcal{I}_\xi^X\) generated by \(v\) subject to \(X_\xi (v) = q^{(\xi, a)} v\) for \(\xi \in \mathbb{C}^n, a \in P\). If \(\xi\) is generic, then the module \(\mathcal{I}_\xi^X\) is \(X\)–semisimple and can be identified with the delta-representation of \(\mathcal{H}\) in the space

\[
\Delta_\xi \overset{\text{def}}{=} \bigoplus_{\bar{w} \in \overline{W}} \mathbb{C}_{q,t} \chi_{\bar{w}}
\]
defined in terms of the characteristic functions $\chi_{\hat{w}}$. The action of the $X$-operators is via their evaluations at $\{q^\xi(\hat{w})\}$:

$$X_a(\chi_{\hat{w}}) \overset{\text{def}}{=} X_a(\hat{w})\chi_{\hat{w}} \quad \text{for} \quad a \in P, \hat{w} \in \hat{W},$$

$$X_a(bw) \overset{\text{def}}{=} X_a(q^{b+w(\xi)}) = q^{(a,b)}X_{w^{-1}(a)}(q^\xi).$$

Let us extend these formulas to the characteristic functions: $\chi_{\hat{w}}(\hat{u}) = \delta_{\hat{u},\hat{w}}$ and $\chi_{\hat{w}}\chi_{\hat{u}} = \delta_{\hat{w},\hat{u}}\chi_{\hat{w}}$ for the Kronecker delta.

The group $\hat{W}$ acts on the characteristic functions through their indices: $\hat{u}(\chi_{\hat{w}}) = \chi_{\hat{w}}\hat{u}$ for $\hat{u}, \hat{w} \in \hat{W}$. Accordingly,

$$T_i(\chi_{\hat{w}}) = \frac{t^{1/2}X^{-1}_\alpha(q^\xi)q^{-(\alpha_i,b)} - t^{-1/2}X^{-1}_\alpha(q^\xi)q^{-(\alpha_i,b)} - 1}{X^{-1/2}_\alpha(q^\xi)q^{-(\alpha_i,b)} - 1} \chi_{\hat{w}} \quad \text{for} \quad \hat{w} = bw \in \hat{W},$$

$$\pi_r(\chi_{\hat{w}}) = \chi_{\pi_r\hat{w}}, \quad \text{where} \quad \pi_r \in \Pi, \quad 0 \leq i \leq n, \quad X_{\alpha_0} = qX_{\theta^{-1}}.$$

The $X$-weight $q^\xi$ is assumed generic in this formula. We follow Section 3.4.2, “Discretization”, from [Ch1].

The delta-functions are defined as $\delta_{\hat{w}}(\hat{u}) = \mu(\hat{u})^{-1}\chi_{\hat{w}}$ for the measure $\mu(\hat{w}) \overset{\text{def}}{=} \mu(\hat{w})/\mu(id)$ in the following inner product:

$$(2.41) \quad \langle f, g \rangle_\bullet = \sum_{\hat{w} \in \hat{W}} \mu(\hat{w})f(\hat{w})g(\hat{w}) = \langle g, f \rangle_\bullet.$$

Here $f, g$ are finite or infinite (provided the convergence) linear combinations of the characteristic functions considered as functions on $\hat{W}$.

The values $\mu(\hat{w})$ are given by formulas in (2.7); replace there $X$ by $q^\xi$ and $\hat{w}$ by $\hat{w}^{-1}$. We see that there is a direct connection with the affine symmetrizer $\mathcal{M} \circ \tilde{\mu}$:

$$\langle f, g \rangle_\bullet = \tilde{\mu}^{-1}\mathcal{M}(\mu fg)(id);$$

recall that $F(X)(id) = F(q^\xi)$ for functions $F$ of $X$ and $\chi_{\hat{w}}(id) = \delta_{\hat{w},id}$.

The anti-involution of $\mathcal{H}$ associated with $\langle , \rangle_\bullet$ is

$$\diamond : T_i \mapsto T_i(i \geq 0), \quad X_a \mapsto X_a(a \in P), \quad \Pi \ni \pi_r \mapsto \pi_r^{-1}.$$

See Section 3.2.2 from [Ch1]; compare with the definition of $\diamond$ from (2.39). By construction, $\langle \chi_{\hat{w}}, \delta_{\hat{u}} \rangle_\bullet = \delta_{\hat{u},\hat{w}}$ for $\hat{u}, \hat{w} \in \hat{W}$.

The (ideal of the) module $\Delta_{\xi}$ and $\diamond_\bullet$ satisfy the Shapovalov property, so the corresponding symmetric form is unique up to proportionality (for sufficiently general $\xi$). By using here $\mathcal{M}$ as in (2.40) instead of
using \( S' \circ \tilde{\mu} \), one readily arrives at the coefficient-wise proportionality of these operators.

**Theorem 2.17.** Let us expand

\[
\mathcal{P}' = \sum_{\hat{w} \in \hat{W}} C_{\hat{w}} \hat{w}
\]

and set

\[
\mathcal{P} = \sum_{\hat{w} \in \hat{W}} C_{\hat{w}} \hat{w} \text{ for } C_{\hat{w}} \overset{\text{def}}{=} C_{\hat{w}} / C_{\text{id}}.
\]

For \( f, g \in \Delta_\xi \) (with the coefficient-wise multiplication),

\[
\langle \mathcal{P}(fg)(id) \rangle = \langle \tilde{\mu}^{-1} S' (\tilde{\mu} fg) \rangle (id).
\]

In particular, \( C_{\hat{w}} \overset{\text{def}}{=} \mu_\ast (\hat{w}^{-1}) \) for any \( \hat{w} \in \hat{W} \) (when \( f = \chi_{\hat{w}^{-1}} = g \) are taken). Thus, \( \mathcal{P}' \) and \( \mathcal{P}' \circ \tilde{\mu} \) are coefficient-wise proportional up to a (common) function of \( X \), which readily results in the exact proportionality claim from (2.15). \( \Box \)

The existence of \( C_{\hat{w}} \) as functions is provided by Theorem 2.9 for \( |t| > q^{1/2} \). Actually, it suffices here to treat these coefficients as formal series in terms of \( X_\hat{w} \) for \( \hat{w} > 0 \) and \( t^{-1} \) (from the original definition). Anyway, the convergence for \( |t| \gg 1 \) is sufficient.

**2.7.3. Higher levels.** Conjugating \( \diamond \) from (2.39) by \( q^{lx^2/2} \) for an integer \( l \geq 0 \), one obtains the following anti-involution:

\[
\diamond_l : T_i \mapsto T_i, \quad (i > 0), Y_b \mapsto q^{-x^2/2} Y_b q^{x^2/2}, \quad (b \in P^v),
\]

\[
X_a \mapsto T_{w_0}^{-1} X_a T_{w_0}, \quad X_a^\varsigma = \varsigma(X_a) = X_{-w_0(a)}, \quad a \in P.
\]

The formulas for \( T_0 \) and \( \pi_x \) can be calculated too but they are not that direct. Let us discuss the invariant forms corresponding to \( \diamond_l \) for \( l > 0 \). The \( \mathcal{H} \)-module will be the polynomial representation \( \mathcal{V} \).

We use that \( \mathcal{H} \) identifies the space of coinvariants \( \mathcal{V} / \mathcal{J}_l(\mathcal{V}) \), from Section 2.4 with the Looijenga space \( L_l \) for generic \( k \). Recall that \( \mathcal{J}_l(\mathcal{V}) \) is the span of linear spaces

\[
q^{-lx^2/2} (T_{w} q^{lx^2/2} - t^{(\hat{w}^{-1})/2}) (\mathcal{V} q^{lx^2/2}) \text{ for } \hat{w} \in \hat{W}.
\]

We see that it is exactly the space of \( \diamond_l \)-coinvariants from (2.36):

\[
\mathcal{H} / (\mathcal{J} + \mathcal{J}_l^{\varsigma_l}) = \mathcal{V} / \mathcal{J}_l(\mathcal{V}), \quad \mathcal{J} = \text{ Ker } (\mathcal{H} \ni A \mapsto A(1) \in \mathcal{V});
\]

the subspaces \( \mathcal{J}_l \) and \( \mathcal{J}_l^{\varsigma_l} \) coincide.

The action of \( \diamond_l \) is trivial in this quotient; use the limit \( t \to 1 \) to see this. Therefore every functional on this space can be used to construct a form associated with \( \diamond_l \), and every such form can be obtained in this way. Using \( \mathcal{H} \), we come to the following extension of Theorem 2.16 from \( l = 0 \) to \( l > 0 \).
Theorem 2.18. Let us assume that $\mathcal{X}$ possesses a nonzero symmetric form $\langle f, g \rangle$ corresponding to the anti-involution $\Diamond_l$ and normalized by $\langle 1, 1 \rangle = 1$. Provided that $\mathcal{F}(\mathcal{X}) = \mathcal{L}_l$, this form can be represented as follows:

$$\langle f, g \rangle = \psi(\mathcal{F}(f T_{w_0}(g^c)))$$

for a proper linear functional $\psi : \mathcal{L}_l \to \mathbb{C}$. When $l = 1$, the resulting symmetric form satisfies the Shapovalov property (following directly from the PBW Theorem).

2.7.4. Analytic theories. Generalizing [Ch1] to arbitrary $l$, the form

$$(2.42) \quad \langle f, g \rangle_l = t^{-l(w_0)/2} \int f T_{w_0}(g^c) \mu' q^{lx^2/2}$$

is symmetric and is served by $\Diamond_l$ for the following major choices of the integration (“theories”):

(a) imaginary integration $\int_{\mathbb{R}^n}$ and the constant term functional;
(b) real integration $\sum_{w \in \mathcal{W}} \int_{w(e)+\mathbb{R}^n}$ for $e \not\in \mathbb{R}^n$ (in progress);
(c) Jackson integration $\int_\xi f = \sum_{\bar{\omega} \in \mathcal{W}} f(q^\bar{\omega}(\xi)) = \mathcal{F}_+ \downarrow_{X=q^e} (f)$.

In the case of (a), we take $l < 0$, $\mu' = \mu$; otherwise $l > 0$ and $\mu' = \bar{\mu}$.

Establishing the connections between these theories is an important problem of harmonic analysis. An equally important problem is in establishing their relation to the corresponding algebraic Shapovalov-type inner products, where there is no integration at all or algebraic substitutes for integration like $\mathcal{F}$ are used. The DAHA-generalization of the Arthur-Heckman-Opdam approach from [HO2] can be stated as finding presentations of algebraically defined inner products in DAHA–modules in terms of integrations.

3. The rank one case

3.1. Polynomial representation.

3.1.1. Basic definitions. Let us consider the root system $A_1$. Following Section 1.2.3, $\mathcal{H}$ is generated by $Y = Y_{\omega_1}, T = T_1, X = X_{\omega_1}$ subject to the quadratic relation $(T - t^{1/2})(T + t^{-1/2}) = 0$ and the cross-relations:

$$(3.1) \quad TXT = X^{-1}, \quad T^{-1}YT^{-1} = Y^{-1}, \quad Y^{-1}X^{-1}YXT^2q^{1/2} = 1.$$ 

Using $\pi \overset{\text{def}}{=} YT^{-1}$, the second relation becomes $\pi^2 = 1$. The field of definition will be $\mathbb{C}(q^{1/4}, t^{1/2})$ although $\mathbb{Z}[q^{\pm 1/4}, t^{\pm 1/2}]$ is sufficient for many constructions; $q^{\pm 1/4}$ will be needed in the automorphisms $\tau_\pm$ below. We will frequently treat $q, t$ as numbers; then the field of definition will be $\mathbb{C}$. 
The following map can be extended to an anti-involution on $\mathcal{H}$: $\varphi : X \mapsto Y^{-1}, T \mapsto T$. The first two relations in (3.1) are obviously fixed by $\varphi$; as for the third, check that $\varphi(Y^{-1}X^{-1}YX) = Y^{-1}X^{-1}YX$.

The conjugation by the Gaussian $q^{x^2}$ can be introduced algebraically as follows:

$$\tau_+(X) = X, \quad \tau_+(T) = T, \quad \tau_+(Y) = q^{-1/4}XY, \quad \tau_+(\pi) = q^{-1/4}X\pi.$$ 

Check that $T^{-1}YT^{-1} = Y^{-1}$ is transformed to $Y^{-1}X^{-1}YXT^2q^{1/2} = 1$ under $\tau_+$. Applying $\varphi$ we obtain an automorphism $\tau_- = \varphi\tau_+\varphi$:

$$\tau_-(Y) = Y, \quad \tau_-(T) = T, \quad \tau_-(X) = q^{1/4}XY.$$ 

The Fourier transform corresponds to the following automorphism of $\mathcal{H}$ (it is not an involution):

$$\sigma(X) = Y^{-1}, \quad \sigma(T) = T, \quad \sigma(Y) = q^{-1/2}Y^{-1}XY = XT^2, \quad \sigma(\pi) = XT,$$

(3.2) $$\sigma = \tau_+\tau_-\tau_+ = \tau_-\tau_+\tau_-.$$ 

Check that $\sigma\tau_+ = \tau_-\sigma, \quad \sigma\tau_+^{-1} = \tau_-\sigma$.

The polynomial representation is defined as $\mathcal{X} = \mathbb{C}_{q,t}[X^\pm]$ over the field $\mathbb{C}_{q,t} = \mathbb{C}(q^{1/4}, t^{1/2})$ with $X$ acting by the multiplication. The formulas for the other generators are

$$T = t^{1/2}s + \frac{t^{1/2} - t^{-1/2}}{X^2 - 1} \circ (s - 1), \quad Y = \pi T$$

in terms of the multiplicative reflection $s(X^n) = X^{-n}$ and $\pi(X^n) = q^{n/2}X^{-n}$ for $n \in \mathbb{Z}$.

The Gaussian $q^{x^2}$ belongs to a completion of $\mathcal{X}$. However the conjugation $A \mapsto q^{x^2}Aq^{-x^2/2}$ for $A \in \mathcal{H}$ preserves $\mathcal{H}$, as we saw, and coincides with $\tau_+$. To see this use that

$$Y = \omega \circ (t^{1/2} + \frac{t^{1/2} - t^{-1/2}}{X^2 - 1} \circ (1 - s)).$$

Recall that $X = q^x$ and

$$s(x) = -x, \quad \omega(f(x)) = f(x - 1/2), \quad \pi = \omega s, \quad \pi(x) = 1/2 - x,$$

$$\omega(q^{x^2}) = q^{1/4}X^{-1}q^{x^2}, \quad Y(q^{x^2}) = \omega(q^{-x^2}) = q^{-1/4}Xq^{-x^2}.$$ 

It is important that $\mathcal{H}$ at $t = 1$ becomes the Weyl algebra defined as the span $\langle X, Y \rangle / (Y^{-1}X^{-1}YXq^{1/2} = 1)$ extended by the inversion $s = T(t = 1)$ sending $X \mapsto X^{-1}$ and $Y \mapsto Y^{-1}$. 
3.1.2. The E-polynomials. Let us assume that $k$ is generic; we set $t = q^k$. The definition is as follows:

$$YE_n = q^{-n_\sharp}E_n \quad \text{for} \quad n \in \mathbb{Z},$$

(3.3)

$$n_\sharp = \begin{cases} \frac{n+k}{2}, & n > 0, \\ \frac{n-k}{2}, & n \leq 0, \end{cases}$$

(3.4)

Note that $0_\sharp = -\frac{k}{2}$.

The normalization is $E_n = \sum \text{"lower terms"}$, where by “lower terms”, we mean polynomials in terms of $X^{\pm m}$ as $|m| < n$ and, additionally, $X^{[n]}$ for negative $n$. It gives a filtration in $\mathcal{A}$; check that $Y$ preserves it, which justifies the definition from (3.3).

The $E_n$ ($n \in \mathbb{Z}$) are called nonsymmetric Macdonald polynomials or simply $E$–polynomials. Obviously, $E_0 = 1, E_1 = X$.

3.1.3. The intertwiners. The first intertwiner comes from the AHA theory:

$$\Phi \overset{\text{def}}{=} T + \frac{t^{1/2} - t^{-1/2}}{Y - 1} : \Phi Y = Y^{-1}\Phi.$$  

The second is $\Pi \overset{\text{def}}{=} q^{1/4}\pi_+(\pi)$; obviously, $\Pi^2 = q^{1/2}$. Explicitly,

$$\Pi = X\pi = q^{1/2}\pi X^{-1} : \Pi Y = q^{-1/2}Y^{-1}\Pi.$$

Use that $\phi(\Pi) = \Pi$ to deduce the latter relation from $\Pi X\Pi^{-1} = q^{1/2}X^{-1}$. The $\Pi$–type intertwiner is due to Knop and Sahi for $A_n$ (the case of arbitrary reduced systems was considered in [Ch3]). Since $\Phi, \Pi$ “intertwine” $\mathcal{A}$, they can be used for generating the $E$–polynomials. Namely,

$$E_{n+1} = q^{n/2}\Pi(E_{-n}) \quad \text{for} \quad n \geq 0,$$

(3.5)

$$E_{-n} = t^{1/2}(T + \frac{t^{1/2} - t^{-1/2}}{q^{2n_\sharp} - 1})E_n$$

(3.6)

and, beginning with $E_0 = 1$, one can readily construct the whole family of $E$–polynomials. For instance,

$$T(X) = t^{1/2}X^{-1} + \frac{(t^{1/2} - t^{-1/2})(X^{-1} - X)}{X^2 - 1} = t^{1/2}X^{-1} - (t^{1/2} - t^{-1/2})X^{-1} = t^{-1/2}X^{-1},$$

$$E_{-1} = t^{1/2}(T + \frac{t^{1/2} - t^{-1/2}}{qt - 1})E_1 = X^{-1} + \frac{1 - t}{1 - tq}X.$$

Using $\Pi$,

$$E_2 = q^{1/2}\Pi E_{-1} = X^2 + q\frac{1 - t}{1 - tq}.$$
Applying $\Phi$ and then $\Pi$,
\[ E_{-2} = X^{-2} + \frac{1-t}{1-tq^2} X^2 + \frac{(1-t)(1-q^2)}{(1-tq^2)(1-q)}, \]
\[ E_3 = X^3 + q^2 \frac{1-t}{1-tq^2} X^{-1} + q \frac{(1-t)(1-q^2)}{(1-tq)(1-q)} X. \]

It is not difficult to find the general formula. See, e.g., (6.2.7) from [Ma4] for integral $k$. However, recalculating these formulas from integral $k$ to generic $k$ can not be too simple; we will provide the exact formulas for the $E$–polynomials below (in the form we need them).

3.1.4. The $E$-Pieri rules. For any $n \in \mathbb{Z}$, we have the evaluation formula
\[ E_n(t^{-1/2}) = t^{-|n|/2} \prod_{0<j<|\tilde{n}|} \frac{1-q^j t^2}{1-q^j t}, \]
where $|\tilde{n}| = |n| + 1$ if $n \leq 0$ and $|\tilde{n}| = |n|$ if $n > 0$.

It is used to introduce the nonsymmetric spherical polynomials
\[ \mathcal{E}_n = \frac{E_n}{E_n(t^{-1/2})}. \]

This normalization is important in many constructions due to the duality formula: $\mathcal{E}_m(q^n) = \mathcal{E}_n(q^m)$. The Pieri rules look the simplest for the $E$–spherical polynomials:
\[ (3.7) \quad X \mathcal{E}_n = t^{-1/2+1} q^{-n} - t^{1/2} q^{-n} \mathcal{E}_{n+1} + \frac{t^{1/2} - t^{-1/2}}{t^{1/2} q^{-n} - 1} \mathcal{E}_{1-n}. \]

Here the sign is $\pm = +$ if $n \leq 0$ and $\pm = -$ if $n > 0$. These formulas give an alternative approach to constructing the $E$–polynomials and establishing their connections with other theories, for instance, with the $p$–adic one.

3.1.5. Rogers' polynomials. Let us introduce the Rogers polynomials for $n \geq 0$:
\[ P_n = (1 + t^{1/2} T)(E_n) = (1 + s) \left( \frac{t - X^2}{1 - X^2 E_n} \right) = E_{-n} + \frac{t - tq^n}{1 - tq^n} E_n. \]

The leading term is $X^n$: $P_n = X^n + \text{"lower terms"}$. They are eigenfunctions of the following well-known operator
\[ (3.8) \quad \mathcal{L} = t^{1/2} X - t^{-1/2} X^{-1} \Gamma + \frac{t^{1/2} X^{-1} - t^{-1/2} X}{X^{-1} - X} \Gamma^{-1}, \]
where we set $\Gamma(f(x)) = f(x + 1/2)$, $\Gamma(X) = q^{1/2} X$, i.e., $\Gamma$ acts as $-\omega$ in $\mathcal{X}$. This operator is the restriction of the operator $Y + Y^{-1}$ to
symmetric polynomials, which is the key point of the DAHA approach to the theory of the Macdonald polynomials.

The exact eigenvalues are as follows:

\[ \mathcal{L}(P_n) = (q^{n/2}t^{1/2} + q^{-n/2}t^{-1/2}) P_n, \quad n \geq 0. \]

The evaluation formula reads:

\[ P_n(t^{\pm 1/2}) = t^{-n/2} \prod_{0 \leq j \leq n-1} \frac{1 - qt^j}{1 - q^jt}. \]

The spherical \( P \)-polynomials \( \mathcal{P}_n \)\( \equiv \) \( P_n/P_n(t^{1/2}) \) satisfy the duality \( \mathcal{P}_n(t^{1/2}q^{m/2}) = \mathcal{P}_m(t^{1/2}q^{n/2}) \).

3.1.6. **Explicit formulas.** Let us begin with the well-known formulas for the Rogers polynomials \( n \geq 0 \):

\[ P_n = X^n + X^{-n} + \sum_{j=1}^{[n/2]} M_{n-2j} \prod_{i=0}^{j-1} \frac{1 - q^{n-i}}{1 - q^{1+i}} \frac{1 - tq^i}{1 - q^{n-i-1}}. \]

where \( M_n = X^n + X^{-n} (n > 0) \) and \( M_0 = 1 \).

The formulas for the \( E \)-polynomials are as follows \( n > 0 \):

\[ E_n = X^{-n} + X^n \frac{1 - t}{1 - tq^n} + \sum_{j=1}^{[n/2]} X^{2j-n} \prod_{i=0}^{j-1} \frac{1 - q^{n-i}}{1 - q^{1+i}} \frac{1 - tq^i}{1 - q^{n-i-1}}. \]

\[ E_n = X^n + \sum_{j=1}^{[n/2]} X^{2j-n} q^{n-j} \frac{1 - q^j}{1 - q^{n-j}} \prod_{i=0}^{j-1} \frac{1 - q^{n-i-1}}{1 - q^{1+i}} \frac{1 - tq^i}{1 - q^{n-i-1}}. \]

3.2. **The p-adic limit.**

3.2.1. **The limits of \( E \)-polynomials.** Let us “separate” \( t \) and \( q \): they will not be connected by the relation \( t = q^k \) in the following theorem. We mainly follow [Ch1], however, with certain technical modifications.

**Theorem 3.1.** The limit \( \mathcal{E}_n^0(X) = \lim_{q \to 0} \mathcal{E}_n \)\( \equiv \) \( \mathcal{E}_n^0 \) exists. The Matsumoto functions \( \varepsilon_n \) from (1.10) are connected with \( \mathcal{E}_n^0 \) as follows:

\[ \varepsilon_n = \mathcal{E}_n^0(t \to t^{-1}, X \to Y). \]
Proof. First of all, \( \lim_{q \to 0} E_n(t^{-1/2}) = t^{-|n|/2} \). For \( n > 0 \), we have
\[
X E_n^0 = t^{-1/2} E_{n+1}^0, \\
X E_n^{-} = t^{1/2} E_{n-1}^0 - (t^{1/2} - t^{-1/2}) E_n^0.
\]
These are exactly the Pieri relations for the Matsumoto functions from (1.6–1.8) upon the substitution \( Y \mapsto X, t \mapsto t^{-1} \).

We know from (1.10) that for \( n \geq 0 \):
\[
\varepsilon_n = t^{-\frac{n}{2}} Y^n, \quad \varepsilon_n^{-} = t^{-\frac{n+1}{2}} (t^{1/2} Y^{-n} + (t^{1/2} - t^{-1/2}) \frac{Y^n - Y^{-n}}{\frac{1}{Y^2} - 1}).
\]
Obtaining them directly from (3.11) and (3.12) may be of some interest.

3.2.2. The limits of \( P \)-polynomials. Formula (3.10) readily gives that
\[
P_n^0 \overset{\text{def}}{=} \lim_{q \to 0} P_n = X^n + X^{-n} + \sum_{j=1}^{[n/2]} M_{n-2j} \prod_{i=0}^{j-1} (X^{n-2j} + X^{2j-n})(1 - t)
= X^n + X^{-n} + (1 - t) \chi_{n-2} = \chi_n - t \chi_{n-2}
\]
for the monomial symmetric functions \( M_n \) and the classical characters \( \chi_n = (X^{n+1} - X^{-n-1})/(X - X^{-1}) \). In the spherical normalization,
\[
P_n^0 = (\chi_n - t \chi_{n-2})^{\frac{t^{n/2}}{1 + t}}.
\]
By letting \( t \to t^{-1} \) and \( X \to Y \), we obtain that \( P_n^0 \) coincides with the spherical function \( \varphi_n \). Recall that, generally, \( P_n = P_n/P_n(t^{1/2}) \) and \( P_n^0(t^{1/2}) = t^{-n/2}(1 + t) \).

Let us obtain this fact directly from the definition of the Rogers polynomials \( P_n \) in terms of the operator \( \mathcal{L} \):
\[
\left( \frac{t^{1/2} X - t^{-1/2} X^{-1}}{X - X^{-1}} \Gamma + \frac{t^{1/2} X^{-1} - t^{-1/2} X^{-1}}{X^{-1} - X} \Gamma^{-1} \right) P_n
= (q^{n/2} t^{1/2} + q^{-n/2} t^{-1/2}) P_n;
\]
see Section 3.1.5.

Indeed, \( \Gamma(X^m) = q^{m/2} X^m \) and \( \lim_{q \to 0} q^{n/2} \Gamma^{\pm 1}(X^{\pm m}) = 0 \) for \( |m| \leq n \) unless
\[
\lim_{q \to 0} q^{n/2} \Gamma(X^{-n}) = X^{-n}, \quad \lim_{q \to 0} q^{n/2} \Gamma^{-1}(X^n) = X^n.
\]
Therefore,
\[
t^{-1/2} P_n^0 = \frac{t^{1/2} X - t^{-1/2} X^{-1}}{X - X^{-1}} X^{-n} + \frac{t^{1/2} X^{-1} - t^{-1/2} X}{X^{-1} - X} X^n.
\]
Using that 

$$P_n^0(t^{1/2}) = t^{-n/2}(1 + t),$$

which is exactly the Macdonald summation formula (1.13) under the substitution \(X \mapsto i, t \mapsto i^{-1}:

\begin{equation}
\varphi_n = \frac{t^{-n/2}}{1+i^{-1}} \left( \frac{1-t^{-1}Y^{-2}}{1-Y^{-2}} Y^{-n} + \frac{1-t^{-1}Y^2}{1-Y^2} Y^n \right).
\end{equation}

(3.13)

We see that (3.13) can be obtained as a limit of the operator \(L\); it is a general fact (true for any root systems).

**Comment.** We expect a similar connection between the *difference-elliptic* symmetric Macdonald-type Looijenga functions and the affine Hall functions. There is no general theory of such Macdonald-Looijenga functions so far; the paper [Ch10] dealt with the difference-elliptic theory only at level of operators. At the operator level, the elliptic Ruijsenaars operators and their generalizations to arbitrary root systems are really connected with the affine symmetrizers in the corresponding limit. There are other approaches to elliptic orthogonal polynomials. The most advanced theory I know is [Ra]; however, it seems that it is not what is needed here.

### 3.3. Coinvariants and symmetrizers.

#### 3.3.1. Coinvariants. Let us prove Theorem 2.14 for the level \(l = 1\) in the case of \(A_1\).

**Theorem 3.2.** For any \(q, t = q^k, \dim_C (\mathcal{X}/J_1(\mathcal{X})) = 1.\)

**Proof.** Let \(\varrho : \mathcal{H} \to \mathbb{C}\) be a functional on \(\mathcal{H}\) such that

\begin{align}
\varrho(\mathcal{H} \cdot (T_{\tilde{w}} - t^{l(\tilde{w})/2})) &= 0 \quad \text{and} \\
\varrho(\tau_+^{-1}(T_{\tilde{w}} - t^{l(\tilde{w})/2}) \cdot \mathcal{H}) &= 0
\end{align}

(3.14) (3.15)

for all \(\tilde{w} \in \widehat{W} = W \ltimes P^\vee = S_2 \ltimes \mathbb{Z}_\omega.\)

**Lemma 3.3.** An arbitrary \(A \in \mathcal{H}\) can be uniquely represented as

\[ A = \sum c_{n,\varepsilon, m} \tau_+^{-1}(Y^n) T^\varepsilon Y^m, \]

where \(\varepsilon = 0\) or \(1\), \(m, n\) are integers and \(c_{n,\varepsilon, m}\) are constants.

**Proof of Lemma 3.3.** We know that the element \(\{X^m T^\varepsilon Y^n\}\) form a PBW basis for \(\mathcal{H}\). Applying \(\tau_+^{-1}\), we obtain that the elements \(\{X^m T^\varepsilon \tau_+^{-1}(Y^n)\}\) also form a basis.
Let \( \varphi \) be the duality anti-involution defined in section 2.6.4, sending \( \varphi : X \leftrightarrow Y^{-1} \) and fixing \( T \). One has:

\[
\tau_+^{-1}(Y) = q^{1/4} X^{-1} Y \quad \text{and} \quad \varphi(\tau_+^{-1}(Y)) = \tau_+^{-1}(Y).
\]

Applying \( \varphi \) to the basis above, we see that \( \{ \tau_+^{-1}(Y^n) T^s Y^{-m} \} \) is a basis too, which completes the proof of the lemma.

Now, for \( A \in \mathcal{H} \), relations (3.14) and (3.15) give that:

\[
\varrho(\tau_+^{-1}(Y^n) A) = t^{n/2} \varrho(A) \quad \text{and} \quad \varrho(AT_{\tilde{w}}) = t^{l(\tilde{w})/2} \varrho(A).
\]

Representing \( A \) as in Lemma 3.3,

\[
\varrho(A) = \sum c_{n,\varepsilon,m} \varrho(\tau_+^{-1}(Y^n) T^s Y^{-m}) = \sum c_{n,\varepsilon,m} t^{n/2 + \varepsilon/2 - m/2}.
\]

Thus, \( \dim_{\mathbb{C}}(\mathcal{F}^\prime/\mathcal{F}(\mathcal{F}^\prime)) = 1 \).

**Comment.** A similar argument can be employed for arbitrary simply-laced root systems (or if the twisted setting is used). A counterpart of Lemma 3.3 is the claim that an arbitrary \( A \in \mathcal{H} \) can be uniquely represented as

\[
A = \sum c_{b,\varepsilon,a} \tau_+^{-1}(Y_b) T_w Y_a,
\]

where \( w \in W, \ a, b \in P \) and \( c_{b,\varepsilon,a} \) are constants.

For any level \( l > 0 \), \( \tau_+^{-l}(Y) = q^{-l/4} X^{-l} Y \). Calculating the space of coinvariants, generally, requires knowing \( \tau_+^{-l}(Y^m) \). The latter can be computed using the relation \( Y^{-1} X^{-1} Y T X^2 q^{1/2} = 1 \), but explicit formulas are involved. However, they can be used for finding the dimension of the space of coinvariants (for arbitrary simply-laced root systems too).

### 3.3.2. The \( \mathbb{P} \)-hat symmetrizer

Let us discuss the rank one version of Theorem 2.7. The explicit list of the elements \( \tilde{w} \in \hat{W} \) (there are four types) and the corresponding \( T_{\tilde{w}}' \overset{\text{def}}{=} t^{-l(\tilde{w})/2} T_{\tilde{w}}^{-1} \), presented in terms of \( Y, T \), is as follows:

1) \( \tilde{w} = m\omega \cdot s (m > 0), \quad l(\tilde{w}) = m - 1, \quad T_{\tilde{w}}' = t^{-\frac{m-1}{2}} Y^{-m} \),

2) \( m\omega (m > 0), \quad m, \quad t^{-\frac{m}{2}} Y^{-m} \),

3) \( -m\omega (m \geq 0), \quad m, \quad t^{-\frac{m}{2}} Y^{-m} T^{-1} \),

4) \( (-m\omega) \cdot s (m \geq 0), \quad m + 1, \quad t^{-\frac{m+1}{2}} Y^{-m} T^{-1} \).

Note that we use the presentation somewhat different from the one used in the justification of this theorem.
Theorem 3.4. The affine symmetrizer $\hat{\mathcal{P}}_+^\prime$ (the prime here indicates that it is without the division by $\hat{\mathcal{P}}(t^{-1})$) can be expressed as follows:

$$\hat{\mathcal{P}}_+^\prime = (1 + t^{1/2}T) \left( \frac{t^{-1/2}Y^{-1}}{1 - t^{-1/2}Y^{-1}} (1 + t^{-1/2}T^{-1}) + t^{-1/2}T^{-1} \right).$$

(3.16)

In particular, $\hat{\mathcal{P}}_+^\prime(1) = 2 \frac{1 + t^{-1}}{1 - t^{-1}} = \hat{\mathcal{P}}(t^{-1}) = 2 + \sum_{m=1}^{\infty} 4t^{-m}$ for $|t| > 1$.

The expansion of $\hat{\mathcal{P}}_+^\prime$ from the theorem in terms of $t^{-1/2}$ is exactly the definition of the $P$–hat symmetrizer upon using (1,2,3,4), as well as the sum $2 + \sum_{m=1}^{\infty} 4t^{-m}$. Note that $(1 + t^{1/2}T)T^{-1/2} = 1 + t^{-1/2}T^{-1}$.

As it was remarked in Section 2.2.5, when treating the right-hand side of (3.16) becomes identically zero when treated as an element of a proper localization of the affine Hecke algebra $H_Y = \langle T, Y^{\pm 1} \rangle$. Indeed, it can be only zero because the localization is not sufficient to construct such affine symmetrizer in $H_Y$ (a completion is needed). This vanishing property can be seen directly using the relation

$$Tf(Y) - f(Y^{-1})T = \frac{t^{1/2} - t^{-1/2}}{Y - 1} (f(Y^{-1}) - f(Y))$$

(3.17)

extended to rational functions $f(Y)$. The justification of this extension is simple; an arbitrary rational function in terms of $Y$ can be represented as a Laurent polynomial divided by a $W$–invariant Laurent polynomial, commuting with $T$. Let

$$U \overset{\text{def}}{=} \frac{t^{-1/2}Y^{-1}}{1 - t^{-1/2}Y^{-1}}, \quad U^+ = U(1 + t^{-1/2}T^{-1}), \quad 1^+ = (1 + t^{-1/2}T^{-1}).$$

Then

$$TU^+ = \frac{t^{-1/2}}{1 - t^{-1/2}Y^{-1}}(1 + t^{-1/2}T^{-1}) = -t^{-1/2}U^+ - t^{-1/2}1^+,$$

therefore, $(1 + t^{1/2}T)U^+ + 1^+ = 0$.

This vanishing property is the key point of a different approach to expressing $\hat{\mathcal{P}}_+^\prime$, with all the nonaffine $T_w$ moved to the right. For integers $M > 0$, let us introduce the truncated symmetrizers

$$\hat{\mathcal{P}}_M^\prime = (1 + t^{1/2}T) \left( \sum_{j=1}^{M} t^{-j/2}Y^{-j}(1 + t^{-1/2}T^{-1}) \right) + 1 + t^{-1/2}T^{-1}.$$

(3.19)
Theorem 3.5. (i) Moving $T$ to the right using relation (3.17) in the affine Hecke algebra $\mathcal{H}_Y = \langle T, Y \rangle$,

\begin{equation}
\hat{\mathcal{P}}'_{M} = \hat{\Sigma}^+_M \overset{\text{def}}{=} \hat{\Sigma}_M (1 + t^{-1/2} T^{-1}), \quad \text{for}
\end{equation}

\begin{equation}
\hat{\Sigma}_M \overset{\text{def}}{=} t^{-\lfloor M/2 \rfloor} + \sum_{j=1}^{M} t^{-\lfloor M-j \rfloor - \frac{1}{2}} (Y^j + Y^{-j}),
\end{equation}

where $[a/b]$ is the integer part.

(ii) The operator $\hat{\mathcal{P}}'_+$ is well defined if and only if the limit $\hat{\Sigma}^+_\infty = \lim_{M \to \infty} \hat{\Sigma}^+_M$ exists. Then these operators coincide. Assuming that $\hat{\Sigma}^+_\infty$ exists, the condition $\lim_{M \to \infty} t^{-M/2} (Y^{-M})^+ = 0$ must hold, which in its turn ensures that $\hat{\Sigma}^+_\infty$ is an affine symmetrizer, i.e., satisfies:

\begin{equation}
Y \hat{\Sigma}^+_\infty = \hat{\Sigma}^+_\infty Y = t^{\frac{1}{2}} \hat{\Sigma}^+_\infty = T \hat{\Sigma}^+_\infty = \hat{\Sigma}^+_\infty T, \quad \text{moreover},
\end{equation}

\begin{equation}
\hat{\mathcal{P}}'_+ = \lim_{M \to \infty} \sum_M^+ \text{ for } \sum_M^+ \overset{\text{def}}{=} \frac{t^{-\lfloor M/2 \rfloor} Y^M + t^{-\lfloor M-1 \rfloor} Y^{M-1}}{1 - t^{-1}}.
\end{equation}

3.3.3. Proof of the Sigma-formula. Only the $t$–powers $t^{-M/2}$ and $t^{(1-M)/2}$ appear in the formula for $\hat{\Sigma}_M^+$:

$$
\hat{\Sigma}_M = t^{-\frac{M}{2}} (Y^M + Y^{-M}) + \frac{1}{2} t^{M} (Y^{-M} + Y^{1-M}) + t^{-\frac{M}{2}} (Y^{M-2} + Y^{2-M}) + \ldots + t^{-\lfloor \frac{M}{2} \rfloor}.
$$

For instance, in the case of even $M$,

$$
\hat{\Sigma}_M(1) = \sum_{j=2l} t^{-M/2}(t^{j/2} + t^{-j/2}) + \sum_{j=2l-1} t^{-M/2+1/2}(t^{j/2} + t^{-j/2})
$$

for $l = 1, 2, \ldots, M/2$. The resulting $t^{-1}$–series is $2 + 2t^{-1} + 2t^{-2} + \ldots$; we obtain that

$$
\lim_{M \to \infty} \hat{\Sigma}_M \cdot (1 + t^{-1/2} T^{-1})(1) = 2 \frac{1 + t^{-1}}{1 - t^{-1}} = \hat{P}(t^{-1}) \quad \text{for} \quad |t| > 1.
$$

Let us check (3.20); we use the truncation $U_M = \sum_{j=1}^{M} t^{-j/2} Y^{-j}$ of the series $U$ introduced in (3.18) and set $U_M^+ = U_M(1 + t^{-1/2} T^{-1})$ for $U_M$ and other operators. Then

$$
\hat{\mathcal{P}}'_M - 1^+ = (1 + t^{1/2} T) U_M^+ = U_M^+ + ts_Y(U_M)^+ + \frac{t - 1}{Y^{-2} - 1}(ts_Y(U_M) - U_M)^+
$$

$$
= \sum_{j=1}^{M} t^{-\frac{j}{2}}((Y^{-j} + tY^j) + (1 - t)(Y^j + Y^{j-2} + \ldots + Y^{2-j}))^+.
$$
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for $s_j(Y^j) = Y^{-j}$. Collecting the terms with $Y^\pm i$, we obtain that

$$
\hat{P}'_M = \sum_{i=1}^{M} \left( \left( \frac{1-t}{1-t^{-1}} t^{-\frac{i}{2}} (1 - t^{-\frac{M-i}{2}}) + t^{\frac{i}{2}} \right) Y^i \right) +
\sum_{i=0}^{M-2} \left( \left( \frac{1-t}{1-t^{-1}} t^{-\frac{i}{2}} (1 - t^{-\frac{M-i}{2}}) + t^{\frac{i}{2}} \right) Y^{-i} \right) +
\left( t^{-\frac{M}{2}} Y^{-M} \right)^+ + \left( t^{\frac{1}{2}} - \frac{M}{2} Y^{-1} \right)^+,
$$

where the last term is present only for $M \geq 2$. For $M = 1$:

$$
\hat{P}'_M = 1^+ + (1 + t^{1/2} T)(t^{-1/2} Y^{-1})^+ = 1^+ + t^{-1/2}(Y + Y^{-1})^+,
$$

which immediately follows from (3.1).

As we have already checked, this sum becomes identically zero as $M \to \infty$. Therefore significant algebraic simplifications are granted; only the terms containing $M$ will contribute.

Finally,

$$
\hat{P}'_M = \left( \sum_{i=1}^{M} t^{-\frac{i}{2}} (1 - t^{-\frac{M-i}{2}}) Y^i \right) + \left( \sum_{i=0}^{M-2} t^{-\frac{i}{2}} (1 - t^{-\frac{M-i}{2}}) Y^{-i} \right) +
\left( t^{-\frac{M}{2}} Y^{-M} \right)^+ + \left( t^{\frac{1}{2}} - \frac{M}{2} Y^{-1} \right)^+,
$$

which can be readily transformed to formula (3.20). Part (i) is checked.

Let us demonstrate that

$$
t^{-\frac{1}{2}} Y^j \hat{\Sigma}^+ = \hat{\Sigma}'_\infty = t^{-\frac{1}{2}} T \hat{\Sigma}^+_\infty
$$

Provided the convergence, the first equality is (formally) equivalent to the condition

$$
\lim_{M \to \infty} t^{-M/2}(Y^{-M})^+ = 0.
$$

Indeed, if $\hat{\Sigma}^+_\infty$ converges, then so does

$$
t^{-1/2} Y \hat{\Sigma}^+_M = \hat{\Sigma}^+_M - (t^{-(M+1/2)} Y^{-M-1} + t^{-M/2} Y^{-M})^+.
$$

Thus the condition $(t^{-(M+1/2)} Y^{-M-1} - t^{-M/2} Y^{-M})^+ \to 0$ as $M \to \infty$ is necessary for the existence of $\hat{\Sigma}^+_\infty$. This condition holds if and only if it holds for each of the two terms separately, i.e., when (3.24) is imposed. Provided the existence of $\hat{\Sigma}^+_\infty$, relation (3.24) is sufficient for its $t^{-\frac{1}{2}} Y$–invariance.
The second of the formulas in (3.23) is an immediate corollary of the $s_Y$–invariance of $\hat{\Sigma}_\infty^+$. Due to
\[
(1 + t^{-1}) \hat{\Sigma}_\infty^+ = \lim_{M \to \infty} (1 + t^{-1/2} T^{-1}) \hat{\Sigma}_M(1 + t^{-1/2} T^{-1}),
\]
$\hat{\Sigma}_\infty^+$ is invariant under the action of the anti-involution of $H_Y$ sending $Y \mapsto Y$ and $T \mapsto T$ (and fixing $t, q$). Applying this anti-involution to (3.23), we arrive at the counterpart of these relations with $\hat{\Sigma}_\infty^+$ placed on the left and $Y, T$ on the right.

Finally, relation (3.24) readily results in (3.22); it completes part (ii) of the theorem.

See Theorem 2.8 for the presentation of the symmetrizer $\hat{\mathcal{P}}'_+$ with all $Y$ on the left for general root systems. It is worth mentioning that the Sigma-formula for $\hat{\mathcal{P}}'_+$ makes it possible to calculate its $C$–coefficients directly and establish the proportionality with $\hat{\mathcal{S}}'_+ \circ \tilde{\mu}$ in the most explicit way.

**Comment.** We note that under the Kac-Moody limit $t \to \infty$, formula (3.22) leads to a presentation of the Kac-Moody characters introduced for affine dominant weights as inductive limits of the corresponding Demazure characters. It can be used of course for arbitrary weights, not necessarily dominant, or even for arbitrary functions provided the convergence.

### 3.3.4. Coefficient-wise convergence

Let us check that Theorem 3.5 holds coefficient-wise in the polynomial representation, where the operators are supposed to be expressed as $C_{\tilde{w}}(X) \hat{w}$ for $\hat{w} \in \hat{W}$ and proper functions $C_{\tilde{w}}$. Let us examine the existence of $C_{\tilde{w}}$ as (meromorphic) functions for $\hat{\mathcal{P}}'_+$ (or $\hat{\Sigma}_\infty^+$).

Treating $\{C_{\tilde{w}}\}$ as meromorphic functions is of course different from considering these coefficients as formal series in terms of $t^{-1}$ and $X_{\tilde{\alpha}}$ for $\tilde{\alpha} \in \tilde{R}_+$ (from the original definition of $\hat{\mathcal{P}}'_+$).

Then, if we know that the $C$–coefficients are meromorphic functions, this does not guarantees that this operator converges in the corresponding space. For instance, when acting in the polynomial representation $\mathcal{X}$, it is well defined at a given Laurent polynomials $P(X)$ only for sufficiently large negative $\Re k$ (depending on $P$), which is significantly worse than the condition $|qt^{-2}| < 1$ (necessary and) sufficient for the coefficient-wise convergence of $\hat{\mathcal{P}}'_+$.

In contrast to the case $l = 0$, the convergence of $\hat{\mathcal{P}}'_+$ in the spaces $\mathcal{X} q^{l^2}$ for $l > 0$ is equivalent to the existence of the corresponding $\{C_{\tilde{w}}\}$ (considered in the next theorem). It is with a reservation concerning
l = 1, where the operator $\hat{\mathcal{T}}_l$ is well defined for any $t$. This fact is not very surprising due to the presence of the Gaussians; the growth of the $C_{\hat{w}}$–coefficients is no greater than exponential in terms of $l(\hat{w})$.

The following theorem is directly related to Theorems 2.9 and 2.11.

**Theorem 3.6.** Continuing to assume that $|q| < 1$, we represent:

\begin{equation}
|t| < 1 : \quad t^{\frac{m}{2}} q^{-\frac{m}{2}} Y^{-m} = \sum_{\hat{w} \in \hat{W}} A_{\hat{w}}^{(-m)}(X) \hat{w}
\end{equation}

and

\begin{equation}
|t| > 1 : \quad t^{-\frac{m}{2}} q^{\frac{m}{2}} Y^{-m} = \sum_{\hat{w} \in \hat{W}} A_{\hat{w}}^{(m)}(X) \hat{w};
\end{equation}

where $m \in \mathbb{Z}_+$. Then, given $\hat{w} \in \hat{W}$, the limits $A_{\hat{w}}^{\pm \infty} = \lim_{m \to \infty} A_{\hat{w}}^{(\pm m)}$ and $B_{\hat{w}}^{\pm \infty} = \lim_{m \to \infty} B_{\hat{w}}^{(\pm m)}$ exist and are meromorphic functions in terms of $X^2$ analytic apart from $0 \neq X^2 \not\in \mathbb{Q}$.

We obtain that the operator $t^{-m/2} Y^{-m}$ for $|t| > 1$ has the coefficients tending to zero as $m \to \infty$. Indeed, given $\hat{w} \in \hat{W}$, the coefficient $B_{\hat{w}}^{(m)}(X)$ approaches $q^{m/2} B_{\hat{w}}^{-\infty}(X)$ in the limit of large $m > 0$. Similarly, $t^{-m/2} Y^{-m}$ has the $A$–coefficients (for $|t| < 1$) convergent to zero as $m \to \infty$ if $|qt^{-2}| < 1$.

It results in formula (3.24) needed above. We obtain that the $C_{\hat{w}}$–coefficients of $\hat{\mathcal{T}}_l$ are meromorphic functions when $|qt^{-2}| < 1$. Here one can use (3.22) or directly (3.16). Note that the case $|t| = 1$ is not covered by Theorem 3.6. In this case, it suffices to know that the $A, B$–coefficients remain bounded for large $-m$ for the analysis of $\hat{\mathcal{T}}_l$.

Compare the theorem with the fact that for any given $n \in \mathbb{Z}_+$, $\lim_{m \to \infty} t^{-m/2} Y^{-m}(X^{\pm n}) = 0$ provided that $|t q^{n/2}| > 1$, which was actually used in Theorem 2.6 (for arbitrary root systems). It can be readily checked by expressing $X_{\pm n}$ in terms of the $E$–polynomials. Recall that $t^{-m/2} Y^{-m}(E_n) = t^{-m} q^{-mn/2} E_{-n}$ for $n \geq 0$ and $t^{-m/2} Y^{-m}(E_n) = q^{mn/2} E_n$ for $n > 0$.

Formulas from (3.26) for the $B$–coefficients and the relations from (3.21) are of clear algebraic nature. Let us demonstrate it. The operators in the following theorem will be considered in the polynomial.
representation as above, however we will treat their coefficients as $q$-series.

**Theorem 3.7.** (i) The $C$-coefficients in the expansion $t^{-m/2}Y^{-m} = \sum_{\bar{w} \in \hat{W}} C^{(-m)}_{w} \bar{w}$ for $m \geq 0$ are from the ring

$$X = \mathbb{Z}[t^{-1}, q^{1/2}, X^{\pm 2}, (1 - q^{l}X^{\pm 2r})^{-1}],$$

where $l, r \in \mathbb{Z}_{+}$, $r > 0$, $l > 0$ for $-2r$. Moreover, the coefficient $C_{w}^{(-m)}$ for $w = 1, s$ and $b = \pm n$ ($m \geq n \geq 0$) belongs to $\mathfrak{g}(m-n)/2 \mathfrak{h} \subset \mathfrak{h}$.

(ii) In particular, the coefficients of $w \cdot (\pm n)$ in the $\hat{w}$-expansions of

$$t^{-1/2}Y \hat{\Sigma}_{M}^{+} - \hat{\Sigma}_{M}^{+}, \quad t^{-1/2}T \hat{\Sigma}_{M}^{+} - \hat{\Sigma}_{M}^{+}$$

(3.27) belong to the ideal $\mathfrak{g}(M-n)/2 \mathfrak{h}$ for $0 \leq n \leq M$. If $n$ is fixed and $M \to \infty$, these coefficients tend to zero with respect to the system of ideals $\mathfrak{g}^{m} \mathfrak{h}$ for $m \to \infty$. □

3.4. $Q$-Hermite polynomials.

3.4.1. Relation to Whittaker functions. As motivation, let us begin with the role of the $q$-Hermite polynomials in the theory of $q$-Whittaker functions. In this section, $|q| < 1$ and $t = q^{k}$. We will use the elementary difference operator $\Gamma(X) = q^{1/2}X$ and also $\Gamma_{k}(X) \overset{\text{def}}{=} t^{k/2}X$.

Etingof states in [Et] (following Ruijsenaars) that

$$\lim_{k \to -\infty} q^{-kx} \Gamma_{k} \mathcal{L} \Gamma_{-k} q^{kx}$$

becomes the so-called $q$-Toda (difference) operator. To be exact, he considered the case of $A_{n}$. It is due to Ruijsenaars in this case, who was also the first to define the difference Toda system in the classical and the quantum (operator) setting; see, e.g., [Rui]. Inozemtsev extended Ruijsenaars’ limiting procedure to the case of differential periodic Toda lattice (which we do not consider here).

The $A_{n}$ is exceptional because all fundamental weights are minus-cule and the formulas for the Macdonald-Ruijsenaars difference QMBP operators are explicit. The justification of this limiting procedure in the case of arbitrary (reduced) root systems (conjectured by Etingof) was obtained in [Ch8]; one can employ the Dunkl operators in the Macdonald theory or use directly the formula for the global $q$-Whittaker function from [Ch8]. Proving the classical integrability (at level of the Poisson brackets) of QMBP and, therefore, the integrability of the classical Toda chain is (generally) significantly simpler.

Following [Ch8], we tend $k$ to $\infty$ ($t \to 0$) in this section. Let

$$\wp(\mathcal{L}) \overset{\text{def}}{=} q^{-kx} \Gamma_{k}^{-1} \mathcal{L} \Gamma_{k} q^{-kx}, \quad \text{RIE}(\mathcal{L}) \overset{\text{def}}{=} \lim_{k \to \infty} \wp(\mathcal{L}),$$


where the second limit is the Ruijsenaars-Inozemtsev-Etingof procedure. At level of functions $F(X)$:

$$RIE(F) = \lim_{k \to \infty} q^{kx} F(q^{-k/2}X) = \lim_{k \to \infty} q^{kx} \Gamma_k^{-1}(F).$$

Generally, the $RIE$ procedure requires very specific functions $F$ to be well defined. Formally, if $L(\Phi) = (\Lambda + \Lambda^{-1})\Phi$, then

$$RIE(L)(W) = (\Lambda + \Lambda^{-1}) W$$

for $W = RIE(\Phi)$ provided its existence.

At level of operators,

$$\tilde{\alpha}(L) = \frac{X - X^{-1}}{t^{-1/2}X - t^{1/2}X^{-1}} + \frac{tX^{-1} - t^{-1}X}{t^{1/2}X^{-1} - t^{-1/2}X} \Gamma^{1/2}$$

(3.28)

Therefore,

$$RIE(L) = \frac{X - X^{-1}}{X - tX^{-1}} \Gamma + \Gamma^{-1} = (1 - X^{-2}) \Gamma + \Gamma^{-1}.$$  

(3.29)

One of the main results of [Ch8] states that the $RIE$–image of the global $q,t$–spherical function (arbitrary reduced root systems; see the definition there) is as follows:

$$W_q(X, \Lambda) = \sum_{m=0}^{\infty} q^{m^2/4} \mathcal{T}_m(\Lambda) X^m \prod_{s=1}^{m} \frac{1}{1 - q^s q^{s^2}} q^s q^{s^2},$$

(3.30)

where $\prod_{s=1}^{0} = 1$, $\Lambda = q^\lambda$ as for $X$, $\mathcal{T}_m$ are the symmetric $q$–Hermite polynomials, to be discussed next.

3.4.2. Definition, major properties. For a $E$–polynomial $E_a$, let us define its two limits:

$$\tilde{E}_a = \lim_{t \to \infty} E_a \quad \text{and} \quad \bar{E}_a = \lim_{t \to 0} E_a.$$

Both limits exist (use the explicit formulas or the intertwining operators) and are closely connected to each other. The following theorem provides the connection.

**Theorem 3.8.** For $n \geq 0$,

$$\tilde{E}_{-n} = (q^{-n} \bar{E}_{-n}(Xq^{1/2})) \bigg|_{q \to q^{-1}}, \quad \tilde{E}_{n} = (q^{-n} \bar{E}_{n}(Xq^{1/2})) \bigg|_{q \to q^{-1}}.$$  

(3.31)

The polynomials $\bar{E}_a$ are nonsymmetric (continuous) $q$–Hermite polynomials (see [Ch8] and references therein). Upon the substitution $X \mapsto X^{-1}$, the polynomials $\bar{E}_a$ are directly connected with the Demazure characters of level one Kac-Moody integrable modules; see [San]
for the $GL_n$-case. Generally, it holds only for the twisted affinization; see [Ion1]. These polynomials also appear naturally when formulas $\hat{\chi}_a^{(l=1)}$ from (2.32) are used for arbitrary $a \in P$; see (2.35).

More systematically, let us define
\[
\overline{T} \overset{\text{def}}{=} \lim_{t \to 0} t^{1/2}T = \frac{1}{1 - X^2} \circ (s - 1), \quad \overline{T}(T + 1) = 0.
\]

Using intertwiners, $E_0 = 1,$
\[
\begin{align*}
\overline{E}_{1+n} &= q^{n/2} \Pi \overline{E}_{-n}, \\
\overline{E}_{-n} &= (\overline{T} + 1) \overline{E}_n
\end{align*}
\]
for $n \geq 0$; the raising operator $\Pi = X\pi$ was defined in Section 1.1.

From the divisibility condition $\overline{T} + 1 = (s + 1) \cdot \{\}$, we obtain that $\overline{E}_{-n}$ is symmetric ($s$-invariant) and $\overline{P}_n = \overline{E}_{-n}$ for $n \geq 0$.

Explicitly,
\[
\begin{align*}
\overline{E}_{-n-1} &= ((\overline{T} + 1)\Pi q^{n/2}) \overline{E}_{-n}, \\
(\overline{T} + 1)\Pi &= \frac{X^2\Gamma - 1 - X^{-2}\Gamma}{X - X^{-1}}.
\end{align*}
\]

The bar-Pieri rules read as follows ($n \geq 0$):
\[(3.32) \quad X^{-1} \overline{E}_{-n} = \overline{E}_{-n-1} - \overline{E}_{n+1},
\]
\[
X^{-1} \overline{E}_n = (1 - q^n) \overline{E}_{n-1} + q^n \overline{E}_{1-n},
\]
\[(3.33) \quad X \overline{E}_{-n} = (1 - q^n) \overline{E}_{1-n} + \overline{E}_{n+1},
\]
\[
X \overline{E}_n = \overline{E}_{n+1} - q^n \overline{E}_{1-n}.
\]

Let $\overline{Y} = \pi \overline{T} = \lim_{t \to 0} t^{1/2} \overline{Y}.$ Recall that
\[
Y E_n = \begin{cases} 
  t^{-1/2} q^{-n/2} E_n, & n > 0, \\
  t^{1/2} q^{n/2} E_n, & n \leq 0.
\end{cases}
\]

In the limit,
\[(3.34) \quad \overline{Y} \overline{E}_n = \begin{cases} 
  q^{-|n|/2} \overline{E}_n, & n > 0, \\
  0, & n \leq 0.
\end{cases}
\]

Since $\overline{Y}$ is not invertible, we need to introduce
\[
\overline{Y}' = \lim_{t \to 0} t^{1/2} Y^{-1} = \lim_{t \to 0} t^{1/2} T^{-1} \pi = \overline{T}' \pi
\]
for $\overline{T}' = \overline{T} + 1$. Then $\overline{Y} \overline{Y}' = 0 = \overline{Y}' \overline{Y}$ and
\[(3.35) \quad \overline{Y}' \overline{E}_n = \begin{cases} 
  q^{-|n|/2} \overline{E}_n, & n \leq 0, \\
  0, & n > 0.
\end{cases}
\]
Finally, see (3.9),
\[ \mathcal{L} = \lim_{t \to 0} t^{1/2} \mathcal{L} = \mathcal{Y}' + \mathcal{Y} = \frac{1}{1 - X^2} \Gamma + \frac{1}{1 - X^{-2}} \Gamma^{-1} \]
and \( \mathcal{P}_n = q^{-n/2} \mathcal{P}_n \), \( n \geq 0 \); recall that \( \mathcal{P}_n = E_{-n} \).

### 3.4.3. Nil-DAHA

We come to the following definition of the nil-DAHA (which can be readily adjusted to any reduced root systems).

**Theorem 3.9.** (i) The nil-DAHA \( \mathcal{H}_+ \) is generated by \( T, \pi_+, X^{\pm 1} \) over the ring \( \mathbb{C}[q^{\pm 1/4}] \) with the defining relations: \( T(T + 1) = 0 \),
\begin{equation}
\pi_+^2 = 1, \quad \pi_+X\pi_+ = q^{1/2}X^{-1}, \quad TX - X^{-1}T = X^{-1}.
\end{equation}
Setting \( Y \overset{\text{def}}{=} \pi_+T \) and \( Y' \overset{\text{def}}{=} T'\pi_+ \) for \( T' \overset{\text{def}}{=} (T + 1) \), (3.36) gives that \( TY - Y'T = -Y \), \( TY' = 0 = YT' \), which results in \( TY' - YT = Y \).

(ii) Similarly, one can define \( \mathcal{H}_- \) subject to \( T(T + 1) = 0 \) and
\begin{equation}
\pi_-^2 = 1, \quad \pi_-Y\pi_- = q^{-1/2}Y^{-1}, \quad TY - Y^{-1}T = -Y.
\end{equation}
Setting \( X \overset{\text{def}}{=} \pi_-T' \), \( X' \overset{\text{def}}{=} T\pi_- \), \( T' = T + 1 \), one has:
\[ TX - X'T = X', \quad T'X' = 0 = XT \Rightarrow TX' - XT = -X'. \]

(iii) The algebra \( \mathcal{H}_- \) is the image of the algebra \( \mathcal{H}_+ \) under the anti-isomorphism
\[ \varphi : T \mapsto T, \quad \pi_+ \mapsto \pi_-, \quad X \mapsto Y^{-1}. \]
Correspondingly, \( \varphi : Y \mapsto X', \quad Y' \mapsto X \). There is also an isomorphism \( \sigma : \mathcal{H}_+ \to \mathcal{H}_- \) sending
\[ \sigma : T \mapsto T, \quad X \mapsto Y, \quad \pi_+ \mapsto \pi_-,
\sigma : Y \mapsto \pi_-T, \quad Y' \mapsto T'\pi_-.
\]

(iv) The automorphism \( \tau_+ \) fixing \( T, X \) and sending \( Y \mapsto q^{-1/4}XY \) acts in \( \mathcal{H}_+ \). Correspondingly, \( \tau_- \overset{\text{def}}{=} \varphi\tau_+\varphi^{-1} \) acts in \( \mathcal{H}_- \) preserving \( T, Y \) and sending \( X \mapsto q^{1/4}XY \). One has the relations
\begin{equation}
\sigma\tau_+ = \tau_-\sigma, \quad \sigma\tau_-^{-1} = \tau_-\sigma,
\end{equation}
matching the identity from (3.2) in the generic case. \( \square \)

Both algebras \( \mathcal{H}_\pm \) satisfy the PBW Theorem, so \( \mathcal{H} \) is their flat deformation. It holds even if \( q \) is a root of unity. However, roots of unity must be avoided in the construction of the \( q \)-Hermite polynomials. The formulas above give an explicit description of the bar-polynomial...
representation of $\overline{\mathcal{H}}_+$ in $\mathcal{B} = \mathbb{C}_q[X^\pm 1]$; recall, $T, \pi_+, X^\pm, Y, Y'$ are mapped to the operators $T, \pi, X^\pm, Y, Y'$.

A surprising fact is that the construction of non-symmetric Whittaker functions naturally leads to a module over $\overline{\mathcal{H}}_-$, which differs significantly from the bar-polynomial representation. We will call it the \textit{hat-polynomial} representation, but it requires using the spinors, to be discussed next.

\textbf{Comment.} Let us mention the relation of our nil-DAHA $\overline{\mathcal{H}}_+$ to the $T$–equivariant $K_T(B)$ for affine flag varieties $B$ from [KK] and the Demazure-type operators on this (commutative) ring considered in this paper. Here $T$ is the maximal torus in the Lie group $G$ constructed by the root system $R$.

The exact $K$–theoretic interpretation of DAHA was obtained in [GG] (see also [GKV]). Namely, $\mathcal{H}$ is essentially $K^{T \times C^*} (\Lambda)$ for a certain canonical Lagrangian subspace $\Lambda \subset T^*(B \times B)$, that is the Grothendieck group of the (derived) category of $T \times C^*$–equivariant coherent sheaves on $\Lambda$.

This interpretation is for arbitrary $q, t$. Switching from $B$ in [KK] to $\Lambda \subset T^*(B \times B)$ is important because it gives the definition of convolution and, therefore, supplies $K^{T \times C^*} (\Lambda)$ with a structure of algebra (isomorphic to $\mathcal{H}$). We note that the Gaussian was added to the definition of DAHA in [GG]. We prefer not to consider the Gaussian as part of the definition of DAHA, treating it as an outer automorphism of $\mathcal{H}$, following the theory of Heisenberg-Weyl algebras and metaplectic representations.

3.5. \textbf{Nonsymmetric Q-Toda theory.} The problem of finding Dunkl operators for the Toda and the $q$–Toda operators, (3.29) is quite natural algebraically. Its solution given below provides a nil-variant of the representation $\mathcal{L} = Y + Y^{-1}$ (upon the restriction to the symmetric functions) for $\mathcal{L}$ from (3.8). The spinor-Dunkl operators make it possible to use DAHA methods at full potential algebraically and in the theory of the $q$–Whittaker functions. The construction can be readily extended to arbitrary root systems. We will begin with the definition of the spinors.

3.5.1. \textbf{The spinors.} Generally, the $W$–spinors are needed in the DAHA theory as it was discussed in the introduction. In the $A_1$–case, we will call them simply \textit{spinors}. In this case, they are really connected with spinors from the theory of the Dirac operator (and with superalgebras). Under the rational degeneration, the Dunkl operator for
$A_1$ becomes the square root of the (radial part of the) Laplace operator, i.e., the Dirac operator. However, this relation (and using super-variables) is a special feature of the root system $A_1$.

For practical calculations with spinors, the language of $\mathbb{Z}_2$–graded algebras can be used in the $A_1$–case (see the differential theory below). However, we prefer to do it in a way that does not rely on the special symmetry of the $A_1$–case and can be transferred to $W$–spinors for arbitrary root systems.

The spinors are simply pairs $\{f_1, f_2\}$ of elements (functions) from a space $\mathcal{F}$ with an action of $s$; the addition or multiplication (if applicable) of spinors is componentwise. The space of spinors will be denoted by $\hat{\mathcal{F}}$.

The involution $s$ on spinors is defined as follows $s\{f_1, f_2\} = \{f_2, f_1\}$, so it does not involve the action of $s$ in $\mathcal{F}$. There is a “natural” embedding $\rho : \mathcal{F} \to \hat{\mathcal{F}}$ mapping $f \mapsto f^\rho = \{f, s(f)\}$ and the diagonal embedding $\delta : \mathcal{F} \to \hat{\mathcal{F}}$ sending $f \mapsto f^\delta = \{f, f\}$. Accordingly, for an arbitrary operator $A$ acting in $\mathcal{F}$, $A^\rho = \{A, s(A)\}, A^\delta = \{A, A\}$. The images $f^\rho$ of $f \in \mathcal{F}$ are called functions (in contrast to spinors) or principle spinors (like for adeles).

For instance, for $\mathcal{F} = \mathbb{R}$,

$$X^\rho : \{f_1, f_2\} \mapsto \{X f_1, X^{-1} f_2\}, \quad \Gamma^\rho : \{f_1, f_2\} \mapsto \{\Gamma(f_1), \Gamma^{-1}(f_2)\},$$
$$X^\delta : \{f_1, f_2\} \mapsto \{X f_1, X f_2\}, \quad \Gamma^\delta : \{f_1, f_2\} \mapsto \{\Gamma(f_1), \Gamma(f_2)\},$$

where, recall, $\Gamma(X) = q^{1/2}X$. We simply put

$$X^\rho = \{X, X^{-1}\}, \quad \Gamma^\rho = \{\Gamma, \Gamma^{-1}\}, \quad X^\delta = \{X, X\}, \quad \Gamma^\delta = \{\Gamma, \Gamma\}.$$ 

Obviously, $s^\rho = s = s^\delta$.

If a function $f \in \mathcal{F}$ or an operator $A$ acting in $\mathcal{F}$ have no super-index $\delta$, then they will be treated as $f^\rho, A^\rho$. I.e., by default, functions and operators are embedded into $\hat{\mathcal{F}}$ and the algebra of spinor operators using $\rho$.

If the operator $A$ is explicitly expressed as $\{A_1, A_2\}$, then $A_1$ and $A_2$ must be applied to the corresponding components of $f = \{f_1, f_2\}$. In the calculations below, $A_i$ may contain $s$. Then $A_i$ must be presented as $A_i^s \cdot s$, where $A_i^s$ contains no $s$; i.e., practically, $s$ must be placed on the right. In the operators in $\mathcal{X}$ we will consider, the commutation relations between $s$ and $X, \Gamma$ must be used when moving $s$. Then the component $i$ of $Af$ will be $A_i^s(f_{3-i})$, i.e., $s$ placed on the right means the switch to the other component before applying $A_i$.

For instance, $\{\Gamma s, s - 1\}(\{f_1, f_2\}) = \{\Gamma(f_2), f_1 - f_2\}$.
We will frequently use the vertical mode for spinors:

\[ \{ f_1, f_2 \} = \left\{ \begin{array}{c} f_1 \\ f_2 \end{array} \right\}, \quad \{ A_1, A_2 \} = \left\{ \begin{array}{c} A_1 \\ A_2 \end{array} \right\}. \]

3.5.2. Q-Toda via DAHA. The \( q \)-Toda spinor operator is the following symmetric (i.e., \( s \)-invariant) difference spinor operator

\[ \hat{\mathcal{L}} = \{ \Gamma^{-1} + (1 - X^{-2}) \Gamma, \Gamma^{-1} + (1 - X^{-2}) \Gamma \}. \tag{3.39} \]

Its first component is the operator \( \text{RIE}(\hat{\mathcal{L}}) \) from Section 3.4.1; we will use the notation and definitions from this section.

We claim that \( \hat{\mathcal{L}} \) can be represented in the form \( \hat{Y} + \hat{Y}^{-1} \) upon the restriction to symmetric spinors, i.e., to \( \{ f, f \} \in \hat{F} \). The construction of the spinor-difference Dunkl operator \( \hat{Y} \) goes as follows.

Let us introduce the following map on the operators in terms of \( X, \Gamma \) and \( s \) with the values in spinor operators:

\[ \alpha^\delta : X \mapsto \tilde{t}^{-1/2} X, \quad \Gamma \mapsto \tilde{t}^{-1/2} \Gamma, \quad s \mapsto s \tag{3.40} \]

for the spinor constant \( \tilde{t}^{1/2} \overset{\text{def}}{=} \{ t^{1/2}, t^{-1/2} \} \). Spinor constants are actually diagonal matrices, which may not commute with \( s \) but commute with \( \Gamma \) and \( X \). The spinor RIE-construction is:

\[ \text{RIE}^\delta : A \mapsto \lim_{t \to 0} \alpha^\delta(A). \]

It is of course very different from the procedure \( \text{RIE}^\rho \) from Section 3.4.1. The spinor-Dunkl operators are \( \hat{Y} = \text{RIE}^\delta(Y) \), \( \hat{Y}' = \text{RIE}^\delta(Y^{-1}) \). They are inverse to each other: \( \hat{Y} \hat{Y}' = 1 \).

**Theorem 3.10.** The map

\[
Y^\pm X^\pm \mapsto \hat{Y}^\pm, \quad \pi_- \mapsto \text{RIE}^\delta(XT), \\
T \mapsto \hat{T} = \text{RIE}^\delta(t^{1/2}T), \quad T' \mapsto \hat{T}' = \text{RIE}^\delta(t^{1/2}T^{-1})
\]

can be extended to a representation of the algebra \( \mathcal{H}_- \) in the space \( \hat{\mathcal{X}} \) of spinors over \( \mathcal{X}^\pm = \mathbb{C}[q^{\pm 1/4}][X^\pm] \). Correspondingly,

\[
X \mapsto \text{RIE}^\delta(t^{1/2}X) = \text{RIE}^\delta(\pi_-) \circ \hat{T}', \\
X' \mapsto \text{RIE}^\delta(t^{1/2}X^{-1}) = \hat{T} \circ \text{RIE}^\delta(\pi_-).
\]

The commutativity of \( T \) and \( Y + Y^{-1} \) in \( \mathcal{H}_- \) results in the \( s \)-invariance of \( \hat{Y} + \hat{Y}^{-1} \) and the \( s \)-invariance of this operator upon its restriction to the space of \( s \)-invariant spinors, which is the one from (3.39). \( \square \)
It is clear from the construction that all hat-operators preserve the space of Laurent polynomials in terms of $X^\pm 1$. We will give below explicit formulas. Upon multiplication by the Gaussian, this $\mathcal{H}_-$ module contains an irreducible submodule, the spinor polynomial representation, isomorphic to the Fourier image of the bar-polynomial representation times the Gaussian; see Section 3.4.2, formula (3.38) and Theorem 3.11 below. The reproducing kernel of the isomorphism between these two modules inducing $\sigma : \mathcal{H}_+ \to \mathcal{H}_-$ at the operator level is given by the nonsymmetric $q$–Whittaker function; its existence was conjectured in [Ch8].

3.5.3. Spinor-Dunkl operators. Let us calculate explicitly the operator $\check{Y} = RLE^\delta(Y) = \lim_{t \to 0} \check{a}^\delta(Y)$. Using formulas (3.40):

$$\check{a}^\delta(Y) = s \cdot (\check{t}^{-1/2} \Gamma) \cdot \left( t^{1/2} s + \frac{t^{1/2} - t^{-1/2}}{t^{-1}X^2 - 1} \cdot (s - 1) \right)$$

$$= t^{1/2} \cdot (\check{t}^{-1/2} \Gamma - 1) + \check{t}^{1/2} \Gamma - 1 \cdot \frac{t^{1/2} - t^{-1/2}}{tX^2 - 1} \cdot (1 - s)$$

Recall that $\check{t}^{1/2} = \{t^{1/2}, t^{-1/2}\}$. A little bit more involved calculation is needed for $\check{Y}' = RLE^\delta(Y^{-1})$:

$$\check{a}^\delta(Y^{-1}) = \left( t^{-1/2} s + \frac{t^{-1/2} - t^{1/2}}{tX^2 - 1} \cdot (s - 1) \right) \cdot (\check{t}^{1/2} \Gamma^{-1} s)$$

$$= \left( \frac{t^{-1/2} \check{t}X^{-2} - t^{1/2}}{tX^2 - 1} \cdot s - \frac{t^{-1/2} - t^{1/2}}{tX^2 - 1} \right) \cdot (\check{t}^{1/2} \Gamma^{-1} s)$$

$$= \frac{t^{-1/2} \check{t}X^{-2} - t^{1/2}}{tX^2 - 1} \Gamma^{-1} s - \frac{t^{-1/2} - t^{1/2}}{tX^2 - 1} \check{t}^{1/2} \Gamma^{-1} s$$

$$= \left\{ \frac{\check{t}X^{-2}}{tX^2 - 1} \Gamma^{-1} - \frac{1}{tX^2 - 1} \Gamma^{-1} s \right\} \xrightarrow{t \to 0} \left\{ \frac{\check{t}X^{-2}}{tX^2 - 1} \Gamma^{-1} - \frac{1}{tX^2 - 1} \Gamma^{-1} s \right\}$$

$$\check{Y}' = \left\{ \frac{1 - X^{-2}}{1} \right\} \Gamma + \left\{ \frac{1}{X^2} \right\} \Gamma^{-1} s.$$
Automatically, $\hat{Y}\hat{Y}' = 1$. Indeed, the $\text{RIE}$–construction is a conjugation followed by taking a limit. Now, as we claimed,

$$\text{RIE}^\delta(Y + Y^{-1}) = \lim_{t \to 0} \alpha^\delta(Y + Y^{-1})$$

$$= \left\{ \begin{array}{c}
\Gamma^{-1}(1 - s) + (1 - X^{-2})\Gamma + \Gamma^{-1}s \\
\Gamma - \Gamma\frac{1}{X^2}(1 - s) + \Gamma^{-1} - \frac{1}{X^2}\Gamma s
\end{array} \right\} \pmod{(s - 1)}.$$

For $X$ and $X^{-1}$, we have

\begin{align*}
\hat{X} = \text{RIE}^\delta(t^{1/2}X) &= \lim_{t \to 0} \alpha^\delta(t^{1/2}X) = \lim_{t \to 0} t^{1/2}\hat{T}^{-1/2}X = \left\{ \begin{array}{c} X \\ 0 \end{array} \right\}, \\
\hat{X}' = \text{RIE}^\delta(t^{1/2}X^{-1}) &= \lim_{t \to 0} \alpha^\delta(t^{1/2}X^{-1}) = \lim_{t \to 0} t^{1/2}\hat{T}^{-1/2}X^{-1} = \left\{ \begin{array}{c} 0 \\ X \end{array} \right\}.
\end{align*}

Obviously, $\hat{X}\hat{X}' = 0$. Next,

\begin{align*}
\hat{T} &= \text{RIE}^\delta(t^{1/2}T) = \lim_{t \to 0} \alpha^\delta(t^{1/2}T) = \left\{ \begin{array}{c} 0 \\ s - 1 \end{array} \right\}, \\
\hat{T}' &= \text{RIE}^\delta(t^{1/2}T^{-1}) = \lim_{t \to 0} \alpha^\delta(t^{1/2}T^{-1}) = \left\{ \begin{array}{c} 1 \\ s \end{array} \right\}.
\end{align*}

It is instructional to check the following relations using the explicit formulas we obtained (they of course follow from Theorem 3.10):

\begin{align*}
\hat{T}' = \hat{T} + 1, \quad \hat{T}\hat{T}' &= 0 = \hat{T}'\hat{T}, \\
\hat{T}\hat{Y} - \hat{Y}^{-1}\hat{T} &= -\hat{Y}, \quad \hat{T}\hat{Y}^{-1} - \hat{Y}\hat{T} = \hat{Y}, \\
\hat{T}\hat{X} - \hat{X}'\hat{T} = \hat{X}', \quad \hat{T}\hat{X}' - \hat{X}\hat{T} = -\hat{X}', \quad \hat{X} + X = X^\delta.
\end{align*}

Relations (3.43) imply that

$$\hat{T}(\hat{Y} + \hat{Y}^{-1}) = (\hat{Y} + \hat{Y}^{-1})\hat{T}.$$

It proves that the spinor operator $\hat{Y} + \hat{Y}^{-1}$ is symmetric (recall that $\hat{Y}' = \hat{Y}$). Indeed, applying (3.45) to a symmetric spinor $\{f, f\}$, let $(\hat{Y} + \hat{Y}^{-1})(\{f, f\}) = \{g_1, g_2\}$. Then $\hat{T}(\{g_1, g_2\}) = 0$, which is possible if and only if $g_1 = g_2$. 

3.5.4. **Using the components.** Explicitly, the action of $\hat{Y}$ and $\hat{Y}'$ on the spinors is as follows:

\[
\hat{Y}(\begin{array}{c} f_1 \\ f_2 \end{array}) = \begin{array}{c} \Gamma^{-1}(f_1 - f_2) \\ \Gamma(f_2) - \Gamma(f_2 - f_1) X \end{array},
\hat{Y}'(\begin{array}{c} f_1 \\ f_2 \end{array}) = \begin{array}{c} (1 - X^{-2})\Gamma(f_1) + \Gamma^{-1}(f_2) \\ \Gamma^{-1}(f_2) - \frac{1}{X^2}\Gamma(f_1) \end{array}.
\]

It is simple but not immediate to check the relation $\hat{Y}\hat{Y}' = \text{id}$ and other identities for $\hat{Y}_{\pm}$. The explicit formulas for $\hat{T}$ and $\hat{T}'$ are:

\[
\hat{T}(\begin{array}{c} f_1 \\ f_2 \end{array}) = \begin{array}{c} 0 \\ f_1 - f_2 \end{array},
\hat{T}'(\begin{array}{c} f_1 \\ f_2 \end{array}) = \begin{array}{c} f_1 \\ f_1 \end{array}.
\]

(3.46)

It readily gives (3.42), (3.43).

It is straightforward to check this. Generally, there is no need to establish and check the formulas for $\hat{X}$ and $\hat{X}'$, as they are simple. From Theorem 3.10,

\[
\hat{X} = \text{RIE}^\delta(\pi_-) \cdot \hat{T}, \quad \hat{X}' = \hat{T} \cdot \text{RIE}^\delta(\pi_-).
\]

Thus we need only to know $\hat{\pi} \overset{\text{def}}{=} \text{RIE}^\delta(\pi_-)$, where $\pi_- = XT$. We have

\[
\alpha^\delta(XT) = (\bar{t}^{-1/2}X)(t^{1/2}s + \frac{t^{1/2} - t^{-1/2}}{t^{-1}X^2 - 1}(s-1))
= \bar{t}^{-1/2}t^{1/2}Xs + \frac{X(\bar{t}^{-1/2}t^{1/2} - \bar{t}^{-1/2}t^{-1/2})}{\bar{t}^{-1}X^2 - 1}(s-1)
= \begin{array}{c} Xs \\ tX^{-1}s \end{array} + \begin{array}{c} \frac{X(1-t^{-1})}{\bar{t}^{-1}X^2 - 1}(s-1) \\ \frac{X^{1-t^{-1}}}{\bar{X}^{1-t^{-1}}}(s-1) \end{array}.
\]

Taking the limit $t \to 0$,

\[
\hat{\pi} = \begin{array}{c} Xs \\ 0 \end{array} + \begin{array}{c} -X^{-1}(s-1) \\ X^{-1}(s-1) \end{array} = \begin{array}{c} Xs - X^{-1}(s-1) \\ X^{-1}(s-1) \end{array}.
\]

Using the components,

\[
(3.47) \quad \hat{\pi} : \begin{array}{c} f_1 \\ f_2 \end{array} \mapsto \begin{array}{c} Xf_2 + \frac{f_1-f_2}{X} \\ \frac{f_1-f_2}{X} \end{array}.
\]

Check directly that $\hat{\pi}^2 = \text{id}$.

This formula completes the “component presentation” of the hat-module of $\overline{H}^{-}$ from Theorem 3.10:

\[
T, \pi_, Y \mapsto \hat{T}, \hat{\pi}, \hat{Y}.
\]

The extension of this Theorem to arbitrary (reduced) root systems is straightforward as well as the justification; we will address it (and the
applications) in further paper(s). The formulas for the $\Upsilon$—operators are of course getting more involved. The justifications in the spinor $q$—Toda theory (including global Whittaker functions) are entirely based on the DAHA-theory. We calculate and check practically everything explicitly in this work mainly to demonstrate the practical aspects of the technique of spinors (and because of novelty of this topic).

3.5.5. Spinor Whittaker function. Let us apply the procedure $RIE^s$ to the global difference spherical function $E_q(x, \lambda)$ from [Ch4], Section 5 (upon the specialization to the case of $A_1$). We do not give here its exact definition and do not discuss the details of the procedure. Actually, the only point that requires comments is using the conjugated $E$—polynomials, $E^*_b$ in the formula for $E_q$ in [Ch4]. Generally, the relation of $\{E_b\}$ and its conjugates is via the action of $T_{w_0}$; compare with Theorem 3.31 in the case of $A_1$.

We arrive at the following spinor nonsymmetric generalization of the function $W_q$ from (3.30) above:

$$\Omega(X, \Lambda) = q^{x^2} q^{\lambda^2} \left( 1 + \sum_{m=0}^{\infty} q^{m^2/4} \left( \frac{E_{-m}(\Lambda)}{\prod_{s=1}^{m} (1 - q^s)} \left\{ \begin{array}{c} X^m \\ q^m X^m \end{array} \right\} \right) \right).$$

(3.48)

Using the Pieri rules from (3.33), we can present it as follows:

$$\Omega = q^{x^2} q^{\lambda^2} \sum_{m=0}^{\infty} q^{m^2/4} \left( \frac{X^m E_{-m}(\Lambda)}{X^m \Lambda^{-1} E_{m+1}(\Lambda)} \right).$$

(3.49)

Either of these two presentation readily gives that the $t$—symmetrization of $\Omega$ is $\{W_q, W_q\}$, i.e., a symmetric spinor with the required components. To see it directly, use the component formula for $\hat{T}'$ from (3.46) when applying the symmetrizer $\mathcal{P}' = T' = T + 1$. Here $\Lambda$ is a (non-spinor) variable independent of $X$.

The spinor $\Omega$ intertwines the bar-representation of $\mathcal{H}_+$ and the hat-representation of $\mathcal{H}_-$. Namely,

$$\hat{Y}(\Omega) = \Lambda^{-1}(\Omega), \quad \hat{X}(\Omega) = \bar{Y}'(\Omega), \quad \hat{X}''(\Omega) = \bar{Y}_\Lambda(\Omega),$$

$$\hat{\pi}(\Omega) = \pi_\Lambda(\Omega), \quad \hat{T}(\Omega) = \bar{T}_\Lambda(\Omega),$$

(3.50)

where $\bar{Y}', \bar{Y}_\Lambda, \pi_\Lambda, \bar{T}_\Lambda$ act on the argument $\Lambda$; the other operators are $X$—operators. These (and other related identities) follow from the general theory for any reduced root systems (at least, in the twisted case). However, in the rank one case (and for $A_n$), one can use the
Pieri rules from (3.32), (3.33) and formulas (3.34), (3.35) for the direct verification.

Let us calculate $\nabla_A'(\Omega)$. First, $\nabla_A'(E_n(\Lambda)) = 0$ for $n > 0$. Second, $q^{-\lambda^2} \nabla_A' q^{\lambda^2} = q^{-1/4} \nabla_A' \Lambda$. For instance,

$$\nabla_A'(q^{\lambda^2}) = q^{-1/4} \nabla_A' (\Lambda) q^{\lambda^2} = \nabla_A'(E_1(\Lambda)) q^{\lambda^2} = 0.$$ 

We see that the second spinor component of $\nabla_A'(\Omega)$ vanishes, as it is supposed to be because the second component of $\hat{X}(\Omega)$ is obviously zero.

The first component reads as follows:

$$\nabla_A'(\Omega) = q^{-x^2} \hat{q}^{-x^2} \frac{\sum_{m=0}^{\infty} q^{m^2/4-1/4} X^m \nabla_A'( \Lambda E_{-m})}{\prod_{s=1}^{m} (1 - q^s)}$$

$$= q^{-x^2} \hat{q}^{-x^2} \frac{\sum_{m=0}^{\infty} q^{m^2/4-1/4-m/2+1/2} X^m (1 - q^m) E_{1-m}}{\prod_{s=1}^{m} (1 - q^s)}$$

$$= q^{-x^2} \hat{q}^{-x^2} X \sum_{m=1}^{\infty} q^{(m-1)^2/4} \frac{X_{1-m} E_{1-m}}{\prod_{s=1}^{m} (1 - q^s)},$$

which coincides with the first component of $\hat{X}(\Omega)$ (its second component is zero). We have used here the nil-Pieri formula:

$$\Lambda E_{-n} = (1 - q^n) E_{1-n} + E_{n+1} \text{ for } n > 0;$$

the second term, $E_{n+1}$, does not contribute to the final formula, since $\nabla_A'(E_{n+1}) = 0$.

The (key) relation $\hat{Y}(\Omega) = \Lambda^{-1}(\Omega)$ can be verified directly in a similar manner. First, $q^{-x^2} \hat{Y} q^{-x^2} = q^{1/4} X^{-1} \hat{Y}$. Therefore

$$(3.51) \quad q^{-x^2} \hat{Y} q^{-x^2} \left\{ \frac{f_1}{f_2} \right\} = q^{1/4} \left\{ X^{-1} \Gamma^{-1}(f_1 - f_2) \right\}.$$ 

Second, $\mathcal{F}_m \overset{\text{def}}{=} E_{-m}(\Lambda) - \Lambda^{-1} E_{m+1}(\Lambda) = (1 - q^m) \Lambda^{-1} E_{1-m}(\Lambda)$ (the Pieri rules). Now, $\Lambda^{-1} q^{-x^2} q^{-\lambda^2} \hat{Y}(\Omega) =$

$$\Lambda^{-1} \sum_{m=0}^{\infty} \frac{q^{m^2/4+1/4}}{\prod_{s=1}^{m} (1 - q^s)} \left\{ \frac{m}{\hat{m}} X^{m+1} \Lambda^{-1} E_{m+1}(\Lambda) + q^{\hat{m}} X^{-1} X^{-1} \Gamma (f_1 - f_2) \right\}$$

$$= \sum_{m=0}^{\infty} \frac{q^{m^2/4+1/4}}{\prod_{s=1}^{m} (1 - q^s)} \left\{ q^{\hat{m}} (1 - q^n) X^{m+1} E_{m+1} + q^{\hat{m}} X^{-1} X^{-1} E_{1-m} \right\}.$$
Collecting the terms with \((1 - q^m)\), we obtain that

\[
\tilde{\mathcal{Y}}(\Omega) = \Lambda^{-1} q x^2 \lambda^2 \sum_{m=1}^{\infty} q^{(m-1)^2/4} \prod_{s=1}^{m-1} (1 - q^s) \left\{ \begin{array}{l} X^{m-1} E_{1-m}(\Lambda) \\ q^{m-1} X^{m-1} E_{1-m}(\Lambda) \end{array} \right\} + \Lambda^{-1} q x^2 \lambda^2 \sum_{m=0}^{\infty} q^{(m+1)^2/4} \prod_{s=1}^{m} (1 - q^s) \left\{ \begin{array}{l} 0 \\ X^{m+1} E_{m+1}(\Lambda) \end{array} \right\},
\]

i.e., exactly the presentation from (3.48) multiplied by \(\Lambda^{-1}\).

Formulas (3.51), (3.47) and (3.46) result in the definition of the spinor-polynomial representation:

\[
\mathcal{X}_{\text{spin}} = \mathbb{C} \oplus \left( \oplus_{m=1}^{\infty} (\mathbb{C} \{X^m, 0\} \oplus \mathbb{C} \{0, X^m\}) \right).
\]

**Theorem 3.11.** The space \(\mathcal{X}_{\text{spin}}\) is an irreducible \(\overline{\mathcal{H}}_-\)-submodule of the space of spinors over \(\mathbb{C}[X^{\pm 1}]\) supplied with the twisted action:

\[
\overline{\mathcal{H}}_- \ni A \mapsto q^{-x^2} \tilde{A} q^{x^2}.
\]

More explicitly, \(\mathcal{X}_{\text{spin}}\) is invariant and irreducible under the action of operators \(\tilde{T}, \tilde{\pi}\) and \(q^{-x^2} \tilde{Y} q^{x^2}\). \(\square\)

General theory of spinor nonsymmetric Whittaker functions will be published elsewhere. Let us now consider the technique of spinors in the differential setting.

### 4. Differential theory

#### 4.1. The degenerate case.

**4.1.1. Degenerate DAHA.** Let us begin with the definition of degenerate double affine Hecke algebra for an arbitrary (reduced) root system \(R\). Recall that \(\tilde{W} = W \ltimes P^\vee\) for the coweight lattice \(P^\vee\).

**Definition 4.1.** The degenerate double affine Hecke algebra \(\tilde{\mathcal{H}}\) is generated by \(\tilde{W}\) (with the corresponding group relations) and pairwise commutative elements \(y_b, b \in P\) satisfying the following relations:

\[
\begin{align*}
s_i y_b - y_{s_i(b)} s_i &= -k(b, \alpha_i^\vee) \quad \text{for } i \geq 1, \\
s_0 y_b - y_{s_0(b)} s_i &= k(b, \theta) \quad \text{and} \quad \pi_r y_b = y_{\pi_r(b)} \pi_r,
\end{align*}
\]

where \(y_{[b, j]} = y_b + j, y_{b+c} = y_b + y_c\).

Note that in contrast to the definition of DAHA from (2.2)), \(y_b\) are labeled by \(b \in P\) (not by \(P^\vee\)). It is convenient because \(X_a\) (to be introduced later) will be naturally labeled by \(a \in P^\vee\).

Due to the additive dependence of \(y_b\) of \(b\), the exact choice (\(P\) or \(P^\vee\)) is not too important here; one can even take \(b \in \mathbb{C}^n\). Similarly, changing
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(b, α∨i) to (b, αi) will simply re-scale the k–parameters. However, the exact choice of the lattice is important to ensure the compatibility of this definition with the limit q → 1 from the q, t–DAHA (see below).

The operators Xa are simply the translations a ∈ P∨i considered as elements of ℋ′. The PBW Theorem holds for {Xa, yb, W}. This algebra appeared for the first time as the limit q → 1 of the q, t–DAHA; see [Ch1], Chapter 2, Section "Degenerate DAHA". There is another approach to its definition via the compatibility and \(\hat{W}\)–equivariance of the affine infinite Knizhnik-Zamolodchikov equation from [Ch3, Ch9]. We note that this equation at critical level is essentially equivalent to the eigenvalue problem for the elliptic deformation of the Heckman-Opdam operators (due to Olshanetsky, Perelomov and others); see [Ch9].

Let us consider the \(A_1\)–case. Then \(ℋ′\) will be generated by \(s, \pi, y\) with the following defining relations:

\[ s^2 = 1, \quad sy + ys = -k, \quad \pi y = \left(\frac{1}{2} - y\right)\pi. \]

Recall that we set \(s = s_1, \quad \omega = \omega_1, \quad \pi = \omega s, \quad y = y_\omega\); for instance, \(\pi(\omega) = \left[-\omega, \frac{1}{2}\right]\).

Letting \(X = \pi s\), one has that \(sXs = X^{-1}, \quad (Xs)y = \left(\frac{1}{2} - y\right)(Xs)\) and, finally,

\[ X(-k - ys) = \left(\frac{1}{2} - y\right)Xs \Rightarrow [y, X] = \frac{1}{2}X + kXs. \]

Similar to DAHA, \(ℋ′\) can be represented as \(\langle y, s, X^{\pm 1}\rangle\) subject to the relations:

\[(4.2) \quad sxs = x^{-1}, \quad sy + ys = -k, \quad s^2 = 1, \quad [y, X] = \frac{1}{2}X + kXs. \]

The corresponding limiting degeneration of \(ℋ\) from (3.1) is as follows. We set \(q = \exp(h), \quad t = q^k = \exp(hk)\). Let \(Y = \exp(-hy)\), \(X = X\) and \(T = s + \frac{k}{2}X\). The letter relation is necessary to ensure that the quadratic relation holds modulo \((h^2)\). Indeed, then

\[ T^2 = 1 + hks = (t^{1/2} - t^{-1/2})T + 1 \mod(h^2). \]

For instance, the coefficient of \(h\) in \(TY^{-1}T = Y\) readily gives that \(sys + ks = -y\).

4.1.2. Polynomial representation. Continuing with the \(A_1\)–case, \(X\) and \(s\) remain the same as in the \(q, t\)–case, however, now we set \(X = e^2\). The generator \(y\) is mapped to the differential operator

\[(4.3) \quad y = \frac{1}{2}d + \frac{k}{1 - X^2}(1 - s) - \frac{k}{2}, \]
called the trigonometric Dunkl or Cherednik-Dunkl operator. It is simple to check directly that \( sy + y = -ks \) and that

\[
[y, X] = \frac{1}{2}X + \frac{k}{1 - X^{-2}}(Xs - X^{-1}s) = \frac{1}{2}X + kXs.
\]

The constant \(-k/2\) in formula (4.3) automatically results from the limiting procedure. However, its importance can be clarified without any reference to DAHA or degenerate DAHA.

**Lemma 4.2.** Let \( \Delta_k \overset{\text{def}}{=} (e^x - e^{-x})^k \). Then

\[
\tilde{y} \overset{\text{def}}{=} \Delta_k y \Delta_k^{-1} = \frac{1}{2} \frac{d}{dx} - \frac{k}{1 - X^{-2}s}.
\]

**Proof.** Indeed, we have

\[
\Delta_k y \Delta_k^{-1} = \frac{1}{2} \frac{d}{dx} - \frac{k}{2} \frac{e^x + e^{-x}}{e^x - e^{-x}} + \frac{k}{1 - X^{-2}}(1 - s) - \frac{k}{2}
\]

\[
= \frac{1}{2} \frac{d}{dx} + \frac{k}{2} \left( 1 - \frac{2e^x}{e^x - e^{-x}} \right) + \frac{k}{1 - X^{-2}}(1 - s) - \frac{k}{2}
\]

\[
= \frac{1}{2} \frac{d}{dx} - \frac{k}{1 - X^{-2}s}.
\]

\[
\square
\]

Thus, the constant \(-k/2\) is necessary to make the conjugation of the trigonometric Dunkl operator by \( \Delta_k \) with pure \( s \) (but then it will not preserve Laurent polynomials). We mention that the trigonometric Dunkl operators were introduced in [Ch11] in terms of \((c - s)\) for an arbitrary constant \( c \) (including \( c = 0 \)) and in the matrix setting. We see that the constant \( c \) can be changed using conjugations by powers of the discriminant.

**Comment.** For complex \( k \), we need to take \(|e^x - e^{-x}|^k\) here. However, the claim of the lemma is entirely algebraic. The best way to proceed here algebraically is to conjugate by the *even spinor*

\[
\{ (e^x - e^{-x})^k, (e^x - e^{-x})^k \}
\]

for any branch of \((e^x - e^{-x})^k\). It is the first appearance of spinors in this part of the paper. \(\square\)

4.1.3. The self-adjointness. Let us first establish the connection of the trigonometric Dunkl operator to the \( k \)-deformation of the Harish-Chandra theory of the radial parts of Laplace operators on symmetric spaces. One has

\[
L' \overset{\text{def}}{=} 2y^2|_{\text{sym}} = \frac{1}{2} \frac{d^2}{dx^2} + k \left( \frac{1 + e^{-2x}}{1 - e^{-2x}} \right) \frac{d}{dx} + \frac{k^2}{2}.
\]
The restriction to symmetric (even) functions simply means that we move all $s$ to the right and then delete them.

In the Harish-Chandra theory, $k$ is one-half of the root multiplicity of the restricted root system corresponding to the symmetric space. For instance, $k = 1$ in the so-called group case. Let us mention the contributions of Koornwinder, Calogero, Sutherland, Heckman, Opdam and van den Ban to developing the theory for arbitrary $k$. See, e.g, [HO1] (we do not need anything beyond this paper in this section).

Lemma 4.2 readily gives that

$$\widetilde{L}' \overset{\text{def}}{=} \Delta_k L' \Delta_k^{-1} = \frac{1}{2} \frac{d^2}{dx^2} + \frac{2k(1 - k)}{(e^x - e^{-x})^2}.$$

Now let us discuss the inner product. We set formally:

$$\langle f, g \rangle \overset{\text{def}}{=} \int f(x)g(-x)\Delta^2_k dx.$$

For instance, the integration here can be taken over $\mathbb{R}$; then $\Delta^2_k$ must be understood as $|e^x - e^{-x}|^{2k}$; the functions $f, g$ must be chosen to ensure the convergence.

The anti-involution $^+$ (formally) corresponding to the “free” inner product $\int f(x)g(-x)dx$ is as follows:

$$x^+ = x, \quad (\frac{d}{dx})^+ = \frac{d}{dx}.$$

Then the anti-involution $A^\circ = \Delta^{-2}_k A^+ \Delta^2_k$ serves $\langle f, g \rangle$.

Lemma 4.3. One has:

$$X^\circ = X^{-1}, \quad y^\circ = y, \quad s^\circ = s,$$

which implies that $(L')^\circ = L'$.

Proof. One can check the self-adjointness of $y$ and $L'$ directly. However, the best way is via Lemma 4.2 (and, first, for $y$ and, second, for $L'$). We use that $\widetilde{y}^+ = \widetilde{y}$:

$$y^\circ = \Delta^{-2}_k (\Delta^{-1}_k \widetilde{y} \Delta_k)^+ \Delta^2_k = \Delta^{-2}_k (\Delta_k \widetilde{y} \Delta^{-1}_k) \Delta^2_k = \Delta^{-1}_k \widetilde{y} \Delta_k = y.$$

4.1.4. The Ruijsenaars-Inozemtsev substitution. The construction is as follows. We begin with $\widetilde{L}' = \frac{1}{2} \frac{d^2}{dx^2} + \frac{2k(1 - k)}{(e^x - e^{-x})^2}$, replace $x$ by $x + M$ and connect $M$ with $k$ by the relation $k(1 - k) = e^{2M}$. Finally, we set $\Re M \rightarrow +\infty$. The resulting operator is $\frac{1}{2} \frac{d^2}{dx^2} + 2e^{-x}$.

Applying this method to arbitrary root systems, one obtains a system of pairwise commutative operators, the Toda operators. In contrast to $L'$, these operators are not $W$–invariant. The Whittaker function is their eigenfunction. Given a weight (the set of eigenvalues), the
dimension of the corresponding space of all eigenfunctions is $|W|$. The “true” Whittaker function can be fixed uniquely using certain decay conditions.

Let us give a reference to paper [Shim], where this procedure was applied to the Heckman-Opdam functions from [HO1]; their limits are, indeed, the true Whittaker ones. Note that $k$ must be arbitrary for the Ruijsenaars-Inozemtsev procedure. It is impossible to obtain the Whittaker function directly from the classical Harish-Chandra spherical function (which is for very special $k$). It is similar to the $p$–theory, where the passage from the Satake-Macdonald spherical function to the $p$–adic Whittaker function can be established only via the $q$–deformation.

4.2. Dunkl operator and Bessel function. Let $X = e^{\varepsilon x}$ with $\varepsilon > 0$. Then the trigonometric Dunkl operator $y$ becomes

$$
\frac{1}{2\varepsilon} \frac{d}{dx} + \frac{k}{2\varepsilon x} (1 - s) - \frac{k}{2} + o(\varepsilon).
$$

Letting $\varepsilon \to 0$,

$$\varepsilon y \to \frac{1}{2} \frac{d}{dx} + \frac{k}{2x} (1 - s).
$$

We will use the same letter $y$ for the right-hand side. However, mainly we will use the Dunkl operator:

$$\mathcal{D} \overset{\text{def}}{=} 2y = \frac{d}{dx} + \frac{k}{x} (1 - s).
$$

This definition is due to Charles Dunkl [Du1], who introduce Dunkl (rational) operators for arbitrary root systems and also for some groups generated by complex reflections.

4.2.1. Rational DAHA.

**Definition 4.4.** The rational double affine Hecke algebra $\mathcal{H}^{\text{rat}}$ is generated by $x, y, s$ with the following relations:

$$sxs = -x, \ sys = -y, \ s^2 = 1, \ [y, x] = \frac{1}{2} + ks.
$$

It is the limit of the relations from (4.2). An abstract (and very general) variant of this definition is actually due to Drinfeld [Dr].

The assignment: $x \to x$, $y \to \mathcal{D}/2$, $s \to s$ defines the polynomial representation of $\mathcal{H}^{\text{rat}}$ in $\mathbb{C}[x]$. It is an induced module from the character of the subalgebra generated by $y, s$ sending $y$ to $y(1) = 0$ and $s$ to $s(1) = 1$. The PBW Theorem is almost immediate in the rational setting (it also follows from the existence of the polynomial representation).
Upon the symmetrization of $\mathcal{D}^2$, we obtain the key operator in the classical theory of Bessel functions:

$$L \overset{\text{def}}{=} \mathcal{D}^2|_{\text{sym}} = \frac{d^2}{dx^2} + \frac{2k}{x} \frac{d}{dx}.$$  

**Lemma 4.5.** (i) One has:

$$x^k \cdot \mathcal{D} \cdot x^{-k} = \tilde{\mathcal{D}} \overset{\text{def}}{=} \frac{d}{dx} - \frac{k}{x} s, \quad x^k \cdot L \cdot x^{-k} = \tilde{L} \overset{\text{def}}{=} \frac{d^2}{dx^2} + \frac{k(1-k)}{x^2}.$$  

(ii) Let $A^\diamond = x^{-2k} \cdot A^* \cdot x^{2k}$, where the anti-involution $*$ is as follows:

$$x^* = x, \quad (\frac{d}{dx})^* = -\frac{d}{dx};$$

the anti-involution $\diamond$ formally serves the form $\langle f, g \rangle = \int f(x)g(x)x^{2k}dx$. Then $\mathcal{D}^\diamond = -\mathcal{D}$, and $L^\diamond = L$.  

4.2.2. **Bessel functions.** Assuming that $\lambda \neq 0$, an arbitrary solution $\varphi^{(k)}_\lambda$ of the eigenvalue problem

$$L\varphi^{(k)}_\lambda = 4\lambda^2 \varphi^{(k)}_\lambda$$

analytic in a neighborhood of $x = 0$ can be represented as

$$\varphi^{(k)}_\lambda(x) = \varphi^{(k)}(x\lambda).$$

Here $\varphi^{(k)}$ can be readily calculated:

$$\varphi^{(k)}(t) = \sum_{m=0}^{\infty} \frac{t^{2m} \Gamma(k + 1/2)}{m! \Gamma(k + n + 1/2)}$$

for the classical Gamma–function: $\Gamma(x + 1) = x\Gamma(x)$, $\Gamma(1) = 1$. The parameter $k$ is arbitrary here provided that $k \neq -1/2 - m$ for $m \in \mathbb{Z}_+$. The function $\varphi^{(k)}(t)$ is a variant of the Bessel $J$–function.

See [O2] (and references therein) for the theory of multi-dimensional Bessel functions.

Notice that

$$\varphi^{(k)}(t) \xrightarrow{k \to -} \sum_{m=0}^{\infty} \frac{(2t)^{2m}}{(2m)!} = \frac{e^{2t} + e^{-2t}}{2},$$

due to the relations:

$$\Gamma(n + 1)\Gamma(n + \frac{1}{2}) = 2^{-2n}(2n)!\sqrt{\pi}, \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$  

Using the passage to the Sturm-Liouville operator $\tilde{L}$, we can control the growth of $\varphi^{(k)}_\lambda$ at infinity.
Lemma 4.6. The differential equation $L \varphi = 4\lambda^2 \varphi$ has the following two fundamental solutions for real $x$. If $\lambda = 0$, then 1 and $x^{1-2k}$ can be taken. If $\lambda \neq 0$, the asymptotic behavior can be used:

$$\varphi^\pm_\lambda = x^{-k} e^{\pm 2\lambda x} (1 + o(1)) \text{ as } x \to +\infty.$$ 

Any solution $\varphi$ is a linear combination of these two. In particular, the growth of any solution as $x \to \pm \infty$ is no greater than exponential, namely, $O(x^{-\Re k} e^{\pm 2\Re \lambda})$ for $\lambda \neq 0$.

We will use this lemma only for justifying that the Gauss-Bessel integrals we need are well defined. The following is the classical formula; see Introduction and Chapter 1 from [Ch1] for a more comprehensive exposition.

4.3. Symmetric master formula.

4.3.1. Hankel transform.

Theorem 4.7.

$$\int_{-\infty}^{+\infty} \varphi^{(k)}_\lambda(x) \varphi^{(k)}_\mu(x) e^{-x^2} |x|^{2k} dx = \Gamma(k + \frac{1}{2}) \varphi^{(k)}_\mu(\lambda) e^{\lambda^2 + \mu^2},$$

where $\Re k > -\frac{1}{2}$. The normalization is given by the Euler integral:

$$\int_{-\infty}^{+\infty} e^{-x^2} |x|^{2k} dx = \Gamma(k + \frac{1}{2}).$$

Here one can set $\int_{-\infty}^{+\infty} = 2 \int_{0}^{+\infty}$, since all functions are even. □

In order to prove Theorem 4.7, we need the following definition.

Definition 4.8. The Hankel transform for even functions $f$ is given by

$$\mathcal{H} f(\lambda) = \frac{1}{\Gamma(k + \frac{1}{2})} \int_{\mathbb{R}} f(x) \varphi^{(k)}_\lambda(x) |x|^{2k} dx$$

in proper functional spaces.

4.3.2. Its properties. Let us denote the operator $L$ acting in the $\lambda$–space by $L_\lambda$; $L$ without suffix $\lambda$ continues to be an operator in terms $x$. Recall that the operator $L$ depends on $k$ (we will sometimes denote it by $L^{(k)}$).

Lemma 4.9. For any functional spaces (not only for even functions), provided the existence,

(a) $\mathcal{H}(L) = 4\lambda^2$, $\mathcal{H}(4x^2) = L_\lambda$;
(b) $e^{-x^2} L e^{x^2} = L + 4x^2 + [L, x^2]$. 

**Proof.** Claim (a) is based the $x \leftrightarrow \lambda$–symmetry of $\varphi^{(k)}_\lambda(x)$ and on the self-adjointness of the operators $L$ and $x^2$ with respect to the measure we consider.

Checking (b) is direct. One can also use the following important connection with the theory of $\mathfrak{sl}(2)$. Setting, $e = x^2, f = -\frac{L}{4}, h = [e, f] = x \frac{d}{dx} + \frac{1}{2} + k,$

we obtain a representation of this Lie algebra. Then $e^{-x^2}Le^{x^2}$ can be interpreted and calculated using the adjoint action of $SL_2$. It must be a priori a linear combination of $e, f, h$; the exact formula is simple. Note that the Hankel transformation becomes the group element $s \in SL_2$ in this interpretation. □

4.3.3. Tilde-Bessel functions. Let us try to apply the master formula to other solutions of the eigenvalue problem (4.4). The proof looks very algebraic; we even did not use that $\varphi^{(k)}_\lambda(x)$ is even.

For $\lambda \neq 0$, there exists another solution $\tilde{\varphi}^{(k)}_\mu(x) = (x \lambda)^{1-2k} \varphi^{(1-k)}_\lambda(x)$ of (4.4). If $\lambda = 0$, let $\tilde{\varphi}^{(k)}_\mu(x) \overset{\text{def}}{=} x^{1-2k}$. We need to assume that $\Re(k) < 1/2$ to avoid the singularity at 0 in these solutions.

Formally, we have proved that

\[(4.7) \quad \mathcal{H}(\tilde{\varphi}^{(k)}_\mu e^{-x^2}) = \tilde{\varphi}^{(k)}_\mu(\lambda)e^{\lambda^2+\mu^2}\]

for a certain solution $\tilde{\varphi}^{(k)}_\mu$ of the same eigenvalue problem, a linear combination of $\varphi^{(k)}_\mu$ and $\tilde{\varphi}^{(k)}_\mu$. If we assume that $0 < \Re(k) < 1/2$ and plug in $\mu = 0$, then $\tilde{\varphi}^{(k)}_0(0) = 0$ and upon obvious cancelations, we come to the following new identity in the theory of Bessel functions:

$$\int_{-\infty}^{+\infty} \varphi^{(k)}_\lambda(x)x|e^{-x^2}dx = e^{\lambda^2}.$$  

*However this formula is wrong.*

Informally, it is wrong simply because no new identities of such kind can be expected in this very classical field. Formally, the error occurred because the integration by parts, necessary for the self-adjointness
claim, requires the convergence at 0 of the first two derivatives of the functions involved. The existence of the starting and the final integral can be insufficient, and, as we see, leads to a mistake.

The following analytic constraints make claim (ii) of Lemma 4.5 rigorous. Actually, these conditions can be somewhat relaxed.

Provided that $f, g \in C^2(\mathbb{R}_+)$ and $f(x)|x|^k$, $g(x)|x|^k$ are absolutely integrable,

$$\int_{-\infty}^{+\infty} L(f)g|x|^{2k}dx = \int_{-\infty}^{+\infty} fL(g)|x|^{2k}dx.$$

4.3.4. Complex analytic theory. The deduction of (4.7) from the properties of the Hankel transform is of course formally correct; it simply gives nothing new in the case of real integration due to the divergence at 0 of the derivatives of the tilde-solution. Let us switch to the Laplace-type integration, which was design exactly to avoid the divergences of this kind.

Let us first re-establish the usual master formula in this setting.

**Theorem 4.10.** For all $k \in \mathbb{C}$ such that $k \neq -\frac{1}{2} - m$, $m \in \mathbb{Z}_+$,

$$\int_{i\varepsilon + \mathbb{R}} \varphi^{(k)}(x)\varphi^{(k)}(x)e^{-x^2}(-x^2)^kdx = \frac{\pi}{\Gamma(\frac{1}{2} - k)}\varphi^{(k)}(\mu)e^{\lambda^2+\mu^2}.$$

Here $\varepsilon > 0$; the condition $k \neq -\frac{1}{2} - m$ is necessary for the existence of $\varphi^{(k)}(x)$. □

For any complex number $k$, the function $(-x^2)^k$ is defined as the function $\exp(k \log(-x^2))$ continued along the integration path $x \in i\varepsilon + \mathbb{R}$ for the usual branch of log with the cutoff at $\mathbb{R}_-$. Using $(-x^2)^k$ is quite standard in classical works on $\Gamma$ and related functions.

Due to the Gamma-term in the right-hand side, this integral must be zero at $k = \frac{1}{2} + m$, $m \in \mathbb{Z}_+$. It is simple to demonstrate directly. Indeed,

$$(-x^2)^{1/2} = -ix \quad \text{along the path } i\varepsilon + \mathbb{R};$$

check the point $x = i\varepsilon$ using that $(\varepsilon^2)^{1/2} = \varepsilon$. The integrand is analytic at zero for such $k$, so we can tend $\varepsilon \to 0$. However the integrand is an odd function on $\mathbb{R}$ and, therefore,

$$\int_{i\varepsilon + \mathbb{R}} \varphi^{(k)}(x)\varphi^{(k)}(x)e^{-x^2}(-ix)^{2m+1}dx = 0.$$
Similarly, for \( \tilde{\varphi}_\lambda(x) \) defined as \(-\lambda^{2/2-k}(-x^2)^{1/2-k}\varphi^{(1-k)}_\lambda(x)\), which is the complex analytic variant of the tilde-solution considered above,

\[
\int_{i\varepsilon + \mathbb{R}} \varphi^{(k)}_\lambda(x)\tilde{\varphi}^{(k)}_\mu(x)(-x^2)^{(k)}e^{-x^2}dx
= \int_{i\varepsilon + \mathbb{R}} \varphi^{(k)}_\lambda(x)\varphi^{(1-k)}_\mu(x)(-x^2)^{1/2}dx
= \int_{\mathbb{R}} \varphi^{(k)}_\lambda(x)\varphi^{(1-k)}_\mu(x)(-ix)dx = 0.
\]

Thus, the standard solution \( \varphi^{(k)}_\lambda(x) \) and the complex-analytic tilde-solution are orthogonal to each other in the master formula.

It is straightforward to calculate the master formula for the tilde-solutions \( \tilde{\varphi}^{(k)}_\lambda(x) \), \( \tilde{\varphi}^{(k)}_\mu(x) \) coupled together in the Gauss-Bessel integral. We will provide the corresponding formulas below when doing the non-symmetric master formula.

4.4. Nonsymmetric theory.

4.4.1. Dunkl eigenvalue problem. We will begin with the eigenvalue problem for the Dunkl operator. It is not a differential operator, but it shares some (but not all) properties with the first order differential operators.

Lemma 4.11. (i) The eigenvalue problem

\[
\mathcal{D}\psi = 2\lambda\psi, \text{ for } \mathcal{D} = \frac{d}{dx} + \frac{k}{x}(1-s)
\]

has a unique analytic at 0 solution \( \psi = \psi^{(k)}_\lambda(x) \) satisfying \( \psi(0) = 1 \) if and only if \( k \notin -1/2 - \mathbb{Z}_+ \).

(ii) Namely, it is \( \psi = 1 \) for \( \lambda = 0 \) and \( \psi(x) = \psi^{(k)}(\lambda x) \) for

\[
\psi^{(k)}(t) = \varphi^{(k)}(t) + \frac{1}{2}(\varphi^{(k)})'(t)
\]

in terms of \( \varphi^{(k)}(t) \) from (4.5).

(iii) When \( \lambda = 0 \) and \( k = -\frac{1}{2} - m \), the space of analytic solutions is generated by \( \psi = 1 \) and \( \psi = x^{2m+1} \). When \( \lambda \neq 0 \) for the same \( k \), the analytic solution \( \psi \) exists and is unique up to proportionality, but vanishes at 0.

The fact that the dimension of the space of solutions of (4.8) can be 2 (for special values of the parameters) requires serious consideration and will eventually lead us to the spinor extension of the space of functions.
4.4.2. Nonsymmetric master formula. For \( k \neq -1/2 - m, m \in \mathbb{Z}_+ \) and the function \( \psi^{(k)}_\lambda(x) = \psi^{(k)}(\lambda x) \) from Lemma 4.11, the following holds.

**Theorem 4.12.** (i) For \( \Re k > -1/2 \),
\[
\int_{\mathbb{R}} \psi^{(k)}_\lambda(x) \psi^{(k)}_\mu(x) e^{-x^2} |x|^{2k} dx = \Gamma(k + \frac{1}{2}) \psi^{(k)}_\lambda(\mu) e^{\lambda^2 + \mu^2}.
\]

(ii) Denote \( \int_{\mathbb{R}}^{\epsilon} \) def. \( = \frac{1}{2}(\int_{i\epsilon+\mathbb{R}} + \int_{-i\epsilon+\mathbb{R}}) \), then
\[
\int_{\mathbb{R}}^{\epsilon} \psi^{(k)}_\lambda(x) \psi^{(k)}_\mu(x) e^{-x^2} (-x^2)^k dx = \frac{\pi}{\Gamma(\frac{1}{2} - k)} \psi^{(k)}_\lambda(\mu) e^{\lambda^2 + \mu^2}.
\]

**Proof.** As in the symmetric theory, the formula readily results from the basic facts concerning the non-symmetric Hankel transform. The (general) definition of this transform is due to Dunkl [Du2]. Its one-dimensional version can be found in Hermite’s works, but it was used only marginally in the classical theory. It is given by:
\[
\mathbb{H}_{ns} f(\lambda) = \frac{1}{\Gamma(k + \frac{1}{2})} \int_{\mathbb{R}} f(x) \psi^{(k)}_\lambda(x) |x|^{2k} dx,
\]
provided the existence. Its theory is actually simpler than that of the classical symmetric Hankel transform (at least, the algebraic aspects). We use the notation \( \mathcal{D}_\lambda \) for the Dunkl operator acting in the \( \lambda \)-space.

**Lemma 4.13.** Provided the existence,

(a) \( \mathbb{H}_{ns}(\mathcal{D}) = 2\lambda, \mathbb{H}_{ns}(2x) = \mathcal{D}_\lambda \);

(b) \( e^{-x^2} \mathcal{D} e^{x^2} = \mathcal{D} + 2x \). \(
\)

The following analytic conditions for the functions \( f, g \) and their derivatives \( f', g' \) are sufficient to ensure that
\[
\int_{\mathbb{R}} \mathcal{D}(f) g dx = -\int_{\mathbb{R}} f \mathcal{D}(g) dx:
\]

(1) \( f(x), g(x) \) are continuous and \( f'(x), g'(x) \) exist in \( \mathbb{R} \setminus 0 \);

(2) the function \( f(x)g(x)|x|^{2k} \) is integrable and continuous at 0;

(3) \( f(x)g(x)|x|^{2k-1}, f'(x)g(x)|x|^{2k}, f(x)g'(x)|x|^{2k}, f(x)g(-x)|x|^{2k} \) are integrable at zero.

It gives (a) for the real integration. Only the integrability at infinity is needed to justify (a) in the case of the integration \( \int_{\mathbb{R}}^{\epsilon} \). The theorem readily follows from the lemma.

**Comment.** Similar to the symmetric case, the integrals from Theorem 4.12 in the complex case are identically zero as \( k \in 1/2 + \mathbb{Z}_+ \). It corresponds to the vanishing condition of the inner products associated with level one coinvariants from Theorem 2.10. See also Section 2.7.4 (the real case).
A NEW TAKE ON SPHERICAL, WHITTAKER AND BESSEL FUNCTIONS

The affine symmetrizer $\hat{\mathcal{I}}$ from (2.24) is a $q,t$–Jackson counterpart of the integration $\int_{i\varepsilon} f(x)(-x^2)^k \, dx$. The zeros of the inner product $\hat{\mathcal{I}}(f \cdot T(g))$ for $A_1$ are exactly in the set $\frac{1}{2} + \mathbb{Z}_+$.

4.5. Spinors.

4.5.1. Basic features. The theory of the nonsymmetric tilde-solutions requires the technique of spinors. They are pairs $f = \{f_1, f_2\}$ of functions defined in an open set $U$ in $\mathbb{R}$ or $\mathbb{C}$. Real spinors are defined for $U = \{x \in \mathbb{R}, x > 0\}$; complex spinors are defined for the set $U = \{x \in \mathbb{C}, \exists x > 0\}$. The operators act naturally on spinors; see Section 3.5.1. For instance,

$$s\{f_1, f_2\} = \{f_2, f_1\}, \quad x\{f_1, f_2\} = \{xf_1, -xf_2\}, \quad \{f_1, f_2\}' = \{f_1', -f_2'\},$$

where here and below $f' \overset{\text{def}}{=} df/dx$.

The super-presentation of a spinor $f$ is defined to be

$$f = [f^0, f^1], \quad \text{where} \quad f^0 = \frac{f_1(x) + f_2(x)}{2}, \quad f^1 = \frac{f_1(x) - f_2(x)}{2}.$$ 

For any two spinors, $f = \{f_1, f_2\}, g = \{g_1, g_2\}$, their product is given by $f \cdot g = \{f_1g_1, f_2g_2\}$. In the super-presentation:

$$f \cdot g = [f^0g^0 + f^1g^1, f^0g^1 + f^1g^0].$$

It is the standard stuff about $\mathbb{Z}_2$–graded algebras.

A spinor $f = \{f_1, f_2\}$ is called a principal spinor (function) if the following holds. There must exist an open connected set $\tilde{U}$ and a function $\tilde{f}$ on $\tilde{U}$ such that $U, U^s \overset{\text{def}}{=} s(U) \subset \tilde{U}$ and $f_1 = \tilde{f}|_U$, $f_2 = s(\tilde{f})|_U$.

The differentiation of spinors $d/dx$ is an odd operator defined by

$$\frac{d}{dx}\{f^0, f^1\} = \{\frac{d}{dx}f^1, \frac{d}{dx}f^0\}.$$ 

The spinor integration is given by

$$\int_\gamma [f^0, f^1] \overset{\text{def}}{=} \int_\gamma f^0,$$

where $\gamma \subset U$ is a path in the set $U$. 
4.5.2. Spinor eigenfunctions. The Dunkl spinor eigenvalue problem is

\( D(\psi) = \llbracket (\psi^1)' + \frac{2k\psi^1}{x}, (\psi^0)' \rrbracket = \llbracket 2\lambda\psi^0, 2\lambda\psi^1 \rrbracket. \)

In the standard representation \( \{\psi_1, \psi_2\} \), it reads as follows:

\[
D(\psi) = \left\{ \begin{array}{l}
\psi_1' + \frac{k(\psi_1 - \psi_2)}{x}, \\
-\psi_2' - \frac{k(\psi_2 - \psi_1)}{x}
\end{array} \right\} = \llbracket 2\lambda\psi_1, 2\lambda\psi_2 \rrbracket.
\]

Lemma 4.14. The space of solutions of the eigenvalue problem (4.10) is always two-dimensional. There are three cases:

1. if \( \lambda \neq 0 \), then all the solutions are in the form \( \psi = \llbracket \varphi, \frac{\varphi'}{2\lambda} \rrbracket \) for \( \varphi \) satisfying \( L\varphi = 4\lambda^2\varphi \), and only one of them (up to proportionality) is a function (i.e., a principle spinor);
2. if \( \lambda = 0 \) and \( k \not\in \mathbb{Z} \) then \( \psi = 1 \) is a solution and also there is an odd spinor solution \( \chi_k \), given by \( \chi_k = \llbracket 0, |x|^{-2k} \rrbracket \) in the real case and \( \chi_k = \llbracket 0, (-x^2)^{-k} \rrbracket \) in the complex case;
3. when \( \lambda = 0 \) and \( k = -1/2 - m \) for \( m \in \mathbb{Z}_+ \), then the solutions are \( 1 \) and \( x^{2m+1} \), i.e., both are principle spinors.

4.5.3. Nonsymmetric tilde-solutions. For \( k \not\in \mathbb{Z}_+ \), the spinor

\[
\tilde{\psi}^{(k)}_{\lambda} = \chi_k(x)\chi_\lambda(\lambda)\psi^{(-k)}_{\lambda}(x)
\]

satisfies (4.10). Actually it is a bi-spinor, in terms of \( x \) and \( \lambda \); we will skip the formal definition.

Let us incorporate the tilde-solution into the master formula. We need to redefine the inner product. Let

\[
x^{2k} \overset{\text{def}}{=} \left\{ \begin{array}{l}
\llbracket |x|^{2k}, 0 \rrbracket, \\
\llbracket (-x^2)^k, 0 \rrbracket
\end{array} \right\}
\]

I.e., both are even spinors (functions, if \( k \in \mathbb{Z} \)). Note that \( \chi_k(x)x^{2k} = \llbracket 0, 1 \rrbracket \) is an odd constant (a spinor of course). The integration will be

\[
\int f(x) \overset{\text{def}}{=} 2 \int_{0}^{+\infty} f^0(x)dx \quad \text{in the real case};
\]

\[
\int f(x) \overset{\text{def}}{=} \int_{i\epsilon+\mathbb{R}} f^0(x)dx \quad \text{in the complex case}.
\]

Let us check that \( \psi \)-solutions and \( \tilde{\psi} \)-solutions are orthogonal to each other in the master formula. Similar to the symmetric case, we have the divergence problem with the integration by parts, so only the complex case can be considered. Then the integral

\[
\int \psi^{(k)}_{\lambda}(x)\tilde{\psi}^{(k)}_{\mu}(x)e^{-x^2}x^{2k}
\]
is proportional to

\[ I = \int_{i\epsilon+\mathbb{R}} e^{-x^2} (\psi^{(k)}_\lambda \psi^{(-k)}_\mu \cdot [0, 1])^0 \, dx = \int_{i\epsilon+\mathbb{R}} e^{-x^2} (\psi^{(k)}_\lambda \psi^{(-k)}_\mu)^1 \, dx. \]

However, \( e^{-x^2} \psi^{(k)}_\lambda(x) \psi^{(-k)}_\mu(x) \) is a principal spinor, i.e., a restriction of an analytic function \( F \). Therefore, the component \( F^1 \) is an odd function on \( \mathbb{R} \). Letting \( \epsilon \to 0 \) in the integration path, we conclude that \( I = 0 \).

4.5.4. Tilde master formulas. Let us list explicitly the Gauss-Bessel integrals for the tilde-solutions.

**Theorem 4.15.** In the real case,

\[
2 \int_0^{+\infty} (\tilde{\psi}^{(k)}_\lambda \tilde{\psi}^{(k)}_\mu)^0 e^{-x^2} |x|^{2k} \, dx = \tilde{\psi}^{(k)}_\lambda(\mu)e^{\lambda^2 + \mu^2} \Gamma\left(\frac{1}{2} - k\right) \text{ for } \Re k < \frac{1}{2}.
\]

In the complex case,

\[
\int_{i\epsilon+\mathbb{R}} (\tilde{\psi}^{(k)}_\lambda \tilde{\psi}^{(k)}_\mu)^0 e^{-x^2} (-x^2)^k \, dx = \frac{\pi}{\Gamma\left(\frac{3}{2} + k\right)} \tilde{\psi}^{(k)}_\lambda(\mu)e^{\lambda^2 + \mu^2} \text{ as } k \notin \frac{1}{2} + \mathbb{Z}_+;
\]

this integral is zero when \( k = -1/2 - m \) for \( m \in \mathbb{Z}_+ \). \( \square \)

We note that the spinors we integrate and those in the right-hand side are actually *bi-spinors*, i.e., spinors in terms of \( x \) and spinors in terms of \( \lambda, \mu \). We will skip exact definitions; they are straightforward.

Let us also provide the symmetric tilde-formulas (no spinors are needed):

\[
2 \int_0^{+\infty} \tilde{\varphi}^{(k)}_\lambda \tilde{\varphi}^{(k)}_\mu e^{-x^2} |x|^{2k} \, dx = \Gamma\left(\frac{3}{2} - k\right) \tilde{\varphi}^{(k)}_\lambda(\mu)e^{\lambda^2 + \mu^2}, \Re k < \frac{3}{2};
\]

\[
\int_{i\epsilon+\mathbb{R}} \tilde{\varphi}^{(k)}_\lambda \tilde{\varphi}^{(k)}_\mu e^{-x^2} (-x^2)^k \, dx = \frac{\pi}{\Gamma\left(-\frac{1}{2} + k\right)} \tilde{\varphi}^{(k)}_\lambda(\mu)e^{\lambda^2 + \mu^2}, k \notin \frac{3}{2} + \mathbb{Z}_+;
\]

and the latter integral is zero at \( k = 1/2 - m \) for \( m \in \mathbb{Z}_+ \).

An obvious problem is in extending the nonsymmetric master formula to all spinor solutions for arbitrary root systems. One cannot expect the formulas to be so simple as for \( A_1 \), because the Weyl groups \( W \) have irreducible representations of higher dimensions. We do not have the general formulas at the moment. Similar questions can be posted for arbitrary, not necessarily symmetric, solutions of the \( L \)-eigenvalue problems in arbitrary ranks, when no spinors are needed.

We mention that in the trigonometric- differential and trigonometric-difference settings, the orthogonality relations for \( \psi \) coupled with \( \tilde{\psi} \).
have counterparts \((Y\text{-semi-simplicity is needed})\). Generally, it can be sufficient to manage the rational case.

4.6. **Affine KZ equations.**

4.6.1. **Degenerate AHA and AKZ.** Let \(R\) be an arbitrary (reduced) root system, \(\mathcal{R}^\vee\) its dual, \(P\) and \(P^\vee\) the corresponding weight and coweight lattices. We set \(z_a = (z,a)\) for \(z \in \mathbb{C}^n\) and define the differentiation \(\partial_b z_a \overset{\text{def}}{=} (b,a)\) for arbitrary vectors \(a,b\) (to be used mainly for \(b \in P\), \(a \in P^\vee\)). We set \(w f(z) = f(w^{-1}(z))\) for \(w \in W\). Let \(s_\alpha\) be the reflections corresponding to the roots \(\alpha\) and \(\{y_b\}\) pairwise commutative elements satisfying \(y_{a+b} = y_a + y_b\) for \(a, b \in P\).

We will follow Definition 4.1 of the degenerate DAHA, but restrict it to the non-DAHA case, i.e., to non-affine reflections \(s_i\), and also replace \(-k\) by \(k\). The relations of degenerate AHA (due to Drinfeld for \(GL_n\) [Dr] and Lusztig [L]) are:

\[(4.12)\]
\[s_i y_b - y_{s_i(b)} s_i = k(b,\alpha_i^\vee), \text{ for } i \geq 1.\]

The corresponding algebra will be denoted by \(\mathcal{H}'\).

Let \(\Phi\) be a function of \(z\) taking its values in the abstract algebraic span \(<s_\alpha, y_b>\). The **affine Knizhnik-Zamolodchikov equation**, AKZ, is the following system of differential equations

\[(4.13)\]
\[
\partial_b(\Phi) = \left( \sum_{\alpha \in R^\vee_+} \frac{k(b,\alpha)s_\alpha}{e^{\alpha} - 1 + y_b} \right) \Phi, \text{ where } b \in P.
\]

Actually, \(b\) can be arbitrary complex vectors here and below.

**Theorem 4.16.** The AKZ is self-consistent and \(W\)-equivariant if and only if the elements \(s_\alpha\) and \(y_b\) satisfy the relations from \((4.12)\). The equivariance means that if \(\Phi\) is a solution of AKZ, then so is \(w(\Phi(z))^n = w(\Phi(w^{-1}(z)))\).

The definition of AKZ and this theorem were the starting point of the DAHA theory; here and below see Chapter 1 of [Ch1]. The following construction is basically from [Ch11], but using the technique of spinors makes it more precise and entirely algebraic. In [Ch11] and other (first) author’s papers, the values of AKZ were considered in \(\mathcal{H}'\)-modules induced from arbitrary finite dimensional representations of \(W\) or induced from the characters of the polynomial algebra \(\mathbb{C}[y] = \mathbb{C}[y_b, b \in P]\). In this paper, we will stick to the modules induced from \(\mathbb{C}[y]\).
4.6.2. Spinor Dunkl operators. The Dunkl operators will be needed here in the following form:

\[ D_0^b = \partial_b - \sum_{\alpha \in R_+^\vee} \frac{k(b, \alpha)\sigma_{a}}{e^{z_{\alpha}} - 1}, \quad \text{where} \quad \sigma_{a}(z_{a}) = z_{s_a(a)}. \]  

(4.14)

Here \( \sigma \) stays for the action on the argument of functions: \( \sigma_{a}(f)(z) = f(u^{-1}z) \) \( u \in W \). The relation to AKZ is established via the spinor Dunkl operators defined as a natural extension of (4.14) to the space of \( W \)–spinors.

The spinors are collections \( \hat{\psi} = \{\psi_w, w \in W\} \) of (arbitrary) scalar functions with component-wise addition, multiplication and the differentiations by \( \partial_b \). The action \( \sigma_u \) for \( u \in W \) is through permutations of the indices:

\[ \sigma_{u}(\hat{\psi}) = \{\psi_{u^{-1}w}, w \in W\}. \]

Notice the sign of \( u^{-1} \), which ensures that it is really a representation of \( W \); the spinors are actually the functions on \( W \) so \( u^{-1} \) is necessary. This definition matches the action of \( W \) on functions \( f \) of \( z \), which will be considered as principle spinors under the embedding

\[ f \mapsto f^\rho \overset{\text{def}}{=} \{f_w = w^{-1}f, w \in W\}. \]

Indeed, we have the commutativity: \( (\sigma_{u}(f))^\rho = \sigma_{u}(f^\rho) \). The definition of \( \rho \) can be naturally extended to the operators acting on functions.

For instance, the function \( z_{a} \) becomes the spinor \( \{z_{w^{-1}(a)}, w \in W\} \) under this embedding, \( (\partial_{b})^\rho = \{\partial_{w^{-1}(b)}, w \in W\} \).

**Theorem 4.17.** For a solution \( \Phi \) of the AKZ with the values in \( \mathcal{H}' \), let us define the spinor \( \hat{\Psi} = \{w(\Phi), w \in W\} \) for the action of \( w \in W \) in \( \mathcal{H}' \) by left multiplications. Then \( \hat{\Psi} \) satisfies the following spinor Dunkl eigenvalue problem:

\[ \mathcal{D}_0^b(\hat{\Psi}) = y_{b} \hat{\Psi}, \quad b \in P. \]  

(4.15)

**Proof.** The \( W \)–equivariance of AKZ readily establishes the equivalence of this theorem with the previous one. Explicitly, \( \sigma_{a}(\hat{\Psi}) = \{s_{a}w(\Phi), w \in W\} \) and the relations for the component \( w = u \) of \( \hat{\Psi} \) read as follows:

\[ \partial_{u^{-1}(b)} u(\Phi) = \sum_{\alpha \in R_+^\vee} \frac{k(b, \alpha)s_{a} u(\Phi)}{\exp(z_{u^{-1}(a)}) - 1} + y_{b} u(\Phi), \quad b \in P. \]

This can be recalculated to the same AKZ system for \( \Phi \) due to the \( W \)–equivariance. □
Comment. In [Ch11], an analytic variant of this construction was used. The algebraic formalization of the argument from [Ch11] can be found in Lemma 3.2 from [O4]; the proof above is very similar to that in [O4]. This “algebraization” can be readily extended to the difference and elliptic theories (considered in [Ch1] and previous first author’s works). From the viewpoint of the applications to the isomorphism theorems, both approaches are equivalent.

As far as the reduction of AKZ to the Dunkl eigenvalue problem is concerned, arbitrary modules of \( H' \) were considered (not only induced) in [Ch11]. The Dunkl operators there were given in terms of the action of \( W \) via the monodromy of AKZ (see below). Treating formally the corresponding \( W \)–orbits as spinors, one makes the construction entirely algebraic (as in Theorem 4.17 and in [O4]).

It is important that the monodromy can be calculated explicitly for the asymptotically free solutions of AKZ. For instance, the explicit formulas were used in Theorem 4.3 from [Ch11] to solve the real Dunkl eigenvalue problem via AKZ (in functions, not only in spinors). The solution found in [Ch11] (using the monodromy approach) is the \( G \)–function that (later) played the key role in paper [O4]. □

4.6.3. The isomorphism theorem. Let us apply Theorem 4.17 to induced representations. Given a one-dimensional representation \( \mathbb{C}_\lambda \) of \( \mathbb{C}[y] \) defined by \( y_b(v_o) = \lambda_b v_o \) for \( \lambda_b = (\lambda, b) \), where \( \lambda \in \mathbb{C}^n \), let

\[
I_\lambda = \text{Ind}_{\mathbb{C}[y]}^{H'} \mathbb{C}_\lambda
\]

be the \( H' \)–module induced from \( \mathbb{C}_\lambda \).

We note that if the space of eigenvectors (pure, not generalized) for the eigenvalue \( \lambda \) is one dimensional, then a rational expression in terms of \( y_b \) can be found serving as a projector of \( I_\alpha \) onto this one dimensional space.

Let \( I_\lambda^* \) be \( \text{Hom}(I_\lambda, \mathbb{C}) \) supplied with the natural action of \( H' \) via the canonical anti-involution of \( H' \) preserving the generators \( s_i, y_b \). We use here that the relations in the degenerate affine Hecke algebra are self-dual. Then we define the linear functional \( \varpi : f \mapsto f(v_o) \) on \( I_\lambda^* \) satisfying the conditions

\[
\varpi((y_b - \lambda_b)I_\lambda^*) = 0 \quad \text{for} \quad b \in P.
\]

Assuming, that the space of \( \lambda \)–eigenvectors in \( I_\lambda \) is one-dimensional, these conditions determine \( \varpi \) uniquely up to proportionality.

The functional \( \varpi \) is nonzero on any nonzero \( H' \)–submodule \( V^* \subset I_\lambda^* \), since \( I_\lambda \) is cyclic generated by \( v_o \). Indeed, if \( \varpi(f) = 0 \) for all \( f \in V^* \) then \( f(H'v_o) = 0 = f(I_\lambda) \) for all such \( f \).
Let $U_0 \subset \mathbb{C}^n$ be a open neighborhood of 0 in $\mathbb{C}^n$; we set $U_0' = \cap_{w \in W} \cdot w(U_0)$. We assume that $U_0$ satisfies the following properties (necessary for the monodromy interpretation below):

1. $U_0$ does not contain any zeros of $\prod_{\alpha \in R^+}(e^{z \alpha} - 1)$;
2. $U_0$ is simply connected and $U'_0/W$ is connected.

Let $U'_0$ be one of the connected components of $U'_0$ (the latter set is a disjoint union of $|W|$ connected open sets).

By $\text{Sol}^\lambda_{\text{AZ}}(U_0)$, we denote the space of $I^*_{\lambda}$–valued analytic solutions $\phi$ of the AKZ equation in $U_0$.

Let $\text{Sol}^\lambda_{\Phi}(U'_0)$ be the space of $W$–spinor solutions $\hat{\psi}$ in $U'_0$ of the scalar eigenvalue problem

\[
D_0 b(\hat{\psi}) = \lambda b \hat{\psi}, \quad b \in P.
\]

The spinors here are collections $\hat{\psi} = \{\psi_w, w \in W\}$ of (arbitrary) scalar analytic functions in $U'_0$.

**Theorem 4.18.** The dimension of the space $\text{Sol}^\lambda_{\Phi}(U'_0)$ equals the cardinality $|W|$ of $W$. There is an isomorphism

\[
\eta : \text{Sol}^\lambda_{\text{AZ}}(U_0) \ni \phi \mapsto \{\varpi(w(\phi)) \downarrow_{U'_0}, w \in W\} \in \text{Sol}^\lambda_{\Phi}(U'_0)
\]

for the action $w$ on the values of $\phi$, which are from $I^*_{\lambda}$.

**Proof.** The claim that $\eta$ is a map between the required spaces of solutions follows from Theorem 4.17. Due to the coincidence of the dimensions of the spaces in (4.17), we need only to check that $\eta$ is injective. As in [Ch11], it follows from the fact that $\varpi$ is nonzero on any $\mathcal{H}'$–submodule of $I^*_{\lambda}$. Note that the construction of $\eta$ is entirely algebraic, so it suffices to assume that $\phi$ is defined in the same open set $U'_0$ in the statement of the theorem.

4.6.4. The monodromy interpretation. Let $\Phi(z)$ be an invertible matrix solution of AKZ in $U_0$ with the values in $\text{Aut}(I^*_{\lambda})$. For any $w \in W$, let us define the monodromy matrix $\mathcal{T}_w$ by

\[
w(\Phi(z)) = \Phi(w(z)) \mathcal{T}_w.
\]

Here $\Phi(w(z))$ is well defined in $U_0 \cap \cdot w^{-1}(U_0)$, so is $\mathcal{T}_w$. The matrix solution $\Phi$ is nothing but a choice of the basis of fundamental solutions in $\text{Sol}^\lambda_{\text{AZ}}(U_0)$ (its columns). Changing the basis conjugates all $\mathcal{T}_w$ by a constant invertible matrix. The matrix-valued functions $\mathcal{T}_w$ have the following properties:

(a) $\mathcal{T}_w$ are defined in $U'_0$ and are locally constant;
(b) $\mathcal{T}_{uw} = w^{-1} \mathcal{T}_u \mathcal{T}_w = \mathcal{T}_u(w(z)) \mathcal{T}_w(z)$ for $u, w \in W$. 


For each \( w \in W \), let us define its \( \sigma' \)-action:

\[
\sigma'_w(F) = w F T_w^{-1} = F(w^{-1}(z)) T_w^{-1}(z).
\]

Then \( \sigma'_1 = 1 \), \( \sigma'_{uw} = \sigma'_u \sigma'_w \) and \( \sigma'_w \partial_a = \partial_{w(a)} \sigma'_w \) for \( u, w \in W \) and \( a \in P \).

We naturally set \( \sigma'_\alpha = \sigma'_{s_\alpha} \) and \( \sigma'_i = \sigma'_{\alpha_i} \). Here \( F \) can be an arbitrary function in \( U'_0 \) with the values in \( \text{Aut}(I^*_\lambda) \).

Introducing

\[
\mathcal{D}'_b \overset{\text{def}}{=} \partial_b - k \sum_{\alpha \in R^+_I} \frac{(\alpha, b) \sigma'_\alpha}{e^{\lambda_\alpha} - 1},
\]

one readily obtains that

\[
y_b \Phi = \left( \partial_b - k \sum_{\alpha \in R^+_I} \frac{(\alpha, b) \sigma'_\alpha}{e^{\lambda_\alpha} - 1} \right) \Phi = \mathcal{D}'_b \Phi.
\]

We simply employ the definition of \( \sigma' \) here. The action of \( \mathcal{D}'_b \) is given in terms of the \( W \)-action on \( z \) and the right multiplications by matrices \( T_s \). So it commutes with \( y_b \), which are left multiplications by constant matrices. Therefore, we can apply the functional \( \varpi \) to \( \Phi \) in (4.18), which gives that (4.18) holds for \( \varpi(\Phi) \). The spinor \( \hat{\Psi} \) from Theorem 4.17 is nothing but \( \{ \Psi_w = \sigma'_{w^{-1}}(\Phi) \downarrow U'_0, w \in W \} \) analytically.

4.6.5. Connection to QMBP. Continuing this construction, one can combine the isomorphism we found with the symmetrization map, which acts from \( \text{Sol}^\lambda(U'_0) \) to the space of solutions of the Heckman-Opdam system (QMBP, using the physics terminology) corresponding to \( \lambda \) in the set \( U'_0 \). To be exact, the map from \( \text{Sol}^\lambda_{\Lambda KZ}(U_0) \) to \( \text{Sol}^\lambda_{\text{QMBP}}(U_0) \) is the projection (of the values) onto the one-dimensional subspace of \( W \)-invariants inside \( I^*_\lambda \). It gives the Matsuo- Cherednik isomorphism theorem from [Mats, Ch11]. The spinors are eliminated from the construction of this map, however, provide the best way to prove the isomorphism theorem.

We note that the relation of the Dunkl-spinor eigenvalue problem above to QMBP is, actually, very similar to Lemma 4.14, which addresses solving the Dunkl eigenvalue problem in spinors. Let us mention Corollary 3.4 from [O4], where a similar extension of the Dunkl eigenvalue problem was considered.

Certain conditions on the module \( I_\lambda \) are necessary to ensure the isomorphism with QMBP. Namely, this module must be assumed spherical, \( \mathcal{H}^r \)-generated by \( \sum_{w \in W} w(v_0) \), correspondingly, \( I^*_\lambda \) will be cospherical. See [Ch11] and [Ch1].
There are relations to the localization functor from [GGOR, VV]. The later is, briefly, taking the monodromy representation of the local systems similar to AKZ; it leads to the modules of $t$–Hecke algebras (i.e., non-degenerate). The monodromy is important in our approach too (the cocycle $\{T_w\}$ does contain $t$). The output is a complete system of eigenfunctions of Dunkl operators in the corresponding $y$–eigenspaces of the initial $\mathcal{H}'$–module. Here we solve the Dunkl eigenvalue problem in spinors treated analytically or algebraically. Adding the multiplication by the (trigonometric) coordinates, one can generate the corresponding DAHA module.

The localization functor is understood completely (so far) only in the rational case and in the differential -trigonometric case (corresponding to the setting of this section); see [GGOR, VV]. Our construction and the isomorphism theorems hold for all known families of AKZ and Dunkl operators (including the elliptic theories). See [Ch11], [Ch9] and Chapter 1 from [Ch1]. The exact connection is still not clarified. Obviously, the monodromy variant of Theorem 4.17 is needed here.

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