ON A NONLINEAR SCHRÖDINGER SYSTEM ARISING IN QUADRATIC MEDIA

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Abstract. We consider the quadratic Schrödinger system
\[
\begin{align*}
    iu_t + \Delta \gamma_1 u + \bar{v}v &= 0, \\
    2iv_t + \Delta \gamma_2 v - \beta v + \frac{1}{2}u^2 &= 0,
\end{align*}
\]
in dimensions $1 \leq d \leq 4$ and for $\gamma_1, \gamma_2 > 0$, the so-called elliptic-elliptic case. We show the formation of singularities and blow-up in the $L^2$-(super)critical case. Furthermore, we derive several stability results concerning the ground state solutions of this system.

Keywords: Nonlinear Schrödinger Systems, Blow-up, Ground States, Stability.

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1. Introduction

In this paper we consider the quadratic Schrödinger system (1)
\[
\begin{align*}
    iu_t + \Delta \gamma_1 u + \bar{v}v &= 0, \\
    2iv_t + \Delta \gamma_2 v - \beta v + \frac{1}{2}u^2 &= 0,
\end{align*}
\]
where $d \leq 4$, $\beta, \gamma_1, \gamma_2 \in \mathbb{R}$ and $\Delta_\gamma = \partial_{x_1}^2 + \cdots + \partial_{x_d}^2 + \gamma \partial_{x_{d+1}}^2$.

This system arises as a model for the interaction of waves propagating in $\chi^{(2)}$ dispersive media. In the case of electromagnetic waves, these media are characterized by a polarization vector $P$ of the form
\[
P = \epsilon_0 \chi^{(1)}(\omega_0)E + \chi^{(2)}(\omega_0)E^2.
\]
Here, $\epsilon_0$ is the vacuum permittivity, $E$ represents the electric field and $\omega_0$ its angular frequency (see [2] for a rigourous derivation of $\Pi$ from the Maxwell-Faraday equation and Ampère’s Law). In fact, the quadratic Schrödinger system $\Pi$ governs the dynamics of propagation in $\chi^{(2)}$ media in other physical contexts, namely in nonlinear optics (see for instance [7], [8], [9]). Despite this wide range of applications, and contrarily to the modelation of propagation in $\chi^{(3)}$ centrosymmetric media, which give rise to the Kerr nonlinearity (and hence to Schrödinger equations with cubic nonlinearities), very few mathematical results concerning quadratic systems are available in the literature.

Very recently, in [2], a rigorous mathematical study of $\Pi$ was undertaken in the $L^2$—subcritical case ($d \leq 2$). After establishing the global well-posedness of $\Pi$ in $H^1(\mathbb{R}^{d+1})$, the authors turn their attention to localized solutions, deriving conditions for their existence (or non-existence). Furthermore, when $\gamma_1, \gamma_2 > 0$, the existence of ground states is shown using the
concentration-compactness principle due to P.L. Lions \[6\]. Finally, for \(d = 2\) and \(\beta = 0\), the authors prove the orbital stability of these ground states.

Before stating the main results of the present paper, we continue this introduction by making some considerations on system (1) and its localized solutions. It is standard to show, for \(d \leq 4\) (that is, in the \(H^1\)–subcritical case) the following local existence result:

**Theorem 1.1** (Local Well-posedness). Let \(d \leq 4\). The IVP (1) with initial data \((u_0, v_0) \in H^1(\mathbb{R}^{d+1}) \times H^1(\mathbb{R}^{d+1})\) admits a unique maximal solution \((u, v) \in C([0, T^*); H^1(\mathbb{R}^{d+1}) \times H^1(\mathbb{R}^{d+1}))\). If \(T^* < +\infty\) then

\[
\lim_{t \to T^*} \|\nabla u(t)\|_2^2 + \|\nabla v(t)\|_2^2 = +\infty.
\]

Also, the following quantities are formally conserved by the flow of (1):

- the mass
  \[
  M(u(t), v(t)) = \int_{\mathbb{R}^{d+1}} \left( |u(t, x)|^2 + 4|v(t, x)|^2 \right) dx
  \]
- the energy
  \[
  E(u(t), v(t)) = \int_{\mathbb{R}^{d+1}} \left( |\nabla u(t, x)|_{\gamma_1}^2 + |\nabla v(t, x)|_{\gamma_2}^2 \right.
  \]
  \[
  \left. + \beta|v(t, x)|^2 - \text{Re} u(t, x)v(t, x) \right) dx,
  \]

where we have put
\[
|\nabla f|_{\gamma}^2 = |\partial_{x_1} f|^2 + \cdots + |\partial_{x_d} f|^2 + \gamma|\partial_{x_{d+1}} f|^2.
\]

In Section 2, using these invariants, we compute two Virial identities which yield the following blow-up results in the critical and supercritical cases:

**Theorem 1.2** (Blow-up, \(d = 3\)). Consider the IVP for system (1) with \(d = 3\), \(\gamma_1 = \gamma_2 := \gamma > 0\) and initial data \((u_0, v_0) \in (H^1(\mathbb{R}^4) \cap L^2(\mathbb{R}^4, |x|^2 dx))^2\). Let

\[
(u, v) \in C\left([0, T^*); (H^1(\mathbb{R}^4) \cap L^2(\mathbb{R}^4, |x|^2 dx))^2\right)
\]

be the corresponding maximal solution. Assume in addition that

\[
E(u_0, v_0) < 0 \quad \text{and} \quad \beta > 0
\]

or

\[
8E(u_0, v_0) + |\beta|M(u_0, v_0) < 0 \quad \text{and} \quad \beta \leq 0.
\]

Then \(T^* < \infty\) and \(\lim_{t \to T^*} \|\nabla u(t)\|_2^2 = +\infty\).
Theorem 1.3 (Blow-up, $d = 4$). Consider the IVP for system (11) with $d = 4$, $\gamma_1, \gamma_2 > 0$ and initial data $(u_0, v_0) \in (H^1(\mathbb{R}^5) \cap L^2(\mathbb{R}^5, |x_1|^2 dx))^2$, where $x_1 = (x_1, \ldots, x_d)$. Let

$$(u, v) \in C \left( [0, T^*); (H^1(\mathbb{R}^5) \cap L^2(\mathbb{R}^5, |x_1|^2 dx))^2 \right)$$

be the corresponding maximal solution. Assume in addition that

\begin{equation}
E(u_0, v_0) < 0 \quad \text{and} \quad \beta > 0
\end{equation}

or

\begin{equation}
8E(u_0, v_0) + |\beta|M(u_0, v_0) < 0 \quad \text{and} \quad \beta \leq 0.
\end{equation}

Then $T^* < \infty$.

For $\gamma_1, \gamma_2, \omega, 4\omega + \beta > 0$ (see [2]), the system (11) admits localized solutions of the form

\begin{equation}
(u(t, x) = P(x)e^{i\omega t}, \quad v(t, x) = Q(x)e^{2i\omega t}.
\end{equation}

The functions $P$ and $Q$ satisfy the system

\begin{equation}
\begin{aligned}
-\omega P + \Delta_{x_1} P + \overline{P}Q &= 0 \\
-(4\omega + \beta)Q + \Delta_{x_2} Q + \frac{1}{2}P^2 &= 0.
\end{aligned}
\end{equation}

Let us denote by $B$ the set of all bound states, that is, the set of all solutions $(P, Q) \in H := H^1(\mathbb{R}) \times H^1(\mathbb{R})$ of the stationary system (9).

We will say that a bound state $(P, Q)$ is a ground state if $(P, Q)$ minimizes the action

\begin{equation}
S(u, v) = E(u, v) + \omega M(u, v)
\end{equation}

among all bound states. That is, putting $G$ the set of all ground states,

$$(P_0, Q_0) \in G \iff (P_0, Q_0) \in B \quad \text{and} \quad \forall (P, Q) \in B, S(P_0, Q_0) \leq S(P, Q).$$

It is not difficult to see that $(P_0, Q_0) \in G$ if and only if $(P_0, Q_0)$ is a minimizer of the problem

\begin{equation}
\inf \{ I(u, v) = K(u, v) + \omega M(u, v) : (u, v) \in W_{(P_0, Q_0)} \},
\end{equation}

where

$$K(u, v) = \int_{\mathbb{R}^{d+1}} \left( |\nabla u(t, x)|^2 \gamma_1 + |\nabla v(t, x)|^2 \gamma_2 + \beta v(t, x) |^2 \right) dx$$

and

$$W_{(P_0, Q_0)} = \left\{ (u, v) \in H : J(u, v) := Re \int_{\mathbb{R}^{d+1}} \nabla v = Re \int_{\mathbb{R}^{d+1}} \overline{P} \right\}.$$

In Section 3 we will show the following instability results concerning ground states in the $L^2$-critical and supercritical cases:

Theorem 1.4 (Strong instability). Let $\gamma_1 = \gamma_2 > 0$, $\beta = 0$ and $d = 3$. Let $B$ be the set of all bound states of (11). Then $B$ is unstable in the following sense: given $(P, Q) \in B$, there exists a sequence $X_{0,k} \to (P, Q)$ such that, for all $k$, the solution $X_k$ of (11) with initial data $X_{0,k}$ blows up in finite time.
Theorem 1.5 (Weak instability). Let $\gamma_1, \gamma_2 > 0$, $d = 3$ and $\beta \neq 0$ or $d \geq 4$. Let $(P, Q) \in G$ and its orbit
\[
\Sigma = \{ f(\theta, y)[P, Q] := (e^{i\theta}P(\cdot + y), e^{2i\theta}Q(\cdot + y)) : \theta \in \mathbb{R}, y \in \mathbb{R}^{d+1} \}.
\]
Then $\Sigma$ is weakly unstable by the flow of (1), in the following sense: there exists $\epsilon > 0$ and a sequence $X_{0,k} \to (P, Q)$ in $H$ such that
- The solution $X_k(t)$ to (1) with initial data $X_{0,k}$ is global and bounded in $H$;
- For all $k$, $T^*_k = \sup\{ T \geq 0 : \forall t \in [0, T], X_k(t) \in \Sigma_\epsilon \} = +\infty$, where $\Sigma_\epsilon$ is the $\epsilon$-neighbourhood of $\Sigma$.

In what concerns stability of ground states, the proof in [2] follows the argument of Cazenave and Lions for the stability of ground states of the nonlinear Schrödinger equation, by showing that the solutions of the minimization problem
\[
(12) \quad \inf\{ E(u, v) : M(u, v) = \nu \}, \quad \nu = M(P, Q), \quad (P, Q) \text{ ground state}
\]
are precisely the ground states of (1). In [2], it was proven that: such a minimization problem has a solution; the solution is a bound state, and so it has an action larger or equal than any ground state; the solution is actually a ground state, by proving that it has the same action as any given ground state. The first and third steps only require that system (1) is $L^2$-subcritical, meaning that $d \leq 2$. However, to show the second step, the procedure used therein only works for $d = 2$ and $\beta = 0$.

Invoking some arguments used in [3], one may actually skip the second step, as long as the energy does not contain any $L^2$ terms (in the present situation, it means that $\beta = 0$). The consequence is a more direct approach, which is also valid for $d = 1$, presented in Section 4 where we prove the following result:

Theorem 1.6. Suppose that $d \leq 2$ and $\beta = 0$. Then the set of ground states $G$ is stable with respect to the flow generated by (1), that is, for each $\delta > 0$, there exists $\epsilon > 0$ such that, if $(u_0, v_0) \in H$ satisfies
\[
\inf_{(P, Q) \in G} \| (u_0, v_0) - (P, Q) \|_H < \epsilon,
\]
then the solution $(u, v)$ of (1) with initial data $(u_0, v_0)$ satisfies
\[
\sup_{t \geq 0} \inf_{(P, Q) \in G} \| (u(t), v(t)) - (P, Q) \|_H < \delta.
\]

2. Virial identity and blow-up

We begin this Section by noticing that the system (1) can be put in the Hamiltonian form
\[
\frac{\partial X}{\partial t}(t) = JE'(X(t)),
\]
where $J$ is the skew-adjoint operator $\begin{bmatrix} -i & 0 \\ 0 & -\frac{i}{2} \end{bmatrix}$ and $X = (u, v)$.

Using this fact, we will derive two global Virial type identities for system (1). Instead of using the standard technique based on several integrations
by parts to calculate the second derivative in time for the variance of the solutions, we use an interesting method presented in [5] that allows to formally understand the evolution of certain real functional along the trajectories of Hamiltonian systems. In subsection 2.1 we describe the general idea of this procedure applied to the system (1). Finally, we use the Virial identities obtained to establish two results about the formation of singularities for system (1) based on classical convexity arguments.

2.1. Dual dynamics for system (1). Consider a real functional $G$, defined on a dense subspace $\mathcal{V}$ of $L^2(\mathbb{R}^d)$, with continuous derivatives in $L^2(\mathbb{R}^d)$. The goal is to study the evolution of $G$ along the trajectories of the dynamical system defined by equation (13).

Recalling that $X(t) = (u(t), v(t))$, the time derivative of $G$ calculated along $X(t)$ is given by

$$
\frac{d}{dt} G(X(t)) = \langle G'(X(t)), J E'(X(t)) \rangle := P(X(t)).
$$

On the other hand, given $\tilde{X}_0 := (\tilde{u}_0, \tilde{v}_0)$, consider the initial value problem

$$
\frac{\partial}{\partial t} \tilde{X}(t) = J G'(\tilde{X}(t)), \quad \tilde{X}(0) = \tilde{X}_0,
$$

which we suppose to be locally well-posed. Thus,

$$
\frac{d}{dt} E(\tilde{X}(t)) = \langle E'(\tilde{X}(t)), \frac{\partial \tilde{X}}{\partial t}(t) \rangle = \langle E'(\tilde{X}(t)), J G'(\tilde{X}(t)) \rangle = -\langle G'(\tilde{X}(t)), J E'(\tilde{X}(t)) \rangle = -P(\tilde{X}(t)).
$$

Therefore, the validation at time $t = 0$ yields

$$
P(\tilde{X}_0) = -\frac{d}{dt} E(\tilde{X}(t)) \bigg|_{t=0},
$$

which determines the evolution of $G$ along the trajectories of (13).

In what follows, we write $x_\perp := (x_1, x_2, \ldots, x_d)$, $\nabla_\perp := (\partial_{x_1}, \ldots, \partial_{x_d})$ and $x_r = (x_\perp, \gamma x_{d+1})$. Also we decompose the energy (14) in the following way:

$$
E(u, v) = E_{\gamma_1}(u) + E_{\gamma_2}(v) + E_\beta(v) - E_{Re}(u, v),
$$

where

$$
E_{\gamma_1}(u) = \frac{1}{2} \int_{\mathbb{R}^{d+1}} |\nabla u|^2_{\gamma_1} dx = \frac{1}{2} \int_{\mathbb{R}^{d+1}} (|\nabla_\perp u|^2 + \gamma_1 |\partial_{x_{d+1}} u|^2) dx,
$$

$$
E_{\gamma_2}(v) = \frac{1}{2} \int_{\mathbb{R}^{d+1}} |\nabla v|^2_{\gamma_2} dx = \frac{1}{2} \int_{\mathbb{R}^{d+1}} (|\nabla_\perp v|^2 + \gamma_2 |\partial_{x_{d+1}} v|^2) dx,
$$

$$
E_\beta(v) = \frac{\beta}{2} \int_{\mathbb{R}^{d+1}} |v|^2 dx,
$$

$$
E_{Re}(u, v) = \frac{1}{2} Re \int_{\mathbb{R}^{d+1}} \bar{u}^2 v dx.
$$

Finally, we set $M_0 := M(u(\cdot, 0), v(\cdot, 0))$ and $E_0 := E(u(\cdot, 0), v(\cdot, 0))$. 
2.2. Virial type identities. In this subsection we prove the following Virial identities:

**Proposition 2.1** (Virial identity). Let $d = 3, 4$ and 
\[(u_0, v_0) \in (H^1(\mathbb{R}^{d+1}) \cap L^2(\mathbb{R}^{d+1}, |x|^2 \, dx))^2.\]

Then, the variance 
\[\mathcal{V}(t) = \mathcal{V}(u(t), v(t)) := \frac{1}{2} \int_{\mathbb{R}^{d+1}} |x|^2 (|u(t)|^2 + 4|v(t)|^2) \, dx\]
is finite on the maximal time interval $[0, T^*)$ and $\mathcal{V} \in C^2([0, T^*))$.
Furthermore, the following identities hold:

(i) \[\frac{d\mathcal{V}}{dt}(t) = 2Im \int_{\mathbb{R}^{d+1}} (x^\gamma \cdot \nabla u \, \Pi + 2x^\gamma \cdot \nabla v \, \overline{\nu}) \, dx,\]

(ii) If $\gamma_1 = \gamma_2 := \gamma$, 
\[\frac{d^2\mathcal{V}}{dt^2}(t) = 4 \int_{\mathbb{R}^{d+1}} (|\nabla u|^2 + |\nabla v|^2) \, dx - (d + \gamma)Re \int_{\mathbb{R}^{d+1}} \bar{u}^2 \, v dx.\]

(iii) In particular, for $d = 3$ and $\gamma_1 = \gamma_2 = 1$, 
\[\frac{d^2\mathcal{V}}{dt^2}(t) = 8E_0 - 4\beta \int_{\mathbb{R}^4} |v|^2 \, dx.\]

**Proposition 2.2** (Transverse Virial identity). Let $d = 3, 4$ and 
\[(u_0, v_0) \in (H^1(\mathbb{R}^{d+1}) \cap L^2(\mathbb{R}^{d+1}, |x|^2 \, dx))^2.\]

Then, the transverse variance 
\[\mathcal{V}_\perp(t) = \mathcal{V}_\perp(u(t), v(t)) := \frac{1}{2} \int_{\mathbb{R}^{d+1}} |x\perp|^2 (|u(t)|^2 + 4|v(t)|^2) \, dx\]
is finite on the maximal time interval $[0, T^*)$ and $\mathcal{V}_\perp \in C^2([0, T^*))$.
Furthermore, the following identities hold:

(i) \[\frac{d\mathcal{V}_\perp}{dt}(t) = 2Im \int_{\mathbb{R}^{d+1}} (x\perp \cdot \nabla u \, \Pi + 2x\perp \cdot \nabla v \, \overline{\nu}) \, dx.\]

(ii) \[\frac{d^2\mathcal{V}_\perp}{dt^2}(t) = 4 \int_{\mathbb{R}^{d+1}} (|\nabla u|^2 + |\nabla v|^2) \, dx - dRe \int_{\mathbb{R}^{d+1}} \bar{u}^2 \, v dx.\]

(iii) In particular, for $d = 4$, we have 
\[\frac{d^2\mathcal{V}_\perp}{dt^2}(t) = 8E_0 - 4\beta \int_{\mathbb{R}^5} |v|^2 \, dx - \int_{\mathbb{R}^5} (\gamma_1 |\partial_{x^d+1} u|^2 + \gamma_2 |\partial_{x^d+1} v|^2) \, dx.\]

**Proof of Proposition 2.1**

**Proof of assertion (i):** We formally apply the technique of dual dynamics to the functional $G(u, v) := \mathcal{V}(u, v)$. The corresponding IVP (15) for this functional is defined by

\[
\begin{align*}
\tilde{u}_t &= -i|x|^2 \tilde{u}, \quad \tilde{u}(0) = \tilde{u}_0, \\
\tilde{v}_t &= -2i|x|^2 \tilde{v}, \quad \tilde{v}(0) = \tilde{v}_0,
\end{align*}
\]
whose solution is given by

\[(\tilde{u}(x, t), \tilde{v}(x, t)) = (e^{-i|x|^2t} \tilde{u}_0, e^{-2i|x|^2t} \tilde{v}_0).\]
Then, from (17), we get
\[ P(\tilde{u}_0, \tilde{v}_0) = -\frac{d}{dt} E(e^{-i|x|^2 t} \tilde{u}_0, e^{-2i|x|^2 t} \tilde{v}_0) \bigg|_{t=0} \]
\[ = -\frac{d}{dt} E_{|z_1}(e^{-i|x|^2 t} \tilde{u}_0) \bigg|_{t=0} - \frac{d}{dt} E_{|z_2}(e^{-2i|x|^2 t} \tilde{v}_0) \bigg|_{t=0} \]
\[ = 2Im \int_{\mathbb{R}^{d+1}} x^{\gamma_1} \cdot \nabla \tilde{u}_0 \overline{u}_0 dx + 4Im \int_{\mathbb{R}^{d+1}} x^{\gamma_2} \cdot \nabla \tilde{v}_0 \overline{u}_0 dx, \]
since \( E_\beta(\tilde{v}) \) and \( E_{\text{Re}}(\tilde{u}, \tilde{v}) \) are independent of time. Thus, it follows from (14) that
\[ \frac{dV}{dt}(t) = 2Im \int_{\mathbb{R}^{d+1}} (x^{\gamma_1} \cdot \nabla u \overline{u} dx + 2x^{\gamma_2} \cdot \nabla v \overline{v}) dx \]
as claimed in (i).

**Proof of assertion (ii):** To prove (ii), we choose instead
\[ G(u, v) := 4m \int_{\mathbb{R}^{d+1}} (2x^\gamma \cdot \nabla u \overline{u} dx + 4x^\gamma \cdot \nabla v \overline{v}) dx. \]
The corresponding IVP (15) is now
\[ \begin{cases} 
\tilde{u}_t = -4x^\gamma \cdot \nabla \tilde{u} - 2(d+\gamma)\tilde{u}, & \tilde{u}(0) = \tilde{u}_0, \\
\tilde{v}_t = -4x^\gamma \cdot \nabla \tilde{v} - 2(d+\gamma)\tilde{v}, & \tilde{v}(0) = \tilde{v}_0,
\end{cases} \]
so that
\[ \tilde{u}(x, t) = e^{-2(d+\gamma)t} \tilde{u}_0(e^{-4t}x, e^{-4t}x_{d+1}), \]
and
\[ \tilde{v}(x, t) = e^{-2(d+\gamma)t} \tilde{v}_0(e^{-4t}x, e^{-4t}x_{d+1}). \]
Now we proceed with the computation of \( P(\tilde{u}_0, \tilde{v}_0) = -\frac{d}{dt} E(\tilde{u}, \tilde{v}) \bigg|_{t=0} \). Using the change of variables \( (x_{\perp}, x_{d+1}) = (e^{4t}y_{\perp}, e^{4t}y_{d+1}) \), we get
\[ E(\tilde{u}, \tilde{v}) = \frac{1}{2} \int_{\mathbb{R}^{d+1}} e^{-8t} |\nabla_{\perp} \tilde{u}_0(y)|^2 + \gamma e^{-8t} |\partial_{x_{d+1}} \tilde{u}_0(y)|^2 dy \]
\[ + \frac{1}{2} \int_{\mathbb{R}^{d+1}} e^{-8t} |\nabla_{\perp} \tilde{v}_0(y)|^2 + \gamma e^{-8t} |\partial_{x_{d+1}} \tilde{v}_0(y)|^2 dy \]
\[ + \frac{\beta}{2} \int_{\mathbb{R}^{d+1}} |\nabla_{\perp} \tilde{v}_0(y)|^2 dy + \frac{e^{-2(d+\gamma)t}}{2} \text{Re} \int_{\mathbb{R}^{d+1}} \tilde{u}_0^2(y) \tilde{v}_0(y) dy. \]
Finally, we conclude that
\[ P(\tilde{u}_0, \tilde{v}_0) = -\frac{d}{dt} E(\tilde{u}, \tilde{v}) \bigg|_{t=0} \]
\[ = 4 \int_{\mathbb{R}^{d+1}} (|\nabla_{\perp} \tilde{u}_0|^2 + \gamma^2 |\partial_{x_{d+1}} \tilde{u}_0|^2 + |\nabla_{\perp} \tilde{v}_0|^2 + \gamma^2 |\partial_{x_{d+1}} \tilde{v}_0|^2) dy \]
\[ - (d+\gamma) \text{Re} \int_{\mathbb{R}^{d+1}} \tilde{u}_0^2 \tilde{v}_0 dy, \]
which implies (ii).

**Proof of assertion (iii):** The identity is an immediate consequence of (ii) combined with the conservation of the energy (3).
Proof of Proposition 2.2

Proof of assertion (i): The proof is similar as the one performed for the case (i) in Proposition 2.1 and follows without major changes.

Proof of assertion (ii): Here we take $G$ defined by
\[ G(u, v) := Im \int_{\mathbb{R}^{d+1}_x} (2x_{\bot} \cdot \nabla_{\bot} u \bar{u} dx + 4x_{\bot} \cdot \nabla_{\bot} v \bar{v}) dx. \]

In this case, the corresponding IVP (15) is written as follows:
\[
\begin{align*}
\bar{u}_t &= -4x_{\bot} \cdot \nabla_{\bot} \bar{u} - 2d\bar{u}, \quad \bar{u}(0) = \bar{u}_0, \\
\bar{v}_t &= -4x_{\bot} \cdot \nabla_{\bot} \bar{v} - 2d\bar{v}, \quad \bar{v}(0) = \bar{v}_0,
\end{align*}
\]
so that
\[(\bar{u}, \bar{v}) = \left( e^{-2dt} \bar{u}_0 (e^{-4t} x_\bot, x_{d+1}), e^{-2dt} \bar{v}_0 (e^{-4t} x_\bot, x_{d+1}) \right) .
\]
Using the change of variables $(x_{\bot}, x_{d+1}) = (e^{4t} y_{\bot}, y_{d+1})$, we have
\[ E(\bar{u}, \bar{v}) = \frac{e^{-8t}}{2} \int_{\mathbb{R}^{d+1}_y} \left( |\nabla_{\bot} \bar{u}_0(y)|^2 + |\nabla_{\bot} \bar{v}_0(y)|^2 \right) dy
\]
\[ + \frac{1}{2} \int_{\mathbb{R}^{d+1}_y} \left( \gamma_1 |\partial_{x_{d+1}} \bar{u}_0(y)|^2 + \gamma_2 |\partial_{x_{d+1}} \bar{v}_0(y)|^2 \right) dy
\]
\[ + \beta \int_{\mathbb{R}^{d+1}_y} \bar{v}_0(y)^2 dy - \frac{e^{-2dt}}{2} Re \int_{\mathbb{R}^{d+1}_y} \bar{u}_0(y) \bar{v}_0(y) dy,
\]
hence
\[ P(\bar{u}_0, \bar{v}_0) = \left. -\frac{d}{dt} E(\bar{u}, \bar{v}) \right|_{t=0}
\]
\[ = 4 \int_{\mathbb{R}^{d+1}_y} \left( |\nabla_{\bot} \bar{u}_0(y)|^2 + |\nabla_{\bot} \bar{v}_0(y)|^2 \right) dy - dRe \int_{\mathbb{R}^{d+1}_y} \bar{u}_0(y) \bar{v}_0(y) dy,
\]
which yields (ii).

Proof of assertion (iii): Once again, this last assertion is a particular case of (ii) combined with the conservation of the energy (3). □

2.3. Proof of the blow-up results. Proof of Theorem 1.2

By rescaling, it is easy to reduce the problem to the case $\gamma = 1$, which can be treated by the classical convexity method, similar to the nonlinear Schrödinger equation (see for instance [1]). The blow-up of $\|\nabla u\|^2$ follows from the energy conservation law and the blow-up alternative presented in Theorem 1.1. Indeed, from (3), Hölder’s inequality and the Sobolev inequality in dimension $n = 4$ it follows that
\[ \|\nabla_{\gamma} u(\cdot, t)\|^2_{L^2} + \|\nabla_{\gamma} u(\cdot, t)\|^2_{L^2} = 2E_0 - \beta \|v(\cdot, t)\|^2_{L^2} + Re \int_{\mathbb{R}^4} \bar{u}^2(\cdot, t)v(\cdot, t)
\]
\[ \leq 2E_0 + \frac{|\beta|}{4} M_0 + c \|\nabla u(\cdot, t)\|_{L^4(\mathbb{R}^4)} \|v(\cdot, t)\|_{L^2(\mathbb{R}^4)}
\]
\[ \leq 2E_0 + \frac{|\beta|}{4} M_0 + c \frac{\sqrt{M_0}}{2} \|\nabla u(\cdot, t)\|_{L^2(\mathbb{R}^4)}.
\] □
We finish by noticing that the Virial identity (iii) in Proposition 2.2 and arguments similar to the ones used in the proof of Theorem 1.2 allow us to establish Theorem 1.3.

**Remark 2.3.** Notice that dimensions $d = 3, 4$ are $L^2$-(super)critical and $H^1$-subcritical. In this situation, the local $H^1 \times H^1$ existence theory allows to prove the persistence of solutions in $H^s \times H^s$, $s > \frac{d + 1}{2}$, provided that the initial data has $H^s \times H^s$ regularity. In this framework, one can show the blow-up

$$
\lim_{t \to T^-} \|v(\cdot, t)\|_\infty = +\infty.
$$

Indeed, for $d = 3$ (a similar computation can be produced for $d = 4$):

$$
\|\nabla_x u(\cdot, t)\|_{L^2}^2 + \|\nabla_x u(\cdot, t)\|_{L^2}^2 = 2E_0 - \beta\|v(\cdot, t)\|_{L^2}^2 + \text{Re} \int_{\mathbb{R}^4} u^2(\cdot, t)v(\cdot, t)
\leq 2E_0 + \frac{|\beta|}{4} M_0 + \|u(\cdot, t)\|_{L^2(\mathbb{R}^4)}^2 \|v(\cdot, t)\|_{L^\infty(\mathbb{R}^4)}
\leq 2E_0 + \frac{|\beta|}{4} M_0 + M_0 \|v(\cdot, t)\|_{L^\infty(\mathbb{R}^4)}.
$$

3. Instability of ground states

The proof of Theorem 1.3 follows from Theorem 1.2 and from the fact that, given a bound state $(P, Q)$, $E(\lambda P, \lambda Q) < 0$ for $\lambda > 1$.

We now show the weak instability of ground state solutions to (1) in the critical ($d = 3$) and supercritical ($d \geq 4$) cases. Let $(P, Q)$ a ground state, that is, a solution of the minimization problem (11).

Noticing that $\int P^2 Q \in \mathbb{R}^+$ (see [2]), we can show that the orbit of every ground state contains an element $(\tilde{P}, \tilde{Q})$, with $\tilde{P}, \tilde{Q} > 0$. More precisely:

**Proposition 3.1.** Let $\gamma_1, \gamma_2 > 0$ and $n \geq 1$.

Then, for every solution $(P, Q) \in H$ of the minimization problem (11):

(i) $(|P|, |Q|)$ is a solution of (11).

(ii) $(|P|, |Q|)$ belongs to the orbit

$$
\Sigma = \{e^{i\theta} P(\cdot + y), e^{2i\theta} Q(\cdot + y) : \theta \in \mathbb{R}, y \in \mathbb{R}^{d+1}\} \text{ of } (P, Q).
$$

**Proof of (i):**

Let $(P, Q) \in H$ a minimizer and take $(\tilde{P}, \tilde{Q}) = (|P|, |Q|)$.

It is straightforward to see that $I(\tilde{P}, \tilde{Q}) \leq I(P, Q)$. Furthermore,

$$
\dot{\mu} = \text{Re} \int \tilde{P} \tilde{Q} = \int \tilde{P} \tilde{Q} = \int |\mathcal{P}|^2 |Q| \geq \int |\mathcal{P}^2 Q| = \text{Re} \int \mathcal{P}^2 Q = \mu.
$$

Now, assume that $\dot{\mu} > \mu$. For $\lambda = \left(\frac{\mu}{\dot{\mu}}\right)^{\frac{1}{2}}$, we put

$$(P_\lambda(\cdot), Q_\lambda(\cdot)) = (\lambda^{\frac{1}{2}} \tilde{P}(\lambda \cdot), \lambda^{\frac{1}{2}} \tilde{Q}(\lambda \cdot)).$$

We get

$$
\int P_\lambda^2 Q_\lambda = \mu \text{ and } I(P_\lambda, Q_\lambda) < I(\tilde{P}, \tilde{Q}) \leq I(P, Q),
$$

which contradicts the minimality of $(P, Q)$. ■
Proof of (ii):
Write \((P, Q) = (|P|e^{i\theta_1(x)}, |Q|e^{i\theta_2(x)})\). Our goal is to show that \(\theta_1, \theta_2\) are constant, and that \(\theta_2 = 2\theta_1\).

We already showed that \(I(P, Q) = I(|P|, |Q|)\), hence
\[
\int \left( \left| \nabla P \right|^2_{\gamma_1} + \left| \nabla Q \right|^2_{\gamma_1} \right) = \int \left( (\left| \nabla |P| \right|_{\gamma_1}^2 + \left| \nabla |Q| \right|_{\gamma_2}^2) \right)
\]
and
\[
\int \left( \left| P \right|^2 \left| \nabla \theta_1 \right|^2_{\gamma_1} + \left| Q \right|^2 \left| \nabla \theta_2 \right|^2_{\gamma_2} \right) = 0.
\]
To conclude that \(\theta_1\) and \(\theta_2\) are constant, one only needs to show that \(|P|\) and \(|Q|\) do not vanish. To show that \(Q\) does not vanish, we use the (real) equation
\[-(4\omega + \beta)Q + \Delta_{\gamma_2}Q = -\frac{1}{2}P^2.\]
Noticing that for \(L = \Delta_{\gamma_2} - (4\omega + \beta), LQ \leq 0\), we can conclude by using the Maximum Principle stated in Theorem 3.5 of [4]). We can also show that \(P\) does not vanish by applying a similar argument to equation
\[-\omega \gamma_1 P + \Delta P = -\frac{1}{2}PQ\]
in a neighborhood of its solution \(|P|\).

Finally, the relation \(\theta_2 = 2\theta_1\) simply comes from the fact that
\[
\int |P|^2 |Q| = \int P^2 Q,
\]
has shown in the proof of (i).

Proof of Theorem 1.5:
For convenience of the notations, we will take \(\gamma_1 = \gamma_2 = 1\), although the exact same proof remains valid for arbitrary \(\gamma_1, \gamma_2 > 0\). In view of Proposition (3.1), we may assume that \(P, Q > 0\). Let
\[
\mathcal{L} = \{ (u, v) \in H : M(u, v) = M(P, Q) \}.
\]
Following [5], it is sufficient to prove the existence of \(\Psi \in H\) such that
(1) \(\Psi\) is tangent to \(\mathcal{L}\) at \((P, Q)\);
(2) \(J^{-1}\Psi\) is \(L^2\)-orthogonal to \(\partial_y f(0, 0)[P, Q] = i(P, 2Q)\) and to \(\nabla_y f(0, 0)[P, Q] = (\nabla P, \nabla Q)\);
(3) \(\partial_y f(0, 0)[P, Q]\) and \(\nabla_y f(0, 0)[P, Q]\) are linearly independent;
(4) \(\langle S''(P, Q)\Psi, \Psi \rangle < 0\).

In order for the present paper to be self-contained, we briefly explain in the next two steps how these four points can be used to prove Theorem 1.5. For details, we refer the reader to [5].

Step 1: Construction of an Auxiliary Dynamical System

From conditions 2. and 3., and for some \(\epsilon > 0\), we build an Auxiliary Dynamical System
\[
\mathcal{H} : \Sigma_\epsilon \rightarrow \mathbb{R}
\]
with the following properties:

- \( \forall (U, V) \in \Sigma, \forall \theta, y \in \mathbb{R} \times \mathbb{R}^n, \mathcal{H}(f(\theta, y) v) = \mathcal{H}(v) \);
- \( \forall (U, V) \in \Sigma, \mathcal{H}'(v) \in H \) and \( \mathcal{H}' : \Sigma \to H \) is \( C^1 \) with bounded derivative;
- \( J\mathcal{H}'(P, Q) = \Psi \).

Indeed, consider the mapping

\[
F : H \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \\
((U, V), \theta, y) \to \frac{1}{2}\|f(\theta, y)(U, V) - (P, Q)\|^2.
\]

Using the fact that \( \partial_0 f(0, 0)[P, Q] \) and \( \nabla_y f(0, 0)[P, Q] \) are linearly independent, one can show, applying the Implicit Function Theorem to \( F \) and arguing by convexity, that for \( (U, V) \) in a neighbourhood \( \mathcal{V} \) of \( (P, Q) \), there exists a function \( G(U, V) = (G_1(U, V), G_2(U, V)) = (\theta(U, V), y(U, V)) \) that minimizes \( F(U, V, \cdot, \cdot) \) in a ball centered in \( (0, 0) \); that is, locally, the \( L^2 \) distance between \( (U, V) \) and the orbit of \( (P, Q) \) is achieved. Furthermore, one can show that, forall \( (\theta, y) \in \mathbb{R} \times \mathbb{R}^n \),

\[
G_1(f(\theta, y)(U, V)) = G_1(U, V) - \theta \mod 2\pi
\]

and

\[
G_2(f(\theta, y)(U, V)) = G_2(U, V) - y
\]

provided that \( f(\theta, y)(U, V) \in \mathcal{V} \). These properties allow to coherently extend the functional

\[
\mathcal{H}(U, V) = \langle J^{-1} \Psi, f(G(U, V))(U, V) \rangle
\]

from \( \mathcal{V} \) to a entire neighbourhood \( \Sigma \) of the orbit of \( (P, Q) \). Furthermore, in view of (20) and (21), it is straightforward that \( \mathcal{H} \) is invariant by the action of \( f \).

Using again the Implicit Function Theorem and the expressions it provides for \( G_1 \) and \( G_2 \), we can check that \( \mathcal{H}'(U, V) \in H \) and that \( \mathcal{H}' \) is \( C^1 \) with bounded derivative.

Finally, from the orthogonality relations expressed in condition 2., one can deduce that \( \mathcal{H}'(P, Q) = J^{-1} \Psi \), that is, \( J\mathcal{H}'(P, Q) = \Psi \).

**Step 2: Instability**

The main idea of the proof is to follow the evolution of the action along the integral curves of the Auxiliary Dynamical System. More precisely, given \( X_0 = (U_0, V_0) \) in a neighbourhood \( \Sigma \) of \( \Sigma \) and for a \( \sigma > 0 \), we consider the path

\[
\phi : s \in ] - \sigma, \sigma [ \to \phi(X_0, s) \in \Sigma
\]

such that

\[
\frac{d}{ds} \phi(X_0, s) = J\mathcal{H}'(X_0, s).
\]

We consider the evolution of the action along this path, \( S(\phi(X_0, s)) \). A simple computation then yields

\[
\frac{d}{ds} S(\phi(X_0, s)) = P((\phi(X_0, s)) \) and \( \frac{d^2}{ds^2} S(\phi(X_0, s)) = R((\phi(X_0, s)),
\]

where

\[
P(U, V) = \langle S(U, V), J\mathcal{H}'(U, V) \rangle
\]
and
\[ R(U, V) = \langle S''(U, V), \mathcal{H}'(U, V), J\mathcal{H}'(U, V) \rangle + \langle S'(U, V), J\mathcal{H}''(U, V), J\mathcal{H}'(U, V) \rangle. \]

Using the Taylor expansion, we obtain the existence of \( \xi \in [0, 1] \) such that
\[ S(\phi(X_0, s)) = S(X_0) + P(X_0)s + \frac{1}{2} R(\phi(X_0, \xi s))s^2. \]

Noticing that \( S'(P, Q) = 0 \) (from (11)) and \( J\mathcal{H}'(P, Q) = \Psi \), we obtain that
\[ R(P, Q) = \langle S''(P, Q)\Psi, \Psi \rangle < 0, \]
yet, from (23), for \( X_0 \) in a neighbourhhood of \( (P, Q) \) and for small \( s \),
\[ S(\phi(X_0, s)) \leq S(X_0) + P(X_0)s. \]

By intersecting the manifold \( \mathcal{W}(P, Q) \) with the trajectories of the Auxiliary Dynamical System, using the Implicit Theorem Function, it is possible to obtain a uniform version of (24), namely, for some \( \epsilon > 0 \),
\[ \forall X_0 \in \Sigma_x, \exists s \in ]-\sigma, \sigma[ \), \( S(\phi(X_0, s)) \leq S(X_0) + P(X_0)s. \]

This means that \( P \) measures the variations of \( S \) (hence of \( E \)) along the trajectories pf the Auxiliary Dynamical System.

The crucial step is now to prove that \( P \) also measures the variations of \( \mathcal{H} \) along the the flow of the initial system (11). More precisely, considering the solution \( (u(t), v(t)) \) of (11) with initial data \( X_0 \), we have
\[ \frac{d}{dt} \mathcal{H}(u(t), v(t)) = -P(u(t), v(t)). \]

This can be achieved by justifying the following formal computation:
\[ \mathcal{H}(u(t), v(t)) - \mathcal{H}(u_0, v_0) = \int_0^t \langle \mathcal{H}'(u(\tau), v(\tau)), (u_\tau(\tau), v_\tau(\tau)) \rangle d\tau \]
\[ = \int_0^t \langle \mathcal{H}'(u(\tau), v(\tau)), J^{-1} E'(u(t), v(t)) \rangle d\tau = -\int_0^t P(u(\tau), v(\tau)) d\tau \]
since mass is conserved along the trajectories of the Auxiliary Dynamical System (11) and \( J^{-1} \) is skew-adjoint.

Finally, setting
\[ \mathcal{P} = \{(U, V) \in \Sigma_x : S(U, V) < S(P, Q) \text{ and } P(U, V) \neq 0\}, \]
it can be shown that, for \( X_0 \in \mathcal{P} \), \( P(u(t), v(t)) \) remains bounded away of the origin as long as the solution \( (u(t), v(t)) \) exists. This implies that solutions of (11) for initial data \( X_0 \in \mathcal{P} \) must leave in finite time any neighbourhood of \( \Sigma \). Indeed,
\[ \left| \frac{d}{dt} \mathcal{H}(u(t), v(t)) \right| = |P(u(t), v(t))| \geq C(X_0) > 0, \]
which contradicts the fact that \( \mathcal{H} \) in bounded in any neighbourhood of \( \Sigma \). (recall \( \mathcal{H} \) is invariant by \( f(\theta, y), \theta \in \mathbb{R}, y \in \mathbb{R}^n \)).

Now, following the action along the trajectory of the Auxiliary Dynamical System that contains \( (P, Q) \) it can be shown that \( \mathcal{P} \) contains points arbitrarily close to \( (P, Q) \) of the form \( \phi((P, Q), s) \), that is, belonging to the considered trajectory.
Also, setting $W(U, V) = Re \int \overline{U}^2 V$ the potential energy and observing that the map

$$A : s \to W(\phi((P, Q), s))$$

is $C^1$ and has a non vanishing derivative at the origin, for small $s$ with the adequate sign,

$$W(\phi((P, Q), s)) < W(P, Q).$$

Putting $X_0 = \phi((P, Q), s)$ and considering the solution $X(t) = (u(t), v(t))$ of (11) with initial data $X_0$, we have, as long as the solution exists,

$$W(X(t)) < W(P, Q).$$

Indeed, if at some point $W(X(t)) = W(P, Q)$ then we would obtain a contradiction with the fact that $X_0 \in \mathcal{P}$ and that $(P, Q)$ is a solution of (11):

$$X(t) \in W(P, Q) \quad \text{and} \quad S(X(t)) \leq S(X_0) < S(P, Q).$$

Since $E$ is conserved by the flow of (1), this is enough to prove that $(u(t), v(t))$ is bounded in $H$ and global. $lacksquare$

**End of the Proof of Theorem 1.5:**

We now exhibit $\Psi \in H$ satisfying properties 1., 2., 3. and 4. Let $\epsilon > 0$. We begin by considering the curve

$$\Gamma : t \in [0, \epsilon] \to \left(\gamma(t)\lambda^2(t)P(\lambda(t)\cdot), \alpha(t)\lambda^2(t)Q(\lambda(t)\cdot)\right)$$

where $\alpha, \gamma$ and $\lambda$ are smooth real functions to be chosen later, such that

$$\alpha(0) = \gamma(0) = \lambda(0) = 1 \quad \text{(that is, } \Gamma(0) = (P, Q)).$$

1. Setting

$$k = \frac{\int P^2}{4 \int Q^2},$$

the condition

$$\gamma^2 k + \alpha^2 = k + 1 \quad \text{(28)}$$

assures that $\Gamma \subset \mathcal{L}$, and, in particular,

$$\Psi = \Gamma'(0) \quad \text{(29)}$$

is tangent to $\mathcal{L}$ at $(P, Q)$.

2. Noticing that

$$\Psi = ((\lambda^2 \gamma)'(0)P + \lambda'(0)\nabla P, (\lambda^2 \alpha)'(0)Q + \lambda'(0)\nabla Q)$$

has real components,

$$i\Psi \perp \nabla_y f(0, 0)[P, Q].$$

Also, $i\Psi \perp i(P, 2Q)$ since $\Psi \in T_{\mathcal{L}}(P, Q)$ and $(P, 2Q) \perp T_{\mathcal{L}}(P, Q)$.

3. Since $i(P, 2Q) \perp (\nabla P, \nabla Q)$, these two vectors are linearly independent.
4. We begin by computing the energy \((3)\) along the path \(\Gamma\):

\[
E(\Gamma(t)) = \gamma^2 \lambda^2 \int |\nabla P|^2 + \alpha^2 \lambda^2 \int |\nabla Q|^2 - \gamma^2 \alpha \lambda^2 \int P^2 Q
\]

\[
= \frac{1}{k} \left((k + 1 - \alpha^2) \lambda^2 \int |\nabla P|^2 + k \alpha^2 \lambda^2 \int |\nabla Q|^2 + k \beta \alpha^2 \int Q^2 - (k + 1 - \alpha^2) \alpha \lambda^2 \int P^2 Q\right).
\]

Differentiating with respect to \(t\),

\[
k \frac{d}{dt} E(\Gamma(t)) = \alpha A(t) + \lambda B(t),
\]

with

\[
A(t) = -2 \alpha^2 \lambda^2 \int |\nabla P|^2 + 2k \alpha \lambda^2 \int |\nabla Q|^2 + 2k \beta \alpha \int Q^2 + (3 \alpha^2 - k - 1) \lambda^2 \int P^2 Q
\]

and

\[
B(t) = 2 \lambda (k + 1 - \alpha^2) \int |\nabla P|^2 + 2k \alpha^2 \lambda \int |\nabla Q|^2 + \frac{n}{2} (\alpha^2 - k - 1) \alpha \lambda^{\frac{n-2}{2}} \int P^2 Q.
\]

Now, observe that since \((P, Q)\) is a solution of \((9)\), \(A(0) = B(0) = 0\) (see \([2]\), (5.2)). Hence, putting \(\alpha_0 = \alpha'(0)\) and \(\lambda_0 = \lambda'(0)\),

\[
k \frac{d^2}{dt^2} E(\Gamma(t)) \big|_{t=0} = \alpha_0 A'(0) + \lambda_0 B'(0)
\]

\[
= \alpha_0^2 \left(-2 \int |\nabla P|^2 + 2k \int |\nabla Q|^2 + 2k \beta \int Q^2 + 6 \int P^2 Q\right)
\]

\[
= \lambda_0^2 \left(2k \int |\nabla P|^2 + 2k \int |\nabla Q|^2 - \frac{n(n-2)}{4} k \int P^2 Q\right)
\]

\[
= 2 \alpha_0 \lambda_0 \left(-4 \int |\nabla P|^2 + 4k \int |\nabla Q|^2 - \frac{(k-2)(4-n)}{2} n \int P^2 Q\right).
\]

Using once again that \(A(0) = B(0) = 0\), this quantity can be re-written in terms of \(\int P^2 Q\) and \(\int Q^2\) exclusively:

\[
k \frac{d^2}{dt^2} E(\Gamma(t)) \big|_{t=0} = \alpha_0^2 \left((k+4) \int P^2 Q\right) + \lambda_0^2 \left(\frac{n(4-n)}{4} k \int P^2 Q\right)
\]

\[
= 2 \alpha_0 \lambda_0 \left(-4k \beta \int Q^2 + \frac{(k-2)(4-n)}{2} \int P^2 Q\right).
\]

The determinant of \(\frac{d^2}{dt^2} E(\Gamma(t)) \big|_{t=0}\) as a quadratic form in \((\alpha_0, \lambda_0)\) is given by

\[
\Delta = (k+4) \frac{n(4-n)}{4} \left(\int P^2 Q\right)^2 - \left(\frac{(k-2)(4-n)}{2} \int P^2 Q - 4k \beta \int Q^2\right)^2.
\]
For $d \geq 4$ ($n \geq 5$), $\Delta < 0$. For $d = 3$ ($n = 4$) and $\beta \neq 0$,  
$$\Delta = -16\beta^2 k \int Q^2 < 0.$$  
Hence, in both these situations, one can choose $\alpha(t), \lambda(t)$ such that  
$$\frac{d^2}{dt^2}E(\Gamma(t))|_{t=0} < 0.$$  
Now, observe that  
$$\frac{d}{dt}S(\Gamma(t)) = \langle S'(\Gamma(t)), \Gamma'(t) \rangle$$  
and  
$$\frac{d^2}{dt^2}S(\Gamma(t)) = \langle S''(\Gamma(t)), \Gamma''(t) + \Gamma'(t)^T \left[ S''(\Gamma(t)) \right] \Gamma'(t) \rangle.$$  
Since $S'(\Gamma(0)) = S'(P, Q) = 0$, setting $t = 0$ yields  
$$\frac{d^2}{dt^2}S(\Gamma(t))|_{t=0} = \langle S''(P, Q)\Psi, \Psi \rangle.$$  
Finally $\Gamma \subset L$, $\frac{d}{dt}M(\Gamma(t)) = 0$ and  
$$\langle S''(P, Q)\Psi, \Psi \rangle = \frac{d^2}{dt^2}E(\Gamma(t))|_{t=0} < 0.$$  

### 4. A Stability Result

Define, for any $(u, v) \in H$ such that $J(u, v) > 0$,  
$$GN(u, v) = \frac{M(u, v)^{\frac{1}{2} - \frac{d+1}{4} + \frac{d+1}{4}}}{J(u, v)}.$$  
This functional is closely related with a vector-valued Gagliardo-Nirenberg inequality: if one sets  
(30)  
$$C_{GN}^\frac{1}{2} := \inf \{ GN(u, v) : J(u, v) > 0 \},$$  
then $C_{GN}$ is the optimal constant of the inequality  
$$\text{Re} \int \overline{w} v \leq C \left( \int |u|^2 + 4|v|^2 \right)^{\frac{1}{2} - \frac{d+1}{4} + \frac{d+1}{2}} \left( \int |\nabla u|^2 + |\nabla v|^2 \right)^{\frac{d+1}{2}}.$$  

**Lemma 4.1.** Suppose that $\beta = 0$. Then the set of solutions for the minimization problem (30) is $G$, up to scalar multiplication and scaling.

**Proof:**

By [2], we know that $G \neq \emptyset$ is the set of solutions of (11). Let $Q \in G$ and $W = (u, z)$ be such that $J(W) > 0$. Recall that $I(Q) = J(Q) > 0$. Define  
$$\nu = \left( \frac{J(Q)M(W)}{M(Q)J(W)} \right)^{\frac{1}{2p}}$$  
and  
$$\zeta = \left( \nu^2 \left( \frac{M(W)}{M(Q)} \right) \right)^{\frac{1}{p}}.$$
Then $Z(x) = \nu W(\zeta x)$ satisfies
\[
J(Z) = J(Q), \quad M(Z) = M(Q) \quad GN(Z) = GN(W).
\]
By the minimality of $Q$, $I(Q) \leq I(Z)$, which implies that $GN(Q) \leq GN(Z) = GN(W)$. Therefore $Q$ is a solution of (30). On the other hand, if $W$ is a solution of (30), then one has necessarily $GN(Z) = GN(Q)$, which implies that $I(Z) = I(Q)$. Therefore $Z \in G$, which concludes our proof. 

**Lemma 4.2.** Suppose that $d \leq 2$ and $\beta = 0$. Then the set of ground states $G$ is the set of solutions of (12).

**Proof:**
Let $W = (w, z) \in H$ be such that $M(W) = \nu$. For any $\lambda > 0$, define $W_\lambda(x) := \lambda^{\frac{d+1}{2}} W(\lambda x)$. Consider the function
\[
\lambda \mapsto f(\lambda) = E(W_\lambda), \quad \lambda > 0
\]
Since $d \leq 2$, $f$ has a unique minimum $\lambda_0$. Let $Z = W_{\lambda_0}$. Then $f'(\lambda_0) = 0$, which implies that $K(Z) = \frac{d + 1}{6} J(Z)$.

Therefore,
\[
E(Z) = \frac{d - 3}{2d + 2} K(Z).
\]
Let $Q \in G$. Notice that, by Pohozaev’s equality, the same relations are valid for $Q$:
\[
K(Q) = \frac{d + 1}{6} J(Q), \quad E(Q) = \frac{d - 3}{2d + 2} K(Q).
\]
By Lemma 4.1, we have
\[
\frac{(d + 1)\nu^{\frac{3 - d + 1}{d + 1}} K(Q)^{\frac{d + 1}{d + 1}}}{6K(Q)} = GN(Q) \leq GN(Z) = \frac{(d + 1)\nu^{\frac{3 - d + 1}{d + 1}} K(Z)^{\frac{d + 1}{d + 1}}}{6K(Z)}
\]
and so $K(Z) \leq K(Q)$. Hence, by (31) and (32),
\[
E(W) \geq E(Z) = \frac{d - 3}{2d + 2} K(Z) \geq \frac{d - 3}{2d + 2} K(Q) = E(Q),
\]
and so $Q$ is a solution of (12).

On the other hand, if $W$ is also a solution of (12), then one must have $Z = W$ and $K(W) = K(Q)$. Again by (31) and (32),
\[
J(W) = \frac{6}{d + 1} K(W) = \frac{6}{d + 1} K(Q) = J(Q).
\]
Moreover, since $M(W) = M(Q)$, one has $I(W) = I(Q)$. This implies that $W$ is a solution of (12), i.e., $W \in G$.

**Sketch of the proof of Theorem 1.6:** The proof follows the same steps as [2, Proposition 4]: suppose, by contradiction, that there exist sequences $(u^n_0, v^n_0)_{n \in \mathbb{N}} \subset H$ and $(t_n)_{n \in \mathbb{N}} \subset \mathbb{R}^+$ with
\[
\inf_{(P, Q) \in G} \|(u^n_0, v^n_0) - (P, Q)\|_H \to 0
\]
and such that the corresponding solutions \((u^n, v^n)\) satisfy
\[
\inf_{(P,Q) \in G} \| (u^n(t_n), v^n(t_n)) - (P, Q) \|_H > \delta.
\]
By (33), for any given \((P, Q) \in G\),
\[
M(u^n_0, v^n_0) \to M(P, Q), \quad E(u^n_0, v^n_0) \to E(P, Q).
\]
Using the conservation of mass and energy, one has
\[
M(u^n(t_n), v^n(t_n)) \to M(P, Q), \quad E(u^n(t_n), v^n(t_n)) \to E(P, Q).
\]
This implies that (up to a normalization) \(\{(u^n(t_n), v^n(t_n))\}_{n \in \mathbb{N}}\) is a minimizing sequence of problem (12). The argument of [2] implies that
\[
(u^n(t_n), v^n(t_n)) \to (\tilde{P}, \tilde{Q}),
\]
where \((\tilde{P}, \tilde{Q})\) is a solution of (12), that is, \((\tilde{P}, \tilde{Q}) \in G\). This convergence contradicts (34), thus finishing the proof.

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