Binary Darboux transformation of the first member of the negative part of the AKNS hierarchy and solitons

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Abstract

Using bidifferential calculus, we derive a vectorial binary Darboux transformation for the first member of the “negative” part of the AKNS hierarchy. A reduction leads to a rather simple nonlinear PDE in two dimensions with a leading mixed third derivative. This PDE may be regarded as describing dynamics of a complex scalar field in one dimension, since it is invariant under coordinate transformations in one of the two independent variables. We exploit the correspondingly reduced binary Darboux transformation to generate multi-soliton solutions of the PDE, also with additional rational dependence on the independent variables, and on a plane wave background.

1 Introduction

The main subject of this work is the third-order nonlinear PDE

$$\left(\frac{f_{xt}}{f}\right)_t + 2\left(f^* f\right)_x = 0, \quad (1.1)$$

where $f$ is a complex function of two independent real variables $x$ and $t$, and $f^*$ is the complex conjugate of $f$. A subscript denotes a partial derivative with respect to one of the independent variables. An evident property of (1.1) is the following.

Proposition 1.1. If $f(x, t)$ solves (1.1), then also $f(\sigma(x), t)$, with an arbitrary differentiable function $\sigma(x)$. □

This expresses the fact that (1.1) is invariant under coordinate transformations $x \mapsto \sigma(x)$ in one dimension, and $f$ can be regarded as a scalar. A generalization of (1.1) to higher dimensions is the system

$$\frac{\partial}{\partial t}(f^{-1} \frac{\partial}{\partial t} \frac{\partial}{\partial x^\mu} f) + 2 \frac{\partial}{\partial x^\mu} (f^* f) = 0 \quad \mu = 1, \ldots, m. \quad (1.2)$$

It behaves as the components of a covector (tensor of type (0,1)) under general coordinate transformations in $m$ dimensions, if $f$ is a scalar, also depending on a parameter $t$. This system thus defines dynamics of a scalar field on an $m$-dimensional differentiable manifold. Obviously, the following holds.

Proposition 1.2. If $f(x, t)$ solves (1.1), then $f(\sigma(x^1, \ldots, x^m), t)$, with an arbitrary differentiable function $\sigma$ of real independent variables $x^\mu$, $\mu = 1, \ldots, m$, solves (1.2). □

The factor 2 can be eliminated by a rescaling of $f$. Writing $f = e^u$, the equation takes the form $u_{xtt} + (u_x u_t)_t + 2(e^{2Re(u)})_x = 0$. A leading mixed third derivative also appears, for example, in the physically relevant Benjamin-Bona-Mahony [1] and the Joseph-Egri equation [2].
can also be written as
\[ \frac{\partial}{\partial t} \left( f^{-1} \frac{\partial f}{\partial t} \right) + 2 d(f^* f) = 0, \]
where \( d \) is the exterior derivative on the \( m \)-dimensional differentiable manifold.

\( (1.1) \) arises, via a reduction, from the first “negative flow” of the AKNS hierarchy, which is related to the complex sine-Gordon equation \( [3] \). In \( [4] \) a relation with the sharp line self-induced transparency (SIT) equations has been observed, also see \( [5, 6] \). Furthermore, it is connected with a 2-component Camassa-Holm equation \( [7, 8] \). Soliton solutions have been found in \( [9] \), using Hirota’s bilinear method. Other methods have been applied in particular in \( [3] \) (see Section 5.1 therein), \( [6] \), and \( [10] \) (see (4.28) therein).

\( (1.1) \) is among the simplest completely integrable PDEs and it has the peculiar property mentioned above. In this work, we derive a binary Darboux transformation (see, e.g., \( [11] \)), by using a general result of bidifferential calculus, which is recalled in Section 4.1 in a self-contained way (see, e.g., \( [12] \) for an introduction to bidifferential calculus and references), and demonstrate its use for finding soliton solutions. Here we proceed beyond the class of “simple solitons”, which are rational expressions built from trigonometric and hyperbolic functions of linear combinations of the independent variables. There are also regular solutions which additionally depend \( \text{rationally} \) on the variables \( x \) and \( t \).

A binary Darboux transformation for \( (1.1) \) has already been obtained in \( [13, 14, 15] \). The advantages of our vectorial binary Darboux transformation, involving a “spectral matrix” instead of a spectral parameter in the underlying linear system, are summarized in Remark 3.3 below.

In Section 2 we recall some useful Lie point symmetries of \( (1.1) \) and traveling wave solutions in terms of elementary Jacobi elliptic functions (cf. \( [16] \) for corresponding solutions of the “positive” part of the NLS hierarchy).

Section 3 presents our version of an \( n \)-fold binary Darboux transformation for \( (1.1) \), which we then exploit to find multi-soliton solutions. This includes solitons superposed on a plane wave background. Section 4 presents a derivation of the binary Darboux transformation, more generally for the first “negative flow” of the AKNS hierarchy. Finally, Section 5 contains some concluding remarks.

### 2 Some symmetries of the PDE and traveling wave solutions

\( (1.1) \) admits the following symmetry transformations:

- \( x \mapsto \sigma(x) \), see Proposition \( 1.1 \)
- \( t \mapsto \pm t + \alpha, \alpha \in \mathbb{R} \)
- \( t \mapsto \pm |\beta| t, f \mapsto \beta f, \beta \in \mathbb{C}, \beta \neq 0 \)
- \( f \mapsto e^{i\varphi_0} f, \varphi_0 \in \mathbb{R} \)
- Complex conjugation of \( f \).

In the following, solutions will typically be presented modulo these symmetries.

Let us assume that \( f \) is real and, in some coordinate \( x \), has the form
\[ f(x, t) = f(x \pm c t), \]

These authors write \( (1.1) \) in the form \( u_{xt} + 2u \partial_x \partial_x^{-1}(|u|^2) = u \) (hence \( x \) and \( t \) are exchanged relative to our notation). But the term on the right hand side should actually be multiplied by an arbitrary function of \( t \), since this is the freedom in the definition of an inverse of \( \partial_x \). It seems that the authors of \( [15] \) wanted to fix the freedom by demanding \( u \to 0 \) as \( |x| \to \infty \). But how can this then be reconciled with exact solutions in their Section 6, having a non-zero constant background? Throughout our work, there will be no need for introducing the inverse of a differential operator.
with a real constant $c > 0$. Then (1.1) reduces to the ODE
\[
\frac{f''}{f} + \frac{2}{c^2} f^2 = k ,
\]
with a real constant $k$. Exclusively in this section, a prime indicates a derivative with respect to the argument of the function $f$. Solutions of this equation are provided by the Jacobi elliptic functions $cn$ and $dn$ (see [18], for example). Indeed,
\[
f_{cn} = \sqrt{c} \sqrt{m} \cn(\frac{1}{\sqrt{c}}(x \pm ct) \mid m)
\]
solves the ODE with $k = (2m - 1)/c$. We note that
\[
f_{cn} = \sqrt{c} \sech(\frac{1}{\sqrt{c}}(x \pm ct)) \quad \text{if} \quad m = 1.
\]
We will recover this solitary wave as a single soliton solution in Section 3. Furthermore,
\[
f_{dn} = \sqrt{c} \dn(\frac{1}{\sqrt{c}}(x \pm ct) \mid m)
\]
satisfies the ODE with $k = (2 - m)/c$. We note that
\[
f_{dn} = \sqrt{c} \sech(\frac{1}{\sqrt{c}}(x \pm ct)) \quad \text{if} \quad m = 1.
\]

3 A binary Darboux transformation and soliton solutions

(1.1) will be treated in the following form,
\[
a_t = (f^* f)_x, \quad f_{xt} + 2a f = 0 ,
\]
where $a$ is a real function. This system is invariant under a coordinate transformation $x \mapsto x'$ if the latter function transforms as $a \mapsto a' = (\partial x/\partial x') a$. We next formulate the main result of this work. A derivation and proof is postponed to Section 4.

Theorem 3.1. Let $a_0, f_0$ be a solution of (3.1). Let $n$-component column vectors $\eta_i, i = 1, 2$, be solutions of the linear system
\[
\Gamma \eta_1 x = a_0 \eta_1 + f_0^* \eta_2, \quad \Gamma \eta_2 x = -a_0 \eta_2 + f_0 \eta_1 ,
\]
\[
\eta_1 t = -\frac{1}{2} \Gamma \eta_1 + f_0^* \eta_2 , \quad \eta_2 t = \frac{1}{2} \Gamma \eta_2 - f_0 \eta_1 ,
\]
where $\Gamma$ is an invertible constant $n \times n$ matrix satisfying the spectrum condition $\text{spec}(\Gamma) \cap \text{spec}(-\Gamma^\dagger) = \emptyset$. Furthermore, let $\Omega$ be an invertible solution of the Lyapunov equation
\[
\Gamma \Omega + \Omega \Gamma^\dagger = \eta_1 \eta_1^\dagger + \eta_2 \eta_2^\dagger ,
\]
where $\dagger$ denotes Hermitian conjugation (transposition and complex conjugation). Then
\[
a = a_0 - (\eta_1^\dagger \Omega^{-1} \eta_1 ) x, \quad f = f_0 - \eta_1^\dagger \Omega^{-1} \eta_2
\]
is also a solution of (3.1). As a consequence, $f$ solves (1.1). □

\[^3\text{This is equation 7.7 in [17].}\]
An application of this theorem essentially reduces to solving the linear system for a given solution $f_0$ of (3.1) and a constant $n \times n$ matrix $\Gamma$, since it is well-known that, under the stated spectrum condition, (3.4) has a unique solution $\Omega$ and there are concrete expressions for it. In order to obtain a more explicit expression for the generated solution $f$, however, one needs to evaluate the inverse of the matrix $\Omega$, which is getting more and more difficult with increasing $n$, of course.

Without restriction of generality, $\Gamma$ can be restricted to Jordan normal form. We also note that $\Omega$ in the preceding theorem is Hermitian (also see (4.18)) and consequently $\det(\Omega)$ is real. If $f_0$ is a regular solution of (1.1) on $\mathbb{R}^2$, a solution $f$ generated via the above theorem can only be singular if $\Omega$ is not invertible somewhere on $\mathbb{R}^2$, i.e., if $\det(\Omega)$ has a zero.

**Remark 3.2.** If $f_0 \neq 0$, the first order system (3.3) can be decoupled into

$$
\eta_{1t} - \frac{f_{0t}}{f_0} \eta_{tt} - \left( \frac{1}{4} \Gamma^2 + \frac{f_{0t}}{2f_0} \Gamma - |f_0|^2 \right) \eta_1 = 0, \hspace{1em} \eta_2 = \frac{1}{f_0} \left( \eta_{tt} + \frac{1}{2} \Gamma \eta_1 \right).
$$

If $f_{0x} \neq 0$, (3.2) can be decoupled correspondingly,

$$
\eta_{1x} - \left( \frac{f_{0xx}}{f_{0x}} \right)^* \eta_{tx} - \left[ (a_0^2 + |f_{0x}|^2) \Gamma^{-2} + \left( a_{0x} - a_0 \left( \frac{f_{0xx}}{f_{0x}} \right)^* \Gamma^{-1} \right) \right] \eta_1 = 0, \hspace{1em} \eta_2 = \frac{1}{f_{0x}} \left( \Gamma \eta_{lx} - a_0 \eta_1 \right).
$$

If $f_0 \neq 0$, but $f_{0x} = 0$, the second of equations (3.1) requires $a_0 = 0$. (3.2) then restricts $\eta_1$ and $\eta_2$ to not depend on $x$. Since any function independent of $x$ solves (1.1), such an $f_0$ is not a useful “seed” for the binary Darboux transformation in Theorem 3.1.

**Remark 3.3.** Writing (3.2) and (3.3) in the form

$$
\begin{pmatrix}
\eta_1 \\
\eta_2
\end{pmatrix}_x = \begin{pmatrix}
a_0 I_n & f_{0x}^* I_n \\
f_{0x} I_n & -a_0 I_n
\end{pmatrix} \Gamma^{-1} \begin{pmatrix}
\eta_1 \\
\eta_2
\end{pmatrix}, \hspace{1em} \begin{pmatrix}
\eta_1 \\
\eta_2
\end{pmatrix}_t = \begin{pmatrix}
-\frac{1}{2} \Gamma & f_{0x}^* I_n \\
f_{0x} I_n & 1/2 \Gamma
\end{pmatrix} \begin{pmatrix}
\eta_1 \\
\eta_2
\end{pmatrix},
$$

where $I_n$ is the $n \times n$ identity matrix, constitutes a “Lax pair” for (3.1), since its integrability conditions are equivalent to $a_0, f_0$ satisfying (3.1). It should be noticed, however, that the usual spectral parameter is here promoted to a *matrix* $\Gamma$. This is the main reason why there is no need in our approach to consider iterations of Darboux transformations (in contrast to the older approach taken, e.g., in [13,14,15]). We obtain the result of an $n$-fold elementary binary Darboux transformation right away in a single step and we can use efficient matrix methods to elaborate concrete solutions. This has been demonstrated in several previous publications, addressing various completely integrable equations, and it will also be demonstrated for the PDE under consideration in the following subsections. Another advantage of our method lies in the fact that important classes of completely integrable equations are directly obtained by choosing the spectral matrix to be a (non-diagonal) Jordan block. This concerns in particular the rogue wave solutions of the nonlinear Schrödinger equation [19]. In other approaches, they can only be obtained indirectly, starting from a solution obtained by a multiple application of an elementary binary Darboux transformation, each time with a different value of the spectral parameter (which then corresponds, in our setting, to taking a diagonal matrix $\Gamma$ with distinct eigenvalues), and then taking suitable coincidence limits of the spectral parameter values and other parameters.

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4If $n > 2$, $\Omega$ can be decomposed into block submatrices and the inverse can be computed using Schur complements, which are (special) quasideterminants. The latter are used in the alternative iterative approach to a binary Darboux transformation in [13,14,15].

5Here $\Gamma^{-1}$ multiplies both, $\eta_1$ and $\eta_2$.

6Nevertheless, the classical iterative (forward, backward and binary) Darboux transformations can also be formulated in bidifferential calculus, see [12].
3.1 Zero seed solutions

If \( f_0 = 0 \), we choose \( a_0 = -1/2 \). The linear system for \( \eta \) then has the solutions

\[
\eta_1 = \exp \left( -\frac{1}{2}(\Gamma^{-1} x + \Gamma t) \right) v, \quad \eta_2 = \exp \left( \frac{1}{2}(\Gamma^{-1} x + \Gamma t) \right) w, \tag{3.6}
\]

where \( v, w \) are constant \( n \)-component column vectors. The ansatz

\[
\Omega = e^{-\frac{1}{2}(\Gamma^{-1} x + \Gamma t)} X e^{-\frac{1}{2}(\Gamma^\dagger^{-1} x + \Gamma^\dagger t)} + e^{\frac{1}{2}(\Gamma^{-1} x + \Gamma t)} Y e^{\frac{1}{2}(\Gamma^\dagger^{-1} x + \Gamma^\dagger t)}, \tag{3.7}
\]

with constant \( n \times n \) matrices \( X, Y \), solves (3.4) if

\[
\Gamma X + X \Gamma^\dagger = vv^\dagger, \quad \Gamma Y + Y \Gamma^\dagger = ww^\dagger. \tag{3.8}
\]

According to Theorem 3.1,

\[
f = \frac{1}{\det(\Omega)} v^\dagger e^{-\frac{1}{2}(\Gamma^\dagger^{-1} x + \Gamma^\dagger t)} \text{adj}(\Omega) e^{\frac{1}{2}(\Gamma^{-1} x + \Gamma t)} w, \tag{3.9}
\]

where \( \text{adj} \) takes the adjugate of a matrix, represents (an infinite set of) exact solutions of (1.1). Here we dropped a global minus sign, since \( f \mapsto -f \) is a symmetry of (1.1). We also note that \( \Gamma \mapsto -\Gamma \) amounts to \( f \mapsto -f \).

3.1.1 Simple multi-soliton solutions

These are obtained by choosing \( \Gamma = \text{diag}(\gamma_1, \ldots, \gamma_n) \), where \( \gamma_i^* \neq -\gamma_j, i, j = 1, \ldots, n \). The solutions of the Lyapunov equations (3.8) are then the Cauchy-like matrices

\[
X = \begin{pmatrix} v_i & v_j^* \\ \gamma_i + \gamma_j^* \end{pmatrix}, \quad Y = \begin{pmatrix} w_i & w_j^* \\ \gamma_i + \gamma_j^* \end{pmatrix}.
\]

Substituting these expressions in (3.7), (3.9) provides us with an exact solution of (1.1) for any \( n \in \mathbb{N} \).

\( n = 1 \). In this case, (3.9) becomes

\[
f = 2 \text{Re}(\gamma) v^* w \left( |v|^2 e^{-(\gamma^{-1} x + \gamma t)} + |w|^2 e^{\gamma^* x + \gamma^* t} \right)^{-1}.
\]

If \( \gamma, v, w \) are real, this can be rewritten, up to a global sign, as

\[
f = \gamma \text{sech}(\gamma^{-1} x + \gamma t + \alpha), \tag{3.10}
\]

where \( \alpha = \ln(|w/v|) \). The solitary wave has the form of the bright soliton of the NLS equation and the solitary wave of the modified KdV (mKdV) equation.

\( n = 2 \). (3.9) yields

\[
f = \frac{1}{2 \text{Re}(\gamma_1)(\gamma_1 + \gamma_2^*) \det(\Omega)} e^{-2 \text{Re}(\gamma_1)(t+x/|\gamma_1|^2) + \text{Im}(\gamma_2)(t-x/|\gamma_2|^2)} \left( (\gamma_2 - \gamma_1^*) |v_1|^2 v_2^* w_2 + (\gamma_1 + \gamma_2^*) v_2^* w_1^2 w_2 e^{2 \text{Re}(\gamma_1)(t+x/|\gamma_1|^2)} - 2 \text{Re}(\gamma_1) v_1^* w_1 |w_2|^2 e^{(\gamma_1 + \gamma_2^*) t+(\gamma_1^{-1} + \gamma_2^*^{-1}) x} \right)
\]
+ \frac{1}{2 \text{Re}(\gamma_2) (\gamma_1^* + \gamma_2)} \det(\Omega) e^{-\text{Re}(\gamma_2)(t+x/|\gamma_2|^2) + i \text{Im}(\gamma_1)(t-x/|\gamma_1|^2)} \left((\gamma_1^* - \gamma_2^*)v_1^* |v_2|^2 w_1 \right. \\
+ (\gamma_1^* + \gamma_2)v_1^* w_1|w_2|^2 e^{2 \text{Re}(\gamma_2)(t+x/|\gamma_2|^2)} - 2 \text{Re}(\gamma_2)v_2^* |w_1|^2 w_2 e^{(\gamma_1 + \gamma_2)t + (\gamma_1^{-1} + \gamma_2^{-1})x},
\right)
\end{equation}

where
\begin{equation}
\det(\Omega) = \frac{e^{-\text{Re}(\gamma_1 + \gamma_2)t - \text{Re}(\gamma_1^{-1} + \gamma_2^{-1})x}}{4 \text{Re}(\gamma_1) \text{Re}(\gamma_2) |\gamma_1 + \gamma_2|^2} \left(|\gamma_1 + \gamma_2|^2 |v_2 w_1 e^{\gamma_1 t + \gamma_1^{-1} x} - v_1 w_2 e^{\gamma_2 t + \gamma_2^{-1} x}|^2 
+ |\gamma_1 - \gamma_2|^2 |v_1 v_2^* + w_1 w_2^* e^{(\gamma_1 + \gamma_2)t + (\gamma_1^{-1} + \gamma_2^{-1})x}|^2 \right),
\end{equation}

which we were able to express in an explicitly non-negative form.

**Proposition 3.4.** The 2-soliton solution is regular if \( \gamma_1, \gamma_2 \neq 0, \gamma_1 \neq \gamma_2, \gamma_1 \neq -\gamma_2^*, \{v_1, w_1\} \neq \{0\} \) and \( \{v_2, w_2\} \neq \{0\} \).

**Proof.** Assuming \( v_2 w_1 w_2 \neq 0 \), for a zero of \( \det(\Omega) \) we would need
\begin{equation}
e^{\varphi_1 - \varphi_2} = \frac{v_1 w_2}{v_2 w_1}, \quad e^{\varphi_1 + \varphi_2} = -\frac{v_1 v_2^*}{w_1 w_2^*},
\end{equation}
where \( \varphi_k = \gamma_k t + \gamma_k^{-1} x, k = 1, 2 \). Hence
\begin{equation}
e^{2 \text{Re}(\varphi_2)} = -\frac{|v_2|^2}{|w_2|^2},
\end{equation}
which contradicts the positivity of the real exponential function. If any of \( v_2, w_1, w_2 \) is zero, a zero of \( \det(\Omega) \) is only possible if either \( v_1 \) and \( w_1 \), or \( v_2 \) and \( w_2 \) are zero, which we excluded. \( \Box \)

**Example 3.5.** Choosing \( n = 2, \gamma_1 = 1, \gamma_2 = 2 \) and \( v_1 = v_2 = w_1 = w_2 = 1 \), (3.9) yields the special real 2-soliton solution
\begin{equation}
f = 6 \frac{\cosh(2t + \frac{1}{2} x) - 2 \cosh(t + x)}{\cosh(3t + \frac{1}{2} x) + 9 \cosh(t - \frac{1}{2} x) - 8}.
\end{equation}

A plot is shown in Fig. 1.
3.1.2 Solitons associated with Jordan block data

In contrast to simple solitons, solutions determined by (3.9), with \( \Gamma \) chosen as a non-diagonal Jordan block, depend also rationally on \( x \) and \( t \). Let \( n = 2 \) and

\[
\Gamma = \begin{pmatrix}
\gamma & 1 \\
0 & \gamma
\end{pmatrix}.
\]

The solution of (3.8) is then

\[
X = \frac{1}{4\text{Re}(\gamma)^2} \begin{pmatrix}
\frac{|v_2|^2}{\text{Re}(\gamma)} + 2\text{Re}(\gamma)|v_1|^2 - v_1\ast v_2 - v_1^\ast v_2 & (2\text{Re}(\gamma)v_1 - v_2) v_2^* \\
2\text{Re}(\gamma) v_1^* v_2 - v_2^* & 2\text{Re}(\gamma)|v_2|^2
\end{pmatrix},
\]

from which \( Y \) is obtained by replacing \( v \) with \( w \). From (3.7) we obtain the following components of the \( 2 \times 2 \) matrix \( \Omega = (\Omega_{ij}) \),

\[
\Omega_{11} = \frac{1}{8|\gamma|^4\text{Re}(\gamma)} \left( 4|\gamma|^4 |w_1|^2 + \frac{2|w_2|^2}{\text{Re}(\gamma)} \left[ \text{Re}(\gamma)^2 x - |\gamma|^4 t + \frac{|\gamma|^4}{\text{Re}(\gamma)} \right] + |w_2|^2 |x - \gamma^2 t|^2 \\
-4\text{Re} \left[ \gamma^2 w_1^* w_2^*(x - \gamma^2 t + \frac{\gamma^2}{\text{Re}(\gamma)}) \right] e^{\text{Re}(\gamma)(t+x/|\gamma|^2)} \\
+ \left( \frac{4|\gamma|^4 |v_1|^2 - 2|v_2|^2}{\text{Re}(\gamma)} \left[ \text{Re}(\gamma)^2 x - |\gamma|^4 t + \frac{|\gamma|^4}{\text{Re}(\gamma)} \right] + |v_2|^2 |x - \gamma^2 t|^2 \\
+ 4\text{Re} \left[ \gamma^2 v_1^* v_2^*(x - \gamma^2 t - \frac{\gamma^2}{\text{Re}(\gamma)}) \right] e^{-\text{Re}(\gamma)(t+x/|\gamma|^2)} \right),
\]

\[
\Omega_{12} = \Omega_{21}^* = \frac{1}{4\gamma^2 \text{Re}(\gamma)^2} \left( \left[ \text{Re}(\gamma)v_2^2(x - \gamma^2 t) - \gamma^2 (v_2 - 2\text{Re}(\gamma)v_1) v_2^* \right] e^{-\text{Re}(\gamma)(t+x/|\gamma|^2)} \\
- \left[ \text{Re}(\gamma)|w_2|^2(x - \gamma^2 t) + \gamma^2 (w_2 - 2\text{Re}(\gamma)w_1) w_2^* \right] e^{\text{Re}(\gamma)(t+x/|\gamma|^2)} \right),
\]

\[
\Omega_{22} = \frac{1}{2\text{Re}(\gamma)} \left( |v_2|^2 e^{-\text{Re}(\gamma)(t+x/|\gamma|^2)} + |w_2|^2 e^{\text{Re}(\gamma)(t+x/|\gamma|^2)} \right).
\]

If we restrict our considerations to solutions with real data, i.e., \( \gamma \in \mathbb{R} \) and \( v_k, w_k \in \mathbb{R}, k = 1,2 \), then (3.9) yields

\[
f = \frac{1}{4\gamma^4 \det(\Omega)} \left( [-\gamma^2 v_1^2 v_2^2 w_2 + \gamma^2 v_3^2 w_1 + v_2^3 w_2(\gamma - x + \gamma^2 t)] e^{-(\gamma t + \gamma^{-1}x)} \\
+ [-\gamma^2 v_2 w_1^2 w_2^2 \gamma^2 v_1 w_3 + w_2^3(\gamma + x - \gamma^2 t)] e^{(\gamma t + \gamma^{-1}x)} \right)
\]  

(3.11)

with

\[
\det(\Omega) = \frac{1}{16\gamma^6} \left( 2\gamma^2 v_2^2 w_2^2 + 4[v_2 w_2(x - \gamma^2 t) + \gamma^2 (v_1 w_2 - v_2 w_1)]^2 \\
+ \gamma^2 v_2^4 e^{-\frac{x}{\gamma^2}(x+x^2)} + \gamma^2 w_2^4 e^{\frac{x}{\gamma^2}(x+x^2)} \right),
\]

which has been cast into an explicitly non-negative form. These solutions are regular if \( \gamma \neq 0 \) and if either \( v_2 \) or \( w_2 \) is different from zero.

**Example 3.6.** Choosing \( \gamma = v_1 = v_2 = w_1 = w_2 = 1 \), (3.11) takes the form

\[
f = 4 \frac{\cosh(x + t) + (x - t) \sinh(x + t)}{1 + 2(x - t)^2 + \cosh(2(x + t))}.
\]

A plot is shown in Fig. 2.
Corresponding results will be obtained if $\Gamma$ is chosen as an $m \times m$ Jordan block with $m > 2$. Here results in [19] about the Lyapunov equation will be useful. More generally, based on (3.9), solutions can be worked out with $\Gamma$ being composed of different Jordan blocks. This leads to (nonlinear) superpositions of different types of solitons.

### 3.2 Solutions with a plane wave background

Choosing as the “seed” $f_0$ the plane wave solution

$$f_0 = Ce^{i(x-t)},$$

with a complex constant $C \neq 0$, (3.1) determines $a_0 = -1/2$ and the linear system (3.2), (3.3) can be decoupled to

$$\eta_{1x} + i\eta_1 - \Gamma^{-2}(|C|^2 + \frac{1}{4} - \frac{i}{2}\Gamma) \eta_1 = 0,$$

$$\eta_{1t} - i\eta_1 + (|C|^2 - \frac{1}{4}\Gamma^2 - \frac{i}{2}\Gamma) \eta_1 = 0,$$

$$\eta_2 = \frac{i}{C^*} e^{i(x-t)} (\Gamma \eta_{1x} + \frac{1}{2} \eta_1) = \frac{1}{C^*} e^{i(x-t)} (\eta_{1t} + \frac{1}{2} \Gamma \eta_1)$$

(cf. Remark 3.2), from which we obtain

$$\eta_1 = e^{-\frac{i}{2}(x-t)} \left( e^{-\frac{i}{2}(\Gamma^{-1} x + i t) \tilde{R} R^\dagger} v + e^{\frac{i}{2}(\Gamma^{-1} x + i t) R^\dagger} \tilde{w} \right),$$

$$\eta_2 = e^{\frac{i}{2}(x-t)} \left( e^{-\frac{i}{2}(\Gamma^{-1} x + i t) \tilde{R} R^\dagger} \tilde{v} + e^{\frac{i}{2}(\Gamma^{-1} x + i t) R} w \right),$$

with constant $n$-component column vectors $v$ and $w$, and

$$\tilde{v} = \frac{1}{2C^*} (\Gamma + i (I - R)) v, \quad \tilde{w} = \frac{1}{2C^*} (\Gamma + i (I + R)) w.$$

Here $R$ is a matrix square root satisfying

$$R^2 = (I - i \Gamma)^2 + 4 |C|^2 I.$$

Inserting the ansatz

$$\Omega = e^{-\frac{i}{2}(\Gamma^{-1} x + i t) \tilde{R} R^\dagger} X_1 e^{-\frac{i}{2}(\Gamma^{-1} x + i t) R} Y e^{\frac{i}{2}(\Gamma^{-1} x - i t) \tilde{R} R^\dagger} + e^{\frac{i}{2}(\Gamma^{-1} x + i t) R} Y^\dagger e^{-\frac{i}{2}(\Gamma^{-1} x - i t) \tilde{R} R^\dagger} + e^{\frac{i}{2}(\Gamma^{-1} x - i t) R} X_2 e^{\frac{i}{2}(\Gamma^{-1} x - i t) \tilde{R} R^\dagger},$$

Figure 2: Plot of the real solution in Example 3.6.
with constant \( n \times n \) matrices \( X_1, X_2, Y, \) in (3.4), requires

\[
\Gamma X_1 + X_1 \Gamma^\dagger = vv^\dagger + \tilde{v} \tilde{v}^\dagger, \quad \Gamma X_2 + X_2 \Gamma^\dagger = ww^\dagger + \tilde{w} \tilde{w}^\dagger, \quad \Gamma Y + Y \Gamma^\dagger = vw^\dagger + \tilde{v} \tilde{w}^\dagger.
\]

If \( \Gamma \) is diagonal, the solutions of these Lyapunov equations are given by the Cauchy-like matrices

\[
X_1 = \left( \frac{v_i v_j^* + \tilde{v}_i \tilde{v}_j^*}{\gamma_i + \gamma_j^*} \right), \quad X_2 = \left( \frac{w_i w_j^* + \tilde{w}_i \tilde{w}_j^*}{\gamma_i + \gamma_j^*} \right), \quad Y = \left( \frac{v_i w_j^* + \tilde{v}_i \tilde{w}_j^*}{\gamma_i + \gamma_j^*} \right).
\]

After inserting these expression in that for \( \Omega \), it remains to compute the inverse matrix \( \Omega^{-1} \) in order to find an explicit form of the new solution given by (3.5).

For \( n = 1 \), we find

\[
\Omega = \frac{1}{2 \Re(\gamma)} \left( (|v|^2 + |\tilde{v}|^2) e^{-\Re(r/\gamma) x + \Im(r) t} + (|w|^2 + |\tilde{w}|^2) e^{\Re(r/\gamma) x - \Im(r) t} + 2 \Re(ww^* + \tilde{v} \tilde{w}^*) e^{-i(\Im(r/\gamma) x + \Re(r) t)} \right)
\]

\[
= \frac{1}{2 \Re(\gamma)} \left( |v| e^{-\frac{1}{2}((r/\gamma) x + i r t) + w e^{\frac{1}{2}((r/\gamma) x + i r t)} + |\tilde{v}| e^{\frac{1}{2}((r/\gamma) x + i r t) + \tilde{w} e^{\frac{1}{2}((r/\gamma) x + i r t)} |^2} \right),
\]

where

\[
\tilde{v} = \frac{1}{\sqrt{2|C|}} (1 - i \gamma - \gamma^{-1} r) v, \quad \tilde{w} = \frac{1}{\sqrt{2|C|}} (1 - i \gamma + \gamma^{-1} r) w,
\]

and \( r \) is a square root of \((1 - i \gamma)^2 + 4|C|^2\). Then we have

\[
f = e^{i(x-t)} \left[ C - \frac{1}{\Omega} e^{-\Re(r/\gamma) x - i \Re(r) t} (\tilde{v} + \tilde{w} e^{(r/\gamma) x + i r t}) (v^* e^{i r t} + w^* e^{(r/\gamma) x}) \right].
\]

We note that \( f = -C e^{-i(x-t)} \) if \( \gamma = 2 |C| - i \). Fig. 3 shows a plot of the real part of the above solution for specified data.

Figure 3: Plot of the real part of \(-f\) for a single soliton on a plane wave background. Here we chose the data \( \gamma = 2, C = v = w = 1 \).

Fig. 4 shows a plot of the real part of a solution with two solitons on the plane wave background.

4 Derivation of the binary Darboux transformation

4.1 Binary Darboux transformations in bidifferential calculus

A graded associative algebra is an associative algebra \( \Omega = \bigoplus_{r \geq 0} \Omega^r \) over a field \( \mathbb{K} \) of characteristic zero, where \( A := \Omega^0 \) is an associative algebra over \( \mathbb{K} \) and \( \Omega^r, \ r \geq 1, \) are \( A \)-bimodules such that
Figure 4: Plot of the real part of $f$ for the solution in Section 3.2 with $n = 2$, diagonal $\Gamma$, and the data $\gamma_1 = 2, \gamma_2 = 4, C = v_1 = v_2 = w_1 = w_2 = 1$.

$\Omega^r \Omega^s \subseteq \Omega^{r+s}$. Elements of $\Omega^r$ will be called $r$-forms. A bidifferential calculus is a unital graded associative algebra $\Omega$, supplied with two $K$-linear graded derivations $d, \bar{d} : \Omega \to \Omega$ of degree one (hence $d\Omega^r \subseteq \Omega^{r+1}$, $\bar{d}\Omega^r \subseteq \Omega^{r+1}$), and such that

$$d^2 = \bar{d}^2 = d\bar{d} + \bar{d}d = 0. \quad (4.1)$$

We refer the reader to [12] for an introduction to this structure and an extensive list of references.

**Theorem 4.1.** Given a bidifferential calculus, let 0-forms $\Delta, \Gamma$ and 1-forms $\kappa, \lambda$ satisfy

$$\bar{d}\Delta + [\lambda, \Delta] = (d\Delta) \Delta, \quad \bar{d}\lambda + \lambda^2 = (d\lambda) \Delta, \quad \bar{d}\Gamma - [\kappa, \Gamma] = \Gamma d\Gamma, \quad \bar{d}\kappa - \kappa^2 = \Gamma d\kappa. \quad (4.2)$$

Let 0-forms $\theta$ and $\eta$ be solutions of the linear equations

$$\bar{d}\theta = A\theta + (d\theta) \Delta + \theta\lambda, \quad \bar{d}\eta = -\eta A + \Gamma d\eta + \kappa \eta, \quad (4.3)$$

where the 1-form $A$ satisfies

$$dA = 0, \quad \bar{d}A = A^2. \quad (4.4)$$

Furthermore, let $\Omega$ be an invertible solution of the linear system

$$\Gamma \Omega - \Omega \Delta = \eta \theta, \quad (4.5)$$

$$\bar{d}\Omega = (d\Omega) \Delta - (d\Gamma) \Omega + \kappa \Omega + \Omega \lambda + (d\eta) \theta. \quad (4.6)$$

Then

$$A' := A - d(\theta \Omega^{-1}\eta) \quad (4.7)$$

also solves (4.4).

**Proof.** Clearly, we have $dA' = 0$. Using (4.4), we obtain

$$\bar{d}A' - A'^2 = \bar{d}\theta(\theta \Omega^{-1}\eta) + A d(\theta \Omega^{-1}\eta) + \theta \Omega^{-1}\eta A - d(\theta \Omega^{-1}\eta) d(\theta \Omega^{-1}\eta).$$

Under suitable assumptions for $\Delta$ and $\Gamma$, these equations arise as integrability conditions of the linear system and “adjoint linear system” given in (4.3), by use of (4.2). In any case, the integrability conditions are satisfied if (4.2) and (4.4) hold.

The equation obtained by acting with $\bar{d}$ on (4.6) is satisfied as a consequence of the preceding equations.
With the help of the linear equations (4.3) and (4.6), we find
\[
\bar{d}(\theta \Omega^{-1} \eta) = A \theta \Omega^{-1} \eta - \theta \Omega^{-1} \eta A + (d \theta) \Delta \Omega^{-1} \eta + \theta \Omega^{-1} \Gamma \triangle \eta - \theta \Omega^{-1} (d \Omega) \eta - \theta \Omega^{-1} (d \eta) \theta \Omega^{-1} \eta.
\]
Eliminating \( \Gamma \) using (4.5), it becomes
\[
\bar{d}(\theta \Omega^{-1} \eta) = A \theta \Omega^{-1} \eta - \theta \Omega^{-1} \eta A + \frac{d(\theta \Delta \Omega^{-1} \eta)}{\Omega} + \theta \Omega^{-1} \eta \frac{d(\theta \Omega^{-1} \eta)}{\eta}.
\]
Inserting this in our first equation leads to \( \bar{d}A' - A'^2 = 0 \).

The preceding theorem, and also the result stated next, remain true if the ingredients are matrices of forms with dimensions chosen in such a way that the required products are all defined. Furthermore, it will be sufficient to have the maps \( d \) and \( \bar{d} \) defined on those matrices that appear in the theorem, but not necessarily on the whole of \( \Omega \).

**Corollary 4.2.** Let (4.2) hold and (4.3) with \( A = d\phi \), where the 0-form \( \phi \) is a solution of
\[
d\bar{d}\phi = d\phi \triangledown \phi. \tag{4.8}
\]
If \( \Omega \) is an invertible solution of (4.5) and (4.6), then
\[
\phi' = \phi - \theta \Omega^{-1} \eta + K, \tag{4.9}
\]
where \( K \) is any d-constant (i.e., \( dK = 0 \)), solves the same equation.

The result in Corollary 4.2 can be regarded as a reduction of that in Theorem 4.1. Corollary 4.2 has been used in many previous applications of bidifferential calculus, see in particular (12, 20, 21). Here we provided short proofs of the above general results. Below, we will use Corollary 4.2 to deduce Theorem 3.1.

### 4.2 An application

Let \( A \) be a unital associative algebra over \( \mathbb{C} \), where the elements are allowed to depend on real variables \( x, t \). Let \( \text{Mat}(A) \) be the algebra of all matrices over \( A \), where the product of two matrices is defined to be zero whenever their dimensions do not fit. We choose
\[
\Omega = \text{Mat}(A) \otimes \bigwedge \mathbb{C}^2, \tag{4.10}
\]
where \( \bigwedge \mathbb{C}^2 \) is the exterior algebra of the vector space \( \mathbb{C}^2 \). It is then sufficient to define \( d \) and \( \bar{d} \) on \( \text{Mat}(A) \), since they extend in an evident way to \( \Omega \), treating elements of \( \bigwedge \mathbb{C}^2 \) as \( d \)- and \( \bar{d} \)-constants.

Let \( \xi_1, \xi_2 \) be a basis of \( \bigwedge^1 \mathbb{C}^2 \). For each \( m \in \mathbb{N} \), let \( J_m \) be a constant \( m \times m \) matrix over \( A \). For an \( m \times n \) matrix \( F \) over \( \mathbb{A} \), let
\[
dF = F_x \xi_1 + \frac{1}{2} (J_m F - F J_n) \xi_2, \quad \bar{d}F = \frac{1}{2} (J_m F - F J_n) \xi_1 + F_t \xi_2
\]
(also see [5][6][22]). Then \( d \) and \( \bar{d} \) satisfy the Leibniz rule on a product of matrices, and the conditions in (4.1) are satisfied. In the linear systems (4.3) we choose a \( 2 \times n \) matrix \( \theta \) and an \( n \times 2 \) matrix \( \eta \). Then \( A \) has to be a \( 2 \times 2 \) matrix of 1-forms. Writing
\[
A = A_1 \xi_1 + A_2 \xi_2, \quad \kappa = \kappa_1 \xi_1 + \kappa_2 \xi_2, \quad \lambda = \lambda_1 \xi_1 + \lambda_2 \xi_2,
\]
\(^9\)On a \( 1 \times 1 \) matrix \( f \), which is an element of \( \mathcal{A} \), we have \( df = f_x \xi_1 \) and \( \bar{d}f = f_t \xi_2 \).
with $2 \times 2$ matrices (over $\mathcal{A}$) $A_1$ and $A_2$. (4.3) reads

$$\frac{1}{2}(J_2\theta - \theta J_n) = A_1\theta + \theta_x\Delta + \theta\lambda_1,$$

$$\theta_t = A_2\theta + \frac{1}{2}(J_2\theta - \theta J_n)\Delta + \theta\lambda_2,$$

$$\frac{1}{2}(J_n\eta - \eta J_2) = -\eta A_1 + \Gamma\eta_x + \kappa_1\eta,$$

$$\eta_t = -\eta A_2 + \frac{1}{2}\Gamma(J_n\eta - \eta J_2) + \kappa_2\eta.$$

Choosing

$$\kappa_1 = \frac{1}{2}J_n, \quad \kappa_2 = -\frac{1}{2}\Gamma J_n, \quad \lambda_1 = -\frac{1}{2}J_n, \quad \lambda_2 = \frac{1}{2}J_n\Delta,$$

the latter system simplifies to

$$\frac{1}{2}J_2\theta = A_1\theta + \theta_x\Delta,$$

$$\theta_t = A_2\theta + \frac{1}{2}J_2\theta\Delta,$$

$$\frac{1}{2}\eta J_2 = \eta A_1 - \Gamma\eta_x,$$

$$\eta_t = -\eta A_2 - \frac{1}{2}\Gamma\eta J_2,$$

which does not involve $J_n$ with $n \neq 2$ anymore. The conditions in (4.2) boil down to

$$\Delta_x = \Delta_t = 0,$$

$$\Gamma_x = \Gamma_t = 0,$$

so that $\Delta$ and $\Gamma$, which are $n \times n$ matrices over $\mathcal{A}$, have to be constant. (4.6) becomes

$$\Omega_x\Delta = -\eta_x\theta,$$

$$\Omega_t = -\frac{1}{2}\eta J_2\theta.$$

In addition, the $n \times n$ matrix $\Omega$ has to satisfy (4.5). Choosing $J_2 = \text{diag}(1, -1)$, where 1 stands for the identity element of $\mathcal{A}$, and writing

$$\phi = \begin{pmatrix} p & f \\ q & -\tilde{p} \end{pmatrix},$$

elaborating Corollary 4.2 we have

$$A = \text{d}\phi = \begin{pmatrix} p_x & f_x \\ q_x & -\tilde{p}_x \end{pmatrix} \xi_1 + \begin{pmatrix} 0 & f \\ -q & 0 \end{pmatrix} \xi_2,$$

and (4.8) takes the form

$$\phi_{xt} = \frac{1}{2}[[J, \phi], \phi_x - \frac{1}{2}J],$$

also see [6]. This results in the system

$$f_{xt} = f - p_x f - f\tilde{p}_x,$$

$$q_{xt} = q - qp_x - \tilde{p}_x q,$$

$$p_{xt} = (fq)_x,$$

$$\tilde{p}_{xt} = (qf)_x.$$

Introducing

$$a := p_x - \frac{1}{2}1,$$

$$\tilde{a} := \tilde{p}_x - \frac{1}{2}1,$$

it reads

$$f_{xt} = -a f - f\tilde{a},$$

$$q_{xt} = -q a - \tilde{a} q,$$

$$a_t = (fq)_x,$$

$$\tilde{a}_t = (qf)_x.$$

(4.11)

The constraint

$$q = \pm f^\dagger,$$

$$a^\dagger = a,$$

reduces the last system to

$$f_{xt} = -a f - f\tilde{a},$$

$$a_t = \pm(f f^\dagger)_x,$$

$$\tilde{a}_t = \pm(f^\dagger f)_x.$$
Remark 4.3. Instead, the reduction \( q = \pm f, \tilde{a} = a \) leads to
\[
f_{xt} = -af - fa, \quad a_t = \pm (f^2)_x.
\]
Choosing the upper sign, this system may be regarded, as has been suggested in [6] (see equation (9) therein), as a matrix version of the self-induced transparency (SIT) equations.

### 4.3 The commutative case

Let \( \mathcal{A} \) now be the commutative algebra of functions on \( \mathbb{R}^2 \). Then we have
\[
\text{tr}(\theta \Omega^{-1} \eta) = \text{tr}(\eta \theta \Omega^{-1}) = \text{tr}((\Gamma \Omega - \Omega \Delta) \Omega^{-1}) = \text{tr}(\Gamma) - \text{tr}(\Delta),
\]
where \( \Gamma \) and \( \Delta \) shall now be \( n \times n \) matrices over \( \mathbb{C} \), \( \theta \) and \( \eta \) of size \( k \times n \) and \( n \times k \), respectively.

Hence
\[
\text{tr}(d(\theta \Omega^{-1} \eta)) = \text{tr}(\theta \Omega^{-1} \eta) \xi_1 = (\text{tr}(\Gamma) - \text{tr}(\Delta)) \xi_1 = 0.
\]

As a consequence,
\[
\text{tr} \mathcal{A} = 0
\]
is a reduction that is consistent with the solution-generating method of Theorem 4.1. The latter reduction means \( \tilde{a} = a \), so that (4.11) becomes
\[
a_t = (f q)_x, \quad f_{xt} = -2af, \quad q_{xt} = -2aq.
\]

This is the first “negative flow” of the AKNS hierarchy (see, e.g., system (12) in [4] and also [5, 10], for example). Soliton solutions of it have, apparently, first been found in [9], using Hirota’s bilinear method.

Since we guaranteed form-invariance of \( \mathcal{A} \) under the transformation given by Corollary 4.2, writing
\[
\theta = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}, \quad \eta = (\eta_1, \eta_2),
\]
we have
\[
d\phi' = A' = A - d(\theta \Omega^{-1} \eta) = \left( \frac{a' + \frac{1}{2} f'_x}{q_x} - a' - \frac{1}{2} \right) \xi_1 + \left( \begin{array}{cc} 0 & f' \\ -q' & 0 \end{array} \right) \xi_2
\]
\[
= \left( a + \frac{1}{2} - (\theta_1 \Omega^{-1} \eta_1)_x \right) \left( f - \theta_1 \Omega^{-1} \eta_2 \right)_x \xi_1 + \left( \begin{array}{cc} 0 & f - \theta_1 \Omega^{-1} \eta_2 \\ -q + \theta_2 \Omega^{-1} \eta_1 & 0 \end{array} \right) \xi_2,
\]
and hence
\[
a' = a - (\theta_1 \Omega^{-1} \eta_1)_x = a + (\theta_2 \Omega^{-1} \eta_2)_x, \quad f' = f - \theta_1 \Omega^{-1} \eta_2, \quad q' = q - \theta_2 \Omega^{-1} \eta_1,
\]
satisfy the same equations as \( a, f, q \). Collecting the main results, we arrive at the following theorem, which expresses a binary Darboux transformation for the system (4.12).

**Theorem 4.4.** Let \( a_0, f_0, q_0 \) be a solution of (4.12). Let \( \theta_i \) and \( \eta_i, i = 1, 2 \), be solutions of the linear system
\[
\begin{align*}
\theta_{1x} \Delta &= -a_0 \theta_1 - f_{0x} \theta_2, & \theta_{2x} \Delta &= a_0 \theta_2 - q_{0x} \theta_1, \\
\theta_{1t} &= \frac{1}{2} \theta_1 \Delta + f_0 \theta_2, & \theta_{2t} &= -\frac{1}{2} \theta_2 \Delta - q_0 \theta_1,
\end{align*}
\]
\[
\Gamma \eta_{1x} = a_0 \eta_1 + q_0 \eta_2, \quad \Gamma \eta_{2x} = -a_0 \eta_2 + f_0 \eta_1, \\
\eta_{1t} = -\frac{1}{2} \Gamma \eta_1 + q_0 \eta_2, \quad \eta_{2t} = \frac{1}{2} \Gamma \eta_2 - f_0 \eta_1,
\]
where \( \Delta \) and \( \Gamma \) are invertible constant \( n \times n \) matrices. Let \( \Omega \) be an invertible solution of the linear equations
\[
\begin{align*}
\Gamma \Omega - \Omega \Delta &= \eta_1 \theta_1 + \eta_2 \theta_2, \\
\Omega_x \Delta &= -\eta_1 x \theta_1 - \eta_2 x \theta_2, \\
\Omega_t &= -\frac{1}{2} \eta_1 \theta_1 + \frac{1}{2} \eta_2 \theta_2.
\end{align*}
\]
Then
\[
a = a_0 - (\theta_1 \Omega^{-1} \eta_1)_x, \quad f = f_0 - \theta_1 \Omega^{-1} \eta_2, \quad q = q_0 - \theta_2 \Omega^{-1} \eta_1,
\]
constitutes also a solution of (4.12).

If we impose the condition
\[
q = \pm f^*, \tag{4.16}
\]
the system (4.12) reduces to (3.1), if we choose the plus sign\textsuperscript{10} It remains to implement the above reduction in the solution-generating method.

4.4 The reduction \( q = f^* \)

Let us set
\[
q = f^*, \quad \theta = \eta^\dagger, \quad \Delta = -\Gamma^\dagger. \tag{4.17}
\]
Then Theorem 4.4 implies Theorem 3.1.

\textbf{Proof of Theorem 3.1}. The linear system in Theorem 4.4 reduces to (3.2) and (3.3), by using (4.17), and (4.13) becomes (3.4). If \( \Gamma \) and \(-\Gamma^\dagger\) have no eigenvalue in common, i.e., \( \text{spec}(\Gamma) \cap \text{spec}(-\Gamma^\dagger) = \emptyset \), the Lyapunov equation (3.4) is known to have a unique solution \( \Omega \). By taking its conjugate, we can then deduce that
\[
\Omega^\dagger = \Omega, \tag{4.18}
\]
which in turn implies that the equations
\[
\Omega_x \Gamma^\dagger = \eta_{1x} \eta^\dagger_1 + \eta_{2x} \eta^\dagger_2, \quad \Omega_t = -\frac{1}{2} \eta_1 \eta^\dagger_1 + \frac{1}{2} \eta_2 \eta^\dagger_2,
\]
resulting from (4.14), are satisfied as a consequence of the equations
\[
\Omega_x \Gamma^\dagger - \eta_x \eta^\dagger + \Gamma \Omega_x - \eta \eta^\dagger = 0, \quad \Gamma (\Omega_t + \frac{1}{2} \eta_1 \eta^\dagger_1 - \frac{1}{2} \eta_2 \eta^\dagger_2) + (\Omega_t + \frac{1}{2} \eta_1 \eta^\dagger_1 - \frac{1}{2} \eta_2 \eta^\dagger_2) \Gamma^\dagger = 0,
\]
obtained by differentiation of (3.4) with respect to \( x \), respectively \( t \), and using (3.3). Furthermore,
\[
(\eta^\dagger \Omega^{-1} \eta)^\dagger = \eta^\dagger \Omega^{-1} \eta,
\]
so that
\[
(\eta^\dagger_1 \Omega^{-1} \eta_2)^\dagger = \eta^\dagger_2 \Omega^{-1} \eta_1,
\]
and the last two equations in (4.15) indeed coincide if (4.17) holds. Furthermore, \( \eta^\dagger \Omega^{-1} \eta_1 \) is real, so that \( a \) is real if \( a_0 \) is real. \( \square \)

\textsuperscript{10}The transformation \( t \mapsto it, x \mapsto ix \) relates the two equations obtained from (4.12) via (4.16).
5 Conclusion

In this work we explored the nonlinear PDE (1.1). We derived a vectorial binary Darboux transformation for this PDE from a universal binary Darboux transformation in bidifferential calculus and exploited it to obtain multi-soliton solutions, also on a plane wave background solution. All these solutions can still be generalized by using Proposition 1.1. More generally, we derived a binary Darboux transformation for the system (4.12), which is the first “negative flow” of the AKNS hierarchy.

Most likely, the multi-soliton solutions admit generalizations in terms of Jacobi elliptic functions. We have seen that there are even two such extensions of the 1-soliton solution.

The plots in this work have been generated using Mathematica [23].

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References

[1] T.B. Benjamin, J.L. Bona, and J.J Mahony. Model equations for long waves in nonlinear dispersive systems. Phil. Trans. R. Soc. London A, 272:47–78, 1972.
[2] R.I. Joseph and R. Egri. Another possible model equation for long waves in nonlinear dispersive systems. Phys. Lett. A, 61:429–430, 1977.
[3] H. Aratyn, L.A. Ferreira, J.F. Gomes, and A.H. Zimerman. The complex sine-Gordon equation as a symmetry flow of the AKNS hierarchy. J. Phys. A: Math. Gen., 33:L331–L337, 2000.
[4] A.M. Kaçhatnov and M.V. Pavlov. On generating functions in the AKNS hierarchy. Phys. Lett. A, 301:269–274, 2002.
[5] A. Dimakis and F. Müller-Hoissen. Bidifferential calculus approach to AKNS hierarchies and their solutions. SIGMA, 6:055, 2010.
[6] A. Dimakis, N. Kanning, and F. Müller-Hoissen. Bidifferential calculus, matrix SIT and sine-Gordon equations. Acta Polytechnica, 51:33–37, 2011.
[7] M. Chen, S.-Q. Liu, and Y. Zhang. A two-component generalization of the Camassa-Holm equation and its solutions. Lett. Math. Phys., 75:1–15, 2006.
[8] H. Aratyn, J.F. Gomes, and A.H. Zimerman. On a negative flow of the AKNS hierarchy and its relation to a two-component Camassa-Holm equation. SIGMA, 2:070, 2006.
[9] J. Ji, J.B. Zhang, and D.-J. Zhang. Soliton solutions for a negative order AKNS equation hierarchy. Commun. Theor. Phys., 52:395–397, 2009.
[10] V.E. Vekslerchik. Functional representation of the negative AKNS hierarchy. J. Nonl. Math. Phys., 19:1–20, 2012.
[11] V.B. Matveev and M.A. Salle. Darboux Transformations and Solitons. Springer Series in Nonlinear Dynamics. Springer, Berlin, 1991.
[12] A. Dimakis and F. Müller-Hoissen. Differential calculi on associative algebras and integrable systems. In S. Silvestrov, A. Malyarenko, and M. Rančić, editors, Algebraic Structures and Applications, volume 317 of Springer Proceedings in Mathematics & Statistics, pages 385–410. Springer, 2020.
[13] H.W.A. Riaz and M. Hassan. Noncommutative negative order AKNS equation and its soliton solutions. Mod. Phys. Lett. A, 33:1850209, 2018.
[14] H.W.A. Riaz. Darboux transformation for a negative order AKNS equation. Commun. Theor. Phys., 71:912–920, 2019.
[15] Z. Amjad and D. Khan. Binary Darboux transformation for a negative-order AKNS equation. Theor. Math. Phys., 206:128–141, 2021.
[16] A. Ankiewicz, D.J. Kedziora, A. Chowdury, U. Bandelow, and N. Akhmediev. Infinite hierarchy of nonlinear Schrödinger equations and their solutions. *Phys. Rev. E*, 93:012206, 2016.

[17] E. Kamke. *Gewöhnliche Differentialgleichungen, Lösungsmethoden und Lösungen*, volume I. Teubner, Stuttgart, 9th edition, 1977.

[18] M. Abramowitz and I.A. Stegun. *Handbook of Mathematical Functions*. Dover Publications, New York, 1965.

[19] O. Chvartatskyi and F. Müller-Hoissen. Nls breathers, rogue waves, and solutions of the Lyapunov equation for Jordan blocks. *J. Phys. A: Math. Theor.*, 50:155204, 2017.

[20] A. Dimakis and F. Müller-Hoissen. Binary Darboux transformations in bidifferential calculus and integrable reductions of vacuum Einstein equations. *SIGMA*, 9:009, 2013.

[21] O. Chvartatskyi, A. Dimakis, and F. Müller-Hoissen. Self-consistent sources for integrable equations via deformations of binary Darboux transformations. *Lett. Math. Phys.*, 106:1139–1179, 2016.

[22] A. Dimakis and F. Müller-Hoissen. Solutions of matrix NLS systems and their discretizations: a unified treatment. *Inverse Problems*, 26:095007, 2010.

[23] Wolfram Research, Inc. Mathematica, Version 13.0. Champaign, IL, 2021.