ON REFINEMENT MASKS OF TIGHT WAVELET FRAMES

E. A. Lebedeva* and I. A. Shcherbakov†

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Sufficient conditions for a trigonometric polynomial to be a refinement mask corresponding to a tight wavelet frame are obtained. The condition is formulated in terms of the roots of a mask. In particular, it is proved that any trigonometric polynomial can serve as a mask if its associated algebraic polynomial has only negative roots (of course at least one of them equals \(-1\)). Bibliography: 4 titles.

1. Introduction

The unitary extension principle (UEP) of Ron and Shen [1] is one of the main tools for the construction of tight wavelet frames. We recall it here for completeness of the presentation.

The unitary extension principle. Let \( \varphi \in L_2(\mathbb{R}) \) be a refinable function, i.e. the following refinement equation holds:

\[
\hat{\varphi}(\xi) = m_0(\xi/2)\hat{\varphi}(\xi/2) \quad \text{a.e.,}
\]

where \( m_0 \in L_2(0, 1) \) is a refinement mask. Suppose \( \hat{\varphi} \) is continuous at zero and \( m_1, \ldots, m_r \) are 1-periodic functions in \( L_2(0, 1) \), called wavelet masks, such that the matrix

\[
M(\xi) = \begin{pmatrix}
m_0(\xi) & m_1(\xi) & \cdots & m_r(\xi) \\
m_0(\xi + 1/2) & m_1(\xi + 1/2) & \cdots & m_r(\xi + 1/2)
\end{pmatrix}
\]

satisfies the equality

\[
M(\xi)M^*(\xi) = I_2, \quad \text{a.e.,}
\]

where \( I_2 \) is the identity matrix of size 2. Define wavelet functions \( \psi^{(1)}, \ldots, \psi^{(r)} \) in the Fourier domain as follows:

\[
\hat{\psi^{(k)}}(\xi) = m_k(\xi/2)\hat{\varphi}(\xi/2) \quad \text{a.e.,} \quad k = 1, \ldots, r.
\]

Then the system of functions \( \psi^{(k)}_{j,k}, j, k \in \mathbb{Z}, k = 1, \ldots, r \), forms a tight frame in \( L_2(\mathbb{R}) \) with frame bounds \( A = B = |\hat{\varphi}(0)|^2 \).

It is well known [2] that the general setup together with the inequality

\[
|m_0(\xi)|^2 + |m_0(\xi + 1/2)|^2 \leq 1 \quad \text{a.e.}
\]

always provides a solution for matrix equation (2) and makes it possible to obtain a frame with two wavelet generators \( \psi^{(1)}, \psi^{(2)} \). Thus, to construct a frame by means of UEP we need to find a function \( \varphi \in L_2(\mathbb{R}) \) such that (1) is fulfilled with a mask \( m_0 \) satisfying (4) and \( \hat{\varphi} \) is continuous at zero. One can construct a wavelet frame starting with a refinement mask. In this case the corresponding refinable function is determined by an infinite product as follows:

\[
\hat{\varphi}(\xi) = \prod_{j=1}^{\infty} m_0(\xi/2^j),
\]

and one needs to check that \( \varphi \in L_2(\mathbb{R}) \). It can be done using the following Mallat theorem.

*St. Petersburg State University, St.Petersburg, Russia, e-mail: ealebedeva2004@gmail.com.
†St. Petersburg State University, St.Petersburg, Russia, e-mail: stscherbakov99@yandex.ru.

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Theorem 1 ([3]). Suppose \( m_0(\xi) = \sum_{k \in \mathbb{Z}} c_k e^{2\pi i k \xi}, m_0(0) = 1, c_k = O(|k|^{-2-\varepsilon}), \varepsilon > 0 \), and (4) holds. Set \( \hat{\varphi}(\xi) = \prod_{j=1}^{\infty} m_0(\xi/2^j) \). Then \( \varphi \in L_2(\mathbb{R}) \), and \( \|\varphi\| \leq 1 \).

Let \( m_0 \) be a trigonometric polynomial such that \( m_0(0) = 1 \). It is well known that the function \( \hat{\varphi}(\xi) = \prod_{j=1}^{\infty} m_0(\xi/2^j) \) is an entire function of exponential type, thus \( \hat{\varphi} \) is continuous at zero. If additionally \( m_0 \) satisfies inequality (4), then, in accordance with the Mallat theorem, the function \( \hat{\varphi}(\xi) = \prod_{j=1}^{\infty} m_0(\xi/2^j) \) is in \( L_2(\mathbb{R}) \), and \( \varphi \) is a refinable function generating a tight wavelet frame because it is constructed by UEP.

The purpose of the paper is to provide sufficient conditions under which a trigonometric polynomial \( m_0 \) is a refinement mask generating a tight frame. As it is seen from the above, inequality (4) is a cornerstone for the constructions of tight wavelet frames. It is worth noting that in the case of orthogonal wavelets assumptions on the mask \( m_0 \) are much more restrictive (see [3, Theorem 4.1.2]). The paper is organized as follows. In Sec. 2, first we consider polynomials of low degrees (2 and 3) and obtain not only a sufficient, but also a necessary condition to satisfy (4). This is done in Propositions 1 and 2. Then in Theorem 2, we consider polynomials of arbitrary degree and obtain a sufficient condition to satisfy (4). In particular, an extremely easily checked case of Theorem 2 is formulated in Corollary 2. In Sec. 3, we present a validation algorithm for the sufficient condition obtained in Theorem 2.

2. Results

2.1. Preliminary. Consider the algebraic polynomial \( P(z) \) associated with \( m_0(\xi) \), that is given by the equation \( m_0(\xi) = P(e^{2\pi i \xi})e^{2\pi i \xi \max\{-N,0\}} \). It is easy to see that the inequality

\[
|P(z)|^2 + |P(-z)|^2 \leq 1 \text{ a.e. on } \mathbb{T}
\]

is equivalent to (4), and \( m_0(0) = 1 \) if and only if \( P(1) = 1 \).

We immediately make a couple of obvious remarks.

Remark 1. If an algebraic polynomial \( P(z) \) satisfies inequality (5) and \( P(1) = 1 \), then this polynomial has a root at the point \( z = -1 \).

Remark 2. If \( P(1) = 1 \), then the polynomial \( P \) can be written in the form \( P(z) = \prod_{i=1}^{n} \frac{z-z_i}{1-z_i} \), where \( z_i \neq 1 \).

Remark 3. A polynomial \( P(z) \) of degree at least 2 satisfies the conditions \( P(1) = 1 \) and (5) if and only if inequality (5) holds true with \( Q(z) = zP(z) \) instead of \( P(z) \).

Let a function \( \psi_{z_0} \) be given by \( z \in \mathbb{T} \rightarrow \psi_{z_0}(z) = \frac{|z-z_0|^2}{1-z_0^2} \), where \( z_0 \neq 1 \). If a polynomial \( P(z) \) has roots \( z_1, \ldots, z_n \) and \( P(1) = 1 \), then, in accordance with Remark 2, \( |P(z)|^2 = \prod_{i=1}^{n} \psi_{z_i}(z) \). Suppose \( \alpha \in \text{Arg}(z-z_0) \) and \( \beta \in \text{Arg}(z_0) \), then it follows from the cosine theorem

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that

\[
\psi_{z_0}(z) = \frac{1 + |z_0|^2 - 2|z_0| \cos \alpha}{1 + |z_0|^2 - 2|z_0| \cos \beta},
\]

\[
\psi_{z_0}(e^{i\varphi}) = \frac{1 + |z_0|^2 - 2|z_0| \cos(\varphi - \beta)}{1 + |z_0|^2 - 2|z_0| \cos \beta}.
\]

Setting \(x = \text{Re } z_0\) and \(y = \text{Im } z_0\), we get a trigonometric first degree polynomial dependent on \(x, y\)

\[
\psi_{z_0}(e^{i\varphi}) = \frac{1 + x^2 + y^2 - 2x \cos \varphi - 2y \sin \varphi}{1 + x^2 + y^2 - 2x}.
\]

Denote by

\[
F_1(x, y) = 1 + \frac{2x}{(x - 1)^2 + y^2},
\]

\[
F_2(x, y) = -\frac{2x}{(x - 1)^2 + y^2},
\]

\[
F_3(x, y) = -\frac{2y}{(x - 1)^2 + y^2}.
\]

the coefficients of this polynomial, then

\[
\psi_{z_0}(z) = \psi_{z_0}(e^{i\varphi}) = F_1(x, y) + F_2(x, y) \cos \varphi + F_3(x, y) \sin \varphi,
\]

and

\[
\psi_{z_0}(-z) = \psi_{z_0}(e^{i(\pi + \varphi)}) = F_1(x, y) - F_2(x, y) \cos \varphi - F_3(x, y) \sin \varphi.
\]

Thus, the left-hand side of inequality (4) takes the form

\[
|P(z)|^2 + |P(-z)|^2 = \prod_{i=1}^{n} \psi_{z_i}(z) + \prod_{i=1}^{n} \psi_{z_i}(-z) =: T(\varphi).
\]

In the sequel, to obtain sufficient conditions for the roots \(z_1, \ldots, z_n\) of the polynomial \(P(z)\) to satisfy inequality (4), we study the trigonometric polynomial \(T(\varphi)\). Note that \(T\) has degree at most \(n\), it is nonnegative and \(\pi\)-periodic. Thus, it only contains monomials with even angles.

### 2.2. The simplest cases

First we assume that the degree of a polynomial \(P\) is equal to 2.

In this case we obtain not only a sufficient, but also a necessary condition to satisfy (4).

**Proposition 1.** Let \(P(z)\) be an algebraic polynomial satisfying the condition \(P(1) = 1\). If the numbers \(z_1 = -1\) and \(z_2\) are all roots of this polynomial, then (5) holds if and only if \(z_2 \leq 0\).

**Proof.** In this case \(F_1(z_1) = F_2(z_1) = \frac{1}{2}\) and \(F_3(z_1) = 0\). Set \(A = F_1(z_2)\) and \(B = F_3(z_2)\). Substituting this into (7), we get

\[
T(\varphi) = \left(\frac{1}{2} + \frac{1}{2} \cos \varphi\right)\left(A + (1 - A) \cos \varphi + B \sin \varphi\right)
\]

\[
+ \left(\frac{1}{2} - \frac{1}{2} \cos \varphi\right)\left(A - (1 - A) \cos \varphi - B \sin \varphi\right) = A + (1 - A) \cos^2 \varphi + B \sin \varphi \cos \varphi
\]

\[
= \frac{1 + A}{2} + \frac{1 - A}{2} \cos 2\varphi + \frac{B}{2} \sin 2\varphi \leq \frac{1 + A}{2} + \frac{1}{2} \sqrt{(1 - A)^2 + B^2}.
\]
Since this inequality is sharp, it follows that inequality (7) is equivalent to
\[
\frac{1 + A}{2} + \frac{1}{2}\sqrt{(1 - A)^2 + B^2} \leq 1,
\]
that is
\[
\sqrt{(1 - A)^2 + B^2} \leq 1 - A,
\]
which is equivalent to the following two conditions
\[
\begin{align*}
1 - A &\geq 0 \iff F_2(z_2) \geq 0 \iff \Re z_2 \leq 0, \\
B &\equiv 0 \iff F_3(z_2) = 0 \iff \Im z_2 = 0.
\end{align*}
\]
\[\square\]

Similarly one can get a necessary and sufficient conditions for \(n = 3\).

**Proposition 2.** Let \(P(z)\) be an algebraic polynomial satisfying the condition \(P(1) = 1\). Suppose the numbers \(\frac{1}{2}, z_1\) and \(z_2\) are all roots of this polynomial, and set \(A_1 = F_1(z_1), A_2 = F_1(z_2), B_1 = F_3(z_1)\) and \(B_2 = F_3(z_2)\). Then (5) holds if and only if
\[
\begin{align*}
1 - A_1A_2 - B_1B_2 &\geq 0, \\
B_1 + B_2 &\equiv 0.
\end{align*}
\]

**Proof.** As in the previous proof, we substitute all the notations into (7) and get
\[
T(\varphi) = \left(\frac{1}{2} + \frac{1}{2}\cos \varphi\right) \left(A_1 + (1 - A_1)\cos \varphi + B_1\sin \varphi\right) \left(A_2 + (1 - A_2)\cos \varphi + B_2\sin \varphi\right)
\]
\[
+ \left(\frac{1}{2} - \frac{1}{2}\cos \varphi\right) \left(A_1 - (1 - A_1)\cos \varphi - B_1\sin \varphi\right) \left(A_2 - (1 - A_2)\cos \varphi - B_2\sin \varphi\right)
\]
\[
= A_1A_2 + \cos^2 \varphi((1 - A_1)(1 - A_2) + (1 - A_1)A_2 + (1 - A_2)A_1) + \sin^2 \varphi \cdot B_1B_2
\]
\[
+ \cos \varphi \sin \varphi((1 - A_1)B_2 + (1 - A_2)B_1 + A_1B_2 + A_2B_1)
\]
\[
= \frac{1 + A_1A_2 + B_1B_2}{2} + \cos \varphi \frac{1 - A_1A_2 - B_1B_2}{2} + \sin \varphi \frac{B_1 + B_2}{2}
\]
\[
\leq \frac{1 + A_1A_2 + B_1B_2}{2} + \sqrt{(1 - A_1A_2 - B_1B_2)^2 + (B_1 + B_2)^2}.
\]

Since this inequality is sharp, it follows that inequality (7) is equivalent to
\[
\frac{1 + A_1A_2 + B_1B_2}{2} + \sqrt{(1 - A_1A_2 - B_1B_2)^2 + (B_1 + B_2)^2} \leq 1.
\]

This can be rewritten as
\[
\sqrt{(1 - A_1A_2 - B_1B_2)^2 + (B_1 + B_2)^2} \leq 1 - A_1A_2 - B_1B_2,
\]
that is equivalent to
\[
\begin{align*}
1 - A_1A_2 - B_1B_2 &\geq 0, \\
B_1 + B_2 &\equiv 0.
\end{align*}
\]
\[\square\]

**Corollary 1.** Let \(P(z)\) be an algebraic polynomial of degree 3 with real roots \(x_1, x_2\) and \(-1\), and let \(P(1) = 1\). Then (5) holds if and only if
\[
x_1x_2(x_1 + x_2 - 2) + x_1 + x_2 \leq 0
\]

**Proof.** Note that for real roots we get \(B_1 = B_2 = 0\), thus, the necessary and sufficient conditions from Proposition 2 can be written as follows:
\[
\left(1 + \frac{2x_1}{(x_1 - 1)^2}\right)\left(1 + \frac{2x_2}{(x_2 - 1)^2}\right) \leq 1 \iff 4x_1x_2 + 2x_1(x_2 - 1)^2 + 2x_2(x_1 - 1)^2 \leq 0
\]
\[
\iff x_1x_2(x_1 + x_2 - 2) + x_1 + x_2 \leq 0
\]
\[\square\]
2.3. The main result. Now we return to polynomials of degree \( n \). We consider a trigonometric polynomial \( m_0 \) and the algebraic polynomial \( P \) associated with \( m_0 \), that is \( m_0(\xi) = P(e^{2\pi i \xi})e^{2\pi i \xi} \max[-N,0] \). Suppose all roots \( z_1, \ldots, z_n \) of the polynomial \( P \) are real. Set \( a_i = F_1(z_i,0), i \in [1, \ldots, n] \) (see (6)). Denote by \( \sigma_k \) the elementary symmetric polynomials
\[
\sum_{S \subseteq [1, \ldots, n], \# S = k} \prod_{j \in S} a_j \text{ for all } k \in [0, \ldots, n],
\]
and set \( \rho_k := \frac{\sigma_k}{k!} \) (see [4, page 73]). Now we are ready to formulate the main theorem.

**Theorem 2.** Let \( T(\varphi) \) be a trigonometric polynomial constructed by the polynomial \( P \) determined in (7). Let all roots \( z_1, \ldots, z_n \) of the polynomial \( P \) be real and at least one of them equal \(-1\). If
\[
\Delta^{2k} \rho_{n-2k} = \sum_{j=n-2k}^{n} \binom{2k}{n-j} (-1)^{n-j} \rho_j \geq 0
\]
for any \( k \in [0, \ldots, \left\lfloor \frac{n}{2} \right\rfloor] \), then \( T(\varphi) \leq 1 \) for any \( \varphi \in \mathbb{R} \).

**Proof.** Taking into account that \( i \in [1, \ldots, n] \) whenever \( z_i \in \mathbb{R} \), we get \( F_2(z_i,0) = 1 - a_i \). Also we have \( F_3(z_i,0) = 0 \) (see (6)). Hence the function \( \psi_{z_i} \) has a simpler form
\[
\prod_{i=1}^{n} \psi_{z_i}(e^{i\varphi}) = \prod_{i=1}^{n} (a_i + (1 - a_i) \cos \varphi) = \sum_{k=0}^{n} \cos^k \varphi \cdot \sum_{S \subseteq [1, \ldots, n], \# S = k} \prod_{j \in S} (1 - a_j) \prod_{j \notin S} a_j.
\]

Therefore, we have
\[
T(\varphi) = \prod_{i=1}^{n} \psi_{z_i}(e^{i\varphi}) + \prod_{i=1}^{n} \psi_{z_i}(e^{i(\pi + \varphi)}) = 2 \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \cos^{2k} \varphi \cdot \sum_{S \subseteq [1, \ldots, n], \# S = 2k} \prod_{j \in S} (1 - a_j) \prod_{j \notin S} a_j. \tag{8}
\]

First we use Euler’s formula for the cosine with \( k \geq 1 \)
\[
\cos^{2k} \varphi = \frac{(e^{i\varphi} + e^{-i\varphi})^{2k}}{2} = \frac{1}{2^{2k}} \sum_{l=0}^{2k} \binom{2k}{l} e^{i\varphi l} e^{-i\varphi (2k-l)}
\]
\[
= \frac{1}{2^{2k}} \left( \left( \frac{2k}{k} \right) + \sum_{l=0}^{k-1} \binom{2k}{l} (e^{2i\varphi (l-k)} + e^{2i\varphi (k-l)}) \right) = \frac{1}{2^{2k}} \left( \binom{2k}{k} + 2 \sum_{l=0}^{k-1} \binom{2k}{l} \cos (l-k) \varphi \right)
\]
\[
= \frac{1}{2^{2k}} \left( \binom{2k}{k} + 2 \sum_{l=1}^{k} \binom{2k}{k-l} \cos 2l \varphi \right).
\]

Second we consider the coefficients of \( T(\varphi) \) as combinations of symmetric polynomials of \( a_1, \ldots, a_n \):
\[
\sum_{S \subseteq [1, \ldots, n], \# S = 2k} \prod_{j \notin S} (1 - a_j) \prod_{j \in S} a_j = \sum_{S \subseteq [1, \ldots, n], \# S = 2k} \prod_{j \notin S} a_j \cdot \sum_{l=0}^{2k} \binom{2k}{l} \sum_{T \subseteq S, \# T = l} \prod_{m \in T} a_m
\]
\[
= \sum_{l=0}^{2k} \binom{2k}{l} \sum_{T \subseteq S, \# T = l} \prod_{m \in T \cup S^c} a_m = \sum_{l=0}^{2k} \binom{2k}{l} \sum_{T \subseteq S, \# T = l} \prod_{m \in T \cup S^c} a_m
\]
Thus, all coefficients of \( \mathcal{F} \) equals \( 1 \), as was to be proved.

\[
T(\varphi) = 2\sigma_n + 2\sum_{k=1}^{\lfloor \frac{n}{2}\rfloor} \frac{1}{2^{2k}} \binom{2k}{k} + 2\sum_{l=1}^{\lfloor \frac{k-1}{2}\rfloor} \frac{2k}{k-l} \cos 2l\varphi \cdot \sum_{m=0}^{2k} (-1)^m \binom{n-m}{2k-m} \sigma_{n-m}
\]

\[
= 2\sigma_n + 2\sum_{k=1}^{\lfloor \frac{n}{2}\rfloor} \sum_{m=0}^{2k} \frac{1}{2^{2k}} \binom{2k}{k} (-1)^m \binom{n-m}{2k-m} \sigma_{n-m}
\]

\[
+ 4\sum_{l=1}^{\lfloor \frac{n}{2}\rfloor} \cos 2l\varphi \left( \sum_{k=1}^{\lfloor \frac{n}{2}\rfloor} \sum_{m=0}^{2k} (-1)^m \frac{1}{2^{2k}} \binom{2k}{k} \binom{n-m}{2k-m} \sigma_{n-m} \right)
\]

To simplify notations, we denote the coefficients by

\[
d_l = 4\sum_{k=l}^{\lfloor \frac{n}{2}\rfloor} \sum_{m=0}^{2k} (-1)^m \frac{1}{2^{2k}} \binom{2k}{k} \binom{n-m}{2k-m} \sigma_{n-m},
\]

where \( l \in [0, \ldots, \lfloor \frac{n}{2}\rfloor] \). Then \( T(\varphi) \) takes the form \( T(\varphi) = d_0 + \sum_{l=1}^{\lfloor \frac{n}{2}\rfloor} d_l \cos 2l\varphi \). Note that

\[
d_l = 4\sum_{k=l}^{\lfloor \frac{n}{2}\rfloor} \frac{1}{2^{2k}} \sum_{m=0}^{2k} (-1)^m \binom{2k}{k} \binom{n-m}{2k-m} \binom{n}{m} \rho_{n-m}
\]

\[
= 4\sum_{k=l}^{\lfloor \frac{n}{2}\rfloor} \frac{1}{2^{2k}} \binom{n}{n-2k,k-l,k+l} \sum_{m=0}^{2k} (-1)^m \binom{2k}{m} \rho_{n-m}
\]

\[
= 4\sum_{k=l}^{\lfloor \frac{n}{2}\rfloor} \frac{1}{2^{2k}} \binom{n}{n-2k,k-l,k+l} \Delta^{2k} \rho_{n-2k} \geq 0.
\]

Therefore, all coefficients of \( T(\varphi) \) are nonnegative and \( T(\varphi) \leq T(0) = \prod_{i=1}^{n} \psi_{z_i}(1) + \prod_{i=1}^{n} \psi_{z_i}(-1) = 1 + \prod_{i=1}^{n} (2a_i - 1) \). Since at least one of \( z_1, \ldots, z_n \) equals \(-1\), it follows that at least one of \( a_1, \ldots, a_n \) equals \( F_1(-1) = 1/2 \). Therefore, \( T(0) = 1 \), as was to be proved. \( \square \)
Corollary 2. Let \( T(\varphi) \) be a trigonometric polynomial constructed by a polynomial \( P \) as it is determined in (7). Let all roots \( z_1, \ldots, z_n \) of the polynomial \( P \) be less or equal to zero and at least one of them equal \(-1\). Then \( T(\varphi) \leq 1 \) for any \( \varphi \in \mathbb{R} \).

Proof. It is clear, that \( z_i \leq 0 \iff a_i \leq 1 \). Then using (8) we get

\[
T(\varphi) = 2 \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \cos^{2k} \varphi \cdot \sum_{S \subset \{1, \ldots, n\}} \prod_{j \in S} (1 - a_j) \prod_{j \notin S} a_j \leq 2 \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{S \subset \{1, \ldots, n\}} \prod_{j \in S} (1 - a_j) \prod_{j \notin S} a_j.
\]

The last expression is exactly \( T(0) = 1 \), as was to be proved. \( \square \)

3. The validation algorithm

Based on the previous theorem we create an algorithm for checking the sufficient condition on \( \mathbb{C}^+ \):

```cpp
#include <iostream>
#include <vector>
#include <cmath>
#include <iomanip>

int main () {
    int n;
    std::cin >> n;

    std::vector<double> x(n+1, 0);
    std::vector<double> a(n+1, 0);
    std::vector<double> sigma (n+1, 1);

    for (size_t i =1; i <= n; i++)  // compute a_i 
    { 
        std::cin >> x[i];
        a[i]=1+2*x[i]/((x[i]-1)*(x[i]-1));
    }

    for (int k =1; k <= n; k++)  // compute the symmetric polynomials 
    { 
        double sum_j = 0;
        for (int j =0; j <= k-1; j++) 
        { 
            double pkj = 0;
            for (int l =1; l <= n; l++) 
            { 
                pkj += std::pow(a[l], k-j); 
            }
            sum_j += pow(-1, k-j-1) * sigma[j] * pkj;
        }
        sigma[k] = sum_j / (double) k;
    }

    std::vector<std::vector<double> >
    c(n+1, std::vector<double>(n+1, 1));
    for(int i = 0; i <= n; ++i) // compute the binomial coefficients 
    { 
        for (int j = 1; j < i; ++j) 
        { 
            c[i][j] = c[i - 1][j - 1] + c[i - 1][j];
        }
    }

    return 0;
}
```

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std::vector<double> ro(n+1, 0);
for (int k = 0; k <= n; k++) // the symmetric means
{
    ro[k] = sigma[k] / c[n][k];
}

std::vector<std::vector<double>> deltaRo(n+1, std::vector<double>(n+1, 0));
for (int i = 0; i <= n; i++)
{
    deltaRo[0][i] = ro[i];
}

for (int i = 1; i <= n; i++) // the divided differences
{
    for (int j = i; j <= n; j++)
    {
        deltaRo[i][j] = deltaRo[i-1][j] - deltaRo[i-1][j-1];
    }
}

for (int i = 0; i <= n; ++i) // output
{
    for (int j = 0; j <= n; ++j)
    {
        std::cout << deltaRo[i][j] << " ";
    }
    std::cout << std::endl;
}

bool isDone = true;
for (int k = 0; k <= n/2; k++)
    if (deltaRo[2 * k][n] < 0)
        isDone = false;

if (isDone)
    std::cout << "[TRUE] The inequality holds" << std::endl;
else
    std::cout << "[FALSE] The criteria doesn’t answer" << std::endl;
return 0;
}

This program takes a positive integer \( n \) (the degree of the polynomial), as an input, and then \( n \) real numbers (the roots of the polynomial) are taken. Note that at least one root must be equal to \(-1\). The values of elementary symmetric polynomials are calculated recursively using Newton’s formula: \( \sigma_k = \frac{1}{k} \sum_{i=0}^{k-1} (-1)^{k-i-1} \sigma_i \sum_{j=1}^{n} a_j^{k-i} \). The screen displays a table of divided differences for \( \rho_k \) and the result of checking the criterion.

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