Conjectured $DXZ$ decompositions of a unitary matrix

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Abstract

For any unitary matrix there exists a ZXZ decomposition, according to a theorem by Idel and Wolf. For any even-dimensional unitary matrix there exists a block-ZXZ decomposition, according to a theorem by Führ and Rzeszotnik. We conjecture that these two decompositions are merely special cases of a set of decompositions, one for every divisor of the matrix dimension. For lack of a proof, we provide an iterative Sinkhorn algorithm to find an approximate numerical decomposition.

Keywords: unitary matrix; matrix decomposition; biunimodular vector; Arnold conjecture.
MSC: 15A21; 15A51; 53D12.

1 Introduction

Recently, two decompositions of an arbitrary $n \times n$ unitary matrix $U$ into a matrix product $DXZ$ of three unitary matrices have been proposed:

- For arbitrary $n$, Idel and Wolf [1] present a decomposition where $D$ and $Z$ are diagonal matrices, whereas $X$ is a matrix with all linesums equal to unity.

- For arbitrary even $n$, Führ and Rzeszotnik [2] present a decomposition where $D$ and $Z$ are block-diagonal matrices, whereas $X$ is a matrix with all block-linesums equal to the $n/2 \times n/2$ unit matrix.

These matrix decompositions have been applied in quantum optics [1], quantum computing [3] [4] [5], and quantum memory [6].
The two matrix decompositions have been proved in a very different way. Whereas the proof of the Idel–Wolf decomposition (based on symplectic topology) is not constructive, the proof of the Führ–Rzeszotnik decomposition (based on linear algebra) is constructive. In the present paper, we conjecture that nevertheless the two decompositions belong to a same set of similar decompositions. We conjecture that there exist as many such decompositions as there are divisors of the number \( n \). We present no proof, as neither the Idel–Wolf proof nor the Führ–Rzeszotnik proof can be easily extrapolated.

## 2 Conjecture

We introduce the following three positive integers:

- \( n \), an arbitrary integer greater than 1,
- \( m \), a divisor of \( n \), distinct from \( n \), and
- \( q \), equal to \( n - m \).

We write \( n = rm \) and \( q = (r - 1)m \). Hence, both \( m \) and \( r \) are divisors of \( n \). They satisfy \( 1 \leq m < n \) and \( 1 < r \leq n \). For convenience, \( n \times n \) matrices will be called ‘great matrices’, \( m \times m \) matrices will be called ‘small matrices’, and \( q \times q \) matrices will be called ‘intermediate matrices’.

**Conjecture 1** Every great unitary matrix \( U \) can be decomposed into three great unitary matrices:  
\[ U = DXZ, \]

where

- \( D \) consists of \( r \) small matrices on its diagonal:  
  \[ D = \text{diag}(D_{11}, D_{22}, D_{33}, \ldots, D_{rr}), \]

- \( Z \) consists of \( r \) small matrices on its diagonal, the upper-left small matrix being equal to the \( m \times m \) unit matrix \( I \):  
  \[ Z = \text{diag}(I, Z_{22}, Z_{33}, \ldots, Z_{rr}), \]

\(^1\)The restriction \( m \neq n \) is merely introduced for convenience. The reader may easily investigate the case \( m = n \). E.g., if \( m = n \), then Conjecture 1 is trivially true: suffice it to choose \( D \) equal to \( U \) and both \( X \) and \( Z \) equal to the \( n \times n \) unit matrix.
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- $X$ consists of $r^2$ small matrices $X_{jk}$, such that all row sums $\sum_{k=1}^{r} X_{jk}$ and all column sums $\sum_{j=1}^{r} X_{jk}$ are equal to the small matrix $I$.

Because $D$ is unitary, automatically all its blocks $D_{jj}$ are unitary; because $Z$ is unitary, automatically all its blocks $Z_{jj}$ are unitary. In contrast, the blocks $X_{jk}$ are not necessarily unitary.

We define the $n \times n$ transformation matrix

$$T = F_r \otimes I,$$

where the matrix $F_r$ is the $r \times r$ discrete Fourier transform. We can easily demonstrate that the product $T^{-1}XT$ is of the form

$$\begin{pmatrix} I & G \end{pmatrix}.$$

We thus have the following property:

$$X = T \begin{pmatrix} I & G \end{pmatrix} T^{-1}.$$

Because both $X$ and $T$ are unitary, automatically $G$ is a unitary intermediate matrix.

We summarize that the decomposition of $U$ corresponds to finding the appropriate $2r - 1$ small unitary matrices and a single appropriate intermediate unitary matrix. This corresponds to find the appropriate

$$(2r - 1)m^2 + q^2 = (2r - 1)m^2 + [(r - 1)m]^2 = r^2m^2$$

real parameters, a number which exactly matches $n^2$, i.e. the number of degrees of freedom of the given matrix $U$.

3 Three special cases

If $m = 1$ (and thus $q = n - 1$), then all small matrices are, in fact, just complex numbers. Both $D$ and $Z$ are diagonal unitary matrices and $X$ is a unit linesum unitary matrix. The transformation matrix $T$ equals the great Fourier matrix $F_n$. In this particular case, the above conjecture has been proposed by De Vos and De Baerdemacker [7] and subsequently proved by Idel and Wolf [1]. The proof is by symplectic topology. Unfortunately, the proof is not constructive and therefore only provides the guarantee that the numbers $D_{11}, D_{22}, ..., D_{nn}$ and $Z_{22}, Z_{33}, ..., Z_{nn}$ exist, without providing
their values. De Vos and De Baerdemacker [7] give a Sinkhorn algorithm that yields numerical approximations of these numbers. Finally, we note that examples of the case $m = 1$ demonstrate that the $DXZ$ decomposition is not always unique.

If $n$ is even and $m$ equals $n/2$ (and thus $q = n/2$), then intermediate matrices are, in fact, small matrices. The transformation matrix $T$ equals $H \otimes I$, where $H = F_2$ is the $2 \times 2$ Hadamard matrix. In this particular case, the above conjecture has been proved by Führ and Rzeszotnik [2]. The proof is constructive and thus gives explicit values for the small matrices $D_{11}, D_{22}, Z_{22},$ and $G$. Also in this special case decomposition is not unique [4] [8].

Finally, if both $m = 1$ and $m = n/2$, i.e. if $n = r = 2$ and $m = q = 1$, then the decomposition is well-known. An arbitrary matrix from $U(2)$ looks like

$$U = \begin{pmatrix} \cos(\varphi)e^{i(\theta+\psi)} & \sin(\varphi)e^{i(\theta+\chi)} \\ -\sin(\varphi)e^{i(\theta-\chi)} & \cos(\varphi)e^{i(\theta-\psi)} \end{pmatrix}.$$  \hspace{1cm} (1)

One possible decomposition is

$$U = \begin{pmatrix} e^{i(\theta+\varphi+\psi)} & ie^{i(\theta+\varphi-\chi)} \\ ie^{i(\theta+\varphi-\chi)} & e^{i(\theta-\varphi-\psi)} \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 + e^{-2i\varphi} & 1 - e^{-2i\varphi} \\ 1 - e^{-2i\varphi} & 1 + e^{-2i\varphi} \end{pmatrix} \begin{pmatrix} 1 \\ -ie^{i(-\varphi+\chi)} \end{pmatrix}. \hspace{1cm} (2)$$

4 \hspace{1cm} Group hierarchy

The matrices $D$ form a group isomorphic to $U(m)^r$, of dimension $rm^2 = nm$. The matrices $Z$ form a group isomorphic to $U(m)^{r-1}$, of dimension $(r-1)m^2 = (n-m)m$. Finally, the matrices $X$ form a group isomorphic to $U(q)$, of dimension $q^2 = (n-m)^2$. We denote these three matrix groups by $DU(n,m)$, $ZU(n,m)$, and $XU(n,m)$, respectively. In particular, the groups $XU(n,1)$ and $ZU(n,1)$ are the groups $XU(n)$ and $ZU(n)$, extensively studied in the past [5, 9].

According to the conjecture, for any $m$, the closure of the groups $DU(n,m)$, $ZU(n,m)$, and $XU(n,m)$ is the unitary group of $U$ matrices. Of course, because $ZU(n,m)$ is a subgroup of $DU(n,m)$, the closure of $DU(n,m)$ and $XU(n,m)$ is also $U(n)$. In fact, the closure of merely $ZU(n,m)$ and $XU(n,m)$ already equals $U(n)$. Indeed, any $DU(n,m)$ matrix can be decomposed into
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If Conjecture 1 is true, then automatically a second conjecture is also true: 

**Conjecture 2** Every great unitary matrix $U$ can be decomposed into three great unitary matrices:

$$U = C \begin{pmatrix} I \\ A \end{pmatrix} Y,$$

where
• **C** is a circulant $n \times n$ matrix, i.e. a unitary matrix consisting of $m \times m$ small blocks, such that two $C_{jk}$ are identical if their two $j-k$ are equal,

• **A** is a $q \times q$ unitary matrix, and

• **Y** is an $n \times n$ circulant matrix, the upper row sum\(^2\) being equal to the $m \times m$ unit matrix $I$.

Indeed, if we apply Conjecture 1, not to the given matrix $U$, but instead to its conjugate

$$u = T^{-1} UT,$$

then we obtain the decomposition

$$u = dxz .$$

This leads to

$$U = T d T^{-1} T x T^{-1} T z T^{-1} .$$

One can easily verify that

• $T d T^{-1}$ is a circulant great matrix,

• $T x T^{-1}$ is of the form $\begin{pmatrix} I & A \end{pmatrix}$, and

• $T z T^{-1}$ is a circulant $\text{XU}(n, m)$ matrix.

Such conjugate decomposition was already noticed before, in both the $m = 1$ case and the $m = n/2$ case \[1\] \[5\] \[9\].

6 **Unitary and biunitary vectors**

For $m = 1$, the DXZ decomposition involves unit-modulus numbers $d_{jj}$ and $z_{jj}$:

$$U = \begin{pmatrix} d_{11} & d_{22} & \cdots & d_{nn} \\
 & \ddots & \vdots & \ddots \\
 & & \ddots & \ddots \\
 & & & d_{nn} \\
\end{pmatrix} X \begin{pmatrix} 1 & \cdots & \cdots & \cdots \\
 & z_{22} & \cdots & \cdots \\
 & & \ddots & \ddots \\
 & & & z_{nn} \end{pmatrix} .$$

\[2\]As the matrix is circulant, all row sums and column sums are equal. Hence $Y$ is a member of $\text{XU}(n, m)$.
If we multiply both sides of the equation by the $n \times 1$ matrix (i.e. column vector) $v = (1, z_{22}^{-1}, z_{33}^{-1}, ..., z_{nn}^{-1})^T$, then we obtain

$$Uv = \begin{pmatrix} d_{11} & d_{22} & & \\
& d_{22} & & \\
& & \ddots & \\
& & & d_{nn} \end{pmatrix} \begin{pmatrix} 1 \\
1 \\
\vdots \\
1 \end{pmatrix}.$$ 

Taking into account that all row sums of $X$ equal unity, we find

$$Uv = w,$$

where $w = (d_{11}, d_{22}, d_{33}, ..., d_{nn})^T$. Both $v$ and $w$ are vectors with all entries having unit modulus. Therefore, they are called unimodular vectors. The unimodular vector $v$ is called biunimodular for the matrix $U$, as $Uv$ is unimodular as well [1] [2]. We say that the Idel–Wolf DXZ decomposition implies the fact that any unitary matrix has at least one biunimodular vector. Moreover, it possesses a biunimodular vector with leading entry 1.

As an example, decomposition (2) of the matrix (1) corresponds with the following biunimodular vector:

$$U \begin{pmatrix} 1 \\
1 \\
\vdots \\
1 \end{pmatrix} = \begin{pmatrix} e^{i(\varphi+\theta+\psi)} \\
e^{i(\varphi+\theta-\chi)} \end{pmatrix}.$$ 

If Conjecture 1 is true for $m > 1$, then we can draw a similar conclusion $UV = W$, however with $V$ and $W$ matrices of size $n \times m$. These matrices consist of $r$ blocks, each a unitary $m \times m$ matrix. Because such blocks have no modulus, we cannot call $V$ and $W$ unimodular vectors. We will instead call them unitary vectors and $V$ a biunitary vector. These unitary vectors reside in an $nm$-dimensional vector space $\mathbb{C}^n \otimes \mathbb{C}^m$, isomorphic to $\mathbb{R}^{2nm}$. A basis for this space consists e.g. of the $nm$ following basis vectors: $a_i \otimes b_j^T$, where the $a_i$ are the $n$ standard basis vectors of $\mathbb{C}^n$ and the $b_j$ are the $m$ standard basis vectors of $\mathbb{C}^m$. We note that a unitary vector $V$ has the property $V^\dagger V = rI$, with $I$ once again the $m \times m$ unit matrix.

If Conjecture 1 is true, then also the following conjecture is true:

**Conjecture 3** Every great unitary matrix $U$ has at least one biunitary vector $V$:

$$UV = W,$$

where

- both $V$ and $W$ consist of $n/m$ unitary $m \times m$ entries and
• *V has leading entry equal to the small unit matrix* $I$.

Suffice it to repeat the above reasoning with $m = 1$ for $m > 1$, the vector $E = (I, I, I, ..., I)^T$ taking over the role of the vector $e = (1, 1, 1, ..., 1)^T$ above.

Important is the fact that not only Conjecture 3 is a consequence of Conjecture 1, but Conjecture 1 is equally a consequence of Conjecture 3. Indeed, if $UV = W$, with both $V$ and $W$ being unitary vectors and $V$ having the unit matrix $I$ as leading entry, then the matrix

$$A = \text{diag } (W_1^{-1}, W_2^{-1}, ..., W_r^{-1}) \ U \ \text{diag } (I, V_2^{-1}, ..., V_r^{-1})$$

belongs to $XU(n, m)$. Proof of this fact consists of two parts:

• Taking into account that $UV = W$, we find that $AE$ equals $E$, such that $A$ has unit row sums.

• Because of $E = AE$, we have $E = \overline{E} = \overline{AE} = \overline{A}E$. Taking into account that $A$ is unitary, we have $A^T \overline{A}$ equal to the $n \times n$ unit matrix. Hence $E = A^T \overline{AE} = A^T E$. Because $A^T E$ thus turns out to equal $E$, we conclude that $A$ has unit column sums.

As $A$ belongs to $XU(n, m)$ and $U$ has decomposition

$$\text{diag } (W_1, W_2, ..., W_r) \ A \ \text{diag } (I, V_2, ..., V_r) ,$$

Conjecture 1 is fulfilled.

We finally note that Conjecture 2 leads to the same Conjecture 3, according to a similar proof, where however the vector $(I, 0, 0, ..., 0)^T$ takes over the role of the vector $E = (I, I, I, ..., I)^T$ above. We conclude:

**Theorem 1** The Conjectures 1, 2, and 3 are equivalent: if one is proved, then all three are proved.

### 7 Group topology

In order to prove the three conjectures, it suffices to prove Conjecture 3. For that purpose, we first give a lemma:

**Lemma 1** If an $n \times n$ unitary matrix $U$ possesses a biunitary $n \times m$ vector, then it possesses a biunitary $n \times m$ vector with leading entry equal to the $m \times m$ unit matrix $I$. 

Indeed, let us suppose that
\[
U \begin{pmatrix} V_1 \\ V_2 \\ V_3 \\ \vdots \\ V_r \end{pmatrix} = \begin{pmatrix} W_1 \\ W_2 \\ W_3 \\ \vdots \\ W_r \end{pmatrix},
\]
with all $V_j$ and $W_j$ are unitary blocks. We multiply to the right with the small matrix $V_1^{-1}$ and thus obtain
\[
U \begin{pmatrix} I \\ V_2 V_1^{-1} \\ V_3 V_1^{-1} \\ \vdots \\ V_r V_1^{-1} \end{pmatrix} = \begin{pmatrix} W_1 V_1^{-1} \\ W_2 V_1^{-1} \\ W_3 V_1^{-1} \\ \vdots \\ W_r V_1^{-1} \end{pmatrix},
\]
a result which proves the lemma.

We consider the vector space $\mathbf{M}$ of vectors $(M_1, M_2, ..., M_r)^T$, where each $M_j$ is a complex $m \times m$ matrix. Let $\mathbf{S}$ be the following submanifold of $\mathbf{M}$:
\[
\mathbf{S} = \{(V_1, V_2, ..., V_r)^T \mid V_j \in U(m)\}.
\]
The Lie group $U(m)^r$ behaves as if it were the following topological product of odd-dimensional spheres [10] [11]:
\[
(S^1 \times S^3 \times S^5 \times \cdots \times S^{2m-1})^r,
\]
where $S^k$ denotes the $k$-sphere. In fact, the Poincaré polynomial of the manifold $\mathbf{S}$ is
\[
P(x) = \left[ (1 + x)(1 + x^3)(1 + x^5)\cdots(1 + x^{2m-1}) \right]^r.
\]
Therefore, the sum of its Betti numbers is
\[
P(1) = (2^m)^r = 2^n,
\]
where $n = mr$.

It is clear that, if
\[
\mathbf{S} \cap US \neq \emptyset,
\]
then there exists at least one unitary vector in $\mathbf{S}$ which is a biunitary vector an which, because of Lemma 1, has a unit leading entry.
One promising approach is to reduce the problem to the Arnold conjecture [12], as has been done in the $m = 1$ case [1]. If $S$ was a Lagrangian submanifold for a symplectic form on $\mathbb{C}^n$ such that $U$ was still a Hamiltonian symplectomorphism, then eqn (3) would be true, provided the Arnold conjecture is true for this particular manifold.

Direct computation suggests that $S$ is no Lagrangian submanifold of $\mathbb{C}^n$ with the standard symplectic form. There are two possible roads to still prove a relation to the Arnold conjecture:

- show that $S$ is a submanifold for some other symplectic structure on $\mathbb{C}^n$ and $U$ is a Hamiltonian symplectomorphism for that particular structure, too
- find a Lagrangian embedding of $S$ into some other manifold such that the mapping of $U$ results in a Hamiltonian symplectomorphism for this other manifold.

Let us start with the first idea: we note that $S$ is a Cartesian product of odd-dimensional spheres and that the Cartesian product of Lagrangian manifolds is a Lagrangian manifold, it might be possible to consider each sphere separately. However this is not true, as no sphere $S^n$ with $n > 1$ can be embedded into $\mathbb{C}^n$ as a Lagrangian manifold according to [13], as no simply-connected manifold can be embedded into $\mathbb{C}^n$. Since $S$ is not simply connected as it contains a product factor of $S^1$, this does not yet rule out the possibility of finding a symplectic structure such that it is a Lagrangian submanifold, but there is no argument we know of.

This leaves us with attempting the second idea: Indeed, using [14], who attributes this idea to [15], we can find a Lagrangian embedding of every odd-dimensional space via:

$$S^{2n+1} \rightarrow \mathbb{P}^n(\mathbb{C}) \times \mathbb{P}^n(\mathbb{C})$$

$$z \mapsto ([z], [\overline{z}])$$

where $\mathbb{P}^n(\mathbb{C})$ denotes the complex projective space and $[z]$ being the canonical projection. Since $S^1$ is a Lagrangian submanifold of $\mathbb{C}$, and products of Lagrangian embeddings are Lagrangian embeddings in the product manifold, we can embed $S$ as a Lagrangian submanifold. To be of help, we also would need $U$ to be mapped to a symplectomorphism. To do that, we note that, if we decompose $z \in S$ as $(z_1^1, z_3^1, z_5^1, \ldots, z_{2m+1}^1)$, then $U$ acts on any factor $z_i^r$ as $U(0, \ldots, 0, z_i^r, 0, \ldots, 0)$, which explains how it must act on $([z], [\overline{z}])$. But this implies that $U$ will mix factors of $\mathbb{P}^n(\mathbb{C})$ in our product manifold, which in turn results in $U$ not being a symplectomorphism.
after direct computation. This does not rule out the second idea either, but shows where the difficulties lie. It is still unclear whether the applicability of symplectic topology to the original problem of a Sinkhorn-like decomposition was a mere coincidence or whether there is a deeper link to unitary decompositions so it seems worthwhile to consider this problem.

We summarize: if the Arnold conjecture is applicable, then the above Conjectures 1, 2, and 3 are true.

8 Numerical approximation

We note that the above three conjectures are not constructive. Only in the case $m = n/2$, do we have explicit expressions for the matrices $D$, $X$, $Z$, $C$, $A$, and $Y$ and for the vectors $V$ and $W$. For other cases, we can only find numerical approximations. Therefore, in the present section, we give a numerical procedure to find, given the matrix $U$, an approximation of the matrices $D$, $X$, and $Z$. It is similar to the Sinkhorn-like method presented earlier for the $m = 1$ case [7].

The successive approximations $X_t$ of $X$ are given by

$$X_0 = U$$

and

$$X_t = L_t X_{t-1} R_t .$$

The diagonal of the left great matrix $L_t$ consists of $r$ small matrices $(L_t)_{jj}$, equal to $\Phi_j^{-1}$, i.e. the inverse of the unitary factor in the polar decomposition $\Phi_j P_j$ of the row sum $r_j = \sum_{k=1}^r (X_{t-1})_{jk}$. The right great matrix $R_t$ consists of $r$ small matrices $(R_t)_{kk}$, equal to $\Upsilon_k^{-1} \Upsilon_1$, with $\Upsilon_k^{-1}$ equal to the inverse of the unitary factor in the polar decomposition $Q_k \Upsilon_k$ of the column sum $c_k = \sum_{j=1}^r (L_{t} X_{t-1})_{jk}$. The extra factor $\Upsilon_1$ in the expression of $(R_t)_{kk}$ guarantees that $(R_t)_{11}$ equals $I$. After a sufficient number (say, $\tau$) of iterations, the product $L = L_\tau L_{\tau-1} \ldots L_1$ and the product $R = R_1 R_2 \ldots R_{\tau}$ yield the desired great matrix $X$:

$$X \approx X_\tau = LUR .$$

The fact that all $(R_t)_{11} = I$ guarantees that $R_{11} = I$ and thus that $R$ belongs to $ZU(n, m)$ instead of merely to $DU(n, m)$. We have

$$D \approx L^{-1} ,$$

$$Z \approx R^{-1} .$$
Exceptionally, a particular row sum \( r_j \) might be singular. Then its polar decomposition is not unique, such that the corresponding matrix \( \Phi_j \) is not determined. In that case, we choose \((L_t)_{jj}\) equal to the unit matrix \( I \). Analogously we choose \((R_t)_{kk} = I\) whenever a particular column sum \( c_k \) is singular.

The progress of the iteration process can be monitored by the following property of a great matrix \( M \):

\[
\Psi(M) = n^2 - |\text{Btr}(M)|^2
\]

where we call \( \text{Btr}(M) \) the ‘block trace’ of \( M \):

\[
\text{Btr}(M) = \sum_{j=1}^{r} \sum_{k=1}^{r} \text{Tr}(M_{jk}) .
\]

Indeed, the quantity \( \Psi(M) \) is zero iff \( M \in \text{XU}(n, m) \). During the iteration process, \( \Psi(X_t) \) becomes smaller and smaller, approaching zero in the limit. See Appendix for details. We note that, if \( m = 1 \), then \( \text{Btr} \) is simply the sum of all \( n^2 \) matrix entries [7].

9 Example

As an example, we choose the following \( \text{U}(6) \) matrix:

\[
U = \frac{1}{12} \begin{pmatrix}
-5 & 6 + 2i & -5 - 5i & -4 + 2i & 2 & -2 - i \\
2 + 2i & -2 - 4i & -4i & -3 - i & 5 + 5i & 2 + 6i \\
-6 - 3i & -2 - 2i & 1 + 3i & -6 & -4 - 2i & 3 + 4i \\
-2 - 4i & -1 - 7i & 2 - 6i & 4 + 3i & -1 - 2i & -2i \\
3 - i & 4 & -4 - 2i & 2 - 2i & -6 + 2i & 7 + i \\
-6i & -1 + 3i & -2 + 2i & 3 + 6i & 5i & -2 + 4i
\end{pmatrix}.
\]

Hence \( n = 6 \). For \( m \), we investigate all different possibilities, i.e. \( m = 1 \), \( m = 2 \), and \( m = 3 \). During the numerical procedure, the progress parameter \( \Psi \) diminishes according to Table \( \text{I} \). We see that, after 36 iterations, \( \Psi \) already approaches 0. Therefore, below we give results for \( \tau = 36 \).

We thus find, after 36 iterations:\footnote{Each iteration, in turn, needs \( 2r \) polar decompositions. These are performed by Hero’s iterative method (a.k.a. Heron’s method). For each, we applied only ten iterations.}
Table 1: Progress parameter $\Psi$ as a function of the number $t$ of iteration steps.

| $t$ | $\Psi_t$ for $m = 1$ | $\Psi_t$ for $m = 2$ | $\Psi_t$ for $m = 3$ |
|-----|-----------------------|-----------------------|-----------------------|
| 0   | 34.889                | 32.000                | 33.743                |
| 1   | 4.407                 | 9.517                 | 6.643                 |
| 2   | 2.573                 | 4.332                 | 2.533                 |
| 3   | 1.381                 | 2.680                 | 1.023                 |
| 4   | 0.586                 | 1.627                 | 0.513                 |
| 5   | 0.213                 | 0.868                 | 0.375                 |
| 6   | 0.084                 | 0.577                 | 0.318                 |
| 7   | 0.042                 | 0.492                 | 0.277                 |
| 8   | 0.027                 | 0.461                 | 0.240                 |
| 9   | 0.020                 | 0.442                 | 0.206                 |
| 10  | 0.016                 | 0.423                 | 0.174                 |
| 11  | 0.014                 | 0.400                 | 0.147                 |
| 12  | 0.012                 | 0.372                 | 0.122                 |
| 13  | 0.010                 | 0.339                 | 0.101                 |
| 14  | 0.009                 | 0.303                 | 0.083                 |
| 15  | 0.008                 | 0.264                 | 0.067                 |
| ... |                      |                       |                       |
| 36  | 0.001                 | 0.001                 | 0.001                 |
\[
X = \begin{pmatrix}
0.27 - 0.31i & -0.27 + 0.45i & 0.58 + 0.13i & 0.28 - 0.24i & 0.07 + 0.15i & 0.07 - 0.17i \\
0.04 + 0.23i & -0.12 - 0.35i & 0.26 - 0.21i & -0.01 - 0.26i & 0.57 + 0.13i & 0.25 + 0.46i \\
0.51 + 0.22i & 0.23 + 0.06i & -0.02 - 0.26i & 0.42 + 0.27i & 0.27 - 0.25i & -0.42 - 0.03i \\
0.37 - 0.03i & 0.57 - 0.14i & 0.28 + 0.45i & -0.42 - 0.04i & 0.06 - 0.18i & 0.16 - 0.06i \\
0.26 + 0.01i & 0.33 - 0.04i & -0.16 - 0.33i & 0.23 + 0.05i & -0.23 + 0.47i & 0.57 - 0.16i \\
-0.49 - 0.12i & 0.26 + 0.02i & 0.06 + 0.23i & 0.51 + 0.22i & 0.26 - 0.33i & 0.37 - 0.03i \\
\end{pmatrix}
\]

with the following row sums and column sums:

\[
\begin{align*}
\text{r}_1 &= 1.002 + 0.000i \\
\text{r}_2 &= 0.998 - 0.001i \\
\text{r}_3 &= 1.007 + 0.001i \\
\text{r}_4 &= 1.007 + 0.002i \\
\text{r}_5 &= 0.998 - 0.000i \\
\text{r}_6 &= 0.988 - 0.002i \\
\text{c}_1 &= 0.989 - 0.002i \\
\text{c}_2 &= 0.998 - 0.001i \\
\text{c}_3 &= 0.997 - 0.000i \\
\text{c}_4 &= 1.006 + 0.001i \\
\text{c}_5 &= 1.002 + 0.001i \\
\text{c}_6 &= 1.008 + 0.001i \\
\end{align*}
\]

\[
\bullet \text{ for } m = 2:
\]

\[
X = \begin{pmatrix}
0.33 - 0.05i & -0.39 - 0.07i & 0.61 + 0.34i & 0.32 + 0.19i & 0.06 - 0.29i & 0.08 - 0.12i \\
-0.30 - 0.16i & 0.35 + 0.36i & 0.07 + 0.16i & 0.08 + 0.09i & 0.23 - 0.01i & 0.57 - 0.45i \\
0.41 - 0.38i & 0.51 - 0.02i & 0.34 + 0.14i & -0.30 - 0.12i & 0.26 + 0.24i & -0.21 + 0.13i \\
0.36 - 0.08i & 0.26 - 0.27i & -0.17 - 0.26i & 0.56 + 0.34i & -0.19 + 0.34i & 0.17 - 0.07i \\
0.26 + 0.43i & -0.12 + 0.08i & 0.06 - 0.48i & -0.02 - 0.06i & 0.68 + 0.04i & 0.13 - 0.02i \\
-0.06 + 0.24i & 0.39 - 0.09i & 0.10 + 0.09i & 0.35 - 0.43i & -0.04 - 0.34i & 0.26 + 0.52i \\
\end{pmatrix}
\]

with row sums and column sums

\[
\begin{align*}
\text{r}_1 &= \begin{pmatrix} 0.994 - 0.001i & 0.003 - 0.002i \\ 0.001 + 0.000i & 0.997 - 0.000i \end{pmatrix} \\
\text{r}_2 &= \begin{pmatrix} 1.003 + 0.002i & -0.001 - 0.005i \\ -0.003 - 0.000i & 1.003 + 0.002i \end{pmatrix}
\end{align*}
\]
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\[
\begin{align*}
  r_3 &= \begin{pmatrix} 1.002 + 0.002i & -0.001 + 0.001i \\ 0.001 - 0.005i & 1.000 + 0.001i \end{pmatrix} \\
  c_1 &= \begin{pmatrix} 1.002 + 0.001i & -0.004 - 0.002i \\ -0.004 - 0.001i & 1.001 + 0.001i \end{pmatrix} \\
  c_2 &= \begin{pmatrix} 1.002 + 0.001i & 0.001 + 0.002i \\ 0.002 - 0.001i & 0.998 - 0.001i \end{pmatrix} \\
  c_3 &= \begin{pmatrix} 0.995 - 0.002i & 0.002 - 0.000i \\ 0.002 + 0.002i & 1.001 + 0.000i \end{pmatrix};
\end{align*}
\]

- for \( m = 3 \):

\[
X = \begin{pmatrix}
  0.54 + 0.37i & -0.10 - 0.25i & 0.16 - 0.13i & 0.45 - 0.36i & 0.10 + 0.24i & -0.16 + 0.13i \\
-0.23 - 0.13i & 0.47 - 0.06i & -0.07 - 0.41i & 0.23 + 0.13i & 0.53 + 0.06i & 0.07 + 0.41i \\
-0.13 - 0.16i & -0.37 - 0.19i & 0.53 + 0.17i & 0.13 + 0.16i & 0.37 + 0.19i & 0.47 - 0.17i \\
0.46 - 0.37i & 0.10 + 0.24i & -0.16 + 0.13i & 0.54 + 0.36i & -0.10 - 0.24i & 0.16 - 0.13i \\
0.23 + 0.13i & 0.53 + 0.06i & 0.07 + 0.41i & -0.23 - 0.14i & 0.47 - 0.06i & -0.07 - 0.41i \\
0.13 + 0.16i & 0.37 + 0.19i & 0.47 - 0.17i & -0.13 - 0.16i & -0.37 - 0.19i & 0.52 + 0.17i
\end{pmatrix}
\]

with row sums and column sums

\[
\begin{align*}
  r_1 &= \begin{pmatrix} 0.997 + 0.001i & -0.004 - 0.001i & -0.002 - 0.002i \\ -0.001 + 0.002i & 0.999 + 0.000i & 0.001 - 0.001i \\ 0.001 + 0.001i & 0.002 - 0.001i & 1.003 - 0.001i \end{pmatrix} \\
  r_2 &= \begin{pmatrix} 1.002 - 0.001i & 0.002 + 0.000i & -0.000 + 0.001i \\ 0.003 - 0.003i & 1.000 - 0.000i & -0.003 + 0.000i \\ 0.001 - 0.002i & -0.001 + 0.001i & 0.997 + 0.001i \end{pmatrix} \\
  c_1 &= \begin{pmatrix} 1.000 + 0.000i & -0.000 - 0.003i & 0.001 - 0.003i \\ 0.002 + 0.002i & 1.000 + 0.000i & 0.000 - 0.002i \\ 0.003 + 0.001i & 0.001 + 0.001i & 1.000 - 0.000i \end{pmatrix} \\
  c_2 &= \begin{pmatrix} 1.000 - 0.000i & 0.000 + 0.003i & -0.001 + 0.003i \\ -0.001 - 0.002i & 1.000 - 0.000i & -0.000 + 0.002i \\ -0.003 - 0.001i & -0.002 - 0.001i & 1.000 + 0.000i \end{pmatrix}.
\end{align*}
\]

For \( m = 2 \), we also give the corresponding binunary vector:

\[
U \begin{pmatrix}
  1.00 - 0.00i & 0.00 + 0.00i \\
  0.00 - 0.00i & 1.00 + 0.00i \\
  0.81 - 0.31i & 0.23 + 0.43i \\
-0.20 + 0.44i & 0.84 + 0.25i \\
-0.34 + 0.77i & -0.37 + 0.38i \\
-0.29 - 0.45i & 0.19 + 0.83i
\end{pmatrix} = \begin{pmatrix}
  -0.95 - 0.16i & 0.24 - 0.14i \\
  -0.14 - 0.24i & -0.91 - 0.31i \\
  0.06 - 0.70i & -0.71 + 0.01i \\
-0.28 - 0.65i & 0.62 - 0.34i \\
-0.12 - 0.73i & 0.67 - 0.03i \\
-0.48 - 0.46i & -0.57 + 0.47i
\end{pmatrix}.
\]
10 Permutation matrices

Although we lack a proof of Conjecture 1 in the case of an arbitrary unitary matrix $U$, we can say that Conjecture 1 is certainly true for the case where $U$ is an arbitrary $n \times n$ permutation matrix. Indeed, any permutation matrix of size $n \times n = mr \times mr$ can be decomposed as a product of three permutation matrices $D$, $X$, and $Z$, the matrix $D$ belonging to the group $DU(n,m)$, the matrix $X$ belonging to the group $XU(n,m)$, and the matrix $Z$ belonging to the group $ZU(n,m)$. In fact, $D$ belongs to a finite subgroup of $DU(n,m)$, of order $(m!)^r$ and isomorphic to the product $S_m^r$ of symmetric groups, $X$ belongs to a finite subgroup of $XU(n,m)$, of order $(r!)^m$ and isomorphic to the product $S_m^r$ of symmetric groups, and $Z$ belongs to a finite subgroup of $ZU(n,m)$, of order $(m!)^{r-1}$ and isomorphic to the product $S_m^{r-1}$. The fact that such a decomposition is always possible [16] [17], is a consequence of Birkhoff’s theorem [18] on doubly stochastic matrices (with rational entries). The decomposition has been applied both in Clos networks of telephone switching systems [19] [20] and in reversible computing [21].

As an example, we choose the following $6 \times 6$ permutation matrix:

$$U = \begin{pmatrix}
0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0
\end{pmatrix}.$$ 

For $m$, we investigate all different non-trivial possibilities: $m = 2$ and $m = 3$. We have:

- for $m = 2$:
  
  $$U = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix},$$

  where indeed the middle matrix has six unit line sums $r_1 = r_2 = r_3 = c_1 = c_2 = c_3 = (1, 1)$.

4We note that, for permutation matrices, not only the case $m = n$ is trivial, but also the case $m = 1$: suffice it to choose both $D$ and $Z$ equal to the $n \times n$ unit matrix and to choose $X$ equal to $U$. 


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• for $m = 3$:

$$
U = \begin{pmatrix}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix},
$$

where indeed the middle matrix has four unit line sums $r_1 = r_2 = c_1 = c_2 = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$.

Because Conjecture 1 is true for any $n \times n$ permutation matrix, it also is true for any $n \times n$ complex permutation matrix (i.e., unitary matrix with only one non-zero entry in every row and column). Such matrices form an $n$-dimensional non-connected subgroup of the $n^2$-dimensional group $U(n)$ (consisting of $n!$ components, each $n$-dimensional). We can indeed decompose such a matrix as $D'P$, where $D'$ is a diagonal unitary matrix and $P$ is a permutation matrix. We decompose $P$ as $D''XZ$, leading to the decomposition $D'D''XZ$ of the complex permutation matrix. Introducing $D = D'D''$, we obtain a desired decomposition $DXZ$.

11 Conclusion

Every $n \times n$ unitary matrix has an Idel–Wolf decomposition. If $n$ is even, then it also has a Führ–Rzeszotnik decomposition. We conjecture that, if $n$ is a composed integer, it has as many similar decompositions as $n$ has divisors. We offer no proof, as generalization of either the Idel–Wolf proof (based on symplectic topology) or the Führ–Rzeszotnik proof (based on linear algebra) is not straightforward. We provide an iterative algorithm for finding a numerical approximation of each of the conjectured decompositions. Finally, we demonstrate that the conjecture is true for $n \times n$ (complex) permutation matrices.

Appendix

In Section 8, multiplying $X_{t-1}$ to the left with $L_t$ increases its block trace:

$$
|\text{Btr}(L_tX_{t-1})| = \left| \sum_{j=1}^{r} \sum_{k=1}^{r} \text{Tr}((L_tX_{t-1})_{jk}) \right| = \left| \sum_{j=1}^{r} \sum_{k=1}^{r} \text{Tr}(\Phi_j^{-1}(X_{t-1})_{jk}) \right|
$$
Analogously, multiplying $L_1X_{t-1}$ to the right with $R_t$ increases its block trace. Hence, we have
\[
|\text{Btr}(X_t)| = |\text{Btr}(L_1X_{t-1}R_t)| \geq |\text{Btr}(L_1X_{t-1})| \geq |\text{Btr}(X_{t-1})|.
\]

The increasing value of $|\text{Btr}(X_t)|$ is bounded by the value $n$. An $n \times n$ unitary matrix $A$ has $|\text{Btr}(A)| = n$ iff it is a member of the group $e^{i\alpha} XU(n,m)$. These two facts are proved by reasoning as in Appendix A of De Vos and De Baerdemacker [7], by considering the following property of the row sums $r_a$ and column sums $c_b$ of $A$:
\[
\sum_{a=1}^{r} \sum_{j=1}^{m} \sum_{k=1}^{m} \left| (r_a)_{jk} \right|^2 = \sum_{b=1}^{r} \sum_{j=1}^{m} \sum_{k=1}^{m} \left| (c_b)_{jk} \right|^2 = n,
\]

a fact which, in turn, is proved by reasoning as in Appendix A of De Vos, Van Laer, and Vandenbrande [22].

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