Higher regularity of homeomorphisms in the Hartman-Grobman theorem and a conjecture on its sharpness

Weijie Lu\textsuperscript{a} Manuel Pinto\textsuperscript{b} Y-H Xia\textsuperscript{a,}\textsuperscript{†}

\textsuperscript{a} College of Mathematics and Computer Science, Zhejiang Normal University, 321004, Jinhua, China
\textsuperscript{b} Departamento de Matemáticas, Universidad de Chile, Santiago, Chile

Email: luwj@zjnu.edu.cn; pintoj.uchile@gmail.com; yhxia@zjnu.cn.

February 7, 2022

Abstract

Hartman-Grobman theorem states that there is a homeomorphism $H$ sending the solutions of the nonlinear system onto those of its linearization under suitable assumptions. Many mathematicians have made contributions to prove Hölder continuity of the homeomorphisms. However, is it possible to improve the Hölder continuity to Lipschitzian continuity? This paper gives a positive answer. We formulate the first result that the homeomorphism is Lipschitzian, but not $C^1$, while its inverse is merely Hölder continuous, but not Lipschitzian. It is interesting that the regularity of the homeomorphism is different from its inverse. Moreover, some illustrative examples are presented to show the effectiveness of our results. Further, motivated by our example, we also propose a conjecture, saying, the regularity of the homeomorphisms is sharp and

\textsuperscript{*}This paper was jointly supported from the National Natural Science Foundation of China under Grant (No. 11931016 and 11671176) and Grant Fondecyt 1170466.

\textsuperscript{†}Corresponding author. Y-H. Xia, xiadoc@outlook.com; yhxia@zjnu.cn. Address: College of Mathematics and Computer Science, Zhejiang Normal University, 321004, Jinhua, China
it could not be improved any more.

**Keywords:** Hartman-Grobman theorem; stable manifolds; linearization; Exponential dichotomies

**MSC2020:** 34C41; 34D09; 34D10

# 1 Introduction and motivation

## 1.1 Brief history of Hartman-Grobman theorem

A pioneering work on the linearization traces back to Poincaré [1]. He proved the analytical conjugation between an analytic diffeomorphism and its linear part near a hyperbolic fixed point in the complex case. Siegel [2], and Yoccoz [3] studied the case that eigenvalues of the linear part lie on the unit circle. A basic contribution to the linearization problem in the real case for autonomous differential equations is the Hartman-Grobman theorem (see [4] and [5]). Palis [6], Pugh [7], Bates and Lu [8], Lu [9], Hein and Prüss [11], and Zgliczyński [10] made contributions to the linearization problem on the infinite dimensional space. In particular, Bates and Lu [8] obtained a Hartman-Groban theorem for Cahn-Hilliard equation and phase field equations. Lu [9] proved a Hartman-Grobman theorem for the scalar reaction-diffusion equations. Hein and Prüss [11] gave a version of Hartman-Grobman theorem for semilinear hyperbolic evolution equation on Banach space. Palmer [12] firstly extended the Hartman-Groban theorem to the nonautonomous case. In order to weaken Palmer’s linearization theorem, various versions of Hartman-Grobman theorem were established, Backes et al. [13] (for nonhyperbolic systems), Barreira and Valls [14] [15] [16] [17] (with nonuniform exponential dichotomies), Huerta et al. [18] [19] (nonuniform exponential contraction), Jiang [20] (generalized exponential dichotomy), Jiang [21] (ordinary dichotomy), Fenner and Pinto [22] and Xia et al. [23] (for impulsive systems), Papaschinopoulos [24] (for differential equations with piecewise constant argument), Pötzche [25] (for dynamic equations on time scales), Reinfelds and Sermone [26], Reinfelds and Steinberga [27] (dynamical equivalence), Shi and Zhang [28] (monograph for linearization), Xia et al. [29] (with unbounded nonlinear term).

Except for the $C^0$ linearization mentioned above, much effort was made to investigate $C^r$ linearization for $C^k$ ($1 \leq r \leq k \leq \infty$) diffeomorphisms. Sternberg [30] [31] initially studied the smooth linearization problem. Recently, the smooth linearization for $C^k$ ($1 \leq k \leq \infty$)
diffeomorphisms are well improved by Sell [32], Belitskil et al. [33, 34], Cuong et al. [35], Dragičević et al. [36, 37], Elbialy [38], Rodrigues and Solà-Morales [39, 40, 41], Zhang et al. [42, 43, 44, 45]. In particular, a set of nice results on the sharp regularity of linearization for hyperbolic diffeomorphisms were established in Zhang et al. [42, 43, 44].

1.2 Motivations and novelty

An important and interesting problem is the regularity of the linearization, which have greatly attracted many mathematicians’ attentions. Among the works on the linearization mentioned above, a lot of papers were devoted to proving the Hölder continuity of the homeomorphisms in the linearization theorem (see Backes et al. [13], Barreira and Valls [14, 15, 16, 17], Dragičević et al. [36, 37], Huerta et al. [18, 19], Hein and Prüss [11], Jiang [20, 21], Pötzche [25], Shi and Zhang [28], Rodrigues and Solà-Morales [39, 40], Xia et al. [23, 29], Zhang et al. [42, 43, 44], Tan [46], Shi and Xiong [47]). For the sake of easier illustration, we restate the Palmer’s linearization theorem [12] which has extended the classical Hartman-Grobman theorem ([4, 5]) to the nonautonomous case. It states that there is a homeomorphism \( H \) sending the solutions of the nonlinear perturbed system

\[
y' = A(t)y + f(t, y).
\]

onto those of its linearization

\[
x'(t) = A(t)x(t)
\]

under suitable assumptions. Many mathematicians have made contributions to prove that both of the homeomorphisms are Hölder continuous. However, is it possible to improve the Hölder continuity to Lipschitzian continuity? Up till now, there is no existing results on the Lipschitzian continuity of the homeomorphisms in the Hartman-Grobman theorem. This paper gave a positive answer. In this paper, we formulate the first result that the homeomorphism \( H \) is Lipschitzian, but not \( C^1 \), while its inverse \( G = H^{-1} \) is merely Hölder continuous, but not Lipschitzian. Moreover, some illustrative examples are presented to show the effectiveness of our results. Further, motivated by our example, we also propose a conjecture, saying, the regularity the homeomorphisms is sharp and it can not be improved any more.

Maybe, one would doubt that the regularity of the homeomorphism \( H \) is different from its
inverse. A simple example gives the answer. If \( H(x) = x^2, (x > 0) \) (locally Lipschitzian), then the inverse is \( G(y) = y^{1/2} \) (Hölder continuous).

It is not standard to prove the Lipschitzian continuity of homeomorphism \( H \). To overcome the difficulty, we have to use the dichotomy inequality as well as the theory of stable manifolds and unstable manifolds.

In the global version of Hartman-Grobmann (type) theorem (for example, Palmer’s linearization theorem), it usually requires that the nonlinear term \( f \) is uniformly bounded and Lipschitzian. In this paper, we also weaken the linearization theorem in two ways: (i) we consider nonlinear terms \( f \) which may be unbounded or not Lipschitzian (see Example 2.3); (ii) we prove that it is enough to assume the boundedness of the Green operator of the coefficients.

1.3 Mechanism of improving the regularity

Standardly, to prove the regularity of the homeomorphisms, one takes direct estimates of the constructing homeomorphisms (e.g. [14, 15, 16, 17]) or employs the Bellman inequality (see e.g. [11, 13, 19, 23, 25, 47]). However, the disadvantage of the Bellman inequality results in an exponential estimate of the form \( e^{\alpha t} (\alpha > 0) \). It is expansive, which leads us to obtain Hölder regularity. Therefore, most of the previous works on the regularity of homeomorphisms of Hartman-Grobman theorem in \( C^0 \) linearization is Hölder continuous.

On the contrary, the advantage of the dichotomy inequality [48, 49, 50, 51] results in an exponential decay of the form \( e^{-\alpha_1 t} (\alpha_1 > 0) \), see Lemma 3.9 in the present paper. Thus, by dichotomy inequality, we can prove the Lipschitz continuity of the conjugacy.

1.4 Organization of the paper

The rest of this paper is organized as follows: In Section 2, we present our main results, i.e. regularity of the linearization and illustrative examples. Also a conjecture on the sharpness is given. In Section 3, rigorous proofs are given to show our main results.
2 Main results, illustrative examples and open conjecture

2.1 Notations and concepts

Consider the following two nonautonomous systems

\[ x' = f(t, x) \]  \hspace{1cm} (2.1)

and

\[ y' = g(t, y) \]  \hspace{1cm} (2.2)

where \( x, y \in \mathbb{R}^n, t \in \mathbb{R} \).

**Definition 2.1.** Suppose that there exists a function \( H : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n \) such that

(i) for each fixed \( t \), \( H(t, \cdot) \) is a homeomorphism of \( \mathbb{R}^n \) into \( \mathbb{R}^n \);

(ii) \( \|H(t, x) - x\| \) is uniformly bounded with respect to \( t \);

(iii) \( G(t, \cdot) = H^{-1}(t, \cdot) \) also has property (ii);

(iv) if \( x(t) \) is a solution of the system (2.1), then \( H(t, x(t)) \) is a solution of the system (2.2);

and if \( y(t) \) is a solution of the system (2.2), then \( G(t, y(t)) \) is a solution of the system (2.1).

If such a map \( H_t := H(t, x(t)) \) exists, then the system (2.1) is topologically conjugated to the system (2.2) and the transformation \( H(t, x) \) is called an equivalent function.

**Definition 2.2.** (Coppel [52]) The linear system \( x' = A(t)x \) is said to possess an exponential dichotomy, if there exist a projection \( P(s) \) and constants \( K > 0, \alpha > 0 \) such that

\[ \|U(t, s)P(s)\| \leq K \exp\{-\alpha(t - s)\}, \quad t \geq s, \]  \hspace{1cm} (2.3)

\[ \|U(t, s)(I - P(s))\| \leq K \exp\{\alpha(t - s)\}, \quad t \leq s, \]

hold; here \( U(t, s) := U(t)U^{-1}(s) \) and \( U(t) \) is a fundamental matrix of linear system \( x' = A(t)x \).

Let the Green function

\[ k(t, s) = \begin{cases} 
U(t, s)P(s), & \text{if } t \geq s \\
-U(t, s)(I - P(s)), & \text{if } t \leq s,
\end{cases} \]

and the Green operator

\[ K(\phi)(t) = \int_{-\infty}^{\infty} k(t, s)\phi(s)ds, \quad t \in \mathbb{R}, \]
where \( \phi : \mathbb{R} \to \mathbb{R}^n \) is a function, \( \| K(\phi) \| \leq \mathcal{L}_\alpha(\| \phi \|) \) with

\[
\mathcal{L}_\alpha(b)(t) = \int_{-\infty}^{\infty} \exp\{-\alpha|t-s|\}b(s)ds, \tag{2.4}
\]

for \( b : \mathbb{R} \to (0, \infty) \) a continuous function.

## 2.2 Dichotomy inequality

The following lemma will be useful in the rest of the present work. It consists of a dichotomic inequality developed by Pinto [48, 49, 50, 51]. They are of the following type

\[
u(t) \leq c \exp\{-\alpha(t-t_0)\} + c_1 \int_{t_0}^{t} \exp\{-\alpha(t-\tau)\}b(\tau)u(\tau)d\tau + c_2 \int_{t}^{s} \exp\{-\alpha(\tau-t)\}b(\tau)u(\tau)d\tau + c_2 \int_{s}^{\infty} \exp\{-\alpha(\tau-t)\}b(\tau)u(\tau)d\tau, \tag{2.5}\]

\[
u(t) \leq c \exp\{-\alpha(s-t)\} + c_1 \int_{t_0}^{t} \exp\{-\alpha(t-\tau)\}b(\tau)u(\tau)d\tau + c_2 \int_{t}^{s} \exp\{-\alpha(\tau-t)\}b(\tau)u(\tau)d\tau + c_2 \int_{s}^{\infty} \exp\{-\alpha(\tau-t)\}b(\tau)u(\tau)d\tau, \tag{2.6}\]

where \( \alpha, c, c_1 \) and \( c_2 \) are positive constants.

For \( \alpha_1 > 0 \), define for \( t \in (t_0, s) \)

\[
\mathcal{L}_{\alpha_1}(b)(t) = c_1 \int_{t_0}^{t} \exp\{-\alpha_1(t-\tau)\}b(\tau)d\tau + c_2 \int_{t}^{s} \exp\{-\alpha_1(\tau-t)\}b(\tau)d\tau. \tag{2.7}\]

Assume that

\[
\sup_{t \in (t_0, s)} \mathcal{L}_{\alpha_1}(b)(t) = \theta_1 < 1. \tag{2.8}\]

**Lemma 2.3.** [First dichotomic inequality] Let \( t_0 \in \mathbb{R}, s \in [t_0, \infty) \) and \( u : [t_0, s) \to [0, \infty) \) be continuous, bounded for \( s = \infty \) functions such that for \( t \in [t_0, s) \) inequality \((2.5)\) holds. Then, \( \forall \alpha_2 < \alpha = \alpha_1 + \alpha_2 \) and \( \forall t \in [t_0, s) \) we have

\[
u(t) \leq \frac{c}{1-\theta_1} \exp\{-\alpha_2(t-t_0)\}.\]

**Lemma 2.4.** [Second dichotomic inequality] Let \( s \in \mathbb{R}, t_0 \in (-\infty, s] \) and \( u : [t_0, s) \to [0, \infty) \) be continuous, bounded for \( t_0 = -\infty \) functions such that for \( t \in (t_0, s] \) inequality \((2.6)\) holds. Then, \( \forall \alpha_2 < \alpha = \alpha_1 + \alpha_2 \) and \( \forall t \in (t_0, s] \) we obtain

\[
u(t) \leq \frac{c}{1-\theta_1} \exp\{-\alpha_2(s-t)\}.\]
Proof. ∀α_2 < α = α_1 + α_2, we use \( \exp\{-\alpha\} = \exp\{-\alpha_2\} \cdot \exp\{-\alpha_1\} \). Since \( \alpha_2(s-t) - \alpha_2(t-\tau) - \alpha_2(s-\tau) = 2\alpha_2(t-\tau) \leq 0 \), for \( t \geq \tau \) and \( \alpha_2(s-t) - \alpha_2(\tau-t) - \alpha_2(s-\tau) = 0 \), inequality (2.6) implies that \( \hat{u}(t) =: u(t) \exp\{\alpha_2(s-t)\} \) satisfies (2.6) with \( \alpha_1 \) instead of \( \alpha \):

\[
\hat{u}(t) \leq c \exp\{-\alpha_1(s-t)\} + c_1 \int_{t_0}^{t} \exp\{-\alpha_1(t-\tau)\} b(\tau) \hat{u}(\tau) d\tau + c_2 \int_{t}^{s} \exp\{-\alpha_1(\tau-t)\} b(\tau) \hat{u}(\tau) d\tau.
\]

Then

\[
\hat{u}(t) \leq c \exp\{-\alpha_1(s-t)\} + \mathcal{L}_{\alpha_1}(b)(t) \cdot \sup_{\tau \in [t_0,s]} \hat{u}(\tau)
\]

and hence

\[
\sup_{\tau \in [t_0,s]} \hat{u}(\tau) \leq \frac{c}{1 - \theta_1}.
\]

Therefore

\[
u(t) \leq \frac{c}{1 - \theta_1} \exp\{-\alpha_2(s-t)\}, \quad \text{for} \ t \in [t_0, s].
\]

Lemma 3 follows in a similar way. \( \square \)

2.3 Main results on the Hartman-Grobman theorem and its regularity

We divide our statements of the main results into two parts. One is on the existence of homeomorphisms, the other is on the higher regularity of homeomorphism.

2.3.1 Existence of homeomorphisms

Consider the system (1.1) where \( y \in \mathbb{R}^n \), \( A(t) \) is a \( n \times n \) continuous matrix defined on \( \mathbb{R} \) and \( f(t, y) \) a continuous function on \( \mathbb{R} \times \mathbb{R}^n \) respectively. The following global linearization theorem is for the existence of the homeomorphisms.

**Theorem 2.5.** (global linearization) Suppose that (1.2) admits an exponential dichotomy of the form (2.3) on \( \mathbb{R} \), and there exist nonnegative integrable functions \( \mu(t), r(t) \) such that for all \( t, x, \bar{x}, f(t, x) \) satisfies

\[
\|f(t, x) - f(t, \bar{x})\| \leq r(t)\|x - \bar{x}\|, \quad (2.9)
\]

\[
\|f(t, x)\| \leq \mu(t),
\]

7
where \( \mu(t), r(t) \) satisfy

\[
\sup_{t \in \mathbb{R}} \mathcal{L}_\alpha(\mu)(t) < \infty \quad \text{and} \quad \sup_{t \in \mathbb{R}} \mathcal{L}_\alpha(r)(t) = \theta < K^{-1}. \tag{2.10}
\]

Then system (1.1) is topologically conjugated to its linear system

\[
x' = A(t)x \tag{2.11}
\]

and the equivalent function \( H(t, x) \) and its inverse \( G(t, x) \) satisfy

\[
\|H(t, x) - x\| \leq K\|\mathcal{L}_\alpha(\mu)\|_\infty, \tag{2.12}
\]

\[
\|G(t, x) - x\| \leq K\|\mathcal{L}_\alpha(\mu)\|_\infty, \tag{2.13}
\]

where

\[
\|\mathcal{L}_\alpha(\mu)\|_\infty = \sup_{t \in \mathbb{R}} \mathcal{L}_\alpha(\mu)(t).
\]

Now we introduce a local version of Hartman-Grobman theorem. The following lemma is elementary.

**Lemma 2.6.** For some \( \epsilon > 0 \), if \( F: \mathbb{R} \times \overline{B}_\epsilon(0) \to \mathbb{R}^n \) satisfies \( F(t, 0) = 0 \) and

\[
\|F(t, x) - F(t, \bar{x})\| \leq r(t)\|x - \bar{x}\|,
\]

where \( \overline{B}_\epsilon(0) \) is a small closed ball around zero and \( r(t) \) is nonnegative local integrable with \( \sup_{t \in \mathbb{R}} \mathcal{L}_\alpha(r)(t) = \theta < K^{-1} \). Then the radial extension \( \tilde{F}(t, x) \) defined by

\[
\tilde{F}(t, x) = \begin{cases} 
F(t, x), & x \in \overline{B}_\epsilon(0), \\
F(t, \epsilon \frac{x}{\|x\|}), & x \in \mathbb{R}^n \backslash \overline{B}_\epsilon(0),
\end{cases}
\]

always satisfies

\[
\|\tilde{F}(t, x) - \tilde{F}(t, \bar{x})\| \leq 2r(t)\|x - \bar{x}\|. \tag{2.14}
\]

**Theorem 2.7.** (Local linearization) Suppose that (1.2) has an exponential dichotomy of the form (2.3) on \( \mathbb{R} \). Furthermore, for some \( \epsilon > 0 \), if \( f: \mathbb{R} \times \overline{B}_\epsilon(0) \to \mathbb{R}^n \) satisfies \( f(t, 0) = 0 \),

\[
\|f(t, x) - f(t, \bar{x})\| \leq 2r(t)\|x - \bar{x}\|,
\]

and such that \( 2\sup_{t \in \mathbb{R}} \mathcal{L}_\alpha(r)(t) = 2\theta < K^{-1} \). Then system (1.1) is topologically conjugated to system (2.11) on \( \overline{B}_\epsilon(0) \).
Remark 2.8. Clearly, Theorem 2.7 is a local version of Hartman-Grobman Theorem. Note that when \( x \in \overline{B}_r(0) \),
\[
\|f(t, x)\| \leq r(t)\|x\| \leq r(t)\varepsilon := \mu(t).
\]
when \( x \in \mathbb{R}^n \setminus \overline{B}_r(0) \),
\[
\|f(t, x)\| \leq 2r(t)\varepsilon := 2\mu(t).
\]
If \( \mu(t) \) satisfies \( \sup_{t \in \mathbb{R}} L_\alpha(\mu)(t) < \infty \), then it satisfies all conditions of Theorem 2.5.

Lemma 2.9. (Coppel [52]) If \( \mu(t), r(t) \) are nonnegative local integrable functions on \( \mathbb{R} \), i.e.,
\[
C_\mu = \sup_{t \in \mathbb{R}} \int_t^{t+1} \mu(s) \, ds, \quad C_r = \sup_{t \in \mathbb{R}} \int_t^{t+1} r(s) \, ds,
\]
then we have
\[
L_\alpha(\mu)(t) \leq 2(1 - \exp\{-\alpha\})^{-1}C_\mu, \quad \text{and} \quad L_\alpha(r)(t) \leq 2(1 - \exp\{-\alpha\})^{-1}C_r. \quad (2.15)
\]
Then the following result is obvious.

Corollary 2.10. Theorem 2.5 or Theorem 2.7 hold if \( \sup_{t \in \mathbb{R}} L_\alpha(\mu)(t) < \infty \) and
\[
\theta := 2(1 - \exp\{-\alpha\})^{-1}C_r < K^{-1}.
\]

2.3.2 Higher regularity of homeomorphism

Now it is the position to state our main result on the regularity of homeomorphisms existing in Theorem 2.5 and 2.7.

Theorem 2.11. (regularity of homeomorphisms) Suppose that the conditions in Theorem 2.5 or Theorem 2.7 are satisfied. If \( \sup_{t \in \mathbb{R}} L_\alpha, r(t) = \tilde{\theta} < K^{-1} \), then the equivalent function \( H \) is Lipschitzian, but its inverse \( G \) is Hölder continuous. More specifically, there exist positive constants \( p, q > 0 \) and \( 0 < \beta < 1 \) such that
\[
\begin{cases}
\|H(t, x) - H(t, \bar{x})\| \leq p\|x - \bar{x}\|, \\
\|G(t, x) - G(t, \bar{x})\| \leq q\|x - \bar{x}\|^{\beta}.
\end{cases} \quad (2.16)
\]

Remark 2.12. In the previous literature [11, 13, 19, 20, 23, 25, 29, 47], it is proven that both of the homeomorphisms are Hölder continuous. It can be restated that there exist positive constants \( p_1, q > 0 \) and \( 0 < \gamma, \beta < 1 \) such that
\[
\begin{cases}
\|H(t, x) - H(t, \bar{x})\| \leq p_1\|x - \bar{x}\|^\gamma, \\
\|G(t, x) - G(t, \bar{x})\| \leq q\|x - \bar{x}\|^{\beta}.
\end{cases}
\]
When \( A(t) \equiv A \), \( A \) is a constant matrix, the systems reduce to the autonomous systems. Then we have the following corollary.

**Corollary 2.13.** Let \( A \) be hyperbolic, i.e., the spectrum of \( A \) has no purely imaginary eigenvalues. If the nonlinear term \( f \) satisfies

\[
\| f(x) - f(\bar{x}) \| \leq r \| x - \bar{x} \|, \quad \| f(x) \| \leq \mu,
\]

for all \( x, \bar{x} \in \mathbb{R}^n \), and such that \( 2rk < \alpha \) (\( k, \alpha \) are given in (2.3)), then the nonlinear autonomous system

\[
x' = Ax + f(x)
\]

is topologically conjugated to

\[
x' = Ax.
\]

Moreover, the homeomorphism \( H(x) \) is Lipschitzian, but the inverse \( G(x) \) is Hölder continuous, i.e., for \( x, \bar{x} \in \mathbb{R}^n \), there exist positive constants \( p, q > 0 \), \( 0 < \beta < 1 \) such that

\[
\begin{align*}
\| H(x) - H(\bar{x}) \| & \leq p \| x - \bar{x} \|, \\
\| G(x) - G(\bar{x}) \| & \leq q \| x - \bar{x} \|^\beta.
\end{align*}
\]

**Corollary 2.14.** In the nonuniform case, that is, the linear system admits a nonuniform exponential dichotomy instead of the uniform exponential dichotomy ([14, 15, 16, 17, 55]), Theorems 2.5, 2.7 and 2.11 are true for

\[
\tilde{\mu}(t) = \mu(t) \exp\{-\epsilon |t|\} \quad \text{and} \quad \tilde{r}(t) = r(t) \exp\{-\epsilon |t|\},
\]

where \( \mu(t) \) and \( r(t) \) are given in (2.9).

In fact, we can see that

\[
\mathcal{L}_\alpha(\tilde{\mu})(t) = \int_{-\infty}^{\infty} \exp\{-\alpha |t - s| + \epsilon |s|\} \mu(s) \exp\{-\epsilon |s|\} ds = \mathcal{L}_\alpha(\mu)(t),
\]

similarly, \( \mathcal{L}_\alpha(\tilde{r})(t) = \mathcal{L}_\alpha(r)(t) \). Thus, all conditions of these theorems in the nonuniform case are satisfied.

**Remark 2.15.** Lemma 2.9 shows that a big class of functions \( \mu, r \) satisfy condition (2.10). \( r, \mu \in L^p, 1 \leq p \leq \infty \), with \( \| r \|_p \) small enough. If \( r \) is uniformly bounded, it is possible to choose \( \alpha = 0 \). Note when \( \mu(t) = \mu \) and \( r(t) = r \) are constants, Theorem 2.5 reduces to the classical Palmer linearization theorem. We note that Palmer did not give a conclusion on the Lipschitz nor the Hölder continuity of \( H \). It should be noted that \( f(t, x) \) in our theorem could be unbounded or not uniformly Lipschitzian. Note \( \mu(t), r(t) \) are locally integrable (satisfying (2.10)), so they could be unbounded.
2.4 Illustrative examples and open conjecture

2.4.1 Illustrative examples to verify the higher regularity of homeomorphisms

Example 2.1. This example on the global linearization shows that the homeomorphism $H$ is Lipschitzian, but its inverse is merely Hölder continuous.

We consider the hyperbolic equations on the unit circle, i.e.,

$$
\begin{pmatrix}
    x_1' \\
    x_2'
\end{pmatrix} =
\begin{pmatrix}
    -1 & 0 \\
    0 & 1
\end{pmatrix}
\begin{pmatrix}
    x_1 \\
    x_2
\end{pmatrix} +
\begin{pmatrix}
    f_1(x_1) \\
    f_2(x_2)
\end{pmatrix},
$$

where $f_1(x_1)$ and $f_2(x_2)$ is given by

$$
f_1(x_1) =
\begin{cases}
    \epsilon x_1, & 0 \leq x_1 \leq 1, \\
    -\epsilon x_1^3, & -1 \leq x_1 < 0,
\end{cases}
$$

and

$$
f_2(x_2) =
\begin{cases}
    -\epsilon x_2, & 0 \leq x_2 \leq 1, \\
    -\epsilon x_2^3, & -1 \leq x_2 < 0.
\end{cases}
$$

It is easy to see that $f$ is bounded and Lipschitzian. Moreover, it is easy to obtain that equation (2.17) is topologically conjugated to its linear part

$$
\begin{pmatrix}
    y_1' \\
    y_2'
\end{pmatrix} =
\begin{pmatrix}
    -1 & 0 \\
    0 & 1
\end{pmatrix}
\begin{pmatrix}
    y_1 \\
    y_2
\end{pmatrix}.
$$

Thus the main purpose here is to construct an explicit formulae for $H = (H_1, H_2)^T$ and its inverse $G = (G_1, G_2)^T = H^{-1}$. For $t \geq 0$, we firstly consider the subsystem

$$
x_1' = -x_1 + f_1(x_1).
$$

Notice that

$$
-x_1 + f_1(x_1) =
\begin{cases}
    < 0, & \text{if } 0 < x_1 \leq 1, \\
    = 0, & \text{if } x_1 = 0, \\
    > 0, & \text{if } -1 \leq x_1 < 0.
\end{cases}
$$

Thus a solution is either always 0, always < 0 or always > 0.

Firstly, $H_1(0) = 0$ since 0 is a solution of (2.19) and $H_1(0)$ is the unique solution of $y_1' = -y_1$. But 0 is such a solution.

Secondly, we consider $0 < x_1(t) \leq 1$. Clearly, $x_1(t)$ is strictly decreasing, i.e., $x_1(t) \to 0$ as $t \to +\infty$; $x_1(t) \to 1$ as $t \to 0$. Therefore, there must exists a unique time $t_0$ such that
$x_1(t_0) = 1$. We set $t_0 = 0$. If $t > 0$, then $0 < x_1(t) < 1$ and so $x'_1(t) = -x_1(t) + \varepsilon x_1(t)$ with $x_1(0) = 1$. Hence,

$$x_1(t) = e^{(-1+\varepsilon)t}, \quad t \geq 0.$$  

We need to find the unique solution $y_1(t)$ of $y'_1 = -y_1$ such that $\|y_1(t) - x_1(t)\|$ is bounded. Looking at $x_1(t)$ when $t \geq 0$, we see that $y_1(t) = (1 - \varepsilon)e^{-t}$. Hence for all $t \geq 0$,

$$H_1(x_1(t)) = (1 - \varepsilon)e^{-t}.$$

Then

$$H_1(1) = H_1(x_1(0)) = 1 - \varepsilon.$$

If $0 < \xi_1 < 1$, then there exists a unique time $t > 0$ such that $x_1(t) = e^{(-1+\varepsilon)t} = \xi_1$. Then

$$H_1(\xi) = H_1(x_1(t)) = (1 - \varepsilon)e^{-t} = (1 - \varepsilon)x_1(t)^{\frac{1}{\varepsilon}} = (1 - \varepsilon)\xi_1^{\frac{1}{\varepsilon}}.$$

Therefore,

$$H(x_1) = (1 - \varepsilon)x_1^{\frac{1}{\varepsilon}}, \quad 0 < x_1 \leq 1.$$

Let us now consider $-1 \leq x_1(t) < 0$, clearly $x_1(t)$ is strictly increasing, i.e., $x_1(t) \to 0$ as $t \to \infty$; $x_1(t) \to -1$ as $t \to 0$. So there must exists a unique time $t_0$ such that $x_1(t_0) = -1$. We set $t_0 = 0$. If $t > 0$, then $-1 < x_1(t) < 0$ and so $x'_1(t) = -x_1(t) + \varepsilon x_1^3(t)$ with $x_1(0) = -1$. Letting $z_1 = x_1^{-2}$, then $z'_1 = 2z - 2\varepsilon$ with $z_1(0) = 1$. Hence, $z_1(t) = (1 - \varepsilon)e^{2t} + \varepsilon$, that is,

$$x_1(t) = -\left[(1 - \varepsilon)e^{2t} + \varepsilon\right]^{-\frac{1}{2}}, \quad t \geq 0.$$  

We need to find the unique solution $y_1(t)$ of $y'_1 = -y_1$ such that $|y_1(t) - x_1(t)|$ is bounded. Looking at $x_1(t)$ when $t \geq 0$, we see that $y_1(t) = (\varepsilon - 1)e^{-t}$. Hence for all $t \geq 0$,

$$H_1(x_1(t)) = (\varepsilon - 1)e^{-t}.$$  

Then

$$H_1(1) = H_1(x_1(0)) = \varepsilon - 1.$$  

If $-1 < \xi_1 < 0$, then there exists a unique time $t > 0$ such that

$$x_1(t) = -\left[(1 - \varepsilon)e^{2t} + \varepsilon\right]^{-\frac{1}{2}} = \xi_1.$$  

Then

$$H_1(\xi_1) = H_1(x_1(t)) = (\varepsilon - 1)e^{-t} = -(1 - \varepsilon)^{\frac{3}{2}}((-x_1(t))^{-2} - \varepsilon)^{-\frac{1}{2}} = -(1 - \varepsilon)^{\frac{3}{2}}((-\xi_1)^{-2} - \varepsilon)^{-\frac{1}{2}}.$$  

12
Thus we obtain that

$$H_1(x_1) = -(1 - \epsilon) \frac{3}{2} ((-x_1)^{-2} - \epsilon)^{-\frac{1}{2}}, \quad -1 \leq x_1 < 0.$$ 

Summarizing we have found that

$$H_1(x_1) = \begin{cases} 
(1 - \epsilon)x_1^\frac{1}{2}, & 0 < x_1 \leq 1, \\
0, & x_1 = 0, \\
-(1 - \epsilon) \frac{2}{3} ((-x_1)^{-2} - \epsilon)^{-\frac{1}{2}}, & -1 \leq x_1 < 0.
\end{cases}$$

We next claim that $H_1$ is a continuous function, but it is not $C^1$. In fact, we only say that $H_1$ is continuous at 0, but is not $C^1$ at 0.

$H_1(x_1)$ is continuous at $x_1 = 0$:

$$\lim_{x_1 \to 0^+} (1 - \epsilon)x_1^\frac{1}{2} = 0,$$

$$\lim_{x_1 \to 0^-} -(1 - \epsilon) \frac{2}{3} ((-x_1)^{-2} - \epsilon)^{-\frac{1}{2}} = 0.$$ 

Hence, $H_1(x_1)$ is continuous, but the following fact proves that $H_1$ is not $C^1$ at 0. Clearly, for $0 < x_1 \leq 1$, $H'_1(x_1) = x_1^\frac{1}{2}$, and for $-1 \leq x_1 < 0$, $H'_1(x_1) = (1 - \epsilon) \frac{2}{3}(1 - \epsilon(x_1)^2)^{-\frac{2}{3}}$. Therefore,

$$\lim_{x_1 \to 0^+} x_1^\frac{1}{2} = 0,$$

$$\lim_{x_1 \to 0^-} (1 - \epsilon) \frac{2}{3}(1 - \epsilon(x_1)^2)^{-\frac{2}{3}} = (1 - \epsilon)^\frac{2}{3},$$

which implies that $H_1(x_1)$ is not in $C^1$. Fortunately, $H_1(x_1)$ is Lipschitz continuous, since $H'_1(x_1)$ is continuous at $x_1$ except for $x_1 = 0$, and it is bounded with $\|H'_1\| \leq 1$. So function $H_1(x_1)$ is globally Lipschitz continuous with Lipschitz constant $L = 1$, but is not in $C^1$.

However, the inverse function $G_1 = H_1^{-1}$ is

$$G_1(y_1) = \begin{cases} 
\frac{y_1}{1-\epsilon}^{1-\epsilon}, & 0 < y_1 \leq 1 - \epsilon, \\
y_1 = 0, & y_1 = 0, \\
-\left((1 - \epsilon)^{-\frac{2}{3}}(y_1)^{-2} + \epsilon\right)^{-\frac{1}{2}}, & -1 + \epsilon \leq y_1 < 0.
\end{cases}$$

Obviously, $G_1(y_1)$ is continuous at 0. So $G_1(y_1)$ is a continuous function. However, this is not Lipschitz continuous since $y_1^{1-\epsilon}$ is not Lipschitz, as $0 < 1 - \epsilon < 1$. 

13
Secondly, for \( t \leq 0 \), we consider the subsystem
\[
x_2' = x_2 + f_2(x_2).
\]

Similar to the procedure just shown, we can obtain that
\[
H_2(x_2) = \begin{cases} 
(1 - \epsilon)\frac{1}{x_2}, & 0 < x_2 \leq 1, \\
0, & x_2 = 0, \\
-(1 - \epsilon)^{\frac{3}{2}} ((-x_2)^{-2} - \epsilon)^{-\frac{1}{2}}, & -1 \leq x_2 < 0,
\end{cases}
\]
and
\[
G_2(y_2) = \begin{cases} 
(y_2)(1-\epsilon)^{1-\epsilon}, & 0 < y_2 \leq 1 - \epsilon, \\
0, & y_2 = 0, \\
-(1 - \epsilon)^{-\frac{3}{2}} ((-y_2)^{-2} + \epsilon)^{-\frac{1}{2}}, & -1 + \epsilon \leq y_2 < 0.
\end{cases}
\]

Hence, \( H_2 \) is Lipschitzian, but \( G_2 \) is only Hölder continuous. Therefore, Theorem 2.11 is verified.

**Example 2.2.** This example on the local linearization is to show that the homeomorphism \( H \) is Lipschitzian, but its inverse is merely Hölder continuous.

We consider the following non-autonomous system
\[
x' = -x + f(t, x),
\]
where \( f(t, x) \) is given by
\[
f(t, x) = \begin{cases} 
0, & \|x\| \leq \epsilon, \\
\frac{2e^{-t}}{e^t + e^{-t}}x, & \|x\| \geq \delta,
\end{cases}
\]
for some arbitrarily chosen \( 0 < \epsilon < \delta \). We assume that \( f \) connects these two value smoothly. Thus the vector field of Eq. (2.20) is nonlinear. When \( \|x\| \leq \epsilon \), it is identical to the linear flow. Hence, we only need to limit ourselves to \( \|x\| \geq \delta \). For \( \|x\| \geq \delta \), we can check that \( x(t) = \frac{2}{e^t + e^{-t}} \) is a bounded solution with the initial value \( x(0) = 1 \).

Now we set \( H(t, x) = \frac{1}{x} - \frac{t^2}{2} \). Notice that
\[
H(t, x(t)) = \frac{1}{x(t)} - \frac{e^t}{2} = \frac{e^t + e^{-t}}{2} - \frac{e^t}{2} = \frac{1}{2} e^{-t},
\]
which implies that \( H(t, x(t)) \) is a solution of \( y' = -y \). To show its regularity, take any \( \|x_1\|, \|x_2\| \geq \delta \), we have

\[
\|H(t, x_1) - H(t, x_2)\| = \left\| \frac{1}{x_1} - \frac{1}{x_2} \right\| \leq \frac{1}{\delta^2} \|x_1 - x_2\|.
\]

It means that \( H \) is Lipschitzian for \( \|x\| \geq \delta \). Moreover for \( \|y\| \leq \frac{1}{\delta}, G := H^{-1} = \frac{2}{e^t + 2y} \) and

\[
\|G(t, y_1) - G(t, y_2)\| = \frac{4\|y_2 - y_1\|}{(e^t + 2y_1)(e^t + 2y_2)},
\]

when \( t \to -\infty, \|G(t, y_1) - G(t, y_2)\| = \frac{\|y_2 - y_1\|}{\|y_1 y_2\|} \to \infty \). Therefore, \( G \) is not Lipschitzian. If we take \( \|y_1 - y_2\| < 1 \), then there exists \( 0 < q < 1 \) such that \( \|G(t, y_1) - G(t, y_2)\| \leq \|y_1 - y_2\|^q \).

**Example 2.3.** The following example shows that \( f(t, x) \) in our conditions (2.9) could be unbounded, nor uniformly Lipschitzian. Thus, it is weaker than previous works on the Palmer’s linearization theorem.

We construct a continuous function \( f(t, x) \) which is unbounded, not uniformly Lipschitzian, but locally integrable. Considering \([0, \infty)\), for any positive constant \( c \) and integer \( m \), let

\[
\bar{g}(t) = \begin{cases} 
0, & \text{if } t \in [0, 1), \\
cm^2t - cm^3, & \text{if } t \in [m, m + \frac{1}{2m}) , \\
-cm^2t + cm^3 + cm, & \text{if } t \in [m + \frac{1}{2m}, m + \frac{1}{m}) , \\
0, & \text{if } t \in [m + \frac{1}{m}, m + 1) .
\end{cases}
\]

Note that \( \bar{g}(t) \) is continuous on \([0, \infty)\). Let \( \mu(t) \) the continuous function on \( \mathbb{R} \):

\[
\mu(t) = \begin{cases} 
\bar{g}(t), & \text{if } t \geq 0, \\
\bar{g}(-t), & \text{if } t < 0.
\end{cases}
\]

Thus,

\[
f(t, x) = \mu(t) \sin(x)
\]

is continuous on \( \mathbb{R} \times \mathbb{R}^2 \). It is easy to see that for any \( (t, x), (t, \bar{x}) \in \mathbb{R} \times \mathbb{R}^2 \),

\[
\|f(t, x) - f(t, \bar{x})\| \leq \mu(t) \|x - \bar{x}\| ;
\]

\[
\|f(t, x)\| \leq \mu(t)
\]
and
\[ \int_t^{t+1} \mu(s) ds \leq c. \]
However, we see that \( \mu \) and \( f \) are unbounded functions, since
\[ \mu \left( m + \frac{1}{2m} \right) \to +\infty, \text{ as } m \to \infty. \]
Consequently, \( f(t, x) \) is not only unbounded, but also \( f(t, x) \) is not uniformly Lipschitzian.

### 2.4.2 Open conjecture

The above two illustrative example show that our main results on the higher regularity of homeomorphisms are correct. That is, the homeomorphism \( H \) is Lipschitzian, and the inverse of the homeomorphism is Hölder continuous. In particular, in Example 2.11 it is shown that homeomorphism \( H \) is Lipschitzian, **but not** \( C^1 \); its inverse is Hölder continuous, **but not** Lipschitzian. Moreover, it is difficult to verify that both the homeomorphism \( H \) and its inverse in this example are unique, respectively. Therefore, from this example, we assert that the homeomorphism \( H \) is Lipschitzian and its inverse is Hölder continuous in the Hartman-Grobman, **and the regularity of the homeomorphisms is sharp. That is to say, the regularity of the homeomorphisms could not be improved any more.** But in this situation, it is only a conjecture from the example. We need a strict proof, but it is an open problem now.

### 3 Preliminary results

#### 3.1 Preliminary results for the existence of homeomorphisms

In what follows, we always suppose that the conditions of Theorem 2.14 are satisfied. Let \( X(t, t_0, x) \) be a solution of system (1.1) satisfying the initial condition \( X(t_0) = x \) and \( Y(t, t_0, y) \) is a solution of system (2.11) satisfying the initial condition \( Y(t_0) = y \). To prove the main results, we divide our proof into several lemmas.

**Lemma 3.1.** For each \((\tau, \xi)\), the system
\[ Z' = A(t)Z - f(t, X(t, \tau, \xi)) \]
has a unique bounded solution \( h(t, (\tau, \xi)) \) with \( \| h(t, (\tau, \xi)) \| \leq K \| L_\alpha(\mu) \|_\infty \).

**Proof.** For any fixed \((\tau, \xi)\), let
\[
h(t, (\tau, \xi)) = -K(f(\cdot, X(\cdot, \tau, \xi))(t) = -\int_{-\infty}^{t} U(t, s)P(s)f(s, X(s, \tau, \xi))ds + \int_{t}^{\infty} U(t, s)(I - P(s))f(s, X(s, \tau, \xi))ds.
\]
Differentiating it, it is easy to see that \( h(t, (\tau, \xi)) \) is a solution of the system (3.1). It follows from (2.3) and (2.9) that
\[
\| h(t, (\tau, \xi)) \| \leq \int_{-\infty}^{t} K\mu(s) \exp\{-\alpha(t-s)\}ds + \int_{t}^{\infty} K\mu(s) \exp\{\alpha(t-s)\}ds
\]
\[
\leq K \| L_\alpha(\mu) \|_\infty,
\]
which implies that \( h(t, (\tau, \xi)) \) is a bounded solution of the system (3.1). We claim that the bounded solution is unique. In fact, for any fixed \((\tau, \xi)\), the system (3.1) is linearly inhomogeneous, and its linear system \( Z' = A(t)Z \) has an exponential dichotomy. This implies that the bounded solution of (3.1) is unique. \( \square \)

**Lemma 3.2.** For each \((\tau, \xi)\), the system
\[
Z' = A(t)Z + f(t, Y(t, \tau, \xi) + Z)
\]
has a unique bounded solution \( g(t, (\tau, \xi)) \), and \( \| g(t, (\tau, \xi)) \| \leq K \| L_\alpha(\mu) \|_\infty \).

**Proof.** Let \( B \) be the complete metric space of all the continuous bounded functions \( Z(t) \), provided of supremum metric, with \( \| Z(t) \| \leq K \| L_\alpha(\mu) \|_\infty \). For each \((\tau, \xi)\) and any \( Z(t) \in B \), define a mapping \( T \) as follows,
\[
TZ(t) = \int_{-\infty}^{t} U(t, s)P(s)f(s, Y(s, \tau, \xi) + Z(s))ds - \int_{t}^{\infty} U(t, s)(I - P(s))f(s, Y(s, \tau, \xi) + Z(s))ds.
\]
A simple computation leads to
\[
\| TZ(t) \| \leq KL_\alpha(\mu)(t),
\]
which implies that \( TB \subset B \). For any \( Z_1(t), Z_2(t) \in B \),
\[
\| TZ_1(t) - TZ_2(t) \| \leq KL_\alpha(\mu)(t) \| Z_1 - Z_2 \|.
\]

17
Now, by (2.10) $K\theta < 1$, then $T$ has a unique fixed point, namely $Z_0(t)$, and

$$Z_0(t) = \int_{-\infty}^{t} U(t, s)P(s)f(s, Y(s, \tau, \xi) + Z_0(s))ds - \int_{t}^{\infty} U(t, s)(I - P(s))f(s, Y(s, \tau, \xi) + Z_0(s))ds.$$

It is easy to see that $Z_0(t)$ is a bounded solution of the system (3.2). From standard argument, the bounded solution is unique. We may call the unique solution $g(t, (\tau, \xi))$. From the above proof, it is easy to see that $\|g(t, (\tau, \xi))\| \leq K \|\mathcal{L}_\alpha(\mu)\|_{\infty}$.

Similarly, we have:

**Lemma 3.3.** Let $x(t)$ be any solution of the system (1.2). Then $Z(t) \equiv 0$ is the unique bounded solution of the system

$$Z' = A(t)Z + f(t, x(t) + Z) - f(t, x(t)).$$

(3.3)

Note the importance in these results of the uniform boundedness of $\mathcal{L}_\alpha(\mu)(t)$.

**Constructing the homeomorphisms:** Now we define two functions as follows

$$H(t, x) = x + h(t, (t, x)), \quad (3.4)$$

$$G(t, y) = y + g(t, (t, y)), \quad (3.5)$$

for $g$ and $h$ as in Lemmas 3.1 and 3.2

By differentiation and similar arguments, we have the following lemmas.

**Lemma 3.4.** For any fixed $(t_0, x)$, $H(t, X(t, t_0, x))$ is a solution of system (2.11).

**Lemma 3.5.** For any fixed $(t_0, y)$, $G(t, Y(t, t_0, y))$ is a solution of system (2.16).

**Lemma 3.6.** For any $t \in \mathbb{R}$, $y \in \mathbb{R}^n$, $H(t, G(t, y)) = y$.

**Proof.** Let $y(t)$ be any solution of linear system (2.11). From Lemma 3.5, $G(t, y(t))$ is a solution of system (1.1). Then by Lemma 3.4 we see that $H(t, G(t, y(t)))$ is a solution of system (2.11), written as $\overline{y}(t)$. Let

$$J(t) = \overline{y}(t) - y(t).$$
To prove this conclusion, we need to show that $J(t) \equiv 0$. In fact, differentiating $J$, we have

$$J'(t) = \overline{y}'(t) - y'(t) = A(t)\overline{y}(t) - A(t)y(t) = A(t)J(t),$$

which implies that $J$ is a solution of the system $Z' = A(t)Z$. From Lemma 3.1 and Lemma 3.2, it follows that

$$\|J(t)\| = \|\overline{y}(t) - y(t)\| = \|H(t, G(t, (t, y(t)))) - y(t)\| \leq \|H(t, G(t, (t, y(t)))) - G(t, (t, y(t))))\| + \|G(t, (t, y(t)))) - y(t)\| \leq 2K \| L_\alpha(\mu) \|_\infty.$$

This implies that $J(t)$ is a bounded solution of the system $Z' = A(t)Z$. However, the linear system $Z' = A(t)Z$ has no nontrivial bounded solution. Hence $J(t) \equiv 0$, that is, $\overline{y}(t) = y(t)$. Thus, $J(t) \equiv 0$, that is,

$$\overline{y}(t) = y(t), \quad \text{or} \quad H(t, G(t, y(t))) \equiv y(t).$$

Since $y(t)$ is an arbitrary solution of linear system (2.11), the proof of Lemma 3.6 is complete.

Lemma 3.7. For any $t \in \mathbb{R}$, $x \in \mathbb{R}^n$, we have

$$G(t, H(t, x)) = x.$$

Proof. The proof is similar to that in Lemma 3.6.

\[\square\]

3.2 Key lemma for the Lipchitzian continuity of homeomorphism

To introduce our key lemma, we begin with a result from [52, 53, 54]. We restate it as follows.

Lemma 3.8. Assume that system (1.2) admits an exponential dichotomy with the form (2.3) on $\mathbb{R}$. 

\[\square\]
(1) If system (1.1) has a bounded solution $X(t,t_0,x)$ on $[t_0,\infty)$ satisfying the initial value $X(t_0) = x$, then $X(t,t_0,x)$ can be expressed by:

$$X(t,t_0,x) = U(t,t_0)P(t_0)x + \int_{t_0}^{t} \Phi_P(t,\tau)f(\tau,X(\tau,t_0,x))d\tau + \int_{t}^{\infty} \Phi_Q(t,\tau)f(\tau,X(\tau,t_0,x))d\tau,$$

where

$$\Phi_P(t,\tau) = U(t,\tau)P(\tau), \quad t \geq \tau, \quad \text{and} \quad \Phi_Q(t,\tau) = -U(t,\tau)(I-P(\tau)), \quad t \leq \tau.$$ 

Conversely, all solutions $X(t,t_0,x)$ of (3.6) on $[t_0,\infty)$ are the solutions of (1.1).

(2) If system (1.1) has a bounded solution $X(t,t_0,x)$ on $(-\infty,t_0]$ satisfying the initial value $X(t_0) = x$, then $X(t,t_0,x)$ can be expressed by:

$$X(t,t_0,x) = U(t,t_0)(I-P(t_0))x + \int_{-\infty}^{t} \Phi_P(t,\tau)f(\tau,X(\tau,t_0,x))d\tau + \int_{t}^{t_0} \Phi_Q(t,\tau)f(\tau,X(\tau,t_0,x))d\tau.$$

Conversely, all solutions $X(t,t_0,x)$ of (3.7) on $(-\infty,t_0]$ are the solutions of (1.1).

The following lemma plays a great role in the proof of Lipchitz continuity of homeomorphism.

**Lemma 3.9.** Denote $X(t,t_0,x)$ is the solution of system (1.1) satisfying $X(t_0) = x \in \mathbb{R}^n$.

(i) For any $\xi_1 \in P(t_0)(\mathbb{R}^n)$, (1.1) has a unique bounded solution $X(t,t_0,x)$ on $[t_0,\infty)$ satisfying $P(t_0)X(t_0) = \xi_1$, which is expressed by (3.6); (ii) For any $\xi_2 \in (I-P(t_0))(\mathbb{R}^n)$, (1.1) has a unique bounded solution $X(t,t_0,x)$ on $(-\infty,t_0]$ satisfying $(I-P(t_0))X(t_0) = \xi_2$, which is expressed by (3.7).

Moreover, if $\alpha = \alpha_1 + \alpha_2$ and $\sup_{t \in \mathbb{R}} L_{\alpha_1}(r)(t) = \bar{\theta} < K^{-1}$, then for any $\alpha_2 < \alpha$ the following conclusions hold:

(1) For $P(t_0)(x - \bar{x}) \in P(t_0)(\mathbb{R}^n)$, we have

$$\|X(t,t_0,x) - X(t,t_0,\bar{x})\| \leq \frac{K}{1 - K\theta}\|P(t_0)(x - \bar{x})\| \exp\{|\alpha_2(t - t_0)\}, \quad t \geq t_0; \quad (3.8)$$

(2) For $(I-P(t_0))(x - \bar{x}) \in (I-P(t_0))(\mathbb{R}^n)$, we have

$$\|X(t,t_0,x) - X(t,t_0,\bar{x})\| \leq \frac{K}{1 - K\theta}\|(I-P(t_0))(x - \bar{x})\| \exp\{\alpha_2(t - t_0)\}, \quad t \leq t_0. \quad (3.9)$$

**Proof.** We claim the first part by means of Banach contraction mapping principle. Let $\mathcal{BC}$ be the set of all bounded continuous functions defined for $t \geq t_0$. If $\mathcal{J}$ is the mapping defined by

$$\mathcal{J}X(t,t_0,x) = U(t,t_0)P(t_0)x + \int_{t_0}^{t} \Phi_P(t,\tau)f(\tau,X(\tau,t_0,x))d\tau + \int_{t}^{\infty} \Phi_Q(t,\tau)f(\tau,X(\tau,t_0,x))d\tau.$$

20
By using (2.9) and (2.10), it is easy to see that $\mathcal{J}X$ is continuous and bounded. In fact,

$$
\| \mathcal{J}X(t, t_0, x) \| \leq K \exp\{ -\alpha(t - t_0) \} \| \xi_1 \| + \int_{t_0}^{\infty} K \exp\{ -\alpha |t - \tau| \} \mu(\tau) d\tau \\
\leq K \| \xi_1 \| + K \sup_{\tau \geq t_0} L_\alpha(\mu)(\tau) < \infty,
$$

where $\xi_1 = P(t_0)X(t_0) = P(t_0)x$. Hence $\mathcal{J}$ maps $BC$ into itself. Note that $K\theta < 1$ (see (2.10)), for any $X_1(t, t_0, x), X_2(t, t_0, x) \in BC$, we have that (also using (2.9) and (2.10))

$$
\| \mathcal{J}X_1(t, t_0, x) - \mathcal{J}X_2(t, t_0, x) \| \leq \int_{t_0}^{\infty} K \exp\{ -\alpha |t - \tau| \} r(\tau) \| X_1(\tau, t_0, x) - X_2(\tau, t_0, x) \| d\tau \\
\leq K\theta \sup_{\tau \geq t_0} \| X_1(\tau, t_0, x) - X_2(\tau, t_0, x) \|,
$$

which implies that $\mathcal{J}$ is a contraction mapping in $BC$, that is, there is a unique fixed point $X^* = \mathcal{J}X^*$ such that $X^*$ is bounded for $t \geq t_0$ and $P(t_0)X(t_0) = \xi_1$. The expression follows from Lemma 3.8 immediately.

Similar to the above procedure for $t \leq t_0$, by replacing the mapping by

$$
\mathcal{J}X(t, t_0, x) = U(t, t_0)\xi_2 + \int_{t_0}^{t} \Phi_Q(t, \tau) f(\tau, X(\tau, t_0, x)) d\tau + \int_{-\infty}^{t} \Phi_P(t, \tau) f(\tau, X(\tau, t_0, x)) d\tau,
$$

where $\xi_2 = (I - P(t_0))x \in (I - P(t_0))(\mathbb{R}^n)$. We can show that $X(t)$ is solution of (1.1) with the required property (ii).

We now show the second part by means of Lemma 2.3 and Lemma 2.4. By (2.9), we conclude that for any initial condition on $P(t_0)(x - \bar{x}) \in P(t_0)(\mathbb{R}^n)$

$$
\| X(t, t_0, x) - X(t, t_0, \bar{x}) \| \leq K \| P(t_0)(x - \bar{x}) \| \exp\{ -\alpha(t - t_0) \} \\
+ \int_{t_0}^{\infty} K \exp\{ -\alpha |t - \tau| \} r(\tau) \| X(\tau, t_0, x) - X(\tau, t_0, \bar{x}) \| d\tau.
$$

(3.10)

Since $K\theta < 1$, and using dichotomic inequality in Lemma 2.3, we obtain that for $t \geq t_0$

$$
\| X(t, t_0, x) - X(t, t_0, \bar{x}) \| \leq \frac{K}{1 - K\theta} \exp\{ -\alpha_2(t - t_0) \} \| P(t_0)(x - \bar{x}) \|.
$$

Finally, analogous analysis for $t \leq t_0$, dichotomic inequality in Lemma 2.4 ends the proof. \qed

### 3.3 An intuitive example to understand Lemma 3.9

**Example 3.1.** In this example, we give an intuitive example to understand Lemma 3.9. It is a key lemma to prove that the homeomorphism $H$ is Lipschitz continuous.
For simplicity, we consider the following planar system

$$x' = Ax + f(x), \quad x \in \mathbb{R}^2,$$

where $f : \mathbb{R}^2 \to \mathbb{R}^2$ and $A = \text{diag}\{-1, 1\}$. If we take $P = \text{diag}\{1, 0\}$ and $I - P = \text{diag}\{0, 1\}$, then

$$Px' = \begin{pmatrix} x'_1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ 0 \end{pmatrix} + \begin{pmatrix} f_1(x_1, x_2) \\ 0 \end{pmatrix},$$

and

$$(I - P)x' = \begin{pmatrix} 0 \\ x'_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ f_2(x_1, x_2) \end{pmatrix}.$$  

Suppose that $f = (f_1, f_2)^T$ satisfies $f(0) = 0$ and $\|f(x)\| \leq \mu$, $\|f(x) - f(\bar{x})\| \leq r\|x - \bar{x}\|$ for all $x, \bar{x} \in \mathbb{R}^2$, where $\mu > 0$ and $r < \frac{1}{2}$. Then we have $K = 1, \alpha = 1, \theta = 2rK/\alpha = 2r < 1$. Thus for $t \geq 0$, it is clear that equation (3.12) has a unique bounded solution $X(t)$ with the initial condition $(x_1(0), 0)^T$ (i.e., $(x_1(0), x_2(0))^T = (x_1(0), 0)^T \in P(\mathbb{R}^2)$). Note

$$X(t) = e^{-t} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1(0) \\ 0 \end{pmatrix} + \int_0^t e^{-(t-s)} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} f_1(x_1(s), x_2(s)) \\ 0 \end{pmatrix} ds,$$

$$- \int_t^\infty e^{t-s} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ f_2(x_1(s), x_2(s)) \end{pmatrix} ds.$$  

Then for $t \geq 0$ and $x, \bar{x}$,

$$\|X(t, 0, x) - X(t, 0, \bar{x})\| \leq c_1e^{-\epsilon t}\|Px - P\bar{x}\|,$$

for some constants $c_1 > 0$, $0 < \epsilon < 1$. For $t \leq 0$, similar to the procedure just shown, we have that $X(t)$ is bounded and

$$\|X(t, 0, x) - X(t, 0, \bar{x})\| \leq c_2e^{\epsilon t}\|(I - P)x - (I - P)\bar{x}\|,$$

for some constants $c_2 > 0$, $0 < \epsilon < 1$. Therefore, Lemma 3.9 always holds.

## 4 Proofs of main results

### 4.1 Proofs of Theorem 2.5

Now we are in a position to prove Theorem 2.5.
Proof of Theorem 2.5. Now we show that \( H(t, \cdot) \) satisfies the four conditions of Definition 2.1. For any fixed \( t \), it follows from Lemma 3.6, 3.7 that \( H(t, \cdot) \) is homeomorphism and \( G(t, \cdot) = H^{-1}(t, \cdot) \). Thus, Condition (i) is satisfied. From (2.9) and Lemma 3.1 we derive \( \|H(t, x) - x\| = \|h(t, (t, x))\| \leq K \| L_{\alpha}(\mu) \|_{\infty} \). Note \( \|H(t, x)\| \to \infty \) as \( \|x\| \to \infty \), uniformly with respect to \( t \). Thus, Condition (ii) is satisfied. From (2.11) and Lemma 3.2, we derive \( \|G(t, y) - y\| = \|g(t, (t, y))\| \). Note \( \|G(t, y)\| \to \infty \) as \( \|y\| \to \infty \), uniformly with respect to \( t \). Thus, Condition (iii) is satisfied. From Lemma 3.4, Lemma 3.5, we know that Condition (iv) is true. Hence, the system (1.1) and its linear system (2.11) are topologically conjugated. This completes the proof of Theorem 2.5.

4.2 Proofs of Theorem 2.11

Now we are in a position to prove Theorem 2.11. We divide the proof into two steps.

Proof. Proof (1). Step 1-1 We are going to use dichotomy inequality to prove the Lipshcitz continuity of the equivalent function \( H \). We claim that

\[
\|H(t, x) - H(t, \bar{x})\| \leq p\|x - \bar{x}\|, \quad p = 1 + \frac{2K^2\bar{\theta}}{1 - K\bar{\theta}}.
\]

By uniqueness \( X(t, (\tau, \xi)) = X(t, (t, X(t, (\tau, \xi)))) \). From Lemma 3.1 it follows that

\[
h(t, (t, \xi)) = -\int_{-\infty}^{t} U(t, s)P(s)f(s, X(s, t, \xi))ds + \int_{t}^{\infty} U(t, s)(I - P(s))f(s, X(s, t, \xi))ds,
\]

which is also equivalent to

\[
P(t)h(t, (t, \xi)) = -\int_{-\infty}^{t} U(t, s)P(s)f(s, X(s, t, \xi))ds,
\]

\[
(I - P(t))h(t, (t, \xi)) = \int_{t}^{\infty} U(t, s)(I - P(s))f(s, X(s, t, \xi))ds.
\]

Thus we get

\[
I_1 := P(t)h(t, (t, \xi)) - P(t)h(t, (t, \bar{\xi})) = \int_{-\infty}^{t} U(t, s)P(s)(f(s, X(s, t, \bar{\xi})) - f(s, X(s, t, \xi)))ds,
\]

\[
I_2 := (I - P(t))h(t, (t, \xi)) - (I - P(t))h(t, (t, \bar{\xi})) = \int_{t}^{\infty} U(t, s)(I - P(s))(f(s, X(s, t, \xi)) - f(s, X(s, t, \bar{\xi})))ds.
\]
In view of Lemma 3.9, for any initial condition on \( P(t_0)(\mathbb{R}^n) \) (or \((I-P(t_0))(\mathbb{R}^n))\) of system (1.1) is bounded on demiaxes \([s, \infty)\) (or \((-\infty, s])\). Then, by condition (2.9), and using Lemma 3.9 (part \( t \leq t_0 \)), we deduce that

\[
\|I_1\| \leq \int_{-\infty}^t K \exp\{-\alpha(t-s)\} \frac{K}{1-K\theta} \exp\{\alpha_2(t-s)\} \| (I-P(t))(\xi - \bar{\xi}) \| r(s) ds
\]

\[
\leq \frac{K^2}{1-K\theta} \| (I-P(t))(\xi - \bar{\xi}) \| \int_{-\infty}^t \exp\{-\alpha_1(t-s)\} r(s) ds,
\]

and similarly, by Lemma 3.9 (part \( t \geq t_0 \)), we have that

\[
\|I_2\| \leq \int_t^\infty K \exp\{\alpha(t-s)\} \frac{K}{1-K\theta} \exp\{-\alpha_2(t-s)\} \| P(t)(\xi - \bar{\xi}) \| r(s) ds
\]

\[
\leq \frac{K^2}{1-K\theta} \| P(t)(\xi - \bar{\xi}) \| \int_t^\infty \exp\{\alpha_1(t-s)\} r(s) ds.
\]

Hence, we conclude that

\[
\|I_1\| + \|I_2\| \leq \frac{K^2}{1-K\theta} \left( \| P(t)(\xi - \bar{\xi}) \| + \| (I-P(t))(\xi - \bar{\xi}) \| \right)
\]

\[
\left( \int_{-\infty}^t \exp\{-\alpha_1(t-s)\} r(s) ds + \int_t^\infty \exp\{\alpha_1(t-s)\} r(s) ds \right).
\]

From (2.10) and the above inequality, it follows that

\[
\|(P(t) + I - P(t))(h(t, (t, \xi)) - h(t, (t, \bar{\xi})))\|
\]

\[
\leq \|I_1\| + \|I_2\| \leq \left( \frac{K^2\theta}{1-K\theta} \right) \left( \| P(t)(\xi - \bar{\xi}) \| + \| (I-P(t))(\xi - \bar{\xi}) \| \right).
\]

By the definition of \( H(t, x) \),

\[
\|H(t, x) - H(t, \bar{x})\| \leq \|x - \bar{x}\| + \frac{K^2\theta}{1-K\theta} \left( \| P(t)(x - \bar{x}) \| + \| (I-P(t))(x - \bar{x}) \| \right)
\]

\[
\leq \left( 1 + \frac{2K^2\theta}{1-K\theta} \right) \|x - \bar{x}\|
\]

\[
\equiv p \|x - \bar{x}\|.
\]

This completes the proof of Step 1-1.

**Step 1-2** We show that there exist positive constants \( q > 0 \) and \( 0 < \beta < 1 \) such that for all \( t, y, \bar{y} \)

\[
\|G(t, y) - G(t, \bar{y})\| \leq q\|y - \bar{y}\|^\beta.
\]
Usually this point is treated with successive approximations. From Lemma 3.2, we know that \( g(t, (\tau, \xi)) \) is a fixed point of the following map \( T \)

\[
(TZ)(t) = \int_{-\infty}^{t} U(t, s) P(s) f(s, Y(s, \tau, \xi) + Z(s)) ds \\
- \int_{t}^{\infty} U(t, s) (I - P(s)) f(s, Y(s, \tau, \xi) + Z(s)) ds.
\] (4.1)

Let \( g_0(t, (\tau, \xi)) \equiv 0 \), and by recursion define

\[
g_{m+1}(t, (\tau, \xi)) = \int_{-\infty}^{t} U(t, s) P(s) f(s, Y(s, \tau, \xi) + g_m(s, (\tau, \xi))) ds \\
- \int_{t}^{\infty} U(t, s) (I - P(s)) f(s, Y(s, \tau, \xi) + g_m(s, (\tau, \xi))) ds.
\]

It is not difficult to show that

\[
g_m(t, (\tau, \xi)) \to g(t, (\tau, \xi)), \quad \text{as} \quad m \to +\infty,
\]

uniformly with respect to \( t, \tau, \eta \).

Note that \( g_0(t, (\tau, \xi)) = g_0(t, (t, Y(t, \tau, \xi))) \). Thus, by induction, it is clear that for all \( m \in \mathbb{N} \), \( g_m(t, (\tau, \xi)) = g_m(t, (t, Y(t, \tau, \xi))) \). Choose \( \lambda > 0 \) sufficiently large and \( \beta > 0 \) sufficiently small such that

\[
\lambda > \frac{3}{1 - \exp\{-a\} + \frac{3}{2(1 - \exp\{-a - M\})}}, \\
\beta < \frac{\alpha}{M + C_\mu}, \\
0 < \frac{2KCr}{1 - \exp\{-(\alpha - M\beta)\}} < \frac{1}{3},
\]

where \( \alpha, K \) are given by (2.3), \( C_\mu, C_r \) are positive constants defined in (2.15) and \( M = \sup_{t \in \mathbb{R}} \|A(t)\| \). Now we first show that if \( 0 < \|\xi - \bar{\xi}\| < 1 \) for all \( m \), we have

\[
\|g_m(t, (t, \xi)) - g_m(t, (t, \bar{\xi}))\| < \lambda\|\xi - \bar{\xi}\|^{\beta}.
\] (4.2)

Obviously, inequality (4.2) holds if \( m = 0 \). Now making the inductive assumption that (4.2) holds. From (4.1), it follows that

\[
g_{m+1}(t, (t, \xi)) - g_{m+1}(t, (t, \bar{\xi})) \\
= \int_{-\infty}^{t} U(t, s) P(s) [f(s, Y(s, t, \xi) + g_m(s, (t, \xi))) - f(s, Y(s, t, \bar{\xi}) + g_m(s, (t, \bar{\xi})))] ds \\
- \int_{t}^{\infty} U(t, s) (I - P(s)) [f(s, Y(s, t, \xi) + g_m(s, (t, \xi))) - f(s, Y(s, t, \bar{\xi}) + g_m(s, (t, \bar{\xi})))] ds \\
\triangleq J_1 + J_2.
\]

\[
\]
We divide $J_1, J_2$ into two parts:

\[
J_1 = \int_{-\infty}^{t-\tau} + \int_{t-\tau}^{t} \triangleq J_{11} + J_{12},
\]

\[
J_2 = \int_{t}^{t+\tau} + \int_{t+\tau}^{\infty} \triangleq J_{21} + J_{22},
\]

where $\tau = \frac{1}{M+C\mu} \ln \frac{1}{\|\xi-\bar{\xi}\|}$. By (2.3), (2.9) and (2.15), we have

\[
\|J_{11}\| \leq \int_{-\infty}^{t-\tau} K \exp\{-\alpha(t-s)\} 2\mu(s)ds
\]

\[
= \sum_{m \in [0, \infty)} \int_{t-\tau-m-1}^{t-\tau-m} 2K\mu(s) \exp\{-\alpha(t-s)\}ds
\]

\[
\leq \sum_{m \in [0, \infty)} 2KC\mu \exp\{-\alpha(\tau+m)\}
\]

\[
\leq 2KC\mu \exp\{-\alpha\}(1 - \exp\{-\alpha\})^{-1}
\]

\[
\leq \frac{2KC\mu}{1 - \exp\{-\alpha\}} \|\xi - \bar{\xi}\| \frac{\alpha}{M+C\mu},
\]

and similarly,

\[
\|J_{22}\| \leq \int_{t+\tau}^{\infty} K \exp\{-\alpha(t-s)\} 2\mu(s)ds
\]

\[
\leq \frac{2KC\mu}{1 - \exp\{-\alpha\}} \|\xi - \bar{\xi}\| \frac{\alpha}{M+C\mu}.
\]

When $0 < \|\xi - \bar{\xi}\| < 1$, $s \in [t-\tau, t]$ and since $\|Y(t, t_0, y) - Y(t, t_0, \bar{y})\| \leq \|y - \bar{y}\| \exp\{M|t-t_0|\}$, we can get

\[
\|Y(s, t, \xi) - Y(s, t, \bar{\xi})\| \leq \|\xi - \bar{\xi}\| \cdot \exp\{M|t-s|\}
\]

\[
\leq \|\xi - \bar{\xi}\| \cdot \exp\{M\tau\}
\]

\[
\leq \|\xi - \bar{\xi}\| \frac{M}{M+C\mu} < 1.
\]

Hence, it is easy to see that

\[
\|g_m(s, (t, \xi)) - g_m(s, (t, \bar{\xi}))\| = \|g_m(s, (s, Y(s, t, \xi))) - g_m(s, (s, Y(s, t, \bar{\xi})))\|
\]

\[
\leq \lambda \|\xi - \bar{\xi}\|^\beta \cdot \exp\{M\beta|t-s|\},
\]
Therefore, we have that

\[ \|J_{12}\| \leq \int_{t-\tau}^{t} K \exp\{-\alpha(t-s)\} r(s) \|\xi - \bar{\xi}\| \]
\[ \cdot \exp\{M(t-s)\} + \lambda \|\xi - \bar{\xi}\|^\beta \cdot \exp\{M\beta(t-s)\} \] ds
\[ = \int_{t-\tau}^{t} K \exp\{(M - \alpha)(t-s)\} r(s) \|\xi - \bar{\xi}\| ds \\
+ \int_{t-\tau}^{t} \lambda K \|\xi - \bar{\xi}\|^\beta \cdot r(s) \cdot \exp\{(M\beta - \alpha)(t-s)\} ds \\
= \sum_{m\in[0,\tau]} K \int_{t-\tau+1}^{t-\tau+m} \exp\{(M - \alpha)(t-s)\} r(s) \|\xi - \bar{\xi}\| ds \\
+ \sum_{m\in[0,\tau]} K\lambda \int_{t-\tau+1}^{t-\tau+m} \exp\{(M\beta - \alpha)(t-s)\} r(s) \|\xi - \bar{\xi}\|^\beta ds \\
\leq \sum_{m\in[0,\tau]} KC_r \exp\{(M - \alpha)(\tau - m)\} \|\xi - \bar{\xi}\| \\
+ \sum_{m\in[0,\tau]} C_r K\lambda \exp\{(M\beta - \alpha)\tau\} \|\xi - \bar{\xi}\|^\beta \exp\{(M\beta - \alpha)(-m - 1)\} \\
\leq C_r K \exp\{(M - \alpha)\tau\} \frac{1}{1 - \exp\{-\alpha(M - \alpha)\}} \|\xi - \bar{\xi}\| \\
+ KC_r \lambda \exp\{(M\beta - \alpha)\tau\} \frac{\exp\{\alpha - M\beta\}(1 - \exp\{\alpha - M\beta\})}{1 - \exp\{\alpha - M\beta\}} \|\xi - \bar{\xi}\|^\beta \\
\leq C_r K \exp\left\{(M - \alpha)\frac{-1}{M + C_\mu} \ln \|\xi - \bar{\xi}\|\right\} \times \frac{1}{1 - \exp\{-(M - \alpha)\}} \|\xi - \bar{\xi}\| \\
+ KC_r \lambda \exp\{(M\beta - \alpha)\tau\} \frac{\exp\{\alpha - M\beta\}\tau}{1 - \exp\{-(\alpha - M\beta)\}} \|\xi - \bar{\xi}\|^\beta \\
= C_r K \|\xi - \bar{\xi}\| \frac{\alpha + C_\mu}{M + C_\mu} \\
\cdot \frac{1}{1 - \exp\{-(M - \alpha)\}} + KC_r \lambda \frac{1}{1 - \exp\{-(\alpha - M\beta)\}} \|\xi - \bar{\xi}\|^\beta.

Note that \(M - \alpha > 0, -\alpha + M\beta < 0\) imply that \(\exp\{(M\beta - \alpha)\tau\} < 1\) and \(\beta < \frac{\alpha + C_\mu}{M + C_\mu}\). Then

\[ \|J_{12}\| \leq KC_r \|\xi - \bar{\xi}\|^\beta \cdot \frac{1}{1 - \exp\{-(M - \alpha)\}} + KC_r \lambda \frac{1}{1 - \exp\{-(\alpha - M\beta)\}} \|\xi - \bar{\xi}\|^\beta \\
= KC_r \left[ \frac{1}{1 - \exp\{\alpha - M\}} + \frac{\lambda}{1 - \exp\{-(\alpha - M\beta)\}} \right] \|\xi - \bar{\xi}\|^\beta.

\[ \|J_{21}\| \leq KC_r \left[ \frac{1}{1 - \exp\{\alpha - M\}} + \frac{\lambda}{1 - \exp\{-(\alpha - M\beta)\}} \right] \|\xi - \bar{\xi}\|^\beta.

27
Hence,

\[
g_{m+1}(t, (t, \xi)) - g_{m+1}(t, (t, \bar{\xi})) \leq \left[ \frac{4KC_{\mu}}{1 - \exp\{-\alpha\}} + \frac{2KC_{r}}{1 - \exp\{\alpha - M\}} + \frac{2KC_{r}\lambda}{1 - \exp\{-(\alpha - M\beta)\}} \right] \|\xi - \bar{\xi}\|^{\beta} \\
\leq \lambda \|\xi - \bar{\xi}\|^{\beta}.
\]

Now by the definition of \(G(t, y)\), if \(0 < \|y - \bar{y}\| < 1\), then we conclude that

\[
\|G(t, y) - G(t, \bar{y})\| \leq \|y - \bar{y}\| + \lambda \|y - \bar{y}\|^{\beta} \leq (1 + \lambda)\|y - \bar{y}\|^{\beta} = q\|y - \bar{y}\|^{\beta}.
\]

Therefore, \(G\) is Hölder continuous. This completes the proof of Step 1-2.

Conflict of interest statement

The authors declare that there is no conflict of interests regarding the publication of this article.

Acknowledgements

This work was jointly supported by Fondecyt project 1170466 and the National Natural Science Foundation of China under Grant (11931016).

References

[1] H. Poincaré, Sur le problème des trois corps et les équations de la dyanamique, Acta Math., 13 (1890) 1–270.

[2] C. Siegel, Iteration of analytic functions, Ann. of Math., 43 (1942) 607–612.

[3] J. Yoccoz, Linéarisation des germes de difféomorphismes holomorphes \(d(C, 0)\), C. R. Acad. Sci. Paris, 36 (1988) 55–58.

[4] P. Hartman, On local homeomorphisms of Euclidean spaces, Bol. Soc. Mat. Mexicana, 5 (1960) 220–241.
[5] D. Grobman, Homeomorphisms of systems of differential equations, *Dokl. Akad. Nauk SSSR*, 128 (1965) 880–881.

[6] J. Palis, On the local structure of hyperbolic points in Banach spaces, *An. Acad. Brasil. Ciênc.*, 40 (1968) 263–266.

[7] C. Pugh, On a theorem of P. Hartman, *Amer. J. Math.*, 91 (1969) 363–367.

[8] P. Bates, K. Lu, A Hartman-Grobman theorem for the Cahn-Hilliard and phase-field equations, *J. Dynam. Differential Equations*, 6 (1994) 101–145.

[9] K. Lu, A Hartman-Grobman theorem for scalar reaction diffusion equations, *J. Differential Equations*, 93 (1991) 364–394.

[10] P. Zgliczyński, Topological shadowing and the Grobman-Hartman theorem, *Topol. Method. Nonl. An.*, 50 (2017), 757-785.

[11] M. Hein, J. Prüss, The Hartman-Grobman theorem for semilinear hyperbolic evolution equations, *J. Differential Equations*, 261 (2016) 4709–4727.

[12] K. Palmer, A generalization of Hartman’s linearization theorem, *J. Math. Anal. Appl.*, 41 (1973) 753–758.

[13] L. Backes, D. Dragučević, K. Palmer, Linearization and Hölder continuity for nonautonomous systems, *J. Differential Equations*, 297 (2021) 536–574.

[14] L. Barreira, C. Valls, A Grobman-Hartman theorem for nonuniformly hyperbolic dynamics, *J. Differential Equations*, 228 (2006) 285–310.

[15] L. Barreira, C. Valls, A Grobman-Hartman theorem for general nonuniform exponential dichotomies, *J. Funct. Anal.*, 257 (2009) 1976–1993.

[16] L. Barreira, C. Valls, Conjugacies between linear and nonlinear non-uniform contractions, *Ergod. Theor. Dyn. Syst.*, 28 (2008) 1–19.

[17] L. Barreira, C. Valls, Conjugacies for linear and nonlinear perturbations of nonuniform behavior, *J. Funct. Anal.*, 253 (2007) 324–358.

[18] Á. Castañeda, I. Huerta, Nonuniform almost reducibility of nonautonomous linear differential equations, *J. Math. Anal. Appl.*, 485 (2020) 123822.
[19] I. Huerta, Linearization of a nonautonomous unbounded system with nonuniform contraction: A spectral approach, *Discrete Contin. Dyn. Syst.*, 40 (2020) 5571–5590.

[20] L. Jiang, Generalized exponential dichotomy and global linearization, *J. Math. Anal. Appl.*, 315 (2006) 474–490.

[21] L. Jiang, Ordinary dichotomy and global linearization, *Nonlinear Anal.*, 70 (2009) 2722–2730.

[22] J. Fenner, M. Pinto, On a Hartman linearization theorem for a class of ODE with impulse effect, *Nonlinear Anal.*, 38 (1999) 307–325.

[23] Y. Xia, X. Chen, V. Romanovski, On the linearization theorem of Fenner and Pinto, *J. Math. Anal. Appl.*, 400 (2013) 439–451.

[24] G. Papaschinopoulos, A linearization result for a differential equation with piecewise constant argument, *Analysis*, 16 (1996) 161–170.

[25] C. Pötzche, Topological decoupling, linearization and perturbation on inhomogeneous time scales, *J. Differential Equations*, 245 (2008) 1210–1242.

[26] A. Reinfelds, L. Sermone, Equivalence of nonlinear differential equations with impulse effect in Banach space, *Latv. Univ. Zint. Raksti.*, 577 (1992) 68–73.

[27] A. Reinfelds, D. Šteinberga, Dynamical equivalence of quasilinear equations, *Int. J. Pure Appl. Math.*, 98 (2015) 355-364.

[28] J. Shi, J. Zhang, The Principle of Classification for Differential Equations, Science Press, Beijing, 2003 (in Chinese).

[29] Y. Xia, R. Wang, K. Kou, D. O’Regan, On the linearization theorem for nonautonomous differential equations, *Bull. Sci. Math.*, 139 (2015) 829–846.

[30] S. Sternberg, Local $C^\alpha$ transformations of the real line, *Duke Math. J.*, 24 (1957) 97–102.

[31] S. Sternberg, Local contractions and a theorem of Poincaré, *Amer. J. Math.*, 79(1957) 809–824.

[32] G. Sell, Smooth Linearization near a fixed point, *Amer. J. Math.*, 107 (1985) 1035–1091.
[33] G. Belitskii, Functional equations and the conjugacy of diffeomorphism of finite smoothness class, *Funct. Anal. Appl.*, 7 (1973) 268–277.

[34] G. Belitskii, V. Rayskinon, The Grobman-Hartman theorem in \( \alpha \)-Hölder class for Banach spaces, preprint.

[35] L. Cuong, T. Doan, S. Siegmund, A Sternberg theorem for nonautonomous differential equations, *J. Dynam. Differential Equations*, 31 (2019) 1279–1299.

[36] D. Dragičević, W. Zhang, W. Zhang, Smooth linearization of nonautonomous difference equations with a nonuniform dichotomy, *Math. Z.*, 292 (2019) 1175–1193.

[37] D. Dragičević, W. Zhang, W. Zhang, Smooth linearization of nonautonomous differential equations with a nonuniform dichotomy, *Proc. London Math. Soc.*, 121 (2020) 32–50.

[38] M. ElBialy, Local contractions of Banach spaces and spectral gap conditions, *J. Funct. Anal.*, 182 (2001) 108–150.

[39] H. Rodrigues, J. Solà-Morales, Linearization of class \( C^1 \) for contractions on Banach spaces, *J. Differential Equations*, 201 (2004) 351–382.

[40] H. Rodrigues, J. Solà-Morales, Smooth linearization for a saddle on Banach spaces, *J. Dynam. Differential Equations*, 16 (2004) 767–793.

[41] H. Rodrigues, J. Solá-Morales, Invertible Contractions and Asymptotically Stable ODE’S that are not \( C^1 \)-Linearizable, *J. Dynam. Differential Equations*, 18 (2006) 961–974.

[42] W. Zhang, W. Zhang, Sharpness for \( C^1 \) linearization of planar hyperbolic diffeomorphisms, *J. Differential Equations*, 257 (2014) 4470–4502.

[43] W. Zhang, W. Zhang, \( \alpha \)-Hölder linearization of hyperbolic diffeomorphisms with resonance, *Ergod. Theor. Dyn. Syst.*, 36 (2016) 310–334.

[44] W. Zhang, W. Zhang, W. Jarczyk, Sharp regularity of linearization for \( C^{1,1} \) hyperbolic diffeomorphisms, *Math. Ann.*, 358 (2014) 69–113.

[45] W. Zhang, K. Lu, W. Zhang, Differentiability of the conjugacy in the Hartman-Grobman Theorem, *Trans. Amer. Math. Soc.*, 369 (2017) 4995–5030.

[46] B. Tan, \( \sigma \)-Hölder continuous linearization near hyperbolic fixed points in \( \mathbb{R}^n \), *J. Differential Equations*, 162 (2000) 251–269.
[47] J. Shi, K. Xiong, On Hartman’s linearization theorem and Palmer’s linearization theorem, *J. Math. Anal. Appl.*, 192 (1995) 813–832.

[48] R. Naulin, M. Pinto, Admissible perturbations of exponential dichotomy roughness. *Nonlinear Anal.*, 31 (1998) 559–571.

[49] A. Coronel, C. Maulén, M. Pinto, D. Sepúlveda, Dichotomies and asymptotic equivalence in alternately advanced and delayed differential systems, *J. Math. Anal. Appl.*, 45 (2017) 1434–1458.

[50] M. Pinto, Perturbations of asymptotically stable differential systems *Analysis*, 4 (1984) 161–175.

[51] M. Pinto, Asymptotic integration of a system resulting from the perturbation of an $h$–system, *J. Math. Anal. Appl.*, 131 (1988) 194–216.

[52] W. Coppel, *Dichotomies in Stability Theory*, Lect. Notes Math., vol. 629, Springer, Berlin/New York (1978).

[53] J. Meiss, *Differential Dynamical Systems*, Society for Industrial and Applied Mathematics, (2007).

[54] W. Zhang, Generalized exponential dichotomies and invariant manifolds for differential equations, *Adv. Math. Chin.*, 22 (1993) 1-45.

[55] J. Chu, F. Liao, S. Siegmund, Y. Xia, W. Zhang, Nonuniform dichotomy spectrum and reducibility for nonautonomous equations, *Bull. Sci. Math.*, 139(2015), 538-557.