ASYMPTOTIC ANALYSIS OF A FAMILY OF NON-LOCAL FUNCTIONALS ON SETS

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Abstract. We study the asymptotic behavior of a family of functionals which penalize a short-range interaction of convolution type between a finite perimeter set and its complement. We first compute the pointwise limit and we obtain a lower estimate on more regular sets. Finally, some examples are discussed.

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1. Introduction

In this paper we study the asymptotic behavior, as $\varepsilon \to 0$, of the family of functionals

$$\mathcal{F}_\varepsilon(E) = \frac{1}{\varepsilon} \int_{E \cap \Omega} f(G_\varepsilon * \chi_{E \cap \Omega}) \, dx,$$

where $\Omega \subset \mathbb{R}^N$ is open and bounded, $N > 1$, $E$ is a set of finite perimeter in $\Omega$, $f$ is given and $G_\varepsilon(z) = \frac{1}{\varepsilon^N} G(\frac{z}{\varepsilon})$ where $G$ is a suitable kernel. Our analysis has been inspired by a paper by Miranda et al. in [8] where the case $f(t) = t$ is considered and $G$ is the Gauss-Weierstrass kernel, namely $G(z) = \frac{1}{(4\pi)^{N/2}} e^{-|z|^2/4}$ (see also [6, 7] for smoother sets and [1] for similar convolution approximation). More precisely, in [8] it is proven that the pointwise limit is, up to a constant, the perimeter of $E$ in $\Omega$. A more general kernel $G$ has been investigated, in the context of optimal partition problems, by Esedoḡlu and Otto [5] where $G$ is smooth and non-negative, radially symmetric and satisfying the following conditions:

$$\int_{\mathbb{R}^N} G(z) \, dz = 1, \quad \int_{\mathbb{R}^N} |z| G(z) \, dz < +\infty, \quad |\nabla G(z)| \lesssim G\left(\frac{z}{2}\right), \quad \nabla G(z) \cdot z \leq 0.$$

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On the other hand, as in [5], Esedoḡlu and Otto consider only the case $f(t) = t$, but they prove a complete $\Gamma$-convergence result for the family $\{F_\varepsilon\}_{\varepsilon > 0}$ on finite perimeter sets with respect to the strong $L^1$-convergence. We point out that for the simpler class of functionals with $f(t) = t$ there is a much simpler proof of gamma convergence in $L^1$ subsequently given by Esedoḡlu and Jacobs in [4]. In particular, they deal with more general convolution kernels, which in particular can be non-radially symmetric, thus including anisotropy, and even change sign to a certain extent, which turns out to be necessary for the approximation of certain anisotropies.

A very similar result has been obtained more recently by Berendsen and Pagliari [3]. As far as we know, the last result is due to Pagliari [9] where he essentially remove the radial symmetry of $G$ and he obtain, as limit, an anisotropic perimeter.

In this paper we try to investigate the general situation. We assume that $G$ is even, non-negative, supported on the unit closed ball and with $\int_{\mathbb{R}^N} G(z) \, dz = 1$. First of all we are able to compute the pointwise limit, as $\varepsilon \to 0$, of $F_\varepsilon(E)$ whenever $f$ is $C^1$, non-decreasing and $f(0) = 0$. It turns out (see Thm. 3.1) that for any $E \subset \mathbb{R}^N$ with finite perimeter in $\Omega$

$$\lim_{\varepsilon \to 0} F_\varepsilon(E) = \int_{\partial^* E \cap \Omega} \int_0^1 f \left( \int_{\{z : \nu_E(x) \geq t\}} G(z) \, dz \right) \, dt \, d\mathcal{H}^{N-1}(x)$$

where $\partial^* E$ is the reduced boundary of $E$ and $\nu_E(x)$ is the outer unit normal at $E$. In view to have a $\Gamma$-convergence result we investigate also the lower estimate. Unfortunately, the technique of Esedoḡlu and Otto [5] does not work in our situation: it is crucial for them to switch the order of integration, that is impossible for us since we have $f$ between the exterior integral and the convolution one. It seems that this difficulty cannot be easily overcome in the general situation. We are able to show (see Thm. 3.2) a $\Gamma$-liminf inequality only on graphs of $C^1$ functions with respect to the $C^1$-uniform convergence. Actually, it is easy to generalize such a inequality in the case of sets which are locally graphs of $C^1$ functions with respect to a suitable convergence (see Rem. 3.3). Finally, we also prove (see Thm. 3.6) that if $f$ is also convex, then the pointwise limit is lower semicontinuous with respect to the strong $L^1$-convergence, which suggests that for $f$ convex the $\Gamma$-limit in the strong $L^1$-convergence should be the pointwise limit. At the end of the paper we will also discuss some examples.

2. Notation and preliminaries

2.1. Notation

In what follows $N \in \mathbb{N}$ with $N \geq 1$. For any $r > 0$ and $x \in \mathbb{R}^N$ the notation $B^d_r(x)$ stands for the open ball in $\mathbb{R}^d$ centered at $x$ with radius $r$, while $S^{N-1} = \partial B^d_1(0)$. If $A \subset \mathbb{R}^N$ we also denote by $\mathcal{H}^k(A)$ the Hausdorff measure of $A$ of dimension $k \in \{0, 1, \ldots, N\}$ ($\mathcal{H}^0$ is the counting measure). If $A_h, A$ are measurable subsets of $\mathbb{R}^N$, then $A_h \to A$ in $L^1(\mathbb{R}^N)$ (or $L^1_{\text{loc}}(\mathbb{R}^N)$) means that $\chi_{A_h} \to \chi_A$ in $L^1(\mathbb{R}^N)$ (respectively $L^1_{\text{loc}}(\mathbb{R}^N)$). Finally, for any $A \subset \mathbb{R}^N$ we let $A^c = \mathbb{R}^N \setminus A$.

2.2. Finite perimeter sets

We recall some notion on finite perimeter sets in euclidean space; for details we refer to [2]. Let $\Omega$ be an open subset of $\mathbb{R}^N$. A measurable set $E \subset \mathbb{R}^N$ is said to be a set of finite perimeter in $\Omega$ if

$$\mathcal{P}(E, \Omega) = \sup \left\{ \int_E \text{div} X(x) \, dx : X \in C^1_c(\Omega; \mathbb{R}^N), \|X\|_\infty \leq 1 \right\} < +\infty.$$
The quantity $\mathcal{P}(E, \Omega)$ is called \textit{perimeter of $E$ in $\Omega$}. Finite perimeter sets have nice boundary in a measure theoretical sense. Precisely, one can define a subset of $E$ as the set of points $x$ where there exists a unit vector $\nu_E(x)$ such that:

$$
\frac{x - E}{r} \to \{ y \in \mathbb{R}^N : y \cdot \nu_E(x) \geq 0 \}, \quad \text{in } L^1_{\text{loc}}(\mathbb{R}^N) \text{ as } r \to 0, \tag{2.1}
$$

and which is referred to as the \textit{outer normal to $E$ at $x$}. The set where $\nu_E(x)$ exists is called the \textit{reduced boundary of $E$} and is denoted by $\partial^* E$. It turns out that, for any $E$ set of finite perimeter in $\Omega$, we have

$$
\mathcal{P}(E, \Omega) = \mathcal{H}^{N-1}(\partial^* E \cap \Omega). \tag{2.2}
$$

Finite perimeter sets satisfy good properties for the Calculus of Variations: for instance, if $E_h, E$ have finite perimeter in $\Omega$ and $E_h \overset{L^1}{\to} E$, then

$$
\mathcal{P}(E, \Omega) \leq \liminf_{h \to +\infty} \mathcal{P}(E_h, \Omega).
$$

### 3. Setting of the problem and main results

Let $N > 1$, let $G : \mathbb{R}^N \to [0, +\infty)$ be of class $C^\infty$ such that

$$
supp G = \overline{B_1^N(0)}, \quad G(-x) = G(x), \quad \int_{\mathbb{R}^N} G(x) \, dx = 1.
$$

For any $\varepsilon > 0$ and for any $x \in \mathbb{R}^N$, let

$$
G_\varepsilon(x) = \frac{1}{\varepsilon^N} G \left( \frac{x}{\varepsilon} \right).
$$

We consider a continuous and non-decreasing function $f : [0, +\infty) \to \mathbb{R}$ with $f(0) = 0$. Let $\Omega \subset \mathbb{R}^N$ be open bounded. We denote by $\mathcal{P}_N(\Omega)$ the set of all sets of finite perimeter in $\Omega$. For any $\varepsilon > 0$, we introduce the functional $\mathcal{F}_\varepsilon : \mathcal{P}_N(\Omega) \to [0, +\infty)$ defined by

$$
\mathcal{F}_\varepsilon(E) = \frac{1}{\varepsilon} \int_{E \cap \Omega} f(G_\varepsilon \ast \chi_{E \cap \Omega}) \, dx. \tag{3.1}
$$

In order to state our main results, we introduce the function $\theta : S^{N-1} \to [0, +\infty)$ given by

$$
\theta(\nu) = \int_0^1 f \left( \int_{\{ x \cdot \nu \geq t \}} G(x) \, dx \right) \, dt. \tag{3.2}
$$

Let $\mathcal{F} : \mathcal{P}_N(\Omega) \to [0, +\infty)$ be the functional given by

$$
\mathcal{F}(E) = \int_{\partial^* E \cap \Omega} \theta(\nu_E(x)) \, d\mathcal{H}^{N-1}(x).
$$
Our first main result concerns the pointwise limit of $\mathcal{F}_\varepsilon$ on $\mathcal{P}_N(\Omega)$.

**Theorem 3.1. (Pointwise limit)** Assume $f$ of class $C^1$. Let $E \in \mathcal{P}_N(\Omega)$. Then

$$\lim_{\varepsilon \to 0} \mathcal{F}_\varepsilon(E) = \mathcal{F}(E).$$

On the other hand, we are also able to prove a lower estimate on graphs.

**Theorem 3.2. (Lower estimate)** Let $D \subset \mathbb{R}^{N-1}$ be open and bounded with Lipschitz boundary, let $u_h, u \in C^1(D)$, with $u_h, u > 0$ on $D$ such that $u_h \to u$ uniformly in $C^1(D)$. Let $E_h, E$ be given by

$$E_h = \{(x, y) \in \mathbb{R}^{N-1} \times \mathbb{R} : x \in D, 0 \leq y \leq u_h(x)\},$$

$$E = \{(x, y) \in \mathbb{R}^{N-1} \times \mathbb{R} : x \in D, 0 \leq y \leq u(x)\}.$$

Then, for any positive infinitesimal sequence $(\varepsilon_h)$ it holds

$$\liminf_{h \to +\infty} \mathcal{F}_{\varepsilon_h}(E_h) \geq \mathcal{F}(E).$$

**Remark 3.3.** It is not difficult to see that Theorem 3.2 can be generalized to uniformly $C^1,1$-regular sets in $\Omega$ with respect to a suitable notion of uniform convergence. Precisely, a set $E \subset \mathbb{R}^N$ is said to be uniformly $C^1,1$-regular set in $\Omega$ if there exist $L, \delta > 0$ such that for every $x \in \partial E \cap \Omega$ there exist $D_x \subseteq \mathbb{R}^{N-1}$ open and a function $u_x \in C^1(D_x)$ such that:

- $\partial E \cap \Omega \cap B^N_\delta(x)$ is the graph of $u_x$;
- $\|\nabla u_x\|_{\infty} \leq L$.

On the set of all uniformly $C^1,1$-regular sets in $\Omega$ we put a convergence of sequences. Precisely, we say that $E_h$ converges to $E$ if there exist $\delta, L > 0$ such that for every $x \in \partial E \cap \Omega$ there exist $D^x \subseteq \mathbb{R}^{N-1}$ open and functions $u^x_h, u^x \in C^1(D^x)$ such that:

- $\partial E_h \cap \Omega \cap B_\delta^N(x), \partial E \cap \Omega \cap B_\delta^N(x)$ are the graphs of $u^x_h, u^x$ respectively;
- $\|\nabla u^x_h\|_{\infty} \leq L$ and $\|\nabla u^x\|_{\infty} \leq L$;
- $u^x_h \to u^x$ uniformly in $C^1(D^x)$.

It is easy to see that with respect to this type of convergence the lower estimate

$$\liminf_{h \to +\infty} \mathcal{F}_{\varepsilon_h}(E_h) \geq \mathcal{F}(E)$$

follows as a simple consequence of Theorem 3.2.

Combining Theorem 3.1 with Theorem 3.2 and Remark 3.3 we eventually obtain a $\Gamma$-convergence result.

**Corollary 3.4.** The family $\{\mathcal{F}_\varepsilon\}_{\varepsilon > 0}$ $\Gamma$-converges to $\mathcal{F}$ as $\varepsilon \to 0$ on uniformly $C^1,1$-regular sets with respect to the convergence introduced in Remark 3.3.

**Remark 3.5.** We do not expect compactness of equibounded sequences of uniformly $C^1,1$-regular sets. Nevertheless, at least if $f(t) \geq mt$ for some $m > 0$, equibounded sequences are compact in $L^1$. Indeed, if $(\varepsilon_h)$ is a
positive and infinitesimal sequence and \((E_h)\) be a sequence in \(\mathcal{P}_N(\Omega)\) with \(\mathcal{F}_{E_h}(E_h) \leq c\) for some \(c \geq 0\), we get

\[
c \geq \mathcal{F}_{E_h}(E_h) \geq \frac{m}{\varepsilon} \int_{E \cap \Omega} G_x \ast \chi_{E \cap \Omega} \, dx
\]

and the compactness follows by Lemma A.4 of [5] (see also [1], Thm. 3.1).

The next and last result suggests that the \(\Gamma\)-limit on \(\mathcal{P}_N(\Omega)\) of the family \((\mathcal{F}_\varepsilon)_{\varepsilon > 0}\) with respect to the \(L^1\)-convergence could be really \(\mathcal{F}\), at least if \(f\) is convex.

**Theorem 3.6.** If \(f\) is convex then the functional \(\mathcal{F}: \mathcal{P}_N(\Omega) \to \mathbb{R}\) is lower semicontinuous with respect to the \(L^1\)-topology.

### 4. The Pointwise Limit

In this section we prove Theorem 3.1. The main idea comes from the technique used in [8]. We divide the proof in some steps.

**Step 1.** We claim that for any \(E \in \mathcal{P}_N\) we have

\[
\mathcal{F}_\varepsilon(E) = \frac{1}{\varepsilon} \int_{\partial^* E} \int_0^\varepsilon X(\eta, x) \cdot \nu_E(x) \, d\eta \, d\mathcal{H}^{N-1}(x),
\]

where for any \(\eta > 0\) and for any \(x \in \partial^* E\)

\[
X(\eta, x) = \frac{1}{\eta^N} \int_{E^c} f'(G_\eta \ast \chi_E(y)) G \left( \frac{y - x}{\eta} \right) \frac{y - x}{\eta} \, dy.
\]

For any \(\eta > 0\) and any \(y \in \mathbb{R}^N\) we have, using the Gauss-Green formula (2.2),

\[
\frac{d}{d\eta} f(G_\eta \ast \chi_E(y)) \\
= -f'(G_\eta \ast \chi_E(y)) \frac{1}{\eta^{N+1}} \int_{\mathbb{R}^N} \left( NG \left( \frac{y - x}{\eta} \right) - \nabla G \left( \frac{y - x}{\eta} \right) \cdot \frac{y - x}{\eta} \right) \chi_E(x) \, dx \\
= f'(G_\eta \ast \chi_E(y)) \frac{1}{\eta^N} \int_{\mathbb{R}^N} \text{div}_x \left( G \left( \frac{y - x}{\eta} \right) \frac{y - x}{\eta} \right) \chi_E(x) \, dx \\
= f'(G_\eta \ast \chi_E(y)) \frac{1}{\eta^N} \int_{\partial^* E} G \left( \frac{y - x}{\eta} \right) \frac{y - x}{\eta} \cdot \nu_E(x) \, d\mathcal{H}^{N-1}(x).
\]

Now notice that, since \(G_\varepsilon \ast \chi_E \to \chi_E\) in \(L^1(\mathbb{R}^N)\) as \(\varepsilon \to 0\), we can say that for any \(y \in \mathbb{R}^N\)

\[
f(G_\varepsilon \ast \chi_E(y)) - f(\chi_E(y)) = \int_0^\varepsilon \frac{d}{d\eta} f(G_\eta \ast \chi_E(y)) \, d\eta.
\]
from which we get, using the fact that $f(0) = 0$,

$$
\mathcal{F}_\varepsilon(E) = \frac{1}{\varepsilon} \int_{E^c} f(G_\varepsilon \ast \chi_E(y)) - f(\chi_E(y)) \, dy \\
= \frac{1}{\varepsilon} \int_{E^c} \int_0^\varepsilon \frac{d}{d\eta} f(G_\eta \ast \chi_E(y)) \, d\eta \, dy \\
= \frac{1}{\varepsilon} \int_{\partial^* E} \int_0^\varepsilon \frac{1}{\varepsilon N} \int_{E^c} f'(G_\eta \ast \chi_E(y)) G\left(\frac{y-x}{\eta}\right) \frac{y-x}{\eta} \, dy \, d\eta \cdot \nu_E(x) \, d\mathcal{H}^{N-1}(x) \\
= \frac{1}{\varepsilon} \int_{\partial^* E} \int_0^\varepsilon X(\eta, x) \cdot \nu_E(x) \, d\eta \, d\mathcal{H}^{N-1}(x)
$$
hence (4.1).

**Step 2.** We claim that for any $x \in \partial^* E$ we have

$$
\lim_{\varepsilon \to 0} X(\varepsilon, x) = \int_{\{z \cdot \nu_E(x) \geq 0\}} f'\left(\int_{\{(v-z) \cdot \nu_E(x) \geq 0\}} G(v) \, dv\right) G(z) z \, dz.
$$

First of all we have

$$
X(\varepsilon, x) = \frac{1}{\varepsilon N} \int_{E^c} f'(G_\varepsilon \ast \chi_E(y)) G\left(\frac{y-x}{\varepsilon}\right) \frac{y-x}{\varepsilon} \, dy \\
= \frac{1}{\varepsilon N} \int_{E^c} f'\left(\int_{E} G\left(\frac{y-w}{\varepsilon}\right) \, dw\right) G\left(\frac{y-x}{\varepsilon}\right) \frac{y-x}{\varepsilon} \, dy.
$$

Performing first the change of variable $y = x + \varepsilon z$ and then $w = x + \varepsilon z - \varepsilon v$, we obtain

$$
X(\varepsilon, x) = \frac{1}{\varepsilon N} \int_{E^c} f'\left(\int_{E} G\left(\frac{x+\varepsilon z-w}{\varepsilon}\right) \, dw\right) G(z) z \, dz \\
= \frac{1}{\varepsilon N} \int_{E^c} f'\left(\int_{\frac{x-E}{\varepsilon} + z} G(v) \, dv\right) G(z) z \, dz.
$$

Passing to the limit as $\varepsilon \to 0$ using (2.1) and applying the Dominated convergence Theorem we easily get (4.2).

**Step 3.** We claim that for any $x \in \partial^* E$ it holds

$$
\int_{\{z \cdot \nu_E(x) \geq 0\}} f'\left(\int_{\{(v-z) \cdot \nu_E(x) \geq 0\}} G(v) \, dv\right) G(z) z \, dz \cdot \nu_E(x) = \theta(\nu_E(x)).
$$

First of all observe any $z \in \mathbb{R}^N$ with $z \cdot \nu_E(x) \geq 0$ can be written in a unique way as $z = \bar{z} + t \nu_E(x)$ with $\bar{z} \cdot \nu_E(x) = 0$ and $t \geq 0$. In particular, $z \cdot \nu_E(x) = (\bar{z} + t \nu_E(x)) \cdot \nu_E(x) = t$. Moreover, since $G$ is supported on
$B_1^N(0)$, we can consider $t \in [0, 1]$ obtaining

$$
\int_{\{z \cdot \nu_E(x) \geq 0\}} f'(\int_{\{(v-z) \cdot \nu_E(x) \geq 0\}} G(v) \, dv) G(z) z \cdot \nu_E(x) \\
= \int_{\{z \cdot \nu_E(x) \geq 0\}} f'(\int_{\{(v-z) \cdot \nu_E(x) \geq 0\}} G(v) \, dv) G(z) \, dz \\
= \int_0^1 \int_{\{\tilde{z} \cdot \nu_E(x) = 0\}} f'(\int_{\{v \cdot \nu_E(x) \geq t\}} G(v) \, dv) (\tilde{z} + t \nu_E(x)) \, t \, dt \, d\tilde{z} \\
= \int_0^1 f'(\int_{\{v \cdot \nu_E(x) \geq t\}} G(v) \, dv) \int_{\{\tilde{z} \cdot \nu_E(x) = t\}} G(z) \, d\mathcal{H}^{N-1}(z) \, t \, dt.
$$

Finally, we remark that

$$
\frac{d}{dt} \int_{\{v \cdot \nu_E(x) \geq t\}} G(v) \, dv \\
= \lim_{h \to 0} \frac{1}{h} \left( \int_{\{v \cdot \nu_E(x) \geq t+h\}} G(v) \, dv - \int_{\{v \cdot \nu_E(x) \geq t\}} G(v) \, dv \right) \\
= - \lim_{h \to 0} \frac{1}{h} \int_{\{t \leq v \cdot \nu_E(x) \leq t+h\}} G(v) \, dv \\
= - \int_{\{v \cdot \nu_E(x) = t\}} G(v) \, dv.
$$

Integrating by parts we finally get

$$
\int_0^1 f' \left( \int_{\{v \cdot \nu_E(x) \geq t\}} G(v) \, dv \right) \int_{\{z \cdot \nu_E(x) = t\}} G(z) \, d\mathcal{H}^{N-1}(z) \, t \, dt \\
= - \int_0^1 \frac{d}{dt} f \left( \int_{\{v \cdot \nu_E(x) \geq t\}} G(v) \, dv \right) \, t \, dt \\
= - f \left( \int_{\{v \cdot \nu_E(x) \geq t\}} G(v) \, dv \right) \bigg|_0^1 + \int_0^1 f \left( \int_{\{v \cdot \nu_E(x) \geq t\}} G(v) \, dv \right) \, dt \\
= \theta(\nu_E(x))
$$

where $\theta$ has been introduced in (3.2). This concludes the proof of (4.3).

**Step 4.** We easily conclude. Using (4.1), (4.2), (4.3), De l’Hôpital rule and the Dominated convergence Theorem.
we deduce that

$$\lim_{\varepsilon \to 0} \mathcal{F}_\varepsilon(E) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{\partial^* E} \int_0^\varepsilon X(\eta, x) \cdot \nu_E(x) \, d\eta \, d\mathcal{H}^{N-1}(x)$$

$$= \int_{\partial^* E} \lim_{\varepsilon \to 0} X(\eta, x) \cdot \nu_E(x) \, d\eta \, d\mathcal{H}^{N-1}(x)$$

$$= \int_{\partial^* E} \theta(\nu_E(x)) \, d\mathcal{H}^{N-1}(x)$$

and this ends the proof of Theorem 3.1.

\[\square\]

**Remark 4.1.** We remark that if \(E\) is a \(C^{1,1}\)-regular set in \(\Omega\) then the computation of the pointwise limit is easier. Indeed, for such sets the following geometric property holds true (for details see [10], Sect. I.2): there exists \(r > 0\) such that the map

\[\Psi_r : \partial E \times [0, r] \to \{y \in E^c : d(y, \partial E) \leq r\}, \quad \Psi_r(x) = x + t\nu_E(x)\]

is a \(C^{1,1}\)-diffeomorphism. Thus, performing change of variable \(x = y - \varepsilon z\) we have

\[\mathcal{F}_\varepsilon(E) = \frac{1}{\varepsilon} \int_{\{y \in E^c : d(y, \partial E) \leq \varepsilon\}} f \left( \frac{1}{\varepsilon} \int_E G \left( \frac{y - x}{\varepsilon} \right) \, dx \right) \, dy \]

\[= \frac{1}{\varepsilon} \int_{\{y \in E^c : d(y, \partial E) \leq \varepsilon\}} f \left( \int_{\frac{y - x}{\varepsilon}} G(z) \, dz \right) \, dy \]

\[= \frac{1}{\varepsilon} \int_{\Psi_r(\partial E \times [0, \varepsilon])} f \left( \int_{\frac{y - x}{\varepsilon}} G(z) \, dz \right) \, dy.\]

For any \((x, t) \in \partial E \times [0, \varepsilon]\) let \(J_\varepsilon(x, t) = |\det D\Psi_\varepsilon(x)|\). Then, using also \(t = \varepsilon s\),

\[\mathcal{F}_\varepsilon(E) = \frac{1}{\varepsilon} \int_{\partial E} \int_0^\varepsilon f \left( \int_{\frac{y - x}{\varepsilon} + \frac{t}{\varepsilon} \nu_E(x)} G(z) \, dz \right) J_\varepsilon(x, t) \, d\mathcal{H}^{N-1}(x) \, dt \]

\[= \int_{\partial E} \int_0^1 f \left( \int_{\frac{y - x}{\varepsilon} + \frac{s}{\varepsilon} \nu_E(x)} G(z) \, dz \right) J_\varepsilon(x, \varepsilon s) \, d\mathcal{H}^{N-1}(x) \, ds.\]

Since the regularity of \(E\) we have

\[\lim_{\varepsilon \to 0} J_\varepsilon(x, \varepsilon s) = 1\]

from which, applying again (2.1),

\[\lim_{\varepsilon \to 0} \mathcal{F}_\varepsilon(E) = \int_{\partial E} \int_0^1 f \left( \int_{\{z : G(z) \geq t\}} G(z) \, dz \right) \, dt \, d\mathcal{H}^{N-1}(x) = \mathcal{F}(E).\]
5. The lower estimate

In this section we will prove our second main result, that is Theorem 3.2. First of all at any $x \in D$ we let

$$
\nu_h(x) = \frac{(-\nabla u_h(x), 1)}{\sqrt{1 + |\nabla u_h(x)|^2}}.
$$

It turns out that $\nu_h(x)$ is the exterior unit normal to $\partial^* E_h$ at $(x, u_h(x))$. For any $\eta > 0$ small enough

$$
D^\eta = \{x \in D : d(x, \partial D) > \eta\}.
$$

It turns out that $D^\eta \not\supset D$ in $L^1$ as $\eta \to 0^+$. If $z \in \mathbb{R}^N$ we will use the notation $z = (\bar{z}, z^N)$. We now divide the proof into several steps.

**STEP 1:** We claim that for any $\sigma > 0$, for any $x \in D^{3\sigma}$ and for any $h \in \mathbb{N}$ with $\varepsilon_h < \sigma$ we have

$$
\overline{B^{N-1}_2(0)} \subset \frac{x - D^\sigma}{\varepsilon_h}.
$$

Indeed, $x \in D^{3\sigma}$ means that $\overline{B^{N-1}_{2\sigma}(x)} \subset D^\sigma$. If now $z \in \mathbb{R}^{N-1}$ and $|z| \leq 2$ then $|x - \varepsilon_h z - x| \leq 2\varepsilon_h < 2\sigma$ which implies that $x - \varepsilon_h z \in D^\sigma$ and then (5.1).

**STEP 2:** For any $x \in D^{3\sigma}$, $s \in [0, 1]$ and $\xi \in \mathbb{R}^{N-1}$ we let

$$
a_h(x, s, \xi) = \frac{u_h(x) - u_h(x + \varepsilon_h s \nu_h(x) - \varepsilon_h \xi)}{\varepsilon_h} + s \nu_h(x)^N.
$$

We claim that

$$
\lim_{h \to +\infty} a_h(x, s, \xi) = \nabla u(x) \cdot (\xi - s \nu(x)) + s \nu(x)^N.
$$

Indeed,

$$
\begin{align*}
&u_h(x + \varepsilon_h s \nu_h(x) - \varepsilon_h \xi) - u_h(x) \\
&= \frac{1}{\varepsilon_h} \int_0^{\varepsilon_h} \frac{d}{dt} u_h(x + t(s \nu_h(x) - \xi)) \, dt \\
&= \frac{1}{\varepsilon_h} \int_0^{\varepsilon_h} \nabla u_h(x + t(s \nu_h(x) - \xi)) \cdot (s \nu_h(x) - \xi) \, dt \\
&= \frac{1}{\varepsilon_h} \int_0^{\varepsilon_h} (\nabla u(x + t(s \nu_h(x) - \xi)) - \nabla u(x + t(s \nu_h(x) - \xi))) \cdot (s \nu_h(x) - \xi) \, dt \\
&\quad + \frac{1}{\varepsilon_h} \int_0^{\varepsilon_h} (\nabla u(x + t(s \nu(x) - \xi)) - \nabla u(x + t(s \nu(x) - \xi))) \cdot (s \nu_h(x) - \xi) \, dt \\
&\quad + \frac{1}{\varepsilon_h} \int_0^{\varepsilon_h} \nabla u(x + t(s \nu(x) - \xi)) \cdot (s \nu_h(x) - \xi) \, dt \\
&=: I_1 + I_2 + I_3.
\end{align*}
$$
Concerning the first integral, we have

\[ |I_1| \leq (s + |\xi|) \| \nabla u_h - \nabla u \|_\infty \to 0 \text{ as } h \to +\infty. \]

On the other hand, if \( L \) is the Lipschitz constant of \( \nabla u \), we get

\[ I_2 \leq L(s + |\xi|) \| \nabla u_h - \nabla u \|_\infty \to 0 \text{ as } h \to +\infty. \]

Finally, for the third integral, let \( g(t) = \nabla u(x + t(s\nu(x) - \xi)) \). Then \( g \) is continuous, hence

\[ \lim_{h \to +\infty} \frac{1}{\varepsilon_h} \int_0^{\varepsilon_h} g(t) \, dt = g(0) \]

from which

\[ \lim_{h \to +\infty} I_3 = \nabla u(x) \cdot (s\nu(x) - \xi) \]

as claimed.

**Step 3:** Let \( M = \sup_h \| u_h \|_\infty \) and let \( \sigma \in (0, M/2) \). We claim that for any \( h \in \mathbb{N} \) with \( \varepsilon_h < \sigma \) it holds

\[ \mathcal{F}_{\varepsilon_h}(E_h) \geq \int_{D^3} \int_0^{1} f \left( \int_{B_{1}^{n-1}(0)} \int_{u_h(x,s,\xi)}^{1} G(\xi, \eta) \, d\eta \, d\xi \right) \, ds \sqrt{1 + |\nabla u_h(x)|^2} \, dx. \]  

(5.3)

Indeed, first of all notice that

\[ \{ z \in E^c_h : B_{\varepsilon_h}(z) \cap E_h \neq \emptyset \} \supset \{ (x - r\nu_h(x), u_h(x) + r\nu_h(x)^N) : x \in D^\sigma, r \in (0, \varepsilon_h) \}, \]

As a consequence,

\[ \mathcal{F}_{\varepsilon_h}(E_h) \geq \int_{D^3} \int_0^{\varepsilon_h} f \left( \int_{B_{1}^{n-1}(0)} \int_{u_h(x,s,\xi)}^{1} G(\xi, \eta) \, d\eta \, d\xi \right) \, dr \sqrt{1 + |\nabla u_h(x)|^2} \, dx. \]

We concentrate now on the term \( G_{\varepsilon_h} \chi_{E_h}(x - r\nu_h(x), u_h(x) + r\nu_h(x)^N) \) and we rewrite it in a suitable way by performing some changes of variables. First of all, by noticing that \( E_h = \{(z, w) \in D \times \mathbb{R} : 0 \leq w \leq u_h(z)\} \) we have

\[ G_{\varepsilon_h} \chi_{E_h}(x - r\nu_h(x), u_h(x) + r\nu_h(x)^N) \geq \int_{D^\sigma} \frac{1}{\varepsilon_h^n} \int_0^{u_h(z)} G \left( \frac{x - r\nu_h(x) - z}{\varepsilon_h}, \frac{u_h(x) + r\nu_h(x)^N - w}{\varepsilon_h} \right) \, dw \, dz. \]

We now perform the change of variables in the following order:

\[ \eta = \frac{u_h(x) + r\nu_h(x)^N - w}{\varepsilon_h}, \quad \xi = \frac{x + r\nu_h(x) - z}{\varepsilon_h}. \]
We obtain
\[ G_{\varepsilon h} \ast \chi E_h(x - r\nu_h(x), u_h(x) + r\nu_h(x)^N) \geq \int_{x - r\nu_h(x) + D\sigma}^{x_h(x) + r\nu_h(x)} \int_{a_h(x, r/\varepsilon h, \xi)}^{u_h(x) + r\nu_h(x)} G(\xi, \eta) \, d\eta \, d\xi. \]

Recalling that \( f \) is non-decreasing and operating the change of variable \( r = \varepsilon h s \) we arrive to
\[ F_{\varepsilon h}(E_h) \geq \int_{D^{3\sigma}} \int_{0}^{1} f \left( \int_{x - D\sigma + s\nu_h(x)}^{u_h(x) + s\nu_h(x)} G(\xi, \eta) \, d\eta \right) \, ds \sqrt{1 + |\nabla u_h(x)|^2} \, dx. \]

Now, since (5.1) we deduce that for any \( x \in D^{3\sigma} \) and for any \( s \in [0, 1] \)
\[ \frac{x - D\sigma}{\varepsilon h} + s\nu_h(x) \supset B_2^{N-1}(0) + s\nu_h(x) \supset B_1^{N-1}(0). \]

Moreover, using \( \sigma < M/2 \) we get also
\[ \frac{u_h(x)}{\varepsilon h} + s\nu_h(x)^N > 1. \]

As a consequence, recalling that \( G \) is supported on \( B_1^{N}(0) \) we obtain (5.3).

**STEP 4:** Passing to the limit as \( h \to +\infty \) in (5.3), using Fatou’s Lemma (5.2) and the Dominated convergence Theorem we obtain
\[ \liminf_{h \to +\infty} F_{\varepsilon h}(E_h) \geq \int_{D} \int_{0}^{1} f \left( \int_{B_2^{N-1}(0)} \int_{\nu u(x) \cdot (\xi - s\nu(x)) + s\nu(x)^N} G(\xi, \eta) \, d\eta \right) \, d\xi \, ds \sqrt{1 + |\nabla u(x)|^2} \, dx. \]

By the arbitrariness of \( \sigma \) small we get
\[ \liminf_{h \to +\infty} F_{\varepsilon h}(E_h) \geq \int_{D} \int_{0}^{1} \left( \int_{B_2^{N-1}(0)} \int_{\nu u(x) \cdot (\xi - s\nu(x)) + s\nu(x)^N} G(\xi, \eta) \, d\eta \right) \, d\xi \, ds \sqrt{1 + |\nabla u(x)|^2} \, dx. \]

**STEP 5:** We conclude the proof showing that
\[ \int_{D} \int_{0}^{1} \left( \int_{B_2^{N-1}(0)} \int_{\nu u(x) \cdot (\xi - s\nu(x)) + s\nu(x)^N} G(\xi, \eta) \, d\eta \right) \, d\xi \, ds \sqrt{1 + |\nabla u(x)|^2} \, dx = \mathcal{F}(E). \]
First of all, we notice that
\[
\eta = \nabla u(x) \cdot (\xi - s\nu(x)) + s\nu(x)^N = \nabla u(x) \cdot \xi + s\sqrt{1 + |\nabla u(x)|^2}
\]
is the equation of an affine hyperplane in \(\mathbb{R}^N\) orthogonal to \(\nu(x)\) whose distance from the origin is
\[
\frac{s\sqrt{1 + |\nabla u(x)|^2}}{\sqrt{1 + |\nabla u(x)|^2}} = s.
\]
As a consequence,
\[
\int_{B_1^{N-1}(0)} \int_0^1 \nabla u(x) \cdot (\xi - s\nu(x)) + s\nu(x)^N G(\xi, \eta) \, d\eta \, d\xi = \int_{\{z \cdot \nu_E(x) \geq s\}} G(z) \, dz
\]
from which we obtain
\[
\tilde{\theta}(v) = \begin{cases} |v| \theta \left( \frac{v}{|v|} \right) \, dt & \text{if} \ v \neq 0, \\ 0 & \text{if} \ v = 0, \end{cases}
\]
is convex. First of all, by direct computation for each \(v \in \mathbb{R}^N\) with \(v \neq 0\) we have
\[
\theta \left( \frac{v}{|v|} \right) = \int_0^1 f \left( \int_{\{z \cdot v \geq |v| t\}} G(z) \, dz \right) \, dt = \frac{1}{|v|} \int_0^{|v|} f \left( \int_{\{z \cdot v \geq s\}} G(z) \, dz \right) \, ds
\]
from which we obtain
\[
\tilde{\theta}(v) = \begin{cases} \int_0^{|v|} f \left( \int_{\{z \cdot v \geq s\}} G(z) \, dz \right) \, ds & \text{if} \ v \neq 0, \\ 0 & \text{if} \ v = 0. \end{cases}
\]
Now it is easy to see that $\tilde{\theta}$ is convex. Indeed, since $f$ is convex there exist $(\alpha_h), (\beta_h)$ such that

$$f = \lim_{h \to +\infty} f_h \text{ uniformly on compact sets, where } f_h(t) = \alpha_h t + \beta_h.$$ 

For any $h \in \mathbb{N}$ let

$$\tilde{\theta}_h(v) = \begin{cases} \int_0^{|v|} f_h \left( \int_{\{z \cdot v \geq s\}} G(z) \, dz \right) \, ds & \text{if } v \neq 0, \\ 0 & \text{if } v = 0. \end{cases}$$

Since $f_h \to f$ uniformly on $[0, 1]$ we can say that $\tilde{\theta}_h \to \tilde{\theta}$ pointwise. In order to conclude it is sufficient to show that $\tilde{\theta}_h$ is convex. For any $v \neq 0$ we let $\hat{v} = \frac{v}{|v|}$. Then

$$\tilde{\theta}_h(v) = \alpha_h \int_0^{|v|} \int_{\{z \cdot v \geq s\}} G(z) \, dz \, ds + \beta_h |v|$$

$$= \alpha_h \int_0^{|v|} \int_{\{\bar{z} \cdot v = 0\}} \int_{s/|v|}^{+\infty} G(\bar{z} + t\hat{v}) \, d\bar{z} \, dt \, ds + \beta_h |v|$$

$$= \alpha_h \int_0^{+\infty} \int_{\{\bar{z} \cdot v = 0\}} \int_0^{|v|} G(\bar{z} + t\hat{v}) \, d\bar{z} \, dt + \beta_h |v|$$

$$= \alpha_h |v| \int_0^{+\infty} \int_{\{\bar{z} \cdot v = 0\}} tG(\bar{z} + t\hat{v}) \, d\bar{z} \, dt + \beta_h |v|$$

$$= \alpha_h \int_{\{\bar{z} \cdot v \geq 0\}} G(z) z \cdot v \, dz + \beta_h |v|$$

$$= \frac{\alpha_h}{2} \int_{\mathbb{R}^N} G(z) |z \cdot v| \, dz + \beta_h |v|$$

where the last equality follows since $G$ is even. Notice that the last expression is convex in $v$ and this ends the proof.

7. Some examples

In this section we characterize the limit functional $\mathcal{F}$ in some interesting cases.

7.1. $G$ radially symmetric

Assume $G(z) = g(|z|)$ for some $g: [0, +\infty) \to \mathbb{R}$. Take $\nu \in S^{N-1}$ and $t \geq 0$. Notice that the quantity

$$\int_{\{z \cdot \nu \geq t\}} G(z) \, dz$$

does not depend on $\nu$. Take now $E \in \mathcal{P}_N$ and $x \in \partial^* E$. We have

$$\int_0^1 f \left( \int_{\{z \cdot \nu(x) \geq t\}} G(z) \, dz \right) \, dt = c$$
where $c$ is a constant that depends only on $N, f$ and $G$. Then

$$\mathcal{F}(E) = c \mathcal{H}^{N-1}(\partial^* E).$$

### 7.2. The case $f(t) = t$

When $f$ is the identity function for any $E \in \mathcal{P}_N$ and for any $x \in \partial^* E$ we have

$$\theta(\nu_E(x)) = \int_0^1 \int_{H_{\nu_E(x)} + t\nu_E(x)} G(z) \, dz \, dt$$

$$= \int_0^1 \int_{\bar{z} \cdot \nu_E(x) = 0}^1 G(\bar{z} + s\nu_E(x)) \, ds \, d\bar{z} \, dt$$

$$= \int_0^1 \int_{\bar{z} \cdot \nu_E(x) = 0}^s G(\bar{z} + s\nu_E(x)) \, dt \, d\bar{z} \, ds$$

$$= \int_{H_{\nu_E(x)}} G(z) z \cdot \nu_E(x) \, dz$$

$$= \frac{1}{2} \int_{\mathbb{R}^N} G(z) |z \cdot \nu_E(x)| \, dz.$$

Then the limit $\mathcal{F}$ is given by

$$\mathcal{F}(E) = \frac{1}{2} \int \partial^* E \int_{\mathbb{R}^N} G(z) |z \cdot \nu_E(x)| \, dz \, d\mathcal{H}^{N-1}(x).$$

This is in accordance to [9].

**Remark 7.1.** If $N > 1$ and $G$ is radially symmetric we have, if $g: [0, +\infty) \rightarrow \mathbb{R}$ is such that $G(z) = g(|z|)$,

$$\frac{1}{2} \int_{\mathbb{R}^N} G(z) |z \cdot \nu_E(x)| \, dz = \frac{1}{2} \int_{\mathbb{R}^N} g(|z|) |z \cdot \nu_E(x)| \, dz$$

$$= \frac{1}{2} \int_0^{+\infty} \int_{S^{N-1}} g(r) |\xi \cdot \nu_E(x)| \, dr \, d\mathcal{H}^{N-1}(\xi)$$

$$= |B_1^{N-1}(0)| \int_0^{+\infty} g(r) \, dr$$

$$= \frac{|B_1^{N-1}(0)|}{\mathcal{H}^{N-1}(S^{N-1})} \int_{\mathbb{R}^N} G(z) |z| \, dz$$

since it is well known that for any $\nu \in S^{N-1}$ it holds

$$\frac{1}{2} \int_{S^{N-1}} |\xi \cdot \nu| \, d\mathcal{H}^{N-1}(\xi) = |B_1^{N-1}(0)|.$$
We thus deduce that
\[ F(E) = c_{N,G} H^{N-1}(E), \quad c_{N,G} = \frac{|B_{1}^{N-1}(0)|}{H^{N-1}(S^{N-1})} \int_{\mathbb{R}^{N}} G(z)|z| \, dz. \]

This is in accordance to [5].

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