Thurston’s norm revisited

Igor V. Nikolaev *

Abstract

We study the Thurston norm on the second homology of a 3-manifold $M$, which is the surface bundle over the circle with a pseudo-Anosov monodromy. A novelty of our approach consists in the application of the $C^*$-algebras to a problem in topology. Namely, one associates to $M$ a $C^*$-algebra, whose $K$-theory gives rise to an algebraic number field $K$. It is shown, that the trace function on the ring of integers of $K$ induces a norm on the second homology of $M$. The norm coincides with the Thurston norm on the second homology of $M$.

Key words and phrases: operator algebras, 3-manifolds

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1 Introduction

In 1986, W. P. Thurston discovered the fundamental measure of complexity of a 3-dimensional manifold, $M$. The measure is a norm on the second homology of $M$, which assigns the non-negative integers to the elements of the group $H_2(M) = H_2(M;\mathbb{Z})$. If $z \in H_2(M)$, then a number $N(z)$ is attached, such that $N(z) = \min_X \{-\chi(X) \mid X \neq S^2$ is compact surface representing class $z\}$, where $\chi$ is the Euler number of $X$. The norm is called a Thurston norm. The function $N$ is linear on the $H_2(M)$ and extends to the real homology group $H_2(M;\mathbb{R})$ as a pseudo-norm (Thurston [10]). The Thurston norm is an important homotopy invariant of the manifold $M$, which can be viewed as a generalization of the genus of a knot.

It is interesting that the abelian groups with a norm arise in the context of the $AF$-algebras (the operator algebras, see Effros [2]). Namely, let $\mathbb{Z}^k \xrightarrow{\lambda} \mathbb{Z}^k \xrightarrow{\lambda} \mathbb{Z}^k \xrightarrow{\lambda} \ldots$, be a stationary dimension group. (We refer the reader to the section 2 for a definition.) By $\lambda_A$ one understands the Perron-Frobenius eigenvalue of the positive integral matrix $A$. Let $K = \mathbb{Q}(\lambda_A)$ be an algebraic

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number field of the degree \( k \), obtained as an extension of the rationals by the algebraic number \( \lambda \). The field \( K \) is known to be an important invariant of the stationary dimension group. Namely, the triple \((K, \alpha, I)\), where \( \alpha \) is an embedding of the field \( K \) and \( I \) is the equivalence class of ideals in the ring of integers of \( K \), is a complete Morita invariant of the stationary dimension group (Bratteli, Jørgensen, Kim & Roush [1], Effros [2], Handelman [6]). Denote by \( O_K \) the ring of integers of the field \( K \) and fix an integral basis \( \omega_1, \ldots, \omega_k \) in \( O_K \). Note that \( O_K \cong \mathbb{Z}^k \) by the Gauss isomorphism. It is well known that the multiplication by \( \alpha \in O_K \) induces a linear operator on the vector space \( O_K \):

\[
\begin{pmatrix}
\alpha \omega_1 \\
\vdots \\
\alpha \omega_k
\end{pmatrix} = 
\begin{pmatrix}
a_{11} & \cdots & a_{1k} \\
\vdots & \ddots & \vdots \\
a_{k1} & \cdots & a_{kk}
\end{pmatrix}
\begin{pmatrix}
\omega_1 \\
\vdots \\
\omega_k
\end{pmatrix},
\]

where \( a_{ij} \) are the rational integers. Define a function \( N : O_K \to \mathbb{Z} \) by the formula \( \alpha \mapsto a_{11} + \cdots + a_{kk} \), where \( \alpha \in O_K \). It is not hard to verify, that \( N \) is a linear function, which is independent of the choice of the integral basis in \( O_K \) (Weyl [13]). Note that the pre-image \( N^{-1}(\mathbb{Z}^+) \) of the semi-group \( \mathbb{Z}^+ = \{0, 1, 2, \ldots\} \) is a cone \( C \subset \mathbb{Z}^k \).

The aim of this note is to show that the algebraic norm \( N \) and the Thurston norm \( \mathcal{N} \) are related. Namely, let \( \varphi : X \to X \) be a pseudo-Anosov diffeomorphism of the compact surface \( X \) of the genus \( g \) and let \( \mathcal{F} \) be the \( \varphi \)-invariant foliation on \( X \) (Thurston [11]). Consider the mapping torus of \( \varphi \), i.e. a 3-dimensional manifold \( M = \{X \times [0, 1] | \ (x, 0) \mapsto (\varphi(x), 1), \ x \in X \} \). (The reader can recognize \( M \) to be the surface bundle over the circle with a monodromy \( \varphi \).) Let us construct the crossed product \( C^* \)-algebra \( A_\varphi = C(X) \rtimes_\varphi \mathbb{Z} \), where \( C(X) \) is the \( C^* \)-algebra of the continuous complex-valued functions on the surface \( X \). It can be shown, that the \( K_0 \)-group of \( A_\varphi \) is a stationary dimension group. Define a map \( \mathcal{N} : H_2(M) \to \mathbb{Z} \) using a natural isomorphism \( O_K \cong H_1(X, \text{Sing} \mathcal{F}; \mathbb{Z}) \cong H_2(M) \), where \( \text{Sing} \mathcal{F} \) is the set of singular points of the foliation \( \mathcal{F} \). Our main result is the following theorem.

**Theorem 1** For every surface bundle \( M \to S^1 \) with a pseudo-Anosov monodromy \( \varphi \), the following is true: (i) the preimage \( \mathcal{N}^{-1}(\mathbb{Z}^+) \) of the semi-group \( \mathbb{Z}^+ = \{0, 1, 2, \ldots\} \) is a cone \( C \subset H_2(M) \) and (ii) the norm \( \mathcal{N} \) coincides with the Thurston norm \( \mathcal{N} \) on the cone \( C \).

The structure of the note is the following. In section 2, the notation is introduced. Theorem 1 is proved in section 3.

## 2 Notation

This section is a brief introduction to the dimension groups, the algebraic number fields and the Thurston norm on the 3-dimensional manifolds. We refer
the reader to O. Bratteli, P. E. T. Jørgensen, K. H. Kim & F. Roush, ([1]), M. Rørdam, F. Larsen & N. Laustsen ([9]), H. Weyl ([13]) and W. Thurston ([10]) for a complete account.

2.1 The dimension group

By the \( C^* \)-algebra one understands a noncommutative Banach algebra with an involution ([9]). Namely, a \( C^* \)-algebra \( A \) is an algebra over \( \mathbb{C} \) with the norm \( a \mapsto ||a|| \) and an involution \( a \mapsto a^* \), \( a \in A \), such that \( A \) is complete with respect to the norm, and such that \( ||ab|| \leq ||a|| ||b|| \) and \( ||a^*a|| = ||a||^2 \) for every \( a, b \in A \). If \( A \) is commutative, then the Gelfand theorem says that \( A \) is isometrically \(*\)-isomorphic to the \( C^* \)-algebra \( C_0(X) \) of the continuous complex-valued functions on a locally compact Hausdorff space \( X \). For otherwise, \( A \) represents a noncommutative topological space.

2.1.1 The ordered abelian groups

Given a \( C^* \)-algebra, \( A \), consider a new \( C^* \)-algebra \( M_n(A) \), i.e. the matrix algebra over \( A \). There exists a remarkable semi-group, \( A^+ \), connected to the set of projections in the algebra \( M_\infty = \bigcup_{n=1}^\infty M_n(A) \). Namely, the projections \( p, q \in M_\infty(A) \) are the Murray-von Neumann equivalent \( p \sim q \), if they can be presented as \( p = v^*v \) and \( q = vv^* \) for an element \( v \in M_\infty(A) \). An equivalence class of the projections is denoted by \([p]\). The semi-group \( A^+ \) is defined to be the set of all equivalence classes of projections in \( M_\infty(A) \) with the binary operation \([p] + [q] = [p \oplus q]\). The Grothendieck completion of \( A^+ \) to an abelian group is called a \( K_0 \)-group of \( A \). The functor \( A \to K_0(A) \) maps the unital \( C^* \)-algebras into a category of the abelian groups, so that the semi-group \( A^+ \subset A \) corresponds to a positive cone \( K_0^+ \subset K_0(A) \) and the unit element \( 1 \in A \) corresponds to an order unit \([1] \in K_0(A) \). The ordered abelian group \((K_0, K_0^+, [1])\) with the order unit is called a dimension group of the \( C^* \)-algebra \( A \).

2.1.2 The \( AF \)-algebras

An \( AF \) (approximately finite-dimensional) \( C^* \)-algebra is defined to be a norm closure of an ascending sequence of the finite dimensional \( C^* \)-algebras \( M_n \)'s, where \( M_n \) is a \( C^* \)-algebra of the \( n \times n \) matrices with the entries in \( \mathbb{C} \). Here the index \( n = (n_1, \ldots, n_k) \) represents a multi-matrix \( C^* \)-algebra \( M_n = M_{n_1} \oplus \ldots \oplus M_{n_k} \). Let \( M_1 \xrightarrow{\varphi_1} M_2 \xrightarrow{\varphi_2} \ldots \) be a chain of the finite-dimensional \( C^* \)-algebras and their homomorphisms. A set-theoretic limit \( A = \lim M_n \) has a natural algebraic structure given by the formula \( a_m + b_k \to a + b \); here \( a_m \to a, b_k \to b \) for the sequences \( a_m \in M_m, b_k \in M_k \). The homeomorphisms of the above (multi-matrix) algebras admit a canonical description (Effros [2]). Suppose that \( p, q \in \mathbb{N} \) and \( k \in \mathbb{Z}^+ \) are such numbers that \( kq \leq p \). Let us define a homomorphism \( \varphi : M_q \to M_p \) by the formula \( a \mapsto a \oplus \ldots \oplus a \oplus 0_h \), where \( p = kq + h \). More
generally, if \( q = (q_1, \ldots, q_s), p = (p_1, \ldots, p_r) \) are vectors in \( \mathbb{N}^s, \mathbb{N}^r \), respectively, and \( \Phi = (\phi_{kl}) \) is a \( r \times s \) matrix with the entries in \( \mathbb{Z}^+ \) such that \( \Phi(q) \leq p \), then the homomorphism \( \varphi \) is defined by the formula:

\[
a_1 \oplus \ldots \oplus a_s \rightarrow \left( \begin{array}{c}
\phi_{11} \oplus \ldots \oplus \phi_{12} \\
\phi_{21} \oplus \ldots \oplus \phi_{22}
\end{array} \right) \left( \begin{array}{c}
(a_1 + a_1 \oplus \ldots) \oplus (a_2 + a_2 \oplus \ldots) \oplus \ldots \oplus 0_{h_1} \\
\oplus \ldots \oplus (a_1 + a_1 \oplus \ldots) \oplus (a_2 + a_2 \oplus \ldots) \oplus \ldots \oplus 0_{h_2} \oplus \ldots
\end{array} \right)
\]

where \( \Phi(q) + h = p \). We say that \( \varphi \) is a canonical homomorphism between \( M_p \) and \( M_q \). Any homomorphism \( \varphi : M_q \rightarrow M_p \) can be rendered canonical ([2]).

2.1.3 The Bratteli diagrams

This graphical presentation of the canonical homomorphism is called a Bratteli diagram. Every block of such a diagram is a bipartite graph with the \( r \times s \) matrix \( \Phi = (\phi_{kl}) \). In general, the Bratteli diagram is given by the vertex set \( V \) and the edge set \( E \), such that \( V \) is an infinite disjoint union \( V_1 \sqcup V_2 \sqcup \ldots \), where each \( V_i \) has a cardinality \( n \). Any pair \( V_{i-1}, V_i \) defines a non-empty set \( E_i \subseteq E \) of the edges with a pair of the range and the source functions \( r, s \), such that \( r(E_i) \subseteq V_i \) and \( s(E_i) \subseteq V_{i-1} \). The non-negative integral matrix of the incidences \( (\phi_{ij}) \) shows how many edges are drawn between the \( k \)-th vertex in the row \( V_{i-1} \) and \( l \)-th vertex in the row \( V_i \). A Bratteli diagram is called stationary, if \( (\phi_{kl}) \) is a constant matrix for all \( i = 1, \ldots, \infty \).

2.2 The number fields

Let \( \mathbb{Q} \) be the field of the rational numbers. Let \( \alpha \not\in \mathbb{Q} \) be an algebraic number over \( \mathbb{Q} \), i.e. root of the polynomial equation \( a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0 = 0 \), \( a_n \neq 0 \), where \( a_i \in \mathbb{Q} \). An algebraic extension of the degree \( n \) is a minimal field \( K = K(\alpha) \), which contains both \( \mathbb{Q} \) and \( \alpha \). Note that the coefficients \( a_i \) can be assumed integer. If \( K \) is an algebraic extension of the degree \( n \) over \( \mathbb{Q} \), then \( K \) is isomorphic to the \( n \)-dimensional vector space (over \( \mathbb{Q} \)) with the basis vectors \( \{1, \alpha, \ldots, \alpha^{n-1}\} \) (Pollard [8]).

2.2.1 The ring of integers

Let \( K \) be an algebraic extension of the degree \( n \) over \( \mathbb{Q} \). The element \( \tau \in K \) is called algebraic integer if there exits a monic polynomial \( \tau^n + a_{n-1} \tau^{n-1} + \ldots + a_0 = 0 \), where \( a_i \in \mathbb{Z} \). It can be easily verified that the sum and the product of two algebraic integers is an algebraic integer. The (commutative) ring \( O_K \subset K \) is called a ring of integers. The elements of the subring \( \mathbb{Z} \subset O_K \) are called the rational integers. An integral basis is a collection \( \omega_1, \ldots, \omega_n \) of the elements of \( O_K \), whose linear span over the rational integers is equal to \( O_K \).
An integral basis exists for any finite extension and therefore $O_K$ is isomorphic to the integral lattice $\mathbb{Z}^n$, where $n$ is the degree of the field $K$ (Weyl [13]).

2.2.2 The trace of an algebraic number

Let $K$ be a number field of the degree $n$ over $\mathbb{Q}$. There exists $n$ isomorphic embeddings (monomorphisms) $K \to \mathbb{C}$ (McCarthy [7]). We denote them by $\sigma_1, \ldots, \sigma_n$. If $\alpha \in K$ then one defines a trace by the formula $^1: \mathcal{N}(\alpha) = \sigma_1(\alpha) + \ldots + \sigma_n(\alpha)$. When $\alpha$ is an algebraic integer, then $\mathcal{N}(\alpha)$ is a rational integer. If $p, q \in \mathbb{Z}$, then $\mathcal{N}(p\alpha + q\beta) = p\mathcal{N}(\alpha) + q\mathcal{N}(\beta)$, for all $\alpha, \beta \in K$. It is not hard to see that the above formula establishes a homomorphism $\mathcal{N} : O_K \cong \mathbb{Z}^n \to \mathbb{Z}$, which does not depend on the choice of an integral basis in $O_K$ (McCarthy [7], Weyl [13]).

2.3 The 3-manifolds

Let $M$ be a compact oriented 3-manifold. Suppose that the second homology group $H_2(M; \mathbb{Z})$ is non-trivial. There exists a linear mapping of the group into the set of the positive integers, which is given by the following construction of W. P. Thurston ([10]).

2.3.1 The Thurston norm

Let $X$ be a connected surface of the genus $g \geq 0$. Denote by $\chi(X)$ its Euler characteristic, i.e. an integer number $2 - 2g$. The negative part of $\chi(X)$ is defined as $\chi_-(X) = \max\{0, -\chi(X)\}$. If $X$ is not connected, one introduces $\chi_-(X)$ as the sum of the negative parts of the connected components of $X$. For a cycle $z \in H_2(M; \mathbb{Z})$, consider a non-negative integer $N(z) = \inf\{\chi_-(X) \mid X$ is an embedded surface representing $z\}$. The $N(z)$ is called a Thurston norm. Given two such cycles $z_1$ and $z_2$, let $X_1$ and $X_2$ be the surfaces representing them. There exists a unique way to mend $X_1$ and $X_2$ together to obtain new embedded surface $X$, such that $\chi_-(X) = \chi_-(X_1) + \chi_-(X_2)$ (Thurston [10]). Thus, $N(z) = N(z_1) + N(z_2)$ extends linearly to the entire group $H_2(M; \mathbb{Z})$.

2.3.2 The pseudo-Anosov diffeomorphisms

Let $X$ be a surface of the genus $g \geq 2$. Denote by Mod $X = Diff X/Diff_0 X$ the mapping class group of $X$, i.e. the group of the isotopy classes of the orientation preserving diffeomorphisms of $X$. The following classification of Mod $X$ is due to J. Nielsen and W. P. Thurston.

**Lemma 1** ([11]) Any diffeomorphism $\varphi \in$ Mod $X$ is isotopic to a diffeomorphism $\varphi'$, such that either (i) $\varphi'$ has finite order, or (ii) $\varphi'$ is pseudo-Anosov

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\(^1\)This notion of $\mathcal{N}$ is equivalent to given in section 1.
(non-periodic) diffeomorphism, or (iii) \( \varphi' \) is reducible by a system of curves \( \Gamma \) surrounded by small tubular neighborhoods \( N(\Gamma) \), such that on \( M \setminus N(\Gamma) \) \( \varphi' \) satisfies either (i) or (ii).

### 2.3.3 The singularity data

Let \( \varphi \) be a pseudo-Anosov diffeomorphism. There exists a pair of \( \varphi \)-invariant measured foliations \( F_\pm \) on \( X \), such that \( \varphi \) expands along \( F_+ \) and contracts along \( F_- \) with dilatation factor \( \mu > 1 \) (Thurston [11]). \( F_+ \) and \( F_- \) are mutually transversal and have common set of singular points, which are saddle points with \( n \geq 3 \) prongs. For brevity, we let \( F = F_+ \). Recall that the index of \( n \)-prong saddle \( s_n \) is \(-\frac{1}{2}(n - 2)\). Therefore \( \sum_{s_n \in \text{Sing } F} s_n = 2g - 2 \), where \( g \) is the genus of surface \( X \). If \( m = |\text{Sing } F| \) is the total number of the singular points of \( F \), then \( 1 \leq m \leq 4g - 4 \), where the minimum is attained by a unique saddle \( s_{4g-2} \) and maximum by the set \( \{s_3, s_3, \ldots, s_3\} \) of \( 4g - 4 \) saddles. We refer to the set \( \{s_{i_1}, \ldots, s_{i_m}\} \) as a singularity data of \( F \).

### 2.3.4 The mapping tori

Let \( \varphi : X \to X \) be a diffeomorphism of the surface \( X \). One can obtain 3-dimensional manifolds \( M = M(\varphi) \) by the formula \( M = \{X \times [0, 1] \mid (x, 0) \mapsto (\varphi(x), 1), x \in X\} \). The manifold \( M \) is called a mapping torus. It is not hard to see that \( M \) is a mapping torus if and only if \( M \to S^1 \) is a fibre bundle over \( S^1 \) with the monodromy \( \varphi \). If the diffeomorphism \( \varphi \in \text{Mod } X \) is of a finite order, then \( M \) will be a Seifert manifold. In the case when \( \varphi \) is pseudo-Anosov, the following result due to W. P. Thurston is true.

**Lemma 2** ([12]) The mapping torus \( M \) admits a hyperbolic structure, if and only if the diffeomorphism \( \varphi \) is pseudo-Anosov.

### 2.3.5 The second homology group of the mapping torus

Let \( \varphi : X \to X \) be a pseudo-Anosov diffeomorphism of genus \( g \geq 2 \) surface. Let \( \text{Sing } F \) be a finite set of the singularities of the \( \varphi \)-invariant foliation \( F \). The relative homology group \( H_1(X, \text{Sing } F; \mathbb{Z}) \) is a torsion-free of the rank \( k = 2g + |\text{Sing } F| - 1 \), where \( |\text{Sing } F| \) is the cardinality of the set \( \text{Sing } F \). Let \( M \) be the mapping torus of \( \varphi \). The 2-cycles of \( M \) are generated by the 1-cycles of \( X \setminus \text{Sing } F \). Indeed, let \( C \in H_1(X, \text{Sing } F; \mathbb{Z}) \) and \( \varphi(C) \) its image under the diffeomorphism \( \varphi \). Let \( X \setminus C \) be a copy of the surface \( X \), which is cut along the closed curve \( C \). Similarly, let \( X \setminus \varphi(C) \) be a copy of \( X \) with a cut along \( \varphi(C) \). The surface \( X = (X \setminus C) \cup (X \setminus \varphi(C)) \), glued along the action of \( \varphi \), belongs to the group \( H_2(M) \), and any non-torsion element of \( H_2(M) \) can be obtained in such a way. Therefore, \( \text{rank } H_2(M) = 2g + m - 1 \), where \( m = |\text{Sing } F| \).
2.3.6 The Thurston norm of the surface bundle

It is interesting to relate the Thurston norm $N$ on the second homology with the geometry of $M$. It turns out, that in this case $N$ can be expressed in terms of the Euler classes of the plane bundle tangent to the fibres of $M$. Namely, let $N^*$ be a dual Thurston norm defined on the first homology $H_1(M;\mathbb{Z})$ by the formula $N^*(z) = \sup_{u \in \text{Hom}(H_1(M;\mathbb{Z}),\mathbb{Z})} u(z)$, where $u \in \text{Hom}(H_1(M;\mathbb{Z}),\mathbb{Z}) \approx H_1(M;\mathbb{Z})$ and $N$ is the Thurston norm on $H_1(M;\mathbb{Z}) \approx H_2(M;\mathbb{Z})$ induced by the Poincaré duality. Then the following lemma is true.

Lemma 3 ([3], ([10]) Let $\tau$ be a subbundle of the tangent bundle $TM$ consisting of the 2-planes tangent to the fiber $X$ of the fibration $M \to S^1$. Let $e(\tau) \in H^2(M;\mathbb{Z})$ be the Euler class of $\tau$, i.e. first obstruction to the cross-section of bundle $\tau$. Then (i) the norm $N^*: H_1(M;\mathbb{Z}) \to \mathbb{Z}^+$ is induced by the cocycle $e(\tau)$, i.e. $N^*(z) = |\int_{\tau} e(\tau)|$; (ii) the set of the (de Rham) cohomology classes $H_1(M;\mathbb{R})$, which is representable by the closed non-singular differential 1-forms on $M$ is a maximal cone $C \subset H_1(M;\mathbb{R})$, where the Thurston norm $N$ can be extended linearly.

3 Proof of theorem 1

(i) Let $\mathcal{N}: H_2(M) \to \mathbb{Z}$ be a linear mapping. To show that the set $C = \{z \in H_2(M) \mid \mathcal{N}(z) > 0\}$ is a cone, we have to establish that

1. $z \in C, c > 0$ implies $cz \in C$
2. $z_1, z_2 \in C$ implies $z_1 + z_2 \in C$.

Indeed, since $\mathcal{N}$ is linear, $\mathcal{N}(cz) = c\mathcal{N}(z)$, where $\mathcal{N}(z) > 0$ by the assumption. Therefore, $c\mathcal{N}(z) > 0$ and the item (1) follows. Similarly, in the item (2), by the linearity of $\mathcal{N}$, we have $\mathcal{N}(z_1 + z_2) = \mathcal{N}(z_1) + \mathcal{N}(z_2) > 0$, since $\mathcal{N}(z_1) > 0, \mathcal{N}(z_2) > 0$ by the assumption. The item (i) is proved.

(ii) The proof of item (ii) is based on a lemma of D. Gabai ([4]). Roughly speaking, we shall estimate the Gromov norm (a simplicial norm) of $H_2(M)$, rather than the Thurston norm itself. This approach gives a technical advantage, because the group of the 2-chains in $M$ has a natural abelian structure. Next we use the Gabai lemma to evaluate the two norms. The Gromov norm was introduced and studied in ([5]).

Let $M$ be a compact manifold and $z \in H_2(M)$ be an element of the second homology group of $M$. A Gromov norm $g(z)$ is given by the formula $g(z) = \inf_{Z \in [z]} \{ \sum |a_i| : Z = \sum a_i \sigma_i \}$, where $[z]$ is the homology class of the 2-chains and $\sigma_1, \ldots, \sigma_n$ is a basis of the simplicial decomposition of $M$. The following lemma is true.
Lemma 4 Suppose \( M \) is a compact oriented 3-manifold. Then the Thurston norm \( N \) and Gromov norm \( g \) are related by the formula \( N(z) = \frac{1}{2} g(z) \), for each \( z \in H_2(M) \) in the domain of definition of the two norms.

Proof. The proof can be found in ([4]), Corollary 6.18. For the sake of clarity, let us outline the main idea. First, notice that if \( M \) is a hyperbolic \( k \)-manifold and \([M]\) its homology class, then we have Gromov's formula \( g([M]) = Vol M / Vol \sigma \), where \( \sigma \) is the largest hyperbolic \( k \)-simplex (Gromov [5]). Thus, for the connected surface \( X \), one has \( g([X]) = \frac{2|\chi(X)|}{\pi} = 2N([X]) \). The formula extends to the case \( X \) with more than one connected component and requires the singular norms \( x_s \) in this case, see Gabai ([4]). Eventually, it can be shown, that \( x_s = N \) and one gets inequality \( g \leq 2N \).

To prove the inequality \( g \geq 2N \), let \( z \in H_2(M) \) and \( Z \in [z] \) be a 2-cycle \( Z = \sum a_i \sigma_i \), where \( a_i \in \mathbb{Z} \). By pasting the singular simplices, one can obtain a proper map \( f : X \to M \), such that \( [f(X)] = z \). By Gromov's formula for the hyperbolic volumes \( 2N([X]) \leq \sum |a_i| = g([X]) \). Lemma 4 follows. \( \square \)

Fix a simplicial basis \( \sigma_1, \ldots, \sigma_n \) in the group \( C \) of the 2-chains of the regular triangulation of \( M \). Consider a subset of \( C \), given by the formula \( K_C = \{ Z \in C | Z = \sum_{i=1}^n a_i \sigma_i, \ a_i > 0 \} \). Denote by \( \tilde{N} : H_2(M) \to \mathbb{Z}^+ \) the mapping \( z \mapsto g(z) \) given by the Gromov norm. It is not hard to verify, that \( \tilde{N} \) is linear on the \( K_C \) and the \( K_C \) is a cone in \( C \). Consider the following commutative diagram of the linear mappings:

\[
\begin{array}{ccc}
K_C & \xrightarrow{\Sigma} & C \\
\downarrow{\tilde{N}} & & \downarrow{N} \\
\mathbb{Z}^+ & \xrightarrow{w} & \mathbb{Z}^+
\end{array}
\]

(Here \( w \) is a doubling map, acting by the formula \( z \mapsto 2z \)). The map \( \Sigma \) on the diagram is linear. The \( \Sigma \) establishes a bijection between the bases \( \{\sigma_i\} \) and \( \{\omega_i\} : \Sigma(\sigma_i) = \omega_i \). It remains to apply lemma 4. The item (ii) of theorem 1 follows. \( \square \)

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THE FIELDS INSTITUTE FOR MATHEMATICAL SCIENCES, TORONTO, ON, CANADA, E-MAIL: igor.v.nikolaev@gmail.com

Current address: 101-315 Holmwood Ave., Ottawa, ON, Canada, K1S 2R2