NATURAL CONNECTION WITH TOTALLY SKEW-SYMMETRIC TORSION ON ALMOST CONTACT MANIFOLDS WITH B-METRIC

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Abstract. A natural connection with totally skew-symmetric torsion on almost contact manifolds with B-metric is constructed. The class of these manifolds, where the considered connection exists, is determined. Some curvature properties for this connection, when the corresponding curvature tensor has the properties of the curvature tensor for the Levi-Civita connection and the torsion tensor is parallel, are obtained.

1. Introduction

The natural connections with totally skew-symmetric torsion (also known as Kähler with torsion (shortly, KT-) connections or Bismut connections) are of particular interest in the string theory [23] and in the Hermitian geometry [3]. The KT-geometry is a natural generalization of the Kähler geometry, since when the torsion is zero the KT-connection coincides with the Levi-Civita connection. The KT-structures with a closed torsion 3-form (i.e. the so-called strong KT-structures) on a Hermitian manifold have been recently studied by many authors and they have also applications in type II string theory and in 2-dimensional supersymmetric σ-models [8, 14, 23].

The connections with totally skew-symmetric torsion arise in a natural way in theoretical and mathematical physics. According to [9, 5], there exists a unique KT-connection on any Hermitian manifold. In [6] and [7], all almost contact metric, almost Hermitian and $G_2$-structures admitting a connection with totally skew-symmetric torsion tensor are described. On almost complex manifolds with Norden metric such a connection is introduced and investigated in [20, 21, 22].

A quaternionic analog of Kähler geometry is given by the hyper-Kähler with torsion (shortly, HKT-) geometry. An HKT-manifold is a hyper-Hermitian manifold admitting the so-called HKT-connection – a hyper-Hermitian connection with totally skew-symmetric torsion, i.e. for which the three KT-connections associated to the three Hermitian structures coincide. This geometry is introduced in [13] and later studied for instance in [10, 4, 11, 2, 24]. As an analogue of the known Hermitian structure on an almost hypercomplex manifold it is introduced and investigated an almost hypercomplex manifold with Hermitian and anti-Hermitian metrics in [13, 12, 17, 19]. On these manifolds with a metric structure, generated by a pseudo-Riemannian metric of neutral signature, a connection of the type of the HKT-connection is defined and investigated in [15].

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The goal of the present work is a consideration of a similar problem of existence of connections of KT-type on almost contact manifolds with B-metric. Our hope is that the wide application of these connections on the several manifolds mentioned above will be extended to manifolds in this case.

In this paper we consider a natural connection $D$ (i.e. preserving the structure) with totally skew-symmetric torsion tensor on almost contact manifolds with B-metric. These manifolds are the odd-dimensional extension of the almost complex manifolds with Norden metric and the case with indefinite metrics corresponding to almost contact metric manifolds.

In Section 2 we give some necessary facts about the considered manifolds.

In Section 3 we define a natural connection $D$ with totally skew-symmetric torsion on an almost contact manifold with B-metric and call it shortly a $\varphi$KT-connection. We determine the class of considered manifolds where $D$ exists.

In Section 4 we characterize the corresponding curvature tensor $K$ of $D$ and related curvature properties in the cases when $K$ has the properties of the curvature tensor $R$ for the Levi-Civita connection $\nabla$ on a $\mathcal{F}_0$-manifold (i.e. a manifold with a $\nabla$-parallel almost contact B-metric structure). We examine also the case of $D$-parallel torsion of $D$. In Subsections 4.1 and 4.2 we consider separately the horizontal and vertical components of this class and specialize the corresponding curvature properties given in the former part of this section.

In Section 5 we construct and characterize a family of 5-dimensional Lie groups as $\mathcal{F}_7$-manifolds with a $D$-parallel torsion of the $\varphi$KT-connection $D$.

2. Almost Contact Manifolds with B-Metric

Let $(M, \varphi, \xi, \eta, g)$ be an almost contact manifold with B-metric or an almost contact B-metric manifold, i.e. $M$ is a $(2n+1)$-dimensional differentiable manifold with an almost contact structure $(\varphi, \xi, \eta)$ consisting of an endomorphism $\varphi$ of the tangent bundle, a vector field $\xi$, its dual 1-form $\eta$ as well as $M$ is equipped with a pseudo-Riemannian metric $g$ of signature $(n, n+1)$, such that the following algebraic relations are satisfied

\begin{align}
\varphi\xi &= 0, \\
\varphi^2 &= -\text{Id} + \eta \otimes \xi, \\
\eta \circ \varphi &= 0, \\
g(\varphi x, \varphi y) &= -g(x, y) + \eta(x)\eta(y)
\end{align}

for arbitrary $x, y$ of the algebra $\mathfrak{X}(M)$ on the smooth vector fields on $M$.

Further, $x, y, z, w$ will stand for arbitrary elements of $\mathfrak{X}(M)$.

The associated metric $\tilde{g}$ of $g$ on $M$ is defined by $\tilde{g}(x, y) = g(x, \varphi y) + \eta(x)\eta(y)$. Both metrics are necessarily of signature $(n, n+1)$. The manifold $(M, \varphi, \xi, \eta, \tilde{g})$ is also an almost contact B-metric manifold.

The structural group of almost contact manifolds with B-metric is $(\text{GL}(n, \mathbb{C}) \cap O(n, n)) \times I_1$, i.e. it consists of real square matrices of order $2n+1$ of the following type

\[
\begin{pmatrix}
A & B & \vartheta^T \\
-B & A & \vartheta^T \\
\vartheta & \vartheta & 1
\end{pmatrix},
\quad
AA^T - BB^T = I_n, \\
AB^T + BA^T = O_n,
\quad
A, B \in \text{GL}(n; \mathbb{R}),
\]

where $\vartheta$ and its transpose $\vartheta^T$ are the zero row $n$-vector and the zero column $n$-vector; $I_n$ and $O_n$ are the unit matrix and the zero matrix of size $n$, respectively.

A classification of the almost contact manifolds with B-metric is given in [7]. This classification is made with respect to the tensor field $F$ of type $(0,3)$ defined
by
\begin{equation}
F(x, y, z) = g((\nabla_x \varphi) y, z),
\end{equation}
where \( \nabla \) is the Levi-Civita connection of \( g \). The tensor \( F \) has the following properties

\begin{equation}
F(x, y, z) = F(x, z, y) = F(x, \varphi y, \varphi z) + \eta(y)F(x, \xi, z) + \eta(z)F(x, y, \xi).
\end{equation}

This classification includes eleven basic classes \( \mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_{11} \). In this work we pay attention to \( \mathcal{F}_3 \) and \( \mathcal{F}_7 \). These basic classes are characterized by the conditions

\begin{equation}
\mathcal{F}_3 \ni \xi, F(x, y, z) = 0, \quad F(\xi, y, z) = F(x, y, \xi) = 0;
\end{equation}

\begin{equation}
\mathcal{F}_7 \ni \xi, F(x, y, z) = 0, \quad F(x, y, z) = -F(\varphi x, \varphi y, z) - F(\varphi x, y, \varphi z),
\end{equation}

where \( \mathcal{S} \) denotes the cyclic sum over three arguments.

The special class \( \mathcal{F}_0 \), belonging to any other class \( \mathcal{F}_i \) \((i = 1, 2, \ldots, 11)\), is determined by the condition \( F(x, y, z) = 0 \). Hence \( \mathcal{F}_0 \) is the class of almost contact B-metric manifolds with \( \nabla \)-parallel structures, i.e. \( \nabla \varphi = \nabla \xi = \nabla \eta = \nabla g = \nabla \bar{g} = 0 \).

The components of the inverse matrix of \( \nabla \) are denoted by \( g^{ij} \) with respect to the basis \( \{ e_i \} \) of the tangent space \( T_p M \) of \( M \) at an arbitrary point \( p \in M \).

By analogy with the square norm of \( \nabla J \) for an almost complex structure \( J \), we define the square norm of \( \nabla \varphi \) by

\begin{equation}
\|\nabla \varphi\|^2 = g^{ij} g^{ks} ((\nabla_{e_i} \varphi) e_k, (\nabla_{e_j} \varphi) e_s).
\end{equation}

It is clear, the equality \( \|\nabla \varphi\|^2 = 0 \) is valid if \((M, \varphi, \xi, \eta, g)\) is a \( \mathcal{F}_0 \)-manifold, but the inverse implication is not always true. An almost contact B-metric manifold having a zero square norm of \( \nabla \varphi \) we call an isotropic-\( \mathcal{F}_0 \)-manifold.

As it is known \[15\], an arbitrary vector field \( \xi \) is Killing if the Lie differential of the metric \( g \) with respect to \( \xi \) is zero. Then, the vector field \( \xi \) is Killing on \((M, \varphi, \xi, \eta, g)\) if and only if

\begin{equation}
(\nabla_x \eta) y + (\nabla_y \eta) x = 0.
\end{equation}

**Lemma 1.** The class \( \mathcal{F}_3 \oplus \mathcal{F}_7 \) is the class of almost contact B-metric manifolds \((M, \varphi, \xi, \eta, g)\) with a Killing vector field \( \xi \) and a zero cyclic sum \( \mathcal{S} \) of \( F \).

**Proof.** Equation (17) and direct calculations yield the statement. \( \square \)

The Nijenhuis tensor \( N \) of the structure is defined by \( N := [\varphi, \varphi] + d\eta \otimes \xi \), where \([\varphi, \varphi](x, y) = [\varphi x, \varphi y] + \varphi^2 [x, y] - \varphi [\varphi x, y] - \varphi [x, \varphi y] \). Hence, \( N \) in terms of the covariant derivatives has the form

\begin{equation}
N(x, y) = (\nabla_{\varphi x} \varphi) y - (\nabla_{\varphi y} \varphi) x - \varphi (\nabla_x \varphi) y + \varphi (\nabla_y \varphi) x
+ (\nabla_x \eta) y. \xi - (\nabla_y \eta) x. \xi.
\end{equation}

The Nijenhuis \((0,3)\)-tensor is determined by \( N(x, y, z) = g(N(x, y), z) \). Then from (8) and (2) we have

\begin{equation}
N(x, y, z) = F(\varphi x, y, z) - F(\varphi y, x, z) - F(x, y, \varphi z) + F(y, x, \varphi z)
+ F(x, \varphi y, \xi) \eta(z) - F(y, \varphi x, \xi) \eta(z).
\end{equation}

In \[16\] there are considered the linear projectors \( h \) and \( v \) over \( T_p M \) which split (orthogonally and invariantly with respect to the structural group) any vector \( x \) into a horizontal component \( h(x) = -\varphi^2 x \) and a vertical component \( v(x) = \eta(x) \xi \). The decomposition \( T_p M = h(T_p M) \oplus v(T_p M) \) generates the corresponding distribution.
of the basic tensors $F$, which gives the horizontal component $\mathcal{F}_3$ and the vertical component $\mathcal{F}_7$ of the class $\mathcal{F}_3 \oplus \mathcal{F}_7$.

Moreover, for the considered classes we have the following

**Lemma 2.** The Nijenhuis tensor $N$ on an almost contact B-metric manifold $(M, \varphi, \xi, \eta, g)$ has the corresponding form:

(i) If $(M, \varphi, \xi, \eta, g) \in \mathcal{F}_3 \oplus \mathcal{F}_7$ then

\[
N(x, y) = 2(\nabla_{\xi} \varphi) y - 2\varphi(\nabla_{\varphi} \eta) y + 2(\nabla_{y} \eta) y, \\
h(N(x, y)) = -2\varphi^2(\nabla_{\xi} \varphi) y - 2\varphi(\nabla_{y} \varphi) y, \\
v(N(x, y)) = 4(\nabla_{y} \eta) y, \\
\]

(ii) If $(M, \varphi, \xi, \eta, g) \in \mathcal{F}_3$, then

\[
N(x, y) = h(N(x, y)) = 2(\nabla_{\xi} \varphi) y - 2\varphi(\nabla_{y} \varphi) y; \\
\]

(iii) If $(M, \varphi, \xi, \eta, g) \in \mathcal{F}_7$, then

\[
N(x, y) = v(N(x, y)) = 4(\nabla_{y} \eta) y, \xi. \\
\]

**Proof.** The expression of $N$ in (i) follows from (2) by virtue of equalities (2)–(7). Hence, $v(N(x, y)) = N(x, y, \xi) \xi = 2F(\varphi, y, \xi) \xi + 2(\nabla_{\varphi} \eta) y, \xi = 4(\nabla_{\xi} \eta) y, \xi$, using that $\xi$ is a Killing vector field. After that, since $h(N(x, y)) = N(x, y) - v(N(x, y))$, we obtain the corresponding expression of $h(N(x, y))$.

In case (ii), according to (4), we have $F(x, y, \xi) = 0$ and therefore $(\nabla_{\varphi} \eta) y = 0$. Then the form of $N$ in (i) implies $N(x, y) = 2(\nabla_{\xi} \varphi) y - 2\varphi(\nabla_{y} \varphi) y$. In addition to that, since $v(N(x, y)) = 0$, then $N(x, y)$ holds.

In case (iii), we have $F(x, y, z) = F(\varphi, y, \varphi, z) - F(\varphi, y, \varphi z)$ from (5). According to properties (3), we get $F(x, y, z) = (\xi, y)F(x, \xi, z) + (\eta, z)F(x, y, \xi)$ and by (2) the relation $(\nabla_{\varphi} \eta) y = -\eta(\varphi) \varphi \nabla_{\xi} \eta - (\nabla_{\xi} \eta) \varphi y, \xi$ holds. Applying the latter equality in the expression of $N$ in (i), we obtain that $N(x, y) = 4(\nabla_{\xi} \eta) y, \xi$. Then the equality $N(x, y) = v(N(x, y))$ follows immediately. \qed

Let $R = [\nabla, \nabla] - \nabla \nabla$ be the curvature tensor of type $(1,3)$ of $\nabla$. We denote the curvature tensor of type $(0,4)$ $R(x, y, z, w) = g(R(x, y), z, w)$ by the same letter.

The Ricci tensor $\rho$ for the curvature tensor $R$ and the scalar curvature $\tau$ for $R$ are defined by the traces $\rho(y, z) = \rho(y, z) = g^{ij}R(e_i, y, z, e_j)$ and $\tau = g^{ij}\rho(e_i, e_j)$, respectively.

Similarly, the Ricci tensor and the scalar curvature are determined for each curvature-like tensor $L$, i.e. for the $(0,4)$-tensor $L$ with the following properties:

\[
L(x, y, z, w) = -L(y, x, z, w) = -L(x, y, w, z), \\
L(x, y, z, w) = 0 \quad (\text{the first Bianchi identity}). \\
\]

A curvature-like tensor we call a $\varphi$-Kähler-type tensor on $(M, \varphi, \xi, \eta, g)$ if the following identity is valid

\[
L(x, y, \varphi z, \varphi w) = -L(x, y, z, w). \\
\]

This property is characteristic of the curvature tensor $R$ on a $\mathcal{F}_0$-manifold. Moreover, (12) is similar to the corresponding property for a Kähler-type tensor with respect to $J$ on an almost complex manifold with Norden metric.
3. $\varphi$KT-connection

**Definition 1.** A linear connection $D$ is called a natural connection on the manifold $(M, \varphi, \xi, \eta, g)$ if the almost contact structure $(\varphi, \xi, \eta)$ and the B-metric $g$ are parallel with respect to $D$, i.e. $D\varphi = D\xi = D\eta = Dg = 0$.

If $T$ is the torsion tensor of $D$, i.e. $T(x, y) = D_x y - D_y x - [x, y]$, then the corresponding tensor field of type $(0,3)$ is determined by $T(x, y, z) = g(T(x, y), z)$.

Let $\nabla$ be the Levi-Civita connection generated by $g$. Then we denote

\begin{equation}
D_x y = \nabla_x y + Q(x, y).
\end{equation}

Furthermore, we use the denotation $Q(x, y, z) = g(Q(x, y), z)$.

**Theorem 3.** The linear connection $D$ is a natural connection on the manifold $(M, \varphi, \xi, \eta, g)$ if and only if

\begin{equation}
Q(x, y, \varphi z) - Q(x, \varphi y, z) = F(x, y, z),
\end{equation}

\begin{equation}
Q(x, y, z) = -Q(x, z, y).
\end{equation}

**Proof.** From (13) we have

\begin{align*}
g(D_x \varphi y, z) &= g(\nabla_x \varphi y, z) + Q(x, \varphi y, z), \\
g(D_x y, \varphi z) &= g(\nabla_x y, \varphi z) + Q(x, y, \varphi z).
\end{align*}

Subtracting the last two equations we immediately obtain

\begin{equation}
g((D_x \varphi)y, z) = F(x, y, z) + Q(x, \varphi y, z) - Q(x, y, \varphi z).
\end{equation}

Then $D\varphi = 0$ is equivalent to (14).

After that we get consecutively

\begin{equation}
(D_x g)(y, z) = xg(y, z) - g(D_x y, z) - g(y, D_x z) = -Q(x, y, z) - Q(x, z, y).
\end{equation}

Therefore, $Dg = 0$ is valid if and only if (16) holds.

Equality (13) implies $g(D_x \xi, z) = g(\nabla_x \xi, z) + Q(x, \xi, z)$. Because of the identity $g(\nabla_x \xi, z) = F(x, \xi, \varphi z)$,

we have that $D\xi = 0$ is equivalent to the relation $F(x, \xi, \varphi z) + Q(x, \xi, z) = 0$. The last equality is a consequence of (14).

Since $\eta(\cdot) = g(\cdot, \xi)$, then supposing $Dg = 0$ we have $D\xi = 0$ if and only if $D\eta = 0$. \qed

**Definition 2.** A natural connection $D$ is called a $\varphi$KT-connection on the manifold $(M, \varphi, \xi, \eta, g)$ if the torsion tensor $T$ of $D$ is totally skew-symmetric, i.e. a 3-form.

For a $\varphi$KT-connection $D$ with torsion tensor $T$ and tensor of deformation $Q$ from $\nabla$ to $D$ determined by (13) we have

\begin{equation}
T(x, y, z) = 2Q(x, y, z).
\end{equation}

Therefore, $Q$ is also a 3-form.

**Lemma 4.** Let $D$ be a $\varphi$KT-connection on $(M, \varphi, \xi, \eta, g)$. Then the torsion $T$ of $D$ and the Nijenhuis tensor $N$ on $(M, \varphi, \xi, \eta, g)$ have the following relation

\begin{equation}
N(x, y, z) = T(x, y, z) + T(x, \varphi y, \varphi z) + T(\varphi x, y, \varphi z) - T(\varphi x, \varphi y, z).
\end{equation}

**Proof.** The statement follows from (9), Theorem 3 and (16). \qed
Proposition 5. If a \(\varphi KT\)-connection exists on \((M, \varphi, \xi, \eta, g)\) then \(\xi\) is a Killing vector field and \(\mathcal{S} F = 0\), i.e. \((M, \varphi, \xi, \eta, g)\) belongs to the class \(\mathcal{F}_3 \oplus \mathcal{F}_7\).

Proof. Let us assume that a \(\varphi KT\)-connection \(D\) exists. Then, according to Theorem 3 the conditions (14) and (15) are valid for \(Q = \frac{1}{4} T\). Taking the cyclic sum in (14), because of (16), we obtain

\[
F(x, y, z) + F(y, z, x) + F(z, x, y) = \frac{1}{2} \left( T(x, y, \varphi z) + T(y, z, \varphi x) + T(z, x, \varphi y) - T(x, \varphi y, z) - T(y, \varphi z, x) - T(z, \varphi x, y) \right).
\]

Since \(T\) is totally skew-symmetric therefore

\[
\mathcal{S} \left( x, y, z \right) F(x, y, z) = 0.
\]

Using (14) and (16) we obtain \(2 F(x, \varphi y, \xi) = T(x, y, \xi)\) and since \((\nabla x \eta) y = \varphi T(x, y, \xi)\) we have

\[
2 (\nabla x \eta) y = T(x, y, \xi).
\]

Then, bearing in mind (7), the statement \(T\) is a 3-form implies that \(\xi\) is Killing and according to Lemma 1, the statement is true. □

Theorem 6. Let \((M, \varphi, \xi, \eta, g)\) be in the class \(\mathcal{F}_3 \oplus \mathcal{F}_7\). Then there exists a \(\varphi KT\)-connection \(D\) determined by

\[
g(D_x y, z) = g(\nabla_x y, z) + \frac{1}{2} T(x, y, z),
\]

where the torsion \(T\) is defined by

\[
T(x, y, z) = (\eta \wedge d \eta) \left( x, y, z \right) + \frac{1}{4} \mathcal{S} \left( x, y, z \right) N(x, y, z)
\]

or equivalently

\[
T(x, y, z) = -\frac{1}{2} \mathcal{S} \left( x, y, z \right) \left\{ F(x, y, \varphi z) - 3 \eta(x) F(y, \varphi z, \xi) \right\}.
\]

Proof. Let us suppose that \((M, \varphi, \xi, \eta, g) \in \mathcal{F}_3 \oplus \mathcal{F}_7\) (i.e. \(\xi\) is a Killing vector field and \(\mathcal{S} F = 0\)) and define a connection \(D\) with torsion \(T\) determined by (19).

We substitute \(\varphi z\) for \(z\) in (19), apply properties (3)–(5) and obtain consequently

\[
T(x, y, \varphi z) = -\frac{1}{2} \left\{ F(x, y, \varphi^2 z) + 3 \eta(x) F(y, z, \xi) + F(y, \varphi z, \varphi x) + 3 \eta(y) F(z, x, \xi) + F(\varphi z, x, \varphi y) \right\}
\]

\[
= -\frac{1}{2} \left\{ - F(x, y, z) + 2 \eta(x) F(y, z, \xi) + 2 \eta(z) F(x, y, \xi) + F(y, z, x) + 3 \eta(y) F(z, x, \xi) + F(\varphi z, x, \varphi y) \right\}
\]

Analogously, from (19) we obtain the following

\[
T(x, \varphi y, z) = -\frac{1}{2} \left\{ F(x, y, z) + 3 \eta(x) F(y, z, \xi) + 2 \eta(y) F(z, x, \xi) + F(\varphi y, z, \varphi x) - F(z, x, y) + 2 \eta(z) F(x, y, \xi) \right\}
\]

Then we subtract (21) from (20) and applying (3)–(5) again, we get

\[
T(x, y, \varphi z) - T(x, \varphi y, z) = 2 F(x, y, z).
\]

Therefore (14) is valid for the connection \(D\) with property \(T = 2 Q\).
Moreover, we form the expression $T(x, y, z) + T(x, z, y)$ using (19). By virtue of the consequence $F(y, \varphi z, \xi) = -F(z, \varphi y, \xi)$ of (7) and the symmetry of $F$ by the latter two arguments known from (3), we obtain

$$T(x, y, z) + T(x, z, y) = \frac{1}{2} \mathcal{S}_{x,y,z} F(\varphi x, y, z).$$

According to (3) and because $\xi$ is Killing, we have

$$T(x, y, z) + T(x, z, y) = \frac{1}{2} \mathcal{S}_{x,y,z} F(\varphi x, \varphi y, \varphi z).$$

Since the cyclic sum of $F$ over three arguments is zero for $F_3 \oplus F_7$ then (15) is valid and $T$ is a 3-form. Moreover, according to Theorem 3, the connection $D$ with torsion $T$ determined by (19) is natural for the almost contact B-metric structure. Therefore $D$ is a $\varphi$KT-connection.

In addition to that, according to (17) and (7), for a $\varphi$KT-connection we have

$$d(\eta(x, y)) = 2(\nabla_x \eta)y = T(x, y, \xi) = 2F(x, \varphi y, \xi).$$

Hence, the following equalities hold

$$(\eta \wedge d\eta)(x, y, z) = \mathcal{S}_{x,y,z} \eta(x)d\eta(y, z) = \mathcal{S}_{x,y,z} \eta(x)T(x, y, \xi)
= \mathcal{S}_{x,y,z} \eta(x)F(x, \varphi y, \xi).$$

By virtue of the formula in Lemma 4, we obtain

$$\mathcal{S}_{x,y,z} N(x, y, z) = 3T(x, y, z) + T(x, \varphi y, \varphi z)
+ T(\varphi x, y, \varphi z) + T(\varphi x, \varphi y, z).$$

Using (22), (24), and (25) and the totally skew-symmetry of $T$, we get the equivalence of (18) and (19). □

Bearing in mind (2) and $F(x, \varphi y, \xi) = (\nabla_x \eta)y = g(\nabla_x \xi, y)$, the torsion $(0,3)$-field $T$ of $D$, given in (19), has the following form of type (1,2)

$$T(x, y) = \frac{1}{2}\{2(\nabla_x \varphi) \varphi y - (\nabla_y \varphi) \varphi x + (\nabla_{\varphi y} \varphi) x
+ 3\eta(x)\nabla_y \xi - 4\eta(y)\nabla_x \xi + 2(\nabla_x \eta)y, \xi\}.$$

Formula (26) implies the following

**Corollary 7.** The $\varphi$KT-connection $D$ determined by (26) has the properties:

$$T(x, \varphi y) = \varphi T(x, y) - 2(\nabla_x \varphi) y, \quad T(\varphi x, y) = \varphi T(x, y) + 2(\nabla_y \varphi) x.$$  

**Proof.** The former equality is a consequence of (22). The latter one follows from the former equality and the totally skew-symmetry of $T$. □

### 4. Curvature properties of the $\varphi$KT-connection

The curvature tensor $K$ for a linear connection $D$ is determined as usual by

$$K(x, y)z = [D_x, D_y]z - D_{[x,y]}z \quad \text{and} \quad K(x, y, z, w) = g(K(x, y)z, w).$$
According to (13) and (15), the curvature tensor $K$ of a connection $D$ with the condition $Dg = 0$ has the form

$$K(x, y, z, w) = R(x, y, z, w) + (D_x Q)(y, z, w) - (D_y Q)(x, z, w)$$
$$+ g(Q(y, z), Q(x, w)) - g(Q(x, z), Q(y, w)) + Q(T(x, y), z, w).$$

(27)

Using (27) and (16), we obtain the following known equality for a connection $D$ with totally skew-symmetric torsion $T$ and a $D$-parallel metric $g$ (cf. [6])

$$K(x, y, z, w) = R(x, y, z, w) + \frac{1}{2} (D_z T)(y, z, w) - \frac{1}{2} (D_y T)(x, z, w)$$
$$+ \frac{1}{4} g(T(x, y), T(z, w)) + \frac{1}{4} \mathfrak{S}_{x,y,z} \{ g(T(x, y), T(z, w)) \}.$$

(28)

Since the scalar curvatures of $D$ and $\nabla$ are determined by the equalities $\tau^D = g^{ij}g^{ks}K(e_i, e_k, e_s, e_j)$ and $\tau = g^{ij}g^{ks}R(e_i, e_k, e_s, e_j)$, respectively, then relation (28) implies

$$\tau^D = \tau - \frac{1}{4} \|T\|^2,$$

where $\|T\|^2 = g^{ij}g^{ks}g(T(e_i, e_k), T(e_j, e_s)).$

**Theorem 8.** The curvature tensor $K$ of a $\varphi KT$-connection $D$ on $(M, \varphi, \xi, \eta, g)$ is of $\varphi$-Kähler type if and only if it has the form

$$K(x, y, z, w) = R(x, y, z, w) + \frac{1}{4} g(T(x, y), T(z, w))$$
$$- \frac{1}{12} \mathfrak{S}_{x,y,z} \{ g(T(x, y), T(z, w)) \}.$$

(30)

**Proof.** Since $K$ is a curvature tensor of a natural connection then property (12) is valid for $K$. Hence, $K$ becomes a $\varphi$-Kähler-type tensor when $\mathfrak{S}K = 0$ is satisfied additionally. The first Bianchi identity for a $\varphi KT$-connection $D$ with torsion $T$ can be written in the form

$$\mathfrak{S}_{x,y,z} K(x, y, z, w) = \mathfrak{S}_{x,y,z} (D_x T)(y, z, w) + \mathfrak{S}_{x,y,z} \{ g(T(x, y), T(z, w)) \},$$

because of $Dg = 0$ and $T$ is a 3-form. Then, $K$ is of $\varphi$-Kähler type if and only if the following is valid

$$\mathfrak{S}_{x,y,z} (D_x T)(y, z, w) = - \mathfrak{S}_{x,y,z} \{ g(T(x, y), T(z, w)) \}.$$

(31)

In this case, according to (28) and (31), $K$ has the form

$$K(x, y, z, w) = R(x, y, z, w) - \frac{1}{2} (D_z T)(x, y, w)$$
$$- \frac{1}{4} g(T(y, z), T(x, w)) - \frac{1}{4} g(T(z, x), T(y, w)).$$

(32)

Hence, we change the positions of $y$ and $w$ in (32) and add the obtained equality to the original form of (32). In the result we exchange $z$ and $w$ and subtract the obtained equality from the original one. In such a way we get (30).
Let us consider the case when \( D \) has a \( D\)-parallel torsion tensor \( T \). Then, using (28), the curvature tensor \( K \) gets the following form
\[
K(x, y, z, w) = R(x, y, z, w) + \frac{1}{4} g(T(x, y), T(z, w)) + \frac{1}{4} S_{x,y,z} \{ g(T(x, y), T(z, w)) \}.
\]

**Theorem 9.** Let a \( \varphiKT \)-connection \( D \) have a \( D\)-parallel torsion tensor \( T \) on \( (M, \varphi, \xi, \eta, g) \). Then the following conditions are equivalent:

(i) The curvature tensor \( K \) is a \( \varphi \)-Kähler-type tensor and it has the form
\[
K(x, y, z, w) = R(x, y, z, w) + \frac{1}{4} g(T(x, y), T(z, w)).
\]

(ii) The torsion 3-form \( T \) is closed;

(iii) The following equality is valid
\[
S_{x,y,z} \{ g(T(x, y), T(z, w)) \} = 0.
\]

**Proof.** Bearing in mind (33) and (30), we get that (i) and (iii) are equivalent, according to Theorem 8.

Since the torsion tensor \( T \) of a \( \varphiKT \)-connection \( D \) is a 3-form, then the exterior derivative of \( T \) has the expression
\[
dT(x, y, z, w) = S_{x,y,z} (D_x T)(y, z, w) - (D_w T)(x, y, z) + 2 S_{x,y,z} \{ g(T(x, y), T(z, w)) \}.
\]

If \( DT = 0 \) then \( dT(x, y, z, w) = 2 S_{x,y,z} \{ g(T(x, y), T(z, w)) \} \) is valid. Therefore, conditions (ii) and (iii) are equivalent. \( \square \)

**Proposition 10.** Let \( (M, \varphi, \xi, \eta, g) \) be a manifold in \( F_3 \oplus F_7 \) with a \( D\)-parallel torsion tensor \( T \) and a curvature tensor \( K \) of \( \varphi \)-Kähler type for the \( \varphiKT \)-connection \( D \). The curvature tensor \( R \) has the following properties with respect to the structure:
\[
R(x, y, \varphi z, \varphi w) = -R(x, y, z, w) - \frac{1}{4} g(T(x, y), T(z, w) + T(\varphi z, \varphi w)),
\]
\[
R(x, y, z, \xi) = \frac{1}{2} g(T(x, y), \nabla_z \xi),
\]
where \( T \) is determined by (26).

**Proof.** Bearing in mind Theorem 9, we apply property (12) for \( K \) determined by (34) and obtain the former equality in the statement. It implies the latter equality since \( \varphi \xi = 0 \) and \( T(z, \xi) = -2 \nabla_z \xi \). \( \square \)

Further, in the next two subsections, we consider curvature properties of the \( \varphiKT \)-connection \( D \) in the two special cases – on the horizontal component \( F_3 \) and on the vertical component \( F_7 \) of the class \( F_3 \oplus F_7 \), where \( D \) exists.
4.1. Curvature properties of the $\varphi$KT-connection on the horizontal component $\mathcal{F}_3$. Since the horizontal restriction of any $\mathcal{F}_3$-manifold is an almost complex manifold with Norden metric in the class $\mathcal{W}_3$ (also known as a quasi-Kähler manifold with Norden metric), then the curvature properties can be obtained in an analogous way as in [20]. Besides, the curvature properties of a $\mathcal{F}_3$-manifold are consequences of the corresponding properties for any manifold in $\mathcal{F}_3 \oplus \mathcal{F}_7$, given in the general part of Section 4.

Bearing in mind (26), the torsion of the $\varphi$KT-connection $D$ on a $\mathcal{F}_3$-manifold has the form

$$T(x, y) = \frac{1}{2} \{ 2 (\nabla_x \varphi) \varphi y - (\nabla_y \varphi) \varphi x + (\nabla_{\varphi y} \varphi) y \}.$$  

According to Theorem 8 the curvature tensor $K$ of this connection on a $\mathcal{F}_3$-manifold $(M, \varphi, \xi, \eta, g)$ is of $\varphi$-Kähler type if and only if $K$ has the form in (30), where $T$ is determined by (36). Hence, by virtue of (29), (36) and (6), we get immediately the following relation between the scalar curvatures for the $\varphi$KT-connection $D$ and the Levi-Civita connection $\nabla$:

$$\tau_D = \tau + \frac{3}{8} \|
abla \varphi\|^2.$$  

The following statement is a direct consequence of the latter equality.

Corollary 11. Let $(M, \varphi, \xi, \eta, g)$ be a $\mathcal{F}_3$-manifold with a curvature tensor of $\varphi$-Kähler type for the $\varphi$KT-connection $D$. Then $(M, \varphi, \xi, \eta, g)$ is an isotropic-$\mathcal{F}_0$-manifold if and only if the scalar curvatures for $D$ and $\nabla$ are equal.

According to Proposition 10 and Corollary 7, we obtain the following

Corollary 12. Let $(M, \varphi, \xi, \eta, g)$ be a $\mathcal{F}_3$-manifold with a $D$-parallel torsion tensor $T$ and a curvature tensor $K$ of $\varphi$-Kähler type for the $\varphi$KT-connection $D$. The curvature tensor $R$ has the following properties with respect to the structure:

$$R(x, y, z, \xi) = 0,$$

$$R(x, y, \varphi z, \varphi w) = -R(x, y, z, w) + \frac{1}{2} g \left( T(x, y), (\nabla_{\varphi z} \varphi) w + (\nabla_{\varphi w} \varphi) z \right),$$

where $T$ is determined by (36).

4.2. Curvature properties of the $\varphi$KT-connection on the vertical component $\mathcal{F}_7$. Now, we will pay attention on the other case when the manifold $(M, \varphi, \xi, \eta, g)$ belongs to the class $\mathcal{F}_7$, i.e. the manifold has the properties: a zero cyclic sum of $F$, a Killing $\xi$ and a nonzero $\nabla \xi$ (or the horizontal component of $F$ is zero). For such a manifold the torsion of the $\varphi$KT-connection $D$ has the form

$$T(x, y) = 2 \{ \eta(x) \nabla_y \xi - \eta(y) \nabla_x \xi + (\nabla_x \eta) y \xi \}$$

or

$$T(x, y, z) = (\eta \wedge d\eta) (x, y, z, z) = 2 \sum_{x, y, z} \{ \eta(x) F(y, \varphi z, \xi) \}.$$
Using (28) and (37), by direct calculations we get the following form of $K$ on a $\mathcal{F}_7$-manifold:

$$K(x, y, z, w) = R(x, y, z, w) + 2(\nabla_x \eta)(\nabla_y \eta)w$$

$$+ (\nabla_y \eta)(\nabla_x \eta)x(\nabla_z \eta)z - \eta(x)(\nabla_y \nabla_z \eta)w - \eta(y)(\nabla_x \nabla_z \eta)z$$

Equations (29), (37) and (38) yield the following relation between the scalar curvatures for $D$ and $\nabla$:

$$\tau^D = \tau + \frac{3}{2}\|\nabla \phi\|^2.$$

The last equality implies the following

**Corollary 13.** Let $(M, \phi, \xi, \eta, g) \in \mathcal{F}_7$. Then $(M, \phi, \xi, \eta, g)$ is an isotropic-$\mathcal{F}_0$-manifold if and only if the scalar curvatures for $D$ and $\nabla$ are equal.

**Proposition 14.** The curvature tensor $K$ of the $\phi$KT-connection $D$ on a $\mathcal{F}_7$-manifold $(M, \phi, \xi, \eta, g)$ is of $\phi$-Kähler type if and only if it has the form

$$K(x, y, z, w) = R(x, y, z, w)$$

$$- \eta(y)\eta(z)g(\nabla_x \xi, \nabla_w \xi) + \eta(x)\eta(z)g(\nabla_y \xi, \nabla_w \xi)$$

Equations (29), (37) and (38) yield the following relation between the scalar curvatures for $D$ and $\nabla$:
\[ \nabla \text{ has the following properties with respect to the structure:} \]
\[ R(x, y, \varphi z, \varphi w) = - R(x, y, z, w) \]
\[ + \eta(y)\eta(z)g(\nabla_x \xi, \nabla_w \xi) - \eta(x)\eta(z)g(\nabla_y \xi, \nabla_w \xi) \]
\[ + \eta(x)\eta(w)g(\nabla_y \xi, \nabla_z \xi) - \eta(y)\eta(w)g(\nabla_x \xi, \nabla_z \xi) \]
\[ + \frac{1}{3} \{(\nabla_x \eta) z (\nabla_y \eta) w - (\nabla_x \eta) w (\nabla_y \eta) z \]
\[ + (\nabla_x \eta) \varphi z (\nabla_y \eta) \varphi w - (\nabla_x \eta) \varphi w (\nabla_y \eta) \varphi z \}, \]
(40)

\[ R(x, y, z, \xi) = \eta(x)g(\nabla_y \xi, \nabla_z \xi) - \eta(y)g(\nabla_x \xi, \nabla_z \xi). \]
(41)

**Proof.** Since \( D\varphi = 0 \) then \( K \) satisfies (12), i.e. \( K \) is a \( \varphi \)-Kähler-type tensor. In (39) we substitute \( \varphi z \) for \( z \) and \( \varphi w \) for \( w \) and add the obtained equality to (39). Then we obtain (40), which gives the defect of property (12) for \( R \). Property (41) follows from (39) using \( K(x, y, z, \xi) = 0 \), because of \( D\xi = 0 \).

Now, let us consider the \( \varphi \)-KT-connection \( D \) with \( D \)-parallel torsion \( T \), i.e. \( DT = 0 \), on almost contact B-metric manifolds \((M, \varphi, \xi, \eta, g)\) in the class \( \mathcal{T}_7 \).

According to Theorem 9, 35 and 37, the following two statements hold.

**Proposition 16.** Let the \( \varphi \)-KT-connection \( D \) have a \( D \)-parallel torsion \( T \) on a \( \mathcal{T}_7 \)-manifold \((M, \varphi, \xi, \eta, g)\). Then the curvature tensor \( K \) has the form
\[ K(x, y, z, w) = R(x, y, z, w) \]
\[ - \eta(y)\eta(z)g(\nabla_x \xi, \nabla_w \xi) + \eta(x)\eta(z)g(\nabla_y \xi, \nabla_w \xi) \]
\[ - \eta(x)\eta(w)g(\nabla_y \xi, \nabla_z \xi) + \eta(y)\eta(w)g(\nabla_x \xi, \nabla_z \xi) \]
\[ + 2(\nabla_x \eta) y (\nabla_y \eta) w + (\nabla_y \eta) z (\nabla_x \eta) w + (\nabla_x \eta) x (\nabla_y \eta) w. \]
(42)

**Proof.** The equality (42) is a consequence of (39), 37 and the following calculations
\[ g(T(x, y), T(z, w)) = 4 \{(\nabla_x \eta) y (\nabla_z \eta) w \]
\[ - \eta(y)\eta(z)g(\nabla_x \xi, \nabla_w \xi) + \eta(x)\eta(z)g(\nabla_y \xi, \nabla_w \xi) \]
\[ - \eta(x)\eta(w)g(\nabla_y \xi, \nabla_z \xi) + \eta(y)\eta(w)g(\nabla_x \xi, \nabla_z \xi) \}
\[ = \sum_{x,y,z} \{g(T(x, y), T(z, w))\} - \sum_{x,y,z} \{(\nabla_x \eta) y (\nabla_z \eta) w \}
\[ = \frac{1}{2} (d\eta \wedge d\eta)(x, y, z, w). \]
(43)

**Corollary 17.** Let the \( \varphi \)-KT-connection \( D \) have a \( D \)-parallel torsion \( T \) on \((M, \varphi, \xi, \eta, g) \in \mathcal{T}_7 \). If the curvature tensor of \( D \) is of \( \varphi \)-Kähler type then
\[ K(x, y, z, w) = R(x, y, z, w) + (\nabla_x \eta) y (\nabla_z \eta) w \]
\[ - \eta(y)\eta(z)g(\nabla_x \xi, \nabla_w \xi) + \eta(x)\eta(z)g(\nabla_y \xi, \nabla_w \xi) \]
\[ - \eta(x)\eta(w)g(\nabla_y \xi, \nabla_z \xi) + \eta(y)\eta(w)g(\nabla_x \xi, \nabla_z \xi). \]
(44)

**Proof.** Expression (44) follows from (42), (43) and (35).
Corollary 18. Let the $\varphi KT$-connection $D$ have a curvature tensor of $\varphi$-Kähler type and a $D$-parallel torsion tensor $T$ on $(M, \varphi, \xi, \eta, g) \in \mathcal{F}_7$. The curvature tensor $R$ has property (11) and the following property:
\[ R(x, y, \varphi z, \varphi w) = - R(x, y, z, w) + \eta(y) \eta(z) g(\nabla_x \xi, \nabla_w \xi) - \eta(x) \eta(z) g(\nabla_y \xi, \nabla_w \xi) + \eta(x) \eta(w) g(\nabla_x \xi, \nabla_z \xi) \cdot \]

5. A Lie group as a 5-dimensional $\mathcal{F}_7$-manifold

Let $G$ be a 5-dimensional real connected Lie group and let $\mathfrak{g}$ be its Lie algebra. If $\{E_i\} \ (i = 1, 2, \ldots, 5)$ is a global basis of left-invariant vector fields of $G$, we define an invariant almost contact structure $(\varphi, \xi, \eta)$ and a left invariant B-metric $g$ on $G$

g as follows:
\[ \varphi E_1 = E_3, \quad \varphi E_2 = E_4, \quad \varphi E_3 = -E_1, \quad \varphi E_4 = -E_2, \quad \varphi E_5 = 0; \]
\[ \xi = E_5; \quad \eta(E_i) = 0 \ (i = 1, 2, 3, 4), \quad \eta(E_5) = 1; \]
\[ g(E_1, E_1) = g(E_2, E_2) = -g(E_3, E_3) = -g(E_4, E_4) = g(E_5, E_5) = 1, \]
\[ g(E_i, E_j) = 0, \quad i \neq j, \quad i, j \in \{1, 2, 3, 4, 5\}. \]

Thus, because of (11), the induced 5-dimensional manifold $(G, \varphi, \xi, \eta, g)$ is an almost contact B-metric manifold.

By analogy with the non-Abelian almost complex structure on an even-dimensional Lie group we introduce the following notion. An almost contact structure on a Lie group $G$ is called non-Abelian if the following property holds with respect to the Lie algebra $\mathfrak{g}$
\[ [\varphi X, \varphi Y] = -[X, Y]. \]

Obviously, the last equation implies $[\xi, Y] = 0$ which is a sufficient condition the vector field $\xi$ to be Killing with respect to $g$.

Proposition 19. Let $(G, \varphi, \xi, \eta, g)$ be an almost contact B-metric manifold with a non-Abelian almost contact structure on the Lie group $G$. Then the manifold $(G, \varphi, \xi, \eta, g)$ belongs to the class $\mathcal{F}_3 \oplus \mathcal{F}_7$.

Proof. The known property of the Levi-Civita connection $\nabla$ of $g$ on $(G, \varphi, \xi, \eta, g)$
\[ 2g(\nabla_X Y, Z) = g([X, Y], Z) + g([Z, X], Y) + g([Z, Y], X) \]
and the equivalent condition $[\varphi X, Y] = [X, \varphi Y]$ of (17) imply
\[ 2F(X, Y, Z) = g([X, \varphi Y] - [\varphi X, Y], Z) + g([X, \varphi Z] - [\varphi X, Z], Y). \]

Therefore, using (17), we obtain
\[ \sum_{x, y, z} F(X, Y, Z) = \sum_{x, y, z} g([X, \varphi Y] - [\varphi X, Y], Z) = 0. \]

Hence and since $\xi$ is Killing, because of (17), we establish that $(G, \varphi, \xi, \eta, g)$ belongs to $\mathcal{F}_3 \oplus \mathcal{F}_7$. \hfill \Box

Corollary 20. Let $(G, \varphi, \xi, \eta, g)$ be an almost contact B-metric manifold with non-Abelian almost contact structure on the Lie group $G$. Then the necessary and sufficient conditions $(G, \varphi, \xi, \eta, g)$ to belong to the component classes of $\mathcal{F}_3 \oplus \mathcal{F}_7$ are respectively:
\[ \mathcal{F}_3 : \eta([X, Y]) = 0; \quad \mathcal{F}_7 : \varphi [\varphi X, Y] = \varphi^2 [X, Y]; \quad \mathcal{F}_0 : [X, Y] = -\varphi [\varphi X, Y]. \]
Proof. From Proposition 19 and 49, using 14, 15 and \(\mathcal{F}_0 = \mathcal{F}_3 \cap \mathcal{F}_7\), we obtain the above conditions.

Now, let us consider the Lie algebra \(\mathfrak{g}\) on \(G\), determined by the following non-zero commutators:

\[
\begin{align*}
[E_1, E_2] &= -[E_3, E_4] = -\lambda_1 E_1 - \lambda_2 E_2 + \lambda_3 E_3 + \lambda_4 E_4 + 2\mu_1 E_5, \\
[E_1, E_4] &= -[E_2, E_3] = -\lambda_3 E_1 - \lambda_4 E_2 - \lambda_1 E_3 - \lambda_2 E_4 + 2\mu_2 E_5,
\end{align*}
\]

where \(\lambda_i, \mu_j \in \mathbb{R} \quad (i = 1, 2, 3, 4; j = 1, 2)\).

Then we prove the following

**Theorem 21.** Let \((G, \varphi, \xi, \eta, g)\) be the almost contact B-metric manifold, determined by (15), (16) and (50). Then it belongs to the class \(\mathcal{F}_7\).

Proof. We check directly that the commutators in (50) satisfy the condition for \(\mathcal{F}_7\) from Corollary 20. Therefore, we establish that the corresponding manifold \((G, \varphi, \xi, \eta, g)\) belongs to the class \(\mathcal{F}_7\). It is not a \(\mathcal{F}_0\)-manifold if and only if \(\mu_1^2 + \mu_2^2 \neq 0\) holds.

Using (15) and (50), we obtain the components of the Levi-Civita connection on the manifold:

\[
\begin{align*}
\nabla_{E_1} E_1 &= -\nabla_{E_4} E_3 = \lambda_1 E_2 - \lambda_3 E_4, \\
\nabla_{E_1} E_2 &= -\nabla_{E_3} E_4 = -\lambda_1 E_1 + \lambda_3 E_3 + \mu_1 E_5, \\
\nabla_{E_1} E_3 &= \nabla_{E_4} E_1 = \lambda_3 E_2 + \lambda_1 E_4, \\
\nabla_{E_1} E_4 &= \nabla_{E_3} E_2 = -\lambda_3 E_1 - \lambda_1 E_3 + \mu_2 E_5, \\
\nabla_{E_2} E_1 &= -\nabla_{E_4} E_3 = \lambda_2 E_2 - \lambda_4 E_4 - \mu_1 E_5, \\
\nabla_{E_2} E_2 &= -\nabla_{E_3} E_4 = -\lambda_2 E_1 + \lambda_4 E_3, \\
\nabla_{E_2} E_3 &= \nabla_{E_4} E_1 = \lambda_4 E_2 + \lambda_2 E_4 - \mu_2 E_5, \\
\nabla_{E_2} E_4 &= \nabla_{E_3} E_2 = -\lambda_4 E_1 - \lambda_2 E_3, \\
\nabla_{E_3} E_1 &= -\nabla_{E_4} E_2 = -\mu_1 E_2 + \mu_2 E_4, \\
\nabla_{E_3} E_2 &= \nabla_{E_4} E_1 = \mu_1 E_1 - \mu_2 E_3, \\
\nabla_{E_3} E_3 &= \nabla_{E_4} E_2 = -\mu_2 E_1 - \mu_1 E_4, \\
\nabla_{E_3} E_4 &= \nabla_{E_4} E_1 = \mu_2 E_2 + \mu_1 E_3, \\
\nabla_{E_3} E_5 &= 0.
\end{align*}
\]

Let us consider the \(\varphi\)KT-connection \(D\) on \((G, \varphi, \xi, \eta, g)\) defined by (15), (16) and (37). Then, by (51) we compute its components as follows:

\[
\begin{align*}
D_{E_1} E_1 &= -D_{E_4} E_3 = \lambda_1 E_2 - \lambda_3 E_4, \\
D_{E_1} E_2 &= -D_{E_3} E_4 = -\lambda_1 E_1 + \lambda_3 E_3, \\
D_{E_1} E_3 &= D_{E_4} E_1 = \lambda_3 E_2 + \lambda_1 E_3, \\
D_{E_1} E_4 &= D_{E_3} E_2 = -\lambda_3 E_1 - \lambda_1 E_4, \\
D_{E_2} E_1 &= -D_{E_3} E_4 = \lambda_2 E_2 - \lambda_4 E_4, \\
D_{E_2} E_2 &= -D_{E_4} E_3 = -\lambda_2 E_1 + \lambda_4 E_3, \\
D_{E_2} E_3 &= D_{E_4} E_1 = \lambda_4 E_2 + \lambda_2 E_4, \\
D_{E_2} E_4 &= D_{E_3} E_2 = -\lambda_4 E_1 - \lambda_2 E_3, \\
D_{E_3} E_1 &= -2\mu_1 E_2 + 2\mu_2 E_4, \\
D_{E_3} E_2 &= 2\mu_1 E_1 - 2\mu_2 E_3, \\
D_{E_3} E_3 &= -2\mu_2 E_2 - 2\mu_1 E_4, \\
D_{E_3} E_4 &= 2\mu_2 E_1 + 2\mu_1 E_3, \\
D_{E_3} E_5 &= 0, \quad (i = 1, 2, 3, 4, 5).
\end{align*}
\]

The basic non-zero components of the torsion of \(D\) are only:

\[
\begin{align*}
T(E_1, E_2, E_5) &= T(E_3, E_4, E_5) = 2\mu_1, \\
T(E_2, E_3, E_5) &= T(E_4, E_1, E_5) = 2\mu_2.
\end{align*}
\]

Hence, using (52), we obtain that the corresponding components of the covariant derivative of \(T\) with respect to \(D\) are zero. Thus, we prove the following
Proposition 22. The $\varphi$-KT-connection $D$ with components (12) on the $\mathcal{T}_7$-manifold $(G, \varphi, \xi, \eta, g)$ has a $D$-parallel torsion $T$.

According to (10) and (11), the curvature tensor $R$ has the following non-zero components $R_{ijkl} = R(E_i, E_j, E_k, E_l)$:

\begin{align*}
R_{1212} &= R_{3434} = (\lambda_1^2 + \lambda_2^2 - \lambda_3^2 - \lambda_4^2) + 3\mu_1^2, \\
R_{1234} &= R_{3412} = - (\lambda_1^2 + \lambda_2^2 - \lambda_3^2 - \lambda_4^2) - 2\mu_1^2 + \mu_2^2, \\
R_{1414} &= R_{2323} = - (\lambda_1^2 + \lambda_2^2 - \lambda_3^2 - \lambda_4^2) + 3\mu_2^2, \\
R_{1423} &= R_{2314} = (\lambda_1^2 + \lambda_2^2 - \lambda_3^2 - \lambda_4^2) + \mu_1^2 - 2\mu_2^2, \\
R_{1214} &= -R_{1223} = -R_{2312} = R_{2334} = R_{1412} = -R_{1434} \\
&= -R_{3414} = R_{3423} = 2(\lambda_1\lambda_3 + \lambda_2\lambda_4) + 3\mu_1\mu_2, \\
R_{1324} &= R_{2413} = -(\mu_1^2 + \mu_2^2), \\
R_{1515} &= R_{2525} = -R_{3535} = -R_{4545} = -\mu_1^2 + \mu_2^2, \\
\end{align*}

and the remaining ones are determined by (53) and properties (10) and (11).

Next we obtain the following basic components $K_{ijkl} = K(E_i, E_j, E_k, E_l)$ of the curvature tensor $K$ of $D$, using (52):

\begin{align*}
K_{1212} &= -K_{1234} = -K_{3412} = K_{3434} = (\lambda_1^2 + \lambda_2^2 - \lambda_3^2 - \lambda_4^2) + 4\mu_1^2, \\
K_{1414} &= -K_{1423} = -K_{2314} = K_{2334} = -(\lambda_1^2 + \lambda_2^2 - \lambda_3^2 - \lambda_4^2) + 4\mu_2^2, \\
K_{1214} &= -K_{1223} = -K_{2312} = K_{2334} = K_{1412} = -K_{1434} \\
&= -K_{3414} = K_{3423} = 2(\lambda_1\lambda_3 + \lambda_2\lambda_4) + 4\mu_1\mu_2.
\end{align*}

Hence, we establish immediately that $K$ is a $\varphi$-Kähler-type curvature tensor if and only if $\mu_1 = \mu_2 = 0$, which is equivalent to the trivial case when $(G, \varphi, \xi, \eta, g)$ is a $\mathcal{T}_0$-manifold. Bearing in mind Theorem 9 we establish that $(G, \varphi, \xi, \eta, g)$ has a weak $\varphi$-KT-structure (i.e. $dT \neq 0$).

Using (53) and (54), we compute the corresponding components of the Ricci tensor $\rho$ and $\rho^D$ and the values of the scalar curvatures $\tau$ and $\tau^D$ for the connections $\nabla$ and $D$, respectively:

\begin{align*}
\rho_{11} &= \rho_{22} = -\rho_{33} = -\rho_{44} = -2(\lambda_1^2 + \lambda_2^2 - \lambda_3^2 - \lambda_4^2) - 2(\mu_1^2 - \mu_2^2), \\
\rho_{13} &= \rho_{24} = -4(\lambda_1\lambda_3 + \lambda_2\lambda_4) - 4\mu_1\mu_2, \\
\rho_{11}^D &= \rho_{22}^D = -\rho_{33}^D = -\rho_{44}^D = -2(\lambda_1^2 + \lambda_2^2 - \lambda_3^2 - \lambda_4^2) - 4(\mu_1^2 - \mu_2^2), \\
\rho_{13}^D &= \rho_{24}^D = -4(\lambda_1\lambda_3 + \lambda_2\lambda_4) - 8\mu_1\mu_2.
\end{align*}

\[\tau = -8(\lambda_1^2 + \lambda_2^2 - \lambda_3^2 - \lambda_4^2) - 4(\mu_1^2 - \mu_2^2),\]
\[\tau^D = -8(\lambda_1^2 + \lambda_2^2 - \lambda_3^2 - \lambda_4^2) - 16(\mu_1^2 - \mu_2^2).\]

Bearing in mind the value of the square norm of $\nabla \varphi$ on $(G, \varphi, \xi, \eta, g)$

\[\|\nabla \varphi\|^2 = -8(\mu_1^2 - \mu_2^2),\]

we obtain that the constructed manifold is an isotropic-$\mathcal{T}_0$-manifold if and only if $\mu_1 = \pm \mu_2$.

Finally, we summarize the latter characteristics in the following
Proposition 23. Let \((G, \varphi, \xi, \eta, g)\) be the \(\mathcal{F}_7\)-manifold determined by (45), (46) and (50) and \(D\) be the \(\varphi\)KT-connection defined by (37). The following conditions are equivalent:

(i) The manifold \((G, \varphi, \xi, \eta, g)\) is an isotropic-\(\mathcal{F}_0\)-manifold;
(ii) The scalar curvatures for \(\nabla\) and \(D\) are equal;
(iii) The vectors \(\nabla_{E_i}\xi\) \((i = 1, 2, 3, 4)\) are isotropic;
(iv) The equality \(\mu_1 = \pm \mu_2\) is valid.

According to (46) and (55) we obtain

Proposition 24. The \(\mathcal{F}_7\)-manifold \((G, \varphi, \xi, \eta, g)\), determined by (45), (46) and (50), is Einsteinian if and only if the following conditions are valid

\[
\mu_1\mu_2 = - (\lambda_1\lambda_3 + \lambda_2\lambda_4), \quad \mu_1^2 - \mu_2^2 = -\frac{1}{3}(\lambda_1^2 + \lambda_2^2 - \lambda_3^2 - \lambda_4^2).
\]

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