Some structural graph properties of the non-commuting graph of a class of finite Moufang loops

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Abstract

For any non-abelian group $G$, the non-commuting graph of $G$, $\Gamma = \Gamma_G$, is graph with vertex set $G \setminus Z(G)$, where $Z(G)$ is the set of elements of $G$ that commute with every element of $G$ and distinct non-central elements $x$ and $y$ of $G$ are joined by an edge if and only if $xy \neq yx$. The non–commuting graph of a finite Moufang loop has been defined by Ahmadidelir. In this paper, we show that the multiple complete split-like graphs and the non-commuting graph of Chein loops of the form $M(D_{2n}, 2)$ are perfect (but not chordal). Then, we show that the non-commuting graph of a non-abelian group $G$ is split if and only if the non-commuting graph of the Moufang loop $M(G, 2)$ is 3—split. Precisely, we show that the non-commuting graph of the Moufang loop $M(G, 2)$, is 3—split if and only if $G$ is isomorphic to a Frobenius group of order $2n$, $n$ is odd, whose Frobenius kernel is abelian of order $n$. Finally, we calculate the energy of generalized and multiple split-like graphs, and discuss about the energy and also the number of spanning trees in the case of the non-commuting graph of Chein loops of the form $M(D_{2n}, 2)$.

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1. Introduction

Let \( Q \) be a set with one binary operation. Then it is a quasigroup if the equation \( xy = z \) has a unique solution in \( Q \) whenever two of the three elements \( x, y, z \in Q \) are specified. A quasigroup \( Q \) is a loop if \( Q \) possesses a neutral element \( e \), i.e., if \( ex = xe = x \) holds for every \( x \in Q \). Moufang loops are loops in which any of the (equivalent) Moufang identities,

\[
\begin{align*}
((xy)x) &= x(y(xz)), \\
x(y(zy)) &= ((xy)z)y, \\
(xy)(zx) &= x((yz)x), \\
(xy)(zx) &= (x(yz))x.
\end{align*}
\]

holds for every \( x, y, z \in Q \). Commutator of \( x \) and \( y \) and the associator of \( x, y \) and \( z \) are defined by \([x, y] = x^{-1}y^{-1}xy \) and \([x, y, z] = ((xy)z)^{-1}(x(yz)) \), respectively. We define the commutant (or Moufang center) \( C(Q) \) of \( Q \) as \( \{ x \in Q \mid xy = yx, \forall y \in Q \} \). The center \( Z(Q) \) of a Moufang loop \( Q \) is the set of all elements of \( Q \) which commute and associate with all other elements of \( Q \). A non-empty subset \( P \) of \( Q \) is called a subloop of \( Q \) if \( P \) is itself a loop under the binary operation of \( Q \), in particular, if this operation is associative on \( P \), then it is a subgroup of \( Q \). A subloop \( N \) of a loop \( Q \) is said to be normal in \( Q \) if \( xN = Nx \) and \( N(xy) = (Nx)y \); for every \( x, y \in Q \). In Moufang loop \( Q \), the subloops \( Z(Q) \) and \( C(Q) \) are normal subloops. For more details about the Moufang loops one may see [8, 16, 13]. In 1974, Chein introduced a class of non-associative Moufang loops \( M(G, 2) \), so called Chein loops. For a group \( G \) and a new element \( u, (u \notin G) \), \( M(G, 2) = G \cup Gu \) such that the multiplication with the new binary operation \( \circ \) is defined as follows:

\[
\begin{align*}
g \circ h &= gh, & g, h \in G, \\
g \circ (hu) &= (hg)u, & g \in G, \ hu \in Gu, \\
(gu) \circ h &= (gh^{-1})u, & gu \in Gu, \ h \in G, \\
(gu) \circ (hu) &= h^{-1}g, & gu, hu \in Gu.
\end{align*}
\]

Clearly, the Moufang loop \( M(G, 2) \) is non-associative if and only if \( G \) is non-abelian, see [8]. In [2], Ahmadidelir has investigated some probabilistic properties of \( M(G, 2) \), such as its commutativity degree.

There are many papers on assigning a graph to a ring or a group in order to investigation of their algebraic properties. For any non-abelian group \( G \) the non-commuting graph of \( G \), \( \Gamma = \Gamma_G \) is a graph with vertex set \( G \setminus Z(G) \), where distinct non-central elements \( x \) and \( y \) of \( G \) are joined by an edge if and only if \( xy \neq yx \). This graph is connected with diameter 2 and girth 3 for a non-abelian finite group and has received some attention in existing literature. For instance, one may see [1, 10, 15, 17]. Similarly, the non-commuting graph of a finite Moufang loop has been defined by Ahmadidelir in [3]. He has defined this graph as follows: Let \( M \) be a Moufang loop, then the vertex set is \( M \setminus C(M) \) and two vertices \( x \) and \( y \) joined by an edge whenever \( [x, y] \neq 1 \). He has shown that this graph is connected (as for groups) and obtained some results related to the non–commuting graph of a finite non-commutative Moufang loop.

We will denote a complete graph with \( n \) vertices by \( K_n \). All graphs considered in this paper are finite and simple. For a graph \( \Gamma \), we denote its vertex and edge sets by \( V(\Gamma) \) and \( E(\Gamma) \), respectively.
The complement of Γ is denoted by \( \bar{\Gamma} \). A graph \( \Gamma = (V, E) \), is called \( k \)-partite where \( k > 1 \), if it is possible to partition \( V \) into \( k \) subsets \( V_1, V_2, \ldots, V_k \), such that every edge of \( E \) joins a vertex of \( V_i \) to a vertex of \( V_j \), \( i \neq j \). A clique in a graph \( \Gamma \) is an induced subgraph whose all vertices are pairwise adjacent. The maximum size of a clique in a graph \( \Gamma \) is called the clique number of \( \Gamma \) and denoted by \( \omega(\Gamma) \). A subset \( X \) of the vertices of \( \Gamma \) is called an independent set (or stable) if the induced subgraph on \( X \) has no edges. The maximum size of an independent set in a graph \( \Gamma \) is called the independence number of \( \Gamma \) and denoted by \( \alpha(\Gamma) \). The vertex chromatic number of a graph \( \Gamma \) is denoted by \( \chi(\Gamma) \), and it is the minimum \( k \) for which \( k \)-vertex coloring of a graph \( \Gamma \) such that no two adjacent vertices have the same color. For a subset \( S \) of \( V(\Gamma) \), \( N_{\Gamma}[S] \) is the set of vertices in \( \Gamma \) which are in \( S \) or adjacent to a vertex in \( S \). If \( N_{\Gamma}[S] = V(\Gamma) \) then \( S \) is said to be a dominating set of the vertices in \( \Gamma \). The minimum size of a dominating set of the vertices in \( \Gamma \) is called the dominating number of \( \Gamma \) and denoted by \( \gamma(\Gamma) \). A vertex cover of a graph \( \Gamma \) is a set \( Q \subseteq V(\Gamma) \) such that contains at least one endpoint of every edge. The minimum size of a vertex cover is denoted by \( \beta(\Gamma) \). Our other used notations about graphs are standard and for more details one may see [6, 7, 11].

There is a relation between \( \alpha(\Gamma) \) and \( \beta(\Gamma) \) as follows:

**Lemma 1.1.** ([7], p. 296) *Let \( \Gamma \) be a graph. Then \( \alpha(\Gamma) + \beta(\Gamma) = n(\Gamma) \), where \( n(\Gamma) \) is the number of vertices of \( \Gamma \).* \( \Box \)

A perfect graph \( \Gamma \), is a graph in which for every induced subgraph its clique number is equal to its chromatic number. A graph \( \Gamma \) is called weakly perfect graph if \( \omega(\Gamma) = \chi(\Gamma) \). So, all perfect graphs are weakly perfect. A chordal graph is one in which all cycles of order four or more have a chord, which is an edge that is not part of cycle but connects two vertices of the cycle. The class of Chordal graphs is a subset of the class of perfect graphs. For more information about these types of graphs, one may see [12, 14]. We have the following Theorem about perfect graphs, called strongly perfect graph theorem, or Berg Theorem.

A graph is called \( k \)-regular, if the vertices of the graph are of the same degree \( k \) and a strongly regular graph \( S \) with parameters \( (n, k, \lambda, \mu) \) is a \( k \)-regular graph of order \( n \) such that each pair of adjacent vertices has \( \lambda \) common neighbors and each pair of non-adjacent vertices has in which \( \mu \) common neighbors. Let \( \Gamma_1 = (V_1, E_1) \) and \( \Gamma_2 = (V_2, E_2) \) be undirected simple graphs. The union \( \Gamma_1 \cup \Gamma_2 \) of graphs \( \Gamma_1 \) and \( \Gamma_2 \) is a graph \( \Gamma = (V, E) \) for which \( V = V_1 \cup V_2 \) and \( E = E_1 \cup E_2 \). The notation \( n\Gamma \) is short for \( \bigcup_{\text{n-times}} \Gamma \).

The complete product \( \Gamma_1 \nabla \Gamma_2 \) of graph \( \Gamma_1 \) and \( \Gamma_2 \) is a graph obtained from \( \Gamma_1 \cup \Gamma_2 \) by joining every vertex of \( \Gamma_1 \) to every vertex of \( \Gamma_2 \). For every \( a, b, n \in N \), a complete split, or simply, a split graph, is the graph \( K_a \nabla K_b \) and denoted by \( CS_b^n \). By a theorem of Földes and Hammer ([12], Theorem 6.3), a graph is (complete) split iff contains no induced subgraph isomorphic to \( 2K_2, C_4 \) or \( C_5 \). Also, an undirected graph is split if and only if its complement is split ([12], Theorem 6.1). Clearly, every split graph is chordal and so perfect, but the converses are not true. More generally, a multiple complete split-like graph is \( K_a \nabla (nK_b) \) and denoted by \( MCS_{b,n}^a \). Specially, in this paper, for \( n = 3 \) we call \( MCS_{b,3}^a \) as a \( 3 \)-split graph.

We generalize the above definitions as follows:
Definition 1.1. The generalized complete split-like graph is $GCS_k^a = \overline{K}_a \nabla S$ such that $S$ is a strongly regular graph with parameters $(n, k, \lambda, \mu)$. The multiple generalized complete split-like graph is $GMCS_{k,m}^a = \overline{K}_a \nabla (mS)$.

The laplacian matrix of a simple graph $\Gamma$ with $n$ vertices, is defined as $L(\Gamma) = D(\Gamma) - A(\Gamma)$, where $A(\Gamma)$ is its adjacency matrix and $D(\Gamma) = (d_1, \ldots, d_n)$ is the diagonal matrix of the vertex degrees in $\Gamma$. For any graph $\Gamma$, the energy of $\Gamma$ is defined as $\xi(\Gamma) = \sum_{i=1}^{n} |\lambda_i|$, where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of the adjacency matrix of $\Gamma$. A spanning tree of a graph $\Gamma$ is an induced subgraph of $\Gamma$, which is a tree and contains every vertex of $\Gamma$.

In this paper, we show that the multiple complete split-like graphs are perfect (but not chordal) and deduce that the non-commuting graph of Chein loops of the form $M(D_{2n}, 2)$ is perfect but not chordal. Then, we show that the non-commuting graph of a non-abelian group $G$ is split if and only if the non-commuting graph of the Moufang loop $M(G, 2)$ is 3–split and then classify all Chein loops that their non-commuting graphs are 3–split. Precisely, we show that for a non-abelian group $G$, the non-commuting graph of the Moufang loop $M(G, 2)$, is 3–split if and only if $G$ is isomorphic to a Frobenius group of order $2^n$, $n$ is odd, whose Frobenius kernel is abelian of order $n$. Finally, we calculate the energy of generalized and multiple split-like graphs, and discuss about the energy and also the number of spanning trees in the case of the non-commuting graph of Chein loops of the form $M(D_{2n}, 2)$. We recall the following Proposition and Theorems in order to provide some tools to these purposes.

Theorem 1.1. ([5], p. 3: Schur complement) Let $A$ be a $n \times n$ matrix partitioned as $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$, where $A_{11}$ and $A_{22}$ are non-singular square matrices. Then the inverse of $A$, $A^{-1}$ can be calculated by the following formula:

$$A^{-1} = \begin{bmatrix} A_{11}^{-1} + A_{11}^{-1}A_{12}(A/A_{11})^{-1}A_{21}A_{11}^{-1} & -A_{11}^{-1}A_{12}(A/A_{11})^{-1} \\ -(A/A_{11})^{-1}A_{21}A_{11}^{-1} & (A/A_{11})^{-1} \end{bmatrix},$$

where

$$A/A_{11} = A_{22} - A_{21}A_{11}^{-1}A_{12}$$

and

$$\det A = \det A_{11} \times \det(A_{22} - A_{21}A_{11}^{-1}A_{12}),$$

such that $\det A$ is the determinant of $A$.

Theorem 1.2. ([14], Theorem 1) For $i = 1, 2$, let $\Gamma_i$ be $r_i$–regular graphs with $n_i$ vertices. Then the characteristic polynomial of the complete product of these two graphs is as follows:

$$P_{\Gamma_1 \nabla \Gamma_2}(\lambda) = \frac{P_{\Gamma_1}(\lambda)P_{\Gamma_2}(\lambda)}{(\lambda - r_1)(\lambda - r_2)[(\lambda - r_1)(\lambda - r_2) - n_1n_2]}.$$
2. Some basic graph properties of the Moufang loop $M(D_{2n}, 2)$

Let $D_{2n}$ denote the dihedral group of order $2n$, which has the following presentation:

$$D_{2n} = \langle a, b \mid a^n = b^2 = (ab)^2 = 1 \rangle.$$ 

In this section, we want to study the non-commuting graph of the Moufang loops $M(D_{2n}, 2)$, simply denoted by $\Gamma$. We will use the following Lemma in next sections.

The following Lemma determines the structure of the non-commuting graph of the Moufang loop $M = M(D_{2n}, 2)$.

**Lemma 2.1.** Let $M = M(D_{2n}, 2)$ and $\Gamma = \Gamma_M$ be its non-commuting graph.

(a) If $n$ is odd then $\Gamma_M \cong \bar{K}_{n-1} \nabla S$, such that $S$ is a strongly regular graph with parameters $(3n, n-1, n-2, 0)$.

(b) If $n$ is even then $\Gamma_M \cong \bar{K}_{n-2} \nabla 3S$, such that $S$ is a strongly regular graph with parameters $(n, n-2, n-3, n-2)$.

**Proof.** a) By Lemma ([3], Lemma 4.4) and the definition of the non-commuting graph, for every odd integer $n$, we can partition the vertices of $\Gamma$ into four sets, as follows:

$$t_1 = \{a, a^2, \ldots, a^{n-1}\}, \quad t_2 = \{b, ab, \ldots, a^{n-1}b\},$$

$$t_3 = \{u, au, \ldots, a^{n-1}u\}, \quad t_4 = \{bu, abu, \ldots, a^{n-1}bu\}.$$ 

For every $0 \leq i, j \leq n-1$, since $a^i a^j = a^j a^i$, $t_1$ is an independent set and from the relations $a^i (a^j b) \neq (a^i b) a^j$, $a^i (a^j u) \neq (a^j u) a^i$ and $a^i (a^j bu) \neq (a^j bu) a^i$, we find that all vertices of $t_1$ are adjacent to all vertices of each of the sets $t_2$, $t_3$ and $t_4$. Also, by the relations $(a^i b) (a^j b) \neq (a^j b) (a^i b)$, the induced subgraph $[t_2]$ of $\Gamma$, is a clique. Similarly, we can show that the induced subgraph $[t_3]$ and $[t_4]$ of $\Gamma$, are cliques. Hence, $\Gamma \cong \bar{K}_{n-1} \nabla 3K_n$ and the graph $\Gamma$ is $3-$split and $3K_n \cong S$, where $S$ is a strongly regular graph with parameters $(3n, n-1, n-2, 0)$.

b) Let $n$ be an even integer. Again, we can partition the vertices of $\Gamma$ into four sets, as follows:

$$t_1 = \{a, a^2, \ldots, a^{n-1}, a^{n+1}, \ldots, a^{n-1}\}, \quad t_2 = \{b, ab, \ldots, a^{n-1}b\},$$

$$t_3 = \{u, au, \ldots, a^{n-1}u\}, \quad t_4 = \{bu, abu, \ldots, a^{n-1}bu\}.$$ 

Since each pair of elements of $t_1$ commute, so the induced subgraph $[t_1]$ is an independent set, that means $[t_1] \cong \bar{K}_{n-2}$. Also, every element in $M$ commutes with its inverse and since, $\forall x \in t_i, (i = 2, 3, 4)$, its inverse $x^{-1}$ belongs to $t_i$. Therefore, every element of $t_i, (i = 2, 3, 4)$ is adjacent to each vertex in $t_i$, $i = 2, 3, 4$, except its inverse. Also any two elements $x, y$ in $t_i, (i = 2, 3, 4)$ commute if and only if $|i - j| = \frac{n}{2}$, where $x = a^i u$ or $a^i b$, $a^i bu$ and $y = a^j u$ or $a^j b$, $a^j bu$. Then $[t_i] \cong S$, where $S$ is a strongly regular graph with parameters $(n, n-2, n-3, n-2)$. Finally, for every $2 \leq i, j \leq 4$ there is no edge of $\Gamma$ such that joins a vertex of $t_i$ to a vertex of $t_j$, $i \neq j$, but each vertex in $t_1$ joins to each vertex in $t_i, (i = 2, 3, 4)$. Therefore, $\Gamma_M \cong \bar{K}_{n-2} \nabla 3S$.

In the following Theorem, we derive some important graph properties of $\Gamma_{M(D_{2n}, 2)}$. 

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Theorem 2.1. Let \( M = M(D_{2n}, 2) \) and \( \Gamma = \Gamma_M \) be its non-commuting graph.

(a) If \( n \) is odd then:

\[
\begin{align*}
\omega(\Gamma) &= n + 1, \\
\alpha(\Gamma) &= n - 1, \\
\beta(\Gamma) &= 3n, \\
(\Gamma) &= 2.
\end{align*}
\]

(b) If \( n \) is even then:

\[
\begin{align*}
\omega(\Gamma) &= \frac{n}{2} + 1, \\
\alpha(\Gamma) &= \begin{cases} 
6, & (n = 6) \\
 n - 2, & (n \geq 8) 
\end{cases}, \\
\beta(\Gamma) &= \begin{cases} 
16, & (n = 6) \\
3n, & (n \geq 8) 
\end{cases}, \\
(\Gamma) &= 2.
\end{align*}
\]

Proof. a) By Lemma 2.1, the non-commuting graph of \( M(D_{2n}, 2) \) is a generalized complete split-like graph for any odd integer \( n \). Then \( \Gamma = \bar{K}_{n-1} \nabla S \) in which \( S \) is a strongly regular graph with parameters \((3n, n - 1, n - 2, 0)\), where \( V(\bar{K}_{n-1}) = \{a, a^2, \ldots, a^{n-1}\} \) and \( S \cong 3K_n \). So this graph is 3-split. By the structure of \( \Gamma \), since every vertex of each copy of \( K_n \) is joined to every vertex of \( \bar{K}_{n-1} \), so we have the complete product \( K_n \nabla [a^i] \), where \( a^i \in \bar{K}_{n-1} \), \( 1 \leq i \leq n - 1 \). Also, \( K_n \nabla [a^i] \) is the largest clique in \( \Gamma \). So, \( \omega(\Gamma) = n + 1 \). We need \( n \) distinct colors for coloring any \( K_n \) and only one color for coloring \( \bar{K}_{n-1} \) which is distinct with the previous ones. So, \( (\Gamma) = n + 1 \). The set of vertices of \( \bar{K}_{n-1} \) is the largest independent set, so \( \alpha(\Gamma) = n - 1 \). By Lemma 1.1, we have \( \beta(\Gamma) = 4n - 1 - (n - 1) = 3n \). Clearly, the set of vertices of \( 3K_n \) has the minimum size of a vertex cover. Any vertex of \( \bar{K}_{n-1} \) is dominating all vertices of \( S \), and any vertex of \( S \) is dominating all vertices in \( \bar{K}_{n-1} \). Thus \( (\Gamma) = 2 \).

b) By Lemma 2.1, the non-commuting graph of \( M(D_{2n}, 2) \), for every even integer \( n \), is a multiple generalized complete split-like graph as \( \Gamma = \bar{K}_{n-2} \nabla 3S \), where \( S \) is a strongly regular graph with parameters \((n, n - 2, n - 3, n - 2)\) and the set of vertices of \( \bar{K}_{n-2} \) is an independent set as follows:

\[ V(\bar{K}_{n-2}) = \{a, a^2, \ldots, a^{n-1}, a^{n+1}, \ldots, a^n - 1\}. \]

In order to find the clique number, we may choose one vertex of \( \bar{K}_{n-2} \) and the other vertices from only one copy of \( S \)’s. By definition, every vertex is not joined to its inverse, so, we can choose \( \frac{n}{2} \) vertices of \( S \) and hence, \( \omega(\Gamma) = \frac{n}{2} + 1 \). The color of every vertex in \( S \) is co-color with its inverse. Therefore, the chromatic number of \( S \) is equal to \( \frac{n}{2} \), and so the maximum color number for all the vertices of \( 3S \) is equal to \( \frac{n}{2} \). By only one color distinct from \( \frac{n}{2} \)-color in \( 3S \), we can color \( \bar{K}_{n-2} \). So, \( (\Gamma) = \frac{n}{2} + 1 \). For \( n = 6 \), \( \bar{K}_{n-2} \) have four independent vertices, but with two non-adjacent vertices chosen from any of the copies of \( S \), we get 6 independent vertices. Therefore, in this case \( \alpha(\Gamma) = 6 \). Now, for \( n \geq 8 \), the set \( \bar{K}_{n-2} \) is the largest independent set and so, \( \alpha(\Gamma) = n - 2 \). By using Lemma 1.1, we have \( \beta(\Gamma) = n(\Gamma) - \alpha(\Gamma) \). Hence, if \( n = 6 \) then \( \beta(\Gamma) = 16 \), else if \( n \geq 8 \) then \( \beta(\Gamma) = 4n - 2 - (n - 2) = 3n \). By choosing any vertex in \( \bar{K}_{n-2} \) and the other in one of the copies of \( S \), the domination set of \( \Gamma \) will be determined. Hence, \( (\Gamma) = 2 \). \( \square \)
3. About perfectness and splitness of the non-commuting graph of a Moufang loop

In this section, first we show that the multiple complete split-like graphs are perfect and then characterize all Chein loops that their non-commuting graphs are 3—split-like.

**Theorem 3.1.** Every multiple complete split-like graph $\text{MCS}_{b,n}^a \cong \overline{K}_a \cup (nK_b)$, $(n \geq 2)$ is perfect, but not chordal. Moreover, every complete split graph $\text{CS}_{b,n}^a \cong \overline{K}_a \cup K_b$ is perfect and also chordal.

**Proof.** Let $\Gamma \cong \overline{K}_a \cup (nK_b)$ and $C$ be an odd cycle. If all vertices of $C$ lie in only one copy of $K_b$’s, clearly this cycle has a chord. Also, if some vertices of $C$ lie in more than one copy of $K_b$’s, then since in this case $C$ has some vertices of $\overline{K}_a$ and also these vertices in $\overline{K}_a$ are adjacent to each vertex of $K_b$, therefore, the cycle has a chord. In addition, the complement graph, $\bar{\Gamma}$, is a disconnected graph of the form $\Gamma \cong K_a \cup S$ such that $S$ is strongly regular graph with parameters $(nb, (n-1)b, (n-2)b, (n-1)b)$ or $S \cong T_{nb,b}$, which is a complete $n-$partite graph with $nb$ vertices, and hence, each part has $b$ vertices. Clearly, any cycle in $K_a$ has a chord. If $C$ be an odd cycle in $S$, then by structure of $S$, there is an intersection of $C$ with more than three sections of $S$ and these vertices are adjacent to any of the vertices in other sections and so, $C$ has a chord. If $C$ has an instruction with only two sections of $S$, then the induced subgraph of these sections will be a bipartite graph such that there is no any odd cycle in it. Now, by Berg Theorem ([9], Theorem 1.2) $\bar{\Gamma}$ is a perfect graph. Let $\Gamma \cong \overline{K}_a \cup (nK_b)$ and $x_1, x_2 \in \overline{K}_a$, $x_1 \neq x_2$. Take $x_3$ and $x_4$ from two distinct copies of $K_b$’s. Now the induced subgraph of $\Gamma$ generated by $x_1, x_2, x_3$ and $x_4$ is a cycle of length four without a chord. So, by definition, $\Gamma$ is not chordal.

Similar to the proof of the first part, $\text{CS}_{b,n}^a \cong \overline{K}_a \cup K_b$ is perfect, but there is no cycle of length four or more without any chord and so this is a chordal graph. This completes the proof. □

**Corollary 3.1.** The non-commuting graph of $M(D_{2n}, 2)$ is perfect but not chordal.

**Proof.** Let $\Gamma = \Gamma(M(D_{2n}, 2))$, where $n$ be an odd integer. Then by Lemma 2.1 (a), $\Gamma \cong \overline{K}_{n-1} \cup \nabla(3K_n)$ and by Theorem 3.1, $\Gamma$ is perfect but not chordal.

If $n$ be an even integer then by Lemma 2.1(b), $\Gamma \cong \overline{K}_{n-2} \cup 3S$ such that $S$ is a strongly regular graph with parameters $(n, n-2, n-3, n-2)$. Assume that $C$ is an odd cycle in $\Gamma$ with length 5 or more, the length of the longest cycle without chord in each copy of $S$ is equal to 4. Then there are some vertices of $\overline{K}_{n-2}$ in $C$, and these vertices are adjacent to each vertex in $3S$. Therefore, $C$ have a chord. On the other hand, $\overline{\Gamma} \cong K_{n-2} \cup (\frac{n}{2}K_2 \cup \frac{n}{2}K_2 \cup \frac{n}{2}K_2)$. Let $C$ be a cycle in $\overline{\Gamma}$. Clearly, every cycle in $K_{n-2}$ have a chord and if $C$ be an odd cycle in $\frac{n}{2}K_2 \cup \frac{n}{2}K_2 \cup \frac{n}{2}K_2$, then $C$ have an intersection with more than two parts of $\frac{n}{2}K_2$, where one of them have more than one vertex in $C$, and these vertices adjacent to all vertices of $C$ in other parts and so, $C$ have a chord and by Theorem ([9], Theorem 1.2), $\Gamma$ is perfect. The induced subgraph consist of any two vertices of $\overline{K}_{n-2}$ and two non-adjacent vertices of $S$ is a cycle with length 4 without chord then $\Gamma$ is not chordal. □

**Remark 3.1.** The generalized multiple complete split-like graph $\text{GMCS}_{b,k}^a$ is not perfect. As a counterexample, let we have a generalized complete split-like graph $\Gamma \cong \overline{K}_a \cup (nS)$ in which $S$ is a Peterson graph. This graph is not perfect, since it has a cycle of length 5 without any chord. Recall that a Peterson graph is a strongly regular graph with parameters $(10, 3, 0, 1)$. 325
Theorem 3.2. Let $G$ be a non-abelian group. Then its non-commuting graph $\Gamma_G$, is split if and only if the non-commuting graph of the Moufang loop $M(G, 2)$, $\Gamma_M$, is 3-split.

Proof. Let $\Gamma_M$ be 3-split of the form $\Gamma_M = I \nabla 3C$, where $I$ is an independent set and $C$ is a complete graph. First we show that $Z(G) = C(M)$. By Lemma[3], Lemma 3.10, $C(M) \subseteq Z(G)$. Let $Z(G) \nsubseteq C(M)$. Then there exists $x \in Z(G)$ such that $x \notin C(M)$. Also, there exists $yu \in Gu$, where $x \circ (yu) \neq (yu) \circ x$, which yields $(yx)u \neq (yx^{-1})u$. Therefore, $x \neq x^{-1}$ and $x \in I$. So, every vertex $y$ in each copy of $C$ is adjacent to $x$ and so $xy \neq yx$. But $x \in Z(G)$ then for every $g \in G$, we have $xg = gx$. Hence $G \subseteq I$. Now, let $g \in G \setminus Z(G)$. So, there exist $t \in G$ such that $tg \neq gt$ but in this case $t, g \in I$ and this is a contradiction, since $I$ is an independent set. So, $G = Z(G)$ and this contradicts with non-abelianity of $G$. Thus $Z(G) = C(M)$. Obviously, every element of $3C$ is an involution. Let $x \in 3C$ and $x \neq x^{-1}$. So, since each element of $Gu$ has order 2 then $x \in G$. Put $3C = C_1 \cup C_2 \cup C_3$, where each $C_i$ is equal to a copy of $C$, $(1 \leq i \leq 3)$. Without loss of generality, let $x \in C_1$ and $x^{-1} \in C_2$ (note that $xx^{-1} = x^{-1}x$). Let $y \in G \setminus Z(G)$ and $y \notin \langle x \rangle$. Then since every element of $G$ which commutes with $x$, also commutes with $x^{-1}$, so if $y \in C_1$ then $xy \neq yx$, and therefore $x^{-1}y \neq yx^{-1}$, but $x^{-1} \in C_2$ and this is a contradiction. Similarly, the case $y \in C_2$ will reach to a contradiction. So, $y \in I$ or $y \in C_3$. Now, consider the element $xy$. By the same reason as above, we have $xy \in I$ or $xy \in C_3$. Trivially, $xy \neq x, x^{-1}$. We have four cases as below:

Case 1. Let $y, xy \in I$. Then $y(xy) = (xy)y \Rightarrow yx = xy$. which is a contradiction, since $y$ is adjacent to every element of $C_1$.

Case 2. Let $y \in I$ and $xy \in C_3$. Then $x \in C_1 \Rightarrow x(xy) = (xy)x$, $(x, y \in G) \Rightarrow xy = yx$ and we have the same contradiction as in case 1.

Case 3. Let $y \in C_3$ and $xy \in I$. Then $(xy)y \neq y(xy) \Rightarrow xy \neq yx$, which is also a contradiction since $y \in C_3$ and $x \in C_1$.

Case 4. Let $y, xy \in C_3$. Then we have $y(xy) \neq (xy)y \Rightarrow xy \neq yx$ and we obtain a similar contradiction as in case 3.

Therefore, every element of $3C$ has order 2. On the other hand, $\Gamma_G$ is always connected and it is the induced subgraph of $\Gamma_M$. Therefore, $\Gamma_G \cong K_m \setminus (K_m \cong C)$ or $\Gamma_G \cong I \nabla nC'$ such that $I' \subseteq I$, $C' \subseteq C$ and $nC' = \cup_{i=1}^n C_i$, where $1 \leq n \leq 3$, and each $C_i$ is a subset of one copy of $C$’s. If $\Gamma_G \cong K_m$, then the order of every element of $G$ will be equal to 2, so $G$ must be abelian, which is absurd. Therefore, we get, $\Gamma_G \cong I' \nabla nC'$. If $n = 1$ then $\Gamma_G$ is split. Suppose that $1 \neq x, y \in G$, $x \in C_1$ and $y \in C_2$, then $xy = yx$ and there exists $z \in I'$ where $yz \neq yz$ and $xz \neq zx$. So, $xy \in G$. If $xy \in I'$, then $x(xy) \neq (xy)x$ and so, $x^2 y \neq x(xy)$. Therefore, $x^2 y \neq x(xy)$ and this is a contradiction. If $xy \in C_1$ then $x(xy) \neq (xy)x$ and $x^2 y \neq x^2 y$, and it is a contradiction, and if $xy \in C_2$ then $y(xy) \neq (xy)y$ and $y^2 x \neq y^2 x$, and it is also a contradiction. Finally, let $xy \in C_3$. Now, $xu \in M(G, 2)$ then:

1) If $xu \in I$ or $xu \in C_1$, then $(xu) \circ x \neq x \circ (xu)$ and so $(xx^{-1})u \neq (xx)u$. Therefore, $u \neq x^2 u$, this is a contradiction. So, every element of $C$ in $\Gamma_M$ is of order 2 therefore, $x^2 = 1$.

2) If $xu \in C_2$ then $(xu) \circ y \neq y \circ (xu)$ and so $(xy^{-1})u \neq (xy)u$. Thus $(xy)u \neq (xy)u$ and this is a contradiction.
3) If \( xu \in C_3 \) then \( (xu) \circ (xy) \neq (xy) \circ (xu) \) and so \( (x(y^{-1}x^{-1}))u \neq (x(y^{-1}x^{-1}))u \neq (x^y)u \). Therefore, \( (x^y)u \neq yu \) and this is a contradiction.

Therefore, \( \Gamma_G \cong I' \nabla C' \) and \( \Gamma_G \) is split.

Conversely, let \( \Gamma_G \) be split. Then \( \Gamma_G \cong I' \nabla C' \). We show that \( \Gamma_M \) is 3-split. By splitness of \( \Gamma_G \) and Lemmas ([4], Lemmas 2.4 and 2.5), we have, \( Z(G) = 1 \) and \( C(M) \subseteq Z(G) \). So, \( C(M) = 1 \).

To prove 3-splitness \( \Gamma_M \), we consider and establish the following claims.

Claim 1. The induced subgraph containing the vertices in \( V(Iu) \) forms a clique.

Suppose that there exist two non-adjacent vertices \( a_i u \) and \( a_j u \). So, \( (a_i u) \circ (a_j u) = (a_j u) \circ (a_i u) \) and then \( a_i a_j^{-1} = a_i a_j^{-1} \) or \( (a_i a_j^{-1})^2 = 1 \). Therefore, by Lemmas ([4], Lemmas 2.4 and 2.5), \( I^* = I \cup \{1\} \) is a maximal subgroup of odd order and there is not any element of even order. So, \( a_i a_j^{-1} \in C \), where in this case \( (a_i a_j^{-1})a_j \neq a_j (a_i a_j^{-1}) \). Then \( a_i \neq a_j (a_i a_j^{-1}) \) and \( a_j^{-1} a_i \neq a_i a_j^{-1} \) and this is a contradiction.

Claim 2. The induced subgraph containing the vertices in \( V(Cu) \) is a clique.

Suppose that there exist two vertices \( b_i u \) and \( b_j u \) such that they are not adjacent. So, \( (b_i u) \circ (b_j u) = (b_j u) \circ (b_i u) \). Therefore, \( b_i b_j^{-1} = b_i b_j \) and \( b_j b_i = b_j b_i \), since, each element of \( C \) is an involution and which yields to a contradiction.

Claim 3. There is no edge between \( V(Iu) \) and \( V(Cu) \).

Suppose that there exist two vertices \( a_i u \) and \( b_j u \) such that \( (a_i u) \circ (b_j u) = (b_j u) \circ (a_i u) \) then \( b_j^{-1} a_i \neq a_i^{-1} b_j \) and \( b_j a_i \neq a_i^{-1} b_j \), therefore \( (b_j a_i)^2 \neq 1 \). On the other hand \( b_j a_i \in G \). So, \( b_j a_i \in I \) or \( b_j a_i \in C \).

1) If \( b_j a_i \in I \) then \( (b_j a_i) a_i = a_i (b_j a_i) \) and \( b_j a_i = a_i b_j \), which yields to a contradiction.

2) If \( b_j a_i \in C \) then \( (b_j a_i)^2 = 1 \) and this is a contradiction. Therefore, any two elements of \( V(Iu) \) and \( V(Cu) \) are non-adjacent.

Claim 4. There is no edge between \( V(C) \) and \( V(Cu) \).

Suppose that there exist two vertices \( b_i \) and \( b_j \) such that \( b_i \circ (b_j u) = (b_j u) \circ b_i \). Then \( (b_j b_i)u \neq (b_j b_i)u \), so, \( (b_j b_i)u \neq (b_i b_j)u \), and this is a contradiction. Therefore any two elements of \( V(C) \) and \( V(Cu) \) are non-adjacent.

Claim 5. There is no edge between \( V(C) \) and \( V(Iu) \).

Suppose that there exist two vertices \( b_i \) and \( a_j \) such that \( b_i \circ (a_j u) \neq (a_j u) \circ b_i \). Then \( (a_j b_i)u \neq (a_j b_i^{-1})u \) and \( a_j b_i \neq a_j b_i \). This is a contradiction. Therefore, any two vertices in \( V(C) \) and \( V(Iu) \) are non-adjacent.

Claim 6. Every vertex in \( V(Iu) \) is adjacent to every vertex in \( V(I) \).
Suppose that there exist two vertices \( a_i \) and \( a_j u \) such that \( a_i \circ (a_j u) = (a_j u) \circ a_i \). Then 
\[
(a_j a_i)u = (a_j a_i^{-1})u \text{ and } a_j a_i = a_j a_i^{-1}.
\]
So, \( a_i = a_i^{-1} \). Therefore, \( a_i^2 = 1 \) and this is a contradiction.

**Claim 7.** Every vertex in \( V(Cu) \) is adjacent to every vertex in \( V(I) \).

Suppose that there exist two vertices \( a_i \in I \) and \( b_j u \in Cu \) such that \( a_i \circ (b_j u) = (b_j u) \circ a_i \). Also, 
\[
(b_j a_i)u = (b_j a_i^{-1})u \text{ then } b_j a_i = b_j a_i^{-1} \text{ and } a_i = a_i^{-1},
\]
therefore \( a_i^2 = 1 \) and this is a contradiction.

Thus the non–commuting graph of \( M(G, 2) \) is 3–split, where the induced subgraphs containing the vertices of \( C \) and \( Cu \) and \( Iu \) are cliques and \( I \) is an independent set.

Now, by using Theorems ([4], Theorem 2.3) and 3.2, we can classify all 3–split Chein loops:

**Corollary 3.2.** Let \( G \) be a non-abelian group. Then the non-commuting graph of the Moufang loop \( M(G, 2) \), is 3–split if and only if \( G \) is isomorphic to a Frobenius group of order \( 2^n, \) \( n \) is odd, whose Frobenius kernel is abelian of order \( n \). \( \square \)

### 4. About the energy and the number of spanning trees of generalized and multiple splite-like graphs

In this section, we are going to calculate the energy of generalized complete and multiple splite-like graphs and derive the energy and also the number of spanning trees in the case of the non-commuting graph of Chein loops of the form \( M(D_{2n}, 2) \).

**Theorem 4.1.** Let \( \Gamma \) be a generalized complete split-like graph, \( \Gamma \cong \tilde{K}_a \nabla (nK_b) \). Then \( \varepsilon(\Gamma) = 2n(b - 1) \).

**Proof.** Let \( P_{K_b}(\lambda) \) be the characteristic polynomial of \( K_b \). Then,
\[
P_{K_b}(\lambda) = (-1)^b(\lambda + 1)^{b-1}(\lambda - b + 1).\]
So,
\[
P_{nK_b}(\lambda) = (-1)^{nb}(\lambda + 1)^{n(b-1)}(\lambda - b + 1)^n
\]
and
\[
P_{K_a}(\lambda) = (-\lambda)^a.
\]
By using Theorem 1.2, we have:
\[
P_{\Gamma}(\lambda) = (-1)^{nb+a}(\lambda + 1)^{a(n-1)}(\lambda - b + 1)^{n-1}\lambda^{a-1}(\lambda^2 - (b - 1)\lambda - nab)
\]
and by definition of the energy of a graph, we get:
\[
\varepsilon(\Gamma) = n(b - 1) + (n - 1)(b - 1) + b - 1.
\]
Hence, \( \varepsilon(\Gamma) = 2n(b - 1) \). \( \square \)

**Corollary 4.1.** Let \( n \) be an odd integer. Let \( G = D_{2n} \) and \( M = M(G, 2) \). Then:
(i) if $n$ is an odd integer, then $\varepsilon(\Gamma_M) = 6(n - 1)$; 

(ii) if $n$ is an even integer, then $\varepsilon(\Gamma_M) = 6(n - 2)$.

Moreover, in both cases, $\varepsilon(\Gamma_G)$ divides $\varepsilon(\Gamma_M)$.

**Proof.** Since, $\Gamma_M \cong \bar{K}_{n-1} \downarrow 3K_n$, by Theorem 4.1, $\varepsilon(\Gamma_M) = 6(n - 1)$. We know that $\Gamma_G \cong \bar{K}_{n-1} \downarrow K_n$ and by Theorem 4.1, we have $\varepsilon(\Gamma_G) = 2(n - 1)$. Thus $\varepsilon(\Gamma_G)$ divides $\varepsilon(\Gamma_M)$.

(ii) Now, let $n$ be an even integer. Then, by Theorem 2.1, $\Gamma_M \cong \bar{K}_{n-2} \downarrow 3S$, in which $S$ is a strongly regular graph with parameters $(n, n - 2, n - 3, n - 2)$. Thus, by Theorems ([5], Theorems 6.2 and 6.22), the adjacency matrix of $S$ has exactly three distinct eigenvalues: $\lambda_1 = n - 2$, whose multiplicity is 1, $\lambda_2 = 0$, whose multiplicity is 1 and $\lambda_3 = -1$, whose multiplicity is $n - 2$. Therefore,

$$P_S(\lambda) = (\lambda - n + 2)(\lambda + 1)^{n-2} \lambda.$$ 

So,

$$P_{3S}(\lambda) = (\lambda - n + 2)^3(\lambda + 1)^{3n-6} \lambda^3$$

and

$$P_{\bar{K}_{n-2}}(\lambda) = \lambda^{n-2}.$$ 

By Theorem 1.2, we have:

$$P_{\Gamma_M}(\lambda) = (\lambda - n + 2)^2(\lambda + 1)^{3n-6} \lambda^{n-2}(\lambda^2 + (2 - n)\lambda - 3n(n - 2)).$$

Thus, $\varepsilon(\Gamma_M) = 6(n - 2)$. We know that $\Gamma_G \cong \bar{K}_{n-2} \downarrow S$, such that $S$ is a strongly regular graph with parameters $(n, n - 2, n - 3, n - 2)$. Therefore, by Theorems ([5], Theorems 6.2 and 6.22),

$$P_{\Gamma_G}(\lambda) = (\lambda + 1)^{n-2}\lambda^{n-2}(\lambda^2 + (2 - n)\lambda - n(n - 2)).$$

So, $\varepsilon(\Gamma_G) = 2(n - 2)$. Thus $\varepsilon(\Gamma_G)$ divides $\varepsilon(\Gamma_M)$. 

Finally, in the following Theorems, we count the number of spanning trees of the non-commuting graph $\Gamma_M$, where $M = M(D_{2n}, 2)$, for odd and even $n$, separately, and they lead us to an important result.

**Theorem 4.2.** The number of spanning trees of the non–commuting graph $\Gamma_M$, where $M = M(D_{2n}, 2)$ and $n$ is odd, is equal to:

$$\kappa(\Gamma_M) = (2n - 1)^{3n-3}(n - 1)^2(3n)^{n-2}.$$ 

**Proof.** There are $4n - 1$ vertices in this graph, such that they are in $t_1$, $t_2$, $t_3$, $t_4$. Each of $t_i$, $2 \leq i \leq 4$, have $n$ vertices of degree $2n - 2$, and $t_1$ have $n - 1$ vertices of degrees $3n$. By the structure of graph $\Gamma_M$ in Lemma 2.1, the matrix of vertex degree, namely $D$ of this graph is equal to:

$$D = \begin{bmatrix} (2n - 2)I_{3n} & 0_{3n(n-1)} \\ 0_{(n-1)3n} & (3n)I_{n-1} \end{bmatrix}$$

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and the adjacent matrix of graph has the form:

\[ A = \begin{bmatrix} (J_n - I_n) \otimes I_3 & J_{3n(n-1)} \\ J_{(n-1)3n} & 0_{n-1} \end{bmatrix}, \]

where, \( \otimes \) denotes the tensor product of matrices. Thus,

\[ L = D - A = \begin{bmatrix} ((2n - 1)I_n - J_n) \otimes I_3 & -J_{3n(n-1)} \\ -J_{(n-1)3n} & (3n)I_{n-1} \end{bmatrix}. \]

Now, to calculate \( \det(L + J) \), we have

\[ L + J = \begin{bmatrix} (2n - 1)I_n & J_n & J_n & 0 \\ J_n & (2n - 1)I_n & J_n & 0 \\ J_n & J_n & (2n - 1)I_n & 0 \\ 0 & 0 & 0 & (3n)I_{n-1} + J_{n-1} \end{bmatrix}. \]

Also, in this case we have

\[ \det(L + J) = \det B \times \det C, \quad (1) \]

where,

\[ B = \begin{bmatrix} (2n - 1)I_n & J_n & J_n \\ J_n & (2n - 1)I_n & J_n \\ J_n & J_n & (2n - 1)I_n \end{bmatrix} \]

and \( C = (3n)I_{n-1} + J_{n-1} \). So,

\[ \det C = (3n)^{n-2}(4n - 1) \quad (2) \]

and

\[ B = \begin{bmatrix} E \\ J_{(2n)} \\ F \end{bmatrix}, \]

where,

\[ E = \begin{bmatrix} (2n - 1)I_n & J_n \\ J_n & (2n - 1)I_n \end{bmatrix} \]

and \( F = (2n - 1)I_n \). By Theorem 1.1, we have

\[ \det B = \det F \times \det(E - JF^{-1}J). \quad (3) \]

So, by using the following relations

\[ \det F = (2n - 1)^n, \quad F^{-1} = \frac{1}{2n-1}I_n, \quad JF^{-1}J = \frac{n}{2n-1}J_{2n}, \quad (4) \]

we have

\[ E - JF^{-1}J = \frac{1}{2n-1} \begin{bmatrix} G & (n-1)J \\ (n-1)J & G \end{bmatrix}, \]
where, \( G = (2n - 1)^2 I - nJ \) and
\[
\det G = (2n - 1)^{2n-2}(n - 1)(3n - 1), \quad G^{-1} = \frac{1}{(2n - 1)^2} (I + \frac{n}{(n - 1)(3n - 1)} J).
\]
(5)

Now,
\[
\det(E - JF^{-1}J) = \left(\frac{1}{2n - 1}\right)^{2n} \det(G) \times \det(G - (n - 1)^2 JG^{-1}J),
\]
(6)

where,
\[
(n - 1)^2 JG^{-1}J = \frac{n(n - 1)}{3n - 1} J
\]
and
\[
G - (n - 1)^2 JG^{-1}J = \frac{1}{3n - 1} ((\alpha - \beta)I + \beta J),
\]
such that, \( \alpha = (n - 1)(2n - 1)(6n - 1) \) and \( \beta = -2n(2n - 1) \). So,
\[
\det(G - (n - 1)^2 JG^{-1}J) = (2n - 1)^{2(n-1)} \frac{8n^3 - 14n^2 + 7n - 1}{3n - 1}.
\]
(7)

By using the relations 5, 6 and 7, we have
\[
\det(E - JF^{-1}J) = (2n - 1)^{2(n-2)}(n - 1)(8n^3 - 14n^2 + 7n - 1)
\]
(8)

and by replacing relations 4 and 8 in 3 we get
\[
\det B = (2n - 1)^{3n-4}(n - 1)(8n^3 - 14n^2 + 7n - 1).
\]
(9)

Now, by replacing relations 2 and 9 in 1, we get
\[
\det(L + J) = (2n - 1)^{3(n-1)}(n - 1)^2(4n - 1)^2(3n)^{n-2}.
\]

By Theorem ([5], Theorem 4.11), we have \( \kappa = \frac{\det(L+J)}{(4n-1)^2} \). Therefore,
\[
\kappa(\Gamma_M) = (2n - 1)^{3(n-1)}(n - 1)^2(3n)^{n-2}.
\]

**Theorem 4.3.** The number of spanning trees of the non-commuting graph \( \Gamma_M \), where, \( M = M(D_{2n}, 2) \) and \( n \) is even, is equal to:
\[
\kappa(\Gamma_M) = 2^{3n-3}(3n)^{n-3}(n - 1)^{3n-3}(n - 2)^{3n-2}\frac{n}{3n-2}.
\]

**Proof.** There are \( 4n - 2 \) vertices in this graph and they are in \( t_1, t_2, t_3, t_4 \). Each of \( t_i, 2 \leq i \leq 4, \) have \( n \) vertices of degree \( 2n - 4 \) and \( t_1 \) have \( n - 2 \) vertices of degree \( 3n \). By the structure of the graph \( \Gamma \) in 2.1, the matrix of the vertex degree namely \( D \), of this graph is:
\[
D = \begin{bmatrix}
2(n - 2)I_{3n} & 0 \\
0 & 3nI_{n-2}
\end{bmatrix}
\]
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and the adjacent matrix of the graph has the form:

\[ A = \begin{bmatrix} X_n & 0 & 0 & J \\ 0 & X_n & 0 & J \\ 0 & 0 & X_n & J \\ J & J & J & 0 \end{bmatrix} \]

By Lemma 2.1, each vertex in every \( t_i \) \( (2 \leq i \leq 4) \), is connected to the other vertices except its inverse element and itself, and so,

\[ X = \begin{bmatrix} J - I & J - I \\ J - I & J - I \end{bmatrix}, \]

such that \( I \) and \( J \) are square matrices of order \( \frac{n}{2} \) in \( X \). So,

\[ L = D - A = \begin{bmatrix} Y_n & 0 & 0 & -J \\ 0 & Y_n & 0 & -J \\ 0 & 0 & Y_n & -J \\ -J & -J & -J & 3nI_{n-2} \end{bmatrix}, \]

such that,

\[ Y = \begin{bmatrix} (2n - 3)I - J & I - J \\ I - J & (2n - 3)I - J \end{bmatrix}. \]

Hence,

\[ L + J = \begin{bmatrix} Z & J & J & 0 \\ J & Z & J & 0 \\ J & J & Z & 0 \\ 0 & 0 & 0 & 3nI + J \end{bmatrix}. \]

We have

\[ Z = Y + J = \begin{bmatrix} (2n - 3)I & I \\ I & (2n - 3)I \end{bmatrix}, \]

in which the order of \( I \) is equal to \( \frac{n}{2} \). Now we obtain

\[ \det(L + J) = \det B \times \det C, \quad (10) \]

where \( C = 3nI_{n-2} + J_{n-2} \) and

\[ B = \begin{bmatrix} Z & J & J \\ J & Z & J \\ J & J & Z \end{bmatrix}. \]

Therefore,

\[ \det C = 2(3n)^{n-3}(2n - 1) \quad (11) \]

and by using Theorem 1.1, we have

\[ \det B = \det Z \times \det(D - JZ^{-1}J), \quad (12) \]

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where,
\[ D = \begin{bmatrix} Z & J \\ J & Z \end{bmatrix} \]
and
\[ \det Z = (4(n - 1)(n - 2))^\frac{n}{2}. \] (13)

Also,
\[ Z^{-1} = \frac{1}{(2n-3)^2-1} \begin{bmatrix} (2n-3)I_{\frac{n}{2}} & -I_{\frac{n}{2}} \\ -I_{\frac{n}{2}} & (2n-3)I_{\frac{n}{2}} \end{bmatrix} \]
and so,
\[ JZ^{-1} = \frac{1}{2(n-1)}J_{2n\times n} \quad \text{and} \quad JZ^{-1}J = \frac{n}{2(n-1)}J_{2n\times 2n}. \] So,
\[ D - JZ^{-1}J = \begin{bmatrix} G & H \\ H & G \end{bmatrix}, \] (14)
such that, \[ H = \frac{n-2}{2(n-1)}J \] and
\[ G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}, \]
where, \[ G_{11} = G_{22} = (2n-3)I - \frac{n}{2(n-1)}J \quad \text{and} \quad G_{12} = G_{21} = I - \frac{n}{2(n-1)}J. \]

By using elementary row or column operations in \( G \) we have
\[ \det G = \det \left( \frac{1}{2(n-1)} \begin{bmatrix} (n-1)4n-6I-nJ & 4(n-2)(1-n)I \\ 4(n-2)(1-n)I & 8(n-1)(n-2)I \end{bmatrix} \right) \]
\[ = (8(n-1)(n-2))^\frac{n}{2} \frac{1}{2(n-1)^n} \det(2(n-1)^2I-nJ). \]

Since,
\[ \det(2(n-1)^2I-nJ) = 2^\frac{n}{2}-2(n-1)^{n-2}(n-2)(3n-2), \]
then
\[ \det G = 2^{n-2}(n-1)^{\frac{n}{2}-2}(n-2)^{\frac{n}{2}+1}(3n-2). \] (15)

By Theorem 1.1, \( G^{-1} \) is as follows:
\[ G^{-1} = \begin{bmatrix} G_{11}^{-1} + (G_{11}^{-1}G_{12})(G/G_{11})^{-1}(G_{12}G_{11}^{-1}) & -G_{11}^{-1}G_{12}(G/G_{11})^{-1} \\ -(G/G_{11})^{-1}G_{12}G_{11}^{-1} & (G/G_{11})^{-1} \end{bmatrix}, \]
such that, \( G/G_{11} = G_{11} - G_{12}G_{11}^{-1}G_{12} \). Therefore,
\[ G_{11}^{-1} = \frac{1}{(n-1)(2n-3)} \left( \frac{1}{2}I + \frac{n}{(n-2)(7n-6)}J \right) \]
and \( G_{12} = I - \frac{n}{2(n-1)}J. \) Then:
\[ G_{12}G_{11}^{-1}G_{12} = \frac{1}{(2n-3)}(2(n-1)I + \frac{n(2n^2-15n+14)}{(7n-6)}J). \]
and
\[ G_{11} - G_{12}G_{11}^{-1}G_{12} = \frac{8(n-1)(n-2)}{2n-3}((n-1)I - \frac{2n}{7n-6}J). \]

Now, we have
\[ G/G_{11} = \frac{8(n-1)(n-2)}{(2n-3)}((n-1)I - \frac{2n}{7n-6}J), \]

and
\[ (G/G_{11})^{-1} = \frac{1}{8(n-1)(n-2)}((2n-3)I + \frac{2n}{3n-2}J). \]

Therefore,
\[ G^{-1} = \frac{1}{4(n-1)(n-2)} \begin{bmatrix} (2n-3)I & -I \\ -I & (2n-3)I \end{bmatrix} + \frac{2n}{3n-2}J. \]

Also, \( HG^{-1}H = \frac{n(n-2)}{2(n-1)(3n-2)}J \) and
\[ G - HG^{-1}H = \begin{bmatrix} (2n-3)I & I \\ I & (2n-3)I \end{bmatrix} - \frac{2n}{3n-2}J. \]

By using elementary row or column operations, we have
\[ \det(G - HG^{-1}H) = \frac{2^n}{(3n-2)(n-2)^{n+1}}(n-1)^{n-1}(2n-1) \quad (16) \]

By relation 14, we get
\[ \det(D - JZ^{-1}J) = \det G \times \det(G - HG^{-1}H). \]

Then, by relations 15 and 16, we have
\[ \det(D - JZ^{-1}J) = 2^{2n-2}(n-1)^{n-3}(n-2)^{n+2}(2n-1). \quad (17) \]

Also, from relations 12, 13 and 17, we obtain
\[ \det B = 2^{3n-2}(n-1)^{3n-3}(n-2)^{3n+2}(2n-1) \quad (18) \]

and by relations 10, 11 and 18, we have
\[ \det(L + J) = 2^{3n-1}(3n)^{n-3}(n-1)^{3n-3}(n-2)^{3n+2}(2n-1)^2, \quad (19) \]

and from replacing 19 in \( \kappa = \frac{\det(L+J)}{(4n-2)^2} \), we get
\[ \kappa(\Gamma_M) = 2^{3n-3}(3n)^{n-3}(n-1)^{3n-3}(n-2)^{3n+2}. \]

\[ \square \]

**Corollary 4.2.** Let \( M = M(G, 2) \), where \( G = D_{2n} \). Then \( \kappa_G \) divides \( \kappa(M) \).
Proof. By Example 1 in [4], the non-commuting graph of $G = D_{2n}$, when in is odd, is a split graph and $\Gamma_G \cong I \nabla C$, where $I$ is an independent set with $n - 1$ vertices and $C \cong K_n$. So, the degree matrix of $\Gamma_G$ has the form:

$$D = \begin{bmatrix} (2n - 2)I_{n-1} & 0 \\ 0 & nI_n \end{bmatrix}$$

and the adjacency matrix of $\Gamma_G$ is equal to:

$$A = \begin{bmatrix} J - I & J \\ J & 0 \end{bmatrix}.$$ 

So,

$$L = D - A = \begin{bmatrix} 2n - 1)I - J & -J \\ -J & nI \end{bmatrix}$$

and

$$L + J = \begin{bmatrix} (2n - 2)I & 0 \\ 0 & nI + J \end{bmatrix}.$$ 

Thus, $\det(L + J) = \det((2n - 1)I) \times \det(nI + J)$ and this gives us:

$$\det(L + J) = (2n - 1)^{n-1}n^{n-2}.$$ 

Therefore,

$$\kappa(\Gamma_G) = \frac{\det(L + J)}{(2n - 1)^2} = (2n - 1)^{n-1}n^{n-2}.$$ 

By Theorem 4.2, $\kappa(\Gamma_M) = (2n - 1)^3(n-1)^2(3n)^{n-2}$. Hence, the proof is complete and $\kappa(\Gamma_G)$ divides $\kappa(\Gamma_M)$, where $n$ is an odd integer.

Now, let $n$ be an even integer. Then $\Gamma_G \cong K_{n-2} \nabla S$, where $S$ is a strongly regular graph with parameters $(n, n - 2, n - 4, n - 2)$. Also, the degree matrix, $D$, of $\Gamma_G$ is equal to:

$$D = \begin{bmatrix} (2n - 4)I & 0 \\ 0 & nI \end{bmatrix}$$

and the adjacency matrix of $\Gamma_G$, namely $A$, has the form:

$$A = \begin{bmatrix} X & J \\ J & 0 \end{bmatrix},$$

where,

$$X = \begin{bmatrix} J - I & J - I \\ J - I & J - I \end{bmatrix},$$

in which, $I$ and $J$ are of order $\frac{n}{2}$. So,

$$L = D - A = \begin{bmatrix} Y & -J \\ -J & nI \end{bmatrix},$$
where,

\[ Y = \begin{bmatrix}
(2n - 3)I - J & I - J \\
I - J & (2n - 3)I - J
\end{bmatrix}. \]

Hence,

\[ L + J = \begin{bmatrix}
Z & 0 \\
0 & nI + J
\end{bmatrix}, \]

where,

\[ Z = \begin{bmatrix}
(2n - 3)I & I \\
I & (2n - 3)I
\end{bmatrix}. \]

Since, \( \det(L + J) = \det Z \times \det(nI + J) \), \( \det Z = (4(n - 1)(n - 2))^{\frac{n}{2}} \) and \( \det(nI + J) = n^{n-3}(2n - 2) \), then

\[ \det(L + J) = 2^{n+1}n^{n-3}(n - 1)^{\frac{n}{2}+1}(n - 2)^{\frac{n}{2}}. \]

Therefore,

\[ \kappa(\Gamma_G) = \frac{\det(L + J)}{(2n - 2)^2} = 2^{n-1}n^{n-3}(n - 1)^{\frac{n}{2}-1}(n - 2)^{\frac{n}{2}}. \]

Also, by Theorem 4.3, we have

\[ \kappa(\Gamma_M) = 2^{3n-3}(3n)^{n-3}(n - 1)^{\frac{3n}{2}-3}(n - 2)^{\frac{3n}{2}+2}. \]

This proves that \( \kappa(\Gamma_G) \) divides \( \kappa(\Gamma_M) \).

\[ \square \]

5. Conclusion

In this research work, we studied some properties of the non–commuting graph of a class of finite Moufang loops. Also, we proved that the multiple complete-like graphs and the non-commuting graph of Chein loops of the form \( M(D_{2n}, 2) \) are perfect, and both graphs are non chordal. Finally, we characterized when the non-commuting graph of Moufang loop \( M(G, 2) \) is 3-splite and we give the energy of generalized and multiple split-like graphs. In future, we will try to study the similar graph properties of the non–commuting graph for the simple Moufang loops and characterize relations between any group \( G \) with the non–commuting graph \( M(G, 2) \).

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