EXPANSION OF ITERATED STRATONOVICH STOCHASTIC INTEGRALS OF MULTIPlicity 2. COMBINED APPROACH BASED ON GENERALIZED MULTIPLE AND ITERATED FOURIER SERIES

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ABSTRACT. The article is devoted to the expansion of iterated Stratonovich stochastic integrals of multiplicity 2 on the base of the combined approach of generalized multiple and iterated Fourier series. We consider two different parts of the expansion of iterated Stratonovich stochastic integrals. The mean-square convergence of the first part is proved on the base of generalized multiple Fourier series converging in the sense of norm in Hilbert space $L^2([t, T]^2)$. The mean-square convergence of the second part is proved on the base of generalized iterated (double) Fourier series converging pointwise. At that, we prove the iterated limit transition for the second part of the expansion on the base of Lebesgue’s Dominated Convergence Theorem. The results of the article can be applied to the numerical integration of Ito stochastic differential equations.

Contents

1. Introduction 1
2. Expansion of Iterated Stratonovich Stochastic Integrals of Multiplicity 2 2
3. Some Recent Results on Expansion of Iterated Stratonovich Stochastic Integrals of Multiplicities 2 to 6 10
4. Theorems 1–6 from Point of View of the Wong–Zakai Approximation 13
References 17

1. Introduction

Let $(\Omega, F, P)$ be a complete probability space, let $\{F_t, t \in [0, T]\}$ be a nondecreasing right-continuous family of $\sigma$-algebras of $F$, and let $f_t$ be a standard $m$-dimensional Wiener stochastic process, which is $F_t$-measurable for any $t \in [0, T]$. We assume that the components $f_t^{(i)} (i = 1, \ldots, m)$ of this process are independent.

Let us consider the following collections of iterated Stratonovich and Ito stochastic integrals
\[ J^*[\psi^{(2)}]_{T,t} = \int_t^T \int_t^{t_2} \psi_2(t_2) \psi_1(t_1) d\mathbf{w}^{(i_1)}_{t_1} d\mathbf{w}^{(i_2)}_{t_2}, \]  

\[ J[\psi^{(2)}]_{T,t} = \int_t^T \int_t^{t_2} \psi_2(t_2) \psi_1(t_1) d\mathbf{w}^{(i_1)}_{t_1} d\mathbf{w}^{(i_2)}_{t_2}, \]  

where every \( \psi_l(\tau) \) \((l = 1, 2)\) is a nonrandom function at the interval \([t, T]\), \(\mathbf{w}_\tau = f^{(i)}_{\tau}\) for \(i = 1, \ldots, m\) and \(\mathbf{w}_\tau^{(0)} = \tau, \ i_1, \ldots, i_k = 0, 1, \ldots, m,\)

\[ \int^* \text{ and } \int \]  

denote Stratonovich and Ito stochastic integrals, respectively (in this paper, we use the definition of the Stratonovich stochastic integral from [1]).

Further, we will denote as \(\{\phi_j(x)\}^\infty_{j=0}\) the complete orthonormal systems of Legendre polynomials and trigonometric functions in the space \(L_2([t, T])\). Also we will pay a special attention on the following well-known facts connecting to these two systems of functions [2].

Suppose that the function \(f(x)\) is bounded at the interval \([t, T]\). Moreover, its derivative \(f'(x)\) is a continuous function at the interval \([t, T]\) except may be the finite number of points of the finite discontinuity. Then the Fourier series

\[ \sum_{j=0}^\infty C_j \phi_j(x), \quad C_j = \int_t^T f(x) \phi_j(x) dx \]  

converges at any internal point \(x\) of the interval \([t, T]\) to the value \((f(x + 0) + f(x - 0))/2\) and converges uniformly to \(f(x)\) on any closed interval of continuity of the function \(f(x)\) laying inside \([t, T]\). At the same time, the Fourier–Legendre series converges if \(x = t\) and \(x = T\) correspondently, and the trigonometric Fourier series converges if \(x = t\) and \(x = T\) to \((f(t + 0) + f(T - 0))/2\) in the case of periodic continuation of the function \(f(x)\).

2. Expansion of Iterated Stratonovich Stochastic Integrals of Multiplicity 2

The use of generalized multiple and iterated Fourier series by various complete orthonormal systems of functions in the space \(L_2([t, T])\) for the expansion of iterated Ito and Stratonovich stochastic integrals is reflected in a number of works of the author [3]-[44]. In these papers, several new approaches to the mean-square approximation of iterated stochastic integrals were proposed and developed. One of the mentioned approaches (the so-called combined approach) for the expansion of iterated Stratonovich stochastic integrals of multiplicities 1 to 4 based on generalized multiple and iterated Fourier series has been considered in [4]. In this article, we consider the case of second multiplicity of iterated Stratonovich stochastic integrals. At that, we prove the mean-square convergence of the expansion of iterated Stratonovich stochastic integrals using the another method in comparison with the method from [4].

Theorem 1 [3] (2013) (also see [8] (Sect. 2.1.1) and references therein). Suppose that \(\{\phi_j(x)\}^\infty_{j=0}\) is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space \(L_2([t, T])\). At the same time \(\psi_2(\tau)\) is a continuously differentiable nonrandom function on \([t, T]\) and
\( \psi_1(\tau) \) is twice continuously differentiable nonrandom function on \([t, T] \). Then the iterated Stratonovich stochastic integral of the second multiplicity

\[
J^*[\psi^{(2)}]_{T,t} = \int_t^T \psi_2(t_2) \int_t^{t_2} \psi_1(t_1) dw_{t_1}^{(i_1)} dw_{t_2}^{(i_2)} \quad (i_1, i_2 = 0, 1, \ldots, m)
\]

is expanded into the multiple series

\[
J^*[\psi^{(2)}]_{T,t} = \lim \sum_{j_1=0}^{P_1} \sum_{j_2=0}^{P_2} C_{j_2,j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)}
\]

that converges in the mean-square sense, where l.i.m. is a limit in the mean-square sense,

\[
\zeta_{j}^{(i)} = \int_t^T \phi_j(\tau) dw^j_{t}
\]

are independent standard Gaussian random variables for various \(i\) or \(j\) (if \(i \neq 0\)),

\[
C_{j_2,j_1} = \int_t^T \psi_2(t_2) \phi_{j_2}(t_2) \int_t^{t_2} \psi_1(t_1) \phi_{j_1}(t_1) dt_1 dt_2
\]

is the Fourier coefficient.

**Remark 1.** It should be noted that Theorem 1 is proved in [3] (2013) (also see [8] and references therein). The proof from [3], [8] (Sect. 2.1.1) is based on double integration by parts. Below we consider another proof of Theorem 1.

**Proof.** Let us consider some auxiliary lemmas from [3] (also see [8] and references therein). At that, we will consider the particular case of these lemmas for \(k = 2\).

Consider the partition \(\{\tau_j\}_{j=0}^N\) of the interval \([t, T]\) such that

\[
t = \tau_0 < \ldots < \tau_N = T, \quad \Delta_N = \max_{0 \leq j \leq N-1} \Delta \tau_j \to 0 \quad \text{if} \quad N \to \infty, \quad \Delta \tau_j = \tau_{j+1} - \tau_j.
\]

**Lemma 1** [3] (also see [8] and references therein). Suppose that every \(\psi_l(\tau) (l = 1, 2)\) is a continuous nonrandom function at the interval \([t, T]\). Then

\[
J[\psi^{(2)}]_{T,t} = \lim_{N \to \infty} \sum_{j_2=0}^{N-1} \sum_{j_1=0}^{j_2-1} \psi_1(\tau_{j_1}) \psi_2(\tau_{j_2}) \Delta w_{\tau_{j_1}}^{(i_1)} \Delta w_{\tau_{j_2}}^{(i_2)} \quad \text{w. p. 1},
\]

where \(J[\psi^{(2)}]_{T,t}\) is the iterated Ito stochastic integral \(\Delta w_{\tau_{j}}^{(i)} = w_{\tau_{j+1}}^{(i)} - w_{\tau_{j}}^{(i)} (i = 0, 1, \ldots, m), \quad \{\tau_{j}\}_{j=0}^N\) is the partition of the interval \([t, T]\) satisfying the condition [3]; hereinafter w. p. 1 means with probability 1.

Let us define the following multiple stochastic integral

\[
\lim_{N \to \infty} \sum_{j_1,j_2=0}^{N-1} \phi(\tau_{j_1}, \tau_{j_2}) \Delta w_{\tau_{j_1}}^{(i_1)} \Delta w_{\tau_{j_2}}^{(i_2)} \quad \text{def} \quad J[\Phi^{(2)}]_{T,t},
\]
where \( \Phi(t_1, t_2) : [t, T]^2 \to \mathbb{R} \) is a nonrandom function (the properties of this function will be specified further), \( \Delta w^{(i)}_{\tau_j} = w^{(i)}_{\tau_{j+1}} - w^{(i)}_{\tau_j} \) \((i = 0, 1, \ldots, m)\), \( \{\tau_j\}_{j=0}^{N} \) is the partition of the interval \([t, T]\) satisfying the condition \([3]\).

Denote

\[
D_2 = \{(t_1, t_2) : t \leq t_1 < t_2 \leq T\}.
\]

We will use the same symbol \(D_2\) to denote the open and closed domains corresponding to the domain \(D_2\) defined by \([3]\). However, we always specify what domain we consider (open or closed).

Also we will write \(\Phi(t_1, t_2) \in C(D_2)\) if \(\Phi(t_1, t_2)\) is a continuous nonrandom function of two variables in the closed domain \(D_2\).

Let us consider the iterated Ito stochastic integral

\[
I[\Phi]_{T,t}^{[2]} \overset{\text{def}}{=} \int_{t}^{T} \int_{t}^{t_2} \Phi(t_1, t_2) dw^{(i_1)}_{t_1} dw^{(i_2)}_{t_2},
\]

where \(\Phi(t_1, t_2) \in C(D_2)\).

**Lemma 2** \([3]\) (also see \([8]\) and references therein). Suppose that \(\Phi(t_1, t_2) \in C(D_2)\) or \(\Phi(t_1, t_2)\) is a continuous nonrandom function in the open domain \(D_2\) and bounded at its boundary. Then

\[
I[\Phi]_{T,t}^{[2]} \overset{\text{def}}{=} \lim_{N \to \infty} \sum_{j_0=0}^{N-1} \sum_{j_1=0}^{j_0-1} \Phi(\tau_{j_1}, \tau_{j_2}) \Delta w^{(i_1)}_{\tau_{j_1}} \Delta w^{(i_2)}_{\tau_{j_2}} \quad \text{w. p. 1},
\]

where \(\Delta w^{(i)}_{\tau_j} = w^{(i)}_{\tau_{j+1}} - w^{(i)}_{\tau_j} \) \((i = 0, 1, \ldots, m)\), \(\{\tau_j\}_{j=0}^{N}\) is the partition of the interval \([t, T]\) satisfying the condition \([3]\).

**Lemma 3** \([3]\) (also see \([8]\) and references therein). Suppose that every \(\varphi_l(\tau) \quad (l = 1, 2)\) is a continuous nonrandom function at the interval \([t, T]\). Then

\[
J[\varphi_1]_{T,t} J[\varphi_2]_{T,t} = J[\Phi]_{T,t}^{[2]} \quad \text{w. p. 1},
\]

where

\[
\Phi(t_1, t_2) = \varphi_1(t_1) \varphi_2(t_2), \quad J[\varphi_l]_{T,t} = \int_{t}^{T} \varphi_l(\tau) dw^{(i)}_{\tau}, \quad (l = 1, 2)
\]

and the stochastic integral \(J[\Phi]_{T,t}^{[2]}\) is defined by the equality \([5]\), \(i_1, i_2 = 0, 1, \ldots, m\).

In accordance to the standard relations between Stratonovich and Ito stochastic integrals we have w. p. 1 \([1]\)

\[
J^*[\psi^{(2)}]_{T,t} = J[\psi^{(2)}]_{T,t} + \frac{1}{2} 1_{\{i_1 = i_2 \neq 0\}} \int_{t}^{T} \psi_1(t_1) \psi_2(t_1) dt_{1},
\]

where \(1_A\) is the indicator of the set \(A\).

Let us define the function \(K^*(t_1, t_2)\) at the square \([t, T]^2\) as follows

\[
K^*(t_1, t_2) = \psi_1(t_1) \psi_2(t_2) \left(1_{\{t_1 < t_2\}} + \frac{1}{2} 1_{\{t_1 = t_2\}}\right) = K(t_1, t_2) + \frac{1}{2} 1_{\{t_1 = t_2\}} \psi_1(t_1) \psi_2(t_2),
\]
where
\[
K(t_1, t_2) = \begin{cases} 
\psi_1(t_1)\psi_2(t_2), & t_1 < t_2 \\
0, & \text{otherwise}
\end{cases}, \quad t_1, t_2 \in [t, T]
\]
and \(1_A\) is the indicator of the set \(A\).

**Lemma 4** \[3\] (also see \[8\] and references therein). **Under the conditions of Theorem 1 the following relation**

\[
J[K^*]^{(2)}_{T,t} = J^*[\psi^{(2)}]_{T,t}
\]

is valid w. p. 1, where \(J[K^*]^{(2)}_{T,t}\) is defined by the equality \[5\].

**Proof.** Substituting \(10\) into \(5\) and using Lemmas 1 and 2, it is easy to see that

\[
J[K^*]^{(2)}_{T,t} = J[\psi^{(2)}]_{T,t} + \frac{1}{2} \sum_{i_1, i_2 \neq 0} \int_t^T \psi_1(t_1)\psi_2(t_1)dt_1 = J^*[\psi^{(2)}]_{T,t} \quad \text{w. p. 1.}
\]

Let us consider the following generalized double Fourier sum

\[
\sum_{j_1 = 0}^{p_1} \sum_{j_2 = 0}^{p_2} C_{j_2,j_1} \phi_{j_1}(t_1)\phi_{j_2}(t_2),
\]

where \(C_{j_2,j_1}\) is the Fourier coefficient of the form

\[
C_{j_2,j_1} = \int_{[t,T]^2} K^*(t_1, t_2)\phi_{j_1}(t_1)\phi_{j_2}(t_2)dt_1dt_2.
\]

Substitute the relation

\[
K^*(t_1, t_2) = \sum_{j_1 = 0}^{p_1} \sum_{j_2 = 0}^{p_2} C_{j_2,j_1} \phi_{j_1}(t_1)\phi_{j_2}(t_2) + K^*(t_1, t_2) - \sum_{j_1 = 0}^{p_1} \sum_{j_2 = 0}^{p_2} C_{j_2,j_1} \phi_{j_1}(t_1)\phi_{j_2}(t_2)
\]

with finite \(p_1\) and \(p_2\) into \(J[K^*]^{(2)}_{T,t}\). Then, using Lemma 3, we obtain

\[
J^*[\psi^{(2)}]_{T,t} = \sum_{j_1 = 0}^{p_1} \sum_{j_2 = 0}^{p_2} C_{j_2,j_1} \phi_{j_1}(t_1)\phi_{j_2}(t_2) + J[R_{p_1,p_2}]^{(2)}_{T,t} \quad \text{w. p. 1},
\]

where the stochastic integral \(J[R_{p_1,p_2}]^{(2)}_{T,t}\) is defined in accordance with \[5\] and

\[
R_{p_1,p_2}(t_1, t_2) = K^*(t_1, t_2) - \sum_{j_1 = 0}^{p_1} \sum_{j_2 = 0}^{p_2} C_{j_2,j_1} \phi_{j_1}(t_1)\phi_{j_2}(t_2),
\]
\[ S_{ij}^{(i)} = \int_{t}^{T} \phi_{j}(\tau) dW_{\tau}^{(i)}, \]

\[ J[R_{p1p2}]^{(2)}_{T,t} = \int_{t}^{t_2} \int_{t}^{T} R_{p1p2}(t_1, t_2) dW_{t_1}^{(i_1)} dW_{t_2}^{(i_2)} + \int_{t}^{t_1} \int_{t}^{T} R_{p1p2}(t_1, t_2) dW_{t_1}^{(i_2)} dW_{t_2}^{(i_1)} + \]

\[ + 1_{\{i_1 = i_2 \neq 0\}} \int_{t}^{T} R_{p1p2}(t_1, t_1) dt_1. \]

Let us consider the case \( i_1, i_2 \neq 0 \) (another cases can be considered absolutely analogously). Using standard estimates for moments of stochastic integrals \cite{23}, we obtain

\[ M \left\{ \left( J[R_{p1p2}]^{(2)}_{T,t} \right)^2 \right\} = \]

\[ = M \left\{ \left( \int_{t}^{T} \int_{t}^{t_2} R_{p1p2}(t_1, t_2) dW_{t_1}^{(i_1)} dW_{t_2}^{(i_2)} + \int_{t}^{t_1} \int_{t}^{T} R_{p1p2}(t_1, t_2) dW_{t_1}^{(i_2)} dW_{t_2}^{(i_1)} \right)^2 \right\} + \]

\[ + 1_{\{i_1 = i_2 \neq 0\}} \left( \int_{t}^{T} R_{p1p2}(t_1, t_1) dt_1 \right)^2 \leq \]

\[ \leq 2 \left( \int_{t}^{T} \int_{t}^{t_2} (R_{p1p2}(t_1, t_2))^2 dt_1 dt_2 + \int_{t}^{T} \int_{t}^{t_1} (R_{p1p2}(t_1, t_2))^2 dt_2 dt_1 \right) + \]

\[ + 1_{\{i_1 = i_2 \neq 0\}} \left( \int_{t}^{T} R_{p1p2}(t_1, t_1) dt_1 \right)^2 = \]

\[ (16) \]

\[ = 2 \int_{[t,T]^2} (R_{p1p2}(t_1, t_2))^2 dt_1 dt_2 + 1_{\{i_1 = i_2 \neq 0\}} \left( \int_{t}^{T} R_{p1p2}(t_1, t_1) dt_1 \right)^2. \]

We have

\[ \int_{[t,T]^2} (R_{p1p2}(t_1, t_2))^2 dt_1 dt_2 = \]

\[ = \int_{[t,T]^2} \left( K^+ (t_1, t_2) - \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} C_{j_2 j_1} \phi_{j_1}(t_1) \phi_{j_2}(t_2) \right)^2 dt_1 dt_2 = \]

\[ = \int_{[t,T]^2} \left( K(t_1, t_2) - \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} C_{j_2 j_1} \phi_{j_1}(t_1) \phi_{j_2}(t_2) \right)^2 dt_1 dt_2. \]
The function $K(t_1, t_2)$ is piecewise continuous in the square $[t, T]^2$. At this situation it is well-known that the generalized multiple Fourier series of the function $K(t_1, t_2) \in L_2([t, T]^2)$ is converging to this function in the square $[t, T]^2$ in the mean-square sense, i.e.

$$\lim_{p_1, p_2 \to \infty} \left\| K(t_1, t_2) - \sum_{j_1 = 0}^{p_1} \sum_{j_2 = 0}^{p_2} C_{j_2, j_1} \prod_{l=1}^{2} \phi_{j_l}(t_l) \right\|_{L_2([t, T]^2)} = 0,$$

where

$$\|f\|_{L_2([t, T]^2)} = \left( \int_{[t, T]^2} f^2(t_1, t_2) dt_1 dt_2 \right)^{1/2}.$$

So, we obtain

$$\lim_{p_1, p_2 \to \infty} \int_{[t, T]^2} (R_{p_1, p_2}(t_1, t_2))^2 dt_1 dt_2 = 0. \quad (17)$$

Note that

$$\int_t^T R_{p_1, p_2}(t_1, t_1) dt_1 =$$

$$= \int_t^T \left( \frac{1}{2} \psi_1(t_1) \psi_2(t_1) - \sum_{j_1 = 0}^{p_1} \sum_{j_2 = 0}^{p_2} C_{j_2, j_1} \phi_{j_1}(t_1) \phi_{j_2}(t_1) \right) dt_1 =$$

$$= \frac{1}{2} \int_t^T \psi_1(t_1) \psi_2(t_1) dt_1 - \sum_{j_1 = 0}^{p_1} \sum_{j_2 = 0}^{p_2} C_{j_2, j_1} \int_t^T \phi_{j_1}(t_1) \phi_{j_2}(t_1) dt_1 =$$

$$= \frac{1}{2} \int_t^T \psi_1(t_1) \psi_2(t_1) dt_1 - \sum_{j_1 = 0}^{p_1} \sum_{j_2 = 0}^{p_2} C_{j_2, j_1} 1_{(j_1 = j_2)} =$$

$$= \frac{1}{2} \int_t^T \psi_1(t_1) \psi_2(t_1) dt_1 - \sum_{j_1 = 0}^{\min\{p_1, p_2\}} C_{j_1, j_1}. \quad (18)$$

From (18) we obtain

$$\lim_{p_1 \to \infty} \lim_{p_2 \to \infty} \int_t^T R_{p_1, p_2}(t_1, t_1) dt_1 =$$

$$= \frac{1}{2} \int_t^T \psi_1(t_1) \psi_2(t_1) dt_1 - \lim_{p_1 \to \infty} \sum_{j_1 = 0}^{p_1} C_{j_1, j_1} = \quad (19)$$
\[ \frac{1}{2} \int_{t}^{T} \psi_1(t_1)\psi_2(t_1)dt_1 - \sum_{j_1,j_1}^{\infty} C_{j_1,j_1} = \]

\[ = \lim_{p_1,p_2 \to \infty} \int_{t}^{T} R_{p_1,p_2}(t_1,t_1)dt_1. \]

(20)

Note that the existence of the limit

\[ \lim_{p_1 \to \infty} \sum_{j_1=0}^{p_1} C_{j_1,j_1} \]

is proved in [8] (Sect. 2.1.1, 2.1.2) for the polynomial and trigonometric cases. If we prove the following relation

(21) \[ \lim_{p_1 \to \infty} \lim_{p_2 \to \infty} \int_{t}^{T} R_{p_1,p_2}(t_1,t_1)dt_1 = 0, \]

then from (20) we get

(22) \[ \frac{1}{2} \int_{t}^{T} \psi_1(t_1)\psi_2(t_1)dt_1 = \sum_{j_1=0}^{\infty} C_{j_1,j_1}, \]

(23) \[ \lim_{p_1,p_2 \to \infty} \int_{t}^{T} R_{p_1,p_2}(t_1,t_1)dt_1 = 0. \]

From (16), (17), and (23) we obtain

\[ \lim_{p_1,p_2 \to \infty} M \left\{ J[R_{p_1,p_2}^{(2)}(t_1,t_1) \right\}^2 = 0 \]

and Theorem 1 will be proved.

Let us expand the function \( K^*(t_1,t_2) \) (see (10)) using the variable \( t_1 \), when \( t_2 \) is fixed, into the generalized Fourier series at the interval \( (t,T) \)

(24) \[ K^*(t_1,t_2) = \sum_{j_1=0}^{\infty} C_{j_1}(t_2) \phi_{j_1}(t_1) \quad (t_1 \neq t,T), \]

where

\[ C_{j_1}(t_2) = \int_{t}^{T} K^*(t_1,t_2) \phi_{j_1}(t_1)dt_1 = \psi_2(t_2) \int_{t}^{t_2} \psi_1(t_1)\phi_{j_1}(t_1)dt_1. \]
The equality (24) is satisfied pointwise in each point of the interval \((t, T)\) with respect to the variable \(t_1\), when \(t_2 \in [t, T]\) is fixed, due to a piecewise smoothness of the function \(K^*(t_1, t_2)\) with respect to the variable \(t_1 \in [t, T]\) \((t_2\) is fixed).

Note also that due to well-known properties of the Fourier–Legendre series and trigonometric Fourier series, the series (24) converges when \(t_1 = t\) and \(t_1 = T\).

Obtaining (24), we also used the fact that the right-hand side of (24) converges when \(t_1 = t_2\) (point of a finite discontinuity of the function \(K(t_1, t_2)\)) to the value

\[
\frac{1}{2} (K(t_2 - 0, t_2) + K(t_2 + 0, t_2)) = \frac{1}{2} \psi_1(t_2)\psi_2(t_2) = K^*(t_2, t_2).
\]

The function \(C_{j_1}(t_2)\) is a continuously differentiable one at the interval \([t, T]\). Let us expand it into the generalized Fourier series at the interval \((t, T)\)

\[
(25) \quad C_{j_1}(t_2) = \sum_{j_2=0}^{\infty} C_{j_2j_1}\phi_{j_2}(t_2) \quad (t_2 \neq t, T),
\]

where

\[
C_{j_2j_1} = \int_t^T C_{j_1}(t_2)\phi_{j_2}(t_2)dt_2 = \int_t^T \psi_2(t_2)\phi_{j_2}(t_2) \int_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2
\]

and the equality (25) is satisfied pointwise at any point of the interval \((t, T)\) (the right-hand side of (25) converges when \(t_2 = t\) and \(t_1 = T\)).

Let us substitute (25) into (24)

\[
(26) \quad K^*(t_1, t_2) = \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} C_{j_2j_1}\phi_{j_1}(t_1)\phi_{j_2}(t_2), \quad (t_1, t_2) \in (t, T)^2.
\]

Furthermore, the series on the right-hand side of (26) converges at the boundary of the square \([t, T]^2\).

From (15) and (26) we obtain

\[
(27) \quad \lim_{p_1 \to \infty} \lim_{p_2 \to \infty} R_{p_1p_2}(t_1, t_1) = 0 \quad \text{when} \quad t_1 \in (t, T).
\]

Since the integral

\[
(28) \quad \int_t^T R_{p_1p_2}(t_1, t_1)dt_1
\]

exists as Riemann integral, then this integral equals to the corresponding Lebesgue integral. Moreover,

\[
(29) \quad \lim_{p_1 \to \infty} \lim_{p_2 \to \infty} R_{p_1p_2}(t_1, t_1) = 0 \quad \text{when} \quad t_1 \in (t, T),
\]

where the left-hand side of (29) is bounded on \([t, T]\).

According to (15), (21)–(26), we have
\[ R_{p_1 p_2}(t_1, t_2) = \left( K^*(t_1, t_2) - \sum_{j_1=0}^{p_1} C_{j_1}(t_2) \phi_{j_1}(t_1) \right) + \]
\[ \left( \sum_{j_1=0}^{p_1} \left( C_{j_1}(t_2) - \sum_{j_2=0}^{p_2} C_{j_2 j_1}(t_2) \phi_{j_2}(t_1) \right) \phi_{j_1}(t_1) \right). \]

Then, applying two times (we mean here an iterated passage to the limit \( \lim_{p_1 \to \infty} \lim_{p_2 \to \infty} \)) the Lebesgue’s Dominated Convergence Theorem to the integral (28), we obtain
\[ \lim_{p_1 \to \infty} \lim_{p_2 \to \infty} \int_{t}^{T} R_{p_1 p_2}(t_1, t_1) dt_1 = 0. \]

For a discussion of the choice of integrable majorants when applying Lebesgue’s Dominated Convergence Theorem to the integral (28) for the polynomial and trigonometric cases, see [8] (Sect. 2.4.1), [43] (Sect. 2).

Note that the development of the approach from this article can be found in [8] (Sect. 2.4), [43].

3. SOME RECENT RESULTS ON EXPANSION OF ITERATED STRATONOVICH STOCHASTIC INTEGRALS OF MULTICILITIES 2 TO 6

Recently, a new approach to the expansion and mean-square approximation of iterated Stratonovich stochastic integrals has been obtained [8] (Sect. 2.10–2.16), [13] (Sect. 13–19), [34] (Sect. 5–11), [35] (Sect. 7–13), [64] (Sect. 4–9). Let us formulate four theorems that were obtained using this approach.

**Theorem 2** [8], [13], [34], [35], [64]. Suppose that \( \{ \phi_j(x) \}_{j=0}^\infty \) is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space \( L^2([t, T]) \). Furthermore, let \( \psi_1(\tau), \psi_2(\tau), \psi_3(\tau) \) are continuously differentiable nonrandom functions on \([t, T]\). Then, for the iterated Stratonovich stochastic integral of third multiplicity
\[ J^*[\psi^{(3)}]_{T,t} = \int_{t}^{T} \int_{t}^{t_3} \psi_3(t_3) \int_{t}^{t_2} \psi_2(t_2) \int_{t}^{t_1} \psi_1(t_1) dw_{t_1}^{(i_1)} dw_{t_2}^{(i_2)} dw_{t_3}^{(i_3)} \] (\( i_1, i_2, i_3 = 0, 1, \ldots, m \))

the following relations
\[ J^*[\psi^{(3)}]_{T,t} = \lim_{p \to \infty} \sum_{j_1, j_2, j_3 = 0}^{p} C_{j_3 j_2 j_1} \zeta_{j_3}^{(i_3)} \zeta_{j_2}^{(i_2)} \zeta_{j_1}^{(i_1)}, \]

\[ M \left\{ \left( J^*[\psi^{(3)}]_{T,t} - \sum_{j_1, j_2, j_3 = 0}^{p} C_{j_3 j_2 j_1} \zeta_{j_3}^{(i_3)} \zeta_{j_2}^{(i_2)} \zeta_{j_1}^{(i_1)} \right)^2 \right\} \leq \frac{C}{p} \]
are fulfilled, where \(i_1, i_2, i_3 = 0, 1, \ldots, m\) in (30) and \(i_1, i_2, i_3 = 1, \ldots, m\) in (31), constant \(C\) is independent of \(p\),

\[
C_{j_3 j_2 j_1} = \int_t^T \psi_3(t_3) \phi_{j_3}(t_3) \int_t^{t_2} \psi_2(t_2) \phi_{j_2}(t_2) \int_t^{t_3} \psi_1(t_1) \phi_{j_1}(t_1) dt_1 dt_2 dt_3
\]

and

\[
\zeta_j^{(i)} = \int_t^T \phi_j(\tau) d\mathbb{P}(\tau)
\]

are independent standard Gaussian random variables for various \(i\) or \(j\) (in the case when \(i \neq 0\)); another notations are the same as in Theorem 1.

**Theorem 3** [8], [13], [34], [35], [64]. Let \(\{\phi_j(x)\}_{j=0}^\infty\) be a complete orthonormal system of Legendre polynomials or trigonometric functions in the space \(L_2([t, T])\). Furthermore, let \(\psi_1(\tau), \ldots, \psi_4(\tau)\) be continuously differentiable nonrandom functions on \([t, T]\). Then, for the iterated Stratonovich stochastic integral of fourth multiplicity

\[
J^*[\psi^{(4)}]_{T,t} = \int_t^T \psi_4(t_4) \int_t^{t_3} \psi_3(t_3) \int_t^{t_2} \psi_2(t_2) \int_t^{t_1} \psi_1(t_1) d\mathbb{W}_{t_1}^{(i_1)} d\mathbb{W}_{t_2}^{(i_2)} d\mathbb{W}_{t_3}^{(i_3)} d\mathbb{W}_{t_4}^{(i_4)}
\]

the following relations

\[
J^*[\psi^{(4)}]_{T,t} = \lim_{p \to \infty} \sum_{j_1, j_2, j_3, j_4 = 0}^p C_{j_4 j_3 j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)};
\]

\[
\max \left\{ J^*[\psi^{(4)}]_{T,t} - \sum_{j_1, j_2, j_3, j_4 = 0}^p C_{j_4 j_3 j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} \right\} \leq \frac{C}{p^{1-\varepsilon}}
\]

are fulfilled, where \(i_1, \ldots, i_4 = 0, 1, \ldots, m\) in (32), (33) and \(i_1, \ldots, i_4 = 1, \ldots, m\) in (34), constant \(C\) does not depend on \(p, \varepsilon\) is an arbitrary small positive real number for the case of complete orthonormal system of Legendre polynomials in the space \(L_2([t, T])\) and \(\varepsilon = 0\) for the case of complete orthonormal system of trigonometric functions in the space \(L_2([t, T])\),

\[
C_{j_4 j_3 j_2 j_1} = \int_t^T \psi_4(t_4) \phi_{j_4}(t_4) \int_t^{t_3} \psi_3(t_3) \phi_{j_3}(t_3) \int_t^{t_2} \psi_2(t_2) \phi_{j_2}(t_2) \int_t^{t_1} \psi_1(t_1) \phi_{j_1}(t_1) dt_1 dt_2 dt_3 dt_4;
\]

another notations are the same as in Theorem 2.
Theorem 4 [8], [13], [34], [35], [64]. Assume that \( \{\phi_j(x)\}_{j=0}^\infty \) is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space \( L_2([t, T]) \) and \( \psi_1(\tau), \ldots, \psi_5(\tau) \) are continuously differentiable nonrandom functions on \([t, T]\). Then, for the iterated Stratonovich stochastic integral of fifth multiplicity

\[
J^*[\psi^{(5)}]_{T,t} = \int_t^{t_2} \psi_5(t_5) \ldots \int_t^{t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_6}^{(i_6)}
\]

the following relations

\[
J^*[\psi^{(5)}]_{T,t} = \lim_{p \to \infty} \; \sum_{j_5, \ldots, j_1 = 0}^p C_{j_5 \ldots j_1} \zeta_{j_5}^{(i_5)} \ldots \zeta_{j_1}^{(i_1)},
\]

\[
M \left\{ \left( J^*[\psi^{(5)}]_{T,t} - \sum_{j_5, \ldots, j_1 = 0}^p C_{j_5 \ldots j_1} \zeta_{j_5}^{(i_5)} \ldots \zeta_{j_1}^{(i_1)} \right)^2 \right\} \leq \frac{C}{p^{1-\varepsilon}}
\]

are fulfilled, where \( i_1, \ldots, i_5 = 0, 1, \ldots, m \) in (35), (36) and \( i_1, \ldots, i_5 = 1, \ldots, m \) in (37), constant \( C \) is independent of \( p \), \( \varepsilon \) is an arbitrary small positive real number for the case of complete orthonormal system of Legendre polynomials in the space \( L_2([t, T]) \) and \( \varepsilon = 0 \) for the case of complete orthonormal system of trigonometric functions in the space \( L_2([t, T]) \),

\[
C_{j_5 \ldots j_1} = \int_t^T \psi_5(t_5) \phi_{j_5}(t_5) \ldots \int_t^{t_2} \psi_1(t_1) \phi_{j_1}(t_1) dt_1 \ldots dt_5;
\]

another notations are the same as in Theorems 2, 3.

Theorem 5 [8], [13], [34], [35]. Suppose that \( \{\phi_j(x)\}_{j=0}^\infty \) is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space \( L_2([t, T]) \). Then, for the iterated Stratonovich stochastic integral of sixth multiplicity

\[
J^{*(i_1 \ldots i_6)}_{T,t} = \int_t^{t_2} \ldots \int_t^{t_2} d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_6}^{(i_6)}
\]

the following expansion

\[
J^{*(i_1 \ldots i_6)}_{T,t} = \lim_{p \to \infty} \; \sum_{j_6, \ldots, j_1 = 0}^p C_{j_6 \ldots j_1} \zeta_{j_6}^{(i_6)} \ldots \zeta_{j_1}^{(i_1)}
\]

that converges in the mean-square sense is valid, where \( i_1, \ldots, i_6 = 0, 1, \ldots, m \),

\[
C_{j_6 \ldots j_1} = \int_t^T \phi_{j_6}(t_6) \ldots \int_t^{t_2} \phi_{j_1}(t_1) dt_1 \ldots dt_6;
\]
another notations are the same as in Theorems 2–4.

Recently the equality \(22\) was proved in \(66\) (also see \(8\) (Sect. 2.1.4)) for the case of an arbitrary complete orthonormal system of functions in \(L^2([t, T])\) and \(\psi_1(\tau), \psi_2(\tau) \in L^2([t, T])\). This means that we have the following generalization of Theorem 1.

**Theorem 6** \(8\) (Sect. 2.1.4). Suppose that \(\{\phi_j(x)\}_{j=0}^\infty\) is an arbitrary complete orthonormal system of functions in the space \(L^2([t, T])\) and \(\psi_1(\tau), \psi_2(\tau)\) are continuous functions on \([t, T]\). Then the iterated Stratonovich stochastic integral of the second multiplicity

\[
J^*_{t, T} = \int_t^T \psi_2(t_2) \int_t^{t_2} \psi_1(t_1) dw^{(i_1)}_{t_1} dw^{(i_2)}_{t_2}
\]

is expanded into the multiple series

\[
J^*_{t, T} = \lim_{p_1, p_2 \to \infty} \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} C_{j_2 j_1} \psi^{(i_1)}_{j_1} \psi^{(i_2)}_{j_2}
\]

that converges in the mean-square sense; where notations are the same as in Theorem 1.

The condition of continuity of the functions \(\psi_1(\tau), \psi_2(\tau)\) is related to the definition of the Stratonovich stochastic integral that we use (see \(\Xi\)).

4. Theorems 1–6 from Point of View of the Wong–Zakai Approximation

The iterated Ito stochastic integrals and solutions of Ito SDEs are complex and important functionals from the independent components \(f_s^{(i)}, \ i = 1, \ldots, m\) of the multidimensional Wiener process \(f_s\), \(s \in [0, T]\). Let \(f_s^{(i)p}, p \in \mathbb{N}\) be some approximation of \(f_s^{(i)}\), \(i = 1, \ldots, m\). Suppose that \(f_s^{(i)p}\) converges to \(f_s^{(i)}\), \(i = 1, \ldots, m\) if \(p \to \infty\) in some sense and has differentiable sample trajectories.

A natural question arises: if we replace \(f_s^{(i)}\) by \(f_s^{(i)p}\), \(i = 1, \ldots, m\) in the functionals mentioned above, will the resulting functionals converge to the original functionals from the components \(f_s^{(i)}\), \(i = 1, \ldots, m\) of the multidimensional Wiener process \(f_s^p\)? The answere to this question is negative in the general case. However, in the pioneering works of Wong E. and Zakai M. \([50, 51]\), it was shown that under the special conditions and for some types of approximations of the Wiener process the answer is affirmative with one peculiarity: the convergence takes place to the iterated Stratonovich stochastic integrals and solutions of Stratonovich SDEs and not to iterated Ito stochastic integrals and solutions of Ito SDEs. The piecewise linear approximation as well as the regularization by convolution \([52–54]\) relate the mentioned types of approximations of the Wiener process. The above approximation of stochastic integrals and solutions of SDEs is often called the Wong–Zakai approximation.

Let \(f_s\), \(s \in [0, T]\) be an \(m\)-dimensional standard Wiener process with independent components \(f_s^{(i)}\), \(i = 1, \ldots, m\). It is well known that the following representation takes place \([53, 54]\)

\[
f_{r}^{(i)} - f_{t}^{(i)} = \sum_{j=0}^\infty \int_t^\tau \phi_j(s) ds \ \zeta_s^{(i)}, \quad \zeta_s^{(i)} = \int_t^\tau \phi_j(\tau) df_{\tau}^{(i)},
\]

where \(\tau \in [t, T], t \geq 0, \{\phi_j(x)\}_{j=0}^\infty\) is an arbitrary complete orthonormal system of functions in the space \(L^2([t, T])\), and \(\zeta_s^{(i)}\) are independent standard Gaussian random variables for various \(i\) or \(j\). Moreover, the series \((39)\) converges for any \(\tau \in [t, T]\) in the mean-square sense.
Let $f^{(i)p}_t - f^{(i)p}_t$ be the mean-square approximation of the process $f^{(i)}_t - f^{(i)}_t$, which has the following form

\begin{equation}
 f^{(i)p}_t - f^{(i)p}_t = \sum_{j=0}^{p} \int_t^\tau \phi_j(s) ds \zeta^{(i)}_j.
\end{equation}

From (40) we obtain

\begin{equation}
 df^{(i)p}_t = \sum_{j=0}^{p} \phi_j(\tau) \zeta^{(i)}_j d\tau.
\end{equation}

Consider the following iterated Riemann–Stieltjes integral

\begin{equation}
 T \int_t^T \psi_k(t_k) \cdots T \int_t^T \psi_1(t_1) dw^{(i_1)p_1}_{t_1} \cdots dw^{(i_k)p_k}_{t_k},
\end{equation}

where $i_1, \ldots, i_k = 0, 1, \ldots, m$, $p_1, \ldots, p_k \in \mathbb{N}$,

\begin{equation}
 dw^{(i)p}_t = \begin{cases}
 df^{(i)p}_t & \text{for } i = 1, \ldots, m \\
 d\tau & \text{for } i = 0
\end{cases},
\end{equation}

and $df^{(i)p}_t$ in defined by the relation (41).

Let us substitute (41) into (42)

\begin{equation}
 \int_t^T \psi_k(t_k) \cdots \int_t^T \psi_1(t_1) dw^{(i_1)p_1}_{t_1} \cdots dw^{(i_k)p_k}_{t_k} = \sum_{j_1=0}^{p_1} \cdots \sum_{j_k=0}^{p_k} C_{j_k \ldots j_1} \prod_{l=1}^k \zeta^{(i_l)}_{j_l},
\end{equation}

where

\begin{equation}
 \zeta^{(i)}_j = \int_t^T \phi_j(\tau) dw^{(i)}_t
\end{equation}

are independent standard Gaussian random variables for various $i$ or $j$ (in the case when $i \neq 0$), $w^{(i)}_s = f^{(i)}_s$ for $i = 1, \ldots, m$ and $w^{(0)}_s = s$,

\begin{equation}
 C_{j_k \ldots j_1} = \int_t^T \psi_k(t_k) \phi_{j_k}(t_k) \cdots \int_t^T \psi_1(t_1) \phi_{j_1}(t_1) dt_1 \cdots dt_k
\end{equation}

is the Fourier coefficient.

Consider the following iterated Stratonovich stochastic integrals

\begin{equation}
 J^*[\psi^{(k)}_{\tau, t}] = \int_t^T \psi_k(t_k) \cdots \int_t^T \psi_1(t_1) dw^{(i_1)}_{t_1} \cdots dw^{(i_k)}_{t_k},
\end{equation}
where every $\psi_l(\tau)$ ($l = 1, \ldots, k$) is a continuously differentiable nonrandom function at the interval $[t, T]$; another notations are the same as in (1).

To best of our knowledge [50]-[52] the approximations of the Wiener process in the Wong–Zakai approximation must satisfy fairly strong restrictions [52] (see Definition 7.1, pp. 480–481). Moreover, approximations of the Wiener process that are similar to (40) for approximations of the Wiener process based on its series expansion (39) should be carried out separately. Thus, the mean-square convergence of the right-hand side of (44) to the appropriate iterated Stratonovich stochastic integral (45) does not follow from the results of the papers [50], [51] (also see [52], Theorems 7.1, 7.2).

However, in [1] (Sect. 5.8, pp. 202–204), [55] (pp. 82-84), [56] (pp. 438-439), [57] (pp. 263-264) the authors use (without rigorous proof) the Wong–Zakai approximation [50]-[52] within the frames of the method of approximation of iterated Stratonovich stochastic integrals based on the Karhunen–Loeve expansion of the Brownian bridge process [58].

From the other hand, Theorems 1–6 from this paper can be considered as the proof of the Wong–Zakai approximation for the iterated Stratonovich stochastic integrals (45) of multiplicities 2 to 6 based on the approximation (40) of the Wiener process. At that, the Riemann–Stieltjes integrals (42) of multiplicities 2 to 6 converge in the mean-square sense to the appropriate Stratonovich stochastic integrals (45). Recall that $\{\phi_j(x)\}_{j=0}^\infty$ (see (39), (40), and Theorems 1–5) is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t, T])$.

To illustrate the above reasoning, consider two examples for the case $k = 2$, $\psi_1(\tau)$, $\psi_2(\tau) \equiv 1$; $i_1, i_2 = 1, \ldots, m$.

The first example relates to the piecewise linear approximation of the multidimensional Wiener process (these approximations were considered in [50]-[52]).

Let $b_{\Delta}^{(i)}(t), t \in [0, T]$ be the piecewise linear approximation of the $i$th component $f_t^{(i)}$ of the multidimensional standard Wiener process $f_t, t \in [0, T]$ with independent components $f_t^{(i)}, i = 1, \ldots, m$, i.e.

$$b_{\Delta}^{(i)}(t) = f_{k\Delta}^{(i)} + \frac{t - k\Delta}{\Delta} f_{k+1\Delta}^{(i)},$$

where

$$\Delta f_{k\Delta}^{(i)} = f_{(k+1)\Delta}^{(i)} - f_{k\Delta}^{(i)}, \quad t \in [k\Delta, (k+1)\Delta), \quad k = 0, 1, \ldots, N - 1.$$

Note that w. p. 1

$$\frac{dB_{\Delta}^{(i)}}{dt}(t) = \frac{\Delta f_{k\Delta}^{(i)}}{\Delta}, \quad t \in [k\Delta, (k+1)\Delta), \quad k = 0, 1, \ldots, N - 1. \quad (46)$$

Consider the following iterated Riemann–Stieltjes integral

$$\int_0^T \int_0^s d\Sigma_{\Delta}^{(i_1)}(\tau)d\Sigma_{\Delta}^{(i_2)}(s), \quad i_1, i_2 = 1, \ldots, m.$$

Using (46) and additive property of the Riemann–Stieltjes integral, we can write w. p. 1

$$\int_0^T \int_0^s d\Sigma_{\Delta}^{(i_1)}(\tau)d\Sigma_{\Delta}^{(i_2)}(s) = \int_0^T \int_0^s \frac{d\Sigma_{\Delta}^{(i_1)}}{dt}(\tau)d\Sigma_{\Delta}^{(i_2)}(s) d\tau =$$
\[ \sum_{l=0}^{N-1} \left( \sum_{q=0}^{l} \frac{\Delta f_{q\Delta}^{(i_1)}}{\Delta} d\tau + \int_{l\Delta}^{s} \frac{\Delta f_{l\Delta}^{(i_2)}}{\Delta} d\tau \right) \frac{\Delta f_{l\Delta}^{(i_2)}}{\Delta} ds = \]

\[ = \sum_{l=0}^{N-1} \sum_{q=0}^{l-1} \Delta f_{q\Delta}^{(i_1)} \Delta f_{l\Delta}^{(i_2)} + \frac{1}{2} \sum_{l=0}^{N-1} \Delta f_{l\Delta}^{(i_1)} \Delta f_{l\Delta}^{(i_2)} \int_{l\Delta}^{s} ds = \]

\[ \sum_{l=0}^{N-1} \sum_{q=0}^{l-1} \Delta f_{q\Delta}^{(i_1)} \Delta f_{l\Delta}^{(i_2)} + \frac{1}{2} \sum_{l=0}^{N-1} \Delta f_{l\Delta}^{(i_1)} \Delta f_{l\Delta}^{(i_2)}. \]

Using (47), it is not difficult to show that

\[ \lim_{N \to \infty} \int_0^T \int_0^s d\mathfrak{D}_{\Delta}^{(i_1)}(\tau) d\mathfrak{D}_{\Delta}^{(i_2)}(s) = \int_0^T \int_0^s d\mathfrak{D}_{\Delta}^{(i_1)} d\mathfrak{D}_{\Delta}^{(i_2)} + \frac{1}{2} \sum_{i_1=i_2}^{m} \mathfrak{H}_{i_1 i_2} s, \]

where \( \Delta \to 0 \) if \( N \to \infty \) (\( N\Delta = T \)).

Obviously, (48) agrees with Theorem 7.1 (see [52], p. 486).

The next example relates to the approximation of the Wiener process based on its series expansion (39) for \( t = 0 \), where \( \{\phi_j(x)\}_{j=0}^{\infty} \) is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space \( L_2([0, T]) \).

Consider the following iterated Riemann–Stieltjes integral

\[ \int_0^T \int_0^s d\mathfrak{D}_{\Delta}^{(i_1)} d\mathfrak{D}_{\Delta}^{(i_2)} = \int_0^T \int_0^s d\mathfrak{D}_{\Delta}^{(i_1)} d\mathfrak{D}_{\Delta}^{(i_2)} + \frac{1}{2} \sum_{i_1=i_2}^{m} \mathfrak{H}_{i_1 i_2} s, \]

where \( d\mathfrak{D}_{\Delta}^{(i_1)} d\mathfrak{D}_{\Delta}^{(i_2)} \) is defined by the relation (41).

Let us substitute (41) into (49)

\[ \int_0^T \int_0^s d\mathfrak{D}_{\Delta}^{(i_1)} d\mathfrak{D}_{\Delta}^{(i_2)} = \sum_{j_1,j_2=0}^{p} C_{j_2j_1} \mathfrak{H}_{j_1 j_2} s, \]

where

\[ C_{j_2j_1} = \int_0^T \phi_{j_2}(s) \int_0^s \phi_{j_1}(\tau) d\tau ds \]

is the Fourier coefficient; another notations are the same as in (41).

As we noted above, approximations of the Wiener process that are similar to (40) were not considered in [50], [51] (also see Theorems 7.1, 7.2 in [52]). Furthermore, the extension of the results of Theorems 7.1 and 7.2 [52] to the case under consideration is not obvious.

On the other hand, we can apply Theorem 1 from this paper and obtain from (50) the desired result.
\[
\begin{align*}
\lim_{p \to \infty} \int_0^T \int_0^s dF_{(i_1)p} \, dF_{(i_2)p} &= \lim_{p \to \infty} \sum_{j_1, j_2 = 0}^p C_{j_2 j_1} \eta^{(i_1)}_{j_1} \eta^{(i_2)}_{j_2} \\
&= \int_0^T \int_0^s dF_{(i_1)} \, dF_{(i_2)}. 
\end{align*}
\]

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