PARTICLES AND FIELDS

Mass Evolution of Schwarzschild Black Holes

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Abstract
In the classical theory of general relativity, black holes can only absorb and not emit particles. When quantum mechanical effects are taken into account, then the black holes emit particles as hot bodies with temperature proportional to \(\kappa\), its surface gravity. This thermal emission can lead to a slow decrease in the mass of the black hole, and eventually to its disappearance, also called black hole evaporation. This characteristic allows us to analyze what happens with the mass of the black hole when its temperature is increased or decreased and how the energy is exchanged with the external environment. This paper has the aim to make a discussion about the mass evolution, due to Hawking radiation emission, of Schwarzschild black holes with different initial masses and external conditions as the empty space, the cosmic microwave background with constant temperature, and with temperature varying in accordance with the eras of the universe. Our main contributions are to take into account these variations in the temperature of the cosmic background radiation and use them into the black holes dynamics and also to analyze the asymptotic behavior of the solutions. As a result, we have the complete evaporation of the black holes in most cases, although their masses can increase in some cases, and even diverge for specific conditions.

Keywords Black hole thermodynamics · Black hole evaporation · Mass evolution of black holes · Schwarzschild black hole

1 Introduction
The theory of general relativity (GR) was published by Albert Einstein in 1915 \(^1\) and changed completely our description of the universe. Since now, gravity is no longer an action-at-a-distance force, but a geometrical property of a curved space-time, i.e., the gravitational field can be interpreted as the result of the space-time curvature \(^2–7\). The black holes are one of the most interesting predictions of the theory, and they appear at the end stage of very massive stars collapse, when no internal process can generate the necessary pressure to balance the gravitational attraction \(^8–10\). These objects are characterized by singularity and an event horizon, from which classically nothing can escape, not even light \(^11–15\).

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Although GR has passed many experimental tests in the last century \(^16, 17\), there are some reasons to suppose it is not the final theory for gravitation, because it explicitly predicts singularities (as the ones inside black holes) \(^18–21\) and it does not agree with the Quantum Field Theory (QFT) in the appropriated limit. Even though there is still no satisfactory candidate to unify GR and QFT, a minimally consistent junction is made by the Quantum Field Theory in Curved Space-time (QFTCS). In this context, the matter fields are quantized, but gravity is still classically described by GR \(^22–27\).

In 1975, Stephen Hawking published his emblematic paper “Particle Creation by Black Holes” \(^28\), where he studied effects of a quantum field in a classical background using the semi-classical approach of QFTCS. It caused scientific upheaval at that time because the result was opposed to the classical idea that black holes are passive objects, in sense that they just absorb matter and never emit them (and hence have zero temperature). By quantizing the scalar field in a spherical symmetric collapse scenario, Hawking proved that black holes emit particles asymptotically in the future, and therefore have non-zero temperature.

Even though the Hawking paper officially inaugurated the so-called black hole thermodynamics, the idea that
black holes should behave as thermodynamic objects appeared previously, in 1973 with Jacob Bekenstein [29]. Based on the Hawking area theorem [30], that says that the area of the event horizon of a black hole cannot decrease, Bekenstein established a formal analogy between the laws of thermodynamics and the laws of black hole mechanics, defining an entropy for the black hole related with its area. This result showed that black holes’ physics is intrinsically irreversible, presenting an arrow of time, just like thermodynamics [31–38]. However, the analogy proposed by Bekenstein was incomplete because it did not include the temperature of the black hole, only the entropy, since classically, all black holes possess zero temperature, and he could not solve this inconsistency. It was Hawking’s paper [28] (using QFTCS) that gave the correct meaning to the temperature and entropy of black holes.

With the expressions for the temperature and entropy defined, the black holes can be analyzed as thermodynamic objects, and an interesting property of Schwarzschild black holes, that will be shown in the present paper, is that they have negative heat capacity, a feature that does not allow them to achieve thermal equilibrium with an energy reservoir. Thus, a Schwarzschild black hole in the well-known [28, 39] empty space will lose energy and increase its temperature until its mass approaches zero. When this happens we have the so-called black hole evaporation, although the final configuration is still unknown [40]. Since a black hole in empty space is an idealization, we will present the study of the mass evolution of Schwarzschild black holes when immersed in the cosmic microwave background radiation with constant temperature and with temperature evolving in time according to the universe eras: radiation, matter, and dark energy, considering only the effects of the Hawking radiation [2, 4, 41].

The objective of this paper is briefly to present the laws of black hole thermodynamics (see Section 2), then show that black holes cannot achieve a thermal equilibrium with an energy reservoir (see Section 3), and analyze the mass evolution of them (see Sections 4, 5, and 6). At the end, we discussed our results.

### Table 1  Thermodynamic laws and classical laws for black holes

| Law | Thermodynamic laws | Black hole laws |
|-----|-------------------|-----------------|
| 0th | Equality of temperatures is a condition for thermodynamic equilibrium between systems or between parts of the same system. | The surface gravity \( \kappa \) of a stationary black hole is constant over its entire horizon. |
| 1st | In an isolated system, the total energy of that system is conserved: \( dU = TdS - dW \). | In an isolated system, including black holes, the total energy is conserved: \( dM = \frac{1}{2\pi}dA + \Omega_H dL + \Phi_H dQ \). |
| 2nd | In an isolated system, during any process, the total entropy of the system increases or remains the same: \( \delta S_T \geq 0 \). | The sum of the entropy of the black hole with its exterior increases or remains the same: \( \delta S_{BH} + \delta S_{ext} \geq 0 \). |
| 3rd | It is impossible to reduce the temperature of a system to zero by a finite number of processes. | It is impossible to reduce the surface gravity \( \kappa \) of a black hole to zero by a finite number of processes. |

### 2 Analogy Between Black Hole Mechanics and Thermodynamics

According to [28], the expression for the entropy \( S \) and temperature \( T \) of a Schwarzschild black hole are:

\[
T = \frac{\kappa}{2\pi} = \frac{1}{8\pi M}, \quad S = \frac{A}{4} = 4\pi M^2,
\]

where \( M \) is the mass, \( A = 16\pi M^2 \) is the event horizon area, and \( \kappa = \frac{1}{4M} \) is the surface gravity evaluated at the horizon of the black hole [2–4, 22, 31, 32, 36]. Also, the black hole mass \( M \) is identified with its energy: \( E = M \). Note that, in the above expressions (and in many other expressions in the present paper), we are going to use Planck units, i.e., we are considering the speed of light \( c \), the universal gravitational constant \( G \), the reduced Planck constant \( \hbar \), and the Boltzmann constant \( k_B \) all equal to 1. We will take those constants when convenient.

With these explicit equations, we are now able to relate the laws of thermodynamics with the laws of black hole mechanics for a Schwarzschild black hole. This analogy can be seen in the Table 1. The 0th law connects the temperature \( T \) of the black hole with its surface gravity \( \kappa \). The 1st law states that the mass of black hole \( M \) represents its total energy, related with the thermodynamic internal energy \( U \) of a system, and that the work term \( dW \) is represented by the variation of angular momentum \( dL \) and charge \( dQ \), with \( \Omega_H \) representing the angular speed and \( \Phi_H \) the electric potential at the event horizon. Finally, the 2nd law links the area \( A \) of the horizon to the entropy \( S \) of the black hole.

### 3 The Inaccessibility of Thermal Equilibrium with an Energy Reservoir and its Implications

In classical thermodynamics, bodies exchanging energy with a thermal reservoir eventually achieve a stable equilibrium, i.e., these bodies acquire the reservoir temperature in a finite time and remain with it. However, for a
Schwarzschild black hole, this thermal equilibrium is never achieved. To show this, let’s first derive the heat capacity of a Schwarzschild black hole [42, 43]:

\[ C = \frac{dE}{dT} = \frac{dM}{dT} = -\frac{1}{8\pi T^2} = -8\pi M^2 < 0. \] (3)

Notice that \( C < 0 \), and the negative sign is responsible for this unusual feature. Let’s interpret this result in terms of energy flux between a thermal reservoir and a black hole. If the black hole, with initial temperature \( T \), is in contact with an energy reservoir with temperature \( T_R \neq T \), we can have two different situations: (i) If \( T > T_R \), the black hole will emit radiation and transfer heat to the reservoir. Since its energy is related with its mass, the black hole will lose mass, in a process called evaporation. And since its temperature is inversely proportional to its mass, the black hole temperature will increase, moving further away from the equilibrium. (ii) If \( T < T_R \), the black hole will receive energy, increase its mass, and decrease its temperature, also moving further away from the equilibrium. Therefore, in both cases, the black hole never achieves the reservoir temperature \( T_R \). If initially \( T = T_R \), there is no net flux, and the temperature of the black hole does not change. Consequently, its mass also remains constant.

### 4 Black Hole and the Empty Space

Consider a Schwarzschild black hole with initial mass \( M_0 \) and initial temperature \( T_0 > 0 \) in empty space. Since this black hole is immersed in the vacuum, we can consider a reservoir with zero temperature, \( T_R = 0 \). The black hole will then emit radiation, lose its mass, and increase its temperature, until the complete evaporation. In order to compute how the mass changes with time, we will use the Stefan-Boltzmann law for the power emitted by the black hole, and we will also assume that the black hole emits as a perfect black body\(^1\) [42, 43]. The power emitted is then:

\[ L = \sigma A T^4 = \frac{\hbar c^2}{15360\pi G^2} M^2, \] (4)

where \( \sigma = \frac{\pi^2 k_B^4}{60h^3 c^2} \) is Stefan-Boltzmann constant, \( A = 4\pi r_S^2 \) is Schwarzschild black hole area with Schwarzschild radius \( r_S = \frac{2GM}{c^2} \), and temperature \( T = \frac{\hbar c}{8\pi k_B G r_S} \). Note that we are recalling the constants to use the international unit system (SI) in order to compare our results with physical masses and temperatures. Thus, the black hole mass decreases in time according to

\[ \frac{d\left(\frac{c^2 M}{t}\right)}{dt} = -L = -\frac{\hbar c^2}{15360\pi G^2} M^2, \] (5)

Equation (5) has a simple analytical solution. Considering the initial time \( t = 0 \), the integration constant is translated as initial mass of black hole \( M(t = 0) = M_0 \). Therefore, we have

\[ M(t) = M_0 \left(1 - \frac{t}{t_V}\right)^{1/3}, \quad t \in [0, t_V) \] (6)

where \( t_V = 5120\pi \frac{G^2}{3\hbar c^2} M_0^3 \) is the lifetime\(^2\) of black hole. The general behavior of the mass decay of a Schwarzschild black hole can be viewed in Fig. 1, with intervals \( \frac{L}{M_0}, \frac{t}{t_V} \in [0, 1) \).

For astronomical values of mass, the lifetime \( t_V \) of a black hole is normally a huge number. Table 2 shows this value for three different initial masses: (i) the mass of Hyperion [44], (ii) the mass of the Sun [45], and (iii) the mass of the super massive black hole in the center of Milky Way Galaxy [46].

| Mass          | Lifetime \( t_V \) (s) |
|---------------|-------------------------|
| Hyperion      | \( 5.6 \cdot 10^{16} \) | 1.5 \cdot 10^{40} |
| Sun           | \( 2.0 \cdot 10^{20} \) | 6.6 \cdot 10^{74} |
| Black hole of Milk Way center | \( 4.5 \cdot 10^6 M_\odot \) | 6.0 \cdot 10^{84} |

\( t_V = 9.0 \cdot 10^{36} \)

Fig. 1 General mass decay of a Schwarzschild black hole in empty space

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1We need to evidence that here, we are making an approximation. We are using the result (1), for a Schwarzschild black hole and considering a perfect black body emission, using the Stefan-Boltzmann law. In order to take this more rigorously, we should consider the angular dependency (from quantized field) and the back-scattering (from probability conservation) in the temperature derivation [22–24]. In this case, we would be dealing with a black hole as a gray body. Reference [47] considers them and estimates a numerical factor \( \alpha = 10^{-4} \) in Eq. 4, rewriting it as \( L = \alpha \sigma A T^4 \).

2By lifetime, we refer to the necessary time until that black hole mass becomes zero, i.e., the necessary time until its complete evaporation.
For a black hole formed in the beginning of the universe to evaporate within the time scale of the universe, the initial mass must be inside the interval \( M_0 \in (10^{-3}, 10^{1}) \) kg. These values were obtained using \( t_V = t_p \simeq 5.4 \cdot 10^{-44} \) s (Planck time [45, 48]) and \( t_V = t_{	ext{uni}} \) respectively.

### 5 Black Hole and the Cosmic Microwave Background

Now, we are going to consider a Schwarzschild black hole with mass \( M \) and temperature \( T \) immersed in cosmic microwave background radiation (CMB). This radiation will act as a thermal reservoir to the black hole. In this first approach, we will consider that the CMB have constant temperature \( T_{\text{CMB}} \simeq 2.7255 K \) [45]. There will be an exchange of energy between the black hole and the CMB, and we can use again the Stefan-Boltzmann law as a thermal current between them\(^3\). So, parallel to Eq. (4), we have

\[
J = \sigma A \left(T^4 - T_{\text{CMB}}^4\right).
\]

As we have made in Eq. (5), we can relate \( J = -\frac{d(Mc^2)}{dt} \), finding

\[
\frac{dM}{dt} = -\frac{a}{M^2} + bM^2 s,
\]

where \( a \equiv \frac{\hbar c^3}{15360\pi G^2} \simeq 4.0 \cdot 10^{15} \text{kg}^3 \text{s}^{-3} \) and \( b \equiv \frac{4\pi \hbar G}{15^2 G^2 T_{\text{CMB}}^4} \simeq 9.6 \cdot 10^{-76} \text{kg}^4 \text{s}^{-2} \). Here, we also have a complete analytical solution for the differential Eq. 8:

\[
t = \frac{1}{4bR^3 a^2} \left[ 2 \arctan \left( \left( \frac{a}{b} \right)^{1/4} M \right) + \log \left( a^{1/4} - b^{1/4} M \right) - \log \left( a^{1/4} + b^{1/4} M \right) \right].
\]

This equation is not so simple as Eq. 6 but it expresses different behaviors for different ranges of mass \( M \), including unphysical ones for negative masses when \( M > -\left( \frac{a}{b} \right)^{1/4} \) (this will be discussed later). This complete analytical solution was used to calibrate the complete numerical one and to check the code used, since in the next section, only the complete numerical solutions will be used (together with the asymptotic analytical solutions, because the complete differential equations will be more complicated). Moreover, according to the initial temperature of the black hole, we have three different situations: (i) the black hole will initially emit energy to CMB if \( T > T_{\text{CMB}} \), (ii) the black hole will initially absorb energy from CMB if \( T < T_{\text{CMB}} \), and (iii) there will be no net flux if \( T = T_{\text{CMB}} \).

\(^3\)Notice that, in the present paper, we are just considering that the Schwarzschild black hole has no other kind of interaction beyond that one provided by Hawking radiation, like gas accretion, for example. For a good reference considering gas accretion into a Schwarzschild black hole (see [49]).

Therefore, the right side (RHS) of Eq. 8 can be negative, positive, or zero, respectively.

Since the temperature of the reservoir does not change with time, we know that (accordingly with the discussion in Section 3) if \( T > T_{\text{CMB}} \), then the black hole temperature will always be greater than \( T_{\text{CMB}} \); if \( T < T_{\text{CMB}} \) the temperature will always be smaller than \( T_{\text{CMB}} \); and if \( T = T_{\text{CMB}} \) the temperature will be constant and equal to \( T_{\text{CMB}} \). Hence, the radiation temperature \( T_{\text{CMB}} \) separates the three different behaviors, and we can compute the mass of the black hole with temperature equivalent to \( T_{\text{CMB}} \) as

\[
M_{\text{CMB}} = \frac{\hbar c^3}{8\pi G k_B T_{\text{CMB}}} \simeq 4.5 \cdot 10^{22} \text{kg}.
\]

In terms of the mass, for initial mass of black hole \( M = M_{\text{CMB}} \), the RHS of Eq. 8 will be zero, and the mass (and also the temperature) will be constant. For \( M < M_{\text{CMB}} \), the RHS of Eq. 8 will be negative, and the mass will decrease with time, eventually evaporating (with the temperature diverging). Finally, for \( M > M_{\text{CMB}} \), the RHS of Eq. 8 will be positive, and the mass will increase with time (as the temperature approaches zero).

Analyzing differential Eq. 8 using stream plot, we obtained Fig. 2, in arbitrary units. The goal of this figure is to show the general behavior of the solutions of Eq. 8. We can see clearly in this figure the three different behaviors for the mass of the black hole discussed above: if \( M < M_{\text{CMB}} \), the mass will decrease to zero (until complete evaporation), if \( M > M_{\text{CMB}} \), the mass will diverge, and if \( M = M_{\text{CMB}} \),

\[\text{Fig. 2}
\](mass versus time) for a Schwarzschild black hole immersed in cosmic microwave background radiation with constant temperature. Note that we used arbitrary units (AU).
the mass will remain constant. The $M = 1$ in the figure plays the role of $M_{\text{CMB}}$ and separates the other two kinds of solutions. The total time that takes for the mass to diverge or reach zero depends of the initial mass $M = M_0$.

To obtain particular solutions to differential Eq. (8), we will analyze its asymptotic behavior, because now it is not so simple as Eq. (5). The energy emission is associated to the first term $-\frac{a}{M^2}$, while the absorption is related to the second term $bM^2$. So, we have the following asymptotic solutions:

1st term: \[ \frac{dM}{dt} = -\frac{a}{M^2} \]

\[ M(t) = M_0 \left(1 - \frac{t}{t_V}\right)^{1/3}, \quad t \in [0, t_V) \] (11)

2nd term: \[ \frac{dM}{dt} = bM^2 \]

\[ M(t) = \frac{M_0}{(1 - bM_0 t)}, \quad t \in [0, \frac{1}{bM_0}) \] (12)

Notice that Eq. (11) is identical to solution (6) for the empty space. Taking the numerical solution of differential (8) and comparing with the asymptotic solutions (11) and (12), we can analyze how the masses of black holes change with time.

In Fig. 3, we used as initial condition Hyperion mass $M = M_0^{\text{hip}} \simeq 5.6 \cdot 10^{18}$ kg, smaller than the critical mass $M_{\text{CMB}}$ (10). Thus, comparing the complete numerical solution (of (8)) with 1st term analytical solution (11), we verify the superposition of these two solutions and the decreasing of mass of this black hole with $M_0^{\text{hip}}$. Also, notice that the time required for this black hole to evaporate is exactly the lifetime for Hyperion, calculated in Table 2, as expected by the 1st term analytical solution (11).

In Fig. 4, we considered the initial condition a Schwarzschild black hole with Sun mass $M = M_0^{\text{⊙}} \simeq 2.0 \cdot 10^{30}$ kg, greater than the critical mass $M_{\text{CMB}}$ (10). Again, comparing the complete numerical solution, of (8), with the 2nd term analytical solution (12), we can see the perfect accordance between this asymptotic solutions and the numerical solution. Moreover, the maximum time achieved by both curves in this figure (i.e., the time required for
the mass to diverge) is exactly the upper limit for the time interval in Eq. (12), that for the Sun is $t_C^\odot = \frac{1}{bM_0^\odot} \simeq 5.21 \cdot 10^{44}$ s.

Therefore, according to the initial mass, these asymptotic solutions (11) and (12) model very well the behavior of the masses of Schwarzschild black holes. Equation (11) can be used when $M < M_{\text{CMB}}$, and (12) when $M > M_{\text{CMB}}$, because in both cases, one of the two terms in the RHS of Eq. (8) will be dominant over the other one, and this approach will be better the further away the initial mass $M = M_0$ is from $M_{\text{CMB}}$ (the convergence will be faster).

Finally, notice that the intervals considered for the time in Eqs. (11) and (12) are open in $\ln t$ and $\frac{1}{bM_0}$, and the reason is that the mass goes to zero or infinity, respectively. Actually, there is no consensus about what happens in these two limits, but we can safely assume that the approximation used in this paper, i.e., perfect black body emitting radiation accordingly to Stefan-Boltzmann law, will no longer be valid, and other physics phenomena must be included in the discussion, but that is beyond the scope of this paper. More details about this discussion can be found in the Section 7.

6 Black Hole and the Thermal Evolution of the Universe

We know that the universe evolve in time and, thus, there is a thermal evolution of the matter and the energy content, mainly for long-time scales. In this way, this thermal evolution is an important point to be considered for the mass evolution of a Schwarzschild black hole. The details of the calculations and approximations used in the present section are found in Appendix.

In this analysis, we will consider that the black hole exchanges energy with the cosmic microwave background and that the CMB temperature is not only evolving with its inherent time evolution but also changing in time according to different universes eras: radiation, matter, and dark energy$^4$, i.e., the CMB temperature satisfies a temporal law inside each era, and this law changes from era to era.

Hence, the thermal evolution of the CMB, in accordance with (45), is

$$T_R(t) = \frac{T_{\text{CMB}}}{\left(\frac{3\pi G}{2}\rho_{\text{Ro}}\right)^{1/4}} \frac{1}{t^{1/2}}, t \in (0, t_{p_1})$$

$$T_M(t) = \frac{T_{\text{CMB}}}{\left(6\pi G \rho_{\text{Mo}}\right)^{1/3}} \frac{1}{t^{2/3}}, t \in (t_{p_1}, t_{p_2})$$

$$T_A(t) = T_{\text{CMB}}\exp\left[-\left(\frac{8\pi G}{3}\rho_{A_0}\right)^{1/2} t\right], t \in (t_{p_2}, \infty)$$

where $t_{p_1} = \left[\frac{3}{32\pi G \rho_{\text{Ro}}} \left(\frac{\rho_{\text{Ro}}}{\rho_{\text{Mo}}}\right)^{4/3}\right]^{1/2} \simeq 9.3 \cdot 10^{11}$ s is the limit time for the radiation era, $t_{p_2} = \left[\frac{1}{6\pi G \rho_{A_0}}\right]^{1/2} \simeq 3.7 \cdot 10^{17}$ s is the limit time for the matter era, $T_{\text{CMB}}$ represents temperature of cosmic microwave background today, and $\rho_{\text{Ro}}, \rho_{\text{Mo}},$ and $\rho_{A_0}$ represents radiation, matter, and dark energy densities today, respectively.

The procedure here will be the same previously used in Section 5, i.e., we are going to use the Stefan-Boltzmann law as a thermal current between the Schwarzschild black hole and the cosmic microwave background in the way that

$$J_i = \sigma A \left(T_i^4 - T_C^4\right),$$

with $i = R$ (radiation), $M$ (matter), or $A$ (dark energy). Again, the black hole will emit ($T > T_i$) or absorb ($T < T_i$) energy, and the CMB will behave as a thermal reservoir$^5$.

Thus, substituting the temperatures (13–15) and already transforming $J = -\frac{d(Mc^2)}{dt}$, we have

Rradiation: \frac{dM}{dt} = -\frac{\alpha}{M^7} + b\alpha \frac{M^2}{t^2}$

Mmatter: \frac{dM}{dt} = -\frac{\alpha}{M^7} + b\alpha \frac{M^2}{t^{7/5}}$

Dark energy: \frac{dM}{dt} = -\frac{\alpha}{M^7} + b\alpha \frac{M^2}{t^{2/3}}$

with $a = \frac{\hbar c^3}{15360 \pi G^2} \simeq 4.0 \cdot 10^{15} \text{kg}^3$, $b = \frac{4\pi^3}{\sqrt{2\pi}} \frac{G^2 T_{\text{CMB}}^4}{3}$ $\simeq 9.6 \cdot 10^{-76} \text{kg}^{-1}$, $a_R = \frac{1}{\left(\frac{3\pi G}{2}\rho\rho_{\text{Ro}}\right)^{4/3}} \simeq 9.6 \cdot 10^{18} \text{ s}^2$, $a_M = \frac{1}{\left(6\pi G \rho_{\text{Mo}}\right)^{1/3}} \simeq 2.0 \cdot 10^{47} \text{ s}^{8/3}$, and $a_A = \left(\frac{128\pi G}{3}\rho_{A_0}\right)^{1/2} \simeq 7.2 \cdot 10^{-18} \text{ s}^{-1}$. Equations (17–19) are more complicated than Eq. (8) because we have direct dependence in $t$. In this way, there is no analytical solution.

$^4$The time evolution of the universe is marked by three eras, i.e., three time intervals that are named according to the dominant content. Considering the “big bang” as the origin of time, the eras and their time intervals are radiation era $t \in (0, 9.3 \cdot 10^{11})$, matter era $t \in (9.3 \cdot 10^{11}, 3.7 \cdot 10^{17})$, and dark energy era $t \in (3.7 \cdot 10^{17}, \infty)$. For more information (see Appendix).

$^5$Even though the CMB temperature is changing with time, we are considering that this change does not interfere with the energetic exchange between the CMB and the black hole, i.e., the CMB is still a thermal reservoir.
In Fig. 5 below, we have constructed the stream plot for differential Eq. (17), in arbitrary units. The goal of this figure is to show the general behavior, not only for solution of Eq. (17) but for all solutions of Eqs. (17–19). The reason is that these three differential equations have the same general behavior for the mass evolution of Schwarzschild black holes. Note that now, these equations do not have a critical mass, as the $M_{\text{CMB}}$ (10) in Section 5. Hence, we can see two different behaviors: the mass will decrease to zero, until complete evaporation (even though it can increase in the beginning) or the mass will diverge, accordingly to its initial mass and universe era. Finally, there is no constant mass behavior because we have the time evolution in all the three cases.

As we have made in Section 5, we can analyze the asymptotic behaviors for Eqs. (17–19). These three equations have two terms that lead to two different asymptotic solutions. The 1st term of differential Eqs. (17–19) (equal for the three equations) can be written as

$$\frac{dM}{dt} = -\frac{a}{M^3},$$

$$\Rightarrow M(t) = M_0 \left[ 1 - \frac{(t - t_0)}{r_i} \right]^{1/3},$$

where the index $i = R, M, \text{or } \Lambda$ represents respectively radiation, matter, or dark energy eras, $M_i(t_0) = M_0$ is the initial mass for the black hole in each era, $t_0_i$ represents the initial time in each era ($t_0_R = 0^+$ for radiation era, $t_0_M = t_{p_1} \approx 9.3 \cdot 10^{11}$s for matter era and $t_0_\Lambda = t_{p_2} \approx 3.7 \cdot 10^{17}$s for dark energy era), and $t_{\nu_i} = 5120 \pi \frac{G^2}{hc^3} M_0^3 t_0^4$ represents the lifetime of black hole$^7$. Equation (20) has the intervals:

$$t \in (t_0, t_{\nu_i} + t_{\nu_i}^R), \text{ if } t_0 + t_{\nu_i} < t_{\nu_i}^R \text{ of era}$$

$$t \in (t_0, t_{\nu_i}^M), \text{ if } t_0 + t_{\nu_i} > t_{\nu_i}^M \text{ of era}$$

where $t_{\nu_i}^R$ is the final time of each era ($t_{\nu_i}^R = t_{p_1} \approx 9.3 \cdot 10^{11}$s for the radiation era, $t_{\nu_i}^M = t_{p_2} \approx 3.7 \cdot 10^{17}$s for the matter era and $t_{\nu_i}^\Lambda \sim \infty$ for the dark energy era). Moreover, Eq. (22) states that if the black hole lifetime $t_{\nu_i}$ exceeds the limit of the considered era, we need to consider the solution relative to the next era, using its final mass (relative to the previous era) as initial mass for the next era. Besides that, the 1st term is still related to energy emission for black holes.

The 2nd term of differential Eqs. (17–19) is different and they can be written as

$$\text{2nd radiation: } \frac{dM}{dt} = b_R \frac{M^2}{t^2},$$

$$\Rightarrow M(t) = \frac{M_0 b_R t_{0_R}^{0.5}}{t_{0_R}^{0.5} + b_R (t_0 - t_0 R)},$$

$$\text{2nd matter: } \frac{dM}{dt} = b_M \frac{M^2}{t^{0.5}},$$

$$\Rightarrow M(t) = \frac{5 M_0 b_M^{1/3} t_i^{1/3} a_{R}^{1/3}}{5 M_0 b_M^{1/3} t_i^{1/3} a_{R}^{1/3} + 5 M_0 b_M^{1/3} t_i^{1/3} a_{R}^{1/3} - 8 s},$$

$$\text{2nd dark energy: } \frac{dM}{dt} = b_M t^{0.5},$$

$$\Rightarrow M(t) = \frac{M_0 a_{M} b^{(e^{a_{M} t} - e^{a_{M} t_0 M})}}{a_{M} + b M_0}.$$
where $t_C^i$, for $i = R, M, \text{or} \Lambda$, represents the critical time (inside each era) for which the mass of the black hole diverges and $t_{\text{end of era}}^i$ has the same interpretation and the same values according to discussion after Eqs. (21) and (22). Thus, we have a second case for the time interval of those three eras given by

$$t \in \left( t_0, t_{\text{end of era}}^i \right), \text{if } t_C^i > t_{\text{end of era}}^i. \quad (29)$$

Equation (29) states that, if the critical time $t_C^i$ exceeds the limit of the considered era, we need to consider the solution relative to the next era, using its final mass (relative to the previous era) as initial mass for the next era. Moreover, the 2nd term is related to energy absorption for black holes. Again, we are going to compare the asymptotic solutions with the complete numerical solutions.

We can analyze the mass of the black hole using the analytical asymptotic solution and the numerical solution of Eqs. (17–19). First, we are going to study the behavior of black holes with initial masses of Hyperion and Sun. As we can see in Figs. 6 and 7, we verify that, for these two cases, the mass remains approximately constant within each universe era. The vertical dotted lines represent the intersection between two respective eras: $t_R \& M = t_{\text{p1}} \simeq 9.3 \cdot 10^{11}$s between radiation and matter eras and $t_{M\&\Lambda} = t_{\text{p2}} \simeq 3.7 \cdot 10^{17}$s between matter and dark energy eras. Note that we have used the logarithm scale in the horizontal axis to get a better representation for the three eras (the time scales of radiation and dark energy eras are small when compared with the matter era scale).

Thus, for those initial conditions, we expect the decrease of the black hole mass for very long periods of time. It is interesting to note that, comparing the two cases with initial Sun mass ($M_0 = M_{\odot}$), for constant CMB temperature and CMB temperature varying accordingly to the universe eras, in the former case, the mass will increase and, in the latter case, it will decrease.

Now, we can analyze the behavior of masses of Hyperion and Sun, still using analytical asymptotic solution and complete numerical solution, but extrapolating the time intervals of each era, i.e., taking them as $t \in (0, t_{v})$, where $t_v$ is the lifetime of each body ($t_{v,\text{Hyperion}} \simeq 1.5 \cdot 10^{40}$s and $t_{v,\odot} \simeq 6.6 \cdot 10^{74}$s). In this way, we can compare those solutions with asymptotic analytical solution of 1st term (20) of differential Eqs. (17–19), which is equivalent to the solution of empty space (6). This comparison is made in Fig. 8, where we see the superposition between the
Fig. 8 Mass evolution comparison of black hole immersed in cosmic microwave radiation with temperature evolving according to the universe eras with initial masses $M_{0}^{\text{HIP}} \simeq 5.6 \cdot 10^{18}$ kg of Hyperion and $M_{0}^{\text{☉}} = 2.0 \cdot 10^{30}$ kg of Sun. The plots show the analytical solution of the 1st term and complete numerical solution to each era.

The mass is decreasing and consequently the temperature increasing, leading to its evaporation due to radiation emission. Hence, the asymptotic solution of the 1st term provides perfectly the behavior of the mass of a black hole for these specific initial conditions.

Thus, for those initial conditions, we expect the decrease of the black hole mass for very long periods of time (in the dark energy era). It is interesting to note that, comparing the two cases with initial Sun mass ($M_0 = M_{\odot}$), for constant CMB temperature and CMB temperature varying accordingly to the universe eras, in the former case, the mass will increase until diverge and in the latter case, it will decrease until complete evaporation.

Following the idea of complete evaporation of a black hole, we computed in Table 3, the initial masses of black holes with lifetime equal to the duration of the universe eras, and we considered, for convenience, the end of dark energy era as the universe lifetime (today date). Then, we verified the expected behavior, of evaporation, for those masses using complete numerical solution of Eqs. (17–19) comparing with analytical solution of the 1st term (20). As we can see in Fig. 9, these plots always match, confirming the previous presented idea. Therefore, for black holes with masses smaller than the masses founded here, we always have the complete evaporation, in each era respectively.

| Table 3 Black hole masses with lifetime equal to the era duration: radiation, matter, and dark energy |
|---------------------------------------------------------------|
| $t_V$ (s) | $M$ (kg) |
| Radiation era | $9.3 \cdot 10^{11}$ | $2.2 \cdot 10^{9}$ |
| Matter era | $3.7 \cdot 10^{17}$ | $1.6 \cdot 10^{11}$ |
| Dark energy era | $6.6 \cdot 10^{16}$ | $9.2 \cdot 10^{10}$ |
Fig. 9 Mass evolution comparison of a black hole immersed in cosmic microwave background radiation with temperature evolving according to the universe eras with initial masses equal to the masses of black holes with lifetime according to the time interval of each universe era (given by Table 3). The plots show the analytical solution of the 1st term and the complete numerical solution to each era, respectively.

We also have found the initial mass, for each era, that makes the black hole to increase its mass initially, i.e., the mass of what we expect to follow the “unstable equilibrium” between the two situations: emission and absorption of...
radiation. We have estimated these values imposing that the second derivative in time of Eqs. (17–19) is null, and these values can be seen in Table 4. See that, one more time, we have used the initial time \( t_0 \) for the radiation era as explained in footnote 6.

This evaluation is an estimate, because this analysis provides the behavior inversion (increasing instead of decreasing mass) for the plots in the initial point of each era and not along all points in question. We can see, for the initial conditions of Table 4, the increasing behavior of the mass for each era in Fig. 10.

Finally, for initial masses greater than those previously found with the anterior estimate, we expect the indefinite increasing of the black hole masses, with radiation absorption and decreasing of temperature, that can be seen in Fig. 11.

In the plots of Fig. 11, we see the comparison of the complete numerical solution for each era with the asymptotic solution of the 2nd term of Eqs. (23–25). In all the cases, we can see the superposition of solutions, concluding that, one more time, the asymptotic solutions of the 2nd term describes with mastery the numerical solution for the appropriate initial conditions. Besides that, the maximum time achieved by both curves, in the three plots, asymptotes the ones expected by the maximum value of the valid interval (26–28) in the solutions of the 2nd term (23–25). They are, respectively, \( t_r^C \approx 0.00100216s \), \( t_m^C \simeq 1.06103 \cdot 10^{12}s \), and \( t_a^C \simeq 1.18256 \cdot 10^{18}s \).

### 7 Final Considerations

Until now, the possibility of a complete evaporation of a black hole is an open question. The point is that, when a black hole evaporates, it loses its event horizon, the entity that hides its inner states. Thus, all of the information inside the black hole disappears with its evaporation. This leads to what is called information loss. It occurs because, as suggested by semi-classical arguments, in the process of black hole formation and evaporation, a pure quantum state will evolve to a mixed state. Some possible alternatives to a non-existence of a complete evaporation of a black hole could be that the black hole evaporates until its length reaches the Planck scale, or the information comes out in a final burst, or that black holes’ quantum tunnels into a white hole, or even that no black hole ever forms. These ideas are discussed in references [40, 50] but they are not explored in the present paper, because we have considered complete evaporation of the black holes, in their appropriate regimes.

There is another feature that we have not considered in the present paper: the “origins” of the black holes that we are working with, i.e., we have considered just the intervals for the masses, and not the formation process for the black holes; in special, we have not discussed if they are primordial or not. Besides that, we have also only considered the semi-classical approach, the Hawking radiation to the energy exchanges between the black hole and the environment. Reference [51] considers the black hole evolution in the cosmological context, reviewing results related to thermodynamics of semi-classical black holes with influx of ambient particles and flux from its evaporation. Reference [52] discuss about evaporation of mini black holes. And a more recent work [53] presents a picture of the quantum mechanics of a collapse-formed, evaporating black hole. This one describes the radiation from the black hole in terms of “hard modes” and “soft modes”, showing an entanglement structure between them and the Hawking radiation. Again, these considerations are out of the scope of this present work, but can be used to complement the analysis made by us.

It is interesting to make a discussion about the divergence of the masses of the black holes that is observed for some values of the parameters. Those situations happen in Sections 5 and 6. In Section 5, we have the black hole immersed in cosmic microwave background with constant temperature, so we expect \( M \to \infty \) for black holes with \( M_0 > M_{CMB} \) (10). In Section 6, we still have the black hole immersed in cosmic microwave background, but with CMB temperature varying in time according to the universe eras. In both cases, it is important to explain two main points. First, neither the complete numerical solutions nor the asymptotic analytical solutions are able to provide the behavior of masses greater than the last pair of points in the respective plots (see Fig. 4 in Section 5 and Fig. 11 in Section 6) - numerically, the software used (Mathematica) do not allow us to see bigger time intervals.

The second point is that the analytical asymptotic solutions ((12) for Section 5 and (23–25) for Section 6) have a finite time interval, i.e., there are critical times \( t_c = \frac{1}{k M_0} \) and \( t_i^C \) (for \( i = R, M, \) or \( \Lambda \)) where the black hole masses diverge in a finite time. For time values greater than the critical time, the masses become negative, and our equations are no longer valid (physically no-reasonable region). In this way, for \( t > t_c \), our model

### Table 4 Minimum black hole masses that present initially increasing behavior for each universe eras

| Era                  | \( t_0 \) (s) | \( M \) (kg) |
|----------------------|---------------|--------------|
| Radiation era        | \( 10^{-3} \) | \( 1.1 \cdot 10^{33} \) |
| Matter era           | \( 9.3 \cdot 10^{11} \) | \( 6.2 \cdot 10^{37} \) |
| Dark energy era      | \( 3.7 \cdot 10^{17} \) | \( 5.4 \cdot 10^{38} \) |

\(^8\text{See valid interval for (12).}\)

\(^9\text{See valid intervals for (26-29).}\)
Fig. 10 Complete numerical solution for the mass evolution of a black hole immersed in cosmic microwave background radiation with temperature evolving according to the universe eras with initial masses that gives initially increasing behavior for each universe era (given by Table 4)

loses its validity. Moreover, physically, we can think that, while we increase the black hole mass, using a finite number of physical processes, we decrease its temperature and, consequently, its surface gravity. Therefore, the 3rd law of black hole thermodynamics arises in this limit and prevents the masses to diverge in a finite time, implying that some other physical effects must take place to prevent the divergence of the mass. Besides all that, when the mass...
Fig. 11  Solutions for the mass evolution of a black hole immersed in cosmic microwave background radiation with temperature evolving according to universe eras with initial masses greater than the ones given by Table 4. The plots show the analytical solution of the 2nd term and complete numerical solutions to each era, respectively. Notice that the curves do not reach the end of the eras, because the masses diverge before
goes to infinity, so does the Schwarzschild radius (the event horizon radius of the black hole), meaning that the black hole is getting arbitrarily big, which is clearly not realistic, since it would interact with other fields and objects, and our assumptions are no longer valid (neither our equations).

In the case of a black hole immersed in cosmic microwave background with temperature evolving in time, we have the division of time intervals in eras. So, we have the same black hole with different initial masses at the beginning of each era, where its final mass in one era is equal to its initial mass in the subsequent era. Therefore, the behavior of the black hole (increase or decrease of its mass) can change between the eras. If we observe Table 4, we can say that, for the black hole to increase its mass in the radiation era, the same has to have an initial mass of $M_{0g} \gtrsim 1.1 \times 10^{33} \text{kg}$, but the mass cannot be much greater than this value, because, if we observe Fig. 11, it will diverge for $M_{0g} \sim 5 \times 10^{35} \text{kg}$. Hence, the allowed interval for the initial mass which causes the mass to increase throughout the radiation era is $1.1 \times 10^{33} \lesssim M_{0g} \lesssim 5 \times 10^{35} \text{kg}$. The same intervals for the matter and dark energy eras are $6.2 \times 10^{47} \lesssim M_{0M} \lesssim 4 \times 10^{48} \text{kg}$ and $5.4 \times 10^{58} \lesssim M_{01} \lesssim 1.1 \times 10^{59} \text{kg}$ respectively.

Notice that these intervals are very narrow, which means that our equations are very sensitive about great values of masses. In order for the masses to grow, but not diverge, throughout each era, the initial masses must satisfy these intervals. In special, if a black hole possess initially $1.1 \times 10^{33} \lesssim M_{0g} \lesssim 5 \times 10^{35} \text{kg}$ in the radiation era, it is possible to reach the matter era with $6.2 \times 10^{47} \lesssim M_{0M} \lesssim 4 \times 10^{48} \text{kg}$, and then reach the dark energy era with $5.4 \times 10^{58} \lesssim M_{01} \lesssim 1.1 \times 10^{59} \text{kg}$. In this case, the mass will always increase, from the beginning of the universe until today.

This paper contains a broad discussion about the mass evolution of Schwarzschild black holes. We have considered the black holes immersed in different environments like empty space and the cosmic microwave background (with constant and evolving in time temperature), and in all cases, we considered the Stefan-Boltzmann law to model the evolution of the masses in time. The result for the empty space was previously known [28, 39], but was included for internal consistency and comparison with the new results for the cosmic microwave background cases. We have analyzed all the equations according to asymptotic solutions and compared the numerical results to them, achieving the same results in all solutions and all cases. Besides that, we have considered several initial mass conditions to illustrate the different behaviors of black holes, with masses increasing or decreasing.

Finally, it is worthy to say that this paper summarizes the Master Degree dissertation of Natali Soler Matubaro de Santi, that can be viewed, in Portuguese, for more details, in [54].

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### Appendix: Thermal Evolution of the Universe

The thermal evolution of the universe can be viewed using the flat universe model ($\kappa = 0$ in the Friedmann’s Eqs. (31) and (32)) [2, 4, 41]. In the present appendix, we are going to introduce the Friedmann-Lemaître-Robertson-Walker (FLRW) metric, Friedmann’s equations, energy momentum tensor for a perfect fluid, and the necessary approximations to this model.

We know that the universe is homogeneous and isotropic and evolves in time. To describe the universe, we use the FLRW metric

$$ds^2 = -dt^2 + a(t)^2 \left[ \frac{dr^2}{1 - \kappa r^2} + r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) \right].$$

where $a(t)$ is the expansion factor of the universe, $r$ is the radial coordinate, $\theta$ and $\phi$ are the angular spherical coordinates, and $\kappa$ is a parameter related with the curvature and with dimension $L^{-2}$. The perfect fluid consideration\(^{11}\) as curvature generator and the Einstein’s equations lead us to the so-called Friedmann’s equations, and these equations define the time evolution of matter and energy. They can be written as

\[
\left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi}{3} \rho - \frac{\kappa}{a^2},
\]

\[
\frac{\ddot{a}}{a} = -\frac{4\pi}{3} (\rho + 3p),
\]

with $\dot{a} = \frac{da}{dt}$, $\rho$ the matter density and $p$ its pressure. Besides Friedmann’s equations, we can use some properties of the energy-momentum tensor and compute

$$\nabla_\mu T^\mu_\nu = \partial_\mu (\rho + p) + \left( \Gamma^1_01 + \Gamma^2_02 + \Gamma^3_03 \right) (-\rho) - \left( \Gamma^1_1 + \Gamma^2_2 + \Gamma^3_3 \right) p. \quad (33)$$

Since $\nabla_\mu T^\mu_\nu = 0$, we have

$$-\partial_\mu \rho - \frac{3}{2} \frac{\partial_\mu a^2}{a^2} \rho = \frac{3}{2} \frac{\partial_\mu a^2}{a^2} p. \quad (34)$$

Note that, writing $\partial_\mu a^3 = 3a^2 \partial_\mu a$ and $\partial_\mu a^2 = 2a \partial_\mu a$, we get the following state equation

$$\frac{da^3}{dt} = -p \frac{da^3}{dt}. \quad (35)$$

\(^{10}\)The curvature parameter $\kappa$ classifies the universe as: closed with positive curvature ($\kappa = 1$), plane and without curvature ($\kappa = 0$) and opened with negative curvature ($\kappa = -1$).

\(^{11}\)We choose the 4 velocities being $u^\mu = (1, 0, 0, 0)$.
It is worthy to say that Eq. (35) does not replace Friedmann’s equations because, although it provides a temporal evolution of the matter and energy content of the universe, it is not straighly related to the universe curvature.

The division of the matter and energy content of the universe is radiation, matter, and dark energy. The radiation includes the photons, the cosmic microwave background, the gravitons, and the neutrinos, i.e., particles that satisfies the state equation $\rho_R = \rho_R(\frac{a_0}{a})^4$. By matter, we refer to hadronic and dark matter\(^\text{12}\), entities whose pressure will be negligible when compared to the density: $\rho_M = 0$. And finally, the dark energy\(^\text{13}\) satisfies $\rho_A = -\rho_A$. Given those state equations, we can integrate Eq. (35) and obtain

\[
\begin{align*}
\rho_R &= \rho_R(\frac{a_0}{a})^4, \quad (36) \\
\rho_M &= \rho_M(\frac{a_0}{a})^3, \quad (37) \\
\rho_A &= \rho_A(\frac{a_0}{a}), \quad (38)
\end{align*}
\]

where $\rho_0$ is the radiation density ($i = R$), matter density ($i = M$) and dark energy density ($i = A$) today and $a_0$ is the expansion factor of the universe today.

Using the experimental values, found in [45], to the radiation, matter, and dark energy, we compute their respective present densities $\rho_R \simeq 4.6 \cdot 10^{-31} \text{kg/m}^3$, $\rho_M \simeq 2.7 \cdot 10^{-27} \text{kg/m}^3$ and $\rho_A \simeq 5.8 \cdot 10^{-27} \text{kg/m}^3$ and we plot the evolution of the densities $\rho_R, \rho_M$, and $\rho_A$ versus the expansion factor of the universe normalized $\frac{a}{a_0}$, viewed in Fig. 12. Observe that we have highlighted three different phases, that we called of radiation era, matter era, and dark energy era. All phases are dominated by matter or energy content of the universe. They are the so-called universe eras. The divisions are represented by the vertical dashed lines, where the densities intersect each other: $p_1 = \frac{a}{a_0} = \frac{\rho_R}{\rho_M} \simeq 1.7 \cdot 10^{-4}$ and $p_2 = \frac{a}{a_0} = \left(\frac{\rho_M}{\rho_A}\right)^{1/3} \simeq 7.7 \cdot 10^{-1}$.

Still using the experimental values of Reference [45], we can say that nowadays the universe is flat, in other words, the curvature parameter $\kappa = 0$. This statement can be viewed rewriting Eq. (31) as

\[
\Omega - 1 = \frac{\kappa}{H^2 a^2}, \quad (39)
\]

where $\Omega = \frac{8\pi G\rho}{3H^2}$, is the density parameter, $\rho_{\text{crit}} = \frac{3H^2}{8\pi G}$ is the critical density, and $H = \frac{\dot{a}}{a}$ the Hubble’s parameter. As $\Omega$ is the sum of the parameters of radiation, matter, and dark energy densities, using their experimental values we can see that

\[
\Omega = \Omega_R + \Omega_M + \Omega_A \simeq 5.46 \cdot 10^{-5} + 0.315 + 0.685 \sim 1, \quad (40)
\]

which implies, by Eq. (39), that $\kappa = 0$.

Now we are going to connect the time evolution of the matter and energy content of the universe with Einstein’s equations. To do this, we just need to replace the results (36-38) in Friedmann’s Eqs. (31) and (32). In order to do this, let’s then consider a flat universe ($\kappa = 0$) not just today but throughout the evolution of the universe (as commented above)\(^\text{14}\). Thus, replacing the densities in the 1st Friedmann’s Eqs. (31), we find

\[
\begin{align*}
\frac{a}{a_0} &= \left(\frac{32\pi}{3}\rho_R\right)^{1/4} t^{1/2}, t \in (0, t_{p_1}) \quad (41) \\
\frac{a}{a_0} &= \left(6\pi\rho_M\right)^{1/3} t^{2/3}, t \in (t_{p_1}, t_{p_2}) \quad (42) \\
\frac{a}{a_0} &= \text{Exp} \left[\left(\frac{8\pi}{3}\rho_A\right)^{1/2} t\right], t \in (t_{p_2}, \infty), \quad (43)
\end{align*}
\]

\(^{12}\)The dark matter means some kind of unhadronic matter that does not interact with electromagnetic radiation.

\(^{13}\)By dark energy, we mean some kind of energy that is still unknown and extends uniformly throughout all the universe and seems like “vacuum energy”. Because of it, one of its proposals is related to non-null cosmological constant $\Lambda$.

\(^{14}\)This choice is our main approximation to this model.
where, recovering the units using SI, $t_{p1} = 9.3 \times 10^{11}$ s and $t_{p2} = 3.7 \times 10^{17}$ s, and these values were obtained replacing $p_1$ and $p_2$ respectively in (41) and (42). Note that we integrate indefinitely, in $a$ and in $t$, the expressions (41) and (42), disregarding any integration constant. We have also integrated indefinitely in (43), defining $\alpha_0$ as integration constant. This is an approximation to the solution of (31) in terms of dominance intervals of radiation, matter, and dark energy, and the result stays undefined due to a lack of the integration constants. However, since we are not interested in the continuity of the solution to Eq. (31), but in its form, our result is convenient to our analysis.

Finally, we need to relate the temperature with the universe densities. We know that a black body emits radiation with energy density according to Stefan-Boltzmann’s law, i.e., $L \propto T^4$. We saw that the density radiation of the universe can be written as $\rho_R \propto \left(\frac{\alpha_0}{a^4}\right)^4$. Thus, doing a dimensional analogy, we can say that the radiation temperature evolves as

$$T_R = T_{CMB} \left(\frac{\alpha_0}{a}\right),$$

where $T_{CMB}$ is the temperature of the cosmic microwave background today. Therefore, considering that the radiation has temperature $T_i$, with $i = R, M, A$, related to the fraction $\frac{\rho_i}{\rho}$, and that this fraction assumes different expressions according to the universe eras: radiation ($R$), matter ($M$), and dark energy ($A$), we have

$$T_R = \frac{T_{CMB}}{\left(\frac{3\pi^2 \rho R_0}{6\pi G \rho M_0}\right)^{1/4}} t^{1/2}, t \in (0, t_{p1}),$$

$$T_M = \frac{T_{CMB}}{6\pi G \rho M_0} \left(\frac{1}{t^{1/3}}, t \in (t_{p1}, t_{p2})\right),$$

$$T_A = T_{CMB} \exp \left[\left(\frac{8\pi}{3} \rho A_0\right)^{1/2}\right], t \in (t_{p2}, \infty).$$

Hence, we can use these results in Section 6 assuming that the black hole is exchanging energy with the universe, considering the three different ways of thermal evolution.

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**Notes:**

- **References** contain titles of books and papers relevant to the content of the text.
- **Equations** are correctly formatted and numbered as per the guidelines.
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