A convex form that is not a sum of squares

James Saunderson

March 7, 2022

Abstract

Every convex homogeneous polynomial (or form) is nonnegative. Blekherman has shown that there exist convex forms that are not sums of squares via a nonconstructive argument. We provide an explicit example of a convex form of degree four in 272 variables that is not a sum of squares. The form is related to the Cauchy-Schwarz inequality over the octonions. The proof uses symmetry reduction together with the fact (due to Blekherman) that forms of even degree, that are near-constant on the unit sphere, are convex. Using this same connection, we obtain improved bounds on the approximation quality achieved by the basic sum-of-squares relaxation for optimizing quaternary quartic forms on the sphere.

1 Introduction

A multivariate polynomial $p$ of degree $2d$ in $n$ variables is homogeneous if it satisfies $p(\lambda x) = \lambda^{2d} p(x)$ for all $x \in \mathbb{R}^n$. Throughout, we use the term form to refer to a homogeneous polynomial. A form is convex if and only if it satisfies the midpoint convexity condition $\frac{1}{2} (p(x) + p(y)) \geq p\left( \frac{1}{2}(x+y) \right)$ for all $x, y \in \mathbb{R}^n$ [AP13]. Convex forms arise naturally, for instance, in the study of polynomial norms [ADKH19]. Every convex form is nonnegative since

$$p(x) = \frac{1}{2} (p(x) + p(-x)) \geq p\left( \frac{1}{2}(x-x) \right) = p(0) = 0 \quad \text{for all } x \in \mathbb{R}^n.$$ 

A natural question, posed by Parrilo in 2007, is whether every convex form is a sum of squares.

Hilbert [Hil88] showed that every nonnegative form in $n$ variables of degree $2d$ is a sum of squares if and only if either $n = 2$ or $2d = 2$ or $(n, 2d) = (3, 4)$. Since every convex form is nonnegative, in these cases convex forms must also be sum of squares. Going beyond this, recently El Khadir [EK20] has established that convex quaternary quartic forms are always sums of squares, despite there being nonnegative quaternary quartic forms that are not sums of squares.

Parrilo’s question was, however, resolved in the negative when Blekherman [Ble12] showed that for any fixed $2d \geq 4$ there is a sufficiently large $n$ such that there exist $n$-variate convex forms of degree $2d$ that are not sums of squares. In particular, he showed that the volume of a particular compact section of the cone of $n$-variate convex forms is strictly larger than the volume of the corresponding compact section of the cone of $n$-variate sums of squares. Blekherman’s argument can be made quantitative, but it only establishes the existence of convex quartic forms (for instance) that are not sums of squares for $n \geq 27179089915$.

A stronger notion of convexity for a form is sum-of-squares (sos) convexity [HN10]. A form $p$ is sos-convex if and only if $\frac{1}{2} (p(x) + p(y)) - p\left( \frac{1}{2}(x+y) \right)$ is a sum of squares as a polynomial in $x$.

*Department of Electrical and Computer Systems Engineering, Monash University, Melbourne VIC 3800, Australia. Email: james.saunderson@monash.edu
and $y$ [AP13]. By restricting to $y = -x$, it follows that every sos-convex form is a sum of squares. Consequently, any example of a convex form that is not a sum of squares must also be an example of a convex form that is not sos-convex. However, all of the currently known examples of forms that are convex but not sos-convex are sums of squares [AP13, Section 6].

The aim of this paper is to present what appears to be the first explicit example of a convex form (of degree 4 in $n = 272 = 16 \times 17$ variables) that is not a sum of squares.

**Optimizing forms on the unit sphere** Our example is also of interest in relation to the fundamental problem of minimizing a form of degree $2d$ over the unit sphere, i.e., computing

$$p_{\min} := \min_{\|x\|^2 = 1} p(x).$$

When $2d \geq 4$, this is a very rich class of optimization problems. It includes NP-hard problems such as finding the size, $\alpha(G)$, of the largest stable set of a graph $G = (V, E)$, since the associated quartic forms $p_G(x) = \sum_{i \in V} x_i^4 + 2 \sum_{\{i, j\} \in E} x_i^2 x_j^2$ satisfy $|p_G|_{\min} = 1/\alpha(G)$ [DKP02, MS65].

In general, a lower bound on $p_{\min}$ is given by the quantity

$$p_{\min}^{\text{sos}} := \max \gamma \quad \text{subject to} \quad p(x) - \gamma \|x\|^{2d} \text{ is a sum of squares.} \quad (1)$$

This optimization problem can be reformulated as a semidefinite program (see, e.g., [Par12]). For forms with a modest number of variables and degree, or forms with additional structure, $p_{\min}^{\text{sos}}$ offers a computationally tractable global lower bound on $p_{\min}$.

Suppose that $p_{\max}$ is the maximum value of a form $p$ of degree $2d$ on the unit sphere. One way to measure the quality of $p_{\min}^{\text{sos}}$ as a lower bound on $p_{\min}$ is via the quantity

$$\text{GAP}(p) := \frac{p_{\max} - p_{\min}^{\text{sos}}}{p_{\max} - p_{\min}} \geq 1.$$ 

For example, if $p$ is the Motzkin form $p(x, y, z) = x^2 y^4 + y^2 x^4 - 3x^2 y^2 z^2 + z^6$ which is nonnegative but not a sum of squares, then a numerical computation reveals that $\text{GAP}(p) \approx 1.0046$, remarkably close to one. For the quartic forms $p_G$ associated with the stable set problem, it is straightforward to show that $\text{GAP}(p_G) < 2$ for all graphs. While general upper bounds are known on $\text{GAP}(p)$ [Nie12, FF20], little appears to be known about explicit forms for which this quantity is large. In this paper we will give an explicit example of a quartic form $p$ for which $\text{GAP}(p) > 2$.

We now describe the connection between forms for which $\text{GAP}(p)$ is large, and forms that are convex but not a sum of squares. In the course of establishing that there are convex forms that are not sums of squares, Blekherman showed that if a form of degree $2d$ is bounded between $1 - \frac{1}{2d-1}$ and $1 + \frac{1}{2d-1}$ on the unit sphere, then it is convex [Ble12, Theorem 4.75]. In Section 2 we derive the following simple consequence of Blekherman’s theorem.

**Proposition 1.1.** Let $p$ be a form of degree $2d$ such that $p_{\min} \|x\|^{2d} \leq p(x) \leq p_{\max} \|x\|^{2d}$ for all $x \in \mathbb{R}^n$. If $\text{GAP}(p) > d$ then $p(x) - [d p_{\min} - (d - 1) p_{\max}] \|x\|^{2d}$ is convex but not a sum of squares.

**The example** The example of a quartic form with $\text{GAP}(\cdot) > 2$ we focus on in this paper is one of a family of quartic forms $c_k^\text{O}$ in $n = 16k$ variables, associated with the Cauchy-Schwarz inequality over the octonions.

Let $\mathbb{D}$ denote a finite dimensional real normed division algebra. By Hurwitz’s theorem [Hur98], $\mathbb{D}$ is either $\mathbb{R}$, $\mathbb{C}$, the quaternions $\mathbb{H}$, or the octonions $\mathbb{O}$. (The necessary background on the octonions is summarized in Section 4.1.) If $x \in \mathbb{D}$, let $\overline{x}$ denote its conjugate and $\|x\| = (\overline{x}x)^{1/2}$.
denote its norm. If \( x, y \in \mathbb{D}^k \) then define \( \|x\|^2 = \sum_{i=1}^k |x_i|^2 \) and \((x, y) = \sum_{i=1}^k x_i y_i \in \mathbb{D}^k\). For all \( k \geq 1 \) we have that

\[
\text{cs}_k^\mathbb{D}(x, y) := \|x\|^2\|y\|^2 - |(x, y)|^2 \geq 0 \quad \text{for all } x, y \in \mathbb{D}^k,
\]
giving a generalization of the Cauchy-Schwarz inequality. For a proof in the octonionic case, see, e.g., \[Kra98\]. All the other cases follow by restricting to a suitable subspace.

The Cauchy-Schwarz form over the reals or the complex numbers is well-known to be a sum of squares. One way to see this is to express the Cauchy-Schwarz form as a determinant and use the Cauchy-Binet formula, giving

\[
\|x\|^2\|y\|^2 - |(x, y)|^2 = \det \begin{bmatrix} \|x\|^2 & (x, y) \\ (x, y) & \|y\|^2 \end{bmatrix} = \sum_{1 \leq i < j \leq k} \left| \det \begin{bmatrix} x_i & x_j \\ y_i & y_j \end{bmatrix} \right|^2.
\]

While the \( 2 \times 2 \) Hermitian determinantal representation of the Cauchy-Schwarz form remains valid over \( \mathbb{H} \) and \( \mathbb{O} \) (for a suitable notion of determinant), the Cauchy-Binet formula does not hold in these cases, and the Cauchy-Schwarz forms over \( \mathbb{H} \) and \( \mathbb{O} \) are, in fact, not sums of squares \[GT18\].

In the quaternionic case, it turns out that \( \text{GAP}(\text{cs}_k^\mathbb{H}) \leq 2 \) for all \( k \), and so that quaternionic Cauchy-Schwarz forms cannot provide the example we are looking for. On the other hand, for sufficiently large \( k \) the octonionic Cauchy-Schwarz forms give rise to convex forms that are not sums of squares.

**Theorem 1.2.** If \( k \geq 17 \) then \( \text{GAP}(\text{cs}_k^\mathbb{O}) > 2 \) and \( \text{cs}_k^\mathbb{O}(x, y) + (1/4)(\|x\|^2 + \|y\|^2)^2 \) is a convex quartic form in \( 16k \) variables that is not a sum of squares.

The forms

\[
q_k^\mathbb{O}(x, y) := \text{cs}_k^\mathbb{O}(x, y) + (1/4)(\|x\|^2 + \|y\|^2)^2
\]

are convex for all \( k \geq 1 \), and are sums of squares exactly for \( k \leq 16 \) (see Theorem 4.11). As such, the smallest example, among this family, of a convex form that is not a sum of squares occurs when \( k = 17 \). Overall, though, there is no reason at all to expect the example is minimal in terms of the number of variables, among all convex forms that are not sums of squares.

The main difficulty in establishing Theorem 1.2 is establishing that \( q_k^\mathbb{O} \) is not a sum of squares for \( k \geq 17 \). To do so, we exploit the symmetry properties of \( q_k^\mathbb{O} \). It turns out that \( q_k^\mathbb{O} \) is invariant under a certain action of the compact Lie group \( \text{Spin}(9) \times O(k) \). Explicitly decomposing the space of quadratic forms in \( 16k \) variables into irreducible representations under this action reveals that the decomposition is multiplicity-free. This means that the problem of deciding whether \( q_k^\mathbb{O} \) is a sum of squares can be reduced to a small linear program by using the symmetry reduction framework of Gatermann and Parrilo \[GP04\].

The paper is organized as follows. In Section 2 we establish the connection between convex forms that are not sums of squares, and sum-of-squares bounds on the minimum of a form on the unit sphere. Using this simple connection we establish that the forms \( q_k^\mathbb{O} \) are convex for all \( k \). In Section 3 we recall basic facts about representation theory and symmetry reduction of semidefinite programs and sum of squares feasibility problems. In Section 4 we apply the general symmetry reduction framework to the specific semidefinite feasibility problem of deciding whether \( q_k^\mathbb{O} \) is a sum of squares. We give an explicit sum-of-squares decomposition of \( q_{16}^\mathbb{O} \) and give an explicit certificate that \( q_{17}^\mathbb{O} \) is not a sum of squares. In Section 5, we discuss some open questions related to the results presented in the paper.
2 Convexity and optimization on the sphere

As part of the proof that there are convex forms that are not sums of squares, Blekherman established the following sufficient condition for a form of even degree to be convex.

**Theorem 2.1** (Blekherman [Ble12]). *If p is a form of degree 2d in n variables and

\[ \|x\|^{2d} \left( 1 - \frac{1}{2d - 1} \right) \leq p(x) \leq \|x\|^{2d} \left( 1 + \frac{1}{2d - 1} \right) \]

for all \( x \in \mathbb{R}^n \), then \( p \) is convex.*

In order to find a convex form that is not a sum of squares, one approach is to find a form \( p \) for which the bound \( \text{sos}^\text{min} \) on its minimum on the sphere, defined in (1), is far from its true minimum on the sphere. The following simple consequence of Theorem 2.1 makes this precise.

**Lemma 2.2.** *Let \( p \) be an form of degree 2d in n variables such that \( \text{min} \|x\|^{2d} \leq p(x) \leq \text{max} \|x\|^{2d} \)

for all \( x \in \mathbb{R}^n \). Then \( p(x) - [d \text{min} - (d - 1) \text{max}] \|x\|^{2d} \) is convex. Furthermore if \( \text{sos}^\text{min} < d \text{min} - (d - 1) \text{max} \) then \( p - [d \text{min} - (d - 1) \text{max}] \|x\|^{2d} \) is not a sum of squares.*

**Proof.** If \( \text{min} \|x\|^{2d} \leq p(x) \leq \text{max} \|x\|^{2d} \) for all \( x \in \mathbb{R}^n \), then a simple computation shows that

\[ \|x\|^{2d} \left( 1 - \frac{1}{2d - 1} \right) \leq \|x\|^{2d} \left( 1 - \frac{1}{2d - 1} \frac{\text{max} + \text{min}}{\text{max} - \text{min}} \right) + \frac{2}{2d - 1} \frac{1}{\text{max} - \text{min}} p(x) \]

\[ \leq \|x\|^{2d} \left( 1 + \frac{1}{2d - 1} \right) \]

for all \( x \in \mathbb{R}^n \).

Applying Theorem 2.1, and multiplying through by the positive quantity \( \frac{1}{2}(2d - 1)(\text{max} - \text{min}) \), establishes that \( p - [d \text{min} - (d - 1) \text{max}] \|x\|^{2d} \) is convex. The final part of the statement follows directly from the definition of \( \text{sos}^\text{min} \).

Proposition 1.1, connecting forms with large \( \text{Gap}() \) values and forms that are convex but not sums of squares, can be readily deduced from Lemma 2.2.

**Proof of Proposition 1.1.** If \( p \) is a form of degree 2d and \( \text{Gap}(p) > d \) then \( \text{max} - \text{sos}^\text{min} > d(\text{max} - \text{min}) \). Therefore, \( \text{sos}^\text{min} < d \text{min} - (d - 1) \text{max} \). Lemma 2.2 then tells us that \( p - [d \text{min} - (d - 1) \text{max}] \|x\|^{2d} \) is not a sum of squares.

Recall the family of nonnegative quartic forms \( c_k^O(x, y) = \|x\|^2\|y\|^2 - |\langle x, y \rangle|^2 \), in \( 16k \) variables, associated with the Cauchy-Schwarz inequality over the octonions. These satisfy

\[ (\|x\|^2 + \|y\|^2)^2(1/4) \geq \|x\|^2\|y\|^2 \geq c_k^O(x, y) \geq 0 \]

for all \( (x, y) \in \mathbb{O}^{2k} \),

where the first inequality follows from the AM-GM inequality. Both inequalities are tight. Applying Lemma 2.2 with \( \text{min} = 0 \) and \( \text{max} = 1/4 \) and \( 2d = 4 \) we can conclude that the forms

\[ q_k^O(x, y) = c_k^O(x, y) + (1/4)(\|x\|^2 + \|y\|^2)^2 \]

are convex, for all \( k \geq 1 \).
2.1 Worst-case bounds for optimization of quartic forms on the unit sphere

In cases where convex forms are sums of squares (but nonnegative forms are not necessarily sums of squares), we can use Proposition 1.1 to give new upper bounds on $\text{GAP}(p)$. In particular, in [EK20], El Khadir showed that every convex quaternary quartic form is a sum of squares. Combining this with the contrapositive of Proposition 1.1 gives the following result.

**Corollary 2.3.** If $p$ is a quaternary quartic form then $\text{GAP}(p) \leq 2$.

We now compare this result with other upper bounds on $\text{GAP}(p)$ from [Nie12, FF20]. We note that the comparison bounds hold in much greater generality, so it is unsurprising that improvements are possible in the very special case considered here. From Nie [Nie12] we have that for any that the comparison bounds hold in much greater generality, so it is unsurprising that improvements are possible in the very special case considered here. From Nie [Nie12] we have that for any quaternary quartic form,

$$\text{GAP}(p) = \frac{p_{\text{max}} - p_{\text{sos}}^{\text{min}}}{p_{\text{max}} - p_{\text{min}}} \leq \frac{1}{\delta_4} \approx \frac{1}{0.0559} \approx 17.88$$

where $\delta_4$ is defined in [Nie12, Equation 2.10] and its numerical value is given in [Nie12, Table 1]. A special case of [FF20, Theorem 6] shows that for any polynomial of degree $2d$ in $n$ variables we have

$$\text{GAP}(p) = \frac{p_{\text{max}} - p_{\text{sos}}^{\text{min}}}{p_{\text{max}} - p_{\text{min}}} \leq 1 + (B_{2d}/2)\rho_{2d}(n,d). \quad (3)$$

In the case $n = 4$ and $2d = 4$ one can compute numerically that $\rho_4(4,2) \approx 5.426$. The quantity $B_4$ is the supremum of $\|f_4\|_\infty/\|f\|_\infty$ where $f$ is a polynomial of degree 4 on the sphere, $f_4$ is the fourth harmonic component of $f$, and $\|f\|_\infty$ is the maximum absolute value of $f$ on the sphere. Restricting to the case of quartic forms in four variables, the argument in [FF20, Appendix B] gives $B_4 \leq 8$. This shows that the right hand side of (3) is at most $1 + 4(5.426) \approx 22.704$. A simple lower bound on $B_4$ is $B_4 \geq 1$. This shows that the right hand side of (3) is at least $3.713$.

3 Background on representation theory and symmetry reduction

In this section we summarize some basic notation, terminology, and facts about representation theory of compact groups over the reals that will be required for what follows. We then briefly review symmetry reduction for semidefinite programming feasibility problems in general, before specializing to feasibility problems related to checking whether a form is a sum of squares.

3.1 Preliminaries on representation theory

Let $G$ be a compact group. A **representation** $(V, \rho_V)$ of $G$ over $\mathbb{R}$ is a real vector space $V$ together with a group homomorphism $\rho_V : G \to GL_\mathbb{R}(V)$ from $G$ to invertible $\mathbb{R}$-linear maps on $V$. We often write $g \cdot x$ instead of $\rho_V(g)x$ when the homomorphism is clear from the context, and refer to $V$, itself, as the representation. Throughout, we assume we are working with finite-dimensional representations. Associated with a compact group $G$ is the **Haar measure**, denoted $\mu_G$, which we normalize so that $\mu_G(G) = 1$.

If $G$ is compact, we can equip any real representation $(V, \rho_V)$ of $G$ over $\mathbb{R}$ with a $G$-invariant inner product, with the property that $\langle g \cdot x, g \cdot y \rangle = \langle x, y \rangle$ for all $x, y \in V$ and $g \in G$. With this choice, the representation is **orthogonal** in the sense that $\rho_V(g)$ is an orthogonal transformation for all $g \in G$.

If $U$ is a representation of $G$ over $\mathbb{R}$, then a subspace $V \subseteq U$ is **invariant** if $g \cdot v \in V$ for all $v \in V$ and $g \in G$. If the only invariant subspaces of $U$ are $\{0\}$ and $U$, then we say that $U$ is an
irreducible representation. If $V$ is an invariant subspace of $U$, then so is its orthogonal complement $V^\perp$ with respect to a $G$-invariant inner product on $U$.

If $(U, \rho_U)$ and $(V, \rho_V)$ are two representations of a compact group $G$ over $\mathbb{R}$, let $\text{Hom}_G(U, V)$ denote the vector space of $\mathbb{R}$-linear maps $A : U \to V$ such that $\rho_V(g)A = A\rho_U(g)$ for all $g \in G$. Schur’s lemma [Sch01] tells us about the structure of these intertwining maps when $U$ and $V$ are irreducible.

**Lemma 3.1** (Schur’s lemma). If $U$ and $V$ are irreducible representations of $G$ over $\mathbb{R}$ and $A \in \text{Hom}_G(U, V)$ then either $A = 0$ or $A$ is invertible.

Two irreducible representations are equivalent if there exists an invertible linear map $A \in \text{Hom}_G(U, V)$. Schur’s lemma tells us that the only intertwiner between inequivalent irreducible representations of $G$ is the zero map. It also tells us about the endomorphism algebra of an irreducible representation $U$, i.e., $\text{End}_G(U) := \text{Hom}_G(U, U)$ consisting of the self-intertwiners. In general, if $U$ is irreducible over $\mathbb{R}$ then the endomorphism algebra is a division algebra over $\mathbb{R}$. As such, by the Frobenius theorem [Fro78], it is isomorphic either to $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$.

The following variation on Schur’s lemma tells us that the only self-adjoint maps from an irreducible representation to itself that commute with the group action are the multiples of identity. Throughout, we use the notation $S^U$ for the self-adjoint linear maps from an inner product space $U$ to itself.

**Proposition 3.2.** Let $U$ be a representation of a compact group $G$ over $\mathbb{R}$ equipped with a $G$-invariant inner product. If $U$ is irreducible then $\text{End}_G(U) \cap S^U = \{ \lambda I : \lambda \in \mathbb{R} \}$.

**Proof.** Suppose that $U$ is irreducible and let $A \in \text{End}_G(U) \cap S^U$. Since $A$ is self-adjoint, all of its eigenvalues are real. Let $\lambda$ denote any eigenvalue of $A$. Let $V$ be the nullspace of $A - \lambda I$. Then $V$ is a non-zero (since it contains an eigenvector) invariant subspace of $U$. Since $U$ is irreducible, $V = U$ and so $A = \lambda I$. \hfill $\square$

Since we can identify self-adjoint maps from $U$ to $U$ with bilinear forms on $U$, Proposition 3.2 tells us that an irreducible representation has a unique (up to scaling) $G$-invariant bilinear form.

One way to study the representations of a group is via their characters. If $(V, \rho_V)$ is a representation of $G$ over $\mathbb{R}$, the associated character is the function $\chi_V : G \to \mathbb{R}$ defined by $\chi_V(g) = \text{tr}(\rho_V(g))$. In particular, the dimension of the space of intertwiners can be computed by integrating characters with respect to Haar measure.

**Lemma 3.3.** If $(U, \rho_U)$ and $(V, \rho_V)$ are two representations of a compact group $G$ over $\mathbb{R}$ then

$$\langle \chi_U, \chi_V \rangle := \int_{g \in G} \chi_U(g)\chi_V(h) \, d\mu(G) = \dim_{\mathbb{R}}(\text{Hom}_G(U, V)).$$

If $(U, \rho_U)$ is a representation of $G$ over $\mathbb{R}$ and $(V, \rho_V)$ is a representation of $H$ over $\mathbb{R}$ then the (external) tensor product $(U \otimes V, \rho_U \otimes \rho_V)$ is a representation of $G \times H$ over $\mathbb{R}$ defined by $(g, h) \cdot (u \otimes v) = (g \cdot u) \otimes (h \cdot v)$ and extending by bilinearity. The corresponding character is $\chi_{U \otimes V}(g, h) = \chi_U(g)\chi_V(h)$. Over $\mathbb{C}$, the external tensor product of irreducible representations is always irreducible. When working over $\mathbb{R}$, a little more care is required. The following result will suffice for our purposes.

**Lemma 3.4.** If $U$ and $V$ are irreducible representations of $G$ and $H$ respectively over $\mathbb{R}$, and $\dim_{\mathbb{R}} \text{End}_G(U) = \dim_{\mathbb{R}} \text{End}_H(V) = 1$, then $\dim_{\mathbb{R}} \text{End}_{G \times H}(U \otimes V) = 1$ and so $U \otimes V$ is an irreducible representation of $G \times H$ over $\mathbb{R}$.
Proof. For the first part of the statement we compute
\[
\dim \mathbb{R} \operatorname{End}_{G \times H}(U \otimes V) = \langle \chi_{U \otimes V}, \chi_{U \otimes V} \rangle = \int_{G \times H} \chi_U(g)^2 \chi_V(h)^2 \, d\mu_G(g) d\mu_H(h) = \dim \mathbb{R} \operatorname{End}_G(U) \dim \mathbb{R} \operatorname{End}_H(V) = 1.
\]
For the second part of the statement, suppose that the endomorphism algebra \( \operatorname{End}_{G \times H}(U \otimes V) \) is one-dimensional (and so contains only multiples of the identity) and let \( W \) be an invariant subspace of \( U \otimes V \). Then the orthogonal projector onto \( W \) is an element of the endomorphism algebra, and so must be a multiple of the identity. As such, the orthogonal projector is either 0 or \( I \), implying that either \( W = \{0\} \) or \( W = U \otimes V \).

3.2 Preliminaries on symmetry reduction

In this section, we will briefly review symmetry reduction for semidefinite programming feasibility problems.

Let \( V \) be a real inner product space and let \( \mathcal{S}^V \subseteq \operatorname{End}(V) \) denote the self-adjoint linear maps from \( V \) to \( V \). After choosing a basis the elements of \( \mathcal{S}^V \) can be identified with symmetric \( \dim(V) \times \dim(V) \) matrices. Let \( \mathcal{S}^V_+ \subseteq \mathcal{S}^V \) denote the cone of positive semidefinite elements of \( \mathcal{S}^V \), i.e., the elements \( X \in \mathcal{S}^V \) that satisfy \( \langle v, X v \rangle \geq 0 \) for all \( v \in V \).

A semidefinite feasibility problem is a problem of the form
\[
\text{find } Y \in \mathcal{S}^V_+ \text{ such that } \mathcal{A}(Y) = b. \tag{4}
\]
where \( \mathcal{A} : \mathcal{S}^V \to W \) is a linear map and \( b \in W \).

Symmetry reduction for semidefinite feasibility problems Let \( G \) be a compact group and let \( V \) and \( W \) be representations of \( G \) over \( \mathbb{R} \) equipped with a \( G \)-invariant inner product. Then \( \mathcal{S}^V \) is also a representation of \( G \) over \( \mathbb{R} \) via the action \( Y \mapsto \rho_Y(g) Y \rho_Y(g)^\top \). If \( \mathcal{A} \in \operatorname{Hom}_G(\mathcal{S}^V, W) \) and \( g \cdot b = b \) for all \( g \in G \) then we say that the semidefinite feasibility problem is \( G \)-invariant. This is because if \( Y \) is feasible for (4) then \( \rho_Y(g) Y \rho_Y(g)^\top \) is also feasible for (4) for all \( g \in G \).

If a semidefinite feasibility problem is \( G \)-invariant, then it is equivalent to the symmetry-reduced feasibility problem
\[
\text{find } Y \in \operatorname{End}_G(V) \cap \mathcal{S}^V_+ \text{ such that } \mathcal{A}(Y) = b. \tag{5}
\]
This is because if \( Y \in \mathcal{S}^V_+ \) is feasible for (4) then
\[
Y^G = \int_{g \in G} \rho_Y(g) Y \rho_Y(g)^\top \, d\mu_G(g) = \int_{g \in G} \rho_Y(g) Y \rho_Y(g^{-1}) \, d\mu_G(g) \in \operatorname{End}_G(V) \cap \mathcal{S}^V_+
\]
and \( \mathcal{A}(Y^G) = b \), so \( Y^G \) is feasible for (5).

To proceed further, we need to consider how \( V \) decomposes into irreducible representations. The general case (for finite groups) is discussed in [GP04] in the language of invariant theory, and, for instance, in [RSST18, Appendix 2] in the language of representation theory. For what follows, we need only focus on the very special case in which \( V \) decomposes as a multiplicity-free direct sum of inequivalent irreducible representations. In this case, it follows from Lemma 3.1 and Proposition 3.2 that \( \operatorname{End}_G(V) \cap \mathcal{S}^V \) is the span of the orthogonal projectors onto the inequivalent irreducible invariant subspaces of \( V \). Then the symmetry-reduced semidefinite feasibility problem (5) becomes a linear programming feasibility problem. Here, and throughout, we use the notation \( \mathbb{R}^k_+ \) to denote the nonnegative orthant in \( \mathbb{R}^k \).
Proposition 3.5. Suppose that $V$ is an orthogonal representation of a compact group $G$ over $\mathbb{R}$, and $V = \bigoplus_{i=1}^{k} V_i$ decomposes as a direct sum of inequivalent irreducible representations $V_i$ of $G$ over $\mathbb{R}$. If the semidefinite feasibility problem (4) is $G$-invariant then it is equivalent to

$$
\text{find } \lambda \in \mathbb{R}_+^k \text{ such that } \sum_{i=1}^{k} A(P_{V_i})\lambda_i = b
$$

where $P_{V_i} : V \to V$ is the orthogonal projector onto the subspace $V_i \subseteq V$.

Symmetry reduction for sum-of-squares feasibility problems Next we specialize to the specific semidefinite feasibility problem that arises when we want to check that a given $G$-invariant form is a sum of squares. Let $p \in \mathbb{R}[x_1, \ldots, x_n]_{2d}$ be a form of degree $2d$ in $n$ variables. Let $V = \mathbb{R}^{(n+d-1)}$, which we think of as the space of coefficients of real forms of degree $d$ in $n$ variables with respect to some fixed basis. With any $c \in V$ we use $c(x)$ to denote the associated form in $\mathbb{R}[x_1, x_2, \ldots, x_n]_d$. Define a linear map $A : \mathcal{S}^V \to W$ on rank one elements by

$$
A(cc^\top) = c(x)^2
$$

and then extend to all of $\mathcal{S}^V$ by linearity of $A$ and by using the fact that an arbitrary element of $\mathcal{S}^V$ can be written as $X = \sum_{i=1}^{\ell} c_i c_i^\top - \sum_{j=\ell+1}^{\ell'} c_j c_j^\top$.

A form $p \in \mathbb{R}[x_1, x_2, \ldots, x_n]_{2d}$ is a sum of squares if and only if the following problem is feasible:

$$
\text{find } Y \in \mathcal{S}^V_+ \text{ such that } A(Y) = p(x).
$$

Suppose, now, that $G$ is a compact group that acts by orthogonal transformations on $\mathbb{R}^n$. The space of forms of degree $d$ becomes a representation of $G$ over $\mathbb{R}$ via $(g \cdot p)(x) = p(gx)$ for all $g \in G$. This induces a representation on $V$, which is orthogonal once we equip $V$ with an invariant inner product. If $p \in \mathbb{R}[x_1, \ldots, x_n]_{2d}$ is fixed by the action of $G$, then the sum-of-squares feasibility problem (6) is $G$-invariant. Suppose that $V$ decomposes in a multiplicity-free way into inequivalent irreducible representations as $V = \bigoplus_{i=1}^{k} V_i$. For $i = 1, 2, \ldots, k$, let $(c_{ij})_{j=1}^{\dim(V_i)}$ be an orthonormal basis for $V_i$. Then Proposition 3.5 tells us that $p$ is a sum of squares if and only if there exist $\lambda_1, \ldots, \lambda_k \in \mathbb{R}_+$ such that

$$
p(x) = \sum_{i=1}^{k} \lambda_i s_i(x) \quad \text{where} \quad s_i(x) = \sum_{j=1}^{\dim(V_i)} c_{ij}(x)^2 \quad \text{for } i = 1, 2, \ldots, k.
$$

This follows directly from the fact that the orthogonal projectors $P_{V_i}$ from Proposition 3.5 can be written explicitly as $P_{V_i} = \sum_{j=1}^{\dim(V_i)} c_{ij} c_{ij}^\top$.

4 Sums of squares

In this section we apply the general results of Section 3 to the specific convex feasibility problem arising from checking that the forms $q_{16}^D$ are sums of squares. We begin in Section 4.1 by recalling basic facts about the octonions, and understanding how the group Spin$(9)$ acts on pairs of octonions. Then, in Section 4.2, we rewrite these forms in a way that reveals that they are invariant under the action of Spin$(9) \times O(k)$ on $\mathbb{O}^k \times \mathbb{O}^k \cong \mathbb{R}^{16 \times k}$. In Section 4.3, we decompose the space of quadratic forms on $\mathbb{R}^{16 \times k}$ into irreducibles under the action of Spin$(9) \times O(k)$. This is the crucial representation-theoretic calculation required to describe the (polyhedral) cone of Spin$(9) \times O(k)$-invariant quartic forms that are sums of squares. Finally, in Section 4.4, we will show that $q_{16}^D$ is a sum of squares and $q_{17}^D$ is not a sum of squares, by studying the resulting linear programming feasibility problem in each case.
4.1 Background on the octonions and Spin(9)

Octonions The octonions are the normed division algebra of dimension 8. They can be explicitly constructed as the span of 8 units \(e_0, e_1, \ldots, e_7\) (with \(e_0\) being the identity) subject to a particular non-associative multiplication (see, e.g., [Tia00, Equation 1.5] for one possible multiplication table arising from a particular labeling of the units).

If \(u = a_0e_0 + \sum_{i=1}^{7} a_ie_i\) then its conjugate is \(\overline{u} = a_0e_0 - \sum_{i=1}^{7} a_ie_i\) and its real part is \(\text{Re}(u) = a_0\). Define a real inner product on \(\mathbb{O}\) via \(\langle u, v \rangle = \text{Re}(|u||v|)\) and let \(|u|^2 = \langle u, u \rangle = \overline{u}u\) denote the corresponding norm. If \(u = \sum_{i=0}^{7} a_i e_i\) then we denote by \([u] \in \mathbb{R}^8\) the coordinate vector of \(u\), i.e., \([u]_i = a_i\) for \(i = 0, 1, \ldots, 7\). Using this notation, \(\langle u, v \rangle = [u]^T[v]\). The squared norm can also be expressed as

\[
|u|^2 = \sum_{i=0}^{7} \langle u, e_i \rangle^2.
\]  

Furthermore, left (resp. right) multiplication by \(a\) is the adjoint of left (resp. right) multiplication by \(\overline{a}\), i.e.,

\[
\langle ax, y \rangle = \langle x, ay \rangle \quad \text{and} \quad \langle xa, y \rangle = \langle x, y\overline{a} \rangle
\]

for all \(x, y, a \in \mathbb{O}\) [Kra98, Equation 2]. Given \(u \in \mathbb{O}\), let \(R_u \in \text{End}_\mathbb{R}(\mathbb{O})\) denote right multiplication by \(u = \sum_{i=0}^{7} a_i e_i\). The matrix representing \(R_u\), in the sense that \([R_u][v] = [vu]\) is [Tia00]

\[
[R_u] = \begin{bmatrix}
    a_0 & -a_1 & -a_2 & -a_3 & -a_4 & -a_5 & -a_6 & -a_7 \\
    a_1 & a_0 & a_3 & -a_2 & a_5 & -a_4 & -a_7 & a_6 \\
    a_2 & -a_3 & a_0 & a_1 & a_6 & a_7 & -a_4 & -a_5 \\
    a_3 & a_2 & -a_1 & a_0 & a_7 & -a_6 & a_5 & -a_4 \\
    a_4 & -a_5 & -a_6 & a_7 & a_0 & a_1 & a_2 & a_3 \\
    a_5 & a_4 & -a_7 & a_6 & -a_1 & a_0 & a_3 & a_2 \\
    a_6 & a_7 & a_4 & -a_5 & -a_2 & a_3 & a_0 & -a_1 \\
    a_7 & -a_6 & a_5 & a_4 & a_3 & -a_2 & a_1 & a_0
\end{bmatrix}
\]

(where we have used the choice of multiplication table in [Tia00]). Observe that \([R_{e_0}] = I\) and that \([R_{\overline{a}}] = [R_u]^T\). Furthermore, \([R_u]\) is skew-symmetric whenever \(u\) is purely imaginary, i.e., whenever \(\text{Re}(u) = 0\).

A model for Spin(9) Define the following collection of \(16 \times 16\) real matrices

\[
S_i := \begin{bmatrix}
0 & R_{e_i} \\
R_{\overline{e}_i} & 0
\end{bmatrix} \quad \text{for } i = 0, 1, \ldots, 7 \text{ and } S_8 := \begin{bmatrix}
I & 0 \\
0 & -I
\end{bmatrix}.
\]

This collection of symmetric matrices form a Clifford system [FKM81], in the sense that

\[
S_i S_j + S_j S_i = 2I \delta_{ij} \quad \text{for } 0 \leq i, j \leq 8.
\]

Let

\[
V_1 := \text{span}\{S_i : i = 0, 1, \ldots, 8\} \subseteq \mathbb{R}^{16 \times 16}
\]

be the 9-dimensional subspace spanned by the \(S_i\). Let \(G\) denote the subgroup of \(SO(16)\) such that \(gV_1g^{-1} = V_1\). This group is isomorphic to \(\text{Spin}(9)\), the simply connected double-cover of \(SO(9)\), and is generated by elements of the form

\[
\begin{bmatrix}
v & R_u \\
R_{\overline{u}} & -v
\end{bmatrix} \quad \text{where } u \in \mathbb{O}, v \in \mathbb{R} \text{ and } |u|^2 + v^2 = 1
\]
Using (8) and (9) we see that

\[ q(X) = \frac{1}{2} \text{tr}(XX^\top)^2 - \frac{1}{4} \sum_{i=0}^8 \text{tr}(X^\top S_i X)^2 = \frac{1}{2} \text{tr}(XX^\top)^2 - 4\|P_V(XX^\top)\|_F^2. \]

**Proof.** Under the identifications in (12) it is straightforward to check that

\[ \|x\|^2 + \|y\|^2 = \sum_{j=1}^k \|[x_j]\|^2 + \|[y_j]\|^2 = \text{tr}(XX^\top) \quad \text{and} \]

\[ \|x\|^2 - \|y\|^2 = \sum_{j=1}^k \|[x_j]\|^2 - \|[y_j]\|^2 = \text{tr}(X^\top S_8 X). \]

Using (8) and (9) we see that

\[ |(x, y)|^2 = \sum_{i=0}^7 \sum_{j=1}^k (x_j y_j, e_i)^2 = \sum_{i=0}^7 \sum_{j=1}^k (y_j, R_{e_i} x_j)^2 = \frac{1}{4} \sum_{i=0}^7 \text{tr}(X^\top S_i X)^2. \]

Combining these observations gives the first equality in the statement of the lemma. For the second, we note that \( S_i/4 \) for \( i = 0, 1, \ldots, 8 \) form an orthonormal basis for \( V_1 \) with respect to the trace inner product. Then \( \sum_{i=0}^8 \text{tr}(X^\top S_i X)^2 = 16 \sum_{i=0}^8 \text{tr}((S_i/4)XX^\top)^2 = 16\|P_V(XX^\top)\|_F^2. \]

Recall that Spin(9) acts on \( \mathbb{R}^{16} \) as described at the end of Section 4.1. The orthogonal group \( O(k) \) acts on \( \mathbb{R}^k \) via the defining representation. As such Spin(9) \( \times O(k) \) acts on \( \mathbb{R}^{16 \times k} \), inducing an action on forms on \( \mathbb{R}^{16 \times k} \) which fixes \( q_k^O \).

**Lemma 4.2.** If \( k \geq 2 \) then \( q_k^O(gXh^\top) = q_k^O(X) \) for all \( g \in \text{Spin}(9) \) and all \( h \in O(k) \).

**Proof.** Clearly \( \text{tr}(gXh^\top hX^\top g^\top)^2 = \text{tr}(XX^\top)^2 \) for all \( g \in \text{Spin}(9) \) and \( h \in O(k) \) since \( \text{Spin}(9) \subseteq SO(16) \). Furthermore, \( \|P_V(gXh^\top hX^\top g^\top)\|_F^2 = \|gP_V(XX^\top)g^\top\|_F^2 = \|P_V(XX^\top)\|_F^2 \) since the orthogonal projector (with respect to a \( G \)-invariant inner product) onto a \( G \)-invariant subspace always commutes with the action of the group. Since \( q_k^O \) is in the span of these two invariant forms, it is also fixed by the action of \( \text{Spin}(9) \times O(k) \).
4.3 Decomposing quadratic forms under Spin(9) \times O(k)

In order to perform symmetry reduction on the problem of deciding whether \( q_{16} \) is a sum of squares, we need to decompose the space of quadratic forms in \( \mathbb{R}^{16 \times k} \cong \mathbb{R}^{16} \otimes \mathbb{R}^k \) under the action of Spin(9) \times O(k). This is equivalent to decomposing the symmetric tensor square of \( \mathbb{R}^{16 \times k} \) into irreducibles. We will do this by first decomposing the full tensor square \( \mathbb{R}^{16 \times k} \otimes \mathbb{R}^{16 \times k} \) into irreducibles and then restricting to the subspace of appropriately symmetric tensors. To achieve this, it is convenient to use the identification \( \mathbb{R}^{16 \times k} \otimes \mathbb{R}^{16 \times k} \cong \mathbb{R}^{16 \times 16} \otimes \mathbb{R}^{k \times k} \) and separately decompose \( \mathbb{R}^{16 \times 16} \) under the action of Spin(9), and \( \mathbb{R}^{k \times k} \) under the action of O(k).

Although the decompositions we will need are classical, it is not so straightforward to find concrete references to the results needed in a form that is broadly accessible, particularly as we are working with representations over \( \mathbb{R} \). As such, we will sketch character-theoretic proofs of the results we need. The approach we take makes use of the following beautiful formula relating integrals over the (special) orthogonal group to moments of Gaussian random variables. This formula is due to Diaconis and Shahshahani [DS94] (for a smaller range of the (special) orthogonal group to moments of Gaussian random variables. This formula is due to Diaconis and Shahshahani [DS94] (for a smaller range of \( n \)) and extended to the range given below for O(k) by Stoltz [Sto05, Theorem 3.4], and for the range given below for SO(k) by Pastur and Vasilchuk [PV04].

**Theorem 4.3.** Fix a positive integer \( r \) and let \((a_1, a_2, \ldots, a_r)\) be an \( r \)-tuple of nonnegative integers. Let \( Z_1, Z_2, \ldots, Z_r \) be independent normally distributed random variables such that

\[
Z_j \sim \mathcal{N}((1 + (-1)^j)/2, j) \quad \text{for} \quad j = 1, 2, \ldots, r.
\]

If \( \sum_{j=1}^r j a_j \leq 2k \) then

\[
\int_{h \in O(k)} \prod_{j=1}^r \operatorname{tr}(h^j)^{a_j} \, d\mu(h) = \prod_{j=1}^r \mathbb{E}[Z_{j-1}^a].
\]

If \( \sum_{j=1}^r j a_j \leq k - 1 \) then

\[
\int_{h \in SO(k)} \prod_{j=1}^r \operatorname{tr}(h^j)^{a_j} \, d\mu(h) = \prod_{j=1}^r \mathbb{E}[Z_j^a].
\]

**Decomposing \( \mathbb{R}^{k \times k} \) under O(k)** Recall that O(k) acts on \( \mathbb{R}^k \otimes \mathbb{R}^k \cong \mathbb{R}^{k \times k} \) via \( X \mapsto hXh^\top \). Let \( U_0 \) denote the span of the \( k \times k \) identity matrix, \( U_1 \) the space of \( k \times k \) traceless symmetric matrices, and \( U_{-1} \) the space of \( k \times k \) skew-symmetric matrices. These are clearly invariant subspaces that span \( \mathbb{R}^{k \times k} \).

**Proposition 4.4.** If \( k \geq 2 \) then the subspaces \( U_j \) for \( j = -1, 0, 1 \) are inequivalent irreducible representations for the action of O(k) on \( \mathbb{R}^{k \times k} \) with dim\( \mathbb{E}(\text{End}_{O(k)}(U_j)) = 1 \).

**Proof.** Let \( \chi_0, \chi_1 \) and \( \chi_{-1} \) denote the character of \( U_0, U_1, \) and \( U_{-1}, \) respectively. By Lemma 3.3 it suffices to establish that \( \|\chi_0\|^2 = \|\chi_1\|^2 = \|\chi_{-1}\|^2 = 1 \). This is a straightforward computation using Theorem 4.3 and the fact that \( \chi_0(h)^2 = 1, \chi_1(h)^2 = [(\operatorname{tr}(h)) + (\operatorname{tr}(h^2))/2 - 1]^2 \) and \( \chi_{-1}(h)^2 = [(\operatorname{tr}(h^2) - \operatorname{tr}(h^2))/2]^2 \).

**Decomposing \( \mathbb{R}^{16 \times 16} \) under Spin(9)** Recall that Spin(9) acts on \( \mathbb{R}^{16} \otimes \mathbb{R}^{16} \cong \mathbb{R}^{16 \times 16} \) via \( X \mapsto gXg^\top \). Given \( J \subseteq \{0, 1, \ldots, 8\} \) with elements \( J = \{j_1, j_2, \ldots, j_k\} \) satisfying \( j_1 < j_2 < \cdots < j_k \), define

\[
S_J = S_{j_1}, S_{j_2}, \ldots, S_{j_k}
\]
if \( J \) is non-empty, and \( S_8 = I \). The following result summarizes the key properties of these matrices that we will need. These follow from the relations (10) that define a Clifford system.

**Proposition 4.5.** The matrices \( S_J \) for \( |J| \in \{0, 1, 2, 3, 4\} \) form an orthonormal basis for \( \mathbb{R}^{16 \times 16} \). Moreover, \( S_J \) is symmetric if \( |J| \in \{0, 1\} \) and \( S_J \) is skew-symmetric if \( j \in \{2, 3\} \).

**Proof.** Clearly \( S_0 = I \) is symmetric and the matrices \( S_J \) are symmetric by construction. If \( J = \{j_1, j_2, \ldots, j_k\} \subseteq \{0, 1, \ldots, 8\} \) with \( j_1 < j_2 < \cdots < j_k \) then

\[
S_J^\top = S_{j_k}^\top S_{j_{k-1}}^\top \cdots S_{j_1}^\top = S_{j_k} S_{j_{k-1}} \cdots S_{j_1} = (-1)^{\binom{k}{2}} S_{j_1} S_{j_2} \cdots S_{j_k} = (-1)^{\binom{k}{2}} S_J
\]

where we have reversed the order of the terms in the product in the last equality by applying (10) \( \binom{k}{2} \) times. Since \( \binom{k}{2} \) is odd if \( k = 2, 3 \), \( S_J \) is skew-symmetric in these cases. Since \( \binom{k}{2} \) is even if \( k = 4 \), \( S_J \) is symmetric in this case.

We now show that the \( S_J \) are mutually orthogonal for \( |J| \in \{0, 1, \ldots, 4\} \). By direct verification, we can check that \( S_0 S_1 \cdots S_8 = -I \). Using this, together with (10), it follows that

\[
\text{tr}(S_J S_{J'}) = 0 \iff \text{tr}(S_{J \triangle J'}) = 0 \iff \text{tr}(S_{(J \triangle J')^c}) = 0
\]

where \( J^c = \{0, 1, \ldots, 8\} \setminus J \) is the complement of \( J \) and \( J \triangle J' \) is the symmetric difference of \( J \) and \( J' \). As such, to show that the \( S_J \) are mutually orthogonal, it is enough to show that \( \text{tr}(S_J) = 0 \) whenever \( 1 \leq |J| \leq 4 \). This clearly holds when \( |J| = 1 \) because the \( S_J \) have trace zero. This also holds for \( k = 2, 3 \) since in these cases \( S_J \) is skew-symmetric. Finally, this holds for \( k = 4 \) since if \( J = \{j_1, j_2, j_3, j_4\} \) then \( S_{j_1} \) is symmetric and \( S_{j_2} S_{j_3} S_{j_4} \) is skew-symmetric and so \( \text{tr}(S_J) = 0 \).

That the \( S_J \) for \( |J| \in \{0, 1, 2, 3, 4\} \) span \( \mathbb{R}^{16 \times 16} \) then follows from the fact that \( 16^2 = 256 = 1 + 9 + 36 + 84 + 126 = \binom{3}{2} + \binom{3}{1} + \binom{3}{1} + \binom{3}{3} + \binom{3}{3} \).

Define the subspaces \( V_k \subseteq \mathbb{R}^{16 \times 16} \) for \( k = 0, 1, 2, 3, 4 \) by

\[
V_k := \text{span}\{S_J : J \subseteq \{0, 1, \ldots, 8\}, \ |J| = k\}.
\]

We have that \( \dim(V_k) = \binom{9}{k} \) and that the \( V_k \) are mutually orthogonal, spanning \( \mathbb{R}^{16 \times 16} \). Moreover, \( V_0, V_1 \) and \( V_4 \) are subspaces of symmetric matrices and \( V_2 \) and \( V_3 \) are subspaces of skew-symmetric matrices.

**Proposition 4.6.** The subspaces \( V_k \) for \( k = 0, 1, 2, 3, 4 \) are inequivalent irreducible representations for the action of Spin(9) on \( \mathbb{R}^{16 \times 16} \) with \( \dim_{\mathbb{R}} \text{End}_{\text{Spin}(9)}(V_k) = 1 \).

**Proof.** Recall the representation \((V_1, \rho)\) of Spin(9) gives rise to a surjective homomorphism \( g \mapsto \rho(g) \) from Spin(9) to SO(9) with kernel \( \{I, -I\} \). The Haar measure on Spin(9) pushes forward to the Haar measure on SO(9) under this action.

Let \( ^k V_1 \) be the \( k \)th anti-symmetric tensor power of \( V_1 \). If \( I = \{i_1, \ldots, i_k\} \) then the map \( S_I : S_{i_1} \wedge \cdots \wedge S_{i_k} \) gives an isomorphism (of Spin(9) representations) between the subspace \( V_k \) and \( ^k V_1 \) for \( k = 0, 1, 2, 3, 4. \)

The \( ^k V_1 \) are inequivalent irreducible representations of Spin(9) with \( \dim_{\mathbb{R}} (\text{End}_{\text{Spin}(9)}(^k V_1)) = 1 \) for all \( k \). To see this, we first note that they are clearly invariant, and are inequivalent because they all have different dimensions. To see why they are irreducible, we note that the character \( \chi_k(h) \) of \( ^k V_1 \) is the elementary symmetric polynomial in the eigenvalues of \( \rho(h) \). Using the Newton identities, we can express this character as a polynomial in the power sum symmetric functions \( p_k(\lambda(\rho(h))) = \text{tr}(\rho(h)^k) \). Then by pushing forward the Haar measure on Spin(9) to the Haar measure on SO(9) and applying Theorem 4.3 we can check that \( \|\chi_k\|^2 = 1 \) for \( k = 0, 1, 2, 3, 4. \)
We now combine the decomposition of $\mathbb{R}^{16 \times 16}$ into $\text{Spin}(9)$ irreducibles, and the decomposition of $\mathbb{R}^{k \times k}$ into $O(k)$ irreducibles to decompose $\mathbb{R}^{16 \times k} \otimes \mathbb{R}^{16 \times k}$ into irreducibles.

**Proposition 4.7.** If $k \geq 2$, the space $\mathbb{R}^{16 \times k} \otimes \mathbb{R}^{16 \times k} \cong \mathbb{R}^{16 \times 16} \otimes \mathbb{R}^{k \times k}$ decomposes under the action $(g, h) \cdot (X \otimes Y) = gXh^\top \otimes gYh^\top$ of $\text{Spin}(9) \times O(k)$ as

$$\mathbb{R}^{16 \times 16} \otimes \mathbb{R}^{k \times k} = \bigoplus_{i \in \{0, 1, 2, 3, 4\}, j \in \{-1, 0, 1\}} (V_i \otimes U_j)$$

where each of the $V_i \otimes U_j$ are inequivalent irreducible representations of $\text{Spin}(9) \times O(k)$.

**Proof.** This follows from Propositions 4.4 and 4.6 and Lemma 3.4.

We identify the coefficients of quadratic forms on $\mathbb{R}^{16 \times k}$ with elements of $\mathbb{R}^{16 \times 16} \otimes \mathbb{R}^{k \times k}$ invariant under the action $(E \otimes F) \mapsto (E^\top \otimes F^\top)$. The identification associates $E \otimes F$ with the quadratic form $\text{tr}(EFX^\top)$. The irreducible representations in the decomposition of quadratic forms under $\text{Spin}(9) \times O(k)$ are exactly those irreducible representations from the decomposition of the tensor square $\mathbb{R}^{16 \times 16} \otimes \mathbb{R}^{k \times k}$ in which either both factors are symmetric, or both factors are skew-symmetric. If we let

$$\Lambda = \{(0, 0), (1, 0), (4, 0), (0, 1), (1, 1), (4, 1), (2, -1), (3, -1)\}$$

then $\Lambda$ indexes exactly these irreducible representations. The following result then follows immediately from Proposition 4.7.

**Proposition 4.8.** The space of coefficients of quadratic form on $\mathbb{R}^{16 \times k}$ decomposes into inequivalent irreducible representations of $\text{Spin}(9) \times O(k)$ over $\mathbb{R}$ as

$$W = \bigoplus_{(i, j) \in \Lambda} W_{ij} \quad \text{where} \quad W_{ij} = \text{span}\{E_i \otimes F_j : E_i \in V_i, F_j \in U_j\} \quad \text{for} \ (i, j) \in \Lambda.$$

### 4.4 Sum of squares feasibility

We now apply the results of the previous two sections to perform symmetry reduction on the problem of deciding whether $q_k^D$ is a sum of squares. We first give a more concrete description for the polynomials that appear in the symmetry-reduced sum of squares decomposition.

Let $(i, j) \in \Lambda$, let $E_1, \ldots, E_{\dim(V_i)}$ be an orthonormal basis (with respect to the trace inner product) for $V_i$ and let $F_1, \ldots, F_{\dim(U_j)}$ be an orthonormal basis (with respect to the trace inner product) for $U_j$. Define

$$s_{ij}(X) = \sum_{i'=1}^{\dim(V_i)} \sum_{j'=1}^{\dim(U_j)} \text{tr}(X^\top E_{i'}XF_{j'})^2.$$

These quartic forms are exactly those that appear in (7), after specializing to the present context. The following is a restatement of (7) specialized to the case of quartic forms on $\mathbb{R}^{16 \times k}$ that are invariant under the action of $\text{Spin}(9) \times O(k)$.

**Proposition 4.9.** Suppose that $f$ is a quartic form in $16k$ variables such that $f(gXh^\top) = f(X)$ for all $(g, h) \in \text{Spin}(9) \times O(k)$. Then $f$ is a sum of squares if and only if there exist $\lambda_{ij} \geq 0$ for $(i, j) \in \Lambda$ such that

$$f(X) = \sum_{(i, j) \in \Lambda} \lambda_{ij} s_{ij}(X) \quad \text{for all} \ X \in \mathbb{R}^{16 \times k}.$$
This gives a linear programming feasibility problem to check whether \( f \) is a sum of squares. To make the linear equality constraints relating the \( \lambda_{ij} \) and \( f \) more concrete, it is helpful to determine the dimension of the span of the \( s_{ij}(X) \) for \( (i, j) \in \Lambda \).

**Proposition 4.10.** The space of \( \text{Spin}(9) \times O(k) \)-invariant quartic forms is three-dimensional and is spanned by \( s_{00}(X) \), \( s_{10}(X) \), and \( s_{40}(X) \).

**Proof.** First we note that \( s_{j0}(X) = \frac{1}{k} \| P_{V_j}(XX^\top) \|_F^2 \) for \( j = 0, 1, 4 \). Recall that the space \( S_4^{16} \) of \( 16 \times 16 \) symmetric matrices decomposes into inequivalent irreducible representations as \( V_0 \oplus V_1 \oplus V_4 \) under the action of \( \text{Spin}(9) \) by \( g \cdot Z \mapsto gZg^\top \). Therefore, the space of \( \text{Spin}(9) \)-invariant quadratic forms on \( S_4^{16} \) is three dimensional, and is spanned by \( Z \mapsto \| P_{V_j}(Z) \|_F^2 \) for \( j = 0, 1, 4 \).

Since \( U_0 \oplus U_1 = S^k \), we have that

\[
\sum_{j'=1}^{\dim(V_j)} \| X^\top E_{j'}X \|_F^2 = \sum_{j'=1}^{\dim(V_j)} \text{tr}(E_{j'}XX^\top E_{j'}XX^\top).
\]

The right hand side is a \( \text{Spin}(9) \)-invariant quadratic form in \( Z = XX^\top \) and so must be in the span of \( \| P_{V_j}(Z) \|_F^2 \) for \( j = 0, 1, 4 \). Therefore \( s_{j1}(X) \) is in the span of \( s_{00}(X) \), \( s_{10}(X) \), and \( s_{40}(X) \) for each \( j = 0, 1, 4 \).

Consider, now, \( s_{2,-1}(X) \) and \( s_{3,-1}(X) \). Since \( U_{-1} \) consists of all skew-symmetric matrices,

\[
s_{j,-1}(X) = \sum_{j'=1}^{\dim(V_j)} \| X^\top E_{j'}X \|_F^2 = \sum_{j'=1}^{\dim(V_j)} \text{tr}(E_{j'}XX^\top E_{j'}XX^\top) \quad \text{for} \quad j = 2, 3.
\]

Again, the right hand side is a \( \text{Spin}(9) \)-invariant quadratic form in \( Z = XX^\top \) and so must be in the span of \( \| P_{V_j}(Z) \|_F^2 \) for \( j = 0, 1, 4 \). Therefore, for \( s_{j,-1}(X) \) is in the span of \( s_{00}(X) \), \( s_{10}(X) \), and \( s_{40}(X) \) for \( j = 2, 3 \).

We are now ready to establish the main result of this section.

**Theorem 4.11.** If \( k \geq 17 \) then \( q_k^\square \) is not a sum of squares. If \( k \leq 16 \) then \( q_k^\square \) is a sum of squares.

**Proof.** If \( k' \leq k \) then \( q_k^\square \) is the restriction of \( q_k^\square \) to a subspace. Therefore, if \( q_{17}^\square \) is not a sum of squares, it follows that the same holds for \( q_k^\square \) with \( k \geq 17 \). Similarly, if \( q_{16}^\square \) is a sum of squares, the same holds for \( q_k^\square \) with \( k \leq 16 \).

Since the space of \( \text{Spin}(9) \times O(k) \)-invariant quartic forms has dimension three, the characterization of \( \text{Spin}(9) \times O(k) \)-invariant sums of squares in Proposition 4.9 is equivalent to

\[
q_{17}^\square(X_\ell) = \sum_{(i,j) \in \Lambda} \lambda_{ij} s_{ij}(X_\ell) \quad \text{for} \quad \ell = 1, 2, 3
\]  

for three points \( X_1, X_2, X_3 \in \mathbb{R}^{16 \times k} \) for which the \( 3 \times 8 \) coefficient matrix has rank three. For \( k \geq 16 \) define

\[
X_1 = \begin{bmatrix}
1 & 0_{1 \times (k-1)} \\
0_{15 \times 1} & 0_{15 \times (k-1)}
\end{bmatrix}, \quad X_2 = \begin{bmatrix}
I_{8 \times 8} & 0_{8 \times (k-8)} \\
0_{8 \times (k-8)} & 0_{8 \times (k-8)}
\end{bmatrix}, \quad \text{and} \quad X_3 = \begin{bmatrix}
I_{16 \times 16} & 0_{16 \times (k-16)}
\end{bmatrix}.
\]

We have that \( q_k^\square(X_1) = 1/4 \), \( q_k^\square(X_2) = 64 \) and \( q_k^\square(X_3) = 128 \). If \( k = 17 \), by explicitly computing the coefficient matrix we see that \( q_{17}^\square \) is a sum of squares if and only if there exists \( \lambda \in \mathbb{R}_+^6 \) such that

\[
\begin{bmatrix}
\frac{1}{3} & 0 & 0 & 0 \\
64 & 16 & 17 & 17 \\
128 & 17 & 17 & 17 \\
\end{bmatrix} \lambda
\]
(where we have ordered the columns according to (13)). To see that this is infeasible, it is enough to multiply both sides on the left by $\begin{bmatrix} 252 & 3 & -2 \end{bmatrix}$ to obtain

$$-1 = \begin{bmatrix} 127/68 & 15/4 & 441/34 & 304/17 & 0 & 6384/17 & 96 & 0 \end{bmatrix} \lambda$$

which clearly cannot be satisfied for any $\lambda \in \mathbb{R}_+^8$.

If $k = 16$, by explicitly computing the coefficient matrix we see that $q^{O}_{16}$ is a sum of squares if and only if there exists $\lambda \in \mathbb{R}_+^8$ such that

$$\begin{bmatrix} \frac{1}{3} \\ 64 \\ 128 \end{bmatrix} = \begin{bmatrix} \frac{1}{102^2} & \frac{1}{102^2} & \frac{14}{102^2} & \frac{15}{102^2} & \frac{15}{102^2} & \frac{210}{102^2} & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 140 & 56 & 56 \\ 1 & 0 & 0 & 0 & 9 & 126 & 36 & 84 \end{bmatrix} \lambda.$$

This is satisfied by choosing $\lambda = \begin{bmatrix} 0 & 0 & 0 & 0 & 64/15 & 0 & 0 & 16/15 \end{bmatrix}$. In other words, we have the sum of squares decomposition

$$q^{O}_{16}(X) = \frac{64}{15}s_{11}(X) + \frac{16}{15}s_{3,-1}(X).$$

\[\square\]

5 Discussion

We conclude with a discussion of some open questions related to this work.

Increasing the degree and number of variables  Clearly, if $q$ is a convex form in $n$ variables of degree $2d$ that is not a sum of squares, then

$$\tilde{q}(x, x_0) = q(x) + x_0^{2d}$$

is convex and not a sum of squares. Therefore, we know that for all $n \geq 16 \times 17$ there is an $n$-variate quartic form that is convex but not a sum of squares.

It would be very interesting to come up with a general construction of explicit degree $2d$ convex forms that are not sums of squares for all $2d \geq 4$. However, it is unclear how to take an existing convex form that is not a sum of squares and increase its degree while maintaining convexity and the property of not being a sum of squares. A similar difficulty was also faced by Ahmadi and Parrilo, when constructing convex forms in all degrees that were not SOS-convex [AP13]. Instead, they managed to come up with a method to directly produce a convex but not sos-convex form of degree $2d + 2$ in $n$ variables, given a non-negative form of degree $2d$ in $n$ variables that is not a sum of squares.

Problem 5.1. Given a convex form that is not a sum of squares, find a way to construct a convex form of higher degree that is also not a sum of squares.

This would immediately yield examples of convex forms of degree $2d \geq 6$ that are not sums of squares.
Gap instances for polynomial optimization on the sphere  While the results of this paper are phrased in terms of convex forms that are not sums of squares, our arguments are really focused on finding examples of quartic forms for which

$$\text{GAP}(p) = \frac{p_{\max} - p_{\sos\min}}{p_{\max} - p_{\min}} > 2.$$  

In general, to find a degree $2d$ form that is convex but not a sum of squares, it suffices to find a form $p$ of degree $2d$ with $\text{GAP}(p) > d$. There has been some study of upper bounds on this quantity \[\text{Nie12, FF20}\], (which grow like $n^{d/2}$ for fixed $d$), and it is known that polynomials with random coefficients give rise to large values of $\text{GAP}(\cdot)$ \[\text{BGL17}\] with high probability. However, very little appears to be known in terms of explicit gap instances.

**Problem 5.2.** Find an explicit family of forms $p_n$ of fixed degree $2d$ with increasing number of variables, such that $\text{GAP}(p_n)$ grows without bound with $n$.

Any such family would eventually give rise to further examples of convex forms that are not sums of squares.

**Explicit gaps for OT-FKM forms**  The Cauchy-Schwarz forms $cs_{k}^{O}$ are closely related to a class of forms known as OT-FKM-type isoparametric forms, which are particular classes of solutions of the Cartan-Münzner equations \[\text{OT75, FKM81}\]. These are forms $F$ such that

$$F(x) = \|x\|^4 - 2 \sum_{i=0}^{m} \langle P_i x, x \rangle^2$$

where the $2\ell \times 2\ell$ matrices $P_i$ form a Clifford system, i.e., they are symmetric matrices satisfying

$$P_i P_j + P_j P_i = 2 \delta_{ij} I.$$  

These forms take values between $-1$ and $1$ on the unit sphere, and so

$$p(x) = \frac{1}{2}(\|x\|^4 + F(x)) = \|x\|^4 - \sum_{i=0}^{m} \langle P_i x, x \rangle^2$$

is nonnegative for all $x$. These vanish (on the sphere) on special isoparametric submanifolds called *focal submanifolds*. The Cauchy-Schwarz forms $cs_{k}^{O}$ studied in this paper correspond (up to a scaling factor) to the case of $p$ where the Clifford system is given by the $16k \times 16k$ matrices $P_i = S_i \otimes I_k$ for $i = 0, 1, \ldots, 8$.

A full classification of when the nonnegative forms $p$ (associated with OT-FKM-type isoparametric forms $F$) are sums of squares is given in \[\text{GT18}\]. It would be interesting to make this classification quantitative.

**Problem 5.3.** Compute $p_{\sos\min}$ for the nonnegative forms $p(x) = \frac{1}{2}(\|x\|^4 + F(x))$ arising from OT-FKM-type isoparametric forms $F$.

By a variation on the proof of Theorem 4.11, it is possible to show that $[cs_{k}^{O}]_{\sos\min} = \frac{-2(k-1)}{8+7k}$ for $k \geq 2$. As such, this family satisfies $\text{GAP}(cs_{k}) = \frac{16k}{8+7k} \to \frac{15}{7}$ as $k \to \infty$. In particular, this does not grow without bound as the number of variables increases.
Examples in fewer variables  Solving the following problem from [AP13] would complete our understanding of when convex quartics are sums of squares.

Problem 5.4. Find the smallest \( n \) such that there is a convex quartic form in \( n \) variables that is not a sum of squares.

We know that such a form must have \( n \geq 5 \) by El Khadir’s work [EK20]. While it should be possible to find explicit examples of forms in fewer than \( 16 \times 17 \) variables for which \( \text{gap}(p) > 2 \), this is only a sufficient condition for a quartic form to be convex but not a sum of squares. Finding minimal examples will likely require working directly with convexity, rather than Blekherman’s sufficient condition for convexity from Theorem 2.1.

Invariant forms  We have seen that convex forms are not always sums of squares when restricted to the three-dimensional subspace of \( \text{Spin}(9) \times O(k) \)-invariant quartic forms in \( 16k \) variables. It would be interesting to investigate the relationship between convexity and sums of squares when restricted to \( G \)-invariant forms in \( n \) variables for other group actions on \( \mathbb{R}^n \). A particularly canonical case would be to study forms invariant under the action of the symmetric group by permuting the variables (symmetric forms). While it is known that there are symmetric nonnegative forms that are not sums of squares whenever there are nonnegative forms that are not sums of squares [GKR16], it remains unclear whether the same is true for convex forms.

Problem 5.5. Determine whether all convex symmetric forms are sums of squares.

More generally, suppose there were an (orthogonal) action of a group \( G \) on \( \mathbb{R}^n \) such that all \( G \)-invariant convex forms of degree \( 2d \) are sums of squares. Then, because \( \|x\|^{2d} \) is \( G \)-invariant, by essentially the same argument as given in Section 2.1, it would follow that \( \text{gap}(p) < d \) for all elements of the subspace of \( G \)-invariant forms.

Acknowledgments

James Saunderson is the recipient of an Australian Research Council Discovery Early Career Researcher Award (project number DE210101056) funded by the Australian Government.

References

[ADKH19] A. A. Ahmadi, E. De Klerk, and G. Hall. Polynomial norms. *SIAM Journal on Optimization*, 29(1):399–422, 2019.

[AP13] A. A. Ahmadi and P. A. Parrilo. A complete characterization of the gap between convexity and SOS-convexity. *SIAM Journal on Optimization*, 23(2):811–833, 2013.

[BGL17] V. Bhattiprolu, V. Guruswami, and E. Lee. Sum-of-squares certificates for maxima of random tensors on the sphere. In *Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques (APPROX/RANDOM 2017)*. Schloss Dagstuhl-Leibniz-Zentrum für Informatik, 2017.

[Ble12] G. Blekherman. Nonnegative polynomials and sums of squares. In *Semidefinite Optimization and Convex Algebraic Geometry*, pages 159–202. SIAM, 2012.
E. De Klerk and D. V. Pasechnik. Approximation of the stability number of a graph via copositive programming. *SIAM Journal on Optimization*, 12(4):875–892, 2002.

P. Diaconis and M. Shahshahani. On the eigenvalues of random matrices. *Journal of Applied Probability*, 31(A):49–62, 1994.

B. El Khadir. On sum of squares representation of convex forms and generalized Cauchy–Schwarz inequalities. *SIAM Journal on Applied Algebra and Geometry*, 4(2):377–400, 2020.

K. Fang and H. Fawzi. The sum-of-squares hierarchy on the sphere and applications in quantum information theory. *Mathematical Programming*, pages 1–30, 2020.

D. Ferus, H. Karcher, and H.-F. Münzner. Cliffordalgebren und neue isoparametrische Hyperflächen. *Mathematische Zeitschrift*, 177(4):479–502, 1981.

G. Frobenius. Über lineare Substitutionen und bilineare Formen. *Journal für die reine und angewandte Mathematik (Crelle’s Journal)*, 1878(84):1–63, 1878.

C. Goel, S. Kuhlmann, and B. Reznick. On the Choi–Lam analogue of Hilbert’s 1888 theorem for symmetric forms. *Linear Algebra and its Applications*, 496:114–120, 2016.

K. Gatermann and P. A. Parrilo. Symmetry groups, semidefinite programs, and sums of squares. *Journal of Pure and Applied Algebra*, 192(1-3):95–128, 2004.

J. Ge and Z. Tang. Isoparametric polynomials and sums of squares. *arXiv preprint arXiv:1811.05587*, 2018.

F. R. Harvey. *Spinors and calibrations*. Elsevier, 1990.

D. Hilbert. Über die Darstellung definiter Formen als Summe von Formenquadraten. *Mathematische Annalen*, 32(3):342–350, 1888.

J. W. Helton and J. Nie. Semidefinite representation of convex sets. *Mathematical Programming*, 122(1):21–64, 2010.

A. Hurwitz. Uber die Composition der quadratischen Formen von beliebig vielen Variablen. *Nach. Ges. Wiss. Göttingen*, 1898:309–316, 1898.

L. Kramer. Octonion hermitian quadrangles. *Bulletin of the Belgian Mathematical Society-Simon Stevin*, 5(2/3):353–362, 1998.

T. S. Motzkin and E. G. Straus. Maxima for graphs and a new proof of a theorem of Turán. *Canadian Journal of Mathematics*, 17:533–540, 1965.

J. Nie. Sum of squares methods for minimizing polynomial forms over spheres and hypersurfaces. *Frontiers of mathematics in china*, 7(2):321–346, 2012.

H. Ozeki and M. Takeuchi. On some types of isoparametric hypersurfaces in spheres I. *Tohoku Mathematical Journal, Second Series*, 27(4):515–559, 1975.

P. A. Parrilo. Polynomial optimization, sums of squares, and applications. In *Semidefinite Optimization and Convex Algebraic Geometry*, pages 47–157. SIAM, 2012.
L. Pastur and V. Vasilchuk. On the moments of traces of matrices of classical groups. *Communications in mathematical physics*, 252(1):149–166, 2004.

A. Raymond, J. Saunderson, M. Singh, and R. R. Thomas. Symmetric sums of squares over k-subset hypercubes. *Mathematical Programming*, 167(2):315–354, 2018.

I. Schur. *Über eine Klasse von Matrizen, die sich einer gegebenen Matrix zuordnen lassen*. Dieterich, 1901.

M. Stolz. On the Diaconis-Shahshahani method in random matrix theory. *Journal of Algebraic Combinatorics*, 22(4):471–491, 2005.

Y. Tian. Matrix representations of octonions and their applications. *Advances in Applied Clifford Algebras*, 1(10):61–90, 2000.