Role of quark-instanton liquid interactions in colour superconductivity phase

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Abstract

The interaction of light quarks and instanton liquid is analyzed at finite density of quark/baryon matter and in the phase of nonzero values of diquark (colour) condensate. It is shown that instanton liquid perturbation produced by such an interaction results in an essential increase of the critical value of quark chemical potential \( \mu_c \) which provokes the perceptible increase of quark matter density around the expected onset of the colour superconductivity phase.

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The intriguing history of studying the structure of strongly interacting matter abounded in many interesting but somewhat contradictory phenomenological results up to the advent of quantum chromodynamics (QCD). Since the time of its discovery the possible existence of quark-gluon plasma (QGP) phase is one of the most fundamental predictions of QCD. This phase is phenomenologically understood as a matter state where quarks and gluons being under extreme conditions of high temperature (T) and(or) quark/baryon density (driven by the corresponding chemical potential \( \mu \)) are able to propagate relatively freely over the considerable (of course, on the hadron scale) distances. Even the simplest quantitative estimates of QGP physical characteristics were so verisimilar that they allowed to launch the process of QGP experimental study in ultrarelativistic heavy ion collisions. And although the first series of experiments at SPS CERN and AGS BNL have provided us with encouraging results a new generation of experiments is necessary to make convincing conclusions. The physical programme for new experimental research certainly needs serious quantitative analysis of the QCD phase diagram.

It seemed lattice QCD could be able to resolve the problem starting even on the first theoretical principles. However, up to now this approach was effective and apparently reliable in producing the results mainly around the temperature axis of the (\( \mu—T \))-plane and, unfortunately, relies on the field theory formulation in imaginary time. Nevertheless, lattice QCD has corroborated the phenomenological expectations that its phase structure is quite rich \cite{1}. In particular, nowadays we know that the chiral symmetry of initial QCD Lagrangian becomes broken and the quarks are getting to be confined when the temperature decreases below a critical value. Moreover, the lattice calculations show us that both physical phenomena result from the corresponding phase transitions and their critical temperatures practically coincide. The latter observation indicates that both fundamental phenomena could be tightly knitted and it should heuristically play an important role to illuminate the confinement mechanism.

The features of the QCD matter at high baryon densities remains poorly understood in spite of the recent splash of renewed activity \cite{2}. One of the major theoretical reasons comes from serious technical difficulties of lattice simulations with Monte-Carlo techniques due to the complex
fermion determinant in QCD at finite quark densities. From the phenomenological side the recent experimental data are not very informative for developing the theory along this direction. It is clear that present and future experiments with heavy ions at higher and higher energies should produce the strong interacting matter rather at higher temperature and closer to the temperature axis of the (μ−T)-plane. In such a situation the phenomenological treatment of studying along the μ-axis should basically rely on the astrophysical observations of the compact stars. In the meantime, it was demonstrated that the diquark pairing in colour anti-triplet attractive channel induced by the instanton interaction vertex should be much more effective than an one-gluon exchange and at sufficiently large quark/baryon densities and small T could lead to colour superconductivity and a sophisticated picture of the QCD phase diagram [2].

The present paper continues our preceding one [3] where the light quark interactions with the instanton liquid (IL) at nonzero values of quark/baryon chemical potentials were investigated in the phase of broken chiral symmetry within the improved method of calculating the generating function of the IL theory [4]. The new treatment of the functional integral is mainly related to an idea that take into account the IL back-reaction upon the quark presence which was always considered negligible. In fact, this effect is pretty weak and should manifest itself in the leading order of expansion in the effective coupling. Nevertheless, it was qualitatively argued that the interaction of IL and quarks could increase the quark matter density around the onset of expected colour superconductivity. As known [3], [4] this density occurred to be surprisingly small (n_q ≈ 0.062 fm^{-3}) and, hence, the diquark phase would come to play already at the density of nuclear matter (n_n ≈ 0.45 fm^{-3} and quark matter density is taken to be three times larger than the conventional nuclear one).

Here we obtain the quantitative estimate of the corresponding critical chemical potential μ_c analyzing the system of the Gorkov equations for the colour superconductor which we further calculate. We do not introduce any simplification of the initial Lagrangian trying to deal with the 'exact' four-quark interaction which is generated by (anti-)instantons. Such an intention is simply compulsory because it is dictated by the precision of problem where the effects of the IL perturbation are expected to be of the same order of magnitude compared to the terms which could usually be ignored.

Let us point out that in the IL approach [7] the generating functional has the following factorized form

$$Z = Z_g \cdot Z_\psi .$$

Here the first factor provides us with information on the gluon condensate whereas the fermion factor Z_ψ serves to describe the quark practice in the instanton environment [3], [4]. In what follows we use the notations of Ref. [3] where the dimensionless variables (motivated by the form of interquark interaction kernel) were introduced, for example, for the chemical potential it looks like μ → μ̅/2 and for the momenta k_i ̅/2 → k_i, i = 1,...,4 where ̅/2 is the average size of pseudoparticle (PP).

The quark determinant Z_ψ may be transformed with the auxiliary integration over the parameter λ to the following form (for the colour SU(3) group with the quarks of two flavours N_f = 2):

$$Z_\psi \simeq \int d\lambda \int D\psi^\dagger D\psi \exp\left\{n V_4 \left(\ln \frac{n ̅^4}{\lambda N} - 1\right)\right\} \times \exp\left\{\int \frac{dk}{\pi} \sum_{f=1}^{2} \psi_f^R(k) 2 \left(-\hat{k} - i\hat{\mu}\right)\psi_f(k) + V\right\} ,$$

$$V = 2\lambda (\psi_1^{\dagger L} L_1 \psi_1^L)(\psi_2^{\dagger L} L_2 \psi_2^L) + 2\lambda (\psi_1^{\dagger R} R_1 \psi_1^R)(\psi_2^{\dagger R} R_2 \psi_2^R) ,$$

where ψ_f^T = (ψ_f^R, ψ_f^L), f = 1, 2 are the quark fields with the spinors of definite chirality, ψ_f^{\dagger R} = P_± ψ_f^R, P_± = \frac{1 ± \gamma_5}{2}, n is the IL density, V_4 is 4-volume which the system occupies, μ_ν = (0, μ) and N is the normalizing factor. For clarity we take it to be equal a unity but, in principle, it could play a role of free model parameter. This factor is inessential for the models with the fixed value of the packing fraction parameter n ̅^4 but in the model where it is admissible for variation the
weak logarithmic dependence on $N$ appears. The factors $2$ in Eq. (1) come with making use of the dimensionless variables. The term of four-fermion quark interaction $V$ may be directly expressed with the chiral components as

$$\langle \psi_f^R | L_f | \psi_f^L \rangle = \int \frac{dp_f dq_f}{\pi^8} \psi_f^L |_{\alpha f i j} \langle p_f | L_{\alpha f i j}^\beta | p_f, q_f, \mu \rangle \psi_f^L |_{\beta f i j} (q_f),$$

it being known that for the right hand fields the substitution $L \to R$ should be performed. The kernels $L_{\alpha f i j}^\beta$ are defined by the functions $h_i, i = 1, \ldots, 4$ and the zero modes (the solutions of the Dirac equation with the chemical potential $\mu$ in the PP field)

$$h_4(k_4, k; \mu) = \frac{\pi}{4k^2} \left\{ ((k - \mu - ik_4)[(2k_4 + i\mu)f_1^- + i(k - \mu - ik_4)f_2^-] + 
+ (k + \mu + ik_4)[(2k_4 + i\mu)f_1^+ - i(k + \mu + ik_4)f_2^+] \right\},$$

$$h_i(k_4, k; \mu) = \frac{\pi k_i}{4k^2} \left\{ ((k - \mu)(k - \mu - ik_4)f_1^- + (2k + \mu)(k + \mu + ik_4)f_1^+ + 
+ 2(k - \mu)(k - \mu - ik_4) - \frac{1}{k^2} \left( (k + \mu + ik_4) \left[ k_4^2 + (p - \mu)^2 \right] f_2^- + 
+ 2(k + \mu)(k + \mu + ik_4) + \frac{1}{k} \left( k + \mu + ik_4 \right) \left[ k_4^2 + (p + \mu)^2 \right] f_2^+ \right) \right\},$$

where $k = |k|$ if the spatial components of 4-vector $k_\nu$ are considered and

$$f_1^\pm = \frac{I_1(z^\pm)K_0(z^\pm)}{z^\pm} - \frac{I_0(z^\pm)K_1(z^\pm)}{z^\pm} \frac{I_1(z^\pm)K_1(z^\pm)}{z^\pm}, \quad f_2^\pm = \frac{I_1(z^\pm)K_1(z^\pm)}{z^\pm}, \quad z^\pm = \frac{\rho}{2} \sqrt{k_4^2 + (k \pm \mu)^2},$$

with the modified Bessel functions $I_i, K_i (i = 0, 1)$. Let us introduce the scalar function $h(k_4, k; \mu)$ related to three-dimensional component $h_i(k_4, k; \mu) = h(k_4, k; \mu) \frac{k_i}{k}, i = 1, 2, 3$ (we omit the arguments of functions $h_i$ when it does not mislead).

$$L_{\alpha i}^\beta (p, q; \mu) = S^{ik} (p; \mu) \epsilon^{kl} U_1^\sigma \epsilon^{\sigma n} S^{+}_{nj} (q; -\mu),$$

$S(p; \mu) = (p + i\mu)^- h^+ (p; \mu), \quad S^+ (p; -\mu) = h^+ (p; -\mu) (p + i\mu)^+$ where it is valid for the conjugated function $\bar{h}(p; -\mu) = h_\mu (p; -\mu)$ and $\epsilon$ is an antisymmetric matrix with $\epsilon_{12} = -\epsilon_{21} = 1$. Here $p^\pm$ and other similar designations are used for the four-vectors spanned by $\sigma^\pm$-matrices, $\sigma^\pm = (\pm i \sigma, 1)$, ($\sigma$ is the three-vector of the Pauli matrices), for example, $p^\pm = p'^\sigma \sigma_\sigma$ and $U$ is a matrix of rotations in the colour space. The similar relations are valid for the right hand components

$$\langle \psi_f^R | R_f | \psi_f^L \rangle = \int \frac{dp_f dq_f}{\pi^8} \psi_f^L |_{\alpha f i j} \langle p_f | R_{\alpha f i j}^\beta | p_f, q_f, \mu \rangle \psi_f^L |_{\beta f i j} (q_f),$$

with the kernel

$$R_{\alpha i}^\beta (p, q; \mu) = T^{ik} (p; \mu) \epsilon^{kl} U_1^\sigma \epsilon^{\sigma n} T^{+}_{nj} (q; -\mu),$$

where $T(p; \mu) = (p + i\mu)^+ h^+ (p; \mu), \quad T^+ (p; -\mu) = h^+ (p; -\mu) (p + i\mu)^-$. The components of matrices $(p + i\mu)^\pm$ and $h^\pm (p; \mu)$ commute because the vector-function $h(p)$ is spanned by the vector $p$ only. Then the following identities are easily understood

$$T(p; \mu) = S^+ (p; -\mu), \quad T^+ (p; -\mu) = S(p; \mu).$$

In what follows we omit the $\mu$-dependence of matrices $T, T^+$ as the chemical potential enters the matrix $T$ being always positive and the matrix $T^+$ being negative only. Besides, two other identities would also be helpful

$$\sigma_2 T^T (p) \sigma_2 = T^+ (p), \quad \sigma_2 T^+ T (p) \sigma_2 = T (p).$$
where $T^T$ means a transposed matrix.

Dealing with the following averages interesting to study the diquark condensates \[^3\]
\[
\langle \psi_{1\alpha i}^L(p)\psi_{2\beta j}^R(q) \rangle = \epsilon_{12} \epsilon_{\alpha\beta} \pi^4 \delta(p+q) \ G_{ij}^{LR}(p),
\]
\[
\langle \psi_{1\alpha i}^R(p)\psi_{2\beta j}^{1R}(q) \rangle = \delta_{f g} \delta_{\alpha\beta} \pi^4 \delta(p-q) \ G_{ij}^{LR}(p),
\]
and making use of the effective action of Eq. \[^1\] one may obtain the Gorkov equations similar to ones treated in Ref. \[^1\]
\[
\begin{align*}
\left[ G_0^+(p) \right]^{-1} F^L(p) - \Sigma^R(p) \ G^{LR}T(-p) &= 0, \\
\left[ G_0^-(p) \right]^{-1} G^{LR}(p) - \Sigma^R(p) \ F^{+R}T(p) &= 1, \\
\left[ G_0^+(p) \right]^{-1} F^R(p) - \Sigma^L(p) \ G^{RL}T(-p) &= 0, \\
\left[ G_0^-(p) \right]^{-1} G^{RL}(p) - \Sigma^L(p) \ F^{+L}T(p) &= 1,
\end{align*}
\]
(2)

where $\left[ G_0^\pm(p) \right]^{-1} = -2 (p + i\mu)^\pm$ means the free Green function, $\Sigma^R(p) = \Delta^R T(p) \epsilon \ T^T(-p)$, $\Sigma^L(p) = \Delta^L T^+(p) \epsilon \ T^{+T}(-p)$ and $\Delta^{LR}$ denotes a gap. The form of $\Sigma$-matrices results from the kernel structure of equations if averaging over colour orientations done (remember that we consider colour stochastic ensemble). In order to complete Eqs. \[^2\] we need the following gap equations
\[
\epsilon \Delta^R = \frac{2\lambda}{N_c(N_c - 1)} \int \frac{dq}{\pi^4} \left[ T^+(q) \ F^R(q) \ T^{+T}(-q) - T^+(q) \ F^R T(q) \ T^{+T}(q) \right],
\]
\[
\epsilon \Delta^L = \frac{2\lambda}{N_c(N_c - 1)} \int \frac{dq}{\pi^4} \left[ T(q) \ F^L(q) \ T^T(-q) - T(q) \ F^L T(q) \ T^T(q) \right].
\]

Apparently, the right hand sides of these equations are proportional $\epsilon$ because they are the differences a matrix and its transposed form. Similar equations are valid for the conjugated matrices
\[
\begin{align*}
F^{+L}T(p) \quad \left[ G_0^+(p) \right]^{-1} - G^{RL}T(-p) \Sigma^{+R}(p) &= 0, \\
G^{RL}(p) \quad \left[ G_0^-(p) \right]^{-1} - F^R(p) \Sigma^{+R}(p) &= 1, \\
F^{+R}T(p) \quad \left[ G_0^+(p) \right]^{-1} - G^{LR}T(-p) \Sigma^{+L}(p) &= 0, \\
G^{LR}(p) \quad \left[ G_0^-(p) \right]^{-1} - F^L(p) \Sigma^{+L}(p) &= 1,
\end{align*}
\]
(3)

with the corresponding gap equations
\[
\epsilon \Delta^{+R} = \frac{2\lambda}{N_c(N_c - 1)} \int \frac{dq}{\pi^4} \left[ T^+(q) \ F^{+R}T(q) \ T(q) - T^+(q) \ F^{+R}(q) \ T(-q) \right],
\]
\[
\epsilon \Delta^{+L} = \frac{2\lambda}{N_c(N_c - 1)} \int \frac{dq}{\pi^4} \left[ T^+(q) \ F^{+L}T(q) \ T(q) - T^+(q) \ F^{+L}(q) \ T(-q) \right],
\]

where $\Sigma^{+R}(p) = \Delta^{+R} T^{+T}(-p) \epsilon T^+(p)$ and $\Sigma^{+L}(p) = \Delta^{+L} T^T(-p) \epsilon T(p)$. In this paper we limit ourselves to treating the diquark condensate only. However, as known \[^3\] the mixed phase of nonzero values both chiral and colour condensates could exist at $\mu_c \sim 300$ MeV realizing the transitional regime for the onset of colour superconducting phase. In order to bring this phase to the play the equation system should be extended including another averages as
\[
\langle \psi_{1\alpha i}^{LR}(p)\psi_{2\beta j}^{1LR}(q) \rangle = \delta_{f g} \delta_{\alpha\beta} \pi^4 \delta(p-q) \ G_{ij}^{LL,RR}(p),
\]
With the auxiliary matrices $C^{+L}(p) = \Delta^{+L} T^+(-p) T(p), \ C^{R}(p) = \Delta^R T(p) T^+(-p)$ we may rewrite the matrices $\Sigma$ as
\[ \Sigma^{+L}(p) = \epsilon \ C^{+L}(p), \ \Sigma^R(p) = C^R(p) \ \epsilon. \]
The identities for $T$-matrix mentioned above help to show the validity of the relations
\[ \epsilon \ C^{+L} T(-p) \ T = C^{+L}(p), \]
\[ \epsilon \ C^R T(-p) \ T = C^R(p). \]
And for the Green function we have
\[ G^{LR}(p) = G^+_0(p) + G^+_0(p) \ C^R(p) \ \epsilon \ G^{LR} T(-p) \ \epsilon \ C^{+L}(p) \ G^+_0(p). \]
Combining the properties of $C$ matrices together with the identities for free Green functions
\[ \sigma_2 \ G^\pm_T(p) \ \sigma_2 = G^\mp_0(p), \]
we are able to get the complete equation for calculating the function $G^{LR}$ in the form
\[ \epsilon \ G^{LR}(-p) \ T = G^-_0(-p) + G^-_0(-p) \ C^{+L}(p) \ G^{LR} T(p) \ C^R(p) \ G^-_0(-p). \]
The vector-function $h(p)$ being spanned by the vector $p$ helps to conclude that all the matrices $G^\pm_0, C^{+L}, C^R$ commute with each other. Then searching the solution for the Green function $G^{LR}$ by iterating one finds immediately that the Green function commutes with those matrices and finally obtains the equation to calculate it in the form
\[ G^{LR}(p) = [1 + H(p)] \ G^+_0(p) + H^2(p) \ G^{LR}(p), \]
or
\[ [1 - H(p)] \ G^{LR}(p) = G^+_0(p), \]
where $H(p) = G^+_0(p) \ C^R(p) \ G^-_0(-p) \ C^{+L}(p)$.
The structure of matrices $H(p)$ allows us to conclude that their sum
\[ H(p) + H(-p) = \alpha(p) \]
and product
\[ H(p) \ H(-p) = \beta(p), \]
are proportional to the unity matrices (it is clear by definition that $\alpha(-p) = \alpha(p), \ \beta(-p) = \beta(p)$).
Introducing notation $g_\nu = h_\nu(-p)$ we are able to present the functions $\alpha(p)$ and $\beta(p)$ in the compact form as
\[ \alpha(p) = 4 \ \Delta^R \ \Delta^{+L} \ [-4 \ A(p) \ (hg) + 2 \ (p^2 + \mu^2) \ (h^2) \ (g^2)] , \]
where
\[ A(p) = (p^2 + \mu^2) \ (hg) - 2i \ \mu \ p (g_4 h - h_4 g). \]
and a scalar product is naturally defined $(hg) = \Sigma_{i=1}^\nu h_i g_i$ together with the functions $(h^2)$ and $(g^2)$ squared. Getting the last term of $A(p)$ we used the designation of scalar function $h$ mentioned above and for the function $\beta$ we have
\[ \beta(p) = 16 \ (\Delta^R)^2 \ (\Delta^{+L})^2 \ (p^2 + \mu^2)^2 \ (h^2)^2 \ (g^2)^2. \]
As both functions $\alpha$ and $\beta$ are spanned by the unit matrices the solution for the Green function may be given in

$$G^{LR}(p) = \frac{G^+_0(p) \left[ 1 - H(-p) \right]}{1 - \alpha(p) + \beta(p)}.$$  

The gap equation then looks like

$$\Delta^L = \frac{2\lambda}{N_c(N_c - 1)} \int \frac{dp}{\pi^4} \frac{\alpha(p) - 2 \beta(p)}{\Delta^+ + 1 - \alpha(p) + \beta(p)}.$$  

We are interested in the solution of the form $\Delta^R = \Delta^+ = \Delta^L = \Delta^+ + R$ at $\lambda < 0$, which is dictated by the symmetries of four-quark interaction kernels. Let us remember that any kernel for every quark sort traditionally carries the imaginary unit factor $i$ \cite{7}, \cite{8}. The sign choice of $\lambda$ relies on this fact but, in principle, there is an alternative $\Delta^R = \Delta^+ = -\Delta^L = -\Delta^+ + R$ for $\lambda > 0$, if the kernels are redefined. An analysis shows that the denominator of Eq. (4) is always positive and, therefore, the solution of this equation does exist at pretty large $\lambda$.

The quark matter state at finite chemical potential is defined by the saddle point of functional Eq. (1) which we treat further maintaining the first nonvanishing contribution which is the figure-eight type diagram (see, for example, \cite{5}),

$$I = 2 \left( N_c - 1 \right) \int \frac{dp}{\pi^4} \frac{\alpha(p) - 2 \beta(p)}{1 - \alpha(p) + \beta(p)}.$$  

In our consideration this contribution to the generating functional ($Z_\psi \sim e^W$) would occur to be

$$W = -n\bar{\rho}^4 \ln \lambda + \lambda \langle Y \rangle, \quad \langle Y \rangle \approx I.$$  

For the simplest situation of constant IL density the saddle point equation reads

$$n\bar{\rho}^4 - \lambda \langle Y \rangle = 0.$$  

Comparing it with the gap equation Eq. (4) one could notice the peculiar feature which is very practical to keep numerical calculations under control. The saddle point equation leads to the condition of gap independent of $\mu$.

It was demonstrated in Refs. \cite{4}, \cite{9} that the quark back-reaction upon IL could be perturbatively estimated by studying the small variations of the IL parameters $\delta n$ and $\delta \rho$ around their equilibrium values $n$ and $\bar{\rho}$. Such variations could be incorporated by the IL theory if the deformable (crumpled) (anti-)instantons of size $\rho$, being the function of $x$ and $z$, i.e. $\rho \rightarrow \rho(x,z)$, are treated as the saturating configurations of the functional integral. Theoretically it is justified by developing the variational principle which makes use the Ritz method based on the multipole expansion of the deformation fields in the following form

$$\rho(x,z) = \rho(z,z) + \frac{\partial \rho}{\partial z_\mu} y_\mu + \frac{\partial^2 \rho}{\partial z_\mu \partial z_\nu} y_\mu y_\nu + \ldots, \quad y = x - z.$$  

Then it is obvious the singular nature of the pseudoparticle solution generates the necessary cut-off at the scale of average instanton size.

In principle, the instanton orientation in the colour space should be treated in similar manner. However, it does not lead to noticeable consequences while in the random instanton medium. Although such a treatment turns out very informative if one considers the polarization characteristics of the IL when the probe colour charge is present.

In addition, the variations of zero modes in the interaction vertices of quark determinant at the transformation $\bar{\rho} \rightarrow \bar{\rho} + \delta \rho$ should be taken into account. Then as the output we have that for the long wave length excitations (for example, $\pi$-mesons) the deformation field describes colour less
scalar excitations of IL with the mass gap $M$ of the order of several hundreds MeV, $M^2 = \frac{\nu}{\kappa}$ where $\nu = b - 4$, $b = \frac{11}{3} N_c - 2 N_f$, $\kappa$ is the kinetic coefficient being derived within the quasi-classical approach. $N_c$ and $N_f$ are the numbers of quark colours and flavours, respectively. Our estimates give for this coefficient value of a few single instanton actions $\beta = \frac{8\pi^2}{g^2}$, i.e. $\kappa \sim c \beta$ (with the factor $c \sim 1.5 \ldots 6$ according to the ansatz taken for the saturating configurations) [9].

Then, besides the diagrams with four legs (see, the term $V$ in (1)) the extra diagrams with the scalar field attached (relatively speaking, the derivative in $\rho$ of the vertex function in which the variation of the functions describing the zero mode $h_i \rightarrow h_i + \frac{\partial h_i}{\partial \rho} \delta \rho$, $i = 1, \ldots, 4$.

$$h_i \rightarrow h_i + \frac{\partial h_i}{\partial \rho} \delta \rho , \quad i = 1, \ldots, 4 , \quad (7)$$

should be performed). Due to the diquark condensate presence handling the Lagrangian is substantially simplified if one restricts oneself with the leading contributions coming from the tadpole diagrams. Indeed, the leading contributions come from the term $V$ and the term shown in Fig. 1 where two vertices are linked with the propagator $\frac{1}{M^2}$ of scalar field.

![Figure 1: The diagram of tadpole approach (see, the text). The solid (dashed) lines correspond to the fermion (scalar) field.](image)

Analyzing the modified Lagrangian it is easy to see that the form of the Gorkov equations would be the same if everywhere $\Sigma$ is meant as an improved one $\Sigma + \delta \Sigma$ in which the correction $\delta \Sigma$ is constructed with the modified functions Eq. (4). Moreover, the result for the Green function is formally the same if the modified functions $\Sigma$ are meant. In particular, the form of gap equation is also retained because its kernel includes again the differences of the matrices which are constructed from the derivatives in $\rho$ of matrices $T$, $T^+$ and their transposed forms. This difference is spanned by the unit vector as well. However, the following substitutions

$$\alpha(p) \rightarrow \alpha(p) + \delta \alpha(p) ,$$

$$\beta(p) \rightarrow \beta(p) + \delta \beta(p) ,$$

where

$$\delta \alpha(p) = J \frac{\partial \alpha}{\partial \rho} ,$$

$$\delta \beta(p) = J \frac{\partial \beta}{\partial \rho} ,$$

are supposed to be performed and here $J$ is the contribution of figure-eight type diagram with the scalar field propagator as an external leg

$$J = \frac{2\lambda}{N_c(N_c - 1)} \frac{1}{n\rho^4\kappa} \frac{1}{4M^2} I .$$
Let us mention that effectively a dependence on the kinetic term $\kappa$ disappears (remember that $M^2 = \frac{G}{\kappa}$) and its precise value is not operative in the approximation developed. The derivative of the figure-eight type diagram which will shortly be necessary to proceed looks like

$$\frac{\partial I}{\partial \rho} = 2 (N_c - 1) \int \frac{dp}{\pi^2} \left\{ \frac{1 - \beta(p)}{(1 - \alpha(p) + \beta(p))^2} \delta \alpha(p) + \frac{\alpha(p) - 2}{(1 - \alpha(p) + \beta(p))^2} \delta \beta(p) \right\}.$$  

And finally the modification of generating functional should be accommodated bringing Eq. (3) to the form

$$W = -n \rho^4 \left( \ln \frac{n \rho^4}{\lambda} - 1 \right) + \lambda \langle Y \rangle.$$  

Then the new (if the variation of the IL density is absorbed) equation to calculate the saddle point reads

$$n \rho^4 - \lambda (n \rho^4) \ln \frac{n \rho^4}{|\lambda|} - \lambda \langle Y \rangle = 0$$  

and the IL density is [4]

$$n \rho^4 = \frac{\nu}{2\beta \xi^2} + \left[ \left( \frac{\nu}{2\beta \xi^2} \right)^2 + \left( \frac{\delta I}{\delta \rho} \right)^2 \frac{\Gamma(\nu + 1/2)}{2\sqrt{\nu} \Gamma(\nu)} \right]^{1/2}, \quad (8)$$  

where the prime available means the derivative in $\lambda$, the constant $\xi^2 = \frac{27}{4} \frac{N_c}{N_c^2 - 1} \pi^2$ is a measure of interaction in the stochastic ensemble of PPs with an average size as

$$\bar{\rho} \Lambda = \exp \left\{ - \frac{2N_c}{2\nu - 1} \right\}.$$  

The derivative in $\lambda$ of the figure-eight type diagram looks like

$$\left( \frac{\delta I}{\delta \rho} \right) = 4 (N_c - 1) \frac{\Delta'}{\Delta} \int \frac{dp}{\pi^4} \left\{ \frac{1 + \alpha - 6\beta + \alpha \beta + \beta^2}{(1 - \alpha + \beta)^3} \delta \alpha + \frac{-4 + 7\alpha + 4\beta - \alpha \beta - \alpha^2}{(1 - \alpha + \beta)^3} \delta \beta \right\}$$  

and the derivative $\Delta'$ is defined by the following equation

$$\frac{N_c (N_c - 1)}{2\lambda^2} = \frac{2\Delta'}{\Delta^3} \int \frac{dp}{\pi^4} \frac{2\beta - 2\beta^2 + 2\alpha \beta - \alpha^2}{(1 - \alpha + \beta)^2}.$$  

Fig. 2 shows the results of calculating the parameter $\lambda$ (which is proportional to the free energy within the precision of one loop approximation) as the function of chemical potential at $N_c = 3$, $N_f = 2$. The dashed lines expose a behaviour in the phase of nonzero values of diquark condensate (the left hand line (lower line) corresponds to the calculations if the quark interaction with IL is ignored) whereas the solid lines show the behaviours in the phase of broken chiral symmetry (upper line for the quark interaction with IL ignored) [3]. The saddle point parameter $\lambda_1$ of Ref. [3] and that exploited in the present paper are related as $\Lambda^2 = -\frac{\lambda}{2(N_c - 1)N_c} n \rho^4$. The crossing points of the curves are fixing the onset of colour superconducting phase. As mentioned above, actually, there is a transitional region of mixed phase where both chiral and diquark condensates develop nonzero magnitudes and the descent is not so steep. However, these details are inessential for further discussion. What is more interesting to be noticed comes from the crossing point of lower solid and dashed lines positioned on the plot.

The first crossing point (along the $\mu$-axis) which corresponds to the approach without the tadpole contributions included looks to be slightly larger in magnitude than that obtained earlier $\mu_c \simeq 300$ MeV in Refs. [3], [4]. However, it does not signal about shifting the critical point to larger values of $\mu$ in the approach dealing with the initial Lagrangian. The IL parameters of various approaches are
Figure 2: The saddle point $\lambda$ as the function of chemical potential $\mu$ at $N_c = 3$ and $N_f = 2$. The solid lines correspond to the phase of broken chiral symmetry (the lower one absorbs the IL perturbation). The dashed lines correspond to the colour superconductivity phase (the left hand one does not include the tadpole contribution whereas the right hand one does).

Figure 3: The quark density being defined by the crossing point of the upper solid and left hand dashed lines in Fig. 2. The solid line corresponds to the stable phase and the dashed one corresponds to the metastable one.

sometimes slightly different and, in principle, could be optimized fitting, for example, $\Lambda_{QCD}$. Thus, the shift of $\mu_c$ value pointed out is well within a precision of the IL theory. Regarding the second crossing point (when IL is perturbed by quarks) the conclusion could be more indicative. Here $\mu_c$ becomes significantly larger. The corresponding quark matter densities as the functions of chemical potential for both approaches are depicted in Fig. 3 and Fig. 4. Apparently, on both plots the density values corresponding to the onset of colour superconductivity phase are appreciably larger than the density of normal nuclear matter. Perhaps, our result could be taken as a general indication on the considerable role of supplementary (insignificant on the instanton background) interactions of light quarks. They are able to change the estimate of critical density for colour diquark condensation drastically.

Now we would like to comment on the crossing point of upper dashed and lower solid lines which is not shown in Fig. 2. It corresponds the unrealistic values of $\mu$ and in that region the IL is hardly applicable. One of the limits for IL approach coming from the very large values of chemical potential points out that with quark matter density increasing the average interquark distances may become
very small and the 'Coulomb' (perturbative) field strengths would occur to be comparable with the (anti-)instanton ones. Then the (anti-)instanton superposition is not a proper configuration to saturate the functional integral. Thus, our conclusion resulting from the approach developed gives a message that the quark perturbation of IL leads the corresponding curves to move apart inherent in the phases of nonzero values of diquark and chiral condensates in Fig. 2. The 'chiral' curve becomes steeper but 'diquark' curve is displaced to the larger values of $\mu$ increasing its critical value. It looks like giving more reliability to our understanding of the perturbative fields role in the IL theory could provide us with more accurate appraisals for the phenomena considered.

Fig. 5 demonstrates the IL density as the function of $\mu$ when the quark back-reaction is incorporated in the phase of broken chiral symmetry (solid curve) and in the phase of nonzero values of diquark condensate (dashed curve). It is interesting to notice that the curves manifest the different character of quark influence on IL. In the phase of broken chiral symmetry the density is almost constant in the suitable interval of $\mu$ whereas in the colour superconducting phase quickly disappears.
It is remarkable that for the former the corresponding analogue of the tadpole contribution \( \left( \frac{\delta I}{\delta \rho} \right)' \) in Eq. (8) is strictly positive and, therefore, can lead to the increase of the IL density (gluon condensate) which is truly insignificant and demonstrates simply the approach sensitivity to the dynamical quark mass variation [3]. In the latter case the sign of tadpole contribution is changing. As seen from Fig. 5 it leads to the drastic change of the IL density behaviour and the gluon condensate is getting weaker. Strictly speaking this effect has already been discussed in [10].

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