A note on the Berman condition

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Abstract. It is established that if a time series satisfies the Berman condition, and another related (summability) condition, the result of filtering that series through a certain type of filter also satisfies the two conditions. In particular it follows that if $X_t$ satisfies the two conditions and if $X_t$ and $a_t$ are related by an invertible ARMA model, then the $a_t$ satisfy the two conditions.

1 Introduction

The condition (on the autocovariances $\gamma_k$ of a stationary time series)

$$\lim_{k \to \infty} |\gamma_k| \ln k = 0 \quad (1.1)$$

was introduced by Berman (1964, Theorem 3.1, p. 510). It appears to have been adopted as a fundamental sufficient condition in proving results about extreme value distributions for correlated data. It is cited for instance in Leadbetter et al. (1983, equation 2.5.1, p. 444), Lindgren and Rootzén (1987, equation 5.1, p. 248), Leadbetter and Rootzén (1988, equation 4.1.1, p. 80), Galambos (1978, Theorem 3.8.2, p. 169; see also p. 198), and in Embrechts et al. (1999, Theorem 4.4.8, p. 217), where it is described as being “very weak.” It appears to be effectively the weakest condition that one can assume and still obtain positive results in this context.

In Chareka, Matarise and Turner (2006) the authors found it necessary to assume, in addition to the Berman condition, another condition

$$\sum_{k=1}^{\infty} \frac{|\gamma_k|}{k^\varepsilon} < \infty \quad \text{for some } \varepsilon < 1. \quad (1.2)$$

This is given as condition (7) on page 598 of Chareka et al. (2006). In that paper the authors find it expedient to deal with the residuals from fitting an ARMA model to the time series $X_t$ under consideration. They are thereby concerned with the innovation terms of such a model. Suppose that $X_t$ and $a_t$ are related via an ARMA model in which the $a_t$ play the role of the innovations. Chareka et al. remark that

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if the time series $X_t$ is a fractionally integrated ARMA (“FARIMA”) time series (whence it satisfies the two conditions of interest (1.1) and (1.2)) then the innovations $a_t$ also form a FARIMA series provided that the model is invertible. Hence the $a_t$ satisfy the two conditions of interest as well.

Chareka et al. assert that more is true: if $X_t$ is any stationary time series satisfying (1.1) and (1.2) and if $X_t$ and a series of innovations $a_t$ are related by an invertible ARMA model, then the $a_t$ will also satisfy these conditions. In this note we present the proof of that claim.

We now remark that interest is focussed on the $a_t$ and these quantities are thought of as being the output of a filter, with the $X_t$ being the input. However the phrasing of the claim, with the $a_t$ being the innovations of an ARMA model, makes it appear as if the $a_t$ are the input to a filter, which is rather confusing. The required condition of invertibility of the ARMA model is also somewhat disconcerting. Finally, it turns out that a slightly stronger claim may be established. We therefore rephrase the assertion to be proven, in a stronger and less confusing form, and state the original claim as a corollary of the rephrased assertion.

## 2 The main result

We state the result to be proven as follows:

**Theorem.** Suppose that $X_t$ is a stationary time series with autocovariances $\gamma_k$ satisfying conditions (1.1) and (1.2) and that the series $Y_t$ is the output of a linear filter with input $X_t$ given as follows:

$$Y_t = \sum_{n=0}^{\infty} \psi_n X_{t-n}.$$  

Suppose that the $\psi_n$ are summable (whence the $Y_t$ form a stationary time series). Furthermore suppose that the $\psi_n$ satisfy the condition

$$|\gamma^W_k| \leq C r^{|k|} \quad \text{for all } k \quad (2.1)$$

for some constants $C$ and $r$, $0 < r < 1$, where

$$\gamma^W_k = \sum_{n=-\infty}^{\infty} \psi_n \psi_{n+k}$$

and where we set $\psi_n = 0$ for $n < 0$ (to simplify the notation). Then the autocovariances $\gamma^Y_k$ of the series $Y_t$ satisfy (1.1) and (1.2) as well.

**Proof.** We remark that the $\gamma^W_k$ are in fact the autocovariances of a time series $W_t$ defined by

$$W_t = \sum_{n=0}^{\infty} \psi_n b_{t-n},$$
where \( b_t \) is white noise with variance 1.

Observe that

\[
\gamma^Y_k = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \psi_n \psi_m \gamma_{m-n+k}
\]

\[
= \sum_{h=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \psi_n \psi_{n+h} \gamma_{k+h}
\]

\[
= \sum_{h=-\infty}^{\infty} \gamma^W_h \gamma_{k+h}.
\]

To show that the \( \gamma^Y_k \) satisfy condition (1.1) we write

\[
\gamma^Y_k = \sum_{h=-\infty}^{k-1} \gamma^W_h \gamma_{k+h} + \sum_{h=-k}^{-1} \gamma^W_h \gamma_{k+h} + \sum_{h=0}^{\infty} \gamma^W_h \gamma_{k+h}
\]

\[
= \sum_{j=1}^{\infty} \gamma^W_{k+j} \gamma_j + \sum_{j=0}^{k-1} \gamma^W_j \gamma_{k-j} + \sum_{j=0}^{\infty} \gamma^W_j \gamma_{k+j}
\]

\[
= \xi_1(k) + \xi_2(k) + \xi_3(k) \quad \text{(say)}.
\]

To deal with \( \xi_1(k) \) we observe that

\[
|\xi_1(k)| \ln k \leq \sum_{j=1}^{\infty} |\gamma^W_{k+j}| |\gamma_j| \ln k \leq C \gamma_0 r^k \ln k \frac{r}{1-r}
\]

using (2.1). This quantity \( \to 0 \) as \( k \to \infty \) since \( r < 1 \).

Similarly

\[
|\xi_3(k)| \ln k \leq \sum_{j=0}^{\infty} |\gamma^W_j| |\gamma_{k+j}| \ln (k+j) \leq C \sum_{j=0}^{\infty} r^j |\gamma_{k+j}| \ln (k+j).
\]

Take \( \delta > 0 \); for sufficiently large \( k \), \( |\gamma_{k+j}| \ln (k+j) \leq \delta \) for all \( j \geq 0 \). Hence

\[
|\xi_3(k)| \ln k \leq \frac{\delta \times C}{1-r}
\]

for sufficiently large \( k \), and since \( \delta \) is arbitrary, \( |\xi_3(k)| \ln k \to 0 \) as \( k \to \infty \).

To deal with the middle term \( \xi_2(k) \) we note that

\[
|\xi_2(k)| \leq C \sum_{j=0}^{k-1} |\gamma_j| r^{k-j}
\]

\[
= C \left[ \sum_{j=0}^{\lfloor k/2 \rfloor} |\gamma_j| r^{k-j} + \sum_{j=\lfloor k/2 \rfloor+1}^{k-1} |\gamma_j| r^{k-j} \right]
\]
\[ \begin{align*}
&\leq C \sum_{j=0}^{[k/2]} \gamma_0 r^{k-j} + C \sum_{j=[k/2]+1}^{k-1} |\gamma_j| r^{k-j} \\
&\leq C \gamma_0 \frac{r^{k/2}}{1-r} + C \gamma_{j^*(k)} \frac{r}{1-r},
\end{align*} \]

where \( j^*(k) = \arg\max \{|\gamma_j| : [k/2] + 1 \leq j \leq k - 1 \} \).

Hence

\[ \begin{align*}
\ln k \times |\xi_2(k)| &\leq C \left[ \gamma_0 \ln k \frac{r^{k/2}}{1-r} + (\ln k/2 + \ln 2) \left( \gamma_{j^*(k)} \frac{r}{1-r} \right) \right] \\
&\leq C \left[ \gamma_0 \ln k \frac{r^{k/2}}{1-r} + (\ln j^*(k) + \ln 2) \left( \gamma_{j^*(k)} \frac{r}{1-r} \right) \right]
\end{align*} \]

which \( \to 0 \) as \( k \to \infty \).

We have thus established that the autocovariances \( \gamma_k^Y \) satisfy (1.1). We now proceed to show that condition (1.2) is satisfied:

\[ \begin{align*}
\sum_{k=1}^{\infty} \frac{|\gamma_k|}{k^\varepsilon} &\leq \sum_{k=1}^{\infty} \left| \sum_{h=-\infty}^{\infty} \gamma_h^W \frac{\gamma_{k+h}}{k^\varepsilon} \right| \\
&\leq \sum_{k=1}^{\infty} \left| \sum_{h=0}^{\infty} \gamma_h^W \frac{\gamma_{k+h}}{k^\varepsilon} \right| \\
&\leq \sum_{j=1}^{\infty} \left| \sum_{k=1}^{\infty} \frac{\gamma_{k-j}}{k^\varepsilon} \right| + \sum_{k=1}^{\infty} \left| \gamma_{k+h} \right| (k+h)^\varepsilon \\
&\leq \sum_{j=1}^{\infty} \left| \sum_{k=1}^{\infty} \frac{\gamma_{k-j}}{k^\varepsilon} \right| + \zeta \sum_{h=0}^{\infty} |\gamma_h^W|(1+h), \quad \text{where } \zeta = \sum_{k=1}^{\infty} \frac{|\gamma_k|}{k^\varepsilon} \\
&\leq \sum_{j=1}^{\infty} \left| \sum_{k=1}^{j} \frac{\gamma_{k-j}}{k^\varepsilon} \right| + \sum_{h=0}^{\infty} \left| \gamma_h^W \right|(1+h) \\
&\leq \sum_{j=1}^{\infty} Cr^j \left[ \sum_{k=1}^{j} \frac{1}{k^\varepsilon} + \sum_{\ell=1}^{\infty} \frac{|\gamma_\ell|}{\ell^\varepsilon} \left( \frac{\ell}{\ell + j} \right)^\varepsilon \right] + \zeta \sum_{h=0}^{\infty} Cr^h (1+h) \\
&\leq C \sum_{j=1}^{\infty} r^j \left[ j \gamma_0 + \sum_{\ell=1}^{\infty} \frac{|\gamma_\ell|}{\ell^\varepsilon} \right] + \zeta C \sum_{h=0}^{\infty} r^h (1+h)
\end{align*} \]
\[
\leq C \sum_{j=1}^{\infty} r^j [j \gamma_0 + \zeta] + \zeta C \sum_{h=0}^{\infty} r^h (1 + h) \\
= C \left[ \gamma_0 \sum_{j=1}^{\infty} j r^j + \zeta \sum_{j=1}^{\infty} j r^j + \zeta \sum_{j=0}^{\infty} j r^j \right] \\
= C \left[ (\gamma_0 + \zeta) \sum_{j=1}^{\infty} j r^j + \zeta \frac{1+r}{1-r} \right] < \infty
\]
since \( \sum_{j=1}^{\infty} j r^j \) converges. (The radius of convergence of this power series is 1, and by assumption \( 0 < r < 1 \).)

3 The original claim

We state the original claim as:

**Corollary.** Suppose that \( X_t \) satisfies conditions (1.1) and (1.2) and that \( X_t \) and \( a_t \) are related by the invertible ARMA model

\[
\phi(B)X_t = \theta(B)a_t,
\]

where \( \phi(z) \) and \( \theta(z) \) are polynomials and \( B \) is the “backshift” operator. Then the \( a_t \) satisfy conditions (1.1) and (1.2) as well.

**Proof.** The invertibility of the model tells us that \( a_t \) can be expressed as

\[
a_t = \sum_{n=0}^{\infty} \psi_n X_{t-n},
\]

where

\[
\frac{\phi(z)}{\theta(z)} = \psi(x) = \sum_{n=0}^{\infty} \psi_n z^n
\]

with the coefficients \( \psi_n \) being summable. (It is more usual in such a context to denote the coefficients of the series expansion as “\( \pi_n \)” rather than “\( \psi_n \)” but to make clear the relationship to the main result we eschew the \( \pi_n \) notation.)

Now if we set

\[
W_t = \sum_{n=0}^{\infty} \psi_n b_{t-n},
\]

where \( b_t \) is white noise (with variance 1) then basic results about ARMA time series (see, e.g., Brockwell and Davis (1991, Chapter 3, problem 3.11)) tell us that the autocovariances \( \gamma_k^W \) of \( W_t \) satisfy (2.1). Hence the claim follows by the theorem proven in Section 2. \( \square \)
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