FAST TRACK COMMUNICATION

Symmetrization of advection-diffusion operators

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Abstract. We present a new method to transform an expanded class of non-selfadjoint advection-diffusion operators into self-adjoint operators. The transform is based on a combination of a point transform and Lie transform in conjunction with an asymptotic expansion in terms of the diffusivity. We illustrate the method in the context of simple shear flow where the expansion is exact and all transformation steps can be performed explicitly.

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Introduction

In this work, we are concerned with the well studied advection-diffusion equation in two space dimensions which we write in the following form:

\[ c_t + 2(u \cdot \nabla)c = \kappa \Delta c , \]  

(1)

where \( u \) is a given incompressible velocity field, \( \kappa \) is a given diffusion constant, \( c \) is the unknown concentration function and the factor of two is included for future algebraic convenience. Most of the difficulties encountered in the study of this equation arise from the fact that the advection operator \( 2u \cdot \nabla \) and the diffusion operator \( \kappa \Delta \) exhibit different symmetries: The advection operator is skew-symmetric while the diffusion operator is symmetric and hence self-adjoint. As a result, the combined advection-diffusion operator possesses, in general, neither symmetry. This lack of symmetry complicates the study of solutions of Eq. (1) especially in the dynamically interesting case of small diffusivity \( \kappa \).

For a very particular class of advection-diffusion operators, namely those associated with irrotational (potential) velocity fields \( u \), it is well-known that a simple point transform maps (1) into a selfadjoint Schrödinger-like problem [2]. Consequently, the transformed equation can be studied using standard techniques from quantum mechanics, in particular WKB approximations, and the results have important implications for the original equation. This approach has been used in the past in the context of Fokker-Planck equations [3]. From both the physical and dynamical-systems points of view, cases where the velocity field is time-dependent are of particular interest. In one spatial dimension, all flows are potential and this fact allows one to apply point transforms to fairly general one-dimensional non-autonomous convection-dominated parabolic equations. In doing so, difficulties arising from the fact that convection-dominated parabolic equations do not satisfy the spectral-gap condition, required for all currently available proofs of the existence of inertial manifolds, are completely circumvented. The existence of inertial manifolds, in turn, allows for a complete decomposition of the convection-diffusion operator in terms of Floquet bundles. Using averaging techniques and inverse scattering theory, it is possible to further transform the non-autonomous parabolic equation, at least in one spatial dimension, into an autonomous Schrödinger-like equation [4].

Point transformations of this kind prove useful not only for linear autonomous and non-autonomous parabolic equations, but also for certain nonlinear parabolic equations. Point transforms have been used to circumvent the spectral-gap condition and prove the existence of inertial manifolds for a class of nonlinear nonlocal Fokker-Planck equations and for a class of viscous Burgers equations with low-wavenumber instability in both one and two space dimensions [5, 6].

For the latter, the necessary point transformation is the well-known Cole-Hopf transformation. The existence of inertial manifolds implies, in particular, that possibly very complicated global attractors can be embedded in smooth finite-dimensional
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manifolds on which the original PDE reduces to a finite system of ODEs. As such, the finite-dimensionality of the global dynamics is rigorously established.

These remarkable results for equations featuring advection by potential flows naturally lead to a search for an extension of the point transform to equations featuring advection by rotational flows. As we show here, no exists no simple point transform that maps Eq. [1] into a Schrödinger-like equation. In what follows, we propose the combination of a point transform with an appropriate Lie-transform in order to obtain a similar result. The approach is perturbative in the diffusivity and leads to a rather intricate system of nonlinear ‘balance’ equations that the sought-after transformation must satisfy. A complete understanding of this system requires methods of nonlinear functional analysis and this continuing work goes beyond the scope of the present paper. Our purpose here is simply to provide the kernel of the combined point-operator transform and illustrate its applicability in the context of a simple, canonical example.

Point and Lie Transforms

We begin this section by assuming that \( u \) is a given, not necessarily incompressible flow, and illustrate how one arrives at the point transform described in the introduction for irrotational flows, and why the same approach fails for flows with non-vanishing curl. For simplicity, let us assume that the velocity field does not depend on time while noting that all results can be extended to the time-dependent case. Consider a simple point transform of the form

\[
c = e^{\phi/\kappa} v .
\]

Substitution of (2) into (1) yields the following equation for \( v \):

\[
v_t + Bv = \frac{1}{\kappa}fv + \kappa \Delta v .
\]

The operator \( (1/\kappa)f + \kappa \Delta \) on the right hand side of the transformed equation is again symmetric, and the potential \( f \) given by

\[
f = \phi_x^2 + \phi_y^2 - 2u_1\phi_x - 2u_2\phi_y = |\nabla \phi|^2 - 2u \cdot \nabla \phi
\]

The new advection operator \( B \) is given by

\[
B = 2(u_1 - \phi_x)\partial_x + 2(u_2 - \phi_y)\partial_y - \Delta \phi ,
\]

and one can easily verify that \( B \) is also skew-symmetric. Now, we can see directly that, if \( u \) is a potential flow, we can choose \( \phi \) such that \( \nabla \phi = u \), and we obtain \( B = 0 \). Thus, we have transformed the advection-diffusion equation involving a skew-symmetric advection operator on the left hand side and a symmetric diffusion operator on the right hand side into a self-adjoint Schrödinger-type problem given by

\[
v_t = \kappa(\frac{1}{\kappa^2}f + \Delta)v .
\]
The above calculation shows also that, for a velocity field with non-vanishing vorticity, a simple point transform (2) cannot be sufficient. The fundamentally new idea of this work is to use a further transform, more precisely a Lie transform,

\[ v = e^{\kappa L} w, \quad L^+ = L \]  

and to choose the function \( \phi \) and the operator \( L \) in a way that, at least for small diffusivities \( \kappa \), the resulting problem becomes self-adjoint. One motivation for choosing a Lie transform is the fact that this allows us to systematically construct approximate solutions in the limit \( \kappa \to 0 \) as we can use the Baker-Campbell-Hausdorff formula [7] to find the following expansion up to the order \( O(\kappa^2) \):

\[
w_t + Bw + \kappa [B, L] w = \left( \frac{1}{\kappa} f + [f, L] + \frac{\kappa}{2} [[f, L], L] \right) w + \kappa \Delta w \quad (8)
\]

Note that, at order \( O(\kappa^2) \) and higher, a variety of terms appears, including commutators that involve not only \( f, B \) and \( L \), but also \( \Delta \) and \( L \). Our aim is to choose \( \phi \) and \( L \) in a manner that the equation for \( w \) involves only self-adjoint operators. Since \( B \) is skew-symmetric and \( L \) is symmetric, one can easily verify that \([B, L]\) is symmetric. On the other hand, \([f, L]\) will be skew-symmetric and \([[[f, L], L]]\) will be symmetric. Therefore, at leading order in the expansion using \( \kappa \) as small parameter, in order to balance the skew-symmetric operators in a manner that their influence on the evolution of \( w \) vanishes, we require

\[ B = [f, L]. \]  

(9)

Assuming that we were able to find a \( \phi \) and an operator \( L \) such that the balance (9) of gradients is satisfied, the resulting equation for \( w \) becomes

\[
w_t = \kappa Aw, \quad A = -\frac{1}{2}[B, L] + \Delta + \frac{1}{\kappa^2} f, \quad (10)
\]

where we have omitted higher order terms. Note that, in general, the commutator \([B, L]\) will not vanish and \( A \) will not be a Schrödinger operator. By construction, however, \([B, L]\) is symmetric and hence \( A \) will be symmetric. In order to satisfy symmetry requirements at higher orders in the expansion in \( \kappa \), it is also possible to expand \( \phi \) and \( L \) in an asymptotic series as

\[
\phi = \sum_k \kappa^k \phi_k, \quad L = \sum_k \kappa^k L_k
\]

in order to derive a hierarchy of equations for \( \phi_k \) and \( L_k \). In this paper, however, we will only consider expansions up to \( O(\kappa^2) \).

So far, the only requirement we have imposed on the operator \( L \) is its symmetry, \( L = L^+ \). Note that, at this stage, we have not imposed any conditions on the function \( \phi \), so that there is great freedom in the choice of \( L \) and \( \phi \). The assumption \( L = L^+ \) can also be modified, to e.g. \( L = -L^+ \), or dropped entirely by writing \( L = L_s + L_a \) where \( L_s \) denotes the symmetric and \( L_a \) the skew-symmetric part. This will change the condition for balancing the gradients in (9). In this paper, we are restricting our attention to the symmetric case.
As $B$ is a first-order differential operator and $f$ is a function, it is reasonable to search for an appropriate $L$ in the family of symmetric second-order differential operators. Symmetry requires $L$ to be of the form

$$L = c_{11} \partial_{xx} + 2c_{12} \partial_{xy} + c_{22} \partial_{yy} + b_1 \partial_x + b_2 \partial_y$$  \hfill (11)

with the conditions

$$b_1 = c_{11} x + c_{12} y, \quad b_2 = c_{12} x + c_{22} y$$  \hfill (12)

One major advantage of choosing a differential operator is that the commutators are relatively easy to calculate. From (9), the balance equations become

$$c_{11} f_x + c_{12} f_y = \phi_x - u_1$$  \hfill (13)

$$c_{12} f_y + c_{22} f_y = \phi_y - u_2.$$  \hfill (14)

Strictly speaking, from (9) we actually obtain a third equation given by

$$Lf = \Delta \phi.$$  \hfill (15)

An easy calculation, however, shows that if (13,14) are satisfied, (15) will be satisfied as well, and the search for the desired transformation reduces to solving the balance equations (13,14) in conjunction with the equation (11) for calculating the potential $f$ from the potential $\phi$. Note that solving (13,14,4) for a given velocity field $u$ means finding functions $\phi, c_{11}, c_{12}, c_{22}$. The balance equations might appear simple at first glance; however they are quite difficult since they are nonlinear due to the fact that $f$ depends on $\phi$ through (4). For purposes of this paper, we shall content ourself with providing an example in which the system has a solution which is easy to determine.

**Simple shear flow**

In this section, we will examine what the equations of the previous section mean for the simplest example of advection by an irrotational flow: simple shear flow. To summarize the previous section, we have shown that, if a solution of the balance equations (13,14) together with (4) can be found, the original advection-diffusion equation (1) can be transformed (up to terms of order $O(\kappa^2)$) into a selfadjoint problem given by (10). Let us consider a velocity field of the form

$$u = \begin{pmatrix} \alpha y \\ \beta x \end{pmatrix}.$$  \hfill (16)

We assume $\alpha \neq \beta$, so that $\nabla \times u \neq 0$ is satisfied. As $u$ is formed by terms that are linear in $x$ and $y$, we assume a quadratic form for $\phi$:

$$\phi = \frac{1}{2} ax^2 + c xy + \frac{1}{2} by^2$$  \hfill (17)

with constants $a, b, c$ to be determined later. Then the expression for $f$ yields

$$f = (a^2 + c^2 - 2\beta c)x^2 + (b^2 + c^2 - 2\alpha c)y^2 + 2(ac + bc - \alpha a - \beta b)xy.$$  \hfill (18)
Due to the particular structure of the velocity field, we can write
\[ \nabla f = F \begin{pmatrix} x \\ y \end{pmatrix}, \quad F = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix} \] (19)
with a symmetric matrix \( F \) whose coefficient can be directly read from the explicit formula for \( f \). The system of equations (13, 14) can now be written in matrix form as
\[ CF = M := \begin{pmatrix} a & c - \alpha \\ c - \beta & b \end{pmatrix} \] (20)
and thus we can express \( C \) in terms of \( \{a, b, c\} \) together with \( \{\alpha, \beta\} \). By solving this system explicitly, we find that the requirement \( \alpha \neq \beta \) yields the condition
\[ c^2 = ab. \] (21)
The explicit formula for \( C \) under this assumption is
\[ 2\gamma^2 C = \begin{pmatrix} \alpha^2 a + \alpha \beta b - \beta cb - \alpha bc & (\alpha + \beta)c^2 - 2\alpha \beta c \\ (\alpha + \beta)c^2 - 2\alpha \beta c & \beta^2 b + \alpha \beta a - \alpha \beta c \end{pmatrix} \] (22)
where we used the abbreviation \( \gamma = aa - \beta b \). For the explicit calculation of \( A \) governing the evolution of \( w \), we find that the coefficients of the commutator \([B, L]/2\) are given by the matrix \( 2MC \). Writing
\[ \Delta - \frac{1}{2}[B, L] = w_{11} \partial_{xx} + 2w_{12} \partial_x \partial_y + w_{22} \partial_{yy} \] (23)
we can easily find all the entries of the corresponding matrix \( W \). Direct calculation shows that the determinant of \( W \) is given by
\[ \det(W) = -\frac{\alpha^2 \beta^2}{\gamma^2} \] (24)
and hence the resulting operator \( W \) is either hyperbolic or parabolic, but not elliptic.

If one of the constants, \( \alpha \) or \( \beta \) vanishes, the velocity field reduces to a simple shear flow. Without loss of generality, let us consider
\[ u = (\alpha y, 0). \] (25)
For this case, we find the matrix \( c_{ik} \) as
\[ c_{11} = \frac{1}{2a} - \frac{cb}{2\alpha a^2}, \quad c_{12} = \frac{b}{2\alpha a}, \quad c_{22} = -\frac{c}{2\alpha a} \] (26)
and
\[ [B, L] = 2 \left( 1 - \frac{b}{a} \right) \partial_{xx} + 4\frac{c}{a} \partial_{xy}. \] (27)
Remember that the evolution of \( w \) is governed by the operator \( A \) defined in (10). For the first two terms in the definition of \( A \) we have found
\[ -\frac{1}{2}[B, L] + \Delta = \frac{b}{a} \partial_{xx} + \partial_{yy} - 2\frac{c}{a} \partial_x \partial_y \] (28)
In this case, it follows from (24) that the evolution of \( w \) is governed by a parabolic equation. The potential \( f \) is given by
\[ f = (a^2 + c^2)x^2 + 2(ac + cb - \alpha a)xy + (c^2 + b^2 - 2ac)y^2 \] (29)
Comparison of explicit solutions

It is instructive to see in detail how the transformation works for a particular initial condition, for which both the original equation (1) as well as the transformed equation can be solved explicitly via the Fourier transform on a finite time interval. This will allow us to compare the two solutions and verify the validity of our method. Let us look for the solution to the original advection-diffusion equation (1) for the shear flow with an initial condition given by

\[ c(t = 0, x, y) = \delta(x)\delta(y). \]  

(30)

Note that, in this case, the analytical solution for \( c(t, x, y) \) is well-known and can be obtained directly from (1) using Fourier transform and the method of characteristics. In order to obtain the same solution using the sequence of the proposed transforms, let us first set \((a, b, c) = (a, 0, 0)\), so that the condition (21) is trivially satisfied. In this case, we obtain for the function \( \phi \) and the operator \( L \) simply

\[ \phi = \frac{1}{2} ax^2, \quad L = \frac{1}{2a} \partial^2 \partial x^2 \]  

(31)

The transformation of the initial conditions is trivial as

\[ w(0, x, y) = e^{-\kappa L} v(0, x, y) = e^{-\kappa L} e^{-\phi/\kappa} \delta(x)\delta(y) \]  

(32)

In order to make this equation meaningful, however, it seems reasonable to assume \( a < 0 \). In this way, \( -\kappa L \) will be regularizing. In frequency domain, we obtain then for \( \hat{w}(0, \omega, \eta) \) obviously

\[ \hat{w}(0, \omega, \eta) = e^{\kappa \omega^2/(2a)} \]  

(33)

which decays, if \( a < 0 \). The next step consists in solving the equation for \( w \) that simplifies to

\[ w_t = \left( \kappa \partial_{yy} + \frac{1}{\kappa} (a^2 x^2 - 2\alpha axy) \right) w \]  

(34)

Using a Fourier transform only in \( y \) and characteristics, we find the explicit solution of this equation as

\[ \tilde{w}(t, x, \eta) = e^{-\kappa \gamma^2 t + a^2 x^2 \kappa + 2a\alpha x \eta^2 + 4a^2 x^2 t^3/(3\kappa)} \tilde{w}(0, x, \eta - 2\alpha x t / \kappa) \]  

(35)

Setting \( \beta = -\kappa/(2a) > 0 \), we can write the initial condition for \( \tilde{w} \) as

\[ \tilde{w}(0, x, \eta) = \frac{1}{2\sqrt{\pi} \beta} e^{-x^2/(4\beta)} \]  

(36)

and obtain the explicit formula for the Fourier transform of the solution, \( \hat{w} = \hat{w}(t, \omega, \eta) \), as

\[ \hat{w}(t, \omega, \eta) = \frac{1}{2\sqrt{\beta\gamma}} e^{-\left(\omega - 2\alpha \eta \kappa^2\right)^2/(4\gamma)} e^{-\kappa \eta^2 t} \]  

(37)

with the abbreviation

\[ \gamma = \frac{-a}{2\kappa} \left( 1 + 2at \left( 1 + \frac{2}{3} \alpha^2 t^2 \right) \right). \]  

(38)
Note here that the Fourier transform method we employ here is valid only as long as \( \gamma > 0 \), i.e., on a finite time interval. After that, the Fourier transform ceases to exist, and the method leading to the explicit solution breaks down. We can interpret this as a loss of regularity in the frequency domain in this particular step of the method, which will actually be gained back in the subsequent step of the transformation. Moreover, \( \gamma > 0 \) is only possible given our initial assumption of \( a < 0 \). We now transform back from \( w \) to \( v \) in Fourier space via

\[
\hat{v}(t, \omega, \eta) = \hat{w}(t, \omega, \eta) e^{-\kappa \omega^2 / (2a)}
\]

In order to obtain the solution \( c(t, x, y) \), we only need to transform the solution (39) back to \((x, y)\)-space and multiply by \( \exp(ax^2/(2\kappa)) \). It is precisely in this multiplication, where we get back the necessary regularity (remember \( a < 0 \)). Moreover, using the explicit formulas for Gaussian integrals, one can easily check that, at the end of the calculation, the parameter \( a \) drops out, as it should since \( a \) was only introduced in the transformation and not present in the original advection-diffusion equation.

For the simple example of spatially homogeneous vorticity, \( L \) and \([B, L]\) are differential operators with constant coefficients and all possible higher order terms in (8) vanish. Therefore, in this particular case, the transformation is exact and holds for any value of \( \kappa \).

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References

[1] A. J. Majda and P. R. Kramer. Simplified models for turbulent diffusion: Theory, numerical modelling and physical phenomena. Physics Reports, 314, 1999.
[2] H. Risken. The Fokker-Planck Equation. Springer Verlag, Berlin, 1984.
[3] M. M. Millonas and L. E. Reichl. Stochastic chaos in a class of fokker-planck equations. Phys. Rev. Lett., 68(21), 1992.
[4] S. Chow, K. Lu, and J. Mallet-Paret. Floquet theory for parabolic differential equations. J. Diff. Equations, 109(1), 1994.
[5] J. Vukadinovic. Finite-dimensional description of the long-term dynamics for the 2d doi-hess model for liquid crystalline polymers in a shear flow. Commun. Math. Sci., 6 (4), 2008.
[6] J. Vukadinovic. Inertial manifolds for a smoluchowski equation on the unit sphere. Commun. Math. Phys., 285, 2009.
[7] T. Schäfer, A. C. Poje, and J. Vukadinovic. Averaged dynamics of time-periodic advection diffusion equations in the limit of small diffusivity. Physica D, 238, 2009.