INVOLUTIONS OF RANK 2 HIGGS BUNDLE MODULI SPACES

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To Nigel Hitchin on the occasion of his 70th birthday

Abstract. We consider the moduli space $\mathcal{H}(2, \delta)$ of rank 2 Higgs bundles with fixed determinant $\delta$ over a smooth projective curve $X$ of genus 2 over $\mathbb{C}$, and study involutions defined by tensoring the vector bundle with an element $\alpha$ of order 2 in the Jacobian of the curve, combined with multiplication of the Higgs field by $\pm 1$. We describe the fixed points of these involutions in terms of the Prym variety of the covering of $X$ defined by $\alpha$, and give an interpretation in terms of the moduli space of representations of the fundamental group.

1. Introduction

Let $X$ be a smooth projective curve of genus $g \geq 2$ over $\mathbb{C}$. A Higgs bundle $(E, \varphi)$ on $X$ consists of a vector bundle $E$ and a twisted endomorphism $\varphi : E \to E \otimes K$, where $K$ is the canonical bundle of $X$. The slope of $E$ is the rational number defined as

$$\mu(E) = \deg E / \text{rank } E.$$ 

A Higgs bundle is said to be stable (resp. semistable) if

$$\mu(F) < (\text{resp.} \leq) \mu(E)$$

for every proper subbundle $F$ of $E$ invariant under $\varphi$ in the sense that $\varphi(F) \subset F \otimes K$. Also, a Higgs bundle $(E, \varphi)$ is polystable if $(E, \varphi) = \bigoplus_i (E_i, \varphi_i)$ where all the $(E_i, \varphi_i)$ are stable and all $E_i$ have the same slope as that of $E$.

Let $\delta$ be a line bundle on $X$. We are interested in the moduli space $\mathcal{H}(n, \delta)$ of isomorphism classes of polystable Higgs bundles $(E, \varphi)$ of rank $n$ with determinant $\delta$ and traceless $\varphi$. This moduli space was constructed analytically by Hitchin [6] and later algebraically via geometric invariant theory by Nitsure [11]. This space is a normal quasi-projective variety of dimension $2(n^2 - 1)(g - 1)$. If the degree of $\delta$ and $n$ are coprime, $\mathcal{H}(n, \delta)$ is smooth.

Let $M(n, \delta)$ be the moduli space of polystable vector bundles of rank $n$ and determinant $\delta$. The set of points corresponding to stable bundles form a smooth open set and the cotangent bundle of it is a smooth, open, dense subvariety of $\mathcal{H}(n, \delta)$.

In this paper, we focus on vector bundles and Higgs bundles of rank 2, leaving the study of those of of higher rank (and indeed of $G$-principal bundles with $G$ reductive) for [11]. There are two kinds of involutions that we consider. Firstly the subgroup $J_2$ of elements of the Jacobian $J$ consisting of elements of order 2 acts on $\mathcal{H}(2, \delta)$ by tensor product.

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We also consider the involutions where in addition, the sign of the Higgs field is changed. More explicitly, for \( \alpha \in J_2 \) we consider the involutions

\[
\iota(\alpha)_{\pm} : \mathcal{H}(2, \delta) \to \mathcal{H}(2, \delta)
\]

\[
(E, \varphi) \mapsto (E \otimes \alpha, \pm \varphi).
\]

We determine the fixed point varieties in all these cases, and their corresponding sub-varieties of the moduli space of representations of the fundamental group of \( X \) (and its universal central extension) under the correspondence between this moduli space and the moduli space of Higgs bundles, established by Hitchin [6] and Donaldson [3]. The case of the involution \((E, \varphi) \mapsto (E, -\varphi)\) is already covered in the beautiful paper of Hitchin [6].

2. Line bundles

To start with, we consider involutions in the case of line bundles. The moduli space of line bundles of degree \( d \) is the Jacobian variety \( J^d \). There is a universal line bundle (called a Poincaré bundle) on \( J^d \times X \) which is unique up to tensoring by a line bundle pulled back from \( J^d \). We will denote \( J^0 \) simply by \( J \).

The involution \( \iota : L \to L^{-1} \) of \( J \) has obviously the finite set \( J_2 \) of elements of order 2, as its fixed point variety.

The Higgs moduli space of line bundles consists of pairs \((L, \varphi)\) where \( L \) is a line bundle of fixed degree and \( \varphi \) is a section of \( K \). The moduli space of rank 1 Higgs bundles of degree \( d \) is thus isomorphic to \( J^d \times H^0(X, K) \). There are a few involutions to consider even in this case. Firstly on the Higgs moduli space of line bundles of degree \( d \), one may consider the involution \((L, \varphi) \mapsto (L, -\varphi)\). The fixed point variety is just \( J^d \) imbedded in the Higgs moduli space by the map \( L \mapsto (L, 0) \) since any automorphism of \( L \) induces identity on the set of Higgs fields on \( L \).

When \( d = 0 \), one may also consider the involution \((L, \varphi) \mapsto (L^{-1}, \varphi)\). This has as fixed points the set \( \{(L, \varphi) : L \in J_2 \text{ and } \varphi \in H^0(X, K)\} \). Also, we may consider the composite of the two actions, namely \((L, \varphi) \mapsto (L^{-1}, -\varphi)\). Again it is obvious that the fixed points are just points of \( J_2 \) with Higgs fields 0.

Finally, translations by elements of \( J_2 \setminus \{0\} \) are involutions without fixed points.

3. Fixed Points of \( \iota(\alpha)_{-} \)

We wish now to look at involutions of \( M = M(2, \delta) \) and \( \mathcal{H} = \mathcal{H}(2, \delta) \). We will often assume that \( \delta \) is either \( \mathcal{O} \) or a line bundle of degree 1. There is no loss of generality, since the varieties \( M \) and \( \mathcal{H} \) for any \( \delta \) are isomorphic (on tensoring with a suitable line bundle) to ones with \( \delta \) as above. In general, we denote by \( \delta \) the degree of \( \delta \).

If \( d \) is odd, the spaces \( M \) and \( \mathcal{H} \) are smooth and the points correspond to stable bundles and stable Higgs bundles, respectively. If \( d \) is even (and \( \delta \) trivial), there is a natural morphism \( J \to M \) which takes \( L \) to \( L \oplus L^{-1} \) and imbeds the quotient of \( J \) by the involution \( \iota \) on \( J \), namely the Kummer variety, in \( M \). This is the non-stable locus (which is also the singular locus if \( g > 2 \)) of \( M \) and has \( J_2 \) as its own singular locus.
Remark 3.1. If \((E, \varphi) \in \mathcal{H}\), but \(E\) is not semi-stable, then there is a line sub-bundle \(L\) of \(E\) which is of degree \(> d/2\). Moreover, it is the unique sub-bundle with degree \(\geq d/2\). Clearly, since \((E, \varphi)\) is semi-stable, \(\varphi\) does not leave \(L\) invariant. Hence \((E, \varphi)\) is actually a stable Higgs bundle. In particular, it is a smooth point of \(\mathcal{H}\).

Before we take up the study of the involutions \(\mathcal{H}\) in general, we note that even when \(\alpha\) is trivial, the involution \(i^- := i(O)^-\) is non-trivial and is of interest. In this case, the fixed point varieties were determined by Hitchin [6] and we recall the results with some additions and clarifications.

**Proposition 3.2.** Polystable Higgs bundles \((E, \varphi)\) fixed by the involution \(i^- : (E, \varphi) \mapsto (E, -\varphi)\) fall under the following types:

(i) \(E \in M = M(2, \delta)\) and \(\varphi = 0\).

(ii) For every integer \(a\) satisfying \(0 < 2a - d \leq 2g - 2\), consider the set \(T_a\) of triples \((L, \beta, \gamma)\) consisting of a line bundle \(L\) of degree \(a\) and homomorphisms \(\beta : L^{-1} \otimes \delta \to L \otimes K\), with \(\gamma \neq 0\) and \(\gamma : L \to L^{-1} \otimes \delta \otimes K\).

(iii) Same as in (ii), but with \(2a = d\) if \(d\) is even.

To every triple as in (ii) or (iii), associate the Higgs bundle \((E, \varphi)\) where

\[
E = L \oplus (L^{-1} \otimes \delta) \quad \text{and} \quad \varphi = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}.
\]

Any type (ii) Higgs bundle \((E, \varphi)\) is stable whereas \(E\) is not even semi-stable.

In type (iii) if \(L^2\) is not isomorphic to \(\delta\), and \(\beta\) and \(\gamma\) are both non-zero, then \((E, \varphi)\) is stable. If \(L^2 \cong \delta\) and \(\beta\) and \(\gamma\) (both of which are then sections of \(K\)) are linearly independent, then \((E, \varphi)\) is stable.

**Proof.** Firstly, if \(E \in M\) and \(\varphi = 0\), it is obvious that it is fixed under the above involution. On the other hand, it is clear that if \((E, \varphi)\) is of type (ii) or (iii), then the automorphism of \(E\)

\[
\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},
\]

takes \(\varphi\) to \(-\varphi\). In type (ii), since \(2a - d > 0\), it follows that \(L\) is the only line sub-bundle of \(E\) of degree \(\geq d/2\). Since \((E, \varphi)\) is semi-stable, \(L\) is not invariant under \(\varphi\), (which is the case if and only if \(\gamma\) is non-zero). Therefore, \((E, \varphi)\) is stable.

Type (iii) is relevant only when \(d\) is even and so we will assume that \(\delta\) is trivial. If \(L^2\) is not trivial, then every line subbundle of \(E\) of degree 0 is either \(L\) or \(L^{-1}\). Since we have assumed that \((E, \varphi)\) is poly-stable, either \(\varphi\) leaves both \(L\) and \(L^{-1}\) invariant or neither, i.e. \(\beta\) and \(\gamma\) are both zero or both non-zero. The former case is covered under type i) and in the latter case, \((E, \varphi)\) is stable. Finally, if \(L^2\) is trivial, then every line sub-bundle of degree 0 is isomorphic to \(L\), and all imbeddings of \(L\) in \(E = L \oplus L\) are given by \(v \mapsto (\lambda v, \mu v)\), with \((\lambda, \mu) \neq 0\). The restriction of \(\varphi\) to \(L\) composed with the projection of \(E \otimes K\) to \((E/L) \otimes K = (L \otimes K)\), is given by \(\lambda \gamma + \mu \beta\). Hence this imbedding
of $L$ is invariant under $\varphi$ if and only if $\lambda \gamma + \mu \beta = 0$, proving that if $\beta$ and $\gamma$ are linearly independent, then $(E, \varphi)$ is stable. Otherwise, $(L, 0)$ is a (Higgs) subbundle of $(E, \varphi)$ and hence it is covered again in i).

Conversely, let $(E, \varphi)$ be a stable Higgs bundle fixed by the involution. Then there exists an automorphism $f$ of $E$ (of determinant 1) which takes $\varphi$ to $-\varphi$. If $E$ is a stable vector bundle, all its automorphisms are scalar multiplications which take $\varphi$ into itself. Hence $\varphi = 0$ in this case. Let $E$ be nonstable. Obviously, then $\varphi$ is non-zero. Since $f^2$ is an automorphism of the stable Higgs bundle $(E, \varphi)$, we have $f^2 = \pm \Id_E$. This implies that $f_x$ is semi-simple for all $x \in X$. If $f^2 = \Id_E$, the eigenvalues of $f_x$ are $\pm 1$ and since $\det(f_x) = 1$ we have $f = \pm \Id_E$ which would actually leave $\varphi$ invariant. So $f_x$ has $\pm i$ as eigenvalues at all points. We conclude that $E$ is a direct sum of line bundles corresponding to the eigenvalues $\pm i$. Thus $f^2 = -\Id_E$ and $E = L \oplus (L^{-1} \otimes \delta)$ with $f|_L = \Id_E$, and $f|(L^{-1} \otimes \delta) = -i\Id_E$. We may assume that $\deg L = a \geq d/2$, replacing $L$ by $L^{-1} \otimes \delta$ (and $f$ by $-f$) if necessary. If $a > d/2$, it also follows that the composite of $\varphi|_L$ and the projection $E \otimes K \to L^{-1} \otimes \delta \otimes K$ is nonzero (since $(E, \varphi)$ is semi-stable) which implies that $a \leq -a + d + 2g - 2$, i.e. $2a - d \leq 2g - 2$. Moreover, from the fact that $f$ takes $\varphi$ to $-\varphi$, one deduces that $\varphi$ is of the form claimed.

If $(E, \varphi)$ is not stable, in which case we may assume $\delta$ is trivial, $(E, \varphi)$ is a direct sum of $(L, \psi)$ and $(L^{-1}, -\psi)$ with $\deg L = 0$. If $\psi$ is nonzero, then $(E, \varphi)$ is isomorphic to $(E, -\varphi)$ if and only if $L \cong L^{-1}$. If then $L \cong L^{-1}$ we may take $g = 1/\sqrt{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ and change the decomposition of $E$ to $g(L) \oplus g(L)$ and see that $(E, \varphi)$ falls under type (iii).

3.1. The set of triples. The above proposition leads us to consider the set of triples as in type (ii) and type (iii) above with $d \leq 2a \leq d + 2g - 2$. Set $m = 2a - d$. To such a triple, we have associated the Higgs bundle $(E, \varphi)$ given by $E = L \oplus (L^{-1} \otimes \delta)$ and $\varphi$ by the matrix in (3.1).

Notice however that this triple and the triple $(L, \lambda^{-1} \beta, \lambda \gamma)$ give rise to isomorphic Higgs bundles. So we consider the set of triples $(L, \beta, \gamma)$ as above, make $\mathbb{C}^*$ act on it, in which $\lambda \in \mathbb{C}^*$ takes $(L, \beta, \gamma)$ to $(L, \lambda^{-1} \beta, \lambda \gamma)$ and pass to the quotient. We have thus given an injective map of this quotient into the $\iota^*$-fixed subvariety of Higgs bundles.

We will equip this quotient with the structure of a variety.

3.2. Construction of the space of triples. Take any line bundle $\mathcal{L}$ on $T \times X$, where $T$ is any parameter variety. For any $t \in T$, denote by $\mathcal{L}_t$ the line bundle $\mathcal{L}|_{t \times X}$. Assume that $\deg(\mathcal{L}_t) = r$ for all $t \in T$. Then we get a (classifying) morphism $c_{\mathcal{L}} : T \to J^r$ mapping $t$ to the isomorphism class of $\mathcal{L}_t$.

There is a natural morphism of $S = S^r(X) \to J^r$ since $S \times X$ has a universal divisor giving rise to a family of line bundles on $X$ of degree $r$, parametrised by $S$. The pull back of any Poincaré bundle on $J^r \times X$ to $S \times X$ is the tensor product of the line bundle given by the universal divisor on $S \times X$ and a line bundle pulled back from $S$. The composite of the projection of this line bundle $U$ to $S$ and the morphism $S \to J$ blows down the zero section of the line bundle to $Z$ and yields actually an affine morphism and the fibre over any $L \in J^r$ can be identified with $H^0(X, L)$, coming up with a section $Z$ of this
affine morphism. Notice that if $r > 2g - 2$, this is actually a vector bundle over $J^r$ of rank $r + 1 - g$ and $Z$ is its zero section.

If $L'$ is a family of line bundles of degree $r$ on $X$, parametrised by $T$ as above the pull back of the morphism $U \to J^r$ by $c_E : T \to J^r$ will be denoted $A(L)$.

If $m > 0$, let $V$ be the pull back by the map $J^a \to J^{2g-2+m}$ given by $L \to K \otimes L^2 \otimes \delta^{-1}$ of the above vector bundle. On the other hand the map $L \to K \otimes L^2 \otimes \delta$ of $J^a \to J^{2g-2-m}$ pulls back the symmetric product $S^{2g-2-m}$ and gives a $2^{2g}$-sheeted étale covering. The inverse image of $V$ tensored with a line bundle on $S^{2g-2-m}$ thus gives the required structure on the quotient of $T_a$ by $\mathbb{C}^*$.

**Proposition 3.3.** For each $m$ with $0 < m < 2g - 2$, consider the pull-back of the map $S^{2g-2-m} \to J^{2g-2-m}$ by the map $L \to K \otimes L^2 \otimes \delta$. A vector bundle over this of rank $g - 1 + m$ is isomorphic to a subvariety of Higgs bundles which are all fixed by $\nu^a$.

We have seen that $M$ imbedded in $H$ by $E$ to $(E, 0)$ is a fixed point variety. It is of course closed and in fact, compact as well.

The set of type (ii) fixed points is the disjoint union of $T_a$ with $d/2 < a < g - 1 + d$ and (disjoint from $M$ as well). Each of these gives an injective morphism of a vector bundle on a $2^d$-sheeted étale covering of $S^a$ into the fixed point subvariety. Since the subvariety of $H$ corresponding to nonstable vector bundles is smooth and closed, this morphism is an isomorphism onto the image.

We need to describe the image of the subvariety $T_a$ when $a = d/2$. We will assume $d = 0$ and $\delta$ is trivial. Consider the natural map of $S^{2g-2}$ onto $J^{2g-2}$. Pull it back to $J$ by the two maps $L \to K \otimes L^2$ and $L \to K \otimes L^{-2}$. Take their fibre product and the quotient by the involution which changes the two factors. There is a natural map of this quotient into $PH^0(K^2)$. Pull back the line bundle $O(1)$ on $PH^0(K^2)$ to this. It is easy to check that this is irreducible and closed.

There are other irreducible components of type (iii) in the case of $g = 2$. Take any line bundle $L$ of order 2 and consider

\[
\begin{pmatrix}
0 & \beta \\
\gamma & 0
\end{pmatrix}
\]

as a Higgs field on $L \oplus L$. Consider the tensor product map $\beta \otimes \gamma$ into $H^0(K^2)$. This is surjective and can be identified with the quotient by $\mathbb{C}^*$ and $\mathbb{Z}/2$ of the fixed point set given by $(L, \beta, \gamma)$ with $L \in J_2$.

### 3.3. An Alternative point of view.

Note that both in Type (ii) and Type (iii) we have a natural morphism of these components into $H^0(X, K^2)$ given by $(\beta, \gamma) \mapsto -\beta \gamma$. Clearly this is the restriction of the Hitchin map. Given a (non-zero) section of $H^0(K^2)$ we can partition its divisor into two sets of cardinality $2g - 2 - m$ and $2g - 2 + m$. They yield elements of $J^{2g-2-m}$ and $J^{2g-2+m}$ together with non-zero sections $\beta$ and $\gamma$ which are defined up to the action of $\mathbb{C}^*$ as we have defined above. Passing to a $2^2g$-sheeted étale covering we get the required set. In particular it follows that except in case i) when the Hitchin map is 0, in all other cases, the Hitchin map is finite and surjective.
Proposition 4.1. Let $E$ be a vector bundle of rank 2 on $X$, and let $\alpha$ be a non-trivial line bundle of order 2 such that $(E \otimes \alpha) \cong E$. Then $E$ is polystable. Moreover if $E$ is not stable it is of the form $L \oplus (L \otimes \alpha)$ with $L^2 \cong \alpha$.

Proof. Assume that $(E \otimes \alpha) \cong E$. If $E$ is not poly-stable, then it has a unique line subbundle $L$ of maximal degree. This implies that $(L \otimes \alpha) \cong L$ which is absurd. If $E$ is of the form $L \oplus M$, then under our assumption, it follows that $M \cong L \otimes \alpha$. \hfill \Box

4. Prym varieties and rank 2 bundles

Let now $\alpha \in J_2 \setminus \{0\}$. To start with, we will determine the fixed points of the involution defined on $M$ defined by tensoring by $\alpha$.

Proposition 4.2. For any line bundle $L$ on $X_\alpha$, the direct image $E = \pi_* L$ is a polystable vector bundle of rank 2 on $X$ such that $E \otimes \alpha \cong E$. If $E$ is not stable, it is of the form $\xi \oplus (\xi \otimes \alpha)$.

Proof. Indeed, if $\xi$ is any line subbundle of $E$, its inclusion in $E$ gives rise to a nonzero homomorphism $\pi^* \xi \to L$, and hence $2\deg \xi = \deg(\pi^* \xi) \leq \deg(L) = \deg(E)$, proving $E$ is semi-stable. If $\deg \xi = \deg E/2$, the homomorphism $\pi^* \xi \to L$ is an isomorphism. But then $\pi_* \xi = \pi_*(\pi^* \xi) = \xi \otimes \pi_* O = \xi \otimes (O \oplus \alpha)$ proving our assertion. \hfill \Box

We have thus a morphism of $\text{Nm}^{-1}(\delta \otimes \alpha)$ into $M(2, \delta)$ which maps $L$ to $\pi_* L$. Let $E$ be stable such that $E \otimes \alpha \cong E$. we may then choose an isomorphism $f : E \to E \otimes \alpha$ such that its iterate $(f \otimes \text{Id}_\alpha) \circ f : E \to E$ is the identity. Indeed this composite is an automorphism of $E$ and hence a non-zero scalar. We can then replace the isomorphism by a scalar multiple so that this composite is $\text{Id}_E$. Now the locally free sheaf $\mathcal{E}$ can be provided a module structure over $O \oplus \alpha$ by using the above isomorphism. This means that it is the direct image of an invertible sheaf on $X_\alpha$. On the other hand, if $E$ is poly-stable but not stable, it is isomorphic to $L \oplus M$. If $E \otimes \alpha$ is isomorphic to $E$, it follows that $L \cong M \otimes \alpha$. Hence we deduce that the above morphism $\text{Nm}^{-1}(\delta \otimes \alpha) \to M(2, \delta)$ is onto the fixed point variety under the action of tensoring by $\alpha$ on $M(2, \delta)$. If $\pi_* L \cong \pi_* L'$, then by applying $\pi^*$ to it, we see that $L'$ is isomorphic either to $L$ or $f^* L$. In other words, the above map descends to an isomorphism of the quotient of $\text{Nm}^{-1}(\delta \otimes \alpha)$ by the Galois involution onto the $\alpha$-fixed subvariety of $M(2, \delta)$. Since the fibres of $\text{Nm}$ are
interchanged by the Galois involution when \( \delta \) is of odd degree, this fixed point variety is isomorphic to a coset of the Prym variety. When \( \delta \) is of even degree, the \( \alpha \)-fixed variety has two connected components, each isomorphic to the quotient of the Prym variety by the involution \( L \to L^{-1} \), that is to say to the Kummer variety of Prym. We collect these facts in the following.

**Theorem 4.3.** Let \( \alpha \) be a non-trivial element of \( J_2(X) \). It acts on \( M(2, \delta) \) by tensor product: \( \iota(\alpha)(E) := E \otimes \alpha \). The fixed point variety \( F_\alpha(\delta) \) is isomorphic to the Prym variety of the covering \( \pi : X_\alpha \to X \) given by \( \alpha \) if \( d = \deg \delta \) is odd, and is isomorphic to the union of two irreducible components, each isomorphic to the Kummer variety of the Prym variety, if \( d \) is even.

**Remarks 4.4.** (1) If \( L \) is a line bundle on \( X_\alpha \) and \( E = \pi_\ast L \), then since \( E \otimes \alpha \cong E \), \( \alpha \) is a line sub-bundle of \( \text{ad}(E) \). Indeed, since \( E \) is poly-stable, \( \alpha \) is actually a direct summand. To see this, interpret \( \text{ad}(E) \) as \( S^2(E) \otimes \det(E)^{-1} \) and notice that there is a natural surjection of \( S^2(\pi_\ast L) \) onto \( \pi_\ast(L^2) \). It follows that \( \pi_\ast(L^2) \det(E)^{-1} \) is contained in \( \text{ad}(E) \). Thus we see that

\[
\text{ad}(\pi_\ast L) \cong \alpha \oplus ((\pi_\ast L^2) \otimes \alpha \otimes \text{Nm}(L^{-1})).
\]

(2) As we have seen above, in the case \( \delta \) is trivial, the fixed point variety intersects the non-stable locus, namely the Kummer variety of the Jacobian at bundles of the form \( \xi \oplus (\xi \otimes \alpha) \), where \( \xi \) is a line bundle with \( \xi^2 \cong \alpha \). Clearly, \( \xi \) and \( \xi \otimes \alpha \) give the same bundle. Thus the intersection of the two copies of the Prym Kummer variety (corresponding to any non-trivial \( \alpha \in J_2 \)) with the Jacobian–Kummer variety is an orbit of smooth points, under the action of \( J_2 \). This geometric fact can be stated in the context of principally polarised abelian varieties and is conjectured to be characteristic of Jacobians. Analytically expressed, this is the Schottky equation.

5. Fixed Points of \( \iota(\alpha)^\pm \) when \( d \) is odd.

If \( (E, \varphi) \) is a polystable Higgs bundle fixed under either of the involutions \( \iota(\alpha)^\pm \), we observe that \( E \) is isomorphic to \( E \otimes \alpha \). This implies that \( E \) is itself polystable. Hence if \( d \) is odd, we have only to consider the action of \( \alpha \) on \( M = M(2, \delta) \), given by \( E \mapsto E \otimes \alpha \), and look at its action on the cotangent bundle. Let \( F_\alpha \) be the fixed point variety in \( M \) under the action of \( \alpha \) (see Theorem 4.3), we have the exact sequence

\[
0 \to N(F_\alpha, M)^* \to T^*(M)|_{F_\alpha} \to T^*(F_\alpha) \to 0,
\]

where \( N(F_\alpha, M) \) is the normal bundle of \( F_\alpha \) in \( M \). This sequence splits canonically since \( \alpha \) acts on the restriction of the tangent bundle of \( M \) to \( F_\alpha \) and splits it into eigen-bundles corresponding to the eigen-values \( \pm 1 \). Clearly the subbundle corresponding to the eigenvalue \( +1 \) (resp. \(-1\)) is \( T(F_\alpha) \) (resp. \( N(F_\alpha, M) \)). Since \( E \otimes \alpha \cong E \) and \( d \) is odd, \( E \) is stable, and we have the following.

**Theorem 5.1.** If \( \deg \delta \) is odd the fixed point subvariety \( F^+_\alpha \) (resp. \( F^-_\alpha \)) of the action of \( \iota(\alpha)^+ \) (resp. \( \iota(\alpha)^- \)) on \( H(2, \delta) \) is the cotangent bundle \( T^*(F_\alpha) \) of \( F_\alpha \) (resp. the conormal bundle \( N(F_\alpha, M)^* \) of \( F_\alpha \)).
6. Fixed points of $\iota(\alpha)^\pm$ when $d$ is even.

We may assume that the determinant is trivial in this case. If $(E, \varphi)$ is fixed by either of the involutions $\iota(\alpha)^\pm$, with $E$ stable, the above discussion is still valid so that we have

(i) The sub-variety of fixed points of $\iota(\alpha)^+$ is $T^*(F_{\alpha}^{\text{stable}})$.

(ii) The sub-variety of fixed points of $\iota(\alpha)^-$ is $N^*(F_{\alpha}^{\text{stable}}, M)$.

Assume then that $(E, \varphi)$ is a fixed point of $\iota(\alpha)^\pm$, where $E$ is polystable of the form $L \oplus L^{-1}$. We have $L^{-1} \cong L \otimes \alpha$ and $\varphi$ is of the form

\begin{equation}
\varphi = \begin{pmatrix} \omega & \beta \\
\gamma & -\omega \end{pmatrix},
\end{equation}

with $\beta, \gamma \in H^0(K \otimes \alpha)$ and $\omega \in H^0(K)$. Since the summands of $E$ are distinct, any isomorphism $f : E \otimes \alpha \rightarrow E$ has to be of the form

\begin{equation}
\begin{pmatrix} 0 & \lambda \\
-\lambda^{-1} & 0 \end{pmatrix},
\end{equation}

with $\lambda \in \mathbb{C}^*$. Also, $f$ takes $\varphi$ to $\pm \varphi$ if and only if

\begin{equation}
\begin{pmatrix} 0 & \lambda \\
-\lambda^{-1} & 0 \end{pmatrix} \begin{pmatrix} \omega & \beta \\
\gamma & -\omega \end{pmatrix} \begin{pmatrix} 0 & -\lambda \\
\lambda^{-1} & 0 \end{pmatrix} = \pm \begin{pmatrix} \omega & \beta \\
\gamma & -\omega \end{pmatrix}.
\end{equation}

In other words,

\begin{equation}
\begin{pmatrix} -\omega & \lambda^{-2} \gamma \\
\lambda^2 \beta & \omega \end{pmatrix} = \pm \begin{pmatrix} \omega & \beta \\
\gamma & -\omega \end{pmatrix}.
\end{equation}

We analyse the cases $\iota(\alpha)^+$ and $\iota(\alpha)^-$ separately.

6.1. Fixed points of $\iota(\alpha)^+$. In the case of $\iota(\alpha)^+$, (6.2) implies that $\omega = 0$ and $\lambda^2 \beta = \gamma$. If $\beta$ or $\gamma$ is 0, the Higgs bundle is $S$-equivalent to $(L \oplus (L \otimes \alpha), 0)$. Hence this fixed point variety is isomorphic to the product of $J/\alpha$ and the space of decomposable tensors in $H^0(K) \otimes H^0(K)$.

Remark 6.1. Since $E$ is of the form $\pi_*(L)$, we conclude that the tangent space at $E$ to $M$ (assuming that $E$ is stable) is

$$H^1(\text{ad}(E)) = H^1(\alpha) \oplus H^1(\pi_*(L^2) \otimes \alpha).$$

It is clear that the first summand here is the tangent space to the Prym variety while the second is the space normal to Prym in $M$. 
6.2. Fixed points of \( \iota(\alpha) \). It is clear that

\[
(6.3) \quad \begin{pmatrix} \omega & \beta \\ \gamma & -\omega \end{pmatrix}
\]

is taken to its negative under the action of

\[
(6.4) \quad \begin{pmatrix} 0 & \lambda \\ -\lambda^{-1} & 0 \end{pmatrix}
\]

if and only if \( \beta \) and \( \gamma \) are multiples of each other, in which case we may as well assume that \( \beta = \gamma \). In other words, \( E \) belongs to the Prym variety and \( \varphi \) belongs to \( H^0(K) \oplus H^0(K \otimes \alpha) \).

7. Higgs bundles and representations of the fundamental group

Let \( G \) be a reductive Lie group, and let \( \pi_1(X) \) be the fundamental group of \( X \). A representation \( \rho : \pi_1(X) \rightarrow G \) is said to be reductive if the composition of \( \rho \) with the adjoint representation of \( G \) in its Lie algebra is completely reducible. When \( G \) is algebraic, this is equivalent to the Zariski closure of the image of \( \rho \) being a reductive group. If \( G \) is compact or abelian every representation is reductive. We thus define the moduli space of representations of \( \pi_1(X) \) in \( G \) to be the orbit space

\[
\mathcal{R}(G) = \text{Hom}^{red}(\pi_1(X), G)/G
\]

of reductive representations. With the quotient topology, \( \mathcal{R}(G) \) has the structure of an algebraic variety.

In this section we briefly review the relation between rank 1 and rank 2 Higgs bundles, and representations of the fundamental group of the surface and its universal central extension in \( \mathbb{C}^*, \mathbb{U}(1), \mathbb{R}^*, \text{SL}(2, \mathbb{C}), \text{SU}(2) \) and \( \text{SL}(2, \mathbb{R}) \). For more details, see [6, 3, 2, 9].

7.1. Rank 1 Higgs bundles and representations. As is well-known \( \mathcal{R}(\mathbb{U}(1)) \) is in bijective correspondence with the space \( J \) of isomorphism classes of line bundles of degree 0. Also, if we identify \( \mathbb{Z}/2 \) with the subgroup \( \pm 1 \) in \( \mathbb{U}(1) \) we get a bijection of \( \mathcal{R}(\mathbb{Z}/2) \) with the set \( J_2 \) of line bundles of order 2.

By Hodge theory one shows that \( \mathcal{R}(\mathbb{C}^*) \) is in bijection with \( T^*J \cong J \times H^0(X, K) \), the moduli space of Higgs bundles or rank 1 and degree 0. The subvariety of fixed points of the involution \( (L, \varphi) \rightarrow (L^{-1}, \varphi) \) in this moduli space is \( J_2 \times H^0(X, K) \) and corresponds to the subvariety \( \mathcal{R}(\mathbb{R}^*) \subset \mathcal{R}(\mathbb{C}^*) \).

7.2. Rank 2 Higgs bundles and representations. The notion of stability of a Higgs bundle \( (E, \varphi) \) emerges as a condition for the existence of a Hermitian metric on \( E \) satisfying the Hitchin equations. More precisely, Hitchin [6] proved the following.

**Theorem 7.1.** An \( \text{SL}(2, \mathbb{C}) \)-Higgs bundle \( (E, \varphi) \) is polystable if and only if \( E \) admits a hermitian metric \( h \) satisfying

\[
F_h + [\varphi, \varphi^* h] = 0,
\]

where \( F_h \) is the curvature of the Chern connection defined by \( h \).
Combining Theorem 7.4 with a theorem of Donaldson [3] about the existence of a harmonic metric on a flat SL(2, \mathbb{C})-bundle with reductive holonomy representation, one has the following non-abelian generalisation of the Hodge correspondence explained above for the rank 1 case [6].

**Theorem 7.2.** The varieties \( \mathcal{H}(2, \mathcal{O}) \) and \( \mathcal{R}(\text{SL}(2, \mathbb{C})) \) are homeomorphic.

The representation \( \rho \) corresponding to a polystable Higgs bundle is the holonomy representation of the flat SL(2, \mathbb{C})-connection given by

\[
D = \partial_E + \partial_h + \varphi + \varphi^h,
\]

where \( h \) is the solution to Hitchin equations and \( \partial_E + \partial_h \) is the SU(2)-connection defined by \( \partial_E \), the Dolbeault operator of \( E \) and \( h \).

**Remark 7.3.** Notice that the complex structures of \( \mathcal{H}(2, \mathcal{O}) \) and \( \mathcal{R}(\text{SL}(2, \mathbb{C})) \) are different. The complex structure of \( \mathcal{H}(2, \mathcal{O}) \) is induced by the complex structure of \( X \), while that of \( \mathcal{R}(\text{SL}(2, \mathbb{C})) \) is induced by the complex structure of SL(2, \mathbb{C}).

Higgs bundles with fixed determinant \( \delta \) of odd degree can also be interpreted in terms of representations. For this we need to consider the universal central extension of \( \pi_1(X) \) (see [4] [6]). Recall that the fundamental group, \( \pi_1(X) \), of \( X \) is a finitely generated group generated by 2\( g \) generators, say \( A_1, B_1, \ldots, A_g, B_g \), subject to the single relation \( \prod_{i=1}^{g} [A_i, B_i] = 1 \). It has a universal central extension

\[
0 \longrightarrow \mathbb{Z} \longrightarrow \Gamma \longrightarrow \pi_1(X) \longrightarrow 1
\]

generated by the same generators as \( \pi_1(X) \), together with a central element \( J \) subject to the relation \( \prod_{i=1}^{g} [A_i, B_i] = J \).

Representations of \( \Gamma \) into SL(2, \mathbb{C}) are of two types depending on whether the central element \( 1 \in \mathbb{Z} \subset \Gamma \) goes to \( +I \) or \( -I \) in SL(2, \mathbb{C}). In the first case the representation is simply obtained from a homomorphism from \( \Gamma/\mathbb{Z} = \pi_1(X) \) into SL(2, \mathbb{C}). The \( +I \) case corresponds to Higgs bundles with trivial determinant as we have seen. The \( -I \) case corresponds to Higgs bundles with odd degree determinant. Namely, let

\[
\mathcal{R}^\pm(\Gamma, \text{SL}(2, \mathbb{C})) = \{ \rho \in \text{Hom}^{red}(\Gamma, \text{SL}(2, \mathbb{C}))/\text{SL}(2, \mathbb{C}) \mid \rho(J) = \pm I \}.
\]

Here a reductive representation of \( \Gamma \) is defined as at the beginning of the section, replacing \( \pi_1(X) \) by \( \Gamma \). Note that \( \mathcal{R}^+(\Gamma, \text{SL}(2, \mathbb{C})) = \mathcal{R}(\text{SL}(2, \mathbb{C})) \). We then have the following [6].

**Theorem 7.4.** Let \( \delta \) be a line bundle over \( X \). Then there are homeomorphisms

(i) \( \mathcal{H}(2, \delta) \cong \mathcal{R}^+(\Gamma, \text{SL}(2, \mathbb{C})) \) if \( \operatorname{deg} \delta \) is even,

(ii) \( \mathcal{H}(2, \delta) \cong \mathcal{R}^-(\Gamma, \text{SL}(2, \mathbb{C})) \) if \( \operatorname{deg} \delta \) is odd.

7.3. **Fixed points of \( \iota(\mathcal{O})^- \) and representations of \( \Gamma \).** For any reductive subgroup \( G \subset \text{SL}(2, \mathbb{C}) \) containing \( -I \) we consider

\[
\mathcal{R}^\pm(\Gamma, G) = \{ \rho \in \text{Hom}^{red}(\Gamma, G)/G \mid \rho(J) = \pm I \}.
\]

In particular we have \( \mathcal{R}^\pm(\Gamma, \text{SU}(2)) \) and \( \mathcal{R}^\pm(\Gamma, \text{SL}(2, \mathbb{R})) \). Note that, since \( \text{SU}(2) \) is compact, every representation of \( \Gamma \) in \( \text{SU}(2) \) is reductive. We can define the subvarieties
$R^\pm_k(\Gamma, \text{SL}(2, \mathbb{R}))$ of $R^\pm(\Gamma, \text{SL}(2, \mathbb{R}))$ given by the representations of $\Gamma$ in $\text{SL}(2, \mathbb{R})$ with Euler class $k$. By this, we mean that the corresponding flat $\text{PSL}(2, \mathbb{R})$ bundle has Euler class $k$. If the $\text{PSL}(2, \mathbb{R})$ bundle can be lift to an $\text{SL}(2, \mathbb{R})$ bundle then $k = 2d$, otherwise $k = 2d - 1$. The Milnor inequality \[7\] says that the Euler class $k$ of any flat $\text{PSL}(2, \mathbb{R})$ bundle satisfies

$$|k| \leq 2g - 2,$$

where $g$ is the genus of $X$.

Hitchin proves the following \[6\].

**Theorem 7.5.** Consider the involution $\iota(O)^-$ of $\mathcal{H}(2, \delta)$. We have the following.

(i) The fixed point subvariety of $\iota(O)^-$ of points $(E, \varphi)$ with $\varphi = 0$ is homeomorphic to the image of $R^\pm(\Gamma, \text{SU}(2))$ in $R^\pm(\Gamma, \text{SL}(2, \mathbb{C}))$, where we have $R^+$ if the degree of $\delta$ is even and $R^-$ if the degree of $\delta$ is odd.

(ii) The fixed point subvariety of $\iota(O)^-$ of points $(E, \varphi)$ with $\varphi \neq 0$ is homeomorphic to the image of $R^\pm(\Gamma, \text{SL}(2, \mathbb{R}))$ in $R^\pm(\Gamma, \text{SL}(2, \mathbb{C}))$, where we have $R^+$ if the degree of $\xi$ is even and $R^-$ if the degree of $\xi$ is odd.

(iii) More precisely, the subvariety of triples $\mathcal{H}_\sigma \subset \mathcal{H}(2, \delta)$ defined in Section 3 is homeomorphic to the image of $R^\pm_{2a}(\Gamma, \text{SL}(2, \mathbb{R}))$ in $R^\pm(\Gamma, \text{SL}(2, \mathbb{C}))$ if the degree of $\delta$ is even or to the image of $R^\pm_{2a-1}(\Gamma, \text{SL}(2, \mathbb{R}))$ in $R^\pm(\Gamma, \text{SL}(2, \mathbb{C}))$ if the degree of $\delta$ is odd.

**Proof.** The conjugations with respect to both real forms, $\text{SU}(2)$ and $\text{SL}(2, \mathbb{R})$, of $\text{SL}(2, \mathbb{C})$ are inner equivalent and hence they induce the same antiholomorphic involution of the moduli space $R^\pm(\Gamma, \text{SL}(2, \mathbb{C}))$, where we recall that the complex structure of this variety is the one naturally induced by the complex structure of $\text{SL}(2, \mathbb{C})$. To be more precise, at the level of Lie algebras, the conjugation with respect to the real form $\mathfrak{su}(2)$ is given by the $\mathbb{C}$-antilinear involution

$$\tau : \mathfrak{sl}(2, \mathbb{C}) \to \mathfrak{sl}(2, \mathbb{C})$$

$$A \mapsto -A^t,$$

while the conjugation with respect to the real form $\mathfrak{sl}(2, \mathbb{R})$ is given by the $\mathbb{C}$-antilinear involution

$$\sigma : \mathfrak{sl}(2, \mathbb{C}) \to \mathfrak{sl}(2, \mathbb{C})$$

$$A \mapsto \overline{A}.$$

Now,

$$\sigma(A) = J\tau(A)J^{-1}$$

for $J \in \mathfrak{sl}(2, \mathbb{R})$ given by

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

This is simply because for every $A \in \mathfrak{sl}(2, \mathbb{R})$, one has that

$$JAJ = -A^t J.$$

Under the correspondence $\mathcal{H}(2, \delta) \cong R^\pm(\Gamma, \text{SL}(2, \mathbb{C}))$, the antiholomorphic involution of $R^\pm(\Gamma, \text{SL}(2, \mathbb{C}))$ defined by $\tau$ and $\sigma$ becomes the holomorphic involution $\iota(O)^-$ of $\mathcal{H}(2, \delta)$

$$(E, \varphi) \mapsto (E, -\varphi),$$

as desired.
where we recall that the complex structure of $H(2,\delta)$ is that induced by the complex structure of $X$. This follows basically from the fact that the $\text{SL}(2,\mathbb{C})$-connection $D$ corresponding to $(\bar{\partial}_E,\varphi)$ under Theorem 7.4 is given by (7.1) and hence

$$\tau(D) = \ast h(\bar{\partial}_E) + \bar{\partial}_E + (\varphi)^* h - \varphi,$$

from which we deduce that $\tau(D)$ is in correspondence with $(E, -\varphi)$. Notice also that $\tau(D) \cong \sigma(D)$.

The proof of (i) follows now from the fact that if $\varphi = 0$ in (7.1) the connection $D$ is an $\text{SU}(2)$ connection. Note that this reduces to the Theorem of Narasimhan and Seshadri for $\text{SU}(2)$.

To prove of (ii) and (iii) one easily checks that the connection $D$ defined by a Higgs bundle in $H^a(\delta)$ is $\sigma$-invariant and hence defines an $\text{SL}(2,\mathbb{R})$-connection. Now, the Euler class $k$ of the $\text{PSL}(2,\mathbb{R})$ bundle is $k = 2d$ if $E = L \oplus L^{-1}$, or $k = 2d - 1 E = L \oplus L^{-1} \delta$, where $d = \text{deg} L$. □

8. Fixed points of $\iota(\alpha)^\pm$ with $\alpha \neq \mathcal{O}$ and representations of $\Gamma$

Consider the normalizer $\text{NSO}(2)$ of $\text{SO}(2)$ in $\text{SU}(2)$. This is generated by $\text{SO}(2)$ and $J = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$. The group generated by $J$ is isomorphic to $\mathbb{Z}/4$ and fits in the exact sequence

$$0 \rightarrow \mathbb{Z}/2 \rightarrow \mathbb{Z}/4 \rightarrow \mathbb{Z}/2 \rightarrow 1,$$

where the subgroup $\mathbb{Z}/2 \subset \mathbb{Z}/4$ is $\{\pm I\}$. We thus have an exact sequence

$$1 \rightarrow \text{SO}(2) \rightarrow \text{NSO}(2) \rightarrow \mathbb{Z}/2 \rightarrow 1. \quad (8.2)$$

The normalizer $\text{NSO}(2,\mathbb{C})$ of $\text{SO}(2,\mathbb{C})$ in $\text{SL}(2,\mathbb{C})$ fits also in an extension

$$1 \rightarrow \text{SO}(2,\mathbb{C}) \rightarrow \text{NSO}(2,\mathbb{C}) \rightarrow \mathbb{Z}/2 \rightarrow 1, \quad (8.3)$$

which is, of course, the complexification of (8.2).

Similarly, we also have that $\text{NSL}(2,\mathbb{R})$, the normalizer of $\text{SL}(2,\mathbb{R})$ in $\text{SL}(2,\mathbb{C})$, is given by

$$1 \rightarrow \text{SL}(2,\mathbb{R}) \rightarrow \text{NSL}(2,\mathbb{R}) \rightarrow \mathbb{Z}/2 \rightarrow 1. \quad (8.4)$$

Note that $\text{NSO}(2)$ is a maximal compact subgroup of $\text{NSL}(2,\mathbb{R})$.

Given a representation $\rho : \Gamma \rightarrow \text{NSO}(2)$ there is a topological invariant $\alpha \in H^1(X, \mathbb{Z}/2)$, which is given by the map

$$H^1(X, \text{NSO}(2)) \rightarrow H^1(X, \mathbb{Z}/2)$$
induced by (8.4). Let

$$\mathcal{R}_\alpha(\Gamma, \text{NSO}(2)) := \{ \rho \in \mathcal{R}_\pm(\Gamma, \text{NSO}(2)) : \text{with invariant } \alpha \in H^1(X, \mathbb{Z}/2) \}.$$

Similarly, we have this $\alpha$-invariant for representations of $\Gamma$ in $\text{NSO}(2,\mathbb{C})$ and in $\text{NSL}(2,\mathbb{R})$, and we can define $\mathcal{R}_\alpha(\Gamma, \text{NSO}(2,\mathbb{C}))$ and $\mathcal{R}_\alpha(\Gamma, \text{NSL}(2,\mathbb{R}))$. 

Theorem 8.1. Let \( \alpha \in J_2(X) = H^1(X, \mathbb{Z}/2) \). Then we have the following.

(i) The subvariety \( F_\alpha \) of fixed points of the involution \( \iota(\alpha) \) in \( M(\delta) \) defined by \( E \mapsto E \otimes \alpha \) is homeomorphic to the image of \( R^\pm(\Gamma, \text{NSO}(2)) \) in \( R^\pm(\Gamma, \text{SU}(2)) \), where we have \( R^+ \) if the degree of \( \delta \) is even and \( R^- \) if the degree of \( \delta \) is odd.

(ii) The subvariety \( F_\alpha^+ \) of fixed points of the involution \( \iota(\alpha)^+ \) of \( \mathcal{H}(\delta) \) is homeomorphic to the image of \( R^+_\alpha(\Gamma, \text{NSO}(2, \mathbb{C})) \) in \( R^+_\alpha(\Gamma, \text{SL}(2, \mathbb{C})) \), where we have \( R^+ \) if the degree of \( \delta \) is even and \( R^- \) if the degree of \( \delta \) is odd.

(iii) The subvariety \( F_\alpha^- \) of fixed points of the involution \( \iota(\alpha)^- \) of \( \mathcal{H}(\xi) \) is homeomorphic to the image of \( R^-\alpha(\Gamma, \text{NSL}(2, \mathbb{R})) \) in \( R^-\alpha(\Gamma, \text{SL}(2, \mathbb{C})) \), where we have \( R^+ \) if the degree of \( \delta \) is even and \( R^- \) if the degree of \( \delta \) is odd.

Proof. The element \( \alpha \in J_2(X) = H^1(X, \mathbb{Z}/2) \) defines a \( \mathbb{Z}/2 \) étale covering \( \pi : X_\alpha \to X \). The strategy of the proof is to lift to \( X_\alpha \) and apply a \( \mathbb{Z}/2 \)-invariant version of the correspondence between Higgs bundles on \( X_\alpha \) and representations of \( \Gamma_\alpha \) — the universal central extension of \( \pi_1(X_\alpha) \). We have a sequence

\[
(8.5) \quad 1 \to \Gamma_\alpha \to \Gamma \to \mathbb{Z}/2 \to 1.
\]

since \( \Gamma_\alpha \) is the kernel of the homomorphism \( \alpha : \Gamma \to \mathbb{Z}/2 \) defined by \( \alpha \).

For convenience, let \( G \) be any of the subgroups \( \text{SO}(2) \subset \text{SU}(2), \text{SO}(2, \mathbb{C}) \subset \text{SL}(2, \mathbb{C}) \) and \( \text{SL}(2, \mathbb{R}) \subset \text{SL}(2, \mathbb{C}) \), and let \( NG \) be its normalizer in the corresponding group. We then have an extension

\[
(8.6) \quad 1 \to G \to NG \to \mathbb{Z}/2 \to 1.
\]

Let \( \text{Hom}_\alpha(\Gamma, NG) \) be the subset of \( \text{Hom}(\Gamma, NG) \) consisting of representations of \( \rho : \Gamma \to NG \) such that the following diagram is commutative

\[
(8.7) \quad \begin{array}{ccc}
1 & \to & \Gamma_\alpha \\
\downarrow & & \downarrow \rho \\
1 & \to & G \\
\end{array} \quad \begin{array}{ccc}
& & \to \\
& & \parallel \\
& & \to \\
1 & \to & NG \\
\to & \to & \mathbb{Z}/2 \\
\to & \to & 1
\end{array}
\]

The group \( NG \) is a disconnected group with \( \mathbb{Z}/2 \) as the group of connected components and \( G \) as the connected component containing the identity. If \( G \) is abelian (\( G = \text{SO}(2), \text{SO}(2, \mathbb{C}) \)), \( \mathbb{Z}/2 \) acts on \( G \) and, since \( \mathbb{Z}/2 \) acts on \( X_\alpha \) (as the Galois group) and hence on \( \Gamma_\alpha \), there is thus an action of \( \mathbb{Z}/2 \) on \( \text{Hom}(\Gamma_\alpha, G) \).

A straightforward computation shows that

\[
(8.8) \quad \text{Hom}_\alpha(\Gamma, NG) \cong \text{Hom}(\Gamma_\alpha, G)^{\mathbb{Z}/2}.
\]

If \( G \) is not abelian (which is the case for \( G = \text{SL}(2, \mathbb{R}) \)), the extension (8.6) still defines a homomorphism \( \mathbb{Z}/2 \to \text{Out}(G) = \text{Aut}(G)/\text{Int}(G) \). We can then take a splitting of the sequence.
(8.9) \[ 1 \to \text{Int}(G) \to \text{Aut}(G) \to \text{Out}(G) \to 1, \]

which always exists [12]. This defines an action on \( \text{Hom}(\Gamma, G) \). However only the action on \( \text{Hom}(\Gamma, G)/G \) is independent of the splitting. In particular, as consequence of (8.8), we have homeomorphisms

\[ R_{\pm}^+(\Gamma, NG) \cong R_{\pm}(\Gamma, G)^{\mathbb{Z}/2}. \]

The result follows now from the usual correspondences between representations of \( \Gamma \) and vector bundles or Higgs bundles on \( X_{\alpha} \), combined with the fact that the fixed point subvarieties \( F_{\alpha}, F_{\pm}^\alpha \) described in Sections 4, 5 and 6 are push-forwards to \( X \) of objects on \( X_{\alpha} \) that satisfy the \( \mathbb{Z}/2 \)-invariance condition (see [4] for more details).

\[ \square \]

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