Entanglement entropy in one-dimensional disordered interacting system: The role of localization

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The properties of the entanglement entropy (EE) in one-dimensional disordered interacting systems are studied. Anderson localization leaves a clear signature on the average EE, as it saturates on length scale exceeding the localization length. This is verified by numerically calculating the EE for an ensemble of disordered realizations using density matrix renormalization group (DMRG). A heuristic expression describing the dependence of the EE on the localization length, which takes into account finite size effects, is proposed. This is used to extract the localization length as function of the interaction strength. The localization length dependence on the interaction fits nicely with the expectations.

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Introduction.— Although more than half a century has passed since the seminal work of Anderson on the absence of diffusion in certain random lattices [1], the localization transition, with an emphasis on understanding its behavior in strongly correlated systems, remains one of the central themes of condensed matter physics. Unlike many other quantum phase transitions in which a gap between the ground state and the excited states appear, the localization transition is more subtle. The canonical manifestation of the localized phase is the exponential dependence of the conductance, $G$, on the linear dimension of the system, $L$. The localization length, $\xi$, (corresponding to the correlation length in the localized regime) is defined through the exponential decrease in the conductance $G(L) \sim \exp(-L/\xi)$ [2].

For non-interacting systems the localization length can be visualized easily as the the length-scale on which the single-electrons wave-function envelope decays. No such transparent interpretation exists for the many-particle wave-function. Indeed, the numerical extraction of the localization length for interacting systems is difficult. Direct calculation of the conductance is computationally taxing since it demands knowledge of the excited states. Ground state properties are easier to calculate, but there one runs into other problems. For example, using the sensitivity to boundary conditions [3] measured, e.g., by the persistent current, depends both on the localization length as well as on the inverse compressibility of the system [4]. Calculating the two requires both a calculation of the sensitivity of the ground state to a magnetic flux in periodic boundary conditions as well as the dependence of the number of electrons in the system on the chemical potential. Alternatively, one may study the response of the system to the introduction of a perturbation [3]. For example, the decay of the Friedel oscillations resulting from the introduction of an additional impurity to the system [3]. Nevertheless, this is strictly speaking justified only for weak disorder, and one must calculate the ground state both in the presence and absence of the impurity.

In this letter we will utilize the entanglement entropy (EE, sometimes referred to as the von Neumann or Shannon entropy) in order to extract the localization length of a 1D interacting disordered system. EE for a many-body system in a pure state is related to partitioning it into two regions: A and B. The entanglement between A and B is measured by the EE $S_{A/B}$ in the following way: Using the Schmidt decomposition the many-body pure state of the entire system, $|\Psi\rangle$, is expressed as the sum of two orthonormal basis sets of regions A ($\{|\phi_{A,i}\rangle\}$) and B ($\{|\phi_{B,j}\rangle\}$), in the following way

$$|\Psi\rangle = \sum_i \alpha_i |\phi_{A,i}\rangle \otimes |\phi_{B,i}\rangle,$$  \hspace{1cm} (1)

with $\sum_i \alpha_i^2 = 1$.

The Schmidt decomposition is related to the eigenbasis of the reduced density operators $\hat{\rho}_{A/B} = \text{Tr}_{B/A} |\Psi\rangle\langle\Psi|$, by

$$\hat{\rho}_{A/B} = \sum_i \alpha_i^2 |\phi_{A/B,i}\rangle \langle \phi_{A/B,i}|.$$  \hspace{1cm} (2)

The EE measures the entanglement between the two regions by the von Neumann entropy of the reduced density matrix:

$$S_{A/B} = -\sum_i \alpha_i^2 \ln(\alpha_i^2),$$  \hspace{1cm} (3)

equivalent to the Shannon entropy of the squared Schmidt coefficients $\alpha_i^2$, clearly $S_A = S_B$.

The behavior of the EE has recently been connected to the presence of quantum phase transitions (QPTs) in condensed matter systems [7-10]. Specifically, the EE of a region of length $L_A$ of a 1D metallic system is expected to grow logarithmically with $L_A$ [11,12], while the EE in the insulating regime should not depend on the regions size. Since for length scales shorter than the localization length the system behaves essentially as a metal, we expect that the EE of a disordered 1D system will show a logarithmic dependence for $L_A \ll \xi$, while it will saturate for $L_A \gg \xi$. Tracing the crossover will be used to determine the localization length $\xi$, even for an many-particle interacting ground state.
Density matrix renormalization group (DMRG) \[14, 15\] is the natural choice for a numerical method to calculate the ground state properties of disordered interacting 1D system. Moreover, since the Schmidt coefficients are calculated in the DMRG in order to decide which states should be truncated, calculating \( S \) has no additional computational overhead. Since DMRG is much more accurate for finite systems with open boundaries than for periodic boundary conditions (another drawback for methods relying on persistent current calculations), we prefer to consider a finite system of size \( L \) in which the EE of a region of length \( L_A \) starting at the edge is calculated. Thus, the EE will depend both on \( L_A \) as well as on \( L \). The explicit function \( S(L_A, L) \) as been derived by several authors \[11–13, 16\], and for open boundaries:

\[
S(L_A, L) = \frac{1}{6} \ln \left( \sin \left( \frac{\pi L_A}{L} \right) \right) + c, \tag{4}
\]

where \( c \) is a non-universal constant which depends also on \( L \) and the boundary entropy \[17\]. This dependence was tested numerically for 1D spin chains \[18\], and found to fit pretty well.

**Model.** We consider a spinless 1D electrons wire of size \( L \) with repulsive nearest neighbor (NN) interactions and on-site disorder. The system’s Hamiltonian is thus given by the Anderson model

\[
H = \sum_{j=1}^{L} \epsilon_j c_j^\dagger c_j - t \sum_{j=1}^{L-1} (c^\dagger_j c_{j+1} + h.c.) + U \sum_{j=1}^{L-1} \left( c^\dagger_j c_j \right)^2 - \frac{1}{2} \left( c^\dagger_{j+1} c_{j+1} - \frac{1}{2} \right),
\]

where \( \epsilon_j \) is the random on-site energy, taken from a uniform distribution in the range \([-W/2, W/2]\), \( U \) is the NN interaction strength \((U \geq 0)\), and \( t = 1 \) is the hopping matrix element between NN. \( c^\dagger_j \) is the creation operator of a spinless electron at site \( j \) in the wire, and a positive background is included in the interaction term. For the non-interacting case the dependence of the localization length on the disorder is known \( \xi(W, U = 0) \approx 105/W^{2} \[19\].

For the interacting case, using renormalization group \[20\] the following dependence of the localization length on interaction strength and disorder is suggested

\[
\xi(W, U) = (\xi(W, U = 0))^{1/(3 - 2g(U))}, \tag{6}
\]

where \( g(U) = \pi/[2 \cos^{-1}(\sqrt{-U/2})] \) is the Luttinger parameter \[21\]. For non-interacting electrons \( g(U = 0) = 1 \). Since for repulsive interactions \( g < 1 \) decreases as a function of the interaction strength, one finds that the localization length always decreases as a function of the interaction strength \[22\].

**Clean systems.** We begin by calculating the EE for clean systems of different sizes as function of the length of region \( A (L_A) \). During the DMRG calculations 192 states are retained. The accuracy may be assessed by comparing \( S(L_A, L) \) to \( S(L - L_A, L) \). The results are presented in Fig. 1. The EE for all system length begin with a logarithmic dependence on \( L_A \) for \( L_A < L \) and tapers out for \( L_A \approx L/2 \). Fitting the numerical results to Eq. (6), where the only fitting parameter is the constant \( c \), works quite well, except for some deviations at small values of \( L_A \). Such deviations were also seen for spin chains with open boundary conditions (see Ref. [18]).

Adding interactions \((U \neq 0)\) to a clean system does not change the conductance and the system remains metallic. Thus, one expects that the EE will behave in a similar way in the presence or absence of interactions. Indeed, this can be seen in the bottom inset of Fig. 1 where the EE as function of \( L_A \) is calculated for different interaction strength \((U = 0, 0.15, 0.3, 0.6) \) corresponding to \( g = (1, 0.954, 0.913, 0.836) \), for \( L = 700 \). Except for a small change in the constant \( c \), the EE shows the same functional behavior for the different values of interaction.

**Disorder.** The study of the influence of disorder on the EE of 1D systems has concentrated on the influence of a single impurity in the middle of the sample \[23–25\]. For the non-interacting Anderson model \[26\] the system remains metallic even in the presence of an impurity and the EE continues to be described by a logarithm, though with a different slope. For the interacting case, (i.e. for a Luttinger liquid) the impurity effectively cuts the system into two, driving the system into an insulating phase \[30\], resulting in a saturation of the EE \[28\].
The multiple impurity situation was only discussed for the non-interacting case [23, 52], where the EE of a single site (i.e., $L_A = 1$) was shown to correspond to the well known measure of the inverse participation ratio.

When disorder is introduced, the main changes anticipated in the behavior of $S(L_A, L)$ are the saturation of the EE for $L > L_A \gg \xi$ and a metallic behavior (i.e., logarithmic, however with a non-universal slope) for $L, \xi \gg L_A$. The following considerations are taken into account while suggesting a functional description of the EE: For small values of $x = L_A/\min(\xi, L/2)$ one expects $S(L_A, \xi, L) \approx S(x) \approx \ln(x)$ while for large values of $x$, $S(x) \approx$ Constant. Thus, it is natural to assume that $S(x)$ takes the form $S(x) \approx \ln(f(x))$, where $f(x) \approx x$ for $x \ll 1$ and $f(x) \approx$ Constant for $x \gg 1$. The most natural choice would be $f(x) = \tanh(x)$, nevertheless it turns out that the crossover between the two regions ($f(x) \approx x$ and $f(x) \approx$ Constant) is quite sharp and $\tanh(x)$ does not describe it well. The straightforward way to sharpen the transition would be to replace the exponential governing $\tanh(x)$ by a stronger decaying function, e.g., a Gaussian, hence $f(x) = \text{erf}(x)$ (the error function) which provides a better description. Thus, we propose to fit the EE to the following heuristic expression:

$$S(L_A, \xi, L) = s(\xi) \ln \left( \text{erf} \left( \frac{L_A}{\zeta(L, \xi)} \right) \right) + c,$$

where $s(\xi) < 1/6$, is a constant which depends on the disorder, $\zeta(L, \xi)$ depends both on $L$ and $\xi$ and $c$ is a non-universal constant. It is important to note that for the disordered case one discusses a disorder averaged value of the EE.

For the clean limit ($\xi \to \infty$), we may infer the behavior of $\zeta_0(L) = \zeta(L, \infty)$ from fitting Eq. (7) to the data we obtained for different system length presented in Fig. 1. The fit is represented by the continuous curves in Fig. 1. As expected $s(\xi \to \infty) \approx 1/6$, while the results for $\zeta_0(L)$ are depicted in the legend. One can see that the heuristic expression for the EE (Eq. (7)) provides a very good description of the numerical data with an even somewhat better fit than Eq. (4) for small values of $L_A$. In the top inset of Fig. 1 $\zeta_0$ as function of $L$ is presented. A linear relation $\zeta_0 = 0.357L$ emerges. Eq. (7) fits also the behavior of the EE for the clean interacting case (bottom inset Fig. 1) where only $c$ depends on the interaction strength.

In the limit of $\xi \ll L$ one would expect that the system size $L$ will have only a rudimentary influence on $\zeta(L, \xi)$, i.e., $\zeta(L \gg \xi) \approx \xi$. We investigate this assertion by studying the behavior of $S(L_A, \xi, L)$ for $W = 0.7$ and $U = 0.6$ for $L > 300$. Using Eq. (7), one expects that the localization length $\xi = 57$, is much smaller than $L$. Prior to examining the behavior of the average value $S(L_A, \xi, L)$, it is important to examine the behavior of the distribution of the EE for different realizations, in order to verify that the concept of an average value of the EE is meaningful, especially since we are in the localized regime. The distribution of $S(L_A, \xi, L)$ for $L = 700$ derived from an ensemble of 400 realizations and different values of $L_A$ is presented in the insets of Fig. 2. In the left inset the distribution is fitted to a normal distribution while in the right inset to a log-normal distribution. While for $L_A = 20$ the distribution follows a normal distribution for $L_A = 330$ a fit to the log-normal distribution works better. Thus, it seems, that while for $L_A \ll \xi$ the EE distribution is normal, for $L_A \gg \xi$ it is log-normal, somewhat similar to the behavior of the distribution of the conductance $\zeta_0$. Thus, a better representation of the typical EE is the median, which is less sensitive to the tails of the distribution. The errors are estimated by the 40 (and 60) percentile of the distribution.

The median EE for samples of length $L = 300, 500, 700$ over ensembles of 600, 400, 400 realizations correspondingly are presented in Fig. 2. It is clear that the EE does not change much for the different length of the samples. Fitting Eq. (7) results in $\zeta = 63.6, 69.6, 63$ for $L = 300, 500, 700$. This is in line with our expectations for $\zeta \approx \xi$ for $L \gg \xi$.

For the case where both the finite-size of the sample and the localization length are of comparable length, one expects that both will influence $\zeta(L, \xi)$. Since the limits are clear, i.e., $\zeta(L \gg \xi) \approx \xi$ and $\zeta(L \ll \xi) = \zeta_0 = 0.357L$.
by the curve, and the numerical results for $\zeta$ of the interpolation formula proposed in Eq. (8) represent the localization length may be achieved by fitting the median EE to Eq. (7) and extracting $\xi$ indicated in the legend. Inset: Comparison of the interpolation formula proposed in Eq. (8) with values of $\zeta$ indicated in the legend.

In Fig. 3 we examine the behavior of the EE for a weakly interacting ($U = 0.15$, $g = 0.954$) case. According to Eq. (5), $\xi = 136$ is close to the values of $\zeta_0$ expected for the system sizes $L = 300, 500, 700$ examined. A fit of the numerical data to Eq. (7) results in $\zeta = 79.9, 92.3, 120.7$ for $L = 300, 500, 700$. As can be seen in the inset, these values of $\zeta$ fit reasonably with the extrapolation of $\zeta$ obtained from Eq. (5).

Now we examine the role played by interactions on the behavior of the localization length. In Fig. 4 we show the EE of the longest samples ($L = 700$) for which it was feasible to obtain an adequate number of realizations. As expected the saturation of the EE begins for smaller values of $L_A$ as the interaction increases. A more quantitative comparison of the behavior of the localization length may be achieved by fitting the median EE to Eq. (4) and extracting $\zeta$. Finite size corrections (especially important when $\zeta$ is comparable to $\zeta_0$, i.e. for weak interactions) can be taken into account by the extrapolation suggested in Eq. (5). The localization length thus obtained is compared to the expectations according to Eq. (4) in the inset. We see that the localization length obtained from the EE fits quite well the expectations for the dependence of $\xi$ on the interaction strength.

Conclusions.—We have studied the properties of the EE in one-dimensional disordered interacting systems. The EE saturates once the length of the segment exceeds the localization length. We propose a heuristic expression which describes the dependence of the EE on the segment length, system size and localization length and compare it to numerical results. By fitting the expression to the numerical data it is possible to extract the localization length of an interacting system while calculating only its ground state many-body wavefunction.

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