INJECTIVITY THEOREMS WITH MULTIPLIER IDEAL SHEAVES
AND THEIR APPLICATIONS

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Abstract. The purpose of this survey is to present analytic versions of the injectivity
theorem and their applications. The proof of our injectivity theorems is based on a
combination of the $L^2$-method for the $\overline{\partial}$-equation and the theory of harmonic integrals.
As applications, we obtain Nadel type vanishing theorems and extension theorems for
pluri-canonical sections of log pairs. Moreover, we give some results on semi-ampleness
related to the abundance conjecture in birational geometry (the minimal model program).

1. Introduction

The Kodaira vanishing theorem is one of the most celebrated results in complex ge-
ometry, and such results play an important role when we consider certain fundamental
problems in algebraic geometry and the theory of several complex variables, including
asymptotics of linear systems, extension problems of holomorphic sections, the minimal
model program, and so on. According to these objectives, the study of vanishing theorems
has been a constant subject of attention in the last decades. This paper is a survey of
recent results in [Mat13b] and [GM13], whose purpose is to present generalizations of the
Kodaira vanishing to pseudo-effective line bundles equipped with singular metrics and their
applications, from the viewpoint of the theory of several complex variables and differential
geometry.

1.1. Analytic versions of the injectivity theorem. In this subsection, we introduce
analytic versions of the injectivity theorem. The injectivity theorem is a generalization
of the vanishing theorem to "semi-positive" line bundles, and it has been studied by sev-
eral authors, for example, Tankeev ([Tan71]), Kollár ([Kol86]), Enoki ([Eno90]), Esnault-
Viehweg ([EV92]), Ohsawa ([Ohs04]), Fujino ([Fuj12a], [Fuj12b]), Ambro ([Amb03], [Amb12]),
and so on. In his paper [Kol86], Kollár gave the following injectivity theorem for semi-
ample line bundles, whose proof depends on the Hodge theory. In [Eno90], Enoki relaxed
his assumption by a different method depending on the theory of harmonic integrals, which
enables us to approach the injectivity theorem from the viewpoint of complex differential
geometry.

Key words and phrases. Injectivity theorems, Vanishing theorems, Singular metrics, Multiplier ideal
sheaves, The theory of harmonic integrals, $L^2$-methods, Extension theorems, Abundance conjecture.
Classification AMS 2010: 14F18, 32L10, 32L20.
**Theorem 1.1** ([Kol86] (resp. [Eno90])). Let $F$ be a semi-ample (resp. semi-positive) line bundle on a smooth projective variety (resp. a compact Kähler manifold) $X$. Then for a (non-zero) section $s$ of a positive multiple $F^m$ of the line bundle $F$, the multiplication map induced by the tensor product with $s$
olimits
\[ \Phi_s : H^q(X, K_X \otimes F) \otimes s \rightarrow H^q(X, K_X \otimes F^m+1) \]

is injective for any $q$. Here $K_X$ denotes the canonical bundle of $X$.

The above theorem can be regarded as a generalization of the Kodaira vanishing theorem to semi-ample (semi-positive) line bundles. On the other hand, the Kodaira vanishing theorem has been generalized by Nadel ([Nad89], [Nad90]). This generalization uses singular metrics with positive curvature and corresponds to the Kawamata-Viehweg vanishing theorem in algebraic geometry ([Kaw82], [Vie82]). Therefore, in the same direction as this generalization, it is natural and of interest to study injectivity theorems for line bundles equipped with “singular metrics”.

\[ \begin{align*}
\text{The Kodaira vanishing} & \quad \text{semi-positivity} \quad \text{Kollár’s injectivity theorem.} \\
\text{cpx. geom: smooth positive metrics} & \quad \text{semi-positivity} \quad \text{cpx. : smooth semi-positive metrics} \\
\text{alg. geom: ample line bundles} & \quad \text{singular metrics} \quad \text{alg. : semi-ample line bundles} \\
\text{singular metrics} & \quad \text{singular metrics} \\
\text{The Nadel, Kawamata-Viehweg vanishing} & \quad \text{semi-positivity} \quad \text{Theorem 1.2} \\
\text{cpx. : singular positive metrics} & \quad \text{semi-positivity} \quad \text{cpx. : singular semi-positive metrics} \\
\text{alg. : big line bundles} & \quad \text{alg. : pseudo-effective line bundles} \\
\end{align*} \]

The following theorem is one of the main results, which can be seen as a generalization of the injectivity theorem and the Nadel vanishing theorem.

**Theorem 1.2** ([Mat13b, Theorem 1.3]). Let $F$ be a line bundle on a compact Kähler manifold $X$ and $h$ be a singular metric with semi-positive curvature on $F$. Then for a (non-zero) section $s$ of a positive multiple $F^m$ satisfying $\sup_X |s|_h^m < \infty$, the multiplication map

\[ \Phi_s : H^q(X, K_X \otimes F \otimes \mathcal{I}(h)) \otimes s \rightarrow H^q(X, K_X \otimes F^{m+1} \otimes \mathcal{I}(h^{m+1})) \]

is (well-defined and) injective for any $q$. Here $\mathcal{I}(h)$ denotes the multiplier ideal sheaf associated to the singular metric $h$.

**Remark 1.3.** The multiplication map is well-defined thanks to the assumption of $\sup_X |s|_h^m < \infty$. When $h$ is a metric with minimal singularities on $F$, this assumption is automatically satisfied for any section $s$ of $F^m$ (see [Dem] for the concept of metrics with minimal singularities).

When we consider the problem of extending (holomorphic) sections from subvarieties to the ambient space, we need to refine the above formulation (see Theorem 2.1). Our injectivity theorem can be seen as an improvement of [Eno90], [Fuj12a], [Kol86], [Mat14]. For the proof, we take an analytic approach for the cohomology groups with coefficients.
in $K_X \otimes F \otimes \mathcal{I}(h)$, which includes techniques of [Eno90], [Fuj12a], [Mat13a], [Mat14], [Ohs04], [Tak97]. The proof is based on a technical combination of the $L^2$-method for the $\bar{\partial}$-equation and the theory of harmonic integrals. To handle transcendental (non-algebraic) singularities, after regularizing a given singular metric, we investigate the asymptotic behavior of the harmonic forms with respect to a family of the regularized metrics. Moreover we establish a method to obtain $L^2$-estimates of solutions of the $\bar{\partial}$-equation by using the Čech complex. See subsection 2.1 for more details.

1.2. Applications to the vanishing theorem. Theorem 1.2 is formulated for singular metrics with transcendental (non-algebraic) singularities, which is one of the advantages of our injectivity theorem. For example, metrics with minimal singularities are important objects, but they do not always have algebraic singularities. By applying Theorem 1.2 to them, we can obtain an injectivity theorem for nef and abundant line bundles (Corollary 1.4) and Nadel type vanishing theorems (Theorem 1.5 and Corollary 1.6).

It is natural to expect the same conclusion as in Theorem 1.1 under the weaker assumption that $F$ is nef. However there is a counterexample to the injectivity theorem for nef line bundles (see for example [Fuj12b, Example 5.1]). If $F$ is nef and abundant (that is, the numerical dimension agrees with the Kodaira dimension), the line bundle $F$ admits a metric $h_{\min}$ with minimal singularities satisfying $\mathcal{I}(h_{\min}^m) = \mathcal{O}_X$ for any $m > 0$. This follows from [Kaw85, Proposition 2.1]. Hence Theorem 1.2 leads to the following corollary.

**Corollary 1.4** ([Mat13b, Corollary 1.5]). Let $F$ be a nef and abundant line bundle on a compact Kähler manifold $X$. Then for a (non-zero) section $s$ of a positive multiple $F^m$ of the line bundle $F$, the multiplication map induced by the tensor product with $s$

$$\Phi_s : H^q(X, K_X \otimes F) \otimes s \rightarrow H^q(X, K_X \otimes F^m+1)$$

is injective for any $q$.

The same statement was proved in [Fuj12a], and a similar conclusion was proved in [EP08], [EV92] by different methods when $X$ is a projective variety. It is worth pointing out that Theorem 1.4 is not sufficient to obtain Corollary 1.4. This is because the above metric $h_{\min}$ may not be smooth and not have algebraic singularities even if $F$ is nef and abundant (see for example [Fuj12b, Example 5.2]).

As another application of Theorem 1.2, we obtain a Nadel type vanishing theorem (Theorem 1.5) and its corollary (Corollary 1.6).

**Theorem 1.5** ([Mat13b, Theorem 3.21] cf. [Mat14, Theorem 5.2]). Let $F$ be a line bundle on a smooth projective variety $X$ and $h$ be a singular metric with semi-positive curvature on $F$. Then

$$H^q(X, K_X \otimes F \otimes \mathcal{I}(h)) = 0 \quad \text{for any } q > \dim X - \kappa_{\text{bdd}}(F, h).$$

See subsection 2.2 or [Mat14, Definition 5.1] for the definition of the bounded Kodaira dimension $\kappa_{\text{bdd}}(F, h)$. 

Corollary 1.6 ([Mat13b, Corollary 1.6] cf. [Mat13a, Theorem 1.2]). Let $F$ be a line bundle on a smooth projective variety $X$ and $h_{\min}$ be a singular metric with minimal singularities on $F$. Then

$$H^q(X, K_X \otimes F \otimes \mathcal{I}(h_{\min})) = 0 \quad \text{for any } q > \dim X - \kappa(F).$$

Here $\kappa(F)$ denotes the Kodaira dimension of $F$.

Since multiplier ideal sheaves are coherent ideal sheaves, the family of multiplier ideal sheaves $\{\mathcal{I}(h^{1+\delta})\}_{\delta > 0}$ has the maximal element, which we denote by $\mathcal{I}_+(h)$ (see [DEL00] for more details). In [Cao12], Cao gave striking results on the Nadel vanishing theorem for the cohomology groups with coefficients in $K_X \otimes F \otimes I_+(h)$. However, the Nadel vanishing theorem for $K_X \otimes F \otimes I(h_{\min})$ is non-trivial even if $F$ is big. In fact, it was first proved in [Mat13a] when $F$ is big.

It is of interest to ask whether or not $\mathcal{I}_+(\varphi)$ agrees with $\mathcal{I}(\varphi)$ for a plurisubharmonic (psh for short) function $\varphi$, which was first posed in [DEL00]. We can easily see that $\mathcal{I}_+(\varphi) = \mathcal{I}(\varphi)$ holds if $\varphi$ has algebraic singularities, but $h_{\min}$ unfortunately does not always have algebraic singularities. It is a natural problem related to the (strong) openness conjecture of Demailly-Kollár (see [DK01]), but it had been an open problem. Recently, Guan-Zhou affirmatively solved the openness conjecture in [GZ13]. Although their celebrated results imply Theorem 1.5, we believe that our techniques are still of interest, since they bring a quite different viewpoint and have further applications.

1.3. Applications to the extension theorem. In this subsection, we give an extension theorem for pluri-canonical sections of log pairs. Our motivation is the abundance conjecture, which is one of the most important conjectures in the classification theory of algebraic varieties. From now on, we freely use the standard notation in [BCHM10], [KaMM87], [KM] and further we interchangeably use the words “Cartier divisors”, “line bundles”, “invertible sheaves”.

Conjecture 1.7 (Generalized abundance conjecture). Let $X$ be a normal projective variety and $\Delta$ be an effective $\mathbb{Q}$-divisor such that $(X, \Delta)$ is a klt pair. Then $\kappa(K_X + \Delta) = \kappa_\sigma(K_X + \Delta)$. In particular, if $K_X + \Delta$ is nef, then it is semi-ample. See [Nak] for the definition of $\kappa(\cdot)$ and $\kappa_\sigma(\cdot)$.

Toward the abundance conjecture, we need to study the non-vanishing conjecture and the extension conjecture (see [DHP13], [Fuj00, Introduction], [FG14, Section 5]). We study the following extension conjecture for dlt pairs formulated in [DHP13, Conjecture 1.3]:

Conjecture 1.8 (Extension conjecture for dlt pairs). Let $X$ be a normal projective variety and $S + B$ be an effective $\mathbb{Q}$-divisor satisfying the following assumptions:

- $(X, S + B)$ is a dlt pair.
- $[S + B] = S$.
- $K_X + S + B$ is nef.
- $K_X + S + B$ is $\mathbb{Q}$-linearly equivalent to an effective divisor $D$ with $S \subseteq \text{Supp} D \subseteq \text{Supp} (S + B)$.
Then the restriction map
\[ H^0(X, \mathcal{O}_X(m(K_X + S + B))) \to H^0(S, \mathcal{O}_S(m(K_X + S + B))) \]
is surjective for sufficiently divisible integers \( m \geq 2 \).

When \( S \) is a normal irreducible variety (that is, \( (X, S + B) \) is a plt pair), Demailly-Hacon-Pâun proved the above conjecture in [DHP13] by using technical methods based on a version of the Ohsawa-Takegoshi \( L^2 \)-extension theorem. This result can be seen as an extension theorem for plt pairs.

By applying Theorem 2.1 instead of the Ohsawa-Takegoshi theorem to the extension problem, we prove the following extension theorem for dlt pairs. Thanks to the injectivity theorem, we can obtain some extension theorems for not only plt pairs but also dlt pairs. This is an advantage of our approach. Even if \( K_X + \Delta \) is semi-positive (that is, it admits a smooth metric with semi-positive curvature), it seems to be very impossible to prove the extension theorem for dlt pairs by the Ohsawa-Takegoshi theorem at least in the current situation, and thus we need our injectivity theorem (Theorem 2.1).

**Theorem 1.9** ([GM13, Corollary 4.5]). Let \( X \) be a compact Kähler manifold and \( S + B \) be an effective \( \mathbb{Q} \)-divisor with the following assumptions:
- \( S + B \) is a simple normal crossing divisor with \( 0 \leq S + B \leq 1 \) and \( |S + B| = S \).
- \( K_X + S + B \) is \( \mathbb{Q} \)-linearly equivalent to an effective divisor \( D \) with \( S \subseteq \text{Supp} \, D \).
- \( K_X + S + B \) admits a singular metric \( h \) with semi-positive curvature.
- The Lelong number \( \nu(h, x) \) is equal to 0 at every point \( x \in S \).

Then, for an integer \( m \geq 2 \) with Cartier divisor \( m(K_X + S + B) \), every section \( u \in H^0(S, \mathcal{O}_S(m(K_X + S + B))) \) can be extended to a section in \( H^0(X, \mathcal{O}_X(m(K_X + S + B))) \).

In particular, Conjecture 1.8 is affirmatively solved under the assumption that \( K_X + \Delta \) admits a singular metric whose curvature is semi-positive and Lelong number is identically zero. This assumption is stronger than the assumption that \( K_X + \Delta \) is nef, but weaker than the assumption that \( K_X + \Delta \) is semi-positive. Let us observe that Verbitsky proved the non-vanishing conjecture on hyperKähler manifolds (holomorphic symplectic manifolds) under the same assumption (see [Ver10]).

As compared to Conjecture 1.8, one of our advances has been to remove the condition \( \text{Supp} \, D \subseteq \text{Supp}(S + B) \). As a benefit of removing the condition \( \text{Supp} \, D \subseteq \text{Supp}(S + B) \) in Conjecture 1.8 we can run the minimal model program while preserving a good condition for metrics (cf. [DHP13, Section 8], [FG14, Theorem 5.9]). By applying the above theorem and techniques of the minimal model program, we obtain results related to the abundance conjecture (see [GM13] for more details).

**Acknowledgement.** The author obtained an opportunity of discussion on the injectivity theorem and extension problem when he attended the conference “The 10th Korean Conference in Several Complex Variables”. He is grateful to the organizers. He would also like to thank the referee for carefully reading the paper and for suggestions. He is partially supported by the Grant-in-Aid for Young Scientists (B) 25800051 from JSPS.
2. Proof of the Main Results

2.1. Proof of Theorem 2.1. In this subsection, we give a proof of the following theorem, which is an improvement of Theorem 1.2 to obtain Theorem 1.9.

**Theorem 2.1.** Let \((F, h_F)\) and \((L, h_L)\) be (singular) hermitian line bundles with semi-positive curvature on a compact Kähler manifold \(X\). Assume that there exists an effective \(\mathbb{R}\)-divisor \(\Delta\) with
\[
h_F = h^0_L \cdot h_\Delta,
\]
where \(a\) is a positive real number and \(h_\Delta\) is the singular metric defined by \(\Delta\).

Then for a (non-zero) section \(s\) of \(L\) satisfying \(\sup_X |s|_{h_L} < \infty\), the multiplication map
\[
\Phi_s : H^q(X, K_X \otimes F \otimes \mathcal{I}(h_F)) \otimes s \rightarrow H^q(X, K_X \otimes F \otimes L \otimes \mathcal{I}(h_F h_L))
\]
is (well-defined and) injective for any \(q\).

**Remark 2.2.**
(1) The case of \(\Delta = 0\) corresponds to Theorem 1.2.

(2) If \(h_L\) and \(h_F\) are smooth on a Zariski open set, the same conclusion holds under the weaker assumption of
\[
\sqrt{-1} \Theta_{h_F}(F) \geq a \sqrt{-1} \Theta_{h_L}(L)
\]
(see [Mat14, Theorem 1.5]).

**Proof.** We give here only the strategy of the proof. See [Mat13b], [GM13] for the precise proof. First of all, we recall Enoki's method to generalize Kollár's injectivity theorem, which gives a proof of the special case where \(h_L\) is smooth and \(\Delta = 0\). In this case, the cohomology group
\[
H^q(X, K_X \otimes F)
\]
is isomorphic to the space of harmonic forms with respect to \(h_F\). For an arbitrary harmonic form \(u \in H^{n,q}(F)_{h_F}\), we can conclude that \(D'_{h_F}^* u = 0\) from the semi-positivity of the curvature and \(h_F = h^0_L\). This step heavily depends on the semi-positivity of the curvature. This implies that the multiplication map \(\Phi_s\) induces the map from \(H^{n,q}(F)_{h_F}\) to \(H^{n,q}(F \otimes L)_{h_F h_L}\), and thus the injectivity is obvious.

When \(h_L\) is smooth on a Zariski open set, the cohomology group \(H^q(X, K_X \otimes F)\) is isomorphic to the space of harmonic forms on the Zariski open set. Therefore we can give a proof similar to Enoki’s proof thanks to the semi-positivity of the curvature (see [Mat14, Theorem 1.5]).

In our situation, we must consider singular metrics with transcendental (non-algebraic) singularities. It is quite difficult to directly handle transcendental singularities, and thus, in Step 1, we approximate a given singular metric \(h_F\) by metrics \(\{h_{\varepsilon}\}_{\varepsilon > 0}\) that are smooth on a Zariski open set. Then we represent a given cohomology class in \(H^q(X, K_X \otimes F \otimes \mathcal{I}(h_F))\) by the associated harmonic form \(u_\varepsilon\) with respect to \(h_\varepsilon\) on the Zariski open set. We want to show that \(s u_\varepsilon\) is also harmonic by using the same method as Enoki. However, the same argument as in [Eno90] fails since the curvature of \(h_\varepsilon\) is not semi-positive. For this reason, in Step 2, we investigate the asymptotic behavior of the harmonic forms \(u_\varepsilon\) with respect to
a family of the regularized metrics \( \{ h_{\varepsilon} \}_{\varepsilon > 0} \). Then we show that the \( L^2 \)-norm \( \| D_{h_{\varepsilon} \Phi_{L,\varepsilon}} \gamma_{\varepsilon} \| \) converges to zero as \( \varepsilon \) tends to zero, where \( h_{L,\varepsilon} \) is a suitable approximation of \( h_L \). Further, in Step 3, we construct solutions \( \gamma_{\varepsilon} \) of the \( \overline{\partial} \)-equation \( \overline{\partial} \gamma_{\varepsilon} = su_{\varepsilon} \) such that the \( L^2 \)-norm \( \| \gamma_{\varepsilon} \| \) is uniformly bounded, by applying the Čech complex with the topology induced by the local \( L^2 \)-norms. In Step 4, we see that

\[
\| su_{\varepsilon} \|^2 = \langle \langle su_{\varepsilon}, \overline{\partial} \gamma_{\varepsilon} \rangle \rangle \leq \| D^{\gamma_{\varepsilon}}_{h_{\varepsilon} \Phi_{L,\varepsilon}} su_{\varepsilon} \| \| \gamma_{\varepsilon} \| \to 0 \quad \text{as} \quad \varepsilon \to 0.
\]

From these observations, we conclude that \( u_{\varepsilon} \) converges to zero in a suitable sense. This completes the proof.

**Step 1 (The equisingular approximation of \( h_F \))**

Throughout the proof, we fix a Kähler form \( \omega \) on \( X \). For the proof, we want to apply the theory of harmonic integrals, but the metric \( h_F \) may not be smooth. For this reason, we approximate \( h_F \) by metrics \( \{ h_{\varepsilon} \}_{\varepsilon > 0} \) that are smooth on a Zariski open set. By [DPS01, Theorem 2.3], we can obtain metrics \( \{ h_{\varepsilon} \}_{\varepsilon > 0} \) on \( F \) satisfying the following properties:

(a) \( h_{\varepsilon} \) is smooth on \( Y := X \setminus Z \), where \( Z \) is a subvariety independent of \( \varepsilon \).
(b) \( h_{\varepsilon_2} \leq h_{\varepsilon_1} \leq h_F \) holds for any \( 0 < \varepsilon_1 < \varepsilon_2 \).
(c) \( \mathcal{I}(h_{F}) = \mathcal{I}(h_{\varepsilon}) \).
(d) \( \sqrt{-1} \Theta_{h_{\varepsilon}}(F) \geq -\varepsilon \omega \).

See [Mat13b, Theorem 2.3] for property (a). By [Fuj12a, Lemma 3.1], we obtain a Kähler form \( \tilde{\omega} \) on \( Y \) satisfying the following properties:

(A) \( \tilde{\omega} \) is a complete Kähler form on \( Y \).
(B) There exists a bounded function \( \Psi \) such that \( \tilde{\omega} = dd^{c} \Psi \) on a neighborhood of \( z \in Z \).
(C) \( \tilde{\omega} \geq \omega \).

In the proof, we mainly consider harmonic forms on \( Y \) with respect to \( h_{\varepsilon} \) and \( \tilde{\omega} \). Let \( L^{n,q}_{(2)}(Y,F)_{h_{\varepsilon},\tilde{\omega}} \) be the space of \( L^2 \)-integrable \( F \)-valued \( (n,q) \)-forms \( \alpha \) with respect to the inner product \( \| \cdot \|_{h_{\varepsilon},\tilde{\omega}} \) defined by

\[
\| \alpha \|^2_{h_{\varepsilon},\tilde{\omega}} := \int_Y |\alpha|^2_{h_{\varepsilon},\tilde{\omega}} \tilde{\omega}^n.
\]

Then we have the following orthogonal decomposition:

\[
L^{n,q}_{(2)}(Y,F)_{h_{\varepsilon},\tilde{\omega}} = \text{Im } \overline{\partial} \oplus \mathcal{H}^{n,q}(F)_{h_{\varepsilon},\tilde{\omega}} \oplus \text{Im } D^{\alpha*}_{h_{\varepsilon}}.
\]

Here the operator \( D^{\alpha*}_{h_{\varepsilon}} \) (resp. \( D^{\alpha*}_{h_{\varepsilon}} \)) denotes the closed extension of the formal adjoint of the \( (1,0) \)-part \( D^{\alpha}_{h_{\varepsilon}} \) (resp. \( (0,1) \)-part \( D^{\alpha}_{h_{\varepsilon}} = \overline{\partial} \)) of the Chern connection \( D_{h_{\varepsilon}} = D^{\alpha}_{h_{\varepsilon}} + D^{\alpha}_{h_{\varepsilon}} \).

Further \( \mathcal{H}^{n,q}(F)_{h_{\varepsilon},\tilde{\omega}} \) denotes the space of harmonic forms with respect to \( h_{\varepsilon} \) and \( \tilde{\omega} \), namely

\[
\mathcal{H}^{n,q}(F)_{h_{\varepsilon},\tilde{\omega}} := \{ \alpha \mid \alpha \text{ is an } F \text{-valued } (n,q) \text{-form with } \overline{\partial} \alpha = D^{\alpha*}_{h_{\varepsilon}} \alpha = 0. \}.
\]

A harmonic form in \( \mathcal{H}^{n,q}(F)_{h_{\varepsilon},\tilde{\omega}} \) is smooth by the regularity theorem for elliptic operators. These results are known to specialists. The precise proof of them can be found in [Fuj12a, Claim 1].
Take an arbitrary cohomology class \( \{ u \} \in H^q(X, K_X \otimes F \otimes \mathcal{I}(h_F)) \) represented by an \( F \)-valued \( (n, q) \)-form \( u \) with \( \| u \|_{h_F, \omega} < \infty \). In order to prove that the multiplication map \( \Phi_s \) is injective, we assume that the cohomology class of \( su \) is zero in \( H^q(X, K_X \otimes F \otimes \mathcal{I}(h_F h_L)) \).

Our goal is to show that the cohomology class of \( u \) is actually zero under this assumption.

By the inequality \( \| u \|_{h_\epsilon, \tilde{\omega}} \leq \| u \|_{h_F, \omega} < \infty \), we can obtain \( u_\epsilon \in H^{n, q}(F)_{h_\epsilon, \tilde{\omega}} \) and \( v_\epsilon \in L_{(2)}^{n,q-1}(Y, F)_{h_\epsilon, \tilde{\omega}} \) such that

\[
u = u_\epsilon + \partial v_\epsilon.
\]

Note that the component of \( \text{Im} D''_{h_\epsilon} \) is zero since \( u \) is \( \partial \)-closed.

At the end of this step, we explain the strategy of the proof. In Step 2, we show that \( \| D''_{h_\epsilon h_L, \epsilon} su_\epsilon \|_{h_\epsilon h_L, \epsilon, \tilde{\omega}} \) converges to zero as \( \epsilon \) tends to zero. Here \( h_{L, \epsilon} \) is the singular metric on \( L \) defined by

\[
h_{L, \epsilon} := h_\epsilon^{1/a} h^{-1/a}.
\]

Since the cohomology class of \( su \) is zero, there are solutions \( \gamma_\epsilon \) of the \( \bar{\partial} \)-equation \( \bar{\partial} \gamma_\epsilon = su_\epsilon \). For the proof, we need to obtain \( L^2 \)-estimates of them. In Step 3, we construct solutions \( \gamma_\epsilon \) of the \( \bar{\partial} \)-equation \( \bar{\partial} \gamma_\epsilon = su_\epsilon \) such that the norm \( \| \gamma_\epsilon \|_{h_{L, \epsilon}, \tilde{\omega}} \) is uniformly bounded. Then we have

\[
\| su_\epsilon \|_{h_{L, \epsilon}, \tilde{\omega}}^2 \leq \| D''_{h_\epsilon h_L, \epsilon} su_\epsilon \|_{h_\epsilon h_L, \epsilon, \tilde{\omega}} \| \gamma_\epsilon \|_{h_{L, \epsilon}, \tilde{\omega}}.
\]

By Step 2 and Step 3, we can conclude that the right hand side goes to zero as \( \epsilon \) tends to zero. In Step 4, from this convergence, we prove that \( u_\epsilon \) converges to zero in a suitable sense, which implies that the cohomology class of \( u \) is zero.

**Step 2 (A generalization of Enoki’s proof)**

By generalizing Enoki’s method, in Step 2, we prove the following proposition:

**Proposition 2.3.** As \( \epsilon \) tends to zero, the norm \( \| D''_{h_\epsilon h_L, \epsilon} su_\epsilon \|_{h_\epsilon h_L, \epsilon, \tilde{\omega}} \) converges to zero.

The same argument as in [Eno90] fails since the curvature of \( h_\epsilon \) is not semi-positive, and further property (d) is not sufficient for the proof of the proposition since there is counterexample to the injectivity theorem for nef line bundles. To overcome these difficulties, we first see the following inequality:

\[
\| u_\epsilon \|_{h_\epsilon, \tilde{\omega}} \leq \| u \|_{h_\epsilon, \tilde{\omega}} \leq \| u \|_{h, \omega}.
\]

This inequality and properties (b), (c) imply the proposition. This step can be considered as a generalization of Enoki’s method.

**Step 3 (A construction of solutions of the \( \bar{\partial} \)-equation via the \( \check{C}ech \) complex)**

In Step 3, we construct solutions of the \( \bar{\partial} \)-equation with suitable \( L^2 \)-norm by using the Čech complex.
Proposition 2.4. There exist $F$-valued $(n, q - 1)$-forms $\alpha_\varepsilon$ on $Y$ satisfying the following properties:

\begin{enumerate}
\item $\overline{\partial} \alpha_\varepsilon = u - u_\varepsilon$.
\item The norm $\|\alpha_\varepsilon\|_{h_\varepsilon, \delta}$ is uniformly bounded.
\end{enumerate}

Remark 2.5. We have already known that there exist solutions $\alpha_\varepsilon$ of the $\overline{\partial}$-equation $\overline{\partial} \alpha_\varepsilon = u - u_\varepsilon$ since $u - u_\varepsilon \in \text{Im} \overline{\partial}$. However, for the proof of the main theorem, we need to construct solutions with uniformly bounded $L^2$-norm.

The strategy of the proof is as follows: The main idea of the proof is to convert the $\overline{\partial}$-equation $\overline{\partial} \alpha_\varepsilon = u - u_\varepsilon$ to the equation $\delta V_\varepsilon = S_\varepsilon$ of the coboundary operator $\delta$ in the space of cochains $\mathcal{C}^*(K_X \otimes F \otimes \mathcal{I}(h_\varepsilon))$, by using the Čech complex and pursuing the De Rham-Weil isomorphism. Here the $q$-cochain $S_\varepsilon$ is constructed from $u - u_\varepsilon$. In this construction, we locally solve the $\overline{\partial}$-equation. The important point is that the space $\mathcal{C}^*(K_X \otimes F \otimes \mathcal{I}(h_\varepsilon))$ is independent of $\varepsilon$ thanks to property (c) of $h_\varepsilon$ although the $L^2$-space $L_{n,q}^2(Y, F)_{h_\varepsilon, \delta}$ depends on $\varepsilon$. Since $\|u - u_\varepsilon\|_{h_\varepsilon, \delta}$ is uniformly bounded, we can observe that $S_\varepsilon$ converges to some $q$-coboundary in $\mathcal{C}^q(K_X \otimes F \otimes \mathcal{I}(h))$ with the topology induced by the local $L^2$-norms with respect to $h$. Further we can observe that the coboundary operator $\delta$ is an open map. Then by these observations we construct solutions $V_\varepsilon$ of the equation $\delta V_\varepsilon = S_\varepsilon$ with uniformly bounded norm. Finally, by using a partition of unity, we conversely construct $\alpha_\varepsilon \in L_{n,q-1}^\infty(Y, F)_{h_\varepsilon, \delta}$ from $S_\varepsilon$ satisfying the properties in Proposition 2.4. This proof gives a new method to obtain $L^2$-estimates of solutions of the $\overline{\partial}$-equation.

Step 4 (The limit of the harmonic forms)

In Step 4, we investigate the limit of $u_\varepsilon$ and complete the proof. By Step 2 and Step 3, we have

$$\|su_\varepsilon\|_{h_\varepsilon, h_{L, \varepsilon}, \delta} \leq \|D_{h_\varepsilon}^* su_\varepsilon\|_{h_\varepsilon, h_{L, \varepsilon}, \delta} \leq \gamma_\varepsilon \|h_{L, \varepsilon}, \delta\| \to 0 \quad \text{as} \quad \varepsilon \to 0.$$}

From this convergence, we can show that $u_\varepsilon$ converges to zero in a suitable sense, which implies that the cohomology class $\{u\}$ of $u$ is zero in $H^q(X, K_X \otimes F \otimes \mathcal{I}(h_\varepsilon))$. By property (c), we obtain the conclusion of Theorem 2.1. □

2.2. Proof of Theorem 1.5. In this subsection, we give a proof of Theorem 1.5 by using Theorem 1.2 and [Mat14, Theorem 4.1].

Proof of Theorem 1.5.) We consider the space of sections with bounded norm defined by

$$H^0_{\text{bdd}, h^m}(X, F^m) := \{s \in H(X, F^m) \mid \sup_X |s|_{h^m} < \infty\}.$$}

The generalized Kodaira dimension $\kappa_{\text{bdd}}(F, h)$ of $(F, h)$ is defined to be $-\infty$ if $H^0_{\text{bdd}, h^m}(X, F^m) = 0$ for any $m > 0$. Otherwise, $\kappa_{\text{bdd}}(F, h)$ is defined by

$$\kappa_{\text{bdd}}(F, h) := \sup \{k \in \mathbb{Z} \mid \limsup_{m \to \infty} \dim H^0_{\text{bdd}, h^m}(X, F^m)/m^k > 0\}.$$}

For a contradiction, we assume that there exists a non-zero cohomology class $\alpha \in H^q(X, K_X \otimes F \otimes \mathcal{I}(h))$. If sections $\{s_i\}_{i=1}^N$ in $H^0_{\text{bdd}, h^m}(X, F^m)$ are linearly independent, then $\{s_i\}_{i=1}^N$
are also linearly independent in $H^q(X, K_X \otimes F^{m+1} \otimes \mathcal{I}(h^{m+1}))$. Indeed, if $\sum_{i=1}^N c_i s_i = 0$ for some $c_i \in \mathbb{C}$, then we know $\sum_{i=1}^N c_i s_i = 0$ by Theorem [1.2]. Since $\{s_i\}_{i=1}^N$ are linearly independent, we have $c_i = 0$ for any $i = 1, 2, \ldots, N$. This yields
\[
\dim H^0_{\text{bdd}, h^m}(X, F^m) \leq \dim H^q(X, K_X \otimes F^{m+1} \otimes \mathcal{I}(h^{m+1})).
\]
On the other hand, by [Mat14, Theorem 4.1], we have
\[
\dim H^q(X, K_X \otimes F^m \otimes \mathcal{I}(h^m)) = O(m^{\dim X - q}) \quad \text{as} \quad m \to \infty
\]
for any $q \geq 0$ (cf. [Dem, (6.18) Lemma]). If $q > \dim X - \kappa_{\text{bdd}}(F, h)$, this is a contradiction. 

2.3. **Proof of Theorem** [1.9]. In this subsection, we give a proof of Theorem [1.9].

**Proof of Theorem [1.9].** For simplicity, we put $\Delta := S + B$ and $G := m(K_X + \Delta)$. We may assume the additional assumption of $h \leq h_D$, where $h_D$ is the singular metric on $\mathcal{O}_X(K_X + \Delta)$ defined by the effective divisor $D$. Indeed, for a smooth metric $g$ on $\mathcal{O}_X(K_X + \Delta)$ and an $L^1$-function $\varphi$ (resp. $\varphi_D$) with $h = g e^{-\varphi}$ (resp. $h_D = g e^{-\varphi_D}$), the metric defined by $g e^{-\max(\varphi, \varphi_D)}$ satisfies the assumptions again.

Consider the following commutative diagram:
\[
0 \to \mathcal{O}_X(G - S) \otimes \mathcal{I}(h^{m-1}h_B) \to \mathcal{O}_X(G) \otimes \mathcal{I}(h^{m-1}h_B) \to \mathcal{O}_S(G) \otimes \mathcal{I}(h^{m-1}h_B) \to 0.
\]
We first prove the induced homomorphism
\[
H^q(X, \mathcal{O}_X(G - S) \otimes \mathcal{I}(h^{m-1}h_B)) \to H^q(X, \mathcal{O}_X(G) \otimes \mathcal{I}(h^{m-1}h_B))
\]
is injective by our injectivity theorem. By the assumption on the support of $D$, we can take an integer $a > 0$ such that $aD$ is a Cartier divisor and $S \leq aD$. Then we have the following commutative diagram:
\[
\begin{array}{ccc}
H^q(X, \mathcal{O}_X(G) \otimes \mathcal{I}(h^{m-1}h_B)) & \xrightarrow{\text{Im (+S)}} & \mathcal{O}_X(G) \\
\xrightarrow{\text{Im (+S)}} & & \xrightarrow{\text{Im (+S)}} \\
H^q(X, \mathcal{O}_X(G - S) \otimes \mathcal{I}(h^{m-1}h_B)) & \xrightarrow{\text{+aD}} & H^q(X, \mathcal{O}_X(G - S + aD) \otimes \mathcal{I}(h^{a+m-1}h_B)),
\end{array}
\]
with a map $+S : H^q(X, \mathcal{O}_X(G - S) \otimes \mathcal{I}(h^{m-1}h_B)) \to H^q(X, \mathcal{O}_X(G) \otimes \mathcal{I}(h^{m-1}h_B))$. In order to show that the upper map on right is injective, we prove that the horizontal map is injective as an application of Theorem [2.1]

By the definition of $G$, we have
\[
G - S = m(K_X + \Delta) - S = K_X + (m - 1)(K_X + \Delta) + B.
\]
Then the line bundle $F := \mathcal{O}_X((m - 1)(K_X + \Delta) + B)$ equipped with the metric $h_F := h^{m-1}h_B$ and the line bundle $L := \mathcal{O}_X(aD)$ equipped with the metric $h_L := h^a$ satisfy the assumptions in Theorem [2.1]. Indeed, we have $h_F = h^{(m-1)/a}h_B$ by the construction, and further the point-wise norm $|s_{aD}|_{h_L}$ is bounded on $X$ by the inequality $h \leq h_D$, where $s_{aD}$
is the natural section of $aD$. Therefore the horizontal map is injective by Theorem 2.1. By the assumption on the Lelong number of $h$, we can conclude that $\mathcal{O}_S \otimes \mathcal{I}(h^{m-1}h_B) = \mathcal{O}_S$. This follows from Skoda’s lemma and Hölder’s inequality. This completes the proof. □

3. Open Problems

In this section, we summarize and give open problems related to the topics mentioned in this survey.

It is of interest to consider the injectivity theorem in the relative situation. The following problem is a relative version of Theorem 1.2. For relative versions of the injectivity theorem and their applications, we refer the reader to [Fuj12b]. In his paper [Fuj12b], Fujino affirmatively solved this problem under the assumption on the regularity of singular metrics, whose proof is based on the Ohsawa-Takegoshi twisted version of Nakano’s identity. To remove this assumption, it seems to be needed to use a combination of his method and the techniques of Theorem 1.2.

Problem 3.1 (cf. [Fuj12b, Problem 1.8]). Let $\pi : X \to Y$ be a surjective holomorphic map from Kähler manifold $X$ to a complex manifold $Y$, and $F$ be a line bundle on $X$ with a singular metric $h$ whose curvature is semi-positive. Then for a (non-zero) section $s$ of a positive multiple $F^m$ satisfying $\sup_X |s|_{h^m} < \infty$, the multiplication map

$$\Phi_s : R^q\pi_* (K_X \otimes F \otimes \mathcal{I}(h)) \otimes \mathcal{O}_X \to R^q\pi_* (K_X \otimes F^m \otimes \mathcal{I}(h^{m+1}))$$

is injective for any $q$. Here $R^q\pi_*(F)$ denotes the higher direct image of a sheaf $F$.

Theorem 2.1 can be expected to hold under the weaker assumption made in the following problem. Indeed, this problem was affirmatively solved in [Mat14] under the regularity assumption on singular metrics. It is also an interesting problem to consider the relative version of this problem in the same direction as Problem 3.1.

Problem 3.2 (cf. [Mat14, Theorem 1.5], [Fuj12a, Theorem 1.2]). Let $(F, h_F)$ and $(L, h_L)$ be (singular) hermitian line bundles with semi-positive curvature on a compact Kähler manifold $X$. Assume there exists a positive real number $a$ such that $\sqrt{-1}\Theta_{h_F}(F) \geq a\sqrt{-1}\Theta_{h_L}(L)$. Then the same conclusion as in Theorem 2.1 holds.

Fujino proposed the following problem, which asks whether one can generalize the injectivity theorem for lc pairs proved by him. The main difficulty in studying this problem is that one must handle lc singularities by analytic methods.

Problem 3.3 (cf. [Fuj11, Theorem 6.1]). Let $D$ be a simple normal crossing divisor and $F$ be a semi-positive line bundle on a compact Kähler manifold $X$. Then, for a (non-zero) section $s$ of a positive multiple $F^m$ whose zero locus $s^{-1}(0)$ contains no lc centers of $(X, D)$, the multiplication map

$$\Phi_s : H^q(X, K_X \otimes F \otimes \mathcal{O}_X(D)) \otimes \mathcal{O}_X \to H^q(X, K_X \otimes F^{m+1} \otimes \mathcal{O}_X(D))$$

is injective for any $q$. 
For a nef line bundle $F$ on a smooth projective variety $X$, it can be proven that
\[ \dim H^q(X, F^m) = O(m^{\dim X - q}) \quad \text{as} \quad m \to \infty. \]

When $X$ is merely supposed to be a compact Kähler manifold, the same conclusion can be expected. This was first posed by Demailly, and proved by Berndtsson under the stronger assumption that $F$ is semi-positive in [Ber12]. The following problem was also proved in [Mat14] when $X$ is a smooth projective variety.

**Problem 3.4** (cf. [Mat14, Theorem 4.1]). Let $F$ be a line bundle on a compact Kähler manifold $X$ and $h$ be a singular metric with semi-positive curvature on $F$. Then, for any vector bundle (or line bundle) $M$, we have
\[ \dim H^q(X, M \otimes F^m \otimes I(h^m)) = O(m^{\dim X - q}) \quad \text{as} \quad m \to \infty. \]

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