PARAMETRIZING THE RAMSEY THEORY OF VECTOR SPACES I: DISCRETE SPACES

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Abstract. We show that the Ramsey theory of block sequences in infinite-dimensional discrete vector spaces can be parametrized by perfect sets. As special cases, we prove combinatorial dichotomies for definable families of partitions and linear transformations on those spaces. We also consider the extent to which analogues of selective ultrafilters in this setting are preserved by Sacks forcing.

1. Introduction

The aim of this article is to present parametrized versions of Ramsey-theoretic dichotomies for infinite-dimensional vector spaces. Our results will typically take the form: For any suitably definable family of partitions, parametrized by reals, of these spaces or related structures, there is an infinite-dimensional subspace on which uncountably many of those partitions all have the same prescribed extremal behavior. We will explore consequences of these results, both mathematical and metamathematical.

Our setting is a countably infinite-dimensional discrete vector space $E$ over a countable, possibly finite, field $F$, having a distinguished basis $(e_n)_{n \in \omega}$. For concreteness, one may take $E = \bigoplus_{n \in \omega} F$ and let $e_n$ be the $n$th unit coordinate vector. The set of nonzero vectors in $E$ will be denoted by $E^*$. For any vector $v \in E^*$, its support is the finite set $\text{supp}(v)$ of indices $i \in \omega$ such that $e_i$ has a nonzero coefficient when $v$ is expressed as a linear combination of the $e_n$’s. By a subspace of $E$, we will always mean a linear subspace.

The primary obstacle to a satisfactory infinitary\(^1\) Ramsey theory for vector spaces is the failure of a natural analogue of the pigeonhole principle: For

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\(^1\)There is an analogue of the finite form of Ramsey’s Theorem, for finite-dimensional vector spaces over finite fields, due to Graham, Leeb, and Rothschild [14].
any finite partition of $E^*$, we would like there to be an infinite-dimensional subspace which is \textit{homogeneous} for the partition, i.e., contained entirely in one piece. When $|F| = 2$, vectors may be identified with their supports in FIN, the set of all nonempty finite subsets of $\omega$, where Hindman’s Theorem \[16\] gives such a principle. However, this fails in all other cases due to the existence of asymptotic pairs.

**Definition 1.1.** (a) A set $A \subseteq E^*$ is \textit{asymptotic} if it meets every infinite-dimensional subspace of $E$.

(b) An \textit{asymptotic pair} is a pair of disjoint asymptotic sets.

**Example 1.2.** Suppose that $|F| > 2$. The \textit{oscillation} of a nonzero vector $v$ is the number of times the nonzero coefficients change in its basis expansion with respect to $(e_n)_{n \in \omega}$, read from left to right. That is, if $\text{supp}(v) = \{n_0, \ldots, n_k\}$ where $n_0 < \cdots < n_k$, and $v = \sum_{i=0}^{k} a_i e_{n_i}$, then

\[
\text{osc}(v) = |\{i < k : a_i \neq a_{i+1}\}|.
\]

It is shown in the proof of Theorem 7 in \[21\] that on any infinite-dimensional subspace of $E$, the range of osc contains arbitrarily long intervals. It follows that the sets

\[
A_0 = \{v \in E^* : \text{osc}(v) \text{ is even}\}
\]

\[
A_1 = \{v \in E^* : \text{osc}(v) \text{ is odd}\}
\]

form an asymptotic pair. Note that the oscillation map, and thus the sets $A_0$ and $A_1$, are invariant under multiplication by nonzero scalars.

This failure of the pigeonhole principle means that we can only assert the following, in general:

\textit{For any $A \subseteq E^*$, either there is an infinite-dimensional subspace $V$ of $E$ such that $A \cap V = \emptyset$ or $A$ is asymptotic.}

Here, we view $A$ as a partition of $E^*$ into $A$ and its complement. While the above statement is trivially true by definition of “asymptotic”, we believe that it is the appropriate “weak pigeonhole principle” for infinite-dimensional vector spaces, something that will become increasingly clear in the parametrized setting (see Section 3 of \[36\] for a related discussion).

The primary results we wish to “parametrize” are dichotomies for definable partitions of certain spaces of sequences in $E$. A \textit{block sequence} in $E$ is a sequence of nonzero vectors $x_n$ such that

\[
\max(\text{supp}(x_n)) < \min(\text{supp}(x_{n+1}))
\]

for all $n$ less than the length of the sequence (which may be finite or $\omega$). Note that any block sequence is linearly independent. By taking appropriate linear combinations of basis vectors and passing to a subsequence, it is easy to see that every infinite-dimensional subspace of $E$ contains an infinite block sequence (Lemma 2.1 in \[34\]).
The set of all infinite block sequences in $E$ will be denoted by $E^{[\infty]}$. It inherits a Polish topology, and thus a structure of Borel and analytic sets, as a subspace of the product space $E^\omega$. The set of all finite block sequences will be denoted by $E^{[<\infty]}$.

We write $\langle X \rangle$ for the linear span of a block sequence $X$, and order block sequences according to their spans: $Y \preceq X$ if $\langle Y \rangle \subseteq \langle X \rangle$, in which case we abuse terminology and say that $Y$ is a block subsequence of $X$. For $n \in \omega$, we write $X/n$ (or $X/\vec{v}$ for a finite sequence of vectors $\vec{v}$) for the tail of $X$ whose supports are strictly above $n$ (or that of $\vec{v}$). We write $Y \preceq^* X$ if $Y/n \preceq X$ for some $n \in \omega$.

The ordering on block sequences has the following property: If $(X_n)_{n \in \omega}$ is a sequence in $E^{[\infty]}$ such that $X_{n+1} \preceq^* X_n$ of all $n \in \omega$, then there is an $X \in E^{[\infty]}$ such that $X \preceq^* X_n$ for all $n \in \omega$. Such an $X$ is called a diagonalization of the sequence $(X_n)_{n \in \omega}$.

The dichotomies for block sequences are phrased in terms of certain games. Given an $X \in E^{[\infty]}$, the asymptotic game played below $X$, $F[X]$, is the two player game where player I goes first and plays a natural number $n_0$, and player II responds with a nonzero vector $y_0 \in \langle X/n_0 \rangle$. They continue in this fashion, alternating with I playing an $n_k \in \omega$ and II playing a $y_k \in \langle X/n_k \rangle$, for each $k \in \omega$. We also demand that II’s moves form a block sequence.

$$F[X] : \begin{array}{cccccc}
I & n_0 & n_1 & n_2 & \cdots \\
II & y_0 & y_1 & y_2 & \cdots 
\end{array}$$

The Gowers game played below $X$, $G[X]$, is defined similarly, with I playing infinite block sequences $Y_k \preceq X$ and II responding with nonzero vectors $y_k \in \langle Y_k \rangle$, again forming a block sequence, for each $k \in \omega$.

$$G[X] : \begin{array}{cccccc}
I & Y_0 & Y_1 & Y_2 & \cdots \\
II & y_0 & y_1 & y_2 & \cdots 
\end{array}$$

Note that $F[X]$ may be viewed as a variant of $G[X]$ where I is restricted to playing tails of $X$. In both games, the outcome is the block sequence $(y_k)_{k \in \omega}$ consisting of II’s moves.

Strategies for the players in these games can be defined as functions from all finite sequences of possible previous moves by the opposing player to valid moves by the current player. Given a set $\Lambda \subseteq E^{[\infty]}$, we say that a player has a strategy for playing into (or out of) $\Lambda$ if they have a strategy all of whose outcomes lie in (or out of) $\Lambda$. We stress again that the “outcomes” here are the result of II’s moves, even when referring to a strategy for I.

Isolating the combinatorial content of a dichotomy for block sequences in Banach spaces, due to Gowers [13], Rosendal proved the following:

**Theorem 1.3** (Rosendal [31]). If $\Lambda \subseteq E^{[\infty]}$ is analytic, then there is an $X \in E^{[\infty]}$ such that either:

1. I has a strategy in $F[X]$ for playing out of $\Lambda$, or
2. II has a strategy in $G[X]$ for playing into $\Lambda$. 


The conclusions (1) and (2) in Theorem 1.3 are mutually exclusive and provide a strategy for the player in the game which is a priori more difficult for that player. If $A$ is a clopen set in $E^{[\omega]}$ which depends only on the first coordinate, then $A$ induces a partition of $E^*$ and the games $F[X]$ and $G[X]$ are decided after one move; in this case, Theorem 1.3 reduces to a form of the weak pigeonhole principle stated above for the subspace spanned by $X$.

In [34], the author showed that Theorem 1.3 could be “localized” in the sense that the witness $X$ to the conclusion could always be found in a prescribed $(p^+)$-family (defined in Section 3 below) of block sequences:

**Theorem 1.4** (Smythe [34]). Let $\mathcal{H}$ be a $(p^+)$-family on $E$. If $\mathcal{A} \subseteq E^{[\omega]}$ is analytic, then there is an $X \in \mathcal{H}$ such that either:

1. I has a strategy in $F[X]$ for playing out of $\mathcal{A}$, or
2. II has a strategy in $G[X]$ for playing into $\mathcal{A}$.

It is further shown in [34] that if the family $\mathcal{H}$ satisfies the additional property of being strategic (also defined in Section 3), then Theorem 1.4 can be extended to all “reasonable definable” sets $\mathcal{A} \subseteq E^{[\omega]}$, i.e., those in the inner model $L(R)$, assuming large cardinal hypotheses. While the set $E^{[\omega]}$ of all block sequences is trivially a strategic $(p^+)$-family, such families may be much smaller; they can (consistently) be filters with respect to the ordering on block sequences. Strategic $(p^+)$-filters can then be characterized as exactly those filters which are generic over $L(R)$ for the partial order $(E^{[\omega]}, \leq)$, in the sense of forcing, and are said to have “complete combinatorics”.

This progression of results mirrors those for the space $[\omega]^{\omega}$ of infinite subsets of the natural numbers $\omega$: Theorems of Galvin and Prikry [10] and Silver [32] established that if $A$ is an analytic subset of $[\omega]^{\omega}$, then there is an $x \in [\omega]^{\omega}$ all of whose infinite subsets are either contained in or disjoint from $A$, a far-reaching generalization of Ramsey’s Theorem. These results were then localized to selective coideals by Mathias [24] and to semiselective coideals by Farah [9]. Todorčević [9] showed that selective ultrafilters are generic over $L(R)$ for $([\omega]^{\omega}, \subseteq)$, under large cardinal hypotheses.

Recall that a subset of a Polish space is perfect if it is closed and has no isolated points. Nonempty perfect sets necessarily contain homeomorphic copies of the Cantor space $2^{\omega}$ and thus are uncountable. The paradigmatic “parametrized” Ramsey theorem is the following result, due independently to Miller and Todorčević:

**Theorem 1.5** (Miller–Todorčević [27]). If $\mathcal{A} \subseteq \mathbb{R} \times [\omega]^{\omega}$ is analytic, then there is a nonempty perfect set $P \subseteq \mathbb{R}$ and an $x \in [\omega]^{\omega}$ such that either $P \times [x]^{\omega} \subseteq \mathcal{A}$ or $(P \times [x]^{\omega}) \cap \mathcal{A} = \emptyset$.

In other words, if we are given a suitably definable family $\{A_t : t \in \mathbb{R}\}$ of partitions of $[\omega]^{\omega}$, then we can find a perfect subfamily $\{A_t : t \in P\}$ and a set $x$ which is homogeneous for all of these perfectly-many partitions simultaneously, and in the same way. By standard facts, $\mathbb{R}$ can be replaced
by any uncountable Polish space here. Theorem 1.5 was localized to semiselective coideals by Farah in [9]. The metamathematical counterpart to Theorem 1.5, due to Baumgartner and Laver, is that selective ultrafilters are preserved by Sacks forcing $\mathbb{S}$:

**Theorem 1.6 (Baumgartner–Laver [3]).** If $\mathcal{U}$ is a selective ultrafilter on $\omega$ and $g$ is $\mathbb{V}$-generic for $\mathbb{S}$, then $\mathcal{U}$ generates a selective ultrafilter in $\mathbb{V}[g]$.

Similar parametrized Ramsey theorems have been established in other settings, see [25], [26], [37], and [40]. Particularly relevant to the present work are results of Zheng [39] on parametrized forms of Milliken’s theorem [28] for the space of infinite block sequences in $\text{FIN}$, and Kawach [18] on parametrized forms of dichotomies (originally due to Gowers [12]) for the spaces $\text{FIN}_k$, $\text{FIN}_{\pm k}$, and the Banach space $c_0$. Calderón, Di Prisco, and Mijares [5] have also recently developed local forms of the parametrized results for $\text{FIN}_k$.

The main result of the present article is the following parametrized version of Theorem 1.4:

**Theorem 1.7.** Let $\mathcal{H}$ be a strategic $(p^+)$-family on $E$. If $\mathcal{A} \subseteq \mathbb{R} \times E^{[\infty]}$ is analytic, then there is a nonempty perfect set $P \subseteq \mathbb{R}$ and an $X \in \mathcal{H}$ such that either:

1. $I$ has a strategy $\sigma$ in $F[X]$ such that $P \times [\sigma] \subseteq \mathcal{A}^C$, or
2. for every $t \in P$, $II$ has a strategy in $G[X]$ for playing into $A_t$.

Here, $\mathcal{A}_t = \{X \in E^{[\infty]} : (t, X) \in \mathcal{A}\}$ is the $t$-slice of $\mathcal{A}$ for $t \in \mathbb{R}$, and $[\sigma]$ the set of all outcomes of $F[X]$ wherein player I follows $\sigma$. We note that even in the non-localized case, where $\mathcal{H} = E^{[\infty]}$ and Theorem 1.7 becomes a parametrized form of Theorem 1.3, our result is new. A major component of our proof will be the following analogue of Theorem 1.6:

**Theorem 1.8.** If $\mathcal{F}$ is a strategic $(p^+)$-filter on $E$ and $g$ is $\mathbb{V}$-generic for $\mathbb{S}$, then $\mathcal{F}$ generates a $(p^+)$-filter in $\mathbb{V}[g]$.

We note some anomalies in our results: First, the conclusions for players I and II in Theorem 1.7 are asymmetric — we will see in Section 6 that this is necessary and these conclusions are appropriately sharp. Second, in Theorem 1.7, we make the additional assumption, not present in Theorem 1.4, that $\mathcal{H}$ is strategic, while in Theorem 1.8 we do not assert that the property of being strategic is preserved. The former may reflect a deficiency in our method of proof, but the latter is necessary.

**Theorem 1.9.** Assume $|F| > 2$. If $\mathcal{F}$ is a strategic $(p^+)$-filter on $E$ and $g$ is $\mathbb{V}$-generic for $\mathbb{S}$, then $\mathcal{F}$ fails to generate a strategic filter in $\mathbb{V}[g]$.

Consequently, it is consistent with ZFC that there are $(p^+)$-filters which are not strategic, answering a question left unresolved by [34]. It follows that there is a breakdown between which properties guarantee a family witnesses Theorem 1.3 for analytic sets and those which are needed to extend beyond
analytic sets under large cardinals. This feature is unique to our setting, as there is no such breakdown on $\omega$ or FIN (equivalently, when $|F| = 2$).

The rest of this article is organized as follows: Section 2 begins by showing that the weak form of the pigeonhole principle mentioned above can be parametrized by perfect sets (Theorem 2.1). We then apply this to obtain a dichotomy for parametrized families of linear transformations on $E$ (Theorem 2.3). Section 2 is entirely self-contained and uses only basic set theory. We review the relevant definitions for families of block sequences from [34] in Section 3. Sacks forcing and the fusion method are reviewed in Section 4.

In Section 5, we prove a local form (Theorem 5.1) of the parametrized weak pigeonhole principle from Section 2 and then use it to establish our main preservation result, Theorem 1.8. In Section 6, we define perfectly strategically Ramsey sets and prove Theorem 1.7 for open sets (Lemma 6.7). This is extended to all analytic sets in Section 7, completing the proof of Theorem 1.7. Theorem 1.9 is proved in Section 8. We conclude in Section 9 with a discussion of possible improvements and limitations of our results.

Throughout, we will make use of standard descriptive set-theoretic facts about Borel and analytic subsets of Polish spaces, all of which can be found in [19]. We will also use the method of forcing and absoluteness, for which standard references are [17] and [20]. While these metamathematical techniques are employed in the proof of Theorem 1.7, we stress that this result is, ultimately, established in ZFC. A familiarity with infinite-dimensional Ramsey theory, as in [37], while helpful, is not necessary. Applications of our results to Banach spaces, including a parametrization of Gowers’s dichotomy, will be in a separate, forthcoming article [33].

2. Partitions and linear transformations

We begin by considering families of partitions of the underlying vector space and establish a parametrized form (Theorem 2.1) of the “weak pigeonhole principle” mentioned above. We then apply this to parametrized families of linear transformations (Theorem 2.3). In both results, the existence of a subspace satisfying the conclusion for a single partition or linear transformation, respectively, is obvious. The novelty here lies in the fact that we obtain the same conclusion for perfectly many partitions or transformations simultaneously, all witnessed by the same subspace. We hope that these preliminary results suggest possible future applications, in addition to being interesting in their own right.

We say that a family $\{A_t : t \in \mathbb{R}\}$ of subsets of $E^*$ is Borel if the function $t \mapsto A_t$ is Borel measurable $\mathbb{R} \to 2^{E^*}$, or equivalently, it is Borel when viewed as a subset $\mathbb{R} \times 2^{E^*}$.

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2We gain nothing here by allowing the family to be analytic, as the graph of a total function is analytic if and only if it is Borel (Theorem 14.12 in [19]).
**Theorem 2.1.** For every Borel family \( \{A_t : t \in \mathbb{R}\} \) of subsets of \( E^* \), there is a nonempty perfect set \( P \subseteq \mathbb{R} \) and an infinite-dimensional subspace \( V \) of \( E \) such that either:

1. for all \( t \in P \), \( A_t \cap V = \emptyset \), or
2. for all \( t \in P \), \( A_t \) is asymptotic below \( V \).

Here, a set \( A \subseteq E^* \) is asymptotic below a subspace \( V \) (or block sequence \( X \), respectively) if \( A \) meets every infinite-dimensional subspace of \( V \) (\( \langle X \rangle \)).

We will need the following technical lemma (cf. Lemma 3.1.1 in [38]).

**Lemma 2.2.** Suppose that \( \{X_\alpha\}_{\alpha<\omega_1} \) and \( \{t_\alpha\}_{\alpha<\omega_1} \) are \( \omega_1 \)-sequences in \( E^{[\infty]} \) and \( \mathbb{R} \), respectively, such that for all \( \alpha < \beta < \omega_1 \), \( X_\beta \preceq^* X_\alpha \) and \( t_\alpha \neq t_\beta \).

Then, there is a countable subset \( \Sigma \) of \( \omega_1 \) and a \( Z \in E^{[\infty]} \) such that:

1. \( \{t_\alpha : \alpha \in \Sigma, \prec\} \) is order isomorphic to \( (\mathbb{Q}, \prec) \), and
2. \( Z \preceq X_\alpha \) for all \( \alpha \in \Sigma \).

**Proof.** Enumerate \( \mathbb{Q} \) as \( \{r_n : n \in \omega\} \). We construct sets \( F_n \in [\omega_1]^n \), \( A_n \in [\omega_1]^\omega_1 \), and vectors \( z_0, \ldots, z_{n-1} \), such that \( F_n \subseteq F_{n+1} \) and \( A_n \supseteq A_{n+1} \), for \( n \in \omega \), as follows: Begin by letting \( F_0 = \emptyset \), \( A_0 = \omega_1 \), and \( z_{-1} = 0 \).

Suppose we have defined \( F_n, A_n, \) and \( z_{n-1} \) as indicated. Choose \( \alpha \in A_n \setminus F_n \) and put \( F_{n+1} = F_n \cup \{\alpha\} \) so that \( \{t_\beta : \beta \in F_{n+1}\}, \prec \cong \{r_i : i \leq n\}, \prec \) and each interval in \( \mathbb{R} \) determined by the \( t_\beta \), for \( \beta \in F_{n+1} \), contains uncountably many of the \( t_\alpha \)'s. Writing

\[
Q = \left\{ z \in E^* : \min(\text{supp}(z)) > \max(\text{supp}(z_{n-1})) \text{ and } z \in \bigcap_{\beta \in F_{n+1}} \langle X_\beta \rangle \right\},
\]

for each \( \alpha \in A_n \), choose \( z_\alpha \in Q \cap \langle X_\alpha \rangle \). Then, for uncountably many \( \alpha \)'s in \( A_n \), the \( z_\alpha \)'s coincide, call their value \( z_n \) and collect these \( \alpha \)'s as \( A_{n+1} \).

Let \( \Sigma = \bigcup_{n \in \omega} F_n \) and \( Z = \langle z_n \rangle_{n \in \omega} \). Property (i) has been ensured by construction. For property (ii), it suffices to show that \( z_n \in \langle Y_\alpha \rangle \) for all \( n \in \omega \) and \( \alpha \in \Sigma \). Suppose that we have verified this for \( z_0, \ldots, z_{n-1} \). \( z_\alpha \) was chosen explicitly to be in \( \langle X_\beta \rangle \) for all \( \beta \in F_{n+1} \), while the choice of \( A_{n+1} \), and the fact that \( \Sigma \setminus F_{n+1} \subseteq A_{n+1} \), ensures that \( z_n \in \langle X_\alpha \rangle \) for all \( \alpha \in \Sigma \setminus F_{n+1} \), which proves the claim. \( \square \)

**Proof of Theorem 2.1.** Suppose that there is no nonempty perfect set \( P \subseteq \mathbb{R} \) and no \( X \in E^{[\infty]} \) such that (2) holds for \( P \) and \( V = \langle X \rangle \). We will construct witnesses to (1). By our assumptions, the set

\[
C_0 = \{ t \in \mathbb{R} : A_t \text{ is asymptotic (for } E) \}
\]

contains no perfect subset.\(^3\) Consequently, its complement is uncountable, as the complement of any countable set must contain a perfect set, and in particular, nonempty. So, we may choose a \( t_0 \in \mathbb{R} \setminus C_0 \) and an \( X_0 \in E^{[\infty]} \)

\(^3\)It is tempting to use the perfect set property to conclude that \( C_0 \) is countable. However, this set is coanalytic, and the perfect set property is not provable for coanalytic sets in ZFC. We do not know if \( C_0 \) is necessarily of lower complexity, i.e., whether it is Borel.
such that \( A_\alpha \cap \langle X_\alpha \rangle = \emptyset \). We continue in this fashion to construct sequences \( t_\alpha \in \mathbb{R} \) and \( X_\alpha \in E^{[\infty]} \), for \( \alpha < \omega_1 \), such that all of the \( t_\alpha \)'s are distinct, \( X_\beta \preceq X_\alpha \) whenever \( \beta < \alpha \), and \( A_{t_\alpha} \cap \langle X_\alpha \rangle = \emptyset \).

Suppose that, for some \( \gamma < \omega_1 \), we have constructed \( t_\alpha \) and \( X_\alpha \) as described for all \( \alpha < \gamma \). If \( \gamma = \beta + 1 \), then consider the set

\[
C_\gamma = \{ t \in \mathbb{R} : A_t \text{ is asymptotic for } X_\beta \}.
\]

By assumption, this set contains no perfect subset, so its complement is uncountable, and we may choose a \( t_\gamma \in \mathbb{R} \setminus C_\gamma \) distinct from \( t_\alpha \), for all \( \alpha < \gamma \), and an \( X_\gamma \preceq X_\beta \) such that \( A_{t_\gamma} \cap \langle X_\gamma \rangle = \emptyset \). If instead, \( \gamma \) is a limit ordinal, first diagonalize to obtain an \( X \in E^{[\infty]} \) such that \( X \preceq X_\alpha \) for all \( \alpha < \gamma \). Then, argue as in the successor case to obtain a \( t_\gamma \) distinct from \( t_\alpha \), for all \( \alpha < \gamma \), and an \( X_\gamma \preceq X \) such that \( A_{t_\gamma} \cap \langle X_\gamma \rangle = \emptyset \).

Let \( \Sigma \) and \( Z \) be as in Lemma 2.2, as applied to the sequences \( (X_\alpha)_{\alpha < \omega_1} \) and \( (t_\alpha)_{\alpha < \omega_1} \) constructed above. Put \( P = \{ t_\alpha : \alpha \in \Sigma \} \), a nonempty perfect subset of \( \mathbb{R} \), and \( V = \langle Z \rangle \). Note that, by possibly shrinking \( P \), we may assume that the Borel map \( t \mapsto A_t \) is actually continuous on \( P \). Take \( t \in P \), say with \( \lim_n t_{\alpha_n} = t \), where each \( \alpha_n \) is in \( \Sigma \). Towards a contradiction, suppose that \( v \in A_t \cap V \), with \( v \neq 0 \). By continuity, for sufficiently large \( n \), we have that \( v \in A_{t_{\alpha_n}} \), but \( Z \preceq X_{\alpha_n} \), so we have that \( v \in A_{t_{\alpha_n}} \cap \langle X_{\alpha_n} \rangle \), and thus \( A_{t_{\alpha_n}} \cap \langle X_{\alpha_n} \rangle \neq \emptyset \), contrary to how \( t_\alpha \) and \( X_\alpha \) were chosen above. Thus, \( A_t \cap V = \emptyset \) for all \( t \in P \), verifying (1).

Let \( L(E) \) denote the set of all linear transformations from \( E \) to itself. \( L(E) \) inherits the structure of an uncountable Polish space when viewed as a subspace of \( E^E \), or equivalently, with the topology of pointwise convergence. A family \( \{ T_t : t \in \mathbb{R} \} \) of transformations in \( L(E) \) is again Borel if the mapping \( t \mapsto T_t \) is Borel measurable.

**Theorem 2.3.** For every Borel family \( \{ T_t : t \in \mathbb{R} \} \) of linear transformations on \( E \), there is a nonempty perfect set \( P \subseteq \mathbb{R} \) and an infinite-dimensional subspace \( V \) of \( E \) such that either:

1. for all \( t \in P \), \( V \subseteq \ker(T_t) \), or
2. for all \( t \in P \), \( T_t \upharpoonright V \) is injective.

**Proof.** For each \( t \in \mathbb{R} \), let

\[
A_t = \{ v \in E^*: T_t(v) \neq 0 \}.
\]

Then, \( \{ A_t : t \in \mathbb{R} \} \) is a Borel family of subsets of \( E^* \). By Theorem 2.1, there is a nonempty perfect set \( P \subseteq \mathbb{R} \) and an infinite-dimensional subspace \( V \) of \( E \) such that either (1') for all \( t \in P \), \( A_t \cap V = \emptyset \), or (2') for all \( t \in P \), \( A_t \) is asymptotic below \( V \). If (1') holds, then for all \( t \in P \) and \( v \in V \), \( T_t(v) = 0 \), which proves (1). So, we may suppose that (2') holds.

For \( t \in P \), if \( T_t \upharpoonright V \) has infinite-dimensional kernel, then \( A_t \cap \ker(T_t \upharpoonright V) = \emptyset \), contradicting that \( A_t \) is asymptotic below \( V \). Thus, \( T_t \upharpoonright V \) has
finite-dimensional kernel for each \( t \in P \). For each \( n \in \omega \), let
\[
P_n = \{ t \in P : \ker(T_t \downarrow V) \subseteq \langle e_0, \ldots, e_n \rangle \}.
\]
Each \( P_n \) is Borel and \( P = \bigcup_{n \in \omega} P_n \), so since \( P \) is uncountable, there is some \( n_0 \in \omega \) for which \( P_{n_0} \) is uncountable and thus contains a nonempty perfect subset, say \( Q \). Let \( W = \{ v \in V : \min(\text{supp}(v)) > n_0 \} \). It follows that for each \( t \in Q \), \( \ker(T_t \downarrow W) = \{ 0 \} \), and so \( T_t \downarrow W \) is injective. \( \square \)

Theorem 2.3 can be extended to allow for transformations \( T : E \to E' \), for any countable \( F \)-vector space \( E' \). One might ask if we can improve (2) to get a perfect family of transformations and an infinite-dimensional subspace \( V \) on which all of the transformations are isomorphisms. This is not possible, as the following example shows.

**Example 2.4.** Let \( \mathcal{A} \) be a nonempty perfect almost disjoint collection of infinite-dimensional subspaces of \( E \), i.e., for any distinct \( X, Y \in \mathcal{A} \), \( X \cap Y \) is finite-dimensional. Such a family exists by Proposition 2.2 of [35]. For each \( X \in \mathcal{A} \), let \( T_X : E \to X \) be an isomorphism. Then, if \( X, Y \in \mathcal{A} \) are distinct and \( V \) is an arbitrary infinite-dimensional subspace, \( T_X \downarrow V \) and \( T_Y \downarrow V \) have almost disjoint ranges. In particular, there cannot be any nontrivial subfamily of the \( T_X \)'s such that every \( T_X \downarrow V \) maps onto \( V \).

### 3. Families of block sequences

Theorem 1.7 is phrased in terms of families of block sequences satisfying certain combinatorial properties originally isolated by the author in [34]. These properties, which we review below, are analogous to selectivity of coideals on \( \omega \) (see [9] or [24]).

**Definition 3.1.** (a) We say that \( \mathcal{H} \subseteq E^{[\omega]} \) is a family on \( E \) if it is nonempty and closed upwards with respect to \( \preceq^* \).

(b) A family \( \mathcal{F} \) is a filter on \( E \) if for all \( X, Y \in \mathcal{F} \), there is a \( Z \in \mathcal{F} \) such that \( Z \preceq X \) and \( Z \preceq Y \).

Given a family \( \mathcal{H} \) and an \( X \in \mathcal{H} \), we let \( \mathcal{H} \upharpoonright X = \{ Y \in \mathcal{H} : Y \preceq X \} \).

**Definition 3.2.** Suppose \( \mathcal{H} \) is a family on \( E \) and \( X \in \mathcal{H} \). A set \( D \subseteq E \) is \( \mathcal{H} \)-dense below \( X \) if for all \( Y \in \mathcal{H} \upharpoonright X \), there is a \( Z \preceq Y \) such that \( \langle Z \rangle \subseteq D \). Likewise, a set \( D \subseteq E^{[\omega]} \) is \( \mathcal{H} \)-dense below \( X \) if for all \( Y \in \mathcal{H} \upharpoonright X \), there is a \( Z \preceq Y \) such that \( \langle Z \rangle \subseteq D \).

Note that if \( \mathcal{F} \) is a filter on \( E \), then a set is \( \mathcal{F} \)-dense below some \( X \in \mathcal{F} \) if and only if it is \( \mathcal{F} \)-dense below every \( X \in \mathcal{F} \); we just call such sets \( \mathcal{F} \)-dense.

The next definition specifies a certain largeness condition for families and plays the role of “coideal” in this setting (and the role of “ ultra” for filters).

**Definition 3.3.** A family \( \mathcal{H} \) on \( E \) is full if whenever \( D \subseteq E \) is \( \mathcal{H} \)-dense below some \( X \in \mathcal{H} \), there is a \( Z \in \mathcal{H} \upharpoonright X \) such that \( \langle Z \rangle \subseteq D \).
While the preceding definition was used in [34], a more enlightening equivalent condition is that full families are exactly those which witness a form of the weak pigeonhole principle described in the Introduction.

**Lemma 3.4** (cf. Proposition 3.3 in [36]). A family \( \mathcal{H} \) on \( E \) is full if and only if for every \( A \subseteq E^* \), there is an \( X \in \mathcal{H} \) such that either \( A \cap \langle X \rangle = \emptyset \) or \( A \) is asymptotic below \( X \).

*Proof.* The proof follows immediately from that fact that if \( A = E^* \setminus D \), then \( D \) is \( \mathcal{H} \)-dense below some \( X \in \mathcal{H} \) if and only if \( A \) is not asymptotic below any \( Y \in \mathcal{H} \restriction X \). \( \square \)

Note that in this criterion, we are only asserting that \( A \) is asymptotic “below” \( X \), rather than globally. This is unavoidable, as \( A \) may be disjoint from some subspace which does not contain any block sequence in \( \mathcal{H} \).

**Definition 3.5.** A family \( \mathcal{H} \) on \( E \) is:

(a) a \((p)\)-family if whenever \((X_n)_{n \in \omega}\) is a \( \preceq \)-decreasing sequence in \( \mathcal{H} \), there is a diagonalization of \((X_n)_{n \in \omega}\) in \( \mathcal{H} \).

(b) a \((p+)\)-family if it is a full \((p)\)-family.

(c) strategic if whenever \( \alpha \) is a strategy for \( \text{II} \) in \( G[X] \), for some \( X \in \mathcal{H} \), there is an outcome of \( \alpha \) in \( \mathcal{H} \).

We have remarked that \( E^{[\omega]} \) is itself a \((p)\)-family, and it is trivially both full and strategic. Less trivial examples, including strategic \((p+)\)-filters, can be constructed using additional set-theoretic assumptions such as the Continuum Hypothesis (CH) or Martin’s Axiom (MA), or by forcing with \((E^{[\omega]}, \preceq)\), however, the existence of full filters in \( E^{[\omega]} \) cannot be proved in ZFC alone; see Sections 5 and 6 of [34].

Fullness of a family \( \mathcal{H} \) tells us that it meets all \( \mathcal{H} \)-dense subsets of \( E^* \). An important consequence of Theorem 1.4 is that, for strategic \((p+)\)-families, we can extend this to all \( \mathcal{H} \)-dense subsets of \( E^{[\omega]} \), provided those subsets are analytic. This is a ZFC instance of the genericity of strategic \((p+)\)-families.

**Lemma 3.6.** Let \( \mathcal{H} \) be a \((p+)\)-family on \( E \) and \( X \in \mathcal{H} \). If \( D \subseteq E^{[\omega]} \) is \( \mathcal{H} \)-dense below \( X \), downwards closed with respect to \( \preceq \), and analytic, then there is a \( Y \in \mathcal{H} \restriction X \) such that \( \text{II} \) has a strategy in \( G[Y] \) for playing into \( D \). Consequently, if \( \mathcal{H} \) is strategic, then \((\mathcal{H} \restriction X) \cap D \neq \emptyset \).

*Proof.* By Theorem 1.4, there is an \( Y \in \mathcal{H} \restriction X \) such that either (1) has a strategy in \( F[Y] \) for playing out of \( D \) or (2) has a strategy in \( G[Y] \) for playing into \( D \). Since \( D \) is \( \mathcal{H} \)-dense below \( X \), there is a \( Z \preceq Y \) such that \( Z \in D \). Then, in the game \( F[Y] \), \( \text{II} \) may play in such a way that their moves always lie in \( \langle Z \rangle \), and as \( D \) is \( \preceq \)-downwards closed, the outcome will also lie in \( D \), contrary to (1). Thus, (2) holds, from which the claim follows. \( \square \)

4. Perfect sets and Sacks forcing

We view the set of all finite binary sequences \( 2^{<\omega} \) as a complete binary tree, ordered by initial segment \( \subseteq \), and \( 2^\omega \) as its set of infinite branches. A
subtree of $2^{<\omega}$ is a subset $p \subseteq 2^{<\infty}$ which is downwards closed with respect to $\sqsubseteq$. We say that $p$ is perfect if for every $s \in p$, there are $t, u \in p$ such that $s \sqsubseteq t, u$ and which are incomparable with respect to $\sqsubseteq$. For such a $p$, we let $[p]$ denote the set of all infinite branches through $p$, a perfect subset of $2^\omega$. Conversely, every perfect subset of $2^\omega$ is of the form $[p]$ for some perfect subtree $p$ of $2^{<\omega}$.

Sacks forcing is the set $S$ of all nonempty perfect subtrees of $2^{<\omega}$, ordered by inclusion: $p \leq q$ if $p \subseteq q$, or equivalently, $[p] \subseteq [q]$. Our presentation of Sacks forcing follows [2]. For a general overview of Sacks forcing, see the survey [11].

For $p \in S$ and $s \in p$, we let 
\[ p|s = \{ t \in p : t \sqsubseteq s \text{ or } s \sqsubseteq t \}, \]
and note that $p|s \in S$ and $p|s \leq p$. We say that $s \in p$ is a branching node of $p$ if both $s \prec 0$ and $s \prec 1$ are in $p$. For $n \in \omega$, a node $s \in p$ has branching level $n$ if there are exactly $n$ branching nodes $t \in p$ such that $t \sqsubseteq s$. We let 
\[ \ell(n, p) = \{ s \in p : s \text{ is a minimal node of branching level } n \}. \]
Note that $|\ell(n, p)| = 2^n$ and $p = \bigcup_{n \in \ell(n, p)} p|s$ for each $n \in \omega$.

We collect a few lemmas about Sacks forcing which will be used later on.

Lemma 4.1 (Lemma 1.5 in [2]). Suppose $p \in S$ and $n \in \omega$. If $q \leq p$, then there is an $s \in \ell(n, p)$ such that $q$ and $p|s$ are compatible.

For $n \in \omega$, we write $p \leq_n q$ if $p \leq q$ and $\ell(n, p) = \ell(n, q)$. A sequence $(p_n)_{n \in \omega}$ of conditions in $S$ is a fusion sequence if $p_{n+1} \leq_n p_n$ for all $n \in \omega$. We call $p_\infty = \bigcap_{n \in \omega} p_n$ the fusion of $(p_n)_{n \in \omega}$. The following important fact is known as the Fusion Lemma:

Lemma 4.2 (Lemma 1.4 in [2]). If $(p_n)_{n \in \omega}$ is a fusion sequence in $S$, then $p_\infty = \bigcap_{n \in \omega} p_n$ is a condition in $S$ and $p_\infty \leq_n p_n$ for all $n \in \omega$.

At the risk of imprecision, we say that a model $M$ is sufficient if it is transitive and satisfies a large enough fragment of ZFC for whatever results are needed. Given such an $M$, if $G \subseteq S$ is an $M$-generic filter for $S$, then there is a unique $g \in \bigcap_{p \in G} [p]$, called an $M$-generic Sacks real. Note that $G$ is definable from $g$ as $G = \{ p \in S : g \in [p] \}$, so $M[g] = M[G]$, see Lemma 1.1 of [2]. The next lemma is folklore, it is close to saying that $S$ is a proper forcing, but we include the proof, as it is a typical application of fusion.

Lemma 4.3. Let $M$ be a sufficient countable model. For any $p \in S \cap M$, there is a $q \leq p$ such that for every $g \in [q]$, $g$ is $M$-generic for $S$.

Proof. Let $\{ D_n : n \in \omega \}$ enumerate those subsets of $S$ in $M$ which are dense open in $S \cap M$. Choose $p_0 \in D_0$ with $p_0 \leq p$.

Suppose we have chosen $p_0, \ldots, p_n$ with $p_{n+1} \leq_i p_i$ for each $i < n$. In order to find $p_{n+1}$, we refine $p_n$ cone-by-cone: Enumerate $\ell(n, p_n)$ as $\{ t_0, \ldots, t_k \}$. For each $\ell \leq k$, choose $q^\ell_{n+1} \in D_{n+1}$ with $q^\ell_{n+1} \leq p_n|t_\ell$. Let $p_{n+1} = \bigcup_{\ell \leq k} q^\ell_{n+1}$. Then, $p_{n+1} \in S$ and $p_{n+1} \leq_n p_n$. 

Let \( q = \bigcap_{n \in \omega} p_n \leq p \), the fusion of \( (p_n)_{n \in \omega} \). Let \( g \in [q] \) and put \( G = \{ r \in S : g \in [r] \} \), the corresponding filter. Since \( g \in [p_0] \) and \( p_0 \in D_0 \), we have that \( G \cap D_0 \neq \emptyset \). More generally, for each \( n \in \omega \), \( g \in [p_{n+1}] \), so there is some \( \ell \in \ell(n, p_n) \) for which \( g \in [q_{n+1}] \). This implies \( q_{n+1} = G \), and since \( q_{n+1} \) was chosen to be in \( D_{n+1} \), we have that \( G \cap D_{n+1} \neq \emptyset \). Hence, \( g \) is \( M \)-generic for \( S \).

Our last lemma says that every sequence of ground model sets in a generic extension by \( S \) is covered by a sequence of finite sets, with prescribed size, from the ground model. This fact, known as the Sacks property, can also be proved using a routine fusion argument.

**Lemma 4.4** (Lemma 27 in \([11]\)). For any \( p \in S \), if \( p \Vdash \bar{f} : \omega \to V \), then there is a sequence of finite sets \( \langle F_n \rangle_{n \in \omega} \in V \) such that \( |F_n| \leq 2^n \) for each \( n \in \omega \) and a \( q \leq p \) such that \( q \Vdash \forall n \in \omega (\bar{f}(n) \in F_n) \).

\[ \square \]

## 5. Localization and Preservation

Our next result is a localized form of Theorem 2.1.

**Theorem 5.1.** Let \( \mathcal{H} \) be a strategic \( (p^+) \)-family on \( E \). For every Borel family \( \{ A_t : t \in \mathbb{R} \} \) of subsets of \( E^* \), there is a nonempty perfect set \( P \subseteq \mathbb{R} \) and an \( X \in \mathcal{H} \) such that either:

1. for all \( t \in P \), \( A_t \cap \langle X \rangle = \emptyset \), or
2. for all \( t \in P \), \( A_t \) is asymptotic below \( X \).

**Proof.** As in the proof of Theorem 2.1, we assume that there is no nonempty perfect set \( P \subseteq \mathbb{R} \) and no \( X \in \mathcal{H} \) such that (2) holds for \( P \) and \( X \). We will produce a witness to (1).

Consider the set

\[ D = \{ Y \in E^{[\omega]} : \exists (t_n)_{n \in \omega} \in \mathbb{R}^{\omega} [\{ t_n : n \in \omega \}, <] \cong (\mathbb{Q}, <) \land \forall n (A_{t_n} \cap \langle Y \rangle = \emptyset) \} \].

Then, \( D \) is analytic and downwards closed with respect to \( \preceq \). We will show that \( D \) is \( \mathcal{H} \)-dense. Once that is accomplished, we can apply Lemma 3.6 to obtain a \( Y \in \mathcal{H} \cap D \) and a witness \( (t_n)_{n \in \omega} \in \mathbb{R}^{\omega} \) such that \( \{ t_n : n \in \omega \}, < \) \( \cong (\mathbb{Q}, <) \) and \( A_{t_n} \cap \langle Y \rangle = \emptyset \) for all \( n \in \omega \). Letting \( P = \{ t_n : n \in \omega \} \), which may further shrink so as to make \( t \mapsto A_t \) continuous on \( P \), we can then argue exactly as in the last part of the proof of Theorem 2.1 to show that for all \( t \in P \), \( A_t \cap \langle Y \rangle = \emptyset \), verifying (1).

To show that \( D \) is \( \mathcal{H} \)-dense, fix an \( X \in \mathcal{H} \). We must find a \( Z \preceq X \) such that \( Z \in D \). First, consider the set

\[ D_0 = \{ Y \preceq X : \exists t \in \mathbb{R} (A_t \cap \langle Y \rangle = \emptyset) \} \].

Clearly, \( D_0 \) is \( \mathcal{H} \)-dense below \( X \). If not, then there is an \( X' \in \mathcal{H} \) such that for all \( Z \preceq X' \) and all \( t \in \mathbb{R} \), \( A_t \cap \langle Y \rangle \neq \emptyset \), but this just says that (2) holds for \( P = \mathbb{R} \) and \( X' \), contrary to our assumptions. Thus, \( D_0 \) is \( \mathcal{H} \)-dense below \( X \) and \( Y \) so by Lemma 3.5, \( (\mathcal{H} \upharpoonright X) \cap D_0 \neq \emptyset \). That is, there is an \( X_0 \in \mathcal{H}_0 \) such that \( X_0 \preceq X \) and \( X_0 \in D_0 \).
and a $t_0 \in \mathbb{R}$ such that $A_{t_0} \cap \langle X_0 \rangle = \emptyset$. We continue in this fashion to construct $X_\alpha \in \mathcal{H} \upharpoonright X$ and $t_\alpha \in \mathbb{R}$, for $\alpha < \omega_1$, such that all of the $t_\alpha$’s are distinct, $X_\alpha \preceq^* X_\beta$ whenever $\beta < \alpha$, and $A_{t_\alpha} \cap \langle X_\alpha \rangle = \emptyset$.

Suppose that, for some $\gamma < \omega_1$, we have constructed $X_\alpha$ and $t_\alpha$ for all $\alpha < \gamma$ as desired. Let $P_\gamma \subseteq \mathbb{R}$ be a nonempty perfect set which does not contain any of the previous countably many $t_\alpha$’s.

If $\gamma = \beta + 1$, then consider the set

$$\mathbb{D}_\gamma = \{ Y \preceq X_\beta : \exists t \in P_\gamma (A_t \cap \langle Y \rangle = \emptyset) \}.$$ 

As above, $\mathbb{D}_\gamma$ is analytic, downwards closed, and $\mathcal{H}$-dense below $X_\beta$; if not, then there is an $X' \in \mathcal{H} \upharpoonright X_\beta$ such that for all $t \in P_\gamma$, $A_t$ is asymptotic below $X'$, contrary to our assumptions. Thus, by Lemma 3.6, there is an $X_\gamma \in \mathcal{H} \upharpoonright X_\beta$ and a $t_\gamma \in P_\gamma$ such that $A_{t_\gamma} \cap \langle X_\gamma \rangle = \emptyset$.

If instead, $\gamma$ is a limit ordinal, first apply the $(p)$-property in $\mathcal{H}$ to obtain a $Y \in \mathcal{H} \upharpoonright X$ such that $Y \preceq^* X_\alpha$ for all $\alpha < \gamma$, then argue as in the successor case to obtain $X_\gamma \in \mathcal{H} \upharpoonright Y$ and $t_\gamma \in P_\gamma$ as required.

We can now apply Lemma 2.2 to the sequences $(X_\alpha)_{\alpha < \omega_1}$ and $(t_\alpha)_{\alpha < \omega_1}$ to obtain a $Z \preceq X$ such that $Z \in \mathcal{D}$, verifying that $\mathbb{D}$ is $\mathcal{H}$-dense, as claimed. □

Clearly, Theorem 2.3 can be localized in a similar fashion. We now turn to the proof of our main preservation result, Theorem 1.8. In fact, we will prove slightly more.

**Definition 5.2.** A filter $\mathcal{F}$ on $E$ has the strong $(p)$-property if whenever $(X_\alpha)_{\alpha \in E[< \infty]}$ is contained in $\mathcal{F}$, there is an $X \in \mathcal{F}$ such that $X/\bar{x} \preceq X_\alpha$ for all $\bar{x} \subseteq X$.

The strong $(p)$-property easily implies the $(p)$-property, and every strategic $(p^+)$-filter is a strong $(p^+)$-filter (Proposition 4.6 in [34]). If $|F| < \infty$, then every $(p^+)$-filter is already a strong $(p^+)$-filter (Corollary 4.5 in [36]), but we do not know if this holds in general. It will be a consequence of the following theorem, together with Theorem 1.9, that when $|F| > 2$, it is consistent that there is a strong $(p^+)$-filter which is not strategic.

**Theorem 5.3.** If $\mathcal{F}$ is a strategic $(p^+)$-filter on $E$ and $g$ is $\mathbf{V}$-generic for $\mathbb{S}$, then $\mathcal{F}$ generates a strong $(p^+)$-filter in $\mathbf{V}[g]$.

**Proof.** All uses of the forcing relation $\Vdash$ and names will be in reference to $\mathbb{S}$. Let $\mathcal{F}$ be a name such that $\Vdash \mathcal{F}$ is the $\preceq$-upwards closure of $\mathcal{F}$.

Our goal, then, is to show that $\Vdash \mathcal{F}$ is a strong $(p^+)$-filter.

Note that $E$ and $E[< \infty]$ remain unchanged by forcing, as is the fact that $\mathcal{F}$ generates a filter.

First, we verify that the strong $(p)$-property is preserved. Suppose that $p \in \mathbb{S}$ is such that

$$p \Vdash \forall \bar{x} \in E[< \infty] (\bar{X}_{\bar{x}} \in \mathcal{F})$$


where each $\hat{X}_x$ is a name for an element of $E^{[\omega]}$. We can then find names $\check{Y}_x$, for each $x \in E^{[<\omega]}$, such that

$$p \vDash \forall x \in E^{[<\omega]}(\check{Y}_x \subseteq \hat{X}_x \land \check{Y}_x \in F).$$

By Lemma 4.4, there is a family $\{F_x : x \in E^{[<\omega]}\}$ of finite subsets of $F$ and a condition $q \leq p$ such that

$$q \vDash \forall x \in E^{[<\omega]}(\check{Y}_x \in F_x).$$

Since $F$ is a filter, for each $x \in E^{[<\omega]}$, we can find a $Z_x \in F$ which is \(\preceq\)-below every element of $F_x$. By the strong ($p$)-property, there is $Z \in F$ with $Z \preceq Z_x$ for all $x \subseteq Z$. But then,

$$q \vDash Z \in F \land \forall x \subseteq Z (Z/\check{z} \preceq \hat{X}_x).$$

Thus, $F$ is forced to have the strong ($p$)-property.

We now turn to verifying that $F$ is forced to be full. To this end, suppose that $p \in S$ and $\check{A}$ is a name such that $p \vDash \check{A} \subseteq E^*$. For $x \in E^*$, we say that a condition $q \leq p$ decides “$x \in \check{A}$” if $q \vDash x \in \check{A}$ or $q \vDash x \notin \check{A}$. Enumerate $E^*$ as $\{v_n : n \in \omega\}$.

**Claim.** There is a $p_\omega \leq p$ such that for all $n \in \omega$ and all $s \in \ell(n, p_\omega)$, $p_\omega \vDash s$ decides “$v_n \in \check{A}$”.

**Proof of claim.** We construct $p_\omega$ using fusion. Let $p_0 \leq p$ decide “$v_0 \in \check{A}$”. Say $\ell(p_0, 1) = \{s, t\}$. Then, there are $q_s \leq p_0|s$ and $q_t \leq p_0|t$ which each decide “$v_1 \in \check{A}$”. Let $p_1 = q_s \cup q_t \leq p_0$. Continue in this fashion, building a fusion sequence $(p_n)_{n \in \omega}$ such that for each $s \in \ell(n, p_n)$, $p_n|s$ decides “$v_n \in \check{A}$”. Then, $p_\omega = \bigcap_{n \in \omega} p_n$ is as claimed. \(\square\)

For each $f \in [p_\omega]$, let

$$A_f = \{v \in E^* : \forall n[v = v_n \rightarrow \exists s \in \ell(n, p_\omega)(s \subseteq f \land p_\omega|s \vDash v_n \in \check{A})]\}.$$ 

The assignment $[p_\omega] \rightarrow 2^{E^*} : f \mapsto A_f$ is continuous. Thus, by Theorem 5.1, there is a $q \leq p_\omega$ and an $X \in F$ such that either (1) for all $f \in [q]$, $A_f \cap \langle X \rangle = \emptyset$, or (2) for all $f \in [q]$, $A_f$ is asymptotic below $X$.

Suppose that (1) holds, but $q \not\vDash \check{A} \cap \langle X \rangle = \emptyset$. Then, there is an $r \leq q$ and a $v = v_n \in \langle X \rangle$ such that $r \vDash v_n \in \check{A}$. Take $s \in \ell(n, r)$. Then, $r|s \leq p_\omega|s$, so by the choice of $p_\omega$, we must have that $p_\omega|s \vDash v_n \in \check{A}$ as well. But then, for $f \in [r] \subseteq [q]$ with $s \subseteq f$, we have that $v_n \in A_f \cap \langle X \rangle$, contrary to (1). So, in this case, we must have that $q \vDash \check{A} \cap \langle X \rangle = \emptyset$.

Next, suppose that (2) holds, but $q \not\vDash \check{A}$ is asymptotic below $X$. Then, there is an $r \leq q$ and a name $\check{Z}$ such that $r \vDash \check{Z} \subseteq X \land \check{A} \cap \langle \check{Z} \rangle = \emptyset$. As in the claim above, we can find an $r_\omega \leq r$ such that for all $n \in \omega$ and all $s \in \ell(n, r_\omega)$, $r_\omega|s$ decides “$v_n \in \langle \check{Z} \rangle$”.

Since $r_\omega \vDash \check{Z}$ is an infinite block sequence, there is an $n_0 \in \omega$ such that for some $t_0 \in \ell(n_0, r_\omega)$, $r_\omega|t_0 \vDash v_{n_0} \in \langle \check{Z} \rangle$. Likewise, there is an $n_1 > n_0$
such that \( \max(\text{supp}(v_{n_0})) < \min(\text{supp}(v_{n_1})) \) and for some \( t_1 \in \ell(n_1, r_\infty) \) extending \( t_0, r_\infty \mathrel{|} t_1 \parallel v_{n_1} \in \langle \mathcal{Z} \rangle \). Continuing in this fashion, we may construct a \( \square \)-increasing sequence \( (t_k)_{k \in \omega} \) in \( r_\infty \) and a block sequence \( (v_{n_k})_{k \in \omega} \subseteq X \) such that for each \( k \in \omega, v_k \in \ell(n_k, r_\infty) \) and \( r_\infty | t_k \parallel v_{n_k} \in \langle \mathcal{Z} \rangle \).

Let \( f = \bigcup_{k \in \omega} t_k \in [r_\infty] \subseteq [q] \). By (2), there is a \( v = v_N \in A_f \cap \langle (v_{n_k})_{k \in \omega} \rangle \). Say \( v_N \in \langle v_{n_0}, v_{n_1}, \ldots, v_{n_t} \rangle \). Take \( s \in \ell(N, r_\infty) \) such that \( s \sqsupseteq f \). Then, \( p_\infty | s \parallel v_N \in A \). If \( s \subseteq t_{n_t} \), then \( r_\infty | t_{n_t} \leq p_\infty | t_{n_t} \leq p_\infty | s \), so \( r_\infty | t_{n_t} \parallel v_N \in \mathcal{A} \cap \langle \mathcal{Z} \rangle \), while if \( t_{n_t} \subseteq s \), then \( r_\infty | s \leq r_\infty | t_{n_t} \) and so again \( r_\infty | s \parallel v_N \in A \cap \langle \mathcal{Z} \rangle \). Either way, this contradicts the fact that \( r_\infty \leq r \) and \( r \parallel \mathcal{A} \cap \langle \mathcal{Z} \rangle = \emptyset \). Hence, \( q \parallel \mathcal{A} \) is asymptotic below \( X \).

As either (1) or (2) must hold, we have proven that either \( q \parallel \mathcal{A} \cap \langle X \rangle = \emptyset \) or \( q \parallel \mathcal{A} \) is asymptotic below \( X \). Thus, \( \mathcal{F} \) is forced to be full. \( \square \)

6. Perfectly strategically Ramsey sets

In looking for an appropriate parametrized form of Theorem 1.3, one might initially arrive at the following conjecture:

\[(\ast) \text{ If } \mathcal{A} \subseteq \mathbb{R} \times E^{[\omega]} \text{ is analytic, then there is a nonempty perfect set } P \subseteq \mathbb{R} \text{ and an } X \in E^{[\infty]} \text{ such that either:}
\]

(1) \( I \) has a strategy \( \sigma \) in \( F[X] \) such that \( P \times [\sigma] \subseteq \mathcal{A}^c \), or

(2) \( II \) has a strategy \( \alpha \) in \( G[X] \) such that \( P \times [\alpha] \subseteq \mathcal{A} \).

Here, \([\sigma]\) denotes the set of all outcomes of \( F[X] \) when \( I \) follows \( \sigma \), and likewise for \([\alpha]\) in \( G[X] \). Unfortunately, unless \(|F| = 2\) (where even stronger dichotomies hold, cf. [39]), (\(\ast\)) is false:

**Example 6.1.** Assume \(|F| > 2\). Let \( A_0 \) and \( A_1 \) be an asymptotic pair, as in Example 1.2. Consider the set

\[ \mathcal{A} = \{ (f, (y_n)_{n \in \omega}) \in 2^\omega \times E^{[\infty]} : \forall n (y_n \in A_{f(n)}) \} . \]

Note that we have replaced \( \mathbb{R} \) with \( 2^\omega \) here, which we may do without loss of generality, and \( \mathcal{A} \) is closed in \( 2^\omega \times E^{[\infty]} \). Suppose that \( P \subseteq 2^\omega \) is a nonempty perfect set and \( X \in E^{[\infty]} \). Then, for any \( f \in P \), \( II \) can ensure that their moves \( y_n \) in \( F[X] \) satisfy \( y_n \in A_{f(n)} \) for all \( n \in \omega \), and thus \( (f, (y_n)_{n \in \omega}) \in \mathcal{A} \).

In particular, there can be no strategy \( \sigma \) for \( I \) in \( F[X] \) for which \( P \times [\sigma] \subseteq \mathcal{A}^c \).

Suppose instead that \( \alpha \) is a strategy for \( II \) in \( G[X] \) such that \( P \times [\alpha] \subseteq \mathcal{A} \). Let \( f, g \in P \) be such that \( f \neq g \) and \( (y_n)_{n \in \omega} \) an outcome of \( \alpha \). Then, for all \( n \in \omega, y_n \in A_{f(n)} \) and \( y_n \in A_{g(n)} \), but this contradicts the disjointness of \( A_0 \) and \( A_1 \). Thus, there is no such strategy \( \alpha \) either. Note that we have only used that \(|P| > 1\) here.

The problem with (\(\ast\)) is the uniformity in the parameter from \( P \): (2) says that there is a single strategy \( \alpha \) for \( II \) in \( G[X] \) such that for all \( f \in P \), every outcome of \( \alpha \) lies in the slice \( \mathcal{A}_f \). It is in dropping this uniformity, in effect, swapping the order of quantifiers in (2), that we find the correct
parametrized form of Theorems 1.3 and 1.4, our Theorem 1.7. Note that in Example 6.1, for every $f \in \mathcal{F}$, II clearly has a strategy in $G[X]$ to play into $\mathcal{A}_f$. However, we are able to maintain the uniformity in (1), which we can see directly in the open case (cf. Theorem 35.23 and Exercise 35.33 in [19]).

**Lemma 6.2.** Let $\mathcal{A} \subseteq 2^\omega \times E^{[\omega]}$ be open, $P \subseteq 2^\omega$ a nonempty perfect set, and $X \in E^{[\omega]}$. If for all $f \in P$, I has a strategy in $F[X]$ for playing out of $\mathcal{A}_f$, then there is a nonempty perfect set $Q \subseteq P$ and a strategy $\sigma$ for I in $F[X]$ such that $Q \times [\sigma] \subseteq \mathcal{A}^c$.

**Proof.** We can view strategies for I in $F[X]$ as functions from the set of all finite block sequences $X^{<\omega}$ in $X$ to $\omega$. Let $\mathcal{S}$ denote the space of all such functions, which we topologize as the product $\omega^{X^{<\omega}}$. Since $\mathcal{A}$ is open, we can fix a countable set $A \subseteq 2^{<\omega} \times E^{<\omega}$ such that for all $(f, Y) \in 2^{<\omega} \times E^{<\omega}$, $(f, Y) \in \mathcal{A}$ if and only if there exists $(t, \bar{x}) \in A$ such that $t \subseteq f$ and $\bar{x} \subseteq Y$.

Consider the set $\mathcal{B} \subseteq P \times \mathcal{S}$ defined by

$$\mathcal{B} = \{(f, \sigma) \in P \times \mathcal{S} : \sigma \text{ is a strategy in } F[X] \text{ for playing out of } \mathcal{A}_f\}.$$ 

So, $(f, \sigma) \in \mathcal{B}$ if and only if for every finite sequence $\bar{y} = (y_0, \ldots, y_n)$ of moves by II in $F[X]$ against $\sigma$, there is no $(t, \bar{x}) \in A$ such that $t \subseteq f$ and $\bar{y} \subseteq \bar{x}$. Thus, $\mathcal{B}$ is Borel. By the Jankov–von Neumann Uniformization Theorem (Theorem 18.1 in [19]), $\mathcal{B}$ admits a $\sigma(\Sigma_1^0)$-uniformization. That is, there is a $\sigma(\Sigma_1^0)$-measurable function $f \mapsto \sigma_f$ on $P$ such that $(f, \sigma_f) \in \mathcal{B}$ for all $f \in P$.

Since $\sigma(\Sigma_1^0)$-measurable functions are Baire measurable, there is a nonempty perfect set $Q \subseteq P$ on which $f \mapsto \sigma_f$ is continuous (cf. Theorem 8.38 in [19]). In particular, the image of the compact set $Q$ under this map is a compact subset of $\mathcal{S}$. It follows that for any $\bar{x} \in X^{<\omega}$, the set $\{\sigma_f(\bar{x}) : f \in Q\}$ is finite, say with maximum value $n_{\bar{x}}$. Let $\sigma$ be the strategy for I in $F[X]$ defined by $\sigma(\bar{x}) = n_{\bar{x}}$ for all $\bar{x} \in X^{<\omega}$. Then, for all $f \in Q$, any outcome in $F[X]$ against $\sigma$ is also a valid outcome against $\sigma_f$, and so $\sigma$ is a strategy for playing out of $\mathcal{A}_f$. Thus, $Q \times [\sigma] \subseteq \mathcal{A}^c$. □

In [34], given a family $\mathcal{H}$ on $E$, we defined a set $\mathcal{A} \subseteq E^{[\omega]}$ to be $\mathcal{H}$-strategically Ramsey if for every $\bar{x} \in E^{[\omega]}$ and $X \in \mathcal{H}$, there is a $Y \in \mathcal{H} \upharpoonright X$ such that either:

1. I has a strategy in $F[\bar{x}, Y]$ for playing out of $\mathcal{A}$, or
2. II has a strategy in $G[\bar{x}, Y]$ for playing in to $\mathcal{A}$.

Here, the games $F[\bar{x}, X]$ and $G[\bar{x}, X]$ are defined in exactly the same way as $F[X]$ and $G[X]$ except that we demand II’s moves $(y_n)_{n \in \omega}$ to be supported above the vectors in $\bar{x}$ and declare the outcome to be $\bar{x}^- (y_n)_{n \in \omega}$. When $\mathcal{H} = E^{[\omega]}$, we just say that $\mathcal{A}$ is strategically Ramsey, as in [31]. Theorems 1.3 and 1.4 follow once it is shown that analytic sets are $\mathcal{H}$-strategically Ramsey, whenever $\mathcal{H}$ is a $(p^+)$-family. The following definition is the parametrized form of being $\mathcal{H}$-strategically Ramsey.
Definition 6.3. Let $\mathcal{H}$ be a family on $E$. A set $A \subseteq 2^\omega \times E^{[\omega]}$ is perfectly $\mathcal{H}$-strategically Ramsey if for every $\bar{x} \in E^{[\omega]}$, $X \in \mathcal{H}$, and $p \in S$, there is a $Y \in \mathcal{H} \upharpoonright X$ and $q \leq p$ such that either:

1. I has a strategy $\sigma$ in $F[\bar{x}, Y]$ such that $[q] \times [\sigma] \subseteq A^c$, or
2. for every $f \in [q]$, II has a strategy in $G[\bar{x}, Y]$ for playing into $A_f$.

When $\mathcal{H} = E^{[\omega]}$, we just say that $A$ is perfectly strategically Ramsey.

Theorem 1.7 will thus follow once we have proved that all analytic subsets of $2^\omega \times E^{[\omega]}$ are perfectly $\mathcal{H}$-strategically Ramsey, whenever $\mathcal{H}$ is a strategic $(p^+)$-family (Theorem 7.6 below).

It will be useful to “finitize” the Gowers game, as in [1]: Given $X \in E^{[\omega]}$, let $G^{<\omega}[X]$ be the two player game where I starts by playing a nonzero vector $x_0^{(0)} \in \langle X \rangle$, and then II responds with either 0 or a nonzero vector $y_0 \in \langle x_0^{(0)} \rangle$. If II plays 0, then I plays a nonzero vector $x_1^{(0)} \in \langle X \rangle$ supported above $x_0^{(0)}$, and II must respond with either 0 or a nonzero vector $y_1 \in \langle x_0^{(0)}, x_1^{(0)} \rangle$. If II plays a nonzero vector, the game “restarts”, with I playing a nonzero vector $x_0^{(1)} \in \langle X \rangle$, and so on. The nonzero plays of II are, again, required to form a block sequence $(y_n)_{n \in \omega}$, which is the outcome of the game. In essence, I is playing a block sequence $X_n \preceq X$ one vector at a time and II is responding with a nonzero vector $y_n \in \langle X_n \rangle$ as soon as enough of $X_n$ has been revealed to generate $y_n$. The game $G^{<\omega}[\bar{x}, X]$, for $\bar{x} \in E^{[<\omega]}$, is defined similarly.

From the point of view of descriptive complexity, $G^{<\omega}[X]$ is a simpler game than $G[X]$ as both players play vectors, which can be coded as natural numbers, rather than infinite sequences of vectors. However, the following says we lose no generality in passing between $G[X]$ and $G^{<\omega}[X]$.

Lemma 6.4 (Theorems 1.1 and 1.2 in [1]). For any $\bar{x} \in E^{[\omega]}$ and $X \in E^{[\omega]}$, the games $G[\bar{x}, X]$ and $G^{<\omega}[\bar{x}, X]$ are equivalent in the sense that for any $A \subseteq E^{[\omega]}$, $I$ (II, respectively) has a strategy in $G[\bar{x}, X]$ to play into $A$ if and only if $I$ (II) has a strategy in $G^{<\omega}[\bar{x}, X]$ to play into $A$.

Our next lemma concerns the complexity of certain sets related to the games $F[X]$ and $G[X]$ (cf. Section 20.D of [19]).

Lemma 6.5. Given $A \subseteq 2^\omega \times E^{[\omega]}$ and $\bar{x} \in E^{[<\omega]}$, consider

$\Lambda_{\bar{x}, I} = \{(f, X) \in 2^\omega \times E^{[\omega]} : I$ has a strategy in $F[\bar{x}, X]$ for playing out of $A_f \}$

$\Lambda_{\bar{x}, II} = \{(f, X) \in 2^\omega \times E^{[\omega]} : II$ has a strategy in $G[\bar{x}, X]$ for playing into $A_f \}$.

If $A$ is open, then $\Lambda_{\bar{x}, I}$ is analytic and $\Lambda_{\bar{x}, II}$ is $\Sigma^1_2$.

Proof. For $\Lambda_{\bar{x}, I}$, $(f, X) \in \Lambda_{\bar{x}, I}$ if and only if there is a strategy $\sigma : X^{[<\omega]} \rightarrow \omega$ for I in $F[X]$ such that $(f, \sigma)$ is in the Borel set $B$ defined (uniformly in $X$) in the proof of Lemma 6.2 above, showing that $\Lambda_{\bar{x}, I}$ is analytic.

For $\Lambda_{\bar{x}, II}$, we use Lemma 6.4 to pass to the finitized Gowers game $G^{<\omega}[\bar{x}, X]$ and see that $(f, X) \in \Lambda_{\bar{x}, II}$ if and only if there is a function $\alpha : X^{<\omega} \rightarrow \langle X \rangle$
such that for any sequence \((x_n)_{n \in \omega}\) of moves by I in \(G^{<\infty}[\vec{x}, X]\), there are infinitely many \(n \in \omega\) for which \(\alpha(x_0, \ldots, x_n) \neq 0\), whenever \(\alpha(x_0, \ldots, x_n) \neq 0\) and \(m\) is the least value \(> n\) such that \(\alpha(x_0, \ldots, x_n) \neq 0\), then \(\alpha(x_0, \ldots, x_n) \in \langle x_{n+1}, \ldots, x_m \rangle\), and the sequence of nonzero values of \(\alpha\) played in response to \((x_n)_{n \in \omega}\) must lie in \(A_f\). This is a \(\Sigma_2^1\) condition.

We can now use metamathematical techniques to prove that open sets are perfectly \(\mathcal{H}\)-strategically Ramsey.

**Lemma 6.6.** Let \(\mathcal{F}\) be a strategic \((p^+)\)-filter on \(E\). Then, open sets are perfectly \(\mathcal{F}\)-strategically Ramsey.

**Proof.** Let \(A \subseteq 2^\omega \times E^{[\infty]}\) be open, and fix \(\vec{x} \in E^{[<\infty]}\), \(X \in E^{[\infty]}\) and \(p \in S\). By Lemma 6.5 and Theorem 25.4 in [17], the statements \((f, Y) \in A_{\vec{x}, 1}\) and \((f, Y) \in A_{\vec{x}, \Omega}\) are upwards absolute for sufficiently rich countable models.

Let \(M\) be the transitive collapse of a countable elementary submodel of \(H(\theta)\), for some sufficiently large \(\theta\), having an extra predicate for \(\mathcal{F}\). By coding and elementarity, we can demand that \(\mathcal{F} \cap M \in M\) and that

\[
M \models \mathcal{F} \cap M \text{ is a strategic } (p^+)\text{-filter}.
\]

Note that \(M\) is sufficient, in the terminology of Lemma 4.3.

Suppose that \(g\) is \(M\)-generic for \(S\). We work in \(M[g]\): Let \(\mathcal{F}\) be the upwards closure of \(\mathcal{F} \cap M\). By Theorem 1.8, \(\mathcal{F}\) is a \((p^+)\)-filter, and so Theorem 1.4 holds for it, applied to the (coded) open set \(A_{\vec{x}, 1}^{M[g]}\) (really, we are using that \(A_{\vec{x}, 1}^{M[g]}\) is \(\mathcal{F}\)-strategically Ramsey in \(M[g]\)). Let \(Y \in \mathcal{F}\) witness this; by passing to a block subsequence, we may assume that \(Y \in \mathcal{F} \cap M\).

Thus, we have that

\[
M[g] \models (Y, g) \in A_{\vec{x}, 1}^{M[g]} \quad \text{or} \quad M[g] \models (Y, g) \in A_{\vec{x}, \Omega}^{M[g]}.
\]

It follows by absoluteness that whenever \(g\) is \(M\)-generic, \((Y, g) \in A_{\vec{x}, 1}\) or \((Y, g) \in A_{\vec{x}, \Omega}\). Taking \(q \leq p\) as in Lemma 4.3 and applying Lemma 6.2 if necessary, \(Y\) and \(q\) witness that \(A\) is perfectly \(\mathcal{F}\)-strategically Ramsey.

**Lemma 6.7.** Let \(\mathcal{H}\) be a strategic \((p^+)\)-family on \(E\). Then, open sets are perfectly \(\mathcal{H}\)-strategically Ramsey.

**Proof.** Let \(A \subseteq 2^\omega \times E^{[\infty]}\) be open, and fix \(\vec{x} \in E^{[<\infty]}\), \(X \in \mathcal{H}\), and \(p \in S\). By Lemma 5.4 in [34], we can force with \((\mathcal{H}, \leq)\) to generically add a strategic \((p^+)\)-filter \(\mathcal{G}\) containing \(X\), and doing so does not add new reals (and thus no new block sequences, perfect subsets of \(2^\omega\), Borel sets, etc.). In \(V[\mathcal{G}]\), \(A\) is perfectly \(\mathcal{G}\)-strategically Ramsey by Lemma 6.6, so there are \(q \leq p\) and \(Y \in \mathcal{G} \upharpoonright X \subseteq \mathcal{H} \upharpoonright X\) witnessing this. As we have added no new reals, \(q\) and \(Y\) are in \(V\), and the fact that they witness the relevant dichotomy involves only quantification over reals, and thus is also true in \(V\). Here we are implicitly using Lemma 6.4 to pass between \(G[\vec{x}, Y]\) to \(G^{<\infty}[\vec{x}, Y]\). Hence, \(A\) is perfectly \(\mathcal{H}\)-strategically Ramsey.
7. Extending to analytic sets

To complete the proof of Theorem 1.7, we will extend Lemma 6.7 to all analytic sets using a combinatorial forcing argument, similar to that used in the proof of Theorems 1.3 in [31].

For the remainder of this section, we fix a strategic \((p^+)\)-family \(\mathcal{H}\) on \(E\). We reserve the notation \(p, X\), \((q, Y)\), etc, for pairs in \(S \times \mathcal{H}\), and write \((q, Y) \leq (p, X)\) when \(q \leq p\) and \(Y \preceq X\). We use \(\vec{x}, \vec{y}\), etc, for finite block sequences in \(E^{[<\infty]}\), and \(m, n\), etc, for natural numbers.

The following definition and Lemmas 7.2 through 7.4 refer implicitly to sets \(A_n \subseteq 2^\omega \times E^{[<\infty]}\), for \(n \in \omega\), and \(A = \bigcup_{n \in \omega} A_n\).

**Definition 7.1.** Given \((p, X)\), \(\vec{x}\), and \(n\), we say that:

(i) \((\vec{x}, n)\) accepts \((p, X)\) if I has a strategy \(\sigma\) in \(F[\vec{x}, X]\) such that \([p] \times [\sigma] \subseteq \mathbb{A}_n^c\).

(ii) \((\vec{x}, n)\) rejects \((p, X)\) if no \((q, Y) \leq (p, X)\) is accepted by \((\vec{x}, n)\).

(iii) \((\vec{x}, n)\) decides \((p, X)\) if it either accepts or rejects \((p, X)\).

**Lemma 7.2.** (a) For every \((p, X)\), \(\vec{x}\), and \(n\), there is a \((q, Y) \leq (p, X)\) such that \((\vec{x}, n)\) decides \((q, Y)\).

(b) For every \((p, X)\), \(\vec{x}\), and \(n\), if \((\vec{x}, n)\) accepts (rejects, respectively) \((p, X)\), \(q \leq p\) and \(Y \preceq^* X\) in \(\mathcal{H}\), then \((\vec{x}, n)\) accepts (rejects) \((q, Y)\) as well.

(c) For every \((p, X)\), \(\vec{x}\), \(n\), and \(m\), if \((\vec{x}, n)\) accepts (rejects) \((p|t, X)\) for all \(t \in \ell(m, p)\), then \((\vec{x}, n)\) accepts (rejects) \((p, X)\).

(d) For every \((p, X)\), \(\vec{x}\), \(n\), and \(m\), there is a \(q \leq m\) and a \(Y \in \mathcal{H} \cap X\) such that for all \(t \in \ell(m, q)\), \((\vec{x}, n)\) decides \((q|t, Y)\).

**Proof.** Parts (a) and (b) are immediate from the definitions above and the nature of the asymptotic game.

(c) If \((\vec{x}, n)\) accepts \((p|t, X)\) for all \(t \in \ell(m, p)\), then by taking the pointwise maximum of each of the finitely many resulting strategies for I in \(F[\vec{x}, X]\) and using that \(p = \bigcup_{t \in \ell(m, p)} p|t\), we have that \((\vec{x}, n)\) accepts \((p, X)\). If \((\vec{x}, n)\) rejects \((p|t, X)\) for all \(t \in \ell(m, p)\), but there is some \((q, Y) \leq (p, X)\) which is accepted by \((\vec{x}, n)\), then by Lemma 4.1, there is a \(t_0 \in \ell(m, p)\) such that \(q \neq p|t_0\) are compatible. So, there is some \(r\) below both \(q\) and \(p|t_0\) such that \((\vec{x}, n)\) accepts \((r, Y)\), contradicting that \((\vec{x}, n)\) rejects \((p|t_0, X)\).

(d) Say \(\ell(m, p) = \{t_0, \ldots, t_k\}\). Apply (a) to \((p|t_i, X)\) successively to obtain \((q_i, Y_i) \leq (p|t_i, X)\) which is decided by \((\vec{x}, n)\), with \(Y_i \preceq Y_{i-1}\), for \(i = 1, \ldots, k\). Let \(q = \bigcup_{i=0}^k q_i \leq m\) and \(Y = Y_k\). Then, \((q, Y)\) is as claimed. \(\square\)

Fix an enumeration of \(E^{[<\infty]}\) via some bijection \(# : E^{[<\infty]} \to \omega\).

**Lemma 7.3.** For every \((p, X)\), there is a \((q, Y) \leq (p, X)\) such that for all \((\vec{y}, n)\), there is an \(m\) such that for all \(t \in \ell(q, m)\), \((\vec{y}, n)\) decides \((q|t, Y)\).

**Proof.** By Lemma 7.2(a), we may find \((q_0, Y_0) \leq (p, X)\) which is decided by \((\vec{y}, 0)\), where \(#(\vec{y}) = 0\). Suppose that \((q_i, Y_i)\) has already been defined
for \( i \leq m \). Let \( \{(\vec{y}^0, \ell_0), \ldots, (\vec{y}^k, \ell_k)\} \) enumerate those finitely many \((\vec{y}, \ell)\) with \( \max\{\#(\vec{y}), \ell\} \leq m + 1 \). Using Lemma 7.2(d), we may successively choose \( q^j \leq m + 1 \) \( q^{j-1} \leq m + 1 \) \( q_m \) and \( Y_j \cong Y^{j-1} \cong Y_m \) in \( \mathcal{H} \) such that for all \( t \in \ell(m + 1, q^j) \), \((\vec{y}^j, \ell_j)\) decides \((q^j | t, Y^j)\), for \( j \leq k \). Let \( q_{m+1} = q_k \) and \( Y_{m+1} = Y_k \). Put \( q = \bigcap_{m \in \omega} q_m \), the fusion of the \( q_m \)'s, and let \( Y \in \mathcal{H} \upharpoonright X \) be a diagonalization of the \( Y_m \)'s. If \( \max\{\#(\vec{y}), n\} \leq m \), then by construction, for all \( t \in \ell(m, q) \), \((\vec{y}, \ell)\) decides \((q | t, Y)\).

\[\square\]

**Lemma 7.4.** For every \((p, X)\) and \( \vec{x} \), there is a \((q, Y) \leq (p, X)\) such that either:

1. \( I \) has a strategy \( \sigma \) in \( F[\vec{x}, Y] \) such that \([q] \times [\sigma] \subseteq \mathbb{A}^c \), or
2. for all \( f \in [q] \), \( II \) has a strategy in \( G[Y] \) for playing \((z_k)_{k \in \omega}\) for which there is an \( n \) and a \( t \sqsupseteq f \) such that \((\vec{y}^\om(z_0, \ldots, z_n), n) \) rejects \((q | t, Y)\).

**Proof.** Let \((q', Y') \leq (p, X)\) be as in Lemma 7.3. Let

\[\mathbb{B} = \{(f, (z_k)_{k \in \omega}) \in [q'] \times E^{[\omega]} : \exists n \exists t \subseteq f[(\vec{y}^\om(z_0, \ldots, z_n), n) \text{ rejects } (q' | t, Y')]\}\]

Note that \( \mathbb{B} \) is relatively open in \([q'] \times E^{[\omega]}\). By Lemma 6.7, there is a \((q, Y) \leq (q', Y')\) such that either I has a strategy \( \sigma \) in \( F[Y] \) such that \([q] \times [\sigma] \subseteq \mathbb{B}^c \), or for all \( f \in [q] \), II has a strategy in \( G[Y] \) for playing into \( \mathbb{B}_f \). In the latter case, we’re done, so we assume the former.

The strategy \( \sigma = \sigma_0 \) produces outcomes \((z_k)_{k \in \omega}\) in \( F[Y] \) such that for all \( n \), \((\vec{x}^\om(z_0, \ldots, z_n), n) \) does not reject, and thus accepts, \((q | t, Y)\) for all \( t \in \ell(m, q) \), where \( m \) is as in Lemma 7.3 for \( \vec{y} = \vec{x}^\om(z_0, \ldots, z_n) \). By Lemma 7.2(c), \((\vec{x}^\om(z_0, \ldots, z_n), n) \) accepts \((q, Y)\). That is, I has a further strategy \( \sigma_{(z_0, \ldots, z_n)} \) in \( F[\vec{x}^\om(z_0, \ldots, z_n), Y] \) such that \([q] \times [\sigma_{(z_0, \ldots, z_n)}] \subseteq \mathbb{A}_c^c \).

We describe a strategy \( \sigma \) for \( I \) in \( F[\vec{x}, Y] \) such that \([q] \times [\sigma] \subseteq \mathbb{A}^c \): If \((z_0, \ldots, z_{n-1})\) has been played by \( II \) thus far, I responds with

\[\max\{\sigma_0(z_0, \ldots, z_{n-1}), \sigma_{(z_0)}(z_1, \ldots, z_{n-1}), \ldots, \sigma_{(z_0, \ldots, z_{n-1})}(\emptyset)\}.

For each \( n \), we have ensured that \([q] \times [\sigma] \subseteq \mathbb{A}_c^c \), and thus \([q] \times [\sigma] \subseteq \mathbb{A}^c \). \(\square\)

Let \( \mathcal{A}_s : s \in \omega^{<\omega} \) be a Suslin scheme of subsets of \( 2^\om \times E^{[\omega]} \), where for each \( s \in \omega^{<\omega} \), \( \mathcal{A}_s = \bigcup_{n \in \omega} \mathcal{A}_{s \upharpoonright n} \). Given \( \vec{x} \), \( n \), and \( s \in \omega^{<\omega} \), we define what it means for \((\vec{x}, s, n)\) to accept or reject \((p, X)\) exactly as above, except in reference to \( \mathcal{A}_s = \bigcup_{n \in \omega} \mathcal{A}_{s \upharpoonright n} \). Given \((q, Y)\), \( \vec{y} \), and \( s \in \omega^{<\omega} \), let

\[\mathbb{B}(\vec{y}, s, q, Y) = \{(f, (z_k)_{k \in \omega}) \in [q] \times E^{[\omega]} : \exists n \exists t \subseteq f[(\vec{y}^\om(z_0, \ldots, z_n), s, n) \text{ rejects } (q | t, Y')]\}\]

**Lemma 7.5.** For every \((p, X)\), there is a \((q, Y) \leq (p, X)\) such that for all \( \vec{y} \) and \( s \in \omega^{<\omega} \), either:

1. \( I \) has a strategy \( \sigma \) in \( F[\vec{y}, Y] \) such that \([q] \times [\sigma] \subseteq \mathbb{A}_c^c \), or
2. for all \( f \in [q] \), \( II \) has a strategy in \( G[Y] \) for playing into \( \mathbb{B}(\vec{y}, s, q, Y)_f \).

**Proof.** For \( \vec{y} = \emptyset \) and \( s = \emptyset \), apply Lemma 7.4 to obtain a \((q_0, Y_0) \leq (p, X)\) such that either I has a strategy \( \sigma \) in \( F[Y_0] \) such that \([q_0] \times [\sigma] \subseteq \mathbb{A}_c^c \), or for all \( f \in [q_0] \), II has a strategy in \( G[Y_0] \) for playing \((z_k)\) for which there is an \( n \) and a \( t \sqsupseteq f \) such that \((\emptyset, \emptyset, n) \) rejects \((q_0 | t, Y_0)\).
Suppose we have defined \((q_i, Y_i)\) for \(i \leq m\). Say \(\ell(m+1, q_m) = \{t_0, \ldots, t_k\}\). Apply Lemma 7.4 to each \((q_m|t_j, Y_m)\) successively to obtain a \(q^j \leq q_m|t_j\) and \(Y^j \preceq Y_m\) in \(\mathcal{H}\) such that for all of the finitely many \(\vec{y}\) with \(\#(\vec{y}) \leq m+1\) and \(s \in \omega^\omega\) with \(\max(s) \leq m+1\), either I has a strategy \(\sigma\) in \(F[\vec{y}, Y^j]\) such that \([q^j] \times [\sigma] \subseteq A^c_s\), or for all \(q \in [q^j]\), II has a strategy in \(G[Y^j]\) for playing into \(B(\vec{y}, s, q^j, Y^j)_f\). Let \(q_{m+1} = \bigcup_{j=0}^k q^j \leq m+1\) and \(Y_{m+1} = Y^k\).

Take \(q = \bigcap_{m \in \omega} q_m\) and \(Y \in \mathcal{H} \mid X\) a diagonalization of the \(Y_m\)'s. Then, \((q, Y)\) is as claimed. \(\square\)

We can now complete the proof of Theorem 1.7.

**Theorem 7.6.** Analytic sets are perfectly \(\mathcal{H}\)-strategically Ramsey.

**Proof.** Let \(A \subseteq 2^\omega \times E^{[\omega]}\) be analytic, as witnessed by a continuous function \(F : \omega^\omega \rightarrow E^{[\omega]}\) such that \(F[\omega^\omega] = A\). For each \(s \in \omega^\omega\), let \(A_s = F[N_s]\), where \(N_s = \{x \in \omega^\omega : s \sqsubseteq x\}\), and note that \(A_s = \bigcup_n A_{s-n}\). All references to acceptance or rejection and the sets \(B(\vec{y}, s, q, Y)\) will be with respect to the Suslin scheme \(\{A_s : s \in \omega^\omega\}\), as above. Let \((p, X)\) and \(\vec{x}\) be given.

By Lemma 7.5, there is a \((q, Y) \leq (p, X)\) such that for all \(\vec{y}\) and \(s \in \omega^\omega\), either:

(1') I has a strategy \(\sigma\) in \(F[\vec{y}, Y]\) such that \([q] \times [\sigma] \subseteq A^c_s\), or

(2') for all \(q \in [q^j]\), II has a strategy in \(G[Y]\) for playing into \(B(\vec{y}, s, q, Y)_f\).

If (1') holds for \(\vec{y} = \vec{x}\) and \(s = \emptyset\), then since \(A_0 = A\), we’re done. Thus, we may suppose that (2') holds for \(\vec{y} = \vec{x}\) and \(s = \emptyset\).

Fix \(q \in [q]\). We will describe a strategy for II in \(G[\vec{x}, Y]\) for playing into \(A_f\). By (2'), II has a strategy in \(G[Y]\) for playing into \(B(\vec{x}, \emptyset, q, Y)_f\); they follow this strategy until \((z_0, \ldots, z_{n_0})\) has been played such that \((\vec{x}^-(z_0, \ldots, z_{n_0}), \emptyset, n_0)\) rejects \((q|t_0, Y)\), where \(t_0 \sqsubseteq f\). By assumption on \((q, Y)\), II must then have a strategy in \(G[Y]\) for playing into \(B((z_0, \ldots, z_{n_0}), (n_0), q|t_0, Y)\). II follows this strategy until a further \((z_{n_0+1}, \ldots, z_{n_0+n_1+1})\) has been played so that \((\vec{x}^-(z_0, \ldots, z_{n_0+1}, \ldots, z_{n_0+n_1+1}), (n_0), n_1)\) rejects \((q|t_1, Y)\), where \(t_1 \sqsubseteq f\), which we may assume properly extends \(t_0\), and so on.

Let \(m_k = (\sum_{j \leq k} n_k) + k\). The outcome of II’s strategy just described will be a sequence \(\vec{x}^* Z\), where

\[
Z = (z_0, z_1, \ldots, z_{m_0}, \ldots, z_{m_1}, \ldots),
\]

such that for all \(k \in \omega\), \((\vec{x}^-(z_0, \ldots, z_{m_k}), (n_0, \ldots, n_{k-1}), (n_k)\) rejects \((q|t_k, Y), t_k \sqsubseteq f\). In particular, there is some \(Z^k\), with \((z_0, \ldots, z_{m_k}) \sqsubseteq Z^k\), and \(f_k \in [q|t_k]\) such that \((f_k, \vec{x}^* Z^k) \in A_{(n_0, \ldots, n_k)} = F[N_{(n_0, \ldots, n_k)}]\). Take \(\beta_k \in N_{(n_0, \ldots, n_k)}\) such that \(F(\beta_k) = (f_k, \vec{x}^* Z^k)\). Then, \(\beta_k\) converges to \((n_0, n_1, \ldots)\), \(f_k\) to \(f\), and \(Z^k\) to \(Z\) as \(k \rightarrow \infty\), so by the continuity of \(F\), \((f, \vec{x}^* Z) = F(n_0, n_1, \ldots) \in A\) and \(\vec{x}^* Z \in A_f\). \(\square\)

Finally, we note that this conclusion for analytic sets is sharp in the sense that it is consistent relative to \(\text{ZFC}\) that there is a coanalytic subset \(A \subseteq E^{[\omega]}\).
for which Theorem 1.3 fails (cf. Section 6 of [23] where this is proved for the analogous dichotomy in Banach spaces).

8. Destroying strategic filters

When $|F| = 2$, $(p^+)$-filters are necessarily strategic (Corollary 2.4 in [36]), and so by Theorem 1.8 above, are preserved by Sacks forcing. Alternatively, in this case nonzero vectors can be identified with their supports in FIN, and $(p^+)$-filters are exactly stable ordered union ultrafilters (Theorem 6.3 in [34]), which are equivalent to the selective ultrafilters on FIN (by Corollary 2.5 of [36]) shown to be preserved by Sacks forcing in [39].

In contrast, we will show in Theorem 1.9 that whenever $|F| > 2$, the strategic property is not preserved. Throughout the remainder of this section, we will assume that $|F| > 2$. Our first lemma is folklore:

**Lemma 8.1.** Let $W$ be any transitive model of ZFC which extends $V$ and suppose that $g \subseteq \omega$ is in $W$ but not $V$. Then, there is an infinite $z \subseteq \omega$ in $W$ such that no infinite subset of $z$ is in $V$.

**Proof.** Fix a recursive bijection $f : [\omega]^\omega \rightarrow \omega$. Setting $z = \{f(g \cap n) : n \in \omega\}$, one can then recursively recover $g$ from any infinite subset of $z$. \hfill \Box

The main idea behind the proof of Theorem 1.9 is to construct, in $V[g]$, a strategy $\alpha$ for player II in a variant of the Gowers game such that every block subsequence of every outcome of $\alpha$ codes a subset of the real $z$ from Lemma 8.1. We will do this using a generalization of asymptotic pairs.

**Definition 8.2.** An asymptotic sequence in $E$ is a sequence $(A_n)_{n \in \omega}$ of pairwise disjoint asymptotic sets.

**Lemma 8.3.** There is a partition $(P_n)_{n \in \omega}$ of $\omega$ such that for each $n \in \omega$, there is an $\ell \in \omega$ such that for every interval $I \subseteq \omega$ of length $\geq \ell$, $I \cap P_n \neq \emptyset$.

**Proof.** Let

\begin{align*}
P_0 &= \{m : m \equiv 0 \mathrm{ mod } 2\} \\
P_1 &= \{2m + 1 : m \equiv 0 \mathrm{ mod } 2\} = \{k : k \equiv 1 \mathrm{ mod } 4\} \\
P_2 &= \{2(2m + 1) + 1 : m \equiv 0 \mathrm{ mod } 2\} = \{k : k \equiv 3 \mathrm{ mod } 8\} \\
P_3 &= \{2(2(2m + 1) + 1) + 1 : m \equiv 0 \mathrm{ mod } 2\} = \{k : k \equiv 7 \mathrm{ mod } 16\}, \\
&\vdots \\
P_n &= \{k : k \equiv (2^n - 1) \mathrm{ mod } 2^{n+1}\},
\end{align*}

for $n \in \omega$. One can check that these sets are disjoint and each $P_n$ has the desired property, taking $\ell = 2^{n+1}$. \hfill \Box

**Lemma 8.4.** $E$ has an asymptotic sequence.
Proof. Let \( \text{osc} : E^* \to \omega \) be as in Example 1.2 and \((P_n)_{n \in \omega}\) as in Lemma 8.3. Define sets \( A_n \) by

\[
A_n = \{ v \in E^* : \text{osc}(v) \in P_n \}
\]

for each \( n \in \omega \). Then, since the range of \( \text{osc} \) contains arbitrarily long intervals on any infinite-dimensional subspace, it follows that \((A_n)_{n \in \omega}\) is an asymptotic sequence. Note that these sets are invariant under nonzero scalar multiplication. \(\square\)

Given a family \( \mathcal{H} \) on \( E \) and an \( X \in \mathcal{H} \), the restricted Gowers game \( G_{\mathcal{H}}[X] \) played below \( X \) is defined exactly as \( G[X] \), except that we require \( I \)'s moves to be from \( \mathcal{H} \mid X \). This game has been extensively studied by Noé de Rancourt and coauthors, see [6] and [7], and enables a variation of Theorem 1.4 for \((p)\)-families without the extra assumption of being full, at the expense of weakening the conclusion for \( II \) from \( G[X] \) to \( G_{\mathcal{H}}[X] \) (Theorem 3.3 in [7]).

It also affords us an alternate criterion for being strategic:

**Definition 8.5.** A family \( \mathcal{H} \) on \( E \) is \(+\)-strategic if whenever \( \alpha \) is a strategy for \( II \) in \( G_{\mathcal{H}}[X] \), where \( X \in \mathcal{H} \), there is an outcome of \( \alpha \) which lies in \( \mathcal{H} \).

**Theorem 8.6** (Theorem 5.5 in [36]). Let \( \mathcal{F} \) be a \((p)\)-filter on \( E \). Then, \( \mathcal{F} \) is \(+\)-strategic if and only if \( \mathcal{F} \) is strategic and full.

The following is the key technical lemma of this section.

**Lemma 8.7.** Suppose that \( \mathcal{F} \) is a filter on \( E \), \( g \) is \( V \)-generic for \( S \), \( \mathcal{F} \) is the filter generated by \( \mathcal{F} \) in \( V[g] \), and \( \alpha \) a strategy for \( II \) in \( G_{\mathcal{F}(E)} \) in \( V[g] \). Denote by \( [\alpha]_{\mathcal{V}}^* \) the set of outcomes of \( \alpha \) restricted to sequences \((X_n)_{n \in \omega}\) of moves by \( I \) such that:

(i) \((X_n)_{n \in \omega} \in \mathcal{F}(E) \cap \mathcal{V} \), and

(ii) for each \( n \in \omega \), \( \max(\text{supp}(\alpha(X_0, \ldots, X_n))) < \min(\text{supp}(X_{n+1})) \)

Then, there is a strategy \( \beta \) for \( II \) in \( G_{\mathcal{F}(E)} \) such that \( [\beta] \subseteq [\alpha]_{\mathcal{V}}^\ast \).

Here, if \( X = (x_k)_{k \in \omega} \) is a block sequence in \( E \), then its support is the sequence \( \text{supp}(X) = (\text{supp}(x_k))_{k \in \omega} \), and so \( \min(\text{supp}(X)) = \min(\text{supp}(x_0)) \).

**Proof.** Let \( \dot{\alpha} \) be a name for \( \alpha \).

**Claim.** There is a name \( \dot{\beta} \) for a strategy for \( II \) in \( G_{\mathcal{F}(E)} \) such that for any sequence of names \((\hat{X}_n)_{n \in \omega}\) for elements of \( \mathcal{F} \) and a \( p \in S \), there is a sequence \((X'_n)_{n \in \omega}\) of elements of \( \mathcal{F} \) in \( V \) and a \( q \leq p \) such that

\[ q \models \forall n (X'_n \leq \hat{X}_n \wedge \max(\text{supp}(\dot{\alpha}(\hat{X}_0, \ldots, \hat{X}_n))) < \min(\text{supp}(X'_{n+1})) \)

\[ \wedge \dot{\beta}(\hat{X}_0, \ldots, \hat{X}_n) = \dot{\alpha}(X'_0, \ldots, X'_n) \]

**Proof of Claim.** Let \((\hat{X}_n)_{n \in \omega}\) and \( p \) be as described. We will define \( \dot{\beta} \) after having found \( q \), but this argument shows that the collection of all such \( q \)'s contains a maximal antichain below \( p \) and thus by defining \( \dot{\beta} \) for each of those \( q \)'s, we can recover a single name which works for all of them.
Start by choosing $p_0 \leq p$ and $X'_0 = Y_0 \in \mathcal{F}$ such that $p_0 \Vdash Y_0 \subseteq \check{X}_0$. Suppose we have defined $p_i \in S$ and $X'_i \in \mathcal{F}$ for $i \leq n$. Enumerate $\ell(n + 1, p_n) = \{t_0, \ldots, t_k\}$. For each $j \leq k$, choose $q_j \leq p_n|t_j$ and $Y_j \in \mathcal{F}$ such that

$$\mathcal{F} \models q_j \Vdash Y_j \subseteq \check{X}_{n+1}/\check{\check{\alpha}}(\check{X}_0, \ldots, \check{X}_n).$$

Let $p_{n+1} = \bigcup_{j=0}^k q_j \leq_n p_n$ and take $X'_{n+1} \in \mathcal{F}$ such that $X'_{n+1} \subseteq Y_j$ for all $j \leq k$.

Let $q = \bigcap_{n\in\omega} p_n$. Then,

$$q \Vdash \forall n(X'_n \in \mathcal{F} \land X'_n \subseteq \check{X}_n/\check{\check{\alpha}}(\check{X}_0, \ldots, \check{X}_n)),$$

so we can define $\hat{\beta}$ to satisfy

$$q \Vdash \forall n(\hat{\beta}(\check{X}_0, \ldots, \check{X}_n)) = \check{\alpha}(X'_0, \ldots, X'_n)).$$

Notice that the choice $X'_0, \ldots, X'_n$, and thus the value of $\hat{\beta}(\check{X}_0, \ldots, \check{X}_n)$, only depends on $\check{X}_0, \ldots, \check{X}_n$, not the rest of the sequence. So, by choosing things uniformly (say, via a fixed well-ordering of $\mathcal{F}$), we can ensure that $\hat{\beta}$ is forced to be a well-defined strategy.

Working in $V[g]$, let $\beta$ be the strategy named by $\hat{\beta}$. Consider a sequence of moves $(X_n)_{n\in\omega}$ in $\mathcal{F}$ by $I$ in $G_{\mathcal{F}}[E]$. By the claim, together with genericity, there is a sequence $(X'_n)_{n\in\omega}$ of elements of $\mathcal{F}$ in $V$ such that for all $n \in \omega$,

$$\max(\text{supp}(\alpha(X_0, \ldots, X_n))) < \min(\text{supp}(X'_{n+1}))$$

and

$$\beta(X_0, \ldots, X_n) = \alpha(X'_0, \ldots, X'_n).$$

Thus, $[\beta] \subseteq [\alpha]_V^*$. □

**Lemma 8.8.** Let $\mathcal{F}$ be a filter on $E$, $\mathcal{W}$ any transitive model of ZFC extending $V$, and $\mathcal{F}$ the filter generated by $\mathcal{F}$ in $\mathcal{W}$. If $\mathcal{F}$ fails to be strategic in $V$, then $\mathcal{F}$ fails to be strategic in $\mathcal{W}$.

**Proof.** We prove the contrapositive. Suppose that $\mathcal{F}$ is strategic in $\mathcal{W}$. By Lemma 4.7 in [34], it suffices to consider the finitized Gowers game (as in Lemma 6.4). Let $\alpha$ be a strategy for $II$ in $G^{<\omega}[X]$, for some $X \in \mathcal{F}$, in $V$. Since $I$'s moves in $G^{<\omega}[X]$ are vectors in $E$, the definition of $\alpha$ being a strategy for $II$ is $\Pi^1_1$, as it must always produce infinite block sequences, and thus absolute between transitive models. So, we may view $\alpha$ as a strategy in $\mathcal{W}$ as well.

Since $\mathcal{F}$ is strategic, there is some $Y \in \mathcal{F}$ which is an outcome of $\alpha$. Let $Z \in \mathcal{F}$ be below $Y$. The statement “$Z$ is below some outcome of $\alpha$” is $\Sigma^1_1$ and thus also absolute between $V$ and $\mathcal{W}$. Hence, there is some $Y' \in E^{[\omega]} \cap V$ which is an outcome of $\alpha$ and with $Z \leq Y'$. Since $\mathcal{F}$ is a family in $V$, $Y' \in \mathcal{F}$. Thus, $\mathcal{F}$ is strategic in $V$, as well. □

We can now prove Theorem 1.9:

**Proof of Theorem 1.9.** Suppose that $\mathcal{F}$ is a strategic $(p^+)$-filter on $E$ and $g$ is $V$-generic for $S$. Working in $V[g]$, we will define a strategy $\alpha$ for $II$ in $G_{\mathcal{F}}[E]$ such that no $Y \in [\alpha]^*_V$ has an infinite block subsequence in $V$. By
Lemma 8.7, this implies that $F$ is not +-strategic, so by Theorems 1.8 and 8.6, it is not strategic either.

Fix a set $z \subseteq \omega$ as in Lemma 8.1 in $V[g]$ and an asymptotic sequence $(A_n)_{n \in \omega}$ in $V$. Note that being an asymptotic sequence is absolute. We may assume that $z$ is enumerated in increasing order as $z(0),z(1),...,\text{etc.}$

We define $\alpha$ as follows: Given $\Gamma$'s first move, $X_0 \in F$, let $\alpha(X_0)$ be the first nonzero element of $\langle X_0 \rangle$ in $A_{z(0)}$, with respect to some fixed enumeration of $E$ in $V$. Having defined $\alpha(X_0,\ldots,X_n)$, let $\alpha(X_0,\ldots,X_n,X_{n+1})$ be the first nonzero element of $\langle X_{n+1} \rangle$ in $A_{z(n+1)}$ which has support above $\alpha(X_0,\ldots,X_n)$. Any infinite subsequence (in the usual sense, not the sense of $\preceq$) of an outcome of $\alpha$ encodes an infinite subset of $z$ and so is not in $V$.

It remains to show that no $Y \in [\alpha]_V^*$ has an infinite block subsequence (in the sense of $\preceq$) in $V$. Take a $Y = (y_n)_{n \in \omega} \in [\alpha]_V^*$, as witnessed by the sequence $(X_n)_{n \in \omega} \in F^\omega \cap V$, where $X_n = (x^i_n)_{i \in \omega}$ and $\max(\supp(y_n)) < \min(\supp(x^i_n+1))$, for each $n \in \omega$. Let $v \in \langle Y \rangle$. It suffices to show how we can compute, from $(X_n)_{n \in \omega}$ and $v$, those $y_i$'s (possibly up to scaling, since the $A_n$'s are invariant under nonzero scalar multiplication) for which $v$ is a nontrivial linear combination. It will follow that, from any infinite block sequence below $Y$, we can compute an infinite subset of $z$.

By the requirement on the supports of the $X_n$'s, we see that there is a least $N \in \omega$ such that $(\bigcup_{n>N} \supp(X_n)) \cap \supp(v) = \emptyset$. As we scan across $v$, from right to left, we will arrive at a point at which the nonzero vector to the right is in $\langle X_N \rangle$ and the vector to the left is either 0 or is strictly below $\supp(x^i_0)$). Consequently, the former is a scalar multiple of $y_N$. Repeating this process, scanning from right to left through $v$, we can compute exactly those $y_i$'s of which $v$ is a linear combination. This completes proof. \[\square\]

Combining this with Theorem 5.3, we have the following:

Corollary 8.9. It is consistent with ZFC that there are strong $(p^+)$-filters on $E$ which fail to be strategic. \[\square\]

It is worth stressing the metamathematical content of these results: A strong $(p^+)$-filter is “$\Pi^1_1$-generic” in the sense that it meets every $\preceq$-dense open, coanalytic subset of $E^{[\omega]}$ (see Section 4 of [34]). However, if such a filter fails to be strategic, there are $\preceq$-dense open analytic subsets of $E^{[\omega]}$ which it fails to meet, namely the set of block sequences below an outcome of some strategy for II in the (finitized) Gowers game.

This is very different from the situation for selective ultrafilters on $\omega$: An ultrafilter on $\omega$ is selective if and only if it meets every dense open, closed subset of $[\omega]^\omega$ (the sets of witnesses for instances of Ramsey’s Theorem for pairs suffices), and if and only if it meets every dense open $\Sigma^1_1 \cup \Pi^1_1$ subset of $[\omega]^\omega$. Similar characterizations hold for stable ordered union ordered ultrafilters on FIN (see [4]) or equivalently, $(p^+)$-filters on $E$ when $|F| = 2$. 
9. Coda

We end here with a discussion of the ways Theorems 1.7 and 1.8 could be improved and some ideas for how to do so. First and foremost, there is the hypothesis of being “strategic” in Theorems 1.7 and 1.8. This assumption is not present in Theorem 1.4, the author’s original local Ramsey theorem for analytic sets of block sequences in $E$, though is required for extending beyond analytic sets under large cardinals.

**Question 1.** Is “strategic” necessary in Theorems 1.7 and 1.8?

This property is only used in one part of the proofs of Theorems 1.7 and 1.8, namely in the proof of Theorem 5.1, the local version of the parametrized weak pigeonhole principle, via an application of Lemma 3.6. Note that fullness (Proposition 3.6 in [34]) and the $(p)$-property (Theorem 4.1 in [36]) are both necessary for the clopen case of Theorem 1.4, and so cannot be removed from the hypotheses of Theorem 1.7 either.

We believe that Question 1 would be resolved by finding the “right” proof of Theorem 1.7. What we have in mind is a combinatorial (rather than metamathematical) argument in the style of Ellentuck’s proof [8] of the Galvin–Prikry and Silver Theorems, and Pawlikowski’s proof [29] of the Miller–Todorčević Theorem, combined with arguments from [31] and [34] specific to the vector space setting.

**Question 2.** Is there an “Ellentuck-style” proof of Theorem 1.7?

The arguments in Section 7 are the remnants of such a proof and show that one can go from open sets being perfectly $H$-strategically Ramsey to analytic sets using only the $(p)$-property of $H$ and fusion in $S$. Thus, it would suffice to give an “Ellentuck-style” proof of Lemma 6.7.

Baumgartner and Laver [3] showed that selective ultrafilters are preserved by iterated Sacks forcing. That is, after an $\omega_2$-length countable support iteration of Sacks forcing, any selective ultrafilters in the ground model still generate selective ultrafilters in the extension.

The presence of “strategic” in the hypotheses of Theorem 1.8, combined with its subsequent destruction when adding a single Sacks real (if $|F| > 2$) in Theorem 1.9, presents an obstacle to proving an analogous result for iterated Sacks forcing here. We do not even know if a strategic $(p^+)$-filter still generates a $(p^+)$-filter after adding two Sacks reals, iteratively. Note that, by Lemma 8.8, once the strategic property is destroyed, it can never be restored in any forcing extension.

**Question 3.** If $F$ is a strategic $(p^+)$-filter on $E$, $\alpha \leq \omega_2$, and $G_\alpha$ is $\mathbb{V}$-generic for an $\alpha$-length countable support iteration $S_\alpha$ of Sacks forcing, must $F$ generate a (strong) $(p^+)$-filter in $\mathbb{V}[G_\omega]$?

If the answer to Question 3 is “Yes”, then, assuming CH holds in the ground model $\mathbb{V}$, there will be many (strong) $(p^+)$-filters in $\mathbb{V}[G_\omega]$, none of which are strategic, a dramatic strengthening of Corollary 8.9.
Theorem 9.1. Assume that the answer to Question \( β \) is “Yes”. If \( \mathbf{V} \models \mathbf{CH} \) and \( G_{\omega_2} \) is \( \mathbf{V} \)-generic for \( S_{\omega_2} \), then no \((p^+)\)-filter in \( \mathbf{V}[G_{\omega_2}] \) is strategic.

Proof. Fix an enumeration of \( E^* \) as \( \{v_n : n \in \omega\} \) in \( \mathbf{V} \). Suppose that \( \mathcal{F} \) is a \((p^+)\)-filter on \( E \) in \( \mathbf{V}[G_{\omega_2}] \). Since \( \mathbf{V}[G_{\omega_2}] \models 2^{\aleph_0} = \aleph_2 \), we can enumerate \( \mathcal{F} \) as \( \{X_\alpha : \alpha < \omega_2\} \) in \( \mathbf{V}[G_{\omega_2}] \). Let \( X_\alpha \) be an \( S_{\omega_2} \)-name for each \( X_\alpha \). For each \( \alpha < \omega_2 \) and \( n \in \omega \), we can find a maximal antichain \( A_{\alpha,n} \in S_{\omega_2} \) such that every element of \( A_{\alpha,n} \) decides whether \( v_n \in \langle X_\alpha \rangle \). Since \( S_{\omega_2} \) satisfies the \( \aleph_2 \)-chain condition (Theorem 3.2 in [3]), for every \( \alpha < \omega_2 \) and \( n \in \omega \), there is a \( \beta < \omega_2 \) such that \( A_{\alpha,n} \subseteq S_\beta \). This ensures that \( \mathcal{F} \cap \mathbf{V}[G_\beta] \in \mathbf{V}[G_\beta] \).

We next use the fact that every real added by forcing with \( S_{\omega_2} \) is added by some initial stage \( S_\alpha \), where \( \text{cf}(\alpha) = \omega \) (Theorem 3.3(a) in [3]). This allows us to choose, in \( \mathbf{V}[G_{\omega_2}] \), a function \( \pi : \omega_2 \to \omega_2 \) such that for each decreasing sequence \( (X_\alpha)_{n \in \omega} \) of block sequences in \( \mathcal{F} \) and subset \( A \subseteq E^* \), coded appropriately as reals in \( \mathbf{V}[G_\alpha] \), there are elements \( X \) and \( Y \) of \( \mathcal{F}_1 \in \mathbf{V}[G_{\pi(\alpha)}] \), such that \( X \) is diagonalization of \( (X_\alpha)_{n \in \omega} \) and either \( A \cap \langle Y \rangle = \emptyset \) or \( A \) is asymptotic below \( Y \). By the \( \aleph_2 \)-chain condition, there is a function \( \rho : \omega_2 \to \omega_2 \) in \( \mathbf{V} \) with \( \pi(\alpha) \leq \rho(\alpha) \) for all \( \alpha < \omega_2 \). Since there is a closed unbounded set of \( \gamma \)'s for which \( \rho(\xi) < \gamma \) for all \( \xi < \gamma \), we may choose such a \( \gamma \) with \( \text{cf}(\gamma) = \omega_1 \) and \( \gamma > \beta \). It follows that \( \mathcal{F} \cap \mathbf{V}[G_\gamma] \in \mathbf{V}[G_\gamma] \) and is a \((p^+)\)-filter in \( \mathbf{V}[G_\gamma] \).

By Theorems 1.8 and 1.9, \( \mathcal{F} \cap \mathbf{V}[G_\gamma] \) generates a \((p^+)\)-filter which fails to be strategic in \( \mathbf{V}[G_{\gamma+1}] = \mathbf{V}[G_\gamma][g] \), where \( g \) is \( \mathbf{V}[G_\gamma] \)-generic for \( S_\mathbf{V}[G_\gamma] \). By our assumption, \( \mathcal{F} \cap \mathbf{V}[G_\gamma] \) generates \( \mathcal{F} \) in \( \mathbf{V}[G_{\omega_2}] \), so by Lemma 8.8 applied the extension \( \mathbf{V}[G_{\omega_2}] \supseteq \mathbf{V}[G_{\gamma+1}] \), \( \mathcal{F} \) fails to be strategic in \( \mathbf{V}[G_{\omega_2}] \). \( \square \)

The above proof also shows that, assuming the answer is “Yes” to Question 3, every \((p^+)\)-filter in \( \mathbf{V}[G_{\omega_2}] \) is \( \aleph_1 \)-generated, so if moreover, \( \mathbf{V} \models 2^{\aleph_1} = \aleph_2 \), then there are exactly \( 2^{\aleph_0} \)-many \((p^+)\)-filters in \( \mathbf{V}[G_{\omega_2}] \) (see Theorem 4.5 in [3], on which the above proof is based).

Next is the question of improving the conclusion of Theorem 1.7. The most natural extension would be to parametrize by a countable sequence of perfect sets, rather than just a single perfect set.

Question 4. Can the Ramsey theory of block sequences in \( E \) be parametrized by countable sequences of perfect sets? That is, given an analytic set \( \mathcal{A} \subseteq \mathbb{R}^\omega \times E^{[\omega]} \), must there exist a sequence \( (P_n)_{n \in \omega} \) of nonempty perfect sets in \( \mathbb{R} \) and an \( X \in E^{[\omega]} \) such that either:

1. \( X \) has a strategy \( \sigma \) in \( F[X] \) such that \( (\prod_{n \in \omega} P_n) \times [\sigma] \subseteq \mathcal{A}^c \), or
2. for every \( T = (t_n)_{n \in \omega} \in \prod_{n \in \omega} P_n \), \( T \) has a strategy in \( G[X] \) for playing into \( \mathcal{A} \)?

If so, can this be localized to a (strategic) \((p^+)\)-family on \( E \)?

Theorem 1.5 can be extended to countable sequences of perfect sets using Laver’s infinite-dimensional form [22] of the Halpern–Läuchli Theorem [15]. Laver used this to show that selective ultrafilters are preserved by forcing with arbitrarily long countable support products, or “side-by-side”, Sacks
forcing. The corresponding facts for finite products of perfects sets and Sacks forcing were proven earlier by Pincus and Halpern [30]. Zheng [39] and Kawach [18] have shown that the Ramsey spaces \( \text{FIN}, \text{FIN}_k, \text{FIN}_\pm k \) can be parametrized by countable sequences of perfect sets, and Zheng also established the corresponding ultrafilter preservation result for \( \text{FIN} \). We ask if such a preservation result is possible in our setting:

**Question 5.** If \( F \) is a strategic \((p^+)\)-filter on \( E, \kappa \) a cardinal, and \( G \) is \( V \)-generic for a \( \kappa \)-length countable support product \( S(\kappa) \) of Sacks forcing, must \( F \) generate a strong \((p^+)\)-filter in \( V[G] \)?

A “Yes” answer to the local form of Question 4 would yield a “Yes” answer to Question 5, though as we have seen in the proofs of Theorems 1.7 and 1.8, their proofs could be intertwined in some way.

Again, we expect that the key to resolving Questions 4 and 5 lies in Question 2. If we were able to find an “Ellentuck-style” proof of Theorem 1.7, then by adapting any fusion arguments in \( S \) to the countable product \( S^{(\omega)} \), as in [2], we should be able to extend the proof to answer Question 4 in the affirmative. Moreover, following Zheng’s proof that stable ordered union ultrafilters in \( \text{FIN} \) are preserved by iterated Sacks forcing (Theorem 5.42 in [40]), we also expect that a “Yes” answer to Question 4 would assist in resolving Question 3.

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