**η-EINSTEIN SASAKIAN IMMERSIONS IN NON-COMPACT SASAKIAN SPACE FORMS**

GIANLUCA BANDE, BENIAMINO CAPPELLETTI–MONTANO, AND ANDREA LOI

**Abstract.** The aim of this paper is to study Sasakian immersions of (non-compact) complete regular Sasakian manifolds into the Heisenberg group and into $\mathbb{B}^N \times \mathbb{R}$ equipped with their standard Sasakian structures. We obtain a complete classification of such manifolds in the $\eta$-Einstein case.

1. Introduction

Sasakian geometry is considered as the odd-dimensional counterpart of Kähler geometry. Despite the Kähler case, where the study of Kähler immersions is well developed, due to the seminal work of Calabi [6] (see also [15] for a modern treatment and an account on the subject), in the Sasakian setting there are few results. Most of the Sasakian results are concerned with finding conditions which ensure that a Sasakian submanifold is totally geodesic or similar geometric properties (see, for instance, [12, 13, 14]).

In [7] the second and the third authors studied Sasakian immersions into spheres. In particular they proved the following classification result:

**Theorem (7).** Let $S$ be a $(2n+1)$-dimensional compact $\eta$-Einstein Sasakian manifold. Assume that there exists a Sasakian immersion of $S$ into $S^{2n+1}$. If $N = n + 2$ then $S$ is Sasaki equivalent to $S^{2n+1}$ or to the Boothby-Wang fibration over $Q_n$, where $Q_n \subset \mathbb{C}P^{n+1}$ is the complex quadric equipped with the restriction of the Fubini–Study form of $\mathbb{C}P^{n+1}$.

Since the (Sasakian) sphere is one of the three “models” of Sasakian space forms, it is quite natural to study as a second step the immersions into Sasakian space forms.

In this paper we give a complete characterisation of Sasakian immersions of complete, regular, $\eta$-Einstein Sasakian manifolds into a non-compact Sasakian space form $M(N, c)$, proving the following:

**Theorem 1.** Let $S$ be a $(2n+1)$-dimensional connected, complete, regular $\eta$-Einstein Sasakian manifold. Suppose that there exists $p \in S$, an open neighborhood $U_p$ of $p$ and a Sasakian immersion $\phi : U_p \to M(N, c)$, where $c \leq -3$. Then $S$ is Sasaki equivalent to $M(n, c)/\Gamma$ where $\Gamma$ is some discrete subgroup of the Sasakian-isometry group of $M(n, c)$. Moreover, if $U_p = S$ then $\Gamma = \{1\}$ and $\phi$ is, up to a Sasakian transformation of $M(N, c)$, given by

$$\phi(z, t) = (z, 0, t + c)$$
Theorem 1 is a strong generalisation of [12, Theorem 3.2] which asserts that a complete, \( \phi \)-invariant, \( \eta \)-Einstein submanifold of codimension 2 of the \((2N + 1)\)-dimensional Heisenberg group is necessarily a totally geodesic submanifold Sasaki-equivalent to a copy of a \((2N - 1)\)-dimensional Heisenberg group and similarly for totally geodesic submanifolds of \( \mathbb{B}^N \times \mathbb{R} \), where \( \mathbb{B}^N \) denotes the unit disc of \( \mathbb{C}^N \) equipped with the hyperbolic metric. In fact in our result there is no restriction on the codimension and we assume that we have a Sasakian immersion instead of a \( \phi \)-invariant submanifold. Moreover the immersion is not necessarily injective and is not assumed to be from the whole space but from an open neighbourhood of a point.

The general philosophy in [7] and in this paper is to take into account the transversal Kähler geometry of the Reeb foliation. When a regular Sasakian manifold is compact as in [7], one can use the so-called Boothby-Wang construction [3], which realises the space of leaves as a Kähler manifold which is the base of a principal \( S^1 \)-fibration. Then one translates the immersion problem into a Kähler immersion problem of the base spaces.

Trying the same trick in the non-compact case is more complicated because the Boothby-Wang construction fails in general, even if the Sasakian manifold is regular. Nevertheless, the Reeb foliation has the strong property to be both a totally geodesic and a Riemannian foliation. Assuming the Sasakian manifold complete, one can appeal to the result of Reinhart [19] which says that the space of leaves is the base space of a fibration, and once again translate the problem into one on Kähler immersions.

The paper contains two other sections. In Section 2 we recall the main definitions and some foliation theory needed in the proof of Theorem 1 to whom Section 3 is dedicated.

2. Preliminaries

A contact metric manifold is a contact manifold \((S, \eta)\) admitting a Riemannian metric \(g\) compatible with the contact structure, in the sense that, defined the \((1,1)\)-tensor \(\phi\) by \(d\eta = 2g(\cdot, \phi \cdot)\), the following conditions are fulfilled

\[
\phi^2 = -Id + \eta \otimes \xi, \quad g(\phi \cdot, \phi \cdot) = g - \eta \otimes \eta,
\]

where \(\xi\) denotes the Reeb vector field of the contact structure, that is the unique vector field on \(S\) such that

\[
i_\xi \eta = 1, \quad i_\xi d\eta = 0.
\]

A contact metric manifold is said to be Sasakian if the following integrability condition is satisfied

\[
N_\phi(X, Y) := [\phi X, \phi Y] + \phi^2[X, Y] - \phi[X, \phi Y] - \phi[\phi X, Y] = -d\eta(X, Y)\xi,
\]

for any vector fields \(X\) and \(Y\) on \(S\).

Two Sasakian manifolds \((S_1, \eta_1, g_1)\) and \((S_2, \eta_2, g_2)\) are said to be equivalent if there exists a contactomorphism \(F : S_1 \rightarrow S_2\) between them which is also an isometry, i.e.

\[
F^*\eta_2 = \eta_1, \quad F^*g_2 = g_1.
\]

One can prove that if (3) holds then \(F\) satisfies also

\[
F_{x_1} \circ \phi_1 = \phi_2 \circ F_{x_2}, \quad F_{x_1} \xi_1 = \xi_2
\]

for any \(x \in S_1\). An isometric contactomorphism \(F : S \rightarrow S\) from a Sasakian manifold \((S, \eta, g)\) to itself will be called a Sasakian transformation of \((S, \eta, g)\).

It is a well-known fact [3] that the foliation defined by the Reeb vector field of a Sasakian manifold \(S\) has a transversal Kähler structure. Using the theory of Riemannian
submersions one can prove that the transverse geometry is Kähler-Einstein if and only if the Ricci tensor of $S$ satisfies the following equality

$$(4) \quad \text{Ric} = \lambda g + \nu \eta \otimes \eta$$

for some constants $\lambda$ and $\nu$. Any Sasakian manifold satisfying (4) is said to be $\eta$-Einstein (see [5] for more details).

A remarkable property of $\eta$-Einstein Sasakian manifolds is that, contrary to Sasaki-Einstein ones, they are preserved by $D_a$-homothetic deformations, that is the change of structure tensors of the form

$$(5) \quad \phi_a := \phi, \quad \xi_a := \frac{1}{a} \xi, \quad \eta_a := a \eta, \quad g_a := ag + a(a - 1) \eta \otimes \eta$$

where $a > 0$.

By a Sasakian immersion (often called invariant submanifolds or Sasakian submanifolds in the literature) of a Sasakian manifold $(S_1, \eta_1, g_1)$ into the Sasakian manifold $(S_2, \eta_2, g_2)$ we mean an isometric immersion $\varphi : (S_1, g_1) \longrightarrow (S_2, g_2)$ that preserves the Sasakian structures, i.e. such that

$$(6) \quad \varphi^* g_2 = g_1, \quad \varphi^* \eta_2 = \eta_1,$$

$$(7) \quad \varphi_* \xi_1 = \xi_2, \quad \varphi_\ast \phi_1 = \phi_2 \circ \varphi_\ast.$$

We refer the reader to the standard references [2, 4] for a more detailed account of Riemannian contact geometry and Sasakian manifolds.

**Sasakian space forms.** Recall that the curvature tensor of a Sasakian manifold is completely determined [2] by its $\phi$-sectional curvature, that is the sectional curvature of plane sections of the type $(X, \phi X)$, for $X$ a unit vector field orthogonal to the Reeb vector field.

A Sasakian space form is a connected, complete Sasakian manifold with constant $\phi$-sectional curvature. According to Tanno [20] there are exactly three simply connected Sasakian space forms depending on the value $c$ of the $\phi$-sectional curvature: the standard Sasakian sphere up $D_a$-homothetic deformation if $c > -3$, the Heisenberg group $\mathbb{C}^n \times \mathbb{R}$ if $c = -3$ and the hyperbolic Sasakian space form $\mathbb{H}^n \times \mathbb{R}$ if $c < -3$. Notice that each simply connected space form admits a fibration over a Kähler manifold and in the non-compact cases the fibration is trivial.

We denote by $M(n, c)$ the simply connected $(2n + 1)$-dimensional Sasakian space form with $\phi$-sectional curvature equal to $c$. Every connected, complete Sasakian space form is Sasakian equivalent to $M(n, c) / \Gamma$, where $\Gamma$ is a discrete subgroup of the Sasakian transformation group of $M(n, c)$.

**Immersions and regular foliations.** We recall some basic concepts from foliation theory (see e.g. [16, 18]). Let $M$ be a smooth manifold of dimension $n$. A foliation can be defined as a maximal foliation atlas on $M$, where a foliation atlas of codimension $q$ on $M$ is an atlas

$$\{ \varphi_i : U_i \longrightarrow \mathbb{R}^n = \mathbb{R}^p \times \mathbb{R}^q \}_{i \in I}$$

of $M$ such that the change of charts diffeomorphisms $\varphi_{ij}$ locally takes the form

$$\varphi_{ij}(x, y) = (g_{ij}(x, y), h_{ij}(y)).$$

Each foliated chart is divided into plaques, the connected components of

$$\varphi_i^{-1}(\mathbb{R}^p \times \{ y \}),$$

where $y \in \mathbb{R}^q$, and the changes of chart diffeomorphism preserve this division.
Definition 2. A foliated map is a map $f : (M, \mathcal{F}) \longrightarrow (M', \mathcal{F}')$ between foliated manifolds which preserves the foliation structure, i.e. which maps leaves of $\mathcal{F}$ into leaves of $\mathcal{F}'$.

Now, let $(M, \mathcal{F})$ and $(M', \mathcal{F}')$ be foliated manifolds and $f : M \longrightarrow M'$ be an immersion. Moreover, assume that $f$ is a foliated map. Thus

$$f_\ast(L(x)) \subset L'(f(x))$$

for each $x \in M$, where $L = T(\mathcal{F})$ and $L' = T(\mathcal{F}')$. In particular, it follows that $\dim(\mathcal{F}) \leq \dim(\mathcal{F}')$. The proof of the following proposition is quite standard and will be omitted:

Proposition 3. $(M, \mathcal{F})$ and $(M', \mathcal{F}')$ be foliated manifolds of dimension $n$ and $n'$, respectively, and $f : M \longrightarrow M'$ be a foliated immersion. Suppose that $\dim(\mathcal{F}) = \dim(\mathcal{F}')$. Then for each $x \in M$ there are charts $\varphi : U \longrightarrow \mathbb{R}^p \times \mathbb{R}^q$ for $M$ about $x$ and $\varphi' : U' \longrightarrow \mathbb{R}^p \times \mathbb{R}^q$ for $M'$ about $f(x)$ such that

\begin{enumerate}[i]
  \item $\varphi(x) = (0, \ldots, 0) \in \mathbb{R}^n$
  \item $\varphi(f(x)) = (0, \ldots, 0) \in \mathbb{R}^{n'}$
  \item $F(x_1, \ldots, x_n) = (x_1, \ldots, x_n, 0, \ldots, 0)$, where $F := \varphi' \circ f \circ \varphi^{-1}$
  \item $L(x) = \text{span}\left\{ \frac{\partial}{\partial x_1}(x), \ldots, \frac{\partial}{\partial x_p}(x) \right\}$
  \item $L'(f(x)) = \text{span}\left\{ \frac{\partial}{\partial x_1}(f(x)), \ldots, \frac{\partial}{\partial x_p}(f(x)) \right\}$
\end{enumerate}

where $p = \dim(\mathcal{F}) = \dim(\mathcal{F}')$, $q = n - p$, $q' = n' - p$.

Let $\mathcal{F}$ be a foliation on a manifold $M$ and let $L$ be a leaf of $\mathcal{F}$. It is well known that $L$ intersects at most a countable number of plaques in a foliated chart $U$. Now we give the following definition.

Definition 4 (\cite{10}). A foliation $\mathcal{F}$ is said to be regular if for any $x \in M$ there exists a foliated chart $U$ containing $x$ such that every leaf of $\mathcal{F}$ intersects at most one plaque of $U$.

The following proposition is a generalisation to the non-compact case and to immersions of \cite{11} Proposition 3.1:

Proposition 5. Let $(M, \mathcal{F})$ and $(M', \mathcal{F}')$ be foliated manifolds such that $\dim(\mathcal{F}) = \dim(\mathcal{F}')$. If there exists a foliated immersion $f : (M, \mathcal{F}) \longrightarrow (M', \mathcal{F}')$ and $\mathcal{F}'$ is regular, then $\mathcal{F}$ is also regular.

\textbf{Proof.} Assume that $\mathcal{F}$ is not regular. Then there exists a point $x \in M$ and a leaf $L$ of $\mathcal{F}$ such that, for any foliated chart $U$ containing $x$, $L$ intersects more than one plaque in $U$. Let us consider the foliated charts $U$ and $U'$, respectively about $x$ and $f(x)$, satisfying the properties stated in Proposition \cite{8}. Then there exist at least two plaques, say $P_1 = \varphi^{-1}(\mathbb{R}^p \times \{y_1\})$ and $P_1 = \varphi^{-1}(\mathbb{R}^p \times \{y_2\})$, such that

$$L \cap P_1 \neq \emptyset, \quad L \cap P_2 \neq \emptyset,$$

where $y_1, y_2 \in \mathbb{R}^q$. Notice that, for each $i \in \{1, 2\}$, $f(P_i)$ is a plaque of $\mathcal{F}'$ in $U' := f(U)$. Indeed, using Proposition \cite{8} we have $f(P_1) = f(\varphi^{-1}(\mathbb{R}^p \times \{y_1\})) = \varphi^{-1}(f(\mathbb{R}^p \times \{y_1\})) = \varphi'^{-1}(\mathbb{R}^p \times \{(y_1, 0, \ldots, 0)\})$. Now, since $f$ is a foliated map, $L' = f(L)$ is a leaf of $\mathcal{F}'$ and from \cite{8} it follows that $L' \cap f(P_1) \neq \emptyset$ and $L' \cap f(P_2) \neq \emptyset$. But this contradicts the regularity of $\mathcal{F}'$. \hfill $\square$
In this Section we prove the main result of this paper, that is the classification of connected, regular $\eta$-Einstein Sasakian manifolds immersed into Sasakian space forms.

**Proof of Theorem** [1]. Let $M(N, c)$ be one of the non-compact simply connected Sasakian space forms and $\pi': M(N, c) \to K'$ the (trivial) fibration over its Kähler quotient. Recall that $K'$ is either $\mathbb{C}^N$ with its flat Kähler metric or the hyperbolic Kähler space form $\mathbb{B}^N$.

Since $S$ is complete and regular, by [19] there exists a fibration $\pi: S \to K$, whose fibers are the leaves of the Reeb foliation of $S$. By assumption $S$ is Sasakian $\eta$-Einstein and then $K$ is necessarily Kähler-Einstein (see [9]). The fibration $\pi: S \to K$ is a Riemannian submersion and since $S$ is complete then $K$ also is (see [17, 9]).

By assumption there exists an open neighbourhood $U_p$ of $p \in S$ and a Sasakian immersion $\varphi: U_p \to M(N, c)$. By Proposition 5 the submanifold $U_p$ is still regular. The restriction of $\pi$ to $U_p$ gives then a projection of the Sasakian $\eta$-Einstein manifold $U_p$ to the Kähler manifold $\pi(U_p) \subset K$.

The Sasakian immersion $\varphi: U_p \to M(N, c)$ covers a Kähler immersion $i(\varphi)$ (see [7, 11]) making the following diagram commutative:

$$
\begin{array}{ccc}
U_p & \xrightarrow{\varphi} & M(N, c) \\
\downarrow{\pi} & & \downarrow{\pi'} \\
\pi(U_p) \subset K & \xrightarrow{i(\varphi)} & K'
\end{array}
$$

We have proved that there exists $q = \pi(p) \in K$, an open neighbourhood $V_q = \pi(U_p)$ of $q$ and a Kähler immersion of $V_q$ in $K'$. By Umehara [21] $V_q$ is flat or complex hyperbolic.

On the other end, by [8] (see also [11, Theorem 5.26]), the Kähler-Einstein manifold $K$ is real analytic and by [6] Theorem 4 and Theorem 10, for every $q \in K$ there exists an open neighbourhood $V'_q$ and a Kähler immersion of $V'_q$ in $K'$. Then $K$ is locally flat and [9] Formula 1.31 implies that the $\phi$-sectional curvature of $S$ is less or equal to $-3$.

By Tanno [20] there exists a discrete group $\Gamma$ of the Sasakian transformations of $S$ such that $S = M(n, c)/\Gamma$ and this proves the first part of the theorem.

For the second part of the theorem, let us suppose $U_p = S$. Reasoning as before, by completeness of $K$ and by Calabi’s Rigidity Theorem [6] for Kähler immersions into Kähler space forms (see also [15]) one obtains the stronger result that either $K = \mathbb{C}^n$ or $K = \mathbb{B}^n$ and the projection is just the trivial fibration because in both cases $K$ contractible. Then, since $S$ is complete, the fibres of the fibration are diffeomorphic either to $\mathbb{R}$ or to $S^1$.

The second case cannot occur because $\varphi$ is a Sasakian immersion and then it restricts to immersions on the leaves of the Reeb foliations of $S$ and $M(N, c)$. But the leaves of $M(N, c)$ are diffeomorphic to $\mathbb{R}$ and if the leaves of $S$ are circles we would obtain immersions of the circle in $\mathbb{R}$ which is not possible.

Now it remains to prove that $\varphi$ is the standard embedding up to Sasakian transformations.

First observe that, again by Calabi’s Rigidity Theorem, the immersion $i(\varphi): K \to K'$ has (up to unitary transformation) the following form:

$$
i(\varphi)(z_1, \ldots, z_n) = (z_1, \ldots, z_n, 0, \ldots, 0).
$$

Because the fibrations are trivial $\varphi$ must have the following expression:

$$\varphi(z_1, \ldots, z_n, t) = (z_1, \ldots, z_n, 0, \ldots, 0, f_N((z_1, \ldots, z_n, t))).$$
Since \( \phi \) is a Sasakian immersion, in particular we have \( \varphi^*(\eta_N) = \eta_h \), where \( \eta_N \) and \( \eta_h \) are the standard contact forms of \( M(N,c) \) and \( M(n,c) \) respectively. Then a direct calculation of \( \varphi^*(\eta_N) = \eta_h \) yields \( \frac{\partial i_x}{\partial t} = 1 \) and \( \frac{\partial i_x}{\partial y_i} = 0 \) for \( i = 1, \ldots n \), where we put \( z_j = x_j + iy_j \).

**Remark 6.** In Theorem 1 the case \( U_p = S \) cannot occur if \( S \) is compact because if \( S \) is compact, a Sasakian immersion cannot exist for otherwise, from the regularity of a compact Sasakian manifold, we would obtain a (compact) Kähler quotient immersed either in \( \mathbb{C}^N \) or in \( \mathbb{B}^N \), which is impossible by the Maximum Principle.

The following result is a variation of Theorem 1:

**Theorem 7.** Let \( S \) be a \((2n + 1)\)-dimensional connected, complete, \( \eta \)-Einstein Sasakian manifold. Suppose that for every \( p \in S \) there exists an open neighbourhood \( U_p \) of \( p \) and a Sasakian immersion \( \phi : U_p \to M(N,c) \), where \( c \leq -3 \). Then \( S \) is Sasaki-equivalent to \( M(n,c)/\Gamma \) where \( \Gamma \) is some discrete subgroup of the Sasaki-isometry group of \( M(n,c) \). Moreover, if \( U_p = S \) then \( \Gamma = \{1\} \) and \( \phi \) is, up to a Sasakian transformation of \( M(N,c) \), given by

\[
\phi(z,t) = (z, 0, t + c)
\]

**Proof.** For every point \( p \in S \) we have an immersion of some \( U_p \). After possibly shrinking the open set \( U_p \) we obtain an open set where the Reeb foliation is given by a fibration over a Kähler base. Then we proceed exactly as in the proof of Theorem 1 and we obtain that \( U_p \) (and then \( S \)) has constant \( \phi \)-sectional curvature at every point. Then \( S \) is Sasaki-equivalent to \( M(n,c)/\Gamma \) where \( \Gamma \) is some discrete subgroup of the Sasaki-isometry group of \( M(n,c) \).

If \( U_p = S \) we cannot directly conclude as in Theorem 1 because a priori we don’t know if \( M(n,c)/\Gamma \) is regular. On the other end we are assuming the existence of an immersion of \( U_p = S \) into the regular Sasakian space form \( M(N,c) \) and then \( S \) is regular by Proposition 5. We can now apply Theorem 1 to conclude. \( \square \)

**References**

[1] Besse, Arthur L., *Einstein manifolds*, Reprint of the 1987 edition. Classics in Mathematics. Springer-Verlag, Berlin, 2008

[2] D. E. Blair, *Riemannian geometry of contact and symplectic manifolds*, Birkhäuser, 2010

[3] W. M. Boothby, H. C. Wang, *On contact manifolds*, Ann. Math. **68** (1958), 721–734

[4] C. P. Boyer, K. Galicki, *Sasakian geometry*, Oxford Mathematical Monographs, Oxford University Press, Oxford, 2008

[5] C. Boyer, K. Galicki, P. Matzeu, *\( \eta \)-Einstein Sasakian geometry*, Comm. Math. Phys. **262** (2006), 177–208

[6] E. Calabi, *Isometric Imbedding of Complex Manifolds*, Ann. Math. **58** (1953), 1–23

[7] B. Cappelletti Montano, A. Loi, *Einstein and \( \eta \)-Einstein Sasakian submanifolds in spheres*, Ann. Mat. Pura Appl. **(4) 198** (2019), no. 6, 2195–2205.

[8] D. M. DeTurck, J. L. Kazdan, *Some regularity theorems in Riemannian geometry*, Ann. Sci. École Norm. Sup. **(4) 14** (1981), 249–260.

[9] M. Falcitelli, S. Ianu¸s, A.M. Pastore, *Riemannian submersions and related topics*, World Scientific Publishing Co., Inc., River Edge, NJ, 2004

[10] M. Harada, *Sasakian space forms immersed in Sasakian space forms*, Bull. Tokyo Gakugei Univ. Ser. IV **24** (1972), 7–11

[11] M. Harada, *On Sasakian submanifolds*, Tôhoku Math. J. **25** (1973), 103–109

[12] K. Kenmotsu, *Invariant submanifolds in a Sasakian manifold*, Tôhoku Math. J. **21** (1969), 495–500

[13] M. Kon, *Invariant submanifolds of normal contact metric manifolds*, Kodai Math. Sem. Rep. **25** (1973), 330–336
[14] M. Kon, *Invariant submanifolds in Sasakian manifolds*, Math. Ann. **219** (1976), 277–290

[15] A. Loi, M. Zedda, *Kähler Immersions of Kähler Manifolds into Complex Space Forms*, Lecture Notes of the Unione Matematica Italiana **23**, Springer, 2018

[16] I. Moerdijk, J. Mrčun, *Introduction to foliations and Lie groupoids*, Cambridge University Press, 2003

[17] B. O’Neill, *The fundamental equations of a submersion*, Michigan Math. J. **13** (1966), 459–469

[18] R. S. Palais, *A global formulation of the Lie theory of transformation groups*, Mem. Amer. Math. Soc. No. 22 (1957)

[19] B. L. Reinhart, *Foliated Manifolds with Bundle-Like Metrics*, Ann. Math. **69** (1959), 119–132

[20] S. Tanno, *Sasakian manifolds with constant φ-holomorphic sectional curvature*, Tôhoku Math. J. **21** (1969), 501–507

[21] M. Umehara, *Einstein-Kähler submanifolds of complex linear or hyperbolic space*, Tôhoku Math. J. **39** (1987), 385–389

Gianluca Bande, Dipartimento di Matematica e Informatica, Università di Cagliari, Italy.

E-mail address: gbande@unica.it

Beniamino Cappelletti–Montano, Dipartimento di Matematica e Informatica, Università di Cagliari, Italy.

E-mail address: b.cappellettimontano@unica.it

Andrea Loi, Dipartimento di Matematica e Informatica, Università di Cagliari, Italy.

E-mail address: loi@unica.it