The Hurwitz Form of a Projective Variety

Bernd Sturmfels

Abstract

The Hurwitz form of a variety is the discriminant that characterizes linear spaces of complementary dimension which intersect the variety in fewer than degree many points. We study computational aspects of the Hurwitz form, relate this to the dual variety and Chow form, and show why reduced degenerations are special on the Hurwitz polytope.

1 Introduction

Many problems in applied algebraic geometry can be expressed as follows. We are given a fixed irreducible projective variety $X \subset \mathbb{P}^n$, defined over the field $\mathbb{Q}$ of rational numbers. Suppose that $X$ has dimension $d$ and degree $p$. We consider various linear subspaces $L \subset \mathbb{P}^n$ of complementary dimension $n - d$, usually defined over the real numbers in floating point representation. The goal is to compute the intersection $L \cap X$ as accurately as possible. If the subspace $L$ is generic then the set $L \cap X$ consists of $p$ distinct points with complex coordinates in $\mathbb{P}^n$. How many of the $p$ points are real depends on the specific choice of $L$.

In this paper we study the discriminant associated with this family of polynomial systems. The precise definition is as follows. Let $\text{Gr}(d, \mathbb{P}^n)$ denote the Grassmannian of codimension $d$ subspaces in $\mathbb{P}^n$, and let $\mathcal{H}_X \subset \text{Gr}(d, \mathbb{P}^n)$ be the subvariety consisting of all subspaces $L$ such that $L \cap X$ does not consist of $p$ reduced points. The sectional genus of $X$, denoted $g = g(X)$, is the arithmetic genus of the curve $X \cap L'$ where $L' \subset \mathbb{P}^n$ is a general subspace of codimension $d - 1$. If $X$ is regular in codimension 1 then the curve $X \cap L'$ is smooth (by Bertini's Theorem) and $g$ is its geometric genus. The following result describes our object.

Theorem 1.1. Let $X$ be an irreducible subvariety of $\mathbb{P}^n$ having degree $p \geq 2$ and sectional genus $g$. Then $\mathcal{H}_X$ is an irreducible hypersurface in the Grassmannian $\text{Gr}(d, \mathbb{P}^n)$, defined by an irreducible element $H_{u_X}$ in the coordinate ring of $\text{Gr}(d, \mathbb{P}^n)$. If the singular locus of $X$ has codimension at least 2 then the degree of $H_{u_X}$ in Plücker coordinates equals $2p + 2g - 2$.

The polynomial $H_{u_X}$ defined here is the Hurwitz form of $X$. The name was chosen as a reference to the Riemann-Hurwitz formula, which says that a curve of degree $p$ and genus $g$ has $2p + 2g - 2$ ramification points when mapped onto $\mathbb{P}^1$. We say that $\text{Hdeg}(X) := \deg(H_{u_X}) = 2p + 2g - 2$ is the Hurwitz degree of $X$. When $X$ is defined over $\mathbb{Q}$ then so is $\mathcal{H}_X$. The Hurwitz form $H_{u_X}$ is unique up to sign, when written in Stiefel coordinates on $\text{Gr}(d, \mathbb{P}^n)$, if we require it to have relatively prime integer coefficients. When written in Plücker coordinates, $\pm H_{u_X}$ is unique only modulo the ideal of quadratic Plücker relations.
The Hurwitz form $H_X$ belongs to the family of higher associated hypersurfaces described by Gel’fand, Kapranov and Zelevinsky in [10 Section 3.2.E]. These hypersurfaces interpolate between the Chow form $Ch_X$ and the $X$-discriminant. The latter is the equation of the dual variety $X^*$. In that setting, the Hurwitz form $H_X$ is only one step away from the Chow form $Ch_X$. An important result in [10 Section 4.3.B] states that these higher associated hypersurfaces are precisely the coisotropic hypersurfaces in $Gr(d, P^n)$, so their defining polynomials are governed by the Cayley-Green-Morrison constraints for integrable distributions.

This article is organized as follows. In Section 2 we discuss examples, basic facts, and we derive Theorem 1.1. Section 3 concerns the Hurwitz polytope whose vertices correspond to the initial Plücker monomials of the Hurwitz form. We compare this to the Chow polytope of [12]. In Section 4 we define the Hurwitz form of a reduced cycle, and we show that this is compatible with flat families. As an application we resolve problems (4) and (5) in [15 §7].

2 Basics

We begin with examples that illustrate Hurwitz forms in computational algebraic geometry.

Example 2.1 (Curves). If $X$ is a curve in $P^n$, so $d = 1$, then $H_X = X^*$ is the hypersurface dual to $X$, and $H_U$ is the $X$-discriminant. For instance, if $X$ is the rational normal curve in $P^n$ then $H_U$ is the discriminant of a polynomial of degree $n$ in one variable. For a curve $X$ in the plane $(n = 2)$, the Hurwitz form is the polynomial that defines the dual curve $X^*$, so the Hurwitz degree $Hdeg(X)$ is the class of $X$, which is $p(p - 1)$ if $X$ is nonsingular.

Example 2.2 (Hypersurfaces). Suppose that $X$ is a hypersurface in $P^n$, so $d = n - 1$, with defining polynomial $f(x_0, x_1, \ldots, x_n)$. The Grassmannian $Gr(n - 1, P^n)$ is a manifold of dimension $2n - 2$ in the projective space $P^{(n+1)\choose 2}$ with dual Plücker coordinates $q_{01}, q_{02}, \ldots, q_{n-1,n}$. We can compute $H_U$ by first computing the discriminant of the univariate polynomial function $t \mapsto f(u_0 + tv_0, u_1 + tv_1, \ldots, u_n + tv_n)$, then removing extraneous factors, and finally expressing the result in terms of $2 \times 2$-minors via $q_{ij} = u_iv_j - u_jv_i$.

We can make this explicit when $p = 2$. Let $M$ be a symmetric $(n + 1) \times (n + 1)$-matrix of rank $\geq 2$ and $X$ the corresponding quadric hypersurface in $P^n$. We write $\wedge_2 M$ for the second exterior power of $M$, and $Q = (q_{01}, q_{02}, \ldots, q_{n-1,n})$ for the row vector of dual Plücker coordinates. With this notation, the Hurwitz form is the following quadratic form in the $q_{ij}$:

$$H_U = Q \cdot (\wedge_2 M) \cdot Q^t.$$  

(1)

For a concrete example let $n = 3$ and consider the quadric surface $X = V(x_0x_3 - x_1x_2)$. Then $Hdeg(X) = 2$ and the Hurwitz form equals $H_U = q_{03}^2 + q_{12}^2 + 2q_{03}q_{12} - 4q_{02}q_{13}$. When expressed in terms of Stiefel coordinates $u_i, v_j$, this is precisely the hyperdeterminant of format $2 \times 2 \times 2$. This is explained by Proposition 2.4 with $X = P^1 \times P^1$ and $Y = X \times P^1 \subset P^7$. ♦

Example 2.3 (Toric Varieties). Consider a toric variety $X_A \subset P^n$, defined by a rank $d$ matrix $A \in \mathbb{Z}^{d \times (n+1)}$ with $(1, 1, \ldots, 1)$ in its row space. Then $H_{X_A}$ is the mixed discriminant $[6]$ of $d$ Laurent polynomials in $d$ variables with the same support $A$. If $A$ is a unit square
then \( H_{X_A} \) is the hyperdeterminant seen above and in \([6, \text{Example 2.3}]\). If \( X_A \) is the \( k \)th Veronese embedding of \( \mathbb{P}^2 \), a surface of degree \( p = k^2 \) in \( \mathbb{P}^{(k^2+2)−1} \), then \( H_{X_A} \) is the classical \textit{tact invariant} which vanishes whenever two plane curves of degree \( k \) are tangent. Its degree is \( \text{Hdeg}(X_A) = 3k^2 − 3k \). An explicit formula for \( k = 2 \) will be displayed in Example 2.7.

Our goal is to develop tools for writing the Hurwitz form \( H_X \) explicitly as a polynomial. There are four different coordinate systems for doing so, depending on how the subspace \( L \in \text{Gr}(d, \mathbb{P}^n) \) is expressed. If \( L \) is the kernel of a \( d \times (n+1) \)-matrix then the entries of that matrix are the \textit{primal Stiefel coordinates} and its maximal minors are the \textit{primal Plücker coordinates}, denoted \( p_{i_1i_2⋯i_d} \). If \( L \) is the row space of an \( (n+1−d) \times (n+1) \)-matrix then the entries of that matrix are the \textit{dual Stiefel coordinates} and its maximal minors are the \textit{dual Plücker coordinates}, denoted \( q_{j_0j_1⋯j_{n−d}} \), as in Example 2.2. In practice, one uses primal coordinates when \( d = \dim(X) \) is small, and one uses dual coordinates when \( n−d = \text{codim}(X) \) is small. The same conventions are customary for writing \textit{Chow forms} \([7, 12, 16]\).

The hypersurface defined by the Chow form of \( X \) is called the \textit{associated variety} in \([10, 18]\). In these references, the Chow form is constructed as the dual of a Segre product. This is the \textit{Cayley trick} of elimination theory. We now do the same for the Hurwitz form of \( X \).

\textbf{Proposition 2.4.} Consider \( Y = X \times \mathbb{P}^{d−1} \) in its Segre embedding in \( \mathbb{P}^{(d+1)^2−1} \). The dual variety \( Y^* \) equals \( H_X \), with defining polynomial \( H_X \) written in primal Stiefel coordinates.

\textbf{Proof.} This is analogous to \([10, \text{Thm. 3.2.17}]\) which concerns the case of the Chow form.

We next prove the statements about dimension and degree of \( H_X \) given in the introduction. The idea is to reduce to the special case of curves, as discussed in Example 2.1.

\textbf{Proof of Theorem 2.1.} Applying Corollary 5.9 in \([10, \text{Section 1.5.D}]\) to the representation in Proposition 2.4, we find that \( H_X \) is a hypersurface if and only if \( \text{codim}(X^*) \leq d \), or \( \dim(X^*) \geq n−d \). In light of Corollary 1.2 in \([10, \text{Section 1.1.A}]\), this happens if and only if \( X \) is not a linear space. So, since we assumed \( p \geq 2 \), this means that \( H_X \) is a hypersurface.

Write \( L = L' \cap H \) where \( H \) is a varying hyperplane, and we think of \( L' \) as fixed. Then \( L \) is in \( H_X \) if and only if the zero-dimensional scheme \( X \cap L = X \cap (L' \cap H) = (X \cap L') \cap H \) is not reduced. This happens if and only if \( H \) is tangent to the curve \( X \cap L' \) if and only if \( H \) is a point in the projective variety dual to \( X \cap L' \). This curve is smooth and irreducible, by Bertini’s Theorem, and it has degree \( p \) and genus \( g \). A classical result states this dual to \( X \cap L' \) is a hypersurface of degree \( 2p+2g−2 \); see, for instance, the paragraph after Theorem 2.14 in \([10, \text{Section 2.2.B}]\). Hence \( H_X \) is a hypersurface of that same degree in \( \text{Gr}(d, \mathbb{P}^n) \).

One motivation for studying the Hurwitz form \( H_X \) comes from the analysis of numerical algorithms for computing \( L \cap X \). An appropriate tubular neighborhood around \( H_X \) in \( \text{Gr}(d, \mathbb{P}^n) \) is the locus where the homotopy methods of numerical algebraic geometry run into trouble. This is quantified by the \textit{condition number} of the algebraic function \( L \mapsto L \cap X \). The quantity \( \text{Hdeg}(X) = 2p+2g−2 \) is crucial for bounding that condition number (cf. \([5]\)).
Example 2.5. This article was inspired by a specific application to multiview geometry in computer vision, studied in \( \square \). The variety of essential matrices is a subvariety \( X \) in the projective space \( \mathbb{P}^8 \) of \( 3 \times 3 \)-matrices. Its real points are the rank 2 matrices whose two nonzero singular values coincide. We have \( d = 5, p = 10, g = 6 \). This implies that \( \text{Hdeg}(X) = 30 \), by Theorem \( \square \) so \( H_X \) is a hypersurface in the Grassmannian \( \text{Gr}(5, \mathbb{P}^8) \) whose defining polynomial \( H_X \) has degree 30 in the \( \binom{9}{3} = 84 \) Plücker coordinates.

\[
\text{Hdeg}(X) = 2p + 2g - 2 = 2 \cdot |h^{(d-1)}|(-1)|.
\]

Indeed, if \( d = 1 \) then this comes from the familiar formula \( h(m) = pm + (1 - g) \) for the Hilbert polynomial of a projective curve. For \( d \geq 2 \), the Cohen-Macaulay hypothesis ensures that the numerator of the Hilbert series remains the same under hyperplane sections, and the Hilbert polynomial is transformed under such sections by taking the derivative.

In \textsc{Macaulay2} [\ref{1}], we can compute the quantity \( |h^{(d-1)}|(-1) = \text{Hdeg}(X)/2 \) from the ideal \( \mathcal{I} \) of \( X \) in two possible ways: take the coefficient of \( P_{d-1} \) in \texttt{hilbertPolynomial}(\mathcal{I}), or add the last two entries in \texttt{genera}(\mathcal{I}). For instance, for the variety in Example 2.5, the former command gives \( 6 * P_3 - 15 * P_4 + 10 * P_5 \) and the latter command gives \( \{0, 0, 0, 0, 6, 9\} \).

If the parameters \( d, n, p, g \) of the given variety are small enough, then we can use computer algebra to determine the Hurwitz form \( H_X \), and to write it as an explicit polynomial in the Plücker coordinates on \( \text{Gr}(d, \mathbb{P}^n) \). The following example will serve as an illustration:

Example 2.7. Let \( X \) be the Veronese surface in \( \mathbb{P}^5 \), defined by the parametrization \( (x : y : z) \mapsto (x^2 : xy : xz : y^2 : yz : z^2) \). Here \( n = 5, d = 2, p = 4, g = 0 \). Following Example 2.3 the Hurwitz form of the classical tach invariant for \( k = 2 \). We have the explicit formula

\[
H_X = 4p_0p_1^2p_2^2p_3 + 4p_1p_2^2p_3^3 + 4p_2p_3^2p_4^3 - 4p_0p_1p_2p_3p_4^4 + 4p_0^2p_1p_3p_4^2 + 4p_0^2p_2p_3p_4^2 - 16p_0p_1p_2p_3^5 - 16p_0p_2p_3p_4^5 - 16p_3p_4p_5p_6 - 256p_0p_1p_3p_4^3.
\]
The first 14 monomials are the initial terms with respect to the torus action on $\text{Gr}(2, \mathbb{P}^5)$. In addition to these 14 weight components, there are 45 other weight components from the non-initial terms. Substituting $p_{ij} = a_ib_j - a_jb_i$ in $H_{\text{deg}}(\mathbb{P}^5)$ gives a sum of 3210 monomials of bidegree $(6, 6)$. This is the taut invariant (cf. Example 2.3) of two ternary quadrics.

Clearly, as the parameters $d, p, n, g$ increase, it soon becomes prohibitive to compute such an explicit expansion for $H_{\text{deg}}(\mathbb{P}^5)$. We can still hope to compute the initial terms from various initial monomial ideals of $X$. This will be studied in the next section. Another possibility is to use numerical algebraic geometry to compute and represent the Hurwitz form $H_{\text{deg}}(\mathbb{P}^5)$. We illustrate the capabilities of Bertini [3] with a non-trivial example from computer vision.

Example 2.8. The calibrated trifocal variety $X$ has dimension $d = 11$ and it lives in $\mathbb{P}^{26}$. This extends the variety of essential matrices in Example 2.5 from two cameras to three cameras. It is obtained from the parametrization of the trifocal variety in [2] by specializing each of the three camera matrices to have a rotation matrix in its left $3 \times 3$-block. Computations in Bertini [3] carried out by Joe Kileel and Jon Hauenstein revealed that $p = 4912$ and $g = 13569$ respectively. The details of this computation, and the equations known to vanish on $X$, will be described elsewhere. Hence for this variety, we have $\text{Hdeg}(X) = 36960$.

3 The Hurwitz polytope

An important combinatorial invariant of hypersurface in a projective space $\mathbb{P}^n$ is its Newton polytope. This generalizes to hypersurfaces in a Grassmannian $\text{Gr}(d, \mathbb{P}^n)$ if we take the weight polytope with respect to the action of the $(n + 1)$-dimensional torus on $\text{Gr}(d, \mathbb{P}^n)$. This torus action corresponds to the $\mathbb{Z}^{n+1}$-grading on the primal Plücker coordinates by

$$\text{deg}(p_{i_1i_2\ldots i_d}) = e_{i_1} + e_{i_2} + \cdots + e_{i_d}.$$  

The weight polytope of the Chow form of a variety $X \subset \mathbb{P}^n$ was studied in [12]. It is known as the Chow polytope. In this section we study the analogous polytope for the Hurwitz form.

We define the Hurwitz polytope, denoted $H_{\text{Po}}$, of a projective variety $X \subset \mathbb{P}^n$ to be the weight polytope of the Hurwitz form $H_{\text{deg}}$. This definition is to be understood as follows. The coordinate ring of $\text{Gr}(d, \mathbb{P}^n)$ is $\mathbb{Z}^{n+1}$-graded. The Hurwitz form $H_{\text{deg}}$ is an element of that ring, so it decomposes uniquely into $\mathbb{Z}^{n+1}$-graded components. The Hurwitz polytope $H_{\text{Po}}$ is the convex hull in $\mathbb{R}^{n+1}$ of all degrees that appear in this decomposition.

Example 3.1. For the Segre variety $X = \mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^3$, the Hurwitz form is invariant under passing from dual Plücker coordinates (as in Example 2.2) to primal Plücker coordinates. The weights occurring in $H_{\text{deg}}$ are $p_{03}^2 + p_{12}^2 + (2p_{03}p_{12} - 4p_{02}p_{13})$ are the three
points \((2,0,0,2), (0,2,2,0)\) and \((1,1,1,1)\) in \(\mathbb{R}^4\). The Hurwitz polytope \(\text{HuPo}_X\) is their convex hull, which is a line segment. We note that the Chow form of our surface equals \(\text{Ch}_X = p_0p_3 - p_1p_2\), in dual Plücker coordinates. This means that \(\text{HuPo}_X\) is twice the segment \(\text{ChPo}_X = \text{conv}\{(1,0,0,1), (0,1,1,0)\}\), which is the Chow polytope of \(X\). 

Integer vectors \(w \in \mathbb{Z}^{n+1}\) represent one-parameter subgroups of the torus action on \(\mathbb{P}^n\). Their action on subvarieties \(X\) is compatible with the construction of Hurwitz forms:

\[
\text{Hu}_{\epsilon w X}(p) = \text{Hu}_X(\epsilon^{-w} p).
\]  

Here \(\epsilon\) is a parameter and \(\epsilon^w = (\epsilon^{w_0}, \epsilon^{w_1}, \ldots, \epsilon^{w_n})\), and the action is the same as that on Chow forms seen in [16, §2.3]. For \(\epsilon \to 0\), the polynomial in (2) is the initial form \(\text{in}_w(\text{Hu}_X)\). For generic \(w\), this initial form is a monomial in the Pücker coordinates. The following example illustrates these initial monomials and how they determine the Hurwitz polytope.

**Example 3.2.** Let \(X\) be the Veronese surface in \(\mathbb{P}^5\). Its Hurwitz form \(\text{Hu}_X\) was displayed explicitly in Example [2,7]. The Hurwitz polytope \(\text{HuPo}_X\) is 3-dimensional and it has 14 vertices, namely the weights of the first 14 Plücker monomials, which were shown in larger font. The weights of these monomials, in the given order, are the following 14 points in \(\mathbb{R}^6\):

\[
(160023), (062103), (302160), (320061), (026301), (106320), (044040),
(222060), (062022), (026220), (160104), (106401), (400161), (400404).
\]

The Chow form and Chow polytope of the Veronese surface \(X\) were displayed explicitly in [16, Section 2, page 270]. This should be compared to the Hurwitz form and Hurwitz polytope. The initial monomials of \(\text{Ch}_X\) correspond to the 14 maximal toric degenerations of \(X\), one for each of the 14 triangulations of the triangle \(2\Delta = \text{conv}\{(200), (020), (002)\}\). The Chow polytope \(\text{ChPo}_X\) is normally equivalent to the Hurwitz polytope \(\text{HuPo}_X\). This means that both polytopes share the same normal fan in \(\mathbb{R}^6\). In particular, they have the same combinatorial type. That type is the 3-dimensional associahedron. The correspondence between the initial monomials of \(\text{Ch}_X\) and those of \(\text{Hu}_X\) is as follows, up to symmetry:

| \(\text{Ch}_X\) | \(p_{012}p_{124}p_{134}p_{345}\) | \(p_{014}p_{024}p_{134}p_{245}\) | \(p_{015}p_{134}p_{145}\) | \(p_{015}p_{035}\) | \(p_{035}\) |
|---|---|---|---|---|---|
| \(\text{Hu}_X\) | \(p_{12}^2p_{14}^2p_{24}^2\) | \(p_{04}^2p_{14}^2p_{24}^2\) | \(p_{01}^2p_{14}^2p_{15}^2\) | \(p_{01}^2p_{13}^2p_{15}^2\) | \(p_{03}^2p_{05}^2p_{35}^2\) |

The first two degenerations are reduced unions of four coordinate planes in \(\mathbb{P}^5\), corresponding to the unimodular triangulations of \(2\Delta\). The last is the degeneration is a plane of multiplicity four, corresponding to the trivial triangulation of \(2\Delta\) into a single large triangle \(\{0,3,5\}\).

**Example 3.2** raises the question whether the vertices of the Hurwitz polytope and the Chow polytope are always in bijection. The answer is “no” as the following example shows.

**Example 3.3** (Plane conics). Let \(X\) be a plane conic in \(\mathbb{P}^2\) defined by the quadratic form

\[
(x_0 \ x_1 \ x_2) \begin{pmatrix} m_{00} & m_{01} & m_{02} \\ m_{01} & m_{11} & m_{12} \\ m_{02} & m_{12} & m_{22} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix},
\]  

(3)
where the $m_{ij}$ are scalars in $\mathbb{Q}$. This equation agrees with the Chow form of $X$, so generally the Chow polytope is the triangle $\text{Ch}_X = 2\Delta$. According to Example 2.1, the hypersurface $H_X$ is the dual conic $X^*$. Hence the Hurwitz form is the quadratic form of the adjoint

$$
H_uX = \left( p_0 \ p_1 \ p_2 \right) \begin{pmatrix}
m_{11}m_{22} - m_{12}^2 & m_{12}m_{20} - m_{10}m_{22} & m_{01}m_{12} - m_{11}m_{02} \\
m_{12}m_{02} - m_{11}m_{22} & m_{00}m_{22} - m_{02}^2 & m_{01}m_{02} - m_{00}m_{12} \\
m_{01}m_{12} - m_{11}m_{02} & m_{01}m_{02} - m_{00}m_{12} & m_{00}m_{11} - m_{01}^2
\end{pmatrix} \begin{pmatrix}
p_0 \\
p_1 \\
p_2
\end{pmatrix}.
$$

(4)

This agrees with formula (1) if we set $p_0 = q_{12} \neq q_{02}$ and $p_2 = q_{01}$. The Hurwitz polytope $H_{uP_0}$ is a subpolytope of the triangle $2\Delta = \text{conv}\{(200), (020), (002)\}$. There are several combinatorial possibilities, depending on which matrix entries in (4) are zero. For instance, if $m_{11}m_{22} = m_{12}^2$ but the $m_{ij}$ are otherwise generic then $H_{uP_0}$ is a quadrilateral. Similarly if $m_{11} = 0$ then $\text{Ch}_{P_0}$ is quadrilateral and $H_{uP_0}$ is a triangle. Hence, there is generally no map from the set of vertices of one polytope to the vertices of the other. ♦

A geometric explanation for this example is given by Katz’ analysis in [13] of the duality of plane curve under degenerations. Suppose that $X_\epsilon$ is a plane curve defined by a homogeneous polynomial of the form $f^\epsilon + \epsilon g$ where $f$ and $g$ are a general homogeneous polynomials of degree $r$ and $p = qr$ respectively. Here $\epsilon$ is a parameter. For $\epsilon \neq 0$, the curve $X_\epsilon$ is smooth, and the Hurwitz form $H_{X_\epsilon}$ defines the dual curve, of degree $p(p - 1) = q^2r^2 - qr$. Now consider the limit $\epsilon \to 0$. The limit curve $X_0$ is $V(f)$ with multiplicity $q$, with no trace of $V(g)$, but the limit of the Hurwitz form remembers the points in the intersection $V(f, g)$.

**Proposition 3.4.** [13] Proposition 1.2] The constant term of $H_{X_\epsilon}$ with respect to $\epsilon$ equals

$$
H_{X_\epsilon}\big|_{\epsilon = 0} = \left( H_{V(f)}\right)^q \cdot \prod_{u \in V(f, g)} (\text{Ch}_u)^{q - 1}.
$$

(5)

Note that the variety $V(f, g)$ consists of $rp = qr^2$ points $u = (u_0 : u_1 : u_2)$. For each of these, the linear form $\text{Ch}_u = u_0p_0 + u_1p_1 + u_2p_2$ appears as a factor. The Hurwitz form $H_{V(f)}$ of the curve $V(f)$ has degree $r(r - 1)$. Hence the right hand side in (5) has degree

$$
r(r - 1)q + (qr^2)(q - 1) = p(p - 1).
$$

Proposition 3.4 is indicative of what happens to the limit of the Hurwitz form $H_{X_\epsilon}$ when a family of irreducible varieties $X_\epsilon$ degenerates to a non-reduced cycle or scheme $X_0$. The limit $H_{X_\epsilon}\big|_{\epsilon = 0}$ remembers information about the family that cannot be recovered from $X_0$.

### 4 Reduced degenerations

In this section we consider families $X_\epsilon$ whose limit object $X_0$ is a reduced cycle, and we show that the indeterminacy seen in Proposition 3.4 no longer happens. Instead, we will have

$$
H_{X_\epsilon}\big|_{\epsilon = 0} = H_{X_0}.
$$

(6)
To make sense of this identity, we now define the Hurwitz form of a reduced cycle $Y$ in $\mathbb{P}^n$. Let $Y = \sum_{i=1}^t Y_i$ where $Y_1, \ldots, Y_t$ are distinct irreducible $d$-dimensional subvarieties in $\mathbb{P}^n$. Let $Z_1, \ldots, Z_m$ be the distinct irreducible varieties of dimension $d - 1$ that arise as components in the pairwise intersections $Y_i \cap Y_j$. We write $\nu_j$ for the multiplicity of the one-dimensional local ring $\mathcal{O}_{Y,Z_j}$ at its maximal ideal. If all pairwise intersections $Y_i \cap Y_j$ are transverse then $\nu_j$ simply counts the number of components $Y_i$ of $Y$ that contain $Z_j$.

We now define the Hurwitz form of the reduced cycle $Y$ as follows:

$$Hu_Y = \prod_{i=1}^t Hu_{Y_i} \cdot \prod_{j=1}^m (Ch_{Z_j})^{2\nu_j - 2}.$$  \hfill (7)

Here $Hu_{Y_i}$ is the Hurwitz form of a $d$-dimensional variety, while $Ch_{Z_j}$ is the Chow form of a $(d - 1)$-dimensional variety, so they both define hypersurfaces in the same Grassmannian $Gr(d, \mathbb{P}^n)$. Our main result in this section states that this is the correct definition for limits.

**Theorem 4.1.** Let $(X_\epsilon)$ be a flat family of subschemes in $\mathbb{P}^n$, where the general fiber (for $\epsilon \neq 0$) is irreducible and the special fiber (for $\epsilon = 0$) is reduced. Then the Hurwitz form of $X_\epsilon$ satisfies the identity (6) with the Hurwitz form $Hu_{X_0}$ of the special fiber defined as in (7).

**Proof.** By intersecting with a general plane $L'$ of codimension $d - 1$ in $\mathbb{P}^n$, as in the proof of Theorem 4.1 we next reduce to the statement to the $d = 1$ case, when $(X_\epsilon)$ is a family of curves. In that case, the Hurwitz form $Hu_{X_\epsilon}$ is the polynomial that defines the hypersurface $(X_\epsilon)^*$ dual to the curve $X_\epsilon$. We can reduce the planar case $n = 2$ by applying a random projection to $X_\epsilon$. By [10] Proposition I.4.1, this is equivalent to intersecting $(X_\epsilon)^*$ with a general plane $\mathbb{P}^2$ inside $\mathbb{P}^n$. For plane curves, our result can now be derived from the General Class Formula in [9] Section A.5.4. Indeed, the $Z_i$ correspond to singular points on the curve, and the $\nu_i$ are the degrees of the singularities in the sense of [9] Section A.5.2. \hfill \Box

An important special case of Theorem 4.1 arises when the limit cycle $Y = X_0$ is an arrangement of linear subspaces of dimension $d$ in $\mathbb{P}^n$. Here, the first factor on the right hand side in (7) disappears, and the Hurwitz form of the subspace arrangement $Y$ equals

$$Hu_Y = \prod_Z (Ch_Z)^{2\nu(Z) - 2}.$$  \hfill (8)

In this formula, $Z$ runs over all strata of dimension $d - 1$ in the subspace arrangement $Y$, and $\nu(Z)$ is the number of $d$-planes $Y_i$ that contain $Z$.

Of particular interest is the situation when $Y$ consists of coordinate subspaces in $\mathbb{P}^n$. Here, $Y$ can be identified with a simplicial complex of dimension $d$ on the vertex set $\{0, 1, \ldots, n\}$. The product (8) is over all $(d - 1)$-simplices $Z$. These correspond to coordinate planes

$$\text{span}(e_{i_1}, e_{i_2}, \ldots, e_{i_d}) = V(x_{j_0}, x_{j_1}, \ldots, x_{j_{n-d}}).$$

These are indexed by set partitions $\{0, 1, \ldots, n\} = \{i_1, i_2, \ldots, i_d\} \cup \{j_0, j_1, \ldots, j_{n-d}\}$, and their Chow forms are just Plücker variables

$$pz := p_{i_1j_2\ldots j_d} = q_{j_0j_1\ldots j_{n-d}} = Ch_Z.$$  \hfill (9)
This situation arises whenever the ideal of a projective variety has a Gröbner bases with a squarefree initial ideal. This is now the Stanley-Reisner ideal of the simplicial complex \( Y \).

**Corollary 4.2.** Let \( I \) be a homogeneous prime ideal in \( K[x_0, x_1, \ldots, x_n] \) and suppose \( M = \text{in}_w(I) \) is a squarefree initial monomial ideal. The initial form \( \text{in}_w(Hu(V(I))) \) of the Hurwitz form equals the monomial \( \prod_Z p_Z^{2 \nu(Z) - 2} \) where \( Z \) runs over codimension 1 simplices in \( V(M) \).

**Proof.** Using (8) and the identification (9), this is a direct consequence of Theorem 4.1. \( \square \)

This corollary allows us to compute the Hurwitz polytope and initial terms of the Hurwitz form for any homogeneous ideal whose initial monomial ideal are all squarefree. One situation where this holds is the ideal generated by the maximal minors of a rectangular matrix of unknowns, by [4, 17]. Here the initial ideals correspond to the coherent matching fields in [17], and each of these determines an initial Plücker monomial in the Hurwitz form for the determinantal variety of maximal minors. We illustrate this with an example.

**Example 4.3** (Ideals of Maximal Minors). The four \( 3 \times 3 \)-minors of the \( 3 \times 4 \)-matrix

\[
\begin{pmatrix}
  x_0 & x_1 & x_2 & x_3 \\
  x_4 & x_5 & x_6 & x_7 \\
  x_8 & x_9 & x_{10} & x_{11}
\end{pmatrix}
\]

form a universal Gröbner basis for the ideal \( I \) they generate [17, Theorem 7.2]. The variety \( X = V(I) \subset \mathbb{P}^{11} \) represents the \( 3 \times 4 \)-matrices of rank \( \leq 2 \). Here, \( n = 11 \), \( d = 9 \), \( p = 6 \), and \( g = 3 \), so the Hurwitz degree equals \( \text{Hdeg}(X) = 16 \). The determinantal ideal \( I \) has 96 initial monomial ideals, all squarefree, in two symmetry classes. Using Corollary 4.2, we read off the corresponding initial Plücker monomials of degree 16 in the Hurwitz form \( Hu_X \):

\[
\begin{align*}
\text{initial monomial ideal} & \quad \text{initial Plücker monomial in } Hu_X \\
72 \text{ ideals like } & \langle x_2 x_5 x_8, x_3 x_5 x_8, x_3 x_6 x_8, x_3 x_6 x_9 \rangle \quad \leadsto \quad q_2^2 q_3 q_5^2 q_8^2 q_9^2 q_{23} q_{35} q_{36} q_{38} q_{39} q_{46} q_{68} q_{69} \\
24 \text{ ideals like } & \langle x_0 x_6 x_9, x_3 x_4 x_9, x_3 x_6 x_9, x_3 x_6 x_9 \rangle \quad \leadsto \quad q_2^2 q_3^2 q_5^2 q_6^2 q_{36} q_{38} q_{39} q_{46} q_{68} q_{69}
\end{align*}
\]

So, while the Hurwitz form \( Hu_X \) is a huge polynomial that is hard to compute explicitly, it is easy to write down the 96 initial Plücker monomials of \( Hu_X \), one for each vertex of the Hurwitz polytope \( \text{HuPo}_X \). This polytope has dimension 6, it is simple, and has the same normal fan as the Chow polytope \( \text{ChPo}_X \). By [17, Theorem 2.8], this is the *transportation polytope* of nonnegative \( 3 \times 4 \)-matrices whose rows sum to 4 and whose columns sum to 3. \( \diamond \)

Here is another important class of varieties all of whose initial ideals are squarefree.

**Example 4.4** (Reciprocal Linear Spaces). Fix a \( (d + 1) \times (n + 1) \)-matrix \( A \) of rank \( d + 1 \) with entries in \( \mathbb{Q} \), and let \( X \) be the reciprocal of the row space of \( A \). Thus \( X \) is the Zariski closure in \( \mathbb{P}^n \) of the set of points \( (u_0 : u_1 : \cdots : u_n) \) with all coordinates nonzero and such that \( (u_0^{-1}, u_1^{-1}, \ldots, u_n^{-1}) \in \text{rowspace}(A) \). Proudfoot and Speyer [14] show that the circuits of \( A \) define a universal Gröbner basis for the ideal of \( X \), and each initial monomial ideal corresponds to the broken circuit complex of \( A \) under some ordering of \( \{0, 1, \ldots, n\} \). For
example, if $A$ is generic, in the sense that all maximal minors of $A$ are nonzero, then $p = \binom{n}{d}$, and the $p$ facets of the broken circuit complex are $\{0, i_1, \ldots, i_d\}$ where $1 \leq i_1 < \cdots < i_d \leq n$. The corresponding initial monomial of the Hurwitz form is

$$
\prod_{(p_0 i_2 \cdots i_d)} (n-d),
$$

where the product is over all $(d-1)$-element subsets $\{i_2, \ldots, i_d\}$ of $\{1, \ldots, n\}$. Therefore,

$$
H_{\text{deg}}(X) = 2 \binom{n}{d-1} (n-d).
$$

If $A$ is not generic then $H_{\text{deg}}(X)$ is given by a matroid invariant that appears in [8] and [15].

Our study of the Hurwitz form furnishes the answer to Question 4 in [15, §7, p. 706]:

**How is the entropic discriminant related to the Gauss curve of the central curve?** The central curve of $A$ is essentially the linear section $X \cap L'$, a smooth curve of degree $p$ and genus $g$, and its Gauss curve has degree $H_{\text{deg}}(X) = 2p + 2g - 2$, by the generalized Plücker formula. The Hurwitz form $H_X$ has that same degree, and hence so does the entropic discriminant:

**Corollary 4.5.** The entropic discriminant of the matrix $A$ equals $(b_1 b_2 \cdots b_d)^2 (n-1)(d-n)$ times the Hurwitz form in primal Stiefel coordinates of $X = \text{rowspace}(A)^{-1}$ evaluated at the matrix

\[
\begin{pmatrix}
 b_1 & -b_0 & 0 & 0 & \cdots & 0 \\
 0 & b_2 & -b_1 & 0 & \cdots & 0 \\
 0 & 0 & b_3 & -b_2 & \cdots & 0 \\
 \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
 0 & 0 & 0 & \cdots & b_d & -b_{d-1}
\end{pmatrix} \cdot A.
\]

**Proof.** The point $(b_0; b_1; \cdots; b_d) \in \mathbb{P}^d$ lies in the entropic discriminant of $A$ if and only if the row space of (10) intersects $X$. The Plücker coordinate vector of that row space can be written as a linear expression in $(b_0, b_1, \ldots, b_d)$. It is is obtained by removing the common factor $b_1 b_2 \cdots b_{d-1}$ from the maximal minors. This extraneous factor has exponent $-H_{\text{deg}}(X)$.

For example, let $d = 2, n = 4$, and consider the surface $X \subset \mathbb{P}^4$ defined by the matrix

$$
A = \begin{pmatrix}
 1 & 0 & 0 & 1 & 1 \\
 0 & 1 & 0 & 1 & 0 \\
 0 & 0 & 1 & 0 & 1
\end{pmatrix}.
$$

Explicitly, $X = V(x_0 x_2 - x_0 x_4 - x_2 x_4, x_0 x_1 - x_0 x_3 - x_1 x_3)$, and $H_{\text{deg}}(X) = 8$. The Hurwitz form (in primal Stiefel coordinates) is a homogeneous polynomial $H_X$ with 46958 terms of degree 16 in the ten entries of a $2 \times 5$-matrix of unknowns. If we substitute (10) into $H_X$ then we obtain $b_1^8$ times the sum of squares listed explicitly in [15, Example 1, page 679].

We close with the remark that Corollary 4.5 can be extended to also answer Question 5 in [15, §7, p.706], as that pertains to translating the variety $X$ via the torus action on $\mathbb{P}^n$. The “Varchenko discriminant” sought in that question is obtained by multiplying the matrix (10) on the right with the diagonal matrix diag$(c_0, c_1, \ldots, c_n)$ before substituting into $H_X$.

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Author's address:
Bernd Sturmfels, Department of Mathematics, University of California, Berkeley, CA 94720-3840, USA, bernd@berkeley.edu