Chern-Simons foam

Steven Willison and Jorge Zanelli
Centro de Estudios Científicos (CECS), Casilla 1469, Valdivia, Chile
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Abstract

Chern-Simons theory can be defined on a cell complex, such as a network of bubbles, which is not a (Hausdorff) manifold. Requiring gauge invariance determines the action, including interaction terms at the intersections, and imposes a relation between the coupling constants of the CS terms on adjacent cell walls. We also find simple conservation laws for charges at the intersections.

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e-mail: steve-at-cecs.cl, z-at-cecs.cl

1 Introduction

Soap bubbles have attracted the attention of physicists and mathematicians for a long time [1]. An ordinary bubble is a simple structure defined by a few elementary rules that can be derived from a minimising principle: it is a surface of minimal area given a certain constraint—a fixed enclosed volume or a fixed contour or boundary. These elementary structures are also the basis for building up more complex structures, like foams. These aggregates of bubbles also obey some simple rules but include some nontrivial discrete topological features which must be taken into account if one tries to derive their laws from an extremal principle.

1.1 Bubbles and foam

Perhaps the most celebrated results in the physics of bubbles and foam are Plateau’s rules for minimum area surfaces. A popular example illustrating these rules are soap films, which, due to surface tension, always tend to form a shape which minimises the surface area. The rules for a bubble network are: i) There are a finite number of pieces of film with smooth curvature, joining at surfaces of smooth intrinsic curvature; ii) The joining can occur in two ways: either three films meet along a smooth curve, or four edges (and six films) meet at a point; iii) When three films meet at a curve, the angle between them is 120°. When four edges meet at a point, the angle at each corner is always a fixed value, given approximately by 109°. Although it has always been assumed that i)-iii) are consequences of the minimum area principle, a rigorous proof of this appeared only fairly recently[2].

The Double bubble theorem is a classic mathematical problem: A minimal surface containing two adjoining cells of unequal volume is composed of three films, each of which is a section of a sphere (Fig. I). The relationship between the three radii of curvature is...
The double bubble is a familiar structure which can be seen in soap bubbles. The diagram shows three soap-films joining at a common edge: the two outer walls (blue and green) and one inner wall (red). The soap films are two-dimensional. The generalisation to a double bubble with three-dimensional films could be the base space for a Chern-Simons theory. Such a structure is not a Hausdorff manifold.

\[ \frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{r_3}, \]  \hspace{1cm} (1)

where films 1 and 2 are curved in the opposite direction to film 3. In other words the central film which forms the dividing wall is curved away from the smaller cell into the larger one. The part of the conjecture that remained unsolved until very recently is to prove that each of the three films must be a piece of a sphere. The curvature rule (1) then follows from this by application of the 120° rule.

**1.2 Gauge theory of foam?**

Many of the spaces which occur in nature, such as foams and cell structures, are not manifolds. Or to be precise, the useful approximation (of zero thickness intersecting films or cell walls) which is often employed, to simplify their study whilst capturing the essential features, means that one is not studying a manifold. So, at least at this level of approximation, non-manifold structures do exist. In this paper, we shall explore the more speculative possibility that the laws of physics themselves can be formulated on space-times which are not manifolds, but which are made up of several manifolds patched together. This is suggested by the special properties of Chern-Simons (CS) gauge theories, which make them amenable to a formulation on cell complexes. One can even conceive of a foam made up of three-dimensional pieces embedded into four dimensions or which has effective (Hausdorff) dimension four. In this way it may be possible to make contact between three-dimensional CS theory and four-dimensional physics. This is an attractive idea as it is widely accepted that at least three of the four interactions of nature are well described by gauge theories and CS provides one of the simplest and most elegant gauge theories that we know of. Unlike most field theories, CS theories demand very little of the spacetimes on which they can be constructed: the CS action doesn’t even need to have a metric defined on the spacetime manifold.

Here we wish to study Chern-Simons theories constructed on non-manifold structures. Roughly speaking, we want to investigate under what conditions a “Chern-Simons foam” could be consis-
tently defined. Instead of minimising the surface area, the foam will be classically described by the extrema of the topological CS lagrangian. We will aim to identify simple rules, the analogues of Plateau’s rules or equations such as (1).

The correspondence between CS theory and gravity in three dimensions suggests that a space-time foam could be described this way. It is perhaps over-optimistic to think that a foam made from pieces of three-dimensional gravity will somehow reproduce four-dimensional gravity, but it may be a hypothesis worth considering. In fact, many current approaches to quantum gravity involve the breakdown of the manifold structure at some small length-scales. In any case, even if it turns out that gravity cannot be described in this way, the study is well-motivated from a theoretical point of view: if CS theory is consistent on non-manifold spaces, it would be artificial and in some sense unnatural to restrict oneself to a theory defined only on a manifold.

1.3 The Chern-Simons three-form

Before going into details of what kind of non-manifold structures we can consider, let us review some basic features of the CS three-form for a gauge field \( A \):

\[ C(A) \equiv \text{Tr} \left( A \wedge dA + \frac{1}{3} A \wedge [A, A] \right). \]  

(2)

Here the gauge field is a one-form which takes values in the Lie algebra \( G \) of some gauge group \( G \). We use \( \text{Tr} \) to represent an invariant bilinear form of the Lie algebra, not necessarily the matrix trace. The Lie bracket of two exterior differential forms, in this case \([A, A]\), is defined as \([A, A] \equiv A^a \wedge A^b [J_a, J_b]\).

Some remarks are in order: i) \( C(A) \) is an exterior three-form that defines the action if integrated over a suitable 3D space. If we were to be conservative, we would insist on a paracompact, oriented topological manifold. Since the CS form contains first derivatives, we would further insist that space-time has a differentiable structure, i.e. it must be a differentiable manifold whose coordinate charts have overlap maps at least once-differentiable. If we were to define some pathological space, \( P \), which is not a \( C^1 \) manifold and just naively write down the action \( I_P = \int_P C \), one might worry that we are doing something ill-defined. We will need to relax these conservative requirements, but not in any arbitrary way: fortunately some aspects of the calculus can be generalised to more general types of spaces, through the notion of integration on chains. Stokes’ theorem is mathematically well-defined on chains and we shall see that this is sufficient for our purposes.

ii) Physically, the CS gauge theory is unusual in that its “kinetic term” is linear in the first derivatives. In the case of a non-abelian gauge group, there is also a cubic “self-interaction term”. As is well known, in all cases this theory has no local propagating degrees of freedom except at the boundary. In fact, the classical field equation is \( F = 0 \), and therefore all classical solutions are locally pure gauge. After imposing gauge fixing (e.g. \( A_0 = 0 \)) and applying the constraints, the reduced phase space is the space of flat connections on the two-dimensional spacelike slice and therefore finite dimensional.

iii) The CS action \( I_{CS} = (k/4\pi) \int_M C(A) \) defined over a manifold without boundary is invariant under infinitesimal gauge transformations \( A \to A + D\lambda \). However, over a manifold with boundary, the action is quasi-invariant: it transforms by a boundary term, \( I_{CS} \to I_{CS} - (k/4\pi) \int_{\partial M} \text{Tr}(\lambda dA) \). The Euler-Lagrange variation is

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1See e.g. [8] for a review including the case of higher dimensional CS theories, where the phase space is infinite dimensional.
\[ \delta_{\text{EL}} I_{\text{CS}} = \frac{k}{4\pi} \int_M 2\text{Tr}(\delta A \wedge F) + \frac{k}{4\pi} \int_{\partial M} \text{Tr}(\delta A \wedge A), \]

where the bulk piece gives the zero curvature field equation. Naively, the boundary piece can be dropped if \( A = 0 \) or if \( \delta A = 0 \), neither of which is a gauge invariant condition. There is no local term at the boundary that can be added to the action to restore the symmetry of the bulk theory. Boundary terms which are not gauge invariant are unsatisfactory since, generically, they make the Noether charges ill defined: under a gauge transformation that is non trivial at infinity, the charges can take unbounded values and may require ad-hoc regularizations as happens, for instance, in CS gravity for asymptotically AdS spaces. This issue was addressed in Refs. [9, 10], where the CS action was supplemented by introducing a second connection \( \bar{A} \) and a boundary term that turns the action functional into a transgression form [4, 11). This expression is gauge invariant (and not quasi invariant as is the case for the integral of a CS form in a manifold with boundary). The transgression action reads

\[ I[A, \bar{A}] = \int_M \mathcal{C}(A) - \int_M \mathcal{C}(\bar{A}) - \int_{\partial M=\partial \bar{M}} \text{Tr}(A \wedge \bar{A}). \]

Here the two connections \( A \) and \( \bar{A} \) have support on two different manifolds with a common boundary where they interact. This functional is invariant (modulo winding number) under independent gauge transformations for \( A \) and \( \bar{A} \),

\[ A \to A' = g^{-1} (A + d) g \]
\[ \bar{A} \to \bar{A}' = g^{-1} (\bar{A} + d) \bar{g} , \]

provided that the gauge transformations for both connections are the same at the boundary, \( g|_{\partial M} = \bar{g}|_{\partial M} \). The suitable generalization of this idea when more than two manifolds meet at a common boundary will be shown to be the appropriate scheme to describe multiple bubbles. We shall pick up on this point again in section 4.1.

iv) For the gauge group \( SO(2,2,\mathbb{R}) \), there is the well known interpretation for this action as equivalent to General Relativity in 2 + 1 dimensions with negative cosmological constant [12, 13] (at least perturbatively [14]). Thus, the action for each wall can describe the geometry of a pseudo-Riemannian surface of constant negative curvature. This is the negative curvature space-time analogue of a conventional double bubble, whose walls are surfaces of constant positive curvature (according to the conjecture. This scheme offers the possibility for matching different three-dimensional spacetime geometries with a common boundary. This construction will be discussed as an explicit example in section 3. As we will see, the form of the interaction between the three geometries at the intersection depends on how one chooses the surface terms.

Geometrically, some key features of the CS form can be understood by considering the characteristic form quadratic in the field strength \( P_4 := \text{Tr}(F \wedge F) \) (Chern character). This four form is closed and therefore in a contractible open patch is exact, \( P_4 = d \mathcal{C} \). Chern-Simons theory on a closed manifold can in this way be interpreted as the integral of a characteristic form over a manifold of one dimension higher. For a concrete example, consider CS theory on a manifold which is topologically a three-sphere \( S^3 \). This can be regarded as the boundary of some four-manifold \( B \). Let \( A \) be a gauge field defined on \( S^3 \). Under suitable topological assumptions, we can define an extension of this gauge field to \( B \), which for convenience we also call \( A \). The CS action on \( S^3 \) is then equal to:
\[ I_{\text{CS}}(S^3) \equiv \frac{k}{4\pi} \int_B \text{Tr}(F \wedge F), \quad (5) \]

The extension of the fiber bundle \( E(S^3) \) to a bundle \( E(B) \) over the four-dimensional space is a non-trivial matter. For example, gauge-related connections on \( S^3 \) may have different extensions in the interior. So, the gauge field on \( B \) cannot really be regarded as completely fixed by the physical data on \( S^3 \). From the fact that the Characteristic form defines an integer cohomology class, it follows that the action will be gauge invariant, modulo some integer multiple of \( 2\pi \), provided the level \( k \) is chosen to be an integer.

Since the characteristic form is closed, it defines a gauge theory intrinsic to the three-sphere. Formally one could say that this describes the surface dynamics of a bubble containing \( B \) as its interior. This is not altogether accurate because \( B \) is not a fixed background. However it does suggest an interesting idea, which we will now outline.

2 Double bubble

Let us consider a double bubble configuration like the one shown in Figure 1. The double-bubble is made up of three different manifolds, which we shall call the walls, joined at a common boundary. In the case of interest the walls are three-dimensional and they meet on their two-dimensional intersection. This edge is a three-way branching surface, so we have a structure which is not a Hausdorff manifold, but rather a more general kind of cell complex. It is sometimes referred to as a rectifiable set [reference], which means that it is arbitrarily close in measure to being a manifold, with the singular set of non-manifold points being of measure zero. The bubble complex can be regarded as the union of boundaries of the interiors of the bubbles, i.e. the union of boundaries of four-dimensional manifolds.

Let us now discuss two different approaches for constructing a Chern-Simons foam or multiple bubble. The first approach (section 2.1) is very natural from a four-dimensional point of view of bubble interiors, with Characteristic forms living in them. This leads to two connections on each wall. In the second approach (section 2.2), the walls themselves play the prominent role (with the four-dimensional interiors being reduced to a kind of metaphysical meaning). It is possible to define a meaningful action for a single connection on each wall. In the rest of the paper, we shall leave aside the first method and concentrate on the second.

2.1 Action as sum of characters of 4D topological spaces

Inspired by equation (5) for the single “bubble” one can postulate an action which is the sum of integrals of a Characteristic form over each of the three four-dimensional interiors:

\[ I[A_1, A_2, A_3] = \frac{k_1}{4\pi} \int_{B_1} \text{Tr}(F_1 \wedge F_1) + \frac{k_2}{4\pi} \int_{B_2} \text{Tr}(F_2 \wedge F_2) + \frac{k_3}{4\pi} \int_{B_3} \text{Tr}(F_3 \wedge F_3) + \text{[Boundary terms]}. \quad (6) \]

\(^2\)This extension to \( E(B) \) can always be found when the cohomology group \( H_3(BG, \mathbb{Z}) \) of the classifying space \( BG \) is trivial. Any bundle over a manifold is the pullback bundle induced by the embedding of the manifold into the classifying space. Therefore, if all three-cycles in the classifying space are boundaries, there always exists a four manifold bounded by \( M_3 \) such that an extension of the bundle onto the interior exists. If \( H_3(BG, \mathbb{Z}) \) is non-trivial, the Chern-Simons theory may still be defined but the concept of an interior manifold may break down.
Figure 2: A slice of the double bubble shows three walls meeting at the intersection. a) The first approach involves introducing a connection on each four-dimensional bubble interior, $B_1$ and $B_2$ and the exterior region $B_3$; b) The second approach, the main subject of this paper, involves a single connection defined intrinsically on each of the walls $M_1$, $M_2$ and $M_3$.

There are three gauge fields, one in each of the four-dimensional regions $B_1$, $B_2$ and $B_3$, as shown in Figure 2.

One could neglect the exterior region, which would amount to fixing the connection $A_3 \equiv 0$, but, it seems more appropriate to be democratic and keep all three connections. Now, since each of the bulk terms in (6) is a closed form, it could be traded for a CS form on the surface that encloses the respective four-volume, $B_i$. In this way one defines an intrinsic CS theory on the walls and intersection of the double bubble. The result would be the sum of three transgression forms defined on the three bubble walls,

$$I[A_1, A_2, A_3] = \frac{k_1}{4\pi} \int_{M_{12}} [C(A_1) - C(A_2)] + \frac{k_2}{4\pi} \int_{M_{23}} [C(A_2) - C(A_3)]$$

$$+ \frac{k_3}{4\pi} \int_{M_{31}} [C(A_3) - C(A_1)] + \int_{M_{123}} \text{[Surface terms]}.$$  (7)

This functional depends on the difference between the CS forms obtained by approaching each wall from both sides. The corresponding connections ($A_i$) induced by their values on the neighboring volumes need not match. There might be interesting cases in which this possibility can be useful. For example, if the curvature two-form $F$ the same on each side of $M_{ij}$, for in that case, the connections must differ at most by a gauge transformation, $A_i = g^{-1}(A_j + d)g$ and the corresponding difference of CS forms is a closed form describing a WZ theory at the two-dimensional boundary.

The doubling of connections on each wall seems somewhat excessive and there is no obvious interpretation of the fields. Even for three-dimensional AdS gravity, it is not essential to introduce the difference of two CS forms: there is an equivalent formulation with a single Chern-Simons form for the AdS group (which can be generalised to higher odd dimensions). So we shall not pursue this approach further in this article.

The action proposed above is one way of formally defining a foam as embedded in an auxiliary
four-dimensional manifold. The resulting action is constructed with two CS forms in the threedimensional walls of the double bubble. This carries the disadvantage of having two dynamically independent connection fields with same quantum numbers defined on each three-surface. Alternatively, we may try a different intrinsic definition, for a single field on each wall (and intersections of them), without reference to any interior regions.

2.2 Intrinsic theory on the bubble walls (Abelian Case)

The action proposed above is one way of formally defining a foam as embedded in an auxiliary four-dimensional manifold. The resulting action is constructed with two CS forms in the three-dimensional walls of the double bubble. This carries the disadvantage of having two dynamically independent connection fields with same quantum numbers defined on each three-surface. Alternatively, we may try a different intrinsic definition, for a single field on each wall (and intersections of them), without reference to any interior regions.

To illustrate the construction, let us look at the simplest case of abelian Chern-Simons theory. We will introduce an action which does not involve any metric or conformal structure on the intersection and which preserves gauge invariance (something which is not possible for a single manifold with boundary, see section 4.1). We will see that this leads to a consistent variational principle and therefore a sensible theory at least at the classical level.

2.2.1 The boundary coupling

Consider three 3-manifolds $M_i$ all sharing the same boundary $\partial M_1 = \partial M_2 = \partial M_3 = \Sigma$. The edge $\Sigma$ is a smooth two-dimensional space where the interaction takes place which is the analogue of a vertex in a Feynman diagram for point particle interactions. On each manifold $M_i$, a connection $A^{(i)}$ is defined. The action is defined as a sum of the corresponding CS functionals with level $k_i$,

$$I[A^{(1)}, A^{(2)}, A^{(3)}] = \sum_{i=1}^{3} \frac{k_i}{4\pi} \int_{M_i} A^{(i)} \wedge dA^{(i)} + \int_{\Sigma} B[A^{(1)}, A^{(2)}, A^{(3)}]. \quad (8)$$

This would be the correct generalization of the transgression (3) if the boundary term were such that the functional be invariant under independent gauge transformations on each $A^{(i)}$, subject to the appropriate matching condition at the edge. Since the interaction lagrangian must be a two-form, it can only be a sum of terms of the form $A^{(i)} \wedge A^{(j)}$.

In what follows we assume the levels $k_i$ to be all positive and allow for an arbitrary sign in front of the kinetic term, $\epsilon_i = \pm 1$ to account for the sign of each level. The level can be eliminated from the action by a suitable rescaling of the connections

$$\hat{A}^{(i)} \equiv \sqrt{k_i} A^{(i)}$$

Thus, the most general possible action with interaction terms can be assumed to be of the form,

$$4\pi I = \sum_{i=1}^{3} \epsilon_i \int_{M_i} \hat{A}^{(i)} \wedge d\hat{A}^{(i)} + \int_{\Sigma} [f_1 \hat{A}^{(2)} \wedge \hat{A}^{(3)} + f_2 \hat{A}^{(3)} \wedge \hat{A}^{(1)} + f_3 \hat{A}^{(1)} \wedge \hat{A}^{(2)}]. \quad (9)$$

The question is now, what restrictions are imposed on $\{f_i, \epsilon_i, k_i\}$ by the requirements of gauge invariance, and that $I$ should have a well posed variational problem.
2.2.2 Gauge Invariance

Under independent gauge transformations of the different connections, \( \hat{A}^{(i)} \rightarrow (\hat{A}')^{(i)} = \hat{A}^{(i)} + d\hat{\lambda}^{(i)} \), the action changes by

\[
\delta I = -\int_{\Sigma} \left[ \left( \varepsilon_1 d\hat{\lambda}^{(1)} + f_3 d\hat{\lambda}^{(2)} - f_2 d\hat{\lambda}^{(3)} \right) \wedge \hat{A}^{(1)} + \left( \varepsilon_2 d\hat{\lambda}^{(2)} + f_1 d\hat{\lambda}^{(3)} - f_3 d\hat{\lambda}^{(1)} \right) \wedge \hat{A}^{(2)} + \left( \varepsilon_3 d\hat{\lambda}^{(3)} + f_2 d\hat{\lambda}^{(1)} - f_1 d\hat{\lambda}^{(2)} \right) \wedge \hat{A}^{(3)} \right].
\]

The right hand side of this equation vanishes identically for arbitrary \( A^{(i)} \) provided the \( \lambda^{(i)} \)'s are such that

\[
\begin{bmatrix}
\varepsilon_1 & f_3 & -f_2 \\
-f_3 & \varepsilon_2 & f_1 \\
f_2 & -f_1 & \varepsilon_3
\end{bmatrix}
\begin{bmatrix}
d\hat{\lambda}^{(1)} \\
d\hat{\lambda}^{(2)} \\
d\hat{\lambda}^{(3)}
\end{bmatrix} = 0. \tag{10}
\]

The existence of non trivial solutions depends on making a reasonable choice of coupling constants. In particular, demanding the vanishing of the determinant requires

\[
\varepsilon_1 \varepsilon_2 \varepsilon_3 + \varepsilon_1 f_1^2 + \varepsilon_2 f_2^2 + \varepsilon_3 f_3^2 = 0. \tag{11}
\]

On the other hand, (10) can be solved as the general vanishing eigenvalue equation

\[
(\eta + f)\bar{\alpha} = 0, \tag{12}
\]

where we have defined the matrices

\[
\eta := \text{diag}(\varepsilon_1, \varepsilon_2, \varepsilon_3); \quad f := \begin{pmatrix}
0 & f_3 & -f_2 \\
-f_3 & 0 & f_1 \\
f_2 & -f_1 & 0
\end{pmatrix}
\]

First we note that since for any solution of (12), \( \bar{\alpha}^T (\eta + f)\bar{\alpha} = 0 \), and in view of the antisymmetry of \( f \), \( \bar{\alpha} \) must satisfy \( \varepsilon_1 (\alpha^{(1)})^2 + \varepsilon_2 (\alpha^{(2)})^2 + \varepsilon_3 (\alpha^{(3)})^2 = 0 \). This could only occur for a nontrivial \( \bar{\alpha} \) if and only if \( \eta \) is an indefinite “metric”, which without loss of generality we take as

\[
\eta = \begin{pmatrix}
+1 & 0 & 0 \\
0 & +1 & 0 \\
0 & 0 & -1
\end{pmatrix}. \tag{13}
\]

Equation (11) reduces to the requirement that the components of a vector \( \bar{f} \) lie on the surface of a hyperboloid of unit space-like distance from the origin, \( (f_1)^2 + (f_2)^2 - (f_3)^2 = 1 \). For later convenience, a general point on this hyperboloid can be parametrised as

\[\text{One could have chosen either } \eta = \text{diag}(+1,+1,-1) \text{ or } \eta = \text{diag}(-1,-1,+1), \text{ but both cases are related by a global reversal of sign convention for orientations.}\]
\[ f_1 = -\sin \Omega + \xi \cos \Omega \]
\[ f_2 = \cos \Omega + \xi \sin \Omega \]
\[ f_3 = \xi \] (14)

As already mentioned, the other consistency condition is that for every nontrivial solution of (12), \( \vec{\alpha} \) must be null, that is \( (\alpha^{(1)})^2 + (\alpha^{(2)})^2 - (\alpha^{(3)})^2 = 0 \). Since the vector \( \vec{\alpha} \) is null, it is determined by (12) up to an arbitrary normalization constant and lies on the null cone through the origin, \( \vec{\alpha} = \alpha_0 \vec{n} \), where

\[ \vec{n} = \begin{pmatrix} \cos \theta \\ \sin \theta \\ 1 \end{pmatrix}. \]

Consistency and solvability of equation (12) relate the angle \( \theta \) and the coupling constants \( f_i \), and one finds

\[ \theta = \Omega, \quad \xi = \text{arbitrary}. \] (15)

From the previous analysis, we conclude that the gauge transformations at the intersection must be such that

\[ \hat{\lambda}^{(1)} = \hat{\lambda}^{(3)} \cos \Omega, \] (16)
\[ \hat{\lambda}^{(2)} = \hat{\lambda}^{(3)} \sin \Omega, \] (17)

in order for the full action (9) to be gauge invariant. Reinstating the coupling constants, and imposing the consistency expressions (14), the action becomes

\[
4\pi I = \sum_{i=1}^{3} \epsilon_i k_i \int_{M_i} A^{(i)} \wedge dA^{(i)} + \int_{\Sigma} \left[ -\sqrt{k_2 k_3} \sin \Omega \ A^{(2)} \wedge A^{(3)} + \sqrt{k_3 k_1} \cos \Omega \ A^{(3)} \wedge A^{(1)} \right] \\
+ \xi \int_{\Sigma} \left[ \sqrt{k_1 k_2} \ A^{(1)} \wedge A^{(2)} + \sqrt{k_2 k_3} \cos \Omega \ A^{(2)} \wedge A^{(3)} + \sqrt{k_3 k_1} \sin \Omega \ A^{(3)} \wedge A^{(1)} \right].
\]

As we have seen, for the abelian CS theory, \( k \) is somewhat of a phoney coupling constant. It can always be set to +1 by using the rescaled connection \( \hat{A} \). But this leads to the rather ad hoc matching of gauge parameters, (16) and (17), at the intersection. It is more natural to require that the “true” gauge parameters \( \lambda^{(i)} = \hat{\lambda}^{(i)} / \sqrt{k_i} \) be continuous at the intersection:

\[ \lambda^{(1)} = \lambda^{(2)} = \lambda^{(3)} \] (18)

This can be achieved by the nontrivial matching for the levels

\[ \sqrt{k_1} = \sqrt{k_3} \cos \Omega, \] (19)
\[ \sqrt{k_2} = \sqrt{k_3} \sin \Omega, \] (20)
which is reminiscent of the matching condition for the tension in a three string junction \[17\]. This relation can be written also more suggestively as a “conservation law” for the levels:

\[ k_1 + k_2 = k_3 . \]  

Finally, the action reduces to

\[
4\pi I = k_1 \left( \int_{M_1} A^{(1)} \wedge dA^{(1)} - \int_{M_3} A^{(3)} \wedge dA^{(3)} - \int_{\Sigma} A^{(1)} \wedge A^{(3)} \right) + k_2 \left( \int_{M_2} A^{(2)} \wedge dA^{(2)} - \int_{M_3} A^{(3)} \wedge dA^{(3)} - \int_{\Sigma} A^{(2)} \wedge A^{(3)} \right) + \xi \sqrt{k_1 k_2} \int_{\Sigma} \left[ A^{(1)} \wedge A^{(2)} + A^{(2)} \wedge A^{(3)} + A^{(3)} \wedge A^{(1)} \right].
\]  

Note that the arbitrary coefficient $\xi$ multiplies a term that is gauge invariant by itself (the origin of this term will be discussed in section \[4.1\]). This term does not contribute to the field equations either and can therefore be dropped from the classical action. The transgression action \[3\] is recovered for $k_2 = 0$ ($k_1 = k_3$, and $\xi = 0$), and therefore expression \[22\] can be regarded as a generalization of the concept of transgression for the case of three manifolds with a common boundary. Note that, for $k_2 \neq 0$, the transgression is not recovered by setting $A^{(2)} = 0$. This is so because the gauge invariance of the action depends upon the existence of all three connections and setting one of them to zero is not a gauge invariant statement.

\subsection*{2.2.3 Matching conditions}

Now we consider the Euler-Lagrange equations for the action \[22\]. Extremising it with respect to independent variations of each gauge field, under the gauge invariant matching conditions

\[ A^{(1)}|_{\Sigma} = A^{(2)}|_{\Sigma} = A^{(3)}|_{\Sigma} , \]  

(\text{where } |_{\Sigma} \text{ denotes the pullback on differential forms onto } \Sigma) \text{ on the edge, one obtains}

\[ F^{(i)} = 0 \]  

on each wall. In other words, the action is stationary with respect to arbitrary infinitesimal variations of the connections on each wall, provided the connections are flat and match continuously at the edge. These matching conditions are the same that guarantee an extremum for the transgression action \[3\].

\subsection*{2.2.4 Comments}

- The matching condition of the gauge parameters \[18\] means that out of the possible $U(1) \times U(1) \times U(1)$ gauge symmetry on $\Sigma$ (the independent gauge transformations $A^{(i)} \rightarrow A^{(i)} + d\lambda^{(i)}$), only a diagonal subgroup $U^d(1)$ is preserved. This is the most that can be achieved without introducing extra fields. This is an exact symmetry of the action and of course is a symmetry of the matching conditions.
• Consistency requires that the sign of one of the $\epsilon$’s must be opposite to the other two (we have taken $\epsilon_3$ to be of opposite sign to $\epsilon_1$ and $\epsilon_2$). We can think of these signs as labeling ingoing and outgoing gauge fields. The consistency conditions furthermore impose the conservation law (21): the net incoming level is always equal to the net outgoing level.
• The matching conditions for $\lambda_{(i)}$ are insensitive to the value of the “coupling constant” $\xi$. Indeed it is easy to check that under (23), the last term in (22) is gauge invariant by itself. Likewise, the matching conditions for $k$ and $A$ are insensitive to $\xi$. For this reason, at least in the classical context, all choices of $\xi$ define the same physical theory. One can fix this coefficient choosing, for instance, $\xi = 0$.
• The interaction term $A^{(1)} \wedge A^{(2)}$ between the two ingoing connections is eliminated by choosing $\xi = 0$.
• For the abelian theory, the choice $\lambda^{(1)} = \lambda^{(2)} = \lambda^{(3)}$ is just a convenient option. It is rather a matter of choice whether we put the nontrivial matching condition into the $k$’s or into the $\lambda$’s and $\hat{A}$’s. In other words $\hat{A}$ is just as good a connection as $A$. When we come to treat the non-abelian theory, however, this will no longer be the case and $A$ should be regarded as the true connection. Therefore, the nontrivial matching of the $k$’s is the preferred interpretation.

2.2.5 Example

In order to understand the physical consequences of the relation among the different gauge transformations, let us examine the case of a double bubble of three-dimensional walls with three $U(1)$ connections. The field equations imply that the connection on each 3-manifold is locally flat, $F^{(i)} = 0$. For instance, a nontrivial locally flat connection could be defined in a spacetime with a topological defect produced by a puncture on the spatial section. Each bubble wall consists of a three-dimensional spacetime manifold $M_{2+1}$ whose spacelike sections have the topology of a disc with a removed point, $M_{2+1} = (D_2 - \{0\}) \times \mathbb{R}$. The action that describes a $U(1)$ connection in this 2+1 manifold is the sum of two CS for 3 and 1 dimensions, respectively,

$$I[A] = k \int_{M^{2+1}} A \wedge dA + k' \int_{M^{0+1}} A. \quad (25)$$

This can also be written in a more familiar form as

$$I[A] = 2k \int_{M^{2+1}} \left[ \frac{1}{2} A \wedge dA - A \wedge j \right], \quad (26)$$

where $j = q \delta^{(2)}(x, y) \, dx \wedge dy$ is the two-form current density source produced by a (magnetic) point charge $q = -\frac{k'}{2k}$. The field equation is $F = j$, and the classical solution takes the form

$$A = \frac{q}{2\pi} \, d\phi. \quad (27)$$

Now we want to put three connections of this sort defined on the three walls of a double bubble. Requiring the action to be invariant under independent gauge transformations of each connection—provided they respect (18) on the common boundary $\Sigma = S^1 \times \mathbb{R}$, implies the conservation law

$$k_1 + k_2 = k_3, \quad (28)$$

and the matching condition for the $A$’s in this case becomes

$$q^{(1)} = q^{(2)} = q^{(3)}. \quad (29)$$
Since $q^{(i)} = -\frac{k_i'}{2k_i}$, there is also a matching for the $k'$s,
\[ k_1' + k_2' = k_3'. \]  

(30)

### 2.3 Non-abelian double bubble

The previous analysis of the conditions at the intersection carries over straightforwardly to the case of non-abelian Chern-Simons theory. There is one subtlety associated with gauge invariance. In the abelian theory we eliminated the coupling constants $k^{(i)}$ on the wall by rescaling the gauge field. In the non-abelian theory the connection in each wall transforms as
\[ A^{(i)} \to g^{-1}^{(i)} A^{(i)} g^{(i)} + g^{-1}^{(i)} dg^{(i)} , \]

which means that we cannot rescale the gauge field $\hat{A} \equiv \sqrt{k} A$ without modifying the gauge transformation correspondingly: $\hat{A}^{(i)} \to g^{-1}^{(i)} \hat{A}^{(i)} g^{(i)} + \sqrt{k_i} g^{-1}^{(i)} dg^{(i)}$. So, it will be more convenient to keep the $k_i$'s explicitly in the action.

It is natural to require that the field equations be invariant under (31) where the gauge parameter $g(x)$ is globally defined over the bubble complex, so that the gauge symmetry group $\mathbb{G}$. We therefore require
\[ g^{(1)}|_{\Sigma} = g^{(2)}|_{\Sigma} = g^{(3)}|_{\Sigma} . \]

(32)

There is no other obvious way to relate the gauge parameters that has a chance of being consistent with the field equations.

Considering infinitesimal gauge transformations leads to exactly the same analysis as in section 2.2.2, and one is led to an action of the form (22),
\[ 4\pi I = k_1 \left( \int_{M_1} C(A^{(1)}) - \int_{M_3} C(A^{(3)}) - \int_{\Sigma} \text{Tr}(A^{(1)} \wedge A^{(3)}) \right) \\
+ k_2 \left( \int_{M_2} C(A^{(2)}) - \int_{M_3} C(A^{(3)}) - \int_{\Sigma} \text{Tr}(A^{(2)} \wedge A^{(3)}) \right) \\
+ \xi \sqrt{k_1 k_2} \int_{\Sigma} \text{Tr} (A^{(1)} \wedge A^{(2)} + A^{(2)} \wedge A^{(3)} + A^{(3)} \wedge A^{(1)}) , \]

(33)

where again, one can choose $\xi = 0$. It can be easily checked that this is invariant under finite gauge transformations, up to a winding number term, as discussed in Appendix 4.2. Then, the Euler variation of the action implies $F^{(i)} = 0$ on the walls, and the gauge fields themselves obey the fairly unexciting relation at the intersection

\[ A^{(1)}|_{\Sigma} = A^{(2)}|_{\Sigma} = A^{(3)}|_{\Sigma} , \]

(34)

\[ \text{One might instead try to demand instead invariance under independent } g_i \text{'s. This would give an enhanced symmetry } G \times G \text{ or } G \times G \times G \text{ at the intersection. It turns out that this is not possible without the addition of extra fields. Indeed, if we wish to have all three gauge fields truly interacting at the intersection, such an enhanced symmetry seems to be unwanted.} \]
and again, the conservation rule \( k_1 + k_2 = k_3 \) applies. Thinking of walls with negative \( \epsilon \) as “ingoing” and walls with positive \( \epsilon \) as “outgoing”, we have the conservation law

\[
(\sum k_i)_{\text{in}} = (\sum k_i)_{\text{out}}.
\]

This concludes the analysis for the three-way intersection. Next we generalise to a four-way intersection which sets the general pattern for intersections of higher order. We will show that the conservation law \([35]\) holds also in that case. Furthermore, we will find an interesting formula which generalises \([34]\).

### 2.4 Example: matching (2+1)-dimensional black holes

To illustrate the non-abelian three-way intersection, we consider the anti-de Sitter gauge group SO(2, 2) (or SO(3, 1), for the Euclideanised case), for which the Chern-Simons construction describes (2+1)-dimensional gravity with negative cosmological constant \([12, 13]\). The anti-de Sitter connection is \( A = \omega^{ab}J_{ab}/2 + e^aJ_{a3}/l \), \( a = 0, 1, 2 \), and the bilinear form is \( \text{Tr}(J_{AB}J_{CD}) \propto \epsilon_{ABCD} \).

The contribution to the action of each wall is therefore an Einstein-Hilbert term with negative cosmological constant \([12, 13]\). The anti-de Sitter space is not a Hausdorff manifold, it is not clear how to relate the co-ordinates of the four walls can be made to match, \( \phi_i = \phi_j \) and \( t_{Ei}/\beta_i = t_{Ej}/\beta_j \). Because this space is not a Hausdorff manifold, it is not clear how to relate the co-ordinates \( r_i \) differeniatibly, \( \frac{dt}{l} \neq \frac{dr}{l} \).

The matching of black-hole geometries takes place on a common intersection \( \Sigma \) at constant radial coordinate, which can in principle take a different value, \( r_i = a_i \), on each wall. The intersection surface \( \Sigma \) has topology \( S^1 \times S^1 \), and using invariance under co-ordinate transformations the angular co-ordinates of the four walls can be made to match, \( \phi_i = \phi_j \) and \( t_{Ei}/\beta_i = t_{Ej}/\beta_j \). Because this space is not a Hausdorff manifold, it is not clear how to relate the co-ordinates \( r_i \) differeniatibly, \( \frac{dt}{l} \neq \frac{dr}{l} \).
but this is not a problem since the matching conditions are for the pullback of $A$ onto $\Sigma$: the $A, dr$ components (in this case $e^l/l$) do not contribute. This is crucial for the consistency of the analysis.

Now, we consider the matching condition for the connection, which for the triple intersection is simply $A^i|_\Sigma = A^j|_\Sigma$. By matching $e^0/l, e^2/l, \omega^{01}$ and $\omega^{12}$ pulled back onto $\Sigma$ one gets:

$$a_i/l_i = a_j/l_j, \quad \beta_i/l_i = \beta_j/l_j, \quad \mu_i = \mu_j.$$  

(38)

It is natural to assume the proper length of the time cycle at the boundary of each wall to be the same, that is, $\beta_i = \beta_j$, which in turn requires $l$ to be a universal constant ($l_i = l_j$). In this way the induced metric on $\Sigma$ is continuous and the temperatures all match. One is therefore matching solutions which are in thermal equilibrium but have different Newton constants. The quantization law $k_i = \text{integer}$ holds for the triple intersection (see section 4.2) and implies a corresponding quantization of the Newton’s constants. Since $m_i = \mu k_i/l$, the masses are not all the same but, using $(\sum k)_{\text{in}} = (\sum k)_{\text{out}}$, we find the conservation law

$$m_1 + m_2 = m_3.$$  

(39)

It is natural to assume that all the Newton constants are positive and account for the signs $\epsilon_i$ by reversing the orientation of $M_3$ with respect to $\Sigma$. If one takes this view, one should match two interior regions $\rho_1 \leq a_1$ and $\rho_2 \leq a_2$ with one exterior $\rho_3 \geq a_3$ or vice versa. The above law then says that the sum of masses in the interiors equals the mass in the exterior. The Newton constants satisfy:

$$\frac{1}{G_1} + \frac{1}{G_2} = \frac{1}{G_3}.$$  

(40)

The problem can be studied in a more generality as the matching of hyperbolic manifolds along homeomorphic boundaries. Here we have only matched Euclidean black holes without angular momentum. They have the topology of a solid torus and we have matched along a surface of topology $S^1 \times S^1$. It would be possible to also include Euclidean black holes with angular momentum, since they have the same topology.

3 Higher order intersections

It is possible to construct an action for more than three films meeting at a two-dimensional intersection. Unlike the three-way intersection, it is possible to have more general matching conditions than $A^i|_\Sigma = A^j|_\Sigma$, which leave the connections less determined. It can be checked that as one goes to higher order, the problem becomes less determined. From the point of view of a static foam, this indefiniteness suggests some kind of instability with respect to the more basic three-way intersection, as occurs in soap bubbles. However, the higher order intersections may be of interest in describing dynamical scattering of films. Here we discuss explicitly only the case of four-way intersections.

3.1 Four-way intersections

Let us consider now the situation where four walls meet at a single 2-dimensional intersection surface $\Sigma$. As before, we choose the orientations such that $\partial M_i = \Sigma$, assume all levels $k_i > 0$, and define $\epsilon_i = \pm 1$. A general ansatz for the interaction term, without introducing other fields or preferred coordinates, will have 6 coupling constants $f_{ij}$.
Figure 3: The four way intersection may be interpreted as a collision, with two incoming and two outgoing walls.

$$4\pi I = \sum_{i=1}^{4} \epsilon_i k_i \int_{M_i} \mathcal{C}(A^{(i)}) + \int_{\Sigma} \sum_{i<j} \sqrt{k_i k_j} f_{ij} \text{Tr}(A^{(i)} \wedge A^{(j)}). \quad (41)$$

Again, the condition of infinitesimal gauge invariance imposes an algebraic constraint at the intersection that can be represented by the matrix equation

$$\left( \eta + f \right) \vec{\beta} = 0, \quad (42)$$

with $\eta = \text{diag}(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4)$, $f$ is the antisymmetric matrix with entries $f_{ij}$, and $\beta^{(i)} = \sqrt{k_i} D\lambda^{(i)}$. By considering $\vec{\beta}^T \left( \eta + f \right) \vec{\beta} = 0$ we find that $\sum_i \epsilon_i k_i (\beta^{(i)})^2 = 0$ so that $\eta$ cannot have Euclidean signature. The matching condition for the connections will also be of the form (42), with $\beta^{(i)} = \sqrt{k_i} A^{(i)}|_{\Sigma}$.

### 3.2 Scattering of Chern-Simons films

There is an interesting possibility which only occurs if $\eta$ has signature $(+, +, -, -)$, which corresponds to the “scattering” of two “incoming” and two “outgoing” walls (see fig. 3). In what follows we restrict our attention to this case; situations with three equal signs are similar in spirit to the three-way intersection discussed above. The coupling constants $f_{ij}$ can be chosen in such a way that the field equations give only two independent equations for the $A^{(i)}$’s. We consider the $f$ matrix of the form

$$f = \begin{pmatrix} \emptyset & -R^T \\ R & \emptyset \end{pmatrix}$$

where $R$ and $\emptyset$ are $2 \times 2$ matrices. Other forms of $f$ can be obtained by an $SO(2, 2)$ rotation from this case. Then, the matching conditions

$$\begin{pmatrix} I & -R^T \\ R & -I \end{pmatrix} \begin{pmatrix} \vec{\beta}_{\text{in}} \\ \vec{\beta}_{\text{out}} \end{pmatrix} = 0,$$

reduce to two independent equations,
provided that $R^T R = I$ i.e. $R$ is an $O(2)$ matrix. We have defined
\[ \vec{\beta}_{\text{in}} := \left( \sqrt{k_1 d\lambda^{(1)}} |_{\Sigma}, \sqrt{k_2 d\lambda^{(2)}} |_{\Sigma} \right) \]
and
\[ \vec{\beta}_{\text{out}} := \left( \sqrt{k_3 d\lambda^{(3)}} |_{\Sigma}, \sqrt{k_4 d\lambda^{(4)}} |_{\Sigma} \right). \]
Now, by an appropriate choice of labels for the walls, $R$ may be taken as an $SO(2)$ matrix:
\[ R = \begin{pmatrix} \cos \Omega & -\sin \Omega \\ \sin \Omega & \cos \Omega \end{pmatrix}. \]

Then, the action is
\[ 4\pi I = \sum_{i=1}^{4} \epsilon_i k_i \int_{M_i} C(A^{(i)}) - \int_{\Sigma} \left\{ \sqrt{k_1 k_3} \text{Tr}(A^{(1)} \wedge A^{(3)}) + \sqrt{k_2 k_4} \text{Tr}(A^{(2)} \wedge A^{(4)}) \right\} \cos \Omega - \int_{\Sigma} \left\{ \sqrt{k_1 k_4} \text{Tr}(A^{(1)} \wedge A^{(4)}) - \sqrt{k_2 k_3} \text{Tr}(A^{(2)} \wedge A^{(3)}) \right\} \sin \Omega. \]

We assume the action to be invariant under gauge transformations that are continuous at the intersection.
\[ g(1) |_{\Sigma} = g(2) |_{\Sigma} = g(3) |_{\Sigma} = g(4) |_{\Sigma}. \]
This implies
\[ \left( \begin{array}{c} \sqrt{k_3} \\ \sqrt{k_4} \end{array} \right) = \left( \begin{array}{cc} \cos \Omega & -\sin \Omega \\ \sin \Omega & \cos \Omega \end{array} \right) \left( \begin{array}{c} \sqrt{k_1} \\ \sqrt{k_2} \end{array} \right). \]

As promised, the conservation law holds. One can use to express $\Omega$ as a function of the $k$'s
\[ \Omega(k_1, k_2, k_3, k_4) = \tan^{-1} \frac{\sqrt{k_1 k_4} - \sqrt{k_2 k_3}}{\sqrt{k_1 k_3} + \sqrt{k_2 k_4}} \]
and therefore $\Omega$ is not an independent coupling constant.

Eliminating $\Omega$ we can write
\[ R = \frac{1}{k_1 + k_2} \left( \begin{array}{cc} \sqrt{k_1 k_3} + \sqrt{k_2 k_4} & -\sqrt{k_1 k_4} + \sqrt{k_2 k_3} \\ \sqrt{k_1 k_4} - \sqrt{k_2 k_3} & \sqrt{k_1 k_3} + \sqrt{k_2 k_4} \end{array} \right) \]
with
\[ k_1 + k_2 = k_3 + k_4. \]

The matching condition for the gauge field at $\Sigma$ is:
\[ \left( \begin{array}{c} \sqrt{k_3} A^{(3)} |_{\Sigma} \\ \sqrt{k_4} A^{(4)} |_{\Sigma} \end{array} \right) = R \left( \begin{array}{c} \sqrt{k_1} A^{(1)} |_{\Sigma} \\ \sqrt{k_2} A^{(2)} |_{\Sigma} \end{array} \right). \]

Finally, it may be helpful to re-express the matching conditions in the form:
\[ \left( \begin{array}{c} A^{(3)} |_{\Sigma} \\ A^{(4)} |_{\Sigma} \end{array} \right) = U \left( \begin{array}{c} A^{(1)} |_{\Sigma} \\ A^{(2)} |_{\Sigma} \end{array} \right), \]

\text{For the abelian theory, this assumption is not necessary and so there is an enhanced gauge symmetry $U(1) \times U(1)$ at the edge, the matching of the gauge parameters being determined only by}.
where the matrix
\[ U \equiv \frac{1}{k_1 + k_2} \left( \begin{array}{cc} k_1 + \sqrt{k_1 k_2 k_4 / k_3} & k_2 - \sqrt{k_1 k_2 k_4 / k_3} \\ k_1 - \sqrt{k_1 k_2 k_3 / k_4} & k_2 + \sqrt{k_1 k_2 k_3 / k_4} \end{array} \right) \]
satisfies the curious relation \( \text{tr}U = 1 + \det U \). It can now be seen that if the \( k \)'s are not all the same there are non-trivial solutions with \( A(i) \neq A(j) \).

This situation can be interpreted as describing the scattering of two “ingoing” and two “outgoing” walls, \( M_1, M_2 \to M_3, M_4 \). Given a set of levels \((k_1, k_2, k_3, k_4)\) satisfying (48), one is free to specify any values of \((A(1), A(2))\) and equation (49) (or equivalently (50)) determines the outgoing values \((A(3), A(4))\). So we can interpret our action as describing an elastic scattering process: what goes out is completely determined by what comes in. The ingoing data is not constrained, except by the bulk field equations \( F(i) = 0 \).

## 4 Extensions

### 4.1 CS foam and Transgression forms

For a single Chern-Simons theory on a manifold with boundary, the presence of the boundary usually breaks the symmetry drastically. One must impose boundary conditions which break the gauge invariance, or else break the diffeomorphism invariance on the boundary down to conformal invariance. A standard construction would be to introduce a preferred complex structure and add a boundary term \( A_z A_{\bar{z}} \), leading to a two-dimensional conformal field theory [22]. Alternatively, one may try to modify the action so as to preserve the topological nature of the theory. This was considered in Refs. [9, 10], by adding a second connection and a boundary term so that the Lagrangian becomes a Transgression form:

\[ T(A^{(1)}, A^{(2)}) \equiv C(A^{(1)}) - C(A^{(2)}) - d\text{Tr}(A^{(1)} \wedge A^{(2)}) \]

This preserves gauge invariance, with \( g_{(1)}|_{\partial M} = g_{(2)}|_{\partial M} \), without introducing any fixed metric or conformal structure on the boundary. Furthermore, since the gauge transformations only need to be related at the boundary, one can take the action

\[ \int_M [C(A^{(1)}) - C(A^{(2)})] - \int_{\Sigma = \partial M} \text{Tr}(A^{(1)} \wedge A^{(2)}) \]

and “pull apart” \( M \) in the middle to produce a blister shaped space \( M_1 \cup M_2 \cup \Sigma \) where \( \partial M_1 = \partial M_2 = \Sigma \). The action

\[ S[A^{(1)}, A^{(2)}] \equiv \int_{M_1} C(A^{(1)}) - \int_{M_2} C(A^{(2)}) - \int_{\Sigma} \text{Tr}(A^{(1)} \wedge A^{(2)}) \]

is still gauge invariant. Now \( \Sigma \) is interpreted as the boundary of our region of space and \( A^{(2)} \) can be interpreted as a gauge field lying beyond the boundary. In the case of AdS gravity, this transgression method was shown to successfully regulate the charges, which would otherwise require ad hoc counterterms.

It is natural to try to generalise this method to more than two connections. An ansatz for the transgression action of a double-bubble would be a linear combination:

\[ I = \alpha_{12} S[A^{(1)}, A^{(2)}] + \alpha_{23} S[A^{(2)}, A^{(3)}] + \alpha_{31} S[A^{(3)}, A^{(1)}] \]
(It may be helpful think of this as the integral of
\[ \alpha_{12} T(A^{(1)}, A^{(2)}) + \alpha_{23} T(A^{(2)}, A^{(3)}) + \alpha_{31} T(A^{(3)}, A^{(1)}) \]
over the chain
\[ C = M_1 + M_2 + M_3 \]
provided we are careful that \( A^{(i)} \) only has its support on the closure of \( M_i \.)

We note the identity
\[ T(A^{(1)}, A^{(2)}) + T(A^{(2)}, A^{(3)}) + T(A^{(3)}, A^{(1)}) = -d \text{Tr}(A^{(1)} \wedge A^{(2)} + A^{(2)} \wedge A^{(3)} + A^{(3)} \wedge A^{(1)}) . \]
which allows us to write
\[ I = (\alpha_{12} - \alpha_{31}) S[A^{(1)}, A^{(2)}] + (\alpha_{23} - \alpha_{31}) S[A^{(2)}, A^{(3)}] - \alpha_{31} \int_\Sigma \text{Tr}(A^{(1)} \wedge A^{(2)} + A^{(2)} \wedge A^{(3)} + A^{(3)} \wedge A^{(1)}) . \]
This is exactly the same as action (22), which we had found by the more constructive approach. What we have done can therefore be seen as a generalisation of the transgression method. We start with a linear combination of transgressions. Then the manifold with boundary is pulled apart into three cobordant manifolds.

The generalisation to more than three connections now seems clear: Assign to a manifold with boundary a lagrangian \( \sum_{i > j} \alpha_{ij} T(A^{(i)}, A^{(j)}) \); Then “pull apart” the manifold into a set of cobordant manifolds, the walls \( M_i \), so that each Chern-Simons form \( C(A^{(i)}) \) is integrated over the corresponding wall \( M_i \). This is sufficient to guarantee gauge invariance. It is also sufficient to guarantee that the consistent matching conditions: \( A^{(i)}|_\Sigma = A^{(j)}|_\Sigma \forall i, j \) will always provide a solution. For certain choices of the coefficients \( \alpha_{ij} \) there may be more interesting matching conditions, as happened in our example of four-way scattering discussed above.

### 4.2 Quantisation of coupling constants

So far we have neglected to mention the winding number contribution to the gauge transformation. Under a general gauge transformation, the double-bubble action (33) transforms by:
\[ -\frac{k_1}{24\pi} \int_{C_{13}} \text{Tr} (a^{-1} da \wedge [a^{-1} da, a^{-1} da]) - \frac{k_2}{24\pi} \int_{C_{23}} \text{Tr} (b^{-1} db \wedge [b^{-1} db, b^{-1} db]) \]
where \( C_{13} = M_1 \cup \Sigma \cup (-M_3) \) and \( a \) is a continous (strictly speaking it must be \( C^1 \) so that \( da \) has at most bounded discontinuity at \( \Sigma \)) gauge parameter which coincides with \( g_1 \) in \( M_1 \) and \( g_3 \) in \( M_3 \), and similarly for \( C_{23} \). Note that it is possible, by choosing a non-Hausdorff topology on the double-bubble, to treat \( C_{13} \) and \( C_{23} \) as closed Hausdorff sub-manifolds, so that these integrals make sense mathematically. Therefore, we can immediately deduce that the double bubble action transforms under a gauge transformation by
\[ 2\pi k_1 n_1 + 2\pi k_2 n_2 \]
with \( n_1 \) and \( n_2 \) integer winding numbers. So the argument regarding quantisation of coupling constants applies to the double bubble just the same as to a closed manifold.

For more general types of bubble network, presumably the quantisation argument should also apply but a less crude proof is required. (It may help to think in terms of cohomology of Chern-Simons forms on chains rather than manifolds.)

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8This is the branching universe topology discussed in Ref. 28.
4.3 Some mathematical subtleties

In the context of topological field theory \[23\] one encounters various situations which are defined on graphs or other non-manifold structures. So the present discussion is not new in that sense. In what sense is our approach different?

A common approach is to assign some “colour” or combinatorial information to the different pieces of a graph. For example, in the Ponzanno-Regge theory \[24\], an irreducible representation of SU(2) is assigned to each of the edges on the skeleton of some triangulated manifold. In another, more recent, proposal \[25\], the states are related to the adjacency matrix of sub-graphs. In contrast to these approaches, we introduced an action depending on local fields on the walls. In this, it is similar to Plateu’s problem, in which one is interested in defining the volume form over bubble networks. Minimising this volume gives a discrete set of solutions, with simple algebraic rules like \[1\] for the intersections.

It is true that our model also gives discrete degrees of freedom on the walls and algebraic relations like \[21\], \[29\] and \[40\] for the intersections, but this is an accident of CS theory in three dimensions. We have seen that an action involving local fields and their derivatives can be well defined. So in principle one can generalise to a theory which has local degrees of freedom on the walls. For many theories one may run into trouble because the space is not a Hausdorff manifold and therefore the derivative of the fields across the intersections is not defined. In a second order field theory, like the Klein-Gordon system, it seems that the situation is hopeless because one encounters the second normal derivative of the field; likewise for 3 dimensional gravity in the second order metric formalism. In the case of CS theory we encounter only a first normal derivative which, due to Stokes’ theorem, does not cause problems. There should be no problems in generalising to higher-dimensional CS theories, which do have local degrees of freedom \[8\], and perhaps to other theories such as GR in its first-order formalism and generalisations thereof \[26\], \[27\].

4.4 Bubbles in a different number of dimensions

The procedure for defining an action for Chern-Simons bubbles is not special to three dimensions. CS theory is defined in all odd dimensions, where one could also expect to find other interesting theories of bubbles. It can be seen that, assuming the form of the action to be given like in \[33\] as a sum of transgressions, the relation \[51\] holds for all higher dimensions as well.

The simplest example of a foam would be a one-dimensional theory for the abelian group U(1). A bubble complex made of one-dimensional CS pieces can be defined as follows. Let \( M_i \) be a collection of 1-dimensional open manifolds. We introduce the co-ordinate \( s_i \) such that the manifold is given by the line interval \( s_i \in (0,1) \). The boundary of \( M_i \) is thus \( \{ s_i = 1 \} - \{ s_i = 0 \} \). To each manifold we assign a single U(1) connection \( A^{(i)} \). The bubble complex is then made by joining pieces together at the boundaries in an arbitrary way. Writing the connection 1-form as \( A = a(s)ds \), the one-dimensional Chern-Simons action for a triple intersection reads

\[
I[A^{(1)}, A^{(2)}, A^{(3)}] = \sum_{i=1}^{3} \epsilon_i k_i \int_{M_i} A^{(i)} .
\]

In this case, there is no need of a boundary term at the vertex. The condition of gauge invariance

\[9\]A space with branching is either: i) a Hausdorff topological space which is not a manifold or; ii) a manifold which is not Hausdorff, depending on how one defines the topology \[29\].
under independent gauge transformations of each $A^{(i)}$ that is continuous at the vertex, implies

$$\sum_{i=1}^{3} \epsilon_i k_i = 0. \quad (51)$$

Again this implies that $\eta$ is an indefinite “metric”.

5 Summary

Chern-Simons theory is normally defined on a manifold, but here we have argued that the theory is also perfectly well defined on a cell complex, such as a network of bubbles, which is not a Hausdorff manifold.

The powerful requirement of gauge invariance determines the action and gives the conservation law for the levels at each intersection:

$$(\sum k)_{\text{out}} = (\sum k)_{\text{in}}$$

where “in” or “out” refer to those walls for which the Chern-Simons term comes with a plus or a minus sign, respectively (with respect to the orientation $\partial M_i = +\Sigma$).

The action principle, with unconstrained independent variation of all gauge fields at the intersection, leads to matching conditions for the connection. In a CS theory for a $U(1)$ connection in the presence of an electromagnetic point source, this matching establishes a relation among the charges. For an $SO(2,2)$ connection in $2 + 1$ dimensions, the black hole masses are related.

We found evidence of a qualitative distinction between branching intersections, with only one “incoming” wall, and scattering intersections, with multiple ingoing and outgoing walls. In the case of a three-way branching the condition was that the connections match. For the example of three walls containing charges, the condition was that all three charges are equal. In the case of a four-way scattering, a more general matching condition (49) is obtained.

Several connections to similar an possible related systems seem to deserve further study. In particular, the curious similarity between the matching for the levels (19,20) and the tension in intersecting $D$-branes as in [17] or [30], in which the Chern-Simons couplings, through requirements of charge conservation, seem to play an important role in determining what kinds of intersections are allowed. There is also some similarity between our work and the Chern-Simons membranes considered in refs [31] and [32], and with work on discontinuous connections and Euler densities [33].

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