A few notes on Lorentz spaces

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1 Introduction

In the sequel, we recall and comment some classical results on the non-increasing rearrangement and Lorentz spaces. There are papers in the existing literature that seemed to have been bypassed as regards its contractive property in $L^p$ spaces. Also, we provide detailed proofs and a few properties that does not seem to arise in the existing literature.

2 Framework of the study

In this section, we have collected properties on the decreasing rearrangement, and on Lorentz spaces $L^{p,q}(\mathbb{R}^d)$, $1 \leq p < +\infty$, $1 \leq q < +\infty$, $d \geq 1$. We have tried to give the detailed proofs.

Définition 2.1. Distribution function

Let us denote by $(E, \mu)$ a measured space. If one denotes by $f$ a real-valued, measurable function, $\mu$–finite a.e., we introduce the positive-valued distribution function $\mu_f$, defined on $\mathbb{R}^+$, such that, for any positive number $\lambda$:

$$\mu_f(\lambda) = \mu(\{x \mid |f(x)| > \lambda\})$$

which can also be written as:

$$\mu_f(\lambda) = \inf_{s > 0} \{s \mid \mu(\{x \mid |f(x)| > s\}) \leq \lambda\}$$

For any strictly positive number $\sigma$, we set:

$$m(\sigma, f) = \text{mes}(\{x \mid |f(x)| > \sigma\})$$

Property 2.1. Properties of the distribution function

The distribution function $\mu_f(\lambda)$ is non-increasing, and right-continuous on $[0, +\infty[$.
Proof. Let us consider a positive number $\lambda_0$. As in [BSSS], we set, for any real positive number $\lambda$:

$$E(\lambda) = \left\{ x \in \mathbb{R}^d \mid |f(x)| > \lambda \right\}$$

As $\lambda$ increases, the sets $E(\lambda)$ decrease; moreover:

$$E(\lambda_0) = \bigcup_{\lambda > \lambda_0} E(\lambda) = \bigcup_{n=1}^{+\infty} E\left(\lambda_0 + \frac{1}{n}\right)$$

In order to prove the right-continuity, one requires to show that:

$$\lim_{\lambda \to 0^+} \mu_f(\lambda_0 + \lambda) = \mu_f(\lambda_0)$$

but since the mapping $\lambda \mapsto \mu_f(\lambda)$ is non-increasing, the monotone convergence theorem yields the right-continuity:

$$\lim_{n \to +\infty} \mu_f\left(\lambda_0 + \frac{1}{n}\right) = \lim_{n \to +\infty} \mu\left(\lambda_0 + \frac{1}{n}\right) = \mu\left(E(\lambda_0)\right) = \mu_f(\lambda_0)$$

$\square$

Définition 2.2. Equimeasurable functions

Let us denote by $(E, \mu)$ and $(F, \nu)$ two measured spaces, $f$ a real-valued function defined on $E$, $g$ a real-valued function defined on $F$. The functions $f$ and $g$ are said to be equimeasurable if:

$$\forall t > 0 : \mu \left\{ x \in E : |f(x)| > t \right\} = \nu \left\{ y \in F : |g(y)| > t \right\}$$

Définition 2.3. Symmetric rearrangement of a set of $\mathbb{R}^d$

Let $V$ be a measurable set of finite volume in $\mathbb{R}^d$. Its symmetric rearrangement $V^*$ is the open centered ball, the volume of which agrees with $V$:

$$V^* = \left\{ x \in \mathbb{R}^d \mid \text{Vol}_{\mathbb{B}_d} \cdot |x|^d < \text{Vol}_V \right\}$$

where $\text{Vol}_{\mathbb{B}_d}$ denotes the volume of the unit ball of $\mathbb{R}^d$:

$$\text{Vol}_{\mathbb{B}_d} = \frac{\pi^d}{\Gamma\left(\frac{d}{2} + 1\right)}$$

Définition 2.4. Symmetric decreasing rearrangement

Let us consider a real-valued, non-negative, measurable function $f$, defined on $\mathbb{R}^d$, which vanishes at infinity, in the sense where

$$\text{mes}\left(\left\{ x \in \mathbb{R}^d \mid f(x) > t \right\}\right) < +\infty \quad \forall t > 0$$

The symmetric decreasing rearrangement of $f$ is the function $f^*$, defined on $\mathbb{R}^d$, positive, measurable, such that, for any $x$ in $\mathbb{R}^d$:
\[ f^*(x) = \int_0^{+\infty} \mathbb{1}_{\{y : f(y) > t\}}(x) \, dt \]

where \( \mathbb{1}_{\{y : f(y) > t\}} \) denotes the characteristic function of the set \( \{y : f(y) > t\} \).

**Définition 2.5. Decreasing rearrangement**

If one denotes by \( f \) a real-valued, measurable function, \( \mu \)-finite a.e., the (scalar) decreasing rearrangement \( f^\# \), is the positive-valued function, defined, for any strictly positive number \( t \), through:

\[
\inf_{\lambda > 0} \left\{ \lambda \left| \begin{array}{l}
\mu_{f}(\lambda) \leq t \\
\end{array} \right. \right\}
\]

with the convention:

\[
\inf\emptyset = +\infty
\]

**Exemple 2.1. Distribution function and rearrangement of a simple function**

Let us denote by \( n \) a natural integer, and \( a_1, \ldots, a_n \) real numbers such that:

\[ a_1 < \ldots < a_n \]

\( I_1, \ldots, I_n \) are pairwise disjoint intervals of \( \mathbb{R} \). We define the function \( f \), from \( \mathbb{R} \) to \( \mathbb{R} \), such that:

\[ f = \sum_{i=1}^{n} a_i \mathbb{1}_{I_i} \]

where, for \( 1 \leq i \leq n_f \), \( \mathbb{1}_{I_i} \) is the characteristic function of \( I_i \):

\[
\forall x \in \mathbb{R}^d : \mathbb{1}_{I_i}(x) = \begin{cases} 
1 & \text{if} \quad x \in I_i \\
0 & \text{otherwise}
\end{cases}
\]

Then:

\( \Rightarrow \) For any \( \lambda \geq a_n \):

\[ \mu_f(\lambda) = 0 \]

\( \Rightarrow \) For any \( \lambda \in [a_{n-1}, a_n] \):

\[ |f(x)| > \lambda \iff x \in I_n \]

which leads to:

\[ \mu_f(\lambda) = \text{mes } I_n \]
For any $\lambda \in [a_{n-2}, a_{n-1}[$:

$$|f(x)| > \lambda \iff x \in I_{n-1} \cup I_n$$

One thus has:

$$\mu_f(\lambda) = \text{mes} I_{n-1} + \text{mes} I_n$$

For any $\lambda \in [a_i, a_{i+1}[$:

$$x \in \{x \mid |f(x)| > \lambda\} \iff x \in I_i \cup \ldots \cup I_n$$

This yields:

$$\mu_f(\lambda) = \text{mes} I_{i+1} + \ldots + \text{mes} I_n$$

If one sets $a_{n+1} = 0$, one gets, by induction:

$$\mu_f = \sum_{i=1}^{n} \left( \sum_{k=i}^{n} \text{mes} I_k \right) \mathbb{1}_{[0,a_i]}$$

which can be written as:

$$\mu_f = \sum_{i=1}^{n} m_i \mathbb{1}_{[a_{i-1},a_i]}$$

with the convention:

$$a_0 = 0$$

and where, for any $i$ in $\{1, \ldots, n\}$:

$$m_i = \sum_{k=i}^{n} \text{mes} I_k$$

One also has:

$$f^\# = \sum_{i=1}^{n} a_i \mathbb{1}_{[m_{i-1},m_i]}$$

with the convention:

$$m_0 = 0$$

Figures 1, 2, 3 display the graphs of $f$, $\mu_f$ et $f^\#$ in the case $n = 3$:
Property 2.2. Some properties of the decreasing rearrangement

Let us denote by $f$ a real-valued, measurable function, $\mu$—finite a.e. Then:

i. For any strictly positive number $t$, and any positive number $\lambda$:

$$f^\#(t) > \lambda \iff \mu_f(\lambda) > t$$

ii. The function $f^\#$ is non increasing.

iii. If the distribution function $\mu_f$ is strictly decreasing and continuous from $I_{\mu_f} \subset \mathbb{R}$ into $\mathbb{R}^+$, then $f^\#$ is the inverse function of $\mu_f$ on $\mathbb{R}^+$. Moreover, the decreasing rearrangement $f^\#$ is right-continuous.
Figure 3: The graph of the decreasing rearrangement $f^\#$ for $n = 3.$

Proof. i. Let us denote by $\lambda$ a positive number. Then, if $\mu_f(\lambda) > t$, one gets, due to the fact that $\mu_f$ is non-increasing:

\[
\lambda > \inf \left\{ \nu \mid \mu_f(\nu) \leq t \right\}
\]

i.e.:

\[
\lambda > f^\#(t)
\]

Conversely, let us assume that $\lambda > f^\#(t)$, i.e.:

\[
\lambda > \inf \left\{ \nu \mid \mu_f(\nu) \leq t \right\}
\]

The distribution function $\mu_f$ being non-increasing, one deduces: $\mu_f(\lambda) > t$.

For any strictly positive number $t$:

\[
f^\#(t) > \lambda \iff \mu_f(\lambda) > t
\]

ii. The function $f^\#$ is decreasing; for $t_1 \leq t_2$:

\[
\{ x \mid |f(x)| > t_1 \} \subseteq \{ x \mid |f(x)| > t_2 \}
\]

Thus:

\[
\mes \left( \{ x \mid |f(x)| > t_1 \} \right) \leq \mes \left( \{ x \mid |f(x)| > t_2 \} \right)
\]

i.e.:

\[
f^\#(t_1) \leq f^\#(t_2)
\]
iii. The fact that the decreasing rearrangement \( f^\# \) is right-continuous follows from the fact that it is the distribution function of \( \mu_f \), with respect to the Lebesgue measure on \( \mathbb{R}^+ \):

\[
\forall t \geq 0 : \quad f^\# (t) = \inf \{ \lambda \mid \mu_f (\lambda) \leq t \} = \sup \{ \lambda \mid \mu_f (\lambda) > t \} = m_{\mu_f} (t)
\]

Thus, it is right-continuous.

\[\square\]

**Property 2.3.** *Hardy-Littlewood inequality* (One may refer to [BS88])

Let us denote by \( f \) and \( g \) two measurable, real-valued functions defined on \( \mathbb{R} \), which vanish at infinity. Then

\[
\int_{\mathbb{R}^d} f(x) g(x) \, dx \leq \int_{\mathbb{R}^+} f^\# (s) g^\# (s) \, ds
\]

**Proof.** The proof is made in the case of non-negative functions \( f \) and \( g \). Due to the monotone convergence theorem, one just needs to consider the case of a simple function \( f \), of the form:

\[
f = \sum_{i=1}^{n_f} f_i \mathbb{1}_{A_i}
\]

where \( n_f \) is a positive integer, and, for \( 1 \leq i \leq n_f \), \( f_i \) is a positive number; \( A_1, \ldots, A_{n_f} \) are measurable sets such that:

\[
A_1 \subset A_2 \subset \ldots \subset A_{n_f}
\]

One has then:

\[
f^\# = \sum_{i=1}^{n_f} f_i \mathbb{1}_{[0, \mu(A_i)]}
\]

which leads to:

\[
\int_{\mathbb{R}^d} f(x) g(x) \, dx = \sum_{i=1}^{n_f} f_i \int_{A_i} g(x) \, dx \\
\leq \sum_{i=1}^{n_f} f_i \int_{[0, \mu(A_i)]} g^\# (t) \, dt \\
\leq \int_{\mathbb{R}^+} \sum_{i=1}^{n_f} f_i \mathbb{1}_{[0, \mu(A_i)]} (t) g^\# (t) \, dt \\
= \int_{\mathbb{R}^+} f^\# (t) g^\# (t) \, dt
\]

\[\square\]
Proposition 2.4. *Contractive properties of the non-increasing rearrangement*

Let us consider two real-valued, measurable functions $f$ and $g$, defined on $\mathbb{R}^d$, which vanishes at infinity. For any strictly positive number $s$:

$$\|f^# - g^#\|_{L^p} \leq \|f - g\|_{L^p}$$

**Remark 2.1.** We would like to point out, as it appears in [LL97], that this specific property is a generalization of a theorem of G. Chiti and C. Pucci [CP79] and [CT80]. It however claimed to be proved for the first time in the 1986 paper [CZR86], which does not mention at all the work of G. Chiti and C. Pucci.

**Property 2.5. Evaluation and algebraic properties**

Let us denote by $f$ and $g$ two real-valued, measurable functions, $\mu$--finite a.e. Then, for all $(t_1, t_2)$ belonging to $\mathbb{R}_+^2$:

$$(fg)^#(t_1 + t_2) \leq f^#(t_1)g^#(t_2), \quad (f + g)^#(t_1 + t_2) \leq f^#(t_1) + g^#(t_2)$$

**Proof.** For any $(t_1, t_2) \in \mathbb{R}_+^2$, one can legitimately assume that the quantities $f^#(t_1)g^#(t_2)$ and $f^#(t_1) + g^#(t_2)$ are respectively finite, since there is, otherwise, nothing to prove.

One has, first:

$$\{x \mid |f(x)g(x)| > \lambda_1 \lambda_2\} \subset \{x \mid |f(x)| > \lambda_1\} \cup \{x \mid |g(x)| > \lambda_2\}$$

Hence:

$$\text{mes} \{x \mid |f(x)g(x)| > \lambda_1 \lambda_2\} \leq \text{mes} \{x \mid |f(x)| > \lambda_1\} \cup \{x \mid |g(x)| > \lambda_2\}$$

i.e.:

$$\text{mes} \{x \mid |f(x)g(x)| > \lambda_1 \lambda_2\} \leq \text{mes} \{x \mid |f(x)| > \lambda_1\} + \text{mes} \{x \mid |g(x)| > \lambda_2\}$$

One has then:

$$\mu_{fg}(\lambda_1 \lambda_2) \leq \mu_f(\lambda_1) + \mu_f(\lambda_2)$$

which, again due to the fact that the rearrangement is non-increasing, leads to:

$$(fg)^#(\mu_f(\lambda_1) + \mu_f(\lambda_2)) \leq (fg)^#(\mu_{fg}(\lambda_1 \lambda_2))$$

i.e.:

$$\mu_{fg}(\lambda_1 \lambda_2) \leq \lambda_1 \lambda_2$$

If one sets $t_1 = \mu_f(\lambda_1)$, $t_2 = \mu_g(\lambda_2)$, or, equivalently:
\[ \lambda_1 = f^\#(t_1) \quad , \quad \lambda_2 = g^\#(t_2) \]

one gets:

\[ (f \, g)^\#(t_1 + t_2) \leq f^\#(t_1) \, g^\#(t_2) \]

Let us notice then that, for any couple of positive numbers \((u, v)\), and for any \((a, b) \in \mathbb{R}^2\), one has:

\[ |a + b| > u + v \Rightarrow |a| > u \quad \text{or} \quad |b| > v \]

since:

\[ |a| \leq u \quad \text{and} \quad |b| \leq v \Rightarrow |a + b| \leq |a| + |b| \leq u + v \]

\[ f^\#(t_1) + g^\#(t_2) < |f(x) + g(x)| < |f(x)| + |g(x)| \]

One has then the natural embedding

\[
\begin{align*}
\{x \mid |f(x) + g(x)| > f^\#(t_1) + g^\#(t_2)\} & \subset \{x \mid |f(x)| > f^\#(t_1)\} \cup \{x \mid |g(x)| > g^\#(t_2)\}
\end{align*}
\]

Thus:

\[
\mu_{f+g} \left( f^\#(t_1) + g^\#(t_2) \right) = \max \left\{ x \mid |f(x) + g(x)| > f^\#(t_1) + g^\#(t_2) \right\} \\
\quad \leq \max \left\{ x \mid |f(x)| > f^\#(t_1) \right\} + \max \left\{ x \mid |g(x)| > g^\#(t_2) \right\} \\
\quad = \mu_f \left( f^\#(t_1) \right) + \mu_g \left( g^\#(t_2) \right) \\
\quad \leq \mu_{f+g} \left( f^\#(t_1) + g^\#(t_2) \right)
\]

Since the rearrangement \((f + g)^\#\) is decreasing, it yields:

\[ (f + g)^\#(t_1 + t_2) \leq (f + g)^\# \left( \mu_{f+g} \left( f^\#(t_1) + g^\#(t_2) \right) \right) = (f + g)^\# \left( \mu_{f+g} \left( f^\#(t_1) + g^\#(t_2) \right) \right) = f^\#(t_1) + g^\#(t_2) \]

\[ \square \]

**Définition 2.6. Maximal function**

Let us denote by \( f \) a function defined on \( \mathbb{R} \), real-valued, measurable, finite a.e. We introduce the maximal function \( f^{**} \), defined, for any strictly positive number \( t \), by:

\[ f^{**}(t) = \frac{1}{t} \int_0^t f^\#(s) \, ds \]

**Property 2.6.** Let us denote by \( f \) a real-valued, measurable function, \( \mu \)-finite a.e. The maximal function \( f^{**} \) is non-increasing on \( \mathbb{R}^*_+ \). Moreover, for any strictly positive number \( t \):

\[ f^\#(t) \leq f^{**}(t) \]
Proof. The maximal function $f^{**}$ is non-increasing on $\mathbb{R}^*_+$ because, for any set of strictly positive numbers $(t_1, t_2)$ such that $t_1 \leq t_2$:

$$f^{**}(t_2) = \frac{1}{t_2} \int_0^{t_2} f^\#(s) \, ds$$

\[= \frac{1}{t_2} \int_0^{t_1} f^\#(s) \, ds + \frac{1}{t_2} \int_{t_1}^{t_2} f^\#(s) \, ds \]

\[\leq \frac{1}{t_2} \int_0^{t_1} f^\#(s) \, ds + \frac{1}{t_2} \int_{t_1}^{t_2} f^\#(t) \, ds \]

\[= \frac{1}{t_2} \int_0^{t_1} f^\#(s) \, ds + \frac{(t_2 - t_1)}{t_2} f^\#(t) \]

\[= \frac{t_1}{t_2} f^{**}(t_1) + \frac{t_2 - t_1}{t_1 t_2} \int_0^{t_1} f^\#(s) \, ds \]

\[\leq \frac{t_1}{t_2} f^{**}(t_1) + \frac{t_2 - t_1}{t_1 t_2} \int_0^{t_1} f^\#(s) \, ds \]

\[= \frac{t_1}{t_2} f^{**}(t_1) + \frac{t_2 - t_1}{t_2} f^{**}(t_1) \]

\[= f^{**}(t_1) \]

One can also note that, for any strictly positive number $t$:

$$\frac{d}{dt} f^{**}(t) = \frac{f^\#(t)}{t} - \frac{1}{t^2} \int_0^t f^\#(s) \, ds$$

\[= \frac{1}{t^2} \left\{ t f^\#(t) - \int_0^t f^\#(s) \, ds \right\} \]

\[\leq \frac{1}{t^2} \left\{ t f^\#(t) - f^\#(t) \int_0^t ds \right\} \]

\[= 0 \]

since the rearrangement $f^\#$ is non-increasing, which yields:

$$\frac{d}{dt} f^{**}(t) \leq 0$$

Finally, one has obviously:

$$f^{**}(t) = \frac{1}{t} \int_0^t f^\#(s) \, ds$$

\[\geq \frac{1}{t} \int_0^t f^\#(t) \, ds \]

\[= \frac{1}{t} t f^\#(t) \]

\[= f^\#(t) \]

\[\square\]

Property 2.7. The maximal function is sub-additive: if $f$ and $g$ are real-valued, measurable function, $\mu$–finite a.e., then:

$$\forall t > 0 : \ (f + g)^{**}(t) \leq f^{**}(t) + g^{**}(t)$$

Proof. Since:
\( \forall (t_1, t_2) \in \mathbb{R}_+^2 : \ (f + g)\# (t_1 + t_2) \leq f\#(t_1) + g\#(t_2) \)

one deduces, for any strictly positive number \( t \):

\[
(f + g)^{**}(t) = \frac{1}{t} \int_0^t (f + g)^\#(s) \, ds
\leq \frac{1}{t} \int_0^t \left\{ f\#\left(\frac{s}{2}\right) + g\#\left(\frac{s}{2}\right)\right\} \, ds
\leq \frac{1}{t} \int_0^t \left\{ f\#(s) + g\#(s)\right\} \, ds
\leq 2 \frac{1}{t} \int_0^t f\#(s) \, ds + \frac{2}{t} \int_0^t g\#(s) \, ds
= \frac{1}{t} \int_0^t f\#(s) \, ds + \frac{1}{t} \int_0^t g\#(s) \, ds
= f^{**}(t) + g^{**}(t)
\]
since the rearrangements \( f\# \) and \( g\# \) are both decreasing functions.

\[\blacksquare\]

**Proposition 2.8. From the decreasing rearrangement towards the symmetric one**

Let us denote by \( f \) a real-valued, measurable function, defined on \( \mathbb{R}^d \), which vanishes at infinity.

If \( \omega_d \) is the volume of the unit ball of \( \mathbb{R}^d \), then, for any \( x \) of \( \mathbb{R}^d \):

\[
f^*(x) = f\#(\omega_d |x|^d)
\]

Especially:

\[
\forall t \in \mathbb{R} : \ f^*(t) = \frac{1}{2} f\#(|t|)
\]

**Property 2.9.** Let us denote by \( f \) a real-valued, measurable function, defined on \( \mathbb{R}^d \), which vanishes at infinity. Then, for any \( t > 0 \):

\[
\text{mes}\left( \left\{ x \in \mathbb{R}^d \mid |f(x)| > t \right\} \right) = \text{mes}\left( \left\{ x \in \mathbb{R}^d \mid |f^*(x)| > t \right\} \right) = \text{mes}\left( \left\{ s \in \mathbb{R} \mid |f\#(s)| > t \right\} \right)
\]

**Property 2.10. Rearrangement and dilatation**

Let us denote by \( f \) a real-valued, measurable function, defined on \( \mathbb{R}^d \), which vanishes at infinity. \( \Lambda \) is a strictly positive real number. One has then:

\[
(f(\Lambda \cdot))\# = f\#(\Lambda^d \cdot)
\]
The decreasing rearrangement $f^*$ is such that:

$$(f(\Lambda \cdot))^* = f^*(\Lambda \cdot)$$

The maximal function $f^{**}$ satisfies:

$$\forall t > 0 : (f(\Lambda \cdot))^{**}(t) = f^{**}(\Lambda^d t)$$

Proof.

$$(f(\Lambda \cdot))^\#(t) = \inf_{\sigma > 0} \{ \sigma \mid m(\sigma, f(\Lambda \cdot)) \leq t \}$$

For any strictly positive number $\sigma$:

$$m(\sigma, f(\Lambda \cdot)) = \mes \left( \{ x \mid |f(\Lambda x)| > \sigma \} \right)$$

$$= \int_{\mathbb{R}^d} 1_{|f(\Lambda x)| > \sigma} dx$$

$$= \int_{\mathbb{R}^d} 1_{|f(x)| > \sigma} \Lambda^{-d} dx$$

$$= \Lambda^{-d} m(\sigma, f)$$

Thus:

$$(f(\Lambda \cdot))^\#(t) = \inf_{\sigma > 0} \{ \sigma \mid m(\sigma, f(\Lambda \cdot)) \leq t \}$$

$$= \inf_{\sigma > 0} \{ \sigma \mid \Lambda^{-d} m(\sigma, f) \leq t \}$$

$$= \inf_{\sigma > 0} \{ \sigma \mid m(\sigma, f(\Lambda \cdot)) \leq \Lambda^d t \}$$

$$= f^\#(\Lambda^d t)$$

Hence:

$$f^*(\Lambda x) = f^\#(\omega_d \Lambda^d |x|^d)$$

Or:

$$(f(\Lambda \cdot))^*(x) = (f(\Lambda \cdot))^\#(\omega_d |x|^d) = f^\#(\omega_d \Lambda^d |x|^d) = f^*(\Lambda x)$$

Also, for any strictly positive number $t$:

$$\int_0^t (f(\Lambda \cdot))^\#(s) ds = \int_0^t f^\#(\Lambda^d s) ds = \int_0^{\Lambda^d t} f^\#(s) \Lambda^{-d} ds$$

$$\frac{1}{t} \int_0^t (f(\Lambda \cdot))^\#(s) ds = \frac{1}{\Lambda^d t} \int_0^{\Lambda^d t} f^\#(s) ds = f^{**}(\Lambda^d t)$$

i.e.:

$$(f(\Lambda \cdot))^{**}(t) = f^{**}(\Lambda^d t)$$

Moreover:

$$m(\sigma, \Lambda f) = \mes \left( \{ x \mid |f(x)| > \frac{\lambda}{\sigma} \} \right)$$

$$f^\#(t) = \inf_{\sigma > 0} \{ \sigma \mid m(\sigma, f) \leq t \}$$

$$(fg)^\#(t) = m(t, fg) = \inf_{\sigma > 0} \{ \sigma \mid m(\sigma, fg) \leq t \}$$
Also:

\[
m(u^p, f) = \text{mes}\left\{ x \mid |f(x)| > u^p \right\} = \text{mes}\left\{ x \mid |f(x)|^p > u \right\} = \inf_{s > 0} \left\{ s \mid \text{mes}\left\{ x \mid |f(x)| > s^p \right\} \leq u \right\} = \inf_{s > 0} \left\{ s^p \mid \text{mes}\left\{ x \mid |f(x)| > s \right\} \leq u \right\} = m^p(u, f) = (f^#)^p(u)
\]

For any strictly positive number \( \Lambda \):

\[
m(t, f(\Lambda \cdot)) = \text{mes}\left\{ x \mid |\Lambda x| > t \right\} = \int_{\mathbb{R}^d} 1_{|f(\Lambda x)| > t} \, dx = \int_{\mathbb{R}^d} 1_{|f(x)| > t} \Lambda^{-d} \, dx = \Lambda^{-d} m(t, f)
\]

\[\square\]

**Property 2.11. Rearrangement and invariance of the \( L^p \) norm**

Let us consider \( p \geq 2 \); we denote by \( f \) a real-valued, measurable function, defined on \( \mathbb{R}^d \), which vanishes at infinity. If the function \( f \) belongs to \( L^p(\mathbb{R}^d) \), the symmetric decreasing rearrangement \( f^* \) belongs to \( L^p(\mathbb{R}^d) \), the decreasing rearrangement \( f^\# \) belongs to \( L^p(\mathbb{R}) \), and, for any \( t > 0 \), one has:

\[
\|f^*\|_{L^p(\mathbb{R}^d)} = \|f^\#\|_{L^p(\mathbb{R})} = \|f\|_{L^p(\mathbb{R})}
\]

\[
\|f\|_{L^p(\mathbb{R}^d)} = \int_0^{+\infty} \left( u^p f^\#(u) \right)^p \frac{du}{u}
\]

**Proof.** The Fubini theorem leads to:

\[
\|f\|_{L^p(\mathbb{R}^d)} = \int_{\mathbb{R}^d} (f(x))^p \, dx = \int_{\mathbb{R}^d} \left\{ \int_0^{+\infty} 1_{(f(x))^p > t} \, dt \right\} \, dx = \int_0^{+\infty} \left\{ \int_{\mathbb{R}^d} 1_{(f(x))^p > t} \, dx \right\} \, dt = \int_0^{+\infty} \text{mes}\left\{ x \mid (f(x))^p > t \right\} \, dt = \int_0^{+\infty} \text{mes}\left\{ x \mid f(x) > t \right\} \, p t^{p-1} \, dt = \int_0^{+\infty} \mu_f(t) \, p t^{p-1} \, dt = \|f^*\|_{L^p(\mathbb{R}^d)}
\]

Also:
An alternate proof of the relations between those norms can be made using the monotone convergence theorem. This way, one just needs to consider the case of a simple function $f$, of the form:

$$f = \sum_{i=1}^{n_f} f_i \mathbb{1}_{A_i}$$

where $n_f$ is a positive integer, and, for $1 \leq i \leq n_f$, $f_i$ is a positive number; $A_1, \ldots, A_{n_f}$ are measurable sets such that:

$$A_1 \subset A_2 \subset \ldots \subset A_{n_f}$$

One has then:

$$f^\# = \sum_{i=1}^{n_f} f_i \mathbb{1}_{[0,\mu(A_i)]}$$

For $1 \leq i \leq n_f$, the equimeasurability of $f_i$ and $f^\#_i$ can be written, for any positive number $\lambda$, as:

$$\mu_{f_i}(\lambda) = \mu_{f^\#_i}(\lambda)$$

which leads to:

$$\mu_f(\lambda) = \mu_{f^\#}(\lambda)$$

and:

$$\|f\|_{L^p(\mathbb{R}^d)}^p = \int_{\mathbb{R}^d} |f(x)|^p \, dx$$

$$= \int_{\mathbb{R}^d} \left\{ \int_0^{|f(x)|} t^{p-1} \, dt \right\} \, dx$$

$$= \int_0^{+\infty} t^{p-1} \left\{ \int_{\mathbb{R}^d} \mathbb{1}_{|f(x)| > t} \, dx \right\} \, dt$$

$$= \int_0^{+\infty} t^{p-1} \text{mes} \left( \{ x \mid |f(x)| > t \} \right) \, dt$$

$$= \int_0^{+\infty} t^{p-1} \inf_{\sigma > 0} \{ \sigma \mid m(\sigma, f^\#) \leq t \} \, dt$$

$$= \int_0^{+\infty} t^{p-1} \inf_{\sigma > 0} \{ \sigma^p \mid m(\sigma^p, f^\#) \leq t \} \, dt$$

$$= \int_0^{+\infty} t^{p-1} m(t, f^\#) \, dt$$

$$= \int_0^{+\infty} m \left( u^\frac{1}{p}, f^\# \right) \, du$$

$$= \int_0^{+\infty} \left( f^\# \right)^p (u) \, du$$

$$= \left\| f^\# \right\|_{L^p(\mathbb{R})}$$

$$= \int_0^{+\infty} u \left( f^\#(u) \right)^p \frac{du}{u}$$

$$= \int_0^{+\infty} \left( \frac{1}{u^p} f^\#(u) \right)^p \frac{du}{u}$$

$$= \int_0^{+\infty} \frac{1}{u^p} f^\#(u) \, du$$

$$= \mu_{f^\#}(\lambda)$$

$$= \mu_f(\lambda)$$

and:

$$\|f\|_{L^p(\mathbb{R}^d)} = \mu_f(\lambda)$$

$$= \mu_{f^\#}(\lambda)$$

$$= \mu_{f^\#}(\lambda)$$

$$= \mu_f(\lambda)$$
\[
\mu_p(\lambda) = \mu_f(\lambda^{\frac{1}{p}}) = \mu_{f^#}(\lambda^{\frac{1}{p}}) = \mu_{(f^#)^p}(\lambda)
\]
i.e.:
\[
\mu \left( \{ x \mid |f^p(x)| > \lambda \} \right) = \mu \left( \{ x \mid |(f^#)^p(x)| > \lambda \} \right)
\]

Starting from the above latter property, Lorentz spaces can be introduced very naturally:

**Définition 2.7.** Let \( p \) and \( q \) denote two strictly positive numbers such that \( p > 1, q \geq 1 \). The Lorentz space \( L^{p,q}(\mathbb{R}^d) \) is defined as the set of real-valued, measurable functions \( f \), defined on \( \mathbb{R}^d \), such that:

\[
\|f\|_{L^{p,q}(\mathbb{R}^d)} = \left( \int_0^{+\infty} \left( t^{\frac{1}{p}} f^#(t) \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} < +\infty
\]

**Remark 2.2.** It is interesting to note that one can also define:

i. the Lorentz space \( L^{p,\infty}(\mathbb{R}^d) \) as the set of real-valued, measurable functions \( f \), defined on \( \mathbb{R}^d \), such that:

\[
\|f\|_{L^{p,\infty}(\mathbb{R}^d)} = \sup_{t>0} t^{\frac{1}{p}} f^#(t) < +\infty
\]

ii. the Lorentz space \( L^{\infty,\infty}(\mathbb{R}^d) \) as the set of real-valued, measurable functions \( f \), defined on \( \mathbb{R}^d \), such that:

\[
\|f\|_{L^{\infty,\infty}(\mathbb{R}^d)} = \sup_{t>0} f^#(t) < +\infty
\]

**Remark 2.3.** It is clear that:

\[
L^{p,p}(\mathbb{R}^d) = L^p(\mathbb{R}^d)
\]
since the \( L^p \) -norm is kept invariant by the rearrangement.

**Remark 2.4.** Lorentz spaces can be considered as "thinner" spaces than the Lebesgue ones; they make it possible to detect logarithmic correction, which can not be done with the classical \( L^p \) spaces.

**Exemple 2.2.** Let us consider, on the unit ball of \( \mathbb{R} \), functions of the form:
\[ f_{\alpha,\beta} : t \neq 0 \mapsto \frac{\left(\ln \frac{1}{|t|}\right)^\beta}{|t|^\alpha} = \frac{(-\ln |t|)^\beta}{|t|^\alpha} \]

\[ \|f_{\alpha,\beta}\|_{L^{p}(\mathbb{R})}^p = \int_{B_{0,1}(\mathbb{R})} f^p_{\alpha,\beta}(t) \, dt \]

\[ = \int_{B_{0,1}(\mathbb{R})} \left(\frac{\ln \frac{1}{|t|}}{|t|^\alpha}\right)^p dt \]

\[ = \int_{B_{0,1}(\mathbb{R})} \left(\frac{-\ln |t|}{t^\alpha}\right)^p dt \]

At stake are Bertrand integrals, of the form

\[ \int_{0}^{1} \frac{dt}{t^{\alpha_0} (-\ln t)^{\beta_0}} \]

When \( \alpha_0 < 1 \), no difference can be seen for distinct values of the parameter \( \beta \). It is not the case if one consider Lorentz norms, as it is illustrated in the following figures.

Figure 4: The \( \| \cdot \|_{L^{1,2}([0,1])}^2 \) norms of the function \( f_{-1,2} \) (in red) and \( f_{-1,4} \) (in green).

Remark 2.5. It is important to note that \( \| \cdot \|_{L^{p,q}(\mathbb{R}^d)} \) is not a norm, since the triangle inequality does not hold, for, in most cases, one cannot have:

\[ (f + g)^* \leq f^* + g^* \]

\( \| \cdot \|_{L^{p,q}(\mathbb{R}^d)} \) is just a quasi-norm.

The space \( L^{p,q}(\mathbb{R}^d) \) is not, thus, a Banach space. In order to norm \( L^{p,q}(\mathbb{R}^d) \), one has to consider, thanks to the sub-additivity of the maximal function:
The (obviously) vectorial space $L^{p,q}(\mathbb{R}^d)$ is, thus, a complete metric space. It is obvious if one considers the mapping

$$(f, g) \in L^{p,q}(\mathbb{R}^d) \times L^{p,q}(\mathbb{R}^d) \mapsto |||f - g|||_{L^{p,q}(\mathbb{R}^d)}$$

which is a distance over $L^{p,q}(\mathbb{R}^d)$. As for the mapping

$$(f, g) \in L^{p,q}(\mathbb{R}^d) \times L^{p,q}(\mathbb{R}^d) \mapsto \|f - g\|_{L^{p,q}(\mathbb{R}^d)}$$

one can also prove that it is a distance over $L^{p,q}(\mathbb{R}^d)$, which follows from the following comparison:

**Property 2.12. Comparison of the Lorentz norm and quasi-norm**

There exists a strictly positive constant $C_{p,q}$ such that, for any $f$ belonging to $L^{p,q}(\mathbb{R}^d)$:

$$|||f|||_{L^{p,q}(\mathbb{R}^d)} \leq C_{p,q} \|f\|_{L^{p,q}(\mathbb{R}^d)} \leq C_{p,q} |||f|||_{L^{p,q}(\mathbb{R}^d)}$$

**Proof.** Let us recall:

$$|||f|||_{L^{p,q}(\mathbb{R}^d)} = \left( \int_{0}^{+\infty} \left( \frac{1}{t^p} f^{**}(t) \right)^q \frac{dt}{t} \right)^{\frac{1}{q}}, \quad ||f||_{L^{p,q}(\mathbb{R}^d)} = \left( \int_{0}^{+\infty} \left( \frac{1}{t^p} f^{#}(t) \right)^q \frac{dt}{t} \right)^{\frac{1}{q}}$$

Due to Hardy’s generalized inequality, for any strictly positive number $T$, and any positive number $\alpha$ such that $\alpha < q - 1$:

$$\int_{0}^{T} \left( \int_{0}^{t} \frac{f^*(u) du}{t} \right)^q t^\alpha \, dt \leq \left( \frac{q}{q - 1 - \alpha} \right)^q \int_{0}^{T} \left( f^#(t) \right)^q t^\alpha \, dt$$
Thus, for $\alpha = \frac{q}{p}$:

$$\int_0^T \left( t^\frac{1}{p} \int_0^t f^*(u) \, du \right)^q \, dt \leq \left( \frac{q}{q - 1 + \frac{2}{p}} \right)^q \int_0^T \left( t^\frac{1}{p} f^#(t) \right)^q \, dt$$

i.e.:

$$\int_0^T \left( t^\frac{1}{p} f^{**}(t) \right)^q \, dt \leq \left( \frac{q}{q - 1 + \frac{2}{p}} \right)^q \int_0^T \left( t^\frac{1}{p} f^#(t) \right)^q \, dt$$

which leads to:

$$\int_0^{+\infty} \left( t^\frac{1}{p} f^{**}(t) \right)^q \, dt \leq \left( \frac{q}{q - 1 + \frac{2}{p}} \right)^q \int_0^{+\infty} \left( t^\frac{1}{p} f^#(t) \right)^q \, dt$$

and:

$$\left( \int_0^{+\infty} \left( t^\frac{1}{p} f^{**}(t) \right)^q \, dt \right)^\frac{1}{q} \leq \frac{q}{q - 1 + \frac{2}{p}} \int_0^{+\infty} \left( t^\frac{1}{p} f^#(t) \right)^q \, dt$$

i.e.:

$$\|f\|_{L^{p,q}(\mathbb{R}^d)} \leq \frac{q}{q - 1 + \frac{2}{p}} \|f\|_{L^{p,q}(\mathbb{R}^d)}$$

or:

$$\|f\|_{L^{p,q}(\mathbb{R}^d)} \leq \frac{p}{p + 1 - \frac{2}{q}} \|f\|_{L^{p,q}(\mathbb{R}^d)} = C_{p,q} \|f\|_{L^{p,q}(\mathbb{R}^d)}$$

The inequality

$$\|f\|_{L^{p,q}(\mathbb{R}^d)} \leq \|f\|_{L^{p,q}(\mathbb{R}^d)}$$

follows from the comparison between $f^#$ and the maximal function $f^{**}$ seen in the above (see property (2.6)).

\[ \square \]

**Property 2.13.** The Lorentz space $L^{p,q}(\mathbb{R}^d)$ is an homogeneous one; for any strictly positive number $\Lambda$, and any $f$ belonging to $L^{p,q}(\mathbb{R}^d)$:

$$\|f(\Lambda \cdot)\|_{L^{p,q}(\mathbb{R}^d)} = \Lambda^{-\frac{d}{p}} \|f\|_{L^{p,q}(\mathbb{R}^d)} \quad , \quad \|f(\Lambda \cdot)\|_{L^{p,q}(\mathbb{R}^d)} = \Lambda^{-d} \|f\|_{L^{p,q}(\mathbb{R}^d)}$$

Moreover:

$$\|\Lambda f\|_{L^{p,q}(\mathbb{R}^d)} = \Lambda \|f\|_{L^{p,q}(\mathbb{R}^d)} \quad , \quad \|\Lambda f\|_{L^{p,q}(\mathbb{R}^d)} = \Lambda \|f\|_{L^{p,q}(\mathbb{R}^d)}$$
Proof.

\[
\| f (\Lambda \cdot) \|^q_{L^p,q(\mathbb{R}^d)} = \int_0^{+\infty} \left( t^\frac{p}{d} (f (\Lambda \cdot))# (t) \right)^q \frac{dt}{t} \\
= \int_0^{+\infty} \left( t^\frac{p}{d} f#(\Lambda^d t) \right)^q \frac{dt}{t} \\
= \int_0^{+\infty} \Lambda^{-\frac{d}{d'}} \left( t^\frac{p}{d} f#(t) \right)^q \Lambda^{-d} \frac{dt}{t} \\
= \Lambda^{-\frac{d}{d'}} \int_0^{+\infty} \left( t^\frac{p}{d} f#(t) \right)^p \frac{dt}{t}
\]

Hence:

\[
\| f (\Lambda \cdot) \|^q_{L^p,q(\mathbb{R}^d)} = \Lambda^{-\frac{d}{p}} \| f \|^q_{L^p,q(\mathbb{R}^d)}
\]

For any strictly positive number \( t \):

\[
(f(\Lambda \cdot))^{**} (t) = \frac{1}{t} \int_0^t (f(\Lambda \cdot))# (s) ds \\
= \frac{1}{t} \int_0^t f#(\Lambda^d s) ds \\
= \frac{1}{t} \int_0^t \Lambda^{-d} f#(s) ds \\
= \Lambda^{-d} f^{**}(t)
\]

Thus:

\[
(f(\Lambda \cdot))^{**} = \Lambda^{-d} f^{**}
\]

and:

\[
\| \| f (\Lambda \cdot) \|^q_{L^p,q(\mathbb{R}^d)} \|_{L^p,q(\mathbb{R}^d)} = \int_0^{+\infty} \left( t^\frac{p}{d} (f (\Lambda \cdot))^{**} (t) \right)^q \frac{dt}{t} \\
= \int_0^{+\infty} \left( t^\frac{p}{d} \Lambda^{-d} f^{**}(t) \right)^q \frac{dt}{t} \\
= \int_0^{+\infty} \Lambda^{-dq} \left( t^\frac{p}{d} f^{**}(t) \right)^p \frac{dt}{t} \\
= \Lambda^{-dq} \| \| f \|^q_{L^p,q(\mathbb{R}^d)}
\]

Also:

\[
\| \Lambda f \|^q_{L^p,q(\mathbb{R}^d)} = \Lambda \| f \|^q_{L^p,q(\mathbb{R}^d)}
\]

\[\square\]

**Property 2.14.** Let us consider \( 0 \leq p < +\infty, 0 \leq q_1 < +\infty, 0 \leq q_2 < +\infty \). Then:

\[q_1 \leq q_2 \Rightarrow L^{p,q_1}(\mathbb{R}^d) \hookrightarrow L^{p,q_2}(\mathbb{R}^d)\]

**Proof.** [BS88]

Due to the fact that the rearrangement \( f# \) is non increasing, one gets, for any real positive number \( t \):
\[
\begin{align*}
\frac{1}{p} f^\#(t) &= \left\{ \frac{p}{q_1} \int_0^t \left( \frac{1}{s^p} f^\#(t) \right)^{q_1} \frac{ds}{s} \right\}^{\frac{1}{q_1}} \\
&\lesssim \left\{ \frac{p}{q_1} \int_0^t \left( \frac{1}{s^p} f^\#(s) \right)^{q_1} \frac{ds}{s} \right\}^{\frac{1}{q_1}} \\
&\lesssim \left( \frac{p}{q_1} \right)^{\frac{1}{q_1}} \|f\| L^{p,q_1}(\mathbb{R}^d) 
\end{align*}
\]

→ If \( q_1 = +\infty \), one gets:

\[
\|f\| L^{p,\infty}(\mathbb{R}^d) \leq \left( \frac{p}{q_1} \right)^{\frac{1}{q_1}} \|f\| L^{p,q_1}(\mathbb{R}^d)
\]

→ If \( q_1 < +\infty \), one gets:

\[
\|f\| L^{p,q_2}(\mathbb{R}^d) = \left\{ \int_0^t \left( \frac{1}{s^p} f^\#(t) \right)^{q_2} \frac{ds}{s} \right\}^{\frac{1}{q_2}} \\
\leq \|f\| L^{q_2,q_1}(\mathbb{R}^d) \left\{ \int_0^t \left( \frac{1}{s^p} f^\#(t) \right)^{q_1} \frac{ds}{s} \right\}^{\frac{1}{q_2}} \\
= \|f\| L^{q_2,q_1}(\mathbb{R}^d) \|f\| L^{q_2,q_1}(\mathbb{R}^d) \\
\leq \left( \frac{p}{q_1} \right)^{\frac{1}{q_1}} \|f\| L^{p,q_1}(\mathbb{R}^d) \|f\| L^{q_2,q_1}(\mathbb{R}^d)
\]

since:

\[
\|f\| L^{p,\infty}(\mathbb{R}^d) = \sup_{t > 0} \frac{1}{p} f^\#(t)
\]

\[\square\]

**Property 2.15.** Let us consider \( 0 \leq p_1 < +\infty, 0 \leq p_2 < +\infty, 0 \leq q < +\infty \). Then:

\[ p_1 \geq p_2 \Rightarrow L^{p_2,q}(\mathbb{R}^d) \hookrightarrow L^{p_1,q}(\mathbb{R}^d) \]

**Proof.** [BS88]

The secondary exponent is not involved, in so far as those inclusions are like the ones of the Lebesgue spaces \( L^p \), and depend on the structure of the underlying measure space.

\[\square\]
3 New properties

Property 3.1. (Cl. David)
For any \((f, g)\) belonging to \(L^{p,q}(\mathbb{R}^d) \times L^{p,q}(\mathbb{R}^d)\):
\[
||| f + g |||_{L^{p,q}(\mathbb{R}^d)} \geq \max \left( ||| f |||_{L^{p,q}(\mathbb{R}^d)}, ||| g |||_{L^{p,q}(\mathbb{R}^d)} \right)
\]

Proof. The proof relies on the following result:

Theorem 3.2. Let us denote by \(f\) a real-valued, positive function, defined on a measurable subset \(A\) of \(\mathbb{R}^d\). Then, \(f\) is the limit of an increasing sequence of positive simple function:
\[
f = \lim_{n \to +\infty} \sum_{i \in I_n} f_{i,n} \mathbb{1}_{A_{i,n}}
\]
where, for any natural integer \(n\), \(I_n\) is a countable set, and where, for any \(i\) belonging to \(I_n\), \(f_{i,n}\) is a positive number, \(A_{i,n}\) a measurable set, and \(\mathbb{1}_{A_{i,n}}\) is the characteristic function of the set \(A_{i,n}\).

Proof. One requires just to examine the case of two simple functions, of the form:
\[
f = a_1 \mathbb{1}_{I_1} \quad , \quad g = a_2 \mathbb{1}_{I_2}
\]
where \(I_1\) and \(I_2\) are intervals of \(\mathbb{R}\), and \(a_1, a_2\), real numbers such that \(a_1 < a_2\).

Then:

\(\Rightarrow\) If \(I_1\) and \(I_2\) are disjoints, due to
\[
f^\# = a_1 \mathbb{1}_{[0,m_1]}
\]
\[
g^\# = a_2 \mathbb{1}_{[0,m_2]}
\]
one has:
\[
(f + g)^\# \geq \max (f^\#, g^\#)
\]

\(\Rightarrow\) If \(I_1 = I_2\), one goes back, for \(f + g\), to a function of the form:
\[
\left( \sum_{i=1}^{2} a_i \right) \mathbb{1}_{I_i}
\]
which leads to:
\[
f^\# = a_1 \mathbb{1}_{[m_0,m_1]}
\]
\[ g' = a_2 \mathbb{1}_{[m_0, m_1]} \]

\[
(f + g)' = \left( \sum_{i=1}^{2} a_i \right) \mathbb{1}_{[0, m_1]} \geq \max(f', g')
\]

Figure 6: The graph of \( f \) (right), and \( g \) (left).

Figure 7: The graph of \( f' \) (right), and \( g' \) (left).

Thus, for any strictly positive number \( t \):
Figure 8: The graph of \((f + g)^\#\).

\[(f + g)^\# \geq \max (f^\#, g^\#)\]

and:

\[\frac{1}{t} \int_0^t (f + g)^\#(s) \, ds \geq \frac{1}{t} \int_0^t \max (f^\#(s), g^\#(s)) \, ds\]

Then:

\[(f + g)^**(t) \geq \max (f**(t), g**(t))\]

and:

\[t^{\frac{1}{p}} \max (f + g)^**(t) \geq t^{\frac{1}{p}} (f**(t), g**(t))\]

For \(q \geq 2\):

\[\left( t^{\frac{1}{p}} (f + g)^**(t) \right)^q \geq \max \left( \left( t^{\frac{1}{p}} f**(t) \right)^q, \left( t^{\frac{1}{p}} g**(t) \right)^q \right)\]

which yields:

\[\|f + g\|_{L^p,q(\mathbb{R}^d)} \geq \max \left( \|f\|_{L^p,q(\mathbb{R}^d)}, \|g\|_{L^p,q(\mathbb{R}^d)} \right)\]

\[\square\]

**Lemme 3.3. Maximal function related to a product of functions (Cl. David)**

Let us denote by \(f\) and \(g\) two real-valued, measurable function, finite a.e. Then:

\[\forall t > 0 \quad (f \cdot g)^** \leq 2 \int_0^t f^\#(s) g^\#(s) \, ds \leq 2 \int_0^t f**(s) g**(s) \, ds\]
Proof. For any positive number \( s \):

\[
(f \ast g)(s) \leq f^\ast \left( \frac{s}{2} \right) g^\ast \left( \frac{s}{2} \right)
\]

then, for any \( t > 0 \):

\[
\frac{1}{t} \int_0^t (f \ast g)(s) \, ds \leq \frac{1}{t} \int_0^t f^\ast \left( \frac{s}{2} \right) g^\ast \left( \frac{s}{2} \right) \, ds = \frac{2}{t} \int_0^{\frac{s}{2}} f^\ast(s) g^\ast(s) \, ds \leq \frac{2}{t} \int_0^{\frac{s}{2}} f^{**}(s) g^{**}(s) \, ds
\]

i.e.:

\[
(f \ast g)^{**}(t) \leq \frac{2}{t} \int_0^t f^\ast(s) g^\ast(s) \, ds
\]

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