FIFA ranking:
Evaluation and path forward

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ABSTRACT
In this work we study the ranking algorithm used by Fédération Internationale de Football Association (FIFA); we analyze the parameters it currently uses, show the formal probabilistic model from which it can be derived, and optimize the latter. In particular, analyzing the games since the introduction of the algorithm in 2018, we conclude that the game’s “importance” (as defined by FIFA) used in the algorithm is counterproductive from the point of view of the predictive capability of the algorithm. We also postulate the algorithm to be rooted in the formal modelling principle, where the Davidson model proposed in 1970 seems to be an excellent candidate, preserving the form of the algorithm currently used. The results indicate that the predictive capability of the algorithm is notably improved by using the home-field advantage and the explicit model for the draws in the game. Moderate, but notable improvement may be attained by introducing the weighting of the results with the goal differential, which although not rooted in a formal modelling principle, is compatible with the current algorithm and can be tuned to the characteristics of the football competition.

1. Introduction
In this work paper we evaluate the algorithm used by Fédération Internationale de Football Association (FIFA) to rank the international men teams. We also propose and study simple modifications to improve the prediction capability of the algorithm.

We are motivated by the fact that the rating and ranking are important elements of sport competitions and the surrounding entertainment environments. The rating consists in assigning the team/player a number, often referred to as “skills” or “strengths”; the ranking is obtained by sorting these numbers and is also referred to as “power ranking”.

The rating has an informative function providing fans and profane observers with a quick insight into the relative strength of the teams. For example, the press is often interested in the “best” teams or the national team reaching some records position in the ranking.

More importantly, the ranking leads to consequential decisions such as a) the seeding, i.e., defining which teams play against each other in the competitions (e.g., used to establish the composition of the groups in the qualification rounds of the FIFA World Cup), b) the promotion/relegation (e.g., determining which teams move between the English Premier League (EPL) and the English Football League Championship, or
teams which move between the Nations Leagues groups), or c) defining the participants in the prestigious (and lucrative) end-of-the-season competitions (such as Champions League in European football, Stanley Cup series in National Hockey League (NHL)).

Most of the currently used sport ratings are based on counting of wins/loses (and draws, when applicable) but in some cases the sport-governing bodies moved beyond these simple methods and implemented more sophisticated rating algorithms where the rating levels attributed to the teams are meant to represent the skills.

In particular, FIFA started a new ranking/rating algorithm in 2018, where the rating level (skills) assigned to the teams are calculated from the game outcome, of course, but also from the skills of the teams before the game. The resulting rating algorithm has a virtue of being simple and defined in a (mostly) transparent manner.

The main objective of this work is to analyze the FIFA ranking using the statistical modelling methodology. Considering that the football association is, by any measure, the most popular sport in the world, it has a value in itself and follows the line of works which analyzed the past strategies of ranking used by FIFA, e.g., (Lasek, Szląvik, & Bhulai, 2013), (Ley, de Wiele, & Eetvelde, 2019). However, the approach we propose can be applied to evaluate other algorithms as well, such as the one used by Fédération Internationale de Volleyball (FIVB) (FIVB, 2020).

In this work we will:

- Derive the FIFA algorithm from the first principles. In particular, we will define the probabilistic model underlying the algorithm and identify the estimation method used to estimate the skills.
- Assess the relevance of the parameters used in the current algorithms. In particular we will evaluate the role played by the change of the adaptation step according to the game’s importance (as defined by FIFA).
- Optimize the parameters of the proposed model. As a result, we derive an algorithm which is equally as simple as the FIFA’s one, but allows us to improve the prediction of the games’ results.
- Propose the modifications of the algorithm which take into account the goals differential, also known as the margin of victory (MOV). We consider legacy-compliant algorithms and a new version of the rating.

Our work is organized as follows. In Sec. 2 we describe the FIFA algorithm in the framework which simplifies the manipulation of the models and the evaluation of the results. This is also where we clarify the data origin and make a preliminary evaluation of the relevance of the game’s importance parameters currently used to control the size of the adaptation step. The algorithm is then formally derived in Sec. 3 where we also discuss the evaluation of the results and the batch estimation approach we use. The incorporation of the MOV into the rating is evaluated in Sec. 4 using two different strategies. In Sec. 5 we return to the on-line rating, evaluating and re-optimizing the proposed algorithms, pointing out to the role of the scale, and commenting on the elements of the FIFA algorithm (the shootout/knockout rules) which seem to be introduced in an ad-hoc manner for which the models are not specified. We conclude the work in Sec. 6 summarizing our findings and in Sec. 6.1 we make an explicit list of recommendations which may be introduced to improve on the current version of the FIFA algorithm.
2. FIFA ranking algorithm

We consider the scenario where there is a total of \( M \) teams playing against each other in the games indexed with \( r = 1, \ldots, T \), where \( T \) is the number of games in the observed period. FIFA ranks \( M = 210 \) international teams and, between June 4, 2018 and October 16, 2021, there were \( T = 2964 \) FIFA-recognized games.

Let us denote the skill of the team \( m = 1, \ldots, M \) before the game \( t \) as \( \theta_{t,m} \), \( t \in T = \{1, \ldots, T\} \) which are gathered in a vector \( \theta_t = [\theta_{t,1}, \ldots, \theta_{t,M}]^\top \), where \((\cdot)^\top\) denotes the transpose. The home and the away teams are denoted by \( i_t \) and \( j_t \) respectively.

The game results \( y_t \in \mathcal{Y} \) are ordinal variables, where the elements of \( \mathcal{Y} = \{H, D, A\} \) represent the win of the home team \((y_t = H)\), the draw \((y_t = D)\), and the win of the away team \((y_t = A)\). These ordinal variables are often transformed into the numerical scores \( \tilde{y}_t = \tilde{y}(y_t) \): \( \tilde{y}(A) = 0 \), \( \tilde{y}(D) = 0.5 \), and \( \tilde{y}(H) = 1 \).

The basic rules of FIFA’s rating for a team \( m \), which plays in the \( t \)-th game are defined as follows

\[
\theta_{t+1,m} \leftarrow \theta_{t,m} + I_c \delta_{t,m}
\]

\[
\delta_{t,m} = y_{t,m} - F(\frac{z_{t,m}}{s})
\]

\[
F(z) = \frac{1}{1 + 10^{-z}}
\]

\[
z_{t,m} = \theta_{t,m} - \theta_{t,n}
\]

where \( s = 600 \) is the scale\(^1\), \( n \) is the index of the team opposing the team \( m \) in the \( t \)-th game, \( \tilde{y}_{t,m} \) is the “subjective” score of the team \( m \) (for the home team, \( m = i_t \), \( \tilde{y}_{t,m} = \tilde{y}_t \), and for the away team, \( m = j_t \), \( \tilde{y}_{t,m} = 1 - \tilde{y}_t \)). The result produced by the logistic function, \( F(z_{t,m}/s) \) in (2) is referred to as the expected score.

When the team \( m \) does not play, its skills do not change, i.e., \( \theta_{t+1,m} \leftarrow \theta_{t,m} \).

Since \( |\delta_{t,m}| \leq 1 \), \( I_0 \) is the maximum allowed update step, where \( c_t \) is the game category (or game “importance”) and we can decompose \( I_c \) into two components

\[
I_c = K\xi_c,
\]

where \( K = 5 \) and \( \xi_c \) is a category-dependent adjustment as defined in Table 1.

| \( c \) | \( I_c \) | \( \xi_c \) | Description | Number |
|-------|-------|-------|-------------|--------|
| 0     | 5     | 1     | Friendlies outside International Match Calendar windows | 436    |
| 1     | 10    | 2     | Friendlies during International Match Calendar windows | 583    |
| 2     | 15    | 3     | Group phase of Nations League competitions | 347    |
| 3     | 25    | 5     | Play-offs and finals of Nations League competitions | 84     |
| 4     | 25    | 5     | Qualifications for Confederations/World Cup finals | 1189   |
| 5     | 35    | 7     | Confederation finals up until the QF stage | 209    |
| 6     | 40    | 8     | Confederation finals from the QF stage onwards | 52     |
| 7     | 50    | 10    | World Cup finals up until QF stage | 56     |
| 8     | 60    | 12    | World Cup finals from QF stage onwards | 8      |

Table 1. Categories, \( c \) of the game and the corresponding update steps \( I_c = K\xi_c \). FIFA (2018), where \( K = 5 \) and \( \xi_c = I_c/I_0 \). The number of the observed categories between June 4, 2018 and October 16, 2021 is also given (total number of games is \( T = 2964 \)).

\(^1\) The role of the scale is to ensure that the values of the skills \( \theta_{t,m} \) are situated in a visually comfortable range; it can be also used when changing the rating algorithm, as is discussed in Sec. 5.2.
The basic equation governing the change of the skills in $I$ is next supplemented with the following rules:

- **Knockout rule**: in the knockout stage of any competition (which follows the group stage), instead of (2) we use
  \[
  \delta_{t,m} \leftarrow \max\{0, \delta_{t,m}\}
  \]
  (6)
  which guarantees that no points are lost by teams moving out of the group stage.

- **Shootout rule**: If the team $m$ wins the game in the shootouts we use
  \[
  \hat{y}_{t,m} \leftarrow 0.75, \quad \hat{y}_{t,n} \leftarrow 0.5,
  \]
  (7)
  where $n$ is the index of the team which lost.

  This rule, however, does not apply in two-legged qualification games if the shoot out is required to break the tie.

We will discuss the effect of the knockout rule later and here we only point out to the fact that by applying the shootout/knockout we always increase $\delta_{t,m}$. Thus, while the basic rules (1)-(2) guarantee that the teams “exchange” the points so the total number of points stays constant, i.e., $\sum_{m=1}^{M} \theta_{t+1,m} = \sum_{m=1}^{M} \theta_{t,m}$ (this is a well-known property of the Elo rating algorithm, (Elo, 1978)), the shootout/knockout rules increase the total number of points which causes an “inflation” of the rating. In fact, in the considered period, there were 124 games where the shootout rule, the knockout rule, or both were applied and this increased the total score by 1739 points (with the initial total being 254680).

The rating we described is published by FIFA since August 2018, roughly on a per-month basis. The algorithm was initialized on the June 4, 2018, with the initialization values $\theta_0$ based on the previous rating system.

To run the algorithm, we need to know the initialization $\theta_0$, the presence of conditions which trigger the use of the knockout/shootout rules, and most importantly, the category/importance of each game, $c_t$. These elements are not officially published so we use here the unofficial data shown in Football Rankings (2021) which keeps track of the FIFA rating since June 2018. Using it, we were able to reproduce the ratings $\theta_t$ with a precision of fractions of rating points which gives us confidence that the categories of the games are assigned according to the FIFA rules.

Before discussing the suitable models and algorithms we ask a very simple question: Are the parameters $I$, defining the “importance” of the game suitably set? If not, how

\[\text{Information provided by Football Rankings (2021) is highly valuable because it is far from straightforward to verify which games are included in the rating and what their importance } I_t \text{ is. In particular, the games in the same tournament can be included or excluded from the rating and in some cases the changes may be done retroactively complicating further the understanding of the rating results. For example, we had to deal with two minor exceptions:}

- We recognized the victory of Guyana (GUY) over Barbados (BRB) in the game played on Sept. 6, 2019 already on the date of the game, while in the FIFA rating, the draw was originally registered and the GUY’s victory was recognized only later, when BRB was disqualified for having fielded an ineligible player.

- We removed the game Côte d’Ivoire (CIV) vs. Zambia (ZAM) played on June 19, 2019, where CIV, the winner and ZAM exchanged 2.21 points. The removal of this game from the FIFA-recognized list seems to be a reason why FIFA changed the ratings of both teams between two official publications on Dec. 19, 2019, and on Feb. 20, 2020. Namely, the CIV’s rating was changed from 1380 to 1378 and ZAM’s from 1277 to 1279. This was done despite both teams not playing at all in this period of time.
should we define them to improve the results? The immediate corollary question is what “improving” the results may mean and, in general, how to evaluate the results produced by the algorithm.

We note here that the concept of the game importance is not unique to the FIFA rating and appears also in the FIVB rating, (FIVB, 2020) and the statistical literature, e.g., (Ley et al., 2019, Sec. 2.1.2).

2.1. Preliminary evaluation of the FIFA rating

A conventional approach in statistics is to base the performance evaluation on a metric, called a scoring function, relating the outcome, \( y_t \) to its prediction obtained from the estimates at hand (here, \( \theta_t \)), (Gelman, Hwang, & Vehtari, 2014).

At this point we want to use only the elements which are clearly defined in the FIFA ranking and the only explicit predictive element defined in the FIFA algorithm is the expected score (3), \( F(z_t/s) = \mathbb{E}[\hat{y}_t | z_t] \), we will base the evaluation on the metric affected by the mean. Later we will abandon this simplistic approach.

Using the squared prediction error
\[
m(z_t, y_t) = (\hat{y}_t - F(z_t/s))^2
\]
(8)

averaged over the large number of games, we obtain the Mean Square Error (MSE) estimate
\[
\text{MSE} = \frac{2}{T} \sum_{t=T/2+1}^{T} m(z_t, y_t),
\]
(9)

where we use the games in the second half of the observation period to attenuate the initialization effects. This truncation is somewhat arbitrary of course but does not affect the results significantly for large \( T \).

The MSE in (9) may be treated as an estimate of the expectation,
\[
\text{MSE} \approx \mathbb{E}_{z_t} \left[ \mathbb{E}_{y_t \mid z_t} \left[ (\hat{y}_t - F(z_t/s))^2 \right] \right] = \text{Var}[\hat{y}_t] + \mathbb{E}_{z_t} \left[ \text{Var}[\hat{y}_t | z_t] \right],
\]
(10)

which highlights the bias-variance decomposition, (Duda, Hart, & Stork, 2001, Ch. 9.3.2) and where \( \text{Var}[\hat{y}_t] = \mathbb{E}_{z_t} \left[ \text{Var}[\hat{y}_t | z_t] \right] \) is the average conditional variance of \( \hat{y}_t \), and \( B(z_t, y_t) = F(z_t/s) - \mathbb{E}[\hat{y}_t | z_t] \) is the estimation bias of the mean.

Therefore, by reducing the (absolute value of the) bias \( B(z_t, y_t) \), that is, by improving the calculation of the expected score \( F(z_t/s) \), should manifest itself in a lower value of the MSE which is calculated as in (9).

Using the MSE we are now able to assess how the values of the importance parameters \( I_c \) (or alternatively, \( K \) and \( \xi_c \)) affect the expected value of the score.

We find the coefficients \( K \) and/or \( \xi_c \) by minimizing the MSE (9) and it turns out that a simple alternate optimization (one variable \( K \) or \( \xi_c \) is optimized at a time, till convergence) leads efficiently to satisfactory solutions. The results are shown in Table 2 and we observe the following:

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3 This was done by a line search as we preferred avoiding more formal, e.g., gradient-based, methods which are not well suited to deal with the complicated functional relationship resulting from the recursive rating algorithm.
• The common update step $K$ increases ten-fold in the optimized setup and it seems that it is the most important contributor to the improvement of the $\text{MSE}$ (which changes from $\text{MSE} = 0.1295$ in the original algorithm to $\text{MSE} = 0.1262$ in the algorithm with fixed-importance games but larger common adaptation step).

• For the games in the categories well represented in the data, i.e., $c \in \{0, 1, 2, 4, 5\}$, the relative importance of the games $\xi_c$ does not seem to be critically different and for sure does not fall in line with the values used in the FIFA algorithm. Overall, the optimized $\xi_c$ yield a very small improvement in the $\text{MSE}$ comparing to the use of fixed $\xi_c$.

Using a very simple criterion of the $\text{MSE}$ derived from the definitions used by the FIFA algorithm, we obtain results which cast a doubt on the optimality/utility of the games’ importance parameters, $I_c$ proposed by FIFA. However, drawing conclusions at this point may be premature. For example, regarding $K$ (which, after optimization should be much larger than 5), it is possible that the relatively short period of observation time (29 months) is not sufficient for small $K$ to guarantee the sufficient convergence but may pay off in a long run, when smaller values of $K$ will improve the performance after the convergence is reached. We cannot elucidate this issue with the data at hand.

On the other hand, to address the concerns regarding the relative importance weights $\xi_c$ the situation is rather different. Even after the convergence, the weights associated with different game categories should affect meaningfully the results. To elucidate this point we will now take a more formal approach and go back to the “drawing board” to derive the rating algorithm from the first principles.

### Table 2.
| $\text{MSE}_{\text{opt}}$ | $K$ | $\xi_0$ | $\xi_1$ | $\xi_2$ | $\xi_3$ | $\xi_4$ | $\xi_5$ | $\xi_7$ | $\xi_8$ |
|--------------------------|-----|---------|---------|---------|---------|---------|---------|---------|---------|
| 0.1295                   | 5   | 1       | 2       | 3       | 5       | 5       | 7       | 8       | 10      | 12      |
| 0.1262                   | 12  | 1       | 2       | 3       | 5       | 5       | 7       | 8       | 10      | 12      |
| 0.1262                   | 55  | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       |
| 0.1250                   | 50  | 1       | 0.6     | 1.8     | 0.8     | 1.2     | 1.1     | 2.4     | 0.1     | 9.9     |

The first row corresponds to the original FIFA algorithm: $K$ and $\xi_c$ are taken from Table 1.
at time $t$, a random variable $y$ takes the value $y_t$, depends on the skills’ difference $z_t = \theta_{t,i} - \theta_{t,j}$, i.e.,

$$\Pr \{y = y_t|\theta_t\} = L(z_t/s; y_t)$$

where $L(z_t/s; y_t)$ is the likelihood of $\theta_t$ (for a given outcome $y_t$) and we define a scheduling vector $x_t = [x_{t,0}, \ldots, x_{t,N-1}]^T$ for the game $t$, as

$$x_{t,m} = I[i_t = m] - I[j_t = m],$$

with $I[a] = 1$ when $a$ is true, and $I[a] = 0$, otherwise. Thus, $x_{t,m} = 1$ if the team $m$ is playing at home, $x_{t,m} = -1$ if the team $m$ is visiting, and $x_{t,m} = 0$ for all teams $m$ which do not play. This notation allows us to a) deal in a compact manner with all the skills $\theta_t$ for each $t$, and b) consider the home-field advantage (HFA) or a lack thereof. As before, $s$ is the scale.

We are interested in the on-line rating algorithms, in which the skills of the participating teams are changed immediately after the results of the game are known. Nevertheless, we will start the analysis with a batch processing, i.e., assuming that the skills $\theta_t$ do not vary in time, $\theta_t = \theta$. This is a reasonable approach if the time window defined $T$ is not too large, so that the skills of the teams may, indeed, be considered approximately constant. The on-line rating algorithms will be then derived as approximate solutions to the batch optimization problem. The purpose of such approach is to a) tie the algorithm used by FIFA with the theoretical assumptions, which are not spelled out when the algorithm is presented, b) remove the dependence on the initialization and/or on the scale, and c) treat the past and present data in the same manner, e.g., avoiding the partial elimination in the performance metrics, see (9).

Assuming that the observations are independent when conditioned on the skills, the rating may be based on the weighted maximum likelihood (ML) estimation principle

$$\hat{\theta} = \arg\min_{\theta} \sum_{t \in T} \xi_c \ell(z_t/s; y_t),$$

where

$$\ell(z_t/s; y_t) = -\log L(z_t/s; y_t),$$

is a (negated) log-likelihood. The weighting with $\xi_c \in (0, 1]$ is used in the estimation literature to take care of the outcomes which are more or less reliable, ([Hu & Zidek, 2001], [Amiguet, 2010]). In our problem, the reliability is associated with the game category, $c_t$, so $\xi_c$ denotes the weight of the category $c$. Since multiplication of $\xi_c$ by a common factor is irrelevant for minimization, we fix $\xi_0 = 1$.

We may solve (14) using the steepest descent

$$\hat{\theta} \leftarrow \hat{\theta} - \mu/s \sum_t x_t \xi_c g(z_t/s; y_t),$$

where

\footnote{The negation in (15) allows us to use a minimization in (14) which is a very common formulation.}
where $\mu$ is the adaptation step and

$$g(z; y) = \frac{d}{dz} \ell(z; y).$$

(17)

The on-line version of (16) is obtained replacing the batch-optimization with the stochastic gradient (SG) which updates the solution each time a new observation becomes available, i.e.,

$$\theta_{t+1} \leftarrow \theta_t - K\xi_t x_t g(z_t/s; y_t),$$

(18)

where the update amplitude is controlled by the weight $\xi_t$ and the step $K$ which absorbs the scale $s$.

3.1. Davidson model and Elo algorithm

The rating depends now on the choice of the likelihood function $L(z; y)$ and we opt here for the Davidson model, (Davidson, 1970), being a particular case of the multinomial model used also in Egidi and Torelli (2021)

$$L(z; H) = \frac{10^{0.5(z+\eta b)}}{10^{0.5(z+\eta b)} + \kappa + 10^{-0.5(z+\eta b)}},$$

(19)

$$L(z; A) = P(-z; H),$$

(20)

$$L(z; D) = \kappa \sqrt{L(z; H)L(z; A)},$$

(21)

where $\eta$ is a HFA modelling the apparent increase in the skills of the local team, the indicator $b = 1$ if the game is played in the home-team country allows us to distinguish between the games played on the home or the neutral venues and $\kappa$ controls for the presence of the draws.

The choice of this model is motivated by the fact that it leads to a simple algorithmic update of the skills generalizing the Elo rating algorithm, (Szczecinski & Djebbi, 2020). It also becomes equivalent to the latter for a particular value of $\kappa = 2$ and $\kappa = 0$. These relationships make possible (or at the least – ease) the comparison with the rating algorithm currently used by FIFA which is also based on the Elo algorithm.

Using (19)-(21) in (17), with straightforward algebra we obtain

$$g(z; y) = \frac{d}{dz} \ell(z; y) = -\ln \tilde{y} - F_\kappa(z),$$

(22)

(23)

where $\tilde{y}$ is the “score” of the game which we already defined, and

$$F_\kappa(z) = \frac{\frac{1}{2} \kappa + 10^{0.5(z+\eta b)}}{10^{0.5(z+\eta b)} + \kappa + 10^{-0.5(z+\eta b)}}$$

(24)

has the meaning of the conditional expected score, $F_\kappa(z) = \mathbb{E}[\tilde{y}|z] = \sum_{y \in Y} \tilde{y} L(z; y)$.

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5Out of $T = 2964$ games we considered, 768 were played on neutral venues. To verify the venue we used (The Room Ba, 2021) and (SoccerWay, 2021).
Therefore, the SG algorithm (18) becomes

$$\theta_{t+1} \leftarrow \theta_t + K_\xi x_t (\hat{y}_t - F_\kappa(z_t/s))$$

(25)

and it obviously has the form of the Elo and FIFA rating algorithms, see (1)-(3), except that we use \(F_\kappa(z)\) while the former use \(F(z)\). Note that the step \(K\) in (25) absorbs the term \(\ln 10\) from (23).

It is easy to see that for \(\eta = 0\) and \(\kappa = 0\) (i.e., when \(L(z; D) = 0\) and the draws are ignored) we have \(F_0(z) = F(z)\) which is simply a logistic function as in the Elo algorithm. Furthermore, for \(\eta = 0\) and \(\kappa = 2\) we obtain \(F_2(z) = F(z/2)\) and thus (25) is again equivalent to the Elo rating algorithm but with the doubled scale value.

While we conclude that the FIFA rating algorithm may be seen as the instance of the maximum weighted likelihood estimation, this is, of course, a “reverse-engineered” hypothesis because the FIFA document, (FIFA, 2018), does not mention any remotely similar concept.

### 3.2. Regularized batch rating

In order to go beyond the limitation of the SG optimization and to avoid the problems related to the removal of the significant portion of the data (meant to eliminate the initialization effects during evaluation, see (9)) we may focus on the original problem defined in (14) for the entire set of data. That is, we ignore now the on-line rating aspect and rather focus on the evaluation of the model and the optimization criterion that underlie the algorithm.

We start noting that the problem (14) is, in general, ill-posed: since the solution depends only on the differences between the skills, \(z_t\), all solutions \(\hat{\theta}\) and \(\hat{\theta} + \theta_0 1\) are equivalent because the differences \(z_t\) are independent from “origin” value \(\theta_0\). To remove this ambiguity we may regularize the problem as

$$\hat{\theta} = \arg\min_{\theta} J(\theta)$$

(26)

$$J(\theta) = \sum_{t \in T} \xi_c \ell(z_t/s; y_t) + \alpha \frac{1}{2s^2} \|\theta\|^2,$$

(27)

where \(\alpha\) is the regularization parameter and we have opted for a so-called ridge regularization (Hastie, Tibshirani, & Friedman, 2009, Ch. 3.4.1).

Under the model (19)-(21), the regularized batch-optimization problem (26) is useful to resolve another difficulty. Namely, if there is a team \(m\) having registered only wins, i.e., when \(\forall i_t = m, y_t = 1\) and \(\forall j_t = m, y_t = 0\), then (14) cannot be solved (or rather, \(\hat{\theta}_m \to \infty\)) because \(J(\theta)\) does not limit the value of \(\theta_m\). Such a solution not only is unattainable numerically but is, in fact meaningless and the regularization (26) settles this issue.\(^6\)

The estimated skills, \(\hat{\theta}\) depend now on the weights \(\xi_c\), on the regularization parameter \(\alpha\), and on the model parameters \(\eta\) and \(\kappa\). If unknown, all these parameters must be optimized.

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\(^6\)The same problem arises, of course, when a team registers a sequences of pure losses. This is not a hypothetical issue and in the official FIFA games three teams registered the streaks of unique wins or losses (without any other results): Tonga (three wins), Eritrea (two losses), and American Samoa (four losses). Thus, the attempt to solve the batch-optimization problem without regularization (i.e., with \(\alpha = 0\)) would yield \(\hat{\theta}_m = \infty\), for \(m\) being index of Tonga.
As for the optimization criterion, we recall that the FIFA algorithm only specified the expected score, so the quadratic error (8) was allowed us to evaluate the algorithm and stay within the boundaries of its definitions. Now, however, with the explicit skills-outcome model, we may go beyond this limitation and may use the prediction metrics known in the machine learning such as the (negated) log-score, (Gelman et al., 2014)

\[ m_{ls}(z_t; y_t) = \ell(z_t/s; y_t), \]  

often preferred due to its compatibility with the log-likelihood used as the optimization criterion, or the accuracy score, (Lasek & Gagolewski, 2020)

\[ m_{acc}(z_t; y_t) = \mathbb{I}[y_t = \arg\max_y L(z_t/s; y)], \]

which equals one if the event with the largest predicted probability was actually observed, otherwise it is zero.

Furthermore, thanks to the batch-rating we are able to consider the entire data set in the performance evaluation by averaging the scoring function (28) or (29) over all games

\[ \text{LS} = \frac{1}{T} \sum_{t \in T} m_{ls}(x_t^\top \hat{\theta}_t, y_t), \]  
\[ \text{ACC} = \frac{1}{T} \sum_{t \in T} m_{acc}(x_t^\top \hat{\theta}_t, y_t), \]

where

\[ \hat{\theta}_t = \arg\min_\theta J_t(\theta), \]  
\[ J_t(\theta) = \sum_{i \in T, i \neq t} \xi_s \ell(x_i^\top \theta/s; y_i) + \frac{\alpha}{2s^2} \|\theta\|^2. \]

In plain words, for given parameters \((\alpha, \kappa, \eta, \xi_c)\), we find the skills \(\hat{\theta}_t\) from all, but the \(t\)-th game [this is (32)–(33)], and next use them to predict the results \(y_t\); we repeat it for all \(t \in T\), summing the obtained scores. This is the well-known leave-one-out (LOO) cross-validation strategy (Hastie et al., 2009, Sec. 2.9); (Duda et al., 2001, Ch. 9.6.2): no data is discarded when calculating the metrics (30)–(31) and this comes with the price of having to find \(\hat{\theta}_t\) for all \(t \in T\). To diminish the computational load, we opt here for the approximate leave-one-out (ALO) cross-validation (Rad & Maleki, 2020) based on the local quadratic approximation of the optimization function defined for all the data. Details are given in Appendix A.

Although both, the average log-score in (30) and the accuracy (31) can be now optimized with respect to \(\alpha, \kappa, \eta, \) and/or \(\xi_c\), we only optimize the log-score whose optimal value is denoted as \(\text{LS}_{opt}\); the resulting accuracy, \(\text{ACC}\) will be also shown. It is, of course, possible to optimize the log-score with respect to any subset of parameters.

Again, we used the alternated minimization: \(\text{LS}\) was minimized with respect to one parameter at a time: \(\alpha, \kappa, \eta, \) or \(\xi_c\), till no improvement was observed. This simple strategy led to the minimum \(\text{LS}_{opt}\) which turned out to be independent of various
starting points we used.

A quick comment may be useful regarding the interpretation of the performance metrics. The accuracy \( \text{ACC} \) is easily understandable: it is an average number of the events which were predicted correctly (as those with the maximum likelihood \( L(z_t/s; y) \)). On the other hand, the metric \( z \) may be represented as \( \exp(-LSS) = \prod_{t=1}^{T} L(z_t/s; y_t) \) which is a geometric mean of the predicted probabilities assigned to the events which were actually observed. While the accuracy metric penalizes the wrong guesses with zero (so \( \text{ACC} \in [0, 1] \)), the log-score penalizes them via the logarithmic function, which may be arbitrarily large (so \( \text{LS} \in (0, \infty) \)).

However, the fundamental difference between the two metrics is that we can use the accuracy without specifying the distribution for all possible outcomes but we cannot calculate the log-score in such a case.

The results obtained are shown in Table 3 and indicate that

- The data does not provide evidence for using a category-dependent weights \( \xi_c \).
  There is actually a slight indication that the optimal weights of the Friendlies within the IMC (category \( c = 1 \)) and Group phase of Nations Leagues (category \( c = 2 \)) are slightly smaller than the weight of the regular Friendlies. This stands in contrast to the FIFA algorithm which doubles the weight \( \xi_1 \) of the Friendlies played in the IMC and triples the weight of \( \xi_2 \).

- In fact, the results obtained using the FIFA weights \( \xi_c \) are worse than those obtained using constant weights \( \xi_c = 1 \) (i.e., essentially ignoring the possibility of weighting). Even with the argument of having a small number of games in some categories (such as a World Cup), it is very unlikely that observing more games will speak in favor of variable weights and almost surely not in favor of the highly disproportionate weights used in the FIFA algorithm.

- A notable improvement in the prediction capacity as measured by the log-score is obtained by considering the FIFA. The value \( \eta \in \{0.3, 0.4\} \) emerges from the optimization fit and we note that \( \eta = 0.25 \) was used in eloratings.net (2020).

- A more important improvement is obtained by optimizing the parameter \( \kappa \) which takes into account the draws and their frequency as discussed in Szczecinski and Djebbi (2020).

\begin{table}[h]
\centering
\begin{tabular}{cccccccccccc}
\hline
\text{LS}_{opt} & \alpha & \eta & \kappa & \xi_0 & \xi_1 & \xi_2 & \xi_3 & \xi_4 & \xi_5 & \xi_6 & \xi_7 & \xi_8 & \text{ACC}\% \\
\hline
0.960 & 1.7 & 0 & 2.0 & 1 & 2.0 & 3.0 & 5.0 & 5.0 & 7.0 & 8.0 & 10.0 & 12.0 & 55 \\
0.948 & 0.2 & 0 & 2.0 & 1 & 1.0 & 1.0 & 1.0 & 1.0 & 1.0 & 1.0 & 1.0 & 1.0 & 56 \\
0.948 & 0.3 & 0 & 2.0 & 1 & 2.0 & 1.0 & 0.8 & 0.9 & 1.2 & 0.7 & 0.8 & 1.1 & 55 \\
0.918 & 0.3 & 0.4 & 2.0 & 1 & 1.0 & 1.0 & 1.0 & 1.0 & 1.0 & 1.0 & 1.0 & 56 \\
0.860 & 0.4 & 0.3 & 0.8 & 1 & 1.0 & 1.0 & 1.0 & 1.0 & 1.0 & 1.0 & 1.0 & 61 \\
0.860 & 0.5 & 0.3 & 0.8 & 1 & 0.8 & 0.7 & 0.8 & 1.0 & 1.2 & 0.8 & 1.0 & 61 \\
\hline
\end{tabular}
\caption{Batch-rating parameters obtained via minimization of the log-score. The parameters (\( \alpha, \kappa, \eta, \xi_0, \xi_2 \)) are either fixed (shadowed cells), or obtained via optimization. The upper-part results correspond to the conventional FIFA algorithm; using \( \kappa = 2 \) and \( \eta = 0.860 \), the expected score is calculated using a logistic function.}
\end{table}

\footnote{Although we cannot prove the solution to be global, in all our observations the log-score functions seemed to be unimodal.}

\footnote{The common confusion is to interpret the function \( F(z_t/s; D) \) as the probability of the home win, and the value \( 1 - F(z_t/s) \), as the probability of an away win. This, of course, implies that the draw probability equals zero. With such an interpretation, we can still calculate the accuracy metric even if we never predict the draw. On the other hand we cannot calculate the log-score, because when the draw occurs, we have undefined metric \( \exp(z_t/s; D) \rightarrow \infty \).}

\footnote{Therein, the unnormalized value \( \eta = 100 \) is reported and since \( s = 400 \), we obtain \( \eta = 0.25 \).}
It is interesting to compare the parameters found by optimization with the simplified formulas proposed in Szczecinski and Djebbi (2020, Sec. 3.2)

\[
\eta = \log_{10} \frac{f_H}{f_A} \\
\kappa = \frac{f_D}{\sqrt{f_Hf_A}} \approx \frac{2f_D}{1-f_D}.
\]

where \(f_y, y \in Y\) are empirical frequencies of outcomes. We can consider separately the games played on the neutral venues and calculate these frequencies as \(f_{\text{neut.}}^H = 0.37, f_{\text{neut.}}^D = 0.24, f_{\text{neut.}}^A = 0.39\), and those played on home venues as \(f_{\text{hfa}}^A = 0.27, f_{\text{hfa}}^D = 0.22, f_{\text{hfa}}^H = 0.51\), which yields

\[
\kappa_{\text{hfa}} = 0.61, \quad \eta_{\text{hfa}} = 0.28
\]

\[
\kappa_{\text{neut.}} = 0.63, \quad \eta_{\text{neut.}} = 0.02.
\]

The parameter \(\eta_{\text{hfa}}\) predicted by (34) is practically equal to the one obtained by optimization. And while the parameters \(\kappa_{\text{hfa}}\) and \(\kappa_{\text{neut.}}\) are slightly different from the one predicted by (35), using them in the rating, we obtained \(L_{\text{opt}} = 0.868\), which is still notably better than using the conventional FIFA rating. This is interesting because finding the parameters \(\eta\) and \(\kappa\) from the frequencies of the games not only avoids optimization but also provides a simple empirical justification.

4. Margin of victory

In the search for a possible improvement of the rating we want to consider now the use of the MOV variable, defined by the difference of the goals scored by each team, and denoted by \(d_t\). With that regard, the most recent works adopt two conceptually different approaches.

The first one keeps the structure of the known rating algorithm (such as FIFA algorithm) and modifies it by changing the adaptation step size as a function of \(d_t\). This was already done in eloratings.net (2020), Hvattum and Arntzen (2010), Silver (2014), Ley et al. (2019), and Kovalchik (2020), and is conceptually similar to the weighting according to the game-category we consider in the previous section.

Second approach changes the model relating the skills to the MOV variable \(d_t\) and was already studied before in Maher (1982), Ley et al. (2019), Lasek and Gagolewski (2020), Szczecinski (2020). We will focus on the simple proposition from Lasek and Gagolewski (2020) building on the formulation of Karlis and Ntzoufras (2008).

4.1. MOV via weighting

For the context, we show in Table 4 the number of games and their percentage of the total, depending on the value of the MOV variable \(d\). While, in principle, it is possible to use directly \(d\), it is customary to consider their absolute value, \(|d|\).

The Elo/FIFA algorithms (25) can be easily modified as follows, to take the MOV
variable into account:

\[ K_{c,d} = K c \xi c \zeta d, \]  

(38)

where, as before, \( K \) is the common step, \( \xi c \) is the weight associated with the game-category \( c \), and \( \zeta d \) is the function of the MOV-variable \( d \).

For example, the ratings.net (2020) uses

\[ \zeta d = \begin{cases} 
1 & |d| \leq 1 \\
1.5 & |d| = 2 \\
1.75 + 0.125(|d| - 3) & |d| \geq 3 
\end{cases}. \]

(39)

The similar propositions may be found in Hvattum and Arntzen (2010) (in the context of association football), in Kovalchik (2020) (to rate the tennis players), or in Silver (2014) (for rating of the teams in American football).

To elucidate how useful such heuristics are, we note that the problem is very similar to the importance-weighting we analyzed before; the difference resides in the fact that the weighting depends now on the product \( \xi c \zeta d \). We may thus reuse our optimization strategy to find the optimal weights for the games with different values of \( |d| \).

To this end we discretize \( |d| \) into \( V + 1 \) MOV-categories, \( v = 0, \ldots, V \) and we use a very simple mapping \( v = |d| \) for \( v < V \) and \( v = V \iff |d| \geq V \). For example, with \( V = 2 \), \( \zeta_0 \) weights the draws (\( |d| = 0 \)), \( \zeta_1 \) weights the games with one goal difference (\( |d| = 1 \)), \( v = 0, 1 \) and \( \zeta_2 \) weights the games with more than one goal difference (\( |d| \geq 1 \)).

Breaking with the predefined functional relationship as the one shown in (39) we are more general than the latter, e.g., treating the cases \( |d| = 0 \) and \( |d| = 1 \) separately. This makes sense since, not only they are the most frequent events, corresponding, respectively, to 22% and 36% of the total, see Table 4, but also they correspond to the events of draw and win/loss treated differently by the algorithm.

On the other hand, we are also less general due to the merging of the events \( |d_t| \geq V \), although this effect will decrease with \( V \), simply because there will be very few observations as may be understood from Table 4. For example, with \( V = 4 \), the weighting \( \zeta_4 \) will be the same for the events with \( |d| = 4 \) and \( |d| > 4 \) but the latter make only 5% of the total.

We consider again the game categories defined in Table 1 and thus we solve now the problem

\[ \hat{\theta} = \arg\min_{\theta} \sum_{t \in T} \xi c_t \zeta v_t \ell(z_t/s; y_t) + \frac{\alpha}{2s^2} \|\theta\|^2, \]

(40)

where \( v_t \) is the index of the MOV variable \( d_t \). To remove ambiguity of the solution, we set \( \xi_0 = 1 \) and \( \zeta_0 = 1 \).

The parameters \( \xi c, \zeta v, \eta, \kappa, \) and \( \alpha \) will be chosen again using the ALO approach we
Table 5. Batch-rating parameters obtained via minimization of the log-score (30) with weighting of the MOV variables. The parameters (α, κ, η, ξ, ζ) are either fixed (shadowed cells), or obtained through optimization; to save space, in the sole case when the parameters ξc are optimized, their optimal values are gathered in the vector ξ = [1,0,1,0,0.9,1.2,1.1,1.8,1.9,1.4,5.6].

described in Sec. 3.2, that is, by optimizing the log-score criterion (30). The results shown in Table 5 allow us to conclude that:

- The optimization of the MOV weights ζv (while keeping ξc = 1) yields LSopt = 0.937 and the optimization of ξc (with ζv = 1) yields LSopt = 0.948 (see Table 3). By comparing them, we see that weighting of the MOV categories is more beneficial than weighting of the game-categories. Therefore, there is little improvement in considering the category-related weights, ξc.
- The optimization indicates that ζv defined by (39) is suboptimal. In particular, the optimal MOV weights, ζv, are monotonically growing (as foreseen by the heuristics) only for |d| ≥ 1 and the draws (i.e., |dt| = 0) have a weight which is more important that the weights of the events |dt| = 1; thus, these two events should not be merged together, nor we should impose a particular functional form for the weights.
- The best improvement in the prediction is obtained again by optimizing the parameters η and κ of the Davidson model together with the MOV weights ζv.

4.2. MOV via modelling

A different approach to deal with the MOV relies on the integration of the latter into the formal model relating the skills θt and the observed MOV variable dt.

A simple approach proposed in Karlis and Ntzoufras (2008) relies on a direct modelling of the goal difference using the Skellam’s distribution

\[
\Pr \{d_t = d|\theta_t\} = L(z_t; d_t)
= e^{-(\mu_{h,t} + \mu_{a,t})} \left(\frac{\mu_{h,t}}{\mu_{a,t}}\right)^{d/2} I_d(2\sqrt{\mu_{h,t} \mu_{a,t}}),
\]

where \(I_v(t)\) is the modified Bessel function of order \(v\) and \(\mu_{h,t}\) and \(\mu_{a,t}\) are means of the Poisson variables modelling the home- and away-goals. The latter are functions of the skills’ difference \(z_t\), (Karlis & Ntzoufras, 2008, Sec. 2.2)

\[
\mu_{h,t} = e^{c+z_t+bn}, \quad \mu_{a,t} = e^{c-z_t-bn},
\]
where is $c$ is a constant and, as before, $\eta$ is the HFA coefficient.  

The model (41) is a particular case of a more general form shown in Karlis and Ntzoufras (2008), which allowed us to model the offensive and the defensive skills. Here, however, we are interested in rating and thus one skill per team should be used. As noted in Ley et al. (2019), Lasek and Gagolewski (2020) this offers a sufficient prediction capability avoiding the problem of over-parametrization due to doubling of the number of skills.

Using (43) in (41), the following log-likelihood is obtained

$$
\ell(z; d) = -\log L(z; d) = (\mu_h + \mu_a) - d(z + b\eta) - 2e^c - \log \tilde{I}_{|d|}(2e^c)
$$

where, for numerical stability it is convenient to use an exponentially modified form of the Bessel function, $\tilde{I}_v(t) = I_v(t)e^{-t}$, available in many computation packages.

The derivative of (44) is given by

$$
g(z; d) = \frac{d}{dz} \ell(z; d) = -(d - \mathcal{F}(z)),
$$

$$
\mathcal{F}(z) = \mu_h - \mu_a = e^c(e^{z+b\eta} - e^{-z-b\eta}).
$$

The batch rating consists then in solving the following problem:

$$
\hat{\theta} = \arg\min_\theta \sum_{t \in T} \ell(z_t/s; d_t) + \frac{\alpha}{2s^2} \|\theta\|^2
$$

and the SG implementation of the ML principle will produce the algorithm

$$
\theta_{t+1} \leftarrow \theta_t + Kx_t(d_t - \mathcal{F}(z_t/s)),
$$

which is again written in a form similar to the FIFA rating algorithm, where the goal difference $d_t$ plays the role of the “score”, and $\mathcal{F}(z_t/s) = \mathbb{E}[d_t|z_t]$ is the expected score. The algorithm (49) can be also obtained by applying the Poisson model to the goals scored by each of the teams (Lasek & Gagolewski, 2020).

To calculate the log-score, we have to merge the events $d < 0$ (away-win) and $d > 0$ (home-win). Since the closed-form formulas do not exist we do it approximately by truncated sums

$$
m^b_s(z; A) = -\log \sum_{d=-D}^{1} L(z; d), \quad m^b_s(z; D) = -\log L(z; 0), \quad m^b_s(z; H) = -\log \sum_{d=1}^{D} L(z; d),
$$

where we used $D = 50$ which guaranteed that $|1 - \sum_{d=-D}^{D} L(z; d)| < 10^{-4}$.

---

10 For the home team we add– and for the away team – subtract $b\eta$ in the exponent. This is different from Karlis & Ntzoufras (2008), Lasek & Gagolewski (2020), where only the home team benefits from the HFA boost while the away team is not penalized, see Karlis & Ntzoufras (2008, Eq. (2.2)-(2.3)). Of course, we can rewrite (43) as $\mu_h,t = e^{c'+z+t+b\eta'}$, $\mu_a,t = e^{c'+z}$ with $c' = c - b\eta$ and $\eta' = 2\eta$ but it makes sense only when the HFA is always present, as then $bn = \eta$. While this condition holds in the context of football leagues considered in Karlis and Ntzoufras (2008) and in Lasek and Gagolewski (2020), this is not the case in the international HFA games, which can be played on the neutral venues.
Table 6. Batch-rating parameters obtained via minimization of the log-score \(30\) using the Skellam’s model \(44\).

| \(|\text{LS}_{\text{opt}}|\alpha\) | \(|\eta|\) | \(|c|\) | \(|\text{ACC}|\)% |
|-----------------|---------|---------|-------------|
| 0.845           | 0.21    | 0.20    | 0           | 61         |

Table 7. Ranking of the top teams: Belgium (BEL), Brazil (BRA), England (ENG), France (FRA), and Italy (ITA). The original FIFA algorithm and its modified rules are considered.

| Original FIFA algorithm | No shootouts rule | No knockouts rule | No shootouts / knockouts rule |
|------------------------|------------------|------------------|-----------------------------|
| BEL (1832.3)           | BEL (1831.0)     | BRA (1775.9)     | FRA (1768.4)                |
| BRA (1820.4)           | BRA (1817.5)     | FRA (1770.1)     | BRA (1767.7)                |
| FRA (1779.2)           | FRA (1778.2)     | BEL (1759.2)     | BEL (1757.4)                |
| ITA (1750.5)           | ITA (1740.2)     | ITA (1730.5)     | ITA (1711.1)                |
| ENG (1750.2)           | ITA (1733.0)     | ENG (1711.9)     | ENG (1701.3)                |

The results shown in Table 6 indicate that, with this very simple approach (with only two parameters of the model which must be optimized) we are able to improve over the MOV-weighting strategy and this should be attributed to the use of a formal skills-outcome model. The price to pay for the improvement lies in the change of the entire algorithm and in abandoning of the legacy of the Elo algorithm.

Moreover, the possible implementation issues may arise since the expected score \(47\) is theoretically unbounded. Whether the improvement of the log-score from \(LS = 0.857\) (in the MOV weighting, see Table 5) to \(LS = 0.845\) in the Skellam’s MOV model is worth the change and the implementation risks, is at least debatable.

5. Online rating

Before starting a metrics-based comparison of the on-line algorithms, in Sec. 5.1 we will address the use of the knockout/shootout rules \(6\) - \(7\) and, in Sec. 5.2 the practical issue of setting the scale.

5.1. Effect of the knockout/shootout rules

Table 7 compares the ranking (of top-five teams) obtained using the FIFA algorithm (first column) to the rating resulting from the modified algorithm in which we a) eliminate the shootout rule (second column), b) eliminate the knockout rule (third column), as well as c) elimination both rules (fourth column). The differences, most notably the removal of Belgium from the first place, are due to the different number of times the teams benefited from the knockout rules (although the shootout rule for sure has an effect on the final rating too).

Indeed, by analyzing the results of the games, we observed that in the original ranking, Belgium (BEL) benefited four times from the knockout rule for a total of 85 points (which would be lost without the rule \(6\)), Brazil (BRA) and England (ENG) benefited twice for a total of 53 and 80 points, respectively, while both France (FRA) and Italy (ITA) benefited only once, gaining 14 points each.\(^{11}\)

What is important is that the points-preserving knockout rule ignores the direct

\(^{11}\) Of course, due to the temporal relationships, eliminating the knockout/shootout rules is not the same as evaluating the points (not lost in the original algorithm) and discarding them from the final results.
comparison between the teams. In fact, the games in which Belgium was not penalized (for loosing in knockout stages) were played against France (twice) and against Italy (twice as well). Thus, despite a direct evidence indicating that France and Italy were able to beat Belgium, the knockout rule preserved the points earned by Belgium in other games.

In fact, such a situation is not surprising and we indeed expect the teams which compete for the top ranking spots to be also likely to make it to the final stages of the important competitions (in case of the Belgium’s games: World Cup 2018, Euro 2020, and UEFA Nations League 2021) and then play against each other. While these games will provide direct comparison results, current knockout rule will preserve the points of the losing team.

Whether this is fair and desirable may be debatable especially considering that the knockout/shootout rules are not rooted in any formal modelling principle, and most likely are introduced to compensate for the increased value of \( I_c \) in the advanced stages of competitions.

### 5.2. Scale adjustment

The scale is obviously irrelevant in the batch optimization and the on-line update can also be written in the scale-invariant manner by dividing (18) by \( s \):

\[
\theta'_{t+1} \leftarrow \theta'_t - K'\xi_c x_t g(z'_t; y_t)
\]

\[
z'_t = z_t/s
\]

\[
\theta'_t = \theta_t/s
\]

\[
K' = K/s;
\]

in other words, for the same scale-invariant initialization \( \theta'_0 \) and using the same step \( K' \) we will obtain the same results \( \theta'_t \).

However, in the FIFA ranking, a non-zero initialization \( \theta_0 \) was determined in advance and thus \( \theta'_0 \) is not scale-invariant. Thus, given the initialization at hand, the question is how to determine the scale? In general, it is, of course, a difficult question but an insight may be gained assuming that the initialization corresponds to the “optimal” solution, e.g., \( \hat{\theta} \) obtained in the batch optimization with a given scale \( s_0 \).

It is easy to see that using \( s > s_0 \) will force the algorithm to significantly change \( \theta_t \) (attainable with large values of the adaptation step, \( K \)); the same will happen for \( s < s_0 \) because the optimal estimates \( \theta_t \) will have to be scaled down.

Since scaling up/down of the skills changes their empirical moments we suggest to choose the scale, \( s \) in a moment-preserving manner. To this end we define the empirical standard deviation of the skills

\[
\sigma_t = \sqrt{\|\hat{\theta}_t - \overline{\theta}_t\|^2/M}
\]

where \( \overline{\theta}_t = (\sum_{m=1}^M \hat{\theta}_{t,m})/M \) is the empirical mean, and postulate that, at the initialization and at the final step, we have \( \sigma_0 \approx \sigma_T \).

In fact, the initialization used by FIFA yields \( \sigma_0 = 220 \) and, after running the FIFA algorithm we obtain \( \sigma_T = 250 \), relatively close to the initial value \( \sigma_0 \).

Changing the scale \( s \), we will obtain different \( \sigma_T \) so the idea is to run the algorithms for different values of the scale \( s \), e.g., as multiples of 100 and to choose the one which
Table 8. Parameters and performance of the on-line rating SG algorithms obtained by minimizing the log-score (56) for a) Davidson model, b) MOV weighting strategy from Sec. 4.1, and c) the MOV modeling strategy from Sec. 4.2.

| Algorithm | LS$_{opt}$ | $V$ | $K$ | $\eta$ | $\kappa$ | $\zeta_0$ | $\zeta_1$ | $\zeta_2$ | $\zeta_3$ | ACC [%] |
|-----------|------------|-----|-----|-------|---------|--------|--------|--------|--------|--------|
| FIFA, $\xi_c$ from Table 11 | 0.951 | 5 | 0 | 2 | 50 |
| FIFA, $\xi_c$ = 1 | 0.933 | 55 | 0 | 2 | 52 |
| SG | 0.917 | 35 | 0 | 2 | 54 |
| | 0.902 | 35 | 0.4 | 2 | 58 |
| | 0.841 | 35 | 0.3 | 0.9 | 61 |
| a) Performance of the algorithms: FIFA ($s = 600$) and Elo-Davidson model with SG ($s = 200$) |

| LS$_{opt}$ | $V$ | $K$ | $\eta$ | $\kappa$ | $\zeta_0$ | $\zeta_1$ | $\zeta_2$ | $\zeta_3$ | ACC [%] |
|------------|-----|-----|-------|---------|--------|--------|--------|--------|--------|
| 0.841 | 1 | 35 | 0.3 | 0.9 | 1.0 | 0.9 | $\times$ | $\times$ | 61 |
| 0.838 | 2 | 35 | 0.3 | 0.9 | 1.0 | 0.6 | 1.3 | $\times$ | 62 |
| 0.837 | 3 | 40 | 0.3 | 0.9 | 1.0 | 0.5 | 0.8 | 1.8 | 62 |
| b) SG with the MOV weighting, $s = 200$ |

| LS$_{opt}$ | $V$ | $K$ | $\eta$ | $c$ | ACC [%] |
|------------|-----|-----|-------|-----|--------|
| 0.827 | 7.5 | 0.2 | $-0.1$ | 62 |
| c) SG implementing Skellam’s model for the MOV, $s = 300$ |

yields a standard deviation $\sigma_T \approx \sigma_0$. In practice it has to be done using historical data before the new rating is deployed but in our case we could do it in the hindsight.

In this manner we found $s = 200$ to be suitable for the Davidson-Elo algorithm (we obtained $\sigma_T = 219$ for the unweighted version and $\sigma_T = 221$ for the MOV-weighted approach), and $s = 300$ well suited for the Skellam’s algorithms (where $\sigma_T = 225$ was obtained). This also indicates that the scale $600$ was too large for the FIFA rating. This can be noted by comparing, in Table 8 the result FIFA with $\xi_c = 1$ to the results of the SG (with $\eta = 0$ and $\kappa = 2$). Both are essentially the same algorithms (although FIFA uses the shootout/knockout rules which have negligible impact on the performance) and the only difference resides in the scale. Since the scale $s = 200$ in the Elo-Davidson algorithm corresponds to the scale $s = 400$ in the FIFA algorithm, the latter would perform better with the scale $s = 400$. This effect, however, appears only due to limited observation window we have at our disposal and will vanish after a sufficiently large number of games.

5.3. Evaluation of the algorithms

To evaluate the SG algorithms for the models studied in the batch context, we will use the log-score and the accuracy metrics defined for the half of the games in the
Table 9. Ranking of the top teams using the proposed algorithms.

|            | Davidson       | MOV weights   | Skellam’s model |
|------------|----------------|---------------|-----------------|
| FRA (1683.5) | FRA (1690.8)   | BRA (1596.0)  |
| BRA (1673.1) | BRA (1677.4)   | ARG (1585.8)  |
| ARG (1668.6) | ARG (1677.0)   | BEL (1546.2)  |
| BEL (1664.9) | BEL (1666.5)   | POR (1541.2)  |
| ITA (1657.7) | ITA (1665.6)   | ESP (1540.7)  |

We consider the SG algorithm based on the Davidson model (Table 8a), the Davidson model with the MOV-weighting (Table 8b), and Skellam’s model algorithm (Table 8c).

In all cases, but in the original FIFA algorithm, we ignore the category-weighting (i.e., we use $\xi_c = 1$) because, as we have already shown, its effect is negligible. This is clearly shown in the first part of Table 8a where we see that using the FIFA weighting we obtain worse results than when the weighting is ignored. This is essentially the same result as the one we have shown in Table 2 but we repeat it here to show the log-score metric which we could not calculate without first introducing the Davidson model underlying the FIFA algorithm.

The results indicate that:

- The most notable improvements are due to, in similar measures, two elements: the introduction of the HFA coefficient, $\eta$ and the explicit use of the Davidson model (and thus, the optimization of the coefficient $\kappa$).
- Additional small, but still perceivable gains are obtained by introducing the MOV-weighting, where from the lesson learnt in Sec. 4.1 we weight independently the draws and the home/away wins.
- The MOV-modelling using the Skellam’s distribution brings again a small benefit.

We present in Table 9 the rating obtained for the top teams via new rating algorithms. Of course, due to smaller scale we used, the skills have smaller values and should not be compared directly to those from Table 7 but the ranking is of interest, where the teams from the FIFA ranking are present (FRA, BRA, BEL) but this time Argentine (ARG), which was on the sixth place in the previous rankings, is now consistently on and above the top-third position. We can also see that the differences between the rating values are much less pronounced.
6. Conclusions

In this work we analyzed the FIFA ranking using the methodology conventionally used in the probabilistic modelling and inference. In the first step, we identified the model relating the outcomes (games results) to the parameters which have to be optimized (skills of the teams). More precisely, we have shown that the FIFA algorithm can be formally derived as the stochastic gradient (SG) optimization of the weighted maximum likelihood (ML) criterion in the Davidson model (Davidson, 1970).

This first step allows us to define the performance metrics related to the predictive performance of the algorithms we study. This is particularly important in the case of the FIFA ranking algorithm because it does not model the outcomes of the game but only explicitly specifies the expected score, which is not sufficient to precisely evaluate the rating results. It also allows us to apply the batch approach to rating and skills' estimation. This conventional machine learning strategy frees us from the considerations related to the scale, initialization, or modeling of the skills’ dynamics.

Using the batch rating, we have shown that the game-category weighting is negligible at best, and counterproductive at worst, which is the case of the weighting used by the FIFA rating. This observation is interesting in its own right because, while on one hand the concept of weighting is also used in the rating literature, e.g., (Ley et al., 2019), on the other, the literature does not show any evidence that it is in any way beneficial and our findings consistently indicate the contrary.

We next considered extensions of the algorithm by including the HFA and optimizing the parameter responsible for the draws. These two elements seem to be particularly important from the point of view of the performance of the rating algorithm. While the HFA is a well-known element, already considered by FIFA in FIFA (2007), the possibility of generalizing the Elo algorithm by using the Davidson’s model, was only recently shown in Szczecinski and Djebbi (2020).

We also evaluated the possibility of using the margin of victory (MOV) given by the goal differential, where we analyzed the weighting strategy and the modelling based on the Skellam’s distribution. These two methods further improve the results at the cost of higher complexity. Here, the formal optimization strategy of the weighting parameters also yield interesting and somewhat counter-intuitive results. Namely, we have shown that the games won with small margin should have smaller weights than the tied games. This stands in net contrast with the weighting strategies proposed before, e.g., in Hvattum and Arntzen (2010), Silver (2014), Kovalchik (2020) which use the weighting with monotonically increasing functions of the margin.

Finally, we evaluated the heuristic shootout/knockout rules which are used in the FIFA ranking. Since their impact on the overall performance is small and they may distort the relationship between the ratings of the strong teams which often face each other in the final stages of the competitions, their usefulness is questionable. In particular, eliminating the knockout rule would strip Belgium from its first place position in the current FIFA ranking due to multiple losses Belgium suffered against the current top teams (e.g., Italy, France).

6.1. Recommendations

Given the analysis and the observations we made, if the FIFA rating was to be changed, the following steps are recommended:

(1) Add the home-field advantage (HFA) parameter to the model because playing
at the home venue is a strong predictor for the victory. Not only this well-known fact is already exploited in Women [FIFA] ranking but such a modification is most likely the simplest and the least debatable element. In our view, it is surprising that the current rating adopted in 2018 does not include the [FIFA].

2. Use explicit model to relate the skills to the outcomes. Not only it would add expressiveness providing the explicit predicted probability for each outcomes, but it also improves the prediction results. Note that the rating algorithm introduced recently by [FIVB] adopts such an approach and specifies the probability for each of the game outcomes. In the context of the [FIFA] ranking, the Davidson model we used in this work is an excellent candidate for that purpose as it results in a natural generalization of the Elo algorithm, preserving the legacy of the current algorithm.

3. Remove the weighting of the games according to their assumed importance because the data does not provide any evidence for their utility, or rather provides the indication that the weighting in its current form is counterproductive. If the concept of the game importance is of extra-statistical nature (such as entertainment), it is preferable to diminish its role, e.g., by shrinking the gap between the largest and the smaller values of $\xi$ used.

4. Remove the shootout and knockout rules which are not rooted in any sound statistical principle.

   As far as the knockout rule is concerned, while the intent to protect the rating of the teams which manage to qualify to the knockout stage is clear, we may argue that the penalty due to losing in the knockout game is aggravated by the increased weighting of these games. Therefore, removing the weight, as we postulate, would also eliminate the very reason to protect the teams’ points with the knockout rule.

   Regarding the shootout rule, a small frequency of events where it can be applied and a marginal changes in the score imposed by the rule, make its impact rather negligible. Its fairness is again debatable because there is little evidence relating the skills of the teams to the outcome of the shootout.

5. If the rating was to consider the MOV the simplest solution lies in weighting the update step using the goal differential. On the other hand, the modification based on the change of the model using the Skellam’s distribution may cause numerical problems and the relatively small performance gains hardly justify the added complexity.

   On the other hand, the MOV may be added using alternative solutions similar to those already considered in the Women’ teams [FIFA] ranking. Again, the latter should be studied, e.g., using the methodology we used in this work and basing the results on a formal probabilistic model.

Appendix A. Approximate leave-one-out cross-validation

Our goal is to calculate in a simple manner the terms $x_i^\top \hat{\theta}_t$ which appear in the scoring function in (30) and in (31).

We start by approximating the maximum a posteriori (MAP) objective function
using the Taylor series
\[
J_t(\theta) = J(\hat{\theta}) + \xi_c \log L(x_t^\top \theta / s; y_t) \\
\approx J(\hat{\theta}) + \xi_c \log L(x_t^\top \hat{\theta} / s; y_t) \\
- \frac{\xi_c}{s} g_t x_t^\top (\theta - \hat{\theta}) + \frac{1}{2} (\theta - \hat{\theta})^\top \left[ \hat{H} - \frac{\xi_c}{s^2} h_t x_t x_t^\top \right] (\theta - \hat{\theta})
\] (A.2)

where \( g_t \equiv g(x_t^\top \hat{\theta} / s; y_t) \) is defined in (23),
\[
\hat{\theta} = \underset{\theta}{\text{argmin}} J(\theta)
\] (A.3)
is the optimal solution for all data, the Hessian at optimum is given by
\[
\hat{H} = \nabla^2_{\theta} J(\theta)_{| \theta = \hat{\theta}} = \sum_{t \in T} \frac{\xi_c}{s^2} h_t x_t x_t^\top + \frac{\alpha}{s^2} I,
\] (A.4)
and we use second derivative \( h_t \equiv h(x_t^\top \hat{\theta} / s) \) with (Szczecinski & Tihon, 2021, Sec. IV)
\[
h(z) = \frac{d}{dz} g(z; y) = \frac{(\ln 10)^2 \kappa 10^{0.5(z+\eta)} + 4 + \kappa 10^{-0.5(z+\eta)}}{(10^{0.5(z+\eta)} + \kappa + 10^{-0.5(z+\eta)})^2}.
\] (A.5)

By equating the gradient of (A.2) to zero, we find the approximate solution to the optimization problem
\[
\hat{\theta}_t \approx \underset{\theta}{\text{argmin}} J_t(\theta) \\
= \hat{\theta} + \frac{\xi_c g_t}{s} \left[ \hat{H} - \frac{\xi_c}{s^2} h_t x_t x_t^\top \right]^{-1} x_t
\] (A.6)
and the terms \( x_t^\top \hat{\theta}_t, t \in T \) which appear as arguments of the metrics (30) and (31) can be now calculated efficiently for all \( t \in T \) once \( \hat{\theta} \) is known (Rad & Maleki, 2020) (Burn, 2020)
\[
x_t^\top \hat{\theta}_t \approx x_t^\top \hat{\theta} + \frac{\xi_c g_t}{s} x_t^\top \left[ \hat{H} - \frac{\xi_c}{s^2} h_t x_t x_t^\top \right]^{-1} x_t
\] (A.7)
\[
= x_t^\top \hat{\theta} + \frac{\xi_c g_t}{s} x_t^\top \hat{H}^{-1} + \frac{\xi_c h_t}{s^2 - \xi_c h_t x_t x_t^\top} \hat{H}^{-1} x_t x_t^\top \hat{H}^{-1} x_t
\] (A.8)
\[
= x_t^\top \hat{\theta} + \frac{\xi_c g_t a_t}{s^2 - \xi_c h_t a_t},
\] (A.9)
where \( a_t = x_t^\top \hat{H}^{-1} x_t \) and to pass from (A.7) to (A.8) we used the matrix inversion lemma (Barber, 2012, Ch. A.1.8).

The advantage of this formulation is clear: instead of solving \( T \) times the optimization problem (32), we only need to solve once the optimization defined in (A.3). In comparison with the latter, the remaining operations of the inversion of the matrix \( H_0 \) and the multiplication required to calculate \( a_t, t \in T \), have a very small complexity.
The identical approach may be used to apply the ALO to the problem (40) but, we have to replace \( \xi_c \) in (A.9) with \( \xi_c, \xi_v \).

In order to apply the ALO to the problem (48) we need a second derivative of (46) which is given by

\[
h(z) = \frac{d}{dz} g(z; d) = e^c(e^{z+b\eta} + e^{-z-b\eta}).
\]

(A.10)

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