Primes In Arithmetic Progressions And Primitive Roots

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Abstract: Let $x \geq 1$ be a sufficiently large number, and let $1 \leq a < q$ be a pair of integers such that $\gcd(a, q) = 1$ and $q = O(\log^c x)$ with $c \geq 0$ constant. This note proves that the counting function for the number of primes $p \in \{p = qn + a : n \geq 1\}$ with a fixed squarefree primitive root $u \neq \pm 1$ has the asymptotic formula $\pi_u(x, q, a) = \delta(u, q, a) \operatorname{li}(x) + O(x/\log^b x)$, where $\delta(u, q, a) > 0$ is the density, and $b = b(c) > 1$ is a constant.

Contents

1 Introduction 1
2 Representation of the Characteristic Function 2
3 Estimates Of Exponential Sums 2
  3.1 Partial And Complete Exponential Sums 3
  3.2 Equivalent Exponential Sums 4
4 Evaluation Of The Main Term 5
5 Estimate For The Error Term 8
6 The Main Result 9
7 References 11

1 Introduction

Let $a \geq 1$ and $q > a$ be a pair of integers such that $\gcd(a, q) = 1$, and let $p \geq 2$ be a prime. The multiplicative order modulo $p$ of an integer $u \neq \pm 1$ is denoted by $\text{ord}_p(u) = p - 1$. The density of the subset of primes in the arithmetic progression $\{p = qn + a : \gcd(a, q) = 1 \text{ and } \text{ord}_p(u) = p - 1\}$ is defined by a real number $\delta(u, q, a) \geq 0$, and the corresponding primes counting function is defined by

$$\pi_u(x, q, a) = \#\{p = qn + a \leq x : \gcd(a, q) = 1 \text{ and } \text{ord}_p(u) = p - 1\},$$  \hspace{1cm}(1)$$

where $x \geq 1$ is a sufficiently large number. This note considers the followings result.

Theorem 1.1. Let $x \geq 1$ be a sufficiently large number. Let $1 \leq a < q$ be integers such that $\gcd(a, q) = 1$ and $q = O(\log^c x)$, with $c \geq 0$ constant. Then, the arithmetic
progression \( \{ p = qn + a : n \geq 1 \} \) has infinitely many primes \( p \geq 2 \) with a fixed squarefree primitive root \( u \neq \pm 1 \). In addition, the corresponding primes counting function has the asymptotic formula

\[
\pi_u(x, q, a) = \delta(u, q, a) \log x + O \left( \frac{x}{\log^b x} \right),
\]

where \( \delta(u, q, a) \geq 0 \) is the density constant depending on the fixed integers \( u, q, a \), and \( b = b(c) > 1 \) is a constant.

The preliminary background results and notation are discussed in Section 2 to Section 5. Section 6 presents a proof of Theorem 1.1.

### 2 Representation of the Characteristic Function

The multiplicative order of an element in the cyclic group \( G = \mathbb{F}_p^\times \) is defined by \( \text{ord}_p(v) = \min \{ k : v^k \equiv 1 \text{ mod } p \} \). Primitive elements in this cyclic group have multiplicative order \( p - 1 = \#G \). The characteristic function \( \Psi : G \to \{0, 1\} \) of primitive elements is one of the standard analytic tools employed to investigate the various properties of primitive roots in cyclic groups \( G \). Many equivalent representations of the characteristic function \( \Psi \) of primitive elements are possible. The standard characteristic function is discussed in \cite{[10, p. 258]}. It detects a primitive element by means of the divisors of \( p - 1 \).

A new representation of the characteristic function for primitive elements is developed here. It detects the order \( \text{ord}_p(u) \geq 1 \) of an element \( u \in \mathbb{F}_p \) by means of the solutions of the equation \( \tau^n - u = 0 \) in \( \mathbb{F}_p \), where \( u, \tau \) are constants, and \( n \) is a variable such that \( 1 \leq n < p - 1 \), and \( \gcd(n, p - 1) = 1 \).

**Lemma 2.1.** Let \( p \geq 2 \) be a prime, and let \( \tau \) be a primitive root mod \( p \). Let \( u \in \mathbb{F}_p \) be a nonzero element, and let, \( \psi \neq 1 \) be a nonprincipal additive character of order \( \text{ord} \psi = p \). Then

\[
\Psi(u) = \sum_{\gcd(n, p-1)=1} \frac{1}{p} \sum_{0 \leq k \leq p-1} \psi ((\tau^n - u)k) = \begin{cases} 1 & \text{if } \text{ord}_p(u) = p - 1, \\ 0 & \text{if } \text{ord}_p(u) \neq p - 1. \end{cases}
\]

**Proof.** Let \( \tau \in \mathbb{F}_p \) be a fixed primitive root. As the index \( n \geq 1 \) ranges over the integers relatively prime to \( p - 1 \), the element \( \tau^n \in \mathbb{F}_p \) ranges over all the primitive roots mod \( p \). Ergo, the equation

\[
\tau^n - u = 0
\]

has a solution if and only if the fixed element \( u \in \mathbb{F}_p \) is a primitive root. Next, replace \( \psi(z) = e^{i2\pi z/p} \) to obtain

\[
\sum_{\gcd(n, p-1)=1} \frac{1}{p} \sum_{0 \leq k \leq p-1} e^{i2\pi (\tau^n - u)k/p} = \begin{cases} 1 & \text{if } \text{ord}_p(u) = p - 1, \\ 0 & \text{if } \text{ord}_p(u) \neq p - 1. \end{cases}
\]

This follows from the geometric series identity \( \sum_{0 \leq n \leq N-1} w^n = (w^N - 1)/(w - 1), w \neq 1 \) applied to the inner sum.
3 Estimates Of Exponential Sums

This section provides simple estimates for the exponential sums of interest in this analysis. There are two objectives: To determine an upper bound, proved in Lemma 3.1, and to establish the asymptotic identity

\[ \sum_{\gcd(n,p-1)=1} e^{i2\pi stn/p} = \sum_{\gcd(n,p-1)=1} e^{i2\pi tn/p} + E(p), \]  

(5)

where \( E(p) \) is an error term, proved in Lemma 3.2. The proofs of these Lemmas are based on established results and elementary techniques.

3.1 Partial And Complete Exponential Sums

Theorem 3.1. \cite{19} Let \( p \geq 2 \) be a large prime, and let \( \tau \in \mathbb{F}_p \) be an element of large multiplicative order \( p - 1 = \text{ord}_p(\tau) \). Let \( x \leq p - 1 \). Then, for any \( s \in [1,p-1] \),

\[ \sum_{n \leq x} e^{i2\pi stn/p} \ll p^{1/2} \log p. \]  

(6)

This appears to be the best possible upper bound. A similar upper bound for composite moduli \( p = m \) is also proved, \cite{op. cit., equation (2.29)}. A simpler proof and generalization of this exponential is is provided in \cite{12}.

Lemma 3.1. Let \( p \geq 2 \) be a large prime, and let \( \tau \) be a primitive root modulo \( p \). Then,

\[ \sum_{\gcd(n,p-1)=1} e^{i2\pi stn/p} \ll p^{1-\varepsilon} \]  

(7)

for any \( s \in [1,p-1] \), and any arbitrary small number \( \varepsilon \in (0,1/2) \).

Proof. Use the inclusion exclusion principle to rewrite the exponential sum as

\[ \sum_{\gcd(n,p-1)=1} e^{i2\pi stn/p} = \sum_{n \leq p-1} e^{i2\pi stn/p} \mu(d) \sum_{d|p-1} \]  

(8)

\[ = \sum_{d|p-1} \mu(d) \sum_{n \leq p-1} e^{i2\pi stn/p} \]  

\[ = \sum_{d|p-1} \mu(d) \sum_{m \leq (p-1)/d} e^{i2\pi stdm/p}. \]

Taking absolute value, and invoking Theorem 3.1 yield

\[ \left| \sum_{\gcd(n,p-1)=1} e^{i2\pi stn/p} \right| \leq \sum_{d|p-1} |\mu(d)| \left| \sum_{m \leq (p-1)/d} e^{i2\pi stdm/p} \right| \]  

\[ \ll \sum_{d|p-1} |\mu(d)| \left( \frac{p-1}{d} \right)^{1/2} \log p \]  

(9)

\[ \ll (p-1)^{1/2} \log(p-1) \sum_{d|p-1} |\mu(d)| d^{1/2} \]  

\[ \ll (p-1)^{1/2+\delta} \log(p-1) \]  

\[ \ll (p-1)^{1/2+\delta}. \]
The last inequality follows from
\[ \sum_{d \mid p-1} \left\lfloor \frac{\mu(d)}{d^{1/2}} \right\rfloor \leq \sum_{d \mid p-1} 1 \ll p^\delta \] (10)
for any arbitrary small number \( \delta > 0 \), and any sufficiently large prime \( p \geq 2 \). This
is restated in the simpler notation \( p^{1/2+\epsilon} \leq p^{1-\epsilon} \) for any arbitrary small number \( \epsilon \in (0, 1/2) \).

A different approach to this result appears in [4, Theorem 6], and related results are given
in [1], [3], [5], and [6, Theorem 1].

3.2 Equivalent Exponential Sums

An asymptotic relation for the exponential sums
\[ \sum_{\gcd(n,p-1)=1} e^{i2\pi s\tau^n/p} \quad \text{and} \quad \sum_{\gcd(n,p-1)=1} e^{i2\pi \tau^n/p}, \] (11)
is provided in Lemma 3.2. This result expresses the first exponential sum in (11) as a sum
of simpler exponential sum and an error term. The proof is based on Lagrange resolvent
\[ (\omega^t, \zeta^s) = \zeta^s + \omega^{-t} \zeta^{s\tau^2} + \cdots + \omega^{-(q-1)t} \zeta^{s\tau^{q-1}}, \] (12)
where \( \omega = e^{i2\pi/q} \), \( \zeta = e^{i2\pi/p} \), and \( 0 \neq s, t \in \mathbb{F}_p \). This is a more general version of the
resolvent based on the two large primes \( p \geq 2 \) and \( q = p + o(p) > p \).

**Lemma 3.2.** Let \( p \geq 2 \) and \( q = p + o(p) > p \) be large primes. If \( \tau \) be a primitive root
modulo \( p \), then,
\[ \sum_{\gcd(n,p-1)=1} e^{i2\pi s\tau^n/p} = \sum_{\gcd(n,p-1)=1} e^{i2\pi \tau^n/p} + O(p^{1/2} \log^3 p), \] (13)
for any \( s \in [1, p-1] \).

**Proof.** Summing (12) times \( \omega^t \) over the variable \( t \in \mathbb{Z}/q\mathbb{Z} \) yields
\[ q \cdot e^{i2\pi s\tau^n/p} = \sum_{0 \leq t \leq q-1} (\omega^t, \zeta^{s\tau}) \omega^t \cdot \tau^n. \] (14)

Summing (14) over the relatively prime variable \( n < p-1 < q-1 \) yields
\[ q \cdot \sum_{\gcd(n,p-1)=1} e^{i2\pi s\tau^n/p} = \sum_{\gcd(n,p-1)=1} \sum_{0 \leq t \leq q-1} (\omega^t, \zeta^{s\tau}) \omega^t \cdot \tau^n \] (15)
\[ = \sum_{1 \leq t \leq q-1} (\omega^t, \zeta^{s\tau}) \sum_{\gcd(n,p-1)=1} \omega^t \cdot \tau^n - \varphi(q). \]

The first index \( t = 0 \) contributes \( \varphi(q) \), see [12 Equation (5)] for similar calculations.
Likewise, the basic exponential sum for \( s = 1 \) can be written as
\[ q \cdot \sum_{\gcd(n,p-1)=1} e^{i2\pi \tau^n/p} = \sum_{0 \leq t \leq p-1} (\omega^t, \zeta^\tau) \sum_{\gcd(n,p-1)=1} \omega^t \cdot \tau^n - \varphi(q), \] (16)
Differencing (15) and (16) produces

\[ S_1 = q \cdot \left( \sum_{\gcd(n,p-1)=1} e^{i2\pi s\tau n/p} - \sum_{\gcd(n,p-1)=1} e^{i2\pi \tau n/p} \right) \]

\[ = \sum_{1 \leq t \leq q-1} \left( (\omega^t, \zeta^{s\tau}) - (\omega^t, \zeta^{\tau}) \right) \sum_{\gcd(n,p-1)=1} \omega^{tn}. \tag{17} \]

Taking absolute value and applying Lemma 3.1 and Lemma 3.2 yield the upper bound

\[ |S_1| \leq \sum_{1 \leq t \leq q-1} \left| (\omega^t, \zeta^{s\tau n}) - (\omega^t, \zeta^{\tau n}) \right| \sum_{\gcd(n,p-1)=1} \omega^{tn} \]

\[ \leq \sum_{1 \leq t \leq p-2} \left( 2q^{1/2} \log q \right) \left( \frac{2q \log p}{\pi t} \right) \]

\[ \leq \left( 4q^{3/2} \log q \log p \right) \sum_{1 \leq t \leq q-1} \frac{1}{t} \]

\[ \leq 8q^{3/2} \log^2 q \log p, \tag{18} \]

Combining (17) and (18) return

\[ q \cdot \left| \sum_{\gcd(n,p-1)=1} e^{i2\pi s\tau n/p} - \sum_{\gcd(n,p-1)=1} e^{i2\pi \tau n/p} \right| \leq |S_1| \]

\[ \leq 16q^{3/2} \log q \log p, \tag{19} \]

where \( q = p + o(p) \). The last inequality implies the claim. \( \blacksquare \)

The same proof works for many other subsets of elements \( A \subset \mathbb{F}_p \). For example,

\[ \sum_{n \in A} e^{i2\pi s\tau n/p} = \sum_{n \in A} e^{i2\pi \tau n/p} + O(p^{1/2} \log c p), \tag{20} \]

for some constant \( c > 0 \).

**Lemma 3.1.** Let \( p \geq 2 \) and \( q = p + o(p) > p \) be large primes, and let \( \omega = e^{i2\pi/q} \) be a qth root of unity. Then,

(i)

\[ \sum_{\gcd(n,p-1)=1} \omega^{tn} = \sum_{d \leq p-1} \mu(d) \frac{\omega^{dt} - \omega^{dp}}{1 - \omega^{dt}}, \]

(ii)

\[ \left| \sum_{\gcd(n,p-1)=1} \omega^{tn} \right| \leq \frac{2q \log p}{\pi t}, \]

where \( \mu(k) \) is the Mobius function, for any fixed pair \( d \mid p-1 \) and \( t \in [1, p-1] \).
Proof. (i) Use the inclusion exclusion principle to rewrite the exponential sum as

\[
\sum_{\gcd(n, p-1) = 1} \omega^{tn} = \sum_{d \leq p-1} \mu(d) \sum_{d \mid n} \omega^{tn}
\]

\[
= \sum_{d \leq p-1} \mu(d) \sum_{m \leq (p-1)/d} \omega^{dmn}
\]

\[
= \sum_{d \leq p-1} \mu(d) \left( \frac{\omega^{dt} - \omega^{dtp}}{1 - \omega^{dt}} \right).
\]

(ii) Observe that the parameters \( q = p + o(p) > p \) is prime, \( \omega = e^{i2\pi/q} \), the integers \( t \in [1, p-1] \), and \( d \leq p-1 < q-1 \). This data implies that \( \pi dt/q \neq k\pi \) with \( k \in \mathbb{Z} \), so the sine function \( \sin(\pi dt/q) \neq 0 \) is well defined. Using standard manipulations, and \( z/2 \leq \sin(z) < z \) for \( 0 < |z| < \pi/2 \), the last expression becomes

\[
\left| \frac{\omega^{dt} - \omega^{dtp}}{1 - \omega^{dt}} \right| \leq \frac{2q}{\pi dt}
\]

for \( 1 \leq d \leq p-1 \). Finally, the upper bound is

\[
\left| \sum_{d \leq p-1} \mu(d) \frac{\omega^{dt} - \omega^{dtp}}{1 - \omega^{dt}} \right| \leq \frac{2q}{\pi t} \sum_{d \leq p-1} \frac{1}{d}
\]

\[
\leq \frac{2q \log p}{\pi t}.
\]

Lemma 3.2. Let \( p \geq 2 \) and \( q = p + o(p) > p \) be large primes. If \( \omega = e^{i2\pi/q} \), \( \zeta = e^{i2\pi/p} \), and \( 0 \neq s, t \in \mathbb{F}_p \), then, the difference of two Lagrange resolvents has the upper bound

\[
\left| (\omega^s, \zeta^{st+dp}) - (\omega^t, \zeta^{st+dp}) \right| \leq 2q^{1/2} \log q.
\]

Proof. The proof for \( \left| (\omega^s, \zeta^{st+dp}) \right| \leq q^{1/2} \log q \) apperas in [12]. Hence, the difference

\[
\left| (\omega^s, \zeta^{st+dp}) - (\omega^t, \zeta^{st+dp}) \right| \leq \left| (\omega^s, \zeta^{st+dp}) \right| + \left| (\omega^t, \zeta^{st+dp}) \right| \leq 2q^{1/2} \log q.
\]

4 Evaluation Of The Main Term

Finite sums and products over the primes numbers occur on various problems concerned with primitive roots. These sums and products often involve the normalized totient function \( \varphi(n)/n = \prod_{p \mid n} (1 - 1/p) \) and the corresponding estimates, and the asymptotic formulas.
Lemma 4.1. (15, Lemma 5) Let \( x \geq 1 \) be a large number, and let \( \varphi(n) \) be the Euler totient function. If \( q \leq \log^c x \), with \( c \geq 0 \) constant, an integer \( 1 \leq a < q \) such that \( \gcd(a, q) = 1 \), then
\[
\sum_{p \leq x \atop p \equiv a \mod q} \frac{\varphi(p-1)}{p-1} = A_q \frac{\text{li}(x)}{\varphi(q)} + O \left( \frac{x}{\log^b x} \right),
\]
where \( \text{li}(x) \) is the logarithm integral, and \( b = b(c) > 1 \) is a constant, as \( x \to \infty \), and
\[
A_q = \prod_{p \mid \gcd(a-1, q)} \left( 1 - \frac{1}{p} \right) \prod_{p \nmid q} \left( 1 - \frac{1}{p(p-1)} \right).
\]
Related discussions for \( q = 2 \) are given in [18, Lemma 1], [14, p. 16], and, [22]. The case \( q = 2 \) is ubiquitous in various results in Number Theory.

Lemma 4.2. Let \( x \geq 1 \) be a large number, and let \( \varphi(n) \) be the Euler totient function. If \( q \leq \log^c x \), with \( c \geq 0 \) constant, an integer \( 1 \leq a < q \) such that \( \gcd(a, q) = 1 \), then
\[
\sum_{p \leq x \atop p \equiv a \mod q} \frac{1}{p} \sum_{p \leq x \atop \gcd(n, p-1) = 1} 1 = A_q \frac{\text{li}(x)}{\varphi(q)} + O \left( \frac{x}{\log^b x} \right),
\]
where \( \text{li}(x) \) is the logarithm integral, and \( b = b(c) > 1 \) is a constant, as \( x \to \infty \), and \( A_q \) is defined in (27).

Proof. A routine rearrangement gives
\[
\sum_{p \leq x \atop p \equiv a \mod q} \frac{1}{p} \sum_{p \leq x \atop \gcd(n, p-1) = 1} 1 = \sum_{p \leq x \atop p \equiv a \mod q} \frac{\varphi(p-1)}{p} \tag{29}
\]
\[
= \sum_{p \leq x \atop p \equiv a \mod q} \frac{\varphi(p-1)}{p-1} - \sum_{p \leq x \atop p \equiv a \mod q} \frac{\varphi(p-1)}{p(p-1)}.
\]
To proceed, apply Lemma 4.1 to reach
\[
\sum_{p \leq x \atop p \equiv a \mod q} \frac{\varphi(p-1)}{p-1} - \sum_{p \leq x \atop p \equiv a \mod q} \frac{\varphi(p-1)}{p(p-1)} = A_q \frac{\text{li}(x)}{\varphi(q)} + O \left( \frac{x}{\log^b x} \right) + \sum_{p \leq x \atop p \equiv a \mod q} \frac{\varphi(p-1)}{p(p-1)} \tag{30}
\]
\[
= A_q \frac{\text{li}(x)}{\varphi(q)} + O \left( \frac{x}{\log^b x} \right),
\]
where the second finite sum
\[
\sum_{p \leq x \atop p \equiv a \mod q} \frac{\varphi(p-1)}{p(p-1)} \ll \log \log x \tag{31}
\]
is absorbed into the error term, \( b = b(c) > 1 \) is a constant, and \( A_q \) is defined in (27).
5 Estimate For The Error Term

The upper bound for exponential sum over subsets of elements in finite fields $\mathbb{F}_p$ stated in the last section will be used here to estimate the error term $E(x)$ arising in the proof of Theorem 1.1.

Lemma 5.1. Let $p \geq 2$ be a large prime, let $\psi \neq 1$ be an additive character, and let $\tau$ be a primitive root mod $p$. If the element $u \neq 0$ is not a primitive root, then,

$$\left| \sum_{x \leq p \leq 2x \atop p \equiv a \mod q} \frac{1}{p} \sum_{0 < k \leq p-1 \atop \gcd(n,p-1)=1} \psi((\tau^n - u)k) \right| \ll \frac{1}{\varphi(q)} \frac{x^{1-\varepsilon}}{\log x}, \quad (32)$$

where $1 \leq a < q$, $\gcd(a,q) = 1$ and $O(\log^c x)$ with $c > 0$ constant, for all sufficiently large numbers $x \geq 1$ and an arbitrarily small number $\varepsilon < 1/16$.

Proof. Let $\psi(z) = e^{i2\pi k z/p}$ with $0 < k < p$, and rearrange the triple finite sum in the form

$$E(x) = \sum_{x \leq p \leq 2x \atop p \equiv a \mod q} \frac{1}{p} \sum_{0 < k \leq p-1 \atop \gcd(n,p-1)=1} \psi((\tau^n - u)k) \quad (33)$$

$$= \sum_{x \leq p \leq 2x \atop p \equiv a \mod q} \frac{1}{p} \sum_{0 < k \leq p-1} e^{-i2\pi uk/p} \sum_{\gcd(n,p-1)=1} e^{i2\pi \tau^n/p}. \quad \text{(34)}$$

Applying Lemma 3.2 yields

$$E(x) = \sum_{x \leq p \leq 2x \atop p \equiv a \mod q} \left( \frac{1}{p} \sum_{0 < k \leq p-1} e^{-i2\pi uk/p} \right) \times \left( \sum_{\gcd(n,p-1)=1} e^{i2\pi \tau^n/p} + O(p^{1/2} \log^2 p) \right)$$

$$= \sum_{x \leq p \leq 2x \atop p \equiv a \mod q} U_p \cdot V_p, \quad \text{(34)}$$

where

$$U_p = \frac{1}{p} \sum_{0 < k \leq p-1} e^{-i2\pi uk/p}, \quad \text{(35)}$$

and

$$V_p = \sum_{\gcd(n,p-1)=1} e^{i2\pi \tau^n/p} + O \left( p^{1/2} \log^2 p \right). \quad \text{(36)}$$

The absolute value of the first exponential sum $U_p$ is given by

$$|U_p| = \left| \frac{1}{p} \sum_{0 < k \leq p-1} e^{-i2\pi uk/p} \right| = \frac{1}{p}. \quad \text{(37)}$$
This follows from \( \sum_{0<k\leq p-1} e^{i2\pi uk/p} = -1 \) for \( u \neq 0 \) and summation of the geometric series. The absolute value of the second exponential sum \( V_p \) has the upper bound

\[
|V_p| = \left| \sum_{\gcd(n,p-1)=1} e^{i2\pi n} + O\left( (p^{1/2} \log^2 p) \right) \right| \\
\leq \left| \sum_{\gcd(n,p-1)=1} e^{i2\pi n} \right| + p^{1/2} \log^2 p \\
\leq p^{1-\varepsilon},
\]

where \( \varepsilon < 1/2 \) is an arbitrarily small number, see Lemma 3.1. A similar application appears in [16, p. 1286].

Now, replace the estimates (37) and (38) into (34), to reach

\[
\left| \sum_{x \leq p \leq 2x} U_p V_p \right| \leq \sum_{x \leq p \leq 2x} |U_p V_p| \\
\leq \sum_{x \leq p \leq 2x} \frac{1}{p} \cdot p^{1-\varepsilon} \\
\leq \frac{1}{x^\varepsilon} \sum_{x \leq p \leq 2x} 1 \\
\leq \frac{1}{\varphi(q) \log x}.
\]

The last finite sum over the primes is estimated using the Brun-Titchmarsh theorem; this result states that the number of primes \( p = qn + a \) in the interval \([x, 2x]\) satisfies the inequality

\[
\pi(2x, q, a) - \pi(x, q, a) \leq \frac{3}{\varphi(q)} x \log x,
\]

see [9, p. 167], [8, p. 157], [13], and [21, p. 83].

### 6 The Main Result

Given a fixed squarefree integer \( u \neq \pm 1 \), the precise primes counting function is defined by

\[
\pi_u(x, q, a) = \#\{p \leq x : p \equiv a \mod q \text{ and } \ord_p(u) = p-1\}
\]

for \( 1 \leq a < q \) and \( \gcd(a, q) = 1 \). The limit

\[
\delta(u, q, a) = \lim_{x \to \infty} \frac{\pi_u(x, q, a)}{\pi(x, q, a)} = a_u \frac{A_q}{\varphi(q)}
\]

is the density of the subset of primes with a fixed squarefree primitive root \( u \neq \pm 1 \).

**Theorem 6.1.** ([11]) Suppose the GRH is true. Then,

\[
\pi_u(x, q, a) = \delta(u, q, a) \frac{x}{\log x} + O\left( \frac{x \log \log x}{\log^2 x} \right).
\]
As explained in [14, Section 8.1], the existing primitive roots counting method fails to prove any unconditional result on primes and primitive roots. To circumvent this obstacle, the proof of Theorem 1.1 below, uses a new primitive roots counting method.

**Proof.** (Theorem 1.1) Suppose that the squarefree integer \( u \neq \pm 1 \) is not a primitive root for all primes \( p \geq x_0 \), with \( x_0 \geq 1 \) constant. Let \( x > x_0 \) be a large number, and \( q = O(\log^c x) \).

Consider the sum of the characteristic function over the short interval \([x, 2x]\), that is,

\[
0 = \sum_{\substack{x \leq p \leq 2x \atop q \equiv a \mod q}} \Psi(u).
\]  

Replacing the characteristic function, Lemma 2.1, and expanding the nonexistence equation (44) yield

\[
0 = \sum_{\substack{x \leq p \leq 2x \atop p \equiv a \mod q}} \Psi(u)
= \sum_{\substack{x \leq p \leq 2x \atop \gcd(n, p-1)=1, 0 \leq k \leq p-1}} \psi\left((\tau^n - u)k\right)
= a_u \sum_{\substack{x \leq p \leq 2x \atop p \equiv a \mod q}} \frac{1}{p} \sum_{\gcd(n, p-1)=1} 1
+ \sum_{\substack{x \leq p \leq 2x \atop \gcd(n, p-1)=1, 0 \leq k \leq p-1}} \psi\left((\tau^n - u)k\right)
\]

where \( a_u \geq 0 \) is a constant depending on the integers \( u \neq \pm 1 \) and \( q \geq 2 \).

The main term \( M(x) \) is determined by a finite sum over the trivial additive character \( \psi = 1 \), and the error term \( E(x) \) is determined by a finite sum over the nontrivial additive characters \( \psi(z) = e^{i2\pi z/p} \neq 1 \).

Take a constant \( b = b(c) > 1 \), depending on \( c \geq 0 \). Applying Lemma 4.2 to the main term, and Lemma 5.1 to the error term yield

\[
0 = \sum_{\substack{x \leq p \leq 2x \atop p \equiv a \mod q}} \Psi(u)
= M(x) + E(x)
= a_u \left(A_q \frac{\log(2x) - \text{li}(x)}{\varphi(q)}\right) + O\left(\frac{x}{\log^b x}\right) + O\left(\frac{1}{\varphi(q) \log x}\right)
= \delta(u, q, a) \left(\log(2x) - \text{li}(x)\right) + O\left(\frac{x}{\log^b x}\right),
\]

where \( \delta(u, q, a) = a_u A_q / \varphi(q) \geq 0 \), and \( a_u \geq 0 \) is a correction factor depending on \( \ell \).
But $\delta(u, q, a) > 0$ contradicts the hypothesis (44) for all sufficiently large numbers $x \geq x_0$.
Ergo, the short interval $[x, 2x]$ contains primes $p = qn + a$ such that the $u$ is a fixed primitive root. Specifically, the counting function is given by

$$\pi_u(x, q, a) = \sum_{\substack{p \leq x \\ p \equiv a \mod q}} \Psi(u) = \delta(u, q, a) \log x + O\left(\frac{x}{\log^b x}\right). \quad (47)$$

This completes the verification. ■

The determination of the correction factor $a_u$ in a primes counting problem is a complex problem, some cases are discussed in [20], and [24].

7 References

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