Bifurcations and limit cycles due to self-excitation in nonlinear systems with joint friction: Initialization of isolated solution branches via homotopy methods

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The main objective of this contribution is finding isolated stationary solutions, e.g. equilibria or limit cycles, in dynamical systems. Usually, NEWTON-type methods are applied for solving the resulting algebraic equation system. Here, the most difficult point is finding adequate initial conditions that are providing a solution on the isolated branch. So, there is a need for a more straight forward manner of initialising the continuation of isolated solutions. Within this contribution, homotopy methods are applied. The crucial point is to define a simplified version of the problem \( F(x; \lambda) \), which can be continuously transformed into the original one. As an example, limit cycles of a friction oscillator including COULOMB damping is discussed and two types of homotopy maps are addressed to obtain a starting point for their continuation.

1 Motivation

Isolated solution branches can occur in various types of dynamical systems. For instance, buckling problems considering an imperfection can involve isolated branches of static equilibria.

Fig. 1 shows the degenerated bifurcation diagram of a buckling column as a function of a load \( p \), where an imperfection \( \left\| \psi_0 \right\| > 0 \) gives rise to an isolated branch. In this case, an analytical solution

\[
p = \frac{\varphi + \kappa \varphi^3}{\sin(\varphi + \psi_0)}
\]

is available and every solution is directly obtained [1]. However, for general problems closed form solutions will usually not be available.

So, there is a need for more generally procedures to calculate these isolated domains. The most evident way would be choosing a similar problem \( G(x; \mu) = 0 \), which has a known or obvious solution. Then, a continuous transformation shall lead from the simplified problem to the original one, \( F(x; \mu) = 0 \). Finally, a solution of the original problem may be obtained, and can be traced using classical path following techniques.

2 Homotopy methods

When applying NEWTON’s method, finding adequate initial conditions may be a crucial point. To guarantee better convergence and expand the basin of attraction of a solution \( x \), homotopy methods can be applied. The general idea is to define a map

\[
H_\lambda(x; \mu) = 0, \quad 0 < \lambda < 1 \quad \text{and} \quad H_0(x; \mu) = G(x; \mu) = 0 \quad \text{(cheap problem)}
\]

where a simplified or cheap problem \( G(x; \mu) = 0 \) is continuously transformed into the complex or expensive problem \( F(x; \mu) \) holding the bifurcation parameter \( \mu \) constant. Within this contribution, for designing the homotopy map \( H_\lambda \), a convex approach was selected, in the form of a weighted linear superposition

\[
H_\lambda(x; \mu) = \lambda F(x; \mu) + (1 - \lambda) G(x; \mu)
\]

related to the homotopy parameter \( \lambda \). Following the path \( \gamma = (x^T, \lambda)^T \), where \( H_\lambda(\gamma) = 0 \) and \( \lambda \in [0, 1] \) holds, a solution of the expensive problem \( F(x; \mu) = 0 \) is obtained. Here, one has to assume, the expensive problem has a solution, \( \gamma \) has finite length and the implicit function theorem holds [2].

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2.1 Selection of the cheap problem

To start the continuation of $\gamma$ in a efficient way, the choice of a suitable $G$ is reasonable. Here, a trivial determination of the cheap problems solution would be feasible. First, the global homotopy, where

$$ G(x; \mu) = F(x; \mu) - F(\alpha; \mu), \quad \alpha \in \mathbb{R}^N $$

holds, provides a trivial solution $x = \alpha$ at $\lambda = 0$ [2]. To make it even more simple, the probability-one homotopy was established [2]. Here, the cheap problem

$$ G(x; \mu) = (x - \alpha), \quad \alpha \in \mathbb{R}^N, $$

provides at $\lambda = 0$ only one solution $x = \alpha$. The nomenclature results from the fact, that it can be mathematically shown that the smooth path $\gamma$ converges to a solution with the probability of one [2].

3 Results

In the following, periodic limit cycles of a friction oscillator including COULOMB damping [3], are approximated using the Harmonic Balance Method with $H = 5$ harmonics. Considering the underlying smooth system, the stability border of the rest position is at $\mu = 0$ and provides the starting point of the periodic solution branch (dashed line). Applying non-smooth COULOMB friction results in a stable rest position for $\mu \in \mathbb{R}$ [3] and homotopy methods are applied to detect the starting point $x_0$ for continuation of the isolated domain, see fig. 2a.

![Bifurcation diagram for the friction oscillator on a belt considering non-smooth COULOMB friction](image)

(2a) Bifurcation diagram for the friction oscillator on a belt considering non-smooth COULOMB friction [3]

![Homotopy path $\gamma$ for two different selections of $G(x; \mu)$.](image)

(2b) Homotopy path $\gamma$ for two different selections of $G(x; \mu)$.

Fig. 2b shows the continuation related to homotopy parameter $\lambda$ for the two different homotopy maps $H_\lambda(x; \mu)$ shown in section 2.1, while $\mu = -0.5$ is kept constant. The path $\gamma = \gamma(s)$ is parametrized by the arc-length $s$ and a classical predictor-corrector scheme is used. The starting point $\alpha$ is the solution of the underlying smooth system for $\mu = -0.5$.

4 Conclusion

This contribution demonstrates the use of homotopy methods to initialize the continuation of isolated branches in a rather straight forward manner. Here, two homotopy methods were been applied. It has been shown that the selection of cheap problems $G(x; \mu)$ has an impact on the homotopy path, while the influence of its trivial solution $\alpha$ on the path will be the objective of future research. The optimal selection of $\alpha$, which may provides a path $\gamma$, where all solutions of the expensive problem $F(x; \mu)$ are included, see [4], will also be objective of future research.

References

[1] D. Gross, W. Hauger and P. Wriggers, Technische Mechanik 4 (Springer Vieweg, 2018).
[2] E.L. Allgower and K. Georg, Introduction to Numerical Continuation Methods (Colorado State University, 1990).
[3] H. Hetzler, Bifurcations in autonomous mechanical systems under the influence of joint damping, J. S. & V. 333, 5953 - 5969 (2014).
[4] M. Kuno and J.D. Seader, Computing all real solutions of Systems of nonlinear equations with a Global Fixed-Point Homotopy, I. & c. c. r. (1988).

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