Alternative Descriptions in Quaternionic Quantum Mechanics

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Abstract

We characterize the quasianti-Hermitian quaternionic operators in QQM by means of their spectra; moreover, we state a necessary and sufficient condition for a set of quasianti-Hermitian quaternionic operators to be anti-Hermitian with respect to a uniquely defined positive scalar product in an infinite dimensional (right) quaternionic Hilbert space. According to such results we obtain two alternative descriptions of a quantum optical physical system, in the realm of quaternionic quantum mechanics, while no alternative can exist in complex quantum mechanics, and we discuss some differences between them.

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1 Introduction

Many attempts have been made in the past in order to formulate quantum mechanics in Hilbert spaces over the skew-field $\mathbb{Q}$ of quaternions. In the early 1960’s a systematic approach began to quaternionic quantum mechanics ($QQM$)\cite{1}; at present, a clear and detailed review of this theory, together with the foundations of quaternionic quantum field theory, can be found in Ref.\cite{2}.

It is worth noting that an important difference exists between complex and quaternionic quantum mechanics about Hamiltonians operators and observables. In both theories, observables are associated with self-adjoint (or Hermitian) operators, whereas Hamiltonians are Hermitian in complex quantum mechanics ($CQM$), but they are anti-Hermitian in QQM, and the same happens for the symmetry generators, like the angular momentum operators. Moreover, in CQM any anti-Hermitian operator can be made Hermitian (and vice versa) by multiplying by $i$. In QQM in contrast, an anti-Hermitian operator cannot be trivially converted to a Hermitian one by multiplying by a c-number; actually in this context in order to obtain such a conversion one needs a ”phase” operator \cite{2}.

Thus, if one wishes to enlarge the theoretical framework and to generalize standard quaternionic Hamiltonians and symmetry generators (as happened in
CQM where pseudo-Hermiticity has been fruitfully introduced), in QQM one rather needs to deal with pseudoanti-Hermitian quaternionic operators.

**Definition [3].** A quaternionic linear operator $H$ is said to be $(\eta)$-pseudoanti-Hermitian if a linear invertible Hermitian operator $\eta$ exists such that

$$\eta H \eta^{-1} = -H^\dagger.$$  

(1)

If Eq. (1) holds with a bounded positive definite $\eta$, $H$ is said quasianti-Hermitian.

Of course, several $\eta$ can exist which verify Eq. (1). The properties of pseudoanti-Hermitian Hamiltonians in QQM are analogous to the ones of pseudo-Hermitian in CQM. In particular, a new inner product in the Hilbert space can be associated with any bounded positive definite $\eta$ which fulfils Eq. (1), and different $\eta$'s give rise to alternative descriptions [4].

In this paper, we preliminarily characterize in sec.2 the subclass of quasianti-Hermitian quaternionic operators with discrete spectrum (in finite dimensional vector spaces), showing that they are necessarily diagonalizable operators with imaginary eigenvalues (and vice versa). Next, facing the unicity problem, we derive in sec.3 a necessary and sufficient condition for a set of quasianti-Hermitian operators to be anti-Hermitian with respect to a uniquely defined scalar product in quaternionic Hilbert spaces. Finally, we consider in sec.4 two alternative descriptions of a physical system in quantum optics, which are possible only in the realm of QQM, according with the previous result, and we discuss some
differences between them.

2 Quasianti-Hermitian quaternionic operators

In this section, we characterize the subclass of quasianti-Hermitian quaternionic operators by means of their spectra, in strict analogy with similar statements in CQM \[5\], \[6\]. The following proposition, which holds in finite dimensional Hilbert spaces, provides a necessary and sufficient condition for a quaternionic operator with discrete spectrum to be quasianti-Hermitian.

**Proposition 1.** Let $H$ be a quaternionic linear operator with discrete spectrum. Then, a definite operator $\eta$ exists such that $H$ is $\eta$-pseudoanti-Hermitian (hence, $\eta$-quasianti-Hermitian) if and only if $H$ is diagonalizable with imaginary spectrum.

**Proof.** Let $H$ be a pseudoanti-Hermitian operator. We preliminarily observe that, being in any case $\eta$ an invertible operator, all its eigenvalues must be different from zero, so that either $\eta$ is definite or it is indefinite. Now, let us suppose that a positive (respectively, negative) definite operator $\eta$ exists which fulfils condition (1); then, an $R$ exists such that $\eta = R^\dagger R$ \[7\] (respectively, $\eta = -R^\dagger R$), and by Eq. (1) we obtain

$$RHR^{-1} = -R^\dagger^{-1}H^\dagger R^\dagger = -(RHR^{-1})^\dagger,$$

i.e., $RHR^{-1}$ is anti-Hermitian, hence it is diagonalizable and it has a imaginary spectrum \[2\]. The same conclusion holds obviously with regard to $H$, since on
a right quaternionic vector space the similarity transformations preserve the properties of the spectrum, in the sense that the real part and the moduli of the imaginary part of the eigenvalues do not change under (quaternionic) similarity transformations.

Conversely, if $H$ is diagonalizable with imaginary spectrum, then by proposition 2 of Ref.[3], a positive definite operator $\eta = SS^\dagger$ exists which fulfils condition (1).

We remark that the above Proposition still holds in infinite dimensional Hilbert spaces $\mathcal{H}^Q$ if one assumes that the eigenvalues of $H$ have finite multiplicity and there is a basis on $\mathcal{H}^Q$ in which $H$ is block diagonal with finite dimensional blocks (see also Ref.[6]).

As a consequence of Proposition 1, any quasianti-Hermitian operator $H$ with discrete spectrum can be written by means of a set of biorthonormal vectors (if we suitably fix their phases) as

$$H = \sum_n \sum_{a=1}^{d_n} |\psi_n, a\rangle i E_n \langle \phi_n, a|, \quad E_n \geq 0,$$

where $d_n$ denotes the degeneracy associated to the $n$th eigenvalue, $a$ is a degeneracy label and the usual relations for a biorthonormal basis hold:

$$\langle \phi_m, b | \psi_n, a \rangle = \delta_{mn} \delta_{ba},$$

$$\sum_n \sum_{a=1}^{d_n} |\psi_n, a\rangle \langle \phi_n, a| = \sum_n \sum_{a=1}^{d_n} |\phi_n, a\rangle \langle \psi_n, a| = 1.$$
3 Alternative descriptions of quantum systems

As we already pointed out in the Introduction, different (alternative) description of the same physical system in a quaternionic Hilbert space $\mathcal{H}^Q$ are possible whenever different $\eta$’s fulfil condition (1). Indeed, for any bounded self-adjoint positive definite $\eta$, the space $\mathcal{H}^Q$ endowed with the scalar product $\langle \varphi \mid \psi \rangle_\eta = \langle \varphi \mid \eta \mid \psi \rangle$ is a Hilbert space $\mathcal{H}^Q_\eta$.

We do not report here the explicit proof of this property, which was already stated in CQM [8]; indeed the proof easily follows from the one in complex case since all the key steps in it still hold in a quaternionic Hilbert space, as for instance the closed graph theorem [9] and the unicity of the decomposition $\eta = S^2$, with $S$ positive and self-adjoint [10].

Hence, an undesirable ambiguity can arise, as we will explicitly show in the next section by means of a physical example; in order to remove that and obtain a proper (quaternionic) quantum mechanical interpretation, we will make resort to the concept of irreducibility of the physical operators on $\mathcal{H}^Q$.

As a preliminary step, we state the following lemma, which actually is very similar to the quaternionic version of the corollary of the Schur Lemma (on the irreducible quaternionic group representations of unitary operators) [10] and can be easily proven in the same way.

**Lemma.** Let $\{H_i\}$ $(i = 1, 2, ..., N)$ be an irreducible set of antiself-adjoint bounded quaternionic linear operators on the (right) quaternionic Hilbert space $\mathcal{H}^Q$. Then, the commutant of $\{H_i\}$, i.e., the set of all bounded quaternionic linear operators which commute with each $H_i$ is composed of the operators $T =$
Thus, the following proposition provides a necessary and sufficient condition for a set of $\eta$-quasianti-Hermitian quaternionic operators to admit a unique positive definite operator $\eta$ which satisfy the quasianti-Hermiticity condition.

**Proposition 2.** Let $\{H_i\}$ be a set of bounded $\eta$-quasianti-Hermitian operators on a right quaternionic Hilbert space $\mathcal{H}^Q$, where $\eta$ denotes a bounded positive selfadjoint operator. Then, $\eta$ is uniquely determined up to a global normalization factor if and only if the set $\{H_i\}$ is irreducible on $\mathcal{H}^Q$.

**Proof.** Firstly, we observe that, by assumption, the set of quasianti-Hermitian observables $H_i$ are bounded both on $\mathcal{H}^Q$ and on $\mathcal{H}^Q_\eta$, since $||H_i x||_\eta = ||SH_i x|| \leq ||SH_i S^{-1}|| ||Sx|| = ||SH_i S^{-1}|| ||x||_\eta$ (where the decomposition $\eta = S^2$, with $S$ positive, self-adjoint has been used [10]); furthermore, they are anti-selfadjoint on $\mathcal{H}^Q_\eta$ because $\eta H_i = -H_i^\dagger \eta \forall i = 1, 2, ..., N$. Assume now that an $\eta'$ exists with the same properties as $\eta$. Then, it follows that $[\eta'^{-1} \eta, H_i] = 0 \forall i = 1, 2, ..., N$. Hence, by the previous lemma, $\eta = \eta'(h1 + aI_a)$. But imposing the Hermiticity condition on $\eta$, one easily obtains $\eta'(h1 + aI_a) = (h1 - aI_a)\eta'$, which implies either $a = 0$ or $\{\eta', I_a\} = 0$. Denoting by $|\eta'\rangle$ an eigenvector of $\eta'$ : $\eta'|\eta'\rangle = \alpha|\eta'\rangle$ (where $\alpha > 0$, since $\eta'$ is positive) the condition $\{\eta', I_a\} = 0$ would imply $\eta'(I_a |\eta'\rangle) = -\alpha(I_a |\eta'\rangle)$, i.e., an eigenvector of $\eta'$ would exist associated with a negative eigenvalue, contradicting thus the hypothesis on the positive definiteness of $\eta'$. Then, $a = 0$ and $\eta = \eta' h$.

The converse is easily proven by merely paraphrasing the analogous proof in complex Hilbert spaces [8]. ■
As a consequence of the above proposition, any reducible set \( \{ H_i \} \) of quasianti-Hermitian operators admits at least two different positive operators \( \eta \) and \( \eta' \) which fulfil the quasianti-Hermiticity condition for any operator belonging to this set. This allows us to construct two different Hilbert spaces \( \mathcal{H}_\eta^Q \) and \( \mathcal{H}_{\eta'}^Q \), endowed with scalar products \( \langle \varphi | \eta | \psi \rangle \) and \( \langle \varphi | \eta' | \psi \rangle \) respectively, such that any \( H_i \) is anti-Hermitian on \( \mathcal{H}_\eta^Q \) as well as on \( \mathcal{H}_{\eta'}^Q \).

In particular, any reducible set of anti-Hermitian operators \( \{ H_i \} \) on \( \mathcal{H}^Q \) will appear at the same time as a set of anti-Hermitian operators on the Hilbert space \( \mathcal{H}_\eta^Q \) where \( \eta \) denotes a bounded, non trivial, positive operator which commutes with any element of \( \{ H_i \} \).

This is just the scenario of the example we will study in next section, which exactly mimics an analogous situation in CQM, where alternative descriptions arise in correspondence with different \( \eta \)'s which fulfil the quasi-Hermiticity condition for a set \( \{ H_i \} \).

### 4 A physical example

Let us consider a two level quantum optical system in the complex Hilbert space \( \mathcal{H} \) whose dynamics is described by the complex anti-Hermitian Hamiltonian

\[
H = 2\Omega_0 J_1 + 2\Omega_1 J_2 + \omega J_3 \quad (\hbar = 1), \quad \Omega_0, \Omega_1, \omega \in \mathbb{R}, \tag{2}
\]

i.e., a (real) linear combination of the anti-Hermitian operators \( J_l \) \( (l = 1, 2, 3) \), which obey the usual rules of commutation of the \( SU(2) \) algebra.
\[ [J_l, J_m] = -\varepsilon_{lmm} J_n. \]

Hamiltonian (2) can be used, for instance, in order to describe the interaction of a chirped classical e.m. field with a two level atomic system in a complex Hilbert space \[ H \]. This model has been extensively studied also to explain the Berry phase \[ [12] \].

As we already noted in the Introduction, \( H \) times \( i \) is of course an observable in CQM, and it coincides with the one introduced in \[ [11] \].

By resorting to the spinorial representation of the \( J \) operators

\[
\begin{align*}
J_1 &= \frac{i}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\
J_2 &= \frac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \\
J_3 &= \frac{i}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\end{align*}
\]

(3)

and putting \( \Omega = \Omega_0 + i\Omega_1 \), we can write the Hamiltonian (2) as a \( 2 \times 2 \) anti-Hermitian (time dependent) complex matrix :

\[
H = i \begin{pmatrix} \frac{\omega(t)}{2} & \Omega^*(t) \\ \Omega(t) & -\frac{\omega(t)}{2} \end{pmatrix}
\]

(4)

By changing the parameters in Eq. (2) or (4), we actually obtain a set of anti-Hermitian complex operators, which is of course irreducible in the 2-dimensional (complex) Hilbert space \( H \), since such is the spinorial representation of the \( J_l \)'s.

From a different point of view, we can interpret the Hamiltonian (4) as a \textit{anti-Hermitian quaternionic operator} in a (right) quaternionic Hilbert space \( H^Q \).
and the dynamics of our quantum system is then described by the Schroedinger equation

\[
\frac{d}{dt}|\Psi\rangle = -H|\Psi\rangle
\]

(5)

where \(|\Psi\rangle\) belongs to \(\mathcal{H}^Q\). (We recall that in QM the eigenvalues of a anti-Hermitian Hamiltonian are imaginary quaternions, whose moduli represent the values of the energy of the system).

Roughly speaking, \(\mathcal{H}^Q\) can be obtained from \(\mathcal{H}\) by simply adding to each complex vector \(|v\rangle \in \mathcal{H}\) a term \(|v\rangle j\), where \(j : j^2 = -1\) is a quaternionic unity different from \(i\); note that \(\dim \mathcal{H}^Q = \dim \mathcal{H} = 2\). Actually the various manner in which one can \textit{quaternionify} a complex Hilbert space are all equivalent to this one \[\text{[13]}\].

Now, let us denote by \(|\Psi\rangle = \begin{pmatrix} \Psi_{\alpha,+} + \Psi_{\beta,+}j \\ \Psi_{\alpha,-} + \Psi_{\beta,-}j \end{pmatrix} = \begin{pmatrix} \Psi_+ \\ \Psi_- \end{pmatrix} \), \((\Psi_{\alpha,\pm}, \Psi_{\beta,\pm} \in \mathbb{C})\) the quaternionic state vector representing the system; the components \(\Psi_-\) and \(\Psi_+\) can be interpreted from a physical point of view as the probability amplitudes for the system of being in the lowest or in the excited state, respectively. From the Schroedinger equation one immediately gets the time evolution of the components \(\Psi_{\pm}\):

\[
\begin{cases}
\Psi'_{\alpha,+} = \frac{i}{2} \omega(t) \Psi_{\alpha,+} + i \Omega^*(t) \Psi_{\alpha,-}, \\
\Psi'_{\alpha,-} = -\frac{i}{2} \omega(t) \Psi_{\alpha,-} + i \Omega(t) \Psi_{\alpha,+},
\end{cases}
\]

(6)
\[ \begin{cases} 
\Psi_{\beta,+}' = \frac{i}{2} \omega(t) \Psi_{\beta,+} + i \Omega^*(t) \Psi_{\beta,-}, \\
\Psi_{\beta,-}' = -\frac{i}{2} \omega(t) \Psi_{\beta,-} + i \Omega(t) \Psi_{\beta,+}.
\end{cases} \]  

(7)

where the prime denotes a time derivative.

Since the systems in (6) and (7) are identical, and they represent a rotation of the vector \( \Psi \) in the complex space, we can write their solutions as a whole using the Cayley-Klein (CK) matrix, independently on the quaternionic or complex character of \( \Psi \pm \):

\[
\begin{pmatrix}
\Psi_+ \\
\Psi_-
\end{pmatrix} =
\begin{pmatrix}
F^* & G \\
-G^* & F
\end{pmatrix}
\begin{pmatrix}
\Psi_+(0) \\
\Psi_-(0)
\end{pmatrix}, \quad (F, G \in \mathbb{C})
\]  

(8)

where \( F(t) \) and \( G(t) \) are complex functions depending on \( \omega \) and \( \Omega \) in a rather involved way; furthermore \( F(0) = 1, G(0) = 0 \), and \( |F|^2 + |G|^2 = 1 \) \[1\].

The CK matrix can be regarded as the matrix representation of the time evolution operator \( U \) associated with the time dependent Hamiltonian (4), and it constitutes a bi-dimensional (complex) unitary representation of the \( SU(2) \) group.

We remark once again that the form of \( U \) in (8) does not depend on the scalar field, \( C \) or \( Q \), adopted. Now, as long as we study the two-level system in \( \mathcal{H} \), the matrix form of \( U \) is clearly irreducible, hence, by the corollary of the Schur Lemma, no non-trivial \( \eta \) exists which commutes with \( U \). Recalling the discussion at the end of previous section, we can conclude that the description of the system in \( \mathcal{H} \) is unique.
On the contrary, if we now consider $U$ as a quaternionic group representation acting on $\mathcal{H}^Q$, it can be proven that this representation is reducible into the direct sum of two equivalent unidimensional irreducible quaternionic representations on $\mathcal{H}^Q$, so that $U$ admits a non-trivial commutant. By a direct computation, the most general quaternionic Hermitian matrix $\eta$ commuting with $U$ (and $H$) is

$$
\eta = \begin{pmatrix} a & jz \\ -jz & a \end{pmatrix}, \quad z \in \mathbb{C}.
$$

(9)

Since its matrix elements are independent of the Hamiltonian, $\eta$ is a secular metric in the sense of [16].

Moreover, $\eta$ is positive definite whenever $a > |z|$, as one can prove by solving the eigenvalue problem associated with it [17].

We can conclude that the group representation $U$ is unitary on $\mathcal{H}^Q$

$$
U^\dagger U = 1,
$$

(10)

and, moreover, it is $\eta$-unitary on $\mathcal{H}^Q_{\eta}$ [18], i.e.,

$$
U^\dagger \eta U = \eta.
$$

(11)

Alternatively, we can say that the Hamiltonian $H$ given in Eq. (4) is anti-Hermitian on $\mathcal{H}^Q$ as well as on the Hilbert space $\mathcal{H}^Q_{\eta}$ endowed with the scalar
product \(\langle \Psi | \eta | \Phi \rangle\), since

\[
H = \eta H \eta^{-1} = -H^\dagger,
\]

(12)

where \(\eta\) is given in Eq. (9).

Then, we may describe the dynamics of our system in \(\mathcal{H}^Q\) or in \(\mathcal{H}_\eta^Q\). Moreover, if the value \(a = 1\) is chosen in Eq. (9), one obtains that for each vector \(|\psi_c\rangle\), with \textit{complex components} \(\langle \psi_c | \psi_c \rangle = \langle \psi_c | \eta | \psi_c \rangle\). The relevant physical quantities with respect to both the alternative descriptions can now be easily computed.

Let us compute firstly the diagonal matrix elements of the angular momentum operators and of the Hamiltonian when the system is described by the vector \(|\pm\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\) and \(|\mp\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\). (In the sequel, by an abuse of language, we will call them \textit{expectation values}). One easily obtains

\[
\langle \pm | J_1 | \pm \rangle = 0, \quad \langle \pm | \eta J_1 | \pm \rangle = \mp \frac{1}{2} k z,
\]

\[
\langle \pm | J_2 | \pm \rangle = 0, \quad \langle \pm | \eta J_2 | \pm \rangle = \mp \frac{1}{2} j z,
\]

\[
\langle \pm | J_3 | \pm \rangle = \langle \pm | \eta J_3 | \pm \rangle = \pm \frac{i}{2},
\]

\[
\langle \pm | H^\dagger | \pm \rangle = \pm \frac{i}{2} \omega, \quad \langle \pm | \eta H^\dagger | \pm \rangle = \frac{i}{2} \omega - k z \Omega, \quad \langle - | \eta H^\dagger | - \rangle = - \frac{i}{2} \omega - k z \Omega^*.
\]

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All these values are obviously imaginary quaternions. In particular the moduli of the mean values of $H$

$$\langle \pm \mid H \mid \pm \rangle = \frac{|\omega|}{2}, \quad \langle \pm \mid \eta H \mid \pm \rangle = \sqrt{\frac{\omega^2}{4} + |z|^2|\Omega|^2}, \quad (13)$$

showing then a sharp difference between the two descriptions, which however vanishes as $|z| \to 0$.

More generally, one can compute all the expectation values associated with any vector $|\Psi\rangle = \begin{pmatrix} \Psi_+ \\ \Psi_- \end{pmatrix}, (\Psi_\pm \in Q)$, being trivially $|\Psi\rangle = |\pm\rangle \Psi_\mp |\Psi\rangle$. The only obvious warning concerns the norm of $|\Psi\rangle$; since (as one can obtain by an easy calculation)

$$\langle \Psi | \eta | \Psi \rangle = |\Psi_+|^2 + |\Psi_-|^2 + 2 \text{Re} \{\overline{\Psi}_+ j z \Psi_-\} \neq (\Psi | \Psi). \quad (14)$$

(Here, $\overline{\Psi}_+ \equiv \text{conjugate of } \Psi_+$).

Finally, making resort to the form (8) of the evolution operator $U$, we can also compute the transition probabilities in both the descriptions. Let us for instance assume that the system is in the excited state $|\pm\rangle$ at $t = 0$; the probability of finding the system in the ground state $|\mp\rangle$ at the time $t$ is given by

$$P_{\pm \to \mp}(t) = |\langle \mp | U | \pm \rangle|^2 = |G|^2 \quad (15)$$
according to the first description, and by [19]

\[ \mathcal{P}'_{+ \rightarrow -}(t) = |\langle -|\eta U|+\rangle|^2 = |z|^2|F|^2 + |G|^2 \]  

(16)

according to the alternative description.

We emphasize in conclusion that the possibility of an alternative description can only occur in QQM, which then appears as a theory intrinsically different from CQM, and not a mere transcription of it.

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