The heavy quark potential in pNRQCD

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The heavy quarkonium static potential is discussed within the framework of potential NRQCD. Some quantitative statements are made in the kinematical situation $mv \gg \Lambda_{\text{QCD}}$ at the level of accuracy of the next-to-leading order in the multipole expansion.

1. INTRODUCTION

In a recent series of papers [1,2] a detailed study of a suitable Effective Field Theory for heavy quark bound states, called potential NRQCD (pNRQCD) [3], has been started. Since several issues have been treated in that context, we address the reader to the quoted literature for a complete overview of the achieved results. Here we only mention the 1-loop matching [4] and the static energies of the hybrids [5] presented in these proceedings. While in the following, due to its considerable importance, we recollect and summarize our understanding of the heavy quarkonium static potential.

Let us define, first, what we mean with heavy quarkonium potential. Being heavy quarkonium a non-relativistic bound system, it is characterized by at least three energy scales: the mass or hard scale, $m$, the momentum or soft scale, $mv$, corresponding to the inverse of the bound-state size and the energy or ultrasoft (US) scale, $mv^2$ ($v$ is the heavy quark velocity). As a consequence, when US degrees of freedom are neglected, heavy quarkonium can be described as a bound state $\phi$ governed by a non-relativistic Schrödinger equation of the type

$$\left(\frac{p^2}{m} + V(r, p, S_1, S_2, m)\right) \phi = E \phi. \quad (1)$$

We will call $V$ the heavy quarkonium potential, which is in general a function of the quark distance $r$, momentum $p$, spin $S_1, S_2$, and mass $m$.

Another relevant quantity for heavy quarkonium physics is expected to be the energy between static sources. For heavy quarkonium in a singlet state this can be defined as

$$E_s(r) = \lim_{T \to \infty} \frac{i}{T} \ln \langle W_\square \rangle, \quad (2)$$

where $W_\square$ is the static Wilson loop of size $r \times T$ and the symbol $\langle \rangle$ means the average over the gauge fields. Often in the literature $E_s$ has been implicitly identified with the static limit of the Schrödinger potential, $\lim_{m \to \infty} V$. While it is reasonable to expect this identification to hold to some extent, there are no general grounds for it to be true in general. Indeed, already long time ago [6] several doubts have been raised on the infrared consistency of that identification at least in perturbation theory. It is the goal of this contribution to make some quantitative statements on the difference $E_s - \lim_{m \to \infty} V$. This will be done in the next section. In the next section we perform, as an intermediate step, the matching in the singlet sector of the pNRQCD Lagrangian at the next-to-leading order in the multipole expansion.

2. pNRQCD

Another scale is relevant in QCD, the scale where nonperturbative effects start to become important. We will call this scale $\Lambda_{\text{QCD}}$ and we will assume that $mv \gg \Lambda_{\text{QCD}}$. For sufficiently heavy quarkonium $v \ll 1$ and therefore the energy scales of the system are widely separated. This allows to systematically integrate out these
scales by matching QCD with simpler but equivalent Effective Field Theories. The integration of the hard scale ($\sim m$) gives rise to the effective theory known as non-relativistic QCD (NRQCD) \cite{9}, whereas the integration of the soft scale ($\sim mv$) gives rise to what we call potential NRQCD. Being $m$ and $mv$ well above $\Lambda_{QCD}$ both matchings can be done perturbatively.

By definition pNRQCD is the effective theory where only degrees of freedom below the soft scale remain dynamical. The surviving fields are quark-antiquark states (with US energy) and gluons with energy and momentum below $mv$. It is convenient to decompose the quark-antiquark states into singlets and octets under colour transformation. The relative coordinate $r$, whose typical size is the inverse of the soft scale, is explicit and can be considered as small with respect to the remaining dynamical lengths in the system. Hence the gluon fields can be systematically expanded in $r$ (multipole expansion). Therefore the pNRQCD Lagrangian is constructed not only by order in $1/m$, but also order by order in $r$. As a typical feature of an effective theory, all the non-analytic behaviour in $r$ is encoded in the matching coefficients.

The most general pNRQCD Lagrangian density that can be constructed with these fields and that is compatible with the symmetries of NRQCD is given at order $1/m^0$ (but we write also explicitly the kinetic energy in the centre-of-mass frame) and at the leading order in the multipole expansion by:

$$L_{pNRQCD} = \text{Tr} \left\{ S^\dagger \left( i\partial_0 - \frac{\mathbf{p}^2}{m} - V_s(r) + \ldots \right) S ight\}$$

$$+ O^\dagger \left( iD_0 - \frac{\mathbf{p}^2}{m} - V_o(r) + \ldots \right) O$$

$$+ gV_A(r)\text{Tr} \{ O^\dagger \mathbf{r} \cdot \mathbf{E} S + S^\dagger \mathbf{r} \cdot \mathbf{E} O \}$$

$$+ gV_B(r)\left( \frac{1}{2} \text{Tr} \left\{ O^\dagger \mathbf{r} \cdot \mathbf{E} O + O^\dagger \mathbf{r} \cdot \mathbf{E} O \right\} \right),$$

where $S = S(\mathbf{r}, \mathbf{R}, t)$ and $O = O(\mathbf{r}, \mathbf{R}, t)$ are the singlet and octet wave functions respectively and $\mathbf{R}$ is the centre-of-mass coordinate of the quark-antiquark system. All the gauge fields in Eq. (3) are evaluated in $\mathbf{R}$ and $t$. In particular $\mathbf{E} = \mathbf{E}(\mathbf{R}, t)$ and $iD_0O = i\partial_0O - g[A_0(\mathbf{R}, t), O]$.

We call $V_s$ and $V_o$ the singlet and octet static matching potentials respectively. By looking at the equations of motion of the Lagrangian (3) it is clear that, as far as higher order terms in the multipole expansion (terms of order $r$ or smaller in (3)) do not give potential-type contributions, $V_s$ and $V_o$ coincide with the static singlet and octet potential to be used in the heavy quarkonium Schrödinger equation. This happens when the US scale $mv^2$ is the next relevant scale of the system (i.e. $\Lambda_{QCD} \lesssim mv^2$). While, in the situation $mv \gg \Lambda_{QCD} \gg mv^2$ one expects to have nonperturbative corrections to the static potential coming from higher order terms in the multipole expansion. Both situations will be discussed in the next section.

Here we sketch the singlet matching at order $1/m^0$ and at the next-to-leading order in the multipole expansion. We refer the reader to \cite{9} for a complete and detailed discussion. The matching is in general done by comparing 2-fermion functions (plus external gluons at a scale below $mv$) in NRQCD and pNRQCD, order by order in $1/m$ and order by order in the multipole expansion. In order to get the singlet potential, we choose the following Green function in NRQCD:

$$I = \delta^3(\mathbf{x}_1 - \mathbf{y}_1)\delta^3(\mathbf{x}_2 - \mathbf{y}_2)\langle W_\Box \rangle,$$

where $W_\Box$ is the rectangular Wilson loop with edges $x_1 = (T/2, r/2), x_2 = (T/2, -r/2), y_1 = (-T/2, r/2)$ and $y_2 = (-T/2, -r/2)$. In pNRQCD we obtain at the next-to-leading order in the multipole expansion

$$I = Z_s(r)\delta^3(\mathbf{x}_1 - \mathbf{y}_1)\delta^3(\mathbf{x}_2 - \mathbf{y}_2)e^{-iTrV_s(r)}$$

$$\times \left( 1 - \frac{g^2}{N_c} T FV_A^2(r) \int_{-T/2}^{T/2} dt \int_{-T/2}^{T/2} dt' e^{-i(t-t')(V_s - V_o)} \right. \times \left( \langle r \cdot \mathbf{E}^a(t)\phi(t, t')_{ab} \mathbf{r} \cdot \mathbf{E}^b(t') \rangle \right),$$

where $\phi_{ab}$ is a Schwinger (straight-line) string in the adjoint representation and fields with only temporal argument are evaluated in the centre-of-mass coordinate. Comparing Eqs. (3) and (3), one gets at the next-to-leading order in the multipole expansion the singlet wave-function normal-
ization $Z_s$ and the singlet static potential $V_s$. $V_A$ and $V_o$ must have been previously obtained from the matching of suitable operators, but for the present purposes we only need the tree-level values: $V_A = 1$ and $V_o = (C_A/2 - C_F) \alpha_s/r$. Let us concentrate here on the matching potential $V_s$. By substituting the chromoelectric field correla-
tions arising from the diagrams studied first in [6], with its perturbative expression we obtain at the next-to-leading order in the multi-
opole expansion and at order $\alpha_s^3 \ln \alpha_s$

$$V_s(r) = E_s(r) \bigg|_{2\text{-loop+NLL}} + C_F \frac{\alpha_s^3}{r} \frac{C_A^3}{12} \ln \frac{C_A \alpha_s}{2r \mu},$$

(6)

where $E_s$ has been defined in Eq. (2). We note that $V_s$ and $E_s$ would coincide in QED and that therefore the effect we are studying here is a genuine QCD feature. The 2-loop contribution to $E_s$ has been calculated in [6]. The NLL contributions arise from the diagrams studied first in [6] and shown below. An explicit calculations gives

\[ \ldots \quad \begin{array}{cccc}
\vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots \\
\end{array} \quad = \frac{-C_F C_A^3}{12 \pi^2} \frac{\alpha_s^3}{2r \mu} \ln \frac{C_A \alpha_s}{2r \mu} + O(1/T) \]

Inserting this and the 2-loop contribution in Eq. (6) we get

$$V_s(r) \equiv -C_F \frac{\alpha_V(r, \mu)}{r},$$

$$\alpha_V(r, \mu) = \alpha_s(r) \left\{ 1 + (a_1 + 2\gamma_E \beta_0) \frac{\alpha_s(r)}{4\pi} \right. \right.$$  

$$+ \frac{\alpha_s^2(r)}{16 \pi^2} \left[ \gamma_E (4a_1 \beta_0 + 2 \beta_1) + \left( \frac{\pi^2}{3} + 4 \gamma_E^2 \right) \beta_0^2 \right]$$  

$$\left. + a_2 + \frac{C_A^3}{12} \frac{\alpha_s^3(r)}{\pi} \ln r \mu \right\},$$

(7)

where $\beta_n$ are the coefficients of the beta function ($\alpha_s$ is in the $\overline{\text{MS}}$ scheme), and $a_1$ and $a_2$ are given in [6].

We conclude this section noticing that the octet matching potential $V_o$ can be calculated in the same way as done for the singlet. In this case the relevant NRQCD Green function could be chosen to be $\delta^3(x_1 - y_1) \delta^3(x_2 - y_2)/(T^a W^T a)$, where the colour matrices are inserted in the endpoint Schwinger strings. Even if this Green function is gauge dependent, as discussed in [6], the matching should guarantee a gauge invariant definition of $V_o$. A 2-loop calculation is still not available, but work is in progress [6].

3. THE STATIC SINGLET POTENTIAL

In the previous section we have established the connection between $E_s$, the energy of static sources and $V_s$ the singlet static matching potential of pNRQCD. Here we discuss in two different kinematical situations the connection of $V_s$ with the static limit of the heavy quarkonium potential defined through the Schrödinger equation.

A) $\Lambda_{\text{QCD}} \lesssim mv^2$. This situation is expected to hold for toponium and for the bottomonium (charmonium?) ground state. As already mentioned, in this situation $V_s$, as given by Eq. (7), is the heavy quarkonium static potential in the sense given in the introduction. The explicit $\mu$ dependence of it originates from the fact that the US dependence (which have the same scale of the kinetic energy and therefore do not belong to the potential) have been explicitly subtracted out to form the static Wilson loop. This fact is not surprising if we understand the heavy quarkonium potential as a matching coefficient of pNRQCD. As a consequence even in a purely perturbative regime the static heavy quarkonium potential (as well as $\alpha_V$) turns out to be an infrared sensitive quantity. In this situation nonperturbative effects are only of non-potential nature (see for instance the Leutwyler–Voloshin type corrections in the situation $\Lambda_{\text{QCD}} \ll mv^2$ [4]). Finally we mention the quite obvious fact that, when calculating any physical observable, the $\mu$ dependence in (7) must cancel against $\mu$-dependent contributions coming from the US gluons (see for instance [1]).

B) $mv \gg \Lambda_{\text{QCD}} \gg mv^2$. Since in this situation there is a physical scale ($\Lambda_{\text{QCD}}$) above the US scale, a potential can be properly defined only once this scale has been integrated out. At the
next-to-leading order in the multipole expansion we get

\[ V(r) = -C_F \frac{\alpha_V(r, \mu)}{r} \]

\[ -i \frac{g^2}{N_c} T_F V_A^2(r) \frac{r^2}{3} \int_0^\infty dt \, e^{-\mu(V_c - V_s)} \times \langle E^a(t) \phi(t, 0)^{adj}_{ab} E^b(0) \rangle(\mu), \quad (8) \]

\[ e^{-\mu(V_c - V_s)} = 1 - it(V_c - V_s) - \frac{t^2}{2} (V_c - V_s)^2 + \ldots \]

Therefore, the heavy quarkonium static potential \( V \) is given in this situation by the sum of the purely perturbative piece calculated in Eq. (3) and a new term carrying also nonperturbative contributions (contained into non-local gluon field correlators). This last one can be organized as a series of power of \( r^n \). We stress that due to the condition \( mv \gg \Lambda_{QCD} \), this expansion makes sense only in the short-range. Typically the nonperturbative piece of Eq. (8) absorbs the \( \mu \) dependence of \( \alpha_V \) (see Eq. 2 for an example) so that the resulting potential \( V \) is now scale independent.

The infrared sensitivity of the static potential can also be expressed in terms of renormalons (see for instance [2]). Rephrasing them in the Effective Field Theory language of pNRQCD we can say that the singlet matching potential \( V_s \), as defined in Eq. (7), suffers from IR renormalons ambiguities with the following structure

\[ V_s(r)|_{\text{IR ren}} = C_0 + C_2 r^2 + \ldots \quad (9) \]

The constant \( C_0 \sim \Lambda_{QCD} \) is known to be cancelled by the IR pole mass renormalon (\( 2m_{\text{pole}}|_{\text{IR ren}} = -C_0 \)). While Eq. (3) provides us with the explicit expression for the operator which absorbs the \( C_2 \sim \Lambda_{QCD}^2 \) ambiguity [3]. More precisely the order \( r^2 \) term on the right-hand side of Eq. (8) suffers from UV and IR renormalons. The UV renormalon ambiguity of it (which can be calculated simply by substituting the chromoelectric field correlator with its perturbative expression and summing up the leading \( \log \) of all the bubble diagrams) cancels exactly the second term in the expansion (8):

\[ -i \frac{g^2}{N_c} T_F V_A^2(r) \frac{r^2}{3} \]

\[ \times \int_0^\infty dt \langle E^a(t) \phi(t, 0)^{adj}_{ab} E^b(0) \rangle(\mu) \bigg|_{\text{UV ren}} = -C_2 r^2. \]

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