Convergence of Distributed Randomized PageRank Algorithms

Wenxiao Zhao, Han-Fu Chen, and Hai-Tao Fang

Abstract—The PageRank algorithm employed by Google quantifies the importance of each page by the link structure of the web. To reduce the computational burden the distributed randomized PageRank algorithms (DRPA) recently appeared in literature suggest pages to update their ranking values by locally communicating with the linked pages. The main objective of the note is to show that the estimates generated by DRPA converge to the true PageRank value almost surely under the assumption that the randomization is realized in an independent and identically distributed (iid) way. This is achieved with the help of the stochastic approximation (SA) and its convergence results.

Index Terms—Distributed randomized PageRank algorithm, stochastic approximation, almost sure convergence.

I. INTRODUCTION

The PageRank algorithm employed by Google quantifies the importance of each page by the link structure of the web and it has achieved a great success as a commercial searching engine. Let us first recall the PageRank problem presented in [3][10]. Consider a web with \( n \) pages. The web is modeled by a direct graph \( G = (\mathcal{V}, \mathcal{E}) \), where \( \mathcal{V} = \{1, 2, \ldots, n\} \) is the index set of the pages and \( \mathcal{E} \subset \mathcal{V} \times \mathcal{V} \) is the set of links representing the structure of the web.

If \((i, j)\in \mathcal{E}\), then page \( i \) has an outgoing link to page \( j \). Without losing generality, we assume \( n > 2 \).

Denote by \( S_i \) the set of those pages which have incoming links from page \( j \), and by \( n_i \) the number of pages in \( S_i \). Thus we have associated with the graph \( G \) a link matrix

\[
A = [a_{ij}]_{n \times n}, \quad a_{ij} = \begin{cases} \frac{1}{n_j}, & j \in \mathcal{L}_i, \\ 0, & \text{otherwise}, \end{cases}
\]

where \( \mathcal{L}_i = \{j : (j, i) \in \mathcal{E}\} \). It is clear that \( \sum_{j=1} a_{ij} \) equals either 1 or 0.

The importance of a page \( i \) is characterized by its PageRank value \( x_i \in [0, 1], \quad i \in \mathcal{V} \). Let us assume \( \sum_{i=1} x_i = 1 \). The basic idea of the PageRank algorithm is that a page which has links from important pages is also important. Mathematically, this suggests to define the PageRank value of page \( i \) by

\[
x_i^* = \sum_{j \in \mathcal{L}_i} \frac{x_j^*}{n_j},
\]

or equivalently, to define \( x^* = [x_1^*, \ldots, x_n^*]^T \) from the following linear algebraic equation

\[
x^* = Ax^*, \quad x_i^* \in [0, 1].
\]

The normalization condition \( \sum_{i=1} x_i^* = 1 \) is possible to be met because if \( x^* \) satisfies \( 3 \) then \( \lambda x^* \) with \( \lambda \in (0, 1) \) also satisfies \( 3 \).

However, since the huge growing size of Internet, 8 billion pages as reported in [10], the matrix \( A \) is of dimension 8 billion × 8 billion, the computation of the PageRank value \( x^* \) becomes a problem. It is reported in [18] that the classical centralized method, such as the Power Method \( 1 \), is rather time-consuming. In this regards, several other approaches, such as the adaptive computation method \( 15 \), distributed randomized method \( 9 \) \( 10 \) \( 11 \) \( 12 \) \( 19 \), asynchronous iteration method \( 14 \) \( 16 \), etc., have been proposed. See also \( 11 \), \( 2 \), \( 4 \), and \( 13 \) among others.

According to the DRPA approach introduced in \( 9 \) \( 10 \) \( 11 \) \( 12 \), the pages can update their PageRank values by locally communicating with the linked pages and the computation load required by this approach is rather mild. It has been shown that the estimates generated by this kind of algorithms converge to the true PageRank value in the mean-square sense. The main objective of the note is to show that the estimates generated by DRPA converge to the true PageRank value with probability one under the assumption that the randomization is realized in an iid way. To achieve this, some results from SA algorithm \( 6 \) are applied.

The rest of the note is arranged as follows. In Section II, DRPA is introduced and the main results of the note are presented. The convergence analysis is given in Section III. The DRPA in a more general case is discussed in Section IV and some concluding remarks are addressed in Section V.

Notations. Denote by \( (\Omega, \mathcal{F}, \mathbf{P}) \) the basic probability space and by \( \omega \in \Omega \) a sample in the probability space. A probability vector \( \mathbf{x} = [x_1, \ldots, x_n]^T \in \mathbb{R}^n \) is defined as \( x_i \geq 0, \quad i = 1, \ldots, n \) and \( \sum_{i=1} x_i = 1 \). While a stochastic matrix \( A = [a_{ij}] \in \mathbb{R}^{n \times n} \) is defined as \( a_{ij} \geq 0, \quad i, j = 1, \ldots, n \) and \( \sum_{i=1} a_{ij} = 1, \quad j = 1, \ldots, n \). Denote by \( S \in \mathbb{R}^{n \times n} \) and \( 1 \in \mathbb{R}^n \) the matrix and vector with all entries being 1. We say that a matrix or a vector is positive if all the entries are positive. By \( \|x\| \) we denote the Euclidean norm of a vector \( x \in \mathbb{R}^n \). Finally, by \( b \in B \) we mean that the element \( b \) does not belong to the set \( B \).

II. DISTRIBUTED RANDOMIZED PAGE RANK ALGORITHM

We recall some basic results in PageRank computation. Notice that in the real world there exist nodes, for example, the dangling nodes, which have no outgoing links to other nodes and thus correspond to zero columns of the link matrix \( A \). To avoid the computational difficulty caused by this, the following assumption A1) is often made on the matrix \( A \).

A1) \( A \in \mathbb{R}^{n \times n} \) is a stochastic matrix.

From \( 1 \) it is clear that the PageRank value of the web is the eigenvector corresponding to eigenvalue 1 of the matrix \( A \). In order the eigenvalue 1 to have multiplicity 1, the following technique is adopted in \( 3 \) and \( 10 \). Define the matrix \( M \in \mathbb{R}^{n \times n} \) by

\[
M = (1 - \alpha)A + \alpha \frac{S}{n},
\]

where \( \alpha \in (0, 1) \).

Lemma 1: \((3)(10)\) If A1) holds, then the following assertions take place.

i) \( M \) is a positive stochastic matrix, whose eigenvalue 1 is with multiplicity 1, and all eigenvalues of \( M \) are in the closed unit disk;

ii) The eigenvectors \( \mathbf{e} \) and \( \mathbf{f} \) of \( M \) corresponding to eigenvalue 1 satisfy \( \mathbf{e} = -\mathbf{f} \), and one of them is positive.

Definition 1: \((3)\) The PageRank value \( x^* \) of web \( G \) is defined by

\[
x^* = Mx^*, \quad x_i^* \in [0, 1], \quad \sum_{i=1} x_i^* = 1.
\]
A widely used solution of the PageRank problem \( \mathbf{5} \) is the Power Method (7) which is recursively computed:

\[
x_{k+1} = Mx_k = (1 - \alpha)Ax_k + \frac{\alpha}{n} 1
\]

with \( x_0 \in \mathbb{R}^n \) being a probability vector.

**Lemma 2:** \((13)(10)\) For the Power Method \( \mathbf{6} \) the following convergence takes place:

\[
x_k \to x^* \quad \text{as} \quad k \to \infty
\]

for any probability vector \( x_0 \).

DRPA considered in \((9)(10)(12)\) makes the link matrices \( \{A_i\} \), to be defined below, to be sparse and thus greatly simplifies the computation.

Consider the web \( \mathcal{G} = (\mathcal{V}, \mathcal{E}) \). The basic idea of DRPA is as follows: At time \( k \), page \( i \) updates its PageRank value by locally communicating with the pages which have incoming links from page \( i \) and/or outgoing links to page \( i \), and page \( i \) which takes the above action is determined in a random manner. To be precise, DRPA is given by

\[
x_{1,i,k+1} = (1 - \alpha_1)A_{\theta(k)}x_{1,k} + \frac{\alpha_1}{n} 1
\]

with \( x_{1,0} \) being an arbitrary probability vector, \( \alpha_1 = \frac{2\alpha}{n - \alpha(n - 2)} \) and the link matrix

\[
(A_{i})_{ij} = \begin{cases} a_{ij}, & \text{if } j = i \lor l = i \\ 1 - a_{ij}, & \text{if } j = l \neq i \\ 0, & \text{otherwise} \end{cases}
\]

for \( i = 1, \ldots, n \), and where \( \{\theta(k)\}_{k=0}^\infty \) is assumed to be a sequence of iid random variables with probability

\[
P(\theta(k) = i) = \frac{1}{n}, \quad i = 1, \ldots, n.
\]

It is clear that matrices \( \{A_i\} \) are sparse.

**Lemma 3:** \((10)\) If A1 holds and \( \alpha_1 = \frac{2\alpha}{n - \alpha(n - 2)} \), then i) the matrix \( M_1 \) defined by

\[
M_1 \triangleq \left(1 - \frac{\alpha_1}{\alpha_0}\right)EA_{\theta(k)} + \frac{\alpha_1}{\alpha_0} S
\]

is a positive stochastic matrix and satisfies

\[
M_1 = \frac{\alpha_1}{\alpha} M + \left(1 - \frac{\alpha_1}{\alpha}\right) I, \quad \text{and} \quad Ex_{1,k+1} = M_1Ex_{1,k};
\]

ii) the average \( \mathbf{1}_{k+1} \) of \( \{x_{1,0}, \ldots, x_{1,k}\} \)

\[
\mathbf{1}_{k+1} = \frac{1}{k+1} \sum_{l=0}^k x_{1,l}
\]

converges to \( x^* \), the PageRank value of the web \( (\mathcal{V}, \mathcal{E}) \), in the mean square sense \( E||\mathbf{1}_{k+1} - x^*||^2 \to 0 \).

The matrix \( A_i \) describes the local link structure of page \( i \). The choice of \( \alpha_1 = \frac{2\alpha_1}{n - \alpha(n - 2)} \) is to make \( M_1 \) to take the form

\[
M_1 = \frac{\alpha_1}{\alpha} M + \left(1 - \frac{\alpha_1}{\alpha}\right) I \text{ so that } M_1 \text{ and } M \text{ share the common eigenvector corresponding to their biggest eigenvalue. This enables } Ex_{1,k} \text{ generated from } Ex_{1,k+1} = M_1Ex_{1,k} \text{ converges to } x^*.
\]

However, \( Ex_{1,k} \) is unavailable and thus the type algorithm \((12)\) is adopted. In the following, we consider the almost sure convergence of \( \{\mathbf{1}_{k+1}\}_{k=0}^\infty \).

Notice that algorithm \((12)\) can be written in a recursive way:

\[
\mathbf{1}_{k+1} = \mathbf{1}_{k+1} - \frac{1}{k+1}(\mathbf{1}_{k+1} - x_{1,k}).
\]

Recall that the SA algorithm (or the Robbins-Monro algorithm \((4)(7)\))

\[
z_{k+1} = z_k + \gamma_k(f(z_k) + \varepsilon_{z,k+1}), \quad k \geq 0,
\]

is used to search the roots of \( f(z) = 0 \), where \( \gamma_k \) is the stepsize, \( f(z) \) is the unknown function with the observation \( f(z_k) + \varepsilon_{z,k+1} \), and \( \varepsilon_{z,k+1} \) is the observation noise at time \( k + 1 \). Comparing \((13)\) and \((14)\), we find that \((13)\) is precisely an SA algorithm with the unknown function \( f(x_{1,k}) = -\langle \mathbf{1}_{k+1} - x^* \rangle \) valued at \( x_{1,k} \) and the observation noise \( \varepsilon_{x,k+1} = -\langle x_{1,k} - x^* \rangle \). This observation motivates us to establish the almost sure convergence of \( \{\mathbf{1}_{k+1}\}_{k=0}^\infty \) by the convergence analysis for SA \((6)\).

Indeed, we have the following results to be proved in Section III.

**Theorem 1:** If A1) holds and \( \alpha_1 = \frac{2\alpha}{n - \alpha(n - 2)} \), then the estimate generated by DRPA \((5)\) and \((13)\) converges to the true PageRank value almost surely:

\[
\mathbf{1}_{k+1} - x^* \to 0 \quad \text{a.s.}
\]

**Remark 1:** By the boundedness of \( \{\mathbf{1}_{k+1}\} \), the strong consistency of \( \{\mathbf{1}_{k+1}\} \) implies its convergence in the mean square sense.

**Theorem 2:** For \( \mathbf{1}_{k+1} \) generated from \((13)\) and with \( \alpha_1 = \frac{2\alpha}{n - \alpha(n - 2)} \), the following convergence rate takes place for any \( \epsilon \in (0, \frac{1}{2}) \):

\[
||\mathbf{1}_{k+1} - x^*|| = o\left(\frac{1}{k^{1/2-\epsilon}}\right) \quad \text{a.s.}
\]

### III. CONVERGENCE ANALYSIS

We first present the basic convergence results of stochastic approximation algorithm with expanding truncations (SAWET), which the proof for Theorems 1 and 2 is essentially based on. More results on SAWET can be found in \((6)\).

Assume the root \( z^0 \) of \( g(\cdot) : \mathbb{R}^n \to \mathbb{R}^n \) is a singleton, i.e., \( g(z^0) = 0 \).

Let \( \{x_k\}_{k=0}^\infty \) be a positive sequence increasingly diverging to infinity and let \( \{z_k\}_{k=0}^\infty \) be given by the following algorithms:

\[
z_{k+1} = [z_k + \gamma_ky_{k+1}] I\|z_k + \gamma_ky_{k+1}\| \leq M_{z_k},
\]

\[
= z^* I\|z_k + \gamma_ky_{k+1}\| > M_{z_k},
\]

\[
\sigma_k = \sum_{i=1}^n I\|z_i + \gamma_iy_{i+1}\| \leq M_{z_k}, \quad \sigma_0 = 0,
\]

\[
y_{k+1} = g(z_k) + \varepsilon_{z,k+1}.
\]

We need the following conditions.

**C1** \( \gamma_k > 0, \quad \lim_{k \to \infty} \gamma_k = 0, \quad \text{and} \quad \lim_{k \to \infty} \gamma_k = \infty. \)

**C2** There is a continuous differentiable function \( v(\cdot) : \mathbb{R}^n \to \mathbb{R} \) such that

\[
\sup_{z \in \mathcal{M}} \nabla v(z) g(z) < 0, \quad \forall \Delta > \delta > 0.
\]

**C3** For the sample path \( \omega \) under consideration:

\[
\lim_{T \to 0} \sup_{k \to \infty} \mathbb{E} m(n_k, T_k) = 0, \quad \forall T_k \in [0, T]
\]

for any \( \{n_k\} \) such that \( \{z_{n_k}\} \) converges, where \( m(k, T) = \max \{m : m \geq 0, \gamma_i \leq T\} \).

**C4** \( g(\cdot) \) is measurable and locally bounded.

**Proposition 1:** (Theorem 2.2.1 in \((6)\)) Assume that C1), C2), and C4) hold. Then \( z_k \to z^0 \) for those \( \omega \) for which C3) holds.

**Remark 2:** (Remark 2.2.6 in \((6)\)) If we know that \( \{z_k\} \) given by an SAWET algorithm evolves in a subspace of \( \mathbb{R}^n \), then it suffices to verify C2) in the subspace in order the corresponding convergence of \( \{z_k\} \) to hold.
Remark 3: Compared with the classical SA algorithm, such as the Robbins-Monro’s algorithm, the conditions required for convergence of SAAWET are significantly weaker. For details we refer to Chapters 1 and 2 of [6].

For the convergence rate of SAAWET, the following conditions are to be used.

C1') \( \gamma_k > 0 \), \( \gamma_k \rightarrow 0 \), \( \sum_{k=0}^{\infty} \gamma_k = \infty \), and \( \frac{\gamma_k}{\gamma_{k+1}} \rightarrow \sigma > 0 \).

C3') For the sample path \( \omega \) under consideration, the noise \( \{e_k\} \) in C3) can be decomposed into two parts \( e_k = e_k^s + e_k^c \) such that
\[
\sum_{k=1}^{\infty} (1 - \delta) e_k + e_k' < \infty, \quad e_k'^{s} = O(\gamma_k)
\]
for some \( \delta \in (0, 1] \).

C4') \( g(\cdot) \) is measurable and locally bounded and is differentiable at \( z^0 \) such that as \( z \rightarrow z^0 \)
\[
g(z) = F(z - z^0) + o(\|z - z^0\|).
\]
The matrices \( F \) and \( F + \sigma M \) are stable, where \( \sigma \) and \( \delta \) are given in C1') and C3'), respectively.

Proposition 2: (Theorem 3.1.1 in [6]) Assume C1'), C2), and C4') hold. Then on those sample paths for which C3') holds, \( z_k \) converges to \( z^0 \) with the following convergence rate:
\[
\|z_k - z^0\| = o(\gamma_k),
\]
where \( \delta \) is the one given in C3').

To apply Propositions 1 and 2, the key point is to verify the convergence of random series like \( \sum_{k=1}^{\infty} \gamma_k k \xi_k \). For this the following proposition plays an important role.

Define \( \{\alpha_k = [k^a]\}_{k \geq 0} \) for some \( a > 1 \), where \( [\cdot] \) denotes the integer part of the number, and define \( I_0(0) \triangleq \{0, \alpha_1, \alpha_2, \cdots\} \) and \( I_j(0) \triangleq \{\alpha_j + i : \alpha_1 + i, \cdots\} \) with \( I_j \triangleright I(0) \), where \( i(2) \triangleright i(1) \).

Proposition 3: (3)]

i) The sets \( \{I_j(0)\} \) defined above are disjoint and
\[
I_0(0) \bigcup \left( \bigcup_{j=1}^{\infty} \left( \bigcup_{i=1}^{\alpha_j} I_i(0) \right) \right) = \{0, 1, 2, 3, \cdots\}.
\]

ii) Let \( \{\xi_k\} \) be a sequence of random vectors with zero mean and \( \text{sup}_{k} \mathbf{E}(\|\xi_k\|)^2 < \infty \). If for any fixed \( j \) and \( i \), \( i(2) \leq i(1) \), the subsequence \( \{\xi_k : k \in I_j(i)\} \) is composed of mutually independent random variables with possible exception of a finite number of \( \xi_k \), then
\[
\sum_{k=1}^{\infty} \frac{1}{k^a} \xi_k < \infty \quad \text{a.s.}
\]
for all \( s > \frac{a}{2} - \frac{1}{2} \).

We now proceed to prove Theorem 1.

Proof of Theorem 1: As discussed in Section II, algorithm [13] can be rewritten as
\[
\mathbf{r}_{1,k+1} = \mathbf{r}_{1,k} + \frac{1}{k+1}(-((\mathbf{r}_{1,k} - x^*) + e_{1,k+1} + e_{2,k+1}), \quad (20)
\]
where \( e_{1,k+1} = -(x^* - E_x x_{k+1}) \) and \( e_{2,k+1} = -(E_x x_{1,k} - x_{1,k+1}) \).

By the fact that both \( x_{1,k+1} \) and \( \mathbf{r}_{1,k+1} \) are probability vectors for all \( k \geq 0 \), thus \( \|\mathbf{r}_{1,k+1}\| \leq 1 \) and the SA algorithm [20] is in fact an SAAWET algorithm whose sequence evolves in a bounded subspace of \( \mathbb{R}^n \). So by Proposition [1] and Remark [2] for [15], we only need to find a Lyapunov function to meet C2) and to verify the noise condition C3).

Define \( f(x) = -(x - x^*) \) and the Lyapunov function \( V(x) \triangleq \|x - x^*\|^2 \). It follows that
\[
\sup_{\delta \leq \|x - x^*\| \leq \Delta, \|x\| \leq 1} \nabla V(x)^T f(x) < 0,
\]
for any \( 0 < \delta < \Delta \). Hence assumption C2) holds.

So, by Proposition 1, to prove [15] it suffices to show that
\[
E_x x_{1,k} - x^* \rightarrow 0 \quad k \rightarrow \infty \quad (22)
\]
and
\[
\sum_{k=0}^{\infty} \frac{1}{k+1} (x_{1,k+1} - E_x x_{1,k+1}) < \infty \quad \text{a.s.} \quad (23)
\]
By Lemma 3, \( M_1 \) and \( M \) share the same eigenvector \( x^* \) corresponding to eigenvalue 1. Then by [11] and Lemma 2, we know that [22] holds. In what follows we show that [23] takes place.

Define the matrix
\[
\Phi(k, l) \triangleq \begin{pmatrix} A_{\theta(k)} A_{\theta(k-1)} \cdots A_{\theta(j)} \end{pmatrix}, \quad \text{if } j \leq k, \quad \mathbf{I}, \quad \text{if } j > k + 1 \quad (24)
\]
Then equation [9] can be rewritten as
\[
x_{1,k+1} = (1 - \alpha_k) k^{1+k} \Phi(k, 0) x_{1,0} + \frac{\alpha_k}{n} \sum_{l=1}^{k} (1 - \alpha_k) k^{1+k} \Phi(k, l) \mathbf{1}, \quad (25)
\]
from which it follows that
\[
x_{1,k+1} = E_{x_{1,k+1}} = (1 - \alpha_k) k^{1+k} \Phi(k, 0) x_{1,0} + \frac{\alpha_k}{n} \sum_{l=1}^{k} (1 - \alpha_k) k^{1+k} \Phi(k, l) \mathbf{1}, \quad (26)
\]
By noticing that \( x_{1,0} \) is a probability vector and \( \{A_{\theta(k)}\} \) are stochastic matrices, it is clear that
\[
\sum_{k=1}^{\infty} \frac{1}{k+1} (1 - \alpha_k) k^{1+k} \Phi(k, 0) x_{1,0} + \frac{\alpha_k}{n} \sum_{l=1}^{k} (1 - \alpha_k) k^{1+k} \Phi(k, l) \mathbf{1} < \infty \quad \text{a.s.} \quad (27)
\]
Thus, for [23] it remains to show that
\[
\sum_{k=1}^{\infty} \frac{1}{k+1} \xi_k < \infty \quad \text{a.s.,} \quad (28)
\]
where
\[
\xi_k \triangleq \sum_{l=1}^{k} (1 - \alpha_k) k^{1+k} \Phi(k, l) \mathbf{1} - E \Phi(k, l) \mathbf{1}. \quad (29)
\]
For any fixed \( a > 1 \), define \( \alpha_k \triangleq [k^a] \), \( k \geq 0 \). Further, as for Proposition 3 define \( I_0(0) \triangleq \{0, \alpha_1, \alpha_2, \cdots\} \) and \( I_j(i) \triangleq \{\alpha_j + i, \alpha_j+1 + i, \cdots\} \) and \( I_j \triangleright I(0) \triangleq \bigcup_{i=1}^{\alpha_j} I_i(0) \), where \( i(2) \triangleright i(1) \).

By Proposition 3, the sets \( \{I_j(i)\} \) are disjoint and
\[
I_0(0) \bigcup \left( \bigcup_{j=1}^{\infty} \left( \bigcup_{i=1}^{\alpha_j} I_i(0) \right) \right) = \{0, 1, 2, 3, \cdots\}.
\]
Take \( \tau \in (0, 1 - \frac{1}{a}) \) and define
\[
\xi_{k+1} \triangleq \sum_{l=1}^{k+1} (1 - \alpha_k) k^{1+k} \Phi(k, l) \mathbf{1} - E \Phi(k, l) \mathbf{1}. \quad (29)
\]
Notice that \( \xi_k \) is not mutually independent. For any fixed \( k \geq 1 \) and \( i \in [\alpha_k - \alpha_k + 1 \cdots \alpha_k - 1] \), let us consider the set \( \{\xi_{k+1} : k+1 \in I_j(i)\} \) and show that \( \xi_{k+1} \) are mutually independent with possible exception of a finite number of \( \xi_{k+1} \).

If \( k+1 \in I_j(i) \), then \( \xi_{k+1} \) is measurable with respect to the \( \sigma \)-algebra \( \sigma\{\theta([m^a] + i - 1), \cdots, \theta([m^a] + i - [(m^a) + i - 1])\}\) for some integer \( m \).
In the set \( \{ \xi_{k+1} : k+1 \in I_{i}(j) \} \), the random vector \( \xi_{[a(m-1)]} \) is with subscript neighboring with \( [a(m-1)] \) and \( i \). Since \( \{ \theta(k) \} \) is iid, for the mutual independence of random vectors in \( \{ \xi_{k+1} : k+1 \in I_{i}(j) \} \), it suffices to show that \( \xi_{[a(m-1)]} \) and \( \xi_{[b(m-1)]} \) are independent. It is clear that for this it suffices to show \( [a(m-1)] > [b(m-1)] \).

Noticing
\[
[a(m-1)] = am^{a-1} + o(m^{a-1}) \quad \text{as} \quad m \to \infty
\]
and \( \tau \in (0, 1 - \frac{1}{a}) \), we find that as \( m \to \infty \)
\[
\frac{[a(m-1)]}{m^{a-1}} = O\left(\frac{m^{a\tau}}{m^{a-1}}\right) = O\left(m^{a\tau+1-a}\right) = o(1).
\]
(30)
Thus, for fixed \( i \) and \( j \) the random vectors in the set \( \{ \xi_{k+1} : k+1 \in I_{i}(j) \} \) are mutually independent with possible exception of a finite number of vectors. Then by noticing \( \sup \mathbb{E} \| \xi_{k} \|^2 < \infty \) from Proposition 3 it follows that
\[
\sum_{k=1}^{\infty} \frac{1}{k+1} \xi_{k+1} < \infty \quad \text{a.s.} \quad (31)
\]

Further, we have
\[
\left\| \sum_{k=1}^{\infty} \frac{1}{k+1} (\xi_{k+1} - \xi_{k+1}) \right\| = \left\| \sum_{k=1}^{\infty} \frac{1}{k+1} \sum_{l=1}^{[k\tau]} (1 - \alpha_1)^{k-l} (\Phi(k,l) \mathbf{1} - E \Phi(k,l) \mathbf{1}) \right\|
\]
\[
= O\left(\sum_{k=1}^{\infty} \frac{1}{k+1} (1 - \alpha_1)^{[k\tau]}\right) = O(1), \quad (32)
\]
which combining with (31) yields (28). Thus, (23) has been proved.

Noticing (20), (22), and (28), by Proposition 1 we derive the assertion of Theorem 1.

**Proof of Theorem 2:** The proof can similarly be carried out as that for Theorem 1 by Propositions 2 and 3. We only outline the key points.

First we have the exponential rate of convergence \( \| E x_{1,k} - x^* \| = O(\rho^k) \) for some \( 0 < \rho < 1 \) (for the analysis we refer to, e.g., [10]).

By Proposition 3 and carrying out a similar discussion as that for (23), (27), and (28), we can also prove that
\[
\sum_{k=0}^{\infty} \frac{1}{k+1} s x_{1,k+1} - E x_{1,k+1} < \infty \quad \text{a.s.} \quad (33)
\]
with \( s > \frac{3}{2} - \frac{1}{a} \) for any fixed \( a > 1 \), which implies \( s > \frac{1}{2} \).

Then by Proposition 2 we obtain (16).

**IV. Extension to Case Where Multiple Pages Updating Simultaneously**

The protocol in Section II is based on the assumption that only one page updates its PageRank value each time. In this section, we discuss the convergence of DRPA for the case where multiple pages update their PageRank values simultaneously. Notice that the problem formulation in this section is precisely the same as that given in [10].

Assume that the sequences of Bernoulli random variables \( \{ \eta_i(k) \}_{i \geq 0} \), \( i = 1, \cdots, n \) are mutually independent and each sequence is iid random variables with probabilities
\[
\mathbb{P}(\eta_i(k) = 1) = \beta, \quad \mathbb{P}(\eta_i(k) = 0) = 1 - \beta.
\]
where \( \beta \in (0, 1) \), if \( \eta_i(k) = 1 \), then page \( i \) updates at time \( k \), sending its PageRank value to the pages that page \( i \) has outgoing links to, and requiring PageRank values from those pages which page \( i \) has incoming links from. While if \( \eta_i(k) = 0 \), no communication is required by page \( i \).

Set \( \eta(k) = (\eta_1(k), \cdots, \eta_n(k)) \). The vector \( \eta(k) \) reflects updating pages at time \( k \). The corresponding link matrix is given by
\[
(A_{p_1, \cdots, p_n})_{ij} \triangleq \begin{cases} a_{ij}, & \text{if } p_i = 1 \text{ or } p_j = 1, \\ 1 - \sum_{h, p_{h-1}} a_{hj}, & \text{if } p_i = 0 \text{ and } i = j, \\ 0, & \text{if } p_i = p_j = 0 \text{ and } i \neq j,
\end{cases}
\]
(36)
where \( (p_1, \cdots, p_n) \) is a realization of \( \eta(k) \). It is clear that \( A_{p_1, \cdots, p_n} \) is a sparse matrix.

Similar to (6) and (12), the DRPA for the multiple pages updating is given by
\[
x_{2,k+1} = (1 - \alpha_2) A_{\eta(k)} x_{2,k} + \frac{\alpha_2}{n} \mathbf{1},
\]
(37)
\[
\xi_{2,k+1} = \frac{1}{k+1} \sum_{i=0}^{k} x_{2,i},
\]
(38)
where \( x_{2,0} \) is an arbitrary probability vector and \( \alpha_2 = \frac{\alpha(1- (1-\beta)^2)}{1- \alpha (1- \beta)^2} \). Clearly, there are \( 2^n \) different link matrices.

**Remark 4:** Define \( M_2 = (1 - \alpha_2) E A_{\eta(k)} + \frac{\alpha_2}{n} \mathbf{1} \). The parameter \( \alpha_2 = \frac{\alpha(1- (1-\beta)^2)}{1- \alpha (1- \beta)^2} \) is to make \( M_2 \) to have the form:
\[
M_2 = \frac{\alpha_2}{n} M + (1 - \frac{\alpha_2}{n}) I \quad (10).
\]
Thus the eigenvector of \( M_2 \) corresponding to its biggest eigenvalue equals the PageRank value \( x^* \) which satisfies \( x^* = M x^* \). Noticing that \( A_{\eta(k)} \) is iid and carrying out the same discussion as that for Theorems 1 and 2, we can show that
\[
\| \xi_{2,k} - x^* \| = O\left(\frac{1}{k^{\frac{3}{2}}}\right) \quad \text{a.s.} \quad \text{for any } \epsilon \in (0, \frac{1}{2}).
\]

**Remark 5:** By carrying out the discussions similar to those done above, the a.s. convergence of DRPA can be established for some other setups, for example, when link failures randomly occur with the assumption that the failure occurring is iid (9).

**V. Concluding Remarks**

In the note, we have shown some further properties of DRPA introduced in [10]. By using some convergence results of SA, the strong consistency of estimates for the PageRank values as well as their convergence rate are established for the cases where one page updates each time and multiple pages update simultaneously.

There are many interesting problems left for future research. For example, a key assumption for strong consistency of DRPA is that the randomization law is iid. How to relax this assumption to the dependent case, for example, to Markov chains, is interesting. Another problem is to investigate the relation between the convergence rate and the size \( n \) of the web. It is also of interest to consider the PageRank computation with communication delay and web aggregation.

**References**

[1] R. Andersen, C. Borgs, J. Chayes, J. Hopcroft, V. S. Mirrokni, and S.-H. Teng, “Local computation of PageRank contributions”, in Algorithms and Models for the Web-Graph. Berlin, Germany: Springer, vol. 4863, Lect. Notes Comp. Sci., pp. 150–165, 2007.

[2] K. Avrachenkov, N. Litvak, D. Nemirovsky, and N. Osipova, “Monte Carlo methods in PageRank computation: When one iteration is sufficient”, SIAM J. Numer. Anal., vol. 45, pp. 890–904, 2007.

[3] S. Brin and L. Page, “The anatomy of a large-scale hypertextual Web search engine”, Computer Networks ISDN Syst., vol. 30, pp. 107–117, 1998.
[4] A. Z. Broder, R. Lempel, F. Maghoul, and J. Pedersen, “Efficient PageRank approximation via graph aggregation”, Inf. Retrieval, vol. 9, pp. 123–138, 2006.

[5] K. Bryan and T. Leise, “The $25,000,000,000 eigenvector: The linear algebra behind Google,” SIAM Rev., vol. 48, pp. 569–581, 2006.

[6] H. F. Chen, Stochastic Approximation and Its Applications. Dordrecht, The Netherlands: Kluwer Academic Publishers, 2002.

[7] R. A. Horn and C. R. Johnson, Topics in Matrix Analysis. Cambridge, U.K.: Cambridge Univ. Press, 1991.

[8] X. L. Hu and H. F. Chen, “Identification for Wiener systems with RTF subsystems”, European J. Control, vol. 6, pp. 581–594, 2006.

[9] H. Ishii and R. Tempo, “Distributed PageRank computation with link failures”, Proc. Amer. Control Conf., pp. 1976–1981, 2009.

[10] H. Ishii and R. Tempo, “Distributed randomized algorithms for the PageRank computation”, IEEE Trans. Automatic Control, vol. 55, no. 9, pp. 1987–2002, 2010.

[11] H. Ishii, R. Tempo and E. W. Bai, “A web aggregation approach for distributed randomized PageRank algorithms”, IEEE Trans. Automatic Control, vol. 57, pp. 2703–2717, 2012.

[12] H. Ishii, R. Tempo and E. W. Bai, “Distributed randomized pageRank algorithms over unreliable channels”, in Developments in Control Theory Towards Global Control (L. Qiu, J. Chen, T. Iwasaki and H. Fujioka, Editors), The Institution of Engineering and Technology, London, pp. 147-156, 2012.

[13] A. Juditsky and B. Polyak, “Robust Eigenvector of a Stochastic Matrix with Application to PageRank”, Proceedings of 51th IEEE Conference on Decision and Control, pp. 3171–3176, 2012.

[14] D. V. de Jager and J. T. Bradley, “Asynchronous iterative solution for state-based performance metrics”, Proc. ACM SIGMETRICS, pp. 373–374, 2007.

[15] S. Kannvar, T. Haveliwala, and G. Golub, “Adaptive methods for the computation of PageRank”, Linear Algebra Appl., vol. 386, pp. 51–65, 2004.

[16] G. Kollias, E. Gallopoulos, and D. B. Szyld, “Asynchronous iterative computations with Web information retrieval structures: The PageRank case”, in Parallel Computing: Current and Future Issues of High-End Computing, G. R. Joubert, Ed. et al. Julich, Germany: John von Neumann-Institut for Computing, vol. 33, NIC Series, pp. 309–316, 2006.

[17] H. J. Kushner and G. G. Yin, Stochastic Approximation and Recursive Algorithms and Applications, Second Edition. New York: Springer-Verlag, 2003.

[18] A. N. Langville and C. D. Meyer, Google’s PageRank and Beyond: The Science of Search Engine Rankings. Princeton, NJ: Princeton Univ. Press, 2006.

[19] Y. Zhu, S. Ye, and X. Li, “Distributed PageRank computation based on iterative aggregation-disaggregation methods,” in Proc. 14th ACM Conf. Inform. Knowledge Manag., pp. 578–585, 2005.