GENERATING SOLUTIONS OF A LINEAR EQUATION AND STRUCTURE OF ELEMENTS OF THE ZELISKO GROUP II

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Abstract. We continue our previous investigation of the Zelisko group of a matrix over Bézout domains. The explicit form of elements of this group over homomorphic image of Bézout domain of stable rank 1.5 is described.

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1. Introduction. Let $R$ be an integral domain in which every finitely generated ideal is principal (Bézout domain) with $1 \neq 0$ and let $R^{n \times n}$ be the ring of $n \times n$ matrices over $R$ in which $n \geq 2$. Let $U(R)$ and $\text{GL}_n(R)$ be groups of units of rings $R$ and $R^{n \times n}$, respectively. The notation $a|b$ in $R$ means that $b = ac$ for some $c \in R$. The greatest common divisor of $a_1, a_2, \ldots, a_n \in R \setminus \{0\}$, which is unique up to associates, is denoted by $(a_1, a_2, \ldots, a_n)$.

To a diagonal matrix $\Phi := \text{diag}(\varphi_1, \ldots, \varphi_k, 0, \ldots, 0) \in R^{n \times n}$, where $\varphi_k \neq 0$, $k \leq n$, and $\varphi_i$ is a divisor of $\varphi_{i+1}$ for $i = 1, \ldots, k - 1$, we associate the following subgroup (see [12, p. 61] and [10, 16])

$$G_\Phi = \{ H \in \text{GL}_n(R) \mid \exists S \in \text{GL}_n(R), \text{ s.t. } H\Phi = \Phi S \} \leq \text{GL}_n(R),$$

which is called the Zelisko group of the matrix $\Phi$. The concept of the Zelisko group as well as its properties, were used by Kazimirskiı [10] for the solution of the problem of extraction of a regular divisor of a matrix over the polynomial ring $F[x]$, where $F$ is an algebraically closed field of characteristic 0. The properties of the group $G_\Phi$ in which $\Phi \in R^{n \times n}$, were explicitly investigated in [12, Chapter 2.3 and Chapter 2.7] and [6, 16].

A ring $K$ has stable range 1.5 (see [14, p. 961] and [13, p. 46]) if for each $a, b \in K$ and $c \in K \setminus \{0\}$ with the property $(a, b, c) = 1$ there exists $r \in K$ such

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that \((a + br, c) = 1\). This notion arose as a modification of the Bass’s concept of the stable range of rings (see [3, p. 498]). Examples of rings of stable range 1.5 are Euclidean rings, principal ideal rings, factorial rings, rings of algebraic integers, rings of integer analytic functions, and adequate rings (see [5, 6] and [12, p. 21]). Note that the commutative rings of stable range 1.5 coincide with rings of almost stable range 1 (see [1, 11]).

Finally, certain properties of the Zelisko group \(\mathbf{G}_\Phi\) are closely related to a factorizability of the general linear group over the ring \(R\) of stable range 1.5 (see [4, 7], [8, Theorem 1.2.2., p. 12], [12, Chapter 2.7], [15, Theorem 3, p. 144]).

Let \(R\) be a commutative Bézout domain of stable range 1.5. For each \(m \in R \setminus \{0, U(R)\}\) we define the homomorphism \(\overline{\cdot} : R \rightarrow R_m = R/mR\).

The explicit appearance of the elements of the group \(\mathbf{G}_\Phi\) in a particular case when \(\Phi := \text{diag}(\varphi_1, \ldots, \varphi_n) \in R_{m}^{n \times n}\) in which \(\varphi_n \neq 0\) and \(\varphi_i\) is a divisor of \(\varphi_{i+1}\) for \(i = 1, \ldots, n-1\) was studied in [6]. In the present article, we continue to investigate the form of elements of the Zelisko group in general case, when some diagonal elements of the matrix \(\Phi\) can be zero.

First of all, we recall some definitions and facts from [6].

A solution of a solvable linear equation \(a \cdot x = b\) in a commutative ring which divides all other solutions is called a generating solution of this equation.

Each solvable linear equation \(a \cdot x = b\) in \(R_m\) has at least one generating solution by [6, Theorem 1(i)] and each two generating solutions \(x_1, x_2\) of this equation are pairwise strongly associates by [6, Theorem 1(ii)], i.e. \(x_1 = x_2 \varepsilon U(R_m)\).

The definition of strongly associates elements is given in [2, Definition 2.1, p. 441]. See also [9].

For each \(c \in R\), let \(\overline{c} := \overline{\cdot}(c) \in R_m\). Using [6, Lemma 3], we have
\[
\overline{c} = \mu_c \overline{c},
\]
where \(\mu_c := (c, m)\) and \(\overline{c} \in U(R_m)\). However, such representation of the element \(\overline{c}\) is not unique (see Example 1 after [6, Lemma 3]). The explicit form of a solution of the linear equation \(\overline{a} \cdot \overline{x} = \overline{b}\) in \(R_m\) depends on the choice of the representation of the elements \(\overline{a}, \overline{b} \in R_m\) in the form (1) (see proof of [6, Theorem 1(i)]). The next example illustrates the relationship between the solutions of the linear equation \(\overline{a} \cdot \overline{x} = \overline{b}\) in \(R_m\) obtained with different representation of elements \(\overline{a}, \overline{b}\) in the form (1).

**Example 1.** Let \(R_m = \mathbb{Z}_{36}\). Consider \(33 \cdot \overline{x} = \overline{30}\). Clearly \(33 = 3 \cdot 11\) and \(30 = 6 \cdot 5\) in which \((33, 36) = 3\), \((30, 36) = 6\), and \(11, 5 \in U(\mathbb{Z}_{36})\). A generating solution of the equation \(33 \cdot \overline{x} = \overline{30}\) is
\[
\overline{x}_0 = \left(\frac{6}{5}\right) \cdot \overline{5} \cdot \overline{11}^{-1} = \overline{2} \cdot \overline{5} \cdot \overline{23} = \overline{14},
\]
where \(\overline{23} = \overline{11}^{-1}\) (see [6, Theorem 1(i)]). Since \(\text{Ann}(33) = (12)\), we conclude that \(14 + \text{Ann}(33) = \{2, 14, 26\}\) is the set of all solutions of \(33 \cdot \overline{x} = \overline{30}\). Let’s now choose another representation of \(33\) and \(30\) in the form (1):

\[
\overline{33} = 3 \cdot \overline{35}, \text{ where } (33, 36) = 3, \text{ and } 35 \in U(\mathbb{Z}_{36}),
\]
\[
\overline{30} = 6 \cdot \overline{35}, \text{ where } (30, 36) = 6, \text{ and } 35 \in U(\mathbb{Z}_{36}).
\]
Clearly, the generating solution is \( x_0 = 2 \). Thus, we obtain another generating solution, which by virtue of [6, Theorem 1(ii)] are associated with each other.

2. Preliminaries. We start our proof with the following.

**Lemma 1.** Let \( T \) be commutative ring with unity \( 1 \neq 0 \). Let \( c_1, c_2, c_3 \) be nonzero elements of \( T \) such that \( c_1 \mid c_2 \mid c_3 \). Let \( \gamma_{21} \) and \( \gamma_{32} \) be two generating solutions of \( c_1 \cdot x = c_2 \) and \( c_2 \cdot x = c_3 \), respectively. Then \( \gamma_{21} \cdot \gamma_{32} \) is a generating solution of \( c_1 \cdot x = c_3 \).

**Proof.** Clearly, \( c_1 \gamma_{21} = c_2 \) and \( c_2 \gamma_{32} = c_3 \), so \( c_1 \gamma_{21} \gamma_{32} = c_3 \). This means that the set of all solutions of \( c_1 \cdot x = c_3 \) has the form \( \gamma_{21} \gamma_{32} + \text{Ann}(c_1) \). Similarly, the set of all solutions of \( c_1 \cdot x = c_2 \) has the form \( \gamma_{21} + \text{Ann}(c_1) \).

Let \( t \in \text{Ann}(c_1) \). Since \( \gamma_{21} \) is a generating solution of \( c_1 \cdot x = c_2 \), the element \( \gamma_{21} \) is a divisor of all elements of the ideal \( \text{Ann}(c_1) \). Thus \( \gamma_{21}t \) and \( t = \gamma_{21}t \) for some \( t \). It follows that

\[
t_1c_2 = t_1c_1\gamma_{21} = c_1(t_1\gamma_{21}) = c_1t = 0,
\]

so \( t_1 \in \text{Ann}(c_2) \).

By analogy, \( \gamma_{32} \) is a generating solution of \( c_2 \cdot x = c_3 \), that is \( \gamma_{32} \) is a divisor of all elements of the ideal \( \text{Ann}(c_2) \). Hence \( t_1 = \gamma_{32}t \) and \( t = \gamma_{21}t_1 = \gamma_{21}\gamma_{32}t \). This yields that \( \gamma_{21} \) a divisor of all elements of the ideal \( \text{Ann}(c_1) \), so all solutions of \( c_1 \cdot x = c_3 \) are divisors too. Consequently, \( \gamma_{21} \) is a generating solution. \( \square \)

Let \( \varphi_1, \varphi_2, \ldots, \varphi_t \in \{R_m \ \{0\}\} \), such that \( \varphi_1 \mid \varphi_2 \mid \cdots \mid \varphi_t \). Denote by \( M_{ij} \) the set of all solutions of the equation \( \varphi_j \cdot x = \varphi_i \), where \( i > j \). By symbol \( \varphi_i / \varphi_j \) we denoted the minimum generating solution from \( M_{ij} \) with respect to some selected relation of order \( \leq \) (see [6, text before Lemma 2, p. 58]). Such choice is not good enough. Therefore, we propose the following improvement of this notation without consideration of any order relation between generating solutions.

In each set \( M_{i+1,i} \) we fix a generating solution denoted

\[
\varphi_{i+1} / \varphi_i \quad (i = 1, \ldots, n - 1).
\]

Fix integers \( p \) and \( q \) such that \( p > q + 1 \geq 2 \). The product

\[
\varphi_{q+1} / \varphi_q \cdot \varphi_{q+2} / \varphi_{q+1} \cdots \varphi_{p-1} / \varphi_{p-2} = \varphi_p / \varphi_{p-1}
\]

of generating solutions of sets \( M_{q+1,q}, M_{q+2,q+1}, \ldots, M_{p-1,p-2}, M_{p,p-1} \), respectively, belongs to \( M_{pq} \) by Lemma 1 and we denote it by \( \varphi_p / \varphi_q \).

Let \( \tau_1, \tau_2 \in \{R_m \ \{0\}\} \). Then \( \tau_i = \mu_i \cdot \varphi_i \), where \( \mu_c_i = (c_i, m) \) and \( \varphi_i \in U(R_m) \) for \( i = 1, 2 \) by (1). Since \( \text{Ann}(\tau_i) = \varphi_i, R_m \), where

\[
\alpha_{c_i} := \frac{m}{\mu_{c_i}}
\]


and \( \mu_{c_1} := (c_1, m) \) by [6, Lemma 5], we set \( \bar{\alpha}_i := \bar{\alpha}_{c_i} \bar{c}_i \). Hence
\[
\bar{\alpha}_i R_m = (\bar{\alpha}_{c_i} \bar{c}_i) R_m = \bar{\alpha}_{c_i} (\bar{c}_i R_m) = \bar{\alpha}_{c_i} R_m = \text{Ann}(\bar{c}_i).
\]
Note that if \( \bar{c}_1 \mid \bar{c}_2 \), then we define \( \frac{\bar{c}_2}{\bar{c}_1} \) as above.

**Lemma 2.** Let \( \bar{c}_1, \bar{c}_2 \in \{ R_m \setminus \{0\} \} \) such that \( \bar{c}_1 \mid \bar{c}_2 \). Then \( \frac{\bar{c}_2}{\bar{c}_1} \) is a generating solution of the equation
\[
\bar{\alpha}_2 \bar{\sigma} = \bar{\alpha}_1,
\]
where each \( \langle \bar{\alpha}_i \rangle R_m = \text{Ann}(\bar{c}_i) \) and \( \alpha_i \) has form (3) for \( i = 1, 2 \).

**Proof.** Since \( \bar{c}_2 = \bar{c}_1 \bar{d} \), where \( \bar{d} \in R_m \), there exist \( c_1, c_2, d \in R \) such that \( c_2 = c_1 d \). This yields that \( \mu_{c_1} := (c_1, m) | (c_2, m) = \mu_{c_2} \) and
\[
\alpha_1 = \alpha_{c_1} e_1 = m_{c_1} e_1 = m_{c_2} \mu_{c_1} e_1 = \alpha_{c_2} \mu_{c_2} e_1 = (\alpha_{c_2} e_2) \mu_{c_1} e_1 e_1^{-1} = \alpha_2 \sigma,
\]
in which \( \sigma := \frac{\mu_{c_2}}{\mu_{c_1}} e_1 e_2^{-1} \).

Thus \( \bar{\alpha}_2 \bar{\sigma} = \bar{\alpha}_1 \), so \( \bar{\sigma} + \text{Ann}(\bar{\alpha}_2) \) is the set of all solutions of (4).

Noting that the preimages of the invertible elements of the ring \( R_m \) are those elements of \( R \) that are relatively prime with \( m \), we obtain that the preimage \( \bar{\alpha}_2 \) in \( R \) is the element \( \alpha_2 e_2 \) where \( (e_2, m) = 1 \). According to [6, Lemma 5] \( \text{Ann}(\bar{\alpha}_2) = \bar{\alpha}_2 R_m \), where
\[
\alpha_2 = \frac{m}{\alpha_2 e_2, m} = \frac{m}{\mu_{c_2}, m} = \frac{m}{\mu_{c_2}} = \mu_{c_2}.
\]
Since \( \sigma = \frac{\mu_{c_2}}{\mu_{c_1}} e_1 e_2^{-1} \), we get \( \mu_{c_2} = \sigma \mu_{c_1} e_2^{-1} e_2 \) and \( \sigma \mu_{c_2} \). Therefore \( \bar{\sigma} \) is the divisor of all elements of \( \text{Ann}(\bar{\alpha}_2) \), so it is a divisor of all solutions of (4) and \( \bar{\sigma} \) is a generating solution.

Using the notation of Lemma 2 we put \( \frac{\bar{c}_1}{\bar{c}_2} := \frac{\bar{c}_2}{\bar{c}_1} \).

**Corollary 1.** Let \( \bar{c}_1, \bar{c}_2 \in \{ R_m \setminus \{0\} \} \) such that \( \bar{c}_1 \mid \bar{c}_2 \). Then \( \frac{\bar{c}_2}{\bar{c}_1} \cdot \bar{\alpha}_2 = \bar{\alpha}_1 \).

**Proof.** Clearly, \( \frac{\bar{c}_2}{\bar{c}_1} \cdot \bar{\alpha}_2 = \frac{\bar{c}_1}{\bar{c}_2} \bar{\alpha}_2 \). The element \( \frac{\bar{c}_1}{\bar{c}_2} \bar{\alpha}_2 \) is a solution of \( \bar{\alpha}_2 \cdot \bar{\sigma} = \bar{\alpha}_1 \), so \( \frac{\bar{c}_2}{\bar{c}_1} \cdot \bar{\alpha}_2 = \bar{\alpha}_1 \) and \( \frac{\bar{c}_2}{\bar{c}_1} \cdot \bar{\alpha}_2 = \bar{\alpha}_1 \).

Note that \( \frac{\bar{c}_2}{\bar{c}_1} \) and \( \frac{\bar{c}_1}{\bar{c}_2} \) are generating solutions of \( \bar{c}_1 \cdot \bar{\sigma} = \bar{\alpha}_2 \) and \( \bar{\alpha}_2 \cdot \bar{\sigma} = \bar{\alpha}_1 \), respectively. It should be noted that sets of solutions of these equations, generally speaking, do not coincide. Moreover, if \( \bar{\alpha}_1, \bar{\alpha}_2 \) have no form (3), then \( \frac{\bar{c}_2}{\bar{c}_1} \) may not be a solution of the equation \( \bar{\alpha}_2 \cdot \bar{\sigma} = \bar{\alpha}_1 \).

**Example 2.** Let \( R_m = \mathbb{Z}_{144} \). Consider the equation \( \bar{4} \bar{\sigma} = \bar{8} \), in which \( \bar{c}_1 = \bar{4} \) and \( \bar{c}_2 = \bar{8} \). Since \( \text{Ann}(\bar{4}) = \{ 0, 36, 72 \} = \bar{36} \mathbb{Z}_{144} \), set of solutions of this equation is \( \bar{2} + \text{Ann}(\bar{4}) = \{ 2, 38, 74 \} \), and all of those solutions is generating. Clearly,

\[
\text{Ann}(\bar{8}) = \{ 0, 18, 36, 54, 72, 90, 108, 126 \} = 18 \mathbb{Z}_{144} = 126 \mathbb{Z}_{144}.
\]
Set $\overline{\alpha_1} = 36$ and $\overline{\alpha_2} = 18$. All solutions of the equation $18\overline{x} = 36$ belong to

$$S = \{2, 10, 18, 26, 34, 42, \ldots, 74, \ldots, 138\}.$$ 

Hence, if $\frac{5}{4} := \overline{38}$, then $\frac{5}{4} \notin S$.

On the other hand, if we put $\overline{\alpha_1} = 36$ and $\overline{\alpha_2} = 126$, then $\overline{38}$ is a solution of the equation $126\overline{x} = 36$.

**Lemma 3.** Let $R$ be a commutative ring and let

$$A = \begin{bmatrix}
1 & 1 & 1 & \ldots & 1 & 1 & 1 \\
1 & 1 & 1 & \ldots & 1 & 1 & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
1 & 1 & 1 & \ldots & 1 & 1 & 1 \\
\end{bmatrix} \in R^{n \times n},$$

in which $b_{ij} = a_{j+1,j}a_{j+2,j+1} \cdots a_{i-1,i-2}a_{i-1,i-1}$, where $i > j + 1$, $i = 3, \ldots, n$, and $j = 1, \ldots, n - 2$.

If $\det(A) = \sum_{\sigma \in S_n} \gamma_\sigma$ and $\det(A^T) = \sum_{\sigma \in S_n} \delta_\sigma$ are classical decompositions by definition of $\det(A)$ and $\det(A^T)$ into terms, respectively, then

$$\gamma_\sigma = \gamma_{\sigma^{-1}} = \delta_\sigma.$$

**Proof.** To each $\sigma = (1 \ 2 \ 3 \ \ldots \ n)$ in $S_n$ we assign the following two sets:

$$\mathcal{I}_1(\sigma) = \{ (p_i, q_{p_i}) \mid p_i > q_{p_i} \} \text{ a column in } \sigma, \ i = 1, \ldots, s;$$

$$\mathcal{I}_2(\sigma) = \{ (\alpha_i, \beta_{\alpha_i}) \mid \alpha_i \leq \beta_{\alpha_i} \} \text{ a column in } \sigma, \ i = 1, \ldots, t;$$

where $s + t = n$. If $A := [c_{ij}]$, then in the decomposition of $\det(A)$ we have

$$\gamma_\sigma = (-1)^{\text{sign}(\sigma)} c_{p_1,q_{p_1}} \cdot c_{p_2,q_{p_2}} \cdots c_{p_s,q_{p_s}},$$

$$\gamma_{\sigma^{-1}} = (-1)^{\text{sign}(\sigma^{-1})} c_{\alpha_1,\beta_{\alpha_1}} \cdot c_{\alpha_2,\beta_{\alpha_2}} \cdots c_{\alpha_t,\beta_{\alpha_t}},$$

in which $c_{p_i,q_{p_i}}$ as well as $c_{\alpha_i,\beta_{\alpha_i}}$ is a product of elements of the first subdiagonal of $A$. According to [6, Lemma 9] we obtain that

$$\prod_{(p_i,q_{p_i}) \in \mathcal{I}_1(\sigma)} \frac{p_i}{q_{p_i}} = \prod_{(\alpha_i,\beta_{\alpha_i}) \in \mathcal{I}_2(\sigma)} \frac{\beta_{\alpha_i}}{\alpha_i},$$

and direct calculation gives $\gamma_\sigma = \gamma_{\sigma^{-1}}$.

Finally, every term $\gamma_\sigma$ appearing in $\det(A)$ corresponds to the term $\gamma_{\sigma^{-1}}$ in $\det(A^T)$, so $\gamma_{\sigma^{-1}} = \delta_\sigma$, as above. □

**Lemma 4.** Let $A, B \in R^{n \times n}$ such that

$$A = \begin{bmatrix}
h_{11} & h_{12} & h_{13} & \ldots & h_{1,n-2} & h_{1,n-1} & h_{1n} \\
ah_{21} & h_{22} & h_{23} & \ldots & h_{2,n-2} & h_{2,n-1} & h_{2n} \\
h_{31} & h_{32} & h_{33} & \ldots & h_{3,n-2} & h_{3,n-1} & h_{3n} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
h_{n1} & h_{n2} & h_{n3} & \ldots & h_{n,n-2} & h_{n,n-1} & h_{nn}
\end{bmatrix}$$

and

$$B = \begin{bmatrix}
h_{11} & h_{12} & h_{13} & \ldots & h_{1,n-2} & h_{1,n-1} & h_{1n} \\
h_{21} & h_{22} & h_{23} & \ldots & h_{2,n-2} & h_{2,n-1} & h_{2n} \\
h_{31} & h_{32} & h_{33} & \ldots & h_{3,n-2} & h_{3,n-1} & h_{3n} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
h_{n1} & h_{n2} & h_{n3} & \ldots & h_{n,n-2} & h_{n,n-1} & h_{nn}
\end{bmatrix}.$$
and

\[
B = \begin{bmatrix}
h_{11} & a_{21}h_{12} & b_{31}h_{13} & \cdots & b_{n-2,1}h_{1,n-2} & b_{n-1,1}h_{1,n-1} & b_{n1}h_{1n} \\
h_{11} & h_{22} & a_{32}h_{23} & \cdots & b_{n-2,2}h_{2,n-2} & b_{n-1,2}h_{2,n-1} & b_{n2}h_{2n} \\
h_{31} & h_{32} & h_{33} & \cdots & b_{n-2,3}h_{3,n-2} & b_{n-1,3}h_{3,n-1} & b_{n3}h_{3n} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
h_{n-1,1} & h_{n-1,2} & h_{n-1,3} & \cdots & h_{n-1,n-2} & h_{n-1,n-1} & a_{n,n-1}h_{n-1,n} \\
h_{n1} & h_{n2} & h_{n3} & \cdots & h_{nn-2} & h_{nn-1} & h_{nn}
\end{bmatrix},
\]

in which \(b_{ij} = a_{j+1,j}a_{j+2,j+1}\cdots a_{i-1,i}a_{i,j-1}\), where \(i > j + 1\), \(i = 3, \ldots, n\), and \(j = 1, \ldots, n-2\).

Then the corresponding terms of \(\det(A)\) and \(\det(B)\) coincide, and \(\det(A) = \det(B)\).

**Proof.** To each \(\sigma = (1 \; 2 \; \ldots \; n) \in S_n\) we associate the following terms

\[
\gamma_\sigma = (-1)^{\text{sign}(\sigma)}\lambda_{1,i_1}h_{1,i_1}\lambda_{2,i_2}h_{2,i_2}\cdots\lambda_{n,i_n}h_{n,i_n}
\]

\[
\mu_\sigma = (-1)^{\text{sign}(\sigma)}(h_{1,i_1}h_{2,i_2}\cdots h_{n,i_n})(\lambda_{1,i_1}\lambda_{2,i_2}\cdots\lambda_{n,i_n});
\]

from the decompositions of \(\det(A)\) and \(\det(B)\), respectively, in which

\[
\lambda_{p,i_p} = \begin{cases} 
    a_{p,i_p}, & \text{if } p - i_p = 1; \\
    b_{p,i_p}, & \text{if } p - i_p > 1; \\
    1, & \text{if } p - i_p < 1.
\end{cases}
\]

It is easy to see that the product \(\lambda_{1,i_1}\lambda_{2,i_2}\cdots\lambda_{n,i_n}\) which arises in the decomposition of the term \(\gamma_\sigma\) in \(\det(A)\), corresponds to the following permutation \(\sigma = (1 \; 2 \; \ldots \; n) \in S_n\). Similarly, the product \(\lambda_{i_1,1}\lambda_{i_2,2}\cdots\lambda_{i_n,n}\) which arises in the decomposition of the term \(\mu_\sigma\) in \(\det(B)\), corresponds to the following permutation \(\sigma^{-1} = (i_1 \; i_2 \; \ldots \; i_n) \in S_n\). By virtue of Lemma 3,

\[
\lambda_{1,i_1}\lambda_{2,i_2}\cdots\lambda_{n,i_n} = \lambda_{i_1,1}\lambda_{i_2,2}\cdots\lambda_{i_n,n}.
\]

Consequently, \(\gamma_\sigma = \mu_\sigma\) for all \(\sigma \in S_n\), so \(\det(A) = \det(B)\). \(\square\)

### 3. Main result and its proof.

To simplify notation, in what follows we omit the bar from above when referring to the elements of the ring \(R_m\).

**Theorem 1.** Let \(R\) be a commutative Bézout domain (with the property \(1 \neq 0\)) of stable range 1.5. Let \(U(R)\) be the group of units of \(R\). For each \(m \in R \setminus \{U(R), 0\}\), we denote the factor ring \(R_m = R/mR\).

Let \(G_\Phi \leq \text{GL}_n(R_m)\) be the Zelisko group of the following matrix:

\[ \Phi := \text{diag}(1, \ldots, 1, \varphi_t, \ldots, \varphi_{t+1}, \ldots, \varphi_k, 0, \ldots, 0) \in R_m^{n \times n} \]

in which \(\varphi_t | \varphi_{t+1} | \cdots | \varphi_k \neq 0\), \(\varphi_t \notin U(R_m)\), \(1 \leq t \) and \(k \leq n\).

Then the group \(G_\Phi\) consists of all invertible matrices \(H\) of the form:
(i) \( H = \begin{bmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ 0 & H_{32} & H_{33} \end{bmatrix} \) for \( 1 < t \) and \( k < n \);

(ii) \( H = \begin{bmatrix} H_{22} & H_{23} \\ H_{32} & H_{33} \end{bmatrix} \) for \( 1 = t \) and \( k < n \);

(iii) \( H = H_{22} \) for \( 1 = t \) and \( k = n \);

(iv) \( H = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \) for \( 1 < t \) and \( k = n \);

(v) \( H = \begin{bmatrix} M_{11} & M_{12} \\ 0 & M_{22} \end{bmatrix} \), in which \( M_{11} \in \text{GL}_s(R) \), \( M_{22} \in \text{GL}_{n-s}(R) \), \( 1 \leq s < n \) and (see (5)) the matrix \( \Phi \) has the following form:

\[
\Phi := \text{diag}(1, \ldots, 1, 0, \ldots, 0) \in R_{m}^{n \times n}.
\]

In all of the above cases (i)-(iv), we have

\[
H_{21} = \begin{bmatrix} \varphi_t h_{t1} & \varphi_t h_{t2} & \cdots & \varphi_t h_{tk} \\ \varphi_{t+1} h_{t+1,1} & \varphi_{t+1} h_{t+1,2} & \cdots & \varphi_{t+1} h_{t+1,k} \\ \cdots & \cdots & \cdots & \cdots \\ \varphi_k h_{k1} & \varphi_k h_{k2} & \cdots & \varphi_k h_{kk} \end{bmatrix},
\]

\[
H_{22} = \begin{bmatrix} h_{tt} & h_{t,t+1} & \cdots & h_{t,k} \\ \frac{\varphi_{t+1}}{\varphi_t} h_{t+1,t} & \frac{\varphi_{t+1}}{\varphi_t} h_{t+1,t+1} & \cdots & \frac{\varphi_{t+1}}{\varphi_t} h_{t+1,k} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\varphi_k}{\varphi_t} h_{k,t} & \frac{\varphi_k}{\varphi_t} h_{k,t+1} & \cdots & \frac{\varphi_k}{\varphi_t} h_{k,k} \end{bmatrix},
\]

\[
H_{32} = \begin{bmatrix} \alpha_t h_{k+1,t} & \alpha_t h_{k+1,t+1} & \cdots & \alpha_k h_{k+1,k} \\ \alpha_t h_{k+2,t} & \alpha_t h_{k+2,t+1} & \cdots & \alpha_k h_{k+2,k} \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_t h_{nt} & \alpha_t h_{n,t+1} & \cdots & \alpha_k h_{nk} \end{bmatrix},
\]

and \( H_{11}, H_{12}, H_{13}, H_{23}, H_{33} \) are some matrices of corresponding sizes.

**Proof.** (i) Let \( \Phi \) be a matrix of the form (5) in which \( 1 < t \) and \( k < n \). A matrix \( H := [p_{ij}] \in G_\Phi \) we represent in the form of a block matrix, in which \( H_{11} \) of size \( t \times t \), \( H_{22} \) of size \( (k-t+1) \times (k-t+1) \) and \( H_{33} \) of size \( (n-k) \times (n-k) \), respectively. There exists \( S = [s_{ij}] \in \text{GL}_n(R_m) \) such that \( H \Phi = \Phi S \) by definition of the Zelisko group. This means that

\[
\varphi_j p_{ij} = \varphi_i s_{ij},
\]

If \( j \geq i \), then we put \( s_{ij} := \frac{\varphi_j}{\varphi_i} p_{ij} \), such that (9) holds for arbitrary value of \( p_{ij} \). It follows that no restrictions are imposed on the elements \( p_{ij} \) from the blocks \( H_{12}, H_{13} \) and \( H_{23} \), because they lie above the main diagonal of \( H \).
Considering that the first $t - 1$ diagonal elements of the matrix $\Phi$ are 1 and the last $n - k$ of its diagonal elements are zeros (see (5)), we get that no restrictions are imposed on the elements $p_{ij}$ from the blocks $H_{11}$ and $H_{33}$.

If $j = 1, \ldots, t - 1$ and $i = t, \ldots, k$ (see (5)), then (9) rewrites as $p_{ij} = \phi_i s_{ij}$. It follows that $H_{21}$ has the form (6).

Consider the block $H_{22}$. Taking into account our definition of the element $\frac{\phi_i}{\phi_j}$ (see after the proof of Lemma 1) and the proof of [6, Theorem 2], we obtain that $H_{22}$ has the form (7).

Consider blocks $H_{31}$ and $H_{32}$. Using the fact that the first $t - 1$ elements of $\Phi$ are 1 and the last $n - k$ elements of $\Phi$ are 0, the equation (9) rewrites as $\phi_j p_{ij} = 0$. This means that $p_{ij} \in \text{Ann}(\phi_j) = \alpha_j R_m$, so $p_{ij} = \alpha_j h_{ij}$. It immediately follows that $H_{31} = 0$ and $H_{32}$ has form (8).

The proofs of (ii)-(v) are particular cases of the previous one.

\begin{enumerate}
\item [\rightarrow] (i) Consider a block matrix $S = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix}$, in which each block $S_{ij}$ has the same size as the size of corresponding block $H_{ij}$ in $H$ and set

\[
S_{12} := \begin{bmatrix}
\phi_t h_{1t} & \phi_{t+1} h_{1,t+1} & \ldots & \phi_k h_{1k} \\
\phi_t h_{2t} & \phi_{t+1} h_{2,t+1} & \ldots & \phi_k h_{2k} \\
\vdots & \vdots & \ddots & \vdots \\
\phi_t h_{(t-1),t} & \phi_{t+1} h_{(t-1),t+1} & \ldots & \phi_k h_{(t-1),k} 
\end{bmatrix},
\]

\[
S_{22} := \begin{bmatrix}
h_{tt} & \phi_{t+1} h_{t,t+1} & \ldots & \phi_k h_{t,k-1} \\
\vdots & \vdots & \ddots & \vdots \\
h_{k-1,t} & h_{k-1,t+1} & \ldots & h_{k-1,k-1} \\
h_{kt} & h_{kt+1} & \ldots & h_{kk} 
\end{bmatrix},
\]

\[
S_{23} := \begin{bmatrix}
\alpha_t h_{t,k+1} & \alpha_t h_{t,k+2} & \ldots & \alpha_t h_{tn} \\
\alpha_{t+1} h_{t+1,k+1} & \alpha_{t+1} h_{t+1,k+2} & \ldots & \alpha_{t+1} h_{t+1,n} \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_k h_{k,k+1} & \alpha_k h_{k,k+2} & \ldots & \alpha_k h_{kn} 
\end{bmatrix}.
\]

It is easy to check that $H \Phi = \Phi S$ for arbitrary $S_{11}, S_{21}, S_{31}, S_{32}$ and $S_{33}$.

Let us prove that $S$ is invertible. Consider the following four cases:

Case 1. Let $p_{ij} \in H_{21}$. Using (2) we have

\[
p_{ij} = \phi_i h_{ij} = \phi_t \frac{\phi_i}{\phi_t} h_{ij} = \phi_t \frac{\phi_{t+1}}{\phi_t} \frac{\phi_{t+2}}{\phi_{t+1}} \ldots \frac{\phi_{i-1}}{\phi_{i-2}} \frac{\phi_i}{\phi_{i-1}} h_{ij}.
\]

Case 2. Let $p_{ij} \in H_{22}$ for $i > j$. Using (2) we obtain that

\[
p_{ij} = \frac{\phi_i}{\phi_j} h_{ij} = \frac{\phi_{j+1}}{\phi_j} \frac{\phi_{j+2}}{\phi_{j+1}} \ldots \frac{\phi_{i-1}}{\phi_{i-2}} \frac{\phi_i}{\phi_{i-1}} h_{ij}.
\]
Case 3. Let \( p_{ij} \in H_{32} \). Using Corollary 1 and (2) we get that
\[
p_{ij} = \alpha_j h_{ij} = \frac{\varphi_k}{\varphi_j} \alpha_k h_{ij} = \frac{\varphi_{j+1}}{\varphi_j} \frac{\varphi_{j+2}}{\varphi_{j+1}} \cdots \frac{\varphi_{k-1}}{\varphi_{k-2}} \frac{\varphi_k}{\varphi_{k-1}} \alpha_k h_{ij}.
\]

Case 4. Finally, let \( p_{ij} \in H_{31} \equiv 0 \). It is easy to check that
\[
p_{ij} = 0 = \varphi_k \alpha_k = \varphi_t \frac{\varphi_{t+1}}{\varphi_t} \frac{\varphi_{t+2}}{\varphi_{t+1}} \cdots \frac{\varphi_{k-1}}{\varphi_{k-2}} \frac{\varphi_k}{\varphi_{k-1}} \alpha_k.
\]
Consequently, all elements of \( H \), which lie below the main diagonal of \( H \), satisfy Lemma 4, so \( S \) is invertible.

The proofs of (ii)-(v) are particular cases of the previous one. \( \square \)

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