A SMÖRGÅSBORD OF SCALAR-FLAT KÄHLER ALE SURFACES

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This article is dedicated to the memory of Egbert Brieskorn.

Abstract. There are many known examples of scalar-flat Kähler ALE surfaces, all of which have group at infinity either cyclic or contained in SU(2). The main result in this paper shows that for any non-cyclic finite subgroup $\Gamma \subset U(2)$ containing no complex reflections, there exist scalar-flat Kähler ALE metrics on the minimal resolution of $\mathbb{C}^2/\Gamma$, for which $\Gamma$ occurs as the group at infinity. Furthermore, we show that these metrics admit a holomorphic isometric circle action. It is also shown that there exist scalar-flat Kähler ALE metrics with respect to some small deformations of complex structure of the minimal resolution. Lastly, we show the existence of extremal Kähler metrics admitting holomorphic isometric circle actions in certain Kähler classes on the complex analytic compactifications of the minimal resolutions.

1. Introduction

In order to state the main theorem, we begin with some preliminary definitions. Let $\Gamma \subset U(2)$ be a finite subgroup. Then the following notions are equivalent.

- $\Gamma$ contains no complex reflections.
- $\Gamma$ acts freely on $S^3$
- $X = \mathbb{C}^2/\Gamma$ has an isolated singularity at the origin.

While an in depth discussion of these groups is given in Section 3.2, for the sake of clarity here, we provide a list of all possible such groups below in Table 1.1. However, before doing so, it is necessary to introduce some notation.

- For $q$ and $p$ relatively prime integers, $L(q, p)$ denotes the cyclic subgroup of $U(2)$ generated by
  $$\begin{pmatrix} \exp(2\pi i/p) & 0 \\ 0 & \exp(2\pi iq/p) \end{pmatrix}. $$
- The binary polyhedral groups (dihedral, tetrahedral, octahedral, icosahedral) are respectively denoted by $D_{4n}^*$, $T^*$, $O^*$, $I^*$.
- The map $\phi: SU(2) \times SU(2) \to SO(4)$ denotes the usual double cover (see (3.1) below).

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Definition 1.1. Let \( \Gamma \subset U(2) \) be a finite subgroup containing no complex reflections. Then, a smooth complex surface \( \tilde{X} \) is called a minimal resolution of \( \mathbb{C}^2/\Gamma \) if there is a mapping \( \pi: \tilde{X} \to \mathbb{C}^2/\Gamma \) such that

1. The restriction \( \pi: \tilde{X} \setminus \pi^{-1}(0) \to \mathbb{C}^2/\Gamma \setminus \{0\} \) is a biholomorphism;
2. \( \pi^{-1}(0) \) is a divisor in \( \tilde{X} \) containing no \(-1\) curves.

These resolutions are minimal in the sense that, given any other resolution \( \pi_Y: Y \to \mathbb{C}^2/\Gamma \), there is a proper analytic map \( p: Y \to \tilde{X} \) such that \( \pi_Y = \pi \circ p \). For each such \( \Gamma \), up to isomorphism there exists a unique such minimal resolution, and in 1968, Brieskorn completely described these resolutions complex-analytically [Bri68]. The exceptional divisor consists of a tree of rational curves with normal crossing singularities. For each non-cyclic subgroup in Table 1.1, it turns out that the exceptional divisor contains a distinguished rational curve which intersects exactly three other rational curves. This curve will be referred to as the central rational curve. The self-intersection number of this curve will be denoted \( -b_\Gamma \). The signature \( \tau(\tilde{X}) \) is given by \( -k_\Gamma \), where \( k_\Gamma \) is the total number of rational curves in the exceptional divisor. The exceptional divisor is described by a tree with three branches attached to the central rational curve, and each branch corresponds to a Hirzebruch-Jung string. We will denote the corresponding graph by

\[
\langle -b_\Gamma; (\alpha_1, \beta_1); (\alpha_2, \beta_2); (\alpha_3, \beta_3) \rangle,
\]

where \( \alpha_i \) and \( \beta_i \), for \( i = 1, 2, 3 \), are integers determined by the group \( \Gamma \), see Section [4].

We are interested in certain metric structures on these spaces. Let \((M, g, J)\) be a Kähler manifold with Kähler form \( \omega = g(J \cdot, \cdot) \). Critical points of the functional

\[
\alpha \mapsto \int_M R_\alpha^2 dV_\alpha,
\]

where \( \alpha \) is a Kähler form in the cohomology class \([\omega]\) and \( R_\alpha \) is the scalar curvature of the metric \( g_\alpha \) determined by \( \alpha \), are called extremal Kähler metrics. These were introduced by Calabi in [Cal82, Cal85]. The Euler-Lagrange equation for a critical point is given by

\[
\overline{\partial} \partial^\# R_\alpha = 0,
\]
where the operator $\partial^\#_g$ denotes the $(1,0)$ part of the gradient of a function with respect to the metric $g_\alpha$. In other words, the $(1,0)$ part of the gradient of the scalar curvature is a holomorphic vector field.

In particular, note that scalar-flat Kähler metrics are extremal. A very important class of scalar-flat Kähler metrics are those of the following type.

**Definition 1.2.** A complete Riemannian manifold $(X^4, g)$ is called **asymptotically locally Euclidean (ALE)** of order $\tau$ if there exists a finite subgroup $\Gamma \subset SO(4)$ which acts freely on $S^3$, a compact subset $K \subset X$, and a diffeomorphism

$$\Psi : X \setminus K \rightarrow (\mathbb{R}^4 \setminus B(0, R))/\Gamma,$$

such that under this identification

$$\left(\Psi^*g\right)_{ij} = \delta_{ij} + O(\rho^{-\tau}),$$

$$\partial^{|k|}(\Psi^*g)_{ij} = O(\rho^{-\tau-k}),$$

for any partial derivative of order $k$, as $\rho \to \infty$, where $\rho$ is the distance to some fixed basepoint. We call $\Gamma$ the **group at infinity**.

We next review briefly the known examples of scalar-flat Kähler ALE surfaces:

- In [LeB88], for all positive integers $n$, LeBrun constructed a $U(2)$-invariant scalar-flat Kähler ALE metric on the total space of the bundle $\mathcal{O}(-n)$ over $\mathbb{CP}^1$, the minimal resolution of $\mathbb{C}^2/L(1, n)$. The $n = 1$ and $n = 2$ cases are the well-known Burns and Eguchi-Hanson metrics respectively [Bur86, EH79].

- The cases in Table 1.1 where $m = 1$ are the ADE-type singularities, and hyperkähler metrics on these minimal resolutions were produced and classified by Kronheimer using the hyperkähler quotient construction [Kro89a, Kro89b]. These generalize earlier examples of Eguchi-Hanson and Gibbons-Hawking, see [EH79, GH78] respectively.

- Using a construction of Joyce, examples of scalar-flat Kähler ALE metrics with a torus action for the general cyclic case $L(m, n)$ were given by Calderbank-Singer [CS04].

Our main result shows that any allowed group may occur as the group at infinity of a scalar-flat Kähler ALE surface:

**Theorem 1.3.** Let $\Gamma \subset U(2)$ be a finite subgroup containing no complex reflections. Then the minimal resolution of $\mathbb{C}^2/L$ admits scalar-flat Kähler ALE metrics. Furthermore, these metrics admit a holomorphic isometric circle action.

The proof of Theorem 1.3 is given in Section 5, and the basic idea is as follows. First, we take quotients of the LeBrun negative mass metrics on $\mathcal{O}(-\ell)$ (for a certain value of $\ell$ determined by $\Gamma$) by certain finite subgroups of $U(2)$. This results in scalar-flat Kähler ALE orbifolds which have only isolated singularities of cyclic type. Then, we resolve these singularities by adapting a gluing theorem for Hermitian anti-self-dual metrics, due to Rollin-Singer, to the situation where the base metric is an ALE orbifold (instead of a compact orbifold), and use this to attach appropriate Calderbank-Singer spaces. Finally, a theorem of Boyer shows that these Hermitian
anti-self-dual metrics are in fact scalar-flat Kähler. To find the holomorphic isometric circle action, we show that our approximate metrics can be chosen to have such an action, and then use an equivariant version of the gluing theorem. We note that this gluing procedure can be thought of as a metric version of Brieskorn’s method of constructing the minimal resolution complex analytically.

1.1. Moduli of scalar-flat Kähler ALE metrics. We next define our notion of moduli space.

**Definition 1.4.** The moduli space of scalar-flat Kähler ALE metrics near a given scalar-flat Kähler ALE metric \( g \) with group at infinity \( \Gamma \subset U(2) \) is the set of scalar-flat Kähler ALE metrics, modulo diffeomorphism, where one allows only diffeomorphisms \( \Phi \) which are asymptotic to the identity transformation in respective ALE coordinates given in Definition [1.2]. That is, we say that \((X, g_1)\) is equivalent to \((X, g_2)\) if there is a diffeomorphism \( \Phi : X \to X \) with \( \Phi^* g_2 = g_1 \), such that there exists ALE coordinates \( \Psi_i \) with respect to \( g_i \) for \( i = 1, 2 \) such that

\[
\Psi_2 \circ \Phi \circ \Psi_1^{-1} = Id + O(r^{-\tau}),
\]

\[
\partial^\alpha (\Psi_2 \circ \Phi \circ \Psi_1^{-1}) = O(r^{-\tau - |\alpha|})
\]

for some \( \tau > 0 \), and for any multi-index \( \alpha \), as \( r \to \infty \).

While not immediately obvious from the definition, the tangent space to the moduli space splits into two summands: those infinitesimal deformations fixing the complex structure and deforming the Kähler class, and those fixing the Kähler class, but deforming the complex structure.

In the cases where \( \Gamma \subset SU(2) \), the hyperkähler quotient construction produces hyperkähler metrics for the minimal resolution complex structure as well as for all small deformations of the minimal resolution complex structure. In the case of the LeBrun negative mass metrics, Honda has shown that all small deformations of the complex structure on \( O(-n) \) also admit scalar-flat Kähler metrics \([Hon13, Hon14]\).

For our next result, we obtain scalar-flat Kähler metrics for some small deformations of the minimal resolution complex structure for the non-cyclic subgroups.

**Theorem 1.5.** If \( \Gamma \subset U(2) \) is a non-cyclic finite subgroup containing no complex reflections, then some small deformations of the minimal resolution of \( \mathbb{C}^2/\Gamma \) admit scalar-flat Kähler ALE metrics. The dimension \( d_\Gamma \) of the moduli space of scalar-flat Kähler ALE metrics near a metric obtained in Theorem 1.3 satisfies the bounds

\[
d_{\Gamma,\min} \leq d_\Gamma \leq d_{\Gamma,\max},
\]

where

\[
0 < d_{\Gamma,\min} = 2(b_\Gamma - 1) + k_\Gamma - 3,
\]

and

\[
d_{\Gamma,\max} = 2\left( \sum_{i=1}^{k_\Gamma} (e_i - 1) \right) + k_\Gamma - 3,
\]

(1.10) and (1.11)
where $-\epsilon_i$ is the self-intersection number of the $i$th rational curve, and $k_\Gamma$ is the number of rational curves in the exceptional divisor as described above.

In the cyclic case, however, it is not known whether there are scalar-flat Kähler metrics for deformations of the minimal resolution complex structure, except of course when these overlap with the aforementioned cases (toric multi-Eguchi-Hanson metrics and LeBrun negative mass metrics). However, we make the following conjecture.

**Conjecture 1.6.** For any finite subgroup $\Gamma \subset U(2)$ containing no complex reflections, all small deformations of the minimal resolution of $\mathbb{C}^2/\Gamma$ admit scalar-flat Kähler ALE metrics.

If the conjecture is true, then $d_{\Gamma, \text{max}}$ is exactly the dimension of the moduli space of Kähler scalar-flat metrics near any of the metrics found in Theorem 1.3. To approach the conjecture, an approach similar to that of [BR12], in which one deforms the complex structure, is likely needed, although in this case one needs to moreover completely understand the scalar-flat Kähler deformations of the Calderbank-Singer metrics.

### 1.2. The hyperkähler case.

We point out that for any non-cyclic $\Gamma \subset SU(2)$, our construction does actually yield hyperkähler metrics with group $\Gamma$ at infinity, see Proposition 8.2. Thus, a corollary is that for any such subgroup, *some* hyperkähler metrics can be obtained by gluing techniques. Furthermore, the only ingredients needed in this case are the Gibbons-Hawking multi-Eguchi-Hanson metrics.

As we will see below in Section 8 using a gluing theorem for anti-self-dual metrics (instead of the Hermitian anti-self-dual gluing theorem), we can actually show that *all* small deformations also admit hyperkähler metrics, and hence obtain an open set in the moduli space. Once again, these metrics are known by Kronheimer’s construction, but we find it interesting that they can be obtained by gluing techniques, using only the multi-Eguchi-Hanson metrics as building blocks.

### 1.3. Extremal Kähler metrics on rational surfaces.

In [LM08, Lemma 4.1], LeBrun-Maskit showed that any scalar-flat Kähler ALE space admits a complex analytic compactification to a rational surface by adding a tree of rational curves at infinity. In Section 7 we identify the surfaces obtained from compactifying the spaces obtained in Theorem 1.3. More precisely, we will show that a compactification can be obtained by adding the tree of rational curves

\begin{equation}
\langle b_\Gamma; (\beta_i - \alpha_1, \beta_1); (\beta_2 - \alpha_2, \beta_2); (\beta_3 - \alpha_3, \beta_3) \rangle.
\end{equation}

We will let $k_i$ denote the length of the Hirzebruch-Jung algorithm for $L(\alpha_i, \beta_i)$ and $\ell_i$ denote the length of the Hirzebruch-Jung algorithm for $L(\beta_i - \alpha_i, \beta_i)$ for $i = 1, 2, 3$ (see Section 2 for a description of this algorithm).

**Theorem 1.7.** Let $\Gamma \subset U(2)$ be a non-cyclic finite subgroup containing no complex reflections. Then the minimal resolution of $\mathbb{C}^2/\Gamma$ has a complex analytic compactification to $\mathbb{C}P^2 \# k \mathbb{C}P^2$, where $k = k_1 + k_2 + k_3 + \ell_1 + \ell_2 + \ell_3 + 1$. 

The complete list of rational curves in the compactification is given by
\begin{align}
[\mathfrak{b} \Gamma], [-\mathfrak{b} \Gamma], \\
[-\mathfrak{b}_i], i = 1, 2, 3, \ell = 1 \ldots \ell_i \\
[-\mathfrak{e}_k], i = 1, 2, 3, k = 1 \ldots k_i,
\end{align}
where notation \([q]\) means that \(q\) is the self-intersection number of the corresponding rational curve.

Next, we have an existence result for extremal \(\text{Kähler} metrics\) in certain \(\text{Kähler classes}\) on the complex analytic compactifications found in Theorem 1.7.

**Theorem 1.8.** There exists an \(\varepsilon_0 > 0\) such that for \(\varepsilon < \varepsilon_0\), and for constants
\begin{align}
a_1 > 0, a_2 > 0, \\
a_i > 0, i = 1, 2, 3, \ell = 1 \ldots \ell_i, \\
c_k > 0, i = 1, 2, 3, k = 1 \ldots k_i,
\end{align}
the compactification in Theorem 1.7 admits extremal \(\text{Kähler metrics}\) with non-constant scalar curvature in the \(\text{Kähler class}\)
\begin{align}
a_1[\mathfrak{b}] + a_2[-\mathfrak{b}] + \sum_{i=1}^{3} \sum_{\ell=1}^{\ell_i} f(i, \varepsilon) a_i[\mathfrak{b}_i] + \sum_{i=1}^{3} \sum_{k=1}^{k_i} f'(i, \varepsilon) c_k[\mathfrak{e}_k],
\end{align}
where
\begin{align}
f(i, \varepsilon) = \begin{cases} 
\varepsilon^2 & \alpha_i \neq 1 \\
\varepsilon^4 & \alpha_i = 1,
\end{cases}
\text{and} \quad f'(i, \varepsilon) = \begin{cases} 
\varepsilon^2 & \alpha_i \neq \beta_i - 1 \\
\varepsilon^4 & \alpha_i = \beta_i - 1.
\end{cases}
\end{align}

The idea of the proof of Theorem 1.8 is similar to that of Theorem 1.3, but instead relies on gluing theorems for extremal \(\text{Kähler metrics}\) due to Arezzo-Pacard-Singer [APS11] and Arezzo-Lena-Mazzieri [ALM14], see also [AP06, AP09, Szé12, Szé13]. The introduction of the functions \(f\) and \(f'\) is necessary because if one of the bubbles is \(\text{Ricci-flat}\), then it must be glued on with a different scaling. This happens because in general \(\text{scalar-flat}\) \(\text{Kähler ALE surfaces}\) are ALE of order 2 [AV12, Str10], but if such a space is moreover \(\text{Ricci-flat}\), then it is ALE of order 4 [BKN89, CT94].

**Remark 1.9.** The case with minimal \(\mathfrak{t}\) is the case of \(\Gamma = D_8^*\) (the quaternion group), for which Theorem 1.8 produces extremal \(\text{Kähler metrics}\) on \(\mathbb{CP}^2 \# 7\mathbb{CP}^2\). The next smallest cases are \(\Gamma = D_{12}^*\) or \(\Gamma = \mathbb{Z}_{23}^2\), which yield examples on \(\mathbb{CP}^2 \# 8\mathbb{CP}^2\).

We refer the reader to [Li14, CH14, RS13] for other results and examples regarding complex analytic compactifications.

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2. Building blocks

Scalar-flat Kähler ALE metrics with cyclic groups at infinity are essential to the work of this paper because, in a sense, they are the “building blocks” of our construction. We introduce several important families here. In Subsection 2.1 and Subsection 2.2, we provide brief descriptions of the LeBrun negative mass metrics and the Calderbank-Singer metrics respectively. The former family of metrics is, in fact, contained in the latter, however we introduce and discuss these negative mass metrics separately, as well as outline their original construction, because it will be necessary to understand the action of the isometry group on these spaces. Also, in Subsection 2.2, we discuss the topology of the minimal resolutions of cyclic quotients of \(\mathbb{C}^2\) on which these metrics lie. In Subsection 2.3, we introduce the Gibbons-Hawking multi-Eguchi-Hanson metrics. The subclass of these given by those with a toric symmetry is actually contained in the family of Calderbank-Singer metrics, so their introduction is not strictly necessary. However, we discuss them separately in light of the results concerning the Kronheimer hyperkähler metrics described in Section 8 below.

2.1. LeBrun negative mass metrics. In \([\text{LeB88}]\), for all positive integers \(n\), LeBrun constructed a U(2)-invariant scalar-flat Kähler ALE metric on the total space of the bundle \(\mathcal{O}(-n)\) over \(\mathbb{CP}^1\), the minimal resolution of \(\mathbb{C}^2/L(1,n)\). The \(n = 1\) and \(n = 2\) cases are the well-known Burns and Eguchi-Hanson metrics respectively \([\text{Bur86, EH79}]\). When \(n > 2\), LeBrun showed that these metrics have negative mass thus providing an infinite family of counterexamples to the generalized positive action conjecture \([\text{HP78}]\). We next briefly describe the method of construction from \([\text{LeB88}]\).

Since any Kähler metric satisfies \(R\omega \wedge \omega = \rho \wedge \omega\), where \(R\) is the scalar curvature and \(\rho\) is the Ricci form, he examined the equation

\[
0 = \rho \wedge \omega. \tag{2.1}
\]

Restricting to those metrics on \(\mathbb{C}^2 \setminus \{0\}\) which result from a U(2)-invariant Kähler potential, i.e. those having Kähler forms \(\omega = \sqrt{-1} \partial \bar{\partial} \phi\) with potential functions \(\phi(z)\), where \(z = |z_1|^2 + |z_2|^2\), on \(\mathbb{C}^2 \setminus \{0\}\), LeBrun extracted a nonlinear ODE from this which he then solved to obtain a family of U(2)-invariant potential functions \(\{\phi_n(z)\}_{n \in \mathbb{Z}^+}\) for which (2.1) is satisfied. For each \(\phi_n(z)\), after defining a new radial coordinate

\[
r = \sqrt{z \frac{\partial \phi_n}{\partial z}}, \tag{2.2}
\]

LeBrun showed that the corresponding metric is

\[
g_{LB} = \frac{dr^2}{1 + \frac{n-2}{r^2} + \frac{1-n}{r^4}} + r^2 \left[ \sigma_1^2 + \sigma_2^2 + \left( 1 + \frac{n-2}{r^2} + \frac{1-n}{r^4} \right) \sigma_3^2 \right], \tag{2.3}
\]

where \(r\) is the radial distance from the origin and \(\sigma_1, \sigma_2, \sigma_3\) are the usual left-invariant coframe on \(\text{SU}(2) = S^3\). This metric is scalar-flat Kähler, hence anti-self-dual, but has an apparently singularity at \(r = 1\). However, redefine the radial coordinate as
\[ r^2 = (r^2 - 1) \] and attaching a \( \mathbb{CP}^1 \) at \( \tilde{r} = 0 \), and then take the \( \mathbb{Z}_n \) quotient of \( \mathbb{C}^2 \setminus \{0\} \) generated by

\[ (z_0, z_1) \mapsto (e^{2\pi i/n}z_0, e^{2\pi i/n}z_1). \]

The metric now extends smoothly over the \( \mathbb{CP}^1 \) at \( \tilde{r} = 0 \), and therefore \( g_{LB} \) defines a \( U(2) \)-invariant scalar-flat Kähler ALE metric on the total space of the bundle \( \mathcal{O}(-n) \) over \( \mathbb{CP}^1 \). Clearly, the group at infinity is \( L(1, n) \).

### 2.2. Calderbank-Singer metrics

For every pair of relatively prime integers \( 1 \leq q < p \), Calderbank-Singer constructed a family of scalar-flat Kähler, hence anti-self-dual, ALE metrics on the minimal resolution of \( \mathbb{C}^2/L(q, p) \), the topology of which is discussed below, \( [CS04] \). For \( 1 < q < p \) these metrics are toric and come in families of dimension \( k - 1 \). For \( q = 1 \) and \( q = p - 1 \), these metrics are the LeBrun negative mass metrics and the toric multi-Eguchi-Hanson metrics respectively, see Remark 2.3.

They used the Joyce ansatz \( [Joy95] \) to find these metrics explicitly, the idea of which is as follows. Let \( (B, h) \) be a spin 2-manifold with constant curvature \(-1\), \( W \to B \) be the spin-bundle with the induced metric (denoted by \( h \) as well), and \( V \) be a two-dimensional vector space with symplectic structure \( \varepsilon(\cdot, \cdot) \). Joyce defined a \( V \)-invariant metric on the product bundle \( B \times V \) associated to each bundle isomorphism \( \Phi : W \to B \times V \). First, he defines a family of metrics on \( V \) by

\[ (v, \tilde{v})_\Phi = h(\Phi^{-1}(v), \Phi^{-1}(\tilde{v})). \]

Notice that this family is parameterized by \( B \). Then, he defines a metric on the total space by

\[ g_\Phi = \Omega^2(h + (\cdot, \cdot)_\Phi), \]

for a nonvanishing conformal factor \( \Omega \in C^\infty(B) \). From the construction \( g_\Phi \) is clearly invariant under the action of \( V \) on \( B \times V \), so it descends to the quotient \( B \times (V/\Lambda) \) where \( \Lambda \subset V \) is any lattice. Joyce then showed that this metric is (anti-)self-dual if \( \Phi \) satisfies a certain linear differential equation. Calderbank-Singer considered \( B \subset \mathcal{H}^2 \), for which Joyce showed that there are scalar-flat Kähler representative in the conformal class of each \( g_\Phi \), and studied the solutions of the Joyce equations. Combining this with work on toric 4-manifolds, see \( [OR70] \), Calderbank-Singer explicitly completed their construction.

**Remark 2.1.** In \( [Wri11] \), Wright showed that if \( (X, g) \) is a toric scalar-flat Kähler ALE space with group at infinity \( \Gamma \) satisfying \( \chi_{orb}(X) > -2/|\Gamma| \), where \( \chi_{orb} \) is the orbifold Euler characteristic (see \( [CS04] \)), then it is isometric to a Calderbank-Singer metric on the minimal resolution of \( \mathbb{C}^2/\Gamma \). Clearly this inequality is satisfied whenever the underlying space is such a minimal resolution, so the Calderbank-Singer spaces are a complete list of the toric scalar-flat Kähler ALE metrics on minimal resolutions.

The minimal resolutions of cyclic quotients of \( \mathbb{C}^2 \) are classical \( [Hir53] \). In order to describe these, we introduce the following modified Euclidean algorithm. For
relatively prime integers $1 \leq q < p$, write
\[
p = e_1q - a_1
\]
\[
q = e_2a_1 - a_2
\]
(2.7)
\[
a_{k-3} = e_{k-1}a_{k-2} - 1
\]
\[
a_{k-2} = e_ka_{k-1} = e_k,
\]
where $e_i \geq 2$, and $0 \leq a_i < a_i - 1 < \cdots < a_1 < q$, for $i = 1, \cdots, k$. We refer to the integer $k$ as the length of the modified Euclidean algorithm. This can also be written as the continued fraction expansion
(2.8)
\[
\frac{q}{p} = \cfrac{1}{e_1 - \cfrac{1}{e_2 - \cdots - \cfrac{1}{e_k}}}
\]
where the $e_i$ and $k$ are as in (2.7).

Now, let $\tilde{X}$ be the minimal resolution of $\mathbb{C}^2/L(q, p)$, where $1 \leq q < p$ are relatively prime integers. The intersection matrix of the exceptional divisor $\tilde{X}$ is shown in Figure 2.2, where the $e_i$ and $k$ are defined above with $e_i \geq 2$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2.1.png}
\caption{Exceptional divisor in the Hirzebruch-Jung resolution.}
\end{figure}

Remark 2.2. For a group $L(q, p)$, where $q$ and $p$ are relatively prime integers with $1 \leq q \leq p - 1$, it will frequently be necessary to consider the length of this modified Euclidean algorithm, which we will denote by $k_{L(q, p)}$.

2.3. Multi-Eguchi-Hanson metrics. In [GH78], for all positive integers $n$, Gibbons and Hawking constructed hyperkähler ALE metrics with cyclic group action $L(-1, n)$ at infinity. These are known as multi-Eguchi-Hanson metrics, and a brief description of the construction is as follows.

Choose $n$ distinct (monopole) points $P = \{p_1, \cdots, p_n\}$ in $(\mathbb{R}^3, g_{\text{Euc}})$, and let $G_{p_i}$ be the fundamental solution of the Laplacian based at $p_i$ with normalization so that $\Delta G_{p_i} = 2\pi \delta_{p_i}$. Then, consider the function $V = \frac{1}{2} \sum_{i=1}^n G_{p_i}$. Since this is harmonic on $\mathbb{R}^3 \setminus P$, the 2-form $*dV$ is closed here and $\frac{1}{2\pi}[*dV] \in H^2(\mathbb{R}^2 \setminus P, \mathbb{Z})$. Let $X_0 \to \mathbb{R}^3 \setminus P$ be the unique principal U(1)-bundle corresponding to this cohomology class. By Chern-Weil theory there is a connection form $\omega$ on $X_0$ with curvature $d\omega = *dV$. The Gibbons-Hawking metric is defined on $X_0$ by
(2.9)
\[
g_{\text{GH}} = Vg_{\text{Euc}} + V^{-1}\omega^2.
\]
Now, define a larger manifold $X$ by attaching points $\tilde{p}_i$ over each $p_i$, over which $g_{GH}$ extends to a smooth Riemannian metric. The space $(X, g_{GH})$ is hyperkähler ALE with group at infinity $L(-1, n)$, and is clearly $S^3$-equivariant.

**Remark 2.3.** When all of the monopole points lie on a common line, these metrics are toric and, in fact, are exactly the Calderbank-Singer metrics for $q = p - 1$. This construction can also have a multiplicity at each point which results in orbifold metrics, see [Via10].

### 3. Group actions

3.1. **Quaternions and $S^3$.** It is our convention to identify $\mathbb{C}^2$ with the space of quaternions, $\mathbb{H} = \{x_0 + x_1i + x_2j + x_3k\}$, by having $(z_1, z_2) \in \mathbb{C}^2$ correspond to $z_1 + z_2j \in \mathbb{H}$. Identifying $S^3$ with the space of unit quaternions in the obvious way, it is easy to see that it acts on itself isometrically, in an orientation preserving way, by both left and right quaternionic multiplication. It is well known that the map $\phi : S^3 \times S^3 \to SO(4)$ defined by

\[
\phi(q_1, q_2)(h) = q_1 * h * q_2,
\]

for $h \in S^3$, is a double cover of $SO(4)$. Clearly, the kernel of $\phi$ is $\{(1, 1), (1, -1)\}$, hence $S^3 \times S^3 /\{(1, 1), (1, -1)\} = SO(4)$. The restriction of $\phi$ to the diagonal subgroup acts isometrically on the space of purely imaginary unit quaternions, which is identified with $S^2$, and therefore induces the double cover $\psi : S^3 \to SO(3)$ defined by

\[
\psi(q)(h) = q * h * \bar{q},
\]

for $h \in S^2$. For the remainder of this work, in a slight abuse of notation, we will write the elements $\phi(\alpha, \beta) \in SO(4)$ as the pair $[\alpha, \beta]$ where $\alpha, \beta \in S^3$, with the action given by $[\alpha, \beta](h) = \alpha * h * \beta$ and composition by $[\alpha_2, \beta_2] \circ [\alpha_1, \beta_1] = [\alpha_2 * \alpha_1, \beta_1 * \beta_2]$.

Now, we briefly introduce the Hopf fibration and examine its behavior under left and right quaternionic multiplication. Writing $S^3 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}$ and $S^2 = \mathbb{C} \cup \{\infty\}$, the Hopf map $\mathcal{H} : S^3 \to S^2$ is given by

\[
\mathcal{H}(z_1, z_2) = z_1/z_2.
\]

Observe, from this, that the typical fiber over $w \in S^2 = \mathbb{C} \cup \{\infty\}$ is the circle

\[
e^{i\theta} \sqrt{|w|^2 + 1} (w, 1) = \frac{e^{i\theta}}{\sqrt{|w|^2 + 1}} (w + \hat{j}) \in S^3, \text{ where } 0 \leq \theta < 2\pi.
\]

Let $h = h_1 + h_2 \hat{j} \in S^3$. First, by left multiplication we have

\[
(h_1 + h_2 \hat{j}) \ast \frac{e^{i\theta}}{\sqrt{|w|^2 + 1}} (w + \hat{j}) = \frac{e^{i\theta} h_1}{\sqrt{|w|^2 + 1}} (w + \hat{j}) + \frac{e^{-i\theta} h_2}{\sqrt{|w|^2 + 1}} (-1 + \bar{w} \hat{j}),
\]

so we see that the Hopf fibration is preserved if and only if $h_1$ or $h_2$ is zero. In the case that $h = h_1$, not only is the Hopf fibration preserved, but clearly the action is just
rotation in the Hopf fiber. When $h = h_2 \hat{j}$, observe that the entire circle is mapped to the Hopf fiber over the point $-1/\bar{w} \in S^2$. Next, by right multiplication we have

$$e^{i\theta} \sqrt{|w|^2 + 1} (w + \hat{j}) \ast (h_1 + h_2 \hat{j}) = e^{i\theta} \sqrt{|w|^2 + 1} (h_1 w - \bar{h}_2, h_2 w + \bar{h}_1).$$

(3.6)

This is the Hopf fiber over the point $(h_1 w - \bar{h}_2)/(h_2 w + \bar{h}_1) \in S^2 = \mathbb{C} \cup \{\infty\}$. Therefore, we see that right quaternionic multiplication always preserves the Hopf fibration.

3.2. **Finite subgroups of $U(2)$ containing no complex reflections.** Recall the following short exact sequence of Lie groups

$$1 \longrightarrow SU(2) \xrightarrow{i} U(2) \xrightarrow{det} U(1) \longrightarrow 1.$$  

(3.7)

Since this splits, we see that $U(2) = U(1) \ltimes SU(2)$; the semidirect product can be seen more explicitly in the double cover

$$\phi : S^1 \times S^3 \rightarrow U(2)$$

(3.8)

obtained by restricting the map $\phi$, from (3.1), to $S^1 \times S^3$ for the particular $S^1 = U(1)$ given by the unit quaternions $\exp(i\theta) \in S^3$ for $0 \leq \theta < 2\pi$, see [Cox91, Cro61, DV64].

First, we consider the finite subgroups of $SU(2)$. There is an isomorphism between $S^3$ and $SU(2)$ via right quaternionic multiplication as follows. Let $h_1 + h_2 \hat{j} \in S^3$, then

$$(z_1 + z_2 \hat{j}) \ast (h_1 + h_2 \hat{j}) \longleftrightarrow \begin{pmatrix} h_1 \\ h_2 \\ \bar{h}_2 \\ \bar{h}_1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}.$$  

(3.9)

Notice, from (3.6), that $SU(2)$ preserves the Hopf fibration. The finite subgroups of $SU(2)$ were determined in [Cox40]. They can be classified by the simply laced affine Dynkin of ADE-type by the McKay correspondence [McK80]. The quotient of $\mathbb{C}^2$ by a finite subgroup of $SU(2)$ yields a Du Val singularity at the origin whose minimal resolution is a tree of smooth rational curves with intersection matrix equivalent to the Cartan matrix of a Dynkin diagram of ADE-type. There has been much exposition on the finite subgroups of $SU(2)$, so here we provide a brief summary and refer the reader to [Ste08] for more details.

The finite subgroups of $SU(2)$ are the cyclic and binary polyhedral groups. For each integer $n \geq 2$, right quaternionic multiplication by $\exp(2\pi i/n)$ generates a cyclic group of order $n$, which we denote by $L(-1,n)$. This is of ADE-type $A_{n-1}$. By (3.9), this generator corresponds to the matrix

$$\begin{pmatrix} \exp(2\pi i/n) & 0 \\ 0 & \exp(-2\pi i/n) \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}.$$  

(3.10)

acting on $\mathbb{C}^2$. A quaternionic representation of the binary polyhedral groups first arose in Coxeter’s classification of the finite subgroups of the multiplicative group of quaternions [Cox40]. In considering a natural generalization of Hamilton’s formula’s for the quaternions group, for all triples of integers $a, b, c$, where $2 \leq a \leq b \leq c$, he introduced a group defined by

$$\langle a, b, c \rangle := \langle R, S, T | R^a = S^b = T^c = RST \rangle,$$  

(3.11)
and proved that it is finite in exactly the following cases

$$a, b, c = \begin{cases} 
2, 2, n & \text{for all } n \in \mathbb{Z}^+ \\
2, 3, 3 \\
2, 3, 4 \\
2, 3, 5.
\end{cases}$$

Coxeter then showed that these are in fact the binary polyhedral groups, and proceeded to find generators for each as a set of unit quaternions acting by right quaternionic multiplication. However, there was a mistake in his work for the binary icosahedral group which Lam later fixed [Lam03]. In Table 3.2 we provide a list of these groups along with their orders and ADE-types. The reader will find a list of generators by setting $m = 1$ in the first four cases of Table 3.3.

| $\langle a, b, c \rangle$ | Binary polyhedral group | Order | ADE-type |
|--------------------------|-------------------------|-------|----------|
| $\langle 2, 2, n \rangle$ | Binary dihedral group   | $D^*_{4n}$ | $D_{n+2}$ |
| $\langle 2, 3, 3 \rangle$ | Binary tetrahedral group| $T^*$  | $E_6$     |
| $\langle 2, 3, 4 \rangle$ | Binary octahedral group | $O^*$  | $E_7$     |
| $\langle 2, 3, 5 \rangle$ | Binary tetrahedral group| $I^*$  | $E_8$     |

Since $RST$ is in the center of $\langle a, b, c \rangle$, Coxeter considered the group quotient by the cyclic subgroup generated by $RST$, which clearly has the presentation

$$\langle a, b, c \rangle := \langle r, s, t \rangle | r^a = s^b = t^c = rst = 1 \rangle.$$  

This presentation, for $a, b, c$ corresponding to the binary polyhedral groups, is that of the polyhedral groups in SO(3); the symmetry groups of the regular $n$-gon, tetrahedron, octahedron and icosahedron. Coxeter showed that $RST = -Id$ in these cases, hence $(a, b, c)$ here is a $\mathbb{Z}_2$ quotient. Equivalently that each binary dihedral groups is a double cover of the corresponding polyhedral group. The polyhedral groups act as orientation preserving isometries on $S^2$, and it is well known the quotient of $S^2$ by one of these groups results in an orbifold with three singular points of orders $a, b, c$ for the corresponding polyhedral group. This is expressed in Table 3.2.

One can see more clearly how the binary polyhedral groups cover the polyhedral groups by examining SU(2)-actions under the Hopf map (3.3). Consider an arbitrary element $\gamma = [\exp(2\pi i/m), h_1 + h_2 \hat{j}] \in U(2)$. Then

$$\mathcal{H}(\gamma(z_1, z_2)) = \mathcal{H}(e^{2\pi i/m}(h_1 \cdot z_1 - \bar{h}_2 \cdot z_2), e^{2\pi i/m}(h_2 \cdot z_1 + \bar{h}_1 \cdot z_2) )$$

$$= \frac{h_1 \cdot z_1 - \bar{h}_2 \cdot z_2}{h_2 \cdot z_1 + \bar{h}_1 \cdot z_2} \in S^2 = \mathbb{C} \cup \{\infty\},$$

so we see that $\gamma$ descends to a Mobius transformation. Notice that left multiplication by the quaternion $\exp(2\pi i/m)$ cancels out under the Hopf map, which is intuitively
**Table 3.2.**

| (a, b, c) | Polyhedral group | Order |
|-----------|------------------|-------|
| (2, 2, n) | Dihedral group    | $D_{2n}$ |
| (2, 3, 3) | Tetrahedral group | $T$    |
| (2, 3, 4) | Octahedral group  | $O$    |
| (2, 3, 5) | Icosahedral group | $I$    |

Sensible since that corresponds to rotation of the Hopf fiber. It will later be important to understand this for a general $U(2)$-action.

**Remark 3.1.** Since $RST = -Id$ for each binary polyhedral group, they can be presented in terms of just the generators $S$ and $T$ as

$$\langle S, T | (ST)^2 = S^b = T^c \rangle,$$

where $b, c$ are as in $\langle a, b, c \rangle$.

Heuristically, from (3.8), one sees that any finite subgroup of $U(2)$ will be generated by some combination of a quaternion of the form $exp(2\pi i/m)$ acting on the left, a finite set of unit quaternions acting on the right, and products thereof.

A classification of the finite subgroups of $U(2)$, up to $U(2)$-conjugacy, was given in [DV64, Cox91]. However, here we are interested in those subgroups contain no complex reflections (those which act freely on $S^3$). These were classified in [Sco83].

Up to certain conditions, they are the image under $\phi$ of the cyclic group $L(1,2m)$ with the binary polyhedral groups, the index–2 diagonal subgroup of $\phi(L(1,4m) \times D_{4n}^*)$ which we denote by $\mathcal{J}_{m,n}^2$, the index–3 diagonal subgroup of $\phi(L(1,6m) \times T^*)$ which we denote by $\mathcal{J}_{m,n}^3$, and the cyclic groups $L(q, p)$. In Table 3.3 we list these groups along with their respective conditions and generators written in the form of $[\alpha, \beta]$ for some unit quaternions $\alpha, \beta \in S^3$ to be consistent with the notation in Section 3.1. From [Cox40, Lam03], a choice of generators in the case of the image under $\phi$ of the product of $L(1,2m)$ with a binary polyhedral group is clear.

However, a set of generators in terms of elements of $S^1 \times S^3$ is not obvious for the subgroups $\mathcal{J}_{m,n}^2, \mathcal{J}_{m,n}^3$ and $L(q, p)$, and we give a proof for these cases below.

**Remark 3.2.** Notice that the groups $\phi(L(1,2m) \times D_{4n}^*)$ and $\mathcal{J}_{m,1}^2$ (the $n = 1$ cases) are in fact cyclic, and are therefore excluded when non-cyclic subgroups are considered.

Generators of $\mathcal{J}_{m,n}^2$ were found in [FP04], however we find it prudent to discuss them here as well. Consider the index–2 normal subgroups, $L(1,2m) \triangleleft L(1,4m)$ and $L(-1,2n) \triangleleft D_{4n}^*$, generated by $[e^{\pi i/m}, 1]$ and $[1, e^{\pi i/n}]$ respectively. Then $\phi(L(1,2m) \times L(-1,2n) \triangleleft \phi(L(1,4m) \times D_{4n}^*)$ is an index–4 subgroup, so to obtain the index–2 diagonal subgroup one only needs to add the generator $[e^{\pi i/(2m)}, \hat{j}]$. In fact, $\mathcal{J}_{m,n}^2$ is generated by $[1, e^{\pi i/n}]$ and $[e^{\pi i/(2m)}, \hat{j}]$ alone since $[e^{\pi i/(2m)}, \hat{j}]^2 = [-e^{\pi i/m}, 1]$ and
Now, from Table 3.3, recall that to obtain the index–3 diagonal subgroup one only needs to add the diagonal element which are both of order 6, and notice that
\[ (3.16) \]

The choice of \((S^2)^{-1}\) is because, via \(\phi\), this generator will multiply by the quaternion \(S^2\) on the right. Note that there are several choices one could make here for the diagonal element to add as a generator, however given the presentations \((3.11)\) and \((3.15)\) it is easy to see that all such choices would yield the same group. Since

\[ [e^{\pi i/(3m)}, (-1 - \hat{i} - \hat{j} + \hat{k})/2]^3 = \exp(\pi i/m), 1], \]

the entire group \(\mathcal{F}_m^3\) can be generated.
with all singularities isolated and specified precisely as follows. The group \( L(q,p) \), where \( 1 \leq q < p \) are relatively prime, is generated by the matrix

\[
\begin{pmatrix}
\exp(2\pi i/p) & 0 \\
0 & \exp(2\pi iq/p)
\end{pmatrix}
\begin{pmatrix}
z_1 \\
z_2
\end{pmatrix}
\]

acting on \( \mathbb{C}^2 \). We know that this generator must have an equivalent representation as

\[
[e^{2\pi ik/p}, h_1 + h_2 \hat{j}]
\]

for some \( k \in \mathbb{Z} \) and some \( h_1 + h_2 \hat{j} \in S^3 \). Therefore

\[
\exp(2\pi ik/p) \begin{pmatrix} h_1 & -\bar{h}_2 \\ h_2 & \bar{h}_1 \end{pmatrix} = \begin{pmatrix} \exp(2\pi i/p) & 0 \\ 0 & \exp(2\pi iq/p) \end{pmatrix},
\]

so \( h_2 = 0, e^{2\pi ik/p} \cdot h_1 = e^{2\pi i/p} \), and \( e^{2\pi ik/p} \cdot \bar{h}_1 = e^{2\pi iq/p} \). From this, observe that \( h_1 = e^{2\pi i(1-k)/p} \). Finally, taking determinants of both sides of (3.19) we find that \( e^{4\pi ik/p} = e^{2\pi i(q+1)/p} \), so \( 2k \equiv (q + 1) \mod p \).

4. ALE orbifold metrics

In this section, for each non-cyclic finite subgroup \( \Gamma \subset U(2) \) containing no complex reflections, we take the essential step of obtaining a scalar-flat Kähler ALE orbifold with group at infinity \( \Gamma \) and with all singularities isolated and of cyclic type. These metrics will be obtained as certain quotients of certain LeBrun negative mass metrics, and all singularities will lie on an orbifold quotient of the \( \mathbb{C}\mathbb{P}^1 \) at the origin.

**Theorem 4.1.** For each non-cyclic finite subgroup \( \Gamma \subset U(2) \) containing no complex reflections, there exists a scalar-flat Kähler ALE orbifold with group at infinity \( \Gamma \) and with all singularities isolated and specified precisely as follows.

| \( \Gamma \subset U(2) \) | Conditions | Orbifold groups |
|------------------------|------------|----------------|
| \( \phi(L(1,2m) \times D_{4n}^*) \) | \((m,2n) = 1\) | \(L(1,2)\) \(L(1,2)\) \(L(-m,n)\) |
| \( \phi(L(1,2m) \times T^*) \) | \((m,6) = 1\) | \(L(1,2)\) \(L(-m,3)\) \(L(-m,3)\) |
| \( \phi(L(1,2m) \times O^*) \) | \((m,6) = 1\) | \(L(1,2)\) \(L(-m,3)\) \(L(-m,4)\) |
| \( \phi(L(1,2m) \times I^*) \) | \((m,30) = 1\) | \(L(1,2)\) \(L(-m,3)\) \(L(-m,5)\) |
| \( T_{m,n}^2 \) | \((m,2) = 2, (m,n) = 1\) | \(L(1,2)\) \(L(1,2)\) \(L(-m,n)\) |
| \( T_m^3 \) | \((m,6) = 3\) | \(L(1,2)\) \(L(1,3)\) \(L(2,3)\) |

**Proof.** Observe, from Table 3.3, that all \( \Gamma \) of interest contain \([e^{\pi i/m}, 1]\) in the set of generators. This acts on \( S^3 \) as rotation by \( \pi/m \) in the Hopf fiber, and generates \( L(1,2m) \). Therefore, we desire to take a quotient of \((\mathcal{O}(-2m), g_{LB})\) by \( \Gamma / L(1,2m) \) as to obtain the action of \( \Gamma \) at ALE infinity. This is equivalent to taking a quotient by \( \Gamma' \subset \Gamma \), where \( \Gamma' \) is the subgroup generated without \([e^{\pi i/m}, 1]\). However, notice that this action is not effective as \( L(1,2m) \cap \Gamma' = -Id \).

From the potential function construction of the LeBrun negative mass metrics discussed in Section 2.1, it is clear that the \( U(2) \)-action on \((\mathcal{O}(-2m), g_{LB})\) descends...
from the usual action on \( \mathbb{C}^2 \setminus \{0\} \) away from the \( \mathbb{CP}^1 \) at the origin. Intuitively the orbifold points of the quotient \( (\mathcal{O}(-2m), g_{LB})/\Gamma' \) can be seen to lie on the \( \mathbb{CP}^1 \) at the origin, and to arise from subgroups of the action on \( S^3/L(1,2m) \) that have some particular Hopf fibers on which they act by rotation. This is so, because as the fibers shrink, singularities occur precisely where some subgroup preserves the fiber as opposed to mapping between different fibers. The map \( \psi : S^3 \to SO(3) \), from (3.2), can be used to find the fixed points of the induced action on the \( \mathbb{CP}^1 \) at the origin. However, to find the actual orbifold groups, one needs to understand in what way the \( L(1,2m) \)-action in the normal directions determines the actual type singularity, and this is not immediately clear. We do so by first, for each \( \Gamma' \), finding the number of orbifold points and their respective orders, and then by using a diagonalization argument along with the Hopf map to find the precise orbifold groups.

For \( \Gamma = \phi(L(1,2m) \times \langle a,b,c \rangle) \), where here the integers \( a, b, c \) are as in Table 3.2, such that they correspond to a binary polyhedral group, observe that \( \Gamma' = \langle a,b,c \rangle \subset SU(2) \). Then, since \( L(1,2m) \) acts on \( S^3 \) by rotation of the Hopf fibers, \( \Gamma' \) descends to the action of the corresponding polyhedral group \( \langle a,b,c \rangle \) on the \( \mathbb{CP}^1 \) at the origin. Recall, from Section 3.2, that \( S^2/(a,b,c) \) has three singularities of orders \( a, b, c \). Since \( L(1,2m) \cap \Gamma' = -Id \), from the presentations (3.11) and (3.13), we see that the effective action of \( \Gamma' \), which is \( \Gamma/\langle a,b,c \rangle \subset U(2)/L(1,2m) \), is that of the corresponding polyhedral group. Therefore, \( (\mathcal{O}(-2m), g_{LB})/(a,b,c) \) has three cyclic singularities of orders \( a, b, c \) on the \( \mathbb{CP}^1 \) at the origin, and the subgroups of \( \Gamma' \) fixing these points are cyclic of order twice that of their respective singularities. Each such subgroup actually fixes two points on the \( \mathbb{CP}^1 \), but there are identifications of this larger set of fixed points in the overall action of \( \Gamma' \) yielding the three singularities. Thus, it is only necessary to understand the action around a representative fixed point of each class to determine the precise orbifold groups. These subgroups, being cyclic in \( SU(2) \), can be respectively diagonalized to \( L(-1,2p) \), generated by \( \gamma(p) = [1, e^{\pi i/p}] \), where \( p \in \{a,b,c\} \) corresponds to the order of the particular singularity. Due to the \( U(2) \)-invariance of the LeBrun negative mass metrics, we can restrict our attention to finding the orbifold group in the quotient by a diagonalized subgroup.

To do this, examine the induced action of \( \gamma(p) \) under the Hopf map as in (3.14):

\[
(4.1) \quad \mathcal{H}(\gamma(p)(z_1, z_2)) = \frac{e^{2\pi i/p} \cdot z_1}{z_2} \in S^2 = \mathbb{C} \cup \{\infty\}.
\]

This fixes the points \( \{0\}, \{\infty\} \in S^2 \). Without loss of generality, we focus on \( \{0\} \). From (4.1) observe that \( \gamma(p) \) induces an action of rotation by \( 2\pi/p \) on the tangent plane to \( \{0\} \subset S^2 \). Since the fixed point corresponds to \( z_1 = 0 \), the normal fiber here is the image of the complex line \( (0, z_2) \) in the quotient \( \mathbb{C}^2/L(1,2m) \). Then, since \( \gamma(p)(0, z_2) = (0, e^{-\pi i/p} \cdot z_2) \), the induced action in the normal directions to the fixed point in \( (\mathcal{O}(-2m), g_{LB}) \) is rotation by \( -2\pi im/p \). Therefore, the orbifold groups of the singularities in \( (\mathcal{O}(-2m), g_{LB})/(a,b,c) \) are \( L(-m,a) \), \( L(-m,b) \) and \( L(-m,c) \).

The proofs of the \( \mathcal{I}_n \) diagonal subgroup cases will follow similarly, however a delicate issue arises due to the fact that not all generators of the respective \( \Gamma' \) here
are in SU(2). Therefore, consider the projection map \( \Pi : U(2) \to SU(2) \) defined by
\[
\Pi([e^{i\theta}, h_1 + h_2 \hat{j}]) = [1, h_1 + h_2 \hat{j}] \in SU(2).
\]
For all \( \gamma \in U(2) \), from (3.14), we see that \( \mathcal{H}(\gamma(z_1, z_2)) = \mathcal{H}(\Pi(\gamma)(z_1, z_2)) \), and therefore, the singularities arising in \( (\mathcal{O}(-2m), g_{LB})/\Gamma' \) here, correspond point-wise, and in order, to those arising in the quotient by \( \Pi(\Gamma') \subset SU(2) \). Also, certain elements of each \( \Gamma' \) contain left quaternionic multiplication, it is necessary to examine their action carefully to determine the orbifold groups.

The subgroup \( \Gamma' \subset \mathbb{Z}_m^2 \) descends to the action of the dihedral group \( (2, 2, n) \) on \( S^2 \), and therefore \( (\mathcal{O}(-2m), g_{LB})/\Gamma' \) has three singularities of orders 2, 2, n as above. Clearly, the order 2 singularities have orbifold groups \( L(1, 2) \). The singularity of order \( n \) arises as a fixed point of the subgroup generated by \( [1, e^{i\pi/n}] \), so the orbifold group will be \( L(-m, n) \) as we saw earlier.

The subgroup \( \Gamma' \subset \mathbb{Z}_m^3 \) descends to the action of the tetrahedral group \( (2, 3, 3) \) on \( S^2 \), and therefore \( (\mathcal{O}(-2m), g_{LB})/\Gamma' \) has three singularities of order 2, 3, 3 as above. Clearly, the order 2 singularity has orbifold group \( L(1, 2) \). The order 3 singularities arise as the fixed points of the subgroup generated by \( [e^{i\pi/3m}, (-1 - i - j + k)/2] \). This diagonalizes to \( [e^{i\pi/3m}, e^{i\pi/3}] \), the induced action of which under the Hopf map fixes the points \( \{0\}, \{\infty\} \subset S^2 = \mathbb{C}^2 \cup \{\infty\} \). However, here these points are not identified in the overall quotient by \( \Gamma' \). This can be seen by examining the action of the tetrahedral group on \( S^2 \). From (1.1) and (1.2), observe that the actions in the tangent directions at \( \{0\} \) and \( \{\infty\} \) are rotations by \( 2\pi/3 \) and \( -2\pi/3 \) respectively. The normal fibers to \( \{0\} \) and \( \{\infty\} \) are the images of the complex lines \( (0, z_2) \) and \( (z_1, 0) \) in \( \mathbb{C}^2/L(1, 2m) \) respectively, and therefore the corresponding actions in the normal direct are rotations by \( 2\pi i(1 - m)/3 \) and \( 2\pi i(1 + m)/3 \). Finally, since \( (m, 6) = 3 \), the orbifold groups are \( L(1 - m, 3) = L(1, 3) \) and \( L(-1 - m, 3) = L(2, 3) \) respectively. \( \square \)

**Remark 4.2.** In Theorem 4.1, there are always precisely 3 singularities, and we will write these as type \( L(\alpha_i, \beta_i) \) for \( i = 1, 2, 3 \), where \( \alpha_i \) is chosen modulo \( \beta_i \) to satisfy \( 1 \leq \alpha_i \leq \beta_i - 1 \).

5. **Scalar-flat Kähler ALE metrics on minimal resolutions**

In this section we construct scalar-flat Kähler metrics ALE metrics on the minimal resolution of \( \mathbb{C}^2/\Gamma \) for each non-cyclic finite subgroup \( \Gamma \subset U(2) \) containing no complex reflections. This will prove Theorem 1.3 as the existence of such metrics for cyclic groups is already known. We begin in Section 5.1 by giving an adaptation of a gluing theorem of Rollin-Singer. Then, we complete the construction in Section 5.2 by using this gluing to resolve the singularities of the ALE orbifolds obtained in Theorem 4.1 by attaching appropriate Calderbank-Singer spaces, and thus obtain smooth scalar-flat Kähler ALE metrics. Finally, we make some remarks about the topology of the minimal resolutions.

5.1. **Scalar-flat Kähler gluing.** An essential component of the proof of Theorem 1.3 is the following adaptation of a gluing theorem due to Rollin-Singer:
Theorem 5.1. Let \((M, \omega)\) be a scalar-flat Kähler ALE orbifold of complex dimension 2 with finitely many cyclic singularities satisfying \(H^1(M; \mathbb{R}) = 0\). Then the minimal resolution \(\hat{M}\) admits scalar-flat Kähler metrics.

The proof is more or less identical to that of Rollin-Singer, with the addition of a weight at infinity, and we will only discuss the few places in the proof that need modification. Conforming to the notation from [RS05], we let \((X_1, g_1)\) be a Calderbank-Singer scalar-flat Kähler ALE space, and let \((X_2, g_2)\) be the scalar-flat Kähler ALE orbifold minus a singular point \(p\). We have holomorphic coordinates \(z\) near infinity on \(X_1\), with

\[
g_1 = |dz|^2 + \eta_1(z),
\]

\[
|\nabla^m \eta_1|_{g_1} = O(|z|^{-m-2}),
\]

as \(|z| \to \infty\), and holomorphic coordinates \(u\) around the singular point \(p\). On \(X_2\)

\[
g_2 = |du|^2 + \eta_2(u),
\]

\[
|\eta_2(u)| = O(|u|^2),
\]

\[
|\nabla \eta_2(u)| = O(|u|),
\]

\[
|\nabla^{m+2} \eta_2(u)| = O(1),
\]

for \(m \geq 0\), as \(|u| \to 0\).

Finally, choose ALE coordinates \(w\) near infinity on \(X_2\) with

\[
g_2 = |dw|^2 + \eta_1(w),
\]

\[
|\nabla^m \eta_1|_{g_1} = O(|w|^{-m-2}),
\]

for \(m \geq 0\), as \(|w| \to \infty\). Note that these coordinates are not necessarily holomorphic.

Let \(r_1 \geq 1\) be a smooth function satisfying

\[
r_1 = \begin{cases} |z| & |z| \geq 2 \\
1 & |z| \leq 1/2,
\end{cases}
\]

and \(r_2 \geq 1/2\) for \(|u| \geq 1/2\) a smooth function satisfying

\[
r_2 = \begin{cases} |u| & |u| \leq 1/2 \\
1 & |u| \geq 2,
\end{cases}
\]

and let \(r_3 \geq 1\) be a smooth function satisfying

\[
r_3 = \begin{cases} |w| & |w| \geq 2 \\
1 & |w| \leq 1/2.
\end{cases}
\]

The weighted space on \(X_1\) is defined exactly as before,

\[
r_1^{\delta_1} B^{n, \alpha}(X_1) = \{ f : r_1^{-\delta_1} f \in B^{n, \alpha}(X_1) \},
\]

with norm

\[
\| f \|_{n, \alpha, \delta_1} = \| r_1^{-\delta_1} f \|_{n, \alpha}
\]
where $C^{n,\alpha}$ are the Hölder spaces with norms defined as on [RS05, page 257].

For $X_2$ we will define a doubly-weighted space by

$$r_2^{-\delta_2}r_3^{-\delta_3}B^{n,\alpha}(X_2) = \{ f : r_2^{-\delta_2}r_3^{-\delta_3}f \in B^{n,\alpha}(X_2) \},$$

with norm

$$\|f\|_{n,\alpha,\delta_2,\delta_3} = \|r_2^{-\delta_2}r_3^{-\delta_3}f\|_{n,\alpha} \tag{5.10}$$

with obvious modification to the definition of the Hölder norm.

Fixing a scalar-flat Kähler metric, the nonlinear map is

$$F : \Lambda_2^- (\cong \Lambda_0^{1,1}) \to \Lambda_2^+, \tag{5.11}$$

defined by

$$F(A) = \Pi_{\Lambda_2^+} \Lambda_+ W_+ (\omega + A),$$

where $W_+$ is the self-dual part of the Weyl tensor, and $\Lambda$ is the adjoint of wedging with $\omega + A$. The zeroes of $F$ correspond to anti-self-dual Hermitian metrics.

The linearized operator of $F$ at $A = 0$ is given by

$$S : \alpha \mapsto d^+ \delta \alpha + \langle \rho, \alpha \rangle \omega, \tag{5.13}$$

where $\rho$ is the Ricci form. The most important step is to show that there is no cokernel for the adjoint of the linearized operator on each of the pieces.

**Proposition 5.2.** Choose $0 < \delta < 2$ and $0 < \alpha < 1$. Then on $X_1$, the linearized operator mapping from

$$S_1 : r_1^{-\delta}B^{2,\alpha}(X_1, \Lambda_2^+) \to r_1^{-\delta - 2}B^{0,\alpha}(X_1, \Lambda_+^{2}) \tag{5.14}$$

has a bounded right inverse $G_1$.

On $X_2$, the linearized operator mapping from

$$S_2 : r_2^{-\delta}r_3^{-\delta}B^{2,\alpha}(X_2, \Lambda_2^-) \to r_2^{-\delta - 2}r_3^{-\delta}B^{0,\alpha}(X_2, \Lambda_+^{2}), \tag{5.15}$$

has a bounded right inverse $G_2$.

**Proof.** The proof is almost identical to that of [RS05, Proposition 4.3.2] with a minor modification. Writing $\phi \in \Lambda_2^+$ as $\phi = f\omega + \alpha$, where $\alpha \in \Lambda_0^{2,0} \oplus \Lambda_0^{0,2}$, a kernel element of the adjoint operator satisfies

$$S^* \phi = d^- \delta (f\omega + \alpha) - f\rho = 0. \tag{5.16}$$

This can be differentiated to yield

$$(\Delta^2 + 2 \text{Ric} \cdot \nabla^2) f = 0, \tag{5.17}$$

which can then be written as

$$(\bar{\partial} \partial^#) (\bar{\partial} \partial^#)^* f = 0, \tag{5.18}$$

where $\partial^#$ is the $(1,0)$ component of the gradient.

The main point now is that by choice of weight forces $f$ to decay, and an integration-by-parts argument shows that

$$(\bar{\partial} \partial^#) f = 0. \tag{5.19}$$
Consequently, $\partial^# f$ is holomorphic, which says that $J\nabla f$ is Killing. However, there are no decaying Killing fields on an ALE space, so $f \equiv 0$, and this implies that $\alpha \equiv 0$. 

The rest of the gluing argument (construction of the approximate metric and application of the implicit function theorem) is almost exactly the same as in [RS05], and the details are omitted. The final ingredient is an adaptation of a result of Boyer to prove the spaces are in fact scalar-flat Kähler. To see this, since $g$ is Hermitian, then the Kähler form $\omega = g(J\cdot, \cdot)$ satisfies
\begin{equation}
\label{eq:5.20}
d\omega + \beta \wedge \omega = 0,
\end{equation}
for a 1-form $\beta$ called the Lee form. In [Boy86], Boyer shows that if $(X, g)$ is anti-self-dual, then $d^+ \omega = 0$. From the choice of weight, an integration-by-parts argument then shows that $d\omega = 0$. Since $H^1(X, \mathbb{R}) = 0$, we have $\omega = df$, and the conformal metric $e^{-f}\omega$ is then Kähler (which is necessarily scalar-flat since it is anti-self-dual).

5.2. Construction of the metrics. For each non-cyclic finite subgroup $\Gamma \subset U(2)$ containing no complex reflections, consider the scalar-flat Kähler ALE orbifold with group at infinity $\Gamma$ obtained in Theorem 4.1. All singularities of this space lie on the $\mathbb{CP}^1$ at the origin and are of cyclic type with the orbifold groups $L(\alpha_i, \beta_i)$ for $i = 1, 2, 3$, recall Remark 4.2. To resolve these singularities, we use Theorem 5.1 to attach a Calderbank-Singer space with group at infinity $L(\alpha_i, \beta_i)$ to the singularity with the corresponding orbifold group. Furthermore, since both the ALE orbifold, obtained as a quotient of a LeBrun negative mass metric, and all of the Calderbank-Singer spaces admit holomorphic isometric circle actions corresponding to rotations of the Hopf fiber, we can impose an $S^1$ symmetry in the gluing argument, that is, we may perform the gluing $S^1$-equivariantly, to obtain a scalar-flat Kähler ALE space with group at infinity $\Gamma$ which admits a holomorphic isometric circle action.

It is straightforward to see that the conditions in Definition 1.1 are satisfied, so we have obtained scalar-flat Kähler ALE metrics on the minimal resolution of $\mathbb{C}^2/\Gamma$. This completes the proof of Theorem 1.3.

Remark 5.3. It would perhaps be possible to use gluing results for constant scalar curvature Kähler metrics, instead of the Hermitian anti-self-dual gluing theorem. We used the latter here since it is easier to generalize the result of Rollin-Singer to the case that the base orbifold is ALE. However, gluing results for extremal Kähler metrics will indeed be used in Section 7.

5.3. Topology of the minimal resolutions. The intersection matrix of the exceptional curve in the minimal resolution is described in Figure 5.1.

In Figure 5.1, each string of $e^i_j$ corresponds to the intersection matrix of the respective Calderbank-Singer space attached to the central rational curve at a particular singularity. Clearly, the signature of the minimal resolution of $\mathbb{C}^2/\Gamma$ is
\begin{equation}
\tau = -b_2^- = -1 - \sum_{i=1}^{3} k_{L(\alpha_i, \beta_i)}.
\end{equation}
The orbifold points which arise on the \( \mathbb{CP}^1 \) at the origin in our work correspond, in fact, to those in Brieskorn’s resolution, and our procedure of resolving singularities by attaching the appropriate Calderbank-Singer spaces is a metric version of Brieskorn’s complex analytic resolution of those singularities by a Hirzebruch-Jung string.

Finally, the self-intersection number of the central rational curve is given by

\[
-b_\Gamma = -\left( \frac{\alpha_1}{\beta_1} + \frac{\alpha_2}{\beta_2} + \frac{\alpha_3}{\beta_3} + \frac{2m}{h} \right),
\]

where \( h \) is the order of the image of the group in \( \text{PGL}(2, \mathbb{C}) \) and \( m \) corresponds to that of the group, see [Bri68]. When \( \Gamma \) is the image under \( \phi \) of the product of \( L(1, 2m) \) with a binary polyhedral group, its image in \( \text{PGL}(2, \mathbb{C}) \) is just the corresponding polyhedral group. Similarly, the images of \( \mathcal{I}_{2m,n} \) and \( \mathcal{I}_{3m} \) in \( \text{PGL}(2, \mathbb{C}) \) are the dihedral and tetrahedral groups respectively. Finally, we note that

\[
b_\Gamma = 2 + \frac{4m}{|\Gamma|} \left[ m - \left( m \mod \frac{|\Gamma|}{4m} \right) \right].
\]

Observe, from this, that \(-b_\Gamma < -1\).

6. Scalar-flat Kähler deformations

We begin by finding the dimension of the space of infinitesimal scalar-flat Kähler deformations of the ALE orbifolds given in Theorem 4.1. An essential ingredient here is Honda’s work in understanding the deformation theory of compactifications of the LeBrun negative mass metrics [Hon13, Hon14]. The space \((\mathcal{O}(-2), g_{LB})\) is the Eguchi-Hanson metric, for which it is well known that there are no such deformations. This is the \( m = 1 \) case, and the groups corresponding to this are exactly the binary polyhedral groups. Therefore it is only necessary to consider \( m > 1 \). In the following proposition we not only find the dimension of the space of such deformations, but in
fact show that it can be given very simply in terms of the respective \( b_\Gamma \), which recall
is negative the self-intersection number of the central rational curve.

**Proposition 6.1.** Let \( \Gamma \subset U(2) \) be a non-cyclic finite subgroup containing no complex
reflections with \( m > 1 \). Then, the dimension of the space of infinitesimal scalar-flat
Kähler deformations of the quotient \(( \mathcal{O}(-2m), g_{LB})/\Gamma' \) is given by

\[
dim(H^s_{\Gamma'}) = 2b_\Gamma - 2 = 2 + \frac{8m}{|\Gamma|} \left[ m - \left( m \mod \frac{|\Gamma|}{4m} \right) \right].
\]

**Proof.** Finding the dimension of scalar-flat Kähler deformation of \(( \mathcal{O}(-2m), g_{LB})/\Gamma' \)
is equivalent to finding the dimension of the space of scalar-flat Kähler deformations
of \(( \mathcal{O}(-2m), g_{LB}) \) that are invariant under the action of \( \Gamma' \). In general, for a representa-
tion \( \rho \) of a finite group \( \Gamma' \), the dimension of the space of invariant elements is
given by

\[
\frac{1}{|\Gamma'|} \sum_{\gamma \in \Gamma'} \chi_\rho(\gamma),
\]

where \( \chi_\rho(\gamma) \) denotes the character of \( \gamma \in \Gamma' \). For \( \gamma = [1, e^{i\theta}] \neq \pm 1 \in SU(2) \), there is the following well-known character identity on \( S^{2k}(\mathbb{C}^2) \) as a representation of \( SU(2) \)

\[
\chi_{S^{2k}(\mathbb{C}^2)}(\gamma) = \sum_{p=0}^{2k} (e^{i\theta})^{2k-2p} \left( \frac{\sin((2k+1)\theta)}{\sin(\theta)} \right) = \sin(2k\theta) \cot(\theta) + \cos(2k\theta).
\]

Notice that \( \chi_{S^{2k}(\mathbb{C}^2)}(\gamma = [1, i]) = \cos(\pi k) \), since \( \sin(\pi k) \cot(\pi/2) = 0 \). It will be
useful here to introduce the sawtooth function which, for any \( x \in \mathbb{R} \), is defined as

\[
((x)) = \begin{cases} 
 x - \lfloor x \rfloor - \frac{1}{2} & \text{when } x \notin \mathbb{Z} \\
 0 & \text{when } x \in \mathbb{Z}, 
\end{cases}
\]

where \( \lfloor x \rfloor \) denotes the greatest integer less than \( x \). When \( \theta = \pi/n \), there is the following useful identity due to Eisenstein, see [Apo90],

\[
\sum_{j=1}^{n-1} \sin(\frac{2\pi k}{n}j) \cot(\frac{\pi}{n}j) = -2n\left( \left( \frac{k}{n} \right) \right)
\]

For \( m > 1 \), Honda showed that the complexification of the space of infinitesimal scalar-flat
Kähler deformations of \(( \mathcal{O}(-2m), g_{LB}) \) is equivalent to

\[
\rho \oplus \overline{\rho} \quad \text{where} \quad \rho = S^{2m-2}(\mathbb{C}^2) \otimes \text{det},
\]

as a representation space of \( U(2) \), see [Hon13]. Using this along with (6.1), we will compute \( \dim(H^s_{\Gamma'}) \) for all appropriate \( \Gamma' \).

Write the maximal torus in \( U(2) \) as \( \text{diag}\{z_1 = e^{i\theta_1}, z_2 = e^{i\theta_2}\} \), and observe that

\[
\chi_\rho(z_1, z_2) = (z_1 z_2)^{2m-2} \sum_{p=0}^{\frac{2m-2}{2}} z_1^{2m-2-p} z_2^p.
\]
where \( \chi_\rho(z_1, z_2) \) denotes the character of any \( \gamma \in \Gamma' \) with eigenvalues \( \{z_1, z_2\} \). Note that \( \dim(H^s_{\Gamma'}) = 2\chi_\rho(1, 1) = 2\chi_\rho(-1, -1) = 4m - 2 \).

Consider when \( \Gamma \) is the image under \( \phi \) of the product of \( L(1, 2m) \) and a binary polyhedral group. (Although the proofs for \( \mathcal{I}_{2m,n} \) and \( \mathcal{I}_3m \) will follow similarly, there is a certain delicate difference.) Here, since the \( \Gamma' \subset SU(2) \) are the respective binary polyhedral groups so any \( \gamma \in \Gamma' \) has eigenvalues \( \{z = e^{i\theta}, z^{-1} = e^{-i\theta}\} \). Thus, from (6.2), the character for \( \gamma \neq \pm 1 \) reduces to

\[
(6.7) \quad \chi_\rho(\gamma) = \sum_{p=0}^{2m-2} (e^{i\theta})^{2m-2-2p} = \sin((2m - 2)\theta) \cot(\theta) + \cos((2m - 2)\theta),
\]

and therefore

\[
(6.8) \quad \dim(H^s_{\Gamma'}) = \frac{2}{|\Gamma'|} \left[ 2 \cdot (2m - 1) + \sum_{\gamma \neq \pm Id \in \Gamma'} \sin((2m - 2)\theta) \cot(\theta) + \cos((2m - 2)\theta) \right].
\]

We will now find \( \dim(H^s_{\Gamma'}) \) in terms of certain greatest integer functions for each binary polyhedral group separately. The conjugacy classes of the binary polyhedral groups are well known, see [Ste08], and from the trace of a representative element one can determine the eigenvalues of the elements in its class. We use this to decompose the binary polyhedral groups into sets of elements having the same eigenvalues. Then, we further group the elements in such a way as to be able to repeatedly use (6.4) to compute \( \dim(H^s_{\Gamma'}) \) in terms of sawtooth functions, which we examine with respect to the possible congruencies of \( m \) and in turn write in terms of greatest integer functions.

**D**\(_{4n}^*\): The binary dihedral group is composed of two disjoint sets of elements; the elements of the cyclic group \( L(-1, 2n) \), and the set of elements

\[
S = \{[1, e^{\pi ik/n} * j] \mid 1 \leq k \leq 2n, 1 \leq j \leq n \}.
\]

Since each element of \( S \) has eigenvalues \( \{i, -i\} \), and because \( ((x/2)) = 0 \) for all \( x \in \mathbb{Z} \), observe that it does not contribute to the sum in (6.8). Decomposing the sum over these sets, we find that

\[
\dim(H^s_{D_{4n}^*}) = \frac{1}{n} \left[ (2m - 1) - 2n \left( \frac{m - 1}{n} \right) \right]
+ n \cos(\pi(m - 1)) + \sum_{j=1}^{n-1} \cos \left( \frac{2\pi}{n} (m - 1)j \right)
= \begin{cases} 
2 \left[ \frac{m-1}{n} \right] + 2 & m \not\equiv 1 \mod n \\
2 \left( \frac{m-1}{n} \right) + 2 & m \equiv 1 \mod n.
\end{cases}
\]

**T**\(^*\): The elements of the binary tetrahedral group are separated into sets according to their eigenvalues as in Table [6.1]. Group together the elements with eigenvalues \( \{e^{\pi i/3}, e^{-\pi i/3}\} \), for \( \ell = 1, 2 \), since these constitute the elements of \( (L(-1, 6)/\mathbb{Z}_2) \setminus \text{Id} \), and ignore the terms having eigenvalues \( \{i, -i\} \) since they do not contribute to the
Table 6.1.

| $\{z_1, z_2\}$ | $\#\{\gamma \in T^* : \text{diag}(\gamma) = \{z_1, z_2\}\}$ |
|-----------------|--------------------------------------------------|
| $\{1, 1\}$     | 1                                                |
| $\{-1, -1\}$   | 1                                                |
| $\{i, -i\}$    | 6                                                |
| $\{e^{\pi i/3}, e^{-\pi i/3}\}$ | 8         |
| $\{e^{2\pi i/3}, e^{-2\pi i/3}\}$ | 8         |

sum in (6.8). Decomposing the sum over these sets, we find that

$$\dim(H_{sf}^{T^*}) = \frac{1}{6} \left[ (2m - 1) - 24 \left( \frac{m - 1}{3} \right) \right]$$

$$+ 3 \cos(\pi(m - 1)) + 4 \sum_{j=1}^{2} \cos \left( \frac{2\pi}{3} (m - 1)j \right)$$

$$= \begin{cases} 4 \left\lfloor \frac{m-1}{3} \right\rfloor - m + 3 & m \equiv 5 \mod 6 \\ \frac{m-1}{3} + 2 & m \equiv 1 \mod 6. \end{cases}$$

$O^*$: The elements of the binary octahedral group are separated into sets according to their eigenvalues as in Table 6.2. Group together the elements with eigenvalues $\{e^{\pi i \ell/3}, e^{-\pi i \ell/3}\}$, for $\ell = 1, 2$, as before. Also, group together the elements with eigenvalues $\{e^{\pi i \ell/4}, e^{-\pi i \ell/4}\}$, for $\ell = 1, 2, 3$, since these constitute $(L(-1, 8)/\mathbb{Z}_2) \setminus \text{Id})$. Since, the $\{i, -i\}$ elements do not contribute to the sum in (6.8), it does not matter that the order of that set differs from the orders of the other sets in this grouping.

Table 6.2.

| $\{z_1, z_2\}$ | $\#\{\gamma \in O^* : \text{diag}(\gamma) = \{z_1, z_2\}\}$ |
|-----------------|--------------------------------------------------|
| $\{1, 1\}$     | 1                                                |
| $\{-1, -1\}$   | 1                                                |
| $\{i, -i\}$    | 18                                               |
| $\{e^{\pi i/3}, e^{-\pi i/3}\}$ | 8         |
| $\{e^{2\pi i/3}, e^{-2\pi i/3}\}$ | 8         |
| $\{e^{\pi i/4}, e^{-\pi i/4}\}$ | 6         |
| $\{e^{3\pi i/4}, e^{-3\pi i/4}\}$ | 6         |
Decomposing the sum over these sets, we find that
\[
\dim(H_{sfk}^{s f k}) = \frac{1}{12}\left[ (2m - 1) - 24\left(\frac{m - 1}{3}\right) - 24\left(\frac{m - 1}{4}\right) + 6 \cos(\pi (m - 1)) \\
+ 4 \sum_{j=1}^{2} \cos\left(\frac{2\pi}{3} (m - 1)j\right) + 3 \sum_{j=1}^{3} \cos\left(\frac{2\pi}{4} (m - 1)j\right) \right]
\]
\[
= \begin{cases} 
2\left\lfloor \frac{m-1}{3} \right\rfloor + 2\left\lfloor \frac{m-1}{4} \right\rfloor - m + 3 & m \equiv 11 \bmod 12 \\
2\left\lfloor \frac{m-1}{3} \right\rfloor + \frac{1-m}{3} + 2 & m \equiv 7 \bmod 12 \\
2\left\lfloor \frac{m-1}{3} \right\rfloor + \frac{1-m}{2} + 2 & m \equiv 5 \bmod 12 \\
\frac{m-1}{6} + 2 & m \equiv 1 \bmod 12.
\end{cases}
\]

\(I^*\) : The elements of the binary icosahedral group are separated into sets corresponding to their eigenvalues as in Table 6.3. Group together elements with eigenvalues \(\{e^{\pi i/3}, e^{-\pi i/3}\}\), for \(\ell = 1, 2\), as before. Also, group together elements with eigenvalues \(\{e^{\pi i/5}, e^{-\pi i/5}\}\), for \(\ell = 1, 2, 3, 4\), since these constitute the elements of \((L(-1, 10)/\mathbb{Z}_2) \setminus \text{Id}\). Ignoring elements with eigenvalues \(\{i, -i\}\) since they do not contribute to the sum in (6.8), and decomposing the sum over these sets, we find that

\[
\dim(H_{sfk}^{s f k}) = \frac{1}{30}\left[ (2m - 1) - 60\left(\frac{m - 1}{3}\right) - 60\left(\frac{m - 1}{5}\right) + 15 \cos(\pi (m - 1)) \\
+ 10 \sum_{j=1}^{2} \cos\left(\frac{2\pi}{3} (m - 1)j\right) + 6 \sum_{j=1}^{4} \cos\left(\frac{2\pi}{5} (m - 1)j\right) \right]
\]
\[
= \begin{cases} 
2\left\lfloor \frac{m-1}{3} \right\rfloor + 2\left\lfloor \frac{m-1}{5} \right\rfloor - m + 3 & m \equiv 17, 23, 29 \bmod 30 \\
2\left\lfloor \frac{m-1}{5} \right\rfloor + \frac{1-m}{3} + 2 & m \equiv 7, 13, 19 \bmod 30 \\
2\left\lfloor \frac{m-1}{3} \right\rfloor + 3\left(\frac{1-m}{5}\right) + 2 & m \equiv 11 \bmod 30 \\
\frac{m-1}{15} + 2 & m \equiv 1 \bmod 30.
\end{cases}
\]
The proof for these groups is completed as follows. We have found each \( \dim(H^{sfk}_T) \) in terms of certain greatest integer functions with respect to the particular congruences \( \mod |\Gamma|/4 \). For any positive integers \( x, y, z \), satisfying \( y < x \mod z \), notice that

\[
\left\lfloor \frac{x - y}{z} \right\rfloor = \frac{x - (x \mod z)}{z} \in \mathbb{Z}.
\]

Using this, the expressions for \( \dim(H^{sfk}_T) \) are easily simplified to obtain \( 2b_T - 2 \).

The idea for the groups \( J^2_{m,n} \) and \( J^3_m \) is similar, however as the authors are unaware of a decomposition of these groups into conjugacy classes, we perform an element-wise decomposition of these groups into sets, from which we are able to find the respective eigenvalue decompositions and proceed as above. We will see that all eigenvalues here will be of the form \( \{z_1, z_2\} = e^{i\theta_1} \{e^{i\theta_2}, e^{-i\theta_2}\} \), for some \( 0 \leq \theta_1, \theta_2 < 2\pi \), and therefore prove the following useful identity before we begin.

\[
\chi_\rho(e^{i(\theta_1+\theta_2)}, e^{i(\theta_1-i\theta_2)}) = e^{i(2\theta_1)} \sum_{p=0}^{2m-2} (e^{i(\theta_1+\theta_2)})^{2m-2-p} (e^{i(\theta_1-\theta_2)})^{p} = e^{i(2m\theta_1)} \sum_{p=0}^{2m-2} (e^{i\theta_2})^{2m-2-2p} = e^{i(2m\theta_1)} \left[ \sin(2(m-1)\theta_2) \cot(\theta_2) + \cos(2(m-1)\theta_2) \right].
\]

\( J^2_{m,n} \): Recall that \( \Gamma' \subset U(2) \) is generated by \( [1, e^{\pi i/n}], [e^{\pi i/(2m)}, j] \). In Section 3.2 we saw that these actually generate the entire group \( J^2_{m,n} \), but in order to obtain a smooth quotient it was necessary to take a quotient of \( \hat{O}(-2m), g_{LB} \), and included \( [e^{\pi i/m}, 1] \) in the set of generators accordingly. To find \( \dim(H^1_{4m,n}) \), we first decompose \( \Gamma' = J^2_{m,n} \) into the two disjoint sets of elements

- \( S_1 = \left\{ \gamma_{1}^{\ell,k} := [e^{\pi i/(2m)}, j]^{2\ell} \cdot [1, e^{\pi ik/n}] \right\} \) \( 0 \leq \ell \leq m-1 \) and \( 0 \leq k \leq 2n-1 \)
- \( S_2 = \left\{ \gamma_{2}^{\ell,k} := [e^{\pi i/(2m)}, j]^{2\ell+1} \cdot [1, e^{\pi ik/n}] \right\} \) \( 0 \leq \ell \leq m-1 \) and \( 0 \leq k \leq 2n-1 \).

Clearly \( S_1, S_2 \subset J^2_{m,n} \) and \( S_1 \cap S_2 = \emptyset \), so since \( |S_1| = |S_2| = 2mn \) and \( |J^2_{m,n}| = 4mn \) we see that \( S_1 \cup S_2 \) constitutes all of \( J^2_{m,n} \). Next, for arbitrary \( \gamma_{1}^{\ell,k} \in S_1 \) and \( \gamma_{2}^{\ell,k} \in S_2 \), as given above, observe that

- The eigenvalues of \( \gamma_{1}^{\ell,k} \in S_1 \) are \( (-1)^{\ell} \{e^{\pi i(\ell+\frac{k}{n})}, e^{\pi i(\ell-\frac{k}{n})}\} \)
- The eigenvalues of \( \gamma_{2}^{\ell,k} \in S_2 \) are \( (-1)^{\ell} \{e^{\pi i(\ell+1+\frac{k}{n})}, e^{\pi i(\ell-1+\frac{k}{n})}\} \).

From (6.10), the characters of these elements are found to be as follows.

- \( \chi_\rho(\gamma_{1}^{\ell,k}) = \begin{cases} \frac{\sin \left( \frac{2\pi}{n} k(m-1) \right)}{2m-1} \cot \left( \frac{2\pi}{n} k \right) + \cos \left( \frac{2\pi}{n} k(m-1) \right) & k \neq 0 \text{ or } n \\ \frac{2}{2m-1} & k = 0 \text{ or } n. \end{cases} \)
- \( \chi_\rho(\gamma_{2}^{\ell,k}) = 1 \) since \( m \) is even.
Therefore, decomposing the sum over these sets, we find that
\[
\dim(H_{3m,n}^{sfk}) = \frac{1}{2mn} \left[ 2m \cdot (2m - 1) - 4mn \left( \frac{m-1}{n} \right) \right] \\
+ 2mn + 2m \sum_{j=1}^{n-1} \cos \left( \frac{2\pi}{n} (m-1)j \right) \\
= \begin{cases} 
2\left( \frac{m-1}{n} \right) + 2 & m \not\equiv 1 \mod n \\
2\left( \frac{m-1}{n} \right) + 2 & m \equiv 1 \mod n.
\end{cases}
\]

Finally, using (6.9), observe that \( \dim(H_{3m,n}^{sfk}) = 2b_{\gamma m,n} - 2 \).

\( \mathcal{J}_m^3 \): Recall that \( \Gamma' \subset U(2) \) is generated by \([1, \hat{i}], [1, \hat{j}], [e^{\pi i/(3m)}, (-1 - \hat{i} - \hat{j} + \hat{k})/2] \).
In Section 3.2, we saw that these actually generate the entire group \( \mathcal{J}_m^3 \), but in order to obtain a smooth quotient it was necessary to take a quotient of \( (\mathcal{O}(-2m), g_{LB}) \), and included \([e^{\pi i/m}, 1]\) in the set of generators accordingly. Therefore, \( \Gamma' = \mathcal{J}_m^3 \) is just \( \phi(D_8^* \times ([e^{\pi i/(3m)}, (-1 - \hat{i} - \hat{j} + \hat{k})/2]) \). To find \( \dim(H_{3m,n}^1) \), we first decompose \( \Gamma' = \mathcal{J}_m^3 \) into twenty-four disjoint sets of elements which we write compactly below as \( S_{\alpha,s} \) where the index \( \alpha \in \{ \pm1, \pm\hat{i}, \pm\hat{j}, \pm\hat{k} \} \) and the index \( s \in \{0, 1, 2\} \):

- \( S_{\pm1,s} = \{ \gamma_{\pm1,s}^r : = \pm[e^{\pi i/(3m)}, (-1 - \hat{i} - \hat{j} + \hat{k})/2]^{3r+s} \}_{0 \leq r < m} \)
- \( S_{\pm i,s} = \{ \gamma_{\pm i,s}^r : [1, \pm\hat{i}] * [e^{\pi i/(3m)}, (-1 - \hat{i} - \hat{j} + \hat{k})/2]^{3r+s} \}_{0 \leq r < m} \)
- \( S_{\pm j,s} = \{ \gamma_{\pm j,s}^r : [1, \pm\hat{j}] * [e^{\pi i/(3m)}, (-1 - \hat{i} - \hat{j} + \hat{k})/2]^{3r+s} \}_{0 \leq r < m} \)
- \( S_{\pm k,s} = \{ \gamma_{\pm k,s}^r : [1, \pm\hat{k}] * [e^{\pi i/(3m)}, (-1 - \hat{i} - \hat{j} + \hat{k})/2]^{3r+s} \}_{0 \leq r < m} \)

Any pair of sets with differing index \( s \) are clearly disjoint, so to see that these sets are all pairwise disjoint it is enough to observe that \( \gamma_{\pm1,1}^0 \neq \gamma_{\pm1,1}^0 \neq \gamma_{\pm1,1}^0 \neq \gamma_{\pm1,1}^0 \neq \gamma_{\pm1,1}^0 \).

Since \( |S_{\alpha,s}| = m \) for all \( \alpha \) and \( s \), and \( |\mathcal{J}_m^3| = 24m \), we see that the disjoint union of these sets constitutes all of \( \mathcal{J}_m^3 \). As \( r \) ranges from 0 to \( m - 1 \), the eigenvalues of the elements in each of these sets can be found to be as follows.

- Eigenvalues for \( S_{\pm1,s} \): \( \pm e^{\pi i/(3m)} \)
  \( \{1, 1\} \) \quad \( s = 0 \)
  \( \{e^{3\pi i/2}, e^{-3\pi i/2}\} \) \quad \( s = 1, 2 \)

- Eigenvalues for \( S_{\pm i,s} \): \( \pm e^{\pi i/(3m)} \)
  \( \{i, -i\} \) \quad \( s = 0 \)
  \( \{e^{\pi i/2}, e^{-\pi i/2}\} \) \quad \( s = 1 \)
  \( \{e^{\pi i}, e^{-\pi i}\} \) \quad \( s = 2 \)

- Eigenvalues for \( S_{\pm j,s} \): \( \pm e^{\pi i/(3m)} \)
  \( \{i, -i\} \) \quad \( s = 0 \)
  \( \{e^{\pi i/2}, e^{-\pi i/2}\} \) \quad \( s = 1 \)
  \( \{e^{\pi i}, e^{-\pi i}\} \) \quad \( s = 2 \)

- Eigenvalues for \( S_{\pm k,s} \): \( \pm e^{\pi i/(3m)} \)
  \( \{i, -i\} \) \quad \( s = 0 \)
  \( \{e^{\pi i/2}, e^{-\pi i/2}\} \) \quad \( s = 1 \)
  \( \{e^{\pi i}, e^{-\pi i}\} \) \quad \( s = 2 \)
• Eigenvalues for $S_{\pm k,s}$: $\pm e^{\frac{\pi i (3r+s)}{3m}}\begin{cases} \{i,-i\} & s = 0 \\ \{e^{\frac{2\pi i}{3}},e^{-\frac{2\pi i}{3}}\} & s = 1,2. \end{cases}$

Using (6.10), the characters are found to be as follows.

• $\chi_\rho(\gamma_{\pm 1,0}) = 2m - 1$
• $\chi_\rho(\gamma_{\pm 1,0}) = \chi_\rho(\gamma_{\pm j,0}) = \chi_\rho(\gamma_{\pm k,0}) = 1$
• $\chi_\rho(\gamma_{\pm 1,1}) = \chi_\rho(\gamma_{\pm 1,1}) = e^{2\pi i/3} \left[ \sin \left(2(m-1)\frac{2\pi}{3}\right) \cot \left(\frac{2\pi}{3}\right) + \cos \left(2(m-1)\frac{2\pi}{3}\right) \right]
= -e^{2\pi i/3}$ since $m \equiv 1 \mod 3$
• $\chi_\rho(\gamma_{\pm 1,1}) = \chi_\rho(\gamma_{\pm j,1}) = \chi_\rho(\gamma_{\pm k,1}) = \chi_\rho(\gamma_{\pm j,1})$
= $e^{4\pi i/3} \left[ \sin \left(2(m-1)\frac{2\pi}{3}\right) \cot \left(\frac{2\pi}{3}\right) + \cos \left(2(m-1)\frac{2\pi}{3}\right) \right]
= -e^{4\pi i/3}$ since $m \equiv 1 \mod 3$.

Finally, we find that

\[
\dim(H_{3m}^{sfk}) = \frac{1}{12m} \left[ 2m \cdot (2m-1) + 6m - 8m \left( e^{2\pi i/3} + e^{4\pi i/3} \right) \right]
= \frac{m}{3} + 1 = 2b_{3m} - 2.
\]

6.1. Proof of Theorem 1.5. We next prove Theorem 1.5. From Proposition 6.4, for any finite non-cyclic subgroup $\Gamma \subset U(2)$ containing no complex reflections, we see that $(\mathcal{O}(-2m),g_{LB})$ always has $H_{3m}^{sfk}$ of positive dimension. As in [Hon13, Hon14], using equivariant deformation theory of the twistor space of LeBrun metrics, the LeBrun metrics consequently have small deformations of complex structure admitting scalar-flat Kähler ALE metrics, which are invariant under the group $\Gamma'$. Choose any such deformed metric, and call it $\tilde{g}$. The quotient of $(\mathcal{O}(-2m),\tilde{g})$ by $\Gamma'$ will then be a complex ALE orbifold with the same types of singular points as before. Consequently, the entire construction in the proof of Theorem 1.3 can be carried out, using one of these deformations as the base metric, to construct scalar-flat Kähler ALE metrics with respect to a small deformation of complex structure (it is important to note that the proof of Theorem 5.1 did not assume the existence of holomorphic coordinates at infinity on $X_2$). Note, the resulting space is diffeomorphic to the minimal resolution, but is no longer a minimal resolution, since the central rational curve is no longer holomorphic.

We next consider the estimates on the dimension of the moduli space of scalar-flat Kähler metrics. Given a minimal resolution $\tilde{X}$ of $\mathbb{C}^2/\Gamma$ with exceptional divisor
\( E = \cup_{i=1}^k E_i \), with rational curves \( E_i \), we have the exact sequence

\[
0 \rightarrow \text{Der}_E(\tilde{X}) \rightarrow \Theta_{\tilde{X}} \rightarrow \bigoplus_{i=1}^k O_{E_i}(E_i) \rightarrow 0,
\]

where \( \text{Der}_E(\tilde{X}) \) is sheaf of vector fields on \( \tilde{X} \) dual to \( \Omega^1_{\tilde{X}}(\log(E)) \), see [Kaw78].

We next examine the following portion of the associated long exact sequence in cohomology

\[
H^1(\tilde{X}, \text{Der}_E(\tilde{X})) \rightarrow H^1(\tilde{X}, \Theta_{\tilde{X}}) \rightarrow \bigoplus_{i=1}^k H^1(O_{E_i}(E_i)) \rightarrow H^2(\text{Der}_E(\tilde{X})).
\]

By Serre duality,

\[
\dim(H^1(O_{E_i}(E_i))) = \dim(H^0(\mathbb{P}^1, K \otimes N^*_E)) = \dim(H^0(O(e_i - 2))) = e_i - 1,
\]

where \( E_i \cdot E_i = -e_i \). Since \( \dim_C(\tilde{X}) = 2 \), by Siu’s vanishing theorem [Sin69],

\[
H^2(\text{Der}_E(\tilde{X})) = 0.
\]

Finally, using results from [Bri68, Lau73, Wah75] (see also [BK87]), there are no nontrivial deformations of \( \tilde{X} \) preserving the entire tree of rational curves, so we have

\[
H^1(\tilde{X}, \text{Der}_E(\tilde{X})) = 0.
\]

Combining the above, this implies that

\[
\dim(H^1(\tilde{X}, \Theta_{\tilde{X}})) = \sum_{i=1}^k (e_i - 1).
\]

In the non-cyclic case, there is a \( \mathbb{C}^* \)-action on the quotient \( \mathbb{C}^2/\Gamma \) which lifts to the resolution. This induced action on \( H^1(O_{E_i}(E_i)) \) is easily seen to generically have orbits of complex dimension 1, so the moduli space of complex structures is of complex dimension no more than

\[
\dim(H^1(\tilde{X}, \Theta_{\tilde{X}})) - 1 = -1 + \sum_{i=1}^k (e_i - 1).
\]

Each of the corresponding deformations of complex structure has a Kähler cone of real dimension \( k \). If one of these Kähler classes admits a scalar-flat ALE metric near the metric constructed in Theorem 1.3 then it is locally unique; this follows from invertibility of the linearized operator shown in the proof of Theorem 1.3. We also use the following: we know there is at least one scalar-flat Kähler ALE metric with respect to the deformed complex structure, so invertibility of the linearized operator as shown in the proof of Theorem 1.3 implies that there is an open set in the Kähler cone of Kähler classes admitting scalar-flat Kähler metrics. This is completely analogous to the situation considered in [LS94a]; the details are similar and are omitted.
The above discussion implies that the space of scalar-flat Kähler metrics (up to scaling) has real dimension at most

\[
d_{\Gamma, \text{max}} = k - 3 + 2 \sum_{i=1}^{k} (e_i - 1).
\]

From Theorem 1.5, similar arguments imply that

\[
d_{\Gamma, \text{min}} \geq \dim(H_{sfk}^r) + k - 3
\]

where \( \dim(H_{sfk}^r) \) is given in Proposition 6.1. Notice that there is again a subtraction of 2 because there is a non-trivial \( \mathbb{C}^* \)-action on \( H^1(O_{E_i}(E_i)) \) arising from \( \mathbb{C}^* \)-action of fiber multiplication on \( O(-2m) \). It is easy to see that these metrics correspond to distinct metrics in the moduli space of scalar-flat Kähler ALE metrics, according to Definition 1.4, and the proof is complete.

### 7. Extremal Kähler metrics on rational surfaces

We next discuss complex analytic compactifications of the Breiskorn minimal resolutions. For this, we require the following.

**Definition 7.1.** For relatively prime integers \( 1 \leq r \leq q \leq p \), the **weighted projective space** \( \mathbb{P}^2_{r,q,p} \) is \( S^5/S^1 \), where \( S^1 \) acts by

\[
(z_0, z_1, z_2) \mapsto (e^{i \theta} z_0, e^{iq \theta} z_1, e^{ip \theta} z_2),
\]

for \( 0 \leq \theta < 2\pi \).

The weighted projective space is a complex orbifold surface, with orbifold point set a subset of \([1,0,0] \cup [0,1,0] \cup [0,0,1]\).

**7.1. Conformal compactifications.** Given an ALE space \((X, g)\), choose a conformal factor \( u : X \to \mathbb{R}_+ \) such that \( u = O(\rho^{-2}) \) as \( \rho \to \infty \). The space \((X, u^2g)\) then compactifies to a \( C^{1,\alpha} \)-orbifold. This is known as a conformal compactification. In the anti-self-dual case, there moreover exists a \( C^\infty \)-orbifold conformal compactification with positive Yamabe invariant, which we denote by \((\hat{X}, \hat{g})\), see [TV05, CLW08]. (Note that this compactification is quite different than the complex analytic compactifications considered below.)

**Remark 7.2.** If \((X, g)\) is an oriented ALE space with group at infinity \( \Gamma \subset \text{SO}(4) \), then the conformal compactification \((\hat{X}, \hat{g})\) has the same group action at the orbifold point, provided we reverse the orientation.

Bryant proved that every weighted projective space admits a Bochner-Kähler metrics [Bry01]. Subsequently, David-Gauduchon gave a direct construction of these metrics and used an argument of Apostolov to show that each such metric is in fact the unique Bochner-Kähler metric on a given weighted projective space [DG06], hence we refer to these as the **canonical Bochner-Kähler metrics** and denote them by \((\mathbb{P}^2_{r,q,p}, g_{BK})\). Since in real dimension four the Bochner-tensor is the same as the anti-self-dual part of the Weyl tensor, these metrics are self-dual Kähler.
Remark 7.3. Of primary importance to us is the fact that the conformal compactification \((\mathcal{O}(-n), \bar{g}_{LB})\) is conformal to \((\mathbb{CP}^2_{(1,1,n)}, g_{BK})\) \([Joy91]\) (also see \([DL14]\) for an explicit conformal equivalence). We emphasize that while both of these metrics are Kähler, they are Kähler with respect to reverse-oriented complex structures.

7.2. Proof of Theorem 1.7. We consider the weighted projective space \(\mathbb{CP}^2_{(1,1,2m)}\), and have the following analogue of Theorem 4.1. The group \(U(2)\) acts as holomorphic automorphisms. For each non-cyclic finite subgroup \(\Gamma \subset U(2)\) containing no complex reflections, the quotient \((\mathbb{CP}^2_{(1,1,2m)}, g_{BK})/\Gamma'\) has four singularities: one at the point \([0,0,1]\) with orbifold group \(\Gamma\), and three on the 2-sphere \(\Sigma = [z_0, z_1, 0]\) (which has self-intersection number \(+b_{\Gamma}\) with respect to the complex orientation on the weighted projective space) with cyclic orbifold groups orientation reversed conjugate to those given in Theorem 4.1 for the corresponding \(\Gamma\), see Remark 7.3. Consequently, these orbifold singularities are of type \(L(\beta_i - \alpha_i, \beta_i)\) for \(i = 1, 2, 3\). It is not difficult to see that \(\{\mathbb{CP}^2_{(1,1,2m)} \setminus \Sigma\}/\Gamma'\) is biholomorphic to \(\mathbb{C}^2/\Gamma\). Let \(b_j^i\) be the Hirzebruch-Jung string associated to \(L(\beta_i - \alpha_i, \beta_i)\) for \(i = 1, 2, 3\), and \(j = 1 \ldots \ell_i\). Then the compactification is found by adding the tree of rational curves at infinity in Figure 7.1.

![Figure 7.1. The compactification divisor.](image)

Note that by \([LM08, Lemma 4.1]\), a scalar-flat Kähler ALE metric always has a complex analytic compactification which is a rational surface. In particular it is diffeomorphic to either \(S^2 \times S^2\) or \(\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}\) for some \(t \geq 0\), and this finishes the proof.

7.3. Proof of Theorem 1.8. We begin by noting that the complex analytic compactification of \(\mathcal{O}(-n)\) is a Hirzebruch surface \(\mathbb{F}_n\), obtained by adding the \([+n]\) curve at infinity. With a minor modification of the above process, we start the process with one of Calabi’s extremal Kähler metrics \(g_C\) on the Hirzebruch surface \(\mathbb{F}_{2m}\) \([Cal82]\).
This has an isometric $U(2)$ action. Arguing as in Theorem 4.1, it is not hard to see that the quotient by the subgroup $\Gamma$ produces 6 singular points. There are 3 on the $[-b]$ curve of type $L(\alpha_i, \beta_i)$ and 3 on the $[+b]$ curve of type $L(\beta_i - \alpha_i, \beta_i)$, for $i = 1, 2, 3$. Note that there is a moduli of Calabi’s metrics, and the area of the $[+b]$ and $[-b]$ curves can be chosen independently to be any two positive real numbers.

The existence result then follows from a slight extension of a recent gluing theorem of Arezzo-Lena-Mazzieri [ALM14], which we only briefly explain here. This theorem is stated for gluing constant scalar curvature metrics, and has a generalization to the extremal Kähler case as in Arezzo-Pacard-Singer. However, we do not need the full generalization for this case (the full generalization will appear in [Len14]). In our case, we simply note that, using the results from [McC02], the quotient of $(\mathbb{F}_{2m}, g_C)$ by $\Gamma$ has identity component of the isometry group exactly $S^1$, so the Lie algebra of Hamiltonian biholomorphisms is 1-dimensional. In this case, it is therefore trivial to overcome the obstructions, and it follows as in Arrezo-Pacard-Singer that we may obtain extremal Kähler metrics on the resolution by attaching Calderbank-Singer metrics at each singular point. Furthermore, the metrics obtained will admit a holomorphic isometric $S^1$-action. We note that the Calderbank-Singer metrics come in moduli, and the area of each rational curve can be chosen arbitrarily. If any of these spaces happens to be Ricci-flat, then the only change is that the scaling of this space must be chosen differently, which accounts for the factor of $\epsilon^4$ in the definition of $f$ and $f'$ above.

Finally, since any Calabi metric $g_C$ on $\mathbb{F}_{2m}$ for $m \geq 1$ does not have constant scalar curvature, and the extremal metrics we obtain are small perturbations of the Calabi metric away from the singularities, these extremal metrics do not have constant scalar curvature either (for sufficiently small gluing parameter $\epsilon$).

Remark 7.4. In the above proof, instead of starting with the Hirzebruch surface, we could have started with a Bochner-Kähler metric on $\mathbb{C}P^2_{(1,1,2m)}$ (see below), taken the quotient by $\Gamma'$, and then used one of the scalar-flat Kähler metrics from Theorem 1.3 to resolve one of the resulting singularities. However, doing this would make the $[-b]$ curve have small area. So starting with the Calabi metric allows us to obtain existence for a larger collection of Kähler classes. This is of course related to the fact that the Calabi metrics on $\mathbb{F}_n$ limit to the Bochner-Kähler metric on $\mathbb{C}P^2_{(1,1,n)}$ as the area of the $[-n]$ curve shrinks to zero [Gau09].

8. Hyperkähler metrics

We will now show that the spaces obtained by our construction for $\Gamma \subset SU(2)$ are in fact hyperkähler (and are therefore ALE of order 4, see [BKN89, CT94]). Note that it follows from [Str10, AV12b] that all of the spaces obtained in the proof of Theorem 1.3 are necessarily ALE of order at least 2. Since these spaces are scalar-flat Kähler, hence anti-self-dual, we begin with the following Lemma.
Lemma 8.1. If \((X, g)\) is a scalar-flat anti-self-dual ALE space with \(H^1(X; \mathbb{R}) = 0\), then
\[
b^-_2(X) \geq 2 - \frac{2}{|\Gamma|} - 3\eta(S^3/\Gamma)
\]
with equality if and only if \(g\) is Ricci-flat and \(b^+_2(X) = 0\).

Proof. Since \((X, g)\) is an ALE space with \(H^1(X; \mathbb{R}) = 0\), it is clear that
\[
\tau_{\text{top}}(X) = b^+_2 - b^-_2 \quad \text{and} \quad \chi_{\text{top}}(X) = 1 + b^+_2 + b^-_2.
\]
Now, consider the ALE versions of the signature and Chern-Gauss-Bonnet theorems, which for an ALE space \((M, g)\) with group at infinity \(\Gamma \subset \text{SO}(4)\), are given as follows.
\[
\tau_{\text{top}}(M) = \tau_{\text{orb}}(M) + \eta(S^3/\Gamma) \quad \text{and} \quad \chi_{\text{top}}(M) = \chi_{\text{orb}}(M) + \frac{1}{|\Gamma|},
\]
where \(\tau_{\text{orb}}(M)\) is the orbifold signature defined by
\[
\tau_{\text{orb}}(M) = \frac{1}{12\pi^2} \int_M \left( |W^+_g|^2 - |W^-_g|^2 \right) dV_g.
\]
\(\eta(S^3/\Gamma)\) is the eta-invariant and \(\chi_{\text{orb}}(M)\) is the orbifold Euler characteristic defined by
\[
\chi_{\text{orb}}(M) = \frac{1}{8\pi^2} \int_M \left( |W^+_g|^2 - \frac{|E_g|^2}{2} + \frac{R^2_g}{24} \right) dV_g.
\]
Then, since \((X, g)\) is anti-self-dual, observe that
\[
3\tau_{\text{top}}(X) + 2\chi_{\text{top}}(X) = 5b^+_2 - b^-_2 + 2 = -\frac{1}{8\pi^2} \int_X |E_g|^2 dV_g + \frac{2}{|\Gamma|} + 3\eta(S^3/\Gamma).
\]
Therefore
\[
b^-_2 = \frac{1}{8\pi^2} \int_X |E_g|^2 dV_g - \frac{2}{|\Gamma|} - 3\eta(S^3/\Gamma) + 2 + 5b^+_2,
\]
so we find that
\[
b^-_2(X) \geq 2 - \frac{2}{|\Gamma|} - 3\eta(S^3/\Gamma),
\]
with equality if and only if \(E_g \equiv 0\), hence \(\text{Ric}_g \equiv 0\) since \(R_g \equiv 0\), and \(b^+_2(X) = 0\). \(\square\)

Proposition 8.2. For \(\Gamma\) a binary polyhedral group, the construction of Theorem 1.3 yields a hyperkähler ALE space which admits a holomorphic isometric circle action.

Proof. The proof of Proposition 8.2 follows easily from Lemma 8.1 below as follows. The spaces obtained in Section 5.2 are scalar-flat Kähler, hence anti-self-dual, and simply connected. From inserting the respective \(\eta(S^3/\Gamma)\), given in [Nak90], and \(b^-_2\) into the inequality in Lemma 8.1 we find that these spaces are Ricci-flat, and therefore hyperkähler, if and only if \(\Gamma \subset \text{SU}(2)\). \(\square\)
Remark 8.3. It is interesting to note that the only ALE spaces needed for in this construction of hyperkähler metrics are the Gibbons-Hawking multi-Eguchi-Hanson metrics.

Remark 8.4. For the weighted projective space \( \mathbb{CP}^2_{(r,q,q+r)} \), by a result of Derdzinski \cite{Der83}, the conformal metric \( R^{-2}g_{BK} \) is Einstein. A simple calculation shows that it is Ricci-flat. This turns out to be the conformal compactification of a multi-Eguchi-Hanson metric with 2 points, one of multiplicity \( q \) and the other of multiplicity \( r \). Again, these are Kähler with respect to reverse-oriented complex structures.

8.1. Anti-self-dual gluing. We first discuss an “inverse” to the process of conformal compactification described above. Given a compact Riemannian orbifold \((\hat{X}, \hat{g})\) with nonnegative Yamabe invariant, and letting \( G_p \) denote the Green’s function of the conformal Laplacian associated to a point \( p \), the space \((\hat{X} \setminus \{p\}, G_p^2 \hat{g})\) is a complete noncompact scalar-flat anti-self-dual ALE orbifold. A coordinate system at infinity arises from using inverted normal coordinates in the metric \( \hat{g} \) around \( p \). We refer to this as a conformal blow-up.

In Proposition \ref{prop:scalar-flat-gluing}, we obtained hyperkähler ALE metrics with groups at infinity the binary polyhedral groups (the “DE” in “ADE”). For these cases, the singularities on the \( \mathbb{CP}^4 \) at the origin of Theorem \ref{thm:ADE-scalar-flat} are all of type \( L(-1, \beta_i) \), and we used the scalar-flat Kähler gluing theorem of Section \ref{sec:scalar-flat-gluing} to glue on the appropriate multi-Eguchi-Hanson metrics (the “A” in “ADE”) to the quotients of \( (\mathcal{O}(-2), g_{LB}) \). However, this construction only produces some hyperkähler metrics. We will next show that we can obtain examples for all small deformations of complex structure, and therefore an open set in the moduli space, by using the following gluing result for (anti-)self-dual orbifolds.

**Theorem 8.5** (\cite{DF89, Flo91, LS94b, KS01, AV12a}). Let \( (X_1, [g_1]) \) and \( (X_2, [g_2]) \) be self-dual conformal structures on compact 4-dimensional orbifolds with \( H_{SD}^2 \) of the respective self-dual deformation complexes vanishing. Let \( p_1 \in X_1 \) and \( p_2 \in X_2 \) be orbifold points with the respective orbifold groups \( \Gamma_1 \) and \( \Gamma_2 \subset SO(4) \) orientation reversed conjugate in the sense that there is an orientation-reversing intertwining map between these groups. Then, the orbifold connect sum \( X_1 \# X_2 \), taken at these points, admits self-dual conformal structures.

We will replace the scalar-flat Kähler gluing result in Theorem \ref{thm:scalar-flat-gluing} with Theorem \ref{thm:ADE-scalar-flat-gluing}. For any binary polyhedral group \( \Gamma' \subset SU(2) \), consider the compact self-dual Kähler orbifold \( (\mathbb{CP}^2_{(1,1,2)}, g_{BK})/\Gamma' \), as discussed above, which has four orbifold points: one with orbifold group \( \Gamma \) at the point of compactification, and three on the 2-sphere at infinity with cyclic orbifold groups \( L(1, \beta_i) \) for \( i = 1, 2, 3 \). Recall that these are just the Kähler conformal compactifications of the corresponding quotients of the Eguchi-Hanson metric. LeBrun-Maskit proved that \( H_{SD}^2 \) of the compactification of any scalar-flat Kähler ALE metric vanishes \cite[Theorem 4.2]{LM08}. Therefore, to each orbifold point with group \( L(1, \beta_i) \) on the 2-sphere at infinity of \( (\mathbb{CP}^2_{(1,1,2)}, g_{BK})/\Gamma' \),
Theorem 8.5 can be used to attach the conformal compactification of the multi-Eguchi-Hanson metric with group $L(-1, \beta_i)$ at infinity, and obtain a self-dual orbifold with a single orbifold point of type $\Gamma$, and positive Yamabe invariant. Since the Yamabe invariant is positive, we can take the conformal blow-up at the orbifold point to obtain a scalar-flat anti-self-dual ALE metric on the minimal resolution of $\mathbb{C}^2 / \Gamma$. Once again, notice that the only building blocks here were the multi-Eguchi-Hanson metrics.

Finally, for each binary polyhedral group $\Gamma$, we examine these scalar-flat anti-self-dual ALE metrics using Lemma 8.1. As in the proof of Proposition 8.2, from inserting $\eta(S^3 / \Gamma)$ given in Nak90 and $b_2$ into the inequality of Lemma 8.1 it is clear that these metrics are necessarily Ricci-flat, and are therefore hyperkähler ALE metrics.

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