On the General Error Event Probability Evaluation of Optical Intensity Modulation Schemes

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Abstract—Wireless intensity-modulated optical channels feature some very specific differences compared to wireless radio channels. These properties have a strong impact on the performance of digital modulation techniques. This paper discusses basics to consider when designing optical intensity modulation schemes. We introduce an error event probability expression for such modulation schemes that is generally applicable for all non-negative waveforms. We indicate the importance of amplitude fluctuations and the influence of pulse shaping on the power efficiency in the optical domain. In numerical examples, we apply the error event probability expression to modifications of orthogonal frequency-division multiplexing and to Hadamard coded modulation.

Index Terms—Error analysis, intensity modulation, optical wireless communication, visible light communication.

I. INTRODUCTION

Optical wireless intensity modulation with direct detection (IM/DD) is a promising technique especially for light emitting diode (LED) based communication. Traditionally, infrared (IR) communication has been studied [1], [2]. Recent interest is in visible light communication (VLC) and positioning (VLP) [3], [4], [5], [6]. Although concepts are mature, the potential of IM/DD with respect to the development of practical communication systems has not been fully exploited. The major benefits of optical IM/DD communication systems are (at least) twofold: Firstly, due to the limited coverage area and tremendous unregulated bandwidth, frequency planning is usually not necessary. Secondly, the optical components like LEDs at the transmitter and PIN photodiodes at the receiver are very cheap and simple to handle. Depending on the application, additional advantages apply – including electromagnetic compatibility and interference immunity, data security on the physical layer, license-free operation, area spectral efficiency, and energy efficiency in joint illumination and communication scenarios [6].

It is well known that IM/DD requires waveforms that are real-valued and non-negative. Still, incoherent optical transmission has some very interesting properties that are unfamiliar to an engineer who is experienced in designing electrical radio communication systems. For instance, it is beneficial to use signals with high amplitude fluctuations because, for a given optical power in the air, they provide a higher electrical power at the receiver. The relation between the optical and the electrical power has to be taken into account in the performance analysis. Some years ago, modified orthogonal frequency-division multiplexing (OFDM) techniques have been developed [7], [8], [9] that turn out to be very suitable for these channels, since OFDM signals are characterized by high amplitude fluctuations.

The intention of this paper is to provide a systematic and unified treatment for the analysis of error probabilities for the intensity modulated optical channel. The error probability depends on the Euclidean distance inside the signal constellation. The Euclidean distance can be expressed by the electrical signal power, and the relation between electrical and optical signal power depends on the statistical properties of the signal. These relations are worked out in detail, and some examples are given to illustrate the analysis.

Novel contributions of this paper include
- a unified generic error event analysis for optical IM/DD transmission via an additive white Gaussian noise (AWGN) channel,
- the inclusion of the effect of pulse shaping,
- an application of the analysis to two OFDM variants as well as to Hadamard coded modulation (HCM), along with a correct error rate expression for HCM.

Due to the specific nature of incoherent wireless optical channels, the generic error rate expression presented next is different from generic error rate formulas that were developed for radio communications, like those in [10]. The generic error rate expression is also suitable for new, future IM/DD modulation schemes.

The remainder of this paper is organized as follows. In Section II, generic formulas are derived for the optical IM/DD channel to express the error event probability by the average optical power. The effects of pulse shaping on the performance is analyzed in Section III. A short review of two popular OFDM variants for IM/DD are given in Section IV, and exact expressions are applied to the error rate analysis of these schemes. In Section V, the analysis is applied to HCM. Finally, in Section VI, conclusions are drawn.

II. GENERIC ERROR RATE EXPRESSION FOR DATA TRANSMISSION VIA INCOHERENT WIRELESS OPTICAL CHANNELS

For optical transmission with intensity modulation and direct detection (IM/DD), the modulated signal is directly given by the
optical intensity which is a non-negative real-valued baseband signal. The received optical power \( \phi(t) \) with average denoted by \( \Phi = E\{\phi(t)\} \) at the photodiode with responsivity \( r \) generates a photocurrent \( x(t) = r\phi(t) \). In this paper, we assume an AWGN channel given by
\[
y(t) = x(t) + n(t),
\]
(1)

which means that \( x(t) \) is corrupted by a noise current \( n(t) \) with the autocorrelation function
\[
E\{n(t)n(t')\} = \frac{N_i}{2} \delta(t-t').
\]
(2)

Since all signals are currents, \( N_i \) is the spectral density of the noise current variance and has dimension \([N_i] = \Lambda^2/\text{Hz}\). The noise is typically modeled as a superposition of shot noise and one or more thermal noise components, see e.g. [2]. The electrical signal-to-noise ratio (SNR) is defined as
\[
\text{SNR} = \frac{E\{x^2(t)\}}{N_i B_n},
\]
(3)

where \( B_n \) is the noise bandwidth. The average optical power \( \Phi \) has to be related to the quantity \( E\{x^2(t)\} \) which is proportional to the average electrical power and determines the error event probability. For this reason, we introduce the shaping gain factor
\[
\kappa \triangleq \frac{E\{x^2(t)\}}{E\{x(t)\}^2}
\]
(4)

and discover that the electrical SNR can be written as
\[
\text{SNR} = \kappa \frac{(r\Phi)^2}{N_i B_n}.
\]
(5)

Since the error event probability is a function of the SNR and the SNR depends on \( \kappa \), the performance for the optical channel is influenced by the shape of the signal. This is in contrast to the electrical radio channel where – for the matched filter receiver and ideal synchronization – the performance does not depend on the pulse shape. The relationships discussed so far are summarized in Fig. 1.

Fig. 1. The optical and the electrical channel.

For the electrical real-valued baseband AWGN channel, the error event probability of any digital modulation scheme can be written as
\[
P_{\text{error}} = \frac{1}{2} \text{erfc} \left( \sqrt{\beta \text{SNR}} \right),
\]
(6)

where \( \beta \) is a constant that can be expressed by the ratio of the minimum squared Euclidean distance between adjacent symbols and the symbol energy of the constellation [11].

For optical IM/DD transmission schemes, we have to express the error rate by the optical Rx power \( \Phi \). We insert the SNR given by (5) into (6) and obtain
\[
P_{\text{error}} = \frac{1}{2} \text{erfc} \left( \sqrt{\frac{\gamma \kappa}{N_i B_n} \frac{r\Phi^2}{N_0 B_n}} \right),
\]
(7)

Whilst the SNR is a quantity that is directly available in measurements, for a fair power efficiency comparison of different schemes, it is often more convenient to write the same error event probability as
\[
P_{\text{error}} = \frac{1}{2} \text{erfc} \left( \sqrt{\frac{E_b}{N_0 B_n}} \right),
\]
(8)

where \( N_0 \) is the noise power spectral density and \( E_b \) is the useful energy per bit. Both quantities are of the same dimension \( W/\text{Hz} \), which is the same as Joule. The constant \( \gamma \) can be interpreted as the electrical power efficiency factor (or the gain) of the constellation because \( E_b = P_s/R_b \) is the electrical signal power \( P_s \) per bit rate \( R_b \) that is needed to achieve a certain error rate. We write
\[
\text{SNR} = \frac{E_b}{N_0 B_n},
\]
(9)

which is the same as the SNR in (3), but multiplied by a resistance in the numerator and the denominator. The above equation yields
\[
\text{SNR} = \frac{R_b E_b}{B_n N_0},
\]
(10)

as the relation between the measurable quantity SNR and the quantity \( E_b/N_0 \), which is needed for a fair evaluation of the power efficiency. Comparing this with (6) and (8), we get the proportionality
\[
\gamma = \beta \frac{R_b}{B_n}.
\]
(11)

To evaluate the optical power efficiency, we must relate \( \Phi \) to the bit rate. Combining (7) and (11) finally yields
\[
P_{\text{error}} = \frac{1}{2} \text{erfc} \left( \sqrt{\frac{\gamma \kappa}{N_i R_b} \frac{r\Phi^2}{N_0 R_b}} \right),
\]
(12)

This is the targeted exact general error rate expression for intensity modulation. It tells us how much squared optical power is needed per bit rate. The factor \( \gamma \kappa \), i.e. the product of the electrical power efficiency factor of the constellation and the optical shaping gain factor, can thus be interpreted as the optical power efficiency factor.

We recall that on-off-keying (OOK) with rectangular non-return-to-zero (NRZ) pulses has the error event probability
\[
P_{\text{error,OOK}} = \frac{1}{2} \text{erfc} \left( \sqrt{\frac{(r\Phi)^2}{N_i R_b}} \right),
\]
(13)
see [1, (5.15)], which means \( \gamma \kappa \)_{QOK} = 1. Being a binary modulation scheme, the bit error probability is the same. (Note the different definition of \( N_i \) by a factor of two in [1].) Hence, \( \gamma \kappa \) is the gain compared to this reference scheme. As will be shown in Section III, expression (13) results as a special case of a more general formula derived there.

### III. Nyquist Pulse Trains and the Effect of Pulse Shaping

Recall that the shaping gain factor defined in (4) is a function of the analogue transmit signal, rather than just of discrete signal constellation points. Consequently, this gain factor is affected by the pulse shaping, usually performed in the electrical domain. In this section, we have a closer look on the effect of pulse shaping on the error performance of IM/DD systems. In most publications, this issue is avoided by assuming rectangular pulse shaping.

Consider an IM/DD signal given by the pulse train

\[
x(t) = \sum_i x_i g(t - iT) \tag{14}
\]

with information carrying symbols \( x_i \) and normalized pulses \( g(t) \). We assume that the 1st Nyquist condition is satisfied and that a matched filter is used at the receiver. Then, the following condition must hold:

\[
\int_{-\infty}^{\infty} g(t - iT)g(t - kT)dt = \delta_{ik} \cdot \tag{15}
\]

We note that if the pulses include negative signal components, a suitable direct current (DC) offset has to be added. For the subsequent analysis, it is convenient to add this offset onto the symbols \( x_i \). Since the pulse train (14) is a cyclostationary rather than a stationary signal, an additional time averaging must be considered in the calculation of expectation values [11]. Thus, in the following treatment, we extend the statistical averages \( E_\{\cdot\} \) by time averages and denote this by \( \langle \cdot \rangle \). It can be shown that

\[
\langle t \rangle = \frac{1}{T} E\{x_i\} \int_{-\infty}^{\infty} g(t)dt \tag{16}
\]

and, by using the Nyquist condition (15),

\[
\langle x^2 \rangle = \frac{1}{T} E\{x_i^2\} \tag{17}
\]

holds. This leads to

\[
\kappa = \frac{1}{\omega} \frac{E\{x_i^2\}}{E\{x_i\}^2} \tag{18}
\]

with the pulse-dependent factor

\[
\omega = \frac{1}{T} \left[ \int_{-\infty}^{\infty} g(t)dt \right]^2 \cdot \tag{19}
\]

We may therefore identify two contributions to the shaping gain factor. One depends on the statistics of the symbols \( x_i \), and the other depends on the pulse shape \( g(t) \). Let us consider three examples.

(i) For the normalized rectangular pulse defined over the interval \([0, T]\) and zero outside, expressed by

\[
g(t) = \frac{1}{\sqrt{T}} \text{rect} \left( \frac{t}{T} \right) \cdot \tag{20}
\]

we obtain \( \omega = 1 \). Therefore, for rectangular pulse shaping, \( \kappa \) is a function of the signal constellation points \( x_i \) only.

(ii) For the sine half-wave

\[
g(t) = \frac{1}{\sqrt{T}} \text{rect} \left( \frac{t}{T} \right) \sin \left( \pi t/T \right) \cdot \tag{21}
\]

we find \( \omega = 8/\pi^2 \), which provides an additional gain of approximately 1 dB.

(iii) Regarding root raised cosine (RRC) pulses, using the general property

\[
\int_{-\infty}^{\infty} g(t)dt = G(0) \tag{22}
\]

for the Fourier transform \( G(f) \) of \( g(t) \) and noting that \( |G(0)|^2 = T \) holds for RRC pulses, we get \( \omega = 1 \). However, because such pulses have negative signal contributions (in the electrical domain), a DC offset must be included in the symbols \( x_i \), which leads to a power efficiency loss.

For the pulse train given by (14), \( E_s = E\{x_i^2\} \) can be interpreted as the symbol energy. Because \( B_s T = 1/2 \) for real-valued baseband transmission (i.e., digital pulse amplitude modulation (PAM) waveforms) with matched filter detection, it follows that [11], [12]

\[
\text{SNR} = \frac{2E_s}{N_0}. \tag{23}
\]

Note that for two-dimensional quadrature amplitude modulation (QAM) schemes, \( \text{SNR} = E_s/N_0 \) holds, which is more popular in the wireless radio community.

For a signal constellation with the minimum Euclidean distance \( \delta \) between the symbol points \( x_i \), the SNR factor in (6) is given by

\[
\beta = \frac{\delta^2}{2E_s}. \tag{24}
\]

Using (18), we find

\[
\beta \cdot \kappa = \frac{1}{\omega} \frac{\delta^2}{2E\{x_i\}^2}. \tag{25}
\]

It is interesting to note that only \( E\{x_i^2\} \) and not \( E\{x_i^2\} \) has to be evaluated for the error rate in the optical channel. For an \( M \)-ary PAM constellation with \( \log_2 M \) bits per symbols and \( R_b/B_n = 2 \log_2 M \), the optical power efficiency factor is given by

\[
\gamma \cdot \kappa = \frac{1}{\omega} \frac{\delta^2 \log_2 M}{E\{x_i\}^2}, \tag{26}
\]

see (11). In case that a constant DC offset has been added onto the symbols \( x_i \) in order to keep the signal \( x(t) \) positive, the above expression must be modified by including this offset in \( E\{x_i\} \).

As an example, for \( M \)-ary amplitude shift keying (ASK) with equiprobable symbols defined over the alphabet \( x_i \in \ldots \).
\(\{0, 2\delta, \ldots, 2(M - 1)\delta\}\), we find \(E\{x_i\} = (M - 1)\delta\) and
\[
\gamma \cdot \kappa = \frac{1}{\varpi} \log_2 M \quad (M - 1)^2.
\]
In this paper, ASK is defined to be unipolar, while PAM may be bipolar. Setting \(M = 2\) and \(\varpi = 1\) (for rectangular pulses) in (27), the OOK-NRZ expression (13) follows from (12) as a special case.

In the next two sections, we discuss some important and illuminating examples. Both examples are based on transforms (Fourier transform and Hadamard transform, respectively), which implies that the pulse shaping gain factor has to be calculated for the transformed signal rather than for the original constellation. This requires the systematic usage of the aforementioned formalism.

IV. ACO-OFDM and PAM-DMT

Because optical IM/DD requires a real-valued non-negative signal, OFDM techniques can thus not directly be applied. In addition to simply adding a DC offset, techniques have been developed to put a constraint on the OFDM signal that makes its negative part redundant and hence allows to clip its negative parts without information loss by the clipping. Asymmetrically clipped optical OFDM (ACO-OFDM) [7], [8] and pulse amplitude modulation with digital multitone transmission (PAM-DMT) [9] are two quite similar variants of OFDM.

In both variants, a real-valued OFDM signal
\[
u(t) = \sqrt{2} \text{Re} \left\{ \sum_{k=1}^{N} u_k e^{j2\pi kt/T} \right\}
\]
(28)
is generated. As explained soon, there are special constraints on the information-carrying Fourier coefficients \(u_k\) to allow the signal to be clipped according to
\[
x(t) = \sqrt{2} u(t) \Theta(u(t))\quad (29),
\]
where \(\Theta(\cdot)\) is the Heaviside step function, and the factor \(\sqrt{2}\) has been introduced to renormalize the power after clipping. The signal \(x(t)\) has a Fourier expansion given by
\[
x(t) = \sqrt{2} \text{Re} \left\{ \sum_{k=0}^{\infty} x_k e^{j2\pi kt/T} \right\}.
\]
(30)

Self-interference of the signal that is introduced by the clipping can be avoided by special constraints on the \(u_k\).

In the first variant [7], [8], only those \(u_k\) with odd indices \(k\) are modulated and the others are set to zero. The corresponding signal \(u(t)\) looks like the signal depicted in Fig. 2. As indicated by the arrow, it has the symmetry property \(u(t + T/2) = -u(t)\). This makes the negative part of the signal redundant, and clipping does not cause any loss of information. One can show that \(x_k = u_k/\sqrt{2}\) holds for odd \(k\). Thus, due to the clipping, half of the energy moves to the \(x_k\) with even \(k\), which do not carry useful information.

In the second variant [9], the constraint is \(\text{Re}\{u_k\} = 0\), i.e., only the imaginary parts of the symbols \(u_k\) are modulated so that the Fourier series (28) consists only of sine waves, and \(u(t)\) becomes an odd function. This again makes the negative part of the signal redundant. Notice that \(\text{Im}\{x_k\} = u_k/\sqrt{2}\). Thus, due to the clipping, half of the energy moves to the real parts of \(x_k\), which do not carry useful information.

Hence, both variants follow the same line: An OFDM signal is generated utilizing only half of the real dimensions. Each dimension can be modulated by a one-dimensional bipolar \(M\)-PAM scheme. Note that for ACO-OFDM, two real \(M\)-PAM symbols form one complex \(M^2\)-QAM symbol. As discussed in [11], the power efficiency factor for \(M\)-PAM and for \(M^2\)-QAM given by
\[
\gamma_{M \cdot \text{PAM}} = \frac{3}{M^2 - 1} \log_2 M
\]
is the same for OFDM and for single carrier modulation. According to the discussion above, for ACO-OFDM, this factor must be replaced by \(\gamma \rightarrow \gamma/2\).

In order to evaluate the performance of ACO-OFDM and PAM-DMT in the optical channel, the shaping gain factor \(\kappa\) has to be calculated. According to the central limit theorem, for a sufficiently high number of subcarriers, the statistics of the signal \(u(t)\) follows a zero-mean Gaussian probability density function (PDF) with variance \(\sigma^2\). Clipping the negative values leads to the following PDF for the signal amplitude:
\[
p(x) = \frac{1}{\sigma} \delta(x) + \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2} x^2\right) \Theta(x).
\]
(32)

From this PDF one calculates \(E\{x\} = \sigma/\sqrt{2\pi}\) and \(E\{x^2\} = \sigma^2/2\) which leads to \(\kappa = \pi\). The optical power efficiency factor is then given by
\[
\gamma \cdot \kappa = \frac{\pi}{2} \gamma_{M \cdot \text{PAM}}.
\]
(33)

We see that the shaping gain factor \(\kappa = \pi\) overcompensates the power loss of \(1/2\) due to the clipping by the factor \(\pi/2 \approx 1.57 \approx 2\) dB. Hence, for \(2\)-PAM modulation in each real dimension with \(\gamma_{2 \cdot \text{PAM}} = 1\), the OFDM variants described above offer a \(2\) dB gain compared to the OOK reference scheme in (13). For \(4\)-PAM modulation in each real dimension with \(\gamma_{4 \cdot \text{PAM}} = 2/5 \approx -4\) dB, we obtain \(\gamma \kappa \approx -2\) dB. From the exact error event probability

![Fig. 2. OFDM signal \(u(t)\) with modulation only in the odd-indexed subcarriers. Such a signal has the symmetry property \(u(t + T/2) = -u(t)\).](image-url)
given by (12), the bit error probability for Gray mapping can be approximated as [11]

\[ P_b \approx \frac{1}{\log_2 M} \frac{M - 1}{M} \text{erfc} \left( \sqrt{\frac{\gamma \kappa}{N_0 R_b}} \right) \quad (34) \]

In the electrical domain, the bit error probability as a function of \( E_b/N_0 \) is approximately given by

\[ P_b \approx \frac{1}{\log_2 M} \frac{M - 1}{M} \text{erfc} \left( \sqrt{\frac{E_b}{N_0}} \right) \quad (35) \]

with \( \gamma \) taken from (31). The approximations in (34) and (35) can be avoided by using a similar calculus as introduced in [10], but this is beyond the scope of this paper.

Bit error rate (BER) simulations have been performed for comparison with expressions (34) and (35). As an example, Fig. 3 shows ACO-OFDM simulations for 96 subcarriers. Each subcarrier is 16-QAM modulated, i.e., 4-PAM modulation is employed in each real dimension. The left-hand side refers to the optical power efficiency and the right-hand side to the electrical power efficiency. The numerically obtained value for \( \kappa \approx 3.139 \) is very close to the theoretical limit \( \kappa = \pi \). The theoretical curves drawn as dashed lines overlap with the solid lines of the simulation. 4-ary PAM-DMT achieves similar performance [9].

Fig. 3. BER performance simulations for the optical channel (left) and the electrical channel (right) in comparison with the theoretical bit error probabilities (dashed lines) for ACO-OFDM/16-QAM with 96 modulated subcarriers. The dashed lines are mostly hidden by the solid lines of the same color. Classical 4-ASK and OOK serve as benchmarks.

Let \( \mathbf{H} \) be the \( N \times N \) Hadamard matrix with entries \( h_{mn} \in \{ \pm 1 \} \). As an example, for \( N = 8 \), this matrix is given by

\[
\mathbf{H} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & -1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 & 1 & -1 & 1 & 1 \end{pmatrix} \quad (36)
\]

We define the orthogonal Hadamard matrix by \( \mathbf{G} = \mathbf{H}/\sqrt{N} \) with the property

\[ \mathbf{G}^T \mathbf{G} = \mathbf{G} \mathbf{G}^T = \mathbf{I}_N \quad (37) \]

where \( \mathbf{I}_N \) is the \( N \times N \) identity matrix. We further define the Cartesian base vector

\[ \mathbf{e}_1 = (1, 0, \ldots, 0)^T \quad (38) \]

of the first dimension and

\[ \mathbf{i}_{DC} = (1, 1, \ldots, 1)^T \quad (39) \]

as the one-dimensional unit DC. Both vectors are of length \( N \). From the structure of the Hadamard matrix, it follows that these vectors are related by

\[ \sqrt{N} \mathbf{G} \mathbf{e}_1 = \mathbf{i}_{DC} \quad (40) \]

Now let \( \mathbf{u} \) be a unipolar \( M \)-ASK symbol vector of length \( N \) with elements

\[ u_i \in \{ 0, 2\delta, \ldots, 2(M - 1)\delta \}, \quad i = 1, \ldots, N \quad (41) \]

For reasons that will become obvious soon, the first symbol \( u_1 = 0 \) is not modulated and set to zero. The other ASK symbols are assumed to be independent and identically distributed. The HCM vector obtained from \( \mathbf{u} \) is given by

\[ \mathbf{x} = \mathbf{G} \mathbf{u} + \sqrt{N}(M - 1)\delta(\mathbf{i}_{DC} - \mathbf{e}_1) \quad (42) \]

V. HADAMARD CODED MODULATION

To apply the concept to a more recently developed digital optical modulation scheme, we consider HCM [13], [14], [15]. Compared to Fourier-transform-based OFDM, HCM is more robust against nonlinearities because it utilizes a discrete set of signal amplitudes. In [14], a transmitter structure is proposed in which light from binary-switched LEDs is superimposed.

The numerically obtained value for \( \mathbf{G}^T \mathbf{G} = \mathbf{G} \mathbf{G}^T = \mathbf{I}_N \). We further define the Cartesian base vector \( \mathbf{e}_1 = (1, 0, \ldots, 0)^T \). As an example, for \( N = 8 \), this matrix is given by

\[
\mathbf{H} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 & 1 & -1 & 1 & 1 \end{pmatrix} \quad (36)
\]

We define the orthogonal Hadamard matrix by \( \mathbf{G} = \mathbf{H}/\sqrt{N} \) with the property

\[ \mathbf{G}^T \mathbf{G} = \mathbf{G} \mathbf{G}^T = \mathbf{I}_N \quad (37) \]

where \( \mathbf{I}_N \) is the \( N \times N \) identity matrix. We further define the Cartesian base vector

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of the first dimension and

\[ \mathbf{i}_{DC} = (1, 1, \ldots, 1)^T \quad (39) \]

as the one-dimensional unit DC. Both vectors are of length \( N \). From the structure of the Hadamard matrix, it follows that these vectors are related by

\[ \sqrt{N} \mathbf{G} \mathbf{e}_1 = \mathbf{i}_{DC} \quad (40) \]

Now let \( \mathbf{u} \) be a unipolar \( M \)-ASK symbol vector of length \( N \) with elements

\[ u_i \in \{ 0, 2\delta, \ldots, 2(M - 1)\delta \}, \quad i = 1, \ldots, N \quad (41) \]

For reasons that will become obvious soon, the first symbol \( u_1 = 0 \) is not modulated and set to zero. The other ASK symbols are assumed to be independent and identically distributed. The HCM vector obtained from \( \mathbf{u} \) is given by

\[ \mathbf{x} = \mathbf{G} \mathbf{u} + \sqrt{N}(M - 1)\delta(\mathbf{i}_{DC} - \mathbf{e}_1) \quad (42) \]
This means that first the normalized Hadamard transform is applied to the vector $u$, and then the constant Tx offset vector $\sqrt{N(M-1)}\delta(i_{DC} - e_1) = \sqrt{N(M-1)}\delta(0,1,\ldots,1)^T$ is added to ensure that the resulting vector has no negative elements. Now, consider the discrete AWGN channel in vector notation,

$$y = x + n,$$  \hspace{1cm} (44)

where $y$ is the receive vector and $n$ is the discrete noise vector with variance $N_i$ in each dimension. Applying $G = G^T$ from the left yields

$$G^T y = G^T Gu + \sqrt{N(M-1)}\delta(G^T i_{DC} - G^T e_1) + G^T n.$$  \hspace{1cm} (45)

Using (37) and (40) and the fact that $G^T$ is an orthogonal transform, and thus does not change the statistical properties of the noise vector $n$, the transformed receive vector $v = G^T y$ can be written as

$$v = u + (M-1)\delta(Ne_1 - i_{DC}) + n.$$  \hspace{1cm} (46)

This is an AWGN channel for the ASK vector $u$ with a constant offset vector

$$(M-1)\delta(Ne_1 - i_{DC}) = (M-1)\delta(N-1,-1,\ldots,-1)^T$$  \hspace{1cm} (47)

that must be subtracted at the receiver. In order to evaluate the performance, the Euclidean half distance $\delta$ of the ASK constellation must be related to the optical power, which is proportional to $E\{x_i\}$. With $x_i = 0$ and the structure of the Hadamard matrix, it follows that

$$0 \leq x_i \leq 2(M-1)(N-1)\delta/\sqrt{N}$$  \hspace{1cm} (48)

and

$$E\{x_i\} = (N-1)(M-1)\delta/\sqrt{N}$$  \hspace{1cm} (49)

holds for the elements $x_i$ of vector $x$, which we may interpret as chips. A sequence of such vectors is then modulated to obtain a pulse train according to (14). If necessary due to the pulse shape, a constant DC offset has to be added to the symbols $x_i$ in order to keep the signal $x(t)$ positive, but we ignore this here in order not to overload the notation. Combining (49) and (25) yields

$$\beta \cdot \kappa = \frac{1}{\pi} \frac{N}{2(N-1)^2} \frac{1}{(M-1)^2}.$$  \hspace{1cm} (50)

Since the HCM scheme maps $(N-1)\log_2 M$ bits to $N$ chips,

$$\frac{R_b}{B_n} = 2\log_2(M) \frac{N-1}{N}$$  \hspace{1cm} (51)

holds, and the optical power efficiency factor is given by

$$\gamma \cdot \kappa = \frac{\log_2 M}{(N-1)(M-1)^2}.$$  \hspace{1cm} (52)

We compare this with the factor for ASK given by (27) and identify a loss in power efficiency by the factor $N-1$. This loss is due to the power that must be spent for the offset vector given by (43).

There is a simple trick to reduce this power loss. We note that the first column of the Hadamard matrix is just the unit DC vector $i_{DC}$. Thus, setting $x_1 = 0$ implies that no information is transmitted in this first dimension and adding any DC component to the transmitted signal does not change the useful information. Correspondingly, we may subtract

$$x_{\min} = \min_{k=1,\ldots,N}\{x_k\}$$  \hspace{1cm} (53)

from each element of the vector $x$ and replace

$$x_i \to x_i - x_{\min}.$$  \hspace{1cm} (54)

This method reduces the transmitted signal power, and thus improves the performance. Though an analytical expression is not known for this DC-reduced (DR-) HCM, the quantity $\gamma \kappa$ can easily calculated from (26) by using an numerical estimate of $E\{x_1\}$.

Unfortunately, the error analysis in [13], [14], [15] is incorrect because reference [10], a generic error rate solution valid for radio channels, was used for IM/DD modulation. This analysis neither considers the SNR loss due to the DC offset that is added to the PAM symbols, nor the offset vector that is added after the Hadamard transform. Our generic solution takes into account both offsets by relating the squared Euclidean distance to the electrical SNR and $E_b/N_0$ by means of the parameters $\beta$ and $\gamma$, respectively, and then by relating the electrical power to the optical power by means of the parameter $\kappa$. The resulting bit error rate (BER) performance is significantly different.

BER simulations have been performed for comparison with the theoretical expression (34) with $\gamma \kappa$ given by (52). As an example, Fig. 4 shows simulations for $N = 8$ and $M = 4$. We first concentrate on the power efficiency in the optical domain, as depicted in the plot on the left-hand side. Theoretical curves are drawn as dashed lines and simulated points are shown by markers. Small deviations can only be found in the low power regime where the approximation for $P_b$ with Gray coding becomes less accurate. We observe that the DC reduction significantly improves the performance of HCM and nearly approaches the curve of classical ASK. The theoretical curve for OOK given by (13) is drawn as a reference. All curves refer to $\varpi = 1$ (rectangular pulses).

Let us now consider the electrical domain. In order to find an expression for the bit error probability (35) as a function of the electrical $E_b/N_0$, $\gamma$ has to be calculated. We first have to evaluate the chip energy $E_c = E\{x_1^2\} = E\{||x||^2\}/N$ from (42). Note that $E_c$ is the energy of the chips, i.e. the symbols on the channel rather than the energy of the $M$-ASK symbols. This calculation results in

$$E\{x_1^2\} = \frac{N-1}{N} \left( (N-1)(M-1)^2 + \frac{1}{3}(M^2-1) \right) \delta^2$$  \hspace{1cm} (55)

and, with (24) but $E_b$ replaced by $E_c$, in

$$\beta = \frac{N}{2(N-1) \left( (N-1)(M-1)^2 + \frac{1}{3}(M^2-1) \right)}.$$  \hspace{1cm} (56)
Fig. 4. BER performance simulations for the optical channel (left) and the electrical channel (right) in comparison with the theoretical bit error probabilities (dashed lines) for HCM and DR-HCM with $N = 8$ and 4-ASK. The dashed lines are mostly hidden by the solid lines of the same color. Classical 4-ASK and OOK serve as benchmarks.

With (51), this yields
\[
\gamma = \log_2 M \frac{(N - 1)(M - 1)^2 + \frac{1}{3}(M^2 - 1)}{2 + \frac{1}{3}(M^2 - 1)}. \tag{57}
\]

We recall that for classical $M$-ASK, the power efficiency factor is given by [6]
\[
\gamma_{\text{ASK}} = \log_2 M \frac{(M - 1)^2 + \frac{1}{3}(M^2 - 1)}{2 + \frac{1}{3}(M^2 - 1)}. \tag{58}
\]

Comparing (57) and (58), we observe that these equations differ by the factor $N - 1$ in the first term of the denominator in (57). This factor can be attributed to the constant offset vector in (42) and leads to a significant performance loss for large values of $N$. HCM suffers from two biases: The first comes from the DC offset inherently included in the ASK symbols, the other from the constant offset vector added after the Hadamard transform. From an error-rate performance point of view, it is indeed better to replace HCM by $M - 1$ binary-switched light sources mimicking ASK.

BER simulation results for $N = 8$ and $M = 4$ are shown in Fig. 4 on the right-hand side. Comparing with the left-hand side, we see that for HCM without DC reduction, the differences between the curves for the optical and the electrical channel is very small. This is in contrast to the other curves, because $\kappa = 1 + \frac{M + 1}{3(N - 1)(M - 1)}$ is very close to one for this modulation scheme.

Compared to the performance results of the preceding section, we identify that HCM is not superior to ACO-OFDM when fixing the same number of two bits per real-valued (PAM or ASK) symbol in the examples under consideration. The theoretical analysis and further simulations not presented here show that this remains true for any other number of bits per symbol. Furthermore, in contrast to ACO-OFDM, the HCM performance decreases with increasing transform length $N$, see (52) and (57). However, HCM is reported to have practical advantages in nonlinear channels [14], [15].

VI. CONCLUSION

In this paper, an exact error event probability expression for optical intensity modulation schemes is introduced that is generally applicable for all real-valued non-negative waveforms. Towards this goal, a waveform-dependent parameter $\kappa$ is introduced to express the error event probability in the optical domain. The so-called shaping gain factor $\kappa$ is the ratio between the average signal power and the squared mean value of the signal. It depends on the discrete symbol constellation as well as on pulse shaping. This parameter is similar to but different from peak-to-average power ratio (PAPR) and crest factor. While in wireless radio communications a minimization of the PAPR typically is targeted, in optical intensity modulation schemes remarkably a maximization of $\kappa$ improves the power efficiency of the optical modulation scheme, given the same received power in the optical domain. Numerical examples of $\kappa$ are provided for commonly used pulse shapes. The $\beta \cdot \kappa$ and $\gamma \cdot \kappa$ products can serve as a measure for evaluation and comparison of different IM/DD modulation schemes, where the constants $\beta$ and $\gamma$ are related to the minimum squared Euclidean distance between adjacent symbols. Finally, the error probability is calculated for two modifications of OFDM and for HCM, both for the optical and the electrical domain, respectively. For the AWGN channel, the OFDM modifications under investigation are shown to outperform HCM, if the number of bits per dimension is fixed. The double DC offset intuitively explains the worse performance. Theoretical error probability results are verified by computer simulations.

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