Exact Formulas for Spherical Photon Orbits Around Kerr Black Holes

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Exact formulas relating the radii of the spherical photon orbits to the black hole’s rotation parameter and the effective inclination angle of the orbit have been known only for equatorial and polar orbits up to now. Here we provide exact formulas for non-equatorial orbits that lie between these extreme limits. For a given rotation parameter of the black hole, there is a critical inclination angle below which there are four null photon orbits. At the critical angle, the radii of these orbits are explicitly found. We also provide approximate but very accurate formulas for any spherical photon orbit. To arrive at these analytical solutions, we carefully study the sextic polynomial in the radius that arises for null spherical geodesics.

Introduction. Observational black hole physics, using gravitational waves or radio waves, is currently thrilling: since the first black hole merger observation via gravitational waves [1], there has been more than a dozen observations, including the most recent groundbreaking detection of a binary black hole system with a total mass of 150 $M_\odot$ [2]. With radio waves, The Event Horizon Telescope provided us with a stunning image of the environment and the shadow of a supermassive black hole [3]. All these black holes are described by the Kerr solution of the vacuum Einstein equation with mass $m$ and angular momentum $J$ [4] and no other parameters. Hence studying physics, especially the motion of ultra-relativistic particles and light around the Kerr black hole metric is extremely relevant to the current observations, especially in the context of imaging the black hole and the ring-down phase of a merger.

Among the light orbits around a rotating black hole, there is a particular class with constant radii that are astrophysically relevant [5]. For this class of light orbits, there are exact expressions of their radii in terms of the black hole rotation or spin parameter and the effective inclination angle of the orbit for only two special cases. These are the equatorial orbits (called the Light Rings) and the polar orbit, but in between these two extremes, called the Spherical Photon Orbits [6], there are no known exact solutions in the Literature. Here we provide exact expressions for an important class of such orbits that are not restricted to the equatorial or polar planes. For these orbits, there is a relation between the inclination angle and the black hole rotation parameter. Moreover, this angle turns out to be critical in the sense that there are four null geodesics below it, albeit two of these are inside the event horizon for non-extremal black holes; one of the external orbits has a non-monotonic dependence on the rotation parameter of the black hole in contrast to the monotonic dependence of the polar and equatorial orbits. On the other hand, above the critical inclination angle, there are only two spherical null geodesics. We shall also give approximate analytical expressions not only for the critical case but also for the case of generic inclination angle.

Null Geodesics. The metric of the Kerr black hole with mass $m$ and angular momentum $J = ma$ in the Boyer-Lindquist coordinates (in the $G = c = 1$ units) is

$$ds^2 = -(1 - \frac{2mr}{\Sigma})dt^2 - \frac{4mar\sin^2 \theta}{\Sigma}dtd\phi + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 + \left( r^2 + a^2 + \frac{2m^2r\sin^2 \theta}{\Sigma} \right) \sin^2 \theta d\phi^2,$$

where $a$ is called the rotation parameter which can be taken to be positive without loss of generality; and

$$\Delta \equiv r^2 - 2mr + a^2, \quad \Sigma \equiv r^2 + a^2\cos^2 \theta.$$

The larger root of $\Delta = 0$ is the event horizon located at

$$r_H = m + (m^2 - a^2)^{1/2}.$$

For the spherical photon orbits, the relevant part of the geodesic equations that we shall be interested in is the radial one [6-9]

$$\Sigma \frac{dr}{d\lambda} = \pm \sqrt{R(r)},$$

where $\lambda$ is an affine parameter along the null geodesics and the radial function is given as

$$R(r) \equiv E^2r^4 + (a^2E^2 - L_z^2 - \mathcal{Q})r^2 + 2m\left((aE - L_z) + \mathcal{Q}\right)r - a^2\mathcal{Q}.$$

Here $E$ is the conserved energy of the photons (related with the $\xi_{(1)} = \frac{\partial}{\partial t}$ Killing vector of the metric), $L_z$ is the conserved $z$-component of the angular momentum of the photons (related to the $\xi_{(\phi)} = \frac{\partial}{\partial \phi}$ Killing vector) and $\mathcal{Q}$ is the Carter’s constant related with a symmetric rank two Killing tensor whose explicit form is not needed here; it suffices to know that it is connected with the motion perpendicular to the equatorial plane and $\mathcal{Q} \geq 0$ for constant radii orbits, satisfying the bound for equatorial orbits.
Two equations must be satisfied, \( R(r) = 0 \) and \( \frac{dR(r)}{dr} = 0 \), for constant radius \( r \) null geodesics. These yield the following (physically viable) equations \([6]\)

\[
\begin{align*}
\frac{L_z}{E} &= -\frac{r^3 - 3mr^2 + a^2r + a^2m}{a(r-m)}, \quad (6) \\
\frac{Q}{E^2} &= -\frac{r^3 (r^3 - 6mr^2 + 9m^2r - 4a^2m)}{a^2(r-m)^2}. \quad (7)
\end{align*}
\]

Defining the effective inclination angle as \([10]\)

\[
\cos i \equiv \frac{L_z}{\sqrt{L_z^2 + Q}} \quad (8)
\]

which is clearly a conserved quantity, and noting the fact that photons with different energies can orbit at the same radius as dictated by the equivalence principle, one can eliminate the energy of the photon in \((6, 7)\) and end up with the following sextic polynomial equation

\[
p(x) \equiv x^6 - 6x^5 + (9 + 2nu)x^4 - 4ux^3 - nu(6-u)x^2 + 2nu^2x + nu^2 = 0, \quad (9)
\]

where we defined the dimensionless variables

\[
x \equiv \frac{r}{m}, \quad u \equiv \frac{a^2}{m^2}, \quad \nu \equiv \sin^2 i. \quad (10)
\]

From now on, we will call \( u \) to be the rotation parameter. So one must solve this polynomial equation as \( x = x(u, \nu) \) for the following intervals

\[
0 < x, \quad 0 \leq \nu \leq 1, \quad 0 \leq u \leq 1. \quad (11)
\]

Let us first briefly discuss the known solutions in the particular cases before we present novel solutions.

**Equatorial and Polar orbits.** To study the equatorial orbits \((Q \equiv 0)\), let \( i = 0 \) or \( \pi \). Then \( \nu = 0 \) and the non-trivial part of the sextic \([9]\) reduces to the cubic

\[
x^3 - 6x^2 + 9x - 4u = 0 \quad (12)
\]

of which the discriminant is \( \Delta = 4u(-1+u) \). For \( u = 1 \), which is the extremal black hole case, there is a double root at \( x = 1 \) and another root at \( x = 4 \). The double root has the same \( r \)-coordinate as the event horizon, but they are not located at the same point as the horizon due to the extended geometry of the extremal black hole’s throat. This tricky case was addressed in \([8, 11]\). For the subextremal case \( u < 1 \) and so \( \Delta < 0 \); there are 3 distinct real roots

\[
x_{\pm} = 2 + 2\cos\left(\frac{2}{3}\cos^{-1}(\pm\sqrt{u})\right) \quad (13)
\]

\[
x_{in} = 4\sin^2\left(\frac{1}{3}\sin^{-1}\sqrt{u}\right), \quad (14)
\]

where \( x_+ \) is the retrograde orbit satisfying \( 3 \leq x_+ \leq 4 \), while \( x_- \) is the prograde orbit satisfying \( 1 \leq x_- \leq 3 \).

The third orbit is inside the event horizon: \( 0 < x_{in} \leq 1 \).

For the polar orbit \((L_z = 0)\), and so \( i = \pm\frac{\pi}{2} \), then \( \nu = 1 \) and \([9]\) reduces to

\[
x^3 - 3x^2 + ux + u = 0. \quad (15)
\]

There are 3 real solutions, but one of the solutions has \( x < 0 \) and hence it is unphysical. The external solution is

\[
x_p = 1 + 2\sqrt{1 - \frac{u}{3}}\cos\left\{\frac{1}{3}\cos^{-1}\left(\frac{1 - u}{(1 - \frac{u}{3})^{3/2}}\right)\right\}, \quad (16)
\]

which lies in the region \( 1 \leq x_p \leq 3 \). The interior solution is

\[
x_{in} = 1 - 2\sqrt{1 - \frac{u}{3}}\sin\left\{\frac{1}{3}\sin^{-1}\left(\frac{1 - u}{(1 - \frac{u}{3})^{3/2}}\right)\right\}, \quad (17)
\]

which lies in the region \( 0 < x_{in} \leq 1 \). The photons in these polar orbits have zero angular momentum but they do rotate with the black hole as a result of the frame-dragging (or the Lens-Thirring) effect.

**Schwarzschild Black Hole.** For \( u = 0 \), the case for which the Kerr black hole has no rotation and reduces to the Schwarzschild black hole, the non-trivial part of \([9]\) for generic \( \nu \) simplifies to

\[
x - 3 = 0, \quad (18)
\]

which is the radius of the circular photon ring (or the Light Ring) independent of the inclination angle due to the spherical symmetry.

**Spherical Photon Orbits** For generic \( \nu \) and \( u \), there are four sign variations in the polynomial \([9]\), and hence Descartes’ rule of sign change says that there can be 4, 2, or 0 positive roots. Using a Tschirnhaus transformation of the form \( y = \alpha x^4 + \beta x^3 + \gamma x^2 + \delta x + \epsilon \), we can bring the sextic to the reduced form \( y^6 + cy^2 + dy + e = 0 \), albeit with very complicated coefficients \( c_i = c_i(u, \nu) \).

The exact expressions of the roots in the generic case of any polynomial beyond the quartic, in terms of a finite number of radicals, is not possible as dictated by the celebrated Abel-Ruffini impossibility theorem. This theorem applies to our sextic in the original or reduced form. However, one can certainly find exact formulas for the roots in terms of special functions such as the Mellin formula \([12]\), Theta functions \([13]\), or the nested radicals. Because of the complexity of exact roots in the most general form, let us concentrate on particular values of \( \nu \) and \( u \) of the sextic for which we can find exact solutions in radicals. Remarkably, this will turn out to be possible.

As we are looking for the real roots of the sextic, we might search for the point in the \((u, \nu)\) parameter space where four real and two complex roots arise as allowed by the Descartes’ rule of sign. It turns out for a given rotation parameter \( u \), there is a critical inclination angle below which there are four real roots and above which there are two real roots. For this critical value, \( \nu_{cr}(u) \), the discriminant of the sextic vanishes, and one can find all the roots of the sextic in an exact form.
To proceed further, it pays to define the following variables which will simplify the final expressions:

\[ \nu \equiv \frac{x}{u}, \quad u \equiv 1 - w^3, \]  

(19)

with \( 0 \leq \xi \leq 1 \) and \( 0 \leq w \leq 1 \). Then it can be shown that the sextic (10) reduces to a quadratic times a quartic at the following critical point

\[ \xi_{\text{cr}} = \frac{3(1 - w)^3}{7 + (w - 5)w}. \]  

(20)

The domain of \( \xi_{\text{cr}} \) is \([0, \frac{2}{7}]\). Note that previously in [14], one particular point in this critical domain was discovered: namely, it was shown that the critical angle corresponding to the extremal black hole \((u = 1)\) is \(\cos i = \sqrt{4/7}\) or \(\nu = \frac{2}{7}\). Here with the formula (20) the critical angle for any rotating parameter is given. So at this critical angle, the sextic factors as

\[ p(x) = (w + x - 1)^2 q(x), \]  

(21)

with the quartic part given as

\[ q(x) = x^4 + \alpha x^3 + \beta x^2 + \gamma x + \sigma, \]  

(22)

where the coefficients are

\[ \alpha = -2(w + 2), \]  

\[ \beta = \frac{3w(w^2(w - 5) + 3w + 8) + 6}{(w - 5)w + 7}, \]  

\[ \gamma = \frac{6(w - 2)(w - 1)(w^2 + w + 1)}{(w - 5)w + 7}, \]  

\[ \sigma = \frac{3(w - 1)^2(w^2 + w + 1)}{(w - 5)w + 7}. \]  

(23)

The quadratic part has a double root at

\[ x = 1 - (1 - u)^{\frac{3}{5}} \]  

(24)

inside the event horizon for finite \( u \) and approaches the event horizon as \( u \to 1 \). The quartic part has generically two real and two complex roots, which can be written in explicit form, but the expressions are rather cumbersome, and so we do not depict them here. These explicit formulas can be used, with the help of a computer, to understand the properties of the orbits exactly. Here we give several such computations.

In Figure 1, we plot the real roots of the sextic as a function of the rotation parameter at the critical inclination angle. One can see how the roots are connected with each other and observe the non-monotonic behavior of the retrograde orbit. Note that, as opposed to the critical retrograde orbit, the equatorial retrograde orbit is monotonic in the rotation parameter; all the other roots are also monotonic. In Figure 2, we plot the dimensionless angular momentum per unit energy, \( \frac{L_z}{mE} \), as given in [6] as a function of \( u \). In Figure 3, we plot \( \frac{d^2 R}{d \phi^2} \) as a function of \( u \) and show that the critical null geodesics are unstable in the radial direction as expected: all spherical photon orbits are unstable in the radial direction; this fact makes them very interesting as they can escape to infinity due to a small perturbation, and this is part of the light detected by the Event Horizon Telescope. In Figure 4, we plot the dimensionless impact parameter for the retrograde orbits at the critical inclination angle and at the equator. Among all the orbits, the retrograde orbit has the highest impact parameter. Figure 5 shows the analogous plot for the prograde orbits.

**Approximate Solutions.** Here we present a method to develop approximate solutions around any of the known solutions presented above. The method is based on the Lagrange-Bürmann inversion theorem [15]. Let us assume \( f \) to be a function of the form

\[ f(x) = \nu, \quad \frac{df(x)}{dx} \bigg|_{x=x_0} \neq 0. \]  

(25)

Then the Lagrange-Bürmann inversion theorem states
that one can write \( x \) as a power series in \( \nu \) of the form

\[
x = x_0 + \sum_{n>0} \frac{g_n}{n!} \left( \nu - f(x_0) \right)^n
\]  

where

\[
g_n = \lim_{x \to x_0} \frac{d^{n-1}}{dx^{n-1}} \left( \frac{x - x_0}{f(x) - f(x_0)} \right)^n, \quad n = 1, 2, \ldots
\]  

To apply this method to the sextic, for a given \( u \), one can solve for \( \nu \) in (26) to get the inclination angle as a function of the radius as

\[
\nu(x) = -\frac{x^4((x-3)x^2-4u)}{u(x+1)^2+2(x^2-3)x^2}.
\]  

For \( x_0 \), as stated above, one can choose any of the known solutions. Here for the sake of simplicity, let us choose the prograde and the retrograde orbits for examples. Then \( x_0 = x_\pm \) and \( \nu(x_0) = 0 \); one can easily compute the approximate solution up to the desired order. Here we shall keep up to \( \mathcal{O}(\nu) \) as it already yields an accurate approximation. Then from (27), one finds

\[
g_1 = \lim_{x \to x_0} \frac{x - x_0}{\nu},
\]

which yields

\[
g_1 = \frac{A(x_0)}{B(x_0)}
\]

\[
A(x_0) = u(u(x_0+1)^2+2x_0^4-6x_0^2)^2
\]

\[
B(x_0) = 4x_0^2(x_0+3)(u(x_0+1)+x_0^3-3x_0^2)
\]

\[
\times \left( u - x_0^3+3x_0^2-3x_0 \right).
\]

Then, the resulting approximate solution is

\[
x = x_0 + g_1\nu + \mathcal{O}(\nu^2).
\]  

which is accurate for small \( \nu \approx 0.1 \) by 2 parts in \( 10^4 \) for slowly rotating black holes, \( u \approx 0.1 \), and by 2 parts in \( 10^5 \) for fast rotating ones, \( u \approx 0.9 \), around the prograde equatorial orbits. The approximation works better for retrograde orbits.

| \( \nu \) | \( u \) | \( x_{\text{approximate}} \) | \( x_{\text{exact}} \) | Error (%) |
|---|---|---|---|---|
| 0.1 | 0.1 | 2.6247277 | 2.6247278 | 4.3 \times 10^{-6} |
| 0.1 | 0.5 | 2.027428 | 2.0274289 | 1.4 \times 10^{-6} |
| 0.1 | 0.9 | 1.411893 | 1.411895 | 7.5 \times 10^{-6} |
| 0.5 | 0.1 | 2.7033 | 2.7038 | 1.9 \times 10^{-2} |
| 0.5 | 0.5 | 2.174 | 2.175 | 6 \times 10^{-2} |
| 0.5 | 0.9 | 1.562 | 1.565 | 2.4 \times 10^{-1} |

Table I: Comparison of approximate and exact numerical results around the prograde orbit at the equator, \( x_0 = x_\pm \). Approximation is done with the Lagrange-Bürmann expansion up to order \( \mathcal{O}(\nu^2) \).

For better accuracy, one needs to compute higher-order terms. We show some examples up to and including...
Approximation is done with the Lagrange-Bürmann expansion up to order $O(\nu^5)$. 

$O(\nu^4)$ in the tables. As the errors show, the approximation is exceptionally accurate. In Table 1, the exact numerical values are compared with the approximate analytical ones obtained via the Lagrange-Bürmann expansion up the retrograde orbit at the equatorial plane. These two interior exact solutions, which are presumably not significant for astrophysics, may nevertheless be important to better explore the geometry of the black hole region. The novel solutions presented here are all unstable in the radial direction, as all the spherical geodesics are unstable; this fact is closely related to the stability and the shadow of the Kerr black hole. An extended discussion of the material here and the discussion for the time-like geodesics along the similar lines will be given elsewhere.

| $\nu$ | $u$ | $x_{\text{approximate}}$ | $x_{\text{exact}}$ | Error (%) |
|-------|-----|--------------------------|-------------------|-----------|
| 0.1   | 0.1 | 3.3249272                | 3.3249271         | $3.3 \times 10^{-6}$ |
| 0.1   | 0.5 | 3.6801618                | 3.6801615         | $7.6 \times 10^{-6}$ |
| 0.1   | 0.9 | 3.8803947                | 3.8803943         | $1.1 \times 10^{-5}$ |
| 0.5   | 0.1 | 3.228                    | 3.227             | $1.6 \times 10^{-2}$ |
| 0.5   | 0.5 | 3.438                    | 3.437             | $3.8 \times 10^{-2}$ |
| 0.5   | 0.9 | 3.533                    | 3.531             | $5.6 \times 10^{-2}$ |

Table II: Comparison of approximate and exact numerical results around the retrograde orbit at the equator, $x_0 = x_{\text{approximate}}$. 

**Conclusions.** The motion of light in the vicinity of a Kerr black hole is important for astrophysics. Up to now, the radii of the polar and equatorial plane light orbits have been known explicitly in terms of the rotation parameter of the black hole. Here we have extended the known analytical solutions by showing that there are two exterior non-equatorial light orbits whose radii can be found analytically for a given black hole with a rotation parameter. One of these orbits is a prograde orbit, which is monotonically decreasing in the rotation parameter and the other one is a retrograde orbit, which is non-monotonic, in contrast to the equatorial retrograde orbit that is monotonic. This feature was observed in the extremal black hole case. The effective inclination angle of these two orbits is critical in the sense that exactly at and below that angle, there are always four spherical null orbits. Two of these are inside the event horizon, and at the critical angle, these two orbits merge into a single orbit with a radius, which can be analytically expressed in terms of the rotation parameter.

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