A New Framework of Multistage Estimation *

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In Memory of My Dear Father Hualong Chen (1933–1990)

Abstract

In this paper, we have established a unified framework of multistage parametric estimation. We demonstrate that a wide spectrum of classical sequential problems such as point estimation with error control, bounded-width confidence intervals, interval estimation following hypothesis testing, construction of confidence sequences, can be cast in the general framework of random intervals. We have developed exact methods for the construction of such random intervals in the context of multistage sampling. Our sampling schemes are unprecedentedly efficient in terms of sampling effort as compared to existing sampling procedures.

Contents

1 Introduction 6

2 General Theory 9
  2.1 Basic Structure of Multistage Estimation 9
  2.2 Truncated Inverse Sampling 10
  2.3 Random Intervals 10
  2.4 Unimodal-Likelihood Estimator 12
  2.5 Principle of Construction of Sampling Schemes 12
  2.6 Multistage Sampling without Replacement 16
  2.7 Asymptotically Unbiased Estimators of Mean Values 17

3 Computational Machinery 18
  3.1 Bisection Coverage Tuning 18
  3.2 Consecutive-Decision-Variable Bounding 18

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3.3 Adaptive Maximum Checking .................................... 19
3.4 Interval Bounding .................................................. 21
3.5 Recursive Computation ............................................. 21
3.6 Domain Truncation .................................................. 22
3.7 Triangular Partition .................................................. 23
3.8 Interval Splitting ..................................................... 24
3.9 Factorial Evaluation ................................................ 25

4 Estimation of Binomial Parameters ................................ 25
4.1 Control of Absolute Error ........................................... 26
  4.1.1 Stopping Rules from CDFs, Chernoff Bounds and Massart’s Inequality ... 26
  4.1.2 Asymptotic Stopping Rule ..................................... 28
  4.1.3 Asymptotic Analysis of Sampling Schemes ....................... 28
4.2 Control of Relative Error .......................................... 29
  4.2.1 Multistage Inverse Sampling ................................... 30
  4.2.2 Asymptotic Stopping Rule ..................................... 33
  4.2.3 Noninverse Multistage Sampling ................................. 34
  4.2.4 Asymptotic Analysis of Multistage Inverse Sampling Schemes ....... 35
  4.2.5 Asymptotic Analysis of Noninverse Multistage Sampling Schemes ....... 36
4.3 Control of Absolute and Relative Errors ......................... 37
  4.3.1 Stopping Rules from CDFs and Chernoff Bounds .................. 37
  4.3.2 Stopping Rule from Massart’s Inequality ........................ 39
  4.3.3 Asymptotic Stopping Rule ..................................... 39
  4.3.4 Asymptotic Analysis of Sampling Schemes ....................... 40

5 Estimation of Bounded-Variable Means ............................ 41
5.1 Control of Absolute Error ........................................... 41
5.2 Control of Relative Error .......................................... 42
5.3 Control of Absolute and Relative Errors ......................... 43
5.4 Using the Link between Binomial and Bounded Variables ............ 44

6 Estimation of Poisson Parameters ................................ 45
6.1 Control of Absolute Error ........................................... 45
  6.1.1 Stopping Rule from CDFs ...................................... 45
  6.1.2 Stopping Rule from Chernoff Bounds ............................ 46
  6.1.3 Asymptotic Analysis of Multistage Sampling Schemes ............. 46
6.2 Control of Relative Error .......................................... 47
  6.2.1 Stopping Rule from CDFs ...................................... 47
  6.2.2 Stopping Rule from Chernoff Bounds ............................ 48
  6.2.3 Asymptotic Analysis of Multistage Sampling Schemes ............. 48
6.3 Control of Absolute and Relative Errors
   6.3.1 Stopping Rules from CDFs and Chernoff Bounds
   6.3.2 Asymptotic Stopping Rule
   6.3.3 Asymptotic Analysis of Multistage Sampling Schemes

7 Estimation of Finite Population Proportion

8 Estimation of Normal Mean
   8.1 Control of Absolute Error
     8.1.1 New Structure of Multistage Sampling
     8.1.2 Exact Construction of Sampling Schemes
   8.2 Control of Relative Error
   8.3 Control of Relative and Absolute Errors

9 Estimation of Exponential Parameters with Relative Precision

10 Exact Bounded-Width Confidence Intervals
   10.1 Construction via Coverage Tuning
   10.2 Bounded-Width Confidence Intervals for Binomial Parameters
     10.2.1 Construction from Clopper-Pearson Intervals
     10.2.2 Construction from Fishman’s Confidence Intervals
     10.2.3 Construction from Explicit Confidence Intervals of Chen et al.
   10.3 Bounded-Width Confidence Intervals for Finite Population Proportion

11 Estimation Following Multistage Tests
   11.1 Clopper-Pearson Type Confidence Intervals
     11.1.1 Finite Population Proportion
   11.2 Confidence Intervals from Coverage Tuning
     11.2.1 Poisson Mean
     11.2.2 Normal Variance
     11.2.3 Exponential Parameters

12 Exact Confidence Sequences
   12.1 Construction via Coverage Tuning
   12.2 Finite Population Proportion
   12.3 Poisson Mean
   12.4 Normal Mean
   12.5 Normal Variance
   12.6 Exponential Parameters
13 Multistage Linear Regression
13.1 Control of Absolute Error .............................................. 74
13.2 Control of Relative Error .................................................. 74

14 Multistage Estimation of Quantile
14.1 Control of Absolute Error .................................................. 75
14.2 Control of Relative Error ................................................... 75
14.3 Control of Absolute and Relative Errors ............................... 76

15 Conclusion ............................................................................ 76

A Preliminary Results
A.1 Proof of Identity [1] .............................................................. 76
A.2 Probability Transform Inequalities ........................................... 78
A.3 Property of ULE ................................................................. 78

B Proof of Theorem [2] ................................................................ 79

C Proof of Theorem [3] ................................................................ 79

D Proof of Theorem [5] ................................................................ 81

E Proof of Theorem [7] ................................................................ 81

F Proof of Theorem [8] ................................................................ 83

G Proof of Theorem [11] ............................................................... 84

H Proof of Theorem [12] ............................................................... 85

I Proofs of Theorems for Estimation of Binomial Parameters
I.1 Proof of Theorem [13] .............................................................. 85
I.2 Proof of Theorem [13] .............................................................. 89
I.3 Proof of Theorem [13] .............................................................. 90
I.4 Proof of Theorem [16] .............................................................. 98
  I.4.1 Proof of Statement (I) ....................................................... 102
  I.4.2 Proof of Statement (II) ...................................................... 103
  I.4.3 Proof of Statement (III) .................................................... 105
I.5 Proof of Theorem [17] .............................................................. 107
I.6 Proof of Theorem [18] .............................................................. 107
I.7 Proof of Theorem [19] .............................................................. 109
I.8 Proof of Theorem [20] .............................................................. 114
1 Introduction

Parameter estimation is a fundamental area of statistical inference, which enjoys numerous applications in various fields of sciences and engineering. Specially, it is of ubiquitous significance to estimate, via sampling, the parameters of binomial, Poisson, hypergeometrical, and normal distributions. In general, a parameter estimation problem can be formulated as follows. Let $X$ be a random variable defined in a probability space $(\Omega, \mathcal{F}, \Pr)$. Suppose the distribution of $X$ is determined by an unknown parameter $\theta$ in a parameter space $\Theta$. In many applications, it is desirable to construct a random interval which includes $\theta$ with a prescribed level of confidence from random samples $X_1, X_2, \cdots$ of $X$. This problem is so fundamental that it has been persistent issues of research in probability, statistics and other relevant fields (see, e.g., [18, 22, 23, 32, 35, 41] and the references therein). Despite the richness of literature devoted to such issues, existing approaches may suffer from the drawbacks of lacking either efficiency or rigorousness. Such drawbacks are due to two frequently-used methods of designing sampling schemes. The first method is to seek a worst-case solution based on the assumption that the true parameter $\theta$ is included in an interval $[a, b] \subseteq \Theta$. Since it is difficult to have tight bounds for the unknown parameter $\theta$, such a worst-case method can lead to overly wasteful sample size if the interval $[a, b]$ is too wide. Moreover, if the true value of $\theta$ is not included in $[a, b]$, the resultant sample size can be misleading. The second
method is to employ asymptotic theories such as large deviations theory, Brownian motion theory, diffusion theory and nonlinear renewal theory in the design and analysis of sampling schemes (see, [17, 28, 37, 40, 42] and the references therein). Undoubtedly, asymptotic techniques may offer approximate solutions and important insight for the relevant problems. Since any asymptotic theory holds only if the sample size tends to infinity and, unfortunately, any practical sampling scheme must be of a finite sample size, it is inevitable for an asymptotic method to introduce unknown error in the resultant approximate solution.

In view of the limitations of existing approaches of parametric estimation, we would like to propose a new framework of multistage estimation. The main characteristics of our new estimation methods is as follows: i) No information of the parameter \( \theta \) is required; ii) The sampling schemes are globally efficient in the sense that the average sample number is almost the same as the exact sample size computed as the true value of \( \theta \) were available; iii) The prescribed level of confidence is rigorously guaranteed. Our new estimation techniques are developed under the spirit that parameter estimation, as an important branch of statistical inference, should be accomplished with minimum cost in sampling and absolute rigorousness in quantifying uncertainty.

The remainder of the paper is organized as follows. In Section 2, we present our general theory for the design and analysis of multistage sampling schemes. Especially, we have established a general theory on coverage probability of random intervals which eliminates the necessity of exhaustive computation of coverage probability for designing sampling schemes. In Section 3, we introduce powerful techniques such as bisection coverage tuning, consecutive-decision-variable bounding, recursive computation, adaptive maximum checking, domain truncation and triangular partition that are crucial for a successful design of a multistage sampling scheme. In Section 4, we present sampling schemes for estimation of binomial parameters. In Section 5, we discuss the multistage estimation of Poisson parameters. In Section 6, we consider the estimation of means of bounded variables. In Section 7, we address the problem of estimating the proportion of a finite population. We consider the estimation of normal mean with unknown variance in Section 8. In Section 9, we discuss the estimation of the parameter of an exponential distribution. In Section 10, we propose our exact methods for the construction of bounded-width confidence intervals. In Section 11, we discuss the interval estimation following hypothesis testing. In Section 12, we consider the exact construction of confidence sequences. In Section 13, we address the problem of multistage linear regression. In Section 14, we investigate the multistage estimation of quantile. Section 15 is the conclusion. The proofs of all theorems are given in Appendices.

Throughout this paper, we shall use the following notations. The set of integers is denoted by \( \mathbb{Z} \). The set of positive integers is denoted by \( \mathbb{N} \). The element of matrix \( A \) in the \( i \)-th row and \( j \)-th column is denoted by \( [A]_{i,j} \). The ceiling function and floor function are denoted respectively by \( \lceil \cdot \rceil \) and \( \lfloor \cdot \rfloor \) (i.e., \( \lceil x \rceil \) represents the smallest integer no less than \( x \); \( \lfloor x \rfloor \) represents the largest integer no greater than \( x \)). The notation \( \text{sgn}(x) \) denotes the sign function which assumes value 1 for \( x > 0 \), value 0 for \( x = 0 \), and value \(-1\) for \( x < 0 \). The gamma function is denoted by \( \Gamma(\cdot) \). For any integer \( m \), the combinatoric function \( \binom{m}{z} \) with respect to integer \( z \) takes value \( \frac{\Gamma(m+1)}{\Gamma(z+1)\Gamma(m-z+1)} \) for \( z \leq m \) and value 0 otherwise. The left limit as \( \epsilon \) tends to 0 is denoted as \( \lim_{\epsilon \downarrow 0} \). The notation
“$\iff$” means “if and only if”. The expectation of a random variable is denoted by $\mathbb{E}[]$. The notation $\Pr\{., \theta\}$ denotes the probability of an event associated with random samples $X_1, X_2, \cdots$ parameterized by $\theta \in \Theta$, where $\theta$ may be dropped if it can be done without introducing confusion. The cumulative distribution function of a Gaussian random variable is denoted by $\Phi(\cdot)$.

For $\alpha \in (0,1)$, let $Z_\alpha$ and $t_{n, \alpha}$ denote, respectively, the $100(1 - \alpha)$% percentiles of a standard normal distribution and a Student $t$-distribution of $n$ degrees of freedom. For $\alpha \in (0,1)$, let $\chi^2_{n, \alpha}$ denote the 100$\alpha$% percentile of a chi-square distribution of $n$ degrees of freedom. In the presentation of our sampling schemes, we need to use the following functions:

\[
S_B(k, n, \theta) = \begin{cases} 
\sum_{i=0}^{k} \binom{n}{i} \theta^i (1 - \theta)^{n-i} & \text{for } \theta \in [0, 1], \\
1 & \text{for } \theta < 0, \\
0 & \text{for } \theta > 1
\end{cases}
\]

\[
S_N(k, n, \theta) = \begin{cases} 
\sum_{i=0}^{k} \left( \frac{\theta^N \chi_{n-1}}{n} \right)^i & \text{for } \theta \in \left\{ \frac{m}{N} : m = 0, 1, \cdots, N \right\}, \\
1 & \text{for } \theta < 0, \\
0 & \text{for } \theta > 1
\end{cases}
\]

\[
S_P(k, \theta) = \begin{cases} 
\sum_{i=0}^{k} \frac{\theta^i e^{-\theta}}{i!} & \text{for } \theta \geq 0, \\
0 & \text{for } \theta < 0
\end{cases}
\]

\[
M(z, \theta) = \begin{cases} 
\frac{2(z - \theta)^2}{z(2(2z - 3))} & \text{for } 0 \leq z \leq 1 \text{ and } \theta \in (0, 1), \\
-\infty & \text{for } 0 \leq z \leq 1 \text{ and } \theta \notin (0, 1)
\end{cases}
\]

\[
\mathcal{M}_B(z, \theta) = \begin{cases} 
z \ln \frac{\theta}{z} + (1 - z) \ln \frac{1 - \theta}{1 - z} & \text{for } z \in (0, 1) \text{ and } \theta \in (0, 1), \\
\ln(1 - \theta) & \text{for } z = 0 \text{ and } \theta \in (0, 1), \\
\ln \theta & \text{for } z = 1 \text{ and } \theta \in (0, 1), \\
-\infty & \text{for } z \in [0, 1] \text{ and } \theta \notin (0, 1)
\end{cases}
\]

\[
\mathcal{M}_1(z, \theta) = \begin{cases} 
\ln \frac{\theta}{z} + \frac{1}{2} - 1 \ln \frac{1 - \theta}{1 - z} & \text{for } z \in (0, 1) \text{ and } \theta \in (0, 1), \\
\ln \theta & \text{for } z = 1 \text{ and } \theta \in (0, 1), \\
-\infty & \text{for } z = 0 \text{ and } \theta \in (0, 1), \\
-\infty & \text{for } z \in [0, 1] \text{ and } \theta \notin (0, 1)
\end{cases}
\]

\[
\mathcal{M}_P(z, \theta) = \begin{cases} 
z - \theta + z \ln \left( \frac{2}{\theta} \right) & \text{for } z > 0 \text{ and } \theta > 0, \\
-\theta & \text{for } z = 0 \text{ and } \theta > 0, \\
-\infty & \text{for } z \geq 0 \text{ and } \theta \leq 0.
\end{cases}
\]

In the design of multistage sampling schemes, we shall use a descending sequence $C_\ell, \ell \in \mathbb{Z}$ such that $C_0 = 1$ and $1 < \inf_{\ell \in \mathbb{Z}} \frac{C_\ell}{C_{\ell+1}} \leq \sup_{\ell \in \mathbb{Z}} \frac{C_\ell}{C_{\ell+1}} < \infty$ to define sample sizes. Throughout the remainder of this paper, $\delta$ and $\zeta$ are reserved, respectively, for the “confidence parameter” and the “coverage tuning parameter”, where these concepts will be illustrated later. It is assumed that $0 < \delta < 1$ and $0 < \zeta < \frac{1}{\delta}$. The other notations will be made clear as we proceed.
2 General Theory

In this section, we shall discuss the general theory of multistage estimation. A central theme of our theory is on the reduction of the computational complexity associated with the design and analysis of multistage sampling schemes.

2.1 Basic Structure of Multistage Estimation

In our proposed framework of multistage estimation, a sampling process consists of $s$ stages, where $s$ can be a finite number or infinity. The continuation or termination of sampling is determined by decision variables. For the $\ell$-th stage, a decision variable $D_\ell = D_\ell(X_1, \cdots, X_{n_\ell})$ is defined in terms of samples $X_1, \cdots, X_{n_\ell}$, where $n_\ell$ is the number of samples available at the $\ell$-th stage. It should be noted that $n_\ell$ can be a random number, depending on specific sampling schemes. The decision variable $D_\ell$ assumes only two possible values 0, 1 with the notion that the sampling process is continued until $D_\ell = 1$ for some $\ell \in \{1, \cdots, s\}$. Since the sampling must be terminated at or before the $s$-th stage, it is required that $D_s = 1$. For simplicity of notations, we also define $D_\ell = 0$ for $\ell < 1$ and $D_\ell = 1$ for $\ell > s$ throughout the remainder of the paper. Let $l$ denote the index of stage when the sampling is terminated. Then, the sample number when the sampling is terminated, denoted by $n$, is equal to $n_l$. Since a sampling scheme with the above structure is like a multistage version of the conventional fixed-size sampling procedure, we call it multistage sampling in this paper.

As mentioned earlier, the number of available samples, $n_\ell$, for the $\ell$-th stage can be a random number. An important case can be made in the estimation of the parameter of a Bernoulli random variable $X$ with distribution $\Pr\{X = 1\} = 1 - \Pr\{X = 0\} = p \in (0, 1)$. To estimate $p$, we can choose a sequence of positive integers $\gamma_1 < \gamma_2 < \cdots < \gamma_s$ and define decision variables such that $D_\ell$ is expressed in terms of i.i.d. samples $X_1, \cdots, X_{n_\ell}$ of Bernoulli random variable $X$, where $n_\ell$ is the minimum integer such that $\sum_{i=1}^{n_\ell} X_i = \gamma_\ell$ for $\ell = 1, \cdots, s$. A sampling scheme with such a structure is called a multistage inverse binomial sampling, which is a special class of multistage sampling schemes and is a multistage version of the inverse binomial sampling (see, e.g., [25, 26] and the references therein).

If the sample sizes of a multistage sampling scheme is desired to be deterministic, the following criteria can be applied to determine the minimum and maximum sample sizes:

(I) The minimum sample size $n_1$ guarantees that $\{D_1 = 1\}$ is not an impossible event.

(II) The maximum sample size $n_s$ guarantees that $\{D_s = 1\}$ is a sure event.

For the purpose of reducing sample number, the minimum and maximum sample sizes should be as small as possible, while satisfying criteria (I) and (II).
2.2 Truncated Inverse Sampling

It should be noted that the conventional single stage sampling procedures can be accommodated in the general framework of multistage sampling. A common stopping rule for single stage sampling procedures is that “the sampling is continued until the sample sum reach a prescribed threshold $\gamma$ or the number of samples reach a pre-specified integer $m$”. Such a sampling scheme is referred to as truncated inverse sampling, for which we have derived the following results.

Theorem 1 Let $\gamma > 1$, $0 < \varepsilon_a < \varepsilon_r < 1$ and $p^* = \frac{\varepsilon_a}{\varepsilon_r}$. Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables such that $0 \leq X_i \leq 1$ and $\mathbb{E}[X_i] = \mu \in (0, 1)$ for any positive integer $i$. Let $m$ be a random variable such that $\{\sum_{i=1}^{n-1} X_i < \gamma \leq \sum_{i=1}^{n} X_i\}$ is a sure event. Let $m = \min\{n, m\}$, where $m$ is a positive integer. The following statements hold true.

(I) $\Pr\{|\frac{1}{m} - \mu| < \varepsilon_a\} > 1 - \delta$ and $\Pr\{|\frac{1}{m} - \mu| < \varepsilon_r\} > 1 - \delta$ provided that $\gamma > \frac{(1+\varepsilon_r)\ln(2/\delta)}{1+\varepsilon_r\ln(1+\varepsilon_r)}$, and $m > \frac{\ln(\delta/2)}{\mathbb{H}(p^*+\varepsilon_r, p^*)}$.

(II) $\Pr\{|\frac{1}{m} - \mu| < \varepsilon_a \text{ or } |\frac{1}{m} - \mu| < \varepsilon_r\} > 1 - \delta$ provided that $p^* + \varepsilon_a \leq \frac{1}{2}$, $\gamma > \frac{1+\varepsilon_a}{\varepsilon_r}$, and $m > \frac{\ln(\delta/2)}{\mathbb{H}(p^*+\varepsilon_a, p^*)}$.

(III) If $X_1, X_2, \ldots$ are i.i.d. Bernoulli variables, then $\Pr\{|\frac{1}{m} - \mu| < \varepsilon_a \text{ or } |\frac{1}{m} - \mu| < \varepsilon_r\} > 1 - \delta$ provided that $p^* + \varepsilon_a \leq \frac{1}{2}$, $\gamma > \frac{\ln(\delta/2)}{\mathbb{H}(p^*+\varepsilon_a, p^*)}$ and $m > \frac{\ln(\delta/2)}{\mathbb{H}(p^*+\varepsilon_a, p^*)}$.

The proof of Theorem 1 can be found in [3, 4].

2.3 Random Intervals

A primary goal of multistage sampling is to construct, based on samples of $X$, a random interval with lower limit $\mathcal{L}(X_1, \ldots, X_n)$ and upper limit $\mathcal{U}(X_1, \ldots, X_n)$ such that, for a priori specified confidence parameter $\delta$,

$$\Pr\{\mathcal{L}(X_1, \ldots, X_n) < \theta < \mathcal{U}(X_1, \ldots, X_n) \mid \theta\} \geq 1 - \delta$$

for any $\theta \in \Theta$. For the $\ell$-th stage, an estimator $\hat{\theta}_\ell$ for $\theta$ can be defined in terms of samples $X_1, \ldots, X_{n_\ell}$. Consequently, the overall estimator for $\theta$, denoted by $\hat{\theta}$, is equal to $\hat{\theta}_\ell$. In many cases, $\mathcal{L}(X_1, \ldots, X_n)$ and $\mathcal{U}(X_1, \ldots, X_n)$ can be expressed as a function of $\hat{\theta}_\ell$ and $n_\ell$. For simplicity of notations, we abbreviate $\mathcal{L}(X_1, \ldots, X_n)$ and $\mathcal{U}(X_1, \ldots, X_n)$ as $\mathcal{L}(\hat{\theta}_\ell, n_\ell)$ and $\mathcal{U}(\hat{\theta}_\ell, n_\ell)$ respectively. Accordingly, $\mathcal{L}(X_1, \ldots, X_n)$ and $\mathcal{U}(X_1, \ldots, X_n)$ are abbreviated as $\mathcal{L}(\hat{\theta}, n)$ and $\mathcal{U}(\hat{\theta}, n)$. In the special case that the lower and upper limits are independent of $n$, we will drop the argument $n$ for further simplification of notations.

In the sequel, we shall focus on the construction of random intervals of lower limit $\mathcal{L}(\hat{\theta}, n)$ and upper limit $\mathcal{U}(\hat{\theta}, n)$ such that $\Pr\{\mathcal{L}(\hat{\theta}, n) < \theta < \mathcal{U}(\hat{\theta}, n) \mid \theta\} \geq 1 - \delta$ for any $\theta \in \Theta$. Such a framework is general enough to address a wide spectrum of traditional problems in parametric estimation. First, it is obvious that the problem of interval estimation following a hypothesis test can be cast in this framework. Second, the issue of error control in the point estimation of parameter $\theta$ can be addressed in the framework of random intervals. Based on different error criteria, the point estimation problems are typically posed in the following ways:
(i) Given a priori margin of absolute error $\varepsilon > 0$, construct an estimator $\hat{\theta}$ for $\theta$ such that $\Pr\{ |\hat{\theta} - \theta| < \varepsilon \mid \theta \} \geq 1 - \delta$ for any $\theta \in \Theta$.

(ii) Given a priori margin of relative error $\varepsilon \in (0, 1)$, construct an estimator $\hat{\theta}$ for $\theta$ such that $\Pr\{ |\hat{\theta} - \theta| < \varepsilon |\theta| \mid \theta \} \geq 1 - \delta$ for any $\theta \in \Theta$.

(iii) Given a priori margin of absolute error $\varepsilon_a \geq 0$ and margin of relative error $\varepsilon_r \in [0, 1)$, construct an estimator $\hat{\theta}$ for $\theta$ such that $\Pr\{ |\hat{\theta} - \theta| < \varepsilon_a \text{ or } |\hat{\theta} - \theta| < \varepsilon_r |\theta| \mid \theta \} \geq 1 - \delta$ for any $\theta \in \Theta$.

Clearly, problem (iii) can be reduced to problems (i) and (ii) by, respectively, setting $\varepsilon_r = 0$ and $\varepsilon_a = 0$. As can be seen from Appendix A.1 putting

$$L(\hat{\theta}, n) = \min \left\{ \hat{\theta} - \varepsilon_a, \frac{\hat{\theta}}{1 + \text{sgn}(\hat{\theta}) \varepsilon_r} \right\}, \quad U(\hat{\theta}, n) = \max \left\{ \hat{\theta} + \varepsilon_a, \frac{\hat{\theta}}{1 - \text{sgn}(\hat{\theta}) \varepsilon_r} \right\},$$

we can show that

$$\{ |\hat{\theta} - \theta| < \varepsilon_a \text{ or } |\hat{\theta} - \theta| < \varepsilon_r |\theta| \} = \{ L(\hat{\theta}, n) < \theta < U(\hat{\theta}, n) \}.$$  \hspace{1cm} (1)

This implies that problems (i)-(iii) can be accommodated in the general framework of random intervals.

Third, the framework of random intervals accommodates an important class of problems concerned with the construction of bounded-width confidence intervals. The objective is to construct lower confidence limit $L(\hat{\theta}, n)$ and upper confidence limit $U(\hat{\theta}, n)$ such that $|U(\hat{\theta}, n) - L(\hat{\theta}, n)| \leq 2\varepsilon$ for some prescribed number $\varepsilon > 0$ and that $\Pr\{ L(\hat{\theta}, n) < \theta < U(\hat{\theta}, n) \mid \theta \} \geq 1 - \delta$ for any $\theta \in \Theta$. Obviously, this class of problems can be cast into the framework of random intervals.

In order to construct a random interval of desired level of confidence, our global strategy is to construct a sampling scheme such that the coverage probability $\Pr\{ L(\hat{\theta}, n) < \theta < U(\hat{\theta}, n) \mid \theta \}$ can be adjusted by some parameter $\zeta$. This parameter $\zeta$ is referred to as “coverage tuning parameter”. Obviously, the coverage probability is a function of the unknown parameter $\theta$. In practice, it is impossible or extremely difficult to evaluate the coverage probability for every value of $\theta$ in the parameter space. Such an issue presents in the estimation of binomial parameters, Poisson parameters and the proportion of a finite population. For the cases of estimating binomial and Poisson parameters, the parameter spaces are continuous and thus the number of parametric values is infinity. For the case of estimating the proportion of a finite population, the number of parametric values can be as large as the population size. To overcome the difficulty associated with the number of parametric values, we have developed a general theory of coverage probability of random intervals which eliminates the need of exhaustive evaluation of coverage probabilities to determine whether the minimum coverage probability achieves the desired level of confidence. In this direction, the concept of Unimodal-Likelihood Estimator, to be discussed in the following subsection, play a crucial role.
2.4 Unimodal-Likelihood Estimator

The concept of maximum-likelihood estimator (MLE) is classical and widely used in numerous areas. However, a MLE may not be unbiased and its associated likelihood function need not be monotone. For the purpose of developing a rigorous theory on coverage probability of random intervals, we shall introduce the concept of *unimodal-likelihood estimator* (ULE) in this paper. For samples \(X_1, \cdots, X_m\) of random length \(m\) with \(X_i\) parameterized by \(\theta\), we say that the estimator \(\varphi(X_1, \cdots, X_m)\) is a ULE of \(\theta\) if \(\varphi\) is a multivariate function such that, for any observation \((x_1, \cdots, x_m)\) of \((X_1, \cdots, X_m)\), the likelihood function is non-decreasing with respect to \(\theta\) no greater than \(\varphi(x_1, \cdots, x_m)\) and is non-increasing with respect to \(\theta\) no less than \(\varphi(x_1, \cdots, x_m)\).

For discrete random samples \(X_1, \cdots, X_m\), the associated likelihood function is \(\Pr\{X_i = x_i, i = 1, \cdots, m \mid \theta\}\). For continuous random samples \(X_1, \cdots, X_m\), the corresponding likelihood function is, \(f_{X_1, \cdots, X_m}(x_1, \cdots, x_m, \theta)\), the joint probability density function of random samples \(X_1, \cdots, X_m\). We emphasize that a MLE may not be a ULE and that a ULE may not be a MLE. In contrast to a MLE, a ULE can assume values not contained in the parameter space.

Clearly, for the cases that \(X\) is a Bernoulli or Poisson variable, \(\varphi(X_1, \cdots, X_m) = \sum_{i=1}^{m} x_i\) is a ULE of \(\theta\) at the \(\ell\)-th stage. As another illustration of ULE, consider the multistage inverse binomial sampling scheme described in Section 2.1. For \(\ell = 1, \cdots, s\), a ULE of \(p\) can be defined as \(\hat{p}_\ell = \frac{z_{\ell}}{n_\ell}\). At the termination of sampling, the estimator, \(\hat{p} = \hat{p}_s\), of \(p\) is also a ULE.

2.5 Principle of Construction of Sampling Schemes

In this subsection, we shall discuss the fundamental principle for the design of multistage sampling schemes. We shall address two critical problems:

(I) Determine sufficient conditions for a multistage sampling scheme such that the coverage probability \(\Pr\{L(\hat{\theta}, n) < \theta < U(\hat{\theta}, n) \mid \theta\}\) can be adjusted by a positive number \(\zeta\).

(II) For a given sampling scheme, determine whether the coverage probability \(\Pr\{L(\hat{\theta}, n) < \theta < U(\hat{\theta}, n) \mid \theta\}\) is no less than \(1 - \delta\) for any \(\theta \in \Theta\).

To describe our sampling schemes, define cumulative distribution functions (CDFs)

\[
F_{\hat{\theta}_\ell}(z, \theta) = \begin{cases} \Pr(\hat{\theta}_\ell \leq z \mid \theta) & \text{for } \theta \in \Theta, \\ 1 & \text{for } \theta < \theta_{\ell}, \\ 0 & \text{for } \theta > \theta_{\ell}, \end{cases} \quad \text{and} \quad G_{\hat{\theta}_\ell}(z, \theta) = \begin{cases} \Pr(\hat{\theta}_\ell \geq z \mid \theta) & \text{for } \theta \in \Theta, \\ 0 & \text{for } \theta < \theta_{\ell}, \\ 1 & \text{for } \theta > \theta_{\ell}, \end{cases}
\]

for \(\ell = 1, \cdots, s\), where \(\theta_{\ell}\) and \(\theta_{\ell}\) represent the infinimum and supremum of \(\theta \in \Theta\) respectively, and \(z\) assumes values in the support of \(\hat{\theta}_{\ell}\). For Theorem 2 and Corollary 11 to be presented in the sequel, we make a common assumption that the relevant random intervals satisfy \(\theta \leq L(\hat{\theta}_\ell, n_\ell) \leq U(\hat{\theta}_\ell, n_\ell) \subseteq \{L(\hat{\theta}_\ell, n_\ell) \leq \theta \}\) and \(\theta \leq U(\hat{\theta}_\ell, n_\ell) \leq U(\hat{\theta}_\ell, n_\ell) \subseteq \{U(\hat{\theta}_\ell, n_\ell) \leq \theta \}\) for \(\ell = 1, \cdots, s\).

Let \(\delta_{\ell} \in (0, 1), \ell = 1, \cdots, s\). For sampling schemes of structure described in Section 2.1, we have the following results on the coverage probability of random intervals.
Theorem 2 Suppose that a multistage sampling scheme satisfies the following requirements:

(i) For \( \ell = 1, \ldots, s \), \( \bar{\theta}_\ell \) is a ULE of \( \theta \).

(ii) For \( \ell = 1, \ldots, s \), \( \{ \mathcal{L}(\bar{\theta}_\ell, n_\ell) \leq \bar{\theta}_\ell \leq \mathcal{U}(\bar{\theta}_\ell, n_\ell) \} \) is a sure event.

(iii) \( \{ D_\ell = 1 \} \subseteq \{ F_{\bar{\theta}_\ell}(\bar{\theta}_\ell, \mathcal{U}(\bar{\theta}_\ell, n_\ell)) \leq \zeta \delta_\ell, \ G_{\bar{\theta}_\ell}(\bar{\theta}_\ell, \mathcal{L}(\bar{\theta}_\ell, n_\ell)) \leq \zeta \delta_\ell \} \) for \( \ell = 1, \ldots, s \).

(iv) \( \{ D_s = 1 \} \) is a sure event.

Then,

\[
\Pr\{ \mathcal{L}(\bar{\theta}, n) \geq \theta \mid \theta \} \leq \sum_{\ell=1}^{s} \Pr\{ \mathcal{L}(\bar{\theta}_\ell, n_\ell) \geq \theta, \ D_\ell = 1 \mid \theta \} \leq \zeta \sum_{\ell=1}^{s} \delta_\ell,
\]

\[
\Pr\{ \mathcal{U}(\bar{\theta}, n) \leq \theta \mid \theta \} \leq \sum_{\ell=1}^{s} \Pr\{ \mathcal{U}(\bar{\theta}_\ell, n_\ell) \leq \theta, \ D_\ell = 1 \mid \theta \} \leq \zeta \sum_{\ell=1}^{s} \delta_\ell,
\]

\[
\Pr\{ \mathcal{L}(\bar{\theta}, n) < \theta < \mathcal{U}(\bar{\theta}, n) \mid \theta \} \geq 1 - 2\zeta \sum_{\ell=1}^{s} \delta_\ell
\]

for any \( \theta \in \Theta \).

See Appendix [A] for a proof. Theorem 2 addresses the first problem posed at the beginning of this subsection. It tells how to define a stopping rule such that the coverage probability of the random interval can be bounded by a function of \( \zeta \) and \( \sum_{\ell=1}^{s} \delta_\ell \). If \( \sum_{\ell=1}^{s} \delta_\ell \) is bounded with respect to \( \zeta \), then, the coverage probability can be “tuned” to be no less than the prescribed level \( 1 - \delta \). This process is referred to as “coverage tuning”, which will be illustrated in details in the sequel. The intuition behind the definition of the stopping rule in Theorem 2 is as follows.

At the \( \ell \)-th stage, in order to determine whether the sampling should stop, two tests are performed based on the observations of \( \bar{\theta}_\ell \), \( \mathcal{L}(\bar{\theta}_\ell, n_\ell) \) and \( \mathcal{U}(\bar{\theta}_\ell, n_\ell) \), which are denoted by \( \vartheta_\ell \), \( L_\ell \) and \( U_\ell \) respectively. The first test is \( \mathcal{H}_0 : \theta < U_\ell \) versus \( \mathcal{H}_1 : \theta \geq U_\ell \), and the second test is \( \mathcal{H}_0' : \theta \leq L_\ell \) versus \( \mathcal{H}_1' : \theta > L_\ell \). Hypothesis \( \mathcal{H}_0 \) is accepted if \( F_{\bar{\theta}_\ell}(\vartheta_\ell, U_\ell) \leq \zeta \delta_\ell \), and is rejected otherwise. On the other side, hypothesis \( \mathcal{H}_0' \) is rejected if \( G_{\bar{\theta}_\ell}(\vartheta_\ell, L_\ell) \leq \zeta \delta_\ell \), and is accepted otherwise. If \( \mathcal{H}_0 \) is accepted and \( \mathcal{H}_0' \) is rejected, then, the decision variable \( D_\ell \) assumes value 1 and accordingly the sampling is terminated. Otherwise, \( D_\ell \) assumes value 0 and the sampling is continued. It can be seen that, if \( \zeta \delta_\ell \) is small, then \( \mathcal{H}_0 \) and \( \mathcal{H}_0' \) are accepted with high credibility and consequently, \( L_\ell < \theta < U_\ell \) is highly likely to be true. Therefore, by making \( \zeta \sum_{\ell=1}^{s} \delta_\ell \) sufficiently small, it is possible to ensure that the coverage probability of the random interval is above the desired level.

For simplicity of stopping boundary, we have established multistage sampling schemes by virtue of Theorem 2 as follows.

Corollary 1 Suppose that a multistage sampling scheme satisfies the following requirements:

(i) For \( \ell = 1, \ldots, s \), \( \bar{\theta}_\ell \) is a ULE of \( \theta \).

(ii) For \( \ell = 1, \ldots, s \), \( \mathbb{E}[e^{i \bar{\theta}_\ell}] \) exists for any real number \( t \).

(iii) For \( \ell = 1, \ldots, s \), \( \{ \mathcal{L}(\bar{\theta}_\ell, n_\ell) \leq \bar{\theta}_\ell \leq \mathcal{U}(\bar{\theta}_\ell, n_\ell) \} \) is a sure event.
Theorem 3

(iv) \( \{D_\ell = 1\} \subseteq \left\{ \mathcal{C}_\ell^+ \left( \hat{\theta}_\ell, \mathcal{L}(\hat{\theta}_\ell, n_\ell) \right) \leq \zeta \delta_\ell, \mathcal{C}_\ell^- \left( \hat{\theta}_\ell, \mathcal{U}(\hat{\theta}_\ell, n_\ell) \right) \leq \zeta \delta_\ell \right\} \) for \( \ell = 1, \ldots, s \), where \( \mathcal{C}_\ell^+ (.,.) \) and \( \mathcal{C}_\ell^- (.,.) \) are functions such that

\[
\mathcal{C}_\ell^+(z, \theta) = \begin{cases} 
\inf_{t > 0} e^{-tz} \mathbb{E}[e^{i\theta t}] & \text{for } \theta \in \Theta, \\
0 & \text{for } \theta < \underline{\theta}, \\
1 & \text{for } \theta > \bar{\theta} 
\end{cases}
\]

\[
\mathcal{C}_\ell^-(z, \theta) = \begin{cases} 
\inf_{t > 0} e^{-tz} \mathbb{E}[e^{i\theta t}] & \text{for } \theta \in \Theta, \\
1 & \text{for } \theta < \underline{\theta}, \\
0 & \text{for } \theta > \bar{\theta} 
\end{cases}
\]

(v) \( \{D_s = 1\} \) is a sure event.

Then,

\[
\Pr\{\mathcal{L}(\hat{\theta}, n) \geq \theta \mid \theta\} \leq \sum_{\ell=1}^s \Pr\{\mathcal{L}(\hat{\theta}_\ell, n_\ell) \geq \theta, D_\ell = 1 \mid \theta\} \leq \zeta \sum_{\ell=1}^s \delta_\ell,
\]

\[
\Pr\{\mathcal{U}(\hat{\theta}, n) \leq \theta \mid \theta\} \leq \sum_{\ell=1}^s \Pr\{\mathcal{U}(\hat{\theta}_\ell, n_\ell) \leq \theta, D_\ell = 1 \mid \theta\} \leq \zeta \sum_{\ell=1}^s \delta_\ell,
\]

\[
\Pr\{\mathcal{L}(\hat{\theta}, n) < \theta < \mathcal{U}(\hat{\theta}, n) \mid \theta\} \geq 1 - 2\zeta \sum_{\ell=1}^s \delta_\ell
\]

for any \( \theta \in \Theta \).

To establish Corollary 11 it suffices to show that the assumption (iv) of Corollary 11 implies the assumption (iii) of Theorem 2 which can be seen from Chernoff bounds

\[
F_{\hat{\theta}_\ell}(z, \theta) = \Pr\{\hat{\theta}_\ell \leq z \mid \theta\} \leq \inf_{t > 0} e^{tz} \mathbb{E}[e^{-i\theta t}] = \inf_{t \leq 0} e^{-tz} \mathbb{E}[e^{i\theta t}] = \mathcal{C}_\ell^-(z, \theta),
\]

\[
G_{\hat{\theta}_\ell}(z, \theta) = \Pr\{\hat{\theta}_\ell \geq z \mid \theta\} \leq \inf_{t > 0} e^{-tz} \mathbb{E}[e^{i\theta t}] = \mathcal{C}_\ell^+(z, \theta)
\]

for \( \theta \in \Theta \) and \( z \) assuming values from the support of \( \hat{\theta}_\ell \).

Now, we turn to consider the second problem posed at the beginning of this subsection. For the sampling schemes of structure described in Section 2.1 we have the following results regarding the coverage probability of random intervals.

**Theorem 3** Let \( X_1, X_2, \ldots \) be a sequence of identical samples of discrete random variable \( X \) parameterized by \( \theta \in \Theta \). For \( \ell = 1, \ldots, s \), let \( \hat{\theta}_\ell = \varphi(X_1, \ldots, X_{n_\ell}) \) be a ULE of \( \theta \). Define estimator \( \hat{\theta} = \hat{\theta}_l \), where \( l \) is the index of stage when the sampling is terminated. Let \( \mathcal{L}(.,.) \) and \( \mathcal{U}(.,.) \) be bivariate functions such that \( \{\mathcal{L}(\hat{\theta}, n) \leq \underline{\theta} \leq \mathcal{U}(\hat{\theta}, n)\} \) is a sure event. Let \( [a, b] \) be a subset of \( \Theta \). Let \( I_{\mathcal{L}} \) denote the intersection of \( [a, b] \) and the support of \( \mathcal{L}(\hat{\theta}, n) \). Let \( I_{\mathcal{U}} \) denote the intersection of \( [a, b] \) and the support of \( \mathcal{U}(\hat{\theta}, n) \). Let \( \mathcal{E} \) be an event determined by the random tuple \( (X_1, \ldots, X_{n_\ell}) \). The following statements hold true:

(I) Both \( \Pr\{\mathcal{L}(\hat{\theta}, n) \geq \theta \mid \theta\} \) and \( \Pr\{\mathcal{L}(\hat{\theta}, n) > \theta \mid \theta\} \) are no-decreasing with respect to \( \theta \) in any open interval with endpoints being consecutive distinct elements of \( I_{\mathcal{L}} \cup [a, b] \). Moreover, both the maximum of \( \Pr\{\mathcal{L}(\hat{\theta}, n) \geq \theta \mid \theta\} \) and
the supremum of \( \Pr\{\mathcal{L}(\hat{\theta}, n) > \theta \text{ and } \mathcal{E} \text{ occurs} \mid \theta \} \) with respect to \( \theta \in [a, b] \) are equal to the maximum of \( \Pr\{\mathcal{L}(\hat{\theta}, n) > \theta \text{ and } \mathcal{E} \text{ occurs} \mid \theta \} \) for \( \theta \in I_{\mathcal{L}} \cup \{a, b\} \).

(II) Both \( \Pr\{\mathcal{U}(\hat{\theta}, n) < \theta \text{ and } \mathcal{E} \text{ occurs} \mid \theta \} \) and \( \Pr\{\mathcal{U}(\hat{\theta}, n) > \theta \text{ and } \mathcal{E} \text{ occurs} \mid \theta \} \) are non-increasing with respect to \( \theta \) in any open interval with endpoints being consecutive distinct elements of \( I_{\mathcal{U}} \cup \{a, b\} \). Moreover, both the maximum of \( \Pr\{\mathcal{U}(\hat{\theta}, n) < \theta \text{ and } \mathcal{E} \text{ occurs} \mid \theta \} \) and the supremum of \( \Pr\{\mathcal{U}(\hat{\theta}, n) < \theta \text{ and } \mathcal{E} \text{ occurs} \mid \theta \} \) with respect to \( \theta \in [a, b] \) are equal to the maximum of \( \Pr\{\mathcal{U}(\hat{\theta}, n) < \theta \text{ and } \mathcal{E} \text{ occurs} \mid \theta \} \) for \( \theta \in I_{\mathcal{U}} \cup \{a, b\} \).

(III) If \( \{\mathcal{L}(\hat{\theta}, n) \geq a\} \subseteq \{\hat{\theta} \geq b\} \), then \( \Pr\{\mathcal{L}(\hat{\theta}, n) \geq a \text{ and } \mathcal{E} \text{ occurs} \mid \theta \} \leq \Pr\{\mathcal{L}(\hat{\theta}, n) \geq \theta \text{ and } \mathcal{E} \text{ occurs} \mid \theta \} \leq \Pr\{\mathcal{L}(\hat{\theta}, n) > a \text{ and } \mathcal{E} \text{ occurs} \mid \theta \} \) and \( \Pr\{\mathcal{L}(\hat{\theta}, n) > \theta \text{ and } \mathcal{E} \text{ occurs} \mid \theta \} \leq \Pr\{\mathcal{L}(\hat{\theta}, n) > a \text{ and } \mathcal{E} \text{ occurs} \mid \theta \} \) for any \( \theta \in [a, b] \). Similarly, if \( \{\mathcal{U}(\hat{\theta}, n) \leq b\} \subseteq \{\hat{\theta} \leq a\} \), then \( \Pr\{\mathcal{U}(\hat{\theta}, n) \leq b \text{ and } \mathcal{E} \text{ occurs} \mid \theta \} \leq \Pr\{\mathcal{U}(\hat{\theta}, n) \leq a \text{ and } \mathcal{E} \text{ occurs} \mid \theta \} \) and \( \Pr\{\mathcal{U}(\hat{\theta}, n) < a \text{ and } \mathcal{E} \text{ occurs} \mid \theta \} \leq \Pr\{\mathcal{U}(\hat{\theta}, n) < \theta \text{ and } \mathcal{E} \text{ occurs} \mid \theta \} \) for any \( \theta \in [a, b] \).

See Appendix C for a proof. In Theorem 3, we have used the concept of support in probability theory. The support of a random variable \( Z \) refers to \( \{Z(\omega) : \omega \in \Omega\} \), which is the set of all possible values of \( Z \).

Based on Theorem 3, in the special case that \( \mathcal{E} \) is a sure event, two different approaches can be developed to address the second problem proposed at the beginning of this subsection.

First, as a consequence of statements (I) and (II) of Theorem 3, it is true that \( \Pr\{\mathcal{L}(\hat{\theta}, n) < \theta < \mathcal{U}(\hat{\theta}, n) \mid \theta \} \geq 1 - \delta \) for any \( \theta \in [a, b] \) provided that

\[
\Pr\{\theta \leq \mathcal{L}(\hat{\theta}, n) \mid \theta \} \leq \frac{\delta}{2}, \quad \forall \theta \in I_{\mathcal{L}} \cup \{a, b\},
\]

\[
\Pr\{\theta \geq \mathcal{U}(\hat{\theta}, n) \mid \theta \} \leq \frac{\delta}{2}, \quad \forall \theta \in I_{\mathcal{U}} \cup \{a, b\}.
\]

As can be seen from the proofs of Theorems 1 and 2, under certain conditions, the probabilities \( \Pr\{\theta \leq \mathcal{L}(\hat{\theta}, n) \mid \theta \} \) and \( \Pr\{\theta \geq \mathcal{U}(\hat{\theta}, n) \mid \theta \} \) can be adjusted by \( \zeta \). Hence, it is possible to obtain appropriate value of \( \zeta \), without exhaustive evaluation of probabilities, such that \( \Pr\{\mathcal{L}(\hat{\theta}, n) < \theta < \mathcal{U}(\hat{\theta}, n) \mid \theta \} \geq 1 - \delta \) for any \( \theta \in [a, b] \).

Second, statements (III) and (IV) of Theorem 3 will be used to develop Adaptive Maximum Checking Algorithm in Section 3.3 to determine an appropriate value of coverage tuning parameter \( \zeta \).

In the special case that the number of stages \( s \) is equal to 1 and that the sample number is a deterministic integer \( n \), we have the following results.

**Theorem 4** Let \( X_1, X_2, \ldots, X_n \) be a sequence of identical samples of discrete random variable \( X \) which is parameterized by \( \theta \in \Theta \). Let \( \hat{\theta} = \varphi(X_1, \ldots, X_n) \) be an estimator of \( \theta \) such that \( \Pr\{\theta \leq \hat{\theta} \leq \bar{\theta} \mid \theta \} \) is a continuous and unimodal function of \( \theta \in \Theta \) for any numbers \( \underline{\theta} \) and \( \overline{\theta} \). Let \( \mathcal{L}(\cdot) \) and \( \mathcal{U}(\cdot) \) be functions such that there exist intervals \( I_{\theta} \) and \( I'_{\theta} \) of real numbers satisfying
\{ \mathcal{L}(\theta) \leq \theta \leq \mathcal{U}(\theta) \} = \{ \hat{\theta} \in I_0 \} \text{ and } \{ \mathcal{L}(\theta) < \theta < \mathcal{U}(\theta) \} = \{ \hat{\theta} \in I_0 \} \text{ for any } \theta \in [a, b], \text{ where [a, b] is a subset of } \Theta. \text{ Let } I_{\mathcal{L}} \text{ denote the intersection of [a, b] and the support of } \mathcal{L}(\hat{\theta}). \text{ Let } I_\mathcal{U} \text{ denote the intersection of [a, b] and the support of } \mathcal{U}(\hat{\theta}). \text{ The following statements hold true:}

(I) The minimum of \( \Pr \{ \mathcal{L}(\hat{\theta}) < \theta < \mathcal{U}(\hat{\theta}) \mid \theta \} \) with respect to \( \theta \in [a, b] \) is attained at the discrete set \{a, b\} \cup \mathcal{U} \cup \mathcal{L}; \text{ (II) The infimum of } \Pr \{ \mathcal{L}(\hat{\theta}) \leq \theta \leq \mathcal{U}(\hat{\theta}) \mid \theta \} \text{ with respect to } \theta \in [a, b] \text{ equals the minimum of the set } \{C(a), C(b) \} \cup \{C_U(\theta) : \theta \in \mathcal{U} \} \cup \{C_L(\theta) : \theta \in \mathcal{L} \}, \text{ where } C(\theta) = \Pr \{ \mathcal{L}(\hat{\theta}) \leq \theta \leq \mathcal{U}(\hat{\theta}) \mid \theta \}, \text{ C}_U(\theta) = \Pr \{ \mathcal{L}(\hat{\theta}) < \theta < \mathcal{U}(\hat{\theta}) \mid \theta \} \text{ and } C_L(\theta) = \Pr \{ \mathcal{L}(\hat{\theta}) < \theta < \mathcal{U}(\hat{\theta}) \mid \theta \}; \text{ (III) For both open and closed random intervals with lower limit } \mathcal{L}(\hat{\theta}) \text{ and upper limit } \mathcal{U}(\hat{\theta}), \text{ the coverage probability is continuous and unimodal for } \theta \in (\theta', \theta''), \text{ where } \theta' \text{ and } \theta'' \text{ are any two consecutive distinct elements of } \{a, b\} \cup \mathcal{U} \cup \mathcal{L}.

The proof of Theorem 4 can be found in [5].

2.6 Multistage Sampling without Replacement

It should be noted that the theories in preceding discussion can be applied to the multistage estimation of the proportion of a finite population, where the random samples are dependent if a sampling without replacement is used. Consider a population of \( N \) units, among which there are \( pN \) units having a certain attribute, where \( p \in \Theta = \{ \frac{M}{N} : M = 0, 1, \ldots, N \} \). In many situations, it is desirable to estimate the population proportion \( p \) by sampling without replacement. The procedure of sampling without replacement can be precisely described as follows:

Each time a single unit is drawn without replacement from the remaining population so that every unit of the remaining population has equal chance of being selected.

Such a sampling process can be exactly characterized by random variables \( X_1, \ldots, X_N \) defined in a probability space \((\Omega, \mathcal{F}, \Pr)\) such that \( X_i \) assumes value 1 if the \( i \)-th sample has the attribute and assumes value 0 otherwise. By the nature of the sampling procedure, it can be shown that

\[ \Pr \{ X_i = x_i, \ i = 1, \ldots, n \} = \frac{pN}{\sum_{i=1}^{n} x_i} \left( \frac{N - pN}{n - \sum_{i=1}^{n} x_i} \right) \left( \frac{\sum_{i=1}^{n} x_i}{n} \right) \]

for any \( n \in \{1, \ldots, N\} \) and any \( x_i \in \{0, 1\}, \ i = 1, \ldots, n \). Clearly, for any \( n \in \{1, \ldots, N\} \), the sample mean \( \frac{\sum_{i=1}^{n} X_i}{n} \) is unbiased but is not a MLE for \( p \in \Theta \). However, we have shown in Appendix D the following result:

**Theorem 5** For any \( n \in \{1, \ldots, N\} \), \( \frac{\sum_{i=1}^{n} X_i}{n} \) is a ULE for \( p \in \Theta \).

Based on random variables \( X_1, \ldots, X_N \), we can define a multistage sampling scheme in the same way as that of the multistage sampling described in Section 2.1. More specifically, we can define decision variables such that, for the \( \ell \)-th stage, \( D_\ell \) is a function of \( X_1, \cdots, X_n \), where the random variable \( n_\ell \) is the number of samples available at the \( \ell \)-th stage. For \( \ell = 1, \cdots, s \), an estimator of \( p \) at the \( \ell \)-stage can be defined as \( \hat{p}_\ell = \frac{\sum_{i=1}^{n_\ell} X_i}{n_\ell} \). Letting \( l \) be the index of stage when the sampling is terminated, we can define an estimator for \( p \) as \( \hat{p} = \hat{p}_l = \frac{\sum_{i=1}^{n} X_i}{n} \), where \( n = n_l \). 

16
is the sample size at the termination of sampling. A sampling scheme described in this setting is referred to as a multistage sampling without replacement in this paper. Regarding the coverage probability of random intervals, we have the following results which are direct consequence of Theorems 3 and 5.

**Corollary 2** Let $\mathcal{L}(\cdot,\cdot)$ and $\mathcal{U}(\cdot,\cdot)$ be bivariate functions such that $\{\mathcal{L}(\hat{p},n) \leq \hat{p} \leq \mathcal{U}(\hat{p},n)\}$ is a sure event and that both $N\mathcal{L}(\hat{p},n)$ and $N\mathcal{U}(\hat{p},n)$ are integer-valued random variables. Let $[a,b]$ be a subset of $\Theta$. Let $I_{\mathcal{L}}$ denote the intersection of $(a,b)$ and the support of $\mathcal{L}(\hat{p},n)$. Let $I_{\mathcal{U}}$ denote the intersection of $(a,b)$ and the support of $\mathcal{U}(\hat{p},n)$. The following statements hold true:

(I) $\Pr\{\mathcal{L}(\hat{p},n) \geq p \mid p\}$ is non-decreasing with respect to $p \in \Theta$ in any interval with endpoints being consecutive distinct elements of $I_{\mathcal{L}} \cup \{a,b\}$. Moreover, the maximum of $\Pr\{\mathcal{L}(\hat{p},n) \geq p \mid p\}$ with respect to $p \in [a,b]$ is achieved at $I_{\mathcal{L}} \cup \{a,b\}$.

(II) $\Pr\{\mathcal{U}(\hat{p},n) \leq p \mid p\}$ is non-increasing with respect to $p \in \Theta$ in any interval with endpoints being consecutive distinct elements of $I_{\mathcal{U}} \cup \{a,b\}$. Moreover, the maximum of $\Pr\{\mathcal{U}(\hat{p},n) \leq p \mid p\}$ with respect to $p \in [a,b]$ is achieved at $I_{\mathcal{U}} \cup \{a,b\}$.

(III) If $\{\mathcal{L}(\hat{p},n) \geq a\} \subseteq \{\hat{p} \geq b\}$, then $\Pr\{\mathcal{L}(\hat{p},n) \geq b \mid a\} \leq \Pr\{\mathcal{L}(\hat{p},n) \geq p \mid p\} \leq \Pr\{\mathcal{L}(\hat{p},n) \geq a \mid b\}$ for any $p \in [a,b]$. Similarly, if $\{\mathcal{U}(\hat{p},n) \leq b\} \subseteq \{\hat{p} \leq a\}$, then $\Pr\{\mathcal{U}(\hat{p},n) \leq a \mid b\} \leq \Pr\{\mathcal{U}(\hat{p},n) \leq p \mid p\} \leq \Pr\{\mathcal{U}(\hat{p},n) \leq b \mid a\}$ for any $p \in [a,b]$.

In the special case that the number of stages $s$ is equal to 1 and that the sample number is a deterministic integer $n$, we have the following results.

**Theorem 6** Let $[a,b]$ be a subset of $\Theta$. Suppose that $\mathcal{L}(\cdot)$ and $\mathcal{U}(\cdot)$ are non-decreasing functions such that both $N\mathcal{L}(\hat{p})$ and $N\mathcal{U}(\hat{p})$ are integer-valued random variables. Then, the minimum of $\Pr\{\mathcal{L}(\hat{p}) < p < \mathcal{U}(\hat{p}) \mid p\}$ with respect to $p \in [a,b]$ is attained at a discrete set $I_{UL}$ which is the union of $[a,b]$ and the supports of $\mathcal{L}(\hat{p})$ and $\mathcal{U}(\hat{p})$. Moreover, $\Pr\{\mathcal{L}(\hat{p}) < p < \mathcal{U}(\hat{p}) \mid p\}$ is unimodal with respect to $p$ in between consecutive distinct elements of $I_{UL}$.

The proof of Theorem 6 can be found in [5].

### 2.7 Asymptotically Unbiased Estimators of Mean Values

Some important distributions are determined by the mean values of associated random variables. Familiar examples are binomial distribution, Poisson distribution, normal distribution, and exponential distribution. To estimate the expectation, $\mu$, of a random variable $X$ based on i.i.d. samples $X_1, X_2, \cdots$, we can use a multistage sampling scheme with a structure described in Section 2.1. Specially, an estimator of $\mu$ can be defined as the sample mean $\hat{\mu} = \frac{\sum_{i=1}^{n} X_i}{n}$, where $n$ is the sample number at the termination of sampling. To justify that the estimator $\hat{\mu}$ is superior than other estimators, we shall show its asymptotic unbiasedness and relevant properties.

For a multistage sampling scheme with deterministic sample sizes $n_1 < n_2 < \cdots < n_s$, we have established the following general results.

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17
Theorem 7 Suppose that $\inf_{\ell>0} \frac{n_{\ell+1}}{n_{\ell}}$ is greater than 1. The following statements hold true.

(I) If $X$ has a finite variance, then $E[\hat{\mu} - \mu]$, $E|\hat{\mu} - \mu|$ and $E|\hat{\mu} - \mu|^2$ tend to 0 as the minimum sample size tends to infinity.

(II) If $X$ is a bounded random variable, then $E[\hat{\mu} - \mu]$ and $E|\hat{\mu} - \mu|^k$, $k = 1, 2, \cdots$ tend to 0 as the minimum sample size tends to infinity.

See Appendix E for a proof.

3 Computational Machinery

3.1 Bisection Coverage Tuning

To avoid prohibitive burden of computational complexity in the design process, we shall focus on a class of multistage sampling schemes for which the coverage probability can be adjusted by a single parameter $\zeta$. Such a parameter $\zeta$ is referred to as the coverage tuning parameter in this paper to convey the idea that $\zeta$ is used to “tune” the coverage probability to meet the desired confidence level. As will be seen in the sequel, we are able to construct a class of multistage sampling schemes such that the coverage probability can be “tuned” to ensure prescribed level of confidence by making the coverage tuning parameter sufficiently small. One great advantage of our sampling schemes is that the tuning can be accomplished by a bisection search method. To apply a bisection method, it is required to determine whether the coverage probability for a given $\zeta$ is exceeding the prescribed level of confidence. Such a task is explored in the following subsections.

3.2 Consecutive-Decision-Variable Bounding

One major problem in the design and analysis of multistage sampling schemes is the high-dimensional summation or integration involved in the evaluation of probabilities. For instance, a basic problem is to evaluate the coverage probabilities involving $\hat{\theta}$ and $n$. Another example is to evaluate the distribution or the expectation of sample number $n$. Clearly, $\hat{\theta}$ depends on random samples $X_1, \cdots, X_n$. Since the sample number $n$ can assume very large values, the computational complexity associated with the high-dimensionality can be a prohibitive burden to modern computers. In order to break the curse of dimensionality, we propose to obtain tight bounds for those types of probabilities. In this regard, we have

**Theorem 8** Let $W(.,.)$ be a bivariate function. Let $\mathcal{R}$ be a subset of real numbers. Then,

$$
\Pr\left\{ W(\hat{\theta}, n) \in \mathcal{R} \right\} \leq \sum_{\ell=1}^{n} \Pr\left\{ W(\hat{\theta}_\ell, n_\ell) \in \mathcal{R}, \ D_\ell = 1 \text{ and } D_j = 0 \text{ for } \max(1, \ell - r) \leq j < \ell \right\},
$$

$$
\Pr\left\{ W(\hat{\theta}, n) \in \mathcal{R} \right\} \geq 1 - \sum_{\ell=1}^{n} \Pr\left\{ W(\hat{\theta}_\ell, n_\ell) \notin \mathcal{R}, \ D_\ell = 1 \text{ and } D_j = 0 \text{ for } \max(1, \ell - r) \leq j < \ell \right\},
$$
for $0 \leq r < s$. Moreover,

$$\Pr\{l > \ell\} \leq \Pr\{D_\ell = 0, D_j = 0 \text{ for } \max(1, \ell - r) \leq j < \ell\},$$

$$\Pr\{l > \ell\} \geq 1 - \sum_{j=1}^{\ell} \Pr\{D_j = 1, D_i = 0 \text{ for } \max(1, j - r) \leq i < j\}$$

for $1 \leq \ell \leq s$ and $0 \leq r < s$. Furthermore, if the number of available samples at the $\ell$-th stage is a deterministic number $n_\ell$ for $1 \leq \ell \leq s$, then $E[n] = n_1 + \sum_{\ell=2}^{s-1} (n_\ell - n_{\ell-1}) \Pr\{l > \ell\}$.

See Appendix F for a proof. As can be seen from Theorem 8, the bounds are constructed by summing up probabilistic terms involving one or multiple consecutive decision variables (CDV). Such general technique is referred to as CDV bounding. A particular interesting special case of CDV method is to construct bounds with every probabilistic term involving consecutive decision variables (i.e., $r = 1$ in Theorem 8). Such method is referred to as double-decision-variable or DDV bounding for brevity. Similarly, the bounds with each probabilistic term involving a single decision variable are referred to as single-decision-variable bounds or SDV bounds (i.e., $r = 0$ in Theorem 8). Our computational experiences indicate that the bounds in Theorem 8 become very tight as the spacing between sample sizes increases. As can be seen from Theorem 8, DDV bounds are tighter than SDV bounds. Needless to say, the tightness of bounds is achieved at the price of computational complexity. The reason that such bounding methods allow for powerful dimension reduction is that, for many important estimation problems, $D_{\ell-1}$, $D_\ell$ and $\hat{\theta}_\ell$ can be expressed in terms of two independent variables $U$ and $V$. For instance, for the estimation of a binomial parameter, it is possible to design a multistage sampling scheme such that $D_{\ell-1}$, $D_\ell$ and $\hat{\theta}_\ell$ can be expressed in terms of $U = \sum_{i=1}^{n_{\ell-1}} X_i$ and $V = \sum_{i=n_{\ell-1}+1}^{n_\ell} X_i$. For the double decision variable method, it is evident that $U$ and $V$ are two independent binomial random variables and accordingly the computation of probabilities such as $\Pr\{\mathcal{W}(\hat{\theta}, n) \in \mathcal{R}\}$ and $\Pr\{l > \ell\}$ can be reduced to two-dimensional problems. Clearly, the dimension of these computational problems can be reduced to one if the single-decision-variable method is employed. As will be seen in the sequel, DDV bounds can be shown to be asymptotically tight for a large class of multistage sampling schemes. Moreover, our computational experiences indicate that SDV bounds are not very conservative.

### 3.3 Adaptive Maximum Checking

A wide class of computational problems depends on the following critical subroutine:

Determine whether a function $C(\theta)$ is smaller than a prescribed number $\delta$ for every value of $\theta$ in interval $[\underline{\theta}, \overline{\theta}]$.

In many situations, it is impossible or very difficult to evaluate $C(\theta)$ for every value of $\theta$ in interval $[\underline{\theta}, \overline{\theta}]$, since the interval may contain infinitely many or an extremely large number of values. To overcome such an issue of computational complexity, we shall propose an Adaptive
Maximum Checking Algorithm, abbreviated as AMCA, to determine whether the maximum of $C(\theta)$ over $[\theta, \overline{\theta}]$ is less than $\delta$. The only assumption required for our AMCA is that, for any interval $[a, b] \subseteq [\theta, \overline{\theta}]$, it is possible to compute an upper bound $\overline{C}(a, b)$ such that $C(\theta) \leq \overline{C}(a, b)$ for any $\theta \in [a, b]$ and that the upper bound converges to $C(\theta)$ as the interval width $b - a$ tends to 0.

Our backward AMCA proceeds as follows:

- Choose initial step size $d > \eta$.
- Let $F \leftarrow 0$, $T \leftarrow 0$ and $b \leftarrow \overline{\theta}$.
- While $F = T = 0$, do the following:
  - Let $st \leftarrow 0$ and $\ell \leftarrow 2$;
  - While $st = 0$, do the following:
    * Let $\ell \leftarrow \ell - 1$ and $d \leftarrow d2^\ell$.
    * If $b - d > \theta$, let $a \leftarrow b - d$ and $T \leftarrow 0$. Otherwise, let $a \leftarrow \theta$ and $T \leftarrow 1$.
    * If $\overline{C}(a, b) < \delta$, let $st \leftarrow 1$ and $b \leftarrow a$.
    * If $d < \eta$, let st $\leftarrow 1$ and $F \leftarrow 1$.
- Return $F$.

The output of our backward AMCA is a binary variable $F$ such that “$F = 0$” means “$C(\theta) < \delta$” and “$F = 1$” means “$C(\theta) \geq \delta$”. An intermediate variable $T$ is introduced in the description of AMCA such that “$T = 1$” means that the left endpoint of the interval is reached. The backward AMCA starts from the right endpoint of the interval (i.e., $b = \overline{\theta}$) and attempts to find an interval $[a, b]$ such that $\overline{C}(a, b) < \delta$. If such an interval is available, then, attempt to go backward to find the next consecutive interval with twice width. If doubling the interval width fails to guarantee $\overline{C}(a, b) < \delta$, then try to repeatedly cut the interval width in half to ensure that $\overline{C}(a, b) < \delta$. If the interval width becomes smaller than a prescribed tolerance $\eta$, then AMCA declares that “$F = 1$”. For our relevant statistical problems, if $C(\theta) \geq \delta$ for some $\theta \in [\theta, \overline{\theta}]$, it is sure that “$F = 1$” will be declared. On the other hand, it is possible that “$F = 1$” is declared even though $C(\theta) < \delta$ for any $\theta \in [\theta, \overline{\theta}]$. However, such situation can be made extremely rare and immaterial if we choose $\eta$ to be a very small number. Moreover, this will only introduce negligible conservativeness in the evaluation of coverage probabilities of random intervals if we choose $\eta$ to be sufficiently small (e.g., $\eta = 10^{-15}$).

To see the practical importance of AMCA in our statistical problems, consider the construction of a random interval with lower limit $\mathcal{L}(\hat{\theta}, n)$ and upper limit $\mathcal{U}(\hat{\theta}, n)$ such that $\Pr\{\mathcal{L}(\hat{\theta}, n) < \theta < \mathcal{U}(\hat{\theta}, n) \mid \theta\} > 1 - \delta$, or equivalently, $C(\theta) < \delta$ for any $\theta \in [\theta, \overline{\theta}]$, where $C(\theta) = \Pr\{\mathcal{L}(\hat{\theta}, n) \geq \theta \mid \theta\} + \Pr\{\mathcal{U}(\hat{\theta}, n) \leq \theta \mid \theta\}$ and $[\theta, \overline{\theta}]$ is a subset of $\Theta$. For our statistical problems, $C(\theta)$ is
dependent on the coverage tuning parameter $\zeta$. By choosing small enough $\zeta$, it is possible to ensure $C(\theta) < \delta$ for any $\theta \in [\hat{\theta}, \overline{\theta}]$. To avoid unnecessary conservativeness, it is desirable to obtain $\zeta$ as large as possible such that $C(\theta) < \delta$ for any $\theta \in [\hat{\theta}, \overline{\theta}]$. This can be accomplished by a computational approach. Clearly, an essential step is to determine, for a given value of $\zeta$, whether $C(\theta) < \delta$ holds for any $\theta \in [\hat{\theta}, \overline{\theta}]$. Here, $C(\theta)$ is defined as the complementary probability of coverage. To reduce computational complexity, $C(\theta)$ can be replaced by its upper bound derived from the consecutive-decision variable bounding method proposed in Section 3.2.

In the case that $\Theta$ is a discrete set, special care needs for $d$ to ensure that $a$ and $b$ are numbers in $\Theta$. The backward AMCA can be easily modified as forward AMCA.

### 3.4 Interval Bounding

By virtue of statements (III) and (IV) of Theorem 3, we have

\[
\begin{align*}
\Pr\{L(\hat{\theta}, n) \geq \theta \text{ or } U(\hat{\theta}, n) \leq \theta \mid \theta\} &\geq \Pr\{b \leq L(\hat{\theta}, n) \mid a\} + \Pr\{a \geq U(\hat{\theta}, n) \mid b\}, \quad (2) \\
\Pr\{L(\hat{\theta}, n) \geq \theta \text{ or } U(\hat{\theta}, n) \leq \theta \mid \theta\} &\leq \Pr\{a \leq L(\hat{\theta}, n) \mid b\} + \Pr\{b \geq U(\hat{\theta}, n) \mid a\} \quad (3)
\end{align*}
\]

for any $\theta \in [a, b]$ provided that

\[
\{a \leq L(\hat{\theta}, n)\} \subseteq \{\hat{\theta} \geq b\}, \quad \{b \geq U(\hat{\theta}, n)\} \subseteq \{\hat{\theta} \leq a\}. \quad (4)
\]

For many problems, if interval $[a, b]$ is narrow enough, then, condition (4) can be satisfied and the upper and lower bounds of $\Pr\{L(\hat{\theta}, n) \geq \theta \text{ or } U(\hat{\theta}, n) \leq \theta \mid \theta\}$ in (2) and (3) can be used to determine whether $\Pr\{L(\hat{\theta}, n) \geq \theta \text{ or } U(\hat{\theta}, n) \leq \theta \mid \theta\} \leq \delta$ for any $\theta \in [a, b]$. This suggests an alternative approach for constructing random intervals to guarantee prescribed confidence level for any $\theta \in [\hat{\theta}, \overline{\theta}]$, where $[\hat{\theta}, \overline{\theta}]$ is a subset of parameter space $\Theta$. The basis idea is as follows:

(i) Construct sampling scheme such that the probabilities $\Pr\{\theta \leq L(\hat{\theta}, n) \mid \theta\}$ and $\Pr\{\theta \geq U(\hat{\theta}, n) \mid \theta\}$ can be adjusted by $\zeta$.

(ii) Partition $[\hat{\theta}, \overline{\theta}]$ as small subintervals $[a, b]$ such that (2) and (3) can be used to determine whether $\Pr\{L(\hat{\theta}, n) \geq \theta \text{ or } U(\hat{\theta}, n) \leq \theta \mid \theta\} \leq \delta$ for any $\theta \in [a, b]$.

### 3.5 Recursive Computation

As will be seen in the sequel, for most multistage sampling plans with deterministic sample sizes $n_1, n_2, \cdots, n_s$ for estimating parameters of discrete variables, the probabilistic terms involving $\hat{\theta}$, $n$ or $\hat{\theta}_\ell$, $n_\ell$ can usually be expressed as a summation of terms $\Pr\{K_i \in \mathcal{K}_i, \ i = 1, \cdots, \ell\}$, $\ell = 1, \cdots, s$, where $K_\ell = \sum_{i=1}^{n_\ell} X_i$ and $\mathcal{K}_i$ is a subset of integers. The calculation of such terms can be performed by virtue of the following recursive relationship:

\[
\begin{align*}
\Pr\{K_i \in \mathcal{K}_i, \ i = 1, \cdots, \ell; \ K_{\ell+1} = k_{\ell+1}\} &= \sum_{k_\ell \in \mathcal{K}_\ell} \Pr\{K_i \in \mathcal{K}_i, \ i = 1, \cdots, \ell - 1; \ K_\ell = k_\ell\} \Pr\{K_{\ell+1} - K_\ell = k_{\ell+1} - k_\ell\}, \quad (5)
\end{align*}
\]
where the computation of probability \( \Pr\{K_{\ell+1} - K_{\ell} = k_{\ell+1} - k_{\ell}\} \) depends on specific estimation problems. For estimating a binomial parameter \( p \) with deterministic sample sizes \( n_1, n_2, \cdots, n_s \), we have

\[
\Pr\{K_{\ell+1} - K_{\ell} = k_{\ell+1} - k_{\ell}\} = \binom{n_{\ell+1} - n_{\ell}}{k_{\ell+1} - k_{\ell}} p^{k_{\ell+1} - k_{\ell}} (1 - p)^{n_{\ell+1} - n_{\ell} - k_{\ell+1} + k_{\ell}}.
\]

For estimating a Poisson parameter \( \lambda \) with deterministic sample sizes \( n_1, n_2, \cdots, n_s \), we have

\[
\Pr\{K_{\ell+1} - K_{\ell} = k_{\ell+1} - k_{\ell}\} = \frac{[(n_{\ell+1} - n_{\ell})\lambda]^{k_{\ell+1} - k_{\ell}} \exp(-(n_{\ell+1} - n_{\ell})\lambda)}{(k_{\ell+1} - k_{\ell})!}.
\]

For estimating the proportion, \( p \), of a finite population using multistage sampling schemes described in Section 2.6 we have

\[
\Pr\{K_{\ell+1} - K_{\ell} = k_{\ell+1} - k_{\ell}\} = \frac{\binom{pN - k_{\ell} \ell}{k_{\ell+1} - k_{\ell}} \binom{N - pN - k_{\ell} \ell}{n_{\ell+1} - n_{\ell} - k_{\ell+1} + k_{\ell}}}{\binom{N - n_{\ell+1}}{n_{\ell+1} - n_{\ell}}}, \tag{6}
\]

where the sample sizes are deterministic numbers \( n_1, n_2, \cdots, n_s \). It should be noted that such idea of recursive computation can be applied to general multistage sampling plans with random sample sizes \( n_1, n_2, \cdots, n_s \). Moreover, the domain truncation technique described in the next subsection can be used to significantly reduce computation.

### 3.6 Domain Truncation

The bounding methods described in the previous subsection reduce the computational problem of designing a multistage sampling scheme to the evaluation of low-dimensional summation or integration. Despite the reduction of dimensionality, the associated computational complexity is still high because the domain of summation or integration is large. The truncation techniques recently established in [7] have the power to considerably simplify the computation by reducing the domain of summation or integration to a much smaller subset. The following result, quoted from [7], shows that the truncation can be done with controllable error.

**Theorem 9** Let \( a_i, b_i, u_i, v_i, \alpha_i, \beta_i, i = 1, \cdots, m \) be real numbers. Suppose that \( \Pr\{Z_i < u_i\} \leq \alpha_i \) and \( \Pr\{Z_i > v_i\} \leq \beta_i \) for \( i = 1, \cdots, m \). Then, \( P' \leq \Pr\{a_i \leq Z_i \leq b_i, i = 1, \cdots, m\} \leq P' + \sum_{i=1}^{m} (\alpha_i + \beta_i) \), where \( P' = \Pr\{a'_i \leq Z_i \leq b'_i, i = 1, \cdots, m\} \) with \( a'_i = \max\{a_i, u_i\} \) and \( b'_i = \min\{b_i, v_i\} \) for \( i = 1, \cdots, m \).

As an example of using the truncation technique, consider probabilistic terms like \( \Pr\{\mathcal{W}(\bar{\theta}, n) \in \mathcal{R}\} \) involved in a multistage sampling scheme. If \( k_{\ell} \) and \( \bar{k}_{\ell} \) can be found such that \( \Pr\{\mathcal{W} \leq \bar{\theta}_{\ell} \leq \underbar{\theta}_{\ell}\} \geq 1 - \frac{\alpha}{s} \) for \( \ell = 1, \cdots, s \), then

\[
\Pr\{\mathcal{W}(\bar{\theta}, n) \in \mathcal{R}\} - \eta \leq \Pr\{\mathcal{W}(\bar{\theta}, n) \in \mathcal{R}, k_{\ell} \leq \bar{\theta}_{\ell} \leq \underbar{\theta}_{\ell}, \ell = 1, \cdots, s\} \leq \Pr\{\mathcal{W}(\bar{\theta}, n) \in \mathcal{R}\}.
\]

For most multistage sampling plans for estimating parameters of discrete variables, the probability \( \Pr\{\mathcal{W}(\bar{\theta}, n) \in \mathcal{R}, \bar{\theta}_{\ell} \leq \bar{\theta}_{\ell} \leq \underbar{\theta}_{\ell}, \ell = 1, \cdots, s\} \) can be evaluated recursively as described in Section 2.6.
3.7 Triangular Partition

As can be seen from the preceding discussion, by means of the double-decision-variable method, the design of multistage sampling schemes may be reduced to the evaluation of probabilities of the form \( \Pr\{ (U, V) \in \mathcal{G} \} \), where \( U \) and \( V \) are independent random variables, and \( \mathcal{G} = \{ (u, v) : a \leq u \leq b, \ c \leq v \leq d, \ e \leq u + v \leq f \} \) is a two-dimensional domain. It should be noted that such a domain can be fairly complicated. It can be an empty set or a polygon with 3 to 6 sides. Therefore, it is important to develop a systematic method for computing \( \Pr\{ (U, V) \in \mathcal{G} \} \). For this purpose, we have

**Theorem 10** Let \( a \leq b, \ c \leq d \) and \( e \leq f \). Let \( \overline{e} = \max\{ e, a + c \}, \ \overline{f} = \min\{ f, b + d \}, \ \underline{a} = \max\{ a, \overline{e} - d \}, \ \underline{f} = \min\{ b, \overline{f} - c \}, \ \underline{b} = \max\{ c, \overline{b} - e \} \) and \( \overline{a} = \min\{ d, \overline{f} - a \} \). Then, for any independent random variables \( U \) and \( V \),

\[
\Pr\{ (U, V) \in \mathcal{G} \} = \Pr\{ \underline{u} \leq U \leq \overline{a} \} \Pr\{ \underline{v} \leq V \leq \overline{b} \} - \Pr\{ U \leq \overline{a}, \ V \leq \overline{b}, \ U + V > \overline{f} \} - \Pr\{ U \geq \underline{u}, \ V \geq \underline{v}, \ U + V < \underline{e} \}.
\]

The goal of using Theorem 10 is to separate variables and thus reduce computation. As can be seen from Theorem 10, random variables \( U \) and \( V \) have been separated in the product and thus the dimension of the corresponding computation is reduced to one. The last two terms on the left side of equality are probabilities that \( (U, V) \) is included in rectangular triangles. The idea of separating variables can be repeatedly used by partitioning rectangular triangles as smaller rectangles and rectangular triangles. Specifically, if \( U \) and \( V \) are discrete random variables assuming integer values, we have

\[
\Pr\{ U \geq i, \ V \geq j, \ U + V \leq k \} = \Pr\{ i \leq U \leq \left\lfloor \frac{k - i + j}{2} \right\rfloor \} \Pr\{ j \leq V < \left\lfloor \frac{k - i + j}{2} \right\rfloor \} + \Pr\{ U > \left\lfloor \frac{k - i + j}{2} \right\rfloor, \ V \geq j, \ U + V \leq k \} + \Pr\{ U \geq i, \ V \geq \left\lfloor \frac{k - i + j}{2} \right\rfloor, \ U + V \leq k \} \quad (7)
\]

for integers \( i, \ j \) and \( k \) such that \( i + j \leq k \); and

\[
\Pr\{ U \leq i, \ V \leq j, \ U + V \geq k \} = \Pr\{ \left\lfloor \frac{k + i - j}{2} \right\rfloor \leq U \leq \left\lfloor \frac{k - i + j}{2} \right\rfloor \} \Pr\{ \left\lfloor \frac{k - i + j}{2} \right\rfloor < V \leq j \} + \Pr\{ U \leq i, \ V \leq \left\lfloor \frac{k - i + j}{2} \right\rfloor, \ U + V \geq k \} + \Pr\{ U < \left\lfloor \frac{k + i - j}{2} \right\rfloor, \ V \leq j, \ U + V \geq k \} \quad (8)
\]

for integers \( i, \ j \) and \( k \) such that \( i + j \geq k \). It is seen that the terms in (7) and (8) correspond to probabilities that \( (U, V) \) is included in rectangular triangles. Hence, the above method of triangular partition can be repeatedly applied. For the sake of efficiency, we can save the probabilities that \( U \) and \( V \) are respectively included in the intervals corresponding to the rectangular sides of a parent triangle, then when partitioning this triangle, it suffices to compute the probabilities that \( U \) and \( V \) are included in the intervals corresponding to two orthogonal sides of the smaller rectangle. The probabilities that \( U \) and \( V \) are included in the intervals corresponding to the rectangular
sides of the smaller triangles can be readily obtained from the results of the smaller rectangle and
the record of the probabilities for the parent triangle. This trick can be repeatedly used to save
computation.

Since a crucial step in designing a sampling scheme is to compare the coverage probability
with a prescribed level of confidence, it is useful to compute upper and lower bounds of the
probabilities that \( U \) and \( V \) are covered by a triangular domain. As the triangular partition
goes on, the rectangular triangles become smaller and smaller. Clearly, the upper bounds of the
probabilities that \((U, V)\) is included in rectangular triangles can be obtained by inequalities

\[
\Pr\{U \geq i, V \geq j, U + V \leq k\} \leq \Pr\{i \leq U \leq k - j\} \Pr\{j \leq V \leq k - i\},
\]

\[
\Pr\{U \leq i, V \leq j, U + V \geq k\} \leq \Pr\{k - j \leq U \leq i\} \Pr\{k - i \leq V \leq j\}.
\]

Of course, the lower bounds can be taken as 0. As the triangular partition goes on, the rectangular
triangles become smaller and smaller and accordingly such bounds becomes tighter. To avoid the
exponential growth of the number of rectangular triangles, we can split the rectangular triangle
with the largest gap between upper and lower bounds in every triangular partition.

### 3.8 Interval Splitting

In the design of sampling schemes and other applications, it is a frequently-used routine to evaluate
the probability that a random variable is bounded in an interval. Note that, for most basic random
variables, the probability mass (or density) functions \( f(.) \) possess nice concavity or convexity
properties. In many cases, we can readily compute inflexion points which can be used to partition
the interval as subintervals such that \( f(.) \) is either convex or concave in each subinterval. By
virtue of concavity or convexity, we can calculate the upper and lower bounds of the probability
that the random variable is included in a subinterval. The overall upper and lower bounds of
the probability that the random variable is included in the initial interval can be obtained by
summing up the upper and lower bounds for all subintervals respectively. The gap between the
overall upper and lower bounds can be reduced by repeatedly partitioning the subinterval with
the largest gap of upper and lower bounds. This strategy is referred to as interval splitting in this
paper.

For a discrete random variable with probability mass function \( f(k) \), we can apply the following
result to compute upper and lower bounds of \( \sum_{k=a}^{b} f(k) \) over subinterval \([a, b]\).

**Theorem 11** Let \( a < b \) be two integers. Define \( r_a = \frac{f(a+1)}{f(a)} \), \( r_b = \frac{f(b-1)}{f(b)} \), \( r_{a,b} = \frac{f(a)}{f(b)} \) and \( j = a + \frac{b-a-(1-r_{a,b})(1-r_a)^{-1}}{1+r_{a,b}(1-r_a)(1-r_b)^{-1}} \). Define \( \alpha(i) = (i+1-a)\left[1 + \frac{(i-a)(a-1)}{2}\right] \) and \( \beta(i) = (b-i)\left[1 + \frac{(b-i-1)(a-1)}{2}\right] \).

The following statements hold true:

1. If \( f(k+1) - f(k) \leq f(k) - f(k-1) \) for \( a < k < b \), then

\[
\frac{(b - a + 1)[f(a) + f(b)]}{2} \leq \sum_{k=a}^{b} f(k) \leq \alpha(i)f(a) + \beta(i)f(b)
\]
for $a < i < b$. The minimum gap between the lower and upper bounds is achieved at $i$ such that $\lfloor j \rfloor \leq i \leq \lceil j \rceil$.

($II$): If $f(k+1) - f(k) \geq f(k) - f(k-1)$ for $a < k < b$, then

$$\frac{(b-a+1)[f(a) + f(b)]}{2} \geq \sum_{k=a}^{b} f(k) \geq \alpha(i)f(a) + \beta(i)f(b)$$

for $a < i < b$. The minimum gap between the lower and upper bounds is achieved at $i$ such that $\lfloor j \rfloor \leq i \leq \lceil j \rceil$.

See Appendix G for a proof. For a continuous random variable with probability density function $f(x)$, we can apply the following result to compute upper and lower bounds of $\int_{a}^{b} f(x)dx$ over subinterval $[a, b]$.

**Theorem 12** Suppose $f(x)$ is differentiable over interval $[a, b]$. The following statements hold true:

($I$): If $f(x)$ is concave over $[a, b]$, then $\frac{f(a)+f(b)}{2} - \frac{f(b)-f(a)}{b-a} \leq \int_{a}^{b} f(x)dx \leq \frac{f(a)+f(b)}{2} + \Delta(t)$, where $\Delta(t) = \left[f'(a) - \frac{f(b)-f(a)}{b-a}\right] \frac{(t-a)^2}{2} - \left[f'(b) - \frac{f(b)-f(a)}{b-a}\right] \frac{(b-t)^2}{2}$.

($II$): If $f(x)$ is convex over $[a, b]$, then $\frac{f(a)+f(b)}{2} - \Delta(t) \leq \int_{a}^{b} f(x)dx \leq \frac{f(a)+f(b)}{2} + \frac{f(a)+f(b)-f(b)}{f'(a) - f'(b)}$.

The minimum of $\Delta(t)$ is achieved at $t = \frac{f'(a) - f'(b)}{f'(a) - f'(b)}$.

See Appendix H for a proof.

### 3.9 Factorial Evaluation

In the evaluation of the coverage probability of a sampling scheme, a frequent routine is the computation of the logarithm of the factorial of an integer. To reduce computational complexity, we can develop a table of $\ln(n!)$ and store it in computer for repeated use. Such a table can be readily made by the recursive relationship $\ln((n+1)!)$ = $\ln(n+1) + \ln(n!)$. Modern computers can easily support a table of $\ln(n!)$ of size in the order of $10^7$ to $10^8$, which suffices most needs of our computation. Another method to calculate $\ln(n!)$ is to use the following double-sized bounds:

$$\ln(\sqrt{2\pi n} n^n) - n + \frac{1}{12n} - \frac{1}{360n^3} < \ln(n!) < \ln(\sqrt{2\pi n} n^n) - n + \frac{1}{12n} - \frac{1}{360n^3} + \frac{1}{1260n^5}$$

for all $n \geq 1$. A proof for such bounds can be available in pages 481-482 of [20].

### 4 Estimation of Binomial Parameters

Let $X$ be a Bernoulli random variable with distribution $\Pr\{X = 1\} = 1 - \Pr\{X = 0\} = p \in (0, 1)$. In this section, we shall consider the multistage estimation of binomial parameter $p$, in the general framework proposed in Section 2.1 based on i.i.d. random samples $X_1, X_2, \ldots$ of $X$. 

25
To describe our estimation methods, we shall introduce the following notations, which will be used throughout this section.

Define $K_\ell = \sum_{i=1}^{n_\ell} X_i$ and $\hat{p}_\ell = \frac{K_\ell}{n_\ell}$ for $\ell = 1, \cdots, s$, where $n_\ell$ is the number of samples available at the $\ell$-th stage. Specially, if the sample sizes are deterministic numbers $n_1 < n_2 < \cdots < n_s$, then $n_\ell = n_\ell$ for $\ell = 1, \cdots, s$. As described in Section 2.1, the stopping rule is that sampling is continued until $D_\ell = 1$ for some $\ell \in \{1, \cdots, s\}$, where $D_\ell$ is the decision variable for the $\ell$-th stage. Let $\hat{p} = \frac{\sum_{i=1}^{n_1} X_i}{n}$, where $n$ is the sample size when the sampling is terminated. Clearly, $\hat{p} = \hat{p}_l$ and $n = n_l$, where $l$ is the index of stage when the sampling is terminated. As mentioned before, the number of stage, $s$, can be a finite number or infinity.

In the development of our multistage sampling schemes, we need to use the following probability inequalities related to bounded variables.

**Lemma 1** Let $\bar{X}_n = \frac{\sum_{i=1}^{n} X_i}{n}$, where $X_1, \cdots, X_n$ are i.i.d. random variables such that $0 \leq X_i \leq 1$ and $\mathbb{E}[X_i] = \mu \in (0, 1)$ for $i = 1, \cdots, n$. Then,

$$
\Pr\{\bar{X}_n \geq z\} \leq \exp(n \cdot \mathcal{M}_B(z, \mu)) < \exp(n \cdot \mathcal{M}(z, \mu))
$$

for any $z \in (\mu, 1)$. Similarly,

$$
\Pr\{\bar{X}_n \leq z\} \leq \exp(n \cdot \mathcal{M}_B(z, \mu)) < \exp(n \cdot \mathcal{M}(z, \mu))
$$

for any $z \in (0, \mu)$.

Inequalities (10) and (12) are classical results established by Hoeffding in 1963 (see, [27]). Inequalities (11) and (13) are recent results due to Massart [29]. In this paper, (10) and (12) are referred to as Hoeffding’s inequalities. Similarly, (11) and (13) are referred to as Massart’s inequalities. If $X_i$ are i.i.d. Bernoulli random variables, then it can be shown that $\exp(n \cdot \mathcal{M}_B(z, \mu)) = \inf_{t > 0} e^{-tz} \mathbb{E}[e^{t \bar{X}_n}]$, which implies that (10) and (12) are actually Chernoff bounds in the special case.

### 4.1 Control of Absolute Error

In this subsection, we shall propose multistage sampling schemes for estimating $p$ with an absolute error criterion. Specifically, for margin of absolute error $\varepsilon \in (0, \frac{1}{2})$, we want to design a multistage sampling scheme such that the estimator $\hat{p}$ satisfies the requirement that $\Pr\{|\hat{p} - p| < \varepsilon \mid p \} > 1 - \delta$ for any $p \in (0, 1)$.

#### 4.1.1 Stopping Rules from CDFs, Chernoff Bounds and Massart’s Inequality

To construct an estimator satisfying an absolute error criterion with a prescribed confidence level, we propose three types of multistage sampling schemes with different stopping rules as follows.
Stopping Rule (i): For \( \ell = 1, \cdots, s \), decision variable \( D_\ell \) assumes value 1 if \( F_{\hat{p}_\ell}(\hat{p}_\ell, \hat{p}_\ell - \varepsilon) \leq \zeta \delta, G_{\hat{p}_\ell}(\hat{p}_\ell, \hat{p}_\ell - \varepsilon) \leq \zeta \delta \); and assumes value 0 otherwise.

Stopping Rule (ii): For \( \ell = 1, \cdots, s \), decision variable \( D_\ell \) assumes value 1 if \( \mathbb{M}_B(\frac{1}{2} - \frac{1}{2} - \hat{p}_\ell) \leq \frac{\ln(\delta)}{n_\ell} \); and assumes value 0 otherwise.

Stopping Rule (iii): For \( \ell = 1, \cdots, s \), decision variable \( D_\ell \) assumes value 1 if \( (|\hat{p}_\ell - \frac{1}{2}| - \frac{2\varepsilon}{n_\ell})^2 \geq \frac{1}{4} + \frac{2^2}{\sigma^2 \ln(\delta)} \); and assumes value 0 otherwise.

Stopping rule (i) is derived by virtue of the CDFs of \( \hat{p}_\ell \). Stopping rule (ii) is derived by virtue of Chernoff bounds of the CDFs of \( \hat{p}_\ell \). Stopping rule (iii) is derived by virtue of Massart’s inequality for the CDFs of \( \hat{p}_\ell \).

For the above three types of multistage sampling schemes, we have the following results.

**Theorem 13** Suppose that the sample size at the \( s \)-th stage is no less than \( \left\lfloor \frac{\ln \frac{1}{2 \varepsilon^2}}{1 \varepsilon^2} \right\rfloor \). Then,

\[
\Pr\{p \leq \hat{p} - \varepsilon \mid p\} \leq \sum_{\ell=1}^{s} \Pr\{p \leq \hat{p}_\ell - \varepsilon, D_\ell = 1 \mid p\} \leq s\zeta \delta,
\]

\[
\Pr\{p \geq \hat{p} + \varepsilon \mid p\} \leq \sum_{\ell=1}^{s} \Pr\{p \geq \hat{p}_\ell + \varepsilon, D_\ell = 1 \mid p\} \leq s\zeta \delta
\]

and \( \Pr\{|\hat{p} - p| \leq \varepsilon \mid p\} \geq 1 - 2s\zeta \delta \) for any \( p \in (0, 1) \).

See Appendix 14 for a proof.

For stopping rules derived from CDFs or Chernoff bounds, we can choose the smallest sample sizes and the largest sample sizes based on the criteria proposed in Section 2.1 such that \( n_1 \geq \frac{\ln(\delta)}{\ln(1 - \varepsilon)} \) and \( n_s \geq \frac{\ln \frac{1}{2 \varepsilon^2}}{1 \varepsilon^2} \). Specifically, the sample sizes \( n_1 < n_2 < \cdots < n_s \) can be be chosen as the ascending arrangement of all distinct elements of

\[
\left\{ \left\lfloor \frac{C_{\tau - \ell} \ln \frac{1}{\varepsilon^2}}{2 \varepsilon^2} \right\rfloor : \ell = 1, \cdots, \tau \right\}, \tag{14}
\]

where \( \tau \) is the maximum integer such that \( \frac{C_{\tau - 1} \ln \frac{1}{\varepsilon^2}}{2 \varepsilon^2} \geq \frac{\ln(\delta)}{\ln(1 - \varepsilon)} \), i.e., \( C_{\tau - 1} \geq \frac{2 \varepsilon^2}{\ln \frac{1}{\varepsilon^2}} \). In a similar manner, for stopping rules derived from Massart’s inequality, the sample sizes \( n_1 < n_2 < \cdots < n_s \) can be defined as \( \{14\} \) with \( \tau \) chosen as the maximum integer such that \( \frac{C_{\tau - 1} \ln \frac{1}{\varepsilon^2}}{2 \varepsilon^2} \geq \frac{24 \varepsilon - 16 \varepsilon^2}{9} \), i.e., \( C_{\tau - 1} \geq \frac{24 \varepsilon - 16 \varepsilon^2}{9} \).

For above sampling methods of choosing sample sizes, we have \( \Pr\{|\hat{p} - p| \leq \varepsilon \mid p\} > 1 - \delta \) for any \( p \in (0, 1) \) if \( \zeta < \frac{1}{2 \varepsilon^2} \), where \( \tau \) is independent of \( \delta \). Hence, we can determine a value of \( \zeta \) as large as possible such that \( \Pr\{|\hat{p} - p| \leq \varepsilon \mid p\} > 1 - \delta \) by virtue of the computational machinery described in Section 15.

To evaluate the coverage probability associated with the stopping rule derived from Chernoff bounds, we need to express events \( \{D_\ell = i\}, \ i = 0,1 \) in terms of \( K_\ell \). This can be accomplished by using the following results.
Theorem 14 Let \( z^* \) be the unique solution of equation \( \ln \frac{(z^* + \varepsilon)(1-z^*)}{(z^* + \varepsilon)(1-z^*)} = \frac{\ln (\zeta \delta)}{n_\ell} \) with respect to \( z \in (\frac{1}{2} - \varepsilon, \frac{1}{2}) \). Let \( n_\ell \) be a sample size smaller than \( \frac{\ln (\zeta \delta)}{M_B(z^*, z^* + \varepsilon)} \). Let \( \bar{z} \) be the unique solution of equation \( M_B(z, z + \varepsilon) = \frac{\ln (\zeta \delta)}{n_\ell} \) with respect to \( z \in [0, z^*] \). Let \( z \) be the unique solution of equation \( M_B(z, z + \varepsilon) = \frac{\ln (\zeta \delta)}{n_\ell} \) with respect to \( z \in (z^*, 1 - \varepsilon) \). Then, \( \{D_\ell = 0\} = \{n_\ell \bar{z} < K_\ell < n_\ell \} \cup \{n_\ell (1 - \bar{z}) < K_\ell < n_\ell (1 - z)\} \).

See Appendix I.2 for a proof.

4.1.2 Asymptotic Stopping Rule

It should be noted that, for a small \( \varepsilon \), we can simplify, by using Taylor’s series expansion formula \( \ln(1 + x) = x - \frac{x^2}{2} + o(x^2) \), the sampling schemes described in Section 4.1.1 as follows:

(i) The sequence of sample sizes \( n_1, \ldots, n_s \) is defined as the ascending arrangement of all distinct elements of \( \left\{ \left\lceil \frac{C_\tau - \ell \ln 1/\zeta \delta}{2\varepsilon^2} \right\rceil : \ell = 1, \ldots, \tau \right\} \), where \( \tau \) is the maximum integer such that \( C_\tau - 1 \geq 2\varepsilon \).

(ii) The decision variables are defined such that \( D_\ell = 1 \) if \( n_\ell \geq \frac{\hat{p}_\ell (1 - \hat{p}_\ell) 2 \ln \frac{1}{\zeta \delta}}{\varepsilon^2} \); and \( D_\ell = 0 \) otherwise.

For such a simplified sampling scheme, we have

\[
\sum_{\ell=1}^{s} \Pr \{ |\hat{p}_\ell - p| \geq \varepsilon, D_\ell = 1 \} \leq \sum_{\ell=1}^{s} \Pr \{ |\hat{p}_\ell - p| \geq \varepsilon \} \leq \sum_{\ell=1}^{\tau} \Pr \{ |\hat{p}_\ell - p| \geq \varepsilon \}
\]

\[
\leq \sum_{\ell=1}^{\tau} 2 e^{-2n_\ell \varepsilon^2} \quad \text{(15)}
\]

\[
< 2 \tau e^{-2n_1 \varepsilon^2} \leq 2 \tau \exp \left( -2 \varepsilon \ln \frac{1}{\zeta \delta} \right), \quad \text{(16)}
\]

where (15) is due to the Chernoff bound. As can be seen from (16), the last bound is independent of \( p \) and can be made smaller than \( \delta \) if \( \zeta \) is sufficiently small. This establishes the claim and it follows that \( \Pr \{ |\hat{p} - p| < \varepsilon \mid p \} > 1 - \delta \) for any \( p \in (0, 1) \) if \( \zeta \) is sufficiently small.

4.1.3 Asymptotic Analysis of Sampling Schemes

In this subsection, we shall focus on the asymptotic analysis of multistage sampling schemes. Throughout this subsection, we assume that the multistage sampling schemes follow stopping rules derived from Chernoff bounds as described in Section 4.1.1. Moreover, we assume that the sample sizes \( n_1, \ldots, n_s \) are chosen as the ascending arrangement of all distinct elements of the set defined by (14).

With regard to the tightness of the DDV bound, we have
Theorem 15 Let $\mathcal{R}$ be a subset of real numbers. Define

$$
\mathcal{P} = \sum_{\ell=1}^{n} \Pr(\hat{p}_\ell \in \mathcal{R}, \mathcal{D}_{\ell-1} = 0, \mathcal{D}_\ell = 1), \quad \mathcal{P} = 1 - \sum_{\ell=1}^{n} \Pr(\hat{p}_\ell \notin \mathcal{R}, \mathcal{D}_{\ell-1} = 0, \mathcal{D}_\ell = 1).
$$

Then, $\mathcal{P} \leq \Pr(\hat{p} \in \mathcal{R}) \leq \mathcal{P}$ and $\lim_{\varepsilon \to 0} |\Pr(\hat{p} \in \mathcal{R}) - \mathcal{P}| = \lim_{\varepsilon \to 0} |\Pr(\hat{p} \in \mathcal{R}) - \mathcal{P}| = 0$ for any $p \in (0, 1)$.

See Appendix I.3 for a proof.

For $\rho > 0$, $d > 0$, $0 < \nu < 1$, define

$$
\Psi(\rho, \nu, d) = \frac{1}{2\pi} \left[ \int_{-\phi_L}^{\phi_U} \exp \left(-\frac{\nu^2 d^2}{2 \cos^2 \phi} \right) d\phi + \int_{\phi_U}^{\phi_L - \phi_p} \exp \left(-\frac{d^2}{2 \cos^2 \phi} \right) d\phi \right]
$$

with $\phi_p = \arctan(\sqrt{\rho})$, $\phi_L = \arctan\left(\frac{1}{\nu} \sqrt{1 + 1 + \frac{1}{\sqrt{\rho}}} \right)$ and $\phi_U = \arctan\left(\frac{1}{\nu} \sqrt{1 + 1 - \frac{1}{\sqrt{\rho}}} \right)$. With regard to the asymptotic performance of the sampling scheme, we have

Theorem 16 Let $N_n(p, \varepsilon) = \mathcal{N}_n\left(\frac{n \ln(\frac{\delta}{\epsilon})}{\sqrt{\ln(\frac{\delta}{\epsilon})} - \ln(\frac{\delta}{\epsilon})}\right) - p \right) > \varepsilon \left|p \right| \mathcal{P}$. Let $\mathcal{N}_1(p, \varepsilon)$ be the minimum sample number $n$ such that $\Pr\left\{\frac{\sum_{i=1}^{\infty} X_i}{n} - p < \varepsilon \left| p \right| \mathcal{P} \right\} > 1 - \zeta \delta$ for a fixed-size sampling procedure. Let $j_p$ be the maximum integer $j$ such that $C_j \geq 4p(1-p)$. Let $\nu = \frac{2}{3}$, $d = \sqrt{2 \ln \frac{1}{\zeta \delta}}$ and $\kappa_p = \frac{C_j}{4p(1-p)}$. Let $\rho_p = \frac{C_{j_p-1}}{4p(1-p)} - 1$ for $\kappa_p = 1$, $j_p > 0$ and $\rho_p = \kappa_p - 1$ otherwise. The following statements hold true:

(I): $\Pr\left\{1 \leq \limsup_{\varepsilon \to 0} \frac{n}{N_n(p, \varepsilon)} \leq 1 + \rho_p \right\} = 1$. Specially, $\Pr\left\{\lim_{\varepsilon \to 0} \frac{n}{N_n(p, \varepsilon)} = \kappa_p \right\} = 1$ if $\kappa_p > 1$.

(II): $\lim_{\varepsilon \to 0} \frac{\mathbb{E}[n]}{N_n(p, \varepsilon)} = \left(\frac{d}{\sqrt{\zeta \delta}}\right)^2 \times \lim_{\varepsilon \to 0} \frac{\mathbb{E}[n]}{N_n(p, \varepsilon)}$, where

$$
\lim_{\varepsilon \to 0} \frac{\mathbb{E}[n]}{N_n(p, \varepsilon)} = \begin{cases} 
\kappa_p & \text{if } \kappa_p > 1, \\
1 + \rho_p \Phi(\nu d) & \text{otherwise}
\end{cases}
$$

and $1 \leq \lim_{\varepsilon \to 0} \frac{\mathbb{E}[n]}{N_n(p, \varepsilon)} \leq 1 + \rho_p$.

(III): If $\kappa_p > 1$, then $\lim_{\varepsilon \to 0} \Pr\{|\hat{p} - p| < \varepsilon \} = 2 \Phi(\epsilon_{\sqrt{\kappa_p}}) - 1 > 2 \Phi(d) - 1 > 1 - 2\zeta \delta$. Otherwise, $2 \Phi(d) - 1 > \lim_{\varepsilon \to 0} \Pr\{|\hat{p} - p| < \varepsilon \} = 1 + \Phi(d) - \Phi(\nu d) - \Psi(\rho_p, \nu, d) > 3 \Phi(d) - 2 > 1 - 3\zeta \delta$.

See Appendix I.4 for a proof.

4.2 Control of Relative Error

In this section, we shall focus on the design of multistage sampling schemes for estimating the binomial parameter $p$ with a relative error criterion. Specifically, for $\varepsilon \in (0, 1)$, we wish to construct a multistage sampling scheme and its associated estimator $\hat{p}$ for $p$ such that $\Pr\{|\hat{p} - p| < \varepsilon \left| p \right| \mathcal{P} \} > 1 - \delta$ for any $p \in (0, 1)$. 

29
4.2.1 Multistage Inverse Sampling

In this subsection, we shall develop multistage sampling schemes, of which the number of stages, s, is a finite number. Let \( \gamma_1 < \gamma_2 < \cdots < \gamma_s \) be a sequence of positive integers. The number, \( \gamma_\ell \), is referred to as the threshold of sample sum of the \( \ell \)-th stage. For \( \ell = 1, \cdots, s \), let \( \hat{p}_\ell = \frac{n_\ell}{n} \), where \( n_\ell \) is the minimum number of samples such that \( \sum_{i=1}^{n_\ell} X_i = \gamma_\ell \). As described in Section 2.2, the stopping rule is that sampling is continued until \( D_\ell = 1 \) for some \( \ell \in \{1, \cdots, s\} \), where \( D_\ell \) is the decision variable for the \( \ell \)-th stage. Define estimator \( \hat{p} = \frac{\sum_{i=1}^{n_i} X_i}{n} \), where \( n \) is the sample size when the sampling is terminated.

The rationale for choosing \( \hat{p} \) as an estimator for \( p \) can be illustrated by the following results.

Theorem 17 Suppose that \( \inf_{\ell>0} \frac{\gamma_{\ell+1}}{\gamma_\ell} > 1 \). Then \( \mathbb{E}[\hat{p} - p] \) and \( \mathbb{E}[\hat{p} - p|^k \), \( k = 1, 2, \cdots \) tend to 0 as the minimum threshold of sample sum tends to infinity.

See Appendix [5] for a proof.

By virtue of the CDFs of \( \hat{p}_\ell \), we propose a class of multistage sampling schemes as follows.

Theorem 18 Suppose that, for \( \ell = 1, \cdots, s \), decision variable \( D_\ell \) assumes values 1 if \( F_{\hat{p}_\ell}(\hat{p}_\ell, \frac{\hat{p}_\ell}{1+\varepsilon}) \leq \zeta \delta \), \( G_{\hat{p}_\ell}(\hat{p}_\ell, \frac{\hat{p}_\ell}{1+\varepsilon}) \leq \zeta \delta \); and assumes 0 otherwise. Suppose that the threshold of sample sum for the \( s \)-th stage is equal to \( \frac{(1+\varepsilon)\ln(\delta)}{\varepsilon - (1+\varepsilon)\ln(1+\varepsilon)} \). Then,

\[
\text{Pr}\left\{ p \geq \frac{\hat{p}}{1-\varepsilon} \mid p \right\} \leq \sum_{\ell=1}^{s} \text{Pr}\{ \hat{p}_\ell \leq (1-\varepsilon)p, \ D_\ell = 1 \mid p \} \leq s\zeta \delta, \tag{17}
\]

\[
\text{Pr}\left\{ p \leq \frac{\hat{p}}{1+\varepsilon} \mid p \right\} \leq \sum_{\ell=1}^{s} \text{Pr}\{ \hat{p}_\ell \geq (1+\varepsilon)p, \ D_\ell = 1 \mid p \} \leq s\zeta \delta \tag{18}
\]

for any \( p \in (0,1) \). Moreover, \( \text{Pr}\left\{ \left| \frac{\hat{p}}{p} - \varepsilon \right| \mid p \right\} \geq 1 - \delta \) for any \( p \in (0,1) \) provided that \( \zeta \) is sufficiently small to guarantee \( 1 - \mathcal{S}_P(\gamma_s - 1, \frac{\gamma_s}{1+\varepsilon}) + \mathcal{S}_P(\gamma_s - 1, \frac{\gamma_s}{1-\varepsilon}) < \delta \) and

\[
\ln(\delta) < \frac{(1+\varepsilon + \sqrt{1+4\varepsilon + \varepsilon^2})^2}{4\varepsilon^2} + \frac{1}{2} \left[ \frac{\varepsilon}{1+\varepsilon} - \ln(1+\varepsilon) \right],
\]

\[
\text{Pr}\left\{ \left| \frac{\hat{p} - p}{p} \right| \leq \varepsilon \mid p \right\} \geq 1 - \delta
\]

for any \( p \in [p^*, 1) \), where \( p^* \in (0, z_{s-1}) \) denotes the unique number satisfying

\[
1 - \mathcal{S}_P(\gamma_s - 1, \frac{\gamma_s}{1+\varepsilon}) + \mathcal{S}_P(\gamma_s - 1, \frac{\gamma_s}{1-\varepsilon}) + \sum_{\ell=1}^{s-1} \exp(\gamma_\ell \mathcal{M}_A(z_\ell, p^*)) = \delta
\]

with \( z_\ell = \min\{ z \in I_{\hat{p}_\ell} : F_{\hat{p}_\ell}(z, \frac{\hat{p}_\ell}{1+\varepsilon}) > \zeta \delta \text{ or } G_{\hat{p}_\ell}(z, \frac{\hat{p}_\ell}{1-\varepsilon}) > \zeta \delta \} \), where \( I_{\hat{p}_\ell} \) represents the support of \( \hat{p}_\ell \), for \( \ell = 1, \cdots, s \).
See Appendix [16] for a proof. Based on the criteria proposed in Section [21], the thresholds of sample sum $\gamma_1 < \gamma_2 < \cdots < \gamma_s$ can be chosen as the ascending arrangement of all distinct elements of

$$\left\{ \left\lfloor \frac{C_{\tau-\ell} (1 + \varepsilon) \ln(\zeta \delta)}{\varepsilon - (1 + \varepsilon) \ln(1 + \varepsilon)} \right\rfloor : \ell = 1, \cdots, \tau \right\}, \tag{19}$$

where $\tau$ is the maximum integer such that $\frac{C_{\tau-1} (1 + \varepsilon) \ln(\zeta \delta)}{\varepsilon - (1 + \varepsilon) \ln(1 + \varepsilon)} \geq \frac{\ln \frac{1}{\varepsilon}}{\ln(1 + \varepsilon)}$, i.e., $C_{\tau-1} \geq 1 - \frac{\varepsilon}{(1 + \varepsilon) \ln(1 + \varepsilon)}$.

By virtue of Chernoff bounds of the CDFs of $\hat{p}_\ell$, we propose a class of multistage sampling schemes as follows.

**Theorem 19** Suppose that, for $\ell = 1, \cdots, s$, decision variable $D_\ell$ assumes values 1 if $M_1(\hat{p}_\ell, \hat{p}_\ell) \leq \frac{\ln(\zeta \delta)}{\eta}$; and assumes 0 otherwise. Suppose that the threshold of sample sum for the $s$-th stage is equal to $\left\lfloor \frac{(1 + \varepsilon) \ln(\zeta \delta)}{\varepsilon - (1 + \varepsilon) \ln(1 + \varepsilon)} \right\rfloor$. Then,

$$\Pr\left\{ p \geq \frac{\hat{p}}{1 - \varepsilon} \mid p \right\} \leq \sum_{\ell = 1}^s \Pr\{ \hat{p}_\ell \leq (1 - \varepsilon)p, D_\ell = 1 \mid p \} \leq s\zeta \delta, \tag{20}$$

$$\Pr\left\{ p \leq \frac{\hat{p}}{1 + \varepsilon} \mid p \right\} \leq \sum_{\ell = 1}^s \Pr\{ \hat{p}_\ell \geq (1 + \varepsilon)p, D_\ell = 1 \mid p \} \leq s\zeta \delta \tag{21}$$

for any $p \in (0, 1)$. Moreover, $\Pr\left\{ \left| \frac{\hat{p} - p}{p} \right| \leq \varepsilon \mid p \right\} \geq 1 - \delta$ for any $p \in (0, 1)$ provided that $\zeta$ is sufficiently small to guarantee $1 - S_p(\gamma_s - 1, \frac{\gamma_s}{1 + \varepsilon}) + S_p(\gamma_s - 1, \frac{\gamma_s}{1 - \varepsilon}) < \delta$ and

$$\ln(\zeta \delta) < \left[ \frac{(1 + \varepsilon + \sqrt{1 + 4\varepsilon + \varepsilon^2})^2}{4\varepsilon^2} + 1 \right] \left[ \frac{\varepsilon}{1 + \varepsilon} \ln(1 + \varepsilon) \right], \tag{22}$$

$$\Pr\left\{ \left| \frac{\hat{p} - p}{p} \right| \leq \varepsilon \mid p \right\} \geq 1 - \delta$$

for any $p \in [p^*, 1)$, where $p^* \in (0, \zeta s_{-1})$ denotes the unique number satisfying

$$1 - S_p(\gamma_s - 1, \frac{\gamma_s}{1 + \varepsilon}) + S_p(\gamma_s - 1, \frac{\gamma_s}{1 - \varepsilon}) + \sum_{\ell = 1}^{s-1} \exp(\gamma_\ell \cdot M_1(z_\ell, p^*)) = \delta$$

where $z_\ell \in (0, 1)$ is the unique number such that $M_1(z_\ell, \frac{\zeta s}{1 + \tau}) = \frac{\ln(\zeta \delta)}{\eta}$ for $\ell = 1, \cdots, s - 1$.

See Appendix [17] for a proof. Based on the criteria proposed in Section [21], the thresholds of sample sum $\gamma_1 < \gamma_2 < \cdots < \gamma_s$ can be chosen as the ascending arrangement of all distinct elements of the set defined by (19).

It should be noted that both $z_\ell$ and $p^*$ can be readily computed by a bisection search method due to the monotonicity of the function $M_1(\cdot, \cdot)$.

By virtue of Massart’s inequality for the CDFs of $\hat{p}_\ell$, we propose a class of multistage sampling schemes as follows.
Theorem 20 Suppose that, for $\ell = 1, \ldots, s$, decision variable $D_\ell$ assumes values 1 if $\hat{p}_\ell \geq 1 + 3(1+\epsilon)\ln(\frac{1}{\delta})$; and assumes 0 otherwise. Suppose the threshold of sample sum for the $s$-th stage is equal to $\left[ \frac{2(1+\epsilon)(3+\epsilon)\ln(\delta)}{3+\epsilon} \right]$. Then,

$$\Pr\left\{ p \geq \frac{\hat{p}}{1-\epsilon} \mid p \right\} \leq \sum_{\ell=1}^{s} \Pr\left\{ \hat{p}_\ell \leq (1-\epsilon)p, \ D_\ell = 1 \mid p \right\} \leq s\zeta\delta,$$

$$\Pr\left\{ p \leq \frac{\hat{p}}{1+\epsilon} \mid p \right\} \leq \sum_{\ell=1}^{s} \Pr\left\{ \hat{p}_\ell \geq (1+\epsilon)p, \ D_\ell = 1 \mid p \right\} \leq s\zeta\delta$$

for any $p \in (0, 1)$. Moreover, $\Pr\left\{ \left| \frac{\hat{p} - p}{p} \right| \leq \epsilon \mid p \right\} \geq 1 - \delta$ for any $p \in (0, 1)$ provided that $\zeta$ is sufficiently small to guarantee $1 - SP(\gamma_s - 1, \frac{\zeta}{1+\epsilon}) + SP(\gamma_s - 1, \frac{\zeta}{1-\epsilon}) < \delta$ and

$$\ln(\zeta\delta) < \left[ \frac{1 + \epsilon + \sqrt{1 + 4\epsilon + \epsilon^2}}{4\epsilon^2} + \frac{1}{2} \right] \left[ \frac{\epsilon}{1+\epsilon} - \ln(1+\epsilon) \right],$$

$$\Pr\left\{ \left| \frac{\hat{p} - p}{p} \right| \leq \epsilon \mid p \right\} \geq 1 - \delta$$

for any $p \in [p^*, 1)$, where $p^* \in (0, z_{s-1})$ denotes the unique number satisfying

$$1 - SP\left( \gamma_s - 1, \frac{\gamma_s}{1+\epsilon} \right) + SP\left( \gamma_s - 1, \frac{\gamma_s}{1-\epsilon} \right) + \sum_{\ell=1}^{s-1} \exp\left( \frac{\gamma_{\ell}}{z_\ell} \mathcal{M}(z_\ell, p^*) \right) = \delta$$

with $z_\ell = 1 + \frac{2\epsilon}{3+\epsilon} + \frac{9\epsilon^2\gamma_s}{2(3+\epsilon)^2\ln(\zeta\delta)}$ for $\ell = 1, \ldots, s-1$.

See Appendix L8 for a proof. Based on the criteria proposed in Section 2.11 the thresholds of sample sum $\gamma_1 < \gamma_2 < \cdots < \gamma_s$ can be chosen as the ascending arrangement of all distinct elements of

$$\left\{ \left\lfloor 2C_{\tau-\ell} \left( \frac{1}{\epsilon} + 1 \right) \left( \frac{1}{\epsilon} + \frac{1}{3} \right) \ln \frac{1}{\zeta\delta} \right\rfloor : \ell = 1, \ldots, \tau \right\},$$

where $\tau$ is the maximum integer such that $2C_{\tau-\ell} \left( \frac{1}{\epsilon} + 1 \right) \left( \frac{1}{\epsilon} + \frac{1}{3} \right) \ln \frac{1}{\zeta\delta} \geq \frac{4(3+\epsilon)}{9\epsilon} \ln \frac{1}{\zeta\delta}$, i.e., $C_{\tau-1} \geq \frac{2\epsilon}{3(1+\epsilon)}$.

It should be noted that $\{ D_\ell = i \}$ can be expressed in terms of $n_\ell$. Specially, we have $D_0 = 0$, $D_s = 1$ and $\{ D_\ell = 0 \} = \{ n_\ell \geq \frac{2\epsilon}{3(1+\epsilon)} \}$ for $\ell = 1, \ldots, s-1$.

To apply the truncation techniques of [7] to reduce computation, we can make use of the bounds in Lemma 25 and a bisection search to truncate the domains of $n_{\ell-1}$ and $n_\ell$ to much smaller sets. Since $n_\ell - n_{\ell-1}$ can be viewed as the number of binomial trials to come up with $\gamma_{\ell-1} - \gamma_{\ell-1}$ occurrences of successes, we have that $n_\ell - n_{\ell-1}$ is independent of $n_{\ell-1}$. Hence, the technique of triangular partition described in Section 3.7 can be used by identifying $n_{\ell-1}$ as $U$ and $n_\ell - n_{\ell-1}$ as $V$ respectively. The computation can be reduced to computing the following types of probabilities:

$$\Pr\{ u \leq n_{\ell-1} \leq v \mid p \} = \sum_{n=u}^{v} \binom{n-1}{\gamma_{\ell-1}-1} \left( \frac{p}{1-p} \right)^{\gamma_{\ell-1}} (1-p)^n,$$

$$\Pr\{ u \leq n_\ell - n_{\ell-1} \leq v \mid p \} = \sum_{n=u}^{v} \binom{n-1}{\gamma_{\ell} - \gamma_{\ell-1}-1} \left( \frac{p}{1-p} \right)^{\gamma_{\ell} - \gamma_{\ell-1}} (1-p)^n$$
where \( u \) and \( v \) are integers.

From the definition of the sampling scheme, it can be seen that the probabilities that \( \hat{p} \) is greater or smaller than certain values can be expressed in terms of probabilities of the form \( \Pr\{n_i \in N_i, i = 1, \ldots, \ell\} \), \( 1 \leq \ell \leq s \), where \( N_1, \ldots, N_s \) are subsets of natural numbers. Such probabilities can be computed by using the recursive relationship

\[
\Pr\{n_i \in N_i, i = 1, \ldots, \ell; n_{\ell+1} = n_\ell\} = \sum_{n_\ell \in N_\ell} p^{\gamma_\ell - \gamma_{\ell-1}} (1 - p)^{n_{\ell+1} - n_\ell} \]

for \( \ell = 1, \ldots, s - 1 \).

With regard to the average sample number, we have

**Theorem 21** For any \( p \in (0, 1) \), \( \mathbb{E}[n] = \frac{\mathbb{E}[\gamma]}{p} \) with \( \mathbb{E}[\gamma] = \gamma_1 + \sum_{\ell=1}^{s-1} (\gamma_{\ell+1} - \gamma_\ell) \Pr\{l > \ell\} \).

See Appendix I.9 for a proof.

### 4.2.2 Asymptotic Stopping Rule

We would like to remark that, for a small \( \varepsilon \), we can simplify, by using Taylor’s series expansion formula \( \ln(1 + x) = x - \frac{x^2}{2} + o(x^2) \), the multistage inverse sampling schemes described in Section 4.2.1 as follows:

(i) The sequence of thresholds \( \gamma_1, \ldots, \gamma_s \) is defined as the ascending arrangement of all distinct elements of \( \left\{ \left\lceil \frac{2C_{\tau-1} \ln \frac{1}{\delta}}{\varepsilon^2} \right\rceil : \ell = 1, \ldots, \tau \right\} \), where \( \tau \) is the maximum integer such that \( C_{\tau-1} \geq \frac{\varepsilon}{2} \).

(ii) The decision variables are defined such that \( D_\ell = 1 \) if \( \gamma_\ell \geq \frac{(1 - \hat{p}) 2 \ln \frac{1}{\delta}}{\varepsilon^2} \); and \( D_\ell = 0 \) otherwise.

For such a simplified sampling scheme, we have

\[
\sum_{\ell=1}^{s} \Pr\{|\hat{p}\ell - p| \geq \varepsilon p, D_\ell = 1\} \leq \sum_{\ell=1}^{s} \Pr\{|\hat{p}\ell - p| \geq \varepsilon p\} \leq \sum_{\ell=1}^{s} \Pr\{|\hat{p}\ell - p| \geq \varepsilon p\} \]

\[
\leq \sum_{\ell=1}^{\tau} 2 \exp\left( -\gamma_\ell \left( \frac{\varepsilon}{1 + \varepsilon} - \ln(1 + \varepsilon) \right) \right) \quad (23)
\]

\[
< 2\tau \exp\left( -\gamma_1 \left( \frac{\varepsilon}{1 + \varepsilon} - \ln(1 + \varepsilon) \right) \right), \quad (24)
\]

where (23) is due to Corollary of [8]. As can be seen from (24), the last bound is independent of \( p \) and can be made smaller than \( \delta \) if \( \zeta \) is sufficiently small. This establishes the claim and it follows that \( \Pr\{|\hat{p} - p| < \varepsilon p \mid p\} > 1 - \delta \) for any \( p \in (0, 1) \) if \( \zeta \) is sufficiently small.
4.2.3 Noninverse Multistage Sampling

In Sections 4.2.1 and 4.2.2, we have proposed a multistage inverse sampling plan for estimating a binomial parameter, $p$, with relative precision. In some situations, the cost of sampling operation may be high since samples are obtained one by one when inverse sampling is involved. In view of this fact, it is desirable to develop multistage estimation methods without using inverse sampling.

In contrast to the multistage inverse sampling schemes described in Sections 4.2.1 and 4.2.2, our noninverse multistage sampling schemes have infinitely many stages and deterministic sample sizes $n_1 < n_2 < n_3 < \cdots$. Moreover, the confidence parameter for the $\ell$-th stage, $\delta_\ell$, is dependent on $\ell$ such that $\delta_\ell = \delta$ for $1 \leq \ell \leq \tau$ and $\delta_\ell = 2^{\tau-\ell}$ for $\ell > \tau$, where $\tau$ is a positive integer.

By virtue of the CDFs of $\hat{p}_\ell$, we propose a class of multistage sampling schemes as follows.

**Theorem 22** Suppose that, for $\ell = 1, 2, \cdots$, decision variable $D_\ell$ assumes values 1 if $F_{\hat{p}_\ell}(\hat{p}_\ell, \frac{z_\ell}{1+z_\ell}) \leq \zeta\delta_\ell$; and assumes 0 otherwise. The following statements hold true.

(I): $\Pr[n < \infty] = 1$ provided that $\inf_{\ell>0} \frac{n_{\ell+1}}{n_\ell} > 1$.

(II): $\mathbb{E}[n] < \infty$ provided that $1 < \inf_{\ell>0} \frac{n_{\ell+1}}{n_\ell} \leq \sup_{\ell>0} \frac{n_{\ell+1}}{n_\ell} < \infty$.

(III): $\Pr \left\{ \left| \frac{\hat{p} - p}{p} \right| < \varepsilon \mid p \right\} \geq 1 - \delta$ for any $p \in (0, 1)$ provided that $\zeta \leq \frac{1}{2(\tau+1)}$.

(IV): Let $0 < \eta < \zeta\delta_\ell$ and $\ell^* = \tau + 1 + \left\lceil \frac{\ln(\delta_\ell/n_\ell)}{\ln 2} \right\rceil$. Then, $\Pr \{ |\hat{p} - p| \geq \varepsilon p \} < \delta$ for any $p \in (0, p^*)$, where $p^*$ is a number such that $0 < p^* < z_\ell$, $\ell = 1, \cdots, \ell^*$, and that $\sum_{\ell=1}^{\ell^*} \exp(n_\ell \mathcal{M}_B(z_\ell, p^*)) < \delta - \eta$ with $z_\ell = \min \{ z \in I_{\hat{p}_\ell} : F_{\hat{p}_\ell}(z, \frac{z}{1+z}) > \zeta\delta_\ell \text{ or } G_{\hat{p}_\ell}(z, \frac{z}{1+z}) > \zeta\delta_\ell \}$, where $I_{\hat{p}_\ell}$ represents the support of $\hat{p}_\ell$, for $\ell = 1, 2, \cdots$. Moreover,

$$
\Pr \left\{ b \leq \frac{\hat{p}}{1+\varepsilon}, \ell \leq \ell^* \mid a \right\} \leq \Pr \left\{ p \leq \frac{\hat{p}}{1+\varepsilon} \mid p \right\} \leq \frac{\eta}{2} + \Pr \left\{ a \leq \frac{\hat{p}}{1+\varepsilon}, l \leq \ell^* \mid b \right\},
$$

$$
\Pr \left\{ a \geq \frac{\hat{p}}{1-\varepsilon}, \ell \leq \ell^* \mid b \right\} \leq \Pr \left\{ p \geq \frac{\hat{p}}{1-\varepsilon} \mid p \right\} \leq \frac{\eta}{2} + \Pr \left\{ b \geq \frac{\hat{p}}{1-\varepsilon}, l \leq \ell^* \mid a \right\}
$$

for any $p \in [a, b]$, where $a$ and $b$ are numbers such that $0 < b < (1+\varepsilon)a < 1$.

(V): Let the sample sizes of the multistage sampling scheme be a sequence $n_\ell = \left\lceil m\gamma^{\ell-1} \right\rceil$, $\ell = 1, 2, \cdots$, where $\gamma \geq 1 + \frac{1}{m} > 1$. Let $0 < \varepsilon < \frac{1}{2}$, $0 < \eta < 1$ and $c = \frac{p(1-\eta)^2}{2}$. Let $\kappa$ be an integer such that $\kappa > \max \left\{ \tau, \frac{1}{\ln m} \ln \left( \frac{\ln(n_\ell)}{\ln 2} \right), 1, \tau + \frac{1}{\ln m} + \frac{\ln(\delta_\ell)}{\ln 2} \right\}$ and $\mathcal{M}_B(\eta p, \frac{p}{1+p}) < \frac{\ln(\delta_\ell)}{n_\ell}$. Then, $\mathbb{E}[n] < \varepsilon + n_1 + \sum_{\ell=1}^\kappa (n_{\ell+1} - n_\ell) \Pr \{ l > \ell \}$.

By virtue of Chernoff bounds of the CDFs of $\hat{p}_\ell$, we propose a class of multistage sampling schemes as follows.

**Theorem 23** Suppose that, for $\ell = 1, 2, \cdots$, decision variable $D_\ell$ assumes values 1 if $\mathcal{M}_B(\hat{p}_\ell, \frac{\hat{p}_\ell}{1+\hat{p}_\ell}) \leq \frac{\ln(\delta_\ell)}{n_\ell}$; and assumes 0 otherwise. The following statements hold true.

(I): $\Pr\{n < \infty\} = 1$ provided that $\inf_{\ell>0} \frac{n_{\ell+1}}{n_\ell} > 1$.

(II): $\mathbb{E}[n] < \infty$ provided that $1 < \inf_{\ell>0} \frac{n_{\ell+1}}{n_\ell} \leq \sup_{\ell>0} \frac{n_{\ell+1}}{n_\ell} < \infty$.

(III): $\Pr \left\{ \left| \frac{\hat{p} - p}{p} \right| < \varepsilon \mid p \right\} \geq 1 - \delta$ for any $p \in (0, 1)$ provided that $\zeta \leq \frac{1}{2(\tau+1)}$.
(IV): Let \( 0 < \eta < \zeta \delta \) and \( \ell^* = \tau + 1 + \left\lfloor \frac{\ln(\zeta \delta / \eta)}{m} \right\rfloor \). Then, \( \Pr\{|\hat{p} - p| \geq \varepsilon p\} < \delta \) for any \( p \in (0, p^*) \), where \( p^* \) is a number such that \( 0 < p^* < z_\ell, \ \ell = \tau, \ldots, \ell^* \) and that \( \sum_{\ell=1}^{\ell^*} \exp(\eta_\ell B(z_\ell, p^*)) < \delta - \eta \) with \( z_\ell \) satisfying \( B(z_\ell, \frac{z_\ell}{1+\varepsilon}) = \frac{\ln(\zeta \delta)}{\eta_\ell} \) for \( \ell = 1, 2, \ldots \). Moreover,

\[
\Pr \left\{ b \leq \frac{\hat{p}}{1 + \varepsilon}, \ l \leq \ell^* \ | \ a \right\} \leq \Pr \left\{ p \leq \frac{\hat{p}}{1 + \varepsilon}, \ l \leq \ell^* \ | \ a \right\} \leq \frac{\eta}{2} + \Pr \left\{ a \leq \frac{\hat{p}}{1 + \varepsilon}, \ l \leq \ell^* \ | \ b \right\}, \\
\Pr \left\{ a \geq \frac{\hat{p}}{1 - \varepsilon}, \ l \leq \ell^* \ | \ b \right\} \leq \Pr \left\{ p \geq \frac{\hat{p}}{1 - \varepsilon}, \ l \leq \ell^* \ | \ b \right\} \leq \frac{\eta}{2} + \Pr \left\{ b \geq \frac{\hat{p}}{1 - \varepsilon}, \ l \leq \ell^* \ | \ a \right\}
\]

for any \( p \in [a, b] \), where \( a \) and \( b \) are numbers such that \( 0 < b < (1 + \varepsilon) a < 1 \).

(V): Let the sample sizes of the multistage sampling scheme be a sequence \( n_\ell = \left\lfloor m^{\gamma \ell - 1} \right\rfloor, \ \ell = 1, 2, \ldots \), where \( \gamma \geq 1 + \frac{1}{m} > 1 \). Let \( 0 < \varepsilon < \frac{1}{2} \), \( 0 < \eta < 1 \) and \( c = \frac{p(1-\eta)^2}{2} \). Let \( \kappa \) be an integer such that \( \kappa \geq \max \left\{ \tau, \frac{1}{\ln \left( \frac{\ln(n \eta)}{m} \right)} + 1, \tau + \frac{1}{\gamma + 1} + \frac{\ln(\zeta \delta)}{\ln 2} \right\} \) and \( B(\eta \rho, \frac{\eta \rho}{1 + \varepsilon}) < \frac{\ln(\zeta \delta)}{\eta_\ell} \).

Then, \( \mathbb{E}[n] < \varepsilon + n_1 + \sum_{\ell=1}^{\kappa} (n_{\ell+1} - n_\ell) \Pr\{l > \ell\} \).

See Appendix A.10 for a proof.

By virtue of Massart’s inequality for the CDFs of \( \hat{p}_\ell \), we propose a class of multistage sampling schemes as follows.

**Theorem 24** Suppose that, for \( \ell = 1, 2, \ldots \), decision variable \( D_\ell \) assumes values 1 if \( \hat{p}_\ell \geq \frac{6(1+\varepsilon)(3+\varepsilon)\ln(\zeta \delta)}{2(3+\varepsilon)^2 \ln(\zeta \delta) - 9\varepsilon^2 m^2} \); and assumes 0 otherwise. The following statements hold true.

(I): \( \Pr\{n < \infty\} = 1 \) provided that \( \inf_{\ell>0} \frac{n_{\ell+1}}{n_\ell} > 1 \).

(II): \( \mathbb{E}[n] < \infty \) provided that \( 1 < \inf_{\ell>0} \frac{n_{\ell+1}}{n_\ell} \leq \sup_{\ell>0} \frac{n_{\ell+1}}{n_\ell} < \infty \).

(III): \( \Pr\{|\hat{p} - p| < \varepsilon \ | \ p\} \geq 1 - \delta \) for any \( p \in (0, 1) \) provided that \( \frac{1}{2(\tau+1)} \).

(IV): Let \( 0 < \eta < \zeta \delta \) and \( \ell^* = \tau + 1 + \left\lfloor \frac{\ln(\zeta \delta / \eta)}{m} \right\rfloor \). Then, \( \Pr\{|\hat{p} - p| \geq \varepsilon p\} < \delta \) for any \( p \in (0, p^*) \), where \( p^* \) is a number such that \( 0 < p^* < z_\ell, \ \ell = \tau, \ldots, \ell^* \) and that \( \sum_{\ell=1}^{\ell^*} \exp(\eta_\ell B(z_\ell, p^*)) < \delta - \eta \) with \( z_\ell = \frac{6(1+\varepsilon)(3+\varepsilon)\ln(\zeta \delta)}{2(3+\varepsilon)^2 \ln(\zeta \delta) - 9\varepsilon^2 m^2} \) for \( \ell = 1, 2, \ldots \). Moreover,

\[
\Pr \left\{ b \leq \frac{\hat{p}}{1 + \varepsilon}, \ l \leq \ell^* \ | \ a \right\} \leq \Pr \left\{ p \leq \frac{\hat{p}}{1 + \varepsilon}, \ l \leq \ell^* \ | \ a \right\} \leq \frac{\eta}{2} + \Pr \left\{ a \leq \frac{\hat{p}}{1 + \varepsilon}, \ l \leq \ell^* \ | \ b \right\}, \\
\Pr \left\{ a \geq \frac{\hat{p}}{1 - \varepsilon}, \ l \leq \ell^* \ | \ b \right\} \leq \Pr \left\{ p \geq \frac{\hat{p}}{1 - \varepsilon}, \ l \leq \ell^* \ | \ b \right\} \leq \frac{\eta}{2} + \Pr \left\{ b \geq \frac{\hat{p}}{1 - \varepsilon}, \ l \leq \ell^* \ | \ a \right\}
\]

for any \( p \in [a, b] \), where \( a \) and \( b \) are numbers such that \( 0 < b < (1 + \varepsilon) a < 1 \).

(V): Let the sample sizes of the multistage sampling scheme be a sequence \( n_\ell = \left\lfloor m^{\gamma \ell - 1} \right\rfloor, \ \ell = 1, 2, \ldots \), where \( \gamma \geq 1 + \frac{1}{m} > 1 \). Let \( 0 < \varepsilon < \frac{1}{2} \), \( 0 < \eta < 1 \) and \( c = \frac{p(1-\eta)^2}{2} \). Let \( \kappa \) be an integer such that \( \kappa \geq \max \left\{ \tau, \frac{1}{\ln \left( \frac{\ln(n \eta)}{m} \right)} + 1, \tau + \frac{1}{\gamma + 1} + \frac{\ln(\zeta \delta)}{\ln 2} \right\} \) and \( B(\eta \rho, \frac{\eta \rho}{1 + \varepsilon}) < \frac{\ln(\zeta \delta)}{\eta_\ell} \).

Then, \( \mathbb{E}[n] < \varepsilon + n_1 + \sum_{\ell=1}^{\kappa} (n_{\ell+1} - n_\ell) \Pr\{l > \ell\} \).

### 4.2.4 Asymptotic Analysis of Multistage Inverse Sampling Schemes

In this subsection, we shall focus on the asymptotic analysis of multistage inverse sampling schemes. Throughout this subsection, we assume that the multistage inverse sampling schemes
follow stopping rules derived from Chernoff bounds as described in Section 4.2.1. Moreover, we assume that the thresholds of sample sum \( \gamma_1, \cdots, \gamma_s \) are chosen as the ascending arrangement of all distinct elements of the set defined by \( \pi \).

With regard to the tightness of the double-decision-variable method, we have

**Theorem 25** Let \( \mathcal{R} \) be a subset of real numbers. Define

\[
P = \sum_{\ell=1}^{s} \Pr\{\hat{p}_\ell \in \mathcal{R}, D_{\ell-1} = 0, D_\ell = 1\}, \quad \bar{P} = 1 - \sum_{\ell=1}^{s} \Pr\{\hat{p}_\ell \notin \mathcal{R}, D_{\ell-1} = 0, D_\ell = 1\}.
\]

Then, \( \bar{P} \leq \Pr\{\hat{p} \in \mathcal{R}\} \leq P \) and \( \lim_{\varepsilon \to 0} |\Pr\{\hat{p} \in \mathcal{R}\} - P| = 0 \) for any \( p \in (0, 1) \).

See Appendix I.11 for a proof.

Recall that \( l \) is the index of stage when the sampling is terminated. Define \( \gamma = \gamma_l \). Then, \( \gamma = \sum_{i=1}^{n} X_i \). With regard to the asymptotic performance of the sampling scheme, we have

**Theorem 26** Let \( \gamma(p, \varepsilon) = \frac{\ln(\zeta \delta)}{\mathcal{N}_{R}(p, \varepsilon)} \). Let \( \mathcal{N}_{R}(p, \varepsilon) \) be the minimum sample number \( n \) such that \( \Pr\{\frac{\sum_{i=1}^{n} X_i}{n} - p < \varepsilon p | p\} > 1 - \zeta \delta \) for a fixed-size sampling procedure. Let \( j_p \) be the maximum integer \( j \) such that \( C_j \geq 1 - p \). Let \( \nu = \frac{\gamma}{\delta}, d = \sqrt{2\ln \frac{1}{\zeta \delta}} \) and \( \kappa_p = \frac{C_j}{1 - p} \). Let \( \rho_p = \frac{C_j - 1}{1 - p} - 1 \) if \( \kappa_p = 1 \) and \( \rho_p = \kappa_p - 1 \) otherwise. The following statements hold true:

1. \( \Pr\left\{1 \leq \limsup_{\varepsilon \to 0} \frac{\gamma(p, \varepsilon)}{\mathcal{N}_{R}(p, \varepsilon)} \leq 1 + \rho_p\right\} = 1 \). Specially, \( \Pr\left\{\lim_{\varepsilon \to 0} \frac{\gamma(p, \varepsilon)}{\mathcal{N}_{R}(p, \varepsilon)} = \kappa_p\right\} = 1 \) if \( \kappa_p > 1 \).
2. \( \lim_{\varepsilon \to 0} \frac{\mathcal{E}[\gamma]}{\mathcal{N}_{R}(p, \varepsilon)} = \left(\frac{d}{2\zeta \delta}\right)^2 \times \lim_{\varepsilon \to 0} \frac{\mathcal{E}[\gamma]}{\gamma(p, \varepsilon)} \), where

\[
\lim_{\varepsilon \to 0} \frac{\mathcal{E}[\gamma]}{\gamma(p, \varepsilon)} = \begin{cases} 
\kappa_p & \text{if } \kappa_p > 1, \\
1 + \rho_p \Phi(\nu d) & \text{otherwise}
\end{cases}
\]

and \( 1 \leq \lim_{\varepsilon \to 0} \frac{\mathcal{E}[\gamma]}{\gamma(p, \varepsilon)} \leq 1 + \rho_p \).

3. If \( \kappa_p > 1 \), then \( \lim_{\varepsilon \to 0} \Pr\{\hat{p} - p < \varepsilon p\} = 2 \Phi(d \sqrt{\kappa_p}) - 1 > 2 \Phi(d) - 1 > 1 + 2 \zeta \delta \).

Otherwise, \( 2 \Phi(d) - 1 > \lim_{\varepsilon \to 0} \Pr\{\hat{p} - p < \varepsilon p\} = 1 + \Phi(d) - \Phi(\nu d) - \psi(\rho_p, \nu, d) > 3 \Phi(d) - 2 > 1 - 3 \zeta \delta \).

See Appendix I.12

### 4.2.5 Asymptotic Analysis of Noninverse Multistage Sampling Schemes

In this subsection, we shall focus on the asymptotic analysis of the noninverse multistage sampling schemes which follow stopping rules derived from Chernoff bounds of CDFs of \( \hat{p}_\ell \) as described in Theorem 23.

We assume that the sample sizes \( n_1, n_2, \cdots \) are chosen as the ascending arrangement of all distinct elements of the set

\[
\left\{ \left[ \frac{C_{\tau - \ell} \ln(\zeta \delta)}{\mathcal{M}_B(p^*, \frac{\nu}{\nu + 1})} \right] : \ell = 1, 2, \cdots \right\}
\]
with $p^* \in (0, 1)$, where $\tau$ is the maximum integer such that $C_{\tau - 1} \ln(\delta) \geq \frac{\ln \frac{1}{\nu}}{\ln(1+\tau)}$, i.e., $C_{\tau - 1} \geq -\frac{\mathcal{A}(p^*, \frac{p^*}{1-p^*})}{\ln(1+\tau)}$.

With regard to the asymptotic performance of the sampling scheme, we have

**Theorem 27** Let $\mathcal{N}_f(p, \varepsilon) = \frac{\ln(\delta)}{\mathcal{A}(p, \frac{p}{1-p})}$. Let $\mathcal{N}_f(p, \varepsilon)$ be the minimum sample number $n$ such that

$\Pr\{\left| \frac{\sum X_n}{n} - p \right| < \varepsilon p \mid p > 1 - \delta\}$ for a fixed-size sampling procedure. Let $j_p$ be the maximum integer $j$ such that $C_j \geq r(p)$, where $r(p) = \frac{p(1-p)}{p(1-p)}$. Let $\nu = \frac{2}{3} \frac{p^*}{1-p^*}$, $d = \sqrt{2 \ln \frac{1}{\delta}}$ and $\kappa_p = \frac{C_j}{r(p)}$. Let $\rho_p = \frac{C_j}{r(p)} - 1$ if $\kappa_p = 1$ and $\rho_p = \kappa_p - 1$ otherwise. For $p \in (p^*, 1)$, the following statements hold true:

(I): $\Pr\left\{ 1 \leq \lim_{\varepsilon \to 0} \frac{n}{\mathcal{N}_f(p, \varepsilon)} \leq 1 + \rho_p \right\} = 1$. Specially, $\Pr\left\{ \lim_{\varepsilon \to 0} \frac{n}{\mathcal{N}_f(p, \varepsilon)} = \kappa_p \right\} = 1$ if $\kappa_p > 1$.

(II): $\lim_{\varepsilon \to 0} \frac{\mathbb{E}[n]}{\mathcal{N}_f(p, \varepsilon)} = \left( \frac{d}{\varepsilon \sqrt{\nu}} \right)^2 \times \lim_{\varepsilon \to 0} \frac{\mathbb{E}[n]}{\mathcal{N}_f(p, \varepsilon)}$, where

$$\lim_{\varepsilon \to 0} \frac{\mathbb{E}[n]}{\mathcal{N}_f(p, \varepsilon)} = \begin{cases} \kappa_p & \text{if } \kappa_p > 1, \\ 1 + \rho_p \Phi(\nu d) & \text{otherwise} \end{cases}$$

and $1 \leq \lim_{\varepsilon \to 0} \frac{\mathbb{E}[n]}{\mathcal{N}_f(p, \varepsilon)} \leq 1 + \rho_p$.

(III): If $\kappa_p > 1$, then $\lim_{\varepsilon \to 0} \Pr\{ |\hat{p} - p| < \varepsilon p \} = \Phi(d) \Phi(\nu d) - 1 + 2 \Phi(d) - 1 > 1 - 2\delta$. Otherwise, $2\Phi(d) - 1 > \lim_{\varepsilon \to 0} \Pr\{ |\hat{p} - p| < \varepsilon p \} = 1 + \Phi(d) - \Phi(\nu d) - \Psi(p, \nu, d) > 3\Phi(d) - 2 > 1 - 3\delta$.

See Appendix I.13 for a proof.

### 4.3 Control of Absolute and Relative Errors

In this section, we shall focus on the design of multistage sampling schemes for estimating the binomial parameter $p$ with a mixed error criterion. Specifically, for $0 < \varepsilon_a < 1$ and $0 < \varepsilon_r < 1$, we wish to construct a multistage sampling scheme and its associated estimator $\hat{p}$ for $p$ such that $\Pr\{ |\hat{p} - p| < \varepsilon_a \mid |\hat{p} - p| < \varepsilon_r p \mid p > 1 - \delta \}$ for any $p \in (0, 1)$. This is equivalent to the construction of a random interval with lower limit $\mathcal{L} (\hat{p})$ and upper limit $\mathcal{U} (\hat{p})$ such that $\Pr\{ \mathcal{L} (\hat{p}) < p < \mathcal{U} (\hat{p}) \mid p > 1 - \delta \}$ for any $p \in (0, 1)$, where $\mathcal{L}(.)$ and $\mathcal{U}(.)$ are functions such that $\mathcal{L}(z) = \min\{z - \varepsilon_a, \frac{z}{1 + \varepsilon_r} \}$ and $\mathcal{U}(z) = \max\{z + \varepsilon_a, \frac{z}{1 + \varepsilon_r} \}$ for $z \in [0, 1]$. In the sequel, we shall propose multistage sampling schemes such that the number of stages, $s$, is finite and that the sample sizes are deterministic numbers $n_1 < n_2 < \cdots < n_s$.

#### 4.3.1 Stopping Rules from CDFs and Chernoff Bounds

To construct an estimator satisfying a mixed criterion in terms of absolute and relative errors with a prescribed confidence level, we have developed two types of multistage sampling schemes with different stopping rules as follows.
Theorem 28 Let \( \varepsilon_a \) and \( \varepsilon_r \) be positive numbers such that \( 0 < \varepsilon_a < \frac{35}{34} \) and \( \frac{70\varepsilon_r}{35-24\varepsilon_a} < \varepsilon_r < 1 \). Suppose that the sample size for the \( s \)-th stage is no less than \( \left\lceil \frac{\ln(\delta)}{\mathcal{M}_B(\hat{p}_1, \mathcal{L}(\hat{p}_1)) + \varepsilon_a, \varepsilon_r} \right\rceil \). Then,

\[
\Pr\{p \leq \mathcal{L}(\hat{p}) \mid p\} \leq \sum_{\ell=1}^{s} \Pr\{p \leq \mathcal{L}(\hat{p}_\ell), D_\ell = 1 \mid p\} \leq s\zeta\delta,
\]

\[
\Pr\{p \geq \mathcal{U}(\hat{p}) \mid p\} \leq \sum_{\ell=1}^{s} \Pr\{p \geq \mathcal{U}(\hat{p}_\ell), D_\ell = 1 \mid p\} \leq s\zeta\delta
\]

and \( \Pr\{|\hat{p} - p| < \varepsilon_a \text{ or } |\hat{p} - p| < \varepsilon_r p \mid p\} \geq 1 - 2s\zeta\delta \) for any \( p \in (0, 1) \).

See Appendix I.14 for a proof. Based on the criteria proposed in Section 2.1, the sample sizes \( n_1 < n_2 < \cdots < n_s \) can be chosen as the ascending arrangement of all distinct elements of the set

\[
\left\{ \left[ \frac{C_{\tau-\ell} \ln(\delta)}{\mathcal{M}_B(\hat{p}_1, \mathcal{L}(\hat{p}_1)) + \varepsilon_a, \varepsilon_r} \right] : \ell = 1, \cdots, \tau \right\}, \tag{26}
\]

where \( \tau \) is the maximum integer such that \( \frac{C_{\tau-\ell} \ln(\delta)}{\mathcal{M}_B(\hat{p}_1, \mathcal{L}(\hat{p}_1)) + \varepsilon_a, \varepsilon_r} \geq \frac{\ln(\delta)}{\ln(1+\varepsilon_a)} \), i.e., \( C_{\tau-1} \geq -\frac{\mathcal{M}_B(\hat{p}_1, \mathcal{L}(\hat{p}_1))}{\ln(1+\varepsilon_a)} \).

For such a choice of sample sizes, as a result of Theorem 28 we have that \( \Pr\{|\hat{p} - p| < \varepsilon_a \text{ or } |\hat{p} - p| < \varepsilon_r p \mid p\} \geq 1 - \delta \) for any \( p \in (0, 1) \) provided that \( \zeta < \frac{1}{2\varepsilon_r} \).

For computing the coverage probability associated with a multistage sampling scheme following a stopping rule derived from Chernoff bounds, events \( \{D_\ell = i\}, i = 0, 1 \) need to be expressed as events involving only \( K_\ell \). This can be accomplished by using the following results.

Theorem 29 Let \( p^* = \frac{\varepsilon_a}{\varepsilon_r} \). For \( \ell = 1, \cdots, s - 1 \), \( \{D_\ell = 0\} = \{\mathcal{M}_B(\hat{p}_\ell, \mathcal{L}(\hat{p}_\ell)) > \frac{\ln(\delta)}{n_\ell}\} \cup \{\mathcal{M}_B(\hat{p}_\ell, \mathcal{U}(\hat{p}_\ell)) > \frac{\ln(\delta)}{n_\ell}\} \) and the following statements hold true:

(I) \( \{\mathcal{M}_B(\hat{p}_\ell, \mathcal{L}(\hat{p}_\ell)) > \frac{\ln(\delta)}{n_\ell}\} = \{n_\ell z^-_a < K_\ell < n_\ell z^+_a\} \) where \( z^+_a \) is the unique solution of equation \( \mathcal{M}_B(z, \frac{1}{\varepsilon_a}) = \frac{\ln(\delta)}{n_\ell} \) with respect to \( z \in (p^* + \varepsilon_a, 1] \), and \( z^-_a \) is the unique solution of equation \( \mathcal{M}_B(z, \varepsilon_a) = \frac{\ln(\delta)}{n_\ell} \) with respect to \( z \in (\varepsilon_a, p^* + \varepsilon_a) \).

(II) \[
\left\{ \mathcal{M}_B(\hat{p}_\ell, \mathcal{U}(\hat{p}_\ell)) > \frac{\ln(\delta)}{n_\ell}\right\} = \begin{cases} 
\{0 \leq K_\ell < n_\ell z^-_a\} & \text{for } n_\ell \leq \frac{\ln(\delta)}{\ln(1+\varepsilon_a)}; \\
\{n_\ell z^-_a < K_\ell < n_\ell z^+_a\} & \text{for } \frac{\ln(\delta)}{\ln(1+\varepsilon_a)} \leq n_\ell \leq \frac{\ln(\delta)}{\mathcal{M}_B(p^* - \varepsilon_a, p^*)}; \\
\emptyset & \text{for } n_\ell \geq \frac{\ln(\delta)}{\mathcal{M}_B(p^* - \varepsilon_a, p^*)}.
\end{cases}
\]
where $z^-_a$ is the unique solution of equation $\mathcal{A}_B(z, \frac{1}{r_a}) = \frac{\ln(\delta)}{n_r}$ with respect to $z \in (p^* - \varepsilon_a, 1 - \varepsilon_r)$, and $z^+_a$ is the unique solution of equation $\mathcal{A}_B(z, \varepsilon_a) = \frac{\ln(\delta)}{n_r}$ with respect to $z \in [0, p^* - \varepsilon_a)$.

See Appendix [L.15] for a proof.

4.3.2 Stopping Rule from Massart’s Inequality

By virtue of Massart’s inequality of the CDFs of $\hat{p}_\ell$, we can construct a multistage sampling scheme such that its associated estimator for $p$ satisfies the mixed criterion. Such a sampling scheme and its properties are described by the following theorem.

**Theorem 30** Let $\varepsilon_a$ and $\varepsilon_r$ be positive numbers such that $0 < \varepsilon_a < \frac{\delta}{8}$ and $\frac{6\varepsilon_a}{3 - 2\varepsilon_a} < \varepsilon_r < 1$. Suppose the sample size for the $s$-th stage is no less than $\left\lceil \frac{\ln(\delta)}{6(1+\varepsilon_r)(3+\varepsilon_r)\ln(\delta)} \right\rceil$. Define

$$D_\ell = \begin{cases} 0 & \text{for } \frac{1}{2} - \frac{2}{3}\varepsilon_a - \sqrt{\frac{1}{4} + \frac{n_s\varepsilon_a^2}{2\ln(\delta)}} < \hat{p}_\ell < \frac{6(1-\varepsilon_r)(3-\varepsilon_r)\ln(\delta)}{2(1+\varepsilon_r)^2\ln(\delta) - 9n_s\varepsilon_a^2} \\ \frac{1}{2} + \frac{2}{3}\varepsilon_a - \sqrt{\frac{1}{4} + \frac{n_s\varepsilon_a^2}{2\ln(\delta)}} < \hat{p}_\ell < \frac{6(1+\varepsilon_r)(3+\varepsilon_r)\ln(\delta)}{2(1+\varepsilon_r)^2\ln(\delta) - 9n_s\varepsilon_a^2} \\ 1 & \text{else} \end{cases}$$

for $\ell = 1, \cdots, s$. Then,

$$\Pr\{p \leq \mathcal{L}(\hat{p}) \mid p\} \leq \sum_{\ell=1}^{s} \Pr\{p \leq \mathcal{L}(\hat{p}_\ell), D_\ell = 1 \mid p\} \leq s\zeta\delta,$$

and $\Pr\{|\hat{p} - p| < \varepsilon_a \text{ or } |\hat{p} - p| < \varepsilon_r p \mid p\} \geq 1 - 2s\zeta\delta$ for any $p \in (0, 1)$.

See Appendix [L.16] for a proof. Based on the criteria proposed in Section 2.1, the sample sizes $n_1 < n_2 < \cdots < n_s$ can be chosen as the ascending arrangement of all distinct elements of

$$\left\{ \left[ 2C_{\tau - \ell} \left( \frac{1}{\varepsilon_a} - \frac{1}{\varepsilon_r} - \frac{1}{3} \right) \left( \frac{1}{\varepsilon_r} + \frac{1}{3} \right) \ln \frac{1}{\zeta\delta} \right] : \ell = 1, \cdots, \tau \right\},$$

where $\tau$ is the maximum integer such that $2C_{\tau - \ell} \left[ \left( \frac{1}{\varepsilon_a} - \frac{1}{\varepsilon_r} - \frac{1}{3} \right) \left( \frac{1}{\varepsilon_r} + \frac{1}{3} \right) \ln \frac{1}{\zeta\delta} \right] \geq \frac{4(3+\varepsilon_r)}{9\varepsilon_a} \ln \frac{1}{\zeta\delta},$

i.e., $C_{\tau-1} \geq \frac{2}{3} \left( \frac{1}{\varepsilon_a} - \frac{1}{\varepsilon_r} - \frac{1}{3} \right)^{-1}$. For such a choice of sample sizes, as a result of Theorem 30 we have that $\Pr\{|\hat{p} - p| < \varepsilon_a \text{ or } |\hat{p} - p| < \varepsilon_r p \mid p\} > 1 - \delta$ for any $p \in (0, 1)$ provided that $\zeta < \frac{1}{2\tau}$.

4.3.3 Asymptotic Stopping Rule

It should be noted that, for small $\varepsilon_a$ and $\varepsilon_r$, we can simplify, by using Taylor’s series expansion formula $\ln(1 + x) = x - \frac{x^2}{2} + o(x^2)$, the sampling schemes described in Section 4.3.1 as follows:

(i) The sequence of sample sizes $n_1, \cdots, n_s$ is defined as the ascending arrangement of all distinct elements of $\left\{ \left[ 2C_{\tau - \ell} \left( \frac{1}{\varepsilon_a} - \frac{1}{\varepsilon_r} \right) \frac{\ln(\delta)}{n_r} \right] : \ell = 1, \cdots, \tau \right\}$ with $\varepsilon_a < \frac{\delta}{2}$, where $\tau$ is the maximum integer such that $C_{\tau-1} \geq \left( \frac{2}{\varepsilon_a} - \frac{2}{\varepsilon_r} \right)^{-1}$. 

39
(ii) The decision variables are defined such that $D_\ell = 1$ if $n_\ell \geq \hat{p}_\ell (1 - \hat{p}_\ell) 2 \ln \frac{1}{\zeta \delta} \max\{\varepsilon_a, (\varepsilon_r p)^2\}$, and $D_\ell = 0$ otherwise.

For such a simplified sampling scheme, we have

$$\sum_{\ell=1}^{s} \Pr \{ |\hat{p}_\ell - p| \geq \max\{\varepsilon_a, \varepsilon_r p\}, D_\ell = 1 \} \leq \sum_{\ell=1}^{s} \Pr \{ |\hat{p}_\ell - p| \geq \max\{\varepsilon_a, \varepsilon_r p\} \} \leq \sum_{\ell=1}^{s} 2 \exp \left( n_\ell \mathcal{M}_B \left( \frac{\varepsilon_a}{\varepsilon_r} + \frac{\varepsilon_r}{\varepsilon_a} \right) \right) \leq 2 \tau \exp \left( n_1 \mathcal{M}_B \left( \frac{\varepsilon_a}{\varepsilon_r} + \frac{\varepsilon_r}{\varepsilon_a} \right) \right),$$

where (27) is due to Theorem 1 of [6]. As can be seen from (28), the last bound is independent of $p$ and can be made smaller than $\delta$ if $\zeta$ is sufficiently small. This establishes the claim and it follows that $\Pr \{|\hat{p} - p| < \varepsilon_a \text{ or } |\hat{p} - p| < \varepsilon_r | p\} > 1 - \delta$ for any $p \in (0, 1)$ if $\zeta$ is sufficiently small.

4.3.4 Asymptotic Analysis of Sampling Schemes

In this subsection, we shall focus on the asymptotic analysis of multistage inverse sampling schemes. Throughout this subsection, we assume that the multistage sampling schemes follow stopping rules derived from Chernoff bounds as described in Section 4.3.1. Moreover, we assume that the sample sizes $n_1, \cdots, n_s$ are chosen as the ascending arrangement of all distinct elements of the set defined by (26).

With regard to the tightness of the double-decision-variable method, we have

**Theorem 31** Let $\mathcal{R}$ be a subset of real numbers. Define

$$P = \sum_{\ell=1}^{s} \Pr \{ \hat{p}_\ell \in \mathcal{R}, D_{\ell-1} = 0, D_\ell = 1 \}, \quad P = 1 - \sum_{\ell=1}^{s} \Pr \{ \hat{p}_\ell \notin \mathcal{R}, D_{\ell-1} = 0, D_\ell = 1 \}.$$

Then, $P \leq \Pr \{ \hat{p} \in \mathcal{R} \} \leq \overline{P}$ and $\lim_{\varepsilon_a \to 0} |\Pr \{ \hat{p} \in \mathcal{R} \} - \overline{P}| = \lim_{\varepsilon_a \to 0} |\Pr \{ \hat{p} \notin \mathcal{R} \} - P| = 0$ for any $p \in (0, 1)$, where the limits are taken under the constraint that $\frac{\varepsilon_a}{\varepsilon_r}$ is fixed.

See Appendix I.17 for a proof.

With regard to the asymptotic performance of the sampling scheme as $\varepsilon_a$ and $\varepsilon_r$ tend to 0, we have

**Theorem 32** Let $N_f(p, \varepsilon_a, \varepsilon_r)$ be the minimum sample number $n$ such that

$$\Pr \left\{ \left| \frac{\sum_{i=1}^{n} X_i}{n} - p \right| < \varepsilon_a \text{ or } \left| \frac{\sum_{i=1}^{n} X_i}{n} - p \right| < \varepsilon_r | p \right\} > 1 - \zeta \delta$$

40
for a fixed-size sampling procedure. Let \( N_m(p, \varepsilon_a, \varepsilon_r) = \frac{\ln(\delta)}{\max\{ \frac{p}{\varepsilon_a, \max\{p + \varepsilon_a, \frac{p}{1-\varepsilon_r} \}}, \frac{p}{\varepsilon_r} \} \} \), where \( p = \min\{ p - \varepsilon_a, \frac{p}{1-\varepsilon_r} \} \) and \( \overline{p} = \max\{ p + \varepsilon_a, \frac{p}{1-\varepsilon_r} \} \). Define \( p^* = \frac{\varepsilon_a}{\varepsilon_r}, \) \( d = \sqrt{2 \ln \frac{1}{\delta}} \), \( \kappa = \frac{C_p}{r(p)}, \) where \( j_p \) is the maximum integer \( j \) such that \( C_j \geq r(p) \). Let \( \rho_p = \frac{C_{p+1}}{r(p)} - 1 \) if \( \kappa_p = 1 \), \( j_p > 0 \) and \( \rho_p = \kappa_p - 1 \) otherwise. The following statements hold true under the condition that \( \frac{\varepsilon_a}{\varepsilon_r} \) is fixed.

(I): \( \Pr \left\{ 1 \leq \liminf_{\varepsilon_a\to 0} \frac{\ln(\delta)}{\max(\frac{p}{\varepsilon_a, \max\{p + \varepsilon_a, \frac{p}{1-\varepsilon_r} \}}, \frac{p}{\varepsilon_r})} \leq 1 + \rho_p \right\} = 1. \) Specially, \( \Pr \left\{ \lim_{\varepsilon_a\to 0} \frac{\ln(\delta)}{\max(\frac{p}{\varepsilon_a, \max\{p + \varepsilon_a, \frac{p}{1-\varepsilon_r} \}}, \frac{p}{\varepsilon_r})} = \kappa_p \right\} = 1 \) if \( \kappa_p > 1 \).

(II): \( \lim_{\varepsilon_a\to 0} \frac{\ln(\delta)}{\max(\frac{p}{\varepsilon_a, \max\{p + \varepsilon_a, \frac{p}{1-\varepsilon_r} \}}, \frac{p}{\varepsilon_r})} = \left\{ \begin{array}{ll} \kappa_p & \text{if } \kappa_p > 1, \\ 1 + \rho_p \Phi(\nu d) & \text{otherwise} \end{array} \right. \)

and \( 1 \leq \lim_{\varepsilon_a\to 0} \frac{\ln(\delta)}{\max(\frac{p}{\varepsilon_a, \max\{p + \varepsilon_a, \frac{p}{1-\varepsilon_r} \}}, \frac{p}{\varepsilon_r})} \leq 1 + \rho_p \).

(III): If \( \kappa_p > 1 \), then \( \lim_{\varepsilon_a\to 0} \Pr \{ |\hat{p} - p| < \varepsilon_a \text{ or } |\bar{p} - p| < \varepsilon_r p \} = 2 \Phi(d, \sqrt{\kappa_p}) - 1 \leq \Phi(\nu d) - 3 \Phi(d) > 2 \Phi(d) - 1 > 3 - 2\delta. \) Otherwise, \( 2 \Phi(d) - 1 > \lim_{\varepsilon_a\to 0} \Pr \{ |\hat{p} - p| < \varepsilon_a \text{ or } |\bar{p} - p| < \varepsilon_r p \} = 1 + \Phi(d) - \Phi(\nu d) - \Phi(\rho_p, \nu, d) > 3 \Phi(d) - 2 > 1 - 3\delta. \)

See Appendix I.18 for a proof.

5 Estimation of Bounded-Variable Means

In the preceding discussion, we have been focusing on the estimation of binomial parameters. Actually, some of the ideas can be generalized to the estimation of means of random variables bounded in interval \([0, 1]\). Formally, let \( X \in [0, 1] \) be a random variable with expectation \( \mu = \mathbb{E}[X] \). We can estimate \( \mu \) based on i.i.d. random samples \( X_1, X_2, \cdots \) of \( X \) by virtue of multistage sampling schemes.

5.1 Control of Absolute Error

To estimate the mean of the bounded variable \( X \in [0, 1] \) with an absolute error criterion, we have multistage sampling schemes described by the following theorems.

**Theorem 33** Let \( 0 < \varepsilon < \frac{1}{2} \). Let \( n_1 < n_2 < \cdots < n_s \) be a sequence of sample sizes such that \( n_s \geq \frac{\ln(\delta)}{4\varepsilon^2} \). Define \( \hat{\mu}_\ell = \frac{\sum_{i=1}^{n_\ell} X_i}{n_\ell} \) for \( \ell = 1, \cdots, s \). Suppose that sampling is continued until \( \mathcal{M}_B\left( \frac{1}{2} - |\hat{\mu}_\ell|, \frac{1}{2} - |\hat{\mu}_\ell| + \varepsilon \right) \leq \frac{1}{n_\ell} \ln\left( \frac{\delta}{\varepsilon^2} \right) \). Define \( \hat{\mu} = \frac{\sum_{i=1}^{n} X_i}{n} \), where \( n \) is the sample size when the sampling is terminated. Then, \( \Pr \{ |\hat{\mu} - \mu| < \varepsilon \} \geq 1 - \delta. \)
5.2 Control of Relative Error

Let \( \gamma \) is a positive integer. As before, define \( \tau = n_\ell \) for \( \ell = 1, \cdots, s \). Suppose that sampling is continued until
\[
\left( |\mu_\ell - \frac{1}{2}| - \frac{2\varepsilon}{3} \right)^2 \geq \frac{1}{4} - \frac{\varepsilon^2 n_\ell}{2\ln(2s/\delta)}
\]
for some \( \ell \in \{1, \cdots, s\} \). Define \( \hat{\mu} = \sum_{i=1}^{n_\ell} X_i \), where \( n \) is the sample size when the sampling is terminated. Then, \( \Pr \{ |\hat{\mu} - \mu| < \varepsilon \} \geq 1 - \delta \).

5.2 Control of Relative Error

To estimate the mean of the bounded variable \( X \in [0,1] \) with a relative precision, we have multistage inverse sampling schemes described by the following theorems.

**Theorem 34** Let \( 0 < \varepsilon < \frac{1}{2} \). Let \( n_1 < n_2 < \cdots < n_s \) be a sequence of sample sizes such that \( n_s \geq \frac{\ln s}{2\varepsilon^2} \). Define \( \hat{\mu}_\ell = \sum_{i=1}^{n_\ell} X_i \) for \( \ell = 1, \cdots, s \). Suppose that sampling is continued until
\[
\left( |\hat{\mu}_\ell - \frac{1}{2}| - \frac{2\varepsilon}{3} \right)^2 \geq \frac{1}{4} - \frac{\varepsilon^2 n_\ell}{2\ln(2s/\delta)}
\]
for some \( \ell \in \{1, \cdots, s\} \). Define \( \hat{\mu} = \sum_{i=1}^{n_\ell} X_i \), where \( n \) is the sample size when the sampling is terminated. Then, \( \Pr \{ |\hat{\mu} - \mu| < \varepsilon \mu \} \geq 1 - \delta \).

See Appendix I.19 for a proof.

**Theorem 35** Let \( 0 < \varepsilon < 1 \). Let \( \gamma_1 < \gamma_2 < \cdots < \gamma_s \) be a sequence of real numbers such that \( \gamma_1 > \frac{1}{\varepsilon} \) and \( \gamma_s \geq \frac{(1+\varepsilon)\ln \frac{\delta}{\varepsilon}}{\ln(1+\varepsilon) - \varepsilon} \). For \( \ell = 1, \cdots, s \), define \( \hat{\mu}_\ell = \frac{n_\ell}{\gamma_\ell} \), where \( n_\ell \) is the minimum sample number such that \( \sum_{i=1}^{n_\ell} X_i \geq \gamma_\ell \). Suppose that sampling is continued until \( \mathcal{M}_B \left( \frac{\nu}{n_\ell}, \frac{\gamma_\ell}{n_\ell(1+\varepsilon)} \right) \leq \frac{1}{n_\ell} \ln \left( \frac{\delta}{2s} \right) \) for some \( \ell \in \{1, \cdots, s\} \). Define \( \hat{\mu} = \frac{n_\ell}{\gamma_\ell} \), where \( l \) is the index of stage when the sampling is terminated. Then, \( \Pr \{ |\hat{\mu} - \mu| < \varepsilon \mu \} \geq 1 - \delta \).

See Appendix I.19 for a proof.

**Theorem 36** Let \( 0 < \varepsilon < 1 \). Let \( \gamma_1 < \gamma_2 < \cdots < \gamma_s \) be a sequence of real numbers such that \( \gamma_1 > \frac{1}{\varepsilon} \) and \( \gamma_s \geq \frac{2(1+\varepsilon)(1+3\varepsilon)\ln \frac{\delta}{\varepsilon}}{2\varepsilon^2} \). For \( \ell = 1, \cdots, s \), define \( \hat{\mu}_\ell = \frac{n_\ell}{\gamma_\ell} \), where \( n_\ell \) is the minimum sample number such that \( \sum_{i=1}^{n_\ell} X_i \geq \gamma_\ell \). Suppose that sampling is continued until \( \mathcal{M} \left( \frac{\nu}{n_\ell}, \frac{\gamma_\ell}{n_\ell(1+\varepsilon)} \right) \leq \frac{1}{n_\ell} \ln \left( \frac{\delta}{2s} \right) \) for some \( \ell \in \{1, \cdots, s\} \). Define \( \hat{\mu} = \frac{n_\ell}{\gamma_\ell} \), where \( l \) is the index of stage when the sampling is terminated. Then, \( \Pr \{ |\hat{\mu} - \mu| < \varepsilon \mu \} \geq 1 - \delta \).

In some situations, the cost of sampling operation may be high since samples are obtained one by one when inverse sampling is involved. In view of this fact, it is desirable to develop multistage estimation methods without using inverse sampling. In contrast to the multistage inverse sampling schemes with different stopping rules as follows.

**Stopping Rule (i):** For \( \ell = 1, 2, \cdots \), decision variable \( D_\ell \) assumes value 1 if \( \mathcal{M}_B (\hat{\mu}_\ell, \frac{n_\ell}{1+x}) \leq \frac{\ln(\delta_\ell)}{n_\ell} \); and assumes value 0 otherwise.

**Stopping Rule (ii):** For \( \ell = 1, 2, \cdots \), decision variable \( D_\ell \) assumes value 1 if
\[
\hat{\mu}_\ell \geq \frac{6(1+\varepsilon)(3+\varepsilon)\ln(\delta_\ell)}{2(3+\varepsilon)^2\ln(\delta_\ell) - 9n_\ell\varepsilon^2}.
\]
and assumes value 0 otherwise.

Stopping rule (i) is derived by virtue of Chernoff-Hoeffding bounds of the CDFs of \( \hat{\mu}_\ell \). Stopping rule (ii) is derived by virtue of Massart’s inequality of the CDFs of \( \hat{\mu}_\ell \).

**Theorem 37** For both types of multistage sampling schemes described above, the following statements hold true:

(I): \( \Pr\{n < \infty\} = 1 \) for any \( \mu \in (0, 1) \) provided that \( \inf_{\ell > 0} \frac{n_{\ell+1}}{n_\ell} > 1 \).

(II): \( \mathbb{E}[n] < \infty \) for any \( \mu \in (0, 1) \) provided that \( 1 < \inf_{\ell > 0} \frac{n_{\ell+1}}{n_\ell} \leq \sup_{\ell > 0} \frac{n_{\ell+1}}{n_\ell} < \infty \).

(III): \( \Pr\left\{ \left| \frac{\hat{\mu} - \mu}{\mu\ell} \right| < \varepsilon_{\mu} \right\} \geq 1 - \delta \) for any \( \mu \in (0, 1) \) provided that \( \varepsilon \leq \frac{1}{2(\tau + 1)} \).

### 5.3 Control of Absolute and Relative Errors

In this subsection, we consider the multistage estimation of the mean of the bounded variable with a mixed error criterion. Specifically, we wish to construct a multistage sampling scheme and its associated estimator \( \hat{\mu} \) for \( \mu = \mathbb{E}[X] \) such that \( \Pr\{\|\hat{\mu} - \mu\| < \varepsilon_a, |\hat{\mu} - \mu| < \varepsilon_r \mu\} > 1 - \delta \). In the special case that the variable \( X \) is bounded in interval \([0, 1]\), our multistage sampling schemes and their properties are described by the following theorems.

**Theorem 38** Let \( 0 < \varepsilon_a < \frac{35}{94} \) and \( \frac{70\varepsilon_a}{n_\ell - 2\varepsilon_a} < \varepsilon_r < 1 \). Let \( n_1 < n_2 < \cdots < n_s \) be a sequence of sample sizes such that \( n_s \geq \frac{\ln(2s/\delta)}{\kappa \varepsilon_a + \varepsilon_r(n_1 + \varepsilon_r)} \). Define \( \hat{\mu}_\ell = \frac{\sum_{i=1}^{n_\ell} X_i}{n_\ell} \) for \( \ell = 1, \cdots, s \). Suppose that sampling is continued until \( \max\{\mathcal{M}(|\hat{\mu}_\ell|, \mathcal{V}(\hat{\mu}_\ell)), \mathcal{M}(|\hat{\mu}_\ell + \varepsilon_a, \hat{\mu}_\ell|, \mathcal{V}(\hat{\mu}_\ell))\} \leq \frac{1}{n_\ell} \ln \frac{1}{\delta} \). Define \( \hat{\mu} = \frac{\sum_{i=1}^{n_s} X_i}{n} \), where \( n \) is the sample size when the sampling is terminated. Then, \( \Pr\{\|\hat{\mu} - \mu\| < \varepsilon_a \text{ or } |\hat{\mu} - \mu| < \varepsilon_r \mu\} > 1 - \delta \).

See Appendix I.21 for a proof.

**Theorem 39** Let \( 0 < \varepsilon_a < \frac{2}{3} \) and \( \frac{6\varepsilon_a}{n_\ell - 2\varepsilon_a} < \varepsilon_r < 1 \). Let \( n_1 < n_2 < \cdots < n_s \) be a sequence of sample sizes such that \( n_s \geq 2 \left( \frac{1}{\varepsilon_a} + \frac{1}{3} \right) \left( \frac{1}{\varepsilon_a} - \frac{1}{3} \right) \ln \left( \frac{2s}{\delta} \right) \). Define \( \hat{\mu}_\ell = \frac{\sum_{i=1}^{n_\ell} X_i}{n_\ell} \) and

\[
D_\ell = \begin{cases} 
0 & \text{for } \frac{1}{3} - \frac{2}{3} \varepsilon_a - \sqrt{\frac{1}{3} + \frac{n_\ell \varepsilon_r^2}{2 \ln(\delta)}} < \hat{\mu}_\ell < \frac{6(1-\varepsilon_a)(3-\varepsilon_r)\ln(\delta)}{2(1-\varepsilon_a)^2\ln(\delta)-9n_\ell} \text{ or } \\
\frac{1}{3} + \frac{2}{3} \varepsilon_a - \sqrt{\frac{1}{3} + \frac{n_\ell \varepsilon_r^2}{2 \ln(\delta)}} < \hat{\mu}_\ell < 
\frac{6(1+\varepsilon_a)(3+\varepsilon_r)\ln(\delta)}{2(3+\varepsilon_r)^2\ln(\delta)-9n_\ell} \text{ or } \\
1 & \text{else}
\end{cases}
\]

for \( \ell = 1, \cdots, s \). Suppose that sampling is continued until \( D_\ell = 1 \) for some \( \ell \in \{1, \cdots, s\} \). Define \( \hat{\mu} = \frac{\sum_{i=1}^{n_s} X_i}{n} \), where \( n \) is the sample size when the sampling is terminated. Then, \( \Pr\{\|\hat{\mu} - \mu\| < \varepsilon_a \text{ or } |\hat{\mu} - \mu| < \varepsilon_r \mu\} > 1 - \delta \).

See Appendix I.22 for a proof.

In the general case that \( X \) is a random variable bounded in \([a, b]\), it is useful to estimate the mean \( \mu = \mathbb{E}[X] \) based on i.i.d. samples of \( X \) with a mixed criterion. For this purpose, we shall propose the following multistage estimation methods.
Theorem 40 Let $\varepsilon_a > 0$ and $0 < \varepsilon_r < 1$. Let $n_1 < n_2 < \cdots < n_s$ be a sequence of sample sizes such that $n_s \geq \frac{(b-a)^2}{2s^2} \ln \left(\frac{2s}{\delta}\right)$. Define $\mu_\ell = \frac{n_\ell}{n}$, $\mu_\ell = a + \frac{1}{b-a} \hat{\mu}_\ell$, $\mu_\ell = a + \frac{1}{b-a} \min \left\{ \hat{\mu}_\ell - \varepsilon_a, \frac{\hat{\mu}_\ell}{1 + \sgn(\hat{\mu}_\ell) \varepsilon_r} \right\}$, and $\mu_\ell = a + \frac{1}{b-a} \max \left\{ \hat{\mu}_\ell + \varepsilon_a, \frac{\hat{\mu}_\ell}{1 - \sgn(\hat{\mu}_\ell) \varepsilon_r} \right\}$ for $\ell = 1, \cdots, s$. Suppose that sampling is continued until $M(\mu_\ell, \mu_\ell) \leq \frac{1}{n_\ell} \ln \frac{\delta}{2s}$ and $M(\mu_\ell, \mu_\ell) \leq \frac{1}{n_\ell} \ln \frac{\delta}{2s}$ for some $\ell \in \{1, \cdots, s\}$. Define $\hat{\mu} = \frac{\sum_{i=1}^{n_s} X_i}{n}$, where $n$ is the sample size when the sampling is terminated. Then, $\Pr \{|\hat{\mu} - \mu| < \varepsilon_a \text{ or } |\hat{\mu} - \mu| < \varepsilon_r |\mu| \} \geq 1 - \delta$.

Theorem 41 Let $\varepsilon_a > 0$ and $0 < \varepsilon_r < 1$. Let $n_1 < n_2 < \cdots < n_s$ be a sequence of sample sizes such that $n_s \geq \frac{(b-a)^2}{2s^2} \ln \left(\frac{2s}{\delta}\right)$. Define $\mu_\ell = \frac{n_\ell}{n}$, $\mu_\ell = a + \frac{1}{b-a} \min \left\{ \hat{\mu}_\ell - \varepsilon_a, \frac{\hat{\mu}_\ell}{1 + \sgn(\hat{\mu}_\ell) \varepsilon_r} \right\}$, and $\mu_\ell = a + \frac{1}{b-a} \max \left\{ \hat{\mu}_\ell + \varepsilon_a, \frac{\hat{\mu}_\ell}{1 - \sgn(\hat{\mu}_\ell) \varepsilon_r} \right\}$ for $\ell = 1, \cdots, s$. Suppose that sampling is continued until $M(\mu_\ell, \mu_\ell) \leq \frac{1}{n_\ell} \ln \frac{\delta}{2s}$ and $M(\mu_\ell, \mu_\ell) \leq \frac{1}{n_\ell} \ln \frac{\delta}{2s}$ for some $\ell \in \{1, \cdots, s\}$. Define $\hat{\mu} = \frac{\sum_{i=1}^{n_s} X_i}{n}$, where $n$ is the sample size when the sampling is terminated. Then, $\Pr \{|\hat{\mu} - \mu| < \varepsilon_a \text{ or } |\hat{\mu} - \mu| < \varepsilon_r |\mu| \} \geq 1 - \delta$.

5.4 Using the Link between Binomial and Bounded Variables

Recently, Chen [10] has discovered the following inherent connection between a binomial parameter and the mean of a bounded variable.

Theorem 42 Let $X$ be a random variable bounded in $[0, 1]$. Let $U$ a random variable uniformly distributed over $[0, 1]$. Suppose $X$ and $U$ are independent. Then, $E[X] = \Pr\{X \geq U\}$.

To see why Theorem 42 reveals a relationship between the mean of a bounded variable and a binomial parameter, we define

$$Y = \begin{cases} 1 & \text{for } X \geq U, \\ 0 & \text{otherwise.} \end{cases}$$

Then, by Theorem 42 we have $\Pr\{Y = 1\} = 1 - \Pr\{Y = 0\} = E[X]$. This implies that $Y$ is a Bernoulli random variable and $E[X]$ is actually a binomial parameter. For a sequence of i.i.d. random samples $X_1, X_2, \cdots$ of bounded variable $X$ and a sequence of i.i.d. random samples $U_1, U_2, \cdots$ of uniform variable $U$ such that that $X_i$ is independent with $U_i$ for all $i$, we can define a sequence of i.i.d. random samples $Y_1, Y_2, \cdots$ of Bernoulli random variable $Y$ by

$$Y_i = \begin{cases} 1 & \text{for } Y_i \geq U_i, \\ 0 & \text{otherwise.} \end{cases}$$

As a consequence, the techniques of estimating a binomial parameter can be useful for estimating the mean of a bounded variable.
6 Estimation of Poisson Parameters

In this section, we shall consider the multistage estimation of the mean, \( \lambda \), of a Poisson random variable \( X \) based on its i.i.d. random samples \( X_1, X_2, \ldots \).

For \( \ell = 1, 2, \cdots \), define \( K_\ell = \sum_{i=1}^{n_\ell} X_i \), \( \widehat{\lambda}_\ell = \frac{K_\ell}{n_\ell} \) where \( n_\ell \) is deterministic and stands for the sample size at the \( \ell \)-th stage. As described in the general structure of our multistage estimation framework, the stopping rule is that sampling is continued until \( D_\ell = 1 \) for some \( \ell \in \{1, \cdots, s\} \).
Define estimator \( \widehat{\lambda} = \widehat{\lambda}_l \), where \( l \) is the index of stage at which the sampling is terminated. Clearly, the sample number at the completion of sampling is \( n = n_l \).

6.1 Control of Absolute Error

In this subsection, we shall focus on the design of multistage sampling schemes for estimating the Poisson parameter \( \lambda \) with an absolute error criterion. Specifically, for \( \varepsilon > 0 \), we wish to construct a multistage sampling scheme and its associated estimator \( \widehat{\lambda} \) for \( \lambda \) such that \( \Pr(\{||\widehat{\lambda} - \lambda|| < \varepsilon \mid \lambda\}) > 1 - \delta \) for any \( \lambda \in (0, \infty) \). As will be seen below, our multistage sampling procedures have infinitely many stages and deterministic sample sizes \( n_1 < n_2 < n_3 < \cdots \). Moreover, the confidence parameter for the \( \ell \)-th stage, \( \delta_\ell \), is dependent on \( \ell \) such that \( \delta_\ell = \delta \) for \( 1 \leq \ell \leq \tau \) and \( \delta_\ell = \delta 2^{\ell-\tau} \) for \( \ell > \tau \), where \( \tau \) is a positive integer.

6.1.1 Stopping Rule from CDFs

By virtue of the CDFs of \( \widehat{\lambda}_\ell \), we propose a class of multistage sampling schemes as follows.

**Theorem 43** Suppose that, for \( \ell = 1, 2, \cdots \), decision variable \( D_\ell \) assumes values 1 if \( F_{\lambda_\ell}(\widehat{\lambda}_\ell, \widehat{\lambda}_\ell + \varepsilon) \leq \zeta \delta_\ell \), \( G_{\lambda_\ell}(\widehat{\lambda}_\ell, \widehat{\lambda}_\ell - \varepsilon) \leq \zeta \delta_\ell \); and assumes 0 otherwise. The following statements hold true.

(I): \( \Pr(n < \infty) = 1 \) provided that \( \inf_{\ell \geq 0} \frac{n_{\ell+1}}{n_\ell} > 1 \).

(II): \( \mathbb{E}[n] < \infty \) provided that \( 1 < \inf_{\ell \geq 0} \frac{n_{\ell+1}}{n_\ell} \leq \sup_{\ell \geq 0} \frac{n_{\ell+1}}{n_\ell} < \infty \).

(III): \( \Pr(\{|\widehat{\lambda} - \lambda\| < \varepsilon \mid \lambda\}) > 1 - \delta \) for any \( \lambda > 0 \) provided that \( \zeta \leq \frac{1}{2(\tau+1)} \).

(IV): Let \( 0 < \eta < \zeta \delta \) and \( \ell^* = \tau + 1 + \left[ \frac{\ln(\zeta \delta / \eta)}{\ln 2} \right] \). Then, \( \Pr(\{|\widehat{\lambda} - \lambda\| \geq \varepsilon \mid \lambda\}) < \delta \) for any \( \lambda \in (\lambda, \infty) \), where \( \lambda \) is a number such that \( \lambda > z_\ell, \ell = 1, \cdots, \ell^* \) and that \( \sum_{\ell=1}^{\ell^*} \exp(n_\ell, \mathbb{M}(z_\ell, \lambda)) < \delta - \eta \) with \( z_\ell = \min\{z \in I_{\lambda_\ell}: F_{\lambda_\ell}(z, z + \varepsilon) > \zeta \delta_\ell \) or \( G_{\lambda_\ell}(z, z - \varepsilon) > \zeta \delta_\ell \} \), where \( I_{\lambda_\ell} \) represents the support of \( \widehat{\lambda}_\ell \), for \( \ell = 1, 2, \cdots \). Moreover,

\[
\Pr\left\{ b \leq \widehat{\lambda} - \varepsilon, \ell \leq \ell^* \mid a \right\} \leq \Pr\left\{ \lambda \leq \widehat{\lambda} - \varepsilon \mid \lambda \right\} \leq \frac{\eta}{2} + \Pr\left\{ a \leq \widehat{\lambda} - \varepsilon, \ell \leq \ell^* \mid b \right\},
\]

\[
\Pr\left\{ a \geq \widehat{\lambda} + \varepsilon, \ell \leq \ell^* \mid b \right\} \leq \Pr\left\{ \lambda \geq \widehat{\lambda} + \varepsilon \mid \lambda \right\} \leq \frac{\eta}{2} + \Pr\left\{ b \geq \widehat{\lambda} + \varepsilon, \ell \leq \ell^* \mid a \right\}
\]

for any \( \lambda \in [a, b] \), where \( a \) and \( b \) are numbers such that \( 0 < b < a + \varepsilon \).

(V): Let the sample sizes of the multistage sampling scheme be a sequence \( n_\ell = \left[ m \gamma^{\ell-1} \right], \ell = 1, 2, \cdots \), where \( \gamma \geq 1 + \frac{1}{m} > 1 \). Let \( \varepsilon > 0, 0 < \eta < 1 \) and \( c = -\mathbb{M}\left(\frac{\eta}{\gamma}, \lambda\right) \). Let \( \kappa \) be an integer such that \( \kappa > \max\left\{ \tau, \frac{1}{\ln \gamma} \ln \left( \frac{1}{c m} \right) + 1, \frac{1}{\ln \gamma} \ln \left( \frac{1}{c m} \ln \frac{1}{c m} \right) + 1, \tau + \frac{1}{\gamma - 1} + \frac{\ln(\zeta) \lambda}{\ln 2} \right\} \) and \( \mathbb{M}\left(\frac{\eta}{\gamma}, \lambda + \varepsilon\right) < \frac{\ln(\zeta \delta_\ell \lambda)}{n_\ell} \).

Then, \( \mathbb{E}[n] < \varepsilon + n_1 + \sum_{\ell=1}^{\ell^*} (n_{\ell+1} - n_\ell) \Pr(\ell > \ell) \).
6.1.2 Stopping Rule from Chernoff Bounds

By virtue of Chernoff bounds of the CDFs of $\hat{\lambda}_\ell$, we propose a class of multistage sampling schemes as follows.

**Theorem 44** Suppose that, for $\ell = 1, 2, \cdots$, decision variable $D_\ell$ assumes values 1 if $M_\ell(\hat{\lambda}_\ell, \lambda_\ell + \varepsilon)$ and assumes 0 otherwise. The following statements hold true.

(I): $\Pr\{n < \infty\} = 1$ provided that $\inf_{\lambda > 0} \frac{\ln(\lambda)}{n_\ell} > 1$.

(II): $\mathbb{E}[n] < \infty$ provided that $1 < \inf_{\lambda > 0} \frac{\ln(\lambda)}{n_\ell} \leq \sup_{\lambda > 0} \frac{\ln(\lambda)}{n_\ell} < \infty$.

(III): $\Pr\{\hat{\lambda} - \lambda < \varepsilon \mid \lambda\} \geq 1 - \delta$ for any $\lambda > 0$ provided that $\zeta \leq \frac{1}{2(\tau + 1)}$.

(IV): Let $0 < \eta < \zeta\delta$ and $\ell^* = \tau + 1 + \left[\frac{\ln(\delta/\eta)}{\ln 2}\right]$. Then, $\Pr\{\hat{\lambda} - \lambda < \varepsilon \mid \lambda\} < \delta$ for any $\lambda \in (\lambda, \infty)$, where $\lambda$ is a number such that $\mathbb{E}[n] < z_\ell$, $\ell = \tau, \cdots, \ell^*$ and that $\sum_{\ell=1}^{\ell^*} \exp(n_{\ell} \cdot M_\ell(z_\ell, \lambda)) < \delta - \eta$ with $z_\ell$ satisfying $M_\ell(z_\ell, \lambda + \varepsilon) = \frac{\ln(\delta/\eta)}{n_\ell}$ for $\ell = 1, 2, \cdots$.

Moreover,

$$
\Pr\left\{b \leq \hat{\lambda} - \varepsilon, l \leq \ell^* \mid a\right\} \leq \Pr\left\{a \leq \hat{\lambda} - \varepsilon, l \leq \ell^* \mid b\right\},
$$

$$
\Pr\left\{a \leq b, l \leq \ell^* \mid a\right\} \leq \Pr\left\{b \leq \hat{\lambda} + \varepsilon, l \leq \ell^* \mid b\right\}
$$

for any $\lambda \in [a, b]$, where $a$ and $b$ are numbers such that $0 < b < a + \varepsilon$.

(V): Let the sample sizes of a multistage sampling scheme be a sequence $n_\ell = \left[\frac{\ln(\delta)}{\ln 2}\right]$, $\ell = 1, 2, \cdots$, where $\gamma \geq 1 + \frac{1}{m} > 1$. Let $\varepsilon > 0$, $0 < \eta < 1$ and $c = -M_\ell(\frac{\lambda}{\eta}, \lambda)$. Let $\kappa$ be an integer such that $\kappa \geq \max\left\{\frac{\tau}{1 + \frac{1}{\ln \eta}} \ln \left(\frac{1}{\eta\lambda}\right) + 1, \frac{1}{\ln \eta} \ln \left(\frac{1}{\lambda}\right) \ln \left(\frac{1}{\eta\lambda}\right) + 1, \tau + \frac{1}{\gamma - 1} + \frac{\ln(\delta)}{\ln 2}\right\}$ and $M_\ell(\frac{\lambda}{\eta}, \lambda + \varepsilon) \leq \frac{\ln(\delta)}{n_\ell}$. Then, $\mathbb{E}[n] < \varepsilon + n_1 + \sum_{\ell=1}^\kappa (n_{\ell+1} - n_\ell) \Pr\{l > \ell\}$.

See Appendix 1.4 for a proof.

6.1.3 Asymptotic Analysis of Multistage Sampling Schemes

In this subsection, we shall focus on the asymptotic analysis of the multistage sampling schemes which follow stopping rules derived from Chernoff bounds of CDFs of $\hat{\lambda}_\ell$ as described in Theorem 44.

Let $\lambda^* > 0$. We assume that the sample sizes $n_1, n_2, \cdots$ are chosen as the ascending arrangement of all distinct elements of the set

$$
\left\{\frac{C_{\tau - \ell} \ln(\delta)}{M_\ell(\lambda^*, \lambda^* + \varepsilon)} : \ell = 1, 2, \cdots\right\},
$$

where $\tau$ is the maximum integer such that $\frac{C_{\tau - 1} \ln(\delta)}{M_\ell(\lambda^*, \lambda^* + \varepsilon)} \geq \frac{\ln(\delta)}{\varepsilon}$, i.e., $C_{\tau - 1} \geq -\frac{M_\ell(\lambda^*, \lambda^* + \varepsilon)}{\varepsilon}$. With regard to the asymptotic performance of the sampling scheme, we have

**Theorem 45** Let $N_\varepsilon(\lambda, \varepsilon) = \frac{\ln(\delta)}{M_\ell(\lambda, \lambda + \varepsilon)}$. Let $N_\varepsilon(\lambda, \varepsilon)$ be the minimum sample number $n$ such that $\Pr\{\frac{\sum_{\ell=1}^n \lambda_{\ell} - \lambda}{n} < \varepsilon \mid \lambda\} > 1 - \zeta\delta$ for a fixed-size sampling procedure. Let $j_\lambda$ be the largest integer $j$ such that $C_j \geq \frac{\lambda^*}{\lambda^* - \lambda}$. Let $\nu = \frac{2}{\lambda^*} \ln \left(\frac{1}{\delta}\right)$, and $d = \sqrt{2 \ln \left(\frac{1}{\varepsilon}\right)}$, and $\kappa\lambda = \frac{\lambda}{\lambda^*}C_j$.$\lambda$. Let $\rho_\lambda = \frac{\lambda^*}{\lambda^*}C_j - 1$ if $\kappa\lambda = 1$ and $\rho_\lambda = \kappa\lambda - 1$ otherwise. For $\lambda \in (0, \lambda^*)$, the following statements hold true:
(I): \( \Pr \left\{ 1 \leq \lim_{\varepsilon \to 0} \frac{n}{N_{a}(\lambda, \varepsilon)} \leq 1 + \rho \lambda \right\} = 1 \). Specially, \( \Pr \left\{ \lim_{\varepsilon \to 0} \frac{n}{N_{a}(\lambda, \varepsilon)} = \kappa \lambda \right\} = 1 \) if \( \kappa \lambda > 1 \).

(II): \( \lim_{\varepsilon \to 0} \frac{\mathbb{E}[n]}{N_{a}(\lambda, \varepsilon)} = \left( \frac{d}{2\varepsilon} \right)^{2} \times \lim_{\varepsilon \to 0} \frac{\mathbb{E}[n]}{N_{a}(\lambda, \varepsilon)} \), where

\[
\lim_{\varepsilon \to 0} \frac{\mathbb{E}[n]}{N_{a}(\lambda, \varepsilon)} = \begin{cases} \kappa \lambda & \text{if } \kappa \lambda > 1, \\ 1 + \rho \lambda \Phi(\nu d) & \text{otherwise} \end{cases}
\]

and \( 1 \leq \lim_{\varepsilon \to 0} \frac{\mathbb{E}[n]}{N_{a}(\lambda, \varepsilon)} \leq 1 + \rho \lambda \).

(III): If \( \kappa \lambda > 1 \), then \( \lim_{\varepsilon \to 0} \Pr \{ |\hat{\lambda} - \lambda| < \varepsilon \} = 2 \Phi \left( d \sqrt{\kappa \lambda} \right) - 1 > 2 \Phi (d) - 1 > 1 - 2 \zeta \delta \). Otherwise, \( 2 \Phi (d) - 1 \geq \lim_{\varepsilon \to 0} \Pr \{ |\hat{\lambda} - \lambda| < \varepsilon \} = 1 + \Phi (d) - \Phi (\nu d) - \Psi (\rho, \lambda, d) > 3 \Phi (d) - 2 > 1 - 3 \zeta \delta \).

See Appendix D for a proof.

6.2 Control of Relative Error

In this subsection, we shall focus on the design of multistage sampling schemes for estimating the Poisson parameter \( \lambda \) with a relative error criterion. Specifically, for \( \varepsilon \in (0, 1) \), we wish to construct a multistage sampling scheme and its associated estimator \( \hat{\lambda} \) for \( \lambda \) such that \( \Pr \{ |\hat{\lambda} - \lambda| < \varepsilon \lambda \mid \lambda \} > 1 - \delta \) for any \( \lambda \in (0, \infty) \). As will be seen below, our multistage sampling procedures have infinitely many stages and deterministic sample sizes \( n_{1} < n_{2} < n_{3} < \cdots \). Moreover, the confidence parameter for the \( \ell \)-th stage, \( \delta_{\ell} \), is dependent on \( \ell \) such that \( \delta_{\ell} = \delta \) for \( 1 \leq \ell \leq \tau \) and \( \delta_{\ell} = \delta 2^{\tau - \ell} \) for \( \ell > \tau \), where \( \tau \) is a positive integer.

6.2.1 Stopping Rule from CDFs

By virtue of the CDFs of \( \hat{\lambda}_{\ell} \), we propose a class of multistage sampling schemes as follows.

**Theorem 46** Suppose that, for \( \ell = 1, 2, \cdots \), decision variable \( D_{\ell} \) assumes values 1 if \( F_{\hat{\lambda}_{\ell}} (\hat{\lambda}_{\ell}, \frac{\hat{\lambda}_{\ell}}{1 + \varepsilon}) \leq \zeta \delta_{\ell} \), \( G_{\hat{\lambda}_{\ell}} (\hat{\lambda}_{\ell}, \frac{\hat{\lambda}_{\ell}}{1 + \varepsilon}) \leq \zeta \delta_{\ell} \), and assumes 0 otherwise. The following statements hold true.

(I): \( \Pr \{ n < \infty \} = 1 \) provided that \( \inf_{\ell > 0} \frac{n_{\ell + 1}}{n_{\ell}} > 1 \).

(II): \( \mathbb{E}[n] < \infty \) provided that \( 1 < \inf_{\ell > 0} \frac{n_{\ell + 1}}{n_{\ell}} \leq \sup_{\ell > 0} \frac{n_{\ell + 1}}{n_{\ell}} < \infty \).

(III): \( \Pr \{ \frac{\hat{\lambda} - \lambda}{\lambda} < \varepsilon \mid \lambda \} \geq 1 - \delta \) for any \( \lambda > 0 \) provided that \( \zeta \leq \frac{1}{2(\tau + 1)} \).

(IV): Let \( 0 < \eta < \zeta \delta \) and \( \ell^{*} = \tau + 1 + \left[ \frac{\ln (\zeta \delta / \eta / 2)}{\ln 2} \right] \). Then, \( \Pr \{ |\hat{\lambda} - \lambda| \geq \varepsilon \lambda \mid \lambda \} \leq \delta \) for any \( \lambda \in (0, \lambda) \), where \( \lambda \) is a number such that \( 0 < \lambda < z_{\ell} \), \( \ell = 1, \cdots, \ell^{*} \) and that \( \sum_{\ell = 1}^{\ell^{*}} \exp (n_{\ell} \cdot 1_{\text{mp}} (z_{\ell}, \lambda)) < \delta - \eta \) with \( z_{\ell} = \min \{ z \in I_{\hat{\lambda}_{\ell}} : F_{\hat{\lambda}_{\ell}} (z, \frac{z}{1 + \varepsilon}) > \zeta \delta_{\ell} \) or \( G_{\hat{\lambda}_{\ell}} (z, \frac{z}{1 + \varepsilon}) > \zeta \delta_{\ell} \} \), where \( I_{\hat{\lambda}_{\ell}} \) represents the support of \( \hat{\lambda}_{\ell} \), for \( \ell = 1, 2, \cdots \). Moreover,

\[
\Pr \left\{ b \leq \frac{\hat{\lambda}}{1 + \varepsilon}, l \leq \ell^{*} \mid a \right\} \leq \Pr \left\{ \lambda \leq \frac{\hat{\lambda}}{1 + \varepsilon} \mid \lambda \right\} \leq \frac{\eta}{2} + \Pr \left\{ a \leq \frac{\hat{\lambda}}{1 + \varepsilon}, l \leq \ell^{*} \mid b \right\},
\]

\[
\Pr \left\{ a \geq \frac{\hat{\lambda}}{1 - \varepsilon}, l \leq \ell^{*} \mid b \right\} \leq \Pr \left\{ \lambda \geq \frac{\hat{\lambda}}{1 - \varepsilon} \mid \lambda \right\} \leq \frac{\eta}{2} + \Pr \left\{ b \geq \frac{\hat{\lambda}}{1 - \varepsilon}, l \leq \ell^{*} \mid a \right\}.
\]
for any $\lambda \in [a, b]$, where $a$ and $b$ are numbers such that $0 < b < (1 + \varepsilon)a$.

(V): $\Pr\{(|\lambda - \tilde{\lambda}| \geq \varepsilon \lambda | \lambda\} < \delta$ for any $\lambda \in (\tilde{\lambda}, \infty)$, where $\tilde{\lambda}$ is a number such that $\tilde{\lambda} > z_1$ and that $2\exp(n_1M_P((1 + \varepsilon)\tilde{\lambda}, \tilde{\lambda})) + \exp(n_1M_P(z_1, \tilde{\lambda})) < \delta$.

(VI): Let the sample sizes of the multistage sampling scheme be a sequence $n_\ell = \lceil m\gamma^{\ell-1} \rceil$, $\ell = 1, 2, \cdots$, where $\gamma \geq 1 + \frac{1}{m} > 1$. Let $\varepsilon > 0$, $0 < \eta < 1$ and $\varepsilon = -M_P(\eta\lambda, \lambda)$. Let $\kappa$ be an integer such that $\kappa > max\{\tau, \frac{\ln\gamma}{\ln m} + 1, \frac{\ln\gamma}{\ln m} \ln \frac{\tilde{\lambda}}{\varepsilon}, \tau + \frac{1}{\gamma} + 1, \frac{\ln(c_0\ell)}{\ln 2}\}$ and $M_P(\eta\lambda, \frac{\eta\lambda}{1 + \varepsilon}) < \frac{\ln(c_0\ell)}{n_\kappa}$. Then, $\mathbb{E}[n] < \varepsilon + n_1 + \sum_{\ell=1}^{n}(n_{\ell+1} - n_\ell) \Pr\{I > \ell\}$.

6.2.2 Stopping Rule from Chernoff Bounds

By virtue of Chernoff bounds of the CDFs of $\tilde{\lambda}_\ell$, we propose a class of multistage sampling schemes as follows.

**Theorem 47** Suppose that, for $\ell = 1, 2, \cdots$, decision variable $D_\ell$ assumes values 1 if $\tilde{\lambda}_\ell \geq \frac{\ln(c_0\ell)}{n_\varepsilon} \varepsilon^{1-(1+\varepsilon)\ln(1+\varepsilon)}$; and assumes 0 otherwise. The following statements hold true.

(I): $\Pr\{n < \infty\} = 1$ provided that $\inf_{\ell > 0} \frac{n_{\ell+1}}{n_\ell} > 1$.

(II): $\mathbb{E}[n] < \infty$ provided that $1 < \inf_{\ell > 0} \frac{n_{\ell+1}}{n_\ell} \leq \sup_{\ell > 0} \frac{n_{\ell+1}}{n_\ell} < \infty$.

(III): $\Pr\{(|\lambda - \tilde{\lambda}| \geq \varepsilon \lambda | \lambda\} < 1 - \delta$ for any $\lambda > 0$ provided that $\zeta \leq \frac{1}{2(\tau+1)}$.

(IV): Let $0 < \eta < \zeta$ and $\ell^* = \tau + 1 + \left\lceil \frac{\ln(c_0\eta\varepsilon)}{\ln 2}\right\rceil$. Then, $\Pr\{(|\lambda - \tilde{\lambda}| \geq \varepsilon \lambda | \lambda\} < \delta$ for any $\lambda \in (0, \tilde{\lambda})$, where $\tilde{\lambda}$ is a number such that $0 < \tilde{\lambda} < z_\ell$, $\ell = \tau, \cdots, \ell^*$ and that $\sum_{\ell=1}^{\ell^*} \exp(n_\ell M_P(z_\ell, \tilde{\lambda})) < \delta - \eta$ with $z_\ell = \frac{\ln(c_0\ell)}{n_\varepsilon} \varepsilon^{1-(1+\varepsilon)\ln(1+\varepsilon)}$ for $\ell = 1, 2, \cdots$. Moreover,

$$\Pr\left\{b \leq \frac{\tilde{\lambda}}{1+\varepsilon}, l \leq \ell^* | a\right\} \leq \Pr\left\{\lambda \leq \frac{\tilde{\lambda}}{1+\varepsilon} | \lambda\right\} \leq \frac{\eta}{2} + \Pr\left\{a \leq \frac{\tilde{\lambda}}{1+\varepsilon}, l \leq \ell^* | b\right\},$$

$$\Pr\left\{a \geq \frac{\tilde{\lambda}}{1-\varepsilon}, l \leq \ell^* | b\right\} \leq \Pr\left\{\lambda \geq \frac{\tilde{\lambda}}{1-\varepsilon} | \lambda\right\} \leq \frac{\eta}{2} + \Pr\left\{b \geq \frac{\tilde{\lambda}}{1-\varepsilon}, l \leq \ell^* | a\right\}$$

for any $\lambda \in [a, b]$, where $a$ and $b$ are numbers such that $0 < b < (1 + \varepsilon)a$.

(V): $\Pr\{(|\lambda - \tilde{\lambda}| \geq \varepsilon \lambda | \lambda\} < \delta$ for any $\lambda \in (\tilde{\lambda}, \infty)$, where $\tilde{\lambda}$ is a number such that $\tilde{\lambda} > z_1$ and that $2\exp(n_1M_P((1 + \varepsilon)\tilde{\lambda}, \tilde{\lambda})) + \exp(n_1M_P(z_1, \tilde{\lambda})) < \delta$.

(VI): Let the sample sizes of the multistage sampling scheme be a sequence $n_\ell = \lceil m\gamma^{\ell-1} \rceil$, $\ell = 1, 2, \cdots$, where $\gamma \geq 1 + \frac{1}{m} > 1$. Let $\varepsilon > 0$, $0 < \eta < 1$ and $\varepsilon = -M_P(\eta\lambda, \lambda)$. Let $\kappa$ be an integer such that $\kappa > max\{\tau, \frac{\ln\gamma}{\ln m} + 1, \frac{\ln\gamma}{\ln m} \ln \frac{\tilde{\lambda}}{\varepsilon}, \tau + \frac{1}{\gamma} + 1, \frac{\ln(c_0\ell)}{\ln 2}\}$ and $M_P(\eta\lambda, \frac{\eta\lambda}{1 + \varepsilon}) < \frac{\ln(c_0\ell)}{n_\kappa}$. Then, $\mathbb{E}[n] < \varepsilon + n_1 + \sum_{\ell=1}^{n}(n_{\ell+1} - n_\ell) \Pr\{I > \ell\}$.

See Appendix J.3 for a proof.

6.2.3 Asymptotic Analysis of Multistage Sampling Schemes

In this subsection, we shall focus on the asymptotic analysis of the multistage sampling schemes which follow stopping rules derived from Chernoff bounds of CDFs of $\tilde{\lambda}_\ell$ as described in Theorem
We assume that the sample sizes $n_1, n_2, \ldots$ are chosen as the ascending arrangement of all distinct elements of the set
\[
\left\{ \frac{C_{\tau-\ell} \ln(\zeta \delta)}{\mathcal{M}_p \left( \lambda', \lambda'' \right)} : \ell = 1, 2, \ldots \right\}
\]
with $0 < \lambda' < \lambda''$, where $\tau$ is the maximum integer such that $\frac{C_{\tau-1} \ln(\zeta \delta)}{\mathcal{M}_p \left( \lambda', \lambda'' \right)} \geq \frac{\ln(\zeta \delta)}{\mathcal{M}_p \left( \lambda', \lambda'' \right)}$, i.e.,
\[
C_{\tau-1} \geq \frac{\mathcal{M}_p \left( \lambda', \lambda'' \right)}{\mathcal{M}_p \left( \lambda', \lambda'' \right)}.
\]
With regard to the asymptotic performance of the sampling scheme, we have

**Theorem 48** Let $\mathcal{N}_f(\lambda, \varepsilon)$ be the minimum sample number $n$ such that $\Pr[|\sum_{i=1}^n X_i - \lambda| < \varepsilon \lambda | \lambda] > 1 - \zeta \delta$ for a fixed-size sampling procedure. Let $j_\lambda$ be the largest integer $j$ such that $C_j \geq \frac{\lambda'}{\lambda}$. Let $d = \sqrt{2 \ln \frac{1}{\varepsilon}}$ and $\kappa_\lambda = \frac{d}{\lambda} C_{j_\lambda}$. Let $\rho_\lambda = \frac{\lambda'}{\lambda} C_{j_\lambda-1} - 1$ if $\kappa_\lambda = 1$ and $\rho_\lambda = \kappa_\lambda - 1$ otherwise. For $\lambda \in (\lambda', \lambda'')$, the following statements hold true:

1. $\Pr \left\{ \sup_{\varepsilon \to 0} \frac{n}{\mathcal{N}_f(\lambda, \varepsilon)} \leq 1 + \rho_\lambda \right\} = 1$. Specially, $\Pr \left\{ \lim_{\varepsilon \to 0} \frac{n}{\mathcal{N}_f(\lambda, \varepsilon)} = \kappa_\lambda \right\} = 1$ if $\kappa_\lambda > 1$.

2. $\lim_{\varepsilon \to 0} \frac{\mathbb{E}[n]}{\mathcal{N}_f(\lambda, \varepsilon)} = \begin{cases} 
\kappa_\lambda & \text{if } \kappa_\lambda > 1, \\
1 + \rho_\lambda & \text{otherwise}
\end{cases}$

and $1 \leq \lim_{\varepsilon \to 0} \frac{\mathbb{E}[n]}{\mathcal{N}_f(\lambda, \varepsilon)} \leq 1 + \rho_\lambda$.

3. $\lim_{\varepsilon \to 0} \Pr \left\{ \hat{\lambda} - \lambda | < \varepsilon \lambda \right\} = 2 \Phi \left( d/\sqrt{\kappa_\lambda} \right) - 1 \geq 2 \Phi \left( d \right) - 1 > 1 - 2 \zeta \delta$.

See Appendix J.4 for a proof.

### 6.3 Control of Absolute and Relative Errors

In this section, we shall focus on the design of multistage sampling schemes for estimating Poisson parameter $\lambda$ with a mixed error criterion. Specifically, for $\varepsilon_a > 0$ and $0 < \varepsilon_r < 1$, we wish to construct a multistage sampling scheme and its associated estimator $\hat{\lambda}$ for $\lambda$ such that $\Pr[|\hat{\lambda} - \lambda| < \varepsilon_a, |\hat{\lambda} - \lambda| < \varepsilon_r \lambda | \lambda] > 1 - \delta$ for any $\lambda \in (0, \infty)$. This is equivalent to the construction of a random interval with lower limit $\mathcal{L}(\hat{\lambda})$ and upper limit $\mathcal{U}(\hat{\lambda})$ such that $\Pr[\mathcal{L}(\hat{\lambda}) < \lambda < \mathcal{U}(\hat{\lambda}) | \lambda] > 1 - \delta$ for any $\lambda \in (0, \infty)$, where $\mathcal{L}(.)$ and $\mathcal{U}(.)$ are functions such that $\mathcal{L}(z) = \min\{z - \varepsilon_a, \frac{z}{1 - \varepsilon_r}\}$ and $\mathcal{U}(z) = \max\{z + \varepsilon_a, \frac{z}{1 - \varepsilon_r}\}$ for $z \in [0, \infty)$. In the sequel, we shall propose multistage sampling schemes such that the number of stages, $s$, is finite and that the sample sizes are deterministic numbers $n_1 < n_2 < \cdots < n_s$.

#### 6.3.1 Stopping Rules from CDFs and Chernoff Bounds

To estimate $\lambda$ with a mixed precision criterion, we propose two types of multistage sampling schemes with different stopping rules as follows.
Theorem 49 Suppose that the sample size for the \( s \)-th stage is no less than \[
\ln(\delta) \left/ \frac{n_{\ell}}{M}\right. ,
\] Then,
\[
\Pr\{\lambda \leq \mathcal{L}(\hat{\lambda}) \mid \lambda\} \leq \sum_{\ell=1}^{s} \Pr\{\lambda \leq \mathcal{L}(\hat{\lambda}_{\ell}), \ D_{\ell} = 1 \mid \lambda\} \leq s \zeta \delta,
\]
\[
\Pr\{\lambda \geq \mathcal{U}(\hat{\lambda}) \mid \lambda\} \leq \sum_{\ell=1}^{s} \Pr\{\lambda \geq \mathcal{U}(\hat{\lambda}_{\ell}), \ D_{\ell} = 1 \mid \lambda\} \leq s \zeta \delta
\]
for any \( \lambda > 0 \). Moreover, \( \Pr\{|\hat{\lambda} - \lambda| < \varepsilon_{a} \text{ or } |\hat{\lambda} - \lambda| < \varepsilon_{r} \mid \lambda\} > 1 - \delta \) for any \( \lambda > 0 \) provided that
\[
\Pr\{\lambda \leq \mathcal{L}(\hat{\lambda}) \mid \lambda\} + \Pr\{\lambda \geq \mathcal{U}(\hat{\lambda}) \mid \lambda\} < \delta \text{ for any } \lambda \in (0, \overline{\lambda}), \text{ where } \overline{\lambda} > 0 \text{ is the unique number satisfying } \sum_{\ell=1}^{s} \exp(n_{\ell} M(\overline{\lambda} + \varepsilon_{r}, \overline{\lambda})) = \frac{\delta}{\overline{\lambda}}.
\]

See Appendix 15 for a proof. Based on the criteria proposed in Section 2.1, the sample sizes \( n_{1} < n_{2} < \cdots < n_{s} \) can be chosen as the ascending arrangement of all distinct elements of
\[
\left\{ \left\lceil \frac{C_{\tau-1} \ln(\delta)}{M \left( \frac{\varepsilon_{a}}{\varepsilon_{r}} + \varepsilon_{a}, \frac{\varepsilon_{r}}{\varepsilon_{a}} \right)} \right\rceil : \ell = 1, \cdots, \tau \right\}, \tag{31}
\]
where \( \tau \) is the maximum integer such that
\[
\frac{C_{\tau-1} \ln(\delta)}{M \left( \frac{\varepsilon_{a}}{\varepsilon_{r}} + \varepsilon_{a}, \frac{\varepsilon_{r}}{\varepsilon_{a}} \right)} \geq \frac{\ln \frac{1}{\delta}}{\varepsilon_{a}}, \text{ i.e., } C_{\tau-1} \geq -\frac{M \left( \frac{\varepsilon_{a}}{\varepsilon_{r}} + \varepsilon_{a}, \frac{\varepsilon_{r}}{\varepsilon_{a}} \right)}{\varepsilon_{a}}. \text{ For such a choice of sample sizes, as a result of Theorem 49, we have that } \Pr\{|\hat{\lambda} - \lambda| < \varepsilon_{a} \text{ or } |\hat{\lambda} - \lambda| < \varepsilon_{r} \mid \lambda\} > 1 - \delta \text{ for any } \lambda > 0 \text{ provided that } \zeta < \frac{1}{2^\tau}.
\]

To evaluate the coverage probability associated with a multistage sampling scheme following a stopping rule derived from Chernoff bounds, we need to express \( \{D_{\ell} = i\} \) in terms of \( K_{\ell} \). For this purpose, the following result is useful.

Theorem 50 Let \( \lambda^{*} = \frac{\varepsilon_{a}}{\varepsilon_{r}} \). Then, \( \{D_{\ell} = 0\} = \{M(\hat{\lambda}_{\ell}, \mathcal{L}(\hat{\lambda}_{\ell})) > \frac{\ln(\delta)}{n_{\ell}}\} \cup \{M(\hat{\lambda}_{\ell}, \mathcal{U}(\hat{\lambda}_{\ell})) > \frac{\ln(\delta)}{n_{\ell}}\} \) for \( \ell = 1, \cdots, s - 1 \) and the following statements hold true:

(I) \( \{M(\hat{\lambda}_{\ell}, \mathcal{L}(\hat{\lambda}_{\ell})) > \frac{\ln(\delta)}{n_{\ell}}\} = \{n_{\ell} \varepsilon_{a} < K_{\ell} < n_{\ell} \varepsilon_{r}\} \) where \( \varepsilon_{r} \) is the unique solution of equation \( M(z, \frac{\varepsilon_{a}}{\varepsilon_{r}}) = \frac{\ln(\delta)}{n_{\ell}} \) with respect to \( z \in (\lambda^{*} + \varepsilon_{a}, \infty) \), and \( \varepsilon_{a} \) is the unique solution of equation \( M(z, \frac{\varepsilon_{a}}{\varepsilon_{r}}) = \frac{\ln(\delta)}{n_{\ell}} \) with respect to \( z \in (\varepsilon_{a}, \lambda^{*} + \varepsilon_{a}) \).
\[
\{ M_P(\lambda, \mathscr{H}(\hat{\lambda})) > \frac{\ln(\zeta \delta)}{n_\ell} \} = \begin{cases}
\{0 \leq K_\ell < n_\ell \ z^-_a\} & \text{for } n_\ell < \frac{\ln \delta}{\varepsilon_a}, \\
\{n_\ell \ z^+_a < K_\ell < n_\ell \ z^-_a\} & \text{for } \frac{\ln \delta}{\varepsilon_a} \leq n_\ell < \frac{\ln(\zeta \delta)}{M_P(\lambda^*, \varepsilon_a, \lambda)}, \\
\emptyset & \text{for } n_\ell \geq \frac{\ln(\zeta \delta)}{M_P(\lambda^*, \varepsilon_a, \lambda)}
\end{cases}
\]

where \(z^-_a\) is the unique solution of equation \(M_P(z, \frac{1}{\tau^*}) = \frac{\ln(\zeta \delta)}{n_\ell}\) with respect to \(z \in (\lambda^* - \varepsilon_a, \infty)\), and \(z^+_a\) is the unique solution of equation \(M_P(z, z + \varepsilon_a) = \frac{\ln(\zeta \delta)}{n_\ell}\) with respect to \(z \in [0, \lambda^* - \varepsilon_a)\).

Theorem 50 can be shown by a variation of the argument for Theorem 29.

### 6.3.2 Asymptotic Stopping Rule

It should be noted that, for small \(\varepsilon_a\) and \(\varepsilon_r\), we can simplify, by using Taylor’s series expansion formula \(\ln(1 + x) = x - \frac{x^2}{2} + o(x^2)\), the sampling schemes as described in Section 6.3.1 as follows:

(i) The sequence of sample sizes \(n_1, \ldots, n_s\) is defined as the ascending arrangement of all distinct elements of \(\{C_{\tau - \ell} \left(\frac{2}{\varepsilon_r} \right) \ln \frac{1}{\xi}\} : \ell = 1, \ldots, \tau\}\), where \(\tau\) is the maximum integer such that \(C_{\tau - 1} \geq \frac{\Delta}{\eta}\).

(ii) The decision variables are defined such that \(D_\ell = 1\) if \(n_\ell \geq \frac{\lambda^* \ 2 \ln \frac{1}{\xi \delta}}{\max\{\varepsilon_a(\varepsilon_a, \lambda)\}^2}\); and \(D_\ell = 0\) otherwise.

For such a simplified sampling scheme, we have

\[
\sum_{\ell=1}^s \Pr\{\hat{\lambda}_\ell - \lambda \geq \max\{\varepsilon_a, \varepsilon_r \lambda\}, \ D_\ell = 1\} \leq \sum_{\ell=1}^s \Pr\{\hat{\lambda}_\ell - \lambda \geq \max\{\varepsilon_a, \varepsilon_r \lambda\}\}
\]

\[
\leq \sum_{\ell=1}^\tau \Pr\{\hat{\lambda}_\ell - \lambda \geq \max\{\varepsilon_a, \varepsilon_r \lambda\}\}
\]

\[
\leq \sum_{\ell=1}^\tau 2 \exp\left(n_\ell \cdot M_P\left(\frac{\varepsilon_a}{\varepsilon_r} + \varepsilon_a, \frac{\varepsilon_a}{\varepsilon_r}\right)\right)
\]

\[
< 2\tau \exp\left(n_1 \cdot M_P\left(\frac{\varepsilon_a}{\varepsilon_r} + \varepsilon_a, \frac{\varepsilon_a}{\varepsilon_r}\right)\right),
\]

where (32) is due to Theorem 1 of [9]. As can be seen from (32), the last bound is independent of \(\lambda\) and can be made smaller than \(\delta\) if \(\zeta\) is sufficiently small. This establishes the claim and it follows that \(\Pr\left\{\hat{\lambda} - \lambda < \varepsilon_a\ or \ \frac{\hat{\lambda} - \lambda}{\lambda} < \varepsilon_r \mid \lambda\right\} > 1 - \delta\) for any \(\lambda \in (0, \infty)\) if \(\zeta\) is sufficiently small.

### 6.3.3 Asymptotic Analysis of Multistage Sampling Schemes

In this subsection, we shall focus on the asymptotic analysis of multistage inverse sampling schemes. Throughout this subsection, we assume that the multistage sampling schemes follow stopping rules derived from Chernoff bounds as described in Section 6.3.1. Moreover, we assume
that the sample sizes \( n_1, \ldots, n_s \) are chosen as the ascending arrangement of all distinct elements of the set defined by (31).

With regard to the tightness of the double-decision-variable method, we have

**Theorem 51** Let \( \mathcal{R} \) be a subset of real numbers. Define

\[
\mathcal{P} = \sum_{\ell=1}^{s} \Pr\{\hat{\lambda}_\ell \in \mathcal{R}, D_{\ell-1} = 0, D_\ell = 1\}, \quad P = 1 - \sum_{\ell=1}^{s} \Pr\{\hat{\lambda}_\ell \notin \mathcal{R}, D_{\ell-1} = 0, D_\ell = 1\}.
\]

Then, \( P \leq \Pr\{\hat{\lambda} \in \mathcal{R}\} \leq \mathcal{P} \) and \( \lim_{\epsilon_n \to 0} |\Pr\{\hat{\lambda} \notin \mathcal{R}\} - \mathcal{P}| = \lim_{\epsilon_n \to 0} |\Pr\{\hat{\lambda} \notin \mathcal{R}\} - P| = 0 \) for any \( \lambda \in (0, \infty) \), where the limits are taken under the constraint that \( \hat{\epsilon}_n \) is fixed.

See Appendix J.6 for a proof.

With regard to the asymptotic performance of the sampling scheme as \( \epsilon_a \) and \( \epsilon_r \) tend to 0, we have

**Theorem 52** Let \( \mathcal{N}_1(\lambda, \epsilon_a, \epsilon_r) \) be the minimum sample number \( n \) such that

\[
\Pr \left\{ \left| \frac{\sum_{i=1}^{n} X_i}{n} - \lambda \right| < \epsilon_a \text{ or } \left| \frac{\sum_{i=1}^{n} X_i}{n} - \lambda \right| < \epsilon_r \lambda \mid \lambda \right\} > 1 - \delta
\]

for a fixed-size sampling procedure. Let \( \mathcal{N}_m(\lambda, \epsilon_a, \epsilon_r) = \frac{\ln(\delta)}{\max\{\epsilon_r(\lambda, \hat{\lambda}), \epsilon_r(\lambda, \lambda)\}} \), where \( \hat{\lambda} = \min\{\lambda - \epsilon_a, \frac{\lambda}{1+\epsilon_r}\} \) and \( \lambda = \max\{\lambda + \epsilon_a, \frac{\lambda}{1+\epsilon_r}\} \). Define \( \lambda^* = \frac{\epsilon_a}{\epsilon_r} \), \( d = \sqrt{2 \ln \frac{1}{\delta}} \),

\[
r(\lambda) = \begin{cases} \frac{\lambda}{\lambda^*} & \text{for } \lambda \in (0, \lambda^*], \\ \frac{\lambda^*}{\lambda} & \text{for } \lambda \in (\lambda^*, \infty) \\ 0 & \text{for } \lambda \in (0, \lambda^*). \end{cases}
\]

Let \( \kappa_\lambda = \frac{C_{\lambda}}{r(\lambda)} \), where \( j_\lambda \) is the maximum integer \( j \) such that \( C_j \geq r(\lambda) \). Let \( \rho_\lambda = \frac{C_{\lambda}}{r(\lambda)} - 1 \) if \( \kappa_\lambda = 1 \), \( j_\lambda > 0 \) and \( \rho_\lambda = \kappa_\lambda - 1 \) otherwise. The following statements hold true under the condition that \( \frac{\hat{\epsilon}_a}{\hat{\epsilon}_r} \) is fixed.

(I): \( \Pr \left\{ 1 \leq \limsup_{\epsilon_n \to 0} \frac{\mathcal{N}_m(\lambda, \epsilon_a, \epsilon_r)}{\mathcal{N}_m(\lambda, \epsilon_a, \epsilon_r)} \leq 1 + \rho_\lambda \right\} = 1. \quad \text{Specially, } \Pr \left\{ \lim_{\epsilon_n \to 0} \frac{\mathcal{N}_m(\lambda, \epsilon_a, \epsilon_r)}{\mathcal{N}_m(\lambda, \epsilon_a, \epsilon_r)} = \kappa_\lambda \right\} = 1 \) if \( \kappa_\lambda > 1 \).

(II): \( \lim_{\epsilon_n \to 0} \frac{\mathbb{E}[n]}{\mathcal{N}_m(\lambda, \epsilon_a, \epsilon_r)} \leq \frac{d}{\epsilon_r} \times \lim_{\epsilon_n \to 0} \frac{\mathbb{E}[n]}{\mathcal{N}_m(\lambda, \epsilon_a, \epsilon_r)}, \) where

\[
\lim_{\epsilon_n \to 0} \frac{\mathbb{E}[n]}{\mathcal{N}_m(\lambda, \epsilon_a, \epsilon_r)} = \begin{cases} \kappa_\lambda & \text{if } \kappa_\lambda > 1, \\ 1 + \rho_\lambda \Phi(\nu d) & \text{otherwise} \end{cases}
\]

and \( 1 \leq \lim_{\epsilon_n \to 0} \frac{\mathbb{E}[n]}{\mathcal{N}_m(\lambda, \epsilon_a, \epsilon_r)} \leq 1 + \rho_\lambda \).

(III): If \( \kappa_\lambda > 1 \), then \( \lim_{\epsilon_n \to 0} \Pr\{|\hat{\lambda} - \lambda| < \epsilon_a \text{ or } |\hat{\lambda} - \lambda| < \epsilon_r \lambda\} = 2\Phi\left(d\sqrt{\kappa_\lambda}\right) - 1 > 2\Phi(d) - 1 > 1 - 2\delta. \)

If \( \kappa_\lambda = 1 \) and \( \lambda \geq \lambda^* \), then \( \lim_{\epsilon_n \to 0} \Pr\{|\hat{\lambda} - \lambda| < \epsilon_a \text{ or } |\hat{\lambda} - \lambda| < \epsilon_r \lambda\} = 2\Phi(d) - 1 > 1 - 2\delta. \)

If \( \kappa_\lambda = 1 \) and \( \lambda < \lambda^* \), then \( 2\Phi(d) - 1 > \lim_{\epsilon_n \to 0} \Pr\{|\hat{\lambda} - \lambda| < \epsilon_a \text{ or } |\hat{\lambda} - \lambda| < \epsilon_r \lambda\} = 1 + \Phi(d) - \Phi(\nu d) - \Psi(\rho_\lambda, \nu, d) > 3\Phi(d) - 2 > 1 - 3\delta. \)

See Appendix J.7 for a proof.


7 Estimation of Finite Population Proportion

In this section, we consider the problem of estimating the proportion of a finite population, which has been discussed in Section 2.4. We shall focus on multistage sampling schemes with deterministic sample sizes \( n_1 < n_2 < \cdots < n_s \). Our methods are described in the sequel.

Define \( K_\ell = \sum_{i=1}^{n_\ell} X_i \), \( \bar{p}_\ell = \frac{K_\ell}{n_\ell} \) for \( \ell = 1, \cdots, s \). Suppose the stopping rule is that sampling without replacement is continued until \( D_\ell = 1 \) for some \( \ell \in \{1, \cdots, s\} \). Define \( \hat{p} = \hat{p}_l \), where \( l \) is the index of stage at which the sampling is terminated.

By using various functions to define random intervals, we can unify the estimation problems associated with absolute, relative and mixed precision. Specifically, for estimating \( D \) the index of stage at which the sampling is terminated.

In this section, we consider the problem of estimating the proportion of a finite population, for \( \ell \) that \( n \) is deterministic sample sizes which has been discussed in Section 2.6. We shall focus on multistage sampling schemes with mixed precision can be cast as the general problem of constructing a random interval with lower limit \( \mathcal{L}(\hat{p}) \) and upper limit \( \mathcal{U}(\hat{p}) \) such that \( \Pr\{\mathcal{L}(\hat{p}) < p < \mathcal{U}(\hat{p})\} \geq 1 - \delta \). For this purpose, making use of Theorems 2 and 5, we immediately obtain the following result.

**Corollary 3** Suppose the sample size of the \( s \)-th stage is no less than the minimum number \( n \) such that \( 1 - S_N(k - 1, n, \mathcal{L}(\frac{k}{n})) \leq \zeta \delta \) and \( S_N(k, n, \mathcal{U}(\frac{k}{n})) \leq \zeta \delta \) for \( 0 \leq k \leq n \). For \( \ell = 1, \cdots, s \), define \( D_\ell \) such that \( D_\ell \) assumes value 1 if \( 1 - S_N(K_\ell - 1, n_\ell, \mathcal{L}(\hat{p}_\ell)) \leq \zeta \delta \), \( S_N(K_\ell, n_\ell, \mathcal{U}(\hat{p}_\ell)) \leq \zeta \delta \); and assumes value 0 otherwise. Then,

\[
\Pr\{p \leq \mathcal{L}(\hat{p}) \mid p \} \leq \sum_{\ell=1}^{s} \Pr\{p \leq \mathcal{L}(\hat{p}_\ell), D_\ell = 1 \mid p \} \leq s \zeta \delta,
\]

\[
\Pr\{p \geq \mathcal{U}(\hat{p}) \mid p \} \leq \sum_{\ell=1}^{s} \Pr\{p \geq \mathcal{U}(\hat{p}_\ell), D_\ell = 1 \mid p \} \leq s \zeta \delta
\]

and \( \Pr\{\mathcal{L}(\hat{p}) < p < \mathcal{U}(\hat{p}) \mid p \} \geq 1 - 2s \zeta \delta \) for any \( p \in \Theta \).

Let

\[
n_{\text{min}} = 1 + \max \left\{ n : 1 - S_N(k - 1, n, \mathcal{L}(\frac{k}{n})) > \zeta \delta \text{ or } S_N(k, n, \mathcal{U}(\frac{k}{n})) > \zeta \delta \text{ for } 0 \leq k \leq n \right\},
\]

\[
n_{\text{max}} = \min \left\{ n : 1 - S_N(k - 1, n, \mathcal{L}(\frac{k}{n})) \leq \zeta \delta \text{ and } S_N(k, n, \mathcal{U}(\frac{k}{n})) \leq \zeta \delta \text{ for } 0 \leq k \leq n \right\}.
\]

53
Based on the criteria proposed in Section 2.1, the sample sizes \( n_1 < n_2 < \cdots < n_s \) can be chosen as the ascending arrangement of all distinct elements of the set \{ \lfloor C_{\tau - \ell} n_{\max} \rfloor : 1 \leq \ell \leq \tau \}, \) where \( \tau \) is the maximum integer such that \( C_{\tau - 1} \geq \frac{\max}{n_{\max}}. \)

Now, define

\[
C(z, p, n, N) = \begin{cases} \binom{Np}{z} n^{-z} \binom{N}{z} \binom{N-n}{n-z} \binom{n-z}{\ell} \binom{N}{\ell} & \text{for } z = 1, \\
\binom{Np}{z} \binom{N}{z} \binom{N-n}{n-z} \binom{n-z}{\ell} \binom{N}{\ell} & \text{for } z \in \left\{ \frac{k}{n} : k \in \mathbb{Z}, 0 \leq k < n \right\}
\end{cases}
\] (34)

where \( p \in \Theta. \) In order to develop multistage sampling schemes with simple stopping boundaries, we have the following results.

**Corollary 4** Suppose the sample size of the \( s \)-th stage is no less than the minimum number \( n \) such that \( C\left( \frac{k}{n}, \mathcal{L}\left( \frac{k}{n} \right), n, N \right) \leq \varepsilon \delta \) and \( C\left( \frac{k}{n}, \mathcal{U}\left( \frac{k}{n} \right), n, N \right) \leq \varepsilon \delta \) for \( 0 \leq k \leq n. \) For \( \ell = 1, \cdots, s, \) define \( D_\ell \) such that \( D_\ell \) assumes value 1 if \( C\left( \tilde{p}_\ell, \mathcal{L}\left( \tilde{p}_\ell \right), n_\ell, N \right) \leq \varepsilon \delta, \) \( C\left( \tilde{p}_\ell, \mathcal{U}\left( \tilde{p}_\ell \right), n_\ell, N \right) \leq \varepsilon \delta; \) and assumes value 0 otherwise. Then,

\[
\Pr\{ p \leq \mathcal{L}(\tilde{p}) \mid p \} \leq \sum_{\ell=1}^{s} \Pr\{ p \leq \mathcal{L}(\tilde{p}_\ell) \mid \mathcal{L}(\tilde{p}_\ell) \leq \varepsilon \delta \}
\]

\[
\Pr\{ p \geq \mathcal{U}(\tilde{p}) \mid p \} \leq \sum_{\ell=1}^{s} \Pr\{ p \geq \mathcal{U}(\tilde{p}_\ell) \mid \mathcal{U}(\tilde{p}_\ell) \leq \varepsilon \delta \}
\]

and \( \Pr\{ \mathcal{L}(\tilde{p}) < p < \mathcal{U}(\tilde{p}) \mid p \} \geq 1 - 2s\varepsilon \delta \) for any \( p \in \Theta. \)

Corollary 4 can be shown by using Theorems 2, 5 and the inequalities obtained by Chen [13] as follows:

\[
\Pr\left\{ \frac{\sum_{i=1}^{n} X_i}{n} \leq z \mid p \right\} \leq C(z, p, n, N) \quad \text{for } z \in \left\{ \frac{k}{n} : k \in \mathbb{Z}, np \leq k \leq n \right\},
\] (35)

\[
\Pr\left\{ \frac{\sum_{i=1}^{n} X_i}{n} \geq z \mid p \right\} \leq C(z, p, n, N) \quad \text{for } z \in \left\{ \frac{k}{n} : k \in \mathbb{Z}, 0 \leq k \leq np \right\}
\] (36)

where \( p \in \Theta. \) Since \( \sum_{i=1}^{n} X_i \) has a hypergeometric distribution, the above inequalities (35) and (36) provide simple bounds for the tail probabilities of hypergeometric distribution, which are substantially less conservative than Hoeffding’s inequalities [27].

It is well known that, for a sampling without replacement with size \( n, \) to guarantee that the estimator \( \hat{p} = \frac{\sum_{i=1}^{n} X_i}{n} \) of the proportion \( p = \frac{M}{N} \) satisfy \( \Pr\{ ||\hat{p} - p\| \leq \varepsilon \} \geq 1 - \delta, \) it suffices to have

\[
n \geq \frac{Np(1-p)}{\varepsilon^2} \left( \frac{1}{n} \right) \left( \frac{1}{N(n-1)} \right) (p(1-p))^2 \leq (N-1)\varepsilon^2 \quad \text{see formula (1) in page 41 of [38].}
\]

Therefore, for a very small margin of absolute error \( \varepsilon, \) we can develop simple multistage sampling schemes based normal approximation as follows.

To estimate the population proportion \( p \in \Theta \) with margin of absolute error \( \varepsilon \in (0, 1), \) we can choose the sample sizes \( n_1 < n_2 < \cdots < n_s \) as the ascending arrangement of all distinct elements
of the set $\left\{ \left\lceil \frac{NC_{\tau - \ell}}{1 + 4(N - 1)\varepsilon_2 / Z_\delta} \right\rceil : \ell = 1, \cdots, \tau \right\}$, where $\tau$ is a positive integer. With such a choice of sample sizes, we define a stopping rule such that sampling is continued until

$$Z^2_\delta \left( \frac{N}{n_{\ell}} - 1 \right) \hat{p}_\ell (1 - \hat{p}_\ell) \leq (N - 1)\varepsilon^2$$

is satisfied at some stage with index $\ell$. Then, $\Pr \{ |\hat{p} - p| \leq \varepsilon \mid p \} \geq 1 - \delta$ for any $p \in \Theta$ provided that the coverage tuning parameter $\zeta$ is sufficiently small.

To estimate the population proportion $p \in \Theta$ with margin of relative error $\varepsilon \in (0, 1)$, we can choose the sample sizes $n_1 < n_2 < \cdots < n_s$ as the ascending arrangement of all distinct elements of the set $\{ NC_{\tau - \ell} : \ell = 1, \cdots, \tau \}$, where $n_\star = \frac{Np(1-p)}{p^*(1-p^*) + (N-1)\varepsilon_r^2 / Z_\delta}$ with $p^* = \frac{\varepsilon_a}{\varepsilon_r} < \frac{1}{2}$. The stopping rule is that sampling is continued until

$$Z^2_\delta \left( \frac{N}{n_{\ell}} - 1 \right) (1 - \hat{p}_\ell) \leq (N - 1)\varepsilon^2 \hat{p}_\ell$$

is satisfied at some stage with index $\ell$. Then, $\Pr \{ |\hat{p} - p| \leq \varepsilon p \mid p \} \geq 1 - \delta$ for any $p \in \Theta$ provided that the coverage tuning parameter $\zeta$ is sufficiently small.

To estimate the population proportion $p \in \Theta$ with margin of absolute error $\varepsilon_a \in (0, 1)$ and margin of relative error $\varepsilon_r \in (0, 1)$, we can choose the sample sizes $n_1 < n_2 < \cdots < n_s$ as the ascending arrangement of all distinct elements of the set $\{ n^\star C_{\tau - \ell} : \ell = 1, \cdots, \tau \}$, where

$$n^\star = \frac{Np^*(1-p^*)}{p^*(1-p^*) + (N-1)\varepsilon_a^2 / Z_\delta}$$

and $p^* = \frac{\varepsilon_a}{\varepsilon_r} < \frac{1}{2}$. The stopping rule is that sampling is continued until

$$Z^2_\delta \left( \frac{N}{n_{\ell}} - 1 \right) \hat{p}_\ell (1 - \hat{p}_\ell) \leq (N - 1) \max \{ \varepsilon_a^2, \varepsilon_r p \hat{p}_\ell \}^2$$

is satisfied at some stage with index $\ell$. Then, $\Pr \{ |\hat{p} - p| \leq \varepsilon_a \text{ or } |\hat{p} - p| \leq \varepsilon_r p \mid p \} \geq 1 - \delta$ for any $p \in \Theta$ provided that the coverage tuning parameter $\zeta$ is sufficiently small.

### 8 Estimation of Normal Mean

Let $X$ be a normal random variable of mean $\mu$ and variance $\sigma^2$. In many situations, the variance $\sigma^2$ is unknown and it is desirable to estimate $\mu$ with predetermined margin of error and confidence level based on a sequence of i.i.d. random samples $X_1, X_2, \cdots$ of $X$.

#### 8.1 Control of Absolute Error

For a priori $\varepsilon > 0$, it is useful to construct an estimator $\hat{\mu}$ for $\mu$ such that $\Pr \{ |\hat{\mu} - \mu| < \varepsilon \} > 1 - \delta$ for any $\mu \in (-\infty, \infty)$ and $\sigma \in (0, \infty)$.

#### 8.1.1 New Structure of Multistage Sampling

Our new multistage sampling method as follows. Define

$$\bar{X}_n = \frac{\sum_{i=1}^{n} X_i}{n}, \quad S_n = \sum_{i=1}^{n} (X_i - \bar{X}_n)^2$$
for $n = 2, 3, \ldots, \infty$. Let $s$ be a positive number. The sampling consists of $s + 1$ stages, of which the sample sizes for the first $s$ stages are chosen as odd numbers $n_\ell = 2k_\ell + 1, \ell = 1, \ldots, s$ with $k_1 < k_2 < \cdots < k_s$. Define $\tilde{\sigma}_\ell = \sqrt{\frac{S_{n_\ell - 1}}{n_\ell - 1}}$ for $\ell = 1, \ldots, s$. Let the coverage tuning parameter $\zeta$ be a positive number less than $\frac{1}{2}$. The stopping rule is as follows:

If $n_\ell < (\tilde{\sigma}_\ell t_{n_{\ell - 1}, \zeta})^2/\varepsilon^2$, $\ell = 1, \ldots, i - 1$ and $n_i \geq (\tilde{\sigma}_i t_{n_{i - 1}, \zeta})^2/\varepsilon^2$ for some $i \in \{1, \ldots, s\}$, then the sampling is stopped at the $i$-th stage. Otherwise, $[(\tilde{\sigma}_s t_{n_{s - 1}, \zeta})^2/\varepsilon^2] - n_s$ more samples of $X$ needs to be taken after the $s$-th stage. The estimator of $\mu$ is defined as $\hat{\mu} = \frac{\sum_{i=1}^n X_i}{n}$, where $n$ is the sample size when the sampling is terminated.

It should be noted that, in the special case of $s = 1$, the above sampling scheme reduces to Stein’s two-stage procedure [36].

**Theorem 53** The following statements hold true.

(I) $\Pr[|\hat{\mu} - \mu| < \varepsilon] > 1 - 2s\zeta\delta$ for any $\mu$ and $\sigma$.

(II) $\lim_{\varepsilon \to 0} \Pr[|\hat{\mu} - \mu| < \varepsilon] = 1 - 2\zeta\delta$.

(III) $\mathbb{E}[n] \leq \frac{(\sigma t_{n_{s - 1}, \zeta})^2}{\varepsilon^2} + n_s$.

(IV) $\limsup_{\varepsilon \to 0} \mathbb{E}[n] \leq \left(\frac{t_{n_{s - 1}, \zeta}}{\varepsilon \zeta \delta}\right)^2$, where $C = \left(\frac{\sigma}{\varepsilon \zeta \delta}\right)^2$.

See Appendix K.1 for a proof.

As can be seen from statement (II) of Theorem 53, to ensure $\Pr[|\hat{\mu} - \mu| < \varepsilon] > 1 - \delta$, it suffices to choose the coverage tuning parameter $\zeta$ to be less than $\frac{1}{2s}$. However, such a choice is too conservative. To reduce sampling cost, it is possible to obtain a value of $\zeta$ much greater than $\frac{1}{2s}$ by an exact computational approach. Such an approach is explored in the sequel.

### 8.1.2 Exact Construction of Sampling Schemes

To develop an exact computational approach for the determination of an appropriate value of coverage tuning parameter $\zeta$, we need some preliminary results as follows.

**Theorem 54** Let $1 = k_0 < k_1 < k_2 < \cdots$ be a sequence of positive integers. Let $0 = z_0 < z_1 < z_2 < \cdots$ be a sequence of positive numbers. Define $h(0, 1) = 1$ and

$$h(\ell, 1) = 1, \quad h(\ell, m) = \sum_{i=1}^{k_\ell} \frac{h(r, i)(z_\ell - z_r)^m}{(m - i)!}, \quad k_r < m \leq k_{r+1}, \quad r = 0, 1, \ldots, \ell - 1$$

for $\ell = 1, 2, \ldots$. Let $Z_1, Z_2, \cdots$ be i.i.d. exponential random variables with common mean unity. Then,

$$\Pr\left\{ \sum_{m=1}^{k_\ell} Z_m > z_j \text{ for } j = 1, \ldots, \ell \right\} = e^{-z_\ell} \sum_{m=1}^{k_\ell} h(\ell, m)$$

for $\ell = 1, 2, \ldots$. Moreover, the following statements hold true.
where $I$ Define and

$$\Pr \left\{ a_j < \sum_{m=1}^{k_j} Z_m < b_j \text{ for } j = 1, \ldots, \ell \right\}$$

$$= \left[ \sum_{i=1}^{2^{\ell-1}} \Pr \left\{ \sum_{m=1}^{k_j} Z_m > [A_i]_{i,j} \text{ for } j = 1, \ldots, \ell \right\} \right] - \left[ \sum_{i=1}^{2^{\ell-1}} \Pr \left\{ \sum_{m=1}^{k_j} Z_m > [B_i]_{i,j} \text{ for } j = 1, \ldots, \ell \right\} \right],$$

where $A_1 = [a_1], B_1 = [b_1]$ and

$$A_{r+1} = \begin{bmatrix} A_r & a_{r+1} I_{2^{r-1} \times 1} \\ B_r & b_{r+1} I_{2^{r-1} \times 1} \end{bmatrix}, \quad B_{r+1} = \begin{bmatrix} B_r & a_{r+1} I_{2^{r-1} \times 1} \\ A_r & b_{r+1} I_{2^{r-1} \times 1} \end{bmatrix}, \quad r = 1, 2, \ldots$$

where $I_{2^{r-1} \times 1}$ represents a column matrix with all $2^{r-1}$ elements assuming value 1.

(II)

$$\Pr \left\{ a_j < \sum_{m=1}^{k_j} Z_m < b_j \text{ for } j = 1, \ldots, \ell, \sum_{m=1}^{k_{i+1}} Z_m > b_{i+1} \right\}$$

$$= \left[ \sum_{i=1}^{2^{\ell-1}} \Pr \left\{ \sum_{m=1}^{k_j} Z_m > [E]_{i,j} \text{ for } j = 1, \ldots, \ell + 1 \right\} \right] - \left[ \sum_{i=1}^{2^{\ell-1}} \Pr \left\{ \sum_{m=1}^{k_j} Z_m > [F]_{i,j} \text{ for } j = 1, \ldots, \ell + 1 \right\} \right],$$

where $E = \begin{bmatrix} A_{\ell} & b_{\ell+1} I_{2^{\ell-1} \times 1} \end{bmatrix}$ and $F = \begin{bmatrix} B_{\ell} & b_{\ell+1} I_{2^{\ell-1} \times 1} \end{bmatrix}$.

(III)

$$\Pr \left\{ a_j < \sum_{m=1}^{k_j} Z_m < b_j \text{ for } j = 1, \ldots, \ell, \sum_{m=1}^{k_{i+1}} Z_m < b_{i+1} \right\}$$

$$= \Pr \left\{ a_j < \sum_{m=1}^{k_j} Z_m < b_j \text{ for } j = 1, \ldots, \ell \right\} - \Pr \left\{ a_j < \sum_{m=1}^{k_j} Z_m < b_j \text{ for } j = 1, \ldots, \ell, \sum_{m=1}^{k_{i+1}} Z_m > b_{i+1} \right\}.$$

For the purpose of computing appropriate coverage tuning parameter $\zeta$, the following results are useful.

**Theorem 55** Let the sample sizes of the sampling scheme be odd numbers $n_{i_\ell} = 2k_{i_\ell} + 1, \ell = 1, \ldots, s$, where $1 = k_0 < k_1 < k_2 < \cdots < k_s$. Let $b_0 = 0$ and $b_\ell = \frac{k_\ell (2k_{\ell+1})^2}{(\sigma_{\ell} t_{2k_{\ell+1}, \zeta})^2}$ for $\ell = 1, \ldots, s$. Define $h(0, 1) = 1, h(\ell, 1) = 1,$

$$h(\ell, m) = \sum_{i=1}^{k_\ell} \frac{h(r, i) (b_\ell - b_r)^{m-i}}{(m-i)!}, \quad k_r < m \leq k_{\ell+1}, \quad r = 0, 1, \ldots, \ell - 1,$$

and $H_\ell(\sigma) = e^{-b_\ell} \sum_{m=1}^{k_\ell} h(\ell, m)$ for $\ell = 1, \ldots, s$. Define $c = \frac{n_{k_{\ell+1}}^2}{(\sigma_{\ell} t_{2k_{\ell+1}, \zeta})^2}, h^*(1) = 1,$

$$h^*(m) = \sum_{i=1}^{k_\ell} \frac{h(r, i) (c - b_r)^{m-i}}{(m-i)!}, \quad k_r < m \leq k_{\ell+1}, \quad r = 0, 1, \ldots, s - 1.$$
and \( H^*(\sigma, n) = e^{-c} \sum_{m=1}^{k_n} h^*(m) \) for \( n \geq n_s \). Then, the following statements hold true.

(1): \( \Pr(|\hat{\mu} - \mu| \geq \varepsilon) = 2 \sum_{n \in \mathcal{S}} \left[ 1 - \Phi \left( \frac{\varepsilon \sqrt{n}}{a} \right) \right] \Pr(n = n), \) where \( \mathcal{S} = \{ n_\ell : 1 \leq \ell \leq s \} \cup \{ n \in \mathbb{N} : n > n_s \} \).

(II): \( \Pr(n = n) = \begin{cases} H_{\ell-1}(\sigma) - H_\ell(\sigma) & \text{for } n = n_\ell, 1 \leq \ell \leq s, \\
* (\sigma, n - 1) - H^*(\sigma, n) & \text{for } n > n_s 
\end{cases} \)

where \( H_0(\sigma) \equiv 1 \).

(III): For any \( \sigma \in [a, b] \),

\[
\Pr(|\hat{\mu} - \mu| \geq \varepsilon) > 2 \sum_{n \in \mathcal{S}} \left[ 1 - \Phi \left( \frac{\varepsilon \sqrt{n}}{a} \right) \right] P_n,
\]

\[
\Pr(|\hat{\mu} - \mu| \geq \varepsilon) < 2 \sum_{n \in \mathcal{S}} \left[ 1 - \Phi \left( \frac{\varepsilon \sqrt{n}}{b} \right) \right] \overline{P}_n + 2 \left[ 1 - \Phi \left( \frac{\varepsilon \sqrt{m}}{b} \right) \right] S_P \left( k_s - 1, \frac{mk_s \varepsilon^2}{(a t_{n_s-1, \zeta})^2} \right),
\]

where

\[
\overline{P}_n = \begin{cases} H_{\ell-1}(b) - H_\ell(a) & \text{for } n = n_\ell, 1 \leq \ell \leq s, \\
* (b, n - 1) - H^*(a, n) & \text{for } n > n_s 
\end{cases}
\]

\[
P_n = \begin{cases} H_{\ell-1}(a) - H_\ell(b) & \text{for } n = n_\ell, 1 \leq \ell \leq s, \\
* (a, n - 1) - H^*(b, n) & \text{for } n > n_s 
\end{cases}
\]

and \( m > n_s \).

(IV):

\[
\mathbb{E}[n] = n_1 + \sum_{\ell=1}^{s-1} (n_{\ell+1} - n_\ell) H_\ell(\sigma) + \sum_{n=n_s}^{\infty} H^*(\sigma, n)
< n_1 + \sum_{\ell=1}^{s-1} (n_{\ell+1} - n_\ell) H_\ell(\sigma) + \sum_{n=n_s}^{m} H^*(\sigma, n) + \frac{3(m \gamma \varepsilon)^v}{\gamma \sqrt{v} e^{m \gamma \varepsilon}},
\]

where \( \gamma = \frac{\varepsilon^2}{(\sigma t_{n_s-1, \zeta})^2}, \) \( v = \frac{n_s - 1}{2} \) and \( m > \max\{\frac{1}{\gamma}, n_s\} \).

See Appendix K.2 for a proof.

The coverage tuning process requires evaluation of the coverage probability \( \Pr(|\hat{\mu} - \mu| < \varepsilon} \) for various values of \( \sigma \). To reduce the evaluation of coverage probability with respect to \( \sigma \) to a finite range of \( \sigma \), we have the following results.

**Theorem 56** Let the sample sizes of the sampling scheme be odd numbers \( n_\ell = 2k_\ell + 1, \) \( \ell = 1, \ldots, s \), where \( 1 < k_1 < k_2 < \cdots < k_s \). Suppose the coverage tuning parameter \( \zeta \) is a positive number less than \( \frac{1}{\gamma} \). Then, there exists a unique number \( \overline{\sigma} \) such that

\[
\sum_{\ell=1}^{s-1} \left[ 1 - S_P \left( k_\ell - 1, \frac{n_\ell k_\ell \varepsilon^2}{(\mathcal{S} n_{\ell+1-\zeta})^2} \right) \right] = (1 - 2\zeta)\delta
\]
and that $\Pr\{\hat{\mu} - \mu \geq \varepsilon\} < \delta$ for $\sigma > \bar{\sigma}$. Similarly, there exists a unique number $\sigma$ such that

$$1 - \Phi\left(\frac{\varepsilon\sqrt{n_\ell}}{\sigma}\right) + \sum_{\ell=1}^{s-2} \left[1 - \Phi\left(\frac{\varepsilon\sqrt{n_{\ell+1}}}{\sigma}\right)\right] S_{\ell}\left(k_\ell - 1, \frac{n_\ell k_\ell \varepsilon^2}{(\sigma^2 t_{n_\ell-1, \zeta\delta})^2}\right) = \left(\frac{1}{2} - \zeta\right) \delta$$

and that $\Pr\{\hat{\mu} - \mu \geq \varepsilon\} < \delta$ for $\sigma < \underline{\sigma}$.

See Appendix K.3 for a proof.

### 8.2 Control of Relative Error

For a priori $\varepsilon > 0$, it is a frequent problem to construct an estimator $\hat{\mu}$ for $\mu$ such that $\Pr\{|\hat{\mu} - \mu| \leq \varepsilon|\mu|\} \geq 1 - \delta$ for any $\mu \in (-\infty, 0) \cup (0, \infty)$ and $\sigma \in (0, \infty)$. For this purpose, we would like to propose a new sampling method as follows.

**Theorem 57** Define $\delta_\ell = \delta$ for $1 \leq \ell \leq \tau$ and $\delta_\ell = \delta 2^{\tau - \ell}$ for $\ell > \tau$, where $\tau$ is a positive integer. For $\ell = 1, 2, \cdots$, let $\hat{\mu}_\ell = \frac{\sum_{i=1}^{n_\ell} X_i}{n_\ell}$ and $\hat{\sigma}_\ell = \sqrt{\frac{1}{n_{\ell-1}} \sum_{i=1}^{n_\ell} (X_i - \hat{\mu}_\ell)^2}$, where $n_\ell$ is deterministic and stands for the sample size at the $\ell$-th stage. Suppose that sampling is continued until $|\hat{\mu}_\ell| \geq \frac{t_{\alpha_{\ell-1}, \zeta\delta}}{\sqrt{n_\ell}} (1 + \frac{1}{\varepsilon^2}) \hat{\sigma}_\ell$ for some stage with index $\ell$. Define estimator $\hat{\mu} = \hat{\mu}_l$, where $l$ is the index of stage at which the sampling is terminated. Then, $\Pr\{l < \infty\} = 1$ and $\Pr\{|\hat{\mu} - \mu| \leq \varepsilon|\mu|\} \geq 1 - \delta$ for any $\mu \in (-\infty, 0) \cup (0, \infty)$ and $\sigma \in (0, \infty)$ provided that $2(\tau + 1)\zeta \leq 1$ and $\inf_{\ell>0} \frac{n_{\ell+1}}{n_\ell} > 1$.

See Appendix K.4 for a proof.

### 8.3 Control of Relative and Absolute Errors

In some situations, it may be appropriate to estimate $\mu$ with a mixed error criterion specified by $\varepsilon_a > 0$ and $\varepsilon_r > 0$. In this respect, we have

**Theorem 58** Define $\delta_\ell = \delta$ for $1 \leq \ell \leq \tau$ and $\delta_\ell = \delta 2^{\tau - \ell}$ for $\ell > \tau$, where $\tau$ is a positive integer. For $\ell = 1, 2, \cdots$, let $\hat{\mu}_\ell = \frac{\sum_{i=1}^{n_\ell} X_i}{n_\ell}$ and $\hat{\sigma}_\ell = \sqrt{\frac{1}{n_{\ell-1}} \sum_{i=1}^{n_\ell} (X_i - \hat{\mu}_\ell)^2}$, where $n_\ell$ is deterministic and stands for the sample size at the $\ell$-th stage. Suppose that sampling is continued until $\max\left(\frac{\varepsilon_a}{\varepsilon_r}, \frac{|\hat{\mu}|}{1 + \varepsilon_r}\right) \geq \frac{t_{\alpha_{\ell-1}, \zeta\delta}}{\sqrt{n_\ell}} \hat{\sigma}_\ell$ for some stage with index $\ell$. Define estimator $\hat{\mu} = \hat{\mu}_l$, where $l$ is the index of stage at which the sampling is terminated. Then, $\Pr\{|\hat{\mu} - \mu| \leq \varepsilon_a$ or $|\hat{\mu} - \mu| \leq \varepsilon_r|\mu|\} \geq 1 - \delta$ for any $\mu \in (-\infty, \infty)$ and $\sigma \in (0, \infty)$ provided that $2(\tau + 1)\zeta \leq 1$ and $\inf_{\ell>0} \frac{n_{\ell+1}}{n_\ell} > 1$.

See Appendix K.5 for a proof.

### 9 Estimation of Exponential Parameters with Relative Precision

For completeness, we shall discuss the estimation of the parameter of an exponential distribution. The material presented in this section is well known (see, [18] and the references therein).
Let $X$ be a random variable of probability density function $f_X(x) = \frac{1}{\theta} \exp\left(-\frac{x}{\theta}\right)$, it is a frequent problem to estimate $\theta$ based on i.i.d. samples $X_1, X_2, \cdots$ of $X$. Let $0 < \varepsilon < 1$. Define $\overline{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i$. The goal is determine the minimum sample size $n$ such that

$$\Pr\left\{\left|\frac{\overline{X}_n - \theta}{\theta}\right| < \varepsilon \mid \theta\right\} > 1 - \delta$$

for any $\theta > 0$. For simplicity of notations, define $Y = \frac{2n\overline{X}_n}{\theta}$. Note that $Y$ has a chi-square distribution of $2n$ degrees of freedom and that

$$\Pr\left\{\left|\frac{\overline{X}_n - \theta}{\theta}\right| < \varepsilon \mid \theta\right\} = \Pr\{2n(1 - \varepsilon) < Y < 2n(1 + \varepsilon)\} = \frac{1}{\theta} S_\theta(n - 1, n(1 - \varepsilon)) - S_\theta(n - 1, n(1 + \varepsilon))$$

for any $\theta > 0$. Therefore, the minimum sample size to ensure is the minimum integer $n$ such that $\frac{1}{\theta} S_\theta(n - 1, n(1 - \varepsilon)) - S_\theta(n - 1, n(1 + \varepsilon)) > 1 - \delta$, which can be easily computed.

### 10 Exact Bounded-Width Confidence Intervals

A classical problem in sequential analysis is to construct a bounded-width confidence interval with a prescribed level of coverage probability. Such a problem can be solved in our framework of multistage estimation described in Section 2.1. Specifically, the problem of constructing a bounded-width confidence interval can be formulated as the problem of constructing a random interval with lower limit $\mathcal{L}(\hat{\theta}, n)$ and upper limit $\mathcal{U}(\hat{\theta}, n)$ such that $\mathcal{U}(\hat{\theta}, n) - \mathcal{L}(\hat{\theta}, n) \leq 2\varepsilon$ and that $\Pr\{\mathcal{L}(\hat{\theta}, n) < \theta < \mathcal{U}(\hat{\theta}, n) \mid \theta\} > 1 - \delta$ for any $\theta \in \Theta$. For this purpose, our computational machinery such as bisection coverage tuning and AMCA can be extremely useful.

#### 10.1 Construction via Coverage Tuning

As an application of Theorem 2 our general theory for constructing bounded-width confidence intervals based on multistage sampling is as follows.

**Corollary 5** Suppose a multistage sampling scheme satisfies the following requirements.

(i) For $\ell = 1, \cdots, s$, $\hat{\theta}_\ell$ is a ULE of $\theta$.

(ii) For $\ell = 1, \cdots, s$, $\{\mathcal{L}(\hat{\theta}_\ell, n_\ell) \leq \hat{\theta}_\ell \leq \mathcal{U}(\hat{\theta}_\ell, n_\ell)\}$ is a sure event.

(iii) For $\ell = 1, \cdots, s$, decision variable $D_\ell$ assumes value 1 if $\mathcal{U}(\hat{\theta}_\ell, n_\ell) - \mathcal{L}(\hat{\theta}_\ell, n_\ell) \leq 2\varepsilon$ and assumes value 0 otherwise.

(iv) $\{D_1 = 1\} \subseteq \left\{\mathcal{F}_{\hat{\theta}_\ell}(\hat{\theta}_\ell, \mathcal{U}(\hat{\theta}_\ell, n_\ell)) \leq \hat{\varepsilon}_\ell, \mathcal{G}_{\hat{\theta}_\ell}(\hat{\theta}_\ell, \mathcal{L}(\hat{\theta}_\ell, n_\ell)) \leq \hat{\varepsilon}_\ell\right\}$ for $\ell = 1, \cdots, s$.

(v) $\{\mathcal{U}(\hat{\theta}_s, n_s) - \mathcal{L}(\hat{\theta}_s, n_s) \leq 2\varepsilon\}$ is a sure event.

Define $\mathcal{L}(\hat{\theta}, n) = \mathcal{L}(\hat{\theta}_1, n_1)$ and $\mathcal{U}(\hat{\theta}, n) = \mathcal{U}(\hat{\theta}_1, n_1)$, where $l$ is the index of stage when the sampling is terminated. Then, $\mathcal{U}(\hat{\theta}, n) - \mathcal{L}(\hat{\theta}, n) \leq 2\varepsilon$ and

$$\Pr\{\mathcal{L}(\hat{\theta}, n) \geq \theta \mid \theta\} \geq \sum_{\ell=1}^{s} \Pr\{\mathcal{L}(\hat{\theta}_\ell, n_\ell) \geq \theta, D_\ell = 1 \mid \theta\} \leq \hat{\varepsilon} \sum_{\ell=1}^{s} \delta_\ell$$

$$\Pr\{\mathcal{U}(\hat{\theta}, n) \leq \theta \mid \theta\} \leq \sum_{\ell=1}^{s} \Pr\{\mathcal{U}(\hat{\theta}_\ell, n_\ell) \leq \theta, D_\ell = 1 \mid \theta\} \leq \hat{\varepsilon} \sum_{\ell=1}^{s} \delta_\ell$$

60
and \( \Pr\{\mathcal{L}(\hat{\theta},n) < \theta < \mathcal{U}(\hat{\theta},n) \mid \theta \} \geq 1 - 2\varepsilon \sum_{\ell=1}^{s} \delta_{\ell} \) for any \( \theta \in \Theta \).

## 10.2 Bounded-Width Confidence Intervals for Binomial Parameters

In this subsection, we provide concrete multistage sampling schemes for the construction of bounded-width confidence intervals for binomial parameters.

### 10.2.1 Construction from Clopper-Pearson Intervals

Making use of Corollary 5 and the Clopper-Pearson confidence interval \([16]\), we have established the following sampling scheme.

**Corollary 6** Let \( 0 < \varepsilon < \frac{1}{\tau} \). Suppose the sample size at the \( s \)-th stage is no less than \( \left\lceil \frac{\ln \frac{1}{\varepsilon}}{2\varepsilon^2} \right\rceil \). For \( \ell = 1, \ldots, s \), let \( \mathcal{L}(\hat{p}_{\ell},n_{\ell}) \) be the largest number such that \( 0 \leq \mathcal{L}(\hat{p}_{\ell},n_{\ell}) \leq \hat{p}_{\ell}, 1 - S_{B}(n_{\ell}\hat{p}_{\ell} - 1, n_{\ell}, \mathcal{L}(\hat{p}_{\ell},n_{\ell})) \leq \zeta \delta \) and let \( \mathcal{U}(\hat{p}_{\ell},n_{\ell}) \) be the smallest number such that \( \hat{p}_{\ell} \leq \mathcal{U}(\hat{p}_{\ell},n_{\ell}) \leq 1, S_{B}(n_{\ell}\hat{p}_{\ell}, n_{\ell}, \mathcal{U}(\hat{p}_{\ell},n_{\ell})) \leq \zeta \delta \), where \( \hat{p}_{\ell} = \frac{\sum_{i=1}^{n_{\ell}} X_{i}}{n_{\ell}} \). For \( \ell = 1, \ldots, s \), define \( D_{\ell} \) such that \( D_{\ell} = 1 \) if \( \mathcal{U}(\hat{p}_{\ell},n_{\ell}) - \mathcal{L}(\hat{p}_{\ell},n_{\ell}) \leq 2\varepsilon \); and \( D_{\ell} = 0 \) otherwise. Suppose the stopping rule is that sampling is continued until \( D_{l} = 1 \) for some \( l \in \{1, \ldots, s\} \). Define \( \mathcal{L}(\hat{p},n) = \mathcal{L}(\hat{p}_{l},n_{l}) \) and \( \mathcal{U}(\hat{p},n) = \mathcal{U}(\hat{p}_{l},n_{l}) \) with \( \hat{p} = \hat{p}_{l} \) and \( n = n_{l} \), where \( l \) is the index of stage when the sampling is terminated. Then, \( \mathcal{U}(\hat{p},n) - \mathcal{L}(\hat{p},n) \leq 2\varepsilon \),

\[
\Pr\{\mathcal{L}(\hat{p},n) \geq p \mid p\} \leq \sum_{\ell=1}^{s} \Pr\{\mathcal{L}(\hat{p}_{\ell},n_{\ell}) \geq p, D_{\ell} = 1 \mid p\} \leq s\zeta \delta,
\]

\[
\Pr\{\mathcal{U}(\hat{p},n) \leq p \mid p\} \leq \sum_{\ell=1}^{s} \Pr\{\mathcal{U}(\hat{p}_{\ell},n_{\ell}) \leq p, D_{\ell} = 1 \mid p\} \leq s\zeta \delta
\]

and \( \Pr\{\mathcal{L}(\hat{\theta},n) < p < \mathcal{U}(\hat{\theta},n) \mid p\} \geq 1 - 2s\zeta \delta \) for any \( p \in (0, 1) \).

Based on the criteria proposed in Section 2.1, the sample sizes \( n_{1} < n_{2} < \cdots < n_{s} \) can be chosen as the ascending arrangement of all distinct elements of

\[
\left\{ \left\lceil \frac{C_{\tau-\ell} \ln \frac{1}{\varepsilon}}{2\varepsilon^2} \right\rceil : \ell = 1, \ldots, \tau \right\},
\]

where \( \tau \) is the maximum integer such that \( \frac{C_{\tau-1} \ln \frac{1}{\varepsilon}}{2\varepsilon^2} \geq \frac{\ln(\delta)}{m(1-\varepsilon)} \), i.e., \( C_{\tau-1} \geq \frac{2\varepsilon^2}{\ln \frac{1}{\varepsilon}} \).

### 10.2.2 Construction from Fishman’s Confidence Intervals

Making use of Corollary 5 and Chernoff-Hoeffding inequalities \([14, 27]\), we have established the following sampling scheme.

**Corollary 7** Let \( 0 < \varepsilon < \frac{1}{\tau} \). Suppose the sample size at the \( s \)-th stage is no less than \( \left\lceil \frac{\ln \frac{1}{\varepsilon}}{2\varepsilon^2} \right\rceil \). For \( \ell = 1, \ldots, s \), let \( \mathcal{L}(\hat{p}_{\ell},n_{\ell}) \) be the largest number such that \( 0 \leq \mathcal{L}(\hat{p}_{\ell},n_{\ell}) \leq \hat{p}_{\ell}, \mathcal{M}_{B}(\hat{p}_{\ell},\mathcal{L}(\hat{p}_{\ell},n_{\ell})) \leq \frac{\ln(\delta)}{n_{\ell}} \) and let \( \mathcal{U}(\hat{p}_{\ell},n_{\ell}) \) be the smallest number such that \( \hat{p}_{\ell} \leq \mathcal{U}(\hat{p}_{\ell},n_{\ell}) \leq 1, \mathcal{M}_{B}(\hat{p}_{\ell},\mathcal{U}(\hat{p}_{\ell},n_{\ell})) \leq \frac{\ln(\delta)}{n_{\ell}}, \) where \( \hat{p}_{\ell} = \frac{\sum_{i=1}^{n_{\ell}} X_{i}}{n_{\ell}} \). For \( \ell = 1, \ldots, s \), define \( D_{\ell} \) such that \( D_{\ell} = 1 \) if \( \mathcal{U}(\hat{p}_{\ell},n_{\ell}) - \mathcal{L}(\hat{p}_{\ell},n_{\ell}) \leq 2\varepsilon \); and \( D_{\ell} = 0 \) otherwise. Suppose the stopping rule is that sampling is continued until \( D_{l} = 1 \) for some \( l \in \{1, \ldots, s\} \). Define \( \mathcal{L}(\hat{p},n) = \mathcal{L}(\hat{p}_{l},n_{l}) \) and \( \mathcal{U}(\hat{p},n) = \mathcal{U}(\hat{p}_{l},n_{l}) \) with \( \hat{p} = \hat{p}_{l} \) and \( n = n_{l} \), where \( l \) is the index of stage when the sampling is terminated. Then, \( \mathcal{U}(\hat{p},n) - \mathcal{L}(\hat{p},n) \leq 2\varepsilon \),

\[
\Pr\{\mathcal{L}(\hat{p},n) \geq p \mid p\} \leq \sum_{\ell=1}^{s} \Pr\{\mathcal{L}(\hat{p}_{\ell},n_{\ell}) \geq p, D_{\ell} = 1 \mid p\} \leq s\zeta \delta,
\]

\[
\Pr\{\mathcal{U}(\hat{p},n) \leq p \mid p\} \leq \sum_{\ell=1}^{s} \Pr\{\mathcal{U}(\hat{p}_{\ell},n_{\ell}) \leq p, D_{\ell} = 1 \mid p\} \leq s\zeta \delta
\]

and \( \Pr\{\mathcal{L}(\hat{\theta},n) < p < \mathcal{U}(\hat{\theta},n) \mid p\} \geq 1 - 2s\zeta \delta \) for any \( p \in (0, 1) \).
Corollary 8  Let \(0 < \varepsilon < \frac{4}{9}\). Suppose the sample size at the \(s\)-th stage is no less than \(\left\lceil \frac{8}{9} \left(\frac{3}{4} + 1\right) \left(\frac{3}{4} - 1\right) \ln \frac{1}{\varepsilon} \right\rceil\). For \(\ell = 1, \cdots, s\), define \(\hat{p}_\ell = \sum_{i=1}^{n_\ell} X_i \) and \(D_\ell\) such that \(D_\ell = 1\) if \(1 - \frac{g_n}{2 \ln(\varepsilon)} \hat{p}_\ell (1 - \hat{p}_\ell) \leq \varepsilon^2 \left[ \frac{4}{9} - \frac{3n_\ell}{2 \ln(\varepsilon)} \right]^2\), and \(D_\ell = 0\) otherwise. Suppose the stopping rule is that sampling is continued until \(D_\ell = 1\) for some \(\ell \in \{1, \cdots, s\}\). Define

\[
\mathcal{L}(\hat{p}_\ell, n_\ell) = \max \left\{ 0, \hat{p}_\ell + \frac{3}{4} \frac{1 - 2\hat{p}_\ell - \sqrt{1 - \frac{2}{n_\ell} \hat{p}_\ell (1 - \hat{p}_\ell)}}{1 - \frac{g_n}{2 \ln(\varepsilon)}} \right\},
\]

\[
\mathcal{V}(\hat{p}_\ell, n_\ell) = \min \left\{ 1, \hat{p}_\ell + \frac{3}{4} \frac{1 - 2\hat{p}_\ell + \sqrt{1 - \frac{2}{n_\ell} \hat{p}_\ell (1 - \hat{p}_\ell)}}{1 - \frac{g_n}{2 \ln(\varepsilon)}} \right\}
\]

for \(\ell = 1, \cdots, s\) and \(\hat{p} = \hat{p}_l\) and \(n = n_l\), where \(l\) is the index of stage when the sampling is terminated. Then, \(\mathcal{V}(\hat{p}, n) - \mathcal{L}(\hat{p}, n) \leq 2\varepsilon\) and

\[
\Pr\{\mathcal{L}(\hat{p}, n) \geq p \mid p\} \leq \sum_{\ell=1}^{s} \Pr\{\mathcal{L}(\hat{p}_\ell, n_\ell) \geq p, D_\ell = 1 \mid p\} \leq s\varepsilon\delta,
\]

\[
\Pr\{\mathcal{V}(\hat{p}, n) \leq p \mid p\} \leq \sum_{\ell=1}^{s} \Pr\{\mathcal{V}(\hat{p}_\ell, n_\ell) \leq p, D_\ell = 1 \mid p\} \leq s\varepsilon\delta
\]

for any \(p \in (0, 1)\).

Based on the criteria proposed in Section 2.1, the sample sizes \(n_1 < n_2 < \cdots < n_s\) can be chosen as the ascending arrangement of all distinct elements of \(\left\{ \left[ C_{\tau - \ell} \left(\frac{1}{2\varepsilon} - \frac{3}{8}\right) \ln \frac{1}{\varepsilon} \right] : 1 \leq \ell \leq \tau \right\}\), where \(\tau\) is the maximum integer such that \(C_{\tau - 1} \left(\frac{1}{2\varepsilon} - \frac{3}{8}\right) \ln \frac{1}{\varepsilon} \geq \left(\frac{1}{2\varepsilon} - \frac{3}{8}\right) \ln \frac{1}{\varepsilon}\), i.e., \(C_{\tau - 1} \geq \frac{4\varepsilon}{3 \varepsilon + 4\varepsilon}\).
10.3 Bounded-Width Confidence Intervals for Finite Population Proportion

In this subsection, we consider the construction of bounded-width confidence intervals for finite population proportion, $p$, based on multistage sampling. Within the general framework described in Sections 2.1 and 2.6, we have established the following method by virtue of Corollary 5 for bounded-width interval estimation.

**Corollary 9** For $z \in \{\frac{k}{n} : 0 \leq k \leq n\}$, define $\mathcal{L}(z, n) = \min\{z, L(z, n)\}$ and $\mathcal{U}(z, n) = \max\{z, U(z, n)\}$, where $L(z, n) = \min\{\theta \in \Theta : 1 - S_N(nz - 1, n, \theta) > \zeta\delta\}$ and $U(z, n) = \max\{\theta \in \Theta : S_N(nz, n, \theta) > \zeta\delta\}$. Suppose the sample size at the $s$-th stage is no less than the smallest number $n$ such that $\mathcal{U}(z, n) - \mathcal{L}(z, n) \leq 2\varepsilon$ for all $z \in \{\frac{k}{n} : 0 \leq k \leq n\}$. For $\ell = 1, \ldots, s$, define $\hat{p}_\ell = \frac{\sum_{i=1}^{n_\ell} X_i}{n_\ell}$ and decision variable $D_\ell$ which assumes values 1 if $\mathcal{U}(\hat{p}_\ell, n_\ell) - \mathcal{L}(\hat{p}_\ell, n_\ell) \leq 2\varepsilon$ and value 0 otherwise. Suppose the stopping rule is that sampling is continued until $D_\ell = 1$ for some $\ell \in \{1, \ldots, s\}$. Define $\hat{p} = \hat{p}_l$ and $n = n_l$, where $l$ is the index of stage when the sampling is terminated. Then, $\mathcal{U}(\hat{p}, n) - \mathcal{L}(\hat{p}, n) \leq 2\varepsilon$,

$$\Pr\{\mathcal{L}(\hat{p}, n) > p \mid p\} = \Pr\left\{\mathcal{L}(\hat{p}, n) - \frac{1}{N} \geq p \mid p\right\} \leq \sum_{\ell=1}^{s} \Pr\left\{\mathcal{L}(\hat{p}_\ell, n_\ell) - \frac{1}{N} \geq p, D_\ell = 1 \mid p\right\} \leq s\zeta\delta,$$

$$\Pr\{\mathcal{U}(\hat{p}, n) < p \mid p\} = \Pr\left\{\mathcal{U}(\hat{p}, n) + \frac{1}{N} \leq p \mid p\right\} \leq \sum_{\ell=1}^{s} \Pr\left\{\mathcal{U}(\hat{p}_\ell, n_\ell) + \frac{1}{N} \leq p, D_\ell = 1 \mid p\right\} \leq s\zeta\delta$$

and $\Pr\{\mathcal{L}(\hat{p}, n) \leq p \leq \mathcal{U}(\hat{p}, n)\} \geq 1 - 2s\zeta\delta$ for all $p \in \Theta$.

Let $n_{\text{max}}$ be the smallest number $n$ such that $\mathcal{U}(z, n) - \mathcal{L}(z, n) \leq 2\varepsilon$ for all $z \in \{\frac{k}{n} : 0 \leq k \leq n\}$. Let $n_{\text{min}}$ be the largest number $n$ such that $\mathcal{U}(z, n) - \mathcal{L}(z, n) > 2\varepsilon$ for all $z \in \{\frac{k}{n} : 0 \leq k \leq n\}$. Based on the criteria proposed in Section 2.1 the sample sizes $n_1 < n_2 < \cdots < n_s$ can be chosen as the ascending arrangement of all distinct elements of $\{\lfloor C_{\tau-1} n_{\text{max}} \rfloor : \ell = 1, \ldots, \tau\}$, where $\tau$ is the maximum integer such that $C_{\tau-1} \geq \frac{n_{\text{min}}}{n_{\text{max}}}$.

In order to develop multistage sampling schemes with simple stopping boundaries, we have the following results.

**Corollary 10** For $z \in \{\frac{k}{n} : 0 \leq k \leq n\}$, define $\mathcal{L}(z, n) = \min\{z, L(z, n)\}$ and $\mathcal{U}(z, n) = \max\{z, U(z, n)\}$, where $L(z, n) = \min\{\theta \in \Theta : C(z, \theta, n, N) > \zeta\delta\}$ and $U(z, n) = \max\{\theta \in \Theta : C(z, \theta, n, N) > \zeta\delta\}$, where $C(z, \theta, n, N)$ is defined by (34). Suppose the sample size at the $s$-th stage is no less than the smallest number $n$ such that $\mathcal{U}(z, n) - \mathcal{L}(z, n) \leq 2\varepsilon$ for all $z \in \{\frac{k}{n} : 0 \leq k \leq n\}$. For $\ell = 1, \ldots, s$, define $\hat{p}_\ell = \frac{\sum_{i=1}^{n_\ell} X_i}{n_\ell}$ and decision variable $D_\ell$ which assumes values 1 if $\mathcal{U}(\hat{p}_\ell, n_\ell) - \mathcal{L}(\hat{p}_\ell, n_\ell) \leq 2\varepsilon$ and value 0 otherwise. Suppose the stopping rule is that sampling is continued until $D_\ell = 1$ for some $\ell \in \{1, \ldots, s\}$. Define $\hat{p} = \hat{p}_l$ and $n = n_l$, where $l$ is the index of stage when the sampling is terminated. Then, $\mathcal{U}(\hat{p}, n) - \mathcal{L}(\hat{p}, n) \leq 2\varepsilon$,

$$\Pr\{\mathcal{L}(\hat{p}, n) > p \mid p\} = \Pr\left\{\mathcal{L}(\hat{p}, n) - \frac{1}{N} \geq p \mid p\right\} \leq \sum_{\ell=1}^{s} \Pr\left\{\mathcal{L}(\hat{p}_\ell, n_\ell) - \frac{1}{N} \geq p, D_\ell = 1 \mid p\right\} \leq s\zeta\delta,$$

$$\Pr\{\mathcal{U}(\hat{p}, n) < p \mid p\} = \Pr\left\{\mathcal{U}(\hat{p}, n) + \frac{1}{N} \leq p \mid p\right\} \leq \sum_{\ell=1}^{s} \Pr\left\{\mathcal{U}(\hat{p}_\ell, n_\ell) + \frac{1}{N} \leq p, D_\ell = 1 \mid p\right\} \leq s\zeta\delta$$

and $\Pr\{\mathcal{L}(\hat{p}, n) \leq p \leq \mathcal{U}(\hat{p}, n)\} \geq 1 - 2s\zeta\delta$ for all $p \in \Theta$. 63
Corollary [10] can be shown by virtue of Corollary [5] and inequalities (35) and (36).

11 Estimation Following Multistage Tests

When a multistage hypothesis test is finished, it is usually desirable to construct a confidence interval for the unknown parameter \( \theta \). In general, multistage test plans can be cast in the framework of sampling schemes described in Section 2.1. We have established various interval estimation methods in the context of multistage tests.

11.1 Clopper-Pearson Type Confidence Intervals

Define cumulative distribution functions \( F_{\hat{\theta}}(z, \theta) \) and \( G_{\hat{\theta}}(z, \theta) \) as (2.5). To construct a confidence interval of Clopper-Pearson type following a multistage test, we have the following results.

**Theorem 59** For \( \ell = 1, \cdots, s \), let \( \hat{\theta}_{\ell} = \varphi(X_1, \cdots, X_{n_{\ell}}) \) be a ULE of \( \theta \). Let \( \hat{\theta} = \hat{\theta}_{l} \) and \( n = n_l \), where \( l \) is the index of stage when the sampling is terminated. Define confidence limits \( \mathcal{L}(\hat{\theta}, n) \) and \( \mathcal{U}(\hat{\theta}, n) \) as functions of \( (\hat{\theta}, n) \) such that, for any observation \((\hat{\theta}, n)\) of \((\hat{\theta}, n)\), \( \mathcal{L}(\hat{\theta}, n) \) is the largest number satisfying \( G_{\hat{\theta}}(\hat{\theta}, \mathcal{L}(\hat{\theta}, n)) \leq \frac{\delta}{2} \) and that \( \mathcal{U}(\hat{\theta}, n) \) is the smallest number satisfying \( F_{\hat{\theta}}(\hat{\theta}, \mathcal{U}(\hat{\theta}, n)) \leq \frac{\delta}{2} \). Then, \( \Pr\{\mathcal{L}(\hat{\theta}, n) < \theta < \mathcal{U}(\hat{\theta}, n) \mid \theta \} \geq 1 - \delta \) for any \( \theta \in \Theta \).

See Appendix L.1 for a proof. It should be noted that, by virtue of our computational machinery, exact computation of confidence intervals is possible for common distributions.

11.1.1 Finite Population Proportion

To construct a confidence interval for the proportion of a finite population after a multistage test in the general framework described in Sections 2.1 and 2.6, we have developed an approach which does not rely on using ULEs as follows.

**Theorem 60** Let \( \hat{p}_{\ell} = \frac{\sum_{i=1}^{n_{\ell}} X_i}{n_{\ell}} \) for \( \ell = 1, \cdots, s \). Let \( \hat{p} = \hat{p}_{l} \) and \( n = n_l \), where \( l \) is the index of stage when the sampling is terminated. Define confidence limits \( \mathcal{L}(\hat{p}, n) \) and \( \mathcal{U}(\hat{p}, n) \) as functions of \( (\hat{p}, n) \) such that, for any observation \((\hat{p}, n)\) of \((\hat{p}, n)\), \( \mathcal{L}(\hat{p}, n) \) is the smallest number in \( \Theta \) satisfying \( \Pr\{\hat{p} \geq \hat{p} \mid \mathcal{L}(\hat{p}, n)\} > \frac{\delta}{2} \) and that \( \mathcal{U}(\hat{p}, n) \) is the largest number in \( \Theta \) satisfying \( \Pr\{\hat{p} \leq \hat{p} \mid \mathcal{U}(\hat{p}, n)\} > \frac{\delta}{2} \). Then, \( \Pr\{\mathcal{L}(\hat{p}, n) \leq p \leq \mathcal{U}(\hat{p}, n) \mid p \} \geq 1 - \delta \) for any \( p \in \Theta \).

See Appendix L.2 for a proof.

11.2 Confidence Intervals from Coverage Tuning

The method of interval estimation described in Section 11.1 suffers from two drawbacks: (i) It is conservative due to the discrete nature of the underlying variable. (ii) There is no closed-form
The coverage probability is always guaranteed for $\lambda$ difficulty, our strategy is to design a confidence interval such that, for a large number be adapted to Poisson variables because the parameter space is not bounded. To overcome such approaches, it seems that the approach described at the beginning of Section 11.2 cannot be applied to the coverage probability is no less than $1 - \alpha$. Let $L(\theta, n) < \theta < U(\theta, n)$ and and $U, L, \ell = 1, \ldots, s$ are ULEs as defined in Theorem 59. Suppose $L(\theta, n) < \theta < U(\theta, n)$ and

$$\Pr\{\theta \leq L(\theta, n) \mid \theta\} \leq \delta L, \quad \Pr\{\theta \geq U(\theta, n) \mid \theta\} \leq \delta L$$

for $\ell = 1, \ldots, s$. Then,

$$\Pr\{\theta \leq L(\theta, n) \mid \theta\} \leq \delta \sum_{\ell=1}^{s} \Pr\{\theta \leq L(\theta, n), D_L = 1 \mid \theta\} \leq \delta s \sum_{\ell=1}^{s} \delta L,$$

$$\Pr\{\theta \geq U(\theta, n) \mid \theta\} \leq \delta \sum_{\ell=1}^{s} \Pr\{\theta \geq U(\theta, n), D_U = 1 \mid \theta\} \leq \delta s \sum_{\ell=1}^{s} \delta L.$$

This implies that it is possible to apply a bisection search method to obtain a number $\delta$ such that the coverage probability is no less than $1 - \delta$. For the purpose of searching $\delta$, we have established tight bounds for $\Pr\{\theta \leq L(\theta, n) < \theta < U(\theta, n) \mid \theta\}$ for $\theta \in [a, b] \subseteq \Theta$ as in Section 3.3. By virtue of such bounds, adaptive maximum checking algorithm described in Section 3.3 can be used to determine an appropriate value of $\delta$.

### 11.2.1 Poisson Mean

At the first glance, it seems that the approach described at the beginning of Section 11.2 cannot be adapted to Poisson variables because the parameter space is not bounded. To overcome such difficulty, our strategy is to design a confidence interval such that, for a large number $\lambda^* > 0$, the coverage probability is always guaranteed for $\lambda \in (\lambda^*, \infty)$ without tuning the confidence parameter and that the coverage probability for $\lambda \in (0, \lambda^*)$ can be tuned to be no less than $1 - \delta$. Such method is described in more details as follows.

Suppose the multistage testing plan can be put in the general framework described in Section 2.1. Let $\alpha \in (0, 1)$ and $\tilde{\lambda}_L = \frac{\sum_{\ell=1}^{n} X_i}{n_{\ell}}$. For every realization, $(\tilde{\lambda}_L, n_{\ell})$, of $(\lambda, n_{\ell})$, let $L = L(\lambda, n_{\ell}, \alpha)$ be the largest number such that $L(\lambda, n_{\ell}, \alpha) \leq \lambda_L$ and $\Pr\{\lambda \geq \lambda_L \mid L\} \leq \alpha$. Let $U = U(\lambda, n_{\ell}, \alpha)$ be the smallest number such that $U(\lambda, n_{\ell}, \alpha) \geq \lambda_L$ and $\Pr\{\lambda \leq \lambda_L \mid U\} \leq \alpha$. One possible construction of $L$ and $U$ can be found in [19]. To eliminate the necessity of evaluating the coverage probability of confidence interval for an infinitely wide range of parameter $\lambda$ in the course of coverage tuning, the following result is crucial.

**Theorem 61** Define

$$L(\lambda, n_{\ell}, \alpha) = \begin{cases} L(\lambda, n_{\ell}, \zeta \delta) & \text{if } U(\lambda, n_{\ell}, \frac{\delta}{2\zeta}) \leq \lambda^*, \\ L(\lambda, n_{\ell}, \frac{\delta}{2\zeta}) & \text{if } U(\lambda, n_{\ell}, \frac{\delta}{2\zeta}) > \lambda^* \end{cases}$$

65
and
\[ \mathcal{L}(\lambda, n) = \left\{ \begin{array}{ll}
U(\lambda, n, \delta) & \text{if } U(\lambda, n, \delta) \leq \lambda^*, \\
U(\lambda, n, \frac{1}{2\delta}) & \text{if } U(\lambda, n, \frac{1}{2\delta}) > \lambda^*.
\end{array} \right. \]

Let the lower and upper confidence limits be, respectively, defined as \( \mathcal{L}(\lambda, n) = \mathcal{L}(\lambda, n) \) and \( \mathcal{H}(\lambda, n) = \mathcal{H}(\lambda, n), \) where \( \ell \) is the index of stage when the sampling is terminated. Then,
\[
\Pr\{\mathcal{L}(\lambda, n) < \lambda < \mathcal{H}(\lambda, n) | \lambda\} \geq 1 - \delta
\]
for any \( \lambda \in (0, \infty) \) provided that (39) holds for any \( \lambda \in (0, \lambda^*]. \)

See Appendix L.3 for a proof.

11.2.2 Normal Variance

A wide class of test plans for the variance of a normal distribution can be described as follows:

Choose appropriate sample sizes \( n_1 < n_2 < \cdots < n_s \) and numbers \( a_\ell < b_\ell, \) \( \ell = 1, \cdots, s. \) Let
\[ \tilde{\sigma}_\ell = \sqrt{\frac{1}{n_\ell} \sum_{i=1}^{n_\ell} (X_i - \bar{X}_n)^2} \]
for \( \ell = 1, \cdots, s. \) Continue sampling until \( \tilde{\sigma}_\ell \leq a_\ell \) or \( \tilde{\sigma}_\ell > b_\ell. \) When the sampling is terminated, accept \( \mathcal{H}_0 \) if \( \tilde{\sigma}_\ell \leq a_\ell; \) reject \( \mathcal{H}_0 \) if \( \tilde{\sigma}_\ell > b_\ell. \)

To construct a confidence interval for \( \sigma \) after the test, we can use a ULE of \( \sigma, \) which is given by \( \bar{\sigma} = \tilde{\sigma}_\ell, \) where \( \ell \) is the index of stage when the test is completed. Accordingly, \( n = n_\ell \) is the sample number when the test is completed. A confidence interval with lower limit \( \mathcal{L}(\tilde{\sigma}, n) \) and upper limit \( \mathcal{H}(\tilde{\sigma}, n) \) can be constructed as follows:

If \( \bar{\sigma} \) assumes value \( \bar{\sigma} \) at the termination of test, the realization of the upper confidence limit is equal to a certain value \( \sigma \) such that \( \Pr\{\bar{\sigma} \leq \bar{\sigma} | \sigma\} = \frac{\delta}{2}. \) Similarly, the realization of the lower confidence limit is equal to a certain value \( \sigma \) such that \( \Pr\{\bar{\sigma} \geq \bar{\sigma} | \sigma\} = \frac{\delta}{2}. \)

To find the value of \( \sigma \) such that \( \Pr\{\bar{\sigma} \leq \bar{\sigma} | \sigma\} = \frac{\delta}{2}, \) it is equivalent to find \( \sigma \) such that
\[
\Pr\{\bar{\sigma} \leq \bar{\sigma} | \sigma\} = \sum_{\ell=1}^{\delta} \Pr\{\bar{\sigma}_\ell \leq \bar{\sigma}, a_j < \bar{\sigma}_j \leq b_j, 1 \leq j < \ell | \sigma\}.
\]

Similarly, to find the value of \( \sigma \) such that \( \Pr\{\bar{\sigma} \geq \bar{\sigma} | \sigma\} = \frac{\delta}{2}, \) it is equivalent to find \( \sigma \) such that
\[
\Pr\{\bar{\sigma} \geq \bar{\sigma} | \sigma\} = \sum_{\ell=1}^{\delta} \Pr\{\bar{\sigma}_\ell \geq \bar{\sigma}, a_j < \bar{\sigma}_j \leq b_j, 1 \leq j < \ell | \sigma\}.
\]

If we choose the sample sizes to be odd numbers \( n_\ell = 2k_\ell + 1, \) \( \ell = 1, \cdots, s, \) we can rewrite (40) and (41) respectively as
\[
\Pr\{\bar{\sigma} \leq \bar{\sigma} | \sigma\} = \sum_{\ell=1}^{\delta} \Pr\left\{ \sum_{m=1}^{k_\ell} Z_m \leq \frac{n_\ell}{2} \left( \frac{\sigma}{\bar{\sigma}} \right)^2, \frac{n_\ell}{2} \left( \frac{a_j}{\sigma} \right)^2 < \sum_{m=1}^{k_\ell} Z_m \leq \frac{n_\ell}{2} \left( \frac{b_j}{\sigma} \right)^2 \text{ for } 1 \leq j < \ell | \sigma\right\}
\]
for any \( \lambda \in (0, \infty) \) provided that (39) holds for any \( \lambda \in (0, \lambda^*]. \)
A wide class of test plans for the variance of a normal distribution can be described as follows:

11.2.3 Exponential Parameters

To find the value of \( \hat{\theta} \) such that \( \Pr\{\theta \leq \hat{\theta} \mid \theta\} = \frac{\delta}{2} \), it is equivalent to find \( \theta \) such that

\[
\Pr\{\theta \leq \hat{\theta} \mid \theta\} = \sum_{\ell=1}^{s} \Pr\left\{ \hat{\theta}_{\ell} \leq \hat{\theta}, a_{j} < \hat{\theta}_{j} \leq b_{j}, 1 \leq j < \ell \mid \theta \right\}. \tag{44}
\]

Similarly, to find the value of \( \theta \) such that \( \Pr\{\hat{\theta} \geq \theta \mid \theta\} = \frac{\delta}{2} \), it is equivalent to find \( \theta \) such that

\[
\Pr\{\hat{\theta} \geq \theta \mid \theta\} = \sum_{\ell=1}^{s} \Pr\left\{ \hat{\theta}_{\ell} \geq \hat{\theta}, a_{j} < \hat{\theta}_{j} \leq b_{j}, 1 \leq j < \ell \mid \theta \right\}. \tag{45}
\]

Let \( Z_{1}, Z_{2}, \ldots \) be i.i.d. exponential random variables with common mean unity. Then, we can rewrite (44) and (45) respectively as

\[
\Pr\{\hat{\theta} \leq \theta \mid \theta\} = \sum_{\ell=1}^{s} \Pr\left\{ \sum_{m=1}^{n_{\ell}} Z_{m} \leq n_{\ell} \left( \frac{\hat{\theta}}{\theta} \right), n_{j} \left( \frac{a_{j}}{\theta} \right) \leq \sum_{m=1}^{n_{j}} Z_{m} \leq n_{j} \left( \frac{b_{j}}{\theta} \right) \text{ for } 1 \leq j < \ell \mid \theta \right\}, \tag{46}
\]

and

\[
\Pr\{\hat{\theta} \geq \theta \mid \theta\} = \sum_{\ell=1}^{s} \Pr\left\{ \sum_{m=1}^{n_{\ell}} Z_{m} \geq n_{\ell} \left( \frac{\hat{\theta}}{\theta} \right), n_{j} \left( \frac{a_{j}}{\theta} \right) \leq \sum_{m=1}^{n_{j}} Z_{m} \leq n_{j} \left( \frac{b_{j}}{\theta} \right) \text{ for } 1 \leq j < \ell \mid \theta \right\}. \tag{47}
\]

As can be seen from (46) and (47), the determination of confidence interval for \( \sigma \) requires the exact computation of the probabilities in the right-hand sides of (46) and (47). For such computational purpose, we can make use of the results in Theorem 54.
12 Exact Confidence Sequences

The construction of confidence sequence is a classical problem in statistics. In this section, we shall consider the problem in a general setting as follows.

Let \( X_1, X_2, \cdots \) be a sequence of samples of random variable \( X \) parameterized by \( \theta \in \Theta \). Consider a multistage sampling procedure of \( s \) stages such that the number of available samples at the \( \ell \)-th stage is a random number \( n_\ell \) for \( \ell = 1, \cdots, s \). Let \( \hat{\theta}_\ell \) be a function of random tuple \( X_1, \cdots, X_{n_\ell} \) for \( \ell = 1, \cdots, s \). The objective is to construct intervals with lower limits \( \mathcal{L}(\hat{\theta}_\ell, n_\ell) \) and upper limits \( \mathcal{U}(\hat{\theta}_\ell, n_\ell) \) such that

\[
\Pr\{\mathcal{L}(\hat{\theta}_\ell, n_\ell) < \theta < \mathcal{U}(\hat{\theta}_\ell, n_\ell), \ell = 1, \cdots, s \mid \theta\} > 1 - \delta
\]

for any \( \theta \in \Theta \).

12.1 Construction via Coverage Tuning

Assume that \( \hat{\theta}_\ell \) is a ULE for \( \ell = 1, \cdots, s \). For simplicity of notations, let

\[
L_\ell = \mathcal{L}(\hat{\theta}_\ell, n_\ell), \quad U_\ell = \mathcal{U}(\hat{\theta}_\ell, n_\ell), \quad \ell = 1, \cdots, s.
\]

As mentioned earlier, our objective is to construct a sequence of confidence intervals \( (L_\ell, U_\ell) \), \( 1 \leq \ell \leq s \) such that \( \Pr\{L_\ell < \theta < U_\ell, 1 \leq \ell \leq s \mid \theta\} \geq 1 - \delta \) for any \( \theta \in \Theta \). Suppose

\[
\Pr\{L_\ell < \theta < U_\ell \mid \theta\} \geq 1 - \zeta \delta, \quad 1 \leq \ell \leq s
\]

for any \( \theta \in \Theta \). By Bonferroni’s inequality, we have \( \Pr\{L_\ell < \theta < U_\ell, 1 \leq \ell \leq s \mid \theta\} \geq 1 - s \delta \) for any \( \theta \in \Theta \). This implies that it is possible to find an appropriate value of coverage tuning parameter \( \zeta \) such that \( \Pr\{L_\ell < \theta < U_\ell, 1 \leq \ell \leq s \mid \theta\} \geq 1 - \delta \) for any \( \theta \in \Theta \).

For this purpose, it suffices to bound the complementary probability \( 1 - \Pr\{L_\ell < \theta < U_\ell, 1 \leq \ell \leq s \mid \theta\} \) and apply the adaptive maximum checking algorithm described in Section 3.3 to find an appropriate value of the coverage tuning parameter \( \zeta \) such that \( 1 - \Pr\{L_\ell < \theta < U_\ell, 1 \leq \ell \leq s \mid \theta\} \leq \delta \) for any \( \theta \in [a, b] \subseteq \Theta \). In this respect, we have

**Theorem 62** Let \( X_1, X_2, \cdots \) be a sequence of identical samples of discrete random variable \( X \) which is parameterized by \( \theta \in \Theta \). For \( \ell = 1, \cdots, s \), let \( \hat{\theta}_\ell = \varphi(X_1, \cdots, X_{n_\ell}) \) be a ULE of \( \theta \). Let \( L_\ell = \mathcal{L}(\hat{\theta}_\ell, n_\ell) \) and \( U_\ell = \mathcal{U}(\hat{\theta}_\ell, n_\ell) \) be bivariate functions of \( \hat{\theta}_\ell \) and \( n_\ell \) such that \( \{L_\ell \leq \hat{\theta} \leq U_\ell\}, \ell = 1, \cdots, s \) are sure events. Let \( [a, b] \) be a subset of \( \Theta \). Let \( I_\mathcal{L} \) denote the intersection of \( [a, b] \) and the union of the supports of \( L_\ell \), \( \ell = 1, \cdots, s \). Let \( I_\mathcal{U} \) denote the intersection of \( [a, b] \) and the union of the supports of \( U_\ell \), \( \ell = 1, \cdots, s \). Define

\[
P_L(\theta) = \sum_{k=1}^{s} \Pr\{L_k \geq \theta, L_\ell < \theta < U_\ell, 1 \leq \ell < k \mid \theta\},
\]

\[
P_U(\theta) = \sum_{k=1}^{s} \Pr\{U_k \leq \theta, L_\ell < \theta < U_\ell, 1 \leq \ell < k \mid \theta\}.
\]
The following statements hold true:

(I): \(1 - \Pr\{L_\ell < \theta < U_\ell, \ 1 \leq \ell \leq s \mid \theta\} = P_L(\theta) + P_U(\theta)\).

(II): \(P_L(\theta)\) is non-decreasing with respect to \(p \in \Theta\) in any interval with endpoints being consecutive distinct elements of \(I_L \cup \{a, b\}\). The maximum of \(P_L(\theta)\) over \([a, b]\) is achieved at \(I_L \cup \{a, b\}\). Similarly, \(P_U(\theta)\) is non-increasing with respect to \(p \in \Theta\) in any interval with endpoints being consecutive distinct elements of \(I_U \cup \{a, b\}\). The maximum of \(P_U(\theta)\) over \([a, b]\) is achieved at \(I_U \cup \{a, b\}\).

(III): Suppose that \(\{L_\ell \geq a\} \subseteq \{\hat{\theta}_\ell \geq b\}\) and \(\{U_\ell \leq b\} \subseteq \{\hat{\theta}_\ell \leq a\}\) for \(\ell = 1, \ldots, s\). Then,

\[
P_L(\theta) \leq \sum_{k=1}^{s} \Pr\{L_k \geq a, L_\ell < b, U_\ell > a, 1 \leq \ell < k \mid b\},
\]

\[
P_U(\theta) \leq \sum_{k=1}^{s} \Pr\{U_k \leq b, L_\ell < b, U_\ell > a, 1 \leq \ell < k \mid a\},
\]

\[
P_L(\theta) \geq \sum_{k=1}^{s} \Pr\{L_k \geq b, L_\ell < a, U_\ell > b, 1 \leq \ell < k \mid a\},
\]

\[
P_U(\theta) \geq \sum_{k=1}^{s} \Pr\{U_k \leq a, L_\ell < a, U_\ell > b, 1 \leq \ell < k \mid b\}
\]

for any \(\theta \in [a, b] \subseteq \Theta\).

Theorem 62 can be established by a similar argument as that of Theorem 3. It should be noted that no need to compute \(s\) terms in the summation independently. Recursive computation can be used.

12.2 Finite Population Proportion

To construct a confidence sequence for the proportion, \(p\), of a finite population described in Section 2.1, we have the following results.

Theorem 63 Let \(L_\ell = L(\hat{p}_\ell, n_\ell)\) and \(U_\ell = U(\hat{p}_\ell, n_\ell)\) be bivariate functions of \(\hat{p}_\ell = \frac{\sum n_i X_i}{n_\ell}\) and \(n_\ell\) such that \(L_\ell \leq \hat{p}_\ell \leq U_\ell\) and that both \(NL_\ell\) and \(NU_\ell\) are integer-valued random variables for \(\ell = 1, \ldots, s\). Let \(a \leq b\) be two elements of \(\Theta = \left\{\frac{m}{N} : m = 0, 1, \ldots, N\right\}\). Let \(I_L\) denote the intersection of \((a, b)\) and the union of the supports of \(L_\ell - \frac{1}{N}, \ell = 1, \ldots, s\). Let \(I_U\) denote the intersection of \((a, b)\) and the union of the supports of \(U_\ell + \frac{1}{N}, \ell = 1, \ldots, s\). Define

\[
P_L(p) = \sum_{k=1}^{s} \Pr\{L_k > p, L_\ell \leq p \leq U_\ell, 1 \leq \ell < k \mid p\},
\]

\[
P_U(p) = \sum_{k=1}^{s} \Pr\{U_k < p, L_\ell \leq p \leq U_\ell, 1 \leq \ell < k \mid p\}.
\]

The following statements hold true.
(I): \( 1 - \Pr\{L_\ell \leq p \leq U_\ell, 1 \leq \ell \leq s \mid p\} = P_L(p) + P_U(p). \)

(II): \( P_L(p) \) is non-decreasing with respect to \( p \in \Theta \) in any interval with endpoints being consecutive distinct elements of \( I_\mathcal{L} \cup \{a, b\} \). The maximum of \( P_L(p) \) over \([a, b]\) is achieved at \( I_\mathcal{L} \cup \{a, b\} \). Similarly, \( P_U(p) \) is non-increasing with respect to \( p \in \Theta \) in any interval with endpoints being consecutive distinct elements of \( I_\mathcal{L} \cup \{a, b\} \). The maximum of \( P_U(p) \) over \([a, b]\) is achieved at \( I_\mathcal{L} \cup \{a, b\} \).

(III): Suppose that \( \{L_\ell \geq a\} \subseteq \{\hat{p}_\ell \geq b\} \) and \( \{U_\ell \leq b\} \subseteq \{\hat{p}_\ell \leq a\} \) for \( \ell = 1, \ldots, s \). Then,

\[
\begin{align*}
P_L(p) & \leq \sum_{k=1}^s \Pr\{L_k > a, L_\ell \leq b, U_\ell \geq a, 1 \leq \ell < k \mid b\}, \\
P_U(p) & \leq \sum_{k=1}^s \Pr\{U_k < b, L_\ell \leq b, U_\ell \geq a, 1 \leq \ell < k \mid a\}, \\
P_L(p) & \geq \sum_{k=1}^s \Pr\{L_k > b, L_\ell \leq b, U_\ell \geq b, 1 \leq \ell < k \mid a\}, \\
P_U(p) & \geq \sum_{k=1}^s \Pr\{U_k < a, L_\ell \leq a, U_\ell \geq b, 1 \leq \ell < k \mid b\}
\end{align*}
\]

for any \( p \in [a, b] \cap \Theta \).

Theorem 63 can be established by a similar argument as that of Theorem 2. It should be noted that our computational machinery such as bisection coverage tuning, AMCA and recursive algorithm can be used.

### 12.3 Poisson Mean

At the first glance, it seems that the adaptive maximum checking algorithm described in Section 3.3 cannot be adapted to Poisson variables because the parameter space is not bounded. To overcome such difficulty, our strategy is to design a confidence sequence such that, for a large number \( \lambda^* > 0 \), the coverage probability is always guaranteed for \( \lambda \in (\lambda^*, \infty) \) without tuning the confidence parameter and that the coverage probability for \( \lambda \in (0, \lambda^*] \) can be tuned to be no less than \( 1 - \delta \). Such method is described in more details as follows.

Let \( \alpha \in (0, 1) \) and \( \hat{\lambda}_\ell = \frac{\sum n_\ell X_\ell}{n_\ell} \). For every realization, \((\hat{\lambda}_\ell, n_\ell)\), let \( L = L(\hat{\lambda}_\ell, n_\ell, \alpha) \) be the largest number such that \( L(\hat{\lambda}_\ell, n_\ell, \alpha) \leq \hat{\lambda}_\ell \) and \( \Pr\{\hat{\lambda}_\ell \geq L \mid L\} \leq \alpha \). Let \( U = U(\hat{\lambda}_\ell, n_\ell, \alpha) \) be the smallest number such that \( U(\hat{\lambda}_\ell, n_\ell, \alpha) \geq \hat{\lambda}_\ell \) and \( \Pr\{\hat{\lambda}_\ell \leq U \mid U\} \leq \alpha \). One possible construction of \( L \) and \( U \) can be found in 19. To eliminate the necessity of evaluating the coverage probability of confidence interval for an infinitely wide range of parameter \( \lambda \) in the course of coverage tuning, the following result is critical.

**Theorem 64** Define

\[
\mathcal{L}(\hat{\lambda}_\ell, n_\ell) = \begin{cases} 
L(\hat{\lambda}_\ell, n_\ell, \zeta \delta) & \text{if } U(\hat{\lambda}_\ell, n_\ell, \frac{M \delta}{2}) \leq \lambda^*, \\
L(\hat{\lambda}_\ell, n_\ell, \frac{M \delta}{2}) & \text{if } U(\hat{\lambda}_\ell, n_\ell, \frac{M \delta}{2}) > \lambda^*
\end{cases}
\]

70
and
\[
\mathcal{U}(\hat{\lambda}_\ell, n_\ell) = \begin{cases} 
U(\hat{\lambda}_\ell, n_\ell, \zeta_\delta) & \text{if } U(\hat{\lambda}_\ell, n_\ell, \frac{\delta}{2s}) \leq \lambda^*, \\
U(\hat{\lambda}_\ell, n_\ell, \frac{\delta}{2s}) & \text{if } U(\hat{\lambda}_\ell, n_\ell, \frac{\delta}{2s}) > \lambda^*.
\end{cases}
\]

Then,
\[
\Pr\{L(\hat{\lambda}_\ell, n_\ell) < \lambda < U(\hat{\lambda}_\ell, n_\ell), \ell = 1, \ldots, s \mid \lambda\} \geq 1 - \delta
\]  
for any \( \lambda \in (0, \infty) \) provided that (48) holds for any \( \lambda \in (0, \lambda^*]. \)

See Appendix M for a proof.

### 12.4 Normal Mean

For normal variable, we have
\[
\Pr\{|X_{n_\ell} - Z_{\zeta_\delta} \sigma/\sqrt{n_\ell} < \mu < X_{n_\ell} + Z_{\zeta_\delta} \sigma/\sqrt{n_\ell}, 1 \leq \ell \leq s\} > 1 - s\zeta_\delta.
\]
Hence, if we choose \( \zeta_\delta \) to be small enough, we have
\[
\Pr\{|X_{n_\ell} - Z_{\zeta_\delta} \sigma/\sqrt{n_\ell} < \mu < X_{n_\ell} + Z_{\zeta_\delta} \sigma/\sqrt{n_\ell}, 1 \leq \ell \leq s\} = 1 - \delta.
\]
To compute the coverage probability of the repeated confidence intervals, there is no loss of generality to assume that \( X_1, X_2, \ldots \) are i.i.d. Gaussian variables with zero mean and variance unity (i.e., \( \mu = 0, \sigma = 1 \)). Hence, it suffices to compute
\[
\Pr\{|X_{n_\ell} - Z_{\zeta_\delta} \sigma/\sqrt{n_\ell} < \mu < X_{n_\ell} + Z_{\zeta_\delta} \sigma/\sqrt{n_\ell}, 1 \leq \ell \leq s\}.
\]
We shall evaluate the complementary probability
\[
1 - \Pr\{|X_{n_\ell} - Z_{\zeta_\delta} \sigma/\sqrt{n_\ell} < \mu < X_{n_\ell} + Z_{\zeta_\delta} \sigma/\sqrt{n_\ell}, 1 \leq \ell \leq s\} = \sum_{r=1}^{s} \Pr\{|X_{n_r} - Z_{\zeta_\delta} \sigma/\sqrt{n_r} < \mu < X_{n_r} + Z_{\zeta_\delta} \sigma/\sqrt{n_r}, 1 \leq \ell \leq r\}
\]
\[
= 2 \sum_{r=1}^{s} \Pr\{|X_{n_r} - Z_{\zeta_\delta} \sigma/\sqrt{n_r} < \mu < X_{n_r} + Z_{\zeta_\delta} \sigma/\sqrt{n_r}, 1 \leq \ell < r\}.
\]
Hence, the bounding method based on consecutive decision variables described in Section 3.2 can be used. Specifically,
\[
1 - \Pr\{|X_{n_\ell} - Z_{\zeta_\delta} \sigma/\sqrt{n_\ell} < \mu < X_{n_\ell} + Z_{\zeta_\delta} \sigma/\sqrt{n_\ell}, 1 \leq \ell \leq s\} \leq 2 \sum_{r=1}^{s} \Pr\{|X_{n_r} - Z_{\zeta_\delta} \sigma/\sqrt{n_r} < \mu < X_{n_r} + Z_{\zeta_\delta} \sigma/\sqrt{n_r}, \max(1, r - k) \leq \ell < r\}
\]
for \( 1 \leq k < s \). Such method can be used for the problem of testing the equality of the mean response of two treatments (see, [33], [39] and the references therein). It can also be applied to the repeated significance tests established by Armitage, McPherson, and Rowe [2].
12.5 Normal Variance

In this section, we shall discuss the construction of confidence sequence for the variance of a normal distribution. Let \( X_1, X_2, \cdots \) be i.i.d. samples of a normal random variable \( X \) of mean \( \mu \) and variance \( \sigma^2 \). Our method of constructing a confidence sequence is follows.

Choose the sample sizes to be odd numbers \( n_\ell = 2k_\ell + 1, \ell = 1, \cdots, s \). Define \( X_{n_\ell} = \frac{\sum_{i=1}^{n_\ell} X_i}{n_\ell} \) and \( S_{n_\ell} = \sum_{i=1}^{n_\ell} (X_i - X_{n_\ell})^2 \) for \( \ell = 1, \cdots, s \). Note that

\[
\Pr \left\{ \frac{S_{n_\ell}}{\chi^2_{n_\ell - 1, -\zeta_\delta}} < \sigma^2 < \frac{S_{n_\ell}}{\chi^2_{n_\ell - 1, \zeta_\delta}}, 1 \leq \ell \leq s \right\} > 1 - 2s\zeta_\delta
\]

and

\[
\Pr \left\{ \frac{S_{n_\ell}}{\chi^2_{n_\ell - 1, 1 - \zeta_\delta}} < \sigma^2 < \frac{S_{n_\ell}}{\chi^2_{n_\ell - 1, \zeta_\delta}}, 1 \leq \ell \leq s \right\} = \Pr \left\{ \frac{\chi^2_{n_\ell - 1, -\zeta_\delta}}{\sigma^2} < \frac{S_{n_\ell}}{\chi^2_{n_\ell - 1, -\zeta_\delta}}, 1 \leq \ell \leq s \right\} = \Pr \left\{ \frac{\chi^2_{n_\ell - 1, -\zeta_\delta}}{\sigma^2} < \sum_{m=1}^{k_s} Z_m < \frac{\chi^2_{n_\ell - 1, -\zeta_\delta}}{\sigma^2}, 1 \leq \ell \leq s \right\},
\]

where \( Z_1, Z_2, \cdots \) are i.i.d. exponential random variables with common mean unity. Therefore, the coverage probability \( \Pr \left\{ \frac{S_{n_\ell}}{\chi^2_{n_\ell - 1, 1 - \zeta_\delta}} < \sigma^2 < \frac{S_{n_\ell}}{\chi^2_{n_\ell - 1, \zeta_\delta}}, 1 \leq \ell \leq s \right\} \) can be exactly computed by virtue of Theorem 54. Consequently, we can obtain, via a bisection search method, an appropriate value of \( \zeta_\delta \) such that

\[
\Pr \left\{ \frac{S_{n_\ell}}{\chi^2_{n_\ell - 1, 1 - \zeta_\delta}} < \sigma^2 < \frac{S_{n_\ell}}{\chi^2_{n_\ell - 1, \zeta_\delta}}, 1 \leq \ell \leq s \right\} = 1 - \delta.
\]

12.6 Exponential Parameters

In this section, we shall consider the construction of confidence sequences for the parameter \( \theta \) of a random variable \( X \) of density function \( f(x) = \frac{1}{\theta} \exp \left( -\frac{x}{\theta} \right) \). Let \( X_1, X_2, \cdots \) be i.i.d. samples of a normal random variable \( X \). Let \( n_1 < n_2 < \cdots < n_s \) be a sequence of sample sizes. Since \( \frac{2n\overline{X}_n}{\theta} \) has a chi-square distribution of 2n degrees of freedom, we have

\[
\Pr \left\{ \chi^2_{2n_\ell, -\zeta_\delta} < \frac{2n_\ell \overline{X}_{n_\ell}}{\theta} < \chi^2_{2n_\ell, 1 - \zeta_\delta}, 1 \leq \ell \leq s \right\} > 1 - 2s\zeta_\delta,
\]

or equivalently,

\[
\Pr \left\{ \frac{2 \sum_{i=1}^{n_\ell} X_i}{\chi^2_{2n_\ell, 1 - \zeta_\delta}} < \theta < \frac{2 \sum_{i=1}^{n_\ell} X_i}{\chi^2_{2n_\ell, -\zeta_\delta}}, 1 \leq \ell \leq s \right\} > 1 - 2s\zeta_\delta.
\]

Note that

\[
\Pr \left\{ \frac{2 \sum_{i=1}^{n_\ell} X_i}{\chi^2_{2n_\ell, 1 - \zeta_\delta}} < \theta < \frac{2 \sum_{i=1}^{n_\ell} X_i}{\chi^2_{2n_\ell, -\zeta_\delta}}, 1 \leq \ell \leq s \right\} = \Pr \left\{ \frac{\chi^2_{2n_\ell, -\zeta_\delta}}{2} < \sum_{i=1}^{n_\ell} Z_i < \frac{\chi^2_{2n_\ell, 1 - \zeta_\delta}}{2}, 1 \leq \ell \leq s \right\},
\]

72
where $Z_1, Z_2, \cdots$ are i.i.d. exponential random variables with common mean unity. Therefore, the coverage probability $\Pr\left\{2\sum_{i=1}^{n_\ell} X_i < \theta < 2\sum_{i=1}^{n_\ell} X_i, 1 \leq \ell \leq s \right\}$ can be exactly computed by virtue of Theorem 54. Consequently, we can obtain, via a bisection search method, an appropriate value of $\zeta$ such that

$$\Pr\left\{2\sum_{i=1}^{n_\ell} X_i < \theta < 2\sum_{i=1}^{n_\ell} X_i, 1 - \zeta \delta < \theta < 2\sum_{i=1}^{n_\ell} X_i, \zeta \delta, 1 \leq \ell \leq s \right\} = 1 - \delta.$$

13 Multistage Linear Regression

Regression analysis is a statistical technique for investigating and modeling the relationship between variables. Applications of regression are numerous and occur in almost every field, including engineering, physical sciences, social sciences, economics, management, life and biological sciences, to name but a few. Consider a linear model

$$y = \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_m x_m + w \quad \text{with} \quad x_1 \equiv 1,$$

where $\beta_1, \cdots, \beta_m$ are deterministic parameters and $w$ is a Gaussian random variable of zero mean and variance $\sigma^2$. A major task of linear regression is to estimate parameters $\sigma$ and $\beta_i$ based on observations of $y$ for various values of $x_i$. In order to strictly control estimation error and uncertainty of inference with as few observations as possible, we shall develop multistage procedures. To this end, we shall first define some variables. Let $\beta = [\beta_1, \cdots, \beta_m]^T$, where the notation “$T$” stands for the transpose operation. Let $w_1, w_2, \cdots$ be a sequence of i.i.d. samples of $w$. Define

$$y_i = \beta_1 x_{i1} + \beta_2 x_{i2} + \cdots + \beta_m x_{im} + w_i \quad \text{with} \quad x_{i1} \equiv 1$$

for $i = 1, 2, \cdots$. Let $n_\ell, \ell = 1, 2, \cdots$ be a sequence of positive integers which is ascending with respect to $\ell$. Define

$$Y_\ell = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n_\ell} \end{bmatrix}, \quad X_\ell = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1m} \\ x_{21} & x_{22} & \cdots & x_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n_{\ell 1}} & x_{n_{\ell 2}} & \cdots & x_{n_{\ell m}} \end{bmatrix} \quad \text{for} \quad \ell = 1, 2, \cdots.$$

Assume that $X_\ell^T X_\ell$ is of rank $m$ for all $\ell$. Define

$$B_\ell = (X_\ell^T X_\ell)^{-1} X_\ell^T Y_\ell, \quad \hat{\sigma}_\ell = \sqrt{\frac{1}{n_\ell - m} \left[ Y_\ell^T Y_\ell - B_\ell^T (X_\ell^T Y_\ell) \right]}$$

for $\ell = 1, 2, \cdots$. For $i = 1, \cdots, m$, let $B_{i,\ell}$ denote the $i$-th entry of $B_\ell$ and let $[(X_\ell^T X_\ell)^{-1}]_{ii}$ denote the $(i,i)$-th entry of $(X_\ell^T X_\ell)^{-1}$. 73
13.1 Control of Absolute Error

For the purpose of estimating the variance $\sigma$ and the parameters $\beta_i$ with an absolute error criterion, we have

**Theorem 65** Let $\varepsilon > 0$ and $\varepsilon_i > 0$ for $i = 1, \cdots, m$. Let $\tau$ be a positive integer. Suppose the process of observing $y$ with respect to $x_i$ and $w$ is continued until $t_{n_{i-m}, \zeta \delta_i} \hat{\sigma}_\ell \sqrt{[(X_\ell^T X_\ell)^{-1}]_{ii}} \leq \varepsilon_i$ for $i = 1, \cdots, m$, and

$$\sqrt{\frac{n_\ell \varepsilon - m}{X^T_{n_{i-m}, \zeta \delta_i} \hat{\sigma}_\ell - \varepsilon} \leq \sqrt{\frac{n_\ell \varepsilon - m}{X^T_{n_{i-m}, 1-\zeta \delta_i} \hat{\sigma}_\ell + \varepsilon}}$$

at some stage with index $\ell$, where $\delta_\ell = \delta$ for $1 \leq \ell \leq \tau$ and $\delta_\ell = \delta 2^{\tau-\ell}$ for $\ell > \tau$. Define $\hat{\sigma} = \hat{\sigma}_l$ and $\hat{\beta} = B_l$, where $l$ is the index of stage at which the observation of $y$ is stopped. For $i = 1, \cdots, m$, let $\hat{\beta}_i$ be the $i$-th entry of $\hat{\beta}$. Then, $\Pr\{|l < \infty\} = 1$ and $\Pr\{|\hat{\sigma} - \sigma| \leq \varepsilon, \ |\hat{\beta}_i - \beta_i| \leq \varepsilon_i \} \geq 1 - \delta$ provided that $2(m+1)(\tau+1) \varepsilon \leq 1$ and that $\inf_{\ell>0} \frac{n_{\ell+1}}{n_\ell} > 1$.

See Appendix N.1 for a proof.

13.2 Control of Relative Error

For the purpose of estimating the variance $\sigma$ and the parameters $\beta_i$ with a relative error criterion, we have

**Theorem 66** Let $0 < \varepsilon < 1$ and $0 < \varepsilon_i < 1$ for $i = 1, \cdots, m$. Let $\tau$ be a positive integer. Suppose the process of observing $y$ with respect to $x_i$ and $w$ is continued until $t_{n_{i-m}, \zeta \delta_i} \sqrt{[(X_\ell^T X_\ell)^{-1}]_{ii}} \leq \frac{\varepsilon_i}{1+\varepsilon_i}$ for $i = 1, \cdots, m$, and $\frac{\hat{\sigma}_\ell}{(1+\varepsilon_i)^2} \leq n_\ell - m \leq \frac{\hat{\sigma}_\ell}{(1-\varepsilon_i)^2}$ at some stage with index $\ell$, where $\delta_\ell = \delta$ for $1 \leq \ell \leq \tau$ and $\delta_\ell = \delta 2^{\tau-\ell}$ for $\ell > \tau$. Define $\hat{\sigma} = \hat{\sigma}_l$ and $\hat{\beta} = \hat{\beta}_l$, where $l$ is the index of stage at which the observation of $y$ is stopped. For $i = 1, \cdots, m$, let $\hat{\beta}_i$ be the $i$-th entry of $\hat{\beta}$. Then, $\Pr\{|l < \infty\} = 1$ and $\Pr\{|\hat{\sigma} - \sigma| \leq \varepsilon \sigma, \ |\hat{\beta}_i - \beta_i| \leq \varepsilon_i \beta_i \} \geq 1 - \delta$ provided that $2(m+1)(\tau+1) \varepsilon \sigma \leq 1$ and that $\inf_{\ell>0} \frac{n_{\ell+1}}{n_\ell} > 1$.

See Appendix N.2 for a proof.

14 Multistage Estimation of Quantile

The estimation of a quantile of a random variable is a fundamental problem of practical importance. Specially, in control engineering, the performance of an uncertain dynamic system can be modeled as a random variable. Hence, it is desirable to estimate the minimum level of performance such that the probability of achieving it is greater than a certain percentage. In general, the problem of estimating a quantile can be formulated as follows.

Let $X$ be a random variable with cumulative distribution function $F_X(.)$. Define quantile $\xi_p = \inf\{x : F_X(x) > p\}$ for $p \in (0, 1)$. The objective is to estimate $\xi_p$ with prescribed precision.
and confidence level based on i.i.d. samples $X_1, X_2, \cdots$ of $X$. To make it possible for the rigorous control of estimation error and uncertainty of inference, we shall propose multistage procedures. For this purpose, we need to define some variables. For an integer $n$, let $X_{i:n}$ denote the $i$-th order statistics of i.i.d. samples $X_1, \cdots, X_n$ of $X$ such that $-\infty = X_{0:n} < X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n} < X_{n+1:n} = \infty$. Let the sample sizes be a sequence of positive integers $n_1, n_2, \cdots$ such that $n_1 < n_2 < n_3 < \cdots$. At the $\ell$-th stage, the decision of termination or continuation of sampling is made based on samples $X_1, \cdots, X_{n_\ell}$.

### 14.1 Control of Absolute Error

For estimating $\xi_p$ with a margin of absolute error $\varepsilon \geq 0$, our sampling procedure can be described as follows.

**Theorem 67** For $\ell = 1, 2, \cdots$, define $\delta_\ell = \delta$ for $1 \leq \ell \leq \tau$ and $\delta_\ell = \delta 2^{\ell - \tau}$ for $\ell > \tau$, where $\tau$ is a positive integer. Let $i_\ell \leq n_\ell$ be the largest integer such that $\sum_{k=0}^{i_{\ell} - 1} \binom{n_\ell}{k} p^k (1 - p)^{n_\ell-k} \leq \zeta_\ell$. Let $j_\ell \geq 0$ be the smallest integer such that $\sum_{k=j_{\ell}}^{n_\ell} \binom{n_\ell}{k} p^k (1 - p)^{n_\ell-k} \leq \zeta_\ell$. Define $\hat{\xi}_{p,\ell}$ such that $\hat{\xi}_{p,\ell} = X_{p\ell_{n_\ell}}$ if $p\ell_{n_\ell}$ is an integer and $\hat{\xi}_{p,\ell} = (\lfloor p\ell_{n_\ell} \rfloor - p\ell_{n_\ell}) X_{\lfloor p\ell_{n_\ell} \rfloor:n_\ell} + (p\ell_{n_\ell} - \lfloor p\ell_{n_\ell} \rfloor) X_{\lfloor p\ell_{n_\ell} \rfloor:n_{\ell}}$ otherwise. Suppose that sampling is continued until $X_{j_{\ell}:n_\ell} - \varepsilon \leq \hat{\xi}_{p,\ell} \leq X_{i_{\ell}:n_\ell} + \varepsilon$ for some stage with index $\ell$. Define estimator $\hat{\xi}_p = \hat{\xi}_{p,1}$ where $1$ is the index of stage at which the sampling is terminated. Then, $\Pr\{1 < \infty\} = 1$ and $\Pr\{|\hat{\xi}_p - \xi_p| \leq \varepsilon\} \geq 1 - \delta$ provided that $2(\tau + 1)\zeta \leq 1$ and $\inf_{\ell > 0} \frac{n_{\ell+1}}{n_{\ell}} > 1$.

See Appendix O.1 for a proof.

### 14.2 Control of Relative Error

For estimating $\xi_p \neq 0$ with a margin of relative error $\varepsilon \in (0, 1)$, our sampling procedure can be described as follows.

**Theorem 68** For $\ell = 1, 2, \cdots$, define $\delta_\ell = \delta$ for $1 \leq \ell \leq \tau$ and $\delta_\ell = \delta 2^{\ell - \tau}$ for $\ell > \tau$, where $\tau$ is a positive integer. Let $i_\ell \leq n_\ell$ be the largest integer such that $\sum_{k=0}^{i_{\ell} - 1} \binom{n_\ell}{k} p^k (1 - p)^{n_\ell-k} \leq \zeta_\ell$. Let $j_\ell \geq 0$ be the smallest integer such that $\sum_{k=j_{\ell}}^{n_\ell} \binom{n_\ell}{k} p^k (1 - p)^{n_\ell-k} \leq \zeta_\ell$. Define $\hat{\xi}_{p,\ell}$ such that $\hat{\xi}_{p,\ell} = X_{p\ell_{n_\ell}}$ if $p\ell_{n_\ell}$ is an integer and $\hat{\xi}_{p,\ell} = (\lfloor p\ell_{n_\ell} \rfloor - p\ell_{n_\ell}) X_{\lfloor p\ell_{n_\ell} \rfloor:n_\ell} + (p\ell_{n_\ell} - \lfloor p\ell_{n_\ell} \rfloor) X_{\lfloor p\ell_{n_\ell} \rfloor:n_{\ell}}$ otherwise. Suppose that sampling is continued until $1 - |\hat{\xi}_{p,\ell}| \leq X_{j_{\ell}:n_\ell} \leq \hat{\xi}_{p,\ell} \leq 1 + |\hat{\xi}_{p,\ell}| \leq X_{i_{\ell}:n_\ell}$ for some stage with index $\ell$. Define estimator $\hat{\xi}_p = \hat{\xi}_{p,1}$ where $1$ is the index of stage at which the sampling is terminated. Then, $\Pr\{1 < \infty\} = 1$ and $\Pr\{|\hat{\xi}_p - \xi_p| \leq \varepsilon|\xi_p|\} \geq 1 - \delta$ provided that $2(\tau + 1)\zeta \leq 1$ and $\inf_{\ell > 0} \frac{n_{\ell+1}}{n_{\ell}} > 1$.

See Appendix O.2 for a proof.
14.3 Control of Absolute and Relative Errors

For estimating $\xi_p$ with margin of absolute error $\varepsilon_a > 0$ and margin of relative error $\varepsilon_r \in (0, 1)$, our sampling procedure can be described as follows.

**Theorem 69** For $\ell = 1, 2, \cdots$, define $\delta_\ell = \delta$ for $1 \leq \ell \leq \tau$ and $\delta_\ell = \delta 2^{\tau-\ell}$ for $\ell > \tau$, where $\tau$ is a positive integer. Let $i_\ell \leq n_\ell$ be the largest integer such that $\sum_{k=0}^{i_\ell-1} \binom{n_\ell}{k} p^k (1-p)^{n_\ell-k} \leq \zeta \delta_\ell$. Let $j_\ell \geq 0$ be the smallest integer such that $\sum_{k=j_\ell}^{n_\ell} \binom{n_\ell}{k} p^k (1-p)^{n_\ell-k} \leq \zeta \delta_\ell$. Define $\xi_{p,\ell}$ such that $\hat{\xi}_{p,\ell} = X_{p_{n_\ell};n_\ell}$ if $p_{n_\ell}$ is an integer and $\hat{\xi}_{p,\ell} = (\lceil p_{n_\ell} \rceil - p_{n_\ell}) X_{\lceil p_{n_\ell} \rceil; n_\ell} + (p_{n_\ell} - \lceil p_{n_\ell} \rceil) X_{\lceil p_{n_\ell} \rceil; n_\ell}$ otherwise. Suppose that sampling is continued until $X_{j_\ell;n_\ell} - \max(\varepsilon_a, \text{sgn}(\hat{\xi}_{p,\ell}) \varepsilon_r X_{j_\ell;n_\ell}) \leq \hat{\xi}_{p,\ell} \leq X_{j_\ell;n_\ell} + \max(\varepsilon_a, \text{sgn}(\hat{\xi}_{p,\ell}) \varepsilon_r X_{j_\ell;n_\ell})$ for some stage with index $\ell$. Define estimator $\hat{\xi}_p = \hat{\xi}_{p,\ell}$ where $\ell$ is the index of stage at which the sampling is terminated. Then, $\Pr\{\ell < \infty\} = 1$ and $\Pr\{|\xi_p - \hat{\xi}_p| \leq \varepsilon_a \text{ or } |\xi_p - \hat{\xi}_p| \leq \varepsilon_r |\xi_p|\} \geq 1 - \delta$ provided that $2(\tau+1)\zeta \leq 1$ and that $\inf_{\ell > 0} \frac{n_{\ell+1}}{n_\ell} > 1$.

See Appendix 3 for a proof.

15 Conclusion

In this paper, we have proposed a new framework of multistage parametric estimation. Specific sampling schemes have been developed for basic distributions. It is demonstrated that our new methods are unprecedentedly efficient in terms of sampling cost, while rigorously guaranteeing prescribed level of confidence.

A Preliminary Results

A.1 Proof of Identity (1)

We claim that

$$\{ |\hat{\theta} - \theta| < \varepsilon_r |\theta| \} \subseteq \left\{ \frac{\hat{\theta}}{1 + \text{sgn}(\hat{\theta}) \varepsilon_r} < \theta < \frac{\hat{\theta}}{1 - \text{sgn}(\hat{\theta}) \varepsilon_r} \right\}. \quad (49)$$

Let $\omega \in \{ |\hat{\theta} - \theta| < \varepsilon_r |\theta| \}$ and $\hat{\theta} = \hat{\theta}(\omega)$. Then, $|\hat{\theta} - \theta| < \varepsilon_r |\theta|$. To show (49), it suffices to show

$$\frac{\hat{\theta}}{1 + \text{sgn}(\hat{\theta}) \varepsilon_r} < \theta < \frac{\hat{\theta}}{1 - \text{sgn}(\hat{\theta}) \varepsilon_r}. \quad (49)$$

In the case of $\hat{\theta} \geq 0$, we have $\hat{\theta} > (\theta - \varepsilon_r |\theta|) \geq 0$ as a result of $|\hat{\theta} - \theta| < \varepsilon_r |\theta|$. Moreover, $\frac{\hat{\theta}}{1 + \text{sgn}(\hat{\theta}) \varepsilon_r} = \frac{\hat{\theta}}{1 + \varepsilon_r} < \theta < \frac{\hat{\theta}}{1 - \text{sgn}(\hat{\theta}) \varepsilon_r} = \frac{\hat{\theta}}{1 - \varepsilon_r}$. In the case of $\hat{\theta} < 0$, we have $\hat{\theta} < (\theta - \varepsilon_r |\theta|) < 0$ as a result of $|\hat{\theta} - \theta| < \varepsilon_r |\theta|$. Moreover, $\frac{\hat{\theta}}{1 + \text{sgn}(\hat{\theta}) \varepsilon_r} = \frac{\hat{\theta}}{1 - \varepsilon_r} < \theta < \frac{\hat{\theta}}{1 - \text{sgn}(\hat{\theta}) \varepsilon_r} = \frac{\hat{\theta}}{1 + \varepsilon_r}$. Therefore, we have established (49).

In view of (49), it is obvious that $\{ |\hat{\theta} - \theta| < \varepsilon_a \text{ or } |\hat{\theta} - \theta| < \varepsilon_r |\theta| \} \subseteq \{ \mathcal{L}(\hat{\theta}, n) < \theta < \mathcal{U}(\hat{\theta}, n) \}$. To complete the proof of identity (1), it remains to show $\{ |\hat{\theta} - \theta| < \varepsilon_a \text{ or } |\hat{\theta} - \theta| < \varepsilon_r |\theta| \} \supseteq \{ \mathcal{L}(\hat{\theta}, n) < \theta < \mathcal{U}(\hat{\theta}, n) \}$. For this purpose, let $\omega \in \{ \mathcal{L}(\hat{\theta}, n) < \theta < \mathcal{U}(\hat{\theta}, n) \}$ and $\hat{\theta} = \hat{\theta}(\omega)$.
Then, \[
\min \left\{ \hat{\theta} - \varepsilon_a, \frac{\hat{\theta}}{1 + \text{sgn}(\hat{\theta})\varepsilon_r} \right\} < \theta < \max \left\{ \hat{\theta} + \varepsilon_a, \frac{\hat{\theta}}{1 - \text{sgn}(\hat{\theta})\varepsilon_r} \right\} \tag{50}
\]
Suppose, to get a contradiction, that \(|\hat{\theta} - \theta| \geq \varepsilon_a\) and \(|\hat{\theta} - \theta| \geq \varepsilon_r|\theta|\). There are 8 cases:

(i) \(\theta \geq 0, \hat{\theta} \geq \theta + \varepsilon_a, \hat{\theta} \geq \theta + \varepsilon_r|\theta|\). In this case, we have \(\hat{\theta} \geq 0, \theta \leq \hat{\theta} - \varepsilon_a\) and \(\theta \leq \frac{\hat{\theta}}{1 + \varepsilon_r} = \frac{\hat{\theta}}{1 + \text{sgn}(\hat{\theta})\varepsilon_r}\), which contradicts the first inequality of (50).

(ii) \(\theta \geq 0, \hat{\theta} \leq \theta - \varepsilon_a, \hat{\theta} \geq \theta + \varepsilon_r|\theta|\). In this case, we have \(\theta + \varepsilon_r|\theta| \leq \hat{\theta} \leq \theta - \varepsilon_a\), which implies that \(\varepsilon_a = 0\) and \(\hat{\theta} \geq 0\). Therefore, the first inequality of (50) can be written as \(\frac{\hat{\theta}}{1 + \varepsilon_r} < \theta\), which contradicts to \(\hat{\theta} \geq \theta + \varepsilon_r|\theta| = (1 + \varepsilon_r)\theta\).

(iii) \(\theta \geq 0, \hat{\theta} \geq \theta + \varepsilon_a, \hat{\theta} \leq \theta - \varepsilon_r|\theta|\). In this case, we have \(\theta + \varepsilon_a \leq \hat{\theta} \leq \theta - \varepsilon_r|\theta|\), which implies that \(\varepsilon_a = 0\) and \(\hat{\theta} \geq 0\). Therefore, the second inequality of (50) can be written as \(\frac{\hat{\theta}}{1 - \varepsilon_r} > \theta\), which contradicts to \(\hat{\theta} \leq \theta - \varepsilon_r|\theta| = (1 - \varepsilon_r)\theta\).

(iv) \(\theta \geq 0, \hat{\theta} \leq \theta - \varepsilon_a, \hat{\theta} \leq \theta - \varepsilon_r|\theta|\). In this case, we have \(\theta + \varepsilon_a \leq \hat{\theta} \leq \theta - \varepsilon_r|\theta|\), which implies that \(\varepsilon_a = 0\) and \(\hat{\theta} \leq 0\). Hence, by the second inequality of (50), we have \(\frac{\hat{\theta}}{1 - \varepsilon_r} \leq \theta < \frac{\hat{\theta}}{1 + \text{sgn}(\hat{\theta})\varepsilon_r}\), which implies \(\hat{\theta}[1 - \text{sgn}(\hat{\theta})\varepsilon_r] < \hat{\theta}(1 - \varepsilon_r)\), i.e., \(\varepsilon_r|\hat{\theta}| > \varepsilon_r\hat{\theta}\). It follows that \(\hat{\theta} < 0\) and thus \(\theta < 0\), which contradicts to \(\theta \geq 0\).

(v) \(\theta < 0, \hat{\theta} \geq \theta + \varepsilon_a, \hat{\theta} \geq \theta + \varepsilon_r|\theta|\). In this case, we have \(\theta \leq \hat{\theta} - \varepsilon_a\) and \(\theta \leq \frac{\hat{\theta}}{1 - \varepsilon_r}\). Hence, by the first inequality of (50), we have \(\frac{\hat{\theta}}{1 - \varepsilon_r} \geq \theta > \frac{\hat{\theta}}{1 + \text{sgn}(\hat{\theta})\varepsilon_r}\), which implies \(\hat{\theta}[1 + \text{sgn}(\hat{\theta})\varepsilon_r] > \hat{\theta}(1 - \varepsilon_r)\), i.e., \(\varepsilon_r|\hat{\theta}| > -\varepsilon_r\hat{\theta}\). It follows that \(\hat{\theta} > 0\) and thus \(\theta > 0\), which contradicts to \(\theta < 0\).

(vi) \(\theta < 0, \hat{\theta} \leq \theta - \varepsilon_a, \hat{\theta} \geq \theta + \varepsilon_a\). In this case, we have \(\theta - \varepsilon_a \geq \hat{\theta} \geq \theta + \varepsilon_a\), which implies that \(\varepsilon_a = 0\) and \(\hat{\theta} < 0\). Therefore, the first inequality of (50) can be written as \(\frac{\hat{\theta}}{1 + \varepsilon_r} < \theta\), which contradicts to \(\hat{\theta} \geq \theta + \varepsilon_r|\theta| = (1 + \varepsilon_r)\theta\).

(vii) \(\theta < 0, \hat{\theta} \geq \theta + \varepsilon_a, \hat{\theta} \leq \theta - \varepsilon_r|\theta|\). In this case, we have \(\theta - \varepsilon_r|\theta| \geq \hat{\theta} \geq \theta + \varepsilon_a\), which implies that \(\varepsilon_a = 0\) and \(\hat{\theta} < 0\). Therefore, the second inequality of (50) can be written as \(\frac{\hat{\theta}}{1 + \varepsilon_r} > \theta\), which contradicts to \(\hat{\theta} \leq \theta - \varepsilon_r|\theta| = (1 + \varepsilon_r)\theta\).

(viii) \(\theta < 0, \hat{\theta} \leq \theta - \varepsilon_a, \hat{\theta} \leq \theta - \varepsilon_r|\theta|\). In this case, we have \(\hat{\theta} < 0, \theta \geq \hat{\theta} + \varepsilon_a\) and \(\theta \geq \frac{\hat{\theta}}{1 + \varepsilon_r} = \frac{\hat{\theta}}{1 - \text{sgn}(\hat{\theta})\varepsilon_r}\), which contradicts the second inequality of (50).

From the above 8 cases, we see that the assumption that \(|\hat{\theta} - \theta| \geq \varepsilon_a\) and \(|\hat{\theta} - \theta| \geq \varepsilon_r|\theta|\) always leads to a contradiction. Therefore, it must be true that either \(|\hat{\theta} - \theta| < \varepsilon_a\) or \(|\hat{\theta} - \theta| < \varepsilon_r|\theta|\). This proves \(|\hat{\theta} - \theta| < \varepsilon_a\) or \(|\hat{\theta} - \theta| < \varepsilon_r|\theta|\). Therefore, the proof of identity (11),
A.2 Probability Transform Inequalities

The well-known probability transform theorem asserts that \( \Pr\{F_Z(Z) \leq \alpha\} = \Pr\{G_Z(Z) \leq \alpha\} = \alpha \) for any continuous random variable \( Z \) and positive number \( \alpha \in [0, 1] \). In the general case that \( Z \) is not necessarily continuous, the probability transform equalities may not be true. Fortunately, their generalizations, referred to as “probability transform inequalities”, can be established as follows.

**Lemma 2** \( \Pr\{F_Z(Z) \leq \alpha\} \leq \alpha \) and \( \Pr\{G_Z(Z) \leq \alpha\} \leq \alpha \) for any random variable \( Z \) and positive number \( \alpha \).

**Proof.** Let \( I_z \) denote the support of \( Z \). If \( \{z \in I_Z : F_Z(z) \leq \alpha\} \) is empty, then, \( \{F_Z(Z) \leq \alpha\} \) is an impossible event and thus \( \Pr\{F_Z(Z) \leq \alpha\} = 0 \). Otherwise, we can define \( z^* = \max\{z \in I_Z : F_Z(z) \leq \alpha\} \). It follows from the definition of \( z^* \) that \( F_Z(z^*) \leq \alpha \). Since \( F_Z(z) \) is non-decreasing with respect to \( z \), we have \( \{F_Z(Z) \leq \alpha\} = \{Z \leq z^*\} \). Therefore, \( \Pr\{F_Z(Z) \leq \alpha\} = \Pr\{Z \leq z^*\} \leq \alpha \) for any \( \alpha > 0 \). By a similar method, we can show \( \Pr\{G_Z(Z) \leq \alpha\} \leq \alpha \) for any \( \alpha > 0 \).

\( \square \)

A.3 Property of ULE

**Lemma 3** Let \( \mathcal{E} \) be an event determined by random tuple \( (X_1, \cdots, X_m) \). Let \( \varphi(X_1, \cdots, X_m) \) be a ULE of \( \theta \). Then,

(i) \( \Pr\{\mathcal{E} \mid \theta\} \) is non-increasing with respect to \( \theta \in \Theta \) no less than \( z \) provided that \( \mathcal{E} \subseteq \{\varphi(X_1, \cdots, X_m) \leq z\} \).

(ii) \( \Pr\{\mathcal{E} \mid \theta\} \) is non-decreasing with respect to \( \theta \in \Theta \) no greater than \( z \) provided that \( \mathcal{E} \subseteq \{\varphi(X_1, \cdots, X_m) \geq z\} \).

**Proof.** We first consider the case that \( X_1, X_2, \cdots \) are discrete random variables. Let \( I_m \) denote the support of \( m \), i.e., \( I_m = \{m(\omega) : \omega \in \Omega\} \). Define \( \mathcal{X}_m = \{(X_1(\omega), \cdots, X_m(\omega)) : \omega \in \mathcal{E}, m(\omega) = m\} \) for \( m \in I_m \). Then,

\[
\Pr\{\mathcal{E} \mid \theta\} = \sum_{m \in I_m} \left( \sum_{(x_1, \cdots, x_m) \in \mathcal{X}_m} \Pr\{X_i = x_i, i = 1, \cdots, m \mid \theta\} \right). \tag{51}
\]

To show statement (i), using the assumption that \( \mathcal{E} \subseteq \{\varphi(X_1, \cdots, X_m) \leq z\} \), we have \( \varphi(x_1, \cdots, x_m) \leq z \) for \( (x_1, \cdots, x_m) \in \mathcal{X}_m \) with \( m \in I_m \). Since \( \varphi(X_1, \cdots, X_m) \) is a ULE of \( \theta \), we have that \( \Pr\{X_i = x_i, i = 1, \cdots, m \mid \theta\} \) is non-increasing with respect to \( \theta \in \Theta \) no less than \( z \). It follows immediately from (51) that statement (i) is true.

To show statement (ii), using the assumption that \( \mathcal{E} \subseteq \{\varphi(X_1, \cdots, X_m) \geq z\} \), we have \( \varphi(x_1, \cdots, x_m) \geq z \) for \( (x_1, \cdots, x_m) \in \mathcal{X}_m \) with \( m \in I_m \). Since \( \varphi(X_1, \cdots, X_m) \) is a ULE of \( \theta \), we
have that $\Pr\{X_i = x_i, \ i = 1, \ldots, m \mid \theta\}$ is non-decreasing with respect to $\theta \in \Theta$ no greater than $z$. It follows immediately from (51) that statement (ii) is true.

For the case that $X_1, X_2, \ldots$ are continuous random variables, we can also show the lemma by modifying the argument for the discrete case. Specially, the summation of likelihood function $\Pr\{X_i = x_i, \ i = 1, \ldots, m \mid \theta\}$ over the set of tuple $(x_1, \ldots, x_m)$ is replaced by the integration of the joint probability density function $f_{X_1,\ldots,X_m}(x_1, \ldots, x_m, \theta)$ over the set of $(x_1, \ldots, x_m)$. This concludes the proof of Lemma 3.

\[\square\]

\section{Proof of Theorem 2}

Making use of assumptions (ii)-(iii), the definition of the sampling scheme and the monotonicity of $F_{\hat{\theta}}(z, \theta)$ as asserted by Lemma 3, we have

$$\Pr\{\theta \geq \mathcal{U}(\hat{\theta}, n) \mid \theta\} = \sum_{\ell=1}^{s} \Pr\{\theta \geq \mathcal{U}(\hat{\theta}_\ell, n_\ell), \ell = \ell' \mid \theta\} \leq \sum_{\ell=1}^{s} \Pr\{\theta \geq \mathcal{U}(\hat{\theta}_\ell, n_\ell) \geq \hat{\theta}_\ell, F_{\hat{\theta}_\ell}(\hat{\theta}_\ell, \mathcal{U}(\hat{\theta}_\ell, n_\ell)) \leq \zeta \delta_\ell \mid \theta\} \leq \sum_{\ell=1}^{s} \Pr\{F_{\hat{\theta}_\ell}(\hat{\theta}_\ell, \theta) \leq \xi \delta_\ell \mid \theta\} \leq \zeta \sum_{\ell=1}^{s} \delta_\ell$$

for any $\theta \in \Theta$, where the last inequality follows from Lemma 2.

Similarly, we can show that $\Pr\{\theta \leq \mathcal{L}(\hat{\theta}, n) \mid \theta\} \leq \zeta \sum_{\ell=1}^{s} \delta_\ell$. Hence, $\Pr\{\mathcal{L}(\hat{\theta}, n) < \theta < \mathcal{U}(\hat{\theta}, n) \mid \theta\} \geq 1 - \Pr\{\theta \leq \mathcal{L}(\hat{\theta}, n) \mid \theta\} - \Pr\{\theta \geq \mathcal{U}(\hat{\theta}, n) \mid \theta\} \geq 1 - 2\zeta \sum_{\ell=1}^{s} \delta_\ell$. This concludes the proof of Theorem 2.

\section{Proof of Theorem 3}

Let $\theta' < \theta''$ be two consecutive distinct elements of $I_{x} \cup \{a, b\}$. Then, $\{\theta \leq \mathcal{L}(\hat{\theta}, n) < \theta''\} \subseteq \{\theta' < \mathcal{L}(\hat{\theta}, n) < \theta''\} = \emptyset$ and it follows that $\{\mathcal{L}(\hat{\theta}, n) \geq \theta\} = \{\mathcal{L}(\hat{\theta}, n) \geq \theta''\} \cup \{\theta \leq \mathcal{L}(\hat{\theta}, n) < \theta''\} = \{\mathcal{L}(\hat{\theta}, n) \geq \theta''\}$ for any $\theta \in (\theta', \theta'')$. Recalling that $\{\hat{\theta} \geq \mathcal{L}(\hat{\theta}, n)\}$ is a sure event, we have $\{\mathcal{L}(\hat{\theta}, n) \geq \theta''\} = \{\hat{\theta} \geq \theta'', \mathcal{L}(\hat{\theta}, n) \geq \theta''\}$. Invoking the second statement of Lemma 3, we have that $\Pr\{\theta \leq \mathcal{L}(\hat{\theta}, n)\}$ and $e'$ occurs $\mid \theta$ = $\Pr\{\mathcal{L}(\hat{\theta}, n) \geq \theta''\}$ and $e'$ occurs $\mid \theta$ = $\Pr\{\hat{\theta} \geq \theta'', \mathcal{L}(\hat{\theta}, n) \geq \theta''\}$ and $e'$ occurs $\mid \theta$ is non-decreasing with respect to $\theta \in (\theta', \theta'')$. This implies that the maximum of $\Pr\{\theta \leq \mathcal{L}(\hat{\theta}, n)\}$ and $e'$ occurs $\mid \theta$ with respect to $\theta \in (\theta', \theta'')$ is equal to $\Pr\{\hat{\theta} \geq \theta'', \mathcal{L}(\hat{\theta}, n) \geq \theta''\}$ and $e'$ occurs $\mid \theta''$. Since the argument holds for arbitrary consecutive distinct elements of $I_{x} \cup \{a, b\}$, we have established statement (I) regarding $\Pr\{\theta \leq \mathcal{L}(\hat{\theta}, n)\}$ and $e'$ occurs $\mid \theta$ for $\theta \in [a, b]$. To prove the statement regarding $\Pr\{\theta < \mathcal{L}(\hat{\theta}, n)\}$ and $e'$ occurs $\mid \theta$, note that $\{\theta < \mathcal{L}(\hat{\theta}, n) < \theta''\} \subseteq \{\theta' < \mathcal{L}(\hat{\theta}, n) < \theta''\} = \emptyset$, which
implies that \( \{ \mathcal{L}(\hat{\theta}, n) > \theta \} = \{ \mathcal{L}(\hat{\theta}, n) \geq \theta'' \} \cup \{ \theta < \mathcal{L}(\hat{\theta}, n) < \theta'' \} = \{ \mathcal{L}(\hat{\theta}, n) \geq \theta'' \} \) for any \( \theta \in [\theta', \theta'') \). Hence, \( \Pr\{ \theta < \mathcal{L}(\hat{\theta}, n) \text{ and } \mathcal{E} \text{ occurs} \mid \theta \} = \Pr\{ \mathcal{L}(\hat{\theta}, n) \geq \theta'' \text{ and } \mathcal{E} \text{ occurs} \mid \theta \} = \Pr\{ \hat{\theta} \geq \theta'' \}, \mathcal{L}(\hat{\theta}, n) \geq \theta'' \text{ and } \mathcal{E} \text{ occurs} \mid \theta \} \) is non-decreasing with respect to \( \theta \in [\theta', \theta'') \). This implies that the supremum of \( \Pr\{ \theta < \mathcal{L}(\hat{\theta}, n) \text{ and } \mathcal{E} \text{ occurs} \mid \theta \} \) with respect to \( \theta \in [\theta', \theta'') \) is equal to \( \Pr\{ \hat{\theta} \geq \theta'' \}, \mathcal{L}(\hat{\theta}, n) \geq \theta'' \text{ and } \mathcal{E} \text{ occurs} \mid \theta'' \} \). Since the argument holds for arbitrary consecutive distinct elements of \( I \cup \{a, b\} \), we have established statement (I) regarding \( \Pr\{ \theta < \mathcal{L}(\hat{\theta}, n) \text{ and } \mathcal{E} \text{ occurs} \mid \theta \} \) for \( \theta \in [a, b] \).

To prove statement (II) regarding \( \Pr\{ \theta \geq \mathcal{L}(\hat{\theta}, n) \text{ and } \mathcal{E} \text{ occurs} \mid \theta \} \), let \( \theta' < \theta'' \) be two consecutive distinct elements of \( I \cup \{a, b\} \). Then, \( \{ \theta' < \mathcal{L}(\hat{\theta}, n) \leq \theta \} \subseteq \{ \theta' < \mathcal{L}(\hat{\theta}, n) < \theta'' \} = \emptyset \) and it follows that \( \{ \mathcal{L}(\hat{\theta}, n) \leq \theta \} = \{ \mathcal{L}(\hat{\theta}, n) \leq \theta' \} \cup \{ \theta' < \mathcal{L}(\hat{\theta}, n) \leq \theta \} = \{ \mathcal{L}(\hat{\theta}, n) \leq \theta' \} \) for any \( \theta \in [\theta', \theta'') \). Recalling that \( \{ \hat{\theta} \leq \mathcal{L}(\hat{\theta}, n) \} \) is a sure event, we have \( \{ \mathcal{L}(\hat{\theta}, n) \leq \theta' \} = \{ \hat{\theta} \leq \theta', \mathcal{L}(\hat{\theta}, n) \leq \theta' \} \). Consequently, \( \Pr\{ \mathcal{L}(\hat{\theta}, n) \leq \theta \text{ and } \mathcal{E} \text{ occurs} \mid \theta \} = \Pr\{ \mathcal{L}(\hat{\theta}, n) \leq \theta' \text{ and } \mathcal{E} \text{ occurs} \mid \theta \} = \Pr\{ \hat{\theta} \leq \theta', \mathcal{L}(\hat{\theta}, n) \leq \theta' \text{ and } \mathcal{E} \text{ occurs} \mid \theta \} \) is non-increasing with respect to \( \theta \in [\theta', \theta'') \) as a result of the first statement of Lemma \[3\]. This implies that the maximum of \( \Pr\{ \mathcal{L}(\hat{\theta}, n) \leq \theta \text{ and } \mathcal{E} \text{ occurs} \mid \theta \} \) for \( \theta \in [\theta', \theta'') \) is equal to \( \Pr\{ \hat{\theta} \leq \theta', \mathcal{L}(\hat{\theta}, n) \leq \theta' \text{ and } \mathcal{E} \text{ occurs} \mid \theta \} \). Since the argument holds for arbitrary consecutive distinct elements of \( I \cup \{a, b\} \), we have established statement (II) regarding \( \Pr\{ \theta > \mathcal{L}(\hat{\theta}, n) \text{ and } \mathcal{E} \text{ occurs} \mid \theta \} \) for \( \theta \in [a, b] \).

To show statement (III), note that \( \Pr\{ \theta \leq \mathcal{L}(\hat{\theta}, n) \text{ and } \mathcal{E} \text{ occurs} \mid \theta \} \) is no greater than \( \Pr\{ a \leq \mathcal{L}(\hat{\theta}, n) \text{ and } \mathcal{E} \text{ occurs} \mid \theta \} \) for any \( \theta \in [a, b] \). By the assumption that \( \{ a \leq \mathcal{L}(\hat{\theta}, n) \} \subseteq \{ \hat{\theta} \geq b \} \), we have \( \Pr\{ a \leq \mathcal{L}(\hat{\theta}, n) \text{ and } \mathcal{E} \text{ occurs} \mid \theta \} = \Pr\{ \hat{\theta} \geq b, a \leq \mathcal{L}(\hat{\theta}, n) \text{ and } \mathcal{E} \text{ occurs} \mid \theta \} \) for any \( \theta \in [a, b] \). As a result of the second statement of Lemma \[3\] we have that \( \Pr\{ \hat{\theta} \geq b, a \leq \mathcal{L}(\hat{\theta}, n) \text{ and } \mathcal{E} \text{ occurs} \mid \theta \} \) is non-decreasing with respect to \( \theta \in [a, b] \). It follows that \( \Pr\{ \hat{\theta} \geq b, a \leq \mathcal{L}(\hat{\theta}, n) \text{ and } \mathcal{E} \text{ occurs} \mid \theta \} \leq \Pr\{ \hat{\theta} \geq b, a \leq \mathcal{L}(\hat{\theta}, n) \text{ and } \mathcal{E} \text{ occurs} \mid b \} \) for any \( \theta \in [a, b] \), which implies that \( \Pr\{ \theta \leq \mathcal{L}(\hat{\theta}, n) \text{ and } \mathcal{E} \text{ occurs} \mid \theta \} \leq \Pr\{ a \leq \mathcal{L}(\hat{\theta}, n) \text{ and } \mathcal{E} \text{ occurs} \mid b \} \) for any \( \theta \in [a, b] \). On the other hand, \( \Pr\{ \theta \leq \mathcal{L}(\hat{\theta}, n) \text{ and } \mathcal{E} \text{ occurs} \mid \theta \} \geq \Pr\{ b \leq \mathcal{L}(\hat{\theta}, n) \text{ and } \mathcal{E} \text{ occurs} \mid \theta \} \) for any \( \theta \in [a, b] \). Recalling that \( \{ \hat{\theta} \geq \mathcal{L}(\hat{\theta}, n) \} \) is a sure event, we have \( \Pr\{ b \leq \mathcal{L}(\hat{\theta}, n) \text{ and } \mathcal{E} \text{ occurs} \mid \theta \} = \Pr\{ b \leq \mathcal{L}(\hat{\theta}, n) \leq \theta \text{ and } \mathcal{E} \text{ occurs} \mid \theta \} \) for any \( \theta \in [a, b] \). Hence, applying the second statement of Lemma \[3\] we have that \( \Pr\{ b \leq \mathcal{L}(\hat{\theta}, n) \leq \theta \text{ and } \mathcal{E} \text{ occurs} \mid \theta \} \geq \Pr\{ b \leq \mathcal{L}(\hat{\theta}, n) \leq \theta \text{ and } \mathcal{E} \text{ occurs} \mid a \} = \Pr\{ b \leq \mathcal{L}(\hat{\theta}, n) \text{ and } \mathcal{E} \text{ occurs} \mid a \} \) for any \( \theta \in [a, b] \), which implies that \( \Pr\{ \theta \leq \mathcal{L}(\hat{\theta}, n) \text{ and } \mathcal{E} \text{ occurs} \mid \theta \} \geq \Pr\{ b \leq \mathcal{L}(\hat{\theta}, n) \text{ and } \mathcal{E} \text{ occurs} \mid a \} \) for any \( \theta \in [a, b] \). So, we have established \( \Pr\{ b \leq \mathcal{L}(\hat{\theta}, n) \text{ and } \mathcal{E} \text{ occurs} \mid a \} \) for any \( \theta \in [a, b] \).
$\mathcal{L}(\hat{\theta}, \mathbf{n})$ and $\mathcal{E}$ occurs $|a| \leq \Pr\{\theta \leq \mathcal{L}(\hat{\theta}, \mathbf{n}) \text{ and } \mathcal{E} \text{ occurs } |\theta| \leq \Pr\{a \leq \mathcal{L}(\hat{\theta}, \mathbf{n}) \text{ and } \mathcal{E} \text{ occurs } |b\}$ for any $\theta \in [a, b]$. In a similar manner, we can show that $\Pr\{b \leq \mathcal{L}(\hat{\theta}, \mathbf{n}) \text{ and } \mathcal{E} \text{ occurs } |a\} \leq \Pr\{\theta \leq \mathcal{L}(\hat{\theta}, \mathbf{n}) \text{ and } \mathcal{E} \text{ occurs } |b\}$ for any $\theta \in [a, b]$.

Statement (IV) can be shown by a similar method as that of statement (III). This concludes the proof of Theorem 5.

D Proof of Theorem 5

It is easy to show that, for $x_i \in \{0, 1\}$, $i = 1, \cdots, n$,

$$\Pr\{X_1 = x_1, \cdots, X_n = x_n\} = h(M, k) \quad \text{where} \quad h(M, k) = \binom{M}{k} \binom{N-M}{n-k} / \left[\binom{n}{k}\binom{N}{n}\right]$$

with $M = pN$ and $k = \sum_{i=1}^{n} x_i$. Note that $h(M, k) = 0$ if $M$ is smaller than $k$ or greater than $N - n + k$. For $k < M \leq N - n + k$, we have $h(M, k) = \frac{h(M-1, k)}{h(M, k)} = M-k \frac{N-M+1}{N-M-n+k+1} \leq 1$ if and only if $M \leq \frac{k}{n}(N+1)$, or equivalently, $M \leq \left\lfloor \frac{k}{n}(N+1) \right\rfloor$. It can be checked that $\frac{k}{n}(N+1) - (N-n+k+1)$ is equal to $(\frac{k}{n}-1)(N+1-n)$, which is negative for $k < n$. Hence, for $k < n$, we have that $\frac{k}{n}(N+1) \leq N-n+k$ and consequently, the maximum of $h(M, k)$ with respect to $M \in \{0, 1, \cdots, N\}$ is achieved at $\left\lfloor (N+1)\frac{k}{n} \right\rfloor$. For $k = n$, we have $h(M, k) = h(M, n) = (\frac{M}{n}) / (\frac{n}{n})$, of which the maximum with respect to $M$ is attained at $M = N$. Therefore, for any $k \in \{0, 1, \cdots, n\}$, the maximum of $h(M, k)$ with respect to $M \in \{0, 1, \cdots, N\}$ is achieved at $\min\{N, \left\lfloor (N+1)\frac{k}{n} \right\rfloor \}$. It follows that $\min\{1, \frac{1}{\sqrt{n}} \left\lfloor \frac{N+1}{n} \sum_{i=1}^{n} X_i \right\rfloor \}$ is a MLE and also a ULE for $p \in \Theta$. For simplicity of notations, let $\hat{p} = \min\{1, \frac{1}{\sqrt{n}} \left\lfloor (N+1)\frac{k}{n} \right\rfloor \}$. We claim that $|\hat{p} - \frac{k}{n}| < \frac{1}{\sqrt{n}}$ for $0 \leq k \leq n$. To prove such claim, we investigate two cases. In the case of $k = n$, we have $\hat{p} = \frac{k}{n} = 1$. In the case of $k < n$, we have $\hat{p} = \frac{1}{\sqrt{n}} \left\lfloor (N+1)\frac{k}{n} \right\rfloor \leq \frac{1}{\sqrt{n}}(N+1)\frac{k}{n} < \frac{k}{n} + \frac{1}{\sqrt{n}}$ and $\hat{p} > \frac{1}{\sqrt{n}} \left[(N+1)\frac{k}{n} - 1\right] = \frac{k}{n} + \frac{1}{\sqrt{n}}(\frac{k}{n} - 1) \geq \frac{k}{n} - \frac{1}{\sqrt{n}}$. The claim is thus proved. In view of this established claim and the fact that the difference between any pair of values of $p \in \Theta$ is no less than $\frac{1}{\sqrt{n}}$, we have that $\frac{\sum_{i=1}^{n} X_i}{n}$ is a ULE for $p \in \Theta$. This completes the proof of the theorem.

E Proof of Theorem 7

Define $\hat{\mu}_\ell = \frac{\sum_{i=1}^{n} X_i}{n\ell}$ and $F_\ell(x) = \Pr\{\hat{\mu}_\ell \leq x, \ l = \ell\}$ for $\ell = 1, \cdots, s$, where $\ell$ is the index of stage when the sampling is terminated. Let $\sigma^2$ denote the variance of $X$. 
To show statement (I), note that

$$\mathbb{E}[\hat{\mu} - \mu] \leq \mathbb{E}|\hat{\mu} - \mu| = \sum_{\ell=1}^{s} \int_{-\infty}^{\infty} |x - \mu| \, dF_\ell(x)$$

$$= \sum_{\ell=1}^{s} \left[ \int_{|x-\mu|<\frac{1}{\sqrt{n_\ell}}} |x - \mu| \, dF_\ell(x) + \int_{|x-\mu|\geq\frac{1}{\sqrt{n_\ell}}} |x - \mu| \, dF_\ell(x) \right]$$

$$= \sum_{\ell=1}^{s} \int_{|x-\mu|<\frac{1}{\sqrt{n_\ell}}} |x - \mu| \, dF_\ell(x) + \sum_{\ell=1}^{s} \int_{|x-\mu|\geq\frac{1}{\sqrt{n_\ell}}} |x - \mu| \, dF_\ell(x)$$

$$\leq \sum_{\ell=1}^{s} \int_{|x-\mu|<\frac{1}{\sqrt{n_\ell}}} \frac{1}{\sqrt{n_\ell}} \, dF_\ell(x) + \sum_{\ell=1}^{s} \int_{|x-\mu|\geq\frac{1}{\sqrt{n_\ell}}} \sqrt{n_\ell}|x - \mu|^2 \, dF_\ell(x)$$

$$\leq \sum_{\ell=1}^{s} \frac{1}{\sqrt{n_\ell}} \int_{|x-\mu|<\frac{1}{\sqrt{n_\ell}}} \, dF_\ell(x) + \sum_{\ell=1}^{s} \sqrt{n_\ell} \int_{-\infty}^{\infty} |x - \mu|^2 \, dF_\ell(x)$$

$$= \sum_{\ell=1}^{s} \frac{1}{\sqrt{n_\ell}} \Pr\left\{ |\hat{\mu}_\ell - \mu| < \frac{1}{\sqrt{n_\ell}}, \ell = \ell \right\} + \sum_{\ell=1}^{s} \sqrt{n_\ell} \mathbb{E}[|\hat{\mu}_\ell - \mu|^2]$$

$$\leq \frac{1}{\sqrt{n_1}} \sum_{\ell=1}^{s} \Pr\{ \ell = \ell \} + \frac{s}{\sqrt{n_1}} \frac{\sigma^2}{n_\ell} = \frac{1}{\sqrt{n_1}} + \frac{\sigma^2}{\sqrt{n_1}} \sum_{\ell=1}^{s} \frac{1}{\sqrt{n_\ell}}$$

By the assumption that $\inf_{\ell>0} \frac{n_{\ell+1}}{n_\ell} > 1$, we have that, there exists a positive number $\rho$ such that $n_\ell \leq (1 + \rho)^{2(\ell-1)} n_1$ for all $\ell > 1$. Hence,

$$\mathbb{E}[\hat{\mu} - \mu] \leq \frac{1}{\sqrt{n_1}} + \frac{\sigma^2}{\sqrt{n_1}} \frac{1}{(1 + \rho)^{\ell-1}} \leq \frac{1}{\sqrt{n_1}} + \frac{\sigma^2}{\sqrt{n_1}} \frac{1 + \rho}{\rho} \to 0$$

as $n_1 \to \infty$. Moreover,

$$\mathbb{E}|\hat{\mu} - \mu|^2 = \sum_{\ell=1}^{s} \int_{-\infty}^{\infty} |x - \mu|^2 \, dF_\ell(x) \leq \sum_{\ell=1}^{s} \mathbb{E}|\hat{\mu}_\ell - \mu|^2$$

$$= \sigma^2 \sum_{\ell=1}^{s} \frac{1}{n_\ell} \leq \sigma^2 \sum_{\ell=1}^{s} \frac{1}{n_1 (1 + \rho)^{2(\ell-1)}} = \sigma^2 \frac{(1 + \rho)^2}{n_1 \rho(2 + \rho)} \to 0$$

as $n_1 \to \infty$. This completes the proof of statement (I).

Now we shall show statement (II). Since $X$ is a bounded variable, there exists a positive number $C$ such that $|X - \mu| < C$. By Chebyshev’s inequality, we have $\Pr\{ |\hat{\mu}_\ell - \mu| \geq \frac{1}{\sqrt{n_\ell}} \} \leq \frac{\sigma^2}{\sqrt{n_\ell}}$
for \( \ell = 1, \ldots, s \). Therefore, for \( k = 1, 2, \ldots \),

\[
\mathbb{E}[|\widehat{\mu} - \mu|^k] = \sum_{\ell=1}^s \int_{-\infty}^{\infty} |x-\mu|^k \, dF_\ell(x)
\]

\[
= \sum_{\ell=1}^s \left[ \int_{|x-\mu| < \frac{1}{n_1}} |x-\mu|^k \, dF_\ell(x) + \int_{|x-\mu| \geq \frac{1}{n_1}} |x-\mu|^k \, dF_\ell(x) \right]
\]

\[
\leq \sum_{\ell=1}^s \left( \frac{1}{\sqrt{n_1}} \right)^k \int_{|x-\mu| < \frac{1}{n_1}} dF_\ell(x) + C^k \sum_{\ell=1}^s \int_{|x-\mu| \geq \frac{1}{n_1}} dF_\ell(x)
\]

\[
= \sum_{\ell=1}^s \left( \frac{1}{\sqrt{n_1}} \right)^k \Pr \left\{ |\widehat{\mu}_\ell - \mu| < \frac{1}{\sqrt{n_1}}, \, l = \ell \right\} + C^k \sum_{\ell=1}^s \Pr \left\{ |\widehat{\mu}_\ell - \mu| \geq \frac{1}{\sqrt{n_1}}, \, l = \ell \right\}
\]

\[
\leq \left( \frac{1}{\sqrt{n_1}} \right)^k \sum_{\ell=1}^s \Pr \{ l = \ell \} + C^k \sum_{\ell=1}^s \Pr \left\{ |\widehat{\mu}_\ell - \mu| \geq \frac{1}{\sqrt{n_1}} \right\}
\]

\[
= \left( \frac{1}{\sqrt{n_1}} \right)^k + C^k \sum_{\ell=1}^s \Pr \left\{ |\widehat{\mu}_\ell - \mu| \geq \frac{1}{\sqrt{n_1}} \right\} \leq \left( \frac{1}{\sqrt{n_1}} \right)^k + C^k \sum_{\ell=1}^s \frac{\sigma^2}{\sqrt{n_1}} \to 0
\]

as \( n_1 \to \infty \). Since \( \mathbb{E}[|\widehat{\mu} - \mu|] \leq \mathbb{E}[|\widehat{\mu} - \mu|] \), we have that \( \mathbb{E}[|\widehat{\mu} - \mu|] \to 0 \) as \( n_1 \to \infty \). This completes the proof of statement (II).

### F Proof of Theorem 8

We only show the last statement of Theorem 8. Note that

\[
n_s - n_1 \Pr\{l = 1\} = n_s \Pr\{l \leq s\} - n_1 \Pr\{l \leq 1\} = \sum_{\ell=2}^s (n_\ell \Pr\{l \leq \ell\} - n_{\ell-1} \Pr\{l \leq \ell - 1\})
\]

\[
= \sum_{\ell=2}^s n_\ell \Pr\{l \leq \ell\} - \Pr\{l \leq \ell - 1\} + \sum_{\ell=2}^s (n_\ell - n_{\ell-1}) \Pr\{l \leq \ell - 1\}
\]

\[
= \sum_{\ell=2}^s n_\ell \Pr\{l = \ell\} + \sum_{\ell=2}^s (n_\ell - n_{\ell-1}) \Pr\{l \leq \ell - 1\},
\]

from which we obtain \( n_s - \sum_{\ell=1}^s n_\ell \Pr\{l = \ell\} = \sum_{\ell=2}^s (n_\ell - n_{\ell-1}) \Pr\{l \leq \ell - 1\} \). Observing that \( n_s = n_1 + \sum_{\ell=2}^s (n_\ell - n_{\ell-1}) \), we have

\[
\mathbb{E}[n] = \sum_{\ell=1}^s n_\ell \Pr\{l = \ell\} = n_s - \left( n_s - \sum_{\ell=1}^s n_\ell \Pr\{l = \ell\} \right)
\]

\[
= n_1 + \sum_{\ell=2}^s (n_\ell - n_{\ell-1}) - \sum_{\ell=2}^s (n_\ell - n_{\ell-1}) \Pr\{l \leq \ell - 1\}
\]

\[
= n_1 + \sum_{\ell=2}^s (n_\ell - n_{\ell-1}) \Pr\{l \geq \ell - 1\} = n_1 + \sum_{\ell=1}^{s-1} (n_{\ell+1} - n_\ell) \Pr\{l > \ell\}.
\]
Proof of Theorem 11

To prove Theorem 11 we shall only provide the proof of statement (I), since the proof of statement (II) is similar. As a consequence of the assumption that \( f(k+1) - f(k) \leq f(k) - f(k-1) \) for \( a < k < b \), we have

\[
\frac{f(b) - f(a)}{b - a} \leq \frac{\frac{f(b) - f(k)}{b - k} + \frac{f(k) - f(a)}{k - a}}{b - a} \leq \frac{\frac{f(k) - f(a)}{k - a}}{b - a} = \frac{f(k) - f(a)}{k - a},
\]

which implies \( f(k) \geq f(a) + \frac{f(b) - f(a)}{b - a}(k - a) \) for \( a \leq k \leq b \) and it follows that

\[
\sum_{k=a}^{b} f(k) \geq (b - a + 1)f(a) + \frac{f(b) - f(a)}{b - a} \sum_{k=a}^{b} (k - a) = \frac{(b - a + 1)[f(b) + f(a)]}{2}.
\]

Again by virtue of the assumption that \( f(k+1) - f(k) \leq f(k) - f(k-1) \) for \( a < k < b \), we have

\[
f(k) - f(a) = \sum_{l=a}^{k-1} [f(l + 1) - f(l)] \leq \sum_{l=a}^{k-1} [f(a + 1) - f(a)] = (k - a)[f(a + 1) - f(a)],
\]

\[
f(k) - f(b) = \sum_{l=k}^{b-1} [f(l) - f(l + 1)] \leq \sum_{l=k}^{b-1} [f(b - 1) - f(b)] = (k - b)[f(b) - f(b - 1)]
\]

for \( a < k < b \). Making use of the above established inequalities, we have

\[
\sum_{k=a}^{b} f(k) = (b - a + 1)f(a) + \sum_{k=a}^{i} [f(k) - f(a)] + \sum_{k=i+1}^{b} [f(b) - f(a)] + \sum_{k=i+1}^{b} [f(k) - f(b)]
\]

\[
\leq (b - a + 1)f(a) + \sum_{k=a}^{i} (k - a)[f(a + 1) - f(a)] + (b - i)[f(b) - f(a)] + \sum_{k=i+1}^{b} (k - b)[f(b) - f(b - 1)]
\]

\[
= \alpha(i)f(a) + \beta(i)f(b)
\]

for \( a < i < b \). Observing that

\[
j = a + \frac{f(b) - f(a) + (a - b)[f(b) - f(b - 1)]}{f(a + 1) + f(b - 1) - f(a) - f(b)} = a + \frac{b - a - (1 - r_{a,b})(1 - r_{b})^{-1}}{1 + r_{a,b}(1 - r_{a})(1 - r_{b})^{-1}}
\]

is the solution of equation \( f(a) + (i - a)[f(a + 1) - f(a)] = f(b) - (b - i)[f(b) - f(b - 1)] \) with respect to \( i \), we can conclude based on a geometric argument that the minimum gap between the lower and upper bounds in [11] is achieved at \( i \) such that \( |j| \leq i \leq |j| \). This completes the proof of Theorem 11.
H Proof of Theorem 12

To prove Theorem 12 we shall only provide the proof of statement (I), since the proof of statement (II) is similar. Define \( g(x) = f(a) + \frac{f(b)-f(a)}{b-a}(x-a) \) and

\[
\begin{cases}
  f(a) + f'(a) (x-a) & \text{if } x \leq t, \\
  f(b) + f'(b) (x-b) & \text{if } x > t
\end{cases}
\]

t for \( t \in (a, b) \). By the assumption that \( f(x) \) is concave over \([a, b]\), we have that \( g(x) \leq f(x) \leq h(x) \) for \( x \in [a, b] \) and it follows that \( \int_a^b f(x)dx \geq \int_a^b g(y)dy = \frac{b-f(a)+f(b)}{2} \) and \( \int_a^b f(x)dx \leq \int_a^b g(y)dy + \int_a^b [h(y) - g(y)]dy \). It can be shown that \( \Delta(t) \) attains its minimum at \( t = \frac{f(b)-f(a)+f'(a)-f'(b)}{f'(a)-f'(b)} \). This completes the proof of Theorem 12.

I Proofs of Theorems for Estimation of Binomial Parameters

I.1 Proof of Theorem 13

We need some preliminary results. The following lemma can be readily derived from Hoeffding’s inequalities stated in Lemma 1.

Lemma 4 \( S_B(k,n,p) \leq \exp(n \cdot A_B(k,n,p)) \) for \( 0 \leq k \leq np \). Similarly, \( 1-S_B(k-1,n,p) \leq \exp(n \cdot A_B(k,n,p)) \) for \( np \leq k \leq n \).

Lemma 5 \( \mathcal{M}_B(z, z - \varepsilon) \leq -2\varepsilon^2 \) for \( 0 < \varepsilon < z < 1 \). Similarly, \( \mathcal{M}_B(z, z + \varepsilon) \leq -2\varepsilon^2 \) for \( 0 < z < 1 - \varepsilon < 1 \).

Proof. It can be shown that \( \frac{\partial \mathcal{M}_B(\mu, \varepsilon)}{\partial \varepsilon} = \ln \left( \frac{\mu}{\varepsilon} - \frac{1}{\mu} - 1 \right) \) and \( \frac{\partial^2 \mathcal{M}_B(\mu, \varepsilon)}{\partial \varepsilon^2} = \frac{1}{\mu(\mu-1)} \) for \( 0 < \varepsilon < 1 - \mu < 1 \). Observing that \( \mathcal{M}_B(\mu, \mu) = 0 \) and \( \frac{\partial \mathcal{M}_B(\mu, \varepsilon)}{\partial \varepsilon} \big|_{\varepsilon=0} = 0 \), by Taylor’s expansion formula, we have that there exists a real number \( \varepsilon^* \in (0, \varepsilon) \) such that \( \mathcal{M}_B(\mu + \varepsilon, \mu) \leq \frac{\varepsilon^2}{2} \left( \frac{1}{(\mu - \varepsilon)(\mu - \varepsilon - 1)} \right) \) where the right side is seen to be no greater than \( -2\varepsilon^2 \). Hence, letting \( z = \mu + \varepsilon \), we have \( \mathcal{M}_B(z, z - \varepsilon) \leq -2\varepsilon^2 \) for \( 0 < \varepsilon < z < 1 \). This completes the proof of the first statement of the lemma.

Similarly, it can be verified that \( \frac{\partial \mathcal{M}_B(\mu - \varepsilon, \mu)}{\partial \varepsilon} = -\ln \left( \frac{\mu}{\mu - \varepsilon} - 1 - \frac{1}{\mu} \right) \) and \( \frac{\partial^2 \mathcal{M}_B(\mu - \varepsilon, \mu)}{\partial \varepsilon^2} = \frac{1}{(\mu - \varepsilon)(\mu - \varepsilon - 1)} \) for \( 0 < \varepsilon < \mu < 1 \). Observing that \( \mathcal{M}_B(\mu, \mu) = 0 \) and \( \frac{\partial \mathcal{M}_B(\mu - \varepsilon, \mu)}{\partial \varepsilon} \big|_{\varepsilon=0} = 0 \), by Taylor’s expansion formula, we have that there exists a real number \( \varepsilon^* \in (0, \varepsilon) \) such that \( \mathcal{M}_B(\mu - \varepsilon, \mu) \leq \frac{\varepsilon^2}{2} \left( \frac{1}{(\mu - \varepsilon)(\mu - \varepsilon - 1)} \right) \) where the right side is seen to be no greater than \( -2\varepsilon^2 \). Therefore, letting \( z = \mu - \varepsilon \), we have \( \mathcal{M}_B(z, z + \varepsilon) \leq -2\varepsilon^2 \) for \( 0 < z < 1 - \varepsilon < 1 \). This completes the proof of the second statement of the lemma.

\[ \square \]

Lemma 6 \( \{ \hat{P}_t(\hat{p}_t, \hat{p}_t + \varepsilon) \leq \zeta \delta, G_{\hat{P}_t}(\hat{p}_t, \hat{p}_t - \varepsilon) \leq \zeta \delta \} \) is a sure event.
Proof. By the definition of sample sizes, we have \( n_s \geq \lceil \frac{\ln(\delta)}{-2\varepsilon^2} \rceil \geq \frac{\ln(\delta)}{-2\varepsilon^2} \) and consequently \( \frac{\ln(\delta)}{n_s} \geq -2\varepsilon^2 \). By Lemmas 4 and 5 we have

\[
\Pr\{ F_{\hat{p}_s}(\hat{p}_s, \hat{p}_s, + \varepsilon) \leq \zeta \delta \} = \Pr\{ S_B(K_s, n_s, \hat{p}_s, + \varepsilon) \leq \zeta \delta \} \\
\geq \Pr\{ M_B(\hat{p}_s, \hat{p}_s, + \varepsilon) \leq \frac{\ln(\delta)}{n_s} \} \geq \Pr\{ M_B(\hat{p}_s, \hat{p}_s, + \varepsilon) \leq -2\varepsilon^2 \} = 1,
\]

\[
\Pr\{ G_{\hat{p}_s}(\hat{p}_s, \hat{p}_s, - \varepsilon) \leq \zeta \delta \} = \Pr\{ 1 - S_B(K_s - 1, n_s, \hat{p}_s, - \varepsilon) \leq \zeta \delta \} \\
\geq \Pr\{ M_B(\hat{p}_s, \hat{p}_s, - \varepsilon) \leq \frac{\ln(\delta)}{n_s} \} \geq \Pr\{ M_B(\hat{p}_s, \hat{p}_s, - \varepsilon) \leq -2\varepsilon^2 \} = 1
\]

which immediately implies the lemma.

Lemma 7 Let \( 0 < \varepsilon < \frac{1}{2} \). Then, \( \mathcal{M}_B(z, z + \varepsilon) \geq \mathcal{M}_B(z, z - \varepsilon) \) for \( z \in \left[ 0, \frac{1}{2} \right] \), and \( \mathcal{M}_B(z, z + \varepsilon) < \mathcal{M}_B(z, z - \varepsilon) \) for \( z \in \left( \frac{1}{2}, 1 \right] \).

Proof. By the definition of the function \( \mathcal{M}_B(., .) \), we have that \( \mathcal{M}_B(z, \mu) = -\infty \) for \( z \in [0, 1] \) and \( \mu \notin (0, 1) \). Hence, the lemma is trivially true for \( 0 \leq z \leq \varepsilon \) or \( 1 - \varepsilon \leq z \leq 1 \). It remains to show the lemma for \( z \in (\varepsilon, 1 - \varepsilon) \). This can be accomplished by noting that \( \mathcal{M}_B(z, z + \varepsilon) - \mathcal{M}_B(z, z - \varepsilon) = 0 \) for \( \varepsilon = 0 \) and that

\[
\frac{\partial \mathcal{M}_B(z, z + \varepsilon) - \mathcal{M}_B(z, z - \varepsilon)}{\partial \varepsilon} = \frac{2\varepsilon^2(1 - 2z)}{(z^2 - \varepsilon^2)((1 - z)^2 - \varepsilon^2)}, \quad \forall z \in (\varepsilon, 1 - \varepsilon)
\]

where the partial derivative is seen to be positive for \( z \in (\varepsilon, \frac{1}{2}) \) and negative for \( z \in (\frac{1}{2}, 1 - \varepsilon) \). □

Lemma 8 \( \{ \mathcal{M}_B \left( \frac{1}{2} - \frac{1}{2} - \hat{p}_s \right), \frac{1}{2} - \frac{1}{2} - \hat{p}_s, + \varepsilon \} \leq \frac{\ln(\delta)}{n_s} \} \) is a sure event.

Proof. To show the lemma, it suffices to show \( \mathcal{M}_B \left( \frac{1}{2} - \frac{1}{2} - \hat{p}_s \right), \frac{1}{2} - \frac{1}{2} - \hat{p}_s, + \varepsilon \leq \frac{\ln(\delta)}{n_s} \) for any \( z \in [0, 1] \), since \( 0 \leq \hat{p}_s(\omega) \leq 1 \) for any \( \omega \in \Omega \). By the definition of sample sizes, we have \( n_s \geq \lceil \frac{\ln(\delta)}{-2\varepsilon^2} \rceil \geq \frac{\ln(\delta)}{-2\varepsilon^2} \) and thus \( \frac{\ln(\delta)}{n_s} \geq -2\varepsilon^2 \). Hence, it is sufficient to show \( \mathcal{M}_B \left( \frac{1}{2} - \frac{1}{2} - \hat{p}_s \right), \frac{1}{2} - \frac{1}{2} - \hat{p}_s, + \varepsilon \leq -2\varepsilon^2 \) for any \( z \in [0, 1] \). This can be accomplished by considering four cases as follows.

In the case of \( z = 0 \), we have \( \mathcal{M}_B \left( \frac{1}{2} - \frac{1}{2} - \hat{p}_s \right), \frac{1}{2} - \frac{1}{2} - \hat{p}_s, + \varepsilon \) = \( \mathcal{M}_B(0, \varepsilon) = \ln(1 - \varepsilon) < -2\varepsilon^2 \), where the last inequality follows from the fact that \( \ln(1 - x) < -2x^2 \) for any \( x \in (0, 1) \).

In the case of \( 0 < z \leq \frac{1}{2} \), we have \( \mathcal{M}_B \left( \frac{1}{2} - \frac{1}{2} - \hat{p}_s \right), \frac{1}{2} - \frac{1}{2} - \hat{p}_s, + \varepsilon \) = \( \mathcal{M}_B(z, z + \varepsilon) \leq -2\varepsilon^2 \), where the inequality follows from Lemma 4 and the fact that \( 0 < z \leq \frac{1}{2} < 1 - \varepsilon \).

In the case of \( \frac{1}{2} < z < 1 \), we have \( \mathcal{M}_B \left( \frac{1}{2} - \frac{1}{2} - \hat{p}_s \right), \frac{1}{2} - \frac{1}{2} - \hat{p}_s, + \varepsilon \) = \( \mathcal{M}_B(z, z + \varepsilon) \leq -2\varepsilon^2 \), where the inequality follows from Lemma 4 and the fact that \( \frac{1}{2} < z < 1 \).

In the case of \( z = 1 \), we have \( \mathcal{M}_B \left( \frac{1}{2} - \frac{1}{2} - \hat{p}_s \right), \frac{1}{2} - \frac{1}{2} - \hat{p}_s, + \varepsilon \) = \( \mathcal{M}_B(0, \varepsilon) = \ln(1 - \varepsilon) < -2\varepsilon^2 \), where the inequality follows from Lemma 4 and the fact that \( \varepsilon < \frac{1}{2} < z < 1 \).

The proof of the lemma is thus completed. □
Lemma 9 \( \{ \mathcal{M}_B \left( \frac{1}{2} - | \frac{1}{2} - \hat{p}_\ell | , \frac{1}{2} - | \frac{1}{2} - \hat{p}_\ell | + \varepsilon \right) \leq \frac{\ln(\zeta \delta)}{n_\ell} \} \subseteq \{ \mathcal{M}_B \left( \hat{p}_\ell, \hat{p}_\ell + \varepsilon \right) \leq \frac{\ln(\zeta \delta)}{n_\ell} \}, \mathcal{M}_B \left( \hat{p}_\ell, \hat{p}_\ell - \varepsilon \right) \leq \frac{\ln(\zeta \delta)}{n_\ell} \} \) for \( \ell = 1, \cdots, s \).

Proof. Let \( \omega \in \{ \mathcal{M}_B \left( \frac{1}{2} - | \frac{1}{2} - \hat{p}_\ell | , \frac{1}{2} - | \frac{1}{2} - \hat{p}_\ell | + \varepsilon \right) \leq \frac{\ln(\zeta \delta)}{n_\ell} \} \) and \( \hat{p}_\ell = \hat{p}_\ell(\omega) \). To show the lemma, it suffices to show \( \max \{ \mathcal{M}_B(\hat{p}_\ell, \hat{p}_\ell + \varepsilon), \mathcal{M}_B(\hat{p}_\ell, \hat{p}_\ell - \varepsilon) \} \leq \frac{\ln(\zeta \delta)}{n_\ell} \) by considering two cases: Case (i) \( \hat{p}_\ell \leq \frac{1}{2} \); Case (ii) \( \hat{p}_\ell > \frac{1}{2} \).

In Case (i), we have \( \mathcal{M}_B(\hat{p}_\ell, \hat{p}_\ell + \varepsilon) = \mathcal{M}_B \left( \frac{1}{2} - | \frac{1}{2} - \hat{p}_\ell | , \frac{1}{2} - | \frac{1}{2} - \hat{p}_\ell | + \varepsilon \right) \leq \frac{\ln(\zeta \delta)}{n_\ell} \). Since \( \hat{p}_\ell \leq \frac{1}{2} \), by Lemma \( \ref{lemma:inequality} \) we have \( \mathcal{M}_B(\hat{p}_\ell, \hat{p}_\ell - \varepsilon) \leq \mathcal{M}_B(\hat{p}_\ell, \hat{p}_\ell + \varepsilon) \leq \frac{\ln(\zeta \delta)}{n_\ell} \).

In Case (ii), we have \( \mathcal{M}_B(\hat{p}_\ell, \hat{p}_\ell - \varepsilon) = \mathcal{M}_B \left( \frac{1}{2} - | \frac{1}{2} - \hat{p}_\ell | , \frac{1}{2} - | \frac{1}{2} - \hat{p}_\ell | + \varepsilon \right) \leq \frac{\ln(\zeta \delta)}{n_\ell} \). Since \( \hat{p}_\ell > \frac{1}{2} \), by Lemma \( \ref{lemma:inequality} \) we have \( \mathcal{M}_B(\hat{p}_\ell, \hat{p}_\ell + \varepsilon) < \mathcal{M}_B(\hat{p}_\ell, \hat{p}_\ell - \varepsilon) \leq \frac{\ln(\zeta \delta)}{n_\ell} \). This completes the proof of the lemma.

\( \square \)

Lemma 10 \( \{ (| \hat{p}_s - \frac{1}{2} | - \frac{2\varepsilon}{3}) \geq \frac{1}{4} + \frac{n_\ell \varepsilon^2}{2 \ln(\zeta \delta)} \} \) is a sure event.

Proof. By the definition of sample sizes, we have \( n_\ell \geq \left\lceil \frac{\ln \left( \frac{1}{2} - \frac{2\varepsilon}{3} \right)}{2 \ln(\zeta \delta)} \right\rceil \geq \frac{\ln \left( \frac{1}{2} - \frac{2\varepsilon}{3} \right)}{2 \ln(\zeta \delta)} \), which implies that \( \frac{1}{4} + \frac{n_\ell \varepsilon^2}{2 \ln(\zeta \delta)} \leq 0 \). Since \( \{ (| \hat{p}_s - \frac{1}{2} | - \frac{2\varepsilon}{3}) \geq 0 \} \) is a sure event, it follows that \( \{ (| \hat{p}_s - \frac{1}{2} | - \frac{2\varepsilon}{3}) \geq \frac{1}{4} + \frac{n_\ell \varepsilon^2}{2 \ln(\zeta \delta)} \} \) is a sure event. This completes the proof of the lemma.

\( \square \)

Lemma 11 \( \{ (| \hat{p}_\ell - \frac{1}{2} | - \frac{2\varepsilon}{3}) \geq \frac{1}{4} + \frac{n_\ell \varepsilon^2}{2 \ln(\zeta \delta)} \} \subseteq \{ \mathcal{M}_B \left( \hat{p}_\ell, \hat{p}_\ell + \varepsilon \right) \leq \frac{\ln(\zeta \delta)}{n_\ell} \}, \mathcal{M}_B \left( \hat{p}_\ell, \hat{p}_\ell - \varepsilon \right) \leq \frac{\ln(\zeta \delta)}{n_\ell} \} \) for \( \ell = 1, \cdots, s \).

Proof. Let \( \omega \in \{ (| \hat{p}_\ell - \frac{1}{2} | - \frac{2\varepsilon}{3}) \geq \frac{1}{4} + \frac{n_\ell \varepsilon^2}{2 \ln(\zeta \delta)} \} \) and \( \hat{p}_\ell = \hat{p}_\ell(\omega) \). Then,

\[
\left( \hat{p}_\ell - \frac{1}{2} - \frac{2\varepsilon}{3} \right)^2 \geq \frac{1}{4} + \frac{n_\ell \varepsilon^2}{2 \ln(\zeta \delta)}. \tag{52}
\]

To show the lemma, it suffices to show \( \mathcal{M} \left( \hat{p}_\ell, \hat{p}_\ell + \varepsilon \right) \leq \frac{\ln(\zeta \delta)}{n_\ell} \) and \( \mathcal{M} \left( \hat{p}_\ell, \hat{p}_\ell - \varepsilon \right) \leq \frac{\ln(\zeta \delta)}{n_\ell} \). For the purpose of proving the first inequality, we need to show

\[
\left( \hat{p}_\ell - \frac{1}{2} + \frac{2\varepsilon}{3} \right)^2 \geq \frac{1}{4} + \frac{n_\ell \varepsilon^2}{2 \ln(\zeta \delta)}. \tag{53}
\]

Clearly, \( \tag{53} \) holds if \( \frac{1}{4} + \frac{n_\ell \varepsilon^2}{2 \ln(\zeta \delta)} \leq 0 \). It remains to show \( \tag{53} \) under the condition that \( \frac{1}{4} + \frac{n_\ell \varepsilon^2}{2 \ln(\zeta \delta)} > 0 \). Note that \( \tag{52} \) implies either

\[
\left| \hat{p}_\ell - \frac{1}{2} \right| - \frac{2\varepsilon}{3} \geq \sqrt{\frac{1}{4} + \frac{n_\ell \varepsilon^2}{2 \ln(\zeta \delta)}}. \tag{54}
\]
or
\[ \left| \hat{p}_\ell - \frac{1}{2} \right| - \frac{2\varepsilon}{3} \leq -\sqrt{\frac{1}{4} + \frac{n_\ell \varepsilon^2}{2 \ln(\zeta \delta)}}. \]  (55)

Since (54) implies either \( \hat{p}_\ell - \frac{1}{2} + \frac{2\varepsilon}{3} \geq \frac{4\varepsilon}{3} + \sqrt{\frac{1}{4} + \frac{n_\ell \varepsilon^2}{2 \ln(\zeta \delta)}} \) or \( \hat{p}_\ell - \frac{1}{2} + \frac{2\varepsilon}{3} \leq -\sqrt{\frac{1}{4} + \frac{n_\ell \varepsilon^2}{2 \ln(\zeta \delta)}} \), it must be true that (54) implies (53). On the other hand, (55) also implies (53) because (55) implies \( \left\| \frac{1}{4} + \frac{n_\ell \varepsilon^2}{2 \ln(\zeta \delta)} \right\| \leq \left| \hat{p}_\ell - \frac{1}{2} + \frac{2\varepsilon}{3} \right| \). Hence, we have established (53).

In the case of \( \hat{p}_\ell + \varepsilon \geq 1 \), we have \( M(\hat{p}_\ell, \hat{p}_\ell + \varepsilon) = -\infty < \frac{\ln(\zeta \delta)}{n_\ell} \). In the case of \( \hat{p}_\ell + \varepsilon < 1 \), we have \(-\frac{1}{2} < \hat{p}_\ell - \frac{1}{2} + \frac{2\varepsilon}{3} < 1 - \varepsilon - \frac{1}{2} + \frac{2\varepsilon}{3} < \frac{1}{2} \) and thus \( \frac{1}{4} - \left( \hat{p}_\ell - \frac{1}{2} + \frac{2\varepsilon}{3} \right)^2 > 0 \). By virtue of (53),
\[ M(\hat{p}_\ell, \hat{p}_\ell + \varepsilon) = -\frac{\varepsilon^2}{2 \left[ \frac{1}{4} - \left( \hat{p}_\ell - \frac{1}{2} + \frac{2\varepsilon}{3} \right)^2 \right]} \lesssim \frac{\ln(\zeta \delta)}{n_\ell}. \]

Now, we shall show the second inequality \( M(\hat{p}_\ell, \hat{p}_\ell - \varepsilon) \lesssim \frac{\ln(\zeta \delta)}{n_\ell} \). To this end, we need to establish
\[ \left( \hat{p}_\ell - \frac{1}{2} - \frac{2\varepsilon}{3} \right)^2 \geq \frac{1}{4} + \frac{n_\ell \varepsilon^2}{2 \ln(\zeta \delta)} \]  (56)
based on (52). It is obvious that (56) holds if \( \frac{1}{4} + \frac{n_\ell \varepsilon^2}{2 \ln(\zeta \delta)} \leq 0 \). It remains to show (56) under the condition that \( \frac{1}{4} + \frac{n_\ell \varepsilon^2}{2 \ln(\zeta \delta)} > 0 \). Since (54) implies either \( \hat{p}_\ell - \frac{1}{2} - \frac{2\varepsilon}{3} \leq -\frac{4\varepsilon}{3} - \sqrt{\frac{1}{4} + \frac{n_\ell \varepsilon^2}{2 \ln(\zeta \delta)}} \) or \( \hat{p}_\ell - \frac{1}{2} - \frac{2\varepsilon}{3} \geq \frac{4\varepsilon}{3} + \sqrt{\frac{1}{4} + \frac{n_\ell \varepsilon^2}{2 \ln(\zeta \delta)}} \), it must be true that (54) implies (56). On the other hand, (55) also implies (56) because (55) implies \( \hat{p}_\ell - \frac{1}{2} - \frac{2\varepsilon}{3} \leq -\sqrt{\frac{1}{4} + \frac{n_\ell \varepsilon^2}{2 \ln(\zeta \delta)}} \). Hence, we have established (56).

In the case of \( \hat{p}_\ell - \varepsilon \leq 0 \), we have \( M(\hat{p}_\ell, \hat{p}_\ell - \varepsilon) = -\infty \leq \frac{\ln(\zeta \delta)}{n_\ell} \). In the case of \( \hat{p}_\ell - \varepsilon > 0 \), we have \(-\frac{1}{2} < \varepsilon - \frac{1}{2} - \frac{2\varepsilon}{3} < \hat{p}_\ell - \frac{1}{2} - \frac{2\varepsilon}{3} \leq 1 - \frac{1}{2} - \frac{2\varepsilon}{3} < \frac{1}{2} \) and thus \( \frac{1}{4} - \left( \hat{p}_\ell - \frac{1}{2} - \frac{2\varepsilon}{3} \right)^2 > 0 \). By virtue of (56),
\[ M(\hat{p}_\ell, \hat{p}_\ell - \varepsilon) = -\frac{\varepsilon^2}{2 \left[ \frac{1}{4} - \left( \hat{p}_\ell - \frac{1}{2} - \frac{2\varepsilon}{3} \right)^2 \right]} \lesssim \frac{\ln(\zeta \delta)}{n_\ell}. \]

Hence, \( \{D_\ell = 1\} \subseteq \{M(\hat{p}_\ell, \hat{p}_\ell + \varepsilon) \leq \frac{\ln(\zeta \delta)}{n_\ell}, M(\hat{p}_\ell, \hat{p}_\ell - \varepsilon) \leq \frac{\ln(\zeta \delta)}{n_\ell}, M_B(\hat{p}_\ell, \hat{p}_\ell + \varepsilon) \leq \frac{\ln(\zeta \delta)}{n_\ell}, M_B(\hat{p}_\ell, \hat{p}_\ell - \varepsilon) \leq \frac{\ln(\zeta \delta)}{n_\ell} \} \) for \( \ell = 1, \ldots, s \). The proof of the lemma is thus completed.

\[ \Box \]

Now we are in a position to prove Theorem 13.

If the stopping rule derived from CDFs is used, then \( \{D_s = 1\} \) is a sure event as a result of Lemma 6. Therefore, the sampling scheme satisfies all the requirements described in Theorem 2 from which Theorem 13 immediately follows.

If the stopping rule derived from Chernoff bounds is used, then \( \{D_s = 1\} \) is a sure event as a result of Lemma 8. Note that \( M_B(z, p) = \inf_{t > 0} e^{-t z} \mathbb{E}[e^{t \hat{p}_\ell}] \) and that \( \hat{p}_\ell \) is a ULE of \( p \) for \( \ell = 1, \ldots, s \). By virtue of these facts and Lemmas 8 and 9, the sampling scheme satisfies all the requirements described in Corollary 1 from which Theorem 13 immediately follows.
If the stopping rule derived from Massart’s inequality is used, then \( \{D_s = 1\} \) is a sure event as a result of Lemma \([10]\). Recall that \( \mathcal{M}_B(z, p) = \inf_{t \geq 0} e^{-t} \mathbb{E}[e^{\hat{p}_t}] \) and that \( \hat{p}_t \) is a ULE of \( p \) for \( \ell = 1, \ldots, s \). By virtue of these facts and Lemmas \([10, 11]\) the sampling scheme satisfies all the requirements described in Corollary \([11]\) from which Theorem \([13]\) immediately follows.

### I.2 Proof of Theorem \([14]\)

Theorem \([14]\) can be shown by applying Lemmas \([12, 13]\) to be established in the sequel.

**Lemma 12** For \( \ell = 1, \ldots, s - 1 \),

\[
\{D_\ell = 0\} = \left\{ \mathcal{M}_B(\hat{p}_\ell, \hat{p}_\ell + \varepsilon) > \frac{\ln(\tilde{\zeta})}{n_\ell} \right\} \cup \left\{ \mathcal{M}_B(\hat{p}_\ell, \hat{p}_\ell - \varepsilon) > \frac{\ln(\tilde{\zeta})}{n_\ell} \right\}.
\]

**Proof.** To show the lemma, by the definition of \( D_\ell \), it suffices to show

\[
\left\{ \mathcal{M}_B(\frac{1}{2} - |\frac{1}{2} - \hat{p}_\ell|, \frac{1}{2} - |\frac{1}{2} - \hat{p}_\ell| + \varepsilon) \leq \frac{\ln(\tilde{\zeta})}{n_\ell} \right\} = \left\{ \mathcal{M}_B(\hat{p}_\ell, \hat{p}_\ell + \varepsilon) \leq \frac{\ln(\tilde{\zeta})}{n_\ell}, \mathcal{M}_B(\hat{p}_\ell, \hat{p}_\ell - \varepsilon) \leq \frac{\ln(\tilde{\zeta})}{n_\ell} \right\}
\]

for \( \ell = 1, \ldots, s - 1 \). For simplicity of notations, we denote \( \hat{p}_\ell(\omega) \) by \( \hat{p}_\ell \) for \( \omega \in \Omega \). First, we claim that \( \mathcal{M}_B(\frac{1}{2} - |\frac{1}{2} - \hat{p}_\ell|, \frac{1}{2} - |\frac{1}{2} - \hat{p}_\ell| + \varepsilon) \leq \frac{\ln(\tilde{\zeta})}{n_\ell} \) implies \( \mathcal{M}_B(\hat{p}_\ell, \hat{p}_\ell + \varepsilon) \leq \frac{\ln(\tilde{\zeta})}{n_\ell} \) and \( \mathcal{M}_B(\hat{p}_\ell, \hat{p}_\ell - \varepsilon) \leq \frac{\ln(\tilde{\zeta})}{n_\ell} \). To prove this claim, we need to consider two cases: (i) \( \hat{p}_\ell \leq \frac{1}{2} \); (ii) \( \hat{p}_\ell > \frac{1}{2} \). In the case of \( \hat{p}_\ell \leq \frac{1}{2} \), we have \( \mathcal{M}_B(\hat{p}_\ell, \hat{p}_\ell - \varepsilon) = \mathcal{M}_B(\hat{p}_\ell, \hat{p}_\ell + \varepsilon) = \mathcal{M}_B(1 - \hat{p}_\ell, 1 - \hat{p}_\ell + \varepsilon) = \mathcal{M}_B(1 - \hat{p}_\ell, 1 - \hat{p}_\ell - \varepsilon) \leq \frac{\ln(\tilde{\zeta})}{n_\ell} \), where the first inequality follows from Lemma \([7]\). Similarly, in the case of \( \hat{p}_\ell > \frac{1}{2} \), we have \( \mathcal{M}_B(\hat{p}_\ell, \hat{p}_\ell + \varepsilon) = \mathcal{M}_B(1 - \hat{p}_\ell, 1 - \hat{p}_\ell + \varepsilon) = \mathcal{M}_B(1 - \hat{p}_\ell, 1 - \hat{p}_\ell - \varepsilon) \leq \frac{\ln(\tilde{\zeta})}{n_\ell} \), where the first inequality follows from Lemma \([7]\). The claim is thus established.

Second, we claim that \( \mathcal{M}_B(\hat{p}_\ell, \hat{p}_\ell + \varepsilon) \leq \frac{\ln(\tilde{\zeta})}{n_\ell} \) and \( \mathcal{M}_B(\hat{p}_\ell, \hat{p}_\ell - \varepsilon) \leq \frac{\ln(\tilde{\zeta})}{n_\ell} \) together imply \( \mathcal{M}_B(\frac{1}{2} - |\frac{1}{2} - \hat{p}_\ell|, \frac{1}{2} - |\frac{1}{2} - \hat{p}_\ell| + \varepsilon) \leq \frac{\ln(\tilde{\zeta})}{n_\ell} \). To prove this claim, we need to consider two cases: (i) \( \hat{p}_\ell \leq \frac{1}{2} \); (ii) \( \hat{p}_\ell > \frac{1}{2} \). In the case of \( \hat{p}_\ell \leq \frac{1}{2} \), we have \( \mathcal{M}_B(\frac{1}{2} - |\frac{1}{2} - \hat{p}_\ell|, \frac{1}{2} - |\frac{1}{2} - \hat{p}_\ell| + \varepsilon) = \mathcal{M}_B(\hat{p}_\ell, \hat{p}_\ell + \varepsilon) \leq \frac{\ln(\tilde{\zeta})}{n_\ell} \). Similarly, in the case of \( \hat{p}_\ell > \frac{1}{2} \), we have \( \mathcal{M}_B(\frac{1}{2} - |\frac{1}{2} - \hat{p}_\ell|, \frac{1}{2} - |\frac{1}{2} - \hat{p}_\ell| + \varepsilon) = \mathcal{M}_B(1 - \hat{p}_\ell, 1 - \hat{p}_\ell + \varepsilon) = \mathcal{M}_B(1 - \hat{p}_\ell, 1 - \hat{p}_\ell - \varepsilon) \leq \frac{\ln(\tilde{\zeta})}{n_\ell} \). This establishes our second claim.

Finally, combining our two established claims leads to \( \{ \mathcal{M}_B(\frac{1}{2} - |\frac{1}{2} - \hat{p}_\ell|, \frac{1}{2} - |\frac{1}{2} - \hat{p}_\ell| + \varepsilon) \leq \frac{\ln(\tilde{\zeta})}{n_\ell} \} = \{ \mathcal{M}_B(\hat{p}_\ell, \hat{p}_\ell + \varepsilon) \leq \frac{\ln(\tilde{\zeta})}{n_\ell}, \mathcal{M}_B(\hat{p}_\ell, \hat{p}_\ell - \varepsilon) \leq \frac{\ln(\tilde{\zeta})}{n_\ell} \} \). This completes the proof of the lemma.

**Lemma 13** For \( \ell = 1, \ldots, s - 1 \),

\[
\left\{ \mathcal{M}_B(\hat{p}_\ell, \hat{p}_\ell + \varepsilon) > \frac{\ln(\tilde{\zeta})}{n_\ell} \right\} = \{ n_\ell < K_\ell < n_\ell \Xi \},
\]

\[
\left\{ \mathcal{M}_B(\hat{p}_\ell, \hat{p}_\ell - \varepsilon) > \frac{\ln(\tilde{\zeta})}{n_\ell} \right\} = \{ n_\ell(1 - \Xi) < K_\ell < n_\ell(1 - \Xi) \}.
\]

**Proof.** Since \( \frac{\partial \mathcal{M}_B(z, z + \varepsilon)}{\partial z} = \ln \frac{(z + \varepsilon)(1 - z - \varepsilon)}{(z - \Xi)(1 - z + \varepsilon - \Xi)} = \ln \frac{1}{z(1 - z) - \Xi(1 - z - \Xi)} \) for \( z \in (0, 1 - \varepsilon) \), it follows that the partial derivative \( \frac{\partial \mathcal{M}_B(z, z + \varepsilon)}{\partial z} \) is equal to 0 for \( z = z^* \). The existence and uniqueness of \( z^* \) can be established by verifying that \( \frac{\partial^2 \mathcal{M}_B(z, z + \varepsilon)}{\partial z^2} = -\varepsilon^2 \left[ \frac{1}{z(1 - z)^2} + \frac{1}{(1 - z)^2(1 - z - \Xi)^2} \right] < 0 \) for any \( z \in (0, 1 - \varepsilon) \) and that

\[
\frac{\partial \mathcal{M}_B(z, z + \varepsilon)}{\partial z} \bigg|_{z=\frac{1}{2}} = \ln \frac{1 + 2\varepsilon}{1 - 2\varepsilon} - \frac{\varepsilon}{\frac{1}{2} - \varepsilon^2} < 0, \quad \frac{\partial \mathcal{M}_B(z, z + \varepsilon)}{\partial z} \bigg|_{z=\frac{1}{2} - \varepsilon} = \ln \frac{1 + 2\varepsilon}{1 - 2\varepsilon} - 4\varepsilon > 0.
\]
Since \( \mathcal{M}_B(z^*, z^* + \varepsilon) \) is negative and \( n_\ell < \frac{\ln(\delta)}{\mathcal{M}_B(z^*, z^* + \varepsilon)} \), we have that \( \mathcal{M}_B(z^*, z^* + \varepsilon) > \frac{\ln(\delta)}{n_\ell} \). On the other hand, by the definition of sample sizes, we have \( n_\ell \geq n_1 = \left\lfloor \frac{\ln(\delta)}{\ln(1-\varepsilon)} \right\rfloor \geq \frac{\ln(\delta)}{\ln(1-\varepsilon)} \), which implies \( \lim_{z \to 0} \mathcal{M}_B(z, z + \varepsilon) \leq \frac{\ln(\delta)}{n_\ell} \). Noting that \( \mathcal{M}_B(z, z + \varepsilon) \) is monotonically increasing with respect to \( z \in (0, z^*) \), we can conclude from the intermediate value theorem that there exists a unique number \( z \in [0, z^*] \) such that \( \mathcal{M}_B(z, z + \varepsilon) = \frac{\ln(\delta)}{n_\ell} \). Similarly, due to the facts that \( \mathcal{M}_B(z^*, z^* + \varepsilon) > \frac{\ln(\delta)}{n_\ell} \), \( \lim_{z \to 1-\varepsilon} \mathcal{M}_B(z, z + \varepsilon) = -\infty < \frac{\ln(\delta)}{n_\ell} \), and that \( \mathcal{M}_B(z, z + \varepsilon) \) is monotonically decreasing with respect to \( z \in (z^*, 1-\varepsilon) \), we can conclude from the intermediate value theorem that there exists a unique number \( z \in (z^*, 1-\varepsilon) \) such that \( \mathcal{M}_B(z, z + \varepsilon) = \frac{\ln(\delta)}{n_\ell} \). Therefore, we have \( \mathcal{M}_B(z, z + \varepsilon) > \frac{\ln(\delta)}{n_\ell} \) for \( z \in (z, 1-\varepsilon) \), and \( \mathcal{M}_B(z, z + \varepsilon) \leq \frac{\ln(\delta)}{n_\ell} \) for \( z \in [0, z] \cup [1-\varepsilon, 1] \). This proves that \( \{\mathcal{M}_B(\hat{\rho}_\ell, \hat{\rho}_\ell + \varepsilon) > \frac{\ln(\delta)}{n_\ell} \} = \{n_\ell \leq K_\ell < n_\ell\varepsilon\} \). Noting that \( \mathcal{M}_B(\frac{1}{2} + \epsilon, \frac{1}{2} + \epsilon - \varepsilon) = \mathcal{M}_B(\frac{1}{2} - \epsilon, \frac{1}{2} - \epsilon - \varepsilon) \) for any \( \epsilon \in (0, \frac{1}{2}) \), we have \( \{\mathcal{M}_B(\hat{\rho}_\ell, \hat{\rho}_\ell - \epsilon) > \frac{\ln(\delta)}{n_\ell} \} = \{n_\ell(1-\varepsilon) < K_\ell < n_\ell(\varepsilon)\} \}. This completes the proof of the lemma.

\( \square \)

I.3 Proof of Theorem 15

We need some preliminary results.

**Lemma 14** \( \lim_{\varepsilon \to 0} \sum_{\ell=1}^{s} n_\ell e^{-n_\ell c} = 0 \) for any \( c > 0 \).

**Proof.** Let \( c \) be a positive number. By differentiation, it can be shown that \( xe^{-xc} \) is monotonically increasing with respect to \( x \in (0, \frac{1}{c}) \) and monotonically decreasing with respect to \( x \in (\frac{1}{c}, \infty) \). As a consequence of the definition of sample sizes, the smallest sample size \( n_1 \) is no less than \( \frac{\ln(\delta)}{\ln(1-\varepsilon)} \), and thus is greater than \( \frac{1}{c} \) for small enough \( \varepsilon > 0 \). Hence, \( \sum_{\ell=1}^{s} n_\ell e^{-n_\ell c} \leq s n_1 e^{-n_1 c} \) if \( \varepsilon > 0 \) is sufficiently small. Let \( \rho = \inf_{\varepsilon > 0} \frac{C_{\varepsilon-1}}{1-\varepsilon} - 1 \). Observing that \( s \leq 1 + \left\lfloor \frac{\ln(\delta)}{\ln(1+\rho)} \right\rfloor = 1 + \frac{\ln(\delta)}{\ln(1+\rho)} \) and \( n_1 \geq \frac{\ln(\delta)}{\ln(1-\varepsilon)} \), we have

\[
\sum_{\ell=1}^{s} n_\ell e^{-n_\ell c} < \left[ 1 + \frac{\ln \left( \frac{1}{c} \ln \frac{1}{1-\varepsilon} \right)}{\ln(1+\rho)} \right] \frac{\ln \left( \frac{1}{c} \ln \frac{1}{1-\varepsilon} \right)}{\ln(1+\rho)} \exp \left( -\frac{c \ln \left( \frac{1}{c} \ln \frac{1}{1-\varepsilon} \right)}{\ln(1+\rho)} \right) = \frac{A(\varepsilon)}{c} + \frac{\ln \left( \frac{1}{c} \ln \frac{1}{1-\varepsilon} \right)}{\ln(1+\rho)} B(\varepsilon)
\]

for small enough \( \varepsilon > 0 \), where \( A(\varepsilon) = \frac{c \ln \left( \frac{1}{c} \ln \frac{1}{1-\varepsilon} \right)}{\ln(1+\rho)} \) and \( B(\varepsilon) = \frac{\ln \left( \frac{1}{c} \ln \frac{1}{1-\varepsilon} \right)}{\ln(1+\rho)} \). Noting that \( \lim_{x \to \infty} xe^{-x} = 0 \) and that \( e^{\ln(\delta)/\ln(1-\varepsilon)} \to \infty \) as \( \varepsilon \to 0 \), we have \( \lim_{\varepsilon \to 0} A(\varepsilon) = 0 \). Now we show that \( \lim_{\varepsilon \to 0} B(\varepsilon) = 0 \). Using Taylor’s expansion formula \( \ln(1+x) = x - \frac{x^2}{2} + o(x^2) \),
Lemma 15

Let $\varepsilon$ be a function of $\varepsilon \in (0, 1)$ such that $0 < a \leq \varepsilon \leq b < 1$. Then,

\[ M_B(\psi, \psi + \varepsilon) = -\frac{e^2}{2\psi_e(1 - \psi_e)} + \frac{e^3}{3(1 - \psi_e)^2} + o(\varepsilon^3), \]

\[ M_1(\psi, \psi + \varepsilon) = -\frac{e^2}{2(1 - \psi_e)} + \frac{e^3}{3(1 - \psi_e)^2} + o(\varepsilon^3), \]

\[ M_B(\psi, \psi + \varepsilon) = -\frac{e^2}{2\psi_e(1 - \psi_e)} + \frac{e^3}{3(1 - \psi_e)^2} + o(\varepsilon^3). \]

Proof. Using Taylor's series expansion formula $\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} + o(x^3)$ for $|x| < 1$, we have

\[ M_B(\psi, \psi + \varepsilon) = \psi_e \ln \left(1 + \frac{\varepsilon}{\psi_e}\right) + (1 - \psi_e) \ln \left(1 - \frac{\varepsilon}{1 - \psi_e}\right) \]

\[ = -\frac{e^2}{2\psi_e(1 - \psi_e)} + \frac{e^3}{3(1 - \psi_e)^2} + o(\varepsilon^3) \]

\[ + \psi_e \times o \left( \frac{e^3}{3(1 - \psi_e)^3} \right) = -\frac{e^2}{2\psi_e(1 - \psi_e)} + \frac{e^3}{3(1 - \psi_e)^2} + o(\varepsilon^3) \]

for $\varepsilon < \psi_e < 1 - \varepsilon$. Since $\lim_{\varepsilon \to 0} \frac{\varepsilon - \psi_e}{1 + \varepsilon - \psi_e} = 0$ and

\[ \lim_{\varepsilon \to 0} \frac{1 - \psi_e}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{1 - \psi_e}{\varepsilon} = 0, \]

we have $\ln \frac{1}{1 - \psi_e} = -\ln(1 - \varepsilon) = \varepsilon + \frac{\varepsilon^2}{2} + o(\varepsilon^2) = \varepsilon + o(\varepsilon)$ and

\[ B(\varepsilon) = \frac{\ln \left(\frac{\varepsilon^2 + \varepsilon + o(\varepsilon^2)}{2\varepsilon^2}\right)}{\varepsilon + o(\varepsilon)} \exp \left(-\frac{c \ln \frac{1}{\varepsilon}}{\varepsilon + \frac{\varepsilon}{2} + o(\varepsilon^2)}\right) = \frac{\ln \left(\frac{\varepsilon^2 + \varepsilon + o(\varepsilon^2)}{\varepsilon + \frac{\varepsilon}{2} + o(\varepsilon^2)}\right)}{\varepsilon + o(\varepsilon)} \exp \left(-\frac{c \ln \frac{1}{\varepsilon}}{\varepsilon + \frac{\varepsilon}{2} + o(\varepsilon^2)}\right) \]

\[ = \frac{\ln \left(\frac{\varepsilon^2 + \varepsilon + o(\varepsilon^2)}{\varepsilon + \frac{\varepsilon}{2} + o(\varepsilon^2)}\right)}{\varepsilon + o(\varepsilon)} \exp \left(-\frac{c \ln \frac{1}{\varepsilon}}{\varepsilon + \frac{\varepsilon}{2} + o(\varepsilon^2)}\right) \]

where $B^*(\varepsilon) = \frac{\ln \frac{1}{\varepsilon}}{\psi} \left(\frac{1}{\varepsilon}\right)^{\frac{\varepsilon}{2} + o(\varepsilon^2)}$. Making a change of variable $x = \frac{1}{\varepsilon}$ and using L'Hôpital's rule, we have

\[ \lim_{\varepsilon \to 0} B^*(\varepsilon) = \lim_{x \to \infty} x \ln \frac{1}{\varepsilon} = \lim_{x \to \infty} x \ln \frac{1}{\varepsilon} \left(\frac{1}{\varepsilon}\right)^{\frac{\varepsilon}{2} + o(\varepsilon^2)} = 0. \]

Therefore, $0 \leq \lim_{\varepsilon \to 0} \sum_{\ell=1}^s n_\ell e^{-\varepsilon c} \leq \frac{1}{c} \lim_{\varepsilon \to 0} A(\varepsilon) + \frac{\ln \frac{1}{\varepsilon}}{\ln(1 + \rho)} \times \left(\frac{1}{\varepsilon}\right)^{\frac{\varepsilon}{2} + o(\varepsilon^2)} \lim_{\varepsilon \to 0} B^*(\varepsilon) = 0$, which implies that $\lim_{\varepsilon \to 0} \sum_{\ell=1}^s n_\ell e^{-\varepsilon c} = 0$. This completes the proof of the lemma. \qed
we have
\[
M_1 \left( \psi, \frac{\psi}{1+\epsilon} \right) = -\ln(1+\epsilon) + \frac{1-\psi}{\psi} \ln \left( 1 + \frac{\epsilon}{1+\epsilon} \frac{\psi}{1-\psi} \right) = -\epsilon + \frac{\epsilon^2}{2} - \frac{\epsilon^3}{3} + \frac{1-\psi}{\psi} \left[ \frac{\epsilon}{1+\epsilon} \frac{\psi}{1-\psi} - \frac{1}{2} \left( \frac{\epsilon}{1+\epsilon} \frac{\psi}{1-\psi} \right)^2 + \frac{1}{3} \left( \frac{\epsilon}{1+\epsilon} \frac{\psi}{1-\psi} \right)^3 \right] + o(\epsilon^3) + \frac{1-\psi}{\psi} \times o \left( \frac{\epsilon}{1+\epsilon} \frac{\psi}{1-\psi} \right)^3 \right)
\]
\[
= \frac{-\epsilon^2}{2} + \frac{e^3}{3} - \frac{\epsilon^2}{1+\epsilon} \frac{1}{2} \left( \frac{\epsilon}{1+\epsilon} \frac{\psi}{1-\psi} \right)^2 + \frac{1}{3} \left( \frac{\epsilon}{1+\epsilon} \frac{\psi}{1-\psi} \right)^3 \left( 1-\psi \right)^2 + o(\epsilon^3)
\]
\[
= \frac{-\epsilon^2}{2(1-\psi)} + \frac{e^3}{3} \frac{2-\psi}{(1-\psi)^2} + o(\epsilon^3).
\]

Since \( \psi \) is bounded in \([a, b]\), we have

\[
M_B \left( \psi, \frac{\psi}{1+\epsilon} \right) = \psi \cdot M_1 \left( \psi, \frac{\psi}{1+\epsilon} \right) = -\frac{\epsilon^2 \psi}{2(1-\psi)} + \frac{e^3 \psi (2-\psi)}{3(1-\psi)^2} + o(\epsilon^3).
\]

\[
\square
\]

**Lemma 16** Let \( 0 < \epsilon < \frac{1}{2} \). Then, there exists a unique number \( z^* \in \left( \frac{1}{2}, \frac{1}{2} + \epsilon \right) \) such that \( M_B(z, z-\epsilon) \) is monotonically increasing with respect to \( z \in (\epsilon, z^*) \) and monotonically decreasing with respect to \( z \in (z^*, 1) \). Similarly, there exists a unique number \( z^* \in \left( \frac{1}{2} - \epsilon, \frac{1}{2} \right) \) such that \( M_B(z, z+\epsilon) \) is monotonically increasing with respect to \( z \in (0, z^*) \) and monotonically decreasing with respect to \( z \in (z^*, 1 - \epsilon) \).

**Proof.** Note that \( \frac{\partial M_B(z, z-\epsilon)}{\partial z} \bigg|_{z=\frac{1}{2}} = \ln \frac{1-2z}{1+2z} + \frac{\epsilon}{2-z} > 0 \) because \( \ln \frac{1-2z}{1+2z} + \frac{\epsilon}{2-z} \) equals 0 for \( \epsilon = 0 \) and its derivative with respect to \( \epsilon \) equals \( \frac{2\epsilon^2}{(4-\epsilon)^2} \) which is positive for any positive \( \epsilon \) less than \( \frac{1}{2} \).

Similarly, \( \frac{\partial M_B(z, z-\epsilon)}{\partial z} \bigg|_{z=\frac{1}{2}+\epsilon} = \ln \frac{1-2z}{1+2z} + 4\epsilon \) equals 0 for \( \epsilon = 0 \) and its derivative with respect to \( \epsilon \) equals \( -\frac{16\epsilon^2}{(4-\epsilon)^2} \) which is negative for any positive \( \epsilon \) less than \( \frac{1}{2} \).

In view of the signs of \( \frac{\partial M_B(z, z-\epsilon)}{\partial z} \) at \( \frac{1}{2} \) \( \frac{1}{2} + \epsilon \) and the fact that \( \frac{\partial^2 M_B(z, z-\epsilon)}{\partial z^2} \bigg|_{z=z^*} = -\frac{\epsilon^2}{(1-z)(1-z+\epsilon)^2} \) \( z=\frac{1}{2} \) \( \frac{1}{2} + \epsilon \) for any \( z \in (\epsilon, 1) \), we can conclude from the intermediate value theorem that there exists a unique number \( z^* \in \left( \frac{1}{2}, \frac{1}{2} + \epsilon \right) \) such that \( \frac{\partial M_B(z, z-\epsilon)}{\partial z} \bigg|_{z=z^*} = 0 \), which implies that \( M_B(z, z-\epsilon) \) is monotonically increasing with respect to \( z \in (\epsilon, z^*) \) and monotonically decreasing with respect to \( z \in (z^*, 1) \).

To show the second statement of the lemma, note that \( \frac{\partial M_B(z, z+\epsilon)}{\partial z} \bigg|_{z=\frac{1}{2}} = \ln \frac{1+2z}{1-2z} - \frac{\epsilon}{2-z} < 0 \) because \( \ln \frac{1+2z}{1-2z} - \frac{\epsilon}{2-z} \) equals 0 for \( \epsilon = 0 \) and its derivative with respect to \( \epsilon \) equals \( -\frac{2\epsilon^2}{(4-\epsilon)^2} \) which is negative for any positive \( \epsilon \) less than \( \frac{1}{2} \).

Similarly, \( \frac{\partial M_B(z, z+\epsilon)}{\partial z} \bigg|_{z=\frac{1}{2}+\epsilon} = \ln \frac{1+2z}{1-2z} - 4\epsilon \) equals 0 for \( \epsilon = 0 \) and its derivative with respect to \( \epsilon \) equals \( \frac{16\epsilon^2}{(4-\epsilon)^2} \) which
Lemma 17 If $\varepsilon$ is sufficiently small, then the following statements hold true.

(I): For $\ell = 1, 2, \cdots, s - 1$, there exists a unique number $z_\ell \in [0, \frac{1}{2} - \varepsilon)$ such that $n_\ell = \frac{\ln(\zeta)}{\mathcal{M}_B(z_\ell + \varepsilon)}$.

(II): $z_\ell$ is monotonically increasing with respect to $\ell$ smaller than $s$.

(III): $\lim_{\varepsilon \to 0} z_\ell = \frac{1 - \sqrt{1 - C_{s-\ell}}}{2}$, where the limit is taken under the restriction that $s - \ell$ is fixed with respect to $\varepsilon$.

(IV) For $p \in (0, \frac{1}{2})$ such that $C_{j_p} = 4p(1 - p)$ and $j_p \geq 1$,

$$\lim_{\varepsilon \to 0} \frac{z_\ell - p}{\varepsilon} = -\frac{2}{3},$$

where $\ell = s - j_p$.

(V): $\{D_\ell = 0\} = \{z_\ell < \hat{p}_\ell < 1 - z_\ell\}$ for $\ell = 1, 2, \cdots, s - 1$.

Proof of Statement (I): By the definition of sample sizes, we have

$$0 < \frac{\ln(\zeta)}{\mathcal{M}_B(0, \varepsilon)} \leq n_\ell \leq \frac{(1 + C_1)n_s}{2} < \frac{1 + C_1}{2} \left( \frac{\ln \frac{1}{\zeta}}{2\varepsilon^2} + 1 \right)$$

for sufficiently small $\varepsilon > 0$. By (57), we have $\frac{\ln(\zeta)}{n_\ell} \geq \mathcal{M}_B(0, \varepsilon)$ and

$$\frac{\ln(\zeta)}{n_\ell} < -2\varepsilon^2 \left( \frac{2}{1 + C_1} - \frac{1}{n_\ell} \right) = -\frac{2\varepsilon^2}{\mathcal{M}_B(1 - \varepsilon, \frac{1}{2})} + \frac{2}{1 + C_1} \mathcal{M}_B(1 - \varepsilon, \frac{1}{2}) + \frac{2\varepsilon^2}{n_\ell}.$$ 

Noting that $\lim_{\varepsilon \to 0} \frac{2\varepsilon^2}{n_\ell} = 0$ and $\lim_{\varepsilon \to 0} \frac{-2\varepsilon^2}{\mathcal{M}_B(1 - \varepsilon, \frac{1}{2})} = 1$, we have $\frac{\ln(\zeta)}{n_\ell} < \mathcal{M}_B(\frac{1}{2} - \varepsilon, \frac{1}{2}) < 0$ for sufficiently small $\varepsilon > 0$. In view of the established fact that $\mathcal{M}_B(0, \varepsilon) \leq \frac{\ln(\zeta)}{n_\ell} < \mathcal{M}_B(\frac{1}{2} - \varepsilon, \frac{1}{2})$ for small enough $\varepsilon > 0$ and the fact that $\mathcal{M}_B(z, z + \varepsilon)$ is monotonically increasing with respect to $z \in (0, \frac{1}{2} - \varepsilon)$ as asserted by Lemma 16, invoking the intermediate value theorem, we have that there exists a unique number $z_\ell \in [0, \frac{1}{2} - \varepsilon)$ such that $\mathcal{M}_B(z_\ell, z_\ell + \varepsilon) = \frac{\ln(\zeta)}{n_\ell}$. This proves Statement (I).

Proof of Statement (II): Since $n_\ell$ is monotonically increasing with respect to $\ell$ for sufficiently small $\varepsilon > 0$, we have that $\mathcal{M}_B(z_\ell, z_\ell + \varepsilon)$ is monotonically increasing with respect to $\ell$ if
\( \varepsilon > 0 \) is sufficiently small. Recalling that \( M_B(z, z + \varepsilon) \) is monotonically increasing with respect to \( z \in (0, \frac{1}{2} - \varepsilon) \), we have that \( z_\ell \) is monotonically increasing with respect to \( \ell \). This establishes Statement (II).

**Proof of Statement (III):** For simplicity of notations, let \( b_\ell = \frac{1 - \sqrt{1 - c_{s-\ell}}}{2} \) for \( \ell = 1, 2, \ldots, s - 1 \). Then, it can be checked that \( 4b_\ell(1 - b_\ell) = C_{s-\ell} \) and, by the definition of sample sizes, we have

\[
\frac{M_B(z_\ell, z_\ell + \varepsilon)}{\varepsilon^2/[2b_\ell(1-b_\ell)]} = \frac{1}{n_\ell} \times \frac{C_{s-\ell} \ln \frac{1}{\zeta \delta}}{2\varepsilon^2} = 1 + o(1) \tag{58}
\]

for \( \ell = 1, 2, \ldots, s - 1 \).

We claim that \( \theta < z_\ell < \frac{1}{2} \) for \( \theta \in (0, b_\ell) \) if \( \varepsilon > 0 \) is small enough. To prove this claim, we use a contradiction method. Suppose the claim is not true, then there exists a set, denoted by \( S_\varepsilon \), of infinite many values of \( \varepsilon \) such that \( z_\ell \leq \theta \) for \( \varepsilon \in S_\varepsilon \). For small enough \( \varepsilon \in S_\varepsilon \), we have \( z_\ell + \varepsilon \leq \theta + \varepsilon < b_\ell + \varepsilon < \frac{1}{2} \). Hence, by (58) and the fact that \( M_B(z, z + \varepsilon) \) is monotonically increasing with respect to \( z \in (0, \frac{1}{2} - \varepsilon) \) as asserted by Lemma 15, we have

\[
1 + o(1) = \frac{M_B(z_\ell, z_\ell + \varepsilon)}{\varepsilon^2/[2b_\ell(1-b_\ell)]} \geq \frac{M_B(\theta, \theta + \varepsilon)}{\varepsilon^2/[2b_\ell(1-b_\ell)]} = \frac{\varepsilon^2/[2\theta(1-\theta)] + o(\varepsilon^2)}{\varepsilon^2/[2\theta(1-\theta)]} = \frac{b_\ell(1-b_\ell)}{\theta(1-\theta)} + o(1)
\]

for small enough \( \varepsilon \in S_\varepsilon \), which implies \( \frac{b_\ell(1-b_\ell)}{\theta(1-\theta)} \leq 1 \), contradicting to the fact that \( \frac{b_\ell(1-b_\ell)}{\theta(1-\theta)} > 1 \). By (58) and applying Lemma 15 based on the established condition that \( \theta < z_\ell < \frac{1}{2} \) for small enough \( \varepsilon > 0 \), we have

\[
\frac{M_B(z_\ell, z_\ell + \varepsilon)}{\varepsilon^2/[2b_\ell(1-b_\ell)]} = \frac{\varepsilon^2/[2z_\ell(1-z_\ell)] + o(\varepsilon^2)}{\varepsilon^2/[2b_\ell(1-b_\ell)]} = 1 + o(1),
\]

which implies \( 1 - \frac{1}{z_\ell(1-z_\ell)} - \frac{1}{z_\ell(1-z_\ell)} = o(1) \) and consequently \( \lim_{\varepsilon \to 0} z_\ell = b_\ell \). This proves Statement (III).

**Proof of Statement (IV):**

Since \( n_\ell = \left\lceil \frac{C_{s-\ell} \ln \frac{1}{\zeta \delta}}{2\varepsilon^2} \right\rceil \) and \( C_{s-\ell} = 4p(1-p) \), we can write

\[
n_\ell = \left\lceil \frac{2p(1-p) \ln \frac{1}{\zeta \delta}}{\varepsilon^2} \right\rceil = \frac{\ln(\zeta \delta)}{M_B(z_\ell, z_\ell + \varepsilon)},
\]

from which we have \( \frac{1}{n_\ell} = o(\varepsilon) \).

\[
1 - o(\varepsilon) = 1 - \frac{1}{n_\ell} < \frac{-2p(1-p) \ln(\zeta \delta)}{\varepsilon^2 \cdot M_B(z_\ell, z_\ell + \varepsilon)} \leq 1
\]

and thus

\[
\frac{-2p(1-p) \ln(\zeta \delta)}{\varepsilon^2 \cdot M_B(z_\ell, z_\ell + \varepsilon)} = \frac{-M_B(z_\ell, z_\ell + \varepsilon)}{\varepsilon^2/[2p(1-p)]} = 1 + o(\varepsilon). \tag{59}
\]

For \( \theta \in (0, p) \), we claim that \( \theta < z_\ell < \frac{1}{2} \) provided that \( \varepsilon \) is sufficiently small. Suppose, to get a contradiction, that the claim is not true. Then, there exists a set of infinite many values of \( \varepsilon \) such that \( z_\ell \leq \theta \) if \( \varepsilon \) in the set is small enough. For \( \varepsilon < \frac{1}{2} - p \), by (59) and the monotonicity of \( M_B(z, z + \varepsilon) \) with respect to \( z \), we have

\[
1 + o(\varepsilon) = \frac{-M_B(z_\ell, z_\ell + \varepsilon)}{\varepsilon^2/[2p(1-p)]} \geq \frac{-M_B(\theta, \theta + \varepsilon)}{\varepsilon^2/[2p(1-p)]} = \frac{\varepsilon^2/[2\theta(1-\theta)] + o(\varepsilon^2)}{\varepsilon^2/[2p(1-p)]} = \frac{p(1-p)}{\theta(1-\theta)} + o(1)
\]

which implies \( \theta(1-\theta) - 1 - \varepsilon^2/[2p(1-p)] = o(1) \), contradicting to the fact that \( \theta(1-\theta) > 1 \).
for small enough $\varepsilon$ in the set, which contradicts to the fact that $\frac{n(1-p)}{n(1-\delta)}>1$. This proves our claim. Since $\theta < z_{\ell_0} < \frac{1}{2}$ is established, by (49) and Lemma (15) we have

$$-\mathcal{M}_B(z_{\ell_0}, z_{\ell_0} + \varepsilon) = \frac{\varepsilon^2 / [2z_{\ell_0}(1-z_{\ell_0})] - \varepsilon^3 (1-2z_{\ell_0})/[3z_{\ell_0}^2(1-z_{\ell_0})^2] + o(\varepsilon^3)}{\varepsilon^2 / [2p(1-p)]} = 1 + o(\varepsilon)$$

and consequently,

$$\frac{1}{z_{\ell_0}(1-z_{\ell_0})} - \frac{1}{p(1-p)} - \frac{2\varepsilon(1-2z_{\ell_0})}{3z_{\ell_0}^2(1-z_{\ell_0})^2} + o(\varepsilon) = 0.$$  \hspace{1cm} (60)

Since $\theta < z_{\ell_0} < \frac{1}{2}$ for small enough $\varepsilon > 0$, by (60), we have $\frac{1}{z_{\ell_0}(1-z_{\ell_0})} - \frac{1}{p(1-p)} = o(1)$, from which it follows that $\lim_{\varepsilon \to 0} z_{\ell_0} = p$. Noting that (60) can be written as

$$\frac{(z_{\ell_0} - p)(z_{\ell_0} + p - 1)}{p(1-p)z_{\ell_0}(1-z_{\ell_0})} - \frac{2\varepsilon(1-2z_{\ell_0})}{3z_{\ell_0}^2(1-z_{\ell_0})^2} + o(\varepsilon) = 0$$

and using the fact that $\lim_{\varepsilon \to 0} z_{\ell_0} = p \in (0, \frac{1}{2})$, we have

$$\frac{z_{\ell_0} - p}{\varepsilon} = \frac{2p(1-p)(1-2z_{\ell_0})}{3(z_{\ell_0} + p - 1)z_{\ell_0}(1-z_{\ell_0})} + o(1)$$

for small enough $\varepsilon > 0$, which implies that $\lim_{\varepsilon \to 0} \frac{z_{\ell_0} - p}{\varepsilon} = -\frac{2}{3}$. This proves Statement (IV).

**Proof of Statement (V):** Note that

$$\{D_\ell = 0\} = \left\{ \mathcal{M}_B \left( \frac{1}{2} - \frac{1}{2} - \widehat{\ell}_t, \frac{1}{2} - \frac{1}{2} - \widehat{\ell}_t \right) > \frac{\ln(\zeta \delta)}{n_\ell}, \widehat{\ell}_t \leq \frac{1}{2} \right\} \cup \left\{ \mathcal{M}_B \left( \frac{1}{2} - \frac{1}{2} - \widehat{\ell}_t, \frac{1}{2} - \frac{1}{2} - \widehat{\ell}_t \right) > \frac{\ln(\zeta \delta)}{n_\ell}, \widehat{\ell}_t > \frac{1}{2} \right\} = \left\{ \mathcal{M}_B \left( \widehat{\ell}_t, \widehat{\ell}_t + \varepsilon \right) > \frac{\ln(\zeta \delta)}{n_\ell}, \widehat{\ell}_t \leq \frac{1}{2} \right\} \cup \left\{ \mathcal{M}_B \left( \widehat{\ell}_t, \widehat{\ell}_t - \varepsilon \right) > \frac{\ln(\zeta \delta)}{n_\ell}, \widehat{\ell}_t > \frac{1}{2} \right\},$$

where we have used the fact that $\mathcal{M}_B(z, z + \varepsilon) = \mathcal{M}_B(1 - z, 1 - z - \varepsilon)$. We claim that

$$\left\{ \mathcal{M}_B \left( \widehat{\ell}_t, \widehat{\ell}_t + \varepsilon \right) > \frac{\ln(\zeta \delta)}{n_\ell}, \widehat{\ell}_t \leq \frac{1}{2} \right\} = \left\{ z_{\ell_0} < \widehat{\ell}_t \leq \frac{1}{2} \right\},$$  \hspace{1cm} (61)

$$\left\{ \mathcal{M}_B \left( \widehat{\ell}_t, \widehat{\ell}_t - \varepsilon \right) > \frac{\ln(\zeta \delta)}{n_\ell}, \widehat{\ell}_t > \frac{1}{2} \right\} = \left\{ \frac{1}{2} < \widehat{\ell}_t < 1 - z_{\ell_0} \right\}$$  \hspace{1cm} (62)

for small enough $\varepsilon > 0$.

To prove (61), let $\omega \in \{ \mathcal{M}_B \left( \widehat{\ell}_t, \widehat{\ell}_t + \varepsilon \right) > \frac{\ln(\zeta \delta)}{n_\ell}, \widehat{\ell}_t \leq \frac{1}{2} \}$ and $\widehat{\ell}_t = \widehat{\ell}_t(\omega)$. Then, $\mathcal{M}_B(\widehat{\ell}_t, \widehat{\ell}_t + \varepsilon) > \frac{\ln(\zeta \delta)}{n_\ell}$ and $\widehat{\ell}_t \leq \frac{1}{2}$. Since $z_{\ell_0} \in (0, \frac{1}{2} - \varepsilon)$, it must be true that $\widehat{\ell}_t > z_{\ell_0}$. Otherwise if $\widehat{\ell}_t \leq z_{\ell_0}$, then $\mathcal{M}_B(\widehat{\ell}_t, \widehat{\ell}_t + \varepsilon) \leq \mathcal{M}_B(z_{\ell_0}, z_{\ell_0} + \varepsilon) = \frac{\ln(\zeta \delta)}{n_\ell}$, leading to a contradiction. This proves $\{ \mathcal{M}_B \left( \widehat{\ell}_t, \widehat{\ell}_t + \varepsilon \right) > \frac{\ln(\zeta \delta)}{n_\ell}, \widehat{\ell}_t \leq \frac{1}{2} \} \subseteq \{ z_{\ell_0} < \widehat{\ell}_t \leq \frac{1}{2} \}$ for small enough $\varepsilon > 0$.

Now let $\omega \in \{ z_{\ell_0} < \widehat{\ell}_t \leq \frac{1}{2} \}$ and $\widehat{\ell}_t = \widehat{\ell}_t(\omega)$. Then, $z_{\ell_0} < \widehat{\ell}_t \leq \frac{1}{2}$. Invoking Lemma (16) that there exists a unique number $z^* \in (\frac{1}{2} - \varepsilon, \frac{1}{2})$ such that $\mathcal{M}_B(z, z + \varepsilon)$ is monotonically increasing with respect to $z \in (0, z^*)$ and monotonically decreasing with respect to $z \in (z^*, 1 - \varepsilon)$, we have

$$\mathcal{M}_B(\widehat{\ell}_t, \widehat{\ell}_t + \varepsilon) > \min \left\{ \mathcal{M}_B \left( z_{\ell_0}, z_{\ell_0} + \varepsilon \right), \mathcal{M}_B \left( \frac{1}{2}, \frac{1}{2} + \varepsilon \right) \right\}.$$  \hspace{1cm} (63)
Noting that \( \lim_{\varepsilon \to 0} \frac{\ln(\delta)}{n_{\varepsilon} \cdot \frac{1}{2} + \varepsilon} = 1 \), we have \( \mathcal{M}_B(\ell, \frac{1}{2} + \varepsilon) > \frac{\ln(\delta)}{n_{\varepsilon}} \) for \( \ell < s \) if \( \varepsilon > 0 \) is small enough. By virtue of (63) and \( \mathcal{M}_B(z_\ell, z_\ell + \varepsilon) = \frac{\ln(\delta)}{n_{\varepsilon}} \), we have \( \mathcal{M}_B(\hat{p}_\ell, \hat{p}_\ell + \varepsilon) > \frac{\ln(\delta)}{n_{\varepsilon}} \). This proves \( \{ \mathcal{M}_B(\hat{p}_\ell, \hat{p}_\ell + \varepsilon) > \frac{\ln(\delta)}{n_{\varepsilon}} \} \) and consequently (61) is established.

To show (62), let \( \omega \in \{ \mathcal{M}_B(\hat{p}_\ell, \hat{p}_\ell + \varepsilon) > \frac{\ln(\delta)}{n_{\varepsilon}} \} \) and \( \hat{p}_\ell > \frac{1}{2} \). Since \( 1 - z_{\ell} \in (\frac{1}{2} + \varepsilon, 1] \) and \( \mathcal{M}_B(z, z - \varepsilon) \) is monotonically decreasing with respect to \( z \in (\frac{1}{2} + \varepsilon, 1) \), it must be true that \( \hat{p}_\ell < 1 - z_{\ell} \). Otherwise if \( \hat{p}_\ell \geq 1 - z_{\ell} \), then \( \mathcal{M}_B(\hat{p}_\ell, \hat{p}_\ell - \varepsilon) \leq \mathcal{M}_B(1 - z_{\ell}, 1 - z_{\ell} - \varepsilon) = \mathcal{M}_B(z_{\ell}, z_{\ell} - z_{\ell} + \varepsilon) = \frac{\ln(\delta)}{n_{\varepsilon}} \), leading to a contradiction. This proves \( \{ \mathcal{M}_B(\hat{p}_\ell, \hat{p}_\ell - \varepsilon) > \frac{\ln(\delta)}{n_{\varepsilon}} \} \subseteq \{ \frac{1}{2} < \hat{p}_\ell < 1 - z_{\ell} \} \).

Now let \( \omega \in \{ \frac{1}{2} < \hat{p}_\ell < 1 - z_{\ell} \} \) and \( \hat{p}_\ell = \hat{p}_\ell(\omega) \). Then, \( \frac{1}{2} < \hat{p}_\ell < 1 - z_{\ell} \). Invoking Lemma 16 that there exists a unique number \( z^* \in (\frac{1}{2}, \frac{1}{2} + \varepsilon) \) such that \( \mathcal{M}_B(z, z - \varepsilon) \) is monotonically increasing with respect to \( z \in (\varepsilon, z^*) \) and monotonically decreasing with respect to \( z \in (z^*, 1) \), we have

\[
\mathcal{M}_B(\hat{p}_\ell, \hat{p}_\ell - \varepsilon) > \min \left\{ \mathcal{M}_B(1 - z_{\ell}, 1 - z_{\ell} - \varepsilon), \mathcal{M}_B\left(\frac{1}{2}, \frac{1}{2} - \varepsilon\right) \right\}.
\]

Recalling that \( \mathcal{M}_B(1, \frac{1}{2} - \varepsilon) = \mathcal{M}_B(\frac{1}{2}, \frac{1}{2} + \varepsilon) > \frac{\ln(\delta)}{n_{\varepsilon}} \) for small enough \( \varepsilon > 0 \), using (64) and \( \mathcal{M}_B(1 - z_{\ell}, 1 - z_{\ell} - \varepsilon) = \mathcal{M}_B(z_{\ell}, z_{\ell} + \varepsilon) = \frac{\ln(\delta)}{n_{\varepsilon}} \), we have \( \mathcal{M}_B(\hat{p}_\ell, \hat{p}_\ell - \varepsilon) > \frac{\ln(\delta)}{n_{\varepsilon}} \). This proves \( \{ \mathcal{M}_B(\hat{p}_\ell, \hat{p}_\ell - \varepsilon) > \frac{\ln(\delta)}{n_{\varepsilon}} \} \supseteq \{ \frac{1}{2} < \hat{p}_\ell < 1 - z_{\ell} \} \) and consequently (62) is established. By virtue of (61) and (62) of the established claim, we have \( \{ D_\ell = 0 \} = \{ z_{\ell} < \hat{p}_\ell \leq \frac{1}{2} \} \cup \{ \frac{1}{2} < \hat{p}_\ell < 1 - z_{\ell} \} = \{ z_{\ell} < \hat{p}_\ell < 1 - z_{\ell} \} \) for small enough \( \varepsilon > 0 \). This proves Statement (V).

**Lemma 18** Let \( \ell_\varepsilon = s - j_\varepsilon \). Then,

\[
\lim_{\varepsilon \to 0} \sum_{\ell = 1}^{\ell_\varepsilon - 1} n_\ell \Pr\{ D_\ell = 1 \} = 0, \quad \lim_{\varepsilon \to 0} \sum_{\ell = \ell_\varepsilon + 1}^{s} n_\ell \Pr\{ D_\ell = 0 \} = 0
\]

for \( p \in (0, 1) \). Moreover, \( \lim_{\varepsilon \to 0} n_{\ell_\varepsilon} \Pr\{ D_{\ell_\varepsilon} = 0 \} = 0 \) if \( C_{j_\varepsilon} > 4p(1 - p) \).

**Proof.** For simplicity of notations, let \( b_\ell = \lim_{\varepsilon \to 0} z_{\ell} \) for \( 1 \leq \ell < s \). The proof consists of three main steps as follows.

First, we shall show that (65) holds for \( p \in (0, \frac{1}{2}) \). By the definition of \( \ell_\varepsilon \), we have \( 4p(1 - p) > C_{s - \ell_\varepsilon + 1} \). Making use of the first three statements of Lemma 17 we have that \( z_{\ell} < \frac{p + b_{\ell - 1}}{2} < p \) for all \( \ell \leq \ell_\varepsilon - 1 \) if \( \varepsilon \) is sufficiently small. By the last statement of Lemma 17 and using Chernoff bounds, we have

\[
\Pr\{ D_\ell = 1 \} = \Pr\{ \hat{p}_\ell \leq z_{\ell} \} + \Pr\{ \hat{p}_\ell \geq 1 - z_{\ell} \} \leq \Pr\{ \hat{p}_\ell \leq \frac{p + b_{\ell - 1}}{2} \} + \Pr\{ \hat{p}_\ell \geq 1 - \frac{p + b_{\ell - 1}}{2} \} \leq \exp\left(-2n_\ell \left( \frac{p - b_{\ell - 1}}{2} \right)^2 \right) + \exp\left(-2n_\ell \left( \frac{2 - 3p - b_{\ell - 1}}{2} \right)^2 \right)
\]

for all \( \ell \leq \ell_\varepsilon - 1 \) provided that \( \varepsilon > 0 \) is small enough. By the definition of \( \ell_\varepsilon \), we have

\[
b_{\ell_\varepsilon - 1} = \frac{1 - \sqrt{1 - C_{s - \ell_\varepsilon + 1}}}{2} < \frac{1 - \sqrt{1 - 4p(1 - p)}}{2} = p,
\]

96
which implies that \((p-b_{\ell+1})^2\) and \((2-3p-b_{\ell+1})^2\) are positive constants independent of \(\varepsilon > 0\) provided that \(\varepsilon > 0\) is small enough. Hence, \(\lim_{\ell \to 0} \sum_{\ell=1}^{s-1} \ell \Pr\{D_\ell = 1\} = 0\) as a result of Lemma 14.

Similarly, it can be seen from the definition of \(\ell_\varepsilon\) that \(4p(1-p) < C_{s-\ell_\varepsilon-1}\). Making use of the first three statements of Lemma 17, we have that \(z_\varepsilon > \frac{p+b_{\ell_\varepsilon+1}}{2} > p\) for \(\ell_\varepsilon + 1 \leq \ell < s\) if \(\varepsilon\) is sufficiently small. By the last statement of Lemma 17 and using Chernoff bound, we have

\[
\Pr\{D_\ell = 0\} = \Pr\{z_\ell < \hat{p}_\ell < 1 - z_\ell\} \leq \Pr\{\hat{p}_\ell > z_\ell\} \leq \Pr\left\{\hat{p}_\ell > p + \frac{b_{\ell+1}}{2}\right\} \leq \exp\left(-2n_\ell \left(\frac{p-b_{\ell+1}}{2}\right)^2\right)
\]

for \(\ell_\varepsilon + 1 \leq \ell < s\) provided that \(\varepsilon > 0\) is small enough. As a consequence of the definition of \(\ell_\varepsilon\), we have that \(b_{\ell_\varepsilon+1}\) is greater than \(p\) and is independent of \(\varepsilon > 0\). In view of this and the fact that \(\Pr\{D_\ell = 0\} = 0\), we can apply Lemma 14 to conclude that \(\lim_{\ell \to 0} \sum_{\ell=\ell_\varepsilon+1}^{s} n_\ell \Pr\{D_\ell = 0\} = 0\).

Second, we shall show that (85) holds for \(p \in (\frac{1}{2}, 1)\). As a direct consequence of the definition of \(\ell_\varepsilon\), we have \(4p(1-p) > C_{s-\ell_\varepsilon+1}\). Making use of the first three statements of Lemma 17, we have that \(z_\varepsilon < \frac{1-p+b_{\ell_\varepsilon+1}}{2} < 1 - p\) for all \(\ell \leq \ell_\varepsilon - 1\) if \(\varepsilon\) is sufficiently small. By the last statement of Lemma 17 and using Chernoff bounds, we have

\[
\Pr\{D_\ell = 1\} = \Pr\{\hat{p}_\ell \leq z_\ell\} + \Pr\{\hat{p}_\ell \geq 1 - z_\ell\} \leq \Pr\{\hat{p}_\ell \leq \frac{1-p+b_{\ell-1}}{2}\} + \Pr\{\hat{p}_\ell \geq \frac{1+p-b_{\ell-1}}{2}\}
\]

\[
\leq \exp\left(-2n_\ell \left(\frac{3p-1-b_{\ell-1}}{2}\right)^2\right) + \exp\left(-2n_\ell \left(\frac{1-p-b_{\ell-1}}{2}\right)^2\right)
\]

for all \(\ell \leq \ell_\varepsilon - 1\) provided that \(\varepsilon > 0\) is small enough. As a result of the definition of \(\ell_\varepsilon\), we have that \(b_{\ell_\varepsilon-1}\) is smaller than \(1 - p\) and is independent of \(\varepsilon > 0\). Hence, by virtue of Lemma 14, we have \(\lim_{\ell \to 0} \sum_{\ell=\ell_\varepsilon-1}^{s} n_\ell \Pr\{D_\ell = 1\} = 0\).

In a similar manner, by the definition of \(\ell_\varepsilon\), we have \(4p(1-p) < C_{s-\ell_\varepsilon-1}\). Making use of the first three statements of Lemma 17, we have that \(z_\varepsilon > \frac{1-p+b_{\ell_\varepsilon+1}}{2} > 1 - p\) for \(\ell_\varepsilon + 1 \leq \ell < s\) if \(\varepsilon\) is sufficiently small. By the last statement of Lemma 17 and using Chernoff bound,

\[
\Pr\{D_\ell = 0\} = \Pr\{z_\ell < \hat{p}_\ell < 1 - z_\ell\} \leq \Pr\{\hat{p}_\ell < 1 - z_\ell\}
\]

\[
\leq \Pr\left\{\hat{p}_\ell < \frac{1+p-b_{\ell+1}}{2}\right\} \leq \exp\left(-2n_\ell \left(\frac{1-p-b_{\ell+1}}{2}\right)^2\right)
\]

for \(\ell_\varepsilon + 1 \leq \ell < s\) provided that \(\varepsilon > 0\) is small enough. Because of the definition of \(\ell_\varepsilon\), we have that \(b_{\ell_\varepsilon+1}\) is greater than \(1 - p\) and is independent of \(\varepsilon > 0\). Noting that \(\Pr\{D_\ell = 0\} = 0\) and using Lemma 14, we have \(\lim_{\ell \to 0} \sum_{\ell=\ell_\varepsilon+1}^{s} n_\ell \Pr\{D_\ell = 0\} = 0\).

Third, we shall show that \(\lim_{\ell \to 0} n_\ell \Pr\{D_{\ell_\varepsilon} = 0\} = 0\) for \(p \in (0, 1)\) such that \(4p(1-p) < C_{j_p}\). For \(p \in (0, \frac{1}{2})\) such that \(4p(1-p) < C_{j_p}\), making use of the first three statements of Lemma 17, we have \(z_{\ell_\varepsilon} > \frac{p+b_{\ell_\varepsilon}}{2} > p\) if \(\varepsilon\) is sufficiently small. By the last statement of Lemma 17 and using Chernoff bound, we have

\[
\Pr\{D_{\ell_\varepsilon} = 0\} = \Pr\{z_{\ell_\varepsilon} < \hat{p}_{\ell_\varepsilon} < 1 - z_{\ell_\varepsilon}\} \leq \Pr\{\hat{p}_{\ell_\varepsilon} > z_{\ell_\varepsilon}\} \leq \Pr\left\{\hat{p}_{\ell_\varepsilon} > \frac{p+b_{\ell_\varepsilon}}{2}\right\} \leq \exp\left(-2n_{\ell_\varepsilon} \left(\frac{p-b_{\ell_\varepsilon}}{2}\right)^2\right)
\]
for small enough $\varepsilon > 0$. As a consequence of the definition of $\ell_\varepsilon$, we have that $b_{\ell_\varepsilon}$ is greater than
$p$ and is independent of $\varepsilon > 0$. It follows that $\lim_{\varepsilon \to 0} n_{\ell_\varepsilon} \Pr\{D_{\ell_\varepsilon} = 0\} = 0$.

Similarly, for $p \in \left(\frac{1}{2}, 1\right)$ such that $4p(1 - p) < C_{g_{\varepsilon}}$, by virtue of the first three statements of
Lemma 17, we have $z_{\ell_\varepsilon} > \frac{1 - p - b_{\ell_\varepsilon}}{2} > 1 - p$ if $\varepsilon$ is sufficiently small. By the last statement of Lemma
17 and using Chernoff bound,

$$\Pr\{D_{\ell_\varepsilon} = 0\} = \Pr\{z_{\ell_\varepsilon} < \hat{p}_{\ell_\varepsilon} < 1 - z_{\ell_\varepsilon}\} \leq \Pr\{\hat{p}_{\ell_\varepsilon} < 1 - z_{\ell_\varepsilon}\}$$

$$\leq \Pr\left\{\hat{p}_{\ell_\varepsilon} < \frac{1 + p - b_{\ell_\varepsilon}}{2}\right\} \leq \exp\left(-2n_{\ell_\varepsilon}\left(\frac{1 - p - b_{\ell_\varepsilon}}{2}\right)^2\right)$$

for small enough $\varepsilon > 0$. Because of the definition of $\ell_\varepsilon$, we have that $b_{\ell_\varepsilon}$ is greater than $1 - p$
and is independent of $\varepsilon > 0$. Hence, $\lim_{\varepsilon \to 0} n_{\ell_\varepsilon} \Pr\{D_{\ell_\varepsilon} = 0\} = 0$.

\[\square\]

Now we are in a position to prove Theorem 15. To show $\lim_{\varepsilon \to 0} |\Pr\{\hat{p} \in \mathcal{R}\} - \bar{P}| = \lim_{\varepsilon \to 0} |\Pr\{\hat{p} \in \mathcal{R}\} - \bar{P}| = 0$, it suffices to show

$$\lim_{\varepsilon \to 0} \sum_{\ell=1}^{s} \Pr\{D_{\ell-1} = 0, D_{\ell} = 1\} = 1.$$  \hspace{1cm} (66)

This is because $\bar{P} \leq \Pr\{\hat{p} \in \mathcal{R}\} \leq \bar{P}$ and $\bar{P} - \bar{P} = \sum_{\ell=1}^{s} \Pr\{D_{\ell-1} = 0, D_{\ell} = 1\} - 1$. Observing that

$$\sum_{\ell=1}^{\ell_{\varepsilon}-1} \Pr\{D_{\ell-1} = 0, D_{\ell} = 1\} \leq \sum_{\ell=1}^{\ell_{\varepsilon}-1} \Pr\{D_{\ell} = 1\} \leq \sum_{\ell=1}^{\ell_{\varepsilon}-1} n_{\ell} \Pr\{D_{\ell} = 1\},$$

$$\sum_{\ell=\ell_{\varepsilon}+2}^{s} \Pr\{D_{\ell-1} = 0, D_{\ell} = 1\} \leq \sum_{\ell=\ell_{\varepsilon}+2}^{s} \Pr\{D_{\ell-1} = 0\} = \sum_{\ell=\ell_{\varepsilon}+1}^{s} \Pr\{D_{\ell} = 0\} \leq \sum_{\ell=\ell_{\varepsilon}+1}^{s} n_{\ell} \Pr\{D_{\ell} = 0\}$$

and using Lemma 18 we have $\lim_{\varepsilon \to 0} \sum_{\ell=1}^{\ell_{\varepsilon}-1} \Pr\{D_{\ell-1} = 0, D_{\ell} = 1\} = 0$ and $\lim_{\varepsilon \to 0} \sum_{\ell=\ell_{\varepsilon}+2}^{s} \Pr\{D_{\ell-1} = 0, D_{\ell} = 1\} = 0$. Hence, to show (66), it suffices to show $\lim_{\varepsilon \to 0} \Pr\{D_{\ell_{\varepsilon}-1} = 0, D_{\ell_{\varepsilon}} = 1\} + \Pr\{D_{\ell_{\varepsilon}} = 0, D_{\ell_{\varepsilon}+1} = 1\} = 1$. Noting that

$$\Pr\{D_{\ell_{\varepsilon}-1} = 0, D_{\ell_{\varepsilon}} = 1\} + \Pr\{D_{\ell_{\varepsilon}-1} = D_{\ell_{\varepsilon}} = 1\} + \Pr\{D_{\ell_{\varepsilon}} = 0, D_{\ell_{\varepsilon}+1} = 1\} + \Pr\{D_{\ell_{\varepsilon}} = D_{\ell_{\varepsilon}+1} = 0\}$$

$$= \Pr\{D_{\ell_{\varepsilon}} = 1\} + \Pr\{D_{\ell_{\varepsilon}} = 0\} = 1,$$

we have

$$\Pr\{D_{\ell_{\varepsilon}-1} = 0, D_{\ell_{\varepsilon}} = 1\} + \Pr\{D_{\ell_{\varepsilon}} = 0, D_{\ell_{\varepsilon}+1} = 1\} = 1 - \Pr\{D_{\ell_{\varepsilon}-1} = D_{\ell_{\varepsilon}} = 1\} - \Pr\{D_{\ell_{\varepsilon}} = D_{\ell_{\varepsilon}+1} = 0\}.$$  

As a result of Lemma 18 we have $\lim_{\varepsilon \to 0} \Pr\{D_{\ell_{\varepsilon}-1} = D_{\ell_{\varepsilon}} = 1\} \leq \lim_{\varepsilon \to 0} \Pr\{D_{\ell_{\varepsilon}-1} = 1\} = 0$ and $\lim_{\varepsilon \to 0} \Pr\{D_{\ell_{\varepsilon}} = D_{\ell_{\varepsilon}+1} = 0\} \leq \lim_{\varepsilon \to 0} \Pr\{D_{\ell_{\varepsilon}+1} = 0\} = 0$. Therefore, $\lim_{\varepsilon \to 0} \sum_{\ell=1}^{s} \Pr\{D_{\ell-1} = 0, D_{\ell} = 1\} = 1$. This completes the proof of Theorem 15.

I.4 Proof of Theorem 16

To prove Theorem 16 we need some preliminary results.

Lemma 19 $\lim_{\varepsilon \to 0} \frac{n_{\ell_\varepsilon}}{Z_{\alpha(p, \varepsilon)}} = \kappa_p$, $\lim_{\varepsilon \to 0} \frac{\varepsilon}{\sqrt{p(1-p)n_{\ell_\varepsilon}}} = d\sqrt{\kappa_p}$. 

98
**Proof.** By the definition of sample sizes, it can be readily shown that \( \lim_{\varepsilon \to 0} \frac{C_{\ell_{\varepsilon}} \ln \frac{1}{\varepsilon}}{2\varepsilon n_{\ell_{\varepsilon}}} = 1 \) for \( 1 \leq \ell < s \) and it follows that

\[
\lim_{\varepsilon \to 0} \frac{n_{\ell_{\varepsilon}}}{N_d(p, \varepsilon)} = \lim_{\varepsilon \to 0} \frac{\mathcal{M}_d\left(\frac{p}{2} - \frac{b_{\ell_{\varepsilon}}}{{\varepsilon}^2}, 1\right) + \mathcal{M}_d\left(\frac{p}{2} - \frac{b_{\ell_{\varepsilon}}}{{\varepsilon}^2}, 0\right)\varepsilon}{\ln(\zeta_d)} = \lim_{\varepsilon \to 0} \left[ \frac{\varepsilon^2}{2p(1-p)} + o(\varepsilon^2) \right] \times \frac{C_{\ell_{\varepsilon}}}{{\varepsilon}^2} = \frac{C_{\ell_{\varepsilon}}}{2p(1-p)} = \frac{C_j}{4p(1-p)} = \kappa_p,
\]

\[
\lim_{\varepsilon \to 0} \frac{\varepsilon}{\sqrt{p(1-p)/n_{\ell_{\varepsilon}}}} = \lim_{\varepsilon \to 0} \frac{\varepsilon}{\sqrt{C_{\ell_{\varepsilon}}}} \frac{1}{\sqrt{2\varepsilon^2 p(1-p)}} \frac{1}{\sqrt{\ln(\zeta_d)}} = \frac{d}{\sqrt{4p(1-p)}} = d\sqrt{\kappa_p}.
\]

\[\square\]

**Lemma 20** Let \( U \) and \( V \) be independent Gaussian random variables with zero means and unit variances. Let \( \ell_{\varepsilon} = s - j_p \). Then, for \( p \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1) \) such that \( C_j = 4p(1-p) \),

\[
\begin{align*}
\lim_{\varepsilon \to 0} \Pr\{\ell = \ell_{\varepsilon}\} &= 1 - \lim_{\varepsilon \to 0} \Pr\{\ell = \ell_{\varepsilon} + 1\} = 1 - \Phi(\nu d), \\
\lim_{\varepsilon \to 0} \Pr\{\{\hat{p}_{\ell_{\varepsilon}} - p \geq \varepsilon, \ell = \ell_{\varepsilon}\} \cup \{\|\hat{p}_{\ell_{\varepsilon}+1} - p \geq \varepsilon, \ell = \ell_{\varepsilon} + 1\}\} &= \Pr\{U \geq d\} + \Pr\{|U + \sqrt{\nu} V| \geq d\sqrt{1 + \nu d}, U < \nu d\}.
\end{align*}
\]

**Proof.** By symmetry, it suffices to show the lemma for \( p \in (0, \frac{1}{2}) \). For simplicity of notations, define

\[
a_{\ell_{\varepsilon}} = \frac{z_{\ell_{\varepsilon}} - p}{\sqrt{p(1-p)/n_{\ell_{\varepsilon}}}}, \quad b_{\ell_{\varepsilon}} = \frac{\varepsilon}{\sqrt{p(1-p)/n_{\ell_{\varepsilon}}}}, \quad U_{\ell_{\varepsilon}} = \frac{\hat{p}_{\ell_{\varepsilon}} - p}{\sqrt{p(1-p)/n_{\ell_{\varepsilon}}}}
\]

for \( \ell = 1, \ldots, s \). Since \( C_j = 4p(1-p) \), we have \( n_{\ell_{\varepsilon}} = \left\lceil \frac{2p(1-p)\ln \frac{1}{\varepsilon^2}}{\varepsilon^2} \right\rceil \) and

\[
\lim_{\varepsilon \to 0} b_{\ell_{\varepsilon}} = \lim_{\varepsilon \to 0} \frac{\varepsilon}{\sqrt{p(1-p)}} = \sqrt{2p(1-p)\ln \frac{1}{\varepsilon^2}} = \sqrt{\frac{2\ln \frac{1}{\varepsilon^2}}{\varepsilon^2}} = \sqrt{2\ln \frac{1}{\varepsilon^2}} = d.
\]

Hence, by Statement (IV) of Lemma [17],

\[
\lim_{\varepsilon \to 0} a_{\ell_{\varepsilon}} = \lim_{\varepsilon \to 0} b_{\ell_{\varepsilon}} \varepsilon \lim_{\varepsilon \to 0} \frac{z_{\ell_{\varepsilon}} - p}{\varepsilon} = d \lim_{\varepsilon \to 0} \frac{z_{\ell_{\varepsilon}} - p}{\varepsilon} = -\frac{2}{3}d = -\nu d.
\]

Let \( \eta > 0 \). Noting that \( \{\hat{p}_{\ell_{\varepsilon}} \leq z_{\ell_{\varepsilon}}\} = \{U_{\ell_{\varepsilon}} \leq a_{\ell_{\varepsilon}}\} \) and \( \{\hat{p}_{\ell_{\varepsilon}} - p \geq \varepsilon\} = \{|U_{\ell_{\varepsilon}}| \geq b_{\ell_{\varepsilon}}\} \), we have

\[
\begin{align*}
\Pr\{U_{\ell_{\varepsilon}} \leq -\nu d - \eta\} &\leq \Pr\{\hat{p}_{\ell_{\varepsilon}} \leq z_{\ell_{\varepsilon}}\} \leq \Pr\{U_{\ell_{\varepsilon}} \leq -\nu d + \eta\}, \\
\Pr\{|U_{\ell_{\varepsilon}}| \geq d + \eta, U_{\ell_{\varepsilon}} \leq -\nu d - \eta\} &\leq \Pr\{|\hat{p}_{\ell_{\varepsilon}} - p \geq \varepsilon, \hat{p}_{\ell_{\varepsilon}} \leq z_{\ell_{\varepsilon}}\} \leq \Pr\{|U_{\ell_{\varepsilon}}| \geq d + \eta, U_{\ell_{\varepsilon}} \leq -\nu d + \eta\}
\end{align*}
\]

for small enough \( \varepsilon > 0 \). Since \( U_{\ell_{\varepsilon}} \) converges in distribution to a Gaussian random variable \( U \) with zero mean and unit variance as \( \varepsilon \to 0 \), it must be true that

\[
\begin{align*}
\Pr\{U \leq -\nu d - \eta\} &\leq \lim_{\varepsilon \to 0} \Pr\{\hat{p}_{\ell_{\varepsilon}} \leq z_{\ell_{\varepsilon}}\} \leq \Pr\{U \leq -\nu d + \eta\}, \\
\Pr\{|U| \geq d + \eta, U \leq -\nu d - \eta\} &\leq \lim_{\varepsilon \to 0} \Pr\{|\hat{p}_{\ell_{\varepsilon}} - p \geq \varepsilon, \hat{p}_{\ell_{\varepsilon}} \leq z_{\ell_{\varepsilon}}\} \leq \Pr\{|U| \geq d + \eta, U \leq -\nu d + \eta\}.
\end{align*}
\]
Since the above inequalities hold true for arbitrarily small \( \eta > 0 \), we have

\[
\lim_{\varepsilon \to 0} \Pr \{ \hat{p}_{\ell_\varepsilon} \leq z_{\ell_\varepsilon} \} = \Pr \{ U \leq -\nu d \} = \Pr \{ U \geq \nu d \} = 1 - \Phi(\nu d), \tag{67}
\]

\[
\lim_{\varepsilon \to 0} \Pr \{ |\hat{p}_{\ell_\varepsilon} - p| \geq \varepsilon, \hat{p}_{\ell_\varepsilon} \leq z_{\ell_\varepsilon} \} = \Pr \{ |U| \geq d, U \leq -\nu d \} = \Pr \{ U \geq d \}. \tag{68}
\]

Now, we shall consider \( \Pr \{ |\hat{p}_{\ell_\varepsilon+1} - p| \geq \varepsilon, \hat{p}_{\ell_\varepsilon} > z_{\ell_\varepsilon} \} \). Note that

\[
\Pr \{ |\hat{p}_{\ell_\varepsilon+1} - p| \geq \varepsilon, \hat{p}_{\ell_\varepsilon} > z_{\ell_\varepsilon} \} = \Pr \{ |U_{\ell_\varepsilon+1}| \geq b_{\ell_\varepsilon+1}, U_{\ell_\varepsilon} > a_{\ell_\varepsilon} \}
\]

and

\[
U_{\ell_\varepsilon+1} = \sqrt{\frac{n_{\ell_\varepsilon}}{n_{\ell_\varepsilon+1}}} U_{\ell_\varepsilon} + \sqrt{1 - \frac{n_{\ell_\varepsilon}}{n_{\ell_\varepsilon+1}}} V_{\ell_\varepsilon}, \quad \text{where} \quad V_{\ell_\varepsilon} = \frac{\sum_{i=n_{\ell_\varepsilon}+1}^{n_{\ell_\varepsilon+1}} X_i - (n_{\ell_\varepsilon+1} - n_{\ell_\varepsilon})p}{\sqrt{p(1-p)(n_{\ell_\varepsilon+1} - n_{\ell_\varepsilon})}}.
\]

For small enough \( \varepsilon > 0 \), we have

\[
\Pr \{ |\hat{p}_{\ell_\varepsilon+1} - p| \geq \varepsilon, \hat{p}_{\ell_\varepsilon} > z_{\ell_\varepsilon} \} \leq \Pr \{ |U_{\ell_\varepsilon+1}| \geq d - \eta, U_{\ell_\varepsilon} > -\nu d - \eta \},
\]

\[
\Pr \{ |\hat{p}_{\ell_\varepsilon+1} - p| \geq \varepsilon, \hat{p}_{\ell_\varepsilon} > z_{\ell_\varepsilon} \} \geq \Pr \{ |U_{\ell_\varepsilon+1}| \geq d + \eta, U_{\ell_\varepsilon} > -\nu d + \eta \}.
\]

Note that \( U_{\ell_\varepsilon} \) and \( V_{\ell_\varepsilon} \) converge in distribution respectively to independent Gaussian random variables \( U \) and \( V \) with zero means and unit variances. Since the characteristic function of \( U_{\ell_\varepsilon+1} \) tends to the characteristic function of \( (U + \sqrt{p}V)/\sqrt{1+p^2} \), we have

\[
\Pr \{ |U_{\ell_\varepsilon+1}| \geq d - \eta, U_{\ell_\varepsilon} > -\nu d - \eta \} \to \Pr \{ |U + \sqrt{p}V| \geq (d - \eta)/\sqrt{1+p^2}, U > -\nu d - \eta \},
\]

\[
\Pr \{ |U_{\ell_\varepsilon+1}| \geq d + \eta, U_{\ell_\varepsilon} > -\nu d + \eta \} \to \Pr \{ |U + \sqrt{p}V| \geq (d + \eta)/\sqrt{1+p^2}, U > -\nu d + \eta \}
\]

as \( \varepsilon \to 0 \). Since \( \eta \) can be arbitrarily small, we have

\[
\lim_{\varepsilon \to 0} \Pr \{ |\hat{p}_{\ell_\varepsilon+1} - p| \geq \varepsilon, \hat{p}_{\ell_\varepsilon} > z_{\ell_\varepsilon} \} = \Pr \{ |U + \sqrt{p}V| \geq d/\sqrt{1+p^2}, U > -\nu d \}
\]

\[
= \Pr \{ |U + \sqrt{p}V| \geq d/\sqrt{1+p^2}, U < \nu d \} \tag{69}
\]

for \( p \in (0, \frac{1}{2}) \) such that \( C_{j,p} = 4p(1-p) \). Noting that

\[
\Pr \{ \hat{p}_{\ell_\varepsilon} \leq z_{\ell_\varepsilon} \text{ or } \hat{p}_{\ell_\varepsilon} \leq 1 - z_{\ell_\varepsilon} \} \geq \Pr \{ l = \ell_\varepsilon \} \geq \Pr \{ \hat{p}_{\ell_\varepsilon} \leq z_{\ell_\varepsilon} \} \text{ or } \hat{p}_{\ell_\varepsilon} \geq 1 - z_{\ell_\varepsilon} \} - \sum_{\ell=1}^{\ell_\varepsilon-1} \Pr \{ D_\ell = 1 \},
\]

\[
\Pr \{ 1 - z_{\ell_\varepsilon} > \hat{p}_{\ell_\varepsilon} > z_{\ell_\varepsilon} \} \geq \Pr \{ l = \ell_\varepsilon + 1 \} \geq \Pr \{ 1 - z_{\ell_\varepsilon} > \hat{p}_{\ell_\varepsilon} > z_{\ell_\varepsilon} \} - \Pr \{ D_{\ell_\varepsilon+1} = 0 \} - \sum_{\ell=1}^{\ell_\varepsilon-1} \Pr \{ D_\ell = 1 \}
\]

and using the result that \( \lim_{\varepsilon \to 0} \left[ \sum_{\ell=1}^{\ell_\varepsilon-1} \Pr \{ D_\ell = 1 \} + \Pr \{ D_{\ell_\varepsilon+1} = 0 \} \right] = 0 \) as asserted by Lemma 18 we have \( \lim_{\varepsilon \to 0} \Pr \{ l = \ell_\varepsilon \} = \lim_{\varepsilon \to 0} \Pr \{ \hat{p}_{\ell_\varepsilon} \leq z_{\ell_\varepsilon} \text{ or } \hat{p}_{\ell_\varepsilon} \geq 1 - z_{\ell_\varepsilon} \} \) and \( \lim_{\varepsilon \to 0} \Pr \{ l = \ell_\varepsilon + 1 \} = \lim_{\varepsilon \to 0} \Pr \{ 1 - z_{\ell_\varepsilon} > \hat{p}_{\ell_\varepsilon} > z_{\ell_\varepsilon} \} \). We claim that \( \lim_{\varepsilon \to 0} \Pr \{ \hat{p}_{\ell_\varepsilon} \geq 1 - z_{\ell_\varepsilon} \} = 0 \) for \( p \in (0, \frac{1}{2}) \). To show this claim, note that \( \lim_{\varepsilon \to 0}(1 - z_{\ell_\varepsilon} - p) = 1 - 2p > 0 \) as a result of Statement (III) of Lemma 17. Therefore, \( 1 - z_{\ell_\varepsilon} - p > \frac{1}{2} - p \) for small enough \( \varepsilon > 0 \). By virtue of the Chernoff bound, we have \( \Pr \{ \hat{p}_{\ell_\varepsilon} \geq 1 - z_{\ell_\varepsilon} \} \leq \exp(-2n_{\ell_\varepsilon}(\frac{1}{2} - p)^2) \) for small enough \( \varepsilon > 0 \), from which the claim immediately follows. This implies that

\[
\lim_{\varepsilon \to 0} \Pr \{ l = \ell_\varepsilon \} = \lim_{\varepsilon \to 0} \Pr \{ \hat{p}_{\ell_\varepsilon} \leq z_{\ell_\varepsilon} \}, \quad \lim_{\varepsilon \to 0} \Pr \{ l = \ell_\varepsilon + 1 \} = \lim_{\varepsilon \to 0} \Pr \{ \hat{p}_{\ell_\varepsilon} > z_{\ell_\varepsilon} \}. \tag{70}
\]
Combining (67) and (70) yields
\[
\lim_{\varepsilon \to 0} \Pr\{l = \ell_\varepsilon\} = 1 - \Phi(\nu d), \quad \lim_{\varepsilon \to 0} \Pr\{l = \ell_\varepsilon + 1\} = \Phi(\nu d).
\]
Noting that
\[
\Pr(|\hat{p}_{\ell_\varepsilon} - p| \geq \varepsilon, \ l = \ell_\varepsilon) \geq \Pr(|\hat{p}_{\ell_\varepsilon} - p| \geq \varepsilon, \ \hat{p}_{\ell_\varepsilon} \notin (z_{\ell_\varepsilon}, 1 - z_{\ell_\varepsilon})) - \sum_{\ell=1}^{\ell_\varepsilon - 1} \Pr\{D_{\ell} = 1\},
\]
\[
\Pr(|\hat{p}_{\ell_\varepsilon+1} - p| \geq \varepsilon, \ l = \ell_\varepsilon + 1) \geq \Pr(|\hat{p}_{\ell_\varepsilon+1} - p| \geq \varepsilon, \ \hat{p}_{\ell_\varepsilon} \in (z_{\ell_\varepsilon}, 1 - z_{\ell_\varepsilon})\}
\]
\[
- \Pr\{D_{\ell_\varepsilon+1} = 0\} - \sum_{\ell=1}^{\ell_\varepsilon - 1} \Pr\{D_{\ell} = 1\}
\]
and using the result that \(\lim_{\varepsilon \to 0} \left[\sum_{\ell=1}^{\ell_\varepsilon - 1} \Pr\{D_{\ell} = 1\} + \Pr\{D_{\ell_\varepsilon+1} = 0\}\right] = 0\), we have
\[
\liminf_{\varepsilon \to 0} \left[\Pr(|\hat{p}_{\ell_\varepsilon} - p| \geq \varepsilon, \ l = \ell_\varepsilon) + \Pr(|\hat{p}_{\ell_\varepsilon+1} - p| \geq \varepsilon, \ l = \ell_\varepsilon + 1)\right]
\geq \lim_{\varepsilon \to 0} \left[\Pr(|\hat{p}_{\ell_\varepsilon} - p| \geq \varepsilon, \ \hat{p}_{\ell_\varepsilon} \notin (z_{\ell_\varepsilon}, 1 - z_{\ell_\varepsilon})) + \Pr(|\hat{p}_{\ell_\varepsilon+1} - p| \geq \varepsilon, \ \hat{p}_{\ell_\varepsilon} \in (z_{\ell_\varepsilon}, 1 - z_{\ell_\varepsilon}))\right].
\]
On the other hand,
\[
\limsup_{\varepsilon \to 0} \left[\Pr(|\hat{p}_{\ell_\varepsilon} - p| \geq \varepsilon, \ l = \ell_\varepsilon) + \Pr(|\hat{p}_{\ell_\varepsilon+1} - p| \geq \varepsilon, \ l = \ell_\varepsilon + 1)\right]
\leq \lim_{\varepsilon \to 0} \left[\Pr(|\hat{p}_{\ell_\varepsilon} - p| \geq \varepsilon, \ \hat{p}_{\ell_\varepsilon} \notin (z_{\ell_\varepsilon}, 1 - z_{\ell_\varepsilon})) + \Pr(|\hat{p}_{\ell_\varepsilon+1} - p| \geq \varepsilon, \ \hat{p}_{\ell_\varepsilon} \in (z_{\ell_\varepsilon}, 1 - z_{\ell_\varepsilon}))\right].
\]
Therefore,
\[
\lim_{\varepsilon \to 0} \left[\Pr(|\hat{p}_{\ell_\varepsilon} - p| \geq \varepsilon, \ l = \ell_\varepsilon) + \Pr(|\hat{p}_{\ell_\varepsilon+1} - p| \geq \varepsilon, \ l = \ell_\varepsilon + 1)\right]
\geq \lim_{\varepsilon \to 0} \left[\Pr(|\hat{p}_{\ell_\varepsilon} - p| \geq \varepsilon, \ \hat{p}_{\ell_\varepsilon} \notin (z_{\ell_\varepsilon}, 1 - z_{\ell_\varepsilon})) + \Pr(|\hat{p}_{\ell_\varepsilon+1} - p| \geq \varepsilon, \ \hat{p}_{\ell_\varepsilon} \in (z_{\ell_\varepsilon}, 1 - z_{\ell_\varepsilon}))\right]
\geq \lim_{\varepsilon \to 0} \left[\Pr(|\hat{p}_{\ell_\varepsilon} - p| \geq \varepsilon, \ \hat{p}_{\ell_\varepsilon} \leq z_{\ell_\varepsilon}) + \Pr(|\hat{p}_{\ell_\varepsilon+1} - p| \geq \varepsilon, \ \hat{p}_{\ell_\varepsilon} > z_{\ell_\varepsilon})\right].
\] (71)

Combining (68), (69) and (71) yields
\[
\lim_{\varepsilon \to 0} \left[\Pr(|\hat{p}_{\ell_\varepsilon} - p| \geq \varepsilon, \ l = \ell_\varepsilon) + \Pr(|\hat{p}_{\ell_\varepsilon+1} - p| \geq \varepsilon, \ l = \ell_\varepsilon + 1)\right]
= \Pr\{U \geq d\} + \Pr\{|U + \sqrt{\rho}V| \geq d\sqrt{1 + \rho}, \ U \leq \nu d\} = \Psi(\rho, \nu, d) + \Phi(\nu d) - \Phi(d) < 3[1 - \Phi(d)].
\]
This completes the proof of the lemma.

\[\Box\]

**Lemma 21** Let \(d > 0\), \(\rho > 0\) and \(0 < \nu < 1\). Let \(U\) and \(V\) be independent Gaussian variables with zero mean and variance unity. Then,
\[
2[1 - \Phi(d)] < \Pr\{U \geq d\} + \Pr\{|U + \sqrt{\rho}V| \geq d\sqrt{1 + \rho}, \ U \leq \nu d\} = \Psi(\rho, \nu, d) + \Phi(\nu d) - \Phi(d) < 3[1 - \Phi(d)].
\]
Proof. Clearly,
\[
\Pr\{|U + \sqrt{\rho V}| \geq d\sqrt{1 + \rho}, U \leq \nu d\} < \Pr\{|U + \sqrt{\rho V}| \geq d\sqrt{1 + \rho}\}
\]
\[
= \Pr\{|U| \geq d\} = 2[1 - \Phi(d)]
\]
Since \(\nu > 0\), we have
\[
\Pr\{|U + \sqrt{\rho V}| \geq d\sqrt{1 + \rho}, U \leq \nu d\} = \Pr\{|U + \sqrt{\rho V}| \geq d\sqrt{1 + \rho}, U < 0\}
\]
\[+
\Pr\{|U + \sqrt{\rho V}| \geq d\sqrt{1 + \rho}, 0 \leq U \leq \nu d\}
\]
\[>
\Pr\{|U + \sqrt{\rho V}| \geq d\sqrt{1 + \rho}, U < 0\} \]
\[= \frac{1}{2}\Pr\{|U + \sqrt{\rho V}| \geq d\sqrt{1 + \rho}\}
\]
\[= \frac{1}{2}\Pr\{|U| \geq d\} = 1 - \Phi(d).
\]
Note that
\[
\Pr\{U \geq d\} + \Pr\{|U + \sqrt{\rho V}| \geq d\sqrt{1 + \rho}, U < \nu d\}
\]
\[=
\Pr\{U \geq d\} + \Pr\{U < \nu d\} - \Pr\{|U + \sqrt{\rho V}| < d\sqrt{1 + \rho}, U < \nu d\}
\]
\[=
\Pr\{U \geq d\} + \Pr\{U < \nu d\} - 1 + \Pr\{|U + \sqrt{\rho V}| \geq d\sqrt{1 + \rho} \text{ or } U \geq \nu d\}
\]
\[=
\Pr\{U \geq d\} - \Pr\{U \geq \nu d\} + \Pr\{|U + \sqrt{\rho V}| \geq d\sqrt{1 + \rho} \text{ or } U \geq \nu d\}
\]
\[=
\Pr\{|U + \sqrt{\rho V}| \geq d\sqrt{1 + \rho} \text{ or } U \geq \nu d\} - \Pr\{\nu d < U \leq d\}
\]
and that \(\Pr\{|U + \sqrt{\rho V}| \geq d\sqrt{1 + \rho} \text{ or } U \geq \nu d\}\) is the probability that \((U, V)\) is included in a domain with a boundary which is visible for an observer in the origin and can be represented in polar coordinates \((r, \phi)\) as
\[
\left\{(r, \phi) : r = \frac{\nu d}{\cos \phi}, -\phi_L \leq \phi \leq \phi_U\right\} \cup \left\{(r, \phi) : r = \frac{d}{\cos(\phi - \phi_p)}, \phi_U \leq \phi \leq 2\pi - \phi_L\right\}.
\]
Hence, by Theorem 6 of [11], we can show that \(\Pr\{|U + \sqrt{\rho V}| \geq d\sqrt{1 + \rho} \text{ or } U \geq \nu d\} = \Psi(\rho, \nu, d)\).
The lemma follows immediately.

\[\square\]

I.4.1 Proof of Statement (I)

First, we shall show that Statement (I) holds for \(p \in (0, \frac{1}{2}]\) such that \(C_{j_p} = 4p(1 - p)\). For this purpose, we need to show that
\[
1 \leq \limsup_{\varepsilon \to 0} \frac{n(\omega)}{N_p(p, \varepsilon)} \leq 1 + \rho_p \quad \text{for any } \omega \in \left\{\lim_{\varepsilon \to 0} \hat{p} = p\right\}.
\]
(72)
To show \(\limsup_{\varepsilon \to 0} \frac{n(\omega)}{N_p(p, \varepsilon)} \geq 1\), note that \(C_s - \ell_\varepsilon + 1 < 4p(1 - p) = C_s - \ell_\varepsilon < C_s - \ell_\varepsilon - 1\) as a direct consequence of the definition of \(\ell_\varepsilon\) and the assumption that \(C_{j_p} = 4p(1 - p)\). By the first three statements of Lemma[17] we have \(\lim_{\varepsilon \to 0} z_\ell < p\) for all \(\ell \leq \ell_\varepsilon - 1\). Noting that \(\lim_{\varepsilon \to 0} \hat{p}(\omega) = p \leq \frac{1}{2}\),
we have $z_\ell < \hat{\rho}(\omega) < 1 - z_\ell$ for all $\ell \leq \ell_\epsilon - 1$ and it follows from the definition of the sampling scheme that $n(\omega) \geq n_\ell$, if $\varepsilon > 0$ is small enough. By Lemma 19 and noting that $\kappa_p = 1$ if $C_{jp} = 4p(1-p)$, we have $\limsup_{\varepsilon \to 0} \frac{n(\omega)}{N_{x}(p,\varepsilon)} \leq \lim_{\varepsilon \to 0} \frac{n(\omega)}{N_{x}(p,\varepsilon)} = \kappa_p = 1$. To show $\limsup_{\varepsilon \to 0} \frac{n(\omega)}{N_{x}(p,\varepsilon)} \leq \lim_{\varepsilon \to 0} \frac{n(\omega)}{N_{x}(p,\varepsilon)} = \kappa_p = 1 + \rho_p$, we shall consider three cases: (i) $\ell_\epsilon = s$; (ii) $\ell_\epsilon = s - 1$; (iii) $\ell_\epsilon < s - 1$. In the case of $\ell_\epsilon = s$, it must be true that $n(\omega) \leq n_s = n_{\ell_\epsilon}$. Hence, $\limsup_{\varepsilon \to 0} \frac{n(\omega)}{N_{x}(p,\varepsilon)} \leq \lim_{\varepsilon \to 0} \frac{n(\omega)}{N_{x}(p,\varepsilon)} = \kappa_p = 1 = 1 + \rho_p$. In the case of $\ell_\epsilon = s - 1$, it must be true that $n(\omega) \leq n_{\ell_\epsilon}$. Therefore, $\limsup_{\varepsilon \to 0} \frac{n(\omega)}{N_{x}(p,\varepsilon)} \leq \lim_{\varepsilon \to 0} \frac{n(\omega)}{N_{x}(p,\varepsilon)} = \lim_{\varepsilon \to 0} \frac{n_{\ell_\epsilon} + \varepsilon}{n_{\ell_\epsilon}} \times \lim_{\varepsilon \to 0} \frac{n_{\ell_\epsilon}}{N_{x}(p,\varepsilon)} = \frac{C_{jp} - 1}{C_{jp}} + 1 + \rho_p$. This establishes (72), which implies $\{1 \leq \limsup_{\varepsilon \to 0} \frac{n(\omega)}{N_{x}(p,\varepsilon)} \leq 1 + \rho_p\} \supseteq \{\lim_{\varepsilon \to 0} \hat{\rho} = p\}$. Applying the strong law of large numbers, we have $1 \geq \text{Pr}\{1 \leq \limsup_{\varepsilon \to 0} \frac{n(\omega)}{N_{x}(p,\varepsilon)} \leq 1 + \rho_p\} \geq \text{Pr}\{\lim_{\varepsilon \to 0} \hat{\rho} = p\} = 1$. This proves that Statement (I) holds for $p \in (0, \frac{1}{2}]$ such that $C_{jp} > 4p(1-p)$.

Next, we shall show that Statement (I) for $p \in (0, \frac{1}{2}]$ such that $C_{jp} > 4p(1-p)$. Note that $C_{s-\ell_\epsilon} < 4p(1-p) < C_{s-\ell_\epsilon} \ell_\epsilon$ as a direct consequence of the definitions of $\ell_\epsilon$ and $j_p$. By the first three statements of Lemma 17 we have $\lim_{\varepsilon \to 0} \frac{n_{\ell_\epsilon} + \varepsilon}{n_{\ell_\epsilon}} \leq \lim_{\varepsilon \to 0} \frac{n_{\ell_\epsilon} + \varepsilon}{n_{\ell_\epsilon}} \times \lim_{\varepsilon \to 0} \frac{n_{\ell_\epsilon}}{N_{x}(p,\varepsilon)} = \frac{C_{jp} - 1}{C_{jp}} = 1 + \rho_p$. In the case of $\ell_\epsilon < s - 1$, it follows from Lemma 17 that $\lim_{\varepsilon \to 0} \frac{n_{\ell_\epsilon} + \varepsilon}{n_{\ell_\epsilon}} \leq \lim_{\varepsilon \to 0} \frac{n_{\ell_\epsilon} + \varepsilon}{n_{\ell_\epsilon}} \times \lim_{\varepsilon \to 0} \frac{n_{\ell_\epsilon}}{N_{x}(p,\varepsilon)} = \frac{C_{jp} - 1}{C_{jp}} = 1 + \rho_p$. This establishes (72), which implies $\{1 \leq \limsup_{\varepsilon \to 0} \frac{n(\omega)}{N_{x}(p,\varepsilon)} \leq 1 + \rho_p\} \supseteq \{\lim_{\varepsilon \to 0} \hat{\rho} = p\}$. Applying the strong law of large numbers, we have $1 \geq \text{Pr}\{1 \leq \limsup_{\varepsilon \to 0} \frac{n(\omega)}{N_{x}(p,\varepsilon)} \leq 1 + \rho_p\} \geq \text{Pr}\{\lim_{\varepsilon \to 0} \hat{\rho} = p\} = 1$. This proves that Statement (I) holds for $p \in (0, \frac{1}{2}]$ such that $C_{jp} > 4p(1-p)$.

In a similar manner, we can show that Statement (I) holds for $p \in (\frac{1}{2}, 1)$. This concludes the proof of Statement (I).

I.4.2 Proof of Statement (II)
In the sequel, we will consider the asymptotic value of $\frac{\mathbb{E}[n]}{N_{x}(p,\varepsilon)}$ in three steps. First, we shall show Statement (II) for $p \in (0, 1)$ such that $C_{jp} = 4p(1-p)$ and $j_p \geq 1$. By the definition of the
sampling scheme, we have

\[
E[u] = \sum_{\ell=1}^{\ell_{\epsilon}-1} n_\ell \Pr\{l = \ell\} + \sum_{\ell=\ell_{\epsilon}+2}^{s} n_\ell \Pr\{l = \ell\} + n_{\ell_{\epsilon}} \Pr\{l = \ell_{\epsilon}\} + n_{\ell_{\epsilon}+1} \Pr\{l = \ell_{\epsilon} + 1\}
\]

and 

\[
E[u] \geq n_{\ell_{\epsilon}} \Pr\{l = \ell_{\epsilon}\} + n_{\ell_{\epsilon}+1} \Pr\{l = \ell_{\epsilon} + 1\}. \]

Making use of Lemma 15 and the assumption that \(\sup_{\ell > 0} \frac{n_{\ell+1}}{n_\ell} < \infty\), we have

\[
\lim_{\varepsilon \to 0} \left[ \sum_{\ell=1}^{\ell_{\epsilon}-1} n_\ell \Pr\{D_\ell = 1\} + \sum_{\ell=\ell_{\epsilon}+1}^{s-1} n_{\ell+1} \Pr\{D_\ell = 0\} \right] \leq \lim_{\varepsilon \to 0} \left[ \sum_{\ell=1}^{\ell_{\epsilon}-1} n_\ell \Pr\{D_\ell = 1\} + \sup_{\ell > 0} \frac{n_{\ell+1}}{n_\ell} \sum_{\ell=\ell_{\epsilon}+1}^{s-1} n_{\ell} \Pr\{D_\ell = 0\} \right] = 0.
\]

Therefore,

\[
\limsup_{\varepsilon \to 0} \frac{E[u]}{N_\alpha(p, \varepsilon)} \leq \lim_{\varepsilon \to 0} \frac{\sum_{\ell=1}^{\ell_{\epsilon}-1} n_\ell \Pr\{D_\ell = 1\} + \sum_{\ell=\ell_{\epsilon}+1}^{s-1} n_{\ell+1} \Pr\{D_\ell = 0\} + n_{\ell_{\epsilon}} \Pr\{l = \ell_{\epsilon}\} + n_{\ell_{\epsilon}+1} \Pr\{l = \ell_{\epsilon} + 1\}}{N_\alpha(p, \varepsilon)} = \lim_{\varepsilon \to 0} \frac{n_{\ell_{\epsilon}} \Pr\{l = \ell_{\epsilon}\} + n_{\ell_{\epsilon}+1} \Pr\{l = \ell_{\epsilon} + 1\}}{N_\alpha(p, \varepsilon)}.
\]

On the other hand,

\[
\liminf_{\varepsilon \to 0} \frac{E[u]}{N_\alpha(p, \varepsilon)} \geq \lim_{\varepsilon \to 0} \frac{n_{\ell_{\epsilon}} \Pr\{l = \ell_{\epsilon}\} + n_{\ell_{\epsilon}+1} \Pr\{l = \ell_{\epsilon} + 1\}}{N_\alpha(p, \varepsilon)}.
\]

It follows that

\[
\lim_{\varepsilon \to 0} \frac{E[u]}{N_\alpha(p, \varepsilon)} = \lim_{\varepsilon \to 0} \frac{n_{\ell_{\epsilon}} \Pr\{l = \ell_{\epsilon}\} + n_{\ell_{\epsilon}+1} \Pr\{l = \ell_{\epsilon} + 1\}}{N_\alpha(p, \varepsilon)}
\]

for \(p \in (0, 1)\) such that \(C_{j_p} = 4p(1-p)\) and \(j_p \geq 1\). Using Lemma 20 and the result \(\lim_{\varepsilon \to 0} \frac{n_{\ell_{\epsilon}}}{N_\alpha(p, \varepsilon)} = \kappa_p\) as asserted by Lemma 19 we have

\[
\lim_{\varepsilon \to 0} \frac{n_{\ell_{\epsilon}} \Pr\{l = \ell_{\epsilon}\} + n_{\ell_{\epsilon}+1} \Pr\{l = \ell_{\epsilon} + 1\}}{N_\alpha(p, \varepsilon)} = \lim_{\varepsilon \to 0} \frac{n_{\ell_{\epsilon}} [1 - \Phi(\nu d)] + n_{\ell_{\epsilon}+1} \Phi(\nu d)}{N_\alpha(p, \varepsilon)} = 1 + \rho_p \Phi(\nu d).
\]

Second, we shall show Statement (II) for \(p \in (0, 1)\) such that \(C_{j_p} = 4p(1-p)\) and \(j_p = 0\). In this case, it must be true that \(p = \frac{1}{2}\). By the definition of the sampling scheme, we have

\[
E[u] = \sum_{\ell=1}^{\ell_{\epsilon}-1} n_\ell \Pr\{l = \ell\} + n_{\ell_{\epsilon}} \Pr\{l = \ell_{\epsilon}\} \leq \sum_{\ell=1}^{\ell_{\epsilon}-1} n_\ell \Pr\{D_\ell = 1\} + n_{\ell_{\epsilon}}
\]

and 

\[
E[u] \geq n_{\ell_{\epsilon}} \Pr\{l = \ell_{\epsilon}\} \geq n_{\ell_{\epsilon}} \left(1 - \sum_{\ell=1}^{\ell_{\epsilon}-1} \Pr\{D_\ell = 1\}\right). \]

Therefore, by Lemma 18

\[
\limsup_{\varepsilon \to 0} \frac{E[u]}{N_\alpha(p, \varepsilon)} \leq \lim_{\varepsilon \to 0} \frac{\sum_{\ell=1}^{\ell_{\epsilon}-1} n_\ell \Pr\{D_\ell = 1\} + n_{\ell_{\epsilon}}}{N_\alpha(p, \varepsilon)} = \lim_{\varepsilon \to 0} \frac{n_{\ell_{\epsilon}}}{N_\alpha(p, \varepsilon)} = \kappa_p = 1.
\]
and thus $\lim_{\varepsilon \to 0} \frac{E_n}{N_\varepsilon(p, \varepsilon)} = \frac{1}{\varepsilon}$ for $p \in (0, 1)$ such that $C_{j_p} = 4p(1 - p)$ and $j_p = 0$.

Third, we shall show Statements (II) for $p \in (0, 1)$ such that $C_{j_p} > 4p(1 - p)$. Note that

$$E[n] = \sum_{\ell = 1}^{\ell - 1} n_\ell \Pr\{l = \ell\} + \sum_{\ell = \ell + 1} s \ n_\ell \Pr\{l = \ell\} + n_\ell \Pr\{l = \ell_\varepsilon\}$$

and $E[n] \geq n_\ell \Pr\{l = \ell_\varepsilon\} \geq n_\ell \left(1 - \sum_{\ell = 1}^{\ell - 1} \Pr\{D_\ell = 1\} - \Pr\{D_\ell = 0\}\right)$. Therefore, by Lemma [18]

$$\lim_{\varepsilon \to 0} \frac{E_n}{N_\varepsilon(p, \varepsilon)} = \frac{1}{\varepsilon}$$

So, $\lim_{\varepsilon \to 0} \frac{E_n}{N_\varepsilon(p, \varepsilon)} = \kappa_p$ for $p \in (0, 1)$ such that $C_{j_p} > 4p(1 - p)$. From the preceding analysis, we have shown $\lim_{\varepsilon \to 0} \frac{E_n}{N_\varepsilon(p, \varepsilon)}$ exists for all $p \in (0, 1)$. Hence, statement (II) is established by making use of this result and the fact that

$$\lim_{\varepsilon \to 0} \frac{\varepsilon}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{N_\varepsilon(p, \varepsilon)}{N_\varepsilon(p, \varepsilon)} \times \lim_{\varepsilon \to 0} \frac{E_n}{n_\varepsilon} = \frac{2\ln \frac{1}{\varepsilon}}{\varepsilon^2} \times \lim_{\varepsilon \to 0} \varepsilon.$$

### I.4.3 Proof of Statement (III)

As before, we use the notations $b_\ell = \frac{\varepsilon}{\sqrt{(p(1 - p))}}$ and $U_\ell = \frac{\varepsilon}{\sqrt{(p(1 - p))}}$.

First, we shall consider $p \in (0, 1)$ such that $C_{j_p} > 4p(1 - p)$. Applying Lemma [18] based on the assumption that $C_{j_p} > 4p(1 - p)$, we have

$$\lim_{\varepsilon \to 0} \Pr\{l < \ell_\varepsilon\} \leq \lim_{\varepsilon \to 0} \sum_{\ell = 1}^{\ell - 1} \Pr\{D_\ell = 1\} \leq \lim_{\varepsilon \to 0} \sum_{\ell = 1}^{\ell - 1} n_\ell \Pr\{D_\ell = 1\} = 0,$$

and thus $\lim_{\varepsilon \to 0} \Pr\{l \neq \ell_\varepsilon\} = 0$. Note that $\Pr\{|\hat{p} - p| \geq \varepsilon\} = \Pr\{|\hat{p}_\ell - p| \geq \varepsilon, \ l = \ell_\varepsilon\} + \Pr\{|\hat{p} - p| \geq \varepsilon, \ l \neq \ell_\varepsilon\}$ and, as a result of the central limit theorem, $U_\ell$ converges in distribution to a standard Gaussian variable $U$. Hence,

$$\lim_{\varepsilon \to 0} \Pr\{|\hat{p} - p| \geq \varepsilon\} = \lim_{\varepsilon \to 0} \Pr\{|\hat{p}_\ell - p| \geq \varepsilon\} = \lim_{\varepsilon \to 0} \Pr\{|U_\ell| \geq b_\ell\} = \Pr\{|U| \geq d_\ell\} = 2\Phi(d_\ell) - 1 > 2\Phi(d) - 1 > 1 - 2\zeta_\delta$$

and $\lim_{\varepsilon \to 0} \Pr\{|\hat{p} - p| < \varepsilon\} = \Pr\{|U| < d_\ell\} = 2\Phi(d_\ell) - 1 > 2\Phi(d) - 1 > 1 - 2\zeta_\delta$ for $p \in (0, 1)$ such that $C_{j_p} > 4p(1 - p)$.
Second, we shall consider $p \in (0, 1)$ such that $C_{jp} = 4p(1 - p)$ and $j_p \geq 1$. In this case, it is evident that $\ell_\epsilon < s$. By the definition of the sampling scheme, we have that $\Pr\{l > \ell_\epsilon + 1\} \leq \Pr\{D_{\ell_\epsilon + 1} = 0\}$ and that $\Pr\{l = \ell\} \leq \Pr\{D_\ell = 1\}$ for $\ell < \ell_\epsilon$. As a result of Lemma 18, we have $\lim_{\epsilon \to 0} \Pr\{l > \ell_\epsilon + 1\} = 0$ and $\lim_{\epsilon \to 0} \Pr\{l < \ell_\epsilon\} \leq \lim_{\epsilon \to 0} \sum_{\ell=1}^{\ell_{\epsilon} - 1} \Pr\{D_\ell = 1\} = 0$. Since

$$
\limsup_{\epsilon \to 0} \Pr\{|\hat{p} - p| \geq \epsilon\} \leq \limsup_{\epsilon \to 0} \left[ \Pr\{|\hat{p}_{\ell_\epsilon} - p| \geq \epsilon, l = \ell_\epsilon\} + \Pr\{|\hat{p}_{\ell_\epsilon + 1} - p| \geq \epsilon, l = \ell_\epsilon + 1\} \right] + \lim_{\epsilon \to 0} \Pr\{l < \ell_\epsilon\} + \lim_{\epsilon \to 0} \Pr\{l > \ell_\epsilon + 1\}
$$

and $\liminf_{\epsilon \to 0} \Pr\{|\hat{p} - p| \geq \epsilon\} \geq \liminf_{\epsilon \to 0} \left[ \Pr\{|\hat{p}_{\ell_\epsilon} - p| \geq \epsilon, l = \ell_\epsilon\} + \Pr\{|\hat{p}_{\ell_\epsilon + 1} - p| \geq \epsilon, l = \ell_\epsilon + 1\} \right]$, we have $\lim_{\epsilon \to 0} \Pr\{|\hat{p} - p| \geq \epsilon\} = \lim_{\epsilon \to 0} \left[ \Pr\{|\hat{p}_{\ell_\epsilon} - p| \geq \epsilon, l = \ell_\epsilon\} + \Pr\{|\hat{p}_{\ell_\epsilon + 1} - p| \geq \epsilon, l = \ell_\epsilon + 1\} \right]$. By Lemma 20, we have $\lim_{\epsilon \to 0} \Pr\{|\hat{p} - p| \geq \epsilon\} = \Pr\{U \geq d\} + \Pr\{|U + \sqrt{\rho p} \geq d\sqrt{1 + \rho p}, U < \nu d\}$ for $p \in (0, 1)$ such that $C_{jp} = 4p(1 - p)$ and $j_p \geq 1$. As a consequence of Lemma 21, Statement (III) must be true for $p \in (0, 1)$ such that $C_{jp} = 4p(1 - p)$ and $j_p \geq 1$.

Third, we shall consider $p \in (0, 1)$ such that $C_{jp} = 4p(1 - p)$ and $j_p = 0$. In this case, it must be true that $p = \frac{1}{2}$. Clearly, $\ell_\epsilon = s$. It follows from the definition of the sampling scheme that $\Pr\{l = \ell\} \leq \Pr\{D_\ell = 1\}$ for $\ell < \ell_\epsilon$. By Lemma 18, we have $\lim_{\epsilon \to 0} \Pr\{l < \ell_\epsilon\} \leq \lim_{\epsilon \to 0} \sum_{\ell=1}^{\ell_{\epsilon} - 1} \Pr\{D_\ell = 1\} = 0$. Therefore, $\lim_{\epsilon \to 0} \Pr\{l = \ell_\epsilon\} = 1$ and

$$
\lim_{\epsilon \to 0} \Pr\{|\hat{p} - p| \geq \epsilon\} = \lim_{\epsilon \to 0} \Pr\{|\hat{p} - p| \geq \epsilon, l = \ell_\epsilon\} = \lim_{\epsilon \to 0} \Pr\{|\hat{p}_{\ell_\epsilon} - p| \geq \epsilon\} = \lim_{\epsilon \to 0} \Pr\{|U_{\ell_\epsilon}| \geq b_{\ell_\epsilon}\} = \Pr\{|U| \geq d\sqrt{\rho p}\} + 2 - 2\Phi(d\sqrt{\rho p})
$$

for $p \in (0, 1)$ such that $C_{jp} = 4p(1 - p)$ and $j_p = 0$.

Note that, for a positive number $z$ and a Gaussian random variable $X$ with zero mean and unit variance, it holds true that $\Phi(z) = 1 - \Pr\{X > z\} = 1 - \inf_{t > 0} \mathbb{E}[e^{t(X - \bar{X})}] = 1 - \inf_{t > 0} e^{-t^2 + \frac{t^2}{2}} = 1 - e^{-\frac{t^2}{2}}$. So, $\Phi(d) = \Phi\left(\sqrt{\frac{2 \ln \frac{1}{\delta}}{d}}\right) > 1 - \zeta \delta$ and consequently, $\liminf_{\epsilon \to 0} \Pr\{|\hat{p} - p| < \epsilon\} > 1 - 2\zeta \delta$. This establishes Statement (III).
I.5 Proof of Theorem 17

Let $I_{\hat{p}_\ell}$ denote the support of $\hat{p}_\ell$ for $\ell = 1, \ldots, s$. Then,

$$
\mathbb{E}[\hat{p} - p]^k = \sum_{\ell=1}^s \sum_{p \in I_{\hat{p}_\ell}} |\hat{p}_\ell - p|^k \Pr\{\hat{p}_\ell = \hat{p}_\ell, l = \ell\}
$$

$$
= \sum_{\ell=1}^s \sum_{p \in I_{\hat{p}_\ell}} |\hat{p}_\ell - p|^k \Pr\{\hat{p}_\ell = \hat{p}_\ell, l = \ell\} + \sum_{\ell=1}^s \sum_{p \in I_{\hat{p}_\ell}} |\hat{p}_\ell - p|^k \Pr\{\hat{p}_\ell = \hat{p}_\ell, l = \ell\}
$$

$$
\leq \sum_{\ell=1}^s \sum_{p \in I_{\hat{p}_\ell}} \Pr\{\hat{p}_\ell = \hat{p}_\ell, l = \ell\} + \sum_{\ell=1}^s \sum_{p \in I_{\hat{p}_\ell}} \Pr\{\hat{p}_\ell = \hat{p}_\ell\}
$$

$$
= \left(\frac{p}{\sqrt{\gamma}}\right)^k + \sum_{\ell=1}^s \Pr\{|\hat{p}_\ell - p| \geq \frac{p}{\sqrt{\gamma}}\} \leq \left(\frac{p}{\sqrt{\gamma}}\right)^k + 2 \exp\left(-\frac{\sqrt{\gamma}}{8}\right) = \left(\frac{p}{\sqrt{\gamma}}\right)^k + 2 \exp\left(-\frac{\sqrt{\gamma}}{8}\right)
$$

for $k = 1, 2, \ldots$, where the last inequality is derived from Corollary 1 of [8], which asserts that

$$
\Pr\{|\hat{p}_\ell - p| \geq \varepsilon p\} \leq 2 \exp\left(-\gamma_\ell \left[\ln(1 + \varepsilon) - \frac{\varepsilon}{1 + \varepsilon}\right]\right) < 2 \exp\left(-\frac{\gamma_\ell \varepsilon^2}{8}\right), \quad \ell = 1, \ldots, s
$$

for $\varepsilon \in (0, 1)$. By the assumption that $\inf_{\ell > 0} \frac{\gamma_{\ell+1}}{\gamma_{\ell}} > 1$, we have that, there exists a positive number $\rho$ such that $\gamma_{\ell} \leq (1 + \rho)^{2(\ell-1)}\gamma_1$ for all $\ell > 1$. Hence,

$$
\mathbb{E}|\hat{p} - p|^k \leq \left(\frac{p}{\sqrt{\gamma}}\right)^k + 2 \sum_{\ell=1}^s \exp\left(-\frac{1}{8}\sqrt{\gamma_\ell}\right) \leq \left(\frac{p}{\sqrt{\gamma}}\right)^k + 2 \sum_{\ell=1}^s \exp\left(-\frac{1}{8}\sqrt{\gamma_\ell}(1 + \rho)\ell^{-1}\right)
$$

$$
\leq \left(\frac{p}{\sqrt{\gamma}}\right)^k + 2 \sum_{\ell=1}^{\infty} \exp\left(-\frac{1}{8}\sqrt{\gamma_\ell}(1 + \rho)\ell^{-1}\right) \leq \left(\frac{p}{\sqrt{\gamma}}\right)^k + 2 \sum_{\ell=1}^{\infty} \exp\left(-\frac{1}{8}\sqrt{\gamma_\ell}(1 + \rho(\ell - 1))\right)
$$

$$
= \left(\frac{p}{\sqrt{\gamma}}\right)^k + 2 \exp\left(-\frac{1}{8}\sqrt{\gamma_\ell}\right) - \exp\left(-\frac{1}{8}\sqrt{\gamma_\ell}\rho\right) \to 0
$$

as $\gamma_1 \to \infty$. Since $\mathbb{E}|\hat{p} - p| \leq \mathbb{E}|\hat{p} - p|$, we have that $\mathbb{E}|\hat{p} - p| \to 0$ as $n_1 \to \infty$. This completes the proof of the theorem.

I.6 Proof of Theorem 18

We need some preliminary results.

**Lemma 22** Let $0 < \varepsilon < 1$. Then, $\mathcal{M}(z, \frac{1}{z+1})$ is monotonically decreasing with respect to $z \in (0, 1)$. 107
Proof. To show that $\mathcal{M}_1(z, \frac{z}{1+\varepsilon})$ is monotonically decreasing with respect to $z \in (0, 1)$, we derive the partial derivative as $\frac{\partial}{\partial \varepsilon} \mathcal{M}_1(z, \frac{z}{1+\varepsilon}) = \frac{\varepsilon}{1+\varepsilon} \ln(1 - \frac{\varepsilon}{1+\varepsilon}) + \frac{\varepsilon}{1+\varepsilon} \ln(1 + \varepsilon)$, where the right side is negative if $\ln(1 - \frac{\varepsilon}{1+\varepsilon}) < -\frac{\varepsilon}{1+\varepsilon}$. This condition is seen to be true by virtue of the standard inequality $\ln(1 - x) < -x$, $\forall x \in (0, 1)$ and the fact that $0 < \frac{\varepsilon}{1+\varepsilon} < 1$ as a consequence of $0 < z < 1$. This completes the proof of the lemma.

\[ \square \]

Lemma 23. $\mathcal{M}_1(z, \frac{z}{1+\varepsilon}) > \mathcal{M}_1(z, \frac{z}{1-\varepsilon})$ for $0 < \varepsilon < 1$.

Proof. The lemma follows from the facts that $\mathcal{M}_1(z, \frac{z}{1+\varepsilon}) = \mathcal{M}_1(z, \frac{z}{1-\varepsilon})$ for $\varepsilon = 0$ and that

$$
\frac{\partial}{\partial \varepsilon} \mathcal{M}_1 \left( z, \frac{z}{1+\varepsilon} \right) = -\frac{\varepsilon}{1+\varepsilon} \frac{1}{1+\varepsilon - z} > \frac{\partial}{\partial \varepsilon} \mathcal{M}_1 \left( z, \frac{z}{1-\varepsilon} \right) = -\frac{\varepsilon}{1-\varepsilon} \frac{1}{1-\varepsilon - z}.
$$

\[ \square \]

Lemma 24. $\{ F_{\hat{p}_s}(\hat{p}_s, \hat{p}_s \frac{1}{1+\varepsilon}) \leq \zeta \delta, G_{\hat{p}_s}(\hat{p}_s, \hat{p}_s \frac{1}{1+\varepsilon}) \leq \zeta \delta \}$ is a sure event.

Proof. By Lemma 4

$$
\Pr \left\{ \hat{G}_{\hat{p}_s} \left( \hat{p}_s, \frac{\hat{p}_s}{1+\varepsilon} \right) \leq \zeta \delta \right\} = \Pr \left\{ 1 - S_B \left( \gamma_s - 1, n_s, \frac{\hat{p}_s}{1+\varepsilon} \right) \leq \zeta \delta \right\}
\geq \Pr \left\{ n_s \mathcal{M}_B \left( \gamma_s \frac{\hat{p}_s}{n_s}, \frac{\hat{p}_s}{1+\varepsilon} \right) \leq \ln(\zeta \delta) \right\} = \Pr \left\{ \frac{n_s \mathcal{M}_B \left( \frac{\gamma_s}{n_s}, \frac{\hat{p}_s}{1+\varepsilon} \right)}{\gamma_s} \leq \ln(\zeta \delta) \right\}.
$$

Making use of Lemma 22 and the fact $\lim_{z \to 0} \mathcal{M}_1(z, \frac{z}{1+\varepsilon}) = \frac{z}{1+\varepsilon} - \ln(1+\varepsilon)$, we have $\mathcal{M}_1(z, \frac{z}{1+\varepsilon}) < \frac{z}{1+\varepsilon} - \ln(1+\varepsilon)$ for any $z \in (0, 1]$. Consequently, $\{ \mathcal{M}_1(\hat{p}_s, \frac{\hat{p}_s}{1+\varepsilon}) \leq \frac{z}{1+\varepsilon} - \ln(1+\varepsilon) \}$ is a sure event because $0 < \hat{p}_s(\omega) \leq 1$ for any $\omega \in \Omega$. By the definition of $\gamma_s$, we have

$$
\gamma_s = \left[ \frac{\ln(\zeta \delta)}{\frac{z}{1+\varepsilon} - \ln(1+\varepsilon)} \right] > \frac{\ln(\zeta \delta)}{\frac{z}{1+\varepsilon} - \ln(1+\varepsilon)}.
$$

Since $\frac{\varepsilon}{1+\varepsilon} - \ln(1+\varepsilon) < 0$ for any $\varepsilon \in (0, 1)$, we have $\frac{\ln(\zeta \delta)}{\gamma_s} \geq \frac{\varepsilon}{1+\varepsilon} - \ln(1+\varepsilon)$. Hence,

$$
\Pr \left\{ \mathcal{M}_1 \left( \hat{p}_s, \frac{\hat{p}_s}{1+\varepsilon} \right) \leq \frac{\ln(\zeta \delta)}{\gamma_s} \right\} \geq \Pr \left\{ \mathcal{M}_1 \left( \hat{p}_s, \frac{\hat{p}_s}{1+\varepsilon} \right) \leq \frac{\varepsilon}{1+\varepsilon} - \ln(1+\varepsilon) \right\} = 1.
$$

Combining (74) and (75) yields $\Pr\{ G_{\hat{p}_s}(\hat{p}_s, \frac{\hat{p}_s}{1+\varepsilon}) \leq \zeta \delta \} = 1$.

Similarly, by Lemmas 4 and 23

$$
\Pr \left\{ F_{\hat{p}_s} \left( \hat{p}_s, \frac{\hat{p}_s}{1-\varepsilon} \right) \leq \zeta \delta \right\} = \Pr \left\{ S_B \left( \gamma_s, n_s, \frac{\hat{p}_s}{1-\varepsilon} \right) \leq \zeta \delta \right\} \geq \Pr \left\{ n_s \mathcal{M}_B \left( \frac{\gamma_s}{n_s}, \frac{\hat{p}_s}{1-\varepsilon} \right) \leq \ln(\zeta \delta) \right\} = \Pr \left\{ \frac{n_s \mathcal{M}_B \left( \frac{\gamma_s}{n_s}, \frac{\hat{p}_s}{1-\varepsilon} \right)}{\gamma_s} \leq \ln(\zeta \delta) \right\}.
$$

Combining (74) and (75) yields $\Pr\{ F_{\hat{p}_s}(\hat{p}_s, \frac{\hat{p}_s}{1-\varepsilon}) \leq \zeta \delta \} = 1$.

Similarly, by Lemmas 4 and 23

$$
\Pr \left\{ F_{\hat{p}_s} \left( \hat{p}_s, \frac{\hat{p}_s}{1-\varepsilon} \right) \leq \zeta \delta \right\} \geq \Pr \left\{ S_B \left( \gamma_s, n_s, \frac{\hat{p}_s}{1-\varepsilon} \right) \leq \zeta \delta \right\} \geq \Pr \left\{ n_s \mathcal{M}_B \left( \frac{\gamma_s}{n_s}, \frac{\hat{p}_s}{1-\varepsilon} \right) \leq \ln(\zeta \delta) \right\} = \Pr \left\{ \mathcal{M}_B \left( \frac{\gamma_s}{n_s}, \frac{\hat{p}_s}{1-\varepsilon} \right) \leq \frac{\ln(\zeta \delta)}{\gamma_s} \right\} = 1.
$$
This completes the proof of the lemma.

Now we are in a position to prove Theorem 18. Clearly, \( \hat{p}_\ell \) is a ULE of \( p \) for \( \ell = 1, \ldots, s \). Define \( \mathcal{L}(\hat{p}_\ell) = \frac{\hat{p}_\ell}{1-\hat{p}_\ell} \) and \( \mathcal{U}(\hat{p}_\ell) = \frac{1}{1-\hat{p}_\ell} \) for \( \ell = 1, \ldots, s \). Then, \( \{ \mathcal{L}(\hat{p}_\ell) \leq \hat{p}_\ell \leq \mathcal{U}(\hat{p}_\ell) \} \) is a sure event for \( \ell = 1, \ldots, s \). By the definition of the stopping rule, we have \( \{ D_\ell = 1 \} = \{ F_{\hat{p}_\ell}(\hat{p}_\ell, \mathcal{U}(\hat{p}_\ell)) \leq \zeta \delta, G_{\hat{p}_\ell}(\hat{p}_\ell, \mathcal{L}(\hat{p}_\ell)) \leq \zeta \delta \} \) for \( \ell = 1, \ldots, s \). By Lemma 24 we have that \( \{ D_s = 1 \} \) is a sure event. So, the sampling scheme satisfies all the requirements described in Theorem 2, from which (17) and (18) of Theorem 18 immediately follows. The other results of Theorem 18 can be shown by a similar method as that of the proof of Theorem 19.

I.7 Proof of Theorem 19

Let \( X_1, X_2, \cdots \) be a sequence of i.i.d. Bernoulli random variables such that \( \Pr\{X_i = 1\} = 1 - \Pr\{X_i = 0\} = p \in (0, 1) \) for \( i = 1, 2, \cdots \). Let \( n \) be the minimum integer such that \( \sum_{i=1}^{n} X_i = \gamma \) where \( \gamma \) is a positive integer. In the sequel, from Lemmas 25 to 30 we shall be focusing on probabilities associated with \( \frac{\gamma}{n} \).

Lemma 25

\[
\Pr \left\{ \frac{\gamma}{n} \leq z \right\} \leq \exp (\gamma \cdot M_1(z, p)) \quad \forall z \in (0, p), \tag{77}
\]

\[
\Pr \left\{ \frac{\gamma}{n} \geq z \right\} \leq \exp (\gamma \cdot M_1(z, p)) \quad \forall z \in (p, 1). \tag{78}
\]

Proof. To show (77), note that \( \Pr \left\{ \frac{\gamma}{n} \leq z \right\} = \Pr \{ n \geq m \} = \Pr \{ X_1 + \cdots + X_m \leq \gamma \} = \Pr \{ \sum_{i=1}^{m} X_i \leq \gamma \} \) where \( m = \lceil \frac{\gamma}{z} \rceil \). Since \( 0 < z < p \), we have \( 0 < \frac{\gamma}{m} = \gamma / \lceil \frac{\gamma}{z} \rceil \leq \gamma / (\frac{\gamma}{z}) = z < p \), we can apply Lemma 1 to obtain \( \Pr \{ \sum_{i=1}^{m} X_i \leq \frac{\gamma}{m} \} \leq \exp (m \cdot M_1 (\frac{\gamma}{m}, p)) = \exp (\gamma \cdot M_1 (\frac{\gamma}{m}, p)) \). Noting that \( 0 < \frac{\gamma}{m} \leq z < p \) and that \( M_1 (z, p) \) is monotonically increasing with respect to \( z \in (0, p) \) as can be seen from \( \frac{dM_1 (z, p)}{dz} = \frac{1}{z^2} \ln \frac{1}{1-p} \), we have \( M_1 (\frac{\gamma}{m}, p) \leq M_1 (z, p) \) and thus \( \Pr \left\{ \frac{\gamma}{n} \leq z \right\} = \Pr \left\{ \sum_{i=1}^{m} X_i \leq \frac{\gamma}{m} \right\} \leq \exp (\gamma \cdot M_1(z, p)) \).

To show (78), note that \( \Pr \left\{ \frac{\gamma}{n} \geq z \right\} = \Pr \{ n \leq m \} = \Pr \{ X_1 + \cdots + X_m \geq \gamma \} = \Pr \{ \sum_{i=1}^{m} X_i \geq \frac{\gamma}{m} \} \) where \( m = \lceil \frac{\gamma}{z} \rceil \). We need to consider two cases: (i) \( m = \gamma \); (ii) \( m > \gamma \). In the case of \( m = \gamma \), we have \( \Pr \left\{ \frac{\gamma}{n} \geq z \right\} = \Pr \{ X_i = 1, \ i = 1, \cdots, \gamma \} = \prod_{i=1}^{\gamma} \Pr \{ X_i = 1 \} = p^\gamma \). Since \( M_1 (z, p) \) is monotonically decreasing with respect to \( z \in (p, 1) \) and \( \lim_{z \to 1} M_1 (z, p) = \ln p \), we have \( \Pr \left\{ \frac{\gamma}{n} \geq z \right\} = p^\gamma < \exp (\gamma \cdot M_1(z, p)) \). In the case of \( m > \gamma \), we have \( 1 > \frac{\gamma}{m} = \gamma / \lceil \frac{\gamma}{z} \rceil \geq \gamma / (\frac{\gamma}{z}) = z > p \). Hence, applying Lemma 1 we obtain \( \Pr \left\{ \sum_{i=1}^{m} X_i \geq \frac{\gamma}{m} \right\} \leq \exp (m \cdot M_1 (\frac{\gamma}{m}, p)) = \exp (\gamma \cdot M_1 (\frac{\gamma}{m}, p)) \). Noting that \( M_1 (z, p) \) is monotonically decreasing with respect to \( z \in (p, 1) \) and that \( 1 > \frac{\gamma}{m} \geq z > p \), we have \( M_1 (\frac{\gamma}{m}, p) \leq M_1 (z, p) \) and thus \( \Pr \left\{ \frac{\gamma}{n} \geq z \right\} = \Pr \left\{ \sum_{i=1}^{m} X_i \geq \frac{\gamma}{m} \right\} \leq \exp (\gamma \cdot M_1(z, p)) \).

The following result, stated as Lemma 26, have recently been established by Mendoza and Hernando 31.
Lemma 26  Let $\gamma \geq 3$ and $\mu_1 \geq \frac{\gamma - 1}{\gamma - \frac{1}{2} - \sqrt{\gamma - \frac{1}{2}}}$. Then, $\Pr(\frac{\gamma - 1}{n} > p \mu_1) < 1 - S_p(\gamma - 1, \frac{\gamma - 1}{\mu_1})$ for any $p \in (0, 1)$.

Since $\Pr\{\frac{\gamma}{n} > (1 + \varepsilon)p\} = \Pr\{\frac{\gamma - 1}{n} \geq \frac{\gamma - 1}{\gamma}(1 + \varepsilon)p\} = \Pr\{\frac{\gamma - 1}{n} \geq p \mu_1\}$ with $\mu_1 = \frac{\gamma - 1}{\gamma}(1 + \varepsilon)$, we can rewrite Lemma 26 as follows:

Lemma 27  Let $0 < \varepsilon < 1$ and $\gamma \geq 3$. Then, $\Pr\{\frac{\gamma}{n} > (1 + \varepsilon)p\} < 1 - S_p(\gamma - 1, \frac{\gamma - 1}{1 + \varepsilon})$ for any $p \in (0, 1)$ provided that $1 + \varepsilon \geq \frac{\gamma - 1}{\gamma - \frac{1}{2} - \sqrt{\gamma - \frac{1}{2}}}$.

The following result stated as Lemma 28 is due to Mendo and Hernando [30].

Lemma 28  Let $\gamma \geq 3$ and $\mu_2 \geq \frac{\gamma + \sqrt{\gamma}}{\gamma - 1}$. Then, $\Pr\{\frac{\gamma - 1}{n} \geq \frac{\gamma - 1}{\gamma}(1 + \varepsilon)p\} \geq 1 - S_p(\gamma - 1, (\gamma - 1)\mu_2)$ for any $p \in (0, 1)$.

Since $\Pr\{\frac{\gamma}{n} \geq (1 - \varepsilon)p\} = \Pr\{\frac{\gamma - 1}{n} \geq \frac{\gamma - 1}{\gamma}(1 - \varepsilon)p\} = \Pr\{\frac{\gamma - 1}{n} \geq p \mu_2\}$ with $\mu_2 = \frac{\gamma - 1}{(\gamma - 1)(1 - \varepsilon)}$, we can rewrite Lemma 28 as follows:

Lemma 29  Let $0 < \varepsilon < 1$ and $\gamma \geq 3$. Then, $\Pr\{\frac{\gamma}{n} \geq (1 - \varepsilon)p\} > 1 - S_p(\gamma - 1, \frac{\gamma - 1}{1 - \varepsilon})$ for any $p \in (0, 1)$ provided that $1 - \varepsilon \geq \frac{1}{\gamma - \frac{1}{2} - \sqrt{\gamma - \frac{1}{2}}}$.

Lemma 30  Let $0 < \varepsilon < 1$ and $\gamma \in \mathbb{N}$. Then, $\Pr\{\sqrt{\frac{\gamma}{n}} - p > \varepsilon p\} < 1 - S_p(\gamma - 1, \frac{\gamma - 1}{1 + \varepsilon}) + S_p(\gamma - 1, \frac{\gamma - 1}{1 - \varepsilon})$ for any $p \in (0, 1)$ provided that $\gamma \geq h(\varepsilon)$, it suffices to prove the following statements:

(i) $1 + \varepsilon \geq \frac{\gamma - 1}{\gamma - \frac{1}{2} - \sqrt{\gamma - \frac{1}{2}}}$ implies $\frac{1}{1 - \varepsilon} \geq 1 + \frac{1}{\sqrt{\gamma}}$;

(ii) $1 + \varepsilon \geq \frac{\gamma - 1}{\gamma - \frac{1}{2} - \sqrt{\gamma - \frac{1}{2}}}$ is equivalent to $\gamma \geq h(\varepsilon)$;

(iii) $\gamma \geq h(\varepsilon)$ implies $\gamma \geq 3$.

To prove statement (i), note that

$$\frac{1}{1 - \varepsilon} \geq 1 + \frac{1}{\sqrt{\gamma}} \iff \varepsilon \geq \frac{1}{\sqrt{\gamma} + 1}, \quad 1 + \varepsilon \geq \frac{\gamma}{\gamma - \frac{1}{2} - \sqrt{\gamma - \frac{1}{2}}} \iff \varepsilon \geq \frac{1}{\gamma - \frac{1}{2} - \sqrt{\gamma - \frac{1}{2}}}.$$

Hence, it suffices to show $\left(\frac{1}{2} + \frac{\gamma - 1}{2}\right) / \left(\gamma - \frac{1}{2} - \sqrt{\gamma - \frac{1}{2}}\right) > \frac{1}{\sqrt{\gamma} + 1}$, i.e., $\frac{\gamma - 1}{2 + \sqrt{\gamma - \frac{1}{2}}} - 2 < \sqrt{\gamma}$. Let $t = \sqrt{\gamma - \frac{1}{2}}$. Then, $\gamma = t^2 + \frac{1}{2}$ and the inequality becomes

$$\gamma > \left(\frac{\gamma - 1}{2 + \sqrt{\gamma - \frac{1}{2}}} - 2\right)^2 \iff t^2 + \frac{1}{2} > \left(\frac{t^2 + \frac{1}{2}}{t + \frac{1}{2}} - 2\right)^2,$$
i.e., $5t^3 - \frac{9}{4}t^2 - \frac{3}{8}t - \frac{1}{6} > 0$ under the condition that $\frac{t^2 + t}{9} - 2 > 0 \iff (t-1)^2 > \frac{4}{9} \iff t > 1 + \sqrt{\frac{2}{3}}$.

Clearly, $5t^3 - \frac{9}{4}t^2 - \frac{3}{8}t - \frac{1}{6} > 5t^3 - \frac{9}{4}t^2 - \frac{3}{8}t = \frac{3}{8}t^3 > 0$ for $t > 1 + \sqrt{\frac{2}{3}}$. It follows that, for $t > 1 + \sqrt{\frac{2}{3}}$, i.e., $\gamma > 5.4$, the inequality holds. It can be checked by hand calculation that it also holds for $\gamma = 1, \ldots, 5$. Hence, the inequality holds for all $\gamma \geq 1$. This establishes statement (i).

To show statement (ii), we rewrite $1 + \varepsilon \geq \frac{7}{\gamma - \frac{3}{8} - \gamma^{\frac{3}{8}}}$ in terms of $t = \sqrt{\gamma - \frac{3}{8} - \gamma^{\frac{3}{8}}}$, which is equivalent to $t^2 - (1 + \varepsilon)t - \frac{1}{2} \geq 0$. Solving this inequality yields $t \geq 1 + \frac{\sqrt{4\varepsilon + 1 + 2\varepsilon}}{2}$.

To show statement (iii), it is sufficient to show that $h(\varepsilon) \geq 3$ for $\varepsilon \in (0, 1]$. Note that $h(\varepsilon) = \frac{1}{2}[1 + g(\varepsilon)]^2 + \frac{1}{2}$ with $g(\varepsilon) = (1 + \sqrt{1 + 4\varepsilon + \varepsilon^2})/\varepsilon$. Since $g'(\varepsilon) = -(\sqrt{1 + 4\varepsilon + \varepsilon^2} + 1 + 2\varepsilon)/(\varepsilon^2\sqrt{1 + 4\varepsilon + \varepsilon^2}) < 0$, the minimum of $h(\varepsilon)$ is achieved at $\varepsilon = 1$, which is $\left(1 + \sqrt{\frac{3}{2}}\right)^2 + \frac{1}{2} > 3$. Hence, $\gamma \geq h(\varepsilon)$ implies $\gamma \geq 3$. This proves statement (iii).

\begin{lemma}
Let $\overline{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i$, where $X_1, \ldots, X_n$ are i.i.d. Poisson random variables with mean $\lambda > 0$. Then, $\Pr\{\overline{X}_n \geq z\} \leq \exp(n.\mathcal{M}(\lambda, z))$ for any $z \in (\lambda, \infty)$. Similarly, $\Pr\{\overline{X}_n \leq z\} \leq \exp(n.\mathcal{M}(\lambda, z))$ for any $z \in (0, \lambda)$.
\end{lemma}

\begin{proof}
Let $Y = n\overline{X}_n$. Then, $Y$ is a Poisson random variable with mean $\theta = n\lambda$. Let $r = nz$. If $z > \lambda$, then $r > \theta$ and, by virtue of Chernoff’s bound [14], we have

$$
\Pr\{\overline{X}_n \geq z\} = \Pr\{Y \geq r\} \leq \inf_{t > 0} \mathbb{E}[e^{t(Y-r)}] = \inf_{t > 0} \sum_{i=0}^{\infty} \frac{e^{t(i-r)}\theta^i}{i!} e^{-\theta} = \inf_{t > 0} e^{-\theta} e^{\theta e^{t-r} t} \sum_{i=0}^{\infty} \frac{(\theta e^{t})^i}{i!} e^{-\theta e^{t-r} t},
$$

where the infimum is achieved at $t = \ln \left(\frac{\theta}{\theta e^{t-r}}\right) > 0$. For this value of $t$, we have $e^{-\theta} e^{\theta e^{t-r} t} = e^{-\theta} \left(\frac{\theta e^r}{r}\right)^r$. Hence, we have $\Pr\{\overline{X}_n \geq z\} \leq e^{-\theta} \left(\frac{\theta e^r}{r}\right)^r = \exp(n.\mathcal{M}(z, \lambda))$.

Similarly, for any number $z \in (0, \lambda)$, we have $\Pr\{\overline{X}_n \leq z\} \leq \exp(n.\mathcal{M}(z, \lambda))$.
\end{proof}

\begin{lemma}
$1 - S_\mathcal{P}(\gamma - 1, \frac{\gamma}{1+\varepsilon}) + S_\mathcal{P}(\gamma - 1, \frac{\gamma}{1-\varepsilon}) < 2 \left[e^\varepsilon (1 + \varepsilon)^{-1+\varepsilon}\right]^{\gamma/(1+\varepsilon)}$.
\end{lemma}

\begin{proof}
Let $K^+$ be a Poisson random variable with mean value $\frac{\gamma}{1+\varepsilon}$. Let $K^-$ be a Poisson random variable with mean value $\frac{\gamma}{1-\varepsilon}$. Then, we have $\Pr\{K^+ \geq \gamma\} = 1 - S_\mathcal{P}(\gamma - 1, \frac{\gamma}{1+\varepsilon})$ and $\Pr\{K^- < \gamma\} = S_\mathcal{P}(\gamma - 1, \frac{\gamma}{1-\varepsilon})$. Applying Lemma 31, we have

$$
\Pr\{K^+ \geq \gamma\} \leq \left[e^\varepsilon (1 + \varepsilon)^{-1+\varepsilon}\right]^{\gamma/(1+\varepsilon)}, \quad \Pr\{K^- < \gamma\} \leq \left[e^{-\varepsilon} (1 - \varepsilon)^{-1-\varepsilon}\right]^{\gamma/(1-\varepsilon)}.
$$

111
It follows that
\[
1 - Sp\left(\gamma - 1, \frac{\gamma}{1 + \varepsilon}\right) + Sp\left(\gamma - 1, \frac{\gamma}{1 - \varepsilon}\right) = Pr\{K^+ \geq \gamma\} + Pr\{K^- < \gamma\}
\leq \left[e^\varepsilon(1 + \varepsilon)^{-1(1+\varepsilon)}\right]^{\gamma/(1+\varepsilon)} + \left[e^{-\varepsilon}(1 - \varepsilon)^{-1(1-\varepsilon)}\right]^{\gamma/(1-\varepsilon)}
\leq 2\left[e^\varepsilon(1 + \varepsilon)^{-1(1+\varepsilon)}\right]^{\gamma/(1+\varepsilon)}.
\]

\[\square\]

**Lemma 33** For \(\ell = 1, \ldots, s - 1\), there exists a unique number \(z_\ell \in (0, 1]\) such that \(\mathcal{M}_1(z_\ell, \frac{z_\ell}{1+\varepsilon}) = \frac{\ln(\zeta\delta)}{\gamma_\ell}\). Moreover, \(z_1 > z_2 > \cdots > z_{s-1}\).

**Proof.** By the definition of \(\gamma_\ell\), we have
\[
\left[\frac{\ln(\zeta\delta)}{-\ln(1+\varepsilon)}\right] \leq \gamma_\ell < \gamma_s = \left[\frac{\ln(\zeta\delta)}{1+\varepsilon} - \ln(1+\varepsilon)\right],
\]
which implies \(\frac{\ln(\zeta\delta)}{-\ln(1+\varepsilon)} \leq \gamma_\ell < \frac{\ln(\zeta\delta)}{1+\varepsilon} - \ln(1+\varepsilon)\). Making use of this inequality and the fact
\[
\lim_{z \to 0} \mathcal{M}_1(z, \frac{z}{1+\varepsilon}) = \frac{\varepsilon}{1+\varepsilon} - \ln(1+\varepsilon) < 0, \quad \lim_{z \to 1} \mathcal{M}_1(z, \frac{z}{1+\varepsilon}) = -\ln(1+\varepsilon) < 0,
\]
we have
\[
\lim_{z \to 1} \mathcal{M}_1(z, \frac{z}{1+\varepsilon}) \leq \frac{\ln(\zeta\delta)}{\gamma_\ell} \leq \lim_{z \to 0} \mathcal{M}_1(z, \frac{z}{1+\varepsilon}).
\]
By Lemma 22, \(\mathcal{M}_1(z, \frac{z}{1+\varepsilon})\) is monotonically decreasing with respect to \(z \in (0, 1]\). Hence, there exists a unique number \(z_\ell \in (0, 1]\) such that \(\mathcal{M}_1(z_\ell, \frac{z_\ell}{1+\varepsilon}) = \frac{\ln(\zeta\delta)}{\gamma_\ell}\).

To show that \(z_\ell\) decreases with respect to \(\ell\), we introduce function \(F(z, \gamma) = \gamma \cdot \mathcal{M}_1(z, \frac{z}{1+\varepsilon}) - \ln(\zeta\delta)\). Clearly,
\[
\frac{dz}{d\gamma} = \frac{\partial}{\partial z} \mathcal{M}_1(z, \frac{z}{1+\varepsilon}) = \frac{\mathcal{M}_1(z, \frac{z}{1+\varepsilon})}{\gamma \mathcal{M}_1(z, \frac{z}{1+\varepsilon})}.
\]
As can be seen from Lemma 22 and the fact \(\lim_{z \to 0} \mathcal{M}_1(z, \frac{z}{1+\varepsilon}) < 0\), we have \(\mathcal{M}_1(z, \frac{z}{1+\varepsilon}) < 0\) and \(\frac{\partial}{\partial z} \mathcal{M}_1(z, \frac{z}{1+\varepsilon}) < 0\) for any \(z \in (0, 1]\). It follows that \(\frac{dz}{d\gamma}\) is negative and consequently \(z_1 > z_2 > \cdots > z_{s-1}\). The proof of the lemma is thus completed.

\[\square\]

**Lemma 34** \(\{D_\ell = 1\} \leq \left\{\mathcal{M}_1(\hat{p}_\ell, \hat{p}_\ell) \leq \frac{\ln(\zeta\delta)}{\gamma_\ell}, \mathcal{M}_1(\hat{p}_\ell, \hat{p}_\ell) \leq \frac{\ln(\zeta\delta)}{\gamma_\ell}\right\}\) for \(\ell = 1, \ldots, s\).

**Proof.** The lemma is a direct consequence of Lemma 23.

\[\square\]

**Lemma 35** \(D_s = 1\).
Lemma 36 \( \{D_\ell = 1\} = \Pr\{\hat{p}_\ell \geq z_\ell\} \) for \( \ell = 1, \ldots, s - 1 \).

Proof. By Lemma 36 for \( \ell = 1, \ldots, s - 1 \), there exists a unique number \( z_\ell \in (0, 1) \) such that \( \mathcal{M}_1(z_\ell, \frac{\gamma_s}{1+\varepsilon}) = \frac{\ln(\delta)}{\gamma_s} \). From Lemma 22, we know that \( \mathcal{M}_1(z, \frac{\gamma_s}{1+\varepsilon}) \) is monotonically decreasing with respect to \( z \in (0, 1) \). It follows that \( \mathcal{M}_1(z, \frac{\gamma_s}{1+\varepsilon}) \leq \frac{\ln(\delta)}{\gamma_s} \) if and only if \( z \geq z_\ell \). This implies that \( \{D_\ell = 1\} = \{\mathcal{M}_1(\hat{p}_\ell, \frac{\gamma_s}{1+\varepsilon}) \leq \frac{\ln(\delta)}{\gamma_s}\} = \Pr\{\hat{p}_\ell \geq z_\ell\} \) for \( \ell = 1, \ldots, s - 1 \). The lemma is thus proved.

Lemma 37 If \( \zeta \) is sufficiently small, then \( 1 - S_p(\gamma_s - 1, \frac{\gamma_s}{1+\varepsilon}) + S_p(\gamma_s - 1, \frac{\gamma_s}{1+\varepsilon}) < \delta \), inequality (22) is satisfied and \( \Pr\left\{\frac{\hat{p} - p}{p} \leq \varepsilon\right\} \geq 1 - \delta \) for any \( p \in (0, p^*) \).

Proof. It is obvious that inequality (22) is satisfied if \( \zeta \) is sufficiently small. By Lemma 22, we have \( 1 - S_p(\gamma_s - 1, \frac{\gamma_s}{1+\varepsilon}) + S_p(\gamma_s - 1, \frac{\gamma_s}{1+\varepsilon}) < \delta \). By the definition of \( \gamma_s \), we have \( \gamma_s = \frac{(1+\varepsilon)\ln(\delta)}{\varepsilon'(1+\varepsilon)} < \frac{(1+\varepsilon)\ln(\delta)}{\varepsilon'(1+\varepsilon)}, \) which implies \( 1 - S_p(\gamma_s - 1, \frac{\gamma_s}{1+\varepsilon}) + S_p(\gamma_s - 1, \frac{\gamma_s}{1+\varepsilon}) < 2\left[\frac{\varepsilon(1+\varepsilon)}{(1+\varepsilon)}\right]^{\gamma_s/(1+\varepsilon)} \leq 2\zeta \delta \). It follows that \( 1 - S_p(\gamma_s - 1, \frac{\gamma_s}{1+\varepsilon}) + S_p(\gamma_s - 1, \frac{\gamma_s}{1+\varepsilon}) < \delta \) if \( \zeta \) is sufficiently small. From now on and throughout the proof of the lemma, we assume that \( \zeta \) is small enough to guarantee \( 1 - S_p(\gamma_s - 1, \frac{\gamma_s}{1+\varepsilon}) + S_p(\gamma_s - 1, \frac{\gamma_s}{1+\varepsilon}) < \delta \) and inequality (22). Applying Lemma 36 and (78) of Lemma 25, we have

\[
\Pr \left\{ \left| \frac{\hat{p} - p}{p} \right| > \varepsilon, \ l = \ell \right\} \leq \Pr \{ I = \ell \} \leq \Pr \{ D_\ell = 1 \} = \Pr \{ \hat{p}_\ell \geq z_\ell \} \leq \exp(\gamma_s \cdot \mathcal{M}_1(z_\ell, p)) \tag{79}
\]

for \( 0 < p < z_{s-1} \) and \( \ell = 1, \ldots, s - 1 \). On the other hand, noting that

\[
\Pr \left\{ \left| \frac{\hat{p} - p}{p} \right| > \varepsilon, \ l = s \right\} = \Pr \left\{ \frac{\hat{p}}{\gamma_s} - \frac{p}{p} \right| > \varepsilon, \ l = s \right\} \leq \Pr \left\{ \frac{\hat{p}}{\gamma_s} - \frac{p}{p} > \varepsilon \right\}
\]

and that \( \gamma_s \geq \left[(1 + \varepsilon + \sqrt{1 + 4\varepsilon + \sigma^2}) / (2\varepsilon) \right]^2 + \frac{1}{2} \) as a consequence of (22) and the definition of \( \gamma_s \), we can apply Lemma 30 to obtain

\[
\Pr \left\{ \left| \frac{\hat{p} - p}{p} \right| > \varepsilon, \ l = s \right\} < 1 - S_p \left( \gamma_s - 1, \frac{\gamma_s}{1+\varepsilon} \right) + S_p \left( \gamma_s - 1, \frac{\gamma_s}{1-\varepsilon} \right) < \delta. \tag{80}
\]

Noting that \( \frac{\partial \mathcal{M}_1(z, p)}{\partial p} = \frac{z - p}{z - p(1-p)} > 0 \) for any \( p \in (0, z) \) and that \( \lim_{p \to 0} \mathcal{M}_1(z, p) = -\infty \), we have that \( \sum_{\ell=1}^{s-1} \exp(\gamma_s \cdot \mathcal{M}_1(z_\ell, p)) \) decreases monotonically to 0 as \( p \) decreases from \( z_{s-1} \) to 0.
Since $1 - SP(\gamma_s - 1, \frac{1}{1 + \varepsilon}) + SP(\gamma_s - 1, \frac{1}{1 - \varepsilon}) < \delta$, there exists a unique number $p^* \in (0, z_{s-1})$ such that $1 - SP(\gamma_s - 1, \frac{1}{1 + \varepsilon}) + \sum_{\ell=1}^{s-1} \exp(\gamma_\ell M_1(z_\ell, p^*)) = \delta$. It follows that $1 - SP(\gamma_s - 1, \frac{1}{1 + \varepsilon}) + \sum_{\ell=1}^{s-1} \exp(\gamma_\ell M_1(z_\ell, p^*)) \leq \delta$ for any $p \in (0, p^*]$. Combining (19) and (50), we have $\Pr \{|\hat{p} - p| > \varepsilon p\} < 1 - SP(\gamma_s - 1, \frac{1}{1 + \varepsilon}) + \sum_{\ell=1}^{s-1} \exp(\gamma_\ell M_1(z_\ell, p)) \leq \delta$ for any $p \in (0, p^*]$. This completes the proof of the lemma.

We are now in a position to prove Theorem 19. Clearly, $\hat{p}_\ell = \frac{\sum_{i=1}^n X_i}{n} + i\varepsilon$ is a ULE of $p$ for $\ell = 1, \ldots, s$. Moreover, $\inf_{t > 0} e^{-t\varepsilon} \mathbb{E}[e^{t\hat{p}_\ell}] = \exp(\gamma_\ell M_1(z_\ell, p))$ for $\ell = 1, \ldots, s$. Define a random interval with lower limit $\mathcal{L}(\hat{p}_\ell) = \frac{\hat{p}_\ell}{1 + \varepsilon}$ and upper limit $\mathbb{U}(\hat{p}_\ell) = \frac{\hat{p}_\ell}{1 - \varepsilon}$ for $\ell = 1, \ldots, s$. Then, $\{\mathcal{L}(\hat{p}_\ell) \leq \hat{p}_\ell \leq \mathbb{U}(\hat{p}_\ell)\}$ is a sure event for $\ell = 1, \ldots, s$. By virtue of these facts and Lemmas 34 and 35, we have that the sampling scheme satisfies requirements (i) – (v) described in Corollary 1 from which (20) and (21) follow immediately. By Lemma 37 there exists a positive number $\zeta_0$ such that $1 - SP(\gamma_s - 1, \frac{1}{1 + \varepsilon}) + \sum_{\ell=1}^{s-1} \exp(\gamma_\ell M_1(z_\ell, p)) < \delta$, inequality (22) is satisfied and $\Pr\{|\hat{p} - p| \leq \varepsilon p | p\} \geq 1 - \delta$ for any $p \in (0, p^*]$ if $0 < \zeta < \zeta_0$. Hence, by restricting $\zeta$ to be less than $\zeta_0$, we can guarantee $\Pr\{|\hat{p} - p| \leq \varepsilon p | p\} \geq 1 - \delta$ for any $p \in (0, 1)$ by ensuring $\Pr\{|\hat{p} - p| \leq \varepsilon p | p\} \geq 1 - \delta$ for any $p \in [p^*, 1]$. This completes the proof of Theorem 19.

I.8 Proof of Theorem 20

We need some preliminary results.

Lemma 38 $\{D_\ell = 1\} \subseteq \{M_1(\hat{p}_\ell, \hat{p}_{\ell+1}) \leq \frac{\ln(\zeta\delta)}{\gamma_\ell}, M_1(\hat{p}_\ell, \hat{p}_{\ell-2}) \leq \frac{\ln(\zeta\delta)}{\gamma_\ell}\}$ for $\ell = 1, \ldots, s$.

Proof. For simplicity of notations, define $M_1(z, \mu) = \frac{M(z, \mu)}{z}$. By tedious computation, we can show that $\{D_\ell = 1\} = \{M_1(\hat{p}_\ell, \hat{p}_{\ell+1}) \leq \frac{\ln(\zeta\delta)}{\gamma_\ell}\}$ for $\ell = 1, \ldots, s$. Noting that

$$M_1\left(z, \frac{z}{1 + \varepsilon}\right) - M_1\left(z, \frac{z}{1 - \varepsilon}\right) = \frac{2\varepsilon^3 (2 - z)}{3 (1 + \varepsilon) (1 + \varepsilon) (1 - \varepsilon) (1 - \varepsilon) (1 - \varepsilon)} > 0$$

for $0 < z < 1 - \varepsilon$, we have

$$\{D_\ell = 1\} = \left\{\begin{array}{cc}
M_1\left(\hat{p}_\ell, \hat{p}_{\ell+1}\right) & \leq \frac{\ln(\zeta\delta)}{\gamma_\ell}, M_1\left(\hat{p}_\ell, \hat{p}_{\ell-2}\right) \leq \frac{\ln(\zeta\delta)}{\gamma_\ell}\end{array}\right\}$$

for $\ell = 1, \ldots, s$. This completes the proof of the lemma.

Lemma 39 $D_s = 1$. 114
Lemma 40

We need the following lemma.

I.10 Proof of Theorem 23

\[ \gamma M \]  
\[ 0 < \rho_{s}(\omega) \leq 1 \] for any \( \omega \in \Omega \) and \( \{ D_{s} = 1 \} = \{ M_{1}(\hat{p}_{s}, \frac{\hat{p}_{s}}{1+\varepsilon}) \leq \frac{\ln(\delta)}{\gamma_s} \} \) as asserted by Lemma 35.

By the definition of sample sizes, we have \( \gamma_s = \frac{\gamma}{(1 + \varepsilon)} \frac{\ln(\delta)}{\gamma_s} \geq 2 (1 + \varepsilon) \left( 1 + \varepsilon \right)^{\frac{\ln(\delta)}{\gamma_s}} \).

Since \( \lim_{z \to 0} M_{1}(z, \frac{z}{1+\varepsilon}) = -\varepsilon^2 \left( \frac{1}{1+\varepsilon} \right)(1 + \varepsilon) \left( 1 + \varepsilon \right)^{\frac{\ln(\delta)}{\gamma_s}} < 0 \), we have \( \lim_{z \to 0} M_{1}(z, \frac{z}{1+\varepsilon}) \leq \frac{\ln(\delta)}{\gamma_s} \).

Note that \( M_{1}(z, \frac{z}{1+\varepsilon}) \) is monotonically decreasing with respect to \( z \in (0, 1) \). Hence, \( M_{1}(z, \frac{z}{1+\varepsilon}) \leq \frac{\ln(\delta)}{\gamma_s} \) for any \( z \in (0, 1) \). Since \( M_{1}(z, \frac{z}{1+\varepsilon}) \) is a continuous function with respect to \( z \in (0, 1) \) and \( M_{1}(1, \frac{1}{1+\varepsilon}) = \lim_{z \to 1} M_{1}(z, \frac{z}{1+\varepsilon}) \), it must be true that \( M_{1}(1, \frac{1}{1+\varepsilon}) \leq \frac{\ln(\delta)}{\gamma_s} \). This completes the proof of the lemma.

Finally, by virtue of the above preliminary results and a similar method as that of Theorem 19, we can establish Theorem 20.

I.9 Proof of Theorem 21

Since \( \Pr\{ n \geq i \} \) depends only on \( X_1, \ldots, X_i \) for all \( i \geq 1 \), we have, by Wald’s equation, \( \mathbb{E}[X_1 + \cdots + X_n] = \mathbb{E}[X_1] \mathbb{E}[n] = p \mathbb{E}[n] \). By the definition of the sampling scheme, \( X_1 + \cdots + X_n = \gamma \), and it follows that \( \mathbb{E}[X_1 + \cdots + X_n] = \gamma \). Hence, \( p \mathbb{E}[n] = \mathbb{E}[\gamma] \), leading to the first identity.

The second identity is shown as follows. Let \( I \) be the index of stage when the sampling is stopped. Then, setting \( \gamma_0 = 0 \), we have

\[
\sum_{i=1}^{s} (\gamma_i - \gamma_{i-1}) \Pr\{ I \geq i \} = \sum_{i=1}^{s} \gamma_i \Pr\{ I \geq i \} - \sum_{i=1}^{s} \gamma_{i-1} \Pr\{ I \geq i \} \\
= \sum_{i=1}^{s} \gamma_i \Pr\{ I \geq i \} - \sum_{j=0}^{s-1} \gamma_j \Pr\{ I \geq j \} + \sum_{j=0}^{s-1} \gamma_j \Pr\{ I = j \} \\
= \gamma_s \Pr\{ I \geq s \} + \sum_{j=0}^{s-1} \gamma_j \Pr\{ I = j \} = \sum_{i=1}^{s} \gamma_i \Pr\{ I = i \} = \mathbb{E}[\gamma] = \mathbb{E}[\gamma].
\]

This completes the proof of Theorem 21.

I.10 Proof of Theorem 22

We need the following lemma.

Lemma 40 \( \mathcal{M}_B(z, \frac{z}{1+\varepsilon}) \) is monotonically decreasing from 0 to \( \ln \frac{1}{1+\varepsilon} \) as \( z \) increases from 0 to 1.

Proof. The lemma can be established by verifying that

\[
\lim_{z \to 0} \mathcal{M}_B \left( z, \frac{z}{1+\varepsilon} \right) = 0, \quad \lim_{z \to 1} \mathcal{M}_B \left( z, \frac{z}{1+\varepsilon} \right) = \ln \frac{1}{1+\varepsilon}, \quad \frac{\partial}{\partial z} \mathcal{M}_B \left( z, \frac{z}{1+\varepsilon} \right) = \ln \frac{1}{1+\varepsilon} + \frac{\varepsilon}{1+\varepsilon} < 0
\]

and \( \frac{\partial^2}{\partial z^2} \mathcal{M}_B \left( z, \frac{z}{1+\varepsilon} \right) = \frac{-3z^2}{(5-1)(1+\varepsilon-z^2)} < 0 \) for any \( z \in (0, 1) \).
Let $0 < \eta < 1$ and $r = \inf_{\ell > 0} \frac{n_{\ell + 1}}{n_{\ell}}$. By the assumption that $r > 1$, we have that there exists a number $\ell' > \max\{\tau, \tau + \frac{2}{\ell - 1} + \frac{\ln(\delta)}{\ln 2}\}$ such that $\frac{n_{\ell + 1}}{n_{\ell}} > \frac{\ell' + 1}{2}$ for any $\ell > \ell'$. Noting that $\frac{\ln(\delta)}{n_{\ell}}$ is negative for any $\ell > 0$ and that

$$\frac{\ln(\delta_{\ell + 1})}{n_{\ell}} < \frac{2}{r + 1} \frac{(\ell + 1 - \tau) \ln 2 - \ln(\delta)}{(\ell - \tau) \ln 2 - \ln(\delta)} = \frac{2}{r + 1} \left( 1 + \frac{1}{\ell - \tau - \ln(\delta)}/\ln 2 \right) < 1$$

for $\ell > \ell'$, we have that $\ln(\delta_{\ell'})$ is monotonically increasing with respect to $\ell$ greater than $\ell'$. In view of such monotonicity and the fact that $\ln(\delta_{\ell'}) = \ln(\delta_{\ell - \tau - \ln(\delta)}) \to 0 > M_B(\eta p, \frac{np}{\ell + 1})$ as $\ell \to \infty$, we have that there exists an integer $\kappa$ greater than $\ell'$ such that $M_B(\eta p, \frac{np}{\ell + 1}) < \ln(\delta_{\ell'})$ for all $\ell \geq \kappa$. For $\ell$ no less than such $\kappa$, we claim that $z > \eta p$ if $M_B(z, \frac{r}{1 + \tau}) > \ln(\delta_{\ell'})$ and $z \in [0, 1]$. To prove this claim, suppose, to get a contradiction, that $z \geq \eta p$. Then, since $M_B(z, \frac{r}{1 + \tau})$ is monotonically decreasing with respect to $z \in (0, 1)$ as asserted by Lemma III, we have $M_B(z, \frac{r}{1 + \tau}) \leq M_B(\eta p, \frac{np}{\ell + 1}) < \ln(\delta_{\ell'})$, which is a contradiction. Therefore, we have shown the claim and it follows that $\{M_B(\frac{K_\ell}{n_{\ell}}, \frac{K_{\ell + 1}}{n_{\ell + 1}}) > \ln(\delta_{\ell'}) \} \subseteq \{K_\ell < \eta p n_\ell\}$ for $\ell \geq \kappa$. So,

$$\Pr\{I > \ell\} \leq \Pr\left\{M_B\left(\frac{K_\ell}{n_{\ell}}, \frac{K_{\ell + 1}}{n_{\ell + 1}}\right) > \ln(\delta_{\ell'})/n_{\ell}\right\} \leq \Pr\{K_\ell < \eta p n_\ell\} < \exp\left(-\frac{(1 - \eta)^2 p n_\ell}{2}\right)$$

for large enough $\ell$, where the last inequality is due to the multiplicative Chernoff bound [24]. Since $\Pr\{I > \ell\} < \exp(-\frac{(1 - \eta)^2 p n_\ell}{2})$ for sufficiently large $\ell$ and $n_\ell \to \infty$ as $\ell \to \infty$, we have $\Pr\{I < \infty\} = 1$ or equivalently, $\Pr\{n < \infty\} = 1$. This completes the proof of statement (I).

### I.10.2 Proof of Statement (II)

In the course of proving Statement (I), we have shown that there exists an integer $\kappa$ such that $\Pr\{I > \ell\} < \exp(-cn_\ell)$ for any $\ell \geq \kappa$, where $c = \frac{(1 - \eta)^2 p}{2}$. Note that

$$\mathbb{E}[n] = n_1 + \sum_{\ell = 1}^{\kappa} (n_{\ell + 1} - n_\ell) \Pr\{I > \ell\} + \sum_{\ell = \kappa + 1}^{\infty} (n_{\ell + 1} - n_\ell) \Pr\{I > \ell\}.$$

Let $R = \sup_{\ell > 0} \frac{n_{\ell + 1}}{n_\ell}$. Then, $n_{\ell + 1} - n_\ell \leq R n_\ell$. Hence, if we choose $\kappa$ large enough such that $cn_1 r^{\kappa} > 1$, then

$$\sum_{\ell = \kappa + 1}^{\infty} (n_{\ell + 1} - n_\ell) \Pr\{I > \ell\} < \sum_{\ell = \kappa + 1}^{\infty} (n_{\ell + 1} - n_\ell) e^{-cn_\ell} \leq R \sum_{\ell = \kappa + 1}^{\infty} cn_\ell e^{-cn_\ell} \leq \frac{R}{c} \sum_{\ell = \kappa}^{\infty} c n_\ell r^\ell \exp(-cn_1 r^\ell)$$

$$< \frac{R}{c} \int_{\kappa - 1}^{\infty} c n_1 r^\ell \exp(-cn_1 r^\ell) d\ell = \frac{R \exp(-cn_1 r^{\kappa - 1})}{\ln r},$$

which implies that $\mathbb{E}[n] < \infty$.

### I.10.3 Proof of Statement (III)

By differentiation with respect to $\varepsilon \in (0, 1)$, we can show that $M_B(z, \frac{r}{1 + \varepsilon}) < M_B(z, \frac{r}{1 + \varepsilon})$ for $0 \leq z < 1 - \varepsilon$. It follows that $\{D_\ell = 1\} = \{M_B(\frac{p_\ell}{1 - \varepsilon}, \frac{p_\ell}{1 + \varepsilon}) \leq \frac{\ln(\delta_{\ell'})}{n_{\ell}}\} \leq \{M_B(\hat{p}_\ell, \frac{p_\ell}{1 + \varepsilon}) \leq \frac{\ln(\delta_{\ell'})}{n_{\ell}}\}$.
\[
\frac{\ln(\delta_1)}{n_\ell}, \quad \mathcal{M}_B(\hat{p}_\ell, \frac{\hat{p}_\ell}{1-\varepsilon}) \leq \frac{\ln(\delta_1)}{n_\ell} \}
\]
for \(\ell = 1, \ldots, s\). Hence, by the definition of the sampling scheme, we have

\[
\Pr \{ |\hat{p}_\ell - p| \geq \varepsilon p, \ l = \ell \ | p \} \leq \Pr \left\{ p \leq \frac{\hat{p}_\ell}{1 + \varepsilon}, \mathcal{M}_B(\hat{p}_\ell, \frac{\hat{p}_\ell}{1 + \varepsilon}) \leq \frac{\ln(\delta_1)}{n_\ell} | p \right\} + \Pr \left\{ p \geq \frac{\hat{p}_\ell}{1 - \varepsilon}, \mathcal{M}_B(\hat{p}_\ell, \frac{\hat{p}_\ell}{1 - \varepsilon}) \leq \frac{\ln(\delta_1)}{n_\ell} | p \right\}
\]
\[
\leq \Pr \left\{ p \leq \frac{\hat{p}_\ell}{1 + \varepsilon}, \mathcal{M}_B(\hat{p}_\ell, p) \leq \frac{\ln(\delta_1)}{n_\ell} | p \right\} + \Pr \left\{ p \geq \frac{\hat{p}_\ell}{1 - \varepsilon}, \mathcal{M}_B(\hat{p}_\ell, p) \leq \frac{\ln(\delta_1)}{n_\ell} | p \right\}
\]
\[
\leq \Pr \{ G_{\hat{p}_\ell}(\hat{p}_\ell, p) \leq \zeta \delta \ell | p \} + \Pr \{ F_{\hat{p}_\ell}(\hat{p}_\ell, p) \leq \zeta \delta \ell | p \}
\]
\[
\leq 2\zeta \delta \ell
\]
for any \(p \in (0, 1)\) and \(\ell = 1, 2, \ldots\). So, \(\sum_{\ell=l^*+1}^\infty \Pr \{ |\hat{p}_\ell - p| \geq \varepsilon p, \ l = \ell \ | p \} \leq 2\zeta \sum_{\ell=l^*+1}^\infty \delta \ell \leq 2(\tau + 1)\zeta \delta\), which implies that \(\Pr \{ |\hat{p} - p| < \varepsilon p | p \} \geq 1 - \delta\) provided that \(\zeta \leq \frac{1}{2(\tau + 1)}\).

### I.10.4 Proof of Statement (IV)

Recall that in the course of proving statement (III), we have shown that \(\Pr \{ |\hat{p}_\ell - p| \geq \varepsilon p, \ l = \ell \ | p \} \leq 2\zeta \delta \ell\) for any \(\ell > 0\). Making use of such result, we have \(\sum_{\ell=l^*+1}^\infty \Pr \{ |\hat{p}_\ell - p| \geq \varepsilon p, \ l = \ell \ | p \} \leq 2\zeta \sum_{\ell=l^*+1}^\infty \delta \ell \leq \eta\) for any \(p \in (0, 1)\). It follows that

\[
\Pr \{ |\hat{p} - p| \geq \varepsilon p | p \} = \sum_{\ell=1}^{l^*} \Pr \{ |\hat{p}_\ell - p| \geq \varepsilon p, \ l = \ell | p \} + \sum_{\ell=l^*+1}^\infty \Pr \{ |\hat{p}_\ell - p| \geq \varepsilon p, \ l = \ell | p \}
\]
\[
\leq \sum_{\ell=1}^{l^*} \Pr \{ |\hat{p}_\ell - p| \geq \varepsilon p, \ l = \ell | p \} + \eta
\]
\[
\leq \sum_{\ell=1}^{l^*} \Pr \{ l = \ell | p \} + \eta \leq \sum_{\ell=1}^{l^*} \Pr \{ \hat{p}_\ell \geq z_\ell | p \} + \eta
\]
\[
\leq \sum_{\ell=1}^{l^*} \exp(n_\ell \mathcal{M}_B(z_\ell, p)) + \eta < \sum_{\ell=1}^{l^*} \exp(n_\ell \mathcal{M}_B(z_\ell, p^*)) + \eta < \delta
\]
for any \(p \in (0, p^*)\).

Now we shall bound \(\Pr \{ p \leq \frac{\hat{p}}{1+\varepsilon} \} \) and \(\Pr \{ p \geq \frac{\hat{p}}{1-\varepsilon} \} \) for \(p \in [a, b] \subseteq (0, 1)\). Observing that \(\{ a \leq \hat{p} \leq b \} \subseteq \{ \hat{p} \geq \hat{b} \}\) as a consequence of \(b < a(1+\varepsilon)\), by statement (III) of Theorem 23, we have

\[
\Pr \left\{ b \leq \frac{\hat{p}}{1+\varepsilon}, \ l \leq \ell^* | a \right\} \leq \Pr \left\{ p \leq \frac{\hat{p}}{1+\varepsilon}, \ l \leq \ell^* | p \right\} \leq \Pr \left\{ a \leq \frac{\hat{p}}{1+\varepsilon}, \ l \leq \ell^* | b \right\}
\]
for any $p \in [a, b]$. On the other hand,

$$
\Pr \left\{ p \leq \frac{\hat{p}}{1 + \varepsilon}, l > \ell^* \mid p \right\} \leq \sum_{\ell = \ell^* + 1}^{\infty} \Pr \left\{ p \leq \frac{\hat{p}_\ell}{1 + \varepsilon}, \mathcal{M}_B \left( \hat{p}_\ell \right) \leq \frac{\ln(\zeta \delta)}{n_\ell} \mid p \right\}
$$

$$
\leq \sum_{\ell = \ell^* + 1}^{\infty} \Pr \left\{ p \leq \frac{\hat{p}_\ell}{1 + \varepsilon}, \mathcal{M}_B(p, \ell) \leq \frac{\ln(\zeta \delta)}{n_\ell} \mid p \right\}
$$

$$
\leq \sum_{\ell = \ell^* + 1}^{\infty} \Pr \left\{ \{G_{\hat{p}_\ell}(\hat{p}_\ell, p) \leq \zeta \delta \mid p\} \leq \zeta \sum_{\ell = \ell^* + 1}^{\infty} \delta_\ell \leq \frac{\eta}{2} \right\}
$$

for any $p \in [a, b]$. Therefore, $\Pr\{b \leq \frac{\hat{p}}{1 + \varepsilon}, l \leq \ell^* \mid a\} \leq \Pr\{p \leq \frac{\hat{p}}{1 + \varepsilon} \mid p\} = \Pr\{a \leq \frac{\hat{p}}{1 + \varepsilon}, l \leq \ell^* \mid b\} + \Pr\{p \leq \frac{\hat{p}}{1 + \varepsilon}, l > \ell^* \mid p\} \leq \Pr\{a \leq \frac{\hat{p}}{1 + \varepsilon}, l \leq \ell^* \mid b\} + \frac{\eta}{2}$ for any $p \in [a, b]$.

Similarly, observing that $\{b \geq \frac{\hat{p}}{1 + \varepsilon}\} \subseteq \{\hat{p} \leq a\}$ as a consequence of $b < a(1 + \varepsilon)$, by statement (IV) of Theorem 3, we have

$$
\Pr \left\{ a \geq \frac{\hat{p}}{1 - \varepsilon}, l \leq \ell^* \mid b \right\} \leq \Pr \left\{ p \geq \frac{\hat{p}}{1 - \varepsilon}, l \leq \ell^* \mid p \right\} \leq \Pr \left\{ b \geq \frac{\hat{p}}{1 - \varepsilon}, l \leq \ell^* \mid a \right\}
$$

for any $p \in [a, b]$. On the other hand,

$$
\Pr \left\{ p \geq \frac{\hat{p}}{1 - \varepsilon}, l > \ell^* \mid p \right\} \leq \sum_{\ell = \ell^* + 1}^{\infty} \Pr \left\{ p \geq \frac{\hat{p}_\ell}{1 - \varepsilon}, \mathcal{M}_B \left( \hat{p}_\ell \right) \leq \frac{\ln(\zeta \delta)}{n_\ell} \mid p \right\}
$$

$$
\leq \sum_{\ell = \ell^* + 1}^{\infty} \Pr \left\{ p \geq \frac{\hat{p}_\ell}{1 - \varepsilon}, \mathcal{M}_B(\hat{p}_\ell, p) \leq \frac{\ln(\zeta \delta)}{n_\ell} \mid p \right\}
$$

$$
\leq \sum_{\ell = \ell^* + 1}^{\infty} \Pr \left\{ \{F_{\hat{p}_\ell}(\hat{p}_\ell, p) \leq \zeta \delta \mid p\} \leq \zeta \sum_{\ell = \ell^* + 1}^{\infty} \delta_\ell \leq \frac{\eta}{2} \right\}
$$

for any $p \in [a, b]$. Therefore, $\Pr\{a \geq \frac{\hat{p}}{1 - \varepsilon}, l \leq \ell^* \mid b\} \leq \Pr\{p \geq \frac{\hat{p}}{1 - \varepsilon} \mid p\} = \Pr\{b \geq \frac{\hat{p}}{1 - \varepsilon}, l \leq \ell^* \mid a\} + \Pr\{p \geq \frac{\hat{p}}{1 - \varepsilon}, l > \ell^* \mid p\} \leq \Pr\{b \geq \frac{\hat{p}}{1 - \varepsilon}, l \leq \ell^* \mid a\} + \frac{\eta}{2}$ for any $p \in [a, b]$. This completes the proof of statement (IV).

### I.10.5 Proof of Statement (V)

We need a preliminary result.

**Lemma 41** Let $p \in (0, 1)$ and $\eta \in (0, 1)$. Let $\kappa$ be an integer greater than $\max\{\tau, \tau + \frac{1}{\gamma - 1} + \frac{\ln(\delta)}{\ln 2}\}$ such that $\mathcal{M}_B(\eta p, \frac{mp}{1 + \varepsilon}) < \frac{\ln(\zeta \delta)}{m_\ell}$. Then, $\Pr\{l > \ell\} < \exp\left(-\frac{(1-\eta)^2 p_{\max}}{2}\right)$ for any $\ell \geq \kappa$.

**Proof.** Let $m_\ell = m_\gamma^{\ell - 1}$ for $\ell = 1, 2, \ldots$. Noting that

$$
\frac{\ln(\zeta \delta)}{m_\ell} = \frac{1}{\gamma} \times \frac{(\ell + 1 - \tau) \ln 2 - \ln(\zeta \delta)}{(\ell - \tau) \ln 2 - \ln(\zeta \delta)} \leq \frac{1}{\gamma} \times \left(1 + \frac{1}{\ell - \tau - \frac{\ln(\zeta \delta)}{\ln 2}}\right) < 1
$$

for $\ell > \max\{\tau, \tau + \frac{1}{\gamma - 1} + \frac{\ln(\delta)}{\ln 2}\}$ and that $\frac{\ln(\zeta \delta)}{m_\ell} = \frac{\ln(\zeta \delta)^{\ell - 1}}{m_\gamma^{\ell - 1}} \rightarrow 0$ as $\ell \rightarrow \infty$, we have that there exists an integer $\kappa$ greater than $\max\{\tau, \tau + \frac{1}{\gamma - 1} + \frac{\ln(\delta)}{\ln 2}\}$ such that $\mathcal{M}_B(\eta p, \frac{mp}{1 + \varepsilon}) < \frac{\ln(\zeta \delta)}{m_\ell}$
for all $\ell \geq \kappa$. Since $m_\ell \leq n_\ell$ and $\mathcal{A}_B(n_\ell, \frac{np}{1+\varepsilon}) < 0$, we have that there exists an integer $\kappa$ greater than $\max\{\tau, \tau + \frac{1}{\gamma - 1} + \frac{\ln(\delta)}{\ln 2}\}$ such that $\mathcal{A}_B(n_\ell, \frac{np}{1+\varepsilon}) < \frac{\ln(\delta)}{n_\ell}$ for all $\ell \geq \kappa$. For $\ell$ greater than such $\kappa$, we claim that $z < np$ if $\mathcal{A}_B(z, \frac{np}{1+\varepsilon}) < \frac{\ln(\delta)}{n_\ell}$ and $z \in [0,1]$. To prove this claim, suppose, to get a contradiction, that $z \geq np$. Then, since $\mathcal{A}_B(z, \frac{np}{1+\varepsilon})$ is monotonically decreasing with respect to $z \in (0,1)$ as asserted by Lemma 40, we have $\mathcal{A}_B(z, \frac{np}{1+\varepsilon}) \leq \mathcal{A}_B(n_\ell, \frac{np}{1+\varepsilon}) = \frac{\ln(\delta)}{n_\ell}$, which is a contradiction. Therefore, we have shown the claim and it follows that $\{\mathcal{A}_B(n_\ell, \frac{K_\ell}{n_\ell}, \frac{K_\ell}{(1+\varepsilon)n_\ell}) > \frac{\ln(\delta)}{n_\ell}\} \subseteq \{K_\ell < npn_\ell\}$ for $\ell \geq \kappa$. So,

$$\Pr\{l > \ell\} \leq \Pr\left\{\mathcal{A}_B\left(\frac{K_\ell}{n_\ell}, \frac{K_\ell}{(1+\varepsilon)n_\ell}\right) > \frac{\ln(\delta)}{n_\ell}\right\} \leq \Pr\{K_\ell < npn_\ell\} < \exp\left(-\frac{(1-\eta)^2pn_\ell}{2}\right),$$

where the last inequality is due to the multiplicative Chernoff bound [21].

\[\square\]

We are now in position to prove statement (V) of the theorem. Note that

$$\mathbb{E}[n] = n_1 + \sum_{\ell=1}^{\kappa} (n_{\ell+1} - n_\ell) \Pr\{l > \ell\} + \sum_{\ell=\kappa+1}^{\infty} (n_{\ell+1} - n_\ell) \Pr\{l > \ell\}.$$

By the definition of $n_\ell$, we have $n_{\ell+1} - n_\ell \leq (\gamma - 1)n_\ell$. By the assumption of $\epsilon$, $\eta$ and $\kappa$, we have $\ln \frac{n_\ell}{n_1} > 1$ and thus $\kappa > \frac{1}{\ln \gamma} \ln \left(\frac{1}{c_{m\gamma}} \ln \frac{n_\ell}{n_1}\right) + 1 > \frac{1}{\ln \gamma} \ln \left(\frac{1}{c_{m\gamma}}\right) + 1$, which implies that $cm\gamma^{\kappa-1} > 1$ and $\exp(-cm\gamma^{\kappa-1}) < \epsilon$. Hence, by Lemma 41, we have

$$\sum_{\ell=\kappa+1}^{\infty} (n_{\ell+1} - n_\ell) \Pr\{l > \ell\} < \sum_{\ell=\kappa+1}^{\infty} (n_{\ell+1} - n_\ell) e^{-cn_\ell} \leq \frac{\gamma - 1}{c} \sum_{\ell=\kappa+1}^{\infty} cn_\ell e^{-cn_\ell} \leq \frac{\gamma - 1}{c} \int_{\kappa}^{\infty} cm\gamma^\ell \exp(-cm\gamma^\ell) d\ell.$$

Making a change of variable $x = cm\gamma^\ell$, we have $d\ell = \frac{1}{\ln \gamma} \frac{dx}{x}$ and

$$\int_{\kappa-1}^{\infty} cm\gamma^\ell \exp(-cm\gamma^\ell) d\ell = \frac{1}{\ln \gamma} \int_{cm\gamma^{\kappa-1}}^{\infty} e^{-x} dx = \frac{\exp(-cm\gamma^{\kappa-1})}{\ln \gamma}.$$

It follows that $\sum_{\ell=\kappa+1}^{\infty} (n_{\ell+1} - n_\ell) \Pr\{l > \ell\} < \frac{\gamma - 1}{c} \frac{\exp(-cm\gamma^{\kappa-1})}{\ln \gamma} < \frac{\gamma}{c} \exp(-cm\gamma^{\kappa-1}) < \epsilon$. This completes the proof of statement (V) of Theorem 23.

I.11 Proof of Theorem 25

We need some preliminary results.

Lemma 42 $\lim_{\varepsilon \to 0} \sum_{\ell=1}^{s} \gamma_\ell \ e^{-\gamma_\ell c} = 0$ for any $c > 0$. 

119
Proof. By differentiation, it can be shown that \( xe^{-xc} \) is monotonically increasing with respect to \( x \in (0, \frac{1}{c}) \) and monotonically decreasing with respect to \( x \in (\frac{1}{c}, \infty) \). Since \( \gamma_\epsilon \geq \gamma_1 \geq \frac{\ln \frac{1}{\epsilon}}{\ln(1+\epsilon)} \) is greater than \( \frac{1}{c} \) for small enough \( \epsilon > 0 \), we have that \( \sum_{\ell=1}^{s} \gamma_\epsilon e^{-\gamma_\epsilon c} \leq s \gamma_1 e^{-\gamma_1 c} \) if \( \epsilon > 0 \) is sufficiently small. Let \( \rho = \inf_{\epsilon>0} \frac{C_{\epsilon}^{1,4}}{C_{\epsilon}^{1,1}} - 1 \). Observing that \( s \leq 1 + \left[ \frac{1}{\ln(1+\rho) \ln \left( \frac{\ln(1+\epsilon)}{\ln(1+\epsilon) - \frac{1}{c\epsilon}} \right) } \right] < 1 + \frac{1}{\ln(1+\rho) \ln \left( \frac{\ln(1+\epsilon)}{\ln(1+\epsilon) - \frac{1}{c\epsilon}} \right) } \) and \( \gamma_1 \geq \frac{\ln \frac{1}{\epsilon}}{\ln(1+\epsilon)} \), we have

\[
\sum_{\ell=1}^{s} \gamma_\epsilon e^{-\gamma_\epsilon c} \leq \left[ 1 + \frac{\ln \left( \frac{\ln(1+\epsilon)}{\ln(1+\epsilon) - \frac{1}{c\epsilon}} \right) }{\ln(1+\rho)} \right] \exp \left( -\frac{c \ln \frac{1}{\epsilon}}{\ln(1+\epsilon)} \right) = A(\epsilon) + \frac{\ln \frac{1}{\epsilon}}{\ln(1+\rho)} B(\epsilon)
\]

for small enough \( \epsilon > 0 \), where \( A(\epsilon) = \frac{c \ln \frac{1}{\epsilon}}{\ln(1+\epsilon)} \exp \left( -\frac{c \ln \frac{1}{\epsilon}}{\ln(1+\epsilon)} \right) \) and \( B(\epsilon) = \frac{\ln \left( \frac{\ln(1+\epsilon)}{\ln(1+\epsilon) - \frac{1}{c\epsilon}} \right) }{\ln(1+\rho)} \exp \left( -\frac{c \ln \frac{1}{\epsilon}}{\ln(1+\epsilon)} \right) \). Noting that \( \lim_{x \to \infty} xe^{-x} = 0 \) and that \( \frac{c \ln \frac{1}{\epsilon}}{\ln(1+\epsilon)} \to \infty \) as \( \epsilon \to 0 \), we have \( \lim_{\epsilon \to 0} A(\epsilon) = 0 \). Now we show that \( \lim_{\epsilon \to 0} B(\epsilon) = 0 \). Using Taylor’s expansion formula \( \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + o(x^3) \), we have

\[
\frac{\ln(1+\epsilon)}{\ln(1+\epsilon) - \frac{1}{c\epsilon}} = \frac{\epsilon - \frac{\epsilon^2}{2} + o(\epsilon^2)}{\epsilon - \frac{\epsilon^2}{2} + \frac{\epsilon^3}{3} + o(\epsilon^3) - \epsilon[1 - \frac{\epsilon^2}{2} + o(\epsilon^2)]} = \frac{\epsilon - \frac{\epsilon^2}{2} + o(\epsilon^2)}{\frac{\epsilon^2}{2} - \frac{\epsilon^3}{3} + o(\epsilon^3)}
\]

and

\[
\frac{\ln \left( \frac{\ln(1+\epsilon)}{\ln(1+\epsilon) - \frac{1}{c\epsilon}} \right) }{\ln(1+\epsilon)} = \frac{\ln \left( \frac{\ln \left( \frac{1}{\epsilon} + o(1) \right) }{\ln(1+\epsilon)} \right) }{\ln(1+\epsilon)} = \frac{\ln \left( \frac{1}{\epsilon} + o(1) \right) }{\ln(1+\epsilon)} = \frac{\ln \left( \frac{1}{\epsilon} \right) }{\ln(1+\epsilon)} + \frac{5}{6} + o(1). \tag{81}
\]

Using (81) and the observation that \( \left[ \frac{\epsilon}{\ln(1+\epsilon)} \right] \exp \left( -\frac{c \ln \frac{1}{\epsilon}}{\ln(1+\epsilon)} \right) = o(1) \), we have

\[
B(\epsilon) = o(1) + \frac{\frac{2}{\epsilon} + o(\epsilon)}{\ln(1+\epsilon)} = \frac{2}{\epsilon} + \frac{1 - \frac{\epsilon}{2} + o(\epsilon)}{1 + \frac{\epsilon}{2} + o(\epsilon)} = \frac{2}{\epsilon} + \frac{5}{6} + o(\epsilon) = o(1) + \frac{B^*(\epsilon)}{1 + o(1)} \left( \frac{1}{\epsilon} \right) \left( \frac{1}{\epsilon} \right) = o(1) + \frac{1}{\epsilon} \left( \frac{1}{\epsilon} \right) = \frac{1}{\epsilon^2}.
\]

Making a change of variable \( x = \frac{1}{\epsilon} \) and using L’Hôpital’s rule, we have

\[
\lim_{\epsilon \to 0} B^*(\epsilon) = \lim_{x \to \infty} x \ln(2x) = \lim_{x \to \infty} \frac{1 + \ln(2x)}{c \ln \frac{1}{\epsilon}} \left( \frac{1}{\epsilon} \right) = 0.
\]

Therefore, \( 0 \leq \lim_{\epsilon \to 0} \sum_{\ell=1}^{s} \gamma_\epsilon e^{-\gamma_\epsilon c} \leq \frac{1}{s \gamma_1} \lim_{\epsilon \to 0} A(\epsilon) + \frac{\ln \frac{1}{\epsilon}}{\ln(1+\rho) \ln \left( \frac{\ln(1+\epsilon)}{\ln(1+\epsilon) - \frac{1}{c\epsilon}} \right) } \times \frac{1}{\epsilon^2} = \lim_{\epsilon \to 0} B^*(\epsilon) = 0 \), which implies that \( \lim_{\epsilon \to 0} \sum_{\ell=1}^{s} \gamma_\epsilon e^{-\gamma_\epsilon c} = 0 \). This completes the proof of the lemma.

\[ \square \]
Lemma 43 If \( \varepsilon \) is sufficiently small, then the following statements hold true.

(I): For \( \ell = 1, \cdots, s - 1 \), there exists a unique number \( z_\ell \in (0, 1) \) such that \( \gamma_\ell = \frac{\ln(\zeta \delta)}{\mathcal{M}_1(z_\ell, \frac{1}{1+\varepsilon})} \).

(II): \( z_\ell \) is monotonically decreasing with respect to \( \ell \).

(III): \( \lim_{\varepsilon \to 0} z_\ell = 1 - C_{s-\ell} \), where the limit is taken under the restriction that \( \ell - s \) is fixed with respect to \( \varepsilon \).

(IV): For \( p \in (0, 1) \) such that \( C_{j_p} = 1 - p \),

\[
\lim_{\varepsilon \to 0} \frac{p - z_{\ell_\varepsilon}}{\varepsilon z_{\ell_\varepsilon}} = \frac{2}{3},
\]

where \( \ell_\varepsilon = s - j_p \).

(V): \( \{D_\ell = 0\} = \{\hat{p}_\ell < z_\ell\} \).

Proof of Statement (I): By the definition of \( \gamma_\ell \), we have

\[
0 < \frac{\ln(\zeta \delta)}{\mathcal{M}_1(1, \frac{1}{1+\varepsilon})} \leq \gamma_1 \leq \gamma_\ell < \frac{(1 + C_1) \gamma_s}{2} < \frac{(1 + C_1) \ln \frac{1}{\zeta \delta}}{(1 + \varepsilon) \ln(1 + \varepsilon) - \varepsilon} + 1
\]  

(82)

for sufficiently small \( \varepsilon > 0 \). By (82), we have \( \frac{\ln(\zeta \delta)}{\gamma_\ell} \geq \mathcal{M}_1(1, \frac{1}{1+\varepsilon}) \) and

\[
\frac{\ln(\zeta \delta)}{\gamma_\ell} < \left[ \frac{\varepsilon}{1 + \varepsilon} - \ln(1 + \varepsilon) \right] \left( \frac{2}{1 + C_1} - \frac{1}{\gamma_\ell} \right) = \frac{\varepsilon}{1 + C_1} - \ln(1 + \varepsilon) \frac{\mathcal{M}_1(0, 0)}{1 + C_1} + \left[ \ln(1 + \varepsilon) - \frac{\varepsilon}{1 + \varepsilon} \right] \frac{1}{\gamma_\ell}.
\]

Noting that \( \lim_{\varepsilon \to 0} (\frac{(1 + \varepsilon) \ln(1 + \varepsilon) - \varepsilon}{(1 + \varepsilon) \gamma_\ell}) = 0 \) and \( \lim_{\varepsilon \to 0} \frac{\varepsilon - (1 + \varepsilon) \ln(1 + \varepsilon)}{(1 + \varepsilon) \mathcal{M}_1(0, 0)} = 1 \), we have \( \frac{\ln(\zeta \delta)}{\gamma_\ell} \leq \mathcal{M}_1(0, 0) \) for sufficiently small \( \varepsilon > 0 \). In view of the established fact that \( \mathcal{M}_1(1, \frac{1}{1+\varepsilon}) \leq \frac{\ln(\zeta \delta)}{\gamma_\ell} < \mathcal{M}_1(0, 0) \) for small enough \( \varepsilon > 0 \) and the fact that \( \mathcal{M}_1(z, \frac{z}{1+\varepsilon}) \) is monotonically decreasing with respect to \( z \in (0, 1) \) as asserted by Lemma 22, invoking the intermediate value theorem, we have that there exists a unique number \( z_\ell \in (0, 1) \) such that \( \mathcal{M}_1(z_\ell, \frac{z_\ell}{1+\varepsilon}) = \frac{\ln(\zeta \delta)}{\gamma_\ell} \), which implies Statement (I).

Proof of Statement (II): Since \( \gamma_\ell \) is monotonically increasing with respect to \( \ell \) for sufficiently small \( \varepsilon > 0 \), we have that \( \mathcal{M}_1(z_\ell, \frac{z_\ell}{1+\varepsilon}) \) is monotonically increasing with respect to \( \ell \) for sufficiently small \( \varepsilon > 0 \). Recalling that \( \mathcal{M}_1(z, \frac{z}{1+\varepsilon}) \) is monotonically decreasing with respect to \( z \in (0, 1) \), we have that \( z_\ell \) is monotonically decreasing with respect to \( \ell \). This establishes Statement (II).

Proof of Statement (III): For simplicity of notations, let \( b_\ell = 1 - C_{s-\ell} \) for \( \ell = 1, 2, \cdots, s-1 \). Then, it can be checked that \( 1 - b_\ell = C_{s-\ell} \) and, by the definition of \( \gamma_\ell \), we have

\[
\frac{(1 - b_\ell)(1 + \varepsilon) \mathcal{M}_1(z_\ell, \frac{z_\ell}{1+\varepsilon})}{\varepsilon - (1 + \varepsilon) \ln(1 + \varepsilon)} = \frac{1}{\gamma_\ell} \times \frac{C_{s-\ell} (1 + \varepsilon) \ln \frac{1}{\zeta \delta}}{(1 + \varepsilon) \ln(1 + \varepsilon) - \varepsilon} = 1 + o(1)
\]  

(83)

for \( \ell = 1, 2, \cdots, s-1 \).

We claim that \( z_\ell < \theta \) for \( \theta \in (b_\ell, 1) \) if \( \varepsilon > 0 \) is small enough. To prove this claim, we use a contradiction method. Suppose the claim is not true, then there exists a set, denoted by \( S_\varepsilon \), of
infinitely many values of $\varepsilon$ such that $z_{\ell} \geq \theta$ for $\varepsilon \in S_{\varepsilon}$. By (S3) and the fact that $M_1(z, \frac{v_{1+\varepsilon}}{1+\varepsilon})$ is monotonically decreasing with respect to $z \in (0, 1)$ as asserted by Lemma 22, we have

$$1 + o(1) = \frac{(1 - b_{\ell})(1 + \varepsilon).M_1(z_{\ell}, \frac{v_{1+\varepsilon}}{1+\varepsilon})}{\varepsilon - (1 + \varepsilon) \ln(1 + \varepsilon)} \geq \frac{(1 - b_{\ell})(1 + \varepsilon).M_1(\theta, \frac{\theta}{1+\varepsilon})}{\varepsilon - (1 + \varepsilon) \ln(1 + \varepsilon)} = \frac{1 - b_{\ell}}{1 - \theta} + o(1)$$

for small enough $\varepsilon \in S_{\varepsilon}$, which implies $\frac{1 - b_{\ell}}{1 - \theta} \leq 1$, contradicting to the fact that $\frac{1 - b_{\ell}}{1 - \theta} > 1$. The claim is thus established. Similarly, we can show that $z_{\ell} > \theta'$ for $\theta' \in (0, b_{\ell})$ if $\varepsilon$ is small enough. Now we restrict $\varepsilon$ to be small enough so that $\theta' < z_{\ell} < \theta$. Applying Lemma 15 based on such restriction, we have

$$\frac{(1 - b_{\ell})(1 + \varepsilon).M_1(z_{\ell}, \frac{v_{1+\varepsilon}}{1+\varepsilon})}{\varepsilon - (1 + \varepsilon) \ln(1 + \varepsilon)} = \frac{(1 - b_{\ell})\left[-\frac{\varepsilon^2}{2(1 - z_{\ell})} + o(\varepsilon^2)\right]}{-\varepsilon^2 + o(\varepsilon^2)} = \frac{1 - b_{\ell}}{1 - z_{\ell}} + o(1). \quad (84)$$

Combining (S3) and (S6) yields $\frac{b_{\ell} - z_{\ell}}{1 - z_{\ell}} = o(1)$, which implies $\lim_{\varepsilon \to 0} z_{\ell} = b_{\ell}$. This proves Statement (III).

**Proof of Statement (IV):**

Since $\gamma_{\ell_{\varepsilon}} = \left[\frac{C_{s - \ell_{\varepsilon}} (1 + \varepsilon) \ln(\delta)}{\varepsilon - (1 + \varepsilon) \ln(1 + \varepsilon)}\right]$ and $C_{s - \ell_{\varepsilon}} = 1 - p$, we can write

$$\gamma_{\ell_{\varepsilon}} = \left[\frac{(1 - p)(1 + \varepsilon) \ln(\delta)}{\varepsilon - (1 + \varepsilon) \ln(1 + \varepsilon)}\right] = \frac{\ln(\delta)}{M_1(z_{\ell_{\varepsilon}}, z_{\ell_{\varepsilon}}/(1 + \varepsilon))},$$

from which we have $\frac{1}{\gamma_{\ell_{\varepsilon}}} = o(\varepsilon)$,

$$1 - o(\varepsilon) = 1 - \frac{1}{\gamma_{\ell_{\varepsilon}}} < \frac{(1 - p)(1 + \varepsilon) \ln(\delta)}{\varepsilon - (1 + \varepsilon) \ln(1 + \varepsilon)} \frac{\ln(\delta)}{M_1(z_{\ell_{\varepsilon}}, z_{\ell_{\varepsilon}}/(1 + \varepsilon))} \leq 1$$

and thus

$$\frac{(1 - p)(1 + \varepsilon) \ln(\delta)}{\varepsilon - (1 + \varepsilon) \ln(1 + \varepsilon)} \frac{\ln(\delta)}{M_1(z_{\ell_{\varepsilon}}, z_{\ell_{\varepsilon}}/(1 + \varepsilon))} = 1 + o(\varepsilon).$$

For $\theta \in (p, 1)$, we claim that $z_{\ell_{\varepsilon}} < \theta$ if $\varepsilon$ is sufficiently small. Suppose, to get a contradiction that the claim is not true. Then, there exists a set of infinitely many values of $\varepsilon$ such that $z_{\ell_{\varepsilon}} \geq \theta$ if $\varepsilon$ in the set is small enough. For such $\varepsilon$, by the monotonicity of $M_1(\cdot, \cdot)$, we have

$$1 + o(\varepsilon) = \frac{(1 - p)(1 + \varepsilon) \ln(\delta)}{(1 + \varepsilon) \ln(1 + \varepsilon) - \varepsilon} \frac{\ln(\delta)}{M_1(z_{\ell_{\varepsilon}}, z_{\ell_{\varepsilon}}/(1 + \varepsilon))} = \frac{(1 - p)(1 + \varepsilon).M_1(z_{\ell_{\varepsilon}}, z_{\ell_{\varepsilon}}/(1 + \varepsilon))}{\varepsilon - (1 + \varepsilon) \ln(1 + \varepsilon)} \geq \frac{(1 - p)(1 + \varepsilon).M_1(\theta, \theta/(1 + \varepsilon))}{\varepsilon - (1 + \varepsilon) \ln(1 + \varepsilon)} = \frac{1 - p}{1 - \theta} + o(1) \quad (85)$$

for small enough $\varepsilon$ in the set, which contradicts to the fact that $\frac{1 - p}{1 - \theta} > 1$. This proves the claim. Similarly, we can show that $z_{\ell_{\varepsilon}} \geq \theta'$ for any $\theta' \in (0, p)$. Now we restrict $\varepsilon$ to be small enough so
that \( \theta' < z_{\ell} < \theta \). By virtue of such restriction, we have

\[
\frac{(1 - p)(1 + \varepsilon)\mathcal{M}(z_{\ell}, z_{\ell}/(1 + \varepsilon))}{\varepsilon - (1 + \varepsilon) \ln(1 + \varepsilon)} = \frac{(1 - p)\left(-\frac{\varepsilon^2}{2(1-z_{\ell})} + \frac{\varepsilon^3(2-z_{\ell})}{3(1-z_{\ell})^2} + o(\varepsilon^3)\right)}{\varepsilon/(1 + \varepsilon) - \ln(1 + \varepsilon)}
\]

Combining (85) and (86) yields

\[
\frac{1-p}{1-z_{\ell}} - \frac{2(1-p)(2-z_{\ell})}{3(1-z_{\ell})^2} + o(\varepsilon)
\]

i.e.,

\[
\frac{p - z_{\ell}}{1-z_{\ell}} = \frac{4\varepsilon}{3} - \frac{2(1-p)(2-z_{\ell})}{3(1-z_{\ell})^2} + o(\varepsilon),
\]

which implies that \( \lim_{\varepsilon \to 0} \frac{p - z_{\ell}}{\varepsilon z_{\ell}} = \frac{4(1-z_{\ell})}{3z_{\ell}} - \frac{2(1-p)(2-z_{\ell})}{3z_{\ell}(1-z_{\ell})} + o(1) \),

**Proof of Statement (V):** Noting that \( \mathcal{M}(z, \frac{z}{1+\varepsilon}) \) is monotonically decreasing with respect to \( z \in (0, 1) \) as asserted by Lemma 22, we have \( \{D_{\ell} = 0\} = \{\mathcal{M}(\hat{p}_{\ell}, \frac{1+\varepsilon}{\hat{p}_{\ell}}) > \frac{\ln(\varepsilon)}{\gamma_{\ell}}\} = \{\hat{p}_{\ell} < z_{\ell}\} \) as claimed by statement (V).

**Lemma 44** Let \( \ell \in [s - j_p, s] \). Then,

\[
\lim_{\varepsilon \to 0} \sum_{\ell=1}^{s-j_p} \gamma_{\ell} \Pr\{D_{\ell} = 1\} = 0, \quad \lim_{\varepsilon \to 0} \sum_{\ell=\ell_{s-1}}^{s} \gamma_{\ell} \Pr\{D_{\ell} = 0\} = 0
\]

for \( p \in (0, 1) \). Moreover, \( \lim_{\varepsilon \to 0} \gamma_{\ell} \Pr\{D_{\ell} = 0\} = 0 \) if \( C_{j_p} > 1 - p \).

**Proof.** For simplicity of notations, let \( \ell_{\varepsilon} = s - j_p \). The proof consists of two main steps as follows.

First, we shall show that (87) holds for any \( p \in (0, 1) \). By the definition of \( \ell_{\varepsilon} \), we have \( 1 - p > C_{s - \ell_{\varepsilon} + 1} \). Making use of the first three statements of Lemma 43, we have \( z_{\ell} > \frac{p + b_{\ell-1}}{2} > p \) for all \( \ell \leq \ell_{\varepsilon} - 1 \) if \( \varepsilon \) is sufficiently small. By the last statement of Lemma 43 and using Lemma 25, we have

\[
\Pr\{D_{\ell} = 1\} = \Pr\{\hat{p}_{\ell} \geq z_{\ell}\} \leq \Pr\left\{\hat{p}_{\ell} \geq \frac{p + b_{\ell-1}}{2}\right\} \leq \exp\left(\gamma_{\ell} \mathcal{M}\left(\frac{p + b_{\ell-1}}{2}, p\right)\right)
\]
for all $\ell \leq \ell_{\varepsilon} - 1$ if $\varepsilon > 0$ is sufficiently small. Since $b_{\ell_{\varepsilon} - 1}$ is greater than $p$ and is independent of $\varepsilon > 0$ as a consequence of the definition of $\ell_{\varepsilon}$, it follows from Lemma 42 that $\lim_{\varepsilon \to 0} \sum_{\ell=1}^{\ell_{\varepsilon} - 1} \gamma_{\ell} \Pr\{D_{\ell} = 1\} = 0$.

Similarly, it can be seen from the definition of $\ell_{\varepsilon}$ that $1 - p < C_{s - \ell_{\varepsilon} - 1}$. Making use of the first three statements of Lemma 43 we have that $z_{\ell} < \frac{p + b_{\ell_{\varepsilon} + 1}}{2} < p$ for $\ell_{\varepsilon} + 1 \leq \ell < s$ if $\varepsilon$ is sufficiently small. By the last statement of Lemma 43 and using Lemma 25 we have

$$\Pr\{D_{\ell} = 0\} = \Pr\{\tilde{p}_{\ell} < z_{\ell}\} \leq \Pr\left\{\tilde{p}_{\ell} < \frac{p + b_{\ell_{\varepsilon} + 1}}{2}\right\} \leq \exp\left(\gamma_{\ell_{\varepsilon}} M_{1}\left(\frac{p + b_{\ell_{\varepsilon} + 1}}{2}, p\right)\right)$$

for $\ell_{\varepsilon} + 1 \leq \ell < s$ if $\varepsilon > 0$ is small enough. By virtue of the definition of $\ell_{\varepsilon}$, we have that $b_{\ell_{\varepsilon} + 1}$ is smaller than $p$ and is independent of $\varepsilon > 0$. In view of this and the fact that $\Pr\{D_{s} = 0\} = 0$, we can use Lemma 42 to conclude that $\lim_{\varepsilon \to 0} \sum_{\ell=\ell_{\varepsilon} + 1}^{s} \gamma_{\ell} \Pr\{D_{\ell} = 0\} = 0$.

Next, we shall show that $\lim_{\varepsilon \to 0} \gamma_{\ell_{\varepsilon}} \Pr\{D_{\ell_{\varepsilon}} = 0\} = 0$ for $p \in (0, 1)$ such that $C_{j_{p}} > 1 - p$. Note that $1 - p < C_{s - \ell_{\varepsilon}}$ because of the definition of $\ell_{\varepsilon}$. Making use of the first three statements of Lemma 43 we have that $z_{\ell_{\varepsilon}} < \frac{p + b_{\ell_{\varepsilon}}}{2} < p$ if $\varepsilon > 0$ is small enough. By the last statement of Lemma 43 and using Lemma 25 we have

$$\Pr\{D_{\ell_{\varepsilon}} = 0\} = \Pr\{\tilde{p}_{\ell_{\varepsilon}} < z_{\ell_{\varepsilon}}\} \leq \Pr\left\{\tilde{p}_{\ell_{\varepsilon}} < \frac{p + b_{\ell_{\varepsilon}}}{2}\right\} \leq \exp\left(\gamma_{\ell_{\varepsilon}} M_{1}\left(\frac{p + b_{\ell_{\varepsilon}}}{2}, p\right)\right)$$

for small enough $\varepsilon > 0$. By virtue of the definition of $\ell_{\varepsilon}$, we have that $b_{\ell_{\varepsilon}}$ is smaller than $p$ and is independent of $\varepsilon > 0$. It follows that $\lim_{\varepsilon \to 0} \gamma_{\ell_{\varepsilon}} \Pr\{D_{\ell_{\varepsilon}} = 0\} = 0$. This completes the proof of the lemma.

Finally, we would like to note that Theorem 25 can be shown by employing Lemma 44 and a similar argument as the proof of Theorem 15.

I.12 Proof of Theorem 26

We need some preliminary results.

**Lemma 45** $\lim_{\varepsilon \to 0} \frac{\gamma_{\ell_{\varepsilon}}}{\gamma(p, \varepsilon)} = \kappa_{p}$, $\lim_{\varepsilon \to 0} \varepsilon \sqrt{\frac{\gamma_{\ell_{\varepsilon}}}{1 - p}} = d \sqrt{\kappa_{p}}$.

**Proof.** By the definition of $\gamma_{\ell}$, we have

$$\lim_{\varepsilon \to 0} \frac{C_{s - \ell} (1 + \varepsilon) \ln(\delta)}{\gamma_{\ell}[\varepsilon - (1 + \varepsilon) \ln(1 + \varepsilon)]} = 1$$

for $1 \leq \ell < s$. It follows that

$$\lim_{\varepsilon \to 0} \frac{\gamma_{\ell_{\varepsilon}}}{\gamma(p, \varepsilon)} = \lim_{\varepsilon \to 0} \frac{M_{1}(p, \frac{p}{1 + \varepsilon}) \times C_{s - \ell_{\varepsilon}} (1 + \varepsilon) \ln(\delta)}{\varepsilon - (1 + \varepsilon) \ln(1 + \varepsilon)} = \lim_{\varepsilon \to 0} \frac{C_{s - \ell_{\varepsilon}} (1 + \varepsilon) \times M_{1}(p, \frac{p}{1 + \varepsilon})}{\varepsilon - (1 + \varepsilon) \ln(1 + \varepsilon)} = \frac{C_{s - \ell_{\varepsilon}} (1 + \varepsilon)}{1 - p} = \frac{C_{j_{p}}}{1 - p} = \kappa_{p}$$
and
\[
\lim_{\varepsilon \to 0} \varepsilon \sqrt{\frac{\gamma \ell_c}{1 - p}} = \lim_{\varepsilon \to 0} \varepsilon \sqrt{\frac{1}{1 - p} \frac{C_{s-\ell_c} (1 + \varepsilon) \ln \frac{1}{\varepsilon}}{1 + \varepsilon}} = d \sqrt{\frac{C_{s-\ell_c}}{1 - p}} = d \sqrt{\varepsilon p}.
\]

\[\Box\]

**Lemma 46** Let \( U \) and \( V \) be independent Gaussian random variables with zero means and unit variances. Then, for \( p \in (0, 1) \) such that \( C_{j_p} = 1 - p \),

\[
\begin{align*}
\lim_{\varepsilon \to 0} \Pr\{l = \ell_{\varepsilon}\} &= 1 - \lim_{\varepsilon \to 0} \Pr\{l = \ell_{\varepsilon} + 1\} = 1 - \Phi(\nu d), \\
\lim_{\varepsilon \to 0} \Pr\{|\hat{p}_{\ell_{\varepsilon}} - p| \geq \varepsilon p, l = \ell_{\varepsilon}\} + \Pr\{|\hat{p}_{\ell_{\varepsilon}+1} - p| \geq \varepsilon p, l = \ell_{\varepsilon} + 1\} \\
&= \Pr\{U \geq d\} + \Pr\{|U + \sqrt{\rho_p}V| \geq d \sqrt{1 + p}, U < \nu d\}.
\end{align*}
\]

**Proof.** By Statement (V) of Lemma 45 we have

\[
\begin{align*}
\Pr\{\hat{p}_{\ell_{\varepsilon}} \geq z_{\ell_{\varepsilon}}\} &\geq \Pr\{l = \ell_{\varepsilon}\} \geq \Pr\{\hat{p}_{\ell_{\varepsilon}} \geq z_{\ell_{\varepsilon}}\} - \sum_{\ell=1}^{\ell_{\varepsilon}-1} \Pr\{D_{\ell} = 1\}, \\
\Pr\{\hat{p}_{\ell_{\varepsilon}} < z_{\ell_{\varepsilon}}\} &\geq \Pr\{l = \ell_{\varepsilon} + 1\} \geq \Pr\{\hat{p}_{\ell_{\varepsilon}} < z_{\ell_{\varepsilon}}\} - \Pr\{D_{\ell_{\varepsilon}+1} = 0\} - \sum_{\ell=1}^{1} \Pr\{D_{\ell} = 1\}.
\end{align*}
\]

Making use of this result and the fact that \( \lim_{\varepsilon \to 0} \left[ \sum_{\ell=1}^{\ell_{\varepsilon}-1} \Pr\{D_{\ell} = 1\} + \Pr\{D_{\ell_{\varepsilon}+1} = 0\} \right] = 0 \) as asserted by Lemma 44, we have

\[
\begin{align*}
\lim_{\varepsilon \to 0} \Pr\{l = \ell_{\varepsilon}\} = \lim_{\varepsilon \to 0} \Pr\{\hat{p}_{\ell_{\varepsilon}} \geq z_{\ell_{\varepsilon}}\}, \\
\lim_{\varepsilon \to 0} \Pr\{l = \ell_{\varepsilon} + 1\} = \lim_{\varepsilon \to 0} \Pr\{\hat{p}_{\ell_{\varepsilon}} < z_{\ell_{\varepsilon}}\}.
\end{align*}
\]

Noting that

\[
\Pr\{|\hat{p}_{\ell_{\varepsilon}} - p| \geq \varepsilon p, l = \ell_{\varepsilon}\} \geq \Pr\{|\hat{p}_{\ell_{\varepsilon}} - p| \geq \varepsilon p, \hat{p}_{\ell_{\varepsilon}} \geq z_{\ell_{\varepsilon}}\} - \sum_{\ell=1}^{\ell_{\varepsilon}-1} \Pr\{D_{\ell} = 1\},
\]

\[
\Pr\{|\hat{p}_{\ell_{\varepsilon}+1} - p| \geq \varepsilon p, l = \ell_{\varepsilon} + 1\} \geq \Pr\{|\hat{p}_{\ell_{\varepsilon}+1} - p| \geq \varepsilon p, \hat{p}_{\ell_{\varepsilon}} < z_{\ell_{\varepsilon}}\} - \Pr\{D_{\ell_{\varepsilon}+1} = 0\} - \sum_{\ell=1}^{1} \Pr\{D_{\ell} = 1\}
\]

and using the result that \( \lim_{\varepsilon \to 0} \left[ \sum_{\ell=1}^{\ell_{\varepsilon}-1} \Pr\{D_{\ell} = 1\} + \Pr\{D_{\ell_{\varepsilon}+1} = 0\} \right] = 0 \), we have

\[
\begin{align*}
\liminf_{\varepsilon \to 0} \left[ \Pr\{|\hat{p}_{\ell_{\varepsilon}} - p| \geq \varepsilon p, l = \ell_{\varepsilon}\} + \Pr\{|\hat{p}_{\ell_{\varepsilon}+1} - p| \geq \varepsilon p, l = \ell_{\varepsilon} + 1\} \right] \\
\geq \lim_{\varepsilon \to 0} \left[ \Pr\{|\hat{p}_{\ell_{\varepsilon}} - p| \geq \varepsilon p, \hat{p}_{\ell_{\varepsilon}} \geq z_{\ell_{\varepsilon}}\} + \Pr\{|\hat{p}_{\ell_{\varepsilon}+1} - p| \geq \varepsilon p, \hat{p}_{\ell_{\varepsilon}} < z_{\ell_{\varepsilon}}\} \right].
\end{align*}
\]

On the other hand,

\[
\begin{align*}
\limsup_{\varepsilon \to 0} \left[ \Pr\{|\hat{p}_{\ell_{\varepsilon}} - p| \geq \varepsilon p, l = \ell_{\varepsilon}\} + \Pr\{|\hat{p}_{\ell_{\varepsilon}+1} - p| \geq \varepsilon p, l = \ell_{\varepsilon} + 1\} \right] \\
\leq \lim_{\varepsilon \to 0} \left[ \Pr\{|\hat{p}_{\ell_{\varepsilon}} - p| \geq \varepsilon p, \hat{p}_{\ell_{\varepsilon}} \geq z_{\ell_{\varepsilon}}\} + \Pr\{|\hat{p}_{\ell_{\varepsilon}+1} - p| \geq \varepsilon p, \hat{p}_{\ell_{\varepsilon}} < z_{\ell_{\varepsilon}}\} \right].
\end{align*}
\]
Therefore,

\[
\lim_{\varepsilon \to 0} \left[ \Pr\{ |\hat{p}_{\ell_\varepsilon} - p| \geq \varepsilon p, l = \ell_\varepsilon \} + \Pr\{ |\hat{p}_{\ell_\varepsilon + 1} - p| \geq \varepsilon p, l = \ell_\varepsilon + 1 \} \right]
= \lim_{\varepsilon \to 0} \left[ \Pr\{ |\hat{p}_{\ell_\varepsilon} - p| \geq \varepsilon p, \hat{p}_{\ell_\varepsilon} \geq z_{\ell_\varepsilon} \} + \Pr\{ |\hat{p}_{\ell_\varepsilon + 1} - p| \geq \varepsilon p, \hat{p}_{\ell_\varepsilon} < z_{\ell_\varepsilon} \} \right].
\]

Since \( \kappa_p = 1 \), by Lemma 15 and Statement (IV) of Lemma 13, we have

\[
\lim_{\varepsilon \to 0} \frac{p - z_{\ell_\varepsilon}}{z_{\ell_\varepsilon}} \sqrt{\frac{\gamma_{\ell_\varepsilon}}{1 - p}} = \lim_{\varepsilon \to 0} \varepsilon \sqrt{\frac{\gamma_{\ell_\varepsilon}}{1 - p}} \lim_{\varepsilon \to 0} \frac{p - z_{\ell_\varepsilon}}{\varepsilon z_{\ell_\varepsilon}} = d \lim_{\varepsilon \to 0} \frac{p - z_{\ell_\varepsilon}}{\varepsilon z_{\ell_\varepsilon}} = - \frac{2}{3} d = -vd.
\]

Note that

\[
\frac{1}{p - 1} - \frac{1}{p} = \frac{n_{\ell_\varepsilon + 1} - 1}{p} = \frac{\gamma_{\ell_\varepsilon}}{\gamma_{\ell_\varepsilon + 1}} \sqrt{\frac{1 - p}{p^2 \gamma_{\ell_\varepsilon}}} U_{\ell_\varepsilon} + \frac{\gamma_{\ell_\varepsilon + 1} - \gamma_{\ell_\varepsilon}}{\gamma_{\ell_\varepsilon + 1}} \sqrt{\frac{1 - p}{p^2 (\gamma_{\ell_\varepsilon + 1} - \gamma_{\ell_\varepsilon})}} V_{\ell_\varepsilon}
\]

where

\[
U_{\ell_\varepsilon} = \left( \frac{1}{p_{\ell_\varepsilon}} - \frac{1}{p} \right) \sqrt{\frac{p^2 \gamma_{\ell_\varepsilon}}{1 - p}}, \quad V_{\ell_\varepsilon} = \left( \frac{n_{\ell_\varepsilon + 1} - n_{\ell_\varepsilon}}{\gamma_{\ell_\varepsilon + 1} - \gamma_{\ell_\varepsilon} - 1} \right) \sqrt{\frac{p^2 (\gamma_{\ell_\varepsilon + 1} - \gamma_{\ell_\varepsilon})}{1 - p}}.
\]

By the central limit theorem, \( U_{\ell_\varepsilon} \to U \) and \( V_{\ell_\varepsilon} \to V \) as \( \varepsilon \to 0 \). Hence,

\[
U_{\ell_\varepsilon + 1} = \left( \frac{1}{p_{\ell_\varepsilon + 1}} - \frac{1}{p} \right) \sqrt{\frac{p^2 \gamma_{\ell_\varepsilon + 1}}{1 - p}} = \left[ \frac{\gamma_{\ell_\varepsilon}}{\gamma_{\ell_\varepsilon + 1}} \sqrt{\frac{1 - p}{p^2 \gamma_{\ell_\varepsilon}}} U_{\ell_\varepsilon} + \frac{\gamma_{\ell_\varepsilon + 1} - \gamma_{\ell_\varepsilon}}{\gamma_{\ell_\varepsilon + 1}} \sqrt{\frac{1 - p}{p^2 (\gamma_{\ell_\varepsilon + 1} - \gamma_{\ell_\varepsilon})}} V_{\ell_\varepsilon} \right] \sqrt{\frac{p^2 (\gamma_{\ell_\varepsilon + 1} - \gamma_{\ell_\varepsilon})}{1 - p}}
\]

as \( \varepsilon \to 0 \). It can be seen that \( \Pr\{ |\hat{p}_{\ell_\varepsilon} - p| \geq \varepsilon p, \hat{p}_{\ell_\varepsilon} < z_{\ell_\varepsilon} \} \) = \( \Pr\{ U_{\ell_\varepsilon} \leq \frac{p - z_{\ell_\varepsilon}}{p z_{\ell_\varepsilon}} \sqrt{\frac{\gamma_{\ell_\varepsilon}}{1 - p}} \} \).

\[
\Pr\{ |\hat{p}_{\ell_\varepsilon + 1} - p| \geq \varepsilon p, \hat{p}_{\ell_\varepsilon} < z_{\ell_\varepsilon} \}
= \Pr\{ \frac{1}{p_{\ell_\varepsilon + 1}} - \frac{1}{p} \leq \frac{\varepsilon}{(1 + \varepsilon)p}, \frac{1}{p_{\ell_\varepsilon}} - \frac{1}{p} > \frac{p - z_{\ell_\varepsilon}}{p z_{\ell_\varepsilon}} \} + \Pr\{ \frac{1}{p_{\ell_\varepsilon + 1}} - \frac{1}{p} \geq \frac{\varepsilon}{(1 - \varepsilon)p}, \frac{1}{p_{\ell_\varepsilon}} - \frac{1}{p} > \frac{p - z_{\ell_\varepsilon}}{p z_{\ell_\varepsilon}} \}
\]

\[
= \Pr\left\{ U_{\ell_\varepsilon + 1} \leq \frac{-\varepsilon}{(1 + \varepsilon)} \sqrt{\frac{\gamma_{\ell_\varepsilon + 1}}{1 - p}}, U_{\ell_\varepsilon} > \frac{p - z_{\ell_\varepsilon}}{z_{\ell_\varepsilon}} \sqrt{\frac{\gamma_{\ell_\varepsilon}}{1 - p}} \right\} + \Pr\left\{ U_{\ell_\varepsilon + 1} \geq \frac{\varepsilon}{(1 - \varepsilon)} \sqrt{\frac{\gamma_{\ell_\varepsilon + 1}}{1 - p}}, U_{\ell_\varepsilon} > \frac{p - z_{\ell_\varepsilon}}{z_{\ell_\varepsilon}} \sqrt{\frac{\gamma_{\ell_\varepsilon}}{1 - p}} \right\}
\]

and

\[
\Pr\{ |\hat{p}_{\varepsilon} - p| \geq \varepsilon p, \hat{p}_{\varepsilon} \geq z_{\ell_\varepsilon} \}
= \Pr\left\{ U_{\ell_\varepsilon} \leq \frac{-\varepsilon}{(1 + \varepsilon)} \sqrt{\frac{\gamma_{\ell_\varepsilon}}{1 - p}}, U_{\ell_\varepsilon} \leq \frac{p - z_{\ell_\varepsilon}}{z_{\ell_\varepsilon}} \sqrt{\frac{\gamma_{\ell_\varepsilon}}{1 - p}} \right\}
+ \Pr\left\{ U_{\ell_\varepsilon} \geq \frac{\varepsilon}{(1 - \varepsilon)} \sqrt{\frac{\gamma_{\ell_\varepsilon}}{1 - p}}, U_{\ell_\varepsilon} \leq \frac{p - z_{\ell_\varepsilon}}{z_{\ell_\varepsilon}} \sqrt{\frac{\gamma_{\ell_\varepsilon}}{1 - p}} \right\}.
\]

Therefore, for \( p \in (0, 1) \) such that \( C_{\varepsilon p} = 1 - p \), we have \( \lim_{\varepsilon \to 0} \Pr\{ l = \ell_\varepsilon \} = 1 - \lim_{\varepsilon \to 0} \Pr\{ l = \ell_\varepsilon + 1 \} = 1 - \Phi(\nu d) \) and

\[
\lim_{\varepsilon \to 0} \Pr\{ |\hat{p}_{\ell_\varepsilon} - p| \geq \varepsilon p, l = \ell_\varepsilon \} + \Pr\{ |\hat{p}_{\ell_\varepsilon + 1} - p| \geq \varepsilon p, l = \ell_\varepsilon + 1 \}
\to \Pr\{ |U| \geq d, U \leq -vd \} + \Pr\{ |U + \sqrt{\rho_p} V| \geq d \sqrt{1 + \rho_p}, U > -vd \}
= \Pr\{ U \geq d \} + \Pr\{ |U + \sqrt{\rho_p} V| \geq d \sqrt{1 + \rho_p}, U < vd \}
\]

126
as \( \varepsilon \to 0 \). This completes the proof of the lemma.

\[ \square \]

### I.12.1 Proof of Statement (I)

First, we shall show that Statement (I) holds for \( p \in (0,1) \) such that \( C_{j_p} = 1-p \). For this purpose, we need to show that

\[
1 \leq \limsup_{\varepsilon \to 0} \frac{\gamma(\omega)}{\gamma(p, \varepsilon)} \leq 1 + \rho_p \quad \text{for any } \omega \in \left\{ \lim_{\varepsilon \to 0} \tilde{p} = p \right\}.
\]

(88)

To show \( \limsup_{\varepsilon \to 0} \frac{\gamma(\omega)}{\gamma(p, \varepsilon)} \geq 1 \), note that \( C_{s-\ell+1} < 1 - p < C_{s-\ell} < C_{s-\ell-1} \) as a direct consequence of the definition of \( \ell_\varepsilon \) and the assumption that \( C_{j_p} = 1-p \). By the first three statements of Lemma 43, we have \( \lim_{\varepsilon \to 0} z_\ell > p \) for all \( \ell \leq \ell_\varepsilon - 1 \). Noting that \( \lim_{\varepsilon \to 0} \tilde{p}(\omega) = p \), we have \( \tilde{p}(\omega) < z_\ell \) for all \( \ell \leq \ell_\varepsilon - 1 \), and it follows from the definition of the sampling scheme that \( \gamma(\omega) \geq \gamma_{\ell_\varepsilon} \) if \( \varepsilon > 0 \) is small enough. By Lemma 41 and noting that \( \kappa_p = 1 \) if \( C_{j_p} = 1 - p \), we have \( \limsup_{\varepsilon \to 0} \frac{\gamma(\omega)}{\gamma(p, \varepsilon)} \geq \lim_{\varepsilon \to 0} \frac{\gamma(\omega)}{\gamma(p, \varepsilon)} = \kappa_p = 1 \).

To show \( \limsup_{\varepsilon \to 0} \frac{\gamma(\omega)}{\gamma(p, \varepsilon)} \leq 1 + \rho_p \), we shall consider two cases: (i) \( \ell_\varepsilon = s - 1 \); (ii) \( \ell_\varepsilon < s - 1 \). In the case of \( \ell_\varepsilon = s - 1 \), it must be true that \( \gamma(\omega) \leq \gamma_s = \gamma_{\ell_\varepsilon + 1} \). Hence, \( \limsup_{\varepsilon \to 0} \frac{\gamma(\omega)}{\gamma(p, \varepsilon)} \leq \lim_{\varepsilon \to 0} \frac{\gamma_{\ell_\varepsilon + 1}}{\gamma(p, \varepsilon)} \lim_{\varepsilon \to 0} \frac{\gamma_{\ell_\varepsilon + 1}}{\gamma_{\ell_\varepsilon}} = 1 + \rho_p \). In the case of \( \ell_\varepsilon < s - 1 \), it follows from the first three statements of Lemma 43 that \( \lim_{\varepsilon \to 0} z_{\ell_\varepsilon + 1} < p \), which implies that \( z_{\ell_\varepsilon + 1} < p \), \( \tilde{p}(\omega) > z_{\ell_\varepsilon + 1} \), and thus \( \gamma(\omega) \leq \gamma_{\ell_\varepsilon + 1} \) for small enough \( \varepsilon > 0 \). Therefore, \( \limsup_{\varepsilon \to 0} \frac{\gamma(\omega)}{\gamma(p, \varepsilon)} \leq \lim_{\varepsilon \to 0} \frac{\gamma_{\ell_\varepsilon + 1}}{\gamma(p, \varepsilon)} = 1 + \rho_p \). This establishes (88) and it follows that \( \{ 1 \leq \limsup_{\varepsilon \to 0} \frac{\gamma}{\gamma(p, \varepsilon)} \leq 1 + \rho_p \} \supseteq \{ \lim_{\varepsilon \to 0} \tilde{p} = p \} \). According to the strong law of large numbers, we have \( 1 \geq \Pr \{ 1 \leq \limsup_{\varepsilon \to 0} \frac{\gamma}{\gamma(p, \varepsilon)} \leq 1 + \rho_p \} \geq \Pr \{ \lim_{\varepsilon \to 0} \tilde{p} = p \} = 1 \). This proves that Statement (I) holds for \( p \in (0,1) \) such that \( C_{j_p} = 1 - p \).

Next, we shall show that Statement (I) holds for \( p \in (0,1) \) such that \( C_{j_p} > 1 - p \). Note that \( C_{s-\ell_\varepsilon + 1} < 1 - p < C_{s-\ell_\varepsilon} \) as a direct consequence of the definition of \( \ell_\varepsilon \) and the assumption that \( C_{j_p} > 1 - p \). By the first three statements of Lemma 43, we have \( \lim_{\varepsilon \to 0} z_{\ell_\varepsilon - 1} > p \) and thus \( z_\ell > p \) for all \( \ell \leq \ell_\varepsilon - 1 \) provided that \( \varepsilon > 0 \) is sufficiently small. Therefore, for any \( \omega \in \left\{ \lim_{\varepsilon \to 0} \tilde{p} = p \right\} \), we have \( \tilde{p}(\omega) < z_\ell \) for all \( \ell \leq \ell_\varepsilon - 1 \) and consequently, \( \gamma(\omega) \geq \gamma_{\ell_\varepsilon} \) provided that \( \varepsilon > 0 \) is sufficiently small. On the other hand, we claim that \( \gamma(\omega) \leq \gamma_{\ell_\varepsilon} \). Such claim can be justified by investigating two cases. In the case of \( \ell_\varepsilon = s \), it is trivially true that \( \gamma(\omega) \leq \gamma_{\ell_\varepsilon} \). In the case of \( \ell_\varepsilon < s \), we have \( p > \lim_{\varepsilon \to 0} z_{\ell_\varepsilon} \) and thus \( p > z_{\ell_\varepsilon} \) provided that \( \varepsilon > 0 \) is sufficiently small. Therefore, for any \( \omega \in \left\{ \lim_{\varepsilon \to 0} \tilde{p} = p \right\} \), we have \( \tilde{p}(\omega) > z_{\ell_\varepsilon} \) and consequently, \( \gamma(\omega) \leq \gamma_{\ell_\varepsilon} \) provided that \( \varepsilon > 0 \) is sufficiently small. This proves the claim and it follows that \( \gamma(\omega) = \gamma_{\ell_\varepsilon} \) if \( \varepsilon > 0 \) is small enough. Applying Lemma 41, we have \( \lim_{\varepsilon \to 0} \frac{\gamma}{\gamma(p, \varepsilon)} = \lim_{\varepsilon \to 0} \frac{\gamma_{\ell_\varepsilon}}{\gamma(p, \varepsilon)} = \kappa_p \), which implies that \( \{ \lim_{\varepsilon \to 0} \frac{\gamma}{\gamma(p, \varepsilon)} = \kappa_p \} \supseteq \{ \lim_{\varepsilon \to 0} \tilde{p} = p \} \). It follows from the strong law of large numbers that \( 1 \geq \Pr \{ \lim_{\varepsilon \to 0} \frac{\gamma}{\gamma(p, \varepsilon)} = \kappa_p \} \geq \Pr \{ \lim_{\varepsilon \to 0} \tilde{p} = p \} = 1 \) and thus \( \Pr \{ \lim_{\varepsilon \to 0} \frac{\gamma}{\gamma(p, \varepsilon)} = \kappa_p \} = 1 \). Since \( 1 \leq \kappa_p \leq 1 + \rho_p \), we have that \( \Pr \{ 1 \leq \limsup_{\varepsilon \to 0} \frac{\gamma}{\gamma(p, \varepsilon)} \leq 1 + \rho_p \} = 1 \) is of course true. This proves that Statement (I) also holds for \( p \in (0,1) \) such that \( C_{j_p} > 1 - p \). The proof of Statement (I) is thus completed.
I.12.2 Proof of Statement (III)

First, we shall consider \( p \in (0, 1) \) such that \( C_{jp} = 1 - p \). In this case, it is evident that \( \ell_\varepsilon < s \). It follows from Lemma 44 and the definition of the sampling scheme that \( \lim_{\varepsilon \to 0} \Pr \{ \ell > \ell_\varepsilon + 1 \} \leq \lim_{\varepsilon \to 0} \Pr \{ D_{\ell_{\varepsilon} + 1} = 0 \} = 0 \) and \( \lim_{\varepsilon \to 0} \Pr \{ \ell < \ell_\varepsilon \} \leq \lim_{\varepsilon \to 0} \sum_{\ell_{\varepsilon}+1}^{s} \Pr \{ D_\ell = 1 \} = 0 \). Since

\[
\limsup_{\varepsilon \to 0} \Pr \{ |\widehat{p} - p| \geq \varepsilon p \} \leq \lim_{\varepsilon \to 0} \left[ \Pr \{ |\widehat{p}_{\ell_\varepsilon} - p| \geq \varepsilon p, \ell = \ell_\varepsilon \} + \Pr \{ |\widehat{p}_{\ell_{\varepsilon}+1} - p| \geq \varepsilon p, \ell = \ell_\varepsilon + 1 \} \right]
\]

and

\[
\liminf_{\varepsilon \to 0} \Pr \{ |\widehat{p} - p| \geq \varepsilon p \} \geq \lim_{\varepsilon \to 0} \left[ \Pr \{ |\widehat{p}_{\ell_\varepsilon} - p| \geq \varepsilon p, \ell = \ell_\varepsilon \} + \Pr \{ |\widehat{p}_{\ell_{\varepsilon}+1} - p| \geq \varepsilon p, \ell = \ell_\varepsilon + 1 \} \right],
\]

we have

\[
\lim_{\varepsilon \to 0} \Pr \{ |\widehat{p} - p| \geq \varepsilon p \} = \lim_{\varepsilon \to 0} \left[ \Pr \{ |\widehat{p}_{\ell_\varepsilon} - p| \geq \varepsilon p, \ell = \ell_\varepsilon \} + \Pr \{ |\widehat{p}_{\ell_{\varepsilon}+1} - p| \geq \varepsilon p, \ell = \ell_\varepsilon + 1 \} \right].
\]

By Lemma 46 we have

\[
\lim_{\varepsilon \to 0} \Pr \{ |\widehat{p} - p| \geq \varepsilon p \} = \Pr \{ U \geq d \} + \Pr \{ |U + \sqrt{p} V| \geq d \sqrt{1 + \rho_p}, U < \nu d \}
\]

for \( p \in (0, 1) \) such that \( C_{jp} = 1 - p \). As a consequence of Lemma 21 Statement (III) must be true for \( p \in (0, 1) \) such that \( C_{jp} = 1 - p \).

Next, we shall consider \( p \in (0, 1) \) such that \( C_{jp} > 1 - p \). Note that \( C_{s-\ell_\varepsilon} < 1 - p < C_{s-\ell_\varepsilon} \). Since \( U_{\ell_\varepsilon} = \left( p / p_{\ell_\varepsilon} - 1 \right) \sqrt{ \frac{\gamma_{\ell_\varepsilon}}{1-p} } \) converges in distribution to a standard Gaussian variable \( U \), \( \lim_{\varepsilon \to 0} \sqrt{ \frac{\gamma_{\ell_\varepsilon}}{1-p} } = d \sqrt{\kappa_p} \) and \( \lim_{\varepsilon \to 0} \Pr \{ \gamma = \gamma_{\ell_\varepsilon} \} = 1 \) as can be seen from Statement (I), we have

\[
\lim_{\varepsilon \to 0} \Pr \{ |\widehat{p} - p| \geq \varepsilon p \} = \lim_{\varepsilon \to 0} \Pr \{ |\widehat{p}_{\ell_\varepsilon} - p| \geq \varepsilon p \}
\]

\[
= \lim_{\varepsilon \to 0} \left[ \Pr \left\{ U_{\ell_\varepsilon} \geq \frac{\varepsilon}{1-\varepsilon} \sqrt{ \frac{\gamma_{\ell_\varepsilon}}{1-p} } \right\} + \lim_{\varepsilon \to 0} \Pr \left\{ U_{\ell_\varepsilon} \leq - \frac{\varepsilon}{1+\varepsilon} \sqrt{ \frac{\gamma_{\ell_\varepsilon}}{1-p} } \right\} \right]
\]

\[
= \lim_{\varepsilon \to 0} \Pr \left\{ |U_{\ell_\varepsilon}| \geq \varepsilon \sqrt{ \frac{\gamma_{\ell_\varepsilon}}{1-p} } \right\} = \Pr \{ |U| \geq d \sqrt{\kappa_p} \}
\]

and consequently, \( \lim_{\varepsilon \to 0} \Pr \{ |\widehat{p} - p| < \varepsilon p \} \geq 2 \Phi \left( d \sqrt{\kappa_p} \right) - 1 > 1 - 2\zeta \delta \) for \( p \in (0, 1) \) such that \( C_{jp} > 1 - p \). This proves Statement (III).

Finally, we would like to note that Statement (II) can be shown by employing Lemma 44 and similar argument as the proof of Statement (II) of Theorem 16.

I.13 Proof of Theorem 27

We need some preliminary results.

Lemma 47 \( \lim_{\varepsilon \to 0} \sum_{\ell_{\varepsilon}=1}^{\gamma} n_{\ell} \varepsilon^{-n_{\ell}c} \) for any \( c > 0 \).

Lemma 47 can be shown by a similar method as that of Lemma 14.
Lemma 40. Invoking the intermediate value theorem, we have that there exists a unique number \( \theta \) such that \( \ell - \tau \) is fixed with respect to \( \epsilon \) for sufficiently small \( \epsilon > 0 \). We claim that \( \theta < z_\ell < 1 \) for \( \theta \in (0, b_\ell) \) if \( \epsilon > 0 \) is small enough. To prove this claim, we use a contradiction method. Suppose the claim is not true, then there exists a set, denoted by \( S_\epsilon \), of infinite many values of \( \epsilon \) such that \( z_\ell \leq \theta \) for \( \epsilon \in S_\epsilon \). Hence, by Lemma 40 and the fact that \( \mathcal{M}_B(z, \frac{z}{1+\epsilon}) \) is monotonically decreasing with respect to \( z \in (0, 1) \) as asserted by Lemma 40, we have that \( z_\ell \) is monotonically decreasing with respect to \( \ell \leq \tau \). This establishes Statement (II).

Proof of Statement (II): Since \( n_\ell \) is monotonically increasing with respect to \( \ell \) for sufficiently small \( \epsilon > 0 \), we have that \( \mathcal{M}_B(z_\ell, \frac{z}{1+\epsilon}) \) is monotonically increasing with respect to \( \ell \leq \tau \) for sufficiently small \( \epsilon > 0 \). Recalling that \( \mathcal{M}_B(z, \frac{z}{1+\epsilon}) \) is monotonically decreasing with respect to \( z \in (0, 1) \) as asserted by Lemma 40, we have that \( z_\ell \) is monotonically decreasing with respect to \( \ell \leq \tau \). This establishes Statement (II).

Proof of Statement (III):

For simplicity of notations, let \( b_\ell = \left[ 1 + (1 - \frac{1}{p^*})C_{\tau - \ell} \right]^{-1} \) for \( \ell = 1, 2, \cdots, \tau \). Then, it can be checked that \( \frac{b_\ell}{b_{\ell-1}} = C_{\tau - \ell} \) for \( 1 \leq \ell \leq \tau \). By the definition of sample sizes, we have

\[
\mathcal{M}_B(z_\ell, \frac{z}{1+\epsilon}) / [2(b_\ell - 1)] = \frac{\ln(\zeta)}{n_\ell} \times \frac{2(p^* - 1)C_{\tau - \ell}}{p^* \epsilon^2} = 1 + o(1)
\]

for \( \ell = 1, \cdots, \tau \), where

\[
n_\ell = \frac{\ln(\zeta)}{\mathcal{M}_B(z_\ell, \frac{z}{1+\epsilon})} = \frac{[1 + o(1)]C_{\tau - \ell} \ln(\zeta)}{\mathcal{M}_B(p^*, \frac{p^*}{1+\epsilon})}.
\]

We claim that \( \theta < z_\ell < 1 \) for \( \theta \in (0, b_\ell) \) if \( \epsilon > 0 \) is small enough. To prove this claim, we use a contradiction method. Suppose the claim is not true, then there exists a set, denoted by \( S_\epsilon \), of infinite many values of \( \epsilon \) such that \( z_\ell \leq \theta \) for \( \epsilon \in S_\epsilon \). Hence, by Lemma 40 and the fact that \( \mathcal{M}_B(z, \frac{z}{1+\epsilon}) \) is monotonically decreasing with respect to \( z \in (0, 1) \) as asserted by Lemma 40, we have

\[
1 + o(1) = \frac{\mathcal{M}_B(z_\ell, \frac{z}{1+\epsilon}) / [2(b_\ell - 1)]}{\epsilon^2 b_\ell / [2(b_\ell - 1)]} \leq \frac{\mathcal{M}_B(\theta, \frac{\theta}{1+\epsilon}) / [2(1 - \theta)]}{\epsilon^2 b_\ell / [2(1 - \theta)]} = \frac{\epsilon^2 \theta / [2(1 - \theta)] + o(\epsilon^2)}{\epsilon^2 b_\ell / [2(1 - \theta)]} = \frac{\theta(1 - b_\ell)}{b_\ell(1 - \theta)} + o(1)
\]

for small enough \( \epsilon \in S_\epsilon \), which implies \( \frac{\theta(1 - b_\ell)}{b_\ell(1 - \theta)} \geq 1 \), contradicting to the fact that \( \frac{\theta(1 - b_\ell)}{b_\ell(1 - \theta)} < 1 \). This proves our claim. In a similar manner, we can show that \( 0 < z_\ell < \theta' \) for \( \theta' \in (b_\ell, 1) \) if \( \epsilon > 0 \) is small enough. By (9) and applying Lemma 12 based on the established condition that

\[
(9)
\]
\[ \theta < z_\ell < \theta' \] for small enough \( \varepsilon > 0 \), we have
\[ \frac{\mathcal{M}_B(z_\ell, \frac{z_\ell}{\ell})}{\mathcal{M}_B(B_z, \frac{2(1-z)}{3})} = \frac{\varepsilon^2 z_\ell / (2(1-z_\ell)) + \eta(\varepsilon)}{\varepsilon^2 b_\ell / (2(1-b_\ell))} = 1 + o(1), \]
which implies \( \frac{z_\ell}{\ell} - \frac{b_\ell}{\ell} = o(1) \) and consequently
\[ \lim_{\varepsilon \to 0} z_\ell = b_\ell. \] This proves Statement (III).

**Proof of Statement (IV):** Noting that \( \mathcal{M}_B(z, \frac{z}{1+\varepsilon}) \) is monotonically decreasing with respect to \( z \in (0, 1) \) as asserted by Lemma 47, we have \( D_\ell = 0 \) \( \mathcal{M}_B(\hat{p}_\ell, \frac{p}{1+\varepsilon}) > \frac{\ln(z_\ell)}{n_\ell} \) \( \mathcal{M}_B \), \( \hat{p}_\ell < z_\ell \) as claimed by statement (IV).

**Lemma 49** Let \( \ell_\varepsilon = \tau - j_\ell p \). Then,
\[ \lim_{\varepsilon \to 0} \sum_{\ell=1}^{\ell_\varepsilon-1} n_\ell \Pr[D_\ell = 1] = 0, \quad \lim_{\varepsilon \to 0} \sum_{\ell=\ell_\varepsilon}^{\tau} n_\ell \Pr[D_\ell = 0] = 0 \] (90)
for \( p \in (p^*, 1) \). Moreover, \( \lim_{\varepsilon \to 0} n_\ell \Pr[D_\ell = 0] = 0 \) for \( p \in (p^*, 1) \) such that \( C_{j_\ell p} > r(p) \).

**Proof.** For simplicity of notations, let \( b_\ell = \lim_{\ell \to 0} z_\ell \) for \( 1 \leq \ell \leq \tau \).

First, we shall show that (90) holds for \( p \in (p^*, 1) \). By the definition of \( \ell_\varepsilon \), we have \( r(p) > C_{\tau-\ell_\varepsilon+1} \). Making use of the first three statements of Lemma 48, we have that \( z_\ell > \frac{p+b_\ell}{2p} \) for all \( \ell \leq \ell_\varepsilon - 1 \) if \( \varepsilon \) is sufficiently small. By the last statement of Lemma 48 and using Chernoff bound, we have
\[ \Pr[D_\ell = 1] = \Pr[\hat{p}_\ell \geq z_\ell] \leq \Pr\left\{ \hat{p}_\ell > \frac{p+b_\ell}{2} \right\} \leq \exp\left( -2n_\ell \left( \frac{p-b_\ell}{2} \right)^2 \right) \]
for all \( \ell \leq \ell_\varepsilon - 1 \) provided that \( \varepsilon > 0 \) is small enough. By the definition of \( \ell_\varepsilon \), we have
\[ b_\ell = 1 + \left( 1 - \frac{1}{p} \right) C_{\tau-\ell_\varepsilon+1}^{-1} > p, \]
which implies that \( \left( \frac{p-b_\ell}{2} \right)^2 \) is a positive constant independent of \( \varepsilon > 0 \) provided that \( \varepsilon > 0 \) is small enough. Hence, \( \lim_{\varepsilon \to 0} \sum_{\ell=1}^{\ell_\varepsilon-1} n_\ell \Pr[D_\ell = 1] = 0 \) as a result of Lemma 47.

Similarly, it can be seen from the definition of \( \ell_\varepsilon \) that \( r(p) < C_{\tau-\ell_\varepsilon} \). Making use of the first three statements of Lemma 48, we have that \( z_\ell < \frac{p+b_\ell}{2} \) for \( \ell_\varepsilon + 1 \leq \ell \leq \tau \) if \( \varepsilon \) is sufficiently small. By the last statement of Lemma 48 and using Chernoff bound, we have
\[ \Pr[D_\ell = 0] = \Pr[\hat{p}_\ell < z_\ell] \leq \Pr\left\{ \hat{p}_\ell < \frac{p+b_\ell+1}{2} \right\} \leq \exp\left( -2n_\ell \left( \frac{p-b_\ell+1}{2} \right)^2 \right) \]
for \( \ell_\varepsilon + 1 \leq \ell \leq \tau \) provided that \( \varepsilon > 0 \) is small enough. As a consequence of the definition of \( \ell_\varepsilon \), we have that \( b_\ell+1 \) is smaller than \( p \) and is independent of \( \varepsilon > 0 \). Therefore, we can apply Lemma 47 to conclude that \( \lim_{\varepsilon \to 0} \sum_{\ell=\ell_\varepsilon}^{\tau} n_\ell \Pr[D_\ell = 0] = 0 \).

Second, we shall show that \( \lim_{\varepsilon \to 0} n_\ell \Pr[D_\ell = 0] = 0 \) for \( p \in (p^*, 1) \) such that \( C_{j_\ell p} > r(p) \). Clearly, \( r(p) < C_{\tau-\ell_\varepsilon+1} \) because of the definition of \( \ell_\varepsilon \). Making use of the first three statements of Lemma 48, we have \( z_\ell < \frac{p+b_\ell}{2} \) if \( \varepsilon \) is sufficiently small. By the last statement of Lemma 48 and using Chernoff bound, we have
\[ \Pr[D_\ell = 0] = \Pr[\hat{p}_\ell < z_\ell] \leq \Pr\left\{ \hat{p}_\ell < \frac{p+b_\ell}{2} \right\} \leq \exp\left( -2n_\ell \left( \frac{p-b_\ell}{2} \right)^2 \right) \]
for small enough \( \varepsilon > 0 \). As a consequence of the definition of \( \ell_\varepsilon \), we have that \( b_\ell \) is smaller than \( p \) and is independent of \( \varepsilon > 0 \). It follows that \( \lim_{\varepsilon \to 0} n_{\ell_\varepsilon} \Pr\{D_{\ell_\varepsilon} = 0\} = 0 \).

\[ \Box \]

**Lemma 50** \( \lim_{\varepsilon \to 0} \sum_{\ell=\tau+1}^{\infty} n_{\ell} \Pr\{l = \ell\} = 0 \) for any \( p \in (p^*, 1) \).

**Proof.** Recalling that the sample sizes \( n_1, n_2, \ldots \) are chosen as the ascending arrangement of all distinct elements of the set defined by (25), we have that

\[
n_\ell = \left\lceil \frac{C_{\tau - \ell} \ln(\zeta \delta)}{\mathcal{M}_B(p^*, \eta)} \right\rceil, \quad \ell = 1, 2, \ldots
\]

for small enough \( \varepsilon \in (0, 1) \). By the assumption that \( \inf_{\zeta \in \mathbb{Z}} \frac{C_{\tau - \ell}}{C_{\ell}} = 1 + \rho > 1 \), we have that

\[
n_\ell > (1 + \rho)^{\ell - \tau - 1} \frac{\ln(\zeta \delta)}{\mathcal{M}_B(p^*, \eta)}, \quad \ell = \tau + 1, \tau + 2, \ldots
\]

for small enough \( \varepsilon \in (0, 1) \). So, we have shown that there exists a number \( \varepsilon^* \in (0, 1) \) such that

\[
n_{\ell, \mathcal{M}_B}(p^*, \eta) < (1 + \rho)^{\ell - \tau - 1} \ln(\zeta \delta), \quad \ell = \tau + 1, \tau + 2, \ldots
\]

for any \( \varepsilon \in (0, \varepsilon^*) \). Observing that there exist a positive integer \( \kappa^* \) such that \( (1 + \rho)^{\ell - \tau - 1} \ln(\zeta \delta) < \ln(\zeta \delta) - (\ell - \tau) \ln 2 = \ln(\zeta \delta) \) for any \( \ell \geq \tau + \kappa^* \), we have that there exists a positive integer \( \kappa^* \) independent of \( \varepsilon \) such that \( \mathcal{M}_B(p^*, \eta) < \ln(\zeta \delta) \eta \) for \( \ell \geq \tau + \kappa^* \) and \( 0 < \varepsilon < \varepsilon^* \). Recall that \( \mathcal{M}_B(z, \frac{1}{1+\varepsilon}) \) is monotonically decreasing with respect to \( z \in (0, 1) \) as asserted by Lemma[10]. For \( \ell \geq \tau + \kappa^* \) and \( 0 < \varepsilon < \varepsilon^* \), as a result of \( \ln(\zeta \delta) > \mathcal{M}_B(p^*, \eta) > \mathcal{M}_B(1, \frac{1}{1+\varepsilon}) = \ln \frac{1}{1+\varepsilon} \), there exists a unique number \( z_\ell \in [0, 1] \) such that \( \mathcal{M}_B(z_\ell, \frac{1}{1+\varepsilon}) = \ln(\zeta \delta) \eta > \mathcal{M}_B(p^*, \eta) \). Moreover, it must be true that \( z_\ell < p^* \) for \( \ell \geq \tau + \kappa^* \) and \( \varepsilon \in (0, \varepsilon^*) \). Therefore, for small enough \( \varepsilon \in (0, \varepsilon^*) \), we have

\[
\sum_{\ell=\tau+1}^{\infty} n_{\ell} \Pr\{l = \ell\} = \sum_{\ell=\tau+1}^{\infty} n_{\ell} \Pr\{D_\tau = 0\} + \sum_{\ell=\tau+1}^{\infty} n_{\ell} \Pr\{D_{\ell-1} = 0\}
\leq \sum_{\ell=\tau+1}^{\tau+\kappa^*} n_{\ell} \Pr\{D_\tau = 0\} + \sum_{\ell=\tau+\kappa^*+1}^{\infty} n_{\ell} \Pr\{D_{\ell-1} = 0\}
= \sum_{\ell=\tau+1}^{\tau+\kappa^*} n_{\ell} \Pr\{D_\tau = 0\} + \sum_{\ell=\tau+\kappa^*}^{\infty} n_{\ell+1} \Pr\{D_\ell = 0\}
< k^* (1 + \rho)^{\kappa^*} n_{\tau} \Pr\{D_\tau = 0\} + (1 + \rho) \sum_{\ell=\tau+\kappa^*}^{\infty} n_{\ell} \Pr\{D_\ell = 0\}
\leq k^* (1 + \rho)^{k^*} n_{\tau} \Pr\{\hat{p}_\tau < z_\tau\} + (1 + \rho) \sum_{\ell=\tau+\kappa^*}^{\infty} n_{\ell} \Pr\{\hat{p}_\ell < z_\ell\}
\leq k^* (1 + \rho)^{k^*} n_{\tau} \exp \left( -\frac{n_\tau (p - p^*)^2}{2} \right) + (1 + \rho) \sum_{\ell=\tau+\kappa^*}^{\infty} n_{\ell} \exp(-2n_\ell(p - p^*)^2) \to 0
\]

131
Lemma 51 \( \lim_{\varepsilon \to 0} \frac{n_{l_{\varepsilon}}}{N_{l}(p, \varepsilon)} = \kappa_{p} \), \( \lim_{\varepsilon \to 0} \frac{\varepsilon p}{\sqrt{(1-p)/n_{l_{\varepsilon}}}} = d \sqrt{\kappa_{p}} \).

Proof. By the definition of sample sizes, it can be readily shown that \( \lim_{\varepsilon \to 0} \frac{2(1-p^*)C_{r-\ell_{\varepsilon}} \ln \frac{1}{\varepsilon \delta}}{p^* \ell_{\varepsilon}^2 n_{l_{\varepsilon}}} = 1 \) for any \( \ell \geq 1 \) and it follows that

\[
\lim_{\varepsilon \to 0} \frac{n_{l_{\varepsilon}}}{N_{l}(p, \varepsilon)} = \lim_{\varepsilon \to 0} \frac{\mathcal{A}_{l_{\varepsilon}}(p, \frac{\varepsilon^2}{1-p}) \ln(\varepsilon \delta)}{\ln(\varepsilon \delta)} \times \frac{2(1-p^*)C_{r-\ell_{\varepsilon}}}{p^* \ell_{\varepsilon}^2 n_{l_{\varepsilon}}} \ln \frac{1}{\varepsilon \delta} = \lim_{\varepsilon \to 0} \frac{2(1-p^*)C_{r-\ell_{\varepsilon}}}{p^* \ell_{\varepsilon}^2 n_{l_{\varepsilon}}} \ln \frac{1}{\varepsilon \delta} = \kappa_{p}.
\]

\[
\lim_{\varepsilon \to 0} \frac{\varepsilon p}{\sqrt{(1-p)/n_{l_{\varepsilon}}}} = \lim_{\varepsilon \to 0} \frac{\varepsilon p}{\sqrt{(1-p)/n_{l_{\varepsilon}}}} \frac{2(1-p^*)C_{r-\ell_{\varepsilon}}}{p^* \ell_{\varepsilon}^2 n_{l_{\varepsilon}}} \ln \frac{1}{\varepsilon \delta} = d \sqrt{\frac{2(1-p^*)C_{r-\ell_{\varepsilon}}}{p^* \ell_{\varepsilon}^2 n_{l_{\varepsilon}}}} \ln \frac{1}{\varepsilon \delta} = d \sqrt{\frac{2(1-p^*)C_{r-\ell_{\varepsilon}}}{p^*(1-p)}} = d \sqrt{\kappa_{p}}.
\]

Lemma 52 Let \( U \) and \( V \) be independent Gaussian random variables with zero means and unit variances. Then, for \( p \in (p^*, 1) \) such that \( C_{j_{p}} = r(p) \),

\[
\lim_{\varepsilon \to 0} \Pr\{ l = \ell_{\varepsilon} \} = 1 - \lim_{\varepsilon \to 0} \Pr\{ l = \ell_{\varepsilon} + 1 \} = 1 - \Phi(\nu d),
\]

\[
\lim_{\varepsilon \to 0} \left[ \Pr\{ \hat{p}_{\ell_{\varepsilon}} - p \geq \varepsilon p, l = \ell_{\varepsilon} \} + \Pr\{ \hat{p}_{\ell_{\varepsilon} + 1} - p \geq \varepsilon p, l = \ell_{\varepsilon} + 1 \} \right]
\]

\[
= \Pr\{ U \geq d \} + \Pr\{ |U + \sqrt{p} V| \geq d \sqrt{1 + \rho_p}, U < \nu d \}.
\]

Lemma 52 can be shown by a similar method as that of Lemma 46.

I.13.1 Proof of Statement (I)

First, we shall show that Statement (I) holds for \( p \in (p^*, 1) \) such that \( C_{j_{p}} = r(p) \). For this purpose, we need to show that

\[
1 \leq \limsup_{\varepsilon \to 0} \frac{n(\omega)}{N_{l}(p, \varepsilon)} \leq 1 + \rho_p \quad \text{for any} \ \omega \in \left\{ \lim_{\varepsilon \to 0} \hat{p} = p \right\}.
\]

To show \( \limsup_{\varepsilon \to 0} \frac{n(\omega)}{N_{l}(p, \varepsilon)} \geq 1 \), note that \( C_{r-\ell_{\varepsilon} + 1} < r(p) = C_{r-\ell_{\varepsilon}} < C_{r-\ell_{\varepsilon} - 1} \) as a direct consequence of the definitions of \( \ell_{\varepsilon} \) and \( j_{p} \). By the first three statements of Lemma 48, we
have \( \lim_{\varepsilon \to 0} z_\ell > p \) for all \( \ell \leq \ell \varepsilon - 1 \). Noting that \( \lim_{\varepsilon \to 0} \hat{p}(\omega) = p \), we have \( \hat{p}(\omega) < z_\ell \) for all \( \ell \leq \ell \varepsilon - 1 \) and it follows from the definition of the sampling scheme that \( n_{\ell \varepsilon} \leq n(\omega) \) if \( \varepsilon > 0 \) is small enough. By Lemma 51 and noting that \( \kappa_p = 1 \) if \( C_{j_p} = r(p) \), we have \( \limsup_{\varepsilon \to 0} n(\omega)/N(\ell \varepsilon, p, \varepsilon) \geq \lim_{\varepsilon \to 0} n(\omega)/N(\ell \varepsilon, p, \varepsilon) = \kappa_p = 1 \).

To show \( \limsup_{\varepsilon \to 0} n(\omega)/N(\ell \varepsilon, p, \varepsilon) \leq 1 + p \rho_p \), note that \( \ell \varepsilon + 1 \leq \tau \) as a result of \( p^* < p < 1 \) and the assumption that \( C_{j_p} = r(p) \). By virtue of Lemma 18, we have \( \lim_{\varepsilon \to 0} z_{\ell \varepsilon + 1} < p \), which implies \( \hat{p}(\omega) > z_{\ell \varepsilon + 1} \) and thus \( n(\omega) \leq n_{\ell \varepsilon + 1} \) for small enough \( \varepsilon \in (0, 1) \). Therefore, \( \limsup_{\varepsilon \to 0} n(\omega)/N(\ell \varepsilon, p, \varepsilon) \leq \lim_{\varepsilon \to 0} n_{\ell \varepsilon + 1} = \lim_{\varepsilon \to 0} n_{\ell \varepsilon + 1} \times \lim_{\varepsilon \to 0} n_{\ell \varepsilon} = 1 + p \rho_p \). This establishes 91, which implies \( \{1 \leq \limsup_{\varepsilon \to 0} n(\omega)/N(\ell \varepsilon, p, \varepsilon) \} \geq \lim_{\varepsilon \to 0} \hat{p} = p \}. Applying the strong law of large numbers, we have \( 1 \geq \Pr\{1 \leq \limsup_{\varepsilon \to 0} n(\omega)/N(\ell \varepsilon, p, \varepsilon) \} \geq \Pr\{\lim_{\varepsilon \to 0} \hat{p} = p \} = 1 \). This proves that Statement (I) holds for \( p \in (p^*, 1) \) such that \( C_{j_p} = r(p) \).

Next, we shall show that Statement (I) holds for \( p \in (p^*, 1) \) such that \( C_{j_p} > r(p) \). Note that \( C_{\tau - \ell \varepsilon + 1} < r(p) \) as a direct consequence of the definition of \( \ell \varepsilon \) and the assumption that \( C_{j_p} > r(p) \). By the first three statements of Lemma 18, we have \( \lim_{\varepsilon \to 0} z_{\ell \varepsilon + 1} > p \geq \lim_{\varepsilon \to 0} z_{\ell \varepsilon} \) and thus \( z_{\ell} > p \geq z_{\ell \varepsilon} \) for all \( \ell \leq \ell \varepsilon - 1 \) provided that \( \varepsilon \in (0, 1) \) is sufficiently small. Therefore, for any \( \omega \in \{\lim_{\varepsilon \to 0} \hat{p} = p \} \), we have \( z_{\ell} > \hat{p}(\omega) > z_{\ell \varepsilon} \) for all \( \ell \leq \ell \varepsilon - 1 \) and consequently, \( n(\omega) = n_{\ell \varepsilon} \) provided that \( \varepsilon \in (0, 1) \) is sufficiently small. Applying Lemma 51, we have \( \lim_{\varepsilon \to 0} n(\omega)/N(\ell \varepsilon, p, \varepsilon) = \lim_{\varepsilon \to 0} n_{\ell \varepsilon}/N(\ell \varepsilon, p, \varepsilon) = \kappa_p \), which implies that \( \{\lim_{\varepsilon \to 0} n(\omega)/N(\ell \varepsilon, p, \varepsilon) = \kappa_p \} \geq \{\lim_{\varepsilon \to 0} \hat{p} = p \} \). It follows from the strong law of large numbers that \( 1 \geq \Pr\{\lim_{\varepsilon \to 0} n(\omega)/N(\ell \varepsilon, p, \varepsilon) = \kappa_p \} \geq \Pr\{\lim_{\varepsilon \to 0} \hat{p} = p \} \) and thus \( \Pr\{\lim_{\varepsilon \to 0} n(\omega)/N(\ell \varepsilon, p, \varepsilon) = \kappa_p \} = 1 \). Since \( 1 \leq \rho_p \leq 1 + \rho_p \), we have that \( \Pr\{1 \leq \limsup_{\varepsilon \to 0} n(\omega)/N(\ell \varepsilon, p, \varepsilon) \} \leq 1 + \rho_p \} \) is of course true. This proves that Statement (I) holds for \( p \in (p^*, 1) \) such that \( C_{j_p} > r(p) \). The proof of Statement (I) is thus completed.

I.13.2 Proof of Statement (II)

In the sequel, we will consider the asymptotic value of \( \mathbb{E}[n]/N(\ell \varepsilon, p, \varepsilon) \) in three steps. First, we shall show Statement (II) for \( p \in (p^*, 1) \) such that \( C_{j_p} = r(p) \). Clearly, \( \ell \varepsilon < \tau \). By the definition of the sampling scheme, we have

\[
\mathbb{E}[n] = \sum_{\ell=1}^{\ell \varepsilon - 1} n_{\ell} \Pr\{l = \ell \} + \sum_{\ell=\ell \varepsilon + 2}^{\tau} n_{\ell} \Pr\{l = \ell \} + \sum_{\ell=\tau + 1}^{\infty} n_{\ell} \Pr\{l = \ell \} + n_{\ell \varepsilon} \Pr\{l = \ell \varepsilon \} + n_{\ell \varepsilon + 1} \Pr\{l = \ell \varepsilon + 1 \}
\leq \sum_{\ell=1}^{\ell \varepsilon - 1} n_{\ell} \Pr\{D_\ell = 1 \} + \sum_{\ell=\ell \varepsilon + 2}^{\tau} n_{\ell + 1} \Pr\{D_\ell = 0 \} + \sum_{\ell=\tau + 1}^{\infty} n_{\ell} \Pr\{l = \ell \} + n_{\ell \varepsilon} \Pr\{l = \ell \varepsilon \} + n_{\ell \varepsilon + 1} \Pr\{l = \ell \varepsilon + 1 \}
\]
and $E[n] \geq n_{t_\varepsilon} \Pr(l = t_\varepsilon) + n_{t_{\varepsilon}+1} \Pr(l = t_\varepsilon + 1)$. Making use of Lemmas 49, 50 and the assumption that $\sup_{\varepsilon > 0} \frac{n_{t_{\varepsilon}+1}}{n_{t_\varepsilon}} < \infty$ for small enough $\varepsilon > 0$, we have

$$
\lim_{\varepsilon \to 0} \left[ \sum_{t=1}^{t_{\varepsilon}-1} n_{t_\varepsilon} \Pr(D_t = 1) + \sum_{t=t_{\varepsilon}+1}^{\tau-1} n_{t_{\varepsilon}+1} \Pr(D_t = 0) + \sum_{t=\tau+1}^{\infty} n_{t_\varepsilon} \Pr(l = t) \right]
\leq \lim_{\varepsilon \to 0} \left[ \sum_{t=1}^{t_{\varepsilon}-1} n_{t_\varepsilon} \Pr(D_t = 1) + \sup_{\varepsilon > 0} \frac{n_{t_{\varepsilon}+1}}{n_{t_\varepsilon}} \sum_{t=t_{\varepsilon}+1}^{\tau-1} n_{t_\varepsilon} \Pr(D_t = 0) + \sum_{t=\tau+1}^{\infty} n_{t_\varepsilon} \Pr(l = t) \right] = 0.
$$

Therefore,

$$
\limsup_{\varepsilon \to 0} \frac{E[n]}{\mathcal{N}_t(p, \varepsilon)} \leq \lim_{\varepsilon \to 0} \frac{n_{t_\varepsilon} \Pr(l = t_\varepsilon) + n_{t_{\varepsilon}+1} \Pr(l = t_{\varepsilon} + 1)}{\mathcal{N}_t(p, \varepsilon)}
$$

and

$$
\liminf_{\varepsilon \to 0} \frac{E[n]}{\mathcal{N}_t(p, \varepsilon)} \geq \lim_{\varepsilon \to 0} \frac{n_{t_\varepsilon} \Pr(l = t_\varepsilon) + n_{t_{\varepsilon}+1} \Pr(l = t_{\varepsilon} + 1)}{\mathcal{N}_t(p, \varepsilon)}.
$$

It follows that

$$
\lim_{\varepsilon \to 0} \frac{E[n]}{\mathcal{N}_t(p, \varepsilon)} = \lim_{\varepsilon \to 0} \frac{n_{t_\varepsilon} \Pr(l = t_\varepsilon) + n_{t_{\varepsilon}+1} \Pr(l = t_{\varepsilon} + 1)}{\mathcal{N}_t(p, \varepsilon)} = \frac{\lim_{\varepsilon \to 0} n_{t_\varepsilon} [1 - \Phi(\nu d)] + n_{t_{\varepsilon}+1} \Phi(\nu d)}{\mathcal{N}_t(p, \varepsilon)} = 1 + \rho \Phi(\nu d).
$$

Second, we shall show Statement (II) for $p \in (p^*, 1)$ such that $C_{j_p} > r(p)$. Note that

$$
E[n] = \sum_{t=1}^{t_{\varepsilon}-1} n_{t_\varepsilon} \Pr(l = t) + \sum_{t=t_{\varepsilon}+1}^{\tau-1} n_{t_{\varepsilon}+1} \Pr(l = t) + \sum_{t=\tau+1}^{\infty} n_{t_\varepsilon} \Pr(l = t) \leq \sum_{t=1}^{t_{\varepsilon}-1} n_{t_\varepsilon} \Pr(D_t = 1) + \sum_{t=t_{\varepsilon}+1}^{\tau-1} n_{t_{\varepsilon}+1} \Pr(D_t = 0) + n_{t_\varepsilon} + \sum_{t=\tau+1}^{\infty} n_{t_\varepsilon} \Pr(l = t)
$$

and $E[n] \geq n_{t_\varepsilon} \Pr(l = t_\varepsilon) \geq n_{t_\varepsilon} \left( 1 - \sum_{t=1}^{t_{\varepsilon}-1} \Pr(D_t = 1) - \Pr(D_t = 0) \right)$. Therefore, by Lemma 49

$$
\limsup_{\varepsilon \to 0} \frac{E[n]}{\mathcal{N}_t(p, \varepsilon)} \leq \lim_{\varepsilon \to 0} \frac{\sum_{t=1}^{t_{\varepsilon}-1} n_{t_\varepsilon} \Pr(D_t = 1) + \sum_{t=t_{\varepsilon}+1}^{\tau-1} n_{t_{\varepsilon}+1} \Pr(D_t = 0) + n_{t_\varepsilon} + \sum_{t=\tau+1}^{\infty} n_{t_\varepsilon} \Pr(l = t)}{\mathcal{N}_t(p, \varepsilon)} = \lim_{\varepsilon \to 0} \frac{n_{t_\varepsilon}}{\mathcal{N}_t(p, \varepsilon)} = \kappa_p.
$$

$$
\liminf_{\varepsilon \to 0} \frac{E[n]}{\mathcal{N}_t(p, \varepsilon)} \geq \lim_{\varepsilon \to 0} \frac{n_{t_\varepsilon} \left( 1 - \sum_{t=1}^{t_{\varepsilon}-1} \Pr(D_t = 1) - \Pr(D_t = 0) \right)}{\mathcal{N}_t(p, \varepsilon)} = \lim_{\varepsilon \to 0} \frac{n_{t_\varepsilon}}{\mathcal{N}_t(p, \varepsilon)} = \kappa_p.
$$

So, $\lim_{\varepsilon \to 0} \frac{E[n]}{\mathcal{N}_t(p, \varepsilon)} = \kappa_p$ for $p \in (p^*, 1)$ such that $C_{j_p} > r(p)$.

From the preceding analysis, we have obtained $\limsup_{\varepsilon \to 0} \frac{E[n]}{\mathcal{N}_t(p, \varepsilon)}$ for all $p \in (p^*, 1)$. Hence, statement (II) is established by making use of this result and the fact that

$$
\lim_{\varepsilon \to 0} \frac{E[n]}{\mathcal{N}_t(p, \varepsilon)} = \lim_{\varepsilon \to 0} \frac{\mathcal{N}_t(p, \varepsilon)}{\mathcal{N}_t(p, \varepsilon)} \times \lim_{\varepsilon \to 0} \frac{E[n]}{\mathcal{N}_t(p, \varepsilon)} = \frac{2 \ln \frac{1}{\varepsilon}}{Z^2_{\delta}} \times \lim_{\varepsilon \to 0} \frac{E[n]}{\mathcal{N}_t(p, \varepsilon)}.
$$
I.13.3 Proof of Statement (III)

First, we shall consider \( p \in (p^*, 1) \) such that \( C_{jp} = r(p) \). In this case, it is evident that \( \ell_\varepsilon < \tau \). By the definition of the sampling scheme, we have that \( \Pr\{l > \ell_\varepsilon + 1\} \leq \Pr\{D_{\ell_\varepsilon + 1} = 0\} \) and that \( \Pr\{l = \ell\} \leq \Pr\{D_\ell = 1\} \) for \( \ell < \ell_\varepsilon \). As a result of Lemma 19, we have \( \lim_{\varepsilon \to 0} \Pr\{l > \ell_\varepsilon + 1\} \leq \lim_{\varepsilon \to 0} \Pr\{D_{\ell_\varepsilon + 1} = 0\} = 0 \) and \( \lim_{\varepsilon \to 0} \Pr\{l < \ell_\varepsilon\} \leq \lim_{\varepsilon \to 0} \sum_{\ell=1}^{\ell_\varepsilon-1} \Pr\{D_\ell = 1\} = 0 \). Since

\[
\limsup_{\varepsilon \to 0} \Pr\{l > \ell_\varepsilon + 1\} \leq \lim_{\varepsilon \to 0} \left[ \Pr\{l = \ell_\varepsilon\} + \Pr\{l = \ell_\varepsilon + 1\} \right] + \lim_{\varepsilon \to 0} \Pr\{l < \ell_\varepsilon\} + \lim_{\varepsilon \to 0} \Pr\{l > \ell_\varepsilon + 1\}
\]

and

\[
\liminf_{\varepsilon \to 0} \Pr\{l > \ell_\varepsilon + 1\} \geq \lim_{\varepsilon \to 0} \left[ \Pr\{l = \ell_\varepsilon\} + \Pr\{l = \ell_\varepsilon + 1\} \right]
\]

we have

\[
\lim_{\varepsilon \to 0} \Pr\{l > \ell_\varepsilon + 1\} = \lim_{\varepsilon \to 0} \left[ \Pr\{l = \ell_\varepsilon\} + \Pr\{l = \ell_\varepsilon + 1\} \right].
\]

By Lemma 21, we have \( \lim_{\varepsilon \to 0} \Pr\{l > \ell_\varepsilon + 1\} = \Pr\{U \geq d\} + \Pr\{|U + \sqrt{\rho_p}V| \geq d\sqrt{1 + \rho_p}, U < \nu d\} \) for \( p \in (p^*, 1) \) such that \( C_{jp} = r(p) \). As a consequence of Lemma 21, Statement (III) must be true for \( p \in (p^*, 1) \) such that \( C_{jp} = r(p) \).

Next, we shall consider \( p \in (p^*, 1) \) such that \( C_{jp} > r(p) \). Applying Lemma 19, we have

\[
\lim_{\varepsilon \to 0} \Pr\{l < \ell_\varepsilon\} \leq \lim_{\varepsilon \to 0} \sum_{\ell=1}^{\ell_\varepsilon-1} \Pr\{D_\ell = 1\} \leq \lim_{\varepsilon \to 0} \sum_{\ell=1}^{\ell_\varepsilon-1} n_\ell \Pr\{D_\ell = 1\} = 0,
\]

and thus \( \lim_{\varepsilon \to 0} \Pr\{l \neq \ell_\varepsilon\} = 0 \). Note that \( \Pr\{l > \ell_\varepsilon + 1\} = \Pr\{l = \ell_\varepsilon\} + \Pr\{l = \ell_\varepsilon + 1\} \) and, as a result of the central limit theorem, \( U_{\ell_\varepsilon} = \frac{\hat{p}_\varepsilon - p}{\sqrt{p(1-p)/n_{\ell_\varepsilon}}} \) converges in distribution to a standard Gaussian variable \( U \). Hence,

\[
\lim_{\varepsilon \to 0} \Pr\{l > \ell_\varepsilon + 1\} = \lim_{\varepsilon \to 0} \Pr\{l = \ell_\varepsilon\} + \Pr\{l = \ell_\varepsilon + 1\} = \lim_{\varepsilon \to 0} \left[ \Pr\{|U_{\ell_\varepsilon}| \geq \frac{\varepsilon p}{\sqrt{p(1-p)/n_{\ell_\varepsilon}}} \right] = \Pr\{|U| \geq d\sqrt{\nu p}\}
\]

and \( \lim_{\varepsilon \to 0} \Pr\{l < \ell_\varepsilon\} = \Pr\{|U| < d\sqrt{\nu p}\} = 2\Phi(d\sqrt{\nu p}) - 2\Phi(d) - 1 > 1 - 2\zeta \delta \) for \( p \in (p^*, 1) \). Here we have used the fact that \( \Phi(z) > 1 - e^{-\frac{z^2}{2}} \) and \( \Phi(d) = \Phi(\sqrt{2\ln \frac{1}{\zeta \delta}}) > 1 - \zeta \delta \). This proves Statement (III).

I.14 Proof of Theorem 28

We need some preliminary results.

Lemma 53. \( \mathcal{M}_B(z, \zeta - \varepsilon) \) is monotonically increasing with respect to \( z \in (\varepsilon, p + \varepsilon) \) provided that \( 0 < \varepsilon < \frac{35}{34} \) and \( 0 < p < \frac{1}{2} - \frac{12}{35} \varepsilon \).
**Proof.** Define $g(\varepsilon, p) = \frac{\varepsilon}{p(1-p)} + \ln \frac{g(1-p-\varepsilon)}{(p+\varepsilon)(1-p)}$ for $0 < p < 1$ and $0 < \varepsilon < 1 - p$. We shall first show that $g(\varepsilon, p) > 0$ if $0 < \varepsilon < \frac{35}{94}$ and $0 < p < \frac{1}{2} - \frac{12}{35}\varepsilon$.

Let $\frac{1}{2} < k < 1$ and $0 < \varepsilon \leq \frac{1}{2(1+k)}$. It can be shown by tedious computation that $\frac{\partial g(\varepsilon, \frac{1}{2} - k\varepsilon)}{\partial \varepsilon} = \frac{16e^{\varepsilon}\left[3\frac{2}{1+k} - 4(1-k)\varepsilon^2 - 4k\varepsilon - \frac{1}{1+k}\right]}{(1-k)^2\varepsilon^2(1-k+1)^2}$, which implies that $g(\varepsilon, \frac{1}{2} - k\varepsilon)$ is monotonically increasing with respect to $\varepsilon \in \left(0, \frac{1}{3\varepsilon}\right)$ and is monotonically decreasing with respect to $\varepsilon \in \left(\frac{1}{2k+1}, \frac{1}{2(1+k)}\right]$. Since $g(0, \frac{1}{2}) = 0$, we have that $g(\varepsilon, \frac{1}{2} - k\varepsilon)$ is positive for $0 < \varepsilon \leq \frac{1}{2(1+k)}$ if $g(\varepsilon, \frac{1}{2} - k\varepsilon)$ is positive for $\varepsilon = \frac{1}{2(1+k)}$. For $\varepsilon = \frac{1}{2(1+k)}$ with $k = \frac{12}{35}$, we have $g(\varepsilon, \frac{1}{2} - k\varepsilon) = 1 + \frac{1}{2k+1} - \ln\left(2 + \frac{1}{k}\right) = 1 + \frac{35}{39} - \ln\left(2 + \frac{35}{39}\right)$, which is positive because $e \times e^{\frac{35}{39}} > 2.718 \times \sum_{i=0}^{4} \frac{1}{(35/39)^i} > 2 + \frac{35}{12}$. It follows that $g(\varepsilon, \frac{1}{2} - \frac{12}{35}\varepsilon)$ is positive for any $\varepsilon \in \left(0, \frac{35}{39}\right]$. Since $\frac{\partial g(\varepsilon, p)}{\partial p} = -\varepsilon^2 \left[\frac{1}{(1-p+\varepsilon)^2} + \frac{1}{(1-p-\varepsilon)^2}\right]$ is negative, we have that $g(\varepsilon, p)$ is positive for $0 < \varepsilon < \frac{35}{39}$ if $0 < p < \frac{1}{2} - \frac{12}{35}\varepsilon$.

Finally, the lemma is established by verifying that $\frac{\partial^2 \mathcal{M}_B(z, z-\varepsilon)}{\partial \varepsilon^2} = -\varepsilon^2 \left[\frac{1}{z(z-\varepsilon)^2} + \frac{1}{(1-z)(1-z+\varepsilon)^2}\right] < 0$ for any $z \in (\varepsilon, 1)$ and that $\frac{\partial \mathcal{M}_B(z, z-\varepsilon)}{\partial z} = g(\varepsilon, p)$. 

\[\square\]

**Lemma 54.** $\mathcal{M}_B(p - \varepsilon, p) < \mathcal{M}_B(p + \varepsilon, p) < -2\varepsilon^2$ for $0 < \varepsilon < p < \frac{1}{2} < 1 - \varepsilon$.

**Proof.** The lemma follows from the facts that $\mathcal{M}_B(p - \varepsilon, p) - \mathcal{M}_B(p + \varepsilon, p) = 0$ for $\varepsilon = 0$ and that

$$\frac{\partial \left[\mathcal{M}_B(p - \varepsilon, p) - \mathcal{M}_B(p + \varepsilon, p)\right]}{\partial \varepsilon} = \ln \left[1 + \frac{\varepsilon^2}{p^2 (1-p)^2 - \varepsilon^2}\right],$$

where the right side is negative for $0 < \varepsilon < p < \frac{1}{2} < 1 - \varepsilon$. By Lemma 5, we have $\mathcal{M}_B(p + \varepsilon, p) < -2\varepsilon^2$ for $0 < \varepsilon < p < \frac{1}{2} < 1 - \varepsilon$. This completes the proof of the lemma. 

\[\square\]

**Lemma 55.** $\mathcal{M}_B(z, \frac{z}{1-\varepsilon})$ is monotonically decreasing from $0$ to $-\infty$ as $z$ increases from $0$ to $1 - \varepsilon$.

**Proof.** The lemma can be shown by verifying that

$$\lim_{z \to 0} \mathcal{M}_B \left( z, \frac{z}{1-\varepsilon} \right) = 0, \quad \lim_{z \to 1 - \varepsilon} \mathcal{M}_B \left( z, \frac{z}{1-\varepsilon} \right) = -\infty, \quad \lim_{z \to 0} \frac{\partial}{\partial z} \mathcal{M}_B \left( z, \frac{z}{1-\varepsilon} \right) = \ln \frac{1}{1-\varepsilon} - \frac{\varepsilon}{1-\varepsilon} < 0$$

and $\frac{\partial^2}{\partial z^2} \mathcal{M}_B \left( z, \frac{z}{1-\varepsilon} \right) = \frac{\varepsilon^2}{(z-1)(1-\varepsilon-z)^2} < 0$ for any $z \in (0, 1 - \varepsilon)$.

\[\square\]

**Lemma 56.** $\mathcal{M}_B(z, \frac{z}{1+\varepsilon}) > \mathcal{M}_B(z, \frac{z}{1-\varepsilon})$ for $0 < z < 1 - \varepsilon < 1$. 

136
Lemma 57 \{ \mathcal{M}_B(\hat{p}_s, \mathcal{L}(\hat{p}_s)) \leq \frac{\ln(\delta)}{n_s}, \mathcal{M}_B(\hat{p}_s, \mathcal{M}(\hat{p}_s)) \leq \frac{\ln(\delta)}{n_s} \} \text{ is a sure event.}

Proof. For simplicity of notations, we denote \( p^* = \frac{\varepsilon_a}{\varepsilon_r} \). In order to show the lemma, it suffices to show

\[
\begin{align*}
\{ \mathcal{M}_B\left(\hat{p}_s, \hat{p}_s, 1 - \varepsilon_r\right) > \frac{\ln(\delta)}{n_s}, \hat{p}_s > p^* - \varepsilon_a \} & = \emptyset, \\
\{ \mathcal{M}_B(\hat{p}_s, \hat{p}_s + \varepsilon_a) > \frac{\ln(\delta)}{n_s}, \hat{p}_s \leq p^* - \varepsilon_a \} & = \emptyset, \\
\{ \mathcal{M}_B\left(\hat{p}_s, \hat{p}_s, 1 + \varepsilon_r\right) > \frac{\ln(\delta)}{n_s}, \hat{p}_s > p^* + \varepsilon_a \} & = \emptyset, \\
\{ \mathcal{M}_B(\hat{p}_s, \hat{p}_s - \varepsilon_a) > \frac{\ln(\delta)}{n_s}, \hat{p}_s \leq p^* + \varepsilon_a \} & = \emptyset.
\end{align*}
\]

By the definition of \( n_s \), we have \( n_s \geq \left\lceil \frac{\ln(\delta)}{\mathcal{M}_B(p^* + \varepsilon_a, p^* + \varepsilon_a)} \right\rceil \geq \frac{\ln(\delta)}{\mathcal{M}_B(p^* + \varepsilon_a, p^*)} \). By the assumption on \( \varepsilon_a \) and \( \varepsilon_r \), we have 0 < \( \varepsilon_a < p^* < \frac{1}{2} \) < 1 - \( \varepsilon_a \). Hence, by Lemma 54, we have \( \mathcal{M}_B(p^* - \varepsilon_a, p^*) < \mathcal{M}_B(p^* + \varepsilon_a, p^*) < 0 \) and it follows that

\[
\frac{\ln(\delta)}{n_s} \geq \mathcal{M}_B(p^* + \varepsilon_a, p^*) > \mathcal{M}_B(p^* - \varepsilon_a, p^*).
\]

By (99),

\[
\left\{ \mathcal{M}_B\left(\hat{p}_s, \hat{p}_s, 1 - \varepsilon_r\right) > \frac{\ln(\delta)}{n_s}, \hat{p}_s > p^* - \varepsilon_a \right\} \subseteq \left\{ \mathcal{M}_B\left(\hat{p}_s, \hat{p}_s, 1 - \varepsilon_r\right) > \mathcal{M}_B(p^* - \varepsilon_a, p^*), \hat{p}_s > p^* - \varepsilon_a \right\}.
\]

Noting that \( \mathcal{M}_B(p^* - \varepsilon_a, p^*) = \mathcal{M}_B\left(p^* - \varepsilon_a, \frac{p^* - \varepsilon_a}{1 - \varepsilon_r}\right) \) and making use of the fact that \( \mathcal{M}_B(z, \frac{z}{1 - \varepsilon}) \) is monotonically decreasing with respect to \( z \in (0, 1 - \varepsilon) \) as asserted by Lemma 55, we have

\[
\left\{ \mathcal{M}_B\left(\hat{p}_s, \hat{p}_s, 1 - \varepsilon_r\right) > \mathcal{M}_B(p^* - \varepsilon_a, p^*) \right\} = \{ \hat{p}_s < p^* - \varepsilon_a \}.
\]

Combining (97) and (98) yields (92). By (96),

\[
\left\{ \mathcal{M}_B(\hat{p}_s, \hat{p}_s + \varepsilon_a) \geq \frac{\ln(\delta)}{n_s}, \hat{p}_s \leq p^* - \varepsilon_a \right\} \subseteq \left\{ \mathcal{M}_B(\hat{p}_s, \hat{p}_s + \varepsilon_a) \geq \mathcal{M}_B(p^* - \varepsilon_a, p^*), \hat{p}_s \leq p^* - \varepsilon_a \right\}.
\]

By the assumption on \( \varepsilon_a \) and \( \varepsilon_r \), we have \( p^* - \varepsilon_a < \frac{1}{2} - \varepsilon_a \). Recalling the fact that \( \mathcal{M}_B(z, z + \varepsilon) \) is monotonically increasing with respect to \( z \in (0, \frac{1}{2} - \varepsilon) \) as asserted by Lemma 16, we have that the
Lemma 58. \( f \) is an impossible event and consequently, \( \{ D_s = 1 \} \) is established. By (95),

\[
\mathcal{M}_B \left( \hat{p}_s, \frac{\hat{p}_s}{1 + \varepsilon_r} \right) > \frac{\ln(\delta)}{n_s}, \quad \hat{p}_s > p^* + \varepsilon_a \}
\]

Noting that \( \mathcal{M}_B (p^* + \varepsilon_a, p^* ) = \mathcal{M}_B \left( p^* + \varepsilon_a, \frac{p^* + \varepsilon_a}{1 + \varepsilon_r} \right) \) and making use of the fact that \( \mathcal{M}_B (z, z + \varepsilon) \) is monotonically decreasing with respect to \( z \in (0, 1) \) as asserted by Lemma 40, we have

\[
\left\{ \mathcal{M}_B \left( \hat{p}_s, \frac{\hat{p}_s}{1 + \varepsilon_r} \right) > \mathcal{M}_B (p^* + \varepsilon_a, p^* ) \right\} = \left\{ \hat{p}_s < p^* + \varepsilon_a \right\}.
\]

Combining (100) and (101) yields (94). By (96),

\[
\left\{ \mathcal{M}_B(\hat{p}_s, \hat{p}_s - \varepsilon_a) > \mathcal{M}_B (p^* + \varepsilon_a, p^* ) \right\} \subseteq \left\{ \mathcal{M}_B(\hat{p}_s, \hat{p}_s - \varepsilon_a) > \mathcal{M}_B (p^* + \varepsilon_a, p^* ) \right\}.
\]

By the assumption on \( \varepsilon_a \) and \( \varepsilon_r \), we have that \( \mathcal{M}_B (z, z - \varepsilon) \) is monotonically increasing with respect to \( z \in (\varepsilon_a, p^* + \varepsilon_a) \) as a result of Lemma 53. Hence, the event in the right-hand side of (102) is an impossible event and consequently, (95) is established. This completes the proof of the lemma.

Now we are in a position to prove Theorem 28. If the multistage sampling scheme follows a stopping rule derived from Chernoff bounds, then \( \{ D_s = 1 \} \) is a sure event as a result of Lemma 57. Note that \( \mathcal{M}_B(\hat{z}, \hat{p}) = \inf_{t > 0} e^{-t} \mathbb{E}[e^{t\hat{p}}] \) and that \( \hat{p}_t \) is a ULE of \( p \) for \( \ell = 1, \ldots, s \). So, the sampling scheme satisfies all the requirements described in Corollary 11 from which Theorem 28 immediately follows.

If the multistage sampling scheme follows a stopping rule derived from CDFs, then, by Lemmas 41 we have

\[
1 \geq \mathbb{P}(G_{\hat{p}_s}(\hat{p}_s, L(\hat{p}_s)) \leq \delta) = \mathbb{P}\left\{ 1 - S_B(K_s - 1, n_s, L(\hat{p}_s)) \leq \delta \right\}
\]

\[
\geq \mathbb{P}\left\{ n_s \mathcal{M}_B (\hat{p}_s, L(\hat{p}_s)) \leq \ln(\delta) \right\} = 1,
\]

and thus \( \mathbb{P}(F_{\hat{p}_s}(\hat{p}_s, U(\hat{p}_s)) \leq \delta) = \mathbb{P}\left\{ S_B(K_s, n_s, U(\hat{p}_s)) \leq \delta \right\}
\]

\[
\geq \mathbb{P}\left\{ n_s \mathcal{M}_B (\hat{p}_s, U(\hat{p}_s)) \leq \ln(\delta) \right\} = 1
\]

and thus \( \mathbb{P}\left\{ F_{\hat{p}_s}(\hat{p}_s, U(\hat{p}_s)) \leq \delta, G_{\hat{p}_s}(\hat{p}_s, L(\hat{p}_s)) \leq \delta \right\} = 1 \), which implies that \( \{ D_s = 1 \} \) is a sure event. So, the sampling scheme satisfies all the requirements described in Theorem 29 from which Theorem 28 immediately follows.

I.15 Proof of Theorem 29

We need some preliminary results.

Lemma 58 \( \{ \mathcal{M}_B(\hat{p}_\ell, \hat{p}_\ell - \varepsilon_a) > \frac{\ln(\delta)}{n_\ell}, \hat{p}_\ell \leq p^* + \varepsilon_a \} = \{ \varepsilon_a < \hat{p}_\ell \leq p^* + \varepsilon_a \}. \)
Proof. By the definition of sample sizes, we have \( n_s = \left\lfloor \frac{\ln(\zeta)}{\mathcal{M}(p^* + \varepsilon, p^* - \varepsilon)} \right\rfloor \) and thus \( n_{\ell} \leq n_s - 1 < \frac{\ln(\zeta)}{\mathcal{M}(p^* + \varepsilon, p^* - \varepsilon)} \), where \( z^* = p^* + \varepsilon \). Since \( \mathcal{M}(z^*, z^* - \varepsilon) \) is negative, we have \( \mathcal{M}(z^*, z^* - \varepsilon) > \frac{\ln(\zeta)}{n_\ell} \). Noting that \( \lim_{z \to -\varepsilon} \mathcal{M}(z, z - \varepsilon) = -\infty \) and that \( \mathcal{M}(z, z - \varepsilon) \) is monotonically increasing with respect to \( z \in (\varepsilon, z^*) \) as asserted by Lemma 55, we can conclude from the intermediate value theorem that there exists a unique number \( z_a^- \in (\varepsilon, p^* + \varepsilon) \) such that \( \mathcal{M}(z_a^-, z_a^- + \varepsilon) = \frac{\ln(\zeta)}{n_\ell} \). Finally, by virtue of the monotonicity of \( \mathcal{M}(z, z - \varepsilon) \) with respect to \( z \in (\varepsilon, z^*) \), the lemma is established.

\[ \mathcal{M}(\hat{p}_\ell, \hat{p}_\ell + \varepsilon) > \frac{\ln(\zeta)}{n_\ell}, \quad \hat{p}_\ell > p^* + \varepsilon \Rightarrow \{ p^* + \varepsilon < \hat{p}_\ell < z_a^+ \}. \]

Proof. Note that \( \mathcal{M}_B(z^*, z^* - \varepsilon) = \mathcal{M}_B(z^*, z^* - \varepsilon) > \frac{\ln(\zeta)}{n_\ell} \). By the definition of sample sizes, we have \( n_1 = \left\lfloor \frac{\ln(\zeta)}{\ln(1/(1+\varepsilon))} \right\rfloor \) and thus \( n_\ell \geq n_1 \geq \frac{\ln(\zeta)}{\ln(1/(1+\varepsilon))} = \frac{\ln(\zeta)}{\mathcal{M}(1/(1+\varepsilon))} \), which implies \( \lim_{\varepsilon \to 0} \mathcal{M}(z, z - \varepsilon) \leq \frac{\ln(\zeta)}{n_\ell} \). Noting that \( \mathcal{M}_B(z^*, z^*) \) is monotonically decreasing with respect to \( z \in (z^*, 1] \), we can conclude from the intermediate value theorem that there exists a unique number \( z_a^+ \in (z^*, 1] \) such that \( \mathcal{M}_B(z_a^+, z_a^+ - \varepsilon) = \frac{\ln(\zeta)}{n_\ell} \). Finally, by virtue of the monotonicity of \( \mathcal{M}_B(z, z - \varepsilon) \) with respect to \( z \in (z^*, 1] \), the lemma is established.

Lemma 60 For \( \ell = 1, \ldots, s - 1 \),

\[ \left\{ \mathcal{M}(\hat{p}_\ell, \hat{p}_\ell + \varepsilon) > \frac{\ln(\zeta)}{n_\ell}, \quad \hat{p}_\ell \leq p^* - \varepsilon \right\} = \begin{cases} \{ 0 \leq \hat{p}_\ell \leq p^* - \varepsilon \} & \text{for } n_\ell < \frac{\ln(\zeta)}{\ln(1 - \varepsilon)}, \\ \{ z_a^+ < \hat{p}_\ell \leq p^* - \varepsilon \} & \text{for } \frac{\ln(\zeta)}{\ln(1 - \varepsilon)} \leq n_\ell < \frac{\ln(\zeta)}{\mathcal{M}(p^* - \varepsilon, p^*)}, \\ \emptyset & \text{for } n_\ell \geq \frac{\ln(\zeta)}{\mathcal{M}(p^* - \varepsilon, p^*)}. \end{cases} \]

Proof. In the case of \( n_\ell < \frac{\ln(\zeta)}{\ln(1 - \varepsilon)} \), it is obvious that \( \ln(1 - \varepsilon) > \frac{\ln(\zeta)}{n_\ell} \). Since \( \lim_{\varepsilon \to 0} \mathcal{M}_B(z, z + \varepsilon) = \ln(1 - \varepsilon) \), we have \( \lim_{\varepsilon \to 0} \mathcal{M}_B(z, z + \varepsilon) > \frac{\ln(\zeta)}{n_\ell} \). Observing that \( \mathcal{M}_B(z^*, z^* + \varepsilon) \) is monotonically increasing with respect to \( z \in (0, p^* - \varepsilon) \), we have \( \mathcal{M}_B(z, z + \varepsilon) > \frac{\ln(\zeta)}{n_\ell} \) for any \( z \in (0, p^* - \varepsilon] \). It follows that \( \{ \mathcal{M}_B(\hat{p}_\ell, \hat{p}_\ell + \varepsilon) > \frac{\ln(\zeta)}{n_\ell}, \quad \hat{p}_\ell \leq p^* - \varepsilon \} = \{ 0 \leq \hat{p}_\ell \leq p^* - \varepsilon \} \).

In the case of \( \frac{\ln(\zeta)}{\ln(1 - \varepsilon)} \leq n_\ell < \frac{\ln(\zeta)}{\mathcal{M}(p^* - \varepsilon, p^*)} \), we have \( n_\ell < \frac{\ln(\zeta)}{\mathcal{M}(p^* - \varepsilon, p^*)} = \frac{\ln(\zeta)}{\mathcal{M}(z^*, z^* + \varepsilon)} \) where \( z^* = p^* - \varepsilon \). Observing that \( \mathcal{M}_B(z^*, z^* + \varepsilon) \) is negative, we have \( \mathcal{M}_B(z^*, z^* + \varepsilon) > \frac{\ln(\zeta)}{n_\ell} \). On the other hand, \( \lim_{\varepsilon \to 0} \mathcal{M}_B(z, z + \varepsilon) \leq \frac{\ln(\zeta)}{n_\ell} \) as a consequence of \( n_\ell \geq \frac{\ln(\zeta)}{\ln(1 - \varepsilon)} \). Noting that \( \mathcal{M}_B(z, z + \varepsilon) \) is monotonically increasing with respect to \( z \in (0, z^*) \) \( \subset (0, \frac{\ln(\zeta)}{n_\ell}) \), we can conclude from the intermediate value theorem that there exists a unique number \( z_a^+ \in (0, p^* - \varepsilon) \) such that \( \mathcal{M}_B(z_a^+, z_a^+ + \varepsilon) = \frac{\ln(\zeta)}{n_\ell} \). By virtue of the monotonicity of \( \mathcal{M}_B(z, z + \varepsilon) \) with respect to \( z \in (0, z^*) \), we have \( \{ \mathcal{M}_B(\hat{p}_\ell, \hat{p}_\ell + \varepsilon) > \frac{\ln(\zeta)}{n_\ell}, \quad \hat{p}_\ell \leq p^* - \varepsilon \} = \{ z_a^+ < \hat{p}_\ell \leq p^* - \varepsilon \} \).

In the case of \( n_\ell \geq \frac{\ln(\zeta)}{\mathcal{M}(p^* - \varepsilon, p^*)} \), we have \( n_\ell \geq \frac{\ln(\zeta)}{\mathcal{M}(p^* - \varepsilon, p^*)} = \frac{\ln(\zeta)}{\mathcal{M}(z^*, z^* + \varepsilon)} \). Due to the fact that \( \mathcal{M}_B(z^*, z^* + \varepsilon) \) is negative, we have \( \mathcal{M}_B(z^*, z^* + \varepsilon) \leq \frac{\ln(\zeta)}{n_\ell} \). Since \( \mathcal{M}_B(z, z + \varepsilon) \) is monotonically increasing with respect to \( z \in (0, z^*) \), we have that \( \mathcal{M}_B(z, z + \varepsilon) \leq \frac{\ln(\zeta)}{n_\ell} \) for any
\(z \in [0, z^*]\). This implies that \(\{M_B(\hat{p}_t, \hat{p}_t + \varepsilon_a) > \frac{\ln(\delta)}{n_{\ell}}, \hat{p}_t \leq p^* - \varepsilon_a\} = \emptyset\). This completes the proof of the lemma.

\(\square\)

**Lemma 61** For \(\ell = 1, \ldots, s - 1\),
\[
\left\{ M_B \left( \hat{p}_t, \frac{\hat{p}_t}{1 - \varepsilon_a} \right) > \frac{\ln(\delta)}{n_{\ell}}, \hat{p}_t > p^* - \varepsilon_a \right\} = \left\{ \{p^* - \varepsilon_a < \hat{p}_t < z^*_r\} \text{ for } n_{\ell} < \frac{\ln(\delta)}{M_M(p^* - \varepsilon_a, p^*)}, \right. \\
\left. \text{for } n_{\ell} \geq \frac{\ln(\delta)}{M_M(p^* - \varepsilon_a, p^*)}. \right\}
\]

**Proof.** In the case of \(n_{\ell} < \frac{\ln(\delta)}{M_M(p^* - \varepsilon_a, p^*)}\), we have \(M_B(z^*, \frac{z^*}{1 - \varepsilon_a}, p^*) = M_B(z^*, z^* + \varepsilon_a) = M_B(p^* - \varepsilon_a, p^*) > \frac{\ln(\delta)}{n_{\ell}}\). Noting that \(\lim_{z \to 1 - \varepsilon} M_B(z, \frac{z}{1 - \varepsilon}) = -\infty < \frac{\ln(\delta)}{n_{\ell}}\) and that \(M_M(z, \frac{z}{1 - \varepsilon})\) is monotonically decreasing with respect to \(z \in (z^*, 1 - \varepsilon_r)\), we can conclude from the intermediate value theorem that there exists a unique number \(z_r^* \in (z^*, 1 - \varepsilon_r)\) such that \(M_B(z_r^*, \frac{z_r^*}{1 - \varepsilon_r}) = \frac{\ln(\delta)}{n_{\ell}}\). By virtue of the monotonicity of \(M_B(z, \frac{z}{1 - \varepsilon})\) with respect to \(z \in (z^*, 1 - \varepsilon_r)\), we have \(\{M_B(\hat{p}_t, \frac{\hat{p}_t}{1 - \varepsilon_a}) > \frac{\ln(\delta)}{n_{\ell}}, \hat{p}_t > p^* - \varepsilon_a\} = \{p^* - \varepsilon_a < \hat{p}_t < z^*_r\} = \emptyset\). The proof of the lemma is thus completed.

\(\square\)

We are now in position to prove Theorem 20. Clearly, it follows directly from the definition of \(D_{\ell}\) that \(\{D_{\ell} = 0\} = \{M_B(\hat{p}_t, \mathcal{L}(\hat{p}_t)) > \frac{\ln(\delta)}{n_{\ell}}\} \cup \{M_B(\hat{p}_t, \mathcal{U}(\hat{p}_t)) > \frac{\ln(\delta)}{n_{\ell}}\}\). It remains to show statements (I) and (II).

With regard to statement (I), invoking the definition of \(\mathcal{L}(\hat{p}_t)\), we have
\[
\left\{ M_B(\hat{p}_t, \mathcal{L}(\hat{p}_t)) > \frac{\ln(\delta)}{n_{\ell}} \right\} = \left\{M_B(\hat{p}_t, \hat{p}_t - \varepsilon_a) > \frac{\ln(\delta)}{n_{\ell}}, \hat{p}_t \leq p^* + \varepsilon_a \right\}
\]
\[
\cup \left\{M_B \left( \hat{p}_t, \frac{\hat{p}_t}{1 + \varepsilon_a} \right) > \frac{\ln(\delta)}{n_{\ell}}, \hat{p}_t > p^* + \varepsilon_a \right\}
\]
\[
= \{z_a^* < \hat{p}_t \leq p^* + \varepsilon_a\} \cup \{p^* + \varepsilon_a < \hat{p}_t < z^*_r\}
\]
\[
= \{z_a^* < \hat{p}_t < z^*_r\} = \{n_{\ell} z_a^* < K_{\ell} < n_{\ell} z_r^*\}
\]
where the second equality is due to Lemma 58 and Lemma 59. This establishes statement (I).

The proof of statement (II) can be completed by applying Lemma 60, Lemma 61, and observing that
\[
\left\{ M_B(\hat{p}_t, \mathcal{U}(\hat{p}_t)) > \frac{\ln(\delta)}{n_{\ell}} \right\} = \left\{M_B(\hat{p}_t, \hat{p}_t + \varepsilon_a) > \frac{\ln(\delta)}{n_{\ell}}, \hat{p}_t \leq p^* - \varepsilon_a \right\}
\]
\[
\cup \left\{M_B \left( \hat{p}_t, \frac{\hat{p}_t}{1 - \varepsilon_a} \right) > \frac{\ln(\delta)}{n_{\ell}}, \hat{p}_t > p^* - \varepsilon_a \right\}
\]
This completes the proof of Theorem 20.

140
I.16 Proof of Theorem 30

We need some preliminary results, especially some properties of function \( \mathcal{M}(z, \mu) \).

**Lemma 62** \( \mathcal{M}(z, z + \varepsilon) \) is monotonically increasing with respect to \( z \in (0, \frac{1}{2} - \frac{2\varepsilon}{3}) \), and is monotonically decreasing with respect to \( z \in (\frac{1}{2} - \frac{2\varepsilon}{3}, 1 - \varepsilon) \). Similarly, \( \mathcal{M}(z, z - \varepsilon) \) is monotonically increasing with respect to \( z \in (\varepsilon, \frac{1}{2} + \frac{2\varepsilon}{3}) \), and is monotonically decreasing with respect to \( z \in (\frac{1}{2} + \frac{2\varepsilon}{3}, 1) \).

**Proof.** The lemma can be established by checking the partial derivatives

\[
\frac{\partial \mathcal{M}(z, z + \varepsilon)}{\partial z} = \frac{\varepsilon^2}{\left[(z + \frac{2\varepsilon}{3})(1 - z - \frac{2\varepsilon}{3})\right]^2} \left(\frac{1}{2} - \frac{2\varepsilon}{3} - z\right),
\]

\[
\frac{\partial \mathcal{M}(z, z - \varepsilon)}{\partial z} = \frac{\varepsilon^2}{\left[(z - \frac{2\varepsilon}{3})(1 - z + \frac{2\varepsilon}{3})\right]^2} \left(\frac{1}{2} + \frac{2\varepsilon}{3} - z\right).
\]

\[\square\]

**Lemma 63** Let \( 0 < \varepsilon < \frac{1}{2} \). Then, \( \mathcal{M}(z, z - \varepsilon) \leq \mathcal{M}(z, z + \varepsilon) \leq -2\varepsilon^2 \) for \( z \in [0, \frac{1}{2}] \), and \( \mathcal{M}(z, z + \varepsilon) < \mathcal{M}(z, z - \varepsilon) \leq -2\varepsilon^2 \) for \( z \in (\frac{1}{2}, 1] \).

**Proof.** By the definition of the function \( \mathcal{M}(., .) \), we have that \( \mathcal{M}(z, \mu) = -\infty \) for \( z \in [0, 1] \) and \( \mu \notin (0, 1) \). Hence, the lemma is trivially true for \( 0 \leq z \leq \varepsilon \) or \( 1 - \varepsilon \leq z \leq 1 \). It remains to show the lemma for \( z \in (\varepsilon, 1 - \varepsilon) \). This can be accomplished by noting that

\[
\mathcal{M}(z, z + \varepsilon) - \mathcal{M}(z, z - \varepsilon) = \frac{2\varepsilon^3(1 - 2z)}{3(z + \frac{2\varepsilon}{3})(1 - z - \frac{2\varepsilon}{3})(z - \frac{2\varepsilon}{3})(1 - z + \frac{2\varepsilon}{3})},
\]

where the right-hand side is seen to be positive for \( z \in (\varepsilon, \frac{1}{2}) \) and negative for \( z \in (\frac{1}{2}, 1 - \varepsilon) \). By Lemma 62, the maximums of \( \mathcal{M}(z, z + \varepsilon) \) and \( \mathcal{M}(z, z - \varepsilon) \) are shown to be \(-2\varepsilon^2\). This completes the proof of the lemma.

\[\square\]

**Lemma 64** \( \mathcal{M}(z, \frac{z}{1 + \varepsilon}) < \mathcal{M}(z, \frac{z}{1 - \varepsilon}) < 0 \) for \( 0 < z < 1 - \varepsilon < 1 \).

**Proof.** It can be verified that

\[
\mathcal{M}\left(z, \frac{z}{1 + \varepsilon}\right) - \mathcal{M}\left(z, \frac{z}{1 - \varepsilon}\right) = \frac{2\varepsilon^3z(2 - z)}{3(1 + \frac{2\varepsilon}{3})\left[1 - z + \varepsilon(1 - \frac{2\varepsilon}{3})\right]\left[1 - z - \varepsilon(1 - \frac{2\varepsilon}{3})\right]},
\]

from which it can be seen that \( \mathcal{M}(z, \frac{z}{1 + \varepsilon}) < \mathcal{M}(z, \frac{z}{1 - \varepsilon}) < 0 \) for \( z \in (0, 1 - \varepsilon) \).

\[\square\]

**Lemma 65** \( \mathcal{M}(\mu - \varepsilon, \mu) < \mathcal{M}(\mu + \varepsilon, \mu) \leq -2\varepsilon^2 \) for \( 0 < \varepsilon < \mu < \frac{1}{2} < 1 - \varepsilon \).
Proof. The lemma follows from Lemma 63 and the fact that
\[ M(\mu - \varepsilon, \mu) - M(\mu + \varepsilon, \mu) = \frac{\varepsilon^3(2\mu - 1)}{3(\mu - \frac{\varepsilon}{3})(1 - \mu + \frac{\varepsilon}{3})}, \]
where the right-hand side is negative for \(0 < \varepsilon < \mu < \frac{1}{2} < 1 - \varepsilon\).

\[ \square \]

Lemma 66. \( M(z, \frac{1}{z}) \) is monotonically decreasing with respect to \( z \in (0, 1) \). Similarly, \( M(z, \frac{1}{1-z}) \) is monotonically decreasing with respect to \( z \in (0, 1 - \varepsilon) \).

Proof. The lemma can be shown by verifying that
\[ \frac{\partial}{\partial z} M \left( z, \frac{z}{1 + \varepsilon} \right) = -\frac{\varepsilon^2}{2 \left( 1 + \frac{\varepsilon}{3} \right)} \times \frac{1 + \varepsilon}{[(1 + \varepsilon)(1 - z) + \frac{2\varepsilon z}{3}]}^2 < 0 \]
for \( z \in (0, 1) \) and that
\[ \frac{\partial}{\partial z} M \left( z, \frac{z}{1 - \varepsilon} \right) = -\frac{\varepsilon^2}{2 \left( 1 - \frac{\varepsilon}{3} \right)} \times \frac{1 - \varepsilon}{[(1 - \varepsilon)(1 - z) - \frac{2\varepsilon z}{3}]}^2 < 0 \]
for \( z \in (0, 1 - \varepsilon) \).

\[ \square \]

Lemma 67. For any fixed \( z \in (0, 1) \), \( M(z, \mu) \) is monotonically increasing with respect to \( \mu \in (0, z) \), and is monotonically decreasing with respect to \( \mu \in (z, 1) \). Similarly, for any fixed \( \mu \in (0, 1) \), \( M(z, \mu) \) is monotonically increasing with respect to \( z \in (0, \mu) \), and is monotonically decreasing with respect to \( z \in (\mu, 1) \).

Proof. The lemma can be shown by checking the following partial derivatives:
\[ \frac{\partial}{\partial \mu} M(z, \mu) = \frac{(z - \mu) [\mu(1-z) + z(1-\mu) + z(1-z)]}{3 \left( \left( \frac{2\mu}{3} + \frac{z}{3} \right) \left( 1 - \frac{2\mu}{3} - \frac{z}{3} \right) \right)^2}, \]
\[ \frac{\partial}{\partial z} M(z, \mu) = \frac{(\mu - z) \left[ \mu(1-\frac{2\mu}{3} - \frac{z}{3}) + \frac{z-\mu}{6} \right]}{\left( \left( \frac{2\mu}{3} + \frac{z}{3} \right) \left( 1 - \frac{2\mu}{3} - \frac{z}{3} \right) \right)^2} = \frac{(\mu - z) \left[ (1-\mu)(\frac{2\mu}{3} + \frac{z}{3}) + \frac{\mu - z}{6} \right]}{\left( \left( \frac{2\mu}{3} + \frac{z}{3} \right) \left( 1 - \frac{2\mu}{3} - \frac{z}{3} \right) \right)^2}. \]

\[ \square \]

Lemma 68. \( \{ D_\ell = 1 \} \subseteq \{ M_b(\hat{p}_\ell, \mathcal{M}(\hat{p}_\ell)) \leq \frac{\ln(\zeta \delta)}{n}, \ M_b(\hat{p}_\ell, \mathcal{L}(\hat{p}_\ell)) \leq \frac{\ln(\zeta \delta)}{n} \} \) for \( \ell = 1, \ldots, s \).
Proof. By the definition of $n_s$, we can show that $n_s \leq \frac{1}{2 \varepsilon_a} \frac{\ln(\varepsilon_a)}{2 \ln(\varepsilon_a)}$, which implies that $\frac{1}{4} + \frac{n_\varepsilon_a^2}{2 \ln(\varepsilon_a)} \geq 0$
for $\ell = 1, \ldots, s$. It can be shown by tedious computation that

\[
\begin{align*}
\mathcal{M}(\hat{p}_\ell, \hat{p}_\ell + \varepsilon_a) > \frac{\ln(\varepsilon_a)}{n_\ell} &= \left\{ \frac{1}{2} - \frac{2}{3} \varepsilon_a - \frac{1}{4} \frac{n_\varepsilon_a^2}{2 \ln(\varepsilon_a)} < \hat{p}_\ell < \frac{1}{2} + \frac{2}{3} \varepsilon_a + \frac{1}{4} \frac{n_\varepsilon_a^2}{2 \ln(\varepsilon_a)} \right\}, \\
\mathcal{M}(\hat{p}_\ell, \hat{p}_\ell - \varepsilon_a) > \frac{\ln(\varepsilon_a)}{n_\ell} &= \left\{ \frac{1}{2} + \frac{2}{3} \varepsilon_a - \frac{1}{4} \frac{n_\varepsilon_a^2}{2 \ln(\varepsilon_a)} < \hat{p}_\ell < \frac{1}{2} - \frac{2}{3} \varepsilon_a + \frac{1}{4} \frac{n_\varepsilon_a^2}{2 \ln(\varepsilon_a)} \right\},
\end{align*}
\]

for $\ell = 1, \ldots, s$. By (103), we have

\[
\begin{align*}
\mathcal{M}(\hat{p}_\ell, \frac{\hat{p}_\ell}{1 + \varepsilon_r}) > \frac{\ln(\varepsilon_a)}{n_\ell}, \hat{p}_\ell > \frac{\varepsilon_a}{\varepsilon_r} - \varepsilon_a} &= \left\{ \frac{\varepsilon_a}{\varepsilon_r} - \varepsilon_a < \hat{p}_\ell < \frac{6(1 - \varepsilon_r)(3 + \varepsilon_r) \ln(\varepsilon_a)}{2(3 + \varepsilon_r)^2 \ln(\varepsilon_a) - 9n_\ell \varepsilon_a^2} \right\}.
\end{align*}
\]

By the assumption that $0 < \varepsilon_a < \frac{3}{8}$ and $\frac{6a}{3 - 2\varepsilon_a} < \varepsilon_r < 1$, we have $\frac{\varepsilon_a}{\varepsilon_r} - \varepsilon_a < \frac{1}{2} - \frac{4\varepsilon_a}{3}$. Hence, by virtue of (103), we have

\[
\begin{align*}
\mathcal{M}(\hat{p}_\ell, \hat{p}_\ell + \varepsilon_a) > \frac{\ln(\varepsilon_a)}{n_\ell}, \hat{p}_\ell \leq \frac{\varepsilon_a}{\varepsilon_r} - \varepsilon_a} &= \left\{ \frac{1}{2} - \frac{2}{3} \varepsilon_a - \frac{1}{4} \frac{n_\varepsilon_a^2}{2 \ln(\varepsilon_a)} < \hat{p}_\ell \leq \frac{\varepsilon_a}{\varepsilon_r} - \varepsilon_a \right\}.
\end{align*}
\]

Therefore, making use of (107) and (108), we have

\[
\begin{align*}
\mathcal{M}(\hat{p}_\ell, \hat{p}_\ell + \varepsilon_a) > \frac{\ln(\varepsilon_a)}{n_\ell}, \hat{p}_\ell \leq \frac{\varepsilon_a}{\varepsilon_r} - \varepsilon_a} &= \left\{ \frac{1}{2} - \frac{2}{3} \varepsilon_a - \frac{1}{4} \frac{n_\varepsilon_a^2}{2 \ln(\varepsilon_a)} < \hat{p}_\ell \leq \frac{\varepsilon_a}{\varepsilon_r} - \varepsilon_a \right\}.
\end{align*}
\]

By the assumption that $0 < \varepsilon_a < \frac{3}{8}$ and $\frac{6a}{3 - 2\varepsilon_a} < \varepsilon_r < 1$, we have $\frac{\varepsilon_a}{\varepsilon_r} + \varepsilon_a < \frac{1}{2} + \frac{2\varepsilon_a}{3}$. Hence, by virtue of (103), we have

\[
\begin{align*}
\mathcal{M}(\hat{p}_\ell, \hat{p}_\ell - \varepsilon_a) > \frac{\ln(\varepsilon_a)}{n_\ell}, \hat{p}_\ell \leq \frac{\varepsilon_a}{\varepsilon_r} + \varepsilon_a} &= \left\{ \frac{1}{2} + \frac{2}{3} \varepsilon_a - \frac{1}{4} \frac{n_\varepsilon_a^2}{2 \ln(\varepsilon_a)} < \hat{p}_\ell \leq \frac{\varepsilon_a}{\varepsilon_r} + \varepsilon_a \right\}.
\end{align*}
\]

Therefore, making use of (111) and (112), we have

\[
\begin{align*}
\mathcal{M}(\hat{p}_\ell, \hat{p}_\ell + \varepsilon_a) > \frac{\ln(\varepsilon_a)}{n_\ell}, \hat{p}_\ell \leq \frac{\varepsilon_a}{\varepsilon_r} + \varepsilon_a} &= \left\{ \frac{1}{2} + \frac{2}{3} \varepsilon_a - \frac{1}{4} \frac{n_\varepsilon_a^2}{2 \ln(\varepsilon_a)} < \hat{p}_\ell \leq \frac{\varepsilon_a}{\varepsilon_r} + \varepsilon_a \right\}.
\end{align*}
\]
It follows from (110) and (114) that
\[
\{ D_\ell = 0 \} = \left\{ M(\hat{p}_t, \mathcal{U}(\hat{p}_t)) > \frac{\ln(\delta)}{n_\ell} \right\} \cup \left\{ M(\hat{p}_t, \mathcal{L}(\hat{p}_t)) > \frac{\ln(\delta)}{n_\ell} \right\},
\]
which implies that \( \{ D_\ell = 1 \} = \left\{ M(\hat{p}_t, \mathcal{U}(\hat{p}_t)) \leq \frac{\ln(\delta)}{n_\ell}, \quad M(\hat{p}_t, \mathcal{L}(\hat{p}_t)) \leq \frac{\ln(\delta)}{n_\ell} \right\} \) for \( \ell = 1, \ldots, s \). So,
\[
\{ D_\ell = 1 \} = \left\{ M(\hat{p}_t, \mathcal{U}(\hat{p}_t)) \leq \frac{\ln(\delta)}{n_\ell}, \quad M(\hat{p}_t, \mathcal{L}(\hat{p}_t)) \leq \frac{\ln(\delta)}{n_\ell} \right\}
\subseteq \left\{ M_B(\hat{p}_t, \mathcal{U}(\hat{p}_t)) \leq \frac{\ln(\delta)}{n_\ell}, \quad M_B(\hat{p}_t, \mathcal{L}(\hat{p}_t)) \leq \frac{\ln(\delta)}{n_\ell} \right\}
\]
for \( \ell = 1, \ldots, s \). This completes the proof of the lemma.

\[ \square \]

**Lemma 69** \( D_s = 1 \).

**Proof.** For simplicity of notations, we denote \( p^* = \frac{\varepsilon_a}{\varepsilon_r} \). In view of (109), (113) and (115), we have that, in order to show \( D_s = 1 \), it suffices to show
\[
\begin{align*}
\left\{ M\left( \hat{p}_s, \frac{\hat{p}_s}{1 - \varepsilon_r} \right) > \frac{\ln(\delta)}{n_s}, \quad \hat{p}_s > p^* - \varepsilon_a \right\} &= \emptyset, \\
\left\{ M\left( \hat{p}_s, \hat{p}_s + \varepsilon_a \right) > \frac{\ln(\delta)}{n_s}, \quad \hat{p}_s \leq p^* - \varepsilon_a \right\} &= \emptyset, \\
\left\{ M\left( \hat{p}_s, \frac{\hat{p}_s}{1 + \varepsilon_r} \right) > \frac{\ln(\delta)}{n_s}, \quad \hat{p}_s > p^* + \varepsilon_a \right\} &= \emptyset, \\
\left\{ M\left( \hat{p}_s, \hat{p}_s - \varepsilon_a \right) > \frac{\ln(\delta)}{n_s}, \quad \hat{p}_s \leq p^* + \varepsilon_a \right\} &= \emptyset.
\end{align*}
\]
By the definition of \( n_s \), we have \( n_s \geq \left\lceil \frac{\ln(\delta)}{M(p^* + \varepsilon_a, p^*)} \right\rceil \geq \frac{\ln(\delta)}{M(p^* + \varepsilon_a, p^*)} \). By the assumption on \( \varepsilon_a \) and \( \varepsilon_r \), we have \( 0 < \varepsilon_a < p^* < \frac{1}{2} < 1 - \varepsilon_a \). Hence, by Lemma 65, we have \( M(p^* - \varepsilon_a, p^*) < M(p^* + \varepsilon_a, p^*) < 0 \) and it follows that
\[
\frac{\ln(\delta)}{n_s} \geq M(p^* + \varepsilon_a, p^*) > M(p^* - \varepsilon_a, p^*).
\]
By (120),
\[
\left\{ M\left( \hat{p}_s, \frac{\hat{p}_s}{1 - \varepsilon_r} \right) > \frac{\ln(\delta)}{n_s}, \quad \hat{p}_s > p^* - \varepsilon_a \right\} \subseteq \left\{ M\left( \hat{p}_s, \frac{\hat{p}_s}{1 - \varepsilon_r} \right) > M(p^* - \varepsilon_a, p^*), \quad \hat{p}_s > p^* - \varepsilon_a \right\}.
\]
Noting that \( M(p^* - \varepsilon_a, p^*) = M\left( p^* - \varepsilon_a, \frac{p^* - \varepsilon_a}{1 - \varepsilon_r} \right) \) and making use of the fact that \( M(z, \frac{z}{1 - \varepsilon}) \) is monotonically decreasing with respect to \( z \in (0, 1 - \varepsilon) \) as asserted by Lemma 66, we have
\[
\left\{ M\left( \hat{p}_s, \frac{\hat{p}_s}{1 - \varepsilon_r} \right) > M(p^* - \varepsilon_a, p^*) \right\} = \{ \hat{p}_s < p^* - \varepsilon_a \}.
\]
Combining (121) and (122) yields (116). By (120),
\[ \left\{ \mathcal{M}(\widehat{p}_s, \hat{p}_s + \varepsilon_a) > \frac{\ln(\delta)}{n_s}, \quad \hat{p}_s \leq p^* - \varepsilon_a \right\} \subseteq \left\{ \mathcal{M}(\widehat{p}_s, \hat{p}_s + \varepsilon_a) > \mathcal{M}(p^* - \varepsilon_a, p^*), \quad \hat{p}_s \leq p^* - \varepsilon_a \right\}. \tag{123} \]

By the assumption on \( \varepsilon_a \) and \( \varepsilon_r \), we have \( p^* - \varepsilon_a < \frac{1}{2} - \frac{2a}{3} \). Recalling the fact that \( \mathcal{M}(z, z + \varepsilon) \) is monotonically increasing with respect to \( z \in (0, \frac{1}{2} - \frac{2a}{3}) \) as asserted by Lemma 62, we have that the event in the right-hand side of (123) is an impossible event and consequently, (117) is established. By (120),
\[ \left\{ \mathcal{M} \left( \widehat{p}_s, \frac{\hat{p}_s}{1 + \varepsilon_r} \right) > \frac{\ln(\delta)}{n_s}, \quad \hat{p}_s > p^* + \varepsilon_a \right\} = \left\{ \mathcal{M} \left( \widehat{p}_s, \frac{\hat{p}_s}{1 + \varepsilon_r} \right) > \mathcal{M}(p^* + \varepsilon_a, p^*), \quad \hat{p}_s > p^* + \varepsilon_a \right\}. \tag{124} \]
Noting that \( \mathcal{M}(p^* + \varepsilon_a, p^*) = \mathcal{M} \left( p^* + \varepsilon_a, \frac{p^* + \varepsilon_a}{1 + \varepsilon_r} \right) \) and making use of the fact that \( \mathcal{M}(z, \frac{z}{1 + \varepsilon}) \) is monotonically decreasing with respect to \( z \in (0, 1) \) as asserted by Lemma 66, we have
\[ \left\{ \mathcal{M} \left( \widehat{p}_s, \frac{\hat{p}_s}{1 + \varepsilon_r} \right) > \mathcal{M}(p^* + \varepsilon_a, p^*) \right\} = \{ \hat{p}_s < p^* + \varepsilon_a \}. \tag{125} \]
Combining (123) and (125) yields (118). By (120),
\[ \left\{ \mathcal{M}(\widehat{p}_s, \hat{p}_s - \varepsilon_a) > \frac{\ln(\delta)}{n_s}, \quad \hat{p}_s \leq p^* + \varepsilon_a \right\} \subseteq \left\{ \mathcal{M}(\widehat{p}_s, \hat{p}_s - \varepsilon_a) > \mathcal{M}(p^* + \varepsilon_a, p^*), \quad \hat{p}_s \leq p^* + \varepsilon_a \right\}. \tag{126} \]
By the assumption on \( \varepsilon_a \) and \( \varepsilon_r \), we have \( p^* + \varepsilon_a < \frac{1}{2} + \frac{2a}{3} \). Recalling the fact that \( \mathcal{M}(z, z - \varepsilon) \) is monotonically increasing with respect to \( z \in (0, \frac{1}{2} + \frac{2a}{3}) \) as asserted by Lemma 62, we have that the event in the right-hand side of (126) is an impossible event and consequently, (119) is established. This completes the proof of the lemma.

Now we are in a position to prove Theorem 30. Note that \( \mathcal{M}_B(z, p) = \inf_{t > 0} e^{-t z} \mathbb{E}[e^{\hat{p}_t}] \) and that \( \hat{p}_t \) is a ULE of \( p \) for \( \ell = 1, \cdots, s \). Moreover, \( \{ D_s = 1 \} \) is a sure event as a result of Lemma 69. So, the sampling scheme satisfies all the requirements described in Corollary 11 from which Theorem 30 immediately follows.

I.17 Proof of Theorem 31

We need some preliminary results.

**Lemma 70** \( \lim_{\varepsilon_a \to 0} \sum_{\ell=1}^s n_{\ell} e^{-n_{\ell} c} = 0 \) for any \( c > 0 \).

**Proof.** For simplicity of notations, define \( p^* = \frac{\varepsilon_a}{c} \) as before. By differentiation, it can be shown that \( xe^{-xc} \) is monotonically increasing with respect to \( x \in (0, \frac{1}{c}) \) and monotonically decreasing with respect to \( x \in (\frac{1}{c}, \infty) \). Since the smallest sample size \( n_1 \geq \frac{\ln \frac{1}{c}}{\ln(1 + \varepsilon_r)} \) is greater than \( \frac{1}{c} \) for
small enough \( \varepsilon_r > 0 \), we have that \( \sum_{\ell=1}^{s} n_\ell e^{-n_\ell c} \leq s n_1 e^{-n_1 c} \) if \( \varepsilon_r > 0 \) is sufficiently small. Let 
\[
\rho = \inf_{\varepsilon_r > 0} \frac{C_r}{C_r} - 1.
\]
Observing that 
\[
s \leq 1 + \frac{\ln \left( \frac{1}{M_B(p^* + \varepsilon_a, p^*)} \ln \frac{1}{1+\varepsilon_r} \right)}{\ln(1+\rho)} < 1 + \frac{\ln \left( \frac{1}{M_B(p^* + \varepsilon_a, p^*)} \ln \frac{1}{1+\varepsilon_r} \right)}{\ln(1+\rho)}
\]
and \( n_1 \geq \frac{\ln \frac{1}{c}}{\ln(1+\varepsilon_r)} \), we have that 
\[
\sum_{\ell=1}^{s} n_\ell e^{-n_\ell c} < \left[ 1 + \frac{\ln \left( \frac{1}{M_B(p^* + \varepsilon_a, p^*)} \ln \frac{1}{1+\varepsilon_r} \right)}{\ln(1+\varepsilon_r)} \right] \ln \frac{1}{c} \exp \left( -\frac{c \ln \frac{1}{c}}{\ln(1+\varepsilon_r)} \right) = A(\varepsilon_r) + \frac{\ln \frac{1}{c}}{\ln(1+\rho)} B(\varepsilon_r),
\]
where \( A(\varepsilon_r) = \frac{\ln \frac{1}{c}}{\ln(1+\varepsilon_r)} \exp \left( -\frac{c \ln \frac{1}{c}}{\ln(1+\varepsilon_r)} \right) \) and \( B(\varepsilon_r) = \frac{\ln \left( \frac{1}{M_B(p^* + \varepsilon_a, p^*)} \ln \frac{1}{1+\varepsilon_r} \right)}{\ln(1+\varepsilon_r)} \exp \left( -\frac{c \ln \frac{1}{c}}{\ln(1+\varepsilon_r)} \right) \). Noting that \( \lim_{x \to \infty} xe^{-x} = 0 \) and that \( \frac{\ln \frac{1}{c}}{\ln(1+\varepsilon_r)} \to \infty \) as \( \varepsilon_r \to 0 \), we have \( \lim_{\varepsilon_r \to 0} A(\varepsilon_r) = 0 \). Now we show that \( \lim_{\varepsilon_r \to 0} B(\varepsilon_r) = 0 \). Using Taylor’s expansion formula \( \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + o(x^3) \), we have \( \ln \frac{1}{1+\varepsilon_r} = - \ln(1+\varepsilon_r) = -\varepsilon_r + \frac{\varepsilon_r^2}{2} + o(\varepsilon_r^2) = -\varepsilon_r + o(\varepsilon_r) \) and 
\[
\mathcal{M}_B(p^* + \varepsilon_a, p^*) = \frac{\varepsilon_a^2}{2(p^* + \varepsilon_a)(1-p^* - \varepsilon_a)} - \frac{\varepsilon_a^3}{3(p^* + \varepsilon_a)^2} + \frac{\varepsilon_a^3}{3(1-p^* - \varepsilon_a)^2} + o(\varepsilon_a^3)
\]
where \( \omega = \frac{1}{2p^* - \frac{1}{2(1-p^*)}} + \frac{2}{3p^*^2} + \frac{2}{3(1-p^*)^2} \). Hence, 
\[
\ln \left( \frac{1}{\mathcal{M}_B(p^* + \varepsilon_a, p^*)} \ln \frac{1}{1+\varepsilon_r} \right) = \ln \left( \frac{-\varepsilon_r + \frac{\varepsilon_r^2}{2} + o(\varepsilon_r^2)}{\varepsilon_a^2 + \omega \varepsilon_a^3 + o(\varepsilon_a^3)} \right) = \ln [2p^*(1-p^*)] + \ln \frac{1}{\varepsilon_a} + \ln \frac{\varepsilon_r - \frac{\varepsilon_r^2}{2} + o(\varepsilon_r^2)}{\varepsilon_r - \frac{\varepsilon_r^2}{2p^*(1-p^*)} + o(\varepsilon_r^2)}
\]
and 
\[
\ln \left( \frac{1}{\mathcal{M}_B(p^* + \varepsilon_a, p^*)} \ln \frac{1}{1+\varepsilon_r} \right) = \ln \left( \frac{2(1-p^*)}{\varepsilon_a} + \frac{2p^*(1-p^*)\omega \varepsilon_a - \frac{\varepsilon_a}{2p^*} + o(\varepsilon_a)}{\ln(1+\varepsilon_r)} \right) = \ln \left( \frac{2(1-p^*)}{\varepsilon_a} + \frac{2p^*(1-p^*)\omega \varepsilon_a - \frac{\varepsilon_a}{2p^*} + o(\varepsilon_a)}{\ln(1+\varepsilon_r)} \right) + \frac{2p^*(1-p^*)\omega \varepsilon_a - \frac{\varepsilon_a}{2p^*} + o(\varepsilon_a)}{\ln(1+\varepsilon_r)}
\]
Making use of (127) and observing that 
\[
\left[ 2p^*(1-p^*)\omega - \frac{1}{2} + o(1) \right] \exp \left( -\frac{c \ln \frac{1}{c}}{\ln(1+\varepsilon_r)} \right) = o(1),
\]
146
\[
\frac{\ln[2(1 - p^*)/p^*]}{\ln(1 + \varepsilon_r)} \exp \left( -\frac{c \ln \frac{1}{\zeta \delta}}{\ln(1 + \varepsilon_r)} \right) = \frac{\ln[2(1 - p^*)/p^*]}{c \ln \frac{1}{\zeta \delta}} \exp \left( \frac{c \ln \frac{1}{\zeta \delta}}{\ln(1 + \varepsilon_r)} \right) = o(1),
\]

we have

\[B(\varepsilon_r) = o(1) + \frac{\ln \frac{1}{\varepsilon_r}}{\ln(1 + \varepsilon_r)} \exp \left( -\frac{c \ln \frac{1}{\zeta \delta}}{\varepsilon_r + o(\varepsilon_r)} \right) = o(1) + \frac{\ln \frac{1}{\varepsilon_r}}{\varepsilon_r + o(\varepsilon_r)} \exp \left( -\frac{c \ln \frac{1}{\zeta \delta}}{\varepsilon_r - \frac{\varepsilon_r}{2} + o(\varepsilon_r^2)} \right)\]

\[= o(1) + \frac{\ln \frac{1}{\varepsilon_r} + o(\varepsilon_r)}{\varepsilon_r + o(\varepsilon_r)} \left( 1 + \frac{\varepsilon_r}{2} + o(\varepsilon_r) \right) \]

\[= o(1) + \frac{B^*(\varepsilon_r)}{1 + o(1)} \left( \frac{1}{\zeta \delta} \right)^{\frac{1}{\varepsilon_r}}\]

where \(B^*(\varepsilon_r) = \frac{\ln \frac{1}{\varepsilon_r}}{\varepsilon_r} \left( \frac{1}{\zeta \delta} \right)^{\frac{1}{\varepsilon_r}}.\) Making a change of variable \(x = \frac{1}{\varepsilon_r}\) and using L’Hôpital’s rule, we have

\[\lim_{\varepsilon_r \to 0} B^*(\varepsilon_r) = \lim_{x \to \infty} \frac{x \ln x}{(\frac{1}{\varepsilon_r})^x} = \lim_{x \to \infty} \frac{1 + \ln x}{(\frac{c \ln \frac{1}{\zeta \delta}}{x})^x} = \lim_{x \to \infty} \frac{1}{(\frac{c \ln \frac{1}{\zeta \delta}}{x})^x} = 0.\]

Therefore, \(0 \leq \lim_{\varepsilon_r \to 0} \sum_{\ell=1}^s n_\ell \exp^{-n_\ell\varepsilon_r} \leq \frac{1}{c} \lim_{\varepsilon_r \to 0} A(\varepsilon_r) + \frac{\ln \frac{1}{\varepsilon_r}}{\ln(1 + p^*)} \times \left( \frac{1}{\zeta \delta} \right)^{-\frac{1}{\varepsilon_r}} \lim_{\varepsilon_r \to 0} B^*(\varepsilon_r) = 0,\)

which implies that \(\lim_{\varepsilon_a \to 0} \sum_{\ell=1}^s n_\ell \exp^{-n_\ell\varepsilon_a} = 0.\) This completes the proof of the lemma.

\[\square\]

**Lemma 71** If \(\varepsilon_a\) is sufficiently small, then the following statements hold true.

(I): For \(1 \leq \ell \leq s,\) there exists a unique number \(z_\ell \in [0, p^* - \varepsilon_a)\) such that \(n_\ell = \frac{\ln(\zeta \delta)}{\mathcal{M}_\varepsilon(z_\ell, z_\ell + \varepsilon_a)}\) for \(n_\ell \geq \frac{\ln(\zeta \delta)}{\ln(1 - \varepsilon_a)}\).

(II): For \(1 \leq \ell \leq s,\) there exists a unique number \(y_\ell \in (p^* + \varepsilon_a, 1]\) such that \(n_\ell = \frac{\ln(\zeta \delta)}{\mathcal{M}_\varepsilon(y_\ell, y_\ell + \varepsilon_a)}\).

(III): \(z_\ell\) is monotonically increasing with respect to \(\ell;\) \(y_\ell\) is monotonically decreasing with respect to \(\ell.\)

(IV): \(\lim_{\varepsilon_a \to 0} z_\ell = \frac{1}{2} \sqrt{1 - 4p^*(1 - p^*) C_{s-\ell}}\) and \(\lim_{\varepsilon_a \to 0} y_\ell = \frac{1}{1 + \left( \frac{1}{\zeta \delta} - 1 \right) C_{s-\ell}}\), where the limits are taken under the constraint that \(\frac{\varepsilon_a}{\varepsilon_r}\) and \(s - \ell\) are fixed with respect to \(\varepsilon_a.\)

(V): Let \(\ell_\varepsilon = s - j_\varepsilon.\) For \(p \in (p^*, 1)\) such that \(C_{j_\varepsilon} = r(p),\)

\[\lim_{\varepsilon_r \to 0} \frac{z_\varepsilon p - p}{\varepsilon_r p} = \frac{2 p - p^*}{3(1 - p^*)}.\]

For \(p \in (0, p^*)\) such that \(C_{j_\varepsilon} = r(p),\)

\[\lim_{\varepsilon_a \to 0} \frac{z_\varepsilon p - p}{\varepsilon_a} = \frac{2 p(1 - p)(1 - 2p^*)}{3p^*(1 - p^*)(1 - 2p) - \frac{2}{3}}.\]
(VI):
\[
\{D_\ell = 0\} = \begin{cases} 
\{z_\ell < \hat{p}_\ell < y_\ell\} & \text{for } n_\ell \geq \frac{\ln(\zeta_\ell)}{\ln(1 - \varepsilon_a)}, \\
\{0 < \hat{p}_\ell < y_\ell\} & \text{for } n_\ell < \frac{\ln(\zeta_\ell)}{\ln(1 - \varepsilon_a)}. 
\end{cases}
\]

Proof of Statement (I): By the definition of sample sizes, we have \(\frac{\ln(\zeta_\ell)}{n_\ell} \geq \mathcal{M}(0, \varepsilon_a)\) and
\[
n_\ell < \frac{(1 + C_1)n_s}{2} < \frac{(1 + C_1)}{2} \left[ \frac{\ln(\zeta_\ell)}{\mathcal{M}(p^* + \varepsilon_a, p^*)} + 1 \right]
\]
for sufficiently small \(\varepsilon_a > 0\). As a consequence of (128), we have
\[
\frac{\ln(\zeta_\ell)}{n_\ell} < \mathcal{M}(p^* + \varepsilon_a, p^*) \left( \frac{2}{1 + C_1} - \frac{1}{n_\ell} \right) = \mathcal{M}(p^* + \varepsilon_a, p^*) \left( \frac{2}{1 + C_1} \right) \mathcal{M}(p^* - \varepsilon_a, p^*) - \mathcal{M}(p^* + \varepsilon_a, p^*) \frac{n_\ell}{\mathcal{M}(p^* - \varepsilon_a, p^*)}
\]
provided that \(\varepsilon_a > 0\) is sufficiently small. Noting that
\[
\lim_{\varepsilon_a \to 0} \mathcal{M}(p^* + \varepsilon_a, p^*) = 1, \quad \lim_{\varepsilon_a \to 0} \frac{\mathcal{M}(p^* + \varepsilon_a, p^*)}{n_\ell} = 0,
\]
we have that \(\frac{\ln(\zeta_\ell)}{n_\ell} < \mathcal{M}(p^* - \varepsilon_a, p^*)\) for small enough \(\varepsilon_a > 0\). In view of the established fact that \(\mathcal{M}(0, \varepsilon_a) \leq \frac{\ln(\zeta_\ell)}{n_\ell} < \mathcal{M}(p^* - \varepsilon_a, p^*)\) and the fact that \(\mathcal{M}(z, z + \varepsilon_a)\) is monotonically increasing with respect to \(z \in (0, p^* - \varepsilon_a)\) as asserted by Lemma 16, invoking the intermediate value theorem, we have that there exists a unique number \(z_\ell \in (0, p^* - \varepsilon_a)\) such that \(\mathcal{M}(z_\ell, z_\ell + \varepsilon_a) = \frac{\ln(\zeta_\ell)}{n_\ell}\), which implies Statement (I).

Proof of Statement (II): By the definition of sample sizes, we have
\[
\frac{\ln(\zeta_\ell)}{\mathcal{M}(1, \frac{1}{1 + \varepsilon_a})} \leq n_1 \leq n_\ell < \frac{(1 + C_1)n_s}{2} < \frac{(1 + C_1)}{2} \left[ \frac{\ln(\zeta_\ell)}{\mathcal{M}(p^* + \varepsilon_a, p^*)} + 1 \right]
\]
and consequently, \(\frac{\ln(\zeta_\ell)}{n_\ell} \geq \mathcal{M}(1, \frac{1}{1 + \varepsilon_a})\).
\[
\frac{\ln(\zeta_\ell)}{n_\ell} < \mathcal{M}(p^* + \varepsilon_a, p^*) \left( \frac{2}{1 + C_1} - \frac{1}{n_\ell} \right) = \mathcal{M}(p^* + \varepsilon_a, p^*) \left( \frac{2}{1 + C_1} \right) \mathcal{M}(p^* + \varepsilon_a, p^*) - \mathcal{M}(p^* + \varepsilon_a, p^*) \frac{n_\ell}{\mathcal{M}(p^* + \varepsilon_a, p^*)}
\]
for sufficiently small \(\varepsilon_a > 0\). Noting that \(\lim_{\varepsilon_a \to 0} \frac{\mathcal{M}(p^* + \varepsilon_a, p^*)}{n_\ell} = 0\), we have \(\frac{\ln(\zeta_\ell)}{n_\ell} < \mathcal{M}(p^* + \varepsilon_a, p^*) \frac{n_\ell}{\mathcal{M}(p^* + \varepsilon_a, p^*)}\) for small enough \(\varepsilon_a > 0\). In view of the established fact that \(\mathcal{M}(1, \frac{1}{1 + \varepsilon_a}) \leq \frac{\ln(\zeta_\ell)}{n_\ell} < \mathcal{M}(p^* + \varepsilon_a, p^* + \varepsilon_a, p^* + \varepsilon_a)\) and the fact that \(\mathcal{M}(z, \frac{z}{1 + \varepsilon_a})\) is monotonically decreasing with respect to \(z \in (0, 1)\) as asserted by Lemma 10, invoking the intermediate value theorem, we have that there exists a unique number \(y_\ell \in (p^* + \varepsilon_a, 1]\) such that \(\mathcal{M}(y_\ell, \frac{y_\ell}{1 + \varepsilon_a}) = \frac{\ln(\zeta_\ell)}{n_\ell}\), which implies Statement (II).

Proof of Statement (III): Since \(n_\ell\) is monotonically increasing with respect to \(\ell\) if \(\varepsilon_a > 0\) is sufficiently small, we have that \(\mathcal{M}(z_\ell, z_\ell + \varepsilon_a)\) is monotonically increasing with respect to \(\ell\) for small enough \(\varepsilon_a > 0\). Recalling that \(\mathcal{M}(z, z + \varepsilon_a)\) is monotonically increasing with respect
to \( z \in (0, p^* - \varepsilon_a) \), we have that \( z_\ell \) is monotonically increasing with respect to \( \ell \). Similarly, \( \mathcal{M}_B(y_\ell, \frac{y}{1+\varepsilon_r}) \) is monotonically increasing with respect to \( \ell \) for sufficiently small \( \varepsilon_a > 0 \). Recalling that \( \mathcal{M}_B(z, \frac{z}{1+\varepsilon_r}) \) is monotonically decreasing with respect to \( z \in (0, 1) \), we have that \( y_\ell \) is monotonically decreasing with respect to \( \ell \). This establishes Statement (III).

**Proof of Statement (IV):** We first consider \( \lim_{\varepsilon_a \to 0} z_\ell \). For simplicity of notations, define \( b_\ell = \frac{1-\sqrt{1-4p^*(1-p^*)C_{s-\ell}}}{2} \) for \( \ell < s \) such that \( n_\ell \geq \ln(1-\varepsilon_a) \). Then, it can be checked that \( \frac{b_\ell(1-b_\ell)}{p^*(1-p^*)} = C_{s-\ell} \) and, by the definition of sample sizes, we have

\[
\frac{b_\ell(1-b_\ell)}{p^*(1-p^*)} \cdot \mathcal{M}_B(z_\ell, z_\ell + \varepsilon_a) = 1 + o(1) \geq \frac{b_\ell(1-b_\ell)}{p^*(1-p^*)} \cdot \mathcal{M}_B(\theta, \theta + \varepsilon_a) = \frac{b_\ell(1-b_\ell)}{\theta(1-\theta)} + o(1)
\]

for \( \ell < s \) such that \( n_\ell \geq \frac{\ln(1-\varepsilon_a)}{\ln(1-\varepsilon_a)} \).

We claim that \( z_\ell > \theta \) for \( \theta \in (0, b_\ell) \) provided that \( \varepsilon_a > 0 \) is sufficiently small. Such a claim can be shown by a contradiction method as follows. Suppose this claim is not true, then there is a set, denoted by \( S_{\varepsilon_a} \), of infinitely many values of \( \varepsilon_a \) such that \( z_\ell \leq \theta \) for any \( \varepsilon_a \in S_{\varepsilon_a} \). For small enough \( \varepsilon_a \in S_{\varepsilon_a} \), it is true that \( z_\ell \leq \theta < b_\ell < \frac{1}{2} - \varepsilon_a \). By (133) and the fact that \( \mathcal{M}_B(z, z + \varepsilon) \) is monotonically increasing with respect to \( z \in (0, \frac{1}{2} - \varepsilon) \) as asserted by Lemma 16, we have

\[
\frac{b_\ell(1-b_\ell)}{p^*(1-p^*)} \cdot \mathcal{M}_B(z_\ell, z_\ell + \varepsilon_a) = 1 + o(1) \geq \frac{b_\ell(1-b_\ell)}{p^*(1-p^*)} \cdot \mathcal{M}_B(\theta, \theta + \varepsilon_a) = \frac{b_\ell(1-b_\ell)}{\theta(1-\theta)} + o(1)
\]

for small enough \( \varepsilon_a \in S_{\varepsilon_a} \), which implies \( \frac{b_\ell(1-b_\ell)}{\theta(1-\theta)} \leq 1 \), contradicting to the fact that \( \frac{b_\ell(1-b_\ell)}{\theta(1-\theta)} > 1 \). This proves the claim. Now we restrict \( \varepsilon_a \) to be small enough so that \( \theta < z_\ell < p^* \). Making use of (133) and applying Lemma 15 based on the condition that \( z_\ell \in (\theta, p^*) \subset (0, 1) \), we have

\[
\frac{b_\ell(1-b_\ell)}{p^*(1-p^*)} \cdot \mathcal{M}_B(z_\ell, z_\ell + \varepsilon_a) = 1 + o(1) \geq \frac{b_\ell(1-b_\ell)}{p^*(1-p^*)} \cdot \mathcal{M}_B(\theta, \theta + \varepsilon_a) = \frac{b_\ell(1-b_\ell)}{\theta(1-\theta)} + o(1)
\]

which implies \( \frac{b_\ell(1-b_\ell)}{z_\ell(1-z_\ell)} = 1 + o(1) \) and thus \( \lim_{\varepsilon_a \to 0} z_\ell = b_\ell \).

We now consider \( \lim_{\varepsilon_a \to 0} y_\ell \). For simplicity of notations, define \( a_\ell = \frac{1}{1+(\frac{1}{p^*}-1)C_{s-\ell}} \) for \( 1 \leq \ell < s \). Then, it can be checked that \( \frac{p^*}{1-p^*} \frac{1-a_\ell}{a_\ell} = C_{s-\ell} \) and, by the definition of sample sizes,

\[
\frac{p^*}{1-p^*} \frac{1-a_\ell}{a_\ell} \cdot \mathcal{M}_B(y_\ell, \frac{y}{1+\varepsilon_r}) = 1 + o(1) \geq \frac{p^*}{1-p^*} \frac{1-a_\ell}{a_\ell} \cdot \mathcal{M}_B(\theta, \frac{\theta}{1+\varepsilon_r}) = \frac{\theta(1-a_\ell)}{a_\ell(1-\theta)} + o(1)
\]

We claim that \( y_\ell < \theta \) for \( \theta \in (a_\ell, 1) \) if \( \varepsilon_r > 0 \) is small enough. To prove this claim, we use a contradiction method. Suppose this claim is not true, then there is a set, denoted by \( S_{\varepsilon_r} \), of infinitely many values of \( \varepsilon_r \) such that \( y_\ell \geq \theta \) for any \( \varepsilon_r \in S_{\varepsilon_r} \). By (132) and the fact that \( \mathcal{M}_B(z, \frac{z}{1+\varepsilon_r}) \) is monotonically decreasing with respect to \( z \in (0, 1) \) as asserted by Lemma 40, we have

\[
\frac{p^*}{1-p^*} \frac{1-a_\ell}{a_\ell} \cdot \mathcal{M}_B(y_\ell, \frac{y}{1+\varepsilon_r}) = 1 + o(1) \geq \frac{p^*}{1-p^*} \frac{1-a_\ell}{a_\ell} \cdot \mathcal{M}_B(\theta, \frac{\theta}{1+\varepsilon_r}) = \frac{\theta(1-a_\ell)}{a_\ell(1-\theta)} + o(1)
\]
for small enough \( \varepsilon_r \in S_{\varepsilon_r} \), which implies \( \theta(1-\alpha_r) - \alpha \theta(1-\theta) \leq 1 \), contradicting to the fact that \( \theta(1-\alpha_r) / \alpha(1-\theta) > 1 \). This proves the claim. Now we restrict \( \varepsilon_r \) to be small enough so that \( p^* < y_r < \theta \). By (132) and applying Lemma 15 based on the condition that \( y_r \in (p^*, \theta) \subset (0, 1) \), we have

\[
\frac{p^*}{1 - p^*} \frac{1 - \alpha_r}{\alpha} \times \frac{\varepsilon_r^2 y_r/[2(y_r - 1)] + o(\varepsilon_r^2)}{\varepsilon_r^2/[2(1 - p^*)(p^* - 1)] + o(\varepsilon_r^2)} = 1 + o(1),
\]

which implies \( \frac{\mu_r - \alpha_r}{\alpha_r(1-y_r)} = o(1) \) and thus \( \lim_{\varepsilon_r \to 0} y_r = \alpha_r \).

**Proof of Statement (V):**

We shall first consider \( p \in (p^*, 1) \). For small enough \( \varepsilon_r > 0 \), there exists \( \varepsilon \ell_r \in (p^*, 1) \) such that

\[
n_{\varepsilon r} = \frac{\ln(\zeta \delta)}{\mathcal{M}_B(z_{\varepsilon r}, z_{\varepsilon r}/(1 + \varepsilon_r))} = \left[ \frac{\mathcal{C}_{s - \varepsilon_r} \ln(\zeta \delta)}{\mathcal{M}_B(p^*, p^*/(1 + \varepsilon_r))} \right] = \left[ \frac{p^* - 1 - p}{1 - p^*} \frac{\ln(\zeta \delta)}{\mathcal{M}_B(p^*, p^*/(1 + \varepsilon_r))} \right] = 1 + o(\varepsilon_r),
\]

which contradicts to the fact that \( \theta(1-p) / p(1-\theta) > 1 \). This proves the claim. Now we restrict \( \varepsilon_r \) to be small enough so that \( p^* < y_r < \theta \). Since \( z_{\varepsilon r} \) is bounded with respect to \( \varepsilon_r \) by (132) and Lemma 15, we have

\[
\frac{p^*}{1 - p^*} \frac{1 - p}{p} \times \frac{-\varepsilon_r^2 z_{\varepsilon r}/[2(1 - z_{\varepsilon r})]}{\varepsilon_r^2/[2(1 - p^*)(p^* - 1)] + o(\varepsilon_r^2)} = 1 + o(\varepsilon_r),
\]

i.e.,

\[
\frac{z_{\varepsilon r}(1-p)}{p(1-z_{\varepsilon r})} - \frac{2\varepsilon_r z_{\varepsilon r}(1-p)(2-z_{\varepsilon r})}{3p(1-z_{\varepsilon r})} = \frac{2\varepsilon_r(2-p^*)}{3(1-p^*)} + o(\varepsilon_r),
\]

i.e.,

\[
\frac{z_{\varepsilon r} - p}{p(1-z_{\varepsilon r})} - \frac{2\varepsilon_r z_{\varepsilon r}(1-p)(2-z_{\varepsilon r})}{3p(1-z_{\varepsilon r})^2} = \frac{2\varepsilon_r(2-p^*)(1-z_{\varepsilon r})}{3(1-p^*)} + o(\varepsilon_r),
\]

i.e.,

\[
\frac{z_{\varepsilon r} - p}{p} - \frac{2\varepsilon_r z_{\varepsilon r}(1-p)(2-z_{\varepsilon r})}{3p(1-z_{\varepsilon r})} = \frac{2\varepsilon_r(2-p^*)(1-z_{\varepsilon r})}{3(1-p^*)} + o(\varepsilon_r),
\]

which implies that \( \lim_{\varepsilon_r \to 0} z_{\varepsilon r} = p \) and consequently,

\[
\lim_{\varepsilon_r \to 0} \frac{z_{\varepsilon r} - p}{\varepsilon_r p} = \frac{2(2-p)}{3} - \frac{2(2-p^*)(1-p)}{3(1-p^*)} = \frac{2 p - p^*}{3(1-p^*)} = \nu \in \left(0, \frac{2}{3}\right).
\]

150
Next, we shall consider \( p \in (0,p^*) \). For small enough \( \varepsilon_a > 0 \), there exists \( z_{\varepsilon} \in (0,p^*) \) such that

\[
n_{\varepsilon} = \frac{\ln(\zeta)}{\mathcal{M}_B(z_{\varepsilon}, z_{\varepsilon} + \varepsilon_a)} = \left[ \frac{C_{s-\varepsilon}}{\mathcal{M}_B(p^*, p^* + \varepsilon_a)} \right] = \left[ \frac{p(1-p)}{p^*(1-p^*) \cdot \mathcal{M}_B(p^*, p^* + \varepsilon_a)} \right].
\]

For \( \theta \in (0,p) \), we claim that \( z_{\varepsilon} > \theta \) if \( \varepsilon_a \) is sufficiently small. Suppose, to get a contradiction, that this claim is not true. Then, there exists a set, denoted by \( S_{\varepsilon_a} \), of infinitely many values of \( \varepsilon_a \) such that \( z_{\varepsilon} \leq \theta \) for any value of \( \varepsilon_a \) in \( S_{\varepsilon_a} \). Noting that

\[
\frac{p(1-p) \cdot \ln(\theta)}{p^*(1-p^*) \cdot \mathcal{M}_B(z_{\varepsilon}, z_{\varepsilon} + \varepsilon_a)} = p(1-p) \cdot \mathcal{M}_B(z_{\varepsilon}, z_{\varepsilon} + \varepsilon_a) = 1 + o(\varepsilon_a),
\]

we have

\[
\frac{p(1-p) \cdot \mathcal{M}_B(z_{\varepsilon}, z_{\varepsilon} + \varepsilon_a)}{p^*(1-p^*) \cdot \mathcal{M}_B(p^*, p^* + \varepsilon_a)} = \frac{p(1-p)}{p^*(1-p^*)} \cdot \mathcal{M}_B(\theta, \theta + \varepsilon_a) = \frac{p(1-p)}{\theta(1-\theta)} + o(1)
\]
for any value of \( \varepsilon_a \) in \( S_{\varepsilon_a} \), which contradicts to the fact that \( \frac{p(1-p)}{\theta(1-\theta)} > 1 \). This proves the claim. Now we restrict \( \varepsilon_a \) to be small enough so that \( \theta < z_{\varepsilon} < p^* \). Since \( z_{\varepsilon} \) is bounded with respect to \( \varepsilon \), by (133) and Lemma 15, we have

\[
\frac{p(1-p) - \varepsilon_a^2/[2z_{\varepsilon}(1-z_{\varepsilon})] + \varepsilon_a^3(1-2z_{\varepsilon})/[3z_{\varepsilon}^2(1-z_{\varepsilon})^2] + o(\varepsilon_a^3)}{1 - 2\varepsilon_a(1 - 2p^*)/[3p^*(1-p^*)] + o(\varepsilon_a)} = 1 + o(\varepsilon_a),
\]
i.e.,

\[
\frac{p(1-p)}{z_{\varepsilon}(1-z_{\varepsilon})} - \frac{2\varepsilon_a p(1-p)(1-2z_{\varepsilon})}{3z_{\varepsilon}^2(1-z_{\varepsilon})^2} + o(\varepsilon_a) = 1 - \frac{2\varepsilon_a(1 - 2p^*)}{3p^*(1-p^*)} + o(\varepsilon_a),
\]
i.e.,

\[
\frac{(z_{\varepsilon} - p)(1 - z_{\varepsilon})}{z_{\varepsilon}(1 - z_{\varepsilon})} = \frac{2\varepsilon_a(1 - 2p^*)}{3z_{\varepsilon}^2(1-z_{\varepsilon})^2} - \frac{2\varepsilon_a p(1-p)(1-2z_{\varepsilon})}{3z_{\varepsilon}^2(1-z_{\varepsilon})^2} + o(\varepsilon_a),
\]
i.e.,

\[
\frac{z_{\varepsilon} - p}{\varepsilon_a} = \frac{2z_{\varepsilon}(1-z_{\varepsilon})(1-2p^*)}{3p^*(1-p^*)(1-z_{\varepsilon})} - \frac{2p(1-p)(1-2z_{\varepsilon})}{3z_{\varepsilon}(1-z_{\varepsilon})(1-z_{\varepsilon} - p)} + o(1),
\]
which implies that \( \lim_{\varepsilon_a \to 0} z_{\varepsilon} = p \) and consequently,

\[
\lim_{\varepsilon_a \to 0} \frac{z_{\varepsilon} - p}{\varepsilon_a} = \frac{2p(1-p)(1-2p^*)}{3p^*(1-p^*)(1-2p^*)} - \frac{2}{3} = - \nu \in \left(-\frac{2}{3}, 0\right).
\]
Proof of Statement (VI): By the definition of the sampling scheme,

\[
\{D_t = 0\} = \left\{ \max \left\{ M_B(\tilde{p}_t, \mathcal{L}(\tilde{p}_t)), M_B(\tilde{p}_t, \mathcal{U}(\tilde{p}_t)) \right\} > \frac{\ln(\zeta)}{n_t}, |\tilde{p}_t - p^*| \leq \varepsilon_a \right\} \\
\bigcup \left\{ \max \left\{ M_B(\tilde{p}_t, \mathcal{L}(\tilde{p}_t)), M_B(\tilde{p}_t, \mathcal{U}(\tilde{p}_t)) \right\} > \frac{\ln(\zeta)}{n_t}, \tilde{p}_t < p^* - \varepsilon_a \right\} \\
\bigcup \left\{ \max \left\{ M_B(\tilde{p}_t, \mathcal{L}(\tilde{p}_t)), M_B(\tilde{p}_t, \mathcal{U}(\tilde{p}_t)) \right\} > \frac{\ln(\zeta)}{n_t}, \tilde{p}_t > p^* + \varepsilon_a \right\} \\
= \left\{ \max \left\{ M_B(\tilde{p}_t, \tilde{p}_t - \varepsilon_a), M_B(\tilde{p}_t, \frac{\tilde{p}_t}{1-\varepsilon_r}) \right\} > \frac{\ln(\zeta)}{n_t}, |\tilde{p}_t - p^*| \leq \varepsilon_a \right\} \\
\bigcup \left\{ M_B(\tilde{p}_t, \tilde{p}_t + \varepsilon_a) > \frac{\ln(\zeta)}{n_t}, \tilde{p}_t < p^* - \varepsilon_a \right\} \\
\bigcup \left\{ M_B(\tilde{p}_t, \frac{\tilde{p}_t}{1+\varepsilon_r}) > \frac{\ln(\zeta)}{n_t}, \tilde{p}_t > p^* + \varepsilon_a \right\} .
\]

We claim that if \( \varepsilon_a > 0 \) is sufficiently small, then it is true that

\[
\left\{ \max \left\{ M_B(\tilde{p}_t, \tilde{p}_t - \varepsilon_a), M_B(\tilde{p}_t, \frac{\tilde{p}_t}{1-\varepsilon_r}) \right\} > \frac{\ln(\zeta)}{n_t}, |\tilde{p}_t - p^*| \leq \varepsilon_a \right\} = \{ |\tilde{p}_t - p^*| \leq \varepsilon_a \},
\]

(134)

\[
\left\{ M_B(\tilde{p}_t, \tilde{p}_t + \varepsilon_a) > \frac{\ln(\zeta)}{n_t}, \tilde{p}_t < p^* - \varepsilon_a \right\} = \{ \varepsilon_a < \tilde{p}_t < p^* - \varepsilon_a \} \quad \text{for} \quad \frac{\ln(\zeta)}{\ln(1 - \varepsilon_a)} \leq n_t < n_s,
\]

(135)

\[
\left\{ M_B(\tilde{p}_t, \frac{\tilde{p}_t}{1+\varepsilon_r}) > \frac{\ln(\zeta)}{n_t}, \tilde{p}_t > p^* + \varepsilon_a \right\} = \{ p^* + \varepsilon_a < \tilde{p}_t < y^t \}.
\]

(137)

To show (134), note that

\[
n_t < \frac{(1 + C_1)n_s}{2} < \frac{(1 + C_1)}{2} \left[ \frac{\ln(\zeta)}{M_B(p^* + \varepsilon_a, p^*)} + 1 \right],
\]

(138)

which implies that

\[
\frac{\ln(\zeta)}{n_t} < \frac{M_B(p^* + \varepsilon_a, p^*)}{M_B(p^* - \varepsilon_a, p^* - \varepsilon_a - \varepsilon_a)} \left( \frac{2}{1 + C_1} \right) M_B(p^* - \varepsilon_a, p^* - \varepsilon_a - \varepsilon_a) - \frac{\ln(\zeta)}{n_t} M_B(p^* + \varepsilon_a, p^*)
\]

if \( \varepsilon_a > 0 \) is sufficiently small. Noting that

\[
\lim_{\varepsilon_a \to 0} \frac{M_B(p^* + \varepsilon_a, p^*)}{M_B(p^* - \varepsilon_a, p^* - \varepsilon_a - \varepsilon_a)} = \lim_{\varepsilon_a \to 0} \frac{\varepsilon_a^2}{2(p^* - \varepsilon_a)(p^* - \varepsilon_a - 1)} + o(\varepsilon_a^2) = 1
\]

and \( \lim_{\varepsilon_a \to 0} \frac{M_B(p^* + \varepsilon_a, p^*)}{n_t} = 0 \), we have

\[
\frac{\ln(\zeta)}{n_t} < \frac{M_B(p^* - \varepsilon_a, p^* - \varepsilon_a - \varepsilon_a)}{M_B(p^* + \varepsilon_a, p^*)}
\]

(139)

for small enough \( \varepsilon_a > 0 \). Again by (138), we have

\[
\frac{\ln(\zeta)}{n_t} < -\frac{M_B(p^* + \varepsilon_a, p^* + \varepsilon_a)}{M_B(p^* + \varepsilon_a, p^*)} \left( \frac{2}{1 + C_1} \right) M_B(p^* + \varepsilon_a, \frac{p^* + \varepsilon_a}{1 - \varepsilon_r}) - \frac{\ln(\zeta)}{n_t} M_B(p^* + \varepsilon_a, p^*)
\]

152
if $\varepsilon_a > 0$ is sufficiently small. Noting that

$$\lim_{\varepsilon_a \to 0} \frac{\mathcal{M}_B(p^* + \varepsilon_a, p^*)}{\mathcal{M}_B(p^* + \varepsilon_a, p^*/\varepsilon_a)} = \lim_{\varepsilon_a \to 0} \frac{\varepsilon_a^2}{2 p^*(p^*-1)} + o(\varepsilon_a^2) = 1$$

and $\lim_{\varepsilon_a \to 0} \frac{\mathcal{M}_B(p^*+\varepsilon_a, p^*)}{n_\ell} = 0$, we have

$$\frac{\ln(\delta)}{n_\ell} < \mathcal{M}_B(p^* + \varepsilon_a, p^*/1 - \varepsilon_r)$$

for small enough $\varepsilon_a > 0$. It can be seen from Lemmas 156 and 159 that, for $z \in [p^* - \varepsilon_a, p^* + \varepsilon_a]$, $\mathcal{M}_B(z, z - \varepsilon_a)$ is monotonically increasing with respect to $z$ and $\mathcal{M}_B(z, \frac{z}{1 - \varepsilon_r})$ is monotonically decreasing with respect to $z$. By (139) and (140), we have $\ln(\delta)/n_\ell < \mathcal{M}_B(z, z - \varepsilon_a)$ and $\ln(\delta)/n_\ell < \mathcal{M}_B(z, \frac{z}{1 - \varepsilon_r})$ for any $z \in [p^* - \varepsilon_a, p^* + \varepsilon_a]$ if $\varepsilon_a > 0$ is small enough. This proves (134).

To show (135), let $\omega \in \{ \mathcal{M}_B(\tilde{p}_\ell, \tilde{p}_\ell + \varepsilon_a) > \ln(\delta)/n_\ell, \tilde{p}_\ell < p^* - \varepsilon_a \}$ and $\tilde{p}_\ell = \tilde{p}_\ell(\omega)$. Then, $\mathcal{M}_B(\tilde{p}_\ell, \tilde{p}_\ell + \varepsilon_a) > \ln(\delta)/n_\ell$ and $\tilde{p}_\ell < p^* - \varepsilon_a$. Since $z_\ell \in (0, p^* - \varepsilon_a)$ and $\mathcal{M}_B(z_\ell, z_\ell + \varepsilon_a)$ is monotonically increasing with respect to $z \in (0, p^* - \varepsilon_a)$, it must be true that $\tilde{p}_\ell > z_\ell$. Otherwise if $\tilde{p}_\ell \leq z_\ell$, then $\mathcal{M}_B(\tilde{p}_\ell, \tilde{p}_\ell + \varepsilon_a) \leq \mathcal{M}_B(z_\ell, z_\ell + \varepsilon_a) = \ln(\delta)/n_\ell$, leading to a contradiction. This proves $\mathcal{M}_B(\tilde{p}_\ell, \tilde{p}_\ell + \varepsilon_a) > \ln(\delta)/n_\ell, \tilde{p}_\ell < p^* - \varepsilon_a \subseteq \{ z_\ell < \tilde{p}_\ell < p^* - \varepsilon_a \}$. Now let $\omega \in \{ z_\ell < \tilde{p}_\ell < p^* - \varepsilon_a \}$ and $\tilde{p}_\ell = \tilde{p}_\ell(\omega)$. Then, $z_\ell < \tilde{p}_\ell < p^* - \varepsilon_a$. Noting that $\mathcal{M}_B(z_\ell, z_\ell + \varepsilon_a)$ is monotonically increasing with respect to $z \in (0, p^* - \varepsilon_a)$, we have that $\mathcal{M}_B(\tilde{p}_\ell, \tilde{p}_\ell + \varepsilon_a) > \mathcal{M}_B(z_\ell, z_\ell + \varepsilon_a) = \ln(\delta)/n_\ell$, which implies $\mathcal{M}_B(\tilde{p}_\ell, \tilde{p}_\ell + \varepsilon_a) > \ln(\delta)/n_\ell, \tilde{p}_\ell < p^* - \varepsilon_a \supseteq \{ z_\ell < \tilde{p}_\ell < p^* - \varepsilon_a \}$. This establishes (135).

Note that, for any $z \in (0, p^* - \varepsilon_a)$, we have $\mathcal{M}_B(z, z + \varepsilon_a) > \mathcal{M}_B(0, \varepsilon_a) = \ln(1 - \varepsilon_a) \geq \ln(\delta)/n_\ell$, which implies (136).

To show (137), let $\omega \in \{ \mathcal{M}_B(\tilde{p}_\ell, \tilde{p}_\ell + \varepsilon_a) > \ln(\delta)/n_\ell, \tilde{p}_\ell > p^* + \varepsilon_a \}$ and $\tilde{p}_\ell = \tilde{p}_\ell(\omega)$. Then, $\mathcal{M}_B(\tilde{p}_\ell, \tilde{p}_\ell + \varepsilon_a) > \ln(\delta)/n_\ell$ and $\tilde{p}_\ell > p^* + \varepsilon_a$. Since $y_\ell \in (p^* + \varepsilon_a, 1]$ and $\mathcal{M}_B(z, \frac{z}{1 + \varepsilon_r})$ is monotonically decreasing with respect to $z \in (p^* + \varepsilon_a, 1)$, it must be true that $\tilde{p}_\ell < y_\ell$. Otherwise if $\tilde{p}_\ell \geq y_\ell$, then $\mathcal{M}_B(\tilde{p}_\ell, \tilde{p}_\ell + \varepsilon_a) \leq \mathcal{M}_B(y_\ell, \frac{y_\ell}{1 + \varepsilon_r}) = \ln(\delta)/n_\ell$, leading to a contradiction. This proves $\mathcal{M}_B(\tilde{p}_\ell, \tilde{p}_\ell + \varepsilon_a) > \ln(\delta)/n_\ell, \tilde{p}_\ell > p^* + \varepsilon_a \subseteq \{ p^* + \varepsilon_a < \tilde{p}_\ell < y_\ell \}$. Now let $\omega \in \{ p^* + \varepsilon_a < \tilde{p}_\ell < y_\ell \}$ and $\tilde{p}_\ell = \tilde{p}_\ell(\omega)$. Then, $p^* + \varepsilon_a < \tilde{p}_\ell < y_\ell$. Noting that $\mathcal{M}_B(z, \frac{z}{1 + \varepsilon_r})$ is monotonically decreasing with respect to $z \in (0, 1)$, we have that $\mathcal{M}_B(\tilde{p}_\ell, \tilde{p}_\ell + \varepsilon_a) > \mathcal{M}_B(y_\ell, \frac{y_\ell}{1 + \varepsilon_r}) = \ln(\delta)/n_\ell$, which implies $\mathcal{M}_B(\tilde{p}_\ell, \tilde{p}_\ell + \varepsilon_a) > \ln(\delta)/n_\ell, \tilde{p}_\ell > p^* + \varepsilon_a \supseteq \{ p^* + \varepsilon_a < \tilde{p}_\ell < y_\ell \}$. This establishes (137).

**Lemma 72** Let $\ell_\varepsilon = s - j_p$. Then, under the constraint that limits are taken with $\varepsilon_a$ fixed,

$$\lim_{\varepsilon_a \to 0} \sum_{\ell_\varepsilon=1}^{\ell_\varepsilon-1} n_\ell \Pr[D_\ell = 1] = 0, \quad \lim_{\varepsilon_a \to 0} \sum_{\ell = \ell_\varepsilon+1}^{s} n_\ell \Pr[D_\ell = 0] = 0$$

(141)

for $p \in (0, 1)$. Moreover, $\lim_{\varepsilon_a \to 0} n_\ell \Pr[D_\ell = 0] = 0$ if $C_{j_p} > r(p)$.
Proof. For simplicity of notations, let $a_\ell = \lim_{\varepsilon \to 0} y_{\ell}$ and $b_\ell = \lim_{\varepsilon \to 0} z_{\ell}$. The proof consists of three main steps as follows.

First, we shall show that (141) holds for $p \in (0, p^*)$. By the definition of $\ell_\varepsilon$, we have $r(p) > C_{s-\ell_\varepsilon+1}$. Making use of the first four statements of Lemma 71, we have that $z_{\ell} < \frac{p + b_{\ell-1}}{2} < p$ for all $\ell \leq \ell_\varepsilon - 1$ with $n_\ell \geq \frac{\ln(\delta)}{\ln(1-\varepsilon_a)}$ and that $y_{\ell} > \frac{p + a_{\ell-1}}{2} > p^*$ for $1 \leq s < s$ if $\varepsilon_a$ is sufficiently small. Therefore, by the last statement of Lemma 71 and using Chernoff bound, we have that

$$
\Pr\{D_\ell = 1\} = \Pr\{\hat{p}_\ell \leq y_{\ell}\} + \Pr\{\hat{p}_\ell \geq y_{\ell}\} \leq \Pr\{\hat{p}_\ell \leq \frac{p + b_{\ell-1}}{2}\} + \Pr\{\hat{p}_\ell \geq \frac{p^* + a_{\ell-1}}{2}\}
$$

for all $\ell \leq \ell_\varepsilon - 1$ with $n_\ell \geq \frac{\ln(\delta)}{\ln(1-\varepsilon_a)}$ and that

$$
\Pr\{D_\ell = 1\} = \Pr\{\hat{p}_\ell \geq y_{\ell}\} + \Pr\{\hat{p}_\ell = 0\} \leq \Pr\{\hat{p}_\ell \geq \frac{p^* + a_{\ell-1}}{2}\} + \Pr\{\hat{p}_\ell = 0\}
$$

for all $\ell$ with $n_\ell < \frac{\ln(\delta)}{\ln(1-\varepsilon_a)}$ if $\varepsilon_a > 0$ is small enough. As a consequence of the definition of $\ell_\varepsilon$, we have that $b_{\ell_\varepsilon-1}$ is smaller than $p$ and is independent of $\varepsilon_a > 0$. Hence, we can apply Lemma 70 to conclude that $\lim_{\varepsilon_a \to 0} \sum_{\ell=1}^{\ell_\varepsilon-1} n_\ell \Pr\{D_\ell = 1\} = 0$.

Similarly, it can be seen from the definition of $\ell_\varepsilon$ that $r(p) < C_{s-\ell_\varepsilon+1}$. Making use of the first four statements of Lemma 71, we have that $z_{\ell} > \frac{p + b_{\ell-1}}{2} > p$ for $\ell_\varepsilon + 1 \leq \ell < s$ if $\varepsilon_a$ is sufficiently small. By the last statement of Lemma 71 and using Chernoff bound, we have

$$
\Pr\{D_\ell = 0\} = \Pr\{z_{\ell} < \hat{p}_\ell < y_{\ell}\} \leq \Pr\{\hat{p}_\ell > z_{\ell}\} \leq \Pr\{\hat{p}_\ell > \frac{p + b_{\ell-1}}{2}\} \leq \exp\left(-2n_\ell \left(\frac{p - b_{\ell-1}}{2}\right)^2\right)
$$

for $\ell_\varepsilon + 1 \leq \ell < s$ if $\varepsilon_a > 0$ is small enough. By virtue of the definition of $\ell_\varepsilon$, we have that $b_{\ell_\varepsilon+1}$ is greater than $p$ and is independent of $\varepsilon_a > 0$. In view of this and the fact that $\Pr\{D_s = 0\} = 0$, we can use Lemma 70 to arrive at $\lim_{\varepsilon_a \to 0} \sum_{\ell=\ell_\varepsilon+1}^{s} n_\ell \Pr\{D_\ell = 0\} = 0$. This proves that (141) holds for $p \in (0, p^*)$.

Second, we shall show that (141) holds for $p \in (p^*, 1)$. As a direct consequence of the definition of $\ell_\varepsilon$, we have $r(p) > C_{s-\ell_\varepsilon+1}$. Making use of the first four statements of Lemma 71, we have that $y_{\ell} > \frac{p + b_{\ell-1}}{2} > p$ for all $\ell \leq \ell_\varepsilon - 1$ and $z_{\ell-1} < \frac{p + a_{\ell-1}}{2} < p^*$ if $\varepsilon_a$ is sufficiently small. By the last statement of Lemma 71 and using Chernoff bound, we have

$$
\Pr\{D_\ell = 1\} \leq \Pr\{\hat{p}_\ell \geq y_{\ell}\} + \Pr\{\hat{p}_\ell \leq z_{\ell-1}\} \leq \Pr\{\hat{p}_\ell \geq \frac{p + a_{\ell-1}}{2}\} + \Pr\{\hat{p}_\ell \leq \frac{p^* + b_{\ell-1}}{2}\}
$$

$$
\leq \exp\left(-2n_\ell \left(\frac{p - a_{\ell-1}}{2}\right)^2\right) + \exp\left(-2n_\ell \left(p - \frac{p^* + b_{\ell-1}}{2}\right)^2\right)
$$

154
for all \( \ell \leq \ell_a - 1 \) provided that \( \varepsilon_a > 0 \) is small enough. As a result of the definition of \( \ell_a \), we have that \( a_{\ell_a - 1} \) is greater than \( p \) and is independent of \( \varepsilon_a > 0 \). Hence, it follows from Lemma 70 that

\[
\lim_{\varepsilon_a \to 0} \sum_{\ell = 1}^{\ell_a - 1} n_\ell \Pr\{D_\ell = 1\} = 0.
\]

In a similar manner, by the definition of \( \ell_a \), we have \( r(p) < C_{s-\ell_a-1} \). Making use of the first four statements of Lemma 71 we have that \( y_\ell < \frac{p+a_{\ell_a+1}}{2} < p \) for \( \ell_a + 1 \leq \ell < s \) if \( \varepsilon_a > 0 \) is sufficiently small. By the last statement of Lemma 71 and using Chernoff bound, we have

\[
\Pr\{D_\ell = 0\} = \Pr\{z_\ell < \hat{p}_\ell < y_\ell\} \leq \Pr\{\hat{p}_\ell < y_\ell\} \leq \Pr\left\{\hat{p}_\ell < \frac{p + a_{\ell_a + 1}}{2}\right\} \leq \exp\left(-2n_\ell \left(\frac{p - a_{\ell_a + 1}}{2}\right)^2\right)
\]

for \( \ell_a + 1 \leq \ell < s \) if \( \varepsilon_a > 0 \) is small enough. Clearly, \( \Pr\{D_s = 0\} = 0 \). As a consequence of the definition of \( \ell_a \), we have that \( a_{\ell_a+1} \) is smaller than \( p \) and is independent of \( \varepsilon_a > 0 \). Hence, it follows from Lemma 70 that \( \lim_{\varepsilon_a \to 0} \sum_{\ell = \ell_a + 1}^s n_\ell \Pr\{D_\ell = 0\} = 0 \). This proves that (141) holds for \( p \in (p^*,1) \).

Third, we shall show \( \lim_{\varepsilon_a \to 0} n_{\ell_a} \Pr\{D_{\ell_a} = 0\} = 0 \) for \( p \in (0,1) \) such that \( C_{j_p} > r(p) \).

For \( p \in (0,p^*) \) such that \( C_{j_p} > r(p) \), we have \( r(p) < C_{s-\ell_a} \) because of the definition of \( \ell_a \). Making use of the first four statements of Lemma 71 we have that \( z_\ell_a < \frac{p + a_{\ell_a}}{2} < p \) if \( \varepsilon_a > 0 \) is small enough. By the last statement of Lemma 71 and using Chernoff bound, we have

\[
\Pr\{D_{\ell_a} = 0\} = \Pr\{z_\ell_a < \hat{p}_{\ell_a} < y_\ell_a\} \leq \Pr\{\hat{p}_{\ell_a} < y_\ell_a\} \leq \Pr\left\{\hat{p}_{\ell_a} < \frac{p + b_{\ell_a}}{2}\right\} \leq \exp\left(-2n_{\ell_a} \left(\frac{p - b_{\ell_a}}{2}\right)^2\right).
\]

Since \( b_{\ell_a} \) is greater than \( p \) and is independent of \( \varepsilon_a > 0 \) due to the definition of \( \ell_a \), it follows that \( \lim_{\varepsilon_a \to 0} n_{\ell_a} \Pr\{D_{\ell_a} = 0\} = 0 \).

For \( p \in (p^*,1) \) such that \( C_{j_p} > r(p) \), we have \( r(p) < C_{s-\ell_a} \) as a result of the definition of \( \ell_a \). Making use of the first four statements of Lemma 71 we have that \( y_\ell_a < \frac{p + a_{\ell_a}}{2} < p \) if \( \varepsilon_a > 0 \) is sufficiently small. By the last statement of Lemma 71 and using Chernoff bound, we have

\[
\Pr\{D_{\ell_a} = 0\} = \Pr\{z_\ell_a < \hat{p}_{\ell_a} < y_\ell_a\} \leq \Pr\{\hat{p}_{\ell_a} < y_\ell_a\} \leq \Pr\left\{\hat{p}_{\ell_a} < \frac{p + a_{\ell_a}}{2}\right\} \leq \exp\left(-2n_{\ell_a} \left(\frac{p - a_{\ell_a}}{2}\right)^2\right).
\]

Since \( a_{\ell_a} \) is smaller than \( p \) and is independent of \( \varepsilon_a > 0 \) as a consequence of the definition of \( \ell_a \), it follows that \( \lim_{\varepsilon_a \to 0} n_{\ell_a} \Pr\{D_{\ell_a} = 0\} = 0 \). This proves \( \lim_{\varepsilon_a \to 0} n_{\ell_a} \Pr\{D_{\ell_a} = 0\} = 0 \) for \( p \in (0,1) \) such that \( C_{j_p} > r(p) \). The proof of the lemma is thus completed.

\[
\square
\]

The proof of Theorem 31 can be accomplished by employing Lemma 72 and a similar argument as the proof of Theorem 15.

I.18 Proof of Theorem 32

As a result of the definitions of \( \kappa_p \) and \( r(p) \), we have that \( \kappa_p > 1 \) if and only if \( r(p) \) is not an integer. To prove Theorem 32, we need some preliminary results.

Lemma 73 \( \lim_{\varepsilon_a \to 0} \frac{n_{\ell_a}}{\sum_{\ell=1}^{\ell_a} n_\ell} = \kappa_p \), \( \lim_{\varepsilon_a \to 0} \varepsilon_a \sqrt{\frac{n_{\ell_a}}{p(1-p)}} = d \sqrt{\kappa_p} \), \( \lim_{\varepsilon_r \to 0} \varepsilon_r \sqrt{\frac{p n_{\ell_a}}{1-p}} = d \sqrt{\kappa_p} \).
Proof. First, we shall consider $p \in (0, p^*)$. By the definition of sample sizes, we have

$$\lim_{\varepsilon_a \to 0} \ell \left( \frac{C_{s-\ell} \ln(\zeta)}{n_{\ell_n} \ell \ln(p, p + \varepsilon_a, p^*)} \right) = 1 \quad \text{(142)}$$

for $1 \leq \ell < s$. It follows that

$$\lim_{\varepsilon_a \to 0} \frac{n_{\ell_n}}{N_m(p, \varepsilon_a, r)} = \lim_{\varepsilon_a \to 0} \frac{\mathcal{M}_B(p, p + \varepsilon_a)}{\mathcal{M}_B(p + \varepsilon_a, p^*)} \frac{C_{s-\ell_n} \ln(\zeta)}{\mathcal{M}_B(p^* + \varepsilon_a, p^*)} = \lim_{\varepsilon_a \to 0} C_{s-\ell_n} \frac{\mathcal{M}_B(p, p + \varepsilon_a)}{\mathcal{M}_B(p^* + \varepsilon_a, p^*)}$$

and

$$\lim_{\varepsilon_a \to 0} \varepsilon_a \sqrt{\frac{n_{\ell_n}}{p(1-p)}} = \lim_{\varepsilon_a \to 0} \varepsilon_a \sqrt{\frac{1}{p(1-p)} \frac{C_{s-\ell_n} \ln(\zeta)}{\mathcal{M}_B(p^* + \varepsilon_a, p^*)}} = \lim_{\varepsilon_a \to 0} \frac{p(1-p)}{p^*(1-p)} \frac{C_{s-\ell_n} \ln(\zeta)}{\mathcal{M}_B(p^* + \varepsilon_a, p^*)}$$

Next, we shall consider $p \in (p^*, 1]$. By virtue of (142), we have

$$\lim_{\varepsilon_a \to 0} \frac{n_{\ell_n}}{N_m(p, \varepsilon_a, \varepsilon_a)} = \lim_{\varepsilon_a \to 0} \frac{\mathcal{M}_B(p, p + \varepsilon_a)}{\mathcal{M}_B(p + \varepsilon_a, p^*)} \frac{C_{s-\ell_n} \ln(\zeta)}{\mathcal{M}_B(p^* + \varepsilon_a, p^*)} = \lim_{\varepsilon_a \to 0} \frac{C_{s-\ell_n} \mathcal{M}_B(p, p + \varepsilon_a)}{\mathcal{M}_B(p^* + \varepsilon_a, p^*)}$$

and

$$\lim_{\varepsilon_a \to 0} \varepsilon_a \sqrt{\frac{p n_{\ell_n}}{1-p}} = \lim_{\varepsilon_a \to 0} \varepsilon_a \sqrt{\frac{p}{1-p} \frac{C_{s-\ell_n} \ln(\zeta)}{\mathcal{M}_B(p^* + \varepsilon_a, p^*)}} = \lim_{\varepsilon_a \to 0} \frac{p}{p^*(1-p)} \frac{C_{s-\ell_n} \ln(\zeta)}{\mathcal{M}_B(p^* + \varepsilon_a, p^*)}$$

Lemma 74 Let $U$ and $V$ be independent Gaussian random variables with zero means and unit variances. Then, for $p \in (0, 1)$ such that $C_{j_p} = r(p)$ and $j_p \geq 1$,

$$\lim_{\varepsilon_a \to 0} \Pr\{l = \ell_{\varepsilon_a}\} = 1 - \lim_{\varepsilon_a \to 0} \Pr\{l = \ell_{\varepsilon_a + 1}\} = 1 - \Phi (\nu d),$$

$$\lim_{\varepsilon_a \to 0} \Pr\{|\hat{M}_{\ell_{\varepsilon_a}} - p| \geq \varepsilon_p, l = \ell_{\varepsilon_a}\} + \Pr\{|\hat{M}_{\ell_{\varepsilon_a + 1}} - p| \geq \varepsilon_p, l = \ell_{\varepsilon_a + 1}\}$$

$$= \Pr\{U \geq d\} + \Pr\{|U + \sqrt{p} V| \geq d\sqrt{1 + p}, U < \nu d\},$$

where $\varepsilon_p = \max\{\varepsilon_a, \varepsilon_r p\}.$
Proof. First, we shall consider $p \in [p^*, 1)$. Since $\kappa_p = 1$, by Statement (V) of Lemma 71, we have
\[
\lim_{\varepsilon \to 0} \frac{z_{\ell_\varepsilon} - p}{\sqrt{p(1-p)/n_{\ell_\varepsilon}}} = \lim_{\varepsilon \to 0} \varepsilon \sqrt{\frac{m_{\ell_\varepsilon}}{1-p}} \lim_{\varepsilon \to 0} \frac{z_{\ell_\varepsilon} - p}{\varepsilon r p} = d \lim_{\varepsilon \to 0} \frac{z_{\ell_\varepsilon} - p}{\varepsilon r p} = \nu d.
\]
Note that
\[
\Pr\{|\hat{p}_{\ell_\varepsilon} - p| \geq \varepsilon, p \leq z_{\ell_\varepsilon}\} = \Pr \left\{ \frac{|\hat{p}_{\ell_\varepsilon} - p|}{\sqrt{p(1-p)/n_{\ell_\varepsilon}}} \geq \varepsilon \sqrt{\frac{m_{\ell_\varepsilon}}{1-p}}, \frac{z_{\ell_\varepsilon} - p}{\sqrt{p(1-p)/n_{\ell_\varepsilon}}} \geq \frac{z_{\ell_\varepsilon} - p}{\sqrt{p(1-p)/n_{\ell_\varepsilon}}} \right\}.
\]
Therefore,
\[
\Pr\{|\hat{p}_{\ell_\varepsilon} - p| \geq \varepsilon, p = \ell_{\varepsilon}\} \to \Pr\{|\hat{p}_{\ell_{\varepsilon}+1} - p| \geq \varepsilon, p = \ell_{\varepsilon} + 1\}
\]
\[
\rightarrow \Pr\{|U| \geq b, U \geq \nu d\} + \Pr\{|U + \sqrt{p_{\varepsilon}}V| \geq d\sqrt{1 + p_{\varepsilon}}, U < \nu d\}
\]
\[
\Rightarrow \Pr\{U \geq d\} + \Pr\{|U + \sqrt{p_{\varepsilon}}V| \geq d\sqrt{1 + p_{\varepsilon}}, U < \nu d\}
\]
for $p \in (p^*, 1)$ such that $C_{j_p} = r(p)$.

Next, we shall consider $p \in (0, p^*)$. Since $\kappa_p = 1$, by Statement (V) of Lemma 71, we have
\[
\lim_{\varepsilon \to 0} \frac{z_{\ell_\varepsilon} - p}{\sqrt{p(1-p)/n_{\ell_\varepsilon}}} = \lim_{\varepsilon \to 0} \varepsilon \sqrt{\frac{m_{\ell_\varepsilon}}{1-p}} \lim_{\varepsilon \to 0} \frac{z_{\ell_\varepsilon} - p}{\varepsilon a} = d \lim_{\varepsilon \to 0} \frac{z_{\ell_\varepsilon} - p}{\varepsilon a} = -\nu d.
\]
Note that
\[
\Pr\{|\hat{p}_{\ell_\varepsilon} - p| \geq \varepsilon, p \leq z_{\ell_\varepsilon}\} = \Pr \left\{ \frac{|\hat{p}_{\ell_\varepsilon} - p|}{\sqrt{p(1-p)/n_{\ell_\varepsilon}}} \geq \varepsilon \sqrt{\frac{n_{\ell_\varepsilon}}{p(1-p)}}, \frac{\hat{p}_{\ell_\varepsilon} - p}{\sqrt{p(1-p)/n_{\ell_\varepsilon}}} \leq \frac{z_{\ell_\varepsilon} - p}{\sqrt{p(1-p)/n_{\ell_\varepsilon}}} \right\}.
\]
Therefore, $\Pr\{D_{\ell_\varepsilon} = 1\} \to \Pr\{U \geq \nu d\}$ and
\[
\Pr\{|\hat{p}_{\ell_\varepsilon} - p| \geq \varepsilon, p = \ell_{\varepsilon}\} \to \Pr\{|\hat{p}_{\ell_{\varepsilon}+1} - p| \geq \varepsilon, p = \ell_{\varepsilon} + 1\}
\]
\[
\rightarrow \Pr\{|U| \geq d, U \leq -\nu d\} + \Pr\{|U + \sqrt{p_{\varepsilon}}V| \geq d\sqrt{1 + p_{\varepsilon}}, U > -\nu d\}
\]
\[
\Rightarrow \Pr\{U \geq d\} + \Pr\{|U + \sqrt{p_{\varepsilon}}V| \geq d\sqrt{1 + p_{\varepsilon}}, U < \nu d\}
\]
for $p \in (0, p^*)$ such that $C_{j_p} = r(p)$.

Now, we shall first show that Statement (I) holds for $p \in (0, p^*)$ such that $C_{j_p} = r(p)$. For this purpose, we need to show that
\[
1 \leq \lim_{\varepsilon \to 0} \frac{\sum_{\omega \in \mathcal{C}_{\varepsilon, p}} \mathbb{B}_\omega}{\mathcal{C}_{\varepsilon, p}} \leq 1 + \rho_p \quad \text{for any } \omega \in \left\{ \lim_{\varepsilon \to 0} \hat{p} = p \right\}.
\]
To show $\lim_{\varepsilon \to 0} \frac{\sum_{\omega \in \mathcal{C}_{\varepsilon, p}} \mathbb{B}_\omega}{\mathcal{C}_{\varepsilon, p}} \geq 1$, note that $C_{\varepsilon - \ell_{\varepsilon} + 1} = C_{\varepsilon - \ell_{\varepsilon}} < C_{\varepsilon - \ell_{\varepsilon} - 1}$ as a direct consequence of the definitions of $\ell_{\varepsilon}$ and $j_p$. By the first four statements of Lemma 71 we have
\[
\lim_{\varepsilon \to 0} z_\ell < p \quad \text{for all } \ell \leq \ell_{\varepsilon} - 1 \quad \text{with } n_\ell \geq \frac{\ln(\varepsilon)}{\ln(1-\varepsilon)}.
\]
Noting that $\lim_{\varepsilon \to 0} \hat{p}(\omega) = p$, we have $\hat{p}(\omega) > z_\ell$ for all $\ell \leq \ell_{\varepsilon} - 1$ with $n_\ell \geq \frac{\ln(\varepsilon)}{\ln(1-\varepsilon)}$ and it follows from the definition of the sampling
scheme that \( n(\omega) \geq n_\epsilon \) if \( \epsilon > 0 \) is small enough. By Lemma 73, and noting that \( \kappa_p = 1 \) if \( C_{jp} = r(p) \), we have \( \limsup_{\epsilon_a \to 0} n_m(p, \epsilon_a, \epsilon, r) \geq \lim_{\epsilon_a \to 0} n_m(p, \epsilon_a, \epsilon, r) = \kappa_p = 1 \).

To show \( \limsup_{\epsilon_a \to 0} n_m(p, \epsilon_a, \epsilon, r) \leq \lim_{\epsilon_a \to 0} n_m(p, \epsilon_a, \epsilon, r) = \kappa_p = 1 + \rho \), we shall consider three cases: (i) \( \ell_\epsilon = s \); (ii) \( \ell_\epsilon = s - 1 \); (iii) \( \ell_\epsilon = s - 1 \). In the case of \( \ell_\epsilon = s \), it must be true that \( n(\omega) \leq n_s = n_\epsilon \). Hence, \( \limsup_{\epsilon_a \to 0} n_m(p, \epsilon_a, \epsilon, r) \leq \lim_{\epsilon_a \to 0} n_m(p, \epsilon_a, \epsilon, r) = \kappa_p = 1 + \rho \). In the case of \( \ell_\epsilon = s - 1 \), it must be true that \( n(\omega) \leq n_s = n_\epsilon + 1 \). Hence, \( \limsup_{\epsilon_a \to 0} n_m(p, \epsilon_a, \epsilon, r) \leq \lim_{\epsilon_a \to 0} n_m(p, \epsilon_a, \epsilon, r) = 1 + \rho \).

In the case of \( \ell_\epsilon = s - 1 \), it follows from Lemma 71 that \( \lim_{\epsilon_a \to 0} z_{\ell_\epsilon + 1} > p \), which implies that \( z_{\ell_\epsilon + 1} > p \), \( \hat{p}(\omega) < z_{\ell_\epsilon + 1} \), and thus \( n(\omega) \leq n_\epsilon + 1 \) for small enough \( \epsilon > 0 \). Therefore, \( \limsup_{\epsilon_a \to 0} n_m(p, \epsilon_a, \epsilon, r) \leq \lim_{\epsilon_a \to 0} n_m(p, \epsilon_a, \epsilon, r) = 1 + \rho \). This establishes (143) and it follows that \( \{1 \leq \limsup_{\epsilon_a \to 0} n_m(p, \epsilon_a, \epsilon, r) \leq 1 + \rho \} \supseteq \{\lim_{\epsilon_a \to 0} \hat{p} = p\} \). According to the strong law of large numbers, we have \( 1 \geq \Pr\{1 \leq \limsup_{\epsilon_a \to 0} n_m(p, \epsilon_a, \epsilon, r) \leq 1 + \rho\} \geq \Pr\{\lim_{\epsilon_a \to 0} \hat{p} = p\} = 1 \). This proves that Statement (I) holds for \( p \in (0, p^*) \) such that \( C_{jp} = r(p) \).

Next, we shall show that Statement (I) holds for \( p \in (0, p^*) \) such that \( C_{jp} = r(p) \). Note that \( C_{s-\epsilon_a + 1} < r(p) < C_{s-\epsilon_a} \) as a direct consequence of the definitions of \( \ell_\epsilon \) and \( j_p \). By the first four statements of Lemma 71, we have \( \lim_{\epsilon_a \to 0} z_{\ell_\epsilon - 1} = p \) and thus \( z_\ell < p \) for all \( \ell \leq \ell_\epsilon - 1 \) with \( n_\ell \geq \frac{n_\ell}{n_\epsilon + 1} \) provided that \( \epsilon > 0 \) is sufficiently small. Therefore, for any \( \omega \in \{\lim_{\epsilon_a \to 0} \hat{p} = p\} \), we have \( z_\ell < \hat{p}(\omega) < y_\ell \) for all \( \ell \leq \ell_\epsilon - 1 \) with \( n_\ell \geq \frac{n_\ell}{n_\epsilon + 1} \) and consequently, \( n(\omega) \geq n_\epsilon \) provided that \( \epsilon > 0 \) is sufficiently small. On the other hand, we can show that \( n(\omega) \geq n_\epsilon \) if \( \epsilon > 0 \) is small enough by investigating two cases. In the case of \( \ell_\epsilon = s \), it is trivially true that \( n(\omega) \leq n_\epsilon \). In the case of \( \ell_\epsilon < s \), we have \( \epsilon < \lim_{\epsilon_a \to 0} z_\ell \) and thus \( p < z_\ell \) provided that \( \epsilon > 0 \) is sufficiently small. Therefore, for any \( \omega \in \{\lim_{\epsilon_a \to 0} \hat{p} = p\} \), we have \( \hat{p}(\omega) < z_\ell \) and consequently, \( n(\omega) \leq n_\ell \) provided that \( \epsilon > 0 \) is sufficiently small. So, we have established that \( n(\omega) = n_\ell \) if \( \epsilon > 0 \) is sufficiently small. Applying Lemma 73, we have \( \lim_{\epsilon_a \to 0} n_m(p, \epsilon_a, \epsilon, r) = \lim_{\epsilon_a \to 0} n_{m(p, \epsilon_a, \epsilon, r)} = \kappa_p \), which implies that \( \lim_{\epsilon_a \to 0} n_n(p, \epsilon_a, \epsilon, r) = \kappa_p \) \( \supseteq \{\lim_{\epsilon_a \to 0} \hat{p} = p\} \) follows from the strong law of large numbers that \( 1 \geq \Pr\{\lim_{\epsilon_a \to 0} n_n(p, \epsilon_a, \epsilon, r) = \kappa_p \} \geq \Pr\{\lim_{\epsilon_a \to 0} \hat{p} = p\} = 1 \) and thus \( \Pr\{\lim_{\epsilon_a \to 0} n_n(p, \epsilon_a, \epsilon, r) = \kappa_p \} = 1 \). Since \( 1 \geq \kappa_p \leq 1 + \rho \), it is of course true that \( \Pr\{1 \leq \limsup_{\epsilon_a \to 0} n_n(p, \epsilon_a, \epsilon, r) \leq 1 + \rho \} = 1 \). This proves that Statement (I) holds true for \( p \in (0, p^*) \) such that \( C_{jp} = r(p) \). Thus, we have shown that Statement (I) holds true for \( p \in (0, p^*) \).

In a similar manner, we can show that Statement (I) is true for \( p \in (p^*, 1) \). This concludes the proof for Statement (I) of the theorem.

To show Statements (II) and (III), we can employ Lemmas 72, 73 and mimic the corresponding arguments for Theorem 10 by identifying \( \epsilon_a \) and \( \epsilon_r \) as \( \epsilon \) for the cases of \( p \leq p^* \) and \( p > p^* \) respectively in the course of proof. Specially, in order to prove Statement (III), we need to make use of the following observation:

\[
\Pr\{\hat{p} - p \geq \epsilon_a, |\hat{p} - p| \geq \epsilon_r, p\} = \begin{cases} \Pr\{\hat{p} - p \geq \epsilon_a\} & \text{for } p \in (0, p^*], \\ \Pr\{\hat{p} - p \geq \epsilon_r, p\} & \text{for } p \in (p^*, 1) \end{cases}
\]
Pr\{∥\hat{p}_t - p\parallel ≥ ε_a\} = Pr\left\{\left|U_\ell\right| ≥ \frac{ε_a}{\sqrt{\frac{np_\ell}{1 - p}}}\right\}, \quad \text{Pr}\{∥\hat{p}_t - p\parallel ≥ ε_r\parallel p\parallel = Pr\left\{\left|U_\ell\right| ≥ \frac{ε_r\parallel p\parallel}{\sqrt{\frac{mp_\ell}{1 - p}}}\right\}

where, according to the central limit theorem, \(U_\ell = \frac{\hat{p}_t - p}{\sqrt{\frac{p(1 - p)}{n_\ell}}}\) converges in distribution to a Gaussian random variable of zero mean and unit variance as \(ε_a \to 0\).

I.19  Proof of Theorem 33

We need some preliminary results.

Lemma 75 Let \(\overline{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i\), where \(X_1, \cdots, X_n\) are i.i.d. random variables such that \(0 ≤ X_i ≤ 1\) and \(\mathbb{E}[X_i] = \mu \in (0, 1)\) for \(i = 1, \cdots, n\). Then, \(\Pr\{\overline{X}_n ≥ \mu, \text{A}_B(\overline{X}_n, \mu) ≤ \frac{\ln α}{n}\} ≤ α\) for any \(α > 0\).

Proof. For simplicity of notations, let \(F_{\overline{X}_n}(z) = \Pr\{\overline{X}_n ≤ z\}\). By Lemma 1 we have that \(\{\overline{X}_n ≥ \mu\} = \{\overline{X}_n ≥ \mu, F_{\overline{X}_n}(\overline{X}_n) ≤ \exp(\frac{n \cdot \text{A}_B(\overline{X}_n, \mu)}{\ln α})\}\). Therefore,

\[
\left\{\overline{X}_n ≥ \mu, \text{A}_B(\overline{X}_n, \mu) ≤ \frac{\ln α}{n}\right\} = \left\{\overline{X}_n ≥ \mu, \text{A}_B(\overline{X}_n, \mu) ≤ \frac{\ln α}{n}, F_{\overline{X}_n}(\overline{X}_n) ≤ \exp(\frac{n \cdot \text{A}_B(\overline{X}_n, \mu)}{\ln α})\right\}
\subseteq \{F_{\overline{X}_n}(\overline{X}_n) ≤ α\}
\]

and thus Lemma 75 follows from Lemma 2.

Lemma 76 Let \(\overline{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i\), where \(X_1, \cdots, X_n\) are i.i.d. random variables such that \(0 ≤ X_i ≤ 1\) and \(\mathbb{E}[X_i] = \mu \in (0, 1)\) for \(i = 1, \cdots, n\). Then, \(\Pr\{\overline{X}_n ≤ \mu, \text{A}_B(\overline{X}_n, \mu) ≤ \frac{\ln α}{n}\} ≤ α\) for any \(α > 0\).

Proof. For simplicity of notations, let \(G_{\overline{X}_n}(z) = \Pr\{\overline{X}_n ≥ z\}\). By Lemma 1 we have that \(\{\overline{X}_n ≤ \mu\} = \{\overline{X}_n ≤ \mu, G_{\overline{X}_n}(\overline{X}_n) ≤ \exp(\frac{n \cdot \text{A}_B(\overline{X}_n, \mu)}{\ln α})\}\). Therefore,

\[
\left\{\overline{X}_n ≤ \mu, \text{A}_B(\overline{X}_n, \mu) ≤ \frac{\ln α}{n}\right\} = \left\{\overline{X}_n ≤ \mu, \text{A}_B(\overline{X}_n, \mu) ≤ \frac{\ln α}{n}, G_{\overline{X}_n}(\overline{X}_n) ≤ \exp(\frac{n \cdot \text{A}_B(\overline{X}_n, \mu)}{\ln α})\right\}
\subseteq \{G_{\overline{X}_n}(\overline{X}_n) ≤ α\}
\]

and thus Lemma 76 follows from Lemma 2.

Now we are in a position to show Theorem 33 By a similar method as that of Lemma 8 we can show that \(\{\text{A}_B\left(\frac{1}{2} - \frac{1}{2} - \hat{\mu}_s\parallel, \frac{1}{2} - \frac{1}{2} - \hat{\mu}_s\parallel + \varepsilon\right) ≤ \frac{\ln(\frac{a}{n_\ell})}{n_\ell}\} \), is a sure event. By a similar method as that of Lemma 9 we can show that \(\{\text{A}_B\left(\frac{1}{2} - \frac{1}{2} - \hat{\mu}_\ell\parallel, \frac{1}{2} - \frac{1}{2} - \hat{\mu}_\ell\parallel + \varepsilon\right) ≤ \frac{\ln(\frac{a}{n_\ell})}{n_\ell}\} \subseteq
We need some preliminary results. For simplicity of notations, let \( F_{X_n}(z) = \Pr \{ X_n \geq z \} \). By Lemma 1, we have that \( \{ X_n \geq \mu \} = \{ X_n \geq \mu, F_{X_n}(X_n) \leq \exp(n,\mathcal{M}(X_n,\mu)) \} \). Therefore,

\[
\{ X_n \geq \mu, \mathcal{M}(X_n,\mu) \leq \frac{\ln \alpha}{n} \} = \{ X_n \geq \mu, \mathcal{M}(X_n,\mu) \leq \frac{\ln \alpha}{n}, F_{X_n}(X_n) \leq \exp(n,\mathcal{M}(X_n,\mu)) \} \subseteq \{ F_{X_n}(X_n) \leq \alpha \}
\]

and thus Lemma 77 follows from Lemma 2.

\[\square\]

\textbf{Lemma 78} Let \( X_n = \sum_{i=1}^n X_i \), where \( X_1, \ldots, X_n \) are i.i.d. random variables such that \( 0 \leq X_i \leq 1 \) and \( \mathbb{E}[X_i] = \mu \in (0,1) \) for \( i = 1, \ldots, n \). Then, \( \Pr \{ \overline{X}_n \leq \mu, \mathcal{M}(\overline{X}_n,\mu) \leq \frac{\ln \alpha}{n} \} \leq \alpha \) for any \( \alpha > 0 \).

\textbf{Proof.} For simplicity of notations, let \( G_{X_n}(z) = \Pr \{ X_n \leq z \} \). By Lemma 1, we have that \( \{ X_n \leq \mu \} = \{ X_n \leq \mu, G_{X_n}(X_n) \leq \exp(n,\mathcal{M}(X_n,\mu)) \} \). Therefore,

\[
\{ X_n \leq \mu, \mathcal{M}(X_n,\mu) \leq \frac{\ln \alpha}{n} \} = \{ X_n \leq \mu, \mathcal{M}(X_n,\mu) \leq \frac{\ln \alpha}{n}, G_{X_n}(X_n) \leq \exp(n,\mathcal{M}(X_n,\mu)) \} \subseteq \{ G_{X_n}(X_n) \leq \alpha \}
\]

and thus Lemma 78 follows from Lemma 2.

\[\square\]

Now we are in a position to show Theorem 34. By a similar method as that of Lemma 10 we can show that \( \{ (\overline{\mu}_n - \frac{1}{n}) - \frac{2\alpha}{3} \}^2 \geq \frac{1}{4} + \frac{n \alpha^2}{2(\ln \alpha)} \} \) is a sure event. By a similar method as that of Lemma
we can show that \( \{ (\hat{\lambda}_{\ell} - \frac{1}{2}) - \frac{\ln n}{3\ell} \} \leq \{ \mathcal{M}_B(\hat{\lambda}_{\ell}, \hat{\lambda}_{\ell} + \varepsilon) \leq \frac{\ln(\hat{\lambda}_{\ell})}{n_{\ell}}, \mathcal{M}_B(\hat{\lambda}_{\ell}, \hat{\lambda}_{\ell} - \varepsilon) \leq \frac{\ln(\hat{\lambda}_{\ell})}{n_{\ell}} \} \) for \( \ell = 1, \cdots, s \). Therefore, by a variation of the argument for Theorem 33 and using Lemmas 74 and 75, we have \( \Pr\{ |\hat{\lambda} - \lambda| \geq \varepsilon \} \leq \delta \), from which Theorem 34 follows.

I.21 Proof of Theorem 38

By a similar method as that of Lemma 68, we can show that \( \{ \mathcal{M}_B(\tilde{\lambda}, \tilde{\lambda}) \} \leq \frac{\ln(\tilde{\lambda})}{n_{\ell}}, \mathcal{M}_B(\tilde{\lambda}, \mathcal{U}(\tilde{\lambda})) \leq \frac{\ln(\tilde{\lambda})}{n_{\ell}} \} \) is a sure event. By Lemmas 75 and 76, we have

\[
\Pr\{ |\hat{\lambda} - \mu| \geq \varepsilon \} \leq \sum_{\ell=1}^{s} \Pr\left\{ \mu \geq \mathcal{U}(\hat{\lambda}_{\ell}), \mathcal{M}_B(\hat{\lambda}_{\ell}, \mathcal{U}(\hat{\lambda}_{\ell})) \leq \frac{\ln(\hat{\lambda}_{\ell})}{n_{\ell}} \right\} + \sum_{\ell=1}^{s} \Pr\left\{ \mu \leq \mathcal{U}(\hat{\lambda}_{\ell}), \mathcal{M}_B(\hat{\lambda}_{\ell}, \mu) \leq \frac{\ln(\hat{\lambda}_{\ell})}{n_{\ell}} \right\} \leq \delta,
\]

from which Theorem 38 follows.

I.22 Proof of Theorem 39

By a similar method as that of Lemma 69, we can show that \( \{ D_s = 1 \} \) is a sure event. By a similar method as that of Lemma 68, we can show that \( \{ D_{\ell} = 1 \} \subseteq \{ \mathcal{M}_B(\hat{\lambda}_{\ell}, \mathcal{U}(\hat{\lambda}_{\ell})) \leq \frac{\ln(\hat{\lambda}_{\ell})}{n_{\ell}} \} \) for \( \ell = 1, \cdots, s \). Therefore, by a variation of the argument for Theorem 38 and using Lemmas 74 and 75, we can establish Theorem 39.

J Proofs of Theorems for Estimation of Poisson Parameters

J.1 Proof of Theorem 44

First, we shall show statement (I). Let \( 0 < \eta < 1 \) and \( r = \inf_{\ell>0} \frac{n_{\ell+1}}{n_{\ell}} \). By the assumption that \( r > 1 \), we have that there exists a number \( \ell' \geq \max\{ \tau, \tau + \frac{2}{r-1} + \frac{\ln(\delta)}{\ln r} \} \) such that \( \frac{n_{\ell+1}}{n_{\ell}} > \frac{r+1}{2} \) for any \( \ell > \ell' \). Noting that

\[
\frac{\ln(\delta)}{n_{\ell}} < \frac{2}{r+1} \times \frac{(\ell + 1 - \tau) \ln 2 - \ln(\delta)}{(\ell - \tau) \ln 2 - \ln(\delta)} = \frac{2}{r+1} \times \left( 1 + \frac{1}{\ell - \tau - \ln(\delta)} - \frac{1}{\ln r} \right) < 1
\]

for \( \ell > \ell' \) and that \( \frac{\ln(\delta)}{n_{\ell}} = \frac{\ln(\delta^r)}{n_{\ell}} \rightarrow 0 \) as \( \ell \rightarrow \infty \), we have that there exists an integer \( k \) greater than \( \ell' \) such that \( \mathcal{M}_p(\frac{\lambda}{\eta}, \frac{\lambda}{\eta} + \varepsilon) > \frac{\ln(\delta^k)}{n_{\ell}} \) for all \( \ell \geq k \). For \( \ell \) no less than such \( k \), we claim that \( z > \frac{\lambda}{\eta} \). If \( \mathcal{M}_p(\frac{\lambda}{\eta}, \frac{\lambda}{\eta} + \varepsilon) < \frac{\ln(\delta^k)}{n_{\ell}} \) and \( z \in [0, \infty) \). To prove this claim, suppose, to get a contradiction, that \( z \leq \frac{\lambda}{\eta} \). Then, since \( \mathcal{M}_p(\frac{\lambda}{\eta}, \frac{\lambda}{\eta} + \varepsilon) \) is monotonically increasing with respect to \( z > 0 \), we have \( \mathcal{M}_p(\frac{\lambda}{\eta}, \frac{\lambda}{\eta} + \varepsilon) < \frac{\ln(\delta^k)}{n_{\ell}} \), which is a contradiction. Therefore, we have shown the claim and it follows that \( \{ \mathcal{M}_p(\frac{K_{\ell}}{n_{\ell}}, \frac{K_{\ell}}{n_{\ell}} + \varepsilon) > \frac{\ln(\delta^k)}{n_{\ell}} \} \subseteq \{ \frac{K_{\ell}}{n_{\ell}} > \frac{\lambda}{\eta} \} \) for \( k \geq \kappa \). So,

\[
\Pr\{ l > \ell \} \leq \Pr\left\{ \mathcal{M}_p\left( \frac{K_{\ell}}{n_{\ell}}, \frac{K_{\ell}}{n_{\ell}} + \varepsilon \right) > \frac{\ln(\delta^k)}{n_{\ell}} \right\} \leq \Pr\left\{ \frac{K_{\ell}}{n_{\ell}} > \frac{\lambda}{\eta} \right\} < \exp(-cn_{\ell})
\]
where $c = -\mathcal{M}_P(\hat{\lambda}, \lambda)$ and the last inequality is due to Chernoff bounds. Since $\Pr\{l > \ell\} < \exp(-cn_\ell)$ and $n_\ell \to \infty$ as $\ell \to \infty$, we have $\Pr\{l < \infty\} = 1$ or equivalently, $\Pr\{n < \infty\} = 1$. This completes the proof of statement (I).

To show statement (II) of Theorem 44, we can use an argument similar to the proof of statement (II) of Theorem 23.

To show statement (III) of Theorem 44, we can use an argument similar to the proof of statement (III) of Theorem 23.

To show statement (IV) of Theorem 44, we can use an argument similar to the proof of statement (IV) of Theorem 23 and make use of the observation that

$$
\Pr\left\{\left|\hat{\lambda} - \lambda\right| \geq \varepsilon \mid \lambda\right\} = \sum_{\ell = 1}^{\ell^*} \Pr\left\{\left|\hat{\lambda}_\ell - \lambda\right| \geq \varepsilon, l = \ell \mid \lambda\right\} + \sum_{\ell = \ell^* + 1}^{\infty} \Pr\left\{\left|\hat{\lambda}_\ell - \lambda\right| \geq \varepsilon, l = \ell \mid \lambda\right\}
$$

where

$$
\Pr\left\{\left|\hat{\lambda}_\ell - \lambda\right| \geq \varepsilon, l = \ell \mid \lambda\right\} \leq \sum_{\ell = 1}^{\ell^*} \Pr\{l = \ell \mid \lambda\} + \eta
$$

and

$$
\sum_{\ell = \ell^* + 1}^{\infty} \Pr\left\{\left|\hat{\lambda}_\ell - \lambda\right| \geq \varepsilon, l = \ell \mid \lambda\right\} + \eta \leq \sum_{\ell = 1}^{\ell^*} \Pr\left\{\hat{\lambda}_\ell \leq z_\ell \mid \lambda\right\} + \eta + \sum_{\ell = 1}^{\ell^*} \exp(n_\ell \mathcal{M}_P(z_\ell, \lambda)) + \eta.
$$

To show statement (V) of Theorem 44, we can use an argument similar to the proof of statement (V) of Theorem 23.

**J.2 Proof of Theorem 45**

Theorem 45 can be established by using a method similar to that of Theorem 27 based on the following preliminary results.

**Lemma 79** Let $\varepsilon > 0$. Then, $\mathcal{M}_P(z, z + \varepsilon)$ is monotonically increasing with respect to $z > 0$.

**Proof.** Note that $\mathcal{M}_P(z, z + \varepsilon) = -\varepsilon + z \ln \left(\frac{z + \varepsilon}{z}\right)$ and

$$
\frac{\partial \mathcal{M}_P(z, z + \varepsilon)}{\partial z} = \ln \left(\frac{z + \varepsilon}{z}\right) - \frac{\varepsilon}{z + \varepsilon} = -\ln \left(1 - \frac{\varepsilon}{z + \varepsilon}\right) - \frac{\varepsilon}{z + \varepsilon} > 0, \quad \forall z > 0
$$

where the inequality follows from $\ln(1-x) \leq -x$, $\forall x \in [0, 1)$.

**Lemma 80** $\lim_{\varepsilon \to 0} \sum_{\ell=1}^{\ell^*} n_\ell e^{-n_\ell c}$ for any $c > 0$.

**Proof.** Lemma 80 can be shown by a similar method as that of Lemma 14.
Lemma 81 If $\varepsilon$ is sufficiently small, then the following statements hold true.

(I): For $\ell = 1, \cdots, \tau$, there exists a unique number $z_{\ell} \in [0, \infty)$ such that $n_\ell = \frac{\ln(\zeta b_\ell)}{\mathcal{M}_P(\varepsilon, z_{\ell} + \varepsilon)}$.

(II): $z_\ell$ is monotonically increasing with respect to $\ell$ no greater than $\tau$.

(III): $\lim_{\varepsilon \to 0} z_\ell = \lambda^* C_{\tau - \ell}$ for $1 \leq \ell \leq \tau$, where the limit is taken under the restriction that $\ell - \tau$ is fixed with respect to $\varepsilon$.

(IV): $\{D_\ell = 0\} = \{\tilde{\lambda}_\ell > z_\ell\}$ for $\ell = 1, \cdots, \tau$.

Proof. Lemma 81 can be shown by a similar method as that of Lemma 88. □

Lemma 82 Define $\ell_\varepsilon = \tau - j_\lambda$, where $j_\lambda$ is the largest integer $j$ such that $C_j \geq \frac{1}{\lambda^*}$. Then,

$$\lim_{\varepsilon \to 0} \sum_{\ell=1}^{\ell-1} n_\ell \Pr\{D_\ell = 1\} = 0, \quad \lim_{\varepsilon \to 0} \sum_{\ell=\ell+1}^{\tau} n_\ell \Pr\{D_\ell = 0\} = 0 \quad (144)$$

for $\lambda \in (0, \lambda^*)$. Moreover, $\lim_{\varepsilon \to 0} n_\ell \Pr\{D_\ell = 0\} = 0$ for $\lambda \in (0, \lambda^*)$ such that $C_{j_\lambda} > \frac{1}{\lambda^*}$.

Proof. For simplicity of notations, let $b_\ell = \lim_{\varepsilon \to 0} z_\ell$ for $1 \leq \ell \leq \tau$.

First, we shall show that (144) holds for $\lambda \in (0, \lambda^*)$. By the definition of $\ell_\varepsilon$, we have $b_{\ell-1} = \lambda^* C_{\tau - \ell - 1} = \lambda^* C_{j_{\lambda+1}} < \lambda$. Making use of the first three statements of Lemma 81, we have that $z_\ell < \frac{\lambda + b_{\ell-1}}{2} < \lambda$ for all $\ell \leq \ell - 1$ if $\varepsilon$ is sufficiently small. By the last statement of Lemma 81, we have

$$\Pr\{D_\ell = 1\} = \Pr\{\tilde{\lambda}_\ell \leq z_\ell\} \leq \Pr\left\{\tilde{\lambda}_\ell < \frac{\lambda + b_{\ell-1}}{2}\right\} \leq \exp\left(n_\ell \mathcal{M}_P\left(\frac{\lambda + b_{\ell-1}}{2}, \lambda\right)\right)$$

for all $\ell \leq \ell - 1$ provided that $\varepsilon > 0$ is small enough. Since $\frac{\lambda + b_{\ell-1}}{2}$ is independent of $\varepsilon > 0$, we have $\lim_{\varepsilon \to 0} \sum_{\ell=1}^{\ell-1} n_\ell \Pr\{D_\ell = 1\} = 0$ as a result of Lemma 80.

Similarly, it can be seen from the definition of $\ell_\varepsilon$ that $b_{\ell+1} = \lambda^* C_{\tau - \ell} = \lambda^* C_{j_{\lambda+1}} > \lambda$. Making use of the first three statements of Lemma 81, we have that $z_\ell > \frac{\lambda + b_{\ell+1}}{2} > \lambda$ for $\ell + 1 \leq \ell \leq \tau$ if $\varepsilon$ is sufficiently small. By the last statement of Lemma 81, we have

$$\Pr\{D_\ell = 0\} = \Pr\{\tilde{\lambda}_\ell > z_\ell\} \leq \Pr\left\{\tilde{\lambda}_\ell > \frac{\lambda + b_{\ell+1}}{2}\right\} \leq \exp\left(n_\ell \mathcal{M}_P\left(\frac{\lambda + b_{\ell+1}}{2}, \lambda\right)\right)$$

for $\ell + 1 \leq \ell \leq \tau$ provided that $\varepsilon > 0$ is small enough. Therefore, we can apply Lemma 80 to conclude that $\lim_{\varepsilon \to 0} \sum_{\ell=\ell+1}^{\tau} n_\ell \Pr\{D_\ell = 0\} = 0$.

Second, we shall show that $\lim_{\varepsilon \to 0} n_\ell \Pr\{D_\ell = 0\} = 0$ for $\lambda \in (0, \lambda^*)$ such that $C_{j_{\lambda}} > \frac{1}{\lambda^*}$. Clearly, $b_\ell = \lambda^* C_{\tau - \ell} = \lambda^* C_{j_{\lambda}} > \lambda$. Making use of the first three statements of Lemma 81, we have $z_\ell > \frac{\lambda + b_{\ell}}{2} > \lambda$ if $\varepsilon$ is sufficiently small. By the last statement of Lemma 81, we have

$$\Pr\{D_\ell = 0\} = \Pr\{\tilde{\lambda}_\ell > z_\ell\} \leq \Pr\left\{\tilde{\lambda}_\ell > \frac{\lambda + b_{\ell}}{2}\right\} \leq \exp\left(n_\ell \mathcal{M}_P\left(\frac{\lambda + b_{\ell}}{2}, \lambda\right)\right)$$

for small enough $\varepsilon > 0$. It follows that $\lim_{\varepsilon \to 0} n_\ell \Pr\{D_\ell = 0\} = 0$. □

Lemma 83 $\lim_{\varepsilon \to 0} \sum_{\ell=\tau+1}^{\infty} n_\ell \Pr\{l = \ell\} = 0$ for any $\lambda \in (0, \lambda^*)$. 

163
Proof. Recalling that the sample sizes \( n_1, n_2, \cdots \) are chosen as the ascending arrangement of all distinct elements of the set defined by (29), we have that
\[
n_\ell = \left\lfloor \frac{C_{\tau - \ell} \ln(\zeta \delta)}{\mathcal{M}_p(\lambda^*, \lambda^* + \epsilon)} \right\rfloor, \quad \ell = 1, 2, \cdots
\]
for small enough \( \epsilon > 0 \). By the assumption that \( \inf_{i \in \mathbb{Z}} \frac{C_{i+1}}{C_i} = 1 + \rho > 1 \), we have that
\[
n_\ell > (1 + \rho)^{\ell - \tau - 1} \frac{\ln(\zeta \delta)}{\mathcal{M}_p(\lambda^*, \lambda^* + \epsilon)}, \quad \ell = \tau + 1, \tau + 2, \cdots
\]
for small enough \( \epsilon > 0 \). So, there exists a number \( \epsilon^* > 0 \) such that
\[
n_{\ell, \mathcal{M}_p}(\lambda^*, \lambda^* + \epsilon) < (1 + \rho)^{\ell - \tau - 1} \ln(\zeta \delta), \quad \ell = \tau + 1, \tau + 2, \cdots
\]
for any \( \epsilon \in (0, \epsilon^*) \). Observing that there exist a positive integer \( \kappa^* \) such that \( (1 + \rho)^{\ell - \tau - 1} \ln(\zeta \delta) < \ln(\zeta \delta) - (\ell - \tau) \ln 2 = \ln(\zeta \delta) \) for any \( \ell \geq \tau + \kappa^* \), we have that there exists a positive integer \( \kappa^* \) independent of \( \epsilon \) such that \( \mathcal{M}_p(\lambda^*, \lambda^* + \epsilon) < \frac{\ln(\zeta \delta)}{n_\ell} \) for \( \ell \geq \tau + \kappa^* \) and \( 0 < \epsilon < \epsilon^* \). Note that \( \mathcal{M}_p(z, z + \epsilon) \) is monotonically increasing with respect to \( z \in (0, \infty) \) as asserted by Lemma 79. For \( \ell \geq \tau + \kappa^* \) and \( 0 < \epsilon < \epsilon^* \), as a result of \( \frac{\ln(\zeta \delta)}{n_\ell} > \mathcal{M}_p(\lambda^*, \lambda^* + \epsilon) \), there exists a unique number \( z_\ell \in [0, \infty) \) such that \( \mathcal{M}_p(z_\ell, z_\ell + \epsilon) = \frac{\ln(\zeta \delta)}{n_\ell} > \mathcal{M}_p(\lambda^*, \lambda^* + \epsilon) \). Moreover, it must be true that \( z_\ell > \lambda^* \) for \( \ell \geq \tau + \kappa^* \) and \( \epsilon \in (0, \epsilon^*) \). Therefore, for small enough \( \epsilon \in (0, \epsilon^*) \), we have
\[
\sum_{\ell=\tau+1}^{\infty} n_\ell \Pr\{I = \ell\} = \sum_{\ell=\tau + 1}^{\tau + \kappa^*} n_\ell \Pr\{I = \ell\} + \sum_{\ell=\tau + \kappa^* + 1}^{\infty} n_\ell \Pr\{I = \ell\} \\
\leq \sum_{\ell=\tau + 1}^{\tau + \kappa^*} n_\ell \Pr\{D_\tau = 0\} + \sum_{\ell=\tau + \kappa^* + 1}^{\infty} n_\ell \Pr\{D_{\ell-1} = 0\} \\
= \sum_{\ell=\tau + 1}^{\tau + \kappa^*} n_\ell \Pr\{D_\tau = 0\} + \sum_{\ell=\tau + \kappa^*}^{\infty} n_{\ell+1} \Pr\{D_\ell = 0\} \\
\leq k^*(1 + \rho)^{k^*} n_\tau \Pr\{D_\tau = 0\} + (1 + \rho) \sum_{\ell=\tau + \kappa^*}^{\infty} n_\ell \Pr\{D_\ell = 0\} \\
\leq k^*(1 + \rho)^{k^*} n_\tau \Pr\{\hat{\lambda}_\tau > z_\tau\} + (1 + \rho) \sum_{\ell=\tau + \kappa^*}^{\infty} n_\ell \Pr\{\hat{\lambda}_\ell > z_\ell\} \\
\leq k^*(1 + \rho)^{k^*} n_\tau \Pr\left\{\hat{\lambda}_\tau > \frac{\lambda^* + \lambda}{2}\right\} + (1 + \rho) \sum_{\ell=\tau + \kappa^*}^{\infty} n_\ell \Pr\{\hat{\lambda}_\ell > \lambda^*\} \\
\leq k^*(1 + \rho)^{k^*} n_\tau \exp\left(n_\tau \mathcal{M}_p\left(\frac{\lambda + \lambda^*}{2}, \lambda\right)\right) \\
+ (1 + \rho) \sum_{\ell=\tau + \kappa^*}^{\infty} n_\ell \exp(n_\ell \mathcal{M}_p(\lambda^*, \lambda)) \\n\to 0
\]
as \( \epsilon \to 0 \), where we have used the assumption that \( \sup_{i \in \mathbb{Z}} \frac{C_{i+1}}{C_i} = 1 + \rho < \infty \). This completes the proof of the lemma. \( \square \)
J.3 Proof of Theorem 47

To show statement (I) of Theorem 47, we can use an argument similar to the proof of statement (I) of Theorem 23.

To show statement (II) of Theorem 47, we can use an argument similar to the proof of statement (II) of Theorem 23.

To show statement (III) of Theorem 47, we can use an argument similar to the proof of statement (III) of Theorem 23.

To show statement (IV) of Theorem 47, we can use an argument similar to the proof of statement (IV) of Theorem 23 and make use of the observation that

\[
\Pr \left\{ \left| \hat{\lambda} - \lambda \right| \geq \varepsilon \lambda \mid \lambda \right\} = \sum_{l=1}^{\ell^*} \Pr \left\{ \left| \hat{\lambda}_l - \lambda \right| \geq \varepsilon \lambda, \; l = \ell \mid \lambda \right\} + \sum_{l=\ell^*+1}^{\infty} \Pr \left\{ \left| \hat{\lambda}_l - \lambda \right| \geq \varepsilon \lambda, \; l = \ell \mid \lambda \right\}
\]

\[
\leq \sum_{l=1}^{\ell^*} \Pr \left\{ \left| \hat{\lambda}_l - \lambda \right| \geq \varepsilon \lambda, \; l = \ell \right\} + \eta
\]

\[
\leq \sum_{l=1}^{\ell^*} \Pr \{ l = \ell \mid \lambda \} + \eta \leq \sum_{l=1}^{\ell^*} \Pr \left\{ \hat{\lambda}_l \geq z \mid \lambda \right\} + \eta \leq \sum_{l=1}^{\ell^*} \exp(n_\ell \mathcal{M}_P(z_\ell, \lambda)) + \eta.
\]

To show statement (V) of Theorem 47, we can use an argument similar to the proof of statement (V) of Theorem 23 and make use of the observation that

\[
\Pr \left\{ \left| \hat{\lambda} - \lambda \right| \geq \varepsilon \lambda \mid \lambda \right\} \leq \Pr \left\{ \left| \hat{\lambda} - \lambda \right| \geq \varepsilon \lambda, \; l = 1 \mid \lambda \right\} + \Pr \left\{ \left| \hat{\lambda} - \lambda \right| \geq \varepsilon \lambda, \; l > 1 \mid \lambda \right\}
\]

\[
\leq \Pr \left\{ \left| \hat{\lambda}_1 - \lambda \right| \geq \varepsilon \lambda \mid \lambda \right\} + \Pr \{ l > 1 \mid \lambda \}
\]

\[
\leq \Pr \left\{ \left| \hat{\lambda}_1 - \lambda \right| \geq \varepsilon \lambda \mid \lambda \right\} + \Pr \left\{ \hat{\lambda}_1 < z \mid \lambda \right\}
\]

\[
\leq 2 \exp(n_1 \mathcal{M}_P((1 + \varepsilon)\lambda, \lambda)) + \exp(n_1 \mathcal{M}_P(z_1, \lambda)).
\]

J.4 Proof of Theorem 48

Theorem 48 can be established by using a method similar to that of Theorem 27 based on the following preliminary results.

**Lemma 84** \( \lim_{\varepsilon \to 0} \sum_{\ell=1}^{\tau} n_\ell e^{-n_\ell c} \) for any \( c > 0 \).

**Proof.** Lemma 84 can be shown by a similar method as that of Lemma 14. \( \square \)

**Lemma 85** If \( \varepsilon \) is sufficiently small, then the following statements hold true.

(I): For \( \ell = 1, \cdots, \tau \), there exists a unique number \( z_\ell \in [0, \infty) \) such that \( n_\ell = \frac{\ln(z_\ell)}{\mathcal{M}_P(z_\ell, \varepsilon \lambda)} \).

(II): \( z_\ell \) is monotonically decreasing with respect to \( \ell \) no greater than \( \tau \).

(III): \( \lim_{\varepsilon \to 0} z_\ell = \frac{\lambda}{\tau + \ell} \) for \( 1 \leq \ell \leq \tau \), where the limit is taken under the restriction that \( \ell - \tau \) is fixed with respect to \( \varepsilon \).

(IV): \( \{ D_\ell = 0 \} = \{ \hat{\lambda}_\ell < z_\ell \} \) for \( \ell = 1, \cdots, \tau \).
Proof. Lemma 85 can be shown by a similar method as that of Lemma 88.

Lemma 86 Define $\ell_\varepsilon = \tau - j_\lambda$, where $j_\lambda$ is the largest integer $j$ such that $C_j \geq \frac{\lambda}{\varepsilon}$. Then,

$$\lim_{\varepsilon \to 0} \ell_\varepsilon^{-1} \sum_{\ell=1}^{\ell_\varepsilon} n_\ell \Pr\{D_\ell = 1\} = 0, \quad \lim_{\varepsilon \to 0} \tau \sum_{\ell = \ell_\varepsilon + 1}^\tau n_\ell \Pr\{D_\ell = 0\} = 0$$

(145)

for $\lambda \in (\lambda', \lambda'')$. Moreover, $\lim_{\varepsilon \to 0} n_\ell \Pr\{D_\ell = 0\} = 0$ for $\lambda \in (\lambda', \lambda'')$ such that $C_{j_\lambda} > \frac{\lambda}{\varepsilon}$.

Proof. For simplicity of notations, let $b_\ell = \lim_{\varepsilon \to 0} z_\ell$ for $1 \leq \ell \leq \tau$.

First, we shall show that (145) holds for $\lambda \in (\lambda', \lambda'')$. By the definition of $\ell_\varepsilon$, we have $b_{\ell_\varepsilon - 1} = \frac{\lambda'}{C_{\tau - \ell_\varepsilon - 1}} = \frac{\lambda'}{C_{j_\lambda - 1}} > \lambda$. Making use of the first three statements of Lemma 85, we have $z_\ell > \frac{\lambda + b_{\ell_\varepsilon - 1}}{\varepsilon} > \lambda$ for $\ell \leq \ell_\varepsilon - 1$ if $\varepsilon$ is sufficiently small. By the last statement of Lemma 85, we have

$$\Pr\{D_\ell = 1\} = \Pr\{\hat{\lambda}_\ell \geq z_\ell\} \leq \Pr\left\{\hat{\lambda}_\ell > \frac{\lambda + b_{\ell_\varepsilon - 1}}{2}\right\} \leq \exp\left(n_{\ell, \hat{\lambda}_\ell} \left(\frac{\lambda + b_{\ell_\varepsilon - 1}}{2}, \lambda\right)\right)$$

for all $\ell \leq \ell_\varepsilon - 1$ provided that $\varepsilon > 0$ is small enough. Since $\frac{\lambda + b_{\ell_\varepsilon - 1}}{2}$ is independent of $\varepsilon > 0$, we have $\lim_{\varepsilon \to 0} \sum_{\ell=1}^{\ell_\varepsilon - 1} n_\ell \Pr\{D_\ell = 1\} = 0$ as a result of Lemma 84.

Similarly, it can be seen from the definition of $\ell_\varepsilon$ that $b_{\ell_\varepsilon + 1} = \frac{\lambda'}{C_{\tau - \ell_\varepsilon + 1}} = \frac{\lambda'}{C_{j_\lambda + 1}} < \lambda$. Making use of the first three statements of Lemma 85, we have $z_\ell < \frac{\lambda + b_{\ell_\varepsilon + 1}}{\varepsilon} < \lambda$ for $\ell_\varepsilon + 1 \leq \ell \leq \tau$ if $\varepsilon$ is sufficiently small. By the last statement of Lemma 85, we have

$$\Pr\{D_\ell = 0\} = \Pr\{\hat{\lambda}_\ell < z_\ell\} \leq \Pr\left\{\hat{\lambda}_\ell < \frac{\lambda + b_{\ell_\varepsilon + 1}}{2}\right\} \leq \exp\left(n_{\ell, \hat{\lambda}_\ell} \left(\frac{\lambda + b_{\ell_\varepsilon + 1}}{2}, \lambda\right)\right)$$

for $\ell_\varepsilon + 1 \leq \ell \leq \tau$ provided that $\varepsilon > 0$ is small enough. Therefore, we can apply Lemma 84 to conclude that $\lim_{\varepsilon \to 0} \sum_{\ell = \ell_\varepsilon + 1}^\tau n_\ell \Pr\{D_\ell = 0\} = 0$.

Second, we shall show that $\lim_{\varepsilon \to 0} n_\ell \Pr\{D_\ell = 0\} = 0$ for $\lambda \in (\lambda', \lambda'')$ such that $C_{j_\lambda} > \frac{\lambda}{\varepsilon}$. Clearly, $b_\ell = \frac{\lambda'}{C_{\tau - \ell_\varepsilon}} = \frac{\lambda'}{C_{j_\lambda}} < \lambda$. Making use of the first three statements of Lemma 85, we have $z_\ell < \frac{\lambda + b_\ell}{\varepsilon} < \lambda$ if $\varepsilon$ is sufficiently small. By the last statement of Lemma 85, we have

$$\Pr\{D_\ell = 0\} = \Pr\{\hat{\lambda}_\ell < z_\ell\} \leq \Pr\left\{\hat{\lambda}_\ell < \frac{\lambda + b_\ell}{2}\right\} \leq \exp\left(n_{\ell, \hat{\lambda}_\ell} \left(\frac{\lambda + b_\ell}{2}, \lambda\right)\right)$$

for small enough $\varepsilon > 0$. It follows that $\lim_{\varepsilon \to 0} n_\ell \Pr\{D_\ell = 0\} = 0$.

Lemma 87 $\lim_{\varepsilon \to 0} \sum_{\ell = \tau + 1}^{\infty} n_\ell \Pr\{l = \ell\} = 0$ for any $\lambda \in (\lambda', \lambda'')$. 

166
Proof. Recalling that the sample sizes \( n_1, n_2, \ldots \) are chosen as the ascending arrangement of all distinct elements of the set defined by (30), we have that
\[
n_{\ell} = \begin{bmatrix} \mathcal{C}_{\tau-\ell} \ln(\delta) \\ \mathcal{M}_p \left( \lambda', \frac{\tau}{1+\tau} \right) \end{bmatrix}, \quad \ell = 1, 2, \ldots
\]
for small enough \( \varepsilon \in (0, 1) \). By the assumption that \( \inf_{\ell \in \mathbb{Z}} \frac{C_{\ell-1}}{C_1} = 1 + p > 1 \), we have that
\[
n_{\ell} > (1 + p)^{\ell-1} \frac{\ln(\delta)}{\mathcal{M}_p \left( \lambda', \frac{\tau}{1+\tau} \right)}, \quad \ell = \tau + 1, \tau + 2, \ldots
\]
for small enough \( \varepsilon \in (0, 1) \). So, there exists a number \( \varepsilon^* \in (0, 1) \) such that
\[
n_{\ell, \mathcal{P}} \left( \lambda', \frac{\lambda}{1+\varepsilon} \right) < (1 + p)^{\ell-\tau-1} \ln(\delta), \quad \ell = \tau + 1, \tau + 2, \ldots
\]
for any \( \varepsilon \in (0, \varepsilon^*) \). Observing that there exist a positive integer \( \kappa^* \) such that \((1 + p)^{\ell-\tau-1} \ln(\delta) < \ln(\delta) - (\ell - \tau) \ln 2 = \ln(\delta_{\ell})\) for any \( \ell \geq \tau + \kappa^* \), we have that there exists a positive integer \( \kappa^* \) independent of \( \varepsilon \) such that \( \mathcal{M}_p \left( \lambda', \frac{\lambda}{1+\varepsilon} \right) < \frac{\ln(\delta_{\ell})}{n_\ell} \) for \( \ell \geq \tau + \kappa^* \) and \( 0 < \varepsilon < \varepsilon^* \). Note that \( \mathcal{M}_p(z, \frac{\lambda}{1+\varepsilon}) = z\left[ \frac{\lambda}{1+\varepsilon} \ln(1+\varepsilon) \right] \) is monotonically decreasing with respect to \( z \in (0, \infty) \). For \( \ell \geq \tau + \kappa^* \) and \( 0 < \varepsilon < \varepsilon^* \), as a result of \( \frac{\ln(\delta_{\ell})}{n_\ell} > \mathcal{M}_p(\lambda', \frac{\lambda}{1+\varepsilon}) \), there exists a unique number \( z_{\ell} \in [0, \infty) \) such that \( \mathcal{M}_p(z_{\ell}, \frac{\lambda}{1+\varepsilon}) = \frac{\ln(\delta_{\ell})}{n_\ell} > \mathcal{M}_p(\lambda', \frac{\lambda}{1+\varepsilon}) \). Moreover, it must be true that \( z_{\ell} < \lambda' \) for \( \ell \geq \tau + \kappa^* \) and \( \varepsilon \in (0, \varepsilon^*) \). Therefore, for small enough \( \varepsilon \in (0, \varepsilon^*) \), we have
\[
\sum_{\ell=\tau+1}^{\infty} n_{\ell} \Pr\{l = \ell\} = \sum_{\ell=\tau+1}^{\tau+\kappa^*} n_{\ell} \Pr\{D_{\tau} = 0\} + \sum_{\ell=\tau+\kappa^*+1}^{\infty} n_{\ell} \Pr\{D_{\ell-1} = 0\}
\leq \sum_{\ell=\tau+1}^{\tau+\kappa^*} n_{\ell} \Pr\{D_{\tau} = 0\} + \sum_{\ell=\tau+\kappa^*+1}^{\infty} n_{\ell} \Pr\{D_{\ell} = 0\}
= \sum_{\ell=\tau+1}^{\tau+\kappa^*} n_{\ell} \Pr\{D_{\tau} = 0\} + \sum_{\ell=\tau+\kappa^*}^{\infty} n_{\ell} \Pr\{D_{\ell} = 0\}
\leq k^*(1 + p)^{\tau} n_{\tau} \Pr\{D_{\tau} = 0\} + (1 + p) \sum_{\ell=\tau+\kappa^*}^{\infty} n_{\ell} \Pr\{D_{\ell} = 0\}
\leq k^*(1 + p)^{\tau} n_{\tau} \Pr\{\lambda_{\tau} < z_{\tau}\} + (1 + p) \sum_{\ell=\tau+\kappa^*}^{\infty} n_{\ell} \Pr\{\lambda_{\ell} < z_{\ell}\}
\leq k^*(1 + p)^{\tau} n_{\tau} \Pr\left\{ \lambda_{\tau} < \frac{\lambda'}{2} \right\} + (1 + p) \sum_{\ell=\tau+\kappa^*}^{\infty} n_{\ell} \Pr\{\lambda_{\ell} < \lambda'\}
\leq k^*(1 + p)^{\tau} n_{\tau} \exp \left( n_{\tau, \mathcal{M}_p} \left( \frac{\lambda}{2}, \lambda' \right) \right)
+ (1 + p) \sum_{\ell=\tau+\kappa^*}^{\infty} n_{\ell} \exp(n_{\ell, \mathcal{M}_p}(\lambda', \lambda)) \to 0
\]
as \( \varepsilon \to 0 \), where we have used the assumption that \( \sup_{\ell \in \mathbb{Z}} \frac{C_{\ell-1}}{C_1} = 1 + p < \infty \). This completes the proof of the lemma. \( \square \)
Lemma 91

Let \( \varepsilon \) be any \( \varepsilon > 0 \). Noting that \( \frac{\partial}{\partial z} M(x) \) where the last inequality follows from \( \ln(1 + z) \).

Lemma 88

\( S_P(0, k, n\lambda) \leq \exp(n M_P(\frac{k}{n}, \lambda)) \) for \( 0 \leq k \leq n\lambda \). Similarly, \( S_P(k, \infty, n\lambda) \leq \exp(n M_P(\frac{k}{n}, \lambda)) \) for \( k \geq n\lambda \).

Lemma 89

\( M_P(\lambda - \varepsilon, \lambda) < M_P(\lambda + \varepsilon, \lambda) < 0 \) for any \( \varepsilon \in (0, \lambda] \).

Proof. In the case of \( \varepsilon = \lambda > 0 \), we have \( M_P(\lambda + \varepsilon, \lambda) = \varepsilon - 2\varepsilon \ln 2 > -\varepsilon = M_P(\lambda - \varepsilon, \lambda) \). In the case of \( 0 < \varepsilon < \lambda \), the lemma follows from the facts that \( M_P(\lambda + \varepsilon, \lambda) = M_P(\lambda - \varepsilon, \lambda) \) for \( \varepsilon = 0 \) and \( \frac{d}{d\varepsilon}[M_P(\lambda + \varepsilon, \lambda) - M_P(\lambda - \varepsilon, \lambda)] = \ln \frac{\lambda^2}{\lambda^2 - \varepsilon^2} > 0 \) for any \( \varepsilon \in (0, \lambda) \). To show \( M_P(\lambda + \varepsilon, \lambda) < 0 \) for any \( \varepsilon \in (0, \lambda] \), note that \( M_P(\lambda + \varepsilon, \lambda) = \varepsilon + (\lambda + \varepsilon) \ln \frac{\lambda^2}{\lambda^2 - \varepsilon^2} < \varepsilon + (\lambda + \varepsilon) \times \frac{\lambda^2}{\lambda^2 - \varepsilon^2} = 0 \). This completes the proof of the lemma.

\[ \square \]

Lemma 90

Let \( \varepsilon > 0 \). Then, \( M_P(z, z - \varepsilon) \) is monotonically increasing with respect to \( z > \varepsilon \).

Proof. Note that \( M_P(z, z - \varepsilon) = \varepsilon + z \ln \left( \frac{z - \varepsilon}{z} \right) \) and

\[ \frac{\partial}{\partial z} M_P(z, z - \varepsilon) = \ln \left( \frac{z - \varepsilon}{z} \right) + \frac{\varepsilon}{z - \varepsilon} = -\ln \left( 1 + \frac{\varepsilon}{z - \varepsilon} \right) + \frac{\varepsilon}{z - \varepsilon} > 0 \]

where the last inequality follows from \( \ln(1 + x) \leq x, \forall x \in [0, 1) \).

\[ \square \]

Lemma 91

Let \( 0 < \varepsilon < 1 \). Then, \( M_P(z, \frac{1}{1+\varepsilon}) < M_P(z, \frac{1}{1-\varepsilon}) \) and \( \frac{d}{d\varepsilon} M_P(z, \frac{1}{1+\varepsilon}) < \frac{d}{d\varepsilon} M_P(z, \frac{1}{1-\varepsilon}) < 0 \) for \( z > 0 \).

Proof. Note that \( M_P(z, \frac{1}{1+\varepsilon}) - M_P(z, \frac{1}{1-\varepsilon}) = \varepsilon g(\varepsilon) \) where \( g(\varepsilon) = \frac{\varepsilon}{1+\varepsilon} + \frac{\varepsilon}{1-\varepsilon} + \ln \frac{1-\varepsilon}{1+\varepsilon} \). Since \( g(0) = 0 \) and \( \frac{d g(\varepsilon)}{d\varepsilon} = \frac{4\varepsilon^2}{(1-\varepsilon)^2} > 0 \), we have \( g(\varepsilon) > 0 \) for \( 0 < \varepsilon < 1 \). It follows that \( M_P(z, \frac{1}{1+\varepsilon}) < M_P(z, \frac{1}{1-\varepsilon}) \).

Using the inequality \( \ln(1 - x) < -x, \forall x \in (0, 1) \), we have \( \frac{d}{d\varepsilon} M_P(z, \frac{1}{1+\varepsilon}) = \frac{4\varepsilon}{(1-\varepsilon)^2} + \ln(1 - \frac{1}{1+\varepsilon}) < 0 \). Noting that \( \frac{d}{d\varepsilon}[M_P(z, \frac{1}{1+\varepsilon}) - M_P(z, \frac{1}{1-\varepsilon})] = g(\varepsilon) > 0 \), we have \( \frac{d}{d\varepsilon} M_P(z, \frac{1}{1+\varepsilon}) < \frac{d}{d\varepsilon} M_P(z, \frac{1}{1-\varepsilon}) < 0 \).

\[ \square \]

Lemma 92

\[ \Pr \{ M_P(\hat{\lambda}_s, L(\hat{\lambda}_s)) \leq \frac{\ln(\varepsilon\delta)}{n_{\varepsilon}}, \ M_P(\hat{\lambda}_s, U(\hat{\lambda}_s)) \leq \frac{\ln(\varepsilon\delta)}{n_{\varepsilon}} \} = 1. \]
Proof. For simplicity of notations, we denote $\lambda^* = \frac{2a}{\epsilon r}$. By the definitions of $\mathcal{L}(\hat{\lambda}_s)$ and $\mathcal{U}(\hat{\lambda}_s)$, we have that, in order to show the lemma, it suffices to show

\[
\begin{align*}
\mathcal{M}_P\left(\lambda_s, \frac{\hat{\lambda}_s}{1 - \epsilon_r}\right) &> \frac{\ln(\epsilon)}{n_s}, \hat{\lambda}_s > \lambda^* - \epsilon a = \emptyset, \\
\mathcal{M}_P(\hat{\lambda}_s, \hat{\lambda}_s + \epsilon a) &> \frac{\ln(\epsilon)}{n_s}, \hat{\lambda}_s \leq \lambda^* - \epsilon a = \emptyset, \\
\mathcal{M}_P\left(\lambda_s, \frac{\hat{\lambda}_s}{1 + \epsilon_r}\right) &> \frac{\ln(\epsilon)}{n_s}, \hat{\lambda}_s > \lambda^* + \epsilon a = \emptyset, \\
\mathcal{M}_P(\hat{\lambda}_s, \hat{\lambda}_s - \epsilon a) &> \frac{\ln(\epsilon)}{n_s}, \hat{\lambda}_s \leq \lambda^* + \epsilon a = \emptyset.
\end{align*}
\]  

(146)

By the definition of $n_s$, we have $n_s \geq \left\lceil \frac{\ln(\epsilon)}{\mathcal{M}_P(\lambda^* + \epsilon a, \lambda^*)}\right\rceil \geq \frac{\ln(\epsilon)}{\mathcal{M}_P(\lambda^* + \epsilon a, \lambda^*)}$. By the assumption on $\epsilon_a$ and $\epsilon_r$, we have $0 < \epsilon_a < \lambda^*$. Hence, by Lemma 89, we have $\mathcal{M}_P(\lambda^* - \epsilon a, \lambda^*) < \mathcal{M}_P(\lambda^* + \epsilon a, \lambda^*) < 0$ and it follows that

\[
\frac{\ln(\epsilon)}{n_s} = \mathcal{M}_P(\lambda^* + \epsilon a, \lambda^*) > \mathcal{M}_P(\lambda^* - \epsilon a, \lambda^*).
\]  

(150)

By (150),

\[
\left\{\mathcal{M}_P\left(\lambda_s, \frac{\hat{\lambda}_s}{1 - \epsilon_r}\right) > \frac{\ln(\epsilon)}{n_s}, \hat{\lambda}_s > \lambda^* - \epsilon a\right\} \subseteq \left\{\mathcal{M}_P\left(\hat{\lambda}_s, \frac{\hat{\lambda}_s}{1 - \epsilon_r}\right) > \mathcal{M}_P(\lambda^* - \epsilon a, \lambda^*), \hat{\lambda}_s > \lambda^* - \epsilon a\right\}.
\]  

(151)

Noting that $\mathcal{M}_P(\lambda^* - \epsilon a, \lambda^*) = \mathcal{M}_P(\lambda^* - \epsilon a, \frac{\lambda^* - \epsilon a}{1 - \epsilon_r})$ and making use of the fact that $\mathcal{M}_P(z, \frac{z}{1 - \epsilon})$ is monotonically decreasing with respect to $z \in (0, \infty)$ as asserted by Lemma 91, we have

\[
\left\{\mathcal{M}_P\left(\hat{\lambda}_s, \frac{\hat{\lambda}_s}{1 - \epsilon_r}\right) > \mathcal{M}_P(\lambda^* - \epsilon a, \lambda^*)\right\} = \{\hat{\lambda}_s < \lambda^* - \epsilon a\}.
\]  

(152)

Combining (151) and (152) yields (146). By (150),

\[
\left\{\mathcal{M}_P(\hat{\lambda}_s, \hat{\lambda}_s + \epsilon a) > \frac{\ln(\epsilon)}{n_s}, \hat{\lambda}_s \leq \lambda^* - \epsilon a\right\} \subseteq \left\{\mathcal{M}_P(\hat{\lambda}_s, \hat{\lambda}_s + \epsilon a) > \mathcal{M}_P(\lambda^* - \epsilon a, \lambda^*), \hat{\lambda}_s \leq \lambda^* - \epsilon a\right\}.
\]  

(153)

By the assumption on $\epsilon_a$ and $\epsilon_r$, we have $\lambda^* - \epsilon a > 0$. Recalling the fact that $\mathcal{M}_P(z, z + \epsilon)$ is monotonically increasing with respect to $z \in (0, \infty)$ as asserted by Lemma 79, we have that the event in the right-hand side of (153) is an impossible event and consequently, (147) is established. By (150),

\[
\left\{\mathcal{M}_P\left(\lambda_s, \frac{\hat{\lambda}_s}{1 + \epsilon_r}\right) > \frac{\ln(\epsilon)}{n_s}, \hat{\lambda}_s > \lambda^* + \epsilon a\right\} = \left\{\mathcal{M}_P\left(\hat{\lambda}_s, \frac{\hat{\lambda}_s}{1 + \epsilon_r}\right) > \mathcal{M}_P(\lambda^* + \epsilon a, \lambda^*), \hat{\lambda}_s > \lambda^* + \epsilon a\right\}.
\]  

(154)

Noting that $\mathcal{M}_P(\lambda^* + \epsilon a, \lambda^*) = \mathcal{M}_P(\lambda^* + \epsilon a, \frac{\lambda^* + \epsilon a}{1 + \epsilon_r})$ and making use of the fact that $\mathcal{M}_P(z, \frac{z}{1 + \epsilon})$ is monotonically decreasing with respect to $z \in (0, \infty)$ as asserted by Lemma 91, we have

\[
\left\{\mathcal{M}_P\left(\hat{\lambda}_s, \frac{\hat{\lambda}_s}{1 + \epsilon_r}\right) > \mathcal{M}_P(\lambda^* + \epsilon a, \lambda^*)\right\} = \{\hat{\lambda}_s < \lambda^* + \epsilon a\}.
\]  

(155)
Combining (154) and (155) yields (148). By (150),
\[
\{M_p(\hat{\lambda}_s,\hat{\lambda}_s - \varepsilon_a) > \frac{\ln(\zeta \delta)}{n_s}, \hat{\lambda}_s \leq \lambda^* + \varepsilon_a\} \subseteq \{M_p(\hat{\lambda}_s,\hat{\lambda}_s - \varepsilon_a) > M_p(\lambda^* + \varepsilon_a, \lambda^*), \hat{\lambda}_s \leq \lambda^* + \varepsilon_a\}.
\] (156)

Recalling the fact that $M_p(z, z - \varepsilon)$ is monotonically increasing with respect to $z \in (\varepsilon, \infty)$ as stated by Lemma 90, we have that the event in the right-hand side of (156) is an impossible event and consequently, (149) is established. This completes the proof of the lemma.

\[\square\]

**Lemma 93** $\Pr\left\{\left|\frac{\hat{\lambda}}{\lambda}\right| \geq \varepsilon_r | \lambda\right\} < \delta$ for $\lambda \in [\lambda, \infty)$.

**Proof.** Note that
\[
\Pr\left\{\left|\frac{\hat{\lambda} - \lambda}{\lambda}\right| \geq \varepsilon_r | \lambda\right\} = \sum_{\ell=1}^{s} \Pr\left\{\left|\frac{\hat{\lambda}_\ell - \lambda}{\lambda}\right| \geq \varepsilon_r, \ell = \ell | \lambda\right\} \leq \sum_{\ell=1}^{s} \Pr\left\{\left|\frac{\hat{\lambda}_\ell - \lambda}{\lambda}\right| \geq \varepsilon_r | \lambda\right\}
\]
\[
\leq \sum_{\ell=1}^{s} \exp(n_{\ell}\cdot M_p(\lambda + \lambda \varepsilon_r, \lambda)) + \exp(n_{\ell}\cdot M_p(\lambda - \lambda \varepsilon_r, \lambda))
\]
\[
< 2 \sum_{\ell=1}^{s} \exp(n_{\ell}\cdot M_p(\lambda(1 + \varepsilon_r), \lambda))
\]
where (157) follows from Lemma 31. Since $\lim_{\lambda \to 0} M_p(\lambda(1 + \varepsilon_r), \lambda) = 0$ and $\lim_{\lambda \to \infty} M_p(\lambda(1 + \varepsilon_r), \lambda) = -\infty$, there exists a unique number $\lambda > 0$ such that $\sum_{\ell=1}^{s} \exp(n_{\ell}\cdot M_p(\lambda(1 + \varepsilon_r), \lambda)) = \frac{\delta}{2}$. Finally, the lemma is established by noting that $M_p(\lambda(1 + \varepsilon_r), \lambda)$ is monotonically decreasing with respect to $\lambda > 0$.

\[\square\]

Now we are in a position to prove Theorem 49. The second statement of Theorem 49 is a result of Lemma 93.

If the multistage sampling scheme follows a stopping rule derived from Chernoff bounds, then \(\{D_s = 1\}\) is a sure event as a result of Lemma 92. Note that $M_p(z, \lambda) = \inf_{t > 0} e^{-t z} \mathbb{E}[e^{t \hat{\lambda}}]$ and that $\hat{\lambda}_\ell$ is a ULE of $p$ for $\ell = 1, \ldots, s$. So, the sampling scheme satisfies all the requirements described in Corollary 11 from which Theorem 49 immediately follows.

If the multistage sampling scheme follows a stopping rule derived from CDFs, then, by Lemmas 88 and 92 we have
\[
\Pr\{G_{\hat{\lambda}}(\hat{\lambda}_s, \mathcal{L}(\hat{\lambda}_s)) \leq \zeta \delta_s\} = \Pr\{1 - S_p(K_s - 1, n_s, \mathcal{L}(\hat{\lambda}_s)) \leq \zeta \delta\} \geq \Pr\{n_s \cdot M_p(\hat{\lambda}_s, \mathcal{L}(\hat{\lambda}_s)) \leq \ln(\zeta \delta)\} = 1,
\]
\[
\Pr\{F_{\hat{\lambda}}(\hat{\lambda}_s, \mathcal{U}(\hat{\lambda}_s)) \leq \zeta \delta_s\} = \Pr\{S_p(K_s, n_s, \mathcal{U}(\hat{\lambda}_s)) \leq \zeta \delta\} \geq \Pr\{n_s \cdot M_p(\hat{\lambda}_s, \mathcal{U}(\hat{\lambda}_s)) \leq \ln(\zeta \delta)\} = 1
\]
and thus $\Pr\{F_{\hat{\lambda}}(\hat{\lambda}_s, \mathcal{U}(\hat{\lambda}_s)) \leq \zeta \delta_s, G_{\hat{\lambda}_s}(\hat{\lambda}_s, \mathcal{L}(\hat{\lambda}_s)) \leq \zeta \delta_s\} = 1$, which implies that $\{D_s = 1\}$ is a sure event. So, the sampling scheme satisfies all the requirements described in Theorem 49 from which Theorem 49 immediately follows.

170
J.6 Proof of Theorem 51

We need some preliminary results.

Lemma 94 \( \lim_{a \to 0} \sum_{\ell=1}^{\rho} n_{\ell} e^{-n_{\ell}c} = 0 \) for any \( c > 0 \).

Proof. For simplicity of notations, define \( \lambda^* = \frac{\ell}{\varepsilon a} \), as before. By differentiation, it can be shown that \( xe^{-x}c \) is monotonically increasing with respect to \( x \in (0, \frac{1}{\varepsilon}) \) and monotonically decreasing with respect to \( x \in (\frac{1}{\varepsilon}, \infty) \). Since the smallest sample size \( n_1 \geq \frac{\ln \varepsilon}{\varepsilon a} \) is greater than \( \frac{1}{\varepsilon} \) for small enough \( \varepsilon > 0 \), we have that \( \sum_{\ell=1}^{\rho} n_{\ell} e^{-n_{\ell}c} \leq s \) if \( \varepsilon a > 0 \) is sufficiently small. Let \( \rho = \inf_{\ell > 0} \frac{C_{\varepsilon a}}{C_{\ell a}} - 1 \). Observing that

\[
\frac{s}{\varepsilon a} + \frac{1}{\varepsilon a} < 1 + \frac{\ln \left( \frac{-\varepsilon a}{\varepsilon a} \right)}{\ln (1 + \rho)} \leq 1 + \frac{\ln \left( \frac{-\varepsilon a}{\varepsilon a} \right)}{\ln (1 + \rho)}
\]

and \( n_1 \geq \frac{\ln a}{\varepsilon a} \), we have

\[
\sum_{\ell=1}^{\rho} n_{\ell} e^{-n_{\ell}c} < \left[ 1 + \frac{\ln \left( \frac{-\varepsilon a}{\varepsilon a} \right)}{\ln (1 + \rho)} \right] \frac{\ln a}{\varepsilon a} \exp \left( - \frac{c \ln a}{\varepsilon a} \right) = \frac{A(\varepsilon a)}{\varepsilon a} + \frac{1}{\varepsilon a} B(\varepsilon a)
\]

for small enough \( \varepsilon a > 0 \), where \( A(\varepsilon a) = \frac{c \ln a}{\varepsilon a} \exp \left( - \frac{c \ln a}{\varepsilon a} \right) \) and \( B(\varepsilon a) = \frac{\ln \left( \frac{-\varepsilon a}{\varepsilon a} \right)}{\varepsilon a} \exp \left( - \frac{c \ln a}{\varepsilon a} \right) \).

Noting that \( \lim_{x \to \infty} xe^{-x} = 0 \) and that \( \frac{c \ln a}{\varepsilon a} \to \infty \) as \( \varepsilon a \to 0 \), we have \( \lim_{\varepsilon a \to 0} A(\varepsilon a) = 0 \). Now we show that \( \lim_{\varepsilon a \to 0} B(\varepsilon a) = 0 \). Using Taylor’s expansion formula \( \ln (1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} + o(x^3) \), we have

\[
\mathcal{M}_P(\lambda^* + \varepsilon a, \lambda^*) = \frac{\varepsilon a^2}{2(\lambda^* + \varepsilon a)} - \frac{\varepsilon a^3}{3(\lambda^* + \varepsilon a)^2} + o(\varepsilon a^3) = -\frac{\varepsilon a^2}{2\lambda^*} + \varpi \varepsilon a + o(\varepsilon a),
\]

where \( \varpi = \frac{1}{2\lambda^*} \). Hence,

\[
\ln \left( \frac{-\varepsilon a}{\mathcal{M}_P(\lambda^* + \varepsilon a, \lambda^*)} \right) = \ln \left( \frac{-\varepsilon a}{\frac{\varepsilon a^2}{2(\lambda^* + \varepsilon a)} - \frac{\varepsilon a^3}{3(\lambda^* + \varepsilon a)^2} + o(\varepsilon a^3)} \right) = \ln(2\lambda^*) + \ln \frac{1}{\varepsilon a} + \ln \frac{1}{1 - 2\lambda^* \varpi \varepsilon a + o(\varepsilon a)}
\]

\[
= \ln(2\lambda^*) + \ln \frac{1}{\varepsilon a} + 2\lambda^* \varpi \varepsilon a + o(\varepsilon a)
\]

and

\[
\ln \left( \frac{-\varepsilon a}{\mathcal{M}_P(\lambda^* + \varepsilon a, \lambda^*)} \right) = \ln(2\lambda^*) + \ln \frac{1}{\varepsilon a} + 2\lambda^* \varpi \varepsilon a + o(1) \tag{158}
\]

Using (158) and the observation that

\[
[2\lambda^* \varpi + o(1)] \exp \left( - \frac{c \ln \frac{1}{\varepsilon a}}{\varepsilon a} \right) = o(1), \quad \frac{\ln(2\lambda^*)}{\varepsilon a} \exp \left( - \frac{c \ln \frac{1}{\varepsilon a}}{\varepsilon a} \right) = \frac{\ln(2\lambda^*)}{\varepsilon a} \exp \left( - \frac{c \ln \frac{1}{\varepsilon a}}{\varepsilon a} \right) = o(1),
\]

171
we have $B(\varepsilon_a) = o(1) + \frac{\ln \frac{1}{\varepsilon_a}}{\varepsilon_a} \exp \left( - \frac{\varepsilon \ln \frac{1}{\varepsilon_a}}{\varepsilon_a} \right)$. Making a change of variable $x = \frac{1}{\varepsilon_a}$ and using L'Hôpital's rule, we have

$$\lim_{\varepsilon_a \to 0} B(\varepsilon_a) = \lim_{x \to \infty} \frac{x \ln x}{1 - \varepsilon_a} = \lim_{x \to \infty} \frac{1 + \ln x}{1 - \varepsilon_a} = \lim_{x \to \infty} \frac{1}{x (1 - \varepsilon_a)} = \frac{1}{x (1 - \varepsilon_a)} = 0.$$

Therefore, $0 \leq \limsup_{\varepsilon_a \to 0} \sum_{\ell=1}^{\varepsilon_a} n_{\ell} e^{-n_{\ell}c} \leq \frac{1}{\varepsilon_a} \lim_{\varepsilon_a \to 0} A(\varepsilon_a) + \frac{\ln \frac{1}{\varepsilon_a}}{n(1 + p)} \times \lim_{\varepsilon_a \to 0} B(\varepsilon_a) = 0$, which implies that $\lim_{\varepsilon_a \to 0} \sum_{\ell=1}^{\varepsilon_a} n_{\ell} e^{-n_{\ell}c} = 0$. This completes the proof of the lemma.

Lemma 95 If $\varepsilon_a$ is sufficiently small, then the following statements hold true.

(I): For $1 \leq \ell < s$, there exists a unique number $z_{\ell} \in [0, \lambda^* - \varepsilon_a)$ such that $n_{\ell} = \frac{\ln(\frac{c}{\varepsilon_a})}{\lambda^* + \varepsilon_a}$.

(II): For $1 \leq \ell < s$, there exists a unique number $y_{\ell} \in (\lambda^* + \varepsilon_a, \infty)$ such that $n_{\ell} = \frac{\ln(\frac{c}{\varepsilon_a})}{\lambda^* + \varepsilon_a}$.

(III): $z_{\ell}$ is monotonically increasing with respect to $\ell$; $y_{\ell}$ is monotonically decreasing with respect to $\ell$.

(IV): $\lim_{\varepsilon_a \to 0} z_{\ell} = \lambda^* C_{s-\ell}$ and $\lim_{\varepsilon_a \to 0} y_{\ell} = \frac{\lambda^*}{C_{s-\ell}}$, where the limits are taken under the constraint that $\frac{n}{\varepsilon_a}$ and $s - \ell$ are fixed with respect to $\varepsilon_a$.

(V): Let $\ell = s - j_{\lambda}$. For $\lambda \in (\lambda^*, \infty)$ such that $C_{j_{\lambda}} = r(\lambda)$,

$$\lim_{\varepsilon_a \to 0} \frac{z_{\ell} - \lambda}{\varepsilon_a} = 0.$$

For $\lambda \in (0, \lambda^*)$ such that $C_{j_{\lambda}} = r(\lambda)$,

$$\lim_{\varepsilon_a \to 0} \frac{z_{\ell} - \lambda}{\varepsilon_a} = \frac{2}{3} \left( \frac{\lambda}{\lambda^*} - 1 \right).$$

(VI): $\{D_{\ell} = 0\} = \{z_{\ell} < \lambda < y_{\ell}\}$ for $1 \leq \ell < s$.

Proof of Statement (I):

By the definition of sample sizes, we have $\frac{\ln(\frac{c}{\varepsilon_a})}{n_{\ell}} \geq \mathcal{M}(0, \varepsilon_a)$ and

$$n_{\ell} \leq \frac{(1 + C_1) n_{\ell}}{2} < \frac{(1 + C_1)}{2} \left[ \frac{\ln(\frac{c}{\varepsilon_a})}{\mathcal{M}(\lambda^* + \varepsilon_a, \lambda^*)} + 1 \right]$$

for sufficiently small $\varepsilon_a > 0$. By (159), we have

$$\frac{\ln(\frac{c}{\varepsilon_a})}{n_{\ell}} \leq \mathcal{M}(\lambda^* + \varepsilon_a, \lambda^*) \left( \frac{2}{1 + C_1} \right) = \frac{2}{1 + C_1} \mathcal{M}(\lambda^* + \varepsilon_a, \lambda^*) \mathcal{M}(\lambda^* + \varepsilon_a, \lambda^*) - \frac{\mathcal{M}(\lambda^* + \varepsilon_a, \lambda^*)}{n_{\ell}}.$$

Noting that

$$\lim_{\varepsilon_a \to 0} \frac{\mathcal{M}(\lambda^* + \varepsilon_a, \lambda^*)}{n_{\ell}} = 1, \quad \lim_{\varepsilon_a \to 0} \frac{\mathcal{M}(\lambda^* + \varepsilon_a, \lambda^*)}{n_{\ell}} = 0,$$

we have that $\frac{\ln(\frac{c}{\varepsilon_a})}{n_{\ell}} \leq \mathcal{M}(\lambda^* - \varepsilon_a, \lambda^*)$ for small enough $\varepsilon_a > 0$. In view of the established fact that $\mathcal{M}(0, \varepsilon_a) \leq \frac{\ln(\frac{c}{\varepsilon_a})}{n_{\ell}} \leq \mathcal{M}(\lambda^* - \varepsilon_a, \lambda^*)$ and the fact that $\mathcal{M}(z, z + \varepsilon_a)$ is monotonically increasing,

172
with respect to \( z > 0 \) as asserted by Lemma 79. Invoking the intermediate value theorem, we have that there exists a unique number \( z_\ell \in (0, \lambda - \varepsilon_a) \) such that \( \mathcal{M}_P(z_\ell, z_\ell + \varepsilon_a) = \frac{\ln(\zeta)}{n_\ell} \), which implies Statement (I).

**Proof of Statement (II):** By (159), we have
\[
\frac{\ln(\zeta)}{n_\ell} < \mathcal{M}_P(\lambda^* + \varepsilon_a, \lambda^*) \left( \frac{2}{1 + C_1} - \frac{1}{n_\ell} \right) = \left( \frac{2}{1 + C_1} \right) \mathcal{M}_P(\lambda^* + \varepsilon_a, \lambda^*) - \frac{\mathcal{M}_P(\lambda^* + \varepsilon_a, \lambda^*)}{n_\ell}.
\]

Noting that \( \lim_{\varepsilon_a \to 0} \mathcal{M}_P(\lambda^* + \varepsilon_a, \lambda^*) = 0 \), we have that \( \frac{\ln(\zeta)}{n_\ell} < \mathcal{M}_P(\lambda^* + \varepsilon_a, \lambda^*) \) for small enough \( \varepsilon_a > 0 \). In view of the established fact that \( \frac{\ln(\zeta)}{n_\ell} < \mathcal{M}_P(\lambda^* + \varepsilon_a, \lambda^* + \varepsilon_a) \) and the fact that \( \mathcal{M}_P(z, \frac{z}{1+\varepsilon_a}) \) is monotonically decreasing to \(-\infty\) with respect to \( z \in (0, \infty) \) as asserted by Lemma 94, invoking the intermediate value theorem, we have that there exists a unique number \( y_\ell \in (\lambda^* + \varepsilon_a, \infty) \) such that \( \mathcal{M}_P(y_\ell, \frac{y_\ell}{1+\varepsilon_a}) = \frac{\ln(\zeta)}{n_\ell} \), which implies Statement (II).

**Proof of Statement (III):** Since \( n_\ell \) is monotonically increasing with respect to \( \ell \) if \( \varepsilon_a > 0 \) is sufficiently small, we have that \( \mathcal{M}_P(z_\ell, z_\ell + \varepsilon_a) \) is monotonically increasing with respect to \( \ell \) for small enough \( \varepsilon_a > 0 \). Recalling that \( \mathcal{M}_P(z, z + \varepsilon_a) \) is monotonically increasing with respect to \( z > 0 \), we have that \( z_\ell \) is monotonically increasing with respect to \( \ell \). Similarly, \( \mathcal{M}_P(y_\ell, \frac{y_\ell}{1+\varepsilon_a}) \) is monotonically increasing with respect to \( \ell \) for sufficiently small \( \varepsilon_a > 0 \). Recalling that \( \mathcal{M}_P(z, \frac{z}{1+\varepsilon_a}) \) is monotonically decreasing with respect to \( z > 0 \), we have that \( y_\ell \) is monotonically decreasing with respect to \( \ell \). This establishes Statement (III).

**Proof of Statement (IV):** We first consider \( \lim_{\varepsilon_a \to 0} z_\ell \). For simplicity of notations, define \( b_\ell = \lambda^* C_{s-\ell} \) for \( \ell < s \). Then, it can be checked that \( \frac{b_\ell}{\lambda^*} = C_{s-\ell} \) and, by the definition of sample sizes, we have
\[
\frac{b_\ell}{\lambda^*} \mathcal{M}_P(z_\ell, z_\ell + \varepsilon_a) = \frac{1}{n_\ell} \times \frac{C_{s-\ell} \ln(\zeta)}{\mathcal{M}_P(\lambda^* + \varepsilon_a, \lambda^*)} = 1 + o(1) \quad (160)
\]
for \( \ell < s \).

We claim that \( z_\ell > \theta \) for \( \theta \in (0, b_\ell) \) if \( \varepsilon_a > 0 \) is small enough. To prove this claim, we use a contradiction method. Suppose this claim is not true, then there is a set, denoted by \( S_{\varepsilon_a} \), of infinitely many values of \( \varepsilon_a \) such that \( z_\ell \leq \theta \) for any \( \varepsilon_a \in S_{\varepsilon_a} \). By (160) and the fact that \( \mathcal{M}_P(z, z + \varepsilon_a) \) is monotonically increasing with respect to \( z > 0 \) as asserted by Lemma 79, we have
\[
\frac{b_\ell}{\lambda^*} \mathcal{M}_P(z_\ell, z_\ell + \varepsilon_a) = 1 + o(1) \geq \frac{b_\ell}{\lambda^*} \frac{\mathcal{M}_P(\theta, \theta + \varepsilon_a)}{\mathcal{M}_P(\lambda^* + \varepsilon_a, \lambda^*)} = \frac{b_\ell}{\theta} + o(1)
\]
for small enough \( \varepsilon_a \in S_{\varepsilon_a} \), which implies \( \frac{b_\ell}{\theta} \leq 1 \), contradicting to the fact that \( \frac{b_\ell}{\theta} > 1 \). This proves the claim. Now we restrict \( \varepsilon_a \) to be small enough so that \( \theta < z_\ell < \lambda^* \). Since \( z_\ell \) is bounded in interval \((\theta, \lambda^*)\), we have \( \mathcal{M}_P(z_\ell, z_\ell + \varepsilon_a) = -\varepsilon_a^2/(2z_\ell) + o(\varepsilon_a^2) \) and by (160), we have
\[
\frac{b_\ell}{\lambda^*} \times \frac{-\varepsilon_a^2/(2z_\ell) + o(\varepsilon_a^2)}{-\varepsilon_a^2/(2\lambda^*) + o(\varepsilon_a^2)} = 1 + o(1),
\]
which implies \( \frac{b_\ell}{z_\ell} = 1 + o(1) \) and thus \( \lim_{\varepsilon_a \to 0} z_\ell = b_\ell \).
We now consider \( \lim_{\varepsilon \to 0} y \ell \). For simplicity of notations, define \( a \ell = \frac{\lambda^*}{\varepsilon_{s-\ell}} \) for \( 1 \leq \ell < s \). Then, it can be checked that \( \frac{\lambda^*}{a \ell} = C_{s-\ell} \) and, by the definition of sample sizes, we have

\[
\frac{\lambda^*}{a \ell} \mathcal{M}(y \ell, \frac{y}{1+\varepsilon}) = \frac{1}{n \ell} \frac{C_{s-\ell} \ln(\zeta \delta)}{\mathcal{M}(\lambda^* + \varepsilon a, \lambda^*)} = 1 + o(1). \tag{161}
\]

We claim that \( y \ell < \theta \) for \( \theta \in (a \ell, \infty) \) if \( \varepsilon_r > 0 \) is small enough. To prove this claim, we use a contradiction method. Suppose this claim is not true, then there is a set, denoted by \( S_{\varepsilon_r} \), of infinitely many values of \( \varepsilon_r \) such that \( y \ell \geq \theta \) for any \( \varepsilon_r \in S_{\varepsilon_r} \). By (161) and the fact that \( \mathcal{M}(z, \frac{z}{1+\varepsilon_r}) \) is monotonically decreasing with respect to \( z \in (0, \infty) \) as asserted by Lemma 91, we have

\[
\frac{\lambda^*}{a \ell} \mathcal{M}(y \ell, \frac{y}{1+\varepsilon}) = 1 + o(1) \geq \frac{\lambda^*}{a \ell} \mathcal{M}(\theta, \frac{\theta}{1+\varepsilon_r}) = \frac{\theta}{a \ell} + o(1)
\]

for small enough \( \varepsilon_r \in S_{\varepsilon_r} \), which implies \( \frac{\theta}{a \ell} \leq 1 \), contradicting the fact that \( \frac{\theta}{a \ell} > 1 \). This proves the claim. Now we restrict \( \varepsilon_r \) to be small enough so that \( \lambda^* < y \ell < \theta \). Since \( y \ell \) is bounded in interval \( (\lambda^*, \theta) \), we have \( \mathcal{M}(y \ell, \frac{y}{1+\varepsilon}) = -\varepsilon r^2 y \ell / 2 + o(\varepsilon r^2) \) and by (161), we have

\[
\frac{\lambda^*}{a \ell} \times \frac{-\varepsilon r^2 y \ell}{2} + o(\varepsilon r^2) = 1 + o(1),
\]

which implies \( \frac{y}{a \ell} = o(1) \) and thus \( \lim_{\varepsilon \to 0} y \ell = a \ell \).

**Proof of Statement (V):**

We shall first consider \( \lambda \in (\lambda^*, \infty) \) such that \( C_{\lambda \ell} = \frac{\lambda^*}{\lambda} \). Let \( \psi \ell \) be a function of \( \varepsilon \in (0, 1) \) such that \( |\psi \ell| \) is bounded from above by a constant independent of \( \varepsilon \). Then, by Taylor’s series expansion formula, we have

\[
\mathcal{M}(\psi \ell, \frac{\psi \ell}{1+\varepsilon}) = \frac{\varepsilon \psi \ell}{1+\varepsilon} - \psi \ell \ln(1+\varepsilon) = \varepsilon \psi \ell \left[ 1 - \varepsilon + e^2 + o(e^2) \right] - \psi \ell \left[ -\frac{e^2}{2} + \frac{e^3}{3} + o(e^3) \right] = -\frac{e^2}{2} + \frac{e^3}{3} + o(e^3)
\]

for \( \varepsilon \in (0, 1) \). By the definition of sample sizes, for small enough \( \varepsilon_r \), there exists \( z \ell \in (\lambda^*, \infty) \) such that

\[
n \ell = \frac{\ln(\zeta \delta)}{\mathcal{M}(z \ell, z \ell (1+\varepsilon_r))} = \frac{C_{s-\ell} \ln(\zeta \delta)}{\mathcal{M}(\lambda^*, \lambda^*/(1+\varepsilon_r))} = \left[ \frac{\lambda^*}{\lambda} \frac{\ln(\zeta \delta)}{\mathcal{M}(\lambda^*, \lambda^*/(1+\varepsilon_r))} \right], \tag{163}
\]

from which we can use an argument similar to the proof of Statement (III) to deduce that \( z \ell \) is smaller than \( \theta \) for \( \theta \in (\lambda, \infty) \) if \( \varepsilon_r > 0 \) is small enough. Hence, by (162) and (163), we have

\[
1 + o(\varepsilon_r) = \frac{\lambda^* \ln(\zeta \delta)}{\lambda \mathcal{M}(\lambda^*, \lambda^*/(1+\varepsilon_r))} = \lambda^* \frac{\ln(\zeta \delta)}{\mathcal{M}(z \ell, z \ell (1+\varepsilon_r))} - \frac{\varepsilon^2 z \ell}{2} + \frac{2 \varepsilon^3 z \ell}{3} + o(\varepsilon^3)
\]

and consequently,

\[
1 + o(\varepsilon_r) = \frac{\lambda^* z \ell - \frac{4 \varepsilon \varepsilon_r z \ell}{3} + o(\varepsilon_r)}{\lambda^* - \frac{4 \varepsilon \varepsilon_r \lambda^*}{3} + o(\varepsilon_r)}.
\]

174
which implies that
\[ \lambda^* \left( z_{\ell_a} - \frac{4\varepsilon_r z_{\ell_a}}{3} \right) = \lambda \left( \lambda^* - \frac{4\varepsilon_r \lambda^*}{3} \right) + o(\varepsilon_r), \]
i.e., \( z_{\ell_a} \left( 1 - \frac{4\varepsilon_r}{3} \right) = \lambda \left( 1 - \frac{4\varepsilon_r}{3} \right) + o(\varepsilon_r) \), i.e., \( z_{\ell_a} = \lambda + o(\varepsilon_r) \). It follows that \( \lim_{\varepsilon_r \to 0} \frac{z_{\ell_a} - \lambda}{\varepsilon_r \lambda} = 0 \) and thus
\[ \lim_{\varepsilon_r \to 0} \frac{z_{\ell_a} - \lambda}{\sqrt{\lambda/n_{\ell_a}}} = b \lim_{\varepsilon_r \to 0} \frac{z_{\ell_a} - \lambda}{\varepsilon_r \lambda} = 0. \]

Next, we shall now consider \( \lambda \in (0, \lambda^*) \) such that \( C_{j\lambda} = \frac{1}{\lambda^*} \). Let \( \psi_{\ell_a} \) be a function of \( \epsilon \in (0, \infty) \) such that \( \frac{1}{|\psi_{\ell_a}|} \) is bounded from above by a constant independent of \( \epsilon \). Then, by Taylor's series expansion formula, we have
\[ \mathcal{M}_p(\psi_{\ell_a}, \psi_{\ell_a} + \epsilon) = -\epsilon + \psi_{\ell_a} \ln \left( 1 + \frac{\epsilon}{\psi_{\ell_a}} \right) = -\epsilon + \psi_{\ell_a} \left[ \frac{\epsilon}{\psi_{\ell_a}} - \frac{\epsilon^2}{2\psi_{\ell_a}^2} + \frac{\epsilon^3}{3\psi_{\ell_a}^3} + o(\epsilon^3) \right] = -\frac{\epsilon^2}{2\psi_{\ell_a}} + \frac{\epsilon^3}{3\psi_{\ell_a}^3} + o(\epsilon^3). \]

By the definition of sample sizes, for small enough \( \varepsilon_a \), there exists \( z_{\ell_a} \in (0, \lambda^*) \) such that
\[ n_{\ell_a} = \frac{\ln(\zeta \delta)}{\mathcal{M}_p(z_{\ell_a}, z_{\ell_a} + \varepsilon_a)} = \left[ \frac{C_{s-\ell} \ln(\zeta \delta)}{\mathcal{M}_p(\lambda^* \lambda^* + \varepsilon_a)} \right] \left[ \frac{\lambda \ln(\zeta \delta)}{\lambda \mathcal{M}_p(\lambda^* \lambda^* + \varepsilon_a)} \right] \]
from which we can use an argument similar to the proof of Statement (III) to deduce that \( z_{\ell_a} \) is greater than \( \theta \) for \( \theta \in (0, \lambda) \) if \( \varepsilon_a > 0 \) is small enough. Hence, by \( 164 \) and \( 165 \), we have
\[ 1 + o(\varepsilon_a) = \frac{\lambda}{z_{\ell_a}} - \frac{2\varepsilon_a \lambda}{3z_{\ell_a}} + O(\varepsilon_a) \]
and consequently,
\[ 1 + o(\varepsilon_a) = \frac{\lambda}{z_{\ell_a}} - \frac{2\varepsilon_a \lambda}{3z_{\ell_a}} + O(\varepsilon_a) \]
which implies that \( \frac{\lambda}{z_{\ell_a}} - \frac{2\varepsilon_a \lambda}{3z_{\ell_a}} = 1 - \frac{2\varepsilon_a}{3\lambda} + o(\varepsilon_a) \), i.e., \( \frac{z_{\ell_a} - \lambda}{\varepsilon_a} = \frac{2z_{\ell_a}}{3\lambda} - \frac{2\lambda}{3z_{\ell_a}} + z_{\ell_a} o(\varepsilon_a) \). So, we have
\[ \lim_{\varepsilon_a \to 0} \frac{z_{\ell_a} - \lambda}{\varepsilon_a} = \frac{2}{3} \left( \frac{\lambda}{\lambda^*} - 1 \right) < 0. \]

**Proof of Statement (VI):** By the definition of the sampling scheme, we have
\[
\{ D_{\ell} = 0 \} = \left\{ \max \left\{ \mathcal{M}_p(\tilde{\lambda}_{\ell_a}, \lambda^*), \mathcal{M}_p(\lambda^*, \lambda^*) \right\} > \frac{\ln(\zeta \delta)}{n_{\ell_a}}, \left| \lambda_{\ell_a} - \lambda^* \right| \leq \varepsilon_a \right\}
\cup \left\{ \max \left\{ \mathcal{M}_p(\tilde{\lambda}_{\ell_a}, \lambda^*), \mathcal{M}_p(\lambda^*, \lambda^*) \right\} > \frac{\ln(\zeta \delta)}{n_{\ell_a}}, \tilde{\lambda}_{\ell_a} < \lambda^* - \varepsilon_a \right\}
\cup \left\{ \max \left\{ \mathcal{M}_p(\tilde{\lambda}_{\ell_a}, \lambda^*), \mathcal{M}_p(\lambda^*, \lambda^*) \right\} > \frac{\ln(\zeta \delta)}{n_{\ell_a}}, \tilde{\lambda}_{\ell_a} > \lambda^* + \varepsilon_a \right\}
= \left\{ \max \left\{ \mathcal{M}_p(\tilde{\lambda}_{\ell_a}, \tilde{\lambda}_{\ell_a} - \varepsilon_a), \mathcal{M}_p\left( \tilde{\lambda}_{\ell_a}, \frac{\tilde{\lambda}_{\ell_a}}{1 - \varepsilon_r} \right) \right\} > \frac{\ln(\zeta \delta)}{n_{\ell_a}}, \left| \tilde{\lambda}_{\ell_a} - \lambda^* \right| \leq \varepsilon_a \right\}
\cup \left\{ \mathcal{M}_p(\tilde{\lambda}_{\ell_a}, \tilde{\lambda}_{\ell_a} + \varepsilon_a) > \frac{\ln(\zeta \delta)}{n_{\ell_a}}, \tilde{\lambda}_{\ell_a} < \lambda^* - \varepsilon_a \right\}
\cup \left\{ \mathcal{M}_p\left( \tilde{\lambda}_{\ell_a}, \frac{\tilde{\lambda}_{\ell_a}}{1 - \varepsilon_r} \right) > \frac{\ln(\zeta \delta)}{n_{\ell_a}}, \tilde{\lambda}_{\ell_a} > \lambda^* + \varepsilon_a \right\}.
We claim that,
\[
\left\{ \max \left\{ \mathcal{M}_P(\hat{\lambda}_\ell, \hat{\lambda}_\ell - \varepsilon_a), \mathcal{M}_P\left( \hat{\lambda}_\ell, \frac{\hat{\lambda}_\ell}{1 - \varepsilon_r} \right) \right\} > \frac{\ln(\zeta \delta)}{n_\ell}, |\hat{\lambda}_\ell - \lambda^*| \leq \varepsilon_a \right\} = \left\{ |\hat{\lambda}_\ell - \lambda^*| \leq \varepsilon_a \right\},
\]
(166)
\[
\left\{ \mathcal{M}_P(\hat{\lambda}_\ell, \hat{\lambda}_\ell + \varepsilon_a) > \frac{\ln(\zeta \delta)}{n_\ell}, \hat{\lambda}_\ell < \lambda^* - \varepsilon_a \right\} = \{ \varepsilon_a \ell < \hat{\lambda}_\ell < \lambda^* - \varepsilon_a \},
\]
(167)
\[
\left\{ \mathcal{M}_P\left( \hat{\lambda}_\ell, \frac{\hat{\lambda}_\ell}{1 + \varepsilon_r} \right) > \frac{\ln(\zeta \delta)}{n_\ell}, \hat{\lambda}_\ell > \lambda^* + \varepsilon_a \right\} = \{ \lambda^* + \varepsilon_a < \hat{\lambda}_\ell < y_\ell \}
\]
(168)
for \(1 \leq \ell < s\) provided that \(\varepsilon_a\) is sufficiently small.

To show (166), note that
\[
n_\ell < \frac{(1 + C_1)n_s}{2} < \frac{1 + C_1}{2} \left( \frac{\ln(\zeta \delta)}{\mathcal{M}_P(\lambda^* + \varepsilon_a, \lambda^*)} + 1 \right),
\]
(169)
from which we have
\[
\frac{\ln(\zeta \delta)}{n_\ell} < \frac{\mathcal{M}_P(\lambda^* + \varepsilon_a, \lambda^*)}{\mathcal{M}_P(\lambda^* - \varepsilon_a, \lambda^* - \varepsilon_a - \varepsilon_a)} \left( \frac{2}{1 + C_1} \right) \mathcal{M}_P(\lambda^* - \varepsilon_a, \lambda^* - \varepsilon_a - \varepsilon_a) - \frac{\mathcal{M}_P(\lambda^* + \varepsilon_a, \lambda^*)}{n_\ell}
\]
Noting that
\[
\lim_{\varepsilon_a \to 0} \frac{\mathcal{M}_P(\lambda^* + \varepsilon_a, \lambda^*)}{\mathcal{M}_P(\lambda^* - \varepsilon_a, \lambda^* - \varepsilon_a - \varepsilon_a)} = \lim_{\varepsilon_a \to 0} \frac{-\varepsilon^2}{2\lambda^*} + o(\varepsilon_a^2) = 1
\]
and \(\lim_{\varepsilon_a \to 0} \frac{\mathcal{M}_P(\lambda^* + \varepsilon_a, \lambda^*)}{n_\ell} = 0\), we have
\[
\frac{\ln(\zeta \delta)}{n_\ell} < \mathcal{M}_P(\lambda^* - \varepsilon_a, \lambda^* - \varepsilon_a - \varepsilon_a)
\]
(170)
for small enough \(\varepsilon_a > 0\). Again by (169), we have
\[
\frac{\ln(\zeta \delta)}{n_\ell} < \frac{\mathcal{M}_P(\lambda^* + \varepsilon_a, \lambda^*)}{\mathcal{M}_P(\lambda^* - \varepsilon_a, \lambda^* + \varepsilon_a)} \left( \frac{2}{1 + C_1} \right) \mathcal{M}_P\left( \lambda^* + \varepsilon_a, \frac{\lambda^* + \varepsilon_a}{1 - \varepsilon_r} \right) - \frac{\mathcal{M}_P(\lambda^* + \varepsilon_a, \lambda^*)}{n_\ell}
\]
Noting that
\[
\lim_{\varepsilon_a \to 0} \frac{\mathcal{M}_P(\lambda^* + \varepsilon_a, \lambda^*)}{\mathcal{M}_P(\lambda^* - \varepsilon_a, \lambda^* + \varepsilon_a)} = \lim_{\varepsilon_a \to 0} \frac{-\varepsilon^2}{2\lambda^*} + o(\varepsilon_a^2) = 1
\]
and \(\lim_{\varepsilon_a \to 0} \frac{\mathcal{M}_P(\lambda^* + \varepsilon_a, \lambda^*)}{n_\ell} = 0\), we have
\[
\frac{\ln(\zeta \delta)}{n_\ell} < \mathcal{M}_P\left( \lambda^* + \varepsilon_a, \frac{\lambda^* + \varepsilon_a}{1 - \varepsilon_r} \right)
\]
(171)
for small enough \(\varepsilon_a > 0\). Note that, for \(z \in [\lambda^* - \varepsilon_a, \lambda^* + \varepsilon_a]\), \(\mathcal{M}_P(z, z - \varepsilon_a)\) is monotonically increasing with respect to \(z\) and \(\mathcal{M}_P(z, \frac{1}{1 - \varepsilon_r})\) is monotonically decreasing with respect to \(z\). By (170) and (171), we have \(\frac{\ln(\zeta \delta)}{n_\ell} < \mathcal{M}_P(z, z - \varepsilon_a)\) and \(\frac{\ln(\zeta \delta)}{n_\ell} < \mathcal{M}_P(z, \frac{1}{1 - \varepsilon_r})\) for any \(z \in [\lambda^* - \varepsilon_a, \lambda^* + \varepsilon_a]\) if \(\varepsilon_a > 0\) is small enough. This proves (166).
To show (167), let \( \omega \in \{ \mathcal{M}_P(\tilde{\lambda}_\ell, \tilde{\lambda}_\ell + \varepsilon_a) > \frac{\ln(\delta)}{n_\ell}, \tilde{\lambda}_\ell < \lambda^* - \varepsilon_a \} \) and \( \tilde{\lambda}_\ell = \tilde{\lambda}_\ell(\omega) \). Then, \( \mathcal{M}_P(\tilde{\lambda}_\ell, \tilde{\lambda}_\ell + \varepsilon_a) > \frac{\ln(\delta)}{n_\ell} \) and \( \tilde{\lambda}_\ell < \lambda^* - \varepsilon_a \). Since \( z_\ell \in [0, \lambda^* - \varepsilon_a] \) and \( \mathcal{M}_P(z, z + \varepsilon_a) \) is monotonically increasing with respect to \( z \in (0, \lambda^* - \varepsilon_a) \), it must be true that \( \tilde{\lambda}_\ell > z_\ell \). Otherwise if \( \tilde{\lambda}_\ell \leq z_\ell \), then \( \mathcal{M}_P(\tilde{\lambda}_\ell, \tilde{\lambda}_\ell + \varepsilon_a) \leq \mathcal{M}_P(z_\ell, z_\ell + \varepsilon_a) = \frac{\ln(\delta)}{n_\ell} \), leading to a contradiction. This proves \( \{ \mathcal{M}_P(\tilde{\lambda}_\ell, \tilde{\lambda}_\ell + \varepsilon_a) > \frac{\ln(\delta)}{n_\ell}, \tilde{\lambda}_\ell < \lambda^* - \varepsilon_a \} \subseteq \{ z_\ell < \tilde{\lambda}_\ell < \lambda^* - \varepsilon_a \} \). Now let \( \omega \in \{ z_\ell < \tilde{\lambda}_\ell < \lambda^* - \varepsilon_a \} \) and \( \tilde{\lambda}_\ell = \tilde{\lambda}_\ell(\omega) \). Then, \( z_\ell < \tilde{\lambda}_\ell < \lambda^* - \varepsilon_a \). Noting that \( \mathcal{M}_P(z, z + \varepsilon_a) \) is monotonically increasing with respect to \( z > 0 \), we have that \( \mathcal{M}_P(\tilde{\lambda}_\ell, \tilde{\lambda}_\ell + \varepsilon_a) > \mathcal{M}_P(z_\ell, z_\ell + \varepsilon_a) = \frac{\ln(\delta)}{n_\ell} \), which implies \( \{ \mathcal{M}_P(\tilde{\lambda}_\ell, \tilde{\lambda}_\ell + \varepsilon_a) > \frac{\ln(\delta)}{n_\ell}, \tilde{\lambda}_\ell < \lambda^* - \varepsilon_a \} \supseteq \{ z_\ell < \tilde{\lambda}_\ell < \lambda^* - \varepsilon_a \} \). This establishes (167).

To show (168), let \( \omega \in \{ \mathcal{M}_P(\tilde{\lambda}_\ell, \frac{\tilde{\lambda}_\ell}{1 + \varepsilon}) > \frac{\ln(\delta)}{n_\ell}, \tilde{\lambda}_\ell > \lambda^* + \varepsilon_a \} \) and \( \tilde{\lambda}_\ell = \tilde{\lambda}_\ell(\omega) \). Then, \( \mathcal{M}_P(\tilde{\lambda}_\ell, \frac{\tilde{\lambda}_\ell}{1 + \varepsilon}) > \frac{\ln(\delta)}{n_\ell} \) and \( \tilde{\lambda}_\ell > \lambda^* + \varepsilon_a \). Since \( y_\ell \in (\lambda^* + \varepsilon_a, \infty) \) and \( \mathcal{M}_P(z, \frac{z}{1 + \varepsilon}) \) is monotonically decreasing with respect to \( z > 0 \), it must be true that \( \tilde{\lambda}_\ell < y_\ell \). Otherwise if \( \tilde{\lambda}_\ell \geq y_\ell \), then \( \mathcal{M}_P(\tilde{\lambda}_\ell, \frac{\tilde{\lambda}_\ell}{1 + \varepsilon}) \leq \mathcal{M}_P(y_\ell, \frac{y_\ell}{1 + \varepsilon}) = \frac{\ln(\delta)}{n_\ell} \), leading to a contradiction. This proves \( \{ \mathcal{M}_P(\tilde{\lambda}_\ell, \frac{\tilde{\lambda}_\ell}{1 + \varepsilon}) > \frac{\ln(\delta)}{n_\ell}, \tilde{\lambda}_\ell > \lambda^* + \varepsilon_a \} \subseteq \{ \lambda^* + \varepsilon_a < \tilde{\lambda}_\ell < y_\ell \} \). Now let \( \omega \in \{ \lambda^* + \varepsilon_a < \tilde{\lambda}_\ell < y_\ell \} \) and \( \tilde{\lambda}_\ell = \tilde{\lambda}_\ell(\omega) \). Then, \( \lambda^* + \varepsilon_a < \tilde{\lambda}_\ell < y_\ell \). Noting that \( \mathcal{M}_P(z, \frac{z}{1 + \varepsilon}) \) is monotonically decreasing with respect to \( z > 0 \), we have that \( \mathcal{M}_P(\tilde{\lambda}_\ell, \frac{\tilde{\lambda}_\ell}{1 + \varepsilon}) > \mathcal{M}_P(y_\ell, \frac{y_\ell}{1 + \varepsilon}) = \frac{\ln(\delta)}{n_\ell} \), which implies \( \{ \mathcal{M}_P(\tilde{\lambda}_\ell, \frac{\tilde{\lambda}_\ell}{1 + \varepsilon}) > \frac{\ln(\delta)}{n_\ell}, \tilde{\lambda}_\ell > \lambda^* + \varepsilon_a \} \supseteq \{ \lambda^* + \varepsilon_a < \tilde{\lambda}_\ell < y_\ell \} \). This establishes (168).

**Lemma 96** Let \( \ell = s - j_\lambda \). Then, under the constraint that limits are taken with \( \frac{\varepsilon_a}{\varepsilon_v} \) fixed,

\[
\lim_{\varepsilon_a \to 0} \sum_{\ell=1}^{\ell_*-1} n_\ell \Pr\{ D_\ell = 1 \} = 0, \quad \lim_{\varepsilon_a \to 0} \sum_{\ell=\ell_*+1}^{s} n_\ell \Pr\{ D_\ell = 0 \} = 0 \tag{172}
\]

for \( \lambda \in (0, \infty) \). Moreover, \( \lim_{\varepsilon_a \to 0} n_\ell \Pr\{ D_\ell = 0 \} = 0 \) if \( C_j \lambda > r(\lambda) \).

**Proof.** Throughout the proof of the lemma, we restrict \( \varepsilon_a \) to be small enough such that \( \frac{\ln(\delta)}{n_\ell} < \frac{\ln(\delta)}{\mathcal{M}_P(\lambda, \frac{\lambda}{1 + \varepsilon})} \). For simplicity of notations, let \( a_\ell = \lim_{\varepsilon_a \to 0} y_\ell \) and \( b_\ell = \lim_{\varepsilon_a \to 0} z_\ell \). The proof consists of three main steps as follows.

First, we shall show that (172) holds for \( \lambda \in (0, \lambda^*) \). By the definition of \( \ell_\xi \), we have \( \lambda^* > C_{s-\ell_*+1} \). Making use of the first four statements of Lemma 95 we have that \( z_\ell < \frac{\lambda + b_{\ell_*-1}}{2} < \lambda \) for all \( \ell \leq \ell_*-1 \) and \( y_{s-1} > \frac{\lambda^* + a_{s-1}}{2} > \lambda^* \) if \( \varepsilon_a \) is sufficiently small. By the last statement of Lemma 95 and using Lemma 81 we have

\[
\Pr\{ D_\ell = 1 \} = \Pr\{ \tilde{\lambda}_\ell \leq z_\ell \} + \Pr\{ \tilde{\lambda}_\ell \geq y_\ell \} \leq \Pr\{ \tilde{\lambda}_\ell \leq z_\ell \} + \Pr\{ \tilde{\lambda}_\ell \geq y_{s-1} \}
\]

\[
\leq \Pr\left\{ \frac{\tilde{\lambda}_\ell + b_{\ell_*-1}}{2} \right\} + \Pr\left\{ \frac{\tilde{\lambda}_\ell + a_{s-1}}{2} \right\}
\]

\[
\leq \exp\left( n_\ell \mathcal{M}_P\left( \frac{\lambda + b_{\ell_*-1}}{2}, \lambda \right) \right) + \exp\left( n_\ell \mathcal{M}_P\left( \frac{\lambda^* + a_{s-1}}{2}, \lambda \right) \right)
\]

for all \( \ell \leq \ell_*-1 \) if \( \varepsilon_a > 0 \) is small enough. Noting that \( b_{\ell_*-1} = \lambda^* C_{s+1}, a_{s-1} = \frac{\lambda^*}{C_1} \),

\[
\frac{\lambda + b_{\ell_*-1}}{2} = \frac{\lambda + \lambda^* C_{s+1}}{2} < \lambda, \quad \frac{\lambda^* + a_{s-1}}{2} = \frac{\lambda^* + \frac{\lambda^*}{C_1}}{2} > \lambda
\]

177
which are constants independent of $\varepsilon_a > 0$. Therefore, both $\mathcal{M}_p\left(\frac{\lambda+b_{\ell-1}}{2}, \lambda\right)$ and $\mathcal{M}_p\left(\frac{\lambda+\alpha_{\ell-1}}{2}, \lambda\right)$ are negative constants independent of $\varepsilon_a > 0$. It follows from Lemma 94 that $\lim_{\varepsilon_a \to 0} \sum_{\ell=1}^{s-1} \varepsilon_{\ell} \mathbb{P}(D_{\ell} = 1) = 0$.

Similarly, it can be seen from the definition of $\ell$, that $\frac{\lambda}{\lambda^*} < C_{s-\ell}$. Making use of the first four statements of Lemma 95, we have that $z_{\ell} > \frac{\lambda+b_{\ell-1}}{2} > \lambda$ for $\ell + 1 \leq \ell < s$ if $\varepsilon_a$ is sufficiently small. By the last statement of Lemma 95 and using Lemma 31, we have

$$\Pr\{D_{\ell} = 0\} = \Pr\{z_{\ell} < \hat{\lambda}_{\ell} < y_{\ell}\} \leq \Pr\{\hat{\lambda}_{\ell} > z_{\ell}\} \leq \Pr\{\hat{\lambda}_{\ell} \geq y_{\ell}\} \leq \exp\left(n_{\ell} \mathcal{M}_p\left(\frac{\lambda+b_{\ell-1}}{2}, \lambda\right)\right)$$

for $\ell + 1 \leq \ell < s$ if $\varepsilon_a > 0$ is small enough. By virtue of the definition of $\ell$, we have that $b_{\ell+1}$ is greater than $b_{\ell-1}$ and is independent of $\varepsilon_a > 0$. In view of this and the fact that $\Pr\{D_s = 0\} = 0$, we can use Lemma 94 to arrive at $\lim_{\varepsilon_a \to 0} \sum_{\ell=\ell+1}^{s} \varepsilon_{\ell} \Pr\{D_{\ell} = 0\} = 0$.

Second, we shall show that (172) holds for $\lambda \in (\lambda^*, \infty)$. As a direct consequence of the definition of $\ell$, we have $\frac{\lambda}{\lambda^*} > C_{s-\ell+1}$. Making use of the first four statements of Lemma 95, we have that $y_{\ell} > \frac{\lambda+\alpha_{\ell-1}}{2} > \lambda$ for all $\ell \leq \ell - 1$ and $z_{s-1} > \frac{\lambda+\alpha_{s-1}}{2}$ if $\varepsilon_a$ is sufficiently small. By the last statement of Lemma 95 and using Lemma 31, we have

$$\Pr\{D_{\ell} = 1\} = \Pr\{\hat{\lambda}_{\ell} \geq y_{\ell}\} + \Pr\{\hat{\lambda}_{\ell} \leq z_{\ell}\} \leq \Pr\{\hat{\lambda}_{\ell} \geq \hat{\lambda}_{\ell}\} + \Pr\{\hat{\lambda}_{\ell} \leq \hat{\lambda}_{s-1}\}$$

$$\leq \Pr\{\hat{\lambda}_{\ell} \geq \frac{\lambda+\alpha_{\ell}}{2}\} + \Pr\{\hat{\lambda}_{\ell} \leq \frac{\lambda^*+b_{s-1}}{2}\}$$

$$\leq \exp\left(n_{\ell} \mathcal{M}_p\left(\frac{\lambda+\alpha_{\ell-1}}{2}, \lambda\right)\right) + \exp\left(n_{\ell} \mathcal{M}_p\left(\frac{\lambda^*+b_{s-1}}{2}, \lambda\right)\right)$$

for all $\ell \leq \ell - 1$ if $\varepsilon_a > 0$ is small enough. By virtue of the definition of $\ell$, we have that $\alpha_{\ell-1}$ is greater than $\lambda$ and is independent of $\varepsilon_a > 0$. Hence, it follows from Lemma 94 that $\lim_{\varepsilon_a \to 0} \sum_{\ell=1}^{s-1} \varepsilon_{\ell} \Pr\{D_{\ell} = 1\} = 0$.

In a similar manner, by the definition of $\ell$, we have $\frac{\lambda}{\lambda^*} < C_{\ell-1}$. Making use of the first four statements of Lemma 95, we have that $y_{\ell} < \frac{\lambda+\alpha_{\ell+1}}{2} < \lambda$ for $\ell + 1 \leq \ell < s$ if $\varepsilon_a$ is sufficiently small. By the last statement of Lemma 95 and using Lemma 31, we have

$$\Pr\{D_{\ell} = 0\} = \Pr\{z_{\ell} < \hat{\lambda}_{\ell} < y_{\ell}\} \leq \Pr\{\hat{\lambda}_{\ell} < y_{\ell}\} \leq \Pr\{\hat{\lambda}_{\ell} \leq \frac{\lambda+\alpha_{\ell+1}}{2}\} \leq \exp\left(n_{\ell} \mathcal{M}_p\left(\frac{\lambda+\alpha_{\ell+1}}{2}, \lambda\right)\right)$$

for $\ell + 1 \leq \ell < s$ if $\varepsilon > 0$ is small enough. As a result of the definition of $\ell$, we have that $\alpha_{\ell+1}$ is smaller than $\lambda$ and is independent of $\varepsilon_a > 0$. In view of this and the fact that $\Pr\{D_s = 0\} = 0$, we can use Lemma 94 to conclude that $\lim_{\varepsilon_a \to 0} \sum_{\ell=\ell+1}^{s} \varepsilon_{\ell} \Pr\{D_{\ell} = 0\} = 0$. This proves that (172) holds for $\lambda \in (\lambda^*, \infty)$.

Third, we shall show that $\lim_{\varepsilon_a \to 0} \varepsilon_a \Pr\{D_{\ell} = 0\} = 0$ if $C_{j_s} > r(\lambda)$.

For $\lambda \in (0, \lambda^*)$ such that $C_{j_s} > r(\lambda)$, we have $\frac{\lambda}{\lambda^*} < C_{\ell-1}$ because of the definition of $\ell$. Making use of the first four statements of Lemma 95, we have that $z_{\ell} > \frac{\lambda+b_{\ell}}{2} > \lambda$ if $\varepsilon_a > 0$ is small enough. By the last statement of Lemma 95 and using Lemma 31, we have
Pr\{D_{\ell} = 0\} = Pr\{z_{\ell} < \hat{\lambda}_{\ell} < y_{\ell}\} \leq Pr\{\hat{\lambda}_{\ell} > z_{\ell}\} \leq Pr\{\hat{\lambda}_{\ell} > \frac{\lambda + a_{\ell}}{2}\} \leq \exp \left(n_{\ell}, \mathcal{M}(\frac{\lambda + a_{\ell}}{2}, \lambda)\right).

Since \(b_{\ell}\) is greater than \(\lambda\) and is independent of \(\varepsilon_a > 0\) due to the definition of \(\ell\), it follows that 
\[
\lim_{\varepsilon_a \to 0} n_{\ell}, \Pr\{D_{\ell} = 0\} = 0.
\]

For \(\lambda \in (\lambda^*, \infty)\) such that \(C_{j\lambda} > r(\lambda)\), we have \(\frac{\lambda}{\lambda^*} < C_{s-\ell}\) as a result of the definition of \(\ell\). Making use of the first four statements of Lemma 95 we have that \(y_{\ell} < \frac{\lambda + a_{\ell}}{2} < \lambda\) if \(\varepsilon_a > 0\) is small enough. By the last statement of Lemma 95 and using Lemma 31 we have
\[
Pr\{D_{\ell} = 0\} = Pr\{z_{\ell} < \hat{\lambda}_{\ell} < y_{\ell}\} \leq Pr\{\hat{\lambda}_{\ell} < y_{\ell}\} \leq Pr\{\hat{\lambda}_{\ell} < \frac{\lambda + a_{\ell}}{2}\} \leq \exp \left(n_{\ell}, \mathcal{M}(\frac{\lambda + a_{\ell}}{2}, \lambda)\right).
\]

Since \(a_{\ell}\) is smaller than \(\lambda\) and is independent of \(\varepsilon_a > 0\) as a consequence of the definition of \(\ell\), it follows that 
\[
\lim_{\varepsilon_a \to 0} n_{\ell}, \Pr\{D_{\ell} = 0\} = 0.
\]

This concludes the proof of the lemma.

\[\square\]

Finally, we would like to note that the proof of Theorem 51 can be completed by employing Lemma 96 and a similar argument as that of Theorem 15.

### J.7 Proof of Theorem 52

As a result of the definitions of \(\kappa_{\lambda}\) and \(r(\lambda)\), we have that \(\kappa_{\lambda} > 1\) if and only if \(C_{j\lambda} > r(\lambda)\). To prove Theorem 52 we need some preliminary results.

**Lemma 97** \(\lim_{\varepsilon_a \to 0} \frac{n_{\ell}}{N_m(\lambda, \varepsilon_a, \varepsilon_r)} = \kappa_{\lambda}\), \(\lim_{\varepsilon_a \to 0} \varepsilon_a \sqrt{\frac{n_{\ell}}{\lambda}} = d \sqrt{\kappa_{\lambda}}\), \(\lim_{\varepsilon_a \to 0} \varepsilon_r \sqrt{\lambda n_{\ell}} = d \sqrt{\kappa_{\lambda}}\).

**Proof.** First, we shall consider \(\lambda \in (0, \lambda^*)\). Note that
\[
\mathcal{M}(z, z + \varepsilon) = -\varepsilon + z \ln \left(1 + \frac{\varepsilon}{z}\right) = -\varepsilon + z \ln \left(1 + \frac{\varepsilon}{2z^2} + o(\varepsilon^2)\right) = \frac{-\varepsilon^2}{2z} + o(\varepsilon^2).
\]

By the definition of sample sizes, we have
\[
\lim_{\varepsilon_a \to 0} \frac{C_{s-\ell} \ln(\zeta\delta)}{n_{\ell}, \mathcal{M}(\lambda^* + \varepsilon_a, \lambda^*)} = 1
\]
for \(1 \leq \ell < s\). It follows that
\[
\lim_{\varepsilon_a \to 0} \frac{n_{\ell}}{N_m(\lambda, \varepsilon_a, \varepsilon_r)} = \lim_{\varepsilon_a \to 0} \frac{\mathcal{M}(\lambda, \lambda + \varepsilon_a) \times C_{s-\ell} \ln(\zeta\delta)}{\mathcal{M}(\lambda^* + \varepsilon_a, \lambda^*)} = \lim_{\varepsilon_a \to 0} \frac{C_{s-\ell} \mathcal{M}(\lambda, \lambda + \varepsilon_a)}{\mathcal{M}(\lambda^* + \varepsilon_a, \lambda^*)}
\]
and
\[
\lim_{\varepsilon_a \to 0} \varepsilon_a \sqrt{\frac{n_{\ell}}{\lambda}} = \lim_{\varepsilon_a \to 0} \varepsilon_a \sqrt{\frac{1}{\lambda} \left(\frac{C_{s-\ell} \ln(\zeta\delta)}{\lambda^* + \varepsilon_a, \lambda^*}\right)}
\]
\[
= \lim_{\varepsilon_a \to 0} \varepsilon_a \sqrt{\frac{1}{\lambda} \left(\frac{C_{s-\ell} \ln(\zeta\delta)}{\lambda^* + \varepsilon_a, \lambda^*}\right)} = d \sqrt{\frac{\lambda^*}{\lambda} C_{s-\ell} = d \sqrt{\kappa_{\lambda}}.}
\]

179
We shall next consider \( \lambda \in (\lambda^*, \infty) \). Note that

\[
\mathcal{M}_P \left( \frac{z}{1+\varepsilon}, \frac{z}{1+\varepsilon} \right) = \frac{\varepsilon}{1+\varepsilon} - z \ln(1+\varepsilon) = \varepsilon z [1 - \varepsilon + o(\varepsilon)] - z \left[ \varepsilon - \frac{\varepsilon^2}{2} + o(\varepsilon^2) \right] = -\frac{\varepsilon^2}{2} + o(\varepsilon^2).
\]

By (1.73), we have

\[
\lim_{\varepsilon \to 0} \frac{n_{\ell \varepsilon}}{N_m(\lambda, \varepsilon_\lambda, \varepsilon_\rho)} = \lim_{\varepsilon \to 0} \frac{\mathcal{M}_P(\lambda, \frac{1}{1+\varepsilon}\ell \varepsilon)}{\ln(\lambda^*)} \frac{C_{\lambda, \ell \varepsilon}}{\mathcal{M}_P(\lambda^* + \varepsilon_\lambda, \lambda^*)} = \lim_{\varepsilon \to 0} \frac{C_{\lambda, \ell \varepsilon}}{\mathcal{M}_P(\lambda^* + \varepsilon_\lambda, \lambda^*)} = \lim_{\varepsilon \to 0} \frac{C_{\lambda, \ell \varepsilon}}{\mathcal{M}_P(\lambda^* + \varepsilon_\lambda, \lambda^*)} = \frac{\lambda}{\lambda^*} C_{\lambda, j\lambda} = \kappa \lambda
\]

and

\[
\lim_{\varepsilon \to 0} \varepsilon R(N_{\ell \varepsilon}) = \lim_{\varepsilon \to 0} \varepsilon R(N_{\ell \varepsilon}) = \frac{\lambda}{\lambda^*} C_{\lambda, j\lambda} = d \sqrt{\frac{\lambda}{\lambda^*}}.
\]

**Lemma 98** Let \( U \) and \( V \) be independent Gaussian random variables with zero means and unit variances. Then, for \( \lambda \in (0, \infty) \) such that \( C_{j\lambda} = r(\lambda) \) and \( j\lambda \geq 1 \),

\[
\lim_{\varepsilon \to 0} \Pr\{ l = \ell \varepsilon \} = 1 - \lim_{\varepsilon \to 0} \Pr\{ l = \ell \varepsilon + 1 \} = 1 - \Phi(\nu d),
\]

\[
\lim_{\varepsilon \to 0} \left[ \Pr\{ \sqrt{\lambda \varepsilon} - \lambda \geq \varepsilon_\lambda, l = \ell \varepsilon \} + \Pr\{ \sqrt{\lambda \varepsilon} - \lambda \geq \varepsilon_\lambda, l = \ell \varepsilon + 1 \} \right] = \Pr\{ U \geq d \} + \Pr\{ U + \sqrt{\rho x V} \geq d \sqrt{1 + \rho \lambda}, U < \nu d \},
\]

where \( \varepsilon_\lambda = \max\{\varepsilon_\lambda, \varepsilon_\rho\} \).

**Proof.** We shall first consider \( \lambda \in (\lambda^*, \infty) \) such that \( C_{j\lambda} = r(\lambda) \). Since \( \kappa \lambda = 1 \), by Statement (V) of Lemma 95, we have

\[
\lim_{\varepsilon \to 0} \frac{z_{\varepsilon \rho} - \lambda}{\sqrt{\lambda / n_{\ell \varepsilon}}} = \lim_{\varepsilon \to 0} \frac{z_{\varepsilon \rho} - \lambda}{\varepsilon_\rho \lambda} = \lim_{\varepsilon \to 0} \frac{z_{\varepsilon \rho} - \lambda}{\varepsilon_\rho \lambda} = 0.
\]

By a similar argument as in the proof of Lemma 20, we can show that

\[
\lim_{\varepsilon \to 0} \Pr\{ l = \ell \varepsilon \} = 1 - \lim_{\varepsilon \to 0} \Pr\{ l = \ell \varepsilon + 1 \} = \lim_{\varepsilon \to 0} \Pr\{ \sqrt{\lambda \varepsilon} - \lambda \geq z_{\varepsilon \rho} \}
\]

\[
\lim_{\varepsilon \to 0} \left[ \Pr\{ \sqrt{\lambda \varepsilon} - \lambda \geq \varepsilon_\lambda, l = \ell \varepsilon \} + \Pr\{ \sqrt{\lambda \varepsilon} - \lambda \geq \varepsilon_\lambda, l = \ell \varepsilon + 1 \} \right]
\]

\[
= \lim_{\varepsilon \to 0} \Pr\{ \sqrt{\lambda \varepsilon} - \lambda \geq \varepsilon_\rho \}, \sqrt{\lambda \varepsilon} - \lambda \geq z_{\varepsilon \rho} \} + \Pr\{ \sqrt{\lambda \varepsilon} - \lambda \geq \varepsilon_\rho, \sqrt{\lambda \varepsilon} - \lambda \geq z_{\varepsilon \rho} \}. \]
Note that

\[ \Pr\{|\hat{\lambda}_\ell - \lambda| \geq \varepsilon_a \lambda, \hat{\lambda}_\ell \geq z_\ell \} = \Pr \left\{ \frac{|\hat{\lambda}_\ell - \lambda|}{\sqrt{\lambda/\ell \varepsilon_a}} \geq \varepsilon_a \sqrt{\lambda n_{\ell \varepsilon}}, \frac{\hat{\lambda}_\ell - \lambda}{\sqrt{\lambda/\ell \varepsilon_a}} \geq \frac{z_\ell - \lambda}{\sqrt{\lambda/\ell \varepsilon_a}} \right\}. \]

Therefore,

\[ \Pr\{|\hat{\lambda}_\ell - \lambda| \geq \varepsilon_a, l = \ell \} + \Pr\{|\hat{\lambda}_{\ell+1} - \lambda| \geq \varepsilon_a, l = \ell + 1\} \]

\[ \to \Pr\{|U| \geq d, U \geq 0\} + \Pr \left\{ |U + \sqrt{\rho \lambda V}| \geq d \sqrt{1 + \rho \lambda}, U < 0 \right\} = \Pr\{|U| \geq d\} \]

for \( \lambda \in (\lambda^*, \infty) \) such that \( C_{j\lambda} = r(\lambda) \).

Next, we shall now consider \( \lambda \in (0, \lambda^*) \) such that \( C_{j\lambda} = r(\lambda) \). Since \( \kappa = 1 \), by Statement (V) of Lemma 95, we have

\[ \lim_{\varepsilon_a \to 0} \frac{z_\ell - \lambda}{\sqrt{\lambda/\ell \varepsilon_a}} \frac{n_{\ell \varepsilon}}{\lambda} = d \lim_{\varepsilon_a \to 0} \frac{z_\ell - \lambda}{\varepsilon_a} = -\nu. \]

Clearly,

\[ \Pr\{|\hat{\lambda}_\ell - \lambda| \geq \varepsilon_a, \hat{\lambda}_\ell \leq z_\ell \} = \Pr \left\{ \frac{|\hat{\lambda}_\ell - \lambda|}{\sqrt{\lambda/\ell \varepsilon_a}} \geq \frac{n_{\ell \varepsilon}}{\lambda}, \frac{\hat{\lambda}_\ell - \lambda}{\sqrt{\lambda/\ell \varepsilon_a}} \leq \frac{z_\ell - \lambda}{\sqrt{\lambda/\ell \varepsilon_a}} \right\}. \]

Therefore,

\[ \Pr\{|\hat{\lambda}_\ell - \lambda| \geq \varepsilon_a, l = \ell \} + \Pr\{|\hat{\lambda}_{\ell+1} - \lambda| \geq \varepsilon_a, l = \ell + 1\} \]

\[ \to \Pr\{|U| \geq d, U \leq -\nu d\} + \Pr \left\{ |U + \sqrt{\rho \lambda V}| \geq d \sqrt{1 + \rho \lambda}, U > -\nu d \right\} \]

\[ = \Pr\{U \geq d\} + \Pr \left\{ |U + \sqrt{\rho \lambda V}| \geq d \sqrt{1 + \rho \lambda}, U < -\nu d \right\}. \]

Finally, we would like to note that the proof of Theorem 52 can be completed by employing Lemma 97 and similar arguments as that of Theorem 16. Specially, we need to restrict \( \varepsilon_a \) to be small enough such that \( \frac{\ln \frac{1}{\varepsilon_a}}{\varepsilon_a} < \frac{\ln (1/\varepsilon_a)}{d(\lambda, 1/\varepsilon)} \). For the purpose of proving Statement (III), we need to make use of the following observation:

\[ \Pr\{|\hat{\lambda} - \lambda| \geq \varepsilon_a, |\hat{\lambda} - \lambda| \geq \varepsilon_a \lambda \} = \begin{cases} \Pr\{|\hat{\lambda} - \lambda| \geq \varepsilon_a \} & \text{for } \lambda \in (0, \lambda^*], \\ \Pr\{|\hat{\lambda} - \lambda| \geq \varepsilon_a \lambda \} & \text{for } \lambda \in (\lambda^*, \infty) \end{cases} \]

\[ \Pr\{|\hat{\lambda}_\ell - \lambda| \geq \varepsilon_a\} = \Pr \left\{ \frac{|\hat{\lambda}_\ell - \lambda|}{\sqrt{\lambda/\ell \varepsilon_a}} \geq \frac{n_{\ell \varepsilon}}{\lambda} \right\}, \quad \Pr\{|\hat{\lambda}_\ell - \lambda| \geq \varepsilon_a \lambda \} = \Pr \left\{ |U| \geq \varepsilon_a \sqrt{\lambda n_{\ell \varepsilon}} \right\} \]

where, according to the central limit theorem, \( U_\ell = \frac{|\hat{\lambda}_\ell - \lambda|}{\sqrt{\lambda/\ell \varepsilon_a}} \) converges in distribution to a Gaussian random variable \( U \) of zero mean and unit variance as \( \varepsilon_a \to 0 \).
K Proofs of Theorems for Estimation of Normal Mean

K.1 Proof of Theorem 53

First, we shall show statement (I) which asserts that \( \Pr\{|\hat{\mu} - \mu| < \varepsilon\} > 1 - 2s\delta \). Define \( m = \max\{n_s, \frac{[(\hat{\sigma}_s) (t_{n_s - 1, \delta})]^2}{\varepsilon^2}\} \). Then, \( \sqrt{m} \geq (\hat{\sigma}_s (t_{n_s - 1, \delta})/\varepsilon) \) is a sure event and by the definition of the sampling scheme,

\[
\Pr\{|\bar{X}_n - \mu| \geq \varepsilon, \; n \geq n_s\} = \Pr\{|\bar{X}_m - \mu| \geq \varepsilon, \; n \geq n_s\} \leq \Pr\{|\bar{X}_m - \mu| \geq \varepsilon\} \leq \Pr\left\{\sqrt{m}|\bar{X}_m - \mu| \geq \varepsilon \times \frac{\hat{\sigma}_s (t_{n_s - 1, \delta})}{\varepsilon}\right\} \\
= \Pr\left\{\frac{\sqrt{m}|\bar{X}_m - \mu|}{\hat{\sigma}_s} \geq t_{n_s - 1, \delta}\right\}. \tag{174}
\]

Note that \( \sqrt{m}(\bar{X}_m - \mu)/\sigma \) is a standard Gaussian variable and that \( \sqrt{m}(\bar{X}_m - \mu)/\sigma \) is independent of \( \hat{\sigma}_s \) because

\[
\Pr\left\{\frac{\sqrt{m}(\bar{X}_m - \mu)}{\sigma} \leq u\right\} = \sum_{m=n_s}^{\infty} \Pr\left\{\frac{\sqrt{m}(\bar{X}_m - \mu)}{\sigma} \leq u, \; m = m\right\} \\
= \sum_{m=n_s}^{\infty} \Pr\left\{\frac{\sqrt{m}(\bar{X}_m - \mu)}{\sigma} \leq u\right\} \Pr\{m = m\} = \sum_{m=n_s}^{\infty} \Phi(u) \Pr\{m = m\} = \Phi(u)
\]

and

\[
\Pr\left\{\frac{\sqrt{m}(\bar{X}_m - \mu)}{\sigma} \leq u, \; \hat{\sigma}_s \leq v\right\} = \sum_{m=n_s}^{\infty} \Pr\left\{\frac{\sqrt{m}(\bar{X}_m - \mu)}{\sigma} \leq u, \; m = m, \; \hat{\sigma}_s \leq v\right\} \\
= \sum_{m=n_s}^{\infty} \Pr\left\{\frac{\sqrt{m}(\bar{X}_m - \mu)}{\sigma} \leq u\right\} \Pr\{m = m, \; \hat{\sigma}_s \leq v\} \\
= \sum_{m=n_s}^{\infty} \Phi(u) \Pr\{m = m, \; \hat{\sigma}_s \leq v\} = \Phi(u) \Pr\{\hat{\sigma}_s \leq v\} \\
= \Pr\{\sqrt{m}(\bar{X}_m - \mu)/\sigma \leq u\} \Pr\{\hat{\sigma}_s \leq v\}
\]

for any \( u \) and \( v \). Therefore, \( \sqrt{m}(\bar{X}_m - \mu)/\hat{\sigma}_s \) has a Student \( t \)-distribution of \( n_s - 1 \) degrees of freedom. It follows from (174) that

\[
\Pr\{|\bar{X}_n - \mu| \geq \varepsilon, \; n \geq n_s\} \leq 2\zeta\delta. \tag{175}
\]

By the definition of the sampling scheme, we have \( \{n = n_\ell\} \subset \left\{ \varepsilon \geq \frac{\hat{\sigma}_\ell (t_{n_\ell - 1, \delta})}{\sqrt{n_\ell}} \right\} \) and thus

\[
\Pr\{|\bar{X}_n - \mu| \geq \varepsilon, \; n = n_\ell\} \leq \Pr\{|\bar{X}_{n_\ell} - \mu| \geq \varepsilon \geq \frac{\hat{\sigma}_\ell (t_{n_\ell - 1, \delta})}{\sqrt{n_\ell}}\} \leq \Pr\left\{\frac{\sqrt{n_\ell} |\bar{X}_{n_\ell} - \mu|}{\hat{\sigma}_\ell} \geq t_{n_\ell - 1, \delta}\right\} = 2\zeta\delta \tag{176}
\]

182
for $\ell = 1, \ldots, s - 1$. Combining (175) and (176) yields

$$\Pr\{\hat{\mu} - \mu \geq \varepsilon\} = \Pr\{|X_n - \mu| \geq \varepsilon, \, n \geq n_s\} + \sum_{\ell=1}^{s-1} \Pr\{|X_n - \mu| \geq \varepsilon, \, n = n_\ell\} \leq 2s\zeta \delta, \quad (177)$$

which implies that $\Pr\{\hat{\mu} - \mu < \varepsilon\} > 1 - 2s\zeta \delta$ for any $\mu$ and $\sigma$. This proves statement (I).

Second, we shall show statement (II) which asserts that $\lim_{\varepsilon \to 0} \Pr\{\hat{\mu} - \mu < \varepsilon\} = 1 - 2\zeta \delta$. Obviously, $\lim_{\varepsilon \to 0} \Pr\{n < n_s\} = 0$. Hence, $\lim_{\varepsilon \to 0} \sum_{\ell=1}^{s-1} \Pr\{|X_n - \mu| \geq \varepsilon, \, n = n_\ell\} = 0$ and

$$\Pr\{\hat{\mu} - \mu \geq \varepsilon\} = \Pr\{|X_n - \mu| \geq \varepsilon, \, n \geq n_s\}$$

$$+ \sum_{\ell=1}^{s-1} \Pr\{|X_n - \mu| \geq \varepsilon, \, n = n_\ell\} \to \Pr\{|X_n - \mu| \geq \varepsilon, \, n \geq n_s\} \quad (178)$$

as $\varepsilon \to 0$. By virtue of (175) and (178), we have $\limsup_{\varepsilon \to 0} \Pr\{\hat{\mu} - \mu < \varepsilon\} \leq 2\zeta \delta$, which implies that

$$\liminf_{\varepsilon \to 0} \Pr\{\hat{\mu} - \mu \geq \varepsilon\} \geq 1 - 2\zeta \delta. \quad (179)$$

On the other hand, by (178) and the fact that $\lim_{\varepsilon \to 0} \Pr\{n \geq n_s\} = 1$, we have

$$\Pr\{\hat{\mu} - \mu < \varepsilon\} \to \Pr\{|X_n - \mu| < \varepsilon, \, n \geq n_s\} = \Pr\{|X_m - \mu| < \varepsilon, \, n \geq n_s\}$$

$$\to \Pr\{|X_m - \mu| < \varepsilon\}$$

$$\leq \Pr\left\{\frac{|X_m - \mu|}{\sqrt{m}} < (1 + \eta) t_{n_s-1, \zeta \delta} \right\} + \Pr\left\{\frac{(1 + \eta) \tilde{\sigma} s t_{n_s-1, \zeta \delta}}{\sqrt{m}} < \varepsilon \right\}$$

as $\varepsilon \to 0$, where $\eta$ is a positive number. Noting that

$$\Pr\left\{\frac{(1 + \eta) \tilde{\sigma} s t_{n_s-1, \zeta \delta}}{\sqrt{m}} < \varepsilon \right\} \leq \Pr\left\{\frac{(1 + \eta) \tilde{\sigma} s t_{n_s-1, \zeta \delta}}{\sqrt{\tilde{\sigma} s t_{n_s-1, \zeta \delta}}} < \varepsilon \right\} = \Pr\left\{\tilde{\sigma} s^2 < \frac{n_s \varepsilon^2}{\eta(2 + \eta)(t_{n_s-1, \zeta \delta})^2} \right\}$$

which tends to 0 as $\varepsilon \to 0$, we have

$$\limsup_{\varepsilon \to 0} \Pr\{|\hat{\mu} - \mu| < \varepsilon\} \leq \Pr\left\{\frac{\sqrt{m}|X_m - \mu|}{\tilde{\sigma} s} < (1 + \eta) t_{n_s-1, \zeta \delta} \right\}.$$ 

Since the above argument holds for arbitrarily small $\eta > 0$, we have

$$\limsup_{\varepsilon \to 0} \Pr\{|\hat{\mu} - \mu| < \varepsilon\} \leq \Pr\left\{\frac{\sqrt{m}|X_m - \mu|}{\tilde{\sigma} s} \leq t_{n_s-1, \zeta \delta} \right\} = 1 - 2\zeta \delta. \quad (180)$$

Combing (179) and (180) yields $\lim_{\varepsilon \to 0} \Pr\{|\hat{\mu} - \mu| < \varepsilon\} = 1 - 2\zeta \delta$. This proves statement (II).

Finally, statements (III) and (IV) can be shown by making use of the observation that $n \leq (\tilde{\sigma} s t_{n_s-1, \zeta \delta})^2/\varepsilon^2 + n_s$. This completes the proof of Theorem 53.
K.2 Proof of Theorem 55

K.2.1 Proof of Statement (I)

Define Helmert transform

\[ U_i = \frac{X_i - \mu}{\sigma}, \quad V_i = \frac{U_1 + \cdots + U_i - iU_{i+1}}{\sqrt{i(i+1)}}, \quad W_i = \frac{U_1 + \cdots + U_i}{\sqrt{i}} \] (181)

for \( i = 1, 2, \ldots, \infty \). Clearly, the \( U_i \) are independent Gaussian variables with zero mean and variance unity. Since the transformation from \((U_1, \cdots, U_i)\) to \((V_1, \cdots, V_{i-1}, W_i)\) is orthogonal for any \( i \geq 2 \), the \( V_i \) are independent Gaussian variables with zero mean and variance unity. It is easily seen that \( \sqrt{n}(X_n - \mu)/\sigma = W_n \) and \( S_n = \sigma^2(\sum_{i=1}^n U_i^2 - W_n^2) = \sigma^2(V_1^2 + \cdots + V_{n-1}^2) \) for \( n = 2, 3, \ldots, \infty \). Hence, by the definition of the sampling scheme, we have that \( \{|X_n - \mu| \geq \varepsilon\} \) is independent of \( \{n = n\} \) for any \( n \in \mathcal{V} \). It follows from such independency and the definition of the sampling scheme that

\[ \Pr(|\bar{\mu} - \mu| \geq \varepsilon) = \sum_{n \in \mathcal{V}} \Pr(|\bar{\mu} - \mu| \geq \varepsilon, n = n) = \sum_{n \in \mathcal{V}} \Pr(|X_n - \mu| \geq \varepsilon, n = n), \]

This proves statement (I).

K.2.2 Proof of Statement (II)

Define \( Z_j = \frac{V_{2j-1} + V_{2j}}{2} \) for \( j = 1, 2, \ldots, \infty \), where \( V_i \) are defined in (181). It is easy to see that \( Z_j \) are identical and independent exponential random variables with density \( e^{-z} \). By the definition of \( \bar{\sigma}_\ell \), we have \( \bar{\sigma}_\ell = \sqrt{\frac{S_{2\ell} + 1}{2k_\ell}} = \sigma \sqrt{\sum_{i=1}^{k_\ell} Z_{i\ell}}/k_\ell \), \( \ell = 1, \ldots, s \) and thus

\[ \left\{ \frac{(\bar{\sigma}_\ell t_{n\ell - 1, \ell})^2}{\varepsilon^2} > n\ell \right\} = \left\{ \sum_{j=1}^{k_\ell} Z_j > b_\ell \right\}, \quad \ell = 1, \ldots, s \] (182)

\[ \left\{ \frac{(\bar{\sigma}_s t_{n_s - 1, s})^2}{\varepsilon^2} > n \right\} = \left\{ \sum_{j=1}^{k_s} Z_j > c \right\}, \quad n \geq n_s. \] (183)

It follows from (182) and the definition of the stopping rule that

\[ \{n > n\ell\} = \left\{ \sum_{j=1}^{k_\ell} Z_j > b_\ell \quad \text{for} \ 1 \leq i \leq \ell \right\} \] (184)

for \( \ell = 1, \ldots, s \). Making use of (184) and Theorem 54 we have

\[ \Pr\{n > n\ell\} = H_\ell(\sigma) \] (185)

for \( \ell = 1, \ldots, s \). Similarly, it follows from (183) and the definition of the stopping rule that

\[ \{n > n\} = \left\{ \sum_{j=1}^{k_s} Z_j > c, \sum_{j=1}^{k_\ell} Z_j > b_\ell \quad \text{for} \ 1 \leq \ell < s \right\} \] (186)

184
for \( n \geq n_s \). Making use of (186) and Theorem 5.1, we have
\[
\Pr\{ n > n \} = H^*(\sigma, n)  \tag{187}
\]
for \( n \geq n_s \). By virtue of (185), we have \( \Pr\{ n = n_1 \} = 1 - \Pr\{ n > n_1 \} = H_0(\sigma) - H_1(\sigma) \) and \( \Pr\{ n = n_\ell \} = \Pr\{ n > n_{\ell-1} \} - \Pr\{ n > n_\ell \} = H_{\ell-1}(\sigma) - H_\ell(\sigma) \) for \( 1 < \ell \leq s \). In a similar manner, using (187), we have \( \Pr\{ n = n \} = \Pr\{ n > n - 1 \} - \Pr\{ n > n \} = H^*(\sigma, n - 1) - H^*(\sigma, n) \) for \( n > n_s \). This completes the proof of statement (II).

**K.2.3 Proof of Statement (III)**

By the established statement (I), we have
\[
\Pr\{ |\hat{\mu} - \mu| \geq \varepsilon \} = 2 \sum_{n \leq m \leq n \leq m} \left[ 1 - \Phi \left( \frac{\varepsilon \sqrt{n}}{\sigma} \right) \right] \Pr\{ n = n \} + 2 \sum_{n > m} \left[ 1 - \Phi \left( \frac{\varepsilon \sqrt{n}}{\sigma} \right) \right] \Pr\{ n = n \}.  \tag{188}
\]

Note that
\[
\sum_{n > m} \left[ 1 - \Phi \left( \frac{\varepsilon \sqrt{n}}{\sigma} \right) \right] \Pr\{ n = n \} \leq \sum_{n > m} \left[ 1 - \Phi \left( \frac{\varepsilon \sqrt{m}}{\sigma} \right) \right] \Pr\{ n = m \} = \left[ 1 - \Phi \left( \frac{\varepsilon \sqrt{m}}{\sigma} \right) \right] \Pr\{ n > m \}
\]

\[
\leq \left[ 1 - \Phi \left( \frac{\varepsilon \sqrt{m}}{\sigma} \right) \right] \Pr\left\{ \frac{\chi^2_{n_{\ell-1} > m(n_s - 1)\varepsilon^2}^{a_\ell > m(n_s - 1)\varepsilon^2}}{(m(n_s - 1)\varepsilon^2)^2} \right\}
\]

\[
= \left[ 1 - \Phi \left( \frac{\varepsilon \sqrt{m}}{\sigma} \right) \right] S_{\ell} - 1, \frac{m(n_s - 1)\varepsilon^2}{(m(n_s - 1)\varepsilon^2)^2} \tag{189}
\]

for any \( \sigma \in [a, b] \), where \( \chi^2_{n_{\ell-1} > m(n_s - 1)\varepsilon^2} \) represents a chi-square random variable of \( n_s - 1 \) degrees of freedom. Observing that \( H_\ell(\sigma) \) is monotonically increasing with respect to \( \sigma \in [a, b] \) for \( 0 \leq \ell \leq s \) and that \( H^*(\sigma, n) \) is monotonically increasing with respect to \( \sigma \in [a, b] \) for \( n \geq n_s \), we have \( \mathcal{P}_n \leq \Pr\{ n = n \} \leq \mathcal{P}_n \) for \( \sigma \in [a, b] \). Therefore,
\[
\sum_{n \leq m} \left[ 1 - \Phi \left( \frac{\varepsilon \sqrt{n}}{a} \right) \right] \mathcal{P}_n \leq \sum_{n \leq m} \left[ 1 - \Phi \left( \frac{\varepsilon \sqrt{n}}{\sigma} \right) \right] \Pr\{ n = n \} \leq \sum_{n \leq m} \left[ 1 - \Phi \left( \frac{\varepsilon \sqrt{m}}{b} \right) \right] \mathcal{P}_n \tag{190}
\]

for \( \sigma \in [a, b] \). So, statement (III) follows from (188), (189) and (190).

**K.2.4 Proof of Statement (IV)**

Applying (185) and (187), we have
\[
\mathbb{E}[n] = n_1 + \sum_{\ell=1}^{\ell=1} (n_{\ell+1} - n_\ell) \Pr\{ n > n_\ell \} + \sum_{n=n_s}^{\infty} \Pr\{ n > n \}
\]

\[
= n_1 + \sum_{\ell=1}^{\ell=1} (n_{\ell+1} - n_\ell) H_\ell(\sigma) + \sum_{n=n_s}^{\infty} H^*(\sigma, n) \tag{191}
\]

185
and
\[ E[n] = n_1 + \sum_{\ell=1}^{s-1} (n_{\ell+1} - n_\ell) H_\ell(\sigma) + \sum_{n=n_s}^{m} H^*(\sigma, n) + \sum_{n=m+1}^{\infty} \Pr\{n > n\}. \]  

(192)

Note that
\[ \Pr\{n > n\} < \Pr\left\{ n^2 > \frac{n(n-1)e^2}{(\sigma n_s-1, \zeta)^2} \right\} = \Pr\left\{ n^2 > (n-1)n\gamma \right\} < \left[ n\gamma e^{-(n\gamma-1)} \right]^\nu \]

for \( n \geq m \), where the last inequality can be deduced from Chernoff bounds. Therefore,
\[ \sum_{n=m+1}^{\infty} \Pr\{n > n\} < \frac{e^\nu}{\gamma} \sum_{n=m+1}^{\infty} g(n\gamma) \gamma, \]

where we have introduced function \( g(x) = (xe^{-x})^\nu \) for simplicity of notations. Note that \( g(x) \) is monotonically decreasing with respect to \( x \) greater than 1 because \( g'(x) = v g(x) \left( \frac{1}{x} - 1 \right) < 0 \) for \( x > 1 \). Making use of the assumption that \( n\gamma \geq m\gamma > 1 \) and the monotone decreasing property of \( g(x) \), we have
\[ \sum_{n=m+1}^{\infty} g(n\gamma) \gamma < \int_{m\gamma}^{\infty} g(x) \, dx = \frac{v!}{\nu v+1} \int_{m\gamma}^{\infty} \frac{\lambda^\nu e^{-\lambda}}{\nu!} \, d\lambda, \]

where
\[ \int_{m\gamma}^{\infty} \frac{\lambda^\nu e^{-\lambda}}{\nu!} \, d\lambda = e^{-m\gamma} \sum_{i=0}^{\nu} \frac{(m\gamma)^i}{i!} = \Pr\{K \leq \nu\} \]
\[ < \inf_{h>0} e^{h\nu} \mathbb{E}[e^{-hK}] = e^{-m\gamma} \left( \frac{m\gamma e}{\nu} \right) = e^{-m\gamma} (m\gamma e)^\nu \]

with \( K \) representing a Poisson random variable with mean \( m\nu\gamma \). It follows that
\[ \sum_{n=m+1}^{\infty} \Pr\{n > n\} < \frac{e^{\nu}v!}{\nu \nu+1} e^{-m\gamma} (m\gamma e)^\nu = \frac{v!}{\nu} \left( \frac{m\gamma}{\nu} \right)^\nu e^{-(m\gamma-2)\nu}. \]

Using inequality \( v! < \sqrt{2\pi} \nu^\nu e^{-\nu+1/24} \), we have
\[ \sum_{n=m+1}^{\infty} \Pr\{n > n\} < \frac{1}{\nu} \sqrt{2\pi} \nu^\nu e^{-\nu+1/24} \left( \frac{m\gamma}{\nu} \right)^\nu e^{-(m\gamma-2)\nu} = \frac{1}{\nu} \sqrt{2\pi} \left( m\gamma \right)^\nu e^{-(m\gamma-1)\nu+1/24} < \frac{3(m\gamma e)^\nu}{\gamma \nu^\nu e^{m\gamma\nu}}. \]

(193)

So, the proof of statement (IV) can be completed by combining (191), (192) and (193).

**K.3 Proof of Theorem 56**

By (175) and (177), we have
\[ \Pr\{\hat{\mu} - \mu > \varepsilon\} \leq 2\zeta\delta + \sum_{\ell=1}^{s-1} \Pr\{\bar{X}_n - \mu > \varepsilon, \, n = n_\ell\}. \]

(194)
By the definition of the sampling scheme, we have

\[
\sum_{\ell=1}^{s-1} \Pr\{|X_n - \mu| \geq \varepsilon, \ n = n_\ell\} \leq \sum_{\ell=1}^{s-1} \Pr\left\{S_{n_\ell} \leq \frac{n_\ell(n_\ell - 1)\varepsilon^2}{t_{n_\ell-1,\zeta\delta}^2} \right\} = \sum_{\ell=1}^{s-1} \Pr\left\{\chi_{n_\ell-1}^2 \leq \frac{n_\ell(n_\ell - 1)\varepsilon^2}{(\sigma t_{n_\ell-1,\zeta\delta})^2} \right\}
\]

\[
= \sum_{\ell=1}^{s-1} \left[1 - S_P \left(k_\ell - 1, \frac{n_\ell k_\ell \varepsilon^2}{(\sigma t_{n_\ell-1,\zeta\delta})^2} \right)\right]
\]

and

\[
\sum_{\ell=1}^{s-1} \Pr\{|X_n - \mu| \geq \varepsilon, \ n = n_\ell\}
\]

\[
\leq \Pr\{|X_{n_1} - \mu| \geq \varepsilon\} + \sum_{\ell=1}^{s-2} \Pr\left\{|X_{n_{\ell+1}} - \mu| \geq \varepsilon, \ S_{n_\ell} > \frac{n_\ell(n_\ell - 1)\varepsilon^2}{t_{n_\ell-1,\zeta\delta}} \right\}
\]

\[
= \Pr\{|X_{n_1} - \mu| \geq \varepsilon\} + \sum_{\ell=1}^{s-2} \Pr\{|X_{n_{\ell+1}} - \mu| \geq \varepsilon\} \Pr\left\{S_{n_\ell} > \frac{n_\ell(n_\ell - 1)\varepsilon^2}{t_{n_\ell-1,\zeta\delta}} \right\} \Pr\left\{\chi_{n_\ell-1}^2 > \frac{n_\ell(n_\ell - 1)\varepsilon^2}{(\sigma t_{n_\ell-1,\zeta\delta})^2} \right\}
\]

\[
= 2 \left[1 - \Phi \left(\frac{\varepsilon \sqrt{n_1}}{\sigma} \right)\right] + 2 \sum_{\ell=1}^{s-2} \left[1 - \Phi \left(\frac{\varepsilon \sqrt{n_{\ell+1}}}{\sigma} \right)\right] S_P \left(k_\ell - 1, \frac{n_\ell k_\ell \varepsilon^2}{(\sigma t_{n_\ell-1,\zeta\delta})^2} \right) \Pr\left\{\chi_{n_\ell-1}^2 > \frac{n_\ell(n_\ell - 1)\varepsilon^2}{(\sigma t_{n_\ell-1,\zeta\delta})^2} \right\}
\]

Combining (194) and (195) yields

\[
\Pr\{|\bar{\mu} - \mu| \geq \varepsilon\} \leq 2\zeta\delta + \sum_{\ell=1}^{s-1} \left[1 - S_P \left(k_\ell - 1, \frac{n_\ell k_\ell \varepsilon^2}{(\sigma t_{n_\ell-1,\zeta\delta})^2} \right)\right], \quad (197)
\]

where the upper bound in the right side of (197) monotonically decreases from \(s - 1 + 2\zeta\delta\) to \(2\zeta\delta\) as \(\sigma\) increases from 0 to \(\infty\). Since \(0 < \zeta < \frac{1}{2}\), there exists a unique number \(\overline{\sigma}\) such that

\[
\sum_{\ell=1}^{s-1} \left[1 - S_P \left(k_\ell - 1, \frac{n_\ell k_\ell \varepsilon^2}{(\sigma t_{n_\ell-1,\zeta\delta})^2} \right)\right] = (1 - 2\zeta)\delta
\]

and that \(\Pr\{|\bar{\mu} - \mu| \geq \varepsilon\} < \delta\) for \(\sigma > \overline{\sigma}\). On the other hand, combining (194) and (196) yields

\[
\Pr\{|\bar{\mu} - \mu| \geq \varepsilon\} \leq 2\zeta\delta + 2 \left[1 - \Phi \left(\frac{\varepsilon \sqrt{n_1}}{\sigma} \right)\right] + 2 \sum_{\ell=1}^{s-2} \left[1 - \Phi \left(\frac{\varepsilon \sqrt{n_{\ell+1}}}{\sigma} \right)\right] S_P \left(k_\ell - 1, \frac{n_\ell k_\ell \varepsilon^2}{(\sigma t_{n_\ell-1,\zeta\delta})^2} \right) \Pr\left\{\chi_{n_\ell-1}^2 > \frac{n_\ell(n_\ell - 1)\varepsilon^2}{(\sigma t_{n_\ell-1,\zeta\delta})^2} \right\}, \quad (198)
\]

where the upper bound in the right side of (198) monotonically increases from \(2\zeta\delta\) to \(s - 1 + 2\zeta\delta\) as \(\sigma\) increases from 0 to \(\infty\). Since \(0 < \zeta < \frac{1}{2}\), there exists a unique number \(\underline{\sigma}\) such that

\[
1 - \Phi \left(\frac{\varepsilon \sqrt{n_1}}{\sigma} \right) + \sum_{\ell=1}^{s-2} \left[1 - \Phi \left(\frac{\varepsilon \sqrt{n_{\ell+1}}}{\sigma} \right)\right] S_P \left(k_\ell - 1, \frac{n_\ell k_\ell \varepsilon^2}{(\sigma t_{n_\ell-1,\zeta\delta})^2} \right) = \left(\frac{1}{2} - \zeta\right)\delta
\]

and that \(\Pr\{|\bar{\mu} - \mu| \geq \varepsilon\} < \delta\) for \(\sigma < \underline{\sigma}\). This completes the proof of Theorem 56.
K.4 Proof of Theorem 57

By the definition of the stopping rule, we have

$$
\Pr \{|\hat{\mu} - \mu| > \varepsilon |\mu|\} \leq \sum_{\ell=1}^{\infty} \Pr \left\{ |\hat{\mu}_\ell - \mu| > \varepsilon |\mu|, |\hat{\mu}_\ell| \geq \frac{t_{n\ell-1}}{\sqrt{n\ell}} \frac{\kappa \delta}{\sqrt{\eta}} \left(1 + \frac{1}{\varepsilon}\right) \hat{\sigma}_\ell \right\}.
$$

By virtue of identity (1), we have

$$
\Pr \left\{ |\hat{\mu}_\ell - \mu| > \varepsilon |\mu|, |\hat{\mu}_\ell| \geq \frac{t_{n\ell-1}}{\sqrt{n\ell}} \frac{\kappa \delta}{\sqrt{\eta}} \left(1 + \frac{1}{\varepsilon}\right) \hat{\sigma}_\ell \right\} = \Pr \left\{ \mu < \frac{\hat{\mu}_\ell}{1 - \text{sgn}(\hat{\mu}_\ell) \varepsilon}, |\hat{\mu}_\ell| \geq \frac{t_{n\ell-1}}{\sqrt{n\ell}} \frac{\kappa \delta}{\sqrt{\eta}} \left(1 + \frac{1}{\varepsilon}\right) \hat{\sigma}_\ell \right\}
$$

$$
+ \Pr \left\{ \mu > \frac{\hat{\mu}_\ell}{1 + \text{sgn}(\hat{\mu}_\ell) \varepsilon}, |\hat{\mu}_\ell| \geq \frac{t_{n\ell-1}}{\sqrt{n\ell}} \frac{\kappa \delta}{\sqrt{\eta}} \left(1 + \frac{1}{\varepsilon}\right) \hat{\sigma}_\ell \right\}
$$

$$
\leq \Pr \left\{ |\hat{\mu}_\ell - \mu| > \frac{\varepsilon |\hat{\mu}_\ell|}{1 + \varepsilon}, |\hat{\mu}_\ell| \geq \frac{t_{n\ell-1}}{\sqrt{n\ell}} \frac{\kappa \delta}{\sqrt{\eta}} \hat{\sigma}_\ell \right\}
$$

$$
\leq \Pr \left\{ \frac{|\hat{\mu}_\ell - \mu|}{\sigma_\ell} > \frac{t_{n\ell-1}}{\sqrt{n\ell}} \frac{\kappa \delta}{\hat{\sigma}_\ell} \right\} = 2 \kappa \delta
$$

for all $\ell > 0$. Therefore, $\Pr \{|\hat{\mu} - \mu| > \varepsilon |\mu|\} \leq 2 \sum_{\ell=1}^{\infty} \kappa \delta \ell = 2(\tau + 1) \kappa \delta$.

The finite stopping property of the sampling scheme can be shown by an argument similar to the proof of statement (I) of Theorem 23.

K.5 Proof of Theorem 58

By the definition of the stopping rule, we have

$$
\Pr \{|\hat{\mu} - \mu| > \max(\varepsilon_a, \varepsilon_r |\mu|)\} \leq \sum_{\ell=1}^{\infty} \Pr \left\{ |\hat{\mu}_\ell - \mu| > \max(\varepsilon_a, \varepsilon_r |\mu|), \max \left( \frac{\varepsilon_a}{1 + \varepsilon_r} |\hat{\mu}_\ell|, \frac{\varepsilon_r |\hat{\mu}_\ell|}{1 + \varepsilon_r} \right) \geq \frac{t_{n\ell-1}}{\sqrt{n\ell}} \frac{\kappa \delta}{\hat{\sigma}_\ell} \right\}.
$$
By virtue of identity (11), we have

\[
\Pr \left\{ |\hat{\mu}_t - \mu| > \max(\varepsilon_a, \varepsilon_r |\mu|), \max \left( \varepsilon_a, \frac{\varepsilon_r |\hat{\mu}_t|}{1 + \varepsilon_r} \right) \geq \frac{t_{n_t - 1, \zeta \delta}}{\sqrt{n_t}} \sigma_t \right\} = \Pr \left\{ \mu < \min \left( \hat{\mu}_t - \varepsilon_a, \frac{\hat{\mu}_t}{1 + \text{sgn}(\hat{\mu}_t) \varepsilon_r} \right), \max \left( \varepsilon_a, \frac{\varepsilon_r |\hat{\mu}_t|}{1 + \varepsilon_r} \right) \geq \frac{t_{n_t - 1, \zeta \delta}}{\sqrt{n_t}} \sigma_t \right\} + \Pr \left\{ \mu \geq \max \left( \hat{\mu}_t + \varepsilon_a, \frac{\hat{\mu}_t}{1 - \text{sgn}(\hat{\mu}_t) \varepsilon_r} \right), \max \left( \varepsilon_a, \frac{\varepsilon_r |\hat{\mu}_t|}{1 + \varepsilon_r} \right) \geq \frac{t_{n_t - 1, \zeta \delta}}{\sqrt{n_t}} \sigma_t \right\} = \Pr \left\{ \hat{\mu}_t - \mu \geq \max \left( \varepsilon_a, \frac{\varepsilon_r |\hat{\mu}_t|}{1 + \varepsilon_r} \right), \max \left( \varepsilon_a, \frac{\varepsilon_r |\hat{\mu}_t|}{1 + \varepsilon_r} \right) \geq \frac{t_{n_t - 1, \zeta \delta}}{\sqrt{n_t}} \sigma_t \right\}
\]

which implies that \( |\hat{\mu}_t - \mu| \) is a sure event and consequently

\[
\Pr \left\{ |\hat{\mu}_t - \mu| > \max(\varepsilon_a, \varepsilon_r |\mu|) \right\} \leq 2 \sum_{t=1}^{\infty} \zeta \delta_t = 2(\tau + 1) \zeta \delta.
\]

The finite stopping property of the sampling scheme can be shown by an argument similar to the proof of statement (I) of Theorem 23.

L  Proofs of Theorems for Estimation Following Tests

L.1  Proof of Theorem 59

Since \( \hat{\theta} \) is a ULE of \( \theta \), by virtue of Lemma 34 we have that \( \Pr \{ \hat{\theta} \leq z \mid \theta \} \) is non-increasing with respect to \( \theta \) no less than \( z \) and that \( \Pr \{ \hat{\theta} > z \mid \theta \} \) is non-decreasing with respect to \( \theta \) no greater than \( z \). This implies that \( \Pr \{ \hat{\theta} \leq z \mid \theta \} \) is non-increasing with respect to \( \theta \in \Theta \) and that \( \Pr \{ \hat{\theta} \geq z \mid \theta \} \) is non-decreasing with respect to \( \theta \in \Theta \). By the definitions of \( F_{\hat{\theta}}(z, \theta) \) and \( G_{\hat{\theta}}(z, \theta) \) given in Section 2.5, we have that \( F_{\hat{\theta}}(z, \theta) \) is non-increasing with respect to \( \theta \in \Theta \) and that \( G_{\hat{\theta}}(z, \theta) \) is non-decreasing with respect to \( \theta \in \Theta \). Recalling the definition of \( \mathcal{V}(\hat{\theta}, n) \), we have that \( \{ F_{\hat{\theta}}(\hat{\theta}, \mathcal{V}(\hat{\theta}, n)) \leq \frac{\delta}{2} \} \) is a sure event and consequently

\[
\{ \theta \geq \mathcal{V}(\hat{\theta}, n) \} = \left\{ \theta \geq \mathcal{V}(\hat{\theta}, n), F_{\hat{\theta}}(\hat{\theta}, \mathcal{V}(\hat{\theta}, n)) \leq \frac{\delta}{2} \right\} \subseteq \left\{ \theta \geq \mathcal{V}(\hat{\theta}, n), F_{\hat{\theta}}(\hat{\theta}, \theta) \leq \frac{\delta}{2} \right\} \subseteq \left\{ F_{\hat{\theta}}(\hat{\theta}, \theta) \leq \frac{\delta}{2} \right\},
\]

which implies that

\[
\Pr \{ \theta \geq \mathcal{V}(\hat{\theta}, n) \} \leq \Pr \left\{ F_{\hat{\theta}}(\hat{\theta}, \theta) \leq \frac{\delta}{2} \right\} \leq \frac{\delta}{2},
\]

where the last inequality follows from Lemma 27. On the other hand, recalling the definition of \( \mathcal{L}(\hat{\theta}, n) \), we have that \( \{ G_{\hat{\theta}}(\hat{\theta}, \mathcal{L}(\hat{\theta}, n)) \leq \frac{\delta}{2} \} \) is a sure event and consequently

\[
\{ \theta \leq \mathcal{L}(\hat{\theta}, n) \} = \left\{ \theta \leq \mathcal{L}(\hat{\theta}, n), G_{\hat{\theta}}(\hat{\theta}, \mathcal{L}(\hat{\theta}, n)) \leq \frac{\delta}{2} \right\} \subseteq \left\{ \theta \leq \mathcal{L}(\hat{\theta}, n), G_{\hat{\theta}}(\hat{\theta}, \theta) \leq \frac{\delta}{2} \right\} \subseteq \left\{ G_{\hat{\theta}}(\hat{\theta}, \theta) \leq \frac{\delta}{2} \right\},
\]

which implies that

\[
\Pr \{ \theta \leq \mathcal{L}(\hat{\theta}, n) \} \leq \Pr \left\{ G_{\hat{\theta}}(\hat{\theta}, \theta) \leq \frac{\delta}{2} \right\} \leq \frac{\delta}{2},
\]

189
where the last inequality follows from Lemma 2. Finally, by virtue of (199) and (200), we have
\[
\Pr\{\mathcal{L}(\hat{\theta}, n) < \theta < \mathcal{U}(\hat{\theta}, n) \mid \theta\} \geq 1 - \Pr\{\theta \geq \mathcal{U}(\hat{\theta}, n)\} - \Pr\{\theta \leq \mathcal{L}(\hat{\theta}, n)\} \geq 1 - \frac{\delta}{2} - \frac{\delta}{2} = 1 - \delta.
\]
This completes the proof of the theorem.

L.2 Proof of Theorem 60

Since \(\hat{p}\) is a ULE of \(p \in \Theta\), by Lemma 3, we can show that that \(\Pr\{\hat{p} \leq z \mid p\}\) is non-increasing with respect to \(p \in \Theta\) and that \(\Pr\{\hat{p} \geq z \mid p\}\) is non-decreasing with respect to \(p \in \Theta\). Define cumulative distribution functions
\[
F_\hat{p}(z, p) = \begin{cases} 
\Pr\{\hat{p} \leq z \mid p\} & \text{for } p \in \Theta, \\
1 & \text{for } p < 0, \\
0 & \text{for } p > 1 
\end{cases}
\]
\[
G_\hat{p}(z, p) = \begin{cases} 
\Pr\{\hat{p} \geq z \mid p\} & \text{for } p \in \Theta, \\
0 & \text{for } p < 0, \\
1 & \text{for } p > 1 
\end{cases}
\]
where \(z\) assumes values from the support of \(\hat{p}\). Then, \(F_\hat{p}(z, p)\) is non-increasing with respect to \(p \in \Theta\) and that \(G_\hat{p}(z, p)\) is non-decreasing with respect to \(p \in \Theta\). Recalling the definition of \(\mathcal{U}(\hat{p}, n)\), we have that \(\{F_\hat{p}(\hat{p}, \mathcal{U}(\hat{p}, n) + \frac{1}{N}) \leq \frac{\delta}{2}\}\) is a sure event and consequently \(\{p > \mathcal{U}(\hat{p}, n)\} = \{p \geq \mathcal{U}(\hat{p}, n) + \frac{1}{N}, F_\hat{p}(\hat{p}, \mathcal{U}(\hat{p}, n) + \frac{1}{N}) \leq \frac{\delta}{2}\} \subseteq \{F_\hat{p}(\hat{p}, p) \leq \frac{\delta}{2}\}\), which implies that
\[
\Pr\{p > \mathcal{U}(\hat{p}, n)\} \leq \Pr\left\{F_\hat{p}(\hat{p}, p) \leq \frac{\delta}{2}\right\} \leq \frac{\delta}{2},
\]
(201)
where the last inequality follows from Lemma 2. On the other hand, recalling the definition of \(\mathcal{L}(\hat{p}, n)\), we have that \(\{G_\hat{p}(\hat{p}, \mathcal{L}(\hat{p}, n) - \frac{1}{N}) \leq \frac{\delta}{2}\}\) is a sure event and consequently \(\{p < \mathcal{L}(\hat{p}, n)\} = \{p \leq \mathcal{L}(\hat{p}, n) - \frac{1}{N}, G_\hat{p}(\hat{p}, \mathcal{L}(\hat{p}, n) - \frac{1}{N}) \leq \frac{\delta}{2}\} \subseteq \{G_\hat{p}(\hat{p}, p) \leq \frac{\delta}{2}\}\), which implies that
\[
\Pr\{p < \mathcal{L}(\hat{p}, n)\} \leq \Pr\left\{G_\hat{p}(\hat{p}, p) \leq \frac{\delta}{2}\right\} \leq \frac{\delta}{2},
\]
(202)
where the last inequality follows from Lemma 2. Finally, by virtue of (201) and (202), we have
\[
\Pr\{\mathcal{L}(\hat{\lambda}_\ell, n_\ell) \leq p < \mathcal{U}(\hat{\lambda}_\ell, n_\ell) \mid p\} \geq 1 - \Pr\{p > \mathcal{U}(\hat{\lambda}_\ell, n_\ell)\} - \Pr\{p < \mathcal{L}(\hat{\lambda}_\ell, n_\ell)\} \geq 1 - \frac{\delta}{2} - \frac{\delta}{2} = 1 - \delta.
\]
This completes the proof of the theorem.

L.3 Proof of Theorem 61

Note that
\[
\Pr\{\mathcal{L}(\hat{\lambda}_\ell, n_\ell) < \lambda < \mathcal{U}(\hat{\lambda}_\ell, n_\ell) \mid \lambda\} 
\geq \Pr\left\{\mathcal{L}(\hat{\lambda}_\ell, n_\ell) < \lambda < \mathcal{U}(\hat{\lambda}_\ell, n_\ell), U\left(\hat{\lambda}_\ell, n_\ell, \frac{\delta}{2s}\right) > \lambda^* \mid \lambda\right\} 
\geq \Pr\left\{L\left(\hat{\lambda}_\ell, n_\ell, \frac{\delta}{2s}\right) < \lambda < U\left(\hat{\lambda}_\ell, n_\ell, \frac{\delta}{2s}\right) \mid \lambda\right\} 
\geq 1 - \frac{\delta}{2s}
\]
for any \( \lambda \in [\lambda^*, \infty) \). Therefore,

\[
\Pr \{ \lambda \notin (L(\hat{\lambda}_t, n_t), W(\hat{\lambda}_t, n_t)) \} = \Pr \{ \lambda \notin (L(\hat{\lambda}_t, n_t), W(\hat{\lambda}_t, n_t)) | \lambda \} \leq \delta
\]

for \( \ell = 1, \cdots, s \) and any \( \lambda \in [\lambda^*, \infty) \). It follows that

\[
\Pr \{ \lambda \notin (L(\hat{\lambda}, n), W(\hat{\lambda}, n)) | \lambda \} = \sum_{\ell=1}^{s} \Pr \{ \lambda \notin (L(\hat{\lambda}_t, n_t), W(\hat{\lambda}_t, n_t)) , t = \ell | \lambda \} \leq \delta
\]

for any \( \lambda \in [\lambda^*, \infty) \). The theorem immediately follows.

**M Proof of Theorem 64**

Note that

\[
\Pr \{ L(\hat{\lambda}_t, n_t) < \lambda < W(\hat{\lambda}_t, n_t) | \lambda \} \geq \Pr \{ L(\hat{\lambda}_t, n_t) < \lambda < W(\hat{\lambda}_t, n_t), U(\hat{\lambda}_t, n_t, \delta/2s) > \lambda^* | \lambda \} = \Pr \{ L(\hat{\lambda}_t, n_t, \delta/2s) < \lambda < U(\hat{\lambda}_t, n_t, \delta/2s), U(\hat{\lambda}_t, n_t, \delta/2s) > \lambda^* | \lambda \} \geq 1 - \frac{\delta}{2s}
\]

for any \( \lambda \in [\lambda^*, \infty) \). The theorem immediately follows.

**N Proofs of Theorems for Multistage Linear Regression**

**N.1 Proof of Theorem 65**

By the definition of the stopping rule,

\[
\Pr \{ |\hat{\beta}_i - \beta_i| > \varepsilon_i \} \leq \sum_{\ell=1}^{\infty} \Pr \left\{ \frac{|B_{i,\ell} - \beta_i|}{\sigma_{\ell}} > \varepsilon_i \geq t_{n_{\ell} - m, \zeta_{\ell}} \sigma_{\ell} \sqrt{\left[(X_{\ell}^\top X_{\ell})^{-1}\right]_{ii}} \right\}
\]

\[
\leq \sum_{\ell=1}^{\infty} \Pr \left\{ \frac{|B_{i,\ell} - \beta_i|}{\sigma_{\ell}} > t_{n_{\ell} - m, \zeta_{\ell}} \sigma_{\ell} \sqrt{\left[(X_{\ell}^\top X_{\ell})^{-1}\right]_{ii}} \right\} \tag{203}
\]

for \( i = 1, \cdots, m \). From the classical theory of linear regression, we know that \( B_{i,\ell} - \beta_i \) is a Gaussian random variable of zero mean, variance \( \sigma^2 \left[(X_{\ell}^\top X_{\ell})^{-1}\right]_{ii} \) and that \( (n_{\ell} - m)(\tilde{\sigma}_{\ell}^{2}) \) is a chi-square variable of \( n_{\ell} - m \) degrees of freedom. Moreover, \( B_{i,\ell} - \beta_i \) is independent of \( (n_{\ell} - m)(\tilde{\sigma}_{\ell}^{2}) \). It follows that \( (B_{i,\ell} - \beta_i) \left( \tilde{\sigma}_{\ell} \sqrt{\left[(X_{\ell}^\top X_{\ell})^{-1}\right]_{ii}} \right)^{-1} \) possesses a Student t-distribution of \( n_{\ell} - m \) degrees of freedom. Hence, by (203), we have

\[
\Pr \{ |\hat{\beta}_i - \beta_i| > \varepsilon_i \} \leq 2 \sum_{\ell=1}^{\infty} \zeta_{\ell} = 2(\tau + 1)\zeta \delta \tag{204}
\]

191
for \( i = 1, \ldots, m \). By the definition of the stopping rule,

\[
\Pr\{|\hat{\sigma} - \sigma| > \epsilon\} \leq \sum_{\ell=1}^{\infty} \Pr\left\{ \hat{\sigma}_\ell - \sigma > \epsilon, \sqrt{\frac{n_{\ell} - m}{\chi^2_{n_{\ell} - m, \zeta \delta_{\ell}}} \hat{\sigma}_\ell - \epsilon} \leq \sqrt{\frac{n_{\ell} - m}{\chi^2_{n_{\ell} - m, 1 - \zeta \delta_{\ell}}} \hat{\sigma}_\ell + \epsilon} \right\}
\]

\[
\leq \sum_{\ell=1}^{\infty} \Pr\left\{ \hat{\sigma}_\ell - \sigma < -\epsilon, \sqrt{\frac{n_{\ell} - m}{\chi^2_{n_{\ell} - m, \zeta \delta_{\ell}}} \hat{\sigma}_\ell - \epsilon} \leq \sqrt{\frac{n_{\ell} - m}{\chi^2_{n_{\ell} - m, 1 - \zeta \delta_{\ell}}} \hat{\sigma}_\ell + \epsilon} \right\}
\]

\[
+ \sum_{\ell=1}^{\infty} \Pr\left\{ \hat{\sigma}_\ell - \sigma > \epsilon, \sqrt{\frac{n_{\ell} - m}{\chi^2_{n_{\ell} - m, \zeta \delta_{\ell}}} \hat{\sigma}_\ell - \epsilon} \leq \sqrt{\frac{n_{\ell} - m}{\chi^2_{n_{\ell} - m, 1 - \zeta \delta_{\ell}}} \hat{\sigma}_\ell + \epsilon} \right\}
\]

\[
\leq \sum_{\ell=1}^{\infty} \Pr\left\{ \sqrt{\frac{n_{\ell} - m}{\chi^2_{n_{\ell} - m, \zeta \delta_{\ell}}} \hat{\sigma}_\ell < \sigma} \right\} + \sum_{\ell=1}^{\infty} \Pr\left\{ \sqrt{\frac{n_{\ell} - m}{\chi^2_{n_{\ell} - m, 1 - \zeta \delta_{\ell}}} \hat{\sigma}_\ell > \sigma} \right\}. \quad (205)
\]

Recalling that \((n_{\ell} - m)(\hat{\sigma}_\ell)^2\) is a chi-square variable of \(n_{\ell} - m\) degrees of freedom, we have

\[
\Pr\left\{ \sqrt{\frac{n_{\ell} - m}{\chi^2_{n_{\ell} - m, \zeta \delta_{\ell}}} \hat{\sigma}_\ell < \sigma} \right\} \leq \zeta \delta_{\ell}, \quad \Pr\left\{ \sqrt{\frac{n_{\ell} - m}{\chi^2_{n_{\ell} - m, 1 - \zeta \delta_{\ell}}} \hat{\sigma}_\ell > \sigma} \right\} \leq \zeta \delta_{\ell} \quad (206)
\]

for all \( \ell > 0 \). Combining (205) and (206) yields

\[
\Pr\{|\hat{\sigma} - \sigma| > \epsilon\} \leq 2 \sum_{\ell=1}^{\infty} \zeta \delta_{\ell} = 2(\tau + 1)\zeta \delta. \quad (207)
\]

By virtue of (204) and (207), we have

\[
\Pr\{|\hat{\sigma} - \sigma| \leq \epsilon, |\hat{\beta}_i - \beta_i| \leq \epsilon_i \text{ for } i = 1, \ldots, m\} \geq 1 - \sum_{i=1}^{m} \Pr\{|\hat{\beta}_i - \beta_i| > \epsilon_i\} - \Pr\{|\hat{\sigma} - \sigma| > \epsilon\}
\]

\[
\geq 1 - 2m(\tau + 1)\zeta \delta - 2(\tau + 1)\zeta \delta = 1 - 2(m + 1)(\tau + 1)\zeta \delta.
\]

The finite stopping property of the sampling scheme can be shown by an argument similar to the proof of statement (I) of Theorem \[23\]. This completes the proof of the theorem.

**N.2 Proof of Theorem 66**

By the definition of the stopping rule,

\[
\Pr\{|\hat{\beta}_i - \beta_i| > \epsilon_i|\beta_i|\} \leq \sum_{\ell=1}^{\infty} \Pr\left\{ |B_{i,\ell} - \beta_i| > \epsilon_i|\beta_i|, t_{n_{\ell} - m, \zeta \delta_{\ell}} \hat{\sigma}_\ell \sqrt{[X_{\ell}^T X_{\ell}]^{-1}}_{ii} \leq \frac{\epsilon_i|B_{i,\ell}|}{1 + \epsilon_i} \right\} \quad (208)
\]
for $i = 1, \ldots, m$. By identity (1), we have

\[
\Pr \left\{ \frac{1}{\sqrt{n}} \left| B_{i, \ell} - \beta_i \right| > \epsilon_i |\beta_i|, \ t_{n_{-m}, \zeta \delta} \frac{\hat{\sigma}_\ell}{\sqrt{\sum_{i=1}^{n} (X_i^\top X_i)^{-1}}_{ii}} \leq \frac{\epsilon_i}{1 + \epsilon_i} |\beta_i| \right\}
\]

\[
= \Pr \left\{ \beta_i < \frac{B_{i, \ell}}{1 + \text{sgn}(B_{i, \ell}) \epsilon_i}, \ t_{n_{-m}, \zeta \delta} \frac{\hat{\sigma}_\ell}{\sqrt{\sum_{i=1}^{n} (X_i^\top X_i)^{-1}}_{ii}} \leq \frac{\epsilon_i}{1 + \epsilon_i} |\beta_i| \right\}
\]

\[
+ \Pr \left\{ \beta_i > \frac{B_{i, \ell}}{1 - \text{sgn}(B_{i, \ell}) \epsilon_i}, \ t_{n_{-m}, \zeta \delta} \frac{\hat{\sigma}_\ell}{\sqrt{\sum_{i=1}^{n} (X_i^\top X_i)^{-1}}_{ii}} \leq \frac{\epsilon_i}{1 + \epsilon_i} |\beta_i| \right\}
\]

\[
\leq \Pr \left\{ \frac{B_{i, \ell} - \beta_i}{\frac{\hat{\sigma}_\ell}{\sqrt{\sum_{i=1}^{n} (X_i^\top X_i)^{-1}}_{ii}}} > t_{n_{-m}, \zeta \delta} \right\}
\]

\[
\leq \frac{2 \zeta \delta}{\sqrt{n}}
\]

(209)

for $i = 1, \ldots, m$, where the last equality (209) follows from the fact that $(B_{i, \ell} - \beta_i) \frac{\hat{\sigma}_\ell}{\sqrt{\sum_{i=1}^{n} (X_i^\top X_i)^{-1}}_{ii}} \sim \chi^2_{n_{-m}}$ possesses a Student $t$-distribution of $n_{-m}$ degrees of freedom. Combining (208) and (209) yields

\[
\Pr \{ |\hat{\beta}_i - \beta_i| > \epsilon_i |\beta_i| \} \leq 2 \sum_{\ell=1}^{\infty} \zeta \delta
\]

(210)

for $i = 1, \ldots, m$. By the definition of the stopping rule,

\[
\Pr \{ |\hat{\sigma} - \sigma| > \epsilon \sigma \} \leq \sum_{\ell=1}^{\infty} \Pr \left\{ |\hat{\sigma}_\ell - \sigma| > \epsilon \sigma, \frac{\lambda_{n_{-m}, 1-\frac{\zeta \delta}{\epsilon}}}{(1 + \epsilon)^2} \leq n_{-m} \leq \frac{\lambda_{n_{-m}, \zeta \delta}}{(1 - \epsilon)^2} \right\}
\]

\[
\leq \sum_{\ell=1}^{\infty} \left[ \Pr \left\{ |\hat{\sigma}_\ell - (1 - \epsilon) \sigma| \leq \frac{\lambda_{n_{-m}, \zeta \delta}}{n_{-m}} \right\} + \Pr \left\{ |\hat{\sigma}_\ell| > (1 + \epsilon) \sigma \geq \frac{\lambda_{n_{-m}, 1-\zeta \delta}}{n_{-m}} \right\} \right]
\]

\[
\leq \sum_{\ell=1}^{\infty} \left[ \frac{n_{-m}}{\lambda_{n_{-m}, \zeta \delta}} \hat{\sigma}_\ell < \sigma \right\} + \Pr \left\{ |\hat{\sigma}_\ell| > \sigma \right\}
\]

\[
= 2(\tau + 1) \zeta \delta
\]

(211)

where (211) follows from an argument similar to that of (206). Making use of (210) and (211), we have

\[
\Pr \{ |\hat{\sigma} - \sigma| \leq \epsilon \sigma, \ |\hat{\beta}_i - \beta_i| \leq \epsilon_i |\beta_i| \text{ for } i = 1, \ldots, m \}
\]

\[
\geq 1 - \sum_{i=1}^{m} \Pr \{ |\hat{\beta}_i - \beta_i| > \epsilon_i |\beta_i| \} - \Pr \{ |\hat{\sigma} - \sigma| > \epsilon \sigma \}
\]

\[
\geq 1 - 2m(\tau + 1) \zeta \delta - 2(\tau + 1) \zeta \delta = 1 - 2(m + 1)(\tau + 1) \zeta \delta.
\]

The finite stopping property of the sampling scheme can be shown by an argument similar to the proof of statement (I) of Theorem 23. This completes the proof of the theorem.
O Proofs of Theorems for Estimation of Quantile

O.1 Proof of Theorem 67

By the definition of the stopping rule,

\[
\Pr\{\hat{\xi}_p - \xi_p > \varepsilon\} \leq \sum_{\ell=1}^{\infty} \Pr\{\hat{\xi}_{p,\ell} - \xi_p > \varepsilon, X_{j_{\ell}:n_{\ell}} - \varepsilon \leq \hat{\xi}_{p,\ell} \leq X_{i_{\ell}:n_{\ell}} + \varepsilon\},
\]

(212)

where

\[
\Pr\{\hat{\xi}_{p,\ell} - \xi_p > \varepsilon, X_{j_{\ell}:n_{\ell}} - \varepsilon \leq \hat{\xi}_{p,\ell} \leq X_{i_{\ell}:n_{\ell}} + \varepsilon\} \\
\leq \Pr\{\xi_p < \hat{\xi}_{p,\ell} - \varepsilon \leq X_{i_{\ell}:n_{\ell}}\} + \Pr\{\xi_p > \hat{\xi}_{p,\ell} + \varepsilon \geq X_{j_{\ell}:n_{\ell}}\} \\
\leq \Pr\{X_{i_{\ell}:n_{\ell}} > \xi_p\} + \Pr\{X_{j_{\ell}:n_{\ell}} < \xi_p\}
\]

(213)

for all \(\ell > 0\).

Now, let \(K_{\ell}\) denote the number of samples among \(X_1, \ldots, X_{n_{\ell}}\) which are no greater than \(\xi_p\). Then, \(\{X_{i_{\ell}:n_{\ell}} > \xi_p\} \subseteq \{K_{\ell} < i_{\ell}\}\) and thus \(\Pr\{X_{i_{\ell}:n_{\ell}} > \xi_p\} \leq \Pr\{K_{\ell} < i_{\ell}\} = \sum_{k=0}^{i_{\ell}-1} \binom{n_{\ell}}{k} p^k (1-p)^{n_{\ell}-k} \leq \zeta\delta_{i_{\ell}},\)

(214)

where the last inequality follows from the definition of \(i_{\ell}\). On the other hand, let \(K_{\ell}^*\) denote the number of samples among \(X_1, \ldots, X_{n_{\ell}}\) which are smaller than \(\xi_p\). Then, \(\{X_{j_{\ell}:n_{\ell}} < \xi_p\} \subseteq \{K_{\ell}^* \geq j_{\ell}\}\) and thus \(\Pr\{X_{j_{\ell}:n_{\ell}} < \xi_p\} \leq \Pr\{K_{\ell}^* \geq j_{\ell}\} = \sum_{k=j_{\ell}}^{n_{\ell}} \binom{n_{\ell}}{k} F_X(\xi_p)^k (1 - F_X(\xi_p))^{n_{\ell}-k},\)

(215)

where \(F_X(\xi_p) = \Pr\{X < \xi_p\}\). By the definition of \(\xi_p\), we have \(F_X(\xi_p) \leq p\). Making use of the fact that \(\sum_{k=m}^{n} \binom{n}{k} \theta^k (1 - \theta)^{n-k}\) is monotonically decreasing with respect to \(\theta \in (0, 1)\), we have that

\[
\Pr\{X_{j_{\ell}:n_{\ell}} < \xi_p\} \leq \sum_{k=j_{\ell}}^{n_{\ell}} \binom{n_{\ell}}{k} p^k (1-p)^{n_{\ell}-k} \leq \zeta\delta_{j_{\ell}},
\]

(216)

where the last inequality follows from the definition of \(j_{\ell}\). Combining (212), (213), (214) and (215) yields \(\Pr\{\hat{\xi}_p - \xi_p > \varepsilon\} \leq 2 \sum_{\ell=1}^{\infty} \zeta\delta_{\ell} = 2(\tau + 1)\zeta\delta\). The finite stopping property of the sampling scheme can be shown by an argument similar to the proof of statement (I) of Theorem 23.

O.2 Proof of Theorem 68

By the definition of the stopping rule,

\[
\Pr\{\hat{\xi}_p - \xi_p > \varepsilon|\xi_p|\} \leq \sum_{\ell=1}^{\infty} \Pr\{\hat{\xi}_{p,\ell} - \xi_p > \varepsilon|\xi_p|, |1 - \text{sgn}(\hat{\xi}_{p,\ell})\varepsilon|X_{j_{\ell}:n_{\ell}} \leq \hat{\xi}_{p,\ell} \leq |1 + \text{sgn}(\hat{\xi}_{p,\ell})\varepsilon|X_{i_{\ell}:n_{\ell}}\},
\]

(216)
By identity (1), we have
\[
\Pr \left\{ \hat{\xi}_{p,\ell} - \xi_p > \varepsilon | \xi_p | \mid X_{j_i:n} \leq \hat{\xi}_{p,\ell} \leq \left[ 1 + \operatorname{sgn}(\hat{\xi}_{p,\ell}) \varepsilon \right] X_{i_i:n} \right\}
\leq \Pr \left\{ \xi_p < \frac{\hat{\xi}_{p,\ell}}{1 + \operatorname{sgn}(\xi_p) \varepsilon}, \mid 1 - \operatorname{sgn}(\hat{\xi}_{p,\ell}) \varepsilon \mid X_{j_i:n} \leq \hat{\xi}_{p,\ell} \leq \left[ 1 + \operatorname{sgn}(\xi_p) \varepsilon \right] X_{i_i:n} \right\}
+ \Pr \left\{ \xi_p > \frac{\hat{\xi}_{p,\ell}}{1 - \operatorname{sgn}(\xi_p) \varepsilon}, \mid 1 - \operatorname{sgn}(\hat{\xi}_{p,\ell}) \varepsilon \mid X_{j_i:n} \leq \hat{\xi}_{p,\ell} \leq \left[ 1 + \operatorname{sgn}(\xi_p) \varepsilon \right] X_{i_i:n} \right\}
\leq \Pr \left\{ \xi_p < \frac{\hat{\xi}_{p,\ell}}{1 + \operatorname{sgn}(\xi_p) \varepsilon} \leq X_{i_i:n} \right\} + \Pr \left\{ \xi_p > \frac{\hat{\xi}_{p,\ell}}{1 - \operatorname{sgn}(\xi_p) \varepsilon} \leq X_{j_i:n} \right\}
\leq \Pr \left\{ X_{i_i:n} > \xi_p \right\} + \Pr \left\{ X_{j_i:n} < \xi_p \right\}
\] (217)
for all \( \ell > 0 \). Combining (214), (215), (216) and (217) yields
\[
\Pr \left\{ \hat{\xi}_p - \xi_p > \varepsilon | \xi_p | \right\} \leq 2 \sum_{\ell=1}^{\infty} 1 \cdot \delta = 2(\tau + 1)\delta.
\]
The finite stopping property of the sampling scheme can be shown by an argument similar to the proof of statement (I) of Theorem 23.

O.3 Proof of Theorem 69

By the definition of the stopping rule and identity (1), we have
\[
\Pr \left\{ \hat{\xi}_p - \xi_p > \max(\varepsilon_a, \varepsilon_r | \xi_p |) \right\}
\leq \sum_{\ell=1}^{\infty} \Pr \left\{ \hat{\xi}_{p,\ell} - \xi_p > \max(\varepsilon_a, \varepsilon_r | \xi_p |), \mid X_{j_i:n} \leq \hat{\xi}_{p,\ell} \leq X_{i_i:n} + \max(\varepsilon_a, \operatorname{sgn}(\hat{\xi}_{p,\ell}) \varepsilon \mid X_{i_i:n} \right\}
\leq \sum_{\ell=1}^{\infty} \Pr \left\{ \xi_p < \min \left( \hat{\xi}_{p,\ell} - \varepsilon_a, \frac{\hat{\xi}_{p,\ell}}{1 + \operatorname{sgn}(\hat{\xi}_{p,\ell}) \varepsilon} \right), \hat{\xi}_{p,\ell} \leq X_{i_i:n} + \max(\varepsilon_a, \operatorname{sgn}(\hat{\xi}_{p,\ell}) \varepsilon \mid X_{i_i:n} \right\}
+ \sum_{\ell=1}^{\infty} \Pr \left\{ \xi_p > \max \left( \hat{\xi}_{p,\ell} + \varepsilon_a, \frac{\hat{\xi}_{p,\ell}}{1 - \operatorname{sgn}(\hat{\xi}_{p,\ell}) \varepsilon} \right), X_{j_i:n} - \max(\varepsilon_a, \operatorname{sgn}(\hat{\xi}_{p,\ell}) \varepsilon \mid X_{j_i:n} \right\} \leq \hat{\xi}_{p,\ell} \right\}
= \sum_{\ell=1}^{\infty} \Pr \left\{ \xi_p < \min \left( \hat{\xi}_{p,\ell} - \varepsilon_a, \frac{\hat{\xi}_{p,\ell}}{1 + \operatorname{sgn}(\hat{\xi}_{p,\ell}) \varepsilon} \right) \leq X_{i_i:n} \right\}
+ \sum_{\ell=1}^{\infty} \Pr \left\{ \xi_p > \max \left( \hat{\xi}_{p,\ell} + \varepsilon_a, \frac{\hat{\xi}_{p,\ell}}{1 - \operatorname{sgn}(\hat{\xi}_{p,\ell}) \varepsilon} \right) \geq X_{j_i:n} \right\}
\leq \sum_{\ell=1}^{\infty} \Pr \left\{ X_{i_i:n} > \xi_p \right\} + \sum_{\ell=1}^{\infty} \Pr \left\{ X_{j_i:n} < \xi_p \right\} \leq 2(\tau + 1)\delta,
\]
where the last inequality follows from (214) and (215). The finite stopping property of the sampling scheme can be shown by an argument similar to the proof of statement (I) of Theorem 23.
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