New Functional Representation for Hubbard Model, Coherent States and Tower of Algebras

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Abstract

New functional representation for the strongly interacting systems is proposed which contains a new type of the coherent state. As a result the new algebraic structure- so called “tower of algebras” appears which gives the tower (or hierarchy) of models.

1 Functional approach.

Consider Hubbard model (lattice version of Thirring model) [1]:

\[ H = -t \sum_{<r,r'>} \alpha_{r\sigma}^+ \alpha_{r'\sigma} + U \sum_r n_{r\uparrow} n_{r\downarrow} = H_{\text{band}} = \]

\[ \sum_r U X_r^{22} + \sum_{A,C,r,r'} t_{AC}(r - r') X_r^{-A} X_{r'}^C = H_{\text{atomic}} \]

where \(<r,r'>\)-denote the sum over the nearest neighbors. Electrons are described by \((\alpha_{r\sigma}^+, \alpha_{r\sigma})\) and live in two dimensional plane \(r = \{x, y\}\) (for example square lattice), \(\sigma\) - spin of the electrons. We give two different forms of this model (the first form is a standard, the second contains the Hubbard operators \(X^A\) (in fact they are projectors \(X^A_{ij} = |ir><jr|\)). These operators act in space of following states: \(0 >, |+>,|->,|2>\), \(A = (i,j); i,j = 0,+,-,2\) (the space of eigenfunctions of Hubbard repulsion-the main and most complicated interaction in many interesting models). \(X^{ij}\) has one nonzero element, sitting in \(i\) row and \(j\) column of \(4 \times 4\) matrix. Take \(H_0 = n_\uparrow n_\downarrow\) as zero field. Then eigenfunctions and eigenvalues are:

\[
\begin{align*}
|0> & \quad E_0 = 0 \\
|+> = a^+_\uparrow |0> & \quad E_+ = -\mu \\
|-> = a^-_\downarrow |0> & \quad E_- = -\mu \\
|2> = a^+_\uparrow a^+_\downarrow |0> & \quad E_2 = U - 2\mu
\end{align*}
\]

here \(\mu\)-is chemical potential. Operators \(a^+_\sigma, a_\sigma\) in this base are:

\[ a^+_\sigma = X^{\sigma,0} + \sigma X^{2,-\sigma} \quad a_\sigma = X^{0,\sigma} + \sigma X^{-\sigma,2} \]

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Local state of system and the Hilbert space of (2) we describe by supercoherent state: \( |G(\chi(r,t'),E(r,t'),h(r,t'))\rangle = E_{\rho}(-iE_{\rho} - i\mathbf{h}s - i\chi a) |0\rangle \) where \( \chi = \{\chi_1, \chi_2\} \) are dynamical two components odd valued grassmann fields, \( E = \{E_1, E_2, E_3\} \) and \( h = \{h_1, h_2, h_3\} \) are three components even valued dynamical fields. \( \chi \) describes electronic degrees of freedoms, \( E \) and \( h \) parametrize charge density \( \rho \) and spin density \( s \) fluctuations consequently. According to this formulation the many particles system are described by following effective functional (2):

\[
L_{\text{eff}} = \frac{\langle G(\mathbf{r}, t') | \left( \frac{\partial}{\partial \mathbf{r}} - H \right) | G(\mathbf{r}, t') \rangle}{\langle G(\mathbf{r}, t') | G(\mathbf{r}, t') \rangle}
\]  

where \( |G\rangle \) -is the supercoherent state, which is expressed through generators of the dynamical superalgebra, \( \{\mathbf{r}, t'\} \) -coordinates of space-time. \( H \) is the Hubbard hamiltonian expressed through the infinite dimensional superalgebra generators \( \{X_r^\alpha\} \) (atomic representation): where \( U \) - on-site coulomb repulsion, \( t_{\alpha,\beta}(\mathbf{r} - \mathbf{r}') \) - "interaction" arisen from kinetic energy. \( |G\rangle \) can be constructed by the following expression:

\[
|G\rangle = \exp \left[ \begin{array}{ccc}
E_z & 0 & 0 \\
\chi_1 & h_z & h^+ \\
\chi_2 & h^- & 0 \\
E^- & -\chi_1 & \chi_2 & -E_z \\
\end{array}\right] |0\rangle = (F\chi, Z(E) + 2B\chi_2\chi_1)
\]

\((h_1, h_2, h_3, E_1, E_2, E_3, \chi_1, \chi_2)\) is the set of bose and fermi fields.

\[
F = a'' + a' E_z + h \sigma^P (a' + a E_z), \quad B = \left(\{0, a\} + \rho(\psi) z(\tau) E^+\right)
\]

where the prime means differentiation with respect to \( \delta, \sigma^P \) -Pauli matrix.

\[
z(\tau) = \{\cos(\tau) + \cos(\theta') \sin(\tau), e^{i\phi'} \sin(\tau) \sin(\theta')\}, \tau = \arcsin\left(\frac{E\psi}{\rho}\right),
\]

\[
\rho = \sqrt{E^2\psi^2 + \psi'^2}, \quad \psi = \delta^5 \frac{\partial f}{\partial E^2}, a = \frac{\partial^2 f}{E^2 \partial^2 \rho} \]

\[
f = -h^2 + \frac{E^2 \left(\frac{\sin(E\delta)}{E^3} - \frac{\sin(h\delta)}{h^3}\right)}{E^2 - h^2}
\]

We use the spherical coordinate system \( \{E, \theta', \phi'\} \) for \( \mathbf{E} \). We put \( \delta = 1 \) in (2) after calculation. The partition function of the system now can be written as:

\[
Z_{\text{hub}} = \int D\mathbf{r}^* D\mathbf{r} D\mathbf{E} D\mathbf{h} e^{-i \int d^2r dt' L(\mathbf{E}, \mathbf{h}, \chi)}
\]

Introduce so called QP-derivative: \( D_{QP}K(x) = (K(Px) - K(Qx))/((P - Q)x) \). Using this derivative one can represent:

\[
K(x) = ch(\sqrt{\alpha})/\alpha; \quad D_{E^2, \chi^2} K(\alpha) = \left(ch(\sqrt{\alpha}E)/(\alpha E^2) - ch(\sqrt{\alpha}h)/(\alpha h^2)\right)/(\alpha(\alpha^2 - h^2)) = f(\sqrt{\alpha})/\alpha^2
\]
It follows here from that all expressions are given as action of two-parameters deformed derivative. As a result we have the appearance of quantum algebra. Our main task will be the investigation of these deformed algebraic structure. Consider several possibility: a) \( E_k = 0, h_k = 0 \), then

\[
| G > = \begin{pmatrix}
1 \\
\chi_1 \\
\chi_2 \\
2\chi_2\chi_1
\end{pmatrix}
\]

this expression coincide with fermionic CS; b) \( \chi = 0, h = 0 \) then \( | G > = Z(E) \) and density operators give us \( \rho_3 \Rightarrow z_0^*z_0 - z_2^*z_2; \rho^- \Rightarrow z_0^*z_0, \rho^+ \Rightarrow z_0^*z_2 \). The set of \( \rho_1, \rho_2, \rho_3 \) gives some algebra. This is a CS for \( SU(2) \) if \( E = \pi/2, 3\pi/2, ... \). In this case \( \rho_3 \Rightarrow \cos(2\theta), \rho^- \Rightarrow e^{i\phi}\sin(2\theta), \rho^+ \Rightarrow e^{-i\phi}\sin(2\theta) \). \( E \) parameterises Casimir operator and gives the value of quasipin. In interval \( 0 < E < \pi/2 \) this is the quantum coherent state for \( SU_q(2) \). For spin operator we have \( SU_q(2) \)

\[
S_q^+ = h\sin h \left( \cos(h) \pm h\sin h \right); \quad S_q^- = \cos^2 h + \frac{\sin^2 h}{h^2} (h_2^2 - h^2 h^-)
\]

Evaluation of Poisson brackets gives us deformed \( SU(2) \) algebra. Thus CS gives two parameters deformed \( SU(2) \ast SU(2) \) (we ignore the difference between \( SU(2) \) and \( SU(1,1) \) for density subsystem), where \( h, E \) (magnetic moment value and density parameter (bandwidth)). Let’s evaluate the expressions for symbols:

\[
\begin{align*}
\rho_3 & \Rightarrow Z^*s_z Z + 2(Z^*s_z B)\chi_2\chi_1 + 2(B^*s_z Z)\chi_1^*\chi_2^* + 4(B^*s_z B)\chi^*2\chi^2 \\
\rho^- & \Rightarrow z_0^*z_0 + 2z_0^*B_2^*\chi_2\chi_1 + 2z_2^*B_0^*\chi_1\chi_2^* + 4B_0^*B_2^*\chi^*2\chi^2 \\
\rho^+ & \Rightarrow z_0^*z_2 + 2z_0^*B_2\chi_2\chi_1 + 2z_2^*B_0\chi_1\chi_2^* + 4B_0^*B_2\chi^*2\chi^2
\end{align*}
\]

Introduce for the spin symbol :\( S' = S/(a'^2 - a^2 E_0^2) \)

\[
\begin{align*}
S'_z & \Rightarrow Q^h_z s_z - h_z Im(m_0)\rho_3 + h^+(Re(m_0) - 2h_z)s^- - h^-(Re(m_0) + 2h_z)s^+ \\
S'^+ & \Rightarrow [-Im(m_0)h^+\rho_3 - h^+h_z s_z - h^+2s^- + (m_0^* + h_z)(m_0 + h_z)s^+] \\
S'^- & \Rightarrow [-Im(m_0)h^-\rho_3 - h^-h_z s_z - h^-2s^+ + (m_0^* - h_z)(m_0 - h_z)s^-]
\end{align*}
\]

here \( Q^h_z = m_0^*m_0 - h_z^2 + h^+h^- \). Symbols for fermionic operators: \( < g | a_+ | g > < g | g > = z^*_0(F\chi)_1 + z_2^*(F\chi)_2 \). Consider the following cases: \( E \to 0, h \to 0 \), thus

\[
F \to \begin{pmatrix}
F_{11} & 0 \\
0 & F_{22}
\end{pmatrix}
\]

and for symbol of creation-annihilation operators one can get:

\[
a_\sigma \to F_{\sigma\sigma} \chi_\sigma; \quad B \to (0, b);
\]

As a result we arrive to the canonical grassmann representation for fermionic systems: \( a_\sigma \to \chi_\sigma \). Let’s obtain the representation for spin. Taking mean value for the density operator one can obtain the expansion for spin:

\[
S^+ = (1 - \alpha)S^+_q + \alpha\chi_1^*\chi_2; \quad S^- = (1 - \alpha)S^-_q + \alpha\chi_2^*\chi_1; \quad S^z = (1 - \alpha)S^z_q + \alpha(\chi_1^*\chi_1 - \chi_2^*\chi_2);
\]

The set \( (\chi_1^*\chi_2, \chi_2^*\chi_1, \chi_1^*\chi_1 - \chi_2^*\chi_2) \) is a classical \( SU(2) \) algebra. This expansion indicates that localized moments are defined as the sum of two parts: classical and
quantum so that the sum of these two parts equal to unity, parameter $1 - \alpha$ is the order parameter of magnetic localization.

**TOWER OF ALGEBRAS**: Consider the expression for $a_\sigma \sim a \chi + b(E + h)\chi + cEh\chi = A_0 + A_1 + A_2$. It is evident that in enveloping algebra there possible only polynomials of three types $A_0, A_1, A_2$: 1) fermionic sector of $A_1$ contains 4 odd operators, 2) fermionic sector of $A_2$ contains 8 odd operators; 3) fermionic sector of $A_0$ contains 2 odd operators. Terms in expansion describe: metal - insulator - spinless noninteracting gas algebras of variables. Thus we have some expansion for $a_\sigma$ through bases of 3 different algebras. This expansion was called "tower of algebras" [3].

**METAL-INSULATOR TRANSITION**: Consider those deformations which do not change a metallic algebra: $a_{\uparrow} = z_1^*(F\chi)_1 - z_2^*(F\chi)_2$, $a_{\downarrow} = z_1^*(F\chi)_2 + z_2^*(F\chi)_1$; Saddle point approximation for energy gives us: $E_H = -(F_1^1F_{11} + F_{2}^2F_{22}) + b^2U$; Evaluation of this expression gives: $E_H = -(E^2 + h^2)[\sin^2E + \sin^2h - b^2] + b^2U$ . Expanding energy on the parameters $E << 1, h << 1$ we arrive to Gutzwiller approximation: $E_H = -\nu(1/2 - \nu) + 24\nu U$ here $\nu = E^2 + h^2 << 1$. We obtain the following interpretation for $\nu$ -it is a combination of two deformation parameters.

## 2 Operator approach

Let’s give the operator’s version of tower of algebras and tower of models. We start from Hubbard model in the atomic representation:

$$H = \sum_{r,p} E_p X_{rp} + \sum_{\alpha' \beta, \gamma' \gamma} t^{-\alpha' \beta}(r - r')X_r^{-\alpha'}X_{r'}^\beta$$

(8)

In atomic base $a_\sigma$ have following nonzero matrix elements: $<0 | a_{\uparrow} | + > <0 | a_{\uparrow} | + > = 1$. One can write the following identity:

$$Op_{\uparrow} = (1 - \alpha)a_{\uparrow} + \alpha(X^{0\uparrow} - X^{12})$$

(9)

here $Op_{\uparrow}$ -operators for which: $\{Op_{\uparrow}^+, Op_{\uparrow}\} = \delta_{\sigma,\sigma'}$ and $0 < \alpha < 1$.

Let’s take $a^+, a$ for which $\{a^+, a\} = 1$, for example: $(a \sim a_\sigma + a_{-\sigma})$. We can construct superalgebra: $A_0 = (a^+, a, n = a^+ a)$. Take $a, a^+$ and construct the following operators: $(\frac{\gamma_2}{2}(\gamma_2 + s_2) + \frac{\gamma_3}{2}(\gamma_2 - s_2) + (\gamma_2 + s_2)\frac{\gamma_3}{2})$, here $\gamma_3 = 1 - n; \gamma_2 = 1 - (\gamma_3)^2; s_z = diag(0,1,-1,0)$. It is seen that: 1) this set is algebra $A_1$; 2) the operators are polynomials on $a, a^+$. Number of fermionic generators equal 4. The third step: taking $\gamma^5 = diag(1,-1,-1,1)$ and $(a_{\uparrow}^+, a_{\sigma}, \rho, s)$ one can construct polynomials of $a, a^+$ (take chiral projection by multiplying $1 \pm \gamma^5$), which gives algebra $A_2 (X^{pq} \sim \gamma^5 a_{\sigma}; \gamma^5 \sim \rho^2 - s^2, thus X \sim a^5)$. The set of $A_0, A_1, A_2$ as linear space gives the universal enveloping of $A_0$. Let’s construct from $A_0, A_1, A_2$ a new set of variables:

$$a_{\sigma}' = (1 - \alpha)a_{\sigma}, X'^{pq} = \alpha X^{pq},$$

(10)

where $X'^{pq}$ -fermi operator, 

$$\{X'^{r+}, X'^{-+}, X'^{12}, X'^{00}, X'^{20}, X'^{00}, X'^{22}\}$$
here $X^{pq} = X^{pq}$.

(Anti)commutation relations for $(a', X')$ are:

$$\{a'_+, a'^+\} = (1 - \alpha)^2 \delta_{\alpha\sigma'} [X^{pq}, X^{pq}] = \alpha^2 X^{pq} [X'^{+}, X'^{-}] = X'^{++} - X'^{-+}$$

$$\{a'_+, X^{t+0}\} = \alpha(1 - \alpha)X'^{++}$$

Structural constants are the functions of $\alpha$. When $\alpha = 0$ we have $A_1$. When $\alpha = 1$, we have insulator algebra $A_2$. Varying $\alpha$ we can obtain the interpolation between metallic and insulator algebras. As a result we can describe two level (or places) of tower. Using these two level algebra one can obtain expression for hamiltonian:

$$H = (1 - \alpha)^2 H_{\text{band}} + \alpha^2 H_{\text{atomic}} + \alpha(1 - \alpha)H_{\text{band}}^{\text{atomic}} + \alpha(1 - \alpha)H_{\text{atomic}}^{\text{band}}$$

It is two-band model: the band $a_\sigma$ describes delocalized electrons, the second band $X^{pq}$ describes atomic states. Thus we obtain from the one-band model an effective two-band model. If $\alpha = 0$ it is weakly-interacting gas, when $\alpha = 1$ we have localized insulator. We have the possibility to describe the following hierarchy of model: spinless gas- fermi liquid - magnetic insulator - two-band model. From the point of view of functional integral we have the following chain (or tower) of dynamical algebras defining local bosonic variables: $U(1) \to U(2) \to U(4) \to U(8)$ . It is obvious that the last structure which is based on $U(8)$ local algebra is very rich from field theoretical point of view. Let's give physical view of functional integral for strongly interacting systems in which the tower of algebras appears. The stabilization of tower means that on the lattice scale we have quantum fluctuations between many ground states described by the tower of models (or the infinite dimensional algebra (quantum many phase state) which contains the infinite number of subalgebras of concrete model of definite ground state - metal, insulator, magnet, superconductor ). It is a high energy regime (the analog of UV regime in the field theory). At medium range of energy as a result of the spontaneous breaking of scale invariance definite dynamical algebra is stabilized and as a result a concrete type of the ground state become stable. In this region the local variables are described by the quantum groups. At the limit of low energy (IR regime in the field theory) we have scenario of the spontaneous breaking of usual (the classical) symmetry. As a result we have very rich structure of the functional integral (the structure of measure, renormalization group, effective action, the methods of evaluation ). This work in part is supported by RFFI under Grant N. 95-03-08287.

References

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