Polarization operator approach to pair creation in short laser pulses

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Abstract—We investigate the nonlinear Breit-Wheeler process inside short laser pulses, i.e. the creation of an electron-positron pair induced by a gamma photon inside a plane-wave background field. To obtain the total pair-creation probability we verify (to leading-order) the cutting rule for the polarization operator in the realm of strong-field QED by an explicit calculation. Furthermore, a double-integral representation for the leading-order contribution to the field-dependent part of the polarization operator is derived. The combination of both results yields a compact expression for the total pair-creation probability inside an arbitrary plane-wave background field. It is shown numerically that with presently available technology pair-creation probabilities of the order of ten percent could be reached for a single gamma photon.

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I. INTRODUCTION

In vacuum the decay of a single photon into a real electron-positron pair is forbidden by energy-momentum conservation, even if the energy $\hbar \omega_\gamma$ of the photon exceeds the threshold $2mc^2$ ($m$ denotes the electron mass). At least one additional interaction is needed to catalyze the process, e.g. a second photon (Breit-Wheeler process) or the Coulomb field of a nucleus (Bethe-Heitler pair creation).

Inside a strong electromagnetic background field the situation is changed, as the field can contribute four-momentum to the reaction. If a constant electric field reaches the critical field strength $E_{cr} = m^2 c^3 / (\hbar |e|) = 1.3 \times 10^{16}$ V/cm, even the spontaneous creation of electron-positron pairs from the vacuum becomes possible ($e < 0$ denotes the electron charge) [5, 8]. Spontaneous pair creation could also be observed if an X-ray laser is focused to its diffraction limit [9, 10].

However, present and near-future laser facilities will not reach the critical field strength ($10^{24} - 10^{25}$ W/cm$^2$) and the critical field corresponds to an intensity of $I_{cr} = \epsilon_0 E_{cr}^2 = 4.6 \times 10^{20}$ W/cm$^2$. Nevertheless, electron-positron pairs could be produced if the process is stimulated by an incoming photon. For example, in the E-144 experiment at SLAC electron-positron pair creation has been observed during the collision of an electron beam with a relativistically intense optical laser via the trident process [14–17].

In this paper we consider the nonlinear Breit-Wheeler process shown in Fig. 1, i.e. pair-creation by a single (on-shell) photon inside a strong (optical) laser pulse. By absorbing multiple low-energy laser photons, the vacuum-forbidden decay into an electron-positron pair becomes feasible. For monochromatic laser fields this process has been considered first in [18, 23]. In the strong-field regime $\xi \gg 1$, where in general many photons can be efficiently absorbed from the laser field, the probability depends non-trivially only on the quantum nonlinearity parameter $\chi$ and is exponentially suppressed for $\chi \ll 1$. Here $\xi = |e| E_0 / (m \omega_c)$ is a gauge and Lorentz invariant measure of the laser intensity [24] and for a head-on collision we obtain $\chi = (2 \hbar \omega_c / m c^3) (E_0 / E_{cr})$.

As existing optical petawatt laser systems reach already $\xi \sim 100$ [25] and GeV-photons are available using Compton backscattering [26, 31], the regime $\chi \gtrsim 1$ could be entered with available technology.

Due to the experimental progress concerning laser development during the last years, the nonlinear Breit-Wheeler process has been recently investigated by several authors [22, 59] (see also the reviews [8, 50]). To achieve the strong field strengths needed to observe the nonlinear Breit-Wheeler process, future experiments will probably use short laser pulses. However, the calculation of the total pair-creation probability is challenging if the phase-space integrals are calculated numerically for an arbitrary plane-wave field [31].

In the present paper we circumvent these difficulties by applying the optical theorem to the polarization operator (see Fig. 2) [51, 55]. In this manner we derive a double-
integral representation for the total pair-creation probability inside a plane-wave laser pulse (for pair creation in combined laser and Coulomb fields the same method was used in [57, 58], see also [51]). The analysis holds for an arbitrarily shaped background field, in particular we focus on the description of experiments with short (optical) laser pulses.

The optical theorem, which relates the total probability for particle production processes to the imaginary part of corresponding loop diagrams, reflects the unitarity of the S-matrix. It is therefore of central importance and has been widely investigated in the literature. On the level of individual Feynman diagrams unitarity leads to so-called “cutting” or “cutkosky rules” [59, 60]. In the derivation for cutting rules in vacuum must be modified inside background fields. Correspondingly, the standard ability inside a plane-wave laser pulse (for pair creation in combined laser and Coulomb fields the same method was used in [57, 58], see also [51]). The analysis holds in strong-field QED.

In the presence of electromagnetic background fields, however, the production of massive particles differs qualitatively from the production in vacuum. In the latter case the total energy must be provided by the incoming particles, which leads to the existence of a production threshold. Since the background field can contribute additional energy and momentum to the process, the production of massive particles is in general also allowed below threshold but exponentially suppressed at low energies and momentum to the process, the production in vacuum. In the latter case the total energy must be provided by the incoming particles, which leads to the existence of a production threshold. Since the background field can contribute additional energy and momentum to the process, the production of massive particles is in general also allowed below threshold but exponentially suppressed at low energies and momentum to the process.

From now on we use natural units $\hbar = c = 1$ and Heaviside-Lorentz units for charge $\alpha = e^2/(4\pi) \approx 1/137$ denotes the fine-structure constant], the notation agrees with [54].

II. OPTICAL THEOREM

A. Pair creation with background fields

The leading-order Feynman diagram for the creation of an electron and a positron with four-momenta $p^\mu$ and $q^\mu$, respectively, by a photon with four-momentum $q^\mu$ is shown in Fig. [1]. In vacuum this process is forbidden, as four-momentum conservation $p^\mu + q^\mu = 0$ cannot be fulfilled if all three particles are on-shell (i.e. $p^2 = q^2 = m^2$, $q^2 = 0$). However, inside a laser field additional laser photons with average four-momentum $k^\mu$ can be absorbed [see Eq. (A20)]

$$p^\mu + q^\mu = q^\mu + nk^\mu$$  (1)

and the process of stimulated pair creation by an incoming photon is feasible (as a non-monochromatic plane-wave laser pulse contains photons with different four-momenta, $n$ must not be an integer in general).

To determine the pair-creation probability we describe the incoming photon by a wave-packet [59, 63, 65]

$$|\Phi_q, \eta\rangle = \int \frac{d^3 q'}{(2\pi)^3 2\epsilon_{q'}} \eta(q') |\Phi_{q'}\rangle,$$  (2)

where $\epsilon_{q'} = \sqrt{q'^2}$. This has the advantage that we do not encounter squared delta functions, which are difficult to interpret in strong-field QED. In Eq. (2) we suppose that all components of the wave-packet have the same polarization (the polarization indices are suppressed) and are on-shell, i.e. $q'^2 = 0$. As we will later assume that the envelope-function $\eta(q')$ is sharply peaked around $q'^\mu = q^\mu$, we have used the label $q$ for the wave-packet state. In Eq. (2) $|\Phi_{q'}\rangle$ denotes a momentum eigenstate of the photon field with relativistic normalization

$$\langle \Phi_q |\Phi_{q'}\rangle = 2\epsilon_q (2\pi)^3 \delta^3 (q - q').$$  (3)

The wave-packet state describes a single particle $|\Phi_q, \eta\rangle|\Phi_q, \eta\rangle = 1$ if the envelope-function obeys the covariant normalization condition

$$\int \frac{d^3 q'}{(2\pi)^3 2\epsilon_{q'}} |\eta(q')|^2 = 1$$  (4)

(this is assumed in the following).

The probability that a single photon decays into an electron-positron pair inside a plane-wave background field is now given by

$$W = \sum_{\sigma, \sigma'} \int \frac{d^3 p d^3 p'}{(2\pi)^6 2\epsilon_p 2\epsilon_{p'}} |\langle \Phi_{p, \sigma, p', \sigma'} | S |\Phi_q, \eta\rangle|^2,$$  (5)

where $|\Phi_{p, \sigma, p', \sigma'}\rangle$ describes an electron and a positron with momenta $p^\mu = (\epsilon_p, p)$ and $p'^\mu = (\epsilon_{p'}, p')$, respectively. $\sigma, \sigma' \in \{1, 2\}$ label the different spin states, $\epsilon_p = \sqrt{m^2 + p^2}$, $\epsilon_{p'} = \sqrt{m^2 + p'^2}$. Eq. (5) holds if the one-particle momentum eigenstates for the electron and the positron are relativistically normalized [see Eq. (3)],

$$\langle \Phi_{p, \sigma} |\Phi_{p', \sigma'}\rangle = 2\epsilon_p (2\pi)^3 \delta^3 (p - p') \delta_{\sigma \sigma'},$$  (6)

as the identity operator (in the one-particle subspace) is then given by

$$1 = \sum_{\sigma=1,2} \int \frac{d^3 p}{(2\pi)^3 2\epsilon_p} |\Phi_{p, \sigma}\rangle \langle \Phi_{p, \sigma}|.$$  (7)

In the following we drop the spin labels and write $|\Phi_{p, \sigma}\rangle = |\Phi_{p, \sigma, p', \sigma'}\rangle$ for simplicity. Note that $W$ is a
probability (not a rate), as the duration of the process is
naturally limited if the background field has only a finite
extend.
Using Eq. (2) we rewrite the squared matrix element in Eq. (5) as

\[ |\langle \Phi_{p,p'} | S | \Phi_q \rangle|^2 = \int \frac{d^3q_1 d^3q_2}{(2\pi)^6 2\epsilon_{q_1} 2\epsilon_{q_2}} \eta(q_1) \eta^*(q_2) \times \mathcal{M}(p, p'; q_1)[\mathcal{M}(p, p'; q_2)]^* \], (8)

where

\[ i\mathcal{M}(p, p'; q) = \langle \Phi_{p,p'} | S | \Phi_q \rangle \] (9)

[for simplicity we often suppress some of the labels, i.e. \( \mathcal{M}(p, \sigma, \sigma'; q) = \mathcal{M}(p, p'; q) = \mathcal{M}(q) \)].

Since in strong-field QED the \( S \)-matrix contains three overall momentum conserving delta-functions [see Eq. (A20)], it is useful to define the reduced matrix element \( \mathcal{M} \) by

\[ i\mathcal{M}(p, p'; q) = (2\pi)^3 \delta^{(-1)}(p + p' - q) \mathcal{M}(p, p'; q) \] (10)

(\(-, \perp\) denote light-cone coordinates, see appendix [A].

After expressing the on-shell momentum integrals in Eq. (8) using light-cone coordinates [see Eq. (A22)], we obtain

\[ |\langle \Phi_{p,p'} | S | \Phi_q \rangle|^2 = \int \frac{dq'_1 dq'_2}{(2\pi)^3} \frac{\theta(q'_1)}{2q_1} |\eta(q_1)|^2 \times \frac{1}{2q_1} |\mathcal{M}(p,p';q_1)|^2 \times (2\pi)^3 \delta^{(-1)}(p + p' - q_1). \] (11)

Finally, the total probability for pair-creation is given by [see Eq. (5)]

\[ W = \int \frac{d^3q'}{(2\pi)^3 2\epsilon_{q'}} |\eta(q')|^2 W(q'), \] (12a)

where

\[ W(q) = \sum_{\text{spin}} \int \frac{d^3p d^3p'}{(2\pi)^6 2\epsilon_{p} 2\epsilon_{p'}} \frac{1}{2q} |\mathcal{M}(p,p';q)|^2 \times (2\pi)^3 \delta^{(-1)}(p + p' - q). \] (12b)

Thus, \( W \) is the average of \( W(q) \) over the momentum distribution of the incoming photon wave packet. Assuming that the matrix element is sufficiently smooth and that the wave-packet is peaked around \( q'' = q' \), we obtain \( W \approx W(q) \) [see Eq. (4)]. We point out that the average over the momentum distribution of the incoming gamma photon may hide substructures in the spectrum, especially if the energy spread of the incoming gamma photon (\( \sim \text{MeV} \)) is much larger then the energy of the colliding laser photons (\( \sim \text{eV} \)).

Using the Feynman rules for QED with plane-wave background fields (see appendix [A]), we obtain the following matrix element for the diagram in Fig. [1]

\[ \mathcal{M}(p, \sigma, p', \sigma'; q) = \epsilon_\mu \bar{u}_{p,\sigma} \Gamma^\mu(p, q, -p') \nu_{p',\sigma'}, \] (13)

where \( \epsilon_\mu \) is the polarization four-vector of the incoming photon (\( \epsilon q = 0, \epsilon^\nu \epsilon^\nu = -1 \)) and \( \Gamma^\mu(p, q, -p') \) the dressed vertex [see Eq. (A10)]. The four-spinors of the electron \( (u_{p,\sigma}) \) and the positron \( (\nu_{p',\sigma'}) \) obey [59, 67]

\[(p - m)u_{p,\sigma} = 0, \quad (p' + m)\nu_{p',\sigma'} = 0. \] (14)

We point out that the matrix element in Eq. (13) represents only the leading-order contribution to the process. In Eq. (13) we assume that the photon wave function is left unchanged during the whole process. However, the true wave function contains radiative corrections (due to the presence of the external field there are finite contributions even after renormalization is carried out [51]. The imaginary part of the polarization operator [which is related to the total emission probability, see Eq. (30)] ensures that the amplitude of the photon wave function decreases if electron-positron pairs are created (a similar situation is encountered for charged particles, see [68]). As long as the pair-creation probability is small, we can assume that the photon wave function is not changed. In general, however, the problem must be solved self-consistently, i.e., radiative corrections must be taken into account (which is beyond the scope of this paper).

\[ \text{Fig. 2. Leading-order contribution to the polarization operator for a photon in a plane-wave background field. The double lines denote the dressed (Volkov) propagator, which takes the classical background field into account exactly. The imaginary part of this diagram is related to the total pair creation probability via the optical theorem [indicated by the dashed line, see Eq. (30)].} \]

\[ \text{Using Eq. (2) we rewrite the squared matrix element given in Eq. (13) into Eq. (12), we obtain the differential pair-creation probability. To determine the total pair-creation probability the phase-space integrals must be evaluated, which is numerically rather demanding [11]. If one is only interested in the total pair-creation probability (and not in its differential structure), these integrals can be taken analytically by applying cutting rules to the polarization operator, as we will demonstrate now (see Fig. 2 and also [51, 57, 58, 69]; for the corresponding proof in vacuum QED see e.g. [60])}. \]

\[ \text{To this end we consider the squared matrix element [see Eq. (13)], which appears in Eq. (8)} \]

\[ \mathcal{M}(p, \sigma, p', \sigma'; q_1)[\mathcal{M}(p, \sigma, p', \sigma'; q_2)]^* \]

\[ = \epsilon_\mu \epsilon_{\nu'}^* \text{tr} \rho_{p,\sigma}^{\mu} \Gamma^\nu(p, q_1, -p') \rho_{p',\sigma'}^{\nu'} \Gamma^\nu(p, q_2, -p') \] (15)
where variables using the cyclic property of the trace and the change of $\Im$ and consider producing two more integrations in by defining the imaginary part of the forward photon scattering amplitude (see also \[51, 57, 58\]) is purely real [see Eq. (A9)], implying

$$\Im [\epsilon_\mu \epsilon_\nu^* \Pi^{\mu\nu}(q, q)] \sim [1 - \text{sign}(p_1^-) \text{sign}(p_2^-)] \pi^2 \delta \delta.$$

Finally, we obtain

$$23 [\epsilon_\mu \epsilon_\nu^* \Pi^{\mu\nu}(q, q)] = \epsilon_\mu \epsilon_\nu^* \int \frac{d^3p d^3p'}{(2\pi)^6 2\epsilon_p 2\epsilon_{p'}} \times (2\pi)^3 \delta^{(-\perp)}(p + p' - q) \text{tr}(p + m) G^\mu(p, q, -p') \times (\gamma' - m) G^\nu(p, q, -p')$$

for $q^- \geq 0$, where $G^\mu$ denotes the non-singular part of the dressed vertex [see Eq. (A20)]. By combining everything, we obtain the following relation between the total nonlinear Breit-Wheeler pair creation probability $W$ and the imaginary part of the photon forward-scattering amplitude (see also \[51, 57, 58\])

$$W(q) = \frac{1}{kq} \Im [\epsilon_\mu \epsilon_\nu^* \Pi^{\mu\nu}(q, q)] \quad (30a)$$

($q^2 = 0$) and [see Eq. (12)]

$$W = \int \frac{d^3q'}{(2\pi)^3 2\epsilon_{q'}} |\eta(q')|^2 W(q') \quad (30b)$$

$W \approx W(q)$ if the wave packet of the incoming photon is sharply peaked around $q'^\mu = q^\mu$, see Eq. (4).

C. Generalization

The optical theorem given in Eq. (30) holds for the decay of a real photon into a lepton pair. We will now generalize this result to the case where the pair-creation
FIG. 3. Leading-order Feynman diagram for the creation of an electron-positron pair by an electron or a positron inside a background field (trident process). The intermediate photon is in general not on-shell (time axis from right to left).

process represents only a part of a more complicated Feynman diagram. The most prominent example is the trident process shown in Fig. 3 [15, 17]. After squaring the matrix element and summing over the final spin states one has to consider the expression [see Eqs. (15) and (29)]

$$\mathcal{M}^{\mu\nu}(q_1, q_2) = \frac{i d^3 p d^3 p'}{(2\pi)^6 2p 2p'} \text{tr}(\gamma + m)\Gamma^\mu(p, q_1, -p')$$

$$\times (\gamma - m)\Gamma^\nu(p, q_2, -p')$$

(31)

where the coefficients [see Eq. (C20)] are evaluated at \(q_1 = q_2 = q\), \(q^2 = 0\) [\(P_Q\) does not contribute for \(q\epsilon = q^2 = 0\)].

### A. Linear polarization

We consider now the special case of a linearly polarized background field \(\xi = \xi_1, \kappa = 0; \psi_1(\phi) = \psi(\phi), \psi_2(\phi) = 0; F^{\mu\nu} = \psi^2(\phi) f^{\mu\nu}, f^{\mu\nu} = k^\mu a^\nu - k^\nu a^\mu\)], where \(P_{12} = P_{21} = 0\). It is then useful to introduce the following two polarization four-vectors [see Eq. (C2)]

$$\epsilon_\parallel = \Lambda^\mu_1, \quad \epsilon_\perp = \Lambda^\mu_2.$$  

(35)

They are real, obey \(\epsilon_\parallel^2 = \epsilon_\perp^2 = -1\), \(\epsilon_\parallel \epsilon_\parallel = 0\) and represent the direction of the electric and the magnetic field of the background field, respectively (the frame where the incoming photon and the laser pulse collide head-on).

Accordingly, we obtain for the total pair-creation probability in a linearly polarized laser pulse by an on-shell photon with polarization four-vector \(\epsilon_\parallel^\mu\) and \(\epsilon_\perp^\mu\) [see Eq. (34)]

$$W_\parallel(q) = -\kappa q \frac{1}{2\pi} \int_{-\infty}^{\infty} dy^{-} \int_{0}^{\infty} \frac{dq}{q} \Re \bar{P}_{11},$$

$$W_\perp(q) = -\kappa q \frac{1}{2\pi} \int_{-\infty}^{\infty} dy^{-} \int_{0}^{\infty} \frac{dq}{q} \Im \bar{P}_{22},$$

(36)

where [see Eq. (C20)]

$$\bar{P}_{11} = -\frac{i}{\kappa q} \left[ W_2(x_1)e^{-i\xi_1} - W_2(x_0)e^{-i\xi_0} \right]$$

$$+ \xi^2 \left[ \frac{1}{2} V W_0(x_1) + 2XW_1(x_1) \right] e^{-i\xi_1},$$

$$\bar{P}_{22} = -\frac{i}{\kappa q} \left[ W_2(x_1)e^{-i\xi_1} - W_2(x_0)e^{-i\xi_0} \right]$$

$$+ \xi^2 \left[ \frac{1}{2} V W_0(x_1) \right] e^{-i\xi_1},$$

(37)

\([V = V_1, X = X_{11}, \text{see Eq. (C25)}; x_0\) and \(x_1\) are defined in Eq. (C22) and \(W_1(x)\) in Eq. (D1)].

### B. Strong fields

As the integrals in Eq. (36) are oscillatory, it is useful to derive non-oscillatory representations for important limits. In this section we consider a strong (\(\xi \gg 1\)), linearly polarized background field. In this case the field-dependent contribution to the polarization operator can be written as

$$i\mathcal{P}^{\mu\nu}(q_1, q_2) - i\mathcal{P}^{\mu\nu}_{a=0}(q_1, q_2) = i(2\pi)^3 \delta^{(-1)}(q_1 - q_2)$$

$$\times \int_{-\infty}^{\infty} dz^{-} e^{i(q_2 - q_1)z} \left[ \pi_1^* (f q)^\mu (f q)^\nu \right.$$

$$\left. + \pi_2^* (f^* q)^\mu (f^* q)^\nu - \frac{\pi_4}{q_1 q_2} G^{\mu\nu} \right],$$

(38)
where
\[
\pi_1 = \alpha \frac{m^2}{3\pi} \int_{-1}^{+1} dv (w - 1) \left[ \frac{\chi(kz)}{w} \right]^{2/3} f'(\rho),
\]
\[
\pi_2 = \alpha \frac{m^2}{3\pi} \int_{-1}^{+1} dv (w + 2) \left[ \frac{\chi(kz)}{w} \right]^{2/3} f'(\rho),
\]
\[
\pi_3 = -\alpha \frac{q_0 q_2}{\pi} \int_{-1}^{+1} dv f_1(\rho),
\]
with \(\frac{1}{w} = \frac{1}{4}(1 - v^2)\), \(\rho = \left[ \frac{w}{\chi(kz)} \right]^{1/2}(1 - \frac{q_0 q_2}{w^2} \frac{1}{w})\), \(\chi(kz) = \chi\psi'/(kz)\) and \(G^{\mu\nu} = g^{\mu\nu} q_0^2 - q_0 q_2 g^{\mu\nu}\) [the Ritus functions are defined in Eq. (E1)]. In Eq. (39) the integration variable can be changed using
\[
\int_{-1}^{+1} dv = 2 \int_{-1}^{1} dv = \int_{-\infty}^{\infty} \frac{4}{w} \frac{dv}{\sqrt{w(w - 4)}}
\]
(valid for symmetric functions in \(v\)).

The expression given in Eq. (38) was obtained by applying suitable approximations to the triple integral representation given in Eq. (C1). It is trying to apply the same approximations now to the double integral representation in Eq. (C19). However, the functions \(W_i\) change over the formation region (\(W_0\) even has a logarithmic singularity at the origin), which means that this is not possible.

To determine the pair-creation probabilities we apply the optical theorem given in Eq. (30) to Eq. (38) and note the identities [see Eq. (C2)]
\[
\Lambda_{\mu\nu} = -\left( \frac{f q}{(f q)^2} \right)^{\mu\nu},
\]
\[
\Lambda_{\mu\nu} = -\left( \frac{f^* q}{(f^* q)^2} \right)^{\mu\nu},
\]
Finally, we obtain for the total probability that a single on-shell photon with four-momentum \(q^\mu\) and polarization four-vector \(\epsilon_{\mu}^\parallel\) or \(\epsilon_{\mu}^\perp\) creates an electron-positron pair inside a strong, \(\chi < \xi\), linearly polarized laser pulse with field tensor \(F^{\mu\nu}(kz) = f^{\mu\nu}\psi'(\phi)\) the following expressions
\[
W_{\parallel}(q) = -\alpha \frac{m^2}{q_0} \int_{-\infty}^{+\infty} d\phi \int_{-1}^{+1} dv \frac{(w - 1)}{3} \frac{A i'(x)}{x},
\]
\[
W_{\perp}(q) = -\alpha \frac{m^2}{q_0} \int_{-\infty}^{+\infty} d\phi \int_{-1}^{+1} dv \frac{(w + 2)}{3} \frac{A i'(x)}{x},
\]
where \(\frac{1}{w} = \frac{1}{4}(1 - v^2)\), \(x = \left[ w/|\chi(\phi)| \right]^{1/2}\) and \(\chi(\phi) = \chi\psi'(\phi)\) (due to \(q_0 = q^2 = 0\) the coefficient \(\pi_3\) does not contribute).

We point out that Eq. (42) holds for an arbitrary shape of the plane-wave background field (\(\chi\) should be such that \(\alpha \chi^{2/3} \ll 1\), otherwise perturbation theory with respect to the radiation field is expected to break down [8, 23]). As the formation region is small for \(\xi > 1\), the total pair-creation probability given in Eq. (42) consists essentially of the probability to create a pair inside a constant-crossed field (see [7], Eq. 64 and [22], chapter 5, Eq. 60; see also [22]), integrated over the pulse shape \(\chi(\phi)\) represents the instantaneous value of the quantum-nonlinearity parameter).

For comparison with the literature we consider now the monochromatic limit of Eq. (42), i.e. \(\psi'(\phi) = \sin(\phi)\) and a counter-propagating photon. As the wave is periodic, we can split the integral in \(\phi\) and consider only a single half-cycle (i.e. \(\phi \in [0, \pi]\)). As the photon is counter-propagating, it passes this half-cycle in the time \(T/4\), where the laser period is given by \(T = 2\pi/\omega\). Correspondingly, the rate for pair-creation by a single photon inside a strong \((\xi > 1)\), linearly polarized, monochromatic plane wave is given by
\[
W_{\parallel}(q) = -\alpha \frac{m^2}{q_0} \int_{-\infty}^{+\infty} d\phi \int_{-1}^{+1} dv \frac{(w - 1)}{3} \frac{A i'(x_m)}{x_m},
\]
\[
W_{\perp}(q) = -\alpha \frac{m^2}{q_0} \int_{-\infty}^{+\infty} d\phi \int_{-1}^{+1} dv \frac{(w + 2)}{3} \frac{A i'(x_m)}{x_m},
\]
where now \(x_m = \left[ w/|\chi_m(\phi)|^{1/2}\right)\), \(\chi_m(\phi) = \chi \sin(\phi)\) \(\frac{1}{w} = \frac{1}{4}(1 - v^2)\). Eq. (43) coincides with the result obtained in [23] (chapter 3, Eq. 35 and chapter 5, Eq. 60). It is also in agreement with the results obtained in [51].

C. Small quantum parameter

For \(\chi \ll 1\) the pair-creation probability is exponentially suppressed. This becomes obvious from the asymptotic expansion of the Airy function [see Eq. (E8)]
\[
A i'(x) \sim -x^{1/4} e^{-2x^{3/2}/2\sqrt{\pi}}.
\]

In this regime we can approximately evaluate the integrals in Eq. (42), resulting in a compact expression for the pair-creation probability. As the pair-creation probability is exponential suppressed, only the region around the peak of the field strength contributes to the integral in \(\phi\). Furthermore, using Eq. (40), we see that the integral in \(w\) is formed around \(w = 4\). Correspondingly, we can use
\[
\int_{4}^{\infty} dw \frac{1}{\sqrt{w - 4}} e^{-x w} = e^{-4x} \sqrt{\frac{\pi}{x}}
\]
(assuming \(x > 0\)) to evaluate the integral in \(w\) approximately. Assuming that \(|\psi'(\phi)| \approx |\sin(\phi)|\) close to a field peak, the contribution from this peak can be approximately taken into account using
\[
\int_{-\infty}^{\pi} d\phi e^{-x/\sin(\phi)} \approx \int_{-\infty}^{+\infty} d\phi e^{-x(1 + \phi^2/2)} = e^{-x} \sqrt{2\pi x}
\]
(assuming \(x > 0\); for different peak-shapes this relation must be modified accordingly). Combining everything,
the pair-creation probability within a single peak of a linearly polarized, plane-wave laser field is in the regime $\xi \gg 1$, $\chi \ll 1$ given by

$$W_\parallel(q) = \alpha \frac{m^2}{kq} \frac{3\sqrt{\pi}}{8} \left(\frac{\chi}{2}\right)^{3/2} e^{-8/(3\chi)}, \quad (47)$$

$W_\perp(q) = 2W_\parallel(q)$. From Eq. (47) the pair-creation rate inside monochromatic fields can be obtained similar as above [see Eq. (13)]. To this end we consider again a photon counter propagating with a monochromatic wave. A counter-propagating photon passes four field maxima during the time of one laser period $T = 2\pi/\omega$. Correspondingly, the pair-creation rate for a single photon is given by (linear polarization, $\xi \gg 1$, $\chi \ll 1$)

$$W_\parallel(q) = \alpha \frac{m^2}{q^0} \frac{3}{8\sqrt{\pi}} \left(\frac{\chi}{2}\right)^{3/2} e^{-8/(3\chi)}, \quad (48)$$

$W_\perp(q) = 2W_\parallel(q)$. The result agrees with [23], chapter 3, Eq. 33 (see also [13]).

IV. NUMERICAL RESULTS

For the numerical calculations we considered a linearly polarized laser pulse with the following pulse-shape [see Fig. 4 and Eq. (A1)]

$$\psi'(\phi) = \sin^2[\phi/(2N)] \sin(\phi + \phi_0) \quad (49)$$

for $\phi \in [0, 2\pi N]$ and zero otherwise. Here $N$ specifies the number of cycles in the pulse and $\phi_0$ its carrier envelope phase (CEP).

Beside the pulse shape parametrized by $N$ and $\phi_0$ we have to chose the classical intensity parameter $\xi$ and the quantum nonlinearity parameter $\chi = \eta \kappa / \sqrt{(kq)^2/m^2}$, see Eq. (A4). For a laser pulse with central angular frequency $\omega$ and peak field amplitude $E_0$ we obtain $\xi = |e|E_0/(m\omega)$. The quantum-nonlinearity parameter is given by $\chi = 2(\omega_\gamma/m)(E_0/E_{cr})$ if the photon and the laser pulse collide head-on ($\omega_\gamma$ denotes the energy of the incoming photon).

As the pair-creation probability is exponentially suppressed for $\chi \ll 1$ [see Eq. (17)], we are mainly interested in the nonlinear quantum regime where $\chi \gtrsim 1$. Existing optical petawatt laser systems reach already $\xi \sim 100$ [24] and photon energies $\sim 1$ GeV are obtainable via Compton backscattering either at conventional facilities like SPring-8 [31] or by using laser wakefield accelerators [26-30]. Hence, it is possible to reach the regime $\chi \gtrsim 1$ with presently available technology.

In the following we will not consider the influence of the incoming photon wave packet and set $W = W(q)$, see Eq. (12). For $\xi \gg 1$ the total pair-creation probability can be calculated using Eq. (42) without further numerical difficulties, as the integrals are non-oscillatory. To verify the validity of Eq. (42), we have compared it with the general expression given in Eq. (36) (the oscillatory integrals have been evaluated numerically as explained...
possible by combining a petawatt laser system (ξ)

\[ \parallel \text{pair-creation probability.} \]

As expected from Eq. (47), the pair-creation probability for perpendicular polarization (W_⊥) scales as \( 1 / \xi \). In the regime \( \xi \rightarrow 1 \), which is not included in Eq. (42), the CEP effect for the total pair-creation probability

\[ \langle W_{\parallel,\perp} \rangle = \frac{1}{2\pi} \int_0^{2\pi} d\phi_0 W_{\parallel,\perp}(\phi_0) \]

and the relative deviation

\[ \Delta W_{\parallel,\perp}(\phi_0) = \frac{W_{\parallel,\perp}(\phi_0) - \langle W_{\parallel,\perp} \rangle}{\langle W_{\parallel,\perp} \rangle} \]

Both are plotted in Fig. (10) for a short pulse (N = 3) of moderate intensity (\( \xi = 10 \)). For \( \chi \approx 0.2 \) the relative CEP effect is of the order of 10\%, but many photons are needed to produce a sufficient amount of electron-positron pairs. In the regime where pair creation is likely (\( \chi \sim 1 \)), the CEP effect for the total pair-creation probability is very small (we point out that this prediction could be changed by higher-order corrections).

V. CONCLUSION

In the present paper we have verified (to leading order) the cutting rule for the polarization operator in general plane-wave background fields, i.e. the relation between the imaginary part of the photon forward scattering amplitude and the total probability for photon-induced electron-positron pair creation nonlinear Breit-Wheeler process, see Eq. (47). Furthermore, we de-
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Appendix A: QED with plane-wave background fields

This appendix summarizes the formalism of strong-field QED with a plane-wave background field, the notation agrees with [53]. A more detailed discussion can be found in the Refs. [8, 23, 75, 81].

We describe a plane-wave background field by the following field strength tensor

\[ F^{\mu \nu}(kx) = f^{\mu \nu}_1(kx) + f^{\mu \nu}_2(kx), \]  

(A1)

where \( f^{\mu \nu}_1 = k^\mu a^\nu - k^\nu a^\mu \) (\( k^\mu \) characterizes the four-momentum of the background-field photons and the prime denotes the derivative with respect to the argument). The field strength tensor is related to the four-potential \( A^\mu \) by

\[ F^{\mu \nu} = \partial^\mu A^\nu - \partial^\nu A^\mu. \]  

(A2)

For a linearly polarized background field [\( \psi_2(\phi) = 0 \)] we drop the index and write \( \psi(\phi) = \psi_1(\phi) \), \( f^{\mu \nu} = f^{\mu \nu}_1 = k^\mu a^\nu - k^\nu a^\mu \). Furthermore, we introduce the integrated field strength tensor

\[ \mathcal{F}^{\mu \nu}(kx) = \int_{-\infty}^{kx} d\phi' \, F^{\mu \nu}(\phi'). \]  

(A3)

The field strength of a plane-wave field can be characterized by the so-called classical intensity parameters \( \xi_i \). The importance of (nonlinear) quantum effects like pair creation is determined by the quantum nonlinearity parameters \( \chi_i \). Both \( \xi_i \) and \( \chi_i \) are gauge and Lorentz invariant [24]

\[ \xi_i = \frac{|e|}{m} \sqrt{-a_i^2}, \quad \chi_i = \frac{|e|}{m^3} \sqrt{q_f^2 q} = \eta \xi_i, \]  

(A4)

where \( \eta = \sqrt{(kq)^2/m^2} \) (\( q^\mu \) is the four-momentum of an incoming particle, in the present context a photon). For linearly polarized background fields [\( \psi_2(\phi) = 0 \), \( \xi_2 = 0 \), \( \chi_2 = 0 \)] we simply write \( \xi = \xi_1 \), \( \chi = \chi_1 \). We always assume that \( \psi_1(kx) \) describe only the shape of the background field and \( \xi_i \) its strength, i.e. we require \( |\psi_1(kx)| \lesssim 1 \).

If a QED process happens inside a plane-wave background field, the Feynman rules must be modified. In-
stead of the free vertex \(-ie\gamma^\mu\) the following dressed vertex must be used \([81]\)

\[
\Gamma^\rho(p', q, p) = -ie \int d^4x e^{-i\vec{k}_x \cdot \vec{E}_{p', x}} \gamma^\rho E_{p, x}.
\] (A5)

Here we have introduced the Ritus matrices \([23]\)

\[
E_{p, x} = \left[ 1 + \frac{e\bar{k}A(kx)}{2kp} \right] e^{iS_p(x)},
\]
\[
\bar{E}_{p, x} = \left[ 1 + eA(kx)k^\mu \right] e^{-iS_p(x)},
\] (A6)

with

\[
S_p(x) = -px - \int_{-\infty}^{kx} d\phi' \left[ \frac{e\mu A(\phi')}{kp} - \frac{e^2A^2(\phi')}{2kp} \right]
\] (A7)

\((S_p\) coincides with the classical action\). We use the following bar notation \(\bar{M} = \gamma^0M\gamma^0\), implying

\[
\Gamma = 1, \quad \bar{\gamma}^5 = -\gamma^5, \quad \bar{\gamma}^\mu = \gamma^\mu,
\]
\[
\left(\bar{i}\gamma^\mu\gamma^5\right) = -i\gamma^\mu\gamma^5, \quad \left(\bar{i}\sigma^\mu\nu\right) = i\sigma^\mu\nu
\] (A8)

and

\[
\Gamma^\rho(p', q, p) = -\Gamma^\rho(p, -q, p').
\] (A9)

The dressed vertex defined in Eq. (A5) can be written as \([53]\)

\[
\Gamma^\rho(p', q, p) = -ie \int d^4x \left[ \gamma_\mu G^{\mu\rho}(kp', kp; kx) + i\gamma_5\gamma_\mu G_5^{\mu\rho}(kp', kp; kx) \right] e^{iS_\Gamma(p', q, p; x)},
\] (A10)

where

\[
S_\Gamma(p', q, p; x) = -S_{p'}(x) - qx + S_p(x)
\]

\[
= (p' - q - p)x + \int_{-\infty}^{kx} d\phi' \left[ \frac{e\mu A_p\bar{A}^{\sigma\mu}(\phi')}{(kp)(kp')} - \frac{e^2A(\phi')^2}{2(kp)(kp')} \right] + \frac{e^2(kp - kp')}{2(kp)^2(kp')^2} \left(\bar{A}_p\bar{A}_{\rho\nu}^\prime\right)(\phi')
\] (A11)

and

\[
G^{\mu\rho} = \delta^{\mu\rho} + G_1\bar{\delta}^{\mu\rho}(kx) + G_2\bar{\delta}^{2\mu\rho}(kx),
\]
\[
G_5^{\mu\rho} = \frac{1}{2}\bar{\delta}^{\mu\rho\nu}(kx),
\]
\[
G_1 = -\frac{ekp + kp'}{2kp'k}, \quad G_2 = \frac{e^2}{2kp'k'}, \quad G_3 = -\frac{ekp - kp'}{2kp'k},
\] (A12)

(\(1 \text{ and } 2 \text{ are also summarized as } \perp\)) where the four-vectors \(k^\mu, \bar{k}^\mu, \bar{e}_1^\mu \) and \(e_2^\mu\) obey

\[
k^2 = \bar{k}^2 = ke_i = \bar{ke}_i = 0, \quad k\bar{k} = 1, \quad e_ie_j = -\delta_{ij},
\] (A15)

\[
e_{\mu\nu}\varepsilon_{\rho}\varepsilon_{\rho} = 1
\] (A16)

(as \(k^\mu \) and \(\bar{k}^\mu \) have dimension of momentum and inverse momentum, respectively, the dimensions of \(v^+ \) and \(v^- \) differ from those of \(v^\mu \)).

For a laser pulse with characteristic angular frequency \(\omega\), propagating along the \(z\)-direction we can, for example, use

\[
k^\mu = \omega(1, 0, 0, 1), \quad \bar{k}^\mu = \frac{1}{2\omega}(1, 0, 0, -1),
\]
\[
e_1^\mu = (0, 1, 0, 0), \quad e_2^\mu = (0, 0, 1, 0).
\] (A17)

In light-cone coordinates the scalar product of two four-vectors is given by

\[
a_\mu a^\mu = a^+b^- + a^-b^+ - a_1^b_1 - a_1^b_1
\] (A18)

(we also use the short notation \(a^\perp b^\perp = a^\perp a^\perp - a_1^b_1\)). Furthermore, the four-dimensional integration measure becomes

\[
\int d^4a = \int d^4a d^4a^- d^4a^\perp, \quad da^\perp = da_1 da_1.
\] (A19)

In light-cone coordinates the dressed vertex [see Eq. (A10)] depends non-trivially only on \(kp = x^-\). Therefore, we can take the integrals in \(dx^+ \) and \(dx^- \) and obtain momentum-conserving delta functions in three of four light-cone components,

\[
\Gamma^\mu(p', q, p) = (2\pi)^3\gamma^\mu(p' - q - p) G^{\mu\rho}(p', q, p).
\] (A20)

Here we defined the non-singular part of the dressed vertex \(G^{\mu\rho}(p', q, p)\) and used the following notation for delta functions in light-cone coordinates

\[
\delta^{(\perp)}(a) = \delta(a^-)\delta(a_1)\delta(a^\perp).
\] (A21)

In contrary to vacuum QED, four-momentum is only conserved up to a multiple of the background-field four-momentum \(k^\mu\) at the dressed vertex.

Finally, we note the following relation between on-shell momentum integrals

\[
\int \frac{d^4p}{(2\pi)^3} \left(\int \frac{d^4p}{(2\pi)^3} \delta(p^2 - M^2) \theta(p^0)f(p)\right)f(p)
\]

\[
= \int \frac{dp^- dp^+}{(2\pi)^3} \frac{\theta(p^-)}{2p^-} f(p).
\] (A22)

Here \(M\) is the particle mass and \(p^2 = M^2\), i.e. \(p^0 = \epsilon_p = \sqrt{M^2 + p^2}\) in the first line and \(p^- = (p^+ + M^2)/(2p^-)\) in the last line. We note that \(p^- > 0\) and \(p^0 = -\epsilon_p\) to \(p^- < 0\) \((p^- = 0\) is only reached in the limit \(\epsilon_p \to \infty\)).

\[
v^- = vk, \quad v^+ = \bar{vk}, \quad v^i = ve_1, \quad v^1 = ve_2,
\] (A14)
Appendix B: Pole structure of the Volkov propagator

To prove Eq. (26), we have to investigate the pole structure of the Volkov-propagator

\[ iG(x, y) = i \int \frac{d^4p}{(2\pi)^4} E_{p,x} \frac{\not{p} + m}{p^2 - m^2 + i0} E_{p,y} \tag{B1} \]

which describes the propagation of a fermion from \( y \) to \( x \) (and correspondingly the propagation of an anti-fermion from \( x \) to \( y \)), taking the plane-wave background field into account exactly. This is most conveniently out in light-cone coordinates, where the integral in \( dp^+ \) has a simple structure \([71]\), as the phase of the propagator depends on \( p^+ \) only via \([see \ Eq. (A7)]\)

\[ \exp \left[ -ip^+(x^- - y^-) \right] \tag{B2} \]

\((A^- = k^- = 0)\). For \( p^- \neq 0 \) and \( x^- - y^- \neq 0 \) we can evaluate the integral in \( p^+ \) using the residue theorem \([53]\).

We note that the point \( p^- = 0 \) does not contribute to the integral. As long as no singularities are encountered, a single point can always be excluded from the integration range. As \( p^+ p^+ + m^2 > 0 \) the integrand never diverges for \( p^- = 0 \). We note that this argument is not invalidated by the singularity in the phase \([see \ Eq. (A7)]\), which leads to strong oscillations in the region around \( p^- = 0 \) (this only indicates that the region around \( p^- = 0 \) does not contribute much to the integral). Therefore, we can assume that \( p^- \neq 0 \).

To take the integral in \( p^+ \) we have to close the contour in the lower complex plane if \( x^- - y^- > 0 \) and in the upper complex plane if \( x^- - y^- < 0 \). The pole of

\[ \frac{1}{p^2 - m^2 + i0} = \frac{1}{2p^+p^- - p^+p^- - m^2 + i0} \tag{B3} \]

is located at

\[ p^+ = \frac{p^+p^- + m^2 - i0}{2p^-}, \tag{B4} \]

i.e. in the lower complex plane for \( p^- > 0 \) and in the upper complex plane for \( p^- < 0 \), in agreement with the Feynman boundary condition.

Following \([60]\) we have to consider also the retarded and advanced propagators, defined by the pole prescriptions

\[ \frac{1}{p^2 - m^2 + \text{sign}(p^-)i0}, \quad \frac{1}{p^2 - m^2 - \text{sign}(p^-)i0}, \tag{B5} \]

respectively. The pole of the former is always located in the lower, the pole of the latter in the upper complex plane. Correspondingly, the propagators vanish for \( x^- < y^- \) and \( x^- > y^- \), respectively.

The polarization operator diagram \([see \ Eq. (19)]\) contains both \( G(x, y) \) and \( G(y, x) \) or in other words the phase factor contains

\[ \exp \left[ -ip_1^+(x^- - y^-) \right] \exp \left[ ip_2^+(x^- - y^-) \right]. \tag{B6} \]

Correspondingly, the contour integrals in \( p_1^+ \) and \( p_2^+ \) must be closed differently. If both propagators of the polarization operator are either replaced by advanced or by retarded propagators, such that for \( x^- - y^- \gtrless 0 \) one propagator is always zero, the contribution of the total diagram vanishes. Using the relation \([see \ Eq. (24)]\)

\[ \frac{1}{p^2 - m^2 + \text{sign}(p^-)i0} = \frac{P}{p^2 - m^2} \]

\[ \mp i \text{sign}(p^-)\pi \delta(p^2 - m^2), \tag{B7} \]

we can now prove the identity \([see \ Eq. (26)]\)

\[ PP = \text{sign}(p_1^-) \text{sign}(p_2^-)\pi^2\delta \]

for \( \Im \epsilon_{\mu\nu} \Pi^{\mu\nu}(q, q) \).

Appendix C: Polarization operator

For plane-wave background fields the polarization operator was first considered in \([51, 52]\) \([see \ Eq. (19), Fig. 2 and also \([53, 54]\) for a recent discussion\]. In \([53]\) the following representation for the field-dependent part of the polarization operator was derived

\[ iP^{\mu\nu}(q_1, q_2) - iP^{\mu\nu}_{\delta=0}(q_1, q_2) = -i\pi e^2 \delta(\tau^- \tau^-) (q_1 - q_2) \]

\[ \times \int_{-1}^{+1} d\tau \int_0^{+\infty} \frac{dr}{r} \int_{-\infty}^{+\infty} dz \left[ b_1 \Lambda_1^a \Lambda_2^a + b_2 \Lambda_2^a \Lambda_1^a \right. \]

\[ + b_3 \Lambda_1^a \Lambda_1^a + b_4 \Lambda_2^a \Lambda_2^a + b_5 Q_1^a Q_2^a \right] \epsilon^{\Phi}, \tag{C1} \]

where \( q_1^a \) and \( q_2^a \) denote the four-momenta of the incoming and outgoing photon, respectively (if they can be used interchangeably due to the momentum-conserving delta functions we simply write \( q^a \)). Here we have introduced the four-vectors

\[ \Lambda_1^a = \frac{f_1^{\mu\nu} q^\nu}{kq\sqrt{-a_1^a}}, \quad \Lambda_2^a = \frac{f_2^{\mu\nu} q^\nu}{kq\sqrt{-a_2^a}}, \tag{C2} \]

\[ Q_1^a = \frac{k^\mu q_1^a - q_1^a kq}{kq}, \quad Q_2^a = \frac{k^\mu q_2^a - q_2^a kq}{kq}, \]

which obey \( \Lambda_1 \Lambda_2 = -\delta_{ij}, \quad k \Lambda_i = q_i \Lambda_j = 0, \quad Q_2^a = -q_2^a, \quad Q_i \Lambda_j = 0 \) and \( q_i \xi_i = 0 \).

The coefficients are given by

\[ b_1 = 2m^2 \xi_1 \xi_2 \frac{\tau}{4\mu} X_{12} \frac{\tau}{4\mu} X_{12} e^{i\tau\beta}, \]

\[ b_2 = 2m^2 \xi_1 \xi_2 \frac{\tau}{4\mu} X_{21} \frac{\tau}{4\mu} X_{21} e^{i\tau\beta}, \]

\[ b_3 = -\left( \frac{i}{\tau} + \frac{q_1 q_2}{2} \right) (e^{i\tau\beta} - 1) \]

\[ + 2m^2 \xi_1 \xi_2 (\xi_1^2 X_{11} + \xi_2^2 X_{22}) \]

\[ b_4 = -\left( \frac{i}{\tau} + \frac{q_1 q_2}{2} \right) (e^{i\tau\beta} - 1) \]

\[ + 2m^2 \xi_1 \xi_2 (\xi_1^2 X_{11} + \xi_2^2 X_{22}) \]

\[ e^{i\tau\beta}, \]
\[ b_5 = -\frac{2\mu}{\tau} \left( e^{i\tau\beta} - 1 \right) \] (C3)

\[ [\mu = \frac{1}{2} \tau (1 - v^2)], \text{ where} \]

\[ e^{i\Phi} = \exp \left( i \left[ (q_2^+ - q_1^+) z^- + \mu q_1 q_2 - \tau m^2 \right] \right), \]
\[ e^{i\tau\beta} = \exp \left[ i \tau m^2 \sum_{i=1.2} \xi_i^2 (I_i^2 - J_i) \right]. \] (C4)

Here

\[ I_i = \frac{1}{2} \int_{-1}^{+1} d\lambda \psi_i(kz - \lambda \mu k), \]
\[ J_i = \frac{1}{2} \int_{-1}^{+1} d\lambda \psi_i^2(kz - \lambda \mu k) \] (C5)

and

\[ X_{ij} = [I_i - \psi_i(kz + \mu k) \overline{[I_j - \psi_j(kz - \mu k)]}], \]
\[ Z_i = \frac{1}{2} [\psi_i(kz - \mu k) - \psi_i(kz + \mu k)]^2 \] (C6)

(for further notational details see also appendix A).

It is often convenient to apply the change of variables from \( \tau = \mu w = qv/kq \) to \( \tau = \mu kq \) \( [\mu = \frac{1}{2} \tau (1 - v^2)], \) \( \frac{1}{w} = \frac{1}{2} (1 - v^2) \) and from \( v \) to \( w \)

\[ \int_{-1}^{+1} dv \int_0^{\infty} \frac{d\tau}{\tau} \int_{-\infty}^{+\infty} dz^- = \int_1^{+\infty} dw \int_0^{w/(w - 4)} \int_0^{\infty} d\varphi \int_{-\infty}^{+\infty} dz^- , \] (C7)

where \( \sigma = \text{sign}(kq) \) \{we assume that the integrand is an even function of \( v \), see Eq. (40)\}. The new variables have a clear physical meaning, as the phases of the creation and the annihilation vertex are given by \( kx = kz - \varrho \) and \( ky = kz + \varrho \), respectively, and the variable \( w \) is related to the momenta \( \rho^2 \) and \( \rho' \) of the created electron and positron, respectively, by \( w = (kq)^2/(kpkp') \) \{the momenta \( \rho^2 \) and \( \rho' \) here differ from those denoted by the same symbols in [33], see Eq. (19)\}.

In terms of the new variables the phases \{see Eq. (C4)\} can be written as

\[ \Phi = (q_2^+ - q_1^+) z^- + \varrho (q_1 q_2/kq) - w (m^2/kq) \varrho, \]
\[ \Phi_1 = (q_2^+ - q_1^+) z^- + \varrho (q_1 q_2/kq) - w (m^2/kq) D(\varrho), \] (C8)

where \( \Phi_1 = \Phi + \tau \beta \) and we defined

\[ D(\varrho, kz) = \varrho \left[ 1 + \sum_{i=1,2} \xi_i^2 (J_i - I_i^2) \right]. \] (C9)

Furthermore, we obtain

\[ I_i(\varrho, kz) = \frac{1}{2\varrho} \int_{kz - \varrho}^{kz + \varrho} d\varphi \psi_i(\varrho), \]
\[ J_i(\varrho, kz) = \frac{1}{2\varrho} \int_{kz - \varrho}^{kz + \varrho} d\varphi \psi_i^2(\varrho). \] (C10)

Thus, after the change of variables given in Eq. (C7), the phases have a very simple dependence on \( w \). Using Eqs. (D2) and (D19) the integral in \( w \) can be calculated analytically and a double-integral representation for the polarization operator is obtained.

The representation given in Eq. (C11) depends on the external momenta via the scalar \( q_1 q_2 \). However, for real incoming or outgoing photons it is more convenient to use a representation which depends only on \( q_1^+ \) or \( q_2^+ \). To obtain such a representation, we use the three-momentum-conserving delta functions and write

\[ q_2^+ = q_1^+ + nk^\mu, \quad n = q_2^+ - q_1^+, \] (C11)

where \( n \) denotes the amount of four-momentum \( k^\mu \) exchanged with the background field \( (n > 0 \text{ corresponds to absorption, } n < 0 \text{ to emission, } n \text{ is in general not an integer}) \). Thus, the integral in \( z^- \) represents a Fourier transform that determines the probability amplitude to absorb \( nk^\mu \) four-momentum from the background field. Using this notation we obtain

\[ q_1 q_2 = q_1^2 + nkq = q_2^2 - nkq. \] (C12)

Correspondingly, the phases \{see Eq. (C8)\} can now be rewritten using

\[ (q_2^+ - q_1^+) z^- + \varrho q_1 q_2/kq = nkz + \varrho q_1 q_2/kq = nkx + \varrho q_2^2/kq. \] (C13)

Thus, by changing the integration variable from \( z^- \) to either \( x^- \) (real outgoing photon) or \( y^- \) (real incoming photon) the phase of the polarization operator simplifies in these cases. Depending on this choice we obtain \{see Eq. (C11)\}

\[ I_i = \int_0^1 d\lambda \psi_i(ky - 2\varphi \lambda) = \int_0^1 d\lambda \psi_i(kx + 2\varphi \lambda), \]
\[ J_i = \int_0^1 d\lambda \psi_i^2(ky - 2\varphi \lambda) = \int_0^1 d\lambda \psi_i^2(kx + 2\varphi \lambda). \] (C14)

Similarly, we can rewrite the preexponent using the following identity

\[ \frac{nkq}{2} \int_{-\infty}^{+\infty} dz^- e^{i\Phi} (e^{i\tau\beta} - 1) = (-i) \frac{kq}{2} \int_{-\infty}^{+\infty} dz^- (e^{i\tau\beta} - 1) \frac{\partial}{\partial z^-} e^{i\Phi} \]
\[ = 2m^2 \tau^2 4\mu \int_{-\infty}^{+\infty} dz^- e^{i\Phi} e^{i\tau\beta} \sum_{i=1,2} \xi_i^2 (Y_i - Z_i), \] (C15)

where

\[ Y_i = [I_i - \psi_i(ky)] [\psi_i(kx) - \psi_i(ky)]. \] (C16)
To prove Eq. (C15) we used integration by parts and
\[
\frac{\partial I_i(q, k_z)}{\partial z^-} = -\frac{1}{2\theta} [\psi_i(k_z - q) - \psi_i(k_z + q)] ,
\]
\[
\frac{\partial I_i(q, k_z)}{\partial z^+} = -\frac{1}{2\theta} [\psi_i^2(k_z - q) - \psi_i^2(k_z + q)] .
\]  
(C17)
Furthermore, it is useful to define
\[
V_i = 2Z_i - Y_i = [I_i - \psi_i(ky - 2\theta)] [\psi_i(ky) - \psi_i(ky - 2\theta)] .
\]  
(C18)

By applying the above relations to the symmetric representation in Eq. (C1), we immediately obtain the representation given in Eq. (109) of [53], which is equivalent to the one in [51]. Similarly, we obtain for the field-dependent part of the polarization operator inside a plane-wave background field the following double-integral representation
\[
\begin{align*}
\sigma \mathcal{P}_{mn}(q_1, q_2) - i\mathcal{P}_{mn}(q_1, q_2) &= -i(2\pi)^3 \delta^{(-1)}(q_1 - q_2) \\
&\times \frac{\alpha}{2\pi} \int_0^{\sigma_{\infty}} d\theta \int_{-\infty}^{\infty} dy I_{12} \Lambda_1 \Lambda_2 + P_{21} \Lambda_1 \Lambda_2 \\
&+ P_{11} \Lambda_1 \Lambda_1 + P_{22} \Lambda_2 \Lambda_2 + P Q Q_l Q_m ,
\end{align*}
\]  
(C19)

with the phases [see Eq. (C8)]
\[
\tilde{\Phi}_0 = (q_2^+ - q_1^+) y^+ + \theta (q_2^+/q_1^+) - 4x_0 ,
\]
\[
\tilde{\Phi}_1 = (q_2^+ - q_1^+) y^- + \theta (q_2^+/q_1^+) - 4x_1 .
\]  
(C21)
Here we have introduced
\[
x_0 = (m^2/q_0^+) \theta , \quad x_1 = (m^2/q_0^+) D(q, k_y) ,
\]  
(C22)
where [see Eq. (C9)]
\[
D(q, k_y) = \theta \left[ 1 + \sum_{i=1,2} \xi_i^2 (I_i - I_i^2) \right]
\]  
(C23)
with [see Eq. (C14)]
\[
I_i = \int_0^1 d\lambda \psi_i(ky - 2\theta\lambda) ,
\]
\[
J_i = \int_0^1 d\lambda \psi_i^2(ky - 2\theta\lambda) .
\]  
(C24)
Furthermore, [see Eqs. (C6) and (C18)]
\[
X_{12} = [I_1 - \psi_1(ky)][I_1 - \psi_1(ky - 2\theta)] ,
\]
\[
V_i = [I_1 - \psi_1(ky - 2\theta)][\psi_1(ky) - \psi_1(ky - 2\theta)] .
\]  
(C25)
The functions \(W_l(x)\) are defined in Eq. (D11) [see also Eq. (D19)].

Having taken the \(w\)-integral analytically, we are left with the integrals in \(y^-\) and \(\theta\). To evaluate these integrals, the precise shape of the background field has to be known. It is therefore reasonable to use numerical methods (see appendix F).

### Appendix D: Polarization operator integrals

In this appendix we consider the non-oscillatory functions
\[
W_l(x) = \int_0^\infty dw \frac{4}{(W + 4)^4 \sqrt{(W + 4)W}} e^{-i W x} \quad \text{(D1)}
\]
\((l = 0, 1, 2, x \geq 0)\), which appear in the following integrals related to the polarization operator
\[
\int_4^\infty dw \frac{4}{w^4 \sqrt{(w - 4)} e^{-iw x}} = e^{-ix} W_l(x)
\]  
(D2)
\((w = W + 4)\).

By rescaling the integration variable \((W x = U\), assuming \(x > 0)\)
\[
W_l(x) = \int_0^\infty dw \frac{4}{(U/x + 4)^4 \sqrt{(U + 4x)U}} e^{-i U}
\]  
(D3)
and noting that for \(x \gg 1\) the integral is formed around \(U = 0\), we obtain the leading-order behavior in the limit \(x \to \infty\)
\[
W_l(x) \sim -\frac{2\sqrt{\pi i}}{4} e^{i\pi/4} \frac{1}{\sqrt{x}} .
\]  
(D4)
Here we used
\[
\int_0^\infty dU \frac{1}{\sqrt{U}} e^{-iU} = \int_{-\infty}^{\infty} dV e^{-iV^2} = -i\sqrt{\pi} e^{\pi/4}
\] (D5)

\((V = \sqrt{U})\). From Eq. (D4) we conclude that the functions \(W_l\) are non-oscillatory and induce a damping for large arguments (i.e., large phases).

We note that the functions \(W_l\) can be represented using the Hankel function \(H^{(2)}_\nu(x) = J_\nu(x) - i Y_\nu(x)\) \((J_\nu\) and \(Y_\nu\) denote the Bessel function of the first and the second kind, respectively) [85]. To this end we apply the change of variables \(W = 2(cosh t - 1)\), which yields together with
\[
cosh t + 1 = 2 \cosh^2(t/2)
\] (D6)

the alternative integral representation
\[
W_l(x) = e^{i2x} \int_0^\infty dt \frac{4}{4t^2} e^{-i2t \cosh t}.
\] (D7)

For \(l = 0\) we obtain, using the following integral representation for the Hankel function [85]
\[
H^{(2)}_\nu(z) = i \frac{2}{\pi} e^{\nu\pi i/2} \int_0^\infty dt e^{-iz \cosh t} \cosh(\nu t)
\] (D8)

(valid for \(-1 < \Re \nu < 1, \ z > 0\))
\[
W_0(x) = (-2\pi i) e^{i2x} H^{(2)}_0(2x).
\] (D9)

With the help of the asymptotic expansion of the Hankel function [85]
\[
H^{(2)}_\nu(z) \sim \sqrt{\frac{2}{\pi z}} e^{-i(3\nu \pi - 4\nu)} \sum_{k=0}^{\infty} (-i)^k a_k(\nu) \frac{1}{z^k},
\] \(a_k(\nu) = \frac{(4\nu^2 - 1^2)(4\nu^2 - 3^2) \cdots (4\nu^2 - (2k - 1)^2)}{k! 8^k}\) (D10)
a0(\nu) = 1, the leading asymptotic behavior given in Eq. (D4) is verified for \(W_0\). We note that \(W_0(x)\) has a logarithmic singularity at \(x = 0\) [85].

We point out that the integral in Eq. (D7) converges also for \(l > 0\). Therefore, we can use the replacement \(x \to x - i \epsilon\) together with the limit \(\epsilon \to 0\), which allows us to apply Eq. (D8) also for \(l = 1\). Using the identity
\[
cosh t - 1 = \tanh(t/2) \sinh t,
\] (D11)

the integrals
\[
\int dt \frac{1}{\cosh^2(t/2)} = 2 \tanh(t/2),
\]
\[
\int dt \frac{1}{\cosh^4(t/2)} = \frac{2}{3} \tanh(t/2) \left[ 2 + \frac{1}{\cosh^2(t/2)} \right]
\] (D12)

and integration by parts (due to the regularization the boundary terms vanish) we obtain
\[
W_1(x) = (-2\pi x) e^{i2x} [H^{(2)}_0(2x) + i H^{(2)}_1(2x)],
\]
\[
W_2(x) = \pi x e^{i2x} [i H^{(2)}_0(4x) - (4x + i) H^{(2)}_1(2x)],
\] (D13)

where
\[
H^{(2)}_0(2x) + i H^{(2)}_1(2x) = J_0(2x) + Y_1(2x) + i [J_1(2x) - Y_0(2x)]
\] (D14)

and
\[
4ix H^{(2)}_0(2x) - (4x + i) H^{(2)}_1(2x) = 4x Y_0(2x) - 4x J_1(2x) - Y_1(2x)
\plus
\[i (4x J_0(2x) + 4x Y_1(2x) - J_1(2x)].
\] (D15)

In order to verify the final result in Eq. (D13) we note that the asymptotic behavior given in Eq. (D4) agrees with the one obtained from Eq. (D13) by applying the asymptotic expansion of the Hankel function given in Eq. (D10). Furthermore, for \(x = 0\) the integrals in Eq. (D7) can be solved using Eq. (D12) and we obtain
\[
W_1(0) = 2, \ W_2 = 1/3.
\] (D16)

These values are also obtained if the limit \(x \to 0\) is considered in Eq. (D13). Finally, the expressions in Eq. (D13) also obey the differential equations
\[
\frac{d}{dx} W_1(x) = 4i W_1(x) - i W_0(x),
\]
\[
\frac{d}{dx} W_2(x) = 4i W_2(x) - i W_1(x),
\] (D17)

obtained by differentiating Eq. (D11) under the integral. This can be verified using the relations [85]
\[
\frac{d}{dx} H^{(2)}_0(x) = -H^{(2)}_1(x),
\]
\[
\frac{d}{dx} H^{(2)}_1(x) = H^{(2)}_0(x) - \frac{1}{x} H^{(2)}_1(x).
\] (D18)
Summarizing, we obtain
\[ W_0(x) = (-2\pi i) e^{2ix} H_0^{(2)}(2x), \]
\[ W_1(x) = (-2\pi i) e^{2ix} \left[ H_0^{(2)}(2x) + i H_1^{(2)}(2x) \right], \]
\[ W_2(x) = \frac{\pi x}{3} e^{2ix} \left[ 4ix H_0^{(2)}(2x) - (4x + i) H_1^{(2)}(2x) \right]. \] (D19)

**Appendix E: Ritus functions**

The Ritus functions are defined by [23, 53, 71]
\[
 f(x) = i \int_0^\infty dt \exp \left[ -itx + t^3/3 \right]
 = \pi Gi(x) + i\pi \text{Ai}(x), \tag{E1a}
\]
\[
 f'(x) = \int_0^\infty tdtd \exp \left[ -itx + t^3/3 \right]
 = \pi Gi'(x) + i\pi \text{Ai}'(x), \tag{E1b}
\]
\[
 f_1(x) = \int_0^\infty \frac{dt}{t} \exp (-itx) \left[ \exp \left( -it^3/3 \right) - 1 \right] \tag{E1c}
\]
and
\[
 f_2(x) = \int_0^\infty \frac{dt}{t} \exp (-itx) \left[ \exp \left( -it^3/3 \right) - 1 \right]
 = -i \left[ xf_1(x) + f'(x) \right], \tag{E1d}
\]
where \( \text{Ai} \) and \( \text{Gi} \) are the Airy and Scorer function, respectively [53]. Note that in Ritus’ work the normalization of the Airy function is different and also changes (see [23], appendix C and [19], Eq. B5). The functions defined in Eq. (E1) obey the following differential equations [23, 53]
\[
 f''(x) = xf(x) - 1, \tag{E2}
\]
\[
 f_1(x) = \frac{1}{x} f(x) - \frac{1}{x^2} f''(x). \tag{E3}
\]

Furthermore, for \( x > 0 \) we obtain [23]
\[
 f_1(x) = \int_0^x dt \left[ f(t) - 1/t \right], \tag{E3a}
\]
\[
 f_1(x) = \ln(x) + \frac{2}{3} \gamma + \frac{1}{3} \ln(3) + i \frac{\pi}{3} - \int_0^x dt f(t). \tag{E3b}
\]
The integral converges, as [53]
\[
 \text{Gi}(x) \sim \frac{1}{\pi x} \sum_{k=0}^\infty \frac{(3k)!}{k!3^k x^{3k}}. \tag{E4}
\]
The imaginary part of \( f_1(x) \) is related to [57]
\[
 \text{Ai}_1(x) = \int_x^\infty dt \text{Ai}(t)
 = \pi \left[ \text{Ai}(x) \text{Gi}'(x) - \text{Ai}'(x) \text{Gi}(x) \right]. \tag{E5}
\]
for \( x \geq 0 \) and
\[
 \text{Ai}_1(x) = \int_{-\infty}^x dt \text{Ai}(t)
 = \pi \left[ \text{Ai}(x) \text{Hi}'(x) - \text{Ai}'(x) \text{Hi}(x) \right] \tag{E6}
\]
for \( x \leq 0 \). Here \( \text{Hi} \) denotes the other Scorer function [53].

The asymptotic behavior of the Airy function and its derivative are given by [53]
\[
 \text{Ai}(z) \sim \frac{e^{-\zeta}}{2\sqrt{\pi} \zeta^{\frac{3}{2}}} \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(3k + \frac{1}{2})}{54^k k! \Gamma(k + \frac{1}{2})} \zeta^{-k} \tag{E7}
\]
and
\[
 \text{Ai}'(z) \sim -z^{\frac{1}{4}} e^{-\zeta} \sum_{k=0}^{\infty} (-1)^k \frac{6k + 1}{2k + 1} \frac{\Gamma(3k + \frac{3}{2})}{54^k k! \Gamma(k + \frac{3}{2})} \zeta^{-k}, \tag{E8}
\]
where \( \zeta \equiv \frac{2}{3}z^{3/2} \).

**Appendix F: Numerical calculation of oscillatory integrals**

The double-integral representation for the polarization operator given in Eq. (C19) contains oscillatory integrals in the variables \( \varrho \) and \( y \). The integral in \( y \) can be calculated using the Fast Fourier Transform (FFT) [88, 89]. The integral in \( \varrho \), however, is more complicated. While the phase \( \Phi_0 \) oscillates regularly, the phase \( \Phi_1 \) is nonlinear [due to the appearance of the field-dependent function \( \mathcal{D}(\varrho, ky) \), see Eq. (C21)]. However, the derivative
\[
 \frac{\partial \mathcal{D}(\varrho, ky)}{\partial \varrho} = 1 + \sum_{i=1,2} \xi_i^2 \left[ \psi_i(ky - 2\varrho) - I_i(\varrho, ky) \right]^2 \tag{F1}
\]
is always positive and therefore the change of variables \( u = \mathcal{D}(\varrho) \) can be applied to obtain an regularly oscillating integral [90]
\[
 \int_0^\infty \frac{d\varrho}{\mathcal{D}(\varrho)} \frac{\varrho g(\varrho)}{e^{-14(m^2/k)\mathcal{D}(\varrho)}}
 = \int_0^\infty \frac{du}{\mathcal{D}(\varrho)} \frac{g(\varrho)}{e^{-14(m^2/k)u}}, \tag{F2}
\]
where the inverse function \( \varrho = \mathcal{D}^{-1}(u) \) is calculated numerically [due to \( \mathcal{D}(\varrho) > 0 \] the map is one-to-one]. Having applied this change of variables, we obtain an ordinary Fourier integral, which can be evaluated with standard methods.

Fourier integrals (with finite limits) can be calculated very fast using Chebyshev series expansions [91, 93]. To this end we write
\[
 \int_a^b dx e^{iwx} g(x) = me^{iwc} \int_{-1}^{+1} dt e^{i\omega t} f(t), \tag{F3}
\]
where we used the change of variables \( x = c + mt \) with \( c = \frac{(a + b)}{2}, \ m = \frac{(b - a)}{2} \) and defined \( f(t) = g[x(t)], \ \Omega = \omega a. \) If the function \( f(t) \) is slowly varying, its expansion into a Chebyshev series is rapidly converging

\[ f(t) = \sum_{n=0}^{\infty} c_n T_n(t), \quad T_n(t) = \cos(n\theta), \quad t = \cos \theta, \]  

(Eq. [F4])

where

\[ c_n = \frac{2}{\pi} \int_{-1}^{1} dt \frac{T_n(t) f(t)}{\sqrt{1 - t^2}} = \frac{2}{\pi} \int_0^{\pi} d\theta \cos(n\theta) f(\cos \theta) \]  

(Eq. [F5])

(the prime at the sum symbol indicates that the first coefficient in the sum is halved). The Chebyshev series coefficients can be calculated using FFT. The absolute error due to the truncation of the Chebyshev series can be estimated from the last series coefficients [94].

Having computed the series coefficients, the Chebyshev moments

\[ C_n(z) = \int_{-1}^{1} dt T_{2n}(t) e^{izt}, \quad S_n(z) = i \int_{-1}^{1} dt T_{2n+1}(t) e^{izt} \]  

(F6)

must be calculated in order to evaluate the integral in Eq. [F3]. To this end we note that they obey the following three-term recurrence relations [92]

\[ z^2(n - 1)(2n - 1)C_{n+1}(z) - (n + 1)(n - 1) \left[ 4z^2 - 8(n + 1)(2n - 1) \right] C_n(z) + z^2(n + 1)(2n + 1)C_{n-1}(z) = -16(n - 1)(n + 1) \cos(z) + 12z \sin(z), \]  

(F7a)

\[ z^2(2n - 1)S_{n+1}(z) - (2n + 3)(2n - 1) \left[ z^2 - 8n(n + 1) \right] S_n(z) + z^2(2n + 3)(n + 1)S_{n-1}(z) = 4(2n - 1)(2n + 3) \sin(z) + 12z \cos(z). \]  

(F7b)

For certain parameters (e.g. for very large frequencies) the Chebyshev moments can be calculated by applying the above relations in the forward direction (e.g. \( S_n \) can be calculated by starting from \( S_0 \) and \( S_1 \)). However, this procedure is in general numerically unstable and Olver’s algorithm must be used [95, 96]. By calculating \( C_n \) and \( S_n \) independently, we can estimate the numerical error of the calculated Chebyshev moments by evaluating the relation [92]

\[ S_n(z) = \frac{\sin z}{2(n + 1)n} - \frac{z}{4n} C_n(z) + \frac{z}{4(n + 1)} C_{n+1}(z). \]  

(F8)

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