Intertwining Laplace Transformations of Linear Partial Differential Equations

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Abstract

We propose a generalization of Laplace transformations to the case of linear partial differential operators (LPDOs) of arbitrary order in $\mathbb{R}^n$. Practically all previously proposed differential transformations of LPDOs are particular cases of this transformation (intertwining Laplace transformation, $ILT$). We give a complete algorithm of construction of $ILT$ and describe the classes of operators in $\mathbb{R}^n$ suitable for this transformation.

Keywords: Integration of linear partial differential equations, Laplace transformation, differential transformation

1 Introduction

In the past decade a number of publications [4, 5, 12, 13, 14, 18, 19, 21, 22] were devoted to application of various differential substitutions to construction of algorithms for closed-form solution of linear partial differential equations or systems of such equations. The obvious drawback was just the vast diversity of such differential substitutions, often considered as absolutely different in properties and necessary tools for their study. As we show in this paper practically all the aforementioned approaches can be naturally unified into a very simple new class of Intertwining Laplace Transformations.

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1
We start with the classical Laplace cascade method. It is well known [1, 2, 3, 4] that second-order linear hyperbolic equations on the plane, for example

$$Lu = u_{xy} + a(x, y)u_x + b(x, y)u_y + c(x, y)u = 0,$$

admit the classical Laplace transformation based on the following equivalent forms of (1):

$$[(D_x + b)(D_y + a) - h(x, y)]u = 0$$  \hspace{1cm} (2)

or

$$[(D_y + a)(D_x + b) - k(x, y)]u = 0,$$

(3)

where $D_x = \frac{\partial}{\partial x}$, $D_y = \frac{\partial}{\partial y}$; $h = a_x + ab - c$, $k = b_y + ab - c$. Equation (2) is equivalent to the first-order system

$$\begin{cases} (D_y + a)u = u_1, \\
(D_x + b)u_1 = hu. \end{cases}$$

(4)

If $h \neq 0$, we can find $u$ from the second equation of the system (4) and substituting it into the first equation of the system (4) we obtain the transformed equation $L_1 u_1 = (u_1)_{xy} + a_1(x, y)(u_1)_x + b(x, y)(u_1)_y + c_1(x, y)u_1 = 0$. The operator $L_1$ is called the Laplace X-transformation of the operator $L$. A simple calculation shows that $L$ and $L_1$ are connected by an intertwining relation $(D_y + a_1)L = L_1(D_y + a)$. Analogously, if the invariant $k \neq 0$, we can define the Laplace Y-transformation of the operator $L$ using (3).

The transformations described above underlie the classical algorithm for finding solutions of certain equations of the form (1) (the Laplace cascade method). Namely, applying for example the Laplace X-transformation several times, in some cases, one can obtain an equation of the form $(D_x + b)(D_y + a)u = 0$, which can be integrated in quadratures. Then with the help of the inverse Laplace Y-transformation, its complete solution can be used to obtain the complete closed form solution of the original equation (1). See [1, 2] for more detail.

In [4] we described a simple method (actually dating back to Legendre, cf. [2]) to apply the Laplace transformation to the general second-order linear hyperbolic equations on the plane

$$Lu = u_{xx} + B(x, y)u_{xy} + C(x, y)u_{yy} + D(x, y)u_x + E(x, y)u_y + F(x, y)u = 0.$$  \hspace{1cm} (5)

Any hyperbolic equation (5) can always be written in the characteristic form

$$Lu = (X_1X_2 - H)u = 0,$$

(6)
where the coefficients of the operators $X_i = D_x + \lambda_i(x, y)D_y + \alpha_i(x, y)$ and the function $H(x, y)$ are constructively found from the coefficients of the original equation (5) (see [4, 5]). Using (6) we see that (5) is equivalent to the system

\[
\begin{aligned}
X_2 u &= v, \\
X_1 v &= Hu.
\end{aligned}
\] (7)

If $H = h(x, y)$ is a nonzero function, then from the second equation (7) we find $u = H^{-1}X_1 v$. Substituting it into the first equation of (7) we obtain the transformed equation

\[
L_1 v = (X_2 X_1 + \omega X_1 - H)v = 0,
\] (8)

where

\[
\omega = -[X_2, H]H^{-1},
\] (9)

where $[,]$ denotes the usual operator commutation. The operator $L_1$ is the result of the so-called $X_1$-Laplace transformation applied to the operator $L$. One can easily check that the operators $L$ and $L_1$ are connected by the intertwining relation

\[
M_1 L = L_1 M,
\] (10)

where $M = X_2$, $M_1 = X_2 + \omega$.

In [5] we have described a generalization of the Laplace transformation to second-order linear partial differential operators in $\mathbb{R}^3$ (and, generally, in $\mathbb{R}^n$) with the principal symbol decomposable into the product of two linear factors. This generalization is based on the fact, that such operators can be always represented in the form (6), but the coefficients $\alpha_i$ of the operators $X_i$ and the term $H$ are elements of the noncommutative ring of differential operators $F[D_z]$.

In dealing with these operators it is reasonable to use the algebraic construction of the noncommutative Ore field ([6, 7]) of formal ratios of differential operators $F(D_z) = \{R \mid R = P^{-1}Q; P, Q \in F[D_z]\}$ (where $F$ is some differential field of functions) with the equivalence relation: $P^{-1}Q \sim K^{-1}N$, if there exist $S, T \in F[D_z]$, $S \neq 0$ and $T \neq 0$ such that $SP = TK$ and $SQ = TN$.

More generally, any noncommutative ring $K$ for which the Ore conditions are satisfied (see below) is isomorphically embedded into the field $T = \{R \mid R = P^{-1}Q; P, Q \in K\}$ with the equivalence relation given above. The Ore conditions on the original noncommutative ring $K$ are as follows:

1. $K$ contains no zero divisors; i.e., if $AB = 0$ for some $A, B \in K$, then $A = 0$ or $B = 0$;
2. \( \forall A, B \in K, A \neq 0, B \neq 0, \exists P \neq 0 \text{ and } Q \neq 0 \text{ such that } PA = QB, \) and \( \exists M \neq 0 \text{ and } N \neq 0 \text{ such that } AM = BN. \)

It can be easily seen that the ring \( F[D_z] \) (and rings of operators with partial derivatives such as \( F[D_x, D_y, D_z] \)) meet the Ore conditions; therefore, \( F[D_z] \) is isomorphically embedded into the above-defined skew field \( F(D_z). \) This allows us to apply to the coefficients of the operator (6) all arithmetical operations, taking into account the property of noncommutativity. This will always result in a differential operator of the same type with coefficients in \( F(D_z). \) The skew field \( F(D_z) \) has external derivations \( D_x \) and \( D_y, \) which can evidently be extended from the initial ring \( F[D_z]. \) In the field \( F(D_z), \) the order of an element is determined correctly by the formula \( \text{ord} (P^{-1}Q) = \text{ord} (Q) - \text{ord} (P). \)

In [5] we showed that formulas (6)–(10) also hold for second-order operators in \( \mathbb{R}^n. \) But constructive results were obtained in [5] only for operators with decomposable principal symbol. For second-order equations in \( \mathbb{R}^3 \) this decomposability means that the principal symbol taken as second-order polynomial in formal commutative variables \( \xi_i = \frac{\partial}{\partial x_i} \) is decomposed into the product of two polynomials that are linear with respect to \( \xi. \) This restriction means that the operator \( H \) in (6) is a first-order operator with respect to \( D_z \) only.

In the present paper we give the definition of a natural generalization of the classical Laplace transformation for arbitrary operators in \( \mathbb{R}^n \) without any restriction on decomposability of the principal symbol. We will call below such generalization an Intertwining Laplace Transformation (ILT). We prove some general properties of such transformations, demonstrate its generality on a wide range of examples and give the general algorithm of construction of ILT in \( \mathbb{R}^n. \)

The paper is organized as follows. We give the general definition of the ILT in Sect. 2 and the general algorithm for its construction in Sect. 3. Section 4 contains a result on non-existence of ILT for a generic second-order operator in \( \mathbb{R}^n \) for \( n \geq 3. \) The generality of the notion of ILT is demonstrated in Sect. 5 on many famous first-order differential transformations of linear ordinary and partial differential equations. In Sect. 6 we discuss surjectivity and invertibility of ILT. Section 7 contains concluding remarks on possible future developments, in particular the statement of a general result on representability of arbitrary intertwining relation (10) with first-order intertwining operator \( M \) and arbitrary linear partial differential operator \( L \) in \( \mathbb{R}^n \) as ILT. The Appendix contains an important technical result establishing a correspondence between existence of an intertwining relation (10) and existence of the left least common multiple of the operators \( L \) and \( M \) in the
ring of linear partial differential operators.

2 Definition of Intertwining Laplace Transformations (ILLT)

Let $L$ be a general linear differential operator of arbitrary order in $\mathbb{R}^n$ with coefficients from some constructive differentially closed field of functions $\mathbb{F}$. Below to simplify notations we set $n = 3$. However, all results are true for arbitrary $n \geq 1$. Let $X_1$, $X_2$ be arbitrary differential operators from the ring of linear partial differential operators $\mathbb{F}[D_x, D_y, D_z]$, then one can always represent the operator $L$ in the following form

$$L = X_1X_2 - H,$$  \hspace{1cm} (11)

where $H = X_1X_2 - L$ is a differential operator in $\mathbb{F}[D_x, D_y, D_z]$ (in general of arbitrary order). We form

$$L_1 = X_2X_1 + \omega X_1 - H,$$  \hspace{1cm} (12)

where

$$\omega = -[X_2, H]H^{-1}$$  \hspace{1cm} (13)

is a (pseudo)differential operator (an element of the skew Ore field $\mathbb{F}(D_x, D_y, D_z)$). It is easy to check that the intertwining relation (10) automatically holds with the operators $M = X_2$ and $M_1 = X_2 + \omega$.

The formulas (10), (11), (12) hold in $\mathbb{F}(D_x, D_y, D_z)$. But it is difficult to use them for transformation of solutions of the equation $Lu = 0$ into solutions of $L_1v = 0$, the latter being a pseudodifferential equation in the general case. So we introduce the definition of Intertwining Laplace Transformation in which we impose the strong condition that $\omega$ should be a differential operator (an element of the subring $\mathbb{F}[D_x, D_y, D_z]$):

**Definition 1** We will say that the differential operators $L$ and $L_1$ defined above by the formulas (11) and (12) with the condition $\omega = -[X_2, H]H^{-1} \in \mathbb{F}[D_{x_1}, \ldots, D_{x_n}]$, are connected by an Intertwining Laplace Transformation (ILLT).

**Lemma 1** If the operators $L$ and $L_1$ are connected by an ILLT then their principal symbols coincide (even if $\text{ord}H \geq \text{ord}L$).

**Proof.** Since $\text{ord}\omega = \text{ord}X_2 - 1$, the principal symbols of the operators $M = X_2$ and $M_1 = X_2 + \omega$ coincide. Then (10) implies $\text{Sym}L = \text{Sym}L_1$. □
It is easy to see that \( \mathcal{ILT} \) is a generalization of the classical Laplace transformation of second-order operators in \( \mathbb{R}^2 \). However, it should be noted that even for dimension two there exist other transformations different from the classical Laplace transformation. They are defined with the help of the intertwining relation (10) with some differential operators \( M \) and \( M_1 \). Such transformations were described in [1] and will be considered in Sect. 5.3.

3 Algorithm of Construction of \( \mathcal{ILT} \) in \( \mathbb{R}^n \)

As we will see below in Sections 4, 5.1-5.7, existence and construction of intertwining relations (10) for a given operator \( L \) is a nontrivial problem. Even the “functional dimension” (number of functions of maximal number of variables) of the set of all possible pairs of operators \( (L, M) \) admitting an intertwining relation (10) is not known in the general case. In this Section we give an algorithm which may be used to construct an arbitrary \( \mathcal{ILT} \) with first-order \( M = X_2 \). It should be noted however that a given intertwining relations (10) may be represented as an \( \mathcal{ILT} \) in a non-unique way (see also Sect. 7).

Lemma 2 For first-order intertwining operator \( M = X_2 \) the element of the skew Ore field \( F(D_x, D_y, D_z) \omega = -[X_2, H]H^{-1} \) is a differential operator (and consequently \( L_1 \) and \( M_1 \) are differential operators) if and only if the operators \( H \) and \( X_2 \) satisfy the relation

\[
HX_2 = (X_2 + \psi(x, y, z))H
\]

with some function \( \psi \in F \).

Proof. Let \( \omega \) be a differential operator then we have \( [H, X_2] = \omega H \). Since \( \text{ord} X_2 = 1 \) we obtain \( \text{ord} \omega = 0 \). Thus \( \omega = \psi(x, y, z) \). This immediately implies (14). The converse is obvious. \( \square \)

From Lemma 2 immediately follows that if an \( \mathcal{ILT} \) connects operators \( L \) and \( L_1 \) with a first-order intertwining operator \( X_2 \) then \( \omega \) is a function \( \psi \in F \). Below in such cases we will denote \( \omega \) as \( \psi \).

It is well known (the theorem on rectification of a vector field in a neighborhood of each nonsingular point—a point where the vector field is nonzero) that an arbitrary first-order operator \( X_2 \) may be locally transformed to the form \( X_2 = D_x + \alpha(x, y, z) \) with \( \alpha \in F \) by an appropriate (nonconstructive!) coordinate transformation in a neighborhood of a generic point. For this we need \( F \) to be large enough to include the necessary for this functions. In the new variables the relation (14) has the form

\[
H(D_x + \alpha) = (D_x + \alpha + \psi)H.
\]
Multiplying (15) on the left and on the right by some functions $\mu(x, y, z)$ and $\rho(x, y, z)$ respectively we obtain

$$
(\mu H \rho)(D_x + \alpha + \rho^{-1}\rho_x) = (D_x - \mu^{-1}\mu_x + \alpha + \psi)(\mu H \rho). 
$$

(16)

So the functions $\mu$ and $\rho$ may be chosen in such a way that (16) will have the form of commutation relation

$$
\tilde{H}D_x = D_x\tilde{H}
$$

(17)

for the operator $\tilde{H} = \mu H \rho$. Again we suppose that $F$ is large enough to include $\rho$ and $\mu$. The following Lemma may be easily proved by explicit computations.

**Lemma 3** The differential operator $\tilde{H}$ in $\mathbb{R}^n$ satisfies the commutation relation (17) if and only if the coefficients of $\tilde{H}$ do not depend on $x$.

Now we can formulate the complete algorithm of construction of arbitrary $\mathcal{ILT}$ in $\mathbb{R}^n$:

1. Take an operator $\tilde{H}$ in $\mathbb{R}^n$ with coefficients not depending on a variable $x$.

2. Form the operator $H = \theta_1\tilde{H}\theta_2$, where $\theta_i$ are arbitrary functions in $F$. Using the relation (16) we find the functions $\alpha$ and $\psi$.

3. Make an arbitrary change of variables in $\mathbb{R}^n$ and find the images of the operators $(D_x + \alpha)$, $H$ and the function $\psi$ in the new variables. They are precisely the operators $X_2$, $H$ and the function $\psi$ in (14).

4. Taking $L = X_1X_2 - H$ and $L_1 = X_2X_1 + \psi X_1 - H$ with an arbitrary operator $X_1$ and $M = X_2$, $M_1 = X_2 + \psi$ we obtain a general example of the $\mathcal{ILT}$.

**Remark.** Note that this algorithm is able to produce only different examples of $\mathcal{ILT}$ with different operators $L$, $L_1$, $M$. The problem of construction of $\mathcal{ILT}$ for a given $L$ is very difficult in the general case and will not be addressed here. Some particular methods of construction of $\mathcal{ILT}$ for some classes of operators $L$ are given in Section 5.

**Example of second-order operators in $\mathbb{R}^3 = \{(x, y, z)\}$ connected by an $\mathcal{ILT}$.** The operator $X_1$ may be chosen arbitrarily: we take $X_1 = x^2D_y + xyD_z + 1$. Following the algorithm we take $H = xD_z^2x^2$ and find $X_2 = D_x + \frac{2}{x}$ and $\omega = \psi = -\frac{2}{x^2}$. We omit the step 3, i.e. we will not change the variables. Finally we obtain the operators

$$
L = X_1X_2 - H = x^2D_xD_y + xyD_zD_x - x^3D_z^2 + D_x + 2xD_y + 2yD_z + 2/x,
$$

7
\[ L_1 = X_2 X_1 + \psi X_1 - H = x^2 D_x D_y + xy D_y D_x - x^3 D_x^2 + D_x + x D_y - 1/x. \]

The intertwining relation (10) has the form \((D_x - 1/x)L = L_1(D_x + 2/x)\).

4 On Non-existence of \(\mathcal{ILT}\) for General Second-Order Operators in \(\mathbb{R}^n\)

**Theorem 1** For a general second-order differential operator \(L\) in \(\mathbb{R}^n\) there are no \(\mathcal{ILT}\) with first-order operators \(M\) for \(n > 2\).

**Proof.** Actually, following the algorithm we see that the number of arbitrary functions of \(n\) variables (we do not take into consideration functions of smaller number of variables) participating in the process of construction of \(\mathcal{ILT}\) do not exceed \(2n + 3\) (two functions of \(n\) variables \(\theta_i\) on step 2, \(n\) functions on step 3 and \((n + 1)\) coefficients of \(X_1\)). On the other hand the number of the coefficients in a second-order differential operator in \(\mathbb{R}^n\) equals \(\frac{n(n+1)}{2} + n + 1 > 2n + 3\), for \(n > 2\). Now the statement of the theorem is obvious. \(\square\)

Note that the given estimate \(2n + 3\) of the functional dimension of the set of all \(\mathcal{ILT}\) for second-order operators in \(\mathbb{R}^n\) is just an upper bound, since different intermediate data (operators \(H, X_2\), functions \(\theta_i\) etc. on the first steps) may result in the same resulting operators \(X_1, X_2, H\) in the final result of the algorithm. It would be interesting to give a precise estimate of this functional dimension.

5 Representation of Different Intertwining Relations as \(\mathcal{ILT}\)

In this section we show that many well known examples of differential transformations of linear differential operators can be represented as particular examples of the \(\mathcal{ILT}\) introduced in the previous Sections. We do this for:

1. gauge transformation \(L \to \lambda^{-1}L\lambda\) where \(\lambda \in \mathbb{F}\) is an arbitrary function;
2. differential substitutions for linear ordinary differential operators and classical Darboux transformation for one-dimensional Schrödinger operator;
3. classical Laplace transformation and Darboux transformations for \(L = D_x D_y + a(x, y)D_x + b(x, y)D_y + c(x, y)\).
4. Euler-Darboux transformation \((3)\) for operators in \(\mathbb{R}^n\) of the form

\[
L = \sum_{i=0}^{k} a_i(x) D_i^x + \sum_{|\alpha| \geq 0} b_\alpha(y) D_\alpha^y,
\]

where \(y = (y_1, \ldots, y_{n-1})\), \(\alpha = (\alpha_1, \ldots, \alpha_{n-1})\), \(D_\alpha^y = \frac{\partial^{|\alpha|}}{(\partial y_1)^{\alpha_1} \cdots (\partial y_{n-1})^{\alpha_{n-1}}}
\)

5. Darboux transformations for parabolic operators \(L = D_x^2 + a(x, y) D_x + b(x, y) D_y + c(x, y)\) on the plane;

6. Petrén transformation \((17)\) for higher-order operators

\[
L = \sum_{i=0}^{n-1} A_i(x, y) D_x D_i^y + \sum_{i=0}^{n-1} B_i(x, y) D_i^y;
\]

7. Dini transformation \((5, 18)\) for second-order operators in \(\mathbb{R}^3\) with decomposable principal symbol.

All aforementioned transformations are usually represented as intertwining relations

\[
M_1 L = L_1 M.
\]

From \((18)\) we conclude that any solution \(u\) of the equation \(Lu = 0\) is transformed into a solution \(v = Mu\) of the transformed equation \(L_1 v = 0\). Usually \((18)\) is considered to be fundamentally different from the classical Laplace transformation described in Sect. 1, since in many cases the mapping \(v = Mu\) of the solution space of the original equation \(Lu = 0\) has a nontrivial kernel (so is not invertible) unlike the classical Laplace transformation (see Sect. 6).

We will use below the following precise definition of intertwining relations:

**Definition 2** Relation \((18)\) with given differential operators \(L, L_1, M, M_1\) is called an intertwining relation between operators \(L\) and \(L_1\) with intertwining operator \(M\) if the following conditions are satisfied:

\[
\text{ord } L = \text{ord } L_1, \quad \text{ord } M = \text{ord } M_1, \quad (19)
\]

\[
\text{Sym } L = \text{Sym } L_1. \quad (20)
\]

In Appendix we discuss the relation of the conditions \((19), (20)\) to the existence of the left least common multiple of the operators \(L\) and \(M\) in the ring of linear partial differential operators.

First we establish a simple general result about representability of \((18)\) as an \(\mathcal{ILT}\).
Proposition 1 Let operators \( L, L_1, M, M_1 \) and its intertwining relation \( M_1 L = L_1 M \) be given. If there exists an operator \( X_1 \) which satisfies the equation
\[
[X_1, X_2] - \omega X_1 = L - L_1 ,
\]
with \( X_2 = M, \omega = M_1 - M \), then \( L \) and \( L_1 \) are connected by an ILT.

Proof. We should prove that all conditions of Def. 1 are satisfied. Suppose \( H = X_1 X_2 - L \) so \( L = X_1 X_2 - H \) with \( X_2 = M \). We should show that \( [H, X_2] = \omega H \). Actually, \( [H, X_2] = [X_1 X_2 - L, X_2] = [X_1, X_2] X_2 - [L, X_2] = (\omega X_1 - L_1) X_2 + X_2 L = \omega X_1 X_2 - (X_2 + \omega) L + X_2 L = \omega H \). The condition \( L_1 = X_2 X_1 + \omega X_1 - H \) follows automatically from (21).

So (21) may be used to find \( X_1 \) in order to represent (18) as an ILT. In fact in many particular cases of intertwining relations we use another trick to find \( X_1 \) directly. This will be explained in detail below.

5.1 Gauge Transformation \( L \to \lambda^{-1} L \lambda, \quad \lambda \in \mathbb{F} \)

In this case the intertwining relation (18) is trivial \( \lambda^{-1} L = L_1 \lambda^{-1} \), so \( M = M_1 = X_2 = \lambda^{-1}, \omega = 0 \) and \( X_1 \) has to be found from the condition (21), i.e.
\[
X_1 \lambda^{-1} - \lambda^{-1} X_1 = L - \lambda^{-1} L \lambda .
\]

Obviously \( X_1 = L \lambda + \varphi \lambda \) satisfies this equation with arbitrary function \( \varphi \). Then \( H = X_1 X_2 - L = \varphi, L_1 = X_2 X_1 - H = \lambda^{-1} L \lambda \), and all the required relations of the ILT are satisfied.

5.2 Differential Substitutions for Linear Ordinary Differential Operators and Classical Darboux Transformation for One-Dimensional Schrödinger Operator

Here we consider first the so-called Loewy-Ore formal theory of linear ordinary differential operators (LODO) which is described in [8]. For any two LODO \( L \) and \( M \) one can determine their right greatest common divisor \( rGCD(L, M) = G \), i.e. \( L = \tilde{L} G, M = \tilde{M} G \) (the order of \( G \) is maximal) and their left least common multiple \( lLCM(L, M) = K \), i.e. \( K = M L = L M \) (the order of \( K \) is minimal). This can be done using the (noncommutative) Euclid algorithm in \( \mathbb{F}[D_x] \). We say that the operator \( L \) is transformed into \( L_1 \) by an operator \( M \), and write \( L \xrightarrow{M} L_1 \), if \( rGCD(L, M) = 1 \) and \( K = lLCM(L, M) = L_1 M = M_1 L \). In this case any solution of \( Ly = 0 \) is
mapped by $M$ into a solution $z = My$ of $L_1z = 0$. Using the extended Euclid
algorithm one may constructively find an operator $N$ such that $L_1 \xrightarrow{N} L,$
$NM = 1 \pmod{L}$. Operators $L, L_1$ are also called similar or of the same kind
(in the given differential field $F$ of their coefficients). So for similar operators
the problem of solution of the corresponding equations $Ly = 0, L_1y = 0$ are
equivalent.

It is easy to represent the transformation $L \xrightarrow{M} L_1$ described above as an
$\mathcal{ILT}$. We consider the case $\text{ord } M = 1$ only. Obviously, using the Euclidean
division we obtain $L = QM + R$, where $R$ is a function and $R \neq 0$, since
$r\text{gcd}(L,M) = 1$. Thus if we take $X_1 = Q, X_2 = M, H = -R, \psi = \parallel H, X_2 \parallel / H$, so we obtain the $\mathcal{ILT}$ with $L = X_1X_2-H, L_1 = X_2X_1+\psi X_1-H,
M_1 = X_2 + \psi$ and the intertwining relation

$$M_1L = L_1M, \quad (22)$$

where $\text{ord } M = \text{ord } M_1 = 1, \text{ord } L = \text{ord } L_1$. Both sides of (22) coincide with
$l\text{lcm}(L, M)$ since it is unique up to a factor $\alpha \in F$.

Now let us consider the one-dimensional Schrödinger operator

$$L = -\frac{d^2}{dx^2} + u(x).$$

Let $\omega$ satisfy the equation $L\omega = 0$. The function $\omega$ determines a factorization
of $L$:

$$L = A^\top A, \quad A = -\frac{d}{dx} + v, \quad A^\top = \frac{d}{dx} + v, \quad v = \frac{\omega_x}{\omega}. \quad (23)$$

The Darboux transformation is simply swapping of $A^\top$ and $A$:

$$L = A^\top A \rightarrow \tilde{L} = AA^\top,$$

or in terms of the potential $u$: $u = u^2 + v_x \rightarrow \tilde{u} = v^2 - v_x = u - 2(\log \omega)_{xx}$. So we have the intertwining relation $AL = \tilde{L}A$. In order to represent this
transformation as the $\mathcal{ILT}$ we take for example $X_1 = \tilde{A}^\top, X_2 = A, H = 0,$
$\psi = 0$, then $L = X_1X_2-H = A^\top A, L_1 = X_2X_1+\psi X_1-H = AA^\top = \tilde{L}$. There is another possibility with $H \neq 0$. Namely, we can take $X_1 = A^\top + 1,$
$X_2 = A, H = A$. Then $\parallel H, X_2 \parallel = [A, A] = 0$, thus $\psi = 0$, and we have
$L_1 = X_2X_1+\psi X_1-H = A(A^\top + 1) - A = AA^\top = \tilde{L}$.

5.3 Classical Laplace Transformations and Darboux Transformations for $L = D_xD_y+a(x,y)D_x+b(x,y)D_y+c(x,y)$

The classical Laplace transformations for these operators are obviously a
particular case of the $\mathcal{ILT}$ (see Sect. [10]). Another type of differential transformation for this class of operators on the plane was studied by Darboux
Such Darboux transformation of order one is constructed using a solution $u_1$ of the original equation $Lu = 0$. Darboux takes $M = D_x + \mu$, $\mu = -(u_1)_x/u_1$ or $M = D_y + \nu$, $\nu = -(u_1)_y/u_1$, so that $Mu_1 = 0$, and proves that there exists an operator $\tilde{L} = D_x D_y + \tilde{a}(x, y)D_x + \tilde{b}(x, y)D_y + \tilde{c}(x, y)$ and $M_1 = D_x + \mu - \alpha x/\alpha$, $\alpha = b + (u_1)_x/u_1$ (with obvious modification for $M = D_y - (u_1)_y/u_1$), which satisfy $M_1 L = \tilde{L} M$. In order to represent this as an $\mathcal{ILT}$ one should solve (21) for the unknown operator $X_1$ with $\psi = -\alpha x/\alpha$, $X_2 = D_x - (u_1)_x/u_1$ and given $L$, $\tilde{L}$. But we use another trick taking into consideration the fact that the following system

\[
\begin{cases}
Lu = 0, \\
X_2 u = 0.
\end{cases}
\]

has a nontrivial solution $u_1(x, y)$. We follow the usual way of reducing (21) to involutive form, simplifying the first equation of (24) using the second one:

\[
L = D_x D_y + a(x, y)D_x + b(x, y)D_y + c(x, y) \rightarrow L - (D_y + a(x, y))X_2 = \alpha(D_y - \frac{(u_1)_y}{u_1}),
\]

where $\alpha = b + \frac{(u_1)_x}{u_1}$. We arrive at the system

\[
\begin{cases}
Hu = 0, \\
X_2 u = 0,
\end{cases}
\]

with $H = -\alpha(D_y - \frac{(u_1)_y}{u_1})$. The system (25) is obviously involutive and has one-dimensional solution space generated by $u_1$ (see the standard techniques of the Riquier-Janet theory for example in [9, 10, 11]). This suggests to take $X_1 = D_y + a$ and $H$ given above. As one can easily check the relation (21) with $L_1 = \tilde{L}$ is satisfied. Applying Prop. 1 we come to the desired representation of this Darboux transformation as an $\mathcal{ILT}$.

Darboux also studied transformations with higher-order operators $M$, $M_1$ (as compositions of first order Darboux transformations). We will limit ourselves to first order transformations and consider the case $M = D_x + q(x, y)D_y + r(x, y)$ (with $q(x, y) \neq 0$). This transformation is defined by two solutions $u_1$, $u_2$ of the original equation $Lu = 0$:

\[
Mu = \begin{vmatrix}
 u & u_y & u_x \\
 u_1 & (u_1)_y & (u_1)_x \\
 u_2 & (u_2)_y & (u_2)_x
\end{vmatrix}
\begin{vmatrix}
 u_1 & (u_1)_y \\
 u_2 & (u_2)_y
\end{vmatrix}^{-1},
\]

with the condition that $\begin{vmatrix}
 u_1 & (u_1)_x \\
 u_2 & (u_2)_x
\end{vmatrix} \neq 0$, $\begin{vmatrix}
 u_1 & (u_1)_y \\
 u_2 & (u_2)_y
\end{vmatrix} \neq 0$. We again form the system

\[
\begin{cases}
Lu = 0, \\
Mu = 0
\end{cases}
\]

(26)
and reduce it to the involutive form

\[
\begin{cases}
Hu = 0, \\
Mu = 0,
\end{cases}
\]  

with \( H = -(L - (D_y + a)M) = \gamma_2(x, y)D_x^2 + \gamma_1(x, y)D_y + \gamma_0(x, y) \). This system is in involution since it has two-dimensional solution space \( \langle u_1, u_2 \rangle \) (cf. [9, 10, 11]). Obviously the commutator \([H, X_2]\) (where \( X_2 = M \)) is a second order operator \( \theta_2 D_x^2 + \theta_1 D_y + \theta_0 \) with some coefficients \( \theta_i = \theta_i(x, y) \) and has the solution space \( \langle \gamma_1(x, y)D_x + \gamma_0(x, y) \rangle \). The last operator has the same solution space as \( H \), so they are proportional: \([H, X_2] = \gamma_2(x, y)D_x^2 + \gamma_1(x, y)D_y + \gamma_0(x, y) \) and has the solution space \( \langle \gamma_1(x, y)D_x + \gamma_0(x, y) \rangle \). Thus we again have represented this Darboux transformation with \( M = D_x + q(x, y)D_y + r(x, y) \) as an \( \mathcal{ILT} \).

Note that all considerations in this subsection are valid for any hyperbolic operator \( L \) with an arbitrary principal symbol \( \Omega \).

5.4 Euler-Darboux Transformation for Higher-Order Operators in \( \mathbb{R}^n \) of the Form \( L = \sum_{i=0}^k a_i(x)D_x^i + \sum_{|\alpha| \geq 0} \gamma_2(x, y)D_x^2 + \gamma_1(x, y)D_y + \gamma_0(x, y) \). This system is in involution since it has two-dimensional solution space \( \langle u_1, u_2 \rangle \) (cf. [9, 10, 11]). Obviously the commutator \([H, X_2]\) (where \( X_2 = M \)) is a second order operator \( \theta_2 D_x^2 + \theta_1 D_y + \theta_0 \) with some coefficients \( \theta_i = \theta_i(x, y) \) and has the solution space \( \langle \gamma_1(x, y)D_x + \gamma_0(x, y) \rangle \). The last operator has the same solution space as \( H \), so they are proportional: \([H, X_2] = \gamma_2(x, y)D_x^2 + \gamma_1(x, y)D_y + \gamma_0(x, y) \) and has the solution space \( \langle \gamma_1(x, y)D_x + \gamma_0(x, y) \rangle \). Thus we again have represented this Darboux transformation with \( M = D_x + q(x, y)D_y + r(x, y) \) as an \( \mathcal{ILT} \).

Note that all considerations in this subsection are valid for any hyperbolic operator \( L \) with an arbitrary principal symbol \( \Omega \).
then \( u_1 = h(x)g(y) \) satisfies (28). The EDT of the operator \( L \) is generated by its solution \( u_1 \). Namely (see [3]) the differential substitution \( w = hD_x h^{-1}u = (D_x - h_x/h)u \) maps solutions \( u \) of (28) into solutions \( w \) of another equation \( \tilde{L}w = 0 \) of the same class \( E_{k,m} \). This implies that the operators \( L \) and \( \tilde{L} \) satisfy the intertwining relation \( M_1L = \tilde{LM} \) with \( M = D_x - \frac{h_x}{h} \) and some first-order operator \( M_1 \). We again do not solve (21) directly and use the same trick as in Sect. 5.3. First we divide \( A \) by \( M \):

\[
A = QM + \phi(x).
\]

Since \( h(x) \) is a solution of (29) we see that \( \phi(x) = const = c \) and \( L = QM + c + B \). Now we can take \( X_1 = Q, X_2 = M, H = -(B+c) \) and obtain the necessary representation for the operator \( L \): \( L = X_1X_2 - H \).

We should check the condition \([H,X_2] = \psi H \) for some function \( \psi \). In fact \([H,X_2] = [-B - c, D_x - \frac{h_x}{h}] = 0 \) since the coefficients of \( B \) do not depend on \( x \). Thus \( \psi = 0 \), and \( L_1 = X_2X_1 - H = MQ + B + c \). Using Prop. 4 (see Appendix) we obtain that \( L_1 = \tilde{L} \).

5.5 Darboux Transformations for Parabolic Operators

\[
L = D_x^2 + a(x,y)D_x + b(x,y)D_y + c(x,y)
\]

We consider here the parabolic operator on the plane of the form

\[
L = D_x^2 + a(x,y)D_x + b(x,y)D_y + c(x,y), b(x,y) \neq 0.
\]

In [14] the authors have proved that for any operator (30) there exist infinitely many differential transformations of the operator \( L \) into the same form operators \( \tilde{L} \) which are defined by intertwining relation

\[
M_1L = \tilde{LM},
\]

with operator \( M \) of arbitrary order \( k \) generated by some set of independent solutions \( z_1(x,y), \ldots, z_k(x,y) \) of the equation \( Lz = 0 \). In contrast to the hyperbolic case considered in Sect. 5.3 there are no other differential transformations similar to the classical Laplace transformations. We limit ourselves to the case of first-order operators \( M = D_x + q(x,y)D_y + r(x,y) \).

**CASE A.** If \( q \neq 0 \) then the operator \( M \) is defined by conditions \( Mz_1 = 0, \ Mz_2 = 0 \) where \( z_1, z_2 \) are arbitrary linearly independent solutions of \( Lu = 0 \) ([14]). Here we call functions \( z_1(x,y), z_2(x,y) \) linearly independent if they satisfy the following conditions:

\[
\begin{vmatrix}
  z_1 (z_1)_x \\
  z_2 (z_2)_x
\end{vmatrix} \neq 0, \quad \begin{vmatrix}
  z_1 (z_1)_y \\
  z_2 (z_2)_y
\end{vmatrix} \neq 0.
\]
We construct this operator $M$ as in Sect. 5.3 using the following analogue of
the Wronskian formula:

$$Mu = \begin{vmatrix} u & u_y & u_x \\ z_1 & (z_1)_y & (z_1)_x \\ z_2 & (z_2)_y & (z_2)_x \end{vmatrix} \cdot \begin{vmatrix} z_1 & (z_1)_y \\ z_2 & (z_2)_y \end{vmatrix}^{-1} = (D_x + q(x, y)D_y + r(x, y))u.$$  

In order to represent (31) as an $\mathcal{ILT}$ we take $X_2 = M = D_x + q(x, y)D_y + r(x, y)$. We always can write the operator (30) in the form $L = QX_2 + R$
where $Q = D_x qD_y + (a - r)$ and $R = q^2D_y^2 + \alpha D_y + \beta$. Here $\alpha$, $\beta$ are expressed in terms of the coefficients of the operators $L$ and $M$ and their derivatives. Setting $X_1 = Q$, $H = -R$ we come to the required form $L = X_1X_2 - H$. The operators $H$ and $[H, X_2]$ are second order operators containing only $D_y$ and satisfying the condition $Hz_i = [H, X_2]z_i = 0$, $i = 1, 2$. The last condition defines both operators up to a functional multiplier. So $[H, X_2] = \psi(x, y)H$. This guarantees that the operators $L$ and $L_1 = X_2X_1 + \psiX_1 - H$ are connected by $\mathcal{ILT}$. By Prop. 4 (see Appendix) we obtain that $L_1 = \tilde{L}$.

**CASE B.** If $q \equiv 0$ then the operator $M$ is defined by one solution $z_1$ of $Lu = 0$ and should satisfy $Mz_1 = 0$ ([13]). The last condition implies that $M = D_x + r(x, y)$ where $r(x, y) = -(z_1)_x/z_1.$ As before we set $X_2 = M = D_x + r(x, y)$ and find the representation for the operator $L$: $L = QX_2 + R$
with $Q = D_x + (a - r)$, $R = bD_y + (c - r_x + r^2 - ar)$. Setting $X_1 = Q$, $H = -R$ we come to the required form $L = X_1X_2 - H$. The operators $H$ and $[H, X_2]$ are first order operators containing only $D_y$ and satisfying the condition $Hz_1 = [H, X_2]z_1 = 0$. This implies as in the Case A that there exists a function $\psi(x, y)$ such that $[H, X_2] = \psi(x, y)H$ and $\tilde{L} = L_1 = X_2X_1 + \psiX_1 - H$ with the help of Prop. 4.

### 5.6 Petrén Transformation ([17]) for Higher-Order Operators

$L = \sum_{i=0}^{n-1} A_i(x, y)D_x D_y^i + \sum_{i=0}^{n-1} B_i(x, y)D_y^i$

In [16] a differential transformation for a class of higher-order operators with two independent variables was proposed. L. Petrén has extensively studied this transformation in her thesis [17]. Below we will call this transformation Petrén transformation.

Petrén transformation applies to differential operators in $\mathbb{R}^2$ of the following form:

$$L = \sum_{i=0}^{n-1} A_i(x, y)D_x D_y^i + \sum_{i=0}^{n-1} B_i(x, y)D_y^i. \quad (32)$$

If we make a differential substitution $v = \alpha_0 D_y \alpha_0^{-1} u = (D_y - (\alpha_0)_y/\alpha_0)u$ for
any solution $u$ of the equation $Lu = 0$ with the function $\alpha_0(x, y)$ such that
\begin{equation}
\sum_{i=0}^{n-1} A_i(x, y) D_y^i \alpha_0 = 0, \tag{33}
\end{equation}
\begin{equation}
L \alpha_0 \neq 0, \tag{34}
\end{equation}
we obtain the transformed equation $\tilde{L}u = 0$ with operator $\tilde{L}$ of the same type (32) (see [17]). As we have already seen this means that $L$ and $\tilde{L}$ are connected by an intertwining relation $M_1 L = \tilde{L} M$, where $M = D_y - (\alpha_0)_y/\alpha_0$ and $M_1$ is some first-order differential operator. It will be shown in Appendix (see Theorem 3 and the paragraph before it) that such operator exists and its coefficients may be found constructively. We take $X_2 = M$. In order to find $X_1$ we first write the operator $L$ in the form
\begin{equation}
L = D_x \sum_{i=0}^{n-1} A_i(x, y) D_y^i + \sum_{i=0}^{n-1} (B_i(x, y) - (A_i(x, y))_x) D_y^i = D_x \hat{A} + \hat{B}. \tag{35}
\end{equation}
Using Euclidean division we can write
\begin{equation}
\hat{A} = \sum_{i=0}^{n-1} A_i(x, y) D_y^i = Q X_2 + q(x, y), \tag{36}
\end{equation}
where $Q$ is a differential operator of order $(n-2)$ and $q(x, y)$ is some function. Since $\hat{A} \alpha_0 = 0$ by (33) and $X_2 \alpha_0 = 0$ we have $q(x, y) \equiv 0$ in (36). Analogously for the operator $\hat{B}$ from (35) we have
\begin{equation}
\hat{B} = R X_2 + r(x, y). \tag{37}
\end{equation}
Substituting (36) and (37) into (35) we come to the form
\begin{equation}
L = (D_x Q + R) X_2 + r(x, y), \tag{38}
\end{equation}
with $r(x, y) \neq 0$ by (34). Setting $X_1 = D_x Q + R$ and $H = h(x, y) = -r(x, y)$ we obtain the required form $L = X_1 X_2 - h(x, y)$. The condition $[H, X_2] = \psi(x, y) H$ is obviously true with $\psi(x, y) = -h_y/h$ since $h(x, y)$ is a function. By Prop. 4 from this follows that $\tilde{L} = L_1 = X_2 X_1 + \psi X_1 - H$. Thus Petrén transformation is represented as an $\mathcal{ILT}$. We note that if $n = 2$ in (32) then we will get the classical Laplace transformation with the intertwining relation
\begin{equation}
(D_y + A_0 - h_y/h) L_1 = L(D_y + A_0)
\end{equation}
where $h = (A_0)_x + A_0 B_1 - B_0$, i.e. $h$ is the Laplace invariant for (1).
5.7 Dini transformation for second-order operators in $\mathbb{R}^3$ with decomposable principal symbols

In [18], an extension of the Dini transformation (hereafter, simply “Dini transformation”) was proposed that can be applied to second-order operators in $\mathbb{R}^3$ with decomposable principal symbols. Applied to such an operator $L = X_1X_2 - H$ with first-order operators $X_1, X_2, H$, this transformation takes a solution $u$ of the original equation $Lu = 0$ into the solution $v$ of the system

\[
\begin{cases}
(X_2 + \nu)u = v, \\
\hat{H}u = (X_1 + \mu)v.
\end{cases}
\]

(39)

where $\hat{H} = H + \mu X_2 + \nu X_1 + [X_1, \nu] + \mu \nu$ and the functions $\mu, \nu \in \mathbb{F}$ are to be chosen in such a way that the old function $u$ can be eliminated from (39) obtaining a second-order equation $\tilde{L}v = 0$, rather than an overdetermined system of equations for $v$ (which is the case for an arbitrary system of the form (39)). The latter means that the commutator $[\hat{H}, (X_2 + \nu)]$ can be expressed in terms of the operators $(X_2 + \nu)$, $\hat{H}$ themselves:

\[
[\hat{H}, (X_2 + \nu)] = \kappa(x, y, z)\hat{H} + \varrho(x, y, z)(X_2 + \nu).
\]

(40)

So we obtain the equation $L_{Dini}v = 0$ with the transformed operator

\[
L_{Dini} = (X_2 + \nu)(X_1 + \mu) - \hat{H} + \kappa(X_1 + \mu) + \varrho.
\]

(41)

Condition (40) differs from our condition (14) by the presence of the second term in the right-hand side and formally seems to be more general than (14). In [5] we have shown that in fact the conditions (40) and (14) for existence of such $\mu, \nu$ are equivalent for the given operator $L$ and the resulting transformed operators $L_{Dini}$ of the Dini transformation and $L_1$ of ILLT are also the same. If we introduce $\hat{X}_1 = X_1 + \mu, \hat{X}_2 = X_2 + \nu$, then $L = \hat{X}_1 \hat{X}_2 - \hat{H}$; (40) converts into

\[
[\hat{H}, \hat{X}_2] = \kappa(x, y, z)\hat{H} + \varrho(x, y, z)\hat{X}_2.
\]

(42)

(41) converts into

\[
L_{Dini} = \hat{X}_2 \hat{X}_1 - \hat{H} + \kappa \hat{X}_1 + \varrho.
\]

(43)

Below we will always omit the hat sign over $\hat{X}_i$ and $\hat{H}$ and write simply $X_i, H$, so for example $L_{Dini} = X_2X_1 - H + \kappa X_1 + \varrho$, the same for (42) and (43). Introducing an extra function $\alpha$ one can write the operator $L$ in the form

\[
L = X_1X_2 - H = (X_1 + \alpha)X_2 - \tilde{H}
\]

(44)
with arbitrary $\alpha \in F$ and $\tilde{H} = H + \alpha X_2$. By Def. 1 $L$ admits ILT if there exists $\alpha$ such that the following condition is satisfied:

$$[\tilde{H}, X_2] = \psi \tilde{H}$$

(45)

with $\psi \in F$.

**Proposition 2** If some first-order operators $H$ and $X_2$ satisfy the condition (42) with some functions $\varpi$ and $\varrho$ then there exists a function $\alpha$ such that $X_2$ and $\tilde{H} = H + \alpha X_2$ satisfy (45) with $\psi = \varpi$. The function $\alpha$ is a solution of the equation

$$[X_2, \alpha] + \varpi \alpha = \varrho.$$  

(46)

**Proof.** Let $\alpha$ be an arbitrary function then obviously $[H + \alpha X_2, X_2] = [H, X_2] + [\alpha, X_2]X_2$. Using (42) for $[H, X_2]$ we see that $[H + \alpha X_2, X_2] = \varpi(H + \alpha X_2) + (\varrho - \varpi \alpha + [\alpha, X_2])X_2$. Thus if $\alpha$ satisfies $\varrho - \varpi \alpha + [\alpha, X_2] = 0$ then (46) is satisfied with $\psi = \varpi$. □

**Theorem 2** Let $L$ be a second-order operator in $\mathbb{R}^n$ with decomposable principal symbol and there exists its representation $L = X_1 X_2 - H$ with first-order operators $X_i$, $H$ such that the condition (42) is satisfied, i.e. $L$ admits Dini transformation with the resulting operator $L_{Dini}$. Then there exists a function $\alpha$ such that the operator $L$ represented in the form $L = (X_1 + \alpha)X_2 - (H + \alpha X_2) = \tilde{X}_1 X_2 - H$ admits ILT with the resulting operator $L_1 = X_2 \tilde{X}_1 + \psi \tilde{X}_1 - H = L_{Dini}$.

**Proof.** From Prop. 2 we obtain $[\tilde{H}, X_2] = [H + \alpha X_2, X_2] = \varpi(H + \alpha X_2)$ with $\alpha$ satisfying (46). So by Def. 1 we can apply to $L$ the ILT and come to the transformed operator $L_1 = X_2(X_1 + \alpha) + \varpi(X_1 + \alpha) - \tilde{H} = X_2 X_1 + \varpi X_1 - (H + \alpha X_2 - X_2 \alpha - \varpi \alpha) = L_{Dini}$ since $\alpha$ satisfies (46). □

It should be noted that there is even a theorem in [18] stating that Dini transformation can be applied to any second-order operators in $\mathbb{R}^3$ with a decomposable principal symbol (i.e., appropriate $\mu$ and $\nu$ in operators $\hat{X}_i$, $\hat{H}$ can always be found). Unfortunately, there is a grave mistake in the proof of that theorem. In fact, Dini transformation can be applied just to those operators to which the Intertwining Laplace Transformation introduced here is applicable. As we have shown in [5], this is not possible for arbitrary second-order operators in $\mathbb{R}^3$ (see also Theorem 1).
6 Mapping of the Solution Spaces for $\mathcal{ILT}$ and their Inverses

If we take the solution space $S(L) = \{u | Lu = 0\}$ and any intertwining relation \((18)\) we obtain a linear mapping $M : S(L) \rightarrow S(L_1)$. Since in many cases considered in Sect. 5 and its subsections the operator $M$ has solutions $z_i \in S(L)$, this mapping of the solution space has a nontrivial kernel for such cases. As we have seen

$$\begin{cases} Lu = 0, \\ Mu = 0, \end{cases} \iff \begin{cases} Hu = 0, \\ Mu = 0, \end{cases}$$

(47)

here $X_2 = M$ and $H = X_1X_2 - L$. If $H$ and $X_2$ are first-order operators with different principal symbols we conclude from the condition $[H, X_2] = \psi(x, y)H$ that \((47)\) is compatible and has one-dimensional solution space in $\mathbb{R}^2$ and infinite-dimensional solution space in $\mathbb{R}^3$ (see the basics of Riquier-Janet theory in \([9, 10, 11]\)). In some cases considered in Sect. 5.3, 5.5 the operator $H$ had order two and the solution space of \((47)\) was two-dimensional.

On the other hand it is easy to prove that in many cases the mapping $M : S(L) \rightarrow S(L_1)$ is surjective. This is true for all cases studied in Sect. 5.1, 5.3–5.7 and obviously not true for the one-dimensional Darboux transformation (Sect. 5.2). In fact we should prove in the aforementioned nontrivial cases that for any $v \in S(L_1)$ there exists $u \in S(L)$ such that $Mu = X_2u = v$. This means that the following system

$$\begin{cases} Lu = 0, \\ X_2u = v \end{cases}$$

(48)

should have a solution iff $L_1v = 0$. This system is obviously equivalent to

$$\begin{cases} Hu = X_1v, \\ X_2u = v. \end{cases}$$

In order to understand its compatibility conditions we again use the Riquier-Janet theory \([9, 10, 11]\). Omitting the technical details one gets the following result: existence of a solution of this system is equivalent to the condition $X_2Hu - HX_2u = X_2X_1v - Hv$ (in fact the result of cross-differentiation of the equations of this system if $GCD(\text{Sym} H, \text{Sym} X_2) = 1$). Since $[H, X_2]u = \psi Hu = \psi X_1v$ we come to the equation $X_2X_1v + \psi X_1v - Hv = L_1v = 0$. Thus the system \((48)\) is compatible so the mapping $M : S(L) \rightarrow S(L_1)$ is surjective.

Let us write the transformed operator $L_1$ in the form

$$L_1 = X_2X_1 + \psi X_1 - H = (X_2 + \psi)X_1 - H = \tilde{X}_2X_1 - H,$$

(49)
where \( \psi = [H, X_2]H^{-1} \). Then we can again apply to \( L_1 \) a formal transformation in the skew Ore field \( F(D_{x_1}, \ldots, D_{x_n}) \) defined by the following formulas:

\[
(X_1 + \sigma)L_1 = \tilde{L}_1 X_1,
\]

(50)

\( \sigma = -[X_1, H]H^{-1}, \tilde{L}_1 = X_1 \tilde{X}_2 + \sigma \tilde{X}_2 - H \). Note that \( \sigma \in F(D_{x_1}, \ldots, D_{x_n}) \) need not to be a differential operator. It is easy to check that \( \tilde{L}_1 = HLH^{-1} \).

Substituting it into (50) we obtain

\[
(X_1 + \sigma)L_1 = HLH^{-1}X_1 \text{ or } H^{-1}(X_1 + \sigma)L_1 = LH^{-1}X_1.
\]

Denoting \( N = H^{-1}(X_1 + \sigma) \), \( N_1 = H^{-1}X_1 \) we get the intertwining relation \( NL_1 = LN_1 \) which defines a formal transformation with \( N, N_1 \in F(D_{x_1}, \ldots, D_{x_n}) \). This may be considered as a pseudodifferential inverse of the \( \mathcal{ILT} \) of the operator \( L \) into \( L_1 \). Note that we had to change \( X_2 \) to \( \tilde{X}_2 \) in the representation of \( L_1 \) in order to obtain this formal inverse. If one uses the representation \( L_1 = X_2X_1 - (H - \psi X_1) = X_2X_1 - \tilde{H} \) and formally follows the intertwining Laplace algorithm then the resulting operator will not coincide with \( L \).

Nevertheless one should note that for some particular operators \( L \) and mapping operators \( M \) there exist differential operators \( N \) mapping the solution space \( S(L_1) \) onto \( S(L) \) even if \( M : S(L) \to S(L_1) \) has a nontrivial kernel. Examples of such operators \( L, L_1, M, N \) can be found in [14] where existence of such differential operators \( N \) was related to famous nonlinear integrable equations for the coefficients of \( L \).

7 Conclusion

As we have demonstrated in the previous sections, the notion of \( \mathcal{ILT} \) unifies many differential transformations of linear partial (and ordinary) differential equations previously considered as fundamentally different.

The methods used in Sect. 5 for representation of various intertwining relations as \( \mathcal{ILT} \) may be used to prove the following general result:

**Proposition 3** Let \( L \) be a linear differential operator of arbitrary order in \( \mathbb{R}^n \) and the following intertwining relation

\[
M_1 L = L_1 M
\]

(51)

holds for first-order operators \( M, M_1, \) and \( \text{Sym} L = \text{Sym} L_1 \). Then \( L \) is transformed by an \( \mathcal{ILT} \) to the operator \( \alpha^{-1}L_1\alpha \) with \( X_2 = \alpha^{-1}M \), where \( \alpha \) is the coefficient at \( D_{x_2} \) in \( M \) (for any chosen \( i \)).

The details of the proof and other developments on construction of \( \mathcal{ILT} \) for a given operator \( L \) will be given elsewhere. This proposition shows that the
results of Sections 3, 4 may be actually formulated for arbitrary intertwining relations with first-order operators $M, M_1$.

A step into the direction of investigation of intertwining relations with higher-order operators $M, M_1$ may be potentially obtained following the recent result [13], where the author had proved that for the particular case of classical Laplace operators $M$ higher-order intertwining relations may be represented as compositions of first-order $\mathcal{ILT}$.

Another important domain of applications for differential transformations is the category of systems of linear partial differential equations, cf. for example [4, 19, 20]. In fact, already Le Roux [16] had noted that it is much more natural to study such transformations, since any differential substitution $v = Mu$ transforms the solution space of a scalar equation $Lu = 0$ for generic $L, M$ into the solution space of a system. Precisely the transition from a higher-order scalar strictly hyperbolic equation $Lu = 0$ in $\mathbb{R}^2$ to an equivalent first-order characteristic system was used in [19] to describe a generalization of the Laplace transformation in this case. For a good definition of general intertwining relations for linear (probably overdetermined or underdetermined) systems we need a deeper understanding of the notion of differential transformation itself since any differential mapping of the solution set of such a general system gives the solution space of another (overdetermined in general) system unlike the case of scalar equations $Lu = 0$ where description of possible intertwining relations is not trivial. Probably a generalization of the notion of $\mathcal{ILT}$ to systems may be of great use. See also [21] for a categorical definition of differential transformations and factorizations for systems of linear partial differential equations.

So far we did not succeed in representing the important Moutard transformation [1, 2, 22] for two-dimensional stationary Schrödinger equation as an $\mathcal{ILT}$. This is a challenging problem since in the categorical treatment Moutard transformation is a natural member of the class of (pseudo)differential transformations in the Serre-Grothendieck factorcategory of systems ([21]). The same similarity of the Moutard transformation with differential transformations was exposed in [22] in terms of the skew Ore field of formal fractions of differential operators.
Appendix: Intertwining Relations and Left Least Common Multiples of Linear Partial Differential Operators

It is well known (cf. for example [8]) that there always exists the left least common multiple (lLCM) for every pair $L$ and $M$ of linear ordinary differential operators. This is not always the case for linear partial differential operators $L$ and $M$. This is related to the algebraic fact that all left (and right) ideals in $\mathbb{F}[D_x]$ are principal ideals, but left ideals in $\mathbb{F}[D_x, D_y]$ are not always principal. In this Appendix we prove that for many (but not all) examples of intertwining relations $M_1L = L_1M$ considered in Sect. 5 in fact $M_1L = L_1M = lLCM(L, M)$. Note that in this case if we know the coefficients of the operators $L$ and $M$ in the intertwining relation (18) we can find constructively the coefficients of the operators $L_1$ and $M_1$ from the corresponding system of algebraic equations.

**Theorem 3** Let $L$ and $M$ be linear partial differential operators in $\mathbb{R}^n$ such that $\text{ord} L \geq 1$, $\text{ord} M = \text{ord} M_1 = 1$ and $L$ is not right divisible by $M$. If $L$ and $M$ satisfy an intertwining relation $M_1L = L_1M$ with

\[ \text{Sym} L = \text{Sym} L_1 \quad (52) \]

then

\[ lLCM (L, M) = M_1L = L_1M. \quad (53) \]

**Proof.** From the conditions of this theorem we conclude that $L$ and $L_1$ are connected by an intertwining relation (see Definition 2 in Sect. 5). Hence $\text{Sym} M = \text{Sym} M_1$. Suppose that

\[ K = PL = QM \quad (54) \]

is some left common multiple of the operators $L$ and $M$. It should be proved that there exists an operator $G$ such that

\[ P = GM_1, \quad Q = GL_1. \quad (55) \]

We prove this by induction on the order of $P$.

**Case 1.** $\text{GCD}(\text{Sym} L, \text{Sym} M) = 1$.

Then using (54) we see that $\text{Sym} P$ should be divisible by $\text{Sym} M$ and $\text{Sym} Q$ should be divisible by $\text{Sym} L$. So we can choose some operator $G_1$ such that $\text{Sym} P = \text{Sym} G_1 \cdot \text{Sym} M = \text{Sym} G_1 \cdot \text{Sym} M_1$ and $\text{Sym} Q = \text{Sym} G_1 \cdot \text{Sym} L = \text{Sym} G_1 \cdot \text{Sym} L_1$. Subtracting from (54) the identity
$G_1M_1L = G_1L_1M$ we obtain $(P - G_1M_1)L = (Q - G_1L_1)M$ with \( \text{ord}(P - G_1M_1) < \text{ord} P, \text{ord}(Q - G_1L_1) < \text{ord} Q. \) So we come to some lower-order left common multiple $K_1 = P_1L = Q_1M$. By induction (if \( \text{ord} P_1 \geq \text{ord} M, \text{ord} Q_1 \geq \text{ord} L \) there exists $G_2$ such that $P_1 = G_2M_1, Q_1 = G_2L_1$, so $P = (G_1 + G_2)M_1, Q = (G_1 + G_2)L_1$.

If $\text{ord} P_1 < \text{ord} M$ and $\text{ord} Q_1 < \text{ord} L$ then obviously in the case $\text{GCD}(\text{Sym} L, \text{Sym} M) = 1$ both $P_1$ and $Q_1$ vanish.

Case 2. $\text{GCD}(\text{Sym} L, \text{Sym} M) \neq 1$.

Since $\text{ord} M = 1$ we can choose some operator $S$ such that $\text{Sym} L = \text{Sym} M \cdot \text{Sym} S$ and $\text{Sym} L_1 = \text{Sym} M_1 \cdot \text{Sym} S$. Then we have

\[
L = SM + T, \quad L_1 = M_1S + T_1,
\]

with $\text{ord} T < \text{ord} L, \text{ord} T_1 < \text{ord} L_1$. From \(58\) and \(59\) we obtain $M_1(SM + T) = (M_1S + T_1)M$ or $M_1T = T_1M$, hence $\text{Sym} T = \text{Sym} T_1$.

If $\text{GCD}(\text{Sym} M, \text{Sym} T) 
eq 1$ we can again simultaneously reduce $T$ and $T_1$ by $M$ in \(56\) until we obtain \(56\) with the condition

\[
\text{GCD}(\text{Sym} M, \text{Sym} T) = 1.
\]

We again take some $\text{LCM}(L, M) = K$ and write it in the form \(54\). Using \(51\), \(56\) and \(57\) we come to $PT = (Q - PS)M$ with $\text{GCD}(\text{Sym} M, \text{Sym} T) = 1$. As it has been proved in the Case 1 there exists $G$ such that $P = GM_1$ and $Q - PS = GT_1$. From \(56\) we obtain $T_1 = L_1 - M_1S$, so $Q - PS = G(L_1 - M_1S)$ or $Q = GL_1$.

This implies the following proposition:

**Proposition 4** Let $L$ and $M$ be linear partial differential operators in $\mathbb{R}^n$ such that $\text{ord} L \geq 1, \text{ord} M = 1, L$ is not right divisible by $M$ and they satisfy two intertwining relations

\[
M_1L = L_1M, \quad (58)
\]

\[
\tilde{M}_1L = \tilde{L}_1M, \quad (59)
\]

where $\text{Sym} L = \text{Sym} L_1 = \text{Sym} \tilde{L}_1$. Then $M_1 = \tilde{M}_1, L_1 = \tilde{L}_1$.

**Proof.** Since $\text{lLCM}(L, M)$ is unique up to a functional multiplier, $\tilde{M}_1 = \phi M_1, \tilde{L}_1 = \phi L_1$ for some $\phi \in F$. From the equality $\text{Sym} L_1 = \text{Sym} \tilde{L}_1$ we see that $\phi \equiv 1$.

Note that we supposed that $L$ is not divisible by $M$. This is not always the case—see Sect. 5.1 and Sect. 5.2 (Darboux transformations for one-dimensional Schrödinger equations).

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