Three talks in Cuautitlan under the general title

Algebraic Topology Based on Knots
Topología algebraica basada sobre nudos
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1 Talk 1: Open problems in knot theory that everyone can try to solve.

Knot theory is more than two hundred years old; the first scientists who considered knots as mathematical objects were A.Vandermonde (1771) and C.F.Gauss (1794). However, despite the impressive grow of the theory, there are simply formulated but fundamental questions, to which we do not know answers. I will discuss today several such open problems, describing in detail the 20 year old Montesinos-Nakanishi conjecture. Our problems lead to sophisticated mathematical structures (I will describe some of them in tommorows talks), but today’s description will be absolutely elementary. Links are circles embedded in our space, $\mathbb{R}^3$, up to topological deformation, that is two links are equivalent if one can be deformed to the other in space without cutting and pasting. We represent links on a plane using their diagrams (we follow the terminology of Lou Kauffman’s talk).

First we should introduce the concept of an $n$ move on a link. One should stress that the move is not topological, so that it can change the type of the link we deal with.

**Definition 1.1** An $n$-move on a link is a local change of the link illustrated in Figure 1.1.

![n-move](n-move.png)

In our convention the part of the link outside of the disk in which the move takes part, is unchanged. For example $\equiv \rightarrow \equiv$ illustrates a 3-move.
Definition 1.2  We say that two links, $L_1, L_2$ are $n$-move equivalent if one can go from one to the other by a finite number of $n$-moves and their inverses ($-n$ moves).

If we work with diagrams of links then the topology of links is reflected by Reidemeister moves, that is two diagrams represent the same link in space if and only if one can go from one to the other by Reidemeister moves:

Thus we say that two diagrams, $D_1$ and $D_2$, are $n$-move equivalent if one can obtain one from the other by $n$-moves, their inverses and Reidemeister moves. To illustrate this let us notice that the move $\begin{array}{c} \text{3-move} \\ \end{array}$ is a consequence of a 3-move and a second Reidemeister move (Fig.1.3).

Conjecture 1.3 (Montesinos-Nakanishi)

Every link is 3-move equivalent to a trivial link.

Yasutaka Nakanishi first considered the conjecture in 1981. José Montesinos analyzed 3-moves before, in connection with 3-fold dihedral branch coverings, and asked a related but different question.
Examples 1.4  
(i) Trefoil knots are 3-move equivalent to the trivial link of two components:

(ii) The figure eight knot is 3-move equivalent to the trivial knot:

We will show later, in this talk, that different trivial links are not 3-move equivalent, however in order to achieve this conclusion we need an invariant of links preserved by 3-moves and different for different trivial links. Such invariant is the Fox 3-coloring. We will introduce it later (today in the simplest form and in the second lecture in a more general context of Fox $n$-colorings and Alexander-Burau-Fox colorings). Now let us present some other related conjectures.

Conjecture 1.5
Any 2-tangle is 3-move equivalent, to one of the four 2-tangles of Figure 1.6. We allow additionally trivial components in the tangles of Fig.1.6.

Montesinos-Nakanishi conjecture follows from Conjecture 1.5. More generally if Conjecture 1.5 holds for some class of 2-tangles, then Conjecture
1.3 holds for a link obtained by closing any tangle from the class, without introducing any new crossing. The simplest interesting tangles for which Conjecture 1.5 holds are algebraic tangles in the sense of Conway (I will call them 2-algebraic tangles and present their generalization in the next talk). For 2-algebraic tangles Conjecture 1.5 holds by an induction and I will leave it as a pleasant exercise for you. The necessary definition is given below:

**Definition 1.6 ([Co, B-S])**

2-algebraic tangles are the smallest family of 2-tangles which satisfies:

0) Any 2-tangle with 0 or 1 crossing is 2-algebraic.

1) If $A$ and $B$ are 2-algebraic tangles then $r^i(A) \ast r^j(B)$ is 2-algebraic; $r$ denotes here the rotation of a tangle by 90° angle along the z-axis, and $\ast$ denotes the (horizontal) composition of tangles.

A link is 2-algebraic if it is obtained from a 2-algebraic tangle by closing its ends without introducing any new crossings.

The Montesinos-Nakanishi conjecture was proven for many special families of links by my students Q. Chen and T. Tsukamoto [Che, Tsu, P-Ts]. In particular Chen proved that the conjecture holds for all five braid links except, possibly one family, represented by the square of the center of the 5-braid group, $(\sigma_1\sigma_2\sigma_3\sigma_4)^{10}$. Chen found 5-braid link 3-move equivalent to it with 20 crossings. It is now the smallest known possible counterexample to Montesinos-Nakanishi conjecture, Fig.1.7.

![Fig. 1.7](image-url)
Previously (1994), Nakanishi suggested the 2-parallel of the Borromean link (with 24 crossings) as a possible counterexample (Fig. 1.8).

We will go back tomorrow to theories motivated by 3-moves, now we will outline conjectures using other elementary moves.

**Conjecture 1.7 (Nakanishi, 1979)**

*Every knot is 4-move equivalent to the trivial knot.*

**Examples 1.8** *Reduction of the trefoil and the figure eight knot is illustrated in Figure 1.9.*

It is not true that every link is 5-move equivalent to a trivial link. One can show, using the Jones polynomial, that the figure eight knot is not 5-move equivalent to any trivial link. One can however introduce a more delicate move, called $(2,2)$-move ($\xrightarrow{\mathcal{K}}$) such that a 5-move is a combination of a $(2,2)$-move and its mirror image $(-2,-2)$-move ($\xleftarrow{\mathcal{K}}$), as illustrated in Figure 1.10 [H-U, P-3].
Conjecture 1.9 (Harikae, Nakanishi, 1992) *Every link is (2, 2)-move equivalent to a trivial link.*

As in the case of 3-moves, an elementary induction shows that the conjecture holds for 2-algebraic links (algebraic in the Conway’s sense). It is also known for all links up to 8 crossings. The key element of the argument is the observation (going back to Conway [Co]) that any link up to 8 crossings (different than 8_{18}) is 2-algebraic. The reduction of the 8_{18} knot to a trivial link of two components by my students, Jarek Buczyński and Mike Veve, is illustrated in Figure 1.11.
The smallest knot, not reduced yet is the 9\textsubscript{49} knot, Figure 1.12. Possibly you can reduce it!

With the next open question, I am much less convinced that the answer is positive, so I will not call it a conjecture. First let define a $(p,q)$ move as a local modification of a link as shown in Figure 1.13. We say that two links,
$L_1, L_2$ are $(p, q)$ equivalent if one can go from one to the other by a finite number of $(p, q), (q, p), (-p, -q)$ and $(-q, -p)$-moves.

![Diagram](image)

**Figure 1.13**

**Problem 1.10 ([Kir]; Problem 1.59(7), 1995)** Is it true that any link is $(2, 3)$ move equivalent to a trivial link?

**Example 1.11** Reduction of the trefoil and the figure eight knots is illustrated in Fig. 1.14. Reduction of the Borromean rings is performed in Fig. 1.15.

![Diagram](image)

**Figure 1.14**
Generally, rather simple inductive argument shows that 2-algebraic links are $(2, 3)$-move equivalent to trivial links. Figure 1.16 illustrates why the Borromean rings are 2-algebraic. By properly filling black dots one can also show that all links up to 8 crossings, but $8_{18}$, are 2-algebraic. Thus, as in the case of $(2, 2)$-equivalence, the only link, up to 8 crossings, which should be still checked is the $8_{18}$ knot. Nobody really tried this seriously, so maybe somebody in the audience will try this puzzle.

\[\text{Figure 1.15}\]

\[\text{To prove that the knot } 8_{18} \text{ is not 2-algebraic one would have to consider the 2-fold cover of } S^3 \text{ with this knot as a branching set and show that it not Waldhausen graph manifold. In fact it is a hypebolic manifold so cannot be a graph manifold.}\]
Fox colorings.
The 3-coloring which we will use to show that different trivial links are not 3-move equivalent, was introduced by Ralph H. Fox in about 1956 when explaining knot theory to undergraduate students at Haverford College (“in an attempt to make the subject accessible to everyone” \cite{C-F}). It is a pleasant method of coding representations of the fundamental group of a knot complement into the group of symmetries of a regular triangle, but this interpretation is not needed in the definition and most of applications of 3-colorings.

**Definition 1.12** (Fox 3-coloring of a link diagram).
Consider a coloring of a link diagram using colors \( r \) (red), \( y \) (yellow) or \( b \) (blue) in such a way that an arc of the diagram (from a tunnel to a tunnel) is colored by one color and at a crossing one uses one or all three colors. Such a coloring is called a Fox 3-coloring. If whole diagram is colored by just one color we say that we have a trivial coloring. Let \( \operatorname{tri}(D) \) denote the number of different Fox 3-colorings of \( D \).

**Example 1.13**
(i) \( \operatorname{tri}(\bigcirc) = 3 \) as the trivial diagram has only trivial colorings.

(ii) \( \operatorname{tri}(\bigcirc\bigcirc) = 9 \), and more generally for a trivial link diagram of \( n \) components, \( U_n \), one has \( \operatorname{tri}(U_n) = 3^n \).
(iii) For a standard diagram of a trefoil knot we have three trivial colorings and 6 nontrivial colorings, one of them is presented in Figure 1.17 (all other differ from this one by permutations of colors. Thus \( \text{tri}( \otimes ) = 3 + 6 = 9 \)

Fig. 1.17; Different colors are marked by lines of different thickness.

Fox 3-colorings were defined for link diagrams, they are however invariants of links. One needs only show that \( \text{tri}(D) \) is unchanged by Reidemeister moves.

The invariance under \( R_1 \) and \( R_2 \) is illustrated in Fig.1.18 and the invariance under \( R_3 \) is illustrated in Fig.1.19.
The next property of Fox 3-colorings is the key in proving that different trivial links are not 3-move equivalent.

**Lemma 1.14 ([P-1](#))** tri(D) is unchanged by a 3-move.

The proof of the lemma is illustrated in Figure 1.20.

![Fig. 1.20](image)

The lemma also explains the fact that the trefoil has nontrivial Fox 3-colorings: the trefoil knot is 3-move equivalent to the trivial link of two components (Example 1.4(i)).

Tomorrow I will place the theory of Fox coloring in more general (sophisticated context), and apply it to the analysis of 3-moves (and (2,2) and (2,3) moves) of n-tangles. Interpretation of tangle colorings as Lagrangians in symplectic spaces is our main (and new) tool. In the second lecture tomorrow, I will discuss another motivation for study 3-moves: to understand skein modules based on their deformation.

# 2 Talk 2: Lagrangian approximation of Fox p-colorings of tangles.

We just have heard beautiful and elementary talk by Lou Kauffman. I hope to follow his example by having my talk elementary and deep at the same time. I will use several results introduced by Lou, like rational tangles and their classification, and will also build on my yesterday’s talk. The talk will culminate in the introduction of the symplectic structure on the boundary of a tangle in such a way that tangles yields Lagrangians in the symplectic space. I could not dream of this connection a year ago; however now, after 10 month perspective, I see the symplectic structure as a natural development.

Let us start slowly from my personal perspective and motivation. In the spring of 1986, I was analyzing behavior of Jones type invariants of links
when a link was modified by a $k$-move (or $t_k$, $\bar{t}_k$ moves in the oriented case). My interest had its roots in the fundamental Conway’s paper [Co]. In July of 1986, I gave a talk at the "Braids" conference in Santa Cruz. I was told by Murasugi and Kawauchi, after my talk, about the Nakanoishi's 3-move conjecture. It was suggested to me by R.Campbell (Kirby’s student in 1986) to consider the effect of 3-moves on Fox colorings. Only several years later, when writing [P-3] in 1993 I realized that Fox colorings can be succesfully used to analyse moves on tangles, by considering not only the space of colorings but also the resulting coloring of boundary points. More of this later, now it is time to define Fox $k$-colorings.

**Definition 2.1** (i) We say that a link (or tangle) diagram is $k$-colored if every arc is colored by one of the numbers $0, 1, \ldots, k-1$ (forming a group $\mathbb{Z}_k$) in such a way that at each crossing the sum of the colors of the undercrossings is equal to twice the color of the overcrossing modulo $k$; see Fig. 2.1.

(ii) The set of $k$-colorings form an abelian group, denoted by $\text{Col}_k(D)$. The cardinality of the group will be denoted by $\text{col}_k(D)$. For an $n$-tangle $T$ each Fox $k$-coloring of $T$ yields a coloring of boundary points of $T$ and we have the homomorphism $\psi: \text{Col}_k(T) \rightarrow \mathbb{Z}_k^{2n}$.

\[
\begin{align*}
\text{c} &= 2a - b \mod(k) \\
\text{a} &\quad \text{b}
\end{align*}
\]

Fig. 2.1

It is a pleasant exercise to show that $\text{Col}_k(D)$ is unchanged by Reidemeister moves and by $k$-moves (Fig.2.2).

\[
\begin{align*}
\text{a} &\quad 2a - b \quad 3a - 2b \quad \ldots \\
\text{b} &\quad \text{a} \quad 2a - b \\
\end{align*}
\]

\[
\begin{align*}
(k+1)a - kb &= a \mod(k) \\
ka - (k-1)b &= b \mod(k)
\end{align*}
\]

Fig. 2.2
Let us look more carefully at the observation that a \( k \)-move preserves the space of Fox \( k \)-colorings and at the unlinking conjectures described till now. We discussed the 3-move conjecture and Nakanishi suggested also 4-move conjecture for knots (it does not hold for links\(^6\)). As I mentioned yesterday not every link can be simplified using 5-moves, but a 5-move is a combination of \((2, 2)\) moves and these moves possibly suffices to reduce every link. Similarly not every link can be reduced using 7-moves, but a 7-move is a combination of \((2, 3)\)-moves\(^7\) which possibly suffice for reduction. We stopped at this point yesterday, but what can one use instead of a general \( k \)-move? Let us consider the case of a \( p \)-move where \( p \) is a prime number. I suggest (and state publicly for the first time) that one should consider rational moves, that is, a rational \( \frac{p}{q} \)-tangle of Conway is substituted in place of the identity tangle\(^8\). The important observation for us is that \( \text{Col}_p(D) \) is preserved by \( \frac{p}{q} \)-moves. Fig.2.3 illustrates the fact that \( \text{Col}_{13}(D) \) is unchanged by a \( \frac{13}{5} \)-move.

\[ \begin{array}{c}
\text{a} \\
\text{c} \end{array} \] \quad \text{13/5 - move} \quad \begin{array}{c}
\text{a} \\
\text{c} \end{array}

Fig. 2.3

We should note also that \((m, q)\)-moves are equivalent to \( \frac{mq+1}{q} \)-moves (Fig.2.4) so the space of Fox \((mq + 1)\)-colorings is preserved.

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\(^2\)Kawauchi suggested that for links one should conjecture that two links are 4-move equivalent iff they are link homotopic.

\(^3\)To be precise, a 7-move is a combination of a \((-3, -2)\) and \((2, 3)\) moves; compare Fig.1.10.

\(^4\)The move was first considered by Y.Uchida [Uch].
We just have heard about the Conway’s classification of rational tangles at the Lou’s talk, so I will briefly sketch definitions and notation. The 2-tangle of Figure 2.5 is called a rational tangle with Conway notation $T(a_1, a_2, \ldots, a_n)$. It is a rational $\frac{p}{q}$-tangle if $\frac{p}{q} = a_n + \frac{1}{a_{n-1} + \ldots + \frac{1}{a_1}}$. 

Conway proved that two rational tangles are ambient isotopic (with boundary fixed) if and only if their slopes are equal (compare [Kaw]).

For a Fox coloring of a rational $\frac{p}{q}$-tangle with boundary colors $x_1, x_2, x_3, x_4$ (Fig.2.5), one has $x_4 - x_1 = p(x - x_1)$, $x_2 - x_1 = q(x - x_1)$ and $x_3 = x_2 + x_4 - x_1$. If a coloring is nontrivial ($x_1 \neq x$) then $\frac{x_4 - x_1}{x_2 - x_1} = \frac{p}{q}$ as explained by Lou.

**Conjecture 2.2**

Let $p$ be a fixed prime number, then

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**Fig. 2.4**

**Fig. 2.5**

\[\frac{5}{q}\] is called the slope of the tangle and can be easily identified with the slope of the meridian disk of the solid torus being the branched double cover of the rational tangle.
(i) Every link can be reduced to a trivial link by rational $\frac{p}{q}$-moves ($q$ any integer).

(ii) There is a function $f(n,p)$ such that any $n$-tangle can be reduced to one of “basic” $f(n,p)$ $n$-tangles (allowing additional trivial components) by rational $\frac{p}{q}$-moves.

First we can observe that it suffices to consider $\frac{p}{q}$-moves with $|q| \leq \frac{p}{2}$, as other $\frac{p}{q}$-moves follow. Then we know that for $p$ odd the $\frac{p}{1}$-move is a combination of $\frac{p}{2}$ and $\frac{p}{2}$-moves (Compare Fig.1.8). Thus, in fact, 3-move, $(2,2)$-move and $(2,3)$-move conjectures are special cases of Conjecture 2.2(i). If we analyze the case $p = 11$ we see that $\frac{11}{2} = 5 + \frac{1}{2}$, $\frac{11}{3} = 4 - \frac{1}{3}$, $\frac{11}{4} = 3 - \frac{1}{4}$, $\frac{11}{5} = 2 + \frac{1}{5}$. Thus:

Conjecture 2.3
Every link can be reduced to a trivial link (with the same space of 11-colorings) by $(2,5)$ and $(4,\mp3)$ moves, they inverses and mirror images.

What about the number $f(n,p)$? We know that because $\frac{p}{q}$-moves preserve $p$-colorings, therefore $f(n,p)$ is bounded from below by the number of subspaces of $p$-colorings of the $2n$ boundary points induced by Fox $p$-colorings of $n$-tangles (that is by the number of subspaces $\psi(\text{Col}_p(T))$ in $Z_p^{2n}$). I noted in [P-3] that for 2-tangles this number is equal to $p+1$ (even in this special case my argument was complicated). For $p = 3$, $n = 4$ the number of subspaces followed from the work of my student Tatsuya Tsukamoto and was equal to 40 [P-15]. The combined effort of Mietek Dąbkowski and Tsukamoto gave the number 1120 for subspaces $\psi(\text{Col}_3(T))$ and 4-tangles. This was my knowledge at the early spring of 2000. On May 2 and 3 I heard talks on Tits buildings (at the Banach Center in Warsaw) by J.Dymara and T.Januszkiewicz. I realized that the topic may have some connection to my work. I asked Tadek Januszkiewicz whether he sees relations and I gave him numbers 4, 40, 1120 for $p = 3$. He immediately answered that most likely I counted the number of Lagrangians in $Z_3^{2n-2}$ symplectic space, and that the number of Lagrangians in $Z_p^{2n-2}$ is known to be equal to $\prod_{i=1}^{n-1}(p^i + 1)$. Soon I constructed the appropriate symplectic form (as did Janek Dymara). I will spend most of this talk on the construction. I will end with discussion of classes of tangles for which it has been proven that $f(n,p) = \prod_{i=1}^{n-1}(p^i + 1)$. 

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Consider 2n points on a circle (or a square) and a field $\mathbb{Z}_p$ of $p$-colorings of a point. Thus colorings of 2n points form $\mathbb{Z}_p^{2n}$ linear space. Let $e_1, \ldots, e_{2n}$ be its basis.

\[ e_i = (0, \ldots, 1, \ldots, 0), \text{ where } 1 \text{ occurs in the } i\text{-th position.} \]

Let $\mathbb{Z}_p^{2n-1} \subset \mathbb{Z}_p^{2n}$ be the subspace of vectors $\sum a_i e_i$ satisfying $\sum (-1)^i a_i = 0$ (alternating condition). Consider the basis $f_1, \ldots, f_{2n-1}$ of $\mathbb{Z}_p^{2n-1}$ where $f_k = e_k + e_{k+1}$. Consider a skew-symmetric form $\phi$ on $\mathbb{Z}_p^{2n-1}$ of nullity 1 given by the matrix

\[
\phi = \begin{pmatrix}
0 & 1 & \ldots & \ldots \\
-1 & 0 & 1 & \ldots \\
\ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & -1 & 0
\end{pmatrix}
\]

that is

\[
\phi(f_i, f_j) = \begin{cases}
0 & \text{if } |j - i| \neq 1 \\
1 & \text{if } j = i + 1 \\
-1 & \text{if } j = i - 1.
\end{cases}
\]

Notice that the vector $e_1 + e_2 + \ldots + e_{2n}$ (= $f_1 + f_3 + \ldots + f_{2n-1} = f_2 + f_4 + \ldots + f_{2n}$) is $\phi$-orthogonal to any other vector. If we consider $\mathbb{Z}_p^{2n-2} = \mathbb{Z}_p^{2n-1}/\mathbb{Z}_p$, where the subspace $\mathbb{Z}_p$ is generated by $e_1 + \ldots + e_{2n}$, that is, $\mathbb{Z}_p$ consists of monochromatic (i.e. trivial) colorings, then $\phi$ descends to the symplectic form $\hat{\phi}$ on $\mathbb{Z}_p^{2n-2}$. Now we can analyze isotropic subspaces.
of \((Z_p^{2n-2}, \hat{\phi})\), that is subspaces on which \(\hat{\phi}\) is 0 \((W \subset Z_p^{2n-2}, \phi(w_1, w_2) = 0 \text{ for } w_1, w_2 \in W)\). The maximal isotropic \((n-1)\)-dimensional subspaces of \(Z_p^{2n-2}\) are called Lagrangian subspaces (or maximal totally degenerated subspaces) and there are \(\prod_{i=1}^{n-1} (p^i + 1)\) of them.

We have \(\psi: Col_pT \to Z_p^{2n}\). Our local condition on Fox colorings (Fig.2.1) guarantees that for any tangle \(T\), \(\psi(Col_pT) \subset Z_p^{2n-1}\). Also \(Z_p\), the space of trivial colorings, always lays in \(Col_pT\). Thus \(\psi\) desents to \(\hat{\psi}: Col_pT/Z_p \to Z_p^{2n-2} = Z_p^{2n-1}/Z_p\) Now we have the fundamental question: Which subspaces of \(Z_p^{2n-2}\) are yielded by \(n\)-tangles? We answer this question below.

**Theorem 2.4** \(\hat{\psi}(Col_pT/Z_p)\) is a Lagrangian subspace of \(Z_p^{2n-2}\) with the symplectic form \(\hat{\psi}\).

The natural question would be whether every Lagrangian subspace can be realized by a tangle. The answer is negative for \(p = 2\) and positive for \(p > 2\) [D-J-P]. As a corollary we obtain a fact which before was considered to be difficult even for 2-tangles.

**Corollary 2.5** For any \(p\)-coloring of a tangle boundary satisfying the alternating property (i.e. an element of \(Z_p^{2n-1}\)) there is an \(n\)-tangle and its \(p\)-coloring yielding the given coloring on the boundary. In other worlds: \(Z_p^{2n-1} = \bigcup_T \psi_T(Col_p(T))\). Furthermore the space \(\psi_T(Col_p(T))\) is \(n\)-dimensional.

We can say that we understand the conjectured value of the function \(f(n, p)\) but when can we prove Conjecture 2.2 with \(f(n, p) = \prod_{i=1}^{n-1} (p^i + 1)\)? In fact we know that for \(p = 2\) not every Lagrangian is realized and actually \(f(n, 2) = \prod_{i=1}^{n-1} (2i + 1)\). For rational 2-tangles Conjecture 2.2 follows almost from the definition and the generalization to 2-algebraic tangles (algebraic tangles in the Conway sense) is not difficult. In order to be able to use induction for tangles with \(n > 2\) we generalize the concept of the algebraic tangle:

**Definition 2.6**

(i) \(n\)-algebraic tangles is the smallest family of \(n\)-tangles which satisfies:

(0) Any \(n\)-tangle with 0 or 1 crossing is \(n\)-algebraic.

(1) If \(A\) and \(B\) are \(n\)-algebraic tangles then \(r^i(A) * r^j(B)\) is \(n\)-algebraic; \(r\) denotes here the rotation of a tangle by \(\frac{2\pi}{2n}\) angle, and * denotes (horizontal) composition of tangles.
(ii) If in the condition (1), $B$ is restricted to tangles with no more than $k$ crossings, we obtain the family of $(n,k)$-algebraic tangles.

(iii) If an $m$-tangle, $T$, is obtained from an $(n,k)$-algebraic tangle (resp. $n$-algebraic tangle) by partially closing its endpoints $(2n - 2m$ of them) without introducing any new crossings then $T$ is called an $(n,k)$-algebraic (resp. $k$-algebraic) $m$-tangle. For $m = 0$ we obtain an $(n,k)$-algebraic (resp. $k$-algebraic) link.

Conjecture 2.2, for $p = 3$, has been proven for 3-algebraic tangles $[\text{P-Ts}] (f(3,3) = 40)$ and $(4,5)$-algebraic tangles $[\text{Tsu}] (f(4,3) = 1120)$. In particular the Montesinos-Nakanishi conjecture holds for 3-algebraic and $(4,5)$-algebraic links. 40 “basic” 3-tangles are shown in Fig. 2.7.

Invertible (braid type) basic tangles

Noninvertible basic tangles

The simplest 4-tangles which cannot be distinguished by 3-coloring for which 3-move equivalence is not yet established are illustrated in Fig. 2.8. With respect to $(2,2)$ moves, equivalence of 2-tangles of Fig. 2.9 is still an open problem.
3 Talk 3: Historical Introduction to Skein Modules.

I will discuss, in my last talk of the conference, *skein modules*, or as I prefer to say more generally, *algebraic topology based on knots*. It is my “brain child” even if the idea was also conceived by other people (most notably Vladimir Turaev), and was envisioned by John H. Conway (as “linear skein”) a decade earlier. Skein modules have their origin in the observation by Alexander [Al], that his polynomials (*Alexander polynomials*) of three links, $L_+, L_-$ and $L_0$ in $R^3$ are linearly related (Fig. 3.1).

For me it started in Norman Oklahoma in April of 1987, when I was enlightened to see that the multivariable version of the Jones-Conway (Homflypt) polynomial analyzed by Hoste and Kidwell is really a module of links in a solid torus (or more generally in the connected sum of solid tori).

I would like to discuss today, in more detail, skein modules related to the (deformations) of 3-moves and the Montesinos-Nakanishi conjecture but first
I will give the general definition and I will make a short tour of the world of skein modules.

Skein Module is an algebraic object associated to a manifold, usually constructed as a formal linear combination of embedded (or immersed) submanifolds, modulo locally defined relations. In a more restricted setting a skein module is a module associated to a 3-dimensional manifold, by considering linear combinations of links in the manifold, modulo properly chosen (skein) relations. It is the main object of the algebraic topology based on knots. In the choice of relations one takes into account several factors:

(i) Is the module we obtain accessible (computable)?

(ii) How precise are our modules in distinguishing 3-manifolds and links in them?

(iii) Does the module reflect topology/geometry of a 3-manifold (e.g. surfaces in a manifold, geometric decomposition of a manifold)?

(iv) Does the module admit some additional structure (e.g. filtration, grading, multiplication, Hopf algebra structure)? Is it leading to a Topological Quantum Field Theory (TQFT) by taking a finite quotient?

One of the simplest skein modules is a $q$-deformation of the first homology group of a 3-manifold $M$, denoted by $S_2(M; q)$. It is based on the skein relation (between unoriented framed links in $M$): $L_+ = qL_0$; it also satisfies the framing relation $L^{(1)} = qL$, where $L^{(1)}$ denote a link obtained from $L$ by twisting the framing of $L$ once in the positive direction. Already this simply defined skein module "sees" nonseparating surfaces in $M$. These surfaces are responsible for torsion part of our skein module [P-I0].

There is more general pattern: most of analyzed skein modules reflect various surfaces in a manifold.

The best studied skein modules use skein relations which worked successfully in the classical knot theory (when defining polynomial invariants of links in $R^3$).

(1) The Kauffman bracket skein module, KBSM.

The skein module based on the Kauffman bracket skein relation, $L_+ = AL_+ + A^{-1}L_\infty$, and denoted by $S_{2,\infty}(M)$, is best understood among the
Jones type skein modules. It can be interpreted as a quantization of the co-ordinate ring of the character variety of $SL(2, C)$ representations of the fundamental group of the manifold $M$, \cite{Bu-2, B-F-K, P-S}. For $M = F \times [0, 1]$, KBSM is an algebra (usually noncommutative). It is finitely generated algebra for a compact $F$, \cite{Bu-1}, and has no zero divisors \cite{P-S}. The center of the algebra is generated by boundary components of $F$, \cite{B-P, P-S}. Incompressible tori and 2-spheres in $M$ yield torsion in KBSM; it is a question of fundamental importance whether other surfaces can yield torsion as well.

(2) Skein modules based on the Jones-Conway (Homflypt) relation.

\[ v^{-1}L_+ - vL_- = zL_0, \]
where $L_+, L_-, L_0$ are oriented links (Fig. 3.1). These skein modules are denoted by $S_3(M)$ and generalize skein modules based on Conway relation which were hinted by Conway. For $M = F \times [0, 1]$, $S_3(M)$ is a Hopf algebra (usually neither commutative nor co-commutative), \cite{Tu-2, P-6}. $S_3(F \times [0, 1])$ is a free module and can be interpreted as a quantization $\cite{H-K, Tu-1, P-3, Tu-2}$. $S_3(M)$ is related to the algebraic set of $SL(n, C)$ representations of the fundamental group of the manifold $M$, \cite{S}.

(3) Skein modules based on the Kauffman polynomial relation

\[ L_{+1} + L_{-1} = x(L_0 + L_\infty) \] (see Fig. 3.3) and the framing relation $L^{(1)} - aL$. It is denoted by $S_{3, \infty}$ and is known to be free for $M = F \times [0, 1]$.

(4) Homotopy skein modules. In these skein modules, $L_+ = L_-$ for self-crossings. The best studied example is the q-homotopy skein module with the skein relation $q^{-1}L_+ - qL_- = zL_0$ for mixed crossings. For $M = F \times [0, 1]$ it is a quantization, $\cite{H-P-1, Tu-2, P-11}$, and as noted by Kaiser they can be almost completely understood using singular tori technique of Lin.

(5) Skein modules based on Vassiliev-Gusarov filtration.
We extend the family of knots, \( K \), by singular knots, and resolve singular crossing by \( K_{cr} = K_+ - K_- \). These allows us to define the Vassiliev-Gusarov filtration: ... \( C_3 \subset C_2 \subset C_1 \subset C_0 = R K \), where \( C_k \) is generated by knots with \( k \) singular points. The \( k \)th Vassiliev-Gusarov skein module is defined to be a quotient: \( W_k(M) = R K / C_{k+1} \). The completion of the space of knots with respect to the Vassiliev-Gusarov filtration, \( \hat{R} K \), is a Hopf algebra (for \( M = S^3 \)). Functions dual to Vassiliev-Gusarov skein modules are called finite type or Vassiliev invariants of knots; [P-7].

(6) Skein modules based on relations deforming n-moves.

\[ S_n(M) = R L / (b_0 L_0 + b_1 L_1 + b_2 L_2 + ... + b_{n-1} L_{n-1}) \]. In the unoriented case, we can add to the relation the term \( b_{\infty} L_{\infty} \), to get \( S_{n,\infty}(M) \), and also, possibly, a framing relation. The case \( n = 4 \), on which I am working with my students, will be described, n greater detail, in a moment.

Examples (1)-(5) gave a short description of skein modules studied extensively until now. I will now spent more time on two other examples which only recently has been considered more detailsly. The first example is based on a deformation of the 3-move and the second on the deformation of the (2,2)-move. The first examples has been studied by my students Tsukamoto and Mike Veve. I denote it by \( S_{4,\infty} \) as it involves (in the skein relation), 4 horizontal positions and the vertical (\( \infty \)) smoothing.

**Definition 3.1** Let \( M \) be an oriented 3-manifold, \( L_{fr} \), the set of unoriented framed links in \( M \) (including the empty knot, \( \emptyset \)) and \( R \) any commutative ring with identity. Then we define the \( (4,\infty) \) skein module as: \( S_{4,\infty}(M;R) = R L_{fr} / I_{(4,\infty)} \), where \( I_{(4,\infty)} \) is the submodule of \( R L_{fr} \) generated by the skein relation:

\[ b_0 L_0 + b_1 L_1 + b_2 L_2 + b_3 L_3 + b_{\infty} L_{\infty} = 0 \] and the framing relation:

\( L^{(1)} = aL \) where \( a, b_0, b_3 \) are invertible elements in \( R \) and \( b_1, b_2, b_{\infty} \) are any fixed elements of \( R \) (see Fig.3.2).
The generalization of the Montesinos-Nakanishi conjecture says that $\mathcal{S}_{4,\infty}(S^3, R)$ is generated by trivial links and that for $n$-tangles our skein module is generated by $f(n, 3)$ basic tangles (with possible trivial components). This would give a generating set for our skein module of $S^3$ or $D^3$ with $2n$ boundary points (an $n$-tangle). In [P-Ts] we analyze extensively the possibilities that trivial links, $T_n$, are linearly independent. This may happen if $b_\infty = 0$ and $b_0b_1 = b_2b_3$. These leads to the following conjecture:

**Conjecture 3.2**  
(1) There is a polynomial invariant of unoriented links, $P_1(L) \in \mathbb{Z}[x, t]$ which satisfies:

- Initial conditions: $P_1(T_n) = t^n$, where $T_n$ is a trivial link of $n$ components.

- Skein relation $P_1(L_0) + xP_1(L_1) - xP_1(L_2) - P_1(L_3) = 0$ where $L_0, L_1, L_2, L_3$ is a standard, unoriented skein quadruple ($L_{i+1}$ is obtained from $L_i$ by a right-handed half twist on two arcs involved in $L_i$; compare Fig.3.2).

(2) There is a polynomial invariant of unoriented framed links, $P_2(L) \in \mathbb{Z}[A^{\pm 1}, t]$ which satisfies:

- Initial conditions: $P_2(T_n) = t^n$;

- Framing relation: $P_2(L^{(1)}) = -A^3P_2(L)$ where $L^{(1)}$ is obtained from a framed link $L$ by a positive half twist on its framing.

- Skein relation: $P_2(L_0) + A(A^2 + A^{-2})P_2(L_1) + (A^2 + A^{-2})P_2(L_2) + AP_2(L_3) = 0$.

The above conjectures assume that $b_\infty = 0$ in our skein relation. Let us consider, for a moment, the possibility that $b_\infty$ is invertible in $R$. Using the “denominator” of our skein relation (Fig.3.3) we get the relation which allows us to compute the effect of adding a trivial component to a link $L$ (we write $t^n$ for the trivial link link $T_n$):

\[(*) \quad (a^{-3}b_3 + a^{-2}b_2 + a^{-1}b_1 + b_0 + b_\infty t)L = 0\]

When considering the “numerator” of the relation and its mirror image (Fig.3.3) we obtain formulas for Hopf link summands, and because the unoriented Hopf link is amphicheiral thus we can eliminate it from our equations.
to get the formula (**):

\[\begin{align*}
  b_3(L\#H) + (ab_2 + b_1t + a^{-1}b_0 + ab_\infty)L &= 0, \\
  b_0(L\#H) + (a^{-1}b_1 + b_2t + ab_3 + a^2b_\infty)L &= 0.
\end{align*}\]

(**) \[((b_0b_1 - b_2b_3)t + (a^{-1}b_0^2 - ab_3^2) + (ab_0b_2 - a^{-1}b_1b_3) + b_\infty(ab_0 - a^2b_3))L = 0.\]

It is possible that (*) and (**) are the only relations in the module. The pleasant substitution which realizes the relation is: \(b_0 = b_3 = a = 1, b_1 = b_2 = x, b_\infty = y\). This may lead to the polynomial invariant of unoriented links in \(S^3\) with value in \(\mathbb{Z}[x, y]\) and the skein relation \(L_3 + xL_2 + xL_1 + L_0 + yL_\infty = 0\).

What about the relations to Fox colorings? One such relation, already mentioned, is the use of 3-coloring to estimate the number of basic \(n\)-tangles (by \(\prod_{i=1}^{n-1}(3^i + 1)\)) for the skein module \(S_{4,\infty}\). I am also convinced that \(S_{4,\infty}(S^3, R)\) contains full information on the space of Fox 7-colorings. It would be a generalization of the fact that the Kauffman bracket polynomial contains information on 3-colorings and the Kauffman polynomial contains information on 5-colorings. In fact, François Jaeger told me that he knew how to get the space of \(p\)-colorings from a short skein relation (of type of \((\frac{p+1}{2}, \infty)\)). Unfortunately François died prematurely in 1997 and I do not know how to prove his statement.

Finally let me describe shortly the skein module related to the \((2, 2)\)-move conjecture. Because a \((2, 2)\)-move is equivalent to the rational \(\frac{5}{2}\)-move, I will denote the skein module by \(S_{\frac{5}{2}}(M; R)\).
Definition 3.3 Let $M$ be an oriented 3-manifold, $\mathcal{L}_{fr}$ the set of unoriented framed links in $M$ (including the empty knot, $\emptyset$) and $R$ any commutative ring with identity. Then we define the $5/2$-skein module as: $S_{5/2}(M; R) = R\mathcal{L}_{fr}/(I_{5/2})$ where $I_{5/2}$ is the submodule of $R\mathcal{L}_{fr}$ generated by the skein relation:

(i) $b_2 L_2 + b_1 L_1 + b_0 L_0 + b_{\infty} L_{\infty} + b_{-1} L_{-1} + b_{-1/2} L_{-1/2} = 0$,

its mirror image:

\(\bar{(i)}\) $b'_2 L_2 + b'_1 L_1 + b'_0 L_0 + b'_{\infty} L_{\infty} + b'_{-1} L_{-1} + b'_{-1/2} L_{-1/2} = 0$

and the framing relation:

$L^{(1)} = aL$, where $a, b_2, b'_2, b_{-1/2}, b'_{-1/2}$ are invertible elements in $R$ and $b_1, b'_1, b_0, b'_0, b_{-1}, b'_{-1}, b_{\infty},$ and $b'_{\infty}$ are any fixed elements of $R$. The links $L_2, L_1, L_0, L_{\infty}, L_{-1}, L_{1/2}$ and $L_{-1/2}$ are illustrated in Fig.3.4.

\[\text{Fig. 3.4}\]

If we rotate the figure from the relation (i) we obtain:

\(\bar{(i')}\) $b_{-1/2} L_2 + b_{-1} L_1 + b_{\infty} L_{0} + b_{0} L_{\infty} + b_{1} L_{-1} + b_{2} L_{-1/2} = 0$

One can use (i) and (i') to eliminate $L_{-1/2}$ and to get the relation:

\[
(b_2^{-2} - b_{-1/2}^{-2}) L_2 + (b_1 b_2 - b_{-1} b_{-1/2}) L_1 + ((b_0 b_2 - b_{\infty} b_{-1/2}) L_0 + (b_{-1} b_2 - b_{1} b_{-1/2}) L_{-1} + (b_{\infty} b_2 - b_{0} b_{-1/2}) L_{\infty} = 0.
\]

Thus either we deal with the shorter relation (essentially that of the fourth skein module described before) or all coefficients are equal to 0 and therefore (assuming that there are no zero divisors in $R$) $b_2 = \varepsilon b_{-1/2}$, $b_1 = \varepsilon b_{-1}$, and $b_0 = \varepsilon b_{\infty}$. Similarly we would get: $b'_2 = \varepsilon b'_{-1/2}$, $b'_1 = \varepsilon b'_{-1}$, and $b'_0 = \varepsilon b'_{\infty}$. Here

\[\text{Our notation is based on Conway's notation for rational tangles. However it differs from it by a sign. The reason is that the Conway convention for a positive crossing is generally not used in the setting of skein relations.}\]
\( \varepsilon = \pm 1 \). Assume for simplicity that \( \varepsilon = 1 \). Further relations among coefficients follows from the computation of the Hopf link component using the amphicheirality of the unoriented Hopf link. Namely, by comparing diagrams in Figure 3.5 and its mirror image we get:

\[
L \# H = -b_2^{-1}(b_1(a + a^{-1}) + a^{-2}b_2 + b_0(1 + T_1))L
\]

\[
L \# H = -b_2'(b_1'(a + a^{-1}) + a^2b_2' + b_0'(1 + T_1))L.
\]

Possibly the above equalities give the only other relation among coefficients (in the case of \( S^3 \)), but I would present below the simpler question (assuming \( a = 1, b_x = b'x \) and writing \( t^n \) for \( T_n \)).

**Question 3.4** There is a polynomial invariant of unoriented links in \( S^3 \), \( P_{2}(L) \in \mathbb{Z}[b_0, b_1, t] \) which satisfies:

(i) Initial conditions: \( P_{2}(T_n) = t^n \), where \( T_n \) is a trivial link of \( n \) components.

(ii) Skein relations

\[
P_{2}(L_2) + b_1P_{2}(L_1) + b_0P_{2}(L_0) + b_0P_{2}(L_{\infty}) + b_1P_{2}(L_{-1}) + P_{2}(L_{-\frac{1}{2}}) = 0.
\]

\[
P_{2}(L_{-2}) + b_1P_{2}(L_{-1}) + b_0P_{2}(L_0) + b_0P_{2}(L_{\infty}) + b_1P_{2}(L_1) + P_{2}(L_{\frac{1}{2}}) = 0.
\]
Notice that by taking the difference of our skein relations one get the interesting identity:

\[ P_2(L_2) - P_2(L_{-2}) = P_2(L_{1/2}) - P_2(L_{-1/2}). \]

Nobody has yet seriously studied the skein module \( S_2(M; R) \) so everything you can get will be a new exploration, even a table of the polynomial \( P_2(L) \) for small links, \( L \).

I wish you luck!

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