Control of dumbbell satellite orbits using moving mass actuators

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Received: 6 March 2022 / Accepted: 7 July 2022 / Published online: 5 September 2022
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Abstract Principles of moving mass control of the orbital parameters of an artificial dumbbell shaped satellite are discussed in the present paper. The rationale is to implement a non-jet principle of actuation by varying the geometry of the satellite through its internal degrees-of-freedom. This can be achieved by spinning the massive parts of the dumbbell and changing their relative distance upon the orbital angle according to the suggested control strategies. The control schemes aim at maintaining a desired satellite size and orientation with respect to the orbital radius in order to take advantage of the variations in the gravitational field along the elliptical orbit. The results demonstrate that the total orbital energy can follow a prescribed temporal profile by controlling the satellite orientation on the orbit to accurately track its desired target. Analytical estimates for the satellite’s energy versus the number of orbital cycles are determined from closed-form solutions. Results from both analytical estimates and numerical integration are in sufficient agreement.

Keywords Satellite orbit control · Moving mass actuators · Dumbbell satellite · Sliding mode control

1 Introduction

A dumbbell-shaped satellite may serve as a simple one-dimensional model for a general rigid-body space object. It enables the two-particle approach to be implemented for representing the dynamics of a rigid-body moving in a central gravitational field. In order to represent a rigid body, some constraint condition must be imposed on the ensemble of the two particles, which is satisfied automatically when deriving the system Lagrangian. Such type of modeling eases analytical manipulations with the differential equations of motion; thus, providing a clear way for designing control algorithms. In addition, the dumbbell model captures important dynamic properties of the so-called tethered satellite systems used in experimental investigations of space related exploitation of the Earth’s magnetic field [1]. During the past decades, different spinning tethered systems were analyzed in connection with specific space missions, including the famous Momentum-Exchange/Electrodynamic-
Reboost (MXER) project by NASA [2]. Furthermore, as noticed in Ref. [3], the developed theoretical tools can help to explain the dynamic behaviors of natural dog-bone shaped bodies, such as asteroids 216 Kleopatra, 4179 Toutatis, and 4761 Castalia [4]. This is the reason for the periodic interest in research of dumbbell-shaped satellites in the past several decades [3, 5–13]. Note that holonomic constraints on the relative motions of the mass particles can be relaxed by assuming some type of physical interaction between them. For instance, in Ref. [9], a dumbbell-shaped spacecraft was modeled as two identical and massive particles connected by a linear elastic spring. The motion in both circular and elliptical orbits of a tethered satellite system consisting of two lumped mass system connected by an ideally flexible massless inextensible tether was analyzed in Ref. [10]. Their formulation accounted for the effects of the earth gravitational field, aerogradient, and aerodynamic pressure and friction. They showed that the aerogradient drag effect can result in a strong spinning of the artificial satellite while the eccentricity may generate a chaotic motion. Their results also revealed that relatively large values of the aerodynamic pressure parameter may yield a domain of aerodynamic stabilization in an elliptic orbit. Celletti and Sidorenko [3] considered planar periodic motions with oscillations around the local vertical to the orbital plane. They proved instability of such type of motions with respect to out-of-plane perturbations. This was also revealed earlier from numerical investigations conducted in Ref. [1]. Recently, Combot et al. [14] analyzed the integrability of a model of elastic satellite whose mass center moves in a circular Keplerian orbit. The satellite was modelled by two point-masses connected with an extensible massless linear elastic spring.

In the present work, the principles of a moving mass control of a satellite orbit by varying the satellite shape and orientation along the orbit are discussed. Such studies are motivated by the long-term operation of large-scale space stations without having to replenish their jet fuels. Ideally, the energy required for powering the control actuators can be produced by solar panels mounted on the space station. The effect of non-jet propulsion of satellites can emerge from the interaction of cyclical variations of the satellite configuration with the nonhomogeneous gravitational field along the orbit. The possibility of such effect was predicted earlier by means of different analytical and numerical modeling [2, 12, 15–17]. Such effects are quite small for typical size artificial satellites. However, they may be sufficient for adjusting the orbital parameters of large articulated space stations designed for long-term operations assuming that the proposed methodology can be further generalized to tethered space structures. The reader is referred to [18, 19] and references therein for types of various designs of tethered structures. The effect of an oblate shaped central body on the dynamics of dumbbell satellites with unequal masses was investigated in Ref. [20]. Using Lindstedt-Poincare’s analytical tool, the authors showed that the trajectory of the satellite mass center is periodic and differs from the classical one whenever the effect of the zonal harmonic parameter is nonzero. In Ref. [21], the stability of relative equilibria of a dumbbell satellite in the gravitational field of an oblate spheroid was analyzed to obtain necessary and sufficient conditions for stable motion in terms of the linear stability criteria. The Lagrangian approach was implemented to derive the differential equations of motion of a dumbbell satellite in the central gravitational field generated by a variable mass object [22]. The rationale was to analyze the mathematical structure of equations including group properties associated with scaling of variables.

The effects of variations in the inertial properties of a satellite on the orbit has been under attention for several years. Such effects may occur due to nonhomogeneity of the gravitational field along the elliptic orbit. A simple illustration of possible mechanism with quantitative estimates was given in Ref. [12] based on the assumption that the radial orientation is maintained, whereas the length is varied upon position on the orbit. For a typical satellite size, the effect is usually either negligible or may require a very long time to be observed. However, the length of a tethered space structure may be about ten kilometers or more as noticed in Ref. [2]. Gratus and Tucker [17] estimated that, by varying the length of a 50-km tether, the system can gain altitude at an approximate rate of three hundred meters per hour in low Earth orbits. Practically, the length of so-called motorized tethers can be controlled by some mechanism fixed on the satellite. The authors of Ref. [2] analyzed the effect of rapid rotations of a tether about its mass center on the evolution of orbital parameters. Lagrange perturbation equations for temporal rates of the orbital parameters
were used. It was stated that the angular momentum transfer between the rotational motion and the orbital motion can be achieved by appropriate variation of the satellite length to implement a non-jet propulsion. For completeness of the reported work, some results from previous publications are reproduced herein. However, the present work differs significantly from previous studies in its approach, physical principles, problem formulation, and model architecture. The objective of the proposed control strategy is to accurately track a desired total orbital energy profile, which includes arbitrarily shaped temporal segments of increasing and decreasing energy levels. The satellite is not assumed to rotate fast about its mass center. Instead, it is only required to change its orientation about the orbital radius synchronously with the orbital rotations based on explicit analytical predictions. The “external inertia torque” needed for controlling the satellite orientation is generated by spinning the circular end masses of the dumbbell satellite according to a controllable angular acceleration about the satellite axis of symmetry. Note that the role of the variable length of the satellite is secondary in the present work. This paper is organized as follows. Section 2 describes a schematic for a dumbbell satellite whose massive components play the role of control actuators. In addition, the equations of motion are derived by assuming the length of the satellite to be small in comparison with the orbital polar radius. Section 3 presents the root causes for possible chaotic rotations of the satellite about its mass center during its uncontrolled motion along an elliptical orbit. In Sect. 4, cyclical variations of the satellite length versus the orbital angle are shown to yield a gradual increase in the eccentricity under either chaotic fluctuations or regular oscillations. In Sect. 5, the size of the satellite is assumed to be fixed while its orientation is considered to depend upon the orbital position. An analytical estimate for the number of orbital cycles required for a prescribed eccentricity adjustment is determined in closed-form based on elementary functions. Section 6 includes a PID controller and a sliding mode controller (SMC) devised to effectively maintain the satellite orientation profile imposed in Sect. 5 for tracking the desired energy level. Section 7 summarizes the main contributions of the current study. Finally, Appendices A and B provide detailed descriptions on the derivations and adaptations of the governing differential equations of motion.

2 Constitutive equations of the dumbbell satellite, and problem formulation

2.1 Assumptions and preliminary model adaptations

Figure 1a illustrates the schematic of a dumbbell satellite operating in a central gravitational field induced by a mass, $M$, as shown in Fig. 1b. Planar dynamics is considered. Two circularly symmetric parts of the satellite, each of mass $m/2$, are moving along a massless stiff rod such that the distance between their mass centers can vary according to a prescribed law, expressed as $2l(t)$, which is specified by the controller. Both parts can synchronously spin about their mass centers with a variable angular velocity $\omega(t)$ in order to generate a necessary inertia torque for controlling the satellite angle $\theta(t)$ relative to the orbital radius-vector of the entire satellite mass center (Fig. 1b). This is possible based on the total angular momentum conservation as discussed subsequently in Sect. 2.2. Below, the Lagrangian approach is implemented in deriving the differential equations of motion with respect to three generalized coordinates, namely, the orbital radius $r = |\mathbf{r}|$, the orbital angle $\varphi$, and the relative angle $\theta$ describing the satellite orientation on the orbit as shown in Fig. 1b. The functions $l(t)$ and $\omega(t)$ play the role of control inputs. The detailed derivations are presented in Appendix A. The basic steps are illustrated here.

With reference to Fig. 1, a normalized Lagrangian per unit mass of the satellite can be represented as

$$
L = \frac{1}{4}[(\dot{\mathbf{r}} + \mathbf{i})^2 + (\dot{\mathbf{r}} - \mathbf{i})^2] + \frac{1}{2}k^2(\dot{\varphi} + \dot{\theta} + \omega)^2 + \frac{GM}{2}|\mathbf{r} + \mathbf{l}|^{-1} + |\mathbf{r} - \mathbf{l}|^{-1}, \tag{1}
$$

$$
\mathbf{r} = r(\cos \varphi + \sin \varphi \mathbf{j}), \quad \mathbf{l} = l(\cos(\varphi + \theta)\mathbf{i} + \sin(\varphi + \theta)\mathbf{j}),
$$

where $G$ is the universal gravitational constant, $\mathbf{r} = \mathbf{r}(t)$ and $\mathbf{l} = \mathbf{l}(t)$ are the position of the satellite mass center and configuration vectors, respectively; $k$ is the
radius of gyration of both spinning parts with respect to their mass centers; and $\omega(t) = \dot{\phi}(t)$ is the angular velocity of the spinning components about the connecting rod. Furthermore, it is assumed that $m/M \ll 1$; thus, causing the mass center of the entire system to be located at the center of the planet $M$. In addition, by considering $l/r < 1$, the following estimate for the Lagrangian (1) holds (see Appendix A):

$$L = \frac{1}{2} \left( \dot{r}^2 + r^2 \dot{\varphi}^2 \right) + \frac{1}{2} \left( \dot{l}^2 + k^2 (\dot{\theta} + \dot{\varphi})^2 \right) + k^2 \dot{\omega}(t)$$

$$+ \frac{GM}{r} + \frac{1}{4} \left( \frac{l}{r} \right)^2 \frac{GM}{r} (1 + 3 \cos 2\theta) + O\left( \frac{l}{r} \right)^3. \quad (2)$$

The corresponding differential equations of motion are given by the Euler–Lagrange equations as

$$\ddot{r} - r \ddot{\varphi}^2 + \frac{GM}{r^2} = - \frac{3}{4} \left( \frac{l}{r} \right)^2 \frac{GM}{r^2} (1 + 3 \cos 2\theta) \quad (3)$$

$$\frac{d}{dt} \left[ r^2 \dot{\varphi} + (l^2 + k^2)(\dot{\theta} + \dot{\varphi}) + k^2 \dot{\omega} \right] = 0 \quad (4)$$

$$\frac{d}{dt} \left[ (l^2 + k^2)(\dot{\theta} + \dot{\varphi}) \right] + \frac{3}{2} \left( \frac{l}{r} \right)^2 \frac{GM}{r} \sin 2\theta + k^2 \dot{\omega} = 0 \quad (5)$$

In the limiting case when $k/l \to 0$ and $l/r \to 0$, system (3)-(5) admits three integrals, calculated per unit mass of the satellite, as

$$\alpha = \frac{E}{m} = \frac{1}{2} \dot{\varphi}^2 + \frac{1}{2} \dot{\varphi}^2 - \frac{GM}{r} - \frac{1}{2} \dot{r}^2 - \frac{1}{2} \frac{h^2}{r^2} - \frac{GM}{r} \quad (6)$$

$$h = r^2 \dot{\varphi} \quad (7)$$

$$q = l^2 (\dot{\theta} + \dot{\varphi}) \quad (8)$$

where $\alpha$ is the total orbital energy, $q$ and $h$ are the angular momenta of the satellite about its mass center and with respect to the center of the planet, respectively. The total angular momentum of the satellite with respect to the fixed origin $M$ is the sum of $q$ and $h$, which is conserved for $\dot{\omega} = 0$, as can be deduced from Eq. (4). Note that $l$ is kept in Eq. (8) to account for the finite satellite size even though the ratio $l/r$ was assumed to be vanishingly small. Note that for $l/r < 1$, $\alpha$, $q$, and $h$ in Eqs. (6)-(8) will no longer be constant but will slowly vary with time. Following the idea of parameter variations, their time derivatives are obtained by direct differentiation of Eqs. (6)-(8) while enforcing Eqs. (3) through (5). Furthermore, since $k^2 \ll l^2$ then $l^2 + k^2 \approx l^2$, which yields the following estimates for the temporal rates of $\alpha$, $h$, and $q$:

$$\dot{\alpha} = - \frac{3}{4} \frac{r^2 \dot{r}^2}{r^2} \left( \frac{l}{r} \right)^2 (1 + 3 \cos 2\theta)$$

$$+ \frac{3}{2} \frac{GM}{r} \left( \frac{l}{r} \right)^2 \sin 2\theta \quad (9)$$
\[ h = \frac{3GM}{2r} \left( \frac{l}{r} \right)^2 \sin 2\theta \] \hspace{1cm} (10)

\[ \dot{q} = -\frac{3GM}{2r} \left( \frac{l}{r} \right)^2 \sin 2\theta - k^2 \dot{\omega} \] \hspace{1cm} (11)

By considering the angle \( \psi \) as the independent variable instead of the time \( t \) and taking into account Eq. (7), one can write the following differentiation rules:

\[ \frac{dr}{dv} = \frac{d}{dv} \left( \frac{d}{dt} \right) = \dot{\theta} \left( \frac{d}{dv} \right) = \frac{d}{dv} \left( \frac{d}{r} \right) = \frac{r^2}{2h} \frac{dv}{dt} \] \hspace{1cm} (12)

Equations (9) and (10) can now be expressed as

\[ \frac{d\psi}{dv} = \frac{3GM}{2h} \left( \frac{l}{r} \right)^2 \sin 2\theta \left( \frac{1}{2} \frac{dr}{dv} \right) \left( 1 + 3 \cos 2\theta \right) \] \hspace{1cm} (13)

\[ \frac{dh}{dv} = \frac{3GM}{2h} \left( \frac{l}{r} \right)^2 r \sin 2\theta \] \hspace{1cm} (14)

If the functions \( l = l(v) \) and \( \theta = \theta(v) \) are known, then Eqs. (13) and (14) can be solved for \( \varphi = \varphi(v) \) and \( h = h(v) \) in order to determine the main geometrical parameters of the elliptical orbit [34]

\[ r = \frac{p}{1 + e \cos \varphi} ; \quad p = \frac{h^2}{GM} ; \quad e = \sqrt{1 + \frac{2eh}{G^2M^2}} \] \hspace{1cm} (15)

where \( r = r(v) \) is the orbital radius, \( p = p(v) \) and \( e = e(v) \) are a slowly varying angular momentum parameter and the eccentricity, respectively; both \( p \) and \( e \) become fixed as \( l/r \to 0 \).

### 2.2 Problem formulation

The main purpose of the present work is to understand the extent to which orbital parameters of the satellite can be controlled through variations of the satellite geometry induced by the satellite length \( l = l(v) \) and the relative angles of the two massive components with respect to the connecting rod (Fig. 1a). Due to the geometrical symmetry of the satellite structure, the synchronous angular displacements of the spinning components are represented by the same angle \( \varphi \), which is measured counterclockwise with respect to the massless rod. It is assumed that the angle \( \varphi \) can be varied by means of electric motors in order to control the satellite’s relative angle \( \theta \) on the orbit (Fig. 1). The related physical principle is based on the angular momentum conservation law, whose simplified illustrating version can be obtained by assuming that position of the satellite’s mass center is “frozen” and the central gravitational force is negligible. In this case, Lagrangian (2) takes the form

\[ L = \frac{1}{2} \dot{\phi}^2 + \frac{1}{2} \left( \dot{\phi}^2 + k^2 \dot{\omega} \right)^2 + k^2 \omega \dot{\theta} \]

where the angular coordinate \( \theta \) is ‘cyclical’. As a result, Noether’s theorem leads to the equation \( \partial L/\partial \dot{\theta} = J_0 = \text{const} \). Solving this equation for \( \omega \) gives the following estimate

\[ \omega = \dot{\phi} \approx J_0 - \left( \frac{1}{k^2} + \frac{\dot{\phi}^2}{k^2} \right) \dot{\theta} \]

This relationship reveals the possibility of controlling the satellite’s angular coordinate \( \theta \) by changing the angle of rotating parts, \( \varphi \). The corresponding control strategies are described below in Sect. 6 based on a more comprehensive model, which is derived from Eq. (5) as follows.

By using the differentiation rules in Eq. (12), the angular velocity can be expressed as

\[ \dot{\omega} = \dot{\psi}(v) = \frac{h}{\dot{\varphi}(v)} \frac{d\varphi}{dv} \]

Furthermore, the incorporation of Eqs. (12) and (15) into Eq. (5), as described in Appendix B, yields the following governing equation for the satellite relative angle \( \theta \):

\[ \frac{d^2 \theta}{dv^2} + 2 \left( \frac{1}{l} \frac{dl}{dv} - \frac{e \sin \varphi}{1 + e \cos \varphi} \right) \left( \frac{d\theta}{dv} + 1 \right) \]

\[ + \frac{3 \sin 2\theta}{2(1 + e \cos \varphi)} \]

\[ = -\frac{k^2 d^2 \varphi}{l^2} \frac{dv^2}{dv^2} \] \hspace{1cm} (16a)

Equations (13), (14), and (16a), along with the algebraic relationships in Eq. (15), represent a dynamical system for predicting the slowly varying responses of \( \varphi(v) \) and \( h(v) \), and the fast response of \( \theta(v) \), respectively. Solutions of the entire dynamical system depend upon the two control inputs, \( l = l(v) \) and \( d^2 \varphi/dv^2 \). In the current study, it is assumed that the satellite design allows for a sufficient accumulation of the solar energy required to power the actuators responsible for generating the control signals. Equation (16a) reveals that both control inputs can directly affect the satellite angle \( \theta \). However, both the energy
and the angular momentum, described by Eqs. (13) and (14), can be indirectly controlled through coupling terms involving \( \theta \) and directly impacted by \( l \). Note that the right-hand sides of Eqs. (13) and (14) are more sensitive to \( \theta \) than to the slowly varying relative length \( l/r \).

### 3 Dynamics of a dumbbell satellite on an elliptical orbit

This section illustrates the effect of a variable length on the dynamics of a dumbbell satellite which can rotate freely about its mass center while moving along an elliptical orbit. The illustration is based on the comparison of Poincaré maps obtained using constant and variable \( l \) under different eccentricities of the orbit. Note that for a fixed length, \( l = \text{const.} \), with no spinning, \( \varphi = \text{const.} \), Eq. (16a) becomes equivalent to the one derived by Beletskii [23] for the motion of an artificial satellite rotating about its mass center with a relative angle defined by \( \delta = 2\theta \). Moreover, for a circular orbit, \( e = 0 \), such an equation is further reduced to that of a simple pendulum, \( d^2\delta/dv^2 + 3\sin\delta = 0 \). The presence of the so-called separatrix loop in the phase plane of the pendulum dictates the nature of possible chaotic dynamics of real satellites in planetary systems [24–27] and of artificial satellites [10, 28]. Specifically, periodic perturbations due to a small eccentricity of the orbit lead to the effect of separatrix splitting with a complicated net of intersecting trajectories in the vicinity of saddle points. Although such an effect was already noticed by H. Poincaré, its geometrical illustration was given by Melnikov [29] who introduced the corresponding perturbation method using the Hamiltonian descriptive function.

The onset and evolution of the so-called chaotic layers near the separatrix loop are illustrated by the corresponding Poincaré maps in Fig. 2, where snapshots are taken once every \( 2\pi \)-period of the orbital angle \( v \). As the eccentricity increases, the stochastic layer becomes thicker and eventually destroys most of the periodic and quasi periodic motions (Figs. 2a through d). Interestingly, a variable satellite length along the orbit can shrink the chaotic layers. Such an effect is easy to predict by assuming the length dependence \( l = l_0/(1 + \zeta \cos v) \) with constant parameters \( l_0 \) and \( \zeta \) in Eq. (16a) and by setting \( d^2\varphi/dv^2 = 0 \). As a consequence, Eq. (16a) takes on the following form:

\[
\begin{align*}
\frac{d^2\theta}{dv^2} & + 2\left( \frac{\zeta \sin v}{1 + \zeta \cos v} - \frac{e \sin v}{1 + e \cos v} \right) \left( \frac{d\theta}{dv} + 1 \right) \\
& + \frac{2(1 + e \cos v)}{3\sin 2\theta} = 0
\end{align*}
\]

(16b)

It is seen that, if \( \zeta \approx e \), the effect of eccentricity is minimized at least through the vanishing middle group of terms in Eq. (16b).

The outcome of the comparison between subplots (c) and (e) of Fig. 2 confirms the shrinking of the stochastic layers near the separatrix loop due to the widening domains of regular motions. Furthermore, Fig. 2f shows a zoomed-in region of regular quasiperiodic oscillations emerging from the chaotic area of Fig. 2d when the satellite’s length is varied along the orbit as \( l = l_0/(1 + 0.48 \cos v) \).

### 4 Freely rotating satellite with a varying length: evolution of orbital parameters

On one hand, Eq. (16b) reveals that the variable length may produce perturbations leading to the effect of separatrix splitting even under zero eccentricity of the orbit, since the term produced by the variable length, \( \zeta \sin v(1 + \zeta \cos v)^{-1} \), still generates a periodic perturbation of the equivalent pendulum equation when \( e = 0 \), due to the gravitational gradient effect. On the other hand, it was shown in Sect. 3 for the case of an elliptical orbit with \( \zeta \approx e \), the same term can slightly balance the influence of eccentricity; thus, reducing the irregularities in the satellite dynamics. The focus of this section is to analyze the numerical integration results of Eq. (3) to Eq. (6) in the time domain in order to illustrate the effect of variable length on the satellite dynamics and on the evolution of orbital parameters. These results were generated by assuming that both spinning actuators are inactive and the satellite can freely rotate about its mass center.
To adapt the system for numerical simulations, the following dimensionless terms are introduced:

\[ r = \frac{R}{R} = \tilde{r}, \quad l = \frac{R}{l} = \tilde{l}, \quad t \approx T = \tilde{t} \]  

and

\[ E = \left( \frac{GMm}{R} \right)^{-1} \bar{E}, \quad \bar{E} = \left( \frac{GM}{R} \right)^{-1} \bar{E}, \quad \bar{T} = \frac{T}{R}, \quad \bar{p} = \frac{p}{R} \]

Fig. 2 Poincaré maps of the satellite orientation states \( \{\theta, \dot{\theta}\} \) obtained from Eq. (16a) with random sets of initial conditions and \( r \) sampled each \( 2\pi \): a through d—constant length, \( \zeta = 0 \), and four different eccentricities: a: \( e = 0 \), b: \( e = 0.01 \), c: \( e = 0.1 \), d: \( e = 0.35 \); e and f—variable length and different eccentricities: e: \( e = 0.1, \zeta = 0.102 \), and f: \( e = 0.35, \zeta = 0.48 \)

\[ \bar{E} = \left( \frac{GMm}{R} \right)^{-1} E, \quad \bar{E} = \left( \frac{GM}{R} \right)^{-1} \bar{E}, \quad \bar{T} = \frac{T}{R}, \quad \bar{p} = \frac{p}{R} \]

where \( R = 6378.17 \cdot 10^3 \text{m}, \quad M = 5.977 \cdot 10^{24} \text{kg}, \quad G = 6.670 \cdot 10^{-1} \text{m}^3/(\text{sec}^2 \cdot \text{kg}), \) and

\[ \bar{E} = \left( \frac{GMm}{R} \right)^{-1} E, \quad \bar{E} = \left( \frac{GM}{R} \right)^{-1} \bar{E}, \quad \bar{T} = \frac{T}{R}, \quad \bar{p} = \frac{p}{R} \]
\[ T = \sqrt{\frac{R^3}{(GM)}} = 806.751 \text{ sec.} \] By assuming \( \omega = 0 \), Eqs. (3) through (6) can be written in unitless form as follows
\[ \frac{d^2 \varpi}{dt^2} - \varpi \left( \frac{dv}{dt} \right)^2 + \frac{1}{\varpi^2} = -\frac{3}{4} \left( \varpi \right)^2 + 3 \cos 2\theta \frac{1}{\varpi^2}, \] (19)

\[ \frac{d}{dt} \left[ \varpi^2 \frac{dv}{dt} + \varpi^2 \frac{d\theta}{dt} + \frac{dv}{dt} \right] = 0, \] (20)

\[ \frac{d}{dt} \left[ \varpi \left( \frac{d\theta}{dt} + \frac{dv}{dt} \right) \right] = -\frac{3}{2} \left( \varpi \right)^2 \sin 2\theta \frac{1}{\varpi}, \] (21)

and
\[ \varpi = \frac{1}{2} \left( \frac{dv}{dt} \right)^2 + \frac{1}{2} \varpi^2 \left( \frac{dv}{dt} \right)^2 - \frac{1}{\varpi} \quad \varpi = \varpi^2 \frac{dv}{dt}. \] (22)

Now the eccentricity is calculated through the unitless quantities \( \varpi \) and \( \varpi \) as
\[ e = \sqrt{1 + 2\varpi^2}. \] (23)

Assume that the satellite length can be controlled in an open-loop way as a function of the orbital angle, \( \varpi = \varpi(v) \). Keeping in mind the form of dependence for the polar radius (15), let us test the following two versions of the variable length:

\[ \varpi = \varpi_0 / (1 + \zeta \sin v), \quad \varpi_0 = 0.05, \quad \zeta = 0.38 \] (24a)

and

\[ \varpi = \varpi_0 / (1 + \zeta \cos v), \quad \varpi_0 = 0.05, \quad \zeta = 0.48. \] (24b)

According to Eq. (24a), the length depends on the polar angle with the phase shift \( \pi/2 \) compared to the polar radius \( r(v) \) in Eq. (15). The parameter \( \varpi_0 \) is chosen to be unrealistically large for clarity of visualization. Note that a much shorter length was used in generating the results in Sect. 6. The expression of \( \varpi \) given in Eq. (24b) assumes the length to vary in-phase with the polar radius so that its minimum value occurs at the perigee and its maximum value at the apogee. A different form of the length variation was discussed in Ref. [12] under the assumption that the satellite rotates in the absolute space to maintain its radial orientation.

Figure 3 represents numerical results generated based on the length variation given in Eq. (24a). Figure 3a illustrates the temporal profile of length variation. Figure 3c, e, and f confirm that the satellite dynamics include some chaotic component whose physical nature has already been explained in Sect. 3. Figure 4 shows the numerical results obtained by using the length variation given in Eq. (24b), which prescribes a length variation in phase with the orbital radius. In this case, the satellite response appears to be more regular in comparison with their counterparts in Fig. 3. Such an effect of regularization was discussed at the end of Sect. 3 by noticing regions of quasiperiodic motions emerging from the stochastic area in Fig. 2f.

5 The effect of controllable satellite orientation at fixed length: analytical estimates

Using the original notations and considering Eqs. (13) through (14) in terms of differentials, and integrating around a complete orbital cycle, yields the following expressions for the changes of the integrals during one orbital cycle:

\[ \Delta x = \frac{3GM}{2} \int_0^{2\pi} \left( \frac{I}{r} \right)^2 \sin 2\theta \left[ \frac{1}{2r} \frac{dv}{dr} \right] \left( 1 + 3 \cos 2\theta \right) \frac{1}{r} dv, \] (25)

\[ \Delta h = \frac{3GM}{2h} \int_0^{2\pi} \left( \frac{I}{r} \right)^2 \sin 2\theta r dv, \] (26)

where the implicitly present orbital parameters on the right-hand sides are assumed to be “frozen” during the orbital cycle, similar in spirit to the method of van der Pol’s averaging [30].

Let us consider first a special case when the relative angle of the satellite can be controlled as

\[ \cos 2\theta = -1 \quad \text{for} \quad \frac{dr}{dv} > 0 \Leftrightarrow 0 < v < \pi, \] (27a)

\[ \cos 2\theta = 1 \quad \text{for} \quad \frac{dr}{dv} < 0 \Leftrightarrow \pi < v < 2\pi, \] (27b)

and thus \( \sin 2\theta = 0 \) along the entire orbit, \( 0 < v < 2\pi \), and therefore, according to (26), the angular momentum \( h \) remains fixed. This requires the idealization of a very rapid change in orientation at apogee and perigee. A geometrical interpretation of these conditions is given by Fig. 5.
Also, assuming that the length of the satellite is fixed, $l = l_0$, and switching the variable of integration to $r$ brings Eq. (25) to the form

$$\Delta t \leq \int_{r_{\text{min}}}^{r_{\text{max}}} \frac{3GMl_0^2}{2r^4} dr - \int_{r_{\text{min}}}^{r_{\text{max}}} \frac{3GMl_0^2}{r^4} dr = \frac{3GMl_0^2(r_{\text{max}}^3 - r_{\text{min}}^3)}{2r_{\text{max}}^3r_{\text{min}}^3}$$

(28)
Recall that
\[ r_{\text{min}} = a(1 - e), \quad r_{\text{max}} = a(1 + e), \]  
(29)
where
\[ a = -GM/(2\varepsilon) \]  
(30)
are semi-major axis and eccentricity, respectively.

Substituting Eq. (29) in Eq. (28), gives
\[ \Delta z \leq 6\varepsilon_0^2 \frac{G^4 M^4}{h^6} \left( 2 + \frac{h^2 \varepsilon}{G^2 M^2} \right) \sqrt{1 + \frac{2h^2 \varepsilon}{G^2 M^2}} \]  
(31)

Then, introducing notations \( z = 2h^2/(GM)^2 \) and \( \varepsilon = l_0 GM/h^2 \) brings Eq. (31) to the unitless form

---

Fig. 4 The example of satellite response to its length variation according to Eq. (24b) and \( \omega = 0 \) under the initial conditions used for the case of Fig. 3.
Δz ≤ 6e²(2 + z)\sqrt{1 + 2z} \quad (32)

where Δz is the change of unitless energy z per orbital cycle. Since Δz is small during a single cycle, then the energy can be considered as a continuous function of the number of cycles, z = z(N). This assumption enables one to make the replacement Δz → dN in Eq. (32). As a result, the number of cycles, during which the energy level changes from its initial level zᵢ to the final level zᶠ, can be estimated by the integral

\[ N \geq \frac{1}{6e²} \int_{zᵢ}^{zᵢ} \frac{dz}{(2 + z)\sqrt{1 + 2z}} = \frac{\sqrt{3}}{9e²} \arctan \left( \frac{1 + 2z}{3} \right) \bigg|_{zᵢ}^{zᵢ} = \frac{\sqrt{3}}{9e²} \arctan \left( \frac{\sqrt{3}(2zᵢ + 1 - \sqrt{2zᵢ + 1})}{\sqrt{2zᵢ + 1}\sqrt{2zᵢ + 1} + 1 + 3} \right) . \quad (33) \]

Further, using the notation z for unitless energy in Eq. (30) gives e = \sqrt{1 + 2z}. The unitless energy therefore can be expressed through the eccentricity at the initial “i” and final “f” levels as zᵢ = (eᵢ² - 1)/2 and zᵢ = (eᵢ² - 1)/2, respectively. Also, taking into account Eq. (30) for the initial state gives e = l₀GM/h² = (l₀/a₁)/(1 - eᵢ²), where a₁ and eᵢ are the initial semi-major axis and eccentricity. As a result, Eq. (33) takes its final form

\[ N ≥ \frac{\sqrt{3}}{9} \left[ (1 - eᵢ²) \frac{a₁}{l₀} \right]^2 \arctan \left( \frac{\sqrt{3}(eᵢ - eᵢ)}{3 + eᵢeᵢ} \right) . \quad (34) \]

The dependence N = N(eᵢ) and its derivative N'(eᵢ) are illustrated in Fig. 6 and are validated below in Sect. 6.1 through numerical simulations using a PID controller for the relative satellite angle θ.

### 6 Closed-loop control strategy

#### 6.1 Description of the control strategy and controlling the energy increase

The purpose of this subsection is to provide numerical validation for the analytical conjectures made in Sect. 5. Thomson [31] discussed methods for controlling spacecraft orientation and provided a thorough treatment of orbital dynamics. In the current study, the orbital angle, v, is considered to be an independent and uncontrolled variable. In line with the assumptions of Sect. 5, the length of the satellite is fixed. The relative angle of the satellite on the orbit, θ, is controlled by a PID controller to track a desired relative angle θᵣ(v) that has been defined below to approximate the expressions given in Eq. (27) as shown in Fig. 7a. For convenience of performing the simulations, Eqs. (13) to (16a) are represented in a unit-less form by rescaling the variables as in Sect. 4 to obtain the following equations:

\[ \frac{dv}{dv} = \frac{3}{2\bar{\theta}} \left( \frac{\bar{\theta}}{\bar{\theta}} \right)^2 \left[ \sin 2\theta - \frac{1}{2\bar{\theta}} \frac{d\bar{\theta}}{dv} (1 + 3 \cos 2\theta) \right] , \quad (35) \]

\[ \frac{\bar{\theta}}{dv} = \frac{3}{2\bar{\theta}} \left( \frac{\bar{\theta}}{\bar{\theta}} \right) \bar{\theta} \sin 2\theta \]

\[ \frac{d^2\theta}{dv^2} + 2 \left( \frac{1}{\bar{\theta}} \frac{d\bar{\theta}}{dv} - \frac{e \sin v}{1 + e \cos v} \right) \frac{d\bar{\theta}}{dv} + 1 \]

\[ = Q(v) , \quad (37) \]

\[ e = \sqrt{1 + 2\bar{\theta}^2} , \quad (38) \]

where
\[ Q(v) = -\frac{k^2}{l^2} \frac{d^2 \varphi}{dv^2} = -\frac{k^2}{l^2} \frac{d\Omega}{dv} \left( \Omega = \frac{d\varphi}{dv}, \quad \ddot{\varphi} = \frac{k}{R} \right), \]

(39)

and \( d\ell/dv = 0 \) according to the assumption of Sect. 5.

In this work, the controlled spinning parts of the satellite play the role of the actuators that are responsible for generating the control actions. As a consequence, the angular rate \( \Omega \) is assigned by the controller. Therefore, to simplify the current formulation, the lumped term \( Q(m) \) was considered to be the control signal generated by the following PID controller:

\[ Q(v) = K_P e_0 + K_D \frac{de_0}{dv} + K_I \int e_0 dv, \]

(40)

Here, the symbol “\( e \)” stands for “error.” In numerical simulations below, \( \bar{l} \) was set to

\[ \bar{l} = l_0 = 7.83699 \times 10^{-4}. \]

(41)

Also, to match the conditions of the above analytical estimates, the following initial conditions were selected: \( \varpi(0) = 1.4, \quad \bar{e}(0) = -0.234086 \) and \( \bar{\varpi}(0) = 1.372 \). Moreover, the desired satellite orientation was defined to be a function of \( v \) (see Fig. 7a):

\[ \theta_d(v) = \frac{1}{4} \pi \left( 1 + \frac{\cos \left( v - \frac{\pi}{2} \right)}{\sqrt{1 - \beta^2 \sin^2 \left( v - \frac{\pi}{2} \right)} } \right). \]

(42)

For \( \beta = 0 \), the above equation yields a shifted cosine wave. However, when \( \beta \rightarrow 1 \) Eq. (42) produces a rectangular pulse train function similar to the one described in Eq. (27) and analytically expressed as

\[ \theta_d(v) = \frac{1}{4} \pi \left[ 1 + \cos \left( v - \frac{\pi}{2} \right) \right] \cos \left( v - \frac{\pi}{2} \right). \]

(43)

The results of Fig. 7 are generated based on a fixed length parameter \( \bar{l} \) given by Eq. (41). When \( b = 0.9999999 \), the desired satellite orientation profile shown in Fig. 7a is close to that described by Eq. (43) mimicking the conditions of Sect. 5 for analytical estimates of the eccentricity growth (Fig. 6). In particular, Fig. 6b shows the rate of change of the eccentricity per unit orbital cycle, which is quite close to the data represented by Fig. 7b and found to be \( \frac{de}{dN} = 1.5625 \times 10^{-6} \).

Note that, even though the controller is capable of bringing the satellite orientation very close to the stepwise discontinuous profile of Fig. 7a, the required control input \( Q(v) \) has localized spikes of unrealistically high amplitudes developed from maxima and minima of the curve shown in Fig. 7e for \( \beta = 0.95 \). The fragments b, c, and d of Fig. 7 illustrate the evolution of satellite dynamics in terms of the eccentricity, energy, and angular momentum, respectively. In particular, these graphs reveal that the analytical estimate in Eq. (34), which is obtained for the limit \( \beta \rightarrow 1 \), provides a lower boundary for the effect; compare the case \( \beta = 0.9999999 \) to \( \beta = 0.1 \) and \( \beta = 0.95 \).
Comparing the ranges of orbital characteristics in fragments (b), (c), and (d) of Figs. 7 and 8 reveals that varying the length along the orbit can enhance the effect, which is achieved by controlling only the relative angle, $\theta$. Note that the choice for dependence $l(v)$ in Eq. (24b) was dictated by the intent to qualitatively match the behavior of orbital radius $r(v)$ in Eq. (15). In particular, such a choice allows for...
cancellation of the non-conservative term in the differential equation for \( h \) whenever \( f = e \). It follows from Eq. (24b) that the parameter \( f \) characterizing the range of length variation as \( l/l_0 = (1 + \zeta \cos \nu)^{-1} \) - unitless length versus orbital angle; b.

Fig. 8 Numerical solutions of Eqs. (35–40) for different values of parameter \( \zeta \) of the satellite variable length: (a) \( l/l_0 = (1 + \zeta \cos \nu)^{-1} \) - unitless length versus orbital angle; b, c and d—eccentricity, energy, and angular momentum responses during the first \( N = 20 \) orbital cycles \( (0 < \nu < 2\pi N) \); the relative angle \( \theta \) follows target (42) at \( \beta = 0.95 \) performing case, \( \zeta = -0.6 \), of Fig. 9, by the length variation estimated as: \( |l(\pi) - l(0)|/l_0 = -1.875 \).

6.2 Energy tracking control

The focus of this subsection is to demonstrate that the energy level can be robustly controlled to track a desired profile, \( \alpha_d(\nu) \). The results in the previous subsection confirmed that a “\(-\pi/2\)” phase shift embedded in the expression of the desired satellite orientation of Eqs. (42) and (43), based on the assumption of Sect. 5, serve to maximize the inflow of energy into the satellite. As a consequence, the satellite orientation along the orbit takes on the orientation profile shown in Fig. 5. Naturally, a “\(\pi/2\)” phase shift in the desired satellite orientation would have the opposite effect, namely maximizing the outflow of energy from the satellite.

This reasoning is at the core of the development of the proposed tracking control strategy for the satellite
energy. Both phase shifts are now incorporated into the following modified expression of the desired satellite orientation:

\[
\theta_d(v) = \frac{1}{4} \pi \left( 1 + \frac{\cos [v - \text{sgn}(\theta_d - \pi)] \frac{\pi}{2}}{\sqrt{1 - \beta^2 \sin^2 [v - \text{sgn}(\theta_d - \pi)] \frac{\pi}{2}}} \right) .
\]

(44)

For \( \theta_d - \pi > 0 \), the energy level should be increased. This is achieved by having \( \text{sgn}(\theta_d - \pi) = 1 \), which yields a “\(-\pi/2\)” phase shift in the above equation. Similarly, \( \theta_d - \pi < 0 \) yields \( \text{sgn}(\theta_d - \pi) = -1 \) which results in a “\(\pi/2\)” phase shift that leads to a decrease in energy level.

For a robust tracking characteristic, a sliding mode controller – SMC [32, 33] – was used to ensure that \( \theta \) accurately tracks \( \theta_d \) as specified in Eq. (44) in spite of structured uncertainties. Equation (37) is rewritten as follows

\[
d^2 \theta / d^2 v = -2 \left( \frac{d^2 \theta}{dv^2} - e \frac{e \sin v}{1 + e \cos v} \right) \left( \frac{d\theta}{dv} + 1 \right) - \frac{3 \sin \theta}{2(1 + e \cos v)} + Q(v)
\]

\[= f(\theta, \frac{d\theta}{dv}, v) + Q(v).\]

(45)

In this section, the length \( l \) will be considered to be a function of the orbital angle as prescribed by Eq. (24b) with \( l_0 = 7.83699 \times 10^{-4} \). The sliding “surface” is defined as

\[s \triangleq \frac{de_1}{dv} + \lambda e_1, \quad e_1 \triangleq \theta(v) - \theta_d(v),\]

(46)

where \( e_1 \) is the error between the actual and desired relative angle of the satellite on the orbit, \( \lambda \) is a control parameter dictating the decay rate of the error along the sliding surface, \( s \). It is assigned a numerical value of 15 in the current study. The structure of the control signal is given by

\[Q(v) = Q_{eq}(v) - K \text{sgn}(s),\]

(47)

where the gain \( K \) is defined below, \( Q_{eq}(v) \) is determined by imposing the condition \( \frac{d\theta}{dv} = 0 \) and replacing \( f(\theta, \frac{d\theta}{dv}, v) \) with its approximate function \( \hat{f}(\theta, \frac{d\theta}{dv}, v) \) due to structured uncertainties inherent in the modeling of the system. The structured uncertainties were included by considering a 10% discrepancy between \( f \) and \( \hat{f} \). It is expressed as

\[Q_{eq}(v) = \frac{d^2 \theta}{dv^2} - \lambda \frac{de_1}{dv} - f(\theta, \frac{d\theta}{dv}, v).\]

(48)

Next, \( K \) is obtained by satisfying the following sliding condition:

\[1 \frac{d(s^2)}{dv^2} \leq - \eta |s|,\]

(49)

which requires \( K \) to satisfy the following inequality:

\[K \geq \eta + F,\]

(50)

where \( F \) is the upper bound on the uncertainties between \( f \) and \( \hat{f} \). It is defined as \( F = \sup|f - \hat{f}| \).

The simulation results were generated based on Eqs. (35–39) and (41). To alleviate the chattering problem induced by the sign function, Eq. (47) was modified and the following expression for the control signal was implemented:

\[Q(v) = Q_{eq}(v) - K \text{sat}\left(\frac{s}{\Phi}\right),\]

(51)

where the boundary layer thickness \( \Phi \) was selected to be 10\(^{-6}\). The term “sat” refers to a saturation function bounded by \( \pm 1 \),

\[\text{sat}(x) = \begin{cases} -1, & x < -1 \\ x, & -1 \leq x \leq 1 \\ 1, & x > 1 \end{cases} .\]

(52)

As in Sect. 6.1, the initial conditions for \( \eta(0), \Phi(0) \) and \( \tilde{\eta}(0) \) were kept the same. The control parameters \( \eta \) and \( \lambda \) were selected to be 90 and 15, respectively. The results are illustrated in Fig. 10. Note that the cyclical fluctuations shown in Fig. 10a, b and c are inevitable since the target satellite orientation is directly linked to the orbital position, and the satellite must complete a full cycle in order to achieve the expected effect. Figure 10d demonstrates the robustness of the SMC controller in making \( \theta \) accurately track \( \theta_d \).

7 Conclusions

A non-jet compensation scheme has been developed to control the orbital parameters of a dumbbell satellite. Its main objective is to maintain an appropriate
The dumbbell satellite is designed to have internal degrees-of-freedom which are controlled to yield a total orbital energy that tracks a desired temporal profile with arbitrarily shaped segments of increasing and decreasing energy levels. Both PID and sliding mode controllers have been used in this study to control the spinning of the massive parts/rotors of the dumbbell about their respective mass centers; thus, producing the effective torque about the mass center of the entire satellite that is required to yield a desired orientation of the satellite with respect to the orbital radius along an elliptical orbit.

A switch from increasing to decreasing segments in the total energy profile is achieved through appropriate phase shift in the satellite relative angle. However, variations in the satellite length were considered in this work and they do not seem to contribute toward the reduction in the total energy level. The results of analytical estimates and numerical integration are in sufficient agreement. Moreover, the results in Sect. 6.1 demonstrate that the effectiveness of the proposed control strategies can be enhanced by considering a variable length of the satellite according to the prescribed open-loop law associated with the behavior of the orbital radius along the orbit. A detailed investigation of combined effects from both actuators is a subject for further work.

**Declarations**

**Conflict of interest** The authors declare that they have no conflict of interest.

**Data availability** Data available by reasonable request.

**Appendix**

**Appendix A: Details on the governing equations.**

Although the derivations below are quite straightforward, they are reproduced here due to the presence of internal degrees-of-freedom whose coordinates are varying in a prescribed way and therefore excluded from the set of generalized coordinates. With
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reference to Fig. 1, the gravitational potential energy and the kinetic energy of the satellite are, respectively,

\[ V = -\frac{GMm}{2} \left( \frac{1}{|\mathbf{r} + \mathbf{l}|} + \frac{1}{|\mathbf{r} - \mathbf{l}|} \right) \quad (A1) \]

and

\[ T = \frac{m}{4} \left[ (\mathbf{r} + \mathbf{l})^2 + (\mathbf{r} - \mathbf{l})^2 \right] + \frac{1}{2} I \Omega^2, \quad (A2) \]

where \( I = mk^2 \) is a combined moment of inertia of rotating parts about their centers of rotation expressed through the total mass \( m \) and the radius of gyration, \( k = \dot{\mathbf{v}} + \dot{\mathbf{v}} + \omega \) is the angular velocity, which is assumed to be the same for both of the spinning parts, and the satellite position vectors are:

\[ \mathbf{r} = r(\cos v + \sin v), \]
\[ \mathbf{l} = l[\cos(v + \theta) + \sin(v + \theta)] \cdot \mathbf{j}. \quad (A3) \]

The system’s Lagrangian per unit mass is used as

\[ L = (T - V)/m. \quad (A4) \]

Substituting Eq. (A3) in Eqs. (A1) and (A2), conducting calculations with the Cartesian vectors, and rearranging terms brings Eq. (A4) to the form

\[ L(r, v, \theta, \dot{r}, \dot{\theta}) = \frac{1}{2} (\dot{r}^2 + r^2 \dot{\theta}^2) \]
\[ + \frac{1}{2} (l^2 + k^2) (\dot{v} + \dot{\theta})^2 + k^2 (\dot{v} + \dot{\theta}) \omega + \frac{1}{2} (l^2 + k^2 \omega^2) \]
\[ + \frac{1}{2} GM \left( \frac{1}{\sqrt{r^2 - 2lr \cos \theta + l^2}} + \frac{1}{\sqrt{r^2 + 2lr \cos \theta + l^2}} \right). \quad (A5) \]

This Lagrangian is a function of the generalized coordinates \( \{r, v, \theta\} \) and velocities \( \{\dot{r}, \dot{\theta}\} \). The additive time dependent term \((l^2 + k^2 \omega^2)/2\) does not include any state variables and thus can be removed from Eq. (A5) with no effect on the corresponding differential equations of motion. Further, taking into account the assumption \( l/r < < 1 \) gives

\[ \frac{1}{\sqrt{r^2 - 2lr \cos \theta + l^2}} = \frac{1}{r} \left[ 1 + \frac{\sqrt{l^2 \cos \theta + (1 + 3 \cos 2\theta)}}{4r^2} \right] + O\left(\frac{l^3}{r^3}\right). \quad (A6) \]

Substituting (A6) in (A5) completes the derivation of Lagrangian in Eq. (2) of Sect. 2. Then Eqs. (3) through (5) follow from Euler–Lagrange equations

\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = 0, \quad \frac{d}{dt} \frac{\partial L}{\partial \dot{v}} - \frac{\partial L}{\partial v} = 0, \quad \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0. \quad (A7) \]

It is seen from both Eq. (A5) and its reduced version (2) that the angular coordinate \( v \) is cyclical, and thus the angular momentum conservation law, \( \partial L/\partial \dot{v} = \text{const.} \), holds as

\[ r^2 \dot{v} + (l^2 + k^2)(\dot{v} + \dot{\theta}) + k^2 \omega = \text{const.} \quad (A8) \]

Appendix B: Equation describing the satellite relative angle

Taking into account the assumption \( k/l < < 1 \) brings Eq. (5) to the form

\[ \frac{d}{dt} \left[ l^2 (\dot{v} + \dot{\theta}) \right] + \frac{3}{2} \left( \frac{l}{r^2} \right)^2 \frac{GM}{r} \sin 2\theta + k^2 \dot{\omega} = 0. \quad (B1) \]

Considering the orbital angle \( v \) as a natural independent variable and thus using the operator of time derivative, \( d/dt \), from Eq. (12) gives

\[ \frac{h}{r^2} \frac{d}{dv} \left[ \frac{h}{r^2} \frac{d\theta}{dv} + \frac{h}{r^2} \right] + \frac{3}{2} \left( \frac{l}{r^2} \right)^2 \frac{GM}{r} \sin 2\theta \]
\[ = -\frac{k^2 d\omega}{r^2 dv}, \quad (B2) \]

where the quantity \( h \) is “frozen” according to Eq. (7).

Multiplying both sides of Eq. (B2) by \( r^4/(hl^2) \) brings it to the form

\[ \frac{r^2}{l^2 dv} \left( \frac{d}{dv} \left( \frac{l}{r^2 dv} + 1 \right) \right) + \frac{3GM}{2h^2 r} r \sin 2\theta = -\frac{k^2 r^2 d\omega}{h^2 dv}. \quad (B3) \]

Finally, substituting the polar radius, \( r = h^2/(GM(1 + e \cos v)) \), on the left-hand side of Eq. (B3) gives
\[
\frac{d^2\theta}{dt^2} + \frac{2}{3\sin 2\theta}\left(\frac{d\theta}{dv} + \frac{e\sin v}{1 + e\cos v}\right)\left(\frac{d\theta}{dv} + 1\right) \\
+ \frac{2(1 + e\cos v)}{k^2r^2}\frac{1}{l^2}\frac{1}{dv}\frac{1}{(r^2dv)}
\]

(B4)

The angle \(\phi\), describing the relative position of both spinning parts, must depend upon the polar angle \(v\) much faster than the polar radius \(r\) in order to generate a sufficient control torque. Therefore, the right-hand side of Eq. (B4) can be estimated as

\[
-\frac{k^2r^2}{l^2}\frac{1}{r^2}\frac{1}{dv}\frac{1}{dv} \approx -\frac{k^2}{l^2}\frac{d^2\phi}{dv^2}
\]

(B5)

Substituting (B5) in (B4) gives Eq. (16a) of Sect. 2.2. Note that the above assumption is not restrictive since, according to the control algorithm of Sect. 6, the entire right-hand side of Eq. (B4) is considered as a control input \(Q(v)\) (see Eq. (37)). Having obtained the function \(Q(v)\), the exact relationship can also be used for determining the dependence \(\phi = \phi(v)\):

\[
-\frac{k^2r^2}{l^2}\frac{1}{r^2}\frac{1}{dv}\frac{1}{dv} = Q(v).
\]

(B6)

The role of control input, \(Q(v)\), is twofold: stabilizing the satellite and changing its orientation on the orbit according to the analytically predicted law. Note that, in case of no spinning, the right-hand side of Eq. (B4) becomes zero, \(Q = 0\), and the system has just one input, \(l = l(v)\). This dependence can also be generated by a closed loop controller. The present hybrid approach however assigns the dependence \(l(v)\) in an empirical way by selection in order to make the main controller more effective. As a guidance for such a selection, one can assume the distance \(l(v)\) to be varying coherently with the orbital radius, for instance, as \(l = l_0/(1 + \zeta\cos v)\), where \(\zeta\) is a constant parameter, \(|\zeta| < 1\). In this case, calculating the quantity \((dl/dv)/l\) and substituting the result in Eq. (B4) with zero right-hand side gives Eq. (16b) of Sect. 3.

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