Mutual information in nonlinear communication channel: Analytical results in large SNR limit

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Introduction. The channel capacity is one of the central concepts of information theory that has its roots in statistical physics. The capacity introduced by Shannon in his seminal work gives the maximum rate at which information can be reliably transmitted through a noisy communication channel. The channel capacity (in bits per symbol) is formally defined as a maximum of the mutual information $I_{P[X]}$ over the input signal probability distribution functional (PDF) $P[X]$: 

$$C = \max_{P[X]} I_{P[X]},$$  

under the condition of a limited average power. The mutual information $I_{P[X]}$ (in continuous-input, a continuous-output channel) is expressed through the path integral over input $X$ and output $Y$ signals:

$$I_{P[X]} = \int DXY P[X]P[Y|X] \ln \frac{P[Y|X]}{P_{out}[Y]},$$

the output signal PDF reads

$$P_{out}[Y] = \int DX P[X]P[Y|X],$$

where $P[Y|X]$ being the conditional probability density, that is, the probability of receiving output signal $Y$ when the input signal is $X$. Capacity in bits per symbol multiplied by the rate at which symbols are transmitted (in symbols per second) gives the error-free information transmission rate in bits per second.

The above definition exemplifies that channel capacity’s close link to information entropy. Mutual information is a difference between the entropy of the output signal

$$H[Y] = -\int DYP_{out}[Y] \ln P_{out}[Y]$$

and conditional entropy

$$H[Y|X] = -\int DXY P[X]P[Y|X] \ln P[Y|X],$$

(having a meaning of a measure of the uncertainty about the output field $Y$ if the input field $X$ is known). When the signal and noise are independent variables and the received signal $Y$ is the sum of the transmitted signal $X$ and the noise, then it can be shown explicitly that the entropy is generated during transmission in noisy channel: $H[Y] \geq H[X]$. In this case, the transmission rate is the entropy of the received signal less the entropy of the noise. The maximum of the functional (channel capacity) can be calculated for such linear channels with an additive white Gaussian noise (AWGN):

$$C \propto \log (1 + \text{SNR}),$$

where SNR is a signal-to-noise power ratio.

This seminal theoretical result is the foundation of communication theory and it has proven its importance in a number of practical applications. To some extent, the Eq. worked so well in so many situations that some engineers cease to distinguish the general Shannon expression for capacity and particular result for the specific linear additive white Gaussian noise channel. However, recent advances in fibre-optic communication where the channel is nonlinear, as opposed to the linear AWGN, changed the situation. To increase the channel capacity over a certain bandwidth with a given accumulated noise of optical amplifiers, one has to increase the signal power, see [4]. This works in the low SNR limit but the effect the refraction index’s dependence on light intensity dramatically changes the propagation properties of the channel at higher optical signal power. In
other words, the fibre-optic channel is nonlinear. Recent studies have shown that the spectral efficiency (that is, the number of bits transmitted per second per Hertz—practical characteristics having the same dimension as channel capacity) of a fibre-optic channel is limited by the Kerr nonlinearity. These studies indicated that observable spectral efficiency always turns out to be less than the Shannon limit of the corresponding linear AWGN channel for equal fibre channel capacity is decreasing with power; see, for example, discussions in [12, 13].

In general, there is a widely spread misconception that nonlinear channel capacity is always less than the capacity of the corresponding linear AWGN channel for equal SNR. However, in Ref. [14–17] the authors note that nonlinearity can be either destructive or constructive. Moreover, in Ref. [12] it was proved that the capacity of certain nonlinear channels could not decrease with SNR. The Shannon capacity of nonlinear fibre channels is still open an problem of great practical and fundamental importance.

In this work, we calculate analytically the first nonzero correction to mutual information of the channel described by the NLSE with additive Gaussian noise. We then calculate the mutual information and estimate the lower bound for channel capacity. Finally, we present the main conclusions of the paper.

The nonlinear channel model and conditional probability
Consider propagation of the signal $\psi_w(z)$ in the channel described by the NLSE with AWGN, that we rewrite, for convenience, in the spectral domain, see [20, 21].

$$\partial_z \psi_\omega(z) = i\beta \omega^2 \psi_\omega(z) + \eta_\omega(z) + i\gamma \int dw_1 dw_2 \frac{1}{(2\pi)^2} \psi_{w_1}(z) \psi_{w_2}(z) \bar{\psi}_{w_3}(z),$$

where $\omega_3 = \omega_1 + \omega_2 - \omega$, $\beta$ is the dispersion coefficient, $\gamma$ is the Kerr nonlinearity coefficient, mean complex conjugation, $\eta_\omega(z)$ is an additive complex white noise with zero mean and correlation function $\langle \eta_\omega(z) \bar{\eta}_\omega(z') \rangle = 2\pi Q\delta(z - z')\delta(\omega - \omega')$, where $Q$ is up to $2\pi$ the noise power per unit frequency per unit length [6, 20]: $P_{\text{noise}} = QW/2\pi$, with $W/(2\pi)$ being a frequency bandwidth.

It is worth emphasizing that in a nonlinear channel, transmitted and received signal bandwidths can be different from each other. Therefore, we assume here that in general, the input $X(\omega)$ and output $Y(\omega)$ signals may have different channel bandwidths $W$ and $W'$, with $W' \supset W$ (though this is not a requirement).

We introduce the dimensionless parameter $\tilde{\gamma} = P_{\text{ave}} \gamma L$ which describes impact of nonlinearity. Here the average power of the signal $X$ reads

$$P_{\text{ave}} = \lim_{T \to \infty} \int DXP[X] \frac{1}{T} \int d\omega |X(\omega)|^2,$$

where $T$ is the (large) time interval containing the whole input signal. It has been shown in Ref. [19] that for NLSE channel governed by Eq. (3) in the large SNR limit $\epsilon = 1/\text{SNR} = QLW/(2\pi P_{\text{ave}}) \ll 1$ the conditional probability density $P[Y(\omega)|X(\omega)]$ to receive $\psi_w(L) = Y(\omega)$ given the input signal $\psi_w(0) = X(\omega)$ can be written as

$$P[Y(\omega)|X(\omega)] \approx \exp\left[-\frac{S[\Psi_w(z)]}{Q}\right] \int_{\phi_w(0)=0} D\phi \exp\left[-\frac{1}{Q} \left\{ S[\Psi_w(z) + \phi_w(z)] - S[\Psi_w(z)] \right\} \right],$$

$$S[\psi] = \int_0^L dz \int \frac{d\omega}{2\pi} |\partial_z \psi_\omega(z) - i\beta \omega^2 \psi_\omega(z) - i\gamma \int dw_1 dw_2 \frac{1}{(2\pi)^2} \psi_{w_1}(z) \psi_{w_2}(z) \bar{\psi}_{w_3}(z)|^2.$$

Here $\Psi_w(z)$ is the so-called “classical trajectory” of the Feynman integral [18], that is, the extremum function of the action $S(\delta S[\psi] = 0$ — see Eq. (14) in [19]) with the boundary conditions $\Psi_w(0) = X(\omega)$, $\Psi_w(L) = Y(\omega)$. At small $\tilde{\gamma}$ we can derive analytically the function $P[Y(\omega)|X(\omega)]$, Eq. (7), using the perturbation theory developed in Ref. [19]. There are two types of terms in the expansion of Eq. (7) in $\tilde{\gamma}$. The first type of pertur-
bative corrections comes from the expansion of exponent
\[ \exp \left[ -\frac{S[\Psi_{\omega}(z)]}{Q} \right] \text{and has the structure} \]
\[
\exp \left[ -\frac{S[\Psi_{\omega}(z)]}{Q} \right] \approx \exp \left[ -\frac{S[\Psi_{\omega}^{(0)}(z)]}{Q} \right] \times \left( 1 + \sum_{p=0}^{\infty} \sum_{k=0}^{\infty} \alpha_{p,k} \left[ \Psi_{\omega}^{(0)}(z) \right] \gamma^{p} \left( \frac{\zeta}{\epsilon} \right)^{k} \right). \tag{9}
\]
Here \( \Psi_{\omega}^{(0)}(z) = e^{i\beta z^{2}} \left[ zB(\omega)/L + X(\omega) \right] \) is the solution of the Eq. (14) of Ref. [19] with the boundary conditions \( \Psi_{\omega}(0) = X(\omega), \Psi_{\omega}(L) = Y(\omega) \), and \( \gamma = 0 \), \( B(\omega) = e^{-i\beta z^{2}}Y(\omega) - X(\omega) \), and \( \alpha_{p,k} \left[ \Psi_{\omega}^{(0)}(z) \right] \) are some constructively defined functionals, see [22]. The second type of corrections originates from the expansion of the path-integral in Eq. (7) and has the form
\[
\Lambda_{QL}^{(M')} \left( 1 + \sum_{p=1}^{\infty} \sum_{k=0}^{\infty} \gamma_{p,k} \left[ \Psi_{\omega}^{(0)}(z) \right] \gamma^{p} \left( \frac{\zeta}{\epsilon} \right)^{k} \right), \tag{10}
\]
where \( \Lambda_{QL}^{(M')} = \left( \frac{\pi M'}{QL} \right)^{M'} \) is the normalization factor, see [19], and \( \gamma_{p,k} \left[ \Psi_{\omega}^{(0)}(z) \right] \) are some functionals, see [22]. Since the parameter \( \epsilon \ll 1 \), the main contribution comes from the expansion of exponent Eq. (9); therefore the expansion of the exponent gives the main contribution to the conditional probability. We substitute the function \( \Psi^{(0)} \) to the right-hand side of Eq. (10), then the obtained result is multiplied by Eq. (10) leading to:
\[
P[Y(\omega)|X(\omega)] \approx P^{(0)}[Y(\omega)|X(\omega)] \left( 1 + \gamma_{1,0} \tilde{\gamma} + \sum_{k=1}^{\infty} \alpha_{0,k} \left( \frac{\tilde{\gamma}}{\epsilon} \right)^{k} + \tilde{\gamma} \sum_{k=1}^{\infty} [\alpha_{1,k} + \alpha_{0,k} \gamma_{1,0}] \left( \frac{\tilde{\gamma}}{\epsilon} \right)^{k} \right). \tag{11}
\]
where
\[
P^{(0)}[Y(\omega)|X(\omega)] = \Lambda_{QL}^{(M')} \exp \left[ -\frac{1}{QL} \int_{W'}^{W} \frac{d\omega}{2\pi} |B(\omega)|^{2} \right]. \tag{12}
\]
In Eq. (11), we keep only leading and next-to-leading order in \( \tilde{\gamma} \) terms for every order in \( \tilde{\gamma}/\epsilon \). This means that the parameter \( \tilde{\gamma}/\epsilon \) can be of order of unity, omitting the dependence of functionals \( \alpha_{p,k} \) and \( \gamma_{p,k} \) on \( \Psi_{\omega}^{(0)}(z) \), but having this in mind. Now nonlinear corrections to the mutual information can be calculated.

**Calculation of the non-linear corrections to the mutual information** To calculate mutual information, we have to calculate the path-integral [2]. In the previous section, we derived the expression (11) for conditional probability function. Let us introduce a probability function of initial signal \( P[X(\omega)] \). In our consideration, the PDF \( P[X(\omega)] \) is chosen to be Gaussian in the spectral domain \( W \) (and zero in \( W' \setminus W \)). The discretized representation of \( P[X(\omega)] \) reads
\[
P[X(\omega)] = \Lambda_{p}^{(M)} \left( \prod_{i=1}^{M} e^{-\frac{\pi}{2H}X_{i}^{2}} \right) \prod_{i=1}^{M'-M} \delta(X_{i}). \tag{13}
\]
where \( \delta(X) = \delta(ReX)\delta(ImX) \) is the \( \delta \)-function, frequency domain \( W' \setminus W \) and is divided by \( M \) \( (M') \) grids spacing \( \delta = W' = W'/2\pi \); \( X_{j} = X(\omega_{j}) \). The coefficient \( \Lambda_{p}^{(M)} = \left( \frac{\delta}{\pi M} \right) \) follows from the normalization condition \( \int DXP[X(\omega)] = 1 \). The measure \( DX \) in (2), and in (3), is understood as \( DX = \prod_{i=1}^{M'} \left( dReX_{j} dImX_{j} \right) \). As follows from the Nyquist-Shannon-Kotelnikov theorem [22] to fully recover analogue fields from the discretization, \( M \) should be chosen greater than \( TW/2\pi \); the limit \( T \to \infty \) in Eq. (3) is equivalent to \( M = TW/2\pi \to \infty \) in the measure. Parameter \( P \) in Eq. (13) describes the signal power per unit of frequency (power spectral density), so the average signal power \( (0) \) is \( P_{av} = PW/2\pi \gg P_{noise} \) and the nonlinearity parameter is \( \tilde{\gamma} = PWL'/(2\pi) \). The calculation of path-integrals in Eq. (9) can be performed using Eq. (31) of [22].

First, let us show that all leading corrections in \( 1/\epsilon \) (corrections of order of \( (\tilde{\gamma}/\epsilon)^{k} \)) to the mutual information are equal to zero for all \( k > 0 \), and there are only sub-leading corrections \( \gamma^{p}(\tilde{\gamma}/\epsilon)^{k} \) with \( p > 0 \). To show that, we substitute expression (11) to Eq. (2) and change the integration variable \( X(\omega) \) to \( B(\omega) \). The function \( P^{(0)}[Y(\omega)|X(\omega)] \) depends only on \( B(\omega) \) and has a variation scale of order \( QL \), the scale of variation of function \( P[X(\omega)] \) is \( P \), which obeys the condition \( P \gg QL \), and the correction of order of \( (\tilde{\gamma}/\epsilon)^{k} \) comes from the third term in the brackets in Eq. (11), but the coefficients \( \alpha_{0,k} \) are proportional to \( B^{k}(\omega) \), see Eq. (13), (15) in [22]. This means that all terms with odd powers \( B(\omega) \) give zero after integration over \( B(\omega) \), while terms proportional to even powers \( B^{k}(\omega) \), after integration over \( B(\omega) \), contribute proportionally to factor \( (QL)^{k/2} \), which reduces the power of \( \tilde{\gamma}/\epsilon \) in the denominator. Therefore, all corrections to the mutual information having the form of a power of \( \tilde{\gamma}/\epsilon \) disappear.

Direct calculation of the mutual information Eq. (2) shows that all linear corrections in \( \tilde{\gamma} \) are equal to zero, see [22]. The first nonzero correction appears in the order \( \tilde{\gamma}^{2}/\epsilon \), and we calculate it below. This correction results from two terms of Eq. (11), with the first proportional to \( \alpha_{0,2} \), and the second to \( \alpha_{1,1} \). The term that contains \( \alpha_{0,1,1,0} \) is proportional to \( B^{2}(\omega) \); therefore, it does not accountably contribute to the order \( \tilde{\gamma}^{2}/\epsilon \) after integration over \( B(\omega) \), but only in the non-leading order \( \tilde{\gamma}^{2} \). To obtain the mutual information in the order \( \gamma^{0} \), and the correction to it which has the order \( \tilde{\gamma}^{2} \) and is enhanced by \( 1/\epsilon \), we substitute the expression
\[
P^{(0)}[Y(\omega)|X(\omega)] \left( 1 + \alpha_{0,2} \left( \frac{\tilde{\gamma}}{\epsilon} \right)^{2} + \alpha_{1,1} \frac{\tilde{\gamma}^{2}}{\epsilon} \right) \tag{14}
\]
instead of Eq. (11), and probability function \( P[X] \) in the form Eq. (13) to the Eq. (2), then perform straightforward, but cumbersome calculations, see Ref.[22], and obtain the following expression for the mutual information:

\[
I_{P[X]} = M \ln \left[ 1 + \frac{1}{\epsilon} \right] + 4MG(\tilde{\beta}) \frac{\tilde{\gamma}^2}{\epsilon},
\]

where the first term in the left-hand side is the well-known Shannon's result (3) for the capacity of the linear channel (1), and the second term is the nonlinear correction to the mutual information. The positive function \( G(\tilde{\beta}) = 1 + \frac{1}{2\tilde{\beta}^2} \int \frac{\sin^2[\tilde{\beta}(y-y_1)(y-y_2)]}{(y-y_1)(y-y_2)^2} \Omega \),

\[
\Omega = \left[ -\frac{1}{2}, \frac{1}{2} \right] \times \left[ -\frac{1}{2}, \frac{1}{2} \right] \times \left[ -\frac{1}{2}, \frac{1}{2} \right] \text{ being a cube in frequency domain. Note that our result (15) does not depend on } W' \text{ but only } W, \text{ since } M = TW/2\pi.
\]

More importantly, since the obtained nonlinear correction in Eq. (15) is positive, there is the region of the parameter \( \epsilon \) where the obtained mutual information of the channel under consideration is greater than Shannon's limit for capacity of linear channel. Since the capacity of the channel is greater or equal to mutual information, see Eq. (1), the Shannon capacity of the considered nonlinear channel exceeds capacity of the linear AWGN channel (4) in some region of SNR. The region of applicability of the result (15) can be found from the two following conditions. First, the method of calculation is correct when SNR = 1/\( \epsilon \gg 1 \). Second, the calculated correction must be much greater than the essential nonlinear correction (of \( \tilde{\gamma}^2/\epsilon^2 \) order), and this condition gives \( \epsilon \gg \tilde{\gamma} \), or SNR \( \gg 1/\tilde{\gamma} \) (i.e. SNR\(_{\text{max}} \sim \frac{1}{\sqrt{\gamma QLW}} \)).

One can see that our result (15) formally depends on discretization parameter \( M \). The reason for that is the definition of mutual information through path-integral (2). Let us introduce spectral efficiency:

\[
\text{SE} = \left( \frac{W}{2\pi} \log 2 \right)^{-1} \lim_{T \to \infty} \frac{1}{T} I_{P[X]},
\]

where \( W/2\pi \) is the signal bandwidth. We substitute Eq. (19) to Eq. (17), and obtain spectral efficiency for the nonlinear channel:

\[
\text{SE} = \frac{I_{P[X]}}{\ln 2} \approx \log_2 \left[ 1 + \frac{1}{\epsilon} \right] + \frac{\tilde{\gamma}^2 G(\tilde{\beta})}{\epsilon \ln 2} + \log_2 \left[ 1 + \text{SNR} \right]
+ \frac{4G(\tilde{\beta})}{\ln 2} \left( \gamma QL^2 \right)^{2/\ln 2} (\text{SNR})^3.
\]

Let us estimate SE for typical fibre optic links [3]: \( \tilde{\beta} = 40 \text{ ps}^2/\text{km}, \ L = 2000 \text{ km}, \ \gamma = 1.31 (\text{Wkm})^{-1}, \ W/2\pi = 50 \text{ GHz}, \ QLW/2\pi = 5.3 \times 10^{-4} \text{mW}. \) For this, the parameters are \( \tilde{\beta} = \beta LW^2 \approx 7900 \) and \( G(\tilde{\beta}) \approx 1.04. \)

\[
\text{SE} \approx \log_2 \left[ 1 + \text{SNR} \right] + 1.16 \times 10^{-5} \times (\text{SNR})^3. \quad (19)
\]

The applicability region of the result Eq.(19) is \( 1 < \text{SNR} < 10.7. \) In the region the correction changes from \( 1.16 \times 10^{-5} \text{ at SNR} = 1, \) to \( 0.014 \text{ at SNR}_{\text{max}} = 10.7. \)

Though this is rather small correction (0.4%), the most important theoretical finding is that this addition is positive leading to a new insight on the role of nonlinearity in fibre channel(s).

Conclusion. We have derived the analytical expression for the mutual information \( I_{P[X]} \) of the NLSE channel in the limit of large SNR (\( \tilde{\gamma} \)) and small nonlinearity \( \tilde{\gamma} \ll 1 \). We have demonstrated that the first nonlinear correction to mutual information is of \( \tilde{\gamma}^2/\epsilon \) order and is positive. This means that the channel capacity of the considered nonlinear channel is greater than the capacity of the classical linear AWGN channel (4). Our result is somewhat counterintuitive and contradicts the spread perception that adding nonlinear propagation effects makes the communication channel suboptimal to the linear AWGN channel. It is also obvious that by solving numerically the nonlinear Schrödinger equation for different realizations of the noise, it is possible to find only some average quantity related to conditional probability density function. It is not possible to cover/explore the whole functional space and this leads to underestimating mutual information in numerical modelling.

Let us stress once again that we have calculated only the mutual information \( I_{P[X]} \) in the case of a Gaussian input signal PDF \( P[X] \) rather than the channel capacity (1). However, the quantity \( I_{P[X]} \) is a natural estimate of a low bound on the channel capacity since it reproduces the Shannon capacity of the linear AWGN channel (4) in the leading order. Theoretically our method allows one to calculate directly the channel capacity that will be the matter of our future considerations.

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[5] R.-J. Essiambre, G. J. Foschini, G. Kramer, and P. J. Winzer, Phys. Rev. Lett. 101, 163901 (2008).
[6] R.-J. Essiambre, G. Kramer, P. J. Winzer, G. J. Foschini, and B. Goebel, J. of Lightwave Technol. 28, 0733, (2010).
[7] A. D. Ellis, J. Zhao, and D. Cotter, J. of Lightwave Technol. 28, 423, (2010).
[8] D. J. Richardson, Science 330, 327, (2010).
[9] R. Killey and C. Behrens, J. Mod. Opt. 58, 1, (2011).
[10] A. D. Ellis and J. Zhao, Impact of Nonlinearities on Fiber Optic Communications, Springer, New York, (2011).
[11] A. Mecozzi, R.-J. Essiambre, J. of Lightwave Technol. 30, 2011, (2012).
[12] E. Agrell, arXiv: 1108.0391v3.
[13] E. Agrell, "Nonlinear Fiber Capacity," presented at the Eur. Conf. Opt. Commun. LondonU.K., paper We.A.D.3 (2013).
[14] E Agrell, A Alvarado, G Durisi, M Karlsson, arXiv:1403.3339
[15] K.S. Turitsyn, S.A. Derevyanko, I.V. Yurkevich, and S.K. Turitsyn, Phys. Rev. Lett. 91, 203901 (2003).
[16] K. S. Turitsyn, K. K. Turitsyn, Opt. Lett. 37, 0146, (2012).
[17] M. A. Sorokina and S. K. Turitsyn, Nat. Comm. 5, 3861 (2014)
[18] R. P. Feynman, A. R. Hibbs, Quantum mechanics and path integrals, McGraw-Hill Book Company, New York, (1965).
[19] I. S. Terekhov, S. S. Vergeles, and S. K. Turitsyn, arXiv:1411.6792
[20] E. Iannoe, F. Matera, A. Mecozzi, and M. Settembre, Nonlinear Optical Communication Networks, John Wiley & Sons, New York, (1998).
[21] S.K. Turitsyn, S.B. Medvedev, M.P. Fedoruk, and E.G. Turitsyna, Phys. Rev. E 61, 3127 (2000).
[22] Supplementary Materials.
[23] C. E. Shannon, Communication in the presence of noise, Proc. Institute of Radio Engineers, vol. 37, 1, (1949).
V. A. Kotelnikov, On the capacity of the ether and wire in Telecommunications, Proceedings of the First All-Union Congress on Technical Reconstruction of Communication. Union Energy Committee. (1933), In Russian.
Supplementary Materials for the article: The analytic calculation of the mutual information in large SNR limit

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EXPANSION OF THE CONDITIONAL PROBABILITY DENSITY FUNCTION

In Ref.[1] we have shown that in the case $1/\epsilon = \text{SNR} \gg 1$ the conditional probability can be written in the form:

$$P[Y(\omega)|X(\omega)] = e^{-S[\Psi_\omega(z)]/Q} \int_{\Psi_\omega(0)=0}^{\Psi_\omega(L)=0} \mathcal{D}\tilde{\psi} e^{-i S[\Psi_\omega(z)+\tilde{\psi}_\omega(z)-S[\Psi_\omega(z)])/Q},$$

(1)

where the measure is defined as

$$\mathcal{D}\tilde{\psi} = \lim_{\delta \to 0} \lim_{\Delta \to 0} \left( \frac{\delta}{\Delta Q} \right)^{NM'} \prod_{j=1}^{M'} \prod_{i=1}^{N-1} d\text{Re}\tilde{\psi}_{i,j} d\text{Im}\tilde{\psi}_{i,j},$$

with $\tilde{\psi}_{i,j} = \tilde{\psi}_\omega(z_i)$ and $\Delta = \frac{L}{M'}$ is the coordinate grids spacing, the action $S[\tilde{\psi}]$ reads:

$$S[\tilde{\psi}] = \int_0^L dz \int \frac{d\omega}{2\pi} |\mathcal{L}[\tilde{\psi}]|^2,$$

(2)

$$\mathcal{L}[\tilde{\psi}] = \mathcal{L}^{(0)}[\tilde{\psi}] - V[\tilde{\psi}],$$

(3)

$$\mathcal{L}^{(0)}[\tilde{\psi}] = \partial_z \tilde{\psi}_\omega(z) - i\beta \omega^2 \tilde{\psi}_\omega(z),$$

(4)

$$V[\tilde{\psi}] = i\gamma \int \frac{d\omega_1 d\omega_2 d\omega_3}{(2\pi)^2} \delta(\omega_1 + \omega_2 - \omega_3 - \omega) \tilde{\psi}_\omega(z_1) \tilde{\psi}_\omega(z_2) \tilde{\psi}_\omega(z_3).$$

(5)

The function $\Psi_\omega(z)$ is the solution of the equation:

$$\left( \partial_z - i\beta \omega^2 \right)^2 \Psi_\omega(z) =$$

$$i\gamma \int \frac{d\omega_1 d\omega_2 d\omega_3}{(2\pi)^2} \delta(\omega_1 + \omega_2 - \omega - \omega_3) \left\{ 4\Psi_{\omega_2}(z) \tilde{\Psi}_{\omega_3}(z) \left[ \left( \partial_z - i\beta \omega_1^2 \right) \Psi_{\omega_1}(z) \right] - \frac{\mu}{L} \Psi_{\omega_1}(z) \tilde{\Psi}_{\omega_2}(z) \tilde{\Psi}_{\omega_3}(z) \right\} -$$

$$3\gamma^2 \int \frac{d\omega_1 d\omega_2 d\omega_3 d\omega_4 d\omega_5 d\omega_6}{(2\pi)^4} \delta(\omega_1 + \omega_2 + \omega_4 - \omega_5 - \omega_6 - \omega) \Psi_{\omega_1}(z) \Psi_{\omega_2}(z) \Psi_{\omega_3}(z) \tilde{\Psi}_{\omega_4}(z) \tilde{\Psi}_{\omega_5}(z) \tilde{\Psi}_{\omega_6}(z) = 0,$$

(6)

with the boundary conditions: $\Psi_\omega(0) = X(\omega), \quad \Psi_\omega(L) = Y(\omega)$, and $\mu = \mu(\omega, \omega_2) = 2i\beta L(\omega - \omega_1)(\omega - \omega_2)$. The equation (6) can be solved using perturbation theory at small $\gamma$. To solve the equation we present the solution $\Psi_\omega(z)$ in the form:

$$\Psi_\omega(z) = \sum_{k=0}^{\infty} \Psi_\omega^{(k)}(z),$$

(7)

where $\Psi_\omega^{(k)}(z)$ is the solution of the Eq. (6) of order of $\gamma^k$. For calculations of the mutual information with precision $\gamma^2$ we need only first two terms of expansion (7). Straightforward solution of Eq. (6) in the leading and next-to-leading order gives:

$$\Psi_\omega^{(0)}(z) = e^{i\beta \omega^2 z} \left[ \frac{z}{L} B(\omega) + X(\omega) \right],$$

(8)
here $B(\omega) = e^{-i\beta \omega^2 t} Y(\omega) - X(\omega)$. The first order correction reads

$$\Psi_\omega^{(1)}(z) = i\gamma e^{i\beta \omega^2 \int_0^L dz' G(z, z') F_\omega(z')}$$

(9)

where $G(z, z') = \frac{z - L}{L} (z' + (z' - z)\theta(z' - z))$ is Green function of the $\partial_z^2$ operator (with the boundary conditions:

$\Psi_\omega^{(1)}(0) = \Psi_\omega^{(1)}(L) = 0$). In Eq. (9) we have

$$F_\omega(z) = \int_{W'} \frac{d\omega_1 d\omega_2}{(2\pi)^2} \frac{e^{-\mu_2/L}}{L} \left[ \frac{z}{L} B(\omega_2) + X(\omega_1) \right] \times \left[ \frac{z}{L} B(\omega_3) + X(\omega_3) \right]$$

(10)

where $\mu = \mu(\omega, \omega_1, \omega_2) = 2i\beta L (\omega - \omega_1)(\omega - \omega_2), \omega_3 = \omega_1 + \omega_2 - \omega$.

The substitution of the solution $\Psi_\omega(z)$ in the form (3) to the Eq. (2) gives

$$S[\Psi_\omega(z)] = \sum_{k=0}^{\infty} S^{(k)}[\Psi_\omega(z)]$$

(11)

where $S^{(k)}[\Psi_\omega(z)]$ is the term of order of $\gamma^k$ of the action (2) expansion in $\gamma$. Now we can expand the exponential prefactor in Eq. (11) and obtain functions $\alpha_{p,k}[\Psi_\omega^{(0)}(z)]$, see Ref. [2], Eq. (9). The expansion of the exponential factor has the form:

$$e^{-S[\Psi_\omega(\omega)/Q} = \exp \left\{ - S^{(0)}[\Psi_\omega^{(0)}(z)] \frac{Q}{\gamma^k} \right\} \left( 1 + \sum_{k=1}^{\infty} \frac{\gamma^k}{k!} \frac{\partial^k}{\partial \gamma^k} \exp \left\{ \sum_{p=1}^{\infty} \frac{S^{(p)}[\Psi_\omega(z)]}{Q} \right\} \right)_{\gamma=0} .$$

(12)

Therefore functions $\alpha_{0,k}[\Psi_\omega^{(0)}(z)]$ and $\alpha_{1,k}[\Psi_\omega^{(0)}(z)]$ have the following form:

$$\alpha_{0,k}[\Psi_\omega^{(0)}(z)] = \frac{(-1)^k}{k!} \left( \frac{LW}{2\pi P_{ave}} \frac{S^{(1)}[\Psi_\omega(z)]}{\gamma} \right)^k ,$$

(13)

$$\alpha_{1,k}[\Psi_\omega^{(0)}(z)] = \frac{(-1)^k}{(k-1)!(k+1)!} \left( \frac{WL}{2\pi P_{ave}} \right)^k \frac{\partial^{k+1}}{\partial \gamma^{k+1}} \left[ \frac{S^{(2)}[\Psi_\omega(z)]}{S^{(1)}[\Psi_\omega(z)]} \right] , \quad k \geq 1,$$

(14)

where

$$S^{(1)}[\Psi_\omega(z)] = 2 \int_0^L \int_{W'} \frac{d\omega}{2\pi} \text{Re} \left\{ \mathcal{L}^{(0)}[\Psi_\omega^{(0)}(z)] \mathcal{V}[\Psi_\omega^{(0)}(z)] \right\} = 2 \int_0^L \int_{W'} \frac{d\omega}{2\pi} \text{Re} \left\{ e^{i\beta \omega^2 z B(\omega) \mathcal{V}[\Psi_\omega^{(0)}(z)]} \right\} ,$$

(15)

$$S^{(2)}[\Psi_\omega(z)] = 2 \int_0^L \int_{W'} \frac{d\omega}{2\pi} \left[ \left| \mathcal{L}^{(0)}[\Psi_\omega^{(1)}(z)] - \mathcal{V}[\Psi_\omega^{(0)}(z)] \right|^2 - 2 \text{Re} \left\{ \mathcal{L}^{(0)}[\Psi_\omega^{(0)}(z)] \mathcal{V}[\Psi_\omega^{(0)}(z)] \right\} \right] .$$

(16)

where $V_1[\Psi, \psi]$ is defined below: see Eq. (15). Here in (15) and (16) we have used that

$$2 \int d\omega\text{Re} \left\{ \mathcal{L}^{(0)}[\Psi_\omega^{(0)}(z)] \mathcal{L}^{(0)}[\Psi_\omega^{(0)}(z)] \right\} = 0$$

by virtue of the boundary conditions: $\Psi_\omega^{(0)}(0) = \Psi_\omega^{(0)}(L) = 0$.

To calculate the path-integral in Eq. (11) we substitute the solution $\Psi_\omega(z)$ to the action. Here we are interested in only leading order in $\epsilon$ terms, therefore we keep only quadratic in $\tilde{\psi}_\omega(z)$ term in the difference $S[\Psi_\omega(z) + \tilde{\psi}_\omega(z)] - S[\Psi_\omega(z)]$ since terms with higher order in $\psi_\omega(z)$ are suppressed in the parameter $\epsilon$. In the leading order in $\epsilon$ the path-integral can be written in the form:

$$\int \mathcal{D} \tilde{\psi} e^{-S[\Psi_\omega(z) + \tilde{\psi}_\omega(z)] - S[\Psi_\omega(z)]} \approx$$

$$\int \mathcal{D} \tilde{\psi} \exp \left\{ - \frac{1}{Q} \int_0^L dz \int \frac{d\omega}{2\pi} \left| \mathcal{L}^{(0)}[\tilde{\psi}] \right|^2 \right\} \exp \left\{ - \Delta S[\Psi, \tilde{\psi}] \right\} ,$$

(17)
where

\[ \Delta S[\Psi, \tilde{\psi}] = \int_0^L dz \int \frac{d\omega}{2\pi} \left[ |V_1[\Psi, \tilde{\psi}] + V_2[\Psi, \tilde{\psi}] + V[\tilde{\psi}]|^2 - 2Re \left\{ \mathcal{L}^{(0)}[\tilde{\psi}] \tilde{V}_1[\Psi, \tilde{\psi}] + \left( \mathcal{L}[\Psi] + \mathcal{L}^{(0)}[\tilde{\psi}] \right) \left( \tilde{V}_2[\Psi, \tilde{\psi}] + \tilde{V}[\tilde{\psi}] \right) \right\} \right]. \]  

(18)

When deriving (18) we have used the equation of motion (19).

\[ V_1[\Psi, \tilde{\psi}] = i\gamma \int \frac{d\omega_1 d\omega_2 d\omega_3}{(2\pi)^2} \delta(\omega_1 + \omega_2 - \omega_3 - \omega) \left( 2\bar{\psi}_{\omega_1} (z) \Psi_{\omega_2}(z) \bar{\psi}_{\omega_3}(z) + \bar{\psi}_{\omega_3}(z) \Psi_{\omega_2}(z) \Psi_{\omega_1}(z) \right), \]  

(19)

\[ V_2[\Psi, \tilde{\psi}] = i\gamma \int \frac{d\omega_1 d\omega_2 d\omega_3}{(2\pi)^2} \delta(\omega_1 + \omega_2 - \omega_3 - \omega) \left( 2\bar{\psi}_{\omega_1} (z) \Psi_{\omega_2}(z) \bar{\psi}_{\omega_3}(z) + \bar{\psi}_{\omega_1}(z) \bar{\psi}_{\omega_3}(z) \bar{\psi}_{\omega_2}(z) \right). \]  

(20)

Below we interested in terms which do not suppressed by parameter \( \epsilon \) (in our case SNR \( \gg 1, \epsilon \ll 1 \)), therefore we must retain only second power in \( \Psi \) terms in \( \Delta S[\Psi, \tilde{\psi}] \). The functional \( \gamma_{p,0} \) has the form

\[ \gamma_{p,0}[\Psi^{(0)}(\omega)] = \frac{1}{\rho^{(0)}[0,0]} \int_{\tilde{\psi}^{(0)}(0)=0}^{\tilde{\psi}^{(L)}(0)=0} D\tilde{\psi} \exp \left\{ -\frac{1}{Q} \int_0^L dz \int \frac{d\omega}{2\pi} \left| \mathcal{L}^{(0)}[\tilde{\psi}] \right|^2 \right\} \frac{1}{p!} \left( \frac{\partial \rho}{\partial \tilde{\psi}} \right) \left( -\frac{\Delta \tilde{S}[\Psi, \tilde{\psi}]}{Q} \right) \right|_{\gamma=0}, \]  

(21)

where

\[ \Delta \tilde{S}[\Psi, \tilde{\psi}] = \int_0^L dz \int \frac{d\omega}{2\pi} \left[ |V_1[\Psi, \tilde{\psi}]|^2 - 2Re \left\{ \mathcal{L}^{(0)}[\tilde{\psi}] \tilde{V}_1[\Psi, \tilde{\psi}] + \mathcal{L}[\Psi] \tilde{V}_2[\Psi, \tilde{\psi}] \right\} \right]. \]  

(22)

\( P^{(0)}[Y|X] \) has the form, see Ref.[1],

\[ P^{(0)}[Y|X] = \Lambda^{(M')}_{QL} \exp \left[ -\frac{1}{QL} \int \frac{d\omega}{2\pi} |B(\omega)|^2 \right], \]  

(23)

\[ \Lambda^{(M')}_{D} = \left( \frac{\delta}{\pi D} \right)^{M'}. \]  

(24)

We need only \( \gamma_{1,0} \) and \( \gamma_{2,0} \) for our calculation. To calculate these functionals we use method developed in [1], direct calculation gives:

\[ \gamma_{1,0}[\Psi^{(0)}(\omega)] = \frac{2W'}{\pi LP_{ave}} \text{Im} \left\{ \int_0^L dz \frac{z(L-z)}{L} \int_{-W'/2}^{W'/2} \frac{d\omega}{2\pi} \mathcal{L}^{(0)} \left[ \Psi^{(0)}(\omega) \right] \tilde{\Psi}^{(0)}(\omega) \right\}, \]  

(25)

\[ \gamma_{2,0}[\Psi^{(0)}(\omega)] = \frac{2W'}{\pi (LP_{ave})^2} \text{Im} \left\{ \int_0^L dz \frac{z(L-z)}{L} \int_{-W'/2}^{W'/2} \frac{d\omega}{2\pi} \left[ \mathcal{L}^{(0)} \left[ \Psi^{(1)}(\omega) \right] \tilde{\Psi}^{(0)}(\omega) + \mathcal{L}^{(0)} \left[ \Psi^{(0)}(\omega) \right] \tilde{\Psi}^{(0)}(\omega) \right] \right\} - \right. \]

\[ \frac{W'}{(LP_{ave})^2} \int_0^L dz \frac{z(L-z)}{L} \int_{-W'/2}^{W'/2} \left[ \frac{5 \delta(\omega_1 + \omega_2 - \omega_3 - \omega_4) \Psi^{(0)}(\omega_1) \Psi^{(0)}(\omega_2) \Psi^{(0)}(\omega_3) \Psi^{(0)}(\omega_4) \tilde{\Psi}^{(0)}(\omega_1) \tilde{\Psi}^{(0)}(\omega_2) \tilde{\Psi}^{(0)}(\omega_3) \tilde{\Psi}^{(0)}(\omega_4)}{(2\pi)^4} \right] \left. \right\}. \]  

(26)
where $\Pi(\Omega) = (1 - \Omega)\theta(1 - \Omega)\theta(\Omega) + (1 + \Omega)\theta(1 + \Omega)\theta(-\Omega)$.

Using Eqs. 13-14 and 21 we can write conditional probability density function with accuracy $\tilde{\gamma}^2$ in the form:

$$P(Y(\omega)|X(\omega)) \approx P^{(0)}[Y(\omega)|X(\omega)\left(1 + \alpha^{(1)}[Y|X] + \alpha^{(2)}[Y|X]\right),$$

where we introduce following notations:

$$\alpha^{(1)}[Y|X] = \gamma_{\alpha,0} \left[\Psi^{(0)}(z)\right] \tilde{\gamma} + \alpha_{\alpha,1} \left[\Psi^{(0)}(z)\right] \frac{\tilde{\gamma}^2}{\epsilon},$$

$$\alpha^{(2)}[Y|X] = \alpha_{\alpha,2} \left[\Psi^{(0)}(z)\right] \frac{\tilde{\gamma}^2}{\epsilon^2} + \left\{\alpha_{\alpha,1} \left[\Psi^{(0)}(z)\right] + \alpha_{\alpha,1} \left[\Psi^{(0)}(z)\right] \gamma_{\alpha,0} \left[\Psi^{(0)}(z)\right]\right\} \frac{\tilde{\gamma}^2}{\epsilon^2}. (29)$$

Now we are ready to calculate the mutual information, see Eq. (2) of [2], with accuracy $\tilde{\gamma}^2$.

**CALCULATION OF THE MUTUAL INFORMATION**

Before calculating mutual information, let us present the auxiliary correlation functions:

$$\langle B(\omega)\overline{B}(\omega')\rangle_{P^{(0)}[Y|X]} = \int\mathcal{D}X P^{(0)}[Y|X]B(\omega)\overline{B}(\omega') = 2\pi QL\delta(\omega - \omega'),$$

$$\langle X(\omega)\overline{X}(\omega')\rangle_{P[X]} = \int\mathcal{D}XP[X]X(\omega)\overline{X}(\omega') = 2\pi P\delta(\omega - \omega')\chi_W(\omega)\chi_W(\omega'),$$

where we’ve used Eq. (28), and (12) of Ref. [2] as the integration weights, $\mathcal{D}X = \prod_{j=1}^{M'} dR_eX_j dI_mX_j$, $\chi_W(\omega) = \theta(\frac{W}{\epsilon} - \omega)\theta(\frac{W}{\epsilon} + \omega)$ stands for the indicator of the (cyclic) frequency domain $W$ in Eq. 31. Equations 30 and 31 can be easily obtained using discrete form of path-integrals. Since functions $P^{(0)}[Y|X] = P^{(0)}[B]$ and $P[X]$ have Gauss form, we can use the following properties (Wick theorem 4):

$$\langle B(\omega_1)\rangle_{P^{(0)}[B]} = 0,$$

$$\langle B(\omega_1)B(\omega_2)\overline{B}(\omega_3)\overline{B}(\omega_4)\rangle_{P^{(0)}[B]} = \langle B(\omega_1)\overline{B}(\omega_3)\rangle_{P^{(0)}[B]}\langle B(\omega_2)\overline{B}(\omega_4)\rangle_{P^{(0)}[B]} + \langle B(\omega_1)\overline{B}(\omega_4)\rangle_{P^{(0)}[B]}\langle B(\omega_2)\overline{B}(\omega_3)\rangle_{P^{(0)}[B]}.$$ (33)

These properties are the consequence of Gauss form of the corresponding path-integral. Averaging $\langle \ldots \rangle_{P[X]}$ has the same properties. Using these properties and Eqs. 30, 31, the calculation of the mutual information with accuracy $\tilde{\gamma}^2$ transforms to the simple calculation of the correlation functions. Note, that now $P_{\text{ase}} = P_{W}/2\pi \gg P_{\text{noise}}$, non-linearity parameter $\tilde{\gamma} = \gamma LPW/(2\pi)$, and SNR = $1/\epsilon = \frac{1}{\sqrt{P}}$.

To calculate the corrections connected with channel non-linearity we substitute the expression 27 to the Eq. (2) of Ref. [2] and obtain the following expansion with accuracy $\tilde{\gamma}^2$ for mutual information:

$$I_{P[X]} \approx \int\mathcal{D}X\mathcal{D}Y P[X]P^{(0)}[Y|X] \left\{ \ln \left[ \frac{P^{(0)}[Y|X]}{P_{\text{out}}^{(0)}[Y]} \right] \left(1 + \alpha^{(1)}[Y|X] + \alpha^{(2)}[Y|X]\right) + \frac{(\alpha^{(1)}[Y|X])^2 - (\beta^{(1)}[Y])^2}{2}\right\}.$$ (34)

where

$$P_{\text{out}}^{(0)}[Y] = \int\mathcal{D}X P[X]P^{(0)}[Y|X],$$

$$\beta^{(1)}[Y(\omega)] = \int\mathcal{D}X P[X]P^{(0)}[Y|X]\alpha^{(1)}[Y|X].$$ (36)
The direct calculation of $P_{\text{out}}^{(0)}[Y]$ and $\beta^{(1)}[Y(\omega)]$ gives

\begin{equation}
\begin{aligned}
P_{\text{out}}^{(0)}[Y] = \Lambda_{QL}^{(M')-M} \exp \left\{ -\frac{1}{QL} \int \frac{d\omega}{2\pi} |Y(\omega)|^2 \right\} \Lambda_{P+QL}^{(M)} \exp \left\{ -\frac{1}{P+QL} \int \frac{d\omega}{2\pi} |Y(\omega)|^2 \right\}.
\end{aligned}
\end{equation}

(37)

\begin{equation}
\beta^{(1)}[Y] = \frac{-2\gamma}{P+QL} \Im \int_0^L \! dz \int \frac{d\omega d\omega_1 d\omega_2}{(2\pi)^3} e^{-\mu z/L} \left( 1 - \frac{z}{L} \frac{QL}{P+QL} \right)^3 \! Y(\omega_1) Y(\omega_2) \overline{Y}(\omega_3) \overline{Y}(\omega).
\end{equation}

(38)

We present the expression (34) in the form:

\begin{equation}
I_P[X] = I_0 + I_1 + I_2 + I_3
\end{equation}

(39)

where

\begin{equation}
I_0 = M \ln \left[ 1 + \frac{P}{QL} \right] \int DXDY P[X] P^{(0)}[Y|X] \left\{ 1 + \alpha^{(1)}[Y|X] + \alpha^{(2)}[Y|X] \right\},
\end{equation}

(40)

\begin{equation}
I_1 = \int DXDY P[X] P^{(0)}[Y|X] \! \int \frac{d\omega}{2\pi} \left( \frac{|B(\omega) - X(\omega)e^{i\beta \omega L}|^2}{P + QL} - \frac{|B(\omega)|^2}{QL} \right) \times \left\{ 1 + \alpha^{(1)}[Y|X] + \alpha^{(2)}[Y|X] \right\}.
\end{equation}

(41)

\begin{equation}
I_2 = \frac{1}{2} \int DXDY P[X] P^{(0)}[Y|X] \left( \alpha^{(1)}[Y|X] \right)^2,
\end{equation}

(42)

\begin{equation}
I_3 = -\frac{1}{2} \int DY P_{\text{out}}^{(0)}[Y] \left( \beta^{(1)}[Y] \right)^2.
\end{equation}

(43)

Here the terms (40) and (41) come from the term which is proportional to $\ln \left[ P^{(0)}[Y|X]/P_{\text{out}}^{(0)}[Y] \right]$.

The direct calculation of $I_0$ shows that path-integral

\begin{equation}
\int DXDY P[X] P^{(0)}[Y|X] \left\{ \alpha^{(1)}[Y|X] + \alpha^{(2)}[Y|X] \right\} = 0.
\end{equation}

(44)

It is the consequence of the normalization condition of the conditional probability $P[Y|X]$: $\int DY P[Y|X] = 1$, see Ref.\[1\] for details. Therefore

\begin{equation}
I_0 = C_{SH} = M \ln \left[ 1 + \frac{P}{QL} \right],
\end{equation}

(45)

coinciding with the classical result for the maximum linear channel capacity $C_{SH}$ (see Shannon-Hartley theorem [3]).

The next contribution to $I_P[X]$ is $I_1$. It is not so obvious but this correction is of $\theta^2$ order and does not contain $\epsilon$ in denominator. We can see this when changing the integration from $Y(\omega)$ to $B(\omega)$. Note that small (in $P$ units) factor $QL$ comes from pairing (37). So the main not $Q$-suppressed term originates only from the inner pairing of the second term in the middle of (42):

\begin{equation}
\frac{1}{QL} \int \frac{d\omega}{2\pi} (|B(\omega)|^2) P^{(0)}[Y|X] = M'.
\end{equation}

(42)

For this case $\alpha^{(1,2)}[Y|X]$ contributions from the last line of (42) vanish owing to the normalization condition of $P[X|Y]$.

The direct calculation of $I_2$ shows that

\begin{equation}
I_2 = 4 MG(\tilde{\gamma}) \frac{\gamma^2}{\epsilon} + O(\gamma^2),
\end{equation}

(46)
where the function $G(\tilde{\beta})$ of $\tilde{\beta} = \beta LW^2$ is defined as

$$G(\tilde{\beta}) = 1 + \frac{1}{2\tilde{\beta}^2} \int_\Omega dy_1 dy_2 \sin^2[\tilde{\beta}(y - y_1)(y - y_2)]$$

with $\Omega = [-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}]$ being a simple cubic region. For the domain of $\tilde{\beta} \geq 0$ variation the function $G(\tilde{\beta})$ is a bounded ($1 < G(\tilde{\beta}) \leq \frac{3}{2}$) and steadily decreasing function with the asymptotical behaviour

$$G(\tilde{\beta}) \sim \begin{cases} \frac{3}{2} - \frac{\tilde{\beta}^2}{180} + \frac{\tilde{\beta}^4}{2341750} + O(\tilde{\beta}^0), & \tilde{\beta} \to 0 \\ 1 + \frac{2}{\tilde{\beta}}(\ln[2\tilde{\beta}] + \gamma_E - 3) + O(\ln[\tilde{\beta}]^{\frac{1}{2}}), & \tilde{\beta} \to \infty, \end{cases}$$

that is good (0.1 percentage accuracy) approximation up to $\tilde{\beta} = 4$ in the first and $\tilde{\beta} \geq 300$ in the second case. Note that our result (46) does not depend on $W'$ but only $W$.

The last component of $I_{P[|X|]}$ is of $\tilde{\gamma}^2$ order as well:

$$I_3 = -\frac{1}{2} \int DYP_{\text{out}}[|Y|](\beta^{(1)}|Y|)^2.$$

Indeed one does obtain $Q$-suppressed expression of $\tilde{\gamma}^2$ order from the expression (38) for $\beta^{(1)}|Y|$ and correlator (30).

Finally we have the following result for the spectral efficiency (SE) for the nonlinear channel:

$$SE = \frac{I_{P[|X|]}}{M \ln 2} \approx \log_2 \left[1 + \frac{1}{\epsilon} \right] + 4 \tilde{\gamma}^2 G(\tilde{\beta}) $$

$$= \log_2 \left[1 + \text{SNR} \right] + \frac{4G(\tilde{\beta})}{\ln 2} \left(\frac{\gamma QL^2 W}{2\pi} \right)^2 (\text{SNR})^3. \quad (49)$$

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[1] I. S. Terekhov, S. S. Vergeles, and S. K. Turitsyn, arXiv:1411.6792
[2] I. S. Terekhov, A. V. Reznichenko, and S. K. Turitsyn
[3] C. Shannon, A mathematical theory of communication, Bell System Techn. J., 27 (1948), 3, 379–423; 27 (1948), 4, 623–656.
[4] C. Itzykson, J.B. Zuber, Quantum Field Theory, McGraw-Hill (1980).