STANLEY–REISNER RINGS FOR QUASI-ARITHMETIC MATROIDS

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Abstract. In this note we define a Stanley–Reisner ring for quasi-arithmetic matroids and more general structures. To this end, we define two types of CW complexes associated with a quasi-arithmetic matroid that generalize independence complexes of matroids. Then we use Stanley’s construction of Stanley–Reisner rings for simplicial posets.

1. Introduction

A (quasi-)arithmetic matroid $A$ is a triple $(E, \Delta, m)$, where $(E, \Delta)$ is a matroid on the ground set $E$ with independence complex $\Delta \subseteq 2^E$ and $m : 2^E \to \mathbb{Z}_{\geq 1}$ is the so-called multiplicity function, that must satisfy certain axioms [8]. In the representable case, i.e. when the arithmetic matroid is determined by a list of integer vectors, this multiplicity function records data such as the absolute value of the determinant of a basis.

A toric arrangement is an arrangement of subtori of codimension one in a real or complex torus (e.g. [11, 16, 18, 23, 30, 38]). While matroids capture a lot of combinatorial and topological information about hyperplane arrangements [40, 48], arithmetic matroids carry similar information about toric arrangements [11, 30, 38]. Toric arrangements are particularly important due to their connection with the problem of counting lattice points in polytopes. This was implicitly discovered in the 1980s by researchers working on splines [17] such as Dahmen and Micchelli. It was recently made more explicit and put in a broader context by De Concini, Procesi, Vergne, and others [20].

Various simplicial complexes can be associated with a matroid in a natural way: the matroid complex (or independence complex), the broken circuit complex, and the order complex of the lattice of flats. As topological spaces, they are interesting in their own right and they can be used to derive interesting information about the underlying matroid (see e.g. [7]). The broken circuit complex has been generalized to arithmetic matroids in the representable case. This construction plays an important role in several papers on the topology of toric arrangements [11, 16, 18].

The first goal of this paper is to define an independence complex for an arithmetic matroid. The $i$th entry of the $f$-vector of this complex should be equal to the weighted number of independent sets of cardinality $i$ in the underlying matroid, where each independent set is weighted by its multiplicity. It is easy to see that this cannot be achieved by a simplicial complex, so we will we construct a CW complex.
instead. In fact, we will provide two constructions: one works for arbitrary quasi-arithmetic matroids (and even more general structures), the other one uses the so-called layer groups that can be associated to a representation of an arithmetic matroid \cite{22, 34}. The \(h\)-vectors of arithmetic matroids (which are by construction also the \(h\)-vectors of the arithmetic independence complexes) and related algebraic structures arising in the theory of vector partition functions have already been studied \cite{15, 33}.

The second goal of this paper is to construct a Stanley–Reisner ring for arithmetic matroids. The construction is straightforward: we are able to prove that our arithmetic independence complexes are simplicial posets. Then we employ a construction of Stanley that associates a Stanley–Reisner ring to any simplicial poset \cite{46}. In fact, our construction not only works for the two types of arithmetic independence complexes that we define, but for a more general structure that we call independence complex defined by a surjective finite abelian group structure on a simplicial complex. Just like the Stanley–Reisner ring of a matroid encodes the \(h\)-vector of the matroid complex, the arithmetic Stanley–Reisner ring encodes the \(h\)-vector of the arithmetic independence complex.

A large number of algebraic structures that can be associated with hyperplane arrangements and matroids have turned out to be very interesting and useful (e. g. \cite{3, 5, 9, 24, 30, 11, 51, 52}). There is a strong interest in inequalities for \(f\)-vectors and \(h\)-vector of matroid and broken circuit complexes (e. g. \cite{12, 27, 29, 49}) and algebraic tools have been crucial for some of the proofs. This includes log-concavity of \(f\)-vectors and \(h\)-vectors of matroid complexes and broken circuit complexes \cite{1, 2, 28, 32}. Using the fact that the Stanley–Reisner ring of a matroid complex is level, one can deduce certain inequalities for their \(h\)-vectors (e. g. \cite{47}, p. 93 and Proposition 3.3(a)). Outside of matroid theory, Stanley–Reisner rings also play an important role. For example, the equivariant cohomology ring of a complete and simplicial toric variety is isomorphic to the Stanley–Reisner ring of the corresponding fan \cite[12.4.14]{13}. The hard Lefschetz property of certain Stanley–Reisner rings allowed Stanley to prove the \(g\)-theorem for rational simplicial polytopes \cite{45}. This result was conjectured by McMullen in 1971 \cite{37} and its many generalisations have received a lot of interest \cite{50}.

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2. Background

In this section we will introduce the mathematical background, i. e. Stanley–Reisner rings of simplicial complexes and simplicial posets, matroids, and arithmetic matroids.

Throughout this paper, \(K\) denotes some fixed field.

2.1. Stanley–Reisner rings of simplicial complexes. An (abstract) simplicial complex \(\Delta\) is a collection of subsets of a finite set \(E\). We denote by \(f_i\) the number of elements of \(\Delta\) of cardinality \(i\). If \(\Delta\) is non-empty, there is a maximal integer \(r\) s. t. \(f_r \neq 0\). Then we say that \(\Delta\) has rank \(r\). The \(f\)-polynomial \(f_\Delta(t)\) and the \(h\)-polynomial \(h_\Delta(t)\) of \(\Delta\) are defined as

\[
\begin{align*}
f_\Delta(t) &:= \sum_{i=0}^{r} f_i t^{r-i}, \\
h_\Delta(t) &:= f_\Delta(t-1) = \sum_{i=0}^{r} h_i t^{r-i}.
\end{align*}
\]

(1) (2)
The vectors \((f_0, \ldots, f_r), (h_0, \ldots, h_r) \in \mathbb{Z}^{r+1}\) are called the \(f\)-vector and the \(h\)-vector of \(\Delta\), respectively. The \(f\)-vector is usually defined in a slightly different way in the topology literature (the index is shifted by 1), but our notation has some advantages and it is used in some articles on matroid theory (e.g. [7]).

Let \(\Delta\) be a simplicial complex on the ground set \(\{1, \ldots, N\}\). The Stanley–Reisner ideal \(I_\Delta\) and the Stanley–Reisner ring (or face ring) \(\mathbb{K}[\Delta]\) are defined as

\[
I_\Delta := \left\{ (s_i, \ldots, s_r : \{i_1, \ldots, i_r \} \not\subseteq \Delta) \right\} \subseteq \mathbb{K}[s_1, \ldots, s_N]
\]

and

\[
\mathbb{K}[\Delta] := \mathbb{K}[s_1, \ldots, s_N]/I_\Delta.
\]

\(\mathbb{K}[\Delta]\) is a graded ring where each variable has degree 1. It is known that its Hilbert series is

\[
\text{Hilb}(\mathbb{K}[\Delta], t) = \frac{h_0 + h_1 t + \ldots + h_r t^r}{(1 - t)^r}. \tag{5}
\]

For more information on these topics, see [17].

2.2. Stanley–Reisner rings of simplicial posets. Closely following Stanley’s paper [46], we will now introduce simplicial posets and their Stanley–Reisner rings. A simplicial poset is a poset \(P\) with a unique minimal element \(\emptyset\) s.t. every interval \([0, y]\) is a boolean algebra [6] [20] [16]. All posets considered in this paper are assumed to be finite. If a simplicial poset \(P\) is in addition a meet-semilattice, then \(P\) is just the face lattice (ordered by inclusion) of a simplicial complex. Simplicial posets are special cases of CW posets, as defined in [6]. This implies that a simplicial poset \(P\) is the face poset of a regular CW complex \(\Gamma\). We may informally regard \(\Gamma\) as a generalized simplicial complex whose faces are still simplices, but we allow two faces to intersect in any subcomplex of their boundaries, rather than just in a single face. For a simplicial poset \(P\) we can define a grading \(\rho\) as follows: for \(y \in P\), \(\rho(y)\) is defined as the rank of the boolean algebra \([0, y]\). Let \(f_i\) denote the number of \(y \in P\) s.t. \(\rho(y) = i\). Then we can define \(f\) and \(h\)-vector/polynomial as above.

**Definition 1** ([46] Definition 3.3). Let \(P\) be a simplicial poset with elements \(\emptyset = y_0, y_1, \ldots, y_p\). We define the Stanley–Reisner ideal of \(P\), \(I_P\) to be the ideal in the polynomial ring \(\mathbb{K}[y_0, y_1, \ldots, y_p]\) generated by the following elements:

\[(S_1)\quad y_i y_j \text{ if } y_i \text{ and } y_j \text{ have no common upper bound in } P, \text{ and}
\]

\[(S_2)\quad y_i y_j - (y_i \wedge y_j)(\sum z), \text{ where } z \text{ ranges over all minimal upper bounds of } y_i \text{ and } y_j, \text{ otherwise;}
\]

\[(S_3)\quad (y_0 - 1).
\]

The Stanley–Reisner ring of \(P\) is defined as \(\mathbb{K}[P] = \mathbb{K}[y_0, y_1, \ldots, y_p]/I_P\).

It is clear that \(y_i \wedge y_j\) exists whenever \(y_i\) and \(y_j\) have an upper bound \(z\), since the interval \([0, z]\) is a boolean algebra and therefore a lattice. It is easy to see that if \(P\) is the face lattice of a simplicial complex \(\Delta\), then \(\mathbb{K}[P] \cong \mathbb{K}[\Delta]\).

Now using the same notation as above, we define a quasi-grading on the ring \(\mathbb{K}[y_0, y_1, \ldots, y_p]\) by setting \(\deg(y_i) := \rho(y_i)\). We do not get an actual graded algebra as we have defined it because \(\deg(y_0) = 0\), so \(\dim_{\mathbb{K}}(\mathbb{K}[y_0, y_1, \ldots, y_p]) = 2\) instead of 1. The relations \((S_1), (S_2)\) and \((S_3)\) are homogeneous with respect to this grading, so \(\mathbb{K}[P]\) inherits a grading from \(\mathbb{K}[y_0, y_1, \ldots, y_p]\). Moreover, since \(\deg(y_0 - 1) = 0\) it follows that \(\dim_{\mathbb{K}}((\mathbb{K}[P])_0) = 1\), so \(\mathbb{K}[P]\) is a genuine graded algebra.

**Proposition 2** ([46] Proposition 3.8]). Let \(P\) be a finite simplicial poset of rank \(r\) with \(h\)-vector \((h_0, h_1, \ldots, h_r)\). With the grading of \(\mathbb{K}[P]\) just defined, we have

\[
\text{Hilb}(\mathbb{K}[P], t) = \frac{h_0 + h_1 t + \ldots + h_r t^r}{(1 - t)^r}. \tag{6}
\]
2.3. Matroids. A matroid complex (or independence complex) is a simplicial complex \( \Delta \) on a finite ground set \( E \) s.t. \( \emptyset \in \Delta \) and if \( A_1, A_2 \in \Delta \) and \( |A_2| > |A_1| \), then there is \( e \in A_2 \setminus A_1 \) s.t. \( A_1 \cup \{e\} \in \Delta \). The pair \((E, \Delta)\) is called a matroid. Matroids can also be defined in several other equivalent ways [42]. The rank function of a matroid \( M = (E, \Delta) \) is the function \( \text{rk} : 2^E \to \mathbb{Z}_{\geq 0} \) given by \( \text{rk}(A) = \max\{S \in \Delta : S \subseteq A\} \). The integer \( r := \text{rk}(E) \) is called the rank of \( M \).

Given a matrix \( X \in \mathbb{R}^{r \times n} \), one obtains the matroid \( M_X = (\{N\}, \Delta_X) \), where \( \Delta_X \) denotes the set of all \( \{S \subseteq [N]\} \) which index a linearly independent set of columns of \( X \). The matrix \( X \) is called a representation of the matroid complex \( \Delta_X \).

The Tutte polynomial \([10]\) of a matroid \( M = (E, \Delta) \) is defined as
\[
\Xi_M(x, y) := \sum_{A \subseteq E} (x-1)^{\text{rk}(A)}(y-1)^{|A| - \text{rk}(A)} \in \mathbb{Z}[x, y].
\]

It is easy to see that \( f_{\Delta}(t) = \Xi_M(t+1, 1) \), hence \( h_{\Delta}(t) = \Xi_M(t, 1) \). Using \([5]\), this implies that the Hilbert series of the Stanley–Reisner ring of a matroid complex is encoded by the Tutte polynomial:
\[
\text{Hilb}(K[\Delta], t) = \frac{t^r \Xi_M(\frac{1}{t}, 1)}{(1-t)^r}.
\]

Matroid complexes have been studied intensively and they have many nice properties \([7, 17]\). In particular, they are shellable \([44]\). This implies that their \( h \)-vectors are non-negative.

2.4. Arithmetic matroids. Arithmetic matroids capture many combinatorial and topological properties of toric arrangements in a similar way as matroids carry information about the corresponding hyperplane arrangements.

Definition 3 \([8, 14]\). An arithmetic matroid is a triple \((E, \Delta, m)\), where \((E, \Delta)\) is a matroid and \( m : 2^E \to \mathbb{Z}_{\geq 1} \) denotes the multiplicity function that satisfies the following axioms:

\begin{enumerate}[(A)]
  \item Let \( R, S \subseteq E \). The set \([R, S] := \{A : R \subseteq A \subseteq S\}\) is called a molecule if \( S \) can be written as the disjoint union \( S = R \cup F_{RS} \cup T_{RS} \) and for each \( A \in [R, S], \text{rk}(A) = \text{rk}(R) + |A \cap F_{RS}| \) holds.

  For each molecule \([R, S]\), the following inequality holds:
\[
\rho(R, S) := \left( (-1)^{|T_{RS}|} \sum_{A \in [R, S]} (-1)^{|S| - |A|} m(A) \right) \geq 0.
\]

  \item For all \( A \subseteq E \) and \( e \in E \): if \( \text{rk}(A \cup \{e\}) = \text{rk}(A) \), then \( m(A \cup \{e\}) | m(A) \).

  Otherwise, \( m(A) | m(A \cup \{e\}) \).

  \item If \([R, S]\) is a molecule and \( S = R \cup F_{RS} \cup T_{RS} \), then
\[
m(R)m(S) = m(R \cup F_{RS})m(R \cup T_{RS}).
\]
\end{enumerate}

A quasi-arithmetic matroid is a triple \((E, \Delta, m)\), where \((E, \Delta)\) is a matroid and the multiplicity function \( m : 2^E \to \mathbb{Z}_{\geq 1} \) satisfies the axioms (A1) and (A2).

The prototypical example of an arithmetic matroid is defined by a list of vectors \( X = (x_1, \ldots, x_N) \subseteq \mathbb{Z}^r \), or equivalently, by a matrix \( X \in \mathbb{Z}^{r \times N} \). In this case, for a subset \( S \subseteq E = [N] := \{1, \ldots, N\} \) of cardinality \( r \) that defines a basis, we have \( m(S) = |\det(S)| \) and in general \( m(S) := |\langle S \rangle_r \cap \mathbb{Z}^r| / |\langle S \rangle| \). Here, \( \langle S \rangle \subseteq \mathbb{Z}^r \) denotes the subgroup generated by \( \{x_e : e \in S\} \) and \( \langle S \rangle_r \subseteq \mathbb{R}^r \) denotes the subspace spanned by the same set.

In fact, one can represent a slightly more general class of arithmetic matroids by a list of elements in a finitely generated abelian group. By the fundamental theorem of finitely generated abelian groups, every finitely generated abelian group \( G \) is isomorphic to \( \mathbb{Z}^d \oplus \mathbb{Z}_{q_1} \oplus \cdots \oplus \mathbb{Z}_{q_n} \) for suitable non-negative integers \( d, n, q_1, \ldots, q_n \).
There is no canonical isomorphism $G \cong \mathbb{Z}^r \oplus \mathbb{Z}_{q_1} \oplus \ldots \oplus \mathbb{Z}_{q_n}$. However, $G$ has a uniquely determined torsion subgroup $G_t \cong \mathbb{Z}_{q_1} \oplus \ldots \oplus \mathbb{Z}_{q_n}$ consisting of all the torsion elements. There is a free group $G := G/G_t \cong \mathbb{Z}^r$. For $X \subseteq G$, we will write $\bar{X}$ to denote the image of $X$ in $G$.

**Definition 4.** Let $A = (E, \Delta, m)$ be an arithmetic matroid. Let $G$ be a finitely generated abelian group and let $X = (x_v)_{v \in E}$ be a list of elements of $G$. For $A \subseteq E$, let $G_A$ denote the maximal subgroup of $G$ s.t. $|G_A/\langle A \rangle|$ is finite. Again, $\langle A \rangle \subseteq G$ is called a representation of $A$ if the matrix $\bar{X} \subseteq \bar{G}$ represents the matroid $(E, \Delta)$ and $m_X(A) = m(A) = |G_A/\langle A \rangle|$ for all $A \subseteq E$. The arithmetic matroid $A$ is called representable if it has a representation $X$.

Note that if $A$ is an independent set, then $m_X(A) = m_X(\bar{X})/|G| \cdot |G_A/\langle A \rangle|$ (cf. [35, Lemma 8.3]). An arithmetic matroid $A = (E, \text{rk}, m)$ is called torsion-free if $m(\emptyset) = 1$. If such an arithmetic matroid is representable, then it can be represented by a list of vectors in lattice, i.e., a finitely generated abelian group which is torsion-free.

The arithmetic Tutte polynomial [11, 35] of an arithmetic matroid $A = (E, \Delta, m)$ is defined as

$$\mathfrak{M}_A(x, y) := \sum_{A \subseteq E} m(A)(x - 1)^{\text{rk}(A)}/(y - 1)^{|A| - \text{rk}(A)}. \quad (11)$$

If an arithmetic matroid $A$ is represented by a list $X \subseteq \mathbb{Z}^r$ that is totally unimodular, then the multiplicity function is constant and equal to 1. Hence the arithmetic Tutte polynomial and the Tutte polynomial are equal in this case.

### 3. Arithmetic independence complexes

In this section we will define several simplicial posets that can be used to define the arithmetic Stanley–Reisner ring using Definition 1. In Subsection 3.1 we will give a very general definition that includes the other two as special cases. In Subsection 3.2 we will construct an arithmetic independence complex, given a representation of an arithmetic matroid. In Subsection 3.3 we will describe another construction that works for quasi-arithmetic matroids and even more general structures. In general, in the representable case, these two constructions yield different complexes.

#### 3.1. Simplicial posets defined by a simplicial complex with a surjective finite abelian group structure

Let $\Delta$ be a simplicial complex on the ground set $E$. Let $\text{FinAb}$ denote the category of finite abelian groups. We say that $\Delta$ has a surjective finite abelian group structure $\mathcal{G}$ if there is a functor $\mathcal{G} : \Delta \to \text{FinAb}$ s.t. the images of all homomorphisms are surjective maps. Here, $\Delta$ is considered as a category in the usual way of turning a poset such as $(\Delta, \subseteq)$ into a category: the objects are the elements of $\Delta$ and for $\sigma_1, \sigma_2 \in \Delta$,

$$\text{hom}(\sigma_1, \sigma_2) = \begin{cases} \{(\sigma_1, \sigma_2)\} & \text{if } \sigma_1 \subseteq \sigma_2, \\ \emptyset & \text{otherwise.} \end{cases} \quad (12)$$

Composition of homomorphisms is defined via $(\sigma_1, \sigma_2) \circ (\sigma_2, \sigma_3) = (\sigma_1, \sigma_3)$.

The definition can be rephrased in the following way without using terminology from category theory. We say that $\Delta$ has a surjective finite abelian group structure $\mathcal{G}$ if there is a map $\mathcal{G} : \Delta \to \text{FinAb}$ s.t. the following conditions are satisfied:

(i) for each $S \in \Delta$ and $a \in E \setminus S$ for which $S \cup \{a\} \in \Delta$, there is a map $\pi^{\mathcal{G}}_{S, a} : \mathcal{G}(S \cup \{a\}) \to \mathcal{G}(S)$.
(ii) for all \( S \in \Delta, a, b \in E \setminus S, a \neq b \) for which \( S \cup \{a, b\} \in \Delta \), the following diagram commutes:

\[
\begin{align*}
\mathcal{G}(S \cup \{a, b\}) & \xrightarrow{\pi_{G(a \cup b), a}} \mathcal{G}(S \cup \{a\}) \\
\pi_{G(a \cup b), a} & \xrightarrow{\pi_{G(a), a}} \mathcal{G}(S).
\end{align*}
\]

We call \( \mathcal{G} \) torsion-free if \(|\mathcal{G}(\emptyset)| = 1\).

**Definition 5.** Let \( \Delta \) be a simplicial complex on the ground set \( E \) with a surjective finite abelian group structure \( \mathcal{G} \). We define the independence poset \( \text{Ind}(E, \Delta, \mathcal{G}) \) as follows: the ground set is

\[
\{(S, g) : S \in \Delta, g \in \mathcal{G}(S)\}
\]

and for \( a \not\in S \) for which \( S \cup \{a\} \in \Delta \), (\( S \cup \{a\}, g \)) covers \((S, h)\) if and only if \( \pi_{G, a}(g) = h \).

\( \text{Ind}(E, \Delta, \mathcal{G}) \) is indeed a poset. If one defines a poset via cover relations, the only thing that needs to be checked is that the cover relations diagram does not contain a cycle, but this is clear from the definition.

For \( S = \{s_1, \ldots, s_k\} \in \Delta \), it will be useful to define the map \( \pi_{S} : \mathcal{G}(S) \to \mathcal{G}(\emptyset) \) by \( \pi_{S}(g) := \pi_{\{s_1, \ldots, s_k\}, s_1}(\ldots(\pi_{\{s_1\}, s_1}(\pi_{\emptyset, s_1}(g))) \ldots) \). Since the diagrams in (13) commute, the order of the elements of \( S \) does not matter.

**Lemma 6.** If \( \mathcal{G} \) is torsion-free, then \( \text{Ind}(E, \Delta, \mathcal{G}) \) is a simplicial poset with minimal element \((\emptyset, 0)\). In general, \( \text{Ind}(E, \Delta, \mathcal{G}) \) is a disjoint union\(^1\) of simplicial posets with minimal elements \( \{(\emptyset, g) : g \in \mathcal{G}(\emptyset)\} \).

Furthermore, the \( f \)-vectors of the connected components of \( \text{Ind}(E, \Delta, \mathcal{G}) \) are pairwise equal.

**Proof.** Let \( S \in \Delta \) and \( h \in \mathcal{G}(S) \). By definition, there is a unique \( g = \pi_{S}(h) \in \mathcal{G}(\emptyset) \) s. t. \((\emptyset, g) \leq (S, h)\). Hence each connected component of the Hasse diagram contains at least one element of type \((\emptyset, g)\) for \( g \in \mathcal{G}(\emptyset) \). Now let \( g_1, g_2 \in \mathcal{G}(\emptyset) \) s. t. \((\emptyset, g_1)\) and \((\emptyset, g_2)\) are in the same connected component. This means that there is a path \( p_0 = (\emptyset, g_1), p_1, \ldots, p_l = (\emptyset, g_2) \) in the Hasse diagram of the poset (considered as an undirected graph) that connects the two elements, i. e. for \( i = 1, \ldots, l \), \( p_i \in \text{Ind}(E, \Delta, \mathcal{G}) \) and either \( p_i \) covers \( p_{i-1} \) or \( p_{i-1} \) covers \( p_i \). Since one of two adjacent elements in the path covers the other one, they must lie above the same minimal element. Hence by induction, all elements in the path must lie above the same minimal element. Thus \( g_1 = g_2 \) and we can deduce that the Hasse diagram has \( \mathcal{G}(\emptyset) \) connected components and each one contains exactly one element of type \((\emptyset, g)\).

Let \( g, h, S \) be as above. We still need to show that \([[(\emptyset, g), (S, h)]\) is a boolean algebra. But this is easy: \([\emptyset, S]\) is the standard example of a boolean algebra. Adding group elements to each set (a suitable image of \( h \)) does not change this poset (for this, the fact that the diagrams commute is used).

Now let \( g_1, g_2 \in \mathcal{G}(\emptyset) \) and let \( S \in \Delta \). To prove the last statement, it is sufficient to show that \( |\pi_{S}(g_1)| \leq |\pi_{S}(g_2)| \). But this is clear since \( \pi_{S} \) is a group homomorphism.

Note that the proof of the lemma relies on the fact that the maps are fixed in advance and everything commutes (‘local’ commutativity as in the case of (non-representable) matroids over a ring \([23]\) is not sufficient).

\(^1\)This means that the Hasse diagrams are disjoint unions in the sense of graph theory.
Let $G$ be a surjective finite abelian group structure on the simplicial complex $\Delta$. We will now construct a torsion-free surjective finite abelian group structure $\tilde{G}$ on the same simplicial complex. Let $S \subseteq E$. We define $G(S)$ as follows:

$$G(S) := (\pi_S^G)^{-1}(0) = \{ h \in G(S) : \pi_S^G(h) = 0 \}. \quad (15)$$

We define the maps $\pi_{S,a}^G : (G(S \cup \{a\}) \to G(S)$ by restricting $\pi_S^G$. By construction, $\pi_{S,a}^G$ is surjective, the image is contained in $G(S)$ and the maps commute as in (13). Hence $\tilde{G}$ is also a surjective finite abelian group structure on $\Delta$.

**Theorem 7.** Let $P = \text{Ind}(E, \Delta, G)$ be an independence poset defined by a surjective finite abelian group structure on a simplicial complex of rank $r$.

Let $\tilde{G}$ be a surjective finite abelian group structure on the simplicial complex $\Delta$. Then the Stanley–Reisner ring $\mathbb{K}[P]$ satisfies

$$\text{Hilb}(\mathbb{K}[P], t) = \frac{h_P(t)}{(1-t)^r}. \quad (16)$$

Now suppose that $G$ has torsion and let $\tilde{P} = \text{Ind}(E, \Delta, \tilde{G})$ denote the independence poset of the corresponding torsion-free surjective finite abelian group structure. Then the Stanley–Reisner ring $\mathbb{K}[\tilde{P}]$ satisfies

$$\text{Hilb}(\mathbb{K}[\tilde{P}], t) = \frac{1}{|G(\emptyset)|} \frac{h_P(t)}{(1-t)^r}. \quad (17)$$

**Proof.** This follows directly from Proposition 2 and Lemma 3. $\square$

Below, we will define two types of independence complexes for (quasi-)arithmetic matroids, which will allow us to define the arithmetic Stanley–Reisner ring whose Hilbert series encodes the arithmetic $h$-vector.

**Theorem 8.** Let $P$ be a poset for which one of the following two statements hold:

- let $X \in \mathbb{Z}^{r \times N}$ be a matrix that represents an arithmetic matroid $A = (E_X, \Delta_X, m_X)$ of rank $r$ and $P = \text{Ind}^{\text{rep}}(E_X, \Delta_X, G_X)$ is the independence poset derived from $X$ (cf. Subsection 3.2).
- let $A = (E, \Delta, m)$ be a (weak quasi-)arithmetic matroid of rank $r$ and $P = \text{Ind}^{\text{cyc}}(E, \Delta, G)$ is its cyclic independence poset (cf. Subsection 3.3).

In both cases, the following statements hold for the arithmetic Stanley–Reisner ring $\mathbb{K}[P]$:

suppose $A$ is torsion-free, i.e. $m(\emptyset) = 1$. Then

$$\text{Hilb}(\mathbb{K}[P], t) = \frac{\mathfrak{M}_A(t, 1)}{(1-t)^r}. \quad (18)$$

In general, we have

$$\text{Hilb}(\mathbb{K}[\tilde{P}], t) = \frac{1}{m(\emptyset)} \frac{\mathfrak{M}_A(t, 1)}{(1-t)^r}, \quad (19)$$

where $\tilde{P}$ denotes the independence poset of the torsion-free structure associated with $A$.

**Proof.** This follows directly from Theorem 7 and Lemma 10 and Lemma 14 below. $\square$

### 3.2. Arithmetic independence complexes derived from a representation.

Let $X \in \mathbb{Z}^{r \times N}$ be a matrix. For $S \subseteq [N]$, let $X[S]$ denote the submatrix of $X$ whose columns are indexed by $S$. Following [22, 34], we define

$$W(S) := X[S]^T \mathbb{R}^r \cap \mathbb{Z}^r, \quad (20)$$

$$I(S) := X[S]^T \mathbb{Z}^r, \quad (21)$$

$$Z(S) := \mathbb{Z}^S/I(S), \quad (22)$$
The projection that forgets the coordinate corresponding to $s$ is the torsion subgroup of $\mathbb{Z}(S)$ \cite{34}. For $s \in [N] \setminus S$, let $\text{pr}_{S,s} : \mathbb{Z}^{S \cup \{s\}} \rightarrow \mathbb{Z}^S$ denote the projection that forgets the coordinate corresponding to $s$. This induces a map $\text{pr}_{S,s} : LG(S \cup \{s\}) \rightarrow LG(S)$ \cite{34, Lemma 2 and Lemma 3}.

Remark 9. Let $X \in \mathbb{Z}^r \times N$ be a matrix. It follows from the definition that if $S \subseteq [N]$ is independent, then the set of lattice points in the half-open parallelepiped $\mathcal{P}_S$ that is spanned by the columns of $X[S]$ is a set of representatives for the elements of $LG(S)$. If $S \cup \{s\}$ is also independent, then $\mathcal{P}_S$ is a facet of $\mathcal{P}_{S \cup \{s\}}$. Note that the projection map $\text{pr}_{S,s}$ sends a point in $\mathcal{P}_{S \cup \{s\}}$ to a point in $\mathbb{Z}^S$ that is not necessarily contained in $\mathcal{P}_S$, but of course it has a representative modulo $I(S)$ that is contained in $\mathcal{P}_S$ (cf. Figure 1).

Lemma 10. Let $X \in \mathbb{Z}^r \times N$ be a matrix. Let $E = [N]$ and let $\Delta_X$ be the matroid complex of the matroid represented by $X$.

For $S \in \Delta_X$, let $\mathcal{G}_X(S) := LG(S)$. Furthermore, for $s \in E \setminus S$ for which $S \cup \{s\} \in \Delta_X$, let $\pi^g_{S,s} := \text{pr}_{S,s} : \mathcal{G}(S \cup \{s\}) \rightarrow \mathcal{G}(S)$.

Then $(E, \Delta_X, \mathcal{G}_X)$ is a surjective finite abelian group structure on $\Delta$.

Proof. By assumption, $S \cup \{s\}$ is independent. Hence by \cite{34, Lemma 5}, $\pi^g_{S,s} : \mathcal{G}_X(S \cup \{s\}) \rightarrow \mathcal{G}_X(S)$ is surjective. To see that the projection maps commute in the required way, note that this is trivial for the projection $\text{pr}_{S,s} : W(S \cup \{s\}) \rightarrow W(S)$ and we can easily pass over to the quotient since $\text{pr}_{S,s}(I(S \cup \{s\})) = I(S)$ \cite{34, Lemma 2}. \hfill \Box

Definition 11 (Arithmetic independence complex derived from a representation). Let $X \in \mathbb{Z}^r \times N$ be a matrix and let $E$, $\Delta_X$, and $\mathcal{G}_X$ be as in Lemma 10. Let $\mathcal{A}_X$ be the arithmetic matroid represented by $X$. Then we call $\text{Ind}^{\text{ar}}(X) := (E_X, \Delta_X, \mathcal{G}_X)$ the arithmetic independence complex of $\mathcal{A}_X$ derived from the representation $X$.

Now let $X$ be a list of elements of a finitely generated abelian group $G$ of rank $r$. Let $\bar{X}$ denote the projection of $X$ to a lattice as explained in Subsection 2.4. Then we call $\text{Ind}^{\text{ar}}_{\text{tor}}(\bar{X}) := (E_X, \Delta_X, \mathcal{G}_X)$ the torsion-free arithmetic independence complex of $\mathcal{A}_X$ derived from the representation $X$.

Example 12. Let us consider the matrix $X = \begin{pmatrix} 2 & -2 \\ 2 & 2 \end{pmatrix}$. We denote the columns by $a$ and $b$. The arithmetic Tutte polynomial is $\mathfrak{M}_X(x, y) = x^2 + 2x + 5$. For the face poset $P$ of the representable independence complex this implies $f_P(t) = \mathfrak{M}_X(t, 1) = t^2 + 4t + 8$ and $h_P(t) = \mathfrak{M}_X(t, 1) = t^2 + 2t + 5$. The $f$-vector $(f_0, f_1, f_2) = (1, 4, 8)$ can also be read off from the corresponding toric arrangement, which is shown in Figure 1 there are $f_2 = 8$ simple vertices (i.e., vertices contained in exactly $r = 2$ 1-dimensional subtori), there are $f_1 = 4$ connected 1-dimensional subtori, and the torus has $f_0 = 1$ connected component.

Now let us construct $\text{Ind}^{\text{ar}}(X)$. Representatives of the group elements of $LG(\{\{1, 2\})$ are given by the lattice points of the half-open parallelepiped spanned by the columns of $X$, i.e. the set $\{(0, 0), (1, 0), (0, 1), (1, 1), (1, 0), (0, 1), (0, 2), (2, 1), (2, 0), (3, 1)\}$. Furthermore, $LG(\{\{1\}) \cong LG(\{2\}) \cong \mathbb{Z}/2\mathbb{Z} = \{0, 1\}$. Now we calculate the images of $LG(\{1, 2\})$ in the other two groups, i.e. we forget a coordinate and reduce the other one modulo 2.

\footnote{Note that while the construction of $\mathcal{G}_X$ requires the choice of a non-canonical isomorphism $G/G_t \cong \mathbb{Z}_t$, where $G/G_t$ denotes the lattice containing the elements of $X$, the torsion-free independence complex is independent of this choice.}
This allows us to construct the poset, which is shown in Figure 2. The Stanley–Reisner ideal is

$$I_P = (a_0C_{-1,1}, a_0C_{1,1}, a_0C_{1,2}, a_0C_{-1,2}, a_1C_0, \ldots, a_0b_0b_1, C_0C_{-1,1}, \ldots, C_0C_{-1,2}, a_0b_0 - (C_0 + C_{0,2}), a_0b_1 - (C_{0,1} + C_{0,3}), a_1b_0 - (C_{-1,2} + C_{1,2}), a_1b_1 - (C_{1,1} + C_{-1,1}))$$

(24)

3.3. Cyclic arithmetic independence complexes. We call $\mathcal{A} = (E, \Delta, m)$ a weak quasi-arithmetic matroid if $(E, \Delta)$ is a matroid and $m : \Delta \to \mathbb{Z}_{\geq 1}$ satisfies $m(S)m(S \cup \{x\})$ for all $S \subseteq E$ and $x \in E$ for which $S \cup \{x\} \in \Delta$, i.e. $m$ is required
to satisfy only one of the axioms of a quasi-arithmetic matroid. We require only this axiom to define the cyclic arithmetic independence complex. The arithmetic Tutte polynomial of a weak quasi-arithmetic matroid is defined as in \([11]\).

**Definition 13** (Cyclic arithmetic independence complex). Let \(\mathcal{A} = (E, \Delta, m)\) be a weak quasi-arithmetic matroid. For \(S \subseteq \Delta\), let \(\mathcal{G}_m(S) := \mathbb{Z}/m(S)\mathbb{Z}\) and for \(a \in E \setminus S\) for which \(S \cup \{a\} \subseteq \Delta\), let \(\pi_{S,a}^m\) be the canonical projection map that sends \(1\) to \(1\).

Then we define the cyclic arithmetic independence complex of \(\mathcal{A}\) as \(\text{Ind}^{\text{cyclic}}(E, \Delta, m) := \text{Ind}(E, \Delta, \mathcal{G}_m)\).

**Lemma 14.** Let \(\mathcal{A} = (E, \Delta, m)\) be a weak quasi-arithmetic matroid. Then \(\mathcal{G}_m(S)\) is indeed a surjective finite abelian group structure on \(\Delta\).

**Proof.** Let \(S \subseteq E\), \(a \in E \setminus S\), and \(S \cup \{a\} \subseteq \Delta\). Since \(m(S)\mathbb{Z}/m(S \cup \{a\})\mathbb{Z} \rightarrow \mathbb{Z}/m(S)\mathbb{Z}\) that sends \(1\) to \(1\) is indeed a surjective group homomorphism. It is easy to see that the commuting squares condition is satisfied. \(\square\)

**Example 15.** Let us consider again the arithmetic matroid \(\mathcal{A}\) that was defined in Example [12]. Its cyclic arithmetic independence complex \(\text{Ind}^{\text{cyclic}}(\mathcal{A})\) is shown in Figure 3. The commuting square is:

\[
\begin{array}{ccc}
\mathbb{Z}/8\mathbb{Z} & & \\
\downarrow & & \downarrow \\
\mathbb{Z}/2\mathbb{Z} & & \mathbb{Z}/2\mathbb{Z} \\
\end{array}
\]

The Stanley–Reisner ideal is

\[I_P = (a_0C_1, a_0C_3, a_0C_5, a_0C_7, b_0C_1, b_0C_3, b_0C_5, b_0C_7, a_1C_0, \ldots)
\[a_0a_1, a_0b_1, b_0a_1, b_0b_1, C_0C_1, C_0C_2, \ldots, C_0C_7,
\]

\[a_0b_0 - (C_0 + C_2 + C_4 + C_6), a_1b_1 - (C_1 + C_3 + C_5 + C_7)).
\]

**Example 16.** Let us consider a quasi-arithmetic matroid whose \(h\)-vector has negative entries. Let \(\mathcal{A} = \{(1, 2), 2^{(1, 2)}, m\}\), with \(m(\emptyset) = 1\) and \(m(\{\emptyset\}) = m(\{1\}) = m(\{2\}) = 2\). The underlying matroid is the uniform matroid \(U_{2, 2}\). The arithmetic Tutte polynomial is \(\mathcal{M}_A(x, y) = x^2 + 2x - 1\).

Let us construct \(\text{Ind}^{\text{cyclic}}(X)\). The commuting square of cyclic groups and the poset are shown in Figure 4. One can read off the \(f\)-vector from the arithmetic Tutte polynomial or from the poset. It is \((1, 4, 2)\) and the \(h\)-vector is \((1, 2, -1)\).

For the Stanley–Reisner ideal and the Stanley–Reisner ring we obtain

\[I_P = (a_0b_1, a_1b_0, a_0a_1, b_0b_1, C_0C_1, a_0C_1, b_0C_1, a_1C_0, b_1C_0, a_0b_0 - C_0, a_1b_1 - C_1)
\]

\[\mathbb{K}[P] \cong \mathbb{K}[a_0, a_1, b_0, b_1]/(a_0b_1, a_1b_0, a_0a_1, b_0b_1).
\)

As a vector space, \(\mathbb{K}[P]\) is isomorphic to \(\mathbb{K}[a_0, b_0] \cup \mathbb{K}[a_1, b_1]\). Thus, its Hilbert series is

\[1 + 2 \sum_{i \geq 1} (i + 1) t^i = 1 + 4t + 6t^2 + 8t^3 + 10t^4 + \ldots
\]

\[= (1 + 2t + 3t^2 + 5t^3 + \ldots)(-t^2 + 2t + 1) = \frac{\mathcal{M}_A(1, 1)}{(1 - t)^2}.
\]
Remark 17. In general, given a representable arithmetic matroid, the two arithmetic independence complexes that we have defined are not isomorphic. For example, \( \hat{0} \) is a cut vertex of the Hasse diagram of the face poset of the cyclic independence complex in Example 15. This is not the case for the representable independence complex in Example 12, which was constructed using the same arithmetic matroid. The representable independence complex appears to be more interesting as it preserves more structure, e.g. the matroid over \( \mathbb{Z} \) structure of the matrix (cf. [25]).

Remark 18. It is possible to compute the poset of layers of a toric arrangement as follows: first construct the arithmetic independence complex derived from a representation and then identify independent sets that correspond to the same flat [34]. It is an open problem whether there is also a 'poset of layers of an arithmetic matroid', in analogy with the lattice of flats of a matroid that exists, even if there is no representation. While we are able to construct an arithmetic independence complex in the non-representable case, the surjective finite abelian group structure on a simplicial complex defined using the cyclic groups can in general not be extended in a way required for the construction of a poset of layers. One would need to assign a cyclic group \( G(S) \) of cardinality \( m(S) \) to each \( S \subseteq E \) s.t. for \( a \in E \setminus S \), there is a surjection \( G(S \cup \{a\}) \rightarrow G(S) \) if \( \text{rk}(S \cup \{a\}) = \text{rk}(S) + 1 \), and an injection \( G(S \cup \{a\}) \hookrightarrow G(S) \) otherwise. In addition, the usual commuting squares conditions would need to be satisfied. In the next paragraph we will see that in general, this is not possible.

Let us consider the arithmetic matroid \( \mathcal{A} = (\{1, 2\}, \emptyset, \{1\}, \{2\}, m) \), where the multiplicity function \( m \) is given by \( m(\emptyset) = m(\{1, 2\}) = 3 \), \( m(\{1\}) = 6 \), and \( m(\{2\}) = 9 \). This arithmetic matroid can be represented by the list \( X = ((2, 0), (3, 0)) \subseteq \mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \). The arithmetic Tutte polynomial is \( \mathfrak{T}_A(x, y) = 3x + 3y + 9 \). The cyclic groups yield the following square:

\[
\begin{array}{c}
\mathbb{Z}/3\mathbb{Z} \\
\mathbb{Z}/6\mathbb{Z} \\
\mathbb{Z}/9\mathbb{Z} \\
\mathbb{Z}/3\mathbb{Z}
\end{array}
\]

Since we require \( \iota_2 \) to be injective, \( \iota_2(1) \in \{3, 6\} \) must hold. Hence \( \pi_2(\iota_2(1)) = 0 \).

On the other hand, \( \iota_1(1) \in \{2, 4\} \), which implies \( \pi_1(\iota_1(1)) \in \{1, 2\} \). Hence the diagram does not commute. A commuting square for this arithmetic matroid can be obtained by replacing the group \( \mathbb{Z}/9\mathbb{Z} \) by \( \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \).
4. Future directions and related work

Remark 19. The construction of arithmetic independence posets and hence, of Stanley–Reisner rings can be extended to the setting of group actions on semimatroids, that where recently introduced by Delucchi and Riedel [22]. This will be done in an upcoming paper of D’Alli and Delucchi.

Remark 20. As stated in the introduction, in the case of matroids, algebraic structures associated to matroids have been used to prove inequalities for their $f$ and $h$-vectors. For arithmetic matroids, only very little is currently known about the shape of these vectors [53]. It would be interesting to prove stronger inequalities. However, one cannot expect results such as log-concavity: let $\alpha \in \mathbb{Z}_{\geq 1}$ and let $X = (e_1, e_2, \ldots, e_{r-1}, e_1 + \ldots + e_{r-1} + \alpha e_d) \subseteq \mathbb{Z}^r$. The arithmetic $f$-polynomial is $f_X(t) = t^r + \binom{r-1}{1}t^{r-1} + \ldots + \binom{r-1}{r-1}t^{1} + \alpha$ [31]. For $r \geq 4$ and sufficiently large $\alpha$, the coefficients of this polynomial are not unimodal and hence also not log-concave.

Proving that our arithmetic independence posets are Cohen–Macaulay would imply that their $h$-vector is positive [46, Theorem 3.10]. In the case of arithmetic matroids, this follows from the positivity of the coefficients of the arithmetic Tutte polynomial, which is known [41, 14].

Remark 21. The Stanley–Reisner ring of the matroid represented by a matrix $X$ is isomorphic to the equivariant cohomology ring of a hypertoric variety associated with $X$ [43, Theorem 3.2.2]. It would be nice to find a similar interpretation of the arithmetic Stanley–Reisner ring defined by an integer matrix $X$. A somewhat similar goal has been achieved in [21], where it was shown that equivariant $K$-theory and cohomology of certain spaces is related to algebraic structures that use Dahmen–Michelli spaces as building blocks. This was done both in the continuous and the discrete case, which correspond to the matroid and the arithmetic matroid case.

Remark 22. Let $X \in \mathbb{R}^{d\times N}$ be a matrix and let $\Delta^*$ be the independence complex of the dual matroid. The quotient of $\mathbb{K}[\Delta^*]$ by a linear system of parameters is isomorphic to the continuous Dahmen–Michelli space $\mathcal{D}(X)$ [19]. This suggests a similar relationship between the Stanley–Reisner ring of an arithmetic matroid that is represented by an integer matrix $X \in \mathbb{Z}^{d\times N}$ and the discrete Dahmen–Michelli space $\text{DM}(X)$ (see [20] for a definition), or its dual, the periodic $P$-space [33]. These two spaces already have the correct Hilbert series, i.e. $t^N - \text{dim}_{\mathbb{K}} \mathcal{D}(X)(1, \frac{1}{t})$. The discrete Dahmen–Michelli space $\text{DM}(X)$ can be decomposed into ‘local’ $\mathcal{D}(X)$-spaces that are attached to the vertices of the toric arrangement [20 (16.1)]. Is there also a decomposition of the arithmetic Stanley–Reisner ring (derived from a representation) into ‘local’ matroid Stanley–Reisner rings attached to the vertices of the toric arrangement?

Remark 23. A recent draft of Martino [36] also suggest a method to construct an arithmetic Stanley–Reisner ring, given a representation of an arithmetic matroid. The approach is somewhat similar to ours, but instead of layer groups, the abelian groups ($\mathbb{Z}$-modules) that Fink and Moci assign to a list of integer vectors in the context of matroids over $\mathbb{Z}$ [25] are used. This structure is in some sense dual to layer groups ([22, Theorem D]), so one could expect that the arithmetic independence complex obtained in this way is isomorphic to our arithmetic independence complex derived from a representation. However, the author was not able to verify this for the following reason: the definition in [36] relies on the fact that every finitely generated abelian group is isomorphic to $\mathbb{Z}^d \times H$ for some $d \in \mathbb{Z}_{\geq 0}$ and a finite group $H$. But this isomorphism is not canonical and it is not specified in [36], hence the poset does not appear to be well-defined.
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