ON AN EQUATION ASSOCIATED WITH THE CONTACT LIE ALGEBRAS

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Abstract
In the framework of a Lie algebraic approach we study a nonlinear equation associated with the contact Lie algebra $K_{m}$. This algebra appears to be relevant for some solvable models of field theory and gravity in higher dimensions.

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In this note we study a nonlinear equation associated in the framework of the group–algebraic approach for nonlinear dynamical systems [1] with the contact Lie algebra $KK_m$ considered in the continuum Lie algebras formulation [2]. It seems that in the same way as the area–preserving diffeomorphisms on two–dimensional torus governs the symmetry of the self–dual Einstein space with the rotational Killing vector, described by the so–called heavenly equation [3], the symmetry realised by the contact Lie algebras has a relation to a number of field theory and gravity models in higher dimensions.

Let us first recall briefly the definition of contact Lie algebras following the notations of ref. [4]. The vector fields transforming the 1-form
\[ \sum_{j=1}^{m} x_j dx_{j+m} + dx_{2m+1} \]  
into a form which differs from (1) by the multiplication by a formal power series, are called contact fields, and the corresponding Lie algebra is denoted $KK_m$. Let $F_n = K[x_1, \ldots, x_n]$, $n = 2m + 1$, be the space of the formal power series, and define on it the structure of a Lie algebra,
\[ [f, g] = -\hat{f}_x g + \frac{\partial f}{\partial x_n} (\hat{D} - 2) g - \frac{\partial g}{\partial x_n} (\hat{D} - 2) f, \]
where
\[ \hat{f}_x \equiv \sum_{j=1}^{m} (\frac{\partial f}{\partial x_j+m} \frac{\partial}{\partial x_j} - \frac{\partial f}{\partial x_j} \frac{\partial}{\partial x_{j+m}}); \quad \hat{D} \equiv \sum_{A=1}^{2m} x_A \frac{\partial}{\partial x_A}. \]
The formula
\[ f \rightarrow \hat{f}_x + (\hat{D} - 2) f \frac{\partial}{\partial x_n} \]
then sets the isomorphism $F_n \rightarrow KK_m$. Being considered as a continuum $\mathbb{Z}$–graded Lie algebra $G(E) = \bigoplus_{p \in \mathbb{Z}} G_p$, $KK_m$ possesses the following defining relations for the elements $X_{0,\pm 1}(f)$, $f \in E$, parametrising its local part $G_{-1} \oplus G_0 \oplus G_{+1}$,
\[ [X_0(f), X_0(g)] = -X_0(\hat{f}_x g), \]  
\[ [X_0(f), X_{\pm 1}(g)] = -X_{\pm 1}(\hat{f}_x g \pm g(\hat{D} - 2) f), \]
\[ [X_{+1}(f), X_{-1}(g)] = -X_0(\hat{f}_x g - (\hat{D} - 4)(fg)). \]

Now, let $\mathcal{M}$ be a two–dimensional manifold endowed with a complex structure, and $A$ be a $KK_m$–valued 1-form on $\mathcal{M}$. Any such 1-form generates a connection form of some connection in the trivial fibre bundle. We suppose that $A$ is flat, so that with a local coordinate $z$ in $\mathcal{M}$, $A = A_+ dz + A_- d\bar{z}$, where $A_\pm$ are some mappings from $\mathcal{M}$ to $KK_m$, satisfying the zero curvature condition
\[ \partial_+ A_- - \partial_- A_+ + [A_+, A_-] = 0. \]
Here and in what follows $\partial_+ \equiv \partial/\partial z, \partial_- \equiv \partial/\partial \bar{z}$. Choosing some basis in $KK_m$ and considering the components of the decomposition of $A_\pm$ over this basis as fields, we can treat (6) as a nonlinear system of partial differential equations for the fields. In accordance with [1], to provide nontriviality of such a system, we impose the so-called grading condition on the connection, so that the $(1,0)$-component $A_+$ of $A$ takes values in $\oplus_{p \geq 0} \mathcal{G}_{+p}$, and the $(0,1)$-component $A_-$ takes values in $\oplus_{p \geq 0} \mathcal{G}_{-p}$. Moreover, we confine ourselves to the case when $A_\pm$ take values only in the local part of the algebra, i.e., in $\mathcal{G}_0 \oplus \mathcal{G}_{\pm 1}$, respectively. Due to the arbitrariness related to the gauge transformations generated by $\mathcal{G}_0$, we can finally take

$$A_+ = X_{+1}(f), \quad A_- = X_0(u) + X_{-1}(g); \quad u, f, g \in E.$$  

(7)

Then it is easy to convince oneself that a $\mathcal{G}_0$-gauge invariant equation arising from (6) with (7), can be represented in the form

$$\partial_+ [(1 - \hat{L})^{-1} \partial_- \rho] = \frac{1}{2} (4 - \hat{D}) e^\rho,$$

(8)

where $2\hat{L} \equiv \hat{D} - \hat{\rho}_x$. Note that here $\rho(z, \bar{z}; x_1, \ldots, x_{2m}) \equiv \log 4f, u = -1/2(1 - \hat{L})^{-1} \partial_- \rho$, while the function $g$ can be taken to be unity, up to inessential transformations. Using the formula

$$\partial \frac{1}{1 - L} = \frac{1}{1 - L} \partial \hat{L} \frac{1}{1 - \hat{L}},$$

one can rewrite equation (8) as

$$\partial_+ \partial_- \rho + \partial_+ \hat{L}(1 - \hat{L})^{-1} \partial_- \rho = \frac{1}{2} (1 - \hat{L})(4 - \hat{D}) e^\rho.$$  

(9)

This equation looks quite complicated, however, the fact that the algebra $KK_m$ is of the finite growth, see e.g. [3], allows us to believe that it can be integrated in this or that sense.

Let us consider special simplifications of the equation in question, based on some additional symmetry conditions imposed on it. It is clear that if the function $\rho$ depends only on $z, \bar{z}$, and the ratios $x_j/x_{j+m}$, then equation (9) is automatically reduced to the Liouville equation, $\partial_+ \partial_- \rho_L = 2e^{\rho_L}$. On the other hand, if $\rho = \rho(z, \bar{z}; r \equiv - \log \sum A x_A^2)$, then we arrive at the form

$$\partial_+ \partial_- \rho = (2 + 3 \partial/\partial r + \partial^2/\partial r^2) e^\rho.$$  

(10)

This equation evidently possesses a self–similar solution

$$e^\rho = e^{\rho_L(z, \bar{z})}(1 + \alpha_1 e^{-r} + \alpha_2 e^{-2r})$$

with arbitrary constants $\alpha_{1,2}$. It plays the same role as the Eguchi–Hanson gravitational instanton for the self–dual Einstein space with the rotational Killing vector,
described by the completely integrable equation \( \partial_+ \partial_- \rho_H = \partial^2 / \partial r^2 e^{\rho_H} \), and related to the continuum Lie algebra \( \mathcal{G}(E; K, \text{id}) \) with the Cartan operator \( K = \frac{\partial^2}{\partial r^2} + 3 \frac{\partial}{\partial r} + 2 \), which in turn is isomorphic to \( S_0 \text{Diff} \mathbb{T}^2 \).

Equation (10) is written in a form of the continuous Toda system with the Cartan operator \( K = \frac{\partial^2}{\partial r^2} + 3 \frac{\partial}{\partial r} + 2 \), and, correspondingly, is associated with the continuum Lie algebra \( \mathcal{G}(E; K, \text{id}) \), to which the algebra \( KK_m \) is reduced in this case. It is interesting to note that \( \mathcal{G}(E; \frac{\partial^2}{\partial r^2} + 3 \frac{\partial}{\partial r} + 2, \text{id}) \) realises a special case \( \{ c_0 = c_1 \neq 0, c_2 = 0 \} \) of the Lie algebra \( W(c_0, c_1, c_2) \) introduced in [6], see also [7], as a modification of \( S_0 \text{Diff} \mathbb{T}^2 \), with the elements satisfying, in a component form, the commutation relations

\[
[Y_m, Y_n] = (c_0 m \times n + c(m - n)) Y_{m+n} + \text{central term}.
\]

Here \( m = (m_1, m_2) \), \( n = (n_1, n_2) \) are two-dimensional integer vectors, \( m \times n \equiv m_1 n_2 - m_2 n_1 \); and \( c = (c_1, c_2) \) is a constant vector.

The role of the (proper) Bäcklund map for equation (10) for the case when the function \( \rho \) depends not on both variables \( z \) and \( \bar{z} \), but on their linear combination, say \( t \equiv \frac{z - \bar{z}}{\sqrt{2}} \), is played by the equation

\[
\frac{\partial \rho}{\partial t} = \left( 1 + \frac{\partial}{\partial r} \right) e^{\rho/2},
\]

which coincides with the classical equation for the Riemann reversal wave written in terms the function \( \Phi(t, \tau \equiv 2e^r) = \frac{\tau}{2} e^{\rho(t, \tau)/2} \). Just the overlap phenomena caused by the equation for \( \Phi \) can lead in turn, in the same way as for the continuous long wave approximation of the Toda system (the heavenly equation), see e.g. [8], to the generation of shock waves.

There are some other symmetry conditions, e.g. also of a scaling type, which considerably simplify equation (13).

Finally, let us note that the system under consideration, described by equations (8) or (9), can be naturally presented in a symmetrical form with respect to the coordinates \( z_+ \) and \( z_- \), namely

\[
\partial_+ [(1 - \hat{L}_-)^{-1} \partial_- \rho] + \partial_- [(1 - \hat{L}_+)^{-1} \partial_+ \rho] + \frac{1}{4} \{ (1 - \hat{L}_-)^{-1} \partial_- \rho, (1 - \hat{L}_+)^{-1} \partial_+ \rho \} P = (4 - \hat{D}) e^\rho,
\]

where \( 2 \hat{L}_\pm \equiv \hat{D} \pm \frac{1}{2} \hat{\rho}_x; \) \( \{ f, g \} P \equiv \hat{f}_x g \).

Moreover, in the case when \( \rho = \rho(t; x_1, \ldots, x_{2m}) \), this equation can be written in the Brockett double commutator form [9] which is relevant for a description of some sorter (analog) systems.

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