Integrable models with boundaries and defects

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1 Introduction

Over the past forty years there has been much effort devoted to understanding various aspects of one-dimensional (that is, one space - one time, generally denoted by $x$ and $t$, respectively) field theories - both from the classical and quantum points of view - and their applications in a variety of contexts. For the most part, attention has been paid to models in the bulk (that is, defined without restriction over the whole space $-\infty < x < \infty$), or models defined on a circle (that is, periodic in $x$). In these two lectures it is intended to describe in a quite straightforward manner some of the ideas and complications when boundaries of various kinds are introduced.

For simplicity, two particular cases will be considered. Firstly, a free massive field will be used to illustrate the basic ideas; secondly, the affine Toda series of models will be used to illustrate the additional features coming into play when boundaries and their associated boundary conditions are required to preserve integrability. It will appear that a number of traditional ideas will need to be adjusted and extended to accommodate this new situation.

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Since the affine Toda field theories as bulk models have some significance in a geometrical context (for example, see [1]), it may be fruitful to wonder about the role their integrable boundary conditions might play.

2 Bulk theories

The simplest of all bulk field theories is a single, free, massive scalar field described classically by the Klein-Gordon equation

\[(\partial^2 + m^2)\phi = 0, \quad \partial^2 \equiv \partial_t^2 - \partial_x^2.\]  

(1)

If the mass parameter \(m\) is zero the field theory is conformal. From a quantum perspective this field describes noninteracting particles of mass \(m\hbar\) (henceforth \(\hbar\) will be set to unity), whereas from a classical point of view (1) is completely solvable using a Fourier transform technique.

Considerably more interesting is the sine-Gordon model whose field equation is

\[\partial^2 \phi = -\frac{m^2}{\beta} \sin \beta \phi,\]  

(2)

where \(m\) provides the mass scale, as before, and \(\beta\) is the bulk coupling. Actually, the latter is only relevant in the quantum field theory since it can be removed from the classical field equation by rescaling the field. If the field \(\phi\) is regarded as ‘small’ then the linearised version of (2) is (1). However, the nonlinearity plays a fundamentally crucial role since it permits the existence of solitons, antisolitons and breathers (see [2, 3] for reviews on solitons in general, and [4] for a treatise on integrability), which means the quantum field theory is much more interesting. The spectrum of particles contains the soliton, the antisoliton, and a collection of breathers whose precise number is coupling dependent. These are not free particles and their mutual scattering properties are fascinating (see, for example, the classic paper [5]). If the sine is replaced by a hyperbolic sine on the right hand side of (2) the spectrum of states simplifies dramatically (essentially because the \(2\pi\)-periodicity in the field is lost) and there is a single scalar particle, though it is not free but scatters with other similar particles.

The sine(sinh)-Gordon model is the first of a series of models, the affine Toda models, each of which is defined in terms of an extended root system. More details on this will be given later but for now consideration of bound-
aries of various kinds will be restricted to fields of type (1) or (2). For a fairly recent review, see [6].

3 One boundary

Consider a free massive scalar field restricted to the left half-line ($x < 0$) by a linear boundary condition at $x = 0$. Its equation of motion and boundary condition are as follows:

\[
\begin{align*}
\left( \partial^2 + m^2 \right) \phi &= 0, & (x < 0) \\
\partial_x \phi &= -\lambda \phi, & (x = 0),
\end{align*}
\]

which follow formally using the Euler-Lagrange equations applied to the Lagrangian density

\[
\mathcal{L} = \theta(-x) \frac{1}{2} \left( (\partial \phi)^2 - m^2 \phi^2 \right) - \delta(x) \frac{1}{2} \lambda \phi^2,
\]

where the last term plays the role of a boundary potential. Note, the boundary condition is assumed to be homogeneous with respect to the field $\phi$; a slightly more general possibility would add a constant to the right hand side of the expression for $\partial_x \phi$ and the effect of that would be to force the lowest energy solution (with $\lambda > 0$) to be $x$-dependent rather than the constant $\phi = 0$.

The pair of equations (3) is solvable and a wave travelling towards the boundary at $x = 0$ will be reflected. Thus

\[
\phi(x, t) = e^{-i\omega t} \left( e^{ikx} + R(k) e^{-ikx} \right) + cc, \quad \omega^2 = k^2 + m^2,
\]

where $R(k)$ is a ‘reflection’ factor and ‘cc’ denotes complex conjugate (necessary to keep $\phi$ real). Clearly, $R$ can be calculated using the boundary condition to find

\[
R(k) = \frac{ik + \lambda}{ik - \lambda}.
\]

On the other hand, if the field $\phi$ is not free, but rather satisfies an equation such as (2), then it is natural to ask what boundary conditions (if any) are compatible with integrability. In those circumstances, the generalisation of (3) is

\[
\begin{align*}
\partial^2 \phi &= -\frac{m^2}{\beta} \sinh \beta \phi, \quad (x < 0) \\
\partial_x \phi &= -\frac{\partial B}{\partial \phi}, \quad (x = 0),
\end{align*}
\]
following in the usual way from the Lagrangian

\[ \mathcal{L} = \theta(-x) \left( \frac{1}{2} (\partial \phi)^2 - \mathcal{V}(\phi) \right) - \delta(x) \mathcal{B}(\phi), \]  

where \( \mathcal{B}(\phi) \) is the boundary potential. For reasons which will be described in more detail later, integrability requires

\[ \mathcal{B}(\phi) = \frac{2}{\beta^2} (b_1 e^{\beta \phi/2} + b_0 e^{-\beta \phi/2}) \]  

\[ \mathcal{V}(\phi) = \frac{m^2}{2 \beta^2} (e^{\beta \phi} + e^{-\beta \phi}) \]

where \( b_0, b_1 \) are arbitrary real constants. This result was first discovered in the case of the sine-Gordon model by Ghoshal and Zamolodchikov [7]. It is interesting to notice that each term in the boundary potential \( \mathcal{B} \) is, up to a constant, the square root of the corresponding term in the bulk potential \( \mathcal{V} \). This particular feature turns out to be universal within the class of affine Toda field theory models once boundaries are incorporated [9, 10], although the number of free parameters introduced via a boundary condition is generally much more severely restricted than it is for the sine/sinh-Gordon model. In fact, these constraints on boundary parameters are somewhat mysterious. For more information on the sinh-Gordon case, including a discussion of specific solutions, see [11]. For further information on the behaviour of classical soliton solutions in the sine-Gordon and other models with a boundary consult [12]. It is also possible to render the boundary dynamical in a consistent manner. To learn about one such possibility, consult [13].

In the quantum field theory we envisage a situation where a particle travels towards the boundary, hits it, and bounces back. Then, the incoming particle state will be related to the outgoing particle state by reversing momentum and by a phase factor (the ‘reflection’ factor). Thus,

\[ |k\rangle_{\text{out}} = R(k; b_1, b_2) | -k\rangle_{\text{in}}. \]

One question to ask concerns the relationship between the \( S \)-matrix, which describes the scattering of two sinh-Gordon particles, and the reflection factor \( R \). This is known for sinh (or sine)-Gordon but remains an open question for almost all the other integrable field theories with boundary. Thus, for the sinh-Gordon model the \( S \)-matrix is given by the rapidity-dependent phase factor

\[ S = -\frac{1}{(B)/\Theta(2-B)/\Theta}; \]  

1
where the notation conceals a certain complexity:

\[
(x)_{\Theta} = \frac{\sinh \left( \frac{\Theta}{2} + \frac{i\pi x}{4} \right)}{\sinh \left( \frac{\Theta}{2} - \frac{i\pi x}{4} \right)},
\]

and

\[
B(\beta) = \frac{2\beta^2}{8\pi + \beta^2},
\]

with \( \Theta = \theta_1 - \theta_2 \), the difference in rapidities of the two particles (\( \theta_1 > \theta_2 \) and \( k_i = m \sinh \theta_i, \ i = 1, 2 \)). On the other hand, using the same notation, the reflection factor for just one particle depends on the two boundary parameters as well as the bulk coupling and the rapidity \( \theta \) of the particle:

\[
R = \frac{(1)_{\theta}(1 + B/2)_{\theta}(2 - B/2)_{\theta}}{(1 + E)_{\theta}(1 - E)_{\theta}(1 + F)_{\theta}(1 - F)_{\theta}}
\]

where

\[
E = (a_0 + a_1)(1 - B/2), \ F = (a_0 - a_1)(1 - B/2), \ b_j = \cos a_j \pi, \ j = 0, 1.
\]

For further details concerning this expression, and further references, see [8, 14, 15, 16].

Returning for a moment to the free field situation there is another phenomenon which is worth examining. Let \( k = -i\lambda \), \( (\lambda < 0) \). Then, provided \( -m < \lambda < 0 \), there is a time-periodic solution to the equations (3) which declines exponentially away from the boundary. In more detail, this is given by

\[
\phi = A \cos \omega t e^{-\lambda x}, \ \omega^2 = m^2 - \lambda^2,
\]

and represents a ‘boundary bound state’. It is clearly a result of the competition between the bulk energy (always positive) and the boundary energy (negative if \( \lambda < 0 \)) and, after quantisation leads to a tower of states equally spaced in energy, actually very similar to the spectrum of a harmonic oscillator. These new kinds of states also exist in the nonlinear models and have been investigated for the sinh-Gordon model in [15, 16]. These states illustrate the much richer content of the nonlinear models once a boundary is introduced.
4 Two boundaries

Once it becomes possible to deal with a single boundary, it is natural to wonder about two of a similar type. Once again it is instructive to look at the free field first, supposing it is defined on the interval $-L < x < L$ with suitable boundary conditions at each end:

$$\begin{align*}
(\partial^2 + m^2)\phi &= 0, \quad (|x| < L) \\
\partial_x \phi &= \mp \lambda_\pm \phi, \quad (x = \pm L),
\end{align*}$$

where $\lambda_\pm$ are two real parameters. As before, it is appropriate to consider a solution with a specific frequency $\omega$, $\omega^2 = m^2 + k^2$ and impose the pair of conditions at the ends of the interval. A little algebra reveals a nice factorised form for a relationship which in effect determines the possible frequencies the field might adopt within the interval. Explicitly, the equation determining the frequencies (via $k$) is

$$e^{ikL} = R_+(k)R_-(k)$$

(14)

where $R_\pm(k)$ are the reflection factors for the reflections at the two ends. Effectively, each boundary works independently of the other.

One important question is whether a similar story will be the case for the nonlinear models such as the sinh-Gordon model. In that case, even classically, the solutions within an interval are not known explicitly, and it has not yet been shown that a similar factorisation to that of (14) will occur in all cases. In fact, to make progress on this it seems likely to require a better understanding of the ‘$N$-zone’ solutions discovered some years ago by Mumford [17] and independently by Dubrovin and Natanzon [18]. These are constructed from $\vartheta$-functions and make essential use of the Fay identities, which appear to fit very naturally into this context. Nevertheless, incorporating the boundary conditions has proved elusive so far. This aspect of the exact solutions is decidedly geometrical in flavour because the relationship between the $\vartheta$-functions and Riemann surfaces plays a central role, and would probably repay closer scrutiny.

5 Defects

Another possibility, which has been explored in [19, 20], introduces an internal boundary or ‘defect’ at which a field may have a discontinuity, or at
which two fields of differing character might meet. Actually, ‘defects’ are a common phenomenon in many areas of physics; one only has to think of a ‘bore’ or ‘hydraulic jump’ at which the level of a fluid flow suddenly changes, a ‘shock’ front at which fluid flow suddenly changes from subsonic to supersonic, or a dislocation within a crystal or other material. In all cases, there is a discontinuity in some physical quantity while others may remain continuous. However, in the context of these lectures the interest is in integrability rather than any specific phenomenon. There is nothing special about one defect, any number of them might be allowed sprinkled along the real line at $x_1, x_2, \ldots$, but, for a first look, consider just one at $x = 0$, and two scalar fields. The field in the region $x < 0$ is denoted by $\phi$ and the field in the region $x > 0$ by $\psi$.

A suitable Lagrangian for the pair of scalar fields $\phi, \psi$ is;

$$ L = \theta(-x) \left\{ \frac{1}{2} (\partial \phi)^2 - V(\phi) \right\} + \theta(x) \left\{ \frac{1}{2} (\partial \psi)^2 - W(\psi) \right\} + \delta(x) \left\{ \frac{1}{2} (\phi \dot{\psi} - \psi \dot{\phi}) - B(\phi, \psi) \right\} $$

(15)

in which $B(\phi, \psi)$ represents the defect potential. Note, this expression has not been derived from any specific physical system; it is a purely theoretical possibility which might (or might not) have any real relevance. The field equations are as follows

$$ \begin{cases} \\
\mathcal{L} = \theta(-x) \left\{ \frac{1}{2} (\partial \phi)^2 - V(\phi) \right\} + \theta(x) \left\{ \frac{1}{2} (\partial \psi)^2 - W(\psi) \right\} \\
+ \delta(x) \left\{ \frac{1}{2} (\phi \dot{\psi} - \psi \dot{\phi}) - B(\phi, \psi) \right\} \\
\end{cases} $$

with boundary conditions at $x = 0$:

$$ \begin{cases} \\
\partial_x \phi = \partial_t \psi - \frac{\partial B}{\partial \phi} \\
\partial_x \psi = \partial_t \phi + \frac{\partial B}{\partial \psi} \\
\end{cases} $$

(16)

The particular form of (15) is really required by integrability and the lack of time reversal invariance is a special feature.

However, since time is limited, and the integrability will not be described, at least not with any details, it is nevertheless worth picking up on a particular point which already captures much of the detail which integrability would imply. For further information consult [19, 20, 22].
Consider the momentum carried by the two fields in their respective domains. This is given by the expression

\[ P = \int_{-\infty}^{0} dx \partial_t \phi \partial_x \phi + \int_{0}^{\infty} dx \partial_t \psi \partial_x \psi \]

and is not expected to be conserved because translation invariance is explicitly broken by placing a defect at \( x = 0 \). However, using the equations of motion in the two domains, and integrating by parts in the usual way, leads to

\[ \dot{P} = \int_{-\infty}^{0} dx \partial_x \left( \frac{1}{2} (\partial_x \psi)^2 - V + \frac{1}{2} (\partial_t \psi)^2 \right) \]

\[ + \int_{0}^{\infty} dx \partial_x \left( \frac{1}{2} (\partial_x \psi)^2 - W + \frac{1}{2} (\partial_t \psi)^2 \right) \]

\[ = \left[ \frac{1}{2} (\partial_x \phi)^2 + \frac{1}{2} (\partial_t \phi)^2 - V \right]_{x=0} - \left[ \frac{1}{2} (\partial_x \psi)^2 + \frac{1}{2} (\partial_t \psi)^2 - W \right]_{x=0} \]

Using the boundary conditions (16), the latter can be rearranged to

\[ \dot{P} = \left[ -\dot{\psi} \frac{\partial B}{\partial \phi} - \dot{\phi} \frac{\partial B}{\partial \psi} + \frac{1}{2} \left( \frac{\partial B}{\partial \phi} \right)^2 - \frac{1}{2} \left( \frac{\partial B}{\partial \psi} \right)^2 - (V(\phi) - W(\psi)) \right]_{x=0} \]

\[ = -\frac{\partial U}{\partial t} \tag{17} \]

where \( U \) is a functional of the fields evaluated at \( x = 0 \) provided the following two equations hold:

\[ \frac{\partial^2 B}{\partial \phi^2} = \frac{\partial^2 B}{\partial \psi^2}, \]

\[ \frac{1}{2} \left\{ \left( \frac{\partial B}{\partial \phi} \right)^2 - \left( \frac{\partial B}{\partial \psi} \right)^2 \right\} = V(\phi) - W(\psi). \]

The first of these is a ‘wave’ equation for \( B \) in terms of the two fields evaluated at \( x = 0 \); the second is a nonlinear condition constraining the relevant solutions. There are several possibilities but the simplest to examine is a pair of free fields with equal mass parameters. Then it is not difficult to check
that

\[ m_\phi = m_\psi = m, \quad B = \frac{m\lambda}{4}(\phi + \psi)^2 + \frac{m}{4\lambda}(\phi - \psi)^2 \]

\[ V(\phi) = \frac{1}{2}m^2\phi^2, \quad W(\psi) = \frac{1}{2}m^2\psi^2. \]

Note that, generally,

\[ \lim_{x \to 0} \phi(x, t) \neq \lim_{x \to 0} \psi(x, t) \]

and the fields can be discontinuous at the location of the defect. Nevertheless, it is perfectly possible to modify slightly the definition of momentum so that the revised momentum \( P + U \) is conserved despite the absence of the usual Noether argument using translation invariance. In a way this is quite surprising. Note also that there is an additional real parameter \( \lambda \) which is free and introduced by the defect; in the limit \( \lambda \to 0 \) the defect disappears since the fields to the left and the right of it match exactly in that limit.

With nonlinear fields there is a variety of possibilities \([19, 20]\), but suffice it to say here that the defect conditions \((16)\), at least in the cases which have been analysed so far, turn out to be Bäcklund transformations frozen at the location of the defect. Thus, to take the sine-Gordon model as an example, the defect boundary conditions turn out to be

\[ \partial_x \phi - \partial_t \psi = -\frac{m}{\beta} \sin \beta \left( \frac{\phi + \psi}{2} \right) - \frac{m}{\beta\lambda} \sin \beta \left( \frac{\phi - \psi}{2} \right) \]  \( (18) \)

\[ \partial_x \psi - \partial_t \phi = \frac{m}{\beta} \sin \beta \left( \frac{\phi + \psi}{2} \right) - \frac{m}{\beta\lambda} \sin \beta \left( \frac{\phi - \psi}{2} \right), \]  \( (19) \)

where \( \lambda \) is again a free parameter. This fact, too, has something of a geometrical flavour about it, given that Bäcklund was interested in spaces of constant negative curvature at the time he developed the transformation bearing his name \([21]\). Also, the solitons of the sine-Gordon model are normally transmitted by a defect - but delayed and possibly converted to anti-solitons - or they may be absorbed. All these possibilities are allowed because the defect can ‘store’ energy/momentum and topological charge. These facts allow the intriguing possibility of controlling solitons (see \([22]\) for some further ideas - an article which appeared after these talks were given).

From the point of view of the quantum field theory, these ideas are not yet fully developed although there is some literature treating aspects of the story (see for example \([23, 24, 25, 26]\)).
6 Generalised Lax pairs

It is not the purpose here to give a full description of classical integrability. For that, the interested reader is recommended to consult one of the books on the subject, for example [3, 4]. Instead, a few words will be said concerning the specific situations mentioned in the previous sections.

There are several routes to demonstrate the classical integrability in the variety of cases mentioned above. One is to systematically check conserved quantities, starting with those of lowest spin (for example, the sine/sinh-Gordon model in the bulk has conserved quantities for each odd spin and it is enough to check the conservation of a modified form of the ‘energy-like’ combination of spin three charges in the presence of a boundary [7]). However, this is a painstaking procedure and it is better to develop a method which generates all conserved quantities simultaneously. The generalised Lax pair provides such a method adaptable to either the boundary or the defect situations.

The formulation of integrability in the presence of a boundary has its origins in the pioneering work of Sklyanin [27], and it was adapted more recently to accommodate the sine(sinh)-Gordon model with a boundary by MacIntyre [28]. However, the procedure which will be described here is somewhat different in style though allowing a direct computation of the crucial element of Sklyanin’s formulation, which otherwise has to be calculated using compatibility with the classical version of the Yang-Baxter relations. Time does not permit a detailed comparison of all these ideas, unfortunately.

The Lax pair idea can be generalised to the full set of affine Toda field theories (for a classification, see [29, 30]), which are defined conveniently in terms of Lie algebra data as follows. There is a set of scalar fields $\phi_a$, $a = 1, 2, \ldots, r$ where $r$ is the rank of a Lie algebra $g$ whose interactions are described by the Lagrangian density

$$
L = \frac{1}{2} \partial_\mu \phi \cdot \partial^\mu \phi - \frac{m^2}{\beta^2} \sum_{i=0}^{r} n_i e^{\beta \alpha_i \cdot \phi},
$$

(20)

where $\alpha_i$, $i = 1, 2, \ldots, r$ is a set of simple roots for $g$, $m$ is a mass scale and $\beta$ is the bulk coupling constant. The vector $\alpha_0$, defined by

$$
\alpha_0 = - \sum_{i=1}^{r} n_i \alpha_i
$$
is the Euclidean part of the additional (affine) root in the Kač classification of the affine root systems (by convention the long simple roots in any root system are taken to have length $\sqrt{2}$). The integers $n_i$, $i = 1, 2, \ldots, r$ are characteristic of a particular root system and $n_0 = 1$. Thus, taking the simplest example of a root system corresponding to $a_1$, one takes $\alpha_1 = \sqrt{2} = -\alpha_0$ and, apart from a rescaling of $\beta$, (20) is identical to the Lagrangian for the sinh-Gordon model. In other words, the sinh-Gordon model is simply part of a large collection of models each possessing the property of integrability (for further references see [6]). It is also worth remarking that if the term in (20) corresponding to $i = 0$ is deleted then what remains is actually a conformal field theory and of considerable interest in its own right.

The field equations following from (20) are

$$\partial^2 \phi = -\frac{m^2}{\beta} \sum_{i=0}^{r} n_i \alpha_i e^{\beta \alpha_i \cdot \phi}$$

and these may be cast into a Lax pair form making use of (some of) the generators of the Lie algebra $g$. To see how to do this, consider defining a two dimensional gauge field $a_t$, $a_x$ as follows:

$$a_t = \frac{1}{2} H \cdot \partial_x \phi + \sum_{i=1}^{r} m_i (\lambda E_{\alpha_i} - \frac{1}{\lambda} E_{-\alpha_i}) e^{\alpha_i \cdot \phi/2}$$

$$a_x = \frac{1}{2} H \cdot \partial_t \phi + \sum_{i=1}^{r} m_i (\lambda E_{\alpha_i} + \frac{1}{\lambda} E_{-\alpha_i}) e^{\alpha_i \cdot \phi/2}.$$  \hspace{1cm} (22)

In (22), $H$ is the Cartan subalgebra of $g$ and $E_{\pm \alpha_i}$ are the step operators corresponding to the simple roots, their negatives, or to $\pm \alpha_0$. The important particular property which allows the Lax pair to work is the fact that the step operators for simple roots satisfy the following

$$[E_{\alpha_i}, E_{-\alpha_j}] = 0, \hspace{0.5cm} i \neq j = 0, 2, \ldots, r$$

together with the basic commutation relations

$$[H, E_{\pm \alpha_i}] = \pm \alpha_i E_{\pm \alpha_i}, \hspace{0.5cm} [E_{\alpha_i}, E_{-\alpha_i}] = \frac{2\alpha_i}{\alpha_i^2} \cdot H.$$  \hspace{1cm} (23)

Armed with these, and a suitable choice of $m_i$ (it is an exercise for you to work it out), the ‘curvature’ built from $a_t$, $a_x$, that is

$$F_{tx} = \partial_t a_x - \partial_x a_t + [a_t, a_x],$$
vanishes if and only if the field equations [21] hold independently of the value of the spectral parameter \( \lambda \).

The Lax pair may be used to generate conserved quantities (see [30]) since the quantity \( Q(\lambda) \) defined by parallel transport over the range \(-\infty < x < \infty\),

\[
Q = \text{tr} \left( P \exp \int_{-\infty}^{\infty} dx a_x \right),
\]

is conserved automatically as a consequence of the zero curvature condition. Its coefficients in an expansion in powers of \( \lambda \) are individually conserved (and indeed are in involution). The details of all this are independently interesting but cannot be pursued here. Instead, a way to develop this idea when there is a boundary (or a defect) will be sketched next.

7 Lax pair with boundary conditions

One needs to proceed slightly differently according to the context. For field theories with a boundary one way to develop this theme is along the lines of reference [10]; on the other hand, for field theories with a defect appropriate references would be [19, 20]. In this lecture, the boundary case will be considered in detail, really to give a flavour of the kind of mathematics involved and to show how constraints on the boundary potential arise. Other cases will not be covered in any detail at all.

For theories with a boundary, it was found to be convenient to construct a field theory on two overlapping halves of the \( x \)-axis: \( R_{\pm} \) defined as follows. The half-line \( R_- \) consists of the portion \(-\infty < x \leq b\) and the half-line \( R_+ \) is the portion \( a \leq x < \infty\), where \( a < 0 < b \). Clearly the two portions overlap on the region \([a, b]\). The field in \( x \geq b \) is defined in terms of the field in \( x \leq a \) via a reflection principle:

\[
\phi(x) = \phi(a + b - x), \quad x \geq b.
\]  

In the case of a defect, no such reflection principle would be imposed, of course.

The next step is to modify the components of two-dimensional gauge
gauge field entering the Lax pair by setting in the two overlapping regions

\[ R_- : \hat{a}^-_t = a_t - \frac{1}{2} \theta(x - a)(\partial_x \phi + \frac{\partial B}{\partial \phi}) \cdot \mathbf{H}, \]
\[ \hat{a}^-_x = \theta(a - x) a_x, \]
\[ R_+ : \hat{a}^+_t = a_t - \frac{1}{2} \theta(b - x)(\partial_x \phi - \frac{\partial B}{\partial \phi}) \cdot \mathbf{H}, \]
\[ \hat{a}^+_x = \theta(x - b) a_x, \]

where \( \theta(x) \) is the usual step function. It is left as an exercise to verify that these do indeed constitute a Lax pair whose zero curvature condition supplies not only the field equations but also the boundary conditions in the two separate regions. However, on the overlap it is clear \( \hat{a}^\pm_x \) vanish identically, by design, and therefore zero curvature requires each of the components \( \hat{a}^\pm_t \) to be constant (i.e., independent of \( x \)), but not necessarily equal. To put it another way, these two quantities need not be equal provided they are related to each other by a gauge transformation. In other words, there should be a group element \( \mathcal{K} \), possibly depending upon \( t \), with the property

\[ \partial_t \mathcal{K} = \mathcal{K} \hat{a}^+_t - \hat{a}^-_t \mathcal{K}, \quad a \leq x \leq b. \]  \hspace{1cm} (25)

Then, provided this is the case, the quantity

\[ Q = \text{tr} \left( P \exp \left\{ \int_{-\infty}^a dx \hat{a}^-_x \right\} \mathcal{K} P \exp \left\{ \int_b^\infty dx \hat{a}^+_x \right\} \right), \]  \hspace{1cm} (26)

will be conserved. Because of the reflection principle, the part involving \( a^+_x \) can be reinterpreted as a parallel transport back along the left half-line. In fact, the expression \( \text{(26)} \) is the starting point for Sklyanin’s analysis \( [27] \). However, following \( [10] \) instead, and making a mild assumption, it is possible to calculate \( \mathcal{K} \) and in the process determine the boundary potential \( B \).

Suppose \( \mathcal{K} \) does not depend on the fields, and suppose further that

\[ \partial_0 \mathcal{K} = 0. \]

At first sight these appear to be strong assumptions. However, an alternative, independent approach, investigating the low spin conserved charges, leads to precisely the same conclusions. For that reason these assumptions are actually quite mild and the boundary potential derived from them seems to be rather general.
Making these two assumptions and using the explicit expressions for the two gauge components $a_i^\pm$, equation (25) becomes the following

$$\frac{1}{2} \left[ \mathcal{K}, \frac{\partial B}{\partial \phi} \cdot H \right]_+ = - \left[ \mathcal{K}, \sum_i \lambda_i \left( \lambda E_{\alpha_i} - \frac{1}{\lambda} E_{-\alpha_i} \right) e^{\alpha_i \phi/2} \right]. \quad (27)$$

It is worth noting, first of all, that there is an anti-commutator on the left hand side and a commutator on the right, and secondly, that although $\mathcal{K}$ depends upon the spectral parameter $\lambda$, the boundary potential $B$ and of course the fields $\phi$ do not. These facts are powerful properties of equation (27).

First of all, if $\mathcal{K} = 1$ the commutator on the right hand side of (27) vanishes identically, while the anti-commutator on the left hand side vanishes only provided

$$\frac{\partial B}{\partial \phi_a} = 0.$$ 

Thus $\mathcal{K} = 1$ is equivalent to the Neumann condition

$$\partial_x \phi_a = 0.$$ 

On the other hand, suppose $\mathcal{K}$ is well-defined at $\lambda = 0$. Then $\mathcal{K}(0)$ will have to commute with all $E_{-\alpha_i}$, otherwise the second term on the right hand side of (27) would make no sense. Hence, $\mathcal{K}(0)$ is a central element of the group and, since (27) is linear in $\mathcal{K}$, one might as well take $\mathcal{K}(0) = 1$. In that case, the group element $\mathcal{K}$ should have an expansion of the form

$$\mathcal{K} = e^{\sum_{n=1}^{\infty} \lambda^n k_n}.$$ 

Using this, equation (27) can be solved iteratively.

For the simplest case $g = a_1$, it is convenient to take $\alpha_1 = \alpha = -\alpha_0$ and proceed directly to find

$$\mathcal{K}(\lambda) = I + \frac{\lambda}{1 - \lambda^4} \begin{pmatrix} 0 & b_1 - \lambda^2 b_0 \\ b_0 - \lambda^2 b_1 & 0 \end{pmatrix},$$ 

with the corresponding boundary potential given by

$$B = b_1 e^{\alpha \phi/2} + b_0 e^{-\alpha \phi/2}.$$ 

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This is, up to rescalings by the bulk coupling, in agreement with (9). Hopefully, all the signs and so on are correct in these expressions but, in any case, it is an exercise to check it! To do so, it is helpful to use the basis

\[ H = \frac{\alpha}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E_\alpha = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_{-\alpha} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \alpha^2 = 2. \]

Returning to the general case consider the terms of \( O(1) \). Balancing these on both sides of (27) gives

\[
\frac{\partial B}{\partial \phi} \cdot H = \left[ k_1, \sum_{i=0}^{r} m_i E_{-\alpha_i} e^{\alpha_i \cdot \phi/2} \right] - \left[ k_1, \sum_{i=0}^{r} m_i E_{\alpha_i} e^{\alpha_i \cdot \phi/2} \right]
\]

where \( m_i^2 = n_i \alpha_i^2 / 8 \). Using the Lie algebra commutation relations (28) it is clear the only solution to this must be to take

\[ k_1 = \sum_{i=0}^{r} c_i E_{\alpha_i}, \]

where \( c_i, i = 0, 1, \ldots, r \), are a set of constants, and

\[
\frac{\partial B}{\partial \phi} = \sum_{i=0}^{r} m_i \alpha_i c_i \frac{2}{\alpha_i^2} e^{\alpha_i \cdot \phi/2}.
\]

Thus, on integrating, the characteristic expression for the boundary potential follows.

\[ B = \sum_{i=0}^{r} b_i e^{\alpha_i \cdot \phi/2} \]

with \( b_i = \sqrt{2n_i \alpha_i^2} c_i \). At this stage it appears as though there are \( r + 1 \) free parameters \( b_i \). However, there are surprises yet to come.

At the next order, \( O(\lambda) \), it is not hard to check that \( k_2 \equiv 0 \). (You should discover that \( k_2 \) has to commute with \( E_{-\alpha_i}, i = 0, \ldots, r \) and therefore contributes a factor in \( \mathcal{K} \) which commutes with everything else. Since, (27) is homogeneous in \( \mathcal{K} \) any such factor can be scaled out, effectively setting \( k_2 \) to zero.) However, that is not yet the end of the story because at \( O(\lambda^2) \) there is a serious-looking equation for \( k_3 \) in terms of \( k_1 \). In detail, it is

\[
[k_3, m_i E_{-\alpha_i}] = \left[ k_1, m_i E_{\alpha_i} + \frac{b_i}{24} [k_1, \alpha_i \cdot H] \right], \quad i = 0, 1, \ldots, r.
\]

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A case by case analysis of (29) is provided in [10] and there are significant differences between the different choices of Lie algebra data. One important class of root systems is ‘simply-laced’, meaning that each root of the Lie algebra has the same length (conventionally taken to be $\sqrt{2}$) and for this class, the consequences of (29) are relatively simple and striking.

Given that $k_1$ according to eq (28) is composed of step operators corresponding to level one roots (the simple roots and, modulo the Coxeter number, $\alpha_0$), and given that the levels must balance across the equation (29), it should be clear that $k_3$ ought to have the form

$$k_3 = \sum_{\text{level 3 roots } \beta} d_\beta E_\beta,$$

where the coefficients $d_\beta$ are to be determined. Using the expressions for $B$ and $k_1$ previously obtained leads not only to expressions for $d_\beta$ but also further constraints on the coefficients $b_i$, $i = 0, \ldots, r$. In fact, for the simply-laced cases, one discovers $b_i^2 = 4n_i$ for every $i = 0, \ldots, r$ - with the only exception being the simplest case $a_1$ (which has no level three roots anyhow). This curious fact was first pointed out for the $a_n$ series of cases in [9] by examining the low spin conserved quantities. To see why this is so, evaluate (29) carefully using the commutation relations of the Lie algebra to obtain

$$\sum_\beta d_\beta m_i \epsilon(\beta, -\alpha_i) E_{\beta - \alpha_i} = \sum_j c_j m_i \epsilon(\alpha_j, \alpha_i) E_{\alpha_j + \alpha_i}$$

$$- \sum_{k \neq l} \frac{b_{ij} c_k c_l}{24} \alpha_i \cdot \alpha_l \epsilon(\alpha_k, \alpha_l) E_{\alpha_k + \alpha_l}.$$  

(30)

Now, for a simply-laced root system $E_{\alpha_j + \alpha_l}$ cannot appear on the left hand side, since that would require $\alpha_j + 2\alpha_l$ to be a root, which it cannot be. Therefore, the coefficient of $E_{\alpha_j + \alpha_l}$ must also vanish on the right hand side, which requires

$$m_i + \frac{b_{ij} c_i}{24} (\alpha_i \cdot \alpha_j - \alpha_i \cdot \alpha_i) = 0.$$

However, the vector $\alpha_j + \alpha_l$ is only a root when $\alpha_i$ and $\alpha_j$ are adjacent on the Dynkin diagram for $g$, and then $\alpha_i \cdot \alpha_j = -1$. Hence, $b_{ij} c_i = 8m_i$ or, $b_i^2 = 4n_i$, for each $i = 0, 1, \ldots, r$.

In other words for each of the cases $a_r$, ($r \geq 2$), $d_r$, ($r \geq 4$), $e_r$, ($r = 6, 7, 8$) the possible boundary potentials for which integrability is maintained
consist of a discrete set \((2^r+1)\) choices) with no free parameters at all. This is very surprising. One might have thought a priori that the more complex the root system the more freedom there might be. In fact, it is only in some of the non-simply-laced cases where some free parameters remain. However, even there, the set of possibilities is severely restricted.

8 Brief discussion

In just two lectures it is difficult to do justice to a topic which by now has a sizeable literature and which is interesting within mathematical physics both from a classical and a quantum field theoretical point of view. Moreover, although there has been interest in Toda theory within the geometry community, largely because of the relationship with harmonic maps, it is not yet clear to what extent the phenomena discussed briefly here will find a geometrical context. Unfortunately, there is also the problem of language! Nevertheless, a possible starting point might be the book by Guest [31]. There is a vast literature on the subject of harmonic maps and perhaps, somewhere, there is a natural home for the affine Toda field theories with one or two boundaries.

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