Entropy of near-extremal dyonic black holes

P. Mitra*
Saha Institute of Nuclear Physics
Block AF, Bidhannagar
Calcutta 700 064, INDIA

hep-th/9712252

Abstract

In this note it is shown that near-extremal four dimensional dyonic black holes, where the dilaton is not constant, can be described by a microscopic model consisting of a one-dimensional gas of massless particles.
1 Introduction

Traditional black hole thermodynamics taught us that a black hole has an entropy given by a quarter of the area of the horizon, but the statistical mechanics of this entropy remained something of a puzzle. This is because of the uncertainties involved in the quantization of gravity. The semiclassical interpretations of the area law were not microscopic enough. Recently, however, in the context of the string theory approach to gravity, a description of black hole states in terms of D-branes has been found to allow a derivation of the area law from a counting of quantum states [1].

Since then, there have been many studies on the microscopic description of black holes. Many of these studies involved extremal and near-extremal Reissner-Nordström black holes in different dimensions. The simpler Schwarzschild black hole could not be accommodated in those studies essentially because it has no extremal version. The so-called dilatonic black hole may come to mind in this connection, but though there are extremal dilatonic black holes, these have zero horizon area. A satisfactory microscopic interpretation of such black holes has not been possible. The next in the order of increasing complication is probably a dyonic black hole with both electric and magnetic charges. Dyonic string-based black holes have often been considered in the literature. Those with at least two magnetic and two electric charges have horizons of non-zero area. In fact the Reissner-Nordström black hole is a special case of this class of black holes with the two magnetic and two electric charges all equal.

The explicit dyonic solutions of the low energy field equations arising from toroidally compactified heterotic string theory were given for the extremal case in [2] and for the non-extremal case in [3]. In [4] it was argued that the number of microscopic (string) states corresponding to a macroscopic state of the extremal black hole would give rise to an entropy equal to a quarter of the area of the horizon, that is, what one could expect from traditional black hole thermodynamics. However, as pointed out in [5], the detailed argument in [4] involved a clustering hypothesis which is known to fail even for the simpler black holes which are only electrically charged. This made the claim quite uncertain. It was further pointed out in [6] that the semiclassical value of the entropy for only a non-extremal black hole is a quarter of the area, that is normally not the case for an extremal black hole. In fact, if the quantum extremal black hole is defined by starting with a classical extremal black hole
and then quantizing it, the (semiclassical) entropy appears to be of degree one in the charges, unlike the area, which is of degree two.

Meanwhile, improved techniques of counting D-brane states indicated that the extremal dyonic black hole indeed can be assigned a microscopic entropy equal to a quarter of the area \( \text{[3]} \). This would seem to be in disagreement with the semiclassical result \( \text{[5]} \), but it was soon realized that the order of the processes of quantization and extremalization is important \( \text{[7]} \), so that if instead of quantizing an extremal, classical black hole, the extremality was imposed after functional integration over both extremal and non-extremal topologies, a different result could be obtained. In the case of Reissner-Nordström black holes, the area formula was explicitly derived in this way \( \text{[7]} \). This indicated that the microscopic area formula could be accommodated with semiclassical ideas even for extremal black holes.

The interest then shifts to non-extremal dyonic black holes. These black holes are dilatonic, and do not fall into the category \( \text{[8]} \) of black hole solutions with constant dilaton, which allow reliable microscopic state counting away from extremality. Although the dilaton is not in general constant in these black holes, it does not diverge near the horizon, so it is not entirely unreasonable to hope for a microscopic calculation to make sense even in the near-extremal case. Indeed, such a calculation has been given for some special ranges of values of the charges \( \text{[9]} \). The purpose of the present note is to confirm that near-extremal dyonic black holes with generic values of the charges can be described by microscopic models.

We shall not make explicit use of any string model to fit the thermodynamics of near-extremal dyonic black holes. As pointed out in \( \text{[10]} \), a gas of massless particles in one space dimension can be used for this purpose. This is essentially the same as using a conformal field theory, as suggested in \( \text{[11]} \). A detailed formulation of the agreement between the gas model and the Reissner-Nordström black hole was presented in \( \text{[12]} \), and it will be extended here to the case of dyonic black holes with generic values of the charges.

In section 2 the reader is reminded about dyonic black holes \( \text{[2, 3]} \). In section 3 the one-dimensional gas model is applied to these black holes. In the concluding discussion some remarks are made about the difference of the formulations given here and in \( \text{[14]} \).
2 Dyonic black holes

A simple four-dimensional dyonic black hole solution of toroidally compactified heterotic string theory can be described by the metric

\[ ds^2 = -\lambda dt^2 + \frac{dr^2}{\lambda} + R^2 d\Omega^2, \quad (1) \]

where

\[ \lambda = \frac{(r + b)(r - b)}{R^2}, \]
\[ R^4 = (r + \hat{Q}_1)(r + \hat{Q}_2)(r + \hat{P}_1)(r + \hat{P}_2), \]
\[ \hat{Q}_1 = \sqrt{Q_1^2 + b^2}, \text{ etc.} \quad (2) \]

It carries two independent electric charges \( Q_1, Q_2 \) with the same sign as well as two independent magnetic charges \( P_1, P_2 \) with the same sign, and there is a real positive parameter \( b \) which specifies the position of the horizon and measures the departure from extremality. The expressions for the 28 gauge fields to which the 28-component charges (with not all components independent in this solution) are coupled will be omitted for simplicity. The expression for the dilaton field is

\[ e^{2\phi} = \frac{(r + \hat{P}_1)(r + \hat{P}_2)}{(r + \hat{Q}_1)(r + \hat{Q}_2)}, \quad (3) \]

which shows that the dilaton does not in general vanish for these solutions, and is not even constant, but goes to zero asymptotically as \( r \to \infty \).

The horizon of such a black hole has a finite area given by

\[ A = 4\pi \sqrt{(b + \hat{P}_1)(b + \hat{P}_2)(b + \hat{Q}_1)(b + \hat{Q}_2)}, \quad (4) \]

and it remains nonvanishing in the extremal limit provided all four charges are nonvanishing. The Hawking temperature is

\[ T_H = \frac{2b}{A}. \quad (5) \]

It vanishes in the extremal limit. The ADM mass of the black hole is

\[ M = \frac{1}{4}(\hat{Q}_1 + \hat{Q}_2 + \hat{P}_1 + \hat{P}_2) \]
\[ \geq M_{ex} = \frac{1}{4}(|Q_1 + Q_2| + |P_1 + P_2|). \quad (6) \]
3 Microscopic model for near-extremal black holes

In this section we seek to use the one-dimensional gas of massless particles (cf. [10, 12]) as a model for the dyonic black holes. The model can work for black holes in any number of dimensions, though it will be used here only for four dimensional black holes. The particles can be either left-moving or right-moving – there is no mixing between the two types. Both bosons and fermions can be present. If the total length of the one-dimensional space is $L$, the entropy and the energy are given by

\[
S = \frac{\pi L}{6\hbar} [n_L T_L + n_R T_R],
\]

\[
E = \frac{\pi L}{12\hbar} [n_L T_L^2 + n_R T_R^2],
\]

(7)

where $n_L(n_R)$ is the number of left(right)-moving bosons plus half the corresponding number of fermions, because a fermion contributes only a half of the contribution a boson makes to the free energy. In the absence of interactions, the left and right degrees of freedom are independent, and the corresponding temperatures can be different. The effective temperature may be defined by $(\frac{dS}{dE})^{-1}$, the differentiation being carried out at constant momentum, i.e., constant difference between $E_L$ and $E_R$. This leads to a temperature

\[
T = \frac{2T_L T_R}{T_L + T_R}
\]

(8)

equal to the harmonic mean of $T_L$ and $T_R$. If $n_L = n_R = n$, these equations get somewhat simplified and by eliminating $T_L, T_R$, one gets the single relation

\[
E = \frac{\pi n L}{12\hbar} \left[ \left( \frac{6hS}{\pi n L} \right)^2 - \frac{6hST}{\pi n L} \right].
\]

(9)

To compare these quantities with those for a near-extremal dyonic black hole in four dimensions, it is necessary to fix two parameters of the gas to be equal to the corresponding parameters of the black hole, and check how the third parameter of the gas compares with that parameter for the black hole. Thus, we could set the temperatures and energies to be equal
and compare the entropies. However, as the entropy of the gas enters the energy quadratically, the expression for the entropy in terms of the energy involves a square root. It can be handled, but it is much simpler to match the temperatures and the entropies and compare the energies. Thus we put

\[
T = T_H(b, Q_1, Q_2, P_1, P_2)
\]

\[
= \frac{b}{2\pi \sqrt{\prod |Q|}} \left( 1 - \frac{b}{2} \sum \frac{1}{|Q|} + \frac{b^2}{8} \left( \sum \frac{1}{|Q|} \right)^2 \right) + \cdots,
\]

\[
S = \frac{A(b, Q_1, Q_2, P_1, P_2)}{4}
\]

\[
= \pi \sqrt{\prod |Q|} \left( 1 + \frac{b}{2} \sum \frac{1}{|Q|} + \frac{b^2}{8} \left( \sum \frac{1}{|Q|} \right)^2 \right) + \cdots \tag{10}
\]

in (9) to get

\[
E = \frac{3\pi \prod Q}{nL} + b \left( \frac{3\pi \prod Q \sum \frac{1}{|Q|}}{nL} - \frac{1}{4} \right) + b^2 \cdot \frac{3\pi \prod Q (\sum \frac{1}{|Q|})^2}{2nL} + \cdots \tag{11}
\]

Here sums and products over \(Q\) are understood to include both the electric and both the magnetic charges and \(b\) is understood to be small. This expression for the energy of the gas shows a quadratic increase with \(b\) (if higher degree terms are neglected), which is similar to the behaviour of the blackhole mass

\[
4M = \sum |Q| + \frac{b^2}{2} \sum \frac{1}{|Q|} + \cdots, \tag{12}
\]

but it involves the characteristic \(nL\) of the model and is not manifestly identical to \(M\). However, for consistency, we must require that the temperature \(\frac{dE}{dS}\) vanish for \(b = 0\), which is the point of extremality, at which the temperature \(T_H\) vanishes. This yields the relation

\[
\frac{3\pi \prod Q \sum \frac{1}{|Q|}}{nL} = \frac{1}{4}, \tag{13}
\]

between the product \(nL\) and the parameters \(Q_1, Q_2, P_1, P_2\) characterizing the family of black holes being considered. \(E\) and \(M\) then agree in both the \(b\) and the \(b^2\) terms, \(i.e.,\) for all \(b\) for which \(b^3\) can be neglected:

\[
E = M + \frac{1}{4} \left( \sum \frac{1}{|Q|} - \sum |Q| \right) + O(b^3). \tag{14}
\]
The difference between $E$ and $M$ represents a zero-point shift depending only on the charges of the black hole and independent of $b$. The gas of massless particles then can be regarded as a model, as far as thermodynamics is concerned, for the family of near-extremal dyonic black holes with a fixed set of values of the charges but varying $b$.

4 Conclusion

A microscopic model has thus been constructed for near-extremal dyonic black holes in terms of the one dimensional gas suggested in [10]. It was of course envisaged in [10] that the model would work for a general class of black holes, but it was noted there that the energy of the gas and the mass of the general black hole with the same temperature and entropy could be made to agree up to a zero point energy, which is apparently dependent on the non-extremality parameter, and thus not constant for the family of near-extremal black holes. The zero point shift in the present formulation is independent of the non-extremality parameter, so that the energy differences match. This is what was demonstrated for the Reissner-Nordström case in [12] and the same behaviour has been shown here for generic dyonic black holes. The new result is to be regarded as a generalization of the earlier one because, as pointed out above, the Reissner-Nordström black hole is a special case of a dyonic black hole. Applications to the Schwarzschild black hole or the dilatonic one, which are also special cases, are not possible because in the former case there is no extremal limit, and in the latter case the limit is singular.

It is instructive to compare the formulation of [10] with that given here and understand why the zero point energies seem to behave differently even though the dyonic black hole is included in the class of black holes considered in [10]. That class is characterized, in four dimensions, for the case of four charges, by the metric

$$ds^2 = -\tilde{\lambda}dt^2 + \frac{d\tilde{r}}{\lambda} + \tilde{R}^2 d\Omega^2,$$

(15)

with

$$\tilde{\lambda} = \frac{\tilde{r}(\tilde{r} - r_0)}{\tilde{R}^2}.$$
\[ \hat{R}^4 = (\hat{r} + r_1)(\hat{r} + r_2)(\hat{r} + r_3)(\hat{r} + r_4). \]  

(16)

To put the dyonic metric (2) in this form, one matches

\[ r^2 - b^2 = \hat{r}(\hat{r} - r_0), \quad r + \hat{Q}_1 = \hat{r} + r_1, \quad etc. \]  

(17)

These imply that

\[ r + b = \hat{r}, \quad 2b = r_0, \quad \hat{Q}_1 - b = r_1, \quad etc. \]  

(18)

The first of these simply means that \( \hat{r} \) is translated with respect to \( r \), but the last one indicates that \( r_1 \) is not just the charge \( Q_1 \) but \( \sqrt{Q_1^2 + b^2 - b} \), which involves both the charge \( Q_1 \) and the non-extremality parameter \( b \). It is because of this that the dependence of the zero point energy on the non-extremality parameter seems to be different in the two formulations. There is in fact no difference, and one can identify the family of near-extremal black holes with fixed charges and the thermal states of the gas.

**Acknowledgment**

It is a pleasure to acknowledge discussions with Amit Ghosh on the gas model.

**References**

[1] A. Strominger and C. Vafa, Phys. Lett. **B379**, 99 (1996)

[2] M. Cvetič and D. Youm, Phys. Rev. **D53**, 584 (1996)

[3] M. Cvetič and D. Youm, [hep-th/9512127]; D. Jatkar, S. Mukherji and S. Panda, Nucl. Phys. **B484**, 223 (1997)

[4] F. Larsen and F. Wilczek, Phys. Lett. **B375**, 37 (1996)

[5] A. Ghosh and P. Mitra, Mod. Phys. Lett. **A11**, 2933 (1996)

[6] J. Maldacena and A. Strominger, Phys. Rev. Lett. **77**, 428 (1996)

[7] A. Ghosh and P. Mitra, Phys. Rev. Lett. **78**, 1858 (1997)
[8] I. Klebanov and A. Tseytlin, Nuc. Phys. B475, 164 (1996)

[9] G. Horowitz, D. Lowe and J. Maldacena, Phys. Rev. Lett. 77, 430 (1996)

[10] S. P. de Alwis and K. Sato, Phys. Rev. D55, 6181 (1997)

[11] J. Maldacena and A. Strominger, Phys. Rev. D56, 4975 (1997)

[12] P. Mitra, hep-th/9704201, to appear in Proceedings of Workshop on Frontiers of Quantum Field Theory, Quantum Gravity and Strings, held at Puri in December 1996.