When Do Gomory-Hu Subtrees Exist?

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Abstract

Gomory-Hu (GH) Trees are a classical sparsification technique for graph connectivity. It is one of the fundamental models in combinatorial optimization which also continually finds new applications, most recently in social network analysis. For any edge-capacitated undirected graph $G = (V, E)$ and any subset of terminals $Z \subseteq V$, a Gomory-Hu Tree is an edge-capacitated tree $T = (Z, E(T))$ such that for every $u, v \in Z$, the value of the minimum capacity $uv$ cut in $G$ is the same as in $T$. Moreover, the minimum cuts in $T$ directly identify (in a certain way) those in $G$. It is well-known that we may not always find a GH tree which is a subgraph of $G$. For instance, every GH tree for the vertices of $K_{3,3}$ is a 5-star. We characterize those graph and terminal pairs $(G, Z)$ which always admit such a tree. We show that these are the graphs which have no terminal-$K_{2,3}$ minor. That is, no $K_{2,3}$ minor whose vertices correspond to terminals in $Z$. We also show that the family of pairs $(G, Z)$ which forbid such $K_{2,3}$ “Z-minors” arises, roughly speaking, from so-called Okamura-Seymour instances. More precisely, they are subgraphs of $Z$-webs. A $Z$-web is built from planar graphs with one outside face which contains all the terminals and each inner face is a triangle which may contain an arbitrary graph. This characterization yields an additional consequence for multiflow problems. Fix a graph $G$ and a subset $Z \subseteq V(G)$ of terminals. Call $(G, Z)$ cut-sufficient if the cut condition is sufficient to characterize the existence of a multiflow for any demands between vertices in $Z$, and any edge capacities on $G$. Then $(G, Z)$ is cut-sufficient if and only if it is terminal-$K_{2,3}$ free.

1 Introduction

The notion of sparsification is ubiquitous in applied mathematics and combinatorial optimization is no exception. For instance, shortest paths to a fixed root vertex in a graph $G = (V, E)$ are usually stored as a tree directed towards the root. Another classical application is that of Gomory-Hu (GH) Trees [4] which encode all of the minimum cuts of an edge-capacitated undirected graph $G = (V, E)$, with capacities $c : E \rightarrow \mathbb{R}^+$. For each $s, t \in V$, we denote by $\lambda(s, t)$ the capacity of a minimum cut separating $s$ and $t$. Equivalently $\lambda(s, t)$ is the maximum flow that can be sent between $s, t$ in $G$.

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Definition 1. Let \( G \) be a graph-terminal pair \((G,Z)\). A \( G \)-tree is a spanning edge-capacitated tree \( T = (V(T),E(T)) \) together with a capacity function \( c' : E' \to \mathbb{R}^+ \). Any edge \( e \in E' \) induces a fundamental cut \( G(A,B) \), where \( A \) and \( B \) are the vertex set of the two components of \( T \setminus e \). Here we use \( G(A,B) \), or occasionally \( \delta(A) = \delta(B) \), to denote the associated cut in \( G \), that is, \( G(A,B) = \{ e \in E(G) : e \text{ has one endpoint in } A \text{ and the other in } B \} \).

A spanning edge-capacitated tree for \( G \) is a spanning tree \( T = (V,E') \) together with a capacity function \( c' : E' \to \mathbb{R}^+ \). Any edge \( e \in E' \) induces a fundamental cut \( G(A,B) \), where \( A \) and \( B \) are the vertex set of the two components of \( T \setminus e \). Here we use \( G(A,B) \), or occasionally \( \delta(A) = \delta(B) \), to denote the associated cut in \( G \), that is, \( G(A,B) = \{ e \in E(G) : e \text{ has one endpoint in } A \text{ and the other in } B \} \).

Theorem 1. Let \( T \) be a spanning edge-capacitated tree. An edge \( e = ab \in E(T) \) is encoding if its fundamental cut \( G(A,B) \) is a minimum ab-cut and its capacity is \( c'(e) \), that is, \( c(G(A,B)) = c'(e) \).

A Gomory-Hu tree (GH tree for concision) is a spanning edge-capacitated tree whose edges are all encoding. In this case, it is an exercise to prove that any minimum cut can be found as follows. For \( s,t \in V \) we have that \( \lambda(s,t) = \min \{ c'(e) : e \in T(st) \} \), where \( T(st) \) denotes the unique path joining \( s \) and \( t \) in \( T \).

In some applications we only specify a subset \( Z \subseteq V \) for which we need cut information. We refer to \( Z \) as the terminals of the instance. The Gomory-Hu method allows one to store a compressed version of the GH Tree which only captures cut values \( \lambda(s,t) \) for \( s, t \in Z \). Namely, a GH \( Z \)-Tree has \( V(T) = Z \).

It is well-known that there may not always exist a GH tree which is a subgraph of \( G \). For instance, every GH tree for the vertices of \( K_{3,3} \) is a 5-star (cf. [S]). Our first main result characterizes the graphs which admit GH subtrees. More precisely, we say that \( G \) has the GH Property if any subgraph \( G' \) of \( G \) with any edge-capacity function \( c \) has a Gomory-Hu tree \( T \) that is a subgraph of \( G' \).

Theorem 1. \( G \) has the GH Property if and only if \( G \) is the 1-sum of outerplanar and \( K_4 \) graphs.

We then turn our attention to the generalized version where we are given a graph-terminal pair \((G,Z)\). Let \( G \) be endowed with edge capacities. A GH \( Z \)-Tree is then a capacitated tree \( T = (V(T),E(T)) \) (cf. [S]). Formally, the vertices of \( T \) form a partition \( \{ B(v) : v \in Z \} \) of \( V(G) \), with \( z \in B(z) \) for all \( z \in Z \). Hence Definition 1 extends as follows. An edge \( B(s)B(t) \) of \( T \) is encoding if its fundamental cut \( (B(S),B(U)) \) induces a minimum \( st \)-cut in \( G \). As before, if all edges are encoding, then \( T \) determines the minimum cuts for all pairs \( s,t \in Z \).

We characterize those pairs \((G,Z)\) which admit a GH \( Z \)-tree as a minor for any edge capacities on \( G \). We call such a tree a GH \( Z \)-minor (a formal definition is delayed to Section 5).

Our starting point is the following elementary observation.

Proposition 1. \( K_{2,3} \) has no Gomory-Hu tree that is a subgraph of itself.

Even if GH \( Z \)-minors always existed in a graph \( G \), it may still contain a \( K_{2,3} \) minor. The proposition implies, however, that it should not have a \( K_{2,3} \) minor where all nodes in the minor are terminals. Given a set \( Z \) of terminals, we say that \( H \) is a terminal minor, or \( Z \)-minor, of \( G \) if nodes
of \( V(H) \) correspond to terminals of \( G \). In other words, it is a minor such that each \( v \in V(H) \) arises by contracting a connected subgraph which contains a vertex from \( Z \). Hence a natural necessary condition for \( G \) to always contain \( GH \) \( Z \)-minors is that \( G \) must not contain a terminal-\( K_{2,3} \) minor. We show that this is also sufficient (see Section 5 for the formal statement).

**Theorem 2.** Let \( Z \subseteq V \). \( G \) admits a Gomory-Hu tree that is a minor, for any capacity function, if and only if \((G, Z)\) is a terminal-\( K_{2,3} \) minor free graph.

Establishing the sufficiency requires a better understanding of terminal minor-free graphs. We show that the family of pairs \( G, Z \) which forbid such terminal-\( K_{2,3} \) minors arises precisely as subgraphs of \( Z \)-webs. \( Z \)-webs are built from planar graphs with one outside face which contains all the terminals \( Z \) and each inner face is a triangle to which we may add an arbitrary graph inside connected to the three vertices; these additional arbitrary graphs are called 3-separated subgraphs. Subgraphs of \( Z \)-webs are called Extended Okamura-Seymour Instances.

**Theorem 3.** Let \( G \) be a 2-connected terminal-\( K_{2,3} \) minor free graph. Then either \( G \) has at most 4 terminals or it is an Extended Okamura-Seymour Instance.

This immediately implies the following.

**Corollary 1.** \( G \) is terminal-\( K_{2,3} \) free if and only if for any 2-connected block \( B \), the subgraph obtained by contracting every edge not in \( B \) is terminal-\( K_{2,3} \) free.

These results also yield the following consequence for multflow problems. Let \( G, H \) be graphs such that \( V(H) \subseteq V(G) \). Call a pair \((G, H)\) cut-sufficient if the cut condition is sufficient to characterize the existence of a multflow for any demands on edges of \( H \) and any edge capacities on \( G \). If \( Z \subseteq V(G) \), we also call \((G, Z)\) cut-sufficient if \((G, H)\) is cut-sufficient for any simple graph on \( Z \).

**Corollary 2.** \((G, Z)\) is cut-sufficient if and only if it is terminal-\( K_{2,3} \) free.

One can compare this to results of Lomonosov and Seymour ([6, 10], cf. Corollary 72.2a [8]) which characterize the class of demand graphs \( H \) such that every supply graph \( G \) “works”, i.e. for which \((G, H)\) is cut-sufficient for any graph \( G \) with \( V(H) \subseteq V(G) \). They prove that any such \( H \) is (a subgraph of) either \( K_4 \), \( C_5 \) or the union of two stars. A related question asks for which graphs \( G \) is it the case that \((G, H)\) is cut-sufficient for every \( H \) which is a subgraph of \( G \); Seymour [11] shows that this is precisely the class of \( K_5 \) minor-free graphs. We refer the reader to [3] for discussion and conjectures related to cut-sufficiency.

The paper is structured as follows. In the next section we prove that every outerplanar instance has a \( GH \) tree which is a subgraph. In Section 3 we present the proof of Theorem 1. In Section 4 we provide the proofs for Theorem 3 and Corollary 2. Section 5 wraps up with a proof of Theorem 2.
1.1 Some Notation and a Lemma

We always work with connected graphs and usually assume (without loss of generality) that the edge capacities $c(e)$ have been adjusted so that no two cuts have the same capacity. In particular, the minimum $st$-cut is unique for any vertices $s,t$. Moreover, we may assume any minimum cut $\delta(X)$ to be central, a.k.a a bond. That is, $G[X], G[V \setminus X]$ are connected. For any $X \subseteq V(G)$ we use shorthand $c(X)$ to denote the capacity of the cut $\delta(X)$, and if $Y \subseteq V(G)$, then $d(X,Y)$ denotes the sum of capacities for all edges with one endpoint in $X$, and the other in $Y$. We consistently use $c'(e)$ to denote the computed capacities on edges $e$ in some Gomory-Hu tree.

As we use the following lemma several times throughout we introduce it now.

**Lemma 1.** Let $t \in V(G)$ and $X,Y$ be disjoint subsets which induce respectively a minimum $xt$-cut and a minimum $yt$-cut where $x \in X, y \in Y$. For any non-empty subset $M$ of $V$ which is disjoint from $X \cup Y \cup \{t\}$, we have $d(M, V \setminus (X \cup Y \cup M)) > 0$.

**Proof.** We have
\[
c(M \cup X) + c(M \cup Y) = c(X) + c(Y) + 2d(M, V \setminus (X \cup Y \cup M)) < c(M \cup X) + c(M \cup Y) + 2d(M, V \setminus (X \cup Y \cup M))
\]
where the second inequality follows from the fact that $\delta(M \cup X)$ (respectively $\delta(Y \cup M)$) separates $t$ from $X$ (respectively $Y$) but $M \cup X \neq X$ (respectively $M \cup Y \neq Y$). \hfill \Box

2 Outerplanar graphs have Gomory-Hu Subtrees

**Theorem 4.** Any 2-connected outerplanar graph $G$ has a Gomory-Hu tree that is a subgraph of $G$.

**Proof.** Let $G$ be an outerplanar graph with outer cycle $C = v_1, v_2, \ldots, v_n$. As discussed in Section 1.1 we assume that no two cuts have the same capacity, so let $T$ be the unique Gomory-Hu tree of $G$. We want to prove that $T$ is a subgraph of $G$.

\footnote{This can be achieved in a standard way by adding multiples of $2^{-\delta}$ where $\delta = O(|E|)$.}
Notice that the shore of any min-cut in $G$ must be a subpath $v_i, v_{i+1}, \ldots, v_{j-1}, v_j$ (indices taken modulo $n$) because we may assume any min-cut $\delta(S)$ to be central (a.k.a. a bond), that is, both $S$ and $V - S$ induce connected subgraphs.

Let $v$ be any vertex and consider the fundamental cuts associated with the edges incident to $v$ in the Gomory-Hu tree. The shores (not containing $v$) of these cuts define a partition $X_1, X_2, \ldots X_k$ of $V \setminus \{v\}$ where each $X_i$ is a subpath of $C$. We may choose the indices such that $v, X_1, \ldots, X_k$ appear in clockwise order on $C$.

Claim 1. For each $i \in \{1, \ldots, k\}$, there is an edge in $G$ from $v$ to some vertex in $X_i$.

Proof. By contradiction, assume there is no edge from $v$ to $X_i$. Notice $i \notin \{1, k\}$ because of the edges of $C$. Let $j \in \{1, \ldots, i - 1\}$ maximum with $d(v, X_j) \neq \emptyset$, and let $j' \in \{i + 1, \ldots, k\}$ minimum with $d(v, X_{j'}) \neq \emptyset$, hence $d(v, M) = \emptyset$ where $M := X_{j+1} \cup X_{j+2} \cup \cdots \cup X_{j'-1}$. By taking $X = X_j, Y = X_{j'}, t = v$, Lemma 1 implies that $d(M, V \setminus (X_j \cup X_{j'} \cup M)) > 0$. However, outerplanarity and the existence of edges from both $X_j$ and $X_{j'}$ to $v$, imply that there is an edge between $v$ and $M$, cf. Figure 2. This contradicts the choice of $i, j$ or $j'$.

Let $xy \in E(T)$ be an edge of the Gomory-Hu tree. We must prove that $xy \in E(G)$. Let $\delta(X)$ be the fundamental cut associated with $xy$, with $x \in X$, define $Y = V \setminus X$. As in the preceding arguments we may use the fundamental cuts associated to edges incident to $x$ and partition $X \setminus \{x\}$ into min-cut shores $X_1, X_2, \ldots, X_k$; we do this by ignoring the one shore $Y$. Similarly, we may partition
$Y \setminus \{y\}$ into min-cut shores $Y_1, Y_2, \ldots, Y_l$. We can label these so that $X_1, X_2, \ldots, X_k, Y_1, \ldots, Y_l$ appear in clockwise order around $C$ - see Figure 2. There is also some $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, l\}$ such that $x$ is between $X_i$ and $X_{i+1}$ (or $Y_1$ if $i = k$) and $y$ is between $Y_j$ and $Y_{j+1}$ (or $X_1$ if $j = l$).

![Figure 1: An arbitrary edge $xy \in T$.](image)

By contradiction suppose $xy \notin E(G)$. By Claim 1 there is an edge $e$ from $x$ to $Y$, let $m \in \{1, \ldots, l\}$ such that $e \in d(x, Y_m)$. If $m \notin \{1, l\}$, by outerplanarity either $d(y, Y_1)$ or $d(y, Y_l)$ is empty; this contradicts Claim 1. By symmetry we may assume $e \in d(x, Y_1)$. By a similar argument there is an edge $e' \in d(y, X_1)$. By Claim 1 there are also two edges $e'' \in d(x, X_1)$ and $e''' \in d(y, Y_1)$.

Let $X' = \{x\} \cup X_2 \cup \ldots \cup X_k$ and $Y' = \{y\} \cup Y_2 \cup \ldots \cup Y_l$, $\delta(X')$ is a cut separating $x$ from $X_1$ and similarly $\delta(Y')$ separates $y$ from $Y_1$. As $\delta(X_1)$ is the fundamental cut between $x$ and $X_1$, we have that $c(X_1) < c(X')$, and similarly $c(Y_1) < c(Y')$. Now, because of the edges $e, e', e'', e'''$, by outerplanarity there is no edge between $X'$ and $Y'$, hence

$$c(X_1) + c(Y_1) = c(X') + c(Y') + 2c(X_1, Y_1) > c(X_1) + c(Y_1) + 2c(X_1, Y_1)$$

a contradiction.

\[\square\]

### 3 Which Instances have Gomory-Hu Subtrees?

The previous result leads to a characterization of graphs with the GH Property: that is, graphs whose capacitated subgraphs always contain a Gomory-Hu Tree as a subtree. In Section 5 we extend this result to the case where a subset of terminals is specified.

We start with a simple observation that $K_{2,3}$ does not have a GH subtree.

**Proposition** If $K_{2,3}$, when all edges have capacity 1, has no Gomory-Hu tree that is a subgraph of itself.
Proof. Let \( \{u_1, u_2\}, \{v_1, v_2, v_3\} \) be the bipartition. Since the minimum \( u_1, u_2 \) cut is of size 3, a GH tree should contain a \( u_1u_2 \) path all of whose edges have capacity at least 3. Suppose this path is \( u_1v_1u_2 \), then the tree’s fundamental cut associated with \( u_1v_1 \) should be a minimum \( u_1v_1 \)-cut. But this is impossible since \( \delta(v_1) \) is a cut of size 2.

This leads to the desired characterization.

**Theorem 1.** \( G \) has the GH Property if and only if \( G \) is the 1-sum of outerplanar and \( K_4 \) graphs.

**Proof.** First suppose that \( G \) is such a 1-sum. Each outerplanar block in this sum has the GH Property by Theorem 4. So consider a \( K_4 \) block and a subgraph \( G' \) with edge capacities. If \( G' \) is \( K_4 \), then clearly any GH tree is a subtree. Otherwise \( G' \) is a proper subgraph of \( K_4 \) and hence is outerplanar. It follows that each block has the GH Property. It is not hard to see that the 1-sum of Gomory-Hu trees of two graphs is a Gomory-Hu tree of the 1-sum of the graphs. Repeating this argument to the blocks we find that \( G \) itself satisfies the GH property.

Suppose now that a 2-connected graph \( G \) has the GH property. By our proposition, \( G \) has no \( K_{2,3} \) minor. Outerplanar graphs are graphs with forbidden minors \( K_{2,3} \) and \( K_4 \). Hence if \( G \) is not outerplanar, then it has a \( K_4 \) minor. Notice that any proper subdivision of \( K_4 \) contains a \( K_{2,3} \), as well as any graph built from \( K_4 \) by adding a path between two distinct vertices. Hence \( G \) must be \( K_4 \) itself. The result now follows.

4 Characterization of terminal-\( K_{2,3} \) free graphs

In this section we prove Theorem 3. Throughout, we assume we have an undirected graph \( G \) with terminals \( Z \subseteq V(G) \). We refer to \( G \) as being \( H \)-terminal free (for some \( H \)) to mean with respect to this fixed terminal set \( Z \).
We first check sufficiency of the condition of Theorem 3. Any graph with at most 4 terminals is automatically terminal-$K_{2,3}$ free and one easily checks that any extended Okamura-Seymour instance cannot contain a terminal-$K_{2,3}$ minor. Hence we focus on proving the other direction: any terminal-$K_{2,3}$ minor-free graph $G$ lies in the desired class. To this end, we assume that $|Z| \geq 5$ and we ultimately derive that $G$ must be an extended OS instance.

We start by excluding the existence of certain $K_4$ minors.

**Proposition 2.** If $|Z| \geq 5$ and $G$ has a terminal-$K_4$ minor, then $G$ has a terminal-$K_{2,3}$ minor.

**Proof.** Let $K_4^+$ be the graph obtained from $K_4$ by subdividing one of its edges. By removing the edge opposite to the subdivided edge, we see that $K_4^+$ contains $K_2,3$. Hence it suffices to prove that $G$ contains a terminal-$K_4^+$ minor.

Consider a terminal-$K_4$ minor on terminals $T' = \{s, t, v, w\}$. Thus we have vertex-disjoint trees $T_x$ for each terminal $x \in T'$, such that for any $x, y \in T'$, there is an edge $e_{xy}$ having one extremity in $T_x$ and one in $T_y$. We may assume that $T_x = \bigcup_{y \in T', \{x\}} P[x, y]$, where $P[x, y]$ is a path from $x$ to $y$. We easily get a terminal-$K_4^+$ minor where $w$ is the terminal which subdivides the minor edge $su$. The last case is where $w$ lies in exactly 2 of the paths, say $P[s, u], P[s, v]$. In this case, one may replace $s$ and use $w$ as the degree 3 vertex of the $K_4$ minor and hence $s$ can play the role of the degree 2 vertex in a terminal-$K_4^+$ minor, a contradiction.

Now we assume that $w$ is not contained in any of the subtrees. Then by 2-connectivity, there are two disjoint paths from $w$ to two vertices $a$ and $b$ in $U$. If $a$ and $b$ are in different subtrees, we easily get a terminal-$K_4^+$ minor, a contradiction. Suppose that $a$ and $b$ is in exactly one of $P[s, u], P[s, v], P[s, t]$ since we could obtain a smaller terminal-$K_4$ minor by replacing $s$ by $w$. If $w$ lies in exactly one of these paths, say $P[s, u]$, then we obtain a terminal-$K_4^+$ minor where $w$ is the terminal which subdivides the minor edge $s$. The last case is where $w$ lies in exactly 2 of the paths, say $P[s, u], P[s, v]$. In this case, one may replace $s$ and use $w$ as the degree 3 vertex of the $K_4$ minor and hence $s$ can play the role of the degree 2 vertex in a terminal-$K_4^+$ minor, a contradiction.

In the last case we may assume that $a, b \in R$ where $R := P[s, u] \cap P[s, v] \cap P[s, t]$. Let $z$ be the end of $R$ that is not $s$, and denote $U' := U \setminus (V(R) \setminus z)$. Let $Q_1, Q_2$ be open vertex-disjoint paths from $s$ to $U'$. Without loss of generality $Q_1$ contains a vertex $z' \in R$ which is closest to $z$ amongst all vertices in $Q_1 \cup Q_2$. Hence replacing $Q_1$ by the path which follows the subpath of $R$ from $z'$ to $z$ also produces a vertex-disjoint pair of paths. Hence we assume the endpoints of $Q_1$ are $s$ and $z$. Now consider following one of the paths from $w$ until it first hits a vertex of $Q_1 \cup Q_2$. If there is no such vertex, then it hits $R$ and we may follow $R$ until it hits $Q_1 \cup Q_2$. In all cases this produces a path from $w$ to $U'$ which is disjoint from exactly one of $Q_1, Q_2$. Let $P', Q'$ be the
resulting vertex-disjoint paths and note that one of them terminates at $z$; without loss of generality $P'$. Hence its other endpoint can now play the role of $s$ as a degree 3 vertex in a terminal-$K_4$ minor. Therefore, the terminal on $Q'$ has a path to $U'$ which is disjoint from $P'$. Thus we are back to one of the previous cases.

Now we have ruled out the existence of terminal-$K_4$ minors, we start building up minors which can be possible.

**Proposition 3.** Any 2-connected graph with terminals $Z$, with $|Z| \geq 3$, has a 2-connected minor $H$ with $V(H) = Z$.

**Proof.** Let $H$ be a minimal 2-connected terminal-minor of $G$ containing $Z$ and assume there is a non-terminal vertex in $H$. In particular we may assume there is an edge $sv$ with $s \in Z$, $v \notin Z$. By minimality, contracting $sv$ decreases the connectivity to 1. Hence, $\{s, v\}$ is a cut separating two vertices $t$ and $t'$. Thus, there are two disjoint $tt'$-paths, one containing $s$ and the other $v$. That is, there is a circuit $C$ containing $s, t, v, t'$ in that order.

By minimality of $H$, we also have that $H - sv$ is not 2-connected. It follows that $H - sv$ contains a cut vertex $\{z\}$ where $s, v$ lie in distinct components of $H - sv - z$. This would contradict the existence of $C$, and this completes the proof.

As $|Z| \geq 5$, the previous lemma implies that there is a terminal-$C_4$ minor. Let $k$ be maximum such that $G$ contains a terminal-$C_k$ minor.

**Proposition 4.** $k = |Z|$.

**Proof.** By Proposition 3 let $H$ be a 2-connected terminal-minor of $G$ with $V(H) = T$. Consider an ear-decomposition of $H$, starting with longest cycle $C_0$ and ears $P_1, \ldots, P_k$. Then all ears are single edges (from which the proposition follows), otherwise let $P_i$ be an ear that is not a single edge, with $i$ minimum. The two ends of $P_i$ are vertices $x, y$ of $C_0$. If $x$ and $y$ are consecutive in $C_0$, this contradicts the maximality of $C_0$. If they are not consecutive, $C_0 \cup P_i$ is a subdivision of $K_{2,3}$.

We let $k = |Z|$ henceforth. A terminal-$C_k$ minor of $G$ can also be represented as a collection of $k$ vertex-disjoint subtrees $T_1, \ldots, T_k$, where each $T_i$ contains exactly one terminal $t_i$. There also exist edges $e_1, \ldots, e_k$, where $e_i$ has one extremity $u_i$ in $T_i$ and the other, $v_{i+1}$, in $T_{i+1}$. The subscript $k+1$ is taken to be 1; the edges in the subtrees are the contracted edges and the edges $e_1, \ldots, e_k$ are the undeleted edges. We define $s_i$ as the only vertex in $V(P[t_i, u_i]) \cap V(P[u_i, v_i]) \cap V(P[v_i, t_i])$, where $V(P[x, y])$ is the vertex set of the path with ends $x$ and $y$ in the tree $T_i$. Thus, $T_i$ is $P[s_i, u_i] \cup P[s_i, v_i] \cup P[s_i, t_i]$.

We denote by $S_i$ the path from $t_i$ to $s_i$ in $T_i$ and we take our representation so that $\sum_{i=1}^k |S_i|$ minimized. We denote by $P_i$ the path from $s_i$ to $s_{i+1}$.

**Proposition 5.** $\sum_{i=1}^k |S_i| = 0$. 

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Figure 3: Reducing $|S_1|$ of finding terminal-$K_{2,3}$ minors depending on the position of $y$.

Proof. By contradiction, suppose $|S_1| > 0$ and so $t_1$ does not lie in the graph induced by $D = P_1 \cup \ldots \cup P_k \cup S_2 \cup \ldots \cup S_k$. By 2-connectivity, there are two disjoint minimal paths from $t_1$ to distinct vertices $x$ and $y$ in $D$. Moreover we can assume that $x = s_1$ lies on $P_k \cup P_1$. To see this, suppose that $z \in S_1$ is the closest vertex to $s_1$ which is used by one of the paths (possibly $z = t_1$). We may then re-route one of the paths to use the subpath of $S_1$ from $z$ to $s_1$.

If $y$ is contained in one of $P_k, P_1$, it is routine to get another representation of the minor where all the $S_i$ are at least as short, and $S_1$ is empty, contradicting the minimality of our choice of representation. A similar argument holds if $y \in S_k \cup S_2$.

So we assume $y \in D \setminus (P_k \cup P_1 \cup S_k \cup S_2)$. We now find a terminal-$K_{2,3}$ minor, and that is again a contradiction. To see this, let $T_i$ be a tree which contains the second vertex $y$. As $k \geq 5$, we may assume either $i \in [4, k-1]$, or $i \in [3, k-2]$. Suppose the latter as the two cases are similar. We obtain a terminal-$K_{2,3}$ where the two degree-3 vertices correspond to the terminals in $T_i$ and $T_k$. The degree-2 vertices will correspond to $t_1, t_2$ and $t_{k-1}$ — see Figure 3.

Hence there is a circuit $C$ containing every terminal, in cyclic order $t_1, t_2, \ldots t_k$.

Proposition 6. There are no two vertex-disjoint paths, one from $t_i$ to $t_{i'}$, the other from $t_j$ to $t_{j'}$, with $i < j < i' < j'$.
Proof. By contradiction. For convenience, let’s denote \( s = t_i, t = t_i', s' = t_j \) and \( t' = t_j' \). Let \( P \) be the \( st \)-path and \( Q \) the \( s't' \)-path. We may assume that we choose \( P \) and \( Q \) to minimize their total number of maximal subpaths disjoint from \( C \).

We consider the set (not multi-set) of edges \( E(C) \cup E(P) \cup E(Q) \), and only keep \( s, s', t, t' \) as terminals. This defines a subgraph \( G' \) of \( G \) of maximum degree 4 by construction. Contract edges in \( E(C) \cap (E(P) \cup E(Q)) \), and then contract edges so that vertices of degree 2 are eliminated. This gives a minor \( H \) where the only vertices not of degree 4 are \( s, t, s', t' \), which have degree 3. \( E(H) \cap E(P) \) induces an \( st \)-path \( P' \) in \( H \), \( E(H) \cap E(Q) \) induces an \( s't' \)-path \( Q' \) in \( H \). \( P' \) and \( Q' \) are again vertex-disjoint. We call the remaining edges of \( E(C) \) in \( H \) \( C \)-edges. They induce a cycle which alternates between vertices of \( P' \) and \( Q' \). To see this, suppose that \( e \) is such an edge joining \( x, y \in V(P') \) (the case for \( Q' \) is the same). We could then replace the subpath of \( P \) between \( x, y \) by the subpath of \( C \) which was contracted to form \( e \). This would reduce, by at least 1, the number of maximal subpaths of \( P \) disjoint from \( C \), a contradiction.

Consider the two vertices \( u' \) and \( v' \) of \( Q' \) adjacent to \( s \), such that \( s', u', v', t' \) appear in that order on \( Q' \). \( u' \) and \( v' \) each has one more incident \( C \)-edge, whose extremities (respectively) are \( u, v \) and must then be on \( V(P') \setminus \{s\} \). We create a terminal-\( K_4 \) minor on \( s, s', t, t' \) as follows — see Figure 4 where \( u, v \) may be in either order on \( P' \). We contract all the edges of \( P' \) except the one \( e_s \) incident to \( s \), and all the edges of \( Q' \) except the one \( e_{u'} \) incident to \( u' \) in the direction of \( t' \), we get a terminal-\( K_4 \) minor with the edges \( su, sv, uu', vv', e_s \) and \( e_{u'} \). One easily checks that this leads to the desired terminal-\( K_4 \) minor This contradiction completes the proof.

\( \square \)

Figure 4: How to get a terminal-\( K_4 \) minor: red parts are contracted into single nodes, the blue edges will then form a \( K_4 \).

To conclude the characterization of terminal-\( K_{2,3} \) minor free graphs, we use (a generalization of) the celebrated 2-linkage theorem. Take a planar graph \( H \), whose outer face boundary is the cycle \( t_1, t_2, \ldots, t_k \), and whose inner faces are triangles. For each inner triangle, add a new clique of arbitrary size, and connect each vertex of the clique to the vertices of the triangle. Any graph built
this way is called a \((t_1, \ldots, t_k)\)-web, or a \(\{t_1, \ldots, t_k\}\)-web if we do not specify the ordering.

Note that a \(Z\)-web, for some set \(Z\), can be described via Okamura-Seymour instances (OS-instance). An OS-instance is a planar graph where all terminals appear on the boundary of the outer face. An Extended OS Instance is obtained from an OS-instance by adding arbitrary graphs, called 3-separated sets, each connected to up to three vertices of some inner face of the Okamura-Seymour instance. We also require that any two 3-separated sets in a common face cannot be crossing each other in that face. Extended OS instances are precisely the \(Z\)-webs.

**Theorem 5** (Seymour [9], Shiloach [12], Thomassen [13]). Let \(G\) be a graph, and \(s_1, \ldots, s_k \in V(G)\).
Suppose there are no two disjoint paths, one with extremity \(s_i\) and \(s_i'\), and one with extremity \(s_j\) and \(s_j'\), with \(i < j < i' < j'\).

Then \(G\) is the subgraph of an \((s_1, s_2, \ldots, s_k)\)-web.

The linkage theorem is usually stated in the special case when \(k = 4\), but the extension presented here is folklore. One can reduce the general case to the case \(k = 4\) by identifying the vertices \(s_1, \ldots, s_k\) with every other inner vertex of a ring grid with 7 circular layers and 2\(k\) rays, and choosing 4 vertices of the outer layer, labelling them \(s, t, s', t'\) in this order, and connecting them in a square — see Figure 5. Is is easy to prove that there are two vertex-disjoint paths, one with extremity \(s\) and \(s'\), the other with extremities \(t\) and \(t'\) in the graph built this way if and only there are two disjoint paths as in the theorem in the original graph (for instance, use the middle layer to route the path from \(s\) to \(s_i\) with only 2 bends, then the remaining graph is a sufficiently large subgrid to route the three other paths). Because the grid is 3-connected, its embedding is unique and we get that \(G\) is embedded inside the inner layer of the ring, from which the general version of the theorem is deduced.

By using Theorem 5 with Proposition 6, we get that any 2-connected terminal-\(K_{2,3}\) free graph is a subgraph of a \(Z\)-web where \(Z\) is the set of terminals.

This now completes the proof of Theorem 4. \qed

We now establish Corollary 1.

**Proof.** If \(G\) is terminal-\(K_{2,3}\) minor-free, then clearly contacting all blocks but one must create a terminal-\(K_{2,3}\) free instance.

Conversely, suppose that \(G\) has a terminal-\(K_{2,3}\) minor. Since this minor is 2-connected, it must be a minor of a graph obtained by contracting or deleting all the edges of every 2-connected component except one. Let’s call that last block \(B\). Hence the terminal-\(K_{2,3}\) minor is a minor of the graph obtained by contracting all the edges not in \(B\). \qed

### 4.1 A Consequence for Multiflows

Recall from the introduction that for a graph \(G\) and \(Z \subseteq V(G)\), we call \((G, Z)\) cut-sufficient if for any multi-flow instance (capacities on \(G\), demands between terminals in \(Z\)), we have feasibility if and only if the cut condition holds.
**Corollary 2.** \((G, Z)\) is cut-sufficient if and only if it is terminal-\(K_{2,3}\) free.

**Proof.** We first establish a lemma which we use again in the next section.

**Lemma 2.** Let \(G\) be an extended OS instance and \(F\) be a 3-separated graph whose attachment vertices to the planar part are \(\{x, y, z\}\). We may define a new graph \(G'\) from \(G\) by removing \(V(F) \setminus \{x, y, z\}\) and add a new vertex \(s\) with edges \(sx, sy, sz\) with capacities \(c_x, c_y, c_z\) so that minimum cuts separating disjoint sets of terminals in \(Z\) have the same capacities in \(G'\) and in \(G\).

**Proof.** For each \(\alpha \in \{x, y, z\}\), let \(c_\alpha\) be the value of a minimum cut in \(F\) separating \(\alpha\) from \(\{x, y, z\} \setminus \{\alpha\}\), for \(\alpha \in \{x, y, z\}\). We use \(S_\alpha\) to denote the shore of such a cut in \(F\), where \(\alpha \in S_\alpha\). We replace \(H\) in \(G\) by a claw where the central vertex is a new vertex \(u_H\), and leaves are \(x, y\) and \(z\), and the capacity of \(u_H\alpha\) is \(c_\alpha\) for any \(\alpha \in \{x, y, z\}\). We claim that this transformation preserves the values of minimum cuts between sets of terminals.

Notice that \(c_\alpha \leq \sum_{\beta \in \{x, y, z\} \setminus \{\alpha\}} c_\beta\), hence a minimum cut \(S\) of \(G'\) containing \(x\) but none of \(y, z\), does not contain \(s\). For a cut \(S'\) in \(G'\) with \(x \in S, s, y, z \notin S\), we may then associate a cut \(S\) in \(G\) with same capacity, by taking \(S := S' \cup S_x\). Reciprocally, given a cut \(S\) of \(G\) with \(x \in S, y, z \notin S\), the cut \(S' := S \setminus V(F) \setminus \{x, y, z\}\) has capacity at most the capacity of \(S\). Thus the values of minimum terminal cuts are preserved.

\end{proof}
Lemma 3. For any extended OS instance $G$ we may replace each 3-separated graph by a degree-3 vertex to obtain an equivalent (planar) OS instance $G'$. It is equivalent in that for any partition $Z_1 \cup Z_2 = Z$, the value of a minimum cut separating $Z_1, Z_2$ in $G$ is the same as it is in $G'$.

We now return to the proof of the corollary. First, if there is a terminal-$K_{2,3}$ minor then we obtain a "bad" multiflow instance as follows. For each deleted edge we assign it a capacity of 0. For each contracted edge we assign it a capacity of $\infty$. The remaining 6 edges have unit capacity. We now define four unit demands. One between the two degree-3 nodes of the terminal minor and a triangle on the remaining three nodes. It is well-known that this instance has a flow-cut gap of $\frac{4}{3}$, cf. [2, 1].

Now suppose that $G$ is terminal-$K_{2,3}$ free and consider a multiflow instance with demands on $Z$. By the preceding corollary, we may replace each 3-separated graph by a degree-3 vertex and this new OS instance will satisfy the cut condition if the old one did. Hence the Okamura-Seymour Theorem [7] yields a half-integral multiflow in the new instance.

We now show that the flow in the modified instance can be mapped back to the original extended OS instance. We do this one 3-separated graph at a time. Consider the total flow on paths that use the new edges through $s$ obtained via the reduction. Let $d(xy), d(yz), d(zx)$ be these values. We claim that these can be routed in the original $F$. First, it is easy to see that this instance on $F$ satisfies the cut condition. Any violated cut $\delta_F(S)$ would contain exactly one of $x, y, z$, say $x$. Hence this cut would have capacity less than $d(xy) + d(zx)$ but since this flow routed through $s$, this value must be at most $c_x$ which is a contradiction. Finally, the cut condition is sufficient to guarantee a multiflow in any graph if demands only arise on the edges of $K_4$, cf. Corollary 72.2a [8]. Hence we can produce the desired flow paths in $F$. 

5 General Case: Gomory-Hu Terminal Trees in terminal-$K_{2,3}$ minor free graphs.

In this section we prove Theorem 2 using the characterization of terminal-$K_{2,3}$ minor free graphs. The high level idea is a reduction to Theorem 1 by contracting away the non-terminal nodes in the graph.

In the following we let $(G, Z)$ denote a connected graph $G$ and terminals $Z \subseteq V(G)$. Recall that the classical Gomory-Hu Algorithm produces a $GH$ Z-Tree $T = (V(T), E(T))$ where formally $V(T)$ is a partition $P = \{B(v) : v \in Z\}$ of $V(G)$. We call $B(v)$ the bag for terminal $v$ and informally one often thinks of $V(T) = Z$. In addition each edge $st \in E(T)$ identifies a minimum $st$-cut in $G$, i.e., it is encoding as per the discussion following Definition 1.

We say that GH Z-tree $T$ occurs as a bag minor in $G$ if (i) each bag induces a connected graph $G[B(v)]$ and (ii) for each $st \in E(T)$, there is an edge of $G$ between $B(s)$ and $B(t)$. We say that $T$
occurs as a weak bag minor if it occurs as a bag minor after deletion of some non-terminal vertices (from its bags and \( G \)).

**Definition 2.** The pair \((G, Z)\) has the GH Minor Property if for any subgraph \( G' \) with capacities \( c' \), there is a GH Tree which occurs as a bag minor in \( G' \). The pair \((G, Z)\) has the weak GH Minor Property if such GH trees occur as a weak bag minor.

An example where we have the weak but not the (strong) property is for \( K_{2,3} \), where \( Z \) consists of the degree-2 vertices and one of the degree-3 vertices, call it \( t \). Clearly this is terminal-\( K_{2,3} \) minor free since it only has 4 terminals. The unique GH Tree \( T \) is obtained from \( G \) by deleting the non-terminal vertex and assigning capacity 2 to all edges in the 3-star. Hence \( T \) is obtained as a minor (in fact a subgraph) of \( G \). However, the bag \( B(t) \) consists of the 2 degree-3 vertices which do not induce a connected subgraph. Hence \( T \) does not occur as a bag minor. Fortunately, such instances are isolated and arise primarily due to instances with at most 4 terminals. We handle these separately.

**Proposition 7.** Let \( G \) be an undirected, connected graph and \( Z \) be a subset of at most 4 terminals. Suppose that edge-capacities are given so that no two central cuts have the same capacity. Then the unique GH Tree \( T \) occurs as a weak bag minor, and if \( T \) is a path, then it occurs as a bag minor.

We omit the to the very end. In the following it is useful to see how a GH Z-Tree bag minor (weak or strong) immediately implies such a minor for some \( Z' \subseteq Z \).

**Lemma 4.** Let \( T \) be a GH Z-Tree bag minor for some capacitated graph \( G \) and let \( v \in Z \). Let \( uv \in T \) be the edge with maximum weight \( c'(uv) \) in \( T \). If we set \( B'(u) = B(v) \cup B(u) \) and \( B'(x) = B(x) \) for each \( x \in Z \setminus \{u, v\} \), then the resulting partition defines a GH \((Z \setminus v)\)-Tree \( T' \) which is a bag minor.

**Proof.** Clearly \( T' \) is a bag minor and every fundamental cut of \( T \), other than \( uv \)'s, is still a fundamental cut of \( T' \). It remains to show that for any \( a, b \in Z \setminus v \), there is a minimum \( ab \)-cut that does not correspond to the fundamental cut of \( uv \). This is immediate if the unique \( ab \)-path \( P \) in \( T \) does not contain \( uv \). If it does contain \( uv \), then since \( a, b \neq v \), the \( ab \)-path in \( T \) contains some edge \( vw \). But since \( c'(vw) \leq c'(uv) \), the result follows. \( \square \)

**Theorem 2.** Let \( G \) be an undirected graph and \( Z \subseteq V \). \((G, Z)\) has the weak GH Minor Property if and only if \((G, Z)\) is a terminal-\( K_{2,3} \) minor free graph. Moreover, if none of \( G \)'s blocks is a 4-terminal instance, then \((G, Z)\) has the GH Minor Property.

**Proof.** If \( G \) has a terminal-\( K_{2,3} \) minor, then by appropriately setting edge capacities to 0, 1 or \( \infty \) we find a case where \( G \) does not have the desired bag minor. So we now assume that \( G \) is a terminal-\( K_{2,3} \) minor free graph. Let \( G' \) be some subgraph of \( G \) with edge capacities \( c(e) > 0 \), perturbed so that all minimum cuts are unique. We show that the unique GH Z-tree occurs as a bag minor.

We deal first with the case where \( G' \) has cut vertices. Note that one may iteratively remove any leaf blocks which do not contain terminals. This operation essentially does not impact the GH
Proof. By contradiction, let embedding of $G$ be a cut vertex in $Z$ in the original graph, and the resulting configuration is still $K_{2,3}$-free. Henceforth we assume that $Z$ includes these extra vertices and show the desired bag minor exists for this terminal set. This is sufficient since we can then retrieve the bag minor for the original terminal set via Lemma $\|$.

One checks that a GH $Z$-Tree is obtained by gluing together the appropriate GH terminal trees in each block. Moreover, since each cut vertex is a terminal, if each of these blocks’ tree is a bag minor (resp. weak bag minor), then the whole tree is a bag minor (resp. weak bag minor). Therefore it is now sufficient to prove the result in the case where $G'$ is 2-connected.

If $G'$ has at most 4 terminals, then Proposition $\|$ asserts that it has a weak bag minor for a GH tree. Moreover, if it has less than 4 terminals, then it’s GH Tree is a path and hence occurs as a bag minor. So we now assume that $G'$ contains at least 5 terminals and hence it is an extended OS instance whose outside face is a simple cycle. Lemma $\|$ implies that we may replace each 3-separated set by a degree-3 vertex and the resulting graph is planar and has the same pairwise connectivities amongst vertices in $Z$. It is easy to check that any $Z$-tree bag minor in this new graph is also such a minor in the original instance. Therefore, it is sufficient to show that any planar OS instance with terminals on the outside face has the desired GH tree bag minor.

Denote by $t_1, t_2, \ldots, t_{|T|}$ the terminals in the order in which they appear on the boundary of the outer face. Let $B(t): t \in Z$ be the bags associated with the (necessarily unique) GH $Z$-tree $T$. We show that (i) each $G'[B(t)]$ is connected and (ii) for any $st \in T$, there is some edge of $G$ between $B(s)$ and $B(t)$.

Consider the fundamental cuts associated with edges incident to some terminal $t$. Let $X_1, X_2, \ldots X_k$ be their shores which do not contain $t$. Since any min-cut is central, each $X_i$ intersects the outside face in a subpath of its boundary. Hence, similar to Claim $\|$ (cf. Figure $\|$), we can order them $X_1, \ldots, X_k$ in clockwise order on the boundary with $t$ between $X_k$ and $X_1$.

The next two claims complete the proof of the theorem.

Claim 2. For each terminal $t$, $G'[B(t)]$ is connected.

Proof. By contradiction, let $C$ be a component of $G' \setminus (X_1 \cup \ldots \cup X_k)$ which does not contain $t$. If $N(C) \subseteq X_i$ for some $i \in \{1, \ldots, k\}$, then $\delta(C \cup X_i)$ is a cut separating $t$ from any vertex in $X_i$ with capacity smaller than $\delta(X_i)$, contradicting the minimality of $X_i$.

Otherwise choose $j < j'$ be such that $N(C) \cap X_j, N(C) \cap X_{j'}$ are non-empty and $j' - j$ is maximized. Call $(j, j')$ the span of $C$. Without loss of generality $C$ has the largest span amongst all components other than $B(t)$ (whose span is $(1, k)$ incidentally). Moreover, amongst those (non $B(t)$) components with span $(j, j')$ we may assume that $C$ was selected to maximize the graph “inside” the embedding of $G'[X_j \cup X_{j'} \cup M']$, where $M' = C \cup X_{j+1} \cup \ldots X_{j'-1}$. In particular, any component $C'$ with neighbours in $M'$ has $N(C') \subseteq M' \cup X_j \cup X_{j'}$. Let $M$ be the union of $M'$ and all such components $C'$. By construction $M$ is non-empty and $t \not\in M$, however $d(M, V \setminus (X_j \cup X_{j'} \cup M)) = 0$ which contradicts Lemma $\|$ if we take $X = X_j, Y = X_{j'}$. \qed
Claim 3. For each \( i \in \{1, \ldots, k\} \), there is an edge from a vertex in \( B(t) \) to a vertex in \( X_i \).

Proof. By contradiction, suppose \( \delta(B(t), X_i) = \emptyset \), for some \( i \in \{1, \ldots, k\} \). Let \( j \) maximum and \( j' \) minimum such that \( j < i < j' \), \( \delta(B(t), X_j) \neq \emptyset \) and \( \delta(B(t), X_{j'}) \neq \emptyset \). Note that \( j \) and \( j' \) are defined because \( X_1 \) and \( X_k \) are adjacent to \( B(t) \) by the outer cycle. If we define \( M := X_{j+1} \ldots \cup X_{j'-1} \), then \( d(M, V \setminus (M \cup X_j \cup X_{j'})) = 0 \), contradicting Lemma \( \square \) where we take \( X = X_j, Y = X_{j'} \). \( \square \)

Finally, we provide the proof for Proposition \( \square \).

Proof. We first consider the case where we have 4 terminals and let \( T \) be the unique GH tree. Suppose that \( T \) is a star with center vertex 1 and let \( B_1, B_2, B_3, B_4 \) be the bags. Since each fundamental cut of \( T \) is central (in \( G \)) we have that \( B_2, B_3, B_4 \) each induces a connected subgraph of \( G \). Let \( Y \subseteq B_1 \) be those vertices (if any) which do not lie in the same component of \( G[B_1] \) as 1. We may try to produce \( T \) as a weak bag minor of \( G \) by deleting \( Y \). This fails only if for some \( j \geq 2 \), \( d(B_1 \setminus Y, B_j) = 0 \); without loss of generality \( j = 2 \). Let \( R = B_2 \cup Y \cup B_3, S = B_2 \cup Y \cup B_4 \). It follows that \( d(R \cap S, V - (R \cup S)) = 0 \) and hence \( c(R) + c(S) = c(R \setminus S) + c(S \setminus R) = c(B_3) + c(B_4) \). But \( \delta(R) \) is a 34-cut and so \( c(R) > c(B_3) \). Similarly, \( c(S) > c(B_4) \). But this now contradicts the previously derived equality.

Consider now the case where \( T \) is a path, say 1, 2, 3, 4. Since each fundamental cut is central, \( G[B_1], G[B_4] \) are connected. Now suppose that \( G[B_2] \) is not connected. Let \( M \) be the set of vertices which do not lie in the same component as 2. If we define \( X = B_1, Y = B_3 \cup B_4 \) and \( t = 2, x = 1, y = 3 \), then Lemma \( \square \) implies that \( d(M, B_2 \setminus M) > 0 \) a contradiction. It remains to show that \( d(B_i, B_{i+1}) > 0 \) for each \( i = 1, 2, 3 \).

Suppose first that \( d(B_1, B_2) = 0 \). Then \( c(B_1 \cup B_2 \cup B_4) \leq c(B_3 \cup B_4) \) contradicting the fact that \( B_3 \cup B_4 \) induces the unique minimum 23 cut. Hence \( d(B_1, B_2) > 0 \) and by symmetry \( d(B_3, B_4) > 0 \). Finally suppose that \( d(B_2, B_3) = 0 \). One then easily checks that \( c(B_1) + c(B_4) \geq c(B_2) + c(B_3) \). But then either \( B_2 \) induces a second minimum 12 cut, or \( B_3 \) induces another minimum 34 cut. In either case, we have a contradiction. The final cases where \( |Z| \leq 3 \) follow easily by the same methods. \( \square \)

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