CHIRAL EXTRAPOLATION OF LIGHT RESONANCES FROM UNITARIZED CHIRAL PERTURBATION THEORY

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Abstract

Both scalar and vector light resonances can be generated from the unitarization of one-loop chiral perturbation theory. This amounts to using in a dispersion relation the chiral expansion, which incorporates the correct QCD quark mass dependence. We can thus predict the quark mass dependence of the poles associated to those light resonances. Our results compare well with some recent lattice results for the $\rho(770)$ mass and can be used as a benchmark for future lattice results on the $\rho(770)$ or the $f_0(600)$ also known as the $\sigma$.

1 Introduction

Light hadron spectroscopy at low energies lies beyond the realm of perturbative QCD, although lattice QCD provides, in principle, a rigorous way to extract non-perturbative quantities from QCD. However, current lattice results are
typically still done for relatively high quark masses. Thus, in order to make contact with experiment, appropriate extrapolation formulas need to be derived. This is typically done by using Chiral Perturbation Theory (ChPT), which provides a model independent description of the dynamics of the lightest mesons, namely, the pions, kaons and etas, which are identified with the Goldstone Bosons (GB) associated to the QCD spontaneous Chiral Symmetry. Hence, ChPT is built out of only those fields, as a low energy expansion of a Lagrangian whose terms respect all QCD symmetries, and in particular its symmetry breaking pattern. Actually, this chiral expansion becomes a series in momenta and meson masses, generically $O(p^2/\Lambda^2)$, when taking into account systematically the small quark masses of the three lightest flavors that can be treated perturbatively. The chiral expansion scale is $\Lambda \equiv 4\pi f_\pi$, where $f_\pi$ denotes the pion decay constant. ChPT is renormalized order by order by absorbing loop divergences in the renormalization of parameters of higher order counterterms, known as low energy constants (LEC). Their values depend on the specific QCD dynamics, and have to be determined either from experiment or from lattice QCD — they cannot be calculated from perturbative QCD.

The relevant remark for us is that, thanks to the fact that ChPT has the same symmetries than QCD and that it couples to different kind of currents in the same way, the orthodox ChPT expansion provides a systematic and model independent description of how the observables depend on some QCD parameters. This is the case for the leading dependence on the number of colors $N_c$ and, more important for our purposes here, the dependence on the quark masses, which can be implemented systematically up to the desired order in the orthodox ChPT expansion.

In this work we focus on the two lightest resonances of QCD, the $\rho(770)$ and the $f_0(600)$. It is therefore enough to work with the two lightest quark flavors $u, d$ in the isospin limit of an equal mass that we take as $\hat{m} = (m_u + m_d)/2$. The pion mass is given by an expansion $m_\pi^2 \sim \hat{m} + \ldots$ (see \[1\] for details). Therefore, studying the quark mass dependence is equivalent to study the pion mass dependence. In $\pi\pi$ scattering at NLO within SU(2) ChPT only four LECs $l_1, \ldots, l_4$ appear. Of course, when changing pion masses we have to take into account that amplitudes are customarily written \[1\] in terms of the $\mu$ independent LECs $\hat{l}$ and the physical pion decay constant $f_\pi = f_0 \left(1 + \frac{m_\pi^2}{16\pi^2 f_0^2} \hat{l}_4 + \ldots \right)$ that depend explicitly on the pion mass, $m_\pi$. 


2 Unitarization and dispersion theory

$S$ matrix unitarity implies, for physical values of $s$, that elastic $\pi\pi$ scattering partial waves $t(s)$ of definite isospin $I$ and angular momenta $J$ should satisfy

$$\text{Im} t(s) = \sigma |t(s)|^2 \Rightarrow \text{Im} \frac{1}{t(s)} = -\sigma(s), \quad \text{with} \quad \sigma(s) = \frac{2p}{\sqrt{s}}$$

and $p$ is the CM momenta. Thus $|t^{IJ}| \leq 1/\sigma$, and interactions are said to become strong precisely when this unitarity bound is saturated.

However, the ChPT low energy expansion $t \simeq t_2 + t_4 + \ldots$, where $t_{2k} \equiv O(p/(4\pi f_\pi)^{2k}$, can only satisfy unitarity perturbatively, i.e:

$$\text{Im} t_2 = 0, \quad \text{Im} t_4 = \sigma t_2^2, \quad \text{etc...}$$

(2)

The one-channel Inverse Amplitude Method (IAM) \cite{2} \cite{3} is a unitarization technique that can be derived within a “naive”, intuitive, approach by noting that eq.(1), *fixes the imaginary part of the inverse amplitude exactly*. If we then use ChPT to write $\text{Re} t^{-1} \simeq t_2^{-2}(t_2 + \text{Re} t_4 + \ldots)$, we find

$$t \simeq \frac{1}{\text{Re} t^{-1} - i\sigma} = \frac{t_2}{1 - t_4/t_2}. \quad (3)$$

However, the above derivation is just formal, since the ChPT series can only be used at low energies. The correct derivation uses dispersion theory, and the fact that the ChPT series of $t$ and $1/t$ beyond leading order have an analytic structure with a “physical cut” from threshold to $\infty$ and a “left cut” from $-\infty$ to $s = 0$. This leads to the following dispersion relation \cite{3} for $t_4$

$$t_4 = b_0 + b_1 s + b_2 s^2 + \frac{s^3}{\pi} \int_{s_{th}}^{\infty} \frac{\text{Im} t_4(s')}{s^3(s' - s - i\epsilon)} ds' + \text{LC}(t_4),$$

(4)

where “LC” stands for a similar integral over the left cut and we have three subtractions to ensure convergence. A similar dispersion relation can be written for the function $G \equiv t_2^2/t$, by simply replacing $t_4$ by $G$ and changing the name of the subtraction constants. Since $t_2$ is real, the functions $G$ and $t_4/t_2^2$ have *exactly* opposite integrals over the physical cut. Their subtractions constants are the value of these functions at $s = 0$ where the ChPT expansion is safe. And finally, they also have opposite left cut contributions up to NNLO ChPT. Such an approximation on the left cut is, of course, only justified for small $|s|$. 
but due to the three subtractions this is precisely the region that dominates
the left cut integrals. Therefore, the IAM derivation is exact for the integrals
over the elastic region and uses ChPT only where it is well justified. The IAM
is even more justified if used sufficiently far from the left cut, as it is usually
done, due to the additional $1/(s - s')$ suppression.

In the scalar channels there are also contributions to the dispersion rela-
tion coming from poles in $1/t$ due to the so-called Adler zeros located well below
threshold. Such contributions lead, formally, to $O(p^6)$ corrections in the IAM,
and are customarily neglected, leading to the standard IAM we have justified
above. However, if not taken into account, the Adler zeros do not appear in the
correct place and also unphysical poles occur in the IAM. Still, the influence
of these unphysical poles is very localized around those Adler zeros and the
standard IAM can be used safely for energies sufficiently far from the Adler
zeros.

Nevertheless, in the next section we will show that resonance poles move
into the subthreshold region for sufficiently large pion masses and it is thus
relevant to include the pole contributions and use a slightly modified IAM,
whose results agree with those of the standard IAM, except in the subthreshold
region, where the modified version is more reliable. Such modified IAM has
already been built and used in $^4$, by adding an ad hoc $O(p^6)$ piece to the
$\text{Re} t^{-1}$ expansion, within the “naive derivation” explained just before Eq.(3).
A rigorous dispersive derivation is in preparation $^5$.

In summary, there are no model dependences in the approach, but just
approximations to a given order in ChPT. Remarkably, the simple formula of
the elastic IAM, Eq.(3), (or the slightly modified one to work in the subthresh-
old region), while reproducing the ChPT expansion at low energies, is also able
to generate both the $\rho(770)$ and $f_0(600)$ resonances with values of the LECS
compatible with standard ChPT $^6$. $^7$. In other words, the IAM generates the
poles $^3$ $^6$ associated to these resonances in the second Riemann sheet. The
fact that resonances are not introduced by hand but generated from first prin-
ciples and data, is relevant because the existence and nature of scalar resonances
is the subject of a long-lasting intense debate.

To be precise, the IAM, when reexpanded, reproduces the orthodox ChPT
series up to the order to which the input amplitude was evaluated and, in par-
ticular the quark mass dependence agrees with that of ChPT up to that order.
A few of the higher order terms are produced correctly by the unitarization but not the complete series— for a discussion of this issue for the scalar pion formfactor see Ref. 8). However, the formalism just described still provides us with a fair estimate of the quark mass dependence of the resonance properties. In this case, we can study, without any a priori assumption, the dependence of the resonances’ positions on QCD parameters like the number of colors $N_c$ or their dependence on the quark masses up to a given order in ChPT.

3 Results

As commented above, since, for the moment, we are only interested in resonances appearing in elastic $\pi\pi$ scattering, we can unitarize SU(2) ChPT at one-loop. For the LECS we take $l_r^3 = 0.8 \pm 3.8$, $l_r^4 = 6.2 \pm 5.7$, directly from 1), (since the partial waves where the $\sigma$ and $\rho$ appear are not very sensitive to these two constants), whereas we use $l_r^1 = -3.7 \pm 0.2$, $l_r^2 = 5.0 \pm 0.4$ obtained from an IAM fit to data up to the resonance region. All LECS are evaluated at $\mu = 770$ MeV, and are in fairly good agreement with standard values.

The highest value of $m_\pi$ we can use is limited since we do not want to spoil the chiral expansion and we want to have some elastic $\pi\pi$ regime below the $K\bar{K}$ threshold. A mass of $m_\pi \leq 500$ MeV satisfies both criteria since we know SU(3) ChPT still works fairly well with a kaon mass that high, and also because if we increase the pion mass to 500 MeV, the kaon mass becomes $\simeq 600$ MeV, and $\pi\pi$ scattering is still elastic for 200 MeV, before reaching the two-kaon threshold. To reach higher masses we would need a coupled-channel IAM, which is feasible, but lies beyond our present scope.

Thus, in Fig.1 we show the $\rho$ and $f_0(600)$ poles movement in the second Riemann sheet as $m_\pi$ increases. Note that in order to follow easily the pole movement relative to the two-pion threshold, which is also increasing, we express all quantities in units of $M_\pi$, so that the two-pion threshold is shown fixed at $\sqrt{s} = 2$. In this way we clearly see that both poles move closer to the two-pion threshold. Let us recall that for narrow resonances, their mass $M$ and width $\Gamma$ are related to the pole position as $\sqrt{s}_{\text{pole}} \simeq M - i\Gamma/2$ and customarily this notation is also kept for broader resonances. Hence, both the $\sigma$ and $\rho$ widths decrease for increasing $m_\pi$, partly due to phase space reduction. In particular, the $\rho$ pole moves toward the real axis and just when the threshold is reached it jumps into the real axis on the first sheet, thus becoming
Figure 1: $\rho$ and $\sigma$ complex plane pole movement with increasing pion mass. To ease the comparison of the pole position relative to the two-pion threshold we normalize by the pion mass that is changing. Note how the sigma pole moves toward the real axis below threshold where it splits in two virtual states, whereas the $\rho$ pole just moves toward threshold.

a traditional bound state, while its conjugate partner remains on the second sheet practically at the very same position as the one in the first. In contrast, when the $\sigma$ mass reaches the two-pion threshold, its poles remain on the second sheet with a non-zero imaginary part before they meet on the real axis and become virtual states. As $m_\pi$ increases further, one of those virtual states moves towards the threshold and jumps onto the first sheet, whereas the other one remains in the second sheet. Although, of course, this happens for very large values of $m_\pi$, such an analytic structure, with two very asymmetric poles in different sheets of an angular momentum zero partial wave, is a signal for a prominent molecular component 10). Differences between P-wave and S-wave pole movements were also found within quark models 11), the latter showing also two second sheet poles on on the real axis below threshold.

In Fig.2 (left) we show in detail, the growth of the $\sigma$ and $\rho$ masses, starting from the chiral limit up to $\sim 500$ MeV. We find that both the $\rho$ and $\sigma$ mass increase, but that of the $\sigma$ grows faster, until it splits in two virtual states. Then one mass keeps growing, whereas the other one decreases.
Finally in Fig.2 (right) we show our central value result for the $\rho$ mass dependence on $m_\pi$ compared with some recent lattice results \cite{12}. Despite our results refer to the $\rho$ “pole-mass” definition and that those results on the lattice have a zero width for the $\rho$, we see a reasonable qualitative agreement between both results, although our dependence seems to be somewhat steeper. We have some preliminary indications that if we decrease the $\rho$ width in our approach (by taking the large $N_c$ limit of ChPT), we reproduce even better those lattice results.

A publication with further details is in preparation \cite{13} including results of the $f_0(600)$ and $\rho(770)$ mass and width evolution with the pion mass as well as a comparison with other works and lattice results. Estimates of uncertainties and possibly an extension to the SU(3) coupled channel case are in progress.

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