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THE INTERSECTION OF PAST AND FUTURE FOR MULTIVARIATE STATIONARY PROCESSES

AKIHKO INOUE, YUKIO KASAHARA, AND MOHSEN POURAHMADI

Abstract. We consider an intersection of past and future property of multivariate stationary processes which is the key to deriving various representation theorems for their linear predictor coefficient matrices. We extend useful spectral characterizations for this property from univariate processes to multivariate processes.

1. Introduction

We write $\mathbb{C}^{m \times n}$ for the set of all complex $m \times n$ matrices. Let $\{X(k) : k \in \mathbb{Z}\}$ be a $\mathbb{C}^{q \times 1}$-valued, centered, weakly stationary process, defined on a probability space $(\Omega, \mathcal{F}, P)$, which we shall simply call a $q$-variate stationary process. Write $X(k) = (X_1(k), \ldots, X_q(k))^T$, and let $M$ be the complex Hilbert space spanned by all the entries $\{X_j(k) : k \in \mathbb{Z}, j = 1, \ldots, q\}$ in $L^2(\Omega, \mathcal{F}, P)$, which has inner product $(Y_1, Y_2)_M := E[Y_1 \overline{Y_2}]$ and norm $\|Y\|_M := (Y, Y)_M^{1/2}$. For $I \subset \mathbb{Z}$ such as $\{n\}$, $(-\infty, n] := \{n, n-1, \ldots\}$, $[n, \infty) := \{n, n+1, \ldots\}$, and $[m, n] := \{m, \ldots, n\}$ with $m \leq n$, we define the closed subspace $M^X_I$ of $M$ by

$$M^X_I := \overline{sp}\{X_j(k) : j = 1, \ldots, q, \ k \in I\}.$$ Notice that $M^X_{[n, n]} = M^X_{(n)} = \overline{sp}\{X_1(n), \ldots, X_q(n)\}$.

In this paper, we are concerned with the following intersection of past and future property of a $q$-variate stationary process $\{X(k)\}$:

$$M^X_{(-\infty, -1]} \cap M^X_{[-n, \infty)} = M^X_{[-n, -1]}, \quad n = 1, 2, 3, \ldots.$$ It is shown in [I1, Theorem 3.1] that a univariate stationary process satisfies (IPF) if it is purely nondeterministic (PND) (see Section 2 below) and has spectral density $w$ such that $w^{-1}$ is integrable. We prove a multivariate analog of this sufficient condition for (IPF). More precisely, we show that a $q$-variate stationary process $\{X(k)\}$ satisfies (IPF) if $\{X(k)\}$ has maximal rank (see Section 2 below) and has spectral density $w$ such that $w^{-1}$ is integrable (see Corollary 3.6 below). We remark that such a process $\{X(k)\}$ is PND.

The importance of (IPF) for univariate stationary processes is that it, combined with von Neumann's Alternating Projection Theorem (cf. [P, §9.6.3]), allows one to derive explicit and useful representations of finite-past prediction error variances ([I1, I2, IK1]), finite-past predictor coefficients ([IK2]), and partial autocorrelations or Verblunsky coefficients ([I3, BIK, KB]), of $\{X(k)\}$. We can extend this approach introduced by [I1] to multivariate stationary processes. In so doing, the sufficient condition for (IPF) stated above plays a crucial role. In our subsequent work,
functions \( f \) to \( H \) and by \( I \)
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defines by
\[ \| f \|_p := \left\{ \int_{-\pi}^{\pi} |f(e^{i\theta})|^p \sigma(d\theta) \right\}^{1/p}. \]

Let \( L_p^{m \times n}(\mathbb{T}) \) be the space of \( C^{m \times n} \)-valued functions on \( \mathbb{T} \) whose entries belong to \( L_p(\mathbb{T}) \).

For \( p \in [1, \infty) \), the Hardy class \( H_p(\mathbb{T}) \) on \( \mathbb{T} \) is the closed subspace of \( L_p(\mathbb{T}) \) defined by
\[ H_p(\mathbb{T}) := \left\{ f \in L_p(\mathbb{T}) : \int_{-\pi}^{\pi} e^{im\theta} f(e^{i\theta}) \sigma(d\theta) = 0 \text{ for } m = 1, 2, \ldots \right\}. \]

Let \( H_p^{m \times n}(\mathbb{T}) \) be the space of \( C^{m \times n} \)-valued functions on \( \mathbb{T} \) whose entries belong to \( H_p(\mathbb{T}) \). Let \( \mathcal{D} \) be the unit open disk in \( \mathbb{C} \), i.e., \( \mathcal{D} := \{ z \in \mathbb{C} : |z| < 1 \} \). For \( p \in [1, \infty) \), we write \( H_p(\mathcal{D}) \) for the Hardy class on \( \mathcal{D} \), consisting of holomorphic functions \( f \) on \( \mathcal{D} \) such that
\[ \sup_{r \in [0, 1]} \int_{-\pi}^{\pi} |f(re^{i\theta})|^p \sigma(d\theta) < \infty. \]
As usual, we identify each function \( f \) in \( H_p(\mathbb{D}) \) with its boundary function
\[
f(e^{i\theta}) := \lim_{r \to 1} f(re^{i\theta}) \quad \sigma \text{-a.e.}
\]
in \( H_p(\mathbb{T}) \) (cf. Rosenblum and Rovnyak [RR]).

A function \( h \) in \( H_2^{q \times q}(\mathbb{T}) \) is called outer if \( \det h \) is a \( \mathbb{C} \)-valued outer function, that is, \( \det h \) satisfies
\[
\log |\det h(0)| = \int_{-\pi}^{\pi} \log |\det h(e^{i\theta})| \sigma(d\theta)
\]
for some outer function \( h(0) \) (see Katsnelson and Kirstein [KK, Definition 3.1]).

Let \( \{X(k)\} \) be a \( q \times q \)-variate stationary process. If there exists a nonnegative \( q \times q \)-valued Hermitian matrix-valued function \( w \) on \( \mathbb{T} \), satisfying \( w \in L_1^{q \times q}(\mathbb{T}) \) and
\[
E[X(m)X(n)^*] = \int_{-\pi}^{\pi} e^{-i(m-n)\theta} w(e^{i\theta}) \sigma(d\theta), \quad n, m \in \mathbb{Z},
\]
then we call \( w \) the spectral density of \( \{X(k)\} \). We say that \( \{X(k)\} \) has maximal rank if
\[
\text{(MR)} \quad \{X(k)\} \text{ has spectral density } w \text{ such that } \det w(e^{i\theta}) > 0 \quad \sigma \text{-a.e.}
\]
(see Rozanov [R, pp. 71–72]). A \( q \times q \)-variate stationary process \( \{X(k)\} \) is said to be purely nondeterministic (PND) if
\[
\text{(PND)} \quad \cap_{n \in \mathbb{Z}} M_{(-\infty, n]}^{X} = \{0\}.
\]
Every PND process \( \{X(k)\} \) has spectral density but it does not necessarily have maximal rank unlike univariate processes (see [R, Theorem 4.1]). So we combine the two to define the condition
\[
\text{(A)} \quad \{X(k)\} \text{ satisfies both (MR) and (PND)}.
\]
A necessary and sufficient condition for (A) is that \( \{X(k)\} \) has spectral density \( w \) such that \( \log \det w \in L_1(\mathbb{T}) \) (see [R, Theorem 6.1]).

Let \( \{X(k)\} \) be a \( q \times q \)-variate stationary process satisfying (A), and let \( w \) be its spectral density. Then, the spectral density \( w \) of \( \{X(k)\} \) has a decomposition of the form
\[
w(e^{i\theta}) = h(e^{i\theta})h(e^{i\theta})^* \quad \sigma \text{-a.e.}
\]
for some outer function \( h \) in \( H_2^{q \times q}(\mathbb{T}) \), and \( h \) is unique up to a constant unitary factor (see, e.g., [R, Chapter II] and Helson and Lowdenslager [HL, Theorem 11]).

**Lemma 2.1.** We assume (A). Then, \( X_j(k), \ k \in \mathbb{Z}, \ j = 1, \ldots, q, \) are linearly independent.

**Proof.** Let \( h(z) = \sum_{n=0}^{\infty} c(n)z^n, \ z \in \mathbb{D}, \) be the power series expansion of \( h, \) where \( \{c(n)\}_n^\infty \) is a \( \mathbb{C}^{q \times q} \)-valued sequence whose entries \( \{c_{i,j}(n)\}_n^\infty, \ i, j = 1, \ldots, q, \) belong to \( \ell^2. \) Then, there exists a \( q \times q \)-variate stationary process \( \{\xi(k)\}, \) called the innovation process of \( \{X(k)\}, \) satisfying \( E[\xi(n)\xi(m)^*] = \delta_{n,m}I_q \) and
\[
X(n) = \sum_{k=-\infty}^{n} c(n-k)\xi(k), \quad n \in \mathbb{Z},
\]
\[
M_{(-\infty, n]}^{X} = M_{(-\infty, n]}^{\xi}, \quad n \in \mathbb{Z},
\]
where $M_{[-\infty,n]}^k := \sp\{\xi_j(k) : k \leq 0, \ j = 1, \ldots, q\}$ in $L^2(\Omega, \mathcal{F}, P)$ (see Theorem 4.3 in [R, Chapter II]).

Suppose $\sum_{k=n}^\infty a(k)X(k) = 0$ for $n, m \in \mathbb{Z}$ with $m \leq n$ and $a(k) \in \mathcal{C}^{1\times q}$, $k = m, \ldots, n$. Let $Q$ be the projection operator from $M$ onto the orthogonal complement $(M_{[-\infty,n-1]}^\infty)^\perp$ of $M_{[-\infty,n-1]}^\infty$. Then,

$$0 = Q \left( \sum_{k=n}^n a(k)X(k) \right) = a(n)c(0)\xi(n).$$

Since $\xi_1(n), \ldots, \xi_q(n)$ are linearly independent, we have $a(n)c(0) = 0$. However, $c(0)$ is invertible by (2.1), whence $a(n) = 0$. In the same way, we also obtain $a(n-1) = \cdots = a(m) = 0$. Thus, $X_j(k)$'s are linearly independent.

In addition to (2.2), $w$ has a decomposition of the form

$$w(e^{i\theta}) = h_\sigma(e^{i\theta})h_\zeta(e^{i\theta}) \quad \text{a.e.}$$

for another outer function $h_\zeta$ in $H^2_\sigma^{1\times q}(\mathbb{T})$, and $h_\sigma$ is also unique up to a constant unitary factor. In fact, for an outer function $g$ in $H^2_\sigma^{1\times q}(\mathbb{T})$ satisfying $w(e^{i\theta})T = g(e^{i\theta})g(e^{i\theta})^* \text{a.e.}$, we may take $h_\sigma = g^T$. It should be noticed that while we may take $h_\sigma = h$ for the univariate case $q = 1$, there is no such simple relation between $h$ and $h_\sigma$ for $q \geq 2$.

We denote by $L(w)$ the complex Hilbert space consisting of all measurable functions $f : \mathbb{T} \rightarrow \mathbb{C}^{1\times q}$ with $\int_0^\pi f(e^{i\theta})w(e^{i\theta})f(e^{i\theta})^*\sigma(d\theta) < \infty$, which has inner product

$$(f, g)_w := \int_0^\pi f(e^{i\theta})w(e^{i\theta})g(e^{i\theta})^*\sigma(d\theta)$$

and norm $\|f\|_w := (f, f)^{1/2}_w$. For $k \in \mathbb{Z}$ and $j = 1, \ldots, q$, we define $e_j(k) \in L(w)$ by

$$e_j(k)(z) := (0, \ldots, 0, z^{-k}, 0, \ldots, 0), \quad z \in \mathbb{T},$$

where $z^{-k}$ is in the $j$-th coordinate. For an interval $I \subset \mathbb{Z}$, let $L_I(w)$ be the closed subspace of $L(w)$ spanned by $\{e_j(k) : k \in I, \ j = 1, \ldots, q\}$. By taking $I_q$ as $w$, we regard $L^{1\times q}_\sigma(\mathbb{T})$ as the complex Hilbert space $L(I_q)$ with inner product $(f, g)_{I_q} := \int_0^\pi f(e^{i\theta})g(e^{i\theta})^*\sigma(d\theta)$ and norm $\|f\|_{I_q} := (f, f)^{1/2}_{I_q}$, and $H^{1\times q}_\sigma(\mathbb{T})$ as its closed subspace.

We put, for $p \in [1, \infty)$,

$$L^{1\times q}_p(\mathbb{T}) := \{f : f \in H^{1\times q}_p(\mathbb{T})\}.$$

Lemma 2.2. We assume (A). Then, for $n \in \mathbb{Z}$ and outer functions $h$ and $h_\zeta$ in $H^{1\times q}_\sigma(\mathbb{T})$ satisfying (2.2) and (2.3), respectively, the following two equalities hold:

$$(2.4) \quad L_{(-\infty,n]}(w) = z^n \cdot H^{1\times q}_\sigma(\mathbb{T}) \cdot h^{-1},$$

$$(2.5) \quad L_{[n,\infty)}(w) = z^{-n} \cdot H^{1\times q}_\sigma(\mathbb{T}) \cdot (h_\zeta^*)^{-1}.$$

Proof. We prove only (2.5); one can prove (2.4) in a similar way. Define an antilinear bijection $G : L(w) \rightarrow L^{1\times q}_\sigma(\mathbb{T})$ by $G(f) := f^\dagger h_\zeta^\dagger$. Since

$$\|G(f)\|_{I_q}^2 = \|f^\dagger h_\zeta^\dagger\|_{I_q}^2 = \int_0^\pi f(e^{i\theta})h_\zeta(e^{i\theta})^* \{f(e^{i\theta})h_\zeta(e^{i\theta})^*\}^* \sigma(d\theta) = \|f\|_w^2,$$

the map $G$ preserves the norms of $f \in L(w)$. Let

$$\mathcal{C}^{1\times q}[z] := \sp\{e_j(k) : k \leq 0, \ j = 1, \ldots, q\}$$
be the space of polynomials with coefficients in $\mathbb{C}^{1 \times q}$. Since $h_4^T$ is also an outer function in $H^2_2(\mathbb{T})$, it follows from the Beurling–Lax–Halmos Theorem that $\mathbb{C}^{1 \times q}[\cdot] \cdot h_4^T$ is dense in $H^2_2(\mathbb{T})$ (cf. [KK, Remark 5.6 and Theorem 5.3]). Moreover, 
$$L_{[n, \infty)}(w) = \mathfrak{sp}\{e_j(k) : k \geq n, \ j = 1, \ldots, q\}$$
and 
$$G(\mathfrak{sp}\{e_j(k) : k \geq n, \ j = 1, \ldots, q\}) = z^n \cdot \mathbb{C}^{1 \times q}[\cdot] \cdot h_4^T.$$
Thus,
$$L_{[n, \infty)}(w) = G^{-1}\left(z^n \cdot H^2_1(\mathbb{T}) \cdot (h_4^T)^{-1}\right),$$
as desired. \hfill \Box

3. The Past and Future

For a $q$-variate stationary process $\{X(k)\}$, the next theorem holds without (A).

**Theorem 3.1.** A $q$-variate CND process satisfies (IPF).

**Proof.** For any $q$-variate stationary process $\{X(k)\}$, we have

$$M^X_{(-\infty, n]} = M^X_{(-\infty, m-1]} + M^X_{[m, n]}, \quad m, n \in \mathbb{Z}, \ m \leq n.$$  

For, the inclusion $\supset$ is trivial, while $M^X_{(-\infty, m-1]}$ is closed and $M^X_{[m, n]}$ is finite-dimensional, whence $M^X_{(-\infty, m-1]} + M^X_{[m, n]}$ is also closed (see Halmos [H, Problem 8]), which implies $\subset$.

For $n \in \mathbb{N}$, let $x \in M^X_{(-\infty, -1]} \cap M^X_{(-n, -1]}$. Since $x \in M^X_{(-\infty, -1]}$, it follows from (3.1) that $x = y + z$ for some $y \in M^X_{(-\infty, -1]}$ and $z \in M^X_{[-n, -1]}$. Since $x, z \in M^X_{(-n, -1]}$, we have

$$y = x - z \in M^X_{(-\infty, -1]} \cap M^X_{(-n, -1]}.$$ 

Therefore, if $\{X(k)\}$ is CND, then $y = 0$ or $x = z \in M^X_{(-n, -1]}$, so that

$$M^X_{(-\infty, -1]} \cap M^X_{(-n, -1]} \subset M^X_{(-n, -1]}.$$ 

Since the converse inclusion $\supset$ is trivial, $\{X(k)\}$ satisfies (IPF). \hfill \Box

**Remark 3.2.** The converse of Theorem 3.1 does not hold without additional assumptions. For example, let $\{Y(k) : k \in \mathbb{Z}\}$ be a univariate CND stationary process; the simplest example is a white noise. Then $\{Y(k)\}$ is PND. Define a two-variate stationary process $\{X(k) : k \in \mathbb{Z}\}$ by $X(k) := (Y(k - 1), Y(k))^T$. For $I \subset \mathbb{Z}$, let $M^Y_I := \mathfrak{sp}\{Y(k) : k \in I\}$ in $L^2(\Omega, \mathcal{F}, P)$. Then, for $n, m \in \mathbb{Z}$ with $n \leq m$, we have

$$M^X_{(-\infty, -1]} \cap M^X_{(-n, -1]} = M^Y_{(-\infty, -1]} \cap M^Y_{(-n, -1]}.$$ 

Since $\cap_n M^X_{(-\infty, -1]} = \cap_n M^Y_{(-\infty, -1]} = \{0\}$, $\{X(k)\}$ is PND. Furthermore, for $n \geq 1$,

$$M^X_{(-\infty, -1]} \cap M^X_{(-n, -1]} = M^Y_{(-\infty, -1]} \cap M^Y_{(-n, -1]} = M^Y_{(-n, -1]},$$ 

whence $\{X(k)\}$ satisfies (IPF). However,

$$M^X_{(-\infty, -1]} \cap M^X_{[0, \infty)} = M^Y_{(-\infty, -1]} \cap M^Y_{[-n, -1)} = M^Y_{[-n, -1]} \neq \{0\},$$ 

whence $\{X(k)\}$ is not CND. Notice that $\{X(k)\}$ has the degenerate spectral density

$$w_X(\xi) = \begin{pmatrix} w_Y(\xi) & \xi w_Y(\xi) \\ \xi^* w_Y(\xi) & w_Y(\xi) \end{pmatrix},$$ 

where $w_Y$ is the spectral density of $\{Y(k)\}$.
We assume (A), and for outer functions \( h \) and \( h_2 \) in \( H^2_2(T) \) satisfying (2.2) and (2.3), respectively, we consider the following two conditions:

(3.2) \[ \left\{ z^{-1} \cdot H^2_2(T) \cdot (h_2^*)^{-1} \right\} \cap \left\{ H^1_2(T) \cdot h^{-1} \right\} = \{0, \ldots, 0\}\]

(3.3) \[ \left\{ H^2_2(T) \cdot (h_2^*)^{-1} \right\} \cap \left\{ H^2_2(T) \cdot h^{-1} \right\} \subseteq C^{1 \times q}\]

For any \( a \in C^{1 \times q} \), we have \( ah^*_2 \in H^2_2(T), ah \in H^2_2(T) \) and

\[ a = ah^*_2(h_2^*)^{-1} = ah^{-1} \]

whence the inclusion \( \supset \) in (3.3) always holds.

Let \( X(k) = \int_{-\pi}^{\pi} e^{-ik\theta} Z(d\theta) \), \( k \in \mathbb{Z} \), be the spectral representation of \( \{X(k)\} \) satisfying (A), where \( Z \) is the random spectral measure such that

\[ E[Z(A_1)Z(A_2)^*] = \int_{A_1 \cap A_2} w(e^{i\theta})\sigma(d\theta), \quad A_1, A_2 \in B([-\pi, \pi]) \]

Define an isometric isomorphism \( S : L(w) \to M \) by

\[ S(f) := \int_{-\pi}^{\pi} f(e^{i\theta})Z(d\theta), \quad f \in L(w) \]

Then, \( S(c_j(k)) = X_j(k) \) for \( k \in \mathbb{Z} \) and \( j = 1, \ldots, q \), whence we have

\[ S(L_I(w)) = M^X_I, \quad I \subseteq \mathbb{Z} \]

**Lemma 3.3.** We assume (A). Then, the following two conditions are equivalent:

1. (3.2) holds.
2. \( M^X_{[0, 0]} \cap M^X_{[1, 1]} = \{0\} \).

**Proof.** By (3.4), (2) is equivalent to \( L_{[(-\infty, 0]}(w) \cap L_{[1, \infty)}(w) = \{0, \ldots, 0\} \), which, in turn, is equivalent to (1) by Lemma 2.2.

**Lemma 3.4.** We assume (A). Then, the following two conditions are equivalent:

1. (3.3) holds.
2. \( M^X_{[0, 0]} \cap M^X_{[0, \infty]} = M^X_{[0]} \).

**Proof.** We have \( L_{[0]}(w) = \text{sp}\{c_j(0) : j = 1, \ldots, q\} = C^{1 \times q} \). Hence, by (3.4), (2) is equivalent to \( L_{[(-\infty, 0]}(w) \cap L_{[0, \infty)}(w) = C^{1 \times q} \), which, in turn, is equivalent to (1) by Lemma 2.2.

Here is the main theorem of this paper.

**Theorem 3.5.** We assume (A). Then, the following five conditions are equivalent:

1. (3.2) holds.
2. (3.3) holds.
3. (CND) holds.
4. \( M^X_{[0, 0]} \cap M^X_{[-\infty, -1]} = M^X_{[-n, -1]} \) for some \( n \in \mathbb{N} \).
5. (IPF) holds.

**Proof.** By Lemma 3.3, (1) and (3) are equivalent. By Lemma 3.4, (2) (resp., (5)) implies (4) (resp., (2)). By Theorem 3.1, (3) implies (5). Suppose (4). Then,

\[ M^X_{[-\infty, -1]} \cap M^X_{[0, 0]} \subseteq M^X_{[-\infty, -1]} \cap M^X_{[-\infty, -1]} = M^X_{[-n, -1]} \]

\[ M^X_{[-\infty, -1]} \cap M^X_{[0, 0]} \subseteq M^X_{[-n, -1]} \cap M^X_{[0, 0]} = M^X_{[0, n-1]} \]
However, by Lemma 2.1, we have $M^X_{[-n,-1]} \cap M^X_{[0,n-1]} = \{0\}$, whence (3). □

The next corollary gives a sufficient condition for (IPF) in terms of the spectral density.

**Corollary 3.6.** We assume (MR) and that the spectral density $w$ of $\{X(k)\}$ satisfies $w^{-1} \in L_1^{1\times q}(T)$. Then $\{X(k)\}$ satisfies (IPF).

**Proof.** Since $(w^{-1})_{i,j} = \sum_{i=1}^{q} (h^{-1}_{i,j})^2$ for $j = 1,\ldots,q$, the condition $w^{-1} \in L_1^{1\times q}(T)$ implies $h^{-1} \in L_1^{1\times q}(T)$. Hence, by [KK, Theorem 3.1] and [RR, Theorem 4.23], $h^{-1} \in H_2^{1\times q}(T)$, so that

$$H_2^{1\times q}(T) \cdot h^{-1} \subset H_1^{1\times q}(T).$$

Similarly, we have $(h_2)^{-1} \in H_2^{1\times q}(T)$, and

$$H_2^{1\times q}(T) \cdot (h_2)^{-1} \subset H_1^{1\times q}(T).$$

However, $H_1^{1\times q}(T) \cap H_1^{1\times q}(T) = C^{1\times q}$, whence (3.3). Therefore, by Theorem 3.5, $\{X(k)\}$ satisfies (IPF). □

**Remark 3.7.** A stationary process $\{X(k)\}$ is said to be minimal if $X(0)$ cannot be interpolated precisely using all the other values of the process. The condition $w^{-1} \in L_1^{1\times q}(T)$ in Theorem 3.5 is known to be necessary and sufficient for the minimality of a stationary process. See Section 10 of [R, Chapter II].

The next corollary gives an example of $\{X(k)\}$ with (IPF) for which $w^{-1}$ is not integrable (compare [BJH, Proposition 3]).

**Corollary 3.8.** Let $B$ be an invertible matrix in $\mathbb{C}^{q\times q}$. Then $\{X(k)\}$ with spectral density $w(e^{i\theta}) = |1 + e^{i\theta}|BB^*$ satisfies (IPF).

**Proof.** We can take $h = (1 + z)^{1/2}B$ and $h_2 = (1 + z)^{1/2}B^*$. Suppose that there exist $f = (f_1,\ldots,f_q)$, $g = (g_1,\ldots,g_q) \in H_2^{1\times q}(T)$ such that

$$z^{-1}f(h_2^*)^{-1} = gh^{-1}.$$

Then, since $(h_2^*)^{-1}h = e^{i\theta/j}I_q$ for $z = e^{i\theta}$ ($-\pi < \theta < \pi$), we have

$$e^{-i\theta} \left( \sum_{j=1}^{q} (e^{i\theta}) \right)^2 = \left( g_j(e^{i\theta}) \right)^2, \quad j = 1,\ldots,q.$$

From $(g_j)^2 \in H_1(T)$, we get

$$\int_{-\pi}^{\pi} e^{im\theta} \left( g_j(e^{i\theta}) \right)^2 \sigma(d\theta) = 0$$

for $m = 1,2,\ldots$, while, from $(f_j)^2 \in H_1(T)$ and (3.5), we see that (3.6) also holds for $m = 0,-1,\ldots$, whence $g_j = 0$ for $j = 1,\ldots,q$. Thus (3.2) holds. Therefore, by Theorem 3.5, $\{X(k)\}$ satisfies (IPF). □
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