Zero-point gravitational field equations

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We study the recently reported $q$-metric (or zero-point-length) expressions of the Ricci (bi)scalar $R(q)$ (namely, expressions of the Ricci scalar in a spacetime with a limit length $L_0$ built in), focusing specifically on the case of null separated events. A feature of these expressions is that, when considered in the coincidence limit $p \to P$, they generically exhibit a dependence on the geodesic along which the varying point $p$ approached $P$, sort of memory of how $p$ went to $P$. This fact demands a deeper understanding of the meaning of the quantity $R(q)$, for this latter tells about curvature of spacetime at $P$ and would not be supposed to depend on whichever vector we might happen to consider at $P$. Here, we try to search for a framework in which these two apparently conflicting aspects might be consistently reconciled. We find a tentative sense in which this could be achieved by endowing spacetime of a specific operational meaning. This comes, however, at the price (or with the benefit) of having a spacetime no longer arbitrary but, in a specific sense, constrained. The constraint turns out to be in the form of a relation between spacetime geometry in the large scale (as compared to $L_0$) and the matter content, namely as sort of field equations. This comes thanks to something which happens to coincide with the expression of balance of (matter and spacetime) exchanged heats, i.e. the thermodynamic variational principle from which the field equations have been reported to be derivable. This establishes a link between (this specific, operational understanding of) the meaning of the limit expression of $R(q)$ on one side and the (large-scale) field equations on the other, this way reconnecting (once more) the latter to a quantum feature.

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I. INTRODUCTION

When trying to combine gravity and quantum mechanics, a variety of results points towards the existence of a lower-limit length $L_0$ (1–14, and 15, 16 for review and further references). The likelihood would then be that the spacetime one is called to consider ought to exhibit, in the small scale, this feature. The approach developed in 17–19, seeks precisely to implement this through modification of the ordinary metric to an effective metric (also called $q$-metric). It aims to provide the specific metric framework that the spacetime should possess, if it has to display a limit length in the small scale.

Among the encouraging results in addressing this way potential quantum features of spacetime 18–24, those related to the Ricci scalar $R$ can be, due to the role this quantity plays in general relativity as gravitational field Lagrangian, somehow directly exploitable to shed light on potential quantum aspects of field equations themselves. In particular, the results 18, 19 showed an expression for the zero-point-length Ricci (bi)scalar $R(q)(p, P)$ (depending on points $P$ and $p$) which in the small scale ($p \to P$) intriguingly differs, in the limit $L_0 \to 0$, from the value of the (classic) Ricci scalar at the given point $P$. This limit expression turned out to be

$$\lim_{L_0 \to 0} \lim_{p \to P} R(q)(p, P) = \epsilon D R_{ab}l^a l^b,$$

(1)

where $D$ is the dimension of spacetime, $R_{ab}$ is the ordinary Ricci tensor (at $P$), $l^a$ is the normalized tangent vector at $P$ to the ordinary geodesic connecting two space or time separated events $P$ and $p$, and $\epsilon \equiv g_{ab}t^a t^b = \pm 1$. In case of null separated events, a recent analogous investigation has given 25

$$\lim_{L_0 \to 0} \lim_{p \to P} R(q)(p, P) = (D - 1) R_{ab}l^a l^b,$$

(2)

with $l^a$ the ordinary null tangent vector at $P$ to the geodesic from $P$ to $p$.

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These expressions clearly show that $\lim_{L_{\alpha}\to 0}\lim_{t_{\alpha}\to P} R_{(q)}(p, P) \equiv R_{\ast} \neq R_{\ast}$ as mentioned. In fact they possess an additional feature, the investigation of which is the focus of present study: they exhibit an explicit dependence on the tangent to the geodesic at $P$, i.e. $R_{\ast} = R_{\ast}(x, t^{\alpha})$ or $R_{\ast} = R_{\ast}(x, l^{\alpha})$, with $x$ concisely denoting the coordinates of $P$. This fact raises the question of how we should interpret a qmetric Ricci scalar at a point $P$, if the value we could think of as assigned to it by continuity along any one direction of approach, does not match with what we find along another one; this, strictly speaking, forbidding to have $R_{(q)}$ smoothly defined at $P$. This demands for further understanding. And as such, this result should be considered not as an issue, but as a virtue. One should consider it as hinting to some deeper and as yet unveiled feature, of a quantum theory of spacetime.

Other curvature-related scalars might be of help. In particular, the Kretschmann scalar might allow to characterize what happens from a minimum-length standpoint in Ricci-flat spacetimes, in which we see from the above that the limit $R_{(q)}$ vanishes with $R_{ab}$. This will deserve scrutiny as soon as a minimum-length expression for the Kretschmann scalar will be available.

The techniques used for extracting an expression for $R_{(q)}$ are point-splitting and coincidence-limit procedures, similar to those used in the works aiming to find regularized expressions for the expectation value of stress-energy tensor $(T_{ab})$ on a curved background $[24–26]$. The works in this latter context, specifically in $[26]$, quantities have been considered (intervening in the expression of $(T_{ab})$) which do exhibit in principle a residual dependence on the separation direction after the coincidence limit is taken, a situation with analogies to what described here. There, the need was to have $(T_{ab})$ a definite quantity assigned at a point, and the way to face the dependence on the separation direction was basically to average over the possible directions $(T_{ab})$ is an expectation value after all).

We might try to do the same, but the focus of present study, meant as a first step towards a more comprehensive analysis, is not on a “need” of a single-valued quantity, but to pause and take note of the fact that in a spacetime endowed with limit length we do not get a single-valued quantity, and consider it like a possible glimpse of some possible underlying truth one might want to try to extract. We do this way, prompted by the fact that it is not at all obvious a priori that a spacetime with a limit length has to show such a multivaluedness for $R_{(q)}$, and as a matter of fact it came as a surprise when it first was found $[18–20]$. In this vein, in the present work we try to gain some insight into this fact looking at it from the following perspective. We ask: Is there some sense in which the coincidence limit of $R_{(q)}$ along some given path, can be considered independent of $l^{\alpha}$ at $P$? Could we distinguish between path independence of the coincidence limit of $R_{(q)}$, a thing which clearly we mathematically do not have, and the operational notion of independence from $l^{\alpha}$ of the value we obtain for the limit of $R_{(q)}$ once a probe of $R_{(q)}$ at $P$ has been taken already? Is there any meaning in this? Our hope is that actually there is, and that something interesting could be extracted through knowledge of whether it is possible to have an independence from $l^{\alpha}$ of the probed $R_{(q)}$ at $P$; and, in case of affirmative answer, one would like to find out what this might mean. This is what we try to do here, considering specifically the case of null separated events.

II. DEFINITION OF THE PROBLEM

Let us consider a point $P$ in $D$-dimensional spacetime $M$ ($D \geq 4$; metric $g_{ab}$ with signature $(-, +, +, ...)$) and consider the zero-point-length metric $q_{ab}(p, P)$ with base $P$ as defined for points $p$ null separated from $P$. It reads $[30]$

$$q_{ab} = A g_{ab} + \left(A - \frac{1}{\alpha}\right)(l_{a} n_{b} + n_{a} l_{b}),$$

where $l^{a}$ is the ordinary (null) tangent, at $p$, to the geodesic connecting $P$ and $p$, $n^{a}$ is an auxiliary null vector with $g_{ab} n^{a} l^{b} = -1$ and $g_{ab} n^{a} e^{b} = 0$ for any spacelike vector $e^{a}$ transverse to $l^{a}$ (i.e. $g_{ab} l^{a} e^{b} = 0$), and $\alpha = \alpha(p, P)$ and $A = A(p, P)$ are biscalars, functions of the difference of affine parameter $\lambda(p, P) \equiv \lambda$ given by

$$\alpha = \frac{d\lambda}{d\lambda}$$

and

$$A = \frac{\dot{\lambda}^{2}}{\lambda^{2}} \left(\frac{\Lambda}{\lambda}\right)^{\frac{\Lambda^{2}}{2\Lambda}},$$
where \( \lambda = \tilde{\lambda}(\lambda) \) is the difference in the qmetric-affine parameter, which has \( \tilde{\lambda} \to L_0 \) for \( p \to P \). The derivative in (1) is thought as taken at \( p \), and

\[
\Delta(p, P) = -\frac{1}{\sqrt{g(p)g(P)}} \det \left[ -\nabla_a(p)\nabla_b(p) \frac{1}{2} \sigma^2(p, P) \right]
\]

(6)

(\( \sigma^2 \) is the squared geodesic distance) is the van Vleck determinant \([26, 27, 31, 32]; \text{ see } [28, 33, 34]\) and

\[
\hat{\Delta}(p, P) = \Delta(\tilde{p}, P),
\]

(7)

with \( \tilde{p} \in \gamma \) such that \( \lambda(\tilde{p}, P) = \lambda \). In the limit \( \lambda/L_0 \to \infty \), \( \alpha \) and \( A \) satisfy \( \alpha \to 1 \) and \( A \to 1 \) \([\ref{30}]\), and the ordinary metric \( g_{ab} \) is recovered.

In this description through the qmetric \( g_{ab} \), all the effects of the degrees of freedom of the as-yet-unknown microscopic theory are supposed to have been encoded in the function \( \lambda = \lambda(\lambda) \) (this for null separations; a function \( S = S(\sigma^2) \), with \( S \) the squared geodesic distance modified according to the qmetric, analogously encodes these effects for time or space separations \([17, 19]\)). This function is conceived as ‘universal’, where we mean with this, any time we have a \( \lambda \) we get a corresponding \( \lambda(\lambda) \) irrespective of the specific geometric characteristics of the spacetime at the point under consideration or the dynamical evolution it is experiencing. The approximation is then such that the biscalar \( \lambda \) has no proper dynamics distinct from the (possible) dynamics of \( \sigma^2 \) or \( g_{ab} \): its evolution is completely determined by that of \( g_{ab} \). Moreover, the main interest within our approach is in taking the coincidence limit \( p \to P \), and the only aspect which matters is the fact that \( \lambda \to L_0 \), with no regard to the details of how \( \lambda \) reaches \( L_0 \). The general idea would then be that \( \lambda(\lambda) \) is actually determined by degrees of freedom pertaining to the unknown description of quantum gravity, but the level of our approximation here is such that the effects of these dofs are, in a sense, meant as frozen in the \( \lambda(\lambda) \) (we do not track their own evolution in connection with \( g_{ab} \) evolution) when \( \lambda \gg L_0 \), and essentially amount in \( \lambda \to L_0 \neq 0 \) when \( \lambda \to 0 \).

We borrow now the expression for the qmetric Ricci scalar \( R_{(q)} \) for null separated events from \([23]\):

\[
R_{(q)}(p, P) = \frac{1}{A} R_\Sigma - 2 \alpha \frac{d\alpha}{d\lambda} K + 2\alpha^2 R_{ab} l^a l^b - (D - 2) \alpha \frac{d\alpha}{d\lambda} \frac{d}{d\lambda} \ln A - (D - 2) \alpha^2 \frac{d^2}{d\lambda^2} \ln A
\]

- \[
\quad - \frac{1}{4}(D - 2)(D - 1) \alpha^2 \left( \frac{d}{d\lambda} \ln A \right)^2 - \alpha^2 K^2 + \alpha^2 K^{ab} K_{ab} - (D - 1) \alpha^2 \left( \frac{d}{d\lambda} \ln A \right) K.
\]

Here, the circumstances are assumed to be that the Ricci scalar at \( P \) is completely described in terms of a congruence of affinely parameterized (parameter \( \lambda \) for \( g_{ab} \) and \( \lambda \) for \( g_{ab} \), with \( g_{ab} \)-tangent \( l^a \)) null geodesics emerging from \( P \), and the expression applies in the limit of \( \lambda \) small. \( \Sigma = \Sigma(P, \lambda) \) is the \((D - 2)\)–surface locus of the points \( p' \), each on a null geodesic from \( P \) and in the future of it, at the \( \lambda \) corresponding to \( p \), i.e. \( \Sigma(P, \lambda) = \{ p' \in L : \lambda(p', P) = \lambda(> 0) \} \), with \( \lambda = \lambda(p, P) \) fixed, where \( L = \{ p' \in M : \sigma^2(p', P) = 0, \text{ and } p' \text{ in the future of } P \} \). All vectors and tensors in expression \([\ref{8}]\) are ordinary –i.e. not qmetric– vectors and tensors and are evaluated, as well as the scalars \( R_\Sigma \) and \( K \), at \( p \), and indices are lowered and raised using \( g_{ab} \) and \( g^{ab} \). \( R_\Sigma \) is the Ricci scalar intrinsic to \( \Sigma \), \( K_{ab} \) the transverse field \( K_{ab} = h^{c} h^{d} \nabla_c l_d \) with \( h_{ab} = g_{ab} + l_a n_b + n_a l_b \), the transverse metric, and \( K = K^a a \).

When \( \lambda \) is small but \( \lambda \gg L_0 \), we have \( \alpha \simeq \text{const} = 1 \) and \( A \simeq \text{const} = 1 \), and several terms on the rhs of \([\ref{8}]\) are vanishing. Writing, in these circumstances,

\[
\alpha = 1 + \epsilon \Phi(\lambda),
\]

(9)

\[
A = 1 + \epsilon \Psi(\lambda)
\]

(10)

with \( \Phi, \Psi \) smooth functions and \( \epsilon \ll 1 \) constant, and assuming that not only the functions \((\alpha - 1)\) and \((A - 1)\) are small but that also their derivatives of every order are small with them when \( \lambda/L_0 \gg 1 \), what we are left with is

\[
R_{(q)}(p, P) = R_\Sigma(p) + 2 R_{ab}(p) l^a(p) l^b(p) - K^2(p) + K^{ab}(p) K_{ab}(p) + O(\epsilon R),
\]

(11)

with \( R \) a typical component of Riemann tensor. Thus, at leading order,
with last equality from \(23\) (equation (45) there, for \(\lambda\) small) (as for Gauss-Codazzi equations, generalized to the case of null hypersurfaces, see e.g. \(35\) and \(36\)).

This result shows that at large scale (meaning this that, even if \(\lambda = \lambda(p, P)\) is small, we have \(\lambda \gg L_0\) the qmetric with base at \(P\) gives for the Ricci scalar at \(P\) an expression with no dependence on the tangent \(l^a\) at \(P\), and this irrespective to the (small) value of \(L_0\). One could argue that this refers strictly speaking to \(p\), not \(P\), and that \(p\) can be after all also far away from \(P\). But, what we just said can anyway be used to tell what the qmetric curvature is at \(P\), precisely. To this end, let us consider the following. Fixed a scale, i.e. assigned a value for \(\lambda/L_0\), the qmetric geometry at \(P\) can be computed using null geodesics connecting different events \(P', P'', \ldots\) with \(P\), each chosen to have \(\lambda'(P, P') = \lambda''(P, P'') = \lambda\) and using as base points \(P', P'', \ldots\). For the qmetric Ricci scalar at \(P\), at leading order this gives

\[
R_{(q)}(P, P') = R_{(q)}(P, P'') = \ldots = R(P)
\]

(13)

i.e., provided the event \(P\) is reached by a null geodesic from an event \(P'\) with \(\lambda(P, P') \gg L_0\), the value of the qmetric Ricci scalar in \(P\) has at leading order no dependence on the chosen geodesic and does coincide with the value there of ordinary Ricci scalar \(R(P)\). As such, it exhibits no dependence on the tangent \(l^a\) to the geodesic in \(P\). We can summarize the results \(\{12\}\) and \(\{13\}\) by saying that \(R(P)\) gives the expression of the (minimum-length) Ricci scalar at \(P\) in the large scale with no dependence in it on the geodesic we may have used to reach \(P\). We can write this as

\[
R_{(q)}^{(\text{Macro})}(P) = R(P),
\]

(14)

having defined

\[
\lim_{\lambda \to \infty} R_{(q)}(P, P') = \lim_{\lambda'' \to \infty} R_{(q)}(P, P'') = \ldots = R_{(q)}^{(\text{Macro})}(P).
\]

(15)

In the small scale, the situation appears very different. What one finds from equation \(\{28\}\) in the limit \(p \to P\), is (see appendix \(A\))

\[
\lim_{p \to P} R_{(q)}(p, P) = (D - 1) (R_{ab} l^a l^b)(\{\bar{p}\}) + \mathcal{O}\left(\frac{L_0}{L_R} R_{ab} l^a l^b(P)\right),
\]

(16)

\[
= (D - 1) (R_{ab} l^a l^b)(\{\bar{p}\}) + \mathcal{O}\left(\frac{L_0}{L_R} R_{ab} l^a l^b(\{\bar{p}\})\right),
\]

with \(\bar{p}\) the event on the null geodesic at \(\lambda(\bar{p}, P) = L_0\), and \(L_R = 1/\sqrt{R_{ab} l^a l^b(P)}\) a length scale associated, for the given \(l^a\), with the assigned curvature at \(P\). In writing this, we assume that our ordinary spacetime obeys the null convergence condition, this then ensuring \(R_{ab} l^a l^b\) is non-negative. With \(L_0\) orders of Planck length, for ordinary curvatures we generically assume that we are at conditions in which \(L_0/L_R \ll 1\) (this implying to say that the event \(\bar{p}\) at \(\lambda = L_0\) is near enough to \(P\) to give \(|g_{ab}(\bar{p})| = \mathcal{O}(R_{abcd} L_0^2 \ll 1\) in a local frame (with Riemann normal coordinates) in which \(\lambda\) is length), and that the effects of the non leading terms in equation \(\{16\}\) result negligible (mathematically, what we are assuming is that \(L_0\) belongs to a right neighbourhood \([0, \bar{L}]\) of \(0\) with \(L\) small enough that this is satisfied). Whether this –at some event in an actual spacetime– can be appropriate or not, must be checked carefully, and how to tag potentially non-negligible terms is discussed in appendix \(A\).

As above, the limiting behaviour of \(R_{(q)}(p, P)\) can be seen as telling us what the qmetric curvature is at \(P\), this time however at a small scale. We have just to look at null geodesics \(\gamma', \gamma'', \ldots\), with \(g_{ab}\)-affine parameters \(\lambda', \lambda'', \ldots\), arriving at \(P\) and having started at points \(P', P'', \ldots\) with \(\lambda'(P, P') = \lambda''(P, P'') = \ldots = L_0\). This gives

\[
R_{(q)}(P, P') \neq R_{(q)}(P, P'') \neq \ldots
\]

(17)

in general, with
\[ R(q)(P, P') = (D - 1)(R_{ab}t^at^b)(P), \]
\[ R(q)(P, P'') = (D - 1)(R_{ab}t'^at'^b)(P), \]
\[ \ldots \]

\( (t^a, t'^a, \ldots) \) are tangents at \( P \) to the geodesics \( \gamma', \gamma'', \ldots \) at leading order. The net result coincides with what one gets considering equation (18) in the limit \( L_0 \rightarrow 0 \), i.e., equation (1).

What we have thus is a situation in which the qmetric Ricci (bi)scalar is probed (through null separations) at a large scale (\( \lambda \gg L_0 \)) at a generic point \( P \), its value does coincide with ordinary Ricci scalar at \( P \) and, of course, does not depend on the path through which we reached \( P \); when instead we probe it at a smaller scale, potentially to the smallest conceivable scale (\( \lambda \rightarrow 0 \)), the qmetric Ricci (bi)scalar deviates from its ordinary value and, moreover, acquires a dependence on the geodesic path followed to reach \( P \). This memory of the path then is not present macroscopically but appears to unavoidably arise in the small scale.

We note that this is not something about inhomogeneity, i.e., what might be expected if one could imagine spacetime as somehow granular at the small scale, we have indeed a dependence on the direction. It is not about anisotropy either, for the dependence of the direction we have is not in the sense of something we get when leaving \( P \) along one direction rather than another, but something defined at \( P \), i.e., pertaining to event \( P \). This entails that \( R(q) \), meant as a function, cannot have a small-scale smooth definition at \( P \); for results, it cannot be continuously prolonged at \( P \); this even if we had decided to give up, at the smallest scale, with the notion of ‘point’ (cf. [37]), for anyway the small-scale value we should assign by continuity to the Ricci scalar would depend on the direction through which we have reached the ‘spot’ that potentially replaces \( P \).

This brings to the following consideration. \( R(q) \) as defined at a point \( P \) might be sort of multiple-valued entity, expressing, from an operational point of view, an unprobed configuration. Different values would correspond to different results of probes at \( P \); the difference in the values would then reflect a difference on the results of measurements, not a dependence of \( R(q) \) itself at \( P \) on direction. Indeed such kind of dependence would seem hardly acceptable, for \( R(q) \) at \( P \) is an intrinsic geometric property of spacetime, and as such we can expect it not to be dependent on whichever vector we can consider at \( P \).

We see, this perspective suggests a description which might be quantum. More explicitly, taking the coincidence limit along a macroscopically assigned geodesic can be thought of as performing a measurement on the quantum system consisting of spacetime at \( P \) of some quantum observable \( \hat{R} \) expressing the Ricci scalar. In connection with the given macroscopic value of \( R_{ab} \) at \( P \), the measurement on the unprobed system is assumed to provide the result (\( D - 1 \))\( R_{ab}l^al^b \) with \( l^a \) the null vector at \( P \) in the direction along which we reach \( P \). In doing so, what is supposed to happen is that, even if we started on a macroscopically assigned geodesic, we actually reach \( P \) microscopically along a specific direction chosen virtually at random among all directions at \( P \), due to the uncertainty in the momentum as we get closer and closer to \( P \). After the measurement, quantum mechanics requires that the quantum system (spacetime at \( P \)) is in an eigenstate associated to the eigenvalue (\( D - 1 \))\( R_{ab}l^al^b \) of \( \hat{R} \), still with a same macroscopic \( R_{ab} \). In any further measurement of the already probed Ricci scalar slightly afterwards, whichever is the tangent \( l'^a \) with which we now reach \( P \), and which, were the system unprobed, would give (\( D - 1 \))\( R_{ab}l'^al^b \), we are required to get that same value (\( D - 1 \))\( R_{ab}l^al^b \).

In this perspective, there should be some mechanism of quantum mechanical origin in action, which, once the system has been already probed, prevents to find as a result of a further measurement something different from what already found. In other words, the probe at \( P \) of this intrinsic geometric quantity may well depend on \( l^a \), but once we get a value, this should be considered as not dependent anymore on the vector \( l^a \) at \( P \), thus representing a geometric property of spacetime as a whole there. This might result somehow puzzling. The following is an attempt to make sense of it. We go to see a potential way this could make sense, this going hand in hand however with the (large scale) spacetimes we are dealing with cannot be given arbitrarily. This will raise the question of which turns out to be the relationship between these spacetimes we get and actual, experimentally probed, spacetime.

III. IRRELEVANCE OF \( l^a \) AFTER A PROBE (EMPTY SPACE)

We have independence from \( l^a \) at \( P \) after a probe, if the above-defined quantity, \( R_\ast \equiv (D - 1)R_{ab}l^al^b \), which has a manifest dependence on \( l^a \), i.e., \( R_\ast = R_\ast(x, l^a) \), does result, as a consequence of constraints on \( R_{ab} \), to be actually independent of \( l^a \). We describe this, writing

\[
\frac{\partial}{\partial l^a} \left[(D - 1) R_{cd}l^c l^d + \mu g_{cd}l^c l^d \right] = 0, \text{ at any } l^a \text{ null.} \quad (19)
\]
Here \( \mu = \mu(x) \) is a scalar which acts as a Lagrange multiplier. Its introduction corresponds to require that the variation is done while keeping \( l^a \) null, i.e. \( g_{ab}l^a l^b = 0 \). This is to be consistent with the fact that the expression \((D - 1) R_{cd} l^c l^d\) is specific to the case of null separations. We would like to emphasize that in writing (19) we are not taking any directional derivative: what we are considering is the quantity \( \Omega \equiv \lim_{p \to P} R(q) \) at \( P \), with this quantity explicitly depending (and this is precisely the item we are addressing) on the way \( p \) approached \( P \). The varied value of \( \Omega \) is always still at \( P \), and is obtained simply varying \( l^a \) in the expression of \( \Omega \), forgetting how \( l^a \) came about.

Equation (19) gives

\[
[(D - 1) R_{ac} + \mu g_{ac}] l^c = 0, \quad \forall l^a \text{ null.}
\]

If this is satisfied, it also is

\[
[(D - 1) R_{ab} + \mu g_{ab}] l^a l^b = 0, \quad \forall l^a \text{ null,}
\]

i.e.

\[
R_{ab} l^a l^b = 0, \quad \forall l^a \text{ null.} \tag{20}
\]

This means that, looking at potential irrelevance of \( l^a \) through equation (19), leaves as unique configuration that which satisfies equation (20). Notice this gives \( L_R = \infty \); we are thus surely at conditions in which in equation (10) we have \( L_0 \ll L_R \).

We can readily inspect the characteristics of this ordinary spacetime. Equation (20) implies

\[
R_{ab} = \xi g_{ab} \tag{21}
\]

with \( \xi = \xi(x) \) a scalar. This gives

\[
G_{ab} = \left( \xi - \frac{1}{2} R \right) g_{ab}, \tag{22}
\]

which, from Bianchi identity, implies \( \partial_a (\xi - \frac{1}{2} R) = 0 \), namely \( \xi - \frac{1}{2} R = \text{const} \), and thus (22) reads

\[
G_{ab} = C g_{ab}, \tag{23}
\]

with \( C \) a constant, independent of \( x \) (as for the mathematical procedure we have followed here, cf. 38 exercise 15.3).

Summing up, in order for equation (19) to hold, equation (23) must hold. We see then that the obtaining of irrelevance of \( l^a \) after a probe, as implemented through equation (19), leaves as unique configuration that which satisfies equation (20). Notice this gives \( L_R = \infty \); we are thus surely at conditions in which in equation (10) we have \( L_0 \ll L_R \).

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R_{ab} l^a l^b = 0, \quad \forall l^a \text{ null.} \tag{20}
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We note that equation (23) has the nature of (vacuum) field equations. Then, independence of \( l^a \) after a given probe of \( R(q) \) at \( P \) results connected to the classical metric not being generic, but obeying instead something which has the status of field equations. From (23), all Einstein spacetimes, i.e. in particular all vacuum solutions to Einstein equations, do admit a quantic description in which the quantum Ricci scalar at \( P \) can be consistently considered (in the operational sense above) an intrinsic geometric property of spacetime as a whole.

We may wonder whether this exhausts all spacetimes which do admit such a consistent quantic description. Clearly, one would not expect this to be the case. Retracing what we have done, it is clear that we did not refer to any potential agent on geometry apart from spacetime itself. No contributors have been allowed to determine the geometry of spacetime; this is to say, what we have considered up to now has been spacetime devoid of any physical agency on it. We need, then, to look at irrelevance of \( l^a \) after a given probe also in a somehow more general context, with matter -which is an obvious missing ingredient- allowed to enter the scene.

**IV. IRRELEVANCE OF \( l^a \) WHEN IN PRESENCE OF MATTER**

A more general context is achieved if we assume that, starting from the small-scale expression \((D - 1) R_{ab} l^a l^b\) for \( R(q)(P,P')\), corresponding to \( R(q) \) probed at \( P \) through an assigned (null) geodesic \( \gamma \) with tangent \( l^a \) at \( P \), any
variation we can have when we further probe \( R_{(q)} \) with a changing \( l^a \) (leaving it null and forgetting how \( l^a \) went about) is cancelled, or absorbed, by the effects of matter. This corresponds to introduce matter as something which acts as needed to endow the small-scale Ricci scalar at \( P \), when considered operationally, a meaning which fits with being a quantity determined by the geometry of spacetime as a whole at \( P \), as specified above. Doing this, implies in particular to consider matter as something somehow capable to affect (large-scale) geometry, i.e. precisely what we learn from general relativity. We express this, associating to matter some entity with geometric significance. It is natural to conceive this as a scalar geometric quantity \( Q \), much the same way as to geometry itself can be associated the Ricci scalar \( R \). In other words, we are thinking of \( Q \) as something which parallels, as for the geometric effects of matter, what \( R \) expresses for geometry itself.

Exactly as it happens for \( R \), we can imagine that this geometrical quantity \( Q \) has a zero-point-length biscal ar counterpart \( Q_{(q)}(p, P) \) at generic base point \( P \), and that, along the geodesic \( \gamma \) connecting \( p \) and \( P \), \( Q_{(q)} \) has the small-scale (and \( L_0 \to 0 \)) limit

\[
\lim_{L_0 \to 0} \lim_{p \to P} Q_{(q)}(p, P) = Q_*,
\]

with \( Q_* \) depending on the geodesic which goes through \( P \), \( Q_* = Q_*(\gamma) \).

Now, irrelevance of \( l^a \) after a probe of \( R_{(q)} \) at \( P \) is introduced as follows. We require that every further variation of the term \( R_{cd} l^c l^d \) for the probed system, when we slightly change \( l^a \), is compensated by an equal and opposite variation induced by matter. In the same way, if in presence of matter we require \( l^a \)-independence of \( R_{cd} l^c l^d \) alone, without taking in due account \( Q_* \), the \( R_{(q)} \) we probed will exhibit at the end a (quite unacceptable) dependence on \( l^a \).

This means that the independence from \( l^a \) after a probe of \( R_{(q)} \) with matter present, is connected with the quantity

\[
F \equiv (D - 1) R_{cd} l^c l^d - Q_*
\]

having vanishing variation with respect to \( l^a \) for variations which keep \( l^a \) null.

When this variation is required to vanish, it is clear that the term \( R_{cd} l^c l^d \), i.e. \( \lim_{p \to P} R_{(q)} \), will keep having the same dependence on \( l^a \) as before. That is absolutely true. Our point however is different. As we said, what we maintain is that, given the value \( V \) that the small-scale Ricci scalar has at \( P \) as first probed through some specific null geodesic through which \( p \) approached \( P \) with tangent \( l^a \) at \( P \), what must happen is, when we change the vector \( l^a \) at \( P \) in the probed spacetime, \( V \) must not change anymore, i.e. its variation must vanish. This happens to be ensured endowing matter with the capability to influence geometry. Had we probed \( R_{(q)} \) at \( P \) for the still unprobed spacetime with a tangent \( l^a \) at \( P \), we would have found a different value \( V' \) of the small-scale Ricci scalar; but that value would have had in turn to remain the same in further probes at \( P \) on the already-probed spacetime.

Assigned \( Q \), we can think of \( Q_* \) as exhibiting: a) no dependence on \( l^a \), i.e. \( Q_* = Q_*(x) \); b) a linear dependence on \( l^a \), i.e. \( Q_* = Q_*(x, l^a) = Q_d l^a \), with \( Q_d \) not depending on \( l^a \); c) a quadratic dependence on \( l^a \), i.e. \( Q_* = Q_*(x, l^a, l^b) = Q_{ab} l^a l^b \), with \( Q_{ab} \) not depending on \( l^a \) and symmetric; d) a cubic dependence on \( l^a \), i.e. \( Q_* = Q_*(x, l^a, l^b, l^c) = Q_{abc} l^a l^b l^c \), with \( Q_{abc} \) not depending on \( l^a \) and symmetric in all its indices without loss of generality; e) a higher power dependence on \( l^a \); f) any combination of the above.

Before we proceed, we make a comment on definition \((23)\). Since \( R_{cd} l^c l^d \) (specifically \((1/L_0) R_{cd} l^c l^d \)) has the physical meaning of heat density \((23)\), a same physical meaning should have \( Q_* \). We see then that the request of irrelevance of \( l^a \) at \( P \) of probed spacetime corresponds to the law of balance, or equilibrium, of two heat densities. In other words, that sort of logical consistency condition we referred to as irrelevance of \( l^a \) after a probe, is automatically satisfied when the physical law of balance of heat densities holds true. This accords with that, if matter sets the geometry, this happens in thermodynamic terms, in particular as an expression of thermodynamic equilibrium.

Let us proceed now to discuss the various cases above. We immediately recognize in \((a) \) the case we have considered in the previous section. This implies that case \((a) \) is equivalent to empty space. Indeed, what we get is the same we get with \( Q_* = Q_{(q)} = Q = 0 \). Clearly, all the cases include in particular case \((a) \); this happens when \( Q_{a...} = 0 \). As for \((b) \), imposing the vanishing of the variation of \( F \) gives

\[
\frac{\partial}{\partial l^a} \left[ (D - 1) R_{cd} l^c l^d - Q_c l^c + \mu g_{cd} l^c l^d \right] = 0, \text{ at any } l^a \text{ null,}
\]

with \( Q_\alpha \) not dependent on \( l^\alpha \). This is the equation which replaces \((19)\). We get

\[
2 \left[ \left((D - 1) R_{ac} + \mu g_{ac}\right) l^c \right] = Q_\alpha, \quad \forall l^a \text{ null,}
\]
which gives $Q_a = 0$ and $Q_al^a = 0$ identically (for (27) has to hold true e.g. both for $l^c$ and $-l^c$), and then nothing more than case (a).

Let us consider case (c), i.e. the case

$$Q_*(x, l^a) = Q_abl^a l^b \neq 0,$$  \hspace{1cm} (28)

with $Q_ab$ independent of $l^a$ and symmetric. To require the vanishing of the variation of $F$ means to impose

$$\frac{\partial}{\partial l^a} \left[ (D - 1) R_{cd} l^c l^d - Q_{cd} l^c l^d + \mu g_{cd} l^c l^d \right] = 0, \text{ at any } l^a \text{ null.} \hspace{1cm} (29)$$

From this we get

$$\left[ (D - 1) R_{ac} - Q_{ac} + \mu g_{ac} \right] l^c = 0, \text{ } \forall l^a \text{ null.} \hspace{1cm} (30)$$

This implies

$$\left[ (D - 1) R_{ab} - Q_{ab} + \mu g_{ab} \right] l^a l^b = 0, \text{ } \forall l^a \text{ null,}$$

which gives

$$\left[ (D - 1) R_{ab} - Q_{ab} \right] l^a l^b = 0, \text{ } \forall l^a \text{ null.} \hspace{1cm} (31)$$

In case (d), $Q_*(x, l^a) = Q_abl^a l^b l^c$, with $Q_abc$ not dependent on $l^a$, and symmetric in all its indices. Starting from $F$ in (25), requiring irrelevance of $l^a$ means in this case

$$\frac{\partial}{\partial l^a} \left[ (D - 1) R_{cd} l^c l^d - Q_{cde} l^c l^d + \mu g_{cde} l^c l^d \right] = 0, \text{ at any } l^a \text{ null.} \hspace{1cm} (32)$$

This gives

$$2 \left[ (D - 1) R_{ac} + \mu g_{ac} \right] l^c = 3 Q_{ac} l^c l^d, \text{ } \forall l^a \text{ null.} \hspace{1cm} (33)$$

Here, when sending $l^a$ in $-l^a$, the rhs does not change, while the lhs flips the sign. We must then have lhs $=$ rhs $= 0$, that is $Q_{ac} l^c l^d = 0$. But this implies $Q_{ac} l^c l^d$ is $= 0$, and we are back to case (a).

Let us dispose now of case (e). Reconsidering what we just said for the case $Q_*(x, l^a) = Q_abl^a l^b l^c$, we notice that for each further choice $Q_*(x, l^a) = Q_abcl^a l^b l^c ...$ with $r \equiv \text{rank}(Q_{abc...}) \text{ odd}$, we get an equation of the kind (33),

$$2 \left[ (D - 1) R_{ac} + \mu g_{ac} \right] l^c = r Q_{ac} l^d l^e \ldots, \text{ with } r - 1 \text{ } l^a \text{ in the rhs},$$

for which the same reasoning just described applies. Then, the same as for the cases with $r = 1$ and $r = 3$, all these further cases with $r$ odd turn out to give nothing more than case (a). When instead we take $Q_*(x, l^a) = Q_abcd l^a l^b l^c l^d$, we have

$$\frac{\partial}{\partial l^a} \left[ (D - 1) R_{cd} l^c l^d - Q_{cde} l^c l^d l^e + \mu g_{cde} l^c l^d l^e \right] = 0, \text{ at any } l^a \text{ null.} \hspace{1cm} (34)$$

Following the by-now usual steps we arrive at

$$\left[ (D - 1) R_{ab} - 2 Q_{abcd} l^c l^d \right] l^a l^b = 0, \text{ } \forall l^a \text{ null.}$$
On the other hand, (34) directly implies

\[(D - 1) R_{ab}^a b = Q_{abcd} l^a l^b l^c l^d + \chi, \quad \forall l^a \text{ null}\]

with \(\chi = \chi(x)\) a scalar not dependent on \(l^a\). Crossing the last two equations, we get

\[Q_{abcd} l^a l^b l^c l^d = \chi, \quad \forall l^a \text{ null}.\]

But this is impossible, unless \(Q_{abcd} l^a l^b l^c l^d = 0\) (from imposing the vanishing of the derivative of the lhs with respect to \(l^a\)). This however would give \(Q_{abcd} l^a l^b l^c l^d = 0\), and then again case (a). One can easily show that the same situation occurs in any further even case, i.e. for \(r > 4\) even.

This concludes the discussion of the situation in which only one single term with the \(l^a\)'s is present. One might wonder that a generic combination of all these terms (the case we called (f)), could lead perhaps to something new with respect to case (c), which we have seen is the only one able to add something to case (a) (including it), i.e. to what we considered in previous Section. It is easily found however that this does not happen; this is detailed in appendix B. At the end, what we have is thus that case (c), namely that with \(r = 2\), exhausts all possibilities through which matter can act to provide irrelevance of \(l^a\) at \(P\) after a probe. As an aside, we notice that the Lagrangian multiplier in (29) had no effect in (31) (as well as the multiplier in (19) had no effect in (20)). This can be interpreted as suggesting that in the analysis above there is no need to restrict the variations of \(l^a\) to give null \(l^a\)'s (\(l^a = l^a + \delta l^a\)), that is we can allow for variations to timelike or spacelike vectors \(v^a\) (with \(l^a\) mapped continuously to \(v^a\)).

Armed with these findings, we come back then to equation (31) which sums up the results of case (c). This equation shows that, thanks to matter acting as ‘generator’ of curvature, we get irrelevance of \(l^a\) at \(P\) after a probe, with \(\lim_{\lambda \to 0} R_{(q)}(P, P') \neq 0\).

The ordinary spacetime corresponding to this underlying minimum-length spacetime is readily found in the same way (and the same maths [38]) we followed in previous Section. Equation (31) implies

\[(D - 1) R_{ab} - Q_{ab} = \zeta g_{ab}\]  \hspace{1cm} (35)

with \(\zeta = \zeta(x)\) a scalar, and this gives

\[G_{ab} = \left( \frac{\zeta}{D - 1} - \frac{1}{2} R \right) g_{ab} + \frac{1}{D - 1} Q_{ab}.\]  \hspace{1cm} (36)

This equation fixes a relation between the metric and matter source terms being the latter expressed by tensor \(Q_{ab}\). But, again, this is what are supposed to do the field equations. We have thus the quite nice fact that any ordinary spacetime obeying the (field) equations (35), does admit a consistent qmetric description, meaning a description in which the qmetric Ricci scalar operationally expresses (according to the criterium we stated above) the intrinsic geometry of spacetime, as due.

If in equation (35) we put \(Q_{ab} = 0\), we see that from Bianchi identity and from the covariant constancy of \(g_{ab}\) we get exactly equation (23); this confirming that what we called ‘empty space’, namely the case considered in the previous Section, is indeed what we obtain using the general equations in presence of matter, with matter removed.

For \(Q_{ab}\) generic, from Bianchi identity we get

\[-\partial_b \left( \zeta - \frac{1}{2} (D - 1) R \right) = \nabla_a Q^a_{\ b}.\]  \hspace{1cm} (37)

If \(Q_{ab}\) is such that (meaning, if the geometric scalar \(Q\) associated to matter is such that)

\[\nabla_a Q^a_{\ b} = 0,\]  \hspace{1cm} (38)

then

\[\zeta - \frac{1}{2} (D - 1) R = \text{const},\]  \hspace{1cm} (39)
and
\[ G_{ab} = C g_{ab} + \frac{1}{D-1} Q_{ab}, \]  
with \( C \) the constant of (23). If we take \( D = 4 \) and \( Q_{ab} = 24\pi GT_{ab} \) (in units making \( c = 1 \) and \( h = 1; \) \( G \) is Newton constant), we see these equations are Einstein' field equations with cosmological constant, implying this in particular that any spacetime which is solution to full Einstein equations does admit this consistent qmetric description. \( Q_{ab} \) is what contains matter degrees of freedom; it can in general depend also on the metric tensor, on functions of it, on derivatives of arbitrary order, as well as on additional fields.

V. DISCUSSION AND CONCLUSIONS

What came out from the above, is that the somehow puzzling aspect of the qmetric Ricci (bi)scalar \( R(q) \) with base at a point \( P \) of having a coincidence limit which depends on the geodesic along which we reach \( P \), could actually be accommodated in a spacetime which is given an operational meaning. Specifically, we discussed that we ought to distinguish between unprobed and probed spacetime at \( P \), the latter being the spacetime we get once a probe (of curvature) at \( P \) of original spacetime is done. The assertion is that \( R(q) \) at \( P \) of unprobed spacetime is, in the small scale, sort of multivalued function, or quantum superposition of different potential values, and the act of probing selects one of these. Logical consistency demands then that a further probe along any geodesic reaching \( P \) on the already-probed spacetime gives that same value obtained in the first probe. What has been shown happens afterwards, is that this requirement goes hand in hand with regarding matter as capable to affect large-scale geometry.

Reconsidering the route we have followed, things go also on the reverse. If a spacetime endowed with a limit length \( L_0 \), does admit an operationally consistent metric-like description in the small scale, then in the large scale (i.e. where it goes to coincide with an ordinary spacetime) it obeys field equations. Thus, if the spacetime we have got to describe has actually a limit length, this implies that in the large scale this spacetime has to obey field equations. This resonates with what expressed in [40] complemented with [41] (cf. also [42, 43]; in present case however in terms of a larger class of possible field equations).

Many have been the attempts which dreamed of the existence of a thermodynamic principle, conceived as more fundamental than field equations themselves, from which the latter could be derived. This study somehow adds to them bolstering the request of balance of exchanged heats as the thermodynamic principle sought-after. Moreover, this thermodynamic principle is pointed out to be connected with a specific requirement of a consistent operational description of spacetime from the large down to the smallest conceivable scale.

The superposition of values of \( R(q) \) for unprobed spacetime, is an effect of a \( L_0 \neq 0 \). The thing is that this feature keeps remaining also in the \( L_0 \to 0 \) limit. We are then confronted with two scenarios which do return inequivalent: i) absence of any limit length (\( l^0 \)-independence at \( P \) obvious, for the coincidence limit would be \( R \); no requirement of large-scale field equations); ii) presence of a vanishingly-small limit length (\( l^0 \)-independence after a probe not obvious;
requirement of large-scale field equations). But, large-scale field equations do exist for actual spacetime (Einstein field equations, to the best of experimental checks), and they indeed foresee a limit length when combined with basic principles of quantum mechanics. This selects scenario (ii), and, at the same time, indicates that the request of existence of large-scale field equations by this scenario is insensitive to the actual (provided very small) value of \( L_0 \). This suggests that field equations, and in particular Einstein field equations, ought to be regarded as quantum in their origin, even if at conditions at which the quantum nature of spacetime can hardly be directly probed by effects small with \( L_0 \) (this adding to what suggested in [47]). Thanks to the persistence of large-scale quantum effects also in the \( L_0 \to 0 \) limit, are the Einstein equations themselves what testifies about spacetime being quantum. In view of this, we might consider the field equations as ‘zero-point’ field equations, with the meaning of something which quantum-mechanically stays there while classically it would not.

Then, the field equations ought not to be considered as the classical limit of a quantum theory of gravity (meaning the equations we would obtain when letting \( L_0 \to 0 \) with \( \hbar \to 0 \)), but a direct prediction of this quantum theory. In other words, we do not get the large-scale field equations (e.g. the Einstein equations) in the \( \hbar \to 0 \) limit; rather, the large-scale field equations arise, find their origin, in an explicitly \( L_0 \neq 0 \) (and then \( \hbar \neq 0 \)), and stay there even when \( L_0 \) becomes exceedingly and unappreciably small. They do not ‘set in’ in the \( L_0 \to 0 \) limit; on the contrary, what they do is to keep staying there also in this limit. They are sort of quantum effect not vanishing with \( \hbar \).

As a closing remark, we would like to emphasize that all this is not about what the field equations ought to become, or what they ought to be replaced by, in the small scale. No word is told about that up to this point in the paper. Everything we described, is only about the connections that the endowing of spacetime with a (lower) limit length seemingly turns out to have with large scale (as compared with Planck length) structure. Clearly, coping with the short scale, would imply to have to do with a length scale at which we can no longer reasonably neglect the own evolution of the dofs of the specific microscopic theory, as we do instead in our approximation. If we apply our model anyway, we notice that when the scale is short enough that \( g_{ab} \) is no longer a good approximation of the qmetric, the constraints (Eq. (36)) keep remaining formally the same, but what they constrain (i.e. \( g_{ab} \)) has no longer the meaning which we operationally assign to a metric (i.e. to give quadratic intervals), and for this we should refer instead to the qmetric. Inverting in these equations from \( g_{ab} \) to the qmetric, would give the evolution equations in the short scale for the (effective) metric.

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Appendix A: Derivation of equalities [16]

The quantity \( \lim_{L_0 \to 0} \lim_{p \to p} R_{(q)}(p, P) \) has been already evaluated in [22]. What we add here, is an expression for \( \lim_{p \to p} R_{(q)}(p, P) \) at finite \( L_0 \), detailing, in a conveniently chosen parameter, the order of magnitude of the non-leading part, this way justifying expressions [16].

To this aim, we start from equation (A3), rewritten in a convenient, slightly modified form (corresponding to merge two of its terms into one, and leave the others unchanged):

\[
R_{(q)}(p, P) = \frac{1}{A} R_S - 2 \alpha \frac{d \alpha}{d \lambda} K + 2 \alpha^2 R_{ab} l^a l^b - (D - 2) \alpha \frac{d}{d \lambda} \left( \alpha \frac{d}{d \lambda} \ln A \right) - \frac{1}{4} (D - 2)(D - 1) \alpha^2 \left( \frac{d}{d \lambda} \ln A \right)^2 - \alpha^2 K^2 + \alpha^2 K_{ab} K_{ab} - (D - 1) \alpha^2 \left( \frac{d}{d \lambda} \ln A \right) K. \tag{A1}
\]

In this formula, first we provide expressions of the (four) terms not containing \( A \). They are [22], cf. [19, 48]

\[
-2 \alpha \frac{d \alpha}{d \lambda} K = -2 \alpha \frac{d \alpha}{d \lambda} \frac{D - 2}{\lambda} + O(\lambda), \tag{A2}
\]

\[
2 \alpha^2 R_{ab} l^a l^b = 2 \alpha^2 E(p), \tag{A3}
\]

\[
-\alpha^2 K^2 = -(D - 2)^2 \alpha^2 \frac{1}{\lambda^2} + \frac{2}{3} (D - 2) \alpha^2 E(p) + O(\lambda), \tag{A4}
\]

\[
\alpha^2 K_{ab} K_{ab} = (D - 2) \alpha^2 \frac{1}{\lambda^2} - \frac{2}{3} \alpha^2 E(p) + O(\lambda). \tag{A5}
\]

with, even where not explicitly indicated, all quantities evaluated at \( p \) and [18] \( E_{ab} \equiv R_{ambn} l^m l^n \) with \( E(p) \equiv E_{ab}(p) = (R_{ab} l^a l^b)(p) \geq 0 \), with the last relation for our spacetime obeys the null convergence condition. Here, we used the expression
\[ K_{ab} = \frac{1}{\lambda} h_{ab} - \frac{1}{3} \lambda h^c_a h^d_b E_{cd} + O(\lambda^2), \]  

and thus also
\[ K = (D - 2) \frac{1}{\lambda} - \frac{1}{3} \lambda E(p) + O(\lambda^2), \]

from \(^{23}\) (but cf. \(^{19, 48}\)). Expressions \(^{A2}\), \(^{A3}\) and \(^{A5}\) have, at leading order in \(\lambda\), factors \(1/\lambda\) or \(1/\lambda^2\) in them, divergent in the \(p \to P\) (i.e. \(\lambda \to 0\)) limit. The actual divergence or not of the whole expressions would depend also on the behavior \(\alpha\) and \(d\alpha/d\lambda\) in the same limit. A divergent \(\alpha\) would also introduce a divergence in the remaining term \(^{A3}\), and also e.g. the \(O(\lambda)\) term in \(^{A2}\) could be diverging in the \(\lambda \to 0\) limit. As we will see, it turns out however that the remaining terms in the expression of \(R(q)(p, P)\) do cancel any \(\lambda^{-2}, \lambda^{-1}, \lambda^0\) term here, i.e. it turns out they do not give any contribution to \(R(q)(p, P)\) whichever is the assumed behaviour of \(\alpha\) and \(d\alpha/d\lambda\) in the \(\lambda \to 0\) limit. It remains an open question however what happens to the \(O(\lambda)\) terms (do the cancellations extend to these, and higher, orders?). To handle this, we assume then that \(\alpha\) and \(d\alpha/d\lambda\) remain finite in the \(\lambda \to 0\) limit. This guarantees that each \(O(\lambda)\) term vanishes with \(\lambda\) (as well as any higher order term). Strictly speaking, the derivation we are providing here is thus for non-divergent \(1/\lambda\) and \(d\alpha/d\lambda\).

As for the terms containing \(\lambda\), from the expression \(^{5}\) for it we see that what we need is the expansion around \(P\) of the van Vleck determinant, which, for \(\lambda_\gamma\) smooth at \(\lambda\), we introduce then a scale length \(\ell_\gamma\) associated to \(\gamma\), with \(\lambda/\ell_\gamma = \tilde{\lambda}\).

The very writing of these expansions comes with the desire that it can happen that any term at an assigned order in powers of \(\lambda\) or in \(\tilde{\lambda}\) turns out generically negligible with respect to that at the previous order. Our first task is to try to characterize this, in the sense of finding a convenient parameter, for which we can say if, at coincidence limit, it is small enough to give what just said. This parameter can be clearly \(\lambda\) itself for the expansion \(^{A3}\), for in the coincidence limit it becomes vanishingly small. It appears perhaps not so clear what happens instead for expansion \(^{A9}\). Here, \(\lambda\) remains finite at coincidence, and we need some reference scale to compare with to establish if \(\lambda\) becomes actually small enough to provide full meaning to the expansion. Let us focus then on expansion \(^{A9}\).

First of all, from the \(\tilde{\lambda}\)-term we see that we need to be at conditions in which \(\tilde{\lambda}^2 E(\tilde{\rho}) \ll 1\). To characterize this, for the given null geodesic \(\gamma\) with tangent \(t^a = dx^a/d\lambda\), we introduce then a scale length \(\ell_R\) associated to curvature (of the assigned spacetime) at any given event (near \(P\)) as \(\ell_R \equiv 1/\sqrt{E}\) at that event (\(E\) non-negative, for null convergence condition assumed to hold). We have to ask that curvature is small, and clearly it is small enough if we have \(\lambda/\ell_R \simeq L_0/\ell_R \ll 1\), where 1st relation comes from assuming to be at conditions \(\lambda \simeq L_0\), i.e. to be near the coincidence limit (\(\lambda \simeq 0\)). This gives indeed \(\tilde{\lambda}^2 E(\tilde{\rho}) = O(\tilde{\lambda}^2/\ell_R^2) \ll 1\). As mentioned in the main text, this corresponds to that, in a local frame in which \(\lambda\) is length, \(\tilde{\rho}\) at \(\lambda(\tilde{\rho}, P) = L_0\) results near enough to \(P\) to give \(|g_{ab}(\tilde{\rho})| = O(R_{abcd}) L_0^2 \ll 1\) (using Riemann normal coordinates).

Next, let us write
\[ \ell_R(\tilde{\rho}) = L_R + C\lambda(\tilde{\rho}, P) + C_2 \frac{\lambda^2(\tilde{\rho}, P)}{L_R} + C_3 \frac{\lambda^3(\tilde{\rho}, P)}{L_R^2} + ..., \]

with \(L_R \equiv 1/\sqrt{E(p)}\), and \(C, C_1, C_2, ...\) constants, and where we have put in evidence as much \(1/L_R\) factors as dimensionally required. This expansion shows that \(\lambda/L_R\) is a parameter which is indeed effective in discriminating how significantly \(\ell_R\) differs from its value \(L_R\) at \(P\). Having this, we consider the next term, the \(\lambda^3\)-term, in \(^{A9}\). We have
\[
\frac{dE}{d\lambda}(\bar{\rho}) = \frac{dE}{d\ell R} \frac{d\ell R}{d\lambda} = -C \frac{1}{\ell R} \hat{\lambda} E(\bar{\rho}) = -C \frac{1}{L_R} E(\bar{\rho}) + \mathcal{O}\left(\frac{\hat{\lambda}}{L_R} \frac{E(\bar{\rho})}{L_R}\right), \tag{A11}
\]

where use has been made of (A10). This gives \(\hat{\lambda} \frac{dE}{d\lambda}(\bar{\rho}) = \mathcal{O}\left(\frac{\hat{\lambda}}{L_R} E(\bar{\rho})\right)\) and \(\hat{\lambda}^3 \frac{dE}{d\lambda}(\bar{\rho}) = \hat{\lambda}^2 \mathcal{O}\left(\frac{\hat{\lambda}}{L_R} E(\bar{\rho})\right)\).

In the \(\hat{\lambda}^4\)-term, we have \(\hat{\lambda}^4 E^2(\bar{\rho}) = \hat{\lambda}^2 E(\bar{\rho}) \frac{\hat{\lambda}^2}{L_R} = \hat{\lambda}^2 E(\bar{\rho}) \frac{\hat{\lambda}^2}{L_R} \left(1 + \mathcal{O}\left(\frac{\hat{\lambda}}{L_R}\right)\right) = \hat{\lambda}^2 \mathcal{O}\left(\frac{\hat{\lambda}^2}{L_R} E(\bar{\rho})\right)\); further, \(\hat{\lambda}^4 E^{ab}(\bar{\rho}) E_{ab}(\bar{\rho}) = \mathcal{O}(\hat{\lambda}^4 E^2(\bar{\rho})) = \hat{\lambda}^2 \mathcal{O}\left(\frac{\hat{\lambda}^2}{L_R} E(\bar{\rho})\right)\) too (from evaluating the scalar \(E^{ab} E_{ab}\) in the local frame in which \(\lambda\) is length); and

\[
\frac{d^2E}{d\lambda^2}(\bar{\rho}) = \frac{d}{d\lambda} \left(-\frac{1}{\ell R} \frac{d\ell R}{d\lambda}\right)(\bar{\rho}) = \left(3 \frac{1}{\ell R} C - \frac{1}{\ell R} \frac{2 C_2}{L_R}\right)(\bar{\rho}) = \frac{1}{L_R} E(\bar{\rho}) \left(3 C - 2 C_2\right) \left(1 + \mathcal{O}\left(\frac{\hat{\lambda}}{L_R}\right)\right) = \mathcal{O}\left(\frac{1}{L_R^2} E(\bar{\rho})\right).
\]

All this, gives \(\hat{\lambda}^2 \mathcal{O}\left(\frac{\hat{\lambda}^2}{L_R} E(\bar{\rho})\right)\) as order of magnitude of the whole \(\hat{\lambda}^4\)-term. We can proceed in a similar manner at any order in \(\hat{\lambda}\). In each \(\hat{\lambda}^n\) term, there will be factors of powers \((E_{ab})^m\) or \(E^m\), derivatives \(d^m E/d\hat{\lambda}^m\), with \(m, m'\) integers \(\geq 0\) such that \(2 m + m' + 2 = n\), as dimensionally required. And this implies that the \(n\)-th order term will be \(\mathcal{O}\left(\frac{\hat{\lambda}^{n-2}}{L_R^2} E(\bar{\rho})\right)\).

Summing all up, we can rewrite (A9) as

\[
\hat{A}^{1/2}(\rho, P) = 1 + \frac{1}{12} \hat{\lambda}^2 E(\bar{\rho}) - \frac{1}{24} \hat{\lambda}^3 \frac{dE}{d\lambda}(\bar{\rho}) + \hat{\lambda}^2 \mathcal{O}\left(\frac{\hat{\lambda}^2}{L_R} E(\bar{\rho})\right) \tag{A12}
\]

\[
= 1 + \frac{1}{12} \hat{\lambda}^2 E(\bar{\rho}) + \hat{\lambda}^2 \mathcal{O}\left(\frac{\hat{\lambda}}{L_R} E(\bar{\rho})\right) + \hat{\lambda}^2 \mathcal{O}\left(\frac{\hat{\lambda}^2}{L_R} E(\bar{\rho})\right), \tag{A13}
\]

where in (A13) we explicitly write the order of magnitude of the 2nd term in the rhs of (A12).

We can proceed now to compute the expressions of the terms containing \(A\) in (A11) in the coincidence limit. For the 1st term, we get

\[
\frac{1}{A} R_{\Sigma} = \frac{\lambda^2 \Delta}{\lambda^2} \left(R(p) + K^2(p) - K^{ab}(p) K_{ab}(p) - 2 E(p)\right) \frac{1}{\lambda^2} \frac{\Delta}{\lambda^2} = \left(D - 2\right) \left(D - 3\right) \mathcal{O}(\lambda^2) \frac{1}{\lambda^2} \left(1 + \frac{1}{3 (D - 2)} \hat{\lambda}^2 E(\bar{\rho}) + \hat{\lambda}^2 \mathcal{O}\left(\frac{\hat{\lambda}}{L_R} E(\bar{\rho})\right)\right) = \left(D - 2\right) \left(D - 3\right) \frac{1}{\lambda^2} + \frac{D - 3}{3} E(\bar{\rho}) + \mathcal{O}\left(\frac{\hat{\lambda}}{L_R} E(\bar{\rho})\right) + \mathcal{O}(\lambda^2), \tag{A14}
\]

where we used of relation (12) (2nd equality), of the expressions (A4) and (A5) for \(K^2\) and \(K^{ab} K_{ab}\), as well as of the expansions of the van Vleck determinant.

As for the 4th term in (A11), we notice first that
\[ \frac{d}{dx} \ln A = \frac{2}{\lambda} - 2 \alpha \frac{1}{3} \frac{d}{dx} \left[ \frac{1}{3(D-2)} \int E(p) - \frac{1}{6(D-2)} \int \frac{dE}{d\lambda}(\tilde{p}) + O\left( \frac{\lambda^2}{L_R} \right) \right] \]

\[ = -\alpha \frac{d}{dx} \left[ -\frac{1}{3(D-2)} \lambda^2 E(p) + O(\lambda^3) \right] \]

\[ = \frac{2}{\lambda} - \frac{2}{3(D-2)} \lambda E(\tilde{p}) + \frac{1}{6(D-2)} \lambda^2 E(\tilde{p}) + O\left( \frac{\lambda^2}{L_R} \right) \]

\[ = -2 \alpha \frac{1}{3} + \frac{2}{3(D-2)} \alpha \lambda E(p) + O(\lambda^2), \quad (A15) \]

where \( \frac{\lambda^2 dE}{d\lambda}(\tilde{p}) = O\left( \frac{1}{L_R} \lambda E(\tilde{p}) \right) \) and we used \( \frac{d}{dx} O\left( \frac{\lambda^2}{L_R} \right) = \frac{1}{\alpha} [O\left( \lambda E(\tilde{p}) \right) + O\left( \frac{\lambda^2}{L_R} \lambda E(\tilde{p}) \right)] \) and similarly at any order. This then gives

\[ -(D-2) \alpha \frac{d}{dx} \left( \alpha \frac{d}{dx} \ln A \right) = 2(D-2) \frac{1}{\lambda^2} + \frac{2}{3} E(\tilde{p}) + O\left( \frac{\lambda}{L_R} E(\tilde{p}) \right) \]

\[ -2(D-2) \alpha \frac{1}{\lambda^2} + 2(D-2) \alpha \frac{d}{dx} \frac{1}{\lambda} - \frac{2}{3} \alpha^2 E(p) + O(\lambda). \quad (A16) \]

As for the remaining two terms in (A1), namely the 5th and the 8th, there is a convenience in treating them together. In fact, we have

\[ -(D-2)(D-1) \alpha^2 \left( \frac{d}{dx} \ln A \right)^2 - (D-1) \alpha^2 \left( \frac{d}{dx} \ln A \right) K \]

\[ = -\frac{1}{4} (D-2)(D-1) \alpha^2 \left( \frac{d}{dx} \ln A \right)^2 - (D-1) \alpha^2 \left( \frac{d}{dx} \ln A \right) \times \]

\[ \left[ \frac{2}{\lambda} - \frac{2}{3(D-2)} \lambda E(\tilde{p}) + \frac{1}{6(D-2)} \lambda^2 E(\tilde{p}) + O\left( \frac{\lambda^2}{L_R} \lambda E(\tilde{p}) \right) - 2 \alpha \frac{1}{3} + \frac{2}{3(D-2)} \alpha \lambda E(p) + O(\lambda^2) \right] \]

\[ = \frac{1}{2} (D-2)(D-1) \alpha^2 \frac{1}{\lambda} \frac{d}{dx} \ln A - \frac{1}{4} (D-2)(D-1) \left\{ \left\{ \ldots \right\} + 2 \alpha \frac{1}{\lambda} \right\} \]

\[ = -(D-2)(D-1) \alpha^2 \frac{1}{\lambda} \frac{d}{dx} \ln A + \frac{D-1}{3} \alpha \lambda E(p) \left\{ \left\{ \ldots \right\} - (D-1) \alpha O(\lambda^2) \left\{ \ldots \right\} \right\} \]

\[ = \frac{1}{4} (D-2)(D-1) \left\{ \left\{ \ldots \right\} \right\}^2 + (D-2)(D-1) \alpha^2 \frac{1}{\lambda^2} - \frac{2}{3} (D-1) \alpha^2 E(p) + O(\lambda) \]

\[ = -(D-2)(D-1) \frac{1}{\lambda^2} + \frac{2}{3} (D-1) E(\tilde{p}) - \frac{D-1}{6} \lambda E(\tilde{p}) + O\left( \frac{\lambda^2}{L_R} E(\tilde{p}) \right) \]

\[ + (D-2)(D-1) \alpha^2 \frac{1}{\lambda^2} - \frac{2}{3} (D-1) \alpha^2 E(p) + O(\lambda) \quad (A17) \]

(\text{where} \ldots) \text{stands for} \alpha \frac{d}{dx} \ln A \text{expanded, i.e. what is written in square brackets in the 1st equality, and} \ldots \text{denotes the quantity in braces in the 2nd equality), and in 3rd equality we see that the} \frac{1}{4} \frac{d}{dx} \ln A \text{terms nicely cancel. The quantity} \lambda \frac{d}{dx} E(\tilde{p}) \text{, we know is} O\left( \frac{1}{L_R} E(\tilde{p}) \right). \)

Putting all this together, i.e. substituting equations (A2-A5) and (A14), (A16), (A17) into equation (A1), we finally get

\[ R_{(q)}(p, P) = (D-1) E(\tilde{p}) + O\left( \frac{\lambda}{L_R} E(\tilde{p}) \right) + O(\lambda). \quad (A18) \]

Then,
\[
\lim_{p \to p'} R_{(q)}(p, P) = (D - 1) E(\tilde{p}) + O\left(\frac{L_0}{L_R} E(\tilde{p})\right)
\]
\[
= (D - 1) \left( E(P) + E(P) O\left(\frac{L_0}{L_R}\right)\right) + O\left(\frac{L_0}{L_R} E(\tilde{p})\right)
\]
\[
= (D - 1) E(P) + O\left(\frac{L_0}{L_R} E(P)\right),
\]
where \(\tilde{p}\) is such that \(\lambda(\tilde{p}, P) = L_0\), and we used \(E(\tilde{p}) = E(P) + E(P) O\left(\frac{1}{L_R}\right)\), which gives \(\frac{L_0}{L_R} E(\tilde{p}) = \frac{L_0}{L_R} E(P) + O\left(\frac{L_0}{L_R} E(P)\right)\). (A19) and (A20) are the equalities (16) of the main text.

Appendix B: Consideration of case \((f)\) (see text)

For the quantity \(Q_*\), i.e. the small scale limit of the quantic quantity which captures the geometrical effects of matter, we already considered in the main text all the cases in which \(Q_*\) has no dependence on \(l^a\), a linear dependence, a quadratic, a cubic, \ldots, separately. Our task here is to establish whether the case of a generic combination of all the cases above adds something or not.

To this aim, let us write

\[
Q_* = \sum_{r=0}^{n} Q_{12...r} l^1 l^2 ... l^r,
\]

where indices \(1, 2, ..., r\) are short for indices \(a_1, a_2, ..., a_r\) with each \(a_i = 1, ..., D\). Tensors \(Q_{12...r}\) do not depend on \(l^a\). They can be taken totally symmetric without loss of generality. Moreover, we think of any coefficient (necessarily independent from \(l^a\)) in the linear combination (B1) as absorbed into \(Q_{12...r}\) themselves.

From \(F\) as in (23), we get path-independence at the path assigned if we require

\[
\frac{\partial}{\partial l^a}\left[(D - 1) \sum_{r=1}^{n} R_{cd} l^d \sum_{r=0}^{n} Q_{12...r} l^1 l^2 ... l^r - \mu l^c\right],
\]

with \(\mu\) a scalar not dependent on \(l^a\). We get

\[
2(D - 1) R_{ac} l^c - \sum_{r=1}^{n} r Q_{a12...r-1} l^1 l^2 ... l^{r-1} - 2 \mu l_a = 0, \quad \forall l^a \text{ null},
\]

whence

\[
2(D - 1) R_{ac} l^c - 2 \mu l_a - \sum_{r=2}^{n} r Q_{a12...r-1} l^1 l^2 ... l^{r-1} = \sum_{r=1}^{n} r Q_{a12...r-1} l^1 l^2 ... l^{r-1}, \quad \forall l^a \text{ null}.
\]

Here we see that sending \(l^a\) in \(-l^a\) the lhs changes sign while the rhs does not. This implies \(\text{lhs} = 0 = \text{rhs} \forall l^a\), then we cannot have terms with \(r\) odd in sum (B1).

We are left with

\[
2(D - 1) R_{ac} l^c - 2 \mu l_a - \sum_{r=2}^{n} r Q_{a12...r-1} l^1 l^2 ... l^{r-1} = 0, \quad \forall l^a \text{ null}
\]

which can be rewritten as

\[
\left[2(D - 1) R_{ac} - 2 \mu g_{ac} - \sum_{r=2}^{n} r Q_{a12...r-2} l^1 l^2 ... l^{r-2}\right] l^c = 0. \quad \forall l^a \text{ null}
\]
If this is true, it is true also

\[
2(D - 1) R_{ab} - 2 \mu g_{ab} - \sum_{r=2}^{n} \left( q_{ab12...r-2} l^1 l^2 ... l^{r-2} \right) t^a t^b = 0, \quad \forall t^a \text{ null},
\]

which is

\[
2(D - 1) R_{ab} - \sum_{r=2}^{n} \left( q_{ab12...r-2} l^1 l^2 ... l^{r-2} \right) t^a t^b = 0, \quad \forall t^a \text{ null}.
\]  \hspace{1cm} (B4)

But equation (B2) (in absence of \( r \)-odd terms from the comment just above equation (B3)) means

\[
(D - 1) R_{ab} t^a t^b = \sum_{r=2}^{n} Q_{12...r} l^1 l^2 ... l^{r} + \eta, \quad \forall t^a \text{ null},
\]  \hspace{1cm} (B5)

with \( \eta = \eta(x) \) a scalar independent of \( t^a \). Crossing this with (B4), gives

\[
\sum_{r=2}^{n} Q_{12...r} l^1 l^2 ... l^{r} + \eta = \sum_{r=2}^{n} \frac{r}{2} Q_{12...r} l^1 l^2 ... l^{r}, \quad \forall t^a \text{ null}.
\]  \hspace{1cm} (B6)

Apart from the trivial case in which all the terms are zero, this equation can be satisfied only if there is one and only one term not zero: that with \( r = 2 \) (also, implying \( \eta = 0 \)) (to be convinced, it suffices to look at what happens if we send \( t^a \) to \( k t^a \), with \( k \) a constant). This shows that we must have \( Q = Q_{ab} t^a t^b \) and we are back to case (c) of the main text, i.e. case (f) adds nothing to case (c).

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