‘Classical’ quantum states

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We show that several classes of mixed quantum states in finite-dimensional Hilbert spaces which can be characterized as being, in some respect, ‘most classical’ can be described and analyzed in a unified way. Among the states we consider are separable states of distinguishable particles, uncorrelated states of indistinguishable fermions and bosons, as well as mixed spin states decomposable into probabilistic mixtures of pure coherent states. The latter were the subject of the recent paper by Giraud et. al. [1], who showed that in the lowest-dimensional, nontrivial case of spin 1, each such state can be decomposed into a mixture of eight pure states. Using our method we prove that in fact four pure states always suffice.

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I. INTRODUCTION

The notion of being the ‘most classical’ among all quantum states of a given system is admittedly vague, and strongly dependent on the particular quantum features we would like to approximate on the classical level – ‘nonclassical’ properties of quantum systems usually encompass a variety of phenomena.

In the situation that we will be concerned with here, there is a Lie group of preferred observables acting on the system. It may be the group SU(N) × SU(M) of local unitaries acting on a composite system, or the SU(2) group one can conveniently realize using linear optics. Then the group action divides the set of states into disjoint orbits, and there will be a special orbit which is in itself a symplectic and indeed a Kähler manifold. It can serve as a classical phase space, and the states in this orbit are our ‘most classical’ states. They are separable states in the first example, and coherent spin states in the second. In general they are the generalized coherent states [2] defined as the orbit in the representation space of the highest weight vector for a particular irreducible representation of the symmetry group of the system. Generalized coherent states also minimize an uncertainty relation [3], or more precisely they minimize the total variance of a state

\[ \delta(\psi) = \sum_i \left( \langle \psi | X_i^2 | \psi \rangle - \langle \psi | X_i | \psi \rangle^2 \right), \]

where the \( \{X_i\} \) form an orthonormal basis in the Lie algebra of the relevant group. Note that this is independent of the choice of the basis \( \{X_i\} \). It is pleasing that both definitions of ‘most classical’ coincide, and moreover, that they coincide with the view that separable states of a composite system are the most classical ones, in the sense that they lack the quantum correlations among subsystems responsible for any EPR-like phenomena.

In the following we call (for shortness) those states ‘classical’ which are ‘most classical’ in the above senses. At the other end of the spectrum we can identify the states in that special orbit on which the symplectic form vanishes as being the ‘most non-classical’ states. The reason why there is such an orbit can be seen by complexifying the group; for composite system this coincides with the maximally entangled states as conventionally understood [4].

Investigations concerning role of pure (generalized) coherent states in exhibiting links between quantum and classical level have a long history, especially for such problems as quantization and classical limits (see eg. [5]). The connections between coherent states and entanglement theory was recently pointed by Klyachko [4, 6] (see also [7, 8, 9]).

The notion of classicality can be extended to mixed states. On a formal level we do it by defining ‘classical’ mixed states as those which can be decomposed in a probabilistic mixture (i.e. a convex linear combination) of pure classical states (here and in the following we identify pure states, i.e. vectors in a Hilbert space, or rather its projectivisation, with one-dimensional projection on their directions). For composite systems it leads to the popular definition of separable (or equivalently non-entangled) states, which can be also characterized on the operational level as those which can be obtained from an initially uncorrelated state of the whole system by local operations, i.e. quantum operations performed only on subsystems, accompanied by possible exchange of classical information between them (e.g. about results of local measurement on which further local operations can be conditioned) [10]. The general practical criteria useful for unambiguous discrimination of mixed separable states among all states of a composite quantum system are not known, but in low-dimensional cases there are several methods (e.g. based on determining the so called concurrence of a state) to achieve the goal.
Equating absence of genuine quantum correlations with the property of being a product state (simple tensor) lacks sense in the case of indistinguishable particles. Indistinguishability forces (anti)symmetrization of wavefunctions, in other words states are vectors not in tensor products of Hilbert spaces but in their (anti)symmetrizations. With the exception bosons occupying the same state, no wave function has the form of a simple tensor. It is reasonable to call ‘nonclassical’ quantum correlations which go beyond those stemming solely from (anti)symmetrization, and consequently states which lack such correlations are treated as quantum uncorrelated or the best candidates for being ‘most classical’. Methods of investigating and characterizing such states were developed in [11] and [12].

For spin-coherent states, when the group is SU(2), the classical mixed states—as is customary in quantum optics—are characterized as having a positive \( P \)-representation. The definition is equivalent to decomposability of a state into a probabilistic mixture of pure coherent states. For the lowest-dimensional nontrivial case (spin 1) Giraud et. al. [1] give a necessary and sufficient condition for classicality of the state. Their proof of the correctness of the criterion is constructive — it gives explicitly a decomposition into eight coherent states.

The observation that all above mentioned cases are instances of the general construction of coherent states allows to analyze them in a unified manner. One may thus use methods known from the theory of coherent states to the separability problem, or vice versa, apply techniques from entanglement theory to exhibit some new features of generalized coherent states. We will mainly follow the latter path — a concrete aim is to refine one of the results of [1] concerning the cardinality of the optimal decomposition of a mixed spin-coherent state into a mixture of pure coherent states. Along the way we present a unified treatment of correlations in two-component quantum systems in terms of bilinear observables. We start by recalling the characterization of entanglement by the (pre)concurrence, defined originally by Wootters [13] and elaborated further by Uhlmann [14]. In this approach one quantifies the entanglement of a (pure) state by the expectation value of some antilinear operator and extends the measure to mixed states by the ‘convex roof’ construction, i.e. by decomposing a mixed state into a probabilistic mixture of pure states and minimizing over all possible decompositions the sum of concurrences of the components. As observed by Uhlmann [14], antilinear operators ‘are intrinsically nonlocal’, and as such are good candidates for a tool probing nonlocal properties like entanglement. On the other hand, they do not correspond to physical observables, hence their expectation values can not be directly measured. Elsewhere [15, 16] we argued that it might be more convenient to use bilinear Hermitian operators to quantify entanglement — such an approach is efficient in higher dimensions and provides measures of entanglement accessible directly in experiments [17]. Remarkably the same approach enables also a unified treatment of the different facets of ‘classicality’ enumerated above.

II. GENERALIZED CONCURRENCES

We start with an elucidation of links between generalized concurrences defined by Uhlmann and bilinear operators. As already advertised in the previous section, both provide tools for characterizing separability and entanglement as we will show in Section IV.

In the following \( \mathcal{H} \) will denote a finite-dimensional complex Hilbert space and \( \{|e_i\rangle\}_{i=1}^{N} \) is an orthogonal basis in \( \mathcal{H} \). Let \( \psi \in \mathcal{H} \) and \( C_\Theta(\psi) \) — the generalized concurrence in the sense of Uhlmann [14]— be defined via an antiunitary operator \( \Theta \), i.e.

\[
C_\Theta(\psi) = |\langle \psi | \Theta | \psi \rangle |. \tag{2}
\]

Each antiunitary \( \Theta \) can be written as \( \Theta = T_\Theta K \), where \( T_\Theta \) is unitary and \( K \) is the complex conjugation in some fixed basis (say the chosen one \( \{|e_i\rangle\}_{i=1}^{N} \)), i.e.

\[
C_\Theta(\psi) = |\langle \psi | T_\Theta | \psi^* \rangle|. \tag{3}
\]

Let us define \( A \in \text{End}(\mathcal{H} \otimes \mathcal{H}) \),

\[
A = (I \otimes \Lambda) |\Phi\rangle \langle \Phi|, \tag{4}
\]

where \( |\Phi\rangle \) is the maximally entangled state in \( \mathcal{H} \otimes \mathcal{H} \),

\[
|\Phi\rangle = \sum_{i=1}^{N} |e_i\rangle \otimes |e_i\rangle, \tag{5}
\]

and for an arbitrary \( \rho \in \text{End}(\mathcal{H}) \)

\[
\Lambda(\rho) = T_\Theta \rho T_\Theta^\dagger, \tag{6}
\]
i.e. $A$ is the image under the Jamiołkowski isomorphism of $\Lambda$ which, in turn, is a completely-positive unitary map on $\text{End}(\mathcal{H} \otimes \mathcal{H})$. Since $\Lambda$ is completely-positive, $A$ is non-negatively definite $[10, 18, 19]$.

The announced connection between the bilinear operator $A$ and the generalized concurrence $C_\Theta$ is given by the following

**Lemma 1**

\[ C_\Theta(\psi) = \langle \psi \otimes \psi | A | \psi \otimes \psi \rangle^{1/2}. \]  

**Proof**

From the definition of $|\Phi\rangle$ and $\Lambda$

\[ A = (I \otimes \Lambda) \sum_{i,j} |e_i\rangle\langle e_j| \otimes |e_i\rangle\langle e_j| = \sum_{i,j} |e_i\rangle\langle e_j| \otimes T_\Theta|e_i\rangle\langle T_\Theta^\dagger e_j|. \]  

Hence, for $|\psi\rangle = \sum_k c_k |e_k\rangle$,

\[ \langle \psi \otimes \psi | A | \psi \otimes \psi \rangle = \sum_{ijklmn} c_i^* c_j^* c_m c_n \langle e_k|e_i\rangle \langle e_j|e_m\rangle \langle e_i|T_\Theta|e_j\rangle \langle T_\Theta^\dagger e_m|e_n\rangle = \sum_{ijklmn} c_i^* c_j^* c_m c_n \delta_{ki} \delta_{jm} \langle e_i|T_\Theta|e_j\rangle \langle T_\Theta^\dagger e_m|e_n\rangle \]

\[ = \sum_{ijlmn} c_i^* c_j^* c_l c_n \langle e_i|T_\Theta|e_j\rangle \langle T_\Theta^\dagger e_n|e_m\rangle = \langle \psi|T_\Theta|\psi^*\rangle \langle \psi^*|T_\Theta^\dagger \psi\rangle = \langle \psi|T_\Theta|\psi^*\rangle \langle \psi|T_\Theta^\dagger \psi^*\rangle = |\langle \psi|T_\Theta|\psi^*\rangle|^2 = C_\Theta(\psi)^2. \]  

Remark 1 We have

\[ \text{Tr}_1 A = I, \]  

and

\[ \text{Tr}_2 A = I, \]  

where $\text{Tr}_{1,2}$ denotes the partial trace over the first or the second copy of $\mathcal{H}$. Indeed, from the unitarity of $T_\Theta$,

\[ \text{Tr}_1 A = \sum_{ijk} \langle e_k|e_i\rangle \langle e_j|e_k\rangle T_\Theta |e_i\rangle \langle e_j| T_\Theta^\dagger |e_k\rangle = \sum_{ijk} \delta_{ki} \delta_{jk} T_\Theta |e_i\rangle \langle e_j| T_\Theta^\dagger |e_k\rangle = T_\Theta \left( \sum_i |e_i\rangle \langle e_i| \right) T_\Theta^\dagger = T_\Theta T_\Theta^\dagger = I, \]  

and

\[ \text{Tr}_2 A = \sum_{ijk} |e_i\rangle \langle e_j| \langle e_k| T_\Theta |e_i\rangle \langle e_j| T_\Theta^\dagger |e_k\rangle = \sum_{ijk} |e_i\rangle \langle e_j| \left( T_\Theta^\dagger \right)_{jk} (T_\Theta)_{ki} = \sum_{ij} |e_i\rangle \langle e_j| (T_\Theta T_\Theta^\dagger)_{ji} \]

\[ = \sum_{ij} |e_i\rangle \langle e_j| \delta_{ji} = \sum_{i} |e_i\rangle \langle e_i| = I. \]  

Observe now that we may proceed in the opposite way, i.e. define the generalized concurrence as in (7),

\[ C_A(\psi) = \langle \psi \otimes \psi | A | \psi \otimes \psi \rangle^{1/2}, \]

where $A$ is some non-negatively defined linear operator on $\mathcal{H} \otimes \mathcal{H}$ with the property $[10]$. Then we define $\Lambda \in \text{End}(\text{End}(\mathcal{H}))$ as the image of $A$ under the inverse of the Jamiołkowski isomorphism, i.e.,

\[ \Lambda(\rho) = \text{tr}_1 \left( (\rho^t \otimes I) A \right), \]

for $\rho \in \text{End}(\mathcal{H})$. Observe that from (10) and (15),

\[ \Lambda(I) = \text{tr}_1 (A) = I. \]

Since $A$ is non-negatively defined, $A$ is completely positive, hence it has a Kraus decomposition $[10, 19]$, \n
\[ \Lambda(\rho) = \sum_{\alpha=1}^s T_\alpha \rho T_\alpha^\dagger. \]
If \( A \) is such that \( s = 1 \), then \( \Lambda \rho = T_1 \rho T_1^\dagger \) and from (16) \( T_1 T_1^\dagger = I \), i.e. \( T_1 \) is unitary. Then we have
\[
C_A(\psi) = |\langle \psi | T_1 | \psi^* \rangle| = C_\Theta(\psi),
\]
where \( \Theta = T_1 K \). To prove the last statement it is enough to check that (15) gives indeed the inverse of the Jamiołkowski isomorphism (4) and perform the calculations in the proof of Lemma 1 in the reverse direction.

It seems to be more appropriate to take (14) with the condition (10) as the definition of the generalized concurrence (see [15]). Generically the non-negative definite matrix \( A \) has the spectral decomposition
\[
A = \sum_{\alpha=1}^{s} \nu_\alpha |v_\alpha \rangle \langle v_\alpha |
\]
with more then one nonvanishing eigenvalues \( \nu_\alpha \). Here \( |w_\alpha \rangle = \sqrt{\nu_\alpha} |v_\alpha \rangle \) are the subnormalized eigenvectors of \( A \). It means that \( A \) defined by (14) is given by (17) with
\[
T_\alpha = (|\Phi \rangle \otimes I)(I \otimes |w_\alpha \rangle).
\]

Only exceptionally (e.g. in low-dimensional cases) \( s = 1 \) and we can write \( C_A \) in terms of an antiunitary operator \( \Theta \).

Performing the same calculations as in (9) for the general case (17) we obtain:
\[
\langle \psi \otimes \psi | A | \psi \otimes \psi \rangle = \sum_{\alpha=1}^{s} \langle \psi | T_\alpha | \psi^* \rangle \langle \psi^* | T_\alpha^\dagger | \psi \rangle = \sum_{\alpha=1}^{s} |\langle \psi | T_\alpha | \psi^* \rangle|^2,
\]
and by polarization,
\[
\langle \psi_2 \otimes \psi_4 | A | \psi_1 \otimes \psi_3 \rangle = \sum_{\alpha=1}^{s} \langle \psi_2 | T_\alpha | \psi_4^* \rangle \langle \psi_4^* | T_\alpha^\dagger | \psi_3 \rangle.
\]

### III. MIXED STATES

The concept of concurrence can be extended to mixed states. The idea is based on the following observation (see also [14, 21]).

Let \( E \) be the set of all extreme points of a compact convex set \( K \) in a finite dimensional real vector space \( V \). For every non-negative function \( f : E \rightarrow \mathbb{R}_+ \) we may define its extension \( f_K : K \rightarrow \mathbb{R}_+ \) by
\[
f_K(x) = \inf_{x = \sum_{p_i} x_i} \sum_{p_i} f(x_i)
\]
where the infimum is taken with respect to all expressions of \( x \) in the form of convex combinations of points \( x_i \) from \( E \). Let now \( E_0 \) be a compact subset of \( E \) with the convex hull \( K_0 = \text{conv}(E_0) \subset K \). If \( f \) is continuous and vanishes exactly on \( E_0 \), then the function \( f_K \) is convex on \( K \) and vanishes exactly on \( K_0 \). In our cases \( K \) is the set of all states, \( E \) — the set of pure states, and \( E_0 \) — the set of pure ‘classical’ states. Observe that due to homogeneity of the generalized concurrence, \( C_A(\alpha \psi) = |\alpha|^2 C_A(\psi) \), we can consider only the decompositions into sums of rank-one operators, defining thus for an arbitrary state \( \rho \),
\[
C_A(\rho) = \min_{\{\phi_k\}} \sum_{k=1}^{K} C_A(\phi_k),
\]
where the minimum is taken over all decompositions of \( \rho \) into a sum of rank-one operators,
\[
\rho = \sum_{k=1}^{K} |\phi_k \rangle \langle \phi_k |.
\]

We took advantage of the finite dimensionality of the Hilbert space and substituted the minimum for the the infimum operation (22) — in the finite-dimensional case there exists always an optimal decomposition, i.e. one which minimizes \( C_A(\rho) \).
A particular example of $\rho$ can be obtained from the spectral decomposition of $\rho$,

$$\rho = \sum_{i=1}^{r} p_i |\eta_i\rangle\langle\eta_i|, \quad \rho |\eta_i\rangle = p_i |\eta_i\rangle, \quad \langle\eta_i|\eta_j\rangle = \delta_{ij}, \quad r = \text{rank}\rho,$$

(26)

by subnormalizing the eigenvectors,

$$|\xi_i\rangle = \sqrt{p_i}|\eta_i\rangle, \quad \rho = \sum_{i=1}^{r} |\xi_i\rangle\langle\xi_i|.$$

(27)

Any other decomposition $\rho$ can be obtained from (27) with the help of a partial isometry $V$,

$$|\phi_k\rangle = \sum_{j=1}^{r} V_{kj}|\xi_j\rangle, \quad k = 1, \ldots, K; \quad V^\dagger V = I.$$

(28)

Hence,

$$C_A(\rho) = \min_{\kappa} \sum_{k} C_A(\phi_k) = \min_{\kappa} \sum_{k} \langle\phi_k \otimes \phi_k|A|\phi_k \otimes \phi_k\rangle^{1/2} = \min_{\kappa} \sum_{k} \left( V_{ki}^* V_{kj} V_{kl} V_{km} \langle\xi_i \otimes \xi_j|A|\xi_k \otimes \xi_l\rangle \right)^{1/2} = \min_{\kappa} \sum_{k} \left( \sum_{\alpha} (V^* \tau_\alpha V^\dagger)_{kk}(V^* \tau_\alpha V^T)_{kk} \right)^{1/2} = \min_{\kappa} \sum_{k} \left( \sum_{\alpha} |(V^* \tau_\alpha V^\dagger)_{kk}|^2 \right)^{1/2},$$

(29)

where $\tau_\alpha$ are $r \times r$ matrices,

$$(\tau_\alpha)_{ij} = \langle\xi_i|T_\alpha|\xi_j^*\rangle,$$

(30)

and the minimum is taken over all partial isometries $V$. This can be a starting point for various estimations of $C_A(\rho)$ in the spirit of $[15]$ and $[16]$.

In the case when $s = 1$ (so the generalized concurrence for pure states can be expressed in terms of an antiunitary operator (23)), the expression (29) reduces to

$$C_A(\rho) = \min_{\kappa} \sum_{k} |(V^* \tau_\alpha V^\dagger)_{kk}|, \quad \tau_{ij} = \langle\xi_i|T_1|\xi_j^*\rangle.$$

(31)

If $T_1$ is symmetric, $T_1 = T_1^T$, (as it happens in all the cases we consider - see below), then $\Theta = T_1 K$ is an antiunitary conjugation, i.e. $\Theta^2 = 1$, and $C_A(\rho)$ can be explicitly calculated,

$$C_A(\rho) = \max \left\{ 0, \mu_1 - \sum_{j=2}^{r} \mu_j \right\},$$

(32)

where $\mu_j$ are singular values of $\tau$ in decreasing order. A short calculation shows that $\mu_j$ are equal to the square roots of the eigenvalues of $\rho \tilde{\rho}$, where $\tilde{\rho} = \Theta \rho \Theta$. The eigenvalues of $R := \rho \tilde{\rho}$ are positive since $R$ is similar to a Hermitian positive-definite matrix $R = \rho^{1/2} R' \rho^{-1/2}$,

$$R' = \rho^{1/2} \Theta \rho \Theta \rho^{1/2} = \rho^{1/2} T_1 \rho^* T_1^\dagger \rho^{1/2} = (\rho^{1/2} T_1 \rho^{*1/2})(\rho^{*1/2} T_1^\dagger \rho^{1/2}) = (\rho^{1/2} T_1 \rho^{*1/2})(\rho^{1/2} T_1 \rho^{*1/2})^\dagger,$$

(33)

since $\rho = \rho^\dagger$ and $T_1^T = T_1$ i.e. $T_1 = T_1^\dagger$.

The proof of (29) was given by Uhlmann [14] (see also [15]) who also showed that the optimal decomposition may be constructed out of $2^{n+1}$ vectors, where $2^n < N \leq 2^{n+1}$. This is a generalization of the Wootters construction for the concurrence of two qubit states [13].

In the following we will call a (pure or mixed) state ‘classical’ if an appropriate generalized concurrence (24) vanishes.

From the arguments in the last two sections it should be clear that although vanishing of an appropriate $C_A(\rho)$ can characterize ‘classical’ (e.g. separable) states in an arbitrary finite dimension, it is only rarely that its calculation can be reduced to (29) via some antilinear $\Theta$. Such situation happens only in low-dimensional cases and this is the reason why e.g. Wootters’ construction [15] gives explicit results only in the two-qubit case. In the next section we will give more examples supporting this observation.
where $\rho_r$ is the reduced (to one of the subsystems) density matrix, vanishes only for separable $|\psi\rangle$, since it is only in this case that the reduced density matrix is pure and the trace of its square equals one. It is now a matter of straightforward calculations to establish that (34) is proportional to the quantity
\[
C^2(\psi) = \langle \psi | \psi \rangle^2 - \text{Tr} \rho_r^2,
\] (34)
where $\rho_r$ is the reduced (to one of the subsystems) density matrix, vanishes only for separable $|\psi\rangle$, since it is only in this case that the reduced density matrix is pure and the trace of its square equals one. It is now a matter of straightforward calculations to establish that (34) is proportional to $\langle \psi \otimes \psi | A | \psi \otimes \psi \rangle$ where $A$ is the projection on $\mathcal{H}_{12}$.

Let us briefly rederive this result using Schmidt decomposition, since later we would like to stress analogies with other examples of ‘classical’ systems. An arbitrary two-partite pure state can be written as
\[
|\psi\rangle = \sum_{i,j} c_{ij} |e_i\rangle \otimes |f_j\rangle,
\] (35)
where $\{|e_i\rangle\}_{i=1}^{N_1}$ and $\{|f_j\rangle\}_{j=1}^{N_2}$ are some orthonormal bases in $\mathcal{H}_1$ and $\mathcal{H}_2$ and $c$ some complex matrix. An arbitrary complex matrix $c$ can be transformed to a diagonal one with non-negative entries by multiplication from left and right by unitary matrices, $c \rightarrow U c V^*$ [24], which amounts to local unitary changes of bases, $|e_i\rangle = \sum_k U_{ki} |e'_k\rangle$, $|f_j\rangle = \sum_l V_{jl} |f'_l\rangle$. Hence, upon an appropriate choice of $U$ and $V$ we obtain $\psi$ in its Schmidt form
\[
|\psi\rangle = \sum_k l_k |e'_k\rangle \otimes |f'_k\rangle, \quad l_k > 0, \quad \sum_k l_k^2 = 1,
\] (36)
where the last condition stems from the normalization $\text{Tr} |\psi\rangle \langle \psi | = 1$. The state $|\psi\rangle$ is a simple tensor (i.e. is separable) if and only if only one among $l_k$ does not vanish. It is now easy to show that this condition is equivalent to the vanishing of $C$ given by (34).

The classical pure states are thus the separable ones. Hence the classical mixed states are those which are decomposable into pure separable states, i.e. (mixed) separable states.

For two qubits, $N_1 = 2 = N_2$, the matrix $A$ reads
\[
A = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}.
\] (37)
It has only one nonvanishing eigenvalue (equal to 4) with the eigenvector
\[ |w_0\rangle = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix}^T. \]
The corresponding matrix \( T \) reads
\[ T = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} = \sigma_y \otimes \sigma_y, \]
and the concurrence is given by the Wootters formula,
\[ C(\rho) = \max \left\{ 0, \mu_1 - \sum_{j=2}^{4} \mu_j \right\}, \]
Here \( \mu_j \) are the singular values of \( \tau_{ij} = \langle \xi_i | T | \xi_j^* \rangle \) and \( |\xi_i\rangle, i = 1, \ldots, 4 \) — the subnormalized eigenvectors of \( \rho \), or equivalently, \( \mu_i \) are the square roots of the eigenvalues of \( \rho \hat{\sigma} \), where \( \hat{\rho} = T \rho^* T \). The optimal decomposition can be constructed out of four vectors, which is also clear from Wootters’ original construction.

B. Entanglement in two-fermion systems

Correlations in systems of two fermions were investigated in [11, 12], where all relevant definitions and proofs can be found. Here we only briefly review the most important findings. In this case \( \mathcal{H} \) is the antisymmetric part of the tensor product of two copies of the single-particle Hilbert space \( \mathcal{H}_{2K} \) of an even dimension \( 2K = 2S + 1 \), where \( S \) is the spin of each particle, \( \mathcal{H} = \mathcal{A}(\mathcal{H}_{2K} \otimes \mathcal{H}_{2K}) \), hence \( \mathcal{H} \) has dimension \( N = 2K(2K - 1)/2 = S(2S + 1) \). An arbitrary pure state can be represented in the form
\[ |\psi\rangle = \sum_{i,j=1}^{2S+1} w_{ij} f_i^\dagger f_j^\dagger |0\rangle, \]
where \( f_i^\dagger \) are fermionic creation operators and \( |0\rangle \) — the vacuum state and \( w \) — a complex antisymmetric matrix. A unitary transformation \( U \) of the single particle space \( \mathcal{H}_S \) leads to the transformation \( f_i^\dagger \mapsto \sum_{ji} U_{ji} f_j^\dagger \) and, consequently,
\[ w \mapsto UwU^T. \]
The analogue of the Schmidt decomposition is now provided by the theorem stating that an arbitrary, complex matrix \( w \) can be brought by an appropriate transformation [12] to a block-diagonal, canonical form,
\[ w' = \text{diag} \left[ Z_1, \ldots, Z_r, Z_0 \right], \quad Z_i = \begin{bmatrix} 0 & z_i \\ -z_i & 0 \end{bmatrix}, \]
with \( z_i \neq 0 \) and \( Z_0 \) - a null matrix [24]. The number \( r \) of non-vanishing \( 2 \times 2 \) blocks is called the Slater rank of \( |\psi\rangle \) [11].

Pure states with the minimal (i.e. equal to one) Slater rank exhibit the minimal allowable quantum correlations [11], so they are the candidates for the ‘classical’ states. Consequently, mixed states are ‘classical’ when they can be decomposed into a convex combination of pure states with Slater rank one. The maximal Slater rank of the components of a pure state decomposition of a mixed state \( \rho \) minimized over all possible decomposition is called the Slater number, by analogy to the Schmidt number defined in a similar manner for distinguishable particles. Using this notion we identify ‘classical’ mixed states with those of the Slater number one.

In order to characterize the ‘classical’ states we make use of a lemma proved in [12]. It provides a general criterion for a pure state in the form [11] to have a Slater rank not exceeding a prescribed value. In particular it states that a two fermion state in \( \mathcal{H} = \mathcal{A}(\mathcal{H}_{2K} \otimes \mathcal{H}_{2K}) \) has the Slater rank one if and only if for all \( 1 \leq \alpha_1 < \cdots < \alpha_{2(K-2)} \leq 2K \)
\[ \sum_{i,j,k,l=1}^{2K} w_{ij} w_{kl} \epsilon^{ijkl} \alpha_1 \cdots \alpha_{2(K-2)} = 0 \]
(44)
where $\varepsilon^{i_1,\ldots,i_{2K}}$ is the totally antisymmetric unit tensor in $\mathcal{H}_{2K}$, which is obviously equivalent to
\[
\sum_{1 \leq \alpha_1 < \cdots < \alpha_{2(K-2)} \leq 2K} \left| \sum_{i,j,k,l=1}^{2K} w_{ij} w_{kl} \varepsilon^{ijkl\alpha_1 \cdots \alpha_{2(K-2)}} \right|^2 = 0. \quad (45)
\]
Each of the terms under the absolute value sign is the Pfaffian $\text{Pf} (w)$ of the $4 \times 4$ matrix obtained from $w$ by deleting the rows and columns with numbers $\alpha_1, \alpha_2, \ldots, \alpha_{2(K-2)}$. It is a matter of simple calculations that the Pfaffian $\text{Pf}(X)$ of a $4 \times 4$ matrix $X$ reads
\[
\text{Pf}(X) = X_{12}X_{34} - X_{13}X_{24} + X_{14}X_{23}, \quad (46)
\]
and
\[
|\text{Pf}(X)|^2 = (x \otimes x)A_{sf}/(x \otimes x), \quad (47)
\]
where $|x\rangle$ is the six-dimensional vector
\[
|x\rangle = \begin{bmatrix} X_{12} \\ X_{13} \\ X_{14} \\ X_{23} \\ X_{24} \\ X_{34} \end{bmatrix}, \quad (48)
\]
and, in the standard basis $|e_1\rangle, \ldots, |e_6\rangle$ in $\mathbb{C}^6$,
\[
A_{sf} = \sum_{i,j=1}^{6} |e_i\rangle\langle e_j| \otimes T|e_i\rangle\langle e_j|T^\dagger, \quad (49)
\]
with
\[
T = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \quad (50)
\]
Substituting for each term in the sum in (45) an appropriate expression (47), involving in each summand a different set of entries of $w$ with the corresponding matrix $A_{sf}$, we rewrite the condition (45)
\[
C_A^2(\psi) := (w \otimes w)A(w \otimes w) = 0. \quad (51)
\]
Here $A$ is a $S^2(2S + 1)^2 \times S^2(2S + 1)^2$ matrix acting in $\mathcal{H} \otimes \mathcal{H}$, built up from the submatrices $A_{sf}$ of each summand, and $|w\rangle$ is the $2S + 1$ dimensional vector of independent entries of the matrix $w$ representing in this way the state vector $|\psi\rangle$. The condition (51) constitutes the appropriate non-entanglement criterion for pure fermionic states in the form of a bilinear generalized concurrence.

For the lowest-dimensional, nontrivial case, i.e. $S = 3/2$, $A$ is a $36 \times 36$ matrix (skipped to save the place) with only one non-vanishing eigenvalue. The corresponding eigenvector has only six non-vanishing elements and the corresponding matrix $T$ (20) is given by (20), as it is also clear from (8) and (19). The result is equivalent to the one of $\mathbb{C}^6$ modulo a change of basis in $\mathbb{C}^6$: $|e'_4\rangle = |e_1\rangle$, $|e'_5\rangle = |e_2\rangle$, $|e'_6\rangle = (|e_3\rangle + |e_4\rangle)/\sqrt{2}$, $|e'_7\rangle = |e_5\rangle$, $|e'_8\rangle = |e_6\rangle$, and $|e'_9\rangle = (|e_3\rangle - |e_4\rangle)/\sqrt{2}$.

The generalized concurrence $C_A(\rho)$ for mixed states is the Slater correlation measure and is given by
\[
C_{Sl}(\rho) = \max \left\{ 0, \mu_1 - \sum_{j=2}^{6} \mu_j \right\}. \quad (52)
\]
The definition of $\mu_1$ is analogous to the one used in the preceding example, the only changes are in the dimensionality and the definition of $T$. As previously $\mu_i$ are equal to the square roots of the eigenvalues of $\rho^\dagger T$, with $\rho = T \rho^* T$, which coincides with the results of $\mathbb{C}^6$. From the result of Uhlmann we infer that the optimal decomposition can be achieved with 8 vectors.
C. Entanglement in two-boson systems

Quantum correlations in bosonic systems were thoroughly investigated in [12]. The line of thought follows *mutatis mutandis* considerations in the fermionic case. For two particles the Hilbert space is the symmetric part of the tensor product of the single-particle space, \( \mathcal{H} = \mathcal{S}(\mathcal{H}_M \otimes \mathcal{H}_M) \), \( \dim \mathcal{H}_M = M \), \( \dim \mathcal{H} = M(M + 1)/2 \). A pure state can be written in the form

\[
|\psi\rangle = \sum_{i,j=1}^{M} b_i^\dagger b_j^\dagger v_{ij}|0\rangle,
\]

with bosonic creation operators \( b_i^\dagger \) and a symmetric complex matrix \( v \) transforming upon unitary map \( U \) in the single particle state according to \( b_i^\dagger \mapsto \sum_i U_{ij} b_j^\dagger \) and, consequently,

\[
v \mapsto U v U^T.
\]

As previously, an appropriate theorem from the linear algebra [24] provides a possibility of using (54) to transform \( v \) to its diagonal, canonical form

\[
v' = \text{diag}(z_1, \ldots, z_r, 0, \ldots, 0), \quad z_i \neq 0.
\]

The number \( r \) of nonvanishing \( z_i \) is dubbed *bosonic Slater rank*, and states with the minimal \( r = 1 \) exhibit the minimal possible amount of purely quantum correlations [12]. Accordingly these are our ‘classical’ pure states, whereas ‘classical’ mixed states are those which can be decomposed into a convex combination of pure states with the Slater rank equal to one. As in the previously considered cases one can define the *bosonic Slater number* of a mixed state as a minimum over all convex decompositions into pure states of the maximal Slater rank among the members of a decomposition. In this terminology, a mixed is ‘classical’ if and only if its bosonic Slater number equals one.

As in the case of fermions there exists a bilinear characterization of pure states with the bosonic Slater rank equal to one [12]. For them

\[
\sum_{i,j,k,l=1}^{M} v_{ij} v_{kl} \varepsilon^{ik\alpha_1 \cdots \alpha_{M-2} \varepsilon jl\alpha_1 \cdots \alpha_{M-2}} = 0
\]

for all \( 1 \leq \alpha_1 < \cdots < \alpha_{M-2} \leq M \), which can be transformed to the desired form (14) along, essentially, the same lines as in the fermionic case. To this end we rewrite (56) in the form

\[
\sum_{1 \leq \alpha_1 < \cdots < \alpha_{M-2} \leq M} \left| \sum_{i,j,k,l=1}^{M} v_{ij} v_{kl} \varepsilon^{ik\alpha_1 \cdots \alpha_{M-2} \varepsilon jl\alpha_1 \cdots \alpha_{M-2}} \right|^2 = 0.
\]

To cut short the connection to the conventions used in [12] it is convenient to represent the vector \( |\psi\rangle \) in \( \mathcal{H} = \mathbb{C}^{M(M+1)/2} \) by the vector

\[
|\tilde{v}\rangle = \begin{bmatrix} v_{11} \\ v_{12} \\ \vdots \\ v_{MM} \end{bmatrix}
\]

of the independent entries of the symmetric matrix \( v \), i.e. \( v_{ij}, i \leq j \), scaling, however, the diagonal entries by \( 1/\sqrt{2} \), \( v'_{ij} = v_{ij}/\sqrt{2} \). The way of representing \( |\psi\rangle \) in terms of \( v \) is, obviously, a matter of convenience, as long as linearity of the representation is observed. The chosen one, besides being in accordance with [12], has the advantage of giving the same ‘weight’ to diagonal and off-diagonal entries of \( v \).

Each summand under the absolute value sign in (57) involves only a \( 2 \times 2 \) submatrix of \( v \),

\[
v(a,b) = \begin{bmatrix} v_{aa} & v_{ab} \\ v_{ba} & v_{bb} \end{bmatrix}, \quad 1 \leq a, b \leq M,
\]

and equals, up to an irrelevant sign, \( 2v_{aa}v_{bb} - v_{ab}^2 = v'_{aa}v'_{bb} - v'_{ab}^2 \). This, in turn, can be rewritten in the form \( \langle \tilde{v} \otimes \tilde{v} \rangle A_{bos} \langle \tilde{v} \otimes \tilde{v} \rangle \), where
\[ |\vec{v}⟩ = |\vec{v}(a, b)⟩ = \begin{bmatrix} v'_{aa} \\ v'_{ab} \\ v'_{bb} \end{bmatrix}, \quad A_{bos} = \sum_{i,j=1}^{3} |e_i⟩⟨e_j| \otimes |e_i⟩⟨e_j| T^1, \quad T = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \]  

(60)

and \{ |e_1⟩, |e_2⟩, |e_3⟩ \} is the standard basis in \( \mathbb{C}^3 \). Again, the non-entanglement condition for pure states can be rewritten as

\[ C_A(ψ) := ⟨v \otimes v | A | v \otimes v⟩ = 0, \]  

(61)

by combining the appropriate \( 6 \times 6 \) matrices \( A_{bos} \) into a \( (M(M+1)/2)^2 \times (M(M+1)/2)^2 \) matrix acting in \( \mathcal{H} \otimes \mathcal{H} \).

In the lowest-dimensional, nontrivial case \( M = 2, N = 4 \) we find \[ 31 \]

\[ A = \begin{bmatrix} 0 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \]  

(62)

The matrix \( A \) has only one non-vanishing eigenvalue with the eigenvector \[ |w_0⟩ = \begin{bmatrix} 0 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & 0 \end{bmatrix}^T, \]  

(63)

to which there corresponds via \( 20 \) the matrix \( T \) given in \( 60 \). In a perfect analogy with the previous examples the generalized concurrence \( C_A(ρ) \) is given in terms of the square roots of the eigenvalues of \( ρ T ρ^T \),

\[ C_A(ρ) = \max \{ 0, μ_1 - \sum_{j=2}^{3} μ_j \}, \]  

(64)

and coincides with the bosonic correlation measure \( 12 \). The optimal decomposition involves at most 4 vectors.

\section*{V. SPIN COHERENT STATES}

Classical spin states were defined in \( 1 \) as those which can be decomposed into a probabilistic mixture of pure spin coherent states. In order to mimic the previous construction we will use the following characterization of pure coherent states, following directly from their minimal-uncertainty property \( 4 \).

\textbf{Theorem 1} A spin-\( S \) state \( |ψ⟩ \in \mathcal{H} \) is coherent if and only if

\[ (L_1 \otimes L_1 + L_2 \otimes L_2 + L_3 \otimes L_3)|ψ \otimes ψ⟩ = S^2 |ψ \otimes ψ⟩, \]  

(65)

where \( L_i \) are operators of the spin components in the spin \( S \) representation of the rotation group.

This is obviously equivalent to

\[ ⟨ψ \otimes ψ | A |ψ \otimes ψ⟩ = 0, \quad A = I - L_1 \otimes L_1 - L_2 \otimes L_2 - L_3 \otimes L_3. \]  

(66)

For the lowest-dimensional nontrivial case \( S = 1 \) we choose

\[ L_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix}, \quad L_2 = \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{bmatrix}, \quad L_3 = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \]  

(67)
Then
\[
A = \begin{bmatrix}
1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\
0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\]  \quad (68)

The matrix \( A \) has eigenvalues 3, 2, 2, 0, 0, 0, 0, 0, hence, at first sight, the previous construction fails. But by inspection of the eigenvectors we find that the corresponding matrices \( T_\alpha \) read
\[
T_1 = I, \quad T_2 = iL_1, \quad T_3 = iL_2, \quad T_4 = iL_3,
\]

hence, due to the antisymmetry of \( T_\alpha \), \( i = 2, 3, 4 \), we have \( \langle \psi | T_i | \psi^* \rangle = 0 \) for \( i = 2, 3, 4 \), and an arbitrary 3-dim vector \( |\psi\rangle \). Thus (cf. (21))
\[
C_A(\psi) = |\langle \psi | \psi^* \rangle|,
\]

and
\[
C_A(\rho) = \max \left\{ 0, \mu_1 - \sum_{j=2}^{3} \mu_j \right\} ,
\]

where \( \mu_i \) are the singular values of \( \tau_{ij} = \langle \xi_i | \xi_j^* \rangle \) and \( |\xi_i\rangle \) are the subnormalized eigenvectors of \( \rho \) (or, equivalently, \( \mu_i \) are the roots of the eigenvalues of \( R = \rho \rho^* \)). The optimal decomposition can be achieved with at most four vectors.

The criterion of classicality based on (74) can be formulated directly in terms of orthogonal invariants (traces of powers) of \( R \). To this end consider a \( 3 \times 3 \) density matrix i.e. \( \rho = \rho^I \), \( \text{Tr} \rho = 1 \) and \( \rho \) — non-negatively definite. In the following we assume that \( \rho \) is nondegenerate, \( \det \rho > 0 \), we also will supplement this by some other (see below) non-degeneracy conditions (all degenerate cases, in general simpler in treatment, can be considered along the same lines).

We decompose \( \rho \) into the real and imaginary part, \( \rho = \rho_R + i \rho_I \). The hermiticity of \( \rho \) implies \( \rho_I^T = -\rho_I \). The real part \( \rho_R \) can be thus diagonalized by an orthogonal transformation which leaves the antisymmetry of \( \rho_I \) unaltered. After that \( \rho \) takes the form
\[
\rho = \begin{bmatrix}
\lambda_1 & -i v_3 & i v_2 \\
i v_3 & \lambda_2 & -i v_1 \\
-i v_2 & i v_1 & \lambda_3
\end{bmatrix}
\]  \quad (72)

Since \( \rho \) is non-negatively definite we have \( \lambda_i \geq 0 \). In the following we assume a further non-degeneracy condition \( \lambda_i > 0 \). We have also \( \lambda_1 + \lambda_2 + \lambda_3 = \text{Tr} \rho = 1 \).

The matrix \( R \) reads now:
\[
R = \rho \rho^* = \begin{bmatrix}
\lambda_1^2 - v_2^2 - v_3^2 & v_1 v_2 + i v_3 (\lambda_1 - \lambda_2) & v_1 v_3 + i v_2 (\lambda_3 - \lambda_1) \\
v_1 v_3 + i v_2 (\lambda_3 - \lambda_1) & \lambda_2^2 - v_1^2 - v_3^2 & v_2 v_3 + i v_1 (\lambda_2 - \lambda_3) \\
v_1 v_3 + i v_2 (\lambda_3 - \lambda_1) & v_2 v_3 + i v_1 (\lambda_2 - \lambda_3) & \lambda_3^2 - v_2^2 - v_1^2
\end{bmatrix}
\]  \quad (73)

From the previous arguments we know that \( R \) has a real non-negative spectrum. We denote the eigenvalues of \( R \) by \( \mu_1^2, \mu_2^2, \) and \( \mu_3^2 \), and assume \( \mu_1 > \mu_2 > \mu_3 > 0 \) (again leaving apart possible degeneracies) and define
\[
x_1 = \mu_1 - \mu_2 - \mu_3, \quad x_2 = \mu_2 - \mu_1 - \mu_3, \quad x_2 = \mu_3 - \mu_1 - \mu_2, \quad x_4 = \mu_1 + \mu_2 + \mu_3.
\]

\]
Hence $x_2 = -(\mu_1 - \mu_2) - \mu_3 < 0$, $x_3 = -(\mu_1 - \mu_3) - \mu_2 < 0$, and $x_4 > 0$. It follows thus that the polynomial

$$P(x) = (x - x_1)(x - x_2)(x - x_3)(x - x_4)$$

has at least two negative roots $x_2$ and $x_3$ and at least one positive root $x_4$. The sign of the remaining one $x_1$ depends on $\lambda_i$ and $\nu_i$.

Straightforward calculations give

$$P(x) = x^4 - (4 \text{Tr} R)x^2 - 8(\det R)^{1/2}x + 2 \text{Tr}(R^2) - (\text{Tr} R)^2$$

(76)

If $2 \text{Tr}(R^2) - (\text{Tr} R)^2 < 0$ then the consecutive signs of the coefficients of $P(x)$ read $++--$, hence from the Descartes rule of signs it has at most one positive root and consequently $x_1 = \mu_1 - \mu_2 - \mu_3 < 0$. On the other hand for $2 \text{Tr}(R^2) - (\text{Tr} R)^2 > 0$ the signs of the coefficients of $P(-x)$ are $+-+-$, hence $P(-x)$ has at most two positive roots, i.e. $P(x)$ has at most two negative roots and, consequently, $x_1$ is positive. (The validity of this assertion can be easily established by considering the graph of $P(x)$ under the conditions that it crosses the $x$-axis in at least two negative and at least one positive point). Summarizing: the sign of $\mu_1 - \mu_2 - \mu_3$ coincides with the sign of $2 \text{Tr}(R^2) - (\text{Tr} R)^2$, or in other words, $\rho$ is ‘classical’ if and only if $2 \text{Tr}(R^2) - (\text{Tr} R)^2 \leq 0$. (The equality in the last formula appears when we relax the non-degeneracy conditions).

It can be proved that (74) is equivalent to the criterion of (3) for the classicality of spin-1 coherent states. The latter is formulated as follows. We rewrite $\rho$ in the form

$$\rho = \frac{1}{2}(I - W + u \cdot L),$$

(77)

where $u$ is a real vector and $L = (L_1, L_2, L_3)$ with $L_i$ given by (67). The matrix $Z$ of Giraud et.al. is then defined as $Z_{ij} = W_{ij} - u_{i}u_{j}$. The state is classical iff $Z$ is positive definite. The proof of the equivalence of both criteria is given in the Appendix.

VI. ‘CLASSICAL’ STATES AS GENERALIZED COHERENT STATES

As already stated in the introduction similarities of all above outlined constructions have their common origin in the fact that ‘classical’ states are special orbits of the Lie group of symmetries characteristic for a particular problem. To make this statement more precise let us recall a few facts from the theory of group representations needed for the definition of generalized coherent states [25]. Let $K$ be a compact semisimple Lie group, $\mathfrak{t}$ its Lie algebra and $g = \mathfrak{t} \oplus i\mathfrak{t}$ - the complexification of $\mathfrak{t}$. As a convenient example we can take $K = SU(N)$, $\mathfrak{t} = \mathfrak{su}_N$, and $g = \mathfrak{sl}_N(\mathbb{C})$. In addition we can imagine all of them as sets of complex matrices in the defining representation in the complex space $\mathbb{C}^N$, then $K = SU(N)$ is the set of unitary $N \times N$ matrices with determinant 1, $\mathfrak{t} = \mathfrak{su}_N$ - the set of traceless antihermitian $N \times N$ matrices, and $\mathfrak{sl}_N(\mathbb{C})$ - the set of complex, traceless $N \times N$ matrices.

The Lie algebra $g$ can be decomposed into the direct sum

$$g = n_- \oplus \mathfrak{t} \oplus n_+,$$

(78)

where $\mathfrak{t}$ is the Cartan subalgebra of $g$ (its maximal commutative subalgebra dimension of which is called the rank of $g$), whereas $n_\pm$ are particular nilpotent subalgebras of $g$. In our $SU(N)$ example realized in $\mathbb{C}^N$ we can choose $\mathfrak{t}$ as the set of traceless diagonal matrices and $n_+$ and $n_-$ as, respectively, upper- and lower-triangular matrices.

As linear spaces $n_+$ and $n_-$ are direct sums of the root spaces

$$n_+ = \bigoplus_\alpha \mathfrak{g}_\alpha, \quad n_- = \bigoplus_\alpha \mathfrak{g}_{-\alpha}.$$  

(79)

Each $\mathfrak{g}_\alpha$ is one-dimensional and is uniquely determined by the commutation relations of its elements with the elements in $\mathfrak{t}$

$$[H, X] = \alpha(H)X, \quad H \in \mathfrak{t}, \quad X \in \mathfrak{g}_\alpha,$$  

(80)

where $\alpha(\cdot)$ is an appropriate linear form on $\mathfrak{t}$. If we choose some basis $\{H_i\}$, $(i = 1, \ldots, r = \text{dim } \mathfrak{t})$ in $\mathfrak{t}$, we can calculate (80) for $H = H_i$ obtaining vectors (roots) $\alpha$ with the components $\alpha_i = \alpha(H_i)$. They span the Euclidean space of dimension equal to the rank of $g$, in which we can choose a basis consisting of positive simple roots — each root $\alpha$ is a linear combination of them with only non-negative (positive roots) or non-positive (negative roots) coefficients. The algebra $g$ is uniquely determined by its positive roots (or, equivalently, by its positive simple roots). The positive
(negative) root vectors are, by definition, the elements $X_{\pm \alpha}$ of $\mathfrak{g}_{\pm \alpha}$, fulfilling, according to [20], $[H_i, X_{\pm \alpha}] = \pm \alpha_i X_{\pm \alpha}$. There is one-to-one correspondence between the negative and positive root vectors.

On a semisimple Lie algebra $\mathfrak{g}$ there exists a nondegenerate bilinear form (the Killing form) defined as

$$(X, Y) = \text{Tr}(\text{ad}X \cdot \text{ad}Y),$$

where $\text{ad}X$ is the linear transformation of $\mathfrak{g}$ into itself given by $\text{ad}X(Y) = [X, Y]$. We can use the Killing form to fix the normalization of root vectors and the elements $H_i$ by choosing $(X_{\alpha}, X_{-\alpha}) = 1$ and $(H_i, H_j) = \delta_{ij}$.

From the root vectors and the elements $H_i$ we can construct the second order Casimir operator

$$C_2 := \sum_{\alpha > 0} (X_{\alpha}X_{-\alpha} + X_{-\alpha}X_{\alpha}) + \sum_{i=1}^{r} H_i^2,$$

which, as straightforward calculations show, commutes with every element of $\mathfrak{g}$. With the help of $C_2$ one can conveniently characterize the set of coherent states.

The group $K$ as well as the algebras $\mathfrak{t}$ and $\mathfrak{g}$ can be irreducibly represented not only in the defining space $\mathbb{C}^N$, as we did with $SU(N)$, but also in spaces of other dimensions. If we choose as the representation space $\mathcal{H} = \mathbb{C}^M$, we can again think of representatives of $K$, $\mathfrak{g}$, etc. as sets of matrices, so we will use the same letter $\pi$ to denote the homomorphism between the sets of matrices in $\mathbb{C}^N$ and $\mathbb{C}^M$ defining the considered irreducible representation. Thus we denote by $\pi(H)$ the matrix representing in $\mathbb{C}^M$ the element $H$ of, say, $\mathfrak{g}$, or by $\pi(U)$ a representative of $U \in K$ etc.

For each irreducible representation of a semisimple $\mathfrak{g}$ there exists a vector $|v_{\text{max}}\rangle$ in $\mathcal{H}$ which is a common eigenvector of all $\pi(H_i)$ annihilated by $\pi(X)$ for $X \in \mathfrak{n}_+$, i.e.

$$\pi(H_i)|v_{\text{max}}\rangle = \lambda_i |v_{\text{max}}\rangle, \quad \pi(X)|v_{\text{max}}\rangle = 0, \quad \text{for } X \in \mathfrak{n}_+.$$  

(83)

The vector $|v_{\text{max}}\rangle$ is called the highest weight vector of the irreducible representation. An irreducible representation of $\mathfrak{g}$, as well as $K$, is uniquely determined by the vector $\lambda = (\lambda_1, ..., \lambda_r)$.

Generalized coherent states for the group $K$, or more precisely, for its irreducible representation in $\mathcal{H}$, are obtained by applying to the highest weight vector $|v_{\text{max}}\rangle$ all possible representatives $\pi(U)$, $U \in K$ [22]. In all cases relevant for the present considerations, we can assume that the representation in question is unitary, so the action of the group $K$ is $\mathfrak{g}$, etc. as sets of matrices, so we will use the same letter $\pi$ to denote the homomorphism between the sets of matrices in $\mathbb{C}^N$ and $\mathbb{C}^M$ defining the considered irreducible representation. Thus we denote by $\pi(H)$ the matrix representing in $\mathbb{C}^M$ the element $H$ of, say, $\mathfrak{g}$, or by $\pi(U)$ a representative of $U \in K$ etc.

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Generalized coherent states for the group $K$, or more precisely, for its irreducible representation in $\mathcal{H}$, are obtained by applying to the highest weight vector $|v_{\text{max}}\rangle$ all possible representatives $\pi(U)$, $U \in K$ [22]. In all cases relevant for the present considerations, we can assume that the representation in question is unitary, so the action of the group $K$ is $\mathfrak{g}$, etc. as sets of matrices, so we will use the same letter $\pi$ to denote the homomorphism between the sets of matrices in $\mathbb{C}^N$ and $\mathbb{C}^M$ defining the considered irreducible representation. Thus we denote by $\pi(H)$ the matrix representing in $\mathbb{C}^M$ the element $H$ of, say, $\mathfrak{g}$, or by $\pi(U)$ a representative of $U \in K$ etc.

For each irreducible representation of a semisimple $\mathfrak{g}$ there exists a vector $|v_{\text{max}}\rangle$ in $\mathcal{H}$ which is a common eigenvector of all $\pi(H_i)$ annihilated by $\pi(X)$ for $X \in \mathfrak{n}_+$, i.e.

$$\pi(H_i)|v_{\text{max}}\rangle = \lambda_i |v_{\text{max}}\rangle, \quad \pi(X)|v_{\text{max}}\rangle = 0, \quad \text{for } X \in \mathfrak{n}_+.$$  

(83)

The vector $|v_{\text{max}}\rangle$ is called the highest weight vector of the irreducible representation. An irreducible representation of $\mathfrak{g}$, as well as $K$, is uniquely determined by the vector $\lambda = (\lambda_1, ..., \lambda_r)$.

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(83)
The set of coherent states $C_S$, i.e. the orbit through the highest weight vector can be uniquely characterized by certain bilinear condition [27]. Let us denote by $\delta$ the vector which is the half of the sum of all positive roots $\alpha$. A state $|u\rangle$ is coherent if and only if

$$L(|u\rangle \otimes |u\rangle) = \langle 2\lambda + 2\delta, 2\lambda \rangle (|u\rangle \otimes |u\rangle),$$  

where $\langle \cdot, \cdot \rangle$ is the scalar product in the $r$-dimensional Euclidean space of vectors $\alpha$ and $\delta$, and

$$L = \pi(C_2) \otimes I + I \otimes \pi(C_2) + \sum_{\alpha>0} (\pi(X_\alpha) \otimes \pi(X_{-\alpha}) + \pi(X_{-\alpha}) \otimes \pi(X_\alpha)) + 2 \sum_{i=1}^{r} \pi(H_i) \otimes \pi(H_i).$$  

We used the above formula, adapted to the case of $SU(2)$, to write the bilinear characterization of spin-coherent states (92), however the other bilinear characteristics of 'classical' states (cf. (34), (51, and (61)) were not straightforwardly derived from (89). The connection between two different bilinear characterizations will be explained elsewhere.

All classes of pure states considered in the preceding sections can be treated as particular instances of the general coherent states. Thus

1. spin $j$ coherent states are generalized coherent states for the $SU(2)$ group irreducibly represented in $\mathbb{C}^{2j+1}$, i.e.

$$K = SU(2), \quad \mathcal{H} = \mathbb{C}^{2j+1},$$  

2. for the fermionic separable states the appropriate identifications read

$$K = SU(N), \quad \mathcal{H} = \mathcal{A}(\mathbb{C}^N \otimes \mathbb{C}^N),$$  

where $\mathcal{A}$ denotes the antisymmetric part of the tensor product,

3. for the bosonic separable states we have

$$K = SU(N), \quad \mathcal{H} = \mathcal{S}(\mathbb{C}^N \otimes \mathbb{C}^N),$$  

where $\mathcal{S}$ denotes the symmetric part of the tensor product,

4. for the separable states of two distinguishable particles,

$$K = SU(N) \times SU(M), \quad \mathcal{H} = \mathbb{C}^N \otimes \mathbb{C}^M.$$

VII. LOW DIMENSIONAL ‘CLASSICAL’ STATES AND ORTHOGONAL SYMMETRIES

Low-dimensional states considered above, apart from the exhibited properties of being effectively characterizable by concurrences based on antilinear operators, have some additional nice representations which we would like to outline shortly in the present section.

Let us start from the following observation. A point in complex projective space of an arbitrary finite dimension (i.e. a pure state of a quantum system with an $N$-dimensional Hilbert space $\mathcal{H}$) can be always represented in the form

$$|\psi\rangle = \cos \theta \mathbf{x} + i \sin \theta \mathbf{y},$$  

where $\mathbf{x}$ and $\mathbf{y}$ are real, unit, orthogonal vectors

$$\mathbf{x}^2 = \mathbf{y}^2 = 1, \quad \mathbf{x} \cdot \mathbf{y} = 0.$$  

Indeed, an arbitrary vector $|\psi\rangle \in \mathcal{H}$ can be decomposed into its real and imaginary parts, $|\psi\rangle = \mathbf{x}' + i\mathbf{y}'$. The normalization condition $1 = \langle \psi | \psi \rangle = \mathbf{x}'^2 + \mathbf{y}'^2$ leads to $\mathbf{x}' = \cos \theta \mathbf{x}$, $\mathbf{y}' = \sin \theta \mathbf{y}$ with real, unit vectors $\mathbf{x}$ and $\mathbf{y}$. Since we work in the projective space $\mathbb{P}\mathcal{H}$, we still have at our disposal a phase factor, $|\psi\rangle \sim e^{i\varphi} |\psi\rangle$, which, as simple calculation shows, is enough to enforce orthogonality of $\mathbf{x}$ and $\mathbf{y}$. Obviously, by alternating the global signs of $\mathbf{x}$ and/or $\mathbf{y}$, we may restrict $\theta$ to the interval $[0, \frac{\pi}{2}]$.

The representation (93) is, in general, not particularly useful or interesting, since it is not invariant under general unitary transformations of $\mathcal{H}$, which are allowed by quantum mechanics, but in general do not leave $\mathbf{x}$ and
y real. If, however, for some reasons the relevant transformations made on the system in question belong to the orthogonal group \( SO(N) \), the situation changes.

For all low dimensional cases considered in the previous sections this is the case. For two qubits, although the Hilbert space is four dimensional, with the ‘natural’ symmetry group \( SU(4) \), the relevant subgroup of the local unitary transformations which do not change the entanglement is \( SU(2) \times SU(2) \), homomorphic to the orthogonal group \( SO(4) \). For two spin-3/2 fermions the Hilbert space is six dimensional but the relevant subgroup of \( SU(6) \) of transformations which respect the antisymmetry of states is \( SU(4) \) — the full symmetry group of the four-dimensional single-particle space (the antisymmetry is not destroyed if both particles undergo the same operation). Here again we have a homomorphism with an orthogonal group: \( SU(4) \sim SO(6) \). Finally, in the case of two bosons in three-dimensional space (two-dimensional single particle space) and coherent spin-1 states the relevant subgroup of \( SU(3) \) is \( SU(2) \) — the full symmetry group in the single-particle space in the case of bosons and the group defining the structure of coherent states, and the relevant homomorphism is the familiar one \( SU(3) \sim SO(3) \).

Throughout we consider the case when a group of transformations leaves the set of ‘classical’ states invariant (or more generally, when the generalized concurrence is constant on the orbits of the group). A recurrent theme is that the group is, on the one hand a unitary subgroup more generally, when the generalized concurrence is constant on the orbits of the group). A recurrent theme is that of four orthogonal Bell states \([13]\), in the fermionic and bosonic cases they are given in \([12]\).

To take direct advantage of the representation \([14]\) one should determine the basis in which the relevant group is represented by real matrices. A moment of reflection suffices to establish that such a basis consists of vectors fulfilling \(|\xi_k\rangle = \Theta|\xi_k\rangle \). Indeed, in this basis a unitary transformation \(|\psi\rangle \to U|\psi\rangle \) leaving \(|\psi_1\rangle (\Theta|\psi_2\rangle \) invariant, fulfills \( U^TU = I \), i.e. \( U \) is orthogonal (and unitary, hence real). Such bases are dubbed ‘magic’—for two qubits the magic basis consists of four orthogonal Bell states \([13]\), in the fermionic and bosonic cases they are given in \([12]\).

The angle \( \theta \) and vectors \( x \) and \( y \) for ‘classical’ states must fulfill \( \langle \psi|T|\psi^*\rangle = (\cos \theta x - i \sin \theta y) \cdot (\cos \theta x - i \sin \theta y) = 0 \). For spin-1 coherent states this gives particularly simple condition \( \cos^2 \theta - \sin^2 \theta = 0 \), i.e. \( \theta = \pi/4 \).

VIII. ACKNOWLEDGMENTS

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IX. APPENDIX

We will prove that positivity of the matrix \( Z \) defined by Giraud et al. is equivalent to \( 2 \text{Tr}(R^2) - (\text{Tr}R)^2 \leq 0 \). For \( \rho \) given by \([72]\) we get

\[
Z = \begin{bmatrix}
-\lambda_1 + \lambda_2 - \lambda_3 - 4 v_1^2 & -4 v_2 v_1 & -4 v_3 v_1 \\
-4 v_2 v_1 & \lambda_1 - \lambda_2 + \lambda_3 - 4 v_2^2 & -4 v_2 v_3 \\
-4 v_3 v_1 & -4 v_2 v_3 & \lambda_1 + \lambda_2 - \lambda_3 - 4 v_3^2
\end{bmatrix},
\]

After a short calculation we establish that the coefficients of characteristic polynomial \( P_2(x) = x^3 + a_2 x^2 + a_1 x + a_0 \) of \( Z \) read

\[
a_2 = 4(v_1^2 + v_2^2 + v_3^2) - 1,
\]

\[
a_1 = 8 \det \rho + 4(\lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_2 \lambda_3) - 8 \lambda_1 \lambda_2 \lambda_3 - 1,
\]

\[
a_0 = 2 \text{Tr}(R^2) - (\text{Tr}R)^2.
\]
We will prove the following

**Lemma 2** Let: \( r_i > 0, s_i \geq 0, i = 1, 2, 3 \) and

\[
\begin{align*}
  r_1 + r_2 + r_3 &= 1 \quad (99) \\
  r_1 r_2 r_3 &\geq r_1 s_1 + r_2 s_2 + r_3 s_3. \quad (100)
\end{align*}
\]

Then

\[
  s_1 + s_2 + s_3 \leq \frac{1}{4}. \quad (101)
\]

**Proof**

Let, eg., \( 1 > r_1 \geq r_2 \geq r_3 > 0 \). Then

\[
  r_1 r_2 r_3 \geq r_3(s_1 + s_2 + s_3) + (r_1 - r_3)s_1 + (r_2 - r_3)s_2 \geq r_3(s_1 + s_2 + s_3). \quad (102)
\]

Hence

\[
  r_1 r_2 \geq s_1 + s_2 + s_3. \quad (103)
\]

From the arithmetic-geometric mean inequality

\[
  \sqrt{r_1 r_2} \leq \frac{r_1 + r_2}{2} < \frac{1}{2}, \quad (104)
\]

hence

\[
  \frac{1}{4} > r_1 r_2 \geq s_1 + s_2 + s_3. \quad (105)
\]

\[
  \square
\]

We substitute in the above lemma \( r_i = \lambda_i \) and \( s_i = v_i^2 \). Then \( (102) \) and \( (103) \) reduce, respectively, to the true statements \( \text{Tr} \rho = 1 \) and \( \det \rho > 0 \). As a consequence we get \( a_2 < 0 \). We have thus the following possibilities

1. \( a_0 < 0, a_1 > 0 \)

   In this case the signs of the coefficients of \( P_Z(x) \) are \(+ -- -\), whereas those of \( P_Z(-x) \) read \(- -- -\), which means that all three roots of \( P_Z(x) \) are positive, hence \( Z \) is positively definite.

2. \( a_0 > 0, a_1 > 0 \)

   The signs of the coefficients of \( P_Z(x) \) read \(+ ++\), i.e. \( P_Z(x) \) has at most two positive roots. Consequently, at least one root is negative and \( Z \) is not positively definite.

3. \( a_0 > 0, a_1 < 0 \)

   The signs of the coefficients of \( P_Z(x) \) read \(+ --\) which again gives at most two positive roots, hence \( Z \) is not positively definite.

The remaining case \( a_0 < 0, a_1 < 0 \) is excluded. Indeed, we will prove \( a_0 < 0 \Rightarrow a_1 > 0 \). To this end assume as previously \( \lambda_1 > \lambda_2 > \lambda_3 > 0 \) and define:

\[
  q_1 = \lambda_2 + \lambda_3 - \lambda_1, \quad q_2 = \lambda_1 + \lambda_3 - \lambda_2, \quad q_3 = \lambda_1 + \lambda_2 - \lambda_3. \quad (106)
\]

The implication we want to prove reduces now to

\[
  q_1 q_2 q_3 > 4 q_1 q_3 v_1^2 + 4 q_1 q_3 v_2^2 + 4 q_2 q_3 v_1^2 \quad (107)
\]

\[
  \downarrow
\]

\[
  q_1 q_2 + q_1 q_3 + q_2 q_3 > 4(q_1 + q_2)v_3^2 + 4(q_1 + q_3)v_2^2 + 4(q_2 + q_3)v_1^2. \quad (108)
\]

We have \( q_1 + q_2 + q_3 = \lambda_1 + \lambda_2 + \lambda_3 = 1 \) and \( q_2 > 0, q_3 > 0 \), whereas the sign of \( q_1 \) can be arbitrary.

If \( q_1 > 0 \) we have from \( (107) \)

\[
  q_1 q_2 q_3 > 4 q_1 q_3 v_3^2 + 4 q_1 q_3 v_2^2 + 4 q_2 q_3 v_1^2 > 4 q_1 q_3 v_2^2 + 4 q_2 q_3 v_1^2, \quad (109)
\]
hence

\[ q_1 q_2 > 4q_1 v_2^2 + 4q_2 v_1^2. \]  \hspace{1cm} (110)

Analogously

\[ q_1 q_3 > 4q_1 v_3^2 + 4q_3 v_1^2, \]  \hspace{1cm} (111)

and

\[ q_2 q_3 > 4q_3 v_2^2 + 4q_2 v_3^2. \]  \hspace{1cm} (112)

Adding (110)-(112) we obtain the desired result.

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[30] Pfaffian of an antisymmetric even-dimensional matrix \( w \) is the polynomial whose square is the determinant of \( A \).
[31] The somehow awkward fact that the dimension of the single particle space is even should not be misleading, it is not determined by the total spin, but rather by the number of available single-particle states.
[32] Perelomov [2] defines generalized coherent states by choosing arbitrary fixed vector \( |u| \) in \( \mathcal{H} \) in place of \( v \). The states obtained by action on \( |v_{\text{max}}| \) are then called ‘closest to classical’.