Renormalization group study of random quantum magnets

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Abstract
We have developed a very efficient numerical algorithm of the strong disorder renormalization group method to study the critical behaviour of the random transverse field Ising model, which is a prototype of random quantum magnets. With this algorithm we can renormalize an $N$-site cluster within a time $N \log N$, independently of the topology of the graph, and we went up to $N \sim 4 \times 10^6$. We have studied regular lattices with dimension $D \leq 4$ as well as Erdős–Rényi random graphs, which are infinite dimensional objects. In all cases the quantum critical behaviour is found to be controlled by an infinite disorder fixed point, in which disorder plays a dominant role over quantum fluctuations. As a consequence the renormalization procedure as well as the obtained critical properties are asymptotically exact for large systems. We have also studied Griffiths singularities in the paramagnetic and ferromagnetic phases and generalized the numerical algorithm for other random quantum systems.

(Some figures in this article are in colour only in the electronic version)

1. Introduction
Quantum phase transitions take place at the temperature $T = 0$ by varying a control parameter, which is involved in the Hamiltonian of the system [1]. Experimental examples in which quantum phase transitions play an important role are, among others, rare-earth magnetic insulators [2], heavy-fermion compounds [3, 4], high-temperature superconductors [5, 6] and two-dimensional electron gases [7, 8]. Generally, when the quantum fluctuations are weak the ground-states of these systems are ordered, whereas for strong quantum fluctuations we are in the quantum disordered region. In between there is a quantum phase transition point, the effect of which is manifested also in the finite temperature in the so called quantum critical region.

A paradigmatic model having a quantum phase transition is the Ising model in the presence of a transverse field of strength $h$, where $h$ plays the role of the control parameter. For $h < h_c (h > h_c)$ the system is ferromagnetic (paramagnetic) and at $h = h_c$ there is a quantum critical point. Experimentally this system is realized by the compound LiHoF$_4$, which is a dipole coupled Ising ferromagnet, and which is placed in a magnetic field transverse to the Ising axis of strength $H$. This magnetic field splits the ground-state doublet and therefore it acts as an effective transverse field of strength $h \sim H^2$. In the above compound one can introduce randomness by substituting the magnetic Ho by a nonmagnetic Y. Then the obtained system LiHo$_x$Y$_{1-x}$F$_4$ is the experimental realization of a random quantum magnet [9]. However, the transverse field also induces a random longitudinal field [10] via the off-diagonal terms of the dipolar interaction; therefore there are several open questions both theoretically and experimentally about the low-temperature behaviour of this compound.

Here we consider a theoretically simpler problem, the random transverse field Ising model (RTIM), which is defined by the Hamiltonian

$$\mathcal{H} = - \sum_{\langle ij \rangle} J_{ij} \sigma_i^x \sigma_j^x - \sum_i h_i \sigma_i^z. \quad (1)$$

Here the $\sigma_i^{x,z}$ are Pauli-matrices at sites $i$ (or $j$) of a lattice and the nearest neighbour couplings, $J_{ij}$, and the transverse fields,
$h_i$, are independent random numbers, which are taken from the distributions $p(J)$ and $q(h)$, respectively. In this paper we have used two different disorder distributions in order to check the universality of the properties at the critical point. For both disorders the couplings are uniformly distributed:

$$p(J) = \Theta(J)\Theta(1-J),$$

where $\Theta(x)$ denotes the Heaviside step-function.

For box-$h$ disorder the distribution of the transverse fields is uniform:

$$q(h) = \frac{1}{h_b} \Theta(h)\Theta(h_b - h),$$

whereas for fixed-$h$ disorder the initial values of the transverse fields are constant:

$$q(h) = \delta(h - h_i).$$

The quantum control parameters are defined as $\theta = \log(h_b)$ and $\theta = \log(h_i)$, respectively.

Theoretical study of the critical behaviour of the RTIM is a complicated issue, since one should treat the joint effect of quantum and disorder fluctuations, as well as non-trivial correlations. Most of the results in this field are known in one dimension (1D), due to a renormalization group (RG) treatment [11], which has been introduced by Ma, Dasgupta and Hu [12] and further developed by Fisher [13]. This RG procedure works in the energy space and in each step the largest local parameter in the Hamiltonian, either a coupling or a transverse field, is decimated out and at the same time new effective parameters are generated perturbatively. Repeating this procedure the energy scale in the system, measured by the largest effective parameter, goes to zero and one studies the distributions of the renormalized couplings and that of the transverse fields at the fixed point.

In 1D where the topology of the system is preserved during renormalization Fisher [13] has solved the RG equations analytically and in this way precise information has been obtained about the behaviour of the RG flow in the vicinity of the fixed point. One important result is that the distribution of the renormalized parameters (couplings and/or transverse fields) at the fixed point is logarithmically broad. This means that disorder fluctuations are completely dominant over quantum fluctuations and the perturbative treatment used in the calculation of the new parameters is asymptotically exact. In this so called infinite disorder fixed point [14] (IDFP) of the 1D model exact results has been derived for the critical exponents, as well as for several scaling functions.

In more complex geometries, even for a ladder with $w > 1$ legs, the topology changes during the RG process and therefore one relies on numerical implementations of the RG procedure. For a ladder with a finite width this procedure can be performed straightforwardly [15], during which the system is renormalized to an effective chain. As a result the critical singularities of ladders are identical to those of a chain; however, from the $w$-dependence of the amplitudes one can deduce cross-over functions and estimate the singularities of the 2D system through finite-size scaling.

If the sample is isotropic, say an $L \times L$ part of a 2D lattice, one cannot successfully renormalize it numerically with a naïve application of the RG rules. This is due to the fact that after $h$ decimations a large number of new effective couplings are generated and the system will soon look like a fully connected cluster, for which the further decimations are very slow. To speed up the process new ideas have been introduced, one of those is the so called maximum rule [16]. This is applied in a situation when between two sites two or more couplings are generated and the maximum of those is taken. This procedure is certainly correct at an IDFP, where the couplings have typically very different magnitudes. Together with the maximum rule one can also use some filtering condition [17] to eliminate the latent, non-decimated couplings of the Hamiltonian, which are at a position such that a larger parallel coupling will be generated. Using these tricks one could numerically study the 2D system [16, 18–21] up to a linear size $L = 128–160$. The critical exponents calculated in this way are in agreement with the finite-size scaling extrapolations obtained in the strip geometry [15] and with the results of quantum Monte Carlo simulations [22].

The correctness of the method is also checked by comparing results of Monte Carlo simulations for the random contact process [23], which is a basic model of reaction–diffusion processes in a random environment. According to an RG analysis the critical behaviour of this model (for strong enough disorder) is identical to that of the RTIM [24], which is indeed found in 1D and 2D calculations.

The model of real random quantum magnets, the RTIM in 3D, cannot be studied even by the above refined RG algorithm, since the available finite sizes of the system are too small to obtain stable estimates for the properties of the fixed point. In early studies the possible presence of an IDFP was expected [16], but no evidence in favour of this conjecture has been presented. Also no studies are available about the random contact process in 3D. In higher dimensions no results of any kind are known, thus it is a completely open question as to whether there is an upper critical dimension, $D_u$, such that for $D < D_u$ infinite disorder scaling works and for $D \geq D_u$ we have conventional random criticality. Furthermore, it is also an open question as to whether for $D \geq 2$ the fixed point is universal and does not depend on the actual form of the disorder.

A possible way to answer the questions presented in the previous paragraph is to improve the numerical algorithm of the RG procedure and make it capable of studying three- and higher dimensional systems. In this paper we describe such an improved algorithm, which uses the maximum rule but otherwise has no further approximations. This means that the results of our algorithm are identical to those of any naıve implementation of the RG method (having also the maximum rule) for any finite graphs, say with $N$ sites and $E$ edges. However, we gain considerable time in the performance of our algorithm: while the naïve method works in $t \sim \mathcal{O}(N^3)$ time, for the improved algorithm $t \sim \mathcal{O}(N \log N + E)$. Having this performance at hand we could treat finite clusters of up to $4 \times 10^6$ sites.

We have used this algorithm to study the properties of the RTIM in different dimensions. Some preliminary results
of these investigations have already been presented elsewhere. The critical behaviour of the 3D and 4D systems is announced in [25]. Here we give details of the determination of the critical parameters as well as an analysis of the scaling behaviour of different quantities in the off-critical region. The model in 2D has been studied by another algorithm in [17]: this algorithm has basically the same performance as the present one in 2D, however it is less effective in higher dimensions. Here the 2D results are merely used to compare with higher dimensional results. We have also studied Erdős–Rényi random graphs [26] with a finite coordination number, which are infinite dimensional objects. In this way we have obtained information about the possible value of the upper critical dimension, \( D_u \), in our system.

Our paper is organized as follows. The RG procedure, the basic decimation rules and the essence of the improved algorithm are presented in section 2. Results about the critical behaviour of the system in different dimensions are presented in section 3. The results are discussed in section 4 and possible extensions of the improved algorithm for other models are given in the appendix.

2. The renormalization group method and the improved algorithm

The so called strong disorder RG (SDRG) method [11] was introduced by Ma et al [12] to study 1D random antiferromagnetic Heisenberg chains. For the RTIM it was Fisher [13] who first used this method and solved the RG equations analytically in 1D. Here we discuss the basic steps of the RG procedure for the RTIM model.

The SDRG method works in the energy space: at each step the largest parameter of the Hamiltonian in equation (1), which is denoted by \( \Omega = \max(J_{ij}, h_i) \), is eliminated. Here we should consider two possibilities.

J-decimation: if the largest term is a coupling, say \( \Omega = J_{ij} \), than the two connected sites \( i \) and \( j \) are merged into a spin cluster having an effective moment \( \tilde{\mu} = \mu_i + \mu_j \) (in the original model \( \mu_i = 1, \forall i \)), which is placed in an effective transverse field of strength \( \tilde{h} = h_i h_j / J_{ij} \). This latter formula is obtained by the second-order perturbation calculation. If a site, say \( k \), was connected to both sites, the maximum of the two couplings, \( \max(J_{ik}, J_{jk}) \) is taken according to the maximum rule.

h-decimation: if the largest term is a transverse field, say \( \Omega = h_i \), than this site brings a negligible contribution to the (longitudinal) susceptibility of the system and is therefore decimated out. At the same time new couplings are generated between all sites, say \( j \) and \( k \), which were nearest neighbours to \( i \). In the second-order perturbation calculation these couplings are given by \( J_{ji} J_{ik} / h_i \). If there is already a coupling, \( J_{jk} > 0 \), between the two sites we use the maximum rule: \( J_{jk} = \max(J_{ij}, J_{ik} / h_i, J_{jk}) \).

In the naive application of the SDRG rules for higher dimensional clusters there is a problem with the h-decimation steps, during which several new couplings are generated. As a result our cluster will be transformed soon into an almost fully connected graph, having \( O(N^2) \) edges. Consequently at any further decimation step one needs to perform \( O(N^2) \) operations, which leads to a performance in time \( t \sim O(N^3) \).

2.1. Improved algorithm

Here and in the following we assume, without loss of generality, that \( J_{ij} \leq 1, \forall i, j \). Then we define the set of local maxima, which consists of those parameters which are larger (not smaller) than any of their neighbouring terms.

Considering that a coupling \( J_{ij} \) is a local maximum, provided that \( J_{ij} \geq h_i, J_{ij} \geq h_j \), and \( J_{ij} \geq J_{ik}, \forall k \), as well as \( J_{ij} \geq J_{kj}, \forall l \). Similarly a transverse field, \( h_i \), is a local maximum \( h_i \geq h_j, \forall j \). We have shown [17] that local maxima can be decimated independently, the renormalization performed in any sequence gives the same final result.

In the improved algorithm we concentrate on the transverse fields, the decimation of which is the most dangerous in respect of the performance of the algorithm. Our strategy is to avoid any \( h \)-decimation during the renormalization. For this purpose we divide the sites into two classes. For ‘inactive’ sites the transverse field is a local maximum and the site has a weight \( l_i = 1 \), whereas all the remaining sites are termed as ‘active’, having weights \( l_i = 0 \). In the log-energy space between sites \( i \) and \( j \) we define a distance, \( d_{ij} \geq 0 \), as

\[
d_{ij} = -\ln J_{ij} + \frac{l_i}{2} \ln h_i + \frac{l_j}{2} \ln h_j.
\] (5)

Having an inactive site, \( k \), between \( i \) and \( j \) and decimating it out the RG rules lead to the additivity property \( \delta_{ij} = \delta_i + \delta_j \), which—according to the maximum rule—should be compared with \( d_{ij} \) in equation (5) and their minimal value is taken. Generally, the true distance between \( i \) and \( j \), denoted by \( \delta_{ij} \), is given by the shortest path which goes over the inactive sites. It is also easy to see that decimating out all or a subset of inactive sites is equivalent to finding in the original problem the shortest paths among the non-decimated sites which go through the decimated sites. This is a well known graph-theoretical problem [27] for which efficient numerical algorithms are available\(^4\). In general, however, one should also deal with the active sites, and therefore \( J \)-decimations should also be performed.

In the following, we concentrate on the active sites, and define for each in the log-energy space a range

\[
r_i = -\ln h_i.
\] (6)

In the improved algorithm we compare the ranges of the active sites with the true distances measured between them. In this respect two possibilities may occur.

- If the true distance between two active sites (\( i \) and \( j \)) is not greater than any of their ranges, i.e. \( \delta_{ij} \leq r_i \) and \( \delta_{ij} \leq r_j \),

\(^4\) The opposite case, in which only coupling decimations occur, is equivalent to another well known graph theoretic problem, the minimum spanning tree problem with link weights \( w_{ij} = -\ln J_{ij} \). In this case the sites are fused together, and the value of the effective transverse field is given by \( \ln h = W + \sum \ln h_i \), where \( W \) is the total weight of the minimum spanning tree.
Figure 1. Snapshots of the na"ive (left) and improved (right) SDRG algorithms for the same critical 20 × 20 sample with box-h disorder and open boundary conditions, where 40% of the spins are decimated out. While in the na"ive algorithm numerous new couplings are generated, in the improved algorithm we merely delete sites.

then $i$ and $j$ are fused together into an effective active site, which has a range

$$\tilde{r} = r_i + r_j - \delta_{ij},$$

Also the distance measured from this effective site to another site, say $k$, is given by $\min(d_{ik}, d_{jk})$.

- If the range, $r_i$, of an active site $i$ is shorter than any of its true distances from active sites, $r_i < \delta_{ij}, \forall j$, then this site cannot be fused together with any other active sites, therefore it is turned into ‘inactive’. Then we set its weight $l_i = 1$ and update the distances, $d_{ij}, \forall j$, in equation (5).

In the above renormalization steps, which can be used in arbitrary order, the number of active sites in the system is reduced by one. Repeating these decimation rules we arrive at a system having only inactive sites and no further fusion steps take place. The complete cluster structure including the excitation energies is readily encoded in this configuration, which can be extracted without further renormalization steps. In any case the final result of the improved algorithm is identical to that obtained by the na"ive SDRG algorithm.

In practice one should measure the true distances between the active sites. This can be done from a selected reference site from which we measure the distances by Dijkstra’s method [27] including one nearby site after another until the range $r_i$ is reached. While repeating the exploration of the shortest paths from all active sites, a given site could be crossed from several directions. However, it can be shown that it is always sufficient to cross it from one direction only [25]. As a consequence, we can delete those sites which have already been explored from the system, because these are no longer needed for the calculation. Therefore in a more efficient algorithm initially we consider all active sites and perform the measurements simultaneously and successively delete the explored sites. The implementation of this algorithm has a time complexity of $O(N \log N + E)$ on any graphs with $N$ sites and $E$ edges. A further speed gain is achieved by recognizing that in this parallel implementation the paths only need to be explored until a length of $r_i/2$ is reached instead of the full $r_i$ range. In figure 1 we compare the topologies of the renormalized systems in the na"ive and improved SDRG algorithms.

We have checked the computational time of the algorithm, $t$, as a function of the size of the hypercubic clusters, $N$, in a log–log scale for 2D, 3D and 4D and for the two different disorders (fixed-h, +; box-h, ). The theoretical prediction, $t \sim N \log N$, is indicated by a dashed line.

Figure 2. Computational time of the algorithm, $t$, as a function of the size of the hypercubic clusters, $N$, in a log–log scale for 2D, 3D and 4D and for the two different disorders (fixed-h, +; box-h, ). The theoretical prediction, $t \sim N \log N$, is indicated by a dashed line.

3. Critical behaviour in different dimensions

3.1. Analytical results in 1D—a reminder

In 1D the position of the critical point is given by the condition [28] $\delta = [\ln h]_{av} - [\ln J]_{av} = 0$, where $[\cdot \cdot \cdot]_{av}$...
stands for averaging over quenched disorder. At the critical point the energy scale, $\Omega$, and the length scale, $L$, which is the linear size of the system, are related as $\log(\Omega_0/\Omega) \sim L^\psi$, with $\psi = 1/2$. This type of unusual dynamical scaling relation is a clear signal of infinite disorder scaling. In the paramagnetic phase, $\delta > 0$, the spin clusters have a finite extent, $\xi$, which in the vicinity of the critical point diverges as $\xi \sim \delta^{-\nu}$, with $\nu = 2$. At the critical point the largest cluster is a fractal, having a moment $\mu \sim L^d$, with a fractal dimension $d_f = (1 + \sqrt{5})/4$. From this the magnetization exponent is expressed by $\beta = x \nu$, with $x = d - d_f$ and $d = 1$. These SDRG results [13], which have been extended to the dynamical properties in the Griffiths phases [29], have been tested by independent analytical [30, 31] and numerical calculations [32–34].

3.2. Numerical results for $D = 3$ and 4

We have studied finite systems of hypercubic lattices in dimensions $D = 3$ and 4, the largest linear sizes of the samples were $L = 128$ and 48, respectively. (In 2D the previously performed numerical investigations [17] went up to $L = 2048$.) For each size we have renormalized typically 40 000 random samples (for each type of disorder), but even for the largest systems we have treated at least 10 000 realizations.

3.2.1. Finite-size critical points. The precise identification of the critical point of a disordered system is a very important issue, since the accuracy of the determination of the critical exponents depends greatly on it. This question is deeply related to the problem of finite-size scaling in random systems [35–40], since estimates for the critical points are generally calculated in finite samples. As has been observed recently [36, 37], the key issue in this point is the scaling behaviour of the finite-size critical points, which are calculated for a large set of samples.

For a given sample, denoted by $\alpha$, this pseudo-critical point, $\theta_{c}(L, \alpha)$, is located at the point where some physical parameter of the system has a maximal value. For example, in a classical random magnet the susceptibility can be used for this purpose, which is divergent at the true critical point of the infinite system.

For a random quantum system the susceptibility is divergent in a whole region, in the so called Griffiths phase, and thus cannot be used to monitor finite-size critical points. Then in 1D random quantum systems the average entanglement entropy turns out to be a convenient quantity, the maximum of which is a good indicator of finite-size criticality [41]. In more complicated topologies, for ladders [15] and in 2D systems [17] the so called doubling method [40] is found to provide an appropriate definition of $\theta_{c}(L, \alpha)$. In this procedure one considers two identical copies of a given sample, $\alpha$, which are joined together by surface couplings, and this replicated sample is denoted by $2\alpha$.

Using the SDRG method one calculates some physical quantity (magnetization or gap) in the original and in the replicated sample, which are denoted by $f(\alpha)$ and $f(2\alpha)$, respectively, and studies their ratio, $r(\alpha) = f(2\alpha)/f(\alpha)$, as a function of the control parameter $\theta$. At $\theta = \theta_{c}(\alpha, L)$ this ratio has a sudden jump, which is identified with the pseudo-critical point of the sample. It has been realized [17] that this jump in the ratio is related to a sudden change in the cluster structure which is generated during the SDRG procedure. For weak quantum fluctuations, $\theta < \theta_{c}(L, \alpha)$, between the replicas correlations are generated during renormalization, which are manifested by the presence of a so called correlation cluster. This contains equivalent sites in the two replicas. In contrast, for stronger quantum fluctuations, $\theta > \theta_{c}(L, \alpha)$, the two replicas are renormalized independently. For $\theta < \theta_{c}(L, \alpha)$ the mass of the correlation cluster, $\mu(L, \alpha, \theta)$, is a monotonically decreasing function of $\theta$. Then we identify $\theta_{c}(L, \alpha)$ as the point where the correlation cluster disappears.

Using this doubling method we have calculated pseudo-critical points in different dimensions and for different forms of the disorder. We illustrate the distributions of the pseudo-critical points in figure 3 for $D = 4$. To analyse these distributions we make use of results of finite-size scaling theory [36, 37], which makes statements about the average value, $\overline{\theta_{c}}(L)$, and the width of the distribution, $\Delta\theta_{c}(L)$, in finite systems of linear size $L$. The average value of the distribution is expected to scale as

$$\left|\theta_{c} - \overline{\theta_{c}}(L)\right| \sim L^{-1/\nu_{c}},$$

(8)

where $\theta_{c}$ is the true critical point of the system and $\nu_{c}$ is the so called shift exponent. On the other hand the width of the distribution is expected to scale as

$$\Delta\theta_{c}(L) \sim L^{-1/\nu_{w}},$$

(9)

with a width exponent $\nu_{w}$. We have checked that these finite-size scaling relations are satisfied in all dimensions, and these relations are used—by comparing results at two different sizes (at $L$ and $L/2$)—to obtain finite-size estimates for the exponents. These are presented in figure 4 for the two different forms of disorder.

For a given dimension the estimated exponents are found to be independent of the form of the disorder; furthermore the
are taken from [17].

In Table 1, are collected in table1, together with the estimates of the critical exponents for 2D, 3D and 4D and for the two different dimensions which is related to the anomalous dimension of the magnetization as $\beta/\nu = d - d_i$. Comparing the average mass of the largest clusters at two finite sizes, we have calculated effective, size-dependent fractal dimensions, as well as effective magnetization scaling dimensions. These are presented in the inset of figure 6 for the different dimensions and using the two disorder distributions in equations (3) and (4). Extrapolating these values the obtained exponents, for a given dimension, do not depend on the form of the disorder. These are presented in table 1.

### Table 1. Critical properties of the RTIM in different dimensions.

| Dimension | 1D | 2D | 3D | 4D |
|-----------|----|----|----|----|
| $\theta_c^{(b)}$ | 0 | 1.6784(1) | 2.5305(10) | 3.110(5) |
| $\theta_c^{(r)}$ | -1 | -0.17034(2) | -0.07627(2) | -0.04698(10) |
| $v_s$ | 2 | 1.24(2) | 0.97(5) | 0.825(40) |
| $v_t$ | 1.25(3) | 0.987(17) | 0.74(4) |
| $x$ | $\frac{3-x}{5}$ | 0.982(15) | 1.840(15) | 2.72(12) |
| $\psi$ | 1/2 | 0.48(2) | 0.46(2) | 0.46(2) |

3.2.2. Magnetization. The magnetization of the system, $m(L, \theta)$, is related to the average mass of the largest effective cluster, $\bar{m}(L, \theta)$, as $m(L, \theta) = \bar{m}(L, \theta)/L^d$. In the thermodynamic limit, $L \to \infty$, in the ferromagnetic phase, $\delta = \theta - \theta_c < 0$, the magnetization is finite and vanishes at the critical point as $\lim_{L \to \infty} m(L, \theta) \sim (-\delta)^{\beta}$, where $\beta$ is the magnetization exponent. At the critical point the largest connected clusters are fractals, which are illustrated in the left panel of figure 5 for 2D and 3D. The mass of these critical clusters scales as $m \sim L^{d_i}$, where $d_i$ is the appropriate fractal dimension which is related to the anomalous dimension of the magnetization as $x = \beta/\nu = d - d_i$. Comparing the average mass of the largest clusters at two finite sizes, we have calculated effective, size-dependent fractal dimensions, as well as effective magnetization scaling dimensions. These are presented in the inset of figure 6 for the different dimensions and using the two disorder distributions in equations (3) and (4). Extrapolating these values the obtained exponents, for a given dimension, do not depend on the form of the disorder. These are presented in table 1.
We have also studied the distribution function of the mass of the clusters, $P_L(\mu)$, which is expected to behave at the critical point as $P_L(\mu) = L^{d_\delta} \tilde{P}(\mu L^{-d_\delta})$. According to scaling theory [42] $\tilde{P}(u)$ for large arguments has a power-law tail, $\tilde{P}(u) \sim u^{-\tau}$, with $\tau = 1 + \frac{d}{d_f}$. In figure 6 we have plotted $P_L(\mu)$ for $D = 2, 3$ and $4$ for $L = 1024, 128$ and $48$, respectively, and good agreement with scaling theory is found.

Close to the critical point the finite-size magnetizations are shown in figure 7 for 3D and 4D. For large $L$ in the ordered phases ($\delta < 0$) the magnetization approaches a finite limiting value, whereas for $\delta > 0$ it tends to zero. In the vicinity of the critical point the finite-size magnetizations can be transformed to a master curve, if one considers the scaled magnetization, $\tilde{m} = m L^x$, as a function of the scaling variable, $\tilde{\delta} = \delta L^{1/\nu}$. This is illustrated in the insets of figure 7 where the exponents $x$ and $\nu$ are taken from table 1.

3.2.3. Dynamical scaling. The dynamical behaviour of the RTIM is related to the low-energy excitations of the system, the energies of which in the SDRG method are given by the values of the effective transverse fields at which a given cluster is eliminated. Generally, for each such eliminated cluster one can define a connected subgraph, which contains the given cluster, and renormalization of this subgraph gives the same energy value. The form of the connected subgraphs of the correlation clusters is illustrated in the right panel of figure 5. The energy parameter of a given sample, which is denoted by $\epsilon(L, \alpha)$, is given by the smallest effective transverse field, not considering the transverse field of the correlation cluster, if it exists in the system. The distribution of the log-energy parameter, $\gamma(L, \alpha) = -\log \epsilon(L, \alpha)$, is shown in the upper panel of figure 8 at the critical points of the 3D and 4D systems.

For correlation clusters the connected subgraphs have the spanning property, whereas for any other clusters the connected subgraphs are non-spanning.
The distributions for both dimensions are broadening with increasing $L$, which is a clear signature of infinite disorder scaling. The typical value of the log-energy parameter grows with the size as $\gamma(L) \sim L^{\psi}$, thus the appropriate scaling combination is given by $\tilde{Y} = (\gamma(L) - \gamma_0)L^{-\psi}$. Here $\psi$ is a scaling exponent and $\gamma_0$ is a non-universal constant. The scaled distributions are shown in the lower panel of figure 8.

We have calculated effective, size-dependent $\psi$ exponents, by comparing the widths of the distributions of the log-energy parameters at two sizes ($L$ and $L/2$). These are given in figure 9 for the different dimensions and for the two different forms of disorder. As for other exponents studied before, the estimates for $\psi$ for a given dimension do not depend on the actual form of the randomness. These are summarized in table 1. Interestingly the $\psi$ exponents for all studied dimensions are close to 1/2, which is the exact value in 1D. This observation can be explained by the fact that the connected subgraphs, which are related to the energy parameter of the sample, are basically one-dimensional objects in all studied dimensions, see the right panel of figure 5. The very small variation of $\psi$ with the dimensionality is probably due to the fact that the renormalized couplings and transverse fields of the connected subgraphs are more and more correlated in higher dimensions.

Finally we note that the thermodynamic singularities at a small temperature, $T$, but $\delta = 0$ are related to the critical exponents in table 1. For example the susceptibility, $\chi$, and the specific heat, $C_V$, behave as $[11, 14] \chi(T) \sim (\log T)^{\delta - 2s + 1/\nu} / T$ and $C_V(T) \sim (\log T)^{-\delta - 1/\nu}$. The similar relations at $T = 0$ but with a small longitudinal field, $H$, are given by $\chi(H) \sim (\log H)^{-s + 1/\nu} / H$ and $C_V(H) \sim (\log H)^{\delta - 1/\nu}$.

### 3.2.4. Griffiths effects

We have also studied the distribution of the low-energy excitations outside the critical point. In the paramagnetic phase the distributions of $\gamma(L)$ for different sizes are shown in figures 10 and 11, for the 3D model ($\delta = 0.37$) and for the 4D model ($\delta = 0.34$), respectively. As seen in these figures the distributions have approximately the same width, they are merely shifted with increasing $L$. This behaviour is in agreement with scaling theory in the disordered Griffiths phase [11], where the typical excitation energy scales with the size as $\epsilon(L) \sim L^{-z}$. Here $z$ is the dynamical exponent, which depends on the distance from the critical point, $\delta$. Then the appropriate scaling combination is $\tilde{Y} = (\gamma(L) - z \log L - \gamma_0)$, in terms of which a scaling collapse of the distributions is found, which is shown in the insets of figures 10 and 11. One way to estimate $z(\delta)$ is to analyse the scaling collapse of the distributions, or equivalently to compare the shift of the distributions with $L$. From this type of analysis we obtain $d/z = 0.75(4)$ for the 3D model and $d/z = 0.80(4)$ for 4D. There is, however, another possibility to calculate $d/z$ from the asymptotic form of the distributions. If the low-energy excitations are localized, which is satisfied for the RTIM, the distribution function of the scaled variable, $\tilde{Y}$, is expected to follow extreme value statistics [43] and be given by the Fréchet distribution [44]

$$ \ln p(\tilde{Y} - \gamma_0) = -\frac{d}{z} \tilde{Y} - \exp \left( -\frac{d}{z} \tilde{Y} \right) + \ln \left( \frac{d}{z} \right). \tag{10} $$

Indeed the scaled distributions in the insets of figures 10 and 11 are well described by this form, having just one free parameter, $\gamma_0$. From equation (10) it follows that the asymptotic slope of $\ln p(\gamma)$ versus $\gamma$ is just $d/z$, which we have measured in figures 10 and 11. The estimates for $d/z$ for different values of $L$, as given in the captions, are in good agreement with our previous estimates from the shifts of the curves.

We have repeated these types of calculations at other points of the disordered Griffiths phase and we have measured $\delta$ dependent dynamical exponents. We could not, however, check the scaling result [11, 14] for small $\delta$, $d/z \propto \delta^{-\nu}$, due to the strong finite-size effects in the vicinity of the critical point. (In 2D this type of analysis has been performed in [17].)
We have also calculated the distribution of the log-energy parameters in the ordered Griffiths phase, which is illustrated in figure 12 for the 3D model at $\delta = -0.33$. In the ordered phase there is a huge magnetization cluster, which has a very small effective field, and the energy parameter is given by the second smallest effective field of the RG process. As in the disordered Griffiths phase the width of the distributions is approximately $L$ independent and the distributions are shifted with $L$. However, the amount of shift with $L$, as well as the shape of the distributions, is different in the two cases. This is related to the scaling result, that the typical value of the excitation energy in the ordered Griffiths phase scales with the size as $\epsilon(L) \sim -\ln^{1/2}(L)$, thus the appropriate scaling combination is $\tilde{\gamma} = \gamma(L) - A \ln^{1/2}(L) - \gamma_0$, with $A$ and $\gamma_0$ being non-universal constants. Using this variable the distributions have a scaling collapse as shown in the inset of figure 12. We should, however, mention that due to finite-size effects we cannot obtain an independent estimate of the exponent of the logarithm, it being theoretically $1/d$. We are facing the same kinds of limitations concerning the shape of the scaling curve, which asymptotically should behave as $\ln p(\tilde{\gamma}) \sim -\tilde{\gamma}^d$ according to scaling theory. Our data, however, are still not in the asymptotic regime.

Finally we note that in the disordered Griffiths phase the singularities of the susceptibility and the specific heat at a small temperature are given by $[11, 16] \chi(T) \sim T^{-d/z}$ and $C_V(T) \sim T^{d/z}$. The same expressions in the ordered Griffiths phase are $\chi(T) \sim \exp(-C|\log T|^{1/\theta})/T$ and $C_V(T) \sim \exp(-C'|\log T|^{d/\theta})$.

### 3.3. Numerical results for Erdős–Rényi random graphs

Here we come back to the question posed in section 1 about the possible value of the upper critical dimension, $D_u$, in the problem. In order to answer this question we consider Erdős–Rényi (ER) random graphs [26] with a finite coordination number, which represent the large-dimensional limit of our lattices. Generally an ER random graph consists of $N$ sites and $kN/2$ edges, which are put in random positions. In order to have a percolating random graph we should have $k > 1$. Here we have used $k = 3$, but some controlling calculations had also been made with $k = 4$. In the actual calculation we have put the RTIM on ER random graphs and study their critical behaviour by our improved algorithm of the SDRG method. Due to the infinite dimensionality of ER clusters we had to modify some parts of the analysis used in section 3.2 for finite $D$.

As for $D \leq 4$ we have calculated sample dependent pseudo-critical points, but now in the doubling method the two identical copies of the sample have been connected by $N/2$ random links. The distributions of the calculated pseudo-critical points for different values of $N$ are shown in figure 13. The general behaviour of the distributions is similar to that for finite $D$, see figure 3 for 4D, but in the present case for large $\theta_c$ values there is an $N$-independent background to the distributions. This background is probably due to the large number of connecting random links between the replicas.
This background, however, has a very small contribution to the distribution and does not influence the analysis of the properties of the pseudo-critical points. Concerning figure 13 we have measured the shift, $\delta_0(N) \sim N^{-1/\omega_0}$, and the width, $\Delta_\delta(N) \sim N^{-1/\omega_0}$, of the distributions, in analogy with the finite dimensional problem. (Compare with equations (8) and (9), as well as with $v_s \rightarrow \omega_s/d$ and $v_w \rightarrow \omega_w/d$, respectively.) From two-point fits we have calculated effective exponents (see figure 14) from which we have obtained the estimates, $\omega_0 = 4.5(1.5)$ and $\omega_w = 7.8(2.0)$, which are valid for both types of randomnesses. We note that the relative error of the estimates is somewhat larger than for finite $D$, but still the two exponents of the distribution agree with each other giving $\omega = 6(2)$. Using the scaled variable, $y = (\delta(N) - \delta_c)N^{1/\omega_0}$, the distributions show a scaling collapse as illustrated in the inset of figure 13.

We have studied the fractal properties of the correlation cluster, the average mass of which is found to scale at the critical point as $\bar{\mu}(N) \sim N^\beta$. From two-point fit we have obtained effective values for $\beta$, which are shown in figure 14 and which are extrapolated to $\beta = 0.17(5)$. We note that according to scaling theory the magnetization exponent of the RTIM on the ER random graph is given by $\beta = \omega(1 - \vartheta) = 5(2)$.

We have also investigated the distribution of the log-energy parameters at the critical point, which is shown in figure 15 for different sizes of the ER clusters. In order to see the possible existence of infinite disorder scaling we have measured the widths of the distributions, which are shown in the inset of figure 15 as a function of log $N$. As seen in the inset, the width of the distribution can be parametrized as $W_0 + W_1 \log^2 N$, where the constant is $W_0 \approx 0$ for fixed-$h$ randomness and it is $W_0 \approx 1.2$ for box-$h$ randomness. In both cases the exponent in the logarithm can be estimated as $\epsilon = 1.3(2)$, thus the increase of the width of the distribution is somewhat larger than linear in log $N$. This fact justifies the fact that even for ER random graphs the critical behaviour of the RTIM is controlled by a (logarithmically)

infinite disorder fixed point\(^6\). Thus we conclude that the upper critical dimension of infinite disorder scaling of the RTIM is $D_u = \infty$. For ER random graphs the singularities of the susceptibility and the specific heat at small temperature are given by $\log[\chi(T)] \sim (2\vartheta - 1) \log T^{1/\epsilon}$ and $\log[C_V(T)] \sim |\log T|^{1/\epsilon}$.

4. Discussion

In this paper we have considered the random transverse field Ising model, which is a basic model of random quantum magnets, and studied its critical behaviour in different dimensions by a variant of the SDRG method. These investigations were made possible because we have developed an improved numerical algorithm for the SDRG method so that we could renormalize clusters with up to $N \sim 4 \times 10^8$ sites irrespective of their dimensionality and topology. We have found strong numerical evidence that the critical behaviour of the RTIM for all dimensions up to $D = 4$ is governed by infinite disorder fixed points. This fact justifies the validity of the use of the SDRG method as well as indicating that the obtained critical properties of the model, which are summarized in table 1, are asymptotically exact. This means that with increasing sizes in the calculation the critical exponents approach their exact values. We have demonstrated by using different disorder distributions in the initial models that the strong disorder fixed points are universal, the critical parameters do not depend on the actual form of the disorder. We have also studied the behaviour of the systems in the vicinity of the critical points and good agreement with scaling considerations was obtained.

We have considered the upper critical dimension of infinite disorder scaling of the RTIM and studied the critical behaviour

\(^6\) The random contact process, which is expected to belong to the same universality class as the RTIM for strong enough disorder \cite{24}, has been simulated recently in ER random graphs. For bimodal disorder the critical behaviour of the system is found to be the same as for the non-random model \cite{45}. This result, which differs from our SDRG results for the RTIM, is probably due to the fact that the applied disorder in the simulations was not in the strong disorder regime.
of the model on ER random graphs by the improved SDRG algorithm. Our results indicate that even in this formally infinite dimensional lattice the critical behaviour of the RTIM is governed by a (logarithmically) infinite disorder fixed point, thus the upper critical dimension is $D_u = \infty$.

Our results presented in this paper are relevant for several other problems too, since the IDF of the RTIM is expected to govern the critical properties of a large class of random systems, at least for strong enough disorder. These are, among others, random quantum ferromagnetic systems having a continuous phase transition at which a discrete symmetry of a non-conserved order parameter is broken. Examples are the quantum Potts and clock models [46] as well as the Ashkin–Teller model [47]. The quantum spin glass (QSG) problem could also be related to the IDF of the RTIM. For the QSG the distribution of couplings in equation (2) contains antiferromagnetic terms too; however, at an IDFP frustration is expected to be irrelevant. Thus the critical exponents in table 1, with some appropriate modifications of the scaling relations in the ordered phase [16], should hold for the QSG, at least for strong enough disorder. Nonequilibrium phase transitions in the presence of quenched disorder are also expected to belong to the universality class of the RTIM [24] and the random walk in a self-affine random potential [48] could be related to the RTIM.

Finally we mention that the ideas about the numerical implementation of the SDRG method in section 2.1 can be generalized for other models. In the appendix we outline the elements of the improved SDRG algorithm for the random quantum Potts model [46], as well as for the model of disordered Josephson junctions [49]. These results can be used to investigate the critical behaviour of these systems in higher dimensions.

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Appendix. Improved SDRG algorithm for other models

The SDRG approach has been applied for a series of random quantum and classical problems mainly in one dimension. In higher dimensions the numerical implementation of the SDRG method for these models has basically the same problems as the naïve algorithm for the RTIM. In these cases one can try to apply and generalize the concept of our improved algorithm. Here we present the appropriate RG rules for two random quantum models: for the disordered $q$-state quantum Potts model and for the disordered quantum rotor model, which is a standard model of granular superconductors and Josephson arrays.

Disordered $q$-state quantum Potts model

In this model at each lattice site $i$ (or $j$) there is a $q$-state spin variable $s_i = 1, 2, \ldots, q$ and the Hamiltonian is given by [46]

$$\mathcal{H} = -\sum_{\langle ij \rangle} J_{ij} \delta(s_i, s_j) - \sum_i h_i \sum_{k=1}^{q-1} M_i^k.$$  \hspace{1cm} (A.1)

Here the first term represents the interaction between the spins and the second term is a generalized transverse field where $M_i$ is a spin-flip operator at site $i$: $M_i | s_i \rangle = | s_i + 1 \mod q \rangle$. As for the random transverse field Ising model, what we recover for $q = 2$, the $J_{ij}$ couplings and the $h_i$ transverse fields are random variables. The SDRG decimation rules are very similar to those of the RTIM as described in section 2, and differ only in an extra factor $\kappa = 2/q$.

$J$-decimation: the effective transverse fields are given by $\tilde{h} = \kappa h_i / J_{ij}$.

$h$-decimation: the effective couplings are given by $\tilde{J}_{jk} = \kappa J_{j} J_{ik} / h_i$, which should be supplemented by the maximum rule.

In the improved algorithm in section 2.1 the distances and ranges in the log-energy space, see equations (5)–(7), are extended by a constant, $d_0 = \ln (q/2) = -\ln \kappa$. These are given by

$$d_{ij} = -\ln J_{ij} + \frac{J_{ij}}{2} - \ln h_i + \frac{J_{ij}}{2} - \ln h_j + d_0.$$  \hspace{1cm} (A.2)

$$r_i = -\ln h_i + d_0.$$  \hspace{1cm} (A.3)

$$\tilde{r} = r_i + r_j - \delta_{ij} + d_0.$$  \hspace{1cm} (A.4)

This generalization works for $2 < q < \infty$.

Disordered Josephson junctions

Here we consider disordered bosons with an occupation operator, $\hat{n}_i$, and a phase variable, $\varphi_i$, at site $i$. The system is described by the following Hamiltonian [49]:

$$\mathcal{H} = -\sum_{\langle ij \rangle} J_{ij} \cos(\varphi_i - \varphi_j) + \sum_i U_i \hat{n}_i^2.$$  \hspace{1cm} (A.5)

with random $J_{ij}$ Josephson couplings and $U_i$ charging energies. The SDRG approach has also been applied to this model resulting in the following RG rules.

$U$-decimation: if the strongest parameter in the Hamiltonian is a grain charging energy $U_i$, then site $i$ is eliminated and effective couplings are generated between the nearest neighbours of $i$, say $j$ and $k$. In the second-order perturbation calculation this is given by $\tilde{J}_{jk} = J_{ij} J_{ik}/U_i$, which has to be supplemented by the maximum rule.

$J$-decimation: if the strongest coupling in the system is a Josephson coupling, $J_{ij}$, then the two sites form a composite
site having an effective charging energy, $\tilde{U}_i$, which does not depend on the value of $J_i$, but is given by $\tilde{U}_i = \frac{1}{2} \sum_j U_{ij} + \frac{1}{2}$. In the improved SDRG algorithm the distance and the range in equations (5) and (6) are modified with the substitution $h_i \rightarrow U_i$ as

$$d_{ij} = -\ln J_j + J_j \ln U_i + \frac{1}{2} \ln U_i, \quad (A.6)$$

$$r_i = -\ln U_i, \quad (A.7)$$

In contrast, the updated range in equation (7) has a different form and is given by

$$\tilde{r} = \ln \left[ \exp (r_j) + \exp (r_j) \right]. \quad (A.8)$$

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