Research Article

Dynamic Programming and Hamilton–Jacobi–Bellman Equations on Time Scales

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Bellman optimality principle for the stochastic dynamic system on time scales is derived, which includes the continuous time and discrete time as special cases. At the same time, the Hamilton–Jacobi–Bellman (HJB) equation on time scales is obtained. Finally, an example is employed to illustrate our main results.

1. Introduction

The stochastic control problem is to find an optimal control such that a cost functional associated with a stochastic system reaches the minimum value. The method of dynamic programming is a powerful approach to solving the stochastic optimal control problems. The dynamic programming is a well-established subject [1–4] to deal with continuous and discrete optimal control problems, respectively, and it has great practical applications in various fields [5, 6]. It is generally assumed that the time is continuous or discrete in the dynamic systems. However, this cannot be guaranteed. In reality, the time scale could be neither continuous nor discrete. There are many processes which are the mixture of continuous time and discrete time, non-uniform discrete time, or the union of disjoint time interval, such as the production and storage process in economics, the investment process in finance, and the population for the seasonal insects. When the time is more complex, it makes the control problem more difficult. How to deal with this problem?

Time scales were first introduced by Hilger [7] in 1988 in order to unite differential and difference equations into a general framework. This allows us to deal with the continuous and discrete analyses from the common point of view. Recently, time scale theory is extensively studied in many works [8–13]. It is well known that the optimal control problems on time scales are an important field for both theory and applications. Since the calculus of variations on time scales was studied by Bohner [14], the results on optimal control problems in the time scale setting and their applications have been rapidly growing. The existence of optimal control for the dynamic systems on time scales was discussed [15–17]. Subsequently, Pontryagin maximum principle on time scales was studied in several works [18, 19], which specifies the necessary conditions for optimality.

The dynamic programming for dynamic systems on time scales is not a simple task to unite the continuous time and discrete time cases because the time scales contain more complex time cases. Seiffertt et al. [20] studied the approximate dynamic programming for the dynamic system in the isolated time scale setting. In addition, Bellman dynamic programming on general time scales for the deterministic optimal control problems was considered in [21, 22]. However, limited work [23, 24] has been done on the linear quadratic stochastic optimal control problem in the time scales setting. That is to say, the general setting of stochastic optimal control problems on time scales is completely open.

Motivated by all these significant works, the purpose of this paper is to study the method of dynamic programming for the stochastic optimal control problems on time scales. As we know, the stochastic dynamic programming principle
is different from the deterministic systems, which reflects the stochastic nature of the optimal problem. So, the method in the deterministic case on time scales cannot be applied to the stochastic case directly. In order to overcome this difficulty, we first give a new form of the chain rule on time scales. Based on this idea, we obtain Itô’s formula for the stochastic process on time scales. Second, we consider a family of optimal control problems with different initial times and states to establish the Bellman optimality principle in the time scale setting. Third, using Itô’s formula and the Bellman optimality principle obtained in the time scales, we also get the associated Hamilton–Jacobi–Bellman (HJB for short) equation on time scales which is a nonlinear second-order partial differential equation involving expectation. If the HJB equation is solvable, then we can get an optimal feedback control. Our work will enrich the dynamic programming problem by providing a more general time framework and make dynamic programming theory to be a powerful tool in tackling the optimal control problem on complexity time.

The organization of this paper is as follows. In Section 2, we show some preliminaries about time scale theory. Section 3 focuses on the Bellman optimality principle and the HJB equations on time scales. By introducing a new symbol, we present Itô’s formula in a new form. On this basis, we get the main results. Finally, an illustrative example is given to show the effectiveness of the proposed results.

2. Preliminaries

A time scale \( \mathbb{T} \) is a nonempty closed subset in real number set \( \mathbb{R} \), and we denote \([0, T]_\mathbb{T} = [0, T] \cap \mathbb{T} \). In this paper, we always suppose \( \mathbb{T} \) is bounded. The forward jump operator \( \sigma \) and backward jump operator \( \rho \) are, respectively, defined by

\[
\sigma (t) = \inf \{ s \in \mathbb{T} : s > t \}, \\
\rho (t) = \sup \{ s \in \mathbb{T} : s < t \},
\]

supplemented by \( \inf \emptyset := \sup \mathbb{T} \) and \( \sup \emptyset := \inf \mathbb{T} \), where \( \emptyset \) denotes the empty set. If \( \sigma (t) = t \) and \( t < \sup \mathbb{T} \), the point \( t \) is called right-dense, while if \( \sigma (t) > t \), the point \( t \) is called right-scattered. Similarly, if \( \rho (t) = t \) and \( t > \inf \mathbb{T} \), the point \( t \) is called left-dense, while if \( \rho (t) < t \), the point \( t \) is called left-scattered. Moreover, a point is called isolated if it is both left-scattered and right-scattered. For a function \( f \), we denote \( f^\sigma (t) = f (\sigma (t)) \) to represent the compositions of the functions \( f \) and \( \sigma \). Similarly, we denote \( f^\rho (t) = f (\rho (t)) \). The definition of the graininess function \( \mu \) is as follows:

\[
\mu (t) = \sigma (t) - t.
\]

We now present some basic concepts and properties about time scales (see [10, 11]).

**Definition 1** (see [10]). Let \( f \) be a function on \( \mathbb{T} \), \( f \) is called a right-dense continuous function if \( f \) is continuous at every right-dense point and has finite left-sided limits at every left-dense point. Similarly, \( f \) is called a left-dense continuous function if \( f \) is continuous at every left-dense point and has finite right-sided limits at every right-dense point.

If \( f \) is right-dense continuous and also is left-dense continuous, then \( f \) is called a continuous function.

Define the set \( \mathbb{T}^s \) as

\[
\mathbb{T}^s = \begin{cases} 
\mathbb{T} \setminus (\rho (\sup \mathbb{T}), \sup \mathbb{T}) & \text{if } \sup \mathbb{T} < \infty, \\
\mathbb{T} & \text{if } \sup \mathbb{T} = \infty.
\end{cases}
\]

**Definition 2** (see [11]). Let \( f : \mathbb{T} \rightarrow \mathbb{R} \) be a function and \( t \in \mathbb{T}^s \). If for all \( \varepsilon > 0 \), there exists a neighborhood \( U \) of \( t \) such that

\[
| f (\sigma (t)) - f (s) - f^\Delta (t)(\sigma (t) - s) | \leq \varepsilon |\sigma (t) - s|, \quad \text{for all } s \in U,
\]

we call \( f^\Delta (t) \) the \( \Delta \)-derivative of \( f \) at \( t \).

We denote by \( C^1 ([0, T]_\mathbb{T}; \mathbb{R}) \) the space of \( \mathbb{R} \)-valued continuously \( \Delta \)-differentiable functions on \([0, T]_\mathbb{T} \) and denote by \( C^{1,2} (\mathbb{T} \times \mathbb{R}^n; \mathbb{R}) \) the family of all functions \( f(t, x) \) defined on \( \mathbb{T} \times \mathbb{R}^n \) such that they are continuously \( \Delta \)-differentiable in \( t \) and twice continuously differentiable in \( x \).

Furthermore, we give the derivation rule of the compound function.

**Lemma 1** (see [25]). Let \( g : \mathbb{T} \rightarrow \mathbb{R} \) be \( \Delta \)-differentiable and \( f : \mathbb{T} \rightarrow \mathbb{R} \) be continuously differentiable. Then \( f(g(t)) \) is \( \Delta \)-differentiable, and

\[
f^\Delta (g(t)) = g^\Delta (t) \int_0^1 f''(g(t) + \lambda h(0)(g^\Delta (t)) \: dh.
\]

In this paper, we adopt the stochastic integral defined by Bohner et al. [26]. Let \( (\Omega, \mathbb{F}, \{ \mathbb{F}_t \}_{t \in [0, T]}, \mathbb{P}) \) be a complete probability space with increasing and continuous filtration \( \{ \mathbb{F}_t \}_{t \in [0, T]} \). We define that \( L^2_x ([0, T]_\mathbb{T}; \mathbb{R}) \) is the set of all \( \mathbb{F}_t \)-adapted, \( \mathbb{R} \)-valued measurable processes \( X(t) \) such that

\[
\mathbb{E} \left[ \int_0^T |X(t)|^2 dt \right] < \infty.
\]

A Brownian motion indexed by time scales \( \mathbb{T} \) was defined by Grow and Sanyal [13]. Although the Brownian motion on time scales is very similar to that on continuous time, there are also some differences between them. For example, the quadratic variation of a Brownian motion on time scales (see [27]) is an increasing process yet, but it is not deterministic. In fact, \( \langle W \rangle_t = \lambda \left( [0, t]_\mathbb{T} + \sum_{s \in \mathbb{T}} (W^s_t - W^s_s) \right) \), where \( \lambda \) is the Lebesgue measure.

Now, we give the definition of the stochastic \( \Delta \)-integral and its properties.

**Definition 3** (see [26]). The random process \( X(t) \) is stochastic \( \Delta \)-integrable on \([0, T]_\mathbb{T} \) if the corresponding \( \bar{X}(t) \) is integrable. Define the integral value of \( X(t) \) as

\[
\int_0^T X(t) \Delta W(t) = \int_0^T \bar{X}(t) dW(t),
\]

where

\[
\bar{X}(t) = X(\sup [0, t]_\mathbb{T}), \quad \text{for all } t \in [0, T],
\]

and the Brownian motion on the right side of (6) is indexed by continuous time.
We also have the following properties.

\begin{align}
(i) \quad & \int_0^T (\alpha X(t) + \beta Y(t))\,dW(t) = \alpha \int_0^T X(t)\,dW(t) + \beta \int_0^T Y(t)\,dW(t), \\
(ii) \quad & E\left[ \int_0^T X(t)\,dW(t) \right] = 0, \\
(iii) \quad & E\left[ \int_0^T X(t)\,dW(t) \right]^2 = E\left[ \int_0^T |X(t)|^2\,dt \right].
\end{align}

where the integral $X$ with respect to the quadratic variation of Brownian motion $\langle W \rangle_t$ is defined by Stieltjes integral as

\[
\int_0^T X_t(\omega)\,d\langle W \rangle_t(\omega).
\]

Let $X$ be an $n$-dimensional stochastic process defined by

\[
\Delta X(s) = a(s, X(s))\Delta s + b(s, X(s))\Delta W(s),
\]

and we have the following Itô's formula.

Lemma 2 (see [28]). Let $f \in C^1\beta (T \times \mathbb{R}^n; \mathbb{R})$ and $X$ satisfy (9), then the following relation holds:

\[
f(t, X(t)) = f(0, X(0)) + \int_0^t f_s(s, X(s))\,ds + \int_0^t \left( \frac{\partial f}{\partial x}(s, X(s)) \right)^T a(s, X(s))\,ds
\]

\[
+ \int_0^t \left( \frac{\partial f}{\partial x}(s, X(s)) \right)^T b(s, X(s))\,dW(s) + \frac{1}{2} \int_0^t \langle b(s, X(s)) \rangle^T \frac{\partial^2 f}{\partial x^2}(s, X(s)) b(s, X(s))\,ds
\]

\[
+ \sum_{s \in \{0,t\}} \left( f(\sigma(s), X^n(s)) - f(\sigma(s), X(s)) \right) - \sum_{s \in \{0,t\}} \left( \frac{\partial f}{\partial x}(s, X(s)) \right)^T a(s, X(s))(\sigma(s) - s)
\]

\[
- \sum_{s \in \{0,t\}} \left( \frac{\partial f}{\partial x}(s, X(s)) \right)^T b(s, X(s))(W^n(s) - W(s)) - \frac{1}{2} \langle b(s, X(s)) \rangle^T \frac{\partial^2 f}{\partial x^2}(s, X(s)) b(s, X(s)).
\]

3. Problem Statement and Main Results

Let $\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in T}, P$ be a given filtered probability space satisfying the usual condition. Consider the stochastic control system

\[
\begin{cases}
\Delta X(s) = a(s, X(s), u(s))\Delta s + b(s, X(s), u(s))\Delta W(s), & s \in [0, \rho(T)]_T, \\
X(0) = x_0 \in \mathbb{R}^n.
\end{cases}
\]

The control $u(\cdot)$ belongs to

\[
\mathcal{U}[0, T]_T = \{u: [0, T]_T \times \Omega \rightarrow \mathbb{R} \mid u \text{ is measurable and } \{\mathcal{F}_t\}_{t \in T} \text{ - adapted} \}.
\]
where $U$ is a convex subset of $\mathbb{R}^m$. And the functions $a(t, x, u)$ and $b(t, x, u)$ satisfy the Lipschitz condition and linear growth condition in $x$. Obviously, equation (11) admits a unique solution (see Bohner et al. [26]).

The cost functional associated with (11) is

$$J(u(\cdot)) = E\left[\int_0^T r(s, X(s), u(s))\Delta s + h(X(T))\right],$$

where the maps $r: [0, T]_\mathbb{T} \times \mathbb{R}^n \times U \longrightarrow \mathbb{R}$ and $h: \mathbb{R}^n \longrightarrow \mathbb{R}$ are continuous.

The optimal control problem is to find $u^\star(\cdot) \in \mathcal{U}[0, T]_\mathbb{T}$ such that

$$J(u^\star(\cdot)) = \inf_{u(\cdot) \in \mathcal{U}[0, T]_\mathbb{T}} J(u(\cdot)).$$

(14)

$u^\star(\cdot)$ is called the stochastic optimal control of the problem, and the corresponding $X(\cdot; x_0, u^\star(\cdot))$ is called an optimal state process.

Now, we consider a family of optimal control problems with different initial times and states. Let $(t, x) \in [0, T]_\mathbb{T} \times \mathbb{R}^n$, consider the state equation

$$\begin{aligned}\Delta X(s) &= a(s, X(s), u(s))\Delta s + b(s, X(s), u(s))\Delta W(s), & s \in [t, T], \\ X(t) &= x, \end{aligned}$$

along with the cost functional

$$J(t, x; u(\cdot)) = E\left[\int_t^T r(s, X(s), u(s))\Delta s + h(X(T))\right].$$

(16)

For any $(t, x) \in [0, T]_\mathbb{T} \times \mathbb{R}^n$, minimize (16) subject to (15) over $\mathcal{U}[t, T]_\mathbb{T}$. The value function of the optimal control problem is defined as

$$f^\Delta_{g^\Delta}(g(t)) = \begin{cases} \lim_{s \rightarrow t} \frac{f(g(t)) - f(g(s))}{g(t) - g(s)}, & t \text{ is right - dense and } g^\Delta \neq 0, \\ \frac{f(g^\star(t)) - f(g(t))}{g^\star(t) - g(t)}, & t \text{ is right - scattered and } g^\Delta \neq 0, \\ 0, & g^\Delta = 0. \end{cases}$$

(18)

Remark 1. Note that $f^\Delta_{g^\Delta}(g(t))$ depends not only on the functions $f$ and $g$ but also on the time scales $\mathbb{T}$. If $t$ is a right-dense point on time scales $\mathbb{T}$, then $f^\Delta_{g^\Delta}(g(t)) = f^\Delta_{g^\Delta}(g(t)) = f^\Delta_{g^\Delta}(g(t))$. On the contrary, if $t$ is a right-scattered point and $g^\sigma(t) = a(g(t))$, we have $f^\Delta_{g^\Delta}(g(t)) = f^\Delta_{g^\Delta}(g(t))$.

The optimal control problem is to find $u^\star(\cdot) \in \mathcal{U}[0, T]_\mathbb{T}$ such that

$$J(u^\star(\cdot)) = \inf_{u(\cdot) \in \mathcal{U}[0, T]_\mathbb{T}} J(u(\cdot)).$$

With the help of the new symbol, we have the following lemma.

Lemma 3. Let $f: \mathbb{T} \times \mathbb{R} \longrightarrow \mathbb{R}$ be continuously differentiable and $x: \mathbb{T} \longrightarrow \mathbb{R}$ and $y: \mathbb{T} \longrightarrow \mathbb{R}$ be $\Delta$-differentiable, then

$$f^\Delta(x(t), y(t)) = f^\Delta_x(x(t), y^\sigma(t))x^\Delta(t) + f^\Delta_y(x(t), y(t))y^\Delta(t), \quad t \in \mathbb{T}^\sigma.$$
Complexity

Proof. If \( x^\Delta(t) = 0 \) or \( y^\Delta(t) = 0 \), it is easy to verify that (19) is true. We only give proofs under conditions \( x^\Delta(t) \neq 0 \) and \( y^\Delta(t) \neq 0 \). If \( t \) is right-dense, one has

\[
f^\Delta(x(t), y(t)) = \lim_{s \to t} \frac{f(x(t), y(t)) - f(x(s), y(s))}{t - s}
\]

\[
= \lim_{s \to t} \frac{f(x(t), y(t)) - f(x(t), y(t)) + f(x(s), y(t)) - f(x(s), y(t))}{t - s}
\]

\[
= \lim_{s \to t} \left( \frac{f(x(t), y(t)) - f(x(s), y(t))}{x(t) - x(s)} \cdot \frac{x(t) - x(s)}{t - s} + \frac{f(x(s), y(t)) - f(x(s), y(s))}{y(t) - y(s)} \cdot \frac{y(t) - y(s)}{t - s} \right)
\]

\[
= f_x^\Delta(x(t), y(t)) x^\Delta(t) + f_y^\Delta(x(t), y(t)) y^\Delta(t).
\]

When \( t \) is right-scattered, then

\[
f^\Delta(x(t), y(t)) = \frac{f(x^\sigma(t), y^\sigma(t)) - f(x(t), y(t))}{\sigma(t) - t}
\]

\[
= \frac{f(x^\sigma(t), y^\sigma(t)) - f(x(t), y^\sigma(t)) + f(x(t), y^\sigma(t)) - f(x(t), y(t))}{\sigma(t) - t}
\]

\[
= \frac{f(x^\sigma(t), y^\sigma(t)) - f(x(t), y^\sigma(t))}{x^\sigma(t) - x(t)} \cdot \frac{x^\sigma(t) - x(t)}{\sigma(t) - t} + \frac{f(x(t), y^\sigma(t)) - f(x(t), y(t))}{y^\sigma(t) - y(t)} \cdot \frac{y^\sigma(t) - y(t)}{\sigma(t) - t}
\]

\[
= f_x^\Delta(x(t), y^\sigma(t)) x^\Delta(t) + f_y^\Delta(x(t), y(t)) y^\Delta(t).
\]

This completes the proof. \( \square \)

Remark 2. Similarly, another form can be expressed as

\[
f^\Delta(x(t), y(t)) = f_x^\Delta(x(t), y(t)) x^\Delta(t) + f_y^\Delta(x^\sigma(t), y(t)) y^\Delta(t).
\]

(22)

Remark 3. In particular, let \( g: \mathbb{T} \to \mathbb{R} \) be \( \Delta \)-differentiable and \( f: \mathbb{R} \to \mathbb{R} \) be continuously differentiable. Then

\[
f^\Delta(g(t)) = f_x^\Delta(g(t)) g^\Delta(t).\]

It is easy to see that this equality is equivalent to (5).

Remark 4. It is not hard for us to get the following result of multidimensions:

\[
F^\Delta(x(t), y(t)) \equiv \left( \begin{array}{c} f_x^\Delta(x(t), y^\sigma(t)) \\ f_y^\Delta(x(t), y(t)) \end{array} \right)^T (x^\Delta(t) + (f_x^\Delta(x(t), y(t)))^T y^\Delta(t),
\]

(23)

where \( F: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R} \) is continuously differentiable and \( x: \mathbb{T} \to \mathbb{R}^n \) and \( y: \mathbb{T} \to \mathbb{R}^m \) are \( \Delta \)-differentiable.

Next, we show Itô's formula in a new form on time scales.
Proposition 1. Let $X$ satisfy (9) and $f \in C^{1,2}(\mathbb{T} \times \mathbb{R}^n; \mathbb{R})$, we have

$$f(t, X(t)) = f(0, X(0)) + \int_0^t f^x_x(s, X(s))\Delta s + \int_0^t \left( f^\pi_x(\sigma(s), X(s)) \right)^T a(s, X(s))ds$$

$$+ \int_0^t \left( f^\pi_x(\sigma(s), X(s)) \right)^T b(s, X(s))\Delta W(s) + \frac{1}{2} \int_0^t I_D(s) (b(s, X(s))^T f^\pi_{xx}(\sigma(s), X(s))b(s, X(s))\Delta s,$$

where $I(\cdot)$ is an indicative function and $D$ is the set of all right-dense points.

Proof. Because of Lemma 2, it is enough to show that

$$\int_0^t f^x_x(s, X(s))\Delta s + \int_0^t \left( \frac{\partial f}{\partial x}(s, X(s)) \right)^T a(s, X(s))\Delta s$$

$$+ \int_0^t \left( \frac{\partial f}{\partial x}(s, X(s)) \right)^T b(s, X(s))\Delta W(s) + \frac{1}{2} \int_0^t (b(s, X(s))^T \frac{\partial^2 f}{\partial x \partial x}(s, X(s))b(s, X(s))\Delta s$$

$$+ \sum_{s \in [0,t]} \left( f(\sigma(s), X^\sigma(s)) - f(\sigma(s), X(s)) \right)$$

$$- \sum_{s \in [0,t]} \left( \frac{\partial f}{\partial x}(s, X(s)) \right)^T b(s, X(s))(W^\sigma(s) - W(s)) - \frac{1}{2} (b(s, X(s))^T \frac{\partial^2 f}{\partial x \partial x}(s, X(s))b(s, X(s))$$

$$= \int_0^t f^x_x(s, X(s))\Delta s + \int_0^t \left( f^\pi_x(\sigma(s), X(s)) \right)^T a(s, X(s))\Delta s$$

$$+ \int_0^t \left( f^\pi_x(\sigma(s), X(s)) \right)^T b(s, X(s))\Delta W(s) + \frac{1}{2} \int_0^t I_D(s) (b(s, X(s))^T f^\pi_{xx}(\sigma(s), X(s))b(s, X(s))\Delta s.$$

By some manipulation, namely,

$$\sum_{s \in [0,t]} \left( f(\sigma(s), X^\sigma(s)) - f(\sigma(s), X(s)) \right)$$

$$= \sum_{s \in [0,t]} \left( f^\pi_x(\sigma(s), X(s)) \right)^T a(s, X(s))(\sigma(s) - s) + \sum_{s \in [0,t]} \left( f^\pi_x(\sigma(s), X(s)) \right)^T b(s, X(s))(W^\sigma(s) - W(s)),$$

it is straightforward to show that the above equation is true. Now, we state the Bellman optimality principle on time scales. □

Theorem 1 (optimality principle). Let $(t, x) \in [0,T]_T \times \mathbb{R}^n$. Then for any $\tilde{t} \in [t, T]_T$, we have

$$V(t, x) = \inf_{u(\cdot) \in \mathcal{W}[t, \tilde{t}]} \mathbb{E} \left[ \int_t^{\tilde{t}} r(s, X(s), u(s))\Delta s + V(\tilde{t}, X(\tilde{t})) \right].$$
Proof. For any ε > 0, there exists a control $u(\cdot) \in \mathcal{U}[t, T]$ such that

$$V(t, x) + \varepsilon \geq J(t, x; u(\cdot))$$

$$= E \left[ \int_t^T r(s, X(s), u(s)) \Delta s + h(X(T)) \right]$$

$$= E \left[ \int_t^T r(s, X(s), u(s)) \Delta s + E \left[ \int_t^T r(s, X(s), u(s)) \Delta s + h(X(T)) | \mathcal{F}_t \right] \right]$$

$$= E \left[ \int_t^T r(s, X(s), u(s)) \Delta s + J(\tilde{t}, X(\tilde{t}); u(\cdot)) \right]$$

$$\geq E \left[ \int_t^T r(s, X(s), u(s)) \Delta s + V(\tilde{t}, X(\tilde{t})) \right]$$

$$\geq \inf_{u(\cdot) \in \mathcal{U}[\tilde{t}, T]} E \left[ \int_{\tilde{t}}^T r(s, X(s), u(s)) \Delta s + V(\tilde{t}, X(\tilde{t})) \right],$$

(28)

where $\mathcal{F}_t$ is the sigma field generated by $\{ W(r) - W(t) : r \in [t, \tilde{t}] \}$.

On the contrary, by the definition of value function (17), we obtain

$$V(t, x) \leq J(t, x; u(\cdot)) = E \left[ \int_t^T r(s, X(s), u(s)) \Delta s + J(\tilde{t}, X(\tilde{t}); u(\cdot)) \right].$$

(29)

Thus, taking infimum over $u(\cdot) \in \mathcal{U}[\tilde{t}, T]$, one has

$$V(t, x) \leq E \left[ \int_t^T r(s, X(s), u(s)) \Delta s + V(\tilde{t}, X(\tilde{t})) \right].$$

(30)

It follows that

$$V(t, x) \leq \inf_{u(\cdot) \in \mathcal{U}[\tilde{t}, T]} E \left[ \int_t^T r(s, X(s), u(s)) \Delta s + V(\tilde{t}, X(\tilde{t})) \right].$$

(31)

Combining with (28) and (31), we get the result. Furthermore, we give the HJB equation on time scales which is similar to continuous and discrete cases.

\[ \square \]

Theorem 2. Let the value function $V \in C^{1,2}([0, T]_T \times \mathbb{R}^n)$, then the value function $V$ satisfies the following HJB equation:

$$\begin{align*}
\left\{ V^x(t, x) + \inf_{u \in \mathcal{U}} E_t^x \left[ H(t, x, u, V^x_x ?(t, x), \nabla^2_x \sigma(t), \nabla^2_x ?(t, x)) \right] = 0, \quad & (t, x) \in [0, \rho(T)]_T \times \mathbb{R}^n, V(T, x) = h(x), \\
\end{align*}$$

(32)

where

$$H(t, x, u, p, P) = r(t, x, u) + p^\top a(t, x, u) + p^\top b(t, x, u) W^x_t(t) + \frac{1}{2} I(t) (b(t, x, u))^\top P b(t, x, u),$$

(33)

and we use the notation
\[ E^x_t \left[ f_x^\pi(X(t)) \right] = E \left[ f_x^\pi(X(t)) | X(t) = x \right], W^\lambda(t) = \begin{cases} W(\epsilon(t) - W(t)) & \text{if } t \text{ is right - scattered}, \\ \mu(t) & \text{if } t \text{ is right - dense.} \end{cases} \] (34)

**Proof.** According to the definition of the value function, \( V(T, x) = h(x) \) is satisfied. Fix \( u \in U \), and let \( X(\cdot) \) be the state process corresponding to the control \( u(\cdot) \equiv u \). By the optimality principle and Itô's formula, for any \( \tilde{t} > t \) and \( t, \tilde{t} \in [0, T] \), we have

\[
0 \leq \frac{E[V(\tilde{t}, X(\tilde{t})) - V(t, x)]}{\tilde{t} - t} + \frac{1}{\tilde{t} - t} E \left[ \int_t^{\tilde{t}} r(s, X(s), u)\Delta s \right] \]

\[
= \frac{1}{\tilde{t} - t} E \left[ \int_t^{\tilde{t}} \left( V^\lambda_s(s, X(s)) + H(s, X(s), u(s), V^\pi_x(s, tX(s)), \lambda^\lambda x_{xx}(s, X(s))) \right) \Delta s \right]. \] (35)

If \( t \) is right-dense, then let \( \tilde{t} \rightarrow t \), while if \( t \) is right-scattered, then let \( \tilde{t} = \sigma(t) \). This leads to

\[
0 \leq V^\lambda_t(t, x) + E^x_t \left[ H\left( t, x, u, V^\pi_x(\sigma(t), tX(n)), tV^\lambda_{xx}(\sigma(t), x) \right) \right]. \] (36)

It follows that

\[
0 \leq V^\lambda_t(t, x) + \inf_{u \in U} E^x_t \left[ H\left( t, x, u, V^\pi_x(\sigma(t), tX(n)), tV^\lambda_{xx}(\sigma(t), x) \right) \right]. \] (37)

Conversely, for any \( \epsilon > 0, \tilde{t} > t \), and \( t, \tilde{t} \in [0, T] \), we can find a control \( u(\cdot) \) such that

\[
\epsilon \geq \frac{E[V(\tilde{t}, X(\tilde{t})) - V(t, x)]}{\tilde{t} - t} + \frac{1}{\tilde{t} - t} E \left[ \int_t^{\tilde{t}} r(s, X(s), u(s))\Delta s \right] \]

\[
= \frac{1}{\tilde{t} - t} E \left[ \int_t^{\tilde{t}} \left( V^\lambda_s(s, X(s)) + H\left(s, X(s), u(s), V^\pi_x(s, tX(n), \sigma(s), X(s)) \right), tV^\lambda_{xx}(\sigma(s), X(s)) \right) \Delta s \right] \]

\[
\geq \frac{1}{\tilde{t} - t} E \left[ \int_t^{\tilde{t}} \left( V^\lambda_s(s, X(s)) + \inf_{u \in U} H\left(s, X(s), u, V^\lambda_x(\sigma(s), tX(n), \sigma(s), X(s)) \right), tV^\lambda_{xx}(\sigma(s), X(s)) \right) \Delta s \right]. \] (38)

By the same argument as that used in the above proof, we get

\[
\epsilon \geq V^\lambda_t(t, x) + \inf_{u \in U} E^x_t \left[ H\left( t, x, u, V^\pi_x(\sigma(t), tX(n)), tV^\lambda_{xx}(\sigma(t), x) \right) \right]. \] (39)
Therefore, (32) follows. The proof is completed. □

Remark 5. It is not surprising that HJB equation (32) on time scales is very similar to the classical HJB equation in continuous and discrete time (see [29, 30]). An intriguing feature of the HJB equation on time scales is that the expectation is involved. When \( T = \mathbb{R}^+ \) or \( T = \mathbb{Z}^+ \), we are able to reduce equation (32) into the classical ones.

Remark 6. Suppose \( b \equiv 0 \), then HJB equation (32) becomes

\[
V^\lambda \left( t, x \right) + \inf_{u \in U} \left\{ \left( V^\lambda_\sigma \left( \sigma \left( t \right), X \left( t \right) \right) \right)^T a \left( t, x, u \right) + r \left( t, x, u \right) \right\}_{X(t)=x} = 0,
\]

(40)

In such a case, it is equivalent to the result in [22].

Remark 7. In particular, in Remark 6, if we further let \( X^\sigma \left( t \right) = \sigma \left( X \left( t \right) \right) \), HJB equation (32) degenerates into

\[
V^\lambda \left( t, x \right) + \inf_{u \in U} \left\{ \left( V^\lambda_\sigma \left( \sigma \left( t \right), x \right) \right)^T a \left( t, x, u \right) + r \left( t, x, u \right) \right\}_{X(t)=x} = 0,
\]

(41)

Then \( F \left( t, x \right) = V \left( t, x \right) \), and \( u \left( t \right) = \phi \left( t, X \left( t \right) \right) \) is an optimal control.

\[
E \left[ H \left( t, X \left( t \right), \phi \left( t, X \left( t \right) \right), TF_x^\sigma n \left( \sigma \left( t \right), X \left( t \right) \right) \right) \right] = \inf_{u \in U} \left[ E \left[ H \left( t, X \left( t \right), u, TF_x^\sigma n \left( \sigma \left( t \right), X \left( t \right) \right) \right) \right] \right].
\]

Then \( F \left( t, x \right) = V \left( t, x \right) \), and \( u \left( t \right) = \phi \left( t, X \left( t \right) \right) \) is an optimal control.

Proof. Let \( u \left( t \right) = \phi \left( t, X \left( t \right) \right) \), we have

\[
E \left[ h \left( X \left( T \right) \right) \right] - F \left( t, x \right)
= E \left[ \int_t^T F_x^\lambda \left( s, X \left( s \right) \right) \left( F_x^\lambda \left( \sigma \left( s \right), X \left( s \right) \right) \right)^T a \left( s, X \left( s \right), u \left( s \right) \right) + \left( F_x^\lambda \left( \sigma \left( s \right), X \left( s \right) \right) \right)^T b \left( s, X \left( s \right), u \left( s \right) \right) \right] W_x^\lambda \left( s \right)
+ \frac{1}{2} E \left[ \int_t^T \left( b \left( s, X \left( s \right), u \left( s \right) \right) \right)^T F_{xx}^\lambda \left( \sigma \left( s \right), X \left( s \right) \right) b \left( s, X \left( s \right), u \left( s \right) \right) \right] \Delta s
= -E \left[ \int_t^T r \left( s, X \left( s \right), u \left( s \right) \right) \Delta s \right].
\]

This yields

\[
F \left( t, x \right) = E \left[ \int_t^T r \left( s, X \left( s \right), u \left( s \right) \right) \Delta s + h \left( X \left( T \right) \right) \right] \geq V \left( t, x \right).
\]

(44)

In addition, for any admissible pair \( \left( X \left( \cdot \right), v \left( \cdot \right) \right) \), we have

\[
E \left[ h \left( X \left( T \right) \right) \right] - F \left( t, x \right)
= E \left[ \int_t^T F_x^\lambda \left( s, X \left( s \right) \right) + \left( F_x^\lambda \left( \sigma \left( s \right), X \left( s \right) \right) \right)^T a \left( s, X \left( s \right), v \left( s \right) \right) + \left( F_x^\lambda \left( \sigma \left( s \right), X \left( s \right) \right) \right)^T b \left( s, X \left( s \right), u \left( s \right) \right) \right] W_x^\lambda \left( s \right)

\]
\[ \frac{1}{2} \int_0^T \left( b(s, X(s), v(s)) + F_{xx}^2(\sigma(s), X(s))b(s, X(s), v(s)) \right) \Delta s \]
\[ = E \left[ \int_0^T \left( F_s^3(s, X(s)) + H(s, X(s), u, F_{x}^2(\sigma(s), X(s)), F_{xx}^2(\sigma(s), X(s))) - r(s, X(s), v(s)) \right) \Delta s \right] \]
\[ \geq E \left[ \int_0^T \left( F_s^3(s, X(s)) + \inf_{u \in U} H(s, X(s), u, F_{x}^2(\sigma(s), X(s)), F_{xx}^2(\sigma(s), X(s))) - r(s, X(s), v(s)) \right) \Delta s \right] \]
\[ = -E \left[ \int_0^T r(s, X(s), v(s)) \Delta s \right]. \]

Namely,
\[ F(t, x) \leq E \left[ \int_t^T r(s, X(s), v(s)) \Delta s + h(X(T)) \right]. \] (46)

By arbitrary for \((X(\cdot), v(\cdot))\), it follows that
\[ F(t, x) \leq V(t, x). \] (47)

Hence, by (44) and (47), one has
\[ F(t, x) = V(t, x). \] (48)

Finally, inequality (44) together with (48) proves the optimality of \(u(\cdot)\).

4. Example

The dynamic programming on time scales contains not only continuous and discrete cases but also other more general cases. In order to illustrate our result, we give an example. Consider the quantum timescale \(T = \left\{ t = q^k : k \in \mathbb{N}, q > 1 \right\}\).

The state equation on time scales \(T\) is as follows:
\[ E \left[ X^2(t) + \phi^2(t, X(t)) + \frac{V(\sigma(t), X^\alpha(t))}{(q - 1)t} \right] = \inf_{u \in U} E \left[ X^2(t) + u^2 + \frac{V(\sigma(t), X^\alpha(t))}{(q - 1)t} \right], \quad t \in \mathbb{T}. \] (51)

We can see that the granularity function \(\mu\) affected the solution of HJB equation (51).

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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References

[1] R. Bellman, “Dynamic programming,” Science, vol. 153, no. 3731, pp. 34–37, 1966.
[2] D. Blackwell, “Discrete dynamic programming,” The Annals of Mathematical Statistics, vol. 33, no. 2, pp. 719–726, 1962.
[3] E. Bandini, A. Cosso, M. Fuhrman, and H. Pham, "Randomized filtering and bellman equation in wasserstein space for partial observation control problem," Stochastic Processes and Their Applications, vol. 129, no. 2, pp. 674–711, 2019.
[4] H. Pham and X. Wei, “Dynamic programming for optimal control of stochastic McKean—Vlasov dynamics,” Siam Journal on Control and Optimization, vol. 55, no. 2, pp. 1069–1101, 2017.
[5] C. Mu, Y. Zhang, H. Jia, and H. He, “Energy-storage-based intelligent frequency control of microgrid with stochastic model uncertainties,” IEEE Transactions on Smart Grid, vol. 11, no. 2, pp. 1748–1758, 2020.
[6] C. Mu and Y. Zhang, “Learning-based robust tracking control of quadrotor with time-varying and coupling uncertainties,” IEEE Transactions on Neural Networks and Learning Systems, vol. 31, no. 1, pp. 259–273, 2020.
[7] S. Hölzer, “Analysis on measure chains—a unified approach to continuous and discrete calculus,” Results in Mathematics, vol. 18, no. 1-2, pp. 18–56, 1990.
[8] G. S. Guseinov, “Integration on time scales,” Journal of Mathematical Analysis and Applications, vol. 285, no. 1, pp. 107–127, 2003.
[9] M. Bohner and G. S. Guseinov, “Partial differentiation on time scales,” Dynamic Systems and Applications, vol. 13, no. 3-4, pp. 351–379, 2004.
[10] M. Bohner and A. Peterson, Dynamic Equations on Time Scales: An Introduction with Applications, Birkhäuser Boston, Boston, MA, USA, 2001.
[11] M. Bohner and A. Peterson, Advances in Dynamic Equations on Time Scales, Birkhäuser Boston, Boston, MA, USA, 2002.
[12] N. H. Du and N. T. Dieu, “The first attempt on the stochastic calculus on time scale,” Stochastic Analysis and Applications, vol. 29, no. 6, pp. 1057–1080, 2011.
[13] D. Grow and S. Sanyal, “Brownian motion indexed by a time scale,” Stochastic Analysis and Applications, vol. 29, no. 3, pp. 457–472, 2011.
[14] M. Bohner, “Calculus of variations on time scales,” Dynamic Systems and Applications, vol. 13, no. 3-4, pp. 339–349, 2004.
[15] Z. Zhan and W. Wei, "On existence of optimal control governed by a class of the first-order linear dynamic systems on time scales," Applied Mathematics and Computation, vol. 215, no. 6, pp. 2070–2081, 2009.
[16] Y. Gong and X. Xiang, "A class of optimal control problems of systems governed by the first order linear dynamic equations on time scales," Journal of Industrial and Management Optimization, vol. 5, no. 1, pp. 1–10, 2009.
[17] Y. Peng, X. Xiang, and Y. Jiang, "Nonlinear dynamic systems and optimal control problems on time scales," ESAIM: Control, Optimisation and Calculus of Variations, vol. 17, no. 3, pp. 654–681, 2011.
[18] Z. Zhan, S. Chen, and W. Wei, "A unified theory of maximum principle for continuous and discrete optimal control problems," Mathematical Control and Related Fields, vol. 2, no. 2, pp. 195–215, 2012.
[19] M. Bohner, K. Kenzhebaev, O. Lavrova, and O. Stanzhytskyi, "Pontryagin’s maximum principle for dynamic systems on time scales," Journal of Difference Equations and Applications, vol. 23, no. 7, pp. 1161–1189, 2017.
[20] J. Seiffertt, S. Sanyal, and D. C. Wunsch, "Hamilton-Jacobi-Bellman equations and approximate dynamic programming on time scales," IEEE Transactions on Systems, Man, and Cybernetics, Part B (Cybernetics), vol. 38, no. 4, pp. 918–923, 2008.
[21] Z. Zhan, W. Wei, and H. Xu, "Hamilton–Jacobi–Bellman equations on time scales," Mathematical and Computer Modelling, vol. 49, no. 9-10, pp. 2019–2028, 2009.
[22] R. Š. Hilšer and V. Zeidan, "Hamilton–Jacobi theory over time scales and applications to linear-quadratic problems," Nonlinear Analysis: Theory, Methods and Applications, vol. 75, no. 2, pp. 932–950, 2012.
[23] Y. Zhu and G. Jia, "Stochastic linear quadratic control problem on time scales," submitted.
[24] Y. Zhu and G. Jia, "Linear feedback of mean-field stochastic linear quadratic optimal control problems on time scales," Mathematical Problems in Engineering, vol. 2020, Article ID 8051918, 11 pages, 2020.
[25] C. Pötzsche, "Chain rule and invariance principle on measure chains," Journal of Computational and Applied Mathematics, vol. 141, no. 1-2, pp. 249–254, 2002.
[26] M. Bohner, O. M. Stanzhytskyi, and A. O. Bratochchina, "Stochastic dynamic equations on general time scales," Electronic Journal of Differential Equations, vol. 2013, no. 57, pp. 1-15, 2013.
[27] D. Grow and S. Sanyal, "The quadratic variation of Brownian motion on a time scale," Statistics and Probability Letters, vol. 82, no. 9, pp. 1677–1680, 2012.
[28] W. Hu, "Itô’s formula, the stochastic exponential, and change of measure on general time scales," Abstract and Applied Analysis, vol. 2017, Article ID 940138, 13 pages, 2017.
[29] M. Bardl and I. Capuzzo-Dolcetta, Optimal Control and Viscosity Solutions of Hamilton-Jacobi-Bellman Equations, Springer Science & Business Media, Berlin, Germany, 2008.
[30] L. Grüne, "Error estimation and adaptive discretization for the discrete stochastic Hamilton–Jacobi–Bellman equation," Numerische Mathematik, vol. 99, no. 1, pp. 85–112, 2004.