A NOTE ON AN OACF-PRESERVING OPERATION BASED ON PARKER’S TRANSFORMATION

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ABSTRACT

Binary sequences with low odd-periodic correlation magnitudes have found important applications in communication systems. It is well known that the nega-cyclic shift and negation preserve the odd-periodic autocorrelation function (OACF) values in general. In this paper, we define a new operation based on Parker’s transformation, which also preserves the OACF values of binary sequences. This enables us to classify Parker’s 16 cases into 8 ones, and may possibly further allow to classify all constructions based on Parker’s transformation.

1 Introduction

A periodic sequence is called binary if every element of the sequence is from \( \mathbb{Z}_2 = \{0, 1\} \). For two binary sequences \( a = (a(i)) \) and \( b = (b(i)) \) of period \( N \), where the sequence indices \( i \) are taken modulo \( N \). The odd-periodic correlation function of \( a \) and \( b \) at the shift \( \tau \) is defined as

\[
\hat{R}_{a,b}(\tau) = \sum_{i=0}^{N-1} (-1)^{a(i)-b(i+\tau)} \lfloor \frac{\tau + i}{N} \rfloor, \quad 0 \leq \tau < N
\]

(1)

where \( \lfloor \frac{\tau + i}{N} \rfloor \) denotes the integer part of \( \frac{\tau + i}{N} \). When the two sequences \( a \) and \( b \) are identical, the odd-periodic correlation function \( \hat{R}_{a,b}(\tau) \) is called negaperiodic autocorrelation function (NACF) or odd-periodic autocorrelation function (OACF) of \( a \), and simply denoted by \( \hat{R}_a(\tau) \). The (even)-periodic correlation function of \( a \) and \( b \) at the shift \( \tau \) is defined as

\[
R_{a,b}(\tau) = \sum_{i=0}^{N-1} (-1)^{a(i)-b(i+\tau)}, \quad 0 \leq \tau < N.
\]

(2)

The (even)-periodic autocorrelation function (PACF) of \( a \) is \( R_{a,a}(\tau) \) and denoted by \( R_a(\tau) \). Binary sequences with low PACF magnitudes have been extensively investigated, and are closely related to combinatorial objects (see, for example, a survey [1]). Very recently, there was progress on binary sequences with low OACF magnitudes (see [2],[3],[4]). The lower bound of OACF magnitudes was given by Pott [5]. In [2], the authors give an alternative proof for the lower bound.

*This paper was presented in part at the 2019 Ninth International Workshop on Signal Design and its Applications in Communications (IWSDA), Dongguan, China, Oct. 20-24, 2019. However, the authors misunderstood that the decimation operation preserves the odd-periodic autocorrelation function (OACF) values. The present extended version clarifies this point and gives a complete description of three OACF-preserving operations, which are exactly the counterparts of the three operations preserving periodic autocorrelation function (PACF) values.
The periodic binary sequence $a$ is called odd-optimal if the lower bound in Eq. (3) is met.

Binary sequences with low OACF magnitudes can be derived from those with low PACF magnitudes by Parker’s transformation [6]. The Parker’s transformation is defined as follows: Let $s$ be a binary sequence with length $N$. Define $u = s \parallel (s \oplus 1) = [s_0, s_1, s_2, \ldots, s_{N-1}, s_0 + 1, s_1 + 1, s_2 + 1, \ldots, s_{N-1} + 1]$, where $\parallel$ denotes the concatenation. Notice that for $0 \leq \tau < N$, we have

$$
\mathcal{R}_u(\tau) = \sum_{k=0}^{2N-1} (-1)^{u(k) \cdot u(k+\tau)} = \sum_{k=0}^{N-\tau-1} (-1)^{u(k) \cdot u(k+\tau)} + \sum_{k=N-\tau}^{2N-1} (-1)^{u(k) \cdot u(k+\tau)} + \sum_{k=N}^{2N-\tau-1} (-1)^{u(k) \cdot u(k+\tau)} + \sum_{k=2N-\tau}^{2N-1} (-1)^{u(k) \cdot u(k+\tau)}
$$

$$
= \sum_{k=0}^{N-\tau-1} (-1)^{s(k) \cdot s(k+\tau)} + \sum_{k=N-\tau}^{N-1} (-1)^{s(k) \cdot s(k+\tau)-1} + \sum_{k=N}^{2N-\tau-1} (-1)^{s(k) \cdot s(k+\tau)-1} + \sum_{k=2N-\tau}^{2N-1} (-1)^{s(k) \cdot s(k+\tau)-1}
$$

$$
= 2(\sum_{k=0}^{N-\tau-1} (-1)^{s(k) \cdot s(k+\tau)} + \sum_{k=N-\tau}^{N-1} (-1)^{s(k) \cdot s(k+\tau)-1})
$$

The OACF values of the sequence $s$ can thereby be computed by the PACF values of the sequence $u$ via the following equation.

$$
\mathcal{R}_s(\tau) = \frac{\mathcal{R}_u(\tau)}{2},
$$

where $0 \leq \tau < N$. Thus, instead of constructing binary sequence $s$ with low OACF magnitudes directly, Parker [6], Li and Yang [3] constructed sequences with low OACF magnitudes by first constructing sequences with low PACF magnitudes of the form $u = s \parallel (s \oplus 1)$, and it follows that $s$ has low OACF magnitudes.

In [6], Parker mentioned that negation, cyclic shift and decimation on $u$ can be translated into operations on $s$, and the translated operations preserve the distribution of OACF magnitudes. The transformation $\delta$ on periodic binary sequence is called OACF-preserving if $s$ and $\delta(s')$ have the identical distribution of OACF values, where $s, s'$ are periodic sequences with same period. The new decimation defined in [7] was proved to be OACF-preserving, based on Parker’s transformation. However, the authors misunderstood the three OACF-equivalent operations mentioned by Parker in [6], and claimed this new decimation (here called nega-decimation) constitutes a fourth OACF-preserving operation. In the present paper, we restate the main results in [7], and make it clear that there are three OACF-preserving operations (negation, nega-cyclic shift, nega-decimation). They are exactly the counterparts of the three PACF-preserving operations, i.e., negation, cyclic shift, and decimation. Furthermore, we classify Parker’s 16 cases [6] into 8 ones, based on the equivalence relation defined on the three OACF-preserving operations.

The remainder of the present paper is organized as follows. In Section 2, we define nega-decimation and prove that nega-decimation preserves the OACF values distribution. Moreover, we give an example to show that decimation is not OACF-preserving. In Section 3, by applying nega-decimation, we are able to classify the sequences conjectured by Parker and confirmed by Li and Yang [3] into fewer cases. Finally, Section 4 concludes this paper.
2 Nega-decimation

In this section, we first review the four well-known operations for periodic binary sequences: negation, cyclic shift, nega-cyclic shift and decimation. Next, we show that decimation is in general not OACF-preserving by giving an example. We then define the nega-decimation and prove that nega-decimation is OACF-preserving. Furthermore, by giving an example, it is shown that the nega-decimation operation cannot be obtained by any combination of negation and nega-cyclic shift in general.

We review the definition of negation, cyclic shift, decimation in the following definitions.

Definition 2.1. Let \( a \) be a binary sequence of period \( N \). The negation on \( a \) is defined as

\[
[a(0) + 1, a(1) + 1, \ldots, a(N - 1) + 1].
\]

and denoted by \( a \oplus 1 \) or \( \bar{a} \).

Definition 2.2. Let \( a \) be a binary sequence of period \( N \). The cyclic shift \( \tau \), denoted by \( L^\tau \), is defined as

\[
L^\tau(a) = [a(\tau), a(\tau + 1), \ldots, a(N - 1), a(0), \ldots, a(\tau - 1)],
\]

where \( 0 \leq \tau < N \).

Definition 2.3. Let \( a \) be a binary sequence of period \( N \). For any integer \( d \) with \( \gcd(d, N) = 1 \), the \( d \)-decimation on \( a \) is a mapping that maps \( a \) to \( b \) with \( b(i) = a(di) \) for all integer \( i \), where \( b \) is a binary sequence of period \( N \). The \( d \)-decimation is denoted by \( D_d(\cdot) \).

Definition 2.4. Let \( a \) be a binary sequence of period \( N \). The nega-cyclic shift \( \tau \) or odd shift \( \tau \), which is denoted by \( \hat{L}^\tau(\cdot) \), is defined as

\[
\hat{L}^\tau(a) = [a(\tau), a(\tau + 1), \ldots, a(N - 1), a(0) + 1, \ldots, a(\tau - 1) + 1],
\]

where \( 0 \leq \tau < N \).

By definition, it is routine to check that the cyclic shift, decimation, and negation operations all preserve the PACF value distribution. Likewise, the nega-cyclic shift, and negation operations preserve the OACF value distribution. Notice that decimation is not OACF-preserving. To explain this, we give the following simple example.

Example 2.1. Define \( s_1 \) and \( s_2 \) as:

\[
s_1 = [1, 1, 1, 0, 1, 0, 0, 1, 1, 1, 0]
\]

\[
s_2 = [1, 0, 0, 1, 1, 0, 1, 1, 1, 0],
\]

where \( s_2 \) is the \( 3 \)-decimation version of \( s_1 \). The OACF values of \( s_1 \) are

\[
[10, 0, -2, -4, 2, 0, -2, 4, 2, 0],
\]

the OACF values of \( s_2 \) are

\[
[10, 0, -6, 0, 6, 0, -6, 0, 6, 0].
\]

We now define the nega-decimation.

Definition 2.5. Let \( s \) be a periodic binary sequence with period \( N \). Define \( u = s ||(s \oplus 1) \). Let \( D_d(u) = [u(0), u(1), \ldots, u(2N - 1)] \) for some positive integer \( d \) co-prime with \( 2N \). The \( d \)-decimation on \( s \) induces a binary sequence \( s' \) with period \( [u(0), u(1), \ldots, u(N - 1)] \), and called nega-decimation, denoted by \( \hat{D}(s)_d = s' \).

By Definition 2.5 we have the following equation:

\[
\hat{D}_d(s)(\tau) = s(d\tau) + \left\lfloor \frac{d\tau}{N} \right\rfloor,
\]

where \( \tau = 0, 1, \ldots N - 1 \). Note that Eq. (5) in fact defines the nega-decimation, and is exactly the counterpart of the normal decimation in Definition 2.3. We gave the following example to demonstrate the new decimation operation.

Example 2.2. Let \( s \) be a binary sequence of period 31.

\[
s = [0, 1, 1, 1, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 1, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0, 1, 1, 0, 1, 1, 0, 0, 1, 1, 0, 1, 1, 0, 1, 1, 0, 1, 0, 1, 1].
\]

Then we have
\[ u = s||u = [0, 1, 1, 1, 1, 0, 1, 0, 0, 0, 1, 1, 0, 0, 1, 1, 1, 0, 1, 0, 0, 1, 1, 1, 0, 0, 0, 1, 1, 1, 1, 0, 0, 0, 1, 1, 1, 1, 1, 1, 1, 1, 0, 1, 1, 0, 1, 0, 0, 0] .\]

Applying 3-decimation on \( u \), we have
\[ D_3(u) = [0, 1, 1, 0, 1, 0, 1, 1, 1, 0, 1, 1, 0, 1, 0, 1, 0, 1, 1, 0, 1, 0, 1, 0, 0, 1, 1, 1, 1, 1, 1, 1, 1, 0, 0, 0, 1, 1, 0, 1, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 1, 0, 0, 0, 0, 0, 1, 1, 1, 1, 0, 1, 1, 1] .\]

We truncate the first half to obtain \( s' = D_3(s) \):
\[ s' = [0, 1, 1, 0, 1, 1, 1, 0, 1, 1, 0, 1, 0, 1, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 0, 1, 0, 0, 0, 0] .\]

The OACF values of \( u \) are
\[ [31, -1, -7, -1, -7, -1, -1, -1, -1, -9, 3, 5, 7, 9, -9, -7, -5, -5, -3, -5, 9, -1, -5, 1, 7, 1, 7, 1, -1] ,\]

and the OACF values of \( s' \) are
\[ [31, -7, -1, 1, 3, 9, -5, -5, 1, -1, 1, 1, -1, -5, 1, -7, 7, 5, -1, -7, -1, 1, -5, -9, 5, -9, -3, -1, 1, 7] .\]

Therefore, the OACF distributions of \( s \) and \( s' \) are both
\[ \{3, -9, 7\} , \{(-3)^3, -3, (-1)^6, 3, 5^3, 7^3, 9^2, 31^*\} ,\]

where the exponent denotes the multiplicity in the multiset.

**Theorem 2.1.** The nega-decimation operation is OACF-preserving for periodic binary sequences, and it cannot be obtained by any combination of negation and nega-cyclic shift in general.

To prove this theorem, we need to prove several lemmas. We first give the following property of PACF distribution.

**Lemma 2.2.** Let \( u \) be a binary sequence of period \( 2N \). If the sequence \( u \) is of the form \( u = s||u = s \) for a certain binary sequence \( s \) of length \( N \), then \( R_u(\tau + N) = -R_u(\tau) \) for all \( \tau = 0, 1, \ldots N - 1 \).

**Proof.** By Eq. (2), we have
\[ R_u(\tau + N) = \sum_{i=0}^{2N-1} (-1)^u(i) \cdot u(i + N + \tau) = \sum_{i=0}^{2N-1} (-1)^u(i) \cdot u(i + \tau - 1) = -\sum_{i=0}^{2N-1} (-1)^u(i) \cdot u(i + \tau) = -R_u(\tau) .\]

The result then follows.

The OACF distribution of a periodic sequence has the following property.

**Lemma 2.3.** Let \( a \) be a binary sequence of period \( N \), then \( \hat{R}_a(\tau) = -\hat{R}_a(N - \tau) \) for all \( 1 \leq \tau \leq N - 1 \).

**Proof.** For \( 1 \leq \tau \leq N - 1 \), by Eq. (1), we have
\[ \hat{R}_a(\tau) = \sum_{i=0}^{N-1} (-1)^a(i) \cdot a(i + \tau) = \sum_{i=N-\tau}^{N-1} (-1)^a(i) \cdot a(i + \tau - 1) = \sum_{i=0}^{\tau-1} (-1)^a(i) \cdot a(i + \tau) - \sum_{i=0}^{\tau-1} (-1)^a(i - \tau) \cdot a(i) .\]

Since \( 1 \leq N - \tau \leq N - 1 \), it then follows that
\[ \hat{R}_a(N - \tau) = \sum_{i=0}^{\tau-1} (-1)^a(i) \cdot a(i - \tau) - \sum_{i=0}^{\tau-1} (-1)^a(i + \tau) \cdot a(i) = -\hat{R}_a(\tau) .\]

The proof is then completed.

\[ \square \]
By Lemma 2.2, Lemma 2.3, and Eq. (4), if two sequences \( s = (s \oplus 1) \) and \( s' = (s' \oplus 1) \) have the same PACF value distribution, then \( s \) and \( s' \) have the same OACF value distribution. This result is stated in the following lemma, which will play a key role in proving Theorem 2.1.

**Lemma 2.4.** Let \( u = s||(s \oplus 1) \) for a binary sequence \( s \) with period \( N \). Let \( u' = s'||(s' \oplus 1) \) for a binary sequence \( s' \) of period \( N \). The distribution of OACF values of \( s' \) are identical to that of \( s \) if the distribution of PACF values of \( u \) and \( u' \) are identical.

**Proof.** Define multisets \( S_u \) and \( S_{u'} \) respectively.

By Lemma 2.2 and Eq. (4), we have

\[
S_u = \{ \ast R_u(i) : i = 0, 1, \ldots, 2N - 1 \ast \} \quad \text{and} \quad S_{u'} = \{ \ast R_{u'}(i) : i = 0, 1, \ldots, 2N - 1 \ast \},
\]

When \( N \) is odd, by Lemma 2.3, we have

\[
\{ \ast 2 \hat{R}_a(i) : i = 1, \ldots, N - 1 \ast \} = \{ \ast 2 \hat{R}_a(i) : i = 1, \ldots, (N - 1)/2 \ast \} \cup \{ \ast - 2 \hat{R}_a(i) : i = 1, \ldots, (N - 1)/2 \ast \},
\]

and

It follows that

\[
\{ \ast R_u(i) : i = 1, \ldots, N - 1 \ast \} = \{ \ast R_{u'}(i) : i = 1, \ldots, N - 1 \ast \}
\]

and

\[
\{ \ast R_{u'}(i) : i = 1, \ldots, N - 1 \ast \} = \{ \ast - R_u(i) : i = 1, \ldots, N - 1 \ast \}
\]

Therefore, we have

\[
S_u = \{ \ast 2N, -2N \ast \} \cup \{ \ast R_u(i)^2 : i = 1, \ldots, N - 1 \ast \},
\]

and

\[
S_{u'} = \{ \ast 2N, -2N \ast \} \cup \{ \ast R_{u'}(i)^2 : i = 1, \ldots, N - 1 \ast \},
\]

where the exponent 2 is the multiplicity. The relation \( S_u = S_{u'} \) forces that \( \{ \ast R_u(i) : i = 1, \ldots, N - 1 \ast \} = \{ \ast R_{u'}(i) : i = 1, \ldots, N - 1 \ast \} \). From Eq. (4) and \( \mathcal{R}_u(0) = \mathcal{R}_{u'}(0) = 2N \), it follows that the OACF values of \( s \) are identical to the OACF values of \( s' \).

When \( N \) is even, \( \hat{R}_u(N/2) = -\hat{R}_u(N/2) = 0 \). By Lemma 2.3, we have

\[
\{ \ast 2 \hat{R}_a(i) : i = 1, \ldots, N - 1 \ast \} = \{ \ast 0 \ast \} \cup \{ \ast 2 \hat{R}_a(i) : i = 1, \ldots, N/2 - 1 \ast \} \cup \{ \ast - 2 \hat{R}_a(i) : i = 1, \ldots, N/2 - 1 \ast \},
\]

and

\[
\{ \ast 2 \hat{R}_{u'}(i) : i = 1, \ldots, N - 1 \ast \} = \{ \ast 0 \ast \} \cup \{ \ast 2 \hat{R}_{u'}(i) : i = 1, \ldots, N/2 - 1 \ast \} \cup \{ \ast - 2 \hat{R}_{u'}(i) : i = 1, \ldots, N/2 - 1 \ast \}.
\]
This implies that
\[ \{ sR_u(i) : i = 1, \ldots, N - 1 \} = \{ s - R_u(i) : i = 1, \ldots, N - 1 \}, \]
and
\[ \{ sR_w(i) : i = 1, \ldots, N - 1 \} = \{ s - R_w(i) : i = 1, \ldots, N - 1 \}. \]
The rest of the proof is the same as the case that \( N \) is odd.

Now we are ready to give a proof of Theorem 2.1. The proof of Theorem 2.1 is based on the PACF distribution of \( u \) and Eq. (4).

Proof of Theorem 2.1. Let \( u = s \cdot (s \oplus 1) \), where \( s \) is a periodic binary sequence with period \( N \). For a positive integer \( d \) that coprime with \( 2N \), denote \( D_d(u) \) by \( u' \), we have the following observation:
\[
u'(i + N) = u(di + dN) = u(di) + 1 = u' + 1, 0 \leq i < N.
\]
Thus, the sequence \( u' \) must has the form \( s' \cdot (s' \oplus 1) \) for a \( s' \) with period \( N \). By Lemma 2.4 we know that nega-\( d \)-decimation is OACF preserving.

It remains to show that nega-decimation cannot be obtained by the previously known operations in general. In Example 2.2 by computer search, the sequence \( s' \) cannot be obtained from any combination of nega-cyclic shift, negation. Thus the nega-decimation is indeed a new operation and the proof of Theorem 2.1 is complete. \qed

Remark 2.1. As far as we are concerned, there are three known OACF-preserving operations: negation, nega-cyclic shift and nega-decimation.

3 Classification of Parker’s 16 cases

In [6], Parker constructed two classes of odd-optimal binary sequences of period \( 2p \) and further conjectured 16 classes of binary sequences’ OACF values of period \( 4p \) with \( p \) prime. These conjectures were recently confirmed by Li and Yang [3]. In this section, we first review Parker’s 16 classes of sequences and then classify them into 8 ones by the newly defined nega-decimation operation.

3.1 Parker’s sequences

Let \( p \) be a prime of the form \( p = x^2 + 4y^2 = 4f + 1 \), where \( x, y, f \) are integers. Fix \( \alpha \) as a generator of the finite field of order \( p \), denoted as \( GF(p) \), and the cyclotomic classes of order 4 with respect to \( GF(p) \) are \( \{ D_0, D_1, D_2, D_3 \} \). Define \( S = \{ G \times \{ 0 \} | G \subseteq Z_8 \} \cup \{ \{ n \} \times A_n | n \in Z_8 \} \) where \( A_n = \cup_{k \in I_n} D_k \) with \( I_n \subseteq Z_4 \). By the Chinese Remainder Theorem (CRT), \( Z_8 \times Z_p \) is isomorphic to \( Z_{8p} \). Let \( \eta \) be an isomorphism from \( Z_8 \times Z_p \) to \( Z_{8p} \) and \( N = 4p \). The characteristic sequence \( u \) of \( \eta(S) \) satisfies \( u(i) = u(i + N) + 1 \) for \( i = 0, 1, \ldots, N - 1 \) if we require that \( A_{n+4} = \cup_{k \in I_{n+4}} D_k \) and \( j + 4 \in (\infty)G \) for all \( j \notin (\infty)G \). Thus, \( u \) is completely described by \( G', A_0, A_1, A_2, A_3 \), where \( G' = \{ g \in G, g \leq 4 \} \). The OACF values of Parker’s 16 classes of sequences were confirmed by Li and Yang [3] via an interleaving argument, and the results are listed in Tables 1 and 2. Each class of sequences is denoted by \( s_i \), for \( i = 1, 2, \ldots, 16 \), and the support of \( s_i \cdot (s_i \oplus 1) \) is determined by \( G' \) and \( \gamma \), where \( \gamma = A_0, A_1, A_2, A_3 \). The sets \( C_i \)’s for \( i = 1, 2, \ldots, 6 \) are defined as follows:

| Notation | \( G' \) | \( \gamma \) | OACF values | \( f \) |
|----------|----------|-------------|-------------|------|
| \( s_1 \) | \{2\} | \( C_3, C_1, C_1, C_1 \) | \( \{0, \pm 2, \pm 2x + 4y\} \) | even |
| \( s_2 \) | \{0, 1, 2\} | \( C_4, C_2, C_1, C_1 \) | \( \{0, \pm 2, \pm 2x + 4y\} \) | even |
| \( s_3 \) | \{3\} | \( C_6, C_1, C_4, C_4 \) | \( \{0, \pm 2, \pm 2x + 4y\} \) | even |
| \( s_4 \) | \{0, 1, 3\} | \( C_1, C_6, C_4, C_4 \) | \( \{0, \pm 2, \pm 2x + 4y\} \) | even |

Table 1: Construction 1
We give a proof of the case that \( s_4 \) can be obtained by \( s_1 \) via a combination of the negation and nega-decimation. The other 14 Parker’s sequences can be handled in a similar way, and the proof is thus omitted.

**Theorem 3.1.** Let \( s_1, s_4 \) be the sequences defined in Table 2. Then the sequence \( s_4 \) can be obtained from \( s_1 \) via a combination of negation, and nega-decimation. More precisely, \( s_4 = D_{\phi^{-1}(1, \alpha^3)}(s_1 \oplus 1) \), where \( \alpha \) is the fixed generator in \( GF(p) \) and \( \phi \) is an isomorphism from \( \mathbb{Z}_{sp} \) to \( \mathbb{Z}_8 \times \mathbb{Z}_p \) that maps \( x \) to \( (x \mod 8, x \mod p) \).

**Proof.** Define \( u_1 = s_1 || (s_1 \oplus 1) \) and \( u_4 = s_4 || (s_4 \oplus 1) \). Let \( S_{u_1}, S_{u_2} \) be the supports of \( u_1, u_2 \) respectively, and

\[
\phi(S_{u_1}) = C_{u_1} = \{(2, 0), (4, 0), (5, 0), (7, 0)\}
\]

\[
\cup (0, C_3) \cup (1, C_4) \cup (2, C_1) \cup (3, C_1)
\]

\[
\cup (4, C_4) \cup (5, C_3) \cup (6, C_6) \cup (7, C_6),
\]

\[
\phi(S_{u_4}) = C_{u_4} = \{(0, 0), (1, 0), (3, 0), (6, 0)\}
\]

\[
\cup (0, C_1) \cup (1, C_6) \cup (2, C_4) \cup (3, C_4)
\]

\[
\cup (4, C_6) \cup (5, C_1) \cup (6, C_3) \cup (7, C_3).
\]

Firstly, we apply negation on \( u_1 \). Then the support of \( u_1 \oplus 1 \) that is denoted by \( S_{u_1 \oplus 1} \) satisfies

\[
\phi(S_{u_1 \oplus 1}) = \mathbb{Z}_8 \times \mathbb{Z}_p \setminus C_{u_1} = \{(0, 0), (1, 0), (3, 0), (6, 0)\}
\]

\[
\cup (0, C_4) \cup (1, C_5) \cup (2, C_6) \cup (3, C_6)
\]

\[
\cup (4, C_3) \cup (5, C_4) \cup (6, C_1) \cup (7, C_1).
\]
We then apply the $d$-decimation on $u_1 \oplus 1$, where $\phi(d) = (1, \alpha^3)$. The support of $D_d(u_1 \oplus 1)$ that is denoted by $S_{D_d(u_1 \oplus 1)}$ satisfies

$$
\phi(S_{D_d(u_1 \oplus 1)}) = (1, \alpha^3) \cdot \phi(S_{u_1 \oplus 1}) = \{(1 \times 0, \alpha^3 \cdot 0), (1 \times 1, \alpha^3 \cdot 0), (1 \times 3, \alpha^3 \cdot 0), (1 \times 6, \alpha^3 \cdot 0)\}
\cup (1 \times 0, \alpha^3 \cdot C_4) \cup (1 \times 1, \alpha^3 \cdot C_3) \cup (1 \times 2, \alpha^3 \cdot C_6) \cup (1 \times 3, \alpha^3 \cdot C_6) \cup (1 \times 4, \alpha^3 \cdot C_5) \cup (1 \times 5, \alpha^3 \cdot C_4) \cup (1 \times 6, \alpha^3 \cdot C_1) \cup (1 \times 7, \alpha^3 \cdot C_1) = \{(0, 0), (1, 0), (3, 0), (6, 0)\} \cup (0, C_1) \cup (1, C_6) \cup (2, C_4) \cup (3, C_4) \cup (4, C_6) \cup (5, C_1) \cup (6, C_3) \cup (7, C_3) = C_{u_4}.
$$

This implies that $S_{D_d(u_1 \oplus 1)} = S_{u_4}$ since $\phi$ is bijective. Therefore, $s_4 = D_{\phi^{-1}(1, \alpha^3)}(s_1 \oplus 1)$. This means that the sequence $s_4$ can be obtained from $s_1$ via a combination of negation and nega-decimation.

**Remark 3.1.** The interleaving-based construction in [3] can be represented by the subsets in $\mathbb{Z}_8 \times \mathbb{Z}_p$ via an isomorphism. The subset has the same format as $C_{u_4}$ in the proof of Theorem 3.7. Thus, we can classify the interleaving based construction by the method in Theorem 3.7.

By applying the same method in Theorem 3.1 we can classify the 16 classes of binary sequences by Parker into 8 classes in Table 4 where $\phi$ and $\alpha$ are defined as in Theorem 3.1.

| Class Index | Sequences | Relation |
|-------------|-----------|----------|
| 1           | $s_1, s_4$| $s_4 = D_{\phi^{-1}(1, \alpha^3)}(s_1 \oplus 1)$ |
| 2           | $s_2, s_3$| $s_3 = D_{\phi^{-1}(1, \alpha^3)}(s_2 \oplus 1)$ |
| 3           | $s_5, s_8$| $s_8 = D_{\phi^{-1}(1, \alpha^3)}(s_5 \oplus 1)$ |
| 4           | $s_6, s_7$| $s_7 = D_{\phi^{-1}(1, \alpha^3)}(s_6 \oplus 1)$ |
| 5           | $s_9, s_{12}$| $s_{12} = D_{\phi^{-1}(1, \alpha^3)}(s_9)$ |
| 6           | $s_{10}, s_{11}$| $s_{11} = D_{\phi^{-1}(1, \alpha^3)}(s_{10})$ |
| 7           | $s_{13}, s_{16}$| $s_{16} = D_{\phi^{-1}(1, \alpha^3)}(s_{13})$ |
| 8           | $s_{14}, s_{15}$| $s_{15} = D_{\phi^{-1}(1, \alpha^3)}(s_{14})$ |

**4 Conclusion**

In this paper, we defined a new OACF-preserving operation called nega-decimation. The nega-decimation operation enables us to classify the 16 Parker’s classes of sequences into 8 ones. The interleaving sequences defined by Li and Yang [3] can also be classified by the new operation. Moreover, we may possibly be able to fully classify all constructions of binary sequences based on Parker’s transformation. This may constitute one of future directions.

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