ON THE CHIRAL RINGS IN $N = 2$ AND $N = 4$
SUPERCONFORMAL ALGEBRAS

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Abstract
We study the chiral primary rings of $N = 2$ and $N = 4$ superconformal algebras (SCA) constructed over triple systems. The chiral primary states of $N = 2$ SCA’s realized over hermitian Jordan triple systems are given. Their coset spaces $G/H$ are hermitian symmetric which can be compact or non-compact. In the non-compact case under the requirement of unitarity of the representations of $G$ we find an infinite discrete set of chiral primary states associated with the holomorphic discrete series representations of $G$ and their analytic continuation. Further requirement that the corresponding $N = 2$ module be unitary truncates this infinite set to a finite subset. There are no chiral primary states associated with the other unitary representations of non-compact groups. Remarkably, the only non-compact groups $G$ that admit holomorphic discrete series unitary representations are such that their quotients $G/H$ with their maximal compact subgroups $H$ are hermitian symmetric. The chiral primary states of $N = 2$ SCA’s constructed over the Freudenthal triple systems are also studied. These algebras have the special property that they admit an extension to $N = 4$ superconformal algebras with the gauge group $SU(2) \otimes SU(2) \otimes U(1)$. We then generalize the concept of chiral rings to these maximal $N = 4$ superconformal algebras. We find four different rings associated with each sector (left or right moving). If one includes both sectors one gets 16 different rings. We also show that our analysis yields all the possible rings of $N = 4$ SCA’s.

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1 Introduction

Extended superconformal algebras have been studied extensively in recent years. They find fundamental applications in superstring theories \cite{1}, integrable systems \cite{2}, topological field theories \cite{3} and in the study of critical phenomena. For example the classical vacua of string theories are described by conformal field theories. In particular, the heterotic string vacua with \( N = 1 \) space-time supersymmetry in four dimensions are described by "internal" \( N = 2 \) superconformal field theories with central charge \( c = 9 \) \cite{4} which have been studied in \cite{5, 6, 7, 8, 9} using coset space methods \cite{10}. 

On the other hand, the \( N = 2 \) space-time supersymmetric vacua of the heterotic string are described by an internal superconformal field theory with four supersymmetries \cite{11}. 

The rings of chiral primary operators \cite{6} play a central role in the understanding and applications of \( N = 2 \) superconformal field theories. In \cite{6} a deep connection between the cohomology rings of Calabi-Yau manifolds and the rings of chiral primary fields of \( N = 2 \) superconformal theories with \( c = 9 \) was established. The existence of two different rings associated with a given (2,2) superconformal theory led to the idea of mirror symmetry of Calabi-Yau manifolds and its natural generalization to other complex manifolds \cite{12}. Furthermore, the chiral primary states of \( N = 2 \) superconformal algebras turn out to be the only physical states of topological field theories that are obtained by the twisting of the corresponding \( \tilde{N} = 2 \) superconformal theories \cite{13}. In this paper we shall study the rings of chiral primary operators of extended superconformal algebras (\( N = 2 \) and \( N = 4 \)). The chiral rings of \( N = 2 \) superconformal algebras (SCA) were introduced and studied in a Cartan-Weyl basis of the underlying current algebras in \cite{16}. Our treatment of the chiral rings of of \( N = 2 \) SCA’s will use their realization over triple systems \cite{14, 15, 16}. After reviewing the necessary background we study the chiral (anti-chiral) primary operators of \( N = 2 \) SCA’s constructed over hermitian Jordan triple systems which correspond to their realization over hermitian symmetric spaces \( G/H \) that may be compact or non-compact. We give a complete characterization of the chiral primary states both in the compact and the non-compact cases. In the non-compact case under the requirement of unitarity of the representations of \( G \) at every level we find an infinite discrete set of chiral (anti-chiral) primary states associated with the holomorphic (anti-holomorphic) discrete series representations of \( G \) and their analytic continuation. However, the requirement of unitarity of the \( N = 2 \) module truncates this infinite discrete set to a finite subset. The uni-
tary representations of the noncompact groups outside the holomorphic (or anti-
holomorphic) discrete series representations and their analytic continuation do not lead to any non-trivial chiral (or anti-chiral) primary states. Remarkably, the only non-compact groups $G$ that admit holomorphic (anti-holomorphic) discrete series representations are those whose quotient $G/H$ with their maximal compact subgroups $H$ are hermitian symmetric. A finite set of purely “fermionic” chiral primary states exist in both the compact and the non-compact cases. Next we study the chiral primary states in $N = 2$ SCA’s constructed over Freudenthal triple systems (FTS). The $N = 2$ SCA’s constructed over FTS’s have the very special property that they allow an extension to $N = 4$ SCA’s with the gauge group $SU(2) \otimes SU(2) \otimes U(1)$ \cite{15,16}. In the second part of the paper we generalize the concept of chiral (anti-chiral) rings to these maximal $N = 4$ SCA’s. We find that the natural extension of the concept of a chiral ring to $N = 4$ SCA’s leads to four different rings associated with each sector (left or right moving). If we include both sectors we find sixteen different rings. We also show that our analysis yields all the possible rings of $N = 4$ SCA’s. Throughout the paper we use the Neveu-Schwarz moding. However, our results can easily be carried over to the Ramond moding using spectral flow.

2 Construction of $N = 2$ Superconformal Algebras over Jordan Triple Systems

In this section we shall review the construction of $N = 2$ superconformal over Jordan triple systems \cite{14} and reformulate it in such a way as to make it easier to relate to the coset space methods that use the Cartan-Weyl basis for Lie algebras. Consider a 3-graded Lie algebra $g$:

$$g = g^{-1} \oplus g^0 \oplus g^1$$  \hspace{1cm} (2 - 1)

where $\oplus$ denotes vector space direct sum and $g^0$ is a subalgebra of maximal rank. We have the formal commutation relations of the elements of various grade subspaces

$$[g^m, g^n] \subseteq g^{m+n} ; m,n = -1, 0, 1$$ \hspace{1cm} (2 - 2)

where $g^{m+n} = 0$ if $|m + n| > 1$. Every simple Lie algebra with such a 3-graded structure can be constructed in terms of an underlying Jordan triple system (JTS) $V$ via the Tits-Kantor-Koecher (TKK) construction \cite{17}. This construction establishes a one-to-one mapping between the grade
+1 subspace of \( g \) and the underlying JTS \( V \):

\[
U_a \in g \iff a \in V
\]  

(2 - 3)

Every such Lie algebra \( g \) admits a conjugation (involutive automorphism) \( \dagger \) under which the elements of the grade +1 subspace get mapped into the elements of the grade −1 subspace.

\[
U^a = U^\dagger_a \in g^{-1}
\]  

(2 - 4)

One then defines

\[
[U_a, U^b] = S^b_a
\]  

(2 - 5)

\[
[S^b_a, U_c] = U_{(abc)}
\]

where \( S^b_a \in g^0 \) and \( (abc) \) is a triple product under which the elements of \( V \) close. Under conjugation \( \dagger \) one finds

\[
(S^b_a)^\dagger = S^a_b
\]

(2 - 6)

\[
[S^b_a, U^c] = -U^{(bac)}
\]

The Jacobi identities in \( g \) are satisfied if and only if the ternary product \( (abc) \) satisfies the defining identities of a JTS:

\[
(abc) = (cba)
\]

(2 - 7)

\[
(ab(cx)) - (cd(ux)) - (a(dch)x) + ((ca)x) = 0
\]

The elements \( S^b_a \) of the grade zero subspace form a subalgebra which is called the structure algebra of \( V \):

\[
[S^b_a, S^d_c] = S^d_{(abc)} - S^b_{(bad)} = S^{(db)}_a - S^b_{(cda)}
\]  

(2 - 8)

Consider now the affine Lie algebra (current algebra) \( \hat{g} \) defined by \( g \). The commutation relations of \( \hat{g} \) can be written as operator products in the basis defined by the underlying JTS:

\[
U_a(z)U^b(w) = \frac{kS^b_a}{(z-w)^2} + \frac{1}{(z-w)}S^b_a(w) + \cdots
\]

\[
S^b_a(z)U_c(w) = \frac{1}{(z-w)}U_{(abc)} + \cdots
\]

\[
S^b_a(z)U^c(w) = \frac{-1}{(z-w)}U^{(bac)} + \cdots
\]

(2 - 9)

\[
S^b_a(z)S^d_c(w) = \frac{kS^d_{(abc)}}{(z-w)^2} + \frac{1}{(z-w)}\{S^d_{(abc)} - S^b_{(bad)}\}(w) + \cdots
\]
where $k$ is the level of $\hat{g}$ and $\Sigma_{ac}^{bd}$ are the structure constants of the JTS $V$:

$$U_{(abc)} = \Sigma_{ac}^{bd} U_d$$  \hspace{1cm} (2 - 10)

We choose a basis for the JTS such that the structure constants satisfy

$$\Sigma_{ae}^{bf} \Sigma_{fc}^{ed} = \hat{g} \Sigma_{ac}^{bd}$$

$$\Sigma_{ac}^{bc} = \hat{g} \delta_a^b$$  \hspace{1cm} (2 - 11)

where $\hat{g}$ is the dual coxeter number of the Lie algebra $g$. We introduce complex Fermi fields labelled by the elements of $V$ that satisfy

$$\psi_a(z) \psi^b(w) = \frac{\delta_b^a}{(z-w)} + \cdots$$  \hspace{1cm} (2 - 12)

$$\psi_a(z) \bar{\psi}_b(w) = \bar{\psi}^a(z) \psi^b(w) = 0 + \cdots$$

The supersymmetry generators defined by the following bilinears of the fermions and the currents labelled by the elements of $V$

$$G(z) = \sqrt{\frac{2}{(k+\hat{g})}} U_a \psi^a(z)$$

$$\bar{G}(z) = \sqrt{\frac{2}{(k+\hat{g})}} U^a \bar{\psi}_a(z)$$  \hspace{1cm} (2 - 13)

generate an $N = 2$ SCA

$$G(z) \bar{G}(w) = \frac{2c}{(z-w)^2} + \frac{2J(w)}{(z-w)^2} + \frac{2T(w) + \partial J(w)}{(z-w)} + \cdots$$  \hspace{1cm} (2 - 14)

with central charge

$$c = \frac{3kD}{(k+\hat{g})}$$  \hspace{1cm} (2 - 15)

where $D$ is the dimension of the JTS $V$. The Virasoro generator $T(z)$ and the U(1) current $J(z)$ of the $N = 2$ SCA are given by

$$T(z) = \left\{ \frac{1}{(k+\hat{g})} \left\{ \frac{1}{2} (U^a U_a + U_a U^a) - \frac{k}{2} (\psi_a \partial \psi^a + \bar{\psi}^a \partial \bar{\psi}_a) + S_a^b \psi^a \bar{\psi}_b \right\} \right\} (z)$$

$$J(z) = \left\{ \frac{1}{(k+\hat{g})} \left\{ S_a^a + k \psi^a \bar{\psi}_a \right\} \right\} (z)$$  \hspace{1cm} (2 - 16)

\(^1\)All local composite operators with a single argument are assumed to be normal ordered. Our conventions for normal ordering are the same as those of reference \[18\].
Note that $S_a^b$ is the trace of the structure algebra of $V$.

If we denote the groups generated by the Lie algebra $g$ and its subalgebra $g^0$ as $G$ and $H$, respectively, then it can be shown that the above realization of $N = 2$ SCA’s over JTS’s is equivalent to their construction over the symmetric spaces $G/H$ [14]. A JTS is called hermitian if it can be given the structure of a complex vector space such that the triple product $(abc)$ is linear in the first and the last arguments and anti-linear in the second argument. The coset spaces $G/H$ corresponding to hermitian JTS’s are hermitian symmetric spaces. The complete list of simple hermitian JTS’s includes four infinite families and two exceptional ones [19]. In the above we implicitly assumed that we are working with JTS’s underlying compact Lie groups. To obtain the realizations of $N = 2$ SCA’s corresponding to non-compact symmetric spaces $G/H$ where $H$ is the maximal compact subgroup of $G$ one needs simply to replace the ternary product $(abc)$ of the compact case by its negative $-(abc)$. This introduces an overall minus sign in the signatures of the Killing metric associated with the coset space generators. The central term of the current algebra $\hat{g}$ must then be taken as $-k$ times the Killing metric in order for the contribution to the central charge from the non-compact generators to be positive for positive $k$. This is then equivalent to replacing the dual coxeter number $\tilde{g}$ by its negative in the expression for the central charge. Therefore the central charge for the non-compact realization of $N = 2$ SCA’s is [8, 9, 14]

$$c = \frac{3kD}{(k - \tilde{g})} \quad (2 - 17)$$

The realization of $N = 2$ superconformal algebras over non-compact coset spaces were studied in [8, 9] and the cohomology of non-compact coset models in [20]. Let us now reformulate the above construction so as to relate it to the one given by Kazama and Suzuki [7]. We first note that the the quadratic Casimir operator of the compact Lie algebra constructed over a JTS, in our normalization, is given by

$$C_2 = \frac{1}{2}(U_aU^a + U^aU_a) + \frac{1}{2\tilde{g}}S_a^bS_b^a \quad (2 - 18)$$

Again to obtain the Casimir operator of the non-compact Lie algebra with maximal compact subalgebra $g^0$ one needs simply to replace the dual Coxeter number $\tilde{g}$ by its negative. The following fermion bilinears

$$\tilde{S}^b_a = -\Sigma^b_{ac}S^c_d(z) \quad (2 - 19)$$
generate the current algebra isomorphic to \( \hat{g}^0 \)

\[
\tilde{S}_a^b(z)\tilde{S}_c^d(w) = \frac{\hat{g} \Sigma_{ac}^{bd}}{(z - w)^2} + \frac{\tilde{S}_d^{(ab)}(w)}{(z - w)} - \frac{\tilde{S}_c^{(bd)}(w)}{(z - w)} + \ldots \tag{2 - 20}
\]

By the symmetric space theorem of [21] one can rewrite the bilinears in the currents \( \tilde{S}_a^b \) as bilinears in the fermions

\[
\frac{1}{\hat{g}} \tilde{S}_a^b \tilde{S}_b^a(w) = -\hat{g}(\bar{\psi}^a \partial \psi_a + \psi_a \partial \bar{\psi}^a)(w) \tag{2 - 21}
\]

where the factor \( \frac{1}{\hat{g}} \) on the left hand side follows from our normalization of the generators. Using these results we can write the term \( S_a^b \bar{\psi}^a \psi_b^a(z) \) that appears in the expression for \( T(z) \) as

\[
S_a^b \bar{\psi}^a \psi_b^a(z) = -\frac{1}{2\hat{g}}(S_a^b + \tilde{S}_a^b)(S_b^a + \tilde{S}_b^a)(z)
+ \frac{1}{2\hat{g}} S_a^b \bar{S}_b^a(z) - \frac{\hat{g}}{2} (\bar{\psi}^a \partial \psi_a + \psi_a \partial \bar{\psi}^a)(z) \tag{2 - 22}
\]

where we used the identity

\[
\Sigma_{ac}^{bd} \Sigma_{ad}^c = \hat{g} S_a^b \tag{2 - 23}
\]

Substituting this in the expression for \( T(z) \) we obtain

\[
T(z) = \frac{1}{(k + g)} \left\{ \frac{1}{2}(U_a U^a + U^a U_a) + \frac{1}{2\hat{g}} S_a^b S_b^a \right\}(z)
- \frac{1}{(k + g)} \left\{ (S_a^b + \tilde{S}_a^b)(S_b^a + \tilde{S}_b^a) \right\}(z)
- \frac{1}{2} \{ \bar{\psi}^a \partial \psi_a + \psi_a \partial \bar{\psi}^a \}(z) \tag{2 - 24}
\]

which shows that it is precisely the Virasoro generator associated with the coset space

\[
\frac{G_k \times SO(2D)}{H_{k+g-h}} \tag{2 - 25}
\]

, thus agreeing with Kazama and Suzuki [7].
3 Chiral Primary States of $N = 2$ Superconformal Algebras realized over Hermitian Jordan Triple Systems

In the Neveu-Schwarz moding the generators of $N = 2$ SCA’s realized over the Hermitian Jordan triple systems are given by

$$G_s = \sqrt{\frac{2}{k+g}} \sum_n U_{an} \psi_s^a_{s-n}$$

$$\bar{G}_s = \sqrt{\frac{2}{k+g}} \sum_n U_{a(s-n)}^a \psi_{a(s-n)}$$

$$J_n = \frac{1}{k+g} \left\{ S_{an}^a + k \sum_s : \psi_{n-s}^a \psi_{as} : \right\}$$

$$T_n = \frac{1}{k+g} \left\{ \frac{1}{2} \sum_m : (U_{(n-m)}^a U_{am} + U_{a(n-m)} U_{m}) : \
+ \frac{k}{2} \sum_s (s + \frac{1}{2}) : (\psi_{a(n-s)} \psi_{s}^a + \psi_{(n-s)} \psi_{as}) : + \sum_{m,s} \sum_{m,n,...} S_{b(m-n)} \psi_{bs} : \right\}$$

$$m, n, ... = 0, \pm 1, \pm 2, ...$$

$$r, s, ... = \pm \frac{1}{2}, \pm \frac{3}{2}, ...$$

(3 - 1)

The primary states $|h, q\rangle$ of the $N = 2$ SCA are defined by the conditions

$$T_0 |h, q\rangle = h |h, q\rangle$$

$$J_0 |h, q\rangle = q |h, q\rangle$$

$$T_n |h, q\rangle = J_n |h, q\rangle = 0, \ n > 0$$

$$G_r |h, q\rangle = \bar{G}_r |h, q\rangle = 0, \ r > 0$$

(3 - 2)

The chiral primary states satisfy the additional condition

$$G_{-\frac{1}{2}} |h, q\rangle = 0$$

(3 - 3)

which implies that

$$h = \frac{q}{2}$$

(3 - 4)
On the other hand, the anti-chiral primary states satisfy

\[ G^{-1/2}_{-\frac{1}{2}}|h, q\rangle = 0 \iff h = -\frac{q}{2} \]  

(3 - 5)

In order to identify the chiral and anti-chiral primary states let us write the operators \( T_0, J_0, G^{\pm 1/2}_+, G^{\pm 1/2}_- \) in a more explicit form

\[
T_0 = \frac{1}{(k+g)} \left\{ \frac{1}{2} \sum_m : (U^a_{-m} U^{a}_{m} + U^a_{-m} U^{a}_{m}) : \right. \\
+ k \sum_{s>0} s : (\psi^a_{-s} \psi^a_{s} + \psi^a_{a(-s)} \psi^a_{s}) : + \sum_{m,s} S^b_{a(-m)} : \psi^a_{(m-s)} \psi^a_{bs} : \left. \right\}
\]

\[
J_0 = \frac{1}{k+g} \left\{ \sum_{s>0} s : (\psi^a_{-s} \psi^a_{s} - \psi^a_{a(-s)} \psi^a_{s}) : \right\}
\]

\[
G^{\pm 1/2}_+ = \sqrt{2/(k+g)} \left\{ U^a_{0} \psi^a_{\pm 1/2} + \sum_{n>0} (U^{a}_{n} \psi^a_{\pm 1/2 - n} + U^{a}_{a(-n)} \psi^a_{\pm 1/2 + n}) \right\}
\]

\[
\bar{G}^{\pm 1/2}_- = \sqrt{2/(k+g)} \left\{ U^a_{0} \psi^a_{a(\pm 1/2)} + \sum_{n>0} (U^{a}_{n} \psi^a_{a(\pm 1/2 - n)} + U^{a}_{a(-n)} \psi^a_{a(\pm 1/2 + n)}) \right\}
\]

(3 - 6)

As is standard in the literature we shall use the term “vacuum representation of \( \hat{g} \)” to refer to the states belonging to a highest weight representation that are annihilated by all the positive mode generators of \( \hat{g} \) and that form a representation of \( g \). The components \(|\Omega\rangle\) of the vacuum representation of \( \hat{g} \) that are annihilated by the grade +1 operators \( U_{a0} \) are chiral primary since

\[
T_0 |\Omega\rangle = \frac{1}{2(k+g)} S^a_{a0} |\Omega\rangle = \frac{p}{2(k+g)} |\Omega\rangle
\]

(3 - 7)

\[
J_0 |\Omega\rangle = \frac{1}{(k+g)} S^a_{a0} |\Omega\rangle = \frac{p}{(k+g)} |\Omega\rangle
\]

where \( p \) is the eigenvalue of the trace current \( S^a_{a0} \) of the structure algebra on the state \(|\Omega\rangle\). Let \(|0\rangle\) denote the tensor product of the Fock vacuum with the scalar vacuum representation of \( \hat{g} \). It satisfies

\[
\psi^a_{r}|0\rangle = \psi^{a}_{ar}|0\rangle = 0, \; r > 0
\]

\[
U_{an}|0\rangle = U^{a}_{n}|0\rangle = 0, \; n \geq 0
\]

(3 - 8)

Note that the generators of the \( N = 2 \) superconformal algebra act on the tensor product of the Fock space of the fermions and the representation space of the affine Lie algebra \( \hat{g} \). Therefore the state \(|\Omega\rangle\) represents the tensor product of the Fock vacuum with the state \( \Omega \) of the vacuum representation of \( \hat{g} \).
Then the states of the form
\[ |a_1, ..., a_N \rangle \equiv \psi_{a_1}^{\frac{1}{2}} \cdots \psi_{a_N}^{\frac{1}{2}} |0\rangle \]  
(3 - 9)
are all chiral primary. The eigenvalues of \( T_0 \) and \( J_0 \) are
\[ T_0 |a_1, ..., a_N \rangle = \frac{kN}{2(k+\bar{g})} |a_1, ..., a_N \rangle \]  
\[ J_0 |a_1, ..., a_N \rangle = \frac{\bar{N}}{(k+\bar{g})} |a_1, ..., a_N \rangle \]  
(3 - 10)
As for the anti-chiral primary states they are of the form
\[ |\bar{a}_1, ..., \bar{a}_N \rangle \equiv \psi_{\bar{a}_1}^{-\frac{1}{2}} \cdots \psi_{\bar{a}_N}^{-\frac{1}{2}} |0\rangle \]  
(3 - 11)
or of the form \( |\bar{\Omega} \rangle \) which are states belonging to the vacuum representation of \( \bar{g} \) annihilated by the grade \(-1\) generators \( U^{a0} \)
\[ U^{a0} |\bar{\Omega} \rangle = 0 \]  
(3 - 12)
For the anti-chiral primary states we find
\[ T_0 |\bar{\Omega} \rangle = \frac{-1}{2(k+\bar{g})} S_a^{a0} |\bar{\Omega} \rangle = \frac{-\bar{p}}{2(k+\bar{g})} |\bar{\Omega} \rangle \]  
\[ J_0 |\bar{\Omega} \rangle = \frac{\bar{p}}{2(k+\bar{g})} |\bar{\Omega} \rangle \]  
(3 - 13)
\[ T_0 |\bar{a}_1, ..., \bar{a}_N \rangle = \frac{kN}{(k+\bar{g})} |\bar{a}_1, ..., \bar{a}_N \rangle \]  
\[ J_0 |\bar{a}_1, ..., \bar{a}_N \rangle = \frac{\bar{N}}{(k+\bar{g})} |\bar{a}_1, ..., \bar{a}_N \rangle \]
Note that the eigenvalues of the trace operator \( S_a^{a0} \) on the states \( |\Omega \rangle \) and \( |\bar{\Omega} \rangle \) are non-negative and non-positive, respectively.

The unique chiral and anti-chiral primary states with \( h = \frac{c}{6} \) are obtained by the action of \( D \) copies of the fermion operators \( \psi_{-\frac{1}{2}} \) and \( \psi_{a(-\frac{1}{2})} \) on the Fock vacuum
\[ |h = \frac{kD}{2(k+\bar{g})}, q = \frac{kD}{(k+\bar{g})} \rangle = |a_1, ..., a_D \rangle \]  
(3 - 14)
\[ |h = \frac{kD}{2(k+\bar{g})}, q = -\frac{kD}{(k+\bar{g})} \rangle = |\bar{a}_1, ..., \bar{a}_D \rangle \]
and are annihilated by \( G_{-\frac{1}{2}} \) and \( \bar{G}_{-\frac{1}{2}} \), respectively. The additional chiral primary states are obtained by tensoring the anti-chiral primary states \( |\bar{\Omega} \rangle \) with the unique chiral primary state \( |a_1, ..., a_D \rangle \)
\[ |\bar{\Omega}, a_1, ..., a_D \rangle \equiv |\bar{\Omega} \rangle \otimes |a_1, ..., a_D \rangle \]  
(3 - 15)
They satisfy

\[ T_0 |\Omega, a_1, ..., a_D \rangle = \frac{1}{2(k+g)} \{ S_{a0}^a + kD \} |\Omega, a_1, ..., a_D \rangle = \frac{(kD-\bar{p})}{2(k+g)} |\Omega, a_1, ..., a_D \rangle \]

\[ J_0 |\Omega, a_1, ..., a_D \rangle = \frac{1}{(k+g)} \{ S_{a0}^a + kD \} |\Omega, a_1, ..., a_D \rangle = \frac{(kD-\bar{p})}{(k+g)} |\Omega, a_1, ..., a_D \rangle \]

where we used the fact that

\[ \psi_a - \frac{1}{2} \psi_b(\frac{1}{2}) |a_1, ..., a_D \rangle = \delta_0^a |a_1, ..., a_D \rangle \]  

(3 - 17)

The corresponding anti-chiral primary states are obtained by tensoring the states \(|\Omega\rangle\) with the unique anti-chiral primary state \(|\bar{\Omega}, \bar{a}_1, ..., \bar{a}_D \rangle\) which satisfy

\[ T_0 |\bar{\Omega}, \bar{a}_1, ..., \bar{a}_D \rangle = \frac{(kD-\bar{p})}{2(k+g)} |\bar{\Omega}, \bar{a}_1, ..., \bar{a}_D \rangle \]

\[ J_0 |\bar{\Omega}, \bar{a}_1, ..., \bar{a}_D \rangle = \frac{(p-kD)}{(k+g)} |\bar{\Omega}, \bar{a}_1, ..., \bar{a}_D \rangle \]  

(3 - 18)

Let us now try to extend the above results to the case when the affine Lie algebra \(\hat{g}\) is non-compact. It is known that the non-compact affine Lie algebras do not admit any non-trivial unitary representations of the highest weight type. The only known method for constructing unitary theories over them appears to be the coset space method [10, 8, 9, 20]. For non-compact groups \(G\) the relevant coset space is \(G/H\) where \(H\) is the maximal compact subgroup of \(G\). However, all such coset spaces are symmetric spaces. Furthermore, the requirement that the unitary realization of the Virasoro algebra admit extension to \(N = 2\) supersymmetry implies that \(G/H\) must be Kaehlerian. The irreducible non-compact Kaehlerian symmetric spaces are precisely the non-compact Hermitian symmetric spaces [22]. Therefore in the non-compact case we need only to restrict ourselves to the \(N = 2\) SCA’s realized over the Hermitian Jordan triple systems. The formalism we developed is very well suited for studying the non-compact case in complete analogy with the compact case. Of course there are many novelties and subtleties that have to be taken into account in the non-compact case. We shall start our discussion by assuming that the only representations of \(G\) of physical relevance are unitary representations. This is a necessary condition for a unitary realization of the superconformal algebra. However, it is by no means sufficient. In fact, it is not even sufficient for unitarity of the representations of the affine Lie algebra \(\hat{g}\). Now for non-compact \(G\) the generators \(U_a\) and \(\bar{U}^a\) of grade +1 and −1 are the non-compact generators.
As was apparent in the discussion of the compact case a large class of chiral (anti-chiral) primary states were defined by those states in the vacuum representation of \( \hat{g} \) that are annihilated by the grade +1 \((-1)\) operators \( U_{a0}(U^{a0}) \). By our assumption the vacuum representation of \( \hat{g} \) is a unitary representation of \( g \). However, in general a unitary representation of a non-compact Lie algebra \( g \) does not contain any states that are annihilated by all the \( U_{a0} \) or all the \( U^{a0} \). This happens precisely when the Lie algebra \( g \) (or rather the non-compact group \( G \) corresponding to it) admits holomorphic or anti-holomorphic unitary representations belonging to the discrete series or its analytic continuation [23]. Remarkably, the necessary and sufficient condition for the existence of such representations is that the quotient space \( G/H \) be hermitian symmetric, where \( H \) is the maximal compact subgroup of \( G \). For each such \( G \) there exists a discrete infinity of holomorphic and anti-holomorphic unitary representations. They can be realized over the Fock spaces of a set of bosons transforming in a certain representation of its maximal compact subgroup \( H \) and have been studied extensively over the last decade in the physics literature [24].

As in the compact case let us denote by \( |\Omega\rangle \) and \( |\bar{\Omega}\rangle \) the states belonging to the vacuum representation that are annihilated by \( U_{a0} \) and \( U^{a0} \), respectively. Then the chiral primary states are of three types

\[
\begin{align*}
&i) & |\Omega\rangle \\
&ii) & |a_1, \ldots, a_N\rangle & , N = 1, \ldots, D \quad (3 - 19) \\
&iii) & |\bar{\Omega}, a_1, \ldots, a_D\rangle
\end{align*}
\]

The eigenvalues of \( T_0 \) and \( J_0 \) on these states are
\[
T_0 |\Omega\rangle = \frac{1}{2(k-g)} S_{a0}^a |\Omega\rangle = \frac{p}{2(k-g)} |\Omega\rangle
\]
\[
J_0 |\Omega\rangle = \frac{1}{(k-g)} S_{a0}^a |\Omega\rangle = \frac{p}{(k-g)} |\Omega\rangle
\]
\[
T_0 |a_1, \ldots, a_n\rangle = \frac{kN}{2(k-g)} |a_1, \ldots, a_n\rangle
\]
\[
J_0 |a_1, \ldots, a_n\rangle = \frac{kN}{k-g} |a_1, \ldots, a_n\rangle
\]
\[
T_0 |\bar{\Omega}, a_1, \ldots, a_D\rangle = \frac{kD-p}{2(k-g)} |\bar{\Omega}, a_1, \ldots, a_D\rangle
\]
\[
J_0 |\bar{\Omega}, a_1, \ldots, a_D\rangle = \frac{kD-p}{2(k-g)} |\bar{\Omega}, a_1, \ldots, a_D\rangle
\]

where the states \(|\Omega\rangle\) and \(|\bar{\Omega}\rangle\) belonging to the vacuum representation of \(\hat{g}\) satisfy
\[
U_{a0} |\Omega\rangle = 0
\]
\[
U_{\bar{a}0}^\dagger |\bar{\Omega}\rangle = 0
\]

and
\[
S_{a0}^a |\Omega\rangle = p |\Omega\rangle
\]
\[
S_{a0}^a |\bar{\Omega}\rangle = -\bar{p} |\bar{\Omega}\rangle
\]

The possible eigenvalues \(p\) and \(\bar{p}\) of \(S_{a0}^a\) on the states \(|\Omega\rangle\) and \(|\bar{\Omega}\rangle\) can be read off from the results of [24] for \(G = SU(m,n), SO^*(2n)\), or \(Sp(2n,\mathbb{R})\). The same methods can be extended to the cases of \(SO(n,2), E_6(-14)\), and \(E_7(-25)\) so as to determine the possible eigenvalues of \(S_{a0}^a\) on the states \(|\Omega\rangle\) and \(|\bar{\Omega}\rangle\). In [24], the states \(|\Omega\rangle\) (\(|\bar{\Omega}\rangle\)) that transform in an irreducible representation of the maximal compact subgroup \(H\) were called the ground states of a highest (lowest) weight unitary irreducible representation of \(G\). Thus the necessary requirement that only the unitary representations of the non-compact group \(G\) be admitted leads to an infinite set of chiral (anti-chiral) primary states of the \(N = 2\) SCA. However if we further impose the condition
\[
0 \leq h \leq \frac{c}{6}
\]
that follows from the unitarity of the \(N = 2\) module [3] we are restricted to a finite subset. We should note that the purely fermionic chiral (or anti-chiral) primary states \(|a_1, \ldots, a_N\rangle\) \((|\bar{a}_1, \ldots, \bar{a}_N\rangle\) all satisfy the bound \(h \leq \frac{c}{6}\), just as in the compact case.
4 The Chiral Primary States of $N = 2$ Superconformal Algebras constructed over Freudenthal Triple Systems

The exceptional Lie algebras $G_2, F_4$ and $E_8$ do not admit a TKK type construction over a Jordan triple system. A generalization of the TKK construction to more general triple systems was given by Kantor [25]. All finite dimensional simple Lie algebras admit a realization over the Kantor triple systems (KTS). The Kantor’s construction of Lie algebras was further developed and generalized to a unified construction of Lie and Lie superalgebras in [26]. This construction proceeds as follows.

Every simple Lie algebra $g$ admits a 5-grading (Kantor structure) with respect to some subalgebra $g^0$ of maximal rank [25, 26]:

$$g = g^{-2} \oplus g^{-1} \oplus g^0 \oplus g^1 \oplus g^2 \quad (4 - 1)$$

One associates with the grade $+1$ subspace of $g$ a triple system $V$ and labels the elements of $g^{+1}$ subspace with the elements of $V$ [25, 26]:

$$U_a \in g^{+1} \iff a \in V \quad (4 - 2)$$

Every simple Lie algebra $g$ admits a conjugation under which the grade $+m$ subspace gets mapped into grade $-m$ subspace:

$$U^a \equiv U^+_a \in g^{-1} \iff U_a \in g^{+1} \quad (4 - 3)$$

One defines the commutators of $U_a$ and $U^b$ as

$$[U_a, U^b] = S^b_a \in g^0$$
$$[U_a, U_b] = K_{ab} \in g^2$$
$$[U^a, U^b] = K^{ab} \in g^{-2}$$
$$[S^b_a, U_c] = U_{(abc)} \in g^{+1} \quad (4 - 4)$$

where $(abc)$ denotes the triple (or ternary) product under which the elements of $V$ close. The remaining non-vanishing commutators of $g$ can all be expressed in terms of the triple product $(abc)$. The Jacobi identities of $g$ follow from the following defining identities of a KTS [25, 26]:

$$(ab(cdx)) - (cd(abx)) - (a(dcb)x) + ((xda)bx) = 0 \quad (4 - 5)$$
\{(axcbd) - ((cbd) xa) + (ab(cx d)) + (c(bax)d)\} - \{c \leftrightarrow d\} = 0 \quad (4 - 6)

In general a given simple Lie algebra admits several inequivalent such constructions corresponding to different choices of the subalgebra \(g^0\) and different KTS’s.

The construction of \(N = 2\) superconformal algebras over KTS’s was given in \([15, 16]\). The realization of \(N = 2\) SCA’s over the KTS’s is equivalent to their realization over the coset spaces \(G/H\) where \(G\) and \(H\) are the groups generated by \(g\) and \(g^0\), respectively.

One can extend the study of the chiral primary states of \(N = 2\) SCA’s associated with hermitian JTS’s to their realization over the more general KTS’s. However, in this paper, we shall restrict our analysis to a very special subset of KTS’s, namely the Freudenthal triple systems (FTS). In \([15, 16]\) it was shown that the \(N = 2\) SCA’s constructed over Freudenthal triple systems can be extended to \(N = 4\) SCA’s with the gauge group \(SU(2) \times SU(2) \times U(1)\). The FTS’s were first introduced by Freudenthal in his investigations of the geometry of the exceptional Lie groups \([27]\) and studied in great detail later \([28, 29, 30, 31]\). There exists a one-to-one correspondence between simple FTS’s with a non-degenerate bilinear form and finite dimensional simple Lie algebras \([31]\). For every simple Lie algebra \(g\) constructed over a Freudenthal triple system the grade \(\pm 2\) subspaces are one dimensional (except for \(SU(2)\) for which the grade \(\pm 2\) subspaces vanish). They generate an \(SU(2)\) subalgebra of \(g\). Furthermore, there is a universal relationship between the dimension \(D\) of a FTS and the dual coxeter number \(\tilde{g}\) of \(g\):

\[\tilde{g} = \frac{D}{2} + 2 \quad (4 - 7)\]

Every FTS admits a symplectic form \(\Omega_{ab}\) such that the elements \(K_{ab}(K^{ab})\) of the grade \(+2(-2)\) subspaces can be represented as \([15, 16]\)

\[K_{ab} = \Omega_{ab}K_+\]
\[K^{ab} = \Omega^{ab}K_+\]

\(\Omega^{ab}\) is the inverse of the symplectic form \(\Omega_{ab}\)

\[\Omega_{ab}\Omega^{bc} = \delta^c_a\]
\[a, b, .. = 1, 2, ..., D \quad (4 - 9)\]
For FT systems the expressions for the generators of \( N = 2 \) SCA take the form \[ 15, 16 \]

\[
G(z) = \sqrt{\frac{2}{k + \hat{g}}} \left\{ U^a \psi^a + K^+ \psi^+ - \frac{1}{2} \Omega_{ab} \psi^a \psi^b \right\}(z) \quad (4 - 10)
\]

\[
\bar{G}(z) = \sqrt{\frac{2}{k + \hat{g}}} \left\{ U^a \psi^a + K^+ \psi^+ - \frac{1}{2} \Omega_{ab} \psi^a \psi^b \right\}(z) \quad (4 - 11)
\]

\[
T(z) = \frac{1}{k + \hat{g}} \left\{ \frac{1}{2} (U^a U^a + U^a U_a) + \frac{1}{2} (K_+ K^+ + K^+ K_+) \right. \\
- \frac{k+1}{2} (\psi_a \partial \psi^a + \psi^a \partial \psi_a) - \frac{1}{2} (k + \hat{g} - 2) (\psi_+ \partial \psi^+ + \psi^+ \partial \psi_+) \\
+ S^b_a \psi^a \psi_b + \frac{1}{\hat{g} - 2} S^a_a \psi^+ \psi_+ + \psi_+ \psi^+ \psi^a \psi_a + \frac{1}{4} \Omega_{ab} \psi^a \psi^b \Omega^{cd} \psi_c \psi_d \left\} \right. (z) \quad (4 - 12)
\]

\[
J(z) = \frac{1}{k + \hat{g}} \left\{ \frac{(\hat{g} - 1)}{(\hat{g} - 2)} S^a_a + (k + 1) \psi^a \psi_a - (k - \hat{g} + 2) \psi^+ \psi_+ \right\}(z) \quad (4 - 13)
\]

where \( U_a(z), K_+(z), \ldots \) etc represent the various graded subspaces of the current algebra \( \hat{g} \). The fermionic fields \( \psi_+ \) and \( \psi^+ \) are associated with the grade \( \pm 2 \) subspaces of \( \hat{g} \) and satisfy

\[
\psi^+ (z) \psi_+(w) = \frac{1}{(z - w)} + \ldots \quad (4 - 14)
\]

The central charge of the \( N = 2 \) SCA defined by a FTS is

\[
c = \frac{6(k + 1)(\hat{g} - 1)}{(k + \hat{g})} - 3 \quad (4 - 15)
\]

where \( k \) is the level of \( \hat{g} \) and \( \hat{g} \) is the dual coxeter number. The above realization of \( N = 2 \) SCA’s over the simple FTS’s correspond to the following coset spaces of simple Lie groups \[ 15, 16 \]
The states $|\Omega\rangle$ belonging to the vacuum representation of $\hat{g}$ that are annihilated by $U_{a0}$

$$U_{a0}|\Omega\rangle = 0 \quad (4 - 16)$$

are all chiral primary. (Note that the above condition implies also that $K_{+0}|\Omega\rangle = 0$). The corresponding $h$ and $q$ values are

$$h = \frac{1}{2} q = \frac{p(\hat{g} - 1)}{2(k + \hat{g})(\hat{g} - 2)} \quad (4 - 17)$$

where $p$ is the eigenvalue of $S^a_{a0}$ on the state $|\Omega\rangle$. The corresponding anti-chiral primary states $|\bar{\Omega}\rangle$ are those states belonging to a highest weight representation $\hat{g}$ that are annihilated by all the positive mode generators of $\hat{g}$ and that satisfy

$$U^a_{0}|\bar{\Omega}\rangle = 0 \quad (4 - 18)$$

Their $h$ and $q$ values are

$$h = -\frac{1}{2} q = \frac{\bar{p}(\hat{g} - 1)}{2(k + \hat{g})(\hat{g} - 2)} \quad (4 - 19)$$

where we denoted the eigenvalue of $S^a_{a0}$ on $|\bar{\Omega}\rangle$ as $-\bar{p}$. Again the eigenvalues $p$ and $-\bar{p}$ of $S^a_{a0}$ can be calculated rather simply for all the cases by using the methods of references [24].

To discuss the purely fermionic chiral primary states simply we choose a basis for the FTS such that

$$\Omega_{A,B} = 0 \quad A, B, \ldots = 1, 2, \ldots, \frac{D}{2} \quad (4 - 20)$$

$$\Omega_{A,B+\frac{D}{2}} = \delta_{A,B}$$
Then the states of the form

\[ |A_1, A_2, ..., A_N \rangle \equiv \psi^{-1}_{-\frac{1}{2}} \psi^{-1}_{-\frac{1}{2}} \cdots \psi^{-1}_{-\frac{1}{2}} |0 \rangle \]

\[ N = 0, 1, 2, ..., \frac{D}{2} \]  

and the state

\[ |\bar{s} \rangle = \psi_{(\bar{s})} |0 \rangle \]

are all chiral primary and have the \( q \) eigenvalues \( \frac{(k+1)N}{(k+\hat{\vartheta})} \) and \( \frac{(k-\hat{\vartheta}+2)}{(k+\hat{\vartheta})} \), respectively. Again the state \( |0 \rangle \) represents the tensor product of the Fock vacuum with the scalar vacuum representation of \( \hat{g} \). The unique chiral primary state with \( h = \frac{c}{6} \) is

\[ |a_1, ..., a_D, s \rangle \equiv \psi^{a_1}_{-\frac{1}{2}} \cdots \psi^{a_D}_{-\frac{1}{2}} \psi^{+}_{-\frac{1}{2}} |0 \rangle \]  

(4 - 23)

Tensoring this state with the states \( |\bar{\Omega} \rangle \) belonging to the vacuum representation of \( \hat{g} \) that are annihilated by \( U^a_0 \) we obtain additional chiral primary states

\[ |\widetilde{\Omega}, a_1, ..., a_D, s \rangle \]

(4 - 24)

with the \( h \) values

\[ h = \frac{c}{6} - \frac{\bar{p}(\bar{g} - 1)}{2(k + \bar{g})(\bar{g} - 2)} \]  

(4 - 25)

The corresponding anti-chiral primary states are simply \( |\Omega, \bar{a}_1, ..., \bar{a}_D, \bar{s} \rangle \).

We should note that the non-compact analogs of the coset spaces \( G/H \) where \( H \) is the maximal compact subgroup of \( G \) associated with FTS’s do not exist as these spaces are not symmetric spaces.

5 The Chiral and Anti-Chiral Rings in \( N = 4 \) Superconformal Algebras

The unitary representations of the maximal \( N = 4 \) superconformal algebras \[ [32, 33] \] were studied in \[ [34] \] and their characters in \[ [35] \]. The maximal \( N = 4 \) SCA has 4 supersymmetry generators, 4 dimension \( \frac{1}{2} \) operators, and \( SU(2) \times SU(2) \times U(1) \) local gauge symmetry generators. Our aim is to generalize the concept of chiral (anti-chiral) rings to the maximal \( N = 4 \) SCAs. Following \[ [33] \], let us denote the supersymmetry generators as \( G^+, G^-, G^{+K}, \) and \( G^{-K} \), the two \( SU(2) \) currents as \( A^+i \) and \( A^{-i} \) \( (i = 1, 2, 3) \), the \( U(1) \) current as \( W \) and the four dimension \( \frac{1}{2} \) operators as \( Q^+, Q^- \),
\( Q^{+K} \) and \( Q^{-K} \). We shall work in the N-S moding and assume the following

hermiticity properties

\[
T_n^\dagger = T_{-n}
\]

\[
(G_s^+)^\dagger = G_{-s}; \quad (G_s^{+K})^\dagger = G_{-s}^{-K}
\]

\[
(A_n^{+i})^\dagger = A_{-n}^{+i}; \quad (A_m^{-i})^\dagger = A_{-m}^{-i}
\]

\[
W_n^\dagger = W_{-n}
\]

\[
(Q_r^+)^\dagger = -Q_{-r}^{-}; \quad (Q_r^{+K})^\dagger = -Q_{-r}^{-K}
\]

where \( m, n, \ldots = 0, \pm 1, \ldots \) and \( r, s = \mp \frac{1}{2}, \mp \frac{3}{2}, \ldots \). \( ^3 \) The central charge of the \( N = 4 \) SCA is given by

\[
c = \frac{6k^+k^-}{k^+ + k^-} = \frac{6k^+k^-}{k}
\]

(5 - 27)

where \( k^+ \) and \( k^- \) are the levels of the two \( SU(2) \) currents and \( k = k^+ + k^- \).

The highest weight state \( |h, \ell^+, \ell^-, u\rangle \) of a unitary representation of the

\( ^3 \) Note that \( SU(2) \times SU(2) \times U(1) \) current generators are hermitian. In \([34]\) they were taken as anti-hermitian.
\( N = 4 \) SCA is defined by the conditions

\[
T_n|h, \ell^+, \ell^-, u\rangle = A_{n}^\pm |h, \ell^+, \ell^-, u\rangle = W_n|h, \ell^+, \ell^-, u\rangle = 0, \quad n > 0
\]

\[
G_{r}^+|h, \ell^+, \ell^-, u\rangle = G_{r}^-|h, \ell^+, \ell^-, u\rangle = 0
\]

\[
G_{r}^{+K}|h, \ell^+, \ell^-, u\rangle = G_{r}^{-K}|h, \ell^+, \ell^-, u\rangle = 0, \quad r > 0
\]

\[
Q_{r}^+|h, \ell^+, \ell^-, u\rangle = Q_{r}^-|h, \ell^+, \ell^-, u\rangle = 0
\]

\[
Q_{r}^{+K}|h, \ell^+, \ell^-, u\rangle = Q_{r}^{-K}|h, \ell^+, \ell^-, u\rangle = 0, \quad r > 0
\]

\[
A_{0}^{++}|h, \ell^+, \ell^-, u\rangle = A_{0}^{-+}|h, \ell^+, \ell^-, u\rangle = 0
\]

\[
T_0|h, \ell^+, \ell^-, u\rangle = h|h, \ell^+, \ell^-, u\rangle
\]

\[
(A_0^+ A_0^-)|h, \ell^+, \ell^-, u\rangle = \ell^+(\ell^+ + 1)|h, \ell^+, \ell^-, u\rangle
\]

\[
(A_0^- A_0^+)|h, \ell^+, \ell^-, u\rangle = \ell^-(\ell^- + 1)|h, \ell^+, \ell^-, u\rangle
\]

\[
W_0|h, \ell^+, \ell^-, u\rangle = u|h, \ell^+, \ell^-, u\rangle
\]

(5 - 28)

where \( A_{0}^{+\mp} = A_{0}^{+1} \mp iA_{0}^{+2}, A_{0}^{-\mp} = A_{0}^{-1} \mp iA_{0}^{-2} \).

The states belonging to the “vacuum representation” of the \( N = 4 \) SCA for a given \( h, \ell^+, \ell^- \) and \( u \) will be denoted as \(|h, \ell_3^+, \ell_3^-, u\rangle\) which satisfy

\[
A_{0}^{+3}|h, \ell_3^+, \ell_3^-, u\rangle = \ell_3^+|h, \ell_3^+, \ell_3^-, u\rangle
\]

\[
A_{0}^{-3}|h, \ell_3^+, \ell_3^-, u\rangle = \ell_3^-|h, \ell_3^+, \ell_3^-, u\rangle
\]

(5 - 29)

where \( \ell_3^+ = -\ell^+, -\ell^+ + 1, \ldots, \ell^+ - 1, \ell^+ \) and \( \ell_3^- = -\ell^-, -\ell^- + 1, \ldots, \ell^- - 1, \ell^- \).

Any generalization of the concept of a chiral or anti-chiral primary ring to \( N = 4 \) SCAs must yield the standard definitions when truncated to the \( N = 2 \) subalgebra. For example, the two supersymmetry generators \( G^+ \) and \( G^- \) generate an \( N = 2 \) subalgebra

\[
\{G_r^+, G_s^-\} = L_{r+s} + \frac{1}{2}(r-s)J_{r+s} + \frac{c}{6}(r^2 - \frac{1}{4})\delta_{r+s,0}
\]

(5 - 30)
where the $U(1)$ current $J(z)$ is given by

$$J(z) = \frac{2}{k}[k^{-}A^{+3}(z) + k^{+}A^{-3}(z)] \quad (5 - 31)$$

When one looks for chiral primary states of the $N = 2$ subalgebra among the states belonging to the highest weight representation of the $N = 4$ SCA, one is generally led to the trivial solution with $h = 0$. This comes about because the highest weight condition for the $N = 4$ algebra requires that the highest weight state be a Fock vacuum of the four fermions $Q$. Such a restriction has no counterpart for the $N = 2$ subalgebra. In fact, one can show that the $N = 2$ generators can be decomposed as follows

$$T(z) = \hat{T}(z) + T_Q(z)$$

$$G^+(z) = \hat{G}^+(z) + G^+_Q(z) \quad (5 - 32)$$

$$G^-(z) = \hat{G}^-(z) + G^-_Q(z)$$

$$J(z) = \hat{J}(z) + J_Q(z)$$

where the operator product of the hatted operators with those labelled by $Q$ are regular. The generators $T_Q$, $G^+_Q$, and $J_Q$ are defined by a “matter multiplet@ of the $N = 2$ algebra [34]:

$$T_Q(z) = -\frac{1}{2}(A \bar{A} + \partial Q \bar{Q} + \partial \bar{Q} Q)(z)$$

$$G^+_Q(z) = -\frac{1}{\sqrt{2}} A \bar{Q}(z)$$

$$G^-_Q(z) = -\frac{1}{\sqrt{2}} A \bar{Q}(z) \quad (5 - 33)$$

$$J_Q = -Q \bar{Q}(z)$$

where

$$A = -\sqrt{\frac{2}{k}}(A^{+3} - A^{-3} + iW)$$

$$\bar{A} = \sqrt{\frac{2}{k}}(A^{+3} - A^{-3} - iW)$$

$$Q = \frac{2}{\sqrt{k}} Q^+$$

$$\bar{Q} = \frac{2}{\sqrt{k}} Q^- \quad (5 - 34)$$
Now if we denote the eigenvalues of $\hat{T}_0$, $T_Q$, $\hat{J}_0$ and $J_Q$, as $\hat{h}$, $h_Q$, $\hat{q}$ and $q_Q$, respectively we have from the results of [6]

\[
\hat{h} \geq \frac{|\hat{q}|}{2},
\]

\[
h_Q \geq \frac{|q_Q|}{2}.
\] (5 - 35)

To have $h = \frac{|q|}{2}$ we must then have $\hat{h} = \frac{|\hat{q}|}{2}$ and $h_Q = \frac{|q_Q|}{2}$. On the highest weight representation space of the $N = 4$ SCA we find that $q_Q = 0$, which requires that $h_Q = 0$ for the chiral primary states. Hence there is no non-trivial contribution from the matter sector to the chiral primary rings of the $N = 2$ subalgebra. Therefore in looking for chiral primary states we need only to restrict ourselves to the $N = 2$ SCA generated by $\hat{T}$, $\hat{J}$ and $\hat{J}$.

There is another truncation of the $N = 4$ SCA to an $N = 2$ subalgebra independent of the above one [33]. It is the $N = 2$ subalgebra generated by $G^+K$ and $G^-K$

\[
\{G^+_r, G^+_s\} = T_{r+s} + \frac{1}{2} (r-s) J'_{r+s} + \frac{c}{6} (r^2 - \frac{1}{4}) \delta_{r+s,0}
\] (5 - 36)

where

\[
J'(z) = \frac{2}{k} (k^- A_3^+ - k^+ A_3^-)(z)
\] (5 - 37)

Again the $N = 2$ generators can be decomposed as

\[
T(z) = \hat{T}'(z) + T_Q(z)
\]

\[
G^{+K}(z) = \hat{G}^{+K}(z) + G^{+K}_{Q'}(z)
\] (5 - 38)

\[
G^{-K}(z) = \hat{G}^{-K}(z) + G^{-K}_{Q'}(z)
\]

\[
J'(z) = \hat{J}'(z) + J_Q'(z)
\]
In this case the “matter multiplet” is
\[ A'(z) = -\sqrt{\frac{2}{k}}(A^+ + A^- + iW)(z) \]
\[ \bar{A}'(z) = \sqrt{\frac{2}{k}}(A^+ + A^- - iW)(z) \]
\[ Q'(z) = \frac{2}{\sqrt{k}}Q^+(z) \]
\[ \bar{Q}'(z) = \frac{2}{\sqrt{k}}Q^-(z) \]

(5 - 39)

To find non-trivial solution for the conditions defining the chiral primary states we need again to restrict ourselves to the hatted \( N = 2 \) subalgebra generated by \( \hat{T}', \hat{G}^+K, \hat{G}^-K, \) and \( \hat{J}'(z) \).

Since we have two independent \( N = 2 \) subalgebras of the \( N = 4 \) SCA (up to automorphisms) the natural generalization of the concept of chirality or anti-chirality to the \( N = 4 \) case would be to look for states that are chiral or anti-chiral with respect to both \( N = 2 \) subalgebras. Thus we can define four rings in the \( N = 4 \) case namely the \( (cc) \), \( (aa) \), \( (ca) \), and \( (ac) \) rings. We shall refer to the subalgebras generated by \{\( \hat{T}, \hat{G}, \hat{J} \}\} and \{\( \hat{T}', \hat{G}', \hat{J}' \)\} as the first and the second \( N = 2 \) superconformal algebra, respectively. The \( (cc) \) ring corresponds to the primary states \( |\phi\rangle \) of the \( N = 4 \) SCA which satisfy
\[ G_{-\frac{1}{2}}^+ |\phi\rangle = G_{-\frac{1}{2}}^+K |\phi\rangle = 0 \]  
(5 - 40)

The \( (aa) \) ring is defined by primary states \( |\phi\rangle \) satisfying
\[ G_{-\frac{1}{2}}^- |\phi\rangle = G_{-\frac{1}{2}}^-K |\phi\rangle = 0 \]  
(5 - 41)

The \( (ca) \) ring is defined by
\[ G_{-\frac{1}{2}}^+ |\phi\rangle = G_{-\frac{1}{2}}^-K |\phi\rangle = 0 \]  
(5 - 42)

and the \( (ac) \) ring by
\[ G_{-\frac{1}{2}}^- |\phi\rangle = G_{-\frac{1}{2}}^+K |\phi\rangle = 0 \]  
(5 - 43)

Thus if we take both left and right-moving sectors into account we have 16 different rings, eight of which are conjugates to the other eight. We shall

\[ \text{note these are not to be confused with the \( (c,c), (a,a), (c,a), \) and \( (a,c) \) rings of an \( N = 2 \) SCA when are considers both left- and right-moving sectors} \]
first give an analysis of these rings within the “massless representations” of the \( N = 4 \) SCA’s for which a complete treatment was given in [34]. Later we shall prove that massless representations yield all possible rings of the \( N = 4 \) SCAs. In [34] massless representations were defined by the condition

\[
\tilde{G}^+_{\frac{1}{2}} |hws\rangle = 0 \tag{5 - 44}
\]

where \( |hws\rangle \) stands for a highest weight vector of the of the non-linear \( N = 4 \) algebra one obtains by factoring out the four dimension \( \frac{1}{2} \) operators and the \( U(1) \) current [34, 36, 37]. \( \tilde{G}^+ \) is one of the supersymmetry generators of the nonlinear algebra. Furthermore, for given levels \( k^+ \) and \( k^- \) of the two \( SU(2) \) currents algebras of the \( N = 4 \) SCA, the allowed angular momentum quantum numbers \( \ell^+ \) and \( \ell^- \) are

\[
\begin{align*}
\ell^- &= 0, \frac{1}{2}, 1, \ldots, \frac{1}{2}(k^- - 1) \\
\ell^+ &= 0, \frac{1}{2}, 1, \ldots, \frac{1}{2}(k^+ - 1) \tag{5 - 45}
\end{align*}
\]

For \( k^- = 1 \) we have \( \ell^- = 0 \) and the restriction to an \( N = 2 \) subalgebra leads to the minimal discrete series representations of the corresponding \( N = 2 \) subalgebra (with \( \hat{c} < 3 \) or \( \hat{c}' < 3 \)). The allowed eigenvalues \( \hat{q} \) of \( \hat{J}_0 \) and \( \hat{h} \) of \( \hat{T}_0 \) on the highest weight representations are

\[
\begin{align*}
\hat{q} &= \frac{2}{(k^+ + 1)^2} \ell_3^+ \\
\hat{h} &= \frac{1}{(k^+ + 1)} \{ \ell^+ (\ell^+ + 1) - (\ell_3^+)^2 \} \tag{5 - 46}
\end{align*}
\]

Thus the chiral primary states of the first \( N = 2 \) subalgebra are the states for which

\[
\ell_3^+ = \ell^+ \tag{5 - 47}
\]

and anti-chiral states are those for which

\[
\ell_3^+ = -\ell^+ \tag{5 - 48}
\]

where \( \ell^+ = 0, \frac{1}{2}, 1, \ldots, \frac{1}{2}(k^+ - 1) \). For the second \( N = 2 \) subalgebra we find the same conditions since

\[
\begin{align*}
\hat{q}' &= \frac{2}{(k^+ + 1)^2} \ell_3^+ \\
\hat{h}' &= \frac{1}{(k^+ + 1)} \{ \ell^+ (\ell^+ + 1) - (\ell_3^+)^2 \} \tag{5 - 49}
\end{align*}
\]

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for $k^- = 1$. Therefore for $k^- = 1$ and $k^+$ arbitrary we find that the (ca) and (ac) rings are trivial, consisting only of the identity element corresponding to the vacuum state with $\ell^+ = \ell^- = h = u = 0$. The (cc) ring has the form

$$(cc) = \{1, x, \ldots, x^{p-1}\} \quad (5 - 50)$$

where $p = 2k^+ - 1$ and $x^p = 0$. The (aa) ring is simply the conjugate of the (cc) ring.

$$(aa) = \{1, \bar{x}, \ldots, \bar{x}^{p-1}\} \quad (5 - 51)$$

The other massless representation of the $N = 4$ SCA with $k^- > 1$ can be obtained from the coset space $[34]

$$\frac{SU(N)}{SU(N-2) \times SU(2) \times U(1) \otimes SU(2) \times U(1)} \quad (5 - 52)$$

with the level of $SU(N)$ taken at $(k^+ - 1)$ and $k^- = N - 1$. The eigenvalues of $\hat{T}_0$, $\hat{J}_0$, $\hat{T}'_0$ and $\hat{J}'_0$ on the highest weight representation space are

$$\hat{h} = \frac{1}{k} \{ (\ell^+ - \ell^-)^2 - (\ell_3^+ - \ell_3^-)^2 + k^- \ell^+ + k^+ \ell^- \}$$

$$\hat{q} = \frac{2}{k} \{ k^- \ell_3^+ + k^+ \ell_3^- \}$$

$$\hat{h}' = \frac{1}{k} \{ (\ell^+ - \ell^-)^2 - (\ell_3^+ + \ell_3^-)^2 + k^- \ell^+ + k^+ \ell^- \}$$

$$\hat{q}' = \frac{2}{k} \{ k^- \ell_3^+ - k^+ \ell_3^- \} \quad (5 - 53)$$

Therefore the states in the highest weight representation that are chiral primary with respect to both $N = 2$ SCAs must satisfy

$$(cc) \iff \ell^- = 0, \ell_3^+ = \ell^+ \quad (5 - 54)$$

Similarly the states that are anti-chiral with respect to both $N = 2$ SCAs satisfy

$$(aa) \iff \ell^- = 0, \ell_3^- = -\ell^+ \quad (5 - 55)$$

The states that are chiral (anti-chiral) with respect to the first $N = 2$ subalgebra and anti-chiral (chiral) with respect to the second satisfy

$$(ca) \iff \ell^+ = 0, \ell_3^- = \ell^- \quad (5 - 56)$$

$$(ac) \iff \ell^+ = 0, \ell_3^- = -\ell^-$$
Hence we have the following rings of $N = 4$ SCAs

\[
\begin{align*}
(cc) &= \{1, x, \ldots, x^{p-1}\} \\
(aa) &= \{1, \bar{x}, \ldots, \bar{x}^{p-1}\} \\
(ca) &= \{1, y, \ldots, y^{q-1}\} \\
(ac) &= \{1, \bar{y}, \ldots, \bar{y}^{q-1}\}
\end{align*}
\]  
(5 - 57)

with the restrictions

\[
\begin{align*}
x^p &= 0 \\
y^q &= 0
\end{align*}
\]  
(5 - 58)

where $p = 2k^+ - 1$ and $q = 2k^- - 1$.

Let us now prove that all the rings associated with $N = 4$ SCAs are of the above form. Consider for example a general (cc) ring. Any primary state belonging to the (cc) ring is annihilated by the operators $\hat{G}^+_\frac{1}{2}$, $\hat{G}^{K+}_\frac{1}{2}$, $\hat{G}^-_\frac{1}{2}$, and $\hat{G}^{-K}_\frac{1}{2}$ and hence by their anti-commutators

\[
\begin{align*}
\{\hat{G}^+_\frac{1}{2}, \hat{G}^-_\frac{1}{2}\} &= \hat{T}_0 - \frac{1}{k}(k^- \hat{\Lambda}^+ + k^+ \hat{\Lambda}^-) \\
\{\hat{G}^{K+}_\frac{1}{2}, \hat{G}^{-K}_\frac{1}{2}\} &= \hat{T}_0' - \frac{1}{k}(k^- \hat{\Lambda}^+ - k^+ \hat{\Lambda}^-) \\
\{\hat{G}^+_\frac{1}{2}, \hat{G}^{K+}_\frac{1}{2}\} &= -\frac{k^-}{k} \hat{\Lambda}^{++} \\
\{\hat{G}^+_\frac{1}{2}, \hat{G}^{-K}_\frac{1}{2}\} &= \frac{k^-}{k} \hat{\Lambda}^{+-} \\
\{\hat{G}^+_\frac{1}{2}, \hat{G}^{+K}_\frac{1}{2}\} &= \frac{k^+}{k} \hat{\Lambda}^{--} \\
\{\hat{G}^-_\frac{1}{2}, \hat{G}^{-K}_\frac{1}{2}\} &= \frac{k^+}{k} \hat{\Lambda}^{+-} \\
\{\hat{G}^-_\frac{1}{2}, \hat{G}^{+K}_\frac{1}{2}\} &= -\frac{k^-}{k} \hat{\Lambda}^{++}
\end{align*}
\]  
(5 - 59)

where the hatted currents $\hat{SU}(2)^+$ and $\hat{SU}(2)^-$ have levels $(k^+ - 1)$ and $(k^- - 1)$, respectively. It is obvious from the above that any state belonging
to the (cc) ring must be an $SU(2)^-$ singlet and satisfy
\[
\ell_3^+ = \ell^+, \ell^- = 0
\]
\[
\hat{h} = \frac{k}{\ell^+} \ell^+ = \frac{\hat{q}}{2}
\]
\[
\hat{h} = \hat{h}', \hat{q} = \hat{q}'
\]
Similarly we find that any state belonging to the (aa) ring must be annihilated by $\hat{G}^{-\frac{1}{2}}$, $\hat{G}^{-K\frac{1}{2}}$, $\hat{G}^{\frac{1}{2}}$, $\hat{G}^{+K\frac{1}{2}}$ and their anti-commutators, which leads to the conditions
\[
\ell_3^+ = -\ell^+; \ell^- = 0
\]
\[
\hat{h} = \frac{k}{\ell^+} \ell^+ = -\frac{\hat{q}}{2}
\]
\[
\hat{h} = \hat{h}'; \hat{q} = \hat{q}'
\]
For the (ca) ring we find that the corresponding primary states must satisfy
\[
\ell_3^- = \ell^-; \ell^+ = 0
\]
\[
\hat{h} = \frac{k}{\ell^-} \ell^- = \frac{\hat{q}}{2}
\]
\[
\hat{h} = \hat{h}'; \hat{q} = -\hat{q}'
\]
and for the (ac) ring
\[
\ell_3^- = -\ell^-; \ell^+ = 0
\]
\[
\hat{h} = \frac{k}{\ell^-} \ell^- = -\frac{\hat{q}}{2}
\]
\[
\hat{h} = \hat{h}'; \hat{q} = -\hat{q}'
\]
The allowed angular momentum quantum numbers $\ell^+$ and $\ell^-$ are
\[
\ell^+ = 0, \frac{1}{2}, \ldots, \frac{1}{2}(k^+ - 1)
\]
\[
\ell^- = 0, \frac{1}{2}, \ldots, \frac{1}{2}(k^- - 1)
\]
Therefore the rings we obtained from the massless unitary highest weight representations of the $N = 4$ SCA exhausts all the allowed values of the quantum numbers and is complete.

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