One-parameter families of functions in the Pick class

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Abstract

In the one-parameter family of power-law maps of the form $f_a(x) = -|x|^{\alpha} + a$, $\alpha > 1$, we give examples of mutually related dynamically determined quantities, depending on the parameter $a$, such that one is a Pick function of the following one. These Pick functions are extendable by reflection through the $(1, +\infty)$ half-axis and have completely monotone derivatives there.

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1 Introduction

The study of the monotonicity of the dynamics in a one parameter family of power-law maps $f_a = -|x|^\alpha + a$ with generally non-integer $\alpha > 1$ and a real parameter $a$ has a long history. This was for that family where the first results pertaining to the uniqueness of some sort of dynamical behavior (like the Feigenbaum dynamics, for instance) were established, though only for $\alpha$’s close enough to 1, cf. [1]. The case of special interest $\alpha = 2$ was later successfully solved with use of quasi-conformal technics.

Roughly speaking, the results for $\alpha$ close to 1 were obtained by perturbations of the well understood linear toy-model. The perturbation was getting out of control with increasing $\alpha$ and the idea that the behavior for larger $\alpha$’s has its roots in the case where there is nearly no nonlinearity present did not enjoy wide recognition since then. We hope this research notes can help revive an interest in that approach to the problem, although the examples described here, that we were able to solve completely, are dynamically very simple ones.

Anyway, as far as we know, what we give here is the first example of carrying the information obtained for $\alpha$ close to 1 up through all greater than 1 exponents, without loss of control of the influence of the perturbation made.

Let us briefly describe the strategy employed. We use arguments from complex variable as well as geometric properties of the root mapping $z \mapsto z^{\frac{1}{\alpha}}$. The former involves properties of the so called Pick functions i.e. the mappings that take the upper half-plane into itself. Whenever such a mapping is extendable by reflection throughout a half-axis and its derivative vanishes at $+\infty$ than this derivative is in the class of completely monotone functions. With some abuse of the language we may say that in what follows we study dynamically determined functions in the parameter space and their complex extensions. The actual parameter is replaced with some related quantity in $[1, +\infty)$. To prove the monotonicity of the derivative of such a function we show the function itself is in the appropriate Pick class. To see the latter we examine geometric properties of the root maps, making use of the contraction of the Poincaré neighborhoods about the rays (cf. the idea stated in [4] in the proof of the sector theorem), as well as of the property of taking rays onto rays. In the course of the proof we show that some regions are transformed in a desired way, eventually leading to the Pick property of our maps in the extended parameter space. Staying in the completely monotone
derivative class while perturbing $\alpha$ provides for the control of the derivative and implies the existence of a strictly monotone in parameter, dynamically determined function for the perturbed $\alpha$ and eventually for all large exponents, because after the perturbation we keep the derivative of the inverse function uniformly bounded away from 0. The prospect of using this kind of technic for more intricate situations depend upon more complete understanding of the geometry of the root maps at the infinitesimal level, for small wedge-shaped neighborhoods. This is a work in progress.

Finally, let us relate the results of this paper to what we previously knew about monotone behavior of dynamically determined quantities in the power-law family. In our work [3] we have found an infinite sequence of monotone functions of the parameter. That sequence had been determined by the whole post-critical orbit rather than by just first few steps as it is the case in Theorems 1 and 2 below. The results of [3] were achieved by purely real variable means and functions in question differed from those investigated in here. Instead, in the current work the information derived from the complex plane extensions is much stronger.

2 Complex variable prerequisites and geometry of the root map.

Consider a complex variable function $\varphi$, analytic in the upper half-plane $\mathbb{H}$. We denote by $P_{(1,+\infty)}$ the class of those Pick functions (ie. with positive imaginary part, cf. [2]) which can be continued by reflection throughout the half-axis $(1, +\infty)$.

Assume that $\varphi \in P_{(1,+\infty)}$ and that $\varphi$ fixes 1, ie. it can be continuously defined along the reals for $z = 1$ and $\varphi(1) = 1$.

We shall denote by $P$ the class of functions satisfying the above and call it the class of Pick Argument Lessening functions, to distinguish them from better recognized argument decreasing maps. The reason for this name explains in Proposition 1 below.

A function in this class will simply be refered to as a *pal*.

**Remark 1** Every function in the Pick class $P_{(a, +\infty)}$, i.e. continuable throughout a half-axis $(a, +\infty)$, that has a finite limit of the derivative at infinity has a completely monotone derivative on that half-axis (cf. [2]).
The above property shall be the key to applying the class of *pals* to the study of the problems of monotonicity of dynamical behavior in a real valued parameter, like in the family $-|x|^{\alpha} + a$, $\alpha > 1$.

Our main point throughout this work will be to show that inverse *pal* functions show up naturally in the one-parameter power-law family as some dynamically defined quantities with the parameter as the variable.

To that goal we will need an appropriate choice of variables, as well as some facts concerning real variable properties of the power-law mapping $x \mapsto |x|^\alpha$. This we postpone for the moment, and instead we focus here on preparatory complex variable arguments.

The following proposition points out to a relevant property of the maps in our class.

**Proposition 1** If $\varphi \in P$ then for every $z \in \mathbb{H}$ $\text{Arg}(\varphi(z) - 1) \leq \text{Arg}(z - 1)$.

**Proof.** For a number $z$ in the upper half-plane $\text{Arg}(z - 1) = \alpha$ means that $z$ lies on the line crossing reals at $z = 1$ at the angle $\alpha$. By Schwarz-Pick lemma a region of the upper half-plane cut out by a circle tangent on the right hand side to that line at the point $z = 1$ is mapped by $\varphi$ into another such a region. Letting the radius of the cutting circle tend to infinity we get the statement of the proposition (cf. [4], proof of the sector theorem).

We now proceed to the study of the simplest dynamical setting where there appear the Pick argument lessening functions.

Let $\alpha > 1$ be a fixed real number. Consider the family of power-law mappings $f_a = -|x|^{\alpha} + a$, $0 \leq a < 1$. The quantity $(1 - a^{\alpha - 1})^{-1}$ is monotone in $a$ and so this quantity itself may well serve as a parameter running the $[1, +\infty)$ half-axis, rather than the interval $[0, 1)$.

Notice that this number, which we shall denote by $p_2$, is precisely the exponent of the length of the interval $(f_a^2(0), f_a(0))$ measured in the Poincaré metric on the positive half-axis.

Before moving on to the geometric part of this section we introduce some more notation.

The interval $(f_a^{n-1}(0), f_a^n(0))$, $n = 3, 4, \ldots$ is contained within the interval $(f_a^{n-2}(0), f_a^{n-3}(0))$.

Denote by $p_n$ the exponent of the Poincaré length of the former interval within the latter, i.e. the value of the cross-ratio.
For a point \( \varphi \in \mathcal{H} \) we denote by \( D_\varphi \) the closed disc bounded by the circle \( C_\varphi \) passing through the points 0, 1, \( \varphi \). The disc \( D_\varphi \) is the image of the upper half-plane under the linear fractional transformation \( \ell_\varphi \):

\[
\ell_\varphi : t \mapsto \frac{t}{1 + \frac{t-1}{\varphi}}.
\]

Points \( t \) in the upper half-plane satisfying the condition

\[
\text{Arg}(t - 1) \geq \text{Arg}(\varphi - 1)
\]

are mapped into the region between the circle \( C_\varphi \) and the circle passing through the points 1, \( \varphi \), tangent to the ray connecting 0 and \( \varphi \). We denote this region by \( D_{\varphi -} \). Here we give a geometric lemma.

**Lemma 1** For every \( \alpha \geq 1 \) the root mapping \( r(z) = z^\frac{1}{\alpha} \) maps the region \( D_{\varphi -} \) into the disc \( D_{r(\varphi)} \).

**Proof:** We will jointly use Proposition 1 and the fact that the root map takes rays originating at 0 onto rays.

The lower part of \( D_\varphi \) (i.e. the part lying below the real axis) is mapped into itself, because the root map is in the \( P_{(0, +\infty)} \) class and fixes 0 and 1. For the same reason the point \( \varphi \) is mapped onto a point \( r(\varphi) \) in the upper part of \( D_\varphi \).

Thus the lower part of \( D_{r(\varphi)} \) contains the lower part of \( D_\varphi \) and so it contains the whole image of the lower part of \( D_\varphi \). To do with the upper part we have to use the fact that the root map is in \( P \). By Proposition 1 the point \( r(\varphi) \) lies to the right-hand side of the ray originating at 1 and passing through \( \varphi \). For any point \( t \in \mathcal{H} \) the circle \( C_t \) is intersected by the ray connecting 0 and \( t \) at the angle equal to \( \text{Arg}(t - 1) \), and so this angle gets decreased when we replace \( t \) by \( r(t) \).

Therefore the part of the disc \( D_{r(\varphi)} \) above the ray connecting 0 and \( r(\varphi) \) contains within itself the arc of the circle that joins 0 and \( r(\varphi) \) and intersects this ray at the angle \( \text{Arg}(\varphi - 1) \). But the image of the part of the disc \( D_\varphi \) lying above the ray passing through 0 and \( \varphi \) is totally contained between the
previously described arc and the ray connecting 0 and \( r(\varphi) \). Here we used the fact that the root map takes rays originating at 0 onto rays and Proposition 1 together.

The image of the interval \((0, \varphi)\) is a concave curve. This completes the proof of the lemma also for the upper part of the disc. □

**Remark 2** The statements of both Proposition 1 and Lemma 1 can be obtained by direct computation in polar coordinates, employing differentiating in \( \alpha \) and Jensen inequality.

However, those computations are rather tedious, whereas our geometrical argument is much simpler.

### 3 The Pick functions in the parameter space

For a linear model of our dynamical system we immediately find a direct formula relating \( p_2 \) to \( p_3 \).

\[
p_2 = \frac{1}{2} + \left( p_3 - \frac{3}{4} \right)^{\frac{1}{\alpha}}.
\]

This function has a holomorphic extension to the upper half-plane and this extension is in the Pick class \( P \).

If the exponent of the power law is greater than 1 we see that the relation between \( p_2 \) and \( p_3 \) is subject to the following functional equation:

\[
p_2 = \left( 1 + \frac{p_3 - 1}{p_2} \right)^{\frac{1}{\alpha}}.
\]

It is clear that for \( \alpha \) close to 1 this equation has a solution which is a real analytic function on \((1, +\infty)\), and that \( p_2(1) = 1 \). We claim this solution is in \( P \).

To see this we will consider a map \( \varphi(z) \) in \( P \) and an operator

\[
R(\varphi)(z) = \left( 1 + \frac{z - 1}{\varphi(z)} \right)^{\frac{1}{\alpha}}.
\]

Notice that this operator takes the class \( P \) into itself. Actually, if \( \varphi \in P \) and \( z \) is a real greater that 1, so is \( R(\varphi)(z) \).
For $z \in \mathbb{H}$

$$\text{Arg}(z - 1) \geq \text{Arg}(\varphi - 1) > \text{Arg}(\varphi) > 0,$$

thus $\frac{z - 1}{\varphi - 1} \in \mathbb{H}$, so $(R(\varphi)(z))^\alpha \in \mathbb{H}$ and so does $R(\varphi)(z)$. It follows that $R(\varphi)$ is well defined for all $z \in \mathbb{H}$, maps the upper half-plane into itself, is extendable by reflection throughout $(1, +\infty)$ and of course $R(\varphi)(1) = 1$. Therefore $R(\varphi) \in P$.

We yet have to investigate the convergence of the series of averages

$$\frac{1}{n} \sum_{i=0}^{n-1} R^i(\varphi)(z). \quad (2)$$

If we begin with $\varphi = Id$, then all summands in $(2)$ will be uniformly bounded on compact subintervals of $(1, +\infty)$. Thus, due to a theorem about convergence of Pick functions (cf. Chapter 2.4 in [2]), we can pick up a sequence of $(2)$, convergent in the Pick class. The limit function of this subsequence solves the functional equation (II), but the solution $p_2(p_3)$ of (II) is a uniquely defined analytic function on $(1, +\infty)$, so it must belong to the Pick class itself.

At infinity, the derivative of $p_2$ tends to 0, so the function $p_2(p_3)$ has a completely monotone derivative, according to Remark[1]. Since the limit value of this derivative at the point 1 is smaller than 1, the inverse function $p_3(p_2)$ has the derivative greater than 1 all the time. Small increase of the exponent $\alpha$ keeps the derivative of $p_3(p_2)$ positive all the time, so the inverse is, by the implicit function theorem, a well defined analytic solution of the equation (II) on $(1, +\infty)$. As we have seen, it again has to be in the Pick class, with its derivative completely monotone. So again its inverse function, that a priori merely had a derivative positive, must actually have a derivative greater than 1 for all its arguments. This shows that the property of bounded away from 1 derivative is kept while increasing $\alpha$, so $\alpha$ can be increased indefinitely and we see that for all $\alpha$’s larger than 1 the solution of the functional equation (II) is a well defined analytic function in $P$. This accounts for the proof of the following:

**Theorem 1** For all exponents $\alpha > 1$ the mapping $p_2(p_3)$ is in the Pick Argument Lessening class. □
With some change in the proof, we could have dealt with all $\alpha$’s greater than 1 simultaneously. We choose the above argument to make the proof of Theorem 1 a preparatory step for Theorem 2 below.

Now we proceed to the proof of the harder part.

**Theorem 2** For all exponents $\alpha > 1$ the mapping $p_3(p_4)$ is in the Pick Argument Lessening class.

**Proof:** The outline of the proof is much like in the previous theorem. However the functional equation we deal with now is somewhat different, and to see that the implied operator on Pick functions has the range in the Pick class also, we shall need the geometric properties of the root map stated in Section 2.

For the linear model of the dynamics, the function $p_3(p_4)$ is simply the identity. For a fixed $\alpha$ close to 1 the map in question is a small perturbation of identity. Due to the dynamics relating $p_4$ to $p_3$ it is elementary to see that the following functional equation must be satisfied:

$$p_3(p_4) = \ell_{p_2(p_3)}^{-1}\left(\left(\frac{p_4}{1 + \frac{p_4 - 1}{p_2(p_3)^\alpha}}\right)^{\frac{1}{\alpha}}\right).$$

(3)

From the proof of Theorem 1 we know not only that $p_2(p_3) \in H$ for $p_3 \in H$, but also $(p_2(p_3))^\alpha \in H$, and $\text{Arg}((p_2(p_3))^\alpha - 1) \leq \text{Arg}(p_3 - 1)$.

Consider a number $w \in H$ and suppose that $z = z(w)$ is a function in $P$. Let $\psi(z) = p_2(z)^\alpha$ be the Pick function defined for that fixed $\alpha$ in the proof of Theorem 1.

We define an operator on Pick functions:

$$\mathcal{R}(z)(w) = \ell_{\psi(z)}^{-1}\left(\left(\frac{w}{1 + \frac{w - 1}{\psi(z)}}\right)^{\frac{1}{\alpha}}\right).$$

As before, we will see that its range is in $P$. For a function $z(w) \in P$ we have

$$\text{Arg}(\psi(z) - 1) \leq \text{Arg}(z - 1) \leq \text{Arg}(w - 1).$$

The linear fractional map $w \mapsto \frac{w\psi(z)}{\psi(z) + w - 1}$ takes the number $w$ into the region $D_{\psi(z)}^-$, which, by Lemma 1, is mapped into the disc $D_{p_2(z)}$, and after applying $\ell_{p_2(z)}^{-1}$ back into the upper half-plane.
So the function $R(z)$ is in the Pick Argument Lessening class and consequently $\arg(\psi(R(z)) - 1) \leq \arg(w - 1)$, which makes the argument iterative.

Having dealt with the problem of remaining in the Pick class, we are now in a position to mimic the further steps of the proof of Theorem 1. We consider the sequence

$$\frac{1}{n} \sum_{i=0}^{n-1} R^i(z)(w),$$

starting with $z(w) = w$. Its consecutive summands stay uniformly bounded on compact subsets of $(1, +\infty)$. This again allows for eliciting a convergent subsequence, which gives that a solution to (3) is in the Pick class. Again, looking at the inverse function, $p_4(p_3)$ on $(1, +\infty)$, we argue that its derivative is bounded away from 1, so we can perturb it by increasing the exponent $\alpha$, still keeping the derivative bounded away from 0. Again, looking at the inverse of a solution to (3) (with that new $\alpha$) we see that it has a non-vanishing derivative. This solution has a completely monotone derivative, starting with a value smaller that 1 and vanishing at $+\infty$, so its inverse actually had a derivative bounded away from 1 rather than merely from 0. This way, lifting up $\alpha$ indefinitely, we get a solution in the Pick class for an arbitrary $\alpha > 1$. □

We clearly see that this kind of argument, that starts with understandable behavior for $\alpha$ sufficiently close to 1, and increasing the exponent while keeping the derivative bounded away from 1 can be applied whenever we know that the derivative remains a monotone function. In Theorems 1 and 2 this was provided for by proving much stronger a statement, namely that it was a completely monotone function.

The main obstacle for iterative use of the ideas from the proofs of Theorems 1 and 2 is that the functional equation we get for further steps requires understanding of the geometry of the root maps acting on circles that pass through 1, but no longer pass through 0. In this case we cannot use the argument of Lemma 1, because the rays originating at 1 no longer are mapped onto rays. So far we have not found a good remedy for this difficulty.

Actually, one might expect that for an arbitrary dynamics in the power-law family the change of behavior while varying the parameter is linked to some underlying completely monotone function.
References

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