Feedback control for stochastic gas flow

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Gas flow in pipe networks has been widely discussed in the engineering community by applying numerous mathematical models [32, 41, 8, 9]. Their meaningfulness depends on the scale of phenomena of interest. Algebraic models may be sufficient to describe average states in a gas network [6]. If there is interest in dynamics on shorter time scales, isothermal Euler equations, which form a 2×2 system of hyperbolic balance laws, provide a suitable model [2]. A mathematical theory for hyperbolic systems on networks has been developed for the Euler equations [11] and for the p-system [10].

Since gas systems are usually operated in a state of equilibrium, one is interested in stable systems, where small perturbations are damped over time. We will consider gas flow on a network with feedback boundary conditions. We focus on isothermal Euler equations that are diagonalizable with Riemann invariants and analyze the stability of a steady state.

Boundary stabilization has been studied intensively in the past years [4]. A well-known approach to prove exponential stability of equilibria is the analysis of dissipative boundary conditions and the construction of suitable Lyapunov functions. Exponential decay of a continuous Lyapunov function under so-called dissipative boundary conditions has been proven in [14, 13, 12, 27]. Also explicit decay rates for numerical schemes have been established [1, 34, 20, 5]. However, if the destabilizing effect of the source term is sufficiently large, the system cannot be controlled by boundary feedback [3, 4, 25].

Most results are based on the assumption that model parameters are known exactly. In practice, there are uncertainties that have to be taken into account. For instance, model parameters are uncertain due to noisy measurements. Moreover, epistemic uncertainties arise, since the speed of sound and the friction factor are usually not constant in a pipe, when the pressure decreases.

When the underlying model is not known exactly, but is given by a probability law or by statistical moments, the deterministic stabilization concept should be extended to the stochastic case. Existence of optimal solutions for some optimization problems with probabilistic

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constraints has been analyzed in [28]. Also the stability of slightly perturbed systems has been established [4]. Here, we study the impact of uncertain propagation speeds and friction factors. We represent stochastic perturbations by series of piecewise orthogonal polynomials, known as generalized polynomial chaos (gPC) expansions [37, 7]. Expansions of the stochastic input are substituted into the governing equations and they are projected to obtain deterministic evolution equations for its coefficients. This approach has been used for hyperbolic equations [16, 17, 33, 22, 21] and for optimal power flow [31]. This chapter introduces a Lyapunov stability analysis for the system of gPC coefficients.

The presented approach allows to model random parameters as Gaussian random fields, which are described by their mean and covariance structure. Then, the gPC expansion coincides with the Karhunen–Loève expansion [38, 29]. However, the orthogonal basis is determined by the correlation of the process such that it is optimal in the sense that it minimizes the total mean squared error. Also measurements can be included by adjusting the covariance structure.

Section 0.1 reviews the deterministic feedback control concept. Section 0.2 is devoted to the representation of stochastic processes with orthogonal functions. In Section 0.3, we derive a deterministic formulation of the stochastic problem, which is stabilized in Section 0.4. Finally, Section 0.5 discusses the applicability to Gaussian random fields.

0.1 Feedback control

We consider a network of gas pipelines as illustrated in Figure 1. The density of the gas \( \rho(t, x) \) and the mass flux \( q(t, x) \) in a pipe \( j = 1, \ldots, n \) are described by the isothermal Euler equations

\[
\frac{\partial}{\partial t} \begin{pmatrix} \rho_j(t, x) \\ q_j(t, x) \end{pmatrix} + \frac{\partial}{\partial x} \begin{pmatrix} q_j^2(t, x) \\ \rho_j(t, x) q_j(t, x) \end{pmatrix} = -\frac{f}{2D} \begin{pmatrix} 0 \\ \rho_j(t, x) \end{pmatrix}
\]

with time and space variables \((t, x) \in \mathbb{R}^+ \times (0, L)\). The parameters are the speed of sound \( a > 0 \), the friction factor \( f > 0 \) and the diameter \( D > 0 \) of a pipe. We write \( y_j := (\rho_j, q_j)^T \) as abbreviation and we describe coupling conditions by

\[
\begin{pmatrix} \rho(t, 0) \\ q(t, L) \end{pmatrix} = K \begin{pmatrix} \rho(t, L) \\ q(t, 0) \end{pmatrix}
\]

with \( \rho = \begin{pmatrix} \rho_1 \\ \vdots \\ \rho_n \end{pmatrix} \), \( q = \begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix} \).
Typical coupling conditions for ingoing (in) and outgoing (out) pipes with the same spatial and physical properties are the following.

(i) **Compressors:** The pressure law can be described by fixing the boost ratio
\[ \rho_{\text{out}}(t,0) = BR \rho_{\text{in}}(t,L), \]
output pressure
\[ a^2 \rho_{\text{out}}(t,0) = \text{OP}. \]
There is conservation of mass, i.e. \( q_{\text{out}}(t,0) = q_{\text{in}}(t,L) \).

(ii) **Gas-fired power plant:** A prescribed withdrawal \( w(t) \) determines the mass flux
\[ q_{\text{out}}(t,0) = q_{\text{in}}(t,L) - w(t). \]
The pressure is preserved, i.e. \( \rho_{\text{out}}(t,0) = \rho_{\text{in}}(t,L) \).

(iii) **Junction:** Ingoing and outgoing pipes are modelled by postulating equality of pressure and conservation of mass.

\[ \begin{align*}
\text{Node Conditions:} \\
\begin{pmatrix}
\rho(t,0) \\
q(t,L)
\end{pmatrix}
= K
\begin{pmatrix}
\rho(t,L) \\
q(t,0)
\end{pmatrix}
\end{align*} \]

Figure 1: Gas network with \( n \) pipes.

These conditions form the boundary control that regulates the steady state in which the network is operated. In the following we employ feedback control to damp small perturbations at steady state over time. Therefore, we introduce the transform into Riemann invariants as \( \mathcal{R}^\pm(y) := -q/\rho \mp a \ln(\rho) \). The linearized transformed system reads as
\[ \partial_t \mathcal{R}_j(t,x) + \bar{\Lambda}_j(x) \partial_x \mathcal{R}_j(t,x) = -\bar{C}_j(x) \bar{\mathcal{R}}_j(t,x) \]
for \( \bar{\Lambda}_j = \text{diag}\{\bar{\lambda}^+_j, \bar{\lambda}^-_j\} = \text{diag}\{\bar{u} + a, \bar{u} - a\}, \quad \bar{C}_j = \frac{\bar{f} |\bar{u}|}{2D} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \).

Here, \( \bar{u}(x) \) denotes the possibly space-varying velocity at steady state. The characteristic speeds satisfy \( \bar{\lambda}^-_j < 0 < \bar{\lambda}^+_j \) for subsonic states. We introduce the notation
\[ \begin{align*}
\mathcal{R} := (\mathcal{R}^+, \mathcal{R}^-)^T, \quad \mathcal{R}^\pm := (\mathcal{R}^+_1, \ldots, \mathcal{R}^+_n)^T, \\
\bar{\Lambda} := (\bar{\lambda}^+, \bar{\lambda}^-)^T, \quad \bar{\lambda}^\pm := (\bar{\lambda}^+_1, \ldots, \bar{\lambda}^+_n)^T
\end{align*} \]
to express the boundary value problem with initial values $\bar{R}(0, x) = \bar{I}(x)$ conveniently as

$$\begin{align*}
&\partial_t \bar{R}(t, x) + \bar{\Lambda}(x) \partial_x \bar{R}(t, x) = -\bar{C}(x) \bar{R}(t, x), \\
&\begin{pmatrix}
\bar{R}^+(t, 0) \\
\bar{R}^-(t, L)
\end{pmatrix} = \bar{G} \begin{pmatrix}
\bar{R}^+(t, L) \\
\bar{R}^-(t, 0)
\end{pmatrix}, \\
&\bar{R}(0, x) = \bar{I}(x).
\end{align*}$$

Since the Riemann invariants $\bar{R}$ describe perturbations at steady state, numerous previous works [4] are devoted to the specification of feedback boundary conditions $\bar{G}$ such that the system is exponentially stable in the sense

$$\int \left\| \bar{R}(t, \cdot) \right\|^2 dx \leq c e^{-\mu t} \int \left\| \bar{R}(0, \cdot) \right\|^2 dx \text{ with decay rate } \mu > 0$$

and positive constant $c > 0$. Figure 2 illustrates a network in Riemann invariants. We note that feedback boundary conditions make explicitly use of the direction of character speeds.

Figure 2: Gas network in Riemann invariants.

### 0.2 Representation of stochastic processes

To account for unknown space-dependent model parameters, we introduce a probability space $(\Omega, \mathcal{F}(\Omega), \mathbb{P})$ and the $L^2$-space

$$L^2(\Omega, \mathbb{P}) = \left\{ Pr : \Omega \to L^2(\mathbb{R}), \omega \mapsto Pr(\cdot; \omega) \mid \| Pr(x; \cdot) \|_p < \infty \right\}$$

for

$$\left\langle Pr(x_1; \cdot), Pr(x_2; \cdot) \right\rangle_p := \int Pr(x_1; \omega) Pr(x_2; \omega) d\mathbb{P}(\omega)$$

and

$$\| Pr(x; \cdot) \|_p := \sqrt{\left\langle Pr(x; \cdot), Pr(x; \cdot) \right\rangle_p}.$$
The expected value and the covariance kernel are defined by

\[
\mathbb{E}\left[ Pr(x_1; \omega) Pr(t, x_2; \omega) \right] := \langle Pr(x_1; \cdot), y(x_2; \cdot) \rangle_{\mathbb{P}},
\]

\[
\mathbb{C}(x_1, x_2) := \text{Cov}\left[ Pr(x_1; \omega), Pr(x_2; \omega) \right] := \mathbb{E}\left[ Pr(x_1; \omega) Pr(x_2; \omega) \right] - \mathbb{E}\left[ Pr(x_1; \omega) \right] \mathbb{E}\left[ Pr(x_2; \omega) \right].
\]

For a centered stochastic processes \( Pr \in L^2 \otimes L^2(\Omega, \mathbb{P}) \) that is mean square continuous, i.e. \( \| Pr(x; \cdot) - Pr(x^*; \cdot) \|_{\mathbb{P}} \to 0 \) for \( x \to x^* \in [0, L] \), the positive definite correlation kernel \( \mathbb{C} \) is continuous and bounded [38].

The Karhunen-Loève decomposition reads as

\[
\mathcal{K}[Pr](x; \omega) := \sum_{k=1}^{N_{KL}} \psi_k(x) \xi_k(\omega) \quad \text{for} \quad \xi_k(\omega) := \int Pr(x; \omega) \psi_k(x) \, dx. \quad (KL)
\]

The eigenfunctions \( \psi_k \) and eigenvalues \( \lambda_k \) of the kernel \( \mathbb{C} \) are given by

the Fredholm integral equation

\[
\int \mathbb{C}(x, \cdot) \psi_k(x) \, dx = \lambda_k \psi_k(\cdot). \quad (FI)
\]

Although the random variables \( \xi_k \) are formally defined, there is in general no practical expression. For Gaussian processes, however, they are given by independent standard normally distributed random variables \( \xi_k \sim \mathcal{N}(0, 1) \), see e.g. [38, 29]. Therefore, Gaussian processes are often used, although it is problematic to represent bounded physical processes, e.g. density of gas.

Dynamic stochastic gas flows, expressed by stochastic Riemann invariants \( Pr(t, x; \omega) = \bar{R}(t, x; \omega) \) cannot be expressed directly in terms of Karhunen-Loève expansions. Firstly, there is a multidimensional covariance structure \( \mathbb{C}(t_1, t_2, x_1, x_2) = \text{Cov}\left[ \bar{R}(t_1, x_1; \omega), \bar{R}(t_2, x_2; \omega) \right]. \)

Then, numerical solutions to the Fredholm integral equation (FI) are in general not feasible [36, 40]. Furthermore, the covariance structure must be known a priori. Here, it is given only implicitly by a hyperbolic boundary value problem. To generalize the Karhunen-Loève expansions, we introduce a generalized polynomial chaos (gPC) as a set of orthogonal subspaces

\[
\mathcal{S}_k \subseteq L^2(\Omega, \mathbb{P}) \quad \text{with} \quad S_K := \bigoplus_{k=0}^{K} \mathcal{S}_k \to L^2(\Omega, \mathbb{P}) \quad \text{for} \quad K \to \infty.
\]

We refer to an orthogonal basis of \( S_K \) as a gPC basis \( \{ \phi_k(\xi) \}_{k=0}^{K} \) with possibly multidimensional germ \( \xi(\omega) \in \mathbb{R}^{N_{KL}} \). Common choices, see e.g. [29, 39] are the following piecewise orthogonal functions:
Legendre polynomials with uniform distribution $\xi \sim U(-1, 1)$

$\phi_0(\xi) = 1$, $\phi_1(\xi) = \xi$, $\phi_{k+1}(\xi) = \frac{2k+1}{k+1}\xi\phi_k(\xi) - \frac{k}{k+1}\phi_{k-1}(\xi)$

Hermite polynomials with Gaussian distribution $\xi \sim N(0, 1)$

$\phi_0(\xi) = 1$, $\phi_1(\xi) = \xi$, $\phi_{k+1}(\xi) = \xi\phi_k(\xi) - k\phi_{k-1}(\xi)$

Then, a dynamic stochastic process $Pr(t, x; \xi)$ is approximated by an orthogonal projection

$G_K[Pr](t, x; \xi) = \sum_{k \in K} \hat{Pr}_k(t, x)\phi_k(\xi)$ with gPC modes ($K_{gPC}$)

$\hat{Pr}_k(t, x) = \frac{\langle Pr(t, x; \cdot), \phi_k(\cdot) \rangle_p}{\|\phi_k\|^2_p}$ and $\phi_k(\xi) := \phi_{k_1}(\xi_1) \cdot \cdots \cdot \phi_{k_M}(\xi_M)$.

Here, $k := (k_1, \ldots, k_M) \in K \subseteq N_0^{N_{KL}}$ is a multi-index. Common choices for index sets are the full tensor and sparse bases

$K = \{ k \in N_0^M \ | \ \|k\|_0 \leq K \}$ with $|K| = (K + 1)^{N_{KL}}$, ($K_T$)

$K = \{ k \in N_0^M \ | \ \|k\|_1 \leq K \}$ with $|K| = \frac{(N_{KL} + K)!}{N_{KL}!K!}$, ($K_S$)

The convergence $\|G_K[Pr](t, x; \cdot) - Pr(t, x; \cdot)\|_p \to 0$ for $K \to \infty$ is shown in [7, 19, 15]. The polynomial chaos approximation ($K_{gPC}$) with Hermite polynomials contains the Karhunen-Loève decomposition (KL) for a Gaussian process $Pr(x; \omega)$ as special case [38, 29], i.e.

$\mathcal{K}[Pr](x; \omega) = \mathbb{E}[Pr(x; \cdot)] + \sum_{k=1}^{N_{KL}} \sqrt{\lambda_k} \psi_k(x)\xi_k(\omega)$

$$= G_K[Pr](x; \xi(\omega)).$$

### 0.3 Stochastic Galerkin formulations

We replace all random quantities in the deterministic formulation (2) by the gPC approximation ($K_{gPC}$), i.e.

$\tilde{\lambda}^\pm(x; \xi) \approx G_K[\tilde{\lambda}^\pm](x; \xi) = \sum_{k \in K} \lambda^\pm_k(x)\phi_k(\xi)$,

$\tilde{C}(x; \xi) \approx G_K[\tilde{C}](x; \xi) = \sum_{k \in K} \tilde{C}_k(x)\phi_k(\xi)$,

$\tilde{R}(t, x; \xi) \approx G_K[\tilde{R}](t, x; \xi) = \sum_{k \in K} \tilde{R}_k(t, x)\phi_k(\xi)$. 
The random system is projected onto the gPC basis \( S_K \subseteq L^2(\Omega, \mathbb{P}) \) by
\[
\left\langle \partial_t G_K \mathbf{R}(t,x;\cdot) + G_K \mathbf{A}(x;\cdot) \partial_x G_K \mathbf{R}(t,x;\cdot) + G_K \mathbf{C}(x;\cdot) G_K \mathbf{R}(t,x;\cdot), \phi_k(\cdot) \right\rangle_\mathbb{P} = 0 \quad \text{for all} \quad k \in \mathbb{K}.
\]
This leads to the stochastic Galerkin formulation
\[
\partial_t \hat{\mathbf{R}}(t,x) + \hat{\mathbf{A}}(x) \partial_x \hat{\mathbf{R}}(t,x) = -\hat{\mathbf{S}}(x) \hat{\mathbf{R}}(t,x) \quad \text{for} \quad \hat{\mathbf{A}} := \begin{pmatrix} \hat{\mathbf{A}}^+ \\ \hat{\mathbf{A}}^- \end{pmatrix}
\]
with
\[
\hat{\mathbf{A}}^\pm(x) := \sum_{k \in \mathbb{K}} \lambda_k^\pm(x) \langle \phi_k, \phi_i \phi_j \rangle_{i,j \in \mathbb{K}}
\]
and
\[
\hat{\mathbf{S}}(x) := \sum_{k \in \mathbb{K}} \hat{\mathbf{C}}_k(x) \langle \phi_k, \phi_i \phi_j \rangle_{i,j \in \mathbb{K}}.
\]
Since the matrices \( \hat{\mathbf{A}}^\pm \) are symmetric, they have an orthogonal eigenvalue decomposition \( \hat{\mathbf{A}}^\pm = \hat{\mathbf{T}}^\pm \hat{\mathbf{D}}^\pm (\hat{\mathbf{T}}^\pm)^T \). This allows to diagonalize the stochastic Galerkin formulation. The IBVP reads as
\[
\partial_t \hat{\mathbf{z}}(t,x) + \hat{\mathbf{D}} \partial_x \hat{\mathbf{z}}(t,x) = -\hat{\mathbf{B}} \hat{\mathbf{z}}(t,x), \quad \hat{\mathbf{z}}(0,x) = \hat{\mathbf{T}}(x)^T \hat{\mathbf{R}}(0,x)
\]
with \( \hat{\mathbf{T}} := \text{diag}\{ \hat{\mathbf{T}}^+, \hat{\mathbf{T}}^- \} \). The characteristic speeds and the source term are
\[
\hat{\mathbf{D}}(x) := \text{diag}\{ \hat{\mathbf{D}}^+(x), \hat{\mathbf{D}}^-(x) \}, \quad \hat{\mathbf{B}}(x) := \hat{\mathbf{T}}(x)^T \hat{\mathbf{C}}(x) \hat{\mathbf{T}}(x) + \hat{\mathbf{D}}(x) \hat{\mathbf{T}}(x)^T \partial_x \hat{\mathbf{T}}(x).
\]
To ensure the wellposedness of the initial boundary value problem (4), we have to guarantee \( \hat{\mathbf{D}}^-(x) < 0 < \hat{\mathbf{D}}^+(x) \) for all \( x \in [0, L] \). The following lemma states two sufficient conditions.

**Lemma 0.1.** Assume that the gPC approximation \((\mathbf{KgPC})\) of the randomly perturbed eigenvalues satisfy one of the following properties:

(i) For all \( x \in [0, L] \) and \( \xi \sim \mathbb{P} \) the gPC approximation satisfies
\[
\mathcal{G}_K[\hat{\lambda}^-](x;\xi) < 0 < \mathcal{G}_K[\hat{\lambda}^+](x;\xi).
\]
(ii) Assume a Karhunen-Loève decomposition of a Gaussian random field that satisfies
\[
\left| \mathbb{E}[\tilde{\lambda}^\pm(x;\xi)] \right| > \sum_{k=1}^{N_{KL}} |\sqrt{\lambda_k} \psi_k(x)|.
\]
Then, the characteristic speeds of the IPVP (4) satisfy
\[
\hat{D}^-(x) < 0 < \hat{D}^+(x) \quad \text{for all} \quad x \in [0,L].
\]

Proof. The first statement is a special case of [35, Th. 2].

(i) For all \( \hat{y} \neq (0,\ldots,0)^T \) and basis functions \( \phi_k \) the equality
\[
\hat{y} \hat{A}^\pm \hat{y} = \pm \int \left( \sqrt{G_K[\tilde{\lambda}^\pm](x;\xi)} \sum_{k \in K} \hat{y}_k \phi_k(\xi) \right)^2 \, d\mathbb{P}
\]
implies \( \hat{y} \hat{A}^+ \hat{y} > 0 \) and \( \hat{y} \hat{A}^- \hat{y} < 0 \). Thus, the symmetric matrix \( \hat{A}^+ \) is strictly positive definite and \( \hat{A}^- \) is strictly negative definite.

(ii) By exploiting the orthogonality \( \langle \xi_k,\xi_\ell \rangle_\mathbb{P} = \delta_{k,\ell} \) in the expansion (3), we obtain the matrices
\[
\hat{A}^\pm = \begin{pmatrix}
|\mathbb{E}[\tilde{\lambda}^\pm(x;\xi)]| & \sqrt{\lambda_1} \psi_1(x) & \cdots & \sqrt{\lambda_{N_{KL}}} \psi_{N_{KL}}(x) \\
\sqrt{\lambda_1} \psi_1(x) & |\mathbb{E}[\tilde{\lambda}^\pm(x;\xi)]| & & \\
\vdots & & \ddots & \\
\sqrt{\lambda_{N_{KL}}} \psi_{N_{KL}}(x) & & & |\mathbb{E}[\tilde{\lambda}^\pm(x;\xi)]|
\end{pmatrix}.
\]
These are strictly positive definite according to Gershgorin circle theorem provided that property (ii) holds.

The presented sufficient conditions are not restrictive in practice, since characteristic speeds are mostly determined by the speed of sound, which is much larger than the velocity for relatively slow subsonic gas flows. The lemma requires that the deviations from the mean of the stochastic process is sufficiently small. We note that the mean and the
variance are directly given by the gPC modes for normalized orthogonal polynomials as
\[
\mathbb{E}\left[G_{K}\hat{\lambda}^\pm(x;\xi)\right] = \lambda_\Theta^\pm(x), \quad \Theta := (0,\ldots,0)^T,
\]
\[
\nabla\left[G_{K}\hat{\lambda}^\pm(x;\xi)\right] = \mathbb{E}\left[G_{K}\hat{\lambda}^\pm(x;\xi)^2\right] - \mathbb{E}\left[G_{K}\hat{\lambda}^\pm(x;\xi)\right]^2
= \sum_{k,\ell \in K} \lambda_k^\pm(x)\lambda_\ell^\pm(x)\langle\phi_k,\phi_\ell\rangle - \lambda_\Theta^\pm(x)^2
= \sum_{k \in K\setminus\{0\}} \lambda_k^\pm(x)^2.
\]
Therefore, property (ii) yields
\[
\left|\mathbb{E}\left[K\hat{\lambda}^\pm(x;\omega)\right]\right| > \sum_{k=1}^{N_{KL}} |\sqrt{\lambda_k}\psi_k(x)| \geq \sqrt{\sum_{k=1}^{N_{KL}} \lambda_k^2}(x)
= \nabla\left[K\hat{\lambda}^\pm(x;\omega)\right]^{1/2}.
\]
Thus, the coefficient of variation (CV) must satisfy
\[
CV\left[K\hat{\lambda}^\pm(x;\omega)\right] := \frac{\nabla\left[K\hat{\lambda}^\pm(x;\omega)\right]^{1/2}}{\mathbb{E}\left[K\hat{\lambda}^\pm(x;\omega)\right]} < 1 \quad \text{for all} \quad x \in [0,L].
\]
Then, finding $L^2$-solutions\(^\dagger\) to the augmented systems (4) is a well-posed problem [4, Th. A.4].

### 0.4 Lyapunov stability analysis

We look for boundary conditions such that the random Riemann invariants are exponentially stable in the sense
\[
\mathbb{E}\left[\|G_{K}[\overline{\mathcal{R}}](t,\cdot;\xi)\|^2\right] \leq c e^{-\mu t}\|\overline{\mathcal{R}}(0,\cdot)\|^2 \quad \text{with decay rate} \quad \mu > 0
\]
and positive constant $c > 0$. Note that this is a relatively strong stabilization concept that makes both the mean and the variance of deviations decay exponentially fast, since the mean squared error of deviations satisfies the expression
\[
\nabla\left[\|G_{K}[\overline{\mathcal{R}}](t,\cdot;\xi)\|\right] + \mathbb{E}\left[\|G_{K}[\overline{\mathcal{R}}](t,\cdot;\xi)\|^2\right] = \mathbb{E}\left[\|G_{K}[\overline{\mathcal{R}}](t,\cdot;\xi)\|^2\right].
\]
\(^\dagger\)The interested reader finds a precise definition in [4, Def. A.3].
By exploiting the orthogonality \( \hat{T}^T = \hat{T}^{-1} \) we obtain the relation
\[
\mathbb{E} \left[ \left\| G_K [\hat{R}] (t, \cdot; \xi) \right\|^2 \right] = \left\| \hat{\mathcal{K}} (t, \cdot) \right\|^2 = \left\| \hat{T} \hat{\mathcal{C}} (t, \cdot) \right\|^2 = \left\| \hat{\xi} (t, \cdot) \right\|^2 .
\]

We observe that the stochastic stabilization follows from the deterministic stabilization of the augmented system (4). However, if we compare the source term \( \bar{C} \) in the underlying deterministic system (1) with the source term \( \hat{B} \) in the augmented system (4), we note that the deterministic source term is positive semidefinite, whereas this property is not transferred to the source term \( \hat{B} \). Thus, the IBVP (4) involves additional instabilities that do not arise in the deterministic case, when the source term acts stabilizing due to friction effects. Recently, conditions were presented for general linear systems of the form (4) with distinct characteristic speeds. We summarize these conditions in the following theorem taken from [4, 20].

**Theorem 0.2.** Define the weights
\[
w_k^+(x) := \frac{h_k^+}{\hat{D}_k^+(x)} \exp \left( - \hat{\mu} \int_0^x \frac{1}{\hat{D}_k^+(s)} \, ds \right),
\]
\[
w_k^-(x) := \frac{h_k^-}{\hat{D}_k^-(x)} \exp \left( \hat{\mu} \int_x^L \frac{1}{\hat{D}_k^-(s)} \, ds \right)
\]
for \( k = 1, \ldots, |K| \) and assume positive values \( \hat{\mu}, h_k^+, h_k^- > 0 \) that satisfy the inequalities
\[
\hat{\mu} - 2 \min_{x \in [0, L]} \min_{k, \ell = 1, \ldots, K} \left\{ \frac{w_k^+(x)}{w_{\ell}^+(x)} \left\| \hat{\mathcal{B}} (x) \right\| \right\} > 0,
\]
\[
e^{\hat{\mu} \lambda_{\min} \left\| D \hat{\mathcal{G}} D^{-1} \right\|} \leq 1
\]
for \( \lambda_{\min} := \min_{x \in [0, L], k = 1, \ldots, |K|} \left\{ \left\| \hat{D}_k^\pm (x) \right\| \right\} \)
and \( D := \text{diag} \{ h_1^+, \ldots, h_{|K|}^+, h_1^-, \ldots, h_{|K|}^- \} \).

If property (i) or (ii) in Lemma 0.1 holds, the gPC expansion, described by the IBVP (4) is exponentially stable and satisfies
\[
\mathbb{E} \left[ \left\| G_K [\hat{R}] (t, \cdot; \xi) \right\|^2 \right] = \left\| \hat{\mathcal{C}} (t, \cdot) \right\|^2 \leq ce^{-\mu t} \left\| \hat{R}_0 \right\|^2 \quad \text{for} \quad \mu, c > 0.
\]
Sketch of proof. The weights (5) define the Lyapunov function

\[ L(t) := \int \hat{\zeta}(t, x)^T W(x) \hat{\zeta}(t, x) \, dx \] with

\[ W(x) := \text{diag}\{W^+(x), W^-(x)\}, \quad W^\pm(x) := \text{diag}\{w_1^\pm(x), \ldots, w_n^\pm(x)\}. \]

We define the matrices

\[ H := \hat{\mathcal{G}}^T \begin{pmatrix} W^+(0) \hat{\mathcal{D}}^+(0) & W^-(L) \hat{\mathcal{D}}^-(L) \\ W^+(L) \hat{\mathcal{D}}^+(L) & W^-(0) \hat{\mathcal{D}}^-(0) \end{pmatrix} \hat{\mathcal{G}} - \begin{pmatrix} W^+(0) \hat{\mathcal{D}}^+(0) & W^-(L) \hat{\mathcal{D}}^-(L) \\ W^+(L) \hat{\mathcal{D}}^+(L) & W^-(0) \hat{\mathcal{D}}^-(0) \end{pmatrix}, \quad (8) \]

\[ M(x) := -\frac{\partial}{\partial x} \left( W(x) \hat{\mathcal{D}}(x) \right) + W(x) \hat{\mathcal{B}}(x) + \hat{\mathcal{B}}(x)^T W(x). \quad (9) \]

It is shown in [20] that the matrix \( \mathcal{H} \) is negative semidefinite if inequality (7) holds and the matrix \( \mathcal{M} \) is strictly positive definite provided that inequality (6) holds. For now, we assume

\[ \hat{\zeta} \in C^1\left([0, \infty) \times [0, L]; \mathbb{R}^{2|\mathcal{K}|}\right). \quad (10) \]

Then, the time derivative of the Lyapunov function is

\[ L'(t) = 2 \int \hat{\zeta}(t, x)^T W(x) \partial_t \hat{\zeta}(t, x) \, dx \]

\[ = -2 \int \hat{\zeta}(t, x)^T W(x) \hat{\mathcal{D}}(x) \partial_x \hat{\zeta}(t, x) \, dx \]

\[ -2 \int \hat{\zeta}(t, x)^T W(x) \hat{\mathcal{B}}(x) \hat{\zeta}(t, x) \, dx \]

\[ = \int \hat{\zeta}(t, x)^T \left[ \frac{\partial}{\partial x} \left( W(x) \hat{\mathcal{D}}(x) \right) - W(x) \hat{\mathcal{B}}(x) - \hat{\mathcal{B}}(x)^T W(x) \right] \hat{\zeta}(t, x) \, dx \]

\[ - \left[ \hat{\zeta}(t, L)^T W(L) \hat{\mathcal{D}}(L) \hat{\zeta}(t, L) - \hat{\zeta}(t, 0)^T W(0) \hat{\mathcal{D}}(0) \hat{\zeta}(t, 0) \right]. \quad (11) \]

Using the linear feedback boundary conditions, the boundary terms (11) read as
We denote the smallest eigenvalue of a matrix as $\sigma_{\text{min}}$ and recall that the matrix $\mathcal{H}$ is negative semidefinite. Then, we obtain the estimates

$$
\mathcal{L}'(t) = \left( \begin{array}{c} \hat{\chi}^+(t, L) \\ \hat{\chi}^-(t, 0) \end{array} \right)^T \mathcal{H} \left( \begin{array}{c} \hat{\chi}^+(t, L) \\ \hat{\chi}^-(t, 0) \end{array} \right) - \int \hat{\chi}(t, x)^T \mathcal{M}(x) \hat{\chi}(t, x) \, dx
$$

$$
\leq - \min_{x \in [0, L]} \left\{ \sigma_{\text{min}} \left\{ W^{-1/2}(x) \mathcal{M}(x) W^{-1/2}(x) \right\} \right\} \mathcal{L}(t)
$$

$$
\implies \mathcal{L}(t) \leq e^{-\mu t} \mathcal{L}(0), \quad \mu := \min_{x \in [0, L]} \left\{ \sigma_{\text{min}} \left\{ W^{-1/2}(x) \mathcal{M}(x) W^{-1/2}(x) \right\} \right\}.
$$

Since the matrix $\mathcal{M}(x)$ is strictly positive definite for all $x \in [0, L]$, there is a positive decay rate $\mu > 0$.

So far, we have assumed differentiable solutions (10) that satisfy boundary conditions. The previous analysis is extended to general $L^2$-solutions in [4, Sec. 2.1.3]. Since initial values $\hat{\chi}(0, x) \in C^4((0, L); \mathbb{R}^{2[K]})$ are dense in $L^2((0, L); \mathbb{R}^{2[K]})$, there exists a differentiable sequence $\hat{\chi}^{(k)}(0, x)$ that converges to initial values $\hat{\chi}(0, x)$ in $L^2((0, L); \mathbb{R}^{2[K]})$. This sequence vanishes for $x = 0$ and $x = L$ and satisfies the linear boundary conditions. It is shown in [4, Th. A.1] that also the solution is differentiable and satisfies

$$
\hat{\chi}^{(k)} \in C^4([0, \infty); L^2((0, L)^{2[K]})) \cap C^0([0, \infty); H^1((0, L)^{2[K]})).
$$

This regularity is sufficient to obtain the sequence of estimates

$$
\mathcal{L}^{(k)}(t) \leq e^{-\mu t} \mathcal{L}^{(k)}(0) \quad \text{for} \quad \mathcal{L}^{(k)}(t) := \int \hat{\chi}^{(k)}(t, x)^T W(x) \hat{\chi}^{(k)}(t, x) \, dx,
$$

which implies $\mathcal{L}(t) \leq e^{-\mu t} \mathcal{L}(0)$ for $k \to \infty$. 

$\square$
Remark 1. Note that positive values $\hat{\mu}, h_+, h_- > 0$, which satisfy the sufficient conditions for exponential stability in Theorem 0.2, may not exist. This may happen if the length of a pipe is very large. Then, the gas flow may not be stabilizable by boundary feedback. These issues arise also in the deterministic case. According to [26, 27], stationary states exist as smooth solutions on a finite space interval only, until a critical length is reached. There, a blow-up in the derivatives occurs. Also explicit conditions have been presented, when certain systems cannot be stabilized [3, 25].

Another problem are approximation errors by the gPC expansion. An overview on approximation errors can be found in [30] and an analysis of the limit $K \rightarrow 0$ is found in [24, 23].

0.5 Applications to Gaussian random fields

Gaussian random fields are completely characterized by their mean and covariance kernel. For any $n \in \mathbb{N}$ and all $z := (z_1, \ldots, z_n)^T \in \mathbb{R}^n$ the distribution of the event \( \{ \omega \in \Omega \mid Pr(x_1; \omega) = z_1, \ldots, Pr(x_n; \omega) = z_n \} \) is given by the multivariate Gaussian probability density

\[
    f(z) = \frac{1}{\sqrt{(2\pi)^n|\Sigma|}} \exp \left( -\frac{1}{2}(z - \mu)^T\Sigma^{-1}(z - \mu) \right)
\]

with $\mu := \left( \mathbb{E}[Pr(x_i; \omega)] \right)_{i=1,\ldots,n}$ and $\Sigma := \left( \mathbb{C}(x_i, x_j) \right)_{i,j=1,\ldots,n}$.

A widely used class of covariance kernels that generate strictly positive definite matrices $\Sigma$ is the Matérn covariance function

\[
    \mathbb{C}_\nu(x_i, x_j) := \sigma^2 \frac{2^{1-\nu}}{\Gamma(\nu)} \left( \sqrt{2\nu \frac{|x_i - x_j|}{\lambda}} \right)^\nu K_\nu \left( \sqrt{2\nu \frac{|x_i - x_j|}{\lambda}} \right).
\]

Here, $\Gamma$ is the gamma function, $K_\nu$ is the modified Bessel function of the second kind, $\sigma^2$ denotes the variance and $\lambda > 0$ is a scaling parameter, which describes the spatial correlation of the stochastic process. The sample paths are $\nu - 1$ times differentiable [38]. The following cases are of special importance.

- **Exponential kernel:** $\mathbb{C}_{0.5}(x, y) = \sigma^2 \exp \left( -\frac{|x - y|}{\lambda} \right)$
- **Gaussian kernel:** $\lim_{\nu \rightarrow \infty} \mathbb{C}_\nu(x, y) = \sigma^2 \exp \left( -\frac{1}{2} \left( \frac{x - y}{\lambda} \right)^2 \right)$
The Gaussian covariance yields smooth sample paths. The exponential covariance kernel describes a stationary Ornstein-Uhlenbeck process, where sample paths are not differentiable.

The upper panels of Figure 3 show a simulation of random characteristic speeds which are mostly determined by the speed of sound \( a \approx 370 \text{ m/s} \). The mean is plotted as dashed, black line, the 0.95 confidence region \( \text{CI}(x) \), satisfying \( P[\text{Pr}(x; \omega) \in \text{CI}(x)] = 0.95 \), is shown in grey.

Since the density at steady state is decreasing, i.e. \( \bar{\rho}_x < 0 \), eigenvalues are increasing due to \( \bar{\lambda}_x^\pm = -\bar{q}\bar{\rho}^{-2}\bar{\rho}_x > 0 \). Therefore, a more realistic simulation is shown in the lower panels of Figure 3, where \( m = 3 \) measurements \( z^* \in \mathbb{R}^m \) at \( x^* = (0, 50, 100)^T \in \mathbb{R}^m \) are included. Intuitively, one may think of simulating from a Gaussian distribution and then rejecting all sample paths that disagree with the measurements. However, the processes restricted to measurements is again a Gaussian
process $P_r^*(x; \omega)$, see e.g. [38, A.2] for the expressions

$$
E[P_r^*(x; \omega)] = \left( C(x, x_i^*) \right)_{i=1,\ldots,m} \left( C(x^*_i, x^*_j) \right)_{i,j=1,\ldots,m} z^*,
$$

$$
\Sigma = \left( \operatorname{Cov}[P_r^*(x_i; \omega), P_r^*(x_j; \omega)] \right)_{i,j=1,\ldots,n}
$$

$$
= \left( C(x_i, x_j) \right)_{i,j=1,\ldots,n}
- \left( C(x_i, x_j^*) \right)_{i=1,\ldots,n,j=1,\ldots,m}
\left( C(x^*_i, x^*_j) \right)_{i,j=1,\ldots,m}
\left( C(x^*_i, x_j) \right)_{i=1,\ldots,m,j=1,\ldots,n}^{-1}
$$

Thus, our presented approach accounts also for measurements.

Figure 4: Karhunen-Loève decomposition for exponential kernel.
The Fredholm integral equation (FI) has been analyzed intensively. In particular, explicit solutions are known for the exponential and Gaussian kernels [38]. Furthermore, there are software packages available, for instance the chebfun-package [18], that allow numerical solutions for general kernels. In any case, the truncation $N_{KL}$ must be specified. Typical indicators are the explained ratio between the
d
$$\text{total variance} \quad \frac{\sum_{k=1}^{N_{KL}} \lambda_k}{\int \mathcal{C}(x,x) \, dx} \quad \text{and pointwise variance} \quad \frac{\sum_{k=1}^{N_{KL}} \lambda_k \psi_k^2(x)}{\mathcal{C}(x,x)}.$$ 

d
Figure 4 and 5 illustrate these error estimates. The first panels show the eigenfunctions $\psi_1, \ldots, \psi_4$. These are smooth functions that approximate the stochastic process as superpositions in the series expansion (KL). Thus, more eigenfunctions are necessary to approximate rough sample paths. Indeed, we observe from the second panels that eigenvalues $\lambda_k$ corresponding to the Gaussian kernel decrease faster
than those corresponding to the exponential kernel. In the case of a Gaussian kernel, more than 0.99 of the variance is explained with the choice $N_{KL} = 4$. In contrast, we would need approximately $N_{KL} = 40$ terms to obtain the same accuracy for the exponential kernel. The full tensor basis then has $|\mathbb{K}_T| = (1 + K)^{40}$ elements. Even if the sparse basis $(\mathbb{K}_S)$ is used, the computational complexity prevents an application to random fields with the exponential kernel. We conclude that although the presented approach accounts for general random fields including measurements, the computational complexity restricts applications to those that can be approximated by a relatively small number $N_{KL}$ of finite random variables.

0.6 Summary

We have considered linear hyperbolic balance law that describe gas flow. Stochastic influences have been introduced by series of orthogonal functions. A deterministic stabilization concept, which makes deviations at steady states decay exponentially fast, has been extended to stochastic influences. These can be described by general Gaussian processes. The computational complexity, however, may prevent an application to certain random fields.

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