New Extensions of Some Known Special Polynomials under the Theory of Multiple $q$-Calculus

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Abstract. In the year 2014, the idea of multiple $q$-calculus was formulated and introduced in the book of Nalci and Pashaev [9] in which this idea is simple but elegant in order to derive new generating functions of some special polynomials that are generalizations of known $q$-polynomials. In this paper, we will use Nalci and Pashaev’s method in order to find a systematic study of new types of the Bernoulli polynomials, Euler polynomials and Genocchi polynomials. Also we will obtain recursive formulas for these polynomials.

Keywords: Quantum calculus, Multiple quantum calculus, $q$-Bernoulli polynomials, $q$-Euler polynomials, $q$-Genocchi polynomials, Generating function.

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1. Introduction

1.1. $q$-Calculus. The usual quantum calculus (or recalled $q$-calculus) has been extensively studied for a long time by many mathematicians, physicists and engineers. The development of $q$-calculus stems from the applications in many fields such as engineering, economics, mathematicians and so on. One of the important branches of $q$-calculus is $q$-special polynomials. For example, Kim [18] constructed $q$-generalized Euler polynomials based on $q$-exponential function. Moreover, Srivastava et al. investigated Apostol $q$-Bernoulli, Apostol $q$-Euler polynomials and Apostol $q$-Genocchi polynomials. This is why $q$-calculus is thought as one of the useful tools to study with special numbers and polynomials. For more information related these issues, see, e.g. [1,2,3,5,6,8-13,16-21].

Before starting at multiple $q$-calculus, we first give some basic notations about $q$-calculus which can be found in [3].

For a real number (or complex number) $x$, $q$-number (quantum number) is known as

\[ [x]_q = \begin{cases} 1 - q^x & \text{if } q \neq 1, \\ x & \text{if } q = 1 \end{cases} \] (1.1)

which is also called non-symmetrical $q$-number. The followings can be easily derived using (1.1):

\[ [x + y]_q = [x]_q + q^x [y]_q \quad (q\text{-addition formula}) \] (1.2)
\[ [x - y]_q = -q^{x-y} [y]_q + [x]_q \quad (q\text{-subtraction formula}) \] (1.3)

\[ xy = [x]_q [y]_q^x \quad (q\text{-product rule}) \] (1.4)
\[ \frac{x}{y} = \frac{[x]_q}{[y]_q} \quad (q\text{-division rule}) \] (1.5)

where $x, y$ are real or complex numbers.

The $q$-binomial coefficients are defined for positive integer $n, k$ as

\[ \binom{n}{k}_q = \prod_{i=0}^{k-1} (1 - q^{-i}) \] (1.6)

where \( [n]_q ! = [n]_q [n-1]_q [n-2]_q \ldots [1]_q \), \( n = 1, 2, \ldots; \)
\( [0]_q ! = 1 \).

The $q$-derivative $D_q f(x)$ of a function $f$ is given as

\[ D_q f(x) = \frac{f(x) - f(qx)}{(1-q)x} \quad , x \neq 0, D_q f(0) = f'(0), \]

provided $f'(0)$ exists.

For any $z \in \mathbb{C}$ with $|z| < 1$, \( e_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{[n]_q !} \) and \( E_q(z) = \sum_{n=0}^{\infty} \frac{\zeta[n]_q}{[n]_q !} \).

For the $q$-commuting variables $x$ and $y$ such as $xy = qxy$, we know that
\[ e_q(x+y) = e_q(x) e_q(y). \]
The \( q \)-integral was defined by Jackson as follows:
\[
\int_{0}^{x} f(y) \, dq_{y} = (1 - q) x \sum_{n=0}^{\infty} f(q^{n} x) q^{n}
\]
provided that the series on right hand side converges absolutely.

1.2. \textbf{Multiple \( q \)-calculus}. All notations and all corollaries written in this part have been taken from the Book of Nalci and Pashaev [9]. Consider basis vector \( \tilde{q} \) with coordinates \( q_{1}, q_{2}, \ldots, q_{N} \) so that the multiple \( q \)-number can be defined as
\[
[n]_{q_{i}, q_{j}} = \frac{q_{i}^{n} - q_{j}^{n}}{q_{i} - q_{j}} = [n]_{q_{j}, q_{i}};
\]
which is symmetric. Hence, we can write \( N \times N \) matrix with \( q \)-numbers elements in the following form:
\[
\begin{pmatrix}
[n]_{q_{1}, q_{1}} & [n]_{q_{1}, q_{2}} & \cdots & [n]_{q_{1}, q_{N}} \\
[n]_{q_{2}, q_{1}} & [n]_{q_{2}, q_{2}} & & [n]_{q_{2}, q_{N}} \\
& \ddots & & \vdots \\
[n]_{q_{N}, q_{1}} & \cdots & & [n]_{q_{N}, q_{N}}
\end{pmatrix}
\]
Diagonal terms of this matrix are defined in the limit \( q_{j} \rightarrow q_{i} \) as
\[
\lim_{q_{j} \rightarrow q_{i}} [n]_{q_{i}, q_{j}} = \lim_{q_{j} \rightarrow q_{i}} \frac{q_{i}^{n} - q_{j}^{n}}{q_{i} - q_{j}} = nq_{i}^{n-1}.
\] (1.8)

So, by (1.8), we see that this symmetric matrix can be shown as
\[
\begin{pmatrix}
nq_{i}^{n-1} & [n]_{q_{1}, q_{2}} & \cdots & [n]_{q_{1}, q_{N}} \\
[n]_{q_{2}, q_{1}} & nq_{2}^{n-1} & & [n]_{q_{2}, q_{N}} \\
& \ddots & & \vdots \\
[n]_{q_{N}, q_{1}} & \cdots & & nq_{N}^{n-1}
\end{pmatrix}
\]
The followings can be easily derived using (1.7):

\( q \)-multiple addition formula
\[
[n + m]_{q_{i}, q_{j}} = q_{i}^{n} [m]_{q_{i}, q_{j}} + q_{j}^{m} [n]_{q_{i}, q_{j}}
\]

\( q \)-multiple subtraction formula
\[
[n - m]_{q_{i}, q_{j}} = -q_{j}^{m} [n]_{q_{i}, q_{j}} - q_{i}^{-m} [m]_{q_{i}, q_{j}}
\]

\( q \)-multiple product rule
\[
[nm]_{q_{i}, q_{j}} = [m]_{q_{i}, q_{j}} [n]_{q_{i}, q_{j}}^{m}
\]

\( q \)-multiple division rule
\[
\frac{[n]}{[m]}_{q_{i}, q_{j}} = \frac{[n]_{q_{i}, q_{j}}^{1}}{[m]_{q_{i}, q_{j}}^{1}} = \frac{[n]_{q_{i}, q_{j}}^{1}}{[m]_{q_{i}, q_{j}}^{1}}
\]

where \( n, m \) are real or complex numbers.

In multiple \( q \)-calculus, multiple \( q \)-derivative with base \( q_{i}, q_{j} \) is given by
\[
D_{q_{i}, q_{j}} f(x) = \frac{f(q_{i} x) - f(q_{j} x)}{(q_{i} - q_{j}) x}
\]
representing \( N \times N \) matrix of multiple \( q \)-derivative operators \( D := (D_{q_{i}, q_{j}}) \) which is sym-metric:
\[
D = \begin{pmatrix}
D_{q_{1}, q_{1}} & D_{q_{1}, q_{2}} & \cdots & D_{q_{1}, q_{N}} \\
D_{q_{2}, q_{1}} & D_{q_{2}, q_{2}} & & D_{q_{2}, q_{N}} \\
& \ddots & & \vdots \\
D_{q_{N}, q_{1}} & \cdots & & D_{q_{N}, q_{N}}
\end{pmatrix}
\]

\textbf{Corollary 1}. For \( N = 1 \) case and \( q_{1} = q_{2} = q \), we have
\[
[n]_{q, q} = nq^{n-1}
\]
and \( D_{q, q} = M_{q} \frac{d}{dx} \) where \( M_{q} = \frac{d}{dx} \). Also, in the case \( q = 1 \), we have the standard number \( [n]_{1, 1} = n \) and the usual derivative
\[
D_{1, 1} = \frac{d}{dx}.
\]

\textbf{Corollary 2}. For \( N = 2 \) case, we have
\[
[n]_{q_{1}, q_{2}} = nq_{1}^{n-1},
\]
\[
[n]_{q_{2}, q_{2}} = nq_{2}^{n-1}
\]
\[
D_{q_{1}, q_{1}} = M_{q_{1}} \frac{d}{dx}, D_{q_{1}, q_{2}} = D_{q_{2}, q_{1}} = M_{q_{1}} - M_{q_{2}} (q_{1} - q_{2}) x,
\]
\[
D_{q_{2}, q_{2}} = M_{q_{2}} \frac{d}{dx}
\]

\textbf{Corollary 3}. Choosing \( q_{1} = 1 \) and \( q_{2} = q \) gives non-symmetrical case as
\[
[n]_{1, q} = n, [n]_{q, 1} = [n]_{q, q} = nq^{n-1},
\]
\[
D_{1, q} = \frac{d}{dx}, D_{q, q} = M_{q} (1 - q) x, D_{q, q} = M_{q} \frac{d}{dx}
\]

\textbf{Corollary 4}. Taking \( q_{1} = q \) and \( q_{2} = \frac{1}{q} \) gives symmetrical case as
\[
[n]_{q, q} = nq^{n-1}, [n]_{\frac{1}{q}, \frac{1}{q}} = \frac{1}{q} n^{n-1}, [n]_{\frac{1}{q}, q} = \frac{1}{q} n^{n-1}, [n]_{q, \frac{1}{q}} = \frac{1}{q} n^{n-1}
\]
\[
D_{q, q} = M_{q} \frac{d}{dx}, D_{\frac{1}{q}, \frac{1}{q}} = D_{1, q} = M_{1} - M_{q} \left( \frac{1}{q} - q \right) x,
\]
\[
D_{\frac{1}{q}, q} = \frac{d}{dx}, D_{q, \frac{1}{q}} = \frac{1}{q} M_{q} \frac{d}{dx}
\]
\[
D_{1, \frac{1}{q}} = M_{1} \frac{d}{dx}.
\]
The multiple $q$-analogue of $(x - a)^n$ is the polynomial
\[
(x + a)^n_{q_i, a_j} := \left\{ \begin{array}{ll}
(x + q_k^{i-1} a)(x + q_k^{j-2} a) \ldots & \text{if } n \geq 1 \\
1 & \text{if } n = 1
\end{array} \right.
\]
or equivalently
\[
(x + a)^n_{q_i, a_j} = \sum_{k=0}^{n-1} \left( \begin{array}{c}
q_k^{i-1} a \\
q_k^{j-2} a
\end{array} \right) \frac{k(k-1)}{2} x^{n-k} a^k
\]
where $x$ and $a$ is commutative, $xa = ax$. $q$-multiple Binomial coefficients and multiple $q$-factorial are defined by
\[
[n]_{q_i, a_j} := \left\{ \begin{array}{ll}
[n]_{q_i, a_j} [n-1]_{q_i, a_j} \ldots [2]_{q_i, a_j} [1]_{q_i, a_j} & \text{if } n \geq 1 \\
1 & \text{if } n = 1
\end{array} \right. (n \in \mathbb{N}).
\]
Two types of multiple $q$-exponential functions are defined by
\[
e^{q_i, a_j}(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]_{q_i, a_j} !} \\
E^{q_i, a_j}(x) = \sum_{n=0}^{\infty} \left( \begin{array}{c}
q_k^{i-1} a \\
q_k^{j-2} a
\end{array} \right) \frac{n(n-1)}{2} \frac{x^n}{[n]_{q_i, a_j} !}
\]
which satisfy the following condition for commutative $x$ and $y$, $xy = yx$
\[
e^{q_i, a_j}(x+y)_{q_i, a_j} = e^{q_i, a_j}(x)E_{q_i, a_j}(y).
\]
The generalization of Jackson’s integral (called multiple $q$-integral) is given by
\[
\int f(x) \frac{d_{q_i, a_j} x}{q_i} = (q_i - q_j) \sum_{k=0}^{\infty} q_k^i x \int_{q_k^{j-1}}^{q_k^{j+1}} f(x) dx.
\]
Let $f(x) = \sum_{k=0}^{\infty} a_k x^k$ be formal power series. Applying multiple $q$-integral to the both sides of $f(x)$ gives
\[
\int f(x) \frac{d_{q_i, a_j} x}{q_i} = \sum_{k=0}^{\infty} a_k \frac{x^{k+1}}{[k+1]_{q_i, a_j}} + C
\]
where $C$ is constant.

In the next section, we will use Nalci and Pashaev’s method in order to find a systematic study of new types of the Bernoulli polynomials, Euler polynomials and Genocchi polynomials. Also we will obtain recursive formulas for these polynomials.

2. Main Results

Recently, analogues of Bernoulli, Euler and Genocchi polynomials were studied by many mathematicians [1,2,5,6,11,12,13,17,18,19,20,21]. We are now ready to give the definition of generating functions, corresponding to multiple $q$-calculus, of Bernoulli type, Euler type and Genocchi type polynomials.

**Definition 1.** Let $n$ be positive integer, we define
\[
S(x, z : q_i, q_j) = \sum_{n=0}^{\infty} \frac{z^n}{[n]_{q_i, a_j} !} e^{q_i, a_j}(z)
\]
and
\[
U(x, z : q_i, q_j) = \sum_{n=0}^{\infty} \frac{z^n}{[n]_{q_i, a_j} !} e^{q_i, a_j}(z)
\]
where $B_n(x|q_i, q_j)$, $E_n(x|q_i, q_j)$ and $G_n(x|q_i, q_j)$ are called, respectively, Bernoulli-type, Euler-type and Genocchi-type polynomials.

**Corollary 5.** Taking $q_i = q_j = 1$ for indexes $i$ and $j$ in the case $N = 1$ in Definition 1, we have
\[
\sum_{n=0}^{\infty} B_n(x|q_i, q_j) \frac{z^n}{[n]_{q_i, a_j} !} = \frac{z}{e^z - 1} e^{q_i, a_j}(z) \quad (|z| < 2\pi)
\]
and
\[
\sum_{n=0}^{\infty} E_n(x|q_i, q_j) \frac{z^n}{[n]_{q_i, a_j} !} = \frac{2}{e^z + 1} e^{q_i, a_j}(z) \quad (|z| < \pi)
\]
and
\[
\sum_{n=0}^{\infty} G_n(x|q_i, q_j) \frac{z^n}{[n]_{q_i, a_j} !} = \frac{2z}{e^z + 1} e^{q_i, a_j}(z) \quad (|z| < \pi)
\]
where $B_n(x)$, $E_n(x)$ and $G_n(x)$ are called Bernoulli polynomials, Euler polynomials and Genocchi polynomials, respectively (see [4,7,14,15,19]).

**Corollary 6.** As a special case of Definition 1, we have
\[
\sum_{n=0}^{\infty} B_n(x|q_i, q_j) \frac{z^n}{[n]_{q_i, a_j} !} = \frac{z}{e^z - 1} e^{q_i, a_j}(z) \quad (|z| < 2\pi)
\]
and
\[
\sum_{n=0}^{\infty} E_n(x|q_i, q_j) \frac{z^n}{[n]_{q_i, a_j} !} = \frac{2}{e^z + 1} e^{q_i, a_j}(z) \quad (|z| < \pi)
\]
and
\[
\sum_{n=0}^{\infty} G_n(x|q_i, q_j) \frac{z^n}{[n]_{q_i, a_j} !} = \frac{2z}{e^z + 1} e^{q_i, a_j}(z) \quad (|z| < \pi)
\]
where $B_n(x|q_i, q_j)$, $E_n(x|q_i, q_j)$ and $G_n(x|q_i, q_j)$ are called $q$-Bernoulli polynomials, $q$-Euler polynomials and $q$-Genocchi polynomials, respectively (see [18,20,21]).

Taking $x = 0$ in the above definition, we have
\[
B_n(0 : q_i, q_j) := B_n(q_i, q_j) \quad \text{(Bernoulli-type number)}
\]
\( \mathcal{E}_n(0: q_i, q_j) := \mathcal{E}_n(q_i, q_j) \) \hspace{1cm} \text{(Euler-type number)}

\( \mathcal{G}_n(0: q_i, q_j) := \mathcal{G}_n(q_i, q_j) \) \hspace{1cm} \text{(Genocchi-type number)}

From Definition 1 and (2.1), we get the following corollary.

**Corollary 7.** The following functional equations hold true:

\[
\mathcal{S}(x, z: q_i, q_j) := \mathcal{S}(z: q_i, q_j) e^{q_i x} e^{q_j z} (xz),
\]

\[
\mathcal{U}(x, z: q_i, q_j) := \mathcal{U}(z: q_i, q_j) e^{q_i x} e^{q_j z} (xz),
\]

\[
\mathcal{M}(x, z: q_i, q_j) := \mathcal{M}(z: q_i, q_j) e^{q_i x} e^{q_j z} (xz).
\]

From Definition 1 and Corollary 7, it becomes

\[
\sum_{n=0}^{\infty} \mathcal{B}_n(x: q_i, q_j) \frac{z^n}{n!} = \left( \sum_{n=0}^{\infty} \mathcal{B}_n(q_i, q_j) \frac{z^n}{n!} \right) \left( \sum_{n=0}^{\infty} \mathcal{B}_n(q_i, q_j) \frac{z^n}{n!} \right).
\]

Comparing the coefficients of \( \frac{z^n}{n!} \) in (2.2), we have

\[
\mathcal{B}_n(x: q_i, q_j) = \sum_{k=0}^{n} \mathcal{B}_k(q_i, q_j) \frac{z^n}{n!}.
\]

From this, we can get similar identities for Euler-type numbers and Genocchi-type numbers. Therefore, we state the following theorem.

**Theorem 1.** The following identities hold true:

\[
\mathcal{B}_n(x: q_i, q_j) = \sum_{k=0}^{n} \mathcal{B}_k(q_i, q_j) x^{n-k},
\]

\[
\mathcal{E}_n(x: q_i, q_j) = \sum_{k=0}^{n} \mathcal{E}_k(q_i, q_j) x^{n-k},
\]

\[
\mathcal{G}_n(x: q_i, q_j) = \sum_{k=0}^{n} \mathcal{G}_k(q_i, q_j) x^{n-k}.
\]

Now we are in a position to investigate some properties of Bernoulli-type numbers and polynomials, Euler-type numbers and polynomials and Genocchi-type numbers and polynomials as follows.
Proof. If we change $x$ by $x+y$ in $S_n(x,z:q_i,q_j)$, then we acquire
\[
\sum_{n=0}^{\infty} B_n(x+y:q_i,q_j) \frac{z^n}{[n]_{q_i,q_j}!} = e_{q_i,q_j}^{-1}(z(t+y)) = e_{q_i,q_j}^{-1}(z(t)) e_{q_i,q_j}^{-1}(t+y) = e_{q_i,q_j}^{-1}(z) e_{q_i,q_j}^{-1}(t) e_{q_i,q_j}^{-1}(y).
\]

By computing the coefficient \( \frac{z^n}{[n]_{q_i,q_j}!} \) of both of side, then we have
\[
B_n(x+y:q_i,q_j) = \sum_{k=0}^{\infty} \binom{n}{k} B_k(x:q_i,q_j) y^{n-k}.
\]

The others can be proved in a like manner.

Now we consider the special cases of Theorem 3 as Corollary 8 and Corollary 9.

**Corollary 8.** Letting $y = 1$ in the Theorem 3, we then get
\[
B_n(x+1:q_i,q_j) = \sum_{k=0}^{\infty} \binom{n}{k} B_k(x:q_i,q_j) y^{n-k}.
\]

**Corollary 9.** Letting $x = 0$ in the Theorem 3, we then get
\[
B_n(x:q_i,q_j) = \sum_{k=0}^{\infty} \binom{n}{k} B_k(q_i,q_j) y^{n-k}.
\]

**Theorem 4.** The following expressions hold true for $n \in \mathbb{N}$
\[
B_n(x+1:q_i,q_j) - B_n(x:q_i,q_j) = [n]_{q_i,q_j} x^{n-1},
\]
\[
E_n(x+1:q_i,q_j) + E_n(x:q_i,q_j) = [2]_{q_i,q_j} x^n,
\]
\[
G_n(x+1:q_i,q_j) + G_n(x:q_i,q_j) = [2]_{q_i,q_j} [n]_{q_i,q_j} x^{n-1}.
\]

**Proof.** By using definitions of these polynomials and numbers, one can easily obtain these relations.

**Theorem 5.** (Identity of Symmetry) The followings hold true for $n \in \mathbb{N}$:
\[
\begin{align*}
B_n(1-x:q_i,q_j) &= (-1)^n B_n(x:q_i,q_j), \\
E_n(1-x:q_i,q_j) &= (-1)^n E_n(x:q_i,q_j), \\
G_n(1-x:q_i,q_j) &= (-1)^{n+1} G_n(x:q_i,q_j).
\end{align*}
\]

**Proof.** Setting $1-x$ instead of $x$ in $S_n(x,z:q_i,q_j)$, we then get
\[
\sum_{n=0}^{\infty} B_n(1-x:q_i,q_j) \frac{z^n}{[n]_{q_i,q_j}!} = \frac{z}{e_{q_i,q_j}(z(t))} = \frac{z}{e_{q_i,q_j}(z)} = \frac{z}{e_{q_i,q_j}(z) - e_{q_i,q_j}(t)} = \sum_{n=0}^{\infty} (-1)^n B_n(x:q_i,q_j) \frac{z^n}{[n]_{q_i,q_j}!}.
\]

Comparing coefficients \( \frac{z^n}{[n]_{q_i,q_j}!} \) of both of side in above equality, we have desired the result. Similar to that of this proof, it can be proved for Euler-type polynomials and Genocchi-type polynomials. So we completed this proof.

**Theorem 6.** (Raabe’s Formula) For $n \in \mathbb{N}$, the followings hold true
\[
\begin{align*}
\mathcal{B}_n(dx:q_i,q_j) &= d^{n-1} \sum_{k=0}^{d-1} \mathcal{B}_n(x+k\frac{d}{d}:q_i,q_j) \quad (d \in \mathbb{Z}^+), \\
\mathcal{E}_n(dx:q_i,q_j) &= d^n \sum_{k=0}^{d-1} (-1)^k \mathcal{E}_n(x+k\frac{d}{d}:q_i,q_j) \quad (d = 1 \text{ mod } 2), \\
\mathcal{G}_n(dx:q_i,q_j) &= d^{n-1} \sum_{k=0}^{d-1} (-1)^k \mathcal{G}_n(x+k\frac{d}{d}:q_i,q_j) \quad (d = 1 \text{ mod } 2).
\end{align*}
\]

**Proof.** By using Definition 1, then we have
\[
\sum_{n=0}^{\infty} \left( d^{n-1} \sum_{k=0}^{d-1} \mathcal{B}_n(x+k\frac{d}{d}:q_i,q_j) \right) \frac{z^n}{[n]_{q_i,q_j}!} = \frac{z}{d} \sum_{k=0}^{d-1} e_{q_i,q_j}(xdz) \frac{z^n}{[n]_{q_i,q_j}!}.
\]

Similarly, we can prove this theorem for Euler-type numbers and Genocchi-type polynomials. So we omit them. Hence, we complete the proof of this theorem.
Theorem 7. The three relations between Euler-type numbers and polynomials and Genocchi-type numbers and polynomials are given by

\[ E_n(q_i, q_j) = \frac{G_{i+1}(q_i, q_j)}{n+1}, \]

\[ E_n(x; q_i, q_j) = \frac{G_{i+1}(x; q_i, q_j)}{n+1}, \]

\[ E_n^*(x; q_i, q_j) = \sum_{k=0}^{\infty} \frac{G_{k+1}(q_i, q_j)}{k+1} x^{n-k}. \]

Proof. By Definition 1, we can easily obtain these relations. So we omit the proof.

Let us now apply the multiple q-derivative \( D_{q_i, q_j} \), with respect to \( x \), on the both sides of Definition 1,

\[ D_{q_i, q_j} \left( \sum_{n=0}^{\infty} E_n(x; q_i, q_j) \right) \frac{z^n}{[n]_{q_i, q_j}!} \]

\[ = \sum_{n=0}^{\infty} D_{q_i, q_j} E_n(x; q_i, q_j) \frac{z^n}{[n]_{q_i, q_j}!} \]

\[ = \frac{z}{e_{q_i, q_j}(x)-1} \left( D_{q_i, q_j} e_{q_i, q_j}(x) \right) \]

\[ = \frac{z^2}{e_{q_i, q_j}(x)-1} D_{q_i, q_j} e_{q_i, q_j}(x) \]

\[ = \sum_{n=0}^{\infty} E_n(x; q_i, q_j) \frac{z^{n+1}}{[n]_{q_i, q_j}!} \]

Matching the coefficients of \( \frac{z^n}{[n]_{q_i, q_j}!} \) gives us

\[ D_{q_i, q_j} E_n(x; q_i, q_j) = [n]_{q_i, q_j} E_{n-1}(x; q_i, q_j). \]

Thus we procure the following theorem.

Theorem 8. The following identities hold true:

\[ D_{q_i, q_j} E_n(x; q_i, q_j) = [n]_{q_i, q_j} E_{n-1}(x; q_i, q_j), \]

\[ D_{q_i, q_j} E_n^*(x; q_i, q_j) = [n]_{q_i, q_j} E_{n-1}^*(x; q_i, q_j) \]

and

\[ D_{q_i, q_j} E_n^*(x; q_i, q_j) = [n]_{q_i, q_j} E_{n-1}^*(x; q_i, q_j). \]

Applying \( k \)-times the operator \( D_{q_i, q_j} \) denoted by \( D_{q_i, q_j}^k \) and the limit \( t \to 0 \), respectively, to the Definition 1, we derive that

\[ E_n\left(x; q_i, q_j\right) = \lim_{t \to 0} D_{q_i, q_j}^k \frac{ze_{q_i, q_j}(xz)}{e_{q_i, q_j}(z)-1}. \]

So we conclude the following theorem.

Theorem 9. For \( k \geq 0 \) and \( n \geq 0 \), we have

\[ E_n\left(x; q_i, q_j\right) = \frac{\mathcal{E}_{n+1}\left(1; q_i, q_j\right)}{[n+1]_{q_i, q_j}}. \]
\[ \int_0^1 G_n \left( \frac{x}{q_i} ; q_i, q_j \right) d \frac{x}{q_i} = \frac{\mathcal{G}_{n+1} \left( 1 : q_i, q_j \right) - \mathcal{G}_{n+1} \left( q_i, q_j \right)}{n!} \].

**Proof.** By using Theorem 1, Definition 2 and for \( \frac{q_j}{q_i} < 1 \), we have

\[ \int_0^1 B_n \left( \frac{x}{q_i} ; q_i, q_j \right) d \frac{x}{q_i} \]

\[ = \sum_{l=0}^{n} \left( \frac{1}{l!} \right) \mathcal{B}_{n-l} \left( q_i, q_j \right) \left( q_i - q_j \right) \sum_{k=0}^{\infty} \frac{q^k}{q_i^{k+1}} \left( \frac{q^k}{q_i^{k+1}} \right)^l \]

\[ = \sum_{l=0}^{n} \left( \frac{1}{l!} \right) \mathcal{B}_{n-l} \left( q_i, q_j \right) \frac{1}{q_i} \sum_{l=0}^{\infty} \frac{q^k}{q_i^{k+1}} \left( \frac{q^k}{q_i^{k+1}} \right)^l \]

\[ = \mathcal{B}_{n+1} \left( 1 : q_i, q_j \right) - \mathcal{B}_{n+1} \left( q_i, q_j \right) \left[ \frac{n+1}{q_i} \right] \]

Similarly, the identities of Euler-type polynomials and Genocchi-type polynomials can be shown. Therefore, we complete the proof of theorem.

### 3. Further Remarks

Here we list a few values of Bernoulli-type, Euler-type and Genocchi-type numbers as follows:

**Bernoulli-type number:**

| \( B_0(q_i,q_j) \) | 1 |
| \( B_1(q_i,q_j) \) | \( \frac{1}{2q_i} \) |
| \( B_2(q_i,q_j) \) | \( \frac{1}{3q_i} + \frac{2}{2q_i} \) |
| \( B_3(q_i,q_j) \) | \( \frac{2}{4q_i} + \frac{[q_i]}{2q_i} \) |
| \( B_4(q_i,q_j) \) | \( \frac{3}{5q_i} + \frac{[q_i]}{2q_i} \) |

As a special case of Table 1, we get

\[ B_0(q_i,q_j) = 1 \]

\[ B_1(q_i,q_j) = \frac{1}{2q_i} \]
From Definition 1 and the Table 1, we easily acquire the first few Bernoulli-type polynomials

Bernoulli-type polynomials

\[ \mathcal{B}_0(x; q_i, q_j) = 1 \]
\[ \mathcal{B}_1(x; q_i, q_j) = x - \frac{1}{[2]_{q_i, q_j}} \]
\[ \mathcal{B}_2(x; q_i, q_j) = x^2 - x - \frac{1}{[2]_{q_i, q_j}} + \frac{1}{[3]_{q_i, q_j}} \]

Usual Bernoulli polynomials

\[ B_0 = 1 \]
\[ B_1 = x - \frac{1}{2} \]
\[ B_2 = x^2 - x - \frac{1}{6} \]

Moreover, the first few Bernoulli-type polynomials can be shown as a \( N \times N \) matrix with \( q \)-numbers elements in the following form

\[
\begin{pmatrix}
\mathcal{B}_0(x; q_i, q_j) & \mathcal{B}_1(x; q_i, q_j) & \mathcal{B}_2(x; q_i, q_j) \\
\mathcal{B}_1(x; q_i, q_j) & \mathcal{B}_0(x; q_i, q_j) & \mathcal{B}_1(x; q_i, q_j) \\
\mathcal{B}_2(x; q_i, q_j) & \mathcal{B}_1(x; q_i, q_j) & \mathcal{B}_0(x; q_i, q_j)
\end{pmatrix}
\]

Euler-type Numbers and Polynomials:

We begin to compute the first few value of \( \mathcal{E}_n(q_i, q_j) \) as follows:

| \( n \) | \( \mathcal{E}_n(q_i, q_j) \) |
|-------|------------------|
| 0     | \( \frac{[2]_{q_i, q_j}}{2} \) |
| 1     | \( -\frac{[2]_{q_i, q_j}}{4} \) |
| 2     | \( \frac{[2]_{q_i, q_j}}{4} + \left( \frac{[2]_{q_i, q_j}}{4} \right)^2 \) |
| 3     | \( \frac{[2]_{q_i, q_j}}{4} + \frac{3[2]_{q_i, q_j}}{4} + \left( \frac{[2]_{q_i, q_j}}{4} \right)^3 \) |
| 4     | \( \frac{[2]_{q_i, q_j}}{4} + \frac{5[2]_{q_i, q_j}}{4} + \frac{15[2]_{q_i, q_j}}{4} + \left( \frac{[2]_{q_i, q_j}}{4} \right)^4 \) |
| 5     | \( \frac{[2]_{q_i, q_j}}{4} + \frac{7[2]_{q_i, q_j}}{4} + \frac{35[2]_{q_i, q_j}}{4} + \left( \frac{[2]_{q_i, q_j}}{4} \right)^5 \) |

As a special case of Table 2, we have
As a special case of Table 2, we have

\[
E_0 (q) = \frac{[2]_q}{2} \\
E_1 (q) = -\frac{[2]_q}{4} \\
E_2 (q) = -\frac{[2]_q}{4} + \frac{([2]_q)^2}{8} \\
E_3 (q) = -\frac{[2]_q}{4} + \frac{[3]_q[2]_q}{4} \left( 1 - \frac{[2]_q}{4} \right) \\
E_4 (q) = -\frac{[2]_q}{4} + \frac{[4]_q[2]_q}{4} + \frac{[4]_q[3]_q[2]_q}{8} \left( \frac{1}{2} \left( \frac{3}{4} \right) \frac{3}{2} \right) + \frac{[2]_q}{4} \\
E_5 (q) = -\frac{[2]_q}{4} + \frac{[5]_q[2]_q}{4} + \frac{[5]_q[4]_q[2]_q}{8} \left( \frac{3}{4} \frac{1}{4} \left( \frac{2}{4} \right) \right) \\
\frac{16}{2} \left( \frac{4}{2} \right) + \frac{3}{2} \left( \frac{2}{4} \right) + \frac{2}{4} 
\]

Moreover the first few Euler-type numbers can be shown \( N \times N \) matrix with \( q \)-numbers elements in the following form

\[
\begin{pmatrix}
q_1 & \frac{[2]_{q_1,q_2}}{2} & \cdots & \frac{[2]_{q_1,q_N}}{2} \\
\frac{[2]_{q_2,q_1}}{2} & q_2 & \cdots & \frac{[2]_{q_2,q_N}}{2} \\
& \ddots & \ddots & \ddots \\
\frac{[2]_{q_N,q_1}}{2} & \cdots & \frac{[2]_{q_N,q_2}}{2} & q_N
\end{pmatrix}
\]

\[
\begin{pmatrix}
\frac{q_1}{2} & -\frac{[2]_{q_1,q_2}}{4} & \cdots & -\frac{[2]_{q_1,q_N}}{4} \\
\frac{[2]_{q_2,q_1}}{4} & \frac{q_2}{2} & \cdots & -\frac{[2]_{q_2,q_N}}{4} \\
& \ddots & \ddots & \ddots \\
-\frac{[2]_{q_N,q_1}}{4} & \cdots & \frac{[2]_{q_N,q_2}}{4} & \frac{q_N}{2}
\end{pmatrix}
\]

From Definition 1 and the Table 2, we easily acquire the first few Euler-type polynomials

**Euler-type polynomials**

\[
\mathcal{E}_0 (x : q_i, q_j) = \frac{[2]_{q_i,q_j}}{2} \\
\mathcal{E}_1 (x : q_i, q_j) = \frac{[2]_{q_i,q_j}}{2} x - \frac{[2]_{q_i,q_j}}{4} \\
\mathcal{E}_2 (x : q_i, q_j) = \frac{[2]_{q_i,q_j}}{2} x^2 - \frac{[2]_{q_i,q_j}}{4} x \\
-\frac{[2]_{q_i,q_j}}{4} + \frac{[2]_{q_i,q_j}}{2} 
\]

**Usual Euler polynomials**

\[
E_0 = 1 \\
E_1 = x - \frac{1}{2} \\
E_2 = x^2 - x
\]

Moreover the first few Euler-type polynomials can be shown \( N \times N \) matrix with multiple \( q \)-numbers elements in the following form

\[
\begin{pmatrix}
q_1 & \frac{[2]_{q_1,q_2}}{2} & \cdots & \frac{[2]_{q_1,q_N}}{2} \\
\frac{[2]_{q_2,q_1}}{2} & q_2 & \cdots & \frac{[2]_{q_2,q_N}}{2} \\
& \ddots & \ddots & \ddots \\
\frac{[2]_{q_N,q_1}}{2} & \cdots & \frac{[2]_{q_N,q_2}}{2} & q_N
\end{pmatrix}
\]

\[
\begin{pmatrix}
\frac{q_1}{2} & -\frac{[2]_{q_1,q_2}}{4} & \cdots & -\frac{[2]_{q_1,q_N}}{4} \\
\frac{[2]_{q_2,q_1}}{4} & \frac{q_2}{2} & \cdots & -\frac{[2]_{q_2,q_N}}{4} \\
& \ddots & \ddots & \ddots \\
-\frac{[2]_{q_N,q_1}}{4} & \cdots & \frac{[2]_{q_N,q_2}}{4} & \frac{q_N}{2}
\end{pmatrix}
\]
Genocchi-type Numbers and Polynomials:
We begin to compute the first few value of $G_i (q_i, q_j)$ as follows:

| $G_0 (q_i, q_j) = 0$ |
|----------------------|
| $G_1 (q_i, q_j) = \frac{[2]_{q_i, q_j}}{2}$ |
| $G_2 (q_i, q_j) = -\frac{[2]_{q_i, q_j}^2}{4}$ |
| $G_3 (q_i, q_j) = -\frac{[3]_{q_i, q_j}}{4} - \frac{[2]_{q_i, q_j}}{2}$ |
| $G_4 (q_i, q_j) = -\frac{[4]_{q_i, q_j}}{4} + \frac{[4]_{q_i, q_j}^2}{8} - \frac{[4]_{q_i, q_j}}{8} \left( -\frac{[2]_{q_i, q_j}}{2} + \frac{[2]_{q_i, q_j}}{2} \right)$ |
| $G_5 (q_i, q_j) = -\frac{[5]_{q_i, q_j}}{4} + \frac{[5]_{q_i, q_j}^2}{8} - \frac{[5]_{q_i, q_j}}{8} \left( 1 + \frac{[2]_{q_i, q_j}}{2} \right)$ |

As a special case of Table 3, we have

| $G_0 (q) = 0$ |
|----------------|
| $G_1 (q) = \frac{[2]_{q}}{2}$ |
\[ \mathcal{G}_2(q) = \frac{([2]_q)^2}{4} \]

\[ \mathcal{G}_3(q) = \frac{[3]_q}{4} \left( -[2]_q + \frac{([2]_q)^2}{2} \right) \]

\[ \mathcal{G}_4(q) = -\frac{[4]_q [2]_q}{4} + \frac{[4]_q [3]_q [2]_q}{8} - \frac{[4]_q [3]_q}{8} \left( -[2]_q + \frac{([2]_q)^2}{2} \right) \]

\[ \mathcal{G}_5(q) = -\frac{[5]_q [2]_q}{4} + \frac{[5]_q [4]_q [2]_q}{8} - \frac{[5]_q [4]_q [3]_q}{8} \left( -1 + \frac{([2]_q)^2}{2} \right) \]

As a special case of Table 3, we have

\[
\begin{align*}
G_0 &= 0 \\
G_1 &= 1 \\
G_2 &= -1 \\
G_3 &= 0 \\
G_4 &= 1 \\
G_5 &= 0 \\
\end{align*}
\]

Moreover the first few Genocchi-type numbers can be shown \( N \times N \) matrix with multiple \( q \)-numbers elements in the following form

\[
\begin{pmatrix}
\mathcal{G}_0(q_1,q_j) \\
\mathcal{G}_1(q_1,q_j) \\
\mathcal{G}_2(q_1,q_j) \\
\end{pmatrix} =
\begin{pmatrix}
0 & 0 & \ldots & 0 \\
0 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 0 \\
\end{pmatrix}
\]

From Definition 1 and the Table 3, we easily acquire the first few Genocchi-type poly-nomials

**Genocchi-type polynomials**

\[
\mathcal{G}_0 \left( x : q_1, q_j \right) = 0
\]

\[
\mathcal{G}_1 \left( x : q_1, q_j \right) = -\frac{[2]_{q_1, q_j}}{2}
\]

\[
\mathcal{G}_2 \left( x : q_1, q_j \right) = \frac{([2]_{q_1, q_j})^2}{4} - \frac{([2]_{q_1, q_j})^2}{2}
\]

**Usual Genocchi polynomials**

\[
G_0 = 0 \\
G_1 = 1 \\
G_2 = 2x - 1
\]

Moreover the first few Genocchi-type polynomials can be shown \( N \times N \) matrix with multiple \( q \)-numbers elements in the following form

\[
\begin{pmatrix}
\mathcal{G}_0(q_1,q_j) \\
\mathcal{G}_1(q_1,q_j) \\
\mathcal{G}_2(q_1,q_j) \\
\end{pmatrix} =
\begin{pmatrix}
0 & 0 & \ldots & 0 \\
0 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 0 \\
\end{pmatrix}
\]
\[
\left( G_2^j (x: q_j, q_{j+1}) \right) = \begin{pmatrix}
\frac{2q_j^2 x - q_j^2}{2} & \frac{\left(2 \right)_{q_1, q_2}}{2} - \frac{\left[2 \right]_{q_1, q_2}}{4} & \ldots & \frac{\left(2 \right)_{q_1, q_N}}{2} - \frac{\left[2 \right]_{q_1, q_N}}{4} \\
\frac{\left(2 \right)_{q_2, q_1}}{2} - \frac{\left[2 \right]_{q_2, q_1}}{4} & \frac{2q_j^2 x - q_j^2}{2} & \ldots & \frac{\left(2 \right)_{q_2, q_N}}{2} - \frac{\left[2 \right]_{q_2, q_N}}{4} \\
\ldots & \ldots & \ldots & \ldots \\
\frac{\left(2 \right)_{q_N, q_1}}{2} - \frac{\left[2 \right]_{q_N, q_1}}{4} & \frac{\left(2 \right)_{q_N, q_2}}{2} - \frac{\left[2 \right]_{q_N, q_2}}{4} & \ldots & \frac{2q_j^2 x - q_j^2}{2}
\end{pmatrix}
\]

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