Causal Structure and Gravitational Waves in Brane World Cosmology

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(Dated: March 27, 2022)

The causal structure of the flat brane universe of RSII type is re-investigated to clarify the boundary conditions for stochastic gravitational waves. In terms of the Gaussian normal coordinate of the brane, a singularity of the equation for gravitational waves appears in the bulk. We show that this singularity corresponds to the “seam singularity” which is a singular subspace on the brane universe. Based upon the causal structure, we discuss the boundary conditions for gravitational waves in the bulk. Introducing a null coordinate, we propose a numerical procedure to solve gravitational waves with appropriate boundary conditions and show some examples of our numerical results. This procedure could be also applied in scalar type perturbations. The problem in the choice of the initial condition for gravitational waves is briefly discussed.

PACS numbers: 04.50.+h, 98.80.Cq, 98.80.-k

I. INTRODUCTION

Since the proposal of a brane world model of our spacetime by Randall and Sundrum \(^1\) (RS), the phenomenology of brane world cosmological models has been the subject of intensive investigations in recent years. In these brane world models, our universe is regarded as a four dimensional boundary (brane) in a higher dimensional spacetime (bulk). Many authors have found more realistic models which include matter fields on the brane and realize the cosmic expansion \(^2\), and tried to constrain on these models by the observational data.

Due to the recent developments in the technology of astronomy, cosmological density perturbations and their observations through large scale structure and cosmic microwave anisotropies (CMB) have become the most stringent test to constrain on models beyond standard cosmologies. In order to examine the constraints on the brane cosmologies by the CMB observations, we have to know the 5-D informations about perturbations on the entire spacetime including the bulk \(^3\), \(^4\).

In addition to the constraint by the CMB, the stochastic gravitational waves will be a promising candidate which provides a direct and even deeper test of such cosmologies. One might see in principle earlier universe than the photon last scattering epochs by gravitational waves. Although equations for gravitational waves are simpler than those of density perturbations, the problem is essentially the same, i.e, one needs to clarify the evolution of gravitational waves not only on the brane but also in the bulk. Therefore, it is also instructive to clarify the minimum information to obtain the evolution of gravitational waves before trying to clarify that of density perturbations in brane cosmologies.

In order to give theoretical predictions of stochastic gravitational waves, many authors have adopted the Gaussian normal (GN) coordinate system in the neighborhood of the brane. In this GN coordinate system, the metric is given by the form

\[
 ds^2 = \frac{\psi^2(\tau, w)}{\varphi(\tau, w)} d\tau^2 + \varphi(\tau, w) a^2(\tau) d\Sigma^2_K + dw^2, \tag{1}
\]

and the equation of gravitational waves has the same form as that of the five-dimensional massless scalar field:

\[
 \square_5 h = 0. \tag{2}
\]

The explicit forms of functions in the metric are given in the main text (see Sec. \(^4\)). To obtain the theoretical spectrum of stochastic gravitational waves, we just solve this equation with appropriate boundary conditions. However, Eq. \(^2\) in terms of the coordinate system \(^1\) includes a singularity at \(w = w_h\) in the bulk, where the metric function \(\varphi\) vanishes. The treatment of Eq. \(^2\) near the singularity is one of difficulties when we obtain the evolution of stochastic gravitational waves \(^5\). In fact, this singularity corresponds to the “seam singularity” discussed by Ishihara \(^5\).

The aim of this paper is to propose a numerical procedure to solve the evolution of cosmological gravitational waves avoiding the above difficulty in GN coordinate system. The procedure proposed here is based on the characteristic initial value problem according to the causal structure of the entire spacetime. This idea is analogous to the analysis of gravitational waves from a non-spherical domain wall by the one of the authors \(^3\). We use a null coordinate instead of the proper time on the
brane. In this procedure, the boundary conditions in the bulk are replaced by the initial condition on a null hypersurface and the above difficulty in the treatment of Eq. (2) near the singularity \( w = w_h \) is resolved if we simply specify the initial spectrum on a null hypersurface.

The organization of this paper is as follows: In Sec. II we briefly review the global structure of a brane world universe and clarify the region covered by the GN coordinate system in terms of the closed chart of the five-dimensional anti-de Sitter spacetime (AdS\(_5\)). In Sec. III we discuss the null hypersurface to clarify the causality of the propagation of gravitational waves in the bulk. In Sec. IV we develop the formulation to obtain the numerical solutions to Eq. (2) and show numerical examples of solutions which are derived by this formulation. The final section (Sec. V) is devoted to summary and discussions.

Throughout this paper, we consider the model without "dark radiation" following discussions in Ref. [9] and we only consider the flat Friedmann-Robertson-Walker (FRW) brane universe which is supported by recent precise measurements of the CMB [10].

II. GN COORDINATE SYSTEM IN AdS\(_5\) BULK

In this section, we first relate GN coordinate system to the flat chart in AdS\(_5\) and then consider the correspondence of GN coordinate system and the closed chart in AdS\(_5\) which covers the entire AdS\(_5\). Through clarifying these relations, we can easily see that the region which is covered by GN coordinate system in the entire AdS\(_5\). Secondly, we find where the singularity of the equation [2] is in the bulk.

Now, we consider a brane universe embedded in the AdS\(_5\) with a negative cosmological constant \( \Lambda_5 = -4/l^2 \).

In terms of the static charts of AdS\(_5\), the metric on AdS\(_5\) is given by

\[
ds^2 = -f_K(r_K)dt^2 + f_K(r_K)^{-1}dr^2 + r_K^2d\Sigma_5^2,\tag{3}
\]

where \( K \) takes the values \(-1, 0, \) and \(+1\), corresponding to the negative, zero, and positive constant curvature of a maximally symmetric three-dimensional space, respectively (for example, see Ref. [7]).

The function \( f_K(r_K) \) in this metric is defined by

\[
f_K(r_K) := K + \frac{r_K^2}{l^2}.\tag{4}
\]

When \( K = +1 \) and 0, the metric \( d\Sigma_5^2 \) is given by

\[
d\Sigma_5^2 = \begin{cases} d\chi_+^2 + \sin^2\chi_+ \{d\theta^2 + \sin^2\theta d\phi^2\} & (K = +1), \\ d\chi_0^2 + \chi_0^2 \{d\theta^2 + \sin^2\theta d\phi^2\} & (K = 0), \end{cases}
\]

respectively. Though we may consider the case \( K = -1 \) which corresponds to the open FRW model, we do not treat this case in this paper. In these static charts, a trajectory of three brane is given by \( r_K = r_K(t_K) := a(\tau) \) and \( t_K = t_K(\tau) \), where \( \tau \) is the proper time of the world volume of the brane and \( a(\tau) \) is a cosmological scale factor on the brane. The equation of the brane motion is given by the generalized Friedmann equation

\[
H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G_N}{3} \rho - \frac{K}{a^2} + \frac{\Lambda_4}{3} + \frac{\kappa_5^4}{36} a^2,\tag{5}
\]

where the dot denotes the derivative with respect to \( \tau \), \( \rho \) is the energy density on the brane, \( \Lambda_4 = \kappa_5^4 \lambda^2/12 + 3\Lambda_5/4 \) is the cosmological constant induced on the brane, \( G_N = k_4 \lambda^4/(48\pi) \) and \( k_5 \) are the four-dimensional and five-dimensional gravitational constants, respectively. Here and hereafter we assume \( Z_2 \) symmetry across the brane.

Cosmological solutions of the RS type brane world in terms of GN coordinate system were found by several authors [2], which is given by Eq. (1). In this metric, functions \( \psi \) and \( \varphi \) are given by

\[
\varphi(\tau, w) = \left[1 - A(1 - e^{\frac{\tau}{w}}) + (A + 1) e^{\frac{-\tau}{w}}\right]^2,\tag{6}
\]

and

\[
\psi(\tau, w) = \frac{1}{2H} \left(\frac{\partial \varphi}{\partial \tau}\right)_{w, z}, \tag{7}
\]

where \( A := \sqrt{1 + l^2 H^2} \). We have chosen the coordinate \( w \) so that \( w = 0 \) on the brane. As noted in Sec. II, this coordinate system has a coordinate singularity at \( w = w_h \), where \( w_h \) is determined by the equation \( \varphi(\tau, w_h) = 0 \) for each \( \tau \). This equation yields

\[
\exp \left(\frac{2w}{l}\right) = \frac{A + 1}{A - 1}.\tag{8}
\]

This is nothing but the singularity in Eq. (2).

To give the explicit coordinate transformations between the metrics (11) and (3) with \( K = 0 \), we show the explicit forms of the functions \( (t_0, r_0) \) in terms of \( (\tau, w) \). The function \( r_0 \) in Eq. (3) corresponds to the volume element on the \( \tau = \text{const.} \) hypersurface in the brane. Then, as shown in Ref. [11], this function is given by

\[
r_0^2 = \varphi(\tau, w) a^2(\tau),\tag{9}
\]

The cosmological scale factor \( a(\tau) \) is given by \( a(\tau) := r_0(\tau, w = 0) \). On the other hand, the explicit expression of \( t_0 \) as a function of \( (\tau, w) \) is given by

\[
t_0 - t_b = \frac{l^2}{a} \sqrt{\frac{A - 1}{A + 1} \left[ \frac{1}{1 - (1 - A) e^{2\frac{\tau}{w}} + 1 + A} \right]},\tag{10}
\]

where \( t_b(\tau) \) is chosen so that \( t_0 = t_b(\tau) \) on the brane \( (w = 0) \) for any \( \tau \). The derivation of this expression is shown in Appendix A. Clearly, the singularity in the equation (2) is just on the region \( r_0 = 0 \) in the flat chart. Actually, in the vicinity of the singularity (\( w \sim w_h \)), \( r_0 \) and \( t_0 \) behave as

\[
r_0 = a(\tau) \sqrt{\frac{A - 1}{A + 1} \left( \xi + \frac{1}{2(A + 1)} \xi^2 \right)} + O(\xi^3),\tag{11}
\]

\[
t_0 = t_b + \frac{l^2}{a(\tau)} \sqrt{\frac{A + 1}{A - 1} \left( \frac{-2}{\xi} + 1 \right)},\tag{12}
\]

where \( \xi = \frac{l^2}{a(\tau)} \sqrt{\frac{A - 1}{A + 1}} \).
where $\xi := (1 - A)e^{2\tilde{\tau}} + 1 + A$. To find where is the singularity in Eq. (2) in the bulk, we first see the region of $r_0 = 0$ in the entire AdS$_5$ using the closed chart. On this closed chart, we can easily specify the point of the singularity in Eq. (2) in AdS$_5$ bulk by tracing the spacelike geodesic which normal to the brane for each $\tau$.

Now, we consider the relation between the flat chart and the closed chart of AdS$_5$. Though this is already given by Ishihara [2], we repeat his arguments to find the event $\varphi(\tau, \omega_h) = 0$ for each $\tau$ in the closed chart. The AdS$_5$ is identified with the universal covering space of a hyperboloid,

$$-Y_0^2 - Y_1^2 + \sum_{i=2}^{5} Y_i^2 = -l^2,$$  \hspace{1cm} (13)

in six-dimensional flat spacetime with the metric,

$$ds^2 = -dY_0^2 - dY_1^2 + \sum_{i=2}^{5} dY_i^2.$$  \hspace{1cm} (14)

The flat chart and the closed chart (Eq. (3)) cover this hyperboloid as follows: For the $K = +1$ case (the closed chart),

$$\begin{pmatrix} Y_0 \\ Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \\ Y_5 \end{pmatrix} = \begin{pmatrix} \sqrt{l^2 + r_0^2 \sin(t_+ / l)} \\ \sqrt{l^2 + r_0^2 \cos(t_+ / l)} \\ r_0 \cos \chi_+ \\ r_0 \sin \chi_+ \cos \theta \\ r_0 \sin \chi_+ \sin \theta \cos \phi \\ r_0 \sin \chi_+ \sin \theta \sin \phi \end{pmatrix}.$$  \hspace{1cm} (15)

For the $K = 0$ case (the flat chart),

$$\begin{pmatrix} Y_2 + Y_0 \\ Y_2 - Y_0 \\ Y_1 \\ Y_3 \\ Y_4 \\ Y_5 \end{pmatrix} = \begin{pmatrix} \left( \frac{r_0}{l^2} - \frac{r_0^2}{\lambda_0^2} \right) + \frac{r_0^2}{\lambda_0^2} \\ \left( \frac{r_0}{l^2} - \frac{r_0^2}{\lambda_0^2} \right) - \frac{r_0^2}{\lambda_0^2} \\ \frac{r_0 \cos \theta}{\lambda_0} \\ \frac{r_0 \sin \theta \cos \phi}{\lambda_0} \\ \frac{r_0 \sin \theta \sin \phi}{\lambda_0} \end{pmatrix}.  \hspace{1cm} (16)$$

We should note that the flat chart does not cover the entire AdS$_5$ since $Y_2 + Y_0 \geq 0$ ($r_0 \geq 0$). Comparing Eqs. (14) with Eqs. (16), we found the coordinate transformations from the flat chart to the closed ones:

$$r^2 = \frac{1}{4} \left( r_0 - \frac{l^2}{r_0} \right)^2 + r_0^2 \lambda_0^2,$$

$$\cos \chi_+ = \frac{r_0 \lambda_0}{2r_+},$$

$$\sin \chi_+ = \frac{r_0 \lambda_0}{r_+},$$

$$\tan \left( \frac{t_+}{l} - \pi \Theta(t_0) \right) = -\frac{1}{2l_0} \left( 1 + \frac{l^2}{r_0^2} - \frac{t_0^2}{r_0^2} + \lambda_0^2 \right),$$

where $\Theta(t_0)$ is the step function, i.e., $\Theta(t_0) = 0$ for $t_0 \leq 0$ and $\Theta(t_0) = 1$ for $t_0 > 0$.

In addition to these coordinate systems, it is convenient to introduce the temporal-radial subspace which makes the metric conformally flat

$$ds^2 = f_+ (r_+) (-dt^2 + dr_+^2) + r_+^2 d\Sigma_+^2,$$  \hspace{1cm} (18)

$$(r_+/l) = \arctan(r_+/l)$$  \hspace{1cm} (19)

for the purpose of the investigation of the causal structure. Note that in these coordinates, radial null rays are represented by straight lines at $\pm 45$ degree. In the coordinate system $(t_+, r_+, \cos \chi_+, r_+ \sin \chi_+)$, the flat FRW brane is embedded as shown in Fig. 1. The $\tau = const$ and $\chi_0 = const$ hypersurfaces on the flat FRW brane is also shown in the same figure, respectively.

Through the relations (9), (10), and (17), we can see how $\tau = const$. hypersurfaces in the coordinate system (1) foliate the entire AdS$_5$ bulk. To do this, we have to consider the spacelike geodesics normal to the brane specifying their starting point on the brane. Note that on each $\tau = const$. hypersurface in the brane ($w = 0$), both functions $t_0(\tau)$ and $r_0(\tau)$ are also constant, while the other coordinates on the brane are arbitrary. Therefore, we may choose a point $(t_0, r_0)$ on the flat chart as a starting point of these spacelike geodesics normal to the brane. Fixing the proper time $\tau$ on the brane, the coordinate functions $r_+, t_+$, and $\chi_+$ behave as

$$r_+ = \frac{l^2}{a(\tau)} \sqrt{A^2 - 1} + t_b,$$

$$\frac{t_+}{l} = \arctan \left[ \frac{r_+}{l} \right] = \frac{r_+}{l}.$$

$$\cos \chi_+ = -1$$

in the limit $w \to w_h$ ($\varphi(\tau, w) \to 0$). This shows that the coordinate singularity at which $\varphi(\tau, \omega_h) = 0$ is just on the “seam singularity” $t_+ = r_+$, $\cos \chi_+ = -1$ (see Fig. 2).

Note that the seam singularity can be replaced by the other regular portion of a spacetime according to the creation scenario of the brane universe. In this sense, we do not have to be afraid of this singularity, or equivalently, the singularity in Eq. (2), seriously. Moreover, when we consider the characteristic initial value problem, the initial conditions are chosen on a null hypersurface and the seam singularity is contained in its causal past. Therefore, the difficulties in the treatment of the singularity in
Eq. (2) are simply reduced to the choice of initial conditions for gravitational waves as seen in below.

Here, we note that the flat chart of AdS5 has the future Cauchy horizon in the entire AdS5. This Cauchy horizon appears as the “infinity” \((t_0, r_0) \to (\pm \infty, +0)\) in the flat chart. The above radial null geodesics from the brane approaches to this Cauchy horizon as the affine parameter \(\lambda\) increases. We choose the affine parameter \(\lambda\) so that \(\lambda = 0\) corresponds to the point on the brane and \(\lambda = \frac{E}{r_b} > 0\) corresponds to that on the Cauchy horizon.

On the other hand, in the closed chart, these null geodesics are given by

\[
\begin{align*}
E_{\tau}^2 &= \frac{1}{4} \left[ \left( 1 + \frac{\zeta^2}{l^2} - \lambda_0^2 \right) r_0 + 2\zeta \right]^2 + r_0^2 \lambda_0^2, \\
\tan \left( \frac{t_+}{l} - \pi \right) &= \frac{l}{l^2 + \zeta r_0} \left\{ \zeta - \frac{1}{2} \left( 1 - \frac{\zeta^2}{l^2} + \lambda_0^2 \right) r_0 \right\}, \\
\cos \chi_+ &= \frac{1}{2r_+} \left[ \left( 1 + \frac{\zeta^2}{l^2} - \lambda_0^2 \right) r_0 + 2\zeta \right],
\end{align*}
\]

where \(\zeta := \frac{\ell^2}{l^2} + t_b\). In Eqs. (23), \(t_0\) and \(r_0\) are parameterized by \(\lambda\) through Eqs. (24). In the limit where the null geodesics from the brane approach to the Cauchy horizon \((t_0, r_0) \to (+\infty, +0)\), Eqs. (23) behave

\[
\begin{align*}
r_+ &= |\zeta| \left( \frac{t_+}{l} \right) = \arctan \left( \frac{\zeta}{l} \right) + \pi, \\
\cos \chi_+ &= \text{sgn}(\zeta). \quad \text{(24)}
\end{align*}
\]

These do represent a point on the future Cauchy horizon of the flat chart.
Eqs. (24) is pointlike when we fix the proper time \( \tau \) on the brane. We also note the fact that Eqs. (24) do not depend on \( \chi_0 \). This implies that all null geodesics with different \( \chi_0 \) from the same \( \tau = \text{const.} \) hypersurface on the brane reach to the same point represented by Eqs. (24). These null geodesics generate a null hypersurface. Hence we have one-to-one correspondence between a \( \tau = \text{const.} \) hypersurface on the brane and a null hypersurface in the bulk. We denote this null hypersurface associated with \( \tau = \text{const.} \) hypersurface on the brane by \( \partial D(\tau) \). We also denote the causal past of \( \partial D(\tau) \) by \( D(\tau) \). We note that \( D(\tau') \subset D(\tau) \) for any \( \tau' < \tau \). In particular, the bulk region covered by the flat chart is foliated by the set of null hypersurfaces \( \{ \partial D(\tau) \mid \tau \text{ is the proper time on the brane} \} \). Since \( \partial D(\tau') \subset D(\tau) \) for \( \tau' < \tau \), the seam singularity, which is also represented by \( \partial D(\tau \to +0) \), is entirely contained in \( D(\tau) \) for any \( \tau \). These situations are schematically depicted in Fig. 3.

Inspecting this causal structure, we can formulate the characteristic initial value problem of gravitational waves. Suppose that the flat FRW brane universe is created at the instance \( \tau = \tau_c \). As depicted in Fig. 3, the initial state of gravitational waves in \( D(\tau_c) \) affects to their evolution only through the null hypersurface \( \partial D(\tau_c) \). Once we specify the state of gravitational waves in the region \( D(\tau_c) \) according to the creation scenario of the flat FRW brane, we can specify the spectrum of gravitational waves on \( \partial D(\tau_c) \). Hence, to determine the evolution of the spectrum of gravitational waves after \( \tau = \tau_c \), we only have to specify the spectrum of gravitational waves on \( \partial D(\tau_c) \) as an initial condition and trace the evolution of them through Eq. (2) as seen in the next section.

IV. EVOLUTION OF GRAVITATIONAL WAVES

At this point, we have a clear strategy to tackle the problem of the evolution of gravitational waves in the brane universe. The singularities \( \varphi(\tau, w_b) = 0 \) in Eq. (2) are entirely contained in \( D(\tau) \), and the evolution of gravitational waves are determined by an initial spectral on its boundary \( \partial D(\tau) \) and boundary conditions on the brane. Then, we do not have to care the singularity \( \varphi(\tau, w_b) = 0 \) any more and there is no need to introduce any artificial boundaries in the bulk to impose boundary conditions at bulk infinity. In the following subsections, we first develop the formulation to obtain the numerical solutions to Eq. (4) in the context of brane world cosmology (in Sec. IV A), and then, we show numerical examples which are obtained by applying the formulation developed here (Sec. IV B).

A. Numerical Formulation

In order to develop the characteristic initial value problem associated with the null hypersurface \( \partial D(\tau) \), we introduce a null coordinate and rewrite down the equation of gravitational waves in a single null coordinate system (see Fig. 4).

Now, we introduce the function \( u \) by

\[
u = t_0 - \frac{l^2}{r_0} - t'_b + \frac{l^2}{r'_b} ,
\]

where \( (t'_b, r'_b) \) determine the zero point of this function \( u \). The function \( u \) is constant on the null hypersurface \( \partial D(\tau) \) for each \( \tau \). We can easily confirm that the one-form \( (du)_a \) is null through the metric \( g \) with \( K = 0 \).

Using the relations listed in the appendix (Eq. (A6)-(A9), see also Ref. [1]), the derivative \( du \) is given by

\[
du = \left( \frac{\partial u}{\partial \tau} \right)_w \, d\tau + \left( \frac{\partial u}{\partial w} \right)_\tau \, dw
= \frac{1}{a} \left( \frac{\partial r_0}{\partial \tau} \right)_w \, F^{-1}(\tau, w) \, d\tau - F^{-1}(\tau, w) \, dw .
\]

where

\[
F(\tau, w) = \frac{f[r_0(\tau, w)]}{\sqrt{f[r_0(\tau, w)] + a^2 + a}} .
\]

In terms of the new coordinate system \( u \) and \( \bar{w} = w \), the five-dimensional metric \( g \) on \( AdS_5 \) is written as

\[
ds^2 = -F(\tau, \bar{w}) du^2 - 2F(\tau, \bar{w}) du d\bar{w} + r^2_0(\tau, \bar{w}) \, d\Sigma^2_0 .
\]

As mentioned in Sec. I, the equation for gravitational waves in the bulk is simply given by that for the five-dimensional massless scalar field Eq. (2). In terms of the coordinate system \( (u, \bar{w}) \) Eq. (2) is given by

FIG. 4: Null coordinate and Gaussian normal coordinate in the flat chart.
\[
\left[ \frac{\partial^2}{\partial r^2} - \frac{2}{F} \frac{\partial}{\partial \tau} \frac{\partial}{\partial u} - \frac{3}{F r_0} \left( \frac{\partial r_0}{\partial u} \right) \frac{\partial}{\partial \tau} + \left( \frac{\partial r_0}{\partial u} \right) \frac{\partial}{\partial u} + \left( \frac{1}{F} \frac{\partial F}{\partial \tau} \right) - \frac{3}{r_0} \frac{\partial r_0}{\partial \tau} \right) \frac{\partial}{\partial u} - \frac{k^2}{r_0^2} \right] h(u, \overline{w}, k) = 0 ,
\]

where \(-k^2\) is the eigen value of the Laplacian of \(d \Sigma_0\).

Eq. (29) can be formally integrated so that
\[
\partial_u h(u, \overline{w}) = r_0^{-3/2}(\tau, \overline{w}) \times \left( \int S(w') r_0^{3/2} dw' + C(u) \right) ,
\]

where
\[
S(\overline{w}) = \frac{F}{2} \left( \frac{\partial^2 h}{\partial \overline{w}^2} - \frac{k^2}{r_0^2} h \right) + \left( \frac{1}{2} \frac{\partial F}{\partial \overline{w}} + \frac{3}{2} \frac{\partial \ln r_0}{\partial \overline{w}} - \frac{3}{2} \frac{\partial \ln r_0}{\partial u} \right) \frac{\partial}{\partial \overline{w}} .
\]

The quantities \(\partial r_0/\partial u, \partial r_0/\partial \overline{w}\) and \(\partial F/\partial \overline{w}\) which appear in the definition (31) are given by known functions as
\[
\left( \frac{\partial r_0}{\partial u} \right)_{\overline{w}} = \dot{a} F ,
\]
\[
\left( \frac{\partial r_0}{\partial \overline{w}} \right)_{\tau} = \dot{\tau} + \left( \frac{\partial r_0}{\partial \overline{w}} \right) ,
\]
\[
\left( \frac{\partial F}{\partial \overline{w}} \right)_{\tau} = \dot{a} \left( \frac{\partial r_0}{\partial \tau} \right)^{-1} \left( \frac{\partial F}{\partial \tau} \right)_{\tau} + \left( \frac{\partial F}{\partial \overline{w}} \right)_{\tau} .
\]

These equations (30) and (31) are the main result of this paper. In these equations, initial data of gravitational waves are set by choosing the function of \(h(u, \overline{w})\) on \(\partial D(\tau)\) where the null coordinate \(u\) is constant on \(\partial D(\tau)\). The function \(C(u)\) in Eq. (30) is determined by the boundary condition at the brane (\(\overline{w} = 0\)) by
\[
\partial_u h(u, 0) = r_0^{-3/2}(\tau, 0) C(u) .
\]

When the anisotropic stress on the brane is not induced due to matter fields on the brane, the boundary condition for gravitational waves \(h\) at the brane is the Neumann type [3]. This is accomplished by imposing
\[
\partial_u h(u, 0) = - \left( \frac{\partial u}{\partial \overline{w}} \right)^{-1} \frac{\partial}{\partial \tau} \frac{\partial}{\partial h} = F(\tau, 0) \frac{\partial}{\partial h} ,
\]

at the brane. Eq. (30) with the boundary condition (36) are easier to solve than Eq. (29) and enough to predict the cosmological evolution of gravitational waves. Note that our method have an important advantage that application to high energy epochs is easy and straightforward. Actually, this advantage is seen in the numerical examples shown in the following subsection (see Figs. 5 and 9).

B. Numerical Scheme and Examples

Here, we show some numerical solutions to Eq. (29) in the context of the brane world cosmology. We first comment on some details of the numerical scheme to obtain solutions to Eq. (29) with the boundary condition Eq. (30).

To solve Eq. (29), we evaluate Eq. (30) on each null hypersurface \(\partial D(\tau(u, \overline{w} = 0))\), numerically. Once \(h(u, \overline{w})\) on \(\partial D(\tau(u, \overline{w} = 0))\) is given as a function of \(\overline{w}\), we can evaluate \(h(u + du, \overline{w})\) on \(\partial D(\tau(u + du, \overline{w} = 0))\) by
\[
h(u + du, \overline{w}) = h(u, \overline{w}) + du \partial_u h(u, \overline{w}) .
\]

Through this evaluation on each \(\partial D(\tau(u, \overline{w} = 0))\), we obtain the gravitational waves \(h(u, \overline{w})\) in the causal future of the initial surface \(\partial D(\tau_{in})\) with an appropriate initial condition. By evaluating \(h(u, 0)\) on the brane, we see the behavior of gravitational waves on our brane universe.

To obtain \(\partial_u h(u, \overline{w})\) by Eq. (30), we have to evaluate Eqs. (32)-(34) on each \(\partial D(\tau(u, \overline{w} = 0))\) and Eq. (30) at the brane. To accomplish this, we evaluate \(\dot{a}(\tau) = a(\tau) \dot{H}(a(\tau))\), \(F(\tau, \overline{w})\), and \(r_0(\tau, \overline{w})\) on each \(\partial D(\tau(u, \overline{w} = 0))\). Since these functions depend on \(u\) only through \(\tau\), we evaluate the change of the function \(\tau(u, \overline{w})\) on each \(\partial D(\tau(u, \overline{w} = 0))\). Further, it is also convenient to use the scale factor \(a(\tau)\) at the brane as an time coordinate when we clarify the behavior of gravitational waves during the cosmic expansion.

We evaluate the change of the function \(a(\tau(u, \overline{w}))\) on \(\partial D(\tau(u, \overline{w} = 0))\) from Eq. (20) by setting \(du = 0\), which yields
\[
\left( \frac{d \ln a}{d \overline{w}} \right) u = a H^2 \left( \frac{\partial r_0}{\partial \tau} \right)^{-1} ,
\]

where \(\partial r_0/\partial \tau\) is directly given by Eq. (19). By integrating Eq. (35) with the boundary condition \(a(\tau) = r_0(\tau, w = 0)\) at the brane, we obtain the function \(a(\tau(u, \overline{w}))\) along \(\partial D(\tau(u, \overline{w} = 0))\). Since the relation between \(a\) and \(\tau\) is given by the explicit integration of the generalized Friedmann equation Eq. (19), we can evaluate \(F(\tau, w)\) and \(r_0(\tau, w)\) on each \(\partial D(\tau(u, \overline{w} = 0))\) by using Eqs. (5) and (9). The relation between the time step \(du\) and \(d\tau\) on the brane is given by
\[
d\tau = a \left( \frac{\partial r_0}{\partial \tau} \right)^{-1} F(\tau, w = 0) du ,
\]

which is led from Eq. (20) by choosing \(dw = 0\) and \(w = 0\). After proceeding the time step, we repeat the above evaluation of \(a(\tau)\) and \(\tau(u, \overline{w})\) on \(\partial D(\tau(u + du, \overline{w} = 0))\). Thus,
we can evaluate $F(\tau, \mathbf{w})$, $r_0(\tau, \mathbf{w})$, and $\dot{a}$ on each null hypersurface $\partial D(\tau(u, \mathbf{w}) = 0))$ from Eqs. (34), (36), and (38), and hence, Eqs. (32) and (39). Through these evaluations, we obtain $\partial_u h(u, \mathbf{w})$ by Eq. (30) and gravitational wave $h(u, \mathbf{w})$ in the causal future of the initial surface $\partial D(\tau_{ini})$.

When we carry out this numerical scheme, we have to specify the zero point of the function $u$. Suppose that we start our numerical calculation from the high energy epoch ($l^2 H^2 \gg 1$, or the $\rho^2$ dominated era). In this epoch, the generalized Friedmann equation and Eq. (A11) lead

$$H^2 \sim \frac{1}{16} \tau^{-2}, \quad (40)$$

$$\frac{a}{a_0} = \left(\frac{2}{3} \kappa_5^2 \rho_\gamma \tau\right)^{1/4} := \alpha \tau^{1/4}, \quad (41)$$

$$t_b(\tau) - t_{b, ini} = -\frac{l^2}{\alpha a_0} \left(\tau^{-1/4} - \tau_{ini}^{-1/4}\right), \quad (42)$$

where $\rho_{\gamma,0}$ and $a_0$ are the energy density of radiations and the cosmic scale factor on the brane at the present universe. $t_{b, ini}$ is the trivial initial value of $t_b$ and we choose so that $t_{b, ini} = 0$. Since the zero points of $u$ and $t_b$ are arbitrary, we choose so that $u = t_b = 0$ and $r'_b = a(\tau_{ini})$ in Eq. (26) at the starting point of our numerical calculation $\tau = \tau_{ini} = 1/4H(a_{ini})$.

Following to these numerical scheme, we obtain numerical examples depicted in Figs. 5 and 6. In these examples, we simply choose $h = const.$ as initial conditions of $h$ on the initial null surface $\partial D(\tau_{ini})$. We take this initial condition only as an example to demonstrate the calculation and investigate how the evolution of gravitational waves can be modified in high energy epochs qualitatively. The wave number is chosen $k = a(\tau_{ini}) H(\tau_{ini})$, which corresponds to the mode just crossing the cosmic horizon at $\tau = \tau_{ini}$. In Figs. 5 and 6 the scale factors of our examples are taken to be $a(\tau_{ini}) = 1.58 \times 10^{-17}$ and $a(\tau_{ini}) = 1.58 \times 10^{-16}$, respectively. The expansion of the universe is completely dominated by radiation in such early epochs. We only consider ordinary radiation (CMB) and three species of massless neutrinos for simplicity neglecting other matter components, and set the curvature scale of AdS5 bulk to be 1 mm. In this cosmology, the transition from the $\rho^2$ dominated universe to the standard radiation dominated one occurs at $a \approx 4 \times 10^{-17}$.

The evolution of gravitational waves in brane world differs that of standard cosmology mainly in two points. First, the amplitude of gravitational waves in brane world models becomes smaller than that in four dimensional standard models. This is due to the fact that decelerating brane motion excites the modes propagating into the bulk. Second, the effective frequency $\omega$ of gravitational waves measured on the brane becomes larger due to their momentum not only along the three dimensional space ($k$ in Eq. (29)) but also to the bulk direction. These effects becomes remarkable in the high energy epoch ($l^2 H^2 \gg 1$).

**FIG. 5:** Evolution of gravitational waves on the brane in brane world cosmology (solid line) which cross the horizon in low energy epochs ($l^2 H^2 \ll 1$). Also shown is standard 4D evolution of gravitational waves (dotted line).

**FIG. 6:** The same as Fig. 5 but for gravitational waves which cross the horizon in high energy epochs ($l^2 H^2 \gg 1$).

**V. SUMMARY AND DISCUSSION**

In this paper, we have carefully investigated the causal structure of the flat FRW model of RS type II brane world and proposed the single null coordinate system to solve the cosmological evolution of gravitational waves. We have explicitly seen that in this null coordinate system, we do not have to care the singularity in Eq. (2). Further, it is not necessary to introduce any artificial boundaries to impose some boundary conditions at the bulk infinity. Thus, we have shown that the problems of the singularity in Eq. (2) and the boundary conditions for gravitational waves at the bulk infinity are resolved if we simply choose an appropriate initial conditions for gravitational waves.

The initial condition for gravitational waves which we considered to obtain the numerical examples (Fig. 5 and...
in the main text might not be realistic one, since 
$h = \text{const.}$ is not an exact solution to Eq. (20), but a mere approximated solution in the long wavelength limit 
k \to 0$. Since the aim of this paper is to propose a num-

erical procedure to solve the evolution of cosmological 
gravitational waves, the details and quantitative studies of

numerical results and the problem on realistic initial 
conditions of stochastic gravitational waves are beyond the 
current scope of this paper and they will be investig-

gated in our forthcoming paper [12]. However, it is in-

teresting to discuss the evolution of gravitational waves

with an appropriate initial conditions and clarify the fi-
nal spectrum of stochastic gravitational waves resulting

from various creation scenarios of the brane world.

The initial conditions for gravitational waves in brane
world cosmologies crucially depend on the creation sce-
nario of the FRW brane. Many kinds of cosmological

scenarios have been proposed so far [13]. Among them,
there are some scenarios where the FRW brane is created
after the inflationary phase. If we adopt these brane in-
flationary scenarios, it might be natural to consider that
the initial spectrum of gravitational waves is determined
in this inflationary phase. The spectrum of gravitational
waves under the deSitter evolution of the brane is dis-
cussed by several authors [14,15,16]. It was pointed
out that gravitational waves decay away except but the
zero-mode, and it approaches to a constant amplitude in
the inflationary phase [14]. These discussions are based
on the GN coordinate system. Since GN coordinate sys-
tem does not cover the entire bulk space, these discuss-
sions seem inappropriate to specify the initial spectrum
of gravitational waves. However, it was also shown that
the vacuum defined on the deSitter slicing asymptoti-
cally approaches to the vacuum defined in terms of the
Poincare coordinates on AdS$_5$ [15]. This will imply that
the vacuum state on the static AdS$_5$ frame is appropriate
as the bulk initial spectrum of gravitational waves when
we consider these inflationary scenarios.

If we do not adopt the inflationary scenarios, we have
to specify the initial spectrum according to the other cre-
ation scenario of the FRW brane. However, in any case,
once given the initial configuration on a null hypersur-
face, our method can be applied to solve the evolution of
gravitational waves. The final spectrum of the stochastic
gravitational waves can be a powerful probe to investi-
gate the existence of extra-dimensions by comparison
with the spectrum in the four-dimensional standard cos-

mology. We leave this problem as a future work.

Besides the evolution of stochastic gravitational waves,
the procedure developed here will be applicable to discuss
the evolution of the density perturbations in the brane
world. The problem in the choice of the initial spectrum
of the density perturbation will also arise as discussed
above and this initial spectrum will also depend on the
creation scenario of the FRW brane. However, accord-
ing to the causality discussed in this paper, we can easily
expect that the density perturbation is completely de-
termined by the initial condition on a null hypersurface

and the boundary conditions at the brane. Though this
expectation should be confirmed by examining the equa-
tions for the density perturbations of the brane world, it
is quite interesting to compare the evolution of the den-
sity perturbations in the brane world scenario with that
in the conventional four-dimensional cosmology. We also
leave this problem as a future work.

Acknowledgments

We would like to thank H. Ishihara and T. Tanaka for useful suggestions. This work has been supported in
part by the Sasakawa Scientific Research Grant from The Japan Science Society. We also thank anonymous
referees for pointing out our misleading presentations and
improving the quality of our paper. KN would like to
thank all members of Department of Physics of Hiyoshi
Campus in Keio University, all members of Division of
Theoretical Astronomy in NAOJ, and the other colleague
for their continuous encouragement.

APPENDIX A: GAUSSIAN NORMAL

GEODESICS

We briefly review the relation between the Gaussian
normal (GN) coordinates and the flat chart in AdS$_5$. To
do this, we first consider the spacelike geodesics normal
to the brane. Let us consider the unit normal vector

$n^a (g_{ab}n^a n^b = 1)$ tangent to these spacelike geodesics.
The existence of the Killing field $(\partial/\partial t_0)^a$ in the bulk
spacetime ensures that the existence of the constant $\mathcal{E} =
-g_{ab}(\partial/\partial t_0)^a (\partial/\partial t_0)^b$ along the geodesics. Using this constant,
the components of the normal vector $n^a$ are given by

\[ n^a =: n^a_0 (\frac{\partial}{\partial t_0})^a + n^{a_0} \left( \frac{\partial}{\partial r_0} \right)^a \]

\[ = \frac{\mathcal{E}}{f(r_0)} \left( \frac{\partial}{\partial t_0} \right)^a - \sqrt{f(r_0) + \mathcal{E}^2} \left( \frac{\partial}{\partial r_0} \right)^a . \]

(A1)

The explicit orbit of these geodesics is given by the
integration of $dx^a = n^a dw$. For $r_0$ component, we obtain

\[ 2r_0^2 + l^2 \mathcal{E}^2 = l^2 \mathcal{E}^2 \cosh \left[ -\frac{2}{l} (w - w_0) \right] , \]

(A2)

where $w_0$ is the additional constant of integration. We
choose $\mathcal{E}$ and $w_0$ by the conditions: (i) the geodesics are
normal to the brane world volume of $r_0 = R(t_0) = a(\tau)$
at $t_0 = t_b$ (i.e. $n_a \propto \nabla_a [r_0 - R(t_0)]$); (ii) $w = 0$ on the
hypersurface. Since $dt_0/d\tau$ along the brane is given by
the definition of the cosmic time $\tau$:

\[ -d\tau^2 = -f(r_0)dt_0^2 + f^{-1}(r_0)dr_0^2 \]

\[ = -f(r_0)dr_0^2 + f^{-1}(r_0)a^2 d\tau^2 , \]

(A3)
the above two conditions lead
\[ E(t_b) = -\dot{a}(\tau), \]
\[ w_0(t_b) = \frac{l}{2} \cosh^{-1} \left[ \frac{2a^2 + l^2E^2(t_b)}{l^2E^2(t_b)} \right]. \]
(A4)

Thus, substituting eqs. (A4) into eq. (A2), we obtain eq. (9). On the other hand, the integration of the \( t_0 \) component of \( dx^\mu = n^\mu dw \), i.e.,
\[ \frac{dt_0}{dw} = n^{t_0} = -\frac{l^2\dot{a}(\tau)}{\varphi(\tau, w) a^2(\tau)}. \]
(A5)
leads Eq. (10).

Finally, we summarize some useful relations. The fact that \( \tau \) is constant along these spacelike geodesics leads
\[ \left( \frac{\partial t_0}{\partial w} \right)_\tau = n^{t_0} = -\frac{\dot{a}(\tau)}{f(r_0)}, \]  
(A6)
\[ \left( \frac{\partial r_0}{\partial w} \right)_\tau = n^{r_0} = -\sqrt{f(r_0) + \dot{a}^2(\tau)} + f(r_0). \]
(A7)

These relations are used in the main text (see Sec.IV).

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