RGB Colors and Ecological Optics

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Object color space is highly structured due to optical constraints (radiant power non-negative, reflectance factors between zero and unity) and ecological context (daylight illuminant). In this setting trichromacy induces a natural geometry through a unique spectral tripartition. Different from null-context colorimetry, one gains two desirable relations: The colorimetric coordinates are coarse-grained spectral reflectance factors and there is a direct link to color experiences, since RGB–coordinates provide ostensive definitions. The framework allows one to deal with subtractive color mixture, source variation, effects of metamerism and relations between scenes and image data in a unified, structured manner.

In ecological contexts, colors are effectively object properties. The formal framework is linear algebra and convex geometry. Applications in human biology, computer graphics, design, etc., are immediate.

Keywords: RGB-color, ecological optics, metamerism, color solid, automatic white balance, color mixtures, color statistics

1 INTRODUCTION

Are colors object properties? Answers depend on the intended ontological roots. Instances are “physical colors” (radiant power spectra and spectral reflectance factors), “colorimetric colors” (like red, nominal; RGB[99|00|00], quantitative; Section 4.2) and “phenomenological colors” (όred, where “ό” stands for experiential quale or ostensive in intersubjective communication, Section 1.1).

The first two bridge physics and physiology, the third one physiology and phenomenology. Meaning derives from the synthesis. Thus art and design, image science, computer graphics (cg) and ecological biology/psychology involve all.

Balanced accounts of such diverse interrelations are rare. We present a minimalist formal account, with various novel developments.

1.1 Minimal Context of “Object Colors”

We use “object color” as relevant in ecological human biology. Visual objects (stones, apples, rabbits, . . .) are due to surface scattering (“reflection”) of environmental radiation (“daylight”). Object colors appear similar to all observers, viewing perspective playing only a minor role. When objects appear in different guises, they count as distinct visual objects. Extreme examples of mismatch between material and visual objects are mirrors and Morpho wings.

Environmental radiation is noticeable in its visual effect on scenes. “White Objects” reveal its nature:

DEFINITION 1 (White object). A white object is a Lambertian scattering surface with unit spectral albedo. Reason: This settles both viewpoint independence and spectral properties.

Examples are white chalk, pressed Ba SO₄ powder, or smoked MgO (Kortüm, 1969). For informal use, white toilet paper will do fine.

We also define the “Black Object.” Its definition is even simpler:
DEFINITION 2 (Black object). A black object has zero spectral albedo.

The standard laboratory implementation is a black hole (Kortüm, 1969). For informal use, black velvet will do. Finally, we define "visual object":

DEFINITION 3 (Generic visual object). A generic visual object is a Lambertian surface with spectral albedo less than that of the white object throughout the spectrum.

For informal experiments one uses colored papers. These definitions involve only physics, no chemistry, psychology or physiology.

We ignore radiometrical intricacies [structure of the radiant field, shapes and other geometrical factors, multiple scattering and vignetting, ... (Koenderink and van Doorn, 1983)]. Objects are scattering surfaces presented in a single plane and are irradiated with a uniform beam. This leaves problems of a phenomenological nature. A carefully designed standard display minimizes these (Supplementary Section S1.1; Supplementary Figure S1). Observers tend to agree on the color of a patch when the overall situational awareness is “natural.” In evolutionary terms that implies steppe or savannah hunter–gatherer existence (Koenderink, 2019). Thus a screen display is less natural than colored papers on a table top, or a bed of flowers.

In this context, people readily learn to associate coarse-grained spectral reflectance factors with “seen colors.” Although such colors are subjective and idiosyncratic, there is no lack of intersubjective agreement. Successful communication is the rule. People recognize hundreds of object colors (Koenderink et al., 2018c). This does not imply being able to name them (Griffin and Mylonas, 2019), but familiarity with color coordinates allows reproduction of hundreds of colors from specifications such as “RGB[99|00|00], an optical object often called 'red' by me and others.” The italic addition serves to indicate the perceived lack of intersubjective understanding.

One cannot deploy Ψ in formal derivations. But the “meaning” of colorimetric description depends upon it. It can only be conveyed through display. Professional users routinely depend upon Ψ and are entirely happy to deploy Ψ−1 as a convenient heuristic.

Thus Ψ is effectively used in both intersubjective communication, as well as in individual, silent thought. Arguing about color in the absence of a shared vision is vacuous talk. Ostension solves that impasse.

2 RADIANT POWER SPECTRA, SPECTRAL REFLECTANCE FACTORS AND COLORIMETRIC COORDINATES

We comment on standard radiometric and colorimetric backgrounds (Bouma, 1946; Wyszecki and Stiles, 1967; Koenderink, 2010; Centore, 2017). Readers are assumed to be familiar with it.

2.1 Radiant Power Spectra

Radiant power spectra are denoted \( S(\lambda) \). Since we only use ratios, physical units are irrelevant. We consider incoherent beams (Feynman et al., 1964; Born and Wolf, 1999) and radiant power on wavelength \( \lambda \) basis. Thus the spectrum of two attenuated (factors \( \xi, \eta \)) and superimposed \((+,^*)\) radiant beams \((P, Q)\) is the appropriate linear combination \( \xi P(\lambda) + \eta Q(\lambda) \). We mainly use this to indicate source spectra. In rare cases we mention the proximate stimulus, that is the scattered beam. The interest is in the distal stimulus, that is to say, the object properties.

Wave length is a continuous coordinate. Radiant spectra are points in a vector space \( \mathbb{R}^\infty \). This “space of beams” \( \mathbb{B} \) is a Hausdorff space, but not a Hilbert space (Arkhangelskii and Pontryagin, 1990). This rules out methods depending on a metric, such as Moore-Penrose pseudoinverses (Axler, 2015), or PDA (Maloney and Wandell, 1986). Such use is common in the literature, but ill considered.

2.2 Spectral Reflectance Factors

The object property to consider is the spectral albedo, specified as spectral reflectance factor. This is the fraction of the scattered power relative to that scattered by the white object in the same attitude at the same location. Spectral reflectance factors are naturally dimensionless. Geometrically, they are represented as diagonal matrices [written \( R(\lambda) \)] with coefficients from the unit interval \( I \). We ignore such processes as fluorescence, which might produce off-diagonal elements. The white object has \( R(\lambda) = I \), the unit matrix, which is usually omitted.

2.3 Colorimetry

Colorimetry considers the equivalence of radiant beams in controlled settings. If members of a pair cannot be
differ by a radiant power spectrum yields a 3-vector, denoted from the Internet (http://www.cvrl.org). This matrix, applied to a null space $N$ (these curves camera. Using humans as null-detectors allows one to measure They are like the spectral sensitivity curves of an electronic camera. Humans as null-detectors allows one to measure these curves modulo arbitrary linear transformations, which is all that counts in colorimetry (Maxwell, 1855). The explanation is that only the ratios of absorptions in three retinal photo-pigments (Stockman et al., 2000) causally affect $\Psi$. They are like the spectral sensitivity curves of an electronic camera. Humans as null-detectors allows one to measure these curves modulo arbitrary linear transformations, which is all that counts in colorimetry (Maxwell, 1855; Koenderink, 2010). The projection operator involves arbitrary linear combinations. For a start we use the CIE–XYZ “color matching” matrix $M_{\text{XYZ}}$ (CIE, 1932; Schanda, 2007). It can be downloaded in table form from the Internet (http://www.cvrl.org). This matrix, applied to a radiant power spectrum yields a 3-vector, denoted “color” in the CIE $\text{XYZ}$–space $C_{\text{XYZ}}$. Note that CIE–XYZ is in no way “special.” Such a color represents an equivalence class of spectra, all mutually differing by elements of $N$.

Since $M_{\text{XYZ}}$ exists as table, analysis depends on numerical procedures. This is all the structure that is required to start building a representation of the space of object colors. Once the natural representation has been set up, the arbitrary CIE XYZ representation may be discarded.

2.3.1 Fundamental Physical Constraints
Formally, colorimetry is linear algebra. There is no particular reason to select a specific basis.

This is true for the physics. Newton’s (Newton, 1704) notion that sunlight is a “confused mixture” of atomic elements is nonsense. In linear spaces there are no atomic elements. The wavelength basis is a convenience. Radiant power spectra imply finite resolution (Born and Wolf, 1999). Monochromatic beams are fictions. Wavelengths stand neither for beams, nor colors.

Purely formally we denote unit power monochromatic beams $M_\lambda(\mu) = \delta(\lambda - \mu)$, in terms of the Dirac delta–function, but all actual spectra we deal with are positive, smooth distributions.

Fundamental physical constraints put highly constraining structure on the amorphic linear continuum. Relevant ones are:

Physics I (Fundamental physical constraints). Two physical constraints structure object color space:

1. radiant power is positive;
2. reflectance factors range over the interval $0 \ldots 1$;
3. the co-dimension of $N$ is 3.

The latter constraint applies to the bulk of the population (roughly one-tenth of the males is dichromatic). Additional constraints that render “color as object property” possible are ecological. We consider these later.

2.3.2 The Spectrum Cone
That radiant power is positive implies that all colors are contained in a conical convex hull, confined to a half-space (Figure 1). An important empirical fact is:

**Empirical fact 1** (The spectrum cone). The generators of the boundary of the spectrum cone are monochromatic beams. Note This is necessarily an ideal limit, since monochromatic beams cannot exist as physical objects.

This is of crucial importance (West and Brill, 1983; Koenderink, 2010). In terms of the sharp map (Eq. 1), the wavelength parameterizes (ignoring trivial matters of resolution) distinct qualia. It fails to cover the range. For instance #purple is a quale for which there is no corresponding wavelength.

In some conventional spectral ranges one encounters minor violations of convexity near the spectrum limits. One may avoid this by limiting the visual range to 390–710 nm. The very minor violations at either end are of no practical consequence (Dropping this clipping makes a
fractional $XYZ$ difference of less than $5 \times 10^{-4}$ relative to white—which is nothing).

The purple sector is best seen in a view from the direction of the color $W$ (equals $\int \lambda \epsilon(\lambda) d\lambda$ [the equal energy spectrum, Figure 2]). This color $W$ tends to appear white. The cone generators (monochromatic beams of unit power $M\lambda$) are labeled by wavelength. Note that there exist pairs $\{\lambda, \overline{\lambda}\}$ that are coplanar with $W$ thus satisfying $[M\lambda, M\overline{\lambda}, W] = 0$, although not all generators are a member of such a pair. The pairs are denoted "complementary."

Complementarity is a formal relation between color triples in colorimetry proper. In the context of object colors complementarity has a formal and a physical meaning. Complementary object colors satisfying $R_1(\lambda) + R_2(\lambda) = I^\infty$ are "bipartitions of white," these might be denoted "supplementary." Supplemetarity implies complementarity, but not vice versa. Supplemetarity cannot be defined in colorimetry proper. In object color space the white object is "given," thereby uniquely defining supplemetarity and thus complementarity (for any illuminant!). In colorimetry proper, complemenarity is an arbitrary convention.

3 OBJECT COLOR SPACE

We proceed to construct object color space. The mere colorimetry of beams, does not offer handles to arrive at a "canonical basis" (CIE $XYZ$–space is a convenience). Ways to go beyond that either turn to physiology (the cone fundamentals as a preferred basis) or psychometry (thresholds, just noticeable differences), or even phenomenology ("eye measure") (Bouma, 1946).

In contradistinction, object color space allows one to establish a preferred basis and even a metric (Section 4.3). That is a decisive step forward.

3.1 The Schrödinger Color Solid

In the standard context of object colors the radiant source is fixed. The generic instance is "average daylight." This is indeed of crucial importance, as evident from the efforts of visual artists (painters through the ages, photographers for almost two centuries, cinematographers for a century, museum directors, and so forth).

Suppose the beam irradiating the gamut of samples is fixed to the equal energy spectrum $\epsilon(\lambda)$. Then the color $C$ of an object with spectral reflectance $R(\lambda)$ is:

$$ C = \int_{\lambda_{min}}^{\lambda_{max}} M R(\lambda) \epsilon(\lambda) d\lambda / \int_{\lambda_{min}}^{\lambda_{max}} M I^\infty \epsilon(\lambda) d\lambda $$

(2)

The beams scattered to the eye of the observer are spectrally dominated by the beam scattered by the white object. The normalization (denominator) achieves $W = [1, 1, 1]$. In the space of beams the set of scattered spectra all lie within a finite volume. It is $I^\infty$, a infinitely dimensional parallelepiped.

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2Here $[a, b, c]$ denotes the conventional scalar triple product.

3Here, as in Eq. 7, we intend the Hadamard (or Schur) element-wise product or quotient.
Thus it is centrally symmetric and convex. One vertex is the origin $\emptyset$.

Since the projection is linear, the gamut of object colors $S$ in color space is also a centrally symmetric, convex volume, connected to the origin. In the infinitesimal environment of the origin (the color $K = M\emptyset$, looks $\text{black}$) it will be tangent to the spectrum cone, whereas, at the “white point” $W$ it will be tangent to the inverted spectrum cone. Otherwise one expects a smooth surface, except for possible curved, dihedral ridges.

Such an object was first intuited by a painter in the early 19th century (Runge, 1810), then by a combination of intuitive, formal and empirical methods approximately constructed by Ostwald (1919), finally formally defined by Schrödinger (1920). Schrödinger proved (also intuited by Ostwald a decade earlier):

Theorem 1 (Schrödinger’s Optimal Colors). The colors on the boundary of $S$ are singled out by two properties:

1. they are characteristic functions in the wavelength domain, that is to say, the spectral reflectance factors are either zero or one;
2. the spectral reflectance factors have no more than two transitions over the visual range.

The proof is immediate (Schrödinger, 1920): With 3–degrees of freedom (DOF) one can move the color away from the origin. Schrödinger’s constraints veto that possibility.

Ostwald (Ostwald 1917a; Ostwald, 1919) proposed a simplifying intuition. If one conceives of the spectrum limits as (colorimetrically) connected (discounting the purple gap), the colors on the boundary are singly connected characteristic functions.

Due to the maximum distance from $K$, Schrödinger (1920) spoke of “optimal colors” (G., Pigmente von größter Leuchtkraft).

One way to visualize $S$ is to compute all optimal colors (a few hundred will do) and construct the convex hull. Another is to generate millions of random object colors and compute the convex hull (the latter action will discard most of the gamut). The result will be the same [(Koenderink, 2010; Centore, 2011), Figure 3].

Of course, the result will depend upon the spectrum (not just the color!) of the radiant environment (daylight say). The consequences of the choice are considered in Section 6. Until......
3.2 The Edge–Color Curves

Although the boundary of the color solid is a surface, thus two-dimensional, it is a very constrained surface. Technically it is a double surface of translation (Koenderink, 1990; Koenderink, 2010). The surface can be constructed from a curve. That curve is the cumulated source radiant power spectrum. Since one may start accumulation from either spectrum end, one may construct two of such curves. These are mutually congruent. The curves are known as “edge–colors,” or boundary colors after the German Kantenspektren. These were accidentally discovered by Goethe (1810) as he playfully looked at a window edge through a prism (Supplementary Section S4; Supplementary Figure S4).

Starting accumulation from the short wavelength end, the curve is [don’t confuse $K$ (black) with $K(\lambda)$ (edge color)].

$$K(\lambda) = \int_{\lambda_{uv}}^{\lambda} ME(\mu) d\mu, \quad \lambda \in (\lambda_{uv}, \lambda_{ir})$$

(3)

where $\mathcal{E}(\lambda)$ is the source spectrum and $\lambda_{uv}$, $\lambda_{ir}$, the spectrum limits near the ultraviolet (uv) and infrared (ir). The other curve $\overline{K}(\lambda)$ may be defined as $\overline{K}(\lambda) = W - K(\lambda)$, where $W$ is the color of the white object.

Supplementary Figure S4 gives an impression of the edge colors as one sees them when looking at a light-dark edge through a prism. Formally the edge–colors are the colors of the accumulated radiant spectrum of the illuminant. Each contains all information present in the combination of the radiant spectrum of the source and the color matching functions. They wrap up all that is needed for object color colorimetry. Note that green and purple are lacking. These are seen when you look at white (spectrum) or black (“inverted spectrum”) bars. Thus Goethe considered the Newtonian spectrum an artifact. Our formalism resolves the dilemma.

Both curves are helices of half a turn (Supplementary Figure S3). Since they are made up of optimal colors, they run over the boundary of the color solid. Any optimal color is either $K(\lambda_{2}) - K(\lambda_{1})$ or $W - [K(\lambda_{2}) - K(\lambda_{1})]$ if we take $\lambda_{2} > \lambda_{1}$.

Theorem 2 (optimal colors). All optimal colors are obtained as chords from one edge–color curve.

Figure 4 shows the cool edge–color curve and the surface defined by its chords. This is one half of the boundary of the Schrödinger $\mathcal{S}$ [The other half is obtained by inverting on the gray point (in the figure), or by plotting the surface of chords of the warm edge–color curve.] This illustrates the relation between the edge–color curve(s) and $\mathcal{S}$. It also shows that the boundary of the Schrödinger $\mathcal{S}$ comes as two, mutually congruent half-shells.

4 SPECTRAL PARTITIONS

4.1 Spectral Bipartitions

If one cuts the spectrum at any wavelength, one has a bichromatic contrast, namely the accumulated spectral radiant power at the short wavelengths side, and the accumulated spectral radiant power at the long wavelengths side. Such bipartitions were explored by Schopenhauer (1816) in the early 19th century—before the advent of colorimetry. Some bipartitions turn out to be special (Supplementary Section S5). Schopenhauer did it by eye. Numerically one considers the chromatic power of the contrast in a meaningful coordinate system (We consider suitable reference frames in Section 4.3).

The largest overall chromatic contrast occurs for a cut at 529 nm. Then the parts are teal and orange. This is evidently the cool-warm contrast universally recognized by visual artists. We consider such cuts in more detail in Section 4.3.

Other special cuts are at 474 nm (blue-yellow) and 577 nm (turquoise-red)—and, trivially, black–white. Schopenhauer also considered spectral extents and would express the chromatic contrasts in terms of (simple!) ratios of spectral extents. From our perspective the latter makes little sense, although the singularity of certain cuts is indeed phenomenologically apparent (Supplementary Figure S5).

Schopenhauer mentions Schwarz-Weiß (black-white), Violet-Gelb (evidently blue-yellow), and Blau-Orange (clearly teal-orange) (Supplementary Figure S6). He also has Grün-Roth, but mentions that the red is really a carmine, so he must indicate our green-purple. This needs two cuts, so it is really a tripartition (Section 4.2). Schopenhauer’s eye-measure is clearly “explained” by our computations.

What is especially relevant here, is that the highly chromatic colors are all but “monochromatic.” These colors have radiant power over large spectral ranges. This is a major step away from the Newtonian notions that still largely determine the understanding of philosophers of color. Color has very little to do with wavelengths. Teal and orange each scatter about half the equal energy spectrum. The best yellow scatters all wavelengths longer than about 474 nm, that is about the whole spectrum! This is an important insight that we will explore in this paper.

4.2 Spectral Tripartitions

A basis of color space involves three mutually independent colors (vectors) $\{A, B, C\}$, say, rooted at the origin $O$. The sum $A + B + C$ is another color and so are the partial sums $A + B$, $B + C$ and $C + A$. The convex hull of the six colors $\{O, A, B, C, A + B, B + C, C + A, A + B + C\}$ is a parallelepiped, that might be called the “crate $\{A, B, C\}$,” say.

Since there is no metric, vectors cannot be compared with respect to length, nor are angles spanned by pairs of vectors comparable. However, there is one comparison that does make sense:

Intuition 1 (Volume–ratio is the invariant of color space). In a linear space the ratio of volumes is the only meaningful, quantitative comparison.

That is an irrelevant fact in generic (context free) colorimetry, simply because there are no volumetric regions to compare. It is the crux of object color formalism. In object color space the color solid provides a unique reference volume.

This suggests a unique crate, the crate of maximum volume inscribed into the color solid. Some geometrical requirements are
obvious: the vectors are optimal colors, the tangent plane at the
color solid at \( A \) must be parallel to the plane \( B \wedge C \), one needs
\( A + B + C = W \), and so forth. The vectors have to be defined by
two spectrum cut loci \( \lambda_1, \lambda_2 \), say, so that \( A = K(\lambda_1) - K(\lambda_\infty) \),
\( B = K(\lambda_2) - K(\lambda_1) \), and \( C = K(\lambda_\infty) - K(\lambda_2) \). One needs the cut-
loci \( \{\lambda_1, \lambda_2\} \) that maximize the volume. Exhaustive search proves
that the extremum is unique (Supplementary Figure S7). Of
course, the location of the extremum depends (slightly) upon the
radiant power spectrum of the illuminant.

This optimum spectral tripartition for average daylight (say)
may well be the most important structure in the colorimetry of
object colors.

Such spectral partitions were pioneered by Schopenhauer
(1816), taking cues from Goethe. He identified certain parts as
atomic “parts of white.” Such parts are hugely different from
Newton’s “spectral atoms.” The Schopenhauer “parts of white
cover broad spectral ranges, they are as different from the
Newtonian atoms as conceptually possible. What they really
are is spectrum bins:

**INTUITION 2** (RGB coordinates). **RGB coordinates are
course-grained spectral reflectance factors** (Formalized below,
Eqs. 4 and 5).

The maximum volume crate \([R, G, B]\) offers the optimum way
to parameterize the interior of the color solid in terms of three
coordinates, optimum in the sense that the gamut with coordinate
values in the range \((0, 1)\) is maximized. The crate yields the best
primaries for a conventional display unit [Figure 5; Foley et al.
(2005)]. This gamut is typical for generic displays. Industry has
zoomed in on to the optimum by trial and error.

The colors of the crate vectors look \( \text{red, green} \) and \( \text{blue} \).
That is what users of display units call them (vastly outnumbering
the vision scientists and philosophers taken together, who object
vehemently to such practice on scientific—actually
philosophical—grounds). So that is what we will call them in
this paper. No excuses offered, although many philosophers are
bound to complain.

“Color fictionalism” (Gatzia, 2010) conveniently enables
one to discuss such tricky topics while holding on to
objective science in which \( \text{colors are not object
properties, but irrelevant mental paint (Gatzia, 2010). For}
academics that is surely a way to go (they don’t do, dabble in
concepts), but users couldn’t care less (they do, dabble in
actualities).

One has a “contraction map” \( \mathcal{C} \):

\[
\mathcal{C} : B \mapsto C_{RGB} \quad \text{thus} \quad \mathcal{C} = M_{RGB}S \quad \text{(a color)} \quad (4)
\]

that maps spectra on RGB colors. One defines an “expansion
map” \( \mathcal{E} \):

\[
\mathcal{E} : C_{RGB} \mapsto B \quad \text{thus} \quad \mathcal{E} = M_{RGB}^{-1}C \quad \text{(a beam)} \quad (5)
\]

that maps RGB colors on spectra. The rows of \( M_{RGB} \) are the color
matching functions, whereas the columns of \( M_{RGB}^{-1} \) (a variety of
“generalized inverse”) are the characteristic functions of the

tripartition (Supplementary Figure S8). The product $M_{\text{RGB}} \cdot M_{\text{RGB}}^{(-1)}$ is the identity in $\mathbb{C}_{\text{RGB}}$. Because $\downarrow\uparrow = I_3$, one has a “generalized left-inverse” (Supplementary Figure S8).

Note that a Moore-Penrose pseudo-inverse (Rao and Mitra, 1971) requires a metric. Its use by Cohen and Kappauf (Cohen and Kappauf, 1982) is unfortunate. One obtains the same advantages using the present formalism. The product $M_{\text{RGB}}^{(-1)} \cdot M_{\text{RGB}}$ maps spectra on their canonical representation. Repeating this will bring no further changes. Thus this product is an involution: $(\downarrow\uparrow)^2 = \downarrow\uparrow$. It has trace three and is the identity on the subspace of canonical spectra.

The intuitive meaning of $\downarrow\uparrow$ is that it “strips the metameristic fluff off spectra.” That was the really nice (unfortunately wrong) notion of Cohen and Kappauf (Cohen and Kappauf, 1982). Stripping twice makes no difference. What is discarded by stripping is “metameristic fluff” in $\mathbb{N}$ (Section 7).

4.3 The RGB Cube Metric

The optimum crate allows comparison of directions and magnitudes in color space. The parts have equal standing as “mutual parts of white” (Koenderink et al., 2018a; Koenderink et al., 2018b), and have disjoint spectral footprints. Geometrically, one treats the crate as the “unit cube” $I_{\text{RGB}}$.

PROPOSITION 1 (The RGB Unit Cube $I_{\text{RGB}}$). The canonical description of the space of object colors is—by way of the maximum volume crate crate $[R, G, B]$—augmented by the Euclidean geometry to make it into the unit cube.

This metric has nothing to do with phenomenology, psychometrics, or eye measure. It derives from colorimetry, physical constraints and a conventional radiant source [$E(\lambda)$ or average daylight serve fine.] Anyone can repeat this construction on a deserted island, starting from any set of color matching functions and will end up with the same result. This takes the arbitrariness out of (object color) colorimetry. Crutches (in our case CIE XYZ) can be discarded after reaching one’s goal.

Note the close connection between colors and spectra. One denotes colors by coordinates, as RGB[90][50][10] (an orange), where the numbers 00 through 99 denote coordinates 0. through 0.99 ≈ 1 in the RGB cube. This color has spectral reflectance factor 0.1 over the range $\lambda_{\text{wr}} - \lambda_{\text{r}}$, 0.5 over the range $\lambda_{\text{r}} - \lambda_{\text{g}}$, and 0.9 over the range $\lambda_{\text{g}} - \lambda_{\text{b}}$. Such a spectrum would indeed cause that color, but so would many others, a point addressed in Section 7. The close relation between spectra and colors (the $\downarrow\uparrow$ operators) is unique to object colors.

Supplementary Figure S4 shows an impression of the “edge spectra,” seen when looking through a prism at any light-dark edge. Unlike the spectrum, the edge spectra are really bright, Goethe’s major point against Newton.

We use an alternative orthonormal frame in $\mathbb{C}_{\text{RGB}}$ defined as $h_1 = [1, 1, 1] \sqrt{3}$, $h_2 = [1, 0, -1] \sqrt{2}$ and $h_3 = [-1, 2, -1] \sqrt{6}$. For the moment, note that $h_1$ is the “achromatic direction,” that is the direction of white, whereas the $h_2 \land h_3$—plane is spanned by purely chromatic dimensions (To be explained later, see Section 5). Supplementary Figure S9 shows the corresponding “opponent color channels.”

Supplementary Figure S10 shows some further properties of the edge—colors. The figure shows projections of the edge—colors on these directions. The achromatic component has two major peaks, yellow (about RGB[99][99][00]) and blue (about RGB[00][00][99]). The chromatic component has three major components, which are just red (RGB[99][00][00]), green (RGB[00][99][00]) and blue (RGB[00][00][99]).

The relation between wavelengths and geometry is seen in a view from the achromatic direction (Figure 2). Spectrum cone generators are labeled by wavelength, but can also be measured by angle in the $h_2 \land h_3$—plane. It is preferable to use the angle metric and consider the wavelengths as arbitrary labels. The relation of complementarity is trivial in the geometry (a diametrical relation, or a 180° shift), but is a complicated, non-linear function in terms of wavelengths (one reason why wavelengths are best regarded as arbitrary labels). The latter relation finds no explanation in the physics of radiation, whereas the former is trivial (but important) in the formal theory of object colors.

Another example of the use of the angle metric is to consider the rate of change of angle of the cone generators with wavelength (Supplementary Figure S11). The relation is nonlinear, with sudden spurts at yellow (RGB[99][99][00]) and turquoise (RGB[00][99][99]). This is visually apparent when you look at the daylight spectrum through a spectroscope. Especially the yellow region looks very narrow. These regions are also apparent in the classical wavelength discrimination curves (Wyszecki and Stiles, 1967; Zhaoping et al., 2011). One discriminates a few nanometers near 570 nm, but several times more near 530 nm. Near the spectrum limits the discrimination deteriorates completely.

4.4 The Semi-chromes, or Full Colors

One uses “dominant wavelength” for the label of a spectrum generator coplanar with the achromatic direction and the color, such that the color is in between. It may or may not exist. Empirically, the dominant wavelengths labels may double as the dominant wavelength (almost black) looks black. Among colors of a dominant wavelength, some stains are off−whites and some shades are close to black.

In terms of optimal colors, all are (roughly) centered on the same spectrum generator, but they come in various spectral widths. Tints may scatter most of the source spectrum, whereas shades may scatter only a narrow region.

Figure 6 shows the example for a dominant wavelength of 530 nm, which appears green. It covers colors from RGB[00][01][01] (almost black) over RGB[00][99][00] (as green as it gets) to RGB[98][99][98] (almost white). The Newtonian “green” (as monochromatic as possible) looks black. The “best” green is an Ostwald semichrome that is all but monochromatic, but scatters a major part of the spectrum. So much for the “color as seeing by wavelength” notion still popular with some philosophers [physical realists; Byrne and

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4Cohen and Kappauf assumed a scalar product where there isn’t any. They implicitly use that when defining a Moore-Penrose pseudo-inverse. “Color space” does not allow for that, it is only a Hausdorff space.
chromatic vertices of the crate (Hilbert (2003)]. The confusion of "dominant wavelengths" with physical (pseudo-)monochromatic beams is nonsensical, but—perhaps unfortunately—common in the literature.

The best color is defined through maximum chromatic content. It has band limits at mutually complementary wavelengths. This is because one adds white when the band width is slightly broadened, whereas one adds black (in the sense of removing white) on narrowing it—always keeping the dominant wavelength fixed. Adding white dilutes the color into a tint, adding black dims the color into a shade.

Because of the complementary band limits, the best colors were denoted "semichromes" by Ostwald. But because they are the "best colors" in the sense of maximum chromatic content he also used the term "full colors" (G., Vollfarben). Empirical semichromes indeed look highly colored, but—especially in the red—some shades actually look a tad redder than the red semichromes. So "semichrome" is the preferred term for what indeed are special ("full") colors.

The semichrome locus geometrically (Supplementary Section S6; Supplementary Figure S12) appears as the "equator" of the color solid in that a circumscribed cylinder with generators parallel to \( \mathbf{h}_1 \) touches the solid along the semichrome locus.

Supplementary Figure S14 shows the achromatic content of the semichromes. It is evident that certain colors are special. We call them "cardinal colors" for that reason. These are close to the chromatic vertices of the crate (Supplementary Figure S27). They evidently come in two types:

- The "primary" cardinal colors correspond to the parts of the tripartition of white. They look \#red, \#green and \#blue, or RGB;
- The "secondary" cardinal colors correspond to the unions of pairs of the tripartition of white. They look \#turquoise, \#purple and \#yellow (Preferred technical terms are cyan, magenta and yellow, or CMY.)

Supplementary Figure S14 right illustrates the relation of the semichrome locus to the \( \mathbf{I}_\text{RGB} \). The curve evidently hugs the R-Y-G-C-B-M-R—edge progression. The RGB cube is a good "summary" of the Schrödinger color solid.

Semichromes come in a periodic sequence that includes the "extra–spectral" \#purple. A theory of object colors enforces this. In the colorimetry of radiant beams purple is an inconvenient oddity. Newton never understood why he didn’t (honestly speaking) see it in the spectrum. His "color circle" (Newton, 1704) was an obvious kludge. The nature of purple was finally cleared up empirically by Helmholtz (Helmholtz, 1855) in the mid 19th century, formally in the early 20th century by Ostwald and Schrödinger.

5 THE OSTWALD HEURISTIC

Wilhelm Ostwald was a chemist (Nobel Prize 1909 on catalysis) who worked on color after his retirement. His color system dominated in education and industry in continental Europe before Ww-II. It has features that are sorely missed in modern accounts, as well as some that need to be amended. Various features are easily included in the formal colorimetry of object colors. A convenient technical and critical account is Bouma (1946).

What is especially valuable is the close connection between object colors and reflectance spectra. The Ostwald system focusses on colors as object properties. It depends on physical measurement.\(^5\) It is ontologically distinct from eye-measure systems such as that of Munsell (1905). Perhaps unfortunately,
conventional colorimetry fails to recognize this ontological chasm.

One basic intuition is illustrated in Figure 7. It concerns the relation of the color circle (the periodic set of hues) to the spectral reflectance factor. The “average daylight” source went understood. We show it as a circular slide rule. The fixed disk is simply our Figure 2, a representation of the spectrum calibrated for human vision. The rotating disk selects one half of the spectrum, that is a semichrome. The two red arrows indicate the pass band limits. The central blue arrow is Ostwald’s intuition: It indicates the dominant wavelength of the semichrome. Note the yellow indicator at the opposite position, it indicates the complementary dominant wavelength of the semichrome. The latter is required when the blue indicator is in the sector of purples. For one thing, this reveals the fundamental reason why the spectrum (a topological linear segment) gives rise to a “color wheel” (a topological circle). In Ostwald’s perspective it is a natural consequence, other “explanations” we know of are essentially arbitrary kludges.

Does it work (Supplementary Section S8)? This requires explicit calculation, which Ostwald does not provide. He used an inverted spectroscope—an excellent idea, perhaps picked up when reading Maxwell (1855) on the “color box.” He could select parts of daylight and look at arbitrary optimal colors. We know few modern color scientists who ever had that opportunity. Electronic display is no alternative. What he saw convinced Ostwald that the intuition applied. We won’t call him wrong—just somewhat sloppy, perhaps “visionary” is the word.

Our calculation is presented in Figure 8 (a more conventional representation of the same structure is Supplementary Figure S13). It is a pity Bouma (1946) doesn’t show it. We are not aware of a depiction in the 1920’s literature; to many scientists at that time it would have looked familiar. The intuition is not perfect, but the deviation is zero at twelve hue angles, whereas the standard deviation of errors in the hue angles is 8.6°. It serves well as a heuristic—even today.

Note that there are four distinct types of semichromes. This is most easily understood by using the slide rule to check when the pass band meets the spectrum limits at either side. These types are the essentially different families of spectral reflectance factors, thus important qualitative object properties:
medium pass bands These look greenish (key example Figure 6);
long pass bands These look orangish. Many flower colors make good examples (example Figure 17 in Section 7);
medium stop bands These look purplish (complementary of Figure 6);
short pass bands These look steaish.

The medium pass and stop band cases are ecologically rare, the long pass bands are common with organic colors (Supplementary Section S12).

There are four basic families, not three or six, as one might intuit on the basis of tripartition. This is due to the closure of the spectrum by way of the sector of purples. Ostwald’s construction is the unique formal explanation of the “color circle,” long familiar to artists on purely phenomenological grounds. Ostwald’s construction comes from the tripartite structure of white augmented with closure of purples. A close look will reveal the four families.

The families come in two mutually complementary types, the medium stop and pass bands (purple–green group), and the short and long pass bands (stea–orange group). Due to ecological factors (Supplementary Section S12) the stea–orange group is at least thrice as common as the purple–green group. Teal-orange is known as the “cool-warm” dimension in the visual arts (Quiller, 1989). The purple–green group is rarely named explicitly. It appears prominently in experimental phenomenological research (Albertazzi et al., 2015).

Hering’s idea (G., Vierfarbentheory) was thought to clash with trichromy, until Schrödinger cleared up the confusion (Schrödinger, 1925). We use the “h” of the \( \{ h_1, h_2, h_3 \} \)-basis to indicate the Hering origin.

The reason why the Hering basis is of great importance is not so much its phenomenological origin, as the fact that it effectively decorrelates spectral reflectance factors ([Buchsbaum and Gottschalk, 1983]; Supplementary Section S12).

For many applications the Hering basis is more appropriate than the RGB tripartition. A good example is the chromaticity diagram. It is possible to base a conventional chromaticity diagram on the tripartition (we use that later), but an orthographic projection along the \( h_3 \) direction on the \( h_2/h_3 \)-plane yields a far more intuitive representation (Figure 9).

It is of some interest to compare the CIE–xy diagram with the orthographic projection. Both are projections of the same objects. The generators of the spectrum cone are infinite half-lines that are based at the origin. In the CIE diagram they appear as points of a curve, the direction image in the plane at infinity. In the orthographic projection the cone generators appear as rays radiating from the origin. Thus a wavelength is indicated by a direction, or half-line. The best colors are found near the semichrome locus in both diagrams.

In the CIE diagram, the spectral locus corresponds to the black object colors. There is no metric and there are no affine notions like bisection of stretches or parallelity. Only coincidence and collinearity is meaningful (if you know how to interpret these). In contrast distinction, in the orthographic projection affine relations count, one even has a Euclidean metric.

In the CIE diagram, a point stands for a color of arbitrary intensity. In the orthographic projection a point stands for a color of arbitrary black-white content.

Figure 10 illustrates an example that clearly shows the power in the structure of the chromaticity diagram in Figure 9. It can be used to compute the diagram of Supplementary Figure S13 or that of Figure 6. Such nomographic calculations were common pre–WW-II. Nowadays one uses straight number-crunching. Figure 10 shows one reason why the proposed chromaticity
The chromaticity diagram (Figure 9) makes sense. Understanding the structure of the chromaticity diagram well yields a lot of power. Moreover, it ensures that one keeps a clear view of the relations between the various colorimetric objects. Such uses were common in the pre-computer era.

**Supplementary Figure S21** summarizes much of the structure. Note the close relation between edge–colors and semichromes, as well as the explicit relation between the semichrome colors and their spectrum. This is all about broadband spectra, the "monochromatic beams" are degraded to mere wavelength calibration in the background.

The object color chromaticity diagram of Figure 9 is much less intuitive for additive mixture (although it can be done) than the conventional chromaticity diagram. This is no big deal, because there is (except for very unlikely settings) no such a thing as the additive mixture of object properties.

In the case of object colors mixture mostly implies union and intersection of characteristic functions, or multiplicative combinations. The chromaticity diagram is well suited for that, since the spectral range is shown in the background.

The main thing to mind is that semichromes are not at all "monochromatic," but actually scatter half of the source spectrum. Combining object colors should always be understood as an operation on spectra, not colors. This two-way relation between the semichromes (points) and band limits (diagonal lines) is a conjugation relation that should always be kept in mind when using the chromaticity diagram.

The chromaticity diagram is less symmetrical than it might seem (Supplementary Section S6). This is intuitively evident from the fact that the color solid at the black and white points has the shape of the spectrum cone. Near the white point you see basically the shape (Supplementary Figure S12) of the boundary of the planar cut of the spectrum cone illustrated in Figure 1 right.

The “chromaticity diagram” introduced in Figure 9 left is actually a view of the color solid from infinity along the Black–White (KW) Hering axis. A complete graphical representation add views along the Teal-Orange (TO) and Green-Purple (GP) Hering axes (Figure 9). This yields a complete representation of a gamut of object colors.

For beam colors the classical chromaticity diagram makes sense, because intensities may run all the way from 0 to ∞, whereas it often makes little sense to note intensities. For instance, a “monochromatic beam” for a wavelength λ₀ will be a spectrum like

\[
S(\lambda) = 0 \quad \text{for} \quad |\lambda - \lambda_0| \leq \frac{\Delta \lambda}{2}
\]

\[
S(\lambda) = I \quad \text{for} \quad |\lambda - \lambda_0| < \frac{\Delta \lambda}{2}
\]

It has a power IΔλ and an intensity I. The spectral width Δλ ∈ R⁺ may be chosen arbitrarily. As long it is small, its actual value is irrelevant. Thus the intensity is essentially undefined ("oo"), but the chromaticity is well defined. Thus chromaticity diagrams are always useful, but 3D-color space plots often are not.

This is very different for objects colors, which have well defined positions in the color solid. Here chromaticity alone leaves out crucial information. That is why the representation that uses three canonical views of the color solid is preferable. The orthographic projection orthogonal to the KW-axis is like a chromaticity diagram. The projections on the KW–TO and KW–GP planes complement that in a useful way. One treats these as three orthogonal views of a 3D-object. This is far more useful than conventional chromaticity plots. The link to spectral composition is as close as can be.

5.1 Ostwald’s Color, White and Black Contents

Wilhelm Oswald proposed a canonical spectral reflectance for any color as the appropriate partitive combination of a

*Thus the material to be shown should be suitably sorted “in depth.”*
semichrome, the standard white, and the standard black. Then any color is parameterized by way of:

- **hue** that is “which” color. One naturally uses something like the angle parameter;
- **color content** that is the attenuation factor for the semichrome, the “color content”;
- **white content** that is the addition of a constant, subject to the constraint that the total reflectance factor may not exceed one;
- **black content** which is simply one minus the sum of the attenuation factor and the white content.

What is special as compared with other systems is that colors come with a unique canonical spectral reflectance factor.

Color content, white content and black content sum to unity. For any hue one has a variety of choices, neatly parameterized by the content in a triangular, barycentric scheme. These became the pages (each hue a page) of the Ostwald atlas (Figure 11).

Ostwald (Ostwald, 1917a; Ostwald, 1917b) actually constructed and published implementations. Although objects of rare beauty, they are only approximations. It was the best a top-notch chemist as Ostwald could do with actual pigments. In contradistinction, the “true” atlas is a *formal* object, not a physical thing. Today, anyone can easily program it on an electronic display. Various industrial designs of “color pickers” essentially implement it.

The Munsell (1905) atlas is today’s *de facto* standard. It is due to eye-measure, not physics. There is no relation to object properties and there is no fundamental formal structure. It cannot be constructed from first principles.

There is room for both ontologically distinct types. The real miracle is that the atlases are remarkably similar. Unfortunately, this is also a reason why the ontological roots tend to be ignored. This “miracle” is another manifestation of the efficaciousness of the sharp map $\Psi$ (Eq. 1).

The Ostwald description was reinvented by Alvy Ray Smith (Smith and Lyons, 1996), of early CG fame. It is the HWB color system. Starting with RGB $[r|g|b]$ one defines the white content as $\min(r,g,b)$, the black content as $1 - \max(r,g,b)$, which then automatically defines the color content as the remainder. For the hue one uses the distance along the edge progression of cardinal colors. This is very convenient and yields an immediate insight in the structure of the canonical spectrum. HWB actually resurrects the Ostwald system (in CG digital form), although nobody notices (not even Smith).

### 5.2 An Ostwald Basis

The Ostwald system uses an over-complete, continuous basis (*Supplementary Section S9*). Thus it is really different from the
tripartite (optimal basis). It overlaps significantly with the RGB basis (Figure 12; Supplementary Figures S25–S27) and exhausts 78.5% of the color solid.

The set of all optimal colors is a continuous basis that captures all colors. In principle such a system is impossible to beat, in practice its implementation would be a nightmare.

5.3 Canonical Spectra

We have mentioned various “canonical spectra”:

- **parts of white** constant reflectance factor over the spectral parts of white, produced from an RGB color through the expansion map, Eq. 5;
- **Ostwald system** an attenuated semichrome, diluted with white;
- **attenuated or diluted optimal color** one might say the attenuated optimal color is Schrödinger’s proposition.

A diluted optimal color (mixture with gray) as mentioned above is a similar notion.

In typical cases, such canonical spectra yield quite good approximations to the actual spectrum (Supplementary Section S7; Supplementary Figure S16).

We show a few examples, using a flower color as example. *Tropaeolum majus* is cultivated for its bright orange flowers. In Figure 13 we show its spectral reflectance factor as compared to the tripartite and Ostwald semichrome canonical spectra. Both canonical spectra are useful for many applications, but—by design—they are different. The difference matters “when the light changes” (Section 6.1).

5.4 The Topology of “Subtractive Color Mixture”

One usually distinguishes “additive” and “subtractive” color mixture, although there are numerous ways to combine spectrally selective scattering. Additive mixture is rarely of interest in the theory of object colors. It is fairly trivial, the main result being that the gamut of beams is confined to the convex hull of the monochromatic beams.

“Subtractive” is actually a misnomer of “multiplicative.” In some circumstances multiplicative combinations of spectral reflectance factors are excellent descriptions, in some cases mere serviceable approximations. It is important to understand the ideal case as a point of departure.

In the additive case it makes no difference whether one adds spectra or colors. In the multiplicative case one has to multiply the spectra. Canonical spectra make a good start. Deviations from canonical may have surprising effects (Section 7.1).

In the cases of the tripartite canonical spectra one may actually multiply the color coordinates, so this is an especially simple case. One has a map of the product of the color circle with itself to the color circle, that means a map from $\mathbb{T}^2 \rightarrow S^1$ (Figure 14; Supplementary Section S3). Intuitive grasp implies having the torus in mind.

As said, deviations from the canonical spectrum may have non-trivial influences. Differences in the source spectrum may have non-trivial consequences too. That is because the proximal stimulus is the product of the source spectrum and the object spectral reflectance factor (Section 7.1).

6 VARIATION OF THE SOURCE AND AUTOMATIC WHITE BALANCE (AWB)

So far, we have fixed the source to $E(\lambda)$. As argued below, this is a good (ecologically relevant) starting point. However, in real life the radiant umwelt may vary greatly. The human observer copes with that and so do modern smart-phone cameras. Object color colorimetry is incomplete without AWB.
6.1 Automatic White Balance

Many electronic cameras use a dedicated sensor to estimate the spectral slope (and perhaps the spectral curvature) of the source spectrum. Even mid-level cameras allow users to enter corrections for both slope and curvature. This allows AWB with optional manual correction. Ideally white objects in the scene will be represented as white in the picture.

AWB can be implemented colorimetrically by a scaling in RGB-space such as to force a spectrum of certain spectral slope and curvature to map on RGB \([99\mid 99\mid 99\)]\. The human visual system apparently does something like this (Kries, 1905), given the observation that white objects will stubbornly appear \(\text{white}\) whatever the luminous environment. In the right settings the white object will look \(\text{white}\) and the black object will look \(\text{black}\), whatever the source spectrum.

We do not consider phenomenology, psychology, or physiology, but simply pursue the effects of AWB. “Proper context” serves to indicate ecological limits. Out-of-limits anything goes and “color vision is impossible” [for a proof (Koenderink and van Doorn, 2020)]. Analytic philosophers who will not accept ecological constraints conclude that colors are mere mental paint or that one only pretends to experience them. Many of such ideas are “right” in the sense of not being wrong, but in ways that are irrelevant if one is out alone in the cold. But that is where all biological agents are, including Homo.

The most basic constraint for an effective AWB is that the white standard should look \(\text{white}\) under any source spectrum. Ideally, one requires

\[
C = \frac{\int_{\lambda_{ir}}^{\lambda_{uv}} M_{\text{RGB}} R(\lambda) S(\lambda) d\lambda}{\int_{\lambda_{ir}}^{\lambda_{uv}} M_{\text{RGB}} S(\lambda) d\lambda} = \Xi_S(R(\lambda))
\]

where \(\Xi_S\) is a linear functional that maps spectral reflectance factors \(R(\lambda)\) to colors, a bit like the contraction map. Thus we define (with an example): \(\Xi_S : P \rightarrow C_{\text{RGB}}\) \(\Xi_S(R_{\text{Tropaeolum Majus}}) = \text{RGB}[79\mid 20\mid 01]\) (8)

FIGURE 13 | The Tropaeolum majus is also known as Indian cress. It is mainly valued for its brilliant orange or red flowers. Here the spectral reflectance factor is approximated by the canonical RGB-reflectance and the Ostwald attenuated and diluted full color spectrum. Both are quite good approximations. All three spectra yield exactly the same color.

FIGURE 14 | Subtractive color mixture is a map \(S_1 \times S_1 \rightarrow S_1\) (or from \(T_2 \rightarrow S_1\)). The color circle is on the diagonal (the dashed white line at left) but the overall topology is hardly intuitive (What people root-learn at elementary school is “blue (actually cyan) and yellow yields green,” and so forth.) For instance, at right, like the green, the red is also a full square. The right figure shows the parameter plane. Both coordinates run over the circle \((0 – 2\pi, that is red to red\). Thus the left side is connected to the right side and the bottom to the top.
white standard suffices and that is available from observation.

The $\Xi_3$–map implements the (object property $\mapsto$ color)–relation. “Colors as object properties” involves some generalized inverse of $\Psi^* \Xi_3$, which is not a trivial issue. We pick up that topic in Section 7.

In this ideal system the white standard will look white under any source. So does (trivially!) the black standard. Because of linearity the whole gray axis remains invariant. Chromatic colors in general change, because the proximal stimulus is the coarse-grained product of the object property and the source spectrum.

That is why featureless source spectra are best. They leave the floor to the object properties so to speak. Source spectra with spectral gaps are effectively useless. Examples include low pressure sodium or mercury vapor lamps.

Generically, in ecologically valid contexts, such a system works remarkably well [(Koenderink, 2010; Centore, 2012; Figure 15].

Some rules of thumb relating to the effects of slope and curvature in source and reflectance are frequently useful, like in “relighting” applications (Debevec, 2020).

**Intuition 3 (Metameric effects of spectral slope and curvature).** These are some useful rules of thumb to keep in mind:

- judge the effect for each of the three parts of white separately. For global effects look at the levels instead;
- the effect scales with the product of the slope of the source and of the reflectance spectrum, including sign;
- slope and curvature do not interact;
- the effect of curvature is mainly due to reflectance, it scales linearly (including sign);
- higher order wriggles may safely be ignored since the spectral envelope modulation falls off by the fourth power of frequency.

In most cases one will be able to foresee effects of metamers quite easily using the hexapartition of white (Section 7). With some experience even quantitative estimates can be made.

Fairly extreme cases are shown in Figure 16 and more in Supplementary Section S13. These examples cover about the range relevant to generic human vision. Notice that \textit{AWB} is evidently not perfect, but amply good enough for the early hominin hunter-gatherer life style.

## 7 Metamerism

The spectrum that hits the eye is the product of the spectral reflectance factor (the “object property”), and the spectrum of the source (the “context”). Since human vision only selects three dimensions out of $\mathbb{R}^3$, there can be infinitely many spectral reflectances and sources that have mutually identical canonical \textit{rgb} spectra. The multiplication plays havoc with that. We consider a simple (arguably the simplest) model that captures the bulk of such effects but still leaves room for intuition and simple numerical simulations.

Split each of the three spectrum parts into two. One may use an arc-length rectification of the edge–color curves to find suitable cut loci. This refines the tripartition of white into a hexapartition. This six-dimensional spectral representation has a three-dimensional null-space, a subspace of $\mathbb{N}$. For convenience, we construct a basis for the null space that is as close to

$$\{+1, -1, 0, 0, 0, 0\}, \{0, 0, +1, -1, 0, 0\}, \{0, 0, 0, +1, -1\}$$

as possible. The \textit{rgb} basis remains simply (essentially a tripartition, not a hexapartition—the additional stuff is a refinement):

$$\{1, 1, 0, 0, 0, 0\}, \{0, 0, 1, 1, 0, 0\}, \{0, 0, 0, 0, 1, 1\}$$

Such a basis is readily constructed (Supplementary Section S14; Supplementary Figure S16 right).
This “hex-basis” may be deployed in many applications. Intuitively, it renders the parts of white sensitive to spectral slope. In that sense, it may be understood as a first order extension of the tripartite system. Further subdivision yields an even better approximation, but only marginally so. The simple hexa-chromatic model easily captures the brunt of the action. Canonical hex-spectra necessitate constraints due to the over-complete basis. Either a maximum entropy “smoothest” approximation or a minimax fit associates any RGB–color with a unique hex-spectrum (Supplementary Section S10). These tend to be excellent approximations of natural spectra (Figure 17; Supplementary Figure S16 right).

Maximum entropy approximations of spectral reflectance factors can be computed on the basis of the RGB color coordinates. If one has an actual spectrum, one may also compute the hex-representation of the true spectrum. For databases of reflectance spectra of various natural objects one finds that the estimations on the basis of colors are very close to veridical. The reason is, no doubt, that natural spectra do not show wild variations over the visual range. Indeed, the estimates are so good that hyperspectral methods might well be overkill for many applications, including effects of metamerism (Koenderink and van Doorn, 2017).

Metamers of $\mathcal{E}(\lambda)$ involve arbitrary amounts of the black components. In the worst case spectral radiance will fall to zero in at least one of the six parts. Such metameric sources are not revealed by the white standard. Phenomenologically, they all provide white light. We prepare twenty-six (all triples of $\{-1, 0, +1\}$, except $\{0, 0, 0\}$, thus $(3^3 − 1)$) of such fake standard sources.

Metamers of the flat central gray reflectance factor, involve arbitrary amounts of the black components. In the worst case the spectral reflectance will be zero or one in at least one of the six parts. Such metameric reflectance factors are not revealed by the standard source. Phenomenologically, they all are central gray. We prepare twenty-six $(3^3 − 1)$ of such fake standard gray objects.

Viewing all fake objects under all fake sources yields 676 $(26^2)$ (worst case!) colors. Only 30 of these lie on the convex hull, which is surprisingly large. It has a volume of 0.83 . . . The vertices of the convex hull are shown in Figure 18; Supplementary Section S14, Supplementary Figures S38–S41.

The result should give ample food for thought. Suppose you are a magician, then this would make a great act: You show a gray object next to a white standard. It looks gray. You switch the source and the object to one of the fakes (a sleight of hand routine). The white will remain white, but the object may...
become any color. Your audience will marvel! Indeed, they should, for it is impossible to foresee such effects by using one’s senses.

So what is “the color of the object” then? It evidently “has none.” This is the reason people say that “colors are not object properties.”

Colors cannot be labels for objects, for given two objects of different colors (red roses and blue violets, say), you may imagine a world with fake objects and sources in which roses turn blue and violets red for no apparent reason. If you are a hunter-gatherer you cannot depend on the experience that blue berries are good to eat whereas red berries will give you a belly ache. Color is useless and one wonders why evolution ever bothered.

Of course, one knows better, for color works pretty well in daily life. Reason is that the fake cases are very rare:

- the visual range is so narrow that spectral radiant power or spectral reflectance do not wildly vary over the range. The correlation range is at least of the order of the width itself. Spectra of any kind are smooth;
- there are no physical processes that let spectral radiant power and spectral reflectance be mutually correlated. Surprises will be extremely rare.

The correlation issue is crucial, for the metameric effect depends on it. The multiplicative process in surface scattering, followed by an averaging by retinal absorption, is just a kind of correlation mechanism. Supplementary Figure S40 demonstrates this.

In a simple simulation one prepares random combinations of the metameric blacks to produce slightly perturbed radiant and reflectance spectra (Figure 19, based on a million samples). The distribution is highly kurtotic (Supplementary Section S16), thus something like the convex hull is not a useful measure. Suppose we compute the 99% covariance ellipsoids.

Their size depends on the magnitude of the perturbation used in computing the fakes. Use a normal deviate with standard deviation $\sigma$ to perturb the source spectrum (1.0 throughout the spectrum) and the object spectral reflectance factor (0.5 throughout the spectrum). Then the volume of the ellipsoid is proportional to $\sigma^6$ [two factors—spectral reflectance factor and source radiant power spectrum—each factor having three (trichromacy!) degrees of freedom].

From the simulation we find $0.259\sigma^6$. For ecologically acceptable values of $\sigma$ (say 20–30%) one finds that there is room to distinguish a thousand objects. Such would serve the hunter-gatherer existence, thus there will be an evolutionary drive.

7.1 Metamerism and Multiplicative Mixture

When mixing stuffs characterized by canonical RGB spectra one obtains unique results. One has a well defined map $\mathbb{R}^2 \rightarrow \mathcal{S}^1$. Metamerism changes this essentially (Supplementary Figure S41).

The figure shows the worst case expectation for mixtures of gray with gray, observed under the standard source $E(\lambda)$. Since the spread is going to be less for more chromatic colors (the optimal colors are not affected at all, Figure 18), there is still ample room for a few dozen or more colors that will not be confused. Because we show effects for the most extreme metomers and moreover include results for very special combinations, the result in the generic environment will be much less.

FIGURE 18 | Metameric confusion regions are space variant (Note that the optimal colors are not affected at all.) This is a simplified sketch of what happens. Near the faces of the RGB cube the confusion region flattens, near the edges they become needle-shaped, near the vertices (optimal colors) they become arbitrarily small spheres.

FIGURE 19 | This shows samples from colors due to perturbation of the standard source $E(\lambda)$ and the canonical gray reflectance with random, isotropic combinations of the six blacks. The distribution is highly kurtotic (Supplementary Section S16) and non-isotropic. The blue ellipsoid is the 99% covariance ellipsoid.
So artists may use their understanding of the deterministic map $T^2 \rightarrow S^4$, based on seen colors with some chance of success. However, they need to reckon with unforeseeable minor deviations. In practice, painters tend to remember the mixture properties of pairs of paints, that is to say, the stuff of the paints, not the colors.

This is a rare example where the fact that colors are not always good indicators of object properties matters.

8 CONCLUSION

Spectral object properties, in relation to human vision, call for a formal “object color colorimetry”. It builds on generic (null-context) colorimetry. It is highly structured due to three factors.

One is a dedicated source spectrum (average daylight, sunlight, . . . ), another is basic constraints of physics, the third are various ecological constraints. Due to these, object color colorimetry is qualitatively different from null-context colorimetry.

Ignoring this, and treating object colors by way of null-context colorimetry necessitates arbitrary constraints and has led to considerable confusion.2

The geometrical structure of object colorimetry is the proper basis for color vision research, fundamental computer vision and image science.

Chromatic contrast is certainly an aid to optical navigation, space perception and so forth, but this is not to be reckoned proper “color vision.” Color vision implies qualia, the use of chromatic spectral structure as an intentional mark, or cue. The cues for many important physicochemical object properties are spectral. This renders the chromatic qualia important to human biological fitness.

The ecologically relevant variations due to metamerism, both in source spectra and spectral reflectance functions of generic objects, dwarf psychophysical thresholds and physiological constraints (Supplementary Sections S14, S15). Color vision as a way to sense object properties is not limited by the physiology. In contradistinction, color as a spatial discrimination device may well run into physiological limits, but does not recognize color as a quale. “Color vision” proper is about object colors as object properties.

The key difference between object color colorimetry and null-context colorimetry is the close connection between (coarse-grained and ANS’d) spectral reflectance functions and experienced colors. That is the concatenation $\Psi \circ \Xi_S$ of the $\Xi_S$-map (Eq. 7) and the sharp map (Eq. 1). This applies especially to ecologically generic cases. In perception one deploys a generalized inverse $(\Psi \circ \Xi_S)^{-1}$. The formal existence is questionable, but there is no doubt of its efficacy in biologically relevant settings.

There are two mutually opposing perspectives on this (Byrne and Hilbert, 2003). One perspective is that ecology has nothing to do with what is the case in the physical world at large (often called “reality”). The upshot is that colors are a quirk of the human mind. They hold no relation to object properties at all. This drives one to fictionalism. Neither are roses red, nor violets blue, but we all politely pretend (Gatzia, 2010).

Another perspective is that the ecological parameters drive evolutionary processes. Color is indeed useful in the human life world. In this context roses indeed are red and violets are blue. Colors are pretty good cues to object properties, at least, as far as the evolution of the human species is concerned.

The former perspective is correct, but irrelevant. The latter conclusion is strictly spoken false, but happens to capture the human condition. It is the biological perspective. The one that counts. Surprises should be reckoned with, but should be rare. An individual failing to cope may die. No big deal if such is a rare event.

This is also of importance to computer vision and image science. The ecological statistics have driven the evolution of human color vision. These statistics also offer constraints (Supplementary Sections S11–S14) that may well be exploited by computer vision (Koenderink and van Doorn, 2017; Koenderink et al., 2020).

In the RGB account, especially when augmented by the hexa-partition of white, most two-way relations between colors and spectra become intuitive. The RGB colorimetric coordinates are coarse-grained spectral reflectance factors (the $\Psi$-map, equation Eq. 4). In most cases the expansion ($\Xi$ map) is a heuristic towards spectral reflectance factors. Of course, the sharp map is usually a necessary bridge (Supplementary Section S2) (Object) colorimetry makes no sense without it.

That is why the tripartite representation is the natural geometry for object color space. The basic formalism presented in this paper serves as the proper base of departure for many applications [Supplementary Sections S12, S14, S17 and (Koenderink and van Doorn, 2017; Koenderink and van Doorn, 2020; Koenderink et al., 2020)].

DATA AVAILABILITY STATEMENT

The original contributions presented in the study are included in the article/Supplementary Material, further inquiries can be directed to the corresponding author.

AUTHOR CONTRIBUTIONS

All authors listed have made a substantial, direct, and intellectual contribution to the work and approved it for publication.

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2E.g., the common enough notion that the best colors (object colors implied) would be monochromatic.
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SUPPLEMENTARY MATERIAL

The Supplementary Material for this article can be found online at: https://www.frontiersin.org/articles/10.3389/fcomp.2021.630370/full#supplementary-material