Actions of groups of diffeomorphisms on one-manifolds by $C^1$ diffeomorphisms

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Abstract. Denote by $\text{Diff}^r_c(M)_0$ the identity component of the group of the compactly supported $C^r$ diffeomorphisms of a connected $C^\infty$ manifold $M$. We show that if $\dim(M) \geq 2$ and $r \neq \dim(M) + 1$, then any homomorphism from $\text{Diff}^r_c(M)_0$ to $\text{Diff}^1(\mathbb{R})$ or $\text{Diff}^1(S^1)$ is trivial.

1. Introduction

É. Ghys [G] asked if the group of diffeomorphisms of a manifold admits a nontrivial action on a lower dimensional manifold. A breakthrough towards this problem was obtained by K. Mann [M] for one dimensional target manifolds. Let $M$ be a connected $C^\infty$ manifold without boundary, compact or not. For $r = 0, 1, 2, \cdots, \infty$, denote by $\text{Diff}^r_c(M)_0$ the identity component of the group of the compactly supported $C^r$ diffeomorphisms (homeomorphisms for $r = 0$) of $M$.

Theorem 1.1. (K. Mann) Any homomorphism from $\text{Diff}^r_c(M)_0$ to $\text{Diff}^2(S^1)$ or to $\text{Diff}^2(\mathbb{R})$ is trivial, provided $\dim(M) \geq 2$ and $r \neq \dim(M) + 1$.

For a simpler proof of this fact, see also [Ma2]. A natural question is whether it is possible to lower the differentiability of the target group. In fact for $r = 0$, E. Militon [Mi] obtained the final result.

Theorem 1.2. (E. Militon) Any homomorphism from $\text{Diff}^0_c(M)_0$ to $\text{Diff}^0(S^1)$ is trivial if $\dim(M) \geq 2$.

Notice that $\text{Diff}^0(\mathbb{R})$ can be considered to be a subgroup of $\text{Diff}^0(S^1)$. So we do not mention in the above theorem the case where the target group is $\text{Diff}^0(\mathbb{R})$.

Even for $r \geq 1$, we have:

Conjecture 1.3. Any homomorphism from $\text{Diff}^r_c(M)_0$ to $\text{Diff}^0(S^1)$ is trivial if $\dim(M) \geq 2$.

The purpose of this paper is to mark one step forward towards this conjecture.

Theorem 1.4. If $\dim(M) \geq 2$ and $r \neq \dim(M) + 1$, any homomorphism from $\text{Diff}^r_c(M)_0$ to $\text{Diff}^1(S^1)$ or $\text{Diff}^1(\mathbb{R})$ is trivial.

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Frequent use of the simplicity of the group $\text{Diff}^r_c(M)_0$ is made in the proof. The condition $r \neq \dim(M) + 1$ is needed for it. As for Theorem 1.1 the proof is built upon a theorem of Kopell and Szekeres about $C^2$ actions of abelian groups on a compact interval, while for Theorem 1.4 upon a theorem of Bonatti, Monteverde, Navas and Rivas about $C^1$ actions of solvable Baumslag-Solitar groups on a compact interval.

By virtue of the fragmentation lemma, Theorem 1.4 reduces to:

**Theorem 1.5.** For $n \geq 2$ and $r \neq n + 1$, any homomorphism from $\text{Diff}^r_c(\mathbb{R}^n)_0$ to $\text{Diff}^1(S^1)$ or $\text{Diff}^1(\mathbb{R})$ is trivial.

In Section 2, we show that the case of target group $\text{Diff}^1(S^1)$ can be reduced to the case $\text{Diff}^1(\mathbb{R})$. In Sections 3 and 4, we establish fixed point results for certain subgroups of $\text{Diff}^\infty(\mathbb{R}^n)_0$. In Section 5, we prove Theorem 1.5 following an argument of E. Militon [M]. Finally we give some sporadic results for $\text{Diff}^0(S^1)$ target in Section 6.

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## 2. Reduction to the case $\text{Diff}^1(\mathbb{R})$

In this section, we show that Theorem 1.5 for the target group $\text{Diff}^1(S^1)$ is reduced to the case of $\text{Diff}^1(\mathbb{R})$.

**Proposition 2.1.** Let $r \neq n + 1$ and $n \geq 1$. Assume that $\Phi : \text{Diff}^r_c(\mathbb{R}^n)_0 \to \text{Diff}^0(S^1)$ is a nontrivial homomorphism. Then the global fixed point set is nonempty: $\text{Fix}(\Phi(\text{Diff}^r_c(\mathbb{R}^n)_0)) \neq \emptyset$.

This proposition enables us to conclude that the image of $\Phi$ is contained in the group of the homeomorphisms of $\mathbb{R}$. In particular, Theorem 1.5 for the target group $\text{Diff}^1(S^1)$ is reduced to the case of $\text{Diff}^1(\mathbb{R})$.

Denote $\mathcal{G} = \text{Diff}^r_c(\mathbb{R}^n)_0$. By the simplicity of the group $\mathcal{G}$, the homomorphism $\Phi$ in the proposition is injective and its image is contained in $\text{Diff}^0_c(S^1)$, the group of the orientation preserving homeomorphisms.

Let $B_0$ be the closed unit ball in $\mathbb{R}^n$ centered at the origin. Define a family $\mathcal{B}$ of the closed balls in $\mathbb{R}^n$ by

$$\mathcal{B} = \{g(B_0) \mid g \in \mathcal{G}\}.$$ 

Also for $B \in \mathcal{B}$, let $\mathcal{G}(B) = \{g \in \mathcal{G} \mid \text{Supp}(g) \subset \text{Int}(B)\}$.

To show Proposition 2.1 it is sufficient to show the following.

**Proposition 2.2.** For any $B \in \mathcal{B}$, the fixed point set $\text{Fix}(\Phi(\mathcal{G}(B)))$ is nonempty.

In fact, choose an increasing sequence of balls, $\{B_k\}_{k \in \mathbb{N}} \subset \mathcal{B}$ such that $\bigcup_k B_k = \mathbb{R}^n$. Then we have $\mathcal{G} = \bigcup_k \mathcal{G}(B_k)$ and $\text{Fix}(\Phi(\mathcal{G})) = \bigcap_k \text{Fix}(\Phi(\mathcal{G}(B_k)))$. Therefore by the compactness of $S^1$, Proposition 2.1 follows from Proposition 2.2.

Now for any $B_1, B_2 \in \mathcal{B}$, the groups $\mathcal{G}(B_1)$ and $\mathcal{G}(B_2)$ are conjugate in $\mathcal{G}$. Therefore their images $\Phi(\mathcal{G}(B_1))$ and $\Phi(\mathcal{G}(B_2))$ are conjugate in $\text{Diff}^0(S^1)$. They are simple. Moreover if $B_1$ and $B_2$ are disjoint, any element of $\Phi(\mathcal{G}(B_1))$ commutes with any element of $\Phi(\mathcal{G}(B_2))$. Therefore Proposition 2.2 reduces to the following.
PROPOSITION 2.3. Let $G_1$ and $G_2$ be simple nonabelian subgroups of $\Diff^1(S^1)$. Assume that $G_2$ is conjugate to $G_1$ in $\Diff^1(S^1)$ and that any element of $G_1$ commutes with any element of $G_2$. Then there is a global fixed point of $G_1$: $\Fix(G_1) \neq \emptyset$.

PROOF. Let $X_2 \subset S^1$ be a minimal set of $G_2$. The set $X_2$ is either a finite set, a Cantor set or the whole of $S^1$. If $X_2$ is a singleton, then $G_2$ admits a fixed point. Since $G_1$ is conjugate to $G_2$, we have $\Fix(G_1) \neq \emptyset$, as is required. So assume for contradiction that $X_2$ is not a singleton.

First if $X_2$ is a finite set which is not a singleton, we get a nontrivial homomorphism from $G_2$ to a finite abelian group, contrary to the assumption of the simplicity. In the remaining case, it is well known, easy to show, that the minimal set is unique. That is, $X_2$ is contained in any nonempty $G_2$ invariant closed subset.

Let $F_1$ be the subset of $G_1$ formed by the elements $g$ such that $\Fix(g) \neq \emptyset$. Let us show that there is a nontrivial element in $F_1$. Assume the contrary. Then $G_1$ acts freely on $S^1$. Consider the group $\hat{G}_1$ formed by any lift of any element of $G_1$ to the universal covering space $\R \to S^1$. Now $\hat{G}_1$ acts freely on $\R$. A theorem of H"older asserts that $\hat{G}_1$ is abelian. See [N] for a short proof, or [Th] for an even shorter proof. The canonical projection $\pi : \hat{G}_1 \to G_1$ is a group homomorphism, and $G_1 = \pi(\hat{G}_1)$ would be abelian, contrary to the assumption of the proposition.

Since $G_1$ and $G_2$ commutes, the fixed point set $\Fix(g)$ of any element $g \in F_1$ is $G_2$ invariant. Therefore we have

\[ (1) \quad X_2 \subset \Fix(g) \text{ for any } g \in F_1. \]

This shows that $F_1$ is in fact a subgroup. By the very definition, $F_1$ is normal. Since $G_1$ is simple and $F_1$ is nontrivial, $F_1 = G_1$. Finally again by (1), $\Fix(G_1) \neq \emptyset$. Then the minimal set of $G_1$ must be a singleton. Since $G_2$ is conjugate to $G_1$, the minimal set $X_2$ of $G_2$ is also a singleton, contrary to the assumption. \hfill \Box

3. Fixed point set of $\Phi(G)$

Again consider $\mathcal{G} = \Diff^r(\R^n)_0$, where $n \geq 1$ and $r \neq n + 1$. We shall show Theorem 1.5 for the target group $\Diff^1(\R)$ by a contradiction. So let us assume that $\Phi : \mathcal{G} \to \Diff^1(\R)$ is a nontrivial homomorphism. By the simplicity of $\mathcal{G}$, $\Phi$ is injective and its image is contained in $\Diff^1_+(\R)$. For the purpose of showing Theorem 1.5 it is no loss of generality to assume the following.

ASSUMPTION 3.1. There is no global fixed point of $\Phi(\mathcal{G})$: $\Fix(\Phi(\mathcal{G})) = \emptyset$.

In fact, we only have to pass from $\R$ to a connected component of $\R \setminus \Fix(\Phi(\mathcal{G}))$. This assumption will be made all the way until the end of the proof of Theorem 1.5.

We consider an embedding of Baumslag-Solitar group $\text{BS}(1, 2)$ into the group $\mathcal{G}(B)$. See Section 2 for the definition of $\mathcal{G}(B)$. Recall that

\[ \text{BS}(1, 2) = \langle a, b \mid aba^{-1} = b^2 \rangle. \]

This group is a subgroup of $GA$, the group of the orientation preserving affine transformations of $\R$, where $a$ corresponds to $x \mapsto 2x$, and $b$ to $x \mapsto x + 1$. The group $GA$ is a subgroup of $\Diff^1_+(\R)$. The group $\Diff^1_+(\R)$ acts on the circle at infinity $S^1_{\infty}$ of the Poincaré upper half plane, where $GA$ is the isotropy subgroup of $\infty \in S^1_{\infty}$. Cutting $S^1_{\infty}$ at $\infty$, we get a $C^\infty$ action of $\text{BS}(1, 2)$ on a compact interval, say $[-1, 1]$. This is called the affine action of $\text{BS}(1, 2)$.
They satsify $\psi$ a compact interval $I$. Define a map $\phi \colon GL$ closed subsets of $R$. Let us show that it belongs to $F$. Since $Fix(\Phi(\mathcal{G}(B)))$ is nonempty, $\Phi(\mathcal{G}(B))$ orbit intersects the compact interval $I$. Since $Fix(\Phi(\mathcal{G}(B)))$ is nonempty, $\Phi(\mathcal{G}(B))$ orbit intersects the compact interval $I$. Now we follow the proof of Proposition 6.1 in [DKNP], to show that there is a unique minimal set $X$ for $\Phi(\mathcal{G}(B))$. In fact we shall show a bit more: there is a nonempty $\Phi(\mathcal{G}(B))$ invariant closed subset $X$ in $R$ which has the property that any nonempty $\Phi(\mathcal{G}(B))$ invariant closed subset contains $X$.

The proof goes as follows. Let $F$ be the family of nonempty $\Phi(\mathcal{G}(B))$ invariant closed subsets of $R$, and $F_I$ the family of nonempty closed subsets $Y$ in $I$ such that $\Phi(\mathcal{G}(B))(Y) \cap I = Y$, where we denote

$$\Phi(\mathcal{G}(B))(Y) = \bigcup_{g \in \mathcal{G}(B)} \Phi(g)(Y).$$

Define a map $\phi : F \to F_I$ by $\phi(X) = X \cap I$, and $\psi : F_I \to F$ by $\psi(Y) = \Phi(\mathcal{G}(B))(Y)$. They satisfy $\psi \circ \phi = \phi \circ \psi = id$.

Let $\{Y_\alpha\}$ be a totally ordered set in $F_I$. Then the intersection $\cap_\alpha Y_\alpha$ is nonempty. Let us show that it belongs to $F_I$, namely,

$$\Phi(\mathcal{G}(B))(\cap_\alpha Y_\alpha) \cap I = \cap_\alpha Y_\alpha.$$

For the inclusion $\subset$, we have

$$\Phi(\mathcal{G}(B))(\cap_\alpha Y_\alpha) \cap I \subset (\cap_\alpha \Phi(\mathcal{G}(B))(Y_\alpha)) \cap I = \cap_\alpha (\Phi(\mathcal{G}(B))(Y_\alpha) \cap I) = \cap_\alpha Y_\alpha.$$
For the other inclusion, notice that
\[ \cap_\alpha Y_\alpha \subset \Phi(\mathcal{G}(B))(\cap_\alpha Y_\alpha) \text{ and } \cap_\alpha Y_\alpha \subset I. \]

Therefore by Zorn’s lemma, there is a minimal element \( Y \in F_I \). The set \( Y \) is not finite. In fact, if it is finite, the set \( X = \psi(Y) \) in \( F \) is discrete, and there would be a nontrivial homomorphism from \( \Phi(\mathcal{G}(B)) \) to \( \mathbb{Z} \), contrary to the fact that \( \mathcal{G}(B) \), and hence \( \Phi(\mathcal{G}(B)) \), is simple.

Now the correspondence \( \phi \) and \( \psi \) preserve the inclusion. This shows that there is no nonempty \( \Phi(\mathcal{G}(B)) \) invariant closed proper subset of \( X = \psi(Y) \). In other words, any \( \Phi(\mathcal{G}(B)) \) orbit contained in \( X \) is dense in \( X \). Therefore \( X \) is either \( \mathbb{R} \) itself or a locally Cantor set. In the former case, any nonempty \( \Phi(\mathcal{G}(B)) \) invariant closed subset must be \( \mathbb{R} \) itself.

Let us show that in the latter case, \( X \) satisfies the desired property: \( X \) is contained in any nonempty \( \Phi(\mathcal{G}(B)) \) invariant closed subset. For this, we only need to show that the \( \Phi(\mathcal{G}(B)) \) orbit of any point \( x \) in \( \mathbb{R} \setminus X \) accumulates to a point in \( X \). In fact, if this is true, then any nonempty \( \Phi(\mathcal{G}(B)) \) invariant closed subset must intersect \( X \). But the intersection must be the whole \( X \) by the above remark.

Let \( (a, b) \) be the connected component of \( \mathbb{R} \setminus X \) that contains \( x \). (If \( x \in X \), there is nothing to prove.) There is a sequence \( g_k \in \mathcal{G}(B) \) \((k \in \mathbb{N})\) such that \( \Phi(g_k)(a) \) accumulates to \( a \) and that \( \Phi(g_k)(a) \)'s are mutually distinct. Then the intervals \( \Phi(g_k)((a, b)) \) are mutually disjoint, and consequently \( \Phi(g_k)(x) \) converges to \( a \). This concludes the proof that \( X \) is contained in any nonempty \( \Phi(\mathcal{G}(B)) \) invariant closed subset.

Choose \( B' \in \mathcal{B} \) such that \( B' \cap B = \emptyset \). Any element of \( \mathcal{G}(B') \) commutes with any element of \( \mathcal{G}(B) \). Define \( \mathcal{F}(B') \) to be the subset of the group \( \mathcal{G}(B') \) consisting of those elements \( g \) such that \( \text{Fix}(\Phi(g)) \neq \emptyset \). By a theorem of H"older, there is a nontrivial element in \( \mathcal{F}(B') \). For any \( g \in \mathcal{F}(B') \), the set \( \text{Fix}(\Phi(g)) \) is closed, nonempty and invariant by \( \Phi(\mathcal{G}(B)) \) by the commutativity. Therefore we have

\[ X \subset \text{Fix}(\Phi(g)) \text{ for any } g \in \mathcal{F}(B'). \]

This shows that \( \mathcal{F}(B') \) is a subgroup of \( \mathcal{G}(B') \), normal and nontrivial. Since \( \mathcal{G}(B') \) is simple, we have \( \mathcal{F}(B') = \mathcal{G}(B') \). Finally again by (3), we get \( \text{Fix}(\Phi(\mathcal{G}(B'))) \neq \emptyset \). Since \( \mathcal{G}(B) \) is conjugate to \( \mathcal{G}(B') \) and \( \mathcal{G} \) is a subgroup of \( \mathcal{G}(B) \), we have \( \text{Fix}(\Phi(\mathcal{G})) \neq \emptyset \), contrary to the assumption. The contradiction concludes the proof of Proposition 3.3. □

4. Fixed point set of \( \Phi(\mathcal{G}_B) \)

For \( B \in \mathcal{B} \), define a subgroup \( \mathcal{G}_B \) of \( \mathcal{G} \) by
\[ \mathcal{G}_B = \{ g \in \mathcal{G} \mid g = \text{id in a neighbourhood of } B \}. \]
Let \( \Phi : \mathcal{G} \to \text{Diff}^1_+(\mathbb{R}) \) be a homomorphism satisfying Assumption 3.1. The purpose of this section is to show the following.

**PROPOSITION 4.1.** For any \( B \in \mathcal{B} \), the fixed point set \( \text{Fix}(\Phi(\mathcal{G}_B)) \) is nonempty.

**PROOF.** Any element of \( \Phi(\mathcal{G}(B)) \) commutes with any element of \( \Phi(\mathcal{G}_B) \). Let us denote \( F = \text{Fix}(\Phi(G)) \), which we have shown to be nonempty in Proposition 3.3. Clearly \( F \) is invariant by any element of \( \Phi(\mathcal{G}_B) \). We shall show that there is a fixed point of \( \Phi(\mathcal{G}_B) \) in \( F \). If \( F \) is bounded to the left or to the right, then the
extremal point will be a fixed point of \( \Phi(G_B) \). So we assume that \( F \) is unbounded towards both directions. That is, any connected component \( U \) of \( \mathbb{R} \setminus F \) is bounded.

Assume that there is \( g \in G_B \) such that \( \Phi(g)(U) \cap U = \emptyset \). (Otherwise \( \Phi(g)(U) = U \) for any \( g \in G_B \), and the proof will be complete.) There is a subgroup \( G' \) of \( G_B \) conjugate to \( G \). By some abuse, denote the generators of \( G' \) by \( a \) and \( b \). They satisfy \( aba^{-1} = b^2 \). Notice that finite products of conjugates of \( b^{\pm1} \) by elements of \( G_B \) form a normal subgroup of \( G_B \). Since \( G_B \) is simple, any element of \( G_B \) can be written as such a product. Writing the above element \( g \) this way, one finds a conjugate of \( b \) whose \( \Phi \)-image displaces \( U \). We may assume that \( \Phi(b)U \cap U = \emptyset \), passing from \( G' \) to its conjugate by an element of \( G_B \) if necessary. (The conjugate is still denoted by \( G' \).)

Let \( V \) be the component of \( \mathbb{R} \setminus \text{Fix}(\Phi(G')) \) that contains \( U \). Since \( G' \) is conjugate to \( G \), \( V \) is a bounded open interval and \( F \cap V \) is a closed nonempty proper subset of \( V \) invariant by \( \Phi(G') \). It is easy to show that \( \Phi(b)|_V \neq \text{id} \) implies that the action \( \Phi(G')|_V \) is faithful. By Theorem 3.2 any \( \Phi(G') \) orbit in \( V \) must be dense in \( V \). This contradicts the fact that \( F \cap V \) is invariant by \( \Phi(G') \). The proof is now complete. \( \blacksquare \)

5. Proof of Theorem 1.5

Again we assume that \( \Phi : G \to \text{Diff}_+^\circ(S^1) \) is a homomorphism satisfying Assumption 3.1. Our purpose here is to get a contradiction. We follow an argument in [Mi].

**Lemma 5.1.** Assume \( B \) and \( B' \) are mutually disjoint balls of \( \mathcal{B} \). Then any \( g \in G \) can be written as \( g = g_1 \circ g_2 \circ g_3 \), where \( g_1 \) and \( g_2 \) belongs to \( G_B \) and \( g_2 \) to \( G_B' \).

**Proof.** Take any \( g \in G \). Then there is an element \( g_1 \in G_B \) such that \( g_1^{-1} \circ g(B) \) is disjoint from \( B' \). Next, there is an element \( g_2 \in G_B' \) such that \( g_2^{-1} \circ g_1^{-1} \circ g \) is the identity in a neighbourhood of \( B \). Thus \( g_3 = g_2^{-1} \circ g_1^{-1} \circ g \) belongs to \( G_B \) and the proof is complete. \( \blacksquare \)

**Lemma 5.2.** Assume \( B \) and \( B' \) are mutually disjoint elements of \( \mathcal{B} \). If two points \( a \) and \( b \) \((a < b) \) belong to \( \text{Fix}(\Phi(G_B')) \), then \( \text{Fix}(\Phi(G_B')) \cap [a,b] = \emptyset \).

**Proof.** Assume a point \( c \) in \([a,b]\) belongs to \( \text{Fix}(\Phi(G_B')) \). Choose an arbitrary element \( g \in G \). There is a decomposition \( g = g_1 \circ g_2 \circ g_3 \) as in Lemma 5.1. Now \( \Phi(g_3)(a) = a \). Since \( \Phi(g_2)(c) = c \) and \( a \leq c \), we have \( \Phi(g_2) \circ \Phi(g_3)(a) \leq c \). Likewise \( \Phi(g)(a) = \Phi(g_1) \circ \Phi(g_2) \circ \Phi(g_3)(a) \leq b \). Since \( g \in G \) is arbitrary, the \( \Phi(G) \) orbit of \( a \) is bounded from the right. Then the supremum of the orbit must be a global fixed point, which is against Assumption 3.1. \( \Phi(G) \) has no global fixed point. \( \blacksquare \)

For any point \( x \in \mathbb{R}^n \), define a subgroup \( G_x \) of \( G \) by

\[
G_x = \{ g \in G \mid g \text{ is the identity in a neighbourhood of } x \}.
\]

**Lemma 5.3.** For any \( x \in \mathbb{R}^n \), the fixed point set \( \text{Fix}(\Phi(G_x)) \) is nonempty.

**Proof.** Notice that for any \( x \in \mathbb{R}^n \), there is an decreasing sequence \( \{B_k\} \) \((k \in \mathbb{N}) \) in \( \mathcal{B} \) such that \( \{x\} = \bigcap_k B_k \). Then \( G_{B_k} \) is an increasing sequence of subgroups of \( G \) such that \( \bigcup_k G_{B_k} = G_x \). Therefore the closed subsets \( \text{Fix}(\Phi(G_{B_k})) \)
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is decreasing and we have

\[ \text{Fix}(\Phi(G)) = \bigcap_k \text{Fix}(\Phi(G_{B_k})). \]

Now it suffices to prove that \( \text{Fix}(\Phi(G)) \) is compact for \( B \in B \). Recall that we have already shown that \( \text{Fix}(\Phi(G)) \) is nonempty. Assume in way of contradiction that \( \sup \text{Fix}(\Phi(G)) = \infty \). (The other case can be dealt with similarly.) Choose \( B' \in B \) such that \( B \cap B' = \emptyset \). Notice that \( \Phi(G) \) consists of orientation preserving diffeomorphisms and \( \Phi(G_{B'}) \) is conjugate to \( \Phi(G_B) \) by such a diffeomorphism. Therefore we also have that \( \sup \text{Fix}(\Phi(G_{B'})) = \infty \). Now one can find points \( a, b \in \text{Fix}(\Phi(G)) \) and a point \( c \in \text{Fix}(\Phi(G_{B'})) \) such that \( a < c < b \). This is contrary to Lemma 5.2. □

We use the assumption \( n \geq 2 \) only in the sequel. Let \( D_0 \) be the unit compact disc centered at 0 in \( \mathbb{R}^{n-1} \subset \mathbb{R}^n \). Define a family \( D \) of closed subsets of \( \mathbb{R}^n \) by

\[ D = \{ g(D_0) \mid g \in G \}. \]

For any \( D \in D \), define a subgroup \( G_D \) of \( G \) by

\[ G_D = \{ g \in G \mid g \text{ is the identity in a neighbourhood of } D \}. \]

Lemma 5.3 implies that \( \text{Fix}(\Phi(G_D)) \neq \emptyset \) for any \( D \in D \).

**Lemma 5.4.** For any \( D \in D \), the set \( \text{Fix}(\Phi(D)) \) is a singleton.

**Proof.** First of all notice that for any \( D, D' \in D \) such that \( D \cap D' = \emptyset \), we have \( \text{Fix}(\Phi(G_D)) \cap \text{Fix}(\Phi(G_{D'})) = \emptyset \). In fact, as is easily shown, \( G_D \) and \( G_{D'} \) generate \( G \). Thus the point of the above intersection would be a global fixed point of \( G \), against Assumption 3.1. This shows that the interior \( \text{Int}(\text{Fix}(\Phi(G_D))) \) is empty. In fact, there are uncountably many mutually disjoint elements of \( D \), while mutually disjoint open subsets of \( \mathbb{R} \) are at most countable.

Assume that \( \text{Fix}(\Phi(G_D)) \) contains more than one points. Since \( \text{Int}(\text{Fix}(\Phi(G_D))) \) is empty, \( \text{Fix}(\Phi(G_D)) \) is not connected. To any \( D \in D \), assign a bounded component \( I_D \) of \( \mathbb{R} \setminus \text{Fix}(\Phi(G_D)) \) in an arbitrary way. This is possible by the axiom of choice. Notice that Lemmata 5.1 and 5.2 for the family \( B \) are valid for \( D \) as well. (No changes of the proofs are needed.) Consequently \( I_D \cap I_{D'} = \emptyset \) if \( D \cap D' = \emptyset \). Again this is contrary to the fact that there are uncountably many mutually disjoint elements of \( D \). □

Finally let us prove Theorem 5.2. Choose any element \( D \in D \) and distinct two points \( x_1, x_2 \in D \) that are contained in \( D \). Then since \( \text{Fix}(\Phi(G_D)) \) is a singleton and \( \text{Fix}(\Phi(G_{x_1})) \) is nonempty, we have \( \text{Fix}(\Phi(G_{x_1})) = \text{Fix}(\Phi(G_{x_2})) \). But \( G_{x_1} \) and \( G_{x_2} \) generate \( G \), and there would be a global fixed point of \( \Phi(G) \), against Assumption 3.1. The contradiction shows that the homomorphism \( \Phi \) must be trivial.

### 6. Sporadic results for \( \text{Diff}^0(S^1) \) target

Let \( M = L \times S^m \) be a closed \( n \)-dimensional manifold such that \( 1 \leq m \leq n \). Then we have the following result.

**Theorem 6.1.** If \( n \geq 2 \) and \( r \neq n + 1 \), there is no nontrivial homomorphism from \( \text{Diff}^r(M)_0 \) to \( \text{Diff}^0(S^1) \).
Proof. Assume that $\Phi : \text{Diff}^r_c(M)_0 \to \text{Diff}^0(S^1)$ is a nontrivial homomorphism. The Lie group $PSL(2, \mathbb{R})$ acts on $S^m$ as Möbius transformations. So it acts on $M = L \times S^m$, trivially on $L$-coordinates. Denote the inclusion by $\iota : PSL(2, \mathbb{R}) \to \text{Diff}^r_c(M)_0$. The simplicity of the group $\text{Diff}^r_c(M)_0$ shows that the homomorphism $\Phi \circ \iota : PSL(2, \mathbb{R}) \to \text{Diff}^0(S^1)$ is nontrivial.

Now Theorem 5.2 in [Ma2] asserts that the homomorphism $\Phi \circ \iota$ is the conjugation of the standard embedding $\iota_0 : PSL(2, \mathbb{R}) \to \text{Diff}^0(S^1)$ by a homeomorphism of $S^1$. It is no loss of generality to assume that $\Phi \circ \iota = \iota_0$, by changing $\Phi$ if necessary. If the dimension of $L$ is positive, then $\text{Diff}^r_c(L)_0$ also acts on $M$, trivially on $S^m$-coordinates. Any element of the group $\Phi(\text{Diff}^r_c(L))$ must commute with any element of $PSL(2, \mathbb{R})$. But there is no nontrivial element in $\text{Diff}^r_c(S^1)$ which commutes with all the element of $PSL(2, \mathbb{R})$. A contradiction.

Let us consider the case where $L$ is a singleton. Then there is an element $g$ in $\text{Diff}^r(S^n)$, $n \geq 2$, which commutes with all the elements of $\iota(SO(2))$ and is not contained in $\iota(SO(2))$. But $\Phi \circ \iota(SO(2)) = SO(2)$, and any element in $\text{Diff}^0(S^1)$ which commutes with all the elements of $SO(2)$ must be an element of $SO(2)$, contradicting the injectivity of $\Phi$.

Remark 6.2. There are a wider class of manifolds for which the above argument holds. For example, if $M$ is the unit tangent bundle of a closed hyperbolic surface and if $r \neq 4$, then any homomorphism from $\text{Diff}^r_c(M)_0$ to $\text{Diff}^0(S^1)$ is trivial.

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