NUMBER THEORETIC APPLICATIONS OF A CLASS OF CANTOR SERIES
FRAC TAL FUNCTIONS PART I

BILL MANCE

Abstract. Suppose that \((P,Q) \in \mathbb{N}_2^2 \times \mathbb{N}_2^2\) and \(x = E_0E_1E_2 \cdots\) is the \(P\)-Cantor series expansion of \(x \in \mathbb{R}\). We define \(\psi_{P,Q}(x) := \sum_{n=1}^{\infty} \frac{E_n}{q_1q_2 \cdots q_n}\). The functions \(\psi_{P,Q}\) are used to construct many pathological examples of normal numbers. These constructions are used to give the complete containment relation between the sets of \(Q\)-normal, \(Q\)-ratio normal, and \(Q\)-distribution normal numbers and their pairwise intersections for fully divergent \(Q\) that are infinite in limit. We analyze the Hölder continuity of \(\psi_{P,Q}\) restricted to some judiciously chosen fractals. This allows us to compute the Hausdorff dimension of some sets of numbers defined through restrictions on their Cantor series expansions. In particular, the main theorem of a paper by Y. Wang et al. [25] is improved.

Properties of the functions \(\psi_{P,Q}\) are also analyzed. Multifractal analysis is given for a large class of these functions and continuity is fully characterized. We also study the behavior of \(\psi_{P,Q}\) on both rational and irrational points, monotonicity, and bounded variation. For different classes of ergodic shift invariant Borel probability measures \(\mu_1\) and \(\mu_2\) on \(\mathbb{N}_2^2\), we study which of these properties \(\psi_{P,Q}\) satisfies for \(\mu_1 \times \mu_2\)-almost every \((P,Q) \in \mathbb{N}_2^2 \times \mathbb{N}_2^2\). Related classes of random fractals are also studied.

1. INTRODUCTION

Let \(\mathbb{N}_k := \mathbb{Z} \cap [k_1, \infty)\). The \(Q\)-Cantor series expansion, first studied by G. Cantor in [4], is a natural generalization of the \(b\)-ary expansion. If \(Q \in \mathbb{N}_2^2\), then we say that \(Q\) is a basic sequence. Given a basic sequence \(Q = (q_n)_{n=1}^\infty\), the \(Q\)-Cantor series expansion of a real \(x \in \mathbb{R}\) is the (unique) expansion of the form

\[(1.1) \quad x = E_0 + \sum_{n=1}^{\infty} \frac{E_n}{q_1q_2 \cdots q_n}\]

where \(E_0 = [x]\) and \(E_n\) is in \(\{0,1, \ldots, q_n-1\}\) for \(n \geq 1\) with \(E_n \neq q_n - 1\) infinitely often. We abbreviate \(1.1\) with the notation \(x = E_0E_1E_2E_3 \cdots \) w.r.t. \(Q\).

Clearly, the \(b\)-ary expansion is a special case of \(1.1\) where \(q_n = b\) for all \(n\). If one thinks of a \(b\)-ary expansion as representing an outcome of repeatedly rolling a fair \(b\)-sided die, then a \(Q\)-Cantor series expansion may be thought of as representing an outcome of rolling a fair \(q_1\) sided die, followed by a fair \(q_2\) sided die, and so on.

Let \(x = E_0E_1E_2 \cdots \) w.r.t. \(P\). If there are no values \(n\) such that \(E_n = 0\) or \(E_n = p_n - 1\), then we let \(\rho_{P,Q}(x) := 0\). Otherwise, set \(\rho_{P,Q}(x) := \sup\{k \in \mathbb{N} : 2n \in \mathbb{N}\) such that \(E_{n+t} \in \{0, p_{n+t} - 1\} \forall t \in [0, k-1]\}\). For \(k \in \mathbb{N} \cup \{0, \infty\}\), put \(\mathcal{W}_{P,Q}^{(k)} := \{x \in \mathbb{R} : \rho_{P,Q}(x) \leq k\}\) and

\[Z_{P,Q}^{(k)} := \{x = 0.E_1E_2 \cdots \text{ w.r.t. } P : E_n < \min(p_n, q_n)\} \cap \mathcal{W}_{P,Q}^{(k)} \cap (\psi_{P,Q})^{-1} \left(\mathcal{W}_{Q,P}^{(k)}\right)\).

Definition 1.1. Let \((P,Q) \in \mathbb{N}_2^2 \times \mathbb{N}_2^2\) and suppose that \(x = E_0E_1E_2 \cdots \) w.r.t. \(P\). We define

\[\psi_{P,Q}(x) := \sum_{n=1}^{\infty} \frac{\min(E_n, q_n - 1)}{q_1q_2 \cdots q_n}\]

and \(\phi_{P,Q}^{(k)} := \psi_{P,Q}\big|_{Z_{P,Q}^{(k)}}\).

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1G. Cantor’s motivation to study the Cantor series expansions was to extend the well known proof of the irrationality of the number \(e = \sum 1/n!\) to a larger class of numbers. Results along these lines may be found in the monograph of J. Galambos [8]. See also [20] and [9].

2Uniqueness can be proven in the same way as for the \(b\)-ary expansions.

3We will use the symbol := only to define notation globally for the whole paper.
The study of the functions $\psi_{P,Q}$ and $\phi^{(k)}_{P,Q}$ and their applications to digital problems involving Cantor series expansions form the core of this paper. Let $Q \in \mathbb{N}_2^*$ and let $\mathbb{N}(Q), \mathbb{R}(Q)$, and $\mathbb{D}(Q)$ be the sets of $Q$-normal numbers, $Q$-ratio normal numbers, and $Q$-distribution normal numbers, respectively.

The original motivation for the author to study the functions $\psi_{P,Q}$ was to study the set $\mathbb{R}(Q) \cap \mathbb{D}(Q) \setminus \mathbb{N}(Q)$ and the sets constructed in the sequel to this paper by B. Li and the author [12]. One of the more surprising applications of the methods introduced in this paper is that for every $k \geq 2$, there exists a basic sequence $Q$ and a real number $x$ that is $Q$-normal of order $k$, but not $Q$ normal of any order $1, 2, \cdots, k - 1$. Explicit examples of computable basic sequences $Q$ and computable real numbers $x$ with this property are given in [12].

The basic sequence $Q$ constructed in Section 3.2 is a computable sequence and the member of $\mathbb{R}(Q) \cap \mathbb{D}(Q) \setminus \mathbb{N}(Q)$ constructed in the same section is a computable real number. No deep knowledge of computability theory will be used and any time we make such a claim there will exist a simple algorithm to compute the number under consideration to any degree of precision. Section 3 is devoted to understanding the relationship between $\mathbb{N}(Q), \mathbb{R}(Q)$, and $\mathbb{D}(Q)$ and intersections thereof. We refer to the directed graph in Figure 1 for the complete containment relationships between these notions when $Q$ is infinite in limit and fully divergent. The vertices are labeled with all possible intersections of one, two, or three choices of the sets $\mathbb{N}(Q), \mathbb{R}(Q)$, and $\mathbb{D}(Q)$. The set labeled on vertex $A$ is a subset of the set labeled on vertex $B$ if and only if there is a directed path from $A$ to $B$. For example, $\mathbb{N}(Q) \cap \mathbb{D}(Q) \subseteq \mathbb{R}(Q)$, so all numbers that are $Q$-normal and $Q$-distribution normal are also $Q$-ratio normal. A block is an ordered tuple of non-negative integers, a block of length $k$ is an ordered $k$-tuple of integers, and block of length $k$ in base $b$ is an ordered $k$-tuple of integers in $\{0, 1, \ldots, b - 1\}$.

The following is the main result of Section 3.

**Theorem 1.2.** Figure 1 represents the complete containment relationship for basic sequences $Q$ that are infinite in limit and fully divergent.

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4 We defer the definition of these sets to Section 3.

5 For a judiciously chosen $Q \in \mathbb{N}_2^*$, we construct an explicit example of a member of $\mathbb{R}(Q) \cap \mathbb{D}(Q) \setminus \mathbb{N}(Q)$ in Section 3.2.

6 It was previously unknown if there are any basic sequences $Q$ such that $\mathbb{R}(Q) \cap \mathbb{D}(Q) \setminus \mathbb{N}(Q) \neq \emptyset$.

The underlying undirected graph in Figure 1 has an isomorphic copy of complete bipartite graph $K_{3,3}$ as a subgraph. Thus, it is not planar and the analogous directed graph that connects two vertices if and only if there is a containment relation between the two labels is more difficult to read.
Suppose that $M = (m_t)$ is an increasing sequence of positive integers. Let $N_{M,n}^Q (B, x)$ be the number of occurrences of the block $B$ at positions $m_t$ for $m_t \leq n$ in the $Q$-Cantor series expansion of $\{x\}$. For $m_t = t$ and $M = (m_t)$, let $N_{n}^Q (B, x) := N_{M,n}^Q (B, x)$. We must also discuss the set of real numbers who have more than one expansion of the form $\{1,1\}$ if we do not restrict $E_n < q_n - 1$ infinitely often. These are precisely the points $x = E_0.E_1E_2\cdots E_n$ w.r.t. $Q$. We note that if $x$ is of this form, then

$$x = E_0 + \sum_{j=1}^{n-1} \frac{E_j}{q_1\cdots q_j} + \frac{E_n - 1}{q_1\cdots q_n} + \sum_{j=n+1}^{\infty} \frac{q_j - 1}{q_1\cdots q_j}.$$  

It should be noted that the distinction between these numbers will play a critical role in studying the properties of $\psi_{P,Q}$ as well as applications towards other problems. Thus, for a basic sequence $Q$, we let $\mathcal{U}_Q := \{x = E_0.E_1E_2\cdots \text{ w.r.t. } Q : E_n \neq 0 \text{ infinitely often}\}$ be the set of points with unique $Q$-Cantor series expansion and let $\mathcal{N}\mathcal{U}_Q := \mathbb{R} \setminus \mathcal{U}_Q$. The following theorem is not difficult to prove but will be of fundamental importance for the normal number constructions in this paper, the sequel to this paper with B. Li [12], and those in planned future projects.

**Theorem 1.3.** Suppose that $M = (m_t)$ is an increasing sequence of positive integers and $Q_1 = (q_1,n), Q_2 = (q_2,n), \cdots, Q_j = (q_j,n)$ are basic sequences and infinite in limit. Set

$$\Psi_j (x) = \left( \psi_{j-1,n} \circ \psi_{j-2,n} \circ \cdots \circ \psi_{1,n} \right) (x).$$

If $x = E_0.E_1E_2\cdots \text{ w.r.t. } Q_1$ satisfies $E_n < \min_{2 \leq r \leq j} (q_r,n-1)$ for infinitely many $n$, then $\Psi_j (x) \in \mathcal{U}_Q$, and for every block $B$

$$N_{M,n}^Q (B, \Psi_j (x)) = N_{M,n}^Q (B, x) + O(1).$$

The functions $\psi_{P,Q}$ and $\phi_{P,Q}^{(k)}$ are interesting in their own right. There is a vast literature studying functions with pathologic properties. An early example due to Weierstrauss is of a class of continuous and nowhere differentiable functions. The study of other functions such as the Cantor function, Minwoski’s question mark function, and the Takagi function also provides motivation for Section 2. We give only a few references as relevant starting points: [1], [6], and [10]. We also mention that other fractal functions defined through Cantor series have been studied by H. Wang and Z. Xu in [23] and [24]. However, these functions are quite different from the $\psi_{P,Q}$ and $\phi_{P,Q}^{(k)}$ functions we study in this paper.

For a set $S \subseteq \mathbb{R}$, we will let $\chi(S)$ denote the Lebesgue measure of $S$ and $\dim_P (S), \dim_H (S), \dim_B (S)$ will denote the Hausdorff, packing, and box dimensions of $S$, respectively. In Section 2 we will examine many properties of the functions $\psi_{P,Q}$ including, but not limited to, rationality, continuity, and bounded variation. We will also study the level sets of $\psi_{P,Q}$ and multifractal analysis of $\psi_{P,Q}$. For simplicity, we will only consider the level sets of $\psi_{P,Q}$ in $[0,1]$ as $\psi_{P,Q}$ is 1-periodic and $\psi_{P,Q} (x) = 0$ if and only if $x \in \mathbb{Z}$. For $w \in (0,1], put

$$\mathcal{L}_{P,Q} (w) := \{x \in (0,1) : \psi_{P,Q} (x) = w\}.$$  

For $\alpha \in [0,1]$, let

$$\mathcal{V}_{P,Q} (\alpha) := \{w \in \psi_{P,Q} ((0,1)) : \dim_H (\mathcal{L}_{P,Q} (w)) = \alpha\}$$  

be a level set of the function $\dim_H (\mathcal{L}_{P,Q} (\cdot))$. Let $\tau_n = n(n+1)/2$ be the $n$'th triangular number. An eventually non-decreasing sequence of real numbers $(s_n)$ grows nicely if

$$\lim_{n \to \infty} \frac{\log s_{\tau_n+2}}{\log s_{\tau_n}} = 1.$$  

\[\text{Corollary [2.17]}\] gives conditions under which $\mathcal{N}\mathcal{U}_Q = \emptyset$.

\[\text{The conclusions of Theorem [1.3] sometimes do not hold without the requirement that } E_n < \min_{2 \leq r \leq j} (q_r,n-1) \text{ for infinitely many } n. \text{ For example, consider } p_n = 3 \text{ and}

$$q_n = \begin{cases} 2 & \text{if } n \equiv 0 \pmod{2} \\ 3 & \text{if } n \equiv 1 \pmod{2} \end{cases}.$$  

Let $x = 7/8 = 0.21\overline{1}$ w.r.t. $P$. Then $\psi_{P,Q} (x) = 1.0$ w.r.t. $Q$ so $N_n^Q ((1), x) = [n/2]$ while $N_n^Q ((1), \psi_{P,Q} (x)) = 0$ for all $n$.\[\text{Note that if } (s_n) \text{ grows nicely, then } \lim_{n \to \infty} s_n = \infty.\]
We will prove the following theorem.

**Theorem 1.4.** For \((P, Q) \in \mathbb{N}_2^N \times \mathbb{N}_2^N\), let \(r_n = p_n - q_n\). If \((p_n), (q_n)\), and \((r_n)\) grow nicely and

\[
\lim_{n \to \infty} \frac{\log r_n}{\log p_n} = \gamma \in (0, 1],
\]

then for all \(\alpha \in [0, 1]\)

\[
\dim_H(\mathcal{V}_{P,Q}(\alpha)) \geq 1 - \frac{\alpha}{\gamma}.
\]

Thus, \(\dim_H(\mathcal{V}_{P,Q}(\alpha)) > 0\) if \(0 \leq \alpha < \gamma\). \[10\]

While some properties such as continuity may easily be described for arbitrary choices of \(P\) and \(Q\), others will be too difficult to analyze for completely arbitrary choices. Thus, for certain classes of ergodic and shift-invariant Borel probability measures \(\mu_1, \mu_2\) on \(\mathbb{N}_2^N\) we will study these properties for \(\mu_1 \times \mu_2\)-almost every \((P, Q) \in \mathbb{N}_2^N \times \mathbb{N}_2^N\). This will naturally give rise to many random fractals that we will consider. We also include graphs of \(\psi_{P,Q}\) for many choices of \(P\) and \(Q\) in Figure 2.

The Hölder and Lipschitz continuity of \(\phi_{P,Q}^{(k)}\) is explored in Section 2.6. This allows us to compute the Hausdorff dimension of some fractals defined through digital restrictions of Cantor series expansions in Section 3. Additionally, we will use the results of Section 2.6 to improve the main theorem in the paper [26] by Y. Wang et al and a result of the author in [13]. For the remainder of this paper, we will assume the convention that the empty sum is equal to 0 and the empty product is equal to 1.

## 2. The functions \(\psi_{P,Q}\) and \(\phi_{P,Q}^{(k)}\)

For \(Q \in \mathbb{N}_2^N\) and a sequence of natural numbers \((a_j)\), define

\[
\mathcal{R}_{(a_j)}(Q) := \{x = 0.E_1E_2 \cdots \text{ w.r.t. } Q : E_j < a_j\}.
\]

We note the following result due to H. Wegmann in [26]:

**Theorem 2.1.** If \(Q = (q_n) \in \mathbb{N}_2^N\) and \(\lim_{n \to \infty} \frac{\log q_n}{\log q_1 \cdots q_n} = 0\), then

\[
\dim_H(\mathcal{R}_{(a_j)}(Q)) = \liminf_{n \to \infty} \frac{\log \prod_{j=1}^n \min(a_j, q_j)}{\log \prod_{j=1}^n q_j}.
\]

The next theorem directly follows from Definition 1.1 and Theorem 2.1.

**Theorem 2.2.** If \((P, Q) \in \mathbb{N}_2^N \times \mathbb{N}_2^N\), then

\[
\psi_{P,Q}(\mathcal{R}) = \mathcal{R}_{\min(p_j, q_j)}(Q) \subseteq \left[0, \sum_{n=1}^\infty \frac{\min(p_n - 1, q_n - 1)}{q_1 \cdots q_n}\right] \quad \text{and} \quad \lambda(\psi_{P,Q}(\mathcal{R})) = \prod_{j=1}^\infty \frac{\min(p_j, q_j)}{q_j}.
\]

Moreover, if \(\lim_{n \to \infty} \frac{\log q_n}{\log q_1 \cdots q_n} = 0\), then

\[
\dim_H(\psi_{P,Q}(\mathcal{R})) = \liminf_{n \to \infty} \frac{\log \prod_{j=1}^n \min(p_j, q_j)}{\log \prod_{j=1}^n q_j}.
\]

Thus, the range of \(\psi_{P,Q}\) can be anywhere from the interval [0, 1] to a Cantor set. Given \(Q \in \mathbb{N}_2^N\), let \(J = (I_n)\), where \(I_n \subseteq \{0, 1, \cdots, q_n - 1\}\). For the rest of this paper, define \(\mathcal{R}_J(Q)\) by

\[
\mathcal{R}_J(Q) := \left\{x = \sum_{n=1}^\infty \frac{E_n}{q_1 q_2 \cdots q_n} : E_j \in I_j\right\}.
\]

The proof of Theorem 2.1 presented in [26] can trivially be modified to arrive at the following generalization of Theorem 2.1 that will frequently be used in this paper.

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[10] The conditions of Theorem 1.4 are not very restrictive. Most monotone sequences \((p_n)\) and \((q_n)\) that do not grow unreasonably fast and where \((p_n)\) dominates \((q_n)\) will satisfy the conditions of Theorem 1.4. For example, \(p_n = 2^n\) and \(q_n = n + 1\) satisfy this condition for \(\gamma = 1\). A graph of \(\psi_{P,Q}\) for these choices of \(P\) and \(Q\) is given in Figure 2a. If \(p_n = n^2 + n\) and \(q_n = n^2 + 1\), then the hypotheses of Theorem 1.4 are satisfied with \(\gamma = 1/2\).
Figure 2. Graphs of $\psi_{P,Q}$ for different choices of $P$ and $Q$ plotted with 500 pixels each. Most graphs without an explicit formula for $p_n$ and $q_n$ were generated randomly.
Theorem 2.3. Suppose that \( Q = (q_n) \in \mathbb{N}^\mathbb{N} \), \( \lim_{n \to \infty} \frac{\log q_n}{\log q_{n+1}} = 0 \), \( I_j \subseteq \{0, 1, \ldots, q_j - 1\} \), and \( J = (I_n) \). Then
\[
\dim_H(\mathcal{R}_J(Q)) = \liminf_{n \to \infty} \frac{\log \sum_{j=1}^{n} |I_j|}{\log \prod_{j=1}^{n} q_j}.
\]

It should be noted that the sets \( \mathcal{R}_J(Q) \) are homogenous Moran sets. Using corollary 3.1 from Feng et al. [7], we have the following result connecting the Hausdorff, packing, and box dimensions of \( \mathcal{R}_J(Q) \).

Lemma 2.4. If
\[
\liminf_{n \to \infty} \frac{\log \sum_{j=1}^{n} |I_j|}{\log \prod_{j=1}^{n} q_j} = \limsup_{n \to \infty} \frac{\log \sum_{j=1}^{n+1} |I_j|}{\log \prod_{j=1}^{n+1} q_j} + \log |I_{n+1}|,
\]
then \( \dim_H(\mathcal{R}_J(Q)) = \dim_P(\mathcal{R}_J(Q)) = \dim_B(\mathcal{R}_J(Q)) \).

Lastly, we give a proof of Theorem 1.3

Proof of Theorem 1.3. Let \( B = (b_1, b_2, \ldots, b_k) \). We use induction on \( j \). The base case \( j = 1 \) is trivial. Suppose now that \( j \geq 2 \) and \( N_{M,n}^{Q_{j-1}}(B, \Psi_{j-1}(x)) = N_{M,n}^{Q_j}(B, x) + O(1) \). Put \( b = \max(b_1, b_2, \ldots, b_k) \) and let \( \Psi_j(x) = F_0.F_1F_2 \ldots \) w.r.t. \( Q_{j-1} \) and \( \Psi_j(x) = G_0G_1G_2 \ldots \) w.r.t. \( Q_j \). Since \( \min(E_n, q_2-1, \ldots, q_j-1) \) \( \leq E_n < q_{n,j} - 1 \) for infinitely many \( n \), we know that \( \Psi_j(x) \in \mathcal{U}_{Q_j} \). Let \( t \) be large enough that \( b < \min_{1 \leq j \leq j}(q_{j,n} - 1) \) for every \( n \geq t \). Since \( \Psi_j(x) \in \mathcal{U}_{Q_j} \), we know for \( n \geq t \) that \( G_n \in \{0, 1, \ldots, b\} \) if and only if \( F_n \in \{0, 1, \ldots, b\} \). Thus,
\[
N_{M,n}^{Q_{j-1}}(B, \Psi_{j-1}(x)) - N_{M,n}^{Q_j}(B, \Psi_j(x)) \leq \left(N_{M,n}^{Q_{j-1}}(B, \Psi_{j-1}(x)) - N_{M,n}^{Q_{j-1}}(B, \Psi_{j-1}(x))\right) + t,
\]
so \( N_{M,n}^{Q_j}(B, \Psi_j(x)) = N_{M,n}^{Q_j}(B, x) + O(1) \) and Theorem 1.3 is proven. \( \square \)

2.1. Level Sets and Multifractal Analysis of \( \psi_{P,Q} \). We wish to examine the range of \( \psi_{P,Q} \) beyond what was discussed in Theorem 2.2. Our main tool will be Theorem 2.3. For this subsection, we will assume that \( \lim_{n \to \infty} \frac{p_n}{q_n} = 0 \) so that we may use Theorem 2.3. We will see in Section 2.3 that the level sets \( \mathcal{L}_{P,Q}(w) \) are always empty, a single point, or a totally disconnected set.

The next theorem follows directly from the definition of the Cantor series expansions and \( \psi_{P,Q} \) and gives a complete characterization of the level sets of \( \psi_{P,Q} \). None of the following statements are difficult to prove so we omit their proofs.

Theorem 2.5. Suppose that \( w = E_0.E_1E_2 \ldots \) w.r.t. \( Q \in (0, 1) \) and \( x = 0.F_1F_2 \ldots \in \mathcal{L}_{P,Q}(w) \).

1. If \( E_n \in [0, q_n - 2] \) and there exists \( m > n \) such that \( E_m \neq 0 \), then \( F_n = E_n \).
2. If \( E_n = q_n - 1 \) and there exists \( m > n \) such that \( E_m \neq 0 \), then \( F_n \in [q_n - 1, p_n - 1] \).
3. If \( w \in \mathcal{U}_Q \cap (0, 1) \) and \( n = \inf \{t \in \mathbb{N} : E_t > 0\} \), then \( \mathcal{L}_{P,Q}(w) = A \cup B \), where
\[ A = \{\zeta = 0.G_1 \cdots G_n \text{ w.r.t. } P : \psi_{P,Q}(\zeta) = w\} \text{ and } \]
\[ B = \left\{ G_1 \cdots G_{n-1}(E_n - 1)G_{n+1} \cdots \text{ w.r.t. } P : G_m \in [q_m - 1, p_m - 1] \forall m > n \wedge \psi_{P,Q}\left(\sum_{j=1}^{n-1} G_j \prod_{j=1}^{n-1} p_j \right) = \sum_{j=1}^{n-1} \frac{E_j}{q_1 \cdots q_j} \right\}. \]

Clearly, the set \( A \) is at most finite although the set \( B \) may be quite large.

4. If \( w \in \mathcal{U}_Q \), then \( \mathcal{L}_{P,Q}(w) = \emptyset \) if and only if there exists a natural number \( n \) such that \( E_n \geq p_n \).
5. If there exists \( n \) with \( p_n < q_n \), then \( \mathcal{L}_{P,Q}(w) = \emptyset \) for all \( w > \sum_{n=1}^{\infty} \min(p_n - 1, q_n - 1) \).
6. If \( p_n > q_n \) for at most finitely many \( n \), then \( \mathcal{L}_{P,Q}(w) \) is finite for all \( w \in \psi_{P,Q}((0, 1)) \).
7. If \( p_n \leq q_n \) for all \( n \), then \( \psi_{P,Q} \) is injective and increasing.

For \( w = 0.E_1E_2 \ldots \) w.r.t. \( Q \), set
\[ \omega_n(w) = \begin{cases} 1 & \text{if } E_n \in [0, q_n - 2] \\ p_n - q_n + 1 & \text{if } E_n = q_n - 1 \\ 0 & \text{if } E_n \geq p_n \end{cases} \]

...
Theorem 2.6. Let \( w = 0.E_1E_2 \cdots \) w.r.t. \( Q \). If \( w \in \mathcal{U}_Q \), then

(2.2) \[
\lambda(\mathcal{L}_{P,Q}(w)) = \prod_{j=1}^{\infty} \frac{\omega_j(w)}{p_j} \quad \text{and} \quad \dim_H(\mathcal{L}_{P,Q}(w)) = \liminf_{n \to \infty} \frac{\log \prod_{j=1}^{n} \omega_j(w)}{\log \prod_{j=1}^{n} p_j}.
\]

If \( w \in \mathcal{U}_Q \), \( M = \inf \{ t \in \mathbb{N} : E_t > 0 \} \), and \( p_n \geq q_n \) for all \( n > M \), then

(2.3) \[
\lambda(\mathcal{L}_{P,Q}(w)) = \left( \prod_{j=1}^{M-1} \frac{\omega_j(w)}{p_j} \right) \cdot \frac{1}{p_M} \left( \prod_{j=M+1}^{\infty} \frac{p_j - q_j + 1}{p_j} \right) \quad \text{and} \quad \dim_H(\mathcal{L}_{P,Q}(w)) = \liminf_{n \to \infty} \frac{\log \prod_{j=M+1}^{n} \omega_j(w)}{\log \prod_{j=M+1}^{n} p_j}.
\]

If \( \mathcal{L}_{P,Q}(w) \neq \emptyset \) and (2.1) holds with

\[
I_n = I_n(w) = \begin{cases} 
E_n & \text{if } E_n \in [0, q_n - 2] \\
q_n - 1, p_n - 1 & \text{if } E_n = q_n - 1
\end{cases},
\]

then \( \dim_H(\mathcal{L}_{P,Q}(w)) = \dim_P(\mathcal{L}_{P,Q}(w)) = \dim_B(\mathcal{L}_{P,Q}(w)) \).

Proof. This follows from Theorem 2.3 and our characterization of the level sets of \( \psi_{P,Q} \) in Theorem 2.4. The last part follows from Lemma 2.4. \( \square \)

We will need the following basic lemmas to help prove Theorem 1.4.

Lemma 2.7. Let \( L \) be a real number and \( (a_n)_{n=1}^{\infty} \) and \( (b_n)_{n=1}^{\infty} \) be two sequences of positive real numbers such that

\[
\sum_{n=1}^{\infty} b_n = \infty \quad \text{and} \quad \lim_{n \to \infty} \frac{a_n}{b_n} = L.
\]

Then

\[
\lim_{n \to \infty} \frac{a_1 + a_2 + \ldots + a_n}{b_1 + b_2 + \ldots + b_n} = L.
\]

Lemma 2.8. Let \( L \) be a real number and \( (a_n)_{n=1}^{\infty} \) and \( (b_n)_{n=1}^{\infty} \) be two sequences of positive integers. Let \( (c_n)_{n=1}^{\infty} \) be an increasing sequence of positive integers. Let \( A_t = \sum_{n=c_t}^{c_t-1} a_n \) and \( B_t = \sum_{n=c_t}^{c_t-1} b_n \). If

\[
\lim_{t \to \infty} \frac{A_t + A_{t+1} + \ldots + A_t}{B_t + B_{t+1} + \ldots + B_t} = L, \quad \sum_{t=1}^{\infty} A_t = \sum_{t=1}^{\infty} B_t = \infty,
\]

and \( \lim_{t \to \infty} \frac{A_t+1}{A_t} = \lim_{t \to \infty} \frac{B_{t+1}}{B_t} = 0 \), then

\[
\lim_{n \to \infty} \frac{a_1 + a_2 + \ldots + a_n}{b_1 + b_2 + \ldots + b_n} = L.
\]

We also need

Lemma 2.9. If \( (s_n) \) grows nicely, then \( \lim_{n \to \infty} \frac{\log s_n}{\log s_1 + \ldots + s_n} = 0 \).

Proof. Let \( m \in \mathbb{N} \). For \( m > m \) and \( n \in [t_{t+1}, t_{t+2}] \)

\[
\frac{\log s_n}{\log s_1 s_2 \cdots s_n} \leq \frac{\log s_{t_{t+2}}}{\log \prod_{j=t_{t+1}}^{t_{t+2}} s_j}.
\]

Since \( (s_n) \) is monotone

\[
\lim_{n \to \infty} \frac{\log s_n}{\log s_1 s_2 \cdots s_n} \leq \lim_{t \to \infty} \frac{\log s_{t_{t+2}}}{\sum_{j=t_{t+1}}^{t_{t+2}} \log s_j} = \frac{1}{m}
\]

and \( \lim_{n \to \infty} \frac{\log s_n}{\log s_1 s_2 \cdots s_n} = 0 \). \( \square \)

Proof of Theorem 1.4. Let \( \alpha < \gamma \) and \( N \) be the smallest integer such that \( p_n > q_n \), \( q_n > 2 \), and \( (p_n), (q_n), \) and \( (r_n) \) are non-decreasing for all \( n > N \). We will describe a set \( S \subseteq (0, 1) \) where \( \dim_H(\mathcal{L}_{P,Q}(w)) = \alpha \) for all \( w \in S \) and \( \dim_H(S) = 1 - \frac{\alpha}{\gamma} \). Let

\[
c_t = \sum_{j=1}^{t} \left( \left\lfloor \frac{1 - \frac{\alpha}{\gamma} \right\rfloor \cdot j \right) + \left\lceil \frac{\alpha}{\gamma} j \right\rceil.
\]
and \( M = \min\{t : c_t > N\} \). Set
\[
I_n = \begin{cases} 
\{0, 1, \ldots, q_n - 2\} & \text{if } n \in \left[ c_t, c_t + \left(1 - \frac{2}{\gamma}\right)t\right] - 1 \\
\{q_n - 1\} & \text{if } n \in \left[c_t + \left(1 - \frac{2}{\gamma}\right)t, c_t + 1 - 1\right]
\end{cases}
\]
Thus, \( \lim_{t \to \infty} A_t = \lim_{n \to \infty} B_t = 0 \) since \((r_n)\) and \((p_n)\) are nice sequences. Since \((r_n)\) is eventually monotone and \( r_n \to \infty \)
\[
\lim_{t \to \infty} \frac{A_{t+1}}{A_t} = \lim_{t \to \infty} \frac{A_1}{M + \cdots + A_t} = \frac{\left[\left(1 - \frac{2}{\gamma}\right)t + \left(1 - \frac{1}{\gamma}\right)t\right]}{\sum_{j=M}^{t} \left(\frac{2}{\gamma} j\right)} \log r_{c_t} = \lim_{t \to \infty} \frac{\left[\left(1 - \frac{2}{\gamma}\right)t + \left(1 - \frac{1}{\gamma}\right)t\right]}{\sum_{j=M}^{t} \left(\frac{2}{\gamma} j\right)} \log r_{c_t} = 0,
\]
Similarly, it can be shown that \( \lim_{t \to \infty} \frac{B_{t+1}}{B_t} = 0 \), so by Lemma 2.7 and Lemma 2.8
\[
\lim_{n \to \infty} \frac{1}{\sum_{j \leq M} \log r_{c_j}} \log r_{c_t} = \lim_{n \to \infty} \frac{1}{\sum_{j \leq M} \log r_{c_j}} \log r_{c_t} = \frac{\left[\left(1 - \frac{2}{\gamma}\right)t + \left(1 - \frac{1}{\gamma}\right)t\right]}{\sum_{j=M}^{t} \left(\frac{2}{\gamma} j\right)} \log r_{c_t} = \frac{\alpha}{\gamma} = \alpha.
\]
By construction, \( w \in \mathcal{U}_Q \). Thus, by (2.2), (2.4), and Lemma 2.9
\[
\dim_H (\mathcal{L}_{P,Q} (w)) = \lim_{n \to \infty} \frac{\log \prod_{j \leq M} \omega_j (w)}{\log \prod_{j \leq M} \rho_j} = \alpha.
\]
Let
\[
v_n = |I_n| = \begin{cases} 
q_n - 1 & \text{if } n \in \left[ c_t, c_t + \left(1 - \frac{2}{\gamma}\right)t\right] - 1 \\
1 & \text{if } n \in \left[c_t + \left(1 - \frac{2}{\gamma}\right)t, c_t + 1 - 1\right]
\end{cases}
\]
Then, by Theorem 2.3 and Lemma 2.9
\[
\dim_H (S) = \lim_{n \to \infty} \frac{\log \prod_{j \leq M} \nu_j (w)}{\log \prod_{j \leq M} \rho_j} = \lim_{n \to \infty} \frac{\log \prod_{j \leq M} \nu_j (w)}{\log \prod_{j \leq M} \rho_j} = \frac{\sum_{k=1}^{\infty} \frac{q_k - 1}{p_k}}{\prod_{j=k+1}^{\infty} \left(1 - \frac{q_j - 1}{p_j}\right)}.
\]
A similar argument using Lemma 2.7 and Lemma 2.8 shows that \( \dim_H (S) = 1 - \frac{2}{\gamma} \). Thus, since \( S \subseteq \mathcal{V}_{P,Q} (\alpha) \), we know that \( \dim_H (\mathcal{V}_{P,Q} (\alpha)) \geq 1 - \frac{2}{\gamma} \) and \( \dim_H (\mathcal{V}_{P,Q} (\alpha)) > 0 \) if \( \alpha < \gamma \).

\textbf{Theorem 2.10.} Suppose that \( p_n \geq q_n \) for all \( n \) and \( \sum \frac{q_n}{p_n} < \infty \). Then \( \lambda (\mathcal{L}_{P,Q} (w)) > 0 \) if and only if \( w \in \mathcal{U}_Q \cap \mathcal{P}_Q ((0, 1)) \). Furthermore,
\[
\sum_{w \in \mathcal{P}_Q ((0, 1))} \lambda (\mathcal{L}_{P,Q} (w)) = \sum_{w \in \mathcal{U}_Q \cap \mathcal{P}_Q ((0, 1))} \lambda (\mathcal{L}_{P,Q} (w)) = \sum_{k=1}^{\infty} \frac{q_k - 1}{p_k} \cdot \prod_{j=k+1}^{\infty} \left(1 - \frac{q_j - 1}{p_j}\right).
\]
Proof. We first note that \( \sum \frac{a_n}{p_n} \) converges if and only if \( \sum \frac{a_n-1}{p_n} \) converges. An argument that shows this is given in the proof of Theorem 2.18. Let \( M = \inf\{t \in \mathbb{N} : E_t > 0\} \). Then by (2.2) for \( w = 0.E_1E_2\ldots E_M \) w.r.t.

\[
\lambda (\mathcal{L}_{P,Q} (w)) = \left( \prod_{j=1}^{M-1} \omega_j (w) / p_j \right) \cdot \frac{1}{p_M} \cdot \left( \prod_{j=M+1}^{\infty} \frac{p_j - q_j + 1}{p_j} \right).
\]

Since \( \sum \frac{a_n-1}{p_n} \) converges,

\[
\prod_{j=M+1}^{\infty} \frac{p_j - q_j + 1}{p_j} = \prod_{j=M+1}^{\infty} \left( 1 - \frac{q_j - 1}{p_j} \right) > 0,
\]

so \( \lambda (\mathcal{L}_{P,Q} (w)) > 0 \). If \( w \in \mathcal{U}_Q \), then

\[
\lambda (\mathcal{L}_{P,Q} (w)) \leq \prod_{1 \leq j < \infty \atop E_j \neq q_j - 1} \frac{1}{p_j} = 0,
\]

by (2.2), so \( \lambda (\mathcal{L}_{P,Q} (w)) = 0 \).

We will now evaluate \( \sum_{w \in \mathcal{U}_Q \cap \mathcal{V}_{P,Q}((0,1))} \lambda (\mathcal{L}_{P,Q} (w)) \). Let

\[
\xi_n (m) = \begin{cases} 
1 & \text{if } m \in [0, q_n - 2] \\
\frac{1}{p_n - q_n + 1} & \text{if } m = q_n - 1 \\
0 & \text{if } m \geq p_n 
\end{cases}
\]

and put \( Y_k = \Pi_{j=k+1}^{\infty} \frac{p_j - q_j + 1}{p_j} = \Pi_{j=k+1}^{\infty} \left( 1 - \frac{q_j - 1}{p_j} \right) > 0 \). Then by (2.2)

\[
\sum_{w \in \mathcal{U}_Q \cap \mathcal{V}_{P,Q}((0,1))} \lambda (\mathcal{L}_{P,Q} (w)) = \sum_{k=1}^{\infty} \sum_{0 \leq E_1 \leq q_1 - 1 \atop 0 \leq E_{k-1} \leq q_{k-1} - 1 \atop 1 \leq E_k \leq q_k - 1} \lambda \left( \mathcal{L}_{P,Q} \left( \sum_{n=1}^{k} \frac{E_n}{q_1 \cdots q_n} \right) \right)
\]

\[
= \sum_{k=1}^{\infty} \sum_{0 \leq E_1 \leq q_1 - 1 \atop 0 \leq E_{k-1} \leq q_{k-1} - 1 \atop 1 \leq E_k \leq q_k - 1} \left( \prod_{j=1}^{k-1} \frac{\xi_j (E_j)}{p_j} \right) \cdot \frac{1}{p_k} \cdot \left( \prod_{j=k+1}^{\infty} \frac{p_j - q_j + 1}{p_j} \right) = \sum_{k=1}^{\infty} \sum_{0 \leq E_1 \leq q_1 - 1 \atop 0 \leq E_{k-1} \leq q_{k-1} - 1 \atop 1 \leq E_k \leq q_k - 1} \left( \prod_{j=1}^{k-1} \frac{\xi_j (E_j)}{p_j} \right) \cdot \left( \prod_{j=k+1}^{\infty} \frac{q_k - 1}{p_k} \cdot \prod_{j=k+1}^{\infty} \left( 1 - \frac{q_j - 1}{p_j} \right) \right).
\]

\[\square\]

2.2. Measures on \( \mathbb{N}_2^N \times \mathbb{N}_2^N \). Let \( \tau : \mathbb{N}_2^N \rightarrow \mathbb{N}_2^N \) be the left shift on \( \mathbb{N}_2^N \) and let \( \mathcal{M}^e \left( \mathbb{N}_2^N \right) \) (resp. \( \mathcal{M}^w \left( \mathbb{N}_2^N \right) \)) be the collection of all ergodic (resp. weakly mixing) \( \tau \)-invariant Borel probability measures on \( \mathbb{N}_2^N \). For \( s, t \in \mathbb{N} \), set \( \sigma_{s,t} = \tau^s \times \tau^t \) and \( \sigma = \sigma_{1,1} \). For \( S \subseteq \mathbb{N} \), we say that \( \nu \in \mathcal{M}^w \left( \mathbb{N}_2^N \right) \) is positive on \( S \) if \( \nu \left( \{ \omega \in \mathbb{N}_2^N : \pi (\omega) \in S \} \right) > 0 \). \( \nu \) is eventually positive if there exists \( M \) such that \( \nu \) is positive on \( \{n\} \) for all \( n \geq M \). If \( \tau \) is weakly mixing, then \( \sigma_{s,t} \) is ergodic and weakly mixing.

Lemma 2.11. Suppose that \( \mu_1, \mu_2 \in \mathcal{M}^w \left( \mathbb{N}_2^N \right), \mu = \mu_1 \times \mu_2 \), and \( \max \left( \int \log \pi_1 (\omega) \ d\mu (\omega), \int \log \pi_2 (\omega) \ d\mu (\omega) \right) < \infty \). If \( \int \log \pi_1 (\omega) \ d\mu (\omega) > \alpha \int \log \pi_2 (\omega) \ d\mu (\omega) \) for \( \alpha > 1 \), then for all integers \( k \geq 0 \) and \( \mu \)-almost every \((P,Q) \in \mathbb{N}_2^N \times \mathbb{N}_2^N \)

\[
\lim_{n \rightarrow \infty} \frac{p_1 \cdots p_n}{q_1 \cdots q_{n+\alpha} + k} = \infty.
\]

If \( \int \log \pi_2 (\omega) \ d\mu (\omega) > \alpha \int \log \pi_1 (\omega) \ d\mu (\omega) \) for \( \alpha > 1 \), then for all integers \( k \geq 0 \) and \( \mu \)-almost every \((P,Q) \in \mathbb{N}_2^N \times \mathbb{N}_2^N \)

\[
\lim_{n \rightarrow \infty} \frac{p_1 \cdots p_{n+\alpha} + k}{q_1 \cdots q_n} = 0.
\]
Proof. For integers $s,t \geq 1$, set
\[
 f_{s,t,u,v}(\omega) = \sum_{j=0}^{t-1} \log \pi_2(\sigma^{j+u}(\omega)) = -\sum_{j=0}^{s-1} \log \pi_1(\sigma^{j+u}(\omega))
\]
Let $t = \lfloor \alpha \rfloor$ and note that
\[
 \log \frac{q_1 \cdots q_{nt+k}}{p_1 \cdots p_n} = \log q_1 \cdots q_k + \sum_{j=1}^{n} \log \left( \prod_{m=1}^{n} q_{(j-1)t+k+m} \right)
\]
But $\sigma_{1,t}$ is ergodic, so for $\mu$-almost every $\omega \in \mathbb{N}_2^N \times \mathbb{N}_2^N$
\[
 \lim_{n \to \infty} \frac{1}{n} \log \frac{\pi_2(\omega) \cdots \pi_2(\sigma^{nt+k-1}(\omega))}{\pi_1(\omega) \cdots \pi_1(\sigma^{n-1}(\omega))} = \lim_{n \to \infty} \left( \frac{1}{n} \cdot \pi_2(\omega) \cdots \pi_2(\sigma^{k-1}(\omega)) + \frac{1}{n} \sum_{i=0}^{n-1} f_{1,t,0,k} \circ \sigma_{1,t}^i(\omega) \right)
\]
= $0 + \int f_{1,t,0,k}(\omega) \, d\mu(\omega) = \int \left( \sum_{j=0}^{t-1} \log \pi_2(\sigma^{j+k}(\omega)) - \log \pi_1(\omega) \right) \, d\mu(\omega)$
\[
 = \left( \sum_{j=0}^{t-1} \log \pi_2(\sigma^{j+k}(\omega)) \right) - \int \log \pi_1(\omega) \, d\mu(\omega) = \left( \sum_{j=0}^{t-1} \int \log \pi_2(\omega) \, d\mu(\omega) \right) - \int \log \pi_1(\omega) \, d\mu(\omega)
\]
= $t \int \log \pi_2(\omega) \, d\mu(\omega) - t \int \log \pi_1(\omega) \, d\mu(\omega) < \alpha \int \log \pi_2(\omega) \, d\mu(\omega) - \int \log \pi_1(\omega) \, d\mu(\omega) < 0$.
Thus, for $\mu$-almost every $\omega \in \mathbb{N}_2^N \times \mathbb{N}_2^N$
\[
 \lim_{n \to \infty} \log \frac{\pi_2(\omega) \cdots \pi_2(\sigma^{nt+k-1}(\omega))}{\pi_1(\omega) \cdots \pi_1(\sigma^{n-1}(\omega))} = -\infty,
\]
so
\[
 \lim_{n \to \infty} \frac{\pi_2(\omega) \cdots \pi_2(\sigma^{nt+k-1}(\omega))}{\pi_1(\omega) \cdots \pi_1(\sigma^{n-1}(\omega))} = 0
\]
and the first assertion follows. The second assertion is proven similarly. \qed

Lemma 2.12. Suppose that $\max \left( \int \log \pi_1(\omega) \, d\mu(\omega), \int \log \pi_2(\omega) \, d\mu(\omega) \right) < \infty$ and $\alpha \in [0,1]$. If $\int \log \pi_1(\omega) \, d\mu(\omega) < \alpha \int \log \pi_2(\omega) \, d\mu(\omega)$, then
\[
 \lim_{n \to \infty} \frac{(p_1 p_2 \cdots p_n)^\alpha}{q_1 q_2 \cdots q_n} \cdot \min(p_n, q_n)^{1-\alpha} = 0
\]
and
\[
 \lim_{n \to \infty} \left( \frac{p_1 p_2 \cdots p_{n+k}}{q_1 q_2 \cdots q_n} \cdot \frac{p_{n+k+1}}{\max(1, p_{n+k+1} - q_{n+k+1})} \right)^\alpha = 0,
\]
for $\mu$-almost every $(P,Q) \in \mathbb{N}_2^N \times \mathbb{N}_2^N$.

Proof. The proof is similar to the proof of Lemma 2.11 after we note that $\min(p_n, q_n)^{1-\alpha} \leq q_n^{1-\alpha} \leq q_n$ and
\[
 \left( \frac{p_{n+k+1}}{\max(1, p_{n+k+1} - q_{n+k+1})} \right)^\alpha \leq p_{n+k+1}^{\alpha}.
\]
\qed

Lemma 2.13. If $\mu_1, \mu_2 \in \mathcal{M}^w(\mathbb{N}_2^N)$, $\mu = \mu_1 \times \mu_2$, and $\int \log \pi_1(\omega) \, d\mu(\omega) < \int \log \pi_2(\omega) \, d\mu(\omega) < \infty$, then for $\mu$-almost every $(P,Q) \in \mathbb{N}_2^N \times \mathbb{N}_2^N$
\[
 \lim_{n \to \infty} \frac{p_1 \cdots p_n}{q_1 \cdots q_n} = 0.
\]

Proof. For $M > 0$, let $A_M = \left\{ \omega \in \mathbb{N}_2^N \times \mathbb{N}_2^N : \lim_{n \to \infty} \frac{\pi_1(\omega) \cdots \pi_1(\sigma^{n-1}(\omega))}{\pi_2(\omega) \cdots \pi_2(\sigma^{n-1}(\omega))} \geq M \right\}$. Assume for contradiction that $\mu(A_M) > 0$. Note that $A_M$ is a $\sigma$-invariant set, so $\mu(A_M) = 1$ by the ergodicity of $\sigma$. Let $f(\omega) = \ldots$
\[ \pi_1(\omega) - \pi_2(\omega) \] and put \( S_n(f)(\omega) = \sum_{i=0}^{n-1} f \circ \sigma^i(\omega) \). Clearly, \( \omega \in A_M \) if and only if \( \lim \inf_{n \to \infty} S_n(f) \geq \log M \). Thus, \( \int \lim \inf_{n \to \infty} S_n(f)(\omega) \, d\mu(\omega) \geq \log M \). However,

\[
\lim \inf_{n \to \infty} \int S_n(f)(\omega) \, d\mu(\omega) = \lim \inf_{n \to \infty} \int \left( \sum_{i=0}^{n-1} f \circ \sigma^i(\omega) \right) \, d\mu(\omega) = \lim \inf_{n \to \infty} \sum_{i=0}^{n-1} \left( \int f \circ \sigma^i(\omega) \, d\mu(\omega) \right)
\]

\[
= \lim \inf_{n \to \infty} \sum_{i=0}^{n-1} \int \left( \pi_1(\sigma^i(\omega)) - \pi_2(\sigma^i(\omega)) \right) \, d\mu(\omega) = \lim \inf_{n \to \infty} \sum_{i=0}^{n-1} \left( \int \pi_1(\omega) - \pi_2(\omega) \right) \, d\mu(\omega)
\]

\[
\leq \lim \inf_{n \to \infty} \sum_{i=0}^{n-1} 0 = 0.
\]

By Fatou’s lemma, \( \int \lim \inf_{n \to \infty} S_n(f)(\omega) \, d\mu(\omega) \leq \lim \inf_{n \to \infty} \int S_n(f)(\omega) \, d\mu(\omega) \), which implies that \( M \leq 0 \), a contradiction.

**Lemma 2.14.** If \( \max \left( \int (\pi_1(\omega))^2 \, d\mu(\omega), \int (\pi_2(\omega))^2 \, d\mu(\omega) \right) < \infty \), then

\[
\sum_{k=1}^{\infty} \sum_{j=k+1}^{\infty} p_k(p_j + q_j) \frac{q_1 \cdots q_j}{q_1^j} < \infty
\]

for \( \mu \)-almost every \((P, Q) \in \mathbb{N}_2^N \times \mathbb{N}_2^N\).

**Proof.** We will show that

\[
(2.5) \quad \int \sum_{k=1}^{\infty} \sum_{j=k+1}^{\infty} \pi_1(\sigma^{k-1}(\omega))(\pi_1(\sigma^{j-1}(\omega)) + \pi_2(\sigma^{j-1}(\omega))) \, d\mu(\omega) < \infty.
\]

Since each term in (2.5) is non-negative, the left hand side of (2.5) is equal to

\[
\sum_{k=0}^{\infty} \sum_{j=k}^{\infty} \int \pi_1(\sigma^k(\omega))(\pi_1(\sigma^j(\omega)) + \pi_2(\sigma^j(\omega))) \, d\mu(\omega) \leq \sum_{k=0}^{\infty} \sum_{j=k}^{\infty} \int \pi_1(\sigma^k(\omega))(\pi_1(\sigma^j(\omega)) + \pi_2(\sigma^j(\omega))) \, d\mu(\omega)
\]

\[
\leq \sum_{k=0}^{\infty} \sum_{j=k}^{\infty} 2^{-j} \left( \int \pi_1(\sigma^k(\omega))^2 \, d\mu(\omega) \right)^{1/2} \left( \int (\pi_1(\sigma^j(\omega)) + \pi_2(\sigma^j(\omega)))^2 \, d\mu(\omega) \right)^{1/2} \text{ by Cauchy-Schwarz}
\]

\[
= \sum_{k=0}^{\infty} \sum_{j=k}^{\infty} 2^{-j} \left( \int \pi_1(\omega)^2 \, d\mu(\omega) \right)^{1/2} \left( \int (\pi_1(\omega) + \pi_2(\omega))^2 \, d\mu(\omega) \right)^{1/2} \leq \sum_{k=0}^{\infty} \sum_{j=k}^{\infty} C \cdot 2^{-j} < \infty.
\]

Lastly, we note the following trivial lemma.

**Lemma 2.15.** If \( \mu_1, \mu_2 \in \mathcal{M}(\mathbb{N}_2^N) \) are eventually positive, then \( p_n < q_n \) infinitely often and \( p_n > q_n \) infinitely often for \( \mu_1 \times \mu_2 \)-almost every \((P, Q) \in \mathbb{N}_2^N \times \mathbb{N}_2^N\).

2.3. **Rationality of \( \psi_{P,Q} \).** We will need the following theorem to discuss the rationality of \( \psi_{P,Q}(x) \) for various \( P, Q \), and \( x \). This theorem and a far more extensive discussion of the irrationality of sums of the form \( \sum_{n=1}^{\infty} \frac{E_n}{q_1 q_2 \cdots q_n} \) may be found in the monograph of J. Galambos \[3\] and is originally due to G. Cantor \[4\].

**Theorem 2.16.** Suppose that \( Q \) has the property that for every positive integer \( m \) there exist infinitely many positive integers \( n \) such that \( m|q_n \). Then \( \sum_{n=1}^{\infty} \frac{E_n}{q_1 q_2 \cdots q_n} \) is rational if and only if \( E_j = q_j - 1 \) for all but finitely many \( j \) or if \( E_j = 0 \), ultimately\[12\].

**Corollary 2.17.** Under the conditions of Theorem 2.16 \( \mathbb{N}Q = Q \) and \( \mathbb{U}Q = \mathbb{R} \setminus \mathbb{Q} \).

\[12\]We remark that this sum isn’t required to be a \( Q \)-Cantor series expansion. That is, we may have \( E_j = q_j - 1 \), ultimately.
Theorem 2.18. Suppose that both $P$ and $Q$ have the property described in Theorem 2.16. Let

$$S = \{ x \in \mathbb{R} \cap Q : \psi_{P,Q}(x) \in Q \}.$$ 

Then

1. $\psi_{P,Q}(\mathbb{R} \cap Q) \subseteq [0, 1] \cap Q$.
2. If $p_n \leq q_n$ infinitely often, then $\psi_{P,Q}(\mathbb{R} \cap Q) \subseteq [0, 1] \cap Q$.
3. If there exists $M = M(P, Q)$ such that $p_n \geq q_n$ for all $n \geq M$, but $p_n \geq q_n + 1$ at most finitely often, then $S$ is countable and $S \cap [0, 1]$ is finite.
4. If there exists $M = M(P, Q)$ such that $p_n \geq q_n$ for all $n \geq M$ and $p_n \geq q_n + 1$ infinitely often, then $S \cap [0, 1]$ is an uncountable meagre set.

In particular, $\lambda(S \cap [0, 1]) = 1$ if and only if $\sum_{j=1}^{\infty} q_j/p_j < \infty$. Also, $\dim_H(S) \leq \liminf_{n \to \infty} \frac{\log \prod_{j=1}^{n} (p_j - q_j)}{\log \prod_{j=1}^{n} p_j}$.

Proof. The first part follows directly from Corollary 2.17. Note that $S = \{ x = E_0 E_1 E_2 \cdots \text{ w.r.t. } P : \exists N \geq M \text{ such that } q_n - 1 \leq E_n \leq p_n - 1 \forall n > N \land E_n \neq p_n - 1 \text{ infinitely often} \}$. $S = \emptyset$ under the conditions of part (2). Part (3) immediately follows from our characterization of $S$. For part (4), we note that

$$\frac{\log \prod_{j=1}^{n} (p_j - q_j)}{\log \prod_{j=1}^{n} p_j} \leq \liminf_{n \to \infty} \frac{\log \prod_{j=1}^{n} (p_j - q_j + 1)}{\log \prod_{j=1}^{n} p_j}.$$ 

The infinite products inside the limits in (2.6) converge if and only if $\sum_{j=1}^{\infty} q_j/p_n$ and $\sum_{j=1}^{\infty} q_j/p_n + 1$ converge, respectively. Since $2/3 < q_j/q_j + 1 < 1$,

$$\sum_{j=\max(M, n)}^{\infty} \frac{q_j}{p_j} < \sum_{j=\max(M, n)}^{\infty} \frac{q_j + 1}{p_j} = \frac{2}{3} \cdot \sum_{j=\max(M, n)}^{\infty} \frac{q_j}{p_j}$$

and either both $\sum_{j=1}^{\infty} q_j/p_j$ and $\sum_{j=1}^{\infty} q_j/p_j + 1$ converge or they both diverge. Thus, if $\sum_{j=1}^{\infty} q_j/p_j$ converges, then

$$\lim_{n \to \infty} \frac{\prod_{j=\max(M, n)}^{\infty} (p_j - q_j)}{p_j} = 1$$

and $\lambda(S) = 1$. Otherwise, $\lambda(S) = 0$ by similar reasoning. The expression for the Hausdorff dimension of $S$ follows by our characterization of $S$ and Theorem 2.3.

\[\square\]

Theorem 2.18 is given as only one example of a result on the rationality of $\psi_{P,Q}(x)$. We should note that there are examples of $(P, Q) \in \mathbb{N}_2 \times \mathbb{N}_2'$ and $x \in Q$ where $\psi_{P,Q}(x) \in [0, 1] \cap Q$. Let $p_n = 3$ and $q_n = n + 1$ for all $n$. Put $x = 0.1111 \cdots$ w.r.t. $P = 1/3$. Then $\psi_{P,Q}(x) = e - 2 \in [0, 1] \cap Q$.

Theorem 2.19. If $\mu_1, \mu_2 \in M(\mathbb{N}_2')$ are eventually positive, then $\psi_{P,Q}(\mathbb{R} \cap Q) \subseteq [0, 1] \cap Q$ and $\psi_{P,Q}(\mathbb{R} \cap Q) \subseteq [0, 1] \cap Q$ for $\mu_1 \times \mu_2$-almost every $(P, Q) \in \mathbb{N}_2 \times \mathbb{N}_2'$.

Proof. This follows immediately from Lemma 2.15 and Theorem 2.18.

\[\square\]
2.4. Continuity of $\psi_{P,Q}$. Let
$$
\mathcal{L}_{P,Q}^L := \{ x \in \mathbb{R} : \psi_{P,Q} \text{ is left continuous at } x \},
\mathcal{L}_{P,Q}^R := \{ x \in \mathbb{R} : \psi_{P,Q} \text{ is right continuous at } x \},
\mathcal{D}_{P,Q}^L := \mathbb{R} \backslash \mathcal{L}_{P,Q}^R, \mathcal{D}_{P,Q}^R := \mathbb{R} \backslash \mathcal{L}_{P,Q}^L.
$$

Lemma 2.20. Suppose that $t$ is a positive integer and $x = E_0.E_1E_2\cdots E_t$ w.r.t. $P$, where $E_t \neq 0$. Then $x \in \mathcal{L}_{P,Q}^L$ if and only if
$$
\min(E_t, q_t - 1) - \min(E_t - 1, q_t - 1) = \sum_{j=t+1}^{\infty} \frac{\min(p_j - 1, q_j - 1)}{q_{t+j}q_{t+j+1}\cdots q_j}.
$$

Proof. We rewrite (2.7) as
$$
\sum_{j=1}^{t-1} \frac{\min(E_j, q_j - 1)}{q_1\cdots q_j} + \frac{\min(E_t, q_t - 1)}{q_1\cdots q_t} = \sum_{j=1}^{t-1} \frac{\min(E_j, q_j - 1)}{q_1\cdots q_j} + \frac{\min(E_t - 1, q_t - 1)}{q_1\cdots q_t} + \sum_{j=t+1}^{\infty} \frac{\min(p_j - 1, q_j - 1)}{q_{j+1}q_{j+2}\cdots q_j}.
$$

Let
$$
y_s = \begin{cases}
(E_0 - 1)(p_1 - 1)(p_2 - 1)\cdots (p_s - 1) & \text{w.r.t. } P \text{ if } x \in \mathbb{Z} \\
E_0E_1E_2\cdots E_{t-1}(E_t - 1)(p_{t+1} - 1)\cdots (p_s - 1) & \text{w.r.t. } P \text{ if } x \notin \mathbb{Z}
\end{cases}
$$
for $s > t$. Clearly, $\lim_{s \to \infty} y_s = x$ and $y_s < x$. We can rewrite (2.8) as
$$
\psi_{P,Q}(x) = \lim_{s \to \infty} \psi_{P,Q}(y_s).
$$
Since $y_s \to x$, $\psi_{P,Q}$ is not left continuous at $x$ if (2.9) does not hold. Now, suppose that (2.9) holds and let $(z_r)$ be any sequence of real numbers in $\mathbb{R}$ such that $z_r < x$ for all $r$ and $\lim_{r \to \infty} z_r = x$. Then there exists a function $f(r)$ such that for large enough $r$, we have
$$
z_r = \begin{cases}
(E_0 - 1)(p_1 - 1)(p_2 - 1)\cdots (p_{f(r)-1} - 1)F_{f(r)+1}F_{f(r)+2}\cdots & \text{w.r.t. } P \text{ if } x \in \mathbb{Z} \\
E_0E_1E_2\cdots E_{t-1}(E_t - 1)(p_{t+1} - 1)\cdots (p_{f(r)-1} - 1)F_{f(r)+1}F_{f(r)+2}\cdots & \text{w.r.t. } P \text{ if } x \notin \mathbb{Z}
\end{cases}
$$
Then $|\psi_{P,Q}(z_r) - \psi_{P,Q}(y_{f(r)})| \to 0$, so $\psi_{P,Q}(z_r) \to \psi_{P,Q}(x)$ by (2.9). Thus, $\psi_{P,Q}$ is left continuous at $x$. \hfill \Box

For a positive integer $t$ and basic sequences $P$ and $Q$, let
$$
A_{P,Q,t} := \{ E_0.E_1E_2\cdots E_t \text{ w.r.t. } P : E_t \geq q_t \};
B_{P,Q,t} := \{ E_0.E_1E_2\cdots E_{t-1} \text{ w.r.t. } P \};
A_{P,Q} := \{ n : p_n > q_n \};
B_{P,Q} := \{ n : p_n < q_n \}.
$$

Theorem 2.21. $\mathcal{D}_{P,Q}^R = \emptyset$ and
$$
\mathcal{D}_{P,Q}^L = \left( \bigcup_{n \in A_{P,Q}} A_{P,Q,n} \right) \cup \left( \bigcup_{n \in B_{P,Q}} B_{P,Q,n} \right) \subseteq \mathcal{U}_P.
$$
Moreover, $\psi_{P,Q}$ is lower semi-continuous on $\mathbb{R}$ if and only if $p_n \leq q_n$ whenever $n \geq 2$. $\psi_{P,Q}$ is upper semi-continuous on $\mathbb{R}$ if and only if it is continuous on $\mathbb{R}$.

Proof. It is not difficult to see that $\psi_{P,Q}$ is continuous at all points in $\mathcal{U}_P$ and right continuous on $\mathbb{R}$. Let $x = E_0.E_1E_2\cdots E_t$ w.r.t. $P$ so
$$
\min(E_t, q_t - 1) - \min(E_t - 1, q_t - 1) = \begin{cases}
0 & \text{if } E_t \geq q_t \\
1 & \text{if } E_t < q_t
\end{cases}
$$
Note that $\sum_{j=t+1}^{\infty} \frac{\min(p_j-1,q_j-1)}{q_{t+j}q_{t+j+1}\cdots q_j} > 0$. If $E_t \geq q_t$, then $x \in \mathcal{D}_{P,Q}^L$ by Lemma 2.20. This can only happen if $p_t > q_t$. In case $E_t < q_t$, we see that $x \in \mathcal{D}_{P,Q}^R$ if and only if there exists some integer $s > t$ such that
Consider the interval shown. Let \( J \) on this interval by applying Theorem 2.24.

2.5. We see that \( n \geq t \) Let \( \psi \) be such that

Theorem 2.23. Suppose that \( \mu_1, \mu_2 \in \mathcal{M}(\mathbb{N}_2^2) \) are eventually positive. Then \( D_{P,Q} = \mathbb{N} \cup P = Q \) for \( \mu_1 \times \mu_2 \)-almost every \( (P,Q) \in \mathbb{N}_2^2 \times \mathbb{N}_2^2 \).

Theorem 2.24. Suppose that \( p_n = q_n \) for all \( n > t \). Then \( \psi_{P,Q} \) is piecewise linear. In particular, for all \( x = E_0E_1E_2 \ldots \) w.r.t. \( P \)

\[
\psi_{P,Q}(x) = \psi_{P,Q}\{x\} = \frac{p_1 \cdots p_t}{q_1 \cdots q_t} \{x\} + \left( \sum_{n=1}^{t} \min(E_n,q_n-1) \frac{p_t \cdots p_1}{q_t \cdots q_1} \right).
\]

Proof. Let \( \alpha = \sum_{n=t}^{\infty} E_n \frac{p_1 \cdots p_t}{q_1 \cdots q_t} \), \( \beta = \sum_{n=t}^{\infty} \frac{p_t \cdots p_1}{q_t \cdots q_1} \), and \( \gamma = \sum_{n=t}^{\infty} E_n \). Since \( \min(E_n,q_n-1) = E_n \) for \( n > t \), we see that \( \psi_{P,Q}(\alpha + \gamma) = \beta + \frac{p_t \cdots p_1}{q_t \cdots q_1} \cdot \gamma \). Thus,

\[
\begin{align*}
\beta + \frac{p_t \cdots p_1}{q_t \cdots q_1} \cdot \gamma &= \frac{p_t \cdots p_1}{q_t \cdots q_1} \cdot \alpha - \frac{p_t \cdots p_1}{q_t \cdots q_1} \cdot \alpha + \beta + \frac{p_t \cdots p_1}{q_t \cdots q_1} \cdot \gamma \\
&= \frac{p_t \cdots p_1}{q_t \cdots q_1} \cdot (\alpha + \gamma) + \beta - \frac{p_t \cdots p_1}{q_t \cdots q_1} \cdot \alpha \\
&= \frac{p_t \cdots p_1}{q_t \cdots q_1} \{x\} + \left( \sum_{n=1}^{t} \min(E_n,q_n-1) \frac{p_t \cdots p_1}{q_t \cdots q_1} \frac{t}{p_t \cdots p_1} \right),
\end{align*}
\]

and the conclusion follows.

2.5. Monotonicity, Bounded Variation, and Approximation of \( \psi_{P,Q} \).

Theorem 2.25. \( \psi_{P,Q} \) is monotone on no intervals if and only if \( p_n > q_n \) infinitely often.

Proof. For simplicity, we only consider intervals contained in \([0,1] \) Suppose that \( p_n > q_n \) infinitely often and let \( J = [a,b] \subseteq [0,1] \) be a closed interval. Then there exists an interval \( I = [c,d] \subseteq J \) and \( n > t \) where \( c = 0E_1E_2 \cdots E_{n-1}(p_n-1) \) w.r.t. \( P \) and \( d = c + \frac{1}{p_1 \cdots p_n} \). Let \( m > n \) be such that \( p_m > q_m \). Set

\[
\begin{align*}
x &= 0E_1E_2 \cdots E_{n-1}(p_n-1) 0 0 0 \cdots 0 (q_m - 1) 1 \quad \text{w.r.t.} \quad P; \\
y &= 0E_1E_2 \cdots E_{n-1}(p_n-1) 0 0 0 \cdots 0 q_m \quad \text{w.r.t.} \quad P.
\end{align*}
\]

Clearly, \( x, y \in I, c < x, \) and \( \psi_{P,Q}(c) < \psi_{P,Q}(x) \). Also, \( x < y, \) but

\[
\psi_{P,Q}(x) = \psi_{P,Q}(y) = \psi_{P,Q}(c) + \frac{q_m - 1}{q_1 \cdots q_m} + \frac{1}{q_1 \cdots q_{m+1}} > \psi_{P,Q}(c) + \frac{q_m - 1}{q_1 \cdots q_m} = \psi_{P,Q}(y).
\]

So, \( \psi_{P,Q} \) is not monotone on the interval \( J \).

Now, suppose that \( p_n > q_n \) at most finitely often. Let \( M \) be large enough that \( p_m \leq q_m \) for all \( m > M \).
Consider the interval \( I = \left[ \frac{p_n - 1}{p_1 \cdots p_n} \sum_{n=1}^{M} p_n - 1, \frac{1}{p_1 \cdots p_N} \right] \). It is easy to verify that \( \psi_{P,Q} \) is increasing on this interval by applying Theorem 2.24.

Corollary 2.26. Suppose that \( p_n > q_n \) infinitely often and \( |L_{P,Q}(w)| \notin (0,1) \). Then \( \mathcal{L}_{P,Q}(w) \) is a totally disconnected set.

Theorem 2.27. Suppose that \( \mu_1, \mu_2 \in \mathcal{M}(\mathbb{N}_2^2) \) are eventually positive. Then \( \psi_{P,Q} \) is monotone on no intervals for \( \mu_1 \times \mu_2 \)-almost every \( (P,Q) \in \mathbb{N}_2^2 \times \mathbb{N}_2^2 \).

Given basic sequences \( P \) and \( Q \), let \( P_t = (p_1, p_2, \ldots, p_t, 2, 2, 2, \ldots) \) and \( Q_t = (q_1, q_2, \ldots, q_t, 2, 2, 2, \ldots) \).
**Theorem 2.28.** The sequence of functions \((\psi_{P,Q_t})\) converges uniformly to \(\psi_{P,Q}\) on \(\mathbb{R}\). \(^{13}\)

**Proof.** Let \(x = E_0E_1E_2 \cdots \text{ w.r.t. } P\) By Theorem 2.24

\[
\begin{align*}
\psi_{P,Q_t}(x) &= \frac{p_1 \cdots p_t}{q_1 \cdots q_t} \cdot \{x\} + \left(\sum_{n=1}^{t} \min(E_n,q_n-1) - \frac{p_1 \cdots p_t}{q_1 \cdots q_t} \cdot \sum_{n=1}^{t} E_n\right) \\
&= \sum_{n=1}^{t} \min(E_n,q_n-1) + \frac{p_1 \cdots p_t}{q_1 \cdots q_t} \sum_{n=t+1}^{\infty} E_n.
\end{align*}
\]

Thus,

\[
|\psi_{P,Q}(x) - \psi_{P,Q_t}(x)| = \sum_{n=t+1}^{\infty} \min(E_n,q_n-1) + \frac{p_1 \cdots p_t}{q_1 \cdots q_t} \sum_{n=t+1}^{\infty} E_n \leq \frac{1}{q_1 \cdots q_t} + \frac{1}{q_1 \cdots q_t} \leq \frac{1}{2^{t-1}}
\]

and \((\psi_{P,Q_t})\) converges uniformly to \(\psi_{P,Q}\). \(\square\)

**Corollary 2.29.**

\[
\begin{align*}
\int_{0}^{1} \psi_{P,Q_t}(x) \, dx &= \frac{1}{2^{q_1 \cdots q_t}} + \frac{p_1 \cdots p_{t-1}}{q_1 \cdots q_t} \sum_{E_1=0}^{p_1-1} \sum_{E_2=0}^{p_2-1} \cdots \sum_{E_t=0}^{p_t-1} \left(\frac{1}{p_1 \cdots p_t} \sum_{n=1}^{t} \min(E_n,q_n-1) - \frac{1}{q_1 \cdots q_t} \sum_{n=1}^{t} E_n\right) \\
\int_{0}^{1} \psi_{P,Q}(x) \, dx &= \lim_{t \to \infty} \sum_{E_1=0}^{p_1-1} \sum_{E_2=0}^{p_2-1} \cdots \sum_{E_t=0}^{p_t-1} \left(\frac{1}{p_1 \cdots p_t} \sum_{n=1}^{t} \min(E_n,q_n-1) - \frac{1}{q_1 \cdots q_t} \sum_{n=1}^{t} E_n\right).
\end{align*}
\]

**Proof.** The first assertion follows from computing the areas of the trapezoids bounded by pieces of the functions \(\psi_{P,Q}\). The latter assertion follows from the former, the dominated convergence theorem, and Theorem 2.28. \(\square\)

We let \(V(I,f)\) denote the total variation of the function \(f\) on the closed interval \(I\). We say that \(f\) is of **bounded variation** on \(I\) if \(V(I,f) < \infty\) and write \(f \in BV(I)\). We will need the following well known theorem from [5].

**Theorem 2.30.** \(V(I,\cdot) : BV(I) \to \mathbb{R}\) is a lower semi-continuous functional. That is, if \((f_n)\) converges to \(f\) pointwise on a closed interval \(I\), then

\[
V(I,f) \leq \liminf_{n \to \infty} V(I,f_n).
\]

We will also need the following lemma which is easily proven.

**Lemma 2.31.** Suppose that \(f : [a,b] \to \mathbb{R}\) is a piecewise monotone function that is right continuous on the non-empty closed interval \([a,b]\) with points of left discontinuity \(x_1, x_2, \cdots, x_r-1\). If \(x_0 = a\) and \(x_r = b\), then

\[
V([a,b],f) = \sum_{j=0}^{r-1} |f(x_j) - f(x_{j+1})| + \sum_{j=1}^{r} |f(x_j) - f(x_{j-1})|.
\]

**Lemma 2.32.** If \(t \geq 2\) and \(p_t \neq q_t\), then

\[
V([0,1],\psi_{P,Q_t}) = \sum_{k=1}^{t} \sum_{k=1}^{p_k-1} \left|\min(E_k,q_k-1) - \min(E_k,q_k-1)\right| - \sum_{j=k+1}^{t} \min(q_j-1,q_j-1) - \frac{1}{q_1 \cdots q_t} \\
+ \psi_{P,Q_t}(1^+) + \frac{p_1 \cdots p_t}{q_1 \cdots q_t} < 2 \cdot \left(\sum_{k=1}^{t} \sum_{j=k+1}^{t} p_k(q_j+q_j) - \frac{p_1 \cdots p_k}{q_1 \cdots q_j} \right) + 2 \cdot \frac{p_1 \cdots p_t}{q_1 \cdots q_t} + 1.
\]

\(^{13}\)Only pointwise convergence of \((\psi_{P,Q_t})\) to \(\psi_{P,Q}\) is used in this paper.
Theorem 2.24. \( \psi_{P,Q} \) is a piecewise linear function with slope \( \frac{p_{n-1} - p_0}{q_{n-1} - q_0} \), which contributes \( \frac{p_{n-1} - p_0}{q_{n-1} - q_0} \cdot (1 - 0) = \frac{p_{n-1} - p_0}{q_{n-1} - q_0} \) to the total variation of \( \psi_{P,Q} \). Thus, by Lemma 2.31 we need only add this term to the sum of the magnitude of the jumps at the points of discontinuity of \( \psi_{P,Q} \). Since \( p_t \neq q_t \), \( \mathcal{D}^L_{P,Q} \subseteq \mathcal{B}_{P,Q,t+1} \) by Theorem 2.21. If \( x = 1 \), then \( \psi_{P,Q}(x) = 0 \), so \( |\psi_{P,Q}(x) - \psi_{P,Q}(x^-)| = \psi_{P,Q}(1^-) \). If \( x = E_0, E_1, E_2, \ldots, E_k \) w.r.t. \( P_t \in \mathcal{B}_{P,Q,t+1} \), where \( E_k \neq 0 \), then \( \psi_{P,Q,1} = \sum_{n=1}^{k-1} \min(\frac{E_n}{q_n}, \frac{E_{n-1}}{q_{n-1}}) + \frac{1}{q_1 \cdots q_t} \)

Thus,

\[
\psi_{P,Q}(x^-) - \psi_{P,Q}(x^-) = \frac{\min(E_k, q_k - 1) - \min(E_k - 1, q_k - 1)}{q_1 \cdots q_k} - \sum_{j=k+1}^{t} \frac{\min(p_j - 1, q_j - 1)}{q_1 \cdots q_j} - \frac{1}{q_1 \cdots q_t}.
\]

So, \( \psi_{P,Q}(x^-) - \psi_{P,Q}(x^-) \) depends only on \( k \) and the value of \( E_k > 0 \). We only need sum over values of \( k \) and \( E_k \) and the first part of the lemma follows.

To prove the inequality, we apply the triangle inequality to the term in the double summation. First, it is clear that \( \min(E_k, q_k - 1) - \min(E_k - 1, q_k - 1) \leq 1 \), so

\[
\sum_{k=1}^{t} \sum_{E=1}^{p_k} \frac{\min(E_k, q_k - 1) - \min(E_k - 1, q_k - 1)}{q_1 \cdots q_k} \leq \sum_{k=1}^{t} \frac{p_k}{q_1 \cdots q_t},
\]

Next, \( \min(p_j - 1, q_j - 1) < p_j + q_j \), so

\[
\sum_{k=1}^{t} \sum_{j=k+1}^{p_k} \frac{\min(p_j - 1, q_j - 1)}{q_1 \cdots q_j} < \sum_{k=1}^{t} \sum_{j=k+1}^{p_k} \frac{p_k (p_j + q_j)}{q_1 \cdots q_j}.
\]

Lastly, \( \psi_{P,Q,1} \leq 1 \) and

\[
\sum_{k=1}^{t} \frac{p_k}{q_1 \cdots q_t} < \frac{p_1 \cdots p_t}{q_1 \cdots q_t},
\]

so the second part of the lemma follows.

\[\square\]

Theorem 2.33. If \( I \subseteq \mathbb{R} \) is a non-empty closed interval, then \( \psi_{P,Q} \in BV(I) \) if

\[
\sum_{k=1}^{\infty} \sum_{j=k+1}^{\infty} \frac{p_k (p_j + q_j)}{q_1 \cdots q_j} < \infty \text{ and } \lim_{t \to \infty} \frac{p_1 \cdots p_t}{q_1 \cdots q_t} = \infty.
\]

Proof. This follows immediately from Theorem 2.28, Theorem 2.30, Lemma 2.32, and the 1-periodicity of \( \psi_{P,Q} \). \[\square\]

Theorem 2.34. Suppose that \( I \subseteq \mathbb{R} \) is a closed interval, \( \mu_1, \mu_2 \in \mathbb{M}^w \left( \mathbb{N}^I \right) \) and \( \mu = \mu_1 \times \mu_2 \). If \( \int \log \pi_1(\omega) \ d\mu(\omega) \leq \int \log \pi_2(\omega) \ d\mu(\omega) \), then \( \psi_{P,Q} \) is of bounded variation for \( \mu \)-almost every \( (P,Q) \in \mathbb{N}^I_2 \times \mathbb{N}^I_2 \).

Proof. This follows from Lemma 2.13, Lemma 2.14, and Theorem 2.33. \[\square\]
2.6. Lipschitz and Hölder continuity of \( \phi^{(k)}_{P,Q} \). We will need to analyze the Hölder continuity of \( \phi^{(k)}_{P,Q} \) in order to prove Theorem 4.5. \( Z^{(k)}_{P,Q} \) will be non-empty as long as \( \lim_{n \to \infty} \min(p_n, q_n) \geq 3 \). Thus, we will require this assumption for every result in this subsection.

Note that
\[
(2.10) \quad Z^{(0)}_{P,Q} \subseteq Z^{(1)}_{P,Q} \subseteq \cdots \subseteq \bigcup_{k=0}^{\infty} Z^{(k)}_{P,Q} \subseteq Z^{(\infty)}_{P,Q} \subseteq [0, 1)
\]
and \( \phi^{(k)}_{P,Q} : Z^{(k)}_{P,Q} \to Z^{(k)}_{Q,P} \).

**Theorem 2.35.** Suppose that \( \lim_{n \to \infty} \min(p_n, q_n) \geq 3 \). Then \( \phi^{(k)}_{P,Q} \) is a homeomorphism from \( Z^{(k)}_{P,Q} \) to \( Z^{(k)}_{Q,P} \) for all \( k \in \mathbb{N}_0 \).

**Proof.** It is easy to see that \( \phi^{(k)}_{P,Q} \) is a bijection. \( \phi^{(k)}_{P,Q} = \psi_{P,Q|a}^{(k)} \) is continuous as \( Z^{(k)}_{P,Q} \) may only be discontinuous on \( \text{Null}_P \) by Theorem 2.21 and \( \text{Null}_P \cap Z^{(k)}_{P,Q} = \emptyset \) for all \( k < \infty \). Additionally, \( \phi^{(k)}_{P,Q}^{-1} \) is continuous as \( \phi^{(k)}_{P,Q} \).

**Lemma 2.36.** Suppose that \( \lim_{n \to \infty} \min(p_n, q_n) \geq 3, \alpha \in (0, 1], k \in \mathbb{N}_0, x, y \in Z^{(k)}_{P,Q}, \) and \( x \neq y \). Then for some constant \( C(\alpha) \),
\[
\frac{|\phi^{(k)}_{P,Q}(x) - \phi^{(k)}_{P,Q}(y)|}{|x - y|^{\alpha}} \leq C(\alpha) \sup_{n \in \mathbb{N}} \left( \left( \frac{p_1 \cdots p_n}{q_1 \cdots q_n} \right)^{\alpha} \cdot \min(p_n, q_n) \right)^{-\alpha} \cdot \left( \frac{p_{n+k+1}}{q_{1} \cdots q_{n+k+1}} \right)^{\alpha}.
\]

**Proof.** Let \( x = 0.E_1E_2 \cdots \) w.r.t. \( P \), \( y = 0.F_1F_2 \cdots \) w.r.t. \( P \), \( t = \min\{s : E_s \neq F_s\} \), and \( G_n = E_n - F_n \). Then
\[
\frac{|\phi^{(k)}_{P,Q}(x) - \phi^{(k)}_{P,Q}(y)|}{|x - y|^{\alpha}} \leq \sum_{n=t}^{\infty} \frac{|G_n|}{p_1 \cdots p_n} \cdot |x - y|^{\alpha} \leq \sum_{n=t}^{\infty} \frac{|G_n|}{p_1 \cdots p_n} \cdot \left( \frac{p_{n+k+1}}{q_1 \cdots q_{n+k+1}} \right)^{\alpha},
\]
which simplifies to
\[
(2.11) \quad \left( \frac{|G_t|+1}{q_1 \cdots q_t} \right)^{\alpha} \cdot \left( \frac{p_{n+k+1}}{\max (1, p_{n+k+1} - q_{n+k+1})} \right)^{\alpha}.
\]

We now consider two cases. First, if \( |G_t| = 1 \), then \( (2.11) \) is equal to
\[
(2.12) \quad \left( \frac{p_{n+k+1}}{q_1 \cdots q_t} \right)^{\alpha} \cdot \left( \frac{p_{n+k+1}}{\max (1, p_{n+k+1} - q_{n+k+1})} \right)^{\alpha}.
\]

Let \( C(\alpha) = \max \left( 2, \sup_{w \geq 2} \frac{w+1}{w^{\alpha} w^{\alpha+1}} \right) \). Clearly, \( w^{\alpha} w^{\alpha+1} \) is continuous for \( w \geq 2 \) and \( \lim_{w \to \infty} w^{\alpha} w^{\alpha+1} = 1 \). Thus, \( 2 \leq C(\alpha) < \infty \). Since \( |G_t| < \min(p_t, q_t) \),
\[
(2.13) \quad \frac{|G_t|+1}{|G_t|-1} \geq |G_t|^{1-\alpha} \cdot |G_t|^{1-\alpha} \leq C(\alpha) |G_t|^{1-\alpha} < C(\alpha) \cdot \min(p_t, q_t)^{1-\alpha}.
\]

Suppose that \( |G_t| > 1 \). Using \( (2.13) \), we may bound \( (2.11) \) above by
\[
(2.14) \quad \left( \frac{p_1 \cdots p_t}{q_1 \cdots q_t} \right)^{\alpha} \cdot \left( \frac{p_{n+k+1}}{\max (1, p_{n+k+1} - q_{n+k+1})} \right)^{\alpha}.
\]
Combining the estimates (2.12) and (2.14) of (2.11), the lemma follows.

\[ \text{Theorem 2.37. Suppose that } k \in \mathbb{N}_0 \text{ and } \liminf_{n \to \infty} \min(p_n, q_n) \geq 3. \text{ Then } \phi^{(k)}_{P,Q} \text{ is Hölder continuous of exponent } \alpha \text{ if} \]

\[ (2.15) \quad \limsup_{n \to \infty} \frac{(p_1 \cdots p_n)^\alpha}{q_1 \cdots q_n} \cdot \min(p_n, q_n)^{1-\alpha} < \infty \]

and

\[ (2.16) \quad \limsup_{n \to \infty} \frac{(p_1 \cdots p_n + k)^\alpha}{q_1 \cdots q_n} \cdot \left( \frac{p_{n+k+1}}{\max(1, p_{n+k+1} - q_{n+k+1})} \right)^\alpha < \infty. \]

Additionally, \( \phi^{(k)}_{P,Q} \) is not Hölder continuous of exponent \( \alpha \) if (2.15) does not hold.

\[ \text{Proof.} \text{ The Hölder continuity of } \phi^{(k)}_{P,Q} \text{ given (2.15) and (2.16) follows directly from Lemma 2.36. Suppose that (2.15) does not hold. Let the sequence } (n_t) \text{ be given such that} \]

\[ (2.17) \quad \frac{(p_1 \cdots p_{n_t})^\alpha}{q_1 \cdots q_{n_t}} \cdot \min(p_{n_t}, q_{n_t})^{1-\alpha} > t. \]

Let \( x_t = \sum_{m=1}^{n_t} \frac{1}{p_1 \cdots p_m} + \sum_{m=1}^{\infty} \frac{1}{p_1 \cdots p_{n_t+2m}} \) and

\[ y_t = \sum_{m=1}^{n_t} \frac{1}{p_1 \cdots p_m} + \frac{\min(p_{n_t-1}, q_{n_t-1})}{p_1 \cdots p_{n_t}} + \sum_{m=1}^{\infty} \frac{1}{p_1 \cdots p_{n_t+2m}}, \]

so \( x_t, y_t \in \mathbb{Z}_P,Q^k \). Then

\[ \left| \phi^{(k)}_{P,Q}(x_t) - \phi^{(k)}_{P,Q}(y_t) \right| = \frac{(p_1 \cdots p_{n_t})^\alpha}{q_1 \cdots q_{n_t}} \cdot \min(p_{n_t-1}, q_{n_t-1})^{1-\alpha} \]

\[ = \frac{(p_1 \cdots p_{n_t})^\alpha}{q_1 \cdots q_{n_t}} \cdot \min(p_{n_t}, q_{n_t})^{1-\alpha} \cdot \frac{\min(p_{n_t-1}, q_{n_t-1})^{1-\alpha}}{\min(p_{n_t}, q_{n_t})^{1-\alpha}} \]

\[ > t \cdot \left( 1 - \frac{1}{\min(p_{n_t}, q_{n_t})} \right)^{1-\alpha} \geq t \cdot \left( \frac{2}{3} \right)^{1-\alpha}. \]

Thus, \( \lim_{t \to \infty} \frac{\left| \phi^{(k)}_{P,Q}(x_t) - \phi^{(k)}_{P,Q}(y_t) \right|}{|x_t - y_t|^\alpha} = 1 \) and \( \phi^{(k)}_{P,Q} \) is not Hölder continuous of exponent \( \alpha \).

\[ \text{A nontrivial application of Theorem 2.37 is given in Lemma 4.4.} \]

\[ \text{Corollary 2.38. Suppose that } k \in \mathbb{N}_0 \text{ and } \liminf_{n \to \infty} \min(p_n, q_n) \geq 3. \text{ Then } \phi^{(k)}_{P,Q} \text{ is Lipschitz if} \]

\[ \limsup_{n \to \infty} \frac{p_1 \cdots p_{n+k}}{q_1 \cdots q_n} \cdot \frac{p_{n+k+1}}{\max(1, p_{n+k+1} - q_{n+k+1})} < \infty. \]

\( \phi^{(k)}_{P,Q} \) is not Lipschitz if

\[ \limsup_{n \to \infty} \frac{p_1 \cdots p_n}{q_1 \cdots q_n} = \infty. \]

\[ \text{Theorem 2.39. Suppose that } \mu_1, \mu_2 \in M^w(\mathbb{N}_0^2), \mu_1 \text{ and } \mu_2 \text{ are not positive on } \{2\}. \text{ Put } \mu = \mu_1 \times \mu_2, \text{ let } \alpha \in (0,1), \text{ and suppose that max} \left( \int \log \pi_1(\omega) \ d\mu(\omega), \int \log \pi_2(\omega) \ d\mu(\omega) \right) < \infty. \text{ If } \int \log \pi_2(\omega) \ d\mu(\omega) > \alpha \int \log \pi_1(\omega) \ d\mu(\omega), \text{ then } \phi^{(k)}_{P,Q} \text{ is Hölder continuous of exponent } \alpha \text{ for all } k \geq 0 \text{ for } \mu \text{-almost every } (P,Q) \in \mathbb{N}_0^2 \times \mathbb{N}_0^2. \text{ If } \int \log \pi_2(\omega) \ d\mu(\omega) > \int \log \pi_1(\omega) \ d\mu(\omega), \text{ then } \phi^{(k)}_{P,Q} \text{ is Lipschitz continuous for } \mu \text{-almost every } (P,Q) \in \mathbb{N}_0^2 \times \mathbb{N}_0^2. \]
3. Normal numbers with respect to the Cantor series expansions

3.1. Introduction. Let

\[ Q_n^{(k)} := \sum_{j=1}^{n} \frac{1}{q_jq_{j+1} \cdots q_{j+k-1}} \]  
and \( T_{Q,n}(x) := \left( \prod_{j=1}^{n} q_j \right) x \pmod{1}. \)

A. Rényi [17] defined a real number \( x \) to be normal with respect to \( Q \) if for all blocks \( B \) of length 1,

\[
\lim_{n \to \infty} \frac{N^Q(B,x)}{Q_n} = 1.
\]

If \( q_n = b \) for all \( n \) and we restrict \( B \) to consist of only digits less than \( b \), then (3.1) is equivalent to simple normality in base \( b \), but not equivalent to normality in base \( b \). A basic sequence \( Q \) is \( k \)-divergent if \( \lim_{n \to \infty} Q_n^{(k)} = \infty \). \( Q \) is fully divergent if \( Q \) is \( k \)-divergent for all \( k \) and \( k \)-convergent if it is not \( k \)-divergent. A basic sequence \( Q \) is infinite in limit if \( q_n \to \infty \).

**Definition 3.1.** A real number \( x \) in \([0,1)\) is \( Q \)-normal of order \( k \) if for all blocks \( B \) of length \( k \),

\[
\lim_{n \to \infty} \frac{N^Q(B,x)}{Q_n^{(k)}} = 1.
\]

We let \( N_k(Q) \) be the set of numbers that are \( Q \)-normal of order \( k \). \( x \) is \( Q \)-normal if \( x \in N(Q) := \bigcap_{k=1}^\infty N_k(Q) \). Additionally, \( x \) is simply \( Q \)-normal if it is \( Q \)-normal of order 1. \( x \) is \( Q \)-ratio normal of order \( k \) (here we write \( x \in RN_k(Q) \)) if for all blocks \( B_1 \) and \( B_2 \) of length \( k \),

\[
\lim_{n \to \infty} \frac{N^Q(B_1,x)}{N^Q(B_2,x)} = 1.
\]

\( x \) is \( Q \)-ratio normal if \( x \in RN(Q) := \bigcap_{k=1}^\infty RN_k(Q) \). A real number \( x \) is \( Q \)-distribution normal if the sequence \((T_{Q,n}(x))_{n=0}^\infty\) is uniformly distributed modulo 1. Let \( DN(Q) \) be the set of \( Q \)-distribution normal numbers.

It is easy to show that \( DN(Q) \) is a set of full Lebesgue measure for every basic sequence \( Q \). For \( Q \) that are infinite in limit, it has been shown that \( N_k(Q) \) is of full measure if and only if \( Q \) is \( k \)-divergent [15]. Early work in this direction has been done by A. Rényi [17], T. Šalát [21], and F. Schweiger [18]. Therefore if \( Q \) is infinite in limit, then \( N(Q) \) is of full measure if and only if \( Q \) is fully divergent.

Note that in base \( b \), where \( q_n = b \) for all \( n \), the corresponding notions of \( Q \)-normality, \( Q \)-ratio normality, and \( Q \)-distribution normality are equivalent. This equivalence is fundamental in the study of normality in base \( b \). It is surprising that this equivalence breaks down in the more general context of \( Q \)-Cantor series for general \( Q \).

It is usually most difficult to establish a lack of a containment relationship. The first non-trivial result in this direction was in [2] where a basic sequence \( Q \) and a real number \( x \) is constructed such that \( x \in N(Q) \backslash DN(Q) \). By far the most difficult of these to establish is the existence of a basic sequence \( Q \) where \( RN(Q) \cap DN(Q) \backslash N(Q) \neq \emptyset \). This case will be considered in the next subsection and requires information about the functions \( \psi_{P,Q} \) established in the previous section. Theorem 3.12 provides a significant improvement over the main result of [2] while Theorem 3.13 and Theorem 3.14 provide simpler proofs of known results using information about \( \psi_{P,Q} \). It was proven in [13] that \( \dim_H(DN(Q) \backslash RN_1(Q)) = 1 \) whenever \( Q \) is infinite in limit. It should be noted that most of the relations in Figure 1 are trivially induced by those in Figure 3.

We note the following fundamental fact about \( Q \)-distribution normal numbers that follows directly from a theorem of T. Šalát [22].

---

14 This real number \( x \) satisfies a much stronger condition than not being \( Q \)-distribution normal: \( T_{Q,n}(x) \to 0 \).

15 The original theorem of T. Šalát says: Given a basic sequence \( Q \) and a real number \( x \) with \( Q \)-Cantor series expansion \( x = [x_1 + \sum_{n=1}^{\infty} \frac{E_n}{q_1q_2 \cdots q_n}] \) if \( \lim_{n \to \infty} \frac{1}{N} \sum_{n=1}^{N} \frac{1}{q_n} = 0 \) then \( x \) is \( Q \)-distribution normal iff \( E_n = [\theta_1q_1, \theta_2q_2, \ldots] \) for some uniformly distributed sequence \( \{\theta_n\} \). N. Korobov [14] proved this theorem under the stronger condition that \( Q \) is infinite in limit. For this paper, we will only need to consider the case where \( Q \) is infinite in limit.
Theorem 3.2. Suppose that $Q = (q_n)$ is a basic sequence and $\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \frac{1}{q_n} = 0$. Then $x = E_0E_1E_2 \cdots$ w.r.t. $Q$ is $Q$-distribution normal if and only if $(E_n/q_n)$ is uniformly distributed mod 1.

The following immediate consequence of Theorem 3.3 will be used in this section.

**Theorem 3.3.** Suppose that $Q_1, Q_2, \cdots, Q_j$ are infinite in limit and $\lim_{n \to \infty} N_n^{Q_i}((0), x) = \infty$. Then

$$\psi_j(\mathcal{RN}_k(Q_j)) \subseteq \mathcal{RN}_k(Q_j) \text{ and } \psi_j(\mathcal{RN}(Q)) \subseteq \mathcal{RN}(Q).$$

It should be noted that $\psi_{P,Q}$ does not preserve normality or distribution normality. We will exploit this fact to construct a basic sequence $Q$ and a member of $\mathcal{RN}(Q) \cap \mathcal{DN}(Q) \setminus \mathcal{N}(Q)$. We will start with a basic sequence $P$ and a real number $\eta$ that is $P$-normal. A basic sequence $Q$ will be carefully chosen so that $\psi_{P,Q}(\eta) \in \mathcal{DN}(Q)$, but $\psi_{P,Q}(\eta) \notin \mathcal{N}(Q)$. Thus, we will be “trading” $P$-normality for $Q$-distribution normality. Theorem 3.3 will guarantee that $\psi_{P,Q}(\eta) \in \mathcal{RN}(Q)$.

We should note that not all constructions in the literature of normal numbers are of computable real numbers. For example, the construction by M. W. Sierpinski in [19] is not of a computable real number. V. Becher and S. Figueira modified M. W. Sierpinski’s work to give an example of a computable absolutely normal number in [3]. Since not every basic sequence is computable we face an added difficulty. Moreover, many of the numbers we construct by using Theorem 4.3 are not computable. Thus, we will indicate when a number we construct is computable.

3.2. Explicit construction of a basic sequence $Q$ and a member of $\mathcal{RN}(Q) \cap \mathcal{DN}(Q) \setminus \mathcal{N}(Q)$.

3.2.1. Some results on construction of distribution normal numbers. Given blocks $B$ and $Y$, we let $N(B,Y)$ be the number of occurrences of the block $B$ in the block $Y$. Given a Borel probability measure $\mu$ on $\mathbb{N}_0^\infty$ and $B = (b_1, \cdots, b_k) \in \mathbb{N}_0^k$, we write

$$[B] = \{\omega = (\omega_1, \omega_2, \cdots) \in \mathbb{N}_0^\infty : \omega_j = b_j \forall j \in [1, k]\} \quad \text{and} \quad \mu(B) = \mu([B]).$$

A block of digits $Y$ is $(\epsilon, k, \mu)$-normal if for all blocks $B$ of length $m \leq k$, we have $(1 - \epsilon)|Y|\mu(B) \leq N(B,Y) \leq (1 + \epsilon)|Y|\mu(B)$. Let $\lambda_0$ be any Borel probability measure on $\mathbb{N}_0$ where $\lambda_0(B) = b^{-k}$ for all blocks $B$ of length $k$ in base $b$. A modular friendly family (MFF), $V$, is a sequence of triples $(l_i, b_i, \epsilon_i)_{i=1}^{\infty}$ such that $(l_i)_{i=1}^{\infty}$ and $(b_i)_{i=1}^{\infty}$ are non-decreasing sequences of non-negative integers with $b_i \geq 2$, such that $(\epsilon_i)_{i=1}^{\infty}$ is a decreasing sequence of real numbers in $(0,1)$ with $\lim_{i \to \infty} \epsilon_i = 0$. A sequence $(X_i)_{i=1}^{\infty}$ of $(\epsilon_i, 1, \lambda_0)$-normal blocks of non-decreasing length with $\lim_{i \to \infty} |X_i| = \infty$ is $V$-nice if $\frac{l_i-1}{l_i} \frac{|X_{i+1}|}{|X_i|} = o(1/i)$ and $\frac{1}{l_i} \frac{|X_{i+1}|}{|X_i|} = o(1)$. Set $L_i = |X_1^{i_1} \cdots X_s^{i_s}| = l_1|X_1| + \cdots + l_i|X_i|$, $s_n = b_i$ for $L_i - 1 < n \leq L_i$, $\Gamma(V,X) := (s_n)_{n=1}^{\infty}$, and $\eta(V,X) := \sum_{n=1}^{\infty} \frac{E_n}{s_n \cdots s_n}$, where $(E_1, E_2, \ldots) = X_1^1X_2^1X_3^2 \cdots$.

**Theorem 3.4.** Let $V = ((l_i, b_i, \epsilon_i))_{i=1}^{\infty}$ be an MFF and suppose that $X = (X_i)_{i=1}^{\infty}$ is $V$-nice. Then $\eta(V,X)$ is $\Gamma(V,X)$-distribution normal.

We will modify the construction of a basic sequence $P$ and a real number $x \in \mathcal{N}(P) \setminus \mathcal{DN}(P)$ given by C. Altmare and the author in [2]. Let $b$ be a positive integer. We define $\nu_b \in M(\mathbb{N}_0^\infty)$ as follows. Put

$$\nu_b((j)) = \begin{cases} \frac{1}{2^b-j+1} & \text{if } 0 \leq j \leq b-1 \\ \frac{2^b - j}{2^b} & \text{if } j = b \\ 0 & \text{if } j > b \end{cases}$$

and for a block $B = (b_1, \ldots, b_k)$, put $\nu_B(B) = \prod_{j=1}^{k} \nu_b((b_j))$. Let $b$ and $w$ be positive integers. Let $V_1, V_2, \ldots, V_{b+1}^{\infty}$ be the blocks in base $b+1$ of length $w$ written in lexicographic order. Put

$$V_{b,w} = V_1^{2^bw} \nu_b(V_1)V_2^{2^bw} \nu_b(V_2)\cdots V_{b+1}^{2^bw} \nu_b(V_{b+1}^{\infty}).$$

With these definitions, we may state the following results from [2].

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16 Our statement of Theorem 3.4 and the preceding definitions have been altered to be more concisely stated than they were in [2]. We also removed some unnecessary hypotheses. It was not stated in [2], but it is not difficult to show that the conclusion of Theorem 3.4 may be strengthened to say that $x(V,X) \in \mathcal{DN}(Q) \cap \mathcal{N}_1(Q)$ by using the main theorem in [14].
Theorem 3.5. For $i \leq 5$, let $X_i = (0,1)$, $b_i = 2$, and $l_i = 0$. For $i \geq 6$, let $X_i = V_{i,2^i}$, $b_i = 2^i$, and $l_i = 2^{4i^2}$. If $V = ((l_i,b_i,\epsilon_i))_{i=1}^{\infty}$ and $X = (X_i)_{i=1}^{\infty}$, then $\eta(V,X) \in N(\Gamma(V,X)) \setminus DN(\Gamma(V,X))$. Moreover, $\lim_{n \to \infty} T_{\Gamma(V,X),n}(\eta(V,X)) = 0$.

3.2.2. The Construction. We need the following lemma from [2].

Lemma 3.6. If $b$ and $w$ are positive integers, then $|W_{b,w}| = w^{2bw}$.

Let $W_i = V_{i,2^i} = (w_{i,t})$, so $|W_i| = i^22^{3i}$ when $i \geq 2$. We first need to define sequences $Q_i = (q_{i,t})_{i=1}^{\infty}$ as follows. Let $1 \leq t \leq |W_i|$. If $w_{i,t} \in \{0,1,\ldots,i-1\}$, set $q_{i,t} = i$. Thus there are

$$i^22^{3i} - \frac{i}{2^i}, i^22^{3i} - i3^32^{3i-1}$$

remaining values $q_{i,t}$ to assign where $w_{i,t} = i$. Since $i((i^22^{3i} - i3^32^{3i-1})$, we may portion these into $i$ classes of $i^22^{3i} - i3^32^{3i-1}$ elements. In the first of these, we let $q_{i,t} = i^3$. If $2 \leq j \leq i$, then we set $q_{i,t} = \lfloor i^2/(j-1) \rfloor$ if $q_{i,t}$ is in the $j$’th grouping. Set $Y_i = (y_{i,t})_{i=1}^{\infty}$, where

$$y_{i,t} = \begin{cases} 0 & \text{if } w_{i,t} = 0 \text{ or } (w_{i,t} = i \text{ and } q_{i,t} = i^3) \\ \alpha & \text{if } w_{i,t} = \alpha \text{ or } (w_{i,t} = i \text{ and } q_{i,t} = \lfloor i^2/\alpha \rfloor) \end{cases}.$$

We note the following lemma which follows immediately from construction.

Lemma 3.7.

$$N((t),W_i) = \begin{cases} \frac{1}{2}|W_i| & \text{if } 0 \leq t \leq i - 1 \\ \frac{2^i-1}{2^i}|W_i| & \text{if } t = i \\ 0 & \text{if } t > i \end{cases} = \begin{cases} i^22^{3i-1} & \text{if } 0 \leq t \leq i - 1 \\ i^22^{3i} & \text{if } t = i \\ 0 & \text{if } t > i \end{cases};$$

$$|\{n : E_{i,n} = t < i\}| = i^22^{3i-1};$$

$$|\{n : E_{i,n} = i \text{ and } q_{i,n} = i^3\}| = i^22^{3i} - i2^23^{3i-1};$$

$$|\{n : E_{i,n} = i \text{ and } q_{i,n} = \lfloor i^2/\alpha \rfloor\}| = i^22^{3i} - i2^23^{3i-1}.$$

Lemma 3.8. $|Y_i| = |W_i| = i^22^{3i}$ and $Y_i$ is $(0,1,\lambda_i)$-normal.

Proof. $|Y_i| = |W_i|$ follows immediately by construction. Let $j \in \{0,\ldots,i-1\}$. By Lemma 3.7, $N(Y_i,(j)) = i^22^{3i} + (i2^3 - i2^23^{3i-1}) = i2^3 = \frac{1}{7}|Y_i|$. \(\square\)

Lemma 3.9. For $i \leq 5$, let $X_i = (0,1)$, $b_i = 2$, and $l_i = 0$. For $i \geq 6$, let $X_i = Y_i$, $b_i = i$, and $l_i = 2^{4i^2}$. Put $V = ((l_i,b_i,\epsilon_i))_{i=1}^{\infty}$ and $X = (X_i)_{i=1}^{\infty}$. Then $\eta(V,X)$ is $\Gamma(V,X)$-distribution normal.

Proof. This follows immediately from Theorem 3.4 and Lemma 3.8. \(\square\)

For the remainder of Section 3.2, we will define $P$ to be the basic sequence constructed in Theorem 3.5 and refer to the number constructed in the same theorem as $\zeta$. We also refer to the number constructed in Lemma 3.9 as $\kappa$ and the basic sequence as $K = (k_n)$. We will write $\kappa = 0.F_1F_2\cdots$ w.r.t. $K$. Clearly, the sequence $(\alpha_n) = (T_{K,n}(\kappa))$ is uniformly distributed mod 1 since $\kappa \in DN(K)$. We will construct a basic sequence $Q$ such that $\beta_n = (T_{Q,n}(\psi_{P,Q}(\zeta)))$ has the property that $\beta_n - \alpha_n \to 0$. This will establish that $\psi_{P,Q}(\zeta)$ is in $\mathbb{DN}(Q)$. Additionally, we will show that for our choice of $Q$, we will have $\psi_{P,Q}(\zeta) \in \mathbb{RN}(Q) \setminus \mathbb{DN}(Q)$. Let $\Delta_{t,i} = \lfloor \frac{1}{2^i}, \frac{t+1}{2^i} \rfloor$.

Lemma 3.10. $\frac{i}{\lfloor i^2/\alpha \rfloor} \in \Delta_{t,i}$.\footnote{This number $\zeta$ has many pathological properties and is a reasonable starting place for constructing counterexamples. A well known property of normal numbers in base $b$ is that $x$ is normal in base $b$ if and only if $rx$ is normal in base $b$ for all rational numbers $r$. It is not difficult to see that $P$-normality is not even preserved by integer multiplication. That is $\zeta$ has the property that $n\zeta$ is not $P$-normal for every integer $n \geq 2$.}
Proof. First, we note that \( \frac{i}{[i^2/\alpha]} \geq \frac{i}{i^2/\alpha} = \frac{\alpha}{i}. \) In order to finish the proof, we need to show that

\[
\frac{i}{[i^2/\alpha]} < \frac{\alpha+1}{i}.
\]

We see that

\[
\frac{i}{[i^2/\alpha]} \leq \frac{i}{\frac{t^2}{\alpha} - 1} = \frac{\alpha i}{i^2 - \alpha},
\]

so

\[
\frac{\alpha+1}{i} - \frac{i}{[i^2/\alpha]} \geq \frac{\alpha+1}{i} - \frac{\alpha i}{i^2 - \alpha} = \frac{i^2 - \alpha^2 - \alpha}{i(i^2 - \alpha)} \geq \frac{i^2 - (i-1)^2 - (i-1)}{i(i^2 - 0)} = \frac{1}{i^2} > 0,
\]

establishing (3.2). \( \square \)

**Theorem 3.11.** Put \( Q = Q^{l_6}_{2}Q^{l_7}_{2}Q^{l_8}_{2} \cdots \), where \( l_i = 2^4 i^2 \). Then \( Q \) is infinite in limit and fully divergent and \( \psi_{P,Q}(\zeta) \in \mathcal{R}_{N}(Q) \cap \mathcal{D}_{N}(Q) \setminus \mathcal{N}(Q) \).

**Proof.** For \( n \in \mathbb{N}, \) let \( i = i(n) \) be the unique integer such that \( l_6|X_6| + \cdots + l_i|X_{i-1}| < n \leq l_i|X_i| + \cdots + l_i|X_i| \). Note that by Lemma 3.10, \( \frac{E_q}{n_I} \in \Delta_{\alpha,i(n)} \) if and only if \( \frac{E_q}{n_I} \in \Delta_{\alpha,i(n)} \). Thus,

\[
|T_{Q,n}(\psi_{P,Q}(\zeta)) - T_{Q,n}(\kappa)| < \frac{E_{n+1}}{q_{n+1}} + \frac{F_{n+1}}{k_{n+1}} + \frac{1}{q_{n+1}} + \frac{1}{k_{n+1}} \leq \frac{1}{i(n+1)} + \frac{1}{q_{n+1}} + \frac{1}{k_{n+1}} \to 0.
\]

Since the sequence \( (T_{K,n}(\kappa)) \) is uniformly distributed mod 1, we may conclude that \( (T_{Q,n}(\psi_{P,Q}(\zeta))) \) is uniformly distributed mod 1. Thus, \( \psi_{P,Q}(\zeta) \in \mathcal{D}_{N}(Q) \). \( \psi_{P,Q}(\zeta) \in \mathcal{R}_{N}(Q) \) follows directly from Theorem 3.3 as \( \zeta \in \mathcal{N}(P) \subseteq \mathcal{R}_{N}(P) \).

Let \( k \) be a positive integer and suppose that \( B \) is a block of length \( k \). We note that for large enough \( n \), \( q_n \leq (\log_2 p_n)^3 \), so \( \lim_{n \to \infty} \frac{p_n}{q_n} = \infty \). Thus, by Theorem 1.3,

\[
\lim_{n \to \infty} \frac{N_n^{(k)}(\psi_{P,Q}(\eta))}{q_n^{(k)}} = \lim_{n \to \infty} \left( \frac{N_n^{P}(\eta) + O(1)}{F_n^{(k)}} \cdot \frac{p_n^{(k)}}{q_n^{(k)}} \right) = 1 \cdot \infty = \infty,
\]

so \( \psi_{P,Q}(\zeta) \notin \mathcal{N}(Q) \). \( Q \) is fully divergent because \( \lim_{n \to \infty} \frac{p_n}{q_n} = \infty \) for all \( k \). \( \square \)

Using Theorem 2.2 it is not difficult to show that \( \dim_{H^*}(\psi_{P,Q}(\mathbb{R})) = 1 \). In fact, we can say even more about \( \psi_{P,Q}(\mathbb{R}) \). Since \( 2^t > t^3 \) for positive integers \( t \) if and only if \( t \geq 10 \), we can show that the Lebesgue measure of \( \psi_{P,Q}(\mathbb{R}) \) is positive: \footnotesize{\(18\)}

\[
\lambda(\psi_{P,Q}(\mathbb{R})) = \prod_{n=1}^\infty \frac{\min(p_n,q_n)}{q_n} = \prod_{t=6}^9 \left( \frac{2^t}{t^3} \right) \cdot \left\{ n : \text{E}_{t,n} = t \text{ and } q_{t,n} = t^3 \right\} \approx 10^{-1.3095 \times 10^{317}} > 0.
\]

Of course, this number is so small that our approximation doesn’t even estimate \( \lambda(\psi_{P,Q}(\mathbb{R})) \) within \( 10^{310} \) orders of magnitude!

3.2.3. **Further Steps.** It should be emphasized that Theorem 3.11 only gives only one example of a basic sequence \( Q \) where \( \mathcal{R}_{N}(Q) \cap \mathcal{D}_{N}(Q) \setminus \mathcal{N}(Q) \neq \emptyset \). It is likely that \( \mathcal{R}_{N}(Q) \cap \mathcal{D}_{N}(Q) \setminus \mathcal{N}(Q) \neq \emptyset \) for every basic sequence \( Q \) that is infinite in limit and fully divergent. The construction in this section makes heavy use of the number \( \zeta \) and estimates pertaining to it from [2] to greatly simplify the proof. It remains to be seen if the methods introduced in this section generalize well to show that \( \mathcal{R}_{N}(Q) \cap \mathcal{D}_{N}(Q) \setminus \mathcal{N}(Q) \) is always non-empty. Moreover, it is likely that \( \dim_{H}(\mathcal{R}_{N}(Q) \cap \mathcal{D}_{N}(Q) \setminus \mathcal{N}(Q)) = 1 \), but it doesn’t seem obvious how this would be proven.

\footnotesize{\(18\)}This approximation is easily obtained by estimating \( \log \lambda \left( \psi_{P,Q}(\mathbb{R}) \right) \).
3.3. The sets $N(Q) \setminus DN(Q)$, $RN(Q) \setminus N(Q)$, and $DN(Q) \setminus RN(Q)$ are always non-empty. In [2], a computable real number $x$ and a computable basic sequence $Q$ were constructed where $x \in N(Q)$, but $T_{Q,n}(x) \to 0$. Unfortunately, the approach taken can only be easily extended to a very restrictive class of basic sequences and the proof and construction require some work. We essentially trivialize the problem of showing that $N(Q) \setminus DN(Q) \neq \emptyset$ with the theory developed in Section 2. The approach used in this subsection is not only simpler, but far stronger than the approach in [2]. Examples of computable members of $DN(Q) \setminus RN(Q)$ are given in [14] for certain classes of computable basic sequences $Q$.

**Theorem 3.12.** Suppose that $Q$ is infinite in limit and fully divergent. Then $N(Q) \setminus DN(Q) \neq \emptyset$.

**Proof.** Let $p_n = \max(\lfloor \log q_n \rfloor, 2)$ and set $P = (p_n)$. By the main theorem of [13], $N(Q) \neq \emptyset$, so let $x \in N(Q)$ and put $y = (\psi_{P,Q} \circ \psi_{Q,P}) (x)$. Then $y$ is $Q$-normal by Theorem 1.3 but $T_{Q,n}(y) \to 0$, so $y$ is not $Q$-distribution normal.

**Theorem 3.13.** If $Q$ is infinite in limit, then $RN(Q) \setminus \bigcup_{k=1}^{\infty} N_k(Q) \neq \emptyset$, so $RN(Q) \setminus N(Q) \neq \emptyset$.

**Proof.** Let $Q$ be $k$-convergent for some $k$, then $N(Q) = \emptyset$, but $RN(Q) \neq \emptyset$ by Proposition 5.1 and Proposition 5.2 in [15]. So suppose that $Q$ is fully divergent. Let $p_n = \max(\lfloor q_n/2 \rfloor, 2)$ and set $P = (p_n)$. Clearly, $P$ is fully divergent. Let $x \in N(P)$ and set $y = \phi^{(k)}_{P,Q}(x)$. Let $k$ be a positive integer and suppose that $B_1$ and $B_2$ are blocks of length $k$. Then by Theorem 1.3

\[
\lim_{n \to \infty} \frac{N^Q_n(B_1,y)}{N^Q_n(B_2,y)} = \lim_{n \to \infty} \frac{N^P_n(B_1,x) + O(1)}{N^P_n(B_2,x) + O(1)} = \lim_{n \to \infty} \frac{N^P_n(B_1,x)/P^{(k)}_n + O(1)}{N^P_n(B_2,x)/P^{(k)}_n + O(1)} = 1.
\]

Thus, $y \in RN(Q)$. Now, suppose that $B$ is some block of length $k$. Then, applying Lemma 2.7 by letting $a_j = p_jp_{j+1}\cdots p_{j+k-1}$ and $b_j = q_jq_{j+1}\cdots q_{j+k-1}$

\[
\lim_{n \to \infty} \frac{N^Q_n(B,y)}{Q^{(k)}_n} = \lim_{n \to \infty} \left( \frac{N^Q_n(B,y)}{P^{(k)}_n} \cdot \frac{P^{(k)}_n}{P^{(k)}_n} \right) = \lim_{n \to \infty} \frac{N^P_n(B,x) + O(1)}{P^{(k)}_n} \cdot \lim_{n \to \infty} \frac{P^{(k)}_n}{Q^{(k)}_n} = 1 \cdot 2^{-k} \neq 1.
\]

So, $y \notin N_k(Q)$ for all $k$. Thus, $RN(Q) \setminus N(Q) \neq \emptyset$.

Using different methods than those used in this paper, it was shown in [13] that $\dim_H(DN(Q) \setminus RN(Q)) = 1$. While the methods of this paper appear to be unable to prove a result that strong, we can still provide an alternate proof that $DN(Q) \setminus RN(Q) \neq \emptyset$.

**Theorem 3.14.** If $Q$ is infinite in limit, then $DN(Q) \setminus RN(Q) \neq \emptyset$.

**Proof.** Let $x \in DN(Q)$ and set $p_n = q_n - 1$. Put $y = (\psi_{P,Q} \circ \psi_{Q,P}) (x) + \sum_{n=1}^{\infty} \frac{1}{q_1\cdots q_n}$. Then the digit $0$ never appears in the $Q$-Cantor series expansion of $y$, so $y \notin RN_1(Q) \supseteq RN(Q)$. We note that $|T_{Q,n-1}(x) - T_{Q,n-1}(y)| \leq \frac{1}{q_n} \to 0$, so the sequence $(T_{Q,n}(y))$ is uniformly distributed mod 1. Thus, $y \in DN(Q) \setminus RN(Q)$.\]

We will use a pair of basic sequences similar to those from Theorem 3.14 in Section 4.2 to sharpen some results on the Hausdorff dimension of $DN(Q) \setminus RN_1(Q)$.

4. The Hausdorff Dimension of some sets

4.1. Refinement of a result concerning Hausdorff dimension. For any sequence $X = (x_n)$ of real numbers, let $\mathcal{A}(X)$ denote the set of accumulation points of $X$. Given a set $D \subseteq [0,1]$, let

\[\mathbb{E}_D(Q) = \{ x = 0.E_1E_2\cdots \text{ w.r.t. } Q : \mathcal{A}(E_n/q_n) = D \} .\]

The following results are proven by Y. Wang, Z. Wen, and L. Xi in [25].

**Theorem 4.1.** If $Q$ is infinite in limit, then $\dim_H(\mathbb{E}_D(Q)) = 1$ for every closed set $D$.

**Corollary 4.2.** Given $0 \leq \delta \leq 1$, let

\[\mathbb{E}_\delta(Q) = \mathbb{E}_{(\delta)}(Q) = \{ x = 0.E_1E_2\cdots \text{ w.r.t. } Q : \lim_{n \to \infty} \frac{E_n}{q_n} = \delta \} .\]

If $Q$ is infinite in limit, then $\dim_H(\mathbb{E}_\delta(Q)) = 1$.\]
For a set $D \subseteq [0, 1]$ and sequence of non-negative integers $(t_n)$, let
\[
\mathbb{E}_{D,(t_n)}(Q) = \mathbb{E}_{D}(Q) \cap \mathbb{R}(q_n-t_n)(Q)
\]
\[
= \{ x = 0, E_1E_2 \cdots \text{ w.r.t. } Q : \mathbb{A}((E_n/q_n)) = D \text{ and } \forall n \ E_n < q_n - t_n \}. 
\]

**Lemma 4.3.** If $\lim_{n \to \infty} \frac{\log q_{n+j+1}}{\log q_1 \cdots q_n} = 0$ for all $j \geq 0$, then $\lim_{n \to \infty} \frac{\log q_{n+j+1} \cdots q_{n+k+1}}{\log q_1 \cdots q_n} = 0$ for all $k \geq 0$.

**Proof.** This follows immediately as $\frac{\log q_{n+1} \cdots q_{n+k+1}}{\log q_1 \cdots q_n} = \frac{\log q_{n+1}}{\log q_1 \cdots q_n} + \cdots + \frac{\log q_{n+k+1}}{\log q_1 \cdots q_n}$.

**Lemma 4.4.** If $Q$ is infinite in limit, $q_n \geq p_n$, $\lim_{n \to \infty} \frac{\log q_{n+1} \cdots q_{n+k+1}}{\log q_1 \cdots q_n} = 0$ for all $j \geq 0$, $\sum_{n=1}^{\infty} \frac{q_n - p_n}{q_n} = 0$, and $\min(p_n, q_n) \geq 3$ for all $n$, then for all $k \geq 0$, $(\phi^{(k)}_{Q,P})^{-1} = \phi^{(k)}_{Q,P}$ is Hölder continuous of exponent $\alpha$ for all $\alpha \in (0, 1)$.

**Proof.** Let $\alpha \in (0, 1)$ and $k \geq 0$. First, we note that
\[
\sum_{j=1}^{\infty} \frac{q_n - p_n}{q_n} < \infty. \text{ Since } \lim_{n \to \infty} \frac{\log q_{n+1} \cdots q_{n+k+1}}{\log q_1 \cdots q_n} = 0 \text{ by Lemma 4.3 and } \lim_{n \to \infty} \frac{q_n - p_n}{q_n} = 0, \text{ we know that } \lim_{n \to \infty} \frac{q_n - p_n}{q_n} = 0, \text{ so }
\]
\[
\lim_{n \to \infty} \frac{q_n+1 \cdots q_{n+k+1}}{p_1 \cdots p_n} = 0.
\]
To verify (2.16)

\[
\limsup_{n \to \infty} \left( \frac{q_1 \cdots q_n}{p_1 \cdots p_n} \right)^{\alpha} \cdot \left( \frac{q_{n+1} \cdots q_{n+k+1}}{p_{n+1}} \right)^{\alpha} \leq \limsup_{n \to \infty} \left( \frac{q_1 \cdots q_{n+k+1}}{p_1 \cdots p_n} \right)^{\alpha} = 0,
\]
by (4.1) and (4.2), verifying (2.16). We can use similar methods to prove that
\[
\limsup_{n \to \infty} \left( \frac{q_1 \cdots q_n}{p_1 \cdots p_n} \right)^{\alpha} \cdot q_{n+1}^{1-\alpha} < \infty,
\]
verifying (2.15).

We prove the following refinement of Theorem 4.1.

**Theorem 4.5.** Suppose that $D \subseteq (0, 1)$ is a closed set and $(t_n)$ is a sequence of non-negative integers. If $Q$ is infinite in limit, $\lim_{n \to \infty} \frac{\log q_{n+j+1}}{\log q_1 \cdots q_n} = 0$ for all $j \in \mathbb{N}$, $\sum_{n=1}^{\infty} \frac{t_n}{q_n} = 0$, and $q_n - t_n \geq 3$ for all $n$, then $\dim H(\mathbb{E}_{D,(t_n)}(Q)) = 1$.

**Proof.** Let $p_n = q_n - t_n$. We will show that
\[
\mathbb{E}_{D,(t_n)}(Q) = \psi_{P,Q}(\mathbb{E}_{D}(P)).
\]
Let $x = \sum_{n=1}^{\infty} \frac{E_n}{q_1q_2 \cdots q_n} \in \psi_{P,Q}(\mathbb{E}_{D}(P))$ and $y \in D$. Thus, for all $\epsilon > 0$, there exists $n$ such that $\left| \frac{E_n}{q_n} - y \right| < \epsilon$. Note that
\[
\left| \frac{E_n}{q_n} - y \right| \leq \left| \frac{E_n}{q_n} - \frac{E_n}{q_n - t_n} \right| + \left| \frac{E_n}{q_n - t_n} - y \right| < \frac{E_n t_n}{q_n(q_n - t_n)} + \epsilon < \frac{t_n}{q_n} + \epsilon.
\]
Since $\sum t_n/q_n < \infty$, we know that $\frac{t_n}{q_n} \to 0$, so $x \in \mathbb{E}_{D,(t_n)}(Q)$. The proof that $\mathbb{E}_{D,(t_n)}(Q) \subseteq \psi_{P,Q}(\mathbb{E}_{D}(P))$ is similar, so (4.3) holds.
We note that $E_D(P) \subseteq \bigcup_{k=0}^{\infty} \sim_{P,Q}^{(k)}$ since $\rho_{P,Q}(x) < \infty$ for all $x \in E_D(P)$ as 0 and 1 are not members of $D$. Similarly, $E_D,(t_\alpha)(Q) \subseteq \bigcup_{k=0}^{\infty} \sim_{P,Q}^{(k)}$. Put $A_k = E_D(P) \cap \sim_{P,Q}^{(k)}$ and $B_k = E_D,(t_\alpha)(Q) \cap \sim_{P,Q}^{(k)}$, so that $E_D(P) = \bigcup_{k=0}^{\infty} A_k$ and $E_D,(t_\alpha)(Q) = \bigcup_{k=0}^{\infty} B_k$. Thus,

\[
\text{dim}_H(\varepsilon_D(P)) = \sup \text{dim}_H(A_k) \text{ and } \text{dim}_H(\varepsilon_D,(t_\alpha)(Q)) = \sup \text{dim}_H(B_k).
\]

By Theorem 4.1 and (4.4), $\sup \text{dim}_H(A_k) = 1$. Let $k \geq 0$. Next, we note that $\phi_{P,Q}^{(k)}(B_k) = A_k$ by (4.3). Thus, by Lemma 4.4, $\text{dim}_H(A_k) \leq \frac{1}{\alpha} \text{dim}_H(B_k)$ for all $\alpha \in (0, 1)$, so $\text{dim}_H(A_k) \geq \text{dim}_H(B_k)$. But then $\sup \text{dim}_H(B_k) \geq \sup \text{dim}_H(A_k) = 1$, so $\sup \text{dim}_H(B_k) = 1$. Thus, $\text{dim}_H(\varepsilon_D,(t_\alpha)(Q)) = 1$ by (4.4).

4.2. The Hausdorff dimension of $(\mathcal{DN}(Q) \setminus \mathcal{RN}(Q)) \cap \mathcal{R}(t_\alpha)$.

**Definition 4.6.** Let $P = (p_n)$ and $Q = (q_n)$ be basic sequences. We say that $P \sim Q$ if $q_n = \prod_{j=1}^{s} p_{n(j-1)+j}$. The following theorem was proven in [13].

**Theorem 4.7.** Suppose that $(Q_j)_{j=1}^{\infty}$ is a sequence of basic sequences that are infinite in limit. Then

\[
\text{dim}_H\left(\bigcap_{j=1}^{\infty} \mathcal{DN}(Q_j) \setminus \mathcal{RN}_1(Q_j)\right) = 1
\]

if either

1. $Q_j$ is 1-convergent for all $j$
2. $Q_1$ is 1-divergent and there exists some basic sequence $S = (s_n)$ with $Q_1 \sim s_1$, $Q_2 \sim s_2$, $Q_3 \sim s_3$, $Q_4 \cdots$.

The following may be proven similarly to Theorem 4.5.

**Theorem 4.8.** Suppose that $(t_\alpha)$ is a sequence of non-negative integers, $(Q_j)_{j=1}^{\infty}$ is a sequence of basic sequences that are infinite in limit, $Q_1 = (q_n)$ is 1-divergent, there exists some basic sequence $S = (s_n)$ with $Q_1 \sim s_1$, $Q_2 \sim s_2$, $Q_3 \sim s_3$, $Q_4 \cdots$, $\sum_{n=1}^{\infty} \frac{t_n}{q_n} = 0$, and $q_n - t_n \geq 3$ for all $n$. Then

\[
\text{dim}_H\left(\mathcal{R}(q_n-t_n) \cap \bigcap_{j=1}^{\infty} \mathcal{DN}(Q_j) \setminus \mathcal{RN}_1(Q_j)\right) = 1.
\]

4.3. The sets $\sim_{P,Q}^{(k)}$. It seems to be difficult to compute the exact Hausdorff dimension of any of the sets $\sim_{P,Q}^{(k)}$, $\bigcup_k \sim_{P,Q}^{(k)}$, or $\sim_{P,Q}^{(\infty)}$. It is likely that an extension of [13] would provide a solution to this problem, but this is beyond the scope of the current paper. However, the following is easily seen to follow from Theorem 2.1.

**Theorem 4.9.** Suppose that $\lim_{n \to \infty} \frac{\log p_n}{\log p_1 \cdots p_n} = 0$. Then for $k \geq 0$

\[
\text{dim}_H\left(\mathcal{N}_{P,Q}^{(\infty)}\right) = \liminf_{n \to \infty} \frac{\log \prod_{j=1}^{n} \min(p_j, q_j)}{\log \prod_{j=1}^{n} p_j} \geq \sup \text{dim}_H\left(\mathcal{N}_{P,Q}^{(j)}\right) = \text{dim}_H\left(\bigcup_{j=0}^{\infty} \mathcal{N}_{P,Q}^{(j)}\right) \geq \text{dim}_H\left(\mathcal{N}_{P,Q}^{(k)}\right) \geq \text{dim}_H\left(\mathcal{N}_{P,Q}^{(0)}\right) = \liminf_{n \to \infty} \frac{\log \prod_{j=1}^{n} \min(p_j-2, q_j-1)}{\log \prod_{j=1}^{n} p_j}.
\]

**Theorem 4.10.** Suppose that $\mu_1, \mu_2 \in M(N_0^N)$ and $\mu_1$ and $\mu_2$ are not positive on $\{2\}$. Put $\mu = \mu_1 \times \mu_2$ and suppose that $\max\left(\int \log \pi_1(\omega) \ d\mu(\omega), \int \log \pi_2(\omega) \ d\mu(\omega)\right) < \infty$. Then for all $k \in N_0$ and $\mu$-almost every $(P, Q) \in N_0^N \times N_0^N$

\[
\frac{\int \log \min(\pi_1(\omega) - 2, \pi_2(\omega) - 1) \ d\mu(\omega)}{\int \log \pi_1(\omega) \ d\mu(\omega)} \leq \text{dim}_H\left(\mathcal{N}_{P,Q}^{(k)}\right) \leq \frac{\int \log \min(\pi_1(\omega), \pi_2(\omega)) \ d\mu(\omega)}{\int \log \pi_1(\omega) \ d\mu(\omega)}.
\]
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