A NEW APPROACH TO NONCOMPACT LATTICE QED
WITH LIGHT FERMIONS.

V. Azcoiti
Departamento de Física Teórica, Facultad de Ciencias, Universidad de Zaragoza,
50009 Zaragoza (Spain)

G. Di Carlo and A.F. Grillo
Istituto Nazionale di Fisica Nucleare, Laboratori Nazionali di Frascati,
P.O.B. 13 - Frascati (Italy).

ABSTRACT

We discuss detailed simulations of the non compact abelian model
coupled to light fermions, using a method previously developed that
includes the effects of the fermionic interactions in an effective action. The
approximations involved are related to an expansion in the flavour number.
We address the problem of the (non) triviality of the theory through a
study of the analytical properties of the effective action as a function of
the pure gauge energy. New numerical results for the plaquette energy,
chiral condensate and a qualitative analysis of the phase diagram are also
presented.
I. Introduction

The study of Quantum Electrodynamics on the lattice, i.e. the theory of fermions coupled to $\mathcal{R}$-valued gauge fields, derives its interest from many reasons. The obvious one is that its continuum limit (if existing), might describe standard QED, which is the most succesfull theory at (low energy) perturbative level.

From a more speculative point of view, this theory presents a challenge to the wisdom that only asymptotically free theories are non-trivial in four dimensions, i.e. the theory can be defined in the infinite cutoff limit without forcing all the renormalized coupling constants to zero.

The question is then posed as whether the renormalized theory obtained in the limit of infinite cutoff of the regularized theory is non trivial.

In recent years, many efforts have been devoted to the study of this problem, in the context of lattice regularization of the model.

Its compact formulation, not possessing a second order phase transition in the bare coupling constant [1,2], is not suitable to define a renormalized continuum theory.

From this point of view, the non compact model is much more interesting; the first numerical investigations of the model, in the quenched approximation [3], have shown the existence of a continuous chiral transition at finite value of the coupling constant. This transition survives after the inclusion of dynamical fermions [4-7] so suggesting that the quantum continuum physics could be reached there. Moreover, it was believed that the non compact regularization of the abelian model is in some sense more nearby the continuum formulation than the compact one.

The theory defined at the critical point as the limit from the broken phase is interesting by itself, being a theory of strongly interacting fermions, with spontaneously broken chiral symmetry[8,9]. In this phase the chiral condensate $<\bar{\psi}\psi>$ is different from zero in the massless limit.

The interest in the first numerical simulations of the non compact QED derived from the approximate solution of the Schwinger-Dyson equations in the quenched ladder approximation, and the consequent prediction of the existence of a chiral transition, with an essential singularity scaling law (Miransky scaling [4,10]), with the aim of testing this result outside the approximations.

Later, various groups have carried out extensive numerical simulations of this model [4-8,11-15] specifically to determine the critical exponents and characterize in this way the nature of the continuum limit, particularly concerning the issue of the triviality.

The actual situation concerning the determination of the critical exponents, can be summarized in the following way [14,15]:

i. Miransky scaling has been disproved, also in the context of
ii. Sensible measurements of the critical exponents are extremely difficult, due to the smallness of the scaling window in which the critical behaviour can be observed. The only relevant results have been obtained in the quenched approximation, for large lattices and small fermion masses. These results contradict the gaussian character of the fixed point, at least in the quenched approximation.

An alternative procedure to assess the triviality of the fixed point consists in the study of the dependence of the renormalized coupling constant on the cut-off. In a recent paper [17] we developed such an analysis, based on general arguments of block spin Renormalization Group approach and the use of a method proposed by us [2] to include the effect of dynamical fermions in numerical simulations of gauge theories.

The aim of this paper is to clarify as much as possible the arguments and results presented in [17], discussing in detail the fundamental characteristics of our simulation, the dependence of the results on the flavour number and fermionic mass and at the same time to present new results for the plaquette energy and the chiral condensate.

Section II is dedicated to a detailed presentation of our method based on the definition of an effective fermionic action. We establish a connection between the functional dependence of the effective fermionic action on the pure gauge energy and the (non) triviality of the fixed point, and show how a second order phase transition reflects itself on a non-analyticity of the effective action.

In Section III we report our results for the effective action, introducing an expansion in the flavour number, and compare our numerical results with the analytical predictions developed in the previous section. Sections IV and V contain our results for the plaquette energy and chiral condensate for various values of the fermionic mass and flavour numbers. In Section VI we develop a qualitative analysis of the phase diagram of the model, while in Section VII we discuss the evaluation of critical indices in the frame of the Equation of State (EOS) approach. Section VIII contains our conclusions.

II. The effective fermionic action and triviality

Addressing the problem of triviality in non compact lattice QED through the determination of the critical exponents is technically a very difficult problem: the scaling region is very small and consequently very large lattices are needed to approach the critical point and characterize the critical behaviour. A complete discussion of this is contained in [14,15].

On the other hand in [14] it has been shown that, in the quenched approximation, critical exponents are definitely distinct from the ones
computed in the Mean Field Theory; it is however not clear whether this result changes with the inclusion of Dynamical Fermions.

Alternatively, triviality can be studied by computing the renormalized coupling constant as a function of the cutoff (or equivalently, as a function of the bare couplings). If the renormalized coupling constant becomes zero when removing the cutoff the corresponding fixed point is gaussian.

Two important examples of gaussian fixed point are $\lambda \phi^4$ theory in 4 dimensions and QCD. The fundamental difference between these two cases is that in the first case it is believed that the renormalized coupling constant becomes zero at a critical value of the cutoff, whereas in QCD $\alpha_R$ is zero only in the infinite cutoff limit. Using the terminology of the Renormalization Group approach, one can say that in $\lambda \phi^4$, $\lambda$ is an irrelevant coupling, whereas in QCD $\alpha$ is relevant.

From a physical point of view, the main difference between these two models is that in the first case ($\lambda \phi^4$) it is not possible to define a quantum continuum limit at the gaussian fixed point which is interacting, whereas in the QCD case this is indeed possible.

Gaussian fixed points are easier to study than non-gaussian ones, since in the first case one can, in a neighbourhood of the fixed point, perturbatively compute the Callan-Symanzik $\beta$ function, which in turn allows to follow the evolution of the renormalized coupling constant as a function of the cutoff.

On the other hand, in Quantum Electrodynamics (QED) and in perturbation theory, the renormalized coupling constant is zero, independently on the value of the bare coupling. This means that the gaussian fixed point of this model is trivial. Hence the only possibility of defining a quantum non trivial continuum limit in this model is that it possesses a non gaussian fixed point, where weak coupling arguments do not apply.

The use of non perturbative techniques is therefore essential for understanding the nature of the continuum limit of the abelian gauge theory. It is, however, extremely difficult from the numerical point of view to analyse the fixed point from the study of the critical exponents [14]; instead, we have performed an analysis of this model based on an indirect study of the evolution of the renormalized coupling constant.

In this section we will introduce the essential theoretical ideas [17] at the basis of the numerical results, which will be presented and discussed in the next sections.

Consider the action of non compact lattice Abelian model coupled to staggered fermions
\[ S = \frac{1}{2} \sum_{x, \mu} \eta_\mu(x) \bar{\chi}(x) \{ U_\mu(x) \chi(x + \mu) - U_\mu^*(x - \mu) \chi(x - \mu) \} + \\
m \sum_x \bar{\chi}(x) \chi(x) + \frac{\beta}{2} \sum_{x, \mu < \nu} F_{\mu \nu}^2(x) \]

\[ F_{\mu \nu}(x) = A_\mu(x) + A_\nu(x + \hat{\mu}) - A_\mu(x + \hat{\nu}) - A_\nu(x) \]

where \( \beta = 1/e^2 \) and the fermions are coupled to the fields \( A_\mu(x) \) through the compact link variable \( U_\mu(x) = e^{i A_\mu(x)} \); the corresponding partition function is

\[ Z = \int [d\chi][d\bar{\chi}] [dA_\mu(x)] e^{-S} = \int [dA_\mu(x)] \det \Delta(m, A_\mu(x)) e^{-S_G} \]

The main steps of our analysis are:

i. Determination of an effective fermionic action as a function of the pure gauge energy, by integrating out all the other operators of which the effective action is function.

ii. The existence of a phase transition is monitored through the appearance of a non analytic behaviour of the effective fermionic action.

iii. We then establish a relation between the effective action as defined in i) and an effective renormalized action. The (non) linearity of the effective action is then related to the (non) vanishing of the renormalized coupling constant.

We first define the density of states at fixed pure gauge (non compact) energy as

\[ N(E) = \int [dA_\mu(x)] \delta \left( \frac{1}{2} \sum_{x, \mu < \nu} F_{\mu \nu}^2(x) - 6VE \right) \]

\[ N(E, M) = \int [dA_\mu(x)] \delta \left( \frac{1}{2} \sum_{x, \mu < \nu} F_{\mu \nu}^2(x) - 6VE \right) e^{-M^2 \sum_{x, \mu} A_\mu(x)^2} \]
The pure gauge energy \( \frac{1}{2} \sum_{x,\mu<\nu} F_{\mu\nu}^2(x) \) is a quadratic form, defined through a real, symmetric matrix, and can therefore be diagonalized by a unitary transformation. The number of zero modes of the quadratic form is \( V + 1 \), so that we can write, in \( d \) dimensions,

\[
N(E, M) = \int \prod_{k=1}^{(d-1)V-1} dB_k \delta \left( \frac{1}{2} \sum_k \lambda_k B_k^2 - \frac{d(d-1)}{2} V E \right) \\
\times \prod_k e^{-M^2 B_k^2} \left[ \int dB e^{-M^2 B} \right]^{V+1} \tag{2.4}
\]

where \( \lambda_k, k = 1, \ldots, (d-1)V-1 \) are the non-zero eigenvalues of the quadratic form. The integral in square brackets in the above formula is gaussian and contains the whole divergence as \( M \to 0 \), while the first factor is finite in the same limit.

The first factor can easily be computed using hyperspherical coordinates in a \( (d-1)V-1 \) dimensional space, leading to

\[
N(E) = C_G E^{(d-1)V-\frac{3}{2}} \tag{2.5}
\]

Since the density of states \( N(E) \) is known analytically, the partition function, as expressed in function of the effective fermionic action \([2,17]\), is now a one-dimensional integral

\[
Z = \int dE N(E) e^{-6\beta V E} e^{-S_{\text{eff}}^F(E,m)} \tag{2.6}
\]

where again the divergence of \( Z \) is contained in \( N(E) \) as a multiplicative constant \( C_G \) of no physical relevance.

The effective fermionic action \( S_{\text{eff}}^F(E,m) \) in (2.6) is related to the logarithm of the average value of the fermionic determinant over gauge configurations of fixed pure gauge energy

\[
e^{-S_{\text{eff}}^F(E,m)} = \frac{\int [dA_\mu(x)] \det \Delta(m,A_\mu(x)) \delta \left( \frac{1}{2} \sum_{x,\mu<\nu} F_{\mu\nu}^2(x) - 6VE \right)}{\int [dA_\mu(x)] \delta \left( \frac{1}{2} \sum_{x,\mu<\nu} F_{\mu\nu}^2(x) - 6VE \right)} \tag{2.7}
\]

Again, numerator and denominator of (2.7) are divergent due to gauge group volume; however, being the fermionic determinant gauge invariant, this divergence cancels in the ratio so that \( S_{\text{eff}}^F(E,m) \) is finite.

The total effective action for this model is therefore

\[
S_{\text{eff}}(E, V, \beta, m) = -\frac{3}{2} V \ln E + 6\beta V E + S_{\text{eff}}^F(E,m) \tag{2.8}
\]
where we have included the contribution from the density of the states in the effective action.

The effective fermionic action is linearly divergent with the lattice volume in the thermodynamical limit; in this limit we can use the saddle point technique to compute $Z$.

Let assume that this model exhibits a second order phase transition at $(\beta_c, m_c)$. The knowledge of the total effective action $S_{eff}$ allows, through (2.6), to compute, in principle exactly, the partition function. What is the manifestation in $S^F_{eff}$ of the phase transition?

Defining $\bar{S}^F_{eff}(E, m)$ the effective action per unit volume, the saddle point technique allows to write for the VEV of the mean plaquette energy and chiral condensate $<\bar{\psi}\psi>$ the following expressions

\[
< E_p > = E_0(m, \beta) \\
< \bar{\psi}\psi > = -\frac{\partial}{\partial m} \bar{S}^F_{eff}(E, m) \bigg|_{E = E_0(m, \beta)} \tag{2.9}
\]

where $E_0(m, \beta)$ is the minimum of the total effective action at given $\beta, m$, i.e. the solution of the following equation

\[
\frac{1}{4E} - \beta - \frac{1}{6E} \bar{S}^F_{eff}(E, m) = 0 \tag{2.10}
\]

From (2.9), (2.10) above we can derive

\[
C_\beta = \frac{\partial}{\partial \beta} < E_p > = -\frac{1}{4E_0^2(m, \beta)} + \frac{1}{6E_0^2} \bar{S}^F_{eff}(E, m) \bigg|_{E = E_0(m, \beta)}^{-1} \\
\frac{\partial}{\partial \beta} < \bar{\psi}\psi > = -C_\beta \frac{\partial^2}{\partial E \partial m} \bar{S}^F_{eff}(E, m) \bigg|_{E = E_0(m, \beta)} \tag{2.11} \\
\frac{\partial}{\partial m} < \bar{\psi}\psi > = -\frac{\partial^2}{\partial m^2} \bar{S}^F_{eff}(E, m) \bigg|_{E = E_0(m, \beta)} + \frac{\partial}{\partial m} E_0(m, \beta) \frac{1}{C_\beta} \frac{\partial}{\partial \beta} < \bar{\psi}\psi >
\]

A second order transition implies a discontinuity of the second derivative of the free energy. In particular, a discontinuity of the specific heat can be produced as well by a zero in the denominator of $C_\beta$ given by (2.11) as by a discontinuity of the second derivative of the fermionic effective action $\frac{\partial^2}{\partial E^2} \bar{S}^F_{eff}(E, m)$ at the value $E_0(m_c, \beta_c)$ corresponding to the critical values of the parameters $\beta_c, m_c$. The first possibility, which indeed happens in the large $n_f$ limit, will be analysed in detail elsewhere. In the next Sections we will show that our numerical results for $\bar{S}^F_{eff}(E, m)$ in the limit $m \to 0$ strongly suggest the existence of a non analyticity in the effective fermionic action.

We conclude this section by discussing point iii) on the connection between the effective fermionic action and the renormalized coupling
constant. Let assume that the continuum limit of the theory is determined by a gaussian, trivial fixed point. This means that, for a sufficiently large value of the cutoff, or, equivalently, sufficiently near to the critical surface, the renormalized coupling constant becomes zero. Therefore the renormalized action near the critical point will consist only of the kinetic term of the gauge fields, apart possibly for an additive constant, i.e. the total, renormalized effective action defined as in (2.8), near the critical point, will be linear in the renormalized energy, apart from the contribution of the density of states proportional to $\ln E_R$.

Therefore the triviality of the continuum limit can be studied from the effective fermionic action once its relation with the renormalized action is known.

This connection can be established in the following way [17]. We first write the partition function as an integral over the plaquette variables $F_{\mu\nu}^2$ in the following way

$$Z = \int [dE_{\mu\nu}(x)] N(E_{\mu\nu}(x))e^{-S(E_{\mu\nu}(x))} \quad (2.12)$$

with

$$e^{-S(E_{\mu\nu}(x))} =$$

$$\frac{\int [dA_\mu(x)][d\bar{\chi}(x)][d\chi(x)] \prod \delta(F_{\mu\nu}^2(x) - E_{\mu\nu}(x))e^{-S}}{\int [dA_\mu(x)][d\bar{\chi}(x)][d\chi(x)] \prod \delta(F_{\mu\nu}^2(x) - E_{\mu\nu}(x))} \quad (2.13)$$

where $S$ in the numerator of (2.13) is the action (2.1) and the denominator of (2.13) is exactly the density of states $N(E_{\mu\nu}(x))$. We next apply linear block spin Renormalization Group transformations in the theory described by the effective action $S(E_{\mu\nu}(x)) - \ln N(E_{\mu\nu}(x))$. Our spin variable is the plaquette variable $E_{\mu\nu}(x)$ which takes values from 0 to $\infty$ and blocking is performed at each $\mu\nu$ plane.

We generate in this way a series of effective actions $S_R(E_{\mu\nu}, N_S)$, where $N_S$ is the number of blocking steps, which are equivalent at large distances since we are integrating out all the short distance details.

The renormalized effective action $S_{R, eff}^e(E_R)$, defined as in (2.6) in function of the renormalized energy $E_R = \frac{1}{V} \sum_{n,\mu<\nu} E_{\mu\nu}(x)$ can therefore be obtained from action (2.6) by the substitution $E \rightarrow X(m, \beta)E_R$, since it has been derived through linear block-spin transformations plus a final linear global transformation.

In particular, apart from the obvious logarithmic contribution from the density of states, linearity of the renormalized action is implied by the linearity of the effective fermionic action.

From a practical point of view, the above arguments tell us that the vanishing of the renormalized coupling constants, as a function of the bare
couplings, can be inferred from the linearity of the effective fermionic action in the corresponding energy interval.

III. Computation of the effective action

The evaluation of the effective fermionic action is obviously not straightforward, owing to the non locality of the fermionic determinant, important especially at small masses.

In the abelian, compact case, a discussion of the reliability of this kind of computation was developed in [2]. In this paragraph this discussion will be expanded, for the non compact case, in greater details, particularly concerning the numerical evaluation of the effective fermionic action (2.7).

To this end, we will expand the effective fermionic action in powers of the number of flavours, and discuss the relative importance of the successive terms of this expansion.

The effective action (2.7), as stated in the previous section, is related to the average of the fermionic determinant, computed over gauge field configurations at fixed pure gauge energy.

To simplify the notation, let write this average as

\[ e^{-S_{\text{eff}}^{E}(E,m)} = \langle \det \Delta(m,A_{\mu}(x)) \rangle_{E} \] (3.1)

Equation (3.1), using staggered fermions, describes the effective action of a gauge field coupled to 4 fermion species. In general, for \( n_{f} \) species, the effective fermionic action will be

\[ e^{-S_{\text{eff}}^{E}(E,m,n_{f})} = \langle e^{n_{f}4\ln \det \Delta(m,A_{\mu}(x))} \rangle_{E} \] (3.2)

\( S_{\text{eff}}^{E} \) can be expanded in cumulants as

\[ -S_{\text{eff}}^{E}(E,m,n_{f}) = \frac{n_{f}}{4} \ln \det \Delta(m,A_{\mu}(x))_{E} \]

\[ + \frac{n_{f}^{2}}{32} \left\{ <(\ln \Delta)^{2}>_{E} - <\ln \det \Delta>^{2}_{E} \right\} \]

\[ + \frac{n_{f}^{3}}{384} \left\{ \langle(\ln \Delta - <\ln \det \Delta>_{E})^{3}\rangle_{E} \right\} + ... \] (3.3)

which is nothing but an expansion in the flavour number of the effective fermionic action.

Since the probability distribution of the logarithm of the fermionic determinant must be computed by numerical methods, the computation of the successive terms in (3.3) will be increasingly difficult with the order of the expansion.
In practice, only a few terms will be computed, so that the numerical results will be affected both by statistical errors, resulting from the numerical determination of the probability distribution of the logarithm of the fermionic determinant, and systematic ones, in consequence of the truncation of expansion (3.3).

Before presenting a detailed discussion of the results for the effective fermionic action, the relevance of systematic errors will be discussed.

Fig. 1 is a plot of the probability distribution of the logarithm of the fermionic determinant in a $8^4$ lattice, $m = 0$ and normalized pure gauge energy $E = 1.20$. This point has been chosen because the statistics here is particularly high (1300 configurations). Every gauge configuration diagonalized is separated from the previous one by 1000 iterations of a canonical MC process, followed by an appropriate rescaling of the gauge fields to bring the energy to the required value. This procedure guarantees the decorrelation of the successive gauge configurations that are diagonalized.

The fermionic matrix associated to these gauge configurations is exactly diagonalized at zero mass through a modified Lanczos algorithm. The knowledge of all the eigenvalues of the fermionic matrix at zero mass allows the computation of the determinant for every value of the mass of the fermions [2].

Coming back to Fig.1, the continuous line is a gaussian fit of the distribution measured numerically. The goodness of the fit is evident ($\chi^2/d.o.f. = 0.487$) and largely independent from the fermion mass. If, from these results, we assume that the probability distribution of the logarithm of the fermionic determinant at fixed pure gauge energy is gaussian, then only the first two contributions to the effective fermionic action (3.3) will be different from zero [2] and no systematic errors will be introduced by the truncation of the expansion.

Moreover, in the thermodynamical limit $V \rightarrow \infty$ it is sufficient to this that the right half (from the maximum) of the distribution is gaussian.

In Figs. 2a,2b,2c we present our results for the first three contributions to $S_{eff}^F$ respectively, as a function of energy at $m = 0$. The results for the third contribution are compatible with zero, according to the previous discussion.

Fig. 3 is a plot of the effective action at $m = 0$ and 4 flavours computed as sum of the first two contributions in (3.3); numerical values are also reported in Table I. From the figure, two different behaviours of the effective action are evident:

i. A small energy regime ($E < 1$), typical of the Coulomb phase, where the effective action shows a linear behaviour as function of the energy.

ii. A large energy regime, typical of the broken chiral symmetry phase, ($E > 1$), where the effective action exhibits a non linear behaviour.

In the previous section we have analysed the implications for the
effective action as function of the pure gauge energy of the existence of a second order transition. Through a saddle point analysis we concluded that a second order transition should manifest as a discontinuity of the second derivative with respect to the energy, at a value of the energy corresponding to the vacuum expectation value of the plaquette energy at the critical value of the parameters \( \beta_c, m_c \).

By fitting the experimental points in Fig.3 with two (different) third degree polynomials, one for \( E \leq 1.007 \) and one for \( E \geq 1.007 \), one gets very good fits (continuous line in this figure), with a gap in the second energy derivative of the effective fermionic action per unit volume, computed at \( E_c \), of 0.35(5). As a result of the fit, we also obtain that the first derivative is continuous at \( E_c \) (notice that a discontinuity in the first energy derivative should imply from equation (2.10) a first order transition) and the derivatives of order larger or equal to the second are zero at \( E < E_c \). The value \( E_c = 1.007(20) \) has been determined from an analysis of the behaviour of the average plaquette energy, as discussed in section VI.

The results of this fit, which do not change by changing the order of the polynomial used for the fit, imply that in effect the second order transition observed in this model [3-7], manifests itself through a non analyticity of the effective fermionic action. Moreover these results also show a change of regime from linear to non linear behaviour when passing from the Coulomb to the chiral symmetry broken phase respectively.

As discussed in the previous section, such a behaviour implies that the renormalized coupling constant is zero in the Coulomb phase for large enough values of the cut-off, while it is non zero in the broken phase, including the infinite cut-off limit. These results do not change qualitatively when the number of flavours varies from 1 to 4.

In Table II we report the values of \( E_c \), of the gap of the second derivative of \( S_{eff}^F \) at \( E_c \) for \( n_f = 1, 2, 3, 4 \) and of the critical coupling \( \beta_c \), as derived from the average plaquette (see Sect. VI).

To complete this section, we will discuss how these results can be affected by finite volume effects. Fig. 4 shows the results for the effective fermionic action per unit volume at three representative values of the energy in lattices \( 4^4, 6^4, 8^4 \) and \( 10^4 \). All the points in this figure have been normalized to the corresponding value in the \( 10^4 \) lattice in such a way that, in absence of finite volume effects, all the points will lie on a line of constant (= 1) ordinate.

The analysis of this figure shows that volume effects in the effective action decrease rapidly going from \( 4^4 \) to \( 10^4 \).
IV. Mean Plaquette Energy

The evaluation of the average plaquette energy, as well as that of other physical observables, is in principle simple in the abelian non compact model since, once the effective fermionic action is known as function of the energy, the average plaquette energy can be expressed as the ratio of one dimensional integrals in $E$.

This simplification derives from the fact that, differently from the compact case, the density of states $N(E)$ is known analytically in the non compact model, so that the average plaquette energy can be written as

$$\langle E_p \rangle = \frac{\int dE N(E)e^{-6\beta V E - S^{\text{eff}}_{\text{F}}(E,m,n_f)}}{\int dE N(E)e^{-6\beta V E - S^{\text{eff}}_{\text{F}}(E,m,n_f)}}$$

In our case we have measured the effective fermionic action for 28 values of the energy with the method described in the previous section (see Table I). Then we have determined the effective fermionic action for $0.3 \leq E \leq 0.7$ using a third order polynomial interpolation, and finally we evaluated numerically the integrals in (4.1).

An alternative method would be to apply directly to (4.1) the saddle point technique, which, as well known, is exact in the $V \to \infty$ limit. In fact we have seen that, in a $8^4$ lattice, the two methods give compatible results.

In order to compare our results with others in the literature and to check in this way the reliability of our method, we report in Tables III and IV our results for $\langle E_p \rangle$ for different fermion masses and flavour number (remember that in our procedure the computations for different masses and flavours are straightforward and practically no time consuming). Statistical errors have been computed using standard Jack-Knife procedure. The agreement with the results reported by other groups [4,6] is extremely good, and implies that systematic effects of the method used, as for plaquette energy is concerned, are entirely under control.

V. The chiral condensate

The vacuum average value of the chiral condensate $\langle \bar{\psi}\psi \rangle$ can be computed as a logarithmic derivative of the partition function (2.6)

$$\langle \bar{\psi}\psi \rangle = -\frac{\int dE e^{-S_{\text{eff}}(E,m,\beta,n_f,V)} \frac{\partial}{\partial m} S^F_{\text{eff}}(E,m,n_f)}{\int dE e^{-S_{\text{eff}}}}$$

where we remind that $S^F_{\text{eff}} = S^F_{\text{eff}}/V$, namely the chiral condensate is the average value of the derivative with respect to the mass of the normalized
effective action, with a probability distribution deriving from the total effective action.

The expansion of the effective fermionic action in powers of the number of flavours (3.3), leads to a similar expansion for the contributions to the chiral condensate

\[-\frac{\partial S_{\text{eff}}^F}{\partial m} = \frac{n_f}{4} < Tr \Delta^{-1} >_E + \]

\[+ \frac{n_f^2}{16} \langle (\ln \det \Delta - < \ln \det \Delta >_E)(Tr \Delta^{-1} - < Tr \Delta^{-1} >_E) \rangle_E + \]

\[+ \frac{n_f^3}{64} \langle (\ln \det \Delta - < \ln \det \Delta >_E)^2(Tr \Delta^{-1} - < Tr \Delta^{-1} >_E) \rangle_E + ... \]

Therefore the chiral condensate is given by the average value over the probability distribution defined in (5.1) of the successive terms in (5.2), normalized by \(V\).

Here, as in the computation of the effective fermionic action, the degree of difficulty in the numerical evaluation of the successive terms in expansion (5.2) increases with the order in the expansion.

In practice also in this case we will be forced to truncate the expansion to a certain order, so also the evaluation of the chiral condensate will be in principle affected by systematic errors due to this approximation.

However, following the analysis done in Section III for the effective fermionic action, the only non zero contributions to the chiral condensate are the first two in (5.2) if the probability distribution of the logarithm of the determinant at fixed pure gauge energy is gaussian.

Table V contains our results for the chiral condensate at different masses and flavour number, computed from the first two contributions to the derivative with respect to the fermionic mass of the effective action (5.2). We have not included the contribution proportional to \(n_f^3\) since our results show it compatible with zero.

VI. The phase diagram

The numerical results for the effective fermionic action reported in Section III suggest, as already discussed, the existence of a phase transition separating a Coulomb phase where the effective action is a linear function of the energy, from a broken phase characterized by a non linear behaviour. This picture has been confirmed by the polynomial fits to the effective action, which predict a gap in its second derivative at the critical value of the energy.

To confirm these results and, at the same time, derive a precise determination of the critical values \(\beta_c, E_c\), we will analyse in this section
the numerical results for the average plaquette energy $< E_p >$ as well as its dependence on $n_f$ and $m$, which will allow to improve our knowledge of the phase diagram of this model.

Fig. 5 is a plot of our results on the average plaquette energy in function of $\beta$ in a $8^4$ lattice at $m = 0$ and 4 flavours. The continuous line in this figure is the plot of $1/4(\beta + h_1(m))$ with $h_1(m) = 0.04032$. This would be the behaviour of the plaquette energy, if the effective action were a linear function of the energy, with $h_1(m) = \frac{1}{6} \times$ the slope of the normalized effective action $\bar{S}^{F}_{\text{eff}}$ [17].

It follows that the numerical results for $< E_p >$ are to a high degree consistent with $1/4(\beta + h_1(m))$ in the weak coupling region while in the strong coupling regime important deviations can be observed.

These results suggest again the existence of a phase transition at an intermediate $\beta_c$ value of $\beta$. For a precise determination of $\beta_c$, we present in Fig.6 the dependence of $h_1(m)$ on $\beta$. If the function $1/4(\beta + h_1(m))$ describes correctly the functional dependence on $\beta, m$ of the plaquette energy, $h_1(m)$ should be independent on $\beta$.

The results reported in Fig. 6 clearly show the existence of two distinct regimes in $\beta$ of the behaviour of $h_1(m)$. In the weak coupling region, i.e. large $\beta$, the results can be fitted with an horizontal straight line, showing that in fact $h_1(m)$ does not depend on $\beta$. On the contrary, the region of strong coupling shows a strong dependence of $h_1(m)$ on $\beta$.

The critical $\beta$ is then defined as the $\beta$ value corresponding to the intersection of the fits represented by the continuous lines in the figure. Once $\beta_c$ is known, the evaluation of $E_c$, the critical plaquette energy, is immediate; this value has been used in Section III to obtain the polynomial fits of the effective fermionic action.

Table II summarizes our results for $\beta_c$ and $E_c$ in a $8^4$ lattice at $m = 0$ and $n_f = 1, 2, 3, 4$.

We next consider the dependence of these results on the bare mass of the fermion. Fig. 7 shows the behaviour in $\beta$ of $h_1(m)$ for two different $m$ values, $m = 0.0125$ (Fig.7a) and $m = 0.1$ (Fig. 7b). The results for $m = 0.0125$ are qualitatively indistinguishable from the results of the massless case. On the other hand, for $m = 0.1$ it is practically impossible to find an interval in $\beta$ in which $h_1(m)$ is constant, at least in the region explored in $\beta$.

Our results suggest that the phase transition present at $m = 0, \beta = 0.208(4)$ continues in the $(\beta, m)$ plane at least up to $m = 0.025$. At larger values of the mass it is extremely difficult to analyse the phase diagram, and, consequently, to establish if the Coulomb and broken phases are analytically connected.

In Table VI we also report the critical values $\beta_c, E_c$ at some representative values of the fermionic mass for 2 and 4 flavours. Fig. 8 presents a tentative phase diagram in the plane $\beta, m$. 
VII. Critical indices and the Equation of State.

Although the numerical results reported in [14,15] show the impossibility of extracting in a meaningful way the critical exponents from simulations in lattices as small as ours, we think interesting to make an analysis of the dependence of the value of $\beta_c$ and of the critical exponents on the extrapolation method used to compute the chiral condensate at zero mass, also in view of the structure of the phase diagram depicted above. Notice that our method allows us to compute the chiral condensate for an arbitrary number of values of $\beta, m$ practically at no extra computer cost, so this analysis is well worth the effort.

As proposed in [12], we can derive critical indices and $\beta_c$ from the study of the Equation Of State (EOS) which describes the response of the order parameter of the theory (i.e. $<\bar{\psi}\psi>$ for QED) to an explicit (chiral) symmetry breaking term.

In the present case the EOS is (using standard notation [12])

$$\frac{<\bar{\psi}\psi>}{m^{1/\delta}} = F\left(\frac{(\beta - \beta_c)}{<\bar{\psi}\psi>^{1/\beta_{mag}}}\right) \quad (7.1)$$

where $F$ is a universal function. In principle, one could derive from Eq. 7.1 both the critical exponents and critical coupling.

We have exploited the universal behaviour of the EOS for our data of the chiral condensate (four flavors) using the expansion (5.2) up to terms in $n_f^2$.

Owing to the smallness of the lattices used, we have only used data for $m \geq 0.0125$ for this analysis, for which we believe our data are fully reliable. We obtain that our data in this mass range can be very well fitted by equation (7.1) with mean field exponents ($\delta = 3, \beta_{mag} = 0.5$) and $\beta_c = 0.190$ (see Fig. 9). This result is not surprising since our data for $<\bar{\psi}\psi>$ are entirely consistent with the data of ref. [6]

Notice however that our $\beta_c$ obtained from the behaviour of the plaquette energy, is inconsistent with equation (7.1). Barring the presence of a second phase transition different from the chiral one, our result suggests, as already stressed in [8] that the determination of critical coupling and indices from the chiral condensate data in small lattices gives inconsistent results, since the minimum achievable mass is relatively high.

On the other hand, our approach to the determination of the chiral condensate based on expansion (5.2) allows to estimate finite size effects on the various terms of the expansion. At masses $0.0025 \leq m \leq 0.0125$, the coefficient of $n_f^2$ cannot be reliably evaluated in the lattices we use (it suffers from strong finite volume effects, the absolute value being decreasing with the volume). However, finite size effects on $Tr\Delta^{-1}$ are small in this mass interval.
We have then decided to investigate the scaling behaviour of the EOS in the previously mentioned mass interval, in terms of the approximation to the chiral condensate consisting in the first term of Eq. (5.2). This approximation should be meaningful, were the critical behavior of the order parameter contained in this term. We find scaling behaviour (see Fig. 10) for $\beta_c = 0.207$ (a value which is consistent with the one derived from the behaviour of the plaquette energy) and $\delta = 2.5, \beta_{mag} = 0.64$. On the other hand, if we impose mean field exponents in eq. (7.1), it is impossible to find such a good scaling behaviour as that of Fig. 10 for values of $\beta_c$ compatible with the one extracted from the behaviour of the mean plaquette energy.

VIII. Conclusions

The first motivation for the work described in the present paper, has been to test, in a different model, the method proposed in [2] for including the dynamical effects of light fermions. Further development of this investigation has shown the possibility of clarifying some physical phenomena, in addition to the verification of the reliability of the numerical method.

To this effect, we have presented theoretical and numerical arguments supporting the fact that a second order phase transition manifests itself in a non analyticity of the fermionic effective action as a function of the pure gauge normalized energy. We have presented numerical evidence for this non analytic behaviour, which in turn allowed the determination of the values for the parameters $\beta_c, m_c$ at the critical point, from the results for the mean plaquette. This determination is completely independent from others based on the study of the chiral condensate and, to our knowledge, it is the first time that the position of the critical point of this model is determined from the results for the plaquette energy. The critical values of $\beta$ at $m = 0$ obtained at $n_f = 2$ and $n_f = 4$ with this method are in perfect agreement with the ones obtained by the Illinois Group [18].

Using general arguments of the renormalization group approach we have related the value of the renormalized coupling constants to the functional dependence of the effective fermionic action on the pure gauge energy $E$. The non linear behaviour shown by the effective action for energies equal or larger than a critical energy indicates, in this approach, that some renormalized coupling constant is different from zero even in the infinite cut off limit, when we approach it from the broken chiral symmetry phase.

Certainly, these general arguments do not allow to identify the renormalized coupling (or couplings) which are different from zero in the infinite cutoff limit [8,9]. Nevertheless, since we define the critical energy
\( E_c \), and consequently coupling \( \beta_c \), at the point where the effective action becomes non linear in the energy, our results can be reconciled with triviality only assuming that the phase transition we have found is different from the chiral one, where the continuum limit of strongly coupled QED is defined. We do not see sign of this behaviour in the lattices we studied.

On the other hand, approaching the critical point from the Coulomb phase, the linearity observed in the effective fermionic action for energies characteristic of this phase indicates a trivial continuum limit associated to the gaussian fixed point.

It is remarkable the fact that, in this phase, the effective fermionic action has a linear behaviour for all the explored energies. According to the Renormalization Group arguments we have presented, this implies that the renormalized coupling constant is zero in this phase for all the values of the cutoff corresponding to the energies here studied, which is in qualitative agreement with an analogous phenomenon reported in [13] and there interpreted as a manifestation of the fact that weakly coupled QED becomes an effective weakly renormalizable theory at a small value of the cutoff.

As for the reliability of the method, we want to stress that the results we have presented for the plaquette energy \( < E_p > \) and chiral condensate \( < \bar{\psi} \psi > \), are entirely consistent with the results presented by other groups. This gives evidence to the accuracy of our method, at least for \( n_f \leq 4 \). The computational cost of our results is however a small fraction of that of standard methods [2], fraction which is difficult to estimate if one takes into account that the fermionic computation has to be performed only once for in principle infinitely many different values of \( \beta, m \).

We believe to have presented in this work further evidences supporting the possibility of a non trivial continuum limit for the strongly coupled lattice regularized QED. We cannot however exclude that a pathological behaviour of the theory could make our results not relevant for this limit.

We close by giving some informations about the computing resources used to produce the results presented in this paper: these informations are useful to give an idea of the power of the method used.

All these simulations have been run on two four-transputers arrays of the Theory Group of LNF, with an estimated total peak computing power of 14.4 Mflops. We have produced with the microcanonical algorithm and completely diagonalized using a modified Lanczos algorithm 7771 configurations in the \( 4^4 \) lattice, 9370 in the \( 6^4 \), 6639 in the \( 8^4 \) and 464 in the \( 10^4 \) (all fully decorrelated) for a total of 173.2 CPU days, roughly corresponding to 255 hours of a Cray running at 240 Mflops.

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TABLE CAPTIONS

I Effective action and accumulated statistics versus pure gauge energy, at $m = 0$, $n_f = 4$, $8^4$ lattice.

II Critical energy, gap of the second derivative of $S_{eff}^{F}$ at $E_c$ and critical coupling for $n_f = 1, 2, 3, 4$, as obtained from the average plaquette data.

III Average plaquette energy, $m = 0$, $n_f = 1...4$.

IV Average plaquette energy, $n_f = 2$, $m = 0.02, 0.04$ and $n_f = 4$, $m = 0.0125, 0.025, 0.05, 0.1$.

V Normalized chiral condensate, $n_f = 2$, $m = 0.02, 0.04$ and $n_f = 4$, $m = 0.0125, 0.025, 0.05, 0.1$.

VI Critical energy and coupling for $n_f = 2, 4$ and $m > 0$. 
FIGURE CAPTIONS

1) Probability distribution of $\log \Delta(m, A_\mu)$ at $m = 0.0$, $E = 1.20$, $n_f = 4$ and $8^4$ lattice. The continuous line is a gaussian fit, with $\chi^2/d.o.f. = 0.487$.

2) a) first, b) second and c) third cumulant of $S_{f}^{\text{eff}}$ versus $E$ at $m = 0$, $8^4$ lattice.

3) Effective fermionic action in a $8^4$ lattice, $m = 0.0$. Errors are not larger than symbols. The continuous line shows the two independent polynomial fits (see text).

4) Finite volume effects on the normalized effective fermionic action at three representative values of the pure gauge energy $E$.

5) Mean plaquette energy versus $\beta$ at $m = 0.0$. The continuous line is a fit of the Coulomb phase data with $E_{pl} = 1/4(\beta + h_1(m))$ and $h_1(0) = 0.04032$.

6) $h_1(m)$ versus $\beta$ at $m = 0.0$. The solid line in the strong coupling phase is a polynomial fit.

7) The same as fig. 6, but for a) $m = 0.0125$ and b) $m = 0.1$.

8) Tentative phase diagram. The continuous line represents second order phase transitions. The dashed line corresponds to values of $m$ where the existence of a phase transition is not clear.

9) $< \bar{\psi}\psi>/m^{1/\delta}$ versus $(\beta_c - \beta)/<\bar{\psi}\psi>^{1/\beta_{mag}}$ with $\beta_c = 0.190$, $\delta = 3$, $\beta_{mag} = 0.5$ and $0.0125 \leq m \leq 0.1$, $0.170 \leq \beta \leq 0.215$. Different symbols correspond to different mass values.

10) $Tr\Delta^{-1}/m^{1/\delta}$ versus $(\beta_c - \beta)/(Tr\Delta^{-1})^{1/\beta_{mag}}$ with $\beta_c = 0.207$, $\delta = 2.5$, $\beta_{mag} = 0.64$ and $0.0025 \leq m \leq 0.01$, $0.170 \leq \beta \leq 0.215$. 