Transitivity of Subtyping for Intersection Types

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Abstract

The subtyping relation for intersection type systems traditionally employs a transitivity rule (Barendregt et al. 1983), which means that the subtyping judgment does not enjoy the subformula property. Laurent develops a sequent-style subtyping judgment, without transitivity, and proves transitivity via a sequence of six lemmas that culminate in cut-elimination (2018). This article presents a subtyping judgment, in regular style, that satisfies the subformula property, and presents a direct proof of transitivity. Borrowing from Laurent’s system, the rule for function types is essentially the $\beta$-soundness property. The main lemma required for the transitivity proof is one that has been used to prove the inversion principle for subtyping of function types. The choice of induction principle for the proof of transitivity is subtle: we use well-founded induction on the lexicographical ordering of the sum of the depths of the first and last type followed by the sum of the sizes of the middle and last type. The article concludes with a proof that the new subtyping judgment is equivalent to that of Barendregt, Coppo, and Dezani-Ciancaglini.

1 Introduction

Intersection types were invented by Coppo, Dezani-Ciancaglini, and Salle, as a tool for studying normalization in the lambda calculus [Coppo et al., 1979]. By varying the subtyping rules and atom types, researchers use intersection type systems to model many different calculi [Coppo and Dezani-Ciancaglini, 1980, Coppo et al., 1981, Engelen, 1981, Coppo et al., 1984, Honsell and Rocca, 1992, Abramsky and Ong, 1993, Plotkin, 1993, Honsell and Lenisa, 1999, Ishihara and Kurata, 2002, Ronchi Della Rocca and Paolini, 2002, Dezani-Ciancaglini et al., 2005, Alessi et al., 2006]. Perhaps the best-known of them is the BCD intersection type system of Barendregt et al. [1983]. For this article we focus on the BCD system, following the presentation of Barendregt et al. [2013]. We conjecture that our results apply to other intersection type systems as well.

The BCD intersection type systems extends the simply-typed lambda calculus with the addition of intersection types, written $A \cap B$, a top type $U$, and an infinite collection of type constants. Figure 1 defines the grammar of types.

The BCD intersection type system includes a subsumption rule which states that a term $M$ in environment $\Gamma$ can be given type $B$ if it has type $A$ and $A$ is
\[ \alpha, \beta ::= \text{atoms} \]
\[ A, B, C, D ::= \alpha \mid A \rightarrow B \mid A \cap B \text{ types} \]

Figure 1: Intersection Types

\[
\begin{align*}
A \leq B \\
\text{(refl)} & \quad A \leq A \\
\text{(trans)} & \quad A \leq B, B \leq C \quad \Rightarrow \quad A \leq C \\
\text{(incl\_L)} & \quad A \cap B \leq A \\
\text{(incl\_R)} & \quad A \cap B \leq B \\
\text{(glb)} & \quad A \leq C, A \leq D \quad \Rightarrow \quad A \leq C \cap D \\
\text{(\_\rightarrow\_\cap)} & \quad C \leq A, B \leq D \quad \Rightarrow \quad (A \rightarrow B) \cap (A \rightarrow C) \leq A \rightarrow (B \cap C) \\
\text{(U\_top)} & \quad \text{A} \leq U, \quad \text{U} \leq \text{C} \rightarrow \text{U} 
\end{align*}
\]

Figure 2: Subtyping of Barendregt, Coppo, and Dezani-Ciancaglini (BCD).

a subtype of \( B \), written \( A \leq B \).

\[
\begin{align*}
\Gamma \vdash M : A & \quad A \leq B \\
\Gamma \vdash M : B & 
\end{align*}
\]

Figure 2 reviews the BCD rules for subtyping. Note that in the (trans) rule, the type \( B \) that appears in the premises does not appear in the conclusion. Thus, the BCD subtyping judgment does not enjoy the subformula property. For other systems, it is straightforward to remove the (trans) rule, modify the other rules, and then prove transitivity [Muehlboeck and Tate, 2018]. Unfortunately, the (\_\rightarrow\_\cap) rule of the BCD system significantly complicates the situation.

The subformula property is a useful one. For example, the author is using intersection types to create a denotational semantics for the ISWIM language, which includes constants and primitive operations [Landin, 1966; Plotkin, 1975; Felleisen et al., 2009]. It seems that doing so requires placing extra conditions on types and it is much easier to do so when subtyping satisfies the subformula property.

Laurent [2018] introduces the ISC sequent-style system, written \( \Gamma \vdash B \), where \( \Gamma \) is a sequence of types \( A_1, \ldots, A_n \). The intuition is that \( A_1, \ldots, A_n \vdash B \) corresponds to \( A_1 \cap \cdots \cap A_n \leq B \). The ISC system satisfies the subformula property and is equivalent to the BCD system. To prove this, Laurent establishes six lemmas that culminate in cut-elimination, from which transitivity follows.

This article presents a more direct route to the subformula property and transitivity. We present a subtyping relation \( A <: B \) and directly prove transitivity.
whether $A$ is a part of $U$ codomain if $A$ is a function type, respectively. If $A$ is an atom, $\text{dom}(A)$ is undefined. Likewise for $\text{cod}(A)$. For example, if $A = (A_1 \rightarrow B_1) \cap \cdots \cap (A_n \rightarrow B_n)$, then $\text{dom}(A) = A_1 \cap \cdots \cap A_n$ and $\text{cod}(A) = B_1 \cap \cdots \cap B_n$.

When $\text{dom}(A)$ or $\text{cod}(A)$ appears in lemma or theorem statement, we implicitly assume that $A$ is a type such that $\text{dom}(A)$ and $\text{cod}(A)$ are defined. The $\text{top}(A)$ predicate identifies types that are equivalent to $U$. The $\text{topInCod}(A)$ predicate identifies types that have $U$ in their codomain. The relation $A \in B$ indicates whether $A$ is syntactically a part of $B$. The relation $A \subseteq B$ holds when every part of $A$ is a part of $B$. We say that $B$ contains $A$ if $A \subseteq B$.

Proposition 1.

If $A \cap B \subseteq C$, then $A \subseteq C$ and $B \subseteq C$.

The new intersection subtyping judgment, $A \ll B$, is defined in Figure 2. First, it does not include the (trans) rule. It also replaces the (refl) rule with reflexivity for atoms (refl$_a$). The most important rule is the one for function types ($\rightarrow'$), which subsumes ($\rightarrow$) and ($\rightarrow \cap$) in BCD subtyping. The ($\rightarrow'$) rule essentially turns the $\beta$-soundness property into a subtyping rule. The ($\rightarrow'$) rule says that a type $A$ is a subtype of a function type $C \rightarrow D$ if a subset of $A$, call it $B$, has domain and codomain that are larger and smaller than $C$ and $D$, respectively. The use of a subset of $A$ enables this rule to absorb uses of (incl$_L$) and (incl$_R$) on the left. The side conditions $\neg \text{top}(B)$ and $\neg \text{topInCod}(D)$ are needed because of the ($U \rightarrow'$) rule, which in turn is needed to preserve types under $\eta$-reduction. In a system that does not involve $\eta$-reduction, the ($U \rightarrow'$) rule can be omitted, as well as those side conditions. The rules (lb$_L$) and (lb$_R$) adapt (incl$_L$) and (incl$_R$) to a system without transitivity, and have appeared many times in the literature [van Bakel, 1995]. The ($U \rightarrow'$) rule generalizes the ($U \rightarrow$) rule, replacing the $U$ on the left with any type $A$, because for transitivity, any type is below $U$. The ($U \rightarrow'$) rule also replaces the $U$ in the codomain on the right with any type $D$ that is equivalent to $U$. 

2 A New Subtyping Judgment

Our new subtyping judgment relies on several auxiliary notions that help us avoid the use of ellipses, which we define in Figure 3. These include the $\text{dom}(A)$ and $\text{cod}(A)$ functions, the $\text{top}(A)$ and $\text{topInCod}(A)$ predicates, and the relations $A \in B$ and $A \subseteq B$. The $\text{dom}(A)$ and $\text{cod}(A)$ functions return the domain or codomain if $A$ is a function type, respectively. If $A$ is an intersection $A_1 \cap A_2$, then $\text{dom}(A)$ is the intersection of the domain of $A_1$ and $A_2$. If $A$ is an atom, $\text{dom}(A)$ is undefined. Likewise for $\text{cod}(A)$. For example, if $A = (A_1 \rightarrow B_1) \cap \cdots \cap (A_n \rightarrow B_n)$, then $\text{dom}(A) = A_1 \cap \cdots \cap A_n$ and $\text{cod}(A) = B_1 \cap \cdots \cap B_n$.

When $\text{dom}(A)$ or $\text{cod}(A)$ appears in lemma or theorem statement, we implicitly assume that $A$ is a type such that $\text{dom}(A)$ and $\text{cod}(A)$ are defined. The $\text{top}(A)$ predicate identifies types that are equivalent to $U$. The $\text{topInCod}(A)$ predicate identifies types that have $U$ in their codomain. The relation $A \in B$ indicates whether $A$ is syntactically a part of $B$. The relation $A \subseteq B$ holds when every part of $A$ is a part of $B$. We say that $B$ contains $A$ if $A \subseteq B$.
\textbf{dom}(A), \textbf{cod}(A)
\[\begin{align*}
\text{dom}(A \to B) &= A \\
\text{dom}(A \cap B) &= \text{dom}(A) \cap \text{dom}(B) \\
\text{cod}(A \to B) &= B \\
\text{cod}(A \cap B) &= \text{cod}(A) \cap \text{cod}(B)
\end{align*}\]

\textbf{top}(A)
\[
\begin{array}{c c c c}
\text{top}(\mathbb{U}) & \text{top}(A) & \text{top}(B) \\
\text{top}(A \to B) & \text{top}(A \cap B) \\
\end{array}
\]

\textbf{A} \subseteq \textbf{B}
\[
\begin{array}{c c c c}
\alpha \in \alpha & A \to B \in A \to B & A \in B & A \in C \\
A \subseteq B & A \subseteq B \cap C & A \subseteq B \cap C \\
\end{array}
\]

\textbf{topInCod}(D)
\[\text{topInCod}(D) = \exists AB. A \to B \in D \text{ and } \text{top}(B)\]

Figure 3: Auxiliary Definitions

\textbf{A} \prec: \textbf{B}
\[
\begin{array}{c}
\text{(refl)} \\
\text{(lb)} \\
\text{(lb)} \\
\text{(glb)} \\
\text{(-)}' \\
\text{(Utop)} \\
\text{(U)}' \\
\end{array}
\]

Figure 4: The New Subtyping Judgment

4
Before moving on, we make note of some basic facts regarding the $<$: relation and the \text{top}(A) predicate.

**Proposition 2** (Basic Properties of $<$:).

1. (reflexivity) $A <: A$
2. If $A <: B \cap C$, then $A <: B$ and $A <: C$.
3. If $A <: B$ and $C \in B$, then $A <: C$.
4. If $A <: B$ and $C \subseteq B$, then $A <: C$.

*Proof.*

1. The proof of reflexivity is by induction on $A$. In the case $A = A_1 \to A_2$, we proceed by cases on whether \text{top}(A_2). If it is, deduce $A_1 \to A_2 <: A_1 \to A_2$ by rule ($U \to'$). Otherwise, apply rule ($\to'$).
2. The proof is by induction on the derivation of $A <: B \cap C$.
3. The proof is by induction on $B$. In the case where $B = B_1 \cap B_2$, either $C \in B_1$ or $C \in B_2$, but in either case part 2 of this proposition fulfills the premise of the induction hypothesis, from which the conclusion follows.
4. The proof is by induction on $C$, using part 3 of this proposition in the cases for atoms and function types.

**Proposition 3** (Properties of \text{top}(A)).

1. If \text{top}(A) then \text{top}(	ext{cod}(A)).
2. If \text{top}(A) and $B \in A$, then \text{top}(B)$.
3. If \text{top}(A) and $B \subseteq A$, then \text{top}(B)$.
4. If \text{top}(A) and $A <: B$, then \text{top}(B)$.
5. If \text{top}(A), then $B <: A$.

*Proof.*

1. The proof is a straightforward induction on $A$.
2. The proof is also a straightforward induction on $A$.
3. The proof is by induction on $B$. The cases for atoms and function types are proved by part 2 of this proposition. In the case for $B = B_1 \cap B_2$, from $B_1 \cap B_2 <: A$, we have $B_1 <: A$ and $B_2 <: A$ (Proposition 1). Then by the induction hypotheses for $B_1$ and $B_2$ we have \text{top}(B_1) and \text{top}(B_2), from which we conclude that \text{top}(B_1 \cap B_2).
4. The proof is by induction on the derivation of $A <: B$. All of the cases are straightforward except for rule $(\rightarrow')$. In that case we have $B = B_1 \rightarrow B_2$ and some $A'$ such that $A' \subseteq A$, $B_1 <: \text{dom}(A')$, $\text{cod}(A') <: B_2$, $\neg \text{top}(B_2)$, and $\neg \text{topInCod}(A')$. From the premise $\text{top}(A)$ and part 3 of this proposition, we have $\text{top}(A')$. Then by part 1 we have $\text{top}(\text{cod}(A'))$. By the induction hypothesis for $\text{cod}(A') <: B$ we conclude that $\text{top}(B)$.

5. The proof is a straightforward induction on $A$. □

Next we turn to the subtyping inversion principle for function types. The idea is to generalize the rule $(\rightarrow')$ with respect to the type on the right, allowing any type that contains a function type. The premises of $(\rightarrow')$ are somewhat complex, so we package most of them into the following definition.

**Definition 4 (factors).** We say $C \rightarrow D$ factors $A$ if there exists some type $B$ such that $B \subseteq A$, $C <: \text{dom}(B)$, $\text{cod}(B) <: D$, and $\neg \text{topInCod}(B)$.

**Proposition 5 (Inversion Principle for Function Types).** If $A <: C'$, $C \rightarrow D \in C'$, and $\neg \text{top}(D)$, then $C \rightarrow D$ factors $A$.

**Proof.** The proof is a straightforward induction on $A <: C'$. □

### 3 Transitivity

The proof of transitivity relies on the following lemma, which is traditionally needed to prove the inversion principle for function types. However, it was not needed for our system because the rule $(\rightarrow')$ is already quite close to the inversion principle. The lemma states that if every function type $C \rightarrow D$ in $A$ factors $B$, then $\text{dom}(A) \rightarrow \text{cod}(A)$ also factors $B$.

**Lemma 6.** If

- for any $C \rightarrow D$, if $C \rightarrow D \in A$ and $\neg \text{top}(D)$, then $C \rightarrow D$ factors $B$, and
- $\neg \text{topInCod}(A)$,

then $\text{dom}(A) \rightarrow \text{cod}(A)$ factors $B$.

**Proof.** The proof is by induction on $A$.

- Case $A$ is an atom. The statement is vacuously true.

- Case $A = A_1 \rightarrow A_2$ is a function type. Then we conclude by applying the premise with $C$ and $D$ instantiated to $A_1$ and $A_2$ respectively.

- Case $A = A_1 \cap A_2$. By the induction hypothesis for $A_1$ and for $A_2$, we have that $\text{dom}(A_1) \rightarrow \text{cod}(A_1)$ factors $B$ and so does $\text{dom}(A_2) \rightarrow \text{cod}(A_2)$. So there exists $B_1$ and $B_2$ such that $B_1 \subseteq B$, $\neg \text{topInCod}(B_1)$, $\text{dom}(A_1) <: B_1$, $\neg \text{topInCod}(B_2)$, $\text{cod}(A_2) <: B_2$, and $\neg \text{top}(B)$.

6
The proof is by well-founded induction on the relation \( \text{dom} \). We proceed by cases on the last rule applied in the derivation of \( B < C \). Thus, we have that \( \text{dom}(B_1 \cap B_2) <: \text{dom}(A) \) and \( \text{cod}(A) <: \text{cod}(B_1 \cap B_2) \), and this case is complete.

}\end{proof}

We now turn to the proof of transitivity, that if \( A <: B \) and \( B <: C \), then \( A <: C \). The proof is by well-founded induction on the lexicographical ordering of the sum of the depths of \( A \) and \( C \) followed by the sum of the sizes of \( B \) and \( C \). To be precise, we define this ordering as follows.

\[
\langle A, B, C \rangle \ll \langle A', B', C' \rangle \iff (\text{depth}(A) + \text{depth}(C) < \text{depth}(A') + \text{depth}(C') \text{ or } \\
\text{depth}(A) + \text{depth}(C) \leq \text{depth}(A') + \text{depth}(C') \text{ and } \text{size}(B) + \text{size}(C) < \text{size}(B') + \text{size}(C'))
\]

where \( \text{size}(A) \) is
\[
\text{size}(\alpha) = 0 \\
\text{size}(A \to B) = 1 + \text{size}(A) + \text{size}(B) \\
\text{size}(A \cap B) = 1 + \text{size}(A) + \text{size}(B)
\]

and \( \text{depth}(A) \) is
\[
\text{depth}(\alpha) = 0 \\
\text{depth}(A \to B) = 1 + \max(\text{depth}(A), \text{depth}(B)) \\
\text{depth}(A \cap B) = \max(\text{depth}(A), \text{depth}(B))
\]

**Theorem 7** (Transitivity of \( <: \)). If \( A <: B \) and \( B <: C \), then \( A <: C \).

**Proof.** The proof is by well-founded induction on the relation \( \ll \). We proceed by cases on the last rule applied in the derivation of \( B <: C \).

**Case (refl)\(_A\)** We have \( B = C = \alpha \). From the premise \( A <: B \) we immediately conclude that \( A <: \alpha \).

**Case (lb\(_L\))** So \( B = B_1 \cap B_2 \), \( B_1 <: C \), and \( A <: B_1 \cap B_2 \). We have \( A <: B_1 \) (Proposition 2 part 2), so we conclude that \( A <: C \) by the induction hypothesis, noting that \( \langle A, B_1, C \rangle \ll \langle A, B, C \rangle \) because \( \text{size}(B_1) < \text{size}(B) \).

**Case (lb\(_R\))** So \( B = B_1 \cap B_2 \), \( B_2 <: C \), and \( A <: B_1 \cap B_2 \). We have \( A <: B_2 \) (Proposition 2 part 2), so we conclude that \( A <: C \) by the induction hypothesis, noting that \( \langle A, B_2, C \rangle \ll \langle A, B, C \rangle \) because \( \text{depth}(B_2) \leq \text{depth}(B) \) and \( \text{size}(B_2) < \text{size}(B) \).
Case (glb) We have $C = C_1 \cap C_2$, $B <: C_1$, and $B <: C_2$. By the induction hypothesis, we have $A <: C_1$ and $A <: C_2$, noting that $\langle A, B, C_1 \rangle \ll\langle A, B, C_2 \rangle \ll\langle A, B, C \rangle$ because $\text{depth}(C_1) \leq \text{depth}(C)$, $\text{depth}(C_2) \leq \text{depth}(C)$, $\text{size}(C_1) < C$, and $\text{size}(C_2) < C$. We conclude $A <: C_1 \cap C_2$ by rule (glb).

Case ($\rightarrow$) So $C = C_1 \rightarrow C_2$, $\neg\text{top}(C_2)$, and there exists $B'$ such that $C_1 <: \text{dom}(B')$, $\text{cod}(B') <: C_2$, $B' \subseteq B$, and $\neg\text{topInCod}(B')$. From $A <: B$ and $B' \subseteq B$, we have $A <: B'$ (Proposition 2 part 4). Thus, for any $B_1 \rightarrow B_2 \in B'$, $B_1 \rightarrow B_2$ factors $A$ (Proposition 5). We have satisfied the premises of Lemma 8 so $\text{dom}(B') \rightarrow \text{cod}(B')$ factors $A$. That means there exists $A'$ such that $A' \subseteq A$, $\neg\text{topInCod}(A')$, $\text{dom}(B') <: \text{dom}(A')$, and $\text{cod}(A') <: \text{cod}(B')$. Then by the induction hypothesis, we have $C_1 <: \text{dom}(A')$ and $\text{cod}(A') <: C_2$ noting that $\langle C_1, \text{dom}(B'), \text{dom}(A') \rangle \ll\langle A, B, C \rangle$ because $\text{depth}(C_1)+\text{depth}(\text{dom}(A')) < \text{depth}(A)+\text{depth}(C)$ and $\langle \text{cod}(A'), \text{cod}(B'), C_2 \rangle \ll\langle A, B, C \rangle$ because $\text{depth}(\text{cod}(A'))+\text{depth}(C_2) < \text{depth}(A)+\text{depth}(C)$. We conclude that $A <: C_1 \rightarrow C_2$ by rule ($\rightarrow$) witnessed by $A'$.

Case ($\cup_{\text{top}}$) We have $C = \cup$ and conclude $A <: \cup$ by rule ($\cup_{\text{top}}$).

Case ($\cup_{\rightarrow}$) We have $C = C_1 \rightarrow C_2$ and $\text{top}(C_2)$. We conclude $A <: C_1 \rightarrow C_2$ by rule ($\cup_{\rightarrow}$).

\[\square\]

4 Equivalence with BCD Subtyping

Having proved (trans), we next prove ($\rightarrow$) and ($\rightarrow\cap$) and then show that $A <: B$ is equivalent to $A \leq B$.

Lemma 8 ($\rightarrow$). If $C <: A$ and $B <: D$, then $A \rightarrow B <: C \rightarrow D$.

Proof. Consider whether $\text{top}(D)$ or not.

Case $\text{top}(D)$ We conclude $A \rightarrow B <: C \rightarrow D$ by rule ($\cup_{\rightarrow}$).

Case $\neg\text{top}(D)$ Consider whether $\text{top}(B)$ or not.

Case $\text{top}(B)$ So $\text{top}(D)$ (Prop. 3 part 4), but that is a contradiction.

Case $\neg\text{top}(B)$ We conclude that $A \rightarrow B <: C \rightarrow D$ by rule ($\rightarrow$).

\[\square\]

Lemma 9 ($\rightarrow\cap$). $(A \rightarrow B) \cap (A \rightarrow C) <: A \rightarrow (B \cap C)$

Proof. We consider the cases for whether $\text{top}(B)$ or $\text{top}(C)$.
Case top(B) and top(C) Then top(B ∩ C) and we conclude that (A → B) ∩ (A → C) <: A → (B ∩ C) by rule (U→′).

Case top(B) and ¬top(C) We conclude that (A → B) ∩ (A → C) <: A → (B ∩ C) by rule (→′), choosing the witness A → C and noting that C <: B by way of Proposition 3 part 5 and C <: C by Proposition 2 part 1.

Case ¬top(B) and top(C) We conclude that (A → B) ∩ (A → C) <: A → (B ∩ C) by rule (→′), this time with witness A → B and noting that B <: B by way of Proposition 3 part 5.

Case ¬top(B) and ¬top(C) Again we apply rule (→′), but with witness (A → B) ∩ (A → C).

We require one more lemma.

Lemma 10. A ≤ dom(A) → cod(A).

Proof. The proof is by induction on A.

Now for the proof of equivalence

Theorem 11 (Equivalence of the subtyping relations).

A <: B if and only if A ≤ B.

Proof. We prove each direction of the if-and-only-if separately.

A <: B implies A ≤ B We proceed by induction on the derivation of A <: B.

Case (refl) We conclude α ≤ α by (refl).

Case (lbL) By the induction hypothesis we have A ≤ C. By (inclL) we have A ∩ B ≤ A. We conclude that A ∩ B ≤ C by (trans).

Case (lbR) By the induction hypothesis we have B ≤ C. By (inclR) we have A ∩ B ≤ B. We conclude that A ∩ B ≤ C by (trans).

Case (glb) By the induction hypothesis we have A ≤ C and A ≤ D, so we conclude that A ≤ C ∩ D by (glb).

Case (→′) By the induction hypothesis we have C ≤ dom(B) and also cod(B) ≤ D. From B ≤ A we have A ≤ B. Then by Lemma 10 we have B ≤ dom(B) → cod(B). Also, we have dom(B) → cod(B) ≤ C → D by rule (→). We conclude that A ≤ C → D by chaining the three prior facts using (trans).

Case (Utop) We conclude that A ≤ U by (Utop).

Case (U→′) We have A ≤ U and U <: C → U. Also, C → U ≤ C → D because U <: D follows from top(D). Thus, applying (trans) we conclude A ≤ C → D.
$A \leq B$ implies $A <: B$. We proceed by induction on the derivation of $A \leq B$.

**Case (refl)** We conclude $A <: A$ by Prop. 2 part 1.

**Case (trans)** By the induction hypothesis, we have $A <: B$ and $B <: C$. We conclude that $A <: C$ by Theorem 7.

**Case (incl\(_L\))** We have $A <: A$ (Prop. 2 part 1), and therefore $A \cap B <: A$ by rule (lb\(_L\)).

**Case (incl\(_R\))** We have $B <: B$ (Prop. 2 part 1), and therefore $A \cap B <: B$ by rule (lb\(_R\)).

**Case (glb)** By the induction hypothesis, we have $A <: C$ and $A <: D$, so we conclude that $A <: C \cap D$ by (glb).

**Case (→)** By the induction hypothesis, we have $C <: A$ and $B <: D$. We conclude that $A \rightarrow B <: C \rightarrow D$ by Lemma 8.

**Case (→∩)** We conclude that $(A \rightarrow B) \cap (A \rightarrow C) <: A \rightarrow (B \cap C)$ by Lemma 9.

**Case (U→)** We conclude that $A <: \top$ by rule (U→).

**Case (U→\top)** We conclude that $A <: \top$ by rule (U→\top).

\[\square\]

5 Conclusion

In this article we present a new subtyping relation $A <: B$ for intersection types that enjoys the subformula property. None of the rules of the new subtyping relation are particularly novel, but the fact that we can prove transitivity directly from them is! We prove that the new relation is equivalent to the subtyping relation $A \leq B$ of Barendregt, Coppo, and Dezani-Ciancaglini.

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