Quantum extension for Newton’s law of motion

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Abstract. The main idea of I. Newton in Principia is a description of the laws of motion by a second-order differential equation. Classical Newtonian physics can describe stability trajectories in Inertial Reference Frames and some Non-Inertial ones. The variation of the action functional $S$ of stability trajectories equals to zero. The observational error is including the influence of the random fields’ background to the particle. Can we use another description in the form of a high-order differential equation? The high-order derivatives can be used as additional variables accounting for the influence of random fields background. Trajectories due the influence of the random fields’ background can be called instability random trajectories. They can be described by high order derivatives. Then the stability classical trajectories to be complemented by additional instability random trajectories. Quantum objects are described by the trajectory with neighborhoods. Quantum Probability can describe quantum objects in random fields. The variation of the action functional $S$ is defined by the Planck constant. For the common description of quantum theory and high-order theory, let us compare $r$-neighborhoods of quantum action functional with $r$-neighborhoods of the action functional of a high-order theory.

1. Introduction - Can classical description of physical reality be considered complete?
Quantum non-locality and quantum correlations are experimentally observable phenomena of quantum physics. At the same time, these experimental facts of quantum experiments have no explanation in classical physics and contradict in some cases to its axiomatic. Should we consider physics not a unified consistent science, but rather, regard classical and quantum physics distinct, independent and unlinked sciences, then there would be no problem. But actually, this is not the case. Physics must be a unified science, with classical physics describing macroscopic objects and quantum physics, microscopic ones. And depending on the situation we apply either former or latter ones. So, their non-contradiction is required for consistency, and vice versa.

A most successful solution in providing physics consistency would be a unified axiomatic of the unified physics underlying both classical and quantum physics. The search for such an axiomatic was for the last century being carried out through the search of suitable quantum axioms. Since 1935 the issue of completeness of the quantum-mechanical description of the reality has been one of the most extensively discussed in physics. Usually the classical axiomatic was considered completed. The psychological
inhibition of the extension of classical axiomatic was proliferated by extensive evidence of Newton's and Einstein's fundamental studies validity. However, a problem of incompleteness of the classical description of the physical reality has never been raised. May we pose such a question? Is the classical description of the physical reality complete?

According to Gödel’s theorem, there exist provisions in any theory that cannot be proved within this theory. It can also be added that no theory is complete. Axioms of any theory are not to be proved, but rather, they are conjectured, so any system of axioms may be replaced with another one. Newton's laws pertain to such improvable provisions. And Classical Physics can be extended to other field of an application.

In short, Newton's laws postulate the description of mechanical system dynamics with second order differential equations. Are there any cases of the reality description with higher order differential equations? The answer is positive, it is known that yes, but it is not Newtonian mechanics of Inertial Reference Frames. As an example of body dynamics description with higher order differential equations let us consider body dynamics in randomly Instability Reference Frames

\[ Q = f\left(t, q, \dot{q}, \ddot{q}, ..., q^{(n)}\right) \]

Here \( \tau \) is a time interval for averaging coordinates \( Q = \langle q(t) \rangle = \frac{1}{\tau} \left[ q(t + \tau) + q(t - \tau) \right] \).

The second example for Classical Physics extension is considering the Quantum Reference Frames.

**Definition.** Quantum Reference Frame is a Reference frames with the transformation of coordinates and time as

\[ Q = \langle q(t) \rangle = \int_{-\tau}^{+\tau} \psi^* q \psi dt \]

Here

\[ \psi = \psi_0 e^{i\left(\frac{\Delta p dt + \Delta t}{\hbar}\right)} = \psi_0 e^{i\frac{\Delta p}{\hbar} dt} = \psi_0 e^{i\frac{\Delta t}{\hbar}} \]

is wave function with the inertial force \( f_0 \) depend of high-order derivatives coordinates on time and \( f_0 \) corresponds of inertial forces and constant force.

An Instability Reference Frames is a method to describe influences of random fields onto both the particle to be described itself and to an observer. Changing from an Instability Reference Frame to an inertial one makes a free particle to randomly oscillate correlating with other free particles oscillations. If we consider single frequency oscillations, then these vibrations look coherent. Inertial frames transformations are prescribed as Galilean transforms (in a relativistic case, Lorentz transforms). Transformations of Instability Reference Frames differ from Galilean-Lorentz transforms by remainder terms in Taylor's expansion. Then free particles in Inertial Reference Frames shall be featured with uncertainty in coordinate and momentum, time and energy equal to remainder terms of Taylor expansion.

Considering a particle in an Inertial Reference Frame instead of Real ones, one shall either introduce inertial forces, that is, change from higher derivative description to the description without higher derivatives, but with inertia forces, or take into account remainder terms of Taylor expansion.

Modern physics (both classical and quantum) is physics of Inertial Reference Frames. The case of a Non-Inertial Reference Frame usually comes down to introducing inertia forces into an Inertial Reference Frame. Application of inertia forces enables reducing problems of physical system dynamics in a Non-Inertial Reference Frame to those in an Inertial Reference Frame through artificial introduction of inertia forces or through application of the d'Alembert's principle. At the same time, an Inertial Reference Frame doesn't exist in reality, as any reference frame is always influenced by small perturbing fields or forces. In the present study, we suggest considering only non-Inertial Instability Reference Frames as real ones. We shall call such non-Inertial Instability Reference Frames as Instability ones. Since the incipience of
d'Alembert's principle and up to now realness of inertia forces is a debatable issue. We believe that the question of inertia forces realness may be reduced to the question of Inertial Reference Frame realness.

In view of the above, there is a question to be answered, how could be physical systems described in non-Inertial Instability Reference Frames without introduction of inertia forces?

Commonly Reference Frames are called Inertial provided Newton's laws hold there; actually, Newton's laws constitute the axiomatic of Classical Physics. They postulate the description of physical systems behavior by second order differential equations. Abandonment of higher time derivatives of coordinates is related to the issue inertia forces in Inertial Reference Frames. So, to answer the above question we have to consider a more general case of higher orders differential equations and to extension Classical Physics with a description employing higher time derivatives of coordinate.

Changing from Inertial Reference Frame to Instability ones without inertia forces introduction means changing from second-order differential equations description of physical systems to their description with higher-order differential equations. Abandonment from employing higher order time derivatives of coordinate in the classical Newtonian physics does not mean they do not exist. They do exist in certain problems. But this is not the Newtonian Physics of Inertial Reference Frames.

Conversion of coordinates of a point particle between Instability Reference Frames provided τ is a time interval for averaging, shall be expressed as

\[
q = q(t) + \dot{q}(t)\tau + \Delta q(t),
\]

\[
\Delta q(t) = \sum_{k=2}^{N} (-1)^{k} \frac{1}{k!} \tau^{k} q^{(k)}(t)
\]  

same holds for momentum

\[
P = p(t) + \dot{p}(t)\tau + \Delta p(t),
\]

\[
\Delta p(t) = \sum_{k=2}^{N} (-1)^{k} \frac{1}{k!} \tau^{k} p^{(k)}(t)
\]

Here, \(\Delta q(t)\) and \(\Delta p(t)\) are remainder terms of the Taylor expansion. The remainder terms \(\Delta q(t)\) and \(\Delta p(t)\) in the Instability Reference Frame may be interpreted as uncertainties of coordinate and momentum of a point particle in this reference frame. In quantum mechanics, uncertainties of coordinate and momentum of a micro particle obey to the rule

\[
\Delta q(t)\Delta p(t) \geq h/2.
\]

In the classical physics can be introduced an uncertainty relation, as there always exist random small fields and forces influencing either the very system to be described or an observer, that is

\[
\left[\sum_{k=2}^{N} (-1)^{k} \frac{1}{k!} \tau^{k} q^{(k)}(t)\right] \left[\sum_{k=2}^{N} (-1)^{k} \frac{1}{k!} \tau^{k} p^{(k)}(t)\right] \geq H/2
\]

inertial one.

The supremum of the difference of the action function in Instability Reference Frames (with higher time derivatives of the generalized coordinate) from the classical mechanics action functions (without higher derivatives) is

\[
\sup [S(q, \dot{q}, \ddot{q}, ..., q^{(n)}, ...) - S(q, \dot{q})] = H/2.
\]

In this case, higher derivatives are non-local additional variables and disclose the sense of the classical analog \(H\) of the Planck's constant. The \(H\) constant defines the supremum of the influence of random fields onto the reference frame and the observer. We shall analyze this case in terms of Instability Reference Frames. In this case, \(H\) defines the supremum of the difference between an Instability
Reference Frame and an Inertial Reference Frame.

Action functions with higher-derivative variables describe instability trajectories of physical systems
dynamics and are accounted for via Instability Reference Frame.

In our case, the classical space is featured by infinite number of variables, same as Hilbertian one.
In the search for a unified axiomatics the classical constant $\hbar$ shall coincide with the quantum one, i.e.
the Planck constant $\hbar$. In this approach, the estimate of the Planck constant may be determined by higher
derivatives, playing the role of non-local hidden variables.

In this case the state of quantum object can be describe

$$\psi(t) = |q, q, \dot{q}, \ddot{q}, ..., q^{(n)}⟩ = |Q(t)⟩. \quad (8)$$

The transfer object from moment 1 to moment 2 can be describe by Lagrange function $L(Q)$

$$⟨Q, t_1|Q_2, t_2⟩ = \int_{t_1}^{t_2} DQ \exp(\frac{i}{\hbar} L(Q)dt. \quad (9)$$

And action function $A_{Q(R)}$ can be represent by Hamiltonian $\widetilde{H}$

$$A_{Q(R)} = \left\{ q(t), q(t), q(t), q(t), q^{(n)}(t) \left\lfloor \left[ \frac{\partial}{\partial t} - \widetilde{H} \right] q(t), q(t), q(t), q(t), q^{(n)}(t) \right\} \right\} \quad (10)$$

2. Stability principle

The condition of stability it is ordinary using in Mechanics and can be extend to other area of Physics.
In this case the condition of stability can be named Stability Principle. The stability principle is a
generalization of basic fundamental physical laws, such as least action principle, Newton’s laws, Euler-
Lagrange equations, Schrodinger equation, et al.

The stability principle may not only generalize, but also logically explain the basic laws of the
Nature. The stability principle enables using the stability of physical objects and their state for
explanation and generalization of such fundamental Nature laws as the least action principle, stability of
atoms, stationarity of possible trajectories, etc. It may be employed as a generalized law explaining such
a fundamental Nature law as the Least Action Principle. Therefore, it can evidently be applied to all
other laws following from the least action principle, such as Newton’s laws, Euler-Lagrange equations,
Schrodinger equation, laws of light and electromagnetic waves propagation, etc.

Stability in mechanical trajectory calculations is dealt with in publications of N.G. Chetayev [6].
According to him, “stability is probably an essentially general phenomenon that has to manifest itself in
principal nature laws”. In his opinion, stability is not a mere casualness, but rather, is a consequence of
system being affected by persistent small perturbations, which, no matter how small, affect the state of
a mechanical system. This definition differs from Lyapunov stability definition.

Let us define a stable state of a physical system through the stability principle.

**Stability principle:** The state of a physical system shall be considered stable if it is preserved
after the influence of external factors.

The state $Φ$ of a physical system shall be considered stable if its deviations from a perturbed state $F$
with small random perturbations are minimal, i.e. for an arbitrarily infinitesimal positive $H$, the
inequality holds:

$$|Φ - F| < H \quad (11)$$

Applying the stability principle for the action function $S$, we obtain [2,3]:
\[ \Delta S(q, \dot{q}, \ldots, \ddot{q}, \ldots) = S(q, \dot{q}, \ddot{q}, \ldots) - S(q, \dot{q}) < H, \] (12)

where small random influences are described with higher time derivatives of generalized coordinates \( q \). In \( S_s(q, \dot{q}, \ddot{q}, \ldots, \dddot{q}, \ldots) \) index \( s \) means both stability \( S(q, \dot{q}) \) and the instability action function \( \Delta S(q, \dot{q}, \ddot{q}, \ldots, \dddot{q}, \ldots) \) being described with the account of a small random influences \( \dot{q}, \ddot{q}, \ldots, \dddot{q}, \ldots \). Euler-Lagrange equations

\[
\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = 0 \tag{13}
\]

describe stability Lagrange function values \( L \) and instability \( \Delta L \), while describe motion at random perturbations. Stability condition in this case is

\[
|\Delta L| = \left| S_s(q, \dot{q}, \ddot{q}, \ldots, \dddot{q}, \ldots) - L(q, \dot{q}) \right| < H \tag{14}
\]

instability Euler-Lagrange equations \( \Delta L(q, \dot{q}, \ddot{q}, \ldots, \dddot{q}, \ldots) \) in Ostrogradsky formulation [4] accounting for random small influences in the form of higher derivatives.

3. Quantum extension for Newtonian dynamics

Ostrogradsky formalism [1] uses Lagrange function is

\[ L = L_s(q, \dot{q}, \ddot{q}, \ldots, \dddot{q}, \ldots) \tag{15a} \]

but not

\[ L = L(q, \dot{q}). \tag{15b} \]

Euler-Lagrange equation in this case is follow from least action principal [1-4]

\[
\delta S = \delta \int L_s(q, \dot{q}, \ddot{q}, \ldots, \dddot{q}, \ldots) dt = \sum_{n=0}^{N} \frac{d^n}{dt^n} L_s \frac{\partial L_s}{\partial q^{(n)}} dt = 0 \tag{16}
\]

or

\[
\frac{\partial L_s}{\partial q} - \frac{d}{dt} \frac{\partial L_s}{\partial \dot{q}} + \frac{d^2}{dt^2} \frac{\partial L_s}{\partial \ddot{q}} - \frac{d^3}{dt^3} \frac{\partial L_s}{\partial \dddot{q}} + \ldots + (-1)^n \frac{d^n}{dt^n} \frac{\partial L_s}{\partial q^{(n)}} + \ldots = 0 \tag{17}
\]

This equation can be written in the form of corrected Newton Second Law of Motion in Random Instability Reference Frames

\[ F - ma + f_0 = 0 \tag{18} \]

Here

\[ f_0 = mw, w = w(t) + w(t)\tau + \ldots + \frac{(-1)^s}{n!} \tau^n w^{(n)} + \ldots \tag{19} \]

is a random inertial force which can be represent by \( w \) Taylor expansion with high-order derivatives coordinates on time

This instability equation can be rewritten as
\[ F - ma + \tau \dot{ma} - \frac{1}{2!} \tau^2 ma^{(2)} + \ldots + (-1)^n \frac{1}{n!} \tau^n ma^{(n)} + \ldots = 0 \]
\[ \dot{a}(t_0) = 0, \ddot{a}(t_0) = 0, \ldots, \alpha^{(n)}(t_0) = 0, \ldots \]  
(20)

\( \tau \) being the time interval, and \( a \) – acceleration, \( t_0 \) is the moment of the time of stability trajectory. The second Newton’s law is a second-derivative differential equation describing stability dynamics
\[ F - ma = 0. \]  
(21)

Ehrenfest Theorem describe the averaging of second Newton Law for micro-objects [5]
\[ \frac{d}{dt} \langle p \rangle = -V'(q) \]  
(22)

After decomposition force
\[ F = -\langle V'(x) \rangle \]  
with \( \Delta q = q - \langle q \rangle \)
\[ m \frac{d^2}{dt^2} \langle q \rangle = F(\langle q \rangle) + \frac{1}{2} F'(\langle q \rangle)(\langle \Delta q \rangle^2) + \ldots \]  
(23)

Then the so-called “characteristic function” (or “moment generating function”) when \( n \) is the even for averaging [5] is
\[ \Phi(t) = \sum_{n=0}^{\infty} \frac{1}{n!} (it)^n \langle q^n \rangle = \int e^{itq} \rho(q) dq \]  
(24)

if \( \tau = it \), \( \rho(q) = \varphi^*(q)\varphi(q) = \delta(q - \langle q \rangle) \), \( q(\tau) = \int q\rho(q) dq \) is a distribution function of the random fields \( \varphi \). Similarly, \( \rho(p) = \varphi^*(p)\varphi(p) = \delta(p - \langle p \rangle) \) Fig. 1.

Then uncertainty relation (6b) can be representing as
\[ |\delta(q)\delta(p)| \leq H / 2. \]  
(6c)

In this case \( H \) defined by measurement units of coordinates \( q \) and momentum \( p = m\dot{q} \). To investigate behavior of bodies in microworld, let's shall compare the constant \( H \) assessing the influence of random perturbations with the Planck’s constant \( \hbar \). Then the maximum value of random variables shall be limited with the Heisenberg uncertainty relation, and we derive from (5)
\[ \sup \left| S(q, \dot{q}, \ddot{q}, \ldots, \dot{q}^{(n)}, \ldots) - S_{\text{av}}(q, \dot{q}) \right| = \hbar / 2 \]  
(25)

In this case, the quantum-mechanical laws (Bohm’s equations) follow from Hamilton-Jacobi equations with additions thereto of random influences in the form of higher derivatives, playing the role of non-local hidden variables. Actually, the Hamilton-Jacobi equation for the action \( S \) accounting for random infinitesimal influences in the form of higher derivatives has the form
\[ H = \frac{\partial S}{\partial \dot{q}} + V(q) = m\dot{q} \]
\[ -\frac{\partial S}{\partial t} = \frac{(\nabla S)^2}{2m} + V(q, \dot{q}) + Q(\dot{q}, \ddot{q}, ...) \]  

(26)

The additional summand \( Q \) accounting for random infinitesimal influences in the form of higher derivatives has in the first approximation the maximum value of \( Q \approx \alpha \dddot{q}^2 \). It is straightforward to compare the summand \( Q \approx \alpha \frac{\dddot{q}^2}{m^2} \) with Bohm’s quantum potential selecting the constant \( \alpha = \frac{i\hbar m}{2} \).

Then considering the latter equation together with continuity equation, it is straightforward to get the Schrodinger equation for the function \( \psi = e^{i\hat{S}} \):

\[ i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + V\psi. \]  

(27)

4. Verification of high-order derivatives as non-local hidden variables

The role of High-Order Derivatives as Hidden Variables can be verified by using the Equivalence Principal when acceleration arises from the influence of any random fields is equal to the gravitational field. Then the correlation factor for entangled photons polarization measurements may be presented as

\[ |M| = \left| \langle (\hat{A}^k g_{ik})(\hat{A}^m B^n g_{mn}) \rangle \right| \]  

(28)

Here, the random variables distribution function may be considered uniform, with the photon polarization varying from 0 to \( \pi \):

\[ \frac{1}{\pi} \int_0^\pi \rho(\phi) \, d\phi = 1. \]  

(29)

According to the definition,

\[ \cos \phi = \frac{\lambda^l A_k g_{ik}}{\sqrt{\lambda^l A_k A_l}} \]  

(30a)

\[ \cos(\phi + \theta) = \frac{\lambda^n B^m g_{mn}}{\sqrt{\lambda^n A_m B_n A_k}} \]  

(30b)

Hence, the correlation factor with the photon polarization varying from 0 to \( \pi \) is

\[ |M| = \left| \frac{1}{\pi} \int_0^\pi \rho(\phi) \cos \phi \cos(\phi + \theta) \, d\phi + \frac{1}{\pi} \int_0^{2\pi} \rho(\phi) \cos \phi \cos(\phi + \theta) \, d\phi \right| \]  

(31)

\[ |M| = |\cos \theta|. \]

From this follows that the Bell’s observable \( |S| < \sqrt{2} \) in our case does not contradict to experimental data. Bell's inequalities are same in either classical or quantum cases of accounting for random fields, forces and waves.

5. Conclusion

Additional terms in the form of higher derivatives may play the role of hidden variables complementing both quantum and classic mechanics. Additional terms have non-local character, which enables their employment for description of nonlocal effects of quantum mechanics. The effect of quantum correlations and non-locality of quantum states can be explained by the random character of interactions if extended particles. Such an approach can be realized within the scope of Ostrogradsky higher derivatives formalism.
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