Clique in $C_4$-free graphs of large minimum degree

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Abstract A graph $G$ is called $C_4$-free if it does not contain the cycle $C_4$ as an induced subgraph. Hubenko, Solymosi and the first author proved (answering a question of Erdős) a peculiar property of $C_4$-free graphs: $C_4$-free graphs with $n$ vertices and average degree at least $cn$ contain a complete subgraph (clique) of size at least $c'n$ (with $c' = 0.1c^2$). We prove here better bounds ($\frac{c'^2n}{2+c}$ in general and $(c - 1/3)n$ when $c \leq 0.733$) from the stronger assumption that the $C_4$-free graphs have minimum degree at least $cn$. Our main result is a theorem for regular graphs, conjectured in the paper mentioned above: $2k$-regular $C_4$-free graphs on $4k + 1$ vertices contain a clique of size $k + 1$. This is the best possible as shown by the $k$th power of the cycle $C_{4k+1}$.

Keywords $C_4$-free graphs · Large cliques · Regular graphs

1 Introduction

A graph is called here $C_4$-free, if it does not contain cycles on four vertices as an induced subgraph. The class of $C_4$-free graphs have been studied from many points of view, for example they appear in the theory of perfect graphs (as families containing chordal graphs). Sometimes the complements of $C_4$-free graphs are investigated, they are the graphs that do not contain $2K_2$ as an induced subgraph, sometimes called a strong matching of size two. Extremal properties of these graphs emerged in works of Bermond et al. [1,2] on
interconnection networks, popularized by Erdős and Nesetril, and generated extremal results, many on the strong chromatic index, for example [3–7].

In this paper we revisit [5] where the following problem (raised by Erdős) was investigated: how large is $\omega(G)$, the size of the largest complete subgraph (clique) in a dense $C_4$-free graph $G$? It was proved in [5] that in a $C_4$-free graph with $n$ vertices and at least $cn^2$ edges, $\omega(G) \geq c'n$, where $c'$ depends on $c$ only. The interest in this result is that as shown in [5], $C_4$ is the only graph with this property (apart from subgraphs of $C_4$). Let $f(c)$ denote the largest $c'$ for which every $C_4$-free graph with $n$ vertices and at least $cn^2$ edges contains a clique of size at least $c'n$. There is no conjecture on $f(c)$, apart from the question in [5] whether $f(1/4) = 1/4$ which is still open. Our main result, Theorem 1.1 gives a positive answer to the the special case of this question for regular graphs (asked also in [5]).

**Theorem 1.1** Every $2k$-regular $C_4$-free graph on $4k + 1$ vertices contains a clique of size $k + 1$.

As shown in [5], Theorem 1.1 is sharp, the cycle on $4k + 1$ vertices with all diagonals of length at most $k$ is a $2k$-regular $C_4$-free graph where the largest clique is of size $k + 1$.

The proof of Theorem 1.1 follows from understanding the work of Paoli et al. [7] on regular $2K_2$-free graphs.

Our other results are improvements over the estimates of [5] under the stronger assumption that the minimum degree $\delta(G)$ is given instead of the average degree.

**Theorem 1.2** For $C_4$-free graphs $\omega(G) \geq \frac{\delta^2(G)}{2n + \delta(G)}$.

Theorem 1.2 improves the estimate $\omega(G) \geq \frac{0.1a^2}{n}$ in [5] where $a$ is the average degree of $G$. For a certain range of $\delta(G)$, one can do better.

**Theorem 1.3** Suppose that $G$ is a $C_4$-free graph with $\delta(G) \leq \frac{11n}{15} \approx 0.733n$. Then $\omega(G) \geq \delta(G) - \frac{n}{3}$.

Note that for $\delta(G) \geq n/2$, Theorem 1.2 gives $\omega(G) \geq n/12$ while Theorem 1.3 gives $\omega(G) \geq n/6$. It seems that the remark “the best estimate we know is $n/6$” in [5] comes from this and it seems an open problem whether $\omega(G) \geq n/6$ follows from $|E(G)| \geq n^2/4$. We also note that for $0.382n \approx \frac{2n}{3+\sqrt{3}} \leq \delta(G)$ the bound of Theorem 1.3 is better than that of Theorem 1.2.

Our last estimate of $\omega(G)$ is for the case when $G$ has a large independent set.

**Theorem 1.4** For every $\varepsilon > 0$ the following holds. Let $G$ be a $C_4$-free graph on $n$ vertices with minimum degree at least $\delta$. Furthermore, let us assume that $G$ contains an independent set of size $t \geq \frac{n^2 - \delta^2}{e\delta^2} + 1$. Then $G$ contains a clique of size at least $(1 - \varepsilon)\delta^2/n$.

Thus we get the following corollary for Dirac graphs (graphs with minimum degree at least $n/2$).

**Corollary 1.5** For every $\varepsilon > 0$ the following holds. Let $G$ be a $C_4$-free graph on $n$ vertices with minimum degree at least $n/2$. Furthermore, let us assume that $G$ contains an independent set of size $t \geq \frac{n}{\varepsilon} + 1$. Then $G$ contains a clique of size at least $(1 - \varepsilon)n/4$.

Corollary 1.5 probably holds in a stronger form: $C_4$-free graphs with $n$ vertices and with minimum degree at least $n/2$ contain cliques of size at least $n/4$. 

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2 Properties of $C_4$-free graphs

The following easy lemma can be essentially found in [3, 4, 7] but we prove it to be self contained. Let $W_5$ denote the 5-wheel, the graph obtained from a five-cycle by adding a new vertex adjacent to all vertices. A clique substitution into a graph $G$ is the replacement of cliques into vertices of $G$ so that between substituted vertices all or none of the edges are placed, depending whether they were adjacent or not in $G$. Substituting an empty clique is accepted as a deletion of the vertex. Clique substitutions into $C_4$-free graphs result in $C_4$-free graphs.

Lemma 2.1 Suppose that $G$ is a $C_4$-free graph with $\alpha(G) \leq 2$. Then one of the following possibilities holds.

- the complement of $G$ is bipartite
- $G$ can be obtained from $W_5$ by clique substitution

Proof If $\overline{G}$, the complement of $G$ is not bipartite then we can find an odd cycle $C$ in $G$. Since $C$ cannot be a triangle, $|C| \geq 5$. However, $|C| \geq 7$ is impossible since $G$ is $C_4$-free. Thus $|C| = 5$. Since $G$ is $C_4$-free and $\alpha(G) = 2$, any vertex not on $C$ must be adjacent to exactly three consecutive vertices of $C$ or to all vertices of $C$. This procedure naturally allows to place all vertices not on $C$ into one of six groups and one can easily check that the groups must be cliques forming the claimed structure. $\Box$

Corollary 2.2 Suppose that $G$ is a $C_4$-free graph with $\alpha(G) \leq 2$. Then $\omega(G) \geq \frac{2n}{5}$.

In the proof of Theorem 1.1 we shall use the following result which is a special case of a more general result on regular $C_4$-free graphs (in [7, Theorem 4 and Lemma 7]). A set $S \subset V(G)$ is dominating if every vertex of $V(G) \setminus S$ is adjacent to some vertex of $S$.

Theorem 2.3 [7] Suppose that $G$ is a $2k$-regular $C_4$-free graph on $4k + 1$ vertices with $\alpha(G) \geq 3$. Then $G$ contains a pair $(u, w)$ of non-adjacent vertices forming a dominating set.

3 Proofs

Proof of Theorem 1.1 The proof comes from Theorem 2.3 and the analysis of Theorem 3 in [7]. We may suppose that $\alpha(G) \geq 3$, otherwise Corollary 2.2 gives a clique of size $\frac{8k+2}{5} \geq k + 1$. Theorem 2.3 ensures a dominating non-adjacent pair $(u, w)$ in $G$. Let $X$ be the set of common neighbors of $u, v$. Then

$$4k - |X| = d(u) + d(w) - |X| = |V(G)| - 2 = 4k - 1,$$

implying that $|X| = 1$. Set $X = \{x\}$, $U = N(u) - \{x\}$, $W = N(w) - \{x\}$, $U_1 = N(x) \cap U$, $W_1 = N(x) \cap W$, $U_2 = U - U_1$, $W_2 = W - W_1$. $\Box$

Claim $U_1, W_1$ span cliques in $G$.

Proof of Claim By symmetry, it is enough to prove the claim for $U_1$. Note that for $w_2 \in W_2, u_1 \in U_1$ we have $(w_2, u_1) \notin E(G)$ otherwise $(w_2, u_1, x, w, w_2)$ would be an induced $C_4$.

Suppose that $y, z \in U_1$ and $(y, z) \notin E(G)$. Let $N$ be the number of non-adjacent pairs $(p, q)$ such that $p \in \{y, z\}, q \notin U_1$.  

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• every $w_1 \in W_1$ contributes at least one to $N$, otherwise $(w_1, y, u, z, w_1)$ is a $C_4$
• every $u_2 \in U_2$ contributes at least one to $N$, otherwise $(u_2, y, x, z, u_2)$ is a $C_4$
• every $w_2 \in W_2$ contributes two to $N$ since $(w_2, u_1) \notin E(G)$ for every $u_1 \in U_1$
• $w$ contributes two to $N$

Therefore we have

$$N \geq |W_1| + |U_2| + 2|W_2| + 2$$

$$= (|W_1| + |W_2|) + (|U_2| + |W_2|) + 2 = (2k - 1) + 2k + 2 = 4k + 1.$$  

However, since $(y, z) \notin E(G)$, $N \leq 2(d_G(y) - 1) = 2(2k - 1) = 4k - 2$, a contradiction, proving that $U_1$ spans a clique in $G$ and the claim is proved. □

Now the two cliques $U_1 \cup \{u, x\}$ and $W_1 \cup \{w, x\}$ cover $A = V(G) \setminus (U_2 \cup W_2)$. Since $|A| = 4k + 1 - 2k = 2k + 1$ and the two cliques intersect in $\{x\}$, one of the cliques has size at least $k + 1$, finishing the proof. □

**Proof of Theorem 1.2** Here we follow the proof of the corresponding theorem in [5] with replacing average degree by minimum degree. Fix an independent set $S = \{x_1, x_2, \ldots, x_t\}$. Let $A_i$ be the set of neighbors of $x_i$ in $G$ and set $m = \max_{i \neq j}|A_i \cap A_j|$. Since $G$ is $C_4$-free, all the subgraphs $G(A_i \cap A_j)$ are complete graphs, and thus $m \leq \omega(G)$. Using that $|A_i| \geq \delta$, we get

$$t \delta \leq \sum_{i=1}^{t} |A_i| < n + \sum_{1 \leq i < j \leq t} |A_i \cap A_j|,$$

implying that

$$\omega(G) \geq m \geq \frac{t \delta - n}{(\frac{t}{2})}.$$  

If $\alpha(G) \geq \frac{2n}{\delta}$ then set $t = \lceil \frac{2n}{\delta} \rceil$ and we get

$$\omega(G) \geq \frac{\left\lceil \frac{2n}{\delta} \right\rceil \delta - n}{\left(\frac{\left\lceil \frac{2n}{\delta} \right\rceil}{2}\right)} \geq \frac{n}{\left(\frac{\left\lceil \frac{2n}{\delta} \right\rceil}{2} + 1\right)}.$$  

If $\alpha(G) \leq \frac{2n}{\delta}$ then of course $\alpha(G) \leq \lceil \frac{2n}{\delta} \rceil$ as well. Now we shall use the following claim:

$$\omega(G) \geq \frac{n}{(\alpha(G) + 1)}.$$  

This follows by selecting an independent set $S$ with $|S| = \alpha(G) = \alpha$.

Using the notation introduced above, the $\left(\frac{\alpha}{2}\right)$ sets $A_i \cap A_j$ and the $\alpha$ sets $\{x_i\} \cup B_i$ cover the vertex set of $G$ where $B_i$ denotes the set of vertices whose only neighbor in $S$ is $x_i$. All of these sets span complete subgraphs because $G$ is $C_4$-free and $S$ is maximal. Now we have

$$\omega(G) \geq \frac{n}{(\alpha(G) + 1)} \geq \frac{n}{\left(\frac{\alpha}{2} + 1\right)}.$$  

Therefore in both cases we have

$$\omega(G) \geq \frac{n}{\left(\frac{\left\lceil \frac{2n}{\delta} \right\rceil}{2} + 1\right)} \geq \frac{n}{\left(\frac{\left\lceil \frac{2n}{\delta} \right\rceil}{2} + 1\right)} = \frac{\delta^2}{2n + \delta}.$$  

□
Proof of Theorem 1.3 If $\alpha(G) \leq 2$ then by Lemma 2.1 and by the upper bound on $\delta(G)$,

$$\omega(G) \geq \frac{2n}{S} \geq \delta(G) - \frac{n}{3}.$$ 

If $\alpha(G) \geq 3$, then select an independent set $\{v_1, v_2, v_3\}$ and let $A_i$ denote the set of neighbors of $x_i$. Then

$$3\delta(G) \leq \sum_{i=1}^{3} |A_i| < n + \sum_{1 \leq i < j \leq 3}|A_i \cap A_j|,$$

implying that for some $1 \leq i < j \leq 3$, the clique induced by $A_i \cap A_j$ is larger than $\delta(G) - \frac{n}{2}$. \hfill $\Box$

Proof of Theorem 1.4 Let $S = \{x_1, x_2, \ldots, x_t\}$ be an independent set in $G$ of size $t \geq \frac{n^2 - \delta^2}{\varepsilon \delta} + 1$. Let $A_i$ be the set of neighbors of $x_i$ in $G$. Note that being induced $C_4$-free implies that for every $i, j, i \neq j$ the set $A_i \cap A_j$ induces a clique in $G$. Thus if we show that there are $i, j, i \neq j$ such that $|A_i \cap A_j| \geq (1 - \varepsilon)\delta^2/n$, then we are done. Assume indirectly, that for every $i, j, i \neq j$ we have $|A_i \cap A_j| < (1 - \varepsilon)\delta^2/n$ and from this we will get a contradiction.

Consider an auxiliary bipartite graph $G_b$ between the sets $S$ and $V = V(G)$, where we connect each $x_i$ with its neighbors in $G$. We will give both a lower and an upper bound for the quantity $\sum_{v \in V} \deg_{G_b}(v)^2$. To get a lower bound we apply the Cauchy–Schwarz inequality and the minimum degree condition:

$$\sum_{v \in V} \deg_{G_b}(v)^2 \geq n \left( \sum_{v \in V} \deg_{G_b}(v) \right)^2 \geq n \left( \sum_{i=1}^{t} \frac{|A_i|}{n} \right)^2 \geq n \left( \frac{t \delta}{n} \right)^2 = \frac{t^2 \delta^2}{n}.$$ 

To get the upper bound we use the indirect assumption:

$$\sum_{v \in V} \deg_{G_b}(v)^2 = \sum_{i=1}^{t} \sum_{j=1}^{t} |A_i \cap A_j| = \sum_{i=1}^{t} |A_i| + \sum_{i \neq j} |A_i \cap A_j|$$

$$\leq nt + (1 - \varepsilon)\frac{\delta^2 t (t - 1)}{n} = \frac{t^2 \delta^2}{n} + nt - \frac{\delta^2 t (t - 1)}{n} - \varepsilon \frac{\delta^2 t (t - 1)}{n} \leq \frac{t^2 \delta^2}{n}$$

(using $t \geq \frac{n^2 - \delta^2}{\varepsilon \delta^2} + 1$), a contradiction. \hfill $\Box$

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