On simple modules of cyclotomic quiver Hecke algebras of type A

Alexander Ferdinand Kerschl

Abstract: We give a classification of the graded simple modules of cyclotomic quiver Hecke algebras of type A using the diagram calculus of the diagrammatic Cherednik algebra. We also obtain a non-trivial lower bound for the dimension of the simple modules which is independent of the field characteristic.

1 Introduction

Khovanov and Lauda [KL09] and, independently, Rouquier [Rou12] introduced a family of diagrammatic algebras that depend on a quiver. These algebras are now known as quiver Hecke or KLR algebras. They are widely studied because they categorify the negative half of the quantum group $U_q^-(g)$, where $g$ is the Kac-Moody Lie algebra associated with the quiver. Recently Webster [Web13] constructed diagrammatic algebras depending on a parameter $\theta$, which are generalisations of the KLR algebras. Hence, he called them weighted KLR algebras but they are also known as diagrammatic Cherednik algebras.

For quivers of type A, each quiver Hecke algebra has a cyclotomic quotient $R_n^\Lambda$ that is isomorphic to the cyclotomic Ariki-Koike Hecke algebra $H_n^\Lambda(q)$, by Brundan and Kleshchev [BK09a]. Ariki [Ari01] showed that the simple modules of $H_n^\Lambda(q)$ are indexed by a certain set $\mathcal{X}^\ell$ of multipartitions and Brundan and Kleshchev [BK09b] used the isomorphism to prove that the same holds for the simple modules of $R_n^\Lambda$. Webster [Web13], on the other hand, showed that the diagrammatic Cherednik algebra contains a subalgebra that is isomorphic to $R_n^\Lambda$ independently of the choice of $\theta$. He [Web13] and Bowman [Bow17] used this isomorphism to construct a family of different graded cellular bases, each depending on $\theta$, for $R_n^\Lambda$. This gives $R_n^\Lambda$ the structure of a graded cellular algebra, as in [HM10]. The corresponding cell modules $S_\theta^\lambda$ of $R_n^\Lambda$ are indexed by multipartitions $\lambda \in \mathcal{P}^\ell(\theta)$ and they come equipped with an inner product. As in [GL96] or [HM10], one can then find all simple modules of $R_n^\Lambda$ by looking at the list of non-zero quotient modules $D_\theta^\lambda = S_\theta^\lambda / \text{Rad}(S_\theta^\lambda)$, where $\text{Rad}(S_\theta^\lambda)$ is the radical of the inner product on $S_\theta^\lambda$. It is a non-trivial problem to determine when $D_\theta^\lambda$ is non-zero.

For each $\theta$ there is a crystal graph that has a recursively defined set $\mathcal{U}^\ell(\theta)$ of multipartitions as vertices. Jacon [Jac18] gives an alternative description of the set $\mathcal{U}^\ell(\theta)$ and allows us to construct special paths $\mathbf{d}$ in a directed graph for each multipartition in $\mathcal{U}^\ell(\theta)$. We can now state our main result:

**Main Theorem.** Let $\lambda \in \mathcal{P}^\ell(\theta)$. Then the following are equivalent:

(a) $D_\theta^\lambda \neq 0$

(b) $\lambda \in \mathcal{U}^\ell(\theta)$

(c) $|\langle c_{\mathbf{d}}^\lambda, c_{\mathbf{d}}^\lambda \rangle_\lambda| = 1$, for any decomposition path $\mathbf{d}$ for $\lambda$
As an immediate application of our constructions, we also obtain a non-trivial lower bound for the graded dimension of the graded simple module $D^\lambda_\theta$ which is independent of the characteristic of the field. In the special case when $\theta$ is ‘well-separated’ (see after Definition 2.11), $\mathcal{H}^\ell = \mathcal{U}^\ell(\theta)$ so this recovers the classification of the simple $\mathcal{R}_n^\Lambda$-modules given by Brundan and Kleshchev [BK09b] and Ariki [Ari01]. In this case we can also prove a generalisation of the Dipper-James-Murphy conjecture [DJM95, Conjecture 8.13] (see also [GL96, Question 5.10 (i)):

**Theorem.** Let $\lambda \in \mathcal{P}_n^\ell(\theta)$ and $\theta$ well-separated. Then $D^\lambda_\theta \neq 0$ if and only if there exists a DJM-tableau of shape $\lambda$.

Ariki and Jacon [AJ07] proved the original conjecture in type B. The generalised conjecture was recently proved by Jacon [Jac18], however, we found an alternative proof using the combinatorial tools we construct in this paper. We conjecture that our proof can be lifted to the case of arbitrary $\theta$.

The paper is structured into 4 sections. In section 2 we start by recalling the theory of graded cellular algebras following [GL96, HM10] and by defining the diagrammatic Cherednik algebra with the necessary combinatorics and diagrams, as in [Web13, Bow17]. For section 3 we define the set $\mathcal{U}^\ell(\theta)$ and recall Jacon’s [Jac18] recent alternative description using staggered sequences. These allow us to construct the decomposition paths $d$ and a family of standard tableaux associated to it. In the key theorem we use the basis elements in $S^\lambda_\theta$ associated to these tableaux to compute the inner product, which implies that $D^\lambda_\theta \neq 0$ if $\lambda \in \mathcal{U}^\ell(\theta)$. This and consequences of this theorem are part of section 4. Section 5 is mostly dedicated to prove the key theorem using diagram calculations.

This paper is written based on Version 4 of Bowman’s paper [Bow17], as available from arXiv.org. Just prior to submitting this paper the author learned that Bowman has recently updated his paper on the ArXiv. The most recent version of his paper uses substantially different notation from the previous versions. This paper uses the notation from version 4 of [Bow17]. In addition, the latest version of Bowman’s paper gives a classification of the simple $\mathcal{R}_n^\Lambda(\theta)$-modules, which gives an independent proof of parts of Main Theorem 4.3.

## 2 Diagrammatic Cherednik Algebra

This section introduces the diagrammatic Cherednik algebras, which are the main objects of study in this paper. The diagrammatic Cherednik Algebras contain a cyclotomic quiver Hecke or KLR Algebra of type A as a subalgebra. Both of these algebras come with bases that give them the structure of a graded cellular algebra. The representation theory of graded cellular algebras was established by Hu and Mathas [HM10], extending Graham and Lehrer’s work in the ungraded case [GL96]. As we recall, cellular algebras come equipped with a family of cell modules and these can be used to construct all simple modules.
2.1 Graded cellular algebras

In this paper graded will always mean \( \mathbb{Z} \)-graded. Throughout let \( K \) be a field of any characteristic. A \textit{graded algebra} \( A \) is a unital associative \( K \)-algebra such that

\[
A = \bigoplus_{j \in \mathbb{Z}} A_j \quad \text{and} \quad A_x A_y \subseteq A_{x+y}, \quad \text{for all } x, y \in \mathbb{Z}.
\]

Elements \( a \in A_x \), for some \( x \in \mathbb{Z} \), are \textit{homogeneous} of degree \( x \). Set \( \deg(a) = x \). Let \( A \) be the ungraded algebra obtained by forgetting the grading.

A graded (right) \( A \)-module \( M \), similarly, has a direct sum decomposition \( M = \bigoplus_{j \in \mathbb{Z}} M_j \) and must satisfy \( M \) being an \( A \)-module and \( M_x A_y \subseteq M_{x+y} \). Given a graded \( A \)-module \( M \) let \( M(x) \) be the graded \( A \)-module that is obtained by shifting the grading on \( M \) up by \( x \), i.e. \( (M(x))_y = M_{y-x} \). The graded version of submodules, left modules, and so on are defined accordingly.

Let \( t \) be an indeterminate over \( \mathbb{N}_0 \). If \( M \) is a graded \( A \)-module such that \( M_j \) is finite dimensional over \( K \), then its \textit{graded dimension} is the Laurent polynomial \( \dim_t(M) = \sum_{j \in \mathbb{Z}} \dim(M_j)t^j \in \mathbb{N}_0[t, t^{-1}] \).

2.1 Definition ([GL96, Definition 1.1],[HM10, Definition 2.1]). Let \( A \) be a finite dimensional graded \( K \)-algebra. A \textit{graded cell datum} for \( A \) is an ordered quadruple \((\mathcal{P}, T, c, \deg)\), where \((\mathcal{P}, \triangleright)\) is a poset, \( T(\lambda) \) a finite set for \( \lambda \in \mathcal{P} \), and

\[
c : \bigcup_{\lambda \in \mathcal{P}} T(\lambda) \times T(\lambda) \rightarrow A : (s, t) \mapsto c_{st}^\lambda, \quad \text{and} \quad \deg : \bigcup_{\lambda \in \mathcal{P}} T(\lambda) \rightarrow \mathbb{Z}
\]

are two functions such that \( c \) is injective and the following hold:

(a) The set \( \{c_{st}^\lambda | \lambda \in \mathcal{P}, s, t \in T(\lambda)\} \) is a \( K \)-basis of \( A \).

(b) Let \( \lambda \in \mathcal{P} \), for any \( s, t \in T(\lambda) \) and \( a \in A \) there exist scalars \( r_{tv}(a) \in K \) not depending on \( s \) such that

\[
c_{st}^\lambda a = \sum_{v \in T(\lambda)} r_{tv}(a)c_{sv}^\lambda \quad (\text{mod } A^v^\lambda),
\]

where \( A^v^\lambda \) is the \( K \)-submodule of \( A \) spanned by \( \{c_{ab}^\mu | \mu \triangleright \lambda \text{ and } a, b \in T(\mu)\} \).

(c) There is a \( K \)-linear anti-isomorphism \( * : A \rightarrow A \) determined by \( (c_{st}^\lambda)^* = c_{ts}^\lambda \), for any \( \lambda \in \mathcal{P} \) and \( s, t \in T(\lambda) \).

(d) Each element \( c_{st}^\lambda \) is homogeneous of degree \( \deg(c_{st}^\lambda) = \deg(s) + \deg(t) \), for \( \lambda \in \mathcal{P} \) and \( s, t \in T(\lambda) \).

A \textit{graded cellular algebra} is a graded algebra with a graded cell datum. The basis \( \{c_{st}^\lambda \} \) is a graded cellular basis of \( A \).

Given a graded cellular basis for an algebra \( A \) we can canonically define special modules using property (b) of the definition.

2.2 Definition. Let \( A \) be a graded cellular algebra and let \( \lambda \in \mathcal{P} \). Then the \textit{graded cell module} \( C^\lambda \) is the graded right \( A \)-module

\[
C^\lambda = \bigoplus_{j \in \mathbb{Z}} C_j^\lambda,
\]
where $C^λ_j$ is the $K$-span of $\{c^λ_t | t ∈ T(λ) \text{ and } \deg(t) = j\}$ and where the $A$-action is given by

$$c^λ_i a = \sum_{v ∈ T(λ)} r_{iv}(a)c^λ_v,$$

and the scalars $r_{iv}(a)$ are the same as in Definition 2.1(b).

As in [GL96, After Definition 1.1], applying $*$ to Definition 2.1(b):

(b') Let $λ ∈ \mathcal{P}$, for any $s, t ∈ T(λ)$ and $a ∈ A$ there exist scalars $r_{ts}(a) ∈ K$ not depending on the choice of $t$ such that

$$ac^λ_{st} = \sum_{v ∈ T(λ)} r_{vs}(a)c^λ_v \pmod{A^λ},$$

where $A^λ$ is the $K$-submodule of $A$ spanned by $\{c^μ_{ab} | μ ▷ λ \text{ and } a, b ∈ T(μ)\}$.

By (b) and (b'), there is a well-defined inner product on $C^λ$ that is uniquely determined by

$$c^λ_{st}c^λ_{uv} ≡ \langle c^λ_s, c^λ_t⟩_{A^λ} c^λ_{uv} \pmod{A^λ}.$$

Equivalently $c^λ_{st}c^λ_{uv} = \langle c^λ_s, c^λ_t⟩_{λ} c^λ_{uv}$.

2.4 Proposition ([HM10, Lemmata 2.6, 2.7]). Let $λ ∈ \mathcal{P}$ and $a ∈ A$, $x, y ∈ C^λ$. Then $\langle , \rangle_λ$ is symmetric, homogeneous of degree 0, and

$$\langle xa, y \rangle_λ = \langle x, ya^* \rangle_λ.$$

It follows that $\text{Rad}(C^λ) = \{x ∈ C^λ | \langle x, y \rangle_λ = 0, \text{ for all } y ∈ C^λ\}$ is a graded submodule of $C^λ$.

2.5 Definition. Suppose $λ ∈ \mathcal{P}$. Let $D^λ := C^λ / \text{Rad}(C^λ)$.

Define the subset $\mathcal{P}_0 = \{λ ∈ \mathcal{P} | D^λ \neq 0\} ⊆ \mathcal{P}$.

2.6 Theorem ([GL96, Theorem 3.4], [HM10, Theorem 2.10]). Let $λ, μ ∈ \mathcal{P}_0$ and $K$ be any field. Then

(a) $D^λ$ is an absolutely irreducible graded $A$-module.

(b) $D^λ ≅ D^μ(⟨x⟩)$, for some $x ∈ \mathbb{Z}$, if and only if $λ = μ$ and $x = 0$.

(c) $\{D^λ(⟨x⟩) | λ ∈ \mathcal{P}_0 \text{ and } x ∈ \mathbb{Z}\}$ is a complete set of pairwise non-isomorphic graded simple $A$-modules.

2.2 Multipartitions

This section sets up some of the combinatorics that we will need.

Fix an integer $e$ such that $e ≥ 2$ or $e = 0$ and define an indexing set $I = \mathbb{Z}/e\mathbb{Z}$. Let $Γ_∞$ be the oriented quiver with vertices $i ∈ I$ and arrows $i ↦ i + 1$, for all $i ∈ I$. So $Γ_∞$ is either the linear quiver of type $A_∞$, if $e = 0$, or the cyclic quiver of type $A^{(1)}_e$, if $e ≠ 0$. To this quiver we associate its symmetric Cartan matrix $(c_{ij})_{i,j ∈ I}$, where $c_{ij} = 2δ_{ij} − δ_{i(j+1)} − δ_{i(j-1)}$.

Following Kac [Kac90], let $\hat{s}_i$, for $e > 0$, and $\hat{s}_i$, for $e = 0$, be the Kac-Moody algebra of $Γ_∞$ with simple roots $\{α_i | i ∈ I\}$ and fundamental weights $\{Λ_i | i ∈ I\}$. Let $( , )$ be the bilinear form determined by

$$(α_i, α_j) = c_{ij} \quad \text{and} \quad (Λ_i, α_j) = δ_{ij}, \text{ for } i, j ∈ I.$$
Let \( P_+ = \bigoplus_{i \in I} \mathbb{N}_0 \Lambda_i \) be the \textbf{dominant weight lattice} and \( Q_+ = \bigoplus_{i \in I} \mathbb{N}_0 \alpha_i \) the \textbf{positive root lattice}. Fix a dominant weight \( \Lambda \in P_+ \) and set \( \ell = \sum_{i \in I} (\Lambda, \alpha_i) \).

Fix an \( n \in \mathbb{N}_0 \). An \( \ell \)-multipartition of \( n \) is an \( \ell \)-tuple \( \lambda = (\lambda^{(1)}, \ldots, \lambda^{(\ell)}) \) of partitions \( \lambda^{(k)} \), such that

\[
|\lambda| = \sum_{k=1}^{\ell} \sum_{j=1}^{\infty} \lambda_j^{(k)} = |\lambda^{(1)}| + \ldots + |\lambda^{(\ell)}| = n.
\]

We identify a multipartition with its \textbf{Young diagram} of boxes with coordinates \((r, c, l)\) of row, column, and component. We visualize these diagrams in the \textbf{Russian style}, where every box has diagonal length \( 2\ell \).

\[\text{2.7 Example. Let } \lambda = ((2, 1), (3, 1)).\]

\[\text{\begin{figure}[h]
\centering
\includegraphics[width=.5\textwidth]{example.png}
\end{figure}}\]

Let \( \mathcal{M}_n^\ell = \{(r, c, l) | r, c, l \in \mathbb{N}, r + c \leq 2n, \text{ and } 1 \leq l \leq \ell\} \) be a set of boxes. Fix an \( e \)-multicharge \( \kappa = (\kappa_1, \ldots, \kappa_\ell) \in I^\ell \) such that \( (\Lambda, \alpha_i) = |\{1 \leq k \leq \ell | \kappa_k = i\}| \). With the multicharge we define the \textbf{residue} of a box \( \alpha \in \mathcal{M}_n^\ell \) to be \( \text{res}(\alpha) = c - r + \kappa_l \). We define an equivalence relation on boxes by \( (r, c, l) \sim (r', c', l') \) if \( l = l' \) and \( c - r = c' - r' \). We refer to its equivalence classes as \textbf{diagonals}. It is easy to see that all boxes in a diagonal have the same residue. A diagonal is an \( i \)-diagonal if the residue of its boxes is \( i \), for \( i \in I \).

A \textbf{loading} is an \( \ell \)-tuple \( \theta = (\theta_1, \ldots, \theta_\ell) \in \mathbb{Z}^\ell \) such that \( \theta_i - \theta_j \notin \ell \mathbb{Z} \), for \( 1 \leq i < j \leq \ell \). Fix a loading \( \theta \) and a quantity \( \varepsilon < \frac{1}{n} \). We define a function \( \theta_\varepsilon : \mathcal{M}_n^\ell \rightarrow \mathbb{R} \), by \( \theta_\varepsilon(r, c, l) = \theta_l + (r - c)\ell + (r + c)\varepsilon \). For a box \( \alpha \in \mathcal{M}_n^\ell \) we then assign to it a real valued coordinate \( x_\alpha = \theta_\varepsilon(\alpha) \in \mathbb{R} \).

\[\text{2.8 Definition ([Bow17, Definition 1.2]). Let } \alpha, \beta \in \mathcal{M}_n^\ell. \text{ Then } \alpha \prec \theta \beta \text{ if } x_\alpha < x_\beta. \text{ We say that } \alpha \text{ is to the left of } \beta \text{ or equivalently } \beta \text{ is to the right of } \alpha.\]

\[\text{2.9 Lemma ([Bow17, Remark 7]). } (\mathcal{M}_n^\ell, \prec_\theta) \text{ is a totally ordered set.}\]

We visualize this order by arranging our Russian diagrams such that for any box \( \alpha \) its top corner aligns vertically with the coordinate \( x_\alpha \).

\[\text{2.10 Example.}\]

\[\text{\begin{figure}[h]
\centering
\includegraphics[width=.5\textwidth]{example.png}
\end{figure}}\]

A \textbf{box configuration} is a subset of \( \mathcal{M}_n^\ell \). Let \( \mathcal{C}_n^\ell(\theta) \) be the set of all box configurations with \( n \) boxes ordered by \( \prec_\theta \). This ordering of boxes defines a partial order on \( \mathcal{C}_n^\ell(\theta) \).
2.11 Definition ([Bow17, Definition 1.2]). Let \( \lambda, \mu \in \mathcal{C}_n^\ell(\theta) \). Then \( \lambda \) dominates \( \mu \) or \( \mu \leq_\theta \lambda \) if for all \( \alpha \in \lambda \cup \mu \)

\[
\{ \beta \in \mu | \text{res}(\beta) = \text{res}(\alpha), \beta \prec_\theta \alpha \} \} \leq \{ \beta \in \lambda | \text{res}(\beta) = \text{res}(\alpha), \beta \prec_\theta \alpha \}\}
\]

Let \( \mathcal{P}_n^\ell(\theta) \subset \mathcal{C}_n^\ell(\theta) \) be the poset of \( \ell \)-multipartitions of \( n \) ordered by the dominance order \( \leq_\theta \). Also we set \( \mathcal{P}_n^\ell(\theta) = \bigcup_{\mathcal{P}_n^\ell(\theta)} \mathcal{P}_n^{\ell}(\theta) \).

As Bowman [Bow17, Example 1.3] remarks, \( \leq_\theta \) coincides with the normal dominance order on multipartitions, if \( \theta_{i+1} - \theta_i > n\ell \), for all \( i = 1, \ldots, \ell - 1 \). Define \( \theta \) to be well-separated in this case.

A box \( \alpha \) is addable for \( \lambda \in \mathcal{P}_n^\ell(\theta) \) if \( \lambda \cup \alpha \in \mathcal{P}_n^{\ell+1}(\theta) \) and it is removable if \( \lambda \setminus \alpha \in \mathcal{P}_n^{\ell-1}(\theta) \).

Let \( \lambda \in \mathcal{P}_n^{\ell}(\theta) \). A standard tableau \( t \) for \( \lambda \) is a bijection \( t : \lambda \rightarrow \{1, \ldots, n\} \) such that

\[
t(r + 1, c, l) > t(r, c, l) \quad \text{and} \quad t(r, c + 1, l) > t(r, c, l),
\]

for all \((r, c, l), (r+1, c, l), (r, c+1, l) \in \lambda \). The shape of a standard tableau \( t \) for \( \lambda \) is \( \text{shape}(t) = \lambda \).

Let \( \text{Std}(\lambda) \) be the set of all standard tableaux of shape \( \lambda \). Let \( t_{ik} \) be the subtableau of \( t \) that consists only of the boxes containing the numbers up to \( k \), for \( 1 \leq k \leq n \). For a standard tableau \( t \), the box containing the number \( k \) is \( t^{-1}(k) \). Then the residue sequence of \( t \), \( \text{res}(t) \) is the \( n \)-tuple \( (\text{res}(t^{-1}(1)), \ldots, \text{res}(t^{-1}(n))) \in \mathbb{P}^n \). More general the multipartition residue sequence \( \text{res}(\lambda) \), for \( \lambda \in \mathcal{P}_n^{\ell}(\theta) \), is the sequence of residues of the boxes of \( \lambda \) ordered from left to right.

Additionally, for a tableau \( t \) of shape \( \lambda \in \mathcal{P}_n^{\ell}(\theta) \) and for some \( 1 \leq k \leq n \) let \( \mathcal{M}^\theta(k) \) and \( \mathcal{R}^\ell(k) \) be the sets of addable and respectively removable boxes of the multipartition \( \text{shape}(t_{ik}) \) of residue \( \text{res}(t^{-1}(k)) \) and are to the right of \( t^{-1}(k) \) with respect to \( \prec_\theta \).

2.12 Definition ([Web13, Definition 2.15],[Bow17, Definition 1.7]). For \( \lambda \in \mathcal{P}_n^{\ell}(\theta) \) and \( t \in \text{Std}(\lambda) \), define the degree of \( t \) to be

\[
\text{deg}_\theta(t) = \sum_{k=1}^{n} (|\mathcal{M}^\theta(k)| - |\mathcal{R}^\ell(k)|).
\]

2.3 Diagrammatic Cherednik Algebra

In this section, following Webster [Web13], we introduce the diagrammatic Cherednik algebra, which is a graded cellular algebra.

To define the Diagrammatic Cherednik algebra, we need to define the diagrams that span this diagrammatic algebra. Let \( x, y \in \mathbb{R} \) then a string of type \((x, y)\) is a coloured path in the Euclidean plane from \((x, 1)\) to \((y, 0)\) such that it is diffeomorphic to the unit interval from \((0, 1)\) to \((0, 0)\). We assign colours to strings to help distinguish them. A solid string of type \((x, y)\) is a black string of type \((x, y)\). A ghost string of a solid string is a dotted grey string and a translation of the solid string by \( l \) units to the left. A red string is a red string of type \((\theta_l, \theta_l)\), for \( 1 \leq l \leq \ell \).

A solid string and its ghost string carry a residue \( i \in I \), we call such strings \( i \)-strings. Red strings of type \((\theta_l, \theta_l)\) carry the residue \( \kappa_l \). Solid strings are allowed to have a finite number of dots on them.

Given \( \lambda \in \mathcal{C}_n^{\ell}(\theta) \), let \( X_\lambda = \{x_\alpha | \alpha \in \lambda \} \subset \mathbb{R} \) be the discrete subset of points on the real line corresponding to all boxes \( \alpha \in \lambda \). Let \( \text{hom}(\lambda, \mu) \) be the set of all bijections from \( X_\lambda \) into \( X_\mu \).

Fix \( \lambda, \mu \in \mathcal{C}_n^{\ell}(\theta) \). A \( \theta \)-diagram of type \((\lambda, \mu)\) is a diagram that

- has \( \ell \) red strings of type \((\theta_l, \theta_l)\), for \( 1 \leq l \leq \ell \),
- has \( n \) solid strings of type \((x_\alpha, f(x_\alpha))\), for \( x_\alpha \in X_\lambda \) and \( f \in \text{hom}(\lambda, \mu) \).
• has \( n \) ghost strings,
• has any finite number of dots on its solid strings,
• has a residue \( i \in I \) assigned to each solid string,
• has at most two strings intersect at any point,
• has no dots on any intersection of strings, and
• has no tangencies of strings.

Two \( \theta \)-diagrams are equivalent if they are the same diagram up to isotopy.

Let \( D \) be a diagram of type \((\lambda, \mu)\). Define the pairs \( \text{top}(D) = (\lambda, i) \) and \( \text{bot}(D) = (\mu, j) \), where \( i, j \in I^n \) are the sequences of residues read from left to right at the top and bottom of the diagram \( D \) respectively.

2.13 Example. Let \( \theta = (0, 8) \), \( \lambda = ((2, 1), (3, 1)) \), and \( \mu = (\emptyset, (1^7)) \). Then a possible \( \theta \)-diagram of type \((\lambda, \mu)\) is:

\[
\begin{array}{ccccccc}
1 & 0 & 2 & 2 & 1 & 0 & 2 \\
\end{array}
\]

Here \( \text{top}(D) = (\lambda, (1, 0, 2, 2, 1, 0, 2)) \) and \( \text{bot}(D) = (\mu, (0, 0, 1, 1, 2, 2, 2)) \).

Note that throughout this paper we draw our diagrams with solid strings that have segments in which they are drawn almost horizontally. It will often look like the solid and its ghost string overlap in these segments. However, this is just due to difficulties in plotting these diagrams. In fact these strings are not overlapping but are just very close to each other in these segments.

Let \( D, E \) be two \( \theta \)-diagrams of types \((\lambda, \mu)\) and \((\nu, \xi)\). Then we construct a diagram \( D \times E \) by translating \( D \) upwards by 1 unit on top of \( E \) and rescaling the stacked diagram vertically by a factor of \( \frac{1}{2} \). We then define a multiplication on \( \theta \)-diagrams by

\[
D \circ E = \begin{cases} 
D \times E, & \text{if } \text{bot}(D) = \text{top}(E) \\
0, & \text{otherwise.}
\end{cases}
\]

If \( D \circ E \neq 0 \) then it is a \( \theta \)-diagram of type \((\lambda, \xi)\).

There is an anti-isomorphism \( * \) on \( \theta \)-diagrams that is given by reflection through the line \( y = \frac{1}{2} \), which turns a \( \theta \)-diagram of type \((\lambda, \mu)\) into a \( \theta \)-diagram of type \((\mu, \lambda)\).

A \( \theta \)-diagram is \textit{unsteady} if there is a solid string \( n\ell \) units to the left of the leftmost red string.

2.14 Definition ([Web13, Definition 4.2]). The diagrammatic Cherednik algebra \( \mathcal{A}_n(\theta) \) is the unital associative algebra spanned by all \( \theta \)-diagrams with \( n \) solid strings and with multiplication \( \circ \) and subject to the following relations:

(a) (Dots and crossings) A dot on an \( i \)-string can pass through any crossing except:
(b) (Double crossings) A double crossing between any two strings can be pulled apart except in the following cases, where \( j = i + 1 \in I \):

(I) \( \begin{array}{c}
\includegraphics[width=0.3\textwidth]{double_crossing_1.png}
\end{array} = 0 \)

(II) \( \begin{array}{c}
\includegraphics[width=0.6\textwidth]{double_crossing_2.png}
\end{array} \)

(III) \( \begin{array}{c}
\includegraphics[width=0.6\textwidth]{double_crossing_3.png}
\end{array} \)

(IV) \( \begin{array}{c}
\includegraphics[width=0.3\textwidth]{double_crossing_4.png}
\end{array} \)

(V) \( \begin{array}{c}
\includegraphics[width=0.3\textwidth]{double_crossing_5.png}
\end{array} \)

(c) (Triple crossings) A string can be pulled through a triple crossing except in the following cases, where \( j = i + 1 \in I \):

(I) \( \begin{array}{c}
\includegraphics[width=0.6\textwidth]{triple_crossing_1.png}
\end{array} \)

(II) \( \begin{array}{c}
\includegraphics[width=0.6\textwidth]{triple_crossing_2.png}
\end{array} \)

(III) \( \begin{array}{c}
\includegraphics[width=0.6\textwidth]{triple_crossing_3.png}
\end{array} \)

(d) (Unsteady) Any unsteady diagram is zero.
The Cherednik algebra comes with a natural grading, which is defined locally.

\[
\begin{align*}
\deg( \begin{array}{c} \ast \end{array} ) &= 2 \\
\deg( \begin{array}{cc} & \ast \end{array} ) &= -2\delta_{ij} \\
\deg( \begin{array}{cc} \ast & \\ \ast & \ast \end{array} ) &= \delta_{i+1,j} = \deg( \begin{array}{cc} \ast & \\ \ast & \ast \end{array} ) \\
\deg( \begin{array}{cc} & \ast \\ \ast & \ast \end{array} ) &= \delta_{ij} = \deg( \begin{array}{cc} \ast & \\ \ast & \ast \end{array} )
\end{align*}
\]

2.15 Definition ([Bow17, Definition 2.1]). Given \( \lambda \in \mathcal{P}_n(\theta) \) and \( \mu \in \mathcal{C}_n(\theta) \) a semistandard tableau \( \Sigma \) of shape \( \lambda \) and weight \( \mu \) is a bijection \( \Sigma : \lambda \rightarrow \lambda' \) such that

(a) \( \Sigma(1,1,l) > \theta_l \),

(b) \( \Sigma(r,c,l) > \Sigma(r-1,c,l) + \ell \),

(c) \( \Sigma(r,c,l) > \Sigma(r,c-1,l) - \ell \).

The set of all semistandard tableau of shape \( \lambda \) and weight \( \mu \) is denoted by \( \text{SSStd}_\theta(\lambda, \mu) \).

Given a semistandard tableau \( \Sigma \) of shape \( \lambda \) and weight \( \mu \) we can associate to it a \( \theta \)-diagram \( C_\Sigma \) of type \( (\lambda, \mu) \), where the solid strings are strings of type \( (x, \Sigma(x)) \) and residue \( \text{res}(x) \), for all \( \alpha \in \lambda \), and such that there is a minimal number of intersections.

This choice of diagram is not unique, so we choose one for every semistandard tableau and fix this choice until further notice. In fact, later we will fix special diagrams for some tableaux. By Bowman [Bow17, 4.1], the results in this section are independent of the choice of diagrams. Taking a pair of semistandard tableaux \( (\Sigma, \Sigma) \in \text{SSStd}_\theta(\lambda, \mu) \times \text{SSStd}_\theta(\lambda, \nu) \), we define the \( \theta \)-diagram \( C_{\Sigma \Sigma} = C_\Sigma S \circ C_\Sigma \), where \( S \) is the anti-isomorphism on \( \mathcal{A}_n^\infty(\theta) \) that flips the diagram about the line \( y = \frac{1}{2} \).

2.16 Theorem ([Bow17, Theorem 4.10]). The algebra \( \mathcal{A}_n^\infty(\theta) \) is a graded cellular algebra with poset \( (\mathcal{P}_n(\theta), \prec_{\theta}) \) and graded cellular basis

\[
\{C^\lambda_{\Sigma \Sigma} | (\Sigma, \Sigma) \in \text{SSStd}_\theta(\lambda, \mu) \times \text{SSStd}_\theta(\lambda, \nu), \lambda \in \mathcal{P}_n(\theta), \text{ and } \mu, \nu \in \mathcal{C}_n(\theta)\},
\]

with \( \deg_\theta(C^\lambda_{\Sigma \Sigma}) = \deg_\theta(\Sigma) + \deg_\theta(\Sigma) \).

Fix the weight \( \omega := (0, \ldots, 0, (1^n)) \in \mathcal{P}_n(\theta) \). Then there is a correspondence \( \Psi : \text{Std}_\theta(\lambda) \rightarrow \text{SSStd}_\theta(\lambda, \omega) \) [Bow17, Proposition 4.4] between standard tableaux of shape \( \lambda \) and semistandard tableaux of shape \( \lambda \) and weight \( \omega \), for any \( \lambda \in \mathcal{P}_n(\theta) \).

For any \( \mu \in \mathcal{P}_n(\theta) \) let \( e_\mu(\mathbf{i}) \) be the \( \theta \)-diagram of type \( (\mu, \mu) \) such that all strings are vertical and the \( k \)-th solid string from the left is an \( i_k \)-string, for \( 1 \leq k \leq n \). Define the idempotent \( E_\omega = \sum_{t \in \Theta} e_\omega(\mathbf{i}) \) and consider the graded cellular subalgebra \( E_\omega \mathcal{A}_n^\infty(\theta) E_\omega \). Define \( c^\lambda_{st} = C^\lambda_{\Sigma \Sigma} \), where \( \Sigma = \Psi(\mathbf{s}) \) and \( \Sigma = \Psi(\mathbf{t}) \), for \( \lambda \in \mathcal{P}_n(\theta) \) and \( s, t \in \text{Std}_\theta(\lambda) \). Then \( E_\omega \mathcal{A}_n^\infty(\theta) E_\omega \) has graded cellular basis

\[
\{C^\lambda_{\Sigma \Sigma} | \lambda \in \mathcal{P}_n(\theta), (\Sigma, \Sigma) \in \text{SSStd}_\theta(\lambda, \omega)\} = \{c^\lambda_{st} | \lambda \in \mathcal{P}_n(\theta), s, t \in \text{Std}(\lambda)\}.
\]

Let \( \mathcal{P}_n^\lambda \) be the **cyclotomic quiver Hecke algebra**, defined by Brundan and Kleshchev [BK09b, 2.3]. As we do not need the actual definition of this algebra for our computations to avoid unnecessary technicalities we refrain from giving it here.
2.17 Theorem ([Web13, Proposition 2.19], [Bow17, Theorem 4.5]). Let $\theta \in \mathbb{Z}^l$ be any loading, then $\mathcal{R}_n^\lambda \cong E_\omega A_n^\phi(\theta)E_\omega$ as graded cellular algebras. In particular, $E_\omega A_n^\phi(\theta)E_\omega$ is independent of $\theta$.

2.18 Corollary ([Web13, Theorem 2.26], [Bow17, Theorem 5.1]). The cyclotomic KLR algebra $\mathcal{R}_n^\lambda$ is a graded cellular algebra with graded cellular basis $\{c^\lambda_{\alpha}\}$, where $\deg_\theta(c^\lambda_{\alpha}) = \deg_\theta(s) + \deg_\theta(t)$.

Henceforth, we set $\mathcal{R}_n^\lambda(\theta) = E_\omega A_n^\phi(\theta)E_\omega$. Define $S^\lambda_\theta$ to be the graded cell module of $\mathcal{R}_n^\lambda(\theta)$ determined by $\lambda$ and the graded cellular basis $\{c^\lambda_{\alpha}\}$ and define $D^\lambda_\theta = S^\lambda_\theta / \text{Rad}(S^\lambda_\theta)$. Note that even though $\mathcal{R}_n^\lambda(\theta)$ does not depend on $\theta$, the basis $\{c^\lambda_{\alpha}\}$ and, hence, $S^\lambda_\theta$ and $D^\lambda_\theta$ do all. For later reference, we want to relate these results to the (ungraded) Ariki-Koike Hecke algebras.

2.19 Definition (see [AK94, Definition 3.1]). Let $1 \neq q \in K^\times$ such that $q^r = 1$. The cyclotomic Ariki-Koike Hecke algebra $\mathcal{H}_n^\lambda(q)$ is the unital associative $K$-algebra with generators $\{T_0, \ldots, T_{n-1}\}$ and relations:

\[(T_0 - q^{s_1}) \ldots (T_0 - q^{s_l}) = 0,\]

\[T_0 T_i T_0 = T_i T_0 T_i,\]

\[(T_i + 1)(T_i - q) = 0, \text{ for } 1 \leq i \leq n - 1,\]

\[T_{i+1} T_i T_{i+1} = T_i T_{i+1} T_i, \text{ for } 1 \leq i \leq n - 2,\]

\[T_i T_j = T_j T_i, \text{ for } 0 \leq i < j - 1 \leq n - 2.\]

2.20 Theorem ([BK09a, Main Theorem]). There is a graded isomorphism $\mathcal{H}_n^\lambda(q) \cong \mathcal{R}_n^\lambda(\theta)$.

3 Decomposition paths

Recently, Jacon [Jac18] introduced staggered sequences in order to better understand the set of Uglov multipartitions. These multipartitions are important because they index the vertices in crystal graphs of the integral highest weight modules for $U_q(\hat{sl}_n)$. We extend Jacon’s ideas to define decomposition paths. We use these paths to compute the inner products in the graded cell modules in section 5.

3.1 Uglov multipartitions and decomposition paths

In this section we define the set of Uglov multipartitions that is originally recursively defined in literature using good boxes; see for example [MM90], [AM00, Introduction], and [Ugl100, Section 2.2]. We recall Jacon’s [Jac18] recent result that gives an alternative recursive description of this set using staggered sequences. We use these sequences to define decomposition paths that help us further understand the elements in this set.

Let $\lambda \in \mathcal{R}_n^\lambda(\theta)$. Define $A_\theta(\lambda)$ the set of addable boxes of $\lambda$ and $R_\theta(\lambda)$ the set of removable boxes of $\lambda$ and let $B_\theta(\lambda) = A_\theta(\lambda) \cup R_\theta(\lambda)$. For $i \in I$, $A^\lambda_{\phi}(\lambda) = \{\alpha \in A_\theta(\lambda) | \text{res}(\alpha) = i\}$ and $R^\lambda_{\phi}(\lambda) = \{\rho \in R_\theta(\lambda) | \text{res}(\rho) = i\}$. Similarly, $B^\lambda_{\phi}(\lambda) = A^\lambda_{\phi}(\lambda) \cup R^\lambda_{\phi}(\lambda)$. $\alpha \in A^\lambda_{\phi}(\lambda)$ is an addable normal box if for all $\rho \in R^\lambda_{\phi}(\lambda)$ with $\rho \prec_\theta \alpha$

\[[\{\tilde{\alpha} \in A^\lambda_{\phi}(\lambda) | \rho \prec_\theta \tilde{\alpha} \prec_\theta \alpha\}] \geq |\{\tilde{\rho} \in R^\lambda_{\phi}(\lambda) | \rho \prec_\theta \tilde{\rho} \prec_\theta \alpha\}|.\]

$\rho \in R^\lambda_{\phi}(\lambda)$ is a removable normal box if for all $\alpha \in A^\lambda_{\phi}(\lambda)$ with $\rho \prec_\theta \alpha$

\[[\{\tilde{\rho} \in R^\lambda_{\phi}(\lambda) | \rho \prec_\theta \tilde{\rho} \prec_\theta \alpha\}] \geq |\{\tilde{\alpha} \in A^\lambda_{\phi}(\lambda) | \rho \prec_\theta \tilde{\alpha} \prec_\theta \alpha\}|.\]
Let
\[ \mathcal{N}A_\theta^i(\lambda) = \{ \alpha \in A_\theta^i(\lambda) | \alpha \text{ is normal} \} \quad \text{and} \quad \mathcal{N}R_\theta^i(\lambda) = \{ \rho \in R_\theta^i(\lambda) | \rho \text{ is normal} \}. \]

\( \alpha \in \mathcal{N}A_\theta^i(\lambda) \) is an **addable good box** if for all \( \alpha' \in \mathcal{N}A_\theta^i(\lambda) \): \( \alpha' \preccurlyeq_\theta \alpha \). \( \rho \in \mathcal{N}R_\theta^i(\lambda) \) is a **removable good box** if for all \( \rho' \in \mathcal{N}R_\theta^i(\lambda) \): \( \rho \preccurlyeq_\theta \rho' \).

### 3.1 Definition
The set of \( \theta \)-Uglov multipartitions \( \mathcal{U}^\ell(\theta) \) is the subset of \( \mathcal{P}^\ell(\theta) \) determined by \( \lambda \in \mathcal{P}^\ell(\theta) \) if either \( \lambda = \emptyset \) or \( \lambda \) has a removable good box \( \rho \) and \( \lambda \setminus \rho \in \mathcal{U}^\ell(\theta) \). Define \( \mathcal{U}_n^\ell(\theta) = \mathcal{U}^\ell(\theta) \cap \mathcal{P}_n^\ell(\theta) \) to be the set of \( \theta \)-Uglov \( \ell \)-multipartitions of \( n \). If \( \theta \) is well-separated then \( \mathcal{K}_n = \mathcal{U}_n^\ell(\theta) \) is the set of Kleshchev multipartitions.

Recently, Jacon [Jac18] gave an alternative recursive description of the set \( \mathcal{U}^\ell(\theta) \).

### 3.2 Definition
Let \( i \in I \) and \( \alpha, \beta \in B_\theta^i(\lambda) \) with \( \alpha \prec_\theta \beta \). Then \( \alpha \) and \( \beta \) are adjacent if when \( \alpha \preccurlyeq_\theta \gamma \preccurlyeq_\theta \beta \), for \( \gamma \in B_\theta^i(\lambda) \), then \( \gamma \in \{ \alpha, \beta \} \). If two boxes are adjacent to each other we write \( \alpha \prec_\theta \beta \).

### 3.3 Definition ([Jac18, Definition 3.1.1])
Let \( i \in I \). A \( (\theta, i) \)-staggered sequence of \( \lambda \) is a non-empty sequence \( s = (s_1, \ldots, s_m) \) of removable boxes in \( B_\theta^i(\lambda) \) such that

1. \( s_i \prec_\theta s_{i+1} \), for \( 1 \leq i < m \),
2. \( \alpha \preccurlyeq_\theta s_m \) for all \( \alpha \in B_\theta^i(\lambda) \),
3. if there is \( \alpha \in B_\theta^i(\lambda) \) with \( \alpha \prec_\theta s_1 \) then \( \alpha \in A_\theta^i(\lambda) \).

We call \( s_1 \) the **leftmost** and \( s_m \) the **rightmost** box of \( s \) and define \( |s| := m \) and \( \text{res}(s) := i \).

Staggered sequences depend on \( \theta \) but we omit this from our notation. If \( s = (s_1, \ldots, s_m) \) is a staggered sequence of \( \lambda \) then define \( \lambda \setminus s := \lambda \setminus \{ s_1, \ldots, s_m \} \). If \( \lambda \) has a staggered sequence and \( s_1, \ldots, s_l \) are all the staggered sequences of \( \lambda \) with respective rightmost boxes \( s_1', \ldots, s'_l \), then there exists a unique \( 1 \leq a \leq l \) such that \( s'_b \prec_\theta s'_a \), for \( 1 \leq b \leq l \), \( b \neq a \). The staggered sequence \( s^\prec := s_a \) is the **rightmost staggered sequence**.

Note that an arbitrary multipartition \( \lambda \) does not need to have a staggered sequence for any \( i \in I \). However, we now recursively define a subset of \( \mathcal{P}_n^\ell(\theta) \) consisting of multipartitions that always have staggered sequences.

### 3.4 Definition ([Jac18, Definition 3.1.2])
The set of \( \theta \)-Staggered multipartitions \( \mathcal{S}^\ell(\theta) \) is the subset of \( \mathcal{P}^\ell(\theta) \) determined by \( \lambda \in \mathcal{S}^\ell(\theta) \) if

1. \( \lambda = \emptyset \) or
2. \( \lambda \) has a staggered sequence \( s = (s_1, \ldots, s_m) \) and \( \lambda \setminus s \in \mathcal{S}^\ell(\theta) \).

Define \( \mathcal{S}_n^\ell(\theta) = \mathcal{S}^\ell(\theta) \cap \mathcal{P}_n^\ell(\theta) \) the set of Staggered \( \ell \)-multipartitions of \( n \).

### 3.5 Theorem ([Jac18, Theorem 5.1.1])
As sets \( \mathcal{S}^\ell(\theta) = \mathcal{U}^\ell(\theta) \).

### 3.6 Theorem
The number of simple \( \mathcal{B}_n^\lambda(\theta) \)-modules is \( |\mathcal{U}_n^\ell(\theta)| \).
So for the remainder of the paper we are allowed to use Definition 3.4 as the definition of \( \mathcal{U}^\ell(\theta) \).

3.7 Definition. Let \( c \geq 2 \) be an integer and \( \lambda = (\lambda^{(1)}, \ldots, \lambda^{(\ell)}) \in \mathcal{P}^\ell(\theta) \). Then \( \lambda \) is \( e \)-restricted if \( \lambda^{(i)}_j - \lambda^{(i)}_{j+1} < c \), for all \( i \geq 1 \) and \( 1 \leq l \leq \ell \).

The following is a well known result for Kleshchev multipartitions but we have not found it in the literature for Uglov multipartitions. However, the proof is short and is analogous to the Kleshchev case.

3.8 Lemma. Let \( \lambda \in \mathcal{U}^\ell_n(\theta) \). Then \( \lambda \) is \( e \)-restricted.

Proof. We argue by induction on \( |\lambda| = n \). If \( n = 1 \), then there is nothing to show. So assume \( n > 1 \) and the statement being true for \( n-1 \). Let \( s = (s_1, s_2, \ldots, s_m) \) be a staggered sequence of \( \lambda \), for some \( m \in \mathbb{N} \), then \( \lambda' = \lambda \setminus s_1 \in \mathcal{U}^\ell_{n-1}(\theta) \) because \( s' = (s_2, \ldots, s_m) \) is a staggered sequence of \( \lambda' \). By induction, \( \lambda' \) is \( e \)-restricted. Assume \( \lambda \) is not \( e \)-restricted and let \( s_1 = (r, c, l) \). Then \( \lambda^{(i)}_r - \lambda^{(i)}_{r+1} = e \). However, this implies \( \alpha = (r+1, c-e, l) \) is an addable box of \( \lambda \) with \( \text{res}(\alpha) = \text{res}(s_1) \) and \( s_1 <_\theta \alpha \). This is a contradiction to \( s \) being a staggered sequence. \( \square \)

For the special case \( \ell = 1 \) the reverse implication is also true.

3.9 Theorem ([MM90]). Let \( \lambda \in \mathcal{P}^1_n(\theta) \). Then \( \lambda \in \mathcal{U}^1_n(\theta) \) if and only if \( \lambda \) is \( e \)-restricted.

3.10 Definition. Let \( \lambda \in \mathcal{U}^\ell_n(\theta) \). The decomposition graph \( \Gamma(\lambda) \) of \( \lambda \) is the edge labelled directed graph with vertex set \( \{ \mu \in \mathcal{U}^\ell(\theta) | \mu \subset \lambda \} \) and edges \( \mu \xrightarrow{s} \nu \) if \( s \) is a staggered sequence of \( \mu \) and \( \nu = \mu \setminus s \). A decomposition path \( d = s_1 s_2 \ldots s_k \) for \( \lambda \) is a finite path

\[
\lambda = \lambda_0 \xrightarrow{s_1} \lambda_1 \xrightarrow{s_2} \ldots \xrightarrow{s_k} \lambda_k = \emptyset
\]

in \( \Gamma(\lambda) \). The length of a decomposition path \( d = s_1 s_2 \ldots s_k \) is \( |d| = k \).

Recall, for every \( \lambda \in \mathcal{U}^\ell(\theta) \) there is its unique rightmost staggered sequence \( s^< \). The rightmost decomposition path \( d^< \) is the unique decomposition path in \( \Gamma(\lambda) \) where every edge is labelled by a rightmost staggered sequence.

3.11 Example. Let \( \lambda = (3, 1, 1), e = 3 \). Then \( \lambda \) has two decomposition paths that are encoded in \( \Gamma(\lambda) \):

![Diagram](image)

The path at the bottom is the rightmost decomposition path. \( \diamond \)
3.2 Decomposition tableaux

This section constructs special standard tableaux for Uglov multipartitions, that will be used for our computations of the inner product on the graded cell modules $S^\lambda_\theta$ in section 5.

3.12 Definition. Let $\lambda$ be an Uglov multipartition and let $d = s_1s_2 \ldots s_k$ be a decomposition path for $\lambda$. The positive decomposition tableau $t_d$ is the unique standard tableau of shape $\lambda$ such that

(a) $t_d(\alpha) < t_d(\beta)$ if $\alpha, \beta \in s_j$, for some $1 \leq j \leq k$, and $\alpha \prec_{\theta} \beta$,

(b) $t_d(\alpha) < t_d(\beta)$ if $\alpha \in s_j$ and $\beta \in s_{j'}$ such that $1 \leq j' < j \leq k$.

Similarly, the negative decomposition tableau $t_d^\circ$ is the unique standard tableau of shape $\lambda$ such that

(a) $t_d^\circ(\alpha) > t_d^\circ(\beta)$ if $\alpha, \beta \in s_j$, for some $1 \leq j \leq k$, and $\alpha \prec_{\theta} \beta$,

(b) $t_d^\circ(\alpha) < t_d^\circ(\beta)$ if $\alpha \in s_j$ and $\beta \in s_{j'}$ such that $1 \leq j' < j \leq k$.

3.13 Example. Let $\lambda = ((2, 1), (3, 1))$, $\epsilon = 3$, $\theta = (0, 8)$, and $\kappa = (0, 0)$. Then there is a unique decomposition path $d = s_1s_2s_3$, where $s_1$ is of residue 2, $s_2$ of residue 1, and $s_3$ of residue 0.

First we give some immediate consequences of this definition. Let $d$ be a decomposition path of length $k$, for $k \in \mathbb{N}_0$, and $m_j = |s_j|$, for $1 \leq j \leq k$. The Young subgroup $S_d := \Pi_{j=1}^k S_{m_j}$ acts on the decomposition tableaux of $d$ by

$$(t_d \cdot \sigma)(\alpha) := \sigma_j(t_d(\alpha)) \quad \text{and} \quad (t_d^\circ \cdot \sigma)(\alpha) := \sigma_j(t_d^\circ(\alpha)),$$

where $\sigma = (\sigma_1, \ldots, \sigma_k) \in S_d$ and $\alpha \in s_j$, for $1 \leq j \leq k$. By definition, staggered sequences are removable boxes for their corresponding multipartition, so $t_d \cdot \sigma$ and $t_d^\circ \cdot \sigma$ are standard tableaux, for $\sigma \in S_d$. Let $\omega_0^d \in S_d$ be the longest element. By definition, $t_d \cdot \omega_0^d = t_d^\circ$. The tableaux in the set $\{t_d \cdot \sigma | \sigma \in S_d\}$ are pairwise distinct.

3.14 Lemma. Let $d$ and $e$ be decomposition paths of $\lambda \in \mathcal{U}^\ell(\theta)$. Then for all $\sigma \in S_d$, $\text{res}(t_d) = \text{res}(t_d \cdot \sigma)$ and $\text{res}(t_d) = \text{res}(t_e) \iff d = e$.

Proof. Let $d = s_1 \ldots s_k$ and $\sigma = (\sigma_1, \ldots, \sigma_k) \in S_d$. $(t_d \cdot \sigma)(\alpha) = \sigma_j(t_d(\alpha))$, for all $\alpha \in s_j$. But by definition, all boxes of $s_j$ are boxes of the same residue. So the residue sequence of $t_d$ does not change under the action of $S_d$.

For the second part, if $d = e$ then $\text{res}(t_d) = \text{res}(t_e)$ trivially. So assume $d \neq e$ and let $d = s_1 \ldots s_k$ and $e = t_1 \ldots t_l$, for some $k, l \in \mathbb{N}$. Then there exists $j \in \mathbb{N}$ minimal such that $s_j \neq t_j$ and $s_i = t_i$ for all $1 \leq i < j$. But this implies the residue of the boxes of $s_j$ are different from those in $t_j$, as there is only one staggered sequence for a given residue. Hence, $\text{res}(t_d) \neq \text{res}(t_e)$. \hfill \Box

3.15 Corollary. Let $d$ and $e$ be decomposition paths of $\lambda \in \mathcal{U}^\ell(\theta)$ and $\sigma \in S_d$ and $\tau \in S_e$. Then $t_d \cdot \sigma = t_e \cdot \tau$ if and only if $d = e$ and $\sigma = \tau$. 

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Next we determine the degree of the tableaux of the form $t_d \cdot \sigma$ for some decomposition path $d$. Let $\ell$ be the usual **length function** on $\mathfrak{S}_d$. Recall **Definition 2.12**, $\deg_\sigma(t) = \sum_{j=1}^n \left( |A_0^\sigma(j)| - |R_0^\sigma(j)| \right)$, where

$$A_0^\sigma(j) = \{ \alpha \in A_0^\sigma(\text{shape}(t_{ij})) | t^{-1}(j) \prec_\sigma \alpha \},$$

$$R_0^\sigma(j) = \{ \rho \in R_0^\sigma(\text{shape}(t_{ij})) | t^{-1}(j) \prec_\sigma \rho \},$$

and $i = \text{res}(t^{-1}(j))$, for some $i \in I$.

**3.16 Lemma.** Let $d = s_1 \ldots s_k$ a decomposition path for $\lambda$. Then

$$\deg(t_d \cdot \sigma) = \ell(\omega_0^d) - 2\ell(\sigma),$$

for all $\sigma \in \mathfrak{S}_d$. In particular,

$$\deg(t_d \cdot \sigma) + \deg(t_d^\sigma \cdot \sigma^{-1}) = 0.$$

**Proof.** First we claim $\deg(t_d) = \ell(\omega_0^d)$. Use induction on $|d| = k$. Assume $k = 1$ then $d = s$, where $s = (s_1, \ldots, s_m)$ with $\text{res}(s_1) = \ldots = \text{res}(s_m) = i$, for some $i \in I$, and $\omega_0^d \in \mathfrak{S}_d = \mathfrak{S}_m$. By definition, $t_d(s_j) < t_d(s_{j+1})$ and $s_j \prec_\sigma s_{j+1}$, for all $1 \leq j < m$. Hence, $t_d(s_j) = j$ and shape$(\{t_d\} \downarrow j) = \{s_1, \ldots, s_j\}$, for $1 \leq j \leq m$. By definition, the boxes of a staggered sequence are the rightmost removable boxes for a given residue. So $|A_0^\sigma(j)| = m - j$ and $|R_0^\sigma(j)| = 0$, for all $1 \leq j \leq m$. Hence,

$$\deg_\sigma(t_d) = \sum_{j=1}^m (m-j) = \sum_{j=1}^{m-1} (m-j) = \frac{m(m-1)}{2} = \ell(\omega_0^d).$$

Now assume $k > 1$, so $d = s_1s_2 \ldots s_k$. Let $|\lambda| = n$, for some $n \in \mathbb{N}$. Let $s_1 = (s_1, \ldots, s_m)$, for some $n > m \in \mathbb{N}$, and $\text{res}(s_1) = \ldots = \text{res}(s_m) = i$, for some $i \in I$. Let $d' = s_2 \ldots s_k$ and $\lambda' = \lambda \setminus s_1$. Then by induction, $\deg_\sigma(t_d) = \ell(\omega_0^d)$, for $\omega_0^d \in \mathfrak{S}_d$. But $\lambda' = \text{shape}(t_{d'}) = \text{shape}(\{t_d\} \downarrow \lambda - m)$, so $\deg_\sigma(t_d) - \deg_\sigma(t_d') = \sum_{j=n-m+1}^n (|A_0^\sigma(j')| - |R_0^\sigma(j')|)$. Also $\ell(\omega_0^d) - \ell(\omega_0^{d'}) = \frac{m(m-1)}{2}$. Now similarly to the base case by definition, $t_d(s_j) < t_d(s_{j+1})$ and $s_j \prec_\sigma s_{j+1}$, for all $1 \leq j < m$. But also by definition, $t_d(\alpha) < t_d(s_j)$, for all $\alpha \in \lambda'$ and $1 \leq j \leq m$. Hence, $t_d(s_j) = n - m + j$ and shape$(\{t_d\} \downarrow n-m+j) = \lambda' \cup \{s_1, \ldots, s_j\}$, for $1 \leq j \leq m$. So $|A_0^\sigma(n-m+j)| = m - j$ and $|R_0^\sigma(n-m+j)| = 0$, for all $1 \leq j \leq m$. Hence,

$$\deg_\sigma(t_d) = \deg_\sigma(t_d') + \sum_{j=n-m+1}^n |A_0^\sigma(j')| = \ell(\omega_0^d) + \sum_{j=1}^m (m-j) = \ell(\omega_0^d).$$

By a similar argument, $\deg(t_d^\sigma) = -\ell(\omega_0^d) = -\deg(t_d)$. So now determine how a simple reflection $(j, j+1) \in \mathfrak{S}_d$, with $\ell(\sigma \cdot (j, j+1)) > \ell(\sigma)$, changes the degree of the element $t_d \cdot \sigma$. $(j, j+1) \in \mathfrak{S}_d$ implies that the boxes $t_d^{-1}(j)$ and $t_d^{-1}(j+1)$ are in the same staggered sequence. It is easy to see that $\deg(t_d \cdot \sigma \cdot (j, j+1)) = \deg(t_d \cdot \sigma) - 2$; see for example [BKW11, Proposition 3.13]. The result now follows because $\ell(\omega_0^d) = \deg(t_d)$ and $\deg(t_d \cdot \omega_0^d) = \deg(t_0^\alpha) = -\deg(t_d)$. 

Let $d = s_1s_2 \ldots s_k$ be a decomposition path for $\lambda \in \mathcal{R}_n^\varepsilon(\theta)$, where $s_1 = (s_1, \ldots, s_m)$. By definition, $t_d^\alpha(\beta) \leq t_d^\alpha(s_1) = n$, for all $\beta \in \lambda$. Then $s_1' = s_1 \setminus s_1$ is a staggered sequence of $\lambda \setminus s_1$ or is empty if $m = 1$ and so $d' = s_1's_2 \ldots s_k$ is a decomposition path for $\lambda \setminus s_1$ with negative decomposition tableau $(t_0^\alpha)_{\downarrow (n-1)}$.

**3.17 Lemma.** Let $d = s_1s_2 \ldots s_k$ be a decomposition path for $\lambda \in \mathcal{R}_n^\varepsilon(\theta)$. Let $\alpha \in \lambda$ such that $t_d^\alpha(\alpha) = n$ and $d' = s_1's_2 \ldots s_k$, where $s_1' = s_1 \setminus \alpha$. Then $t_0^\alpha = (t_0^\alpha)_{\downarrow (n-1)}$. 

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4 Classification Theorem and Applications

In this section we use the standard tableaux of shape \( \lambda \in \mathcal{U}_n^{\ell}(\theta) \) defined in section 3. They allow us to state the key theorem of this section. This theorem then implies the classification of the graded simple \( \mathcal{R}_n^\Lambda(\theta) \)-modules and gives us information about the graded dimensions of the graded simple \( \mathcal{R}_n^\Lambda(\theta) \)-modules. However, the proof of this key theorem, which requires diagram calculations, will be done in section 5.

We have seen that \( \mathcal{R}_n^\Lambda(\theta) \) is a graded cellular algebra with graded cellular basis \( \{c_{n,t}^{\lambda}| \lambda \in \mathcal{P}_n^{\ell}(\theta) \text{ and } t \in \text{Std}(\lambda)\} \). Its graded cell modules are \( S_n^\lambda \) with basis \( \{c_{t}^{\lambda}| \lambda \in \mathcal{P}_n^{\ell}(\theta) \text{ and } t \in \text{Std}(\lambda)\} \) and its simple modules are \( D_n^\lambda \) for \( \lambda \in \mathcal{P}_n^{\ell}(\theta) \) with \( D_n^\lambda \neq 0 \). By Definition 2.5, if we find \( x, y \in S_n^\theta \) such that \( (x, y)_\lambda \neq 0 \) then \( D_n^\lambda \neq 0 \).

We are now able to state the following theorem. The proof of this theorem, which requires diagram calculations, will be done in section 5.

4.1 Theorem. Let \( \lambda \in \mathcal{U}_n^{\ell}(\theta) \) be an Uglov multipartition and \( d \) a decomposition path for \( \lambda \). Then \( |\langle c_{d t}^\lambda, c_{d t}^\lambda \rangle_\lambda| = 1 \). Consequently, \( D_n^\lambda \neq 0 \).

4.1 Classification Theorem and DJM Conjecture

In this section we use Theorem 4.1 to classify the graded simple \( \mathcal{R}_n^\Lambda(\theta) \)-modules. We also use the tableaux defined in subsection 3.2 to give an alternative proof of a recent result by Jacon [Jac18, Corollary 10.1.2] on a known conjecture.

4.2 Corollary. Let \( \lambda \in \mathcal{P}_n^{\ell}(\theta) \). Then \( D_n^\lambda \) is simple if and only if \( \lambda \in \mathcal{U}_n^{\ell}(\theta) \).

Proof. If \( \lambda \in \mathcal{U}_n^{\ell}(\theta) \) then \( D_n^\lambda \) is simple, by Theorem 4.1. On the other hand, by Theorem 3.6, the number of simple \( \mathcal{R}_n^\Lambda(\theta) \)-modules is \( |\mathcal{U}_n^{\ell}(\theta)| \). Hence, the result follows.

Now we can prove our main result:

4.3 Main Theorem. Let \( \lambda \in \mathcal{P}_n^{\ell}(\theta) \). Then the following are equivalent:

(a) \( D_n^\lambda \neq 0 \)

(b) \( \lambda \in \mathcal{U}_n^{\ell}(\theta) \)

(c) \( |\langle c_{d t}^\lambda, c_{d t}^\lambda \rangle_\lambda| = 1 \), for any decomposition path \( d \) of \( \lambda \)

Proof. By Corollary 4.2, parts (a) and (b) are equivalent and, by Theorem 4.1 (b) implies (c). On the other hand, (c) implies (a) by Definition 2.5.

4.4 Corollary (Classification). The set \( \{D_n^\lambda(x)| \lambda \in \mathcal{U}_n^{\ell}(\theta), \text{ and } x \in \mathbb{Z}\} \) is a complete set of pairwise non-isomorphic simple \( \mathcal{R}_n^\Lambda(\theta) \)-modules.

4.5 Definition. A standard tableau \( t \) is a Dipper-James-Murphy tableau, or a DJM-tableau, if whenever there exists a standard tableau \( s \) such that \( \text{res}(s) = \text{res}(t) \), then \( \text{shape}(s) \not\subset \theta \text{ shape}(t) \).

In type B, Dipper, James, and Murphy [DJM95, Conjecture 8.13] conjectured that \( D_n^\lambda \neq 0 \) if and only if there exists a DJM-tableau of shape \( \lambda \). Graham and Lehrer [GL96, Question 5.10 (i)] asked if this holds in general for the simple modules of the Ariki-Koike algebras (when \( \theta \) is well-separated).

Let \( d = s_1 \ldots s_k \) be a decomposition path for \( \lambda \in \mathcal{U}_n^{\ell}(\theta) \). Assume there is a standard tableau \( s \) such that \( \text{res}(s) = \text{res}(t_{d s}^\lambda) \). Then for every staggered sequence \( s_j \) of \( d \), for \( j = 1, \ldots, k \), there
exists a corresponding set of boxes $r_j \subset \text{shape}(s)$ such that $t^*_d(s_j) = s(r_j)$, for $1 \leq j \leq k$. The disjoint union of sets of boxes $\bigcup_{j=1}^k r_j = \text{shape}(s)$ is the \textit{partitioning of $s$ along $d$}.

For the rest of this section we assume that $\theta$ is well-separated. Recall from Definition 3.4 that the set of Kleshchev multipartitions is $\mathcal{K}_n^\ell = \mathcal{U}_n^\ell(\theta)$, in this case.

4.6 Lemma. Let $\lambda \in \mathcal{K}_n^\ell$, $d^\prec = s_1 \ldots s_k$ the rightmost decomposition path for $\lambda$, and $s_1 = (s_1, \ldots, s_m)$. Then $s_m$ is the rightmost removable box in $\lambda$.

\textbf{Proof.} Let $\gamma = (r,c,l)$ be the rightmost removable box in $\lambda$ and $\text{res}(\gamma) = i$, for $i \in I$. Then $\lambda$ is maximal such that $\lambda^{(l)} \neq \emptyset$ and in the component $\lambda^{(l)}$ it is the removable box such that $r$ is maximal compared to all removable boxes in $\lambda^{(l)}$. By Lemma 3.8, this row has at most $e - 1$ boxes, so there is no addable $i$-box to the right of $\gamma$. Hence, $\gamma$ is a box of a staggered sequence of $\lambda$ and the result follows. \hfill $\square$

4.7 Lemma. Let $\lambda \in \mathcal{K}_n^\ell$, $d^\prec = s_1 \ldots s_k$ be its rightmost decomposition path, and $s_1 = \{s_1, \ldots, s_m\}$. If $\alpha \in \lambda$ with $\text{res}(\alpha) = \text{res}(s_m)$ then $\alpha \prec_\theta s_m$.

\textbf{Proof.} Assume that there exists a box $\alpha \in \lambda$ such that $\text{res}(\alpha) = \text{res}(s_m)$ and $s_m \prec_\theta \alpha$. If there is more than one pick the rightmost one. But this implies that $\alpha$ is not removable in $\lambda$ since otherwise $\alpha \in s_1$, a contradiction. By Lemma 4.6, $s_m$ is the rightmost removable box of $\lambda$, so $\alpha$ must be in the same row as $s_m$. Hence, this row has more than $e$ boxes. So, by Lemma 3.8, $\lambda \notin \mathcal{K}_n^\ell$, a contradiction. \hfill $\square$

4.8 Lemma. Let $\lambda \in \mathcal{K}_n^\ell$, $d^\prec = s_1 \ldots s_k$ its rightmost decomposition path, and $s_j = (s^{(j)}_1, \ldots, s^{(j)}_{m_j})$, for $1 \leq j \leq k$. If there is a standard tableau $s$ such that $\text{res}(s) = \text{res}(t^*_d)$ then $\alpha \prec_\theta s^{(j)}_{m_j}$, for all $\alpha \in r_j$, and $1 \leq j \leq k$, where $\bigcup_{j=1}^k r_j = \text{shape}(s)$ is the partitioning of $s$ along $d^\prec$.

\textbf{Proof.} We prove this statement by induction on $|d^\prec| = k$. If $k = 1$ then $d^\prec = s_1$, where $s_1$ is a staggered sequence of residue $i \in I$. Then the boxes of $s_1$ are, by definition, the rightmost addable $i$-boxes in the empty multipartition. Hence, $\alpha \preceq_\theta s^{(1)}_{m_1}$, for all $\alpha \in r_1$.

Now let $k > 1$ and assume the statement is true for $k - 1$. Let $\alpha_j$ be the rightmost box in $r_j$, for $j = 1, \ldots, k$. Let $\lambda = \lambda \setminus s_1$ and $\mu = \text{shape}(s) \setminus r_1$. So $s_2 \ldots s_k$ is the rightmost decomposition path for $\lambda$. By induction, $\alpha_j \preceq_\theta s^{(j)}_{m_j}$, for $j = 2, \ldots, k$. Assume, by way of contradiction, $s^{(1)}_{m_1} \prec_\theta \alpha_1$ and $\text{res}(\alpha_1) = \text{res}(s^{(1)}_{m_1}) = i$, for some $i \in I$. By Lemma 4.7, $\alpha_1 \notin \lambda$. Moreover, $\alpha_1$ cannot be addable in $\lambda$ since otherwise $s_1$ would not be a staggered sequence. But this implies that $s^{(j)}_{m_j} \prec_\theta \alpha_j$, for some $j$ with $2 \leq j \leq k$, a contradiction. \hfill $\square$

Recently, Jacon [Jac18] proved the following result. We give an alternative proof here.

4.9 Theorem (Generalized Dipper-James-Murphy Conjecture; see [Jac18, Corollary 10.1.2]). Let $\lambda \in \mathcal{P}_n^\ell(\theta)$ and $\theta$ well-separated. Then $D^\ell_\theta \neq 0$ if and only if there exists a DJM-tableau of shape $\lambda$.

\textbf{Proof.} Let $t$ be a DJM-tableau. By [BK09b, Theorem 5.10], the only composition factors in $S^\lambda_\theta$ are simple modules $D^\mu_\theta$ with $\mu \preceq_\theta \lambda$. Assuming $D^\lambda_\theta = 0$ then implies there exists $\mu \preceq_\theta \lambda$ and $s \in \text{Std}(\mu)$ with $\text{res}(s) = \text{res}(t)$, a contradiction to our assumption. Hence, $D^\lambda_\theta \neq 0$.

Now let $D^\lambda_\theta \neq 0$. By Corollary 4.2, this implies $\lambda \in \mathcal{K}_n^\ell$. So there exists the rightmost decomposition path $d^\prec = s_1 \ldots s_k$ of $\lambda$ and we claim that $t_{d^\prec}$ is a DJM-tableau. Assume, by way of contradiction, that there is a standard tableau $s$ such that $\text{res}(s) = \text{res}(t_{d^\prec})$. Then let $\bigcup_{j=1}^k r_j = \text{shape}(s)$ be the partitioning of $s$ along $d^\prec$. Let $r_j = (r^{(j)}_1, \ldots, r^{(j)}_{m_j}) \subset \text{shape}(s)$, for $1 \leq j \leq k$. Without loss of generality we can assume that $s(r^{(j)}_a) > s(r^{(j)}_{a+1})$, for $1 \leq a \leq m_j$, for all $1 \leq j \leq k$. 

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We finish the argument constructively. Let \( q \in \mathbb{N} \) with \( 1 \leq q \leq n \). Set \( \alpha_q = (t_q^\lambda)^{-1}(q) \in \lambda \) and \( \beta_q = s^{-1}(q) \in \text{shape}(s) \) be the boxes with the number \( q \) in it and let \( \lambda_q = \text{shape}((t_q^\lambda)_{\lambda q}) \) and \( \mu_q = \text{shape}(s_{\lambda q}) \), for \( 1 \leq q \leq n \), and set \( \lambda_0 = \mu_0 = \emptyset \). Let \( q_1 \) be minimal such that \( \alpha_{q_1} \neq \beta_{q_1} \) and \( j_1 \) such that \( \alpha_{q_1} \in s_{j_1} \). Then \( \lambda_{q_1 - 1} = \mu_{q_1 - 1} \). By definition, \( \alpha_{q_1} \) must be the rightmost addable box of its residue in \( \lambda_{q_1 - 1} \) but since \( \alpha_{q_1} \neq \beta_{q_1} \) we get \( \beta_{q_1} \prec_\theta \alpha_{q_1} \) and \( \mu_q \not\prec_\theta \lambda_q \), for all \( q \leq q_1 \). Moreover, \( \gamma \in \lambda_{q_1} \) and \( \delta \in \mu_{q_1} \), for all \( \gamma \in \mu_{q_1 - 1}, \delta \in \lambda_{q_1} \) such that \( \alpha_{q_1} \prec_\theta \gamma, \delta \). The rest of the boxes in \( s_{j_1} \) and \( r_{j_1} \) are added to the left of \( \alpha_{q_1} \). By Lemma 4.7, we know that the rightmost box of a staggered sequence \( s_j \) of residue \( \text{res}(\alpha_{q_1}) \) with \( j < j_1 \) is to the right of \( \alpha_{q_1} \).

So continuing with the boxes of \( r_{j_1 - 1} \) and \( s_{j_1 - 1} \). Let \( q_2 > q_1 \) be minimal such that \( \alpha_{q_2} \neq \beta_{q_2} \) and \( \alpha_{q_2} \prec_\theta \alpha_{q_2} \). Then as argued above \( \beta_{q_2} \prec_\theta \alpha_{q_2} \) and all boxes in \( \lambda_{q_2} \) and \( \mu_{q_2} \) to the right of \( \alpha_{q_2} \) are the same. Moreover, \( \mu_q \not\prec_\theta \lambda_q \), for \( q \leq q_2 \).

Continuing this argument yields \( \mu_q \not\prec_\theta \lambda_q \), for \( q \leq n \), proving the claim. \( \square \)

**Note** that the argument above would work for general \( \theta \) if we could show that there always exists a decomposition path \( d^* \) of \( \lambda \) satisfying the conditions of Lemma 4.8. Unfortunately, for general \( \theta \) one can find examples that show \( d^* \) does not always satisfy these conditions. We conjecture that such a decomposition path always exists.

### 4.2 Graded dimension of simple modules

This section gives an application of Main Theorem 4.3. We compute a non-trivial lower bound on the graded dimension of \( D^\lambda_\theta \). First we need the following technical Proposition.

**4.10 Proposition.** Let \( \lambda \in \mathcal{W}^l_n(\theta) \), \( d \) a decomposition path for \( \lambda \), and \( \sigma, \tau \in \mathcal{S}_d \). Then

\[
|\langle c_{\lambda, \sigma}^\lambda, c_{\lambda, \tau}^\lambda \rangle|_\lambda = \delta_{\sigma^{-1}, \tau}.
\]

The proof of this statement requires technical diagram calculations, so we defer the proof to section 5. Proposition 4.10 implies the following result.

**4.11 Corollary.** Let \( \lambda \in \mathcal{W}^l_n(\theta) \) and \( d \) a decomposition path for \( \lambda \). Then the elements in the set \( \{c_{\lambda, \sigma}^\lambda + \text{Rad}(S^\lambda_\theta) \in D^\lambda_\theta \mid \sigma \in \mathcal{S}_d \} \) are linearly independent.

**Proof.** Assume we have scalars \( s_\sigma \in K \) such that

\[
\sum_{\sigma \in \mathcal{S}_d} s_\sigma c_{\lambda, \sigma}^\lambda \in \text{Rad}(S^\lambda_\theta).
\]

Note that \( c_{\lambda, \sigma}^\lambda \not\in \text{Rad}(S^\lambda_\theta) \), by Corollary 4.11. Let \( \tau \in \mathcal{S}_d \). Taking the inner product with \( c_{\lambda, \tau^{-1}}^\lambda \) shows that \( s_\tau = 0 \), by Corollary 4.11. Hence, \( s_\sigma = 0 \), for all \( \sigma \in \mathcal{S}_d \), and the elements are linearly independent. \( \square \)

Let \( q \in \mathbb{N}_0 \) and \( t \) an indeterminate over \( \mathbb{N}_0 \). We define the polynomial

\[
[q]_t = \sum_{i=0}^{q-1} t^{-(q-1)+2i} = t^{-(q-1)} + t^{-(q-1)+2} + \ldots + t^{(q-1)-2} + t^{q-1}
\]

and set \( [q]_t = \prod_{i=1}^{q} [i]_t = [1]_t [2]_t \ldots [q]_t \). Let \( p, \dot{p} \in \mathbb{N}_0[t] \) and write \( p \leq \dot{p} \) if \( \dot{p} - p \in \mathbb{N}_0[t] \). Note that \( [q]_1 = q \) and \( [q]_1! = q! \).

**4.12 Theorem.** Let \( \lambda \in \mathcal{W}^l_n(\theta) \) and \( d = s_1 \ldots s_k \) a decomposition path for \( \lambda \) with \( |s_j| = m_j \).

Then

\[
\dim(D^\lambda_\theta, \ell_{\omega}(\text{res}(t^\lambda_d))) \geq \prod_{j=1}^{k} [m_j]_t!\]

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Proof. By Corollary 4.11, for any decomposition path \( d = s_1 \ldots s_k \) of \( \lambda \) there is a set \( \{ c_{\lambda}^{e} \sigma + \text{Rad}(S_{\theta}) | \sigma \in \mathcal{S}_d \} \) of linearly independent elements in \( D_{\theta}^{\lambda} \) of size \( |\mathcal{S}_d| = \prod_{j=1}^{k} m_j! \). These elements have the same residue sequence as \( t_d \), by Lemma 3.14. By Lemma 3.16 and Corollary 2.18, the degree of these elements is \( \deg_{\theta}(c_{\lambda}^{e} \sigma + \text{Rad}(S_{\theta})) = \deg_{\theta}(t_d \cdot \sigma) = \ell(S_{\theta}^{d}) - 2\ell(\sigma) \). The following equation is a variation of a well-known result about Poincaré polynomials; see for example [Hum90, Section 1.11]

\[
\sum_{\sigma \in \mathcal{S}_m} \ell(\omega_{0}) - 2\ell(\sigma) = [m]_{\ell}!
\]

where \( \omega_{0} \in \mathcal{S}_m \) is the longest element.

But then the result follows because \( \mathcal{S}_d = \mathcal{S}_{m_1} \times \ldots \times \mathcal{S}_{m_k} \) is a direct product. \( \square \)

Note that we did not make a specific choice for the field \( K \) and its characteristic. But the graded dimension of the simple module does depend on the characteristic. Hence, Theorem 4.12 implies that we are looking at a specific piece of the simple module which is independent of the characteristic of the field \( K \).

5 Inner products

This section gives the proof of Theorem 4.1 and Proposition 4.10 using Webster’s diagram calculus. This completes the proof of Main Theorem 4.3, which gives a classification of the simple \( \mathcal{R}_{\Lambda}^n(\theta) \)-modules.

5.1 Diagram Calculus

In this section we prove some technical results that hold locally for the diagrammatic Cherednik algebra.

Partly following Bowman [Bow17, Section 3], we first introduce some useful notation. Let \( \Delta = \{ \delta_0, \ldots, \delta_k \} \) be an \( i \)-diagonal in some multipartition \( \lambda \) such that \( \delta_j = (r_0 + j, c_0 + j, l) \) for some \( 1 \leq l \leq \ell \). The diagonal starts at \( \delta_0 = (r_0, c_0, l) \) and ends at \( \delta_k \). An \( i \)-diagonal is addable if \( (r_0 + (k+1), c_0 + (k+1), l) \) is an addable box, removable if \( \delta_k \) is a removable box, and negligible otherwise. There are two types of negligible \( i \)-diagonals. A negligible \( i \)-diagonal is \((i+1)\)-negligible if \( (r_0 + k, c_0 + (k+1), l) \in \lambda \) and \((i-1)\)-negligible if \( (r_0 + k, c_0 + (k+1), l) \notin \lambda \). An \( i \)-diagonal is a left diagonal if \( x_\delta < \theta_l \), for all \( \delta \in \Delta \). Similarly, it is a centred diagonal if \( x_\delta - \ell < \theta_l < x_\delta \) and a right diagonal if \( \theta_l < x_\delta - \ell \).

Consequently, there are 3 types of addable and removable \( i \)-diagonals and 6 types of negligible \( i \)-diagonals.

5.1 Example. Let \( e = 3 \) and \( \kappa = (0) \). Then for \( \lambda = (9, 7, 6, 6, 5, 3, 2) \) the 0-diagonals are:

![Diagram of 0-diagonals](image-url)
From left to right we have an addable left diagonal, a 2-negligible left diagonal, a removable centred diagonal, a removable right diagonal, and a 1-negligible right diagonal. 

5.2 Proposition. Let $i \in I$ then locally the following relation holds:

\[
\begin{array}{c}
\begin{array}{cccc}
\chi & \chi & = & \circ \circ \\
\end{array}
\end{array}
\]

Proof. Applying relations 2.14(a) and then 2.14(b)(I) gives us

\[
\begin{array}{c}
\begin{array}{cccc}
\circ \circ & = & \chi & + \\
\end{array}
\end{array}
\]

5.3 Lemma. Let $i \in I$ then the following three local relations hold:

\([a]\) \begin{array}{c}
\begin{array}{cccc}
\chi & \chi & \chi & = & \circ \circ \circ \\
\end{array}
\end{array}
\]

\([b]\) \begin{array}{c}
\begin{array}{cccc}
\chi & \chi & \chi & = & (-1) \circ \circ \\
\end{array}
\end{array}
\]

\([c]\) \begin{array}{c}
\begin{array}{cccc}
\chi & \chi & \chi & = \circ \circ \circ \\
\end{array}
\end{array}
\]

Proof. For \([a]\) we apply Proposition 5.2 and then relation 2.14(b)(IV):

\[
\begin{array}{c}
\begin{array}{cccc}
\chi & \chi & \chi & = \circ \circ \circ \\
\end{array}
\end{array}
\]

For \([b]\) we apply again Proposition 5.2 and then relations 2.14(b)(I) and 2.14(b)(III):

\[
\begin{array}{c}
\begin{array}{cccc}
\chi & \chi & \chi & = \circ \circ \circ \\
\end{array}
\end{array}
\]

The proof for \([c]\) is similar to \([b]\) except that we use relation 2.14(b)(II) instead of 2.14(b)(III).
5.4 Lemma. For \( i \in I \) we have the following decomposition:

\[
\begin{array}{c}
\langle x \rangle = \langle x \rangle + \langle x \rangle - \langle x \rangle + \langle x \rangle - \langle x \rangle
\end{array}
\]

Proof. First, apply relation 2.14(c)(II) to get:

\[
\begin{array}{c}
\langle x \rangle = \langle x \rangle + \langle x \rangle - \langle x \rangle + \langle x \rangle - \langle x \rangle
\end{array}
\]

Next, apply relation 2.14(c)(I) to the first summand giving

\[
\begin{array}{c}
\langle x \rangle = \langle x \rangle + \langle x \rangle - \langle x \rangle + \langle x \rangle - \langle x \rangle
\end{array}
\]

Now applying relations 2.14(b)(II) and 2.14(b)(III) to the second summands of the two equations above gives us the result. 

5.5 Lemma. For \( i \in I \) we have the following decomposition:

\[
\begin{array}{c}
\langle x \rangle = \langle x \rangle + \langle x \rangle - \langle x \rangle + \langle x \rangle - \langle x \rangle
\end{array}
\]

Proof. First we apply Lemma 5.4 to get:

\[
\begin{array}{c}
\langle x \rangle = \langle x \rangle + \langle x \rangle - \langle x \rangle + \langle x \rangle - \langle x \rangle
\end{array}
\]
By relation \(2.14(\text{b})(\text{I})\), the second, fourth, and fifth summand are zero and by relation \(2.14(\text{c})\) the first summand becomes:

Finally, by relation \(2.14(\text{a})\), the remaining summand, which is the third, becomes:

Collecting the surviving terms completes the proof.

Note that in the following Lemma the vertical strings in the diagrams on the left-hand side of the equations correspond to boxes in an addable \(i\)-diagonal.

5.6 Lemma. Let \(i \in I\) and \(k \in \mathbb{N}\). Locally the following hold:

Proof. All three diagrams on the left hand side of the equations share the following subdiagram:
Applying Lemma 5.5 gives us

\[
\begin{align*}
\sum_{k} & \cdot \ldots \cdot k \cdot \ldots \cdot k \\
\sum_{k} & \cdot \ldots \cdot k \cdot \ldots \cdot k
\end{align*}
\]

We see that the second summand on the right-hand side of the equation contains the diagram on the left-hand side of the equation as a subdiagram, except that the number of vertical strings is decreased by 1. Hence, we can apply Lemma 5.5 again on this subdiagram. As for the first summand, we can apply relations 2.14(c)(I) and (II) to get

\[
\begin{align*}
\sum_{k} & \cdot \ldots \cdot k \cdot \ldots \cdot k \\
\sum_{k} & \cdot \ldots \cdot k \cdot \ldots \cdot k
\end{align*}
\]

Note that relations 2.14(c)(I) and (II) have a second summand with vertical strings. Replacing the \(i\)-crossing in the diagram with vertical strings yields a double \(i\)-crossing, hence, this diagram vanishes, by 2.14(b)(I).

So consequently, we can repeat applying Lemma 5.5 until we have done so \(k\)-times. In total, we then collect the following terms:

\[
\begin{align*}
\sum_{k} & \cdot \ldots \cdot k \cdot \ldots \cdot k \\
\sum_{k} & \cdot \ldots \cdot k \cdot \ldots \cdot k
\end{align*}
\]

The difference for the three equations of the Lemma lies in either an extra solid \((i + 1)\)-, ghost \((i - 1)\)-, or red \(i\)-string to the left of the \(i\)-crossings in the decomposition above. The following
is the case of the additional red $i$-string:

\[
\begin{align*}
&\quad = (-1)^k.
\end{align*}
\]

Applying relation 2.14(c)(III) to all diagrams in the sum above yields

\[
\begin{align*}
&\quad + \sum_{j=0}^{k-1} (-1)^j.
\end{align*}
\]

Where the last summand above vanishes by relation 2.14(b)(I). This proves the result.

The other two cases are similar, except that 2.14(c)(III) is replaced by 2.14(c)(I) or (II) respectively.

Note that in the next Lemma the vertical strings in the diagrams on the left- and right-hand side of the equations correspond to boxes in a negligible $i$-diagonal.

\textbf{5.7 Lemma.} Let $i \in I$ and $k \in \mathbb{N}$. Locally, for a crossing of solid $i$-strings, the following local
identities hold:

\[ \begin{array}{c}
\vdots \\
+ \quad +
\vdots \\
\quad \quad \quad \vdots \\
\vdots \\
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\end{array} \]  

Proof. Each of these equations is proved using a similar argument to that of Lemma 5.6. When the \( i \)-crossing is dragged leftward over the previous vertical solid \( i+1 \)-string, ghost \( i-1 \)-string, or red \( i \)-string, respectively, we apply relations 2.14(c)(I), (II), or (III) respectively. Similarly, as argued in the proof of Lemma 5.6 by application of the relations there is one diagram with the crossing dragged over the vertical string and one diagram where the crossing is replaced by vertical \( i \)-strings locally. Because of the double \( i \)-crossings the later diagram vanishes, by 2.14(b)(I). Hence, in each case, an \( i \)-crossing can be dragged leftwards through all the vertical strings, as claimed. 

Note that in the following Lemma the vertical strings in the diagrams on the left-hand side of the equations correspond to boxes in a negligible \( i \)-diagonal.
5.8 Lemma. Let $i \in I$ and $k \in \mathbb{N}$. Locally, for a curved solid $i$-string, the following hold:

\[
\sum_{j=0}^{k-1} (-1)^j \cdot \sum_{j=0}^{k-1} (-1)^{j+1} = (-1)^{k-1}.
\]
Proof. The proof for all six equations is similar, so we prove the first equation carefully and then sketch how to prove the other five. The diagram on the left-hand side of the first equation contains the following subdiagram

\[
\begin{array}{c}
i \\
\hline
i & i-1 \\
\hline
\end{array}
\]

Applying 2.14(b)(II) to this subdiagram gives us

\[
\begin{array}{c}
i \\
\hline
i & i-1 \\
\hline
\end{array}
= \begin{array}{c}
i \\
\hline
i & i-1 \\
\hline
\end{array}
- \begin{array}{c}
i \\
\hline
i & i-1 \\
\hline
\end{array}
\]

The first summand vanishes by relation 2.14(b)(I). Applying 2.14(a) to the second summand gives us then

\[
\begin{array}{c}
i \\
\hline
i & i-1 \\
\hline
\end{array}
= \begin{array}{c}
i \\
\hline
i & i-1 \\
\hline
\end{array}
- \begin{array}{c}
i \\
\hline
i & i-1 \\
\hline
\end{array}
\]

Again here the first summand vanishes by relation 2.14(b)(I). So combining both equations gives us

\[
\begin{array}{c}
i \\
\hline
i & i-1 \\
\hline
\end{array}
= \begin{array}{c}
i \\
\hline
i & i-1 \\
\hline
\end{array}
\]
Replacing the subdiagram in with the one on the right-hand side above in the original diagram gives us

\[
\begin{array}{c}
\text{i  \ldots  \ldots  k} \\
\text{k  \ldots  \ldots  k} \\
\text{k  \ldots  \ldots  k}
\end{array}
= 
\begin{array}{c}
\text{i  \ldots  \ldots  k} \\
\text{k  \ldots  \ldots  k} \\
\text{k  \ldots  \ldots  k}
\end{array}
\]

Now we see that the diagram on the right-hand side above contains a subdiagram as in the first equation of Lemma 5.6. Applying Lemma 5.6 on this subdiagram completes the proof for the first equation.

The proof for the third and fifth equation follows the same argument. For the proof of the second, fourth, and sixth equation we instead look at the subdiagram

\[
\begin{array}{c}
i + 1 \\
i \\
\end{array}
\]

By a similar chain of arguments we get

\[
\begin{array}{c}
i + 1 \\
i \\
i \\
\end{array}
= (-1)
\]

The result then follows again by applying Lemma 5.6.

5.2 Computing inner products

This section proves the key Theorem for the classification of all simple \( \mathcal{R}_n^\Lambda(\theta) \)-modules. Before giving the proof we explain how to construct certain diagrams that we need to compute inner products. We give two different constructions of these diagrams and show that these different diagrams represent the same element in \( \mathcal{R}_n^\Lambda(\theta) \).

Fix a small positive quantity \( \delta < \frac{1}{2n} \). Let \( \lambda \in \mathcal{U}_n^\ell(\theta) \) and \( d = s_1 \ldots s_k \) a decomposition path for \( \lambda \). Recall \( \omega = (\emptyset, \ldots, \emptyset, (1^n)) \in \mathcal{P}_n^\ell(\theta) \) and let \( \omega = \{ \beta_1, \ldots, \beta_n \} \) considered as a set of boxes, such that \( \beta_a \prec \theta \beta_b \), for \( 1 \leq a < b \leq n \). Let \( s_j = (s_j^{(1)}, \ldots, s_j^{(n)}) \), for \( 1 \leq j \leq k \). Let \( \sigma \in \mathcal{S}_d \) and consider the tableau \( t_d \cdot \sigma \). We now describe a construction of a diagram of the element \( c^\lambda_{t_d \cdot \sigma} \) satisfying the definition of \( \theta \)-diagrams; see beginning of subsection 2.3.

**Construction A:** First we draw the red strings vertically then we need to draw the solid and their ghost strings. We do this recursively and assume we have already drawn the solid and ghost strings for \( (t_d \cdot \sigma)_{i-1} \) already, for \( 1 \leq i-1 < n \). Let \( \alpha \in \lambda \) be the box containing the number \( i \) in \( t_d \cdot \sigma \) then we need to draw a solid string of type \( (x_\alpha, x_\beta) \). This path is a union of straight line segments between the following four points:

\[
(x_\alpha, 1) \rightarrow (x_\alpha, \frac{i}{n+1}) \rightarrow (x_\beta, \frac{i}{n+1} - \delta) \rightarrow (x_\beta, 0)
\]
Together with its ghost string we get the following strings:

\[ x_{\alpha}, x_{\beta_1}, x_{\beta_i}, x_{\beta_{i-1}} \]

Define \( A_{\lambda}^{\mathbf{t}_d} \) to be the resulting diagram for \( c_{\mathbf{t}_d}^{\lambda} \) using Construction A. The benefit of this construction is that in the horizontal strip between \( y_1 = \frac{i_1 - 1}{n+1} \) and \( y_2 = \frac{i_2}{n+1} \), for \( 1 \leq i_1 < i_2 \leq n \), all solid strings of type \( (x_{\alpha}, x_{\beta_j}) \) and their ghost strings are vertical, for \( 1 \leq j \leq n \) such that \( j \notin [i_1, i_2] \). By construction of \( \mathbf{t}_d \cdot \sigma \), this allows us to partition the diagram into \( k \) horizontal strips according to the \( k \) staggered sequences in \( \mathbf{d} \).

5.9 Example. Let \( \theta = (0, 8) \) and \( \lambda = ((2, 1), (3, 1)) \). Let \( e = 3 \) and \( \kappa = (0, 0) \). Then there is only one staggered sequence \( \mathbf{d} \); see Example 3.13. The diagram \( A_{\lambda}^{\mathbf{t}_d} \) of \( c_{\mathbf{t}_d}^{\lambda} \) is:

The three different shadings highlight the three horizontal strips

We can now prove the main result of this section. Recall from Definition 3.12 that we have a pair of decomposition tableaux \( \mathbf{t}_d \) and \( \mathbf{t}_d^* \), for each decomposition path \( \mathbf{d} \).

5.10 Theorem. Let \( \lambda \in \mathcal{U}^\ell(\theta) \) be an Uglov multipartition and \( \mathbf{d} \) any decomposition path for \( \lambda \). Then \( |\langle A_{\mathbf{t}_d}^\lambda, A_{\mathbf{t}_d^*}^\lambda \rangle_\lambda| = 1 \).

Proof. Let \( \lambda \in \mathcal{U}^\ell(\theta) \) and \( \mathbf{d} = s_1 \ldots s_k \) a decomposition path for \( \lambda \). Recall 2.3:

\[ \langle c_{\mathbf{t}_d}^{\lambda}, c_{\mathbf{t}_d^*}^{\lambda} \rangle_\lambda c_{\mathbf{t}_d}^{\lambda} = c_{\mathbf{t}_d}^{\lambda} \circ (c_{\mathbf{t}_d^*}^{\lambda})^* \circ c_{\mathbf{t}_d}^{\lambda} \]

Hence, the inner product is determined by the diagram \( A_{\mathbf{t}_d}^\lambda \circ (A_{\mathbf{t}_d^*}^\lambda)^* \). We show that this diagram ‘straightens out’ to \( \pm c_\lambda(\text{res}(\mathbf{t}_d)) \) modulo terms of more dominant shape. This will show that the inner product is \( \pm 1 \), giving the result.

We can describe the diagram \( A_{\mathbf{t}_d}^\lambda \circ (A_{\mathbf{t}_d^*}^\lambda)^* \) recursively as the diagram

\[ \begin{array}{ccc}
\lambda & \backslash & s_1 \\
& \vdots & \\
\mathbf{i}_1 & \ldots & \mathbf{i}_k \\
\end{array} \]
where $\mu = \lambda \setminus s_1$ is a smaller Uglov multipartition. The shaded part of the diagram above is the diagram $A_{d'}^\mu \circ (A_{d'}^\mu)^*$, where $d' = s_2 \ldots s_k$. By construction, in the diagram above all strings not belonging to the staggered sequence $s_1$ are vertical when they are above and below the shaded component. This recursive description of this diagram allows us to argue by induction on $|d| = k$.

Let $k = 0$ then $\lambda = \emptyset$ and there is nothing to prove. So let $k > 0$ and assume the statement is true for $k - 1$. Hence, by induction, the diagram above simplifies to:

![Diagram](image)

where all of the strings inside the shaded area are now vertical. So it remains to show that the non-vertical strings, which correspond to $s_1$, can be straightened. We assume that $s_1 = (s_1, \ldots, s_m)$ is a sequence of residue $i \in I$ and that $s_1$ has $|s_1| = m$ solid strings.

We now proceed by induction on $m$. Let $m = 1$ then the diagram is:

![Diagram](image)

where the black line is the vertical line at coordinate $x_{s_1} - \ell$. This is the place where the solid and its ghost string become vertical. We refer to the curved $i$-string in the diagram above as an $i$-loop for our argument. Because $s_1$ is a staggered sequence there is no addable or removable $i$-boxes to the right of this line in $\lambda$. Hence, to the right of this line there can only be vertical $i$-strings corresponding to boxes in negligible $i$-diagonals. If there are no such strings then there are also no vertical solid $i + 1$-strings and ghost $i - 1$-strings. So the $i$-loop can be straightened with no extra terms, by relation 2.14(b).

If there are such strings, then the diagram contains a subdiagram that is equal to one of the diagrams in Lemma 5.8. By Lemma 5.8, this subdiagram can be replaced with a sum of terms. One term contains the $i$-loop to the left of all the vertical $i$-strings and all other terms contain an $i$-crossing to the left of all the vertical $i$-strings such that the two left exiting strings of the crossing are connected to the points $(x_{s_1}, 1)$ and $(x_{s_1}, 0)$. Denote the first term as $D_1$ and the other terms $X_j^{(1)}$, for $j \in J_1$ some finite indexing set. So $A_{d'}^\lambda \circ (A_{d'}^\lambda)^* = D_1 + \sum_{j \in J_1} X_j^{(1)}$. If in the diagram $D_1$ there are still vertical $i$-strings corresponding to boxes in a negligible $i$-diagonal.
to the left of the $i$-loop and to the right of the vertical line at the coordinate $x_{s_1} - \ell$, then we can apply Lemma 5.8 to $D_1$ again. Hence, $D_1 = D_2 + \sum_{j \in J_2} X_j^{(2)}$, where $D_2$ is the diagram where the $i$-loop is to the left of the vertical $i$-strings and $X_j^{(2)}$, for $j \in J_2$ a finite indexing set, a diagram with an $i$-crossing to the left of the vertical $i$-strings such that the two left exiting strings of the crossing are connected to the points $(x_{s_1}, 1)$ and $(x_{s_1}, 0)$. So if in $\lambda$ there are $c$ negligible $i$-diagonals to the right of $s_1$ then $A^\lambda_{t_4} \circ (A^\lambda_{t_4})^* = D_c + \sum_{l=1}^c \sum_{j \in J_l} X_j^{(l)}$, by Lemma 5.8. By the argument used in the case $c = 0$, the $i$-loop in the diagram $D_c$ can be straightened with no extra terms, by 2.14(b). So $D_c$ can be transformed into

It remains to show that the terms $X_j^{(l)}$ 'vanish'. In the diagram $X_j^{(l)}$, for $1 \leq l \leq c$ and $j \in J_l$, there are $c - l$ negligible $i$-diagonals to the left of the position of the $i$-crossing and to the right of the vertical line at $x_{s_1} - \ell$. By Lemma 5.7, we can drag this crossing through all the vertical $i$-strings corresponding to the boxes in these $c - l$ negligible $i$-diagonals. By relation 2.14(c), we can drag the $i$-crossing up to the position $x_{s_1}$ without extra terms. Locally, the diagram then contains a subdiagram that is equal to one of the diagrams in Lemma 5.3. By Lemma 5.3, we can replace this subdiagram with one that contains two $i$-crossings. In the resulting diagram we take a small horizontal strip between the two $i$-crossings, where all strings are vertical. Consider this strip as a subdiagram. Then this subdiagram is a diagram of type $(\lambda', \lambda')$, where $\lambda \prec_\theta \lambda'$, because $\lambda$ and $\lambda'$ are equal up to one vertical $i$-string that is to the right of $x_{s_1}$ in $\lambda$ and to the left of $x_{s_1}$ in $\lambda'$. Hence, the whole diagram $X_j^{(l)}$ vanishes because it is of more dominant shape. Implieding that all these terms vanish. So $A^\lambda_{t_4} \circ (A^\lambda_{t_4})^* = D_c = \pm e_\lambda(\text{res}(t_4))$, completing the proof of the base case $m = 1$ of the induction.

Now let $m > 1$ and assume the statement is true for $m - 1$. Because $m > 1$ we have at least one $i$-crossing of the solid strings of the staggered sequence and the diagram is:
However, by the same argument as for case \( m = 1 \), this diagram can be transformed into:

\[
\begin{array}{c}
\includegraphics{diagram}
\end{array}
\]

where the circles indicate important \( i \)-crossings. Note that their location is at the respective coordinate \( x_{s_l} \), for \( 2 \leq l \leq m \). Because \( s_1 \) is a staggered sequence there are no addable \( i \)-boxes to the right of \( s_1 \) in \( \lambda \) and \( \{ s_l | 2 \leq l \leq m \} \) is the complete set of removable \( i \)-boxes to the right of \( s_1 \) in \( \lambda \). So the circles are at the locations of all removable \( i \)-diagonals to the right of \( x_{s_1} \).

The two left exiting strings of the rightmost circled \( i \)-crossing are connected to the points \((x_{s_1}, 1)\) and \((x_{s_{m-1}}, 0)\). At its position we have a subdiagram that is equal to one of the diagrams in Lemma 5.6. By Lemma 5.6, this subdiagram can be replaced with a sum of terms.

One term contains an \( i \)-loop to the left of all the vertical \( i \)-strings corresponding to the boxes in the removable \( i \)-diagonal such that it is connected to the points \((x_{s_1}, 1)\) and \((x_{s_{m-1}}, 0)\). Each of the other terms contain an \( i \)-crossing to the left of all the vertical \( i \)-strings corresponding to the boxes in the removable \( i \)-diagonal such that the two left exiting strings of the crossing are connected to the points \((x_{s_1}, 1)\) and \((x_{s_{m-1}}, 0)\). Denote the first term as \( D^{(1)}_0 \) and the other terms \( X^{(0,1)}_{j} \), for \( j \in J_{(0,1)} \) a finite indexing set. ByLemma 5.8. The diagram \( D^{(1)}_0 \) is the diagram

\[
\begin{array}{c}
\includegraphics{diagram}
\end{array}
\]

In \( D^{(1)}_{c_1} \) at the rightmost circled crossing we have a subdiagram that is equal to one of the diagrams in Lemma 5.6. By Lemma 5.6, we get \( D^{(1)}_{c_1} = D^{(2)}_0 + \sum_{j \in J_{(0,2)}} X^{(0,2)}_j \), for a finite indexing set \( J_{(0,2)} \). By repetition of the argument we get

\[
A^{\lambda}_{td} \circ (A^\lambda_{td})^* = D^{(m-1)}_{c_{m-1-1}} + \sum_{b=1}^{m-1} \sum_{a=0}^{c_b} \sum_{j \in I_{(a,b)}} X^{(a,b)}_j.
\]

31
where $J_{(a,b)}$ is a finite indexing set, for $1 \leq b \leq m - 1$ and $0 \leq a \leq c_b$. The diagram $D_{c_{m-1}}^{(m-1)}$ is

Without the vertical $i$-string this diagram corresponds to strings of the staggered sequence $s'_1 = (s_2, \ldots, s_m)$. So $|s'_1| = m - 1$. By induction, $D_{c_{m-1}}^{(m-1)}$ is equal to

It remains to show that the terms $X_{j}^{(a,b)}$, for $1 \leq b \leq m - 1$, $0 \leq a \leq c_b$, and $j \in J_{(a,b)}$ a finite indexing set, ‘vanish’. First let $b = m - 1$. The diagram $X_{j}^{(a,m-1)}$, for $1 \leq a \leq c_{m-1}$ and $j \in J_{(a,m-1)}$ a finite indexing set, contains an $i$-crossing such that its position is to the left of $x_{s_2}$ and that its two exiting strings at the left are connected to the points $(x_{s_1}, 1)$ and $(x_{s_1}, 0)$. Between its position and $x_{s_1}$ there are $c_{m-1} - a$ many negligible $i$-diagonals and no removable $i$-diagonal. Then following a similar argument to the one used in the base case, $X_{j}^{(a,m-1)}$ vanishes.

So let $1 \leq b < m - 1$ and assume all terms with $b + 1$ vanish. The diagram $X_{j}^{(a,b)}$, for $1 \leq a \leq c_b$ and $j \in J_{(a,b)}$ a finite indexing set, contains an $i$-crossing such that its position is to the left of $x_{s_{m-1}}$ and that its two exiting strings at the left are connected to the points $(x_{s_1}, 1)$ and $(x_{s_{m-1}}, 0)$. Between its position and $x_{s_{m-1}}$ there are $c_b - a$ many negligible $i$-diagonals and between its position and $x_{s_1}$ there are $m - (b + 1)$ many removable $i$-diagonals. By Lemma 5.7,
we can drag the crossing to the position $x_{s_{m-b}}$. In $X^{(a,b)}_j$ we have the following subdiagram

where the left circled crossing is the same circled crossing in the diagram $A^{(a)}_t \circ (A^{(a)}_t)^*$ at the position $x_{s_{m-b}}$. The shaded area on the left corresponds to vertical $i$-strings of boxes in a removable $i$-diagonal. The shaded area in the middle contains all the vertical $i$-strings of the boxes in the $c_b - a$ many negligible $i$-diagonals. And the shaded area on the right contains the vertical $i$-strings corresponding to boxes in a removable $i$-diagonal. Note that these vertical $i$-strings in general do not belong to boxes in a removable $i$-diagonal in $\lambda$, however, see the diagrams in the sums in Lemma 5.6 and Lemma 5.8 to check that these vertical $i$-strings are equivalent to ones of a removable $i$-diagonal. Locally, this diagram has a subdiagram at the right circled $i$-crossing that is equal to one in Lemma 5.6. By Lemma 5.6, $X^{(a,b)}_j = E^{(a,b)}_{(j,0)} + \sum_{w \in W^{(a,b)}_{(j,v)}} Y^{(a,b)}_{(j,w,v)}$, for a finite indexing set $W^{(a,b)}_{(j,0)}$. The diagram $E^{(a,b)}_{(j,0)}$ is the one containing an $i$-loop to the left of the vertical $i$-strings in the shaded area on the right such that it is connected to the points $(x_{s_1}, 1)$ and $(x_{s_{m-(b+1)}}, 0)$. Each diagram $Y^{(a,b)}_{(j,w,v)}$ contains an $i$-crossing to the left of the vertical $i$-strings in the shaded area on the right such that its two exiting strings on the left are connected to the points $(x_{s_1}, 1)$ and $(x_{s_{m-(b+1)}}, 0)$. Similarly as before, $E^{(a,b)}_{(j,0)} = E^{(a,b)}_{(j,c_b-a)} + \sum_{v=1}^{c_b-a} \sum_{w \in W^{(a,b)}_{(j,v)}} Y^{(a,b)}_{(j,w,v)}$, by Lemma 5.8. The subdiagram in $E^{(a,b)}_{(j,c_b-a)}$ is

which vanishes, by relation 2.14(b)(I). By Lemma 5.7, the $i$-crossing in the diagram $Y^{(a,b)}_{(j,w,v)}$, for $1 \leq v \leq c_b - a$ and $w \in W^{(a,b)}_{(j,v)}$ a finite indexing set, can be dragged to the position $x_{s_{m-b}} + \varepsilon$, where $\varepsilon < \frac{1}{2n}$ is the small quantity fixed in subsection 2.2; see above Definition 2.8. So the subdiagram in $Y^{(a,b)}_{(j,w,v)}$ is
Applying relation 2.14(c) gives the subdiagram

The shaded area on the left of this diagram contains vertical i-strings corresponding to boxes in a removable i-diagonal. Hence, considering the circled i-crossing we have locally a subdiagram that is equal to one of Lemma 5.6. By Lemma 5.6, we can replace this subdiagram with a sum of terms such that in this sum there is only one term in which the rightmost i-string in the subdiagram is not straight; see Lemma 5.6. All terms where the rightmost i-string is straight correspond to a diagram where we can replace the circled i-crossing above with two vertical i-strings. By relation 2.14(b)(I), all of these terms vanish. So the subdiagram in $Y^{(a,b)}_{(j,v)}$ is

But this is a diagram with an i-crossing such that to its left and to the right of $x_{s_1}$ there are only $m - (b + 2)$ many removable i-diagonals. Such a diagram is equivalent to one of the form $X^{(a',b+1)}_{j'}$, hence, it vanishes by induction. So

$$X^{(a,b)}_j = E^{(a,b)}_{(j,c_b-a)} + \sum_{v=0}^{c_b-a} \sum_{w \in W^{(a,b)}_{(j,v)}} Y^{(a,b)}_{(j,w,v)} = 0 + \sum_{v=0}^{c_b-a} \sum_{w \in W^{(a,b)}_{(j,v)}} 0 = 0$$
Therefore,

\[ A_{\lambda}^{\tau} \circ (A_{\lambda}^{\lambda})^* = D_{c_{m-1}}^{(m-1)} + \sum_{b=1}^{m-1} \sum_{a=0}^{c_b} \sum_{j \in J(a,b)} X_j^{(a,b)} = D_{c_{m-1}}^{(m-1)} = \pm e_{\lambda}(\text{res}(t_d)), \]

which is what we wanted to show.\[ \square \]

We now describe a second, a priori, different construction of a diagram of the element \( c^\lambda_{\text{td},\sigma} \).

**Construction B:** Again draw the red strings vertically. For the solid and ghost strings we first take the diagram \( A_{\lambda}^\tau \) of \( c^\lambda_{\text{td}} \) and scale it vertically by the factor \( \frac{1}{2} \). Then we use this diagram as the top half of this construction for \( c^\lambda_{\text{td},\sigma} \). In the bottom half we add the permutation \( \sigma = (\sigma_1, \ldots, \sigma_k) \in S_d^\tau \) as a braid diagram with minimal crossings such that the corresponding solid and ghost strings stay within the boxed areas below. Note, that \( \sigma_j \) is the permutation operating on the staggered sequence \( s_j \), hence, the boxed areas only contain solid and ghost strings of the same residue as \( s_j \):

Define \( B^\lambda_{\text{td},\sigma} \) to be the diagram for \( c^\lambda_{\text{td},\sigma} \) obtained using Construction B. The benefit of this construction is that it highlights the action of the permutation \( \sigma \in S_d^\tau \). Note that the minimal braid diagram for the permutation \( \sigma_j \), for \( 1 \leq j \leq k \), is not unique up to isotopy but recall that a reduced expression is unique up to braid moves. Since \( \sigma_j \) only permutes boxes of the same residue, the choice of minimal braid diagram does not matter, by relation 2.14(c).

**5.11 Example.** Let everything be as in Example 5.9. Then the diagram \( B^\lambda_{\text{td}} \) of \( c^\lambda_{\text{td}} \) is:

The three shadings highlight the boxed areas containing the braid diagrams of the permutations \( \sigma_1, \sigma_2, \) and \( \sigma_3 \) in \( \sigma = (\sigma_1, \sigma_2, \sigma_3) \).

\[ \diamond \]

**5.12 Lemma.** Let \( \lambda \in \U_n^\ell(\theta) \) and \( d \) a decomposition path for \( \lambda \). As elements of \( A_{\text{nd}}^\tau(\theta) \),

\[ A_{\text{td},\sigma}^\lambda = B_{\text{td},\sigma}^\lambda, \text{ for all } \sigma \in S_d. \]

**Proof.** We start with Construction A and transform it into Construction B. We show how it works when \( \sigma = \omega_0^d \), as for all other permutations \( \sigma \in S_d^\tau \) the argument is similar.

By the definition of a staggered sequence, all of the non-vertical strings in the horizontal strips of Construction A have the same residue. Hence, we first focus on the solid and ghost
strings of a single staggered sequence inside the diagram. This is a subdiagram of the form:

\[ \lambda_t \cdot \omega_d \]

By changing the heights of the almost horizontal strings, using relation 2.14(b), this subdiagram can be transformed into:

\[ \lambda_t \cdot \omega_d \]

However, the same manipulation actually works in the whole diagram \( A^\lambda_{\lambda_t \cdot \omega_d} \) because by using Construction A all other strings, in particular those of other residues, are vertical in the horizontal strip we are working in. Hence, when we apply relation 2.14(c), the braid relation applies exactly and no correction terms are introduced.

The shaded area in the diagram above indicates where the permutation \( \sigma_j \) acts on the strings of the staggered sequence. The left and right bounds of it are exactly the left and right bound of the boxed area for the braid diagram of \( \sigma_j \) in Construction B. Using isotopy to transform the diagram into:

\[ \lambda_t \cdot \omega_d \]

Note that this works in the whole diagram \( A^\lambda_{\lambda_t \cdot \omega_d} \) because, by using Construction A, the area in \( A^\lambda_{\lambda_t \cdot \omega_d} \), corresponding to the shaded area indicated above, only contains the solid and ghost strings of the staggered sequence. But the diagram above is now in form of Construction B. Hence, applying these manipulations to the strings of all other staggered sequences gives us the diagram \( B^\lambda_{\lambda_t \cdot \omega_d} \).
5.13 Proposition. Let $\lambda \in \mathcal{V}_n^\ell(\theta)$, $d$ a decomposition path for $\lambda$, and $\sigma, \tau \in \mathfrak{S}_d$. Then $|\langle B_{t_{d}}^{\lambda}, B_{t_{d}}^{\lambda} \rangle_{\lambda}| = \delta_{\sigma^{-1}, \tau}$.

Proof. Recall that $t_{d}^{\sigma} = t_{d} \cdot \omega_{\lambda}^{d}$ and choose a reduced expression for $\omega_{\lambda}^{d} \cdot \tau$. The diagram for the inner product is $B_{t_{d}}^{\lambda} \cdot \sigma$:

Where the shaded part at the top and bottom are by construction the diagrams $A_{t_{d}}^{\lambda}$ and $(A_{t_{d}}^{\lambda})^*$ and the boxed areas in the middle correspond to the braid diagrams of the permutations $\sigma$ and $\omega_{\lambda}^{d} \cdot \tau$. So combined the middle part of the diagram corresponds to the permutation $\eta := \omega_{\lambda}^{d} \cdot \tau \cdot \sigma$.

By Theorem 5.10 and Lemma 5.12, if $\eta = \omega_{\lambda}^{d}$ then the inner product is $\pm 1$. Now there are three different cases:

(a) If $\tau \neq \sigma^{-1}$ and $\ell(\omega_{\lambda}^{d} \cdot \tau) + \ell(\sigma) \neq \ell(\omega_{\lambda}^{d})$ then, by Proposition 2.4, the inner product is 0 because the degrees of the elements, see Lemma 3.16, do not add up to 0.

(b) If $\tau \neq \sigma^{-1}$ and $\ell(\omega_{\lambda}^{d} \cdot \tau) + \ell(\sigma) = \ell(\omega_{\lambda}^{d})$ then $\ell(\eta) < \ell(\omega_{\lambda}^{d})$ because $\tau \neq \sigma^{-1}$. However, this means that there is a simple reflection $s \in \mathfrak{S}_d$ such that $s^2$ appears in the product $\eta$. In the diagram this means a double $i$-crossing for some $i \in I$, so by relation 2.14(b)(I) the inner product is 0. Remember that we can find $s$ because we use Construction B and so the braid diagram does not depend on the choice of reduced expressions for $\sigma$ and $\omega_{\lambda}^{d} \cdot \tau$, as we can use braid relations here; see above Example 5.11.

(c) If $\tau = \sigma^{-1}$ then $\eta = \omega_{\lambda}^{d}$ and the inner product is $\pm 1$, by Theorem 5.10 and Lemma 5.12.

This yields the desired result. \hfill \Box

Note that we made a special choice of diagram for the elements $c_{t_{d}}^{\lambda}$ and a priori the inner product could be different for a different choice of diagram. By Corollary 2.18, Definition 2.14(c) and [Bow17, Proof of Theorem 4.2], for an arbitrary diagram of $c_{t_{d}}^{\lambda}$ we get

$$c_{t_{d}}^{\lambda} = B_{t_{d}}^{\lambda} + \sum_{\sigma' \in \mathfrak{S}_d, \ell(\sigma') < \ell(\sigma)} k_{\sigma'} c_{t_{d}}^{\lambda} \sigma'$$

modulo terms of more dominant shape, for scalars $k_{\sigma'} \in K$. So Proposition 5.13 implies Proposition 4.10.

Proposition. Let $\lambda \in \mathcal{V}_n^\ell(\theta)$, $d$ a decomposition path for $\lambda$, and $\sigma, \tau \in \mathfrak{S}_d$. Then $|\langle c_{t_{d}}^{\lambda}, c_{t_{d}}^{\lambda} \rangle_{\lambda}| = \delta_{\sigma^{-1}, \tau}$.

In particular, this establishes Theorem 4.1.
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