Exact learning for infinite families of concepts

Mikhail Moshkov

Abstract

In this paper, based on results of exact learning, test theory, and rough set theory, we study arbitrary infinite families of concepts each of which consists of an infinite set of elements and an infinite set of subsets of this set called concepts. We consider the notion of a problem over a family of concepts that is described by a finite number of elements: for a given concept, we should recognize which of the elements under consideration belong to this concept. As algorithms for problem solving, we consider decision trees of five types: (i) using membership queries, (ii) using equivalence queries, (iii) using both membership and equivalence queries, (iv) using proper equivalence queries, and (v) using both membership and proper equivalence queries. As time complexity, we study the depth of decision trees. In the worst case, with the growth of the number of elements in the problem description, the minimum depth of decision trees of the first type either grows as a logarithm or linearly, and the minimum depth of decision trees of each of the other types either is bounded from above by a constant or grows as a logarithm, or linearly. The obtained results allow us to distinguish seven complexity classes of infinite families of concepts.

Keywords: exact learning, test theory, rough set theory, decision trees, complexity classes.

1 Introduction

Decision trees are widely used in many areas of computer science. They are studied, in particular, in exact learning initiated by Angluin [1], test theory initiated by Chegis and Yablonskii [8], and rough set theory initiated by Pawlak [19]. In some sense, these theories deal with dual objects: for example, attributes from test theory and rough set theory correspond to membership queries from exact learning. In contrast to test theory and rough set theory, in exact learning besides membership queries, equivalence queries are also considered.

Hegedûs in [9] generalized some bounds from [13] obtained in the framework of test theory to the case of exact learning with membership and equivalence queries. Similar results were obtained independently and in the other way by Hellerstein et al. [10]. In [15], we moved in the opposite direction: we added to the model considered in test theory and rough set theory the notion of a hypothesis that allowed us to use an analog of equivalence queries, and studied decision trees using various combinations of attributes, hypotheses, and proper hypotheses (an analog of proper equivalence queries). The paper [15] did not contain proofs.
The proofs and some new results were considered in [17, 18]. The aim of the present paper is to translate the results obtained in [15, 17, 18] into exact learning.

In [15, 17, 18], based on the results of exact learning [1, 2, 11, 12] and test theory and rough set theory [13, 14, 16], we investigated infinite binary information systems each of which consists of an infinite set of elements \( A \) and an infinite set \( F \) of functions (attributes) from \( A \) to \( \{0, 1\} \). We defined the notion of a testing problem described by a finite number of attributes \( f_1, \ldots, f_n \) from \( F \): for a given element \( a \in A \), we should recognize the tuple \((f_1(a), \ldots, f_n(a))\). To this end, we can use decision trees based on two types of queries. We can ask about the value of an attribute \( f_i \in \{f_1, \ldots, f_n\} \). We will obtain an answer of the kind \( f_i(a) = \delta \), where \( \delta \in \{0, 1\} \). We can also ask if a hypothesis \( f_1(a) = \delta_1, \ldots, f_n(a) = \delta_n \) is true, where \( \delta_1, \ldots, \delta_n \in \{0, 1\} \). Either this hypothesis will be confirmed or we will obtain a counterexample \( f_i(a) = -\delta_i \), which is chosen nondeterministically. The considered hypothesis is called proper if there exists an element \( b \in A \) such that \( f_1(b) = \delta_1, \ldots, f_n(b) = \delta_n \). As time complexity of a decision tree, we considered its depth, which is equal to the maximum number of queries in a path from the root to a terminal node of the tree.

For an arbitrary infinite binary information system, we studied five functions of Shannon type, which characterize the dependence in the worst case of the minimum depth of a decision tree solving a testing problem on the number of attributes in the problem description. The considered five functions correspond to the following five types of decision trees:

1. Only attributes are used in decision trees.
2. Only hypotheses are used in decision trees.
3. Both attributes and hypotheses are used in decision trees.
4. Only proper hypotheses are used in decision trees.
5. Both attributes and proper hypotheses are used in decision trees.

We proved that the first function has two possible types of behavior: logarithmic and linear. Each of the remaining functions has three possible types of behavior: constant, logarithmic, and linear. We also studied joint behavior of all five functions and described seven complexity classes of infinite binary information systems.

An exact learning problem is defined by a domain \( X \) and a concept class \( B \) [2]. The domain \( X \) is a nonempty finite set. A concept is any subset of \( X \), and a concept class \( B \) is a nonempty set of concepts. For a given (but hidden) concept \( c \in B \), we should recognize it using two types of queries. A membership query: for an element \( x \in X \), we ask if \( x \in c \). The answer is either \( x \in c \) or \( x \notin c \). An equivalence query: for a concept \( c' \subseteq X \), we ask if \( c' = c \). The answer is either yes or no. In the latter case, we receive a counterexample \( x \in (c' \setminus c) \cup (c \setminus c') \), which is chosen nondeterministically. The concept \( c' \) can be considered as a hypothesis. This hypothesis and corresponding equivalence query are called proper if \( c' \in B \).

To translate the results obtained in [15, 17, 18] into exact learning, we introduce the notion of an infinite family of concepts consisting of an infinite set of elements \( U \) and an infinite set \( C \) of subsets of \( U \) called concepts. Each nonempty finite subset \( X \) of the set \( U \) describes a
problem of exact learning with the domain $X$ and the concept class $\{c \cap X : c \in C\}$. We study the following modification of this problem: for a given (but hidden) concept $c \in C$, we should recognize the concept $c \cap X$ using the same queries as for the initial problem. Both the problem and its modification have the same sets of decision trees solving them.

In the present paper, for each infinite family of concepts, we study five functions of Shannon type, which characterize the dependence in the worst case of the minimum depth of a decision tree solving an exact learning problem on the number of elements in the problem description. The considered five functions correspond to the following five cases:

1. Only membership queries are used in decision trees.
2. Only equivalence queries are used in decision trees.
3. Both membership and equivalence queries are used in decision trees.
4. Only proper equivalence queries are used in decision trees.
5. Both membership and proper equivalence queries are used in decision trees.

As in the case of testing problems, the first function has two possible types of behavior – logarithmic and linear, and each of the remaining functions has three possible types of behavior – constant, logarithmic, and linear. We also consider joint behavior of these five functions and distinguish seven complexity classes of infinite families of concepts.

The translation of the results obtained in [15, 17, 18] into exact learning will be helpful for researchers in this area: the duality of the two directions is simple (attributes and elements from test theory and rough set theory correspond to elements and concepts from exact learning, respectively) but working out the details takes some effort.

The study of decision trees with hypotheses is interesting not only from the theoretical point of view. Experimental results obtained in [3, 4, 5, 6, 7] show that such decision trees can have less complexity than the conventional decision trees. These results open up some prospects for using decision trees with hypotheses as a means for knowledge representation.

The rest of the paper is organized as follows. Sections 2 and 3 present basic notions and main results. Sections 4–6 contain proofs, and Section 7– short conclusions.

2 Basic Notions

Let $U$ be a nonempty set and $C$ be a nonempty set of subsets of $U$ called concepts. The pair $F = (U, C)$ is called a family of concepts. If $U$ and $C$ are infinite sets, then the pair $F = (U, C)$ is called an infinite family of concepts. For each element $u \in U$, we define a function $u : C \rightarrow \{0, 1\}$ with the same name as follows: for any $c \in C$, $u(c) = 1$ if and only if $u \in c$.

A problem over $F$ is an arbitrary $n$-tuple $z = (u_1, \ldots, u_n)$, where $n \in \mathbb{N}$, $\mathbb{N}$ is the set of natural numbers $\{1, 2, \ldots\}$, and $u_1, \ldots, u_n \in U$. The problem $z$ is as follows: for a given concept $c \in C$, we should recognize the tuple $z(c) = (u_1(c), \ldots, u_n(c))$. The number $\dim z = n$ is called the dimension of the problem $z$. Denote $U(z) = \{u_1, \ldots, u_n\}$ and
$C(z) = \{ c \cap U(z) : c \in C \}$. Note that the problem $z$ can also be interpreted as the exact learning problem with the domain $U(z)$ and the concept class $C(z)$. We denote by $P(F)$ the set of problems over the family of concepts $F$.

A system of equations over $F$ is an arbitrary equation system of the kind

$$\{v_1(x) = \delta_1, \ldots, v_m(x) = \delta_m\},$$

where $m \in \mathbb{N} \cup \{0\}$, $v_1, \ldots, v_m \in U$, and $\delta_1, \ldots, \delta_m \in \{0, 1\}$ (if $m = 0$, then the considered equation system is empty). This equation system is called a system of equations over $z$ if $v_1, \ldots, v_m \in U(z)$. The considered equation system is called consistent (on $C$) if its set of solutions on $C$ is nonempty. The set of solutions of the empty equation system coincides with $C$.

As algorithms for problem $z$ solving, we consider decision trees with two types of queries – membership and equivalence. We can choose an element $u_i \in U(z)$ and ask about value of the function $u_i$. This membership query has two possible answers $\{u_i(x) = 0\}$ and $\{u_i(x) = 1\}$. We can formulate a hypothesis over $z$ in the form $H = \{u_1(x) = \delta_1, \ldots, u_n(x) = \delta_n\}$, where $\delta_1, \ldots, \delta_n \in \{0, 1\}$, and ask about this hypothesis. This equivalence query has $n + 1$ possible answers: $H, \{u_1(x) = -\delta_1\}, \ldots, \{u_n(x) = -\delta_n\}$, where $-1 = 0$ and $-0 = 1$. The first answer means that the hypothesis is true. Other answers are counterexamples. This hypothesis and the corresponding equivalence query are called proper (for $F$) if the system of equations $H$ is consistent on $C$.

A decision tree over $z$ is a marked finite directed tree with the root in which

- Each terminal node is labeled with an $n$-tuple from the set $\{0, 1\}^n$.
- Each node, which is not terminal (such nodes are called working), is labeled with an element from the set $U(z)$ (a membership query) or with a hypothesis over $z$ (an equivalence query).
- If a working node is labeled with an element $u_i$ from $U(z)$, then there are two edges, which leave this node and are labeled with the systems of equations $\{u_i(x) = 0\}$ and $\{u_i(x) = 1\}$, respectively.
- If a working node is labeled with a hypothesis $H = \{u_1(x) = \delta_1, \ldots, u_n(x) = \delta_n\}$ over $z$, then there are $n + 1$ edges, which leave this node and are labeled with the systems of equations $H, \{u_1(x) = -\delta_1\}, \ldots, \{u_n(x) = -\delta_n\}$, respectively.

Let $\Gamma$ be a decision tree over $z$. A complete path in $\Gamma$ is an arbitrary directed path from the root to a terminal node in $\Gamma$. We now define an equation system $S(\xi)$ over $F$ associated with the complete path $\xi$. If there are no working nodes in $\xi$, then $S(\xi)$ is the empty system. Otherwise, $S(\xi)$ is the union of equation systems assigned to the edges of the path $\xi$. We denote by $C(\xi)$ the set of solutions on $C$ of the system of equations $S(\xi)$ (if this system is empty, then its solution set is equal to $C$).
We will say that a decision tree \( \Gamma \) over \( z \) solves the problem \( z \) relative to \( F \) if, for each concept \( c \in C \) and for each complete path \( \xi \) in \( \Gamma \) such that \( c \in C(\xi) \), the terminal node of the path \( \xi \) is labeled with the tuple \( z(c) \).

We now consider an equivalent definition of a decision tree solving a problem. Denote by \( \Delta_F(z) \) the set of tuples \( (\delta_1, \ldots, \delta_n) \in \{0,1\}^n \) such that the system of equations \( \{u_1(x) = \delta_1, \ldots, u_n(x) = \delta_n\} \) is consistent. The set \( \Delta_F(z) \) is the set of all possible solutions to the problem \( z \). Let \( \Delta \subseteq \Delta_F(z) \), \( u_{i_1}, \ldots, u_{i_m} \in \{u_1, \ldots, u_n\} \), and \( \sigma_1, \ldots, \sigma_m \in \{0,1\} \). Denote

\[
\Delta(u_{i_1}, \sigma_1) \cdots (u_{i_m}, \sigma_m)
\]

the set of all \( n \)-tuples \( (\delta_1, \ldots, \delta_n) \in \Delta \) for which \( \delta_{i_1} = \sigma_1, \ldots, \delta_{i_m} = \sigma_m \).

Let \( \Gamma \) be a decision tree over the problem \( z \). We correspond to each complete path \( \xi \) in the tree \( \Gamma \) a word \( \pi(\xi) \) in the alphabet \( \{(u_i, \delta) : u_i \in U(z), \delta \in \{0,1\}\} \). If the equation system \( S(\xi) \) is empty, then \( \pi(\xi) \) is the empty word. If \( S(\xi) = \{u_{i_1}(x) = \sigma_1, \ldots, u_{i_m}(x) = \sigma_m\} \), then

\[
\pi(\xi) = (u_{i_1}, \sigma_1) \cdots (u_{i_m}, \sigma_m).
\]

The decision tree \( \Gamma \) over \( z \) solves the problem \( z \) relative to \( F \) if, for each complete path \( \xi \) in \( \Gamma \), the set \( \Delta_F(z) \pi(\xi) \) contains at most one tuple and if this set contains exactly one tuple, then the considered tuple is assigned to the terminal node of the path \( \xi \).

As time complexity of a decision tree, we consider its depth that is the maximum number of working nodes in a complete path in the tree or, which is the same, the maximum length of a complete path in the tree. We denote by \( h(\Gamma) \) the depth of a decision tree \( \Gamma \).

Let \( z \in P(F) \). We denote by \( h_F^{(1)}(z) \) the minimum depth of a decision tree over \( z \), which solves \( z \) relative to \( F \) and uses only membership queries. We denote by \( h_F^{(2)}(z) \) the minimum depth of a decision tree over \( z \), which solves \( z \) relative to \( F \) and uses only equivalence queries. We denote by \( h_F^{(3)}(z) \) the minimum depth of a decision tree over \( z \), which solves \( z \) relative to \( F \) and uses both membership and equivalence queries. We denote by \( h_F^{(4)}(z) \) the minimum depth of a decision tree over \( z \), which solves \( z \) relative to \( F \) and uses only proper equivalence queries. We denote by \( h_F^{(5)}(z) \) the minimum depth of a decision tree over \( z \), which solves \( z \) relative to \( F \) and uses both membership and proper equivalence queries.

For \( i = 1, \ldots, 5 \), we define a function of Shannon type \( h_F^{(i)}(n) \) that characterizes dependence of \( h_F^{(i)}(z) \) on \( \dim z \) in the worst case. Let \( i \in \{1, \ldots, 5\} \) and \( n \in \mathbb{N} \). Then

\[
h_F^{(i)}(n) = \max\{h_F^{(i)}(z) : z \in P(F), \dim z \leq n\}.
\]

### 3 Main Results

Let \( F = (U, C) \) be an infinite family of concepts and \( r \in \mathbb{N} \). We will say that the family of concepts \( F \) is \( r \)-reduced if, for each consistent on \( C \) system of equations over \( F \), there exists a subsystem of this system that has the same set of solutions on \( C \) and contains at most \( r \) equations. We denote by \( \mathcal{R} \) the set of infinite families of concepts each of which is \( r \)-reduced for some \( r \in \mathbb{N} \).

The next theorem follows from the results obtained in [14], where we studied closed classes of test tables (decision tables). It also follows from the results obtained in [16], where we investigated the weighted depth of decision trees for testing problems.
Theorem 1. Let $F$ be an infinite family of concepts. Then the following statements hold:

(a) If $F \in \mathcal{R}$, then $h_F^{(1)}(n) = \Theta(\log n)$.

(b) If $F \notin \mathcal{R}$, then $h_F^{(1)}(n) = n$ for any $n \in \mathbb{N}$.

Let $F = (U, C)$ be an infinite family of concepts. A subset $\{u_1, \ldots, u_m\}$ of $U$ is shattered by $C$ if, for any $\delta_1, \ldots, \delta_m \in \{0, 1\}$, the system of equations $\{u_1(x) = \delta_1, \ldots, u_m(x) = \delta_m\}$ is consistent on the set $C$. This definition is equivalent to the standard one: a subset $\{u_1, \ldots, u_m\}$ of $U$ is shattered by $C$ if, for any subset $X$ of $\{u_1, \ldots, u_m\}$, there exists a concept $c \in C$ such that $X = \{u_1, \ldots, u_m\} \cap c$. The empty set of elements is shattered by $C$ by definition. We now define inductively the notion of $k$-dimension or concept $c$.

Theorem 2. Let $F$ be an infinite family of concepts. Then the following statements hold:

(a) If $F \in \mathcal{C}$, then $h_F^{(2)}(n) = O(1)$ and $h_F^{(3)}(n) = O(1)$.

(b) If $F \notin \mathcal{C}$, then $h_F^{(2)}(n) = \Theta(\log n)$, $h_F^{(3)}(n) = \Omega\left(\frac{\log n}{\log \log n}\right)$, and $h_F^{(3)}(n) = O(\log n)$.

(c) If $F \notin \mathcal{D}$, then $h_F^{(2)}(n) = n$ and $h_F^{(3)}(n) = n$ for any $n \in \mathbb{N}$.

From Lemma B below it follows that $\mathcal{C} \subseteq \mathcal{D}$. Therefore, for any infinite family of concepts $F$, either $F \in \mathcal{C}$, or $F \in \mathcal{D} \setminus \mathcal{C}$, or $F \notin \mathcal{D}$.

Let $F = (U, C)$ be an infinite family of concepts and $r \in \mathbb{N}$. We will say that the family of concepts $F$ is $r$-i-reduced if, for each inconsistent on $C$ system of equations over $F$, there exists a subsystem of this system that is inconsistent and contains at most $r$ equations. We denote by $\mathcal{I}$ the set of infinite families of concepts each of which is $r$-i-reduced for some $r \in \mathbb{N}$.

Since $\mathcal{C} \subseteq \mathcal{D}$, for any infinite family of concepts $F$, either $F \in \mathcal{C} \cap \mathcal{I}$, or $F \in (\mathcal{D} \setminus \mathcal{C}) \cap \mathcal{I}$, or $F \in \mathcal{D} \setminus \mathcal{I}$, or $F \notin \mathcal{D}$.

Theorem 3. Let $F$ be an infinite family of concepts. Then the following statements hold:

(a) If $F \in \mathcal{C} \cap \mathcal{I}$, then $h_F^{(4)}(n) = O(1)$ and $h_F^{(5)}(n) = O(1)$.
(b) If $F \in (\mathcal{D} \setminus \mathcal{C}) \cap \mathcal{I}$, then $h_F^{(4)}(n) = \Theta(\log n)$, $h_F^{(5)}(n) = \Omega(\frac{\log n}{\log \log n})$, and $h_F^{(5)}(n) = O(\log n)$.

(c) If $F \in \mathcal{D} \setminus \mathcal{I}$ and $i \in \{4, 5\}$, then $h_F^{(i)}(n) \geq n - 1$ for infinitely many $n \in \mathbb{N}$ and $h_F^{(i)}(n) \leq n$ for any $n \in \mathbb{N}$.

(d) If $F \notin \mathcal{D}$, then $h_F^{(4)}(n) = n$ and $h_F^{(5)}(n) = n$ for any $n \in \mathbb{N}$.

Let $F$ be an infinite family of concepts. We now consider the joint behavior of the functions $h_F^{(1)}(n), \ldots, h_F^{(5)}(n)$. It depends on the belonging of the family of concepts $F$ to the sets $\mathcal{R}, \mathcal{D}, \mathcal{C}$, and $\mathcal{I}$. We correspond to the family of concepts $F$ its indicator vector $\text{ind}(U) = (e_1, e_2, e_3, e_4) \in \{0, 1\}^4$ in which $e_1 = 1$ if and only if $F \in \mathcal{R}$, $e_2 = 1$ if and only if $F \in \mathcal{D}$, $e_3 = 1$ if and only if $F \in \mathcal{C}$, and $e_4 = 1$ if and only if $F \in \mathcal{I}$.

Table 1: Possible indicator vectors of infinite families of concepts

| $\mathcal{R}$ | $\mathcal{D}$ | $\mathcal{C}$ | $\mathcal{I}$ |
|---------------|---------------|---------------|---------------|
| 1             | 0             | 0             | 0             |
| 2             | 0             | 0             | 1             |
| 3             | 0             | 1             | 0             | 0             |
| 4             | 0             | 1             | 0             | 1             |
| 5             | 0             | 1             | 1             | 0             |
| 6             | 0             | 1             | 1             | 1             |
| 7             | 1             | 1             | 0             | 1             |

Theorem 4. For any infinite family of concepts, its indicator vector coincides with one of the rows of Table 1. Each row of Table 1 is the indicator vector of some infinite family of concepts.

Table 2: Summary of Theorems 1–4

| $\mathcal{F}_i$ | $\mathcal{R}$ | $\mathcal{D}$ | $\mathcal{C}$ | $\mathcal{I}$ | $h_F^{(1)}(n)$ | $h_F^{(2)}(n)$ | $h_F^{(3)}(n)$ | $h_F^{(4)}(n)$ | $h_F^{(5)}(n)$ |
|-----------------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|
| $\mathcal{F}_1$ | 0             | 0             | 0             | 0             | $n$           | $n$           | $n$           | $n$           | $n$           |
| $\mathcal{F}_2$ | 0             | 0             | 0             | 1             | $n$           | $n$           | $n$           | $n$           | $n$           |
| $\mathcal{F}_3$ | 0             | 1             | 0             | 0             | $n$           | $\Theta(\log n)$ | $\approx \log n$ | $\approx n$ | $\approx n$ |
| $\mathcal{F}_4$ | 0             | 1             | 0             | 1             | $n$           | $\Theta(\log n)$ | $\approx \log n$ | $\Theta(\log n)$ | $\approx \log n$ |
| $\mathcal{F}_5$ | 0             | 1             | 1             | 0             | $n$           | $O(1)$        | $O(1)$        | $\approx n$ | $\approx n$ |
| $\mathcal{F}_6$ | 0             | 1             | 1             | 1             | $n$           | $O(1)$        | $O(1)$        | $O(1)$        | $O(1)$        |
| $\mathcal{F}_7$ | 1             | 1             | 0             | 1             | $\Theta(\log n)$ | $\Theta(\log n)$ | $\approx \log n$ | $\Theta(\log n)$ | $\approx \log n$ |

For $i = 1, \ldots, 7$, we denote by $\mathcal{F}_i$ the class of all infinite families of concepts, which indicator vector coincides with the $i$th row of Table 1. Table 2 summarizes Theorems 1–4. The first column contains the name of complexity class $\mathcal{F}_i$. The next four columns describe the indicator vector of families of concepts from this class. The last five columns
Proof of Theorem 1
(a) Let $F = (U, C) \in \mathcal{R}$. First, we prove that $h_F^{(1)}(n) = O(\log n)$. Since $F \in \mathcal{R}$, there exists a natural $r$ such that, for each consistent on $C$ system of equations over $F$, there exists a subsystem of this system, which has the same set of solutions on $F$ and contains at most $r$ equations.

Let $z = (u_1, \ldots, u_m)$ be a problem over $F$. The number of equation systems over $z$ containing at most $r$ equations is at most the number of $r$-tuples of equations of the kind $u_i = \delta$, where $u_i \in \{u_1, \ldots, u_m\}$ and $\delta \in \{0, 1\}$. The latter number is equal to $(2m)^r$. Therefore $|\Delta_F(z)| \leq (2m)^r$.

We consider a decision tree $\Gamma$, which solves the problem $z$ relative to $F$ and uses only membership queries. This tree is constructed by a halving algorithm that is similar to proposed in [13]. We will describe the work of $\Gamma$ for an arbitrary concept $c$ from $C$. This work consists of the following steps.

Denote $\Delta = \Delta_U(z)$. If $|\Delta| = 1$, then the only $n$-tuple from $\Delta$ is the solution $z(c)$ of the problem $z$ for the concept $c$. Let $|\Delta| \geq 2$. For $i = 1, \ldots, m$, we denote by $\delta_i$ a number from $\{0, 1\}$ such that $|\Delta(u_i, \delta_i)| \geq |\Delta(u_i, \neg \delta_i)|$. Let $k$ be the maximum number from $\{1, \ldots, m\}$ for which the equation system $\{u_i(x) = \delta_1, \ldots, u_k(x) = \delta_k\}$ is consistent on $C$. This system contains a subsystem $\{u_{i_1}(x) = \delta_{i_1}, \ldots, u_{i_t}(x) = \delta_{i_t}\}$, which has the same set of solutions on $C$ and for which $t \leq r$. We now show that $|\Delta(u_{i_1}, \delta_{i_1}) \cdots (u_{i_t}, \delta_{i_t})| \leq |\Delta|/2$. If $k = m$, then $|\Delta(u_{i_1}, \delta_{i_1}) \cdots (u_{i_t}, \delta_{i_t})| \leq |\Delta|/2$. Let $k < m$. Then the equation system $\{u_{i_1}(x) = \delta_{i_1}, \ldots, u_{i_t}(x) = \delta_{i_t}, u_{k+1}(x) = \delta_{k+1}\}$ is inconsistent. Therefore $\Delta(u_{i_1}, \delta_{i_1}) \cdots (u_{i_t}, \delta_{i_t}) \subseteq \Delta(u_{i_1}, \delta_{i_1}) \cdots (u_{i_t}, \delta_{i_t}) \cdots (u_{k+1}, \neg \delta_{k+1})$ and $|\Delta(u_{i_1}, \delta_{i_1}) \cdots (u_{i_t}, \delta_{i_t})| \leq |\Delta|/2$. We sequentially compute values of the functions $u_{i_1}, \ldots, u_{i_t}$ for the concept $c$. If $u_{i_1}(c) = \delta_{i_1}, \ldots, u_{i_t}(c) = \delta_{i_t}$, then during the next step we will work with the set of tuples $\Delta' = \Delta(u_{i_1}, \delta_{i_1}) \cdots (u_{i_t}, \delta_{i_t})$ for which $|\Delta'| \leq |\Delta|/2$. If, for some $p \in \{1, \ldots, t - 1\}$, $u_{i_1}(c) = \delta_{i_1}, \ldots, u_{i_p}(c) = \delta_{i_p}$ and $u_{i_{p+1}}(c) = \neg \delta_{i_{p+1}}$, then during the next step we will work with the set of tuples $\Delta'' = \Delta(u_{i_1}, \delta_{i_1}) \cdots (u_{i_p}, \delta_{i_p})(u_{i_{p+1}}, \neg \delta_{i_{p+1}})$.

It is clear that $|\Delta''| \leq |\Delta|/2$. At this step, we compute values of at most $r$ functions (make at most $r$ membership queries) and reduce the number of $m$-tuples (possible solutions) by half.
Let during the work with the concept \( c \), the decision tree \( \Gamma \) take \( q \) steps. After \( (q - 1) \)th step, the number of remaining \( m \)-tuples will be at least two and at most \( (2m)^r / 2^{q-1} \). Therefore \( 2^q \leq (2m)^r \) and \( q \leq r \log_2(2m) \). So during the processing of the concept \( c \), the decision tree \( \Gamma \) computes values of at most \( r^2 \log_2(2m) \) functions (makes at most \( r^2 \log_2(2m) \) membership queries). Since \( c \) is an arbitrary concept from \( C \), the depth of \( \Gamma \) is at most \( r^2 \log_2(2m) \). Since \( z \) is an arbitrary problem over \( F \), we obtain \( h^{(1)}_F(n) = O(\log n) \).

We now show that \( h^{(1)}_F(n) = \Omega(\log n) \). We prove by induction on \( n \) that, for any natural \( n \), there is a problem \( z_n = (u_1, \ldots, u_n) \) over \( F \) such that \( |\Delta_F(z_n)| \geq n + 1 \). Since \( C \) is an infinite set of concepts, there exist two concepts \( c_1, c_2 \in C \) and an element \( u_1 \in U \) such that \( u_1 \in c_1 \) and \( u_1 \notin c_2 \). Therefore \( |\Delta_F(z_1)| \geq 2 \), where \( z_1 = (u_1) \). Let, for some natural \( n \), there exist a problem \( z_n = (u_1, \ldots, u_n) \) over \( F \) such that \( |\Delta_F(z_n)| \geq n + 1 \). Since \( C \) is an infinite set of concepts, there exist two concepts \( c_3, c_4 \in C \) and an element \( u_{n+1} \in U \) such that \( \{u_1, \ldots, u_n\} \cap c_3 = \{u_1, \ldots, u_n\} \cap c_4, u_{n+1} \in c_3, \) and \( u_{n+1} \notin c_4 \). One can show that \( |\Delta_F(z_{n+1})| \geq n + 2 \), where \( z_{n+1} = (u_1, \ldots, u_n, u_{n+1}) \). Let \( n \in \mathbb{N}, z_n = (u_1, \ldots, u_n) \) be a problem over \( F \) such that \( |\Delta_F(z_n)| \geq n + 1 \) and \( \Gamma \) be a decision tree solving the problem \( z \) relative to \( F \). Then \( \Gamma \) should have at least \( n+1 \) terminal nodes. One can show that the number of terminal nodes in the tree \( \Gamma \) is at most \( 2^{h(\Gamma)} \). Therefore \( n + 1 \leq 2^{h(\Gamma)} \), \( h(\Gamma) \geq \log_2(n+1) \), and \( h^{(1)}_F(z) \geq \log_2(n+1) \). Thus, \( h^{(1)}_F(n) = \Omega(\log n) \) and \( h^{(1)}_F(n) = \Theta(\log n) \).

(b) Let \( F = (U, C) \notin \mathcal{R} \). One can show that, for any \( n \in \mathbb{N} \), there is a consistent system of equations \( S = \{u_1(x) = \delta_1, \ldots, u_n(x) = \delta_n\} \) over \( F \) for which each proper subsystem has different solution set than \( S \). Let \( \Gamma \) be a decision tree solving the problem \( z = (u_1, \ldots, u_n) \) relative to \( F \). Then there is a complete path \( \xi \) in \( \Gamma \) such that \( |\Delta_F(z_\xi)|| \geq \{\delta_1, \ldots, \delta_n\} \). From here it follows that \( S = \mathcal{S}(\xi) \). Therefore the complete path \( \xi \) contains at least \( n \) working nodes and \( h(\Gamma) \geq n \). Taking into account that \( \Gamma \) is an arbitrary decision tree solving \( z \) relative to \( F \), we obtain \( h^{(1)}_F(z) \geq n \) and \( h^{(1)}_F(n) \geq n \). It is easy to show that, for each problem \( z \) over \( U \), the inequality \( h^{(1)}_F(z) \leq \dim z \) holds: to solve the problem \( z \), it is enough to compute values of functions corresponding to all elements from the set \( U(z) \). Thus, \( h^{(1)}_F(n) = n \) for any \( n \in \mathbb{N} \).

\( \square \)

We precede the proof of Theorem 2 by two lemmas.

Let \( d \in \mathbb{N} \). A \textit{d-complete tree over the family of concepts} \( F = (U, C) \) is a marked finite directed tree with the root in which

- Each terminal node is not labeled.

- Each nonterminal node is labeled with an element \( u \in U \). There are two edges leaving this node that are labeled with the systems of equations \( \{u(x) = 0\} \) and \( \{u(x) = 1\} \), respectively.

- The length of each complete path (path from the root to a terminal node) is equal to \( d \).

For each complete path \( \xi \), the equation system \( \mathcal{S}(\xi) \), which is the union of equation systems assigned to the edges of the path \( \xi \), is consistent.
Let $G$ be a $d$-complete tree over $F$ and $U(G)$ be the set of all elements attached to the nonterminal nodes of the tree $G$. The number of nonterminal nodes in $G$ is equal to $2^0 + 2^1 + \ldots + 2^{d-1} = 2^d - 1$. Therefore $|U(G)| \leq 2^d$.

The results mentioned in the following lemma are obtained by methods similar to used by Littlestone [11], Maass and Turán [12], and Angluin [2].

**Lemma 1.** Let $F = (U, C)$ be a family of concepts, $d \in \mathbb{N}$, $G$ be a $d$-complete tree over $F$, and $z$ be a problem over $U$ such that $U(G) \subseteq U(z)$. Then

1. $h_F^{(2)}(z) \geq d$.
2. $h_F^{(3)}(z) \geq \frac{d}{\log_2(2d)}$.

**Proof.** (a) We prove the inequality $h_F^{(2)}(z) \geq d$ by induction on $d$. Let $d = 1$. Then the tree $G$ has the only one nonterminal node, which is labeled with an element $u$ such that the function $u$ is not constant on $C$. Therefore $|\Delta_F(z)| \geq 2$ and $h_F^{(2)}(z) \geq 1$. Let, for $t \in \mathbb{N}$ and for any natural $d$, $1 \leq d \leq t$, the considered statement hold. Assume now that $d = t + 1$, $G$ is a $d$-complete tree over $F$, $z$ is a problem over $F$ such that $U(G) \subseteq U(z)$, and $\Gamma$ is a decision tree over $z$ with the minimum depth, which solves the problem $z$ and uses only equivalence queries. Let $u$ be the element attached to the root of the tree $G$ and $H$ be the hypothesis attached to the root of the decision tree $\Gamma$. Then there is an edge, which leaves the root of $\Gamma$ and is labeled with the equation system $\{u(\sigma) = \delta\}$, where the equation $u(\sigma) = -\delta$ belongs to the hypothesis $H$. This edge enters to the root of the subtree of $\Gamma$, which will be denoted by $\Gamma_u$. There is an edge, which leaves the root of $G$ and is labeled with the equation system $\{u(\sigma) = \delta\}$. This edge enters to the root of the subtree of $G$, which will be denoted by $G_\delta$. One can show that the decision tree $\Gamma_u$ solves the problem $z$ relative to the family of concepts $F' = (U, C(u, \delta))$ and $G_\delta$ is a $t$-complete tree over $F'$. It is clear that $U(G_\delta) \subseteq U(z)$. Using the inductive hypothesis, we obtain $h(\Gamma_u) \geq t$. Therefore $h(\Gamma) \geq t + 1 = d$ and $h_F^{(2)}(z) \geq d$.

(b) We now prove the inequality $h_F^{(3)}(z) \geq \frac{d}{\log_2(2d)}$. Let $z = (u_1, \ldots, u)$ and $\Gamma$ be a decision tree over $z$ with the minimum depth, which solves the problem $z$ and uses both membership and equivalence queries. The $d$-complete tree $G$ has $2^d$ complete paths $\xi_1, \ldots, \xi_{2^d}$. For $i = 1, \ldots, 2^d$, we denote by $c_i$ a solution of the equation system $S(\xi_i)$. Denote $B = \{c_1, \ldots, c_{2^d}\}$. We now show that the decision tree $\Gamma$ contains a complete path, which length is at least $\frac{d}{\log_2(2d)}$. We describe the process of this path construction beginning with the root of $\Gamma$.

Let the root of $\Gamma$ be labeled with an element $u_{i_0}$. For $\delta \in \{0, 1\}$, we denote by $B^\delta$ the set of solutions on $B$ of the equation system $\{u_{i_0}(x) = \delta\}$ and choose $\sigma \in \{0, 1\}$ for which $|B^\sigma| = \max\{|B^0|, |B^1|\}$. It is clear that $|B^\sigma| \geq \frac{|B|}{2} \geq \frac{|B|}{2^d}$. In the considered case, the beginning of the constructed path in $\Gamma$ is the root of $\Gamma$, the edge that leaves the root and is labeled with the equation system $\{u_{i_0}(x) = \sigma\}$, and the node to which this edge enters.

Let as assume now that the root of $\Gamma$ is labeled with a hypothesis $H = \{u_1(x) = \delta_1, \ldots, u_n(x) = \delta_n\}$. We denote by $\xi_H$ the complete path in $G$ for which the system of equations $S(\xi_H)$ is a subsystem of $H$. Let the nonterminal nodes of the complete path $\xi_H$ be labeled with the elements $u_{i_1}, \ldots, u_{i_d}$. For $j = 1, \ldots, d$, we denote by $B_j$ the set of solutions on $B$ of the equation system $\{u_{i_j}(x) = -\delta_{i_j}\}$. It is clear that $|B_1| + \ldots + |B_d| \geq |B| - 1$. Therefore there exists $l \in \{1, \ldots, d\}$ such that $|B_l| \geq \frac{|B| - 1}{d} \geq \frac{|B|}{2^d}$. In the considered case, the
beginning of the constructed path in \( \Gamma \) is the root of \( \Gamma \), the edge that leaves the root and is labeled with the equation system \( \{u_i(x) = -\delta_i\} \), and the node to which this edge enters.

We continue the construction of the complete path in \( \Gamma \) in the same way such that after the \( t \)th query we will have at least \( \frac{|B|}{2d^t} \) elements from \( B \). The process of path construction will continue at least until \( \frac{|B|}{2d^t} \leq 1 \), i.e., at least until \( \log_2 |B| \leq t \log_2(2d) \). Since \( |B| = 2^d \), we have \( h(\Gamma) \geq t \geq \frac{d}{\log_2(2d)} \) and \( h_F^{(3)}(z) \geq \frac{d}{\log_2(2d)} \).

**Lemma 2.** Let \( F = (U, C) \) be a family of concepts, \( k \in \mathbb{N} \cup \{0\} \), and \( F \) be not \( m \)-family of concepts for \( m = 0, \ldots, k \). Then there exists a \((k + 1)\)-complete tree over \( F \).

**Proof.** We prove the considered statement by induction on \( k \). Let \( k = 0 \). In this case, \( F \) is not \( 0 \)-family of concepts. Then there exists an element \( u \in U \) for which the function \( u \) is not constant on \( C \). Using this element, it is easy to construct 1-complete tree over \( F \).

Let the considered statement hold for some \( k, k \geq 0 \). We now show that it also holds for \( k + 1 \). Let \( F = (U, C) \) be a family of concepts, which is not \( m \)-family of concepts for \( m = 1, \ldots, k + 1 \). Then there exists an element \( u \in U \) such that, for any \( \delta \in \{0, 1\} \), the information system \( F_\delta = (U, C(u, \delta)) \) is not \( m \)-information system for \( m = 1, \ldots, k \). Using the inductive hypothesis, we conclude that, for any \( \delta \in \{0, 1\} \), there exists a \((k + 1)\)-complete tree \( G_\delta \) over \( F \). Denote by \( G \) a directed tree with root in which the root is labeled with the element \( u \) and, for any \( \delta \in \{0, 1\} \), there is an edge that leaves the root, is labeled with the equation system \( \{u(x) = \delta\} \), and enters the root of the tree \( G_\delta \). One can show that the tree \( G \) is a \((k + 2)\)-complete tree over \( F \).

**Proof of Theorem 2.** It is clear that \( h_F^{(3)}(z) \leq h_F^{(2)}(z) \) for any problem \( z \) over \( F \). Therefore \( h_F^{(3)}(n) \leq h_F^{(2)}(n) \) for any \( n \in \mathbb{N} \).

(a) Let \( k \in \mathbb{N} \cup \{0\} \). We now show by induction on \( k \) that, for each \( k \)-family of concepts \( F \) (not necessary infinite) for each problem \( z \) over \( F \), the inequality \( h_F^{(2)}(z) \leq k \) holds. Let \( F = (U, C) \) be a \( 0 \)-family of concepts and \( z \) be a problem over \( F \). Since all functions corresponding to elements from \( U(z) \) are constant on \( C \), the set \( \Delta_F(z) \) contains only one tuple. Therefore the decision tree containing only one node labeled with this tuple solves the problem \( z \) relative to \( F \), and \( h_F^{(2)}(z) = 0 \).

Let \( k \in \mathbb{N} \cup \{0\} \) and, for each \( m, 0 \leq m \leq k \), the considered statement hold. Let us show that it holds for \( k + 1 \). Let \( F = (U, C) \) be a \((k + 1)\)-family of concepts and \( z = (u_1, \ldots, u_n) \) be a problem over \( F \). For \( i = 1, \ldots, n \), choose a number \( \delta_i \in \{0, 1\} \) such that the family of concepts \( (U, C(u_i, -\delta_i)) \) is \( m_i \)-family of concepts, where \( 1 \leq m_i \leq k \). Using the inductive hypothesis, we conclude that, for \( i = 1, \ldots, n \), there is a decision tree \( \Gamma_i \) over \( z \), which uses only equivalence queries, solves the problem \( z \) over \( (U, C(u_i, -\delta_i)) \), and has depth at most \( m_i \). We denote by \( \Gamma \) a decision tree in which the root is labeled with the hypothesis \( H = \{u_1(x) = \delta_1, \ldots, u_n(x) = \delta_n\} \), the edge leaving the root and labeled with \( H \) enters the terminal node labeled with the tuple \( (\delta_1, \ldots, \delta_n) \), and for \( i = 1, \ldots, n \), the edge leaving the root and labeled with \( \{u_i(x) = -\delta_i\} \) enters the root of the tree \( \Gamma_i \). One can show that \( \Gamma \) solves the problem \( z \) relative to \( F \) and \( h(\Gamma) \leq k + 1 \). Therefore, \( h_F^{(2)}(z) \leq k + 1 \) for any problem \( z \) over \( F \).
Let $F \in \mathcal{C}$. Then $F$ is $k$-family of concepts for some natural $k$ and, for each problem $z$ over $F$, we have $h^{(3)}_F(z) \leq h^{(2)}_F(z) \leq k$. Therefore $h^{(2)}_F(n) = O(1)$ and $h^{(3)}_F(n) = O(1)$.

(b) Let $F = (U, C) \in \mathcal{D} \setminus \mathcal{C}$. First, we show that $h^{(2)}_F(n) = O(\log n)$. Let $z = (u_1, \ldots, u_n)$ be an arbitrary problem over $F$. From Lemma 5.1 it follows that $|\Delta_F(z)| \leq (4n)^{VC(F)}$. The proof of this lemma is based on the results similar to ones obtained by Sauer [20] and Shelah [21]. We consider a decision tree $\Gamma$ over $z$, which solves $z$ relative to $F$ and uses only equivalence queries. This tree is constructed by the halving algorithm [11][11]. We describe the work of this tree for an arbitrary concept $c$ from $C$. Set $\Delta = \Delta_F(z)$. If $|\Delta| = 1$, then the only $n$-tuple from $\Delta$ is the solution $z(c)$ of the problem $z$ for the concept $c$. Let $|\Delta| \geq 2$. For $i = 1, \ldots, m$, we denote by $\delta_i$ a number from $\{0, 1\}$ such that $|\Delta(u_i, \delta_i)| \geq |\Delta(u_i, \neg \delta_i)|$. The root of $\Gamma$ is labeled with the hypothesis $H = \{u_1(x) = \delta_1, \ldots, u_n(x) = \delta_n\}$. After this query either the problem $z$ is solved (if the answer is $H$) or we halve the number of tuples in the set $\Delta$ (if the answer is a counterexample $\{u_i(x) = \neg \delta_i\}$). In the latter case, set $\Delta = \Delta_F(z)(u_i, \neg \delta_i)$. The decision tree $\Gamma$ continues to work with the concept $c$ and the set of $n$-tuples $\Delta$ in the same way. During the work with the concept $c$, the considered decision tree make $q$ queries. After the $(q - 1)$th query, the number of remaining $n$-tuples in the set $\Delta$ is at least two and at most $(4n)^{VC(F)}/2^{q-1}$. Therefore $2^q \leq (4n)^{VC(F)}$ and $q \leq VC(F) \log_2(4n)$. So during the processing of the concept $c$, the decision tree $\Gamma$ makes at most $VC(F) \log_2(4n)$ queries. Since $c$ is an arbitrary concept from $C$, the depth of $\Gamma$ is at most $VC(F) \log_2(4n)$. Since $z$ is an arbitrary problem over $F$, we obtain $h^{(2)}_F(n) = O(\log n)$. Therefore $h^{(3)}_F(n) = O(\log n)$.

Using Lemma 2 and the relation $F \notin \mathcal{C}$, we obtain that, for any $d \in \mathbb{N}$, there exists $d$-complete tree $G_d$ over $F$. Let $U(G_d) = \{u_1, \ldots, u_{n_d}\}$. We know that $n_d \leq 2^d$. Denote $z_d = (u_1, \ldots, u_{n_d})$. From Lemma 1 it follows that $h^{(2)}_F(z_d) \geq d$ and $h^{(3)}_F(z_d) \geq d \log_2(2d)$. As a result, we have $h^{(2)}_F(2^d) \geq d$ and $h^{(3)}_F(2^d) \geq \frac{d \log_2(2d)}{\log_2(2d)}$. Let $n \in \mathbb{N}$ and $n \geq 8$. Then there exists $d \in \mathbb{N}$ such that $2^d \leq n < 2^{d+1}$. We have $d > \log_2 n - 1$, $h^{(2)}_F(n) \geq \log_2 n - 1$, $h^{(2)}_F(n) = \Omega(\log n)$, and $h^{(2)}_F(n) = \Theta(\log n)$. It is easy to show that the function $\frac{x}{\log_2(2x)}$ is nondecreasing for $x \geq 2$. Therefore $h^{(3)}_F(n) \geq \frac{\log_2 n - 1}{\log_2(2(\log_2 n - 1))}$ and $h^{(3)}_F(n) = \Omega(\frac{\log n}{\log \log n})$.

(c) Let $F = (U, C) \notin \mathcal{D}$. We now consider an arbitrary problem $z = (u_1, \ldots, u_n)$ over $F$ and a decision tree over $z$, which uses only equivalence queries and solves the problem $z$ over $F$ in the following way. For a given concept $c \in \mathcal{C}$, the first query is about the hypothesis $H_1 = \{u_1(x) = 1, \ldots, u_n(x) = 1\}$. If the answer is $H_1$, then the problem $z$ is solved for the concept $c$. If, for some $i \in \{1, \ldots, n\}$, the answer is $\{u_i(x) = 0\}$, then the second query is about the hypothesis $H_2$ obtained from $H_1$ by replacing the equality $u_i(x) = 1$ with the equality $u_i(x) = 0$, etc. It is clear that after at most $n$ equivalence queries the problem $z$ for the concept $c$ will be solved. Thus, $h^{(2)}_F(z) \leq n$ and $h^{(3)}_F(z) \leq n$. Since $z$ is an arbitrary problem over $F$, we have $h^{(2)}_F(n) \leq n$ and $h^{(3)}_F(n) \leq n$ for any $n \in \mathbb{N}$.

Let $n \in \mathbb{N}$. Since $F \notin \mathcal{D}$, there exist elements $u_1, \ldots, u_n \in U$ such that, for any $(\delta_1, \ldots, \delta_n) \in \{0, 1\}^n$, the equation system $\{u_1(x) = \delta_1, \ldots, u_n(x) = \delta_n\}$ is consistent on $C$. We now consider the problem $z = (u_1, \ldots, u_n)$ and an arbitrary decision tree $\Gamma$ over $z$, which solves the problem $z$ over $F$ and uses both membership and equivalence queries. Let
us show that $h(\Gamma) \geq n$. If $n = 1$, then the considered inequality holds since $|\Delta_F(z)| \geq 2$.
Let $n \geq 2$. It is easy to show that an equation system over $z$ is inconsistent if and only if it contains equations $u_i(x) = 0$ and $u_i(x) = 1$ for some $i \in \{1, \ldots, n\}$. For each node $v$ of the decision tree $\Gamma$, we denote by $S_v$ the union of systems of equations attached to edges in the path from the root of $\Gamma$ to $v$. A node $v$ of $\Gamma$ will be called consistent if the equation system $S_v$ is consistent.

We now construct a complete path $\xi$ in the decision tree $\Gamma$, which nodes are consistent. We will start constructing the path from the root that is a consistent node. Let the path reach a consistent node $v$ of $\Gamma$. If $v$ is a terminal node, then the path $\xi$ is constructed. Let $v$ be a working node labeled with an element $u_i \in U(z)$. Then there exists $\delta \in \{0, 1\}$ for which the system of equations $S_v \cup \{u_i(x) = \delta\}$ is consistent. Then the path $\xi$ will pass through the edge leaving $v$ and labeled with the system of equations $\{u_i(x) = \delta\}$. Let $v$ be labeled with a hypothesis $H = \{u_1(x) = \delta_1, \ldots, u_n(x) = \delta_n\}$. If there exists $i \in \{1, \ldots, n\}$ such that the system of equations $S_v \cup \{u_i(x) = -\delta\}$ is consistent, then the path $\xi$ will pass through the edge leaving $v$ and labeled with the system of equations $\{u_i(x) = -\delta\}$. Otherwise, $S_v = H$ and the path $\xi$ will pass through the edge leaving $v$ and labeled with the system of equations $H$.

Let all edges in the path $\xi$ be labeled with systems of equations containing one equation each. Since all nodes of $\xi$ are consistent, the equation system $S(\xi)$ is consistent. We now show that $S(\xi)$ contains at least $n$ equations. Let us assume that this system contains less than $n$ equations. Then the set $\Delta_F(z)\pi(\xi)$ contains more than one $n$-tuple, which is impossible. Therefore the length of the path $\xi$ is at least $n$. Let there be edges in $\xi$, which are labeled with hypotheses, and the first edge in $\xi$ labeled with a hypothesis $H$ leaves the node $v$. Then $S_v = H$ and the length of $\xi$ is at least $n$. Therefore $h(\Gamma) \geq n$, $h_F(3)(z) \geq n$, and $h_F(2)(z) \geq n$. As a result, we obtain $h_F(3)(n) \geq n$ and $h_F(2)(n) \geq n$. Thus, $h_F(2)(n) = n$ and $h_F(3)(n) = n$ for any $n \in \mathbb{N}$.

5 Proof of Theorem 3

In this section, we prove Theorem 3. First, we consider several auxiliary statements.

**Lemma 3.** Let $F = (U, C)$ be a family of concepts, $z$ be a problem over $F$, and $\Gamma_1$ be a decision tree over $z$ that solves the problem $z$ relative to $F$ and uses both membership and proper equivalence queries. Then there exists a decision tree $\Gamma_2$ over $z$ that solves the problem $z$ relative to $F$, uses only proper equivalence queries, and satisfies the inequality $h(\Gamma_2) \leq 2^{h(\Gamma_1)} - 1$.

**Proof.** We prove this statement by the induction on the depth $h(\Gamma_1)$ of the decision tree $\Gamma_1$. Let $h(\Gamma_1) = 0$. Then, as the decision tree $\Gamma_2$, we can take the decision tree $\Gamma_1$. It is clear that $h(\Gamma_2) = 2^{h(\Gamma_1)} - 1$. We now assume that the considered statement is true for any family of concepts, any problem over this family, and any decision tree over the considered problem that solves this problem, uses both membership and proper equivalence queries, and has depth at most $k$, $k \geq 0$. 

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Lemma 4. Let \( F = (U, C) \) be an infinite family of concepts. Then \( h_F^{(3)}(n) \leq h_F^{(5)}(n) \leq 2^{h(\Gamma_1)} - 1 \).

Let \( F = (U, C) \) be a family of concepts, \( z = (u_1, \ldots, u_n) \) be a problem over \( F \), and \( \Gamma_1 \) be a decision tree over \( z \) that solves the problem \( z \) relative to \( F \), uses both membership and proper equivalence queries, and satisfies the condition \( h(\Gamma_1) = k + 1 \). We now show that there exists a decision tree \( \Gamma_2 \) over \( z \), which solves the problem \( z \) relative to \( F \), uses only proper equivalence queries, and which depth is at most \( 2^{k+1} - 1 \).

Let the root of \( \Gamma_1 \) be labeled with a proper hypothesis \( H = \{ u_1(x) = \delta_1, \ldots, u_n(x) = \delta_n \} \).

Then there are \( n + 1 \) edges, which leave the root, are labeled with the systems of equations \( H, \{ u_1(x) = -\delta_1 \}, \ldots, \{ u_n(x) = -\delta_n \} \), and enter the roots of subtrees \( G_0, G_1, \ldots, G_n \) of the tree \( \Gamma_1 \), respectively. It is clear that, for \( i = 1, \ldots, n \), \( G_i \) is a decision tree over \( z \), which solves the problem \( z \) relative to the family of concepts \( F_i = (U, C(u_i, -\delta_i)) \), uses only membership queries and proper equivalence queries for \( F_i \), and satisfies the inequality \( h(G_i) \leq k \). Using the inductive hypothesis, we obtain that, for \( i = 1, \ldots, n \), there exists a decision tree \( G_i' \) over \( z \) that solves the problem \( z \) relative to \( F_i \), uses only proper equivalence queries for \( F_i \), and satisfies the inequalities \( h(G_i') \leq 2^{h(G_i)} - 1 \leq 2^k - 1 \). Let \( G_0' \) be the decision tree, which contains only one node labeled with the tuple \( (\delta_1, \ldots, \delta_n) \). We denote by \( \Gamma_2 \) the decision tree over \( z \) that is obtained from the decision tree \( \Gamma_1 \) by replacing the subtrees \( G_0, G_1, \ldots, G_n \) with the subtrees \( G_0', G_1', \ldots, G_n' \). It is easy to show that \( \Gamma_2 \) is a decision tree over \( z \), which solves the problem \( z \) relative to \( F \), uses only proper equivalence queries for \( F \), and satisfies the inequalities \( h(\Gamma_2) \leq 2^k - 1 + 1 \leq 2^{h(\Gamma_1)} - 1 \).

Let the root of \( \Gamma_1 \) be labeled with an element \( u_i \). Then there are two edges, which leave the root, are labeled with the equation systems \( \{ u_i(x) = 0 \} \) and \( \{ u_i(x) = 1 \} \), and enter the roots of subtrees \( T_0 \) and \( T_1 \) of the tree \( \Gamma_1 \), respectively. It is clear that, for \( p = 0, 1 \), \( T_p \) is a decision tree over \( z \), which solves the problem \( z \) relative to the family of concepts \( F_p = (U, C(u_i, p)) \), uses only membership queries and proper equivalence queries for \( F_p \), and satisfies the inequality \( h(T_p) \leq k \). Using the inductive hypothesis, we obtain that, for \( p = 0, 1 \), there exists a decision tree \( T_p' \) over \( z \) that solves the problem \( z \) relative to \( F_p \), uses only proper equivalence queries for \( U_p \), and satisfies the inequalities \( h(T_p') \leq 2^{h(T_p)} - 1 \leq 2^k - 1 \). We denote by \( T \) the decision tree obtained from the decision tree \( T_0' \) by replacing each terminal node of \( T_0' \) with the decision tree \( T_1' \).

Denote by \( \Gamma_2 \) the decision tree obtained from \( T \) by the following transformation of each complete path \( \xi \) in \( T \). If \( C(\xi) = \emptyset \), then we keep the path \( \xi \) untouched. Let \( C(\xi) \neq \emptyset \), \( \delta = (\delta_1, \ldots, \delta_n) \) be the tuple that was attached to the terminal node of the tree \( T_0' \) through which the path \( \xi \) passes, and \( \sigma = (\sigma_1, \ldots, \sigma_n) \) be the tuple attached to the terminal node of \( \xi \). Since \( C(\xi) \neq \emptyset \), at least one of the tuples \( \delta \) and \( \sigma \) belongs to the set \( \Delta_F(z) \). Let, for the definiteness, \( \delta \in \Delta_F(z) \). Denote \( H = \{ u_1(x) = \delta_1, \ldots, u_n(x) = \delta_n \} \). We replace the terminal node of the path \( \xi \) with the working node labeled with the hypothesis \( H \), which is proper for \( F \). There are \( n + 1 \) edges that leave this node and are labeled with the systems of equations \( H, \{ u_1(x) = -\delta_1 \}, \ldots, \{ u_n(x) = -\delta_n \} \), respectively. The edge labeled with \( H \) enters to the terminal node labeled with the tuple \( \delta \). All other edges enter to terminal nodes labeled with the tuple \( \sigma \). One can show that \( \Gamma_2 \) is a decision tree over \( z \) that solves the problem \( z \) relative to \( F \), uses only proper equivalence queries for \( U \), and satisfies the relations \( h(\Gamma_2) \leq 2(2^k - 1) + 1 = 2^{h(\Gamma_1)} - 1 \).
\(h_F^{(4)}(n) \leq n \text{ and } h_F^{(2)}(n) \leq h_F^{(4)}(n) \text{ for any } n \in \mathbb{N}.

Proof. It is clear, that \(h_F^{(3)}(z) \leq h_F^{(5)}(z) \leq h_F^{(4)}(z)\) and \(h_F^{(2)}(z) \leq h_F^{(4)}(z)\) for any problem \(z\) over \(U\). Therefore \(h_F^{(3)}(n) \leq h_F^{(5)}(n) \leq h_F^{(4)}(n)\) and \(h_F^{(2)}(n) \leq h_F^{(4)}(n)\) for any \(n \in \mathbb{N}\).

We now consider an arbitrary problem \(z = (u_1, \ldots, u_n)\) over \(F\) and a decision tree over \(z\), which uses only proper equivalence queries for \(F\) and solves the problem \(z\) relative to \(F\) in the following way. For a given concept \(c \in C\), the first query is about an arbitrary proper hypothesis \(H_1 = \{u_1(x) = \delta_1, \ldots, u_n(x) = \delta_n\}\) for \(F\). If the answer is \(H_1\), then the problem \(z\) is solved for the concept \(c\). If, for some \(i \in \{1, \ldots, n\}\), the answer is \(\{u_i(x) = -\delta_i\}\), then the second query is about a proper hypothesis \(H_2 = \{u_1(x) = \sigma_1, \ldots, u_n(x) = \sigma_n\}\) such that \(\sigma_i = -\delta_i\). If the answer is \(H_2\), then the problem \(z\) is solved for the concept \(c\). If, for some \(j \in \{1, \ldots, n\}\), the answer is \(\{u_j(x) = -\sigma_j\}\), then the third query is about a proper hypothesis \(H_3 = \{u_1(x) = \gamma_1, \ldots, u_n(x) = \gamma_n\}\) such that \(\gamma_i = -\delta_i\) and \(\gamma_j = -\sigma_j\), etc. It is clear that after at most \(n\) queries the problem \(z\) relative to the concept \(c\) will be solved. Thus, \(h_F^{(4)}(z) \leq n\). Since \(z\) is an arbitrary problem over \(F\), we have \(h_F^{(4)}(n) \leq n\) for any \(n \in \mathbb{N}\). □

Proof of Theorem 3 (a) Let \(r \in \mathbb{N}\). We now show by induction on \(k \in \mathbb{N} \cup \{0\}\) that, for each \(r\)-i-reduced \(k\)-family of concepts \(F\) (not necessary infinite) for each problem \(z\) over \(F\), the inequality \(h_F^{(5)}(z) \leq rk\) holds.

Let \(F = (U, C)\) be a \(r\)-i-reduced 0-family of concepts and \(z\) be a problem over \(F\). Since all functions corresponding to elements from \(U(z)\) are constant on \(C\), the set \(\Delta_F(z)\) contains only one tuple. Therefore the decision tree consisting of one node labeled with this tuple solves the problem \(z\) relative to \(F\), and \(h_F^{(5)}(z) = 0\).

Let \(k \in \mathbb{N} \cup \{0\}\) and, for each \(m_i\), \(0 \leq m \leq k\), the considered statement hold. Let us show that it holds for \(k + 1\). Let \(F = (U, C)\) be a \(r\)-i-reduced \((k + 1)\)-family of concepts and \(z = (u_1, \ldots, u_n)\) be a problem over \(F\). For \(i = 1, \ldots, n\), choose a number \(\delta_i \in \{0, 1\}\) such that the family \(U, C(u_i, -\delta_i)\) is \(m_i\)-family of concepts, where \(1 \leq m_i \leq k\). It is easy to show that \((U, C(u_i, -\delta_i))\) is \(r\)-i-reduced family of concepts. Using the inductive hypothesis, we conclude that, for \(i = 1, \ldots, n\), there is a decision tree \(\Gamma_i\) over \(z\), which uses both membership queries and proper equivalence queries for \((U, C(u_i, -\delta_i))\), solves the problem \(z\) relative to \((U, C(u_i, -\delta_i))\), and has depth at most \(rm_i\).

Let the hypothesis \(H = \{u_1(x) = \delta_1, \ldots, u_n(x) = \delta_n\}\) be proper for \(F\). We denote by \(T_1\) a decision tree in which the root is labeled with the hypothesis \(H\), the edge leaving the root and labeled with \(H\) enters the terminal node labeled with the tuple \((\delta_1, \ldots, \delta_n)\), and for \(i = 1, \ldots, n\), the edge leaving the root and labeled with \(\{u_i(x) = -\delta_i\}\) enters the root of the tree \(\Gamma_i\). One can show that \(T_1\) is a decision tree over \(z\), which uses both membership queries and proper equivalence queries for \(F\), solves the problem \(z\) relative to \(F\), and satisfies the inequalities \(h(T_1) \leq rk + 1 \leq r(k + 1)\).

Let the hypothesis \(H\) be not proper for \(F\). Then the equation system \(\{u_1(x) = \delta_1, \ldots, u_n(x) = \delta_n\}\) is inconsistent on \(C\), and there exists its subsystem \(\{u_i(x) = \delta_i, \ldots, u_i(x) = \delta_i\}\), which is inconsistent on \(C\) and for which \(t \leq r\). We denote by \(G\) a decision tree over \(z\) with \(2^t\) terminal nodes in which each terminal node is labeled with \(n\)-tuple \((0, \ldots, 0)\), and each complete path contains \(t\) working nodes labeled with elements \(u_1, \ldots, u_t\) starting from the root. We denote by \(T_2\) a decision tree obtained from the decision tree \(G\) by transformation of each
complete path $\xi$ in $G$. Let $\{u_{i_1}(x) = \sigma_1, \ldots, u_{i_t}(x) = \sigma_t\}$ be equation systems attached to edges leaving the working nodes of $\xi$ labeled with the elements $u_{i_1}, \ldots, u_{i_t}$, respectively. If $(\sigma_1, \ldots, \sigma_t) = (\delta_{i_1}, \ldots, \delta_{i_t})$, then we keep the path $\xi$ untouched. Otherwise, let $j$ be the minimum number from the set $\{1, \ldots, t\}$ such that $\sigma_j = \neg \delta_{i_j}$. In this case, we replace the terminal node of the path $\xi$ with the root of the decision tree $\Gamma_{i_j}$. One can show that $T_2$ is a decision tree over $z$, which uses both membership and proper equivalence queries, solves the problem $z$ relative to $F$, and satisfies the inequalities $h(T_2) \leq rk + t \leq r(k + 1)$. Therefore, $h_F(z) \leq r(k + 1)$ for any problem $z$ over $F$.

Let $F \in \mathcal{C} \cap \mathcal{I}$. Then $F$ is $r$-i-reduced $k$-family of concepts for some natural $r$ and $k$, and $h_F(z) \leq rk$ for each problem $z$ over $F$. From Lemma 3 it follows that $h_F(z) \leq 2^r k - 1$ for each problem $z$ over $F$. Therefore $h_F(n) = O(1)$ and $h_F(n) = O(1)$.

(b) Let $F = (U, C) \in (D \setminus C) \cap \mathcal{I}$. By Lemma 4, $h_F(n) \geq h_F(n)$ and $h_F(n) \geq h_F(n)$ for any $n \in \mathbb{N}$. Using the fact that $U \in D \setminus C$ and Theorem 2 we obtain $h_F(n) = \Omega(\log n)$ and $h_F(n) = \Omega(\log n)$. Therefore $h_F(n) = \Omega(\log n)$ and $h_F(n) = \Omega(\log n)$.

Since the family of concepts $F$ belongs to the set $\mathcal{D}$, it has finite VC-dimension $VC(F)$. Since $F \in \mathcal{I}$, the family of concepts $F$ is $r$-i-reduced for some natural $r$. We assume that $r \geq 2$. We can do it because each $t$-i-reduced family of concepts, $t \in \mathbb{N}$, is $(t + 1)$-i-reduced.

We now show that $h_F(n) = O(\log n)$. Let $z = (u_1, \ldots, u_n)$ be an arbitrary problem over $F$. From Lemma 5.1 [16] it follows that $|\Delta_F(z)| \leq (4n)^{VC(F)}$.

We consider a decision tree $\Gamma$ over $z$, which solves the problem $z$ relative to $F$ and uses only proper equivalence queries. This tree is constructed by a variant of the halving algorithm [2, 9, 10]. We describe the work of this tree for an arbitrary concept $c$ from $C$. Set $\Delta = \Delta_F(z)$. If $|\Delta| = 1$, then the only $n$-tuple from $\Delta$ is the solution $z(c)$ to the problem $z$ for the concept $c$. Let $|\Delta| \geq 2$. For $i = 1, \ldots, n$, we denote by $\delta_i$ a number from $\{0, 1\}$ such that $|\Delta(u_i, \delta_i)| \geq |\Delta(u_i, \neg \delta_i)|$.

Let the system of equations $H = \{u_1(x) = \delta_1, \ldots, u_n(x) = \delta_n\}$ be consistent on $C$. In this case, the root of $\Gamma$ is labeled with the proper hypothesis $H$. After this query, either the problem $z$ will be solved (if the answer is $H$) or the number of remaining tuples in $\Delta$ will be at most $|\Delta|/2$ (if the answer is a counterexample $\{u_i(x) = \neg \delta_i\}$).

Let the system of equations $H$ be inconsistent on $C$. For any inconsistent subsystem $B$ of $H$, there exists a subsystem $D$ of $B$, which is inconsistent and contains at most $r$ equations. Then the system $D$ contains at least one equation $u_i(x) = \delta_i$ such that $|\Delta(u_i, \neg \delta_i)| \geq |\Delta|/r$. If we assume the contrary, we obtain that the system $D$ is consistent, which is impossible. Let $u_i \in \{u_1, \ldots, u_n\}$. The element $u_i$ is called balanced if $|\Delta(u_i, \neg \delta_i)| \geq |\Delta|/r$, and unbalanced if $|\Delta(u_i, \neg \delta_i)| < |\Delta|/r$.

We denote by $H_u$ the subsystem of $H$ consisting of all equations $u_i(x) = \delta_i$ from $H$ with unbalanced elements $u_i$. We now show that the system $H_u$ is consistent. Let us assume the contrary. Then it will contain at least one equation for balanced element, which is impossible. Let $b$ be a solution from $C$ to the system $H_u$, and $u_1(b) = \sigma_1, \ldots, u_n(b) = \sigma_n$. Then the system of equations $P = \{u_1(x) = \sigma_1, \ldots, u_n(x) = \sigma_n\}$ is consistent on $C$.

In the considered case, the root of $\Gamma$ is labeled with the proper hypothesis $P$. After this query, either the problem $z$ will be solved (if the answer is $P$), or the number of remaining
tuples in $\Delta$ will be less than $|\Delta|/r$ (if the number is a counterexample $\{u_i(x) = -\sigma_i\}$ and $u_i$ is an unbalanced element), or the number of remaining tuples in $\Delta$ will be at most $|\Delta|/2$ (if the answer is a counterexample $\{u_i(x) = -\sigma_i\}$, $\sigma_i = \delta_i$, and $u_i$ is a balanced element), or the number of remaining tuples in $\Delta$ will be at most $|\Delta|(1 - 1/r)$ (if the answer is a counterexample $\{u_i(x) = -\sigma_i\}$, $\sigma_i = -\delta_i$, and $u_i$ is a balanced element).

After the first query ($H$ or $P$) of the decision tree $\Gamma$, either the problem $z$ will be solved or the number of remaining tuples in $\Delta$ will be at most $|\Delta|(1 - 1/r)$. In the latter case when the answer is a counterexample of the kind $\{u_i(x) = -\gamma_i\}$ ($\gamma_i = \delta_i$ if the first query is $H$ and $\gamma_i = \sigma_i$ if the first query is $P$) set $\Delta = \Delta_U(z)(u_i, -\gamma_i)$. It is easy to show that the family of concepts $(U,C(u_i,-\gamma_i))$ is also $r$-i-reduced. The decision tree $\Gamma$ continues to work with the concept $c$ and the set of $n$-tuples $\Delta$ in the same way.

Let during the work with the concept $c$, the decision tree $\Gamma$ make $q$ queries. After the $(q - 1)$th query, the number of remaining $n$-tuples in the set $\Delta$ is at least two and at most $(4n)^{VC(F)}(1 - 1/r)^{q - 1}$. Therefore $(1 + 1/(r - 1))^q \leq (4n)^{VC(F)}$ and $q \ln(1 + 1/(r - 1)) \leq VC(F) \ln(4n)$. Taking into account that $\ln(1 + 1/m) > 1/(m + 1)$ for any natural $m$, we obtain $q < rVC(F) \ln(4n)$. So during the processing of the concept $c$, the decision tree $\Gamma$ makes at most $rVC(F) \ln(4n)$ queries. Since $c$ is an arbitrary concept from $C$, the depth of $\Gamma$ is at most $rVC(F) \ln(4n)$ and $h_F^{(4)}(z) \leq rVC(F) \ln(4n)$. Since $z$ is an arbitrary problem over $F$, we obtain $h_F^{(4)}(n) = O(\log n)$. By Lemma 4 $h_F^{(5)}(n) = O(\log n)$.

(c) Let $F = (U,C) \in D \setminus I$. From Lemma 4 it follows that $h_F^{(5)}(n) \leq h_F^{(4)}(n) \leq n$ for any $n \in \mathbb{N}$. We now show that, for any $m \in \mathbb{N}$, there exists a natural $n$ such that $n \geq m$, $h_F^{(4)}(n) \geq n - 1$, and $h_F^{(5)}(n) \geq n - 1$.

Let $m \in \mathbb{N}$. Since $F \notin I$, there exists a system of equations $P$ over $U$ with $n \geq m$ equations such that $P$ is inconsistent but each proper subsystem of $P$ is consistent on $C$. Let, for the definiteness, $P = \{u_1(x) = 0, \ldots, u_n(x) = 0\}$. Consider the problem $z = (u_1, \ldots, u_n)$ over $F$. Then, for $i = 1, \ldots, n$, the set $\Delta_F(z)$ contains $n$-tuple $\bar{\delta}_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ in which all digits with the exception of the $i$th one are equal to 0.

Let $\Gamma$ be a decision tree over $z$ that solves the problem $z$ relative to $F$ and uses both membership and proper equivalence queries. We consider a complete path $\xi$ in $\Gamma$ in which each edge is labeled with an equation system of the kind $\{u_i(x) = 0\}$, where $u_i \in U(z)$. Such complete path exists since $P$ is not a proper hypothesis. Let $\pi(\xi) = (u_{i_1}, 0) \cdots (u_{i_t}, 0)$ for some $u_{i_1}, \ldots, u_{i_t} \in U(z)$. Since $\Gamma$ solves the problem $z$, the set $\Delta_F(z)\pi(\xi)$ contains at most one tuple. If we assume that $t < n - 1$, we obtain that $\Delta_F(z)\pi(\xi)$ contains at least two tuples. Therefore $t \geq n - 1$ and $h(\Gamma) \geq n - 1$. Thus, $h_F^{(5)}(z) \geq n - 1$, $h_F^{(5)}(n) \geq n - 1$ and, by Lemma 4 $h_F^{(4)}(n) \geq n - 1$.

(d) Let $F \notin D$. From Lemma 4 it follows that $h_F^{(3)}(n) \leq h_F^{(5)}(n) \leq h_F^{(4)}(n) \leq n$ for any $n \in \mathbb{N}$. By Theorem 2 $h_F^{(3)}(n) = n$ for any $n \in \mathbb{N}$. Thus, $h_F^{(5)}(n) = h_F^{(4)}(n) = n$ for any $n \in \mathbb{N}$.

6 Proof of Theorem 4

First, we prove several auxiliary statements.
Lemma 5. \( \mathcal{R} \subseteq \mathcal{D} \).

Proof. Let \( F \in \mathcal{R} \). By Theorem 7 \( h_F^{(1)}(n) = \Theta(\log n) \). Let us assume that \( F \not\in \mathcal{D} \). Then, for any \( n \in \mathbb{N} \), there exists a problem \( z = (u_1, \ldots, u_n) \) over \( F \) such that \( |\Delta_F(z)| = 2^n \). Let \( \Gamma \) be a decision tree over \( z \), which solves the problem \( z \) relative to \( F \) and uses only membership queries. Then \( \Gamma \) should have at least \( 2^n \) terminal nodes. One can show that the number of terminal nodes in the tree \( \Gamma \) is at most \( 2^{k(F)} \). Then \( 2^n \leq 2^{k(\Gamma)} \), \( h(\Gamma) \geq n \), and \( h_F^{(1)}(z) \geq n \). Therefore \( h_F^{(1)}(n) \geq n \) for any \( n \in \mathbb{N} \), which is impossible. Thus, \( \mathcal{R} \subseteq \mathcal{D} \).

Lemma 6. \( \mathcal{C} \subseteq \mathcal{D} \).

Proof. Let \( F \in \mathcal{C} \). By Theorem 2 \( h_F^{(2)}(n) = O(1) \). Let us assume that \( F \not\in \mathcal{D} \). Then, by Theorem 2 \( h_F^{(2)}(n) = n \) for any \( n \in \mathbb{N} \), which is impossible. Therefore \( \mathcal{C} \subseteq \mathcal{D} \).

Lemma 7. \( \mathcal{R} \cap \mathcal{C} = \emptyset \).

Proof. Assume the contrary: \( \mathcal{R} \cap \mathcal{C} \neq \emptyset \) and \( F = (U, C) \in \mathcal{R} \cap \mathcal{C} \). Let \( r, k, l, m \in \mathbb{N} \) and \( F \) be \( r \)-reduced \( k \)-family of concepts. We now consider an arbitrary problem \( z = (u_1, \ldots, u_n) \) over \( F \) and describe a decision tree \( \Gamma \) over \( z \), which uses only membership queries, solves the problem \( z \) over \( F \), and has depth at most \( kr \).

For \( i = 1, \ldots, n \), let \( \delta_i \) be a number from \( \{0, 1\} \) such that \( (U, C(u_i, \neg \delta_i)) \) is \( m_i \)-family of concepts with \( 0 \leq m_i < k \). Let \( t \) be the maximum number from the set \( \{1, \ldots, n\} \) such that the system of equations \( S = \{u_1(x) = \delta_1, \ldots, u_t(x) = \delta_t\} \) is consistent. Then there exists a subsystem \( \{u_i(x) = \delta_i, \ldots, u_p(x) = \delta_p\} \) of the system \( S \), which has the same set of solutions as \( S \) and for which \( p \leq r \). For a given \( c \in C \), the decision tree \( \Gamma \) computes sequentially values \( u_{i_1}(c), \ldots, u_{i_p}(c) \).

If, for some \( q \in \{1, \ldots, p\} \), \( u_{i_q}(c) = \delta_{i_q}, \ldots, u_{i_q-1}(c) = \delta_{i_{q-1}}, \) and \( u_{i_q}(c) = \neg \delta_{i_q} \), then the decision tree \( \Gamma \) continues to work with the problem \( z \) and the family of concepts \( F' = (U, C') \), where \( C' \) is the set of solutions on \( C \) of the equation system \( \{u_1(x) = \delta_1, \ldots, u_{q-1}(x) = \delta_{i_{q-1}}, u_q(x) = \neg \delta_{i_q}\} \). One can show that \( F' \) is \( l' \)-family of concepts for some \( l' \leq m_{i_q} < k \).

Let \( u_{i_1}(c) = \delta_{i_1}, \ldots, u_{i_p}(c) = \delta_{i_p} \). If \( t = n \), then \( (\delta_1, \ldots, \delta_n) \) is the solution of the problem \( z \) for the considered concept \( c \). Let \( t < n \). Then the decision tree \( \Gamma \) continues to work with the problem \( z \) and the family of concepts \( F'' = (U, C'') \), where \( C'' \) is the set of solutions on \( C \) of the equation system \( \{u_1(x) = \delta_1, \ldots, u_p(x) = \delta_p\} \). We know that the equation system \( \{u_1(x) = \delta_1, \ldots, u_{t+1}(x) = \delta_{t+1}\} \) is inconsistent. Therefore the system \( \{u_1(x) = \delta_1, \ldots, u_{p}(x) = \delta_{p}, u_{t+1}(x) = \delta_{t+1}\} \) is inconsistent. Hence \( C'' \subseteq C(u_{t+1}, \neg \delta_{t+1}) \) and \( F'' \) is \( l'' \)-family of concepts for some \( l'' \leq m_{t+1} < k \).

As a result, after at most \( r \) membership queries, we either solve the problem \( z \) or reduce the consideration of the problem \( z \) over \( k \)-family of concepts \( F \) to the consideration of the problem \( z \) over some \( l \)-family of concepts, where \( l < k \). After at most \( rk \) membership queries, we solve the problem \( z \) since each problem over 0-family of concepts has exactly one possible solution. Therefore \( h_F^{(1)}(z) \leq rk \) and \( h_F^{(1)}(n) = O(1) \). Therefore \( h_F^{(1)}(n) = \Theta(\log n) \). The obtained contradiction shows that \( \mathcal{R} \cap \mathcal{C} = \emptyset \).

Lemma 8. \( \mathcal{R} \subseteq \mathcal{I} \).
Proof. Let $F = (U, C) \in \mathcal{R}$. Then $F$ is $r$-restricted for some natural $r$. We now show that $F$ is $(r + 1)$-i-restricted. Let $S$ be an arbitrary inconsistent on $C$ equation system over $F$ and $S'$ be a subsystem of $S$ with the maximum number of equations that is consistent. Since $F$ is $r$-restricted, the system $S'$ has a subsystem $S''$ with at most $r$ equations and the same set of solutions on $C$ as the system $S'$. It is clear that there exists an equation $u(x) = \delta$ from $S$ such that the system of equations $S' \cup \{u(x) = \delta\}$ is inconsistent. Then the subsystem $S'' \cup \{u(x) = \delta\}$ of $S$ with at most $r + 1$ equations is inconsistent. Therefore $F$ is $(r + 1)$-i-restricted and $F \in \mathcal{I}$.

Table 3: All 3-tuples from the set $\{0, 1\}^3$

|   | $\mathcal{R}$ | $\mathcal{D}$ | $C$ |
|---|--------------|-------------|-----|
| 1 | 0            | 0           | 0   |
| 2 | 0            | 1           | 0   |
| 3 | 0            | 1           | 1   |
| 4 | 1            | 1           | 0   |
| 5 | 1            | 0           | 0   |
| 6 | 0            | 0           | 1   |
| 7 | 1            | 0           | 1   |
| 8 | 1            | 1           | 1   |

Let $F$ be an infinite family of concepts and its indicator vector $\text{ind}(F)$ be equal to $(e_1, e_2, e_3, e_4)$. The vector $(e_1, e_2, e_3)$ will be called the restricted indicator vector for the family of concepts $F$ and will be denoted $\text{rind}(U)$. In this vector, $e_1 = 1$ if and only if $F \in \mathcal{R}$, $e_2 = 1$ if and only if $F \in \mathcal{D}$, and $e_3 = 1$ if and only if $F \in C$.

Lemma 9. For any infinite family of concepts, its restricted indicator vector coincides with one of the rows of Table 3 with numbers 1-4.

Proof. Table 3 contains as rows all 3-tuples from the set $\{0, 1\}^3$. We now show that rows with numbers 5-8 cannot be restricted indicator vectors of infinite families of concepts. Assume the contrary: there is $i \in \{5, 6, 7, 8\}$ such that the row with the number $i$ is the restricted indicator vector of an infinite family of concepts $F$. If $i = 5$, then $F \in \mathcal{R}$ and $F \notin \mathcal{D}$, but this is impossible since, by Lemma 5 $\mathcal{R} \subseteq \mathcal{D}$. If $i = 6$, then $F \in \mathcal{C}$ and $F \notin \mathcal{D}$, but this is impossible since, by Lemma 5 $\mathcal{C} \subseteq \mathcal{D}$. If $i = 7$, then $F \in \mathcal{R}$ and $F \notin \mathcal{D}$, but this is impossible since, by Lemma 7 $\mathcal{R} \cap \mathcal{C} = \emptyset$. Therefore, for any infinite family of concepts, its restricted indicator vector coincides with one of the rows of Table 3 with numbers 1-4.

Lemma 10. For any infinite family of concepts, its indicator vector coincides with one of the rows of Table 1.

Proof. Let $F$ be an infinite family of concepts and $\text{ind}(F) = (e_1, e_2, e_3, e_4)$. Then $\text{rind}(F) = (e_1, e_2, e_3)$. By Lemma 9 $(e_1, e_2, e_3)$ is one of the rows of Table 3 with numbers 1-4. Therefore,
Table 4: All extensions of rows 1-4 of Table 3

| R | D | C | I |
|---|---|---|---|
| 1 | 0 | 0 | 0 | 0 |
| 2 | 0 | 0 | 0 | 1 |
| 3 | 0 | 1 | 0 | 0 |
| 4 | 0 | 1 | 0 | 1 |
| 5 | 0 | 1 | 1 | 0 |
| 6 | 0 | 1 | 1 | 1 |
| 7 | 1 | 1 | 0 | 1 |
| 8 | 1 | 1 | 0 | 0 |

for each infinite family of concepts, its indicator vector is an extension of one of the rows of Table 3 with numbers 1-4: it can be obtained from the row by adding the fourth digit, which is equal to 0 or 1. Table 4 contains all extensions of rows of Table 3 with numbers 1-4. We now show that the row with number 8 cannot be the indicator vector of an infinite family of concepts. Assume the contrary: there is an infinite family of concepts $F'$ such that $\text{ind}(F') = (1, 1, 0, 0)$. Then $F' \in R$ and $F' \notin I$, but this is impossible since, by Lemma 8, $R \subseteq I$. Therefore, for any infinite family of concepts, its indicator vector coincides with one of the rows of Table 4 with numbers 1-7. Thus, it coincides with one of the rows of Table 1.

We now define seven infinite families of concepts $F_1, \ldots, F_7$ and prove that these families belong to the complexity classes $F_1, \ldots, F_7$, respectively.

Define an infinite family of concepts $F_1 = (U_1, C_1)$ as follows: $U_1$ is the set of all infinite sequences $u = (u(i))_{i \in \mathbb{N}}$, where $u(i) \in \{0, 1\}$ for any $i \in \mathbb{N}$, $C_1 = \{c_i : i \in \mathbb{N}\}$ and, for any $u = (u(i))_{i \in \mathbb{N}} \in U_1$ and $c_i \in C_1$, $u \in c_i$ if and only if $u(i) = 1$.

**Lemma 11.** The family of concepts $F_1$ belongs to the class $F_1$.

**Proof.** It is easy to show that the family of concepts $F_1$ has infinite VC-dimension. Therefore $F_1 \notin D$. We now show that $F_1 \notin I$. Let $n \in \mathbb{N}$. We now define elements $u_0 = (u_0(i))_{i \in \mathbb{N}}, u_1 = (u_1(i))_{i \in \mathbb{N}}, \ldots, u_n = (u_n(i))_{i \in \mathbb{N}} \in U_1$. For any $i \in \mathbb{N}$, $u_0(i) = 1$ if and only if $i \in \{1, \ldots, n\}$. For $j = 1, \ldots, n$, $u_j(i) = 1$ if and only if $j = i$. It is easy to show that the equation system \{ $u_0(x) = 1, u_1(x) = 0, \ldots, u_n(x) = 0$ \} is inconsistent on $C_1$ but each proper subsystem of this system is consistent. Therefore $F_1 \notin I$. Using Lemma 11 we obtain $\text{ind}(F_1) = (0, 0, 0, 0)$, i.e., $F_1 \in F_1$. \qed

Define an infinite family of concepts $F_2 = (U_2, C_2)$ as follows: $U_2 = \mathbb{N}$ and $C_2$ is the set of all subsets of the set $\mathbb{N}$.

**Lemma 12.** The family of concepts $F_2$ belongs to the class $F_2$.

**Proof.** It is easy to show that the family of concepts $F_2$ has infinite VC-dimension. Therefore $F_2 \notin D$. Let $S$ be a system of equations over $U_2$. It is clear that $S$ is inconsistent if and only
if, for some $i \in \mathbb{N}$, the system $S$ contains equations $i(x) = 0$ and $i(x) = 1$. Therefore $F_2$ is 2-i-restricted and $F_2 \in \mathcal{I}$. Using Lemma 10 we obtain $ind(U_2) = (0, 0, 0, 1)$, i.e., $F_2 \in \mathcal{F}_2$. \hfill \Box

Define an infinite family of concepts $F_3 = (U_3, C_3)$ as follows: $U_3 = \{p_i : i \in \mathbb{N}\} \cup \{l_i : i \in \mathbb{N}\}$ and $C_3 = \{c_i : i \in \mathbb{N}\}$, where $c_1 = \{p_1\}$ and, for $i \geq 2$, $c_i = \{p_i, l_1, \ldots, l_{i-1}\}$.

**Lemma 13.** The family of concepts $F_3$ belongs to the class $\mathcal{F}_3$.

*Proof.* For $n \in \mathbb{N}$, denote $S_n = \{p_1(x) = 0, \ldots, p_n(x) = 0, l_n(x) = 0\}$. It is easy to show that the equation system $S_n$ is inconsistent on $C_3$ and each proper subsystem of $S_n$ is consistent. Therefore $F_3 \notin \mathcal{I}$. By Lemma 8, $F_3 \notin \mathcal{R}$. Using elements from the set $\{l_i : i \in \mathbb{N}\}$, we can construct $d$-complete tree over $F_3$ for each $d \in \mathbb{N}$. By Lemma 1 and Theorem 2, $F_3 \notin \mathcal{C}$. One can show that $VC(F_3) = 1$. Therefore $F_3 \in \mathcal{D}$. Thus, $ind(U_3) = (0, 1, 0, 0)$, i.e., $F_3 \in \mathcal{F}_3$. \hfill \Box

Define an infinite binary information system $F_4 = (U_4, C_4)$ as follows: $U_4 = \{u_i : i \in \mathbb{N}\} \cup \{u_{i,j} : i, j \in \mathbb{N}\}$ and $C_4 = \{c_{p,q} : p, q \in \mathbb{N}\}$. For any $u_i \in U_4$ and any $c_{p,q} \in C_4$, $u_i \in c_{p,q}$ if and only if $p > i$. For any $u_{i,j} \in U_4$ and any $c_{p,q} \in C_4$, $u_{i,j} \in c_{p,q}$ if and only if $(p, q) = (i, j)$.

**Lemma 14.** The family of concepts $F_4$ belongs to the class $\mathcal{F}_4$.

*Proof.* Let $n \in \mathbb{N}$ and $S_n = \{u_{1,1}(x) = 0, \ldots, u_{1,n}(x) = 0\}$. It is easy to show that the system $S_n$ is consistent and each proper subsystem of $S_n$ has another set of solutions on $C_4$ than the system $S_n$. Therefore $F_4 \notin \mathcal{R}$.

Using elements from the set $\{u_i : i \in \mathbb{N}\}$, we can construct $d$-complete tree over $F_4$ for each $d \in \mathbb{N}$. By Lemma 1 and Theorem 2, $F_4 \notin \mathcal{C}$.

Let $S$ be an equation system over $F_4$. One can show that $S$ is inconsistent if and only if $S$ contains at least one of the following pairs of equations:

- $u_{i,j}(x) = 0$ and $u_{i,j}(x) = 1$;
- $u_{i,j}(x) = 1$ and $u_{k,l}(x) = 1, (i, j) \neq (k, l)$;
- $u_{i,j}(x) = 1$ and $u_{k}(x) = 0, i > k$;
- $u_{i,j}(x) = 1$ and $u_{k}(x) = 1, i \leq k$;
- $u_{i}(x) = 0$ and $u_{j}(x) = 1, i \leq j$.

Therefore $F_4$ is 2-i-restricted and $F_4 \in \mathcal{I}$. One can show that $VC(F_4) = 1$. Therefore $F_4 \in \mathcal{D}$. Thus, $ind(F_4) = (0, 1, 0, 1)$, i.e., $F_4 \in \mathcal{F}_4$. \hfill \Box

Define an infinite family of concepts $F_5 = (U_5, C_5)$ as follows:

$$U_5 = \bigcup_{i \in \mathbb{N}} \{u_i, u_{i,1}, \ldots, u_{i,i}\}$$

and $C_5 = \bigcup_{i \in \mathbb{N}} \{c_{i,1}, \ldots, c_{i,i}\}$. For any $u_i \in U_5$ and any $c \in C_5$, $u_i \in c$ if and only if $c \in \{c_{i,1}, \ldots, c_{i,i}\}$. For any $u_{i,j} \in U_5$ and any $c \in C_5$, $u_{i,j} \in c$ if and only if $c = c_{i,j}$.

**Lemma 15.** The family of concepts $F_5$ belongs to the class $\mathcal{F}_5$. 

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Based on the results of exact learning, test theory, and rough set theory, for an arbitrary infinite family of concepts, we studied five functions, which characterize the dependence in the worst case of the minimum depth of a decision tree solving a problem of exact learning on the number of elements in the problem description. These five functions correspond to (i) decision trees using membership queries, (ii) decision trees using equivalence queries, (iii) decision trees using both membership and equivalence queries, (iv) decision trees using proper
equivalence queries, and (v) decision trees using both membership and proper equivalence queries. We described possible types of behavior for each of these five functions. We also studied join behavior of these functions and distinguished seven complexity classes of infinite family of concepts. In the future, we are also planing to study partial equivalence queries [12].

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References

[1] Angluin, D.: Queries and concept learning. Mach. Learn. 2(4), 319–342 (1988)
[2] Angluin, D.: Queries revisited. Theor. Comput. Sci. 313(2), 175–194 (2004)
[3] Azad, M., Chikalov, I., Hussain, S., Moshkov, M.: Entropy-based greedy algorithm for decision trees using hypotheses. Entropy 23(7), 808 (2021). URL https://doi.org/10.3390/e23070808
[4] Azad, M., Chikalov, I., Hussain, S., Moshkov, M.: Minimizing depth of decision trees with hypotheses. In: S. Ramanna, C. Cornelis, D. Ciucci (eds.) Rough Sets - International Joint Conference, IJCRS 2021, Bratislava, Slovakia, September 19–24, 2021, Lecture Notes in Computer Science, vol. 12872, pp. 123–133. Springer (2021)
[5] Azad, M., Chikalov, I., Hussain, S., Moshkov, M.: Minimizing number of nodes in decision trees with hypotheses. In: J. Watrobski, W. Salabun, C. Toro, C. Zanni-Merk, R.J. Howlett, L.C. Jain (eds.) 25th International Conference on Knowledge-Based and Intelligent Information & Engineering Systems, KES 2021, September 8–10, 2021, Szczecin, Poland, Procedia Computer Science, vol. 192, pp. 232–240. Elsevier (2021). URL https://doi.org/10.1016/j.procs.2021.08.024
[6] Azad, M., Chikalov, I., Hussain, S., Moshkov, M.: Optimization of decision trees with hypotheses for knowledge representation. Electronics 10(13), 1580 (2021). URL https://doi.org/10.3390/electronics10131580
[7] Azad, M., Chikalov, I., Hussain, S., Moshkov, M.: Sorting by decision trees with hypotheses (extended abstract). In: H. Schlingloff, T. Vogel (eds.) 29th International Workshop on Concurrency, Specification and Programming, CS&P 2021, Berlin, Germany, September 27-28, 2021, CEUR Workshop Proceedings, vol. 2951, pp. 126–130. CEUR-WS.org (2021). URL http://ceur-ws.org/Vol-2951/paper1.pdf
[8] Chegis, I.A., Yablonskii, S.V.: Logical methods of control of work of electric schemes. Trudy Mat. Inst. Steklov (in Russian) 51, 270–360 (1958)
[9] Hegedűs, T.: Generalized teaching dimensions and the query complexity of learning. In: W. Maass (ed.) Eighth Annual Conference on Computational Learning Theory, COLT 1995, Santa Cruz, California, USA, July 5–8, 1995, pp. 108–117. ACM (1995)

[10] Hellerstein, L., Pillaiapakkamnatt, K., Raghavan, V., Wilkins, D.: How many queries are needed to learn? J. ACM 43(5), 840–862 (1996)

[11] Littlestone, N.: Learning quickly when irrelevant attributes abound: A new linear-threshold algorithm. Mach. Learn. 2(4), 285–318 (1988)

[12] Maass, W., Turán, G.: Lower bound methods and separation results for on-line learning models. Mach. Learn. 9, 107–145 (1992)

[13] Moshkov, M.: Conditional tests. In: S.V. Yablonskii (ed.) Problemy Kibernetiki (in Russian), vol. 40, pp. 131–170. Nauka Publishers, Moscow (1983)

[14] Moshkov, M.: On depth of conditional tests for tables from closed classes. In: A.A. Markov (ed.) Combinatorial-Algebraic and Probabilistic Methods of Discrete Analysis (in Russian), pp. 78–86. Gorky University Press, Gorky (1989)

[15] Moshkov, M.: Test theory and problems of machine learning. In: International School-Seminar on Discrete Mathematics and Mathematical Cybernetics, Ratminio, Russia, May 31–June 3, 2001, pp. 6–10. MAX Press, Moscow (2001)

[16] Moshkov, M.: Time complexity of decision trees. In: J.F. Peters, A. Skowron (eds.) Trans. Rough Sets III, Lecture Notes in Computer Science, vol. 3400, pp. 244–459. Springer (2005)

[17] Moshkov, M.: Exact learning and test theory. CoRR abs/2201.04506 (2022). URL https://arxiv.org/abs/2201.04506

[18] Moshkov, M.: On the depth of decision trees with hypotheses. Entropy 24(1), 116 (2022). URL https://www.mdpi.com/1099-4300/24/1/116

[19] Pawlak, Z.: Rough sets. Int. J. Parallel Program. 11(5), 341–356 (1982)

[20] Sauer, N.: On the density of families of sets. J. of Combinatorial Theory (A) 13, 145–147 (1972)

[21] Shelah, S.: A combinatorial problem; stability and order for models and theories in infinitary languages. Pacific J. of Mathematics 41, 241–261 (1972)

[22] Vapnik, V.N., Chervonenkis, A.Y.: On the uniform convergence of relative frequencies of events to their probabilities. Theory Probab. Appl. 16(2), 264–280 (1971)