THE IMPRIMITIVE FAITHFUL COMPLEX CHARACTERS OF THE SCHUR COVERS OF THE SYMMETRIC AND ALTERNATING GROUPS

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Abstract. Using combinatorics and character theory, we determine the imprimitive faithful complex characters, i.e., the irreducible faithful complex characters which are induced from proper subgroups, of the Schur covers of the symmetric and alternating groups. Furthermore, for every imprimitive character we establish all its minimal block stabilizers. As a corollary, we also determine the monomial faithful characters of the Schur covers.

1. Introduction

In his seminal paper [1], Aschbacher gives a subgroup structure theorem for the classical groups by defining eight collections $C_1, \ldots, C_8$ of natural subgroups, as well as a class $S$ of almost simple subgroups satisfying certain ‘irreducibility’ conditions. Building on that, in [10] Kleidman and Liebeck have determined, given a finite almost simple classical group $G$ of dimension at least 13, which members of $C_1, \ldots, C_8$ constitute maximal subgroups of $G$. This leaves only the maximal members of $S$ to be determined, in order to obtain a classification of the maximal subgroups of these classical groups.

One strategy is to determine those subgroups in $S$ which fail to be maximal. By Aschbacher’s theorem they are contained in maximal subgroups which lie in one of the collections $C_2, C_4, C_7$, or again in $S$ (see [10, §1.2]).

This note is a contribution to the analysis of the case $C_2$, as by definition of this collection, its successful treatment amounts to classifying all absolutely irreducible imprimitive representations (respectively their characters) of the quasisimple groups and their automorphism groups. Recall that an irreducible character $\chi$ of a finite group $G$ is called imprimitive, if it is induced from a proper subgroup $H$ of $G$. In this case we call $H$ a block stabilizer of $\chi$.

In their articles [5] and [6], Djoković and Malzan have already determined all imprimitive ordinary characters of the symmetric and the alternating groups, allowing us to focus on the faithful characters of the covers of these groups. The Schur covers of the symmetric groups are non-split central extensions

$$1 \rightarrow Z \rightarrow \tilde{S}_n \xrightarrow{\theta} S_n \rightarrow 1,$$

for which the Schur multiplier $Z$ has order two, and $Z \leq \tilde{S}_n'$, provided $n$ is at least four. For $n < 4$ the Schur multiplier is trivial, so let $n \geq 4$ hold throughout the present paper. There are two isoclinic Schur covers of $S_n$, which are only isomorphic if $n = 6$. In this note we adhere to Schur’s choice (see [14]) and consider the group

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defined by the presentation
\[ \hat{S}_n \cong \langle z, t_1, \ldots, t_{n-1} | z^2 = 1, t_i^2 = z, 1 \leq i \leq n - 1, \]
\[ (t_it_{i+1})^3 = z, 1 \leq i \leq n - 2, \]
\[ t_it_j = zt_jt_i, 1 \leq i < j \leq n - 1, |i - j| > 1 \}, \]
i.e., transpositions lift to order four. Therefore, given the presentation
\[ S_n \cong \langle s_1, \ldots, s_{n-1} | s_i^2 = 1, 1 \leq i \leq n - 1, \]
\[ (s_is_{i+1})^3 = 1, 1 \leq i \leq n - 2, \]
\[ s_is_j = s_js_i, 1 \leq i < j \leq n - 1, |i - j| > 1 \}
of \( S_n \), the epimorphism \( \theta \) is defined by mapping \( s_i \) to \( t_i \) for all \( 1 \leq i \leq n - 1 \). For any \( g \in S_n \) we define its standard lift \( \hat{S}_n \) to be the element which is obtained by replacing \( s_i \) in the product expansion of \( g \) into transpositions by \( t_i \) for every \( 1 \leq i \leq n - 1 \).

Note that even though we are focusing on \( \hat{S}_n \), we also obtain information on the imprimitive characters of the isoclinic group \( \hat{S}_n \): By [4, Section 6.7] (see also [3, III.5] for details) the characters of \( \hat{S}_n \) are obtained by tensoring with an appropriate linear character to give the characters of the isoclinic cover \( \hat{S}_n \). Hence if a character of \( \hat{S}_n \) is imprimitive, so is its corresponding character of \( S_n \).

Except for \( n = 6 \) or \( n = 7 \) the Schur covers of the alternating groups are simply given by the derived subgroups of the Schur covers of the symmetric groups. As isoclinic groups possess isomorphic derived subgroups, there is always only one cover up to isomorphism. Hence we may define \( A_n := \hat{S}_n' \). When \( n = 6 \) or \( n = 7 \), the Schur multiplier of \( A_n \) is exceptional: In these cases it is cyclic of order six, giving rise to the two non-split extensions \( 6.A_6 \) and \( 6.A_7 \).

Before we give the main result of this paper, let us briefly review some basic facts of the faithful complex characters of \( \hat{S}_n \) to introduce some notation in the process:

Of course, when dealing with symmetric groups, the proper combinatorial objects to consider are partitions of \( n \), whose set we denote by \( P_n \), and whose parts we assume to be ordered ascendingly. Furthermore, if \( \pi \in P_n \) is the cycle type of some \( g \in S_n \), then we set \( \sigma(g) := n - \ell(\pi) \) (mod 2), where \( \ell(\pi) \) is the length of \( \pi \). Hence, \((-1)^{\sigma(g)}\) gives the familiar sign character \( sgn \), and we extend \( \sigma \) to \( \hat{S}_n \) by setting \( \sigma(\hat{g}) := \sigma(\theta(\hat{g})) \) for \( \hat{g} \in \hat{S}_n \). Also note that, as \( \hat{A}_n \) is the full \( \theta \)-preimage of the alternating group \( A_n \), we have \( \ker \sigma = \hat{A}_n \). For any subgroup \( G \) of \( \hat{S}_n \) containing the center \( Z \) the set of its ordinary characters faithful on \( Z \) is denoted by \( Irr_-(G) \), and the elements of \( Irr_-(\hat{S}_n) \) and \( Irr_-(\hat{A}_n) \) are called spin characters. The spin characters of \( \hat{S}_n \) are parameterized by strict partitions, i.e., by partitions of \( n \) all of whose parts are distinct. We write \( D_n \) for the set of strict partitions of \( n \), \( D_n^+ \) for all even strict partitions, and \( D_n^- \) for all strict partitions which are odd. Given \( \lambda \in D_n \) we denote the corresponding spin character by \( \langle \lambda \rangle \). If \( \lambda \) is an odd partition, it gives rise to two associate characters \( \langle \lambda \rangle \) and \( \langle \lambda \rangle^\alpha := \text{sgn} \otimes \langle \lambda \rangle \).

With a minimum of notation in place, we can now state our result on the imprimitive faithful ordinary characters of \( \hat{S}_n \) and \( \hat{A}_n \). We do so by giving all triples \((H, \varphi, \chi)\), where \( \chi \) is an imprimitive faithful character of \( G \in \{ \hat{S}_n, \hat{A}_n \} \) such that \( \chi = \varphi^G \) for some \( \varphi \in Irr_-(H) \) and \( H \leq G \) is a subgroup minimal with this property, i.e., \( H \) does not properly contain a block stabilizer of \( \chi \). In the following, we refer to such triples as minimal triples.
Theorem 1.1. For $\tilde{S}_n$, the minimal triples $(H, \varphi, \chi)$ are the following:

(i) $H = \tilde{A}_{n-1}$, $\varphi$ is a constituent of $<\lambda>_{\tilde{A}_{n-1}}$ for some $\lambda \in D^{+}_n$ and $\chi = <\lambda>$, except when $n = 6$ and $\lambda = (4, 2)$, or $n = 9$ and $\lambda = (6, 2, 1)$.

(ii) $H = \tilde{S}_{n-1}$, $\varphi = <\mu>$ for $\mu = (m, m-1, \ldots, 1) \in D^{+}_{n-1}$ with $m \equiv 2, 3 \pmod{4}$, and $\chi = <\lambda>$ with $\lambda = (m+1, m-1, m-2, \ldots, 1) \in D^{+}_n$.

(iii) For $n = 6$ we have $H = 3^2 : 8$, $\varphi$ is an extension of either linear character of order four of the subgroup $3^2 : 4$, and $\chi = <4, 2>$ is the unique spin character of degree 20.

(iv) For $n = 9$: $H = 2 \times L_2(8) : 3$, $\varphi$ is a linear character of order six, and $\chi = <6, 2, 1>$ is the unique spin character of degree 240.

Theorem 1.2. For $\tilde{A}_n$ the minimal triples $(H, \varphi, \chi)$ are as follows:

(i) $H = \tilde{A}_{n-1}$, $\varphi$ is either constituent of $<\mu>_{\tilde{A}_{n-1}}$ for $\mu = (m, m-1, \ldots, 1) \in D^{+}_{n-1}$ and $\chi = <\lambda>_{\tilde{A}_n}$ with $\lambda = (m+1, m-1, \ldots, 1) \in D^{+}_n$.

(ii) $n = 6$: $H = 3^2 : 8$, $\varphi$ is an extension of either linear character of order four of the subgroup $3^2 : 4$. Both linear characters have a pair of extensions in $H$. The members of each pair induce to the constituents of $<4, 2>_{\tilde{A}_6}$, i.e., the two faithful characters of degree 10.

(iii) $n = 9$: $H = 2 \times L_2(8) : 3$, taking $\varphi$ to be one of the irreducible linear characters of order six of $H$ gives either of the two faithful characters of degree 120, i.e., $\chi$ is a constituent of $<6, 2, 1>_{\tilde{A}_9}$.

The proofs of Theorems 1.1 and 1.2 are given in Sections 4 and 5. At the end of Section 5 we also classify the imprimitive faithful characters of the exceptions $6A_6$, $6A_7$, $3A_6$ and $3A_7$ (see Theorem 5.8).

Our strategy is as follows: We begin in Section 2 by determining the proper subgroups of $\tilde{S}_n$ and $\tilde{A}_n$ which are viable candidates for block stabilizers. Of course, by transitivity of induction, it is sufficient to restrict our attention to maximal subgroups. As these are $\theta$-preimages of maximal subgroups of $\tilde{S}_n$ or $\tilde{A}_n$, they form three classes of subgroups, distinguished by the type of the natural action of their image under $\theta$: We analyze intransitive, imprimitive and primitive subgroups separately. For the first two classes this analysis is straightforward, shortening our list of viable candidates considerably for the imprimitive subgroups. For the primitive subgroups, whose classification in general is still open, we make use of a classification by Kleidman and Wales in [11], who determined the primitive subgroups of $\tilde{S}_n$ whose order is at least $2^{n-4}$.

In Section 3 we lay the combinatorial foundations needed to successfully tackle the double covers of the symmetric groups: The main tools are the Branching Rule and an analogue of the Littlewood-Richardson rule for spin characters due to Stembridge (see [15]). The latter provides a means to describe the constituents of a faithful character induced from a maximal imprimitive subgroup of $\tilde{S}_n$ (Stembridge refers to such a character as a projective outer product).

With the combinatorics in place, in Section 4 we complete Bessenrodt’s classification of the projective outer products which are multiplicity free (see [2]) with a result of the first author’s diploma thesis [13]. As a consequence we obtain the imprimitive characters of $\tilde{S}_n$ whose block stabilizers are maximal imprimitive subgroups. Furthermore, this information is used to analyze the situation for imprimitive subgroups of $\tilde{S}_n$: Together with character theoretic arguments and Clifford
Theory we are able to classify the imprimitive characters induced from maximal imprimitive subgroups.

In Section 5 we employ Clifford Theory again to apply our results of the previous section to the double covers of the alternating groups. Lastly, we deal with the exceptional Schur covers $\widetilde{A}_6$ and $\widetilde{A}_7$ with the help of the ATLAS [4] and some GAP-calculations (see [7]). The triple covers $3\widetilde{A}_6$ and $3\widetilde{A}_7$ are also considered.

2. Reductions

In order to determine the imprimitive irreducible characters of any finite group $G$, it is sufficient to consider the irreducible characters of its maximal subgroups, and single out those which induce irreducibly. To this end let us begin by stating two results which allow us to narrow down the list of subgroup candidates for block stabilizers in $G$. First, by Lemma 2.1 the orders of the candidates must not be too small.

**Lemma 2.1.** Let $G$ be a finite group, $H \leq G$ and $\chi \in \text{Irr}(H)$. If $|G : H|^2 > |G|$, then $\chi^G$ is reducible.

**Proof.** Since $\chi^G(1) = \chi(1)|G : H|$, the hypothesis forces $\chi^G(1)^2 > |G|$, hence $\chi^G$ cannot be irreducible. □

Secondly, the following consequence of Mackey’s Theorem, which is already used implicitly in [5, 6], enables us to eliminate candidates, too.

**Lemma 2.2.** Let $G$ be a finite group, $H \leq G$ and $\chi \in \text{Irr}(H)$. If there exists a non-trivial element $t \in G$ which centralizes $H \cap H_t$, then $\chi^G$ is reducible.

**Proof.** Let $M$ be the $CH$-module affording $\chi$, and consider the endomorphism ring $E := \text{End}_{CH}(M^G)$. By adjointness and Mackey’s Theorem we have

$$\dim E = \dim \left( \bigoplus_{y \in H \setminus G/H} \text{Hom}_{C(H \cap H)}(M|_{H \cap H}, M^y|_{H \cap H}) \right).$$

By the hypothesis $M^y|_{H \cap H} \cong M|_{H \cap H}$, thus taking one of the double coset representatives to be $t$, we obtain

$$\dim E = \sum_{y \in H \setminus G/H} \dim \text{End}_{C(H \cap H)}(M|_{H \cap H}, M^y|_{H \cap H}) \geq \dim \text{End}_{C(H \cap H)}(M|_{H \cap H}) + \dim \text{End}_{C(H \cap H)}(M|_{H \cap H}) \geq 2.$$

□

From now on let $G = \hat{S}_n$ or $G = \hat{A}_n$. As the maximal subgroups of $G$ are the preimages under $\theta$ of the maximal subgroups of $\theta(G)$, we construct our list of viable candidates by considering subgroups of the symmetric and alternating groups first.

The subgroups of $S_n$ and $A_n$ fall into three main classes distinguishable by the type of their natural action on the set $\{1, \ldots, n\}$: The intransitive subgroups, the subgroups which are imprimitive and transitive, and those which act primitively.

The intransitive maximal subgroups of $S_n$ are the maximal parabolic subgroups isomorphic to a group of the form $S_l \times S_{n-l}$ for $1 \leq l \leq \lfloor \frac{n}{2} \rfloor$. As every intransitive
subgroup of $\mathcal{A}_n$ is contained in an intransitive maximal subgroup of $S_n$, the intransitive maximal subgroups of $\mathcal{A}_n$ are given by $(S_i \times S_{n-i}) \cap \mathcal{A}_n$. Hence we obtain the following lemma.

**Lemma 2.3.** Let $H$ be a maximal subgroup of $G$ such that $\theta(H)$ acts intransitively, then $H$ is the preimage of $(S_i \times S_{n-i}) \cap \theta(G)$.

The imprimitive and transitive maximal subgroups of $S_n$ are of the form $S_{n/k} \wr S_k$ for $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$ with $k \mid n$. Again as every imprimitive and transitive subgroup of $\mathcal{A}_n$ is contained in an imprimitive and transitive maximal subgroup of $S_n$, the imprimitive and transitive maximal subgroups of $\mathcal{A}_n$ have the form $(S_{n/k} \wr S_k) \cap \mathcal{A}_n$, where $k$ is as above. However, not all of these maximal subgroups are viable candidates, as we can rule out the majority of cases with the help of Lemma 2.2.

**Lemma 2.4.** If $H$ is the block stabilizer of some irreducible and imprimitive character $\chi \in \text{Irr}(G)$, and $H$ is the preimage of a transitive and imprimitive maximal subgroup of $G$, then $n$ is even and $H$ is the preimage of $(S_{n/2} \wr S_2) \cap \theta(G)$.

**Proof.** Assume $n = kl$ and let $H$ be the preimage of $(S_l \wr S_k) \cap \theta(G)$ for some $k \geq 3$. Without loss of generality let $B_i := \{(i-1)+1, \ldots, il\}$ for $i = 1, \ldots, k$ be the blocks of the action of $\theta(H)$ on $\{1, \ldots, n\}$. Furthermore, let $t := (1, l+1, 2l+1) \in \theta(G)$. Then the centralizer of $\theta(H)^t \cap \theta(H)$ in $\theta(G)$ contains $t$. As the order of $t$ is odd, there exists an element $\tilde{t} \in \theta^{-1}(t)$ which is of odd order, too. Now, for any $h \in C_G(H^t \cap H)$ we have $\theta(h^t) = \theta(h)^t = \theta(h)$, and hence $h^t \in \{h, zh\}$. By our choice of $\tilde{t}$, we conclude $h^t = h$, i.e., the element $\tilde{t}$ centralizes $C_G(H^t \cap H)$. Therefore Lemma 2.2 gives the claim.

In the case of a primitive action, for our treatment when $n \geq 5$ we employ a classification of the primitive subgroups of $S_n$ which do not contain $\mathcal{A}_n$, and whose order is at least $2^{n/2}$. This is provided by Kleidman and Wales in [11, Proposition 6.2] on the basis of the O’Nan-Scott Theorem (see, for example, [12]). Together with Lemma 2.1 we can derive the following result on irreducible characters induced from primitive subgroups.

**Lemma 2.5.** Let $H \neq \mathcal{A}_n$ be the preimage of a primitive subgroup of $G$ such that $H$ is the block stabilizer of some irreducible and imprimitive $\chi \in \text{Irr}_-(G)$. Then $n = 9$ and $H \cong 2 \times L_2(8) : 3$. Let $\phi$ and $\phi'$ denote the two linear irreducible characters of order six of $H$. For $G = \mathcal{A}_9$ we obtain $\chi = \phi_9^G$ or $\chi = \phi'^G$ and for $G = \mathcal{S}_9$ we obtain $\chi = \phi_9^G = \phi'^G$.

**Proof.** If $n = 4$ then it is easily verified that no maximal subgroup of $G$ fulfilling the hypothesis is primitive. Therefore let $n \geq 5$, and we commence by considering the case $|\theta(H)| \geq 2^{n/2}$. Let $G = \mathcal{S}_n$. With the help of Lemma 2.1 we can work through the list of [11, Proposition 6.2] and verify for most groups that they are not block stabilizers of some $\chi \in \text{Irr}_-(G)$. There remains a small list of groups for which we have to take a closer look. These are the preimages of $\mathcal{A}_5$ and $\mathcal{S}_5$ for $n = 6$, $L_2(7)$ for $n = 7$, $2^3 : L_3(2)$ and $L_3(2).2$ for $n = 8$, $3^2 : \mathcal{S}_4$ and $L_2(8)$: 3 for $n = 9$, $M_{11}$ for $n = 11$, and finally $M_{12}$ for $n = 12$. Except in the case $L_2(8)$: 3 for $n = 9$, the character table of $G$ shows that there is no spin character whose degree is divisible by the index of the given groups in $S_n$ (which of course is the index of the respective preimage in $G$). The preimage of $L_2(8)$: 3 is the subgroup $2 \times L_2(8)$: 3 $\leq G$. Its commutator factor group is cyclic of order six, and with the aid of GAP we verify...
that the inflation of its two faithful characters induce irreducibly to give the same character of $G$.

Let $G = \bar{A}_n$. As the primitive subgroups of $\mathcal{A}_n$ are obtained by intersection those of $\mathcal{S}_n$ with $\mathcal{A}_n$, we may argue as in the case $G = \bar{S}_n$ by going through the corresponding subgroups of $\mathcal{A}_n$ using [11, Proposition 6.2]. Again only the subgroup $2 \times L_2(8)$, which is maximal in $\bar{A}_9$, gives rise to two irreducible and imprimitive characters of $\bar{A}_9$: Here both inflated faithful characters of the commutator factor group induce to two non-isomorphic characters.

On the other hand, if $|H| < 2^{n-4}$, then as $|H|^2 < |G|$ for both $G = \bar{S}_n$ and $G = \bar{A}_n$, we have $|G : H|^2 > |G|$, and no such group is a block stabilizer of some irreducible and imprimitive character of $\bar{S}_n$. □

If $G = \bar{S}_n$, Lemma 2.5 leaves the case $H = \bar{A}_n$, i.e., it remains to decide which characters of $\bar{S}_n$ are induced from $\bar{A}_n$. But this is well known (see, for example, [8, Theorem 4.2]): A spin character $\langle \lambda \rangle \in \text{Irr}(\bar{S}_n)$ is induced from $\bar{A}_n$ if and only if $\lambda$ is an even partition. Therefore we will not concern ourselves with it any further in the sequel.

3. Combinatorics

In this section we establish the combinatorial framework needed in Sections 4 and 5. We fix the notation and for the convenience of the reader we collect the necessary theorems on which further proofs are based.

One of them is the Branching Rule for spin characters. To state it and to provide further notation which will be useful for some arguments used in Sections 4 and 5, we have to describe possible enlargements of partitions. Therefore for a strict partition $\lambda$ of $n$, let $N(\lambda)$ be the subset of $D_{n+1}$ consisting of all partitions which can be obtained by adding 1 to one of the parts of $\lambda$. If $\lambda$ does not contain 1 as a part, let $\lambda^+ := \lambda \cup \{1\}$, and for two partitions $\lambda$ and $\mu$ we denote by $\lambda \cup \mu$ the partition obtained by forming the union of both sets of parts. Note that $\lambda^+$ is not in $N(\lambda)$. Furthermore defining

$$\langle \mu \rangle^* = \begin{cases} \langle \mu \rangle, & \text{if } \mu \text{ is even}, \\ \langle \mu \rangle + \langle \mu \rangle^a, & \text{if } \mu \text{ is odd}. \end{cases}$$

for a strict partition $\mu$, allows us to state the Branching Rule for inducing spin characters from $\bar{S}_n$ to $\bar{S}_{n+1}$.

**Theorem 3.1** ([8, Theorem 10.2]). Let $\lambda$ be a strict partition of $n$. Then

$$\langle \lambda \rangle^{\bar{S}_{n+1}} = (1 - \delta_{1, \lambda(\lambda)}) \langle \lambda^+ \rangle + \sum_{\mu \in N(\lambda)} \langle \mu \rangle^*.$$  

An analogous formula holds for the associate character $\langle \lambda \rangle^a$. In this case the character $\langle \lambda^+ \rangle$ is replaced by its associate.

As stated in the previous section, in Section 4 it will be important to consider the spin characters of the subgroups $\theta^{-1}(\mathcal{S}_l \times \mathcal{S}_{n-l})$ for $1 \leq l \leq n - 1$. By [15, Theorem 4.3] every irreducible spin character of $\theta^{-1}(\mathcal{S}_l \times \mathcal{S}_{n-l})$ is a reduced Clifford product $\langle \mu \rangle \otimes_2 \langle \nu \rangle$ (see [15, Section 4] for a definition) for two strict partitions $\mu \in D_l$ and $\nu \in D_{n-l}$. The character values of $\langle \mu \rangle \otimes_2 \langle \nu \rangle$ are readily determined from the values of $\langle \mu \rangle \in \text{Irr}(\mathcal{S}_l)$ and $\langle \nu \rangle \in \text{Irr}(\mathcal{S}_{n-l})$ (see, for example, [8, Table 5.7] for a tabulation).
The \textit{projective outer product} is then simply defined to be the induced character

\[ <\mu> \hat{\otimes} <\nu> := ( <\mu> \otimes_z <\nu> )_{\mathfrak{S}_n}. \]

To decide which projective outer products are irreducible, we need information on their constituents. Thanks to the work of Stembridge in [15] these may be determined by an analogue of the Littlewood-Richardson Rule (see [9, 2.8.13]) for spin characters. In order to state this rule, we have to expand a little:

To a partition \( \lambda \in \mathcal{D}_n \) we associate a \textit{shifted diagram}

\[ S(\lambda) = \{ (i, j) \mid 1 \leq i \leq \ell(\lambda), i \leq j \leq \lambda_i + i - 1 \} \]

whose elements we interpret as coordinates in a matrix style notation. Let \( A' = \{ 1' < 1 < 2' < 2 < \ldots \} \) be an ordered alphabet. The letters \( 1', 2', 3', \ldots \) are said to be \textit{marked}, the others are \textit{unmarked}. We write \( \mathcal{A} \) for the set of unmarked letters of \( A' \). A \textit{shifted tableau} \( T \) of shape \( \lambda \) is a map \( T : \lambda \rightarrow A' \) satisfying

\begin{enumerate}
\item \( T(i, j) \leq T(i + 1, j) \) for all \( i, j \) with \( (i, j), (i + 1, j) \in S(\lambda) \); \quad \text{(nondecreasing columns)}
\item \( T(i, j) \leq T(i, j + 1) \) for all \( i, j \) with \( (i, j), (i, j + 1) \in S(\lambda) \); \quad \text{(nondecreasing rows)}
\item every \( k \in \{ 1, 2, \ldots \} \) appears at most once in each column of \( T \);
\item every \( k' \in \{ 1', 2', \ldots \} \) appears at most once in each row.
\end{enumerate}

The \textit{content} of \( T \) is a sequence of integers \( (c_1, c_2, \ldots) \), where \( c_k \) counts the number of nodes \( (i, j) \in S(\lambda) \) such that \( T(i, j) = k \) or \( k' \).

For two strict partitions \( \lambda \) and \( \mu \) with \( S(\mu) \subseteq S(\lambda) \) the \textit{skew shifted diagram} \( S(\lambda/\mu) \) of shape \( \lambda/\mu \) is the set \( S(\lambda) \setminus S(\mu) \). The corresponding \textit{skew shifted tableau} of shape \( \lambda/\mu \) is given by restricting the shifted tableau of shape \( \lambda \) to this set. Reading the rows of a (skew) shifted Tableau \( T \) from left to right and from bottom to top, gives its \textit{tableau-word} over \( A' \).

We now count the number of skew shifted tableaux of shape \( \lambda/\mu \) with content \( \nu \) whose words are such that for all \( a \in \mathcal{A} \) the leftmost letter of \( \{ a', a \} \) in \( w \) is unmarked, and which fulfill a set of further combinatorial conditions (called the \textit{lattice property}) as detailed in [15, Section 8]. For brevity, we say that such a tableau satisfies (TP).

Denoting the number of tableaux of shape \( \lambda/\mu \) with content \( \nu \) satisfying (TP) by \( \text{st}(\lambda; \mu, \nu) \), and setting

\[ \varepsilon_\alpha := \begin{cases} 1, & \text{if } \alpha \text{ is even}, \\ \sqrt{2}, & \text{if } \alpha \text{ is odd.} \end{cases} \]

for a partition \( \alpha \), allows us to finally state the Littlewood-Richardson Rule.

**Theorem 3.2** ([15, Theorems 8.1, 8.3], [8, Corollary 14.4]). Let \( \mu \in \mathcal{D}_l, \nu \in \mathcal{D}_{n-l} \) and \( \lambda \in \mathcal{D}_n \). Then we have

\[ \langle \mu> \hat{\otimes} <\nu>, <\lambda> \rangle = \frac{1}{\varepsilon_{\lambda/\mu} \varepsilon_{\mu \cup \nu}} 2^{|\lambda|} \left( \ell(\mu) + \ell(\nu) - \ell(\lambda) \right) \text{st}(\lambda; \mu, \nu), \]

unless \( \lambda \) is odd and equal to \( \mu \cup \nu \). In this case, either \( <\lambda> \) or \( <\lambda>^{\alpha} \) is constituent of \( <\mu> \hat{\otimes} <\nu> \) with multiplicity 1. If \( \ell(\lambda) > \ell(\mu) + \ell(\nu) \) then \( \text{st}(\lambda; \mu, \nu) \) is zero, and \( <\lambda> \) is not a constituent.
4. The Symmetric Groups

In this section we determine the irreducible spin characters of $\hat{S}_n$ which are induced characters of proper subgroups.

In Section 2 we gathered that there are two types of subgroups left to consider: On the one hand, by Lemma 2.3, there are the preimages of the maximal parabolic subgroups of $S_n$, i.e., subgroups of the form $\hat{S}_{l,n-1} := \theta^{-1}(S_l \times S_{n-1})$. On the other hand there are the subgroups $\hat{S}_{n,2} := \theta^{-1}(S_{n/2} \wr S_2)$ of Lemma 2.4.

4.1. Characters Induced from $\hat{S}_{l,n-1}$. Let us begin by considering spin characters induced from subgroups of the first type. In other words, given two irreducible spin characters $<\mu> \in \hat{S}_l$ and $<\nu> \in \hat{S}_{n-1}$, we have to determine if their projective outer product $<\mu> \hat{\otimes} <\nu>$ is again irreducible. To decide this question, we make use of a classification of the multiplicity-free projective outer products of two spin characters which is mainly due to Bessenrodt.

To this end we need the following notation: A staircase is a partition of the form $(k,k-1,\ldots,2,1)$, for some $k \in \mathbb{N}$. A fat staircase is a partition of the form $(k+r,k-1+r,\ldots,2+r,1+r)$ for some $k \in \mathbb{N}$, $r \geq 0$. In particular, a staircase is also a fat staircase. A hook staircase is the concatenation of a fat staircase and a staircase (where one of them may be empty).

In [2, Theorem 3.2] Bessenrodt determined almost all multiplicity-free projective outer products of $\hat{S}_n$. Together with the first author’s diploma thesis, this leads to the following theorem.

**Theorem 4.1.** Let $\mu$ and $\nu$ be strict partitions of $l$ and $n-l$, respectively. The projective outer product $<\mu> \hat{\otimes} <\nu>$ is multiplicity-free if and only if it is as in one of the following cases:

1. $<\text{something}> \hat{\otimes} <1>$.
2. $<\text{hook staircase}> \hat{\otimes} <2,1>$, and the hook staircase is in $D^+_{n-3}$.
3. $<\text{staircase}> \hat{\otimes} <m>$ for $m \in \mathbb{N}$.
4. $<\text{staircase}> \hat{\otimes} <m-1,1>$ for $m \in \mathbb{N}$, $m > 3$, such that the staircase and $(m-1,1)$ have different signs.
5. $<\text{staircase}> \hat{\otimes} <m+1,m>$ for $m \in \mathbb{N}$, and the staircase is in $D^+_{n-(2m+1)}$.
6. $<\text{fat staircase}> \hat{\otimes} <m>$ for $m \in \mathbb{N}$, $m > 1$, such that the fat staircase and $(m)$ have different signs.

While the proofs of cases (i) to (v) of Theorem 4.1 were established in [2], the proof of case (vi), which we present here, was first conceived in [13].

**Proof.** Let $\mu = (\mu_1, \ldots, \mu_{\ell(\mu)})$ and $\nu = (\nu_1, \ldots, \nu_{\ell(\nu)})$ be strict partitions, and assume without loss of generality that $\ell(\mu) \geq \ell(\nu)$. Then the partition

$$\lambda = \mu + \nu = (\mu_1 + \nu_1, \ldots, \mu_{\ell(\mu)} + \nu_{\ell(\mu)}, \mu_{\ell(\mu)} + 1, \ldots, \mu_{\ell(\mu)} + \ell(\nu))$$

yields a constituent of $<\mu> \hat{\otimes} <\nu>$, and the factor $2^{\frac{1}{2}(\ell(\mu) + \ell(\nu) - \ell(\lambda))}$ in formula (⋆) of the Littlewood-Richardson rule simplifies to $2^{\frac{1}{2}\ell(\nu)}$. Now, assuming that $<\mu> \hat{\otimes} <\nu>$ is multiplicity-free, and since there is exactly one tableau of shape $\lambda/\mu$ with content $\nu$, the factor $2^{\frac{1}{2}\ell(\nu)}$ is at most two, and therefore $1 \leq \ell(\nu) \leq 2$.

Also, as $<\mu> \hat{\otimes} <\nu>$ is multiplicity free, the inequality $st(\lambda; \mu, \nu) \leq 2$ holds for all $\lambda \in D_n$. In particular, if we have $st(\lambda; \mu, \nu) \leq 1$ for all $\lambda \in D_n$, then
one of the cases (i)-(v) of Theorem 4.1 applies, and we are done by [2, Theorem 3.2]. Thus we consider the previously untreated case that there exists a \( \lambda_0 \in \mathcal{D}_n \) such that \( \text{st}(\lambda_0;\mu,\nu) = 2 \). This implies that the coefficient of \((*)\) simplifies to 

\[
\frac{1}{\varepsilon_{\lambda_0} \varepsilon_{\mu,\nu}} \frac{1}{\sqrt{2}} 2^{\ell(\mu)+\ell(\nu)-\ell(\lambda_0)} = \frac{1}{2},
\]

and hence \( \ell(\lambda_0) = \ell(\mu)+\ell(\nu) \) and \( \varepsilon_{\lambda_0} = \varepsilon_{\mu,\nu} = \sqrt{2} \), i.e., the partitions \( \mu \) and \( \nu \) have different signs.

Let us assume \( \ell(\nu) = 2 \). If \( \mu \) is a staircase or \( \nu = (2,1) \), there is no partition \( \lambda \) with \( \text{st}(\lambda;\mu,\nu) > 0 \) and \( \ell(\lambda) = \ell(\mu) + 2 \). On the other hand, if \( \mu \) is not a staircase and \( \nu \neq (2,1) \), there exists a partition \( \lambda \) with \( \text{st}(\lambda;\mu,\nu) \geq 2 \) and \( \ell(\lambda) < \ell(\mu) + 2 \), giving \((\langle \lambda \rangle,\langle \mu \rangle \hat{\otimes} \langle \nu \rangle) > 1 \).

Therefore we conclude \( \ell(\nu) = 1 \), and we may assume that we are not in one of the Cases (i) to (v). In particular, we can assume that \( \mu \) is not a staircase. Now, if \( \mu_{i+1} + 1 < \mu_i \) for some \( i < \ell(\mu) \), there exists at least one partition \( \lambda \) with \( \text{st}(\lambda;\mu,\nu) \geq 2 \) and \( \ell(\lambda) < \ell(\mu) + 1 \), again giving rise to constituent whose multiplicity exceeds one. Thus \( \mu \) has to be a fat staircase, and we are in case (vi).

For the converse, it is easy to see that in Cases (i) to (vi) the projective outer product \( \langle \mu \rangle \hat{\otimes} \langle \nu \rangle \) is multiplicity-free. \( \square \)

As a corollary to the Branching Rule 3.1 and the previous theorem, we can now determine the irreducible projective outer products \( \langle \mu \rangle \hat{\otimes} \langle \nu \rangle \).

**Lemma 4.2.** Let \( \mu \in \mathcal{D}_l \) and \( \nu \in \mathcal{D}_{n-l} \).

(a) If \( l = n-1 \), then the spin character \( \langle \mu \rangle \hat{\otimes} \langle 1 \rangle = \langle \mu \rangle \hat{\otimes} \langle 1 \rangle = \langle \lambda \rangle^\wedge_n \) is irreducible if and only if \( \mu = (m,m-1,\ldots,1) \) for some \( m \in \mathbb{N} \) with \( m \equiv 2 \) or \( 3 \) (mod 4). In this case we have \( \mu \in \mathcal{D}_{n-1}^- \) and \( \langle \mu \rangle \hat{\otimes} \langle 1 \rangle = \langle \lambda \rangle \) with \( \lambda = (m+1,m-1,m-2,\ldots,1) \). In this case \( \langle \lambda \rangle \hat{\otimes} \langle 1 \rangle \) simplify to

(b) If \( l = n \) and \( l \) are larger than 1, then \( \langle \mu \rangle \hat{\otimes} \langle \nu \rangle \) is a reducible character of \( \hat{\mathcal{S}}_n \). In particular, \( \langle \mu \rangle \hat{\otimes} \langle \nu \rangle \) contains (at least) two non-associate irreducible constituents, except when the partition \( \mu = (m,m-1,\ldots,1) \) is an even staircase and \( \nu = (2,1) \). In this case \( \langle \mu \rangle \hat{\otimes} \langle \nu \rangle \) is reducible with \( \lambda = (m+2,m-2,\ldots,2,1) \).

**Proof.** (a) Let \( \langle \mu \rangle \) be a spin character of \( \hat{\mathcal{S}}_{n-1} \) such that \( \langle \mu \rangle \hat{\otimes} \langle 1 \rangle = \langle \lambda \rangle^\wedge_n \) is irreducible. Then in the terminology of Theorem 3.1 we have \( |N(\mu)| = 1 \) and \( \mu_{\ell(\mu)} = 1 \), i.e., \( \mu = (m,m-1,\ldots,1) \) for \( m = \ell(\mu) \). Since \( \langle \lambda \rangle^\wedge \) appears as a summand in \( \langle \mu \rangle \hat{\otimes} \langle 1 \rangle \) by the Branching Rule, we have \( \lambda \in \mathcal{D}_n^+ \) and hence \( \mu \in \mathcal{D}_{n-1}^- \).

It follows that \( \lambda = (m+1,m-1,m-2,\ldots,1) \). Since \( n = 1 + \sum_{i=1}^m i = m(m+1)/2+1 \) and \( n-m = (m(m-1))/2+1 \) is even, \( m \) is congruent to 2 or 3 modulo 4.

(b) We consider the Cases (ii)-(vi) of Theorem 4.1.

Case (ii): Let \( \mu \in \mathcal{D}_{n-3}^+ \) be a hook staircase and \( \nu = (2,1) \). We write \( \mu = \mu^{(1)} \cup \mu^{(2)} \) for a fat staircase \( \mu^{(1)} = (k+r,k-1+r,\ldots,1+r) \) and a staircase \( \mu^{(2)} = (m,m-1,\ldots,1) \). Note that one of these parts may be empty. If \( \mu^{(1)} \) is not empty then \( r \geq 1 \); If both \( \mu^{(1)} \) and \( \mu^{(2)} \) are non-empty then \( \mu^{(1)} = 1 + r > m + 1 = \mu^{(2)} + 1 \).

Assume that \( \mu^{(1)} \) is not empty and let \( k \geq 2 \), then the spin character \( \langle \lambda \rangle \) is a constituent of \( \langle \mu \rangle \hat{\otimes} \langle 2,1 \rangle \) for

(a) \( \lambda = (\mu_1^{(1)} + 2,\mu_2^{(1)} + 1,\mu_3^{(1)},\ldots) \), and

(b) \( \lambda = (\mu_1^{(1)} + 2,\mu_2^{(1)} + 1,\mu_3^{(1)},\mu_4^{(1)},\ldots,\mu_k^{(1)},\mu_1^{(2)} + 1,\mu_2^{(2)},\mu_3^{(2)},\ldots,\mu_m^{(2)}) \)

(with \( \lambda = (\mu_1^{(1)} + 2,\mu_2^{(1)} + 1,\mu_3^{(1)},\ldots,\mu_k^{(1)},1) \) if \( \mu^{(2)} \) is empty).
Let $k = 1$, i.e., $\mu^{(1)} = (r + 1)$. If $\mu^{(2)}$ is not empty, then the spin character $\langle \lambda \rangle$ for $\lambda$ as in (h) above is again a constituent of $\langle \mu \rangle \hat{\otimes} \langle 2, 1 \rangle$. Furthermore, $\langle \lambda \rangle$ with $\lambda = (\mu_1^{(1)} + 1, \mu_2^{(1)} + 1, \mu_3^{(2)}, \ldots, \mu_m^{(2)})$ is a constituent, too. If $\mu^{(2)}$ is empty, the characters $\langle \mu_1^{(1)} + 2, 1 \rangle$ and $\langle \mu_1^{(1)} + 1, 2 \rangle$ are constituents of $\langle \mu \rangle \hat{\otimes} \langle 2, 1 \rangle$.

Now assume that $\mu^{(1)}$ is empty. Let $\mu = \mu^{(2)} = (m, m - 1, \ldots, 1)$ be an even staircase. Hence $m$ is congruent to 0 or 1 modulo 4, and therefore $m \geq 4$. In this case only $\lambda = (m + 2, m, m - 2, \ldots, 1)$ yields a constituent of the outer projective product. The corresponding spin character $\langle \lambda \rangle$ is not self-associate and therefore we have $\langle \mu \rangle \hat{\otimes} \langle 2, 1 \rangle = \langle \lambda \rangle^+ + \langle \lambda \rangle^-$. 

Case (iii): Let $\mu = (k, k - 1, \ldots, 1)$ be a staircase and $\nu = (m)$ for $k$ and $m > 1$. In this case, the partitions $(k + m, k - 1, \ldots, 1)$ and $(k + m - 1, k, k - 2, \ldots, 1)$ yield two non-associate constituents of $\langle \mu \rangle \hat{\otimes} \langle m \rangle$. 

Case (iv): If $\mu = (k, k - 1, \ldots, 1)$ is again a staircase with $k > 1$ and $\nu = (m - 1, 1)$ for $m > 3$, then the two partitions $(k + m - 1, k, k - 2, \ldots, 1)$ and $(k + m - 2, k + 1, k - 2, \ldots, 1)$ yield constituents of $\langle \mu \rangle \hat{\otimes} \langle m - 1, 1 \rangle$.

Case (v): Let $\mu = (k, k - 1, \ldots, 1)$ be an even staircase and $\nu = (m + 1, m)$ for $m > 1$ (we have already considered the case $\nu = (2, 1)$ and $\mu$ a staircase). By the same argument as in Case (ii) above, we have $k \equiv 0$ or 1 (mod 4). Since $k > 1$, this implies $k \geq 4$. The partitions $(k + m + 1, (k - 1) + m, k - 2, \ldots, 1)$ and $(k + m + 1, k + m - 2, k - 1, k - 3, \ldots, 1)$ yield two non-associate constituents of $\langle \mu \rangle \hat{\otimes} \langle m + 1, 1 \rangle$. 

Case (vi): Let $\mu = (k + r, k - 1 + r, \ldots, 1 + r)$ be a fat staircase for some $k, r \geq 1$ and $\nu = (m)$ for some $m > 1$. Then the partitions $(k + r + m, k - 1 + r, \ldots, 1 + r) = (k + r + m - 1, k - 1 + r, \ldots, 1 + r, 1)$ yield two non-associate constituents of $\langle \mu \rangle \hat{\otimes} \langle \nu \rangle$. \hfill \qed

4.2. Characters Induced from $\hat{S}_{n, 2}$. We will now consider the imprimitive and transitive subgroups $\hat{S}_{n, 2}$ of $\hat{S}_n$. Let $n = 2m$, where $m \geq 2$. The subgroup $\hat{S}_{m, m}$ has index two in $\hat{S}_{n, 2}$, and therefore the irreducible characters of $\hat{S}_{n, 2}$ may be determined through an elementary application of Clifford Theory. Every irreducible character of $\hat{S}_{m, m}$ gives rise to one or two irreducible characters of $\hat{S}_{n, 2}$, and all irreducible characters of $\hat{S}_{n, 2}$ arise in this manner: A character is either invariant under the conjugation action of $\hat{S}_{n, 2}$ and thus possesses two distinct extensions to $\hat{S}_{n, 2}$, or its inertia subgroup is $\hat{S}_{m, m}$ and the character fuses with its conjugate to give a single irreducible character of $\hat{S}_{n, 2}$ by induction.

For $\langle \mu \rangle$, $\langle \nu \rangle \in \text{Irr}_{\text{red}}(\hat{S}_m)$ let $(\langle \mu \rangle \otimes \langle \nu \rangle)^+$ and $(\langle \mu \rangle \otimes \langle \nu \rangle)^-$ denote the two distinct extensions of the reduced Clifford product $\langle \mu \rangle \otimes \langle \nu \rangle$, if the latter is invariant. Analogously, we denote by $\langle \mu \rangle \otimes \langle \nu \rangle^0$ the irreducible induced character $(\langle \mu \rangle \otimes \langle \nu \rangle) |_{\hat{S}_{n, 2}}$, if $\langle \mu \rangle \otimes \langle \nu \rangle$ is not invariant.

The following first result is immediate.

Corollary 4.3. Let $\langle \mu \rangle \otimes \langle \nu \rangle^0 \in \text{Irr}_{\text{red}}(\hat{S}_{n, 2})$. Then the induced character $\langle \mu \rangle \otimes \langle \nu \rangle^0 |_{\hat{S}_n}$ is always reducible.

Proof. This is a simple consequence of Lemma 4.2 and the transitivity of induction. \hfill \qed

The analysis if either of the characters $(\langle \mu \rangle \otimes \langle \nu \rangle)^+$ and $(\langle \mu \rangle \otimes \langle \nu \rangle)^-$ induces irreducibly to $\hat{S}_n$ is slightly more involved. As both are extensions of an invariant character of $\hat{S}_{m, m}$ we will determine these first.
Let $S_{[1,...,m]} := \langle s_1,\ldots,s_{m-1} \rangle$ and $S_{[m+1,...,2m]} := \langle s_{m+1},\ldots,s_{2m-1} \rangle$. The image of $S_{n,2}$ under $\theta$ is isomorphic to $(S_{[1,...,m]} \times S_{[m+1,...,2m]}) \times \langle \tau \rangle$, for which the action of $\tau$ induces the outer automorphism which maps $s_i$ to $s_{i+m}$, if the indices are taken modulo $m$. Let $\tilde{\tau}$ denote the standard lift of $\tau$ in $\tilde{S}_n$. Then a character of $\tilde{S}_{m,m}$ is invariant under the action of $\tilde{S}_{n,2}$ if and only if it is invariant under conjugation by $\tilde{\tau}$. Therefore we consider the action of $\tilde{\tau}$ on the conjugacy classes of $\tilde{S}_{m,m}$.

As $t_i t_j = z t_j t_i$ for $|i - j| > 1$, any element $y$ of $\tilde{S}_{m,m}$ may be written as a product $gh$ for some $g \in \tilde{S}_{[1,...,m]} := \langle t_1,\ldots,t_{m-1} \rangle$ and $h \in \tilde{S}_{[m+1,...,2m]} := \langle t_{m+1},\ldots,t_{2m-1} \rangle$, and we have $gh = z^\sigma(g)\sigma(h)hg$. If the two elements $y$ and $zy$ are not conjugate in $\tilde{S}_{m,m}$, the full preimage of $\theta(y)\tilde{S}_{m,m}$ is the union of the two classes containing $y$ and $zy$. In this case we say that the conjugacy class of $\theta(y)$ splits, or that the classes of $y$ and $zy$ are split. As a class of $\tilde{S}_m \times \tilde{S}_m$ is naturally parameterized by a pair $(\pi,\mu)$ of partitions of $m$, the class $y\tilde{S}_{m,m}$ is therefore also parameterized by $(\pi,\mu)$, if the cycle types of $\theta(g)$ and $\theta(h)$ are $\pi$ and $\mu$, respectively. By a slight abuse of notation we denote conjugacy classes by their parameters. If there are two classes with the same parameter, as is the case for split classes, we affix subscripts to distinguish them.

We begin by determining the classes of $\tilde{S}_m \times \tilde{S}_m$ which split. Let $O_m$ denote the set of all odd part partitions of $m$.

**Lemma 4.4.** A class $(\pi,\mu)$ of $\tilde{S}_m \times \tilde{S}_m$ splits if and only if $(\pi,\mu)$ is an element of $O_m \times O_m$, $D^-_m \times D^+_m$, or $D^+_m \times D^-_m$.

**Proof.** If $(\pi,\mu) \in O_m \times O_m$, then $\theta(gh)$ has odd order. Hence we may assume that $gh$ has odd order, too. For any $y \in \theta^{-1}(C_{\tilde{S}_m \times \tilde{S}_m}(\theta(gh)))$ we have that $(gh)^y \in \{gh, zgh\}$, so as the order of $gh$ is odd, we conclude $(gh)^y = gh$. Therefore the centralizer of $gh$ in $\tilde{S}_{m,m}$ is the full preimage of $C_{\tilde{S}_m \times \tilde{S}_m}(\theta(gh))$ and the class of $\theta(gh)$ splits.

If $(\pi,\mu) \in D^-_m \times D^+_m$, then $\pi$ possesses an odd number $o$ of even parts, and $\mu$ has an even number $e$ of even parts. For $i = 1,\ldots,\ell(\pi)$ let $\tilde{C}_{\pi_i} \in \tilde{S}_{[1,...,m]}$ denote a lift of a $\pi_i$-cycle. Likewise, for $j = 1,\ldots,\ell(\mu)$ let $\tilde{C}_{\mu_j} \in \tilde{S}_{[m+1,...,2m]}$ be a lift of a $\mu_j$-cycle, and set $g := \tilde{C}_{\pi_1} \cdots \tilde{C}_{\pi_{\ell(\pi)}}$ and $h := \tilde{C}_{\mu_1} \cdots \tilde{C}_{\mu_{\ell(\mu)}}$. If $\pi_i$ is odd, then $\tilde{C}_{\pi_i}gh = gh\tilde{C}_{\pi_i}$, and if $\pi_i$ is even, we have $\tilde{C}_{\pi_i}gh = z^o + c - 1 gh\tilde{C}_{\pi_i} = gh\tilde{C}_{\pi_i}$, as $o + e - 1$ is even. Thus $\tilde{C}_{\pi_i} \in C_{\tilde{S}_{m,m}}(gh)$ for all $i = 1,\ldots,\ell(\pi)$. The same holds for all $\tilde{C}_{\mu_j}$, $j = 1,\ldots,\ell(\mu)$. Therefore $C_{\tilde{S}_{m,m}}(gh)$ is again the full preimage of the centralizer of $\theta(gh)$, and thus the class of $\theta(gh)$ splits.

Interchanging the roles of $\pi$ and $\mu$ above, yields the result for $(\pi,\mu) \in D^+_m \times D^-_m$. By [8, Theorem 5.9] there are exactly $|D^-_m|^2 + 2|D^+_m||D^-_m|$ splitting classes. As $|O_m| = |D_m|$ we are done. $\square$

In order to give the action of $\tilde{\tau}$ on the conjugacy classes of $\tilde{S}_{m,m}$, we first study its effect on the generators $t_1,\ldots,t_{m-1},t_{m+1},\ldots,t_{2m-1}$ of $\tilde{S}_{m,m}$. Note that the group $\tilde{S}_{m,m}$ also possesses the outer automorphism $\iota$ which maps $t_i$ to $t_{i+m}$ where the indices are again taken modulo $m$. There is a subtle difference between the actions of $\tilde{\tau}$ and $\iota$ depending on the parity of $m$.

**Lemma 4.5.** Let $1 \leq i \leq m - 1$. Then $t_i^\tilde{\tau} = z^m t_{i+m}$ and $t_{i+m}^\tilde{\tau} = z^m t_i$. 

Proof. For $1 \leq k \leq m$ we set 
\[ \tilde{\tau}_k := t_{k+m-1}t_{k+m-2} \cdots t_{k+1}t_k \cdots t_1. \]
Then $\tilde{\tau} = \tilde{\tau}_1 \cdots \tilde{\tau}_m$. First note that we have $\tilde{\tau}_i \tilde{\tau}_j = z\tilde{\tau}_j \tilde{\tau}_i$ for all $1 \leq i, j \leq m$. The element $\theta(\tilde{\tau}_i \tilde{\tau}_j)$ of $S_n$ has cycle type $(2^2, 1^{2m-4}) \notin O_n \cup D_n^-$. Thus $\tilde{\tau}_i \tilde{\tau}_j$ is conjugate to $t_i t_j$ and therefore has order four. This yields $\tilde{\tau}_i \tilde{\tau}_j = z \tilde{\tau}_j^{-1} \tilde{\tau}_i^{-1}$, which is $z \tilde{\tau}_j \tilde{\tau}_i$ as every $\tilde{\tau}_k$ is the product of $2m - 1$ factors. Using the relations given in the presentation of $S_n$ we obtain $\tilde{\tau}_i^{-1} t_i \tilde{\tau} = z^m \tilde{\tau}_i \tilde{\tau}_{i+1}$ for $1 \leq i \leq m$. Further meticulous applications of these relations give that the latter is equal to $\tilde{\tau}_i^{-1} t_i \tilde{\tau} = z^m t_i + m$, as claimed. Therefore the equation $\tilde{\tau}_i^{-1} t_i + m \tilde{\tau} = z^m \tilde{\tau}_i \tilde{\tau}_j$ follows. \hfill $\square$

Lemma 4.6. Let $\iota$ denote the automorphism of $\tilde{S}_{m,m}$ taking $t_i$ to $t_i + m$ (where the indices are taken modulo $2m$). In other words, for $g \in \tilde{S}_{1,\ldots,m}$ and $h \in \tilde{S}_{(m+1),\ldots,2m}$ we have $(gh)^\iota = z^\sigma(g)\sigma(h)h^\iota$. Hence $\iota$ maps the conjugacy class containing the standard lift of $\theta(gh)$ to the conjugacy class containing the standard lift of $\theta(h^\iota)$. 

Proof. For a non-split class the assertion is clear. Let $gh$ lie in a split class, and without loss of generality assume that $gh$ is the standard lift of $\theta(gh)$, i.e., either both $g$ and $h$ are the standard lifts of $\theta(g)$ and $\theta(h)$, or both are not. The same holds for the image under $\iota$. By the definition of $\iota$ both $g^\iota$ and $h^\iota$ are standard lifts if and only if both $g$ and $h$ are standard lifts. Now as the element $\theta(gh)$ lies in a conjugacy class of $S_m \times S_m$ with parameter $(\pi, \mu)$, where $(\pi, \mu)$ is an element of either $O_m \times O_m$, $D_m^- \times D_m^+$, or $D_m^+ \cap D_m^-$, we have that $\sigma(g) = 0$ or $\sigma(h) = 0$. Therefore $(gh)^\iota$ is the standard lift. \hfill $\square$

Corollary 4.7. For a splitting class of $S_m \times S_m$ denoted by its parameter $(\pi, \mu)$ let $(\pi, \mu)_1$ be the class of $\tilde{S}_{m,m}$ containing the standard lift $gh$ of the canonical representative of $(\pi, \mu)$. Accordingly, let $(\pi, \mu)_2$ denote the class containing $zgh$.

If $m$ is even, then $(\pi, \mu)_1^\tilde{\tau} = (\mu, \pi)_1$.

If $m$ is odd,
\[ (\pi, \mu)_1^\tilde{\tau} = \begin{cases} (\mu, \pi)_1, & \text{if } (\pi, \mu) \in O_m \times O_m, \\ (\mu, \pi)_2, & \text{if } (\pi, \mu) \in D_m^- \times D_m^+ \text{ or } D_m^+ \times D_m^- \text{.} \end{cases} \]

Proof. If $m$ is even, then by Lemma 4.5 we see that $\tilde{\tau}$ and $\iota$ act identically, and Lemma 4.6 gives the claim. If $m$ is odd, by Lemma 4.5 we have 
\[ (gh)^\tilde{\tau} = z^\sigma(g) + \sigma(h)h^\iota. \]
Thus if $gh \in (\pi, \mu)_1$ for $(\pi, \mu) \in D_m^- \times D_m^+$ or $D_m^+ \times D_m^-$, then $(gh)^\tilde{\tau} = zh^\iota g^\iota$. Hence by Lemma 4.6 we have $(gh)^\tilde{\tau} = z(gh)^\iota \in (\mu, \pi)_2$. \hfill $\square$

Summing up, we can give the action of $\tilde{\tau}$ on the irreducible characters of $\tilde{S}_{m,m}$ as follows:

Corollary 4.8. For $\chi \in \text{Irr}_-(\tilde{S}_{m,m})$ the action of $\tilde{\tau}$ is given by
\[ \chi^\tilde{\tau} = \begin{cases} \chi^\iota, & \text{if } m \text{ is even,} \\ (\chi^\iota)^m, & \text{if } m \text{ is odd}. \end{cases} \]

Proof. This is now immediate by Corollary 4.7. \hfill $\square$
With the action of $\tilde{\tau}$ on $\text{Irr}_-(\hat{S}_{m,m})$ known, it is easy to determine which characters are $\tilde{\tau}$-invariant.

**Corollary 4.9.** Let $\mu, \nu \in D_m$. Then $<\mu> \otimes_z <\nu>$ is $\tilde{\tau}$-invariant if and only if, $\mu = \nu$.

**Proof.** Let $\chi := <\mu> \otimes_z <\nu>$, then by Corollary 4.8 we have $\chi^\sharp |_{\ker \sigma} = \chi |_{\ker \sigma}$ independent of the parity of $m$. If $\chi$ is $\tilde{\tau}$-invariant, this implies $(<\mu> \otimes_z <\nu>) |_{\ker \sigma} = (<\nu> \otimes_z <\mu>) |_{\ker \sigma}$. We may now argue as in [8, proof of 5.9]: From the character values of the reduced Clifford product (cf. [8, Table 5.7]) it follows that $<\mu> |_{\ker \sigma} = <\nu> |_{\ker \sigma}$. Hence $<\mu> = <\nu>$ or $<\mu> = <\nu>^a$, so $\mu = \nu$.

The converse follows from the fact that for any spin character $<\mu> \in \text{Irr}_-(\hat{S}_m)$ the reduced Clifford product $<\mu> \otimes_z <\mu>$ is self-associate. Hence it is $\tilde{\tau}$-invariant by Corollary 4.8.

Having the $\hat{S}_{n,2}$-invariant characters of $\hat{S}_{m,m}$ at our disposal, we can study their behavior under induction.

**Lemma 4.10.** Let $\chi \in \text{Irr}_-(\hat{S}_{m,m})$ be invariant under the conjugation action of $\hat{S}_{n,2}$, and let $\chi^\pm_{\hat{S}_{n,2}} = \chi^+ + \chi^-$, where $\chi^+, \chi^- \in \text{Irr}_-(\hat{S}_{n,2})$ are two extensions of $\chi$. Then for all $g \in \hat{S}_n$ we have $\chi^+ |_{\hat{S}_{n}}(g) = \chi^- |_{\hat{S}_{n}}(g)$, except possibly when $\theta(g)$ has cycle type $2\pi$. In this case $\chi^- |_{\hat{S}_{n}}(g) = \chi^+ |_{\hat{S}_{n}}(zg) = -\chi^+ |_{\hat{S}_{n}}(g)$.

**Proof.** Let $C$ denote a conjugacy class of $\hat{S}_{n}$. The value of $\chi^+ |_{\hat{S}_{n}}$ on $C$ is determined by the values of $\chi^+$ on the conjugacy classes of $\hat{S}_{n,2}$ lying in $C$. The inner and outer classes do not interfere in the following sense: If there are outer classes of $\hat{S}_{n,2}$ lying in $C$, then the character value of $\chi^+ |_{\hat{S}_{n}}$ on $C$ depends only on the values of $\chi^+ \ up to a constant the product of the values of the spin character $<\lambda> \in \text{Irr}_-(\hat{S}_n)$ on the corresponding classes of $\hat{S}_n$ with parameters $\pi$ and $\mu$. Since a spin character of $\hat{S}_m$ only takes nonzero values on classes parameterized by elements of $O_m$ or $D_m$, we conclude that $\chi^+$ is zero on all inner classes which fuse into $C$.

It is now immediate that $\chi^+ |_{\hat{S}_{n}}(g) = \chi^- |_{\hat{S}_{n}}(g)$ for all $g \in \hat{S}_n$, except when $\theta(g)$ has cycle type $2\pi$, i.e., when an outer class of $\hat{S}_{n,2}$ lies in the $\hat{S}_n$-conjugacy class of $g$. Now, if $\theta(g)$ has cycle type $2\pi \notin D_n$, the elements $g$ and $zg$ are conjugate in $\hat{S}_n$, and therefore $\chi^+ |_{\hat{S}_{n}}(g) = 0 = \chi^- |_{\hat{S}_{n}}(g)$. So let $\theta(g)$ have cycle type $2\pi \in D_n$ for an element $g \in \hat{S}_{n,2}$. Then the two classes $g_{\hat{S}_{n,2}}$ and $(zg)_{\hat{S}_{n,2}}$ fuse into the two classes $g_{\hat{S}_{n}}$ and $(zg)_{\hat{S}_{n}}$, respectively. Setting $e := [C_{\hat{S}_{n}}(g) : C_{\hat{S}_{n,2}}(g)]$, we arrive at $\chi^- |_{\hat{S}_{n}}(g) = e\chi^-(g) = -e\chi^+(g) = \chi^+ |_{\hat{S}_{n}}(zg)$, as the extensions $\chi^+$ and $\chi^-$ fulfill $\chi^- (g) = -\chi^+(g)$.

In the light of Lemma 4.10 it is not surprising that either both induced extensions $(<\mu> \otimes_z <\mu>)^+$ and $(<\mu> \otimes_z <\mu>)^-$ are irreducible, or both are not.

**Corollary 4.11.** Under the hypothesis of Lemma 4.10 the character $\chi^+ |_{\hat{S}_{n}}$ is irreducible if and only if $\chi^- |_{\hat{S}_{n}}$ is irreducible.
Proof. By Lemma 4.10 the induced characters $\chi^+|_{\tilde{S}_n}$ and $\chi^-|_{\tilde{S}_n}$ have the same norm.

Lemma 4.12. Let $\mu \in D_m$. The spin character $<\mu> \circlearrowleft <\mu>$ contains (at least) two non-associate, distinct constituents or one constituent with multiplicity four, except when $m = 3$ and $\mu = (2, 1)$, or $m = 6$ and $\mu = (3, 2, 1)$. In these cases we have $<2, 1> \circlearrowleft <2, 1> = 2 <4, 2>$ and $<3, 2, 1> \circlearrowleft <3, 2, 1> = 2(<6, 4, 2> + <6, 4, 2>^{a})$.

Proof. Taking $\lambda = 2\mu$ yields a constituent $<\lambda>$ of $<\mu> \circlearrowleft <\mu>$, and the factor $2^{\frac{1}{2}(2\ell(\mu) - \ell(\lambda))}$ in formula (8) of 3.1 simplifies to $2^{\frac{1}{2}\ell(\mu)}$. Note that $\varepsilon_{\mu, \mu} = 1$ since $\mu \subseteq \mu$ is an even partition. If $\ell(\mu) \geq 4$ then the coefficient $2^{\frac{1}{2}\ell(\mu)}$ in (8) is at least 4. If $\ell(\mu) = 1$, i.e., $\mu = (m)$ for some $m \geq 2$, then $(2m - 1, 1)$ yields a second constituent not associate to $<2m>$. If $\ell(\mu) = 2$ then $\lambda = 2\mu$ is an even partition and the factor 2 appears in formula (8). Hence, we may assume $st(2\mu; \mu, \mu) = 1$.

By [2, Theorem 2.2] this leaves $\mu = (2, 1)$, and the corresponding projective outer product is $2 <4, 2>$. If $\ell(\mu) = 3$ an analogous argument gives $\mu = (3, 2, 1)$, and we have $<3, 2, 1> \circlearrowleft <3, 2, 1> = 2(<6, 4, 2> + <6, 4, 2>^a)$.

Corollary 4.13. If $n = 6$ and $\mu = (2, 1)$ we have $(<\mu> \circlearrowleft <\mu>)^+ \circlearrowleft |_{\tilde{S}_6} = (<\mu> \circlearrowleft <\mu>)^{-} \circlearrowleft |_{\tilde{S}_6} = <4, 2> \in \text{Irr}(\tilde{S}_6)$. In all other cases, both $(<\mu> \circlearrowleft <\mu>)^+$ and $(<\mu> \circlearrowleft <\mu>)^-$ of $\text{Irr}(\tilde{S}_6)$ induce reducibly to $\tilde{S}_n$.

Proof. We have $<\mu> \circlearrowleft <\mu> = (<\mu> \circlearrowleft <\mu>)^+ \circlearrowleft |_{\tilde{S}_n} + (<\mu> \circlearrowleft <\mu>)^- \circlearrowleft |_{\tilde{S}_n}$. By Corollary 4.11 either both extensions induce irreducibly, or both do not. The only possibility for $\mu$ such that $<\mu> \circlearrowleft <\mu>$ has norm 2 is given by Lemma 4.12 to be $\mu = (2, 1)$.

With the help of Lemma 2.5, Lemma 4.2 and Corollary 4.13 we may readily identify all imprimitive faithful characters of $\tilde{S}_n$, and state the corresponding maximal subgroups which occur as their block stabilizers. For convenience, we summarize our current standing in the following corollary.

Corollary 4.14. Let $<\lambda>$ be an imprimitive spin character of $\tilde{S}_n$. Then

(i) $\lambda \in D_+^n$, and $<\lambda>$ is the induced of either constituent of $<\lambda>|_{\tilde{A}_n}$, i.e., $\tilde{A}_n$ is a block stabilizer.

Furthermore,

(ii) if $\lambda = (m + 1, m - 1, \ldots, 1) \in D_+^n$ for $m = 2, 3$ (mod 4), then the subgroup $\tilde{S}_{n-1}$ is also a block stabilizer, as $<\lambda> = <\mu>|_{\tilde{S}_n}$ for the partition $\mu = (m, m - 1, \ldots, 1) \in D_-^{n-1}$.

And in particular,

(iii) if $n = 6$ then $<4, 2>$ is the induced of an irreducible character of degree 2 of $\tilde{S}_{6, 2} \cong S_3^2: Q_8$, and if $n = 9$ then $<6, 2, 1>$ is an induced linear character of order six of the subgroup $2 \times L_2(8)$: 3.

In order to prove Theorem 1.1, we have to verify the minimality of the triples given. As this involves information on the imprimitive faithful characters of $\tilde{A}_n$, we make use of our results in Section 5. Note that these are independent of Theorem 1.1.

Proof of Theorem 1.1. With the help of Corollaries 4.14 and 5.6 we recursively trace an imprimitive character to a subgroup from which it is induced until we find
a minimal triple. Let $\lambda \in D_+^+$. If a constituent of $<\lambda>_\tilde{A}_n$ were an imprimitive character of $\tilde{A}_n$, then $\lambda$ would be an odd partition, or one of the exceptional cases would hold by Corollary 5.6. Hence the triples of Part (i) of Theorem 1.1 are minimal. If $\lambda$ is as in Case (ii) of Corollary 4.14, then $\tilde{S}_{n-1}$ is another block stabilizer, and $<\lambda>$ is the induced of a not-self-associate spin character of this group. The latter cannot be imprimitive by Corollary 4.14. Note that if $n = 4$, then $\mu = (2,1)$ and $<\mu>$ is linear. This proves the minimality of the triples in Case (ii) of Theorem 1.1. Lastly, we have to consider the exceptional cases of Part (iii) of Corollary 4.14: With GAP it is elementary to check that the character of degree 2 of $3^2$ : $Q_8$ 2 which induces to $<4,2>$ is in fact an induced linear character of $3^2$: 8, giving the claimed minimal triple. Likewise, we confirm the minimality of the remaining triple. □

It is now immediate which irreducible spin characters of $\tilde{S}_n$ are monomial.

**Corollary 4.15.** Let $\chi \in \text{Irr}_-(\tilde{S}_n)$ be an imprimitive monomial spin character. Then we have

(i) $n = 4$ and $\chi = <3,1>$, or
(ii) $n = 6$ and $\chi = <4,2>$, or
(iii) $n = 9$ and $\chi = <6,2,1>$.

**Proof.** Going through our list of minimal triples $(H, \varphi, \chi)$ in Theorem 1.1, we determine which $\varphi$ are linear. For $n \geq 4$ there are no faithful non-trivial linear characters in $\text{Irr}_-(\tilde{A}_n)$. In Case (ii) of Theorem 1.1 the partition $\mu = (m,m-1,\ldots,1)$ of $n-1$ is an odd staircase. Hence, if $m \geq 3$, then the degree of $<\mu>$ is at least two. Only if $m = 2$ the resulting spin character $<\mu> = <2,1>$ is linear. The exceptional imprimitive characters of Parts (iii) and (iv) are evidently monomial. □

5. The Alternating Groups

Using the results of the previous section, we may now determine the irreducible spin characters of $\tilde{A}_n$, which are induced spin characters of subgroups, with the help of Clifford Theory.

In analogy to Section 4, we have two types of maximal subgroups to consider: On the one hand there are the intersections with the preimages of the maximal parabolic subgroups of $\tilde{S}_n$, i.e., the subgroups $\tilde{A}_{l,n-1} := \tilde{S}_{l,n-1} \cap \tilde{A}_n$. On the other hand there are the subgroups $\tilde{A}_{m,2} := \tilde{S}_{m,2} \cap \tilde{A}_n$.

5.1. **Characters Induced from $\tilde{A}_{l,n-1}$.** We begin by considering the spin characters of the subgroups $\tilde{A}_{l,n-1}$. With the help of Lemma 4.2 it is now straightforward to determine which characters induce irreducibly.

**Lemma 5.1.** A faithful irreducible character $\chi$ of $\tilde{A}_n$ is induced from a subgroup $\tilde{A}_{l,n-1}$ if and only if $l = n-1$, in other words $\tilde{A}_{l,n-1} = \tilde{A}_{n-1}$, and $\chi = <\lambda>_{\tilde{A}_n}$ with $\lambda = (m+1,m-1,m-2,\ldots,1) \in D_n$ for some $m \in \mathbb{N}$ with $m \equiv 0$ or 1 (mod 4). In this case $\chi$ is the induced of either constituent of $<\mu>_{\tilde{A}_{n-1}}$ with $\mu = (m,m-1,\ldots,1) \in D^+_{n-1}$.

**Proof.** Let $\alpha$ be a spin character of $\tilde{A}_{n-1,l}$ such that $\alpha|_{\tilde{A}_n}$ is irreducible. If $\alpha|_{\tilde{S}_n} = <\lambda>$ is irreducible, so is $\alpha|_{\tilde{S}_{n-1}}$. Therefore by Lemma 4.2 we have $l = n-1$ and $\alpha|_{\tilde{S}_{n-1}} = <\mu>$ for some $\mu \in D^-_{n-1}$. Hence $\alpha|_{\tilde{S}_{n-1}}$ is not self-associate, which is a
contradiction. Therefore \( \alpha^0 \otimes \tilde{\alpha} = \langle \lambda \rangle + \langle \lambda \rangle^a \) for a not self-associate character \( \langle \lambda \rangle \in \text{Irr}_-(\tilde{S}_n) \).

Let \( l, n - l > 1 \). Suppose \( \alpha \otimes \tilde{\alpha}^{l-1} \) is irreducible. Then Lemma 4.2 forces \( l = 3 \) and \( \alpha^0 \otimes \tilde{\alpha}^{n-3} \) is not self-associate, giving again a contradiction. Therefore \( \alpha^0 \otimes \tilde{\alpha}^{n-1} = \varphi + \varphi^a \) for a not self-associate \( \varphi \in \text{Irr}_-(\tilde{S}_{n-l}) \), and \( \varphi \) induces to \( \langle \lambda \rangle \) or \( \langle \lambda \rangle^a \). By Lemma 4.2 this is impossible if \( l, n - l > 1 \).

So let \( l = n - 1 \) and suppose that \( \alpha^0 \otimes \tilde{\alpha}^{l-1} = \langle \mu \rangle \) for some \( \mu \in \mathcal{D}_{n-1}^+ \). Hence \( \langle \mu \rangle \otimes \tilde{\alpha}^{n} = \langle \lambda \rangle + \langle \lambda \rangle^a \), so by the Branching Rule we obtain \( \mu = (m, m-1, \ldots, 1) \) where \( m \equiv 2, 3 \) (mod 4). On the other hand, if \( \alpha^0 \otimes \tilde{\alpha}^{n-1} = \langle \mu \rangle + \langle \mu \rangle^a \) for some \( \mu \in \mathcal{D}_n^- \), we obtain again that \( \langle \mu \rangle \otimes \tilde{\alpha}^n \) induces to either \( \langle \lambda \rangle \) or \( \langle \lambda \rangle^a \), and thus by Lemma 4.2 the partition \( \lambda \) is even, giving another contradiction.

\( \square \)

5.2. Characters Induced from \( \tilde{A}_{n-2} \). Let \( n = 2m \), where \( m \geq 2 \). By Clifford Theory, an irreducible character \( \alpha \in \text{Irr}_-(\tilde{A}_{m,m}) \) is either invariant under conjugation in \( \tilde{A}_{n-2} \) or its inertia subgroup is \( \tilde{A}_{m,m} \). In the first case it is \( \alpha = \alpha^+ + \alpha^- \) for two extensions of \( \alpha \) to \( \tilde{A}_{n-2} \). In the second case \( \alpha = \alpha^0 \) is irreducible.

The following result is an immediate consequence of Corollary 5.1.

**Lemma 5.2.** For \( \alpha^0 \in \text{Irr}_-(\tilde{A}_{n-2}) \) the induced character \( \alpha^0 \otimes \tilde{\alpha}^n \) is always reducible.

We now consider the case when the inertia subgroup of \( \alpha \in \text{Irr}_-(\tilde{A}_{m,m}) \) is \( \tilde{A}_{m-2} \), and obtain an analogue to Corollary 4.11.

**Lemma 5.3.** Let \( \alpha \in \text{Irr}_-(\tilde{A}_{m,m}) \) be invariant under the action of \( \tilde{A}_{m-2} \), i.e., \( \alpha^0 \otimes \tilde{\alpha}^{m-2} = \alpha^+ \otimes \tilde{\alpha}^{m-2} + \alpha^- \otimes \tilde{\alpha}^{m-2} \) and both summands are self-associate. Assume that \( \alpha \) is invariant in \( \tilde{S}_{m,m} \). Then we have \( \alpha \otimes \tilde{\alpha}^{m} = \psi + \psi^a \) for some irreducible not self-associate \( \psi \in \text{Irr}_-(\tilde{S}_{m,m}) \) by [8, Theorem 5.10]. Say \( \psi = \langle \mu \rangle \otimes \zeta \langle \nu \rangle \) for some \( \mu, \nu \in \mathcal{D}_m \) with \( \sigma(\mu) \neq \sigma(\nu) \). In particular, neither \( \psi \) nor \( \psi^a \) are invariant in \( \tilde{S}_{m-2} \). Therefore, both \( \psi \otimes \tilde{\alpha}^{m-2} \) and \( (\psi^a) \otimes \tilde{\alpha}^{m-2} \) are irreducible. By the initial decomposition of \( \alpha \otimes \tilde{\alpha}^{m} \) both constituents are self-associate, so \( \psi \otimes \tilde{\alpha}^{m-2} = (\psi \otimes \tilde{\alpha}^{m-2})^a \). But by Corollary 4.8 \( \langle \mu \rangle \otimes \zeta \langle \nu \rangle \) and either \( \langle \nu \rangle \otimes \zeta \langle \mu \rangle \) or \( \langle \nu \rangle \otimes \zeta \langle \mu \rangle^a \) are the only constituents of \( \psi \otimes \tilde{\alpha}^{m-2} \otimes \tilde{\alpha}^{m} \), a contradiction. Therefore \( \alpha \) is not invariant in \( \tilde{S}_{m,m} \) and \( \alpha \otimes \tilde{\alpha}^{m} \) is irreducible under the conjugation action of \( \tilde{S}_{m,m} \). As \( \alpha^+ \otimes \tilde{\alpha}^{m-2} \) and \( \alpha^- \otimes \tilde{\alpha}^{m-2} \) are self-associate the equality \( \alpha^+ \otimes \tilde{\alpha}^{m} = \alpha^- \otimes \tilde{\alpha}^{m} \) follows from Lemma 4.10.

\( \square \)

**Corollary 5.4.** Under the hypothesis of Lemma 5.3 we have that \( \alpha^+ \otimes \tilde{\alpha}^n \) is irreducible if and only if \( \alpha^- \otimes \tilde{\alpha}^n \) is irreducible.

**Proof.** Without loss of generality let \( \alpha^+ \otimes \tilde{\alpha}^n \) be irreducible. Induction to \( \tilde{S}_n \) gives that either \( \alpha^+ \otimes \tilde{\alpha}^n \) is irreducible, which forces \( \alpha^- \otimes \tilde{\alpha}^n \) to be irreducible too by Lemma 5.3, or \( \alpha^+ \otimes \tilde{\alpha}^n = \langle \lambda \rangle + \langle \lambda \rangle^a \) for some \( \lambda \in \mathcal{D}_n^- \). Were \( \alpha^- \otimes \tilde{\alpha}^n \) not irreducible in the latter case, it would have two constituents inducing irreducibly to either \( \langle \lambda \rangle \) or \( \langle \lambda \rangle^a \). But then \( \langle \lambda \rangle \) would be self-associate, giving a contradiction.

\( \square \)

**Corollary 5.5.** Let the hypothesis of Lemma 5.3 hold. The characters \( \alpha^+ \otimes \tilde{\alpha}^n \) and \( \alpha^- \otimes \tilde{\alpha}^n \) are irreducible if and only if \( n = 6 \) and \( \alpha \otimes \tilde{\alpha}^{1,1} = \langle 2,1 \rangle \otimes \zeta \langle 2,1 \rangle \). In
and therefore by Corollary 5.4 also \( \alpha^{-}\mathcal{A}_n \), be irreducible. By Lemma 5.3 we have \( \alpha^{+}\mathcal{A}_n = \alpha^{-}\mathcal{S}_n \) and \( \alpha \) is a constituent of \( \langle \mu \rangle \otimes < \chi \rangle \mathcal{A}_{m,n} \) for some \( \mu \in \mathcal{D}_m \). As \( \alpha^{+}\mathcal{S}_n \) is either irreducible or the sum of an associate pair of characters, Lemma 4.12 gives that \( \mu \) is either \((2,1)\) or \((3,2,1)\). But if \( \mu = (3,2,1) \), both constituents of \( \langle \mu \rangle \otimes < \chi \rangle \mathcal{A}_{6,6} \) are not invariant in \( \mathcal{A}_{12,2} \), hence induce irreducibly to \( \alpha^0 \). This leaves the case \( n = 6 \) and \( \mu = (2,1) \) in which both constituents of \( \langle \mu \rangle \otimes < \chi \rangle \mathcal{A}_{6,3} \) are invariant in \( \mathcal{A}_{6,2} \). In this case the extensions \( \alpha^{+} \) and \( \alpha^{-} \) induce to the two conjugate constituents of \( <4,2>\mathcal{A}_6 \).

In analogy to Corollary 4.14, we summarize the conclusions of Lemma 2.5, Lemma 5.1, and Corollary 5.5 in the following corollary.

**Corollary 5.6.** Let \( \chi \) be an imprimitive spin character of \( \mathcal{A}_n \). Then

(i) \( \chi \) is the restriction to \( \mathcal{A}_n \) of the not-self-associate spin character \( \langle \lambda \rangle \) of \( \mathcal{S}_n \) with \( \lambda = (m+1,m-1,m-2,\ldots,1) \) for some \( m \equiv 0,1 \) (mod 4). It is the induced of either constituent of \( \langle m,m-1,\ldots,1 \rangle \mathcal{A}_{n-1} \) and hence \( \mathcal{A}_{n-1} \) is a block stabilizer.

Additionally, we have the following exceptions:

(ii) If \( n = 6 \) then \( \chi \) is either constituent of \( <4,2>\mathcal{A}_6 \). Both are induced extensions to \( \mathcal{A}_{6,2} \cong 3^2 : 8 \) of linear characters of the subgroup \( \mathcal{A}_{3,3} \cong 3^2 : 4 \) of order four.

(iii) If \( n = 9 \) then \( \chi \) is either constituent of \( <6,2,1>\mathcal{A}_9 \). Both are induced linear characters of order six of the subgroup \( 2 \times L_2(8) : 3 \).

With the help of Corollary 5.6 it is now straightforward to establish the veracity of the claims made in Theorem 1.2.

**Proof of Theorem 1.2.** If \( \lambda \) is as in Case (i) of Corollary 5.6, then the staircase partition \((m,m-1,\ldots,1)\) of \( n-1 \) is an even partition. Hence the constituents of \( \langle m,m-1,\ldots,1 \rangle \) are not imprimitive, giving the minimality of the triples in Part (i) of Theorem 1.2. The minimality of the exceptional triples is immediate, as the corresponding imprimitive characters are monomial.

Going through the list of minimal triples \((H,\varphi,\chi)\) in 1.2 we can easily identify the monomial characters of \( \mathcal{A}_n \).

**Corollary 5.7.** Let \( \chi \in \text{Irr}_- (\mathcal{A}_n) \) be an imprimitive, monomial faithful character. Then

(i) \( n = 6 \) and \( \chi \) is a constituent of \( <4,2>\mathcal{A}_6 \), or

(ii) \( n = 9 \) and \( \chi \) is a constituent of \( <6,2,1>\mathcal{A}_9 \).

**Proof.** The constituents of \( \langle \mu \rangle \mathcal{A}_{n-1} \) for \( \mu = (m,m-1,\ldots,1) \in \mathcal{D}_{n-1}^+ \) with \( m \geq 4 \) have degree at least 4. On the other hand, we have already mentioned, that the characters \( \varphi \) of the exceptions (ii) and (iii) of Theorem 1.2 are linear.
5.3. The Sixfold Covers of $\mathcal{A}_6$ and $\mathcal{A}_7$. As we have pointed out in the introduction, the Schur covers of the alternating groups $\mathcal{A}_6$ and $\mathcal{A}_7$ are sixfold covers (see [8, Theorem 2.11]) in contrast to the double covers we have considered so far. Hence to complete our classification of the imprimitive faithful characters, we still have to consider the non-split extensions $6\mathcal{A}_6$ and $6\mathcal{A}_7$. For the sake of completeness we will also consider the triple covers $3\mathcal{A}_6$ and $3\mathcal{A}_7$.

The characters and the maximal subgroups of the groups considered are given in the ATLAS [4] or in the GAP character table library. In the sequel, if the character table of a group is given in the ATLAS, we denote its characters as they are denoted there. Note that, as in each of the ATLAS tables considered, of a pair of complex conjugate faithful characters only one member is printed, by a slight abuse of notation, more than one character may have the same label. Here we distinguish these characters by writing $\bar{\chi}$ for the character whose proxy is $\chi$.

In the case of $6\mathcal{A}_6$ we deduce from the character degrees that any faithful imprimitive character is necessarily induced from a maximal subgroup isomorphic to $3 \times \mathcal{A}_5$. Indeed, an analysis of the character table of this group with GAP shows that each faithful character of degree two induces to either $\chi_{21}$ or $\chi_{22}$ (or its complex conjugates). Since there is no maximal subgroup of $3 \times \mathcal{A}_5$ of index two, this yields minimal triples in the sense of Theorems 1.1 and 1.2.

By the same arguments in the case of $3\mathcal{A}_6$, we have two isomorphism types of maximal subgroups to consider: For $3 \times \mathcal{A}_5$ the two non-trivial linear characters induce irreducibly to give both faithful characters $\chi_{16}$ and $\bar{\chi}_{16}$. Further imprimitive characters arise by inducing both linear characters of $3 \times S_4$ which have order six. This yields $\chi_{18}$ and $\bar{\chi}_{18}$ of $3\mathcal{A}_6$.

Our treatment of $6\mathcal{A}_7$ and $3\mathcal{A}_7$ is the same: Again by examining the character degrees of $6\mathcal{A}_7$, we conclude that no faithful character is imprimitive. The situation is somewhat different for $3\mathcal{A}_7$, however. Here we have three isomorphism types of maximal subgroups to consider which contribute to imprimitive characters of $3\mathcal{A}_7$: Inducing any faithful character of degree 3 of $3\mathcal{A}_6$ gives an irreducible character of degree 21. More precisely, both characters $\chi_{14}$ and $\bar{\chi}_{15}$ of $3\mathcal{A}_6$ induce to the character $\chi_{21}$ of $3\mathcal{A}_7$. This again yields minimal triples, since there is no maximal subgroup of $3\mathcal{A}_6$ of index 3. Also, the characters $\chi_{19}$ and $\bar{\chi}_{20}$ of $3\mathcal{A}_7$ are induced linear characters of order six of $3 \times S_5$. And lastly, the faithful characters $\chi_{19}$ and $\bar{\chi}_{19}$ of $3\mathcal{A}_7$ are induced non-trivial linear characters of a maximal subgroup isomorphic to $3 \times L_2(7)$.

We summarize the above in the following theorem.

**Theorem 5.8.** For $3\mathcal{A}_6$ the minimal triples $(H, \varphi, \chi)$ are as follows:

(i) $H = 3 \times \mathcal{A}_5$, $\varphi$ is a non-trivial linear character, and $\chi$ is either $\chi_{16}$ or $\bar{\chi}_{16}$.
(ii) $H = 3 \times S_4$, $\varphi$ is a linear character of order six, and $\chi$ is $\chi_{18}$ or $\bar{\chi}_{18}$.

For $3\mathcal{A}_7$ we have

(i) $H = 3\mathcal{A}_6$, $\varphi \in \{\chi_{14}, \chi_{15}\}$, and $\chi = \chi_{21}$, or $\varphi \in \{\bar{\chi}_{14}, \bar{\chi}_{15}\}$ and $\chi = \bar{\chi}_{21}$.
(ii) $H = 3 \times S_5$, $\varphi$ is a linear character of order six, and $\chi$ is either $\chi_{20}$ or $\bar{\chi}_{20}$.
(iii) $H = 3 \times L_2(7)$, $\varphi$ is a non-trivial linear character, and $\chi$ is $\chi_{19}$ or $\bar{\chi}_{19}$.

For $6\mathcal{A}_6$ we have

(i) $H = 3 \times \mathcal{A}_5$, $\varphi$ has degree two, and $\chi$ is either $\chi_{21}$ or $\chi_{22}$ or one of their complex conjugates.

None of the faithful ordinary characters of $6\mathcal{A}_7$ are imprimitive.
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