The permanent functions of tensors

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Abstract By a tensor we mean a multidimensional array (matrix) or hypermatrix over a number field. This article aims to set an account of the studies on the permanent functions of tensors. We formulate the definitions of $1$-permanent, $2$-permanent, and $k$-permanent of a tensor in terms of hyperplanes, planes and $k$-planes of the tensor; we discuss the polytopes of stochastic tensors; at end we present an extension of the generalized matrix function for tensors.

AMS Classification: 15A15, 15A02, 52B12

Keywords: Birkhoff-von Neumann Theorem, doubly stochastic matrix, hypermatrix, matrix of higher order, multidimensional array, permanent, polytope, stochastic tensor, tensor

1 Introduction

The study on multidimensional arrays (or matrices) may date back as early as the nineteenth century by Cayley \cite{7, 8}. Jurkat and Ryser revived the topic in their seminal paper \cite{21} in 1968 in which they investigated configurations and decompositions for multidimensional arrays. Jurkat and Ryser’s work was followed by a great deal of research on the topic, mainly on the combinatorial aspects of certain types (such as stochasticity) of multidimensional arrays; see, e.g., Brualdi and Csima \cite{3, 5}. In recent years, multidimensional arrays are found applications in practical fields such as image processing (see, e.g., Qi and Luo \cite{32}), theory of computing (see, e.g., Cifuentes and Parrilo \cite{12}), and physics (see, e.g., Tichy \cite{36}). We are concerned with the permanent functions of multidimensional arrays. Our purpose is to set an account on the specific topic based on publications, including, in particular, the ones by Dow and Gibson \cite{19} and Taranenko \cite{35}. The results are expositiorily presented with explanations other than in the format of theorem-proofs. Some results are easy observations; they are not necessarily new. For the determinants of multidimensional arrays, hyperdeterminants, and related topics, see, e.g., \cite{19, 20, 26, 34}.

Let $n_1, n_2, \ldots, n_d$ be positive integers. We write $A = (a_{i_1i_2\cdots i_d})$, $i_k = 1, 2, \ldots, n_k, k = 1, 2, \ldots, d$, for an $n_1 \times n_2 \times \cdots \times n_d$ multidimensional array or hypermatrix of order $d$ (the number of indices). Multidimensional arrays, or
hypermatrices, or matrices of higher orders, are referred to as tensors; see, e.g., [15, 24, 32]. So, by a tensor we mean a multidimensional array. The tensors of order 1 (i.e., \( d = 1 \)) are vectors in \( \mathbb{R}^{n_1} \), while the 2nd order tensors are just regular \( n_1 \times n_2 \) matrices. A 3rd order tensor, i.e., an \( n_1 \times n_2 \times n_3 \) tensor, may be viewed as a book of \( n_3 \) pages (slices), each page is an \( n_1 \times n_2 \) matrix.

If \( n_1 = n_2 = \cdots = n_d = n \), we say that \( A \) is of order \( d \) and dimension \( n \) or we say that \( A \) is an \( n \times \cdots \times n \) tensor. We also call an \( n \times n \times n \) tensor (i.e., of order 3 and dimension \( n \)) a tensor cube or a 3D matrix. We refer to the permanents of multidimensional arrays as the permanents of tensors, or hyperpermanents.

Following the line of Dow and Gibson [16], we will begin with the definitions of 1-permanent, 2-permanent, and \( k \)-permanent of tensors. 1-permanents, the most modest ones, are useful in studying hypergraphs (see, e.g., [16, 35]), the 2-permanents with \( d = 3 \) or of special relation of \( d \) and \( n \) are found applications in projective planes (see, e.g., [16]) and polytope theory (see, e.g., [9, 14, 25]), while \( k \)-permanents are certainly an object in combinatorics themselves.

**Remark 1** We adopt Lim’s terminology in [26] (see also [24, 32]), calling an \( n \times \cdots \times n \) tensor a tensor of order \( d \) and dimension \( n \). Such a tensor is also said to be of order \( n \) and dimension \( d \) in the literature; see, e.g., [5, 35].

### 2 The definitions of permanents of tensors

#### 2.1 1-permanent and 2-permanent

Let \( A = (a_{i_1 \ldots i_d}) \) be an \( n_1 \times \cdots \times n_d \) tensor of order \( d \) with real entries. Dow and Gibson [16] defined (over a commutative ring) the permanent of \( A \) as

\[
\text{per}(A) = \sum_{\sigma} \prod_{i=1}^{n_1} a_{\sigma_1(i) \ldots \sigma_d(i)},
\]

where the summation runs over all one-to-one functions \( \sigma_k \) from \( \{1, 2, \ldots, n_1\} \) to \( \{1, 2, \ldots, n_k\} \), \( k = 2, 3, \ldots, d \), with \( \text{per}(A) = 0 \) if \( n_1 > n_k \) for some \( k \).

Note: under the definition (1), if \( A \) is an \( n_1 \times n_2 \) matrix and \( n_1 > n_2 \), then \( \text{per}(A) = 0 \), but \( \text{per}(A^t) \) need not be 0, where \( A^t \) is the transpose of \( A \). This is not in agreement with the fact that a matrix (i.e., order 2 tensor) and its transpose have the same permanent. We may slightly modify and extend the definition (1) as follows. Let \( n = \min\{n_1, n_2, \ldots, n_d\} = n_j \) for some \( j \). Then

\[
\text{per}(A) = \sum_{\sigma} \prod_{i=1}^{n} a_{\sigma_1(i) \ldots \sigma_j(i) \ldots \sigma_d(i)},
\]

where the summation runs over all one-to-one functions \( \sigma_k \) from \( \{1, 2, \ldots, n\} \) to \( \{1, 2, \ldots, n_k\} \), \( k \neq j \), and \( \sigma_j \) is the identity map. (2) reduces to (1) if \( n = n_1 \). It is immediate by definition (2) that \( \text{per}(A) = \text{per}(A^t) \) for rectangular matrices.
When \( n_1 = n_2 = \cdots = n_d = n \), (1) can be written in a symmetric form

\[
\text{per}(A) = \frac{1}{n!} \sum_{\pi_1, \ldots, \pi_d \in S_n} \prod_{i=1}^{n} a_{\pi_1(i) \cdots \pi_d(i)},
\]

(3)

where \( S_n \) is the symmetric group of degree \( n \).

If \( d = 2 \), (3) reduces to the usual permanent for square matrices.

The definition (1) of permanent is in fact the so-called 1-permanent (or 1-per for short) of the tensor \( A \). 1-per (v.s. \( k \)-per; see the definition below or see [16, Sec. 4]) of \( A \) is the sum of all products of \( n_1 \) entries of \( A \) no two of which are taken from the same hyperplane (of order \( d - 1 \); see Sec. 2.2). In the case of 3D matrices, the planes of \( A \) are the submatrices obtained by fixing one of \( i, j, k \), and the lines of \( A \) are the submatrices obtained by fixing two of \( i, j, k \).

Figure 1: 2 \( \times \) 2 \( \times \) 2 tensors and their flattened frontal slices

\[ A = \begin{pmatrix} a_{111} & a_{121} & a_{112} & a_{122} \\ a_{211} & a_{221} & a_{212} & a_{222} \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}. \]

For example, take \( A = (a_{ijk}) \), \( i, j, k = 1, 2 \). Then

\[
1\text{-per}(A) = a_{111}a_{222} + a_{211}a_{122} + a_{121}a_{212} + a_{221}a_{112}, \\
2\text{-per}(A) = a_{111}a_{221}a_{122}a_{212} + a_{211}a_{121}a_{112}a_{222}.
\]

Note that we use the same symbol for the tensor and its frontal slice flattening as there is no confusion will be caused in this paper.

By a \((0,1)\)-tensor, we mean a tensor in which every entry is either 0 or 1. For the 2 \( \times \) 2 \( \times \) 2 \((0,1)\)-tensor \( B \) in Fig. 1 on the right hand side, 1-per \((B) = 0\), 2-per \((B) = 1\). If \( I_3 \) is the 3 \( \times \) 3 \( \times \) 3 identity tensor, i.e., the entry in the \((i, i, i)\) position is 1 for every \( i \) and everywhere else is 0, then 1-per \((I_3) = 1\) and 2-per \((I_3) = 0\). Let \( J_3 \) be the 3 \( \times \) 3 \( \times \) 3 tensor of 1s (i.e., all entries are 1). Then 1-per \((J_3) = 9 \cdot 4 = (3!)^2 = 36\) and 2-per \((J_3) = 6 \cdot 2 = 12\).

1-permanent and 2-permanent are the most important permanent functions of tensors. We write per \((\cdot)\) for 1-per \((\cdot)\) and Per \((\cdot)\) for 2-per \((\cdot)\). We simply call 1-permanent permanent in the sense of (1) unless otherwise stated.
The following observations are immediate for permanents (i.e., 1-permanents):
(i) the permanent function of tensors is linear with respect to each hyperplane;
(ii) interchange of two hyperplanes does not change the permanent; (iii) if
\[ A = (a_{i_1i_2 \ldots i_d}) \]
is a tensor of order \( d \) and dimension \( n \) and \( A^\sigma = (a_{\sigma(i_1)i_2 \ldots i_{\sigma(d)}}) \) is the \( \sigma \)-transpose of \( A \), where \( \sigma \in S_d \), then \( A \) and \( A^\sigma \) have the same permanent;
(iv) if \( A = (a_{i_1i_2 \ldots i_d}) \) and \( B = (b_{i_1i_2 \ldots i_d}) \) are nonnegative tensors of the same size and \( A \leq B \) entrywise, then the permanent of \( A \) is less than or equal to the permanent of \( B \); and (v) the Laplace expansion theorem holds.

The classic Frobenius-König theorem (see, e.g., [38, p. 158]) states that for an \( n \times n \) nonnegative matrix \( A \), the permanent of \( A \) vanishes if and only if \( A \) contains an \( r \times s \) zero submatrix such that \( r + s = n + 1 \). The following result is an analog for tensors.

**Proposition 2 (Dow and Gibson [16])** Let \( A \) be an \( n_1 \times n_2 \times \cdots \times n_d \)-tensor. If \( A \) contains an \( m_1 \times m_2 \times \cdots \times m_d \) zero sub-tensor such that \( \sum_{k=1}^d m_k = 1 + \sum_{k=2}^d n_k \), then \( \per (A) = 0 \); but not conversely.

Consequently, for a nonnegative tensor of order \( d \) and dimension \( n \), if 1-permanent is positive, then \( \sum_{k=1}^d m_k \leq (d - 1)n \). The positivity of the permanent of a nonnegative tensor of order \( d \) and dimension \( n \) is characterized in terms of term rank in [6]: it is positive if and only if the term rank is \( n \).

Lower and upper bounds for the permanents of \((0,1)\)-tensors (or matrices, or Latin squares, etc) are always interesting and challenging. Shown below is a lower bound of the permanent, given the number of zero entries.

**Proposition 3 (Dow and Gibson [16])** Let \( A \) be an \( n \times n \times \cdots \times n \) \((0,1)\)-tensor of order \( d \) with exactly \( t \) 0s. Then \( \per (A) \geq \left( n^{d-1} - t \right) \left( (n - 1)! \right)^{d-1} \).

The well-known Minc-Bregman theorem on a \((0,1)\)-matrix gives an upper bound for the permanent of the \((0,1)\)-matrix in terms of the numbers of 1s on each row (or column). For tensors, we have the following.

**Proposition 4 (Dow and Gibson [17])** Let \( A = (a_{ijk}) \) be an \( n \times n \times n \) \((0,1)\)-tensor. Let \( r_i = \sum_{j,k} a_{ijk} \) for \( i = 1, 2, \ldots, n \). Then the Minc-Bregman type inequality for 1-permanent holds:

\[ \per (A) \leq \prod_{i=1}^n (r_i l)^{1/r_i} . \]

**Proposition 5 (Dow and Gibson [17])** Let \( A = (a_{ijk}) \) be an \( n \times n \times n \) \((0,1)\)-tensor. Let \( r_{ij} = \sum_k a_{ijk} \) for \( i, j = 1, 2, \ldots, n \). Then the Minc-Bregman type inequality for 2-permanent holds:

\[ \Per (A) \leq \prod_{i=1}^n (r_{ij} l)^{1/r_{ij}} . \]
A permutation tensor is a (0,1)-tensor in which every hyperplane contains one and only one 1. In particular, the usual permutation matrices are permutation tensors of order 2; the identity tensor $I_n$ (all entries on the main diagonal $(i,i,i)$, $i = 1, 2, \ldots, n$, are 1) is a permutation tensor of order 3. (Note: permutation tensors are defined differently in the literature; see, e.g., [27].) Let $\Omega_n^d$ be the convex hull of the permutation tensors of order $d$ and dimension $n$. An analog of the Van der Waerden conjecture (see, e.g., [30]) for tensors is surely appealing. Dow and Gibson [16] conjectured that if $A = (a_{i_1i_2\ldots i_d}) \in \Omega_n^d$, then
\[
\text{per}(A) \geq \frac{(n! / n^n)^{d-1}}{n^{d-1}}
\]
with equality if and only if $A = (1/n^{d-1})J_n$, where $J_n$ is the tensor of all 1s.

This is disproved by Taranenko [35, Proposition 4, p. 590]. Taranenko presented as many as 13 conjectures concerning permanents and stochastic polytopes in [35]. We single out a couple that are easily stated and understood.

**Conjecture 6 (Taranenko [35])** Let $A$ be a $d \times \cdots \times n$ line-stochastic tensor. If $d$ is even, then $\text{per}(A) > 0$.

**Conjecture 7 (Taranenko [35])** Let $A$ be a $d \times \cdots \times n$ line-stochastic tensor. If $n$ is odd, then $\text{per}(A) > 0$.

For more discussions on this, see Theorems 19 and 22 of [35].

**2.2 $k$-permanent and the Hadamard product**

Let $A = (a_{i_1i_2\ldots i_d})$ be a tensor of order $d$. For $1 \leq k \leq d$, let $f = d - k$. If we fix $f$ of the indices $i_1, i_2, \ldots, i_d$ and let the rest $k$ indices vary, then we obtain a sub-tensor of $A$. We call such a sub-tensor a $k$-plane of $A$ (see [16, 35]). 1-plane (1 free index) is referred to as a line (or fiber or tube); 2-plane (2 free indices) is simply a plane; a $(d-1)$-plane of an order $d$ tensor is usually called a hyperplane.

Dow and Gibson [16] defined the $k$-permanent of $A$, denoted by $k\text{-per}(A)$, to be the sum of all possible products of $n^k$ entries of $A$ so that no two entries are taken from the same $(d-k)$-plane [16, p. 142]. If such a selection of entries of $A$ does not exist, then we write $k\text{-per}(A) = 0$.

**Remark 8** The existence of such selections of the entries of $A$ is extensively studied (see, e.g., [5, 11, 21, 28, 33]) and it is in the area of configurations and block designs in combinatorics (see, e.g., [13]).

For $2 \times 2 \times 2$ tensors, we have demonstrated 1-per and 2-per in the previous examples. Permanents defined by [1] always exist. Let $A = (a_{ijst})$ be a $2 \times 2 \times 2$ tensor. The 2-per$(A)$ is the sum of products of $n^k = 2^2 = 4$ entries of $A$ that are not in the same $d - k = 4 - 2 = 2$-plane. Such a selection of entries is
impossible for four sequences $i, j, s, t$ of length 4 whose components are 1 or 2: $a_{i_1i_2i_3i_4}a_{j_1j_2j_3j_4}a_{s_1s_2s_3s_4}a_{t_1t_2t_3t_4}$. Thus,
\[2\text{-per } (A) = \sum_{i} a_{i}a_{j}a_{s}a_{t} = 0.\]

Let $A = (a_{i_1i_2...i_d})$ be an $n \times \cdots \times n$ tensor and let $1 \leq k < d$. A $k$-per diagonal of $A$ consists of $n^k$ entries of $A$; each entry is from a $(d-k)$-plane and no two entries are from the same $(d-k)$-plane. A 1-per diagonal is simply called a diagonal; that is, a diagonal of a tensor of dimension $n$ consists of $n$ entries, no two are from the same hyperplane. For $d = 2$, $k = 1$, a 1-per diagonal of $A$ consists of $n$ entries of $A$ from different lines (i.e., rows and columns). For $d = 3$, $k = 1$, a 1-per diagonal of $A$ consists of $n$ entries of $A$, each of which is from 1-plane, no two fall on the same plane. For $d = 3$, $k = 2$, a 2-per diagonal of $A$ consists of $n^2$ entries of $A$ each plane contains exactly $n$ entries of $A$.

We say that $A$ is $k$-per feasible if it is possible to choose $n^k$ entries of $A$, no two in the same $(d-k)$-plane. Such a selection of the $n^k$ entries comprises of a $k$-per diagonal of $A$. The $k$-per diagonal of $A$ can be extracted by the Hadamard (Schur or entrywise) product of $A$ with a $(0,1)$-tensor $D$ of the same size (order and dimension) as $A$ in which the $k$-per diagonal entries of $D$ in the same positions as the $k$-per diagonal of $A$ are 1s and 0s elsewhere. We call such a $(0,1)$-tensor $D$ a $k$-per index tensor (or a $k$-per cell). That is, a $k$-per index tensor is a $(0,1)$-tensor of order $d$ and dimension $n$ which contains $n^k$ 1s so that no two 1s are located in the same $(d-k)$-plane. Let $\prod(A \circ D)$ be the product of the $k$-per diagonal entries of $A$ indexed by $D$. Denote by $\mathcal{P}_{d,n,k}$, or simply $\mathcal{P}_k$, the set of $k$-per index tensors. (Note: again, the existence of a $k$-per index tensor for a given $k$ is a problem of configuration which is not a concern of this paper. For the study of the existence of $(0,1)$-tensors with a fixed number of 1s on a $k$-plane, see, e.g., [11, 13, 33].)

We formulate the $k$-permanent of tensor $A$ [10] as follows.

**Proposition 9** Let $A = (a_{i_1i_2...i_d})$ be an $n \times \cdots \times n$ tensor, $1 \leq k < d$. Then
\[k\text{-per } (A) = \sum_{D \in \mathcal{P}_k} \prod_{i} (A \circ D).\]

**Proposition 10** Let $A = (a_{i_1i_2...i_d})$ be an $n \times \cdots \times n$ tensor, $1 \leq k < d$. Then
\[k\text{-per } (cA) = c^k \left( k\text{-per } (A) \right), \quad \text{where } c \text{ is a constant.}\]

The following result states that every $k$-per can be converted to a 1-per.

**Proposition 11** (Dow and Gibson [10]) Let $A$ be a tensor of order $d$ and dimension $n$, $1 \leq k < d$. Then there exists an $n^k \times n^k \times \cdots \times n^k$ tensor $B$ of order $\binom{d}{k}$ whose nonzero entries are equal to the nonzero entries of $A$ such that
\[k\text{-per } (A) = 1\text{-per } (B).\]
Remark 12 Different generalizations of the permanents from matrices to tensors exist. Taranenko [35] defined \( r \)-permanents, \( \text{per}_r \), of multidimensional matrices by the Maximum Distance Separable (MDS) codes with distance \( r \). In [35], the permanent is in fact the \( d \)-permanent, that is, \( \text{per}(A) = \text{per}_d(A) \), which is the same as the 1-permanent in [16], namely our (1). More generally, if \( r + s = d + 1 \), then the \( r \)-permanent in [35] is just the \( s \)-permanent in [16]. For \( n \times n \times n \) tensors, the 2-permanents defined in [16] and in [35] turn out to be the same, namely, 2-per \( \text{per}(A) = \text{per}_2(A) \). However, for \( 2 \times 2 \times 2 \times 2 \) tensors, 2-per \( \text{per}(A) \neq \text{per}_2(A) \).

2.3 Permanents and the Hamming distance

Let \( x = (x_1, x_2, \ldots, x_n) \), \( y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n \). The Hamming distance, denoted by \( \rho(x, y) \), of \( x \) and \( y \) is the number of nonzero components of \( x - y \).

Take \( x = (1, 2, 3) \), \( y = (1, 3, 2) \). Then \( x - y = (0, -1, 1) \). Thus, \( \rho(x, y) = 2 \).

Denote \( \mathcal{I}_n^d = \{(i_1, i_2, \ldots, i_d)\} \), where \( 1 \leq i_k \leq n \) for \( k = 1, 2, \ldots, d \). Let \( A = (a_{i_1i_2\ldots i_d}) \). We write \( A = (a_i) \), where \( a_i = a_{i_1i_2\ldots i_d} \), \( i = (i_1, i_2, \ldots, i_d) \in \mathcal{I}_n^d \). For \( i = (i_1, \ldots, i_d) \) and \( j = (j_1, \ldots, j_d) \), \( \rho(i, j) = d \) implies that the corresponding components of \( i \) and \( j \) are pairwise distinct. Taranenko [35] defined the permanent of a tensor using Hamming distance, which is essentially the same as (1), i.e., the 1-permanent of [16], namely the \( d \)-permanent of [35].

Proposition 13 Let \( A = (a_{i_1i_2\ldots i_d}) \) be an \( n \times \ldots \times n \) tensor. Then

\[
\text{per}(A) = \sum_{\alpha, \alpha' \in \mathcal{I}_n^d, \rho(\alpha, \alpha') = d, i \neq j} a_{\alpha_1}\alpha_2 \cdots a_{\alpha_n}. \tag{4}
\]

Let \( A = (a_{ij}) \) be a \( 2 \times 2 \times 2 \times 2 \) tensor. Then \( \text{per}(A) \) is the sum of all products of \( n^k = 2^4 = 16 \) entries of \( A \) that are not in the same \( d - k = 4 - 1 = 3 \)-plane. It follows from (4) that

\[
\text{per}(A) = \sum_{\rho(\alpha, \beta) = 4} a_\alpha a_\beta = a_{1111}a_{2222} + a_{1112}a_{2221} + a_{1121}a_{2212} + a_{1211}a_{2122} + a_{2111}a_{1222} + a_{1122}a_{2111} + a_{1212}a_{2121} + a_{1221}a_{2112}.
\]

Note that the 2-per \( \text{per}(A) \) is the sum of all products of \( n^k = 2^2 = 4 \) entries of \( A \) that are not in the same \( d - k = 4 - 2 = 2 \)-plane. It is impossible for four sequences \( \alpha, \beta, \gamma, \delta \) of length 4 whose components are 1 or 2 have \( \rho(p, q) \geq 3 \) for all pairs \( p \) and \( q \) from \{\( \alpha, \beta, \gamma, \delta \} \). Thus, \( \text{Per}(A) = 0 \).

2.4 The permanents of 3D matrices (i.e., \( n \times n \times n \) tensors)

For \( \alpha, \beta \in S_n \), we may regard \( \alpha \) and \( \beta \) as sequences in \( \mathbb{R}^n \): \( \alpha = (\alpha(1), \ldots, \alpha(n)) \) and \( \beta = (\beta(1), \ldots, \beta(n)) \). If \( \rho(\alpha, \beta) = n \), then \( \alpha(i) \neq \beta(i) \) for all \( i \).
Let \( A = (a_{ijk}) \) be a 3D matrix (i.e., a tensor of order 3 and dimension \( n \)), namely, \( A \) is an \( n \times n \times n \) tensor. A diagonal of \( A \) consists of \( n \) entries, each of which is taken from a plane and no two entries are from the same plane (as \( d - k = 3 - 1 = 2 \)); A triagonal (\cite[p. 181]{18}) of \( A \) consists of \( n^2 \) entries, each of which is taken from a line and no two are from the same line. Then 1-per (\( A \)) is the sum of products of diagonal entries. Thus (\cite{17})

\[
\text{1-per}(A) = \sum_{\alpha, \beta \in S_n} \prod_{i=1}^{n} a_{\alpha(i)\beta(i)}, (5)
\]

while 2-per (\( A \)) is the sum of products of triagonal entries. So,

\[
\text{2-per}(A) = \sum \prod(n^2 \text{ entries of } A; \text{no two are collinear}).
\]

For \( 3 \times 3 \times 3 \) tensors, 1-permanent is the sum of all products of 3 elements, no two are on the same plane (frontal, lateral or horizontal \cite{24}), i.e., one element from each plane, while 2-permanent is the sum of all products of 9 entries any two of which are non-collinear (in any direction), i.e., one entry from each line.

Let \( A = (a_{ijk}) \) be an \( n \times n \times n \) tensor. If we denote (or label) the \( k \)th frontal page of \( A \) by \( \pi_k \in S_n \), we can write a triagonal \( a_\pi \) of \( A \) as

\[
a_\pi = (a_{\pi_1}, a_{\pi_2}, \ldots, a_{\pi_n})
\]

where \( \rho(\pi_i, \pi_j) = n \) whenever \( i \neq j \). Let \( D(a_\pi) = \prod_{i=1}^{n} D(a_{\pi_i}) \) for the product of the triagonal entries. Then (see, e.g., \cite{9 14}), we have

**Proposition 14** Let \( A = (a_{ijk}) \) be an \( n \times n \times n \) tensor. Then

\[
\text{Per}(A) = 2\text{-per}(A) = \sum_{\pi} \prod D(a_\pi) = \sum_{\pi_1, \pi_2, \ldots, \pi_d \in S_n} \prod_{i=1}^{n} D(a_{\pi_i}).
\]

It is easy to see that there are \( n^2 \cdot (n - 1)^2 \ldots 2^2 \cdot 1^2 = (n!)^2 \) permutation tensors of order 3 and dimension \( n \). Let \( L_n \) be the number of Latin squares of order \( n \) and let \( J^3_n \) denote the \( n \times n \times n \) tensor of all 1s. Then \( \text{Per}(J^3_n) \) is equal to the number of triagonals of \( A = (a_{ijk}) \). Observe that every triagonal of \( J^3_n \) corresponds solely to a Latin square of order \( n \) (see, e.g., \cite{21}).

**Proposition 15** A 3D matrix of dimension \( n \) has \( L_n \) triagonals.

## 3 Stochastic tensors

### 3.1 Line, plane, \( k \)-stochastic, and permutation tensors

Recall the celebrated Birkhoff-von Neumann theorem on the polytope of doubly stochastic matrices (see, e.g., \cite[p.159]{38}). It states that an \( n \times n \) matrix is doubly
stochastic if and only if it is a convex combination of some \( n \times n \) permutation matrices. This result is about the matrices that are 2-way stochastic. What would be the mathematical objects that are 3-way stochastic?

Let \( A = (a_{ijk}) \) be an \( n \times n \times n \) tensor. \( A \) is said to be triply line stochastic \[14\] (or stochastic semi-magic cube \[1\]) if all \( a_{ijk} \geq 0 \) and

\[
\sum_{i=1}^{n} a_{ijk} = 1, \quad \sum_{j=1}^{n} a_{ijk} = 1, \quad \sum_{k=1}^{n} a_{ijk} = 1.
\]

That is, each of horizontal, lateral and frontal slices (see \[24\]) is a doubly stochastic matrix. For a nonnegative tensor \( A = (a_{i_1i_2...i_d}) \) of order \( d \) and dimension \( n \), we say \( A \) is line-stochastic \[18\] if the sum of the entries on each line is 1:

\[
\sum_{i=1}^{n} a_{i_1i_2...i_d} = 1.
\]

Equivalently, every plane (i.e., 2-plane) of \( A \) is doubly stochastic, namely, for \( e = (1, 1, \ldots, 1)^t \in \mathbb{R}^n \), every \( n \times n \) matrix with \((i, j)\) entry \( a_{i_1i_2...i_d} \) satisfies

\[
(a_{i_1i_2...i_d})e = e, \quad e^t(a_{i_1i_2...i_d}) = e^t.
\]

We say that \( A \) is plane-stochastic \[4\] if the sum of all elements on every plane is equal to 1, that is,

\[
\sum_{i,j=1}^{n} a_{i_1i_2...i_d} = 1.
\]

More generally, let \( A \) be a nonnegative tensor of order \( d \) and dimension \( n \) and let \( 1 \leq k \leq d \). If the sum of the entries of \( A \) on every \( k \)-plane is 1, then \( A \) is said to be \( k \)-stochastic (see, e.g., \[3\] \[33\]). A \( k \)-stochastic \((0,1)\)-tensor is called a \( k \)-permutation tensor (or a permutation tensor of degree \( k \); for its existence, see Remark \[8\]). Being line stochastic is 1-stochastic; being 2-stochastic is plane-stochastic; and a 1-permutation tensor is nothing but a line-permutation tensor, while a 2-permutation tensor is a plane-permutation tensor. In case of \( n \times n \times n \), 1-permutation tensor has 1s on its diagonal and 2-permutation tensor has 1s on its triagonal. The \((d-1)\)-permutation tensors (of order \( d \) and dimension \( n \)) are simply called permutation tensors \[16\].

Let \( P \) and \( Q \) be \( n \times n \) permutation matrices. We say that \( P \) and \( Q \) are diagonally disjoint (or Hadamard orthogonal) if no 1 appears in the same (overlapping) position of \( P \) and \( Q \), that is, the Hadamard product \( P \circ Q = 0 \).

**Proposition 16** Let \( P_1, P_2, \ldots, P_n \) be \( n \times n \) permutation matrices and \( \pi_1, \pi_2, \ldots, \pi_n \) be the corresponding elements (via group isomorphism) in the symmetric group \( S_n \). The following statements are equivalent:

1. The tensor with frontal slice flattening \([P_1|P_2|\cdots|P_n]\) is an \( n \times n \times n \) line-permutation tensor.
2. $P_1, P_2, \ldots, P_n$ are mutually diagonally disjoint.

3. $\rho(\pi_i, \pi_j) = n$ for all $i \neq j$.

4. $P_1 + P_2 + \cdots + P_n = J$ (where $J$ is the matrix of 1s).

For an analog for $n \times n \times n$ plane-permutation tensors, let $Q_1, Q_2, \ldots, Q_n$ be $n \times n$ permutation matrices. Then the $n \times n \times n$ (0,1)-tensor $R$ with frontal slice flattening $[Q_1|Q_2| \cdots |Q_n]$ is a plane-permutation tensor if and only if each of the plus-projections (by adding the elements) $f_i(R)$, $f_j(R)$, and $f_k(R)$ of $R$ along $i$, $j$, and $k$-axes is an $n \times n$ permutation matrix.

### 3.2 Polytopes of stochastic tensors

The Birkhoff-von Neumann Theorem asserts that the set of the doubly stochastic matrices and the convex hull of the permutation matrices coincide. In other words, the permutation matrices are precisely the vertices (extreme points) of the polytope of doubly stochastic matrices. This is usually proven by the Frobenius-König theorem (see, e.g., [38, p.158]).

The Birkhoff-von Neumann Theorem does not generalize to tensors of higher dimensions. The $3 \times 3 \times 3$ line-stochastic tensor $D$ in Fig. 2 is not a combination of line-permutation tensors; in fact, it is an extreme point of the polytope of $3 \times 3 \times 3$ line-stochastic tensors. Moreover, $\text{Per}(D) = 0$. Let

- $\Delta^n_\ell$ be the convex hull of $n \times n \times n$ line-permutation tensors.
- $\Delta^n_\wp$ be the convex hull of $n \times n \times n$ plane-permutation tensors.
- $\Omega^n_\ell$ be the set of all $n \times n \times n$ line-stochastic tensors.
- $\Omega^n_\wp$ be the set of all $n \times n \times n$ plane-stochastic tensors.

The $\Delta$s and $\Omega$s are polytopes in $\mathbb{R}^{n^3}$. It is tempting to know the structures and the numbers of the extreme points of the polytopes $\Delta$s. Obviously,

$$\Delta^n_i \subseteq \Omega^n_i,$$ where $i = \ell$ or $\wp$.

The following facts are known or easy to obtain:

1. For $n = 2$, $\Delta^2_\ell = \Omega^2_\ell$. That is to say, every $2 \times 2 \times 2$ line-stochastic tensor is a convex combination of the two (0,1) line-stochastic tensors.

2. For $n = 2$, $\Delta^2_\wp$ is a proper subset of $\Omega^2_\wp$. Take $C = (c_{ijk})$ with $c_{111} = c_{121} = c_{112} = c_{222} = \frac{1}{2}$, and 0 everywhere else. $C$ is not a convex combination of the plane-permutation tensors. $\Omega^2_\wp$ has 6 extreme points, 4 of which are (0,1)-tensors and 2 are non-(0,1) (with entries 1/2); see [4, 10, 23, 33].

The cube on the left in Fig. 2 represents the tensor $C$, while shown below is its frontal slice flattening. (Likewise, the other cube is for tensor $D$.)

$$C = \frac{1}{2} \begin{pmatrix} 0 & 1 & \vdots & 1 & 0 \\ 1 & 0 & \vdots & 0 & 1 \end{pmatrix}.$$
3. For $n = 3$, the polytope $\Omega_3^\ell$ has 66 vertices, of which 12 are line-permutation tensors (due to the fact that there are 12 Latin squares of order 3), 54 are non-(0,1) (with entries $1/2$). $\Delta_3^\ell$ is a proper subset of $\Omega_3^\ell$ because tensor $D$ is not a convex combination of line-permutation tensors (see, e.g., [9]). Moreover, for the line-stochastic tensor $D$, we have $\text{Per}(D) = 0$. This says, unlike the permanent of a doubly stochastic matrix, that the 2-permanent (i.e., $\text{Per}$) of a triply line-stochastic tensor may vanish.

$$D = \frac{1}{2} \begin{pmatrix} 0 & 1 & 1 \vdots & 1 & 1 & 0 \vdots & 1 & 0 & 1 \\ 1 & 1 & 0 \vdots & 0 & 1 & 1 \vdots & 1 & 0 & 1 \\ 1 & 0 & 1 \vdots & 1 & 0 & 1 \vdots & 0 & 2 & 0 \end{pmatrix}.$$ 

4. For $n = 3$, $\Delta_3^\ell$ is a proper subset of $\Omega_3^\ell$. A complete list of the extreme points of $\Omega_3^\ell$, up to equivalence, is available in [3].

**Question 17** What would be the minimums of the permanents on the sets $\Delta_s$?

**Proposition 18** Let $A = (a_{ijk})$ be an $n \times n \times n$ nonnegative tensor. If a plus-projection (by adding the elements) $f_i(A)$, $f_j(A)$, or $f_k(A)$ of $A$ along $i$, $j$, or $k$-axis contains a 0, then $\text{Per}(A) = 0$. (The converse is not true.)

If $R$ is a nonnegative tensor such that ($k$-per or) $\text{Per}(R) > 0$, then for any nonnegative tensor $S$ of the same size, ($k$-per, resp.) $\text{Per}(R + S) \geq \text{Per}(R) > 0$.

**Proposition 19** Let $P_0 = \{A \in \Omega_n^\ell \mid \text{Per}(A) = 0\}$ and let $P$ and $Q$ be in $P_0$. Then either everything between $P$ and $Q$ is contained in $P_0$ (i.e., $tP + (1-t)Q \in P_0$ for all $0 < t < 1$), or nothing between $P$ and $Q$ lies in $P_0$ (i.e., $tP + (1-t)Q \not\in P_0$ for all $0 < t < 1$).

4 Generalized tensor functions

Let $A = (a_{ij})$ be an $n \times n$ matrix. Let $G$ be a subgroup of $S_n$ and $\chi$ be a character on $G$. The classic determinant, permanent, and generalized matrix
function of $A$ are respectively defined by

$$\det A = \sum_{\beta \in S_n} (-1)^{\text{sgn}(\beta)} \prod_{i=1}^{n} a_{i(\beta(i))} = \frac{1}{n!} \sum_{\alpha, \beta \in S_n} (-1)^{\text{sgn}(\alpha)\text{sgn}(\beta)} \prod_{i=1}^{n} a_{\alpha(i)(\beta(i))},$$

$$\text{per} A = \sum_{\beta \in S_n} \prod_{i=1}^{n} a_{i(\beta(i))} = \frac{1}{n!} \sum_{\alpha, \beta \in S_n} \prod_{i=1}^{n} a_{\alpha(i)(\beta(i))},$$

$$d^n_G A = \sum_{\beta \in G} \chi(\beta) \prod_{i=1}^{n} a_{i(\beta(i))} = \frac{1}{|G|} \sum_{\alpha, \beta \in G} \chi(\alpha)\chi(\beta) \prod_{i=1}^{n} a_{\alpha(i)(\beta(i))}.$$

For a tensor $A = (a_{i_1,i_2,\ldots,i_d})$ of order $d$ and dimension $n$, Cayley’s combinatorial (v.s. geometric) hyperdeterminant of $A$ is defined to be

$$\det A = \frac{1}{n!} \sum_{\pi_1,\ldots,\pi_d \in S_n} \text{sgn}(\pi_1)\ldots\text{sgn}(\pi_d) \prod_{i=1}^{n} a_{\pi_1(i)\ldots\pi_d(i)}. \quad (6)$$

The reader is referred to [19, 26, 34, 20] for hyperdeterminants or the determinants of multidimensional matrices (tensors).

For a tensor $A = (a_{i_1,i_2,\ldots,i_d})$ of order $d$ and dimension $n$, the permanent (1-permanent) of $A$ is defined analogously as in (1) and (3). We now give a try to extend the notation to generalized tensor functions.

Let $A = (a_{i_1,i_2,\ldots,i_d})$ be a tensor of order $d$ and dimension $n$. Let $G = (G_1,G_2,\ldots,G_d)$ and $\chi = (\chi_1,\chi_2,\ldots,\chi_d)$, where $G_i$ is a subgroup of $S_n$ and $\chi_i$ is a character on $G_i$, $i = 1,2,\ldots,d$. We define

$$d^n_G A = \sum_{\beta \in G} \chi(\beta) \prod_{i=1}^{n} a_{i(\beta(i))} = \frac{1}{|G|} \sum_{\alpha, \beta \in G} \chi(\alpha)\chi(\beta) \prod_{i=1}^{n} a_{\alpha(i)(\beta(i))}.$$

(7)

Apparent, the determinant (6) and permanent (3) are special cases of (7). Like 2-permanent for $n \times n \times n$ tensors, we may define 2-$d^n_G$ as follows:

$$2-d^n_G A = \sum_{\rho(\pi_1,\pi_j) = n, i \neq j} \prod_{i=1}^{n} \chi_i(\pi_i) a_{\pi_i}. \quad (8)$$

Additionally, in regard to the $k$-permanent, we may define the $k$-generalized tensor functions ($k$-gtf). Let $f_k$ be a scalar-valued function defined on a domain that contains all $k$-per diagonals $A \circ D$ of $A$, where $D \in \mathcal{P}_k$ (see Sec. 2.2). Then

$$k$-gtf$A = \sum_{D \in \mathcal{P}_k} f_k(A \circ D) \prod (A \circ D). \quad (9)$$

The work of Merris [29] may be a hint for the study in this direction.

**Acknowledgement.** The work was done while the second author was visiting Shanghai University during his sabbatical leave from Nova Southeastern
University. The work of Wang was partially supported by the Natural Science Foundation of China (11571220); the work of Zhang was partially supported by an NSU PFRDG Research Scholar grant. This expository article was written based on the second author’s presentation at ICMAA in Da Nang, Vietnam, June 14-18, 2017. The authors appreciate the communications with C. Bu, L. Cui, S. Hu, L. Qi, A. Taranenko, Y. Wei, and G. Yu during the preparation of the manuscript.

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