Cosmological Production of Fermions in a Flat Friedman Universe with Linearly Growing Scale Factor: Exactly Solvable Model

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We consider an exactly solvable model for production of fermions in the Friedman flat universe with a scale factor linearly growing with time. Exact solution expressed through the special functions admit an analytical calculation of the number density of created particles. We also discuss in general the role of the phenomenon of the cosmological particle production in the history of universe.

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I. INTRODUCTION

Although a cosmological particle production [1–4] is a well known phenomenon [5–7], exactly solvable models [8–10] are of interest because they would allow comparing the different approximate calculation methods for the number of produced particles. One of the problems in this field is to define the in- and out-vacuum states if the metric has not an asymptotic form corresponding to that of the Minkowski space-time [5, 11].

As the approximate methods, the adiabatic vacuum [5], WKB series [12] and instantaneous Hamiltonian diagonalization [6] methods are widely used. Evidently, the later method does not define an exact vacuum state. As is shown in Ref. [13], a certain superposition of the states, which diagonalize the instantaneous Hamiltonian, is needed to obtain the exact vacuum state.

The method has been suggested in Ref. [14], which allows finding the exact vacuum state by minimization of some functional. Here, on basis of this method, we have found the exact vacuum states for the case of a flat universe with a linearly growing scale factor. Such a model has appeared in Ref. [15], where the universe driven by the vacuum has been considered. Although, that model in its present version contradicts the nucleogenesis theory, which insists on an existence of the radiation epoch, it is of theoretical interest as a toy model. It should be noted, that the scalar particle creation in such a Universe was considered earlier [16] (See e.g. Ref. [17]) on basis of the WKB approximation, and father exact solution have been obtained [10]. For fermions energy-momentum tensor have been calculated in Ref. [18].

Let us also emphasize, that the flat universe with a linearly growing scale factor differs substantially from the Milne-like empty closed universe in a sense that the later can be transformed to the region of the Minkowski space-time [19] and does not exhibit a particle production.

II. FERMION PRODUCTION

Let us consider the fermion in the expanding universe. After decomposition of the bispinor \( \psi(r, \eta) \) in the complete set of modes \( \psi_k(\eta) e^{ikr} \), the Lagrangian of the fermion field in the expanding universe [20] takes the form

\[
L = \sum_k \left( \frac{i a^3}{2} \psi_k^+ \partial_\eta \psi_k - \frac{i a^3}{2} \partial_\eta \psi_k^+ \psi_k^+ \right) - a^3 \psi_k^+ (\alpha k) \psi_k - a^4 m \psi_k^+ \beta \psi_k,
\]

where \( \eta \) is the conformal time, \( a(\eta) \) is the universe scale factor.

The equation of motion is

\[
iv'_k - (\alpha k) \psi_k + \frac{3a'}{2a} \psi_k - ma \beta \psi_k = 0,
\]

where the bispinor is quantized as

\[
\hat{\psi}_k = \hat{b}_{-k,s} \psi_{-k,s} + \hat{a}_{k,s} \psi_{k,s},
\]

where the bispinor is:

\[
\psi_{k,s}(\eta) = i \frac{\chi_{k,s}^+}{\sqrt{a^{3/2}}} \left( \begin{array}{c} \varphi_+^s \chi_s^+ \varphi_s^+ \\ \chi_s \varphi_+^s \varphi_s^+ \end{array} \right),
\]

where \( \varphi_+ = \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \) and \( \varphi_- = \left( \begin{array}{c} 0 \\ 1 \end{array} \right) \).
The bispinor $v_{k,s}$ is expressed as $v_{k,s} = i \gamma^0 \gamma^2 (\bar{u}_{k,s})^T$, where the symbol $T$ denotes a transposed vector, $\bar{u} = u^+ \gamma^0$ and the representation of the Dirac matrices is the same as in Refs. [21, 22]. The functions $\chi_k(\eta)$ satisfy

\[
\chi''_k + (k^2 + m^2 a^2 - ima') \chi_k = 0,
\]

\[
k^2 \chi_k \chi_k^* + (am\chi_k^* - i\chi_k') (am\chi_k + i\chi_k') = 1,
\]

that corresponds to a time-dependent oscillator with a complex frequency.

Linearly growing scale factor

\[
a(t) = a_0 H_0 t \tag{4}
\]
corresponds to the exponential dependence $a(\eta) = a_0 \exp(\eta H)$ in the conformal time $d\eta = dt/a$, where the conformal Hubble constant is $H = a_0 H_0$. To describe the particles production, one should define the in- and out-vacuums. There is a lot of solutions connected by the Bogoliubov transformation [22].

For the fermions with the momentum $k$ directed along $z$-axes, the Bogoliubov transformations are:

\[
U_+(k) = \cos r_k u_+(k) - \sin r_k e^{i\delta_k} v_-(k),
\]

\[
V_+(k) = \cos r_k u_+(k) + \sin r_k e^{-i\delta_k} v_-(k),
\]

\[
U_-(k) = \cos r_k u_-(k) - \sin r_k e^{i\delta_k} v_+(k),
\]

\[
V_-(k) = \cos r_k u_-(k) + \sin r_k e^{-i\delta_k} v_+(k), \tag{5}
\]

where $u_{\pm}(k), v_{\pm}(k), U_{\pm}(k), V_{\pm}(k)$ denote $u_{\pm,k}, v_{\pm,k}, U_{\pm,k}, V_{\pm,k}$ with $k$ directed along $z$-axes. From Eqns. [5] it follows that the functions $\chi_k$, $\chi_k'$ corresponding to the different vacuums are connected as

\[
\chi_k = \cos r_k \chi_k(\eta) - e^{i\delta_k} \sin r_k (ma(\eta)\chi_k(\eta) - i\chi_k'(\eta)) / k, \tag{6}
\]

\[
\chi_k' = \cos r_k \chi_k'(\eta) - e^{-i\delta_k} \sin r_k (ma(\eta)\chi_k(\eta) + i\chi_k'(\eta)) / k.
\]

According to [14], the quantity

\[
\sigma_k(\eta) = \frac{1}{2} (\chi_k'(\eta) \chi_k(\eta) + \chi_k''(\eta) \chi_k(\eta)) \tag{6}
\]
is monotonic in the past but has oscillating behavior in the future for the in-vacuum. For the out-vacuum state, it is oscillating in the past but is monotonic in the future.

With the help of the functional minimization method [14], we obtain that

\[
\chi_k = \sqrt{\pi} \sqrt{\frac{\cosh (2\pi k/H)}{2\pi k/H}} \exp (-im a_0 e^{\mp \eta H}/H - i\eta k)
\]

\[
L_{ik/H-1}^{(-2ik/H)} (2im a_0 e^{\mp \eta H}/H) \tag{7}
\]

for the in-vacuum state and function $\sigma$ shown in Fig. 1 (a) has the form

\[
\sigma = -\frac{\pi ma_0}{4\mathcal{H} k} e^{\mp \mathcal{H} \eta} \text{sech} \left( \frac{\pi k}{\mathcal{H}} \right) \left( J_{1/2 - i\mathcal{H}} \left( e^{\mp \mathcal{H} \eta} / \mathcal{H} \right) J_{1/2 + i\mathcal{H}} \left( e^{\mp \mathcal{H} \eta} / \mathcal{H} \right) \right). \tag{8}
\]

For the out-vacuum state (see Fig. 1 (b))

\[
\chi_k = \frac{1}{\mathcal{H}} \exp \left( \frac{\pi k}{\mathcal{H}} - im a_0 e^{\mp \eta H}/H - i\eta k \right)
\]

\[
U \left( -\frac{ik - \mathcal{H} \eta}{\mathcal{H}}, 1 - \frac{2ik}{\mathcal{H}} - \frac{2i e^{\mp \eta H} m a_0}{\mathcal{H}} \right). \tag{9}
\]

Here $U(a, b, c)$ is the confluent hypergeometric Kummer’s function and $L_{m}^{(\nu)} (z)$ is the generalized Laguerre function [24].

\[
\text{Square of sin } r_k \text{ can be expressed as}
\]

\[
\sin^2 r_k = \left| \frac{k (\chi_k \chi_k' - \chi_k' \chi_k)}{im^2 a^2 \chi_k \chi_k' - \chi_k \chi_k' + \chi_k \chi_k' + i k^2 \chi_k \chi_k' + i \chi_k \chi_k'} \right|^2 = \frac{1}{1 + \exp (2\pi k/H)}. \tag{10}
\]

III. DISCUSSION AND CONCLUSION

It is interesting to compare the number of the produced fermions with the result of Refs. [12, 23], where the
dependence

\[ a(t) = a_0 t^q = \tilde{a}_0 p, \]  \hspace{1cm} \text{(12)}

\[ \tilde{a}_0 = \sigma_0^{1/(1-q)} (1 - q)^{q/(1-q)}, \]

\[ p = q/(1 - q), \hspace{0.5cm} 0 < q < 1 \]  \hspace{1cm} \text{(13)}

was considered. The density of the produced fermions as well as scalar particles have been estimated as \[6, 25\]

\[ n \sim m^3 (mt)^{-3q}. \]  \hspace{1cm} \text{(14)}

Under \( q = 1 \) one has \( n \sim t^{-3} \). Up to a numerical multiplier this coincides with our exact result given by \[4, 10, 11\]. The numerical multiplier equals approximately to \( \sigma_0 \approx 2.5 \times 10^{-4} \).

Now we want to discuss the moment of time when the particles of some mass \( m \) are created. According to Ref. \[14\], the moment of time, from which the function \( \sigma \) given by \[9\] acquires oscillating behavior corresponds to the particle creation.

From the analysis of the expression \[8\] it follows that the particles of mass \( m \) are created by time \( \eta_1 \) determined by the equation \( ma_0 \exp (\mathcal{H} \eta_1) / \mathcal{H} \approx 1 \) that is \( m / \mathcal{H} (t_1) \approx 1 \). By that moment of time according to \[11\] total density of a particles approximately equals \( n_1 \approx H(t_1)^3 \) (or equally \( n_1 \approx m^3 \)) and further it is simply reduces with time as the \( n(t) \approx n_1 \frac{a(t)}{a^3(t)} \). From dimension arguments it can be argued that for arbitrary \( a(t) \) density number of the created particles can be estimated as

\[ n(t) \sim m^3 \frac{a^3(t_1)}{a^3(t)}, \hspace{0.5cm} H(t_1) \approx m. \]  \hspace{1cm} \text{(15)}

Let us check this formula for the dependence given by \[12\]. For the time of particle creation we have \( t_1 \approx \frac{2}{\mathcal{H} } \) and finally come to \[14\].

It is also interesting to discuss in general the possible role of the cosmological particle production in the creation of matter in the Universe. According to the modern view decelerating fast-roll stage preceded the inflationary stage \[20\]. At this stage Hubble constant decreased down to the value \( H(t_1) \) right up to the time \( t_1 \) when the inflation begins. Let us estimate amount of matter created before the inflation stage. At the inflation stage typical value of the Hubble constant is \( H_1 = m \), where \( m \) is the inflanton mass. It is an order of \( m \sim 10^{-6} M_p \), where \( M_p \sim 10^{18} \text{ GeV} \) is the Plank mass. Density of the inflantons produced under the background of the coherent inflaton field is give by \[15\]. During inflation and further expansion their density is reduced in \( a_i^3/a_0^3 \) times where \( a_0 \) is the scale factor of the present universe. Present time matter density created due to this effect can be estimated as

\[ \rho = m n_1 \frac{a_1^3}{a_0^3} = (M_p 10^{-6})^4 \frac{a_1^3}{a_0^3}, \]  \hspace{1cm} \text{(16)}

where for simplicity we suggest that the created inflantons decay to the massive particles, so that their general amount of mass is conserved. Comparing this quantity with the observed density of matter in the universe \( \rho_c = M_0^2 H_0^2 = M_p^4 10^{-122} \) we find that the expansion rate has to be \( \frac{a_0}{a_i} = 10^{33} \) in order to the densities be equal.

That is the matter arising due to cosmological creation of inflantons will be negligible only if the expansion rate of the universe during inflation and further expansion exceeds \( 10^{33} \) (76 e-foldings).

To summarize, we have presented the new exactly solvable model for fermion production in the flat universe with a scale factor linearly growing with time. Spectrum of the produced fermions over physical momentums \( p = k / a \)

\[ n(p) = \frac{1}{\exp(2\pi p / H(t)) + 1} \]

is thermal at large \( p \), and the Hubble constant divided by \( 2\pi \) plays a role of the temperature. The number density of the created particles does not depend on the particle mass, and particles of large mass can be easy created. However, the function \( \sigma \) begins to oscillate later in the time, when the mass decreases.

The oscillations of this function mean that the real particles appear \[14\]. Thus, in the limit of \( m \rightarrow 0 \), the particles would be created infinitely late.
[1] L. Parker. Quantized fields and particle creation in expanding universes. *Phys. Rev.* **183**, 1057 (1969).

[2] R. U. Sexl and H. K. Urbantke. Production of particles by gravitational fields. *Phys. Rev.* **179**, 1247 (1969).

[3] Ya. Zel’dovich and A. Starobinsky. Particle creation and vacuum polarization in an anisotropic gravitational field. *Zh. Eksp. Teor. Fiz.* **61**, 2161 (1971) [Sov. Phys.- JETP **34**, 1159 (1972)].

[4] A.A. Grib and S.G. Mamaev. On field theory in the Friedman space. *Yad.Fiz.* **10**, 1276-1281 (1969).

[5] N.D. Birrell and P. C. W. Davis, *Quantum Fields in Curved Space* (Univ. Press, Cambridge, 1982).

[6] A. A. Grib, S. G. Mamaev and V. M. Mostepanenko. Vacuum quantum effects in strong fields. (Friedmann Laboratory Publishing, St. Petersburg, 1994)

[7] S.P. Gavrilov, D.M. Gitman and A.E. Goncalves. Quantum spinor field in the FRW universe with a constant electromagnetic background. *Int.J.Mod.Phys.* **A16**, 4235-4260 (2001).

[8] C. Bernard and A. Duncun. Regularization and renormalization of quantum field theory in curved space-time. *Ann. Phys. (N. Y.)* **107**, 201 (1977).

[9] H.E. Kandrup. Generating a thermal spectrum from nothing. *Phys. Lett. B* **215**, 473 (1988).

[10] W.-H. Huang. Particle Creation in Kaluza-Klein Cosmology. *Phys.Lett. A* **140**, 280 (1989).

[11] S.A. Fulling. Remarks on positive frequency and hamiltonians in expanding universes. *Gen. Rel. Grav.* **10**, 807 (1979).

[12] S. Winitzki. Cosmological particle production and the precision of the WKB approximation. *Phys. Rev.* **D72**, 104011 (2005).

[13] S.V. Anischenko. Violation of the virial theorem for the ground state of the time-dependent oscillator. *Vestnik Belarus State U., ser. Fiz.-Mat.* **2**, 43 (2008)[in Russian].

[14] S.V. Anischenko, S.L. Cherkas and V.L. Kalashnikov. Functional minimization method addressed to the vacuum finding for an arbitrary driven quantum oscillator. *Nonlin. Phen. in Compl. Syst.* **12**, 16 (2009).

[15] S.L. Cherkas, V.L. Kalashnikov. Universe driven by the vacuum of the scalar field: VFD model. *Proc. Int. Conf. "Problems of Practical Cosmology", held at Russian Geographical Society, 23-27 June 2008, Saint Petersburg. Ed. Yu. V. Burgshev, I.N. Taganov, P. Teerskorsps, 2, 135 (2008), arXiv: astro-ph/0611795.

[16] D. M. Chitre and J. B. Hartle. Path-integral quantization and cosmological particle production. An example. *Phys Rev D* **16**, 251 (1977).

[17] H. Narai and T. Azuma. Propagators for a quantum scalar fields in some isotropic universe. *Progr. Theor. Phys.* **59**, 1522 (1978).

[18] V. Sahni. Regularized quantum energy-momentum tensor for spinor fields in the Chitre-Hartle and Milne metrics. *Class. Quant. Grav.* **1**, 579 (1984).

[19] P.S. Flores. The Robertson-Walker metrics expressible in static form. *Gen. Rel. Grav.* **12**, 563-574 (1980).

[20] S. L. Cherkas and V. L. Kalashnikov. Determination of the UV cut-off from the observed value of the Universe acceleration. *JCAP 01*, 028 (2007).

[21] V. B. Berestetskii, E. M. Lifshitz and L.P. Pitaevskii. *Quantum electrodynamics*. (Pergamon Press, Oxford, 1982).

[22] S. L. Cherkas. About spin motion in the external field. *Proc. Acad. Sci. Belarus, ser. Fiz.-Mat.* **2**, 70 (1994)[in Russian].

[23] N.N. Bogoliubov. Concerning a new method in the theory of superconductivity. *Zh. Eksp. Teor. Fiz.* **34**, 58 (1958)[,Sov. Phys. JETP **34**, 41 (1958)].

[24] M. Abramowitz and I.A. Stegun. *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*. (Dover, New York, 1972).

[25] S. G. Mamaev and V. M. Mostepanenko. Renormalization of gravitational constant and creation of fermions by nonstationary gravitational field. *Yad.Fiz.* **28**, 1640 (1978).

[26] C. Destri, H. J. de Vega and N. G. Sanchez. The pre-inflationary and inflationary fast-roll eras and their signatures in the low CMB multipoles. arXiv:0912.2994.