Abstract
Proceeding from nonlinear realizations of (super)conformal symmetries, we explicitly demonstrate that adding the harmonic oscillator potential to the action of conformal mechanics does not break these symmetries but modifies the transformation properties of the (super)fields. We also analyze the possibility to introduce potentials in $N = 4$ supersymmetric mechanics by coupling it with auxiliary fermionic superfields. The new coupling we considered does not introduce new fermionic degrees of freedom - all our additional fermions are purely auxiliary ones. The new bosonic components have a first order kinetic term and therefore they serve as spin degrees of freedom. The resulting system contains, besides the potential term in the bosonic sector, a non-trivial spin-like interaction in the fermionic sector. The superconformal mechanics we constructed in this paper is invariant under the full $D(2,1;\alpha)$ superconformal group. This invariance is not evident and is achieved within modified (super)conformal transformations of the superfields.
1 Introduction

The conformal symmetry in one dimension is so powerful, that in the one particle case it completely fixes the potential to be $1/x^2$ [1]. The supersymmetric extensions of conformal mechanics [2, 3] add to the theory some fermions interacting with the single bosonic field $x$, but in the bosonic sector the potential is still the same. So it seems that the question whether we can add something to the bosonic potential without breaking (super)conformal invariance has a unique answer - no. Nevertheless, it is not completely true. Indeed, the standard description of the one dimensional conformal invariance consists in the statement that the Hamiltonian

$$H = \frac{1}{2}p^2 + \frac{g^2}{2x^2}$$  \hspace{1cm} (1.1)

forms the $so(1, 2)$ algebra together with the generators of the dilatation $D$ and conformal boost $K$ defined as

$$D = xp, \quad K = x^2,$$  \hspace{1cm} (1.2)

with respect to the canonical Poisson bracket

$$\{x, p\} = 1.$$  \hspace{1cm} (1.3)

Now it is completely clear that the modified Hamiltonian $\hat{H}$

$$\hat{H} = H + m^2 K$$  \hspace{1cm} (1.4)

will do the same job perfectly forming the same $so(1, 2)$ algebra with generators $D$ and $K$ [1, 2]. Thus, the additional harmonic oscillator potential is also admissible without breaking conformal symmetry. It has been firstly shown in [4] that this additional harmonic oscillator potential just modifies the realization of the conformal group, keeping the resulting action invariant under conformal group transformations. Unfortunately, the approach presented in [4] is not fully appropriate and the nice idea of the modification of conformal group transformations could not be immediately extended to the $N = 2, 4$ supersymmetric cases. In the present paper (Sections 2-4) we will demonstrate that this additional harmonic oscillator term can be easily obtained within the nonlinear realizations framework in which the conformal mechanics (together with its $N = 2, 4$ superextensions) describes the motion along geodesics in the group space of the $d = 1$ (super)conformal group [5, 3]. In such approach, the almost trivial bosonic case (Section 2) has natural and straightforward $N = 4$ superconformal analogs.

Another issue we analyzed in this paper is the generation of bosonic potentials in $N = 4$ supersymmetric mechanics through coupling it with the auxiliary fermionic supermultiplet. This additional supermultiplet enters the action in a rather special manner:

- the fermionic components $\psi^a, \bar{\psi}_a$ appear in the action only hatted by the time derivative. Thus the corresponding equations of motion are nothing but the conservation laws, and all fermionic components could be expressed in terms of the remaining components;

- the bosonic components $w^i, \bar{w}_i$ have a first order kinetic term and therefore serve as spin degrees of freedom;

- the coupling constant $g$ in the bosonic potential is the square of the norm of the spin degrees of freedom $w^i : g = w^i \bar{w}_i = const.$

The idea to generate the bosonic potential by coupling with additional supermultiplets has been firstly proposed in [6]. In this paper the Authors introduced coupling with a supermultiplet containing physical fermions, thus finishing with a system with a doubled number of fermions, in contrast with our case where all fermions are auxiliary. Our approach in this paper is very close to those one recently proposed in [7]. The component actions we constructed in this paper have to coincide with the ones from [7], because the main ingredient - the action describing the coupling of the basic $(1, 4, 3)$ supermultiplet with the auxiliary fermionic $(0, 4, 4)$ one - is unique and is completely fixed by $N = 4$ Poincaré supersymmetry (up to an overall constant).
2 Bosonic case: justification of the idea

The standard conformal algebra in \( d = 1 \) is the \( \text{so}(1, 2) \) one spanned by the generators of translations \((P)\), dilatations \((D)\) and conformal boosts \((K)\) \((A.1)\). One of the simplest ways to construct a one-dimensional system which is conformally invariant is to use the method of covariant reduction \([5]\). In application to this simplest case this method includes the following steps:

- realization of the conformal group \( \text{SO}(1, 2) \) in some coset;
- building the Cartan forms;
- imposing invariant constraints on the Cartan forms which result in the desired equations of motion.

Let us choose the following parametrization of the \( \text{SO}(1, 2) \) group space

\[
g = e^{itP} e^{izK} e^{iuD} \tag{2.5}
\]

where the coordinates \( u \) and \( z \) are functions of the time variable \( t \). Thus, we are dealing with the nonlinear realization of \( \text{SO}(1, 2) \) in its group space. The Cartan forms for the group element \( g \) \((2.5)\) read

\[
g^{-1} dg = i\omega_P P + i\omega_K K + i\omega_D D, \tag{2.6}
\]

where

\[
\omega_P = e^{-u} dt, \quad \omega_D = du - 2z dt, \quad \omega_K = e^u \left[dz + z^2 dt\right]. \tag{2.7}
\]

In the present case the coset coincides with the group space, therefore all Cartan forms \((2.7)\) are invariant under \( \text{SO}(1, 2) \) transformations realized as left multiplications of the group element \( g \) \((2.5)\). The set of constraints we are going to impose on the forms \((2.7)\) reads \([3]\)

\[
\omega_D = 0, \quad \omega_K = g^2 \omega_P, \tag{2.8}
\]

where \( g \) is a free parameter with the dimension of the mass. The first constraint in \((2.8)\)

\[
\omega_D = du - 2z dt = 0 \quad \Rightarrow \quad z = \frac{1}{2} \dot{u} \tag{2.9}
\]

is just a simplest version of the Inverse Higgs phenomenon \([8]\). Its meaning is rather simple – we do not need an independent field \( z(t) \) in order to realize the conformal invariance in the group space. Instead we may use the time derivative of the dilaton \( \dot{u} \) which has the same transformation properties with respect to the \( \text{SO}(1, 2) \) group.

The second constraint in \((2.8)\) is a dynamical one. Using the \((2.9)\) it may be rewritten in the familiar form as

\[
\ddot{x} = \frac{g^2}{x^3}, \quad \text{where} \quad x \equiv e^\dot{u}. \tag{2.10}
\]

Clearly, the equation of motion \((2.10)\) follows from the action of conformal mechanics \([1]\)

\[
S = \int dt \left( \frac{\dot{u}^2}{2} - \frac{g^2}{2x^2} \right). \tag{2.11}
\]

It is not unexpected now that the action \((2.11)\) is invariant under conformal transformations

\[
\delta t = f(t) = a + bt + ct^2, \quad \delta u = \dot{f}. \tag{2.12}
\]

Let us stress that the explicit form of \((2.12)\) simply follows from the left action of the conformal group \( \text{SO}(1, 2) \) on the group element \( g \) \((2.5)\).

All this is not new and has been known for a long time. In order to learn something new, let us change the parametrization \((2.5)\) and associate the time variable \( t \) with the generator \( P + m^2 K \), where \( m \) is an additional parameter with the dimension of the mass as

\[
\tilde{g} = e^{it(P + m^2 K)} e^{izK} e^{iuD}. \tag{2.13}
\]
The explicit relation between the coset element $\hat{g}$ and $g$ is given by
\[
e^{it(P+m^2K)}e^{i\alpha K}e^{iuD} = e^{i\tan(\alpha m)P}e^{i(z \cos^2[mt]+m \cos[mt] \sin[mt])K}e^{i(u-2 \log[\cos[mt]])D}.
\]
(2.14)

The Cartan forms are slightly changed to be
\[
\bar{\omega}_P = e^{-u}dt, \quad \bar{\omega}_D = du - 2z dt, \quad \bar{\omega}_K = e^u [dz + z^2 dt + m^2 dt].
\]
(2.15)

Now we impose the same constraints as before (2.8). As a result, we are ending up with the following equation of motion:
\[
\ddot{x} = -m^2 x + \frac{g^2}{x^3}, \quad \text{where again} \quad x = e^{\tilde{x}}.
\]
(2.16)

Clearly, this equation follows from the following action:
\[
\tilde{S} = \int dt \left[ \frac{x^2}{2} - \frac{m^2 x^2}{2} - \frac{g^2}{2x^2} \right].
\]
(2.17)

Thus, the conformal mechanics with the added harmonic oscillator term (1-particle Calogero-Moser system) is invariant under the $SO(1,2)$! It is not too hard to find the corresponding realization of this symmetry
\[
\delta t = \tilde{f}(t) = a (1 + \cos(2mt)) + \frac{b}{2m} \sin(2mt) + \frac{c}{2m^2} (1 - \cos(2mt)), \quad \delta u = \tilde{f},
\]
(2.18)

where the parameters $(a, b, c)$ are, as before, the parameters for translations, conformal boosts and dilatations, respectively. The function $\tilde{f}(t)$ which collects all parameters of the $SO(1,2)$ transformations obeys, in view of (2.18), the constraint
\[
\frac{d}{dt} \left[ \tilde{f} + 4m^2 \tilde{f} \right] = 0.
\]
(2.19)

Thus, the conformal mechanics in the added harmonic oscillator term (1-particle Calogero-Moser system) is invariant under the $SO(1,2)$ group which has a non-canonical realization on the time variable $t$.

Two comments have to be added here. Firstly, in the limit $m \rightarrow 0$ the transformations (2.18) are reduced to the standard ones (2.12), as it should be. Secondly, the kinetic term in the action (2.17) is invariant under (2.18) only together with the oscillator term, while the conformal potential is invariant by itself.

The invariance of the action (2.17) with respect the $SO(1,2)$ transformations (2.18) has been firstly demonstrated in [4]. Although we do not use the realization of the Virasoro group in the corresponding coset, which was the key ingredient in [4], it makes sense to consider the present approach as a further development of the nice ideas of this paper.

In the next Sections we will extend our description to the $N=2$ and $N=4$ supersymmetric cases.

### 3 N=2 superconformal mechanics

The simplest nontrivial extension of the conformal mechanics corresponds to the $N=2$ case with $SU(1,1|1)$ superconformal group [2]. The geometric construction of $N=2$ superconformal mechanics within the framework of the nonlinear realization of $SU(1,1|1)$ in the coset $SU(1,1|1)/U(1)$ has been carried out in [3]. Without deeply going in the details, our extension of the consideration in [3] looks as follows. As usual, we are starting from the $N=2$ superconformal group $SU(1,1|1)$. Its superalgebra includes the four bosonic generators $\{P, D, K, V_3\}$ and the four fermionic ones $\{Q^1, Q^1, S^1, S^1\}$. Their commutators follow from the general formulas given in Appendix (A.1)-(A.2) with $\alpha = -1$.

We will realize this group in the coset $SU(1,1|1)/U(1)$ parameterized as
\[
g = e^{it\hat{P}}e^{i\theta \hat{Q} + i\bar{\theta} \hat{\bar{Q}}}e^{izK}e^{i\phi S^1 + i\bar{\phi} \bar{S}^1}e^{iuD},
\]
(3.1)

where
\[
\hat{P} = P + 2imV_3 + m^2 K, \quad \hat{Q} = Q^1 + imS^1, \quad \hat{\bar{Q}} = \bar{Q}^1 - im\bar{S}^1.
\]
(3.2)
and all the coordinates \((u, z, \psi, \bar{\psi})\) are \(N = 2\) superfields which depend on \((t, \theta, \bar{\theta})\). One should stress that the \(U(1)\) generator is anti-hermitian \((A.5)\), so the operator \(\hat{P}\) \((3.2)\) is hermitian. The only difference with the presentation in \([3]\) is given by the \(m\)-dependent terms in \((3.1), (3.2)\).

In what follows we shall need the explicit structures of several Cartan forms in the expansion \(g^{-1}dg\) over the generators,

\[
\begin{align*}
\omega_P &= e^{-u} \left( dt + i\theta d\bar{\theta} + i\bar{\theta} d\theta \right), \quad \omega_{Q1} = e^{-\frac{i}{2}u} \left( d\bar{\theta} - \psi d\tilde{t} \right), \\
\omega_D &= du - 2x d\bar{\theta} - 2i\bar{\theta} d\psi - 2i d\theta \bar{\psi}, \\
\omega_{S1} &= e^{\frac{i}{2}u} \left( d\psi - z d\theta - i\psi d\tilde{\theta} + (z + im)\psi d\tilde{t} + im d\tilde{\theta} \right). \\
\end{align*}
\]

The constraints we impose on the Cartan forms are the same as in \([3]\)

\[
\omega_D = 0, \quad \omega_{S1} = ig\omega_Q1,
\]

where, as before, the arbitrary parameter \(g\) has the dimension of the mass. As a result we will get two sets of equations which follow from \((3.3)\)

\[
z = \frac{1}{2} \dot{u}, \quad \psi = -\frac{i}{2} D\bar{u}, \quad \bar{\psi} = -\frac{i}{2} Du,
\]

and

\[
[D, \bar{D}] X = 2mX + \frac{2g}{X}, \quad X \equiv e^{\frac{i}{2}u},
\]

where the semi-covariant (fully invariant only under Poincaré supersymmetry) spinor derivatives are defined by

\[
D = \frac{\partial}{\partial \theta} + i\bar{\psi} \partial_{\bar{t}}, \quad \bar{D} = \frac{\partial}{\partial \bar{\theta}} + i\psi \partial_t, \quad \{D, \bar{D}\} = 2i\partial_t.
\]

The equations \((3.5)\) express unessential Goldstone superfields \((z, \psi, \bar{\psi})\) in terms of the super-dilaton \(u\), while the equation \((3.6)\) is the dynamical one. Clearly, it can be obtained from the following superfield action:

\[
S = \int dt d\theta \left[ DX \bar{D} X + mX^2 + 2g \log(X) \right].
\]

It is worth to note that the action \((3.8)\), despite the presence of the harmonic oscillator potential \(mX^2\), has to be invariant under the full \(SU(1, 1|1)\) superconformal group. In order to clarify this point, let us write the transformations as follows:

\[
\delta t = E - \frac{1}{2} \bar{D} E - \frac{1}{2} \partial_{\bar{t}} E, \quad \delta \theta = -\frac{i}{2} \bar{D} E, \quad \delta \bar{\theta} = -\frac{i}{2} \partial_{\bar{t}} E, \quad \delta u = \dot{E},
\]

where \(E\) is a superfunction collecting all parameters of the \(SU(1, 1|1)\) group. In the standard realization (with \(m = 0\)) this function obeys the following constraint: \([3]\)

\[
\frac{\partial}{\partial t} \left[ D, \bar{D} \right] E = 0,
\]

which leaves in \(E\) just the parameters of the \(SU(1, 1|1)\) group. In the present case this constraint is modified to become

\[
\frac{\partial}{\partial t} \left[ D, \bar{D} \right] E = 4m\dot{E}.
\]

One may easily check that the solution of the modified equation \((3.11)\) also contains the set of parameters corresponding to the transformations of the \(SU(1, 1|1)\) group. Nevertheless, the realization of this group on the superspace \((t, \theta, \bar{\theta})\) and superfield \(u\) \((3.4)\) is different now. The action \((3.8)\) is invariant with respect to these transformations. In a full analogy with the bosonic case, the term \(g \log(X)\) is invariant by itself, while the kinetic term is invariant only together with the harmonic potential. Thus, we explicitly demonstrated that the \(N = 2\) superconformal mechanics, being equipped with the harmonic oscillator potential, admits the same invariance with respect to \(N = 2\) superconformal symmetry \(SU(1, 1|1)\), but with a different realization.

To conclude, it makes sense to note that in principle one could ignore the coset construction and just ask which invariance does the superfield action \((3.8)\) possess. Looking for the answer one may write the general transformations in the form \((3.9)\) and then immediately get the constraint \((3.11)\) on the superfunction \(E\) which selects just the \(SU(1, 1|1)\) group.
4 N = 4 superconformal mechanics

The extension of our previous consideration to the case of N = 4 superconformal mechanics goes almost straightforwardly. We will start with the su(1, 1|2) superalgebra which is isomorphic to D(2, 1; −1) (see Appendix). Next, similarly to [3], we will realize SU(1, 1|2) in the coset SU(1, 1|2)/SU(2) parameterized as

\[ g = e^{-i \hat{P} \theta + i \hat{Q} \psi} e^{i z K} e^{i \psi S^i + i \tilde{\psi} \tilde{S}^i} e^{i u D}, \]  

(4.1)

where

\[ \hat{P} = P + 2i m V_3 + m^2 K, \quad \hat{Q}^1 = Q^1 + i m S^1, \quad \hat{Q}^2 = Q^2 - i m S^2, \quad \hat{Q}_1 = \overline{Q}_1 - i m \overline{S}_1, \quad \hat{Q}_2 = \overline{Q}_2 + i m \overline{S}_2 \]  

(4.2)

and all the coordinates (u, z, \( \psi \), \( \tilde{\psi} \)) are now N = 4 superfields which depend on (t, \( \theta \), \( \tilde{\theta} \)). With our choice (4.2) the generators \( \{ \hat{P}, \hat{Q}^i, \hat{Q}_i \} \) span the N = 4, d = 1 super Poincaré algebra.

In order to construct the equations of motion, we have to impose covariant constraints on the Cartan forms. We will choose the same constraints as in [3], namely

\[ \omega_D = 0, \quad \omega_{S^i} = i g \omega_{Q^i}. \]  

(4.3)

Using explicit expressions for the Cartan forms

\[ \omega_D = du - 2 z dt - 2 i \tilde{\theta} \xi - 2 i \theta \tilde{\xi}, \quad \omega_{Q^i} = e^{-i \tilde{\xi}} d \theta_i + d \tilde{\xi} (\ldots), \]

\[ \omega_{S^1} = e^{i \tilde{\xi}} (d \xi - i z \tilde{\xi}) d \theta_1 + (i m - z) d \theta_1 + d \tilde{\xi} (\ldots), \]

\[ \omega_{S_2} = e^{i \tilde{\xi}} (d \xi - i \tilde{\xi}) d \theta_2 + (i m + z) d \theta_2 + d \tilde{\xi} (\ldots), \]  

(4.4)

where the covariant differential of t is defined as

\[ d \tilde{t} = dt - i (d \tilde{\theta}_i + d \theta_\tilde{i}) \]

(4.5)

we will get the following set of equations from (4.3):

\[ z = \frac{1}{2} \tilde{u}, \quad \xi_i = - \frac{i}{2} \overline{\nabla}_i u, \quad \tilde{\xi}^i = - \frac{i}{2} D^i u, \]  

(4.6)

\[ D^i D_j e^u = 0, \quad \overline{D}_i \overline{D}_j e^u = 0, \quad [D^i, \overline{D}_j] e^u = -8 g, \]  

(4.7)

\[ D^i \overline{D}_2 u = D^2 \overline{D}_i u = 0, \quad (D^i \overline{D}_1 - D^2 \overline{D}_2) u = 4 m. \]  

(4.8)

Here, we introduced the spinor covariant derivatives as

\[ D^i \equiv \frac{\partial}{\partial \theta_i} + i \tilde{\theta}_i \partial_i, \quad \overline{D}^i \equiv \frac{\partial}{\partial \theta^i} + i \tilde{\theta} \partial_i, \quad \{ D^i, \overline{D}_j \} = 2 i \partial_i. \]  

(4.9)

The meaning of the equations (4.6)–(4.8) is clear:

- the equations (4.6) express unessential superfields \( \{ z, \xi, \tilde{\xi} \} \) in terms of the superdilaton u;
- the constraints (4.7) are off-shell irreducibility conditions: they reduce the components content of the N = 4 superfield u to 1 physical and 3 auxiliary bosonic fields and four fermionic fields [3];
- equations (4.8) are dynamical: they serve to eliminate the triplet of auxiliary fields and give rise to equations of motion for the physical fields.

The component action has a very simple form

\[ S = \int dt \left[ \frac{\tilde{y}_1^2}{2} + \frac{i}{2} (\tilde{\psi}_1 \psi^i - \tilde{\psi}^i \psi^1) - \frac{m^2}{2} y^2 - \frac{y^2}{2y^2} + m \tilde{\psi}_1 \psi^1 + \frac{y}{y^2} (\tilde{\psi}_1 \psi^1 - \tilde{\psi}_2 \psi^2) + 3 \frac{\tilde{\psi}_1 \psi^1 \tilde{\psi}_2 \psi_2}{y^2} \right] \]  

(4.10)

where

\[ y = e^{\tilde{\xi}} |_{\theta = \tilde{\theta} = 0}, \quad \psi^1 = D^1 e^{\tilde{\xi}} |_{\theta = \tilde{\theta} = 0}, \quad \psi^2 = \overline{D}_2 e^{\tilde{\xi}} |_{\theta = \tilde{\theta} = 0}. \]  

(4.11)

By construction the action (4.11) is invariant with respect to the su(1, 1|2) superalgebra realized in the modified coset (4.11).
5 Potentials in \( N=4 \) superconformal mechanics

One of the most restrictive features of \( N = 4 \) supersymmetric mechanics based on the \((1, 4, 3)\) supermultiplet is a specific generation of potential terms from constants in the defining constraints. It has been shown many years ago [3] that the constraints defining the irreducible \( N = 4, d = 1 \) supermultiplet with \((1, 4, 3)\) components content are uniquely fixed to be

\[
D^i D_i X = gf, \quad \overline{D}^i \overline{D}_i X = g\bar{f}, \quad [D^i, \overline{D}_i] X = 2gc,
\]

where the set of constants \( f, \bar{f}, c \) are related by

\[
c^2 + f\bar{f} = 1.
\]

Clearly, one may always choose

\[
c = 0, \quad f = -\bar{f} = i
\]
to have

\[
D^i D_i X = ig, \quad \overline{D}^i \overline{D}_i X = -ig, \quad [D^i, \overline{D}_i] X = 0.
\]

Now, the most general action of the one-particle \( N = 4 \) supersymmetric mechanics reads

\[
S = -\int dt d^4\theta F(X),
\]

with \( F(X) \) being an arbitrary function of the superfield \( X \). It is not hard to get the bosonic part of the component action (with auxiliary fields excluded by their equations of motion)

\[
S_B \sim \int dt \left[ F'' \dot{x}^2 + g^2 F'' \right], \quad x \equiv X|_{\theta = \bar{\theta} = 0}, \quad F(x) \equiv F(X)|_{\theta = \bar{\theta} = 0}.
\]

Finally, one could bring the kinetic term to the flat one

\[
S_B \sim \int dt \left[ y'^2 + g^2(y')^2 \right], \quad F'' = (y')^2,
\]

where \( y'(y) \) is considered as a function of \( y \). The additional requirement of conformal invariance, i.e. \( y' \sim 1/y \), completely fixes everything, unambiguously restoring \( F(X) \sim X \log X \). Thus, everything is strictly fixed by \( N = 4 \) superconformal invariance (\( SU(1,1|2) \) in the case at hands).

Another possibility to get the potential term in \( N = 4 \) supersymmetric mechanics has been proposed in [6]. The main idea was to couple the \((1, 4, 3)\) supermultiplet to the fermionic \((0, 4, 4)\) one. The price one has to pay for this is a doubled up number of physical fermions in the resulting system. Here we will use the same idea of coupling but with a different action. In our action the fermions appear only through the time derivatives which can be replaced, without breaking of supersymmetry, by auxiliary fermions. Moreover, the bosonic fields in the action have the kinetic term which is linear in the time derivatives, and therefore these bosonic fields acquire the interpretation of spin degrees of freedom.

Our starting point is the \((1, 4, 3)\) supermultiplet \( X \) with \( g = 0 \) [5,4] and the fermionic \((0, 4, 4)\) supermultiplet \( \Psi^a, \overline{\Psi}_a \) defined by the constraints [6]

\[
D^i \Psi^1 = 0, \quad D^i \Psi^2 + \overline{D}^i \overline{\Psi}^1 = 0, \quad \overline{D}_i \overline{\Psi}^2 = 0.
\]

We introduce the coupling of these supermultiplets by considering the following action:

\[
S = S_1 + S_2 = -\frac{1}{32} \int dt d^4\theta F(X) - \frac{1}{32} \int dt d^4\theta X \Psi^a \overline{\Psi}_a.
\]

After integration over theta’s, the components action which follows from \([5,9]\) reads

\[
S = \int dt \left[ \frac{1}{8} F'' \dot{x}^2 + \frac{1}{16} F'' A^{ij} \dot{A}_{ij} + \frac{i}{8} F'' (\dot{\eta}^i \dot{\bar{\eta}}_i - \eta^i \dot{\bar{\eta}}_i) + \frac{i}{8} F'' (\dot{\bar{\eta}}^i \dot{\eta}_i - \bar{\eta}^i \dot{\eta}_i) - \frac{1}{32} F^{(4)} \eta^i \dot{\bar{\eta}}_i \dot{\bar{\eta}}_j \dot{\bar{\eta}}_j \right] + \int dt \left[ -x \left( \dot{\psi}^1 \dot{\bar{\psi}}^2 - \dot{\bar{\psi}}^2 \dot{\psi}^1 \right) + \frac{i}{4} x \left( \dot{\psi}^i \dot{\bar{v}}^j - \dot{\bar{v}}^j \dot{\psi}^i \right) + \frac{1}{4} A_{ij} \dot{v}^i \dot{v}^j + \frac{1}{2} \eta^i \left( \dot{\bar{v}}^i \dot{\bar{v}}^j + \dot{\bar{v}}^j \dot{\bar{v}}^i \right) \right],
\]

\[\text{\([5.10]\)}

If we combine the spinor derivatives \( D^i, \overline{D}^i \) in the quartet of spinor derivatives \( \nabla^{\alpha\beta} = \{ D^i, \overline{D}^i \} \) then the constraints acquire the familiar form \( \nabla^{(\alpha} \Psi^{\beta)} = 0 \).
where
\[ x \equiv X, \ A_{(ij)} = \frac{1}{2} [D_i, D_j] X, \quad \eta^i = -i D^i X, \ \bar{\eta}_i = -i \overline{D}_i X, \]
\[ \psi^a \equiv \Psi^a, \quad \bar{\psi}^a \equiv \overline{\Psi}^a, \]
\[ \psi^a \equiv \Psi^a, \quad \bar{\psi}^a \equiv \overline{\Psi}^a, \quad \bar{\psi}^a = \bar{\Psi}^a. \] (5.11)

and, as usual, \(|\bar{\psi}^a\rangle\) in the r.h.s. denotes the \( \theta = \bar{\theta} = 0 \) limit. In the component action \( \bar{\mathcal{L}}_{10} \) the fermionic fields \( \psi^a \) and \( \bar{\psi}^a \) enter only through the time derivatives. Let us replace these time derivatives by new fermionic fields \( \xi^a \) and \( \bar{\xi}^a \) as
\[ \xi^a = \dot{\psi}^a, \quad \bar{\xi}^a = \dot{\bar{\psi}}^a. \] (5.12)

This is nothing but the reduction from the \((0, 4, 4)\) supermultiplet to the auxiliary \((4, 4, 0)\) one \( \bar{\mathcal{L}}_{10} \). This reduction is compatible with \( N = 4 \) supersymmetry. Indeed, the components of the \( \Psi^a \) have the following transformation properties under \( N = 4 \) Poincaré supersymmetry
\[ \delta \psi^1 = -\epsilon^i \bar{\psi}_i, \quad \delta \psi^2 = \epsilon_i \bar{\psi}^i, \quad \delta v^1 = -2i \epsilon^i \bar{\psi}^i + 2i \epsilon^i \psi^i, \quad \delta v^2 = -2i \epsilon_i \psi^1 + 2i \epsilon_i \psi^2. \] (5.13)

From (5.13) we learn the transformation properties of the new fermions \( \xi^a, \bar{\xi}^a \)
\[ \delta \xi^1 = -\epsilon^i \bar{\xi}_i, \quad \delta \xi^2 = \epsilon_i \bar{\xi}^i, \quad \delta v^1 = -2i \epsilon^i \bar{\xi}^i + 2i \epsilon^i \xi^i, \quad \delta v^2 = -2i \epsilon_i \xi^1 + 2i \epsilon_i \xi^2. \] (5.14)

Now one may easily check that the action
\[ S = \int dt \left[ \frac{1}{16} \frac{F''}{F''} x^2 + \frac{1}{10} \int \overline{D}_l D_l x^2 \right] + \int dt \left[ -x \left( \xi^i \xi^2 - \xi^2 \xi^i \right) + \frac{i}{4} \left( \bar{\psi}_i \psi^i + \psi_i \bar{\psi}^i \right) \right] + \frac{1}{2} \eta_i (v_i \bar{\psi}^i + v^i \bar{\psi}^i) + \frac{1}{2} \eta_i (v_i \bar{\xi}^i + v^i \bar{\xi}^i), \] (5.15)
is invariant under (5.14), provided the components of the \( X \) supermultiplet transform in a standard way as
\[ \delta x = -i \epsilon_i \eta^i - i \epsilon^i \bar{\eta}_i, \quad \delta \eta^i = -\epsilon^i \bar{\psi}_i - i \epsilon^i \psi_i, \quad \delta \bar{\eta}_i = -\epsilon_i \bar{\psi}^i + i \epsilon_i \psi^i, \quad \delta A_{ij} = -\epsilon_i \bar{\psi}^j + i \epsilon_i \psi^j. \] (5.16)

In the action (5.15) the fields \( \xi^a, \bar{\xi}^a \) and \( A_{ij} \) are auxiliary ones. Eliminating them by their equations of motion
\[ \xi^1 = \frac{1}{2x} \eta_i \bar{\psi}_i, \quad \xi^2 = -\frac{1}{2x} \eta_i \bar{\psi}_i, \quad A_{ij} = \frac{F''}{F''} \eta_i \bar{\eta}_j + \frac{2}{F''} v_{(i} \bar{v}_{j)}, \] (5.17)
we will get the following action:
\[ S = \int dt \left[ \frac{1}{8} \frac{F''}{F''} x^2 + \frac{1}{8} \int \overline{D}_l D_l x^2 \right] + \frac{1}{32} \left( \frac{F''}{F''} \right)^2 \eta_i \bar{\eta}_j \bar{\eta}_j + \frac{i}{4} \left( \bar{w}_i \bar{w}^i - w_i \bar{w}^i \right) + \frac{1}{8} \eta_i \bar{\eta}_j \left( \bar{w}^i \bar{w}^i + w_i \bar{w}^i \right), \] (5.18)
where, in order to bring the kinetic term for \( v^i \) to the standard form we introduce the new fields \( w^i \) as
\[ w^i = \frac{1}{\sqrt{2}} v^i. \] (5.19)

Thus, we see that from our fermionic superfields \( \Psi^a \), there survived only bosonic components \( w^i, \bar{w}_i \) which enter the Lagrangian only through first time-derivatives. After quantization these variables become purely internal spin degrees of freedom. Moreover, from the equations of motion for \( w^i, \bar{w}_i \) one may conclude that
\[ w^i \bar{w}_i = g = \text{constant}, \] (5.20)
where \( g \) is the constant of the internal spin degrees of freedom.
and therefore, the last term in (5.18) is just the bosonic potential for \( x \)

\[
V_B = \frac{g^2}{8F''x^2}.
\]  

Thus, we conclude that indeed by coupling our \((1, 4, 3)\) superfield \( X \) with the fermionic \((0, 4, 4)\) auxiliary supermultiplet \( \Psi^a \) one may generate the bosonic potential for the physical bosonic field \( x \), together with terms describing the spin interaction of the fermionic components \( \eta^i, \bar{\eta}_i \).

It is quite interesting to understand whether the action (5.9) could possess any type of superconformal symmetry. The key point is to achieve the invariance of the second term \( S_2 \) in the action (5.9), because the first one \( S_1 \) can be always chosen to be superconformally invariant. Indeed, the superfield \( X \) obeys the constraints (5.3) with \( g = 0 \). The invariance of these constraints under the \( D(2, 1; \alpha) \) group forces \( X \) to transform as \( [12] \)

\[
\delta X = 2i\alpha (\epsilon_i \bar{\theta}^i + \bar{\epsilon}^i \theta_i) X (5.22)
\]

while the superspace measure transformation reads

\[
\delta dtd^4\theta = 2i (\epsilon_i \theta^i + \bar{\epsilon}^i \theta_i) dtd^4\theta. (5.23)
\]

Clearly, the superconformally invariant action for the supermultiplet \( X \) has the following form:

\[
S_{1_{Conf}}^f = -\frac{1}{32} \int dtd^4\theta (X)^{-\frac{1}{2}}, \quad \alpha \neq -1
\]

or \( [3] \)

\[
S_{1_{Conf}}^f = -\frac{1}{32} \int dtd^4\theta X \log X, \quad \alpha = -1. (5.25)
\]

The invariance of the second term \( S_2 \) needs to be considered more carefully. First of all, one may check that the action \( S_2 \) in the form of (5.15) is invariant under the \( D(2, 1; \alpha) \) group for arbitrary \( \alpha \), provided the components transform under conformal supersymmetry as

\[
\begin{align*}
    \delta v^i &= -2it (\bar{\epsilon}^i \bar{\xi}^2 - \bar{\xi}^i \bar{\xi}^2), \\
    \delta \bar{\xi}^i &= -\alpha \bar{\epsilon}_i v^i + t \bar{\epsilon}_i \bar{\epsilon}^i, \\
    \delta x &= -t (\bar{\epsilon}_i \eta^i + \bar{\epsilon}^i \bar{\eta}_i), \\
    \delta \eta^i &= -2t \bar{\epsilon}^i \bar{x} - \bar{\epsilon}_i \bar{t} x - t \bar{\epsilon}_i \bar{t}, \\
    \delta A_{ij} &= -2(1 + 2\alpha) (\bar{\epsilon}_i \eta_j - \bar{\epsilon}_j \eta_i) - 2t (\bar{\epsilon}_i \bar{\eta}_j - \bar{\epsilon}_j \bar{\eta}_i).
\end{align*}
\]  

(5.26)

Thus, the action (5.15) with the properly chosen superpotential \( F \) as in (5.24), (5.24) is invariant with respect to the full \( N = 4 \) superconformal group \( D(2, 1; \alpha) \).

The crucial point in proving the superconformal invariance of our action, was its form (5.15) obtained after reduction (5.12). If we instead would check the invariance of the action (5.9) and limit ourselves by considering the local transformations of fermionic superfields \( \Psi^a \) we would get that

\[
\delta (\Psi^a \bar{\Psi}_a) = 2i(1 + \alpha) (\epsilon_i \bar{\theta}^i + \bar{\epsilon}^i \theta_i) (\Psi^a \bar{\Psi}_a). (5.28)
\]

Therefore the full action (5.9) will be invariant only for \( \alpha = -1 \) which corresponds just to the \( SU(1, 1|2) \) group! Clearly, with this value of \( \alpha \) the first term is also fixed to be \( F = X \log X \). As we already proved this is not correct. The subtle point is the locality of the transformation properties of \( \Psi^a \). Indeed, from the explicit form of \( \delta \xi^a (5.26) \), it follows that they can be explicitly integrated only for \( \alpha = -1 \). For any other value of parameter \( \alpha \) the integrated \( \delta \xi^a \) which is just \( \delta \psi^a \) will contain non-local term. Thus, we see that similarly to the preceding Sections the action does not seem to be conformally invariant, but it possesses this invariance by modification of the transformation properties of the involved fields.

To conclude, let us make several comments.

Funnily enough, but in contrast with the standard action (5.9), by fixing the bosonic potential in the action (5.9) to be \( 1/y^2 \) in flat coordinates, one does not completely fix the prepotential \( F \). Indeed, rewriting (5.9) in flat coordinates \( y(x) \) with \( F'' = (y')^2 \) we get the condition

\[
\frac{dy}{dx} = ay \quad \Rightarrow \quad y(x) = x^a. (5.29)
\]
Thus, any polynomial superpotential \( F \sim X^\alpha \), will give rise to a \( N = 4 \) supersymmetric mechanics with inverse square potential term in the bosonic sector.

One of the most interesting examples of such superpotentials are the superconformally invariant ones (5.24) and (5.26). Thus, the actions of \( D(2,1;\alpha) \) superconformal invariant mechanics read

\[
S_\alpha = \int dt \left[ \frac{\eta^2}{2} + (1 + \alpha) \frac{i}{8} \left( \eta^{\dagger} \eta - \eta^{\dagger} \eta \right) + \frac{i}{4} \left( \tilde{w}_i \tilde{w}_i - w_i \tilde{w}_i \right) - \frac{\alpha^2}{8(1 + \alpha)y^2} (w^i \tilde{w}_i)^2 - \right.
\]

\[
\frac{\alpha}{8y^2} \eta^{\dagger} \eta \left( w^i \tilde{w}^j + w^j \tilde{w}^i \right) + \frac{(1 + \alpha)(1 + 2\alpha)}{64y^2} \eta^{\dagger} \eta \right], \quad \alpha \neq -1, 0 \tag{5.30}
\]

where

\[
y = x^{- \frac{\alpha}{2}}, \quad \tilde{\eta}^i = x^{- \frac{\alpha}{2\alpha}} \eta^{\dagger} \eta^i \tag{5.31}
\]

and

\[
S_{-1} = \int dt \left[ \frac{\tilde{\eta}^2}{2} + i \left( \eta^{\dagger} \tilde{\eta} - \eta^{\dagger} \tilde{\eta} \right) + \frac{i}{4} \left( \tilde{w}_i \tilde{w}_i - w_i \tilde{w}_i \right) - \frac{1}{8y^2} \eta^{\dagger} \eta \right], \quad \alpha = -1, \quad y = \sqrt{x}, \quad \tilde{\eta}^i = \frac{\eta^i}{\sqrt{x}}. \tag{5.32}
\]

Now it is clear that the simplest case of \( N = 4 \) superconformal invariant mechanics corresponds just to the \( \alpha = -1/2 \) case, i.e. the \( OSp(4|2) \) superconformal group. Indeed, it follows from (5.30), that with \( \alpha = -1/2 \) the four-fermionic interaction disappears from the Lagrangian. This means that the corresponding supercharges contain the fermions only linearly, similarly to the \( N = 2 \) supersymmetric case.

Finally, one should note that our consideration in this Section is very close to the one presented in the recent paper \([7]\). Of course, here we considered only the one-particle case and used the standard \( N = 4, d = 1 \) superspace, in contrast with the harmonic superspace approach \([13, 6, 14]\) advocated in \([7]\). It seems that our action (5.9) is a more economical one. In any case, the final component action (5.18) has to coincide with the one which could be obtained from the harmonic superspace action presented in \([7]\) upon gauge fixing, integration over theta’s and harmonics, and elimination of the auxiliary components.

6 Conclusion

In this paper we demonstrated that (super)conformal mechanics with an additional harmonic oscillator term in the bosonic sector possesses the same superconformal symmetry as the standard one. The main difference between systems with and without oscillator term is the modification of the transformation laws in such a way that the kinetic term is invariant under the (super)conformal group only together with the oscillator potential. The treatment of the bosonic case has natural and straightforward extensions to \( N = 2 \) and \( N = 4 \) superconformal symmetry. Probably, our approach could be also extended to the case of \( N \)-extended supersymmetric mechanics with \( Osp(1,1|N/2) \) superconformal group \([8]\). Another interesting question for further investigation is the generalization of this approach to the \( n \)-particles superconformal mechanics \([15]\).

We also analyzed in this paper the generation of the bosonic potentials in \( N = 4 \) supersymmetric mechanics through coupling it with auxiliary fermionic supermultiplet. In contrast with \([6]\) the coupling we introduced does not advocate new fermionic degrees of freedom - all our additional fermions are purely auxiliary ones. The additional bosonic components have a first order kinetic term and therefore they serve as spin degrees of freedom. The new coupling we introduced in this paper is invariant under the full \( D(2,1;\alpha) \) superconformal group. This invariance is not evident, because the starting action possesses only \( SU(1,1|2) \) invariance, provided we limit ourselves by considering local transformation properties of involved superfields. The invariance under the \( D(2,1;\alpha) \) superconformal group is achieved within non-local transformations, which become local in terms of new variables. Thus, similarly to the situation with oscillator type potentials, the key ingredient for the construction of the most general \( N = 4 \) superconformal mechanics is the modification of (super)conformal transformations.

Our approach in this paper is very close to the one recently proposed in \([7]\). It would be interesting to compare our action with the one-particle action (with all fermionic terms included) presented in \([7]\). They
have to coincide, because the main ingredient - the action describing the coupling of the basic \((1, 4, 3)\) supermultiplet \(X\) with the auxiliary bosonic \((4, 4, 0)\) one is unique and is completely fixed by \(N = 4\) Poincaré supersymmetry (up to an overall constant). The full component action we constructed therefore, the corresponding supercharges are linear over fermionic components, similarly to the case of \(N = 2\) supersymmetry.

Finally, the way to deal with spin degrees of freedom proposed in the present work could be relevant for a proper supersymmetric generalization of the system with Yang monopole recently analyzed in [16].

Acknowledgments

We acknowledge discussions with A. Shcherbakov.

S.K. thanks the Laboratori Nazionali di Frascati for the warm hospitality extended to them during the course of this work. This work was partially supported by INTAS under contract 05-7928, RFBR grants 08-02-90490-Ukr, 06-02-16684 and DFG grant 436 Rus 113/669/03.

Appendix. \(N=4, d=1\) Superconformal algebra

The most general \(N=4, d=1\) superconformal algebra is the superalgebra \(D(2, 1; \alpha)\). We use the standard definition of this superalgebra [17] with the notations of refs. [18, 12]. It contains nine bosonic generators which form a direct sum of \(sl(2)\) with generators \(P, D, K\) and two \(su(2)\) subalgebras with generators \(V, \nabla, V_3\) and \(T, \bar{T}, T_3\), respectively:

\[
i [D, P] = P, \ i [D, K] = -K, \ i [P, K] = -2D, \ i [V_3, V] = -V, \ i [V_3, \nabla] = \nabla; \\
i [V, \nabla] = 2V_3, \ i [T_3, T] = -T, \ i [T_3, \bar{T}] = \bar{T}, \ i [T, \bar{T}] = 2T_3. \tag{A.1}
\]

The eight fermionic generators \(Q^i, \bar{Q}_i, S^i, \bar{S}_i\) are in the fundamental representations of all bosonic subalgebras (in our notation only one \(su(2)\) is manifest):

\[
i [D, Q^i] = \frac{1}{2} Q^i, \ i [D, S^i] = -\frac{1}{2} S^i, \ i [P, S^i] = -Q^i, \ i [K, Q^i] = S^i, \\
i [V_3, Q^1] = \frac{1}{2} Q^1, \ i [V_3, Q^2] = -\frac{1}{2} Q^2, \ i [V_3, Q^3] = Q^3, \ i [V, \bar{Q}_1] = -\bar{Q}_1, \\
i [V_3, S^1] = \frac{1}{2} S^1, \ i [V_3, S^2] = -\frac{1}{2} S^2, \ i [V_3, S^3] = S^3, \ i [V, \bar{S}_1] = -\bar{S}_1, \\
i [T_3, Q^i] = \frac{1}{2} Q^i, \ i [T_3, S^i] = \frac{1}{2} S^i, \ i [T, Q^i] = \bar{Q}_i, \ i [T, S^i] = \bar{S}_i. \tag{A.2}
\]

The fermionic generators \(Q^i, \bar{Q}_k\) together with \(P\) form the \(N = 4, d = 1\) super Poincaré subalgebra, while \(S^i, \bar{S}_k\) generate superconformal translations:

\[
\{Q^i, \bar{Q}_j\} = -2\delta^i_j P, \ \{S^i, \bar{S}_j\} = -2\delta^i_j K. \tag{A.3}
\]

The non-trivial dependence of the superalgebra \(D(2, 1; \alpha)\) on the parameter \(\alpha\) manifests itself only in the cross-anticommutators of the Poincaré and conformal supercharges

\[
\{Q^i, S^j\} = -2(1 + \alpha)\delta^{ij} T, \ \{Q^1, \bar{S}_2\} = 2\alpha \nabla, \ \{Q^1, \bar{S}_3\} = -2D - 2\alpha V_3 + 2(1 + \alpha)T_3, \\
\{Q^2, \bar{S}_1\} = -2\alpha V, \ \{Q^2, \bar{S}_3\} = -2D + 2\alpha V_3 + 2(1 + \alpha)T_3. \tag{A.4}
\]

The generators \(P, D, K\) are chosen to be hermitian, and the remaining ones obey the following conjugation rules:

\[
(T)^\dagger = \bar{T}, \ (T_3)^\dagger = -T_3, \ (V)^\dagger = \nabla, \ (V_3)^\dagger = -V_3, \ (Q^j)^\dagger = \bar{Q}_j, \ (S^i)^\dagger = \bar{S}_i. \tag{A.5}
\]

The parameter \(\alpha\) is an arbitrary real number. At \(\alpha = 0\) and \(\alpha = -1\) one of the \(su(2)\) algebras decouples and the superalgebra \(su(1, 1|2) \oplus su(2)\) is recovered.
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