Regular Objects, Multiplicative Unitaries and Conjugation

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Abstract

The notion of left (resp. right) regular object of a tensor $C^*$–category equipped with a faithful tensor functor into the category of Hilbert spaces is introduced. If such a category has a left (resp. right) regular object, it can be interpreted as a category of corepresentations (resp. representations) of some multiplicative unitary. A regular object is an object of the category which is at the same time left and right regular in a coherent way. A category with a regular object is endowed with an associated standard braided symmetry.

Conjugation is discussed in the context of multiplicative unitaries and their associated Hopf $C^*$–algebras. It is shown that the conjugate of a left regular object is a right regular object in the same category. Furthermore the representation category of a locally compact quantum group has a conjugation. The associated multiplicative unitary is a regular object in that category.

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1 Introduction

In this paper we look at the theory of multiplicative unitaries from the standpoint of their categories of representations and corepresentations. As is well known, multiplicative unitaries just express the fundamental property of the regular representation. Our approach therefore starts with a tensor category which may be thought of as the tensor category of (unitary) representations of some quantum group. It is regarded as a concrete category in the sense that it is equipped with a faithful tensor functor into the tensor category of Hilbert spaces. Once this tensor category has a “regular object” we will see that it allows an interpretation as a category of representations of a multiplicative unitary and at the same time as a category of corepresentations of another multiplicative unitary. It is instructive to compare this result with the Tannaka--Krein duality theorem or perhaps better with Woronowicz's duality theorem [16]. In fact, our result starts with an embedded tensor category and constructs a multiplicative unitary and hence, if the multiplicative unitary is regular, two Hopf $C^*$–algebras [1]. However, by requiring the existence of a regular object, we are imposing a restriction that may not be easy to verify in practice and presupposes what a good duality theorem should prove. In fact, our result is close in spirit to Tatsuuma's duality theorem for locally compact groups [14] where the group elements are identified in the regular representation using the multiplicative unitary.

Another aspect of the representation theory of multiplicative unitaries that has not received the attention it deserves is the conjugation structure. We work here with multiplicative unitaries arising as the left regular representations of locally compact quantum groups. These are left regular objects in the category of corepresentations and we show that there is a canonical choice of conjugate which is a right regular object. In fact, the multiplicative unitary of a locally compact quantum group is a regular object in its representation category. Furthermore, we define the conjugate of any corepresentation up to unitary equivalence and the corresponding antilinear involution on intertwiners. This forms the subject matter of Section 5.

In this paper we prefer to work with strictly associative tensor products and a simple way of achieving this is to use as the underlying Hilbert spaces the Hilbert spaces in some fixed von Neumann algebra since these are objects in a strict tensor $W^*$–category. We will be concerned here with the representation categories of multiplicative unitaries and recall the basic definitions from [1]. If $K$ is such a Hilbert space then a unitary $V$ on the tensor square $K^2$ is said to
be multiplicative if
\[ V_{12}V_{13}V_{23} = V_{23}V_{12}, \]
where we use the usual convention regarding indices and tensor products. A representation of \( V \) on a Hilbert space \( H \) is a unitary \( W \in (HK, HK) \) such that
\[ W_{12}W_{13}V_{23} = V_{23}W_{12}, \quad \text{on} \quad HK^2. \]

If \( W \) and \( W' \) are representations of \( V \) on \( H \) and \( H' \) respectively, we say that \( T \in (H, H') \) intertwines \( W \) and \( W' \) and write \( T \in (W, W') \) if \( T \times 1_K W = W' T \times 1_K \). We define the tensor product of \( W \) and \( W' \) to be the representation \( W \times W' \) on \( HH' \) given by \( W \times W' := W_{13}W'_{23} \). The usual tensor product of intertwiners is again an intertwiner and in this way we get a strict tensor \( W^* \)-category \( \mathcal{R}(V) \) of representations of \( V \). In fact this assertion does not depend on \( V \) being multiplicative. When it is then \( V \) itself is a representation of \( V \) called the regular representation.

A corepresentation of \( V \) on \( H \) is a unitary \( W \in (KH, KH) \) such that
\[ V_{12}W_{13}W_{23} = W_{23}V_{12}, \quad \text{on} \quad K^2H. \]

If \( W \) and \( W' \) are corepresentations on \( H \) and \( H' \) respectively, we say that \( T \in (H, H') \) intertwines \( W \) and \( W' \) and write \( T \in (W, W') \) if \( 1_K \times T W = W' T \times 1_K \). The tensor product \( W \times W' \) of corepresentations is defined by \( W \times W' := W_{12}W'_{13} \). Just as in the case of representations we get a strict tensor \( W^* \)-category now denoted by \( \mathcal{C}(V) \). If \( \vartheta = \vartheta_{K,K} \) denotes the flip on \( K^2 \) then \( \vartheta V^* \vartheta \) is again a multiplicative unitary and the mapping \( W \mapsto \tilde{W} := \vartheta_{H,K} W^* \vartheta_{K,H} \) defines a 1–1 correspondence between representations of \( V \) and corepresentations of \( \vartheta V^* \vartheta \). However, it does not define an isomorphism of tensor \( W^* \)-categories since \( W \times W' \mapsto \tilde{W}_{13} \tilde{W}_{12} \) and so leads to an alternative definition of the tensor product of corepresentations. In fact the two expressions for the tensor product will be equal if and only if \( \vartheta_{W,W'} \in (W \times W', W' \times W) \), cf. Prop. 2.5 in [15].

## 2 Regular Objects and Multiplicative Unitaries

The main aim of this section will be to provide characterizations of categories of representations and corepresentations of multiplicative unitaries and in particular to study tensor categories which are simultaneously a tensor category of representations of a multiplicative unitary and of corepresentations of some
regular object. The main idea is to replace the notion of multiplicative unitary by that of regular object. Thus multiplicative unitaries are seen as intertwining operators, taking us back to the origins of the theory. They are therefore seen not as determining a category of representations or corepresentations but as being a structural element in some tensor category. This helps us to understand the degree to which they are not unique and to see the tensor categories that are simultaneously a category of representations and a category of corepresentations as being tensor categories with a left and right regular object.

Here is our motivating example. Let \( \mathcal{H} \) denote the strict tensor \( W^* \)-category of Hilbert spaces in a von Neumann algebra \( M \) and \( \vartheta \) its unique permutation symmetry. Let \( K \) be an object of \( \mathcal{H} \) and \( V \) a multiplicative unitary on \( K^2 \). We let \( R(V) \) and \( C(V) \) be the tensor \( W^* \)-categories of representations and corepresentations of \( V \) on Hilbert spaces of \( \mathcal{H} \). These are to be considered as equipped with the forgetful functor \( \iota \) into \( \mathcal{H} \) itself, regarded as the subcategory of trivial representations or corepresentations. Thus \( \iota \) is an idempotent tensor \( * \)-functor.

We now ask the following question: when can a strict tensor \( W^* \)-category \( T \) equipped with a faithful idempotent tensor \( * \)-functor \( \iota_T := \iota \) onto a tensor \( W^* \)-subcategory of Hilbert spaces be interpreted as a category of representations or corepresentations of a multiplicative unitary? Note that \( \vartheta_{H,H'} \) is an intertwining operator in \( R(V) \) for a tensor product of representations \( W \) on \( H \) and \( W' \) on \( H' \) whenever either \( W \) or \( W' \) is a trivial representation. Any full tensor subcategory \( \mathcal{T} \) of \( R(V) \) containing the regular representation \( V \) has the striking property that for any object \( W \), \( W \times V \) is a, possibly infinite, direct sum of copies of \( V \) but more is true: we set \( \eta_W := W \in (W \times V, \iota(W) \times V) \) then \( \eta \in (R, R \iota) \), where \( R \) denotes the functor of tensoring on the right by \( V \), is a natural unitary transformation such that

\[
\eta_{W \times W'} = (\eta_W)_{13} \circ 1_W \times \eta_{W'},
\]

for each pair \( W, W' \) of objects of \( \mathcal{T} \).

To formalize the essential aspects of the above situation we consider a strict tensor \( W^* \)-category \( \mathcal{T} \) equipped with a faithful idempotent tensor \( * \)-functor \( \iota_T := \iota \). The tensor subcategory \( \iota(\mathcal{T}) \) is equipped with a (permutation) symmetry \( \vartheta \). We further suppose that given objects \( W \) and \( W' \) of \( \mathcal{T} \) there are arrows \( \vartheta_{W,\iota(W')} \in (W \times \iota(W'), \iota(W') \times W) \) and \( \vartheta_{\iota(W),W'} \in (\iota(W) \times W', W' \times \iota(W)) \), necessarily unique, whose image under \( \iota \) is \( \vartheta_{\iota(W),\iota(W')} \). We call a right regular
object of \( T \) a pair \((V, \eta)\) consisting of an object \( V \) of \( T \) and a unitary natural transformation \( \eta \in (R, R\iota) \), where \( R \) denotes the functor of tensoring on the right by \( V \), satisfying (2.1) above for each pair \( W, W' \) of objects of \( T \). Here \((\eta_W)_1^{13}\) is to be understood as \( 1_{\iota(W)} \times \vartheta_{V, \iota(W')} \circ \eta_W \times 1_W \times \vartheta_{\iota(W'), V} \). (2.1) implies that \( \eta \) evaluated on the tensor unit \( \mathbb{C} \) is \( 1_{\mathbb{C}} \).

The following result now provides an answer to the above question.

2.1 Theorem Any tensor \( W^* \)-category equipped with a faithful idempotent tensor \( ^* \)-functor into a tensor subcategory of Hilbert spaces and with a right regular object is isomorphic to a tensor \( ^* \)-subcategory of \( \mathcal{R}(V) \) for some multiplicative unitary \( V \).

Proof. Let \( \eta \) denote a natural transformation in \((R, R\iota)\) making an object \( V \) into a right regular object then \( \iota(\eta_W) \) is a unitary for each object \( W \). If \( T \in (W, W') \) then the naturality of \( \eta \) shows that \( \iota(T) \) intertwines \( \iota(\eta_W) \) and \( \iota(\eta_W') \), once we know that these are representations of \( \iota(\eta_V) \). In particular, taking \( \eta_W \) as \( T \), naturality gives

\[
\iota(\eta_W) \times 1_V \circ \eta_W \times V = \eta_{\iota(W)} \times V \circ \eta_W \times 1_V.
\]

Equation (2.1) tells us that the tensor product in the category corresponds to the tensor product of representations. Bearing this in mind, (2.2) tells us that \( \iota(\eta_W) \) is a representation of \( \iota(\eta_V) \) and the particular case \( W = V \) tells us that \( \iota(\eta_V) \) is indeed a multiplicative unitary.

The notion of multiplicative unitary and Theorem 2.1 can be easily generalized replacing a multiplicative unitary on a Hilbert space by a multiplicative invertible in a monoidal category. We refrain from spelling this out to keep a uniform setting for this paper.

Notice that the isomorphism in question is even canonical, given \( \eta \), and commutes with \( \iota \). But there are several other comments to be made about this result. First, (2.2) has the structure of an associative law: it equates two ways of passing from \( RR \) to \( R\iota R\iota \). Secondly, there is an analogous result for corepresentations. We define a notion of left regular object by dualizing in \( T \) with respect to the composition law \( \times \). Our category \( T \) is isomorphic to a tensor subcategory of the category of corepresentations of a multiplicative unitary if it admits a left regular object. If \( \xi \) denotes a natural transformation rendering \( V \) a left regular object \((\xi, V)\), then the unitary associated with an object \( W \) is \( \iota(\xi_W^{-1}) \). The appearance of an inverse here is just an artefact of conventions.

One might have thought of basing a definition of right regular object on a different familiar property of the regular representation, namely that \( W \times V \)
is a, possibly infinite, direct sum of copies of \( V \) for each object \( W \) of \( \mathcal{T} \). This property is too weak in that it does not imply the coherence properties of the previous definition and furthermore puts an unwanted emphasis on the notion of infinite direct sum. In fact, our definition implies the second property once we specify, as we now do, that \( \iota(\mathcal{T}) \) is just a category of Hilbert spaces, i.e. a (strict) tensor \( W^\ast \)-category with unit reducing to the complex numbers where every object is a (possibly infinite) direct sum of the unit. Since \( \iota(W) \) is a direct sum of copies of \( \mathbb{C} \), \( W \times V \cong \iota(W) \times V \) is a direct sum of copies of \( V \). Note that if \( V_r \) is right regular and \( V_\ell \) is left regular then \( V_\ell \times V_r \) is a direct sum of copies of both \( V_\ell \) and \( V_r \). It follows that, if we have both a left and a right regular object, then these objects are unique up to quasiequivalence in the \( W^\ast \)-category in question. In a \( \sigma \)-finite \( W^\ast \)-category a left or right regular object with infinite multiplicity will then be unique up to unitary equivalence.

The following variant on the definition of a right regular object is worth noting. Consider a tensor \( W^\ast \)-category \( \mathcal{T} \) and unit \( \mathbb{C} \), but where the endofunctor \( \iota \) is not a priori defined. Suppose for each object \( W \), there is a unitary arrow \( \eta_W \in (W \times V, \iota(W) \times V) \) such that \( \iota(W) \) is a (possibly infinite) direct sum of the tensor unit. Suppose (2.1) holds and

\[
\eta_{W'} \circ T \times 1_V \circ \eta_W^{-1} \in (\iota(W'), \iota(W)) \times 1_V, \quad T \in (W, W').
\]

Then setting \( \iota(T) := \eta_{W'} \circ T \times 1_V \circ \eta_W^{-1} \), we get a tensor \( ^* \)-endofunctor from \( \mathcal{T} \) into a tensor subcategory of Hilbert spaces. If \( \eta_{W(W)} = 1_{\iota(W) \times V} \) for each object \( W \) of \( \mathcal{T} \), \( \iota \) is even idempotent. This illustrates the role of (2.1) in guaranteeing that a tensor \( W^\ast \)-category can be embedded into a tensor category of Hilbert spaces.

We now make some further remarks on the notion of regular object, supposing for the moment that our category \( \mathcal{T} \) has sufficient irreducibles in the sense that every object is a (possibly infinite) direct sum of irreducibles and \( \mathcal{T} \) is closed under finite direct sums. Suppose further that the full subcategory \( \mathcal{T}_f \) whose objects are finite direct sums of irreducibles is a tensor subcategory and that \( \iota_W \) is finite dimensional for each irreducible \( W \). A dimension function \( d \) on \( \mathcal{T}_f \) assigns to each object \( W \) of \( \mathcal{T}_f \) a \( d(W) \in \mathbb{R}_+ \) such that

\[
d(W \oplus W') = d(W) + d(W'),
\]

\[
d(W \times W') = d(W)d(W').
\]

Note that if \( F : \mathcal{T}_f \to \mathcal{T}'_f \) is a tensor \( ^* \)-functor and \( d' \) is a dimension function on \( \mathcal{T}'_f \), then \( F \circ d' \) is a dimension function on \( \mathcal{T}_f \). Thus, our category \( \mathcal{T}_f \) has an
integer-valued dimension function induced by the tensor ∗-functor ι from the Hilbert space dimensions. If \( \mathcal{T}_f \) has conjugates, then there is another dimension function, not necessarily integer-valued, given intrinsically by its structure as a tensor \( C^* \)-category \([10]\). Let \( I \) be an index set labelling the equivalence classes of irreducibles and \( W_i \) an irreducible of class \( i \in I \). Then a simple computation shows that if \( d \) is a dimension function and \( d_i := d(W_i) \) then
\[
d_j d_k = \sum_i m^j_{ki} d_i, \quad i, j, k \in I,
\]
where \( m^j_{ki} \) denotes the dimension of \( (W_i, W_j \times W_k) \). Thus the dimension function, which is determined by the \( d_i, i \in I \), gives an eigenvector with positive entries of the matrix \( m^j \) corresponding to the eigenvalue \( d_j \) and simultaneously an eigenvector of \( m_k \) with eigenvalue \( d_k \). Conversely, any such simultaneous eigenvalue does arise in this way. Suppose that \( V \) is a left regular object of \( \mathcal{T} \) such that \( (W, V) \) is finite dimensional for each irreducible \( W \) and hence for each object \( W \) of \( \mathcal{T}_f \). Let \( v_i \) be the dimension of \( (W_i, V) \) and let \( d(W) \) be defined so that \( V \times W \) is a direct sum of \( d(W) \) copies of \( V \). Then \( d \) is an integer-valued dimension function and
\[
d_j v_i = \sum_k m^k_{ji} v_k.
\]
The case of a right regular object can be treated similarly. If \( \mathcal{T}_f \) has conjugates then we have a corresponding involution \( i \mapsto \overline{i} \) on \( I \) and
\[
m^i_{jk} = m^i_{kj} = m^i_{ji}.
\]
If \( d \) is a dimension function, there is a conjugate dimension function \( \overline{d} \) such that \( \overline{d}(W) = d(W) \) for each object \( W \) of \( \mathcal{T}_f \). The interesting dimension functions, such as the intrinsic dimension function of a tensor \( C^* \)-category with conjugates \([10]\), are self-conjugate.

Now, there is another natural transformation implicitly involved in (2.1), namely, \( \theta \in (R_L, L_R) \), defined by
\[
\theta_W := \theta_{(W), V}.
\]
This brings us to the concept of braided symmetry, developed in the Appendix of \([3]\). Let \( \varepsilon \) be a braided symmetry relative to a left regular object \( V \) of \( \mathcal{T} \). Thus \( \varepsilon \) is a unitary natural transformation from the functor \( R \) of tensoring on the right by \( V \) to the functor \( L \) of tensoring on the left by \( V \) such that
\[
\varepsilon_{W \times W'} = \varepsilon_W \times 1_{W'} \circ 1_W \times \varepsilon_{W'}.
\]
Note that $\varepsilon_{\iota(W)} = \theta_{\iota(W),V} = \theta_W$. Since $V \times W$ is just a multiple of $V$ and the functor $L$ is faithful, $\varepsilon_W$ is uniquely determined by $\varepsilon_V$ using

$$\varepsilon_{V \times W} = \varepsilon_V \circ 1_V \times \varepsilon_W.$$ 

The index notation for tensor products will now be taken to refer to the braided symmetry. This is consistent with its use in (2.1). Using the braided symmetry, we get a unitary natural transformation $\eta$ from $R$ to $R\iota$ defined by

$$\eta_W = \varepsilon_W^{-1} \varepsilon_W,$$

where $\xi$ is the unitary natural transformation from $L$ to $L\iota$ making $V$ into a left regular object. We ask whether $\eta$ makes $V$ into a right regular object. This question is addressed in the following results.

2.2 Proposition Let $(\xi, V)$ be a left regular object of $\mathcal{T}$. The braided symmetries $\varepsilon$ for $\mathcal{T}$ relative to $V$ are in 1–1 correspondence with invertible natural transformations $\eta$ from $R$ to $R\iota$ such that

$$\eta_{W \times W'} = (\xi_{W'})_{32}(\eta_{W'})_{13}(\xi_W^{-1})_{32}(\eta_W)_{23}. \quad (2.3)$$

$\varepsilon$ and $\eta$ are related by

$$\varepsilon_W = \xi_W^{-1} \theta_W \eta_W. \quad (2.4)$$

Proof. Given $\varepsilon$, equation (2.4) defines a natural unitary transformation $\eta$ and (2.3) follows by direct computation. Conversely, given $\eta$, equation (2.4) defines a natural unitary transformation $\varepsilon$ which is a braided symmetry by virtue of (2.3).

If we take the images under $\iota$ of the terms in (2.3) then the computation leading to (2.3) can be modified to show that the analogous identity holds with the tensor product notation now referring to the permutation of Hilbert spaces. Note, too, that (2.3) can be used to compute $\eta$ in terms of $\eta_V$.

2.3 Theorem Given functors $L$ and $R$ of tensoring on the left and right, respectively, by an object $V$ of $\mathcal{T}$ and invertible natural transformations $\xi \in (L, L\iota)$, $\eta \in (R, R\iota)$ and $\varepsilon \in (R, L)$ such that $\xi \varepsilon = \theta \eta$, consider the following four conditions:

a) $(\xi, V)$ is a left regular object,

b) $(V, \eta)$ is a right regular object,
c) $\varepsilon$ is a braided symmetry relative to $V$, 

\[ d) \eta_W \times 1_{i(W')} \circ 1_W \times \xi_{W'} = 1_{i(W)} \times \xi_{W'} \circ \eta_W \times 1_{W'}, \text{ for each pair } W, W' \text{ of objects of } \mathcal{T}. \]

Then any three of these conditions imply the fourth.

**Proof.** We see from Proposition 2.2 that given a) and c), b) is equivalent to requiring that each pair $(\eta_W)_{13}$ and $(\xi_{W'})_{32}$ commute. Interchanging 2 and 3 using the braided symmetry, we see that, given a) and c), b) and d) are equivalent. Similarly, given b) and c), a) and d) are equivalent. It remains to show that a), b) and d) imply c). However, given a), b) and d), (2.3) follows from (2.1), since d) implies that $(\eta_W)_{13}$ and $(\xi_{W'})_{32}$ commute. Thus $\varepsilon$ is a braided symmetry, completing the proof.

An alternative way of proving the above theorem is by arguing in terms of a diagram with ten vertices, where the conditions a), b), c) and d) are expressed as the commutativity of subdiagrams. The reader is urged to draw the diagram for himself. Begin with an outer square whose sides are used as an hypotenuse for the conditions on $\eta, \xi, \varepsilon$ and $\theta$ respectively with d) as a rhombus in the middle of the square.

We may also strengthen one of the implications in the above theorem.

**2.4 Lemma** Under the hypotheses of Theorem 2.3, the conditions a), b) and

\[ d') \eta_V \times 1_{i(V)} \circ 1_V \times \xi_V = 1_{i(V)} \times \xi_V \circ \eta_V \times 1_V \]

imply that $\varepsilon$ is a braided symmetry.

The necessary computations can be found in the proof of Theorem A.2 in [3]. This proof can be rewritten entirely in terms of compositions in the tensor category $\mathcal{T}$ and this is recommended to the reader as an exercise. The computations also show that a), c) and d) with $V$ in place of $W'$ imply b) and that b), c) and d) with $V$ in place of $W$ imply a).

We will refer to a braided symmetry fulfilling the conditions of Theorem 2.3 as being a *standard* braided symmetry. In the presence of a standard braided symmetry we have an object which is at the same time a left and right regular object in a coherent way in that it fulfills d) of Theorem 2.3. We call such an object a *regular* object. Under these circumstances we have the following corollary of Theorem 2.1.

**2.5 Corollary** A tensor $W^*$–category equipped with a faithful idempotent tensor $^*$–functor into a tensor subcategory of Hilbert spaces and with a regular
object \((\xi, V, \eta)\) is isomorphic to a tensor subcategory of both \(\mathcal{R}(\iota(\eta_V))\) and \(\mathcal{C}(\iota(\eta_V)^{-1})\). There is an associated standard braided symmetry given by (2.4).

In particular, if a multiplicative unitary \(V\) considered as a corepresentation and hence a left regular object in \(\mathcal{C}(V)\) is even a right regular object, there is a multiplicative unitary \(V\) on the same Hilbert space such that \(\mathcal{C}(V)\) is canonically isomorphic as a tensor \(W^\star\)-category to a tensor \(*\)-subcategory of \(\mathcal{R}(\hat{V})\). We do not know when the image coincides with \(\mathcal{R}(\hat{V})\).

Given a left regular object \(V\) of \(\mathcal{T}\), we would like to analyse in how many different ways we may choose \(\eta\) and \(\varepsilon\) so as to fulfill the conditions of Theorem 2.3. Of course \(\varepsilon\) determines \(\eta\) uniquely so our question amounts to parametrizing the standard braided symmetries relative to \(V\). It is convenient to rephrase this problem in terms of the associated natural transformations (Prop. 2.2).

In the remark following Proposition 2.2, we have noted that any such natural transformation is uniquely determined by its value in \(V\). Therefore we first consider the situation where \(\mathcal{T}\) is \(\mathcal{C}(V)_V\), the full tensor subcategory of \(\mathcal{C}(V)\) generated by \(V\).

**2.6 Lemma** Let \(\xi\) be the natural unitary transformation from \(L\) to \(L\) making \(V\) into a left regular object of \(\mathcal{C}(V)_V\), and let \(\eta_V\) be a unitary operator on the Hilbert space \(K^2\) of \(V\). Then there is a natural unitary transformation \(\eta\) from \(R\) to \(R\) taking the value \(\eta_V\) at \(V\) and defining a standard braided symmetry \(\varepsilon\) on \(\mathcal{C}(V)_V\) if and only if \(\eta_V\) satisfies

\[
\begin{align*}
&\text{a) } \eta_V \times 1_{\iota(V)} \circ 1_V \times \xi_V = 1_{\iota(V)} \times \xi_V \circ \eta_V \times 1_V, \\
&\text{b) } \eta_V \circ T \times 1_V = \iota(T) \times 1_V \circ \eta_V, \quad T \in (V, V), \\
&\text{c) } \eta_V \in (V^{\times 2}, \iota(V) \times V), \\
&\text{d) } \iota(\xi_V) \times 1_V \circ (\eta_V)_{13} \circ 1_V \times \eta_V = (\eta_V)_{13} \circ \xi_V \times 1_V.
\end{align*}
\]

**Proof.** If \(\eta\) is the natural unitary transformation associated with the standard braided symmetry \(\varepsilon\) then, by Proposition 2.2, \(\eta \in (R, Ru)\), thus, evaluating in \(V\), we get c); a) is a special case of d) in Theorem 2.3. Since \(\eta\) is natural, given any pair \(W, W' \in \mathcal{C}(V)_V\) and any \(T \in (W, W')\), \(\iota(T) \times 1_V \circ \eta_V = \eta_W \circ T \times 1_V\), thus choosing \(W = W' = V\) we obtain b). On the other hand \(V \in (V \times \iota(V), V^{\times 2})\), therefore, as \(\eta\) makes \(V\) into a right regular object, d) follows from the naturality of \(\eta\). Conversely, by virtue of a) and Lemma 2.4, it suffices to show that \(\eta_V^{\times r} := (\eta_V)_{1r+1} \ldots (\eta_V)_{rr+1}\) is a natural unitary transformation from \(R\) to \(Ru\) making \(V\) into a right regular object. It is easy to see that c) yields \(\eta_V^{\times r} \in (V^{\times r} \times V, \iota(V^{\times r}) \times V)\), and we must show the naturality
of \( \eta \), i.e. that for any \( T \in (V^{\times r}, V^{\times s}) \), \( \eta_{V^{\times r}} \circ T \times 1_V = \iota(T) \times 1_V \circ \eta_{V^{\times r}} \). We note that if this relation holds for \( r+1 \) and \( s+1 \) then it holds for \( r \) and \( s \) as well, since \((V^{\times r}, V^{\times s})\) embeds in \((V^{\times r+1}, V^{\times s+1})\) via \( R \). Therefore, it suffices to assume \( r, s \) sufficiently large. Now by b) the relation holds for \( r = s = 1 \). We regard the Hilbert space \( K \) of the corepresentation \( V \) as a space of bounded linear operators from \( K \) to \( K^2 \) by letting the elements of \( K \) act by tensoring on the left. By the multiplicativity of \( V \), \( V \vartheta_{K,K} K \) is a Hilbert space \( \tilde{K} \) of intertwiners of \( \mathcal{C}(V)_V \), contained in \((V,V^{\times 2})\) and property d) shows that the desired relation holds for elements of \( \tilde{K} \). For \( s \geq 2 \), on the other hand, \((V^{\times r}, V^{\times s})\) is generated as a weakly closed subspace of \((K^r, K^*)\) by elements of the form \( \psi \times 1_{V^{\times s-2}} \circ T \), with \( T \in (V^{\times r}, V^{\times s-1}) \) and \( \psi \in \tilde{K} \). The relations therefore hold for a generating set of intertwiners in \( \mathcal{C}(V)_V \), and hence for all the intertwiners, completing the proof.

It emerges from the proof that property d) has the role of ensuring the naturality of \( \eta \) for elements of the Hilbert space \( \tilde{K} \subset (V,V^{\times 2}) \). \( \tilde{K} \) could be replaced by any other Hilbert space with support \( I \) in \((V,V^{\times 2})\). Choosing \( \tilde{K} := \eta_V^{-1} K \) amounts to replacing d) by

\[ d') \eta_V \text{ is a multiplicative unitary on } K^2. \]

Furthermore, we note that b) characterizes the elements of \((V,V) \subset (K,K)\). Indeed, if \( T \in (K,K) \) satisfies \( \eta_V \circ T \times 1_V = T \times 1_V \circ \eta_V \) then by c) \( T \times 1_V = \eta_V^* \circ T \times 1_V \circ \eta_V \in (V^{\times 2},V^{\times 2}) \), so \( T \in (V,V) \).

To parametrize the standard braided symmetries, we shall need two further notions: let \( W \) be an object of \( \mathcal{C}(V) \) acting on \( H \) and set

\[ G_W := \{ U \in \mathcal{U}(H) : TU^{\times r} = U^{\times s} T , \ T \in (W^{\times r}, W^{\times s}) \} \]

\[ 2.7 \text{ Proposition } \text{If } V \text{ is the regular corepresentation then } G_V = \{ U \in (V,V)' : \delta(U) : VU \times 1_V V^{-1} = U \times U \} \]

**Proof.** \( U \in G_V \) implies \( \delta(U) = U \times U \) since \( V \in (V \times 1_V , V \times V) \) and \( U \in (V,V)' \).

For the converse, note that the above two conditions suffice to conclude that \( U \in G_V \) by the fundamental property of the regular corepresentation.

We regard the dual multiplicative unitary \( V^d := \vartheta_{K,K} V^{-1} \vartheta_{K,K} \) as a left regular object of \( \mathcal{C}(V^d) \). We call \( V \) weakly irreducible if \( (V,V) \cap (V^d,V^d) = CI \).

If \( V \) is irreducible in the sense of \( \mathcal{L} \) then it is weakly irreducible.

\[ 2.8 \text{ Theorem } \text{Let } V \text{ be a weakly irreducible multiplicative unitary, and let } \eta, \tilde{\eta} \in (R,R^c) \text{ be natural unitary transformations defining standard braided symmetries on } \mathcal{C}(V)_V. \text{ Then there is a unique unitary } U \in G_V \text{ such that} \]
\[ \tilde{\eta}_V = U \times 1_V \circ \eta_V. \] Conversely, given any unitary \( U \in G_V \) and any natural transformation \( \eta \) as above there is a unique \( \tilde{\eta} \) such that \( \tilde{\eta}_V = U \times 1_V \circ \eta_V. \)

**Proof.** Let \( \eta \) and \( \tilde{\eta} \) define standard braided symmetries on \( \mathcal{C}(V)_V \). By virtue of the commutation relation a) of Lemma 2.6, for any \( \omega \in (K, K)_* \), \( \omega \otimes \iota(\eta_V) \circ V = V \circ \omega \otimes \iota(\eta_V) \in (V^d, V^d) \). On the other hand, by c) of the same lemma, \( \tilde{\eta}_V \eta_V^{-1} \in (\iota(V) \times V, \iota(V) \times V) = (\iota_V, \iota_V) \otimes (V, V), \)

so \( \tilde{\eta}_V \eta_V^{-1} \in (K, K) \otimes (V, V) \cap (V^d, V^d) = (K, K) \otimes \mathbb{I} \) since \( V \) is weakly irreducible. Now, by b), \( \iota \otimes \omega(\eta_V) \in (V, V)', \) so both \( \eta_V, \tilde{\eta}_V, \) and therefore \( \eta_V \eta_V^{-1} \) belong to \( (V, V)' \otimes (K, K) \). We conclude that there is a \( U \in (V, V)' \) with \( \tilde{\eta}_V = U \times 1_V \eta_V. \) Finally, comparing d) for \( \eta_V \) and \( \tilde{\eta}_V, \) we conclude that \( U \) satisfies \( \tilde{\eta}_V = U \times 1_V \eta_V = U \times U \circ 1_V, \) i.e. \( U \in G_V. \)

Conversely, a straightforward computation shows that any unitary of the form \( \tilde{\eta}_V := U \times 1_V \eta_V, \) with \( U \in G_V, \) satisfies the properties stated in the previous lemma.

**2.9 Lemma** Let \( V \) be a multiplicative unitary and \( \eta_V \) a unitary operator on \( K^2 \) satisfying properties a) and c) of Lemma 2.6. If \( V \) is regular in the sense of \( \mathbb{R} \), then the algebra generated by \( (V, V) \) and \( (V^d, V^d) \) acts irreducibly on \( K. \)

**Proof.** For any pair \( \psi \) and \( \varphi \) of elements of \( K \) we may write

\[
\psi^* \times 1_V \varphi \times 1_V = \sum_i \psi^* \times 1_V \eta_V^* \varphi_i \times 1_V \varphi_i^* \times 1_V \eta_V V \varphi \times 1_V,
\]

where \( \varphi_i \) is an orthonormal basis of \( K. \) Now

\[
K^* \times 1_V \eta_V^* K \times 1_V \subseteq (V^d, V^d)
\]
as \( \eta_{V_{12}} \) and \( V_{23} \) commute. On the other hand \( K^* \times 1_V \eta_V V \varphi K \times 1_V \subseteq (V, V), \) therefore \( K^* \times 1_V \varphi \times 1_V \) is a subspace of the weak closure of the algebra generated by \( (V, V) \) and \( (V^d, V^d). \) On the other hand this subspace generates the compact operators since \( V \) is regular, completing the proof.

**2.10 Theorem** If \( V \) is a multiplicative unitary and the algebra generated by \( (V, V) \) and \( (V^d, V^d) \) acts irreducibly on \( \mathcal{C}(V)_V \) then any braided symmetry \( \epsilon \) on \( \mathcal{C}(V)_V \) extends uniquely to a braided symmetry on \( \mathcal{C}(V) \), standard if \( \epsilon \) is standard.

**Proof.** We have already noted that any braided symmetry on \( \mathcal{C}(V) \) is determined uniquely by \( \eta_V. \) The explicit relation is

\[
1_V \times \eta_W = (\xi_W)_{12}(\eta_V^{-1})_{13}(\xi_W^{-1})_{12}(\xi_W^{-1})_{13}(\xi_W)_{13}(\xi_W)_{12}.
\]

We show that the right hand side does define a natural unitary transformation from \( R \) to \( R \) in \( \mathcal{C}(V) \) satisfying (2.3). For brevity, we write \( W^{-1} \) for \( \xi_W. \)
Let $H$ be the Hilbert space of the corepresentation $W$. The key idea is to show that the unitary operator $X_W$ on $KHK$ defined by the right hand side acts trivially on the first factor, by showing that its first component lies in the commutant of the algebra generated by $(V, V)$ and $(V^d, V^d)$, this being the complex numbers, by assumption. We first show that the first component of $X_W$ is in the commutant of $(V^d, V^d)$. Since the first component of $W$ is in $(V^d, V^d)$, and $W^{-1}$ acts trivially on the first factor, it is enough to prove the claim for the first component of $(\eta^{-1})_{13}W_{32}W_{12}(\eta_{V^d})_{13}$. However, by the corepresentation relation $W_{32}W_{12} = V_{31}^{-1}W_{12}V_{31}$, we are thus reduced to showing that the first component of $(\eta^{-1})_{13}V_{31}^{-1}W_{12}V_{31}(\eta_{V^d})_{13}$ is contained in the commutant of $(V^d, V^d)$. Now, this holds in the special case $W = V$ since this operator coincides with $V_{32}(\eta_{V^d})_{23}V_{12}$. In the general case, since the first component of $W$ is contained in $(V^d, V^d)'$ we deduce that we can approximate $W$ weakly by finite sums of operators of the form $1_K \times A^*V1_K \times B$, with $A, B \in (H, K)$. Hence it suffices if the first component of

$$(\eta^{-1})_{13}V_{31}^{-1}1_K \times A^* \times 1_K V_{12}1_K \times B \times 1_K V_{31}(\eta_{V^d})_{13} = 1_K \times A^* \times 1_K (\eta^{-1})_{13}V_{31}^{-1}V_{12}V_{31}(\eta_{V^d})_{13}1_K \times B \times 1_K,$$

is in $(V^d, V^d)'$ and this is now clear.

On the other hand by b) of Lemma 2.6, $\eta_V \in (V, V)' \otimes (K, K)$. To prove the claim it remains to show that the first component of $W_{12}(\eta_{V^d})_{13}W_{12}^{-1}$ is in $(V, V)'$. Now

$$1_{KH} \times K^*W_{12}(\eta_{V^d})_{13}W_{12}^{-1}1_{KH} \times K = W_{1KH} \times K^*(\eta_{V^d})_{13}1_{KH} \times KW^{-1}$$

$$\subset W(V, V)' \otimes CW^{-1} \subset (V, V)' \otimes (W, W)'$$

by the corepresentation relation, as claimed. Let $\eta_{W^d}$ be the unitary on $HK$ defined by $(\eta_{W^d})_{23} = X_{W^d}$. A straightforward computation shows that $X_{W^d} \in (V \times W \times V, V \times \iota(W) \times V)$, thus $\eta_{W^d} \in (W \times V, \iota(W) \times V)$, and that $W \in C(V) \mapsto \eta_{W^d}$ is a natural transformation from $R$ to $R_l$. We now check that (2.3) holds.

$$(\eta_{W^d})_{23} = X_{W^d} = W_{43}^{-1}W_{42}^{-1}(\eta_{V^d})_{14}W_{42}W_{43}W_{12}W_{13}(\eta_{V^d})_{14}W_{13}^{-1}W_{12}^{-1} =$$

$$W_{43}^{-1}W_{42}^{-1}(\eta_{V^d})_{14}W_{42}W_{12}(\eta_{V^d})_{14}W_{43}(\eta_{W^d})_{34}W_{12}^{-1} =$$

$$W_{43}^{-1}(\eta_{W^d})_{24}W_{43}(\eta_{W^d})_{34}.$$
Finally, we prove the last statement. Let us assume that $\varepsilon$ is standard on $\mathcal{C}(V)$, so a) of Lemma 2.6 holds. Hence $\varepsilon$ is standard on $\mathcal{C}(V)$ by Lemma 2.4, completing the proof.

We now describe one way of getting standard braided symmetries on $\mathcal{C}(V)$.

2.11 Proposition Let $V$ be a multiplicative unitary and $U \in \mathcal{U}(K)$ such that

$$\hat{V} = I \times U \vartheta V \vartheta I \times U^*$$

is multiplicative, with $\vartheta = \vartheta_{K,K}$. If $[\hat{V}_{12}, V_{23}] = 0$ and $W \in \mathcal{C}(V)$, then there is a standard braided symmetry $\varepsilon$ on $\mathcal{C}(V)$ defined by:

$$\varepsilon_W := WU \times IWU^* \times I\vartheta_{H,K}.$$

The corresponding natural transformation $\eta$ making $V$ into a right regular object is given by

$$\eta_W := I \times U\vartheta_{K,H}W\vartheta_{H,K}I \times U^*.$$

Proof. It is obvious from the form of $\varepsilon$ that we have a natural transformation. Hence $\eta$ will be a natural transformation, too and a simple computation shows that it makes $V$ into a right regular object. Since $d'$ of Lemma 2.4 holds, $\varepsilon$ is a standard braided symmetry.

If $U \in \mathcal{U}(K)$ has the properties listed in [1] to make $V$ an irreducible multiplicative unitary then all the conditions of the above proposition are satisfied. In particular, $\mathcal{C}(V)$ has a canonical standard braided symmetry if $V$ comes from a Kac–von Neumann algebra as in [1] or is any regular discrete or compact multiplicative unitary. If $V$ is derived from a locally compact group $G$, the corresponding braided symmetry is that derived from the usual permutation symmetry on the representation category of $G$ interchanging the order of factors in the tensor product of two representations.

3 Conjugation

Our aim is to discuss conjugation in the context of multiplicative unitaries and their associated Hopf algebras. Although this aspect was not discussed in [1], relevant related work can be found in a number of publications, and we refer, in particular, to the work of Woronowicz in the context of compact quantum groups [10].
However, some of the relevant problems can be seen at the level of the representation theory of $C^*$-algebras and von Neumann algebras and it is hence wise to discuss them in this simplified setting. We therefore begin with $C^*$-categories and $W^*$-categories. If $\mathcal{F}$ is a $C^*$-category, then a conjugation on $\mathcal{F}$ is an extension $\mathcal{F}'$ of $\mathcal{F}$ with the same objects to include antilinear arrows with the property that any object is the source of an antiunitary. To formalize the structure involved, we define a semilinear $C^*$-category to be a $C^*$-category where for each pair of objects $\rho, \sigma$ in addition to the linear space $(\rho, \sigma)$ of “linear” arrows there is a second linear space $(\rho, \sigma)_a$ of “antilinear” arrows. The composition of two arrows is antilinear if and only if precisely one of them is antilinear. Identity arrows are, of course, linear and we have

$$\mu s \circ \lambda r = \mu \lambda s \circ r,$$

$$\mu s \circ \lambda r = \mu \overline{s} \circ r,$$

according as $s$ is linear or antilinear. The adjoint $r \mapsto r^*$ is a contravariant involution leaving objects fixed and being antilinear on linear arrows and linear on antilinear arrows. The spaces $(\rho, \sigma)$ and $(\rho, \sigma)_a$ are equipped with a norm making them into Banach spaces and having the $C^*$-property:

$$\|r\|^2 = \|r^* \circ r\|.$$

If we forget the antilinear arrows, we get an ordinary $C^*$-category and the norm is determined by its values on that subcategory.

An antiunitary arrow in a semilinear $C^*$-category is an arrow $J$ in some $(\rho, \sigma)_a$ such that $J^* \circ J = 1_\rho$ and $J \circ J^* = 1_\sigma$. Two objects $\rho$ and $\overline{\rho}$ are said to be conjugates if there exists an antiunitary $J \in (\rho, \overline{\rho})_a$. Conjugates, if they exist, are defined up to unitary equivalence.

The above definition would seem to be the most natural from the categorical point of view. However, if conjugates exist, we may wish to make a choice, $\rho \mapsto J_\rho$, of antiunitary for each object $\rho$ and then there is an associated antilinear $^*$-functor on $\mathcal{F}$ defined by

$$\mathcal{T} := J_\sigma \circ T \circ J_\rho^* \in (\overline{\rho}, \overline{\sigma}), \quad T \in (\rho, \sigma).$$

It can be extended to $\mathcal{F}'$ by defining

$$\overline{R} \circ J_\rho := \overline{R} \circ J_\sigma$$

on antilinear arrows. In addition there is an associated natural unitary transformation $d_\rho : (\rho, \overline{\rho})$ defined by

$$d_\rho := J_\overline{\sigma} \circ J_\rho.$$
and satisfying $\overline{d_\rho} = d_\rho$.

More interestingly, we can also go in the other direction. If we are given an antilinear $\ast$–functor and a unitary natural transformations $d$, as above, we may define a semilinear $C^\ast$–category as follows. For each object $\rho$, we introduce an antiunitary arrow $J_\rho \in (\rho, \overline{\rho})$. A general antilinear arrow in $(\rho, \sigma)$ can now be written uniquely in the form $R \circ J_\rho$, where $R \in (\overline{\rho}, \sigma)$. Composition with a linear arrow $P \in (\pi, \rho)$ is defined by

$$R \circ J_\rho \circ P := R \circ \overline{P} \circ J_\pi.$$  

Composition with a linear arrow $S \in (\sigma, \tau)$ is defined by

$$S \circ R \circ J_\rho := (S \circ R) \circ J_\rho.$$  

Finally, composition with an antilinear arrow $S \circ J_\sigma$, where $S \in (\overline{\sigma}, \tau)$, is defined by

$$S \circ J_\rho \circ R \circ J_\rho := S \circ \overline{R} \circ d_\rho.$$  

Routine computations verify that we get a $\ast$–category and indeed a semilinear $C^\ast$–category if we define the norms of antilinear arrows in the only way compatible with $J_\rho$ being antiunitary, namely by setting

$$\|R \circ J_\rho\| := \|R\|.$$  

It should be noted that in the above construction of $\mathcal{T}^a$ if $U_\rho \in (\overline{\rho}, \tilde{\rho})$ is a unitary natural transformation between two antilinear $\ast$–functors then mapping antilinear arrows by $R \circ J_\rho \mapsto R \circ U_\rho \circ J_\rho$ and leaving linear arrows invariant is an isomorphism of the constructed semilinear tensor $C^\ast$–categories. Two different choices, $\rho \mapsto J_\rho$ and $\rho \mapsto \tilde{J}_\rho$, within $\mathcal{T}^a$ lead to a unitary natural equivalence $U_\rho := \tilde{J}_\rho \circ J_\rho^\ast$ between the associated antilinear $\ast$–functors.

We may want our conjugation to have additional properties. The following definition would seem to describe the best possible situation. We call a strict involutive conjugation an involutive antilinear covariant functor on $\mathcal{T}$ commuting with the adjoint, taking an object $\rho$ to $\overline{\rho}$ and an arrow $T$ to $\overline{T}$. If we now adjoin to the category, as a special case of the above construction, an antiunitary $J_\rho$ for each object $\rho$ with the property that $J_\rho^\ast = J_\rho$ and

$$J_\sigma T = \overline{T} J_\rho, \quad T \in (\rho, \sigma)$$  

then we will have constructed a conjugation on $\mathcal{T}$. This special case corresponds to being able to take $d$ as the identity natural transformation. Looked at from the point of view of $\mathcal{T}^a$, it means that $J_\rho$ can be chosen so that $J_\rho = J_\rho^\ast$. 

To give a simple example: let \( \mathcal{H} \) be a category of Hilbert spaces then we get a conjugation on \( \mathcal{H} \) by adding to the arrows all bounded antilinear mappings between the respective objects. Such a category will be denoted \( \mathcal{H}^a \) and referred to as a category of Hilbert spaces with conjugation. Pick an orthonormal basis for each Hilbert space \( H \) in the category and let \( J_H \) denote the antiunitary involution on \( H \) leaving this basis fixed. Then define for \( T \in (H,K) \), \( \mathcal{T} := J_K T J_H \) and we have a strict involutive conjugation on \( \mathcal{H} \) yielding \( \mathcal{H}^a \) as the associated conjugation.

A second simple example is provided by a \( C^* \)-algebra \( A \) equipped with a conjugation \( j \), i.e. an antilinear involutive \( * \)-homomorphism. Consider the representation theory of \( A \) on the objects of \( \mathcal{H} \). If \( \pi \) is such a representation, we write \( J_\pi \) and define
\[
\pi(A) := J_\pi \pi(j(A)) J_\pi, \quad A \in A,
\]
\[
\mathcal{T} = J_\sigma T J_\rho, \quad T \in (\rho, \sigma).
\]
In this way, we get a strict involutive conjugation on the \( C^* \)-category of representations of \( A \) on the objects of \( \mathcal{H} \). The forgetful functor into \( \mathcal{H} \) preserves the strict involutive conjugation in the obvious sense.

There is also a simple result going in the other direction. We recall that if \( H : \mathcal{T} \to \mathcal{H} \) is a \( * \)-functor of a \( C^* \)-category into the category of Hilbert spaces then the bounded natural transformations from \( H \) to \( H \) form a von Neumann algebra denoted \( (H,H) \) and called the commutant of \( H \). The evaluation maps \( \eta \mapsto \eta_\rho \) are normal representations of \( (H,H) \). When \( \mathcal{T} \) is a \( W^* \)-category and \( H \) is faithful and normal, then \( \mathcal{T} \) can be interpreted as a category of normal representations of \( (H,H) \).

3.1 Lemma Let \( \mathcal{T}^a \) be a conjugation on \( \mathcal{T} \) and \( H^a : \mathcal{T}^a \to \mathcal{H}^a \) be a \( * \)-functor into a category of Hilbert spaces with conjugation and \( H \) its restriction to \( \mathcal{H} \). Given \( \eta \in (H,H) \), set
\[
j(\eta)_\rho := H^a(J_\rho^*)^\eta H^a(J_\rho),
\]
where \( J_\rho \) is an antiunitary from \( \rho \) to \( \overline{\rho} \) in \( \mathcal{T}^a \). Then \( j \) is a conjugation on \( (H,H) \).

Proof. Given \( T \in (\rho, \sigma) \), \( \mathcal{T} := J_\sigma T J_\rho^* \in (\overline{\rho}, \overline{\sigma}) \) and a simple computation shows that \( j(\eta) \in (H,H) \). Two different choices of \( J_\rho \) differ by a unitary in \( \mathcal{T} \). But \( \eta \) is a natural transformation, so \( j \) does not depend on the choice of \( J_\rho \) and this makes it obvious that \( j \) is an involution.
We next show how, given a conjugation on a $C^*$–algebra, the GNS construction provides canonical antiunitary intertwiners between conjugate representations.

**3.2 Lemma** Let $j$ be a conjugation on a $C^*$–algebra $A$ and let $\phi$ denote a lower semicontinuous densely defined weight on $A$ and let $\overline{\phi} := \phi \circ j$. Let $N_\phi$ and $N_{\overline{\phi}}$ be the associated scalar product spaces mapped by $\phi$ into the associated Hilbert spaces, $L^2(A, \phi)$ and $L^2(A, \overline{\phi})$, respectively. Then there is a canonical antiunitary operator $J_\phi$ from $L^2(A, \overline{\phi})$ to $L^2(A, \phi)$ defined by

$$J_\phi \hat{N} = \hat{j(N)}, \quad N \in N_\phi,$$

and we have

$$J_\phi \pi_\phi(A) = \pi_{\overline{\phi}} \circ j(A) J_\phi, \quad A \in A.$$

If $\phi$ extends to a faithful normal weight on $\pi_\phi(A)'^\prime$ then

$$J_\phi S_\phi = S_{\overline{\phi}} J_\phi,$$

where the operators $S$ are the closed operators derived from the adjoint.

**Proof.** $J_\phi$ is uniquely defined as an antiunitary operator since

$$\phi(N^*N) = \overline{\phi}(j(N)^* j(N)), \quad N \in A.$$

Furthermore, the intertwining property holds since

$$j(AN) = j(A) j(N), \quad A \in A, \quad N \in N_\phi.$$

The final relation follows from

$$j(N^*) = j(N)^*, \quad N \in N_\phi.$$

## 4 Conjugation and Tensor Products

After this general discussion of conjugation which already illustrates the basic problems involved, we turn to conjugation on tensor $C^*$–categories, the structures arising in the representation theory of Hopf $C^*$–algebras, locally compact quantum groups and multiplicative unitaries.

In a semilinear tensor $C^*$–category, the tensor product is defined separately for linear and antilinear arrows. If $R \in (\rho, \sigma)$ and $R' \in (\rho', \sigma')$ then $R \times R' \in (\rho \rho', \sigma \sigma')$ and the map $R, R' \mapsto R \times R'$ is bilinear. If $R \in (\rho, \sigma)_a$ and
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If \( R' \in (\rho', \sigma')_a \), then \( R \times R' \in (\rho \rho', \sigma \sigma')_a \) and \( R, R' \mapsto R \times R' \) is again bilinear. If \( S \in (\sigma, \tau) \) and \( S' \in (\sigma', \tau') \), then

\[
S' \times S \circ R \times R' = (S \circ R) \times (S' \circ R').
\]

If \( P \in (\pi, \rho) \) and \( P' \in (\pi', \rho') \) then

\[
R \times R' \circ P \times P' = (R \circ P) \times (R' \circ P').
\]

If \( S \in (\sigma, \tau)_a \) and \( S' \in (\sigma', \tau')_a \) then again

\[
S' \times S \circ R \times R' = (S \circ R) \times (S' \circ R').
\]

Finally, \((R \times R')^* = R'^* \times R^*\) for \( R, R' \) antilinear.

A simple example of a semilinear tensor \( C^*\)-category is got by considering the linear and antilinear intertwining operators between a set of unitary representations of a group \( G \), closed under tensor products, where an antilinear intertwining operator \( R \in (\rho, \sigma)_a \) is a bounded antilinear operator from \( H(\rho) \) to \( H(\sigma) \), the underlying Hilbert spaces, such that

\[
R \rho(g) = \sigma(g) R, \quad g \in G.
\]

The only point to note is that the tensor product \( R \times R' \) is not the usual tensor product \( R \otimes R' \) of antilinear operators, but is given by

\[
R \times R' = \vartheta(\sigma, \sigma') \circ R \otimes R' = R' \otimes R \circ \vartheta(\rho, \rho')
\]

where \( \vartheta \) is the symmetry on the underlying tensor \( C^*\)-category of Hilbert spaces.

The idea of conjugation in §3 nows adapts to a tensor \( C^*\)-category \( \overline{T} \). It is an extension \( \overline{T}^a \) of \( \overline{T} \) to a semilinear tensor \( C^*\)-category where every object is the source of an antiunitary. At this point we make contact with the notion of conjugation introduced by Hayashi and Yamagami\(^6\). If we pick antiunitaries \( J_\rho \) for each object \( \rho \) and set

\[
c_{\rho,\sigma} := J_\rho \circ (J_\rho \times J_\sigma)^*,
\]

we get a natural unitary equivalence from \( \overline{\mathcal{P}} \) to \( \overline{\mathcal{P}\sigma} \). A computation shows that

\[
c_{\pi,\rho} \circ c_{\pi,\rho} \times 1_\pi = c_{\pi,\rho} \circ 1_\pi \times c_{\pi,\rho},
\]

\[
\overline{\mathcal{P}}_{\rho,\sigma} \circ c_{\pi,\sigma} \circ d_\rho \times d_\sigma = d_{\rho\sigma},
\]

where \( d \) is the natural unitary equivalence of §3. Conversely, given an antilinear functor \( T \mapsto \overline{T} \) and the natural equivalences \( c \) and \( d \) satisfying the above relations, then the semilinear \( C^*\)-category \( \overline{T}^a \) constructed in §3 can be made into
a tensor $C^*$–category by using the following definition of the tensor product of antilinear arrows:

$$(R \circ J_\rho) \times (R' \circ J_{\rho'}) := R' \times R \circ c_{\rho\rho'}^{-1} \circ J_{\rho\rho'}.$$  

Rather than using semilinear structure, Hayashi and Yamagami define a conjugation as an antilinear $^*$–functor equipped with the natural transformations $c$ and $d$.

They also introduce the notion of a strict conjugation on a tensor $C^*$–category requiring $c$ and $d$ to be identities. In terms of antiunitary operators, this obviously corresponds to requiring that there is a choice of $J$ such that

$$J_{\rho\sigma} = J_\rho \times J_\sigma \in (\rho\sigma, \sigma\rho)$$

for each pair of objects $\rho$ and $\sigma$ and we refer to this case as a strict involutive conjugation of tensor $C^*$–categories.

We give an example of a strict tensor $W^*$–category of Hilbert spaces with conjugation. Let $M$ be a von Neumann algebra equipped with a conjugation $j$. Let the objects of the category be the Hilbert spaces in $M$. If $H$ is such a Hilbert space then its conjugate is $j(H)$. If $T \in (H, H')$ then its conjugate is $j(T) \in (j(H), j(H'))$. As this conjugation is involutive, we may take the natural unitary equivalence $d$ to be the identity. The natural unitary equivalence $c$ from $j(K)j(H)$ to $j(HK)$ is defined by

$$c_{H,K} := \theta(j(K), j(H)).$$

Since $j(c_{K,H}) \circ c_{j(K),j(H)} = 1_{KH}$, we may construct a semilinear tensor $W^*$–category with conjugation, as explained above. This construction is realized concretely by taking as antiunitary arrows $J_H$, the mapping $\psi \mapsto j(\psi)$ for $\psi \in H$, and defining

$$J_H \times J_K := c_{H,K}^{-1} \circ J_{HK}.$$  

For a second example, the category of matrices with complex entries is a $C^*$–category in a natural way and becomes a strict tensor $C^*$–category when the tensor product is defined using lexicographical ordering. However, this cannot be made into a strict tensor $C^*$–category with a strict involutive conjugation. In fact, labelling the objects by the integers in the obvious way, the equation

$$J_2 \times J_3 = J_3 \times J_2$$

cannot be satisfied. On the other hand, our axioms require $\overline{\rho\sigma}$ rather than $\overline{\sigma\rho}$ to be the conjugate of $\rho\sigma$. If we use the ordinary tensor product of antilinear
operators, denoted by $\otimes$, then we can satisfy

$$J_m \otimes J_n = J_{mn} = J_n \otimes J_m, \quad m, n \in \mathbb{N}$$

by defining

$$J_m e_i := e_{m+1-i},$$

with respect to the natural orthonormal basis $e_i$, $i = 1, 2, \ldots, m$.

For a third example, we consider a strict tensor $C^*$-category with conjugates

Then, we can solve the conjugate equations for an object $\rho$ of $\mathcal{T}$, then they also solve the corresponding equations in the category of Hilbert spaces and there is an invertible antilinear operator $T$ from $H$ to $\overline{H}$, the underlying Hilbert space of $\rho$ and $\overline{\rho}$, such that

$$R(1) = \sum_i T e_i \otimes e_i, \quad \overline{R}(1) = \sum_i e_i \otimes T^* e_i,$$

where $e_i$ is an orthonormal basis of $H$. We set $T := T^*$. If we pick, for each object $\rho$ of $\mathcal{T}$, a standard solution of the conjugate equations and denote the antilinear operator by $T_\rho$, then there is an antilinear functor $S \mapsto \overline{S}$ commuting with the adjoint defined by

$$\overline{S} := T_\rho ST_\rho^{-1}, \quad S \in (\rho, \sigma),$$

$$f_\rho := T_\rho^* T_\rho$$

is independent of the choice of $T_\rho$, $\rho \mapsto f_\rho$ is a natural transformation of the embedding functor to itself and

$$f_{\rho\sigma} = f_\rho \otimes f_\sigma.$$

One can similarly define an antilinear functor $S \mapsto \tilde{S}$ associated to the antilinear operators $\overline{T}_\rho := T_\rho^* T_\rho^{-1}$ by

$$\tilde{S} := \overline{T}_\sigma S \overline{T}_\rho^{-1}, \quad S \in (\rho, \sigma).$$

**Theorem 4.1**

a) Setting

$$(\rho, \sigma)_a := (\overline{\rho}, \sigma) \circ T_\rho$$

and

$$T_\rho \times T_\sigma := T_\sigma \otimes T_\rho \circ \theta(H, K),$$
where $H$ and $K$ denote the underlying Hilbert spaces of $\rho$ and $\sigma$, respectively, gives an embedded semilinear tensor category. Set $d_\rho := T_\rho \circ T_\rho$ and $c_{\rho,\sigma} := T_{\rho\sigma} \circ (T_\rho \times T_\sigma)^{-1}$. Then $d$ and $c$ are unitary natural transformations satisfying the identities needed to yield a semilinear tensor $C^*$-category with conjugation $\mathcal{T}^\sigma$.

b) Setting 

\[(\rho, \sigma)_{\overline{\pi}} := (\overline{\rho}, \sigma) \circ \overline{T}_\rho\]

\[\text{gives, in a similar way, another semilinear tensor category with conjugation } \mathcal{T}^\pi \text{ with the corresponding properties.}\]

In general, $T_\rho \notin (\rho, \overline{\rho})_{\overline{\pi}}$. The category $\mathcal{T}^\sigma$ (resp. $\mathcal{T}^\pi$) may be identified with the embedded semilinear tensor category after the adjoint of antilinear arrows has been redefined so as to make the $T_\rho$ (resp. $\overline{T}_\rho$) antiunitary. It is independent of the embedding of $\mathcal{T}$ into a tensor $C^*$-category of Hilbert spaces.

**Proof.** The observation on the embedded semilinear category is already made in [12]. We get a semilinear tensor category since $T_\rho \times T_\sigma$ would be a possible choice of $T_{\rho\sigma}$. Since $S \mapsto \overline{S}$ is defined in terms of $\rho \mapsto T_\rho$, $d$ and $c$ are obviously natural transformations satisfying the required identities. $d_\rho$ is unitary since $T_\rho^{-1}$ would be a possible choice of $T_{\overline{\rho}}$. Similarly, $c_{\rho,\sigma}$ is unitary since $T_\rho \times T_\sigma$ is a possible choice of $T_{\rho\sigma}$. The semilinear tensor $C^*$-category determined by this data is obviously $\mathcal{T}^\sigma$, where the adjoint of antilinear arrows has been changed to make the $T_\rho$ antiunitary. Since different choices of the $T_\rho$ differ by a unitary, the resulting category is independent of the embedding. The statements relative to the category $\mathcal{T}^\pi$ can be proved similarly.

If the semilinear tensor $C^*$-category with conjugation constructed above can be embedded in the semilinear tensor category of Hilbert spaces, then the $T_\rho$ are antiunitary for this embedding and the intrinsic dimensions coincide with the dimensions of the underlying Hilbert spaces.

We can learn more from the above construction of a conjugation. The natural transformations $c$ and $d$ have here been defined in terms of the invertible antilinear operators $T_\rho$ which in turn were defined using standard solutions $R_\rho$ and $\overline{R}_\rho$ of the conjugate equations. Expressing $c$ and $d$ in terms of the $R_\rho$ and $\overline{R}_\rho$, we find

\[c_{\rho,\sigma} = 1_{\overline{\pi} \sigma} \times (\overline{R}_{\rho} \circ 1_{\rho} \times \overline{R}_{\sigma}) \times 1_{\overline{\pi}} \circ R_{\rho \sigma} \times 1_{\overline{\pi} \sigma} \times 1_{\overline{\pi} \tau},\]

\[d_\rho = 1_{\overline{\pi} \rho} \times R_{\rho} \circ R_{\overline{\pi} \tau} \times 1_{\rho} \times 1_{\overline{\pi} \tau} \times 1_{\overline{\pi} \tau}.\]
These expressions no longer make reference to an ambient Hilbert space. Defining the conjugate linear $\ast$–functor by
\[ S \times 1_\rho \circ R_\rho = 1_\sigma \circ S^* \circ R_\sigma, \quad S \in (\rho, \sigma), \]
c and d become natural transformations. Furthermore, after a somewhat lengthy calculation, the identities between c and d can be verified, leading to the following result.

4.2 Theorem Any strict tensor $C^*$–category with conjugates admits a canonical conjugation defined, as above, in terms of standard solutions of the conjugate equations.

Proof. The only point still to be checked is that the conjugation does not depend on the choice of standard solutions of the conjugate equations. However, a second choice $\rho \mapsto \tilde{R}_\rho$ is related to the first by $\tilde{R}_\rho = U_\rho \times 1_\rho \circ R_\rho$, where $U_\rho \in (\rho, \tilde{\rho})$ is unitary. We then have $\tilde{d}_\rho = U_\rho \circ U_\rho$ and $\tilde{c}_{\rho,\sigma} = U_{\rho \sigma} \circ (U_\sigma \times U_\rho)^*$. As we have seen this leads to the same conjugation.

The reader’s attention is drawn to a result of Yamagami’s, Theorem 3.6 of [17], where he achieves more, at the cost of passing to an equivalent tensor $C^*$–category in the course of the proof. We also remark that
\begin{align*}
c_{\rho,\sigma} &= 1_\sigma \times (R^*_\sigma \circ 1_\sigma \times R^*_\rho \times 1_\sigma) \circ 1_\sigma \times 1_\sigma \times T_{\rho\sigma}, \\
d_\rho &= R^*_\rho \times 1_\rho \circ 1_\rho \times T_{\rho}. 
\end{align*}

We recall from the beginning of the previous section that, in the presence of a conjugation $j$ on a $C^*$–algebra $A$, the representation theory relative to a category of Hilbert spaces with a strict involutive conjugation has a strict involutive conjugation given by
\[ \bar{\pi}(A) := J_{\pi} \pi(j(A)) J_{\pi}, \quad A \in A, \]
\[ \bar{T} = J_{\sigma} T J_{\rho}, \quad T \in (\rho, \sigma), \]
where $J_{\pi} = J_{H_\pi}$ and $H_\pi$ is the Hilbert space of $\pi$. If $A$ is a Hopf $C^*$–algebra with coproduct $\delta$ satisfying
\[ \delta \circ j = j \otimes j \circ \theta \circ \delta, \]
and we consider representations relative to a strict tensor $W^*$–category of Hilbert spaces with a conjugation, then $J_{\pi \times \rho} = J_{\rho} \otimes J_{\pi}(H_\pi, H_\rho)$ defines $\bar{\rho} \times \bar{\sigma}$ as a conjugate for $\pi \times \rho$ and we get a conjugation on the tensor $W^*$–category of representations of $A$. If the underlying category of Hilbert spaces has a strict
involutive conjugation then the same is true for the category of representations of $A$.

We now come to other cases where conjugates can be defined in terms of antiunitary arrows but where we need to make a simple extension of our formalism. Instead of starting with a strict tensor $C^*$–category, we need to adjoin antilinear 2–arrows to a $2–C^*$–category. A formal definition of $2–C^*$–category can be found in [10] but the examples given below should be self–explanatory. We consider a set of von Neumann algebras. These form the 0–arrows. The bimodules (correspondences) on this set form the 1–arrows, whilst the bimodule homomorphisms form the 2–arrows. Compositions are defined in the obvious manner. What we get is not a $2–C^*$–category but what might be called a bi–$C^*$–category, because the composition of 1–arrows is defined only up to equivalence. Now there is no problem in adjoining antilinear 2–arrows in a natural way because there is a natural notion of an antilinear bimodule homomorphism. An antilinear bimodule homomorphism from an $M$–$N$–bimodule to an $N$–$M$–bimodule is simply a bounded antilinear map $A$ between the underlying Hilbert spaces such that

$$A(M \cdot \psi \cdot N) = N^* \cdot (A\psi) \cdot M^*, \ M \in M, \ N \in N.$$  

Adding these antilinear 2–arrows, we get a semilinear bi–$C^*$–category, where every 1–arrow is the source of an antiunitary 2–arrow. In fact, conjugating an $M$–$N$–bimodule with an antiunitary operator yields an $N$–$M$–bimodule, a conjugate bimodule unique up to equivalence.

In the second example, we deal with morphisms of von Neumann algebras and whilst it is not immediately evident that we can define antilinear intertwining operators between such morphisms, the close links between morphisms and bimodules suggest that it must be possible. Furthermore, there is a definition of conjugation for such morphisms going back to Longo. These considerations lead us to consider a separable Hilbert space and a set of von Neumann algebras represented standardly on that Hilbert space. We denote by $J_M$ the corresponding modular conjugation of the von Neumann algebra $M$ and let $j_M$ denote $\text{Ad}J_M$. Then if $\rho : M \to N$ and $\sigma : N \to M$. Then we write $A \in (\rho, \sigma)_a$ if $A$ is a (bounded) antilinear operator on our Hilbert space such that

$$A\rho(M) = j_M(M)A, \ M \in M,$$

$$Aj_J(N) = \sigma(N)A, \ N \in M.$$  

As these conditions may look surprising, it is perhaps worth observing that if $\tau : M \to N$ then the condition for a (bounded) linear operator $T$ to be in $(\rho, \tau)$
is that
\[ T_\rho(M) = \tau(M) T, \quad M \in \mathcal{M}, \]
\[ T_\sigma(N) = j_N(N) T, \quad N \in \mathcal{N}. \]
Composition of these 2–arrows can be defined in the obvious fashion. The tensor product for linear 2–arrows is well known. For antilinear arrows, we proceed as follows: if \( A \) is as above and \( A' \in (\rho', \sigma'_a) \), where \( \rho' : \mathfrak{P} \to \mathcal{M} \) and \( \sigma' : \mathcal{M} \to \mathfrak{P} \), then
\[ A \times A' := A' J_M A \in (\rho \rho', \sigma' \sigma)_a. \]
It is easy to verify that we get a semilinear 2–\( C^\ast \)–category in this way. It is also easy to recover Longo’s result on the existence of conjugates. Given \( \rho \) as above, we want to show that \( \rho \) is the source of an antiunitary arrow \( A \). The first of the equations that \( A \) has to satisfy can be solved by taking \( A := J_M U^* \), where \( U \) is a unitary implementing \( \rho \). We now set
\[ \overline{\rho}(N) := A j_N(N) A^*, \]
and can check that \( \overline{\rho} : N \to N \).

We now adapt the above formalism to the case of \( C^\ast \)–algebras by replacing the above von Neumann algebras by weakly dense unital \( C^\ast \)–algebras \( \mathcal{A}, \mathcal{B}, \mathcal{C}, \ldots \). We write \( J_\mathcal{A} \) for the modular conjugation of the weak closure of \( \mathcal{A} \). The difference is that we now no longer have the analogue of Longo’s result on the existence of conjugates. We therefore consider a 2–\( C^\ast \)–category \( \mathcal{T} \) whose 0–arrows are this set of \( C^\ast \)–algebras whose 1–arrows are morphisms between these \( C^\ast \)–algebras and whose 2–arrows are intertwiners between these morphisms. We suppose we may pick for each 1–arrow \( \rho : \mathcal{A} \to \mathcal{B} \) an antiunitary operator \( J_\rho \) in such a way that
\[ a) \ J_\rho^* J_\mathcal{A} \text{ induces } \rho, \]
\[ b) \ J_\rho J_\mathcal{B} \text{ induces a 1–arrow } \overline{\rho} : \mathcal{B} \to \mathcal{A}. \]
It follows that a 1–arrow \( \rho \), being unitarily implemented, extends to a morphism \( \hat{\rho} \) between the weak closures and is a 1–arrows in the semilinear 2–\( C^\ast \)–category, \( \mathcal{C}_a \) say, of morphisms and intertwiners between the weak closures. It follows easily from a) and b) that \( J_\rho \in (\hat{\rho}, \overline{\rho}). \) We give conditions allowing us to construct a conjugation on \( \mathcal{T} \) as a semilinear 2–\( C^\ast \)–subcategory of \( \mathcal{C}_a \). The antiunitaries \( J_\rho \) are not unique and two choices \( J_\rho \) and \( \tilde{J}_\rho \) are said to be equivalent if \( J_\rho^* \tilde{J}_\rho \in (\rho, \rho) \) and \( J_\rho \tilde{J}_\rho^* \in (\overline{\rho}, \overline{\rho}) \). The equivalence class of \( J_\rho \) is denoted by \([ J_\rho ]\). We now require
4.3 Proposition

We consider a strict tensor $W^*$–category $\mathcal{T}$ equipped with a faithful idempotent tensor $^*$–functor $\iota$ onto a tensor $W^*$–subcategory of Hilbert spaces. We suppose that $\mathcal{T}$ admits a conjugation $J$ such that for each pair $W$, $W'$ of objects of $\mathcal{T}$

$$J_{\iota(W \times W')} \circ J^{-1}_{\iota(W')} \times J^{-1}_{\iota(W)} = \iota(J_{W \times W'} \circ J^{-1}_W \times J^{-1}_W).$$

Then if $V$ is a left regular object of $\mathcal{T}$, its conjugate $\overline{V}$ is a right regular object.

**Proof.** Let $R^\overline{V}$ denote the functor of tensoring on the right by $\overline{V}$ and let $\xi \in (L, L\iota)$ denote the natural unitary transformation making $V$ into a left regular object. We show that there is a natural unitary transformation in $(R^\overline{V}, R^\overline{V} \iota).$ We have, for every object $W$ of $V$, an antiunitary arrow $J_W$ defining a conjugate $\overline{W}$ of $W$. There is a unitary $\eta_{\overline{W}} \in (W \times \overline{V}, \iota(W) \times \overline{V})$ defined explicitly by

$$\eta_{\overline{W}} = J_V \times J_{\iota(W)}^{-1} \circ \xi_{\overline{W}} \circ J_W \times J_V^{-1}.$$ 

$\eta$ is a natural transformation. In fact, if $T \in (W, W')$ then

$$\eta_{\overline{W'}} \circ T \times 1_{\overline{V}} = J_V \times J_{\iota(W')}^{-1} \circ \xi_{\overline{W}} \circ 1_V \times \overline{T} \circ J_W \times J_V^{-1}.$$
= J_V \times J_W^{-1} \circ 1_V \times \iota(T) \circ \xi_{W \times W'} \circ J_W \times J_W^{-1} = \iota(T) \times 1_W \circ \eta_{W'}.

We have to show that $\eta$ satisfies the coherence relation

$$\eta_{W \times W'} = (\eta_{W})_{13}(\eta_{W'})_{23}.$$

Now $J_W \times J_W' \circ J_W^{-1} \times J_W' \in (W \times W', W' \times W)$. Thus by the naturality of $\xi$, we have

$$\xi_{W \times W'} \circ 1_V \times (J_W \times J_W' \circ J_W^{-1} \times J_W') = 1_V \times (J_W \times J_W' \circ J_W^{-1} \times J_W') \circ \xi_{W \times W'}.$$

Hence, we may express $\eta_{W \times W'}$ using $\xi_{W \times W'}$ rather than $\xi_{W' \times W}$ and, after simplifying, this yields

$$\eta_{W \times W'} = J_V \times J_W^{-1} \circ J_W \times J_W'^{-1} \circ J_W \times J_W' \times J_V^{-1}.$$

Finally, writing

$$\xi_{W \times W'} = (\xi_{W})_{13} \circ J_W \times J_W' \times J_V^{-1} \circ J_V \times J_W^{-1} \circ J_W \times J_W' \times J_V^{-1},$$

we get the required result.

## 5 Conjugation for locally compact quantum groups

Conjugation for the representation theory of a locally compact group provides motivation for the generalization which follows but is too well known to merit discussion. Therefore we turn to consider a locally compact quantum group, $(\mathcal{A}, \delta, \phi, \psi, R, \tau)$, a concept which, after initial work of Masuda, Nakagami and Woronowicz, has perhaps now received its final definition and denomination (cf. [7], [9]). Here $\delta : \mathcal{A} \to M(\mathcal{A} \otimes \mathcal{A})$ is a coassociative coproduct such that $\delta(\mathcal{A}) \mathcal{A} \otimes C = \delta(\mathcal{A})C \otimes \mathcal{A} = \mathcal{A} \otimes \mathcal{A}$, $\phi$ (or $\psi$) is called left (right) Haar measure and is a faithful, lower semicontinuous, densely defined, left (right) invariant KMS weight in the $C^*$–algebraic sense. $R$ is an involutive $^*$–antiautomorphism and $\tau$ is a pointwise norm continuous one–parameter automorphism group of $\mathcal{A}$ commuting with $R$. $R$ and $\tau$ together should implement the coinverse, in the sense that the coinverse of $\mathcal{A}$ should be $\kappa := R\tau_{-i/2}$, where $\tau_{-i/2}$ is the analytic generator of $\tau$. We focus attention, not on the antiautomorphism $R$ but on the conjugation $j := R \circ *$. This defines for us the conjugation on the representation theory and makes it clear that the problems of defining a conjugation have been avoided by a judicious choice of axioms. On the other hand it must be stressed that Woronowicz [15] in effect overcame these difficulties in the case of compact
matrix pseudogroups by proving that they are locally compact quantum groups in a natural way.

We refer to [9] for the general definition, recalling here, instead, the properties we need. First, we explain left invariance. Let $M_\phi$ denote as usual the dense $\ast$–algebra linearly spanned by the $a \in A^+$ such that $\phi(a) < \infty$ and $N_\phi$ the associated left ideal. One has:

$$\phi(\omega \otimes i(\delta(a))) = \phi(a)\omega(I), \quad \omega \in A_+^*, a \in M_\phi^+.$$ 

The following property, referred to as strong left invariance, is shown to hold in a locally compact quantum group ([9]): for all $a, b \in N_\phi$, $x := \iota \otimes \phi(\delta(a^*)I \otimes b)$ lies in the domain of $\kappa$ and

$$\kappa(x) = \iota \otimes \phi(I \otimes a^*\delta(b)).$$

$M_{i \otimes \phi}$ and $M_{i \otimes \phi}$ will denote the domains of $i \otimes \phi$ and its natural extension to the multiplier algebra $M(A \otimes A)$, cf. [8]. In particular we have

$$\phi(a^* \omega \otimes i(\delta(b))) = \phi(\omega \kappa \otimes i(\delta(a^*))b), \quad a, b \in N_\phi,$$

with $\omega$ in the set of those elements $\omega \in A^*$ for which $\omega \kappa \in A^*$. The above expression can also be written as

$$\omega \otimes \phi(I \otimes a^*\delta(b)) = \omega \kappa \otimes \phi(\delta(a^*)I \otimes b).$$

Then the following relations also hold.

$$\phi \circ \tau_t = \nu^{-t} \phi, \quad \text{for some } \nu \in \mathbb{R},$$

$$\delta \circ j = j \otimes j \circ \theta \circ \delta$$

$$\tau_t \otimes \tau_t \circ \delta = \delta \circ \tau_t,$$

$\nu$ is referred to as the scaling constant.

Right invariance can be derived formally from left invariance choosing $\phi_r = \phi R$, using the antimultiplicativity of $R$ and the relation $\delta R = \theta R \otimes R \delta$.

We now turn to multiplicative unitaries starting with a construction of [1]. Let $V$ be a multiplicative unitary on $K^2$, and define an associative product on $(K, K)_*$ by $\omega * \omega'(T) = \omega \otimes \omega'(V^{-1} I \otimes TV), T \in (K, K)$ making $(K, K)_*$ into a Banach algebra.

5.1 Proposition ([1]) For any corepresentation $W$ of $V$, $\omega \in (K, K)_* \mapsto \omega \otimes i(W) \in (H_W, H_W)$ is an algebra homomorphism. If $V$ is a regular multiplicative unitary, the closure $A(W)$ of its image is a $C^*$–algebra.
We endow \((K, K)_*\) with the maximal \(C^*\)-seminorm determined by all (unitary) corepresentations of \(V\) and denote the completion of \((K, K)_*\) in this seminorm by \(\mathcal{A}_{\text{max}}(V)\). We denote by \(\pi_W\) the \(*\)-representation of \(\mathcal{A}_{\text{max}}(V)\) obtained extending the homomorphism defined in the above proposition.

For an operator \(X \in (KH,KH)\), we denote the norm closure of \(\{\omega \otimes i(X) : \omega \in (K, K)_*\}\) by \(\mathcal{A}(X)\) and the closure of \(\{i \otimes \omega(X) : \omega \in (H, H)_*\}\) by \(\hat{\mathcal{A}}(X)\) where \(i\) denotes the appropriate identity map. Note that \(\omega \in (K, K)_*\) has zero seminorm if and only if \(\omega\) annihilates every \(\hat{\mathcal{A}}(W)\). Let us assume that \(V\) is a regular multiplicative unitary, or, more generally, that \(\hat{\mathcal{M}}(V)_*\) is a \(C^*\)-algebra. Since \(W\) is a corepresentation of \(V\), \(\hat{\mathcal{A}}(W)\) is contained in \(\hat{\mathcal{M}}(V)_*\), the von Neumann algebra generated by \(\hat{\mathcal{A}}(V)\). Therefore \(\omega\) will annihilate \(\hat{\mathcal{A}}(W)\) if it annihilates \(\hat{\mathcal{M}}(V)\). Hence \(\mathcal{A}_{\text{max}}(V)\) can also be defined as the completion of \(\hat{\mathcal{M}}(V)_*\), the predual of \(\hat{\mathcal{M}}(V)_*\), in the \(C^*\)-norm \(\|\omega\| = \sup_{W \in \mathcal{C}(V)} \|\omega \otimes i(W)\|\).

5.2 Theorem ([1]) If \(V\) is a regular multiplicative unitary, the map \(W \rightarrow \pi_W\) defines a faithful \(*\)-functor from the \(C^*\)-category \(\mathcal{C}(V)\) of unitary corepresentations of \(V\) onto the \(C^*\)-category of nondegenerate \(*\)-representations of \(\mathcal{A}_{\text{max}}(V)\).

It is also shown in [1] that \(\mathcal{A}_{\text{max}}(V)\) is equipped with a natural coassociative coproduct \(\delta\). One can easily check, in analogy with the group case, that the map \(\rho, \sigma \in \text{Rep}(\mathcal{A}_{\text{max}}(V)) \rightarrow \rho \otimes \sigma \circ \delta \in \text{Rep}(\mathcal{A}_{\text{max}}(V))\) makes \(\text{Rep}(\mathcal{A}_{\text{max}}(V))\) into a tensor \(C^*\)-category in such a way that \(\pi\) is a tensor functor.

Recalling our aim of defining a conjugate representation \(\overline{W}\) of a given unitary representation \(W\) of \(V\), Theorem 5.2 tells us that it suffices to determine the associated \(*\)-representation \(\pi_{\overline{W}}\) of \(\mathcal{A}_{\text{max}}(V)\) and we know from the discussion in Section 4 that we should look for a suitable conjugation on \(\text{Rep}(\mathcal{A}_{\text{max}}(V))\) or some related \(C^*\)-algebra.

If \(\mathcal{A}_{\text{max}}(V)\) is equipped with a (densely defined, unbounded) coinverse \(\kappa\), this would be the natural starting point and in view of the duality between \(\hat{\mathcal{M}}(V)\) and \(\hat{\mathcal{M}}(V)_*\), the coinverse \(\kappa'\) and the \(*\)-involution of \(\mathcal{A}_{\text{max}}(V)\) are indeed related by:

\[ * \circ \kappa'(\omega) = \overline{\omega}, \quad \omega \in \hat{\mathcal{M}}(V)_*. \]

There are difficulties involved as we shall, in general, be dealing with an antilinear involution that does not commute with the adjoint and is only densely defined.

However, it suggests giving a definition in terms of an, in general, unbounded antilinear operator. We say that a unitary \(\overline{W}\) on a Hilbert space of the form \(KH\overline{W}\) is a conjugate of \(W\) if there is a densely defined closed antilinear invertible
operator $T : H_W \to H_{\overline{W}}$ such that
\[ \omega \otimes i(W)T < T \overline{W} \otimes i(W), \quad \omega \in (K, K)^* \]

5.3 Lemma A conjugate $\overline{W}$ of a corepresentation $W$ is itself necessarily a corepresentation.

Proof. Let $T : H_W \to H_{\overline{W}}$ be densely defined closed antilinear and invertible, $W$ a corepresentation of $V$ and $\overline{W}$ a unitary on $KH_{\overline{W}}$ such that
\[ \omega \otimes i(W)T < T \overline{W} \otimes i(W), \quad \omega \in (K, K)^* \]

Pick $\phi \in D_T$ and $\omega_1, \omega_2 \in (K, K)^*$ then
\[ \omega_1 \otimes \omega_2 \otimes i(V_{12}\overline{W}_{13}W_{23})T\phi = T \overline{W}_1 \otimes \overline{W}_2 \otimes i(W_{13}W_{23}V_{12}^*)\phi = \]
\[ T \overline{W}_1 \otimes \overline{W}_2 \otimes i(V_{12}^*W_{23})\phi = \omega_1 \otimes \omega_2 \otimes i(\overline{W}_{23}V_{12})T\phi. \]

Since $T$ is invertible
\[ \omega_1 \otimes \omega_2 \otimes i(V_{12}\overline{W}_{13}W_{23}) = \omega_1 \otimes \omega_2 \otimes i(\overline{W}_{23}V_{12}) \]
and since $\omega_1, \omega_2 \in (K, K)^*$ are arbitrary,
\[ V_{12}\overline{W}_{13}W_{23} = \overline{W}_{23}V_{12}. \]

Thus $\overline{W}$ is a corepresentation of $V$.

It suffices to verify the intertwining relation for $\omega$ in a total set of $(K, K)^*$, e.g. the set $\{\omega_{\xi, \eta}\}$. Hence, $\overline{W}$ is a conjugate of $W$ if and only if for $\eta' \in D_T$, $\xi' \in D_{T^*}$, $\xi, \eta \in K$,
\[ (\xi \otimes \xi', \overline{W} \eta \otimes T\eta') = (W \xi \otimes \eta', \eta \otimes T^*\xi'). \quad (5.1) \]

In practice, this will be checked for $\eta'$ and $\xi'$ in a core of $T$ and $T^*$, respectively. In analogy with the classical situation this equation can be interpreted from the standpoint of the Banach algebra $\hat{A}(V)$ associated with the multiplicative unitary as in $[\text{R}]$: it asserts that the adjoint of the “matrix coefficient” $i \otimes \omega_{\xi, \eta'}(W)$ is given by $i \otimes \omega_{\eta', -i\xi', T \eta'}(\overline{W})$. In particular, if the regular representation is selfconjugate, $\hat{A}(V)$ is $^*$–invariant, hence a $C^*$–algebra. We shall see later that the left regular representation of a locally compact quantum group is selfconjugate in this sense. Other examples arise from compact quantum groups $[\text{R}]$, Hopf–von Neumann algebras $[\text{R}]$ and Kac systems $[\text{R}]$ as pointed out in $[\text{R}]$. It raises the question of whether the existence of conjugate representations might
prove a more effective postulate than regularity in developing the theory of multiplicative unitaries.

On the other hand, the above definition of conjugate, with its reliance on unbounded operators with unspecified domains, is difficult to work with. It is clear that if \( W \) is a conjugate of \( W \), then \( W \) is a conjugate of \( W \) since we may use \( T^{-1} \) in place of \( T \). But it is not even clear whether a conjugate is unique up to unitary equivalence. Nevertheless, as we shall see in the sequel, we can, in special cases, relate this notion of conjugate to the other notions used in this paper.

We begin by considering the category of finite dimensional representations of a compact quantum group and add antilinear operators to get a semilinear category. \( \mathcal{T} \in (\mathcal{W}, \mathcal{W})_a \) if \( T \) is an antilinear operator with

\[
\omega \otimes i(\mathcal{W}) T = T \mathcal{W} \otimes i(W), \quad \omega \in (K, K)^*. 
\]

Defining tensor products by

\[
T \times T' := \theta(\mathcal{H}, \mathcal{H}') \circ T \otimes T',
\]

where \( T \in (\mathcal{W}, \mathcal{W})_a \) and \( T' \in (\mathcal{W}', \mathcal{W}')_a \), \( H \) and \( H' \) are the underlying Hilbert spaces of \( W \) and \( W' \) and \( \mathcal{H} \) and \( \mathcal{H}' \) are the underlying Hilbert spaces of \( \mathcal{W} \) and \( \mathcal{W}' \). Adding these antilinear intertwiners, we get a semilinear tensor category of bounded intertwiners that is not in general self-adjoint. We have already met this phenomenon in Theorem 4.1 and can make this more precise using the antilinear operators \( T \) and \( \mathcal{T} := T^{*-1} \) associated with solutions \( \mathcal{R}, \mathcal{R} \) of the conjugate equations for \( W \) and \( \mathcal{W} \) as discussed in conjunction with that theorem. We then have

\[
\begin{align*}
\mathcal{W}_1 \circ 1_K \times \mathcal{R} &= \mathcal{W}_1 \circ 1_K \times \mathcal{R}, \\
\mathcal{W}_3 \circ 1_K \times \mathcal{R} &= \mathcal{W}_2 \circ 1_K \times \mathcal{R},
\end{align*}
\]

where \( K \) denotes the Hilbert space of the regular corepresentation. Writing these equations in terms of \( T \) instead, we get

\[
\begin{align*}
(\xi \otimes \xi'', W \eta \otimes T^* \xi') &= (\mathcal{W} \xi \otimes \xi', \eta \otimes T^* \xi''), \quad (5.2) \\
(\xi \otimes \xi', \mathcal{W} \eta \otimes T^{*-1} \xi'') &= (W \xi \otimes \xi'', \eta \otimes T^{-1} \xi'), \quad (5.3)
\end{align*}
\]

These equations imply that \( T^* \in (\mathcal{W}, \mathcal{W})_a \) and \( T^{*-1} \in (W, \mathcal{W})_a \). Thus \( \mathcal{W} \) is a conjugate of \( W \) in the sense of Lemma 5.3, too. However \( T \) is not an intertwiner,
here reflecting the fact that the antilinear involution $^* \circ \kappa'$ on $A_{\max}(V)$ does not commute with the adjoint.

Up till now, we have not explicitly exhibited interesting examples of infinite dimensional conjugate corepresentations. It is not surprising that this can be done for multiplicative unitaries derived from the regular representations of locally compact quantum groups.

In fact, the map

$$a \otimes b \rightarrow \delta(b)a \otimes I \quad \text{for } a, b \in \mathcal{N}_\phi$$

defines a bounded linear operator $U$ on $L^2(A, \phi) \otimes L^2(A, \phi)$. The left invariance of $\phi$ implies that $U$ is isometric:

$$(Ua \otimes b, Uc \otimes d) = \phi(\delta(b^*d)c \otimes I) =$$

$$\phi(a^* \otimes \delta(b^*d))c = \phi(a^* \kappa(\delta(b^*d)))c = (a \otimes b, c \otimes d).$$

and therefore $U$ is unitary, since its range is dense (recall that in a locally compact quantum group $\delta(A)A \otimes I$ and $\delta(A)I \otimes A$ are assumed to be dense in $A \otimes A$). We next compute the Hilbert space adjoint $V := U^*$ (which will be the standard multiplicative unitary associated to $\phi$). For $a, b, c, d \in \mathcal{N}_\phi$,

$$(Va \otimes b, c \otimes d) = (a \otimes b, Uc \otimes d) = (a \otimes b, \delta(d)c \otimes I) =$$

$$\phi(a^* \otimes \phi(I \otimes b^* \delta(d)))c = \phi(a^* \kappa(\delta(b^*I \otimes d)))c$$

by strong left invariance of $\phi$. Note that both in the proof of being an isometry and in the computation of the adjoint of $U$ only the left invariance of the second factor of $L^2(A, \phi) \otimes L^2(A, \phi)$ has been used.

Using a right invariant measure $\phi_r$ (e.g. $\phi_r = \phi \circ R$) the map $a \otimes b \mapsto \delta(a)I \otimes b$ defines another multiplicative unitary operator on $L^2(A, \phi_r) \otimes L^2(A, \phi_r)$. But we want the right regular corepresentation, $V_r$, instead, a unitary operating on $L^2(A, \phi) \otimes L^2(A, \phi)$ and this is derived from the map $a \otimes b \mapsto \partial \circ \delta(b)a \otimes 1$, where $\partial$ permutes the factors in the tensor product.

When we have a locally compact quantum group, we can form a two–parameter (pointwise norm continuous) group $\omega_{s,t} = \nu^{s} \tau_s \sigma_t$ generated by the modular group $\sigma$ of the left Haar measure $\phi$, and the scaling group $\tau$ (which commutes with $\sigma$). It is well known that the set $I(\omega)$ of norm entire elements for $\omega$ is a dense $^*$–subalgebra of $A$, stable under all the maps $\omega_z$, $z \in \mathbb{C}^2$. Thus $I(\omega)$ is a natural common core for the analytic generators of both $\sigma$ and $\tau$.\[\Box\]
However, we regard $I(\omega)$ as a subspace of $L^2(A, \phi)$ so that $\omega$ becomes a unitary group on $L^2(A, \phi)$. We then look for a canonical subspace of $I(\omega)$ dense in $L^2(A, \phi)$ which is at the same time a common core for the generators of the unitary group.

5.4 Lemma Let $\phi$ be a lower semicontinuous densely defined KMS weight on a $C^*$–algebra $A$ and let $\omega : \mathbb{R}^n \to \text{Aut}(A)$ be a pointwise norm continuous $\phi$–invariant automorphism group of $A$ containing the modular group of $\phi$ as a coordinate subgroup. Then $\mathcal{B}_{\phi,\omega} := \bigcap_{z \in \mathbb{C}} \omega_z((N_{\phi} \cap N_{\phi}^*) \cap I(\omega))$, is dense in $L^2(A, \phi)$ and $\omega$ acts on $L^2(A, \phi)$ as a strongly continuous unitary group, denoted $\Omega$. $\mathcal{B}_{\phi,\omega}$ is a common core for the positive selfadjoint operators $\Delta_1, \ldots, \Delta_n$ on $L^2(A, \phi)$ generating, by Stone’s Theorem, $\Omega$ by

$$
\Omega_{(t_1, \ldots, t_n)} = \Delta_1^{it_1} \cdots \Delta_n^{it_n}.
$$

Proof. As $\omega$ is a $\phi$–invariant automorphism group, it acts as a unitary group $\Omega$ on $L^2(A, \phi)$. $\mathcal{B} := \mathcal{B}_{\phi,\omega}$ is the greatest subset of $I(\omega) \cap N_{\phi} \cap N_{\phi}^*$ invariant under the $\omega_z$, $z \in \mathbb{C}$. $\mathcal{B}$ is a $^*$–algebra invariant under the $\sigma_z$, $z \in \mathbb{C}$, such that $\mathcal{B}^2$ is dense in $\mathcal{B}$ in the 2–norm. We show that $\mathcal{B}$ is dense in $L^2(A, \phi)$. For any element $x$ in $N_{\phi} \cap N_{\phi}^*$ (which is dense in $L^2(A, \phi)$),

$$
x(\alpha) := (\alpha/\pi)^{n/2} \int_{\mathbb{R}^n} e^{-\alpha|x|^2} \omega_t(x) dt,
$$

is still an element in $N_{\phi} \cap N_{\phi}^*$ which approximates $x$ as $\alpha \to +\infty$ provided we show that $\Omega$ is strongly (or weakly) continuous. One can show that, if $x \in N_{\phi} \cap N_{\phi}^*$ then $\sigma_z(x(\alpha)) \in N_{\phi} \cap N_{\phi}^*$, $z \in \mathbb{C}$, so that actually $x(\alpha) \in \mathcal{B}$. We now show that $\Omega$ is weakly continuous. We first consider the case of the modular group $\sigma$. Since $\phi$ is a KMS weight, the function $t \to \phi(x^* \sigma_t(x))$ is continuous for $x \in N_{\phi}$. In the general case, it is enough to show that the functions $\phi(\omega_t(x^*)y)$ are continuous when $x$, $y$ range over a subset of $N_{\phi}$ dense in the 2–norm. Now $\omega$ is pointwise norm continuous, therefore $\phi(x^* \omega_t(a)y)$ is continuous for $a \in A$, $x, y \in N_{\phi}$. In particular $a, x, y \in \mathcal{B}$,

$$
\phi(x^* \omega_t(a)y) = \phi(\sigma_{i/2}(\omega_t(a)y)\sigma_{i/2}(x)^*) = \phi(\omega_t(\sigma_{i/2}(a))\sigma_{i/2}(y)\sigma_{-i/2}(x^*))
$$

by the KMS property. We can then consider the positive selfadjoint operators $\Delta_1, \ldots, \Delta_n$ that generate the $n$–parameter group $\Omega$, by Stone’s Theorem, by $\Omega_{(t_1, \ldots, t_n)} = \Delta_1^{it_1} \cdots \Delta_n^{it_n}$. Arguments similar to those previously used show that if $x, y \in \mathcal{B}^2$, then $z \to \phi(x^* \omega_z(y))$ is an entire function coinciding with
\( \phi(\omega_{-z}(x^{*})y) \) by the uniqueness principle of analytic functions. \( \mathcal{B} \) is a common core for the \( \Delta_{j} \)'s. It follows in particular that \( \Omega_{z_{1}, \ldots, z_{n}} \), regarded as an operator \( \Delta_{1}^{iz_{1}} \cdots \Delta_{n}^{iz_{n}} \) with domain \( \mathcal{B} \) is a densely defined preclosed operator.

We shall consider the two subspaces \( \mathcal{B}_{\phi,\omega} \subset L^{2}(A, \phi) \) and \( \mathcal{B}_{\phi,\omega} \subset L^{2}(A, \phi) \) as natural common cores for certain unbounded bijections naturally arising from the locally compact quantum group. For example, let us write \( S \) and \( S_{r} \) for the closed operators \( S_{\phi} \) and \( S_{\phi_{r}} \) defined by the adjoint, denoting their polar decompositions by \( S = J \Gamma^{1/2} \) and \( S_{r} = J_{r} \Gamma_{r}^{1/2} \). \( S \) and \( S_{r} \) determine each other in the sense that there is an antiunitary operator, \( Y : L^{2}(A, \phi_{r}) \to L^{2}(A, \phi) \) such that \( Ya = R(a)^{*}, a \in A \), intertwining them: \( SY = YS_{r} \). Hence by the uniqueness of polar decompositions, we have \( JY = JY_{r}, \Gamma^{1/2}Y = \Gamma_{r}^{1/2} \) as well. The next result describes other unbounded bijections arising from the coinverse. We denote by \( \Delta \) and \( \Delta_{r} \) the generators of the scaling group \( t \mapsto \nu^{2} \tau_{t} \) when regarded as a strongly continuous unitary group on \( L^{2}(A, \phi) \) and \( L^{2}(A, \phi_{r}) \) respectively. Clearly \( \Delta Y = Y \Delta_{r} \).

5.5 Theorem Let \( \sigma \) denote the modular group of \( \phi \), \( \tau \) the scaling automorphism group of the locally compact quantum group \( A \), and let \( \omega \) be the 2-parameter group \( \omega_{s,t} = \nu^{2} \tau_{s} \sigma_{t} \). The following operators

\( \mathcal{B}_{\phi,\omega} \subset L^{2}(A, \phi_{r}) \to \mathcal{B}_{\phi,\omega} \subset L^{2}(A, \phi) \)

taking \( a \) to \( \kappa(a), \kappa(a)^{*} \) and \( \kappa(a^{*}) \) are densely defined and preclosed. Denoting the closures of the first operator by \( K \), the closures of the second and third are \( SK \) and \( KS_{r} \), respectively. We have the following polar decompositions:

\( \nu^{+} SK = Y \Delta_{r}^{1/2}, \)
\( \nu^{-} KS_{r} = Y \Delta_{r}^{-1/2}, \)
\( \nu^{-} K = JY^{1/2} \Delta_{r}^{1/2}. \)

Furthermore,

\( (KS_{r})^{*} = \nu^{1/2}(SK)^{-1}, \)
\( (SK)^{*} = \nu^{-1/2}(KS_{r})^{-1}. \)

Proof. It follows from the previous discussion that \( \Delta_{r}^{-1/2}, \Delta_{r}^{1/2} \) and \( \Gamma_{r}^{1/2} \) are bijective, densely defined, positive and essentially selfadjoint on the indicated domains. In fact they are connected with 1-parameter subgroups of \( \omega_{s,t} \). Since \( Y \) is antiunitary, it is clear that \( SK \) and \( KS_{r} \) are bijections between the indicated domains. The polar decomposition of \( SK \) and \( KS_{r} \) on \( \mathcal{B}_{\phi,\omega} \), now follows
from the data of the locally compact quantum group. Furthermore $SK$ and $KS_r$ form part of an essentially selfadjoint pair in the sense explained in the appendix. As $S$ and $S_r$ also form part of an essentially selfadjoint pair, it follows from Lemma A.2 that the operators in question can be denoted by $K$, $SK$ and $KS_r$. On the indicated domain one has:

$$K = S(SK) = \nu^{i/2} J^{1/2} Y \Delta^{1/2} = \nu^{i/2} J Y \Gamma^{1/2} \Delta^{1/2}.$$

Since $B_{\phi,\omega}$ is $^\ast$–invariant and $\sigma_z$–invariant, $z \in \mathbb{C}$, one deduces, looking at the polar decomposition of $S$, that $J$ acts bijectively on $B_{\phi,\omega}$ too, and therefore $K$ is a bijection from $B_{\phi,\omega}$ to $B_{\phi,\omega}$ as well. It remains to show that $(KS_r)^\ast = \nu^{i/2}(SK)^{-1}$ and $(SK)^\ast = \nu^{-i/2}(KS_r)^{-1}$. We prove the latter relation, as the former follows by taking inverses and adjoints. The polar decompositions show that $I(\omega) \cap N_{\phi}$ is a core for both $(KS_r)^\ast$ and $\nu^{i/2}(SK)^{-1}$. But the two operators coincide on the core, completing the proof.

5.6 Corollary $K$ is a closed intertwiner from $V_r$ to $V$,

$$VI \otimes K = I \otimes KV_r.$$

In particular, if $U := \nu^{i/2} JY : L^2(A, \phi_r) \to L^2(A, \phi)$ denotes the polar part of $K$,

$$I \otimes UV_r = VI \otimes U.$$

Thus $V_r$ is a corepresentation of $V$. Moreover

$$V(SK \otimes S) = (SK \otimes S)W,$$

where $W$ is the unitary on $L^2(A, \phi_r) \otimes L^2(A, \phi)$ defined by $Wa \otimes b = \delta(b)a \otimes I$.

In particular, taking the polar decomposition of $(SK \otimes S)^\ast$,

$$V \Delta^{1/2} \otimes \Gamma^{-1/2} = \Delta^{1/2} \otimes \Gamma^{-1/2}V,$$

$$VY \otimes J = Y \otimes JW.$$

Proof. We identify $K^\ast$ on a suitable $^\ast$–invariant core $B$ of jointly analytic vectors for $\sigma$ and $\tau$ contained in $N_{\phi}$ (see Lemma 5.4). The polar decomposition of $K$ described in the previous Theorem shows that a subset $B \subset A \subset L^2(A, \phi)$ satisfies these properties if $R(B)^\ast = Y^*B$ is a core for $K$ in $L^2(A, \phi_r)$ satisfying similar properties. Consider the right invariant weight $\phi_r = \phi \circ R$, with modular group $x \mapsto \sigma^r_1(x) = R\sigma_{-1}(R(x))$. For $a \in B$, $b \in R(B)^\ast$,

$$(a, K(b)) = \phi(a^* \kappa(b)) = \nu^{i/2} \phi_r(b \kappa(a)^*) =$$
$\nu^{i/2}\phi_r(\sigma^r_{-i}(\kappa(a))^*b)$

thus, for $a \in \mathcal{B}$, $K^*(a) = \nu^{-i/2}\sigma^r_{-i}(\kappa(a))$. We need to show the following relations

\[ VI \otimes K < I \otimes KV_r, \quad (5.4) \]
\[ V_r I \otimes K^* < I \otimes K^*V. \quad (5.5) \]

We start from (5.4). We claim that it suffices to prove that, for $d \in \mathcal{B}$, $a, c \in \mathcal{N}_\phi$,

\[ (Va \otimes \kappa(b), c \otimes d) = (V_r a \otimes b, c \otimes K^*d). \quad (5.6) \]

Indeed, the algebraic tensor product $\mathcal{N}_\phi \otimes \mathcal{B}$ is a core for $I \otimes K^*$, so equation (5.6) shows that $V_r \mathcal{N}_\phi \otimes R(\mathcal{B})^*$ lies in the domain of $(I \otimes K^*)^* = I \otimes K$ and for $x \in \mathcal{N}_\phi \otimes R(\mathcal{B})^*$, $I \otimes KV_rx = VI \otimes Kx$. On the other hand $\mathcal{N}_\phi \otimes R(\mathcal{B})^*$ is a core for $I \otimes K$, so the claim follows. We compute the left hand side of (5.6).

\[
\phi \otimes \phi(a^* \otimes \kappa(b)^*\delta(d) c \otimes I) = \phi(a^* \iota \otimes \phi(I \otimes \kappa(b)^*\delta(d))c) = \\
\nu^{-i/2}\phi(a^* \iota \otimes \phi_r(I \otimes \kappa(d) \theta \circ \delta(b^*)) c) = \\
\nu^{-i/2}\phi(a^* \iota \otimes \phi_r(I \otimes \kappa(d) \theta \circ \sigma^r_{-i} \kappa^r(b^*)) c) = \\
\nu^{-i/2}\phi_r(a^* \iota \otimes I \theta \circ \delta(b^*) c \otimes \sigma^r_{-i} \kappa^r(d)) = (V_r a \otimes b, c \otimes K^*d).
\]

Arguing in a similar way, we see that (5.5) will be a consequence of

\[ (Va \otimes b, c \otimes \kappa(d)) = (V_r a \otimes K^*(b), c \otimes d), \quad (5.7) \]

for $a, c \in \mathcal{N}_\phi$, $b \in \mathcal{B}$, $d \in R(\mathcal{B})^*$. The r.h.s. equals

\[ \nu^{i/2}\phi \otimes \phi_r(a^* \otimes I \theta \circ \delta \circ \sigma^r_{-i} \circ \kappa^{-1}(b^*))c \otimes d) \]

while the l.h.s. equals

\[ \phi \otimes \phi(a^* \otimes b^* \delta(\kappa(d)) c \otimes I), \]

therefore it suffices to show that

\[ \iota \otimes \phi(I \otimes b^* \delta(\kappa(d))) = \nu^{i/2}\iota \otimes \phi_r(\delta \circ \delta \circ \sigma^r_{-i} \circ \kappa^{-1}(b^*) I \otimes d). \]

Using successively

\[ \sigma^r_{-i} \circ \kappa^{-1} = \kappa^{-1} \circ \sigma_{-i}, \]
\[ \theta \circ \delta \circ \kappa^{-1} = \kappa^{-1} \circ \kappa^{-1} \circ \delta \]
and

\[ \nu^{i/2} \phi \kappa^{-1} = \phi \]

we see that the r.h.s. of the previous relation equals

\[ \kappa^{-1} \iota \otimes \phi(I \otimes \kappa(d) \delta(\sigma_{-i}(b^*))) \]

which in turn equals

\[ \iota \otimes \phi(\delta(\kappa(d))I \otimes \sigma_{-i}(b^*)) \]

and the proof of (5.7) is completed by the KMS property of \( \phi \).

We briefly sketch the second part of the proof. We need to show the two relations

\[ VSK \otimes S < SK \otimes SW, \]
\[ W(SK)^* \otimes S^* < (SK)^* \otimes S^*V \]

which will follow respectively from

\[ (a \otimes b, VSK \otimes Sc \otimes d) = (Wc \otimes d, (SK)^* \otimes S^*a \otimes b), \]

for \( a, b, d \in \mathcal{B}, c \in R(\mathcal{B})^* \),

\[ (a \otimes b, W(SK)^* \otimes S^*c \otimes d) = (Vc \otimes d, SK \otimes Sa \otimes b) \]

for \( b, c, d \in \mathcal{B}, a \in R(\mathcal{B})^* \). These equations can be obtained, in turn, by computations similar to those of the first part of the proof.

From the general structure of the quantum groups under consideration, it follows that the dual Hopf algebra is equipped with the same structure as \( \hat{A} \). We have already noted that the coinverse \( \kappa' \) of \( A_{max}(V) \) serves to define the adjoint on \( \hat{A} \) and consequently that \( \kappa' \) is uniquely determined by \( \kappa \) via \( \kappa'(\omega)(a) = \omega(\kappa(a)) \), \( \omega \in \hat{M}(V)^*_\omega, a \in \hat{A} \), where \( \hat{A} \) is being considered as a dense subspace of \( \hat{M}(V) \). Taking the square of \( \kappa' \), which coincides with the square of the analytic generator of \( \tau' \), it follows that the coinverse data \( R' \) and \( \tau' \) of \( A_{max}(V) \) are determined by those of \( \hat{A} \) and similar formulas hold. In particular, \( \kappa'(\omega)^* = \varpi \). We may thus write

\[ \varpi = \kappa'(\omega)^* = \tau'_{i/2}(R'(\omega)^*), \quad \omega \in \hat{M}(V)^*_\omega. \]

In fact, \( \tau'_{i/2} \) is spatially implemented in the regular representation.

5.7 Proposition Let \( \tau' \) be the scaling group of \( A_{max}(V) \) as defined above. Then for \( \omega \in D_{\tau'_{i/2}}, \xi' \in D_{\Gamma^{-1/2}} \) and \( \eta' \in D_{\Gamma^{1/2}} \)

\[ (\xi', \pi_V(\tau'_{i/2}(\omega))\eta') = (\Gamma^{-1/2}\xi', \pi_V(\omega)\Gamma^{1/2}\eta'). \]
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Proof. It follows from Corollary 5.6 that

\[ \Delta^{it} \otimes \Gamma^{-it} V = V \Delta^{it} \otimes \Gamma^{-it}, \]

Hence

\[ \tau_t \otimes i(V) = I \otimes \Gamma^{it} VI \otimes \Gamma^{-it}, \]

Recalling that \( \pi_V(\omega) = \omega \otimes i(V) \) and that \( \tau_t^* \) is the transpose of \( \tau_t \), we deduce that

\[ \pi_V(\tau_t^*(\omega)) = \Gamma^{it}\pi_V(\omega)\Gamma^{-it}, \quad \omega \in \hat{\mathcal{M}}(V). \]

Now with \( \omega, \xi' \) and \( \eta' \) as in the statement of the proposition, we have functions \( z \mapsto (\xi', \pi_V(\tau_2(z)\eta')) \) and \( z \mapsto (\Gamma^{-iz}\xi', \pi_V(\omega)\Gamma^{-iz}\eta') \), analytic in the strip \( 0 < \Re z < \frac{1}{2} \). Their boundary values agree on the real line and taking the boundary values at \( z = \frac{1}{2} \), we get the required result.

Now \( j' := R' \circ \ast \) is a conjugation defined on \( \mathcal{A}_{\text{max}}(V) \) and satisfies \( \delta \circ j' = j' \otimes j' \circ \theta \circ \delta \). Thus we know from Section 4 that \( j' \) defines a conjugation on the tensor category of representations of \( \mathcal{A}_{\text{max}} \).

5.8 Corollary A conjugate for \( V \) in the conjugation defined by \( j' \) is a conjugate for \( V \) in the sense of Equation (5.1). The converse holds if \( T \) in (5.1) has the form \( T = J\Gamma^{1/2} \) with \( J \) antiunitary.

Proof. Let \( J \) be an antiunitary operator then \( \omega \mapsto J\pi_V(j'(\omega)J^{-1}, \omega \in \hat{\mathcal{M}}(V)_* \), defines a representation of \( \mathcal{A}_{\text{max}}(V) \) and a conjugate \( \overline{V} \) of \( V \) in the conjugation defined by \( j' \) with

\[ \omega \otimes i(\overline{V}) = J\pi_V(j'(\omega)J^{-1}. \]

Now let \( T := J\Gamma^{1/2} \) and pick \( \eta' \in \mathcal{D}_T \) and \( \xi' \in \mathcal{D}_{T'}, \) i.e. \( \eta', J\xi' \in \mathcal{D}_{\Gamma^{1/2}}, \) then

\[ (\xi \otimes \xi', \overline{\eta} \otimes T\eta') = (\xi', \omega_{\xi,\eta} \otimes i(\overline{V})T\eta') = (\xi', J\pi_V(j'(\omega_{\xi,\eta})J^{-1}\eta') \]

\[ = (\xi', J\pi_V(j'(\omega_{\xi,\eta}))\Gamma^{1/2}\eta', \Gamma^{-1/2}\Gamma^{1/2}T^*\xi'). \]

Now \( j'(\omega_{\xi,\eta}) \in \mathcal{D}_{\Gamma^{1/2}}, \) in fact \( \tau_{1/2}j'(\omega_{\xi,\eta}) = \overline{\xi_{\eta}} = \omega_{\eta,\xi}. \) Hence, by Proposition 5.7,

\[ (\pi_V(j'(\omega_{\xi,\eta}))\Gamma^{1/2}\eta', \Gamma^{-1/2}\Gamma^{1/2}T^*\xi') = (\pi_V(\tau'_{1/2}j'(\omega_{\xi,\eta})\eta', T^*\xi') = \]

\[ (\pi_V(\omega_{\eta,\xi})\eta', T^*\xi'). \]

We conclude that

\[ (\xi \otimes \xi', \overline{\eta} \otimes T\eta') = (V\xi \otimes \overline{\eta}, \eta \otimes T^*\xi'). \]
so that $\bar{V}$ is a conjugate for $V$ in the sense of (5.1), as required. The converse follows by reversing the argument.

Notice that, since $\Delta^{1/2}\Gamma^{-1/2}$ is a self-intertwiner of $V$, the result remains valid if we take $T$ to be of the form $T = J\Delta^{1/2}$ with $J$ antiunitary. We could replace $V$ by a corepresentation $W$ in the above argument if we knew that the automorphism group $\tau$ were unitarily implemented in the representation $\pi_W$.

We now show, in analogy with the classical case, that the right regular corepresentation is a canonical choice of conjugate for the left regular one. We therefore look for a unitary corepresentation $\bar{V}$ and an antiunitary operator $J$ such that

$$(\xi \otimes \xi', \nabla \eta \otimes J\Delta^{1/2}\eta') = (V\xi \otimes \eta', \eta \otimes \Delta^{1/2}J^*\xi'),$$

for $J^*\xi', \eta' \in \mathcal{D}_{\Delta^{1/2}}, \xi, \eta \in L^2(A, \phi)$.

**5.9 Theorem** The pair $\bar{V} = V_r, T = Y^*\Delta^{1/2} = \Delta^{1/2}Y^*$ solves the conjugate equation (5.1) for the regular corepresentation $V$ and shows that $V_r$ is a conjugate for $V$ in the conjugation defined by $j'$.

**Proof.** In view of the previous discussion, it suffices to verify the above equation and we begin by computing the l.h.s. $L$ making the change of variable $\hat{\xi} := Y\Delta^{1/2}\xi'$ and using Theorem 5.5.

$$L = (\xi \otimes \Delta^{1/2}Y^*\hat{\xi}, V_r\eta \otimes \Delta^{1/2}Y^*\eta') = (\xi \otimes (KS_r)^*\hat{\xi}, V_r\eta \otimes (KS_r)^{-1}\eta') =$$

$$= \nu^{-i/2}(\xi \otimes (SK)^{-1}\hat{\xi}, V_r\eta \otimes (KS_r)^{-1}\eta').$$

We choose $\hat{\xi}, \eta'$ belonging to a suitable common core of the indicated operators contained in $\mathcal{N}_\phi$ and write $(SK)^{-1}\xi = \kappa(\hat{\xi})^* = (KS_r)^{-1}\eta' = \kappa(\eta'^*)$. Then, recalling the definition of $V_r$, we have

$$L = \nu^{-i/2}(\xi \otimes \kappa(\hat{\xi})^*, V_r\eta \otimes \kappa(\eta'^*)) = \nu^{-i/2}\phi \otimes \phi_r(\xi^* \otimes \kappa(\hat{\xi})^* \delta(\kappa(\eta'^*)I \otimes \bar{\xi})), $$

where the inner product refers to $L^2(A, \phi) \otimes L^2(A, \phi_r)$. Thus

$$L = \nu^{-i/2}\phi(\xi^* \otimes \phi_r(I \otimes \kappa(\hat{\xi})^* \delta(\kappa(\eta'^*)I \otimes \bar{\xi})))\eta = \nu^{-i/2}\phi(\xi^* \otimes \phi_r(\kappa \otimes \kappa(\delta(\eta'^* I \otimes \bar{\xi})))\eta).$$

Since $\phi_r \circ \kappa = \nu^{i/2}\phi$ and $\kappa(\iota \otimes \phi(\delta(\eta'^* I \otimes \bar{\xi}))) = \iota \otimes \phi(I \otimes \eta'^* \delta(\bar{\xi}))$, we have

$$L = \phi(\xi^* \kappa(\iota \otimes \phi(\delta(\eta'^* I \otimes \bar{\xi})))\eta) = \phi(\xi^* \iota \otimes \phi(I \otimes \eta'^* \delta(\bar{\xi})))\eta =$$

$$= (\xi \otimes \eta', V^*\eta \otimes \bar{\xi}) = (V\xi \otimes \eta', \eta \otimes \bar{\xi}),$$

completing the proof.
We have seen above that we have a conjugation $j' := R' \circ *$ defined on $A_{\text{max}}(V)$ and satisfying $\delta \circ j' = j' \otimes j' \circ \theta \circ \delta$, and we may, as described earlier, define a conjugation on its tensor category of representations. We state this as a theorem.

**5.10 Theorem** Pick for each object $H$ in the image of the trivializing endofunctor $\iota$ on $\mathcal{C}(V)$ an antiunitary operator $J_H : H \to \overline{H}$. Let $\pi_W$ be the representation of $A_{\text{max}}(V)$ on $H$ corresponding to a $W \in \mathcal{C}(V)$. Let $\overline{W}$ denote the corepresentation of $V$ defined by the representation $A \mapsto J_H \pi_W(j'(A)) J_H^{-1}$ of $A_{\text{max}}(V)$. Set

$$J_W := J_{i(W)},$$

whence

$$\overline{i(W)} = i(\overline{W}).$$

Given $T \in (W,W')$, we set $\overline{T} := J_W T J_W^{-1} \in (\overline{W}, \overline{W'})$. Then $T \in (W,W') \mapsto \overline{T} \in (\overline{W}, \overline{W'})$ is an antilinear functorial equivalence. We get a semilinear tensor $W^*$–category with conjugation with natural unitary equivalences

$$d_W := J_{\overline{T}} \circ J_W,$$

$$c_{W,W'} := J_{W \times W'} \circ (J_W \times J_{W'})^*,$$

where $J_W \times J_{W'} := \theta(\overline{W}, \overline{W'}) \circ J_W \otimes J_{W'}$.

Now that we have equipped the category of unitary corepresentations of a multiplicative unitary with a conjugation, it follows from Proposition 4.3 that the conjugate of a left regular object is a right regular object of the same category.

On the other hand, we also see from Theorem 5.10 that the tensor $W^*$–category of finite–dimensional representations of a compact quantum group admits two canonical conjugations, which are in general quite distinct. The one coming from Theorem 5.10 is embedded in a semilinear tensor $W^*$–category of Hilbert spaces and is related to the antilinear involution $j'$ which commutes with the adjoint. The other cannot be embedded unless the intrinsic dimensions are integral (Corollary 4.2) and is related to the antilinear involution $\ast \circ \kappa'$ which does not commute with the adjoint in general. In the case of $SU_q(2)$, it is clear that the conjugation on objects is the same in both cases since the fusion rules imply that each irreducible is self–conjugate. We shall show that this holds for all compact quantum groups. Now $\ast \circ \kappa' = \tau_{i/2} \circ j' = j' \circ \tau_{-i/2}'$, so the relation between the two conjugations should be describable in terms of $\tau'$. 
To this end, we consider the category of finite-dimensional unitary representations of the compact quantum group. As is well known, there is an associated Hopf-von Neumann algebra. Its elements are most conveniently described as bounded natural transformations of the embedding functor $F$ into the underlying Hilbert spaces. Now we have identified a natural transformation $\rho \mapsto f_\rho$ prior to Theorem 4.1. In general $f$ is not an element of the Hopf-von Neumann algebra $(F,F)$ since the natural transformation is not bounded in general. Nevertheless, its bounded functions such as $\rho \mapsto f_\rho$ will be elements of $(F,F)$ and these induce the inner automorphisms $\tau'_\rho$ of $(F,F)$. It follows that $f$ itself induces $\tau'_\rho$. The relation between the two conjugations is now described in the following proposition.

5.11 Proposition Adjoin to the tensor $W^*$-category of finite dimensional representations of a compact quantum group two sets of antilinear intertwiners: $X \in (\rho,\sigma)_\kappa$ defined by

\[ X_\rho(\kappa'(A)) = \sigma(A)X, \quad X \in D_{\omega\kappa}, \]

and $Y \in (\rho,\sigma)_j$ defined by

\[ Y_\rho(\j'(A)) = \sigma(A)Y, \quad A \in (F,F). \]

Then we obtain two conjugations $\mathcal{T}_\kappa$ and $\mathcal{T}_j$. The second has the induced $^*$-structure, whilst the first has the $^*$-structure $\mathcal{T}_{\overline{\pi}}$ described in Theorem 4.1 b). There is an isomorphism of tensor categories which is the identity on linear arrows and takes $X \in (\rho,\sigma)_\kappa$ into $X f_\rho^{-1/2} = f_{\sigma}^{1/2}X$. This isomorphism does not commute with the adjoint in general.

Proof. $\mathcal{T}_j$ is the conjugation defined by $j'$ and has the induced $^*$-structure. To see that $\mathcal{T}_\kappa$ is the conjugation $\mathcal{T}_{\overline{\pi}}$ described in Theorem 4.1 b), it suffices to show that the invertible antilinear operators $\mathcal{T}_\rho$, introduced in connection with Theorem 4.1 are in $(\rho,\overline{\rho})_\kappa$. Now this has already been noted after (5.3). A computation shows that $X \in (\rho,\sigma)_\kappa$ if and only if $X f_\rho^{-1/2} \in (\rho,\sigma)_j$, so we will have the desired isomorphism once we show that $X f_\rho^{-1/2} = f_{\sigma}^{1/2}X$. However $T_\rho T_\rho = f_\rho$ and $T_\rho^{-1} T_\rho^{-1} = f_{\overline{\pi}}$. Hence $f_{\overline{\pi}} T_\rho = T_\rho^{-1} f_\rho^{-1} = f_{\sigma}^{1/2}$, so $f_{\overline{\pi}} T_\rho = T_\rho f_\rho^{-1}$ as required. From this we deduce that if $X \in (\pi,\rho)_\kappa$, then $T_\rho \circ X \in (\pi,\overline{\rho})$ so

\[ T_\rho X f_\rho^{-1} = f_{\overline{\pi}}^{-1} T_\rho X = T_\rho f_\rho X, \]

leading to the desired result.

We see, therefore, that for compact quantum groups the conjugations $\mathcal{T}_\kappa$ and $\mathcal{T}_j$ although corresponding to different notions of antilinear intertwiner lead to the same notions of conjugate object.
We conclude this section asking, in view of Theorem 2.3 and Proposition 4.3, whether there is a relationship between conjugation and standard braided symmetries for $\mathcal{T}$. Assume that $\mathcal{T}$ admits a conjugation $J$ assigning to a left regular object $V$ a conjugate $\overline{V}$ equivalent to $V$ itself. By Proposition 4.3 $\overline{V}$ is a right regular object, so $V$ is a right regular object of $\mathcal{T}$ as well. Explicitly, if $U \in (\overline{V}, V)$ is a unitary and $\eta_{\overline{V}} \in (R_{\overline{V}}', R_V')$ is the natural unitary transformation derived from $J$ as in the proof of Proposition 4.3, then

$$\eta_W := 1_{\iota(W)} \times U \circ \eta_{\overline{V}} W \circ 1_W \times U^* = 1_{\iota(W)} \times U \circ J_U \times J_{\iota(W)^{-1}} \circ \xi_W^{-1} \circ J_W \times J_V^{-1} \circ 1_W \times U^*$$

makes $V$ into a right regular object. To have a standard braided symmetry we further need the commutation relation $d')$ of Lemma 2.4

$$\eta_V \times 1_{\iota(V)} \circ 1_V \times \xi_V = 1_V \times \xi_V \circ \eta_V \times 1_{\iota(V)}$$

which expresses a precise relationship between the antilinear conjugation arrows $J_U, J_{\iota(V)}$, the unitary intertwiner $U \in (\overline{V}, V)$ and $\xi_V$. We next interpret this relation from the viewpoint of the Banach Hopf algebra associated to $V$.

5.12 Proposition Let $\mathcal{T}$ be a tensor $C^*$--category with a left regular object $V$ and conjugation as described in Proposition 4.3. Assume furthermore that there is a unitary $U \in (\overline{V}, V)$ such that $UJ_U\hat{A}(V^*)(UJ_U)^{-1}$ and $\hat{A}(V)$ commute. Then $\mathcal{T}$ has a standard braided symmetry.

Proof Let as usual $\mathcal{A}(X)$ and $\hat{A}(X)$ stand for the Banach spaces derived from a unitary operator $X$ on a tensor product Hilbert space compressing its first and the second component respectively. First note that, $V$ and $\overline{V}$ being unitarily equivalent, $\hat{A}(\overline{V}) = \hat{A}(V)$, so

$$\mathcal{A}(\eta_{\overline{V}}) = J_U\hat{A}(\overline{V})J_U^{-1} = J_U\hat{A}(V^*)J_U^{-1},$$

therefore

$$\mathcal{A}(\eta_V) = UA(\eta_{\overline{V}})U^* = UJ_U\hat{A}(V^*)(UJ_U)^{-1}.$$ Recalling now that $d')$ of Lemma 2.4 is equivalent to requiring that $\mathcal{A}(\eta_V)$ and $\hat{A}(V)$ commute.

In the case where $J$ is a conjugation taking $V$ to a conjugate $\overline{V}$ solving the conjugate equation, we have already noted that $\hat{A}(V) = \hat{A}(V^*)$, therefore one is reduced to requiring that

$$UJ_U\hat{A}(V)(UJ_U)^{-1} \subset \hat{A}(V)'.$$
We now show that for locally compact quantum groups there are canonical choices of $U$ and $J_V$ for which the above commutation relation holds. Indeed in this case a solution of the conjugate equation for the regular corepresentation is given by $J_V = Y^*$ and $\overline{V} = V_r$ (Theorem 5.9). Furthermore the polar part $U = JY$ of $K$ is a unitary intertwiner in $(\overline{V}, V)$ (Corollary 5.6), therefore $UJ_V = J$ is the polar part of the Tomita operator on $L^2(A, \phi)$, which takes, via its adjoint action, $\hat{A}(V) = A$ into its commutant. We can thus conclude that the corepresentation category of $V$ has a standard braided symmetry. Making use of the previous observation, we compute explicitly the associated natural unitary transformation $\eta$ making $V$ into a right regular object. First we have that

$$\eta_V = J_V \times \tilde{J}_{(V)}^* \xi_{\overline{V}} \times \tilde{J}_V^* = \vartheta_{H,K} Y^* \otimes Y V_r^* Y \otimes Y^* \vartheta_{K,H}$$

where $H = L^2(A, \phi_r)$ and $W$ is the operator defined in Corollary 5.6. The last equality follows from the commutation relations between $V$, $V_r$ and $W$ obtained in that corollary. We next have that $\eta$ is determined by

$$\eta_V = 1_{i(V)} \times \vartheta_{H,K} \vartheta \times \ind \times \vartheta^*$$

where $U$ is the polar part of $K$. This equation may be understood as a categorical interpretation in terms of the conjugation structure in $\mathcal{C}(V)$ of the usual way of getting standard braided symmetries for quantum groups described in Proposition 2.11. (Unitaries of the form of $\eta_V$ had previously appeared in [1] in the context of irreducible multiplicative unitaries). We have thus shown the following result.

5.13 Proposition Let $(A, \delta, R, \tau, \phi)$ be a locally compact quantum group, and let us endow $\mathcal{C}(V)$ with a conjugation as described in Theorem 5.10 and let $V$ be the usual multiplicative unitary associated to it. Then there is an associated standard braided symmetry $\varepsilon$ on $\mathcal{C}(V)$ whose evaluation in $V$ is given by

$$\varepsilon_V = VU \times IW^* U^* \times I\vartheta,$$

where $W$ is the unitary on $L^2(A, \phi_r) \otimes L^2(A, \phi)$ defined in Corollary 5.6.

6 Appendix. Essentially self–adjoint pairs

The basic difficulty in defining conjugation in the context of quantum groups or multiplicative unitaries is that one starts from an antilinear involution which is
not, in general, a conjugation on a $C^*$-algebra in that it may not commute with the adjoint and may only be densely defined and unbounded. In the theory of Woronowicz [15, 16] this involution defines the contragredient representation and he shows how to pass to the conjugate representation in the context of compact quantum groups. This involves a variant of modular theory and, to prepare for this, we begin with a simple result on densely defined semilinear bijections, where semilinear is understood to mean linear or antilinear. If $s$ is a densely defined semilinear mapping between Hilbert spaces $H$ and $K$ and $f$ a densely defined semilinear mapping from $K$ to $H$ then we call the pair $s, f$ Hermitian, essentially self–adjoint or selfadjoint if the matrix

$$\begin{pmatrix} 0 & f \\ s & 0 \end{pmatrix}$$

defines a semilinear mapping on $H \oplus K$ with the corresponding property.

A.1 Lemma Let $s, f$ be an essentially selfadjoint pair of semilinear bijections between dense subspaces of Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$. Let $\hat{s}$ and $\hat{f}$ denote the semilinear involutions given by the matrices

$$\begin{pmatrix} 0 & s^{-1} \\ f & 0 \end{pmatrix}, \begin{pmatrix} 0 & f \\ f^{-1} & 0 \end{pmatrix}.$$

Then $\hat{s} \subset \hat{f}^*$. The eigenspaces of $\hat{s}$ and $\hat{f}$ corresponding to the eigenvalue $\pm 1$ consist of vectors of the form

$$\begin{pmatrix} h \\ \pm sh \end{pmatrix}, \text{ and } \begin{pmatrix} h' \\ \pm f^{-1}h' \end{pmatrix},$$

for $h \in \mathcal{D}_s$ and $h' \in \mathcal{D}_{f^{-1}}$. Let the closure of these eigenspaces be denoted $M^s_\pm$ and $M^f_\pm$ then $M^f_\pm = M^s_\mp$. Writing $\hat{S}$ for the closure of $\hat{s}$ and denoting its polar decomposition by

$$\hat{S} = \hat{J}\Delta^{1/2},$$

$\hat{S}$ and $\hat{J}$ are involutions, $\hat{J}\Delta^{1/2} = \Delta^{-1/2}\hat{J}$ and $\hat{J}(M^s_\pm) = M^f_\mp$. Furthermore $\hat{S}^* = \hat{F}$, the closure of $\hat{f}$.

Proof $\hat{s}$ and $\hat{f}$ are obviously involutions with eigenspaces as stated and $\hat{f} \subset \hat{s}^*$. The closures of these eigenspaces are obviously the eigenspaces of the closures $\hat{S}$ and $\hat{F}$. The eigenspaces of the involution $\hat{S}^*$ are then $M^s_\pm$ and $M^s_\mp$. But $\hat{S}^* = \hat{F}$ since the pair $s, f$ is essentially selfadjoint so $M^f_\pm = M^s_\mp$. Finally, $\hat{J}(M^s_\pm) = M^f_\mp$ since $\hat{S}\hat{J} = \hat{J}\hat{S}$ and this and the remaining assertions just follow from the fact that $\hat{S}$ is a closed densely defined involution.

The formalism here is best known in the antilinear case, through modular theory. In this case, it is usual to formulate the result on the eigensubspaces
by saying that $M_+^s$ is a standard subspace and $M_+^f$ is its symplectic complement. The above formulation brings out the similarities between the linear and antilinear case. We draw the obvious conclusions about the original operators thus

$$
\hat{S} = \begin{pmatrix} 0 & S^{-1} \\ S & 0 \end{pmatrix}, \quad \hat{\Delta} = \begin{pmatrix} S^* S & 0 \\ 0 & S^{-1} S^{-1} \end{pmatrix}.
$$

Finally, letting $J \Delta^{1/2}$ be the polar decomposition of $S$, we have

$$
\hat{J} = \begin{pmatrix} 0 & J^{-1} \\ J & 0 \end{pmatrix}.
$$

To generalize the above version of modular theory so as to treat compositions of mappings, we let $\mathcal{V}$ be a subcategory of the category of vector spaces with linear (or antilinear) maps, and pick, for each object $V$ of $\mathcal{V}$, an object $V'$ of $\mathcal{V}$ such that $V$ and $V'$ are vector spaces in duality, via a bilinear or sesquilinear form. Let $s \in (V_1, V_2)$ be an arrow endowed with a transposed arrow $f = s' \in (V_2', V_1')$ with respect to the duality. For example, if $s$ is linear, $f$ is the linear map defined by

$$(k_2', sk_1) = (fk_2', k_1), \quad k_1 \in V_1, k_2' \in V_2'.$$

While, if $s$ is antilinear, $f$ is the antilinear map defined by

$$(k_2', sk_1) = \overline{(fk_2', k_1)}, \quad k_1 \in V_1, k_2' \in V_2'.$$

This would seem to be a natural notion to study in the theory of dual spaces. In the present context we need a more restrictive notion, replacing vector spaces by (positive–definite) scalar product spaces and requiring the duality to make the scalar product space $V'$ a dense subspace of the completion of $V$. A pair of the form $(s, f)$ will in this case be called an Hermitian pair of $\mathcal{V}$. A particular instance of this situation would be to fix a set of Hilbert spaces, each Hilbert space $H$ having two distinguished dense subspaces $H^s$ and $H^f$, say, and take an arrow in the category to be a linear (or just semilinear) mapping from some $H^s_1$ into some $H^f_2$ and maps it into $H^f_1$.

**A.2 Lemma** Let $\mathcal{SP}^a$ be the category of scalar product spaces with semilinear mappings. Then for each hermitian pair $(s, f)$ of $\mathcal{SP}^a$, we let $\overline{s}$ and $\overline{f}$ denote the closure of $s$ and $f$, respectively. If $s$'s defines an essentially self–adjoint pair, then $\overline{s}$'s is the closure of $\overline{s}$'s, and $\overline{ff'}$ the closure of $\overline{f} \overline{f'}$.

**Proof.** Let $h \in \mathcal{D}_s$, $\overline{s}h \in \mathcal{D}_{\overline{s}}$ and $k$ in the domain of $f'$, then

$$(k, \overline{s}sh) = (f'k, \overline{s}h) = (ff'k, h),$$
since $f' \subset s'^*$, $f \subset s^*$. Hence $s's \subset s'^* \subset (ff')^* = s^*s$ since $s's$ and $ff'$ is an essentially self-adjoint pair, completing the proof.

An instructive example of the situation of Lemma A.1 is got by considering a von Neumann algebra $\mathcal{M}$ with cyclic and separating vector $\Omega$. Then $\mathcal{M}$ and $\mathcal{M}'$ are in (sesquilinear) duality via $(\mathcal{M}'\Omega, \mathcal{M}\Omega)$. We may now take $\mathcal{C}$ to be the category of semilinear mappings between $\mathcal{M}$ and $\mathcal{M}'$ with dual. For example if $\kappa$ is a linear mapping from $\mathcal{M}$ to $\mathcal{M}$ then $\kappa$ has a dual if there is a linear mapping $\kappa'$ from $\mathcal{M}'$ to $\mathcal{M}'$ such that

$$(\kappa'(\mathcal{M}'\Omega, \mathcal{M}\Omega) = (\mathcal{M}'\Omega, \kappa(\mathcal{M})\Omega), \quad M \in \mathcal{M}, \mathcal{M}' \in \mathcal{M}',$$

and a linear mapping $\lambda$ from $\mathcal{M}$ to $\mathcal{M}'$ has a dual if there is a linear mapping $\lambda'$ from $\mathcal{M}$ to $\mathcal{M}'$ such that

$$(\lambda'(\mathcal{M}\Omega, N\Omega) = (\mathcal{M}\Omega, \lambda(N)\Omega), \quad M, N \in \mathcal{M}.$$
We are led to a simple example of the situation in Lemma A.2. Consider the situation of Lemma 5.4. We let $C_{\varphi,\omega}$ be the set of semilinear mappings of $B_{\varphi,\omega}$ whose adjoint is defined on $B_{\varphi,\omega}$ and maps it into itself. Elements of $C_{\varphi,\omega}$ include the restrictions of the $\omega_z$, $z \in \mathbb{C}$ as well as the restriction of $S_\varphi$. Taking the closures, we recover the $\omega_z$ and $S_\varphi$.

We give a rather more complicated example involving two objects $B_{\varphi,\omega}$ and $B_{\varphi,\omega}$ motivated by Theorem 5.5.

**A.4 Lemma** Let $\varphi$ be a lower semicontinuous densely defined weight on a $C^*$-algebra $A$ equipped with a conjugation $j$. Let $\omega : \mathbb{R}^n \to \text{Aut}(A)$ be a pointwise norm continuous $\varphi$-invariant automorphism group of $A$ and suppose that $j_\omega = j_{\omega_{\mathbb{R}^n}}$. Let $\omega := \varphi \circ j$ then $j(B_{\varphi,\omega}) = B_{\varphi,\omega}$. Now let $C$ denote the category of semilinear mappings between the two objects $B_{\varphi,\omega}$ and $B_{\varphi,\omega}$ whose adjoints restrict to a mapping in the other direction. Then $C$ contains $C_{\varphi,\omega}$ and $C_{\varphi,\omega}$ as categories in a natural way and the map $B \mapsto j(B)$, $B \in B_{\varphi,\omega}$ is an antilinear bijection.

**Proof.** We have $j(N_\varphi) = N_{\varphi}$ and since $j$ normalizes $\omega_{\mathbb{R}^n}$, $j(B_{\varphi,\omega}) = B_{\varphi,\omega}$. The remaining statements are now obvious.

Theorem 5.5 corresponds to taking for $\varphi$ the left Haar weight on a locally compact quantum group and a conjugation $j = R_\varphi^*$.

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