The Comparison of Fourth Order Runge-Kutta and Homotopy Analysis Method for Solving Three Basic Epidemic Models

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Abstract. All epidemic models include a system of non-linear differential equation. Mostly the analytical solution of epidemic model is difficult to obtain. There are many different methods to solve non-linear differential equation, one of them is homotopy analysis method. The homotopy analysis method is an analytic approximation method using series solution for highly non-linear equations. The advantage of this method is a guarantee the convergence of approximation power series solution by choosing suitable values of the auxiliary parameter. In this paper, we consider three epidemic models in a closed population without demographics; SIR, SIR, and SEIR models. We find the solutions of the models by homotopy analysis method and then compare the numerical results with fourth order Runge-Kutta method. The homotopy analysis method gives a good result for the solution of the epidemic models with a few iterations and the solutions obtained from this method are good as compared to fourth order Runge-Kutta numerical method.

1. Introduction
Mathematical modeling is a process to build a mathematical model for describing change of a system [1]. Mathematical modeling is always related to other fields, such as in physics, biology, chemistry, engineering [2], medicine [3], politics [4], and epidemiology [5]. There are several transmission models with different states used in epidemiology. These include SI (Susceptible-Infected), SIR (Susceptible-Infected-Recovered), and SEIR (Susceptible-Exposed-Infected-Recovered) [6]. The susceptibles are those who are not infected, not immune, and they are able to catch the disease. The exposed are those who have the disease but cannot transmit it to susceptible individuals. The infected are those who are infected and can transmit the disease to susceptible individuals. The recovered are those who have been infected and assumed to be immune for life.

In epidemic models, it is very hard to get an analytic solution for such models, so it is solved numerically. There are many numerical methods used to solve the differential equation, such as Euler method, Taylor method, midpoint method, and Runge-Kutta method. Usually, the problem in epidemic models which involved the non-linear differential equations are solved using fourth order Runge-Kutta (RK4) method. Unlike the Euler method which gives high truncation error, this method gives the better solution with relatively small of truncation error.

Homotopy analysis method (HAM) is a method for approximating analytically to solve the non-linear differential equation [7]. There are two operators used in this method, linear and non-linear operators. Non-linear operator is stated based on the form of the non-linear differential equation. In order to apply HAM, we need to construct the homotopy equation. This homotopy equation needs the
auxiliary parameter for controlling the convergence of solution. The HAM gives the solution in the form of power series.

This paper discusses the formulation of HAM in finding the analytic approximate solution of some epidemic models. The results will be compared with the numerical solution of epidemic models using RK4 method. Numerical simulation with varies auxiliary parameter will be shown to investigate the effect of this parameter to the models. In the next section, the basic idea of the HAM will be described.

2. Homotopy Analysis Method

Homotopy analysis method (HAM) is one of the method for finding the solution of the non-linear problems. The HAM is first time introduced by Liao in 1992 [8]. In this section, we explain the basic idea of HAM and then we apply it to our proposed epidemic models; $SI$, $SIR$, and $SEIR$. The HAM has the following advantages [9]. HAM is independent of any small/large physical parameters at all. Unlike all other analytical techniques, HAM provides us a convenient way to guarantee the convergence of solution series. HAM provides us extremely large freedom to choose the auxiliary linear operator and base functions.

Let $N$ is non-linear operator, $t$ is independent variable, and $\vartheta(t)$ is unknown function with $\vartheta_0(t)$ denote an initial guess of the exact solution $\vartheta(t)$ on the following equation

$$N[\vartheta(t)] = 0$$

(1)

When $\vartheta(t) = 0$, the auxiliary linear operator has the property $L[\vartheta(t)] = 0$. Construct homotopy equation $\bar{H}$ with embedding parameter $\rho \in [0,1]$, auxiliary parameter $h \neq 0$, and auxiliary function $H(t) \neq 0$ as follows

$$\left(1 - \rho\right)L[\phi(t; \rho) - \vartheta_0(t)] - \rho h H(t)N[\phi(t; \rho)] = \bar{H}[\phi(t; \rho); \vartheta_0(t), H(t), h, \rho]$$

(2)

We have freedom to choose the initial guess $\vartheta_0(t)$, the auxiliary linear operator $L$, the non-zero auxiliary parameter $h$, and the non-zero auxiliary function $H(t)$. Equating the homotopy equation (2) to zero,

$$\bar{H}[\phi(t; \rho); \vartheta_0(t), H(t), h, \rho] = 0$$

(3)

which is called as the zero-order deformation equation.

When $\rho = 0$, the equation (3) becomes

$$L[\vartheta(t; 0) - \vartheta_0(t)] = 0$$

(4)

When $\rho = 1$, since $h \neq 0$ and $H(t) \neq 0$, the equation (3) becomes

$$N[\phi(t; 1)] = 0$$

(5)

or equivalent to

$$\vartheta(t; 1) = \vartheta(t)$$

According to equations (4) and (5), as the embedding parameter $\rho$ increases from 0 to 1, $\phi(t; \rho)$ varies continuously from the initial approximation $\vartheta_0(t)$ to the exact solution $\vartheta(t)$. Using Taylor series approximation with respect to $\rho$, $\phi(t; \rho)$ can be written as follows

$$\phi(t; \rho) = \vartheta_0(t) + \sum_{m=1}^{\infty} \vartheta_m(t) \rho^m$$

(6)

where

$$\vartheta_m(t) = \frac{1}{m!} \frac{\partial^m \phi(t, \rho)}{\partial \rho^m} \bigg|_{\rho = 0}$$

(7)

The solution (6) of the zero-order deformation equation (3) exists for all $\rho \in [0,1]$ and the deformation derivative $\frac{\partial^m \phi(t, \rho)}{\partial \rho^m}$ exists for $m = 1, 2, 3, \ldots$

If the linear operator $L$, the initial guess $\vartheta_0(t)$, the auxiliary parameter $h$, and the auxiliary function $H$ are properly chosen such that the power series (6) converges at $\rho = 1$, i.e.
\[
\phi(t; 1) = \vartheta_0(t) + \sum_{m=1}^{\infty} \vartheta_m(t)
\]

Differentiating the zero-order deformation equation (3) \( m \) times with respect to \( \rho \), dividing by \( m! \), and setting \( \rho = 0 \), we have the so-called \( m \)-order deformation equation

\[
L(\vartheta_m(t) - \chi_m \vartheta_{m-1}(t)) = h H(t) \mathcal{R}_m(\vartheta_{m-1}(t))
\]

where

\[
\mathcal{R}_m(\vartheta_{m-1}(t)) = \frac{1}{(m-1)!} \frac{\partial^{m-1}N[\phi(t; \rho)]}{\partial \rho^{m-1}}
\]

and

\[
\chi_m = \begin{cases} 0, & m \leq 1 \\ 1, & m > 1 \end{cases}
\]

Using equation (8), it can be found the analytic approximate solution for any non-linear operator \( N \). In the next section, the numerical results of three epidemic models using RK4 method and convergence of the HAM solutions will be discussed.

3. Results and Discussion

The epidemic models will be used in this paper are SI, SIR, and SEIR models in a closed population without demographics. In each model, the analytic approximate solutions are obtained using HAM. The compartmental diagram of the epidemic models are presented in Figure 1.

![Figure 1](image)

**Figure 1.** Scheme of (a) SI, (b) SIR, and (c) SEIR epidemic models.

In Fig. 1, the infected individuals \( I \) infect the susceptible individuals \( S \) with constant rate \( \beta \). The number of individuals enter to \( I \) is \( \beta SI \). The SI epidemic model can be mathematically written as

\[
\begin{aligned}
\frac{dS}{dt} &= -\beta SI \\
\frac{dI}{dt} &= \beta SI
\end{aligned}
\]

with initial value condition

\[
S_0(t) = N_S, I_0(t) = N_I
\]

Choosing \( h = -1 \) and \( H(t) = -1 \), the solutions for SI epidemic model using HAM are

\[
\begin{align*}
S_m(t) &= \chi_m S_{m-1}(t) - \int_0^t \left[ S_{m-1}'(\tau) + \beta \sum_{k=0}^{m-1} S_k(\tau) I_{m-1-k}(\tau) \right] d\tau \\
I_m(t) &= \chi_m I_{m-1}(t) - \int_0^t \left[ I_{m-1}'(\tau) - \beta \sum_{k=0}^{m-1} S_k(\tau) I_{m-1-k}(\tau) \right] d\tau
\end{align*}
\]
In Fig. 2, the infected individuals $I$ infect the susceptible individuals $S$ with constant rate $\beta$. The infected individuals move into the recovered individuals because of immune to the disease with constant rate $\gamma$. The number of individuals enter to $I$ is $\beta SI$ and the number of individuals enter to $R$ is $\gamma I$. The $SIR$ epidemic model can be mathematically described by

$$\begin{cases}
\frac{dS}{dt} = -\beta SI \\
\frac{dI}{dt} = \beta SI - \gamma I \\
\frac{dR}{dt} = \gamma I
\end{cases}$$

with initial value condition

$S_0(t) = N_S, I_0(t) = N_I, R_0(t) = N_R$

Choosing $h = -1$ and $H(t) = -1$, the solutions for $SIR$ epidemic model using HAM are

$$S_m(t) = \chi_m S_{m-1}(t) - \int_0^t \left[ S'_{m-1}(\tau) + \beta \sum_{k=0}^{m-1} S_k(\tau) I_{m-1-k}(\tau) \right] d\tau$$

$$I_m(t) = \chi_m I_{m-1}(t) - \int_0^t \left[ I'_{m-1}(\tau) - \beta \sum_{k=0}^{m-1} S_k(\tau) I_{m-1-k}(\tau) + \gamma I_{m-1}(\tau) \right] d\tau$$

$$R_m(t) = \chi_m R_{m-1}(t) - \int_0^t \left[ R'_{m-1}(\tau) - \gamma I_{m-1}(\tau) \right] d\tau$$

In Fig. 3, the infected individuals $I$ infect the susceptible individuals $S$ with constant rate $\beta$. The exposed individuals move into the infected individuals with constant rate $\omega$. The infected individuals move into the recovered individuals because of immune to the disease with constant rate $\gamma$. The number of individuals enter to $E$ is $\beta SI$, the number of individuals enter to $I$ is $\omega I$, and the number of individuals enter to $R$ is $\gamma I$. The $SEIR$ epidemic model can be mathematically written as

$$\begin{cases}
\frac{dS}{dt} = -\beta SI \\
\frac{dE}{dt} = \beta SI - \omega E \\
\frac{dI}{dt} = \omega E - \gamma I \\
\frac{dR}{dt} = \gamma I
\end{cases}$$

with initial value condition

$S_0(t) = N_S, E_0(t) = N_E, I_0(t) = N_I, R_0(t) = N_R$

Choosing $h = -1$ and $H(t) = -1$, the solutions for $SEIR$ epidemic model using HAM are

$$S_m(t) = \chi_m S_{m-1}(t) - \int_0^t \left[ S'_{m-1}(\tau) + \beta \sum_{k=0}^{m-1} S_k(\tau) E_{m-1-k}(\tau) \right] d\tau$$

$$E_m(t) = \chi_m E_{m-1}(t) - \int_0^t \left[ E'_{m-1}(\tau) - \beta \sum_{k=0}^{m-1} S_k(\tau) I_{m-1-k}(\tau) + \omega E_{m-1}(\tau) + \omega E_{m-1}(\tau) \right] d\tau$$

$$I_m(t) = \chi_m I_{m-1}(t) - \int_0^t \left[ I'_{m-1}(\tau) - \omega E_{m-1}(\tau) + \gamma I_{m-1}(\tau) \right] d\tau$$
\[ R_m(t) = \chi_m R_{m-1}(t) - \int_0^t [R'_{m-1}(\tau) - \gamma I_{m-1}(\tau)]d\tau \]

Figure 2 show the solutions of \( SI \) epidemic model using RK4 method and HAM for \( h = -1.0, -0.5, -0.2, -0.1 \), with parameter \( \beta = 0.01 \) and initial values condition are \( N_S = 20 \) and \( N_I = 10 \).

Figure 2. Solutions of \( SI \) epidemic model using RK method and HAM for different \( h \); (a) \( S \) and (b) \( I \)

Five terms approximation for solutions of \( SI \) epidemic model are
\[
S_S(t) = 20 - 2t - 0.1t^2 + 0.01t^3 + 0.00125t^4 - 0.000035t^5
\]
\[
I_S(t) = 10 + 2t + 0.1t^2 - 0.01t^3 - 0.00125t^4 + 0.000035t^5
\]

From Table 1, we can see that the solutions of \( SI \) epidemic model given by both methods are same (in four decimal places) from \( t = 1 \) to \( t = 3 \) (for \( S \) and \( I \)).

**Table 1.** The solutions of \( SI \) model based on RK4 and HAM (\( h = -1 \)).

| \( t \) | \( S_{RK4} \) | \( S_{HAM} \) | \( I_{RK4} \) | \( I_{HAM} \) |
|---|---|---|---|---|
| 1 | 17.9112 | 17.9112 | 12.0888 | 12.0888 |
| 2 | 15.6981 | 15.6981 | 14.3019 | 14.3019 |
| 3 | 13.4541 | 13.4541 | 16.5459 | 16.5459 |
| 4 | 11.2779 | 11.2782 | 18.7221 | 18.7218 |
| 5 | 9.2568 | 9.2591 | 20.7432 | 20.7409 |
| 6 | 7.4537 | 7.4677 | 22.5463 | 22.5323 |
| 7 | 5.9019 | 5.9637 | 24.0981 | 24.0363 |
| 8 | 4.6072 | 4.8175 | 25.3928 | 25.1825 |
| 9 | 3.5546 | 4.1326 | 26.4454 | 25.8674 |
| 10 | 2.7167 | 4.0263 | 27.2833 | 25.9737 |
Figure 3 show the solutions of SIR epidemic model using RK4 method and HAM for $h = -1.0, -0.5, -0.2, -0.1$, with parameters $\beta = 0.01$ and $\gamma = 0.02$. The initial values condition are $N_S = 20$, $N_I = 10$, and $N_R = 5$.

![Graphs](image)

**Figure 3.** Solutions of SIR epidemic model using RK4 method and HAM for different $h$; (a) $S$, (b) $I$, and (c) $R$

Five terms approximation for solutions of SIR epidemic model are

- $S_S(t) = 20 - 2t - 0.08t^2 + 0.01053333t^3 + 0.000954t^4 - 0.00005489t^5$
- $I_S(t) = 10 + 1.8t + 0.062t^2 - 0.01094666t^3 - 0.0008992666t^4 + 0.00005849t^5$
- $R_S(t) = 5 + 0.2t + 0.018t^2 + 0.00041333t^3 - 0.00005473t^4 - 0.00000359t^5$

From Table 2, we can see that the solutions of SIR epidemic model given by both methods are same (in four decimal places) from $t = 1$ to $t = 3$ (for $S$ and $I$) and from $t = 1$ to $t = 5$ (for $R$).
Table 2. The solutions of SIR model based on RK4 and HAM (h = −1).

| t   | S_{RK4} | S_{HAM} | I_{RK4} | I_{HAM} | R_{RK4} | R_{HAM} |
|-----|---------|---------|---------|---------|---------|---------|
| 1   | 17.9314 | 17.9314 | 11.8502 | 11.8502 | 5.2184  | 5.2184  |
| 2   | 15.7772 | 15.7772 | 13.7485 | 13.7485 | 5.4743  | 5.4743  |
| 3   | 13.6223 | 13.6223 | 15.6096 | 15.6096 | 5.7680  | 5.7680  |
| 4   | 11.5510 | 11.5511 | 17.3511 | 17.3510 | 6.0979  | 6.0979  |
| 5   | 9.6342  | 9.6344  | 18.9050 | 18.9049 | 6.4608  | 6.4608  |
| 6   | 7.9205  | 7.9201  | 20.2270 | 20.2278 | 6.8526  | 6.8521  |
| 7   | 6.4341  | 6.4235  | 21.2976 | 21.3105 | 7.2682  | 7.2660  |
| 8   | 5.1775  | 5.0998  | 22.1197 | 22.2064 | 7.7028  | 7.6938  |
| 9   | 4.1371  | 3.7581  | 22.7114 | 23.1214 | 8.1515  | 8.1205  |
| 10  | 3.2896  | 1.8389  | 23.1005 | 24.6440 | 8.6099  | 8.5171  |

Figure 4 show the solutions of SEIR epidemic model using RK4 method and HAM for \( h = -1.0, -0.5, -0.2, -0.1 \), with parameters \( \beta = 0.01, \gamma = 0.02, \) and \( \omega = 0.1 \). The initial values condition are \( N_S = 20, N_E = 15, N_I = 10, \) and \( N_R = 5 \).

Five terms approximation for solutions of SEIR epidemic model are

\[
S_5(t) = 20 - 2t - 0.03t^2 + 0.00866666t^3 - 0.0000685t^4 - 0.00001156t^5
\]
\[
E_5(t) = 15 + 0.5t + 0.005t^2 - 0.00903333t^3 + 0.00029433t^4 + 0.00000679t^5
\]
\[
I_5(t) = 10 + 1.3t + 0.012t^2 + 0.00086666t^3 - 0.00022626t^4 + 0.00000679t^5
\]
\[
R_5(t) = 5 + 0.2t + 0.013t^2 + 0.00008t^3 + 0.0000043t^4 - 0.0000099t^5
\]

From Table 3, we can see that the solutions of SEIR epidemic model given by both methods are same (in four decimal places) from \( t = 1 \) to \( t = 5 \) (for \( S \)), from \( t = 1 \) to \( t = 6 \) (for \( E \) and \( I \)), and from \( t = 1 \) to \( t = 8 \) (for \( R \)).

Table 3. The solutions of SEIR model based on RK4 and HAM (h = −1).

| t   | S_{RK4} | S_{HAM} | E_{RK4} | E_{HAM} | I_{RK4} | I_{HAM} | R_{RK4} | R_{HAM} |
|-----|---------|---------|---------|---------|---------|---------|---------|---------|
| 1   | 17.9788 | 17.9788 | 15.4963 | 15.4963 | 11.3119 | 11.3119 | 5.2131  | 5.2131  |
| 2   | 15.9494 | 15.9494 | 15.9527 | 15.9527 | 12.6453 | 12.6453 | 5.4526  | 5.4526  |
| 3   | 13.9606 | 13.9606 | 16.3267 | 16.3267 | 13.9937 | 13.9937 | 5.7190  | 5.7190  |
| 4   | 12.0556 | 12.0556 | 16.5850 | 16.5850 | 15.3470 | 15.3470 | 6.0124  | 6.0124  |
| 5   | 10.2710 | 10.2710 | 16.7040 | 16.7040 | 16.6922 | 16.6922 | 6.3328  | 6.3328  |
| 6   | 8.6345  | 8.6344  | 16.6705 | 16.6705 | 18.0151 | 18.0151 | 6.6799  | 6.6799  |
| 7   | 7.1645  | 7.1643  | 16.4815 | 16.4818 | 19.3008 | 19.3008 | 7.0532  | 7.0532  |
| 8   | 5.8704  | 5.8692  | 16.1432 | 16.1447 | 20.5348 | 20.5348 | 7.4516  | 7.4516  |
| 9   | 4.7525  | 4.7474  | 15.6694 | 15.6756 | 21.7040 | 21.7040 | 7.8741  | 7.8740  |
| 10  | 3.8041  | 3.7859  | 15.0794 | 15.1009 | 22.7971 | 22.7943 | 8.3193  | 8.3189  |

From Fig. 2, Fig. 3, and Fig. 4, it can be seen that the solutions converge when \( h \in [-1, -0.5] \).
Figure 4. Solutions of SEIR epidemic model using RK4 method and HAM for different $h$; (a) $S$, (b) $E$, (c) $I$, and (d) $R$.

4. Conclusion
In this paper, we considered SI, SIR, and SEIR epidemic models and applied the homotopy analysis method. It is found that the HAM is effective because it could be used to find the approximate solutions of three epidemic models with small error. The solutions of three epidemic models converge when the auxiliary parameter $h \in [-1, -0.5]$. The HAM is efficient because it use only a few iterations to find the convergent series solutions. We also solved numerically the epidemic models and compared with RK4 method. The solutions of epidemic models obtained from homotopy analysis method are good as compared to RK4 numerical method. For further research, other analytical approximation methods can be used to find the solutions of epidemic models.

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Acknowledgements
This research was supported by Universitas Katolik Parahyangan, with Monodisiplin research grant scheme 2017 (Number: III/LPPM/2017-01/20-P).