The first-passage time of the Brownian motion to a curved boundary: an algorithmic approach

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January 29, 2015

Abstract

Under some weak conditions, the first-passage time of the Brownian motion to a continuous curved boundary is an almost surely finite stopping time. Its probability density function (pdf) is explicitly known only in few particular cases. Several mathematical studies proposed to approximate the pdf in a quite general framework or even to simulate this hitting time using a discrete time approximation of the Brownian motion. The authors study a new algorithm which permits to simulate the first-passage time using an iterating procedure. The convergence rate presented in this paper suggests that the method is very efficient.

Key words and phrases: first-passage time, Brownian motion, potential theory, randomized algorithm.

2010 AMS subject classifications: primary 65C05; secondary 65N75, 60G40.

Introduction

Modeling biological or physical systems often requires to handle with one-dimensional diffusion processes. The marginal probability distribution of such processes, at a fixed time, permits to describe quite precisely the model. Nevertheless, in many applications, this information is insufficient and the description of the whole paths becomes crucial. This is namely the case for variety problems related to neuronal sciences, financial derivatives with barriers, ruin probability of an insurance fund, optimal stopping problems,... In these frameworks, the main task is the description of the first passage time densities for time-dependent boundaries (for level-crossing problems see [1, 2]).

For instance, let us focus our attention on a simple interpretation of the neural transmission. When a neuron is stimulated by pressure, heat, light, or chemical information, its membrane voltage changes as time elapses and, as soon as it reaches a constant threshold, the depolarization phenomenon occurs and

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*This work has been supported by the Agence National de la Recherche through the ANR Project MANDy “Mathematical Analysis of Neuronal Dynamics”, ANR-09-BLAN-0008-01.
the voltage is reset to a resting potential. The family of integrate-and-fire spiking neuron models is based on this simple interpretation. The firing time therefore corresponds to the first-passage time of the membrane potential, represented by a stochastic mean-reverting process (usually the Ornstein-Uhlenbeck process) to the neural threshold (for introduction of noise in neuron systems see Part I Chapter 5 in [9], for the integrate-and-fire model see Chapter 10 in [8]).

Our main motivation is to emphasize an algorithmic approach in order to approximate the first-passage time of the Brownian motion to curved boundaries. Using simple time transformations will also permit to apply the results to the Ornstein-Uhlenbeck process (see Section 3).

In order to describe approximations of the first-passage time of the Brownian motion, we assume that this stopping time is almost surely finite. In this way, we introduce particular conditions for this property to be satisfied. Let us consider a continuous function \( \varphi : \mathbb{R}_+ \to \mathbb{R} \) satisfying the following hypothesis:

\[
\varphi(0) > 0 \quad \text{and} \quad \limsup_{t \to \infty} \frac{\varphi(t)}{\sqrt{2t \log \log t}} < 1. \tag{H1}
\]

We then define the hitting time

\[
\tau_\varphi = \inf \{ t > 0 : B_t = \varphi(t) \}
\]

where \((B_t, t \geq 0)\) stands for a standard one-dimensional Brownian motion. Under (H1), the a.s. finiteness of \( \tau_\varphi \) is an obvious consequence of the law of the iterated logarithm (see e.g. [11, Th.9.23 p.112]). It is quite difficult to obtain precise informations about this stopping time in general situations.

Durbin [6, 7] proposed to approximate the first-passage distribution \( p(t) dt = P(\tau_\varphi \in dt) \) of the Brownian motion as follows: \( p \) can be represented by a serie expansion

\[
p(t) = \sum_{j=1}^{k} (-1)^{j-1} q_j(t) + (-1)^k r_k(t), \quad k \geq 1,
\]

where \( q_j \) for \( 1 \leq j \leq k \) and \( r_k \) are defined by multiple integrals depending on the boundary \( \varphi \). The approximation simply consists in truncating the series. Let us note that the first term corresponds in fact to the tangent approximation of Strassen [15] and Daniels [3]. The convergence of the series and the error bounds can be precised if the curved boundary is wholly concave or wholly convex.

For particular cases, the probability density function can be computed explicitly. Lerche [12] used the method of images in order to obtain explicit expressions of the p.d.f. \( p \). However, in practice, for general boundaries, the expression emphasized by Durbin does not permit to simulate easily the hitting time.

One way to approximate \( \tau_\varphi \) and especially to compute the probability for the hitting time to be smaller than some given \( T > 0 \), is to use a time discretization of the Brownian motion on \([0,T] \). The time interval is then split into \( n \) small intervals of the kind \([ (k-1)T/n, kT/n] \), with \( 1 \leq k \leq n \). At each endpoint \( kT/n \), the event \( B_{kT/n} < \varphi(kT/n) \) has to be tested and at the same time we need to know if, given the Brownian values at the endpoints, the Brownian paths on the small intervals (the conditional distribution corresponds therefore to the Brownian bridge one) hit the curved boundary. The probability not to hit the boundary on a small interval for the Brownian bridge can be approximated [10].
This method can become onerous if the observed time interval \([0,T]\) is large. Let us also note that the time-splitting can be replaced by a space-splitting namely for the first time the Brownian motion exits from a given interval [3].

The aim of this study is to present a new method of approximation of \(\tau_\varphi\) by some families \((\tau_\varphi^\epsilon)_{\epsilon \geq 0}\) which satisfy exact simulation and such that \(\tau_\varphi^\epsilon\) converges toward \(\tau_\varphi\) in distribution as \(\epsilon\) tends to 0. Stopping such sequence at a small level \(\epsilon = \epsilon_0\) induces the error term in the approximation. Two different families of sequences will be emphasized and the associated convergence rates are estimated. The first algorithm developed in Section 1 concerns increasing curved boundaries and the second one, Section 2, permits to deal with quite general boundaries provided that its derivative is bounded. In the last section, we present different examples in order to illustrate the algorithm efficiency.

1 First-passage time to non-decreasing boundaries

Let us assume that the boundary \(\varphi\) satisfies (H1) and that the following additional conditions hold

\[
\varphi : \mathbb{R}_+ \to \mathbb{R} \text{ is a non-decreasing } C^1\text{-continuous function}, \quad \text{(H2)}
\]

\[
2\varphi'(t)\sqrt{1+t} \leq 1, \quad \forall t \geq 0. \quad \text{(H3)}
\]

We introduce the algorithm associated to the hitting time \(\tau_\varphi\) defined by (0.1).

**Algorithm (A1).** Let \(\epsilon > 0\) be a small parameter and \((G_n)_{n \geq 0}\) a sequence of independent standard Gaussian distributed random variables.

Initialization: \(T_0 = 0, T_1 = 0, T_2 = (\varphi(0)/G_0)^2\) and \(N_\epsilon = 1\).

While \(\varphi(T_2) - \varphi(T_1) \geq \epsilon\) do:

\[
\begin{align*}
(T_0, T_1, T_2) &\leftarrow (T_1, T_2, T_2 + (\varphi(T_2) - \varphi(T_1))^2/G_{N_\epsilon}^2) \\
N_\epsilon &\leftarrow N_\epsilon + 1.
\end{align*}
\] (1.1)

Outcome: \(\tau_\varphi^\epsilon \leftarrow T_2\) and \(N_\epsilon\).

Let us just note that Algorithm (A1) is very simple to use since each step only requires one Gaussian distributed random variable. Moreover it is a approximation of the first-passage time:

**Theorem 1.1.**

1. Let us assume that the boundary function \(\varphi\) satisfies (H1), (H2) and (H3) then the random variable \(\tau_\varphi^\epsilon\) defined in Algorithm (A1) converges in distribution towards \(\tau_\varphi\) defined by (0.1) as \(\epsilon\) tends to zero. More precisely

\[
F_\epsilon(t - \epsilon) - \frac{3\sqrt{\epsilon}}{\sqrt{2\pi}} \leq F(t) \leq F_\epsilon(t), \quad \text{for any } t \geq \epsilon, \quad (1.2)
\]

where \(F\) (resp. \(F_\epsilon\)) is the cumulative distribution function of \(\tau_\varphi\) (resp. \(\tau_\varphi^\epsilon\)).

2. There exists a constant \(C > 0\) such that the random number of iterations \(N_\epsilon\) defined in Algorithm (A1) satisfies:

\[
\mathbb{E}[N_\epsilon] \leq C\sqrt{\log \epsilon}. \quad (1.3)
\]
The parameter $\epsilon$ describes the precision of the approximation. The number of steps in the Algorithm (A1) is very small (even smaller than usual results obtained for algorithms based on random walks on spheres, which are close to Algorithm (A1) see [14]): in fact the constant appearing in (1.3) can be explicitly computed: for any constant $0 < \kappa < 1/2$, there exists $\epsilon_0(\kappa) > 0$ such that (1.3) is satisfied as soon as $\epsilon < \epsilon_0$, with the particular constant

$$C = \frac{1}{m\kappa}, \quad m = \log(4) + \frac{2\sqrt{2}}{\sqrt{\pi}} \mu$$

and

$$\mu = \int_0^\infty (\log|x|) e^{-x^2} \, dx. \quad (1.4)$$

The proof of Theorem 1.1 is based on a main argument developed in the following proposition: each step of Algorithm (A1) has to be related to a particular part of the Brownian paths before hitting the boundary.

**Proposition 1.2.** Let $(B_t, t \geq 0)$ be a standard one-dimensional Brownian motion. We define the following sequence of stopping times:

$$s_0 := T_0 = 0 \quad \text{and} \quad s_n := \inf \{t \geq 0 : B_t + T_{n-1} = \varphi(T_{n-1})\} \quad \text{and} \quad T_n := s_1 + \ldots + s_n, \quad (1.5)$$

where the function $\varphi$ satisfies [H1], [H2] and [H3]. Then the following properties hold:

1. $(T_n)_{n \geq 0}$ is a non-decreasing sequence which almost surely converges towards $T$.
2. Let $n \geq 1$, then the probability distribution of $s_{n+1}$ given the $\sigma$-algebra $\mathcal{F}_n := \sigma \{s_1, \ldots, s_n\}$ is identical as $(\varphi(T_n) - \varphi(T_{n-1}))^2/G_n^2$ where $(G_n)_{n \geq 0}$ is a sequence of independent standard Gaussian random variables. Moreover $s_1 \overset{(d)}{=} (\varphi(0)/G_0)^2$.
3. Let $\mathcal{M}_\epsilon := \inf \{n \geq 1 : \varphi(T_n) - B_{T_n} \leq \epsilon\}$, then $T_{\mathcal{M}_\epsilon}$ and $T_{\mathcal{M}_\epsilon}$, defined in Algorithm (A1), are identically distributed, so are $\mathcal{M}_\epsilon$ and $\mathcal{N}_\epsilon$.

Let us note that the mean of each random variable $s_n$ defined by (1.5) is infinite since $\mathbb{E}[G^{-2}] = +\infty$ where $G$ is a standard Gaussian variable. Proposition 1.2 suggests that the first-passage time can be obtained as a sum of positive random variables of infinite average, we easily deduce $\mathbb{E}[T]$ = $+\infty$. In the particular case of increasing boundaries $\varphi$, the sum has infinitely many terms.

**Proof of Proposition 1.2.**

**Step 1.** By construction, the sequence $(T_n)_{n \geq 0}$ is non-decreasing and non-negative: it converges almost surely to $T_\infty$. Since $\varphi$ is a non-decreasing boundary, $T_n \leq T_\varphi$ for any $n \geq 0$. In particular $T_\infty$ is less than $T_\varphi$ which is a finite stopping time due to the law of the iterated logarithm, see [H1] followed by discussion. Consequently, the random variable $B_{T_\infty}$ is well defined. Since $\varphi$ is non-decreasing, we get $B_{T_n} = \varphi(T_{n-1})$ for any $n \geq 1$. Taking the large $n$ limit leads to $B_{T_\infty} = \varphi(T_\infty)$, the Brownian paths and the function $\varphi$ being continuous. We deduce that $T_\infty = T$.
Step 2. Let us first consider the stopping time $s_1$. Using the reflection principle of the Brownian paths and a scaling property, we obtain:

$$
P(s_1 > t) = \mathbb{P}\left( \sup_{0 \leq u \leq t} B_u < \varphi(0) \right) = \mathbb{P}(|B_t| < \varphi(0)) = \mathbb{P}(B_t^2 < \varphi(0)^2 / t) = \mathbb{P}(\varphi(0)^2 / G_0^2 > t), \quad t \geq 0.
$$

The general $n$-th case can be proven using similar arguments combined with the Markov property of the Brownian motion:

$$
P(s_{n+1} > t | F_n) = \mathbb{P}\left( \sup_{T_n \leq u \leq T_n + t} B_u < \varphi(T_n) | F_n \right) = \mathbb{P}\left( B_{u+T_n} - B_{T_n} < \varphi(T_n) - \varphi(T_n-1) | F_n \right) = \mathbb{P}\left( \sup_{0 \leq u \leq t} \tilde{B}_u < \varphi(T_n) - \varphi(T_n-1) | F_n \right),$$

where $\tilde{B}$ is a Brownian motion independent of $F_n$.

Step 3. Using the results developed in Step 2, we observe that $(s_n)_{n \wedge M}$ and the sequence of values $T_2$, defined in Algorithm (A1), have the same distribution. It is therefore obvious that $T_M$ and $\tau_\varphi$ are identically distributed. Indeed the stopping time can be rewritten as follows:

$$M = \inf\{n \geq 1 : \varphi(T_n) - \varphi(T_n-1) \leq \epsilon\}. \quad (1.6)$$

\begin{proof} of Theorem 1.1 \end{proof}

Step 1. Let us recall that $T_n$ is defined by (1.5). By Proposition 1.2, $T_n \leq \tau_\varphi$ for any $n \geq 0$ and in particular $T_M \leq \tau_\varphi$. Hence

$$
P(T_M \leq t) \geq \mathbb{P}(\tau_\varphi \leq t), \quad \forall t \geq 0.
$$

Since $\tau_\varphi$ has the same distribution as $T_M$, we obtain

$$F_\epsilon(t) \geq F(t), \quad \forall t \geq 0, \quad (1.7)$$

where $F_\epsilon$ and $F$ are the associated cumulative distribution functions. Let us now prove the second bound in (1.2). For $t \geq \epsilon$,

$$F_\epsilon(t - \epsilon) = \mathbb{P}(\tau_\varphi^\epsilon \leq t - \epsilon) = \mathbb{P}(T_M \leq t - \epsilon) \leq \mathbb{P}(T_M \leq t - \epsilon, \tau_\varphi > t) + \mathbb{P}(\tau_\varphi \leq t) \leq \mathbb{P}(|T_M - \tau_\varphi| > \epsilon) + F(t). \quad (1.8)$$

Combining the Markov property of the Brownian motion and the reflection
Using Hypothesis (H3) and straightforward computations permits to obtain

\[ P_2 := \Pr(|T_{M_\epsilon} - \tau_\varphi| > \epsilon) \leq 1 - \Pr\left( \sup_{0 \leq u \leq \epsilon} B_{T_{M_\epsilon} + u} \leq \sup_{0 \leq u \leq \epsilon} \varphi(T_{M_\epsilon} + u) \right) \]

\[ \leq 1 - \Pr\left( \sup_{0 \leq u \leq \epsilon} B_{T_{M_\epsilon} + u} - B_{T_{M_\epsilon}} \geq \sup_{0 \leq u \leq \epsilon} \varphi(T_{M_\epsilon} + u) - \varphi(T_{M_\epsilon}) + \epsilon \right) \]

\[ \leq 1 - \Pr\left( \sup_{0 \leq u \leq \epsilon} B_{T_{M_\epsilon} + u} - B_{T_{M_\epsilon}} \geq \varphi(T_{M_\epsilon} + \epsilon) - \varphi(T_{M_\epsilon}) + \epsilon \right) \]

Using Hypothesis (H3) implies

\[ \Pr(|T_{M_\epsilon} - \tau_\varphi| > \epsilon) \leq 1 - \Pr(\hat{B}_1 \geq 3\epsilon/2) \leq 3 \frac{\epsilon}{2\pi}. \quad (1.9) \]

The lower bound in (1.2) holds due to both (1.8) and (1.9).

**Step 2.** Let us now focus our attention to the efficiency of this algorithm. We need to estimate the number of steps which depends on the small parameter \( \epsilon \).

Let us first estimate the previous upper-bound. We introduce a sequence of independent standard Gaussian random variables \((G_n)_{n \geq 0}\) and define

\[ X_n = \log(4G_n^2), \quad \Xi_n = \sum_{k=0}^{n} X_k \quad \text{and} \quad Z_n = \sum_{k=0}^{n} \Xi_k. \quad (1.11) \]

Let us define \( \Pi(n, \epsilon) := \Pr(s_n > 2\epsilon) \). By Proposition 1.2 we know that the random variables \( s_{n+1} \) are related to \( G_n \) and therefore

\[ \Pi(1, \epsilon) = \Pr(2G_1^2 < \varphi(0)^2) = \Pr\left( \log(4G_1^2) < -\log(\epsilon) + \log(2) + 2\log \varphi(0) \right) \]

\[ = \Pr\left( Z_0 < -\log(\epsilon) + \log(2) + 2\log \varphi(0) \right). \]

Let us prove that, for \( n \geq 1 \), we have the general formula:

\[ \Pi(n, \epsilon) \leq \Pr(Z_{n-1} < -\log(\epsilon) + (2n - 1)\log(2) + (2n)\log \varphi(0)). \quad (1.12) \]

By Proposition 1.2 we have for \( n \geq 2 \),

\[ \Pi(n, \epsilon) = \Pr\left( (\varphi(T_{n-1}) - \varphi(T_{n-2}))^2 > 2\epsilon G_{n-1}^2 \right). \quad (1.13) \]
Hence for upper-bound holds for $n \geq 2$:

$$\varphi(T_{n-1}) - \varphi(T_{n-2}) \leq \frac{T_{n-1} - T_{n-2}}{2 \sqrt{1 + T_{n-2}}} \leq \frac{s_{n-1}}{2 \sqrt{1 + s_{n-2}}}.$$ \hspace{1cm} (1.14)

Hence for $n = 2$, (1.13) and (1.14) imply

$$\Pi(2, \epsilon) \leq \mathbb{P}\left( \frac{s_2^2}{2} > 2\epsilon G_1^2 \right) = \mathbb{P}(\epsilon(2G_1^2)(2G_0^2) < \varphi(0)^4)
= \mathbb{P}(2X_0 + X_1 < -\log(\epsilon) + 3\log(2) + 4\log \varphi(0))
= \mathbb{P}(Z_1 < -\log(\epsilon) + 3\log(2) + 4\log \varphi(0)).$$

Using the lower-bound $1+s_{n-1} \geq s_{n-1}$ and similar arguments as those developed previously, the general case is expressed as follows:

$$\Pi(n, \epsilon) \leq \mathbb{P}\left( \frac{s_{n-1}^2}{2} > 2\epsilon G_{n-1}^2 \right) \leq \mathbb{P}\left( \frac{s_{n-2}^2}{22s_{n-3}} > 2\epsilon G_{n-1}^2 G_{n-2}^4 \right)
\leq \mathbb{P}\left( \frac{s_{n-1}^2}{s_{n-2}^2} > 2\epsilon G_{n-1}^2 G_{n-2}^4 \cdots G_{2}^{2(n-2)} \right)
\leq \mathbb{P}\left( \epsilon G_1^{2n} > 2\epsilon G_{n-1}^2 G_{n-2}^4 \cdots G_{2}^{2n} \right)
\leq \mathbb{P}\left( Z_{n-1} < -\log(\epsilon) + (2n - 1)\log(2) + (2n)\log \varphi(0) \right).$$

**Step 2.2.** By (1.10) and the arguments developed in Step 2.1, we obtain

$$\mathbb{P}(X_\epsilon > n) \leq \mathbb{P}(s_n > 2\epsilon) \leq \mathbb{P}(Z_{n-1} - \mathbb{E}Z_{n-1} < \eta(\epsilon, n) - \mathbb{E}Z_{n-1}),$$

where

$$\eta(\epsilon, n) := -\log(\epsilon) + (2n - 1)\log(2) + (2n)\log \varphi(0).$$

Let us observe that, for any $n \geq 0$, $m := \mathbb{E}[X_n] = \log(4) + \frac{2\sqrt{\pi}}{\sqrt{\mu}} \mu > 0$ where $\mu$ is defined by (1.13). Hence

$$\mathbb{E}[Z_n] = \sum_{k=0}^{n} \mathbb{E}[X_n] = \sum_{k=0}^{n} \sum_{j=0}^{k} \mathbb{E}[X_j] = m \sum_{k=0}^{n} (k + 1) = \frac{m(n + 1)(n + 2)}{2}.$$

Thus, for $n$ large enough, $\eta(\epsilon, n) - \mathbb{E}Z_{n-1} < 0$. Introducing $d_n := |mn(n + 1)/2 - \eta(\epsilon, n)|$, we observe that, for any $0 < \kappa < 1/2$ there exists $\mathbb{N}(\kappa, \epsilon) \in \mathbb{N}$ such that $d_n > mn^2(1/2 - \kappa)$ for $n$ sufficiently large that is $n \geq \mathbb{N}(\kappa, \epsilon)$. After straightforward computations, we can choose

$$\mathbb{N}(\kappa, \epsilon) := \left\lfloor \frac{\log(2\epsilon)}{m\kappa} + \frac{1}{2\kappa} - \frac{\log(2\varphi(0))}{m\kappa} \right\rfloor + 1. \hspace{1cm} (1.15)$$

Markov’s inequality leads to

$$\mathbb{P}(X_\epsilon > n) \leq \mathbb{P}(|Z_{n-1} - \mathbb{E}Z_{n-1}| > d_n) \leq \frac{\mathbb{E}[(Z_{n-1} - \mathbb{E}Z_{n-1})^4]}{d_n^4}. \hspace{1cm} (1.16)$$
Let us note that $\overline{X}_j := X_j - m$ are i.i.d. random variables with finite moments of any order. We denote $m_k := E[\overline{X}_j^k]$. Therefore we obtain

\[ Z_{n-1} := E[(Z_{n-1} - E[Z_{n-1}])^4] = E \left[ \left( \sum_{k=0}^{n-1} \sum_{j=0}^{k} \overline{X}_j \right)^4 \right] = E \left[ \left( \sum_{j=0}^{n-1} (n-j)\overline{X}_j \right)^4 \right] \]

\[ = \sum_{j=0}^{n-1} (n-j)^4 m_4 + 2 \sum_{0 \leq j < k \leq n-1} (n-j)^2 (n-k)^2 m_2^2 \]

\[ \leq \frac{m_4}{30} n(n+1)(6n^3 + 9n^2 + n - 1) + \frac{m_2^2}{36} n^2(n + 1)^2(2n + 1)^2. \quad (1.17) \]

Hence, there exist a constant $C_0 > 0$ such that $E[(Z_{n-1} - E[Z_{n-1}])^4] \leq C_0 n^6$. Combining the previous inequality with (1.15) and (1.16) leads to

\[ \mathbb{P}(\mathcal{N}_e > n) \leq \frac{C_0}{m^4(1/2 - \kappa)^4} \frac{1}{n^2}, \quad \text{for} \quad n \geq R(\kappa, \epsilon). \]

Consequently, the following upper-bound holds

\[ E[\mathcal{N}_e] = \sum_{n \geq 0} \mathbb{P}(\mathcal{N}_e > n) \leq R(\kappa, \epsilon) + \frac{C_0}{m^4(1/2 - \kappa)^4} \sum_{n \geq R(\kappa, \epsilon)} \frac{1}{n^2}. \]

In order to conclude, it suffices to note that $R(\kappa, \epsilon) \to \infty$ as $\epsilon \to 0$, the second term in the previous inequality therefore becomes small as $\epsilon \to 0$: the leading term is finally $R(\kappa, \epsilon)$ which is equivalent to $\sqrt{\log(2\epsilon)}/(m \kappa)$ by (1.15).

\[ \square \]

2 First-passage time to boundaries with bounded derivative

The algorithm presented in Section 1 is simple to achieve (it only requires independent Gaussian random variables) and efficient: the averaged number of steps is of the order $\sqrt{\log \epsilon}$ where $\epsilon$ stands for the small parameter appearing in the rejection sampling (see Theorem 1.1). In order to apply Algorithm (A1) the curved boundary, the Brownian motion is going to hit, has to satisfies suitable conditions: (H1), (H2) and (H3). Asking for the monotonicity of the function $\phi$ is quite restrictive, that’s why we present an extension of the algorithm which is of course less efficient (even if the average number of steps is still very small) but which permits to deal with more general boundaries. Let us introduce the following assumption: there exist two constants $\rho_+ > 0$ and $\rho_- > 0$ such that

\[ \phi : \mathbb{R}_+ \to \mathbb{R} \text{ is a } C^1 \text{-continuous function satisfying} \]

\[ \sup_{t \geq 0} \phi'(t) \leq \rho_+ \quad \text{and} \quad \inf_{t \geq 0} \phi'(t) \geq -\rho_. \quad (H4) \]

For such boundaries, we present an algorithm which permits for any $K \in \mathbb{R}^+$ to approximate the hitting time $\tau_{\phi}^K = \tau_{\phi} \wedge K$, where $\tau_{\phi}$ is defined in (0.1). Let us introduce some notations: the inverse Gaussian distribution of parameters $\mu > 0$ and $\lambda > 0$ will be denoted by $I(\mu, \lambda)$ and is defined by its the probability distribution function:

\[ f(x) = \sqrt{\frac{\lambda}{2\pi x^3}} \exp \left\{ -\frac{(\lambda(x - \mu)^2}{2\mu^2 x} \right\} \mathbb{1}_{\{x \geq 0\}}. \]

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Algorithm (A2). Let $\epsilon > 0$ be a small parameter and $r > \rho$ where $\rho$ is defined in (H4).
Initialization: $(T, H) = (0, \varphi(0))$ and $N_{\epsilon,K} = 0$.

While $H > \epsilon$ and $T < K$, simulate $\hat{G}$ an inverse Gaussian random variable with distribution $I(H/r, H^2)$ and do:
\[
\begin{cases}
H \leftarrow \varphi(T + \hat{G}) - \varphi(T) + r \hat{G}, \\
T \leftarrow \hat{G} + T, \\
N_{\epsilon,K} \leftarrow N_{\epsilon,K} + 1.
\end{cases}
\] 
(2.1)

Outcome: $\tau_{\varphi,K} \leftarrow T \wedge K$ and $N_{\epsilon,K}$.

Algorithm (A2) is quite simple, it only requires the simulation of inverse Gaussian distributed random variables. Let us recall the following scaling property: if $\hat{G} \sim I(H/r, H^2)$ then $\frac{r\hat{G} - H}{H} \sim I(1,rH)$. Moreover $(r\hat{G} - H)^2$ is Chi-squared distributed with one degree of freedom (the square of a standard Gaussian random variable). In order to simulate an inverse Gaussian random variable, we suggest to use the algorithm introduced by Michael, Schucany and Haas (see [13] or [5, p. 149]). Let us now state the efficiency of Algorithm (A2).

Theorem 2.1. 1. Let us assume that the boundary function $\varphi$ satisfies (H4) then the random variable $\tau_{\tau_{\varphi,K}}$ defined in Algorithm (A2) converges in distribution towards $\tau_{\varphi,K} = \tau_{\varphi} \wedge K$ where $\tau_{\varphi}$ is defined by (0.1) as $\epsilon$ tends to zero. More precisely
\[
F_{\epsilon,K}(t - \epsilon) - (1 + \rho)\sqrt{\frac{2\epsilon}{\pi}} \leq F_{K}(t) \leq F_{\epsilon,K}(t), \quad \text{for any } t \geq \epsilon, \tag{2.2}
\]
where $F_K$ (resp. $F_{\epsilon,K}$) is the cumulative distribution function of $\tau_{\varphi,K}$ (resp. $\tau_{\varphi,K}$).

2. There exist positive constants $a$, $b$, $\kappa_0$, $\kappa_1$ and $\epsilon_0$ such that: for any $\rho_+ \leq \kappa_0$ and any $(K, r)$ satisfying $(r + \kappa_0)K \leq \kappa_1$, the random number of iterations $N_{\epsilon,K}$ defined in Algorithm (A2) satisfies the following upper bound
\[
E[N_{\epsilon,K}] \leq (a + br)\log \epsilon, \quad \forall \epsilon \leq \epsilon_0. \tag{2.3}
\]

3. For non increasing functions $\varphi$: there exists two positive constants $a$ and $\epsilon_0$ such that
\[
E[N_{\epsilon,K}] \leq ar^2K\log \epsilon, \quad \forall \epsilon \leq \epsilon_0. \tag{2.4}
\]

This theorem is based on the following intermediate statement which is a modification of Proposition 1.2.
**Proposition 2.2.** Let \((B_t, t \geq 0)\) be a standard one-dimensional Brownian motion. We introduce the following stopping times: \(s_0 = T_0^K = 0\) and for any \(n \geq 1:\)

\[
s_n := \inf \left\{ t \geq 0 : B_{t+T_n^K} = \varphi(T_n^K) - rt \right\} \quad \text{and} \quad T_n^K := (s_1 + \ldots + s_n) \land K,\]

where the boundary \(\varphi\) satisfies \((H4)\). Then the following properties hold:

1. \((T_n^K)_{n \geq 0}\) is a non-decreasing sequence which almost surely converges towards \(\tau^K\).

2. On the event \(\{s_1 + \ldots + s_n < K\}\), the probability distribution of \(s_{n+1}\) given the \(\sigma\)-algebra \(\mathcal{F}_n := \sigma\{T_1^K, \ldots, T_n^K\}\) is the inverse Gaussian distribution \(I(\mathcal{H}_n/r, \mathcal{H}_n^2)\) with

\[
\mathcal{H}_n := \varphi(T_n^K) - \varphi(T_{n-1}^K) + rs_n.
\]

3. Let \(\mathcal{M}_e := \{n \geq 1 : \varphi(T_n^K) - B_t \leq \epsilon\}\), \(\mathcal{M}_K := \{n \geq 1, T_n^K = K\}\), \(\mathcal{M}_e^K = \mathcal{M}_e \land \mathcal{M}_K\). Then \(\mathcal{M}_e^K\) and \(\mathcal{M}_{\varphi,K}^{e,K}\), defined in Algorithm (A2), are identically distributed, so are \(\mathcal{M}_e^K\) and \(\mathcal{N}_{\varphi,K}\).

**Proof of Proposition 2.2.** The first and the third part of the proof are left to the reader. They need similar arguments as those presented in Proposition 1.2. Here the monotonicity property is just replaced by \((H4)\) which permits easily to prove that \(T_n^K \leq \tau^K\).

Let us now focus our attention to the second part of the statement. Due to the definition of \(s_{n+1}\) and since \(\{T_n^K < K\}\), we get \(B_t \leq \varphi(T_{n+1}^K) - rs_n\). Hence, we have

\[
s_{n+1} = \inf \{ t \geq 0 : B_{t+T_n^K} - B_{T_n^K} = \varphi(T_n^K) - B_{T_n^K} - rt \} = \inf \{ t \geq 0 : W_t = \mathcal{H}_n - rt \},
\]

where \(W_t = B_{t+T_n^K} - B_{T_n^K}\) is a standard Brownian motion independent of \(\mathcal{F}_n\) and the \(\mathcal{F}_n\) adapted r.v. \(\mathcal{H}_n\) is defined by \((2.6)\). The distribution of \(s_{n+1}\) corresponds to the distribution of the first passage time of the standard Brownian motion with drift at the constant level \(\mathcal{H}_n\). The probability distribution is well known (see, for instance [11, p. 197]):

\[
P(s_{n+1} \in dt|\mathcal{F}_n) = \frac{\mathcal{H}_n}{\sqrt{2\pi t^3}} \exp \left\{ \frac{(\mathcal{H}_n - rt)^2}{2t} \right\} dt,
\]

we can consequently identify the inverse Gaussian distribution \(I(\mathcal{H}_n/r, \mathcal{H}_n^2)\). \(\square\)

**Proof of Theorem 2.1.** We can prove the convergence in distribution of \(\tau^K_{\varphi}\) towards \(\tau^K\) using similar arguments as those presented in the proof of Theorem 1.1. The upper-bound in \((2.2)\) is an adaptation of \((1.7)\) which requires that \(T_n^K \leq \tau^K\) and that \(\mathcal{M}_{\varphi,K}\) and \(\mathcal{M}_{\varphi,K}^{e,K}\) are identically distributed. These conditions are satisfied, see Proposition 2.2. For the lower-bound in \((2.2)\), we obtain

\[
F_{\epsilon,K}(t - \epsilon) \leq P(|\mathcal{M}_{\varphi,K} - \tau^K| > \epsilon) + F_{\epsilon,K}(t),
\]

and
leads to Combining Hypothesis (H4) and the reflection principle of the Brownian motion \( \tilde{\varphi} \).

Let us consider the function \( f \) it is then obvious that the infinitesimal generator associated to the Markov chain \( P \) out of the domain 2.2 can also be interpreted as the first time the Markov chain defined by (2.5) and \( \tau \) see (1.8) for the details. Let us note that by (H4), the density function:

\[
\varphi(t + \hat{\varphi}) - \varphi(t) + r \hat{\varphi}
\]

where \( \hat{\varphi} \) is an inverse Gaussian distributed random variable with the following density function:

\[
p(x) = \frac{h}{\sqrt{2\pi x^3}} \exp \left\{ -\frac{(h - rx)^2}{2x} \right\}, \quad x \geq 0.
\]

By (H4), \( \varphi(t + \hat{\varphi}) - \varphi(t) \leq \rho_+ \hat{\varphi} \), we get

\[
Pf(t,h) - f(t,h) \leq \log \left( 1 + \frac{\rho_+}{r} \right) + \mathbb{E} \left[ \log \left( \frac{r \hat{\varphi}}{h} \right) \right]. \quad (2.7)
\]

Let us find now an explicit upper bound of \( Pf - f \). Using first the change of variables \( u = rx/h \) and secondly \( u \mapsto 1/u \), we get

\[
\mathbb{E} \left[ \log \left( \frac{r \hat{\varphi}}{h} \right) \right] = \int_0^\infty \log \left( \frac{rx}{h} \right) \frac{h}{\sqrt{2\pi x^3}} \exp \left( -\frac{(h - rx)^2}{2x} \right) dx
\]

\[
= \frac{hr}{2\pi} \int_0^\infty \frac{\log(u)}{u^{3/2}} \exp \left( -\frac{hr(1 - u)^2}{2u} \right) du
\]

\[
= \frac{hr}{2\pi} \int_1^\infty \frac{(1 - u) \log(u)}{u^{3/2}} \exp \left( -\frac{hr(1 - u)^2}{2u} \right) du. \quad (2.8)
\]

It is then obvious that \( \mathbb{E} \left[ \log \left( \frac{r \hat{\varphi}}{h} \right) \right] < 0 \). Let us now give a more precise upper-bound. We set \( \alpha = hr \), then (2.8) emphasizes that \( \mathbb{E} \left[ \log \left( \frac{r \hat{\varphi}}{h} \right) \right] \) only depends
Figure 1: Monte Carlo approximation of the function $\psi$

on the parameter $\alpha$, this dependence being continuous. Let us therefore denote this function $\psi(\alpha)$ (see Figure 1 below representing $\psi$ obtained with the Monte-Carlo method sample size: 10000).

Simple computations lead to

$$\psi(\alpha) := E\left[\log \left( \frac{r^G}{\pi} \right) \right] = -\sqrt{\frac{\alpha}{2\pi}} \int_0^\infty u \log(1 + u) \exp -\frac{\alpha u^2}{2(1 + u)} \, du \quad (2.9)$$

$$\leq -\sqrt{\frac{\alpha}{2\pi}} \int_0^\infty u \log(1 + u) \exp -\frac{\alpha u}{2} \, du$$

$$\leq -\frac{1}{\sqrt{2\pi}} \int_0^\infty w \log(1 + w/\alpha) \exp -\frac{w}{2} \, dw$$

$$\leq -\frac{1}{\sqrt{2\pi}} \int_{1/2}^\infty w \log(1 + w/\alpha) \exp -\frac{w}{2} \, dw.$$ 

Using the inequality $(\alpha + w) \leq (1 + 2\alpha)w$, we get

$$\psi(\alpha) \leq -\frac{\log(1 + (2\alpha)^{-1})}{(1 + 2\alpha)^{3/2}} \frac{1}{2\pi} \int_{1/2}^\infty \frac{1}{\sqrt{w}} \exp -\frac{w}{2} \, dw$$

$$\leq -\frac{\log(1 + (2\alpha)^{-1})}{(1 + 2\alpha)^{3/2}} P(G \geq 1/2),$$

where $G$ is a standard gaussian r.v. and so $P(G \geq 1/2) \approx 0.3085$.

We deduce from the previous upper-bound that $\lim_{\alpha \to 0^+} \psi(\alpha) = -\infty$. Moreover the right hand side is a non decreasing function with respect to the variable $\alpha$. Hence

$$\psi(\alpha) \leq -\frac{\log(3/2)}{3\sqrt{3}} P(G \geq 1/2) \approx -0.0241, \quad \text{for } \alpha \leq 1. \quad (2.10)$$

Let us observe what happens for large values of the variable $\alpha$. The Laplace method implies that

$$\psi(\alpha) \sim -\frac{1}{2\alpha} \quad \text{as } \alpha \to \infty.$$
Let us prove now that there exists a constant \( c > 0 \) such that

\[
\psi(\alpha) \leq -\frac{c}{\alpha}, \quad \text{for any } \alpha \geq 1. \tag{2.11}
\]

For \( \alpha \geq 1 \), we get

\[
\psi(\alpha) \leq -\sqrt{\frac{\alpha}{2\pi}} \int_0^\infty \frac{u \log(1 + u)}{(1 + u)^{3/2}} \exp -\frac{\alpha u^2}{2} du \\
\leq -\sqrt{\frac{\alpha}{2\pi}} \int_0^1 \frac{u \log(1 + u)}{(1 + u)^{3/2}} \exp -\frac{\alpha u^2}{2} du.
\]

Due to the convexity property of the logarithm function \((\log(1 + u) \geq \log(2)u)\) and the Cauchy-Schwarz inequality, we obtain

\[
\psi(\alpha) \leq -\frac{\log(2)}{\alpha^{3/2}} \left( \frac{1}{2} \mathbb{E}[G^2] - \mathbb{E}[G^2 \mathbb{I}_{\{G \geq \sqrt{\alpha}\}}] \right) \\
\leq -\frac{\log(2)}{\alpha^{3/2}} \left( \frac{1}{2} - \sqrt{\mathbb{E}[G^4]} \sqrt{\mathbb{P}(G \geq \sqrt{\alpha})} \right) \\
\leq -\frac{\log(2)}{\alpha^{3/2}} \left( \frac{1}{2} - \frac{\sqrt{3}}{2} e^{-\alpha} \right) \leq -\frac{\log(2)}{\alpha^{3/2}} (1 - \sqrt{3}e^{-1}), \quad \text{for } \alpha \geq 1.
\]

We deduce that \( \psi(\alpha) \leq -c/\alpha \) with \( c \approx 0.0445 \) when \( \alpha \geq 1 \). Combining both inequalities (2.10) and (2.11) leads to the existence of a constant \( c > 0 \) such that

\[
\psi(\alpha) \leq -c \left( \frac{1}{\alpha} \wedge 1 \right). \tag{2.12}
\]

By (2.7), the following upper-bound holds: for \( f(x, y) = \log(y) \),

\[
P f(t, h) - f(t, h) \leq \log \left( 1 + \frac{\rho_+}{r} \right) - c \left( \frac{1}{Kr} \wedge 1 \right) \\
\leq \frac{\rho_+}{r} - c \left( \frac{1}{Kr} \wedge 1 \right), \quad h \geq 0, \ t \geq 0. \tag{2.13}
\]

Due to the definition of \( \rho_+ \), we know that

\[
h \leq \varphi(0) \lor (r + \rho_+)t \leq \varphi(0) \lor (r + \rho_+)K,
\]

where \( \varphi \) is the boundary the process has to hit. In other words, there exist two constants \( \kappa_0 > 0 \) and \( \kappa_1 > 0 \) such that for any \( \rho_+ \leq \kappa_0 \) and any \( (K, r) \) satisfying \((r + \kappa_0)K \leq \kappa_1 \) the following bound holds \( \rho_+ \leq \frac{\varepsilon}{2} \left( \frac{1}{K} \wedge r \right) \). Hence:

\[
P f(t, h) - f(t, h) \leq -c \left( \frac{1}{\varphi(0) \wedge \kappa_1} \wedge r \right) =: -R^{-1}(r).
\]

We deduce that the function \( g(t, h) \) defined by \( g(t, h) = R(r) (f(t, h) - \log \epsilon) \) satisfies \( g(t, h) \geq 0 \) for any \((t, h) \in E \) and \( Pg(t, h) - g(t, h) \leq -1 \) on \( E \). The potential theory therefore implies:

\[
\mathbb{E}[N_{\epsilon,K}] \leq g(0, \varphi(0)) \leq R(r)(\log(\varphi(0)) - \log(\epsilon)).
\]

We finally deduce the existence of \( a > 0 \) and \( b > 0 \) such that \( \mathbb{E}[N_{\epsilon,K}] \leq (a + br)|\log \epsilon| \) for \( \epsilon \) small enough.

For the particular case of a non increasing boundary function it suffices to vanish \( \rho_+ \) in (2.13) and to apply the same arguments of the potential theory in order to get (2.4). \( \square \)
3 Examples and numerics.

In this section, we present three different examples which nicely illustrate the efficiency of these new algorithms \{A1\} and \{A2\}.

3.1 Brownian hitting time of $\varphi(t) = \sqrt{1 + \alpha t}$

Let us first consider an application of Theorem 1.1. We observe that $\varphi(t) = \sqrt{1 + \alpha t}$ is an increasing function satisfying (H1), (H2) and (H3) for $\alpha \in [0, 1]$. Consequently Algorithm \{A1\} converges and permits to obtain an approximation of the hitting time $\tau_{\varphi}$. In the figures, we present the link between the averaged number of steps and $\epsilon$ which characterizes the approximation error size.

The first figure (resp. the second one) concerns: $\alpha = 1$ (resp. $\alpha = 0.01$), $\epsilon = 0.5^n$ ($n$ is represented on the horizontal axis) and the number of simulation in order to estimate the averaged number of steps is 10000.

![Figure 2: $E(N_{\epsilon})$: mean number of steps for $\epsilon = 0.5^n$ as a function of $n$. The boundary is $\varphi(t) = \sqrt{1 + \alpha t}$.](image)

Let us now present the approximate distribution of the hitting time.

![Figure 3: Empirical distribution of the approximate first hitting time of the boundary $\varphi(t) = \sqrt{1 + \alpha t}$.](image)
3.2 Brownian hitting time of $\varphi(t) = \alpha + \beta \cos(\omega t)$

Let us now consider the first time the Brownian motion hits the periodic boundary $\varphi(t) = \alpha + \beta \cos(\omega t)$. Since the boundary is not an increasing function, we shall use Algorithm [A2]. Theorem 2.1 ensures that the algorithm converges. Let us therefore use the Monte-Carlo method in order to estimate precisely the average number of steps. As explained in the previous section, the simulation procedure permits to approximate the stopping time $\tau_\varphi \wedge K$ for some given fixed time $K$. Figure 4 illustrates the approximation $\tau_\varphi$ by $\tau_\varphi^K$, where the parameters are fixed at $\alpha = 3.5$, $\beta = 3$ and $\omega = \pi/2$. The maximal time are $K = 20$ on one hand and $K = 100$ on the other hand and the error rate is given by $\epsilon = 0.5^n$, for $1 \leq n \leq 10$. A sample of $10^6$ paths has been simulated to approximate the mean. We know that the mean number of steps is a decreasing function of $\epsilon$ and

(a) $\mathbb{E}(N_{1/2^n,K})$ versus $n$

(b) Distribution of $\tau_\varphi^{1/2_n,K}$. $n = 10$, $K = 20$.

Figure 4: Approximation of $\tau_\varphi$ with $\varphi(t) = 3.5 + 3 \cos(\pi t/2)$

Figure 5: $\mathbb{E}(N_{\varphi,K})$ as a function of $K$.

an increasing function of $K$. Figure 5 gives the evolution of the mean number of steps as a function of the truncation $K$. In practice, we obtained easily an impressively accurate approximation of $\tau_\varphi$. 

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3.3 The first time the Ornstein Uhlenbeck process hits the boundary \( \varphi(t) = \alpha + \beta \cos(\omega t) \)

The last example concerns the one-dimensional Ornstein-Uhlenbeck process defined as the unique solution of the following stochastic differential equation:

\[
\text{d}X_t = -\lambda X_t \text{d}t, \quad X_0 = x_0, \quad (3.1)
\]

where \((B_t, t \geq 0)\) is the standard Brownian motion. The aim is to approximate the first passage time through the curved boundary \( \varphi(t) = \alpha + \beta \cos(\omega t) \) where \( \varphi(0) > x_0 \). Since the Ornstein-Uhlenbeck process can be represented as a time-changed Brownian motion, the question is directly related to the main results of this study. Indeed the solution of (3.1) is given by

\[
X_t = e^{-\lambda t} \left( x_0 + \int_0^t e^{\lambda s} \text{d}B_s \right), \quad t \geq 0.
\]

Using Levy’s theorem, \((X_t, t \geq 0)\) has the same distribution as \((Y_t, t \geq 0)\) defined by

\[
Y_t := e^{-\lambda t} \left( x_0 + W_{u(t)} \right), \quad t \geq 0,
\]

with \( u(t) := \frac{1}{\lambda^2} (e^{2\lambda t} - 1) \) and \( W \) a standard Brownian motion. We deduce that

\[
T_{\varphi} := \inf \{ t \geq 0 : X_t = \varphi(t) \}
\]

has the same distribution as

\[
\hat{T}_{\varphi} := \inf \{ t \geq 0 : e^{-\lambda t} \left( x_0 + W_{u(t)} \right) = \varphi(t) \}
\]

\[
= \inf \left\{ u^{-1}(s) \geq 0 : W_s = \varphi(u^{-1}(s)) e^{\lambda u^{-1}(s) - x_0} \right\}
\]

\[
= u^{-1} (\tau_{\psi}),
\]

where

\[
\tau_{\psi} := \inf \{ t \geq 0 : W_t = \psi(t) \}, \quad \psi(t) := \sqrt{1 + 2\lambda t} \varphi \left( \frac{\log(1 + 2\lambda t)}{2\lambda} \right) - x_0.
\]

Consequently, in order to simulate the Ornstein-Uhlenbeck hitting time \( T_{\varphi} \land K \) for some \( K \), we simply use Algorithm \([A2]\) and propose an approximation of the Brownian hitting time \( \tau_{\psi} \land \hat{K} \) with \( \hat{K} := u(K) = (e^{2\lambda K} - 1)/(2\lambda) \).

Let us note that a straightforward computation leads to the following upper-bound:

\[
|\psi'(t)| \leq \frac{\lambda \alpha + \lambda \beta + \omega \beta}{\sqrt{1 + 2\lambda t}} \leq \lambda \alpha + \lambda \beta + \omega \beta, \quad t \geq 0.
\]

In other words, the continuous curve \( \psi \) satisfies Hypothesis \([H4]\): Algorithm \([A2]\) therefore converges and Theorem \([2.1]\) can be applied.

In the following numerical experiences, we will choose \( r = 0.5 + \lambda \alpha + \lambda \beta + \omega \beta \). Figures \([6]\) and \([7]\) concern the following choice of parameters: \( x_0 = 0, \alpha = 2, \beta = 1, \omega = \pi/5, \lambda = 0.5 \). We have chosen \( K = 5 \) for Figure \([6]\) and \( K = 10 \) for Figure \([7]\). In both cases, the first figure represents the average number of steps as a function of \( n \) where the approximation parameter \( \epsilon \) is chosen as \( 0.5^n \), for \( n = 1, \cdots, 10 \). The average has been estimated using 5,106 simulations. The second figure represents the distribution of \( T_{\varphi} \land \hat{K} \) for \( n = 10 \).
We observe that the change of time \( \tilde{K} = (e^{2\lambda K} - 1)/(2\lambda) \) increases very fast with \( K \) and the number becomes quite large when \( K \) increases. Note however that the number of random variables we have to simulate keeps relatively small in comparison with the use of a classical stopped Euler scheme usually used to approximate \( T_\varphi \).

![Graphs showing the distribution of first hitting times.](image)

Figure 6: First hitting time of \( \varphi(t) = \alpha + \beta \cos(\omega t) \) by an Ornstein Uhlenbeck process solution of (3.1) \( (\alpha = 2, \beta = 1, \omega = \pi/5, \lambda = 0.5, K = 5) \).

![Graphs showing the distribution of first hitting times.](image)

Figure 7: First hitting time of \( \varphi(t) = \alpha + \beta \cos(\omega t) \) by an Ornstein Uhlenbeck process solution of (3.1) \( (\alpha = 2, \beta = 1, \omega = \pi/5, \lambda = 0.5, K = 10) \).

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