A GENERALIZATION OF REFLEXIVE RINGS

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Abstract. In this paper, we introduce a class of rings which is a generalization of reflexive rings and $J$-reversible rings. Let $R$ be a ring with identity and $J(R)$ denote the Jacobson radical of $R$. A ring $R$ is called $J$-reflexive if for any $a, b \in R$, $aRb = 0$ implies $bRa \subseteq J(R)$. We give some characterizations of a $J$-reflexive ring. We prove that some results of reflexive rings can be extended to $J$-reflexive rings for this general setting. We conclude some relations between $J$-reflexive rings and some related rings. We investigate some extensions of a ring which satisfies the $J$-reflexive property and we show that the $J$-reflexive property is Morita invariant.

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1. Introduction

Throughout this paper all rings are associative with identity unless otherwise stated. We write $M_n(R)$ for the ring of all $n \times n$ matrices and $T_n(R)$ for the ring of all $n \times n$ upper triangular matrices over a ring $R$. Also we write $R[x], R[[x]], N(R), U(R)$ and $J(R)$ for the polynomial ring, the power series ring over a ring $R$, the set of all nilpotent elements, the set of all invertible elements and the Jacobson radical of a ring $R$, respectively. Also $\mathbb{Z}$ denote the ring of integers.

In [6], Mason introduced the reflexive property for ideals. Let $R$ be a ring (without identity) and $I$ an ideal of $R$. Then $I$ is called reflexive, if $aRb \subseteq I$ for $a, b \in R$ implies $bRa \subseteq I$. It is clear that every semiprime ideal is reflexive. Also, the ring $R$ is called reflexive, if 0 is a reflexive ideal (i.e $aRb = 0$ implies $bRa = 0$ for $a, b \in R$). In [4], Kwak and Lee studied reflexive rings. They investigate reflexive property of rings related to matrix rings and polynomial rings. According to Cohn [2], a ring $R$ is said to be
reversible if for any \(a, b \in R\), \(ab = 0\) implies \(ba = 0\). It is clear that every reversible ring is reflexive. Recently, as a generalization of a reversible ring, so-called \(J\)-reversible ring has been studied in [1]. A ring \(R\) is called \(J\)-reversible, \(ab = 0\) implies that \(ba \in J(R)\) for \(a, b \in R\). As an application it is shown that every \(J\)-clean ring is directly finite. Motivated by these studies, we introduce a class of rings which generalize \(J\)-reversible rings and reflexive rings. A ring \(R\) is called \(J\)-reflexive, \(bRa \subseteq J(R)\) whenever \(aRb = 0\) for \(a, b \in R\).

We summarize the contents of this paper. In Section 2, we study main properties of \(J\)-reflexive rings. We give some characterizations of \(J\)-reflexive rings. We prove that every \(J\)-reversible ring is \(J\)-reflexive and we supply an example (Example 2.4) to show that the converse is not true in general. Moreover, we see that if \(R\) is a Baer ring, then \(J\)-reversible rings are \(J\)-reflexive. It is clear that reflexive rings are \(J\)-reflexive. Example 2.6 shows that \(J\)-reflexive rings need not be reflexive. We give a necessary and sufficient condition for a quotient ring to be \(J\)-reflexive. Also we conclude some results which investigate relations between \(J\)-reflexive rings and some class of rings. With our finding, we prove that every uniquely clean ring is \(J\)-reflexive and quasi-duo rings are \(J\)-reflexive. Moreover, we shows that the converse is not true in general. Being Morita invariant property is very important for class of rings. A ring-theoretic property \(\mathcal{P}\) is Morita invariant if and only if whenever a ring \(R\) satisfies \(\mathcal{P}\) so does \(eRe\), for any full idempotent \(e\) and \(M_n(R)\) for any \(n > 1\). There are a lot of studies on Morita invariant property of rings. In Section 3, we prove that the \(J\)-reflexive property is Morita invariant. Furthermore, we study the \(J\)-reflexive property in several kinds of ring extensions (Dorroh extension, upper triangular matrix ring, Laurent polynomial ring, trivial extension etc.).

2. \(J\)-Reflexive Rings

In this section we define the \(J\)-reflexive property of a ring. We investigate some properties of \(J\)-reflexive rings and exert relations between \(J\)-reflexive rings and some related rings.

**Definition 2.1.** A ring \(R\) is called \(J\)-reflexive, \(aRb = 0\) implies that \(bRa \subseteq J(R)\), for \(a, b \in R\).
For a nonempty subset \( X \) of a ring \( R \), the set \( r_R(X) = \{ a \in R : Xa = 0 \} \) is called the right annihilator of \( X \) in \( R \) and the set \( l_R(X) = \{ b \in R : bX = 0 \} \) is called the left annihilator of \( X \) in \( R \).

Now we give our main characterization for \( J \)-reflexive rings.

**Theorem 2.2.** The following are equivalent for a ring \( R \).

1. \( R \) is \( J \)-reflexive.
2. For all \( a \in R \), \( r_R(aR)Ra \subseteq J(R) \) and \( aRl_R(Ra) \subseteq J(R) \).
3. \( IRK = 0 \) implies \( KRI \subseteq J(R) \) for every nonempty subsets \( I, K \) of \( R \).
4. \( < a > < b >= 0 \) implies \( < b > < a > \subseteq J(R) \) for any \( a, b \in R \).
5. \( IK = 0 \) implies \( KI \subseteq J(R) \) for every right (left) ideals \( I, K \) of \( R \).
6. \( IK = 0 \) implies \( KI \subseteq J(R) \) for every ideals \( I, K \) of \( R \).

**Proof.** (1) \( \Rightarrow \) (2) Let \( b \in r_R(aR) \). Then \( aRb = 0 \) for \( a, b \in R \). Since \( R \) is \( J \)-reflexive, \( bRa \subseteq J(R) \). So we have \( r_R(Ra)Ra \subseteq J(R) \). Similarly, one can show that \( aRl_R(Ra) \subseteq J(R) \).

(2) \( \Rightarrow \) (1) Assume that \( aRb = 0 \) for \( a, b \in R \). Then, \( b \in r_R(aR) \). By (2) we have \( bRa \subseteq J(R) \). So \( R \) is a \( J \)-reflexive ring.

(3) \( \Rightarrow \) (4) \( \Rightarrow \) (5) \( \Rightarrow \) (6) It is clear.

(6) \( \Rightarrow \) (1) Let \( aRb = 0 \) for \( a, b \in R \). Then \( RaRRaR = 0 \). By hypothesis, \( RbRRaR \subseteq J(R) \). As \( bRa \subseteq RbRRaR \), we have \( bRa \subseteq J(R) \).

(1) \( \Rightarrow \) (3) Assume that \( IRK = 0 \) for nonempty subsets \( I, K \) of \( R \). Then for any \( a \in I \) and \( b \in K \), \( aRb = 0 \). As \( R \) is \( J \)-reflexive, \( bRa \subseteq J(R) \). This implies that \( KRI \subseteq J(R) \). \( \square \)

Examples of \( J \)-reflexive rings are abundant. All reduced rings, symmetric rings, reversible rings and reflexive rings are \( J \)-reflexive. In the sequel, we show that every \( J \)-reversible ring, uniquely clean ring and every right (left) quasi-duo ring is \( J \)-reflexive.

**Proposition 2.3.** Every \( J \)-reversible ring is \( J \)-reflexive.

**Proof.** Let \( R \) be a \( J \)-reversible ring and \( aRb = 0 \) for some \( a, b \in R \). Then \( ab = 0 \) and \( abr = 0 \) for any \( r \in R \). As \( R \) is \( J \)-reversible, \( bra \in J(R) \). Hence, \( bRa \subseteq J(R) \). \( \square \)
The converse statement of Proposition 2.3 is not true in general as the following example shows.

**Example 2.4.** Consider the ring \( R = M_2(\mathbb{Z}) \). It can be easily shown that \( R \) is a \( J \)-reflexive ring. Let \( A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \in R \). Although

\[
AB = 0, \quad BA = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \notin J(R).
\]

So \( R \) is not \( J \)-reversible.

Recall that a ring \( R \) is called **Baer** if the right (left) annihilator of every nonempty subset of \( R \) is generated by an idempotent (see for details [3]). We show that the converse statement of Proposition 2.3 is true for Baer rings.

**Theorem 2.5.** Let \( R \) be a Baer ring. Then the following are equivalent.

1. \( R \) is a \( J \)-reversible ring.
2. \( R \) is a \( J \)-reflexive ring.

**Proof.** (1) \( \Rightarrow \) (2) It is clear by Proposition 2.3.

(2) \( \Rightarrow \) (1) Let \( ab = 0 \) for \( a, b \in R \). Then \( abR = 0 \) and so \( a \in l_R(bR) \). As \( R \) is a Baer ring, there exists an idempotent \( e \in R \) such that \( l_R(bR) = eR \). Then we have \( eRbR = 0 \). Since \( R \) is \( J \)-reflexive, \( bReR \subseteq J(R) \) and so \( ba \in J(R) \), as desired. \( \square \)

Though reflexive rings are \( J \)-reflexive, \( J \)-reflexive rings are not reflexive as the following example shows.

**Example 2.6.** Let \( R \) be a commutative ring. Consider the ring

\[
S = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} : a, b, c, d \in R \right\}.
\]

By [1 Proposition 3.7], \( S \) is \( J \)-reversible and by Proposition 2.3, it is \( J \)-reflexive. For \( A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in S \), \( ASB = 0 \) but \( BSA \neq 0 \). Thus, \( S \) is not a reflexive ring.
The following result can be easily obtain by the definition of $J$-reflexive rings.

**Corollary 2.7.** The following are equivalent for a ring $R$.

1. If $R/J(R)$ is reflexive, then $R$ is $J$-reflexive.
2. If $R/J(R)$ is commutative, then $R$ is $J$-reflexive.

An element $a$ in a ring $R$ is called *uniquely clean* if $a = e + u$ where $e^2 = e \in R$ and $u \in U(R)$ and this representation is unique. A ring $R$ is called a *uniquely clean ring* if every element of $R$ is uniquely clean (see [7]).

**Corollary 2.8.** Every uniquely clean ring is $J$-reflexive.

**Proof.** Assume that $R$ is uniquely clean. Then $R/J(R)$ is Boolean by [7, Theorem 20]. Hence, $R$ is $J$-reflexive by Corollary 2.7. □

The converse statement of Corollary 2.8 is not true in general as the following example shows.

**Example 2.9.** For a commutative ring $R$, consider the ring $M_2(R)$. Since $M_2(R)$ is not an abelian ring, $M_2(R)$ is not a uniquely clean ring. Also, it can be easily shown that $M_2(R)$ is a $J$-reflexive ring by Theorem 3.1.

**Proposition 2.10.** Let $R$ be a ring. If $N(R) \subseteq J(R)$, then $R$ is $J$-reflexive.

**Proof.** Assume that $aRb = 0$ for $a, b \in R$. Then for any $r \in R$, $arb = 0$ and so $ab = 0$. Hence, $(bra)^2 = brabra = 0$ for all $r \in R$. So $bra \in N(R)$. By hypothesis we have $bra \in J(R)$, as asserted. □

A ring $R$ is called *right (left) quasi-duo* if every right (left) maximal ideal of $R$ is an ideal [5].

**Corollary 2.11.** Every right (left) quasi-duo ring is $J$-reflexive.

**Proof.** It is clear by [8, Lemma 2.3]. □

We now give a necessary and sufficient condition for a quotient ring to be $J$-reflexive.

**Theorem 2.12.** Let $R$ be a ring and $I$ a nilpotent ideal of $R$. Then $R$ is $J$-reflexive if and only if $R/I$ is $J$-reflexive.
Proof. Let $R/I = \overline{R}$, $a + I = \overline{a} \in \overline{R}$ and $\overline{aRb} = \overline{0}$ for $\overline{a}, \overline{b} \in \overline{R}$. So $aRb \subseteq I$. As $I$ is nilpotent there exists $k \in \mathbb{Z}^+$ such that $(RaRb)^k = 0$. $(RbRaR)^k \subseteq J(R)$, since $R$ is $J$-reflexive. Thus $RbRaR \subseteq J(R)$ as Jacobson radical is semiprime. Hence $\overline{RbRaR} \subseteq J(R)/I = \overline{J(R)}$. So $\overline{bRa} \subseteq \overline{J(R)}$.

Conversely, assume that $aRb = 0$ for $a, b \in R$. Then $\overline{aRb} = \overline{0}$. So $aRb \subseteq I$ and $RaRbR \subseteq I$. Therefore there exists $k \in \mathbb{Z}^+$ such that $I^k = 0$, and so $(RaRb)^k = RaRbRaRb \cdots RaRbR = 0$. Hence, $(RaRb)^k = 0$.

Since $R/I$ is $J$-reflexive, $(RaRbRaR \cdots RaRbR) = (RaRbR)^k \subseteq J(R)$. As Jacobson radical is semiprime ideal, we have $RaRb \subseteq J(R)$. Thus, $bRa \subseteq J(R)$. Hence, for all $r \in R$, we have $1 - (bra)x \in U(R)$ for some $x \in J(R)$. Then, there exists $s \in R$ such that $(1 - (bra)x)s = 1$. Hence, $1 - (1 - bra)x s \in I$. As every nilpotent ideal is nil, $1 - (brax)s \in U(R)$. This implies that $bra \in J(R)$ and so $bRa \subseteq J(R)$, as desired. □

Corollary 2.13. Let $R$ be a ring. Then the following are satisfied.

(1) If $J(R)$ is a nilpotent ideal, then $R$ is $J$-reflexive if and only if $R/J(R)$ is $J$-reflexive.

(2) If $R$ is an Artinian ring, then $R$ is $J$-reflexive if and only if $R/J(R)$ is $J$-reflexive.

Proof. (1) It is clear by Theorem 2.12.

(2) Since Jacobson radical of Artinian ring is nilpotent, it is clear by (1). □

Proposition 2.14. Let $R$ be a ring and $I$ an ideal of $R$ with $I \subseteq J(R)$. If $R/I$ is $J$-reflexive, then $R$ is $J$-reflexive.

Proof. Let $\overline{R} = R/I$ and for $\overline{a} = a + I \in R/I$. Assume that $aRb = 0$ for $a, b \in R$. So $\overline{aRb} = \overline{0}$. Since $R$ is $J$-reflexive, $\overline{bRa} \subseteq J(\overline{R})$ and $\overline{bra} \in J(\overline{R})$ for any $\overline{a} \in \overline{R}$. Thus, for all $\overline{a} \in \overline{R}$ we have $\overline{1} - (bra)\overline{x} \in U(\overline{R})$. Then, there exists $\overline{x} \in \overline{R}$ such that $(\overline{1} - (bra)\overline{x})\overline{s} = \overline{1}$. Hence, $1 - (1 - brax)s \in I$. As $I$ is contained $J(R)$, $(1 - brax)s \in U(R)$. This implies that $bra \in J(R)$ and so $bRa \subseteq J(R)$, as desired. □

Proposition 2.15. Let $R$ be a ring and $I$ a reflexive ideal of $R$. Then $R/I$ is $J$-reflexive.
Proof. Let \( \overline{R} = R/I \) and for \( \overline{a} = a + I \in R/I \). Suppose that \( \overline{a}\overline{b} = \overline{0} \) for \( \overline{a}, \overline{b} \in \overline{R} \). Then \( aRb \subseteq I \). Since \( I \) is a reflexive ideal, we have \( bRa \subseteq I \). Hence, \( bRa = \overline{0} \in J(\overline{R}) \). □

**Theorem 2.16.** Every subdirect product of \( J \)-reflexive ring is \( J \)-reflexive.

Proof. Let \( R \) be a ring, \( I, K \) ideals of \( R \) and \( R \) a subdirect product of \( R/I \) and \( R/K \). Assume that \( R/I \) and \( R/K \) are \( J \)-reflexive. Let \( aRb = 0 \) for \( a, b \in R \). Then \( \overline{a}\overline{b} = \overline{0} \) in \( R/I \) and \( R/K \). Since \( R/I \) and \( R/K \) are \( J \)-reflexive, \( bRa \subseteq J(R/I) \) and \( bRa \subseteq J(R/K) \). Then for each \( x \in R \) we have \( 1- brax \in U(R/I) \) and \( 1- brax \in U(R/K) \). Hence, there exist \( y \in R/I \) and \( z \in R/K \) such that \( (1- brax)y = \overline{1} \in R/I \) and \( (1- brax)z = \overline{1} \in R/K \). So \( 1- (1- brax)y \in I \) and \( 1- (1- brax)z \in K \). If we multiply the last two elements, we have \( (1- (1- brax)y)(1- (1- brax)z) \in IK \subseteq I \cap K = 0 \). Thus, \( 1- (1- brax)t = 0 \) and \( (1- brax)t = 1 \). This implies that \( bRa \subseteq J(R) \). □

**Corollary 2.17.** Let \( I \) and \( K \) be ideals of a ring \( R \). If \( R/I \) and \( R/K \) are \( J \)-reflexive, then \( R/I \cap K \) is \( J \)-reflexive.

Proof. Let \( \alpha : R/(I \cap K) \to R/I \) and \( \beta : R/(I \cap K) \to R/K \) where \( \alpha(r + (I \cap K)) = r + I \) and \( \beta(r + (I \cap K)) = r + K \). It can be shown that \( \alpha \) and \( \beta \) surjective ring homomorphisms and \( \ker \alpha \cap \ker \beta = 0 \). Hence \( R/(I \cap K) \) is subdirect product of \( R/I \) and \( R/K \). Therefore, \( R/(I \cap K) \) by Theorem 2.16. □

**Corollary 2.18.** Let \( R \) be a ring and \( I, K \) of ideals of \( R \). If \( R/I \) and \( R/K \) are \( J \)-reflexive, then \( R/IK \) is \( J \)-reflexive.

Proof. Assume that \( R/I \) and \( R/K \) are \( J \)-reflexive. Since
\[
R/I \cap K \cong (R/IK)/(I \cap K/IK)
\]
and \((I \cap K/IK)^2 = 0\), we complete the proof by Theorem 2.12. □

3. Extensions of \( J \)-reflexive rings

In this section we show that several extensions (Doroh extension, upper triangular matrix ring, Laurent polynomial ring, trivial extension etc.) of a \( J \)-reflexive ring are \( J \)-reflexive. In particular, it is proved that the \( J \)-reflexive condition is Morita invariant.
Two rings $R$ and $S$ are said to be *Morita equivalent* if the categories of all right $R$-modules and all right $S$-modules are equivalent. Properties shared between equivalent rings are called *Morita invariant properties*. $\mathcal{P}$ is Morita invariant if and only if whenever a ring $R$ satisfies $\mathcal{P}$, then so does $eRe$ for every full idempotent $e$ and so does every matrix ring $M_n(R)$ for every positive integer $n$.

Next result shows that the property of $J$-reflexivity is Morita invariant.

**Theorem 3.1.** Let $R$ be a ring. Then we have the following.

1. If $R$ is $J$-reflexive, then $eRe$ is $J$-reflexive for all idempotent $e \in R$.
2. $R$ is a $J$-reflexive ring if and only if $M_n(R)$ is $J$-reflexive for any positive integer $n$.

**Proof.** (1) Assume that $R$ is a $J$-reflexive ring. Let $a, b \in eRe$ with $aeReb = 0$. As $R$ is $J$-reflexive, $ebRea = ebRea \subseteq J(eRe) = eJ(R)e$. This implies that $eRe$ is a $J$-reflexive ring.

(2) Assume that $M_n(R)$ is $J$-reflexive ring. It is clear that $R$ is $J$-reflexive by (1). Conversely, suppose that $R$ is $J$-reflexive and $I, K$ are ideals of $M_n(R)$ such that $IK = 0$. Then, there exist $I_1, K_1$ ideals of $R$ such that $I = M_n(I_1)$ and $K = M_n(K_1)$. So $0 = IK = M_n(I_1)M_n(K_1) = M_n(I_1K_1)$. Thus, $I_1K_1 = 0$. Since $R$ is $J$-reflexive, $K_1I_1 \subseteq J(R)$. This implies that $KI = M_n(K_1)M_n(I_1) = M_n(K_1I_1) \subseteq J(M_n(R)) = M_n(J(R))$. This completes the proof. □

**Corollary 3.2.** Let $M$ be a finitely generated projective modules over a $J$-reflexive ring $R$. Then $\text{End}_R(M)$ is $J$-reflexive.

**Proof.** It is obvious by Theorem 3.1. □

**Proposition 3.3.** The following are equivalent for a ring $R$.

1. $R$ is $J$-reflexive.
2. $M = \left\{ \begin{pmatrix} r & x_{12} & \cdots & x_{1n} \\ 0 & r & \cdots & x_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & r \end{pmatrix} : r \in R, \ x_{ij} \in R \right\}$ is $J$-reflexive.
Proof. (1) \(\Leftrightarrow\) (2) Take \(I = \begin{pmatrix} 0 & x_{12} & \cdots & x_{1n} \\ 0 & 0 & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix}\). The proof is clear by Theorem 2.12.

Recall that the trivial extension of \(R\) by an \(R\)-module \(M\) is the ring denoted by \(R \otimes M\) with underlying additive group is \(R \oplus M\) with multiplication given by \((r, m)(r', m') = (rr', rm' + mr')\). The ring \(R \otimes M\) is isomorphic to \(S = \left\{ \begin{pmatrix} x & y \\ 0 & x \end{pmatrix} : x \in R, y \in M \right\}\) under the usual matrix operations.

**Proposition 3.4.** The following are equivalent for a ring \(R\).

1. The trivial extension \(R \otimes R\) of the ring \(R\) is \(J\)-reflexive.
2. \(R\) is a \(J\)-reflexive ring.

**Proof.** (1) \(\Rightarrow\) (2) Assume that \(R \otimes R\) is \(J\)-reflexive. Let \(aRb = 0\) for \(a, b \in R\). Then, for \(A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, B = \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix}\) \(\in R \otimes R\), we have \(A(R \otimes R)B = \begin{pmatrix} aRb & aRb \\ 0 & aRb \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\). As \(R \otimes R\) is \(J\)-reflexive, \(B(R \otimes R)A \subseteq J(R \otimes R)\). Hence, \(bRa \subseteq J(R)\).

(2) \(\Rightarrow\) (1) Suppose that \(R\) is \(J\)-reflexive. Let \(A(R \otimes R)B = 0\) for \(A = \begin{pmatrix} a & x \\ 0 & a \end{pmatrix}, B = \begin{pmatrix} b & y \\ 0 & b \end{pmatrix}\) \(\in R \otimes R\). Then for any \(M = \begin{pmatrix} s & t \\ 0 & s \end{pmatrix}\) \(\in R \otimes R\), we have \(AMB = \begin{pmatrix} asb & asy + atb + xsb \\ 0 & asb \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\). Since \(R\) is \(J\)-reflexive and \(aRb = 0\), we conclude that \(bsa \in J(R)\) for any \(s \in R\). Note that \(J(R \otimes R) = \begin{pmatrix} J(R) & R \\ 0 & J(R) \end{pmatrix}\). Hence, \(B(R \otimes R)A \subseteq J(R \otimes R)\), as asserted.

**Proposition 3.5.** Let \(\{R_i\}_{i \in \mathcal{I}}\) indexed set of the ring \(R_i\). Then \(R_i\) is \(J\)-reflexive for all \(i \in \mathcal{I}\) if and only if \(\prod_{i \in \mathcal{I}} R_i\) is \(J\)-reflexive.
Proof. \(\Rightarrow\) Let \(\prod_{i \in I} M_i K_i = 0\) for ideals \(\prod_{i \in I} M_i, \prod_{i \in I} K_i\) of \(\prod_{i \in I} R_i\). Then \(\prod_{i \in I} M_i K_i = 0\). Therefore, \(M_i K_i = 0\) for all \(i \in I\). Since \(R_i\) is \(J\)-reflexive, \(K_i M_i \subseteq J(R_i)\) for all \(i \in I\). So \(\prod_{i \in I} K_i \prod_{i \in I} M_i = \prod_{i \in I} K_i M_i \subseteq J(\prod_{i \in I} R_i) = \prod_{i \in I} J(R_i)\).

\(\Leftarrow\) Assume that \(M_\phi K_\phi = 0\) for ideals \(M_\phi, K_\phi\) of \(R_\phi\). Choose \(M = (M_\phi)_{\phi \in I}\) and \(K = (K_\phi)_{\phi \in I}\) as only \(\phi\) components are nonzero ideal. So \(M\) and \(K\) are ideals of \(\prod_{i \in I} R_i\). Also we have \(MK = 0\). As \(\prod_{i \in I} R_i\) is \(J\)-reflexive, \(KM \subseteq J(\prod_{i \in I} R_i)\). Thus, \(K_\phi M_\phi \subseteq J(R_\phi)\). \(\Box\)

**Proposition 3.6.** The following are equivalent for a ring \(R\).

1. \(R\) is a \(J\)-reflexive ring.
2. \(T_n(R)\) is \(J\)-reflexive for any \(n \in \mathbb{Z}^+\).

*Proof.* (1) \(\Rightarrow\) (2) For \(n = 1\) it is clear. Consider the ring \(T_2(R)\). Choose the ideal \(I = \begin{pmatrix} 0 & R \\ 0 & 0 \end{pmatrix}\). It is clear that \(I^2 = 0\). So \(T_2(R)/I \cong R \times R\).

By Proposition 3.5, \(T_2(R)/I\) is \(J\)-reflexive. Hence \(T_2(R)\) is \(J\)-reflexive, by Theorem 2.12. By induction, \(T_n(R)\) is \(J\)-reflexive for any \(n \in \mathbb{Z}^+\).

(2) \(\Rightarrow\) (1) It is evident from Theorem 3.1(1). \(\Box\)

**Proposition 3.7.** Let \(R\) be a ring and \(e^2 = e \in R\) is central. Then, \(R\) is a \(J\)-reflexive ring if and only if \(eR\) and \((1 - e)R\) are \(J\)-reflexive.

*Proof.* The necessity is obvious by Theorem 3.1. For the sufficiency suppose that \(eR\) and \((1 - e)R\) are \(J\)-reflexive for a central idempotent \(e \in R\). It is well-known that \(R \cong eR \times (1 - e)R\). By Proposition 3.5, \(R\) is \(J\)-reflexive. \(\Box\)

For an algebra \(R\) over a commutative ring \(S\), the *Dorroh extension* \(I(R; S)\) of \(R\) by \(S\) is the additive abelian group \(I(R; S) = R \oplus S\) with multiplication \((r, v)(s, w) = (rs, rw + vs + vw)\).

**Proposition 3.8.** Let \(R\) be a ring and \(M = I(R; S)\) Dorroh extension of \(R\) by commutative ring \(S\). Assume that for all \(s \in S\) there exists \(s' \in S\) such that \(s + s' + ss' = 0\). Then the following are equivalent.

1. \(R\) is \(J\)-reflexive.
2. \(M\) is \(J\)-reflexive.
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Proof. (1) ⇒ (2) Let \((a_1, b_1)M(a_2, b_2) = (0, 0)\) for \((a_1, b_1), (a_2, b_2) \in M\). So for any \((x, y) \in M\), we have \((a_1, b_1)(x, y)(a_2, b_2) = (0, 0)\). Then \((a_1x_2, a_1x_2 + a_1y_2 + a_1y_2 + a_1y_2 + a_2x_2 + a_1y_2 + b_1y_2 + b_1y_2) = (0, 0)\). Hence, \(a_1x_2 = 0\) and \(a_1x_2 + a_1y_2 + a_1y_2 + a_1y_2 + b_1y_2 = 0\). As \(R\) is \(J\)-reflexive, \(a_2x_1 \in J(R)\) for any \(x \in R\). Thus, \((a_2, b_2)(x, y)(a_1, b_1) = (a_2x_1, +)\). By hypothesis, \((0, S) \subseteq J(M)\). It can be easy to show that \((a_2x_1, 0) \in J(M)\) for each \(x \in R\). Therefore, \((a_2, b_2)S(a_1, b_1) \subseteq J(M)\).

(2) ⇒ (1) Let \(aRb = 0\) for \(a, b \in R\). Then \((a, 0)M(b, 0) = (0, 0)\). Since \(M\) is \(J\)-reflexive, \((b, 0)M(a, 0) \subseteq J(M)\). By hypothesis, \((0, S) \subseteq J(M)\). This implies that \((bRa, 0) \subseteq J(S)\). Hence, \(bRa \subseteq J(R)\). □

If \(R\) is a ring and \(f : R \rightarrow R\) is a ring homomorphism, let \(R[[x, f]]\) denote the ring of skew formal power series over \(R\); that is all formal power series in \(x\) with coefficients from \(R\) with multiplication defined by \(xr = f(r)x\) for all \(r \in R\). Note that \(J(R[[x, f]])) = J(R) + < x >\). Since \(R[[x, f]] \cong I(R; < x ))\) where \(< x >\) is the ideal generated by \(x\), we have the following result.

**Corollary 3.9.** Let \(R\) be a ring and \(f : R \rightarrow R\) a ring homomorphism. Then the following are equivalent.

1. \(R\) is a \(J\)-reflexive ring.
2. \(R[[x, f]]\) is \(J\)-reflexive.

If \(f\) is taken \(f = 1_R : R \rightarrow R\) (i.e. \(1_R(r) = r\) for all \(r \in R\)), we have \(R[[x]] = R[[x, 1_R]]\) is the ring of formal power series over \(R\).

**Corollary 3.10.** The following are equivalent for a ring \(R\).

1. \(R\) is a \(J\)-reflexive ring.
2. \(R[[x]]\) is \(J\)-reflexive.

Let \(R\) be a ring and \(u \in R\). Recall that \(u\) is right regular if \(ur = 0\) implies \(r = 0\) for \(r \in R\). Similarly, left regular element can be defined. An element is regular if it is both left and right regular.

**Proposition 3.11.** Let \(R\) be a ring and \(M\) multiplicatively closed subset of \(R\) consisting of central regular elements. Then the following are equivalent.

1. \(R\) is \(J\)-reflexive.
2. \(S = M^{-1}R = \{ \frac{a}{b} : a \in R, b \in M\} \) is \(J\)-reflexive.
Proof. (1) ⇒ (2) Let \(aSb = 0\) for \(a, b \in S\). So there exist \(a_1, b_1 \in R\) and \(u^{-1}, v^{-1} \in M\) such that \(a = a_1u^{-1}\) and \(b = b_1v^{-1}\). Then \(0 = aSb = a_1u^{-1}Sb_1v^{-1} = a_1Sbv^{-1}\). Hence for any \(rs^{-1} \in S\) we have \(a_1rs^{-1}b\). Thus, \(a_1rb_1 = 0\) for each \(r \in R\). As \(R\) is \(J\)-reflexive, \(b_1ra_1 \in J(R)\). This implies that \(b_1v^{-1}rs^{-1}a_1u^{-1} \in J(R)\). As \(J(R) \subseteq J(S)\), \(aSb \subseteq J(S)\).

(2) ⇒ (1) Let \(aRb = 0\) for \(a, b \in R\) and \(u, v \in M\). So we have \(auRbv = 0\). Then for any \(m \in M\) and \(r \in R\) \(aurmbv = 0\). Since \(S\) is \(J\)-reflexive, \(bvrmau \in J(S)\). If we multiply \(bvrmau\) with inverses of \(u, m, v\), then we have \(bra \in J(R)\) for any \(r \in R\). This completes the proof. \(\square\)

The following result is a direct consequence of Proposition 3.11.

**Corollary 3.12.** Let \(R\) be a ring. Then the following are equivalent.

1. \(R[x]\) is \(J\)-reflexive.
2. \(R[x, x^{-1}]\) is \(J\)-reflexive.

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