Algorithm on rainbow connection for maximal outerplanar graphs

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Abstract

In this paper, we consider rainbow connection number of maximal outerplanar graphs (MOPs) on algorithmic aspect. For the (MOP) $G$, we give sufficient conditions to guarantee that $rc(G) = diam(G)$. Moreover, we produce the graph with given diameter $d$ and give their rainbow coloring in linear time. X.Deng et al. [5] give a polynomial time algorithm to compute the rainbow connection number of MOPs by the Maximal fan partition method, but only obtain a compact upper bound. J. Lauri [19] proved that, for chordal outerplanar graphs given an edge-coloring, to verify whether it is rainbow connected is NP-complete under the coloring, it is so for MOPs. Therefore we construct Central-cut-spine of MOP $G$, by which we design an algorithm to give a rainbow edge coloring with at most $2rad(G) + 2 + c, 0 \leq c \leq rad(G) - 2$ colors in polynomial time.

Keywords: Rainbow connection number, maximal outerplanar graph, diameter, algorithm.

1 Introduction

Graphs considered are finite, simple and connected in this paper. Notations and terminologies not defined here, see West [30]. The concept of rainbow connection was introduced by Chartrand, Johns, McKeon and Zhang in 2008 [14]. Let $G$ be a nontrivial finite simple connected graph on which is assigned a coloring $c : E(G) \to \{1, 2, \cdots, n\}, n \in \mathbb{N}$, where adjacent edges may have same color. A rainbow path in $G$ is a path with different colors on it. If for any two vertices of $G$, there is a rainbow path connecting them, then $G$ is called rainbow connected and $c$ is called a rainbow coloring. Obviously, any $G$ has a trivial rainbow coloring by coloring each edge with different colors. Chartrand et al. [14] defined the rainbow connection number $rc(G)$ of graph $G$ as the smallest number of colors needed to make $G$ rainbow connected. For any two vertices $u$ and $v$ in $G$, the length of a shortest path between them is their distance, denoted by $d(u, v)$. The eccentricity of a vertex $v$ is $ecc(v) := max_{x \in V(G)} d(v, x)$. The diameter of $G$ is $diam(G) := max_{x \in V(G)} ecc(x)$. The radius of $G$ is $rad(G) := min_{x \in V(G)} ecc(x)$. Distance between a vertex $v$ and a set $S \subseteq V(G)$ is $d(v, S) := min_{x \in S} d(v, x)$. The k-step open neighbourhood of a set $S \subseteq V(G)$ is $N_k(S) := \{ x \in V(G) | d(x, S) = k \}, k \in \{ 0, 1, 2, \ldots \}$. The degree of a vertex $v$ is $degree(v) := | N_1(v) |$. The maximum degree of $G$ is $\Delta(G) := max_{x \in V(G)} degree(x)$. The girth of a graph $G$ is $g(G) :=$ the length of maximal induced cycle in $G$. A vertex is called pendant if its degree is 1. Let $n(G) = | V(G) |$ and $e(G)$ be the size of $G$. Obviously $diam(G) \leq rc(G) \leq e(G)$. From [14], we know that rainbow connection number of any complete graph is 1 and that of a tree is its size.

Obviously, we know that cut-edges must have distinct colours when $G$ is rainbow connected. Thus stars have arbitrarily large rainbow connection number while having diameter 2. Therefore, it is significant to seek upper bound on $rc(G)$ in terms of $diam(G)$ in 2-edge-connected graphs. Chandran et al. [19] showed that $rc(G) \leq rad(G)(rad(G) + 2)$ when $G$ is 2-edge-connected, and hence $rc(G) \leq diam(G)(diam(G) + 2)$.

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Li et al. [21] proved that $rc(G) \leq 5$ when $G$ is a 2-edge-connected graph with diameter 2. Li et al. [22] proved that $rc(G) \leq 9$ when $G$ is a 2-edge-connected graph with diameter 3.

Recalling an outerplanar graph is a planar graph which has a plane embedding with all vertices placed on the boundary of a face, usually taken to be the exterior one. A MOP is an outerplanar graph which can not be added any line without losing outerplanarity.

By [2], a MOP can be recursively defined as follows: (a) $K_3$ is a MOP, (b) For a MOP $H_1$ embedded in the plane with vertices lying in the exterior face $F_1$, $H_2$ is obtained by joining a new vertex to two adjacent vertices on $F_1$. Then $H_2$ is a MOP. (c) Any MOP can be constructed by finite steps of (a) and (b).

Note each inner face of a MOP $H$ is a triangle and the connectivity $\kappa(H) = 2$. Moreover, $H$ can be represented by two line arrays $High(1), High(2), \ldots, High(n)$ and $Low(1), Low(2), \ldots, Low(n)$. Here for any vertex $i$, $High(i)$ and $Low(i)$ are labels of its two neighbors whose labels are less than $i$, and $High(i) > Low(i)$; and $High(1), Low(1)$ and $Low(2)$ are undefined, and $High(2) = 1$. Figure 1 illustrates a MOP and its canonical representation.

![Figure 1: Example](https://example.com/figure1.png)

**Property (A)** A graph is outerplanar if and only if it has no $K_4$ or $K_{2,3}$ minor.

We summarize some results for the rainbow connection number of graphs in the following.

Huang et al. proved that if $G$ is a bridgeless outerplanar graph of order $n$ and $diam(G) = 2$, then $rc(G) \leq 3$ and the bound is tight. Moreover they proved that if $diam(G) = 3$, then $rc(G) \leq 6$, in [25].

**Theorem 1.1** [14]. For cycle $C_n$, we have

$$rc(C_n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even}, \\ \lceil \frac{n}{2} \rceil & \text{if } n \text{ is odd}. \end{cases}$$

Chandran et al. [13] studied the relation between rainbow connection numbers and connected dominating sets, and they obtained the following results:

(1) For any bridgeless chordal graph $G$, $rc(G) \leq 3rad(G)$. Moreover, the result is tight.

(2) For any unite interval graph $G$ with $\delta(G) \geq 2$, $rc(G) = diam(G)$.

A finite simple connected graph $G$ is called a Fan if it is $P_n \vee K_1$ (the join of $P_n$ and $K_1$), denoted by $Fan_n$, for some $n \in \mathbb{N} \setminus \{1\}$. Here the vertex $v$ of $K_1$ is called central vertex, the edges $v_i, v_{i+1} (1 \leq i \leq n-1)$ of $P_n = (v_1, v_2, \ldots, v_n)$ are called path edges, and the edges $v_iv$ between $P_n$ and $K_1$ are called spoke edges.

**Theorem 1.2** [5]. The rainbow connection number of $Fan_n$ satisfies

$$rc(Fan_n) = \begin{cases} 1 & \text{if } n = 2, \\ 2 & \text{if } 3 \leq n \leq 6, \\ 3 & \text{if } n \geq 7. \end{cases}$$

**Theorem 1.3** [6]. Let $G$ be a bridgeless outerplanar graph of order $n$.

1. If $diam(G) = 2$, then

$$rc(G) = \begin{cases} 3 & \text{if } G = F_n \text{ (n \geq 7) or } C_5, \\ 2 & \text{otherwise}. \end{cases}$$

2. If $diam(G) = 3$, then

$$rc(G) = \begin{cases} 3 & \text{if } G = F_n \text{ (n \geq 7) or } C_5, \\ 2 & \text{otherwise}. \end{cases}$$
2. If \( \text{diam}(G) = 3 \), then \( 3 \leq \text{rc}(G) \leq 4 \) and the bound is tight.

Following, we give a theorem on edge, vertex cut set and their rainbow connection for a connected graph \( G \).

**Theorem 1.4.** Let \( G \) be a connected graph and \( S_1, S_2 \) be two disjoint edge cuts, then they must be colored by at least two different colors in order to make \( G \) rainbow connected.

**Proof.** Let \( V_{i1}, V_{i2} \) be the end vertex sets of \( S_i \) (\( i = 1, 2 \)), then \( X_i, X' \) be the vertex sets separated by \( S_i \), (\( i = 1, 2 \)). Clearly, there are no red edges in the graph \( G \) as shown in Figure 2. Since \( S_2 \) is an edge cut of \( G \), no edges between \( V_{i1} \) and \( V_{i2} \) and so no edges between \( V_{i2} \cap (X_2 \setminus V_{21}) \) and \( V_{i1} \cap (X_2 \setminus V_{22}) \), between \( V_{i1} \cap (X_2 \setminus V_{21}) \) and \( V_{i2} \cap X_2 \), between \( V_{i1} \cap (X_2 \setminus V_{22}) \) and \( V_{i2} \cap X_2 \). If \( (X_1 \setminus V_{12}) \cap (X_2 \setminus V_{22}) \) and \( (X_1 \setminus V_{11}) \cap (X_2 \setminus V_{21}) \) are not empty, thus must have rainbow path through \( S_1 \) and \( S_2 \) in order to make \( G \) rainbow connected, therefore Theorem 1.4 is correct. Other cases can be proved by the same method. \( \square \)

![Figure 2: Partition of V determined by the edge cuts S1 and S2](image)

### 2 \( \text{rc}(G) \) of minimum maximal outerplanar graph (MMOP) \( G \)

We give a sufficient condition for some graphs of maximal outerplanar graphs whose rainbow connection numbers equal their strong rainbow connection numbers, just their diameters in this section.

The MMOP \( G \) is a MOP with given diameter having minimum vertices. Recall a MOP is outerplanar, therefore \( G \) does not contain \( K_4 \) or \( K_{2,3} \) minor ([12]). We construct the MMOPs with diameter 2 by its recursive definition through the following three steps.

**Step 1.** \( K_3 \) with vertices \( a, b, c \) (as Figure 3 showing) as the start of the process constructing the MMOPs with diameter 2.

**Step 2.** By the symmetry of \( K_3 \), we could add one new vertex \( d \) joining any two vertexes of \( a, b, c \) (For example as Figure 4 showing).

![Figure 3: \( H_1 = K_3 \)](image)

**Step 3.** Since Figure 4 has diameter 2 and only \( a, d \) has distance two, in the process of the construction we add a new vertex \( e \) and joining any one of edges of exterior face of Figure 4. By symmetry, we obtain the following MOP (Figure 5) with diameter 2.

Since Figure 4 and Figure 5 have diameter 2 and the resulting graph of deleting any one of the vertices which have distance 2 in Figure 4 is \( K_3 \) with diameter 1, thus Figure 4 is the MMOP with diameter 2.
In Figure 5, the two vertex pairs \( a, d \) and \( a, e \) have distance 2. We add a new vertex adjacent to \( d \) and \( e \) and obtain an outerplanar graph with diameter 3.

Note that \( Lad_3 \) and \( Lad_3 + \) have diameter 3 and \( Lad_3 \) is the MMOP with diameter 3. Following, we call the MMOP with diameter \( d \) \( Lad_d \), \( Lad_d + \) is obtained by adding a new vertex to \( Lad_d \) as \( Lad_3 \) and \( Lad_3 + \) showing.

**Theorem 2.1.** If \( G \) is \( Lad_d \) or \( Lad_d + \) with diameter \( d \), then \( rc(G) = src(G) = d \).

**Proof.** We prove the theorem by induction method. Through the above three steps constructing the MMOPs with diameter 2 and 3, we know that the theorem is corrected when \( d = 2, 3 \).

Now suppose that we obtain the MMOP \( G \) with diameter \( d - 1 \) and \( rc(G) = src(G) = d - 1 \). Then \( G \) has a vertex pair, denoted by \( a, b \), having distance \( d - 1 \) and there is a \( d - 1 - path \) connecting them. Since \( G \) is a MMOP with diameter \( d - 1 \), therefore there is only one vertex pair with distance \( d - 1 \).

Following, we construct the MMOP with diameter \( d \). \( G \) has an \( ab - path \) with distance \( d - 1 \) and is a MOP, then we add a new vertex \( e \) to \( G \) adjacent to \( b \) and it’s preceding vertex \( c \), on the \( ab - path \). The edges \( ec, eb \) are colored by \( d - 1 \) respectively. Finally, we add another vertex \( f \) to the above graph and adjacent it to \( b, e \). The new graph has diameter \( d \) and has minimum vertex. The edges \( fe, fb \) are colored by \( d \). We give the construction process in Figure 7.

Now, we prove the colouring of Figure 7 is a rainbow and strong rainbow coloring of the graph. The part \( A \) of Figure 7 is a MMOP with diameter \( d - 1 \) and has a strong rainbow coloring with \( d - 1 \) colors. When \( e \) is added to \( A \), we connected it to \( b, c \) which are colored by \( d - 1 \) with the same color of the edge \( cb \). Therefore there is rainbow path between \( e \) and other vertices of \( A \) through the edge \( ce \). Obviously, the two vertex pairs \( ec, eb \) are strong rainbow connected by the coloring. By the same reason, we can prove the resulting graph added the vertex \( f \) and \( fe, fb \) colored by \( d \) is strong rainbow connected, which is \( Lad_d \). Moreover, we add a new vertex \( g \) which is adjacent to \( e \) and \( f \), the edges \( ge, gf \) is colored by \( d \). The resulting graph is \( Lad_d + \). They can be obtained by the following algorithm. \( \square \)
Algorithm 1: Giving the graphs \((\text{Lad}_d, \text{Lad}_{d+})\) with diameter \(d\) and their rainbow(strong rainbow) colorings

Input: A number \(d \geq 4\).

Output: The MMOP \(G\) with diameter \(d\) and a strong rainbow coloring of it.

Step 1: Given a \(K_3\) with vertices 1, 2, 3, \(d(1, 1) = 0, d(1, 2) = 1, d(1, 3) = 1\). Adding vertex 4 to \(K_3\) and 4 is adjacent to 2, 3, then \(d(1, 4) = 2\). Now, adding a new vertex 5 to the above graph with vertices \{1, 2, 3, 4\}, which is adjacent to 4 and it’s preceding neighbor on a \(2\) − \(length\)−\(path\), then \(d(1, 5) = 2\). Adding a vertex 6 adjacent to vertices 4, 5 to the graph with vertices \{1, 2, 3, 4, 5\}, then \(d(1, 6) = 3\).

Step 2: In this step, we construct the MMOP \(\text{Lad}_d\) with diameter \(d\) and it’s strong rainbow coloring.

begin

for \(1 \leq i \leq 2d\) in \(G\), if \(d(1, i - 1) = d(1, i) - 1 < d\), add a new vertex \((i + 1)\) connecting to \(i - 1, i\) then \(d(1, i + 1) = d(1, i)\), Color the edges \((i - 1, i + 1)(i, i + 1)\) by \(i + 1\);

do \(i \leftarrow i + 1\).

if \(d(1, i - 1) = d(1, i) < d\), add a new vertex \((i + 1)\) connecting to \(i - 1, i\) then \(d(1, i + 1) = d(1, i) + 1\), Color the edges \((i - 1, i + 1)(i, i + 1)\) by \(i - 1\);

do \(i \leftarrow i + 1\).

Step 3: Adding a vertex \(2d + 1\) to the above graph \(G\) with \(2d\) vertices and connecting it to \(2d - 1, 2d\). Coloring the new edges \((2d + 1, 2d), (2d + 1, 2d - 1)\) by \(2d\), then we obtain the graph \(\text{Lad}_{d+}\) and it’s strong rainbow coloring.

don

Problem 2.2. Characterize those graphs \(G\) with \(rc(G) = \text{diam}(G)\), or give some sufficient conditions to guarantee that \(rc(G) = \text{diam}(G)\). Similar problems for the parameter \(src(G)\) can be proposed.

Obviously, it is very difficult to give the sufficient conditions for general graphs. Up to now, we only know very few graphs whose rainbow connection numbers and their strong rainbow connection numbers are their diameters, which are particular graph classes. The MMOP is a graph class whose rainbow and strong rainbow connection numbers are their diameters. Figure 8 gives three MOPs with diameter 3 and rainbow coloring using 3 colors. Therefore the condition of theorem 2.1 is not necessary.

3 Central-cut-spine of MOP

In this section, Our main result is to give the definition of Central- cut-spine of MOP, which play a key role in algorithm 3. Following we give algorithm 2 to compute the Central-cut-spine of a MOP.

A graph \(G\) is called 2-degenerate if any of its subgraph has a vertex with degree 2 or less.
Theorem 3.1\textsuperscript{[15]}. For a connected graph $G$ with at least 2 vertices, it is an MOP iff the following hold:
(a) for any vertex $v$ of $G$, its neighbors induce a path in $G$, (b) $G$ is 2-degenerate.

It is well known that a MOP $G$ can be embedded in the plane such that every vertex lies on the boundary of the exterior face, all exterior edges form a Hamiltonian cycle $C = (v_1v_2 \cdots v_nv_1)$. In $G$, a Hamiltonian degree sequence $D = (d_1, d_2, \cdots, d_n, d_1)$ is the degree sequence of vertices $v_1, v_2, \cdots, v_n, v_1$. In [3], T. Beyer et al. gave an Algorithm which takes a Hamiltonian degree sequence and produces the unique corresponding MOP in linear time. For any edge $v_iv_j$ of $G$, which is a chordal edge of $G$ when $|s-t| \geq 2$, the two vertices are a 2-vertex-cut set. Since $g(G) = 3$, any two vertices incident to a chord of $G$ have exactly two common neighbors, while two vertices incident to an outer edge of $G$ have exactly one common neighbor. Now if $G$ has order $n$. Using the fact that the boundary of the exterior region of $G$ is a hamiltonian cycle and the boundary of every interior region of $G$ is a triangle, which follows that $G$ has $2n-3$ edges and $n-1$ regions by Euler’s formula. Thus $G$ has $n-3$ chords and $n-2$ interior triangles.

Theorem 3.2\textsuperscript{[3]}. A MOP $G$ is determined uniquely up to isomorphisms by its Hamiltonian degree sequence $D = (d_1, d_2, \cdots, d_n, d_1)$.

A graph is chordal if every cycle of length greater than three has a chord, which is meaning that there is an edge joining two nonconsecutive vertices of the cycle.

Theorem 3.3\textsuperscript{[4]}. $2rad(G) - 2 \leq diam(G) \leq 2rad(G)$ for any connected chordal graph $G$. Moreover, if $2rad(G) - 2 = diam(G)$, then $G$ has a 3-sun as an induced subgraph.

Let $\eta(G)$ be the smallest integer such that every edge of $G$ belongs to a cycle of length at most $\eta(G)$.

Theorem 3.4\textsuperscript{[26]}. For every bridgeless graph $G$, $rc(G) \leq \sum_{i=1}^{rad(G)} \min\{2i+1, \eta(G)\} \leq rad(G)\eta(G)$.

The Central-cut-spine(CCS) of a MOP $G$ is a tree generated by the following method: First, we choose an center vertex, denoted by $v$, marked red, on the unique hamiltonian cycle of $G$. Second, give the sets $N_i(v) := \{x \in V(G)|d(x,v) = i\}, i \in \{0, 1, 2, \ldots, rad(G)\}$. Third, for $1 \leq i \leq rad(G) - 1$, if $|N_i(v)| = 2$ and they are adjacent, then shrink the corresponding edge to obtain a new vertex, marked green, adjacent to the new vertex obtained by the vertices in $N_{i-1}(v)$ pertinent to $N_i(v)$; If $|N_i(v)| \geq 3$, then we partition them into $N_{ij}(v), 1 \leq j \leq |N_i(v)|$, for each $j$, the vertices of $N_{ij}(v)$ forming all the path edges of a Fan-structure, for any edge $v_i v_j, |i_j - i_j| \geq 2$ and they have neighbors in $N_{i+1}(v)$, then shrink the corresponding edge to obtain a new vertex, named $v_{ij}v_i v_j$, marked green, and adjacent to the new vertex $v_{ij}v_i v_j v_{i+1}v_{i+2}$ pertinent to $v_i v_j$.

An example: we discuss the property of Figure 9 and give it’s CCS. Notice that it has 40 vertices and the center vertex $v$, radius 5 and diameter 10. By Theorem 3.3, we know that $rc(Figure 9) \leq 15$. In Figure 9 we give it a $12 - \text{rainbow coloring}$.

The red vertices, edges and green vertices constitute it’s CCS which is computed by the following operations: We choose an eccentricity vertex, denoted by $v$, with maximum degree and the vertices $N_1(v)$. Choose the adjacent vertices in $N_1(v)$, which are cut sets of $G$ and replaced by new vertices, which are marked green and adjacent to $v$. We proceed the same operation on $N_i(v), 2 \leq i \leq 4$ as $N_1(v)$. The last step we choose the vertices, in $N_4(v) = N_{\text{radius}-1}(v)$, whose neighbors forming a Fan-structure.
with them as central vertices, which are marked red and adjacent to the preceding new vertices. Now we obtained the Central-cut-spine of Figure 9 denoted by CCS(Figure 9).

Furthermore, we consider the Ladₙs and their edge cuts of Figure 9. By theorem 1.4, in order to give Figure 9 a rainbow coloring, we know that every edge cut has a color which is different from other edge-cuts having in Ladₙ. Since it has four Ladₙ, at least two Ladₙ are colored by eight different colors. Obviously, rc(Figure 9) \( \geq 10 \). When the Ladₙs are replaced by Ladₙs, the resulting graph, denoted by \( \text{Fan}_7 + 4 - \text{sun} + \text{Lad}_n \), has diameter \( 2n + 2 \) and its rainbow connection number is at least \( 2n + 2 \). With the same method of coloring in Figure 9, we can rainbow color \( \text{Fan}_7 + 4 - \text{sun} + \text{Lad}_n \) by \( 2(n - 1) + 6 = 2n + 4 \), which is less \( n - 1 \) than the 3rad(\( \text{Fan}_7 + 4 - \text{sun} + \text{Lad}_n \)). As \( n \to \infty \), we know that the upper bound 3rad in Theorem 3.3 may be arbitrarily far from \( 2n + 4 \). Therefore, the upper bound is not good for rainbow connection number of MOPs. Figure 11 gives another graph with the Central-cut-spine formed by red edges, vertices and green vertices, which has a rainbow coloring with 12 colors.

Figure 10: \( \text{Fan}_7 + 4 - \text{sun} + \text{FLad}_4 \)

Figure 11: diameter \( - 3 + a9b9c9 \)

Figure 11 gives an example of MOP with diameter 3, where the maximal Fan-structures have 10 vertices and the red vertices and edges give it’s CCS(Figure 11). The Fan-structures can be replaced by any \( F_n \) with \( n \geq 2 \), and the resulting graph \( G \) can be rainbow colored by at most 4 colors with the same method as Figure 11 showing. As the above description, we know that the MOPs having any length hamiltonian cycles. We may color the hamiltonian cycle making it rainbow connected by \( \frac{3}{2}n(G) \) colors. However, as \( n \to \infty \), the number of colors \( \to \infty \). Thus, rainbow coloring the hamiltonian cycle is not an effective method for MOPs. Moreover, when we compute the upper bound of \( rc(G) \) by the method of Maximal fan partition method showed in [5], the colors needed will increase to 9 as \( n \) increasing. The method is not effective either.

Following, we give an effective algorithm to compute an upper bound of rainbow connection number for MOP with the help of it’s CCS.

**Theorem 3.5.** If \( G \) is a MOP and \( v_r \) is the root of CCS(G), then we have the following results:

1. Each \( v_r - path \) in CCS(G) corresponding to two edge disjoint paths in G.
2. The green vertices of CCS(G) are corresponding to cut edges or the edges connecting to the vertices which are central vertices in the construction of CCS(G).

3. If v_r is a red vertex of CCS(G), then it is a vertex of G and a central vertex in the construction of CCS(G).

Proof: 1. By the construction of the CCS(G) of G, we know that v_r is a red vertex and the root of CCS(G).

Case1: The leaf vertex is a red vertex in CCS(G), then it is a vertex of G. Since G is a 2-connected graph, there are two edge disjoint paths between it and v_r in G, one of which has length at most rad(G) − 1. Since all edges on the path are situated in triangles, there is another path with length no more than 2(rad(G) − 1).

Case2: If the leaf is a green vertex in CCS(G), then it is corresponding to two adjacent vertices which are a vertex cut set and situated in N(rad(G)−1) or N(rad(G)−2). Therefore there are paths, with length at most rad(G) − 1, between v_r and the two adjacent vertices which are corresponding to the leaf. If they are edge disjoint then we choose them as the corresponding two paths, otherwise we choose one path with length rad(G) − 1 or rad(G) − 2. Since any edge of the path located in one triangle and every common edge can be replaced by two adjacent edges in the corresponding triangle, thus we can obtain a path between v_r and the another vertex with length at most (rad(G) − 1) + c_1, where c_1 is the number of common edges on the two paths.

By the construction of CCS(G), we produce the corresponding two paths in G for a v_r-path of CCS(G):

1. v_r is the start point of the two paths;
2. If the successor of v_r is a green vertex on one v_r-path, then whose corresponding vertices in G are the successors of the two paths respectively;
3. Now we consider the vertex having distance two to v_r. If it is a green vertex v_{i2v_1v_2} of CCS(G), which is corresponding to v_{i1} and v_{i2} in G. We choose the vertex having maximum degree, w.l.o.g. v_{i1}. Since v_{i1} is situated in N_2(v_r), there is an path between it and v_r with length 2. Moreover any edge of the path located in a triangle, we could choose an edge disjoint path with the 2-length path between v_{i2} and v_r with length at most 4. Following we produces the above process.

By the construction of CCS(G), 2. 3. are correct Obviously. □

A. Farley et al. introduced the notion of edge eccentricities by the separation property of an edge for outerplanar graphs in [8].

Definition 3.6 [8]. Let p = (s, t) be any edge of an outerplanar graph G, showing as in Figure [12]. A. Farley et al. defined four values e(p, x, S), which are called edge eccentricity of p, one for each vertex x (s or t) and side S of the edge p. The absolute value of e(p, x, S) equals the eccentricity of the vertex x in the induced subgraph G[S ∪ {s, t}] The value e(p, x, S) is negative iff all vertices of S ∪ {s, t} at distance d = e(p, x, S) from x lie at distance d − 1 from the other end vertex of p.

In [8], A. Farley et al. gave an edge eccentricity algorithm: Given a MOP G, calculate the eccentricities of its edges as follows:

(a) For all edges p = (s, t) on the Hamiltonian cycle of G, assign the value −1 to e(p, x, ∅), where x ∈ {s, t}.

(b) For each triangle (s, t, w), the values e(a, s, S_1), e(a, w, S_1), e(b, t, S_2), and e(b, w, S_2) are defined, one assign values of e(p, s, S) and e(p, t, S) according to the following rules:

Given an edge p = (s, t) of a MOP with a non-empty side S, let e_1, e_2 and r represent the values of e(b, w, S_2), e(b, t, S_2) and the eccentricity of s in the subgraph G[S_2 ∪ {s, t, w}], respectively.

\[ r = \begin{cases} 
-(1 + e_2), & e_2 > 0, \\
| e_2 |, & \text{otherwise.}
\]
Given an edge \( p = (s, t) \) with a non-empty side \( S \), let \( e_3 \) and \( d_1 \) represent the values \( e(a, s, S_1) \) and \( e(p, s, S) \), respectively. Let \( r \) be the eccentricity of \( s \) in the graph \( G[S_2 \cup \{s, t, w\}] \).

\[
d_1 = \begin{cases} e_3 & |e_3| \geq |r|, \\ r, & \text{otherwise}. \end{cases}
\]

If an edge \( p = (s, t) \) with a non-empty side \( S \),

\[
q = \begin{cases} -(1 + e(a, s, S_1)) & \text{if } e(a, s, S_1) > 0, \\ e(a, s, S_1) & \text{otherwise}. \end{cases}
\]

The eccentricity \( d_2 = e(p, t, S) = \begin{cases} |e(b, t, S_2)| & \text{if } |e(b, t, S_2)| \geq |q|, \\ |e(b, t, S_2)| & \text{otherwise}. \end{cases} \]

![Figure 12: An edge \( p = (s, t) \) with a nonempty side \( S \).](image)

By the above algorithm, A. Farley et al. give the following algorithm, denoted by Farley DRC algorithm, to compute the diameter and center for outerplanar graph \( G \).

1. For every vertex \( u \) of \( G \), choose an edge \( p \) incident with \( u \), then \( ecc(u) := \) the maximum of the absolute values of the two pertinent edge eccentricities of \( p \).

2. \( diam(G) := \max_{x \in V(G)} ecc(x), \quad rad(G) := \min_{x \in V(G)} ecc(x). \)

3. The center of \( G \) is the vertex set \( C(G) := \{v : ecc(v) = rad(G), v \in V(G)\} \).

The time complexity of Farley DRC algorithm for computing all vertex eccentricities in an outerplanar graphs is \( O(n) \), where \( n = |V(G)| \).

A vertex with whose neighbors inducing a clique in \( G \) is called simplicial. A simplicial elimination ordering (perfect elimination ordering) is a vertex ordering \( v_n, \ldots, v_2, v_1 \) for which each vertex \( v_i \) is simplicial in the induced graph by \( \{v_j, \ldots, v_i\} \). Its reverse is called a simplicial construction ordering of \( G \). The simplicial construction ordering of a chordal graph \( G \) can be found by Maximum Cardinality Search (MCS) in time \( O(n(G) + e(G)) \). The MCS algorithm is a simple linear time algorithm that choose a vertex \( x \) and let \( f(x) = 1 \), where \( f : V(G) \rightarrow \{1, \ldots, n(G)\} \) is a function, and produces an elimination ordering in reverse. For each vertex \( v \), it maintains an integer weight \( l(v) \) that is the cardinality of the already processed neighbors of \( v \); and produces a simplicial construction ordering when a chordal graph is input.

Following we give a polynomial time algorithm to compute the Central-cut-spine of MOP.

**Algorithm 2 producing the Central-cut-spine of MOP (CCS-Algorithm):**

**Input:** A MOP \( G \).

**Output:** The Central-cut-spine of \( G \).

**Step 1:** Finding a Hamiltonian degree sequence \( D = (d_1, d_2, \ldots, d_n, d_1) \) which is respond to the vertex sequence \( (v_1, v_2, \ldots, v_n, v_1) \) of \( G \).
Step 2: In this step, we have a simplicial construction ordering of vertices.

\[
\begin{align*}
&\text{begin} \\
&\text{for all vertices } v \text{ in } G \text{ do } l(v) = 0; \\
&\text{for } i = 1 \text{ up to } n \text{ do} \\
&\quad \text{Choose an unnumbered vertex } z \text{ of maximum weight}; f(z) = i; \\
&\text{for all unnumbered vertices } y \in N(z) \text{ do } l(y) = l(y) + 1; \\
&\end{align*}
\]

Step 3: In this step, we have all maximal Fans of \( G \).

\[
\begin{align*}
&\text{begin} \\
&\text{for } i = 1 \text{ up to } n \text{ do } i : N (f^{-1}(i)) = \{f^{-1}(i)\}; \\
&\text{for } i = 1 \text{ up to } n \text{ do} \\
&\quad \text{if } |N (f^{-1}(i))| < d_{f^{-1}(i)} + 1 \text{ do} \\
&\quad \quad \text{for } j = 1, \ldots , i - 1, i + 1, \ldots , n \text{ do} \\
&\quad \quad \quad \text{if } f^{-1}(j) \text{ and } f^{-1}(i) \text{ are adjacent, } N (f^{-1}(i)) \leftarrow N (f^{-1}(i)) \cup \{f^{-1}(j)\}, j \leftarrow j + 1; \\
&\quad \quad \quad \text{otherwise } j \leftarrow j + 1; \\
&\quad \quad \text{else } i \leftarrow i + 1; \\
&\end{align*}
\]

Step 4: Finding the center vertices, denoted by \( C(G) \), of \( G \) by the Farley DRC algorithm.

Step 5: Choose a vertex in \( C(G) \) with minimum degree, denoted by \( v_r \), \( 1 \leq r \leq n \). Giving a BFS \( v_r \) - tree \( T \) in \( G \) with predecessor function \( p \), a level function \( \ell \) such that \( \ell(v) = d_G(v_r, v) \) for all \( v \in V \).

Step 6: In this step, we produce the Central-cut-spine of \( G \).

\[
\begin{align*}
&\text{begin} \\
&\text{Let } S_r \text{ be the successor function, } P_r \text{ be the predecessor function and } l_r \text{ be level function of } \text{CCS}(G). \\
&\text{for } i = 0 \text{ up to } rad(G), \text{ do } i : N_i(v_r) := \{x \in V(G) | \ell(x) = i\}, i \in \{0, 1, 2, \ldots , rad(G)\}; \\
&\text{for } i = 0, \text{ then } v_r \text{ and } N_1(v_r) \text{ forming a Fan-structure of } G. \text{ Let } v_r \text{ be the root of } \text{CCS}(G) \text{ and is marked red. For any edge } v_1, v_2, \text{ of it, } v_1, v_2, \in N_1(v_r), \text{ if } |1_s - 1_t| \geq 2, \text{ then shrink the corresponding edge to obtain new vertex, named } v_{1_{i_1}, v_{1_{i_2}}}, \text{ marked green and } S_r(v_r) \leftarrow v_{1_{i_1}, v_{1_{i_2}}}; \\
&\text{for } i = 2 \text{ up to } rad(G) - 1 \text{ do} \\
&\quad \text{if } |N_i(v_r)| = \{v_{i_1}, v_{i_2}\} \geq 2, |i_1 - i_2| \geq 2 \text{ and they are adjacent, then shrink the corresponding edge to obtain a new vertex, named } v_{i_1, v_{i_2}}, \text{ marked green, adjacent to the new vertex } v_{1_{i_1}, v_{1_{i_2}}} \text{ obtained by the vertices in } N_{i-1}(v_r) \text{ which are pertinent to } N_i(v_r); S_r(v_{i_1, v_{i_2}}) \leftarrow v_{i_1, v_{i_2}}, P_r(v_{i_1, v_{i_2}}) = v_{1_{i_1}, v_{1_{i_2}}}, l_r(v_{i_1, v_{i_2}}) \leftarrow i, i \leftarrow i + 1, \\
&\quad \text{if } |N_i(v_r)| \geq 3, \text{ then we partition them into } N_j(N_i(v_r)), 1 \leq j \leq |N_i(v_r)|, \text{ for each } j, \text{ the vertices of } N_j(N_i(v_r)) \text{ forming all the path edges of a Fan structure, for any edge } v_{i_1, v_{i_2}} \text{ and } |i_1 - i_2| \geq 2, \text{ then shrink the corresponding edge to obtain a new vertex, named } v_{ij, v_{ij}}, \text{ marked green, and adjacent to the new vertex } v_{i-1, v_{i-1}}, \text{ pertaining to } v_{i_1, v_{i_2}}; S_r(v_{i_1, v_{i_2}}) \leftarrow v_{ij, v_{ij}}, P_r(v_{i_1, v_{i_2}}) \leftarrow v_{1_{i_1}, v_{1_{i_2}}}, l_r(v_{i_1, v_{i_2}}) \leftarrow i, i \leftarrow i + 1, \\
&\quad \text{if } i = rad(G), \text{ then we partition them into } N_j(N_i(v_r)), 1 \leq j \leq |N_i(v_r)|, \text{ for each } j, \text{ the vertices of } N_j(N_i(v_r)) \text{ forming all the path edges of a Fan structure, we know their common neighbor } v_{i_1}, 1 \leq s \leq n, \text{ is the central vertex of it according to the step 2, then } v_s \text{ is a vertex of } \text{CCS}(G) \text{ and marked red, which is adjacent to a vertex generated by the vertices in } N_{i-2}(v_r); \\
&\quad \text{If there are central vertices whose subscripts adjacent, then we shrink the corresponding edge to be a green vertex connected a vertex generated by the pertinent vertices in } N_{i-2}(v_r); \text{ The produced new vertex is adjacent to the pertinent vertex generated by the corresponding vertices in } N_{rad(G) - 2}. \\
&\end{align*}
\]

The time complexity of CCS-Algorithm computing the CCS(G) is \( O(n^3) \), where \( n = |V(G)| \), for a MOP G.
4 Polynomial time algorithm for rainbow connection number of MOP

Our main results in this section is to give a polynomial time algorithm computing rainbow coloring of MOPs with at most $3\text{rad}(G)$ colors. Specially, we can give $2\text{rad}(G) + 3$ rainbow coloring for some MOPs by our algorithm.

In order to give Algorithm 3, we first give some notations for $\text{CCS}(G)$. Since $\text{CCS}(G)$ is a $v_r$ root tree and any two vertices are connected by exactly one path. Assuming that the $\text{CCS}(G)$ has $m$ leaves. Therefore there are $m$ paths between $v_r$ and it’s leaves, denoted by $P_k$, $1 \leq k \leq m$. The length $L(P_k) =$: the number of edges on $P_k$. Let $P_1$ be the minimum length path of all the $m$ paths.

Given a MOP $G$, algorithm 2 gives the $\text{CCS}(G)$. We perform the following operations: First, giving the significant Fan-structures, whose central vertices are corresponding to the vertices of $\text{CCS}(G)$, of $G$, pertinent to $\text{CCS}(G)$ : (1) we choose $v_r$ and it’s neighbors as the first Fan-structure, denoted by $F_{v_r}$. If a vertex of $\text{CCS}(G)$ is red, then it is a vertex of $G$ and forms up a Fan-structure as a central vertex with it’s neighbors; If a vertex $v_{i_1}v_{i_2}$ green, then we choose the vertex corresponding to the $\text{max}(|v_{i_1}|, |v_{i_2}|)$ as central vertices of $G$, which form Fan-structures with it’s neighbors. If there are neighbors in $N(\text{max}\{\text{degree}(v_{i_1}), \text{degree}(v_{i_2})\}) \setminus N(\text{min}\{\text{degree}(v_{i_1}), \text{degree}(v_{i_2})\})$, then we choose $v_{i_1}, v_{i_2}$ as the central vertices of Fan-structures in $G$. Second, giving a rainbow coloring using at most $3\text{rad}(G)$ colors.

Algorithm 3 giving rainbow coloring of MOP:

Input: A MOP $G$.
Output: A rainbow coloring of $G$.
Step 1: Giving the $\text{CCS}(G)$ of $G$ by Algorithm 2.
Step 2: Give three color sets: $C_1 = \{7, 8, \cdots, 2\text{rad}(G) + 4\}$ and $C_2 = \{\text{rad}(G) + 5, \text{rad}(G) + 7, \cdots 2\text{rad}(G) + 2\}$, $C_3 = \{2\text{rad}(G) + 3, \cdots, 3\text{rad}(G)\}$.
Step 3: We produce the corresponding two path for every $P_k$, $1 \leq k \leq m$ by the procedures of Theorem 3.4. Now we color the edges of the shorter paths by $C_1$ and the other by the the unused minimum colors in $C_2$ and $C_3$.
Step 4: In this step, we give a $2\text{rad}(G) + 2 + c$, where $c \leq \text{rad}(G) - 2$, rainbow coloring of $G$.
(1) Choose $r$ and it’s neighbors as the first Fan, denoted by $F_{v_r}$. Whose spoke edges are colored by colors 4, 5 alternatively and according to the clockwise around the central vertex and uncolored path edges are colored by 6.
(2)begin
for $k = 1$ up to $m$ do
if $v_c \in V(P_k)$ and is a red vertex of $\text{CCS}(G)$, then it is a vertex of $G$ and forms up a Fan-structure as a central vertex with it’s neighbors. According to the step 2 of Algorithm 2, we know the Fan-structure. Whose uncolored spoke edges are colored by colors 1, 2 alternatively and according to the clockwise around the central vertex and uncolored path edges are colored by 3;
if $v_c \in V(P_k)$ and is a green vertex $v_{i_1}v_{i_2}$, first choose the vertex corresponding to the $\text{max}\{\text{degree}(v_{i_1}), \text{degree}(v_{i_2})\}$ and it’s neighbors which forming a Fan-structure in $G$, second, if there are neighbors in $N(\text{min}\{\text{degree}(v_{i_1}), \text{degree}(v_{i_2})\}) \setminus N(\text{max}\{\text{degree}(v_{i_1}), \text{degree}(v_{i_2})\})$ then we choose $v_{i_1}, v_{i_2}$ as the central vertices of Fan-structures in $G$. According to the step 2 of Algorithm 2, we know the Fan-structures. They are colored with 1, 2, 3 by the method as above shown.
end

Theorem 4.1. If $G$ is a MOP, then the edge coloring given by algorithm 3 is a rainbow coloring of $G$. Moreover it uses colors at most $3\text{rad}(G)$.

Proof.
Case 1: For any two vertices, which locate in a Fan-structure whose central vertex corresponding the vertex of $\text{CCS}(G)$. Because it’s spoke edges are colored by colors 1, 2 alternatively and according to the clockwise around the central vertex and path edges are colored by 3, other colors, if exist, belong to $C_1, C_2$ or $C_3$. Obviously, there is a rainbow path between them.
If two vertices locate in different Fan-structures whose central vertices corresponding two vertices, which are situated on a $v_r$-path, of CCS($G$). Then there is a rainbow path, which use colors of $C_1, C_2$ or $C_3$ connecting their central vertices in $G$. Moreover the two Fan-structures are colored with the above method, therefore the two vertices can be rainbow connected under the coloring given by Algorithm 3.  

**Case 2:** We know that $v_r$ with it’s neighbors forming a Fan-structure, whose spoke edges are colored by 4, 5 alternatively and according to the clockwise around it and uncolored path edges are colored by 6. Obviously any two vertices of the Fan structure are rainbow connected under the coloring.

Any two vertices whose central vertices are saturated on different $v_r$-paths are rainbow connected through Fan$_{v_r}$. Since there are two edge disjoint rainbow paths connecting the central vertices of the corresponding Fan-structures and the Fan Fan$_{v_r}$.

For any longer path of Step 3, since their first edges are colored by 4 or 5 and other edges no more than $2(rad(G) − 2)$, which can be rainbow colored by the colors of $C_2$ and $C_3$. □

5 Examples of rainbow connection algorithms for MOPs

X.Deng et al. [5] gave a polynomial time algorithm to compute the rainbow connection number of MOPs, but only obtain an compact upper bound, by the Maximal fan partition method. By the method, they proved that $rc(Figure\ 13) \leq 6$.

![Figure 13: An example](image)

Obviously, we know that the vertex $v$ is the central vertex of Figure 13 and the red and green vertices and red edges are CCS(Figure 13), then we know that $rc(Figure\ 13) \leq rc(CCS(Figure\ 13)) + 3 = 5$ by the coloring of Algorithm 3.

Figure 11 is a MOP with diameter 3, where the Maximal fan structures having 9 vertices and the red vertices and edges give the CCS(Figure 11). When the Fan structures are replaced by any $F_n$ with $n \geq 9$, the resulting graphs can be rainbow colored by 4 colors with the same method as Figure 11 showing. If we color the graph by the Maximal fan partition method, then the colors needed increase to 9 with the rising of $n$.

**Definition 5.1** [13]. (Two-way dominating set). A dominating set $D$ in a graph $G$ is called a two-way dominating set, if every pendant vertex of $G$ is included in $D$. In addition, if $G[D]$ is connected, we call $D$ a connected two-way dominating set.

**Theorem 5.2** [13]. If $D$ is a connected two-way dominating set in a graph $G$, then $rc(G) \leq rc(G[D]) + 3$.

By the two-way dominating set and induction on radius, they proved the following result:

**Theorem 5.3** [13]. If $G$ is a bridge-less chordal graph, then $rc(G) \leq 3rad(G)$. Moreover, there exists a bridge-less chordal graph with $rc(G) = 3rad(G)$.

For Figure 13, the minimum connected two-way dominating set is a 4-length path, thus $rc(Figure\ 13) \leq 3rad(Figure\ 13) = 6$ by the theorem 5.3.

Another example Figure 9, we know that Figure 9 is a MOP, obviously a chordal graph, and has a hamiltonian cycle with 40 vertices. By Theorem 1.1, we know that $rc(Figure\ 9) \leq 12$. It is smaller than
it’s $3\text{rad}(\text{Figure} 9) = 15$ and bigger 2 than it’s diameter.

If a MOP $G$ is $\text{Fan}_n, n \geq 7$, then $\text{rad}(G) = 1, \text{rc}(G) = 3\text{rad}(G) = 3$. Therefor the upper bound of the algorithm given is sharp for MOPs, so is the Theorem 5.3. But above all, the bound given by our algorithm is better to the theorem 5.3 obtained for some MOPs. For example, Figure 9 showing.

6 Concluding remarks

Recall algorithm 1, we know that the MOPs $Lad_d$ and $Lad_{d+}$, with diameter $d$, have rainbow connection number $d$. Producing the graphs and giving their strong rainbow connection numbers take time at most $O(d)$.

In algorithm 2: The Hamiltonian cycle $C_G$ of MOP $G$ can be obtained by a linear time algorithm presented in [24] through the canonical representation of $G$. Then select any vertex of $G$ as the initial vertex of $C_G$, we can obtain a Hamiltonian degree sequence in time $O(n^3)$, where $n = n(G)$. Note the time of Step 2 is $O(n + e)$, where $e = e(G)$ and $e(G) = 2n - 3$, which is $O(n)$, Step 3 takes time at most $O(n^3)$ and Step 4 at most $O(n)$. For MOP, Step 5 takes time $O(n + e)$, which is $O(n)$. Step 6 at most $O(n^2)$.

For algorithm 3: The time of Step 3 is no more than $O(n)$, Step 4 uses time at most $O(n^2)$. If $G$ is a MOP, the algorithm gives a tight upper bound of $\text{rc}(G)$ which is no more than $2\text{rad}(G) + 2 + c$, where $0 \leq c \leq \text{rad}(G) - 2$ and needs time at most $O(n^3)$.

In the future, we will consider the rainbow connection numbers for outerplanar and general planar graphs on algorithm aspect. It is interesting to study the rainbow connection for planar graphs on algorithm aspect.

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