A non-associative quaternion scalar field theory

Sergio Giardino

Instituto de Matemática, Estatística e Computação Científica, Universidade Estadual de Campinas
Rua Sérgio Buarque de Holanda 651, 13083-859, Campinas, SP, Brazil

Paulo Teotônio-Sobrinho

Instituto de Física, Universidade de São Paulo,
CP 66318, 05315-970 São Paulo, SP, Brazil.

A non-associative Groenewold-Moyal plane is constructed using quaternion-valued function algebras. The symmetrized multi-particle states, the scalar product, the annihilation/creation algebra and the formulation in terms of a Hopf algebra are also developed. Non-associative quantum algebras in terms of position and momentum operators are given as the simplest examples of a framework whose applications may involve string theory and non-linear quantum field theory.

PACS numbers:

I. INTRODUCTION

Non-commutative geometry [1] has a wide range of applications in quantum field theory [2, 3], in the construction of non-commutative physical models. These non-commutative theories are associative. A more general framework could be conceived whereby, in addition to non-commutativity, the algebra is also non-associative. Our aim is to find an example where non-commutative and non-associative algebra appears naturally in the context of field theory. Since most field theories are based on associative algebra, our aim is to obtain a deformation parameter \( \theta \) such that associativity is recovered when \( \theta \) goes to zero.

In the following pages, this goal is achieved by means of construction: we start with a field theory where the base space is comprised of \( \mathbb{R}^D \) and target space is comprised of quaternions \( \mathbb{H} \). The second step is to deform \( \mathbb{R}^D \) into non-commutative algebra such that \([x^\mu, x^\nu] = i \theta^{\nu\mu}\). It turns out that the resulting algebra of fields is non-associative. As expected, when \( \theta^{\mu\nu} \) goes to zero, associativity is recovered.

Let us consider a quaternion-valued field theory, and write the field \( \mathcal{F} : \mathbb{R}^D \to \mathbb{H} \) in a symplectic notation as \( \mathcal{F} = f_0 + j f_1 \), so that \( f_{i=0,1} : \mathbb{R}^D \to \mathbb{C} \). In this theory, the sources of non-commutativity are the quaternion complex units \( i, j \) and \( k = ij = -ji \). By deforming the commutative multiplication of the complex-valued functions \( f_{i=0,1} \) to a non-commutative, we obtain a theory with non-associativity as a byproduct of the superposition of the these two different non-commutativities.

Non-associative phenomena appear in many places, and further information can be found in reviews on the subject [4, 5]. However, while non-associativity is common place in algebra [6], examples of non-associativity in physics are a collection of disconnected problems. The most obvious proposals for finding a physical phenomenon that may be described by non-associativity involve the octonion field [7–11]. Although octonion algebra is a standard example of non-associativity, it does not have an associative limit. Recently, non-associative structures have appeared in general relativity, [13–17], string theory [18–20] and brane theory [21–26]. The model proposed in this article is an attempt to obtain a very simple example of non-associativity where associativity can be recovered at a suitable limit.

The field theory described in this article has a natural interpretation since its target space may be understood as the tangent space of a hyper-complex manifold. In the same way a complex manifold is locally complex, a hyper-complex manifold is locally quaternionic. In the context of super-symmetric models, there can be various types of complex and hyper-complex manifolds as found in super-symmetric extensions of non-linear sigma models [27–31], string compactification on \( K3 \) surfaces [32], generalized hyper-Kähler applied to string theory [33, 34], and even speculations on the nature of time [35]. Therefore, the model presented here can be understood as a linearized version of such non-linear sigma models.

Our results are related to quaternion quantum mechanics and quantum field theory [35–41]. However, these latter models do not consider multi-particle states, and consequently in these theories it is impossible to build states with

---

*Electronic address: giardino@ime.unicamp.br
†Electronic address: teotonio@fma.if.usp.br
particle statistics, a problem that has been solved here by defining annihilation and creation operators of symmetrized states.

This paper is organized as follows: in the second section we present the non-deformed quaternion scalar field theory and its multi-particle states and statistics. In the third section a deformed algebra of functions is formulated according to the Groenewold-Moyal procedure. We then show that the resulting algebra is non-associative. Examples of non-associative quantum algebra obtained from introducing quaternion unity are presented as well. The last section contains our conclusions and future perspectives.

II. THE QUATERNION SCALAR FIELD THEORY

The afore mentioned quaternion field theories have only one-particle states. This means that multi-particle states cannot be built according to boson-fermion statistics. In this section this void is filled in the mathematical structure of quaternion field theory following the Hopf algebra formalism of [42], where the Poincaré group acts on the Groenewold-Moyal plane with a deformed coproduct. In this section the deformation is the hyper-complex quaternion structure. A second deformation, in the usual multiplication, is introduced in the third section.

Poincaré invariance

If $g$ is an element belonging to the Poincaré algebra, the action of the symmetry algebra ($\triangleright$) on space-time functions $\mathcal{F}, \mathcal{G} \in \mathbb{H}$ must obey

$$g \triangleright (\mathcal{F} \cdot \mathcal{G}) = (g \triangleright \mathcal{F}) \cdot (g \triangleright \mathcal{G}),$$

where the dot represents ordinary multiplication. It is adopted the symplectic notation for quaternionic functions, so that $\mathcal{F} = f_0 + f_1j$, with $f_i \in \mathbb{C}$–functions, and $j$ is the complex element of quaternion algebra, and thus $ij = -ji$.

In terms of Hopf algebras, the action of the elements of an algebra over a product of complex functions is determined by the co-product. By way of example, the translation generator $\hat{p} = i\partial_x$ of the Poincaré group acts on complex function algebra according to the co-product

$$\Delta(\hat{p}) = 1 \otimes \hat{p} + \hat{p} \otimes 1,$$

which, acting on $f, g \in \mathbb{C}$ with multiplication $m$, is subject to the consistency constraint $m(\Delta(\hat{p})(f \otimes g)) = \hat{p}(f \cdot g)$, where $m$ takes the elements of the tensor product and multiplies them. On the other hand, taking the quaternion functions $fj$ and $gj$, again with $f, g \in \mathbb{C}$–functions, we obtain $m(\Delta(\hat{p})(f \otimes g)) \neq p(f \cdot g)$. This difficulty is solved by defining a quaternion tensor product, namely

$$(f \otimes g) \cdot (m \otimes n) = (f \cdot m) \otimes (g \cdot n),$$

$$(f \otimes gj) \cdot (mj \otimes n) = (f \cdot mj) \otimes (gj \cdot n),$$

$$(f \otimes g) \cdot (mj \otimes n) = (f \cdot mj) \otimes (gj \cdot n).$$

$f, g, m$ and $n$ are complex-valued functions and barred functions are the complex conjugates. This result follows for $f, n \in \mathbb{H}$ as well. This kind of structure is similar to the $\mathbb{Z}_2$ tensor product found in Lie super-algebras [43]. Adopting this tensor product the coproduct satisfies the identity $\Delta(\hat{p} \hat{q}) = \Delta(\hat{p})\Delta(\hat{q})$, and the first element of a multi-particle quaternionic state is given: a well defined co-product. As the co-product of the translation operator of the Poincaré algebra has the expression (2), either in the quaternion case or in the complex case, it will have the same behavior when the multiplication operation is deformed according to the Moyal procedure in both cases. Thus the deformed coproduct of the rotation operator of the Poincaré group in the non-commutative complex function algebra [44] is valid for the deformed quaternion spaces discussed in the next section as well.

State statistics

States endowed with well-defined statistics have a permutation operator which interchanges the positions of the functions describing the particles in a state. As the particles are represented by quaternion functions, for a generic quaternion state

$$\mathcal{F} \otimes \mathcal{G} = f_0 \otimes g_0 + f_0 \otimes g_1j + f_1j \otimes g_0 + f_1j \otimes g_1j,$$
the following operators are defined:

\[
\hat{s} \triangleright (\mathcal{F} \otimes \mathcal{G}) = (\mathcal{G} \otimes \mathcal{F})
\]
\[
\hat{r} \triangleright (\mathcal{F} \otimes \mathcal{G}) = g_0 \otimes f_0 + g_1 \otimes \tilde{f}_0 + \bar{g}_0 \otimes f_1 j + \bar{g}_1 j \otimes \tilde{f}_1 j
\]
\[
\hat{\kappa} \triangleright (\mathcal{F} \otimes \mathcal{G}) = f_0 \otimes g_0 + \tilde{f}_0 \otimes g_1 j + f_1 j \otimes \tilde{g}_0 + \tilde{f}_1 j \otimes \tilde{g}_1 j,
\]

so that \( \hat{s}^2 = \tilde{\hat{s}}^2 = 1 \) and \( \hat{\kappa} = \tilde{\hat{\kappa}} \). Symmetrized states and anti-symmetrized states are defined as follows

\[
|\mathcal{F}_1, \mathcal{F}_2\rangle_{\pm} = \frac{1}{2}(1 \pm \hat{\tau}) \triangleright |\mathcal{F}_1, \mathcal{F}_2\rangle,
\]

with \( |\mathcal{F}_1, \mathcal{F}_2\rangle = \mathcal{F}_1 \otimes \mathcal{F}_2 \), the defined states are eigen-states of the permutation operator \( \hat{\tau} \) according to

\[
\hat{\tau}|\mathcal{F}_1, \mathcal{F}_2\rangle_{\pm} = \pm |\mathcal{F}_1, \mathcal{F}_2\rangle_{\pm}.
\]

After defining symmetrized states and anti-symmetrized states, the particle statistics is guaranteed, and a scalar product is needed, which is presented below.

The scalar product

By expressing a one-particle state as \( |\mathcal{F}\rangle = |f_0\rangle + |f_{1j}\rangle \), so that the orthogonality condition \( \langle f | g \rangle = \langle f j | g \rangle = 0 \) holds, a complex-valued scalar product is obtained as a sum of usual scalar products of complex functions

\[
\langle \mathcal{F}, \mathcal{G} \rangle = \langle f_0 | g_0 \rangle + \langle f_{1j} | g_{1j} \rangle = \langle f_0 | g_0 \rangle + \langle g_1 | f_1 \rangle.
\]

In the above scalar product, \( \langle z \mathcal{F} | \mathcal{G} \rangle \neq \langle \mathcal{F} | z \mathcal{G} \rangle \), where \( z \in \mathbb{C} \) and \( \mathcal{F}, \mathcal{G} : \mathbb{R}^4 \to \mathbb{H} \). As a consequence, this fact will result in the splitting of the creation/annihilation operator algebra, as shown in the next item. On the other hand, when defining the scalar product of two-particle states as

\[
\langle \mathcal{F}, \mathcal{G} | \mathcal{M}, \mathcal{N} \rangle = (|\mathcal{F}, \mathcal{G}\rangle, \hat{\kappa}| \mathcal{M}, \mathcal{N}\rangle)
\]
\[
= \langle f_0 | m_0 \rangle \langle g_0 | n_0 \rangle + \langle f_{1j} | m_1 j \rangle \langle g_{1j} | n_{1j} \rangle,
\]

it is observed that if \( \mathcal{F} = \mathcal{M} \) and \( \mathcal{G} = \mathcal{N} \), then

\[
|\mathcal{F} \otimes \mathcal{G}|^2 = |\mathcal{F}|^2 |\mathcal{G}|^2.
\]

The scalar product also obeys the necessary self-adjointness condition

\[
(\hat{\tau}|\mathcal{F}, \mathcal{G}, | \mathcal{M}, \mathcal{N}\rangle = (|\mathcal{F}, \mathcal{G}\rangle, \hat{\tau}| \mathcal{M}, \mathcal{N}\rangle).
\]

Thus, the scalar product constructed above is valid for multi-particles, something which has not been observed in previous quaternion quantum theories.

Creation and annihilation operators

In principle, the creation \( a_{\mathcal{F}}^\dagger \) operator and annihilation operator \( a_{\mathcal{F}} \) of a quaternionic state are

\[
a_{\mathcal{F}}^\dagger = a_{f_0}^\dagger + a_{f_{1j}}^\dagger \quad \text{and} \quad a_{\mathcal{F}} = a_{f_0} + a_{f_{1j}}.
\]

However, as the scalar product constructed above is such that \( \langle z \mathcal{F} | \mathcal{G} \rangle \neq \langle \mathcal{F} | z \mathcal{G} \rangle \), where \( z \in \mathbb{C} \) and \( \mathcal{F}, \mathcal{G} : \mathbb{R}^4 \to \mathbb{H} \), the creation/annihilation algebra will be built in terms of \( a_{f_0}^\dagger, a_{f_{1j}}^\dagger, a_{f_0}, \) and \( a_{f_{1j}} \). These operators create complex fields, and thus satisfy commutation rules with the quaternionic unity \( j \), namely

\[
za_{f_0} = a_{f_0} z, \quad ja_{f_0} = a_{f_0} j, \quad za_{f_{1j}} = a_{f_{1j}} \bar{z} \quad \text{and} \quad ja_{f_{1j}} = a_{f_{1j}} \bar{j}.
\]
The operator creates/annihilates either a bosonic or a fermionic state, thus the wave-function must be either symmetrized or anti-symmetrized. In order to construct the algebra, the scalar product must have the same result as that obtained by the creation annihilation operators. The necessary scalar products are
\[\pm \langle f \otimes g, m \otimes n \rangle = \langle f, m \rangle \langle g, n \rangle \pm \langle f, n \rangle \langle g, m \rangle\] (20)
\[\pm \langle f \otimes gj, m \otimes nj \rangle = \langle f, mj \rangle \langle gj, nj \rangle \pm \langle f, nj \rangle \langle gj, mj \rangle\] (21)
\[\pm \langle fj \otimes gj, mj \otimes nj \rangle = \langle fj, mj \rangle \langle gj, nj \rangle \pm \langle fj, nj \rangle \langle gj, mj \rangle\] (22)
so that the plus sign corresponds to the symmetric bosonic states and the minus sign corresponds to the fermionic anti-symmetric states. For the bosonic case, the operator algebra reproduces the above results as
\[\{ a_f, a_g \} = [ a_f^\dagger, a_g^\dagger ] = 0 \] (23)
\[a_f a_gj - a_gj a_f = a_f^\dagger a_g^\dagger j - a_g^\dagger a_f^\dagger = 0 \] (24)
\[a_fj a_gj - a_gj a_fj = a_fj^\dagger a_gj^\dagger - a_gj^\dagger a_fj^\dagger = 0 \] (25)
\[a_fj a_gj^\dagger - a_gj^\dagger a_fj = 0 \] (26)
\[[ a_f, a_g ] = \langle f \mid g \rangle \] (27)
\[a_fj a_gj^\dagger - a_gj^\dagger a_fj = \langle fj \mid gj \rangle. \] (28)
remembering that f and g are complex-valued functions, and that \(\langle a, z b \rangle = z \langle a, b \rangle\) and \(\langle a, z b \rangle = \bar{z} \langle a, j b \rangle\) are adopted. On the other hand, for an anti-symmetric fermionic state, the operator algebra is
\[\{ a_f, a_g \} = \{ a_f^\dagger, a_g^\dagger \} = 0 \] (29)
\[a_f a_gj + a_gj a_f = a_f^\dagger a_g^\dagger j + a_g^\dagger a_f^\dagger = 0 \] (30)
\[a_fj a_gj + a_gj a_fj = a_fj^\dagger a_gj^\dagger + a_gj^\dagger a_fj^\dagger = 0 \] (31)
\[a_fj a_gj^\dagger + a_gj^\dagger a_fj = 0 \] (32)
\[\{ a_f, a_g^\dagger \} = \langle f \mid g \rangle \] (33)
\[a_fj a_gj^\dagger + a_gj^\dagger a_fj = \langle fj \mid gj \rangle. \] (34)
Thus the constructed quaternionic scalar field theory has all the structure necessary: Poincaré invariant one-particle and multi-particle states, symmetrized and anti-symmetrized states with well-defined statistics, a scalar product and a creation/annihilation operator algebra. This theory can be deformed according to the Groenewold-Moyal procedure generalizing the well-known non-commutative complex field theories, and this is carried out in the next section.

### III. THE DEFORMED PRODUCT

Non-commutative geometry is obtained by changing the ordinary commutative product of complex-valued functions f and g into the Groenewold-Moyal (GM) deformed product
\[f(x) \star g(x) = f(x)g(x) + \sum_{n=1}^{\infty} \left( \frac{i}{2} \right) \frac{1}{n} \theta^{i_1j_1} \ldots \theta^{i_nj_n} \partial_{i_1} \ldots \partial_{i_n} f(x) \partial_{j_1} \ldots \partial_{j_n} g(x), \] (35)
so that \(\theta^{ij}\) is anti-symmetric in its indices. Linear functions generate the commutator between coordinates
\[x^i \star x^j - x^j \star x^i = \theta^{ij}, \] (36)
and in the limit where \(\theta^{ij} \to 0\) the commutative geometry is recovered. Both the commutative product and the non-commutative product of functions are associative.

A more general picture may be obtained by deforming the usual product of quaternion-valued functions according to the GM prescription. The quaternionic-valued functions \(\mathcal{F} : \mathbb{R}^D \to \mathbb{H}\) over a D-dimensional Euclidean space with coordinates \(x^i\) are represented by
\[\mathcal{F} = f_1 + f_2 j, \quad \text{so that} \quad f_{a=1,2} : \mathbb{R}^D \to \mathbb{C}. \] (37)
$f_a$ are defined on a Schwarz space, thus allowing a Fourier transform $\tilde{F}$, where

$$\tilde{F} = \tilde{f}_1 + \tilde{f}_2 j,$$

and

$$\tilde{f}_a(k) = \int d^Dx e^{-ik\cdot x} f_a(x).$$

(38)

Accordingly, the Weyl symbol of a quaternion function may be introduced as well, so that

$$\hat{W}[\mathcal{F}] = \hat{W}[f_1] + \hat{W}[f_2]j,$$

and

$$\hat{W}[f_a] = \frac{d^Dk}{(2\pi)^D} e^{-ik\cdot x} \hat{f}_a.$$  

(39)

The Weyl symbol allows the GM product to be introduced, thus replacing the usual multiplication, so that the complex-valued functions obey $\hat{W}[f_a \ast f_b] = \hat{W}[f_a] \ast \hat{W}[f_b]$, which results in

$$\mathcal{F} \ast \mathcal{G} = (f_1 \ast g_1) + (f_1 \ast g_2)j + j(f_2 \ast g_1) + j((f_2 \ast g_2)j),$$

(40)

where the bar means the conjugate of the complex function. This star product of quaternion functions is, of course, non-commutative; nevertheless, it is also non associative, so that

$$(\mathcal{F} \ast \mathcal{G}) \ast \mathcal{H} - \mathcal{F} \ast (\mathcal{G} \ast \mathcal{H}) \neq 0.$$  

(41)

This is an interesting byproduct for introducing a non-commutative local structure on a former non-commutative complex structure. This simple theory has a number of possible applications, as cited in the introduction of this paper.

**Non-associative quaternion quantum algebras**

The simplest example of a non-associative deformed theory comes from quantum mechanics and its celebrated commutation relation

$$[\hat{x}, \hat{p}] = i\hbar,$$

(42)

whose $\hbar \to 0$ limit, or classical limit, turns the operators into a commutative algebra. Introducing the quaternion complex unity $j$ naturally generates a non-associative structure. As $j$ does not commute with $[\hat{x}, \hat{p}]$, it does not associate with the products of the commutator anymore. The associator $(\hat{x}, \ hat{p}) = (\hat{x} j) \hat{p} - \hat{x} (j \hat{p})$ may be calculated in the specific case where the quantum quaternion algebra is an alternative algebra. Using the Moufang identities [6], the resulting associator is

$$(\hat{x}, \hat{p}) = kj\hbar.$$  

(43)

so that $k = ij$. This example in which quantum mechanics turns to a non-associative theory is somewhat surprising, but it shows very simply how combining non-commutative structures generates a non-associative one. In this case, the associative limit goes to a commutative complex theory, but this is a classical one. In this sense, the commutativity and associativity are coupled. A non-coupled case comes in the more general framework discussed previously.

On the other hand, it is possible to further extend the quantum algebra. Defining the operators

$$\hat{z}^\dagger = \frac{1}{\sqrt{2}}(p + ix)$$

and

$$\hat{z} = \frac{1}{\sqrt{2}}(p - ix),$$

(44)

so that $[\hat{z}, \hat{z}^\dagger] = \hbar$, and with the use of the associator (43),

$$(\hat{z}, j, \hat{z}^\dagger) = (\hat{z}^\dagger, j, \hat{z}) = (\hat{z}, \hat{z}, j) = (j, \hat{z}, \hat{z}) = (\hat{z}^\dagger, \hat{z}, j) = (j, \hat{z}^\dagger, \hat{z}) = 0$$

(45)

$$(\hat{z}^\dagger, \hat{z}, j) = - (j, \hat{z}^\dagger, \hat{z}) = -(\hat{z}, \hat{z}^\dagger, j) = (j, \hat{z}, \hat{z}^\dagger) = - (\hat{z}, \hat{z}, j) = (\hat{z}^\dagger, j, \hat{z}^\dagger) = j.$$  

(46)

This is also a non-associative and non-commutative algebra, although it is not alternative as that formed by $\hat{x}, \hat{p}$ and $j$, but its classical limit is also a quaternion classical quaternion theory as expected. The above examples are the simplest examples of the deformed algebras, whose geometry is to be analyzed in forthcoming studies.
IV. CONCLUSION

In this article two novel quaternion quantum scalar field theories have been presented. Both of them are non-commutative because of the quaternion nature of their fields. In one of them ordinary commutative multiplication is defined, and in this case a multi-particle quaternion scalar field theory has been constructed. The second theory is a deformation of the former one according to the Groenenwold-Moyal procedure. This second theory is a non-commutative and non-associative one. These theories are well-defined, and may be used in a number of physical applications, as the models are quite general. Developments in quaternion scalar fields and non-associative geometry are the most immediate applications. We expect that results derived from this linear model will be useful when applied to hyper-Kähler structures in string theory and non-linear sigma models.

Acknowledgements Sergio Giardino is grateful for the support offered by the Departamento de Física Matemática of the Universidade de São Paulo and also for the financial support provided by Capes. Both of the authors thank B. Chandrasekhar, A. P. Balachandran and T. R. Govindarajan for invaluable discussions.

[1] Alain Connes. “Noncommutative geometry.” Academic Press (1994).
[2] E. Akofor. “Quantum Theory, Noncommutativity and Heuristics”. (2010) arXiv:1012.5133 [hep-th].
[3] R. J. Szabo. “Quantum field theory on noncommutative spaces”. Phys.Rep., 378:207–299, (2003) hep-th/0109162.
[4] S. Okubo. “Introduction to Octonion and Other Non-Associative Algebras in Physics”. Cambridge University Press (2005).
[5] J. Lohmus; E. Paal; L.Sorgsepp. “About Nonassociativity algebras in mathematics and physics”. Acta Math. Phys., 50:3–31, (1998).
[6] R. Schafer. “An Introduction to Nonassociative Algebras”. Dover (1996).
[7] T. Kugo and P. K. Townsend. “Supersymmetry and the Division Algebras”. Nucl. Phys., B221:357, (1983).
[8] R. Foot and G. C. Joshi. “Nonstandard signature of space-time, superstrings and the split composition algebras”. Lett. Math. Phys., 19:65, (1990).
[9] M. Gunaydin; S. V. Ketov. “Seven sphere and the exceptional N=7 and N=8 superconformal algebras”. Nucl.Phys., B467:215–246, (1996) hep-th/9601072.
[10] J. C. Baez. “The octonions”. Bull. Am. Math. Soc., 39:145–205, (2002) math/0105155.
[11] M. Gogberashvili “Octonionic version of Dirac equations”. Int. J. Mod. Phys., A21:3513-3524, (2006) hep-th/0505101.
[12] V. Dzhunushaliev “Non-associativity, supersymmetry and 'hidden variables’”. J. Math. Phys., 49:042108, (2008) arXiv:0712.1647 [quant-ph].
[13] A. I. Nesterov; L. V. Sabinin. “Nonassociative geometry and discrete structure of space-time”. Comment. Math. Univ. Carolinae, (1999) hep-th/0003238.
[14] A. I. Nesterov; L. V. Sabinin. “Nonassociative geometry: Towards discrete structure of space-time”. Phys.Rev., D62:081501, (2000) hep-th/0010159.
[15] A. I. Nesterov; L. V. Sabinin. “Nonassociative geometry: Friedman-Robertson-Walker spacetime”. Int. J. Geom. Meth. Mod. Phys., 3:1481–1492, (2006) hep-th/0406229.
[16] L. Sbitneva. “Nonassociative geometry of special relativity”. Int. J. Theor. Phys., 40:359–362, (2001).
[17] A. I. Nesterov. “Principal loop bundles: Toward nonassociative gauge theories”. Int. J. Theor. Phys., 40:339–350, (2001).
[18] R. Blumenhagen; E. Plauschinn. “Nonassociative Gravity in String Theory?”. J. Phys., A44:015401, (2011) arXiv:1010.1263 [hep-th].
[19] F. R. Blumenhagen; A. Deser; D. Lust; E. Plauschinn; F. Rennecke. “Non-geometric Fluxes, Asymmetric Strings and Nonassociative Geometry”. J.Phys., A44:385401, (2011) arXiv:1106.0316.
[20] D. Mylonas; P. Schupp; R. J. Szabo. “Membrane Sigma-Models and Quantization of Non-Geometric Flux Backgrounds”. JHEP, 1209:012, (2012) arXiv:1207.0926 [hep-th].
[21] J. Bagger and N. Lambert. “Modeling multiple M2’s”. Phys. Rev., D75:045020, (2007) [hep-th/0611108].
[22] J. Bagger and N. Lambert. “Gauge Symmetry and Supersymmetry of Multiple M2-Branes”. Phys. Rev., D77:065008, (2008) arXiv:0711.0955[hep-th].
[23] J. Bagger and N. Lambert. “3−algebras and N = 6 Chern-Simons gauge theories”. Phys. Rev., D79:025002, (2009) arXiv:0807.0163[hep-th].
[24] A. Gustavsson. “Algebraic structures on parallel M2−branes”. Nucl. Phys., B811:66–76, (2009) arXiv:0709.1260[hep-th].
[25] M. Yamazaki. “Octonions, G2 and generalized Lie 3−algebras”. Phys. Lett., B670:215–219, (2008) arXiv:0809.1650[hep-th].
[26] S. Giardino; H. L. Carrion. “Lie 3−algebra and super-affinization of split-octonions”. Mod. Phys. Lett., A26:2663–2675, (2011) arXiv:1004.4228[math-ph].
[27] B. Zumino. “Supersymmetry and Kahler Manifolds”. Phys. Lett., B87:203, (1979).
[28] L. Alvarez-Gaume; D. Z. Freedman. “Ricci Flat Kahler Manifolds And Supergeometry”. Phys. Lett., B94:171, (1980).
[29] L. Alvarez-Gaume; D. Z. Freedman. “Kahler Geometry and the Renormalization of Supersymmetric Sigma Models”. Phys. Rev., D22:846, (1980).
[30] L. Alvarez-Gaume; D. Z. Freedman. “Geometrical Structure and Ultraviolet Finiteness in the Supersymmetric Sigma
Model". Commun. Math. Phys., 80:443, (1981).
[31] M. Rocek. “Supersymmetry and non-linear sigma models”. Physica, 15D:75–82, (1985).
[32] P. S. Aspinwall. “K3 surfaces and string duality”. (1996) hep-th/9611137.
[33] M. Zabzine. “Lectures on Generalized Complex Geometry and Supersymmetry”. Archivum Math., 42:119–146, (2006) hep-th/0605148.
[34] B. Ezhuthachan; D. Ghoshal. “Generalised hyperKahler manifolds in string theory”. JHEP, 0704:083, (2007) hep-th/0608132.
[35] G. W. Gibbons. “The emergent nature of time and the complex numbers in quantum cosmology”. (2011) arXiv:1111.0457 [gr-qc].
[36] P. Rotelli. “The Dirac equation on the quaternion field”. Mod. Phys. Lett., A4:933, (1989).
[37] S. L. Adler. “Quaternionic Quantum Mechanics and Quantum Fields”. Oxford University Press (1995).
[38] S. L. Adler. “Quaternionic quantum field theory”. Commun. Math. Phys., 104:611, 1986.
[39] S. L. Adler. “Quaternionic quantum field theory”. Phys. Rev. Lett., 55:783–786, 1985.
[40] S. De Leo; P. Rotelli. “The Quaternion scalar field”. Phys. Rev., D45:575–579, 1992.
[41] A. Gsponer; J.-P. Hurni. “Quaternions in mathematical physics. I. Alphabetical bibliography”. (2005) math-ph/0510059.
[42] A. P. Balachandran; A. Pinzul; B. A. Qureshi. “Twisted Poincare Invariant Quantum Field Theories”. Phys.Rev., D77:025021, (2008) arXiv:0708.1779 [hep-th].
[43] M. Scheunert. “The theory of Lie super-algebras: an introduction. Springer Verlag (1979). Lecture Notes in Mathematics 716.
[44] M. Chaichian; P. P. Kulish; K. Nishijima; A. Tureanu. “On a Lorentz-invariant interpretation of noncommutative space-time and its implications on noncommutative QFT”. Phys. Lett., B604:98–102, (2004) hep-th/0408069.