Non-Abelian Born-Infeld Action and
Type I – Heterotic Duality (II):
Nonrenormalization Theorems

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Abstract

Type I – heterotic duality in $D=10$ predicts various relations and constraints on higher order $F^n$ couplings at different string loop levels on both sides. We prove the vanishing of two-loop corrections to the heterotic $F^4$ terms, which is one of the basic predictions from this duality. Furthermore, we show that the heterotic $F^5$ and (CP even) $F^6$ couplings are not renormalized at one loop. These results strengthen the conjecture that in $D=10$ any Tr$F^{2n}$ coupling appears only at the disk tree-level on type I side and at $(n-1)$-loop level on the heterotic side. Our non-renormalization theorems are valid in any heterotic string vacuum with sixteen supercharges.
1. Introduction

In the recent paper [1], we discussed a class of higher-derivative $SO(32)$ gauge boson interactions in the framework of $D=10$ type I - heterotic duality. On type I side, these interactions are related to the non-Abelian completion of the Born-Infeld action (NBI) which, in more general context, describes systems of D-branes and orientifold planes. We considered the NBI action as a power expansion in the gauge field strength $F$ and examined the $F^6$ interactions in a way suggested by superstring duality which, for this class of terms, relates type I at the classical level to the heterotic theory at two loops. We performed explicit two-loop computations of the scattering amplitudes and derived the corresponding constraints on the heterotic $F^6$ interactions. The constraints originate from Riemann identities reflecting supersymmetry of the underlying theory, and lead to an unexpected conclusion that the heterotic $F^6$ terms are not related in any simple way to Born-Infeld theory. Namely when the gauge bosons are restricted to the $SO(32)$ Cartan subalgebra generators, the result is different from the expression obtained by expanding the Abelian Born-Infeld action. This creates an interesting problem how to reconcile such a discrepancy with superstring duality.

Type I - heterotic correspondence is a strong-weak coupling duality so it is guaranteed to work in a straightforward manner only for quantities that are subject to non-renormalization theorems. Hence it is very important to investigate possible corrections to $F^n$ interactions coming from both higher and lower number of loops. In this paper, we discuss the cases of $n=4, 5, 6$ in heterotic superstring theory. We derive several non-renormalization theorems that are directly related to our discussion of the NBI action. These theorems are also interesting per se as they apply to the heterotic perturbation theory.

We begin by recalling the relation between the $F^n$ couplings of type I and heterotic theories. Duality is manifest in the Einstein frame where the perturbative expansions of the respective actions can be written as

$$S_{H, \text{Einstein}} = \int d^{10}x \sqrt{-g_H^E} \left[ R - \frac{1}{4} e^{-\Phi_H/4} F^2 - \sum_{n>2} \sum_{m} b_{mn} e^{-m\Phi_H/4} F^n + \ldots \right]$$

$$S_{I, \text{Einstein}} = \int d^{10}x \sqrt{-g_I^E} \left[ R - \frac{1}{4} e^{\Phi_I/4} F^2 - \sum_{n>2} \sum_{m} b_{mn} e^{m\Phi_I/4} F^n + \ldots \right].$$

(1.1)

These equations are symbolic in the sense that $F^n$ denotes collectively any Lorentz contraction and any group theoretical contraction of $n$ gauge field strengths. The integer $m$
governs the dilaton dependence of each $F^n$ coupling while the corresponding coefficients $b_{mn}$ are constant numbers. In this basis, type I - heterotic duality is manifest \[2\]:

$$\Phi_H = -\Phi_I \; , \; g^E_H = g^E_I \; , \; A_{\mu}^{a,H} = A_{\mu}^{a,I} \; .$$

(1.2)

Transforming (1.1) into the string basis with $g^E_{\mu\nu} = e^{-\Phi/4}g^S_{\mu\nu}$ gives

$$S_{H, \text{string}} = \int d^{10}x \sqrt{-g^S_H} \left[ e^{-\Phi_H} R - \frac{1}{4} e^{-\Phi_H} F^2 - \sum_{n>2} \sum_{m} b_{mn} e^{\Phi_H(n-m-5)/4} F^n + \ldots \right]$$

$$S_{I, \text{string}} = \int d^{10}x \sqrt{-g^S_I} \left[ e^{-\Phi_I} R - \frac{1}{4} e^{-\Phi_I/2} F^2 - \sum_{n>2} \sum_{m} b_{mn} e^{\Phi_I(n+m-5)/4} F^n + \ldots \right]$$

(1.3)

The string loop counting is determined by the dilaton factor $e^{-\chi\Phi/2}$, with the Euler number $\chi = 2 - 2g$ for closed strings and $\chi = 2 - 2g - b - c$ for open strings, where $g$ is the genus of Riemann surface and $b, c$ are the numbers of holes and crosscaps, respectively. Depending on the value of $m$, we obtain different chains of $F^n$ couplings at heterotic $(n - m - 1)/4$ loops related to the corresponding type I couplings at order $e^{\Phi_I(n+m-5)/4}$. For instance, the chain associated to $m = 3 - n$ relates the heterotic $F^n$ couplings at $(n - m - 1)/4$ loops to type I couplings at the tree level (disk with $\chi = 1$). Changing the value of $m$ by two units gives other chains. We obtain the following dictionary:

| Type I | Heterotic |
|--------|-----------|
| $e^{-\Phi_I/2}$ | $\mathbf{F^2}$ | $\mathbf{F^4}$ | $F^6$ | $F^8$ | $F^{10}$ | $3 - n$ |
| 1 | $F^3$ | $F^5$ | $F^7$ | $F^9$ | $F^{11}$ | $5 - n$ |
| $e^{\Phi_I/2}$ | $F^4$ | $F^6$ | $F^8$ | $F^{10}$ | $F^{12}$ | $7 - n$ |
| $e^{\Phi_I}$ | $F^5$ | $F^7$ | $F^9$ | $F^{11}$ | $F^{13}$ | $9 - n$ |
| $e^{3/2\Phi_I}$ | $F^6$ | $F^8$ | $F^{10}$ | $F^{12}$ | $F^{14}$ | $11 - n$ |

**Table:** Possible $F^n$ couplings of type I and heterotic string theory in $D = 10$.

Since the tree-level type I action that originates from the disk diagram has a single group (Chan-Paton) trace from its boundary, here we will be mostly interested in this type of group contractions. These gauge group traces will be denoted by Tr while the Lorentz group traces by tr. In this case, the couplings written in the Table in bold face are already

\[1\] Here, the string tension is normalized by $2\pi\alpha' = 1$ and $\Phi = 2\phi$, where $\phi$ is the standard dilaton.
known to be consistent with duality. In particular, $\text{Tr}F^3$ couplings vanish as a consequence of supersymmetry \textsuperscript{3}. The tree-level heterotic $\text{Tr}F^4$ couplings are also zero, as shown in \textsuperscript{4}. The heterotic one-loop $\text{Tr}F^4$ has been calculated in \textsuperscript{5,6} and agrees with type I \textsuperscript{3,7} at the tree level. Finally, the absence of one-loop corrections (from the annulus and Möbius strip) to $\text{Tr}F^4$ has been demonstrated in type I theory in \textsuperscript{8,9}.

The first chain, $m = 3 - n$, is particularly interesting. The $g$-loop heterotic $\text{Tr}F^{2g+2}$ couplings are related by duality to classical type I theory, hence to the NBI action \textsuperscript{10}. This observation offers a tool for computing the NBI action by using heterotic perturbation theory. It is quite remarkable that the tree-level, open string amplitudes are encoded in the heterotic theory at higher genus. In fact, this motivated us to perform the two-loop computations in \textsuperscript{1}.

The dual actions (1.3) can be compared order by order in perturbation theory only for quantities that receive contributions from a limited number of loops. Then some couplings at a given loop level on one side are simply forbidden because they would imply a “negative loop order” on the dual side. For instance, the heterotic $\text{Tr}F^4$ couplings must vanish at two loops. In general, by inspecting the Table, we see that a $\text{Tr}F^{2g+2}$ coupling can appear on the heterotic side only up to the $g$-loop order, but not beyond. Furthermore, if the NBI action does not receive corrections beyond the disk level, one would expect that a $\text{Tr}F^{2g+2}$ term appears on the heterotic side only at a $g$-loop order. This would indicate a topological nature of these couplings, similar to the amplitudes discussed in \textsuperscript{11}. However, in view of our findings in \textsuperscript{1}, this connection may hold not for all couplings, but only for those which are BPS saturated and in some way related to a higher loop generalization of the elliptic genus \textsuperscript{12,5}. In this paper, we prove a number of perturbative non-renormalization theorems for the heterotic superstring, all of them consistent with the duality conjecture. In Section 2, we demonstrate the absence of two-loop corrections to $F^4$. In Section 3, we show that all one-loop contributions to $F^5$ are zero. In Section 4, we extend our one-loop analysis to $F^6$ terms, which is the case most relevant to \textsuperscript{1}. Here again, we show that all one-loop contributions vanish. In Section 5, we place these non-renormalization theorems in a broader context of all-order perturbation theory and explain their implications for the results of \textsuperscript{1}. The paper includes two appendices. In Appendix A, we present some useful tools for computing the one-loop amplitudes. Finally, in the self-contained Appendix B, we show the absence of two-loop corrections to $F^4$ in a formulation which is independent on the choice of gauge slice.
2. Vanishing of heterotic two-loop $F^4$

The heterotic $\text{Tr} F^4$ terms have been calculated at one loop in $D = 10$ in [5,6]. It has been speculated for a long time that the two-loop correction to $\text{Tr} F^4$ may be absent, because $t_8 \text{Tr} F^4$ appears in the same superinvariant as the Green-Schwarz anomaly term $\epsilon_{10} B \text{Tr} F^4$. According to Adler-Bardeen theorem, anomalies are not renormalized: at two loops, this has been shown explicitly for Green-Schwarz anomaly in [13].

In this section we will prove that the two-loop correction to $\text{Tr} F^4$ does indeed vanish. To that end, we consider the correlator of four gauge bosons:

$$\langle V_{A_1^{(a)}}(z_1, \bar{z}_1) V_{A_2^{(b)}}(z_2, \bar{z}_2) V_{A_3^{(c)}}(z_3, \bar{z}_3) V_{A_4^{(d)}}(z_4, \bar{z}_4) Y(x_1) Y(x_2) \rangle, \quad (2.1)$$

with the corresponding vertex operators taken in the zero-ghost picture,

$$V_{A_\mu}^a(z, \bar{z}) = \epsilon_\mu : J^a(z) \left[ \partial X^\mu + i(k_\nu \psi^{\nu}) \psi^\mu \right] e^{ik_\nu X^\nu(z, \bar{z})} : \quad (2.2)$$

and with two necessary picture changing operator (PCO) insertions $Y$ at arbitrary points $x_1$ and $x_2$, see [1]. The gauge currents can be fermionized as:

$$J^a(z) = (T^a)_{ij} \psi^i(z) \psi^j(z), \quad (2.3)$$

where $T^a$ are the $SO(32)$ gauge group generators in the defining representation. The space-time part of this amplitude has been already discussed in [14,15]. In our proof, we will combine it with the gauge part.

Essentially, there are two different ways to tackle two-loop calculations. One way is the so-called $\theta$-function approach, where the partition function and correlators are expressed in terms of genus two $\theta$-functions. After choosing the unitary gauge, this method proves to be very useful e.g. for identifying the combinations of amplitudes that vanish due to Riemann identities, as demonstrated in [1]. The other way, which is more suitable for the present discussion, uses the hyperelliptic formalism. This approach allows a more transparent treatment of the ambiguity in choosing the gauge slices, i.e. fixing the PCO positions. In the hyperelliptic formalism, the genus two surface is represented by a two-sheet covering of the complex plane,

$$y(z)^2 = \prod_{i=1}^{6} (z - a_i), \quad (2.4)$$

\footnote{The unitary gauge is a special choice of the PCO insertion points. Different choices are related by total derivatives w.r.t. the moduli of the two-loop Riemann surface [13]. These contributions are zero provided these derivative terms vanish at the boundaries of the moduli space. This has to be verified for each amplitude. A very useful approach to handle these complications has been recently elaborated by D’Hoker and Phong [17]. In Appendix B we will address the vanishing of two-loop $\text{Tr} F^4$ within that framework.}
with six branch (ramification) points \( a_i \). The string correlators are functions on this hyperelliptic surface. We refer the reader to [14,18] and references therein for more information.

Let us introduce the basic ingredients of hyperelliptic formalism. The space-time zero mode contribution \( \det \text{Im} \Omega \) is related to the corresponding quantity \( T \) on the hyperelliptic surface:

\[
T(a_i, \bar{a}_i) = \int d^2u d^2v \frac{|u - v|^2}{|y(u)y(v)|^2} = 2|\det K|^2\det \text{Im} \Omega .
\]  

(2.5)

Here, \( \Omega \) are the moduli of the genus two Riemann surface, defined by integrals \( \Omega_{ij} = \oint_{b_j} \omega_i \) of the canonical one-forms \( \omega_i \) over the \( b \)-cycles \( b_j \). The determinant factor \( |\det K|^2 \) arises because we are working with the non-canonically normalized Abelian differentials \( \tilde{\omega}_i = \omega_j K_{ji} \), with \( \tilde{\omega}_1(z) = \frac{dz}{y(z)} \) and \( \tilde{\omega}_2(z) = \frac{z \, dz}{y(z)} \), respectively. On the hyperelliptic Riemann surface, even spin structures \( \delta \) are in one-to-one correspondence with the splittings of six branch points \( a_i \) into two non-intersecting sets \( \{A_1, A_2, A_3\} \) and \( \{B_1, B_2, B_3\} \). The chiral fermion determinant is given by

\[
\det\delta \partial_1/2 = \alpha^{-1} \prod_{i<j} (A_i - A_j)^{1/4} (B_i - B_j)^{1/4} \equiv \frac{Q_\delta^{1/4}}{\alpha} ,
\]  

(2.6)

where \( \alpha \) represents the oscillator contributions:

\[
\alpha = \prod_{i<j} (a_i - a_j)^{1/8} .
\]  

(2.7)

The quantity \( Q_\delta \) is related to the familiar genus two \( \theta \)-functions \( \theta_\delta(0, \Omega) \) through the Thomae formula [19]:

\[
\theta_\delta(0, \Omega) = (\det K)^{1/2} Q_\delta^{1/4} .
\]

For even spin structure \( \delta \), the Szegő kernel takes the form [20]:

\[
\langle \psi(z_1) \psi(z_2) \rangle_{\delta} = \frac{1}{2} \frac{dz_1^{1/2} dz_2^{1/2}}{z_1 - z_2} \frac{u_\delta(z_1) + u_\delta(z_2)}{\sqrt{u_\delta(z_1) u_\delta(z_2)}} .
\]  

(2.8)

Here, the functions \( u_\delta(z) \) are introduced as

\[
u_\delta(z) = \frac{(z - A_1)(z - A_2)(z - A_3)}{y(z)} ,
\]  

(2.9)

where \( A_i \) are the three branch points \( a_i \) associated to a given spin structure \( \delta \). Finally, the (three-dimensional) genus two measure \( d\mu \) can be expressed in terms of integrals over three branch points:

\[
d\mu = \frac{d^2a_1 d^2a_2 d^2a_3 |a_{45} a_{56} a_{46}|^2}{6 \prod_{i<j} |a_{ij}|^2} .
\]  

(2.10)
The space-time part of the two-loop amplitude under consideration has been previously evaluated within the hyperelliptic formalism by several authors [14]. The result is:

\[
\Delta_{t^8 Tr F^4}^{2-\text{loop}} = \int d\mu \ T^{-5} \prod_{l=1}^{4} \frac{d^2 z_l(x - z_l)}{y(z_l)} \ I(x) \ F([\bar{\pi}_m]; \{z_n\}) .
\]  

(2.11)

Here, \(I(x)\) summarizes all contributions from PCOs whose two positions have been chosen at the same point \(x\) on the upper and lower sheets. The amplitude is independent of this choice, as any change amounts to a total derivative on the moduli space [16]. The kinematics of the four gauge boson amplitude corresponds to the familiar \(t^8\)-tensor. All eight space-time fermions from (2.2) are contracted in (2.11). This gives already \(O(k^4)\) in momenta. Fewer fermion contractions would give a vanishing result. The expression (2.11) assumes zero momenta in the exponentials of the gauge vertices (2.2) as they would bring down more momentum factors. This step needs some care because there may be potential singularities which would decrease the power of momentum. We shall give a justification of this step later. Finally, the function \(F([\bar{\pi}_m]; \{z_n\})\) encodes the gauge part from the right-moving sector. It comprises the \(SO(32)\) gauge partition function (cf. Eqs. (2.6) and (2.7))

\[
Z_{SO(32)}(a) = \alpha^{-16} \sum_{\beta \text{ even}} Q_\beta^4 ,
\]  

(2.12)

supplemented with the four gauge current correlator \(\langle J^{a_i}(z_i)J^{a_j}(z_j)J^{a_k}(z_k)J^{a_l}(z_l)\rangle\) that we shall determine now.

In order to obtain the group theoretical structure \(\text{Tr}(T^{a_i}T^{a_j}T^{a_k}T^{a_l})\), the gauge fermions of (2.3) have to be contracted in such a way that their four vertex positions \(z_i\) form a closed loop (square):

\[
B_\beta(z_i, z_j, z_k, z_l) \equiv \langle \psi(z_i)\psi(z_j)\psi(z_k)\psi(z_l)\rangle_{\beta} \langle \psi(z_j)\psi(z_k)\psi(z_l)\rangle_{\beta} \langle \psi(z_k)\psi(z_l)\rangle_{\beta} \langle \psi(z_l)\psi(z_i)\rangle_{\beta}
\]

\[
= \frac{1}{16} \frac{dz_1dz_2dz_3dz_4}{z_{ij}z_{jk}z_{kl}z_{li}} \frac{1}{u_\beta(z_i)u_\beta(z_j)u_\beta(z_k)u_\beta(z_l)}
\]

\[
\times [u_\beta(z_i) + u_\beta(z_j)][u_\beta(z_j) + u_\beta(z_k)][u_\beta(z_k) + u_\beta(z_l)][u_\beta(z_l) + u_\beta(z_i)] .
\]  

(2.13)

Due to the symmetry of the space-time part (2.11), it is sufficient to focus on one specific contraction, say \(B(z_1, z_2, z_3, z_4)\). However, it is more convenient to take the combination

\[
\frac{1}{3}[B(z_1, z_2, z_3, z_4) + B(z_1, z_2, z_4, z_3) + B(z_1, z_3, z_4, z_2)]
\]

which shows a particularly simple
dependence on the vertex positions $z_i$:

\[
\frac{1}{3} \left[ B_\beta(z_1, z_2, z_3, z_4) + B_\beta(z_1, z_2, z_4, z_3) + B_\beta(z_1, z_3, z_4, z_2) \right]
\]

\[
= \frac{1}{48 \, y(z_1)y(z_2)y(z_3)y(z_4)} \left[ c_0(a_i) + c_1(a_i) \sum_i z_i + c_2(a_i) \sum_{i<j} z_i z_j \right. \\
+ c_3(a_i) \sum_{i<j<k} z_i z_j z_k + c_4(a_i) z_1 z_2 z_3 z_4 \right]
\]

(2.14)

with the spin structure dependent coefficients $c_j(a_i)$ being polynomials in $a_i$. Their explicit form is not important for our arguments. Furthermore, for later use, we note that the bracket of (2.14) scales with the weight $\lambda^8$ under the simultaneous rescalings $a_i \to \lambda a_i$ and $z_i \to \lambda z_i$. The r.h.s. of (2.14) involves linear combination of two Abelian differentials $\frac{dz}{y(z)}$ and $\frac{dz}{y(z)}$, introduced earlier. The above equation represents one of identities that can be found in [21], rewritten in the hyperelliptic formalism. The total gauge part $F(\{a_m\}; \{z_n\})$ from (2.11) is obtained by combining (2.14) with the $SO(32)$ lattice sum (2.12):

\[
F(\{a_m\}; \{z_n\}) = \frac{1}{48 \, y(z_1)y(z_2)y(z_3)y(z_4)} \alpha^{16} \sum_\beta Q_\beta^4 \left[ c_0(a_i) + c_1(a_i) \sum_i z_i + \\
+ c_2(a_i) \sum_{i<j} z_i z_j + c_3(a_i) \sum_{i<j<k} z_i z_j z_k + c_4(a_i) z_1 z_2 z_3 z_4 \right].
\]

(2.15)

This expression is particular useful, since in this form all possible singularities at $z_i \to z_j$ are eliminated. This justifies setting the momenta of the exponentials (2.2) to zero at the beginning.

By following the same steps as in [22], using simultaneous $SL(2, \mathbb{C})$ transformations on $a_i, z_i$ and $x$, the expression (2.11) can be rewritten in an explicitly modular invariant form:

\[
\Delta_{\text{loop}}^{2\text{-loop}} = \int_{\mathcal{C}} d\tilde{\mu} \prod_{l=1}^4 \frac{d^2 z_l (x-z_l)}{|y(z_l)|^2} \prod_{i=1}^3 \delta^2 (z_i - z_0^i) \prod_{j<k}^3 |z_j^0 - z_k^0| I_M(x) \, \bar{F}(\{a_m\}; \{z_n\}),
\]

(2.16)

where

\[
\bar{F}(\{a_m\}; \{z_n\}) = y(z_1)y(z_2)y(z_3)y(z_4) \, F(\{a_m\}; \{z_n\}),
\]

(2.17)

and the measure is

\[
d\tilde{\mu} = \frac{\prod_{i=1}^6 d^2 a_i}{T^5 \prod_{i<j}^6 |a_i - a_j|^2}.
\]

(2.18)

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3 We call the $SL(2, \mathbb{C})$ transformed values again $a_i, z_i$ and $x$. 

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Modular invariance manifests in (2.19) as an invariance under the permutations of $a_i$. Now, three vertex positions $z_i$ are fixed to $z^0_i$ ($i = 1, 2, 3$) at the cost of allowing for integration over all six branch points $a_i$ rather than three. The function $I_M(x)$ denotes the PCO contributions [22]:

$$I_M(x) = \frac{1}{4} \sum_{i=1}^{6} \frac{1}{(x-a_i)^2} - \frac{1}{4} \sum_{i<j}^{6} \frac{1}{x-a_i - x-a_j} - \frac{1}{8} \sum_{i=1}^{6} \frac{1}{x-a_i} \sum_{l=1}^{4} \frac{1}{x-z_l} + \frac{1}{4} \sum_{k<j=1}^{4} \frac{1}{x-z_k - x-z_j} - \frac{5}{4} \sum_{i=1}^{6} \frac{1}{x-a_i} \partial \ln T . \quad (2.19)$$

The amplitude $\Delta_{t_sTrF^4}^{2-\text{loop}}$ is independent of the point $x$, which is the insertion point of the two PCOs. Any change in $x$ amounts to a total derivative in moduli space [23].

A great simplification occurs when we choose $x$ equal to the vertex position $z_1$ ($x \to z_1^0$):

$$\Delta_{t_sTrF^4}^{2-\text{loop}} = V \int_C d\tilde{\mu} \int_C d^2z \frac{z_1^0 - z}{|y(z_1^0)y(z_2^0)y(z_3^0)y(z)|^2} I_\infty \tilde{F}(\{\tilde{\alpha}_m\}; \{\tilde{\omega}_n, \tilde{\varphi}\}) , \quad (2.20)$$

with

$$I_\infty = \frac{1}{8} \sum_{i=1}^{6} \frac{1}{z_1^0 - z} + \frac{1}{4} \frac{1}{z_1^0 - z_2^0} + \frac{1}{4} \frac{1}{z_1^0 - z_3^0} + \frac{1}{4} \frac{1}{z_1^0 - z} , \quad (2.21)$$

$$V = (z_1^0 - z_2^0)^2(z_1^0 - z_3^0)^2(\tilde{\omega}_1 - \tilde{\omega}_2)^2(z_2^0 - z_3^0)^2(\tilde{\varphi}_1 - \tilde{\varphi}_2)^2(\tilde{\varphi}_1 - \tilde{\varphi}_3)^2 .$$

In the limit $z \to z_1^0$ the integrand in (2.21) remains finite. However, it is not clear what happens when $a_i \to z_i^0$. This limit can be analyzed by inspecting a potentially more singular expression,

$$\int_C d^2\tilde{\mu} \int_C d^2z \frac{1}{|y(z_1^0)y(z_2^0)y(z_3^0)|^2} . \quad (2.22)$$

This is the same expression that appears in the context type IIA two-loop $R^4$ terms, for which the finiteness of (2.22) has been verified in the limit $a_i \to z_j^0$ [22].

The integrand of (2.20) is invariant under the modular transformation

$$z_i^0 \to \frac{az_i^0 + b}{cz_i^0 + d} , \quad ad - bc = 1 . \quad (2.23)$$

This allows to fix three positions. A convenient choice is:

$$z_1^0 = 0 , \quad z_2^0 = \infty , \quad z_3^0 = x \neq 0 , \quad \infty . \quad (2.24)$$

\[4\] The gauge function $\tilde{F}(\{\tilde{\alpha}_m\}; \{\tilde{\varphi}_n\})$ becomes at most zero for some or all $a_i$ approaching $z_1^0$.  

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Therefore in the integrand of (2.27) we may choose allowing the changes

\[ \Delta \equiv \frac{1}{2} \sum_{i=1}^{6} \frac{1}{a_i - x - \frac{1}{z}} \]

\[ F_\infty (\{ \alpha_m \}; x, z) , \]

where

\[ F_\infty (\{ a_m \}; x, z) \equiv \frac{\partial}{\partial z^2} \tilde{F} (\{ a_m \}; \{ z_n \}) \bigg|_{z_0 = x, z_1 = 0} . \]

Observing \( \partial_{a_i} y(0)^{-1} = -\frac{1}{2} a_i^{-1} y(0)^{-1} \), \( \sum_{a_i} \partial_{a_i} y(x)^{-1} = \sum_i \frac{1}{2} (x-a_i)^{-1} y(x)^{-1} = -\partial_{a_i} y(x)^{-1} \), allows us to rewrite \( \Delta^{2\text{-loop}}_{ts\text{Tr} F^4} \) in a more convenient form after partial integrations:

\[ \Delta^{2\text{-loop}}_{ts\text{Tr} F^4} = \frac{1}{4} |x|^2 \left( x \frac{\partial}{\partial x} + 1 \right) \int_C \frac{z}{d^2 z} \left| \frac{z}{y(z) y(0) y(x)} \right|^2 F_\infty (\{ \alpha_m \}; x, z) . \]

The whole expression (2.27) (or (2.25)) is independent of \( x \). This can be verified by allowing the changes \( x \rightarrow \lambda x \), \( z \rightarrow \lambda z \) and \( a_i \rightarrow \lambda a_i \) under which the transformation rules \( T \rightarrow \lambda^{-3} \bar{\lambda}^{-3} T \), \( d\tilde{\mu} \rightarrow \lambda^6 \bar{\lambda}^6 d\tilde{\mu} \), and \( F_\infty (\{ \lambda a_m \}; \lambda x, \lambda z) \rightarrow \lambda F_\infty (\{ a_m \}; x, z) \) follow. Therefore in the integrand of (2.27) we may choose \( x = 1 \):

\[ \Delta^{2\text{-loop}}_{ts\text{Tr} F^4} = \frac{1}{4} |x|^2 \left( x \frac{\partial}{\partial x} + 1 \right) \int_C \frac{z}{d^2 z} \left| \frac{z}{y(z) y(0) y(1)} \right|^2 F_\infty (\{ \alpha_m \}; 1, z) . \]

However, due to \( (x \frac{\partial}{\partial x} + 1)x^{-1} = 0 \), we conclude:

\[ \Delta^{2\text{-loop}}_{ts\text{Tr} F^4} = 0 . \]

For the group contraction \( (\text{Tr} F^2)^2 \), instead of (2.15), the relevant gauge part is:

\[ F (\{ a_m \}, \{ z_n \}) = \frac{1}{16} \frac{\alpha^{-16}}{(z_1 - z_2)^2 (z_3 - z_4)^2} \sum_{\beta} Q^4 \frac{[u_\beta (z_1) + u_\beta (z_2)][u_\beta (z_3) + u_\beta (z_4)]}{u_\beta (z_1) u_\beta (z_2) u_\beta (z_3) u_\beta (z_4)} . \]

Again, \( \tilde{F} (\{ a_m \}, \{ z_n \}) \) (defined through (2.17)), which enters (2.19), shows the previously encountered behaviour. Namely its \( z_2 \)-independent part (defined by (2.26)) transforms as \( F_\infty (\{ \lambda a_m \}; \lambda x, \lambda z) \rightarrow \lambda F_\infty (\{ a_m \}; x, z) \) under the rescalings \( x \rightarrow \lambda x \), \( z \rightarrow \lambda z \) and \( a_i \rightarrow \lambda a_i \). Aside from these properties of the gauge part, the essential steps to prove (2.24) affected only the space-time part. Therefore, we derive also for the group contraction \( (\text{Tr} F^2)^2 \):

\[ \Delta^{2\text{-loop}}_{ts(\text{Tr} F^2)^2} = 0 . \]

Thus we have established the two-loop non-renormalization theorems (2.24) and (2.31) in \( D=10 \) heterotic string theory.

---

5 The following steps are similar to those of Ref. 23 demonstrating the vanishing of two-loop corrections to the ts\text{Tr} R^4 term in type IIA/B theories.

6 The poles appearing in the integrand (2.17) in the limit for \( z_i \rightarrow z_j \) can be analytically continued to a finite value.
3. Vanishing of heterotic one-loop $F^5$

In the past, only 1/2 BPS-saturated one-loop amplitudes have been discussed in heterotic string vacua with sixteen supercharges. They describe couplings which are related by supersymmetry to eight-fermion terms. Their characteristic feature is that they depend on the ground state only of the right-moving sector. The latter then contributes just as a constant, and one is left with world-sheet torus integrals over anti-holomorphic functions representing the contributions of the left-moving sector. This will no longer be the case once we consider non 1/2 BPS-saturated amplitudes involving more than eight fermions, which generically receive also non-constant contributions from the right-moving sector.

In this section, we prove that the one-loop corrections to the two possible $F^5$ space-time contractions, $\text{tr} F^5$ (denoted by $P$) and $\text{tr} F^3 \text{tr} F^2$ (denoted by $S$), vanish exactly for any gauge contraction. To that end, we consider the five-point gauge boson amplitude,

$$\langle V_{A_{a_1}^a}^\alpha(z_1, \bar{z}_1) \cdots V_{A_{a_5}^a}(z_5, \bar{z}_5) \rangle_{\text{even}}$$

(3.1)

and extract the relevant kinematic pieces. The gauge boson vertex operator (in the zero-ghost picture) is given in (2.2) and the gauge currents are fermionized according to (2.3). The one-loop fermion propagator for even spin structure $\vec{\alpha} = (\alpha_1, \alpha_2)$ is

$$G^F_{\vec{\alpha}}(z_{ij}) \delta^{\mu\nu} = \langle \psi^\mu(z_1) \psi^\nu(z_2) \rangle_{\vec{\alpha}} \frac{\theta_{\vec{\alpha}}(z_{12}, \tau)}{\theta_{1}(z_{12}, \tau) \theta_{\vec{\alpha}}(0, \tau)} \delta^{\mu\nu}.$$ (3.2)

In the following, we shall first discuss the gauge pentagon, $\text{Tr} F^5$ case. In order to yield the corresponding group theoretical factor $\text{Tr}(T^{a_i} T^{a_j} T^{a_k} T^{a_l} T^{a_m})$, the world-sheet gauge fermions (2.3) must be contracted in such a way that the vertex positions $z_i$ form a pentagon. Hence the gauge part for the spin structure $\vec{\beta} = (\beta_1, \beta_2)$ becomes

$$f_{\vec{\beta}}(\vec{\tau}, \{z_n\}) = G^F_{\vec{\beta}}(z_{ij}) G^F_{\vec{\beta}}(z_{jk}) G^F_{\vec{\beta}}(z_{kl}) G^F_{\vec{\beta}}(z_{lm}) G^F_{\vec{\beta}}(z_{mi}).$$ (3.3)

Due to the periodicity properties of the fermion propagator, $G^F_{\beta}(z+1, q) = -e^{2\pi i \beta_1} G^F_{\beta}(z, q)$ and $G^F_{\beta}(z + \tau, q) = -e^{-2\pi i \beta_2} G^F_{\beta}(z, q)$, we conclude that the expression (3.3) is periodic under the transformations $z_i \rightarrow z_i + 1$ and $z_i \rightarrow z_i + \tau$.

Space-time supersymmetry requires that at least eight world-sheet fermions from the five vertex operators (2.2) are taken into account. Each vertex operator supplies a pair of fermions at the same position $z_i$. Therefore we have to contract all ten fermions. This gives already the required $O(k^5)$ order in momentum and we may set the momenta of the exponentials in (2.3) to zero. We shall comment on this step at the end of this section.

---

7 With the exception of Ref.[24], where also 1/4 BPS saturated amplitudes have been considered.
Two space-time kinematics $K_P$ and $K_S$ are possible, depending on how the ten fermions are contracted.\\

\[
g^{K_S}_{\vec{\alpha}}(q, \{z_n\}) = -G^F_{\vec{\alpha}}(z_{12}) G^F_{\vec{\alpha}}(z_{23}) G^F_{\vec{\alpha}}(z_{31}) G^F_{\vec{\alpha}}(z_{45})^2, \\
g^{K_P}_{\vec{\alpha}}(q, \{z_n\}) = -G^F_{\vec{\alpha}}(z_{12}) G^F_{\vec{\alpha}}(z_{23}) G^F_{\vec{\alpha}}(z_{34}) G^F_{\vec{\alpha}}(z_{45}) G^F_{\vec{\alpha}}(z_{51}). \tag{3.4}
\]

Thus in total, the two different kinematics $K$ receive the following one-loop corrections:

\[
\Delta^{1\text{-loop}}_{K\text{Tr}F^3} = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \frac{1}{\eta(q)^{24}} \frac{1}{\eta(q)^{12}} \sum_{\vec{\alpha},\vec{\beta}} s_{\vec{\alpha}} \theta_{\vec{\alpha}}(q)^4 \theta_{\vec{\beta}}(q)^4 \int_{\mathcal{I}_r} d^2 z_i g^{K}_{\vec{\alpha}}(q, \{z_n\}) f_{\vec{\beta}}(\underline{q}, \{\bar{z}_n\}) .
\tag{3.5}
\]

Here, the sum over $\vec{\alpha}$ represents the even spin structure sum (with the phases $s_{\vec{\alpha}} = (-1)^{2\alpha_1+2\alpha_2}$) and the sum over even $\vec{\beta}$ is the $SO(32)$ gauge lattice sum. In general, the world-sheet torus integrals (with the integration region $\mathcal{I}_r = \{ z \mid -\frac{1}{2} \leq \text{Re}(z) \leq \frac{1}{2}, 0 \leq \text{Im}(z) \leq \tau_2 \}$) over the five positions imply a coupling between the left-moving $f_{\vec{\beta}}(\underline{q}, \{\bar{z}_n\})$ and right-moving $g^{K}_{\vec{\alpha}}(q, \{z_n\})$ parts, which complicates the procedure. In fact, so far, only purely antiholomorphic, $z_i$-independent world-sheet torus integrals have been discussed in the literature. This is the case when the amplitude represents a BPS-saturated coupling. Then the right–moving sector is in the ground state and contributes a constant to the full amplitude, without a holomorphic position $z_i$-dependence.

After recalling the relation of the square of the fermionic propagator

\[
G^F_{\vec{\alpha}}(z)^2 = \left( \frac{\theta_{\vec{\alpha}}(z, \tau) \theta'_{\vec{\alpha}}(0, \tau)}{\theta_{\vec{\alpha}}(z, \tau) \theta'_{\vec{\alpha}}(0, \tau)} \right)^2 = \frac{\partial^2}{\partial z^2} \ln \theta_{\vec{\alpha}}(0, \tau) + \frac{\pi}{\tau_2} - \partial^2 G_B(z) , \tag{3.6}
\]

with the bosonic Greens function $G_B$

\[
\partial G_B(z) = \partial \ln \theta_{\vec{\alpha}}(z, \tau) + \frac{2\pi i}{\tau_2} \text{Im}(z) , \tag{3.7}
\]

we proceed to the evaluation of the spin structure sum $\vec{\alpha}$ in (3.3). We obtain

\[
\tilde{g}^{K_S}(q, \{z_n\}) = -\frac{(2\pi)^3}{\eta^3} \sum_{\vec{\alpha}} s_{\vec{\alpha}} \frac{\partial^2}{\partial z^2} \theta_{\vec{\alpha}}(0, \tau) \frac{\theta_{\vec{\alpha}}(z_{12}) \theta_{\vec{\alpha}}(z_{23}) \theta_{\vec{\alpha}}(z_{31})}{\theta_{\vec{\alpha}}(z_{12}) \theta_{\vec{\alpha}}(z_{23}) \theta_{\vec{\alpha}}(z_{31})} ,
\]

\[
\tilde{g}^{K_P}(q, \{z_n\}) = -\frac{(2\pi)^5}{\eta^3} \sum_{\vec{\alpha}} s_{\vec{\alpha}} \frac{1}{\theta_{\vec{\alpha}}(0)} \frac{\theta_{\vec{\alpha}}(z_{12}) \theta_{\vec{\alpha}}(z_{23}) \theta_{\vec{\alpha}}(z_{34}) \theta_{\vec{\alpha}}(z_{45}) \theta_{\vec{\alpha}}(z_{51})}{\theta_{\vec{\alpha}}(z_{12}) \theta_{\vec{\alpha}}(z_{23}) \theta_{\vec{\alpha}}(z_{34}) \theta_{\vec{\alpha}}(z_{45}) \theta_{\vec{\alpha}}(z_{51})} . \tag{3.8}
\]

\footnote{Due to kinematical reasons, there are no CP odd $P^5$ couplings in $D = 10$.}

\footnote{Diagrammatically, the corresponding Lorentz contractions can be represented by a pentagon ($P$) and by a triangle plus one line ($S$), respectively.}

\footnote{The whole right-moving part of the integrand (3.3) is put into $\tilde{g}$.}
for the kinematics $\mathcal{K}_S$ and $\mathcal{K}_P$, respectively. Since the last term $\partial^2 G_B(z)$ of (3.6) is a periodic function and the gauge part $f_{\tilde{g}}(q, \{z_n\})$ is also periodic, it will give a vanishing contribution to (3.5) after performing the position integral:

$$
\int_{\mathcal{I}_r} d^2 z_j d^2 z_k \, \partial^2 G_B(z_j - z_k) \, f_{\tilde{g}}(q, \{z_n\}) = 0 .
$$

(3.9)

This is why we dropped that term in $\tilde{g}^{K_S}(q, \{z_n\})$. We shall simplify the sums (3.8) in the appendix A.4 by a combined action of Riemann and Fay trisector identities. These manipulations result in the following expression for $\Delta^{1-\text{loop}}_{\mathcal{K}\mathcal{T}_{F^5}}$

$$
\Delta^{1-\text{loop}}_{\mathcal{K}\mathcal{T}_{F^5}} = \int_{\mathcal{I}_r} \frac{d^2 \tau}{\tau_2^6} \sum_{\beta} \frac{\theta_{\tilde{g}}(q)}{\eta(q)^2} \int_{\mathcal{I}_r} d^2 z_i \, \tilde{g}^{K_S}(q, \{z_n\}) \, f_{\tilde{g}}(q, \{z_n\}) ,
$$

(3.10)

with:

$$
\tilde{g}^{K_S}(q, \{z_n\}) = -(2\pi)^4 \left[ \partial G_B(z_1 - z_2) + \partial G_B(z_2 - z_3) + \partial G_B(z_3 - z_1) \right],
$$

$$
\tilde{g}^{K_P}(q, \{z_n\}) = (2\pi)^4 \left[ \partial G_B(z_1 - z_2) + \partial G_B(z_2 - z_3) + \partial G_B(z_3 - z_4) + \partial G_B(z_4 - z_5) + \partial G_B(z_5 - z_1) \right].
$$

(3.11)

This form of the spin structure-dependent piece of (3.5) is very convenient for performing the integrations over the positions $z_i$. Indeed, since the gauge part $f_{\tilde{g}}(q, \{z_n\})$ is periodic at the boundary of $\mathcal{I}_r$, the integral (3.10) vanishes after partial integration:

$$
\int_{\mathcal{I}_r} d^2 z_j \, \partial G_B(z_j - z_k) \, f_{\tilde{g}}(q, \{z_n\}) = 0 .
$$

(3.12)

We conclude:

$$
\Delta^{1-\text{loop}}_{\mathcal{K}_S, P\mathcal{T}_{F^5}} = 0 .
$$

(3.13)

Since it is the form of the space-time part (3.11) which leads to the vanishing of $\mathcal{T}_{F^5}$ couplings, we may conclude the same for the other group-theoretical contraction: $\Delta^{1-\text{loop}}_{\mathcal{K}_S, P\mathcal{T}_{F^2}\mathcal{T}_{F^3}} = 0$.

The form of (3.11) is very useful for analyzing eventual singularities that could appear when the vertex positions $z_i, z_j$ approach each other. Due to supersymmetry, there are no singularities from three, four or five points colliding. However, the so-called pinch effects appear in certain regions of the integration domain $\mathcal{I}_r$ in the limit $z_i \to z_j$. Then the momenta of the exponentials of (2.2) cannot be a priori neglected. In this limit, the fermionic correlators behave as $G\tilde{\alpha}(z_{ij}) \to 1/z_{ij}, G\tilde{\beta}(\bar{z}_{ij}) \to 1/\bar{z}_{ij}$, while the exponentials $\langle e^{ik_i X(z, \bar{z})} e^{ik_j X(z, \bar{z})} \rangle \sim |z_{ij}|^{\alpha' k_1 k_2}$, and we encounter:

$$
\int_{|z_{ij}|<\epsilon} \frac{|z_{ij}|^{\alpha' k_1 k_2}}{|z_{ij}|^2} = \frac{2\pi}{\alpha' k_1 k_2} .
$$

(3.14)

This signals poles from massless particle exchanges in one-particle reducible diagrams [23]. Setting the momenta of the exponentials to zero means that we neglect such reducible contributions, which is the right thing to do when discussing the effective action.
4. Vanishing of heterotic one-loop $F^6$

In this section, we prove that the one-loop corrections to the (CP even) space-time kinematics, $\text{tr}F^6$, $\text{tr}F^4\text{tr}F^2$, $(\text{tr}F^2)^3$ and $(\text{tr}F^3)^2$, vanish exactly for any gauge configuration. These four space-time contractions will be denoted by $H$, $S$, $L$ and $T$, respectively, referring to their diagrammatic representation, see $\blacksquare$. We will consider the six-point gauge boson amplitude:

$$
\langle V_{\mu_1}^a(z_1, \overline{z}_1) \ldots V_{\mu_6}^a(z_6, \overline{z}_6) \rangle_{\text{even}}
$$

(4.1)

and extract the relevant kinematic pieces. The gauge boson vertex operator is given in (2.2) and the gauge currents are fermionized according to (2.3). The one-loop fermion propagator is written for even spin structure in (3.2). In the following, we shall first discuss the gauge hexagon $\text{Tr}F^6$ case. In order to yield the corresponding group-theoretical factor $\text{Tr}(T^{a_i}T^{a_j}T^{a_k}T^{a_m}T^{a_n})$, the world-sheet gauge fermions (2.3) must be contracted in such a way that the vertex positions $z_i$ form a hexagon. Thus for the spin structure $\vec{\beta} = (\beta_1, \beta_2)$, the gauge part becomes

$$
f_{\vec{\beta}}(q, \{z_n\}) = G_{\vec{\beta}}^F(z_{ij}) G_{\vec{\beta}}^F(z_{jk}) G_{\vec{\beta}}^F(z_{kl}) G_{\vec{\beta}}^F(z_{lm}) G_{\vec{\beta}}^F(z_{mn}) G_{\vec{\beta}}^F(z_{ni}) .
$$

(4.2)

Space-time supersymmetry requires that at least eight fermions from the six vertex operators (2.2) are taken into account. Thus, we have to consider two cases: contracting eight fermions or all twelve fermions. Let us discuss the latter case first. Then the respective parts of vertices yield the desired $O(k^6)$ order in momentum. Thus we may neglect the exponentials as they would increase the power of momentum. Depending on the way how these twelve fermions are contracted, four different kinematical configurations $\mathcal{K}_L, \mathcal{K}_S, \mathcal{K}_H$ and $\mathcal{K}_T$ arise. They correspond to the following contractions:

$$
\begin{align*}
\tilde{g}_{\vec{\alpha}}^{\mathcal{K}_L}(q, \{z_n\}) &= G_{\vec{\alpha}}^F(z_{12})^2 G_{\vec{\alpha}}^F(z_{34})^2 G_{\vec{\alpha}}^F(z_{56})^2 , \\
\tilde{g}_{\vec{\alpha}}^{\mathcal{K}_S}(q, \{z_n\}) &= -G_{\vec{\alpha}}^F(z_{12}) G_{\vec{\alpha}}^F(z_{23}) G_{\vec{\alpha}}^F(z_{34}) G_{\vec{\alpha}}^F(z_{41}) G_{\vec{\alpha}}^F(z_{56})^2 , \\
\tilde{g}_{\vec{\alpha}}^{\mathcal{K}_H}(q, \{z_n\}) &= -G_{\vec{\alpha}}^F(z_{12}) G_{\vec{\alpha}}^F(z_{23}) G_{\vec{\alpha}}^F(z_{34}) G_{\vec{\alpha}}^F(z_{45}) G_{\vec{\alpha}}^F(z_{56}) G_{\vec{\alpha}}^F(z_{61}) , \\
\tilde{g}_{\vec{\alpha}}^{\mathcal{K}_T}(q, \{z_n\}) &= G_{\vec{\alpha}}^F(z_{12}) G_{\vec{\alpha}}^F(z_{23}) G_{\vec{\alpha}}^F(z_{31}) G_{\vec{\alpha}}^F(z_{45}) G_{\vec{\alpha}}^F(z_{56}) G_{\vec{\alpha}}^F(z_{64}) ,
\end{align*}
$$

(4.3)

respectively. Thus in total, the four different kinematics $\mathcal{K}$ receive the following one-loop corrections:

$$
\Delta_{\mathcal{K} \text{Tr}F^6}^{1-\text{loop}} = \int \frac{d^2 \tau}{\tau_2} \frac{1}{\eta(q)^8} \frac{1}{\eta(q)^8} \sum_{\vec{\alpha}, \vec{\beta}} s_{\vec{\alpha}} \theta_{\vec{\alpha}}(q)^4 \theta_{\vec{\beta}}(\overline{q})^{16} \int_{I_r} \prod_{i=1}^6 d^2 z_i \tilde{g}_{\vec{\alpha}}^{\mathcal{K}}(q, \{z_n\}) f_{\vec{\beta}}(q, \{z_n\}) .
$$

(4.4)

---

11 See also the comment made at the end of the previous section.
4.1. Symmetric Trace in the gauge combination

We will first investigate one special combination of $F^6$ couplings, the symmetric trace $\text{STr} F^6$. The string amplitude (4.1) includes all permutations of gauge group generators $T^a$; any such permutation is equal to $\text{Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4}T^{a_5}T^{a_6})$ up to some commutator terms. The commutators can be discarded if one appropriately symmetrizes in the positions of gauge currents. Thus extracting the (gauge) hexagonal $\text{STr} F^6$ term from (4.1) amounts to averaging over 60 permutations:

$$f_\beta(q) = \frac{1}{60} \sum_{\text{60 permutations}} G^F_\beta(z_{ij})G^F_\beta(z_{jk})G^F_\beta(z_{kl})G^F_\beta(z_{lm})G^F_\beta(z_{mn})G^F_\beta(z_{ni}) = -\frac{1}{120} \frac{\partial^6}{\partial z^6} \ln \theta_\beta(0, \tau),$$

with the overall group-theoretical factor $\text{Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4}T^{a_5}T^{a_6})$. The last equality is a generalization of the identity [21]

$$B_\beta(z_{12}, z_{23}, z_{34}) + B_\beta(z_{12}, z_{34}, z_{23}) + B_\beta(z_{12}, z_{23}, z_{34}) = -\frac{1}{2} \frac{\partial^4}{\partial z^4} \ln \theta_\beta(0, \tau), \quad (4.6)$$

where

$$B_\beta(z_{12}, z_{23}, z_{34}) \equiv G^F_\beta(z_{12})G^F_\beta(z_{23})G^F_\beta(z_{34})G^F_\beta(z_{41}). \quad (4.7)$$

Thanks to the relation (4.5), the left-moving part $f_\beta(q, \{z_n\})$ of (4.4) does not depend on the vertex positions and now we may permute them also in the functions $g^K_\alpha(q, \{z_n\})$ without affecting the integral. Thus we may borrow (4.6) and (4.5) to simplify the space-time parts $g^K_\alpha(q, \{z_n\})$ by appropriate symmetrizations:

$$g^K_\alpha(q, \{z_n\}) = \frac{1}{6} \frac{\partial^4}{\partial z^4} \ln \theta_\alpha(0, \tau) \ G^F_\alpha(z_{56})^2, \quad (4.8)$$

For the square of the fermionic propagator $G^F_\alpha(z_{56})^2$ in $g^K_\alpha(q, \{z_n\})$ we use the identity (3.4). With the same argument as outlined after Eq. (3.8), we may drop its last term.

---

12 For a given hexagonal diagram, which contracts the twelve fermions in the order $(i, j, k, l, m, n)$, there exist 5 equivalent diagrams: $(j, k, l, m, n, i), \ldots, (n, i, j, k, l, m)$, corresponding to the cyclic permutations. Furthermore, changing their orientation results in twelve equivalent diagrams. Thus, from the 6! possible permutations we only take into account $720/12 = 60$ hexagonal diagrams.

13 This identity can also be proven for higher genus $\theta$-functions [26].

14 Because of this property, it was justified to set the momenta of the exponentials of (2.3) to zero from the beginning. With these exponentials no poles in momenta (in the sense of [25]) that would decrease the total power in momenta are generated.
Thus the one-loop corrections to the four space-time kinematics become:

$$\Delta^{1-\text{loop}}_{\mathcal{K}\text{STrf}^6} = -\frac{2^6}{120} \int d^2 \tau \tilde{g}^{\mathcal{K}}(q) \sum_{\beta} \frac{\theta_{\beta}(\eta)}{\eta(\eta)^2} \frac{\partial^6}{\partial z^6} \ln \theta_{\beta}(0) ,$$

with the functions:

\[
\tilde{g}_L^\mathcal{K}(q) = \frac{1}{\eta^{12}} \sum_{\alpha} s_{\tilde{\alpha}} \theta_{\tilde{\alpha}}(q) \left( \frac{\partial^2}{\partial z^2} \ln \theta_{\tilde{\alpha}}(0, \tau) + \frac{\pi}{\tau_2} \right)^3 = \frac{(2\pi)^6}{4} \hat{E}_2 ,
\]

\[
\tilde{g}_S^\mathcal{K}(q) = \frac{1}{6 \eta^{12}} \sum_{\alpha} s_{\tilde{\alpha}} \theta_{\tilde{\alpha}}(q) \left( \frac{\partial^4}{\partial z^4} \ln \theta_{\tilde{\alpha}}(0, \tau) \left( \frac{\partial^2}{\partial z^2} \ln \theta_{\tilde{\alpha}}(0, \tau) + \frac{\pi}{\tau_2} \right) \right) = -\frac{(2\pi)^6}{12} \hat{E}_2 ,
\]

\[
\tilde{g}_H^\mathcal{K}(q) = \frac{1}{120 \eta^{12}} \sum_{\alpha} s_{\tilde{\alpha}} \theta_{\tilde{\alpha}}(q) \frac{\partial^6}{\partial z^6} \ln \theta_{\tilde{\alpha}}(0, \tau) = 0 ,
\]

\[
\tilde{g}_T^\mathcal{K}(q) = 0 .
\]

Note the relations:

$$L = -3S \quad , \quad H = 0 ,$$

with $L = \tilde{g}_L^\mathcal{K}(q)$, $S = \tilde{g}_S^\mathcal{K}(q)$ and $H = g_H^\mathcal{K}(q)$. These are exactly the same relations as they appear at two loops \[ as a solution of the constraints implied by Riemann identities. It is remarkable that they can be derived at one loop directly.

Let us now consider the second contribution, where only eight fermions of the gauge boson vertex operators (2.3) are contracted. The eight fermions stemming from those four vertex operators give rise to the $t^8$ structure of space-time kinematics. The other two momenta arise from the two exponentials of the remaining two gauge vertex operators (labeled by $i$ and $j$), contracted with their $\partial X_i$, $\partial X_j$:

\[
(k_i \epsilon_j)(k_j \epsilon_i) \langle \partial X(z_i)X(z_j) \rangle \langle \partial X(z_j)X(z_i) \rangle = -(k_i \epsilon_j)(k_j \epsilon_i) [\partial_{z_i} G_B(z_{ij})]^2 .
\]

Thus these contractions will give additional contributions to the kinematics $\mathcal{K}_L$ and $\mathcal{K}_S$, but not to $\mathcal{K}_H$. The $t^8$ part is the same correlator that appears in the four gauge boson amplitude. In particular, this means that all (holomorphic) position dependence in $z_k \neq z_i, z_j$ drops out after applying Riemann identity on the spin structure sum involving a

\[ The following equations are obtained by using the Riemann identity (A.5), $\frac{\partial^3}{\partial z^3} \theta_1(0, \tau) = -\frac{1}{32\pi \tau} \eta^3 E_2$, and $\hat{E}_2 = E_2 - \frac{3}{\pi^2}$. \]
product of four fermionic Green's functions whose positions form two lines or a square. They give the constants $\mp (2\pi)^4$, respectively. The only $z$-integral to be done is [5]:

$$
\int_{I_r} d^2 z_i d^2 z_j \partial z_i G_B(z_i - z_j) \partial z_j G_B(z_j - z_i) = \frac{4\pi^2}{3} \tau_2^2 \tilde{E}_2(q) ,
$$

(4.13)
giving rise to additional contributions to $\tilde{g}^{K_L}$ and $\tilde{g}^{K_S}$ of Eq.(4.11). In order to compare these contributions with the previous ones, we should perform the (trivial) integral over two points $z_i$ and $z_j$ in (4.10), which gives a factor of $(2\tau_2)^2$. Finally, we have to take into account that there are three possibilities to obtain a given $L$ kinematics from contracting only eight fermions. Multiplying all factors, we see that for a given $L$ or $S$ kinematics, the contributions of twelve-fermion contractions cancel against those of eight-fermion contractions. To summarize, our final result is:

$$
\Delta_{K_{L,S,H,T}}^1 = 0 .
$$

(4.14)

This result could have also been anticipated by observing the following identity in the gauge sector:

$$
\sum_\beta \theta_\beta(\vec{q})^6 \frac{\partial^6}{\partial z^6} \ln \theta_\beta(0) = 0 .
$$

(4.15)

Therefore, the vanishing of the one-loop corrections to STf$^6$ for SO(32) gauge group has two independent explanations: one relying on the cancellations in the gauge fermion sector and another one originating from the cancellations in the space-time fermion sector. Of course, the latter cancellations are more general, and allow us to generalize our findings – it was only the application of the symmetric trace “prescription” on the gauge part, resulting in (4.5), which allowed further simplifications of the space-time part, finally resulting in the elimination of any position dependence in the integrand (4.9). This procedure does not depend on the gauge group or on group-theoretical contractions. All our arguments from above can be applied to show that also:

$$
\Delta_{K_{L,S,H,T}}^{\text{1-loop}} = 0 ,
$$

$$
\Delta_{K_{L,S,H,T}}^{\text{1-loop}}(\text{Tr} F^2)^3 = 0 .
$$

(4.16)

4.2. Generic gauge combination

One of the main properties of the one-loop STf$^6$ coupling, calculated in the previous subsection, was the independence of the gauge part $f_\beta(\vec{q}; \{z_n\})$ of the vertex positions $z_i$. This is a consequence of symmetry and protects us from possible poles (of the kind (3.14)) in the integrand, due to massless particle exchange. Furthermore, this position independence of the gauge part allowed substantial simplification of the space-time part as well.
When we do not impose the symmetrized gauge trace on the gauge part, the dependence of \((4.2)\) on the vertex positions does not simplify as in Eq.\((4.5)\), and the left- and right-moving parts are coupled through the position integral \((4.4)\). When one performs these integrals explicitly, there appears one obvious complication: the \(z\)-integrals under consideration contain fermion propagators \(G^F(z)\) with \(z\)-arguments that may take values outside of the fundamental domain \(\mathcal{I}_\tau\). The fermionic Green’s functions have the periodicity behaviour: \(G^F_\alpha(z+1,q) = -e^{2\pi i \alpha_1} G^F_\alpha(z,q)\) and \(G^F_\alpha(z+\tau,q) = -e^{2\pi i \alpha_2} G^F_\alpha(z,q)\) under \(z \rightarrow z+1\) and \(z \rightarrow z + \tau\). Thus we pick up phases when leaving \(\mathcal{I}_\tau\). The expression for \(G^F(z)\) as a power series, used so far in the literature (see e.g. [3]),

\[
G^F_\alpha(z) = 2\pi i \sum_{n \in \mathbb{Z}} \frac{e^{2\pi i (n+\alpha_1+\frac{1}{2})z}}{1 + (-1)^{2\alpha_2} q^{n+\alpha_1+\frac{1}{2}}} \tag{4.17}
\]

is not appropriate to capture these complications. It represents a convergent power series only inside \(\mathcal{I}_\tau\). These problems can be overcome if we perform a double Fourier expansion of \((4.17)\). Introducing \(x\) and \(y\), with \(z = x + \frac{1}{\tau_2} y + iy\), i.e. \(x = \text{Re}(z) - \frac{\alpha_1}{\tau_2} \text{Im}(z)\), \(y = \text{Im}(z)\), we perform a Fourier expansion w.r.t. \(x\) with period 2 and w.r.t. \(y\) with period \(2\tau_2\), to obtain:

\[
G^F_\alpha(z) = \sum_{(m,n)} \Lambda(s) \frac{1}{m + \frac{1}{2} + \alpha_2 + (n + \alpha_1 + \frac{1}{2}) \tau} e^{2\pi i x(n+\alpha_1+\frac{1}{2})} e^{2\pi i (m+\frac{1}{2}+\alpha_2) \frac{y}{\tau_2}}. \tag{4.18}
\]

With the regulator

\[
\Lambda(s) = \frac{1}{|m + \frac{1}{2} + \alpha_2 + (n + \alpha_1 + \frac{1}{2}) \tau|^s}, \tag{4.19}
\]

for \(s > 1\), the function \((4.18)\) transforms manifestly covariantly under modular transformations, and \(G^F_\alpha(z)\) is defined by analytic continuation to \(s = 0\). In this form, \(G^F_\alpha(z)\) furnishes the desired properties under the shifts \(x \rightarrow x + 1\) and \(y \rightarrow y + \tau_2\) corresponding to \(z \rightarrow z+1\) and \(z \rightarrow z + \tau\), respectively.

The Fourier expansion \((4.18)\) is particularly convenient if the space-time part decouples from the gauge part. In that case we find a generating function for the integral \((d^2 z = 2dxdy)\):

\[
\int \prod_{i=1}^N d^2 z_1 G^F_{\beta}(z_{12}) \ldots G^F_{\beta}(z_{N1}) = \sum_{(m,n)} \frac{(2\tau_2)^N}{m + \frac{1}{2} + \beta_2 + (n + \frac{1}{2} + \beta_1) \tau} \left[ 2^{2\beta_1} N E_N(4^{2\beta_1} \frac{\tau}{2} + \beta_1 + \beta_2 + \frac{1}{2}) - E_N(\tau) \right] \tag{4.20}
\]

\[
= -\frac{(2\tau_2)^N}{(N-1)!} \frac{\partial^N}{\partial \theta^N} \ln \theta_\beta(0, \tau),
\]
valid for even \( N > 2 \). For odd \( N \) the integral vanishes. This relation should be compared with the identities (4.3) and (4.5).

After these preliminaries, let us first discuss the case with only eight space-time fermions contracted. This case gives contributions only to the kinematics \( \mathcal{K}_L \) and \( \mathcal{K}_S \), whose dependence on the vertex positions is given by (cf. (4.12)):

\[
[ \partial z_i G_B(z_{ij}) ]^2 \ G^E_\beta(\bar{z}_{12}) \ G^E_\beta(\bar{z}_{23}) G^E_\beta(\bar{z}_{34}) \ G^E_\beta(\bar{z}_{45}) \ G^E_\beta(\bar{z}_{56}) \ G^E_\beta(\bar{z}_{61}). \tag{4.21}
\]

Here \( i,j \) denote those two gauge boson vertex operators in (4.11) whose exponentials and \( \partial X(z) \) contribute instead of their fermion pairs. To perform the integral

\[
\mathcal{R}_{ij} = - \int \prod_{k=1}^{6} d^2 z_k \ [ \partial z_i G_B(z_{ij}) ]^2 \ G^E_\beta(\bar{z}_{12}) \ G^E_\beta(\bar{z}_{23}) G^E_\beta(\bar{z}_{34}) \ G^E_\beta(\bar{z}_{45}) \ G^E_\beta(\bar{z}_{56}) \ G^E_\beta(\bar{z}_{61}) \tag{4.22}
\]

we use the explicit expression (4.18) for \( G^E_\beta(z) \) and the Fourier expansion for \( \partial G_B(z) \), which may be found in [3]:

\[
\partial G_B(z) = \sum_{(M,N) \neq (0,0)} \frac{1}{M + N \tau} \frac{1}{|M + N \tau|^s} e^{2 \pi i M x} e^{2 \pi i M y/\tau_2}. \tag{4.23}
\]

In the following, let us evaluate \( \mathcal{R}_{ij} \) for \( i < j \). The integral \( \mathcal{R}_{ij} \) (performed in the variables \( x, y \)) leads to various projections on the integers of the sums \( \partial G_B(z_{ij}) \) and \( G_F(z_{rs}) \). To this end we arrive at:

\[
\mathcal{R}_{ij} = (2\tau_2)^6 \sum_{(M,N) \neq (0,0)} \sum_{k,l \neq (-M,-N)} \frac{1}{M + N \tau} \frac{1}{M + k + (N + l) \tau} \frac{1}{|M + N \tau|^s} \frac{1}{|M + k + (N + l) \tau|^s} \times \sum_{m,n \in \mathbb{Z}} \frac{1}{|m - k + \frac{1}{2} + \beta_2 + (n - l + \frac{1}{2} + \beta_1) \tau|^{j+i}} \frac{1}{|m + \frac{1}{2} + \beta_2 + (n + \frac{1}{2} + \beta_1) \tau|^{6-j-i}}. \tag{4.24}
\]

For finite \( (k,l) \neq (0,0), (-M,-N) \) the sum over \( M,N \) can be expressed as partial fraction

\[
\sum_{M,N \neq (0,0)} \left( \frac{1}{M + N \tau} - \frac{1}{M + k + (N + l) \tau} \right) \frac{1}{k + l \tau},
\]

which converges and vanishes. Therefore, non-vanishing contributions arise only for \( (k,l) = (0,0) \):

\[
\mathcal{R}_{ij} = (2\tau_2)^6 \sum_{(M,N) \neq (0,0)} \frac{1}{(M + N \tau)^2} \frac{1}{|M + N \tau|^s} \sum_{m,n \in \mathbb{Z}} \frac{1}{|m + \frac{1}{2} + \beta_2 + (n + \frac{1}{2} + \beta_1) \tau|^{6}}
\]

\[
= - \frac{(2\tau_2)^6 \pi^2}{120} \frac{\partial^6}{\partial z^6} \ln \theta_\beta(0, \tau). \tag{4.25}
\]
Thus, after performing the integral \((4.22)\), the space-time part becomes decoupled from the gauge part. The latter is described by the second sum in \((4.25)\) and may be evaluated with \((4.20)\). In that form, it becomes obvious that it will lead to a vanishing gauge lattice sum \((4.15)\).

To obtain something potentially non-vanishing we shall take into account all twelve fermions contracted. Similarly as in the \(\text{Tr}F^5\) case, we first simplify the spin structure sums involving the correlators \((4.3)\) for the four possible space-time kinematics:

\[
\tilde{g}^{K_L}(q, \{z_n\}) = \frac{1}{\eta^{12}} \sum_\alpha s_\alpha \theta_\alpha(0) \left( \frac{\partial^2}{\partial z^2} \theta_\alpha(0, \tau) + \frac{\pi}{\tau_2} \right)^3 = \frac{(2\pi)^6}{4} \tilde{E}_2 ,
\]

\[
\tilde{g}^{K_S}(q, \{z_n\}) = - (2\pi)^4 \sum_\alpha s_\alpha \frac{1}{\theta_\alpha(0)} \left( \frac{\partial^2}{\partial z^2} \theta_\alpha(0, \tau) + \frac{\pi}{\tau_2} \right) \times \frac{\theta_\alpha(z_{12}) \theta_\alpha(z_{23}) \theta_\alpha(z_{34}) \theta_\alpha(z_{41})}{\theta_1(z_{12}) \theta_1(z_{23}) \theta_1(z_{34}) \theta_1(z_{41})} ,
\]

\[
\tilde{g}^{K_H}(q, \{z_n\}) = - (2\pi)^6 \eta^6 \sum_\alpha s_\alpha \frac{1}{\theta_\alpha(0)^2} \frac{\theta_\alpha(z_{12}) \theta_\alpha(z_{23}) \theta_\alpha(z_{34})}{\theta_1(z_{12}) \theta_1(z_{23}) \theta_1(z_{34})} \times \frac{\theta_\alpha(z_{45}) \theta_\alpha(z_{56}) \theta_\alpha(z_{61})}{\theta_1(z_{45}) \theta_1(z_{56}) \theta_1(z_{61})} ,
\]

\[
\tilde{g}^{K_T}(q, \{z_n\}) = (2\pi)^6 \eta^6 \sum_\alpha s_\alpha \frac{1}{\theta_\alpha(0)^2} \frac{\theta_\alpha(z_{12}) \theta_\alpha(z_{23}) \theta_\alpha(z_{31})}{\theta_1(z_{12}) \theta_1(z_{23}) \theta_1(z_{31})} \times \frac{\theta_\alpha(z_{45}) \theta_\alpha(z_{56}) \theta_\alpha(z_{64})}{\theta_1(z_{45}) \theta_1(z_{56}) \theta_1(z_{64})} .
\]

Again, for the square \(G_F(z)^2\) we used \((3.6)\) and dropped its second term. The latter gives a vanishing contribution \((3.9)\) after partial integrations over the positions \(z_i\) due to the periodicity of both \(G_B(z)\) and \(f_\beta(\vec{q}, \{\vec{z}_n\})\) at the boundary of \(I_\tau\). In the appendix A.5, we simplify the sums \((4.26)\) by a combined action of Riemann and Fay trisecant identities. These manipulations result in the following expression:

\[
\Delta^{\text{1-loop}}_{K_{\text{Tr}F^6}} = \int \frac{d^2\tau}{\tau_0} \sum_\beta \frac{\theta_\beta(\vec{q})^{16}}{\eta(\vec{q})^{24}} \int_{I_\tau} \prod_{i=1}^6 d^2z_i \tilde{g}^K(q, \{z_n\}) f_\beta(\vec{q}, \{\vec{z}_n\}) ,
\]

with the functions \(\tilde{g}^K(q, \{z_n\})\) given in Eqs. (A.16), (A.17) and (A.18) for the kinematics \(K_S, K_H\) and \(K_T\), respectively. With these expressions and noting

\[
\int_{I_\tau} d^2z_j d^2z_k \partial^2 G_B(z_j - z_k) f_\beta(\vec{q}, \{\vec{z}_n\}) = 0
\]

\[
\int_{I_\tau} d^2z_j d^2z_l \partial G_B(z_j - z_k) \partial G_B(z_l - z_m) f_\beta(\vec{q}, \{\vec{z}_n\}) = 0
\]

\[
\int_{I_\tau} d^2z_j d^2z_l \partial G_B(z_j - z_k) \partial G_B(z_l - z_k) f_\beta(\vec{q}, \{\vec{z}_n\}) = 0 ,
\]
it is straightforward to show that the integrals (4.27) for $K_H$ and $K_T$ vanish after partial integrations over $I_\tau$. The last two terms of the function $\tilde{g}^{K_S}$, shown in Eq.(A.16), do not give zero after integrating them with the gauge part (by applying (4.25)). However, they give a contribution which is cancelled again by the relevant eight-fermion contraction $R_{ij}$ after taking into account the right factors, as discussed in the previous section. Finally, the contribution from $\tilde{g}^{K_L}$ is cancelled against the term coming from the eight-fermion contractions (4.25).

We conclude:

$$\Delta^{1-\text{loop}}_{K_L,S,H,T,\text{Tr} F^6} = 0$$

(4.29)

for a general hexagonal gauge contraction and the space-time kinematics $K_L, K_S, K_H$ and $K_T$. As in the previous section, the vanishing is an effect of cancellations in the space-time fermion sector. Thus it holds for any gauge group. All the previous steps, together with some partial integrals of the kind (3.9), can be repeated to prove the same thing for other group-theoretical contractions:

$$\Delta^{1-\text{loop}}_{K_L,S,H,T,\text{Tr} F^4 \text{Tr} F^2} = 0$$,     $$\Delta^{1-\text{loop}}_{K_L,S,H,T(\text{Tr} F^2)^3} = 0$$.

(4.30)

Finally, let us briefly comment on the one-loop corrections to CP-odd $F^6$ couplings which appear in the discussion of anomaly cancellation. These corrections have been calculated in [5] and, except for the correction to $\text{Tr} F^6$ which vanishes as a result of (4.15), they receive non-vanishing contributions from the boundary of the fundamental domain.

5. Conclusions

Type I - heterotic duality in $D=10$ predicts various relations and constraints on $F^n$ couplings at different string loop levels on both sides, as shown in the Table displayed in the Introduction. One of the basic predictions of this duality is the vanishing of two-loop corrections to the heterotic $F^4$. We proved that this is indeed the case in Section 2 by using the hyperelliptic approach to genus two Riemann surfaces. This result is related by supersymmetry to Adler-Bardeen theorem for Green-Schwarz anomaly.

Furthermore, in Section 3, we showed that all heterotic $F^5$ terms vanish at one loop. In type I theory, there is a convincing evidence [27,28] that non-vanishing $F^5$ terms appear already at the classical level. Formally, this corresponds to 1.5 loops on the heterotic side, hence an order by order comparison may not be appropriate in this case.

Similarly, all one-loop contributions to the heterotic (CP even) $F^6$ are zero, as shown in Section 4. Matching na"ively to the dual side, this excludes such type I couplings at order $e^{\Phi_1/2}$. Apart from the tree-level $\text{Tr} F^6$, which was the focus of [1], the only room left for such terms in type I theory is at order $e^{3\Phi_1/2}$. It corresponds to a tree-level coupling on the heterotic side and probably vanishes on similar grounds as $\text{Tr} F^4$ does [3]. If this is indeed
the case, our results support the conjecture that any tree-level, NBI type I Tr$F^{2n}$ coupling appears in $D=10$ only at $n-1$ loops on the heterotic side. Furthermore, the classical NBI action should not receive quantum corrections, at least for $n \leq 3$.

Several comments are in order here. The computations of [1] indicate that some Tr$F^6$ terms are basically different from the conventional BPS-saturated quantities, therefore they may escape a naïve duality argument. It may be a general pattern, that only certain kinematic structures, summarized in superinvariants, are useful objects in the framework of strong-weak coupling duality. In fact, a classification into several superinvariants – one class, which is sensitive only to BPS states and receives corrections at a specific loop order, and another class, which is sensitive to the full string spectrum and is renormalized at various loop orders – has been proposed for eight-fermion terms in [10]. The heterotic tree-level coupling $J_0 = t_8 t_8 R^4 - \frac{1}{8} \epsilon_{10} \epsilon_{10} R^4$, whose coupling constant is proportional to $\zeta(3)$ [4], receives higher order corrections and is not appropriate for a duality comparison in the above sense, in contrast to five other superinvariants which are related to anomaly cancellation terms. Two independent superinvariants have been argued to exist for non-Abelian Tr$F^6$ couplings in $N = 4$, $D = 4$ gauge theories [29] (see also [30]). The recently calculated tree-level Tr$F^5$ couplings on type I side [28], which are also proportional to $\zeta(3)$, are renormalized at one-loop [31]. On the other hand, following the Table and the results from section 3, such couplings cannot exist on the heterotic side. As argued before about $J_0$, Tr$F^5$ couplings on the type I side are not appropriate for a duality comparison. Thus the comparison order by order in coupling constant may be justified only for a certain subclass of couplings. We plan to carefully discuss this problem in the near future [26].

The vanishing of the heterotic $F^4$ at two loops, and of $F^5$ and $F^6$ at one loop is a consequence of supersymmetry encoded in Riemann identities. Compactifications on tori do not change these identities, therefore our results extend to arbitrary gauge groups and group-theoretical contractions in any heterotic string vacua with sixteen supercharges. In particular they hold for $D = 4$, $N = 4$ and $D = 3$, $N = 8$ heterotic vacua. This is in agreement with field theoretical arguments about $F^4$ couplings, which forbid corrections beyond one-loop in $D = 4$, $N = 4$ [34] and the absence of higher loop corrections in $D = 3$, $N = 8$ field theories [35].

Finally, what can we say more about the two-loop heterotic versus tree-level type I Tr$F^6$? In this paper, we have essentially eliminated one possibility, that the mismatch is due to some perturbative corrections complicating the comparison. Furthermore, in $D=10$ there are no instanton corrections from NS5 branes. Hence we are confident that we have a complete result, at least on the heterotic side [1]. However, there may be a subtlety on

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16 $F^4$ couplings have been discussed in the framework of heterotic - type I duality also in $D = 8$ [32,33]. Our heterotic results, valid in $D = 8$, could be useful in this context.
type I side. The Born-Infeld action describes open strings stretched between D9-branes while the heterotic action considered here maps via duality onto the full type I theory. The latter includes also non-perturbative states, not included in the Born-Infeld action. More work is necessary in order to understand how do they affect the low-energy interactions.

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Appendix A. Tools for one-loop amplitudes

A.1. Riemann identity

The genus one Riemann identity [19] reads:17

$$\sum_{\delta} s_{\delta} \theta_{\delta}(z_1) \theta_{\delta}(z_2) \theta_{\delta}(z_3) \theta_{\delta}(z_4) = 2 \theta_1(z'_1) \theta_1(z'_2) \theta_1(z'_3) \theta_1(z'_4), \quad (A.1)$$

with

$$\begin{pmatrix} z'_1 \\ z'_2 \\ z'_3 \\ z'_4 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix}, \quad (A.2)$$

and the phases $s_{(0,0)} = 1$, $s_{(\frac{1}{2},\frac{1}{2})} = -1$, $s_{(\frac{1}{2},0)} = -1$ and $s_{(\frac{1}{2},\frac{1}{2})} = 1$. The sum in (A.1) runs over both even and odd spin-structures. When one focuses on a CP even string amplitude one would like to have a similar formula with a sum over the even spin-structures only. A slight modification of (A.1) is the identity

$$\sum_{\delta} \tilde{s}_{\delta} \theta_{\delta}(z_1) \theta_{\delta}(z_2) \theta_{\delta}(z_3) \theta_{\delta}(z_4) = -2 \theta_1(z''_1) \theta_1(z''_2) \theta_1(z''_3) \theta_1(z''_4), \quad (A.3)$$

with the transformation

$$\begin{pmatrix} z''_1 \\ z''_2 \\ z''_3 \\ z''_4 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix}, \quad (A.4)$$

17 We refer the reader to Ref. [30] for an account of one-loop \(\theta\)-functions.
and the phases $\tilde{s}_{(0,0)} = 1$, $\tilde{s}_{(0,\frac{1}{2})} = -1$, $\tilde{s}_{(\frac{1}{2},0)} = -1$, and $\tilde{s}_{(\frac{1}{2},\frac{1}{2})} = -1$. We may combine Eqs. (A.1) and (A.3) to a sum over even spin structures only:

$$
\sum_{\delta \text{ even}} s_{\delta} \theta_{\delta}(z_1) \theta_{\delta}(z_2) \theta_{\delta}(z_3) = \theta_1(z'_1) \theta_1(z'_2) \theta_1(z'_3) \theta_1(z'_4) 
$$

(A.5)

$$
- \theta_1(z''_1) \theta_1(z''_2) \theta_1(z''_3) \theta_1(z''_4),
$$

with $z'_i$ and $z''_i$ given in (A.2) and (A.4), respectively.

### A.2. Fay trisecant identity and odd $\theta$-function relations

In this subsection we derive some useful $\theta$-function relations. We start from Fay trisecant identity [21]:

$$
\det_{i,j} Z_{\alpha}(x_i - y_j + D) = (-1)^{\frac{1}{2}n(n-1)} Z_{\alpha}(D)^{n-1} \left. Z_{\alpha}(\sum_{i=1}^{n} x_i - \sum_{i=1}^{n} y_i + D) \right|_{\xi = 0},
$$

(A.6)

with some divisor $D = \sum_i q_i \xi_i$ of weight zero ($\sum_i q_i = 0$),

$$
Z_{\alpha}(\sum_{i=1}^{n} x_i - \sum_{i=1}^{n} y_i) = \theta_{\alpha}(\sum_{i=1}^{n} x_i - \sum_{i=1}^{n} y_i) \frac{\prod_{i,j=1, i \neq j}^n E(x_i, x_j) \prod_{i,j=1, i \neq j}^n E(y_i, y_j)}{\prod_{i=1}^n E(x_i, y_i)},
$$

(A.7)

and the (one-loop) “prime form”:

$$
E(x, y) = \frac{\theta_1(x - y, \tau)}{\theta_1'(0, \tau)}.
$$

(A.8)

Eq. (A.6) holds for both even and odd spin structures $\alpha$. Here we shall be interested in the odd case, i.e. $\alpha = (\frac{1}{2}, \frac{1}{2})$. We seek for identities, which relate $\theta$-functions with multiple arguments, as they usually arise after applying the Riemann identity (A.1) to objects with fewer arguments. Since for $D = 0$ (A.6) becomes trivial ($Z_{\alpha}(0) \equiv \theta_{\alpha}(0)$, $Z_1(z) \equiv Z_{(\frac{1}{2}, \frac{1}{2})}(z)$, $\theta_1(z) \equiv \theta_{(\frac{1}{2}, \frac{1}{2})}(z)$) for odd $\alpha$, we shall first choose $D = \xi_1 - \xi_2$ and get rid of it later. For this choice we may find an useful relation for the case $n = 2$. We first multiply Eq. (A.6) by $E(\xi_1, \xi_2)^2$. Then we differentiate the resulting equation one times w.r.t. $\xi_1$ and take the limit $\xi_1 \to \xi_2$: Because in this limit $\theta_1(\xi_1 - \xi_2)$ becomes zero, the derivative has to act on the latter. Thus we obtain ($Z_1'(0) \equiv \theta_1'(0)$):

$$
Z_1(x_1 + x_2 - y_1 - y_2)Z_1'(0) = 
- \left. \frac{\partial}{\partial \xi} \left\{ \frac{\theta_1(x_1 - y_1 + \xi)}{E(x_1, y_1)} \frac{\theta_1(x_2 - y_2 + \xi)}{E(x_2, y_2)} - \frac{\theta_1(x_2 - y_1 + \xi)}{E(x_2, y_1)} \frac{\theta_1(x_1 - y_2 + \xi)}{E(x_1, y_2)} \right\} \right|_{\xi = 0}
$$

(A.9)

$$
= \theta_1'(0)^2 [g(x_2 - y_1) + g(x_1 - y_2) - g(x_1 - y_1) - g(x_2 - y_2)].
$$
Here, the function $g(x - y)$ is defined by

$$g(x - y) = \partial \ln \theta_1(x - y) .$$  \hspace{1cm} \text{(A.10)}$$

It is related to the Green’s function $\langle \tilde{\psi}(z)\tilde{\psi}(w) \rangle = \partial G_B(z - w)$ for the non-zero modes $\tilde{\psi}$ of odd fermions through the equation (cf. (3.7)):

$$g(z) = \partial G_B(z) - \frac{\pi}{\tau_2}(z - \bar{z}) .$$ \hspace{1cm} \text{(A.11)}$$

Similarly, we may proceed in the case $n = 3$. After multiplying (A.6) with $E(\xi_1, \xi_2)^3$, differentiating two times w.r.t. $\xi_1$ and taking the limit $\xi_1 \to \xi_2$ we obtain:

$$Z_1(x_1 + x_2 + x_3 - y_1 - y_2 - y_3)Z'_1(0)^2 = -\frac{1}{2} \frac{\partial^2}{\partial \xi^2} \det_{i,j} \frac{\theta_1(x_i - y_j + \xi)}{E(x_i, y_j)} \bigg|_{\xi=0}$$

$$= -\theta'_1(0)^3 \left[ -g(x_2 - y_1)g(x_1 - y_2) + g(x_3 - y_2)g(x_1 - y_2) + g(x_2 - y_1)g(x_2 - y_2) \
- g(x_3 - y_1)g(x_2 - y_2) - g(x_1 - y_1)g(x_3 - y_2) + g(x_2 - y_1)g(x_3 - y_2) \
+ g(x_2 - y_1)g(x_1 - y_3) - g(x_3 - y_1)g(x_1 - y_3) - g(x_2 - y_2)g(x_1 - y_3) \
+ g(x_3 - y_2)g(x_1 - y_3) - g(x_1 - y_1)g(x_2 - y_3) + g(x_3 - y_1)g(x_2 - y_3) \
+ g(x_1 - y_2)g(x_2 - y_3) - g(x_3 - y_2)g(x_2 - y_3) + g(x_1 - y_1)g(x_3 - y_3) \
- g(x_2 - y_1)g(x_3 - y_3) - g(x_1 - y_2)g(x_3 - y_3) + g(x_2 - y_2)g(x_3 - y_3) \right].$$  \hspace{1cm} \text{(A.12)}$$

\textbf{A.3. Inversion formula}

For $n = 2$ and even $\alpha$, Eq.\text{(A.6)} can be inverted:

$$Z_{\bar{\alpha}}(z_1 - z_2)Z_{\bar{\alpha}}(z_3 - z_4) =$$

$$\frac{1}{2} Z_{\bar{\alpha}}(0) [Z_{\alpha}(z_1 - z_2 + z_3 - z_4) - Z_{\bar{\alpha}}(z_1 + z_2 - z_3 - z_4) + Z_{\bar{\alpha}}(z_1 - z_2 - z_3 + z_4)] .$$  \hspace{1cm} \text{(A.13)}$$

We shall use this relation in our spin structure sums as a preparation for applying the Riemann identity \textbf{(A.5)}.

\textbf{A.4. Spin-structure sums for $F^5$}

In this subsection, we perform the spin structure sum on the space-time part. In all our equations, whenever $g$ appears, we may simply replace it by $\partial G_B$ without introducing extra terms. This reinstates the correct periodicity and modular behaviour.
Kinematics $\mathcal{K}_S$

Applying Riemann identities to the sum $\tilde{g}^{\mathcal{K}_S}$ of (3.8) gives:

$$-\tilde{g}^{\mathcal{K}_S}(q, \{z_n\}) = \left. \frac{(-2\pi)^3}{\eta^3} \frac{\partial^2}{\partial z^2} \sum_{\bar{\alpha} \text{ even}} s_{\bar{\alpha}} \theta_{\bar{\alpha}}(z) \theta_{\bar{\alpha}}(z_{12}) \theta_{\bar{\alpha}}(z_{23}) \theta_{\bar{\alpha}}(z_{31}) \right|_{z=0}$$

$$= -\frac{8\pi^3}{\eta^3} \left\{ 2\theta_1(z/2) \frac{\theta_1(z_1 - z_2 + \frac{\eta}{2}) \theta_1(z_2 - z_3 + \frac{\eta}{2}) \theta_1(z_3 - z_1 + \frac{\eta}{2})}{\theta_1(z_1 - z_2) \theta_1(z_2 - z_3) \theta_1(z_3 - z_1)} - 2\theta_1(z/2) \frac{\theta_1(z_1 - z_2 - \frac{\eta}{2}) \theta_1(z_2 - z_3 - \frac{\eta}{2}) \theta_1(z_3 - z_1 - \frac{\eta}{2})}{\theta_1(z_1 - z_2) \theta_1(z_2 - z_3) \theta_1(z_3 - z_1)} \right\}_{z=0}$$

$$= (2\pi)^4 \left[ g(z_1 - z_2) + g(z_2 - z_3) + g(z_3 - z_1) \right].$$

(A.14)

Kinematics $\mathcal{K}_P$

After applying the inversion formula (A.13) for the two products $\theta_\alpha(z_{12})\theta_\alpha(z_{34})$ and $\theta_\alpha(z_{23})\theta_\alpha(z_{45})$ in $\tilde{g}^{\mathcal{K}_P}$ of (3.8) we are ready to use Riemann identities for the spin structure sum:

$$-\tilde{g}^{\mathcal{K}_P}(q, \{z_n\}) = \left. \frac{(-2\pi)^5}{\eta^3} \sum_{\bar{\alpha} \text{ even}} s_{\bar{\alpha}} \frac{1}{\theta_{\bar{\alpha}}(0)} \theta_{\bar{\alpha}}(z_{12}) \theta_{\bar{\alpha}}(z_{23}) \theta_{\bar{\alpha}}(z_{34}) \theta_{\bar{\alpha}}(z_{45}) \theta_{\bar{\alpha}}(z_{51}) \right|_{z=0}$$

$$= -\frac{4\pi^3}{\eta^3} \left[ -Z_1(z_1 + z_2 - z_3 - z_5) - Z_1(z_1 - z_2 + z_3 - z_5) + Z_1(z_1 + z_2 - z_4 - z_5) - Z_1(z_1 + z_3 - z_4 - z_5) - Z_1(z_1 - z_2 + z_4 - z_5) - Z_1(z_1 - z_3 + z_4 - z_5) \right]$$

$$= (2\pi)^4 \left[ -g(z_1 - z_2) - g(z_2 - z_3) - g(z_3 - z_4) - g(z_4 - z_5) - g(z_5 - z_1) \right].$$

(A.15)

In the last equality we used (A.3) for $Z_1$'s.

A.5. Spin-structure sums for $F^6$

Let us now proceed to the spin structure sums in the $F^6$ couplings. Again, whenever $g$ appears, we may simply replace it by $\partial G_B$ without introducing extra terms.

Kinematics $\mathcal{K}_S$

After using the identity (A.13) for the two products $\theta_\alpha(z_{12})\theta_\alpha(z_{34})$ and $\theta_\alpha(z_{23})\theta_\alpha(z_{45})$ in
After applying the inversion formula (A.13) for the three products $K_{s}$ of (4.26), we apply Riemann identities for the spin structure sum:

$$
-\frac{\tilde{g}^{K_{s}}(q, \{z_{n}\})}{(2\pi)^{4}} = -\frac{\pi}{\tau_{2}} + \frac{\partial^{2}}{\partial z^{2}} \sum_{\tilde{\alpha} \text{ even}} s_{\tilde{\alpha}} \theta_{\tilde{\alpha}}(z) \frac{\theta_{\tilde{\alpha}}(z_{12}) \theta_{\tilde{\alpha}}(z_{23}) \theta_{\tilde{\alpha}}(z_{34}) \theta_{\tilde{\alpha}}(z_{41})}{\theta_{\tilde{\alpha}}(0) \theta_{1}(z_{12}) \theta_{1}(z_{23}) \theta_{1}(z_{34}) \theta_{1}(z_{41})} \bigg|_{z=0}
$$

$$
= \frac{1}{4} \theta_{1}'(0)^{-2} \left[ -Z_{1}(z_{1} + z_{2} - z_{3} - z_{4})^{2} + Z_{1}(z_{1} - z_{2} + z_{3} - z_{4})^{2} - Z_{1}(z_{1} - z_{2} - z_{3} + z_{4})^{2} \right] - \frac{1}{2} [\partial G_{B}(z_{1} - z_{3}) + \partial G_{B}(z_{2} - z_{4})]
$$

$$
= -\frac{1}{2} \left[ -g(z_{1} - z_{2})g(z_{1} - z_{3}) - g(z_{1} - z_{2})g(z_{2} - z_{3}) - g(z_{1} - z_{3})g(z_{2} - z_{3}) + g(z_{1} - z_{2})g(z_{1} - z_{4}) - g(z_{1} - z_{3})g(z_{1} - z_{4}) + 2g(z_{2} - z_{3})g(z_{1} - z_{4}) + g(z_{1} - z_{2})g(z_{2} - z_{4}) - g(z_{1} - z_{3})g(z_{2} - z_{4}) - g(z_{1} - z_{4})g(z_{2} - z_{4}) - 2g(z_{1} - z_{2})g(z_{3} - z_{4}) + g(z_{1} - z_{3})g(z_{3} - z_{4}) - g(z_{2} - z_{3})g(z_{3} - z_{4}) - g(z_{1} - z_{4})g(z_{3} - z_{4}) - g(z_{2} - z_{4})g(z_{3} - z_{4}) \right] - \frac{1}{2} [\partial G_{B}(z_{1} - z_{3}) + \partial G_{B}(z_{2} - z_{4})]
$$

$$
= -\frac{1}{2} \left[ g(z_{1} - z_{3})^{2} + g(z_{2} - z_{4})^{2} \right].
$$

(A.16)

For the last equality we made use of (A.9). As it turns out in section 4.2, the last two terms play an important rôle.

**Kinematics $K_{H}$**

After applying the inversion formula (A.13) for the three products $\theta_{\alpha}(z_{12})\theta_{\alpha}(z_{45})$, $\theta_{\alpha}(z_{23})\theta_{\alpha}(z_{56})$ and $\theta_{\alpha}(z_{34})\theta_{\alpha}(z_{61})$ in $\tilde{g}^{K_{s}}$ of (4.26), we use Riemann identities for the
\[-g^{\mathcal{K}_T}(q, \{z_n\}) = (2\pi)^6 \eta^6 \sum_{\vec{\alpha}} s_{\vec{\alpha}} \frac{1}{\theta_\alpha(0)^2} \frac{\theta_\alpha(z_{12}) \theta_\alpha(z_{23}) \theta_\alpha(z_{34}) \theta_\alpha(z_{45}) \theta_\alpha(z_{56}) \theta_\alpha(z_{61})}{\theta_1(z_{12}) \theta_1(z_{23}) \theta_1(z_{34}) \theta_1(z_{45}) \theta_1(z_{56}) \theta_1(z_{61})} \]
\[= \frac{1}{2} (2\pi)^4 [\sum_{\vec{\alpha}} s_{\vec{\alpha}} \frac{1}{\theta_\alpha(0)^2} \frac{\theta_\alpha(z_{12}) \theta_\alpha(z_{23}) \theta_\alpha(z_{34}) \theta_\alpha(z_{45}) \theta_\alpha(z_{56}) \theta_\alpha(z_{61})}{\theta_1(z_{12}) \theta_1(z_{23}) \theta_1(z_{34}) \theta_1(z_{45}) \theta_1(z_{56}) \theta_1(z_{61})} \]
\[= \frac{1}{2} (2\pi)^4 \left[ -g(z_1 - z_2)g(z_1 - z_3) - g(z_1 - z_2)g(z_2 - z_3) - g(z_1 - z_3)g(z_2 - z_3) \right. \]
\[\left. - g(z_2 - z_3)g(z_2 - z_4) - 2g(z_1 - z_2)g(z_3 - z_4) - g(z_2 - z_3)g(z_3 - z_4) \right. \]
\[\left. - g(z_2 - z_4)g(z_3 - z_4) + g(z_1 - z_3)g(z_1 - z_5) - g(z_1 - z_3)g(z_3 - z_5) \right. \]
\[\left. - g(z_3 - z_4)g(z_3 - z_5) + g(z_1 - z_5)g(z_3 - z_5) - 2g(z_1 - z_2)g(z_4 - z_5) \right. \]
\[\left. - 2g(z_2 - z_3)g(z_4 - z_5) - g(z_3 - z_4)g(z_4 - z_5) - g(z_3 - z_5)g(z_4 - z_5) \right. \]
\[\left. + g(z_1 - z_2)g(z_1 - z_6) + 2g(z_2 - z_3)g(z_1 - z_6) + 2g(z_3 - z_4)g(z_1 - z_6) \right. \]
\[\left. - g(z_1 - z_5)g(z_1 - z_6) + 2g(z_4 - z_5)g(z_1 - z_6) + g(z_1 - z_2)g(z_2 - z_6) \right. \]
\[\left. + g(z_2 - z_4)g(z_2 - z_6) - g(z_1 - z_6)g(z_2 - z_6) - g(z_2 - z_4)g(z_4 - z_6) \right. \]
\[\left. - g(z_4 - z_5)g(z_4 - z_6) + g(z_2 - z_6)g(z_4 - z_6) - 2g(z_1 - z_2)g(z_5 - z_6) \right. \]
\[\left. - 2g(z_2 - z_3)g(z_5 - z_6) - 2g(z_3 - z_4)g(z_5 - z_6) + g(z_1 - z_5)g(z_5 - z_6) \right. \]
\[\left. - g(z_4 - z_5)g(z_5 - z_6) + g(z_1 - z_6)g(z_5 - z_6) - g(z_4 - z_6)g(z_5 - z_6) \right] \]
\[\text{(A.17)} \]

In the last equality, we made use of (A.9) and (A.12).

**Kinematics \(K_T\)**

We use the inversion formula (A.13) for the two products \(\theta_\alpha(z_{12})\theta_\alpha(z_{45})\) and \(\theta_\alpha(z_{23})\theta_\alpha(z_{56})\)
in $g^{K_T}$ of (1.26) and then apply Riemann identities for the spin structure sum:

$$
\widetilde{g}^{K_T}(q, \{z_n\}) = (2\pi)^6 \eta^6 \sum_{\alpha \text{ even}} s_\alpha \frac{1}{\theta_\alpha(0)^2} \frac{\theta_\alpha(z_{12}) \theta_\alpha(z_{23}) \theta_\alpha(z_{31}) \theta_\alpha(z_{45}) \theta_\alpha(z_{56}) \theta_\alpha(z_{64})}{\theta_1(z_{12}) \theta_1(z_{23}) \theta_1(z_{31}) \theta_1(z_{45}) \theta_1(z_{56}) \theta_1(z_{64})}
$$

$$
= \frac{1}{2} (2\pi)^4 \left[ -g(z_1 - z_2)g(z_1 - z_3) + g(z_1 - z_2)g(z_2 - z_3) - g(z_1 - z_3)g(z_2 - z_3) \\
+ g(z_2 - z_3)g(z_2 - z_4) - g(z_2 - z_3)g(z_3 - z_4) + g(z_2 - z_4)g(z_3 - z_4) \\
+ g(z_1 - z_2)g(z_1 - z_5) - g(z_1 - z_2)g(z_2 - z_5) - g(z_2 - z_4)g(z_2 - z_5) \\
+ g(z_1 - z_5)g(z_2 - z_5) - 2g(z_1 - z_2)g(z_4 - z_5) + 2g(z_1 - z_3)g(z_4 - z_5) \\
- 2g(z_2 - z_3)g(z_4 - z_5) + g(z_2 - z_4)g(z_4 - z_5) - g(z_2 - z_5)g(z_4 - z_5) \\
+ g(z_1 - z_3)g(z_1 - z_6) - g(z_1 - z_5)g(z_1 - z_6) - g(z_1 - z_3)g(z_3 - z_6) \\
- g(z_3 - z_4)g(z_3 - z_6) + g(z_1 - z_6)g(z_3 - z_6) + 2g(z_1 - z_2)g(z_4 - z_6) \\
- 2g(z_1 - z_3)g(z_4 - z_6) + 2g(z_2 - z_3)g(z_4 - z_6) + g(z_3 - z_4)g(z_4 - z_6) \\
+ g(z_4 - z_5)g(z_4 - z_6) - g(z_3 - z_6)g(z_4 - z_6) - 2g(z_1 - z_2)g(z_5 - z_6) \\
+ 2g(z_1 - z_3)g(z_5 - z_6) - 2g(z_2 - z_3)g(z_5 - z_6) + g(z_1 - z_5)g(z_5 - z_6) \\
- g(z_4 - z_5)g(z_5 - z_6) - g(z_1 - z_6)g(z_5 - z_6) + g(z_4 - z_6)g(z_5 - z_6) \right].
$$

(A.18)

Again, in the last equality we made use of (A.9) and (A.12).

Appendix B. Two-loop $\text{Tr}F^4$ for a different gauge slice

The path integral for a $g$-loop string amplitude contains an exponential with a coupling

$$
\int d^2 z \chi T_F
$$

of the world-sheet gravitino $\chi$ to the fermionic part of the stress tensor $T_F$\textsuperscript{13}. Expanding the gravitino w.r.t. the basis $\{\chi^{(a)} = \delta^{(2)}(z - x_a) ; a = 1, \ldots, 2g - 2\}$ of 3/2 differentials, and integrating over the supermoduli, brings down $2g - 2$ supercurrent operators $T_F(x_a)$ inserted at arbitrary positions $x_a$ in the amplitude. In other words, in this gauge, the result of integrating over the supermoduli is the appearance of $2g - 2$ insertions of the supercurrent $T_F(x_a)$. The points $x_a$ are arbitrarily chosen on the Riemann surface. Different choices are related by total derivatives w.r.t. to the moduli of the Riemann surface. The final expression for the amplitude does not depend on these points \textsuperscript{14}, however in practice, it is difficult to find a convenient choice. Furthermore, the total derivatives encountered after changing the points $x_a$, are not globally well defined in the moduli space and, if their boundary contributions do not vanish, they cause problems.

\textsuperscript{13} The PCO $Y$, discussed in Section 2, is related to the supercurrent $T_F$ by $Y(z) \equiv e^{\phi(z)}T_F(z)$, with the background charge operator $e^\phi$. 

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In a recent beautiful series of papers [17] a new method for descending from the supermoduli to the moduli space has been developed. In this way, the ambiguities in choosing the gauge slice are avoided and the invariance under changing the gauge slice becomes manifest. Essentially, it introduces an additional coupling to the stress tensor $T(z)$, in addition to the supercurrent $T_F(z)$ insertion. The reinstated gauge slice-independence allows an arbitrary choice of the insertion points $x_a$. One particular choice, the so called split gauge (defined by the vanishing of the fermion propagator $G_F(x_1, x_2) = 0$ at these points) has proven to be very efficient: the amplitudes become independent on $x_a$ at any point in the moduli space. This results in an extremely simple expression for the heterotic two-loop cosmological constant (in $D=10$) including the combined effect of supermoduli, superconformal ghost system and background ghost charge. In that case, the integral over moduli and supermoduli is expressed as a spin structure dependent modular function [17].

Due to the additional stress tensor insertions, the correlators with $n$ vertex operators will now also involve couplings of the vertex operators to $T(z)$ aside from the usual couplings to $T_F(z)$. Two cases are possible: the split and the non-split. In the first case, the vertex operators do not interact with $T(z)$ and $T_F(z)$, while in the second case they do interact. In the split case, a formula has been derived [17] for $n=4$:

$$S_{t_8} = \frac{1}{\chi_{10}(\Omega)} \omega_i(z_1)\omega_j(z_2)\omega_k(z_3)\omega_l(z_4) \sum_\delta \Xi_\delta^z(0, \Omega) \theta_\delta(0, \Omega)^3 \partial_i \partial_j \partial_k \partial_l \theta_\delta(0, \Omega),$$  

(B.1)

which can be applied e.g. to the space-time part of a four gauge boson amplitude (2.1). Here $\Xi_\delta(\Omega)$ is a complicated modular function of weight six, defined in [17]. The piece $S_{t_8}$ accounts for eight-fermion contractions coming from four vertex operators in the zero-ghost picture in addition to the pieces coming from the $T(z)$ and $T_F(z)$. Thus it gives order $O(k^4)$ in momentum and comprises the $t_8$ tensor in ten dimensions. We accomplished to write (B.1) in a somewhat simpler way, by using Siegel modular forms only (depending on arguments $\vec{z} = (z^1, z^2)$):

$$S_{t_8} = \frac{1}{\chi_{10}(\Omega)} \omega_i(z_1)\omega_j(z_2)\omega_k(z_3)\omega_l(z_4)$$

$$\times \partial_\xi \partial_\eta \partial_{\xi'} \partial_{\eta'} [8 E_{8,4}(\vec{z}, \Omega) - \frac{2}{3} E_4(\Omega) E_{4,4}(\vec{z}, \Omega) - \frac{4}{3} E_{4,2}(\vec{z}, \Omega)^2] \big|_{\vec{z}_i=0}$$  

(B.2)

with

$$E_{8,4}(\vec{z}, \Omega) = \sum_{\vec{\alpha} \text{ even}} \theta_{\vec{\alpha}}(0, \Omega)^{12} \theta_{\vec{\alpha}}(\vec{z}/2, \Omega)^4,$$

$$E_{4,4}(\vec{z}, \Omega) = \sum_{\vec{\alpha} \text{ even}} \theta_{\vec{\alpha}}(0, \Omega)^4 \theta_{\vec{\alpha}}(\vec{z}/2, \Omega)^4,$$

$$E_{4,2}(\vec{z}, \Omega) = \sum_{\vec{\alpha} \text{ even}} \theta_{\vec{\alpha}}(0, \Omega)^6 \theta_{\vec{\alpha}}(\vec{z}/2, \Omega)^2,$$  

(B.3)

However, in the description of [17], there appears another, non-symmetric kinematics which is believed to be cancelled by a similar term from the non-split contribution.
and the Siegel forms \( E_4(\Omega) = \sum_{\alpha \text{ even}} \theta_4(0, \Omega)^8 \) and \( \chi_{10}(\Omega) = \prod_{\alpha \text{ even}} \theta_4(0, \Omega)^2 \). The latter represents the oscillator partition function.

To calculate the split contribution to the two-loop corrections to \( \text{Tr} F^4 \), one first observes that \( S_{t_8} \) is completely symmetric in the vertex positions \( z_i \). Similarly to Section 2, this allows to symmetrize over the positions in the gauge part and to take the same combination of gauge contractions as in Eq.(2.14). In terms of genus two \( \theta \)-functions, Eq.(2.13) reads

\[
B_{\beta}(z_1, z_2, z_3, z_4) = \frac{1}{\theta_{\beta}(0, \Omega)^4 E(z_1, z_2) E(z_2, z_3) E(z_3, z_4) E(z_4, z_1)},
\]

with the two-loop Szegö kernel

\[
\langle \psi(z_1) \psi(z_2) \rangle_{\beta} = \frac{1}{\theta_{\beta}(0, \Omega) E(z_1, z_2)},
\]

and the prime form

\[
E(z_1, z_2) = \frac{\theta_\alpha(z_1 - z_2, \Omega)}{h_\alpha(z_1) h_\alpha(z_2)}, \quad h_\alpha(z) = \sqrt{\partial_i \theta_\alpha(0, \Omega) \omega_i(z)}
\]

for any odd spin-structure \( \alpha \). Then, the analogue of Eq. (2.14) becomes [21]:

\[
B_{\beta}(z_1, z_2, z_3, z_4) + B_{\beta}(z_1, z_2, z_4, z_3) + B_{\beta}(z_1, z_3, z_4, z_2) = -\frac{1}{2} \omega_i(z_1) \omega_j(z_2) \omega_k(z_3) \omega_l(z_4) \partial_{z_2} \partial_{z_3} \partial_{z_4} \partial_{z_1} \ln \theta_{\beta}(0, \Omega),
\]

with the canonical one-forms \( \omega_i(z) \), \( i = 1, 2 \). This expression has to be inserted into the gauge partition function \( \chi_{10}^{-1} \sum_{\beta} \theta_{\beta}(0, \Omega)^{16} \). Thus, in the split case, the final expression for the two-loop corrections to \( t_8 \text{Tr} F^4 \) becomes:

\[
\Delta^{\text{2-loop}}_{t_8 \text{Tr} F^4} = -\frac{1}{6} \int_{\mathcal{F}_2} \frac{d^2 \Omega_{12} d^2 \Omega_{212} d^2 \Omega_{12}}{[\det \text{Im}(\Omega)]^5} \frac{1}{|\chi_{10}(\Omega)|^2} \times \int \int \omega_i(z_1) \omega_j(z_2) \omega_k(z_3) \omega_l(z_4) \wedge \omega_i(\overline{z}_1) \omega_j(\overline{z}_2) \omega_k(\overline{z}_3) \omega_l(\overline{z}_4) \times \partial_{z_1} \partial_{z_2} \partial_{z_3} \partial_{z_4} [8 E_{8,4}(\overline{\Omega}) - \frac{2}{3} E_4(\overline{\Omega}) E_{4,4}(\overline{\Omega}) - \frac{4}{3} E_{4,2}(\overline{\Omega})^2] \big|_{z_1=0} \times \sum_{\beta} \theta_{\beta}(0, \Omega)^{16} \frac{\partial_\overline{\Omega}}{\partial_{\overline{z}_i}} \frac{\partial_\overline{\Omega}}{\partial_{\overline{z}_j}} \frac{\partial_\overline{\Omega}}{\partial_{\overline{z}_k}} \frac{\partial_\overline{\Omega}}{\partial_{\overline{z}_l}} \ln \theta_{\beta}(0, \Omega),
\]
with the fundamental region $\mathcal{F}_2$ of the genus two Riemann surface. The evaluation of the $z_i$ integrals is straightforward. After taking a closer look at the space-time part, one can derive the following remarkable identities:

\[
\partial^4_{\bar{z}_1} \left[ 8E_{8,4}(\bar{z}, \Omega) - \frac{2}{3} E_4(\Omega)E_{4,4}(\bar{z}, \Omega) - \frac{4}{3} E_{4,2}(\bar{z}, \Omega)^2 \right] \mid_{\bar{z}_i = 0} = 0 ,
\]
\[
\partial^3_{\bar{z}_1} \partial_{\bar{z}_2} \left[ 8E_{8,4}(\bar{z}, \Omega) - \frac{2}{3} E_4(\Omega)E_{4,4}(\bar{z}, \Omega) - \frac{4}{3} E_{4,2}(\bar{z}, \Omega)^2 \right] \mid_{\bar{z}_i = 0} = 0 , 
\)
\[
\partial^2_{\bar{z}_1} \partial^2_{\bar{z}_2} \left[ 8E_{8,4}(\bar{z}, \Omega) - \frac{2}{3} E_4(\Omega)E_{4,4}(\bar{z}, \Omega) - \frac{4}{3} E_{4,2}(\bar{z}, \Omega)^2 \right] \mid_{\bar{z}_i = 0} = 0 .
\]

This proves that $\triangle^{2\text{-loop}}_{t_s \text{Tr} F^4} = 0$ for the split case. Since this result has its origin in the cancellations in the space-time sector, one may conclude the same for the two-loop corrections to other couplings: $(\text{Tr} F^2)^2$, $(\text{Tr} F)^2 R^2$, $R^4$, $(R^2)^2$. However, as already mentioned, this result is not complete because it proves the vanishing of two-loop corrections for split contributions (w.r.t. the gauge choice of [17]) only. In Section 2, we proved the vanishing of two-loop corrections to the $\text{Tr} F^4$ coupling in the hyperelliptic approach. Thus – by an indirect argument – the non-split contributions must vanish, too.
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