A General Framework for Multi-level Subsetwise Graph Sparsifiers

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Abstract

Given an undirected weighted graph \( G(V, E) \), a subsetwise sparsifier over a terminal set \( T \subset V \) is a subgraph \( G' \) having a certain structure which connects the terminals. Examples are Steiner trees (minimal-weight trees spanning \( T \)) and subsetwise spanners (subgraphs \( G'(V', E') \)) such that for given \( \alpha, \beta \geq 1 \), \( d_{G'}(u, v) \leq \alpha d_G(u, v) + \beta \) for \( u, v \in T \). Multi-level subsetwise sparsifiers are generalizations in which terminal vertices require different levels or grades of service.

This paper gives a flexible approximation algorithm for several multi-level subsetwise sparsifier problems, including multi-level graph spanners, Steiner trees, and \( k \)-connected subgraphs. The algorithm relies on computing an approximation to the single level instance of the problem for the subsetwise spanner problem, there are few existing approximation algorithms for even a single level; consequently we give a new polynomial time algorithm for computing a subsetwise spanner for a single level. Specifically, we show that for \( k \in \mathbb{N}, \varepsilon > 0 \), and \( T \subset V \), there is a subsetwise \((2k-1)(1+\varepsilon)\)-spanner with total weight \( O(|T|^2 W(ST(G, T))) \), where \( W(ST(G, T)) \) is the weight of the Steiner tree of \( G \) over the subset \( T \). This is the first algorithm and corresponding weight guarantee for a multiplicative subsetwise spanner for nonplanar graphs.

We also generalize a result of Klein to give a constant approximation to the multi-level subsetwise spanner problem for planar graphs. Additionally, we give a polynomial-size ILP for optimally computing pairwise spanners of arbitrary distortion (beyond linear distortion functions), and provide experiments to illustrate the performance of our algorithms.

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1 Introduction

Graph sparsifiers and sketches have become a fundamental object of study due to the breadth of applications which benefit from them and the power that such techniques have for data compression, computational speedup, network routing, and many other tasks \[25\]. Some canonical examples of graph sparsifiers are spanners \[15\], spectral sparsifiers \[27\], spanning trees, and Steiner trees \[19\]. Sparsifier constructions attempt to delete as much edge weight from the graph as possible while maintaining certain properties of the underlying graph; for instance, spanners are subgraphs which approximately preserve the shortest path distances in the initial graph, whereas spectral sparsifiers maintain the properties of the graph Laplacian matrix and are useful in fast solvers for linear systems \[28\].

Multi-level graph representations have been increasingly utilized, as many networks contain within them a notion of priority of nodes. Indeed, displaying a map of a road network with varying detail based on zoom level is an instance where vertices (intersections) are given priority based on the size of the roads. Alternatively, following a natural disaster, a city might designate priorities for rebuilding infrastructure to connect buildings; buildings with higher priority will be connected first, for example the city will first ensure that routes from major hubs to hospitals are repaired. Multi-level variants of the Steiner tree problem and the multiplicative spanner problem were studied in \[2\] and \[3\], respectively.

In this paper, we give some general and flexible algorithms for finding certain types of graph sparsifiers for multi-level graphs (which may also be cast as grade-of-service problems). The class of sparsifiers includes many variants of Steiner trees and spanners, among others. In the spanner case, the algorithms presented here necessitate a solution to the subsetwise spanner problem in which one seeks to find a subgraph in which distances between certain prescribed vertices are preserved up to some small distortion factor. Specifically, given a distortion function \( f : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) satisfying \( f(x) \geq x \) (typically \( f(x) = \alpha x + \beta \) for \( \alpha, \beta \geq 1 \)), and a subset \( T \subset V \), a subsetwise spanner of \( G \) is a subgraph \( G' \) such that \( d_{G'}(u,v) \leq f(d_G(u,v)) \) for all \( u,v \in T \), where \( d_G(u,v) \) denotes the length of the shortest \( u-v \) path in \( G \). We describe a compact integer linear program (ILP) formulation to this problem, and also give a novel and simple polynomial time approximation algorithm and prove corresponding bounds on the weight of multiplicative subsetwise spanners (Theorems 6 and 7); such bounds have thus far not been considered in the literature. These weight bounds are both in terms of the optimal solution to the problem and of the weight of the Steiner tree over the given subset. In particular, for \( k \in \mathbb{N}, T \subset V, \) and \( \varepsilon > 0 \), we obtain a subsetwise \((2k-1)(1+\varepsilon)\)-spanner (i.e. \( \alpha = (2k-1)(1+\varepsilon) \) and \( \beta = 0 \)) with weight \( O(|T|^\frac{1}{2}) \) times the weight of the Steiner tree over \( T \) (which is the minimum-weight tree spanning the vertices \( T \)). In the subsetwise spanner case, this is a stronger notion of lightness than previous spanner guarantees in terms of the weight of the minimal spanning tree of the graph. Moreover, this weight bound is for arbitrary graphs, whereas the only previous bound of this type was for planar graphs \[22\].

We also provide numerical experiments to illustrate our algorithms in the appendix, and show the effect on the single level subroutines used in our main algorithm.

1.1 Problem Statement

First, we quantify the kinds of sparsifiers that our algorithms are flexible enough to handle. In what follows, graphs \( G(V,E) \) will be weighted, connected, and undirected. We say that \( G'(V',E') \) is an admissible graph sparsifier of \( G(V,E) \) over terminals \( T \subseteq V \) provided \( G' \) is a connected subgraph of \( G \) and \( T \subseteq V' \). We typically assume that \( G' \) has some sort of
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structure, such as it is a tree, or it approximately maintains distances between terminals. Moreover, we assume that for a given type of sparsifier, there exists a merging operation \( \oplus \) such that if terminals \( T_1, T_2 \) are not disjoint, and \( G_1, G_2 \) are admissible sparsifiers of \( G \) over \( T_1 \) and \( T_2 \), respectively, then \( G_1 \oplus G_2 \) is an admissible sparsifier of \( G \) over \( T_1 \cup T_2 \). The multi-level sparsifier problem under consideration is as follows.

\[\text{Problem 1 (Multi-level Admissible Graph Sparsifier (MLAGS) Problem)).} \text{ Given a weighted, connected, undirected graph } G(V, E, w), \text{ a nested sequence of terminals } T_i \subseteq T_{i-1} \subseteq \cdots \subseteq T_1 \subseteq V, \text{ and an edge cost function for each level } c_i : E \rightarrow \mathbb{R}^+, \text{ compute a minimum-cost sequence of admissible sparsifiers } G_{\ell} \subseteq \cdots \subseteq G_1 \text{ where } G_i(V_i, E_i) \text{ is an admissible sparsifier (of the same type) for } G \text{ over } T_i. \]  

The cost of a solution is defined to be  

\[\text{COST}(G_{\ell}, \ldots, G_1) = \sum_{i=1}^{\ell} \sum_{e \in E_i} c(e).\]

There is an equivalent formulation of this problem in terms of grades of service; see \cite{9} for a full explanation.

1.2 Examples

One example of an admissible sparsifier over a set of terminals is a Steiner tree. A Steiner tree of \( G \) over \( T \) is a subtree of \( G \) that spans \( T \), possibly including other vertices, and is denoted \( \text{ST}(G, T) \). The classical Steiner tree problem (ST) is to find a minimum-weight edge set \( E' \subseteq E \) that spans all terminals. ST is one of Karp’s initial NP–hard problems \cite{19}, is APX–hard \cite{7}, and cannot be approximated within a factor of 96/95 unless \( \text{P} = \text{NP} \) \cite{12}. The edge-weighted ST problem admits a simple 2-approximation \cite{17} by computing a minimum spanning tree on the metric closure \footnote{Given \( G = (V, E) \), the metric closure of \( T \subseteq V \) is the complete graph \( K_{|T|} \), where edge weights are equal to the lengths of shortest paths in \( G \).} of \( T \). The LP–based approximation algorithm of Byrka et al. \cite{9} guarantees a ratio of \( \ln(4) + \varepsilon < 1.39 \). Several other variants of the ST problem on graphs are admissible sparsifiers including \( \varepsilon \)-dense ST, prize-collecting ST, and bottleneck ST. Details about these ST problem variants can be found in the online compendium and references therein \cite{18}.

Another example of admissible sparsifier is a \( k \)-connected subgraph \cite{24}, in which (similar to the ST problem) a set of terminal vertices are given, and the goal is to find the minimum weight subgraph such that each pair of the terminals is connected with at least \( k \) vertex disjoint paths.

Another example of an admissible sparsifier is the pairwise spanner with arbitrary distortion. Multiplicative spanners, initially introduced by Peleg and Schäffer \cite{15} are subgraphs which approximately preserve distances up to a multiplicative factor, i.e., \( d_{G'}(u, v) \leq td_G(u, v) \) for all \( u, v \in V \). The most general type of spanner is the pairwise spanner, where one is given a set of pairs \( P \subseteq V \times V \), and a distortion function \( f: \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) satisfying \( f(x) \geq x \), and one attempts to find the subgraph \( G'(V', E') \) of smallest weight or minimal number of edges such that \( d_{G'}(u, v) \leq f(d_G(u, v)), \ (u, v) \in P \).

Any subgraph \( G' \) satisfying this condition is called a pairwise spanner of \( G \) with distortion \( f \), or a subsetwise \( f \)-spanner for short. Common distortion functions are \( f(x) = tx \) (multiplicative...
spanners), \( f(x) = x + \beta \) (additive spanners), or \( f(x) = \alpha x + \beta \) (linear, or \((\alpha, \beta)\)-spanners). In the case that \( P = V \times V \), we simply use the term spanner, whereas if \( P = T \times T \), for some \( T \subseteq V \), then we use the term subsetwise spanner.

The merging operator \( \oplus \) is sparsifier specific. For example, in case of spanners and \( k \)-connected subgraphs, it would simply be the union of the subgraphs, while for Steiner trees it would be to take the union and prune edges to enforce the tree structure.

Note that spectral sparsifiers are not admissible since they are not subgraphs as edges can be added to obtain a spectral sparsifier of a graph.

1.3 Related Work

There are several results known about multi-level or grade-of-service Steiner tree problems for weighted graphs. Balakrishnan et al. \[6\] give a \( \frac{4}{3} \rho \)-approximation algorithm for the 2-level network design problem with proportional edge costs where \( \rho = \ln 4 + \varepsilon \). Charikar et al. \[10\] describe a simple \( 4\rho \)-approximation for the Quality-of-Service (QoS) Multicast Tree problem with proportional edge costs (termed the rate model), which was later improved to \( e\rho \) by randomized doubling. Karpinski et al. \[20\] use an iterative contraction scheme to obtain a \( 2.454\rho \)-approximation. Ahmed et al. \[2\] have further improved the approximation ratio to \( 2.351\rho \), by combining top-down and bottom-up strategies.

Among the few discussions of multi-level graph spanners is \[3\], where level-dependent approximation guarantees are given assuming a single level subsetwise spanner oracle. However, it would be a significant improvement if approximation algorithms for subsetwise spanners were used instead of an oracle in \[3\]. Coppersmith et al. \[13\] study the subsetwise distance preserver problem (the case \( f(x) = x \)) and show that given an undirected weighted graph and a subset of size \( O(n^{1+\varepsilon}) \), one can construct a linear size preserver in polynomial time. Cygan et al. \[14\] give polynomial time algorithms to compute subsetwise and pairwise additive spanners for unweighted graphs and show that there exists an additive pairwise spanner of size \( O(n|T|^{\frac{1}{2}} \log n) \) with 4 additive stretch. They also show how to construct \( O(n|T|^{\frac{1}{2}}) \) size subsetwise additive 2-spanners. Abboud et al. \[1\] improved that result by showing how to construct \( O(n|T|^{\frac{1}{2}}) \) size subsetwise additive 2-spanners. Kavitha \[21\] shows that there is a polynomial time algorithm which constructs subsetwise spanners of size \( O(n|T|^{\frac{1}{2}}) \) and \( O(n|T|^{\frac{1}{2}}) \) for additive stretch 4 and 6, respectively. Bodwin et al. \[8\] give an upper bound on the size of subsetwise spanners with polynomial additive stretch factor. To the authors’ knowledge, there are no existing guarantees for multiplicative subsetwise spanners except those of Klein \[22\] who gives a polynomial time algorithm that computes a subsetwise multiplicative spanner of an edge weighted planar graph for a constant stretch factor with constant approximation ratio.

Hardness of approximation of multi-level spanners follows from the single level case. Peleg and Schäffer \[15\] show that determining if there exists a \( t \)-spanner of \( G \) with \( m \) or fewer edges is NP-complete. Further, it is NP-hard to approximate the (unweighted) \( t \)-spanner problem for \( t > 2 \) to within a factor of \( O(\log |V|) \) even when restricted to bipartite graphs \[23\].

2 A General Rounding Up Approach

Here, we give a very general and flexible extension of the rounding approach of Charikar et al. \[11\] for designing Steiner trees, which we apply to compute approximate solutions to the MLAGS problem based on solving the problem at a subset of levels. Furthermore, we also consider a more general cost model compared to \[3\] and \[2\] for multi-level design. We assume the cost of an edge \( e \) on a given level \( i \) can be decomposed as \( c_i(e) = g(i)w(e) \) for all
where \( g : \{1, \ldots, \ell\} \to \mathbb{R}^+ \) is a cost scaling function. That is, the cost scales uniformly over edges for a given level. Hence our cost model is more general than the case considered in [3] where \( g(i) = i \) as \( g \) is an arbitrary non-decreasing function.

Additionally, consider a rounding up function, or “level quantizer,”

\[
q : \{1, \ldots, \ell\} \to Q = \{i_1 = 1, \ldots, i_m\} \subset \{1, \ldots, \ell\},
\]

which satisfies \( q(i) \geq i \). Suppose also that there exists a positive constant \( A = A(Q) \)

\[
g(q(i)) \leq Ag(i), \quad i = 1, \ldots, \ell, \tag{1}
\]

and there exists a constant \( B = B(Q) > 0 \) such that if \( e \) appears on level \( i_k \), then

\[
\sum_{j=1}^k g(i_j)w(e) \leq Bg(i_k)w(e). \tag{2}
\]

The latter can also be written as \( \sum_{j=1}^k c_{i_j}(e) \leq Bc_{i_k}(e) \).

Our general rounding up algorithm for computing a MLAGS is as follows. For each \( i \in Q \), compute an admissible sparsifier over terminals \( T_{i} \), and let \( H_1, \ldots, H_m \) be the subgraphs returned. Then for \( i = 1, \ldots, \ell \) we will set \( G_i \) as follows:

\[
G_i = \left( \bigoplus_{j > i, j \in Q} H_j \right) \oplus H_k \tag{3}
\]

where \( k \) is defined as the largest element in \( Q \) less than or equal to \( i \). In other words, the graph on level \( i \) is the merging (in the sense of the operator \( \oplus \)) of all computed sparsifiers \( H_j \) on higher levels, as well as the computed sparsifier \( H_k \). For example, if \( \ell = 6 \) and \( Q = \{1, 4, 6\} \), then \( G_6 = H_6 \), \( G_4 = G_5 = H_4 \oplus H_6 \), and \( G_1 = G_2 = G_3 = H_1 \oplus H_4 \oplus H_6 \).

The rounding up approach of Charikar et al. [11] is a special case where \( Q = \{1, 2, 4, \ldots, 2^k\} \).

Algorithm 1 Rounding Approximation Algorithm for MLAGS(\( G, T_1, \ldots, T_\ell, Q \))

\[
\begin{align*}
&\text{for } i = 1, \ldots, m \text{ do} \\
&\quad H_i \leftarrow \text{an } s(i)\text{-approximation to the optimal admissible sparsifier of } G \text{ over } T_i \\
&\text{end for} \\
&\text{for } i = 1, \ldots, \ell \text{ do} \\
&\quad G_i \leftarrow \left( \bigoplus_{j > i, j \in Q} H_j \right) \oplus H_k \text{ (as in (3))} \\
&\text{end for} \\
&\text{return } G_1, \ldots, G_\ell
\end{align*}
\]

Theorem 1. Given a graph \( G \), terminal sets \( T_\ell \subseteq \cdots \subseteq T_1 \subseteq V \), a level cost function \( g \), and a rounding function \( q \) with rounding set \( Q \) which satisfies conditions [1] and [2], Algorithm [2] yields an \( ABs \)-approximation to the MLAGS problem, where \( s = \max_k \{s(k)\} \).

Proof. By assumption [1], if we consider \( \text{OPT}_Q \) to be the optimal solution to the rounded MLAGS problem where edges are assigned levels \( q(i) \) instead of \( i \), then an edge \( e \) costs at most \( A \) times what it would have in the optimal solution for the unrounded MLAGS problem.
By summing over $E$, this implies that $\text{OPT}_Q \leq A\text{OPT}$, where $\text{OPT}$ is the cost of the optimal solution to the full MLAGS problem.

Again consider the optimal solution to the rounded problem with cost $\text{OPT}_Q$. Given any edge $e$ in this solution, if its level is $a_k$, then we may replace the edge with $k$ edges on levels $a_1, \ldots, a_k$. The total cost of these edges is then

$$\sum_{j=1}^{k} g(a_j)w(e) \leq Bg(a_k)w(e) = Bc_{a_k}(e)$$

by (2). At each rounded level, Algorithm 1 computes an $s$–approximation to the optimal subsetwise spanner; it follows that the spanner returned by the algorithm has cost no worse than $Bs\text{OPT}_Q$ which is at most $ABs\text{OPT}$ from the previous analysis, and the proof is complete.

Note that the proof of Theorem 1 is independent on the type of sparsifier desired, and hence the algorithm is quite flexible. A similar approach was used to approximate minimum-cost multi-level Steiner trees in [2], but the above analysis shows this approach also works for spanners and $k$–connected subgraphs, for example.

Now we can compute the approximation guarantee given $g(i)$. Let $\text{MIN}_i$ denote the cost of a minimum sparsifier of $G$ over $T_i$. As $T_\ell \subseteq T_{\ell-1} \subseteq \cdots \subseteq T_1$, we have $\text{MIN}_\ell \leq \text{MIN}_{\ell-1} \leq \cdots \leq \text{MIN}_1$. Then the following holds ($R(Q)$ is the cost of the output of Algorithm 1 for a given $Q$).

**Lemma 2.** For any set $Q$, we have $R(Q) \leq \sum_{k=1}^{m} g(i_{k+1} - 1)\text{MIN}_{i_k}$, where $i_{m+1} = \ell + 1$.

**Proof.** This follows from the merging bound; note that the edges of the subgraph $H_{i_m}$ appear on all $m$ levels. The edges from the subgraph $H_{i_{m-1}}$ appear on $i_m - 1$ levels, namely $1, 2, \ldots, i_m - 1$.

Then applying Lemma 2 for a particular $Q$, we have

$$\frac{R(Q)}{\text{OPT}} \leq \frac{\sum_{k=1}^{m} g(i_{k+1} - 1)\text{MIN}_{i_k}}{\sum_{i=1}^{\ell} \text{MIN}_i},$$

where we have used the observation that $\text{OPT} \geq \sum_{i=1}^{\ell} \text{MIN}_i$. Therefore, the right-hand side of the above inequality provides a approximation guarantee assuming knowledge of $\text{MIN}_1, \ldots, \text{MIN}_\ell$. We can also find a generic bound by looking at the worst-case scenario for $\text{MIN}_1, \ldots, \text{MIN}_\ell$. Without loss of generality, we may assume that $\sum_{i=1}^{\ell} \text{MIN}_i = 1$, so that $\text{OPT} \geq 1$. Since $g(i)$ is an increasing function, the worst case of level costs will be of the form $\text{MIN}_1 = \text{MIN}_2 = \cdots = \text{MIN}_h = \frac{1}{h}$ and $\text{MIN}_{h+1} = \cdots = \text{MIN}_\ell = 0$ for some $h$. Therefore, the general approximation guarantee (regardless of costs $\text{MIN}_i$ of the subset sparsifiers of each level) is

$$\min_h \frac{\sum_{k=1}^{h} g(i_{k+1} - 1)}{i_h}$$

### 2.1 Examples

A natural example is to take $g(i) = i$, i.e., a linear cost growth along levels. Following Charikar et al. [11], we may take $g(i) = 2^{\log_2 i}$. In this case, using an oracle to compute the subsetwise spanner at each level yields a $4$–approximation to the MLAGS problem for a...
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multiplicative $t$–spanner. The same approximation holds in this case for multi-level Steiner trees [2]. Indeed, $q(i) \leq 2i$, whence we may choose $A = 2$, and if edge $e$ gets its rate rounded to $2^j$, then

$$
\sum_{j=1}^{k} i_j w(e) = \sum_{j=1}^{k} 2^j w(e) \leq 2 \cdot 2^j w(e) = 2c(e),
$$

whence $B = 2$, thus yielding a 4–approximation. This yields the following corollary improves on Theorems 1 and 3 in [3], as the approximation ratio is independent of the number of levels.

\textbf{Corollary 3.} Let $t \geq 1$. If an oracle which computes the optimal subsetwise multiplicative $t$–spanner of $G$ over terminals $T_i$ is used in Algorithm 1, and $Q = \{1, 2, \ldots, 2^m\}$, then Algorithm 1 produces a 4–approximation to the optimal multi-level graph spanner problem.

Note that the use of an oracle such as the ILP given in Appendix A is costly; this issue will be addressed in Section 3.

It is of interest to note that choosing $q(i) = b^{\lceil \log_b i \rceil}$ for some other base $b > 0$ does not improve the approximation ratio. In this case, one can show that $A = b$, and $B$ is approximately $\frac{b}{b-1}$, and thus the approximation ratio is $\frac{b^2}{b^2-1}$ which is minimized when $b = 2$.

Using a coarser quantizer instead yields a worse approximation. Consider the coarsest quantizer which sets $q(i) = \ell$ for all $i > 1$ and $q(1) = 1$, with $g(i) = i$ as before. In this case, $A = \ell$, and $B = 1$, which means that the best one can do is an $\ell$s–approximation. This approach corresponds to the Bottom Up approach described in [3].

If no rounding is done (i.e., $q(i) = i$), then one computes a sparsifier at each level and merges them together according to the operation $\oplus$ going down the levels. This can be considered a Top Down approach to the problem, and yields an upper bound on the approximation ratio of $\frac{\ell + 1}{2}$.

2.2 Composite Algorithm

One way to find a good approximation algorithm is to run Algorithm 1 for all $2^\ell - 1$ subsets $Q$ containing 1, and then choosing the multi-level sparsifier with the smallest cost. This requires $\ell$ sparsifier computations followed by finding the minimum over $2^\ell - 1$ solutions. Consequently, this method is costly to implement; however, it provides the lowest cost and the best guarantees possible using single-level solvers. We call this the composite algorithm.

\textbf{Algorithm 2} Composite($G, T_1, \ldots, T_\ell$)

\begin{algorithmic}
\For {all $Q \subset \{1, 2, \ldots, \ell\}$ with $1 \in Q$}
\State Compute MLAGS($G, T_1, \ldots, T_\ell, Q$) via Algorithm 1
\EndFor
\Return solution with minimum cost.
\end{algorithmic}

We can find the approximation guarantee of the composite algorithm using the following linear program (LP):

Find $\max_{t, \text{MIN}_1, \ldots, \text{MIN}_\ell} t$ subject to

$$
t < \sum_{k=1}^{m} g(i_{k+1} - 1) \text{MIN}_{i_k}, \text{ for all } Q = \{i_1 = 1, \ldots, i_m\} \subset \{1, 2, \ldots, \ell\}
$$
This optimization problem suggested above is similar to that in [2]. In particular, for linear costs \( g(i) = i \) and \( \ell \leq 100 \), the solution returned by the composite algorithm has cost no worse than \( 2.351 \text{OPT} \) assuming an oracle that computes sparsifiers optimally. This approach uses \( \ell \) sparsifier computations, and computes the minimum over \( O(2^\ell - 1) \) candidate solutions.

### 2.3 Computing the Best Rounding Set \( Q \)

Suppose having computed all single-level solutions, we are interested to find what \( Q \) would provide the least multi-level solution. We formulate this as a minimization problem. Define binary variables \( \theta_{ij} \) such that \( \theta_{i{i+1}} = 1 \) and 0 otherwise. For example if \( \ell = 3 \) and \( Q = \{1,3\} \), \( \theta_{13} = \theta_{34} = 1 \) and \( \theta_{12} = \theta_{24} = 0 \).

**Lemma 4.** Given a vector \( y = [y_1, \ldots, y_{\ell}] \), the choice of \( Q^* = \{i, \text{s.t.}, \theta_{ij} = 1\} \) from the following ILP minimizes \( \sum_{k=1}^{m} g(i_k+1-1) y_i \), where \( i_k \) is the \( k \)-th smallest element of \( Q^* \) and \( i_{m+1} = \ell + 1 \).

\[
\min \sum_{i=1}^{\ell} \sum_{j=i+1}^{\ell+1} g(j-1) \theta_{ij} y_i \\
\text{subject to} \quad \sum_{j>i} \theta_{ij} \leq 1, \quad \sum_{i<j} \theta_{ij} \leq 1, \quad i \in \{1, \ldots, \ell\}, j \in \{2, \ldots, \ell + 1\} \\
\sum_{i<k} \theta_{ik} = \sum_{j>k} \theta_{kj} \quad k \in \{2, \ldots, \ell\} \\
\sum_{1<j} \theta_{ij} = 1, \quad \sum_{i<\ell+1} \theta_{i(\ell+1)} = 1
\]

**Proof.** Using the indicator variables, the objective function can be expressed as \( \sum_{i=1}^{\ell} \sum_{j=i+1}^{\ell+1} g(j-1) \theta_{ij} y_i \) because \( \theta_{i{i+1}} = 1 \) and the other \( \theta_{ij} \)'s are zero. In the above formulation, the first constraint indicates that for every given \( i \) or \( j \), at most one \( \theta_{ij} \) is equal to one. The second constraint indicates that for a given \( k \), if \( \theta_{ik} = 1 \) for some \( i \), then there is also a \( j \) such that \( \theta_{kj} = 1 \). In other words, the result determines a proper choice of levels by ensuring continuity of \([i,j]\) intervals. The last constraint guarantees that 1 \( \in Q^* \). ◀

### 3 Metric Closure Multi-level Spanners

Observe that when the admissible sparsifier is a subsetwise spanner, the composite algorithm proposed above necessitates having a good algorithm for computing a subsetwise spanner on a single level with a given distortion. Unfortunately, there are precious few algorithms that provably approximate the subsetwise spanner problem. In this section, the cost at each level \( i \) is assumed to be \( c(e) = i \ w(e) \).

**Definition 5 (Metric Closure).** Given a graph \( G(V,E) \) and a set of terminals \( T \subset V \), the metric closure of \( G \) over \( T \) is defined as the complete weighted graph on \( |T| \) vertices with the weight of edge \((u,v)\) given by \( d_G(u,v) \).
Here, we give a new and simple algorithm for computing a subsetwise spanner with general distortion function \( f \). We will use an \( f \)-spanner subroutine in this algorithm which must work for weighted graphs. Note that most spanner algorithms for weighted graphs are usually of the multiplicative type.

### 3.1 Metric Closure Subsetwise Spanners

Algorithm 3 describes a subsetwise spanner construction; later we will generalize this to multi-level spanners in Algorithm 4, but even for the case of a single level this provides a novel approximation for a subsetwise spanner.

**Algorithm 3** Metric Closure Subsetwise Spanner \((G, T, f)\)

\[
\begin{align*}
\tilde{G} & \leftarrow \text{metric closure of } G \text{ over } T \\
\tilde{G}' & \leftarrow \text{spanner of } \tilde{G} \text{ with distortion } f \\
E' & \leftarrow \text{edges in } G \text{ corresponding to } \tilde{G}' \\
\text{return } E'
\end{align*}
\]

Note that the spanner obtained in the second step can come from any known algorithm which computes a spanner for a weighted graph, which adds an element of flexibility. This also allows us to obtain some new results on the weight of multiplicative subsetwise spanners.

It is known (see [4]) that if \( k \) is a positive integer, then one can construct a \((2k-1)(1+\varepsilon)\)–spanner for a given graph (over all vertices, not a subset) in \( O(\sqrt{k^{2+\varepsilon}}) \) time, which has weight \( O(n^{1\frac{k}{2}}W(MST(G))) \), where \( MST(G) \) is a minimum spanning tree of \( G \). Using this in the second stage of the Algorithm 3, we can conclude the following.

**Theorem 6.** Let \( G(V, E) \), \( k \in \mathbb{N} \), \( \varepsilon > 0 \), and \( T \subset V \) be given. Using the spanner construction of [4] as a subroutine, Algorithm 3 yields a subsetwise \((2k-1)(1+\varepsilon)\)–spanner \( \tilde{G}'(V', E') \) of \( G \) in \( O(|T|^{2+\varepsilon}) \) time with total weight \( W(E') = O(|T|^{\frac{1}{k}})OPT \). Moreover, \( W(E') = O(|T|^{\frac{1}{k}})W(ST(G,T)) \).

**Proof.** Let \( \tilde{G} \) be the metric closure of \( G \) over \( T \), and let \( G^*(V^*, E^*) \) be the minimum weight subsetwise \( f \)–spanner for \( G \), where \( f \) is any distortion function. Note that \( G^* \) must contain a tree, \( G_0^* \), which spans \( T \), whence

\[
W(G^*) \geq W(G_0^*) \geq W(ST(G,T))
\]

by definition of Steiner trees. From the approximation result for Steiner trees (see [29]) we have

\[
W(MST(\tilde{G})) \leq 2W(ST(G,T)).
\]

It follows that \( W(MST(\tilde{G})) \leq 2W(G^*) = 2OPT \). Now, using the results of [4] for the special case where \( f(x) = tx \) and \( t = (2k-1)(1+\varepsilon) \), we can construct a \((2k-1)(1+\varepsilon)\)–spanner \( \tilde{G}' \) of \( G \) which satisfies

\[
W(\tilde{G}') = O(|T|^{\frac{1}{k}})W(MST(\tilde{G})).
\]

Combining these we have the desired estimate

\[
W(\tilde{G}') \leq O(|T|^{\frac{1}{k}})W(G^*) = O(|T|^{\frac{1}{k}})OPT.
\]

Finally, \( W(E') = O(|T|^{\frac{1}{k}})W(ST(G,T)) \) follows from (5).
Both bounds given in Theorem 6 are interesting for different reasons. The first stated bound shows that Algorithm 4 yields an $O(|T|^{1/k})$–approximation to the optimal solution. The second bound gives a better notion of lightness of a subsetwise spanner. When $T = V$, the minimal spanning tree and the Steiner tree over $V$ are the same, and hence lightness bounds for spanners in this case are stated in terms of the weight of MST($G$). However, for $T \subseteq V$ it is not generally true that MST($G$) = ST($G, T$), but by definition the Steiner tree has smaller weight. Thus, this notion of lightness for subsetwise spanners stated in terms of the weight of the Steiner tree is more natural. Klein [22] uses this notion of lightness for subsetwise spanners of planar graphs, but the results presented here are the first for general graphs.

The following gives a precise (as opposed to asymptotic) bound for a subsetwise spanner by utilizing the greedy algorithm of Althöfer et al. [5].

**Theorem 7.** Let $G(V, E)$, $t > 0$, and $T \subset V$ be given. Using the greedy spanner algorithm as a subroutine, Algorithm 3 yields a subsetwise $(2t + 1)$–spanner $G'(V', E')$ of $G$ in $O(|T|^{3 + 1/k})$ time with total weight $W(E') \leq (2 + |T|/t)OPT$. Moreover, $W(E') \leq (2 + |T|/t)W(ST(G, T))$.

**Proof.** The greedy algorithm takes $O(mn^{1+1/k})$ time, and the rest is similar to the proof of previous theorem. ▶

Note that in the greedy spanner algorithm of Althöfer et al. [5], the minimum spanning tree is a subgraph of the solution. However, the optimal Steiner tree might not necessarily be a subset of the final solution. Nevertheless, the produced subset spanner will include a Steiner tree with cost at most twice the optimal one.

### 3.2 Multi-level Metric Closure Spanner

Here, we propose the multi-level version of the spanner in Algorithm 4. Note this is different than using Algorithm 3 as a subroutine in Algorithm 1.

**Algorithm 4** Metric Closure Multilevel Spanner($G, T_1, \ldots, T_\ell, f$)

\[
\begin{align*}
\tilde{G} &\leftarrow \text{metric closure of } G \text{ over } T_1 \\
\tilde{G}'_1 &\leftarrow \text{spanner of } G' \text{ with distortion } f \\
E'_1 &\leftarrow \text{edges in } G \text{ corresponding to } \tilde{G}'_1 \\
\text{for } j = 2 \ldots \ell &\text{ do} \\
&\quad \text{For any } (u, v) \in T_j \times T_j \text{ add the shortest path } P(u, v) \text{ in } E'_j \text{ to } E'_j \end{align*}
\]

end for

return $E'_1, \ldots, E'_\ell$

Here we provide a general bound which is also tight in certain cases and can be derived easily.

**Proposition 8.** Given a graph $G(V, E)$ and terminals $T_\ell \subseteq \cdots \subseteq T_1$, and a multiplicative stretch factor $t \geq 1$ (so the distortion is $f(x) = tx$), let $C$ be the total cost of the multilevel $t$–spanner of Algorithm 4. Then $C = O(\ell t |T_1|^2 \text{diam}(G))$. Moreover, if $|T_{k+1}| \leq \delta|T_k|$ then $C = O\left(\frac{t |T_1|^2 \text{diam}(G)}{1-\delta^2}\right)$.

**Proof.** The bounds follow from the fact that $W(E'_k) \leq \binom{|T_k|}{2} \text{diam}(G)t$ and $c = \sum_{k=1}^{\ell} W(E'_k)$. ▶
Now, we show that the bound $C = O(\ell t |T_1|^2 \text{diam}(G))$ is tight for the case $t = 1$. Consider a graph $G$ with terminal sets $T_1 = T_2 = \cdots = T_\ell$ such that there is a unique shortest path of length $\text{diam}(G)$ between any pairs of terminals. Assume that we do not have any other vertex in $G$ beyond the ones appearing on these shortest paths. The diameter of $G$ is clearly $2\text{diam}(G)$ and we need $\binom{|T_1|}{2}$ shortest paths to be copied across all levels. Therefore, we end up with the cost $C = 2\ell \binom{|T_1|}{2} \text{diam}(G)$ for the multilevel spanner.

4 Polynomial Time Approximation Algorithms

Solving a single level ILP takes much less time than solving a multi-level ILP, especially as the number of levels increases. This fact was one of the motivations behind the rounding algorithm. If a single level ILP is used as an oracle subroutine as in Corollary 3 then a constant approximation ratio is obtained, but at the expense of the subroutine being exponential time.

It is natural to consider what happens when the subsetwise spanner Algorithm 3 is used as the approximation algorithm in Algorithm 1. By Theorem 6 Algorithm 3 is a $O((|T_j|)^\frac{1}{2})$-approximation on any single level, and hence combining this with Theorem 1 we find that Algorithm 1 yields an $AB \cdot O((|T_j|)^\frac{1}{2})$-approximation to the multi-level graph spanner problem with stretch factor $(2k - 1)(1 + \varepsilon)$ for all levels.

If the input graph is planar then instead of using Algorithm 3 we can use the algorithm provided by Klein [22] to compute a subsetwise spanner for the set of levels we get from the rounding up algorithm. The polynomial time algorithm in [22] has constant approximation ratio, assuming that the stretch factor is constant. Hence, we have the following Corollary.

► Corollary 9. Let $G(V, E)$ be a weighted planar graph with terminals, $T_\ell \subseteq T_{\ell - 1} \subseteq \cdots \subseteq T_1 \subseteq V$. Let $t > 1$ be a constant, and suppose the level cost function is $g(i) = i$. Then there exists polynomial time, constant approximation (independent of the number of levels) for the optimal multi-level $t$-spanner problem.

For additive spanners there exists algorithms to compute subsetwise spanner of size $O(n|T|^{\frac{1}{2}})$, $\tilde{O}(n|T|^{\frac{1}{2}})$ and $O(n|T|^{\frac{1}{2}})$ for additive stretch 2, 4 and 6, respectively [1, 21]. If we use these algorithms in Algorithm 1 to compute subsetwise spanners for different levels, then we have the following Corollary.

► Corollary 10. Let $G(V, E)$ be an unweighted graph with exponentially decreasing terminals, $T_\ell \subseteq T_{\ell - 1} \subseteq \cdots \subseteq T_1 \subseteq V$ such that $|T_i| = \frac{|T_{i-1}|}{2}$, $i > 1$. Then there exists polynomial time algorithms to compute multi-level graph spanners with additive stretch 2, 4 and 6, of size $O(n|T_1|^{\frac{1}{2}})$, $\tilde{O}(n|T_1|^{\frac{1}{2}})$, and $O(n|T_1|^{\frac{1}{2}})$, respectively.

5 Experimental results

To evaluate the performance of different variants of Algorithm 1 we use several experiments. This requires optimal single-level solvers for Algorithm 1 which we obtain using an ILP formulation of the problem; see Appendix A. We generate graphs using the Erdős–Rényi random graph model [16]; more details for the experiment are in Appendix B.

We then consider the multiplicative graph spanner version of the MLAGS problem, and analyze the effect of using an ILP vs. the metric closure subsetwise spanner Algorithm 3 for the single level approximation therein. Figure 1 shows the impact of different parameters (number of vertices $|V|$, number of levels $\ell$, and stretch factors $\ell$) using box plots for the two approaches corresponding to 3 trials for each parameter. In general, the performance of both
variants decreases as the number of levels and stretch factor increases. It is notable that the metric closure approximation algorithm yields very similar results to those from ILP solver.

In the Appendix, we consider three variants for the rounding set: \( Q = \{1, \ell\} \), which we call bottom-up (BU), \( Q = \{1, \ldots, \ell\} \), which we call top-down (TD), and the optimal \( Q^* \), which we call (CMP). Varying the size of the input graph, number of levels, and stretch factor \( t \), we test the performance of Algorithm 1. If the ILP is used in the single level solver, we call these oracle TD, BU, and CMP, and otherwise we call them the metric closure TD, BU, and CMP solutions.

Figures are shown in Appendix B and we only summarize the results here. Generally, both the runtime and the approximation ratio for the multi-level graph spanner problem increase as the size of the parameter increases for the oracle TD, BU, and CMP algorithms. We cannot compute the exact solution for large graphs, but the metric closure variant scales well. Thus for large graphs the estimation of the approximation ratio is given to be \( \frac{\text{BU}}{\min\{\text{BU}, \text{TD}, \text{CMP}\}} \), for example. In this case, the runtime increases with larger input graphs, but interestingly increasing the number of levels and stretch factors does not have a strong impact on the performance.

6 Conclusions and Future Work

We have given a general framework for solving multi-level graph sparsification problems utilizing single level approximation algorithms as a subroutine. When an oracle is used for the single level instances, our algorithm can yield a constant approximation to the optimal multi-level solution that is independent of the number of levels. Using the metric closure subsetwise spanner algorithm as a subroutine for the multi-level spanner algorithm, we derive an approximation algorithm which depends on the size of the terminal set but not the number of levels. It would be natural to look for multi-level algorithms that do not rely on single level instances of the problem but build the solution simultaneously on all levels.

Additionally, we gave a new algorithm for finding subsetwise spanners of weighted graphs via the metric closure over the subset, and gave novel weight bounds in the case of multiplicative spanners. It would be worthwhile to explore new approximation algorithms for the subsetwise spanner problem, which would strengthen the results in this paper.
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A Integer Programming Formulation for Pairwise Spanners

Here we describe an ILP solver which gives a minimum cost solution to the pairwise spanner problem with arbitrary distortion function $f$. This may be used as an oracle subroutine in Algorithm 1.

Sigurd and Zachariasen [26] give an integer linear programming (ILP) formulation for the minimum cost pairwise $t$-spanner problem, where individual paths are decision variables (hence the ILP has exponentially many parameters in terms of the number of edges). In [23], a compact flow-based formulation for the minimum-cost pairwise $t$-spanner problem using $O(|E||V|^2)$ variables and constraints is given; however, it turns out that the ILP is valid for generic distortion functions. Let $P \subseteq V \times V$ be the subset of vertex pairs on which distances are desired to be preserved, and let $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be any function satisfying $f(x) \geq x$ for all $x$ (note the function need not be continuous). Let $x_e = 1$ if $e$ is included in the spanner, and 0 otherwise. Given $G = (V, E)$, let $\bar{G} = (V, \bar{E})$ be the bidirection graph of $G$ obtained by replacing every edge $e = uv$ with two directed edges $(u, v)$ and $(v, u)$ each of weight $w(e)$ (thus $|\bar{E}| = 2|E|$). Given $(i, j) \in \bar{E}$, and an unordered pair of vertices $(u, v) \in P$, define indicator variables by $x_{(i,j)}^{uv} = 1$ if edge $(i, j)$ is included in the selected $u-v$ path in the spanner $G'$, and 0 otherwise. Given $f(u, v)$, and $Out(v)$ denote the set of incoming and outgoing edges from $v$, respectively.

Next we select a total order of all vertices so that the path constraints (9)-(10) are well-defined. In (11)-(12) we assume $u < v$ in the total order, so spanner paths are from $u$ to $v$. The ILP is as follows.

Minimize $\sum_{e \in E} w(e)x_e$ subject to

$$\sum_{(i,j) \in E} x_{(i,j)}^{uv}x_e \leq f(d_G(u, v)) \quad \forall (u, v) \in P, u < v; e = (i, j) \in E$$ (8)

$$\sum_{(i,j) \in Out(i)} x_{(i,j)}^{uv} - \sum_{(j,i) \in In(i)} x_{(j,i)}^{uv} = \begin{cases} 1 & i = u \\ -1 & i = v \\ 0 & \text{else} \end{cases} \quad \forall (u, v) \in P, u < v; \forall i \in V$$ (9)

$$\sum_{(i,j) \in Out(i)} x_{(i,j)}^{uv} \leq 1 \quad \forall (u, v) \in P, u < v; \forall i \in V$$ (10)

$$x_{(i,j)}^{uv} + x_{(j,i)}^{uv} \leq x_e \quad \forall (u, v) \in P, u < v; \forall (i, j) \in E$$ (11)

$$x_e, x_{(i,j)}^{uv} \in \{0, 1\}$$ (12)

Ordering the edges induces $2|E||P|$ binary variables, or $2|E|\binom{|V|}{2} = 2|E||V||(|V| - 1)$ variables in the full spanner problem where $P = V \times V$. Note that if $u$ and $v$ are connected by multiple paths in $G'$ of length $\leq f(d_G(u, v))$, we need only set $x_{(i,j)}^{uv} = 1$ for edges along some path. The following is the main theorem for this ILP.

**Theorem 11.** Given a graph $G = (V, E)$, a subset $P \subseteq V \times V$, and any distortion function $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ which satisfies $f(x) \geq x$ for all $x$, the solution to the ILP given by (7)-(12) is an optimally light (or sparse if $G$ is undirected) pairwise spanner for $G$ with distortion $f$.

**Proof.** Let $G^* = (V, E^*)$ denote an optimal pairwise spanner of $G$ with distortion $f$, and let $OPT$ denote the cost of $G^*$ (number of edges or total weight if $G$ is unweighted or weighted, respectively). Let $OPT_{ILP}$ denote the minimum cost of the objective function in the ILP
First we notice that from the minimum cost spanner $G^*$, a solution to the ILP can be constructed as follows: for each edge $e \in E^*$, set $x_e = 1$. Then for each unordered pair $(u, v) \in P$ with $u < v$, compute a shortest path $\rho_{uv}$ from $u$ to $v$ in $G^*$, and set $x_{(i,j)}^{uv} = 1$ for each edge along this path, and $x_{(i,j)}^{uv} = 0$ if $(i, j)$ is not on $\rho_{uv}$.

As each shortest path $\rho_{uv}$ necessarily has cost at most $f(d_G(u, v))$, constraint (5) is satisfied. Constraints (6) and (10) are satisfied as $\rho_{uv}$ is a simple $u$-$v$ path. Constraint (11) also holds as $\rho_{uv}$ cannot traverse the same edge twice in opposite directions. In particular, every edge in $G^*$ appears on some shortest path; otherwise, removing such an edge yields a pairwise spanner of lower cost. Hence $\text{OPT}_{\text{ILP}} \leq \text{OPT}$.

Conversely, an optimal solution to the ILP induces a feasible pairwise spanner $G'$ with distortion $f$. Indeed, consider an unordered pair $(u, v) \in P$ with $u < v$, and the set of decision variables satisfying $x_{(i,j)}^{uv} = 1$. By (9) and (10), these edges form a simple path from $u$ to $v$. The sum of the weights of these edges is at most $f(d_G(u, v))$ by (5). Then by (11), the chosen edges corresponding to $(u, v)$ appear in the spanner, which is induced by the set of edges $e$ with $x_e = 1$. Hence $\text{OPT} \leq \text{OPT}_{\text{ILP}}$.

Combining the above observations, we see that $\text{OPT} = \text{OPT}_{\text{ILP}}$, and the proof is complete.

In the multiplicative spanner case, the number of ILP variables can be significantly reduced (see [3] for more details). These reductions are somewhat specific to multiplicative spanners, and so it would be interesting to determine if other simplifications are possible for more general distortion.

Note that the distortion $f$ does not have to be continuous, which allows for tremendous flexibility in the types of pairwise spanners the above ILP can produce.

## B Experiments

### B.1 Setup

We use the Erdős–Rényi [16] model to generate random graphs. Given a number of vertices, $n$, and probability $p$, the model $\text{ER}(n, p)$ assigns an edge to any given pair of vertices with probability $p$. An instance of $\text{ER}(n, p)$ with $p = (1 + \varepsilon) \ln n / n$ is connected with high probability for $\varepsilon > 0$ [16]. For our experiments we allow $n$ to range from 5 to 300, and set $\varepsilon = 1$.

For experimentation, we consider only the multiplicative graph spanner version of the MLAGS problem, hence we abbreviate this as MLGS; for similar experimental results on multi-level Steiner trees, see [2]. An instance of the MLGS problem is characterized by four parameters: the graph generator, the number of vertices $|V|$, the number of levels $\ell$, and stretch factor $t$. As there is randomness involved, we generated 3 instances for every choice of parameters (e.g., ER, $|V| = 80$, $\ell = 3$, $t = 2$).

We generated MLGS instances with 1 to 6 levels ($\ell \in \{1, 2, 3, 4, 5, 6\}$), where terminals are selected on each level by randomly sampling $\lfloor |V| \cdot (\ell - i + 1)/(\ell + 1) \rfloor$ vertices on level $i$ so that the size of the terminal sets decreases linearly. As the terminal sets are nested, $T_i$ can be selected by sampling from $T_{i-1}$ (or from $V$ if $i = 1$). We used four different stretch factors in our experiments, $t \in \{1, 2, 1.4, 2, 4\}$. Edge weights are randomly selected from $\{1, 2, 3, \ldots, 10\}$.

### B.2 Algorithms and outputs

We implemented several variants of Algorithm [1] which yield different results based on the rounding set $Q$ as well as the single level approximation algorithm. In our experiment we...
used three setups for $Q$: bottom-up (BU) in which $Q = \{1, \ell\}$, top-down (TD) in which $Q = \{1, \ldots, \ell\}$, and composite (CMP) which selects the optimal set of levels $Q^*$ as in Section 2. We used Python 3.5, and utilized the same high-performance computer for all experiments (Lenovo NeXtScale nx360 M5 system with 400 nodes). When using an oracle for single levels in Algorithm 1, we use the ILP formulation provided in Appendix A using CPLEX 12.6.2.

For each instance of the MLGS problem, we compute the costs of the MLSG returned using the BU, TD, CMP approaches, and also compute the minimum cost MLGS using the ILP in Appendix A. For the first set of experiments, we use the ILP as an oracle to find the minimum weight spanner for each level; in this case we refer to the results as Oracle BU, TD, and CMP. In the second set of experiments, we use the metric closure subsetwise spanner Algorithm 3 as the single level subroutine, which we refer to as Metric Closure BU, TD, and CMP. We show the performance ratio for each heuristic in the $y$-axis (defined as the heuristic cost divided by OPT), and how the ratio depends on the input parameters (number of vertices $|V|$, number of levels $\ell$, and stretch factors $t$).

Finally, we discuss the running time of the algorithms. All box plots show the minimum, interquartile range and maximum, aggregated over all instances using the parameter being compared.

### B.3 Results

Figures 2–5 show the results of the oracle TD, BU, and CMP. We show the impact of different parameters (number of vertices $|V|$, number of levels $\ell$, and stretch factors $t$) using line plots for the three approaches separately in Figures 2–4. Figure 5 shows the performance of the three variants together in box plots. In Figure 2 we can see that all variants perform better when the size of the vertex set $|V|$ increases. Figure 3 shows that all variants perform worse as the number of levels $\ell$ increases. In Figure 4 we see that in general, performance decreases as the stretch factor $t$ increases.

The most time consuming part of the experiment is the execution time of the ILP for solving MLGS instances optimally. Hence, we first show the running times of the exact solution of the MLGS instances in Figure 6 with respect to the number of vertices $|V|$, number of levels $\ell$, and stretch factors $t$. For all parameters, the running time tends to increase as the size of the parameter increases. In particular, the running time with stretch factor 4 (Fig. 6 right) was much worse. We can reduce the size of the ILP by removing some constraints based on different techniques discussed in [3]. However, these size reduction techniques are less effective as the stretch factor increases. We show the running times of computing oracle bottom-up, top-down and composite solutions in Figure 7.

Notice that, although the running
Figure 3 Performance of oracle bottom-up, top-down and composite on Erdős–Rényi graphs w.r.t. the number of levels.

Figure 4 Performance of oracle bottom-up, top-down and composite on Erdős–Rényi graphs w.r.t. the stretch factors.

Figure 5 Performance of oracle bottom-up, top-down and composite on Erdős–Rényi graphs w.r.t. the number of vertices, the number of levels, and the stretch factors.
time of composite should be worse, sometimes top down takes more time. The reason is that we have an additional edge-pruning step after computing subsetwise spanner. In top down, every level has this pruning step, which is causing additional computation time and affecting the runtime especially when the graph is large.

![Figure 6](image1.png) Experimental running times for computing exact solutions w.r.t. the number of vertices, the number of levels, and the stretch factors.

![Figure 7](image2.png) Experimental running times for computing oracle bottom-up, top-down and combined solutions w.r.t. the number of vertices, the number of levels, and the stretch factors.

The ILP is too computationally expensive for larger input sizes and this is where the heuristic can be particularly useful. We now consider a similar experiment using the metric closure algorithm to compute subsetwise spanners, as described in Section 3. We show the impact of different parameters in Figures 8–10. Figure 11 shows the performance of the three algorithms together in box plots. We can see that the heuristics perform very well in practice.

![Figure 8](image3.png) Performance of heuristic bottom-up, top-down and composite on Erdős–Rényi graphs w.r.t. the number of vertices.

Our final experiments test the heuristic performance on a set of larger graphs. We generated the graphs using the Erdős–Rényi model, with $|V| \in \{100, 200, 300\}$. We evaluated more levels ($\ell \in \{2, 4, 6\}$) with stretch factors $t \in \{1.2, 1.4, 2, 4\}$. Here, the ratio is determined by dividing the BU, TD and CMP cost by $\min(BU, TD, CMP)$ (as computing the optimal
Figure 9 Performance of heuristic bottom-up, top-down and composite on Erdős–Rényi graphs w.r.t. the number of levels.

Figure 10 Performance of heuristic bottom-up, top-down and composite on Erdős–Rényi graphs w.r.t. the stretch factors.

Figure 11 Performance of heuristic bottom-up, top-down and composite on Erdős–Rényi graphs w.r.t. the number of vertices, the number of levels, and the stretch factors.
MLGS would be too time consuming). Figure 12 shows the performance of the bottom-up, top-down and composite algorithms with respect to $|V|$, $\ell$ and $t$. Figure 13 shows the aggregated running times per instance, which significantly worsen as $|V|$ increases. The results indicate that while running times increase with larger input graphs, the number of levels and the stretch factors seem to have little impact on performance. Notably when the metric closure algorithm is used in place of the ILP for the single level solver (Fig. 13), the running times decrease for larger stretch factors.

**Figure 12** Performance of heuristic bottom-up, top-down and composite on large Erdős–Rényi graphs w.r.t. the number of vertices, the number of levels, and the stretch factors. The ratio is determined by dividing the objective value of the combined (min(BU, TD, CMP)) heuristic.

**Figure 13** Experimental running times for computing heuristic bottom-up, top-down and composite solutions on large Erdős–Rényi graphs w.r.t. the number of vertices, the number of levels, and the stretch factors.