WHAT CAN THE ALIGNMENTS OF THE VELOCITY MOMENTS TELL US ABOUT THE NATURE OF THE POTENTIAL?

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ABSTRACT

We prove that, if the time-independent distribution function $f(v; r)$ of a steady-state stellar system is symmetric under velocity inversion such that $f(-v_1, v_2, v_3; r) = f(v_1, v_2, v_3; r)$ and the same is true for $v_2$ and $v_3$, where $(v_1, v_2, v_3)$ is the velocity component projected onto an orthogonal frame, then the potential within which the system is in equilibrium must be separable (i.e., the Stäckel potential). Furthermore, we find that the Jeans equations imply that, if all mixed second moments of the velocity vanish; that is, $\langle v_i v_j \rangle = 0$ for any $i \neq j$, in some Stäckel coordinate system and the only non-vanishing fourth moments in the same coordinate are those in the form of $(v_i^2)$ or $(v_i^2 v_j^2)$, then the potential must be separable in the same coordinates. Finally we also show that all second and fourth velocity moments of tracers with an odd power to the radial component $v_r$ being zero is a sufficient condition to guarantee the potential to be of the form $\Phi = f(r) + r^{-2}g(\theta, \phi)$.

Key words: Galaxy: kinematics and dynamics – methods: analytical

1. INTRODUCTION

With the advent of large data sets for stellar kinematics in the solar neighborhood, growing evidence suggests that the velocity ellipsoids constructed from the local halo stars are aligned along the coordinate frame directions of the spherical polar coordinate centered at the Galactic center to a good approximation (e.g., Smith et al. 2009; Bond et al. 2010; King et al. 2015). Smith et al. (2009) have claimed that this alignment implies the sphericity of the underlying potential due to the Galactic dark matter halo. Their argument is based on the theorem that, if all orbits in the given potential respect an integral of motion that is independent of the sign for the radial component of the velocity, then the radial coordinate can always be separated off in the Hamilton–Jacobi equation (HJE) for the system. Thanks to the Jeans theorem, this indicates that steady-state populations with a distribution symmetric under the parity of the radial motion are allowed only if the gravitational potential is spherical or in the form of the one due to a pure dipole, while the velocity ellipsoids resulting from such populations must be aligned radially in the direction of the spherical coordinate frames.

Nevertheless this reasoning is incomplete because there may exist distribution functions that fail the symmetry condition, but still produce velocity ellipsoids that are aligned radially. Since the velocity ellipsoid is defined for any distribution irrespective of its symmetry, it is always possible to find such a distribution locally. However, the answer to the question as to whether it is possible to construct a global steady-state distribution function that does not possess symmetry under reversal of the radial velocity, but which nonetheless has radially aligned velocity ellipsoids everywhere, is still unclear at the moment. Note that Binney & McMillan (2011) have provided a three-integral distribution function in a highly flattened axisymmetric potential with the velocity ellipsoids at some high-latitude locations aligned radially, albeit not globally. In fact, they have argued that the explicit connection between the behavior of the velocity ellipsoids and the shape of the potential can only be drawn under the prior assumption of the Stäckel potential, but not with an arbitrary potential.

We then would like to ask what are the observational constraints to guarantee the underlying potential to be separable in the given coordinate. Classically, a sufficient condition for the Stäckel potential is that the distribution function is given by a function of a quadratic polynomial of velocities (other than combinations of the energy and the square of the angular momentum) (c.f. Eddington 1915; Chandrasekhar 1939). Very recently, Evans et al. (2015) have also shown that there actually exists a weaker sufficient condition on the distribution for separable potentials; namely, if the even part of the distribution is symmetric under each separate parity transform of a single momentum component in a spherical, cylindrical, or spheroidal/ellipsoidal coordinate, then the potential must be in the separable form in the corresponding coordinate. Although this does not settle the original question regarding whether the alignment of the velocity ellipsoids can by itself imply the separability of the potential, we conjecture that “alignments” of even velocity moments in every order actually can. This idea will be formalized rigorously and proven in this paper.

Extending this, we also seek the possibility of relaxing the requirement for every order to some finite subsets of the velocity moments. An obvious line of approach would be using the Jeans equations, which directly relate velocity moments and their spatial gradients to the underlying potential. While there have been many investigations based on the Jeans equation to find a model with aligned velocity ellipsoids in an arbitrary potential, most, if not all of these studies have only considered the behavior of the velocity dispersions (i.e., the second moments). By contrast, the symmetric distribution considered by Evans et al. (2015) actually generates a specifically constrained set of velocity moments of higher order, too. Inspired by this, we consider what constraints on the potential may be deduced if additional conditions on the alignments of higher order moments are imposed. In the end, we discover that the alignments of the fourth moments as well as the second moments are actually a sufficient assumption to deduce the separability of the potential.

In the following section (Section 2), we first review the principal concepts related to the Stäckel coordinates and the
separable potential. Compared to the standard approach typically found in the astrophysical literature (see, e.g., de Zeeuw 1985; de Zeeuw & Lynden-Bell 1985), the point of view here is slightly more abstract and somewhat more formal, which is more suitable for our purpose and also affords us more general conclusions. For the sake of the self-containedness, we include more materials than what is absolutely necessary. In Section 3, we then generalize the result of Evans et al. (2015) and provide its more formal proof. In addition, we also introduce a precise statement concerning the alignment of the higher order velocity moments in terms of vanishing cross-term. In Section 4, we then derive the Jeans equations in every order from the moment integrals on the collisionless Boltzmann equation (CBE), which are the basis of the proof found in the subsequent sections. The next section (Section 5) provides the proof of the primary result of this paper, Theorem 6; that is, the alignments of the second and fourth moment in the Stäckel coordinate implies the separability of the potential in the same coordinate. In Section 6, we shift our focus to specific three-dimensional cases and find that the translational, rotational, or spherical symmetry of the potential may be inferred from only a subset of the alignment requirement for the second and fourth moments.

2. PRELIMINARIES
2.1. The Stäckel Coordinate

In the astrophysical literature, the Stäckel coordinates are usually considered as synonymous with the confocal ellipsoidal coordinates (including their degenerate limits). Although this approach is not necessarily incorrect, it is still unsatisfactory because it does not inform us about their defining characteristics. Instead, we consider a pedagogical definition of the Stäckel coordinate, namely the orthogonal curvilinear coordinate in which the HJE for the geodesic (i.e., force-free) motion is solvable through additive separation of variables.

Provided that the Hamiltonian \( \mathcal{H}(q^1, ..., q^n; p_1, ..., p_n) \) does not explicitly depend on the time, the HJE is reducible to the partial differential equation on the Hamilton characteristic function, \( W(q^1, ..., q^n) \); that is, \( \mathcal{H}(q^1, ..., q^n; W, ..., W, n) = E \), where \( W \equiv \partial W/\partial q^i \) and \( E \) is a constant. The reduced HJE then implies \( \partial^2 \mathcal{H}/\partial q^j \partial q^i \mathcal{H} \big|_{p_i = W_i} + \sum_{i=1}^{n} W_{ik} \partial^2 \mathcal{H}/\partial p_i \partial q^i \big|_{p_i = W_i} = 0 \)

for any \( k \). Here \( \partial \mathcal{H} = \partial \mathcal{H}(q^1, ..., q^n; p_1, ..., p_n) \partial q^k \) and \( \partial^2 \mathcal{H}/\partial q^j \partial q^i = \partial \mathcal{H}(q^1, ..., q^n; p_1, ..., p_n) \partial p_j \partial p_i \). The Hamilton–Jacobi method of integrating the equation of motions seeks a complete solution of this equation such that \( W = \sum_{i=1}^{n} w_i(q^i) \) (so \( W_i = 0 \) for all \( i \neq j \)). The existence of a complete solution is equivalent to the existence of the solution set \( \{p_1, ..., p_n\} \) for the overdetermined system of the partial differential equations,

\[
\frac{\partial \mathcal{H}}{\partial q^i} = \frac{\partial \mathcal{H}}{\partial q^j}; \quad \frac{\partial \mathcal{H}}{\partial q^j} = 0 \quad (j \neq k).
\]

In order for this system to be integrable, the compatibility condition; that is, \( (\partial/\partial q_k)(\partial \mathcal{H}/\partial q_j) = (\partial/\partial q_j)(\partial \mathcal{H}/\partial q_k) = 0 \) for all \( j \neq k \) needs to be satisfied. This then results in

\[
(\partial^2 \mathcal{H})(\partial q^j)(\partial q^j) + (\partial^2 \mathcal{H})(\partial q^j)(\partial q^k) = (\partial^2 \mathcal{H})(\partial q^j)(\partial q^j) + (\partial^2 \mathcal{H})(\partial q^j)(\partial q^j)
\]

for any \( j \neq k \), which is known as the Levi-Civita separability condition after Levi-Civita (1904).

In an orthogonal curvilinear coordinate \( (q^1, ..., q^n) \) with the scale factors \( \{h_1, ..., h_n\} \), the Hamiltonian of the geodesic motion is given by \( \mathcal{H} = \sum_{i=1}^{n} p_i^2/(2h_i^2) \). The Levi-Civita condition on this Hamiltonian reduces to

\[
\sum_{i=1}^{n} p_i^2 \partial \mathcal{H}(h_i^{-2}) = 0 \quad (j \neq k),
\]

where \( \partial \mathcal{H}(f) \) is the differential operator acting on a function \( f(q^1, ..., q^n) \), defined to be

\[
\partial \mathcal{H}(f) = \frac{\partial^2 f}{\partial q^j \partial q^i} + \frac{\partial \ln h_i^2}{\partial q^j} \partial_{q^i} f + \frac{\partial \ln h_j^2}{\partial q^i} \partial_{q^j} f.
\]

Note that \( \partial \mathcal{H} = \partial_{q^i} \). Hence the necessary and sufficient condition for the HJE of the geodesic Hamiltonian in the chosen orthogonal coordinate to be soluble through separation of variable is \( \partial \mathcal{H}(h_i^{-2}) = 0 \) for all triplets of indices \( (i, j, k) \) with \( j \neq k \), which is referred to as the Stäckel coordinate condition after Stäckel (1891, 1893). That is to say, Stäckel coordinates are any orthogonal curvilinear coordinates whose scale factors satisfy the Stäckel coordinate condition.

Like the Levi-Civita condition, the Stäckel coordinate condition is also understood to be the integrability condition for the existence of the solution set to a system of differential equations. In particular, suppose that there exists a set of \( n \) independent functions \( \{u_1(q^1), ..., u_n(q^n)\} \) such that \( \mathcal{U}(q^1, ..., q^n) = \sum_{i=1}^{n} u_i(q^i)/h_i^2 \) is constant. Then \( \nabla \mathcal{U} = 0 \), or

\[
\frac{\partial \mathcal{U}}{\partial q^k} = \sum_{i=1}^{n} \frac{\partial h_i^{-2}}{\partial q^k} u_i(q^i) + \frac{u_i^{(k)}}{h_i^2} = 0,
\]

for any \( k \). Thus the set of functions \( u_i \) must be the solution of

\[
\frac{\partial u_i}{\partial q^k} = -\delta_{i}^{k} h_k^{-2} \sum_{i=1}^{n} \partial h_i^{-2}/\partial q^k u_i,
\]

where \( \delta_{i}^{k} \) is the Kronecker delta. The integrability condition on this set of partial differential equations then results in

\[
\frac{\partial}{\partial q^j} \frac{\partial \mathcal{U}^{\mathcal{K}}}{\partial q^k} = \frac{\partial}{\partial q^k} \frac{\partial \mathcal{U}_i}{\partial q^j} = h_i^{-2} \sum_{i=1}^{n} \partial \mathcal{H}(h_i^{-2}) u_i = 0 \quad (j \neq k).
\]

In other words, the condition that \( \partial \mathcal{H}(h_i^{-2}) = 0 \) for all \( j \neq k \) and any \( i \) implies the existence of the set of functions \( u_i(q^i) \) such that \( \sum_{i=1}^{n} u_i(h_i^{-2}) \) is constant. Moreover, the Frobenius theorem further indicates that there actually exist \( n \) such linearly independent solution sets \( \{u_1^{(q^1)}, ..., u_n^{(q^n)}\} \) where \( j \in \{1, ..., n\} \). Hence, if \( h_i \)’s are the scale factors of the Stäckel coordinate, there exists an invertible \((n \times n)\)-matrix of
functions $[S'_i(q')]$ that satisfy constraints:

$$
\sum_{i=1}^{n} S'_i(q') h_i^{-2} \left\{ \begin{array}{l}
1 \quad (j = 1) \\
0 \quad (j = 2, \ldots, n)
\end{array} \right.
$$

(9)

This is equivalent to insisting that $h_i^{-2} = C_i^j/|S|$, where $C_i^j$ is the co-factor of the matrix $[S'_i(q')]$ and $|S| = \det [S'_i(q')] = \sum_{i} C_i^j/|S|$ is its determinant (known as the Stäckel determinant). The existence of such invertible matrices of functions $[S'_i(q')]$ may be considered as an alternative definition of the Stäckel coordinate (c.f. Goldstein 1980), which is closer to Stäckel’s original approach.

The Stäckel coordinate condition is the system of partial differential equations on the scale factors of an orthogonal coordinate. It is fairly straightforward to demonstrate that the scale factors of the confocal ellipsoidal coordinate as well as all of its degenerate limits satisfy the Stäckel coordinate condition. On the other hand, the differential equation system due to the Stäckel coordinate condition can in principle be soluble to obtain the general expression (including some arbitrary functions) for the scale factors of the Stäckel coordinates. In the flat Euclidean space, the general solution actually results in the scale factors of the confocal ellipsoidal coordinate, up to arbitrary scaling functions (Levi-Civita 1904). The same result was also found by Eddington (1915) and Lynden-Bell (1962), although their respective assumptions upon which the derivation of the differential equations equivalent to the Stäckel coordinate condition is based are distinct from the consideration here.

2.2. The Separable or Stäckel Potentials

Next let us consider the condition for the HJE of a natural dynamical system with the potential $\Phi(q^1, \ldots, q^n)$ to be soluble via separation of variables. In an orthogonal coordinate, the Hamiltonian of a natural system is $H = \sum_{i=1}^{n} p_i^2/(2h_i^2) + \Phi$, and the Levi-Civita condition simplifies to

$$
\frac{1}{2} \sum_{i=1}^{n} h_i^{-2} D_{jk}(h_i^{-2}) + D_{jk}(\Phi) = 0 \quad (j \neq k).
$$

(10)

Hence the corresponding HJE admits a complete integral if the chosen orthogonal coordinate is the Stäckel coordinate and the potential is the solution of the differential equation $D_{jk}(\Phi) = 0$ for all $j \neq k$ with $D_{jk}$ given by Equation (5).

With $\Phi = \sum_{i=1}^{n} f_i(q')/h_i^2$, where $f_i(q')$ is an arbitrary function of the coordinate component $q^i$ alone and $h_i$’s are the scale factors of the Stäckel coordinate, it is straightforward to show $D_{jk}(\Phi) = 0$ for all $j \neq k$. If the specific expressions for the scale factors are given, the opposite implication is also shown to hold by solving the differential equation. For general cases, however, one needs to find the system of partial differential equations, whose integrability condition leads to $D_{jk}(\Phi) = 0$. In particular, if we assume the existence of the set of functions $[f'_i(q')]$ such that $\Phi = \sum_{i=1}^{n} f'_i(q')/h_i^2$, then

$$
\frac{\partial \Phi}{\partial q^j} = \frac{f'_i(q')}{h_i^2} + \sum_{i=1}^{n} \frac{\partial h_i^{-2}}{\partial q^j} f_i(q'),
$$

(11)

and so $[f'_1, \ldots, f'_n]$ must be the solution set of the system

$$
\frac{\partial f'_i}{\partial q^j} = \delta^j_i h_i^{-2} \left( \frac{\partial \Phi}{\partial q^j} - \sum_{i=1}^{n} \frac{\partial h_i^{-2}}{\partial q^j} f_i(q') \right)
$$

(12)

According to the Frobenius theorem, the necessary and sufficient condition for the solution to exist is the compatibility condition $(\partial/\partial q^j)(\partial f'_i/\partial q^j) = (\partial/\partial q^k)(\partial f'_i/\partial q^k)$ for any $i, j, k$ to hold. Here the only non-trivial conditions among them are

$$
\frac{\partial}{\partial q^j} \left( \frac{\partial f'_i}{\partial q^j} \right) = h_i^{-2} \left( D_{jk}(\Phi) - \sum_{i=1}^{n} D_{jk}(h_i^{-2}) f'_i \right) = 0
$$

(13)

for all $j = k$. The condition $D_{jk}(\Phi) = 0$ is thus the necessary condition for the existence of the solution set $[f'_1, \ldots, f'_n]$, while the same condition is also sufficient for the (local) existence of such solution sets with the scale factors of the Stäckel coordinate satisfying $D_{jk}(h_i^{-2}) = 0$ for $j \neq k$. In other words,

$$
\Phi(q^1, \ldots, q^n) = \sum_{i=1}^{n} f'_i(q')/h_i^2
$$

(14)

is the general solution of $D_{jk}(\Phi) = 0$ for all $j \neq k$, given $D_{jk}(h_i^{-2}) = 0$. Henceforth, we shall refer the potential in the form of Equation (14) to be separable in the particular Stäckel coordinate, whereas the potential shall be referred to as the Stäckel potential if there exists a Stäckel coordinate in which the potential is expressible as in Equation (14).

By definition, the HJE of the natural dynamical system with a Stäckel potential is soluble through the separation of variables in the Stäckel coordinate in which the potential is separable. Less abstractly, the Stäckel potential admits a set of $n$ independent integrals of motion. In particular, let

$$
\alpha_j = \sum_{i=1}^{n} \left[ p_i^2 + 2 f'_i(q') \right] T'_j(q^1, \ldots, q^n),
$$

(15)

where $(T'_j)$ is the inverse matrix of $(S'_i)$ in Equation (9) for the chosen coordinate (i.e., $T'_j = C'_i/|S|$). Here $T'_i = C'_i/|S| = h_i^{-2}$ and so $\alpha_1 = 2H$. Next consider the Poisson brackets

$$
\{\alpha_j, \alpha_k\} = \sum_{i=1}^{n} 2p_i \left( p_i^2 + 2 f'_i(q') \right) \left( T'_k \frac{\partial T'_j}{\partial q^i} - T'_j \frac{\partial T'_k}{\partial q^i} \right).
$$

(16)

However, since $\sum_{i=1}^{n} S'_i T'_j = \delta^j_i$ and $S'_i = S'_i(q')$, we find

$$
\sum_{i=1}^{n} S'_i T'_j = \delta^j_i \quad (i = k), \quad R'_k = - \sum_{i=1}^{n} \frac{dS'_i T'_j}{dq^i}
$$

(17)

which further implies that

$$
\frac{\partial T'_j}{\partial q^i} = \sum_{m, k=1}^{n} T'_{jm} S'_{ik} \frac{\partial T'_k}{\partial q^i} = T'_{jk} R'_k,
$$

(18)

and thus $\{\alpha_j, \alpha_k\} = 0$ for any $j, k$. In other words, $\alpha_i$’s are all functionally independent—thanks to $(T'_i)$ being invertible—integrals of motion (note $\alpha_1 = 2H$) that are in involution and so all orbits within the Stäckel potential are Liouville-integrable (i.e., all bounded orbits are quasi-periodic).
We note that every integral of motion \( \alpha_i \) is a linear function of \( p_i^2 \)'s. What is more interesting is the converse: namely,

**Theorem 1.** If the natural dynamical system admits an integral of motion expressible in an orthogonal coordinate as \( \mathcal{J} = \sum_{i=1}^{n} \zeta_i v_i^2 + \Xi \), where \( \zeta_i \)'s and \( \Xi \) are smooth functions of positions and all \( \zeta_i \)'s are distinct, then the coordinate must be a Stäckel coordinate and the potential is separable in the same coordinate.

**Proof:** First let the Hamiltonian be \( \mathcal{H} = \sum_{i=1}^{n} p_i^2/(2\hbar^2) + \Phi \). Then \( v_i^2 = p_i^2/\hbar^2 \) and so \( \mathcal{J} \) is an integral of motion if

\[
\{\mathcal{J}, \mathcal{H}\} = \sum_{i=1}^{n} \left( \sum_{j=1}^{n} \alpha_{ij} p_i^2 + \beta_{ij} \right) p_i^2 / \hbar^2, \tag{19}
\]

identically vanishes, where

\[
\alpha_{ij} = \frac{\partial}{\partial q^j} \left( \zeta_i h_i^2 \right) - \zeta_i \frac{\partial h_i}{\partial q^j}; \quad \beta_{ij} = \frac{\partial \Xi}{\partial q^j} - 2\zeta_i \frac{\partial \Phi}{\partial q^j}. \tag{20}
\]

This requires all \( \alpha_{ij} = 0 \) and \( \beta_{ij} = 0 \): that is, for any \( i, j \),

\[
\frac{\partial \zeta_i}{\partial q^j} - \zeta_i \frac{\partial h_i}{\partial q^j} = 0; \quad \frac{\partial \Xi}{\partial q^j} = 0. \tag{21}
\]

Here \( \alpha_{ij} = 0 \) results in a system of differential equations on \( \zeta_i \)'s, and so in order for the solution to exist, the compatibility condition should again be satisfied: namely,

\[
\frac{\partial}{\partial q^k} \frac{\partial \zeta_i}{\partial q^j} - \frac{\partial}{\partial q^j} \frac{\partial \zeta_i}{\partial q^k} = 0. \tag{22}
\]

Provided that \( \zeta_j = \zeta_k \) for \( j = k \), the Stäckel coordinate condition is therefore indeed necessary for \( \{\mathcal{J}, \mathcal{H}\} = 0 \). Similarly the compatibility condition on \( \Xi \) results \( ^3 \) in

\[
\frac{\partial}{\partial q^j} \frac{\partial \Xi}{\partial q^k} - \frac{\partial}{\partial q^k} \frac{\partial \Xi}{\partial q^j} = 0, \tag{23}
\]

and so this indicates that \( D_{ij}(\Phi) = 0 \) for all \( i = j \), assuming \( \zeta_i = \zeta_j \) for any \( i \neq j \). QED.

This theorem was implicit in Eddington (1915), who derived Equation (21) for three-dimensions (his Equation (13)) under the so-called Schwarzschild ellipsoidal hypothesis; that is to say, \( \mathcal{F} \propto \exp(-\mathcal{J}) \), where \( \mathcal{F} \) is the phase-space distribution function. He then showed that, in the three-dimensional flat Euclidean space, this implies (i) the coordinate surfaces are confocal quadrics, so the coordinate must be a confocal ellipsoidal coordinate or one of its degenerate limits and (ii) the potential must be able to be expressible in the form of Equation (14). Given the Jeans (1915) theorem, the distribution \( \mathcal{F} \) is an integral of motion, and therefore the ellipsoidal hypothesis implies that \( \mathcal{J} \) is an integral. In fact, his results were due to \( \mathcal{J} \) being an integral and do not rely on the assumed form of \( \mathcal{F} \). It was not until Lynden-Bell (1962) that it was explicitly stated that the Stäckel potential is implied by the existence of an integral of motion in a specific nature rather than the particular form of the distribution function.

Chandrasekhar (1939) investigated a nominally weaker assumption than that of Eddington (1915); that is, the existence of a distribution \( \mathcal{F}(\mathcal{J}) \) depending on the single integral \( \mathcal{J} \). However, there appears to be some incompleteness in Chandrasekhar’s analysis, as he failed to identify the Stäckel potentials as solutions, although he did find an unusual (albeit somewhat academic) stellar system with helical symmetry—see Evans (2011) for a historical review. Technically, the form of the integral \( \mathcal{J} \) considered by Chandrasekhar (1939) is more relaxed than that of Theorem 1, as it is a quadratic polynomial of the velocity components. Thanks to the time reversal symmetry of the natural dynamical system, the even and odd parts of any integral of motion are also independent integrals, and so his assumption is basically equivalent to the existence of an integral of the form \( \mathcal{J} = q(r) + \Xi \) with \( q(r) \) being a homogeneous degree-two polynomial (i.e., a quadratic form) of the velocities. However, an arbitrary quadratic form (over the reals) can always be diagonalized\(^4\) to bring it into the form considered in Theorem 1, although the principal values, \( \zeta_i \)'s of the integral are not necessarily all distinct. In other word, Theorem 1 actually indicates that the existence of any integral that is a quadratic function of the velocity (excluding some degenerate cases corresponding to the Hamiltonian or the squares of the momenta) necessarily implies that the potential is of Stäckel (see Makarov et al. 1967; Evans 1990).

### 3. DISTRIBUTIONS IN THE STÄCKEL POTENTIAL

Evans et al. (2015) have shown that the sufficient condition for the integral of motion to guarantee the separability of the potential can be weaker than that of Theorem 1. In particular, they found that the symmetry of the integral under velocity inversion actually suffices: that is,

**Theorem 2.** Suppose that \((q^1, \ldots, q^n)\) is an orthogonal coordinate, and \((p_1, \ldots, p_n)\) is the conjugate set of the momenta. If the dynamical system governed by the Hamiltonian \( \mathcal{H} = \sum_{i=1}^{n} p_i^2/(2\hbar^2) + \Phi(q^1, \ldots, q^n) \) observes an integral of motion of the form \( \mathcal{I} = \mathcal{I}(p_1^2, \ldots, p_n^2; q^1, \ldots, q^n) \) with \( \zeta_j = \zeta_k \) for all \( j = k \), where \( \zeta_i = 2\hbar^2 \partial^2 \mathcal{I}/\partial (p_i^2) \), then the orthogonal coordinate must be a Stäckel coordinate and the potential \( \Phi \) is separable in the same coordinate.

**Proof:** Since \( \{\mathcal{I}, \mathcal{H}\} = \sum_{i=1}^{n} p_i^2 \mathcal{E}_i(p_1^2, \ldots, p_n^2; q^1, \ldots, q^n) \) is an integral of motion,

\[
\{\mathcal{I}, \mathcal{H}\} = \sum_{i=1}^{n} p_i^2 \mathcal{E}_i(p_1^2, \ldots, p_n^2; q^1, \ldots, q^n) = 0, \tag{24}
\]

where (note \( \partial \mathcal{I}/\partial p_i = 2p_i [\partial^2 \mathcal{I}/\partial (p_i^2)] \))

\[
\mathcal{E}_i = \frac{1}{\hbar^2} \frac{\partial \mathcal{I}}{\partial q^i} - 2 \frac{\partial \mathcal{H}}{\partial q^i} \frac{\partial \mathcal{I}}{\partial (p_i^2)} \tag{25}
\]
Since both \( I \) and \( H \) are invariant under \( p_i \rightarrow -p_i \) for any \( j \), all \( \xi_j \)'s are also invariant under the same transforms. Hence it follows that \( \sum_{i=1}^n p_i \xi_i = 0 \) and that \( \xi_j = 0 \) for all \( i \). Specifically

\[
\frac{\partial I}{\partial q^j} = \xi_i \frac{\partial H}{\partial q^i} = 2h^2 \frac{\partial I}{\partial (p^j)}.
\]

Here, we first note that, for any \( i, j \),

\[
\frac{\partial \zeta_i}{\partial q^j} = 2h^2 \left( \frac{\partial I}{\partial q^j} \right) + \frac{\partial h^2}{\partial q^j} \frac{\partial I}{\partial (p^j)} = 2h^2 \left( \frac{\partial I}{\partial q^j} \right) + \frac{\partial h^2}{\partial q^j} \frac{\partial I}{\partial (p^j)} = 4h^2 \frac{\partial h^2}{\partial q^j} \frac{\partial I}{\partial (p^j)} + \frac{\partial \ln h^2}{\partial q^j}.
\]

Thus, we must have \( \zeta_i \equiv \zeta_j \) for all \( i, j \). Consequently, if \( \zeta_i \equiv \zeta_j \) then the orthogonal coordinate must be a St"uckel coordinate and the potential is separable in the same coordinate.

Here, we have replaced the non-degeneracy condition on the integral, \( h^2 \left( \frac{\partial I}{\partial q^j} \right) = h^2 \left( \frac{\partial I}{\partial (p^j)} \right) \) for all \( i \neq j \) by the velocity second moment tensor with all distinct principal axes. This is allowed because the general solution of \( h^2 \left( \frac{\partial I}{\partial q^j} \right) = h^2 \left( \frac{\partial I}{\partial (p^j)} \right) \) for \( i \neq j \) is \( F^i = F^j(v^j) \), where \( v^j = \left( p_i / h^2 \right)^2 + \left( p_j / h^2 \right)^2 = v^j + v^j \); in other words, the dependences of \( F^j \) on \( p_i, p_j \) are only through \( v^j \) (i.e., \( F^j \) becomes isotropic within \( v_i-v_j \) plane), which implies \( \langle v^j \rangle = \langle v^j \rangle \).

Note that \( \langle v^j \rangle \) becomes isotropic within \( v_i-v_j \) plane, which implies \( \langle v^j \rangle = \langle v^j \rangle \). However, construction of the full distribution is challenging. Instead the usual constraints on the distribution are typically given as the set of velocity moments in the orthogonal coordinate: namely (here \( \varphi \equiv \int dv F \) is the local density)

\[
\varphi \left( \prod_{i=1}^n v_i^{m_i} \right) = \int dv \left( \prod_{i=1}^n v_i^{m_i} \right) F
\]

Note that \( \mathcal{F}(v) \rightarrow \mathcal{F}(-v) \) results in \( \int dv \mathcal{F}(-v) (\prod_{i=1}^n v_i^{m_i}) = \int dv \mathcal{F}(v)(\prod_{i=1}^n (-v_i)^{m_i}) = (-1)^{\sum_{i=1}^n m_i} \varphi \left( \prod_{i=1}^n v_i^{m_i} \right) \), and so

\[
\int dv \left( \prod_{i=1}^n v_i^{m_i} \right) F^+ = \begin{cases} 
\varphi \left( \prod_{i=1}^n v_i^{m_i} \right) & \text{if } \sum_{i=1}^n m_i \text{ is even } \\
0 & \text{if } \sum_{i=1}^n m_i \text{ is odd }
\end{cases}
\]

If \( \mathcal{F}(v_1, \ldots, v_n) = \mathcal{F}^+(v_1, \ldots, v_n) \) \( \mathcal{F}(v) \rightarrow \mathcal{F}(-v) \) results in \( \int dv \mathcal{F}(-v) (\prod_{i=1}^n v_i^{m_i}) = \int dv \mathcal{F}(v)(\prod_{i=1}^n (-v_i)^{m_i}) = (-1)^{\sum_{i=1}^n m_i} \varphi \left( \prod_{i=1}^n v_i^{m_i} \right) \), and so on. In fact, the converse also holds:

**Lemma 4.** Suppose that \( \mathcal{F}(v_1, \ldots, v_n) \) is a phase-space distribution with \( (v_1, \ldots, v_n) \) the conjugate momentum set of an orthogonal coordinate \( (q^1, \ldots, q^n) \). Then \( \mathcal{F}(v_1, \ldots, v_n) = \mathcal{F}^+(v_1, \ldots, v_n) \) if and only if \( \langle v_1^{m_1} v_2^{m_2} \cdots v_n^{m_n} \rangle = (-1)^{m_1} \langle v_1^{m_1} v_2^{m_2} \cdots v_n^{m_n} \rangle \) for even \( \sum_{i=1}^n m_i \). Thus, the only non-vanishing even velocity moments resulting from the distribution satisfying the condition of Corollary 3 are in the form of \( \prod_{i=1}^n v_i^{2m_i} \)---specifically \( \langle v_i^2 \rangle \) for the second moments, \( \langle v_i^4 \rangle \) and \( \langle v_i^6 \rangle \) for the fourth moments and so on. In fact, the converse also holds:

**Corollary 5.** If the only non-vanishing even velocity moments of the steady-state tracers in an orthogonal coordinate are of the form \( \prod_{i=1}^n v_i^{2m_i} \) and the second moments are all distinct, then the orthogonal coordinate must be a St"uckel coordinate and the potential is separable in the same coordinate.

The remaining “if”-part of Lemma 4 may be proven utilizing the characteristic function \( \varphi \); that is, consider the Fourier transform of the distribution (here \( p \cdot k \equiv \sum_{i=1}^n p_i k_i \)).

\[
\varphi(k^1, \ldots, k^n) = \int \cdots dv_1 \cdots dv_n e^{\pi k \cdot \mathcal{F}(v_1, \ldots, v_n)}. \tag{32}
\]
The partial derivatives of $\varphi$ with respect to $k^j$'s result in
\[\varphi^{(m_1, \ldots, m_n)}(k^1, \ldots, k^n) = \prod_{j=1}^n \left( \frac{\partial}{\partial k_j} \right)^{m_j} \varphi = \]
\[i^m \times \int \cdots \int dp_1 \cdots dp_n e^{i\varphi(k)} \frac{1}{p_j^{m_j}} \mathcal{F}, \tag{33}\]
where $m = \sum_{j=1}^n m_j$. Evaluating at $k = 0$, this results in
\[\varphi^{(m_1, \ldots, m_n)}(0) = i^m \left( \prod_j h_j^{m_j+1} \right) \varphi \left( \prod_j v_j^{m_j} \right). \tag{34}\]
In other words, all the velocity moments are essentially the coefficients of the MacLaurin–Taylor series expansion of $\varphi$ (at $k = 0$) and vice versa. Moreover, if all velocity moments with an odd power of $v_1$ vanish, then $\varphi$ is symmetric under the transform $k^1 \leftrightarrow -k^1$, for all the coefficients in the MacLaurin series for the odd-power terms of $k^1$ vanish. On the other hand, the same symmetry for the real part $R\varphi$ (which is also the even part if $\mathcal{F}$ is real) of $\varphi$ is similarly deduced only with recovered velocities. Finally, the distribution function is recovered through the inverse Fourier transform,
\[\mathcal{F} = \frac{1}{(2\pi)^n} \int \cdots \int dk^1 \cdots dk^n e^{-i\varphi(k)} \); \]
\[\mathcal{F}^+ = \frac{1}{(2\pi)^n} \int \cdots \int dk^1 \cdots dk^n \cos(p \cdot k) R\varphi(k), \tag{35}\]
where $\mathcal{F}$ is assumed to be real. Then $\mathcal{F}(-p_1, \ldots, -p_n) = \int d^k k e^{i\varphi(k)} \varphi(k^1, \ldots, k^n)$ and $\mathcal{F}^+(-p_1, \ldots, -p_n) = \int d^k k \cos(p^j k_j) R\varphi(-k^1, \ldots, k^n)$. Thus, if $\varphi$ is symmetric under $k^1 \leftrightarrow -k^1$, then $\mathcal{F}$ is also symmetric under $p_1 \leftrightarrow -p_1$, whereas the even part $\mathcal{F}^+$ is symmetric under $p_1 \leftrightarrow -p_1$ if the real part $R\varphi$ of the characteristic function is symmetric under $k^1 \leftrightarrow -k^1$, which completes the proof of Lemma 4. Since the argument is valid irrespective of the label for the index, Corollary 3 and Lemma 4 together then imply Corollary 5.

4. THE JEANS EQUATIONS

Corollary 5 is still of little practical use, as it refers to the infinite set of all of the even velocity moments, which is difficult to constrain from observables. Instead, we would like to seek the sufficient condition for the Stäckel potential referring only to a finite subset of the velocity moments. For this, we need to establish the explicit relations among the velocity moments of the system in equilibrium first. This may be achieved through taking the moment integrals of the CBE (Binney & Tremaine 2008). The resulting first moment equations correspond to the usual Jeans equations. Here, we derive all of the $m$th moment equations in an arbitrary coordinate system.

4.1. In an Arbitrary Coordinate System

Suppose that $(q^1, \ldots, q^n)$ is an arbitrary coordinate, with $g_{\mu\nu}$ being its metric coefficient (so that the line element is $ds^2 = g_{\mu\nu} dq^\mu dq^\nu$); throughout this section, the Einstein summation convention for Greek indices are assumed). Let us think of the Hamiltonian of the form $\mathcal{H} = \frac{1}{2} g_{\mu\nu} p_\mu p_\nu + \Phi(q^1, \ldots, q^n)$, where $g_{\mu\nu}$ is the inverse metric. Then the CBE (assuming $\partial \mathcal{F}/\partial t = 0$) in the canonical phase-space coordinate is equivalent to
\[\{\mathcal{F}, \mathcal{H}\} = g_{\mu\nu} \frac{\partial \mathcal{F}}{\partial p_\mu} \left( \frac{1}{2} \frac{\partial g^{\lambda\xi}}{\partial q^\nu} p_\lambda p_\xi + \frac{\partial \Phi}{\partial q^\nu} \right) \frac{\partial \mathcal{F}}{\partial p_\nu} = 0. \tag{36}\]
Next consider integrating this over the momentum space after multiplying by $\int p_i$, where $(1, \ldots, n)$ is a sequence of indices with $i \in \{1, \ldots, n\}$, and also utilizing integration by parts
\[\int d^p \mathcal{F} \prod_i p_i = \int d^p \frac{\partial}{\partial p_i} \left( \mathcal{F} \prod_i p_i \right) \frac{\partial}{\partial p_i}, \tag{37}\]
where $d^p \equiv dp_1 \cdots dp_n$. Since $(q^1, \ldots, q^n, p_1, \ldots, p_n)$ is a canonical phase-space coordinate, $d^q dq dp_1 \cdots dp_n = d^q d^p$ where $d^q$ and $d^p$ are the volume $n$-forms for the configuration and the velocity spaces. Therefore $d^q p = h dq dq^\lambda dq^{\mu_2} dq^{\mu_n}$ where $h = \det(g_{ij})$, and so this gives the Jacobian determinant as $d^q p = h d^q$. Therefore
\[\int d^q \mathcal{F} \prod_i p_i = h \int d^q \mathcal{F} \prod_i p_i = h \mathcal{F} \prod_i p_i, \tag{38}\]
and the $m$th moment integrals of Equation (36) (divided by $h \partial$) result in
\[\sum_{i=1}^n \frac{\partial \Phi}{\partial q^i} V^{(m-1)}_{\mu_{i+1} \cdots \mu_n} + \frac{g_{\mu\nu}}{h \partial} \frac{\partial}{\partial q^i} \left( h \Phi V^{(m+1)}_{\mu_{i+1} \cdots \mu_n} \right) + \frac{\partial g^{\lambda\xi}}{\partial q^i} V^{(m+1)}_{\lambda_{i+1} \cdots \mu_n} = 0. \tag{39}\]
where the slash through the index represents skipping the particular index, while $V^{(m)}_{\mu_{i+1} \cdots \mu_n} = \{p_{i+1} \cdots p_n \}$ is the $m$th momentum moment, which forms a symmetric $(0, m)$-tensor. Thanks to the relation (see Arfken & Weber 2005, Section 2.11)
\[\frac{\partial \ln h^2}{\partial q^i} = g^{\lambda\xi} \frac{\partial g_{\lambda\xi}}{\partial q^i}. \tag{40}\]
Equation (39) is further reducible to
\[\sum_{i=1}^n \frac{\partial \Phi}{\partial q^i} V^{(m-1)}_{\mu_{i+1} \cdots \mu_n} + \frac{1}{2} \frac{\partial}{\partial q^i} \left( g^{ij} V^{(m)}_{\mu_{i+1} \cdots \mu_n} \right) + \frac{g^{ij}}{2} \frac{\partial g_{\mu\nu}}{\partial q^i} V^{(m-1)}_{\mu_{i+1} \cdots \mu_n} = 0, \tag{41}\]
where $V^{(m)}_{\mu_{i+1} \cdots \mu_n} = \{q^i p_{i+1} \cdots p_n \}$, which is basically the same tensor as $V^{(m+1)}_{\lambda_{i+1} \cdots \mu_n}$ but one of the index raised, $V^{(m)}_{\mu_{i+1} \cdots \mu_n} = g^{ij} V^{(m+1)}_{\lambda_{i+1} \cdots \mu_n}$. Also used are $g_{\mu\nu}(\partial g^{\lambda\xi}/\partial q^i) = -g^{ij}(\partial g_{\mu\nu}/\partial q^i)$, which follows

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\[ g^{\kappa \lambda} g_{\mu \nu} = \delta^\kappa_\mu. \] Utilizing the covariant derivative of the tensor
\[ \nabla_\mu T^{\kappa \lambda \cdots}_{\nu \cdots} = \frac{\partial T^{\kappa \lambda \cdots}_{\nu \cdots}}{\partial q^\mu} + \sum_j \Gamma^\mu_{\kappa \lambda} T^{j \lambda \cdots}_{\nu \cdots} - \sum_i \Gamma^\mu_{\nu \lambda} T^{\kappa \cdots\cdot j}_{\nu \cdots} \] defined with the Levi-Civita connection coefficients
\[ \Gamma^\mu_{\kappa \lambda} = \frac{g^{\mu \nu}}{2} \left( \frac{\partial g_{\kappa \nu}}{\partial q^\lambda} + \frac{\partial g_{\lambda \nu}}{\partial q^\kappa} - \frac{\partial g_{\kappa \lambda}}{\partial q^\nu} \right). \] (42)

Equation (41) finally simplifies to
\[ \nabla_\mu \left( \rho \left( q^\mu p_n \cdots p_m \right) \right) + \rho \sum_{(i)} \frac{\partial \Phi}{\partial q^i} (p_n \cdots p_m) = 0, \] (44)
where the sum is over all cyclic permutations through the indices. Here, the \( \nabla_\mu \) term is in fact the divergence of the \( (1, m) \)-tensor field \( \left( q^\mu p_n \cdots p_m \right) \), which, given \( \nabla_\mu g^{\kappa \lambda} = 0 \), is also equivalent to \( \nabla_\mu \left( \rho \left( q^\mu p_n \cdots \right) \right) = \rho \nabla_\mu (\rho \left( q^\mu \right) - \rho \left( q^\mu \right) \) \).

Since Equation (44) is symmetric with respect to any permutation of free indices among \( \{ t_1, \ldots, t_m \} \), there are \( \binom{n}{m} \) independent equations for a fixed \( m \) in \( n \) dimensions—here \( \binom{n}{m} \) is the m combination out of \( n \) elements with repetition, and \( \binom{n}{m} = \prod_{j=0}^{m-1} (n + j) \) is the rising sequential product. The single equation for \( m = 0 \) that is, \( \nabla_\mu (\rho (q^\mu)) = 0 \), is simply the continuity equation \( \nabla^\mu (\rho \Phi) = 0 \) for the time-independent density field, whereas the \( m = 1 \) equation, \( \nabla_\mu (\rho (q^\mu p_n)) + \rho (\partial \Phi/\partial q^i) = 0 \), basically corresponds to the static Euler equation with an anisotropic stress tensor \( \mathbf{P} \) (i.e., the Cauchy or Navier–Stokes momentum equation in fluid mechanics or the Jeans equation in stellar dynamics); namely \( \nabla_\mu \Phi + \rho \nabla_\mu \Phi = 0 \).

### 4.2. In An Orthogonal Coordinate

In an orthogonal coordinate with scale factors \( h_i \), the metric is diagonal as in \( g_{ij} = 0 \) for \( i \neq j \) and \( g_{ii} = h_i^2 \). Since the velocity component \( v_i \) projected onto the orthogonal frame is related to the specific momentum via \( p_i = h_i v_i \) and \( v_i = h_i q_i \), the tensor components in the orthogonal coordinates related to the orthogonal velocity moments through
\[ \langle v_j v_i \cdots v_k \rangle = \frac{V^{(m+1)}_{i j k \cdots}}{h_j h_i \cdots h_k} = \frac{h_j V^{(i)}_{i j} \cdots}{h_j h_i \cdots h_k}. \] (45)

Then Equation (39) or (41) reduces to (here \( h = \prod_{j=1}^{n} h_j \))
\[ \sum_{i=1}^{n} \rho \left( v_i v_i \cdots v_k \right) \frac{\partial \Phi}{h_i h_k \cdots h_k} + \sum_{j=1}^{n} \rho \left( v_i v_i \cdots v_k \right) \frac{\partial \Phi}{h_i h_k \cdots h_k} \] \[ + \sum_{j=1}^{n} \rho \left( v_i v_i \cdots v_k \right) \frac{\partial \ln \left( h_j h_i \cdots h_k \right)}{h_j h_i \cdots h_k} \] \[ - \sum_{j=1}^{n} \rho \left( v_i v_i \cdots v_k \right) \frac{\partial \ln h_j}{h_j h_i \cdots h_k} = 0. \] (46)

in the orthogonal coordinate. For \( m = 0 \), this becomes
\[ \sum_{j=1}^{n} \rho \left( v_i v_i \right) \frac{\partial \ln h_j}{h_j h_i \cdots h_k} = 0. \] (47)

The \( m = 1 \) case results in the Jeans equation
\[ \sum_{j=1}^{n} \rho \left( v_i v_i v_k \right) \frac{\partial \ln h_j}{h_j h_i \cdots h_k} = - \frac{\partial \Phi}{h_i h_i h_i}. \] (48)

with a fixed \( \ell \in \{ 1, \ldots, n \} \). The Jeans equations in an arbitrary three-dimensional curvilinear coordinate system have been derived before (Lynden-Bell 1960; Evans & Lynden-Bell 1989), although the usual expressions typically involve the second moments decomposed into those due to random and coherent motions; namely, \( \langle v_i v_j \rangle = \langle v_i \rangle \langle v_j \rangle + \sigma_{ij} \) etc.

For our purpose here, we also require the expression for the \( m = 3 \) equations: with fixed \( j, k, \ell \)
\[ \sum_{i=1}^{n} \rho \left( v_i v_i v_k \right) \frac{\partial \ln h_j}{h_j h_i \cdots h_k} = - \frac{\partial \Phi}{h_i h_i h_i}. \] (49)

### 5. The Second and Fourth Moments in the Stäckel Potentials

Now we are ready to prove the main finding:

**Theorem 6.** Suppose in the Stäckel coordinate \( (q^1, \ldots, q^n) \) that all mixed second moments of the steady-state tracer velocities vanish and the remaining second moments are all distinct (i.e., \( \langle v_i v_j \rangle = 0 \) and \( \langle v_i^2 \rangle = \langle v_i \rangle^2 \) for all \( i \neq j \) and only nonvanishing fourth velocity moments of the tracers are those in the form of \( \langle v_i^3 \rangle \) or \( \langle v_i^2 v_j \rangle \). Then the potential must be separable in the given Stäckel coordinate.

**Proof:** Under the given condition, Equation (48) simplifies to
\[ \frac{\partial \rho}{\partial q^i} \rho + \rho \left( v_i^2 \right) \frac{\partial \ln h_j}{h_j^2} - \sum_{i=1}^{n} \rho \left( v_i^2 \right) \frac{\partial \ln h_j}{h_j^2} + \rho \frac{\partial \Phi}{h_j^2} = 0. \] (50)

while Equation (49), with \( j = k = \ell \), reduces to
\[ \frac{\partial \rho}{\partial q^i} \rho \left( v_j^4 \right) + \rho \left( v_j^2 \right) \frac{\partial \ln h_i^2}{h_i^2} - \sum_{i=1}^{n} \rho \left( v_i^2 \right) \frac{\partial \ln h_i^3}{h_i^3} + 3 \rho \left( v_i^2 \right) \frac{\partial \Phi}{h_i^3} = 0; \] (51)
and those with \( j = k = \ell \) to

\[
\frac{\partial \langle v_j^2 v_k^2 \rangle}{\partial q^j} + \rho \langle v_j^2 v_k^2 \rangle \frac{\partial \ln (HH^2)}{\partial q^j} - \frac{n}{i=1} \rho \langle v_i^2 v_k^2 \rangle \frac{\partial \ln h_i}{\partial q^j} + \rho \langle v_k^2 \rangle \frac{\partial \Phi}{\partial q^j} = 0. \tag{52}
\]

Differentiating Equation (52) with respect to \( q^k \) results in

\[
\frac{\partial^2 \rho \langle v_j^2 v_k^2 \rangle}{\partial q^k \partial q^j} + \frac{\partial \rho \langle v_j^2 v_k^2 \rangle}{\partial q^k} \frac{\partial \ln (HH^2)}{\partial q^j} + \rho \langle v_j^2 v_k^2 \rangle \frac{\partial^2 \ln (HH^2)}{\partial q^k \partial q^j} - \frac{n}{i=1} \rho \langle v_i^2 v_k^2 \rangle \frac{\partial \ln h_i}{\partial q^j} + \rho \langle v_k^2 \rangle \frac{\partial^2 \Phi}{\partial q^k \partial q^j} + \rho \langle v_k^2 \rangle \frac{\partial \Phi}{\partial q^k} = 0 \quad (j \neq k). \tag{53}
\]

Note that the same equation with indices \( j \mapsto k \) switched also holds. Hence the second derivative term \( \partial^2 \rho \langle v_j^2 v_k^2 \rangle/(\partial q^k \partial q^j) \), which is symmetric under \( j \mapsto k \), can be eliminated by subtracting this from the \( j \mapsto k \) switched equation: that is,

\[
\frac{\partial \rho \langle v_j^2 v_k^2 \rangle}{\partial q^k} \frac{\partial \ln (HH^2/h_h)}{\partial q^j} - \frac{\partial \langle v_j^2 v_k^2 \rangle}{\partial q^k} \frac{\partial \ln (HH^2/h_h)}{\partial q^j} \]

\[
+ \rho \langle v_j^2 v_k^2 \rangle \frac{\partial^2 \ln (h_h^2/h_h^2)}{\partial q^k \partial q^j} - \frac{n}{i=1} \rho \langle v_i^2 v_k^2 \rangle \frac{\partial \ln h_i}{\partial q^j} \frac{\partial \rho \langle v_j^2 \rangle}{\partial q^k} \frac{\partial \Phi}{\partial q^j} + \rho \langle v_k^2 \rangle \frac{\partial^2 \Phi}{\partial q^k \partial q^j} + \rho \langle v_k^2 \rangle \frac{\partial \Phi}{\partial q^k} = 0. \tag{54}
\]

where the remaining spatial derivatives of the moments can be replaced by means of Equations (50)–(52). After tedious but trivial algebra, we then obtain

\[
\left( \langle v_j^2 \rangle - 3 \langle v_j^2 v_k^2 \rangle \right) h_j^2 D_{jk}(h_j^{-2}) - \left( \langle v_k^2 \rangle - 3 \langle v_j^2 v_k^2 \rangle \right) h_k^2 D_{jk}(h_k^{-2})
\]

\[
+ \frac{n}{i=1} \left( \langle v_i^2 v_k^2 \rangle - \langle v_i^2 v_k^2 \rangle \right) h_i^2 D_{jk}(h_i^{-2}) + 2 \left( \langle v_k^2 \rangle - \langle v_i^2 \rangle \right) D_{jk}(\Phi) = 0, \tag{55}
\]

where \( D_{jk} \) is as defined in Equation (5). In the Stäckel coordinate such that \( D_{jk}(h_j^{-2}) = 0 \) for any \( j \neq k \) and all \( i \), Equation (55) then indicates \( \langle v_j^2 \rangle - \langle v_k^2 \rangle D_{jk}(\Phi) = 0 \). So given \( \langle v_j^2 \rangle = \langle v_k^2 \rangle \) for all \( j \neq k \), the potential must be separable in the given Stäckel coordinate.

### 6. PARTIALLY SEPARABLE POTENTIALS

#### IN THREE-DIMENSIONAL SPACE

Theorem 6 provides a sufficient condition for the potential to be separable in the given Stäckel coordinate in terms of the second and fourth velocity moments of the tracers. In three-dimensional space, the separable potential satisfies \( D_{12}(\Phi) = D_{13}(\Phi) = D_{23}(\Phi) = 0 \). However, \( D_{25}(\Phi) = 0 \) does not explicitly involve \( q^j \) and so one might expect that the condition \( D_{12}(\Phi) = D_{13}(\Phi) = 0 \) may be implied by only those moments involving \( v_j \). In fact, we can establish:

**Theorem 7.** Let \( (q^1, q^2, q^3) \) be the Stäckel coordinate with the scale factors \( (h_1, h_2, h_3) \) satisfying \( \langle \partial \rho \partial q^j \rangle (h_j/h_3) = 0 \). If all the second and fourth velocity moments of the steady-state tracers with an odd power to \( v_j \) vanish (i.e., \( \langle v_1 v_3 \rangle = 0 \), \( \langle v_1^2 v_3 \rangle = \langle v_3^2 \rangle = 0 \), \( \langle v_3 v_1^2 \rangle \rangle = \langle v_3 v_3 \rangle \rangle = 0 \), and \( \langle (v_1 - v_3) (v_1 - v_3) \rangle = 0 \), then the potential satisfies the partial differential equations \( D_{12}(\Phi) = D_{13}(\Phi) = 0 \), where \( D_{ij}(\Phi) \) is as defined in Equation (5).

Here we provide only a sketch of the proof. First, consider Equation (49) with \( \{ j, k, \ell \} = \{ 1, 2, 3 \}, \{ 1, 2, 2 \}, \{ 1, 1, 2 \} \) under the given conditions:

\[
\frac{\partial \rho \langle v_j^2 v_k^2 \rangle}{\partial q^j} + \rho \langle v_j^2 v_k^2 \rangle \frac{\partial \ln (h_j h_k^2)}{\partial q^j} - \rho \langle v_k^2 \rangle \frac{\partial \ln h_j}{\partial q^j} = 0; \tag{56}
\]

\[
\frac{\partial \rho \langle v_j^2 v_k^2 \rangle }{\partial q^j} + \rho \langle v_j^2 v_k^2 \rangle \frac{\partial \ln (h_j h_k^2)}{\partial q^j} - \rho \langle v_k^2 \rangle \frac{\partial \ln h_j}{\partial q^j} = 0; \tag{57}
\]

\[
\frac{\partial \rho \langle v_j^2 v_k^2 \rangle }{\partial q^j} + \rho \langle v_j^2 v_k^2 \rangle \frac{\partial \ln (h_j h_k^2)}{\partial q^j} - \rho \langle v_k^2 \rangle \frac{\partial \ln h_j}{\partial q^j} = 0. \tag{58}
\]

Among the partial derivatives of Equation (56) with respect to \( q^3 \), Equation (57) with respect to \( q^1 \), and Equation (58) with respect to \( q^1 \), the two second derivatives of the fourth moments, \( \partial^2 \rho \langle v_j^2 v_k^2 \rangle )/(\partial q^3 \partial q^j) \) and \( \partial^2 \rho \langle v_j^2 v_k^2 \rangle )/(\partial q^1 \partial q^j) \) can be eliminated, which leaves a single equation relating the second
and fourth moments and their first derivatives. It turns out all the first derivatives in the resulting equation can be replaced by means of the Jeans Equations (48) and (49), except \( \partial \)(\( v_i^2 v_j v_k \))/(\( \partial q^4 \)). Following lengthy algebra, we arrive at

\[
A_1 h_1^2 D_{12}(h_1^{-2}) - A_2 h_2^2 D_{12}(h_2^{-2}) + B h_1^2 D_{13}(h_1^{-2}) - 4 \left( v_1^2 v_2 v_3 \right) h_1^2 D_{13}(h_1^{-2}) + \left( v_1^2 v_2 v_3 \right) h_2^2 D_{13}(h_2^{-2})
\]

\[
+ C h_3^2 D_{13}(h_3^{-2}) \]

\[
+ 4 \left( \frac{\partial }{\partial q^3} \right) \left( v_1^2 - v_2^2 \right) D_{12}(\Phi) - \left( v_2 v_3 \right) h_2^2 D_{13}(\Phi) \right] = 0,
\]

\( (59) \)

where \( A_1 \equiv (v_1^4) - 3 (v_1^2 v_2^2), \quad A_2 \equiv (v_2^4) - 3 (v_1^2 v_2^2), \quad B \equiv (v_1^2 v_2^2) - (v_1^2 v_2^2), \quad \text{and} \quad C \equiv (v_2^2 v_3^2) - 2 (v_1^2 v_2 v_3).

It is obvious that the same equation with the indices 2 \( \leftrightarrow 3 \) switched holds too. In the Stäckel coordinate with \((\partial / \partial q^3)(h_2 / h_3) = 0\), these two then simplify to

\[
h_1 \left( \left( v_1^2 \right) - \left( v_2^2 \right) \right) D_{12}(\Phi) = h_2 \left( v_2 v_3 \right) D_{13}(\Phi); \]

\[
h_2 \left( \left( v_1^2 \right) - \left( v_2^2 \right) \right) D_{13}(\Phi) = h_3 \left( v_2 v_3 \right) D_{13}(\Phi).
\]

\( (60) \)

Provided that \((v_1^2 - v_2^2)(v_1 - v_2) \approx (v_2 v_3)^2\), they are linearly independent, implying \(D_{12}(\Phi) = D_{13}(\Phi) = 0\).

The importance of this result becomes clearer with a concrete choice of the Stäckel coordinate. In particular,

**Corollary 8.** If the steady-state tracer velocity moments in the Cartesian coordinate \((x, y, z)\) are constrained such that \((v_x v_x) = (v_y v_y) = 0, \quad (v_x v_y) = (v_x v_z) = 0\) and \((v_x v_z) = (v_y v_z) = 0\), the potential must be in the form of \(\Phi(x, y, z) = f(x, y) + g(z)\), provided that \((v_1^2 - v_2^2)(v_1 - v_2) = (v_2 v_3)^2\).

The scale factors of the Cartesian coordinate are \(h_x = h_y = h_z = 1\) and so \(\partial (h_i / h_z) / \partial z = 0\). By Theorem 7, the condition then implies \(D_{12}(\Phi) = \partial^2 \Phi / (\partial x \partial z) = 0\) and \(D_{13}(\Phi) = \partial^2 \Phi / (\partial y \partial z) = 0\); that is, \(\partial \Phi / \partial z\) is a function of \(z\) alone and so its general solution is \(\Phi(x, y, z) = f(x, y) + g(z)\).

It is clear that a similar result also holds for any three-dimensional Stäckel coordinate that is translation-symmetric along the \(z\)-direction (including the cylindrical-polar, elliptic-cylindrical, and parabolic-cylindrical coordinates). In fact, this is true even for any three-dimensional coordinate resulting from the linear duplication of a two-dimensional coordinate (not necessarily Stäckel). That is to say, if \((q^1, q^2, z)\) is an orthogonal coordinate for the three-dimensional Euclidean space such that the coordinate surfaces for a fixed \(z\) are parallel planes with \((q^1, q^2)\) being a translationally invariant coordinate system on each of the planes, then the scale factors \(h_1\) and \(h_2\) must be independent of \(z\), while \(h_3\) is a function of \(z\) alone (which can always be set to the unity after rescaling). Therefore \(D_{12} = \partial^2 / (\partial q^1 \partial z)\) for \(i \in \{1, 2\}\) in such a coordinate and so it follows that \(D_{12}(h_i^{-2}) = D_{12}(h_i^{-2}) = 0\). Since \(\partial (h_i / h_2) / \partial z = 0\), if all second and fourth moments with the odd power to \(v_i\) vanish, Equation (59) in this coordinate still implies Equation (60), and thus the potential must satisfy \(\partial^2 \Phi / (\partial q^1 \partial z)^2\); \(\partial^2 \Phi / (\partial q^2 \partial z)^2\); that is, the potential being decomposable into a function of the height alone and that of the mid-plane coordinates; namely \(\Phi = f(q^1, q^2) + g(z)\).

Separable potentials in Cartesians are rather unrealistic, and so we turn to the condition for axisymmetric potentials,

**Corollary 9.** The steady-state tracer population with \((v_\phi v_\phi) = (v_\theta v_\theta) = 0, \quad (v_\phi v_\theta) = (v_\theta v_\phi) = 0\) and \((v_\phi v_\phi) = (v_\theta v_\theta) = (v_\phi v_\phi) = 0\) in the cylindrical-polar coordinate \((R, \phi, z)\) implies that the potential must be in the form of \(\Phi(R, \phi, z) = R^{-2}f(\phi) + g(R, z)\), provided that \((v_\phi^2 - v_\theta^2)(v_\phi^2 - v_\theta^2) \approx (v_\phi v_\phi)^2\).

The scale factors of the cylindrical-polar coordinate are \(h_R = h_\phi = 1\) and \(h_\phi = R\), and so \(\partial (h_\phi / h_\phi) / \partial \phi = 0\). According to Theorem 7, the condition implies that

\[
D_{R,\phi}(\Phi) = \frac{\partial^2 \Phi}{\partial R \partial \phi} = \frac{1}{R^2} \frac{\partial \Phi}{\partial \phi} \left( R^2 \frac{\partial \Phi}{\partial \phi} \right) = 0;
\]

\[
D_{\phi,\phi}(\Phi) = \frac{\partial^2 \Phi}{\partial \phi \partial \phi} = \frac{1}{R^2} \frac{\partial \Phi}{\partial \phi} \left( R^2 \frac{\partial \Phi}{\partial \phi} \right) = 0.
\]

\( (61) \)

That is to say, \(R^2 (\partial \Phi / \partial \phi) = F(\phi)\) is a function of \(\phi\) alone. Integrating \(R^2 F(\phi)\) over \(\phi\), the general solution is therefore of the form \(\Phi(R, \phi, z) = R^{-2}f(\phi) + g(R, z)\).

Provided that \(\Phi(R, \phi, z)\) is single-valued, \(f(\phi)\) must be \(2\pi\)-periodic. If \(f(\phi) = \sum_k a_k \cos(k(\phi - \phi_k))\) is the Fourier series expansion, the density profile for \(\Phi \propto f / R^2\) behaves like

\[
\nabla^2 \left( \frac{f}{R^2} \right) = \frac{1}{R^2} \sum_k (4 - k^2) a_k \cos(k(\phi - \phi_k)).
\]

\( (62) \)

Unless \(a_k = 0\) for all \(k \neq 0\), this is unintegrable as \(R \rightarrow 0\), which is considered unphysical. The case \(f(\phi) \propto \cos[2(\phi - \phi_0)]\) is technically allowed but this potential is only due to the choice of the boundary condition and no actual source in otherwise empty space can generate such a potential. Hence we may in fact further infer that the potential resulting from the corollary is axisymmetric (i.e., \(f = 0\)).

Similar to the translation symmetric cases, the result is in fact applicable for any three-dimensional coordinate constructed by rotating a reflection-symmetric two-dimensional coordinate along its symmetry axis (e.g., the cylindrical-polar, spherical-polar, rotational-parabolic, and oblate and prolate spheroidal coordinates). In particular, in the three-dimensional coordinate \((q^1, q^2, \phi)\) with the scale factors \(h_1\) and \(h_2\) that are independent of \(\phi\)—so \(\partial (h_1 / h_2) / \partial \phi = 0\)—and \(h_\phi = R(q^1, q^2)\), we have

\[
D_{\phi,\phi}(f) = \frac{\partial^2 f}{\partial q^1 \partial \phi} \left( R^2 \frac{\partial f}{\partial \phi} \right) + \frac{\partial \ln R^2}{\partial q^1} \frac{\partial f}{\partial \phi} = \frac{1}{R^2} \frac{\partial}{\partial q^1} \left( R^2 \frac{\partial f}{\partial \phi} \right)
\]

\( (63) \)

for \(i \in \{1, 2\}\). Given that all scale factors are assumed to be independent of \(\phi\), if all second and fourth moments with the odd power to \(v_\phi\) vanish, we then infer that \(F = R^2 (\partial \Phi / \partial \phi)\) is a function of \(\phi\) alone (i.e., \(D \Phi / \partial q^1 = 0\) for \(q^1 = 0\)) and subsequently the potential must be axisymmetric.
Here we also note that the vanishing moments are the true moments but not the central moments (such as the co-variance or co-kurtosis etc.), which may be seen by the fact that the underlying potential only determines the orbit of individual tracers and does not distinguish between coherent and random motions for groups of tracers.

Lastly, we consider the case of spherical-polar coordinates for which we have

**Corollary 10.** The steady-state tracer population with \( \langle v_i v_j \rangle = 0 \), \( \langle v'_i v'_j \rangle = 0 \) and \( \langle v'_i v'_j \rangle = 0 \) in the spherical-polar coordinate \((r, \theta, \phi)\) implies that the potential must be in the form of \( \Phi(r, \theta, \phi) = f(r) + r^{-2}g(\theta, \phi) \), provided that \( \left((\langle v^2_i \rangle - \langle v'_i v'_i \rangle)(\langle v^2_j \rangle - \langle v'_j v'_j \rangle) = \langle v'_i v'_j \rangle^2 \right) \).

Since \((h_r, h_\theta, h_\phi) = (1, r, r \sin \theta)\), we have \( h_\phi/h_\theta = \sin \theta \), which is independent of \( r \). Theorem 7 therefore indicates

\[
D_{\text{rel}}(\Phi) = \frac{\partial^2 \Phi}{\partial r \partial \theta} + \frac{2}{r^2} \frac{\partial \Phi}{\partial \theta} + \frac{2}{r^2} \frac{\partial^2 \Phi}{\partial \theta^2} = 0;
\]

\[
D_{\text{rel}}(\Phi) = \frac{\partial^2 \Phi}{\partial r \partial \phi} + \frac{2}{r^2} \frac{\partial \Phi}{\partial \phi} + \frac{2}{r^2} \frac{\partial^2 \Phi}{\partial \phi^2} = 0, \quad (64)
\]

and so follows that \( \partial (r^2 \Phi) / (\partial r) \) is a function of \( r \) alone. Consequently, the general solution for \( \Phi \) is given by \( \Phi(r, \theta, \phi) = f(r) + r^{-2}g(\theta, \phi) \).

The same result is also obtained in any three-dimensional coordinate \((r, q^2, q^3)\) such that the coordinate surfaces of constant \( r \) consist of the set of concentric spheres (of the radius \( r \)) and \( (q^2, q^3) \) corresponds to the coordinate on the unit sphere. The notable example of such coordinates other than the spherical coordinate is the conical coordinates (see Morse & Feshbach 1953). The scale factors for such a coordinate system are found to be \( h_r = 1, h_\theta = nh_\theta(q^2, q^3) \) and \( h_\phi = nh_\phi(q^2, q^3) \), for which \( h_2/h_3 = h_2/h_3 = \text{independent of } r \) and so

\[
D_{\text{rel}}(f) = \frac{\partial^2 f}{\partial r \partial q^2} + \frac{2}{r} \frac{\partial f}{\partial q^2} + \frac{1}{r^2} \frac{\partial^2 f}{\partial q^2} \left( \frac{\partial^2 (r^2 f)}{\partial \theta^2} \right) = 0, \quad (65)
\]

where \( i \in \{2, 3\} \). Then \( r^2 D_{\text{rel}}(h_i^{-2}) = \frac{\partial^2 (h_i^{-2})}{(\partial q^2)(\partial r)} = 0 \) for \( i = 2, 3 \), whereas \( r^2 D_{\text{rel}}(h_r^{-2}) = 0 \). Again from Equation (59), we then find that, if all second and fourth moments with an odd power of \( v_j \) vanish in such a coordinate, \( \partial (r^2 \Phi) / (\partial r) \) is a function of \( r \) alone and so \( \Phi = f(r) + r^{-2}g(q^2, q^3) \).

In general, any single-valued (smooth) function on the unit sphere \( g(\theta, \phi) \) may be expressed as the sum over the spherical harmonics, as in \( g(\theta, \phi) = \sum_{\ell,m} c_{\ell m} Y^m_\ell(\theta, \phi) \), for which

\[
\nabla^2 \left( \frac{g}{r^2} \right) = \frac{1}{r^2} \sum_{\ell, \ell} (1 - \ell)(2 + \ell) \left( \sum_{m = -\ell}^{\ell} c_{\ell m} Y^m_\ell \right). \quad (66)
\]

Similar to the axisymmetric case, the \( r^{-4} \)-density singularity as \( r \to 0 \) is again unphysical, as it is unintegrable. Provided that the tracer population includes the orbits passing the center, physical potentials consistent with \( \partial (r^2 \Phi) / (\partial r) \) being a function of \( r \) alone should thus be spherically symmetric. Although the dipole potential (corresponding to \( \ell = 1 \)) is formally allowed, there is no real source generating such potentials. In principle, masses within a fixed boundary may be arranged in such a way that the potential outside the boundary becomes dipole-like. However, for such cases, the potential within the boundary cannot be dipole-like without it possessing the \( r^{-4} \)-singularity at the center. Thus, in order for it to avoid the unintegrable singularity, the potential within the boundary should no longer be separable in the spherical coordinate. This may still be acceptable if all orbits in the tracer population are restricted to the outside of the boundary (see, e.g., Evans et al. 2015, Appendix A).

7. CONCLUSIONS

This paper has shown that the alignments of the velocity moments of a stellar system can provide powerful constraints on the potential. This idea may be traced back to the classical work of Eddington (1915) and Chandrasekhar (1939). Their papers, however, muddled the issue by assuming unnecessarily restrictive forms for the distribution. By contrast, Evans et al. (2015) have shown that, if the velocity distribution in a steady state possesses planes of reflection symmetry such that \( \mathcal{F}(v_x, v_y, v_z; r) = \mathcal{F}(v_x, v_y, v_z; r) \) and similarly for \( v_x \) and \( v_z \), then the potential must be of Stäckel type. We can recast this result in terms of the velocity moments. Suppose all the mixed second moments vanish (i.e., \( \langle v_i v_j \rangle = 0 \) for \( i \neq j \)) so that the “stress tensor” is aligned in some Stäckel coordinate system. Then, if the only non-vanishing fourth moments are those in the form of \( \langle v^4_i \rangle \) or \( \langle v^2_i v^2_j \rangle \), then the potential must be separable in the same coordinate. Although our conclusions are superficially very similar to those of Eddington (1915), our work is much more general in its scope, as nothing has been assumed about the distribution other than some basic symmetries.

Our work has been motivated by the stellar halo of the Galaxy, for which the second moments do appear to be close to spherical alignment. It is worth stating the form of our result explicitly in the spherical-polar coordinate system. If the second velocity moments are spherically aligned and all the fourth velocity moments with the radial component \( v_r \) being either linear or cubic vanish, the potential must be separable in the spherical coordinate. An alternative way to state this is, if the second velocity moments are spherically aligned and the velocity distribution is symmetric with respect to \( v_r \), then the potential must be \( \Phi = f(r) + r^{-2}g(\theta, \phi) \). Although current observational studies (e.g., Smith et al. 2009; Bond et al. 2010; King et al. 2015) seem to indicate that the condition required to infer the Galactic potential to be separable in the spherical coordinate is present, a word of caution is still warranted before any definite conclusion on the shape of the Galactic halo is reached. Most of these data only cover a relatively small volume of the Galaxy and there is a large extrapolation to go from the local velocity ellipsoid being radially aligned (within observational uncertainties) to the global alignments of the velocity moments. However, with upcoming availability of large data sets including proper motions for many stars in substantial local volumes, it is within our grasp in the near future to test symmetry properties of the “global” velocity distribution explicitly for the stellar halo (c.f. Evans et al. 2015).

Eddington’s paper is now a hundred years old. It is a testimony to his greatness that it remains a fruitful avenue for research still today. Perhaps the most interesting topic for
future exploration is to understand how insights from the Stäckel models with their exact alignment can be applied to numerical models based on orbital tori or made-to-measure (Binney & McMillan 2011; Evans et al. 2015). In the framework of the Hamiltonian perturbation theory, the invariant orbital tori of regular orbits found in non-Stäckel potentials may be understood as the result of perturbation to the exactly integrable Stäckel models. In fact, some recent works (e.g., Binney 2012; Sanders & Binney 2014; Bienaymé et al. 2015) have used the Stäckel potentials to find an approximate third integral of motion (which sometimes doubles as an action integral) to define regular orbits in more realistic (non-Stäckel) potentials. It is then an interesting question how the conclusion of the present paper relates to the behavior of the distribution consisting of these regular orbits in non-Stäckel potentials. It appears that each regular orbit in these potentials observes its own Stäckel coordinate system, which seems to suggest that these systems are able to evade the global constraint required for the Stäckel models, but any definite statement should follow more careful studies. In this regard, we hypothesize that the Stäckel potentials (including those due to Noether–Killing symmetries) are the only potentials in which all (bound) initial conditions result in a regular orbit, whereas, in other potentials, there must exist some initial conditions that lead to an irregular (which can be chaotic or ergodic) orbit.

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