1. Introduction and main results

1.1. Quivers, Coxeter functors and preprojective algebras. Let $Q$ be a finite
connected acyclic quiver, and let $H = KQ$ be the path algebra of $Q$ with coefficients
in a field $K$. The following five results, all proved in the 1970’s, form an
essential part of the foundations of modern representation theory of
finite-dimensional algebras.

(1) Gabriel’s Theorem [Ga1]: The quiver $Q$ is representation-finite if and only if $Q$
is a Dynkin quiver of type $A_n$, $D_n$, $E_6$, $E_7$, $E_8$. In this case, there is a
bijection between the isomorphism classes of indecomposable representations
of $Q$ and the set of positive roots of the corresponding simple complex Lie algebra.

(2) Bernstein, Gelfand and Ponomarev’s [BGP] discovery of Coxeter functors

$$C^\pm(-) = F_{i_n}^\pm \circ \cdots \circ F_{i_1}^\pm : \text{rep}(H) \to \text{rep}(H),$$

which are defined as compositions of reflection functors. They lead to a more
conceptual proof of Gabriel’s Theorem. Applied to the indecomposable projective
(resp. injective) representations they yield a family of indecomposable representations, called \textit{preprojective} (resp. \textit{preinjective}) representations.

(3) Gabriel’s Theorem \cite{Ga3} saying that there are functorial isomorphisms \(TC^\pm(-) \cong \tau^\pm(-)\), where \(T\) is a twist functor and \(\tau(-)\) is the Auslander-Reiten translation (see also the comment below on an earlier result by Brenner and Butler).

(4) Auslander, Platzeck and Reiten’s Theorem \cite{APR} saying that there are functorial isomorphisms \(F^\pm_k(-) \cong \text{Hom}_H(T,-)\), where \(F^\pm_k\) is a BGP-reflection functor and \(T\) is the associated APR-tilting module. This result is considered as the starting point of tilting theory.

(5) Gelfand and Ponomarev’s \cite{GP} discovery of the preprojective algebra \(\Pi(Q)\) of a quiver, and their result that \(\Pi(Q)\), seen as a module over \(H\), is isomorphic to the direct sum of all preprojective \(H\)-modules, hence the name \textit{preprojective algebra}. It was proved independently by Crawley-Boevey \cite{CB2} and Ringel \cite{Ri3} that \(\Pi(Q)\) is isomorphic to the tensor algebra \(T_H(\text{Ext}_H^1(DH,H))\), where \(D\) denotes the duality with respect to the base field \(K\).

The above results hold for arbitrary ground fields \(K\). At the price of quite strong assumptions on \(K\) they were generalized from quivers to the more general setup of modulated graphs. For the finite type situation, this extended the theory from the simply laced root systems of types \(A_n, D_n, E_6, E_7\) and \(E_8\) to the non-simply laced root systems \(B_n, C_n, F_4\) and \(G_2\). The definition of a modulated graph (also called \textit{species}) and of its representations is due to Gabriel \cite{Ga2}. The theory itself has been developed by Dlab and Ringel, who generalized (1), (2) and (5) to modulated graphs \cite{D, DR1, DR2, DR3}. Brenner and Butler \cite{BB} proved an earlier result closely related to (3), which is also valid for modulated graphs. (They don’t treat \(C^\pm\) as endofunctors, and the twist automorphism \(T\) does not appear.)

1.2. \textbf{Hereditary, selfinjective and Iwanaga-Gorenstein algebras}. In this section, by an \textit{algebra} we mean a finite-dimensional \(K\)-algebra.

An algebra \(A\) is \textit{hereditary} if all \(A\)-modules have projective and injective dimension at most 1. The representation theory of quivers and species corresponds to the representation theory of finite-dimensional hereditary algebras.

An algebra \(A\) is \textit{selfinjective} if the classes of projective and injective \(A\)-modules coincide. This implies that all modules (except the projective-injectives) have infinite projective and injective dimension. Despite being opposite homological extremes, hereditary and selfinjective algebras are often intimately linked. For example the path algebra \(KQ\) is always hereditary, and in contrast, if \(Q\) is a Dynkin quiver, then the closely related preprojective algebra \(\Pi(Q)\) is selfinjective. Also, the classification of representation-finite selfinjective algebras shows striking similarities to the classification of representation-finite hereditary algebras.

An algebra \(A\) is \textit{\(m\)-Iwanaga-Gorenstein} if

\[
\text{inj.dim}(A) \leq m \quad \text{and} \quad \text{proj.dim}(DA) \leq m.
\]

These algebras were first studied in \cite{I}. In this case, \(\text{inj.dim}(A) = \text{proj.dim}(DA)\), and for any \(A\)-module \(M\) the following are equivalent:

(i) \(\text{proj.dim}(M) \leq m\);
(ii) \(\text{inj.dim}(M) \leq m\);
(iii) \(\text{proj.dim}(M) < \infty\);
(iv) $\text{inj. dim}(M) < \infty$.

Note that with this definition a given algebra can be $m$-Iwanaga-Gorenstein for different values of $m$. An algebra is selfinjective if and only if it is 0-Iwanaga-Gorenstein. All hereditary algebras and also all selfinjective algebras are 1-Iwanaga-Gorenstein. Now let $A$ be a 1-Iwanaga-Gorenstein algebra. Then there are two subcategories of the category $\text{rep}(A)$ of finite-dimensional $A$-modules which are of interest:

(a) The subcategory
$$\mathcal{H}(A) := \{M \in \text{rep}(A) \mid \text{proj. dim}(M) \leq 1 \text{ and } \text{inj. dim}(M) \leq 1\}.$$  

(b) The subcategory
$$\mathcal{GP}(A) := \{M \in \text{rep}(A) \mid \text{Ext}^1_A(M,A) = 0\}$$

of Gorenstein-projective modules.

Let $\mathcal{P}(A)$ be the subcategory of projective $A$-modules. We have
$$\mathcal{P}(A) = \mathcal{H}(A) \cap \mathcal{GP}(A).$$

For each $M \in \text{rep}(A)$ there are short exact sequences
$$0 \to H_M \to G_M \to M \to 0$$
and
$$0 \to M \to H^M \to G^M \to 0$$
with $H_M, H^M \in \mathcal{H}(A)$ and $G_M, G^M \in \mathcal{GP}(A)$, see [AB, 8.1].

The category $\mathcal{H}(A)$ carries the homological features of module categories of hereditary algebras, whereas $\mathcal{GP}(A)$ is a Frobenius category, thus displaying the homological features of module categories of selfinjective algebras. We have $\mathcal{GP}(A) = \mathcal{P}(A)$ and $\mathcal{H}(A) = \text{rep}(A)$ if and only if $A$ is hereditary, and in the other extreme we have $\mathcal{GP}(A) = \text{rep}(A)$ and $\mathcal{H}(A) = \mathcal{P}(A)$ if and only if $A$ is selfinjective.

The stable category of $\mathcal{GP}(A)$ is a triangulated category, which is triangle equivalent to the singularity category
$$D_{\text{sg}}(A) := D^b(A)/K^b(A)$$
defined and studied by Buchweitz [Bu], see also [O]. (Here $D^b(A)$ denotes the bounded derived category, and $K^b(A)$ is the bounded homotopy category of finite-dimensional $A$-modules.) It follows that $D_{\text{sg}}(A) = 0$ if and only if $A$ is hereditary.

Thus the class of 1-Iwanaga-Gorenstein algebras can be seen as an intermediary class sitting between the hereditary and the selfinjective algebras, and the singularity category $D_{\text{sg}}(A)$ can be considered as a measure of how far $A$ is from being hereditary.

1.3. 1-Iwanaga-Gorenstein algebras attached to Cartan matrices. To each symmetrizable generalized Cartan matrix $C$ and an orientation $\Omega$ of $C$ we attach an infinite series of 1-Iwanaga-Gorenstein algebras $H = H(C, D, \Omega)$ indexed by the different symmetrizers $D$ of $C$. These algebras are defined by quivers with relations over an arbitrary field $K$.

If $C$ is symmetric and connected, then $(C, \Omega)$ corresponds to a connected acyclic quiver $Q$, and the series of algebras $H$ consists of the algebras of the form
$$A_m \otimes_K KQ, \quad (m \geq 1),$$
where $A_m := K[X]/(X^m)$ is a truncated polynomial ring. Representations of such algebras are nothing else than representations of $Q$ over the ground rings $A_m$. 
In the general case of a symmetrizable matrix $C$, the algebras $H$ can be identified with tensor algebras of modulations of the oriented valued graph $\Gamma$ corresponding to $(C,\Omega)$. However, in contrast with the classical notion of a modulation, the rings attached to vertices of $\Gamma$ are truncated polynomial rings instead of division rings.

We also introduce a series of algebras $\Pi = \Pi(C,D)$, again defined by quivers with relations, which can be regarded as preprojective algebras of quivers (or more generally of modulated graphs) over truncated polynomial rings.

We show that analogues of all five results mentioned in Section 1.1 also hold for our algebras $H$ and $\Pi$. However certain definitions must be adapted. For example, we say that $H$ has finite $\tau$-representation type if its Auslander-Reiten quiver has only finitely many $\tau$-orbits consisting entirely of modules of finite homological dimension. The analogue of (1) states that $H$ has finite $\tau$-representation type if and only if $C$ is of Dynkin type. In this case, there is a bijection between the isomorphism classes of indecomposable modules sitting on these $\tau$-orbits and the positive roots of the simple Lie algebra associated with $C$. So for each Cartan matrix $C$ of Dynkin type we get an infinite family of new representation-theoretic incarnations of the root system of $C$. Let us stress that even in the non-simply laced case these incarnations are defined without any assumption on the ground field $K$.

To prove this theorem, we define analogues of the reflection functors and Coxeter functors of (2), and we give an analogue of Gabriel’s Theorem (3) for the subcategory of $H$-modules of finite homological dimension. This yields alternative descriptions of the preprojective algebra $\Pi$ similar to (5). We also obtain an analogue of (4) describing the reflection functors in terms of certain tilting $H$-modules.

In the rest of this section we give precise definitions of the algebras $H$ and $\Pi$, and we state our main results in more detail. We then point out some previous appearances of the algebras $H$ and $\Pi$ in the literature.

1.4. Definition of $H$ and $\Pi$. A matrix $C = (c_{ij}) \in M_n(\mathbb{Z})$ is a symmetrizable generalized Cartan matrix provided the following hold:

(C1) $c_{ii} = 2$ for all $i$;
(C2) $c_{ij} \leq 0$ for all $i \neq j$;
(C3) $c_{ij} \neq 0$ if and only if $c_{ji} \neq 0$.
(C4) There is a diagonal integer matrix $D = \text{diag}(c_1, \ldots, c_n)$ with $c_i \geq 1$ for all $i$ such that $DC$ is symmetric.

The matrix $D$ appearing in (C4) is called a symmetrizer of $C$. The symmetrizer $D$ is minimal if $c_1 + \cdots + c_n$ is minimal. From now on, by a Cartan matrix we always mean a symmetrizable generalized Cartan matrix. In this case, define for all $c_{ij} < 0$

$$g_{ij} := |\gcd(c_{ij}, c_{ji})|, \quad f_{ij} := |c_{ij}|/g_{ij}, \quad k_{ij} := \gcd(c_i, c_j).$$

Note that we have

$$g_{ij} = g_{ji}, \quad k_{ij} = k_{ji}, \quad c_i = k_{ij}f_{ji}.$$

Let $C = (c_{ij}) \in M_n(\mathbb{Z})$ be a Cartan matrix. An orientation of $C$ is a subset $\Omega \subset \{1,2,\ldots,n\} \times \{1,2,\ldots,n\}$ such that the following hold:

(i) $\{(i,j), (j,i)\} \cap \Omega \neq \emptyset$ if and only if $c_{ij} < 0$;
(ii) For each sequence $((i_1, i_2), (i_2, i_3), \ldots, (i_t, i_{t+1}))$ with $t \geq 1$ and $(i_s, i_{s+1}) \in \Omega$ for all $1 \leq s \leq t$ we have $i_1 \neq i_{t+1}$.
For an orientation $\Omega$ of $C$ let $Q := Q(C, \Omega) := (Q_0, Q_1, s, t)$ be the quiver with the set of vertices $Q_0 := \{1, \ldots, n\}$ and with the set of arrows

$$Q_1 := \{\alpha_{ij} \in j \to i \mid (i, j) \in \Omega, 1 \leq g \leq g_{ij}\} \cup \{\varepsilon_i : i \to i \mid 1 \leq i \leq n\}.$$  

(Thus we have $s(\alpha_{ij}) = j$ and $t(\alpha_{ij}) = i$ and $s(\varepsilon_i) = t(\varepsilon_i) = i$, where $s(a)$ and $t(a)$ denote the starting and terminal vertex of an arrow $a$, respectively.) If $g_{ij} = 1$, we also write $\alpha_{ij}$ instead of $\alpha_{ij}^{(1)}$. We call $Q$ a quiver of type $C$. Let $Q^\circ := Q^\circ(C, \Omega)$ be the quiver obtained from $Q$ by deleting all loops $\varepsilon_i$. Clearly, $Q^\circ$ is an acyclic quiver.

Throughout let $K$ be a field. For a quiver $Q = Q(C, \Omega)$ and a symmetrizer $D = \text{diag}(c_1, \ldots, c_n)$ of $C$, let

$$H := H(C, D, \Omega) := KQ/I$$

where $KQ$ is the path algebra of $Q$, and $I$ is the ideal of $KQ$ defined by the following relations:

(H1) For each $i$ we have the nilpotency relation

$$\varepsilon_i^{c_i} = 0.$$  

(H2) For each $(i, j) \in \Omega$ and each $1 \leq g \leq g_{ij}$ we have the commutativity relation

$$\varepsilon_i^{(g)}(\alpha_{ij}) = \alpha_{ij}^{(g)}(\varepsilon_i).$$

The following remarks are straightforward.

(i) $H$ is a finite-dimensional $K$-algebra.

(ii) $H$ depends on the chosen symmetrizer $D$. But note that the relation (H2) does not depend on $D$.

(iii) The relation (H2) becomes redundant for all $(i, j) \in \Omega$ with $k_{ij} = 1$.

(iv) If $C$ is symmetric and if $D$ is minimal, then $H$ is isomorphic to the path algebra $KQ^\circ$.

The opposite orientation of an orientation $\Omega$ is defined as $\Omega^* := \{(j, i) \mid (i, j) \in \Omega\}$. Let $\overline{\Omega} := \Omega \cup \Omega^*$. For later use, let us define

$$\Omega(i, -) := \{j \in Q_0 \mid (i, j) \in \Omega\}, \quad \Omega(-, j) := \{i \in Q_0 \mid (i, j) \in \Omega\},$$

$$\overline{\Omega}(i, -) := \{j \in Q_0 \mid (i, j) \in \overline{\Omega}\}, \quad \overline{\Omega}(-, j) := \{i \in Q_0 \mid (i, j) \in \overline{\Omega}\}.$$  

Observe that $\overline{\Omega}(i, -) = \overline{\Omega}(-, i)$.

For $(i, j) \in \overline{\Omega}$ define

$$\text{sgn}(i, j) := \begin{cases} 1 & \text{if } (i, j) \in \Omega, \\ -1 & \text{if } (i, j) \in \Omega^*. \end{cases}$$

For $Q = Q(C, \Omega)$ and a symmetrizer $D = \text{diag}(c_1, \ldots, c_n)$ of $C$, we define an algebra

$$\Pi := \Pi(C, D, \Omega) := KQ/I$$

as follows. The double quiver $\overline{Q} = \overline{Q}(C)$ is obtained from $Q$ by adding a new arrow $\alpha_{ji}^{(g)} : i \to j$ for each arrow $\alpha_{ij}^{(g)} : j \to i$ of $Q^\circ$. (Note that we did not add any new loops to the quiver $Q$.) The ideal $\overline{I}$ of the path algebra $K\overline{Q}$ is defined by the following relations:

(P1) For each $i$ we have the nilpotency relation

$$\varepsilon_i^{c_i} = 0.$$
(P2) For each \((i, j) \in \Omega\) and each \(1 \leq g \leq g_{ij}\) we have the commutativity relation
\[ \varepsilon_i^{f_{ij}}(g) = \alpha_i^f \varepsilon_j^{f_{ij}}. \]

(P3) For each \(i\) we have the mesh relation
\[ \sum_{j \in \Omega(-i)} g_{ji} \varepsilon_i^{f_{ij}}(g) = 1 - \sum_{f=0}^{g_{ij}-1} \text{sgn}(i, j) \alpha_i^f \alpha_j^{g_{ij}} \varepsilon_i^{f_{ij}} - 1 - \varepsilon_i^{f_{ij}} = 0. \]

We call \(\Pi\) a preprojective algebra of type \(C\). Here are again some straightforward remarks:

(i) Up to isomorphism, the algebra \(\Pi := \Pi(C, D) := \Pi(C, D, \Omega)\) does not depend on the orientation \(\Omega\) of \(C\).

(ii) In general, \(\Pi\) can be infinite-dimensional.

(iii) \(\Pi\) depends on the chosen symmetrizer \(D\). But note that the relations (P2) and (P3) do not depend on \(D\).

(iv) If \(C\) is symmetric and if \(D\) is minimal, then \(\Pi\) is isomorphic to the classical preprojective algebra \(\Pi(Q^o)\) associated with \(Q^o\).

For an example illustrating the above definitions, see below Section 12.1.

1.5. Main results. Let \(e_1, \ldots, e_n\) be the idempotents in \(H\) (resp. \(\Pi\)) corresponding to the vertices of \(Q\) (resp. \(Q^o\)). Define \(H_i := e_i H e_i\). Clearly, \(H_i\) is isomorphic to the truncated polynomial ring \(K[\varepsilon_i]/(\varepsilon_i^n)\). For each representation \(M\) of \(H\) or \(\Pi\) we get an \(H_i\)-module structure on \(M_i := e_i M\). Let \(\text{rep}_{l.f.}(H)\) (resp. \(\text{rep}_{l.f.}(\Pi)\)) be the subcategory of all \(M \in \text{rep}(H)\) (resp. \(M \in \text{rep}(\Pi)\)) such that \(M_i\) is a free \(H_i\)-module for every \(i\). In this case, \(M\) is called locally free.

**Theorem 1.1.** The algebra \(H\) is a 1-Iwanaga-Gorenstein algebra. For \(M \in \text{rep}(H)\) the following are equivalent:

(i) \(\text{proj.dim}(M) \leq 1\);
(ii) \(\text{inj.dim}(M) \leq 1\);
(iii) \(\text{proj.dim}(M) < \infty\);
(iv) \(\text{inj.dim}(M) < \infty\);
(v) \(M\) is locally free.

Let \(M\) be a locally free module. For each \(i \in Q_0\) let \(r_i\) be the rank of the free \(H_i\)-module \(M_i\). Thus \(\dim_K(M_i) = r_i c_i\). We put
\[ \text{rank}(M) := (r_1, \ldots, r_n). \]

Let \(\tau\) be the Auslander-Reiten translation for the algebra \(H\), and let \(\tau^-\) be the inverse Auslander-Reiten translation. An indecomposable \(H\)-module \(M\) is preprojective (resp. preinjective) if there exists some \(k \geq 0\) such that \(M \cong \tau^{-k}(P)\) (resp. \(M \cong \tau^k(I)\)) for some indecomposable projective \(H\)-module \(P\) (resp. indecomposable injective \(H\)-module \(I\)). Let us warn the reader that the usual definition of a preprojective or preinjective module \(M\) requires some additional conditions on the Auslander-Reiten component containing \(M\).

In general, the Auslander-Reiten translates \(\tau^k(M)\) of an indecomposable locally free \(H\)-module \(M\) are not locally free. An indecomposable \(H\)-module \(M\) is called \(\tau\)-locally free, if \(\tau^k(M)\) is locally free for all \(k \in \mathbb{Z}\).

A module \(M\) over an algebra \(A\) is called rigid if \(\text{Ext}_A^1(M, M) = 0\).
The following result is an analogue for the algebras $H = H(C,D,\Omega)$ of Gabriel’s Theorem (1) for quivers and of its generalization by Dlab and Ringel to modulated graphs.

**Theorem 1.2.** There are only finitely many isomorphism classes of $\tau$-locally free $H$-modules if and only if $C$ is of Dynkin type. In this case, the following hold:

1. \(\text{The map } M \mapsto \text{rank}(M) \) yields a bijection between the set of isomorphism classes of $\tau$-locally free $H$-modules and the set $\Delta^+(C)$ of positive roots of the semisimple complex Lie algebra associated with $C$.

2. For an indecomposable $H$-module $M$ the following are equivalent:
   - $M$ is preprojective;
   - $M$ is preinjective;
   - $M$ is $\tau$-locally free;
   - $M$ is locally free and rigid.

Note that the algebras $H$ are usually representation infinite, even if $C$ is a Cartan matrix of Dynkin type with a minimal symmetrizer. Already for $C$ of type $B_3$ with minimal symmetrizer, there exist indecomposable locally free $H$-modules $M$ with $\text{rank}(M) \notin \Delta^+(C)$, see Section [2.7]. Furthermore, for $C$ of type $B_5$ with minimal symmetrizer there exists a $K^*$-family of pairwise non-isomorphic non-isomorphic indecomposable locally free $H$-modules, all having the same dimension vector.

Inspired by the classical theory for path algebras and modulated graphs we define Coxeter functors

\[ C^+, C^- : \text{rep}(H) \to \text{rep}(H) \]

as products of reflection functors, see Section [6]. Let

\[ T : \text{rep}(H) \to \text{rep}(H) \]

be the twist automorphism induced from the algebra automorphism $H \to H$ defined by $\varepsilon_i \mapsto \varepsilon_i$ and $\alpha_{ij}^{(g)} \mapsto -\alpha_{ij}^{(g)}$. In other words, $T$ sends a representation $(M_i, M(\alpha_{ij}^{(g)}), M(\varepsilon_i))$ of $H$ to $(M_i, -M(\alpha_{ij}^{(g)}), M(\varepsilon_i))$. The following theorem, analogous to Gabriel’s Theorem (3), relates Coxeter functors to the Auslander-Reiten translation, and provides the main step in proving Theorem [1.2].

**Theorem 1.3.** For $M \in \text{rep}(H)$ there are functorial isomorphisms

\[ D \text{Ext}^1_H(M,H) \cong TC^+(M) \quad \text{and} \quad \text{Ext}^1_H(DH, M) \cong TC^-(M). \]

Furthermore, if $M \in \text{rep}_{lf}(H)$, then there are functorial isomorphisms

\[ \tau(M) \cong TC^+(M) \quad \text{and} \quad \tau^-(M) \cong TC^-(M). \]

Vice versa, if $\tau(M) \cong TC^+(M)$ or $\tau^-(M) \cong TC^-(M)$ for some $M \in \text{rep}(H)$, then $M \in \text{rep}_{lf}(H)$.

Recall that $H$ is a 1-Iwanaga-Gorenstein algebra, and that $\mathcal{GP}(H)$ denotes the subcategory of Gorenstein-projective $H$-modules.

**Corollary 1.4.** For $M \in \text{rep}(M)$ the following are equivalent:

1. $C^+(M) = 0$;
2. $M \in \mathcal{GP}(H)$.

For an algebra $A$ and an $A$-$A$-bimodule $M$, let $T_A(M)$ denote the corresponding tensor algebra. Theorem [1.3] implies the following description of the preprojective algebra $\Pi = \Pi(C, D)$ associated with $H$. 

\[ \text{Corollary 1.4.} \]
Theorem 1.5. $\Pi \cong T_H(\text{Ext}_H^1(DH,H))$.

The algebra $\Pi$ contains $H$ as a subalgebra in an obvious way. Let $\mu H$ be the algebra $\Pi$ considered as a left module over $H$. The following result says that $\mu H$ is isomorphic to the direct sum of all preprojective $H$-modules. This justifies that $\Pi$ is called a preprojective algebra.

Theorem 1.6. We have $\mu H \Pi \cong \bigoplus_{m \geq 0} \tau^{-m}(H H)$. In particular, $\Pi$ is finite-dimensional if and only if $C$ is of Dynkin type.

Finally, we obtain the following analogue for locally free $\Pi$-modules of the classical important Ext-symmetry of preprojective algebras.

Theorem 1.7. For $M, N \in \text{rep}_{l.f.}(\Pi)$ we have a functorial isomorphism

$$D \text{Ext}_H^1(M,N) \cong \text{Ext}_H^1(N,M).$$

1.6. Previous appearances of $H(C,D,\Omega)$.

1.6.1. Let $Q$ be a quiver without oriented cycles. Ringel and Zhang [RiZ] study representations of $Q$ over the algebra $A := K[X]/(X^2)$ of dual numbers. This can be interpreted as the category of $\Lambda$-modules with $\Lambda := A \otimes_K KQ$. It is shown in [RiZ] that $\Lambda$ is a 1-Iwanaga-Gorenstein algebra, and that the stable category of $\mathcal{GP}(\Lambda)$ is triangle equivalent to the orbit category $D^b(KQ)/[1]$ of the bounded derived category $D^b(KQ)$ of the path algebra $KQ$ modulo the shift functor $[1]$. In our setup, if we take symmetric Cartan matrices $C$ with symmetrizer $D = \text{diag}(2,\ldots,2)$, then the class of algebras $H(C,D,\Omega)$ coincides with the class of algebras studied by Ringel and Zhang. Fan [F] studies the Hall algebra of representations of $Q$ over $K[X]/(X^m)$ with $m \geq 1$. Again this is a special case of our setup with $C$ symmetric and $D = \text{diag}(m,\ldots,m)$. For $Q$ a quiver of type $A_2$, $A = K[X]/(X^m)$ and $\Lambda := A \otimes_K KQ$ the category $\mathcal{GP}(\Lambda)$ is studied in work of Ringel and Schmidmeier [RS]. Note also that in this case we have $\Lambda \cong T_2(A)$, where $T_2(A)$ is the algebra of upper triangular $2 \times 2$-matrices with entries in $A$. More generally, the algebras $T_2(A)$ with $A$ a Nakayama algebra have been studied by Skowroński [S], and the algebras $T_n(A)$ have been studied by Leszczyński and Skowroński [LS].

1.6.2. A general framework for studying cluster structures arising from 2-Calabi-Yau categories with loops has been provided by [BMV]. As an example they study the cluster category $\mathcal{C} := D^b(T_n)/\tau^{-1}[1]$ of the mesh category of a tube $T_n$ of rank $n \geq 2$. The endomorphism algebras of the maximal rigid objects in $\mathcal{C}$ have been studied by Vatne and Yang [VY]. It turns out that there exists a maximal rigid object $T$ in $\mathcal{C}$ such that $\text{End}_\mathcal{C}(T)$ is isomorphic to one of our algebras $H(C,D,\Omega)$, where $C$ is of Dynkin type $C_{n-1}$ and $D$ is minimal. (We identify the types $C_1 = A_1$ and $C_2 = B_2$.)

1.6.3. Let $Q$ be a Dynkin quiver of type $E_8$, and let $F := S^4 \Sigma^{-4}$, where $S$ is the Serre functor and $\Sigma$ is the translation functor for the bounded derived category $D^b(Q)$ of the path algebra $KQ$. Ladkani [La] Section 2.6] studies the orbit category $\mathcal{C} := D^b(Q)/F$. He shows that $\mathcal{C}$ is a triangulated 2-Calabi-Yau category containing exactly 6 cluster-tilting objects. Ladkani shows that $\mathcal{C}$ categorifies a cluster algebra of Dynkin type $G_2$. He also shows that the cluster tilting-objects in $\mathcal{C}$ have an endomorphism algebra isomorphic to $A = KQ/I$, where $Q$ is a quiver of the form

```
\begin{tikzpicture}
    \node (1) at (0,0) {$1$};
    \node (2) at (1,0) {$2$};
    \draw (1) -- (2);
    \node (e) at (0.5,1) {$\varepsilon$};
\end{tikzpicture}
```
and $I$ is generated by $e^3$. Note that the algebras $A$ are isomorphic to the algebras $H(C, D, \Omega)$ with $C$ of type $G_2$ and $D$ minimal.

1.7. Previous appearances of $\Pi(C, D)$.

1.7.1. Let $C$ be a Cartan matrix of Dynkin type. In [HL], an algebra $A = A(C)$ was introduced, by means of an infinite quiver with potential. Certain finite-dimensional $A$-modules (the generic kernels $K^{(i)}_{k,m}$, see [HL] Definition 4.5) were shown to encode the $q$-characters of the Kirillov-Reshetikhin modules of the quantum loop algebra $U_q(Lg)$, where $g$ is the complex simple Lie algebra with Cartan matrix $C$. The connection with the algebras considered in this paper is the following: Let $\tilde{\Pi}(C)$ denote the algebra $KQ/\tilde{I}$, where $\tilde{I}$ is the two-sided ideal defined by the relations (P2) and (P3) only. (Thus, the nilpotency relation (P1) is omitted.) Then $A(C)$ is a truncation of a $\mathbb{Z}$-covering of $\tilde{\Pi}(C^*)$, where $C^*$ is the transposed Cartan matrix, in other words, the Cartan matrix of the Langlands dual $g^L$ of $g$. In particular, for $m \ll 0$ the generic kernels $K^{(i)}_{c_i,m}$ of $A(C)$ coincide with the indecomposable projective $\Pi(C^*, D)$-modules regarded as $\mathbb{Z}$-graded $\tilde{\Pi}(C^*)$-modules (compare for instance [HL, Section 6.5] to Figure 10 below). This generalizes [HL, Example 4.7].

1.7.2. The algebras $\tilde{\Pi}(C)$ mentioned in Section 1.7.1 were defined and studied independently by Cecotti [C, Section 3.4] and Cecotti and del Zotto [CD, Section 5.1]. In [C] they are called generalized preprojective algebras.

1.7.3. For $(C, \Omega)$ let

$$W(C, \Omega) := \sum_{(j,i) \in \Omega} \sum_{s=1}^{g_{ji}} \text{sgn}(i,j) x_i^{f_{ji}} \alpha_{ij}^{(g)} \alpha_{ji}^{(g)} .$$

Then the cyclic derivatives of the potential $W(C, \Omega)$ yield the defining relations (P2) and (P3) of $\Pi(C, D)$, compare [C, CD, HL], where these relations are also encoded via potentials.

1.8. Future directions. This article intends to provide the foundation for generalizing many of the connections between path algebras, preprojective algebras, Lie algebras and cluster algebras from the symmetric to the symmetrizable case.

In particular, since the algebras $H$ and $\Pi$ are defined via quivers with relations, one can study their module varieties over an arbitrary field $K$. Taking $K = \mathbb{C}$, one can hope to generalize Lusztig’s nilpotent varieties and Nakajima quiver varieties from the symmetric to the symmetrizable case.

As a first step in this direction, in a forthcoming publication we will construct the positive part of the enveloping algebra of a complex Lie algebra of non-simply laced Dynkin type as an algebra of constructible functions on varieties of locally free $H$-modules.

1.9. The paper is organized as follows. In Section 2 we recall some definitions and basic facts on Cartan matrices, quadratic forms and Weyl groups. A description of the projective and injective $H$-modules and some fundamental results on locally free $H$-modules are obtained in Section 3. In particular, Section 3 contains the proof of Theorem 1.1 (combine Proposition 3.5 and Corollary 3.7). In Section 4 we show that the quadratic form $q_C$ associated with a Cartan matrix $C$ coincides with the restriction of the homological Euler form of $H$ to the subcategory of locally free $H$-modules. The representation theory of the
algebras $H$ and $\Pi$ can be reformulated in terms of a generalization of the representation theory of modulated graphs. This is explained in Section 5. Section 6 contains some fundamental properties of generalizations of BBK-reflection functors to our algebras $\Pi$. (The letters BBK stand for Baumann and Kamnitzer [BK, BKT] and Bolten [Bo]. Independently from each other they introduced reflection functors for the classical preprojective algebras associated with quivers.) An interpretation of $H$ and $\Pi$ as tensor algebras is discussed in Section 7. Section 8 contains the construction of bimodule resolutions of the algebras $H$ and $\Pi$. These resolutions play an important part in the study of locally free representations. In particular, Section 8.5 contains the proof of Theorem 1.7 (see Theorem 8.11). The intimate relation between Coxeter functors and the Auslander-Reiten translation for $H$ is studied in Section 9. Theorem 1.3 follows from Theorem 9.1 and Proposition 10.9. We also prove some crucial properties of the algebras $\Pi$. In particular, Theorem 1.5 corresponds to Corollary 9.6. In Section 10 we use the previous constructions for proving Theorem 1.2 (see Theorem 10.10). The proof of Theorem 1.6 can be found in Section 10.3. We also obtain some first results on the Auslander-Reiten theory of $H$.

Section 11 deals with showing that the reflection functors defined for $H$ are generalized versions of APR-tilting functors. Finally, Section 12 contains a collection of examples.

1.10. Notation. By a subcategory we always mean a full subcategory which is closed under isomorphisms and direct summands. By an algebra we mean an associative $K$-algebra with 1. For a $K$-algebra $A$ let $\text{mod}(A)$ be the category of finite-dimensional left $A$-modules. If $A = KQ/I$ is the path algebra of a quiver $Q$ modulo some ideal $I$, then $\text{rep}(A)$ denotes the category of finite-dimensional representations of $(Q, I)$. By definition these are the representations of $Q$ which are annihilated by $I$. We often identify $\text{mod}(A)$ and $\text{rep}(A)$. Let $D := \text{Hom}_K(-, K)$ be the usual $K$-duality. For a finite-dimensional algebra $A$ let $\tau(-) = \tau_A(-)$ be the Auslander-Reiten translation of $A$. For a module $X$ we denote by $\text{add}(X)$ the subcategory of modules which are isomorphic to finite direct sums of direct summands of $X$. As a general reference for the representation theory of finite-dimensional algebras we refer to the books [ARS, Ri2]. The composition of maps $f: X \to Y$ and $g: Y \to Z$ is denoted by $gf: X \to Z$.

2. Cartan matrices and the Weyl group

2.1. Cartan matrices and valued graphs. Let $C = (c_{ij}) \in M_n(\mathbb{Z})$ be a Cartan matrix, and let $D = \text{diag}(c_1, \ldots, c_n)$ be a symmetrizer of $C$. The valued graph $\Gamma(C)$ of $C$ has vertices $1, \ldots, n$ and an (unoriented) edge between $i$ and $j$ if and only if $c_{ij} < 0$. An edge $i \longrightarrow j$ has value $(|c_{ji}|, |c_{ij}|)$. In this case, we display this valued edge as $i \overset{|c_{ji}|,|c_{ij}|}{\longrightarrow} j$ and we just write $i \longrightarrow j$ if $(|c_{ji}|, |c_{ij}|) = (1, 1)$.

A Cartan matrix $C$ is connected if $\Gamma(C)$ is a connected graph. In this case, the symmetrizer $D$ is uniquely determined up to multiplication with a positive integer. More precisely, if $D$ is a minimal symmetrizer of a connected Cartan matrix $C$, then the other symmetrizers of $C$ are given by $mD$ with $m \geq 1$.

2.2. The quadratic form. Define the quadratic form $q_C : \mathbb{Z}^n \to \mathbb{Z}$ of $C$ by

$$ q_C := \sum_{i=1}^{n} c_i X_i^2 - \sum_{i<j} c_i |c_{ij}| X_i X_j. $$
(Recall that $c_i|c_{ij} = c_j|c_{ji}$.) The quadratic form $q_C$ plays a crucial role in the representation theory of the quivers of Cartan type $C$ and more generally of the species (see for example [DR1]) of type $C$.

A Cartan matrix $C$ is of Dynkin or Euclidean type if $q_C$ is positive definite or positive semidefinite, respectively. It is well known that $C$ is of Dynkin type if and only if $\Gamma(C)$ is a disjoint union of Dynkin graphs. (The Dynkin graphs are listed in Section 12.2.)

2.3. The Weyl group. As before let $C = (c_{ij})$ be a Cartan matrix, and let $\alpha_1, \ldots, \alpha_n$ be the positive simple roots of the Kac-Moody algebra $\mathfrak{g}(C)$ associated with $C$. For $1 \leq i, j \leq n$ define

$$s_i(\alpha_j) := \alpha_j - c_{ij}\alpha_i.$$  

This yields a reflection $s_i : \mathbb{Z}^n \to \mathbb{Z}^n$ on the root lattice

$$\mathbb{Z}^n = \sum_{i=1}^n \mathbb{Z}\alpha_i$$

where we identify $\alpha_i$ with the standard basis vector of $\mathbb{Z}^n$. The Weyl group $W(C)$ of $\mathfrak{g}(C)$ is the subgroup of $\text{Aut}(\mathbb{Z}^n)$ generated by $s_1, \ldots, s_n$. The Weyl group is finite if and only if $C$ is of Dynkin type.

2.4. Roots. Let

$$\Delta_{\text{re}}(C) := \bigcup_{i=1}^n W(\alpha_i)$$

be the set of real roots of $C$.

Let

$$(-,-)_C : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$$

be the symmetric bilinear form of $C$ defined by $(\alpha_i, \alpha_j)_C := c_i c_{ij}$. The fundamental region of $C$ is defined by

$$F := \{d \in \mathbb{N}^n \mid d \neq 0, \text{supp}(d) \text{ connected}, (d, \alpha_i)_C \leq 0 \text{ for all } 1 \leq i \leq n\},$$

where $\text{supp}(d)$ is the full subgraph of $\Gamma(C)$ given by the vertices $i$ with $d_i \neq 0$. Then

$$\Delta_{\text{im}}(C) := W(F) \cup W(-F)$$

is the set of imaginary roots of $C$.

Let

$$\Delta_{\text{re}}^+(C) := \Delta_{\text{re}}(C) \cap \mathbb{N}^n \quad \text{and} \quad \Delta_{\text{im}}^+(C) := \Delta_{\text{im}}(C) \cap \mathbb{N}^n$$

be the set of positive real roots and positive imaginary roots, respectively. It turns out that

$$\Delta_{\text{re}}(C) = \Delta_{\text{re}}^+(C) \cup -\Delta_{\text{re}}^+(C) \quad \text{and} \quad \Delta_{\text{im}}(C) = \Delta_{\text{im}}^+(C) \cup -\Delta_{\text{im}}^+(C).$$

Finally, let

$$\Delta(C) := \Delta_{\text{re}}(C) \cup \Delta_{\text{im}}(C)$$

be the set of roots of $C$, and

$$\Delta^+(C) := \Delta(C) \cap \mathbb{N}^n = \Delta_{\text{re}}^+(C) \cup \Delta_{\text{im}}^+(C)$$

is the set of positive roots.

By definition, for $w \in W(C)$ and $d \in \Delta(C)$ we have $w(d) \in \Delta(C)$. We have $q_C(d) = c_i$ if $d \in W(\alpha_i)$ is a real root, and $q_C(d) \leq 0$ if $d$ is an imaginary root. The following are equivalent:
Lemma 2.2. Assume that $q$ is of Dynkin type.

We get

(i) $C$ is of Dynkin type;
(ii) $\Delta(C)$ is finite;
(iii) $\Delta_{\text{re}}(C) = \Delta(C)$.

2.5. Coxeter transformations. A sequence $i = (i_1, \ldots, i_n)$ is a $+-$admissible sequence for $(C, \Omega)$ if $\{i_1, \ldots, i_n\} = \{1, \ldots, n\}$, $i_1$ is a sink in $Q^r(C, \Omega)$ and $i_k$ is a sink in the acyclic quiver $Q^r(C, s_{i_{k-1}} \cdots s_{i_1}(\Omega))$ for $2 \leq k \leq n$. For such a sequence $i$ let

$$\beta_{i,k} := \beta_k := \begin{cases} \alpha_{i_1} & \text{if } k = 1, \\ s_{i_1} s_{i_2} \cdots s_{i_{k-1}}(\alpha_{i_k}) & \text{if } 2 \leq k \leq n \end{cases}$$

where $s_1, \ldots, s_n \in W(C)$. Similarly, define

$$\gamma_{i,k} := \gamma_k := \begin{cases} s_{i_n} \cdots s_{i_{k+1}}(\alpha_{i_k}) & \text{if } 1 \leq k \leq n - 1, \\ \alpha_n & \text{if } k = n. \end{cases}$$

Let

$$c^+ := s_{i_n} s_{i_{n-1}} \cdots s_{i_1} : \mathbb{Z}^n \to \mathbb{Z}^n \text{ and } c^- := s_{i_1} s_{i_2} \cdots s_{i_n} : \mathbb{Z}^n \to \mathbb{Z}^n.$$

be the Coxeter transformations. For $k \in \mathbb{Z}$ we set

$$c^k := \begin{cases} (c^+)^k & \text{if } k > 0, \\ (c^-)^{-k} & \text{if } k < 0, \\ \text{id} & \text{if } k = 0. \end{cases}$$

We get

$$c^+(\beta_{i,k}) = (s_{i_n} s_{i_{n-1}} \cdots s_{i_1})(s_{i_1} s_{i_2} \cdots s_{i_{k-1}}(\alpha_{i_k})) = s_{i_n} s_{i_{n-1}} \cdots s_{i_k}(\alpha_{i_k}) = -s_{i_n} s_{i_{n-1}} \cdots s_{i_{k-1}}(\alpha_{i_k}) = -\gamma_{i,k}.$$

The following two lemmas are well known. For example, they are a consequence of the study of preprojective and preinjective representations of species without oriented cycles.

Lemma 2.1. Suppose $C$ is not of Dynkin type. Then the elements $c^- r(\beta_i)$ and $c^s(\gamma_j)$ with $r, s \geq 0$ and $1 \leq i, j \leq n$ are pairwise different elements in $\Delta^+_0(C)$.

Let $C$ be of Dynkin type. For $1 \leq i \leq n$ let $p_i \geq 1$ be minimal with $c^{-p_i}(\beta_i) \notin \mathbb{N}^n$, and let $q_j \geq 1$ be minimal such that $c_{\beta_j}(\gamma_j) \notin \mathbb{N}^n$. It is well known that such $p_i$ and $q_j$ exist. The elements $c^{-r}(\beta_i)$ with $1 \leq i \leq n$ and $0 \leq r \leq p_i - 1$ are pairwise different, and the elements $c^s(\gamma_j)$ with $1 \leq i \leq n$ and $0 \leq s \leq q_j - 1$ are pairwise different.

Lemma 2.2. Assume that $C$ is of Dynkin type. Then

$$\Delta^+(C) = \{c^{-r}(\beta_i) \mid 0 \leq r \leq p_i - 1\} = \{c^s(\gamma_j) \mid 0 \leq s \leq q_j - 1\}.$$

3. Locally free $H$-modules

For the whole section let $H = H(C, D, \Omega)$ and $Q = Q(C, \Omega)$.
3.1. Description of the projective and injective modules. The algebra $H$ is by definition a path algebra modulo an admissible ideal generated by zero relations and commutativity relations. This implies that each indecomposable projective $H$-module $P_i := He_i$ has a basis $B_i$ with the following properties: For each path $p$ in $Q$ and each $b \in B_i$ we have $p \cdot b \in B_i \cup \{0\}$. In particular, we can visualize $P_i$ by drawing a graph with vertices the elements in $B_i$, and an arrow $b \xrightarrow{a} b'$ if for an arrow $a \in Q_1$ and $b, b' \in B_i$ we have $a \cdot b = b'$. We say that $P_i$ has a multiplicative basis. Similarly, the indecomposable injective $H$-modules $I_i := D(e_iH)$ have a multiplicative basis.

Let $i$ be a vertex of $Q$. We define an $H$-module $-H_i$ as follows: A basis of $e_i - H_i$ is given by vectors $a_{i,c}$ with $1 \leq c \leq c_i$, and for $j \in \Omega(-,i)$ a basis of $e_j - H_i$ is given by vectors $b_{j,c}^{f,g}$ with $1 \leq c \leq c_j$, $1 \leq f \leq f_{ji}$ and $1 \leq g \leq g_{ji}$, and for $s \notin \Omega(-,i)$ define $e_s - H_i := 0$. The arrows of $Q$ act as follows:

$$\varepsilon_ia_{i,c} := \begin{cases} a_{i,c-1} & \text{if } c \geq 2, \\ 0 & \text{if } c = 1, \end{cases}$$
$$\varepsilon_jb_{j,c}^{f,g} := \begin{cases} b_{j,c-1}^{f,g} & \text{if } c \geq 2, \\ 0 & \text{if } c = 1, \end{cases}$$

and for $0 \leq k < k_{ji}$, $0 \leq f < f_{ji}$ and $1 \leq g \leq g_{ji}$ we have

$$\alpha_{i,c}^{(g)}a_{i,c}-k_{f_{ji}} := b_{j,c}^{f_{ji}-f,g}.$$

For $c_i = 9$, $c_j = 6$, $f_{ji} = 3$, $f_{ij} = 2$ (and therefore $k_{ji} = 3$) we display a part of the module $(e_i + e_j)_-H_i$ in Figure 1. The module $-H_i$ has one $i$-column with basis $(a_{i,1}, \ldots, a_{i,c_i})$, and for each $j \in \Omega(-,i)$ it has a $j$-column with basis $(b_{j,1}^{f,g}, \ldots, b_{j,c_j}^{f,g})$ for each $1 \leq f \leq f_{ji}$ and each $1 \leq g \leq g_{ji}$. By definition we have

$$\dim -H_i = c_i + \sum_{j \in \Omega(-,i)} f_{ji}g_{ji}c_j.$$  

The number of $j$-columns of $-H_i$ is $f_{ji}g_{ji} = |c_{ji}|$.

Suppose $j$ is a sink in $Q$. The module $P_j = e_jP_j$ has a basis $a_{j,1}, \ldots, a_{j,c_j}$ such that

$$\varepsilon_ja_{j,c} := \begin{cases} a_{j,c-1} & \text{if } c \geq 2, \\ 0 & \text{if } c = 1. \end{cases}$$

Then $(a_{j,1}, \ldots, a_{j,c_j})$ is the $j$-column of $P_j$.

Next, assume that $i$ is a vertex of $Q$ such that for each $j \in \Omega(-,i)$ the projective module $P_j$ is already constructed, and $P_j$ has a distinguished basis including a $j$-column $(a_{j,1}, \ldots, a_{j,c_j})$, which forms a basis of $e_jP_j$.

Then $P_i$ is constructed as follows: We take for each $j \in \Omega(-,i)$, $1 \leq f \leq f_{ji}$ and $1 \leq g \leq g_{ji}$ a copy $P_j^{f,g}$ of $P_j$ and identify the $j$-column $(b_{j,1}^{f,g}, \ldots, b_{j,c_j}^{f,g})$ of $-H_i$ with the $j$-column $(a_{j,1}, \ldots, a_{j,c_j})$ of $P_j^{f,g}$. The resulting module is our indecomposable projective $H$-module $P_i$, and by definition its $i$-column is the $i$-column $(a_{i,1}, \ldots, a_{i,c_i})$ of the module $-H_i$.

The indecomposable injective $H$-modules $I_j$ are constructed dually by gluing modules $-H_i$, which are dual to the modules $-H_i$. Again, for $c_i = 9$, $c_j = 6$, $f_{ji} = 3$, $f_{ij} = 2$ and $k_{ji} = 3$ we display a part of the module $(e_i + e_j)_jH_-$ in Figure 2.
Figure 1. Construction of the $H$-module $(e_i + e_j) - H_i$ with $c_i = 9$, $c_j = 6$, $f_{ji} = 3$, $f_{ij} = 2$ and $k_{ji} = 3$.

3.2. Projectives and injectives are locally free. Let $S_1, \ldots, S_n$ be the simple $H$-modules with $\dim(S_i) = \alpha_i$, and let $E_1, \ldots, E_n$ be the (indecomposable) locally free $H$-modules with $\operatorname{rank}(E_i) = \alpha_i$. (Here $\alpha_1, \ldots, \alpha_n$ is the standard basis of $\mathbb{Z}^n$.) Thus $E_i$ corresponds to the regular representation of $H_i$. More precisely, we have $E_i = e_i E_i$, and $e_i E_i$ has a basis $a_{i,1}, \ldots, a_{i,c_i}$ such that

$$\varepsilon_i a_{i,c} := \begin{cases} a_{i,c-1} & \text{if } c \geq 2, \\ 0 & \text{if } c = 1. \end{cases}$$
In particular, if \( i \) is a sink in \( Q^0(C, \Omega) \), then \( E_i = P_i \). Dually, if \( i \) is a source in \( Q^0(C, \Omega) \), then \( E_i = I_i \).

Our construction of \( P_i \) and \( I_i \) in Section 3.1 yields the following result.

**Proposition 3.1.** For every \( i \in Q_0 \),

\[
0 \to \bigoplus_{j \in \Omega(-,i)} P_j^{I_{j|i|}} \to P_i \to E_i \to 0
\]

is a minimal projective resolution of \( E_i \), and

\[
0 \to E_i \to I_i \to \bigoplus_{j \in \Omega(i,-)} P_j^{I_{j|i|}} \to 0
\]
is a minimal injective resolution of $E_i$. This implies that $P_i, I_i \in \operatorname{rep}_{lf}(H)$.

### 3.3. The dimension vectors of projective and injective modules

Assume that $i = (i_1, \ldots, i_n)$ is a $+$-admissible sequence for $(C, \Omega)$. Without loss of generality, assume that $i_k = k$ for $1 \leq k \leq n$. Recall that we defined some positive roots $\beta_k, \gamma_k \in \Delta^+(C)$ in Section 2.5.

**Lemma 3.2.** We have $\operatorname{rank}(P_k) = \beta_k$.

**Proof.** By our construction of the indecomposable projective $H$-modules $P_i$ we get

$$\operatorname{rank}(P_k) = \operatorname{rank}(E_k) + \sum_{j \in \Omega(-, k)} g_{jk} f_{jk} \operatorname{rank}(P_j)$$

$$= \operatorname{rank}(E_k) + \sum_{j \in \Omega(-, k)} |c_{jk}| \operatorname{rank}(P_j).$$

For $k = 1$ we have $P_k \cong E_k$. Thus we have $\operatorname{rank}(P_k) = \alpha_1 = \beta_1$. For $k \geq 2$ we have

$$\beta_k = s_1 \cdots s_{k-2}(\alpha_k - c_{k-1,k} \alpha_{k-1})$$

$$= s_1 \cdots s_{k-2}(\alpha_k) - c_{k-1,k} \beta_{k-1}$$

$$= \alpha_k - \sum_{j \in \Omega(-, k)} c_{jk} \beta_j$$

$$= \alpha_k + \sum_{j \in \Omega(-, k)} |c_{jk}| \beta_j$$

The claim follows by induction. $\square$

The proof of the next result is similar to the proof of Lemma 3.2.

**Lemma 3.3.** We have $\operatorname{rank}(I_k) = \gamma_k$.

As a consequence of Lemmas 3.2 and 3.3 we get the following result.

**Proposition 3.4.** We have $\operatorname{rank}(P_i), \operatorname{rank}(I_i) \in \Delta^+_r(C)$.

### 3.4. The Coxeter matrix

The Cartan matrix $C_H$ of $H$ is the $(n \times n)$-matrix with $k$th column the dimension vector of $P_k$, $1 \leq k \leq n$. (This is not to be confused with the Cartan matrix $C$.) The matrix $C_H$ is invertible. (We can choose a numbering of the vertices of $Q(C, \Omega)$ such that $C_H$ is an upper triangular matrix with non-zero entries on the diagonal.) The Coxeter matrix of $H$ is defined as

$$\Phi_H := -C_H^T C_H^{-1}$$

where $C_H^T$ denotes the transpose of $C_H$. It can also be defined by $\Phi_H(\dim(P_k)) := -\dim(I_k)$. (Here we treat $\dim(P_k)$ again as a column vector.)

It follows from Section 2.5 and Lemmas 3.2 and 3.3 that if we express $\Phi_H$ on the basis given by the vectors $\dim(E_k)$, we can identify $\Phi_H$ with the Coxeter transformation $c^+.$

### 3.5. Homological characterization of locally free modules

**Proposition 3.5.** For an $H$-module $M$ the following are equivalent:

(i) $\operatorname{proj}. \dim(M) \leq 1$;

(ii) $\operatorname{inj}. \dim(M) \leq 1$;

(iii) $\operatorname{proj}. \dim(M) < \infty$;
(iv) \( \text{inj. dim}(M) < \infty \);  
(v) \( M \) is locally free.

**Proof.** Assume that \( M \) is locally free, and let \( i \) be a sink in \( Q^\circ(C, \Omega) \). We get a short exact sequence  
\[
0 \to e_i M \to M \to (1 - e_i)M \to 0
\]
with \( e_i M \) and \( (1 - e_i)M \) locally free. Using Proposition 3.1 and induction (on the dimension of \( M \) or the number of vertices of \( Q^\circ \)) we know that the projective and the injective dimension of \( e_i M \) and \( (1 - e_i)M \) are at most one. Thus the same hold for \( M \).

Next assume that \( M \) is not locally free. Let \( i \) be a vertex of \( Q \) such that \( e_i M \) is not a projective \( H_i \)-module. Any projective resolution  
\[
\cdots \to P_2 \to P_1 \to P_0 \to M \to 0 \tag{3.1}
\]
of \( M \) yields a projective resolution of \( H_i \)-modules  
\[
\cdots \to e_iP_2 \to e_iP_1 \to e_iP_0 \to e_iM \to 0. \tag{3.2}
\]
But \( H_i \) is a selfinjective algebra, and \( e_i M \) is not a projective \( H_i \)-module. Thus the resolution \((3.2)\) and therefore also the resolution \((3.1)\) has to be infinite. This implies \( \text{proj. dim}(M) = \infty \). Dually, one shows that \( \text{inj. dim}(M) = \infty \). \( \square \)

For a finite-dimensional algebra \( A \), let \( \tau = \tau^A \) denote its Auslander-Reiten translation. Recall that for \( X, Y \in \text{mod}(A) \) there are functorial isomorphisms  
\[
\text{Ext}^1_A(X, Y) \cong D \text{Hom}_A(Y, \tau(X)) \cong D \text{Hom}_A(\tau^{-}(Y), X),
\]
see for example [Ri3, Section 2.4] for details. These isomorphisms are often referred to as Auslander-Reiten formulas. If \( \text{proj. dim}(X) \leq 1 \), we get a functorial isomorphism  
\[
\text{Ext}^1_A(X, Y) \cong D \text{Hom}_A(Y, \tau(X)),
\]
and if \( \text{inj. dim}(Y) \leq 1 \), then  
\[
\text{Ext}^1_A(X, Y) \cong D \text{Hom}_A(\tau^{-}(Y), X).
\]
Recall that an \( A \)-module \( X \) is \( \tau \)-rigid (resp. \( \tau^{-} \)-rigid) if \( \text{Hom}_A(X, \tau(X)) = 0 \) (resp. \( \text{Hom}_A(\tau^{-}(X), X) = 0 \)) [AIR]. Clearly, if \( X \) is \( \tau \)-rigid or \( \tau^{-} \)-rigid, then \( X \) is rigid.

**Corollary 3.6.** For \( M \in \text{rep}_{\text{lf}}(H) \) the following are equivalent:

(i) \( M \) is rigid;  
(ii) \( M \) is \( \tau \)-rigid;  
(iii) \( M \) is \( \tau^{-} \)-rigid.

Combining Propositions 3.1 and 3.5 yields the following result.

**Corollary 3.7.** The algebra \( H \) is a 1-Iwanaga-Gorenstein algebra.

**Lemma 3.8.** The subcategory \( \text{rep}_{\text{lf}}(H) \) is closed under extensions, kernels of epimorphisms and cokernels of monomorphisms.

**Proof.** Let  
\[
0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0
\]
be a short exact sequence in \( \text{rep}(H) \). For each \( 1 \leq i \leq n \) this induces a short exact sequence  
\[
0 \to e_i X \to e_i Y \to e_i Z \to 0
\]
of $H$-modules. Recall that $M \in \text{rep}(H)$ is locally free if and only if $e_i M$ is a projective (and therefore also an injective) $H$-module for all $i$. It follows that if any two of the three modules $e_i X, e_i Y$ and $e_i Z$ are projective $H$-modules, then the third module is also projective as an $H$-module. This finishes the proof. □

For the following definitions, see for example [AS]. Let $A$ be a finite-dimensional $K$-algebra, and let $\mathcal{U}$ be a subcategory of $\text{mod}(A)$. Then $\mathcal{U}$ is a resolving subcategory if the following hold:

(i) $A A \in \mathcal{U}$;
(ii) $\mathcal{U}$ is closed under extensions (i.e. for a short exact sequence $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$ of $A$-modules, if $X, Z \in \mathcal{U}$, then $Y \in \mathcal{U}$);
(iii) $\mathcal{U}$ is closed under kernels of epimorphisms.

Dually, $\mathcal{U}$ is coresolving if

(i) $D(A A) \in \mathcal{U}$;
(ii) $\mathcal{U}$ is closed under extensions;
(iii) $\mathcal{U}$ is closed under cokernels of monomorphisms.

For $X \in \text{mod}(A)$ a homomorphism $f : X \rightarrow U$ is a left $\mathcal{U}$-approximation of $X$ if $U \in \mathcal{U}$ and

$$\text{Hom}_A(U, U') \xrightarrow{\text{Hom}_A(f, U')} \text{Hom}_A(X, U') \rightarrow 0$$

is exact for all $U' \in \mathcal{U}$. Dually, a homomorphism $g : U \rightarrow X$ is a right $\mathcal{U}$-approximation of $X$ if $U \in \mathcal{U}$ and

$$\text{Hom}_A(U', U) \xrightarrow{\text{Hom}_A(U', g)} \text{Hom}_A(U', X) \rightarrow 0$$

is exact for all $U' \in \mathcal{U}$. The subcategory $\mathcal{U}$ is covariantly finite if every $X \in \text{mod}(A)$ has a left $\mathcal{U}$-approximation. Dually, $\mathcal{U}$ is contravariantly finite if every $X \in \text{mod}(A)$ has a right $\mathcal{U}$-approximation. Finally, $\mathcal{U}$ is functorially finite if $\mathcal{U}$ is covariantly and contravariantly finite.

**Theorem 3.9.** The subcategory $\text{rep}_{l.f.}(H)$ is resolving, coresolving and functorially finite. In particular, $\text{rep}_{l.f.}(H)$ has Auslander-Reiten sequences.

**Proof.** By Lemma 3.8 and Proposition 3.1 we get that $\text{rep}_{l.f.}(H)$ is a resolving and coresolving subcategory of $\text{rep}(H)$. Furthermore, by Proposition 3.5 we know that $\text{rep}_{l.f.}(H)$ coincides with the subcategory of all $H$-modules with projective dimension 1. Thus $\text{rep}_{l.f.}(H)$ is covariantly finite by [AR, Proposition 4.2]. Since $\text{rep}_{l.f.}(H)$ also coincides with the subcategory of all $H$-modules with injective dimension 1, the dual of [AR, Proposition 4.2] yields that $\text{rep}_{l.f.}(H)$ is contravariantly finite. Thus $\text{rep}_{l.f.}(H)$ is functorially finite in $\text{rep}(H)$. Now it follows from [AS, Theorem 2.4] that $\text{rep}_{l.f.}(H)$ has Auslander-Reiten sequences. □

### 4. The homological bilinear form

As before, let $H = H(C, D, \Omega)$. For $M, N \in \text{rep}_{l.f.}(H)$ define

$$\langle M, N \rangle_H := \dim \text{Hom}_H(M, N) - \dim \text{Ext}_H^1(M, N),$$

$$\langle M, N \rangle_H := \langle M, N \rangle_H + \langle N, M \rangle_H,$$

$$q_H(M) := \langle M, M \rangle_H.$$
**Proposition 4.1.** For \( M, N \in \text{rep}_{1,f}(H) \) we have
\[
\langle M, N \rangle_H = \sum_{i=1}^{n} c_i a_i b_i - \sum_{(j,i) \in \Omega} c_i |c_{ij}| a_i b_j
\]
in which \( \text{rank}(M) = (a_1, \ldots, a_n) \) and \( \text{rank}(N) = (b_1, \ldots, b_n) \).

**Proof.** Let \( Q = Q(C, \Omega) \). Let \( i_1 \) be a sink in \( Q^\circ \), and let \( i_n \) be a source of \( Q^\circ \). We get short exact sequences
\[
0 \to E_{i_1}^{n_1} \xrightarrow{f_1} M \xrightarrow{f_2} M' \to 0 \tag{4.1}
\]
and
\[
0 \to N' \xrightarrow{g_1} N \xrightarrow{g_2} E_{i_n}^{n_n} \to 0 \tag{4.2}
\]
where \( f_1 \) is the obvious canonical inclusion, \( f_2 \) is the canonical projection onto \( \text{Cok}(f_1) \), \( g_2 \) is the obvious canonical projection, and \( g_1 \) is the canonical inclusion of \( \text{Ker}(g_2) \). Applying \( \text{Hom}_H(-, N) \) to sequence (4.1) and \( \text{Hom}_H(M, -) \) to the sequence (4.2) yields the long exact cohomology sequences
\[
0 \to \text{Hom}_H(M', N) \to \text{Hom}_H(M, N) \to \text{Hom}_H(E_{i_1}^{n_1}, N) \to \text{Ext}_1^H(M', N) \to \text{Ext}_1^H(M, N) \to \text{Ext}_1^H(E_{i_1}^{n_1}, N) \to 0 \tag{4.3}
\]
and
\[
0 \to \text{Hom}_H(M, N') \to \text{Hom}_H(M, N) \to \text{Hom}_H(M, E_{i_n}^{n_n}) \to \text{Ext}_1^H(M, N') \to \text{Ext}_1^H(M, N) \to \text{Ext}_1^H(M, E_{i_n}^{n_n}) \to 0. \tag{4.4}
\]
For the exactness of the first cohomology sequence we used that \( \text{inj. dim}(N) \leq 1 \), and for the second sequence we needed that \( \text{proj. dim}(M) \leq 1 \), compare Proposition 3.5. The first sequence implies that
\[
\langle M, N \rangle_H = \langle M', N \rangle_H + \langle E_{i_1}^{n_1}, N \rangle_H,
\]
and the second sequence yields
\[
\langle M, N \rangle_H = \langle M, N' \rangle_H + \langle M, E_{i_n}^{n_n} \rangle_H.
\]
Thus by induction we get
\[
\langle M, N \rangle_H = \sum_{1 \leq i, j \leq n} a_i b_j \langle E_i, E_j \rangle_H.
\]
For \( 1 \leq j \leq n \) we have
\[
\text{dim} \text{Hom}_H(E_i, E_j) = \text{dim} \text{Hom}_H(P_i, E_j) = \begin{cases} 
  c_i & \text{if } i = j, \\
  0 & \text{otherwise}. 
\end{cases}
\]
Recall that the minimal projective resolution of \( E_i \) has the form
\[
0 \to \bigoplus_{j \in \Omega(-i)} P_j^{c_{ji}} \to P_i \to E_i \to 0. \tag{4.5}
\]
Applying \( \text{Hom}_H(-, E_j) \) for \( 1 \leq j \leq n \) yields
\[
\text{dim} \text{Ext}_1^H(E_i, E_j) = \begin{cases} 
  c_j |c_{ji}| & \text{if } j \in \Omega(-i), \\
  0 & \text{otherwise.}
\end{cases}
\]
Since \( c_j c_{ji} = c_i c_{ij} \), the result follows. \( \square \)

Proposition 4.1 shows that for \( M, N \in \text{rep}_{1,f}(H) \) the number \( \langle M, N \rangle_H \) depends only on the rank vectors \( \text{rank}(M) \) and \( \text{rank}(N) \). This implies:
Corollary 4.2. The map \((M,N) \mapsto \langle M,N \rangle_H\) descends to the Grothendieck group \(Z^n\) of \(\operatorname{rep}_{1,1}(H)\) and induces a bilinear form \(Z^n \times Z^n \rightarrow \mathbb{Z}\) still denoted by \(\langle -,- \rangle_H\). This bilinear form is characterized by \(\langle \alpha_i, \alpha_j \rangle_H = \langle E_i, E_j \rangle_H\), where \(\alpha_1, \ldots, \alpha_n\) is the standard basis of \(Z^n\). \(\square\)

Let

\[\langle -,- \rangle_H : Z^n \times Z^n \rightarrow \mathbb{Z}\]

be the symmetrization of \(\langle -,- \rangle_H\) defined by \((a,b)_H := \langle a,b \rangle_H + \langle b,a \rangle_H\), and let \(q_H : Z^n \rightarrow \mathbb{Z}\) be the quadratic form defined by \(q_H(a) := \langle a, a \rangle_H\). The forms \(q_H\) and \(\langle -,- \rangle_H\) are called the homological bilinear forms of \(H\).

Corollary 4.3. We have \(q_H = q_C\) and \(\langle -,- \rangle_H = \langle -,- \rangle_C\).

Proof. By definition we have

\[q_C := \sum_{i=1}^{n} c_i X_i^2 - \sum_{i<j} c_i |c_{ij}| X_i X_j,\]

and we know from Proposition 4.1 that

\[q_H = \sum_{i=1}^{n} c_i X_i^2 - \sum_{(j,i) \in \Omega} c_i |c_{ij}| X_i X_j.\]

Note that \(q_H\) does not depend on the orientation \(\Omega\), since \(c_i c_{ij} = c_j c_{ji}\) for all \(i,j\). Thus we have \(q_H = q_C\). Similarly, one also shows easily that \(\langle -,- \rangle_H = \langle -,- \rangle_C\). \(\square\)

5. An analogy to the representation theory of modulated graphs

5.1. The bimodules \(jH_i\). Let \(C = (c_{ij}) \in M_n(Z)\) be a Cartan matrix with symmetrizer \(D = \text{diag}(c_1, \ldots, c_n)\), and let \(\Omega\) be an orientation of \(C\), and let \(\Omega^*\) be the opposite orientation. Let \(H := H(C,D,\Omega)\) and \(H^* := H(C,D,\Omega^*)\). Recall that for \(1 \leq i \leq n\) we have

\[H_i := e_i He_i = K[e_i]/(e_i e_i).\]

In the following we write \(\otimes_i\) for a tensor product \(\otimes H_i\) over \(H_i\). If there is no danger of misunderstanding, we also just write \(\otimes\) instead of \(\otimes_i\).

For \((j,i) \in \Omega\) we define

\[jH_i := H_j \text{Span}_K(\alpha^{(g)}_{ji} | 1 \leq g \leq g_{ji})H_i = \text{Span}_K(\varepsilon^f_j \alpha^{(g)}_{ji} \varepsilon^f_i | f_j, f_i \geq 0, 1 \leq g \leq g_{ji}).\]

Our considerations in Section 5.1 show that \(jH_i\) is an \(H_j\-H_i\)-bimodule, which is free as a left \(H_j\)-module and free as a right \(H_i\)-module. Let \(iH_j\) be the corresponding \(H_i\-H_j\)-bimodule coming from \(H^*\). We get

\[jH_i = \bigoplus_{g=1}^{g_{ji}} \bigoplus_{f=0}^{f_{ji} - 1} H_j(\alpha^{(g)}_{ji} \varepsilon^f_i) = \bigoplus_{g=1}^{g_{ji}} \bigoplus_{f=0}^{f_{ji} - 1} (\varepsilon^f_j \alpha^{(g)}_{ji}) H_i\]

and

\[iH_j = \bigoplus_{g=1}^{g_{ji}} \bigoplus_{f=0}^{f_{ji} - 1} H_i(\alpha^{(g)}_{ij} \varepsilon^f_j) = \bigoplus_{g=1}^{g_{ji}} \bigoplus_{f=0}^{f_{ji} - 1} (\varepsilon^f_i \alpha^{(g)}_{ij}) H_j.\]
So we have
\[ H_j(iH_i) \cong H_j^{c_{ij}} \cong (iH_j)H_j, \]
\[ H_i(iH_j) \cong H_i^{c_{ij}} \cong (jH_i)H_i. \]

Define
\[
\begin{align*}
jL_i &:= \{ \alpha_{ji}^{(g)}, \alpha_{ji}^{(g)} e_i, \ldots, \alpha_{ji}^{(g)} e_i^{f_{ji}-1} | 1 \leq g \leq g_{ij} \}, \\
iL_j &:= \{ \alpha_{ij}^{(g)}, \alpha_{ij}^{(g)} e_j, \ldots, \alpha_{ij}^{(g)} e_j^{f_{ji}-1} | 1 \leq g \leq g_{ij} \}, \\
jR_i &:= \{ \alpha_{ji}^{(g)}, e_j \alpha_{ji}^{(g)}, \ldots, e_j^{f_{ij}-1} \alpha_{ji}^{(g)} | 1 \leq g \leq g_{ij} \}, \\
iR_j &:= \{ \alpha_{ij}^{(g)}, e_i \alpha_{ij}^{(g)}, \ldots, e_i^{f_{ij}-1} \alpha_{ij}^{(g)} | 1 \leq g \leq g_{ij} \}.
\end{align*}
\]
Then \( jL_i \) (resp. \( jR_i \)) is a basis of \( jH_i \) as a left \( H_j \)-modules (resp. as a right \( H_i \)-module). We have \( |jL_i| = |jR_i| = |c_{ij}| \) and \( |iL_j| = |jR_i| = |c_{ij}| \).

Let \((jL_i)^*\) and \((jR_i)^*\) be the dual basis of \( \text{Hom}_{H_j}(jH_i, H_j) \) and \( \text{Hom}_{H_i}(jH_i, H_i) \), respectively. For \( b \in jL_i \) or \( b \in jR_i \) let \( b^* \) be the corresponding dual basis vector. Similarly, define \((iL_j)^*\) and \((iR_j)^*\).

There is an \( H_i \)-\( H_j \)-bimodule isomorphism
\[
\rho: iH_j \to \text{Hom}_{H_j}(jH_i, H_j)
\]
given by
\[
\rho \left( \varepsilon_i^{f_{ji}-1-f} \alpha_{ji}^{(g)} \right) = \left( \alpha_{ji}^{(g)} \varepsilon_i^{f_{ji}} \right)^*
\]
for \( 0 \leq f \leq f_{ji} - 1 \) and \( 1 \leq g \leq g_{ij} \). Indeed, for the left \( H_i \)-module structure on \( \text{Hom}_{H_j}(jH_i, H_j) \) one has
\[
\varepsilon_i \cdot \left( \alpha_{ji}^{(g)} \varepsilon_i^{f_{ji}-1} \right)^* = \begin{cases} 
(\alpha_{ji}^{(g)} \varepsilon_i^{f_{ij}})^* & \text{if } f > 0, \\
(\alpha_{ji}^{(g)} \varepsilon_i^{f_{ij}})^* \cdot \varepsilon_i^{f_{ij}} & \text{if } f = 0.
\end{cases}
\]
Similarly there is an \( H_i \)-\( H_j \)-bimodule isomorphism
\[
\lambda: jH_j \to \text{Hom}_{H_i}(jH_i, H_i)
\]
given by
\[
\lambda \left( \alpha_{ij}^{(g)} \varepsilon_j^{f_{ij}-1-f} \right) = \left( \varepsilon_j^{f_{ij}} \alpha_{ij}^{(g)} \right)^*
\]
for \( 0 \leq f \leq f_{ij} - 1 \) and \( 1 \leq g \leq g_{ij} \). In particular, we get \( \rho(jR_i) = (jL_i)^* \) and \( \lambda(iL_j) = (jR_i)^* \). In the following, we sometimes identify the spaces \( \text{Hom}_{H_j}(jH_i, H_j), iH_j \) and \( \text{Hom}_{H_i}(jH_i, H_i) \) via \( \rho \) and \( \lambda \). For example, for \( b \in jL_i \), we consider \( b^* \in \text{Hom}_{H_j}(jH_i, H_j) \) as an element in \( jH_j \).

If \( N_j \) is an \( H_j \)-module, then we have a natural isomorphism of \( H_i \)-modules
\[
\text{Hom}_{H_j}(jH_i, N_j) \to iH_j \otimes_j N_j
\]
defined by
\[
f \mapsto \sum_{b \in jL_i} b^* \otimes_j f(b).
\]
Now, if in addition \( M_i \) is an \( H_i \)-module, the adjunction map gives an isomorphism of \( K \)-vector spaces:
\[
\text{Hom}_{H_j}(jH_i \otimes_i M_i, N_j) \to \text{Hom}_{H_i}(M_i, \text{Hom}_{H_j}(jH_i, N_j)).
\]
Combining these two maps we get a functorial isomorphism of $K$-vector spaces

$$\text{ad}_{ji} := \text{ad}_{ji}(M_i, N_j) : \text{Hom}_{H_i}(jH_i \otimes_i M_i, N_j) \rightarrow \text{Hom}_{H_i}(M_i, iH_j \otimes_j N_j)$$

given by

$$f \mapsto \left(f^\vee : m \mapsto \sum_{b \in_j L_i} b^* \otimes_j f(b \otimes_i m)\right).$$

The inverse $\text{ad}_{ji}^{-1}$ of $\text{ad}_{ji}$ is given by

$$g \mapsto \left(g^\vee : h \otimes_i m \mapsto \sum_{b \in_j L_i} b^*(h)(g(m))_b\right)$$

where the $K$-linear maps $(g(m))_b$ are uniquely determined by

$$g(m) = \sum_{b \in_j L_i} b^* \otimes_j (g(m))_b.$$  

Here we used that each element $x$ in $iH_j \otimes_j N_j$ can be written uniquely as a sum of the form

$$x = \sum_{b \in_j L_i} b^* \otimes_j x_b.$$

### 5.2. Representation theory of modulated graphs.

The tuple $(H_i, iH_j, jH_i)$ defined in Section 5.1 is called a modulation of $C$ and is denoted by $\mathcal{M}(C, D)$.

For an orientation $\Omega$ of $C$, a representation $M = (M_i, M_{ij})$ of $\mathcal{M}(C, D), \Omega)$ is given by a finite-dimensional $H_i$-module $M_i$ for each $1 \leq i \leq n$ and an $H_i$-linear map

$$M_{ij} : iH_j \otimes_j M_j \rightarrow M_i$$

for each $(i, j) \in \Omega$. A morphism $f : M \rightarrow N$ of representations $M = (M_i, M_{ij})$ and $N = (N_i, N_{ij})$ of $\mathcal{M}(C, D), \Omega)$ is a tuple $f = (f_i)_i$ of $H_i$-linear maps $f_i : M_i \rightarrow N_i$ for $1 \leq i \leq n$ such that for each $(i, j) \in \Omega$ the diagram

$$\begin{array}{ccc}
  iH_j \otimes_j M_j & \xrightarrow{f_{ij}} & iH_j \otimes_j N_j \\
  \downarrow M_{ij} & & \downarrow N_{ij} \\
  M_i & \xrightarrow{f_i} & N_i
\end{array}$$

commutes. One easily checks that the representations of $(\mathcal{M}(C, D), \Omega)$ form an abelian category $\text{rep}(C, D, \Omega)$.

For $(M_i, M_{ij}) \in \text{rep}(C, D, \Omega)$ define a representation

$$(M_i, M(\alpha_{ij}^{(g)}, M(\varepsilon_i))$$

of $\mathcal{H}(C, D, \Omega)$ as follows: Define a $K$-linear map $M(\varepsilon_i) : M_i \rightarrow M_i$ by

$$M(\varepsilon_i)(m) := \varepsilon_i m.$$  

(Here we use that $M_i$ is an $H_i$-module.) Let $(i, j) \in \Omega$. Recall that $iH_j$ has an $H_i$-basis

$$iL_j = \{\alpha_{ij}^{(g)}, \alpha_{ij}^{(g)} \varepsilon_j, \ldots, \alpha_{ij}^{(g)} \varepsilon_j^{g_{ij}-1} \mid 1 \leq g \leq g_{ij}\}.$$  

Define a $K$-linear map $M(\alpha_{ij}^{(g)}) : M_j \rightarrow M_i$ by

$$M(\alpha_{ij}^{(g)})(m) := M_{ij}(\alpha_{ij}^{(g)} \otimes_j m).$$
Now one can check that the relations (H1) and (H2) are satisfied. In other words, 
\((M_i, M(\alpha_{ij}^{(q)}), M(\varepsilon_i))\) is a representation of \(H(C, D, \Omega)\).

Conversely, let \((M_i, M(\alpha_{ij}^{(q)}), M(\varepsilon_i))\) be a representation of \(H(C, D, \Omega)\). Note that \(M_i\) is an \(H_i\)-module via the map \(M(\varepsilon_i)\). For \((i, j) \in \Omega\) define an \(H_i\)-linear map 
\[M_{ij} : iH_j \otimes j M_j \rightarrow M_i\]
by 
\[M_{ij}(\alpha_{ij}^{(q)} \varepsilon_j^f \otimes m) := (M(\alpha_{ij}^{(q)})) \circ M(\varepsilon_j)^f(m).\]

Then \((M_i, M_{ij}) \in \text{rep}(C, D, \Omega)\).

These two constructions yield obviously mutually inverse bijections between the representations of \((M(C, D), \Omega)\) and \(H(C, D, \Omega)\). It is also clear how to associate to a morphism in \(\text{rep}(C, D, \Omega)\) a morphism in \(\text{rep}(H(C, D, \Omega))\) and vice versa. Now it is straightforward to verify the following statement.

**Proposition 5.1.** The categories \(\text{rep}(C, D, \Omega)\) and \(\text{rep}(H(C, D, \Omega))\) are isomorphic.

Thus the representation theory of the algebras \(H(C, D, \Omega)\) shows a striking analogy to the representation theory of modulated graphs in the sense of Dlab and Ringel [DR1]. The main difference is that in Dlab and Ringel’s theory, the rings \(H_i\) would be division rings, whereas in our case they are commutative symmetric algebras, or more precisely, truncations of polynomial rings. Generalizations of the representation theory of modulated graphs have been formulated already in [Li].

5.3. Next, we want to interpret the category \(\text{rep}(\Pi(C, D))\) of finite-dimensional representations of \(\Pi(C, D)\) as a category of representations of modulated graphs. Let \(\text{rep}(C, D, \Omega)\) be the category with objects \(M = (M_i, M_{ij}, M_{ji})\) with \((i, j) \in \Omega\) such that \((M_i, M_{ij}) \in \text{rep}(C, D, \Omega)\) and \((M_i, M_{ji}) \in \text{rep}(C, D, \Omega^*)\). Given two such objects \(M\) and \(N\) a tuple \(f = (f_{ij})_{ij}\) is a homomorphism \(f : M \rightarrow N\) if \(f\) is both a homomorphism \((M_i, M_{ij}) \rightarrow (N_i, N_{ij})\) in \(\text{rep}(C, D, \Omega)\) and a homomorphism \((M_i, M_{ji}) \rightarrow (N_i, N_{ji})\) in \(\text{rep}(C, D, \Omega^*)\).

For an object \(M = (M_i, M_{ij}, M_{ji})\) in \(\text{rep}(C, D, \Omega)\) let 
\[M_{i,\text{in}} := (\text{sgn}(i, j)M_{ij})_j : \bigoplus_{j \in \Omega(i, -)} iH_j \otimes M_j \rightarrow M_i\]
and 
\[M_{i,\text{out}} := (M_{ji}^\vee)_j : M_i \rightarrow \bigoplus_{j \in \Omega(-, i)} iH_j \otimes M_j.\]

These are both \(H_i\)-module homomorphisms. (Recall that \(M_{ji}^\vee = \text{adj}_{ji}(M_{ji})\), see Section 5.1.) Set 
\[\tilde{M}_i := \bigoplus_{j \in \Omega(i, -)} iH_j \otimes M_j.\]

Since \(\Omega(i, -) = \Omega(-, i)\), we have 
\[\bigoplus_{j \in \Omega(i, -)} iH_j \otimes M_j = \bigoplus_{k \in \Omega(-, i)} iH_k \otimes M_k.\]

Thus we get a diagram 
\[
\begin{array}{ccc}
\tilde{M}_i & \xrightarrow{M_{i,\text{in}}} & M_i \\
\downarrow & & \downarrow \\
\tilde{M}_i & \xrightarrow{M_{i,\text{out}}} & M_i
\end{array}
\]
Proposition 5.2. The category \( \text{rep}(\Pi(C, D)) \) is isomorphic to the full subcategory of \( \text{rep}(C, D, \Omega) \) with objects \( M = (M_i, M_{ij}, M_{ji}) \) such that

\[ M_{i,\text{in}} \circ M_{i,\text{out}} = 0 \]

for all \( i \).

Proof. For an object \( M = (M_i, M_{ij}, M_{ji}) \) in \( \text{rep}(C, D, \Omega) \), the composition

\[ M_{i,\text{in}} \circ M_{i,\text{out}} = \sum_{j \in \Pi(-,i)} \text{sgn}(i, j) M_{ij} \circ M_{ji}' \]

is in \( \text{End}_{H_i}(M_i) \) and maps an element \( m \in M_i \) to

\[ M_{i,\text{in}} \circ M_{i,\text{out}}(m) = \sum_{j \in \Pi(-,i)} \text{sgn}(i, j) \sum_{b \in jL_i} M_{ij}(b^* \otimes_j M_{ji}(b \otimes_i m)) . \]

Let \( b \in jL_i \). Thus we have \( b = \alpha_{ij}^f \varepsilon_i f_{ji}^{-1} - f \) for some \( 0 \leq f \leq f_{ji} - 1 \). This implies that \( b^* = \varepsilon_i^f \alpha_{ij}^{f_{ji}^{-1} - f} \in iR_j \). It follows that

\[ \text{sgn}(i, j) M_{ij}(b^* \otimes_j M_{ji}(b \otimes_i m)) = \text{sgn}(i, j) M(e_i)^f M(\alpha_{ij}^f) M(\alpha_{ji}^{f_{ji}^{-1} - f}) M(e_i)^{f_{ji}^{-1} - f}(m) . \]

In view of the defining relation (P3) of \( \Pi \), this yields the result. \( \square \)

6. Reflection functors

In this section, let \( H = H(C, D, \Omega) \) and \( \Pi = \Pi(C, D) \).

6.1. Reflection functors for \( \Pi \). Let \( M \in \text{rep}(\Pi) \). Thus we have \( M = (M_i, M_{ij}, M_{ji}) \), where \((i,j)\) runs over \( \Omega \), such that \( M_{i,\text{in}} \circ M_{i,\text{out}} = 0 \) for each \( i \). Hence for every \( i \), we have \( M_{i,\text{out}}(M_i) \subseteq \text{Ker}(M_{i,\text{in}}) \).

Generalizing the construction in [BK] Section 2.2, see also [Bo], we fix some vertex \( i \) and construct a new \( \Pi \)-module by replacing the diagram

\[ \tilde{M}_i \xrightarrow{M_{i,\text{in}}} M_i \xrightarrow{M_{i,\text{out}}} \tilde{M}_i . \]

by

\[ \tilde{M}_i \xrightarrow{\tilde{M}_{i,\text{out}}M_{i,\text{in}}} \text{Ker}(M_{i,\text{in}}) \xrightarrow{\text{can}} \tilde{M}_i \]

where \( \tilde{M}_{i,\text{out}} : M_i \to \text{Ker}(M_{i,\text{in}}) \) is induced by \( M_{i,\text{out}} \) and \( \text{can} \) is the canonical inclusion. Gluing this new datum with the remaining part of \( M \) gives a new \( \Pi \)-module \( \Sigma^+_i(M) \).

Similarly, replacing

\[ \tilde{M}_i \xrightarrow{M_{i,\text{in}}} M_i \xrightarrow{M_{i,\text{out}}} \tilde{M}_i . \]

by

\[ \tilde{M}_i \xrightarrow{\text{can}} \text{Cok}(M_{i,\text{out}}) \xrightarrow{M_{i,\text{out}}\tilde{M}_{i,\text{in}}} \tilde{M}_i \]

where \( \text{can} : \text{Cok}(M_{i,\text{out}}) \to M_i \) is induced by \( M_{i,\text{in}} \) and \( \text{can} \) is the canonical projection. Gluing this new datum with the remaining part of \( M \) gives a new \( \Pi \)-module denoted by \( \Sigma^-_i(M) \).

The above constructions are obviously functorial. It is straightforward to show that \( \Sigma^+_i \) is left exact, and \( \Sigma^-_i \) is right exact. Both functors are covariant, \( K \)-linear and additive.
The commutative diagram

\[ \begin{array}{c}
\tilde{M}_i \\
\downarrow \quad \downarrow \\
M_i \\
\downarrow \quad \downarrow \\
\tilde{M}_i \end{array} \xrightarrow{\text{can}} \begin{array}{c}
\text{Cok}(M_{i,\text{out}}) \xrightarrow{M_{i,\text{out}}} M_{i,\text{in}} \xrightarrow{\text{can}} \tilde{M}_i \\
\downarrow \quad \quad \quad \downarrow \\
M_i \xrightarrow{M_{i,\text{in}}} M_i \xrightarrow{M_{i,\text{out}}} \tilde{M}_i \\
\downarrow \quad \quad \quad \downarrow \\
\text{Ker}(M_{i,\text{in}}) \xrightarrow{\text{can}} \tilde{M}_i \end{array} \]

of \( H_i \)-module homomorphisms summarizes the situation and also shows the existence of canonical homomorphisms \( \Sigma^-_i(M) \to M \to \Sigma^+_i(M) \).

For \( M \in \text{rep}(\Pi) \) let \( \text{sub}_i(M) \) be the largest submodule \( U \) of \( M \) such that \( e_i U = U \), and let \( \text{soc}_i(M) \) be the largest submodule \( V \) of \( M \) such that \( V \) is isomorphic to a direct sum of copies of \( S_i \). For example, we have \( \text{sub}_i(E_i) = E_i \) and \( \text{soc}_i(E_i) \cong S_i \). Dually, let \( \text{fac}_i(M) \) be the largest factor module \( M/U \) of \( M \) such that \( e_i (M/U) = M/U \), and let \( \text{top}_i(M) \) be the largest factor module \( M/V \) of \( M \) such that \( M/V \) is isomorphic to a direct sum of copies of \( S_i \). All these constructions are functorial.

The proof of the following proposition follows almost word by word the proof of Baummann and Kamnitzer [BK, Proposition 2.5], who deal with classical preprojective algebras associated with Dynkin quivers. One difference is that we need to work with \( \text{sub}_i \) and \( \text{fac}_i \) instead of \( \text{soc}_i \) and \( \text{top}_i \).

**Proposition 6.1.** For each \( i \) the following hold:

(i) The pair \( (\Sigma^-_i, \Sigma^+_i) \) is a pair of adjoint functors, i.e. there is a functorial isomorphism

\[ \text{Hom}_\Pi(\Sigma^-_i(M), N) \cong \text{Hom}_\Pi(M, \Sigma^+_i(N)). \]

(ii) The adjunction morphisms \( \text{id} \to \Sigma^+_i \Sigma^-_i \) and \( \Sigma^-_i \Sigma^+_i \to \text{id} \) can be inserted in functorial short exact sequences

\[ 0 \to \text{sub}_i \to \text{id} \to \Sigma^+_i \Sigma^-_i \to 0 \]

and

\[ 0 \to \Sigma^-_i \Sigma^+_i \to \text{id} \to \text{fac}_i \to 0. \]

**Proof.** To establish (i), it is enough to define a pair of mutually inverse bijections between \( \text{Hom}_\Pi(\Sigma^-_i(M), N) \) and \( \text{Hom}_\Pi(M, \Sigma^+_i(N)) \) for any \( \Pi \)-modules \( M \) and \( N \), which are functorial in \( M \) and \( N \). The construction looks as follows. Consider a morphism \( f: M \to \Sigma^+_i(N) \). By definition, this is a collection of \( H_j \)-module homomorphisms

\[ f_j: M_j \to (\Sigma^+_i(N))_j \]

with \( 1 \leq j \leq n \) such that the diagram

\[ \begin{array}{c}
iH_j \otimes f_j M_j \xrightarrow{1 \otimes f_j} iH_j \otimes (\Sigma^+_i(N))_j \\
\downarrow M_{ij} \quad \quad \downarrow (\Sigma^+_i(N))_{ij} \\
M_i \xrightarrow{f_i} \Sigma^+_i(N_i) \end{array} \]
commutes for all \((i, j) \in \Omega\). Recall that
\[
\tilde{M}_i = \bigoplus_{j \in \Omega(i, -)} i H_j \otimes_j M_j.
\]
Set
\[
\tilde{f}_i := \bigoplus_{j \in \Omega(i, -)} 1 \otimes f_j : \tilde{M}_i \to \tilde{N}_i.
\]

In the diagram
\[
\begin{array}{ccccccccc}
\tilde{M}_i & \xrightarrow{M_{i,\text{in}}} & M_i & \xrightarrow{M_{i,\text{out}}} & M_i & \xrightarrow{\pi} & \text{Cok}(M_{i,\text{out}}) & \xrightarrow{M_{i,\text{out}} \tilde{M}_{i,\text{in}}} & \tilde{M}_i \\
\tilde{N}_i & \xrightarrow{\tilde{N}_{i,\text{out}} N_{i,\text{in}}} & \text{Ker}(N_{i,\text{in}}) & \xrightarrow{\iota} & \tilde{N}_i & \xrightarrow{N_{i,\text{in}}} & N_i & \xrightarrow{N_{i,\text{out}}} & \tilde{N}_i \\
\tilde{f}_i & \downarrow f_i & \tilde{f}_i & \downarrow g_i & \tilde{f}_i & \downarrow g_i & \\
\tilde{g}_i & \downarrow f_i & \tilde{g}_i & \downarrow g_i & \tilde{g}_i & \downarrow g_i & \\
\tilde{N}_i & \xrightarrow{\tilde{N}_{i,\text{out}} N_{i,\text{in}}} & \text{Ker}(N_{i,\text{in}}) & \xrightarrow{\iota} & \tilde{N}_i & \xrightarrow{N_{i,\text{in}}} & N_i & \xrightarrow{N_{i,\text{out}}} & \tilde{N}_i
\end{array}
\]
the two left squares commute.

There is thus a unique map \(g_i\) making the third square commutative. (Observe that
\(N_{i,\text{in}} \tilde{f}_i M_{i,\text{out}} = N_{i,\text{int}} f_i = 0\). Thus \(N_{i,\text{in}} \tilde{f}_i\) factors through the cokernel of \(M_{i,\text{out}}\).)

The fourth square also commutes. Thus if we set \(g_j := f_j\) for all vertices \(j \neq i\) we get a homomorphism \(g : \Sigma_i^- (M) \to N\). Conversely, consider a homomorphism \(g : \Sigma_i^- (M) \to N\) and set
\[
\tilde{g}_i := \bigoplus_{j \in \Omega(-, i)} 1 \otimes g_j : \tilde{M}_i \to \tilde{N}_i.
\]

In the diagram
\[
\begin{array}{ccccccccc}
\tilde{M}_i & \xrightarrow{M_{i,\text{in}}} & M_i & \xrightarrow{M_{i,\text{out}}} & M_i & \xrightarrow{\pi} & \text{Cok}(M_{i,\text{out}}) & \xrightarrow{M_{i,\text{out}} \tilde{M}_{i,\text{in}}} & \tilde{M}_i \\
\tilde{N}_i & \xrightarrow{\tilde{N}_{i,\text{out}} N_{i,\text{in}}} & \text{Ker}(N_{i,\text{in}}) & \xrightarrow{\iota} & \tilde{N}_i & \xrightarrow{N_{i,\text{in}}} & N_i & \xrightarrow{N_{i,\text{out}}} & \tilde{N}_i \\
\tilde{g}_i & \downarrow f_i & \tilde{g}_i & \downarrow g_i & \tilde{g}_i & \downarrow g_i & \\
\tilde{g}_i & \downarrow f_i & \tilde{g}_i & \downarrow g_i & \tilde{g}_i & \downarrow g_i & \\
\tilde{N}_i & \xrightarrow{\tilde{N}_{i,\text{out}} N_{i,\text{in}}} & \text{Ker}(N_{i,\text{in}}) & \xrightarrow{\iota} & \tilde{N}_i & \xrightarrow{N_{i,\text{in}}} & N_i & \xrightarrow{N_{i,\text{out}}} & \tilde{N}_i
\end{array}
\]
the two right squares commute. Thus there is a unique map \(f_i\) making the second square commutative. The first square then also commutes. Thus if we set \(f_j := g_j\) for all the vertices \(j \neq i\), we get a morphism \(f : M \to \Sigma_i^+ (N)\). To establish (ii), one checks that \(\Sigma_i^- \Sigma_i^+ (M)\) is the \(\Pi\)-module obtained by replacing in \(M\) the part
\[
\begin{array}{ccc}
\tilde{M}_i & \xrightarrow{M_{i,\text{in}}} & M_i & \xrightarrow{M_{i,\text{out}}} & \tilde{M}_i
\end{array}
\]
with
\[
\begin{array}{ccc}
\tilde{M}_i & \xrightarrow{M_{i,\text{in}}} & \text{Im}(M_{i,\text{in}}) & \xrightarrow{M_{i,\text{out}}} & \tilde{M}_i
\end{array}
\]
and that \(\Sigma_i^+ \Sigma_i^- (M)\) is the \(\Pi\)-module obtained by replacing in \(M\) the part
\[
\begin{array}{ccc}
\tilde{M}_i & \xrightarrow{M_{i,\text{in}}} & M_i & \xrightarrow{M_{i,\text{out}}} & \tilde{M}_i
\end{array}
\]
with
\[
\begin{array}{ccc}
\tilde{M}_i & \xrightarrow{M_{i,\text{in}}} & \text{Im}(M_{i,\text{out}}) & \xrightarrow{M_{i,\text{out}}} & \tilde{M}_i
\end{array}
\]
It remains to observe that as vector spaces, \(\text{fac}_i (M) \cong \text{Cok}(M_{i,\text{in}})\) and \(\text{sub}_i (M) \cong \text{Ker}(M_{i,\text{out}})\). \(\square\)
Corollary 6.2. The functors $\Sigma^+_i: \mathcal{T}_i \rightarrow \mathcal{S}_i$ and $\Sigma^-_i: \mathcal{S}_i \rightarrow \mathcal{T}_i$ define inverse equivalences of the subcategories

$$\mathcal{T}_i := \{ M \in \text{rep}(\Pi) \mid \text{top}_i(M) = 0 \}$$

and

$$\mathcal{S}_i := \{ M \in \text{rep}(\Pi) \mid \text{soc}_i(M) = 0 \}.$$ 

Corollary 6.3. For $M, N \in \text{rep}(\Pi)$ the following hold:

(i) If $M, N \in \mathcal{T}_i$, then $\Sigma^+_i$ induces an isomorphism

$$\text{Ext}^1_\Pi(M, N) \cong \text{Ext}^1_\Pi(\Sigma^+_i(M), \Sigma^+_i(N)).$$

(ii) If $M, N \in \mathcal{S}_i$, then $\Sigma^-_i$ induces an isomorphism

$$\text{Ext}^1_\Pi(M, N) \cong \text{Ext}^1_\Pi(\Sigma^-_i(M), \Sigma^-_i(N)).$$

Proposition 6.4. For $M \in \text{rep}(\Pi)$ the following are equivalent:

(i) $\text{top}_i(M) = 0$;
(ii) $M \cong \Sigma^-_i \circ \Sigma^+_i(M)$.

Furthermore, if $M \in \text{rep}_{lf}(\Pi)$, then (i) and (ii) are equivalent to the following:

(iii) $\text{rank}(\Sigma^+_i(M)) = s_i(\text{rank}(M)).$

Dually, the following are equivalent:

(i) $\text{soc}_i(M) = 0$;
(ii) $M \cong \Sigma^+_i \circ \Sigma^-_i(M)$.

Furthermore, if $M \in \text{rep}_{lf}(\Pi)$, then (i) and (ii) are equivalent to the following:

(iii) $\text{rank}(\Sigma^-_i(M)) = s_i(\text{rank}(M)).$

Proof. The equivalence of (i) and (ii) follows directly from Proposition 6.1 and Corollary 6.2.

Suppose (i) holds for some vertex $i$ of $Q(C, \Omega)$. Let $a = (a_1, \ldots, a_n) = \text{rank}(M)$. Recall that we have the $H_i$-module homomorphism

$$M_{i,\text{in}}: \bigoplus_{j \in \Pi(i,-)} iH_j \otimes_j M_j \rightarrow M_i.$$ 

Since $\text{top}_i(M) = 0$, the map $M_{i,\text{in}}$ is surjective. This implies that $\Sigma^+_i(M)$ is again locally free with

$$\text{rank}(\Sigma^+_i(M))_i = \sum_{j \in \Pi(i,-)} |c_{ij}|a_j - a_i = (s_i(\text{rank}(\Sigma^+_i(M))))_i.$$ 

(Here we used that $iH_j \otimes_j M_j$ is a free $H_i$-module of rank $|c_{ij}|a_j$.) Thus (iii) holds.

Vice versa, the equality

$$\text{rank}(\Sigma^+_i(M))_i = (s_i(\text{rank}(\Sigma^+_i(M))))_i$$

implies that $M_{i,\text{in}}$ is surjective. Thus (iii) implies (i).

Let $M = (M_i, M_{ij}, M_{ji}) \in \text{rep}(C, D, \Pi)$. We say that $M_{i,\text{in}}$ (resp. $M_{i,\text{out}}$) splits if the image of $M_{i,\text{in}}$ (resp. $M_{i,\text{out}}$) is a free $H_i$-module. The following lemma is straightforward.
Lemma 6.5. Let $M \in \text{rep}(\Pi)$ be locally free. For each $i$ the following hold:

(i) $\Sigma_+^i(M)$ is locally free if and only if $M_{i,\text{in}}$ splits;

(ii) $\Sigma_-^i(M)$ is locally free if and only if $M_{i,\text{out}}$ splits.

6.2. Reflection functors for $H$. For an orientation $\Omega$ of $C$ and some $1 \leq k \leq n$ let

$$s_k(\Omega) := \{(r, s) \in \Omega \mid k \notin \{r, s\}\} \cup \{(s, r) \in \Omega^* \mid k \in \{r, s\}\}.$$ 

This is again an orientation of $C$. Define

$$s_k(H) := s_k(H(C, D, \Omega)) := H(C, D, s_k(\Omega)). \quad (6.1)$$

Now let $k$ be a sink in $Q^\circ(C, \Omega)$. Then $\Sigma_+^k$ obviously restricts to a reflection functor

$$F_+^k: \text{rep}(H) \to \text{rep}(s_k(H))$$

which can also be described as follows. Let $M = (M_i, M_{ij}) \in \text{rep}(C, D, \Omega)$. Recall that

$$M_{k,\text{in}} = (\text{sgn}(k, j) M_{kj}) j : \bigoplus_{j \in \Omega(k,-)} k H_j \otimes M_j \to M_k.$$ 

(Nota that $\Omega(k,-) = \overline{\Omega}(k,-)$, since $k$ is a sink.) Let $N_k := \text{Ker}(M_{k,\text{in}})$. We obtain an exact sequence

$$0 \to N_k \to \bigoplus_{j \in \Omega(k,-)} k H_j \otimes j M_j \xrightarrow{M_{k,\text{in}}} M_k.$$ 

Let us denote by $(N^\nu_j)_j$ the inclusion map $N_k \to \bigoplus_{j \in \Omega(k,-)} k H_j \otimes j M_j$. Then we have $F_+^k(M) = (N_r, N^\nu_r)$ with $(r, s) \in s_k(\Omega)$, where

$$N_r := \begin{cases} M_r & \text{if } r \neq k, \\ N_i & \text{if } r = k \end{cases} \quad \text{and} \quad N^\nu_r := \begin{cases} M^\nu_r & \text{if } (r, s) \in \Omega \text{ and } r \neq k, \\ (N^\nu_{s'})^\vee & \text{if } (r, s) \in \Omega^* \text{ and } s = k. \end{cases}$$

Similarly, if $k$ is a source in $Q^\circ(C, \Omega)$, then $\Sigma_-^k$ restricts to a reflection functor

$$F_-^k: \text{rep}(H) \to \text{rep}(s_k(H)).$$

Proposition 6.6. Let $M$ be locally free and rigid in $\text{rep}(H)$. Then $F_+^k(M)$ is locally free and rigid.

Proof. Without loss of generality assume that $1$ is a sink and $n$ is a source in $Q^\circ(C, \Omega)$. Let $M$ be a locally free and rigid $H$-module. To get a contradiction, assume that $F_+^1(M)$ is not locally free and that $M$ is of minimal dimension with this property. Recall that $F_+^1(E_1) = 0$. Thus by the minimality of its dimension, $M$ does not have any direct summand isomorphic to $E_1$. We can also assume that $e_i M \neq 0$ for all $1 \leq i \leq n$. (Otherwise $M$ can be considered as a module over an algebra $H(C', \Omega')$ with fewer vertices.) Since $F_+^1(M)$ is not locally free, we get $\text{Hom}_H(M, E_1) \neq 0$, see Lemma 6.5.

We have a short exact sequence

$$0 \to M' \to M \to e_n M \to 0 \quad (6.2)$$

where $M' := (e_1 + \cdots + e_{n-1}) M$. Clearly, $M'$ and $e_n M$ are both locally free. In particular, $e_s M \cong E_n^s$ for some $s \geq 1$. We have $\text{Hom}_H(M', e_n M) = 0$. Thus applying $\text{Hom}_H(M', -)$ to (6.2) we get an embedding $\text{Ext}_H^1(M', M') \to \text{Ext}_H^1(M', M)$. Applying $\text{Hom}_H(-, M')$ to (6.2) and using that $\text{Ext}_H^2(e_n M, M) = 0$ we get $\text{Ext}_H^1(M', M) = 0$. This shows that $M'$ is rigid. Applying $\text{Hom}_H(-, E_1)$ to the sequence (6.2) yields that $\text{Hom}_H(M', E_1) \neq 0$. Since $M'$ is locally free and rigid, the minimality of $M$ implies that $M' \cong E_n^r \oplus U$ for some $r \geq 0$.
and some locally free and rigid module $U$ with $\text{Hom}_H(U, E_1) = 0$. This yields short exact sequences

$$0 \to U \xrightarrow{f} M \xrightarrow{g} V \to 0 \quad (6.3)$$

and

$$0 \to E_1^r \to V \to e_n M \to 0 \quad (6.4)$$

where $f$ is the obvious embedding, and $g$ is the obvious projection onto $V := \text{Cok}(f)$. Note that $V$ is also locally free, and that $\text{Hom}_H(U, E_n) = 0$. Applying $\text{Hom}_H(U, -)$ to $(6.4)$ implies that $\text{Hom}_H(U, V) = 0$. Now we apply $\text{Hom}_H(M, -)$ and $\text{Hom}_H(-, V)$ to $(6.3)$ and get similarly as before that $V$ is rigid. Applying $\text{Hom}_H(-, E_1)$ to $(6.3)$ implies $\text{Hom}_H(M, E_1) \cong \text{Hom}_H(V, E_1) \neq 0$. By the minimality of $M$ it now follows that $U = 0$ and $V = M$. Thus we have $n = 2$.

Since $M$ is locally free, Proposition 6.3 implies that it has a minimal projective resolution of the form

$$0 \to P'' \to P' \to M \to 0.$$  

Since $M$ is rigid and locally free, we know that $M$ is also $\tau$-rigid, see Corollary 6.6. Thus by [AIR, Proposition 2.5] we get that $\text{add}(P') \cap \text{add}(P'') = 0$. (It can be easily checked that [AIR, Proposition 2.5] is true for arbitrary ground fields.) We obviously have $\text{Hom}_H(M, E_2) \neq 0$, since 2 is a source in $Q^o(C, \Omega)$. By assumption we have $\text{Hom}_H(M, E_1) \neq 0$. This implies that $\text{Hom}_H(M, S_i) \neq 0$ for the simple $H$-modules $S_i$, where $i = 1, 2$. Thus $P'$ contains both $P_1$ and $P_2$ as direct summands. Since $\text{add}(P') \cap \text{add}(P'') = 0$, it follows that $P'' = 0$. In other words, $M$ is projective. But $P = E_1$ is the only indecomposable projective $H$-module with $\text{Hom}_H(P, E_1) \neq 0$. Thus $M$ contains a direct summand isomorphic to $E_1$, a contradiction.

Altogether we proved that $\text{Hom}_H(M, E_1) = 0$ for any locally free and rigid $H$-module $M$, which does not have a direct summand isomorphic to $E_1$. In this case, we have $\text{top}_1(M) = 0$ and $M_{1, \text{in}}$ is surjective. Thus $F^+_1(M)$ is rigid by Corollary 6.3 and $F^+_1(M)$ is locally free by Lemma 6.5. The corresponding statement for $F^-_n(M)$ is proved dually. This finishes the proof. 

6.3. Coxeter functors. Let $Q = Q(C, \Omega)$.

Given a $+$-admissible sequence $(i_1, \ldots, i_n)$ for $(C, \Omega)$ let

$$C^+ := F^+_1 \circ \cdots \circ F^+_i : \text{rep}(H) \to \text{rep}(H).$$

Dually, one defines $-$-admissible sequences $(j_1, \ldots, j_n)$ and $C^- := F^-_{j_n} \circ \cdots \circ F^-_{j_1}$. We call $C^+$ and $C^-$ Coxeter functors. Similarly as in the classical case one proves the following result, compare [BCP].

Lemma 6.7. The functors $C^+$ and $C^-$ do not depend on the chosen admissible sequences for $(C, \Omega)$.

The next lemma is a consequence of Proposition 6.1, Corollary 6.2 and Lemma 6.5.

Lemma 6.8. Let $M$ be an indecomposable locally free $H$-module. Let $(i_1, \ldots, i_n)$ be a $+$-admissible sequence for $(C, \Omega)$. Assume that $F^+_{i_s} \cdots F^+_{i_1}(M)$ is locally free and non-zero for some $1 \leq s \leq n$. Then we have

$$F^+_{i_k} \cdots F^+_{i_1}(M) \cong F^-_{i_k+1} \cdots F^-_{i_s} F^+_{i_s} \cdots F^+_{i_{k+1}} F^+_{i_k} \cdots F^+_{i_1}(M)$$

for $1 \leq k \leq s - 1$, and $F^+_{i_k} \cdots F^+_{i_1}(M)$ is indecomposable and locally free for $1 \leq k \leq s$.

There is also an obvious dual of Lemma 6.8.
7. The algebras \( H \) and \( \Pi \) are tensor algebras

Let \( A \) be a \( K \)-algebra, and let \( M = \_A M _A \) be an \( A \-A \)-bimodule. The tensor algebra \( T _A (M) \) is defined as

\[
T _A (M) := \bigoplus _{k \geq 0} M ^{\otimes k}
\]

where \( M ^0 := A \), and \( M ^{\otimes k} \) is the \( k \)-fold tensor product of \( M \) for \( k \geq 1 \). The multiplication of \( T _A (M) \) is defined as follows: For \( r, s \geq 1 \), \( m _i, m _i ^' \in M \) and \( a, a ^' \in A \) let

\[
(m _1 \otimes \cdots \otimes m _r) \cdot (m _1 ^' \otimes \cdots \otimes m _s ^') := (m _1 \otimes \cdots \otimes m _r \otimes m _1 ^' \otimes \cdots \otimes m _s ^')
\]

and

\[
a(m _1 \otimes \cdots \otimes m _r)a ^' := (am _1 \otimes \cdots \otimes m _r a ^').
\]

Recall that the modules over a tensor algebra \( T _A (M) \) are given by the \( A \)-module homomorphisms \( M \otimes _A X \to X \), where \( X \) is an \( A \)-module.

Let \( A \) be a \( K \)-algebra, \( A _0 \) a subalgebra and \( A _1 \) an \( A _0 \-A _0 \)-subbimodule of \( A \). Following [BSZ] we say that \( A \) is freely generated by \( A _1 \) over \( A _0 \) if the following holds: For every \( K \)-algebra \( B \) and any pair \( (f _0, f _1) \) with \( f _0 : A _0 \to B \) an algebra homomorphism, and \( f _1 : A _1 \to B \) an \( A _0 \-A _0 \)-bimodule homomorphism (with the \( A _0 \-A _0 \)-module structure on \( B \) given by \( f _0 \)) there exists a unique \( K \)-algebra homomorphism \( f : A \to B \) which extends \( f _0 \) and \( f _1 \). The following two lemmas can be found in [BSZ, Section 1].

**Lemma 7.1.** For any \( K \)-algebra \( A \) and any \( A \-A \)-bimodule \( M \) the tensor algebra \( T _A (M) \) is freely generated by \( M \) over \( A \).

**Lemma 7.2.** Let \( A \) be a \( K \)-algebra which is freely generated by \( A _1 \) over \( A _0 \). Then \( A \) is isomorphic to the tensor algebra \( T _{A _0} (A _1) \).

Let \( Q \) be a finite quiver, and let \( w : Q _1 \to \{0, 1\} \) be a map assigning to each arrow of \( Q \) a degree. Then the path algebra \( K Q \) is naturally \( \mathbb{N} \)-graded: Each path gets as degree the sum of the degrees of its arrows. By definition the paths of length 0 have degree 0. Let \( r _1, \ldots, r _m \) be a set of relations for \( K Q \) which are homogeneous with respect to this grading. Suppose that there is some \( 1 \leq l \leq m \) such that \( \deg (r _i) = 0 \) for \( 1 \leq i \leq l \) and \( \deg (r _j) = 1 \) for \( l + 1 \leq j \leq m \).

Let \( A := K Q / I \), where \( I \) is the ideal generated by \( r _1, \ldots, r _m \). Clearly, \( A \) is again \( \mathbb{N} \)-graded. Let \( A _i \) be the subspace of elements with degree \( i \). Observe that \( A _1 \) is naturally an \( A _0 \-A _0 \)-bimodule. Now Lemmas 7.1, 7.2 yield the following result.

**Proposition 7.3.** \( A \) is isomorphic to the tensor algebra \( T _{A _0} (A _1) \).

As before, let \( H = H (C, D, \Omega) \). Define

\[
S := \prod _{i=1} ^n H _i \text{ and } B := \bigoplus _{(i,j) \in \Omega} i H _j.
\]

Clearly, \( B \) is an \( S \-S \)-bimodule.

**Proposition 7.4.** \( H \cong T _S (B) \).

**Proof.** The algebra \( H \) is graded by defining \( \deg (\varepsilon _i) := 0 \) and \( \deg (\alpha _{ij} ^{(g)}) = 1 \) for all \( (i,j) \in \Omega \) and all \( g \). The defining relations for \( H \) are homogeneous, \( S \) is the subalgebra of elements of degree 0, and \( B \) is the subspace of elements of degree 1. Now we can apply Proposition 7.3. \( \square \)
Let $\Pi = \Pi(C, D, \Omega)$ be the preprojective algebra. Define $\deg(\varepsilon_i) := 0$ for all $i$, and for $(i, j) \in \Omega$ let $\deg(\alpha_{ij}^{(g)}) := 0$ and $\deg(\alpha_{ji}^{(g)}) := 1$ for all $g$. Let

$$\Pi_1 := \Pi(C, D, \Omega)$$

be the subspace of $\Pi$ consisting of the elements of degree 1. Note that $\Pi_1$ is an $H-H$-bimodule. Again we can apply Proposition 7.3 and get the following result.

**Proposition 7.5.** $\Pi \cong T_H(\Pi_1)$. □

Define

$$\overline{B} := \bigoplus_{(i,j) \in \Omega} i H_j.$$

Next, for $1 \leq i \leq n$ let

$$\rho_i := \sum_{j \in \Omega(-,i)} \text{sgn}(i,j) \sum_{b \in j L_i} b \otimes b \in T_S(\overline{B}).$$

Every $b \in j L_i$ is of the form $b = \alpha_{ji}^{(g)} \varepsilon_{i}^{f} - f^{j}_{i} - f^{j}_{i}$ for some $0 \leq f \leq f^{j}_{i} - 1$ and $1 \leq g \leq g_{ij}$. Then $b^* \in i R_j$ is equal to $\varepsilon_{i}^{f} \alpha_{ij}^{(g)}$. Thus $\rho_i$ translates to the defining relation (P3)

$$\sum_{j \in \Omega(-,i)} \sum_{g=1}^{g_{ij}} \sum_{f=0}^{f^{j}_{i} - 1} \text{sgn}(i,j) \varepsilon_{i}^{f} \alpha_{ij}^{(g)} \alpha_{ji}^{(g)} \varepsilon_{i}^{f} = 0.$$

of $\Pi$.

The algebra $T_S(\overline{B})/(\rho_1, \ldots, \rho_n)$ is an analogue of Dlab and Ringel’s [DR3] definition of a preprojective algebra of a modulated graph.

**Proposition 7.6.** $\Pi \cong T_S(\overline{B})/(\rho_1, \ldots, \rho_n)$.

**Proof.** Similarly as in the proof of Proposition 7.3 one shows that $T_S(\overline{B})$ is isomorphic to the path algebra $K\overline{Q}$ modulo the defining relations (P1) and (P2) of $\Pi$.

Let $M$ be a module over the tensor algebra $T_S(\overline{B})$. Then $M$ is defined by the structure maps

$$M_{ij} : i H_j \otimes j M_j \rightarrow M_i$$

for each $(i, j) \in \Omega$. This yields maps

$$M_{iji} := M_{ij} \circ (\text{id}_{i} H_j \otimes j M_i) : i H_j \otimes j H_i \otimes i M_i \rightarrow M_i.$$

Now $M$ is a module over $T_S(\overline{B})/(\rho_1, \ldots, \rho_n)$ if and only if for each vertex $i$ and each $m \in M_i$ we have $\rho_i m = 0$. This is equivalent to

$$\sum_{j \in \Omega(-,i)} \text{sgn}(i, j) M_{iji}(\sum_{b \in j L_i} b^* \otimes b \otimes i m) = 0.$$

It follows from the definitions that

$$\sum_{j \in \Omega(-,i)} \text{sgn}(i, j) M_{iji}(\sum_{b \in j L_i} b^* \otimes b \otimes i m) = (M_{i,in} \circ M_{i,out})(m).$$

Now Proposition 5.2 yields the result. □
8. Projective resolutions

8.1. The trace pairing for homomorphisms between free $H_i$-modules. For each $i$ we have the $K$-linear map

$$t_i^{\max}: H_i \rightarrow K$$

defined by

$$\sum_{j=0}^{c_i-1} \lambda_j \varepsilon_i^j \mapsto \lambda_{c_i-1}.$$ 

For finitely generated free $H_i$-modules $U$ and $V$ the trace pairing is the map

$$\text{tr} := \text{tr}_{U,V}: \text{Hom}_{H_i}(U,V) \times \text{Hom}_{H_i}(V,U) \rightarrow K$$

defined by

$$(f,g) \mapsto t_i^{\max}(\text{tr}_{H_i}(f \circ g)).$$

This is clearly a non-degenerate $K$-bilinear form.

Let $W$ be another finitely generated free $H_i$-module, and let $f \in \text{Hom}_{H_i}(V,W)$. The trace pairings $\text{tr}_{U,V}$ and $\text{tr}_{U,W}$ induce isomorphisms

$$D \text{Hom}_{H_i}(U,V) \rightarrow \text{Hom}_{H_i}(U,W) \quad \text{and} \quad D \text{Hom}_{H_i}(U,W) \rightarrow \text{Hom}_{H_i}(W,U).$$

The following lemma is easily verified:

**Lemma 8.1.** Under the isomorphisms above, the transpose of the map

$$\text{Hom}_{H_i}(U,f): \text{Hom}_{H_i}(U,V) \rightarrow \text{Hom}_{H_i}(U,W)$$

defined by $g \mapsto f \circ g$ gets identified with the map

$$\text{Hom}_{H_i}(f,U): \text{Hom}_{H_i}(W,U) \rightarrow \text{Hom}_{H_i}(V,U)$$

defined by $g' \mapsto g' \circ f$. \hfill $\Box$

Let $(a_1, \ldots, a_p)$ and $(b_1, \ldots, b_q)$ be fixed $H_i$-bases of $U$ and $V$, respectively. Define homomorphisms $\varepsilon_i^k E_i^m \in \text{Hom}_{H_i}(U,V)$ by

$$(\varepsilon_i^k E_i^m)(a_V) := \delta_{l',m} \varepsilon_i^k b_m.$$ 

Thus

$$(\varepsilon_i^k E_i^m \mid 0 \leq k \leq c_i - 1, \ 1 \leq l \leq p, \ 1 \leq m \leq q)$$

is a $K$-basis of $\text{Hom}_{H_i}(U,V)$. Similarly, we have a $K$-basis of $\text{Hom}_{H_i}(V,U)$ denoted by

$$(\varepsilon_i^k E_i^m \mid 0 \leq k \leq c_i - 1, \ 1 \leq m \leq q, \ 1 \leq l \leq p).$$

The isomorphism $D \text{Hom}_{H_i}(U,V) \rightarrow \text{Hom}_{H_i}(V,U)$ induced by the trace pairing $\text{tr}_{U,V}$ maps the dual basis $((\varepsilon_i^k E_i^m)^*)$ of $D \text{Hom}_{H_i}(U,V)$ to a permutation of the basis $(\varepsilon_i^k E_i^l)$ of $\text{Hom}_{H_i}(V,U)$, namely

$$(\varepsilon_i^k E_i^m)^* \mapsto \varepsilon_i^{c_i-1-k} E_i^l.$$ 

Indeed, one calculates easily that

$$\text{tr}_{H_i}(\varepsilon_i^k E_i^m \circ \varepsilon_i^{k'} E_i^{m'}) = \delta_{l',m} \delta_{m,m'} \varepsilon_i^{k+k'},$$

hence

$$\text{tr}_{U,V}(\varepsilon_i^k E_i^m, \varepsilon_i^{k'} E_i^{m'}) = \delta_{l',m} \delta_{m,m'} \delta_{k',c_i-1-k}.$$
8.2. Adjunction and trace pairing. Let now

\[ M = \bigoplus_{s=1}^{p} H_s a_s \quad \text{and} \quad N = \bigoplus_{t=1}^{q} H_j b_t \]

be a free \( H_i \)-module and a free \( H_j \)-module of finite ranks. Recall from \( \mathbf{5.1} \) the isomorphisms of \( K \)-vector spaces

\[ \text{ad}_{ji} = \text{ad}_{ji}(M, N) : \text{Hom}_{H_j}(j H_i \otimes_i M, N) \to \text{Hom}_{H_i}(M, j H_j \otimes_j N) \]

and

\[ \text{ad}_{ij} = \text{ad}_{ij}(N, M) : \text{Hom}_{H_i}(i H_j \otimes_j N, M) \to \text{Hom}_{H_j}(N, j H_i \otimes_i M). \]

Clearly,

\[ (\alpha_{ji}^g f_i \otimes a_s | 1 \leq g \leq g_{ij}, \ 0 \leq f \leq f_{ji} - 1, \ 1 \leq s \leq p) \]

is an \( H_j \)-basis of the free \( H_j \)-module \( j H_i \otimes_i M \). Similarly,

\[ (\alpha_{ij}^g f_j \otimes b_t | 1 \leq g \leq g_{ij}, \ 0 \leq f \leq f_{ij} - 1, \ 1 \leq t \leq q) \]

is an \( H_i \)-basis of the free \( H_i \)-module \( i H_j \otimes_j N \). Let

\[ (\varepsilon_u^t E_{g,f,s} | 0 \leq u \leq c_j - 1, \ 1 \leq g \leq g_{ij}, \ 0 \leq f \leq f_{ji} - 1, \ 1 \leq s \leq p, \ 1 \leq t \leq q) \]

be the \( K \)-basis of \( \text{Hom}_{H_j}(j H_i \otimes_i M, N) \) defined by

\[ (\varepsilon_u^t E_{g,f,s}) (\alpha_{ij}^{(g)} f_i \otimes a_s) := \delta_{f,f'} \delta_{g,g'} \delta_{s,s'} \varepsilon_{ij}^{u t} b_t. \]

Similarly, let

\[ (\varepsilon_v^t E_{g,f,s} | 0 \leq v \leq c_i - 1, \ 1 \leq g \leq g_{ij}, \ 0 \leq f \leq f_{ij} - 1, \ 1 \leq s \leq p, \ 1 \leq t \leq q) \]

be the \( K \)-basis of \( \text{Hom}_{H_i}(i H_j \otimes_j N) \) defined by

\[ (\varepsilon_v^t E_{g,f,s}) (a_s') := \delta_{s,s'} \varepsilon_{ij}^{v t} (\alpha_{ij}^{(g)} f_i \otimes b_t). \]

**Lemma 8.2.** We have

\[ \text{ad}_{ji}(\varepsilon_u^t E_{g,f,s}) = \varepsilon_{fi}^{t-g+f+q_{ij}} E_{a_s'} \]

where \( q_{ij} \) and \( r_{ij} \) are the quotient and the remainder of the division of \( u \) by \( f_{ij} \).

**Proof.** By definition of \( \text{ad}_{ji} \) one has

\[ (\text{ad}_{ji}(\varepsilon_u^t E_{g,f,s}))(a_s') = \sum_{f',g'} (\alpha_{ji}^{(g') f'} f_i) \varepsilon_{ij}^{u t} E_{g,f,s} (\alpha_{ij}^{(g') f'} f_i \otimes a_s') \]

\[ = \delta_{s,s'} (\alpha_{ij}^{(g') f'} f_i) \varepsilon_{ij}^{v t} b_t \]

\[ = \delta_{s,s'} \varepsilon_{ij}^{f_{ij} - 1 - f} \alpha_{ij}^{(g') f'} \varepsilon_{ij}^{u t} b_t \]

\[ = \delta_{s,s'} \varepsilon_{ij}^{f_{ij} - 1 - f + q_{ij}} \alpha_{ij}^{(g') r_{ij} t} \]

\[ = (\varepsilon_{fi}^{t-g+f+q_{ij}} E_{a_s'}). \]

\[ \square \]

Similarly we define \( K \)-bases \( (\varepsilon_u^t E_{g,f,s}) \) and \( (\varepsilon_v^t E_{g,f,t}) \) of \( \text{Hom}_{H_j}(N, j H_i \otimes M) \), and \( \text{Hom}_{H_j}(i H_j \otimes_j N, M) \), respectively, by

\[ (\varepsilon_u^t E_{g,f,s})(b_{v'}) := \delta_{t,t'} \varepsilon_{ij}^{u t} (\alpha_{ij}^{(g') f_i} \otimes a_s), \]

\[ (\varepsilon_v^t E_{g,f,t})(\alpha_{ij}^{(g') f_i} \otimes b_{v'}) := \delta_{f,f'} \delta_{g,g'} \varepsilon_{ij}^{v t} a_s. \]

The analogue of Lemma \( \mathbf{8.2} \) is:
Lemma 8.3. We have
\[ \text{ad}_{ij}(\xi^t_iE_{g,f,s}) = \xi^{f_{ij} - 1 - f + f_{ji}q_{ji}}_jE_{ij}^g_{r_{ij},s}, \]
where \( q_{ji} \) and \( r_{ji} \) are the quotient and the remainder of the division of \( v \) by \( f_{ji} \).

Proposition 8.4. Let \( \text{ad}^*_i : D\text{Hom}_{H_i}(N, jH_i \otimes_i M) \to D\text{Hom}_{H_i}(iH_j \otimes_j N, M) \) denote the transpose of \( \text{ad}_{ij} \). We have a commutative diagram
\[
\begin{array}{ccc}
\text{Hom}_{H_i}(jH_i \otimes_i M, N) & \xrightarrow{\text{ad}_{ij}} & \text{Hom}_{H_i}(M, iH_j \otimes_j N) \\
\downarrow & & \downarrow \\
D\text{Hom}_{H_i}(N, jH_i \otimes_i M) & \xrightarrow{\text{ad}^*_i} & D\text{Hom}_{H_i}(iH_j \otimes_j N, M)
\end{array}
\]
where the vertical arrows are the isomorphisms induced by the trace pairings.

Proof. By Lemma 8.2 we have \( \text{ad}_{ij}(\xi^u_jE_{g,f,s}) = \xi^{f_{ji} - 1 - f + f_{ji}q_{ji}}_iE_{g,r_{ij},t}. \) This element of \( \text{Hom}_{H_i}(M, jH_j \otimes_j N) \) is mapped via the trace pairing to the linear form
\[
\psi \mapsto t_{ij}^\text{max}(\text{tr}_{H_i}(\xi^{f_{ij} - 1 - f + f_{ji}q_{ji}}_iE_{g,r_{ij},t} \circ \psi))
\]
on \( \text{Hom}_{H_i}(iH_j \otimes jN, M) \). It is easy to check that
\[
t_{ij}^\text{max}(\text{tr}_{H_i}(\xi^{u_j}_jE_{g,f,s} \circ \text{ad}_{ij} \circ \xi^{v_j}_iE_{g',f',t'})) = \delta_{g,s,t}^{g',t'}(\delta_{s,t}^{r_{ij}}(\delta_{f,f'}^{v_{ij}}(\delta_{v+v',c_i-1}))).
\]
Thus, taking \( \psi = \xi^{u_j}_jE_{g',f',t'} \), the right-hand side of (8.1) is equal to
\[
\delta_{g,s,t}^{g',t'}(\delta_{s,t}^{r_{ij}}(\delta_{f,f'}^{v_{ij}}(\delta_{v+v',c_i-1}))).
\]
We need to show that (8.2) is equal to
\[
t_{ij}^\text{max}(\text{tr}_{H_i}(\xi^{u_j}_jE_{g,f,s} \circ \text{ad}_{ij} \circ \xi^{v_j}_iE_{g',f',t'})).
\]
Using Lemma 8.3 we find that (8.3) is equal to
\[
\delta_{g,s,t}^{g',t'}(\delta_{s,t}^{r_{ij}}(\delta_{f,f'}^{v_{ij}}(\delta_{v+v',c_i-1}))).
\]
Now, if \( v + f_{ji} - f + f_{ij}q_{ij} = c_i \) then \( r_{ji} = f \) because \( c_i \) is divisible by \( f_{ji} \). Similarly, if \( u + f_{ij} - f' + f'_{ij}q_{ij} = c_j \) then \( r_{ij} = f' \). It follows that (8.3) is indeed equal to (8.5).

8.3. Projective resolutions of \( H(C, D, \Omega) \)-modules.

Proposition 8.5. We have a short exact sequence of \( H \)-\( H \)-bimodules
\[
P_\bullet : 0 \to \bigoplus_{(j,i) \in \Omega} He_j \otimes_j jH_i \otimes_i e_i H \xrightarrow{d} \bigoplus_{k=1}^n He_k \otimes_k e_k H \xrightarrow{\text{mult}} H \to 0
\]
where
\[
d(p \otimes_j h \otimes_i q) := ph \otimes_i q - p \otimes_j hq.
\]

Proof. We know that \( H = T_S(B) \). The sequence \( P_\bullet \) is isomorphic to the sequence
\[
0 \to H \otimes S B \otimes S H \xrightarrow{d} H \otimes S H \xrightarrow{\text{mult}} H \to 0
\]
of \( H \)-\( H \)-bimodules, where \( d(h \otimes b \otimes h') := (hb \otimes h' - h \otimes bh') \). Now the statement follows from [Sch] Theorems 10.1 and 10.5.]
The components of $P_\bullet$ are projective as left $H$-modules and as right $H$-modules. However, the components are not projective as $H$-$H$-bimodules. (Except, if $S$ is semisimple, then the first two components are in fact projective bimodules.) In any case, viewed as a short exact sequence of left or right modules, $P_\bullet$ splits as an exact sequence of projective modules.

**Corollary 8.6.** If $M \in \text{rep}_{\mathbb{A},f}(H)$, then $P_\bullet \otimes_H M$ is a projective resolution of $M$. Explicitly, $P_\bullet \otimes_H M$ looks as follows

$$0 \to \bigoplus_{(j,i) \in \Omega} He_j \otimes jH_i \otimes M_i \xrightarrow{d \otimes M} \bigoplus_{k=1}^n He_k \otimes M_k \xrightarrow{\text{mult}} M \to 0 \quad (8.7)$$

where

$$(d \otimes M)(p \otimes j h \otimes_i m) = ph \otimes_i m - p \otimes_j M_{ji}(h \otimes_i m).$$

(Here $M_{ji} : jH_i \otimes M_i \to M_j$ is the $H_j$-linear structure map of $M$ associated with $(j,i) \in \Omega).$

**Proof.** By the remarks above, $P_\bullet \otimes_H M$ is always exact. If $M$ is locally free, then $e_kH \otimes_H M = e_kM$ and $jH_i \otimes e_iH \otimes_H M = jH_i \otimes M_i$ are free $H_k$- resp. $H_j$-modules. Thus the relevant components of $P_\bullet \otimes_H M$ are indeed projective. \qed

### 8.4. Projective resolutions of $\Pi(C, D)$-modules

Let $H = H(C, D, \Omega)$. Recall that we have defined the $S$-$S$-bimodule $B := \bigoplus_{(i,j) \in \Omega} iH_j,$

and we have

$$\Pi = \Pi(C, D) = T_S(B)/J,$$

where $J$ is the ideal of $T_S(B)$ which is generated by the elements $\rho_i \in B \otimes_S B$ for $i \in I$. For the next result we follow closely the ideas from [CBSH, Lemma 3.1].

**Proposition 8.7.** There is an exact sequence of $\Pi$-$\Pi$-bimodules

$$P_\bullet : \bigoplus_{i \in I} \Pi e_i \otimes e_i \Pi \xrightarrow{f} \bigoplus_{(j,i) \in \Omega} \Pi e_j \otimes jH_i \otimes_i e_i \Pi \xrightarrow{g} \bigoplus_{i \in I} \Pi e_i \otimes e_i \Pi \xrightarrow{h} \Pi \to 0 \quad (8.8)$$

where

$$f(e_i \otimes e_i) := \rho_i \otimes e_i + e_i \otimes \rho_i,$$

$$g(e_j \otimes h \otimes e_i) := he_i \otimes e_i - e_j \otimes e_j h,$$

$$h(m \otimes m') := mm'.$$

**Proof.** Observe first, that the above complex can be written more compactly as

$$\Pi \otimes_S \Pi \xrightarrow{f} \Pi \otimes_S \overline{B} \otimes_S \Pi \xrightarrow{g} \Pi \otimes_S \Pi \xrightarrow{h} \Pi \to 0$$

Note that we have a surjective $\Pi$-$\Pi$-bimodule homomorphism

$$\bigoplus_{i=1}^n \Pi e_i \otimes e_i \Pi \xrightarrow{\tau} J/J^2$$

defined by $e_i \otimes e_i \mapsto \rho_i$. Moreover, we have a canonical map

$$J/J^2 \xrightarrow{\text{can}} \Pi \otimes_S \overline{B} \otimes_S \Pi$$
given by \( \mathfrak{T} \mapsto \ddot{x}_1 \otimes 1 + 1 \otimes \ddot{x}_r \) coming from the compositions of

\[
J \xrightarrow{i_{J,\ell}} \mathop{\bigoplus}_{k \geq 1} (B^\otimes k \otimes_S B) \xrightarrow{\text{proj} \otimes \text{id}} \Pi \otimes_S B
\]

and

\[
J \xrightarrow{i_{J,r}} \mathop{\bigoplus}_{k \geq 1} (B \otimes_S B^\otimes k) \xrightarrow{id \otimes \text{proj}} B \otimes_S \Pi,
\]

respectively, where the maps \( i_{J,\ell} \) and \( i_{J,r} \) are the obvious inclusions. Note that both compositions vanish on \( J^2 \). It is easy to see that \( f = \text{can} \circ r \). Thus we only have to show that the sequence

\[
J/J^2 \xrightarrow{\text{can}} \Pi \otimes_S B \otimes_S B \xrightarrow{u} \Pi \otimes_S \Pi \xrightarrow{\text{mult}} \Pi \to 0
\]

where \( u(1 \otimes b \otimes 1) := b \otimes 1 - 1 \otimes b \) is exact. This is a special case of a combination of results by Schofield \([\text{Sch}, \text{Theorems 10.1, 10.3, 10.5}]\). \( \square \)

**Corollary 8.8.** For each \( M \in \text{rep}_{I,\ell}(\Pi) \) the complex \( P_\bullet \otimes_\Pi M \) is the beginning of a projective resolution of \( M \).

**Proof.** The components of \( P_\bullet \) are projective as left and as right modules. In fact, for example \( H^m_i \otimes e_i \Pi \cong (e_i \Pi)^m \) is a projective right \( \Pi \)-module. Now, \( jH_i \) is a free right \( H_i \)-module and \( H_i e_j \) is a free right \( H_j \)-module and thus \( H_i e_j \otimes_j H_i \) is also a free right \( H_i \)-module. Altogether, \( H_i e_j \otimes_j H_i \otimes_j \Pi \) is a projective right \( \Pi \)-module. A similar argument shows that \( H_i e_j \otimes H_i \Pi \) is projective as a right \( \Pi \)-module. Thus as a sequence of right modules the sequence \( P_\bullet \) splits. This implies that the sequence \( P_\bullet \otimes_\Pi M \) is exact. Now, if \( M \in \text{rep}_{I,\ell}(\Pi) \), then the relevant components of \( P_\bullet \otimes_\Pi M \) are evidently projective. \( \square \)

Let us write the complex \( P_\bullet \otimes_\Pi M \) explicitly:

\[
\bigoplus_{i \in I} \Pi e_i \otimes e_i M \xrightarrow{f_M} \bigoplus_{(j,i) \in \Pi} \Pi e_j \otimes_j H_i \otimes_i e_i M \xrightarrow{g_M} \bigoplus_{i \in I} \Pi e_i \otimes e_i M \xrightarrow{h_M} M \to 0 \tag{8.9}
\]

and the maps \( f_M, g_M, h_M \) act on generators as follows:

\[
f_M(e_j \otimes m_j) = \sum_{i \in \Omega(j, -)} (l^* \otimes l \otimes m_j + e_j \otimes l^* \otimes M_{ij}(l \otimes m_j))
\]

\[
- \sum_{k \in \Omega(j, -) \atop r \in jH_k} (r \otimes r^* \otimes m_j + e_j \otimes r \otimes M_{kj}(r^* \otimes m_j)),
\]

\[
g_M(e_i \otimes h \otimes m_j) = h \otimes m_j - e_i \otimes M_{ij}(h \otimes m_j),
\]

\[
h_M(e_i \otimes m_i) = m_i.
\]

**8.5. Symmetry of extension groups.** Let \( M = (M_i, M_{ij}, M_{ji}) \) and \( N = (N_i, N_{ij}, N_{ji}) \) be in \( \text{rep}_{I,\ell}(\Pi) \). Let \( Q_\bullet(M, N) \) be defined by

\[
\bigoplus_{k \in I} \text{Hom}_{H_k}(M_k, N_k) \xrightarrow{\text{proj}_{M,N}} \bigoplus_{(i,j) \in \Pi} \text{Hom}_{H_i}(iH_j \otimes_j M_j, N_i) \xrightarrow{g_{M,N}} \bigoplus_{k \in I} \text{Hom}_{H_k}(M_k, N_k)
\]

where \( \text{proj}_{M,N} \) is defined by

\[
\left( \text{proj}_{M,N}([\psi_{ij}]_{(i,j) \in \Pi}) \right)_k := \sum_{j \in \Pi(-,-k)} \text{sgn}(j, k)(N_{kj} \circ \text{adj}_{j,k}([\psi_{ij}]_{(i,j) \in \Pi}) - \psi_{k,j} \circ \text{adj}_{j,k}(M_{jk}))
\]

\[
\left( \text{proj}_{M,N}([\psi_{ij}]_{(i,j) \in \Pi}) \right)_k := \sum_{j \in \Pi(-,-k)} \text{sgn}(j, k)(N_{kj} \circ \text{adj}_{j,k}([\psi_{ij}]_{(i,j) \in \Pi}) - \psi_{k,j} \circ \text{adj}_{j,k}(M_{jk}))
\]
and \( \tilde{g}_{M,N} \) is defined by

\[
(\tilde{g}_{M,N}((\phi_k)_{k \in I}))_{(i,j)} := N_{ij} \circ (\text{id}_{H_j} \otimes \phi_j) - \phi_i \circ M_{ij}.
\]

Let \( \tilde{f}_{M,N}^* \) denotes the dual of \( f_{M,N} \) under the trace pairing.

Define

\[
ad_{N,M} := \bigoplus_{(i,j) \in \Omega} \text{sgn}(i,j)\, \text{ad}_{ji}:
\]

\[
\bigoplus_{(i,j) \in \Omega} \text{Hom}_{H_i}(jH_i \otimes_i N_i, M_j) \to \bigoplus_{(i,j) \in \Omega} \text{Hom}_{H_i}(N_{i}, H_j \otimes_j M_j).
\]

We know that \( ad_{N,M} \) is an isomorphism.

**Lemma 8.9.** For \( M, N \in \text{rep}_{l,\Pi}(\Pi) \) the following hold:

(a) The complex \( \text{Hom}_{\Pi}(P_\cdot \otimes_{\Pi} M, N) \) is isomorphic to \( Q_\bullet(M, N) \).

(b) We have \( \tilde{f}_{M,N}^* = ad_{N,M} \circ \tilde{g}_{N,M}. \)

**Proof.** Part (a) is straightforward, and for (b) observe that \( \tilde{f}_{M,N}^* \) is a map

\[
\tilde{f}_{M,N}^*: \bigoplus_{k=1}^{n} \text{Hom}_{H_i}(N_k, M_k) \to \bigoplus_{(i,j) \in \Omega} \text{Hom}_{H_i}(N_{i}, H_j \otimes_j N_j).
\]

According to our discussion of the trace pairing in Sections 8.1 and 8.2, we have

\[
(\tilde{f}_{M,N}^*((\lambda_k)_{k}))_{ij} = \text{sgn}(i,j)(\text{ad}_{i,j}^*(\lambda_j \circ N_{ji}) - \text{ad}_{j,i}(M_{ji}) \circ \lambda_i)
\]

\[
= \text{sgn}(i,j)(\text{ad}_{j,i}(\lambda_j \circ N_{ji} - M_{ji} \circ (\text{id}_{H_i} \otimes \lambda_i)))
\]

where the second equality follows from Proposition 8.3 and the third equality is just the definition of \( \text{ad}_{ji}. \)

**Proposition 8.10.** For \( M, N \in \text{rep}_{l,\Pi}(\Pi) \) we have the following functorial isomorphisms:

(a) \( \text{Ker}(\tilde{g}_{M,N}) = \text{Hom}_{\Pi}(M, N); \)

(b) \( \text{Ker}(\tilde{f}_{M,N})/\text{Im}(\tilde{g}_{M,N}) \cong \text{Ext}_{\Pi}^1(M, N); \)

(c) \( \text{Cok}(\tilde{f}) \cong D \text{Hom}_{\Pi}(N, M). \)

**Proof.** Part (a) follows directly from the definition of the complex \([8.10]\). Part (b) follows from the functorial isomorphism of \([8.10]\) with \( \text{Hom}_{\Pi}(P_\cdot \otimes_{\Pi} M, N) \). For (c) we just note that by (a) and Lemma 8.9 (b) we have \( \text{Ker}(\tilde{f}_{M,N}^*) \cong \text{Hom}_{\Pi}(N, M). \)

**Theorem 8.11.** For locally free \( \Pi \)-modules \( M \) and \( N \) there is a functorial isomorphism

\[
D \text{Ext}_{\Pi}^1(M, N) \cong \text{Ext}_{\Pi}^1(N, M).
\]

Moreover, we have

\[
\dim \text{Ext}_{\Pi}^1(M, N) = \dim \text{Hom}_{\Pi}(M, N) + \dim \text{Hom}_{\Pi}(N, M) - (M, N)_H.
\]

**Proof.** By Lemma 8.9 (b) and Proposition 8.3 we have

\[
\tilde{f}_{N,M} = \tilde{g}_{M,N}^* \circ \text{ad}_{M,N}^* = \tilde{g}_{N,M}^* \circ \text{ad}_{N,M}.
\]

Thus,

\[
(\text{id}_{\text{Hom}_{H_k}(N_k, M_k)}, \text{ad}_{N,M}, \text{id}_{\text{Hom}_{H_k}(N_k, M_k)})
\]
is an isomorphism between $Q_{\bullet}(N,M)$ and $Q_{\bullet}(M,N)^*$. The middle cohomology of the first complex is $\text{Ext}^1_H(N,M)$ and the middle cohomology of the second one is functorially isomorphic to $D\text{Ext}^1_H(M,N)$.

Finally, we observe that by Proposition 8.10 we obtain from the complex 8.10 the equality

$$\dim \text{Hom}_H(N,M) - \dim \text{Ext}^1_H(M,N) + \dim \text{Hom}_H(M,N)$$

which is equivalent to our claim. \qed

The last statement in Theorem 8.11 generalizes Crawley-Boevey’s formula in [CB3, Lemma 1].

9. Coxeter functors and Auslander-Reiten translations

9.1. Overview. As before we fix $H = H(C,D,\Omega)$. Our aim is to compare the Coxeter functors $C^+$ and $C^-$ introduced in Section 6.3 with the Auslander-Reiten translations $\tau$ and $\tau^-$. Without loss of generality we assume that for each $(i,j) \in \Omega$ we have $i < j$. Thus,

$$C^+ = F^+_n \circ \cdots \circ F^+_1.$$ Recall that we defined the twist automorphism $T$ of $H$ by

$$T(\varepsilon_i) = \varepsilon_i, \quad T(\alpha_{ij}^{(g)}) = -\alpha_{ij}^{(g)}, \quad (i \in Q_0, (i,j) \in \Omega, 1 \leq g \leq g_{ij}).$$

The twist by $T$ defines an automorphism of $\text{rep}(H)$ which we denote also by $T$. More explicitly, for $M = (M_i, M_{ij}) \in \text{rep}(H)$ we have $(TM)_i = M_i$ and $(TM)_{ij} = -M_{ij}$.

Following [Ga3, Section 5], we will start by constructing a new algebra $\tilde{H}$ containing two subalgebras $H(0)$ and $H(1)$ canonically isomorphic to $H$. Denoting by

$$\text{Res}_a : \text{rep}(\tilde{H}) \to \text{rep}(H(a)), \quad (a \in \{0,1\})$$

the corresponding restriction functors, we will show that $C^+ \cong \text{Res}_1 \circ \text{Res}_0^*$, where

$$\text{Res}_0^* : \text{rep}(H(0)) \to \text{rep}(\tilde{H})$$

is right adjoint to $\text{Res}_0$. This will follow from a factorization

$$\text{Res}_0^* = \text{Res}_{(n-1)}^* \circ \cdots \circ \text{Res}_{(1,2)}^* \circ \text{Res}_{(0,1)}^*$$

similar to the definition of $C^+$, and from a comparison of the functors $\text{Res}_{(i-1,1)}^*$ and $F^+_i$ obtained in Lemma 9.2.

After that, we will give a different description of the adjoint functor $\text{Res}_0^*$, which will allow to show that, for $M \in \text{rep}_{1,f}(H)$, the $H$-module $\text{Res}_1 \circ \text{Res}_0^*(M)$ is the kernel of a certain map $d^*_M$. On the other hand, it follows from Corollary 8.6 that $\tau(TM)$ is the kernel of the map $D\text{Hom}_H(d \otimes TM,H)$. We will then show that, under the trace pairings, the maps $d^*_M$ and $D\text{Hom}_H(d \otimes TM,H)$ can be identified, hence

$$C^+(M) \cong \text{Res}_1 \circ \text{Res}_0^*(M) \cong \tau(TM).$$

A more detailed statement of our results will be given in Theorem 9.11 whose proof is carried out in Sections 9.3 to 9.5.
The remaining subsections present direct applications of Theorem \(9.1\). In \(9.6\) we give another description of the preprojective algebra \(\Pi = \Pi(C, D)\) as a tensor algebra. In \(9.7\) we adapt to our setting a description of the category \(\text{rep}(\Pi)\) due to Ringel in the classical case, in terms of \(H\)-module homomorphisms \(M \to TC^+(M)\). Finally, in \(9.8\) we show that the subcategory of Gorenstein-projective \(H\)-modules coincides with the kernel of the Coxeter functor \(C^+\).

9.2. An analogue of the Gabriel-Riedtmann construction. The following is an adaptation of [Ga3, Section 5] to our situation.

9.2.1. The algebra \(\tilde{H}\). To our fixed datum \((C, D, \Omega)\) we attach a new algebra \(\tilde{H}\) defined by a quiver with relations. The quiver \(\tilde{Q}\) has set of vertices

\[ \tilde{Q}_0 := \{(i, a) \mid i \in Q_0, \ a \in \{0, 1\}\}, \]

and set of arrows

\[ \tilde{Q}_1 := \{\alpha_{(i,a)(j,a)}^{(g)} : (j, a) \to (i, a) \mid (i, j) \in \Omega, \ 1 \leq g \leq g_{ij}, \ a \in \{0, 1\}\} \]

\[ \cup \ \{\alpha_{(j,0)(i,1)}^{(g)} : (i, 1) \to (j, 0) \mid (i, j) \in \Omega, \ 1 \leq g \leq g_{ij}\} \]

\[ \cup \ \{\varepsilon_{(i,a)} : (i, a) \to (i, a) \mid (i, a) \in \tilde{Q}_0\}. \]

Accordingly we put

\[ \tilde{\Omega} := \{((i, a), (j, a)), ((j, 0), (i, 1)) \mid (i, j) \in \Omega, a \in \{0, 1\}\}. \]

Let

\[ \tilde{H} := K\tilde{Q}/\tilde{I} \]

where \(\tilde{I}\) is the ideal of \(K\tilde{Q}\) defined by the following relations:

(\(\tilde{H}1\)) For each \((i, a) \in \tilde{Q}_0\) we have

\[ \varepsilon_{(i,a)}^c = 0. \]

(\(\tilde{H}2\)) For each \((i, a), (j, b)\) \(\in \tilde{\Omega}\) and each \(1 \leq g \leq g_{ij}\) we have

\[ \varepsilon_{(i,a)}^{f_{ji}} \alpha_{(i,a)(j,b)}^{(g)} = \alpha_{(i,a)(j,b)}^{(g)} \varepsilon_{(j,b)}^{f_{ij}}. \]

(\(\tilde{H}3\)) For each \(i \in Q_0\) we have

\[ \sum_{j \in \Omega(i, \_)} \sum_{g = 1}^{g_{ij}} \sum_{f = 0}^{f_{ji} - 1} \varepsilon_{(i,0)}^{f} \alpha_{(i,0)(j,0)}^{(g)} \alpha_{(j,0)(i,1)}^{(g)} \varepsilon_{(i,1)}^{f_{ji} - 1 - f} 
+ \sum_{j \in \Omega(_i, i)} \sum_{g = 1}^{g_{ij}} \sum_{f = 0}^{f_{ji} - 1} \varepsilon_{(i,0)}^{f} \alpha_{(i,0)(j,1)}^{(g)} \alpha_{(j,1)(i,1)}^{(g)} \varepsilon_{(i,1)}^{f_{ji} - 1 - f} = 0. \]

When \(C\) is symmetric and \(D\) is minimal the algebra \(\tilde{H}\) coincides with the bounded quiver denoted by \(\tilde{Q}\tilde{Q}\) in [Ga3, Section 5.3].
9.2.2. Example. Let \( H = H(C, D, \Omega) \) be defined by the quiver

\[
\begin{array}{ccc}
\varepsilon_1 & \varepsilon_2 & \varepsilon_3 \\
\alpha_{12} & 1 & \alpha_{13} \\
2 & 1 & 3
\end{array}
\]

with relations \( \varepsilon_2 = 0, \varepsilon_1^2 = \varepsilon_3^2 = 0 \) and \( \varepsilon_1 \alpha_{13} = \alpha_{13} \varepsilon_3 \). Here \( C \) is a Cartan matrix of Dynkin type \( B_3 \), and \( D \) is the minimal symmetrizer. Then \( \bar{H} \) is defined by the quiver

\[
\begin{array}{ccc}
\varepsilon_{(2,0)} & \varepsilon_{(1,1)} & \varepsilon_{(2,1)} \\
\alpha_{(1,0)(1,1)} & 1 & \alpha_{(1,1)(2,1)} \\
(2, 0) & 1 & (2, 1)
\end{array}
\]

\[
\begin{array}{ccc}
\varepsilon_{(1,0)} & \varepsilon_{(1,1)} & \varepsilon_{(3,1)} \\
\alpha_{(2,0)(1,1)} & 1 & \alpha_{(3,0)(1,1)} \\
(1, 0) & 1 & (3, 1)
\end{array}
\]

bound by the relations

\[
\begin{align*}
\varepsilon_{(2,a)} &= 0, \\
\varepsilon_{(1, a)}^2 &= \varepsilon_{(3, a)}^2 = 0, \\
\varepsilon_{(1, a)} \alpha_{(1, a)(3, a)} &= \alpha_{(1, a)(3, a)} \varepsilon_{(3, a)}, \\
\varepsilon_{(3, 0)} \alpha_{(3, 0)(1, 1)} &= \alpha_{(3, 0)(1, 1)} \varepsilon_{(1, 1)},
\end{align*}
\]

with \( a \in \{0, 1\} \), and

\[
\alpha_{(2,0)(1,1)} \alpha_{(1,1)(2,1)} = 0,
\]

\[
\begin{align*}
\varepsilon_{(1,0)} \alpha_{(1,0)(2,0)} \alpha_{(2,0)(1,1)} &+ \alpha_{(1,0)(2,0)} \alpha_{(2,0)(1,1)} \varepsilon_{(1,1)} \\
+ \varepsilon_{(1,0)} \alpha_{(1,0)(3,0)} \alpha_{(3,0)(1,1)} &+ \alpha_{(1,0)(3,0)} \alpha_{(3,0)(1,1)} \varepsilon_{(1,1)} = 0,
\end{align*}
\]

\[
\begin{align*}
\varepsilon_{(3,0)} \alpha_{(3,0)(1,1)} \alpha_{(1,1)(3,1)} &+ \alpha_{(3,0)(1,1)} \alpha_{(1,1)(3,1)} \varepsilon_{(3,1)} = 0.
\end{align*}
\]

9.2.3. \( \bar{H} \) as a quotient of a tensor algebra. It will be useful to have a more intrinsic description of \( \bar{H} \) in the spirit of Section 7. Define \( \bar{C} = (\bar{c}_{(i,a),(j,b)}) \in \mathbb{Z}^{\mathbb{Q}_0 \times \mathbb{Q}_0} = M_{2n}(\mathbb{Z}) \) by

\[
\bar{c}_{(i,a),(j,b)} := \begin{cases} 
  c_{ij} & \text{if } (a = b), \\
  0 & \text{otherwise}.
\end{cases}
\]

Clearly, \( \bar{C} \) is a Cartan matrix with symmetrizer \( \bar{D} = \text{diag}(c_1, \ldots, c_n, c_1, \ldots, c_n) \), where \( D = \text{diag}(c_1, \ldots, c_n) \) is our symmetrizer for \( C \), and \( \bar{\Omega} \) is an orientation of \( \bar{C} \). Moreover, if \( \bar{c}_{(i,a),(j,b)} < 0 \) then

\[
\bar{g}_{(i,a),(j,b)} = g_{ij}, \quad \bar{f}_{(i,a),(j,b)} = f_{ij}, \quad \bar{k}_{(i,a),(j,b)} = k_{ij}.
\]

As before, one defines the corresponding algebra \( H(\bar{C}, \bar{D}, \bar{\Omega}) \). Let

\[
H_{(i,a)} = K[\varepsilon_{(i,a)}]/(\varepsilon_{(i,a)}^{c_{ij}}).
\]

We have isomorphisms

\[
\eta_{(i,a)} : H_i \rightarrow H_{(i,a)}
\]
defined by $\varepsilon_i \mapsto \varepsilon_{(i,a)}$, and as before for each $((i, a), (j, b)) \in \tilde{\Omega}$ we get an $H_{(i,a)} - H_{(j,b)}$-bimodule $(i,a)H_{(j,b)}$ and an $H_{(j,b)} - H_{(i,a)}$-bimodule $(j,b)H_{(i,a)}$. There are bimodule isomorphisms

$$(i,a)H_{(j,b)} \cong \left\{ \begin{array}{ll} iH_j & \text{if } a = b \text{ and } (i, j) \in \Omega, \\ iH_j & \text{if } (a, b) = (0, 1) \text{ and } (i, j) \in \Omega^*. \end{array} \right.$$

via $\eta_{(i,a)}$ and $\eta_{(j,b)}$.

Set

$$\tilde{S} := \prod_{(i,a) \in \tilde{Q}_0} H_{(i,a)}.$$ 

Then

$$\tilde{B} := \bigoplus_{(i,j) \in \tilde{\Omega}} iH_j$$ 

is an $\tilde{S} - \tilde{S}$-bimodule, and we have an isomorphism

$$T_{\tilde{S}}(\tilde{B}) \cong H(\tilde{C}, \tilde{D}, \tilde{\Omega}).$$

In case $(i, j) \in \Omega$ we abbreviate $iR^0_j$ for the standard right basis of $(i,0)H_{(j,0)}$ and $iL^1_j$ for the standard left basis of $(i,1)H_{(j,1)}$. Moreover, in this case we can identify in a obvious way $(j,0)H_{(i,1)}$ with $\text{Hom}_{H_{(i,0)}}((i,0)H_{(j,0)},H_{(j,0)})$ and obtain an $H_{(j,0)}$-basis $(r^+_{\ell_i})_{\ell_i \in iR^0_j}$ of $(j,0)H_{(i,1)}$ which is under this identification dual to $iR^0_j$. Similarly, we obtain a dual $H_{(i,1)}$-basis $(\ell^*_{\ell_i})_{\ell_i \in iL^1_j}$ of $(j,0)H_{(i,1)}$.

For $j \in Q_0$, define

$$\tilde{\rho}_j := \sum_{i \in \Omega(-,j)} \ell^*_{\ell_i} \otimes \ell_i + \sum_{k \in \Omega(j,-)} r \otimes r^*_{\ell_k}.$$ 

We have $\tilde{\rho}_j \in e_{(j,0)}\tilde{B} \otimes \tilde{S}\tilde{B} e_{(j,1)}$. Now, arguing as in Section 4 we obtain:

$$\tilde{H} \cong T_{\tilde{S}}(\tilde{B})/(\tilde{\rho}_j \mid j \in Q_0).$$

Similarly to the case of preprojective algebras, for $M \in \text{rep}(H(\tilde{C}, \tilde{D}, \tilde{\Omega}))$ and $j \in Q_0$ we can define maps

$$\tilde{M}_{j,\text{in}} = (M_{(j,0),(i,a)})_{(k,c)} : \bigoplus_{(k,c) \in \tilde{\Omega}((j,0),-)} (j,0)H_{(k,c)} \otimes H_{(k,c)} M_{(k,c)} \rightarrow M_{(j,0)},$$

$$\tilde{M}_{j,\text{out}} = (\text{ad}_{(i,a),(j,1)}(M_{(i,a),(j,1)}))_{(i,a)} : M_{(j,1)} \rightarrow \bigoplus_{(i,a) \in \tilde{\Omega}(-,(j,1))} (j,1)H_{(i,a)} \otimes H_{(i,a)} M_{(i,a)}.$$ 

Note that $\tilde{\Omega}((j,0),-) = \tilde{\Omega}(-,(j,1))$ and thus, if we identify by a slight abuse $H_{(j,0)}$ with $H_{(j,1)}$, we can write

$$\bigoplus_{(k,c) \in \tilde{\Omega}((j,0),-)} (j,0)H_{(k,c)} \otimes H_{(k,c)} M_{(k,c)} = \bigoplus_{(i,a) \in \tilde{\Omega}(-,(j,1))} (j,1)H_{(i,a)} \otimes H_{(i,a)} M_{(i,a)}.$$ 

With this setup $M \in \text{rep}(H(\tilde{C}, \tilde{D}, \tilde{\Omega}))$ belongs to $\text{rep}(\tilde{H})$ if and only if

$$\tilde{M}_{j,\text{in}} \circ \tilde{M}_{j,\text{out}} = 0$$ 

for all $j \in Q_0$. 
9.2.4. The subalgebras $H(0)$ and $H(1)$. For $a = 0, 1$ set
\[ \mathbb{1}_a := \sum_{i \in I} e_{(i,a)}, \quad H(a) := \mathbb{1}_a \bar{H} \mathbb{1}_a. \]

Clearly, $H(a)$ is a (non-unitary) subalgebra of $\bar{H}$, and we have natural isomorphisms $\eta_a: H \to H(a)$ with $\eta_a(\varepsilon_i) = \varepsilon_{(i,a)}$ and $\eta_a(\alpha_{ij}^{(g)}) = \alpha_{(i,a)(j,a)}^{(g)}$.

We obtain for $a \in \{0, 1\}$ exact restriction functors
\[ \text{Res}_a: \text{rep}(\bar{H}) \to \text{rep}(H(a)), \quad M \mapsto \mathbb{1}_a \bar{H} \otimes \bar{H} M = \mathbb{1}_a M. \]

We will use several times the elementary fact that the functor
\[ \text{Res}_0: \text{rep}(H(0)) \to \text{rep}(\bar{H}), \quad M \mapsto \text{Hom}_{H(0)}(\mathbb{1}_0 \bar{H}, M) \]
is uniquely characterized up to isomorphism as the right adjoint of $\text{Res}_0$. It is not hard to see that
\[ \text{Res}_0 \circ \text{Res}_0^* \cong \text{id}_{\text{rep}(H(0))}. \]

9.2.5. The $H$-$H$-bimodule $X$. Define
\[ X = X(C, D, \Omega) := \mathbb{1}_0 \bar{H} \mathbb{1}_1. \]

We regard $X$ as an $H$-$H$-bimodule via the maps $\eta_0$ and $\eta_1$, that is,
\[ hxh' := \eta_0(h)x\eta_1(h'), \quad (h, h' \in H, \ x \in X). \]

Similarly, using $\eta_0$ and $\eta_1$ we can regard $\text{Res}_1 \circ \text{Res}_0^*$ as a functor from $\text{rep}(H)$ to $\text{rep}(H)$. Then it is easy to see that we have an isomorphism of functors:
\[ \text{Res}_1 \circ \text{Res}_0^* \cong \text{Hom}_H(X, -). \]

**Theorem 9.1.** The following hold:

(a) For each $M \in \text{rep}(H)$ we have a functorial isomorphism
\[ \text{Hom}_H(X, M) \cong C^+(M). \]

(b) For each $M \in \text{rep}_{1,1}(H)$ we have functorial isomorphisms
\[ \text{Hom}_H(X, TM) \cong TC^+(M) \cong \tau(M). \]

(c) For each $M \in \text{rep}(H)$ we have a functorial isomorphism
\[ X \otimes_H M \cong C^-(M). \]

(d) For each $M \in \text{rep}_{1,1}(H)$ we have functorial isomorphisms
\[ X \otimes_H TM \cong TC^-(M) \cong \tau^-(M). \]

9.3. **Proof of Theorem 9.1(a).** We follow the hints from [Ga3, Section 5.5]. For $l \in Q_0 \cup \{0\}$ we define idempotents in $\bar{H}$
\[ \mathbb{1}^{(l)} := \sum_{i > l} e_{(i,0)} + \sum_{i \leq l} e_{(i,1)}, \quad \mathbb{1}^{(l)}_0 := \mathbb{1}_0 + \sum_{i \leq l} e_{(i,1)}, \]
and the corresponding (non-unitary) subalgebras
\[ H^{(l)} := \mathbb{1}^{(l)} \bar{H} \mathbb{1}^{(l)}, \quad \bar{H}^{(l)} := \mathbb{1}^{(l)}_0 \bar{H} \mathbb{1}^{(l)}_0. \]

Clearly, $H^{(0)} = \bar{H}^{(0)} = H(0)$, $H^{(1)} = H(1)$, $\bar{H}^{(1)} = \bar{H}$, and an easy calculation shows that, using the notation of equation (6.1),
\[ H^{(l)} = s_1 \cdots s_{2s_1}(H^{(0)}), \quad (l \in Q_0). \]
Moreover \( \mathbf{1}^{(l)} \in \widetilde{H}^{(l)} \) and thus \( H^{(l)} \subset \widetilde{H}^{(l)} \supset \widetilde{H}^{(l-1)} \) for \( l \in Q_0 \). We study the corresponding restriction functors:

\[
\text{Res}^{(l)}: \text{rep}(\widetilde{H}) \to \text{rep}(H^{(l)}), \quad M \mapsto \mathbf{1}^{(l)}\widetilde{H} \otimes \widetilde{H} M,
\]

\[
\text{Res}_{(l,m)}: \text{rep}(\widetilde{H}^{(m)}) \to \text{rep}(\widetilde{H}^{(l)}), \quad M \mapsto \mathbf{1}^{(l)}\widetilde{H}^{(m)} \otimes \widetilde{H}^{(m)} M, \quad (l < m).
\]

Obviously, \( \text{Res}_{(l,m)} \) admits a right adjoint

\[
\text{Res}_{(l,m)}^*(-) = \text{Hom}_{\widetilde{H}^{(l)}}(\mathbf{1}^{(l)}\widetilde{H}^{(m)}, -)
\]

and

\[
\text{Res}_0 = \text{Res}_{(0,1)} \circ \text{Res}_{(1,2)} \circ \cdots \circ \text{Res}_{(n-1,n)}.
\]

Thus we have

\[
\text{Res}_0^* = \text{Res}_{(n-1,n)}^* \circ \cdots \circ \text{Res}_{(1,2)}^* \circ \text{Res}_{(0,1)}^*.
\]

**Lemma 9.2.** With the above notations we have functorial isomorphisms

\[
\text{Res}^{(i)} \circ \text{Res}_{(i-1,i)}^*(M) \cong F_i^+ \circ \text{Res}^{(i-1)}(M)
\]

for all \( M \in \text{rep}(\widetilde{H}^{(i-1)}) \) and \( i \in Q_0 = \{1, \ldots, n\} \).

**Proof.** Note that naturally

\[
\text{Res}_{(i-1,i)} \circ \text{Res}^{(i-1,i)}_*(M) \cong M
\]

for all \( M \in \text{rep}(\widetilde{H}^{(i-1)}) \). Now there is a unique functor

\[
R_{(i-1,i)}^*: \text{rep}(\widetilde{H}^{(i-1)}) \to \text{rep}(\widetilde{H}^{(i)})
\]

satisfying the two following conditions for all \( M \in \text{rep}(\widetilde{H}^{(i-1)}):

\[
\text{Res}_{(i-1,i)} \circ R_{(i-1,i)}^*(M) = M, \quad \text{Res}^{(i)} \circ R_{(i-1,i)}^*(M) = F_i^+ \circ \text{Res}^{(i-1)}(M).
\]

Indeed, the first condition fixes the restriction of \( R_{(i-1,i)}^*(M) \) to \( \widetilde{H}^{(i-1)} \) and the second one fixes the restriction of \( R_{(i-1,i)}^*(M) \) to \( H^{(i)} \). Because of the definitions of \( H^{(i)} \) and \( \widetilde{H}^{(i-1)} \), this determines completely the structure of \( R_{(i-1,i)}^*(M) \), and gives uniqueness. Note that the quivers of \( \widetilde{H}^{(i-1)} \) and of \( H^{(i)} \) contain some common arrows, but the representations \( M \) and \( F_i^+ \circ \text{Res}^{(i-1)}(M) \) are the same for those arrows, by definition of \( F_i^+ \). So \( R_{(i-1,i)}^*(M) \) is indeed a representation of \( H(\widetilde{C}, \widetilde{D}, \widetilde{Q}) \), supported on the vertices and arrows of \( \widetilde{H}^{(i)} \).

Finally, this representation satisfies the relation \((9.2)\) for \( j = i \), because again of the definition of \( F_i^+ \), so \( R_{(i-1,i)}^*(M) \in \text{rep}(\widetilde{H}^{(i)}) \).

To prove the lemma, we have to show that the above functor \( R_{(i-1,i)}^* \) is isomorphic to \( \text{Res}_{(i-1,i)}^* \), or equivalently, that \( R_{(i-1,i)}^* \) is right adjoint to \( \text{Res}_{(i-1,i)} \). To do so, let \( N \in \text{rep}(\widetilde{H}^{(i)}) \) and \( M \in \text{rep}(\widetilde{H}^{(i-1)}) \), and consider the natural map

\[
\text{Hom}_{\widetilde{H}^{(i)}}(N, R_{(i-1,i)}^*(M)) \to \text{Hom}_{\widetilde{H}^{(i-1)}}(\text{Res}_{(i-1,i)}(N), M)
\]

obtained by restricting \( f: N \to R_{(i-1,i)}^*(M) \) to \( \text{Res}_{(i-1,i)}(N) \). We have to show that this restriction is in fact bijective. That is, for \( g \in \text{Hom}_{\widetilde{H}^{(i-1)}}(\text{Res}_{(i-1,i)}(N), M) \) we have to show that there exists a unique \( g_{(i,1)} \in \text{Hom}_{H_i}(N_{(i,1)}, \text{Ker} \overline{M}_{i,1}) \) which lifts \( g \) to an element of \( \text{Hom}_{\widetilde{H}^{(i)}}(N, R_{(i-1,i)}^*(M)) \).
Now, let

\[ N_{i,+} := \bigoplus_{(k,c) \in \Omega((i,0),-)} (1,0) H(k,c) \otimes H(k,c) N(k,c) \]

denote the domain of \( \tilde{N}_{i,\text{in}} \), and similarly let \( M_{i,+} \) denote the domain of \( \tilde{M}_{i,\text{in}} \). By the definition of \( \tilde{H}^{(i-1)} \)-homomorphisms, we have a commutative diagram

\[
\begin{array}{ccc}
N_{(i,1)} & \xrightarrow{\tilde{N}_{i,\text{out}}} & N_{i,+} & \xrightarrow{\tilde{N}_{i,\text{in}}} & N_{(i,0)} \\
0 & \xrightarrow{\text{Ker}(\tilde{M}_{i,\text{in}})} & M_{i,+} & \xrightarrow{\tilde{M}_{i,\text{in}}} & M_{(i,0)}
\end{array}
\]

where the bottom row is exact by construction, and in the top row the composition is zero since \( N \) is a \( \tilde{H}^{(i)} \)-module. Thus \( \tilde{M}_{i,\text{in}} \circ g_{i,+} \circ \tilde{N}_{i,\text{out}} = 0 \). By the universal property of \( \text{Ker} \tilde{M}_{i,\text{in}} \) there exists a unique morphism of \( H_i \)-modules \( g_{(i,1)} : N_{(i,1)} \to \text{Ker} \tilde{M}_{i,\text{in}} \) which makes the left-hand square commutative.

We can now finish the proof of Theorem 9.1(a). Using \( n \) times Lemma 9.2 for \( M \in \text{rep}(H) \) (regarded as a representation of \( H_0 \)) we have

\[
\text{Hom}_H(X, M) = \text{Res}_1 \circ \text{Res}_0^* (M) \\
= \text{Res}^{(n)} \circ \text{Res}_0^{(n-1)} \circ \cdots \circ \text{Res}_0^*(M) \\
= F_n^+ \circ \text{Res}^{(n-1)} \circ \text{Res}_0^{(n-2,n-1)} \circ \cdots \circ \text{Res}_0^*(M) \\
= \cdots \\
= F_n^+ \circ \cdots \circ F_1^+ \circ \text{Res}^{(0)}(M) \\
= C^+(M).
\]

9.4. Proof of Theorem 9.1(b). We follow the idea of [Ga3, Section 5.4], and start by giving an alternative description of \( \text{Res}_0^* \). This is done by constructing in two steps a functor \( R_0^* : \text{rep}(H_0) \to \text{rep}(\tilde{H}) \), and then showing that \( R_0^* \) is right adjoint to \( \text{Res}_0 \).

Let \( M \in \text{rep}(H_0) \). We first define \( \tilde{M} \in \text{rep}(H(C,\tilde{D},\Omega)) \) by requiring that

\[
\text{Res}_0(\tilde{M}) = M, \\
\text{Res}_1(\tilde{M}) = \bigoplus_{(k,l) \in \Omega} \text{Hom}_H((l,0) H(k,1) \otimes H(k,1)) e_{(k,1)} H_1, M(l,0)).
\]

(Note that \( H_{(0)} \) and \( H_{(1)} \) can also be regarded as subalgebras of \( H(C,\tilde{D},\Omega) \), so we allow ourselves, by some abuse of notation, to continue to denote the restriction functors \( \text{rep}(H(C,\tilde{D},\Omega)) \to \text{rep}(H_{(a)}) \) by \( \text{Res}_{a,*} \).) It remains to define, for \( (i,j) \in \Omega \), the structure map

\[
\tilde{M}_{(j,0),(i,1)} : (j,0) H_{(1)} \otimes H_{(i,1)} \tilde{M}_{(i,1)} \to \tilde{M}_{(j,0)} = M_{(j,0)}.
\]
This is given by the following composition:

\[
\begin{aligned}
(j,0)H_{(i,1)} \otimes H_{(i,1)} & \left( \bigoplus_{(k,l) \in \Omega} \text{Hom}_{H_{(i,0)}}(H_{(k,1)} \otimes e_{(k,1)}H_{e_{(i,1)}}(i), M_{(l,0)}) \right) \\
\text{proj}_{(j,0)H_{(i,1)}} & \left( (j,0)H_{(i,1)} \otimes H_{(i,1)} \text{Hom}_{H_{(j,0)}}(j,0)H_{(i,1)} \otimes e_{(i,1)}H_{e_{(i,1)}}, M_{(j,0)}) \right) \\
= & \text{proj}_{(j,0)H_{(i,1)}} \left( (j,0)H_{(i,1)} \otimes H_{(i,1)} \text{Hom}_{H_{(j,0)}}(j,0)H_{(i,1)}, M_{(j,0)}) \right) \text{eval} \rightarrow M_{(j,0)},
\end{aligned}
\]

where the first map is the projection on the direct summand indexed by \((k,l) = (i,j)\) and the second map is the evaluation \(h \otimes \varphi \rightarrow \varphi(h)\).

Secondly, we define a subrepresentation \(R_0^*(M)\) of \(\widetilde{M}\) as follows. We set

\[
(R_0^*(M))_{(i,0)} = \widetilde{M}_{(i,0)} = M_{(i,0)}, \quad (i \in Q_0),
\]

and we define \((R_0^*(M))_{(h,1)}\) as the subspace of \(\widetilde{M}_{(h,1)}\) consisting of all

\[
(\mu^h_{k,l})_{(k,l) \in \Omega} \in \bigoplus_{(k,l) \in \Omega} \text{Hom}_{H_{(i,0)}}(H_{(k,1)} \otimes e_{(k,1)}H_{e_{(i,1)}}, M_{(l,0)})
\]

such that, for all \(l \in Q_0\) and \(n^{(1)} \in e_{(l,1)}H_{e_{(i,1)}}\) the following relation holds:

\[
\sum_{k \in \Omega_{(l,-)}} \mu^h_{k,l}(\ell^\ast \otimes \ell \cdot n^{(1)}) + \sum_{m \in \Omega_{(l,-)}} M_{(l,0),\mu^h_{k,m}(r^\ast \otimes n^{(1)})} = 0. \tag{9.3}
\]

Here, we use the notation from Section 9.2.3. It is straightforward to check that \(R_0^*(M)\) is an \(H(\tilde{C}, D, \tilde{\Omega})\)-subrepresentation of \(M\). Moreover, \(R_0^*(M)\) is in fact a representation of \(\widetilde{H}\). To see this, we check the defining relations (9.1) with the help of the special case \(n^{(1)} = e_{(l,1)}\). In fact, if we apply \(\rho_j\) to \(\mu^{(j)} = (\mu^j_{k,l})_{(k,l) \in \Omega} \in R_0^*(M)_{(j,1)}\) we deduce from the definitions that

\[
\sum_{i \in \Omega_{(l,-)}} \tilde{M}_{(j,0),(i,1)}(\ell^\ast \otimes M_{(i,1),(j,1)}(\ell \cdot \mu^{(j)})) + \sum_{k \in \Omega_{(l,-)}} M_{(j,0),(k,0)}(r \otimes \tilde{M}_{(k,0),(j,1)}(r^\ast \otimes \mu^{(j)}))
\]

\[
= \sum_{i \in \Omega_{(l,-)}} \mu^j_{i,j}(\ell^\ast \otimes \ell \cdot e_{(j,1)}) + \sum_{k \in \Omega_{(l,-)}} M_{(j,0),(k,0)}(r \otimes \mu^j_{k,0}(r^\ast \otimes e_{(j,1)})) = 0,
\]

as required.

Thus, we have obtained a functor \(R_0^* : \text{rep}(H(0)) \rightarrow \text{rep}(\tilde{H}), M \mapsto R_0^*(M)\). It will follow from the next lemma that \(R_0^*\) is isomorphic to \(\text{Res}_0^*\).

**Lemma 9.3.** \(R_0^*\) is right adjoint to \(\text{Res}_0^*\).

**Proof.** Let \(N \in \text{rep}(\tilde{H})\) and \(M \in \text{rep}(H(0))\). Consider \(\chi \in \text{Hom}_{\tilde{H}}(N, R_0^*(M))\). Thus, \(\chi\) is given by a family of maps

\[
\chi^{(i,a)} \in \text{Hom}_{H_{(i,a)}}(N_{(i,a)}, (R_0^*(M))_{(i,a)}), \quad ((i,a) \in \tilde{Q}_0),
\]
subject to the usual commutativity relations. By the construction of \( R_0^*(M) \) we have more explicitly for all \( i \in Q_0 \) and \( n_{(i,1)} \in N_{(i,1)}:\)

\[
\chi_{(i,0)}^{(i,0)} \in \text{Hom}_{H_{(i,0)}}(N_{(i,0)}, M_{(i,0)}),
\]

\[
\chi_{(i,1)}^{(i,1)}(n_{(i,1)}) \in \bigoplus_{(k,l) \in \Omega} \text{Hom}_{H_{(i,0)}}((i,0) H_{(k,1)} \otimes e_{(k,1)} H_{(i,1)}, M_{(i,0)}).
\]

Let us denote by \( \chi_{k,l}^{(i,1)}(-, n_{(i,1)}) \) the \((k,l)\)-component of \( \chi_{(i,1)}^{(i,1)}(n_{(i,1)}) \). These maps are subject to the following relations for \((i,j) \in \Omega, \ell(a) \in iL_j, r \in iR_0^1:\)

\[
\chi^{(i,0)}((i,0),(j,0)) \ell(0) = M_{(i,0),(j,0)}(\ell(0) \otimes \chi_{(i,0)}^{(i,0)}(n_{(j,0)})) = (9.4)
\]

\[
\chi^{(j,0)}((j,0),(i,1)) r(1) = \chi_{i,j}^{(i,1)}(r(1) \otimes e_{(i,1)} n_{(i,1)}), \quad (9.5)
\]

\[
\chi_{k,l}^{(i,1)}(-, (i,1)) r(1) = \chi_{k,l}^{(i,1)}(\ell(1) \otimes n_{(i,1)}), \quad (9.6)
\]

Equation \( (9.4) \) means that we have indeed a well-defined restriction

\[
r_{N,M} : \text{Hom}_{\tilde{H}}(N, R_0^*(M)) \to \text{Hom}_{H_{(0)}}(\text{Res}_0(N), M).
\]

Combining \( (9.5) \) and \( (9.6) \) we see that the maps \( \chi_{k,l}^{(i,1)} \) for \((k,l) \in \Omega \) and \( j \in Q_0 \) are determined by the maps \( \chi^{(i,0)} \) with \( i \in Q_0 \), in other words \( r_{N,M} \) is injective.

By the same token we see that for each \( \chi^{(0)} \in \text{Hom}_{H_{(0)}}(\text{Res}_0(N), M) \) there exists \( \tilde{\chi} \in \text{Hom}_{H_{(\tilde{C}, \tilde{D}, \tilde{\Omega})}}(N, \tilde{M}) \) which restricts to \( \chi^{(0)} \). We leave it as an exercise to show that if \( N \in \text{rep}(H) \) then \( \text{Im}(\tilde{\chi}) \subset R_0^*(M) \). Thus, \( r_{N,M} \) is bijective.

\[\square\]

**Proposition 9.4.** For \( M \in \text{rep}_{\text{I},I}(H) \) we have

\[
\tau(TM) \cong \text{Res}_1 \circ R_0^*(M),
\]

where in the right-hand side \( H_{(0)} \) and \( H_{(1)} \) are identified with \( H \) by means of the isomorphisms \( \eta_0 \) and \( \eta_1 \).

**Proof.** Since \( M \) is locally free, \( TM \) is also locally free and Corollary 8.6 provides a projective resolution:

\[
0 \to \bigoplus_{(j,i) \in \Omega} He_j \otimes j H_i \otimes (TM)_i \xrightarrow{d \otimes TM} \bigoplus_{k=1}^n He_k \otimes (TM)_k \xrightarrow{\text{mult}} TM \to 0
\]

Therefore, by definition of the Auslander-Reiten translation \( \tau \), we know that \( \tau(TM) \) is isomorphic to \( \text{Ker}(D \text{Hom}_H(d \otimes TM, H)) \).

On the other hand, the construction of \( R_0^*(M) \) shows that \( \text{Res}_1 \circ R_0^*(M) \) can be identified with the kernel of the map

\[
d_M^* : \bigoplus_{(j,i) \in \Omega} \text{Hom}_{H_i}(j H_i^* \otimes e_j H, M_i) \to \bigoplus_{k=1}^n \text{Hom}_{H_k}(e_k H, M_k)
\]

whose \((j,i)\)-component is defined by

\[
\varphi_{(j,i)} \mapsto \sum_{\ell \in j L_i} \varphi_{(j,i)}(\ell^* \otimes j \ell \cdot -) + \sum_{r \in j R_i} M_{ji}(r \otimes \varphi_{(j,i)}(r^* \otimes j -)), \quad (9.7)
\]
Indeed, \( \text{Res}_1 \circ R^*_0(M) \) is the subspace of \( \text{Res}_1(\tilde{M}) \) defined by equations (9.3). Our goal is to identify under the trace pairing the map \( D \text{Hom}_H(d \otimes TM, H) \) with \( d^*_M \). For \( (j, i) \in \Omega \), the restriction \( d \otimes TM : H e_j \otimes j H_l \otimes M_i \to H e_i \otimes M_l \oplus H e_j \otimes M_j \) is given by

\[
(d \otimes TM)(p \otimes h \otimes m) = ph \otimes m + p \otimes M_{ji}(h \otimes m). \tag{9.8}
\]

(Note the plus sign, coming from the twist map \( T \)). Using adjunction we have

\[
\text{Hom}_H(He_i \otimes M_l, H) \cong \text{Hom}_H(M_l, \text{Hom}_H(He_i, H)) \cong \text{Hom}_{H_i}(M_l, e_i H),
\]

so under the trace pairing we get

\[
D \text{Hom}_H(He_i \otimes M_l \oplus He_j \otimes M_j, H) \cong \text{Hom}_{H_i}(e_i H, M_l) \oplus \text{Hom}_{H_j}(e_j H, M_j).
\]

Similarly,

\[
D \text{Hom}_H(He_j \otimes j H_l \otimes M_i) \cong \text{Hom}_{H_j}(e_j H, j H_l \otimes M_i) \cong \text{Hom}_{H_i}(j H_j \otimes e_j H, M_i),
\]

where the second isomorphism is given by \( \text{ad}^{-1}_{ij} \). Hence \( D \text{Hom}_H(d \otimes TM, H) \) can be identified with a map from \( \oplus_{(j, i) \in \Omega} \text{Hom}_{H_l}(i H_j \otimes e_j H, M_i) \) to \( \oplus_k \text{Hom}_{H_k}(e_k H, M_k) \), and comparison between (9.7) and (9.8) shows that this map is indeed \( d^*_M \).

Now we are ready to prove part (b) of Theorem 9.1. By Lemma 9.3 and the uniqueness of adjoint functors we have a functorial isomorphism \( R^*_0(-) \cong \text{Res}^*_0(-) \). Hence, by Proposition 9.4 if \( M \in \text{rep}_{\text{lf}}(H) \) we have

\[
\text{Hom}_H(X, TM) = \text{Res}_1 \circ \text{Res}^*_0(TM) \cong \text{Res}_1 \circ R^*_0(TM) \cong \tau(T^2 M) = \tau(M).
\]

This proves Theorem 9.1 (b).

9.5. Proof of Theorem 9.1(c),(d). Clearly, (c) follows from (a) since \( C^- \) is left adjoint to \( C^+ \). In order to show (d), let \( M, N \in \text{rep}_{\text{lf}}(H) \). Recall, that this implies that both, \( M \) and \( N \), have projective and injective dimension at most 1. We obtain functorial isomorphisms

\[
\text{Hom}_H(\tau^{-}(M), N) \cong \text{Hom}_H(M, \tau(N)) \\
\cong \text{Hom}_H(M, C^+(TN)) \\
\cong \text{Hom}_H(C^-(TM), N).
\]

The first isomorphism is obtained from the Auslander-Reiten formulas, the second follows from (b), and the third isomorphism is just the adjunction map.

With the usual \( H-H \)-bimodule structure on \( DH \) we obtain a functorial isomorphism of right \( H \)-modules

\[
DX \cong \text{Hom}_H(X, DH)
\]

for all (left) \( H \)-modules \( X \). Now, in our situation \( DH \) is locally free, thus taking \( N = DH \) in the above chain of functorial isomorphisms we get

\[
\tau^{-}(M) \cong C^{-}(TM) \cong X^T \otimes_H M
\]

where the last isomorphism comes from (c). This proves Theorem 9.1 (d).
9.6. **Another description of $\Pi(C, D)$ as a tensor algebra.** Let $\Pi = \Pi(C, D)$. Recall from Section 7 that $\Pi_1$ is the subspace of $\Pi$ of elements of degree 1. Let $X^T$ be the twisted version of the $H$-$H$-bimodule $X$, where the bimodule structure is defined by

$$hxh' := hxT(h'), \quad (h, h' \in H, \, x \in X).$$

**Theorem 9.5.** We have isomorphisms of $H$-$H$-bimodules

$$\Pi_1 \cong X^T \cong \text{Ext}^1_H(DH, H).$$

**Proof.** Note that the bimodule isomorphism $\Pi_1 \cong X^T$ follows directly from the definitions. On the other hand, we have by Theorem 9.1(d) for locally free modules $M$ a functorial isomorphism

$$X^T \otimes_H M \cong \tau^-(M) \cong \text{Ext}^1_H(DH, M). \quad (9.9)$$

Note that the functor

$$\text{Ext}^1_H(DH, -) : \text{rep}(H) \to \text{rep}(H)$$

is right exact since $\text{proj. dim}(DH) \leq 1$. For $M \in \text{rep}(H)$ let

$$P_1 \to P_0 \to M \to 0 \quad (9.10)$$

be a projective presentation of $M$. Applying the right exact functors $X^T \otimes_H -$ and $\text{Ext}^1_H(DH, -)$ to (9.10) yields a functorial commutative diagram

$$\begin{array}{c}
X^T \otimes_H P_1 \quad \text{Ext}^1_H(DH, P_1) \\
\downarrow{\eta_{P_1}} \quad \downarrow{\eta_M} \\
X^T \otimes_H P_0 \quad \text{Ext}^1_H(DH, P_0) \quad \text{Ext}^1_H(DH, M) \\
\downarrow{\eta_{P_0}} \quad \downarrow{\quad} \\
X^T \otimes_H M \quad 0
\end{array}$$

with exact rows. Since the restrictions of $X^T \otimes_H -$ and $\text{Ext}^1_H(DH, -)$ to $\text{rep}_l(H)$ are isomorphic, we get that $\eta_{P_1}$ and $\eta_{P_0}$ are isomorphisms. This implies that $\eta_M$ is an isomorphism as well. It follows that the functors $X^T \otimes_H -$ and $\text{Ext}^1_H(DH, -)$ are isomorphic.

From the canonical isomorphism of $H$-$H$-bimodules $\text{Ext}^1_H(DH, H) \otimes_H H \cong \text{Ext}^1_H(DH, H)$ we conclude that the right exact functors $\text{Ext}^1_H(DH, -)$ and $\text{Ext}^1_H(DH, H) \otimes_H -$ are isomorphic. This implies that $\text{Ext}^1_H(DH, H)$ and $X^T$ are isomorphic as $H$-$H$-bimodules. \(\square\)

**Corollary 9.6.** We have $K$-algebra isomorphisms

$$\Pi \cong T_H(X^T) \cong T_H(\text{Ext}^1_H(DH, H)).$$

**Proof.** Combine Theorem 9.5 and Proposition 7.5 \(\square\)

**Corollary 9.7.** For $M \in \text{rep}(H)$ there are functorial isomorphisms

$$\text{Hom}_H(X^T, M) \cong \text{D Ext}^1_H(M, H) \quad \text{and} \quad X^T \otimes_H M \cong \text{Ext}^1_H(DH, M).$$

**Proof.** We get the second isomorphism from the proof of Theorem 9.5. The first isomorphism follows then by adjunction. \(\square\)
9.7. **The morphism categories** \( \mathcal{C}(1, TC^+) \) and \( \mathcal{C}(TC^-, 1) \). As before let \( \Pi = \Pi(C, D) \). Following a definition due to Ringel [Ri4], we define a category \( \mathcal{C}(1, TC^+) \) as follows. Its objects are the \( H \)-module homomorphisms \( M \to TC^+(M) \), where \( M \in \text{rep}(H) \) and the morphisms in \( \mathcal{C}(1, TC^+) \) are given by commutative diagrams

\[
\begin{array}{ccc}
M & \xrightarrow{f} & TC^+(M) \\
h & & \downarrow \text{TC}^+(h) \\
N & \xrightarrow{g} & TC^+(N).
\end{array}
\]

Similarly, let \( \mathcal{C}(TC^-, 1) \) be the category with objects the \( H \)-module homomorphisms \( \text{TC}^-(M) \to M \).

**Theorem 9.8.** The categories \( \text{rep}(\Pi), \mathcal{C}(1, TC^+) \) and \( \mathcal{C}(TC^-, 1) \) are isomorphic.

**Proof.** It follows from Proposition 6.1 that \( (TC^-, TC^+) \) is a pair of adjoint functors \( \text{rep}(H) \to \text{rep}(H) \). Now [Ri4, Lemma 1] implies that the categories \( \mathcal{C}(1, TC^+) \) and \( \mathcal{C}(TC^-, 1) \) are isomorphic. It follows from Theorem 9.1(c) that there is a functorial isomorphism \( X_T \otimes H \cong \text{TC}^-(M) \). Now [Ri4, Lemma 12] gives an isomorphism of categories \( \mathcal{C}(TC^-, 1) \cong \text{rep}(\Pi(C, D)) \). □

One can also adapt Ringel’s proof of [Ri4, Theorem B] to obtain a more direct proof of Theorem 9.8.

9.8. **The kernel of the Coxeter functor.** As before, let \( H = H(C, D, \Omega) \). Recall that \( H \) is a 1-Iwanaga-Gorenstein algebra with the subcategory \( \mathcal{GP}(H) = \{ M \in \text{rep}(H) \mid \text{Ext}^1_H(M,H) = 0 \} \) of Gorenstein-projective modules.

As an immediate consequence of Theorem 9.1(a), Corollary 9.7 and the definition of \( C^+(\cdot) \) we get the following result. Here, the map \( M_{i, \text{in}} \) is defined as in Section 5.3 since we can regard the \( H \)-module \( M \) also as a module over \( \Pi(C, D) \).

**Theorem 9.9.** For an \( H \)-module \( M \) the following are equivalent:

(i) \( M \in \mathcal{GP}(H) \);
(ii) \( C^+(M) = 0 \);
(iii) \( M_{i, \text{in}} \) is injective for all \( 1 \leq i \leq n \).

If \( C \) is symmetric, then the equivalence of (i) and (iii) in Theorem 9.9 is a special case of [LuZ, Theorem 5.1]. For \( C \) symmetric and \( D = \text{diag}(2, \ldots, 2) \) the category \( \mathcal{GP}(H) \) has been studied in detail in [RiZ].

10. **\( \tau \)-locally free \( H \)-modules**

10.1. Let \( M \) be an indecomposable \( H \)-module. Recall that \( M \) is \( \tau \)-locally free if \( \tau^k(M) \) is locally free for all \( k \in \mathbb{Z} \). Furthermore, \( M \) is called preprojective (resp. preinjective) if there exists some \( k \geq 0 \) such that \( M \cong \tau^{-k}(P) \) (resp. \( M \cong \tau^k(I) \)) for some indecomposable projective \( H \)-module \( P \) (resp. indecomposable injective \( H \)-module \( I \)). A \( \tau \)-locally free \( H \)-module \( M \) is regular if \( M \) is neither preprojective nor preinjective.

Let \( C \) be a \( K \)-linear category. The **stable category** \( \mathcal{C} \) (resp. \( \overline{\mathcal{C}} \)) is the quotient category of \( \mathcal{C} \) modulo the ideal of all morphisms factoring through projective (resp. injective) objects.
Proposition 10.1. The restriction of $\tau(-)$ yields an equivalence of stable categories

$$\operatorname{rep}_{lf}(H) \to \mathcal{F}(H)$$

where $\mathcal{F}(H) := \{M \in \operatorname{rep}(H) \mid \operatorname{Hom}_H(DH, M) = 0\}$, and $\tau(-)$ yields an equivalence of stable categories

$$\operatorname{rep}_{lf}(H) \to \mathcal{G}(H)$$

where $\mathcal{G}(H) := \{M \in \operatorname{rep}(H) \mid \operatorname{Hom}_H(M, H) = 0\}$.

**Proof.** Combine Proposition 3.5 with [AR, Lemma 4.1] and its dual. \(\Box\)

Corollary 10.2. For an indecomposable $M \in \operatorname{rep}(H)$ the following are equivalent:

(i) $M \in \operatorname{rep}_{lf}(H)$;
(ii) $\operatorname{Hom}_H(\tau^{-}(M), H) = 0$;
(iii) $\operatorname{Hom}_H(DH, \tau(M)) = 0$.

Corollary 10.3. For an indecomposable $M \in \operatorname{rep}(H)$ the following are equivalent:

(i) $M$ is $\tau$-locally free;
(ii) $\operatorname{Hom}_H(\tau^{-}(\tau^k(M)), H) = 0$ for all $k \in \mathbb{Z}$;
(iii) $\operatorname{Hom}_H(DH, \tau(\tau^k(M))) = 0$ for all $k \in \mathbb{Z}$.

**Proof.** Recall that for an indecomposable module $M$ we have $\tau(\tau^{-}(M)) \cong M$ if and only if $M$ is not injective, and $\tau^{-}(\tau(M)) \cong M$ if and only if $M$ is not projective. Now the statement is a direct consequence of Corollary 10.2. \(\Box\)

Proposition 10.4. Let $M \in \operatorname{rep}_{lf}(H)$ be indecomposable and rigid. Then $M$ is $\tau$-locally free and $\tau^k(M)$ is rigid for all $k \in \mathbb{Z}$.

**Proof.** By Theorem 9.1 we know that $\tau^k(M) \cong T^kC^k(M)$. Now the result follows from Proposition 6.6. \(\Box\)

Recall from Section 3.4 the Coxeter matrix $\Phi_H$.

Proposition 10.5. For a $\tau$-locally free module $M \in \operatorname{rep}(H)$ the following hold:

(i) If $\tau^k(M) \neq 0$ for some $k \in \mathbb{Z}$, then $\dim(\tau^k(M)) = \Phi^k_H(\dim(M))$.
(ii) If $\tau^k(M) \neq 0$ for some $k \in \mathbb{Z}$ and $\operatorname{rank}(M)$ is contained in $\Delta^+_{\text{re}}(C)$ or $\Delta^+_{\text{im}}(C)$, then $\operatorname{rank}(\tau^k(M))$ is in $\Delta^+_{\text{re}}(C)$ or $\Delta^+_{\text{im}}(C)$, respectively.

**Proof.** Let $i$ be a sink (resp. source) in $Q^\circ(C, \Omega)$. Then for a $\tau$-locally free module $M$ with $M \not\cong E$, the map $M_{i,\text{in}}$ is surjective (resp. $M_{i,\text{out}}$ is injective). Now the result follows from Proposition 6.4 and Theorem 9.1(b),(d). \(\Box\)

Proposition 10.6. Let $M$ be a preprojective or preinjective $H$- module. Then the following hold:

(i) $M$ is $\tau$-locally free and rigid;
(ii) $\operatorname{rank}(M) \in \Delta^+_{\text{re}}(C)$;
(iii) If $M$ and $N$ are preprojective or preinjective $H$-modules with $\dim(M) = \dim(N)$, then $M \cong N$. 
Proof. By definition we have $M \cong \tau^{-k}(P_i)$ or $M \cong \tau^k(I_i)$ for some $k \geq 0$ and some $1 \leq i \leq n$. The modules $P_i$ and $I_i$ are indecomposable, locally free and rigid. Thus by Proposition 10.3, the module $M$ is $\tau$-locally free and rigid. We know from Section 3.3 that $\text{rank}(P_i), \text{rank}(I_i) \in \Delta_{\text{re}}(C)$. Now part (ii) follows from Proposition 10.5(ii), and (iii) is a consequence of Lemmas 2.1 and 2.2. 

Lemma 10.7. Assume $C$ is connected and not of Dynkin type. Let $X$ be a preprojective, $Y$ a regular and $Z$ a preinjective $H$-module. Then we have $\text{Hom}_H(Z,Y) = 0$, $\text{Hom}_H(Y,X) = 0$ and $\text{Hom}_H(Z,X) = 0$.

Proof. We have $X \cong \tau^{-k}(P_i)$ for some $1 \leq i \leq n$ and some $k \geq 0$. We get $\text{Hom}_H(Y,X) \cong \text{Hom}_H(\tau^k(Y),P_i)$ and $\text{Hom}_H(Z,X) \cong \text{Hom}_H(\tau^k(Z),P_i)$. Now Corollary 10.3 yields that these homomorphism spaces are zero. Similarly, one shows that $\text{Hom}_H(Z,Y) = 0$. 

A sequence $((i_1,p_1),\ldots,(i_t,p_t))$ with $1 \leq i_k \leq n$ and $p_t \in \{+,-\}$ is admissible for $(C,\Omega)$ if the following hold:

(i) Either $i_1$ is a sink in $Q^o(C,\Omega)$ and $p_1 = +$, or $i_1$ is a source in $Q^o(C,\Omega)$ and $p_1 = -$;

(ii) For each $2 \leq k \leq t$, either $i_k$ is a sink in $Q^o(C,s_{i_{k-1}}\cdots s_{i_1}(\Omega))$ and $p_k = +$, or $i_k$ is a source in $Q^o(C,s_{i_{k-1}}\cdots s_{i_1}(\Omega))$ and $p_k = -$.

Proposition 10.8. For an indecomposable locally free $M \in \text{rep}(H)$ the following are equivalent:

(i) $M$ is $\tau$-locally free;

(ii) For each admissible sequence $((i_1,p_1),\ldots,(i_t,p_t))$ for $(C,\Omega)$ the module

$$F_{i_1}^{p_1} \cdots F_{i_t}^{p_t}(M)$$

is locally free.

Proof. Assume $M$ is $\tau$-locally free. Let $i$ be a sink in $Q^o(C,\Omega)$. We want to show that $F_{i}^+(M)$ is $\tau$-locally free. We can assume that $\tau(M) \neq 0$. (Otherwise $M$ is projective and therefore $\tau$-locally free.) There clearly exists a $+\text{-admissible}$ sequence $(i_1,\ldots,i_n)$ for $(C,\Omega)$ with $i_1 = i$. Using that $M$ is $\tau$-locally free and applying Theorem 9.1, we get

$$\tau(M) \cong TC^+(M) \cong TF_{i_n}^+ \cdots F_{i_1}^+ (M).$$

By Lemma 6.8, the module $F_{i}^+(M)$ is indecomposable and locally free.

Let $k > 0$. There exists a $+\text{-admissible}$ sequence $(j_1,\ldots,j_n)$ for $(C,s_i(\Omega))$ with $j_n = i$. It follows that $(i,j_1,\ldots,j_{n-1})$ is a $+\text{-admissible}$ sequence for $(C,\Omega)$. We have

$$\tau(F_{i}^+(M)) \cong T(F_{j_n}^+ \cdots F_{j_1}^+(M)) \cong F_{i}^+(\tau(M)),$$

and this module is indecomposable and locally free since $\tau(M)$ is $\tau$-locally free. Now it follows by induction that $\tau^k(F_{i}^+(M)) \cong F_{i}^+(\tau^k(M))$ is indecomposable and locally free for each $k > 0$.

Next, let $k < 0$. Then there is a $-\text{-admissible}$ sequence $(j_1,\ldots,j_n)$ for $(C,s_i(\Omega))$ with $j_1 = i$. Then $(j_2,\ldots,j_n,i)$ is a $-\text{-admissible}$ sequence for $(C,\Omega)$. We can assume that $\tau^-(F_{i}^+(M)) \neq 0$. (Otherwise $F_{i}^+(M)$ is injective and therefore $\tau$-locally free.) We get

$$\tau^-(F_{i}^+(M)) \cong F_{j_n}^- \cdots F_{j_1}^- F_{i}^+(M) \cong F_{j_n}^- F_{j_1}^- (M)$$

and

$$F_{i}^+(\tau^-(M)) \cong F_i^+ F_{j_n}^- \cdots F_{j_1}^- (M) \cong F_{i}^+ F_{j_n}^- (M).$$
(For the last isomorphism we used the dual of Lemma 6.8.) Again by induction we get that \( \tau^k(F^+_i(M)) \cong F^+_i(\tau^k(M)) \) is indecomposable and locally free for each \( k < 0 \).

Altogether we showed that \( F^+_i(M) \) is \( \tau \)-locally free. Dually, one shows that \( F^-_j(M) \) is \( \tau \)-locally free for each source \( j \) in \( Q^0(C, \Omega) \). This implies (ii).

To show the other direction, assume that (ii) holds. It follows that \( T^kC^k(M) \) is locally free for all \( k \in \mathbb{Z} \). Now we can apply Theorem 9.1 and get \( \tau^k(M) \cong T^kC^k(M) \). Thus \( M \) is \( \tau \)-locally free.

**Proposition 10.9.** For an \( H \)-module \( M \) the following are equivalent:

(i) \( M \) is locally free;

(ii) \( \tau(M) \cong TC^+(M) \);

(iii) \( \tau^-(M) \cong TC^-(M) \).

**Proof.** By Theorem 9.1(b) we know that (i) implies (ii). Now suppose that (ii) holds. Let \( f: M \to N \) be a monomorphism from \( M \) to a locally free \( H \)-module \( N \). (For example we could just take the injective envelope of \( M \).) Since \( TC^+ \) is a left exact functor we get an exact sequence

\[
0 \to TC^+(M) \to TC^+(N).
\]

By Theorem 9.1(b) and assumption (ii) we get an exact sequence

\[
0 \to \tau(M) \to \tau(N).
\]

By Proposition 10.1 we have \( \text{Hom}_H(DH, \tau(N)) = 0 \), since \( N \) is locally free. Applying \( \text{Hom}_H(DH, -) \) to the exact sequence above gives \( \text{Hom}_H(DH, \tau(M)) = 0 \). Again by Proposition 10.1 this implies that \( M \) is locally free. The equivalence of (i) and (iii) is proved dually.

**10.2. Finite type classification for \( \tau \)-locally free modules.**

**Theorem 10.10.** For a Cartan matrix \( C \) of Dynkin type the following hold:

(i) The map \( M \mapsto \text{rank}(M) \) yields a bijection between the set of isomorphism classes of \( \tau \)-locally free \( H \)-modules and the set \( \Delta^+(C) \) of positive roots of the Lie algebra associated with \( C \).

(ii) For an indecomposable \( H \)-module \( M \) the following are equivalent:

(a) \( M \) is preprojective;

(b) \( M \) is preinjective;

(c) \( M \) is \( \tau \)-locally free;

(d) \( M \) is locally free and rigid.

**Proof.** Let \( i = (i_1, \ldots, i_n) \) be a +-admissible sequence for \( (C, \Omega) \). We have \( \text{rank}(P_{ik}) = \beta_{i,k} \) for \( 1 \leq k \leq n \), compare Section 3.3. Since \( C \) is of Dynkin type, we get all elements of \( \Delta^+(C) \) by applying the Coxeter transformation \( c^{-s} \) to the \( \beta_{i,k} \) with \( s \geq 0 \) and \( 1 \leq k \leq n \), compare Lemma 2.2. In particular, the preprojective \( H \)-modules and the preinjective \( H \)-modules coincide. Now Proposition 10.6 implies part (i). We also get that (a) and (b) in part (ii) are equivalent, and that (a) and (b) implies (c) and (d). By Proposition 10.4 we know that (d) implies (c). Now let \( M \in \text{rep}(H) \) be \( \tau \)-locally free. Then there exists an injective \( H \)-module \( I_i \) with \( \text{Hom}_H(M, I_i) \neq 0 \). Since \( C \) is of Dynkin type, we know that \( P_j = \tau^k(I_i) \) for some \( k \geq 0 \) and some \( 1 \leq j \leq n \). If \( \tau^s(M) = 0 \) for some \( s \geq 0 \), then \( M \) is preprojective and we are done. Thus assume that \( \tau^s(M) \neq 0 \) for all \( s \geq 0 \). Then we have \( \text{Hom}_H(M, I_i) \cong \text{Hom}_H(\tau^k(M), \tau^k(I_i)) \neq 0 \). (Note that all
modules appearing here have projective and injective dimension at most one. Thus the stable homomorphism spaces are equal to the ordinary homomorphism spaces.) It follows that \( \text{Hom}_H(\tau^-(\tau^{k+1}(M)), P_j) \neq 0 \), a contradiction to Corollary 10.3. This finishes the proof. □

Combining the results in Section 10.1, Theorem 10.10 and Lemmas 2.1 and 2.2 we get the following result.

**Theorem 10.11.** There are finitely many isomorphism classes of \( \tau \)-locally free \( H \)-modules if and only if \( C \) is of Dynkin type.

10.3. **The algebra \( \Pi \) as a module over \( H \).** Let \( \Pi = \Pi(C, D) \).

**Theorem 10.12.** \( H \Pi \cong \bigoplus_{m \geq 0} \tau^{-m}(\mu H) \).

**Proof.** By Proposition 10.6 we know that \( \tau^{-m}(H) \) is locally free for all \( m \geq 0 \). Thus we have \( \tau^{-}(H) \cong \text{Ext}^1_H(DH, H) \) and

\[
\tau^{-}(\tau^{-}(m-1)(H)) \cong \text{Ext}^1_H(DH, \tau^{-}(m-1)(H)) \\
\cong \text{Ext}^1_H(DH, H) \otimes_H \tau^{-}(m-1)(H) \\
\cong \text{Ext}^1_H(DH, H) \otimes_H \text{Ext}^1_H(DH, H) \otimes (m-1)
\]

where the last isomorphism follows by induction. By Corollary 9.6 we know that \( \Pi \cong T_H(\text{Ext}^1_H(DH, H)) \). The result follows. □

**Corollary 10.13.** \( \Pi \) is finite-dimensional if and only if \( C \) is of Dynkin type.

**Proof.** This follows directly from Theorems 10.10 and 10.12 and the fact that \( \Delta^+(C) \) is finite if \( C \) is of Dynkin type. □

10.4. **Regular components of the Auslander-Reiten quiver.** A connected component \( C \) of the Auslander-Reiten quiver of \( H \) is regular if it consists only of regular modules.

**Proposition 10.14.** For a connected component \( C \) of the Auslander-Reiten quiver of \( H \) the following are equivalent:

(i) \( C \) contains a regular module;

(ii) \( C \) is regular.

**Proof.** Trivially, (ii) implies (i). For the other direction assume that \( M \) is a regular module in \( C \). Let \( 0 \to \tau(M) \to E \to M \to 0 \) be the Auslander-Reiten sequence ending in \( M \). Applying \( \tau^k(-) \) yields again an Auslander-Reiten sequence

\[
0 \to \tau^{k+1}(M) \to \tau^k(E) \to \tau^k(M) \to 0
\]

for each \( k \in \mathbb{Z} \). Here we used that \( \tau^{k+1}(M) \) and \( \tau^k(M) \) and therefore also \( \tau^k(E) \) have projective and injective dimension equal to 1. It follows that \( \tau^k(N) \) is locally free for each indecomposable direct summand \( N \) of \( E \). Now (ii) follows by induction. □

Let \( C \) be a connected component of the Auslander-Reiten quiver of \( H \). Suppose \( C \) contains an indecomposable projective module \( P_c \) with \( c_i \geq 2 \). Then \( \text{rad}(P_c) \) is obviously not locally free. Thus \( \text{rad}(P_i) \) contains an indecomposable direct summand \( R \), which is not locally free. Since the inclusion \( \text{rad}(P_i) \to P_i \) is a sink map, there is an arrow \( [R] \to [P_i] \) in the Auslander-Reiten quiver of \( H \). Thus \( C \) contains a module, which is not locally free.
Ringel [Ri2] proved that the regular components of the Auslander-Reiten quiver of a wild hereditary algebra are always of type $\mathbb{Z}A_\infty$. An alternative proof is due to Crawley-Boevey [CB1, Section 2] and can easily be adapted to obtain the following theorem.

**Theorem 10.15.** Assume that $C$ is connected and neither of Dynkin nor of Euclidean type. Let $C$ be a regular component of the Auslander-Reiten quiver of $H$. Then $C$ is a component of type $\mathbb{Z}A_\infty$.

### 11. Reflection functors and APR-tilting

The following result is a generalization of Auslander, Platzeck and Reiten’s [APR] ground breaking interpretation of BGP-reflection functors as homomorphism functors of certain tilting modules. Their result was the beginning of tilting theory. Let $H = H(C, D, \Omega)$ and $Q = Q(C, \Omega)$.

**Theorem 11.1.** For each sink $i$ in $Q^\circ$ there is a functorial isomorphism

$$F_i^+(-) \cong \text{Hom}_H(T, -) : \text{rep}(H) \to \text{rep}(s_i(H))$$

where $T := H/E_i \oplus \tau^-(E_i)$.

**Proof.** Let $E'_i$ be the right $H$-module such that $D(E'_i) \cong E_i$. Similarly to Proposition 3.1 there is a canonical exact sequence

$$0 \to \bigoplus_{j \in \Omega(i,-)} iH_j \otimes_j e_j H \xrightarrow{\text{can}} e_i H \xrightarrow{p} E'_i \to 0 \quad (11.1)$$

where $p$ is the canonical projection, and for $j \in \Omega(i,-)$ the $j$th component of the map can is just the multiplication map

$$H_{ij} : iH_j \otimes_j e_j H \to H_i.$$

(The maps $H_{ij}$ are the structure maps of the (left) $H$-module structure on $H$.)

Recall that we have an isomorphism $jH_i \cong D(iH_j)$ of $H_j$-$H_i$-bimodules. Under the isomorphism

$$D(e_i H) \otimes_j D(iH_j) \to D(iH_j \otimes_j e_j H)$$

defined by

$$\iota \otimes \eta \mapsto (h \otimes e_j h' \mapsto \eta(h)e_j h')$$

the dual of the multiplication map $H_{ij}$ is identified with the map

$$H^*_{ij} : D(e_i H) \to D(e_j H) \otimes_j D(iH_j)$$

defined by

$$\varphi \mapsto \sum_{r \in iR_j} \varphi(r \cdot -) \otimes r^\vee.$$

We have $D(e_i H) \cong I_i$ and $D(e_j H) \otimes_j D(iH_j) \cong I_j \otimes_j jH_i$. Applying the duality $D$ to (11.1) we get a minimal injective resolution

$$0 \to E_i \to I_i \xrightarrow{(H^*_{ij})_{j \in \Omega(i,-)}} \bigoplus_{j \in \Omega(i,-)} I_j \otimes_j jH_i \to 0.$$

Set

$$\theta_{ij} := \nu^{-1}_H(H^*_{ij}) : P_i \to P_j \otimes_j jH_i$$

which is given by $he_i \mapsto \sum_{r \in iR_j} hr \otimes r^\vee$. 
For $M \in \text{rep}(H)$ we have a functorial isomorphism
\[ \eta_{ij}^M : \text{Hom}_H(P_j \otimes_j jH_i, M) \to iH_j \otimes j M_j \]
defined by
\[ f \mapsto \sum_{r \in R_j} r \otimes f(e_j \otimes r^\vee). \]
Now for each $j \in \Omega(i, -)$ we get a commutative diagram
\[
\begin{array}{ccc}
\text{Hom}_H(P_j \otimes_j jH_i, M) & \xrightarrow{\text{Hom}_H(\theta_{ij}, M)} & \text{Hom}_H(P_i, M) \\
\downarrow{\eta_{ij}^M} & & \downarrow{\eta_i^M} \\
iH_j \otimes j M_j & \xrightarrow{M_{ij}} & M_i
\end{array}
\]
where $\eta_i^M$ is the evaluation map $g \mapsto g(e_i)$. This follows from the observation that for $f \in \text{Hom}_H(P_j \otimes_j jH_i, M)$ and $r \in iR_j$ we have
\[ f(r \otimes r^\vee) = M_{ij}(r \otimes f(e_j \otimes r^\vee)). \]
We obtain a commutative diagram
\[
\begin{array}{ccc}
0 & \xrightarrow{\eta} & \bigoplus_j \text{Hom}_H(P_j \otimes_j jH_i, M) \\
0 & \xrightarrow{\eta} & \text{Ker}(M_{i,in})
\end{array}
\begin{array}{ccc}
\xrightarrow{\text{Hom}_H(\theta, M)} & \text{Hom}_H(P_i, M) \\
\bigoplus_j \eta_{ij}^M & M_{i,in} & \eta_i^M
\end{array}
\]
with exact rows, where $\theta := (\theta_{ij})_{j \in \Omega(i, -)}$ and the direct sums are taken over all $j \in \Omega(i, -)$. The maps $\bigoplus_j \eta_{ij}^M$ and $\eta_i^M$ are isomorphisms. Thus $\eta$ is an isomorphisms as well. This implies the functorial isomorphism $F_i^+(-) \cong \text{Hom}_H(T, -)$.

We leave it as an exercise to formulate a dual version of Theorem 11.1.

12. Examples

12.1. The matrix
\[ C = \begin{pmatrix} 2 & -4 & 0 \\ -6 & 2 & -3 \\ 0 & -9 & 2 \end{pmatrix} \]
is a Cartan matrix, and $D = \text{diag}(9, 6, 2)$ is the minimal symmetrizer of $C$. Let $\Omega = \{(1, 2), (2, 3)\}$. This is an orientation of $C$. We have $f_{12} = 2$, $f_{21} = 3$, $f_{23} = 1$, $f_{32} = 3$, $g_{12} = 2$ and $g_{23} = 3$. The algebra $H = H(C, D, \Omega)$ is given by the quiver
\[
\begin{array}{ccc}
\epsilon_1 & \epsilon_2 & \epsilon_3 \\
1 & 2 & 3
\end{array}
\]
with relations
\[ \epsilon_1^0 = 0, \quad \epsilon_2^0 = 0, \quad \epsilon_3^0 = 0, \]
\[ \epsilon_1^3 = (g) = \alpha_{12}^3 \epsilon_2^2, \quad (g = 1, 2), \]
\[ \epsilon_2^3 = (g) = \alpha_{23}^3 \epsilon_3^2, \quad (g = 1, 2, 3). \]
(Recall that $\alpha_{ij}^g$ denotes an arrow $j \to i$.)
The preprojective algebra \( \Pi = \Pi(C, D) \) is given by the double quiver \( \overline{Q}(C) \) with relations

\[
\begin{align*}
\varepsilon_1^9 &= 0, \\
\varepsilon_2^6 &= 0, \\
\varepsilon_3^2 &= 0,
\end{align*}
\]

\[
\begin{align*}
\varepsilon_3^3 \alpha_{12}^g &= \alpha_{12}^g \varepsilon_2^g, \\
\varepsilon_2^2 \alpha_{21}^g &= \alpha_{21}^g \varepsilon_1^g, \\
\varepsilon_3^2 \alpha_{23}^g &= \alpha_{23}^g \varepsilon_3^g,
\end{align*}
\]

\[
\begin{align*}
\sum_{g=1}^{2} \left( \alpha_{12}^g \alpha_{21}^g \varepsilon_1^g + \varepsilon_1 \alpha_{12}^g \alpha_{21}^g \varepsilon_1^g + \varepsilon_1 \alpha_{12}^g \alpha_{21}^g \varepsilon_1^g \right) &= 0,
\end{align*}
\]

\[
\begin{align*}
\sum_{g=1}^{2} \left( -\alpha_{21}^g \alpha_{12}^g \varepsilon_2^g - \varepsilon_2 \alpha_{21}^g \alpha_{12}^g \varepsilon_2^g - \varepsilon_2 \alpha_{21}^g \alpha_{12}^g \varepsilon_1^g \right) + \sum_{g=1}^{3} \left( \alpha_{23}^g \alpha_{32}^g \varepsilon_2^g + \varepsilon_2 \alpha_{23}^g \alpha_{32}^g \varepsilon_2^g + \varepsilon_2 \alpha_{23}^g \alpha_{32}^g \varepsilon_2^g \right) &= 0,
\end{align*}
\]

\[
\sum_{g=1}^{3} \alpha_{23}^g \alpha_{32}^g = 0.
\]

12.2. Cartan matrices of Dynkin type. Figure [3] shows a list of valued graphs called Dynkin graphs. By definition each of the graphs \( A_n, B_n, C_n, \) and \( D_n \) has \( n \) vertices. The graphs \( A_n, D_n, E_6, E_7, E_8 \) are the simply laced Dynkin graphs. A Cartan matrix \( C \) is of Dynkin type if the valued graph \( \Gamma(C) \) is isomorphic (as a valued graph) to a disjoint union of Dynkin graphs.

12.3. Finite representation type. Let \( H = H(C, D, \Omega) \) with \( D = \text{diag}(c_1, \ldots, c_n) \). Without loss of generality assume that \( C \) is connected. We only sketch the proof of the following proposition.

**Proposition 12.1.** The algebra \( H \) is representation-finite if and only if we are in one of the following cases:

(i) \( C \) is of Dynkin type \( A_n, C_n, D_n, E_6, E_7, E_8, B_2, B_3 \) or \( G_2 \), and \( D \) is minimal;
(ii) \( C \) is of Dynkin type \( A_1 \);
(iii) \( C \) is of Dynkin type \( A_2 \), and we have \( (c_1, c_2) = (2, 2) \) or \( (c_1, c_2) = (3, 3) \);
(iv) \( C \) is of Dynkin type \( A_3 \), and we have \( (c_1, c_2, c_3) = (2, 2, 2) \).

**Proof.** Assume that \( D \) is minimal. For Dynkin types \( A_n, D_n, E_6, E_7, E_8 \), the algebra \( H \) is representation-finite by Gabriel’s Theorem. For type \( C_n \), the algebra \( H \) is a representation-finite string algebra. The Auslander-Reiten quiver of \( H \) for types \( B_2, B_3 \) and \( G_2 \) can be computed by covering theory and the knitting algorithm for preprojective components. They all turn out to be finite.

If \( C \) is of type \( A_1 \), then the symmetrizers are \( D = (m) \) with \( m \geq 1 \). Then \( H \cong K[\varepsilon_1]/(\varepsilon_1^m) \) is just a truncated polynomial ring, which is obviously representation-finite.

If \( C \) is of type \( A_2 \) and \( (c_1, c_2) = (2, 2) \) or \( (c_1, c_2) = (3, 3) \), then \( H \) is a representation-finite algebra, see Bongartz and Gabriel’s list *Maximal algebras with 2 simples modules* in [BG] Section 7.

If \( C \) is of type \( A_3 \) with \( (c_1, c_2, c_3) = (2, 2, 2) \), then one can again use covering theory and the knitting algorithm to check that \( H \) is representation-finite.
It is straightforward to check that these are all representation-finite cases. (One first compiles the list of all minimal algebras $H$, which are not mentioned in (i), (ii), (iii) and (iv). These are the algebras $H = H(C, D, \Omega)$ of types

- $A_2$ with $D = \text{diag}(4, 4)$;
- $A_3$ with $D = \text{diag}(3, 3, 3)$;
- $A_4$ with $D = \text{diag}(2, 2, 2, 2)$;
- $B_2$ with $D = \text{diag}(4, 2)$;
- $B_4$ with $D$ minimal;
- $D_4$ with $D = \text{diag}(2, 2, 2, 2)$;
- $F_4$ with $D$ minimal.

Then one uses covering theory and the Happel-Vossieck list to check that these minimal algebras are representation infinite.)

12.4. Notation. In the following subsections we discuss several examples. We also display the Auslander-Reiten quivers of some representation-finite algebras $H$. The $\tau$-locally free $H$-modules are marked with a double frame, the locally free $H$-modules, which are not
τ-locally free, are marked with a single frame, and the Gorenstein-projective $H$-modules, which are not projective, are encircled.

12.5. **Dynkin type $A_2$**. Let

$$C = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

with symmetrizer $D = \text{diag}(2, 2)$ and $\Omega = \{(1, 2)\}$. Thus $C$ is a Cartan matrix of Dynkin type $A_2$ with a non-minimal symmetrizer. We have $f_{12} = f_{21} = 1$. Thus $H = H(C, D, \Omega)$ is given by the quiver

$$1 \overset{\varepsilon_1}{\underset{\alpha_{21}}{\leftrightarrow}} 2 \overset{\varepsilon_2}{\underset{\alpha_{12}}{\leftrightarrow}} 1$$

with relations $\varepsilon_1^2 = \varepsilon_2^2 = 0$ and $\varepsilon_1 \alpha_{12} = \alpha_{12} \varepsilon_2$. The Auslander-Reiten quiver of $H$ is displayed in Figure 4. The numbers in the figure correspond to composition factors and basis vectors. (The three modules in the left most column have to be identified with the three modules in the right most column.) Note that $P_2 \cong I_1$ is projective-injective.

![Figure 4](image)

**Figure 4.** The Auslander-Reiten quiver of $H(C, D, \Omega)$ of type $A_2$ with $D = \text{diag}(2, 2)$.

The preprojective algebra $\Pi = \Pi(C, D)$ is given by the quiver

$$1 \overset{\varepsilon_1}{\underset{\alpha_{21}}{\leftrightarrow}} 2 \overset{\varepsilon_2}{\underset{\alpha_{12}}{\leftrightarrow}} 1$$

with relations $\varepsilon_1^2 = \varepsilon_2^2 = 0$, $\varepsilon_1 \alpha_{12} = \alpha_{12} \varepsilon_2$, $\varepsilon_2 \alpha_{21} = \alpha_{21} \varepsilon_1$, $\alpha_{12} \alpha_{21} = 0$ and $-\alpha_{21} \alpha_{12} = 0$. The indecomposable projective $\Pi$-modules are shown in Figure 5. (The arrows indicate when an arrow of the algebra $\Pi$ acts with a non-zero scalar on a basis vector.)

12.6. **Dynkin type $B_2$**. Let

$$C = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}$$

with symmetrizer $D = \text{diag}(2, 1)$ and $\Omega = \{(1, 2)\}$. The graph $\Gamma(C)$ looks as follows:

$$1 \overset{(2,1)}{\leftrightarrow} 2$$
Thus $C$ is a Cartan matrix of Dynkin type $B_2$. We have $f_{12} = 1$ and $f_{21} = 2$. Then $H = H(C, D, \Omega)$ is given by the quiver

$$
\begin{array}{ccc}
\varepsilon_1 & \varepsilon_2 \\
1 & 2 \\
\end{array}
$$

with relations $\varepsilon_1^2 = 0$ and $\varepsilon_2 = 0$. The Auslander-Reiten quiver of $H$ is shown in Figure 6. The numbers in the figure correspond to composition factors and basis vectors. (In the last two rows the two modules on the left have to be identified with the corresponding two modules on the right.)

The preprojective algebra $\Pi = \Pi(C, D)$ is given by the quiver

$$
\begin{array}{ccc}
\varepsilon_1 & \varepsilon_2 \\
1 & 2 \\
\end{array},
$$

with relations $\varepsilon_1^2 = 0$, $\varepsilon_2 = 0$, $\alpha_{12}\alpha_{21}\varepsilon_1 + \varepsilon_1\alpha_{12}\alpha_{21} = 0$ and $-\alpha_{21}\alpha_{12} = 0$. The indecomposable projective $\Pi$-modules are shown in Figure 7. (The arrows indicate when an arrow of the algebra $\Pi$ acts with a non-zero scalar on a basis vector.)

12.7. Dynkin type $B_3$. Let

$$
C = \begin{pmatrix}
2 & -1 & 0 \\
-1 & 2 & -1 \\
0 & -2 & 2
\end{pmatrix}
$$
with symmetrizer $D = \text{diag}(2, 2, 1)$ and $\Omega = \{(1, 2), (2, 3)\}$. The graph $\Gamma(C)$ looks as follows:

Thus $C$ is a Cartan matrix of Dynkin type $B_3$. We have $f_{12} = f_{21} = 1$, $f_{23} = 1$ and $f_{32} = 2$. Thus $H = H(C, D, \Omega)$ is given by the quiver

with relations $\varepsilon_1^2 = \varepsilon_2^2 = 0$, $\varepsilon_3 = 0$ and $\varepsilon_1 \alpha_{12} = \alpha_{12} \varepsilon_2$. The Auslander-Reiten quiver of $H$ is shown in Figure 8. As vertices we have the graded dimension vectors (arising from the obvious $\mathbb{Z}$-covering of $H$) of the indecomposable $H$-modules. (In the last three rows the three modules on the left have to be identified with the corresponding three modules on the right.) The indecomposable $H$-module $M$ with graded dimension vector

is locally free. (It is a direct summand of an extension of locally free modules.) We have $\text{rank}(M) = (1, 2, 1)$. In the root lattice of $C$ this corresponds to $\alpha_1 + 2\alpha_2 + \alpha_3$. Thus we have $\text{rank}(M) \notin \Delta^+(C)$. 

**Figure 7.** The indecomposable projective $\Pi(C, D)$-modules for type $B_2$ with $D$ minimal.
Figure 8. The Auslander-Reiten quiver of $H(C, D, \Omega)$ of type $B_3$ with $D$ minimal.
12.8. **Dynkin type $C_3$.** Let

$$C = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix}$$

with symmetrizer $D = \text{diag}(1,1,2)$ and $\Omega = \{(1,2),(2,3)\}$. The graph $\Gamma(C)$ looks as follows:

Thus $C$ is a Cartan matrix of Dynkin type $C_3$. We have $f_{12} = f_{21} = 1$, $f_{23} = 2$ and $f_{32} = 1$. Then $H = H(C,D,\Omega)$ is given by the quiver

$$\begin{array}{c}
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\downarrow \\
\alpha_{12} \\
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12.9. **Dynkin type** $G_2$. Let

$$C = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$$

with symmetrizer $D = \text{diag}(1, 3)$ and $\Omega = \{(1, 2)\}$. The graph $\Gamma(C)$ looks as follows:

Thus $C$ is a Cartan matrix of Dynkin type $G_2$. We have $f_{12} = 3$ and $f_{21} = 1$. Thus $H = H(C, D, \Omega)$ is given by the quiver

with relations $\varepsilon_1 = 0$ and $\varepsilon_2^3 = 0$. The Auslander-Reiten quiver of $H$ is displayed in Figure 11. As vertices we have the graded dimension vectors (arising from the obvious $\mathbb{Z}$-covering of $H$) of the indecomposable $H$-modules. (The three modules in the left most column have to be identified with the three modules in the right most column.)

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