Form Factors and Semileptonic Decay of $D_s^+ \rightarrow \phi \bar{\ell} \nu$ From QCD Sum Rule

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Abstract

We calculate $D_s^+ \rightarrow \phi$ transition form factors $V$, $A_0$, $A_1$ and $A_2$, and study semileptonic decay of $D_s^+ \rightarrow \phi \bar{\ell} \nu$ based on QCD sum rule method. We compare our results of the ratios of $V(0)/A_1(0)$, $A_2(0)/A_1(0)$, $\Gamma_L/\Gamma_T$, and the total decay branching ratio of $D_s^+ \rightarrow \phi \bar{\ell} \nu$ with experimental data, and find that they are consistent.

1. Introduction

Semileptonic decay of charm meson is important for studying strong and weak interactions. It can be used to test techniques developed for solving perturbative and nonperturbative problems in Quantum Chromodynamics (QCD), and to extract elements of Cabibbo-Kobayashi-Maskawa (CKM) matrix. Semileptonic decay is simpler than hadronic decay of charm meson because leptons do not involve strong interaction. The amplitude of semileptonic decay can be decomposed into several transition form factors due to Lorentz property of the hadronic matrix element. The form factors include all the nonperturbative effects. Several methods can be used to treat these problems, such as quark model, QCD sum rule and Lattice QCD, among which, QCD sum rule and Lattice QCD are based on the first principle of QCD.

The method of QCD sum rules [1] has been widely used in hadronic physics since its establishment in the late 1970s. For semileptonic decays of charm meson, $D^+ \rightarrow K^0 e^+ \nu_e$ was firstly studied in QCD sum rule method with three-point correlation function [2]. Several years later, QCD sum rule method was extended to semileptonic decays of $B$ meson, $B \rightarrow D(D^*) \bar{\ell} \nu$ [3] and $B \rightarrow \pi e \nu$ [4]. In these works, form factors $f_+(q^2)$ and $f_V(q^2)$ are calculated at the point $q^2 = 0$, where $q^2$ is the momentum transfer squared. For the whole physical region of $0 \leq q^2 \leq q^2_{\text{max}}$, the form factors are either assumed to be pole dominance $f(0)/(1 - q^2/m_{\text{pol}}^2)$, or a linear approximation...
was used. In Refs [5, 6], $D \to \bar{K}^0 e^+ \nu_e$, $\bar{K}^0 e^+ \nu_e$ and $D \to \pi e^+ \bar{\nu}_e$, $\bar{K}^0 \ast e^+ \nu_e$ and $D \to \pi e^+ \bar{\nu}_e$ were studied, where QCD sum rule method was extended to a very large value of $q^2$ with a careful treatment of non-Landau-type singularities. $D_s$ decays to $\eta$ and $\eta'$ final states were studied in [7].

In this work, we study $D_s^+ \to \phi \bar{\ell} \nu$ in QCD sum rule method. This decay mode has been measured in experiment long time before [8, 9, 10, 11]. Now it is necessary to analyze it theoretically. We calculate up to contributions of operators of dimension 6 in the operator product expansion (OPE) and keep the mass of s-quark. In our result, the large contributions come from unit operator $I$ (result of perturbative diagram) and condensate of operators of dimension 3. Operator of dimension 5 gives smaller contribution. The contributions of operators of dimension 6 are negligible. When calculating contribution of perturbative diagram and gluon condensate (operator of dimension 4), Cutkosky’s rule has been used. Therefore subtraction of continuum contribution is conveniently performed not only for perturbative diagram but also for contribution of gluon condensate. After some long steps of calculation, we find the contributions of diagrams for the gluon condensate cancel each other, so there is no gluon-condensate contribution in $D_s^+ \to \phi$ transition. This is our new finding.

There are two independent Borel parameters $M_1^2$ and $M_2^2$ in manipulating three-point correlation functions. In general, to simplify the numerical analysis, a fixed ratio of $M_1^2/M_2^2$ was taken in recent references. In this work, we make the numerical analysis in the whole region of independent $M_1^2$ and $M_2^2$ to select the stability “window”, so it is different from taking a fixed ratio of these two Borel parameters.

We calculate $D_s \to \phi$ transition form factors $V$, $A_0$, $A_1$, $A_2$ and the branching ratio of $D_s^+ \to \phi \bar{\ell} \nu$. Our result of the ratios of $V(0)/A_1(0)$, $A_2(0)/A_1(0)$, $\Gamma_L/\Gamma_T$ ($\Gamma_L$ and $\Gamma_T$ denote decay width of $D_s^+$ to $\phi$ meson in longitudinal and transverse polarization, respectively), and the total branching fraction are in agreement with experimental data.

Recently, just before this work is finished, we find that $D_s^+ \to \phi \bar{\ell} \nu$ was also calculated in Ref.[12]. However, their analysis is very different from ours. First, by carefully choosing the requirement that the double Borel parameters $M_1^2$ and $M_2^2$ should not be too large for keeping the continuum contribution small, and at the same time, $M_1^2$ and $M_2^2$ should not be too small for keeping the truncated OPE series effective, i.e., keeping the contributions of higher dimension operators small, we get very different stability “window” for the Borel parameters. Second, our results of the transition form factors are different from theirs. Especially for $A_2$, they got negative value, however, we get positive. Using their values of the form factors, although one can get the total branching ratio of $D_s^+ \to \phi \bar{\ell} \nu$ to be compatible with experimental result, the ratio of $\Gamma_L/\Gamma_T$ will be too large. But in our case, $\Gamma_L/\Gamma_T = 0.99 \pm 0.43$ which is consistent with world average $0.72 \pm 0.18$.

The paper is organized as follows. In section 2, we briefly introduce the QCD sum rule method used in this work. Section 3 is the calculation. Section 4 is the numerical analysis and discussion. Section 5 is devoted to the summary.
2. The method

To calculate the transition form factors of semileptonic $D_s$ meson decays, the standard procedure in QCD sum rule method is to consider the three-point correlation function defined as

$$\Pi_{\mu\nu} = i^2 \int d^4x d^4y e^{ip_0x - ip_1y} \langle 0| T\{ j^\phi_\nu(x) j^\mu_\nu(0) j^D_5(y)\} |0\rangle,$$

(1)

with the currents having the same quantum numbers as the relevant mesonic states under consideration, which are defined by: 1) the current of $D_s$ channel, $j^D_5(y) = \bar{c}(y)i\gamma_5s(y)$; 2) the current of weak transition: $j_\mu(0) = s\gamma_\mu(1 - \gamma_5)c$; 3) the current of $\phi$ channel: $j^\phi_\mu(x) = \bar{s}(x)\gamma_\mu s(x)$. On one hand, inserting a complete set of intermediate hadronic states into the correlation function, and using the double dispersion relation, one can express the correlation function in terms of a set of hadronic states,

$$\Pi_{\mu\nu} = \int ds_1 ds_2 \frac{\rho(s_1, s_2, q^2)}{(s_1 - p_1^2)(s_2 - p_2^2)},$$

(2)

where

$$\rho(s_1, s_2, q^2) = \sum_{XY} \langle 0| j^\phi_\mu(X) \langle X| j_\mu|Y\rangle \langle Y| j^D_5|0\rangle \delta(s_1 - m^2_X) \delta(s_2 - m^2_Y) \theta(p_0^X) \theta(p_0^Y),$$

with $X$ and $Y$ denote the complete set of hadronic states of $\phi$ and $D_s$ channels, respectively. $p_X$ and $p_Y$ are the four-momentum of $X$ and $Y$ states, $s_1 = p_X^2$, $s_2 = p_Y^2$, and $q = p_1 - p_2$. Integrate over $s_1$ and $s_2$ in Eq. (2), we can obtain

$$\Pi_{\mu\nu} = \sum_{XY} \frac{\langle 0| j^\phi_\mu|\phi\rangle \langle \phi| j_\mu|D_s\rangle \langle D_s| j^D_5|0\rangle}{(m^2_\phi - p_1^2)(m^2_X - p_2^2)} + \text{continuum states.}$$

(3)

Separate the ground states of $D_s$ and $\phi$ channels apparently, the above equation becomes

$$\Pi_{\mu\nu} = \frac{\langle 0| j^\phi_\mu|\phi\rangle \langle \phi| j_\mu|D_s\rangle \langle D_s| j^D_5|0\rangle}{(m^2_{D_s} - p_1^2)(m^2_\phi - p_2^2)} + \text{higher resonances and continuum states.}$$

(4)

The weak transition matrix element $D_s \to \phi$ can be decomposed as

$$\langle \phi(\varepsilon, p_2)| j_\mu|D_s(p_1)\rangle = \varepsilon_{\mu\alpha\beta} \varepsilon^{*\nu\gamma} p_1^\alpha p_2^\beta \frac{2V(q^2)}{m_{D_s} + m_\phi}$$

$$-i(\varepsilon^*_\mu - \frac{\varepsilon^*}{q^2} q_\mu) (m_{D_s} + m_\phi) A_1(q^2)$$

$$+ i[(p_1 + p_2)_\mu - \frac{m^2_{D_s} - m^2_\phi}{q^2} q_\mu] \varepsilon^* \cdot q \frac{A_2(q^2)}{m_{D_s} + m_\phi}$$

$$- i \frac{2m_\phi \varepsilon^* \cdot q}{q^2} q_\mu A_0(q^2),$$

(5)
where \( q = p_1 - p_2 \). The vacuum-to-meson transition amplitudes can be parameterized through defining the corresponding decay constants,

\[
\langle 0 | \bar{s}_i \gamma_\nu s | \phi \rangle = m_\phi f_\phi \varepsilon_\nu^{(A)},
\]

\[
\langle 0 | \bar{s}_i \gamma_5 c | D_s \rangle = \frac{f_D m_D^2}{m_c + m_s}.
\]  

(6)

Finally the correlation function can be expressed in terms of meson decay constants and \( D_s \rightarrow \phi \) transition matrix element,

\[
\Pi_{\mu \nu} = \frac{m_\phi f_\phi \varepsilon_\nu^{(A)} \langle \phi (\varepsilon_\mu^{(A)}, p_2) | j_\mu | D_s(p_1) \rangle f_D m_D^2}{(m_D^2 - p_1^2)(m_\phi^2 - p_2^2)(m_c + m_s)} + \text{higher resonances and continuum states.}
\]  

(7)

On the other hand, the correlation function of Eq. (1) can be evaluated at negative values of \( p_1^2 \) and \( p_2^2 \) by the operator-product expansion in QCD, in which the time-ordered current operators in Eq. (1) is expanded in terms of a series of local operators with increasing dimensions,

\[
i^2 \int d^4 x d^4 y e^{i p^2 x - i p_1 y} T \{ j_\mu(x) j_\mu(0) j_\nu^D(y) \} = C_{0 \mu \nu} I + C_{3 \mu \nu} \bar{\Psi} \Psi + C_{4 \mu \nu} G_{\alpha \beta}^a G^{a\alpha \beta} + C_{5 \mu \nu} \bar{\Psi} \sigma_{\alpha \beta} T^a G^{a\alpha \beta} \Psi + \cdots,
\]  

(8)

where \( C_{i \mu \nu} \)'s are Wilson coefficients, \( I \) is the unit operator, \( \bar{\Psi} \Psi \) is the local Fermion field operator of light quarks, \( G_{\alpha \beta}^a \) is gluon strength tensor, \( \Gamma \) and \( \Gamma' \) are the matrices appearing in the procedure of calculating the Wilson coefficients. Sandwich the left and right hand sides of Eq. (8) between two vacuum states, we get the correlation function in terms of Wilson coefficients and condensates of local operators,

\[
\Pi_{\mu \nu} = i^2 \int d^4 x d^4 y e^{i p^2 x - i p_1 y} \langle 0 | T \{ j_\mu(x) j_\mu(0) j_\nu^D(y) \} | 0 \rangle = C_{0 \mu \nu} I + C_{3 \mu \nu} \langle 0 | \bar{\Psi} \Psi | 0 \rangle + C_{4 \mu \nu} \langle 0 | G_{\alpha \beta}^a G^{a\alpha \beta} | 0 \rangle + C_{5 \mu \nu} \langle 0 | \bar{\Psi} \sigma_{\alpha \beta} T^a G^{a\alpha \beta} \Psi | 0 \rangle + \cdots
\]  

(9)

For later convenience, we shall reexpress the above equation. In general, it can be expressed in terms of six independent Lorentz structures

\[
\Pi_{\mu \nu} = -f_0 \varepsilon_{\mu \alpha \beta} p_1^\alpha p_2^\beta - i (f_1 p_1 \mu p_1 \nu + f_2 p_2 \mu p_2 \nu + f_3 p_1 \nu p_2 \mu + f_4 p_1 \mu p_2 \nu + f_5 g_{\mu \nu}).
\]  

(10)

Each \( f_i \) includes perturbative and condensate contributions

\[
f_i = f_i^{\text{pert}} + f_i^{(3)} + f_i^{(4)} + f_i^{(5)} + f_i^{(6)} + \cdots
\]  

(11)

where \( f_i^{(3)}, \cdots, f_i^{(6)} \) are contributions of condensates of dimension 3, 4, 5, 6, \cdots in Eq. (10). In next section we can see that perturbative contribution and gluon condensate contribution can be finally written in the form of dispersion integration,

\[
f_i^{\text{pert}} = \int ds_1 ds_2 \frac{\rho_i^{\text{pert}}(s_1, s_2, q^2)}{(s_1 - p_1^2)(s_2 - p_2^2)},
\]

\[
f_i^{(4)} = \int ds_1 ds_2 \frac{\rho_i^{(4)}(s_1, s_2, q^2)}{(s_1 - p_1^2)(s_2 - p_2^2)}.
\]
We approximate the contribution of higher resonances and continuum states as integrations over some thresholds $s_1^0$ and $s_2^0$ in the above equations. Then equate the two representations of the correlation function in Eq. (7) and (10), we can get an equation for the form factors. To improve such equation, we make Borel transformation over $p_1^2$ and $p_2^2$ in both variables, which can further suppress higher resonance contribution. The definition of Borel transformation to any function $f(p^2)$ is

$$
\hat{B}_{p^2,M^2} f(p^2) = \lim_{n \to \infty} \lim_{p^2 \to -\infty} \frac{(-p^2)^n}{(n-1)!} \partial^p \frac{\partial^n}{p^2} f(p^2).
$$

Some examples of Borel transformation is given in the following:

$$
\hat{B}_{p^2,M^2} (s - p^2) = \frac{1}{(k-1)!} \frac{1}{M^2} e^{-s/M^2}, \quad \hat{B}_{p^2,M^2} (p^2)^k = 0, \text{ for any } k \geq 0.
$$

Equating the two representations of the correlation function, subtracting the higher resonances and continuum contribution, and performing Borel transformation in both variables $p_1^2$ and $p_2^2$, we finally obtain the sum rules for the form factors,

$$
V(q^2) = \frac{(m_e + m_s)(m_{D_s} + m_{\phi})}{2m_{\phi} f_{D_s} m_{D_s}^2} e^{m_{D_s}^2/M^2} e^{m_{\phi}^2/M^2} M_1^2 M_2^2 \hat{B} f_0,
$$

$$
A_1(q^2) = -\frac{(m_e + m_s)}{m_{\phi} f_{D_s} m_{D_s}^2 (m_{D_s} + m_{\phi})} e^{m_{D_s}^2/M^2} e^{m_{\phi}^2/M^2} M_1^2 M_2^2 \hat{B} f_5,
$$

$$
A_2(q^2) = \frac{(m_e + m_s)(m_{D_s} + m_{\phi})}{m_{\phi} f_{D_s} m_{D_s}^2} e^{m_{D_s}^2/M^2} e^{m_{\phi}^2/M^2} M_1^2 M_2^2 \frac{1}{2} \hat{B} (f_1 + f_3), \quad (12)
$$

$$
A_0(q^2) = -\frac{(m_e + m_s)}{2m_{\phi}^2 f_{D_s} m_{D_s}^2} e^{m_{D_s}^2/M^2} e^{m_{\phi}^2/M^2} M_1^2 M_2^2 \hat{B} (f_1 + f_3) \frac{m_{D_s}^2 - m_{\phi}^2}{2}
$$

$$
+ \hat{B} (f_1 - f_3) \frac{q^2}{2} + f_5,
$$

where $\hat{B} f_i$ denotes Borel transforming $f_i$ in both variables $p_1^2$ and $p_2^2$, $M_1$ and $M_2$ are Borel parameters. Because we have subtracted the higher resonance and continuum contribution, now the dispersion integration for perturbative and gluon condensate contribution should be performed under the threshold,

$$
\begin{align*}
\rho_{pert}^{(1)}(s_1, s_2, q^2) &= \int_{s_1}^{s_2} ds_1 \int_{s_2}^{s_1} ds_2 \rho_{pert}(s_1, s_2, q^2) \frac{1}{(s_1 - p_1^2)(s_2 - p_2^2)}, \\
\rho_{pert}^{(4)}(s_1, s_2, q^2) &= \int_{s_1}^{s_2} ds_1 \int_{s_2}^{s_1} ds_2 \rho_{pert}(s_1, s_2, q^2) \frac{1}{(s_1 - p_1^2)(s_2 - p_2^2)}.
\end{align*}
$$

In the next section, we will explain the technique of calculating the Wilson coefficients and given the resulted form of the sum rules for the form factors.
3. The Calculation of the Wilson Coefficients

In this work, we first calculate the Wilson coefficients in the operator-product expansion [13], then extract the relevant terms $f_i$'s for the sum rules of the form factors in Eq. (12). We will not present the result of each Wilson coefficient here because their forms are very tedious. We only give the results of the form factors according to the contribution of each condensate.

3.1 The Calculation of the Perturbative part

The diagram for the perturbative contribution is depicted in Fig.1. The leading order in $\alpha_s$ expansion is considered here. This contribution amounts to Wilson coefficient $C_0$ in OPE representation of the correlation function in Eq. (9). We can write down this amplitude (see Fig.1),

![Diagram for perturbative contribution.](image)

$$C_0 = i^2 \int \frac{d^4k}{(2\pi)^4} (-1) Tr \left[ i\gamma_5 \frac{i(k + m_s)}{k^2 - m_s^2 + i\varepsilon} \frac{i(k + p_2 + m_s)}{(k + p_2)^2 - m_s^2 + i\varepsilon} \frac{i(k + p_1 + m_s)}{(k + p_1)^2 - m_c^2 + i\varepsilon} \right].$$ (13)

The above integration can be performed according to Cutkosky’s rule [14]. That is, to write the integration of Eq. (13) in the form of dispersion integration,

$$C_0 = \int ds_1 ds_2 \frac{\rho(s_1^2, s_2^2, q^2)}{(s_1 - p_1^2)(s_2 - p_2^2)}.$$ (14)

The spectral density $\rho(s_1, s_2, q^2)$ can be directly calculated by substituting the denominators of the quark propagators for $\delta$ functions, i.e., putting all the quark lines on-mass-shell,

$$\frac{1}{k^2 - m_s^2 + i\varepsilon} \rightarrow -2\pi i \delta(k^2 - m_s^2), \text{ etc.},$$ (15)
then the spectral density can be calculated from,
\[
\rho(s_1, s_2, q^2) = \frac{1}{(2\pi)^3} (-2\pi i)^3 \int \frac{d^4k}{(2\pi)^4} \text{Tr}[\gamma_5 (k + m_s)\gamma_\nu (k + p_2 + m_s)\gamma_\mu (1 - \gamma_5)]
\]
\[
\times (k + p_1 + m_u)\delta(k^2 - m^2)\delta((k + p_1)^2 - m_1^2)
\times \delta((k + p_2)^2 - m_2^2) \left| \begin{array}{c}
p_1^2 \rightarrow s_1 \\
p_2^2 \rightarrow s_2 \end{array} \right|
\] (16)

To perform the above integration, some basic formulas are needed. Part of them have been given in Ref. [15] without the quark mass, here we calculate them with the quark masses included,
\[
I = \int d^4k \delta(k^2 - m^2)\delta((k + p_1)^2 - m_1^2)\delta((k + p_2)^2 - m_2^2) = \frac{\pi}{2\sqrt{\lambda}}, \quad (17)
\]
\[
I_\mu = \int d^4k k_\mu \delta(k^2 - m^2)\delta((k + p_1)^2 - m_1^2)\delta((k + p_2)^2 - m_2^2)
\equiv a_1p_{1\mu} + b_1p_{2\mu}, \quad (18)
\]
\[
\begin{cases}
a_1 = -\frac{\pi}{2\sqrt{\lambda}} [s_2(-s_1 + s_2 - q^2) + (s_1 + s_2 - q^2)(m^2 - m_1^2) \\
- 2s_2(m^2 - m_1^2)], \\
b_1 = -\frac{\pi}{2\sqrt{\lambda}} [s_1(-s_2 + s_1 - q^2) + (s_1 + s_2 - q^2)(m^2 - m_1^2) \\
- 2s_1(m^2 - m_1^2)],
\end{cases}
\]
\[
I_{\mu\nu} = \int d^4k k_\mu k_\nu \delta(k^2 - m^2)\delta((k + p_1)^2 - m_1^2)\delta((k + p_2)^2 - m_2^2)
\equiv a_2p_{1\mu}p_{1\nu} + b_2p_{2\mu}p_{2\nu} + c_2(p_{1\mu}p_{2\nu} + p_{1\nu}p_{2\mu}) + d_2g_{\mu\nu}, \quad (19)
\]
\[
\begin{cases}
D_1 \equiv s_1 - m_1^2 + m^2, \quad D_2 \equiv s_2 - m_2^2 + m^2, \\
a_2 = \frac{\pi}{\lambda} m^2 s_2 + \frac{1}{\lambda} [3s_2 D_1 a_1 - (s_1 + s_2 - q^2)D_2 b_1 + s_2 D_3 b_1] \\
b_2 = \frac{\pi}{\lambda} m^2 s_1 + \frac{1}{\lambda} [s_1 D_1 a_1 - (s_1 + s_2 - q^2)D_1 b_1 + 3s_1 D_3 b_1] \\
c_2 = -\frac{\pi}{\lambda} m^2 \frac{1}{2}(s_1 + s_2 - q^2) \\
- \frac{1}{\lambda} [\frac{1}{2}(s_1 + s_2 - q^2)D_1 a_1 - 2s_2 D_1 b_1 + \frac{3}{2}(s_1 + s_2 - q^2)D_2 b_1] \\
d_2 = \frac{\pi}{\sqrt{\lambda}} + \frac{1}{4} [D_1 a_1 + D_2 b_1]
\end{cases}
\]

where \(\lambda(s_1, s_2, q^2) = (s_1 + s_2 - q^2) - 4s_1s_2\), and in Eqs. (17) \(\sim (19)\), substitutions \(p_1^2 \rightarrow s_1\) and \(p_2^2 \rightarrow s_2\) have been indicated.

### 3.2 The Contribution of bi-quark Operators \(\bar{\Psi}(x)\Psi(y)\), \(\bar{\Psi}(0)\Psi(x)\)

The diagrams for the contributions of \(\bar{\Psi}(x)\Psi(y)\) and \(\bar{\Psi}(0)\Psi(x)\) are shown in Fig. 2. The contribution of Fig. 2(b) is zero after double Borel transformation in both
variables $p_1^2$ and $p_2^2$ because only one variable appears in the denominator \(1/(p_2^2 - m_s^2)\). So we will not consider Fig. 2(b) in the following. The contribution of Fig. 2(a) to the correlation function is

\[
\Pi_{\mu\nu}^{2a} = i^2 \int d^4x d^4y e^{ip_2 \cdot x - ip_1 \cdot y} \langle 0 | \bar{\Psi}(x) \gamma_\mu iS_F^a(x) \gamma_\nu (1 - \gamma_5) iS_F(c)(-y) i\gamma_5 \Psi(y) | 0 \rangle, \tag{20}
\]

where $iS_F(x)$ and $iS_F(-y)$ are the propagators of $s$ and $c$ quarks, respectively. Move the quark field operators $\bar{\Psi}(x)$ and $\Psi(y)$ together, we get

\[
\Pi_{\mu\nu}^{2a} = i^2 \int d^4x d^4y e^{ip_2 \cdot x - ip_1 \cdot y} \langle 0 | \bar{\Psi}(x) \gamma_\mu iS_F^a(x) \gamma_\nu (1 - \gamma_5) iS_F(c)(-y) i\gamma_5 \Psi(y) | 0 \rangle, \tag{21}
\]

where $\alpha$ and $\beta$ are Dirac spinor indices. The matrix element $\langle 0 | \bar{\Psi}(x) \Psi(y) | 0 \rangle$ can be dealt with in the fixed-point gauge \[16\]. We expand it up to the order of $x^3$ and $y^3$ using the technique explained in \[16\].

\[
\langle 0 | \bar{\Psi}(x) \gamma_\mu iS_F^a(x) \gamma_\nu (1 - \gamma_5) iS_F(c)(-y) i\gamma_5 \Psi(y) | 0 \rangle = \delta_{ab} \left[ \langle \bar{\Psi}(\Psi) \right] \left( \frac{1}{12} \delta_{\beta\alpha} + i \frac{m}{48} (\beta - \gamma_5)_{\beta\alpha} - \frac{m^2}{96} (x - y)^2 \delta_{\beta\alpha} 
\right.
\]

\[
- \frac{i m^3}{3! 96} (x - y)^2 (\beta - \gamma_5)_{\beta\alpha} + g \langle \bar{\Psi}(T G \sigma) \Psi \rangle \left( \frac{1}{192} (x - y)^2 \delta_{\beta\alpha} \right.
\]

\[
+ \frac{i m}{3! 192} (x - y)^2 (\beta - \gamma_5)_{\beta\alpha} - \frac{i g^2}{3! 3^4 \times 2^4} \langle \bar{\Psi}(\Psi) \rangle^2 (x - y)^2 (\beta - \gamma_5)_{\beta\alpha} + \cdots \right]. \tag{22}
\]

\(a\) and \(b\) in the above are the color indices, \(m\) is the quark mass, and the ellipsis stands for terms of higher orders in \(x\) and \(y\) expansion. From Eq.\[22\] we know that Fig.\[2\](a) contributes to the coefficients of quark condensate $\langle \bar{\Psi}(\Psi) \rangle$, mixed quark-gluon condensate $g \langle \bar{\Psi}(T G \sigma) \Psi \rangle$ and the four-quark condensate $\langle \bar{\Psi}(\Psi) \rangle^2$. Substitute Eq.\[22\] into \[21\] and integrate over the coordinates \(x\) and \(y\), we can obtain explicitly the coefficients of these condensates contributed by Fig.\[2\](a).
3.3 Contributions of bi-Gluon Operator $G^a_{\mu\nu} G^{a\mu\nu}$

The diagrams for the contribution of bi-gluon operator are depicted in Fig.3. They are calculated in the fixed-point gauge, in which the gauge fixing condition is taken to be $x^\mu A^a_\mu(x) = 0$ [13]. Then the external gauge field can be expressed directly in terms of the color field strength tensor [18],

$$A^a_\mu(x) = \int_0^1 d\alpha x^\rho G^a_{\rho\mu}(\alpha x),$$

which is expanded to the first order to be,

$$A^a_\mu(x) = \frac{1}{2} x^\rho G^a_{\rho\mu}(0) + \cdots.$$  \hfill (24)

In the following calculation, it is convenient to transform $A^a_\mu(x)$ to the momentum space,

$$A^a_\mu(k) = -i \frac{(2\pi)^2}{2} \frac{\partial}{\partial k_\rho} \delta^4(k) G^a_{\rho\mu}(0) + \cdots.$$  \hfill (25)

![Diagrams](image)

Figure 3: Diagrams for contributions of bi-gluon operator.

Then the amplitude can be written down in the momentum space by following the standard Feynman rule. Again, as what we did in the previous subsections, we move the gluon strength tensor operator together: $G^a_{\alpha\sigma} G^{b}_{\beta\rho}$. Then using the following decomposition to obtain the bi-gluon condensate,

$$\langle 0| G^a_{\alpha\sigma} G^{b}_{\beta\rho}|\rho \rangle = \frac{1}{96} \langle GG \rangle \delta_{ab} (g_{\alpha\beta} g_{\sigma\rho} - g_{\alpha\rho} g_{\sigma\beta}),$$

\hfill (26)
in which \langle GG \rangle is the abbreviation of \langle 0 | C^a_{\mu\nu} G^{a\mu\nu} | 0 \rangle.

In the evaluation of the diagrams of Fig. 3 some types of loop integrals encountered are treated at first by derivatives with respect to the quark masses, then transform them to dispersion integrals by using Cutkosky’s rule and with help of \( I, I_{\mu} \) and \( I_{\mu\nu} \) functions given previously. For instance,

\[
\int d^4 k \frac{k_{\mu} k_{\nu}}{(k^2 - m^2)^2[(k + p_2)^2 - m_2^2][(k + p_1)^2 - m_1^2]^2} = \frac{\partial}{\partial m_2^2} \frac{\partial}{\partial m_1^2} \int d^4 k \frac{k_{\mu} k_{\nu} k \cdot p_2}{(k^2 - m^2)^2[(k + p_2)^2 - m_2^2][(k + p_1)^2 - m_1^2]^2}
\]

\[
= -2\pi i \frac{\partial}{\partial m_2^2} \frac{\partial}{\partial m_1^2} \int ds_1 ds_2 \frac{I_{\mu\nu}}{(s_1 - p_1^2)(s_2 - p_2^2)}, \quad (27)
\]

and

\[
\int d^4 k \frac{k_{\mu} k_{\nu} k \cdot p_2}{(k^2 - m^2)^2[(k + p_2)^2 - m_2^2][(k + p_1)^2 - m_1^2]^2} = \frac{\partial}{\partial m_2^2} \frac{\partial}{\partial m_1^2} \int d^4 k \frac{k_{\mu} k_{\nu} k \cdot p_2}{(k^2 - m^2)^2[(k + p_2)^2 - m_2^2][(k + p_1)^2 - m_1^2]^2}
\]

\[
= -2\pi i \frac{\partial}{\partial m_2^2} \frac{\partial}{\partial m_1^2} \int ds_1 ds_2 \frac{-\frac{1}{2}(s_2 - m_2^2 + m^2) I_{\mu\nu}}{(s_1 - p_1^2)(s_2 - p_2^2)}, \quad (28)
\]

where the term \(-\frac{1}{2}(s_2 - m_2^2 + m^2)\) comes from the \( \delta \) functions \( \delta(k^2 - m^2) \delta[(k + p_2)^2 - m_2^2] \) with the substitution \( p_2^2 \rightarrow s_2 \) when using the Cutkosky’s rule.

After some long steps of calculation, we finally find that the contributions of the diagrams (a) to (f) in Fig. 3 cancel each other. Therefore there are no contributions of gluon condensate in \( D_s \rightarrow \Phi \) transition.

3.4 Contributions of Quark-Gluon mixing and Four-Quark Operators: \( \bar{\Psi}(x)\Psi(y)G^a_{\mu\nu} \) and \( \langle \bar{\Psi} \Psi \rangle^2 \)

The diagrams for quark-gluon mixing and four-quark contributions are depicted in Fig. 4 and Fig. 5, respectively. The techniques are similar to that explained in previous subsections. We only give some different points here.

The vacuum average of non-local quark-gluon mixing operator \( \bar{\Psi}(x)\Psi(y)G^a_{\mu\nu} \) is calculated to be

\[
\langle 0 | \bar{\Psi}_\alpha^i(x) \Psi_\beta^j(y) G^a_{\mu\nu} | 0 \rangle
= \frac{1}{192} \langle \bar{\Psi} \sigma T G \Psi \rangle \langle \sigma_{\mu\nu} \rangle_{\beta\alpha} T^a_{ji} + \left[ -\frac{g}{96 \times 9} \langle \bar{\Psi} \Psi \rangle^2 (g_{\rho\mu} \gamma_\nu - g_{\rho\nu} \gamma_\mu)(x + y)^\rho \\
+i(y - x)^\rho \left( \frac{g}{96 \times 9} \langle \bar{\Psi} \Psi \rangle^2 + \frac{m}{96 \times 4} \langle \bar{\Psi} \sigma T G \Psi \rangle \right) \varepsilon_{\rho\mu\nu\sigma} \gamma_5 \gamma^\sigma \right]_{\beta\alpha} T^a_{ji}, \quad (29)
\]

where \( \langle \bar{\Psi} \sigma T G \Psi \rangle \) and \( \langle \bar{\Psi} \Psi \rangle^2 \) are the abbreviations of \( \langle 0 | \bar{\Psi} \sigma_{\mu\nu} T^a G^{a\mu\nu} \Psi | 0 \rangle \) and \( \langle 0 | \bar{\Psi} \Psi | 0 \rangle^2 \) respectively. \( g \) is the strong coupling.
Because we calculate up to the condensate of dimension-six operators, the external gluon field \( A^a_\mu(x) \) in Fig.4 should be expanded up to the second term, which will contribute a dimension-six operator,

\[
A^a_\mu(x) = \int_0^1 d\alpha x^\rho G^a_\rho\mu(\alpha x) = \frac{1}{2} x^\rho G^a_\rho\mu(0) + \frac{1}{3} x^\alpha x^\rho \hat{D}_\alpha G^a_\rho\mu(0) + \cdots ,
\]

where \( \hat{D}_\alpha \) is the covariant derivative in the adjoint representation, \((\hat{D}_\alpha)^{mn} = \partial_\alpha \delta^{mn} - g f^{ann} A^a_\alpha\). Then another vacuum matrix element needed is

\[
\langle 0 | \bar{\Psi}_i \gamma_\xi G_{\alpha \rho}^a | 0 \rangle = -\frac{g}{3^3} \times \frac{1}{2^4} \langle \bar{\Psi} \Psi \rangle^2 (g_{\xi\rho} \gamma_\sigma - g_{\xi\sigma} \gamma_\rho)_{\beta\alpha} T^a_{ji} .
\]

We calculate these diagrams and find that the contributions of Fig.4(c), (d) and
Fig. 5(c), (d) vanish after double Borel transformation in two variables $p_1^2$ and $p_2^2$, because only one variable appearing in the denominator, for instance, $\frac{1}{q^2(p_2^2-m_2^2)}$. The Borel transformation in $p_1^2$ will kill such terms.

Following the above method, after some tedious algebraic derivation with the software MATHEMATICA, we obtain the coefficients $f_0$, $f_1 + f_3$, $f_1 - f_3$ and $f_5$ needed in Eq. (12). They are listed in Appendix.

4. Numerical Analysis and Discussion

In the numerical analysis the standard values of the condensates at the renormalization point $\mu = 1\text{GeV}$ are taken [11, 20],

$$\langle \bar{q}q \rangle = -(0.24 \pm 0.01\text{GeV})^3, \quad \langle \bar{s}s \rangle = m_3^2 \langle \bar{q}q \rangle,$$

$$g \langle \bar{\Psi} \sigma T \Psi \rangle = m_0^2 \langle \bar{\Psi} \Psi \rangle, \quad \alpha_s \langle \bar{\Psi} \Psi \rangle^2 = 6 \times 10^{-5}\text{GeV}^6,$$

(32)

$$m_0^2 = 0.8 \pm 0.2\text{GeV}^2.$$

The quark masses are fixed to be $m_s = 140\text{MeV}$, $m_c = 1.3\text{GeV}$ [21], and the decay constant of $\phi$ meson is extracted from experimental data $f_\phi = 0.228$ [22]. For the decay constant of $D_s$ meson we take $f_{D_s} = 0.214 \pm 0.038\text{GeV}$ [21].

The Borel parameters $M_1$ and $M_2$ are not physical parameters. The physical result should not depend on them if the operator product expansion can be calculated up to infinite order. However, OPE has to be truncated to some finite orders in practice. Therefore, Borel parameters have to be selected in some “windows” to get the best stability of the physical results. The requirement to select the stable “windows” is: the Borel parameters can not be too large, or, contributions of higher resonance and continuum states can not be effectively suppressed; at the same time, they should not be too small, or, the truncated OPE would fail because the series in OPE generally depend on Borel parameters in the denominator $1/M$. We find the optimal stability with the requirements shown in Table 1 and the thresholds $s_1^0$, $s_2^0$ in the ranges $s_1^0 = 5.8 - 6.2\text{GeV}^2$, $s_2^0 = 1.9 - 2.1\text{GeV}^2$. The regions of Borel parameters which satisfies the requirements of Table 1 are shown in Fig 6 in two-dimensional diagram of $M_1^2$ and $M_2^2$. We find good stability of the form factors within these regions.

Because it is not easy to show the contribution of each term of OPE in two-dimensional regions of $M_1^2$ and $M_2^2$, we show the contributions of perturbative and condensate terms in Table 2 at a representative point $(M_1^2, M_2^2)$ in the stable region of $M_1^2$ and $M_2^2$. In general the higher the dimension of the operators, the smaller the relevant contributions of the condensates. The main contributions to $V(0)$, $A_1(0)$ and $A_2(0)$ are from perturbative and quark condensate term. For $A_0(0)$, the largest two contributions are from perturbative term and mixed quark-gluon condensate. Contributions of four-quark condensate are less than a few percent, therefore contributions of operator of dimension 6 are negligible.
Figure 6: Selected regions of $M_1^2$ and $M_2^2$: (a) for $V$; (b) for $A_0$; (c) for $A_1$; (d) for $A_2$. 
Table 1: Requirements to select Borel Parameters $M_1^2$ and $M_2^2$ for each form factors $V(0)$, $A_0(0)$, $A_1(0)$ and $A_2(0)$

| Form Factors | contribution of condensate | continuum of $D_s$ channel | continuum of $\phi$ channel |
|--------------|----------------------------|-----------------------------|-----------------------------|
| $V(0)$       | $\leq 49\%$               | $\leq 5\%$                 | $\leq 26\%$                |
| $A_0(0)$     | $\leq 29\%$               | $\leq 16\%$                | $\leq 31\%$                |
| $A_1(0)$     | $\leq 49\%$               | $\leq 18\%$                | $\leq 22\%$                |
| $A_2(0)$     | $\leq 11\%$               | $\leq 27\%$                | $\leq 5\%$                 |

Table 2: Contributions of perturbative and condensate terms in the operator-product expansion to the form factors $V(0)$, $A_0(0)$, $A_1(0)$ and $A_2(0)$, at a representative point $(M_1^2, M_2^2)$ in the stable region of $M_1^2$ and $M_2^2$. $f^{pert}$: perturbative; $f^{(3)}$: quark condensate; $f^{(4)}$: gluon condensate; $f^{(5)}$: mixed quark-gluon condensate; $f^{(6)}$: four-quark condensate.

| Form Factors | total | $f^{pert}$ | $f^{(3)}$ | $f^{(4)}$ | $f^{(5)}$ | $f^{(6)}$ | $(M_1^2, M_2^2)$ GeV$^2$ |
|--------------|-------|------------|-----------|-----------|-----------|-----------|--------------------------|
| $V(0)$       | 1.20  | 0.63       | 0.66      | 0         | $-0.10$   | 0.01      | (2.2, 1.4)               |
| $A_0(0)$     | 0.43  | 0.28       | $-0.10$   | 0         | 0.23      | 0.02      | (1.7, 1.5)               |
| $A_1(0)$     | 0.53  | 0.28       | 0.20      | 0         | 0.04      | 0.01      | (2.0, 1.2)               |
| $A_2(0)$     | 0.57  | 0.22       | 0.44      | 0         | $-0.09$   | 0.00      | (3.6, 1.5)               |

The final results for the form factors at $q^2 = 0$ are

$$V(0) = 1.21 \pm 0.33, \quad A_0(0) = 0.42 \pm 0.12,$$

$$A_1(0) = 0.55 \pm 0.15, \quad A_2(0) = 0.59 \pm 0.17, \quad (33)$$

$$r_V \equiv \frac{V(0)}{A_1(0)} = 2.20 \pm 0.85, \quad r_2 \equiv \frac{A_2(0)}{A_1(0)} = 1.07 \pm 0.43.$$

We compare our results for the ratios of form factors with experimental data in Table 3. It shows that the results are consistent with experimental data.

The physical region for $q^2$ in $D_s \to \phi \bar{\nu}$ decay extends from 0 to $(m_{D_s} - m_{\phi})^2 \simeq 0.9$ GeV$^2$. In the range $q^2 < 0.4$ GeV$^2$, there is no non-Landau-type singularity with the thresholds $s_0^1$ and $s_0^2$ chosen in this paper. The $q^2$ dependence of the form factors is shown in Fig. 7 in the range $-0.4$ GeV$^2 < q^2 < 0.4$ GeV$^2$. Within this range, the behavior of $V(q^2)$ and $A_0(q^2)$ is well compatible with the pole-model,

$$V(q^2) = \frac{V(0)}{1 - q^2/m_{pole}^V}.$$

While the $q^2$ dependence of and $A_1(q^2)$ and $A_2(q^2)$ is very weak.
Table 3: Comparison of our results of $r_V$ and $r_2$ with experimental data: E791 is from Ref.[11], CLEO from Ref.[10], E687 from Ref.[9] and E653 from Ref.[8].

|       | $r_V$             |       |
|-------|-------------------|-------|
| E791  | $2.27 \pm 0.35 \pm 0.22$ | $1.57 \pm 0.25 \pm 0.19$ |
| CLEO  | $0.9 \pm 0.6 \pm 0.3$   | $1.4 \pm 0.5 \pm 0.3$   |
| E687  | $1.8 \pm 0.9 \pm 0.2$   | $1.1 \pm 0.8 \pm 0.1$   |
| E653  | $2.3_{-0.9}^{+1.1} \pm 0.4$ | $2.1_{-0.5}^{+0.6} \pm 0.2$ |
| Average | $1.92 \pm 0.32$   | $1.60 \pm 0.24$   |
| our result | $2.20 \pm 0.85$ | $1.07 \pm 0.43$ |

Figure 7: $q^2$ dependence of the form factors from QCD sum rule. The solid curve is for $V(q^2)$, the short dashed curve for $A_0(q^2)$, the long dashed curve for $A_1(q^2)$, and the dotted one is for $A_2(q^2)$.

We fit $V(q^2)$ and $A_0(q^2)$ by the pole model in the range $-0.4 \text{GeV}^2 < q^2 < 0.4 \text{GeV}^2$, and extrapolate the fitted result to the whole physical region. The fitted pole masses are,

$$m_{V\text{pole}} = 2.08 \pm 0.13 \text{ GeV},$$

$$m_{A_0\text{pole}} = 1.9 \pm 0.2 \text{ GeV}. \quad (34)$$

The form factors calculated in QCD sum rule in this paper are used to calculate the differential and total decay rate of $D_s \to \phi \ell \nu$ decay. There are three polarization states for $\phi$ meson: one longitudinal state, two transverse polarization states (right-handed and left-handed). The differential decay rate to longitudinally polarized $\phi$ meson is

$$\frac{d\Gamma_{L}}{dq^2} = \frac{G_F^2|V_{cs}|^2}{192\pi^3m_{D_s}^3} \sqrt{\lambda(m_{D_s}^2, m_{\phi}^2, q^2)} \left| \frac{1}{2m_{\phi}} \left[ (m_{D_s}^2 - m_{\phi}^2 - q^2)(m_{D_s} + m_{\phi})A_1(q^2) \right. \right.$$  

$$- \left. \frac{\lambda(m_{D_s}^2, m_{\phi}^2, q^2)}{m_{D_s} + m_{\phi}}A_2(q^2) \right|^2,$$  

where $G_F$ is Fermi constant, $V_{cs}$ is CKM matrix element for $c \to s$ transition, and

$$\lambda(m_{D_s}^2, m_{\phi}^2, q^2) \equiv (m_{D_s}^2 + m_{\phi}^2 - q^2)^2 - 4m_{D_s}^2m_{\phi}^2.$$
The differential decay rate to transverse state is

\[ \frac{d\Gamma_T^\pm}{dq^2} = \frac{G_F^2 |V_{cs}|^2}{192\pi^3 m_{D_s}^2} \lambda(m_{D_s}^2, m_\phi^2, q^2) \left| \frac{V(q^2)}{m_{D_s} + m_\phi} \mp \frac{(m_{D_s} + m_\phi)A_1(q^2)}{\sqrt{\lambda(m_{D_s}^2, m_\phi^2, q^2)}} \right|^2, \]  

(36)

where the symbol “ + ” and “ − ” denote right and left-handed states, respectively. Finally, the combined transverse and total differential decay rates are

\[ \frac{d\Gamma_T}{dq^2} = \frac{d}{dq^2}(\Gamma_T^+ + \Gamma_T^-), \quad \frac{d\Gamma}{dq^2} = \frac{d}{dq^2}(\Gamma_L + \Gamma_T). \]  

(37)

Figure 8: Differential decay widths of $D_s^+ \rightarrow \phi \bar{\ell} \nu$ as a function of momentum transfer squared $q^2$ in unit of $10^{-14}\text{GeV}^{-1}$.

The differential decay widths as a function of momentum transfer squared $q^2$ are shown in Fig. 8. Integrate them over $q^2$ in the whole physical region from $q^2 = 0$ to $(m_{D_s} - m_\phi)^2$, we get the integrated decay widths

\[ \Gamma_T^+ = (1.39 \pm 0.75) \times 10^{-15}\text{GeV}, \quad \Gamma_T^- = (1.05 \pm 0.22) \times 10^{-14}\text{GeV}, \]

\[ \Gamma_L = (1.18 \pm 0.43) \times 10^{-14}\text{GeV}, \quad \Gamma_T = (1.19 \pm 0.29) \times 10^{-14}\text{GeV}, \]  

(38)
and the ratio of $\Gamma_L/\Gamma_T$ is

$$\Gamma_L/\Gamma_T = 0.99 \pm 0.43,$$  \quad (39)

which is consistent with the averaged experimental data ($\Gamma_L/\Gamma_T)^{exp} = 0.72 \pm 0.18$ \cite{22}. The detailed comparison of this ratio with experimental data is shown in Table 4.

Table 4: Comparison of our results of $\Gamma_L/\Gamma_T$ with experimental data: CLEO from Ref.\[10\], E687 from Ref.\[9\] and E653 from Ref.\[8\].

|        | $\Gamma_L/\Gamma_T$ |
|--------|---------------------|
| CLEO   | 1.0 ± 0.3 ± 0.2     |
| E687   | 1.0 ± 0.5 ± 0.1     |
| E653   | 0.54 ± 0.21 ± 0.10  |
| Average| 0.72 ± 0.18         |
| our result | 0.99 ± 0.43         |

We use the total decay width of $D_s$ meson $\Gamma_{D_s} = 1.34 \times 10^{-12}$ \cite{22} to obtain the branching ratio of $D_s^+ \rightarrow \phi \ell \nu$, our result is

$$Br(D_s^+ \rightarrow \phi \ell \nu) = (1.8 \pm 0.5)\%,$$  \quad (40)

which is in good agreement with experimental data $Br(D_s^+ \rightarrow \phi \ell \nu)^{exp} = (2.0 \pm 0.5)\%$.

5. Summary

We calculate the transition form factors for $D_s \rightarrow \phi$ transition in the region $q^2 \leq 0.4\text{GeV}^2$ in QCD sum rule, where no non-Landau-type singularity occurs. Then fit the result from QCD sum rule in this region of momentum transfer, and extrapolate it to the whole physical region in the decay $D_s^+ \rightarrow \phi \ell \nu$. We treat the two Borel parameters $M_1^2$ and $M_2^2$ as independent parameters, and select the allowed region for $M_1^2$ and $M_2^2$ by requiring that the higher resonance and continuum contributions in $D_s$ and $\phi$ channels are not large, at the same time requiring that the condensate of higher dimension operators do not contribute too much. We find good stability for the transition form factors $V$, $A_0$, $A_1$ and $A_2$ in the relevant two-dimensional regions of $M_1^2$ and $M_2^2$. We obtain the results of the transition form factors $V$, $A_0$, $A_1$ and $A_2$ in these regions of $M_1^2$ and $M_2^2$. Our result of the ratios of these form factors $r_V$ and $r_2$ are well consistent with experimental data.

We studied the process $D_s^+ \rightarrow \phi \ell \nu$ with the form factors calculated from QCD sum rule. For the transverse polarization state of the final $\phi$ meson, the rate of $D_s$ decaying to right-hand state is almost an order smaller than decaying to left-hand state. The ratio of $\Gamma_L/\Gamma_T$ and the branching ratio of $D_s^+ \rightarrow \phi \ell \nu$ are in good agreement with the experimental data within the error bars of both the present experimental data and theoretical calculation.
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Appendix

Borel transformed Coefficients of perturbative and nonperturbative contributions to the transition form factors in Eq. (12) are given here. The contributions of condensate of dimension-six operator \(\langle \bar{\Psi} \Psi \rangle^2\) are numerically negligible, whereas their expressions are more tedious, therefore we do not present all of them here.

1) Results for \(f_0\):

\[
\hat{B}f_0 = \hat{B}f_0^{\text{pert}} + \hat{B}f_0^{(3)} + \hat{B}f_0^{(5)} + \hat{B}f_0^{(6)},
\]

with

\[
\hat{B}f_0^{\text{pert}} = \int_{4m_2^2}^{s_0^L} ds_2 \int_{s_1^L}^{s_0^L} ds_1 \frac{3e^{-s_1/M_1^2 - s_2/M_2^2}}{4M_1^2 M_2^2 \pi^2 \lambda^3/2} [2s_2 m_3^2 - 2s_2 m_s^2 - s_2(2m_s^2 + s_1 - s_2 + q^2)m_c + m_s(-s_2^2 + 2m_s^2 s_2 + s_1 s_2 + q^2 s_2 + \lambda)],
\]

where \(\lambda = (s_1 + s_2 - q^2)^2 - 4s_1 s_2\). The lower integration limit \(s_1^L\) is determined by the condition that all internal quarks are on their mass shell \([19]\),

\[
s_1^L = -\frac{m_2^2}{m_2^2 - q^2 s_2 + m_c^2}.
\]

\[
\hat{B}f_0^{(3)} = -\frac{e^{-m_2^2/M_1^2 - m_2^2/M_2^2}}{6M_1^2 M_2^2} [(6M_1^2 M_2^2 - 3(M_1^2 + q^2)m_2^2 M_2^2 + (4M_1^2 + M_2^2) + q^2)m_2^2 M_2^2 - (M_1^2 + M_2^2)m_s^2 M_1^2 + M_2^2 (M_1^2 + M_2^2)m_s^2 M_1^2 (3M_2^2 - m_s^2) M_2^2
\]

\[
- (M_1^2 + M_2^2)m_s^2 M_1^2 - M_2^4 (M_1^2 + M_2^2)m_s^2 M_1^2 \times \langle \bar{s}s \rangle,
\]

\[
\hat{B}f_0^{(5)} = -\frac{e^{-m_2^2/M_1^2 - m_2^2/M_2^2}}{12M_1^4 M_2^4} [(3M_2^6 - M_1^4 M_2^2 + (M_1^2 + M_2^2)m_s^4
\]

\[
-(3M_2^6 + 5M_1^4 M_2^2)m_s^2 M_1^2 - M_2^2 (M_1^2 + M_2^2)m_s^2 (3M_2^2 - m_s^2) M_1^2
\]

\[
+ M_2^4 m_s^2 (2M_1^2 - 2M_2^2) M_1^2 + (M_1^2 + M_2^2)m_s^2 M_1^2 + M_2^2 (M_1^2 + M_2^2)m_s^2 M_1^2 \times \langle \bar{s}T\bar{s} \rangle,
\]

\[
+ q^2 (M_1^4 M_2^2 (3M_2^2 - m_s^2) - M_1^4 M_2^2 m_s^2) \times g\langle \bar{s}T\bar{s} \rangle,
\]
\[
\hat{B}_f^{(6)}(s) = \frac{e^{-m_2^2/M_1^2 - m_2^2/M_2^2}}{81M_1^6 M_2^6 (m_s^2 - q^2)m_s^3} \left\{ 18 (-1 + e^{-m_2^2/M_2^2}) M_1^6 (2m_c - m_s)m_s M_2^6 \\
+ (M_1^2 + M_2^2) m_s^2 m_s^2 M_2^2 + M_1^2 (M_1^2 + M_2^2) m_s^4 M_2^2 \\
+ M_1^2 q^4 m_s^2 (m_s M_1^2 + M_2^2 m_c) M_2^2 + M_1^2 m_s^3 (-13M_1^2 + 2M_1^2 M_2^2) \\
+ (M_1^2 + M_2^2) m_s^4 M_2^2 + M_1^2 m_s^2 (-54 (-1 + e^{-m_2^2/M_2^2}) M_1^6 M_2^6) \\
+ 54 M_1^2 m_s^2 M_2^2 - (M_1^2 + 10 M_2^2) m_s^2 M_2^2 + (M_1^2 + M_2^2) m_s^6 \right\} \times g^2 \langle \bar{s}s \rangle^2 .
\]

(A4)

2) Results for \( f_1 + f_3 \):

\[
\hat{B}(f_1 + f_3) = \hat{B}f^{pert}_+ + \hat{B}f^{(3)}_+ + \hat{B}f^{(5)}_+ + \hat{B}f^{(6)}_+ ,
\]

with

\[
\hat{B}f^{pert}_+ = \int_{4m_s^2}^{\infty} ds_2 \int_{s_2}^{\infty} \frac{3 e^{-s_1/M_1^2 - s_2/M_2^2}}{4 M_1^2 M_2^2 \pi^2 \lambda^2} \left\{ -6s_2 (-s_1 + s_2 + q^2) m_s^5 + 6s_2 (-s_1) \\
+ s_2 + q^2) m_s m_s^4 + 2s_2 (-4s_2^2 + 8s_2 s_1 - 4s_2^2 + 2q^2 - 6(s_1 - s_2) m_s^2 \\
+ \lambda + 2q^2 (3m_s^2 + s_1 + s_2) m_s^3 + 2s_2 m_s^4 s_1^2 - 8s_2 s_1 + 4s_2^2 - 2q^4 \\
+ 6(s_1 - s_2) m_s^4 - 3\lambda - 2q^2 (3m_s^2 + s_1 + s_2) m_s^3 + 6(s_1) \\
- s_2) s_2 m_s^4 + 2(4s_2^3 - 8s_1 s_2^3 + 4s_2^4) s_2 - 2\lambda s_2 + s_1 \lambda) m_s^2 + (s_1) \\
- s_2) s_2 (2s_2^2 - 4s_2 s_1 + 2s_2^2 - \lambda) - 2s_2 q^2 (2m_s^2 + 4s_2^2 + s_1) \\
+ s_2) + q^2 (-6s_2 m_s^4 - 2(2s_2^2 + 2s_1 s_2 + \lambda) m_s^2 + s_2 (2s_2^3 - 6s_2 s_1) \\
+ 4s_2^2 - \lambda) \} m_c + m_s [2s_2^2 - 6s_2 s_2^2 + 6s_2^2 s_2^2 - 3\lambda s_2^2 + 6(s_2) \\
- s_1) m_s^4 s_2 - s_2^2 s_2 + 3s_1 \lambda s_2 + 2q^2 (2m_s^2 + 2s_1 + s_2) s_2 + \lambda^2 \\
- 2(4s_2 - 8s_1 s_2^3 + 4s_2^4 s_2 - 4\lambda s_2 + s_1 \lambda) m_s^4 + q^2 (6s_2 m_s^4 + 2(2s_2^3 \\
+ 2s_1 s_2 + \lambda) m_s^4 + s_2 (-2s_2^2 + 6s_2 s_1 - 4s_2^2 + 3\lambda) \} \}
\]

(A5)

\[
\hat{B}f^{(3)}_+ = -\frac{e^{-m_2^2/M_1^2 - m_2^2/M_2^2}}{6 M_1^8 M_2^8} \left\{ [6 M_1^6 M_2^6 - 3(M_1^2 - 2M_2^2 + q^2) m_s^2 M_2^4] + (4(M_1^2 \\
+ M_2^2) + q^2) m_s^4 M_2^2 - (M_1^2 + M_2^2) m_s^4 M_1^4 + M_2^2 (M_1^2 + M_2^2) m_s^4 M_2^2 (3M_2^2 \\
- m_2^2) M_2^2 - M_2^2 m_c m_s (3M_1^2 M_2^2 + (M_1^2 + M_2^2 - q^2) m_s^2 M_2^2 \\
+ (M_1^2 + M_2^2) m_c^4 M_1^4 - M_2^2 (M_1^2 + M_2^2) m_s^4] \times \langle \bar{s}s \rangle \right\}
\]

(A6)
\[ \hat{B} f^{(5)} = \frac{e^{-m_c^2/M_1^2 - m_s^2/M_2^2}}{12M_1^2 M_2^2} \{ \left[ (M_1^2 + 3M_2^2)M_4^2 - (M_1^2 + M_2^2)M_s^2 + (7M_2^4 + 5) \right. \\
\left. + 5M_1^2 M_2^2 M_s^2 \right] M_4^2 + M_2^2 (M_1^2 + M_2^2)M_s^2 (3M_2^2 - M_1^2) M_4^2 - M_2^2 m_s \} \\
\left[ 2(M_1^2 + 2M_2^2)M_4^2 + (M_2^2 + M_7^2) m_s^2 M_4^2 - M_2^2 (M_1^2 + M_2^2) m_s^2 + q^2 \left[ M_2^2 (m_s^2 - 3M_2^2) M_4^2 + M_2^2 m_s m_s M_7^2 \right] \right] \times g(\bar{s}\sigma T G s), \]

\[ \hat{B} f^{(6)} = \frac{e^{-m_c^2/M_1^2 - m_s^2/M_2^2}}{81M_1^2 M_2^6 (m_s^2 - q^2)^3} \left\{ -18(-1 + e^{-m_c^2/M_1^2}) M_4^2 m_s^2 M_7^2 + (7M_2^2 - 2M_1^2 + M_2^2) M_4^2 m_s^2 M_7^2 + M_2^2 m_s M_4^2 \left[ -26 M_2^4 + 4M_1^2 M_2^2 + (M_1 - 2M_2^2) m_s^2 \right] \right. \\
\left. -2M_1^2 M_2^2 M_s^2 + (M_1^2 + M_2^2) m_s^2 \right] + q^2 \left[ (1 + e^{-m_c^2/M_1^2}) M_7^2 M_4^2 \right. \\
\left. -4M_1^2 M_2^2 M_4^2 + (M_1^2 + M_2^2) m_s^2 \right] + q^2 \left[ (1 + e^{-m_c^2/M_1^2}) M_7^2 M_4^2 \right. \\
\left. \right] + 4M_1^2 (4M_4^2 - 2M_2^4) M_4^2 - (2M_2^4 + M_4^2) m_s^2 M_7^2 + M_2^2 m_s \left\{ -4 M_1^2 + 26 M_2^2 M_4^2 + (M_1^2 - 7M_2^2) m_s^2 \right\} \} \times g^2(\bar{s}s^2). \]

3) Results for \( f_1 - f_3 \):

\[ \hat{B}(f_1 - f_3) = \hat{B} f^{pert} + \hat{B} f^{(3)} + \hat{B} f^{(5)} + \hat{B} f^{(6)}, \]

with

\[ \hat{B} f^{pert} = \int_{v(t)}^{v(0)} dv = \int_{v(t)}^{v(0)} ds \int_{0}^{v(t)} v \frac{3e^{-s_1/M_1^2 - s_2/M_2^2}}{4M_1^2 M_2^2 \pi^2 \lambda^{5/2}} \left\{ 6s_2(s_1 + 3s_2 - q^2) m_s^5 - 6s_2(s_1 + 3s_2 - q^2) m_s^5 \right. \\
\left. + 3s_2 - q^2 \right] m_s m_s^4 + 2s_2[-4s_1^2 - 4s_2 s_1 + 8s_2^2 + 2q^4 - 6(s_1 + 3s_2) \\
\right. m_s^2 + \lambda + 2q^2(3m_s^2 + 4s_2 s_2 + 2s_2 m_s) \left[ 4s_2^2 + 4s_2 s_1 - 8s_2^2 \right. \\
\left. - 2q^2 + 6(s_1 + 3s_2) m_s^2 + \lambda - 2q^2(3m_s^2 + 4s_1 s_1 + \lambda) m_s^4 + (s_1 - s_2) s_2 \left( 2s_1^2 \\
\right. - 2s_1 - \lambda + 2s_2 q^2(-2m_s^2 - 2s_2 + 2s_2 + s_2 q^2) + q^2(-6s_2 m_s^4 - 2(-10s_2^2 \\
\right. + 2s_1 s_2 + \lambda) m_s^4 - s_2(-2s_1^2 + 10s_2 s_1 + 4s_2^2 + \lambda) m_s^4 \\
\left. - 2s_1 s_2^2 - 2s_1^2 \right] - \lambda s_2 + 6(s_1 + 3s_2) m_s^2 s_2 + 2s_1^2 s_2 + s_1 \lambda s_2 \\
\right. + 2q^2(-2m_s^2 - 2s_1 + 2s_2) - \lambda^2 + 2(-8s_2^3 + 4s_1 s_2^2 + 4s_1 s_2 \\
\left. + 4s_2 s_2 + 4s_1 s_1) m_s^4 + 2(-10s_2^2 + 2s_1 s_2 + \lambda) m_s^4 \\
\left. + s_2(2s_1^2 + 10s_2 s_1 - 4s_2^2 + \lambda) \right\}. \]
\[
\dot{B}_f^{(3)} = \frac{e^{-m_2^2/M_1^2 - m_3^2/M_2^2}}{6M_1^2 M_2^2} \left\{ 16M_1^2 M_2^2 - 3(M_1^2 + 2M_2^2 + q^2)m_{s_1}^2 M_4^4 + (4(M_1^2 + M_2^2 + q^2)m_{s_3}^2 M_4^4 \right. \\
+ M_1^2 + q^2)M_2^4 - (M_1^2 + M_2^2 + q^2)M_1^4 + M_2^2(M_1^2 + M_2^2)M_2^4 \\
- m_2^2(3M_2^2 - m_2^2)M_2^4 - M_2^2 m_c m_s (3M_1^2 M_2^2 + (M_2^2 - 3M_2^2 - q^2)m_{s_1}^2 \\
M_2^2 + (M_1^2 + M_2^2)M_2^4) - M_2^2 (M_1^2 + M_2^2) m_{s_1}^2 \right\} \times \langle s s \rangle ,
\]
\[
\dot{B}_f^{(5)} = \frac{e^{-m_2^2/M_1^2 - m_3^2/M_2^2}}{12M_1^2 M_2^2} \left\{ (M_1^2 - 9M_2^2)M_2^4 - (M_1^2 + M_2^2)m_{s_1}^4 + (5M_1^2 M_2^4 \\
- M_1^2 m_{s_1}^4 M_4^4 + M_2^2 (M_1^2 + M_2^2) m_{s_1}^2 (3M_2^2 - m_2^2) M_1^4 \\
- M_2^4 (M_1^2 + M_2^2) m_{s_1}^4 - m_c [2M_1^4 m_{s_1}^4 + M_2^4 (M_1^2 + M_2^2) m_{s_1}^4 M_2^4] \\
+ q^2 [M_2^4 (m_{s_1}^2 - 3M_2^2) M_1^4 + M_2^4 m_c m_s M_2^4] \right\} \times g(\sigma T G s) ,
\]
\[
\dot{B}_f^{(6)} = \frac{e^{-m_2^2/M_1^2 - m_3^2/M_2^2}}{81M_1^4 M_2^8 (m_{s_1}^2 - q^2)m_{s_3}^2} \left\{ (54(-1 + e^{-m_2^2/M_2^2})M_1^6 m_{s_1}^2 M_2^6 + (7M_2^6 \\
- 2M_1^2 m_{s_1}^4 M_2^4 + M_1^2 (M_1^2 + M_2^2) m_{s_1}^2 M_2^4 + M_1^2 q^4 m_{s_1}^2 (M_1^2 m_c \\
- 2M_2^2 m_c) M_2^4 + M_2^4 m_{s_1}^2 M_1^4 - 18M_1^4 + 4M_1^2 M_2^2 + (M_1 - 2M_2^2) M_2^4 \\
- 2M_1^4 m_{s_1}^2 M_2^4 + (M_1 + M_2^2) m_{s_1}^2 M_2^4 + 54M_1^4 m_{s_1}^2 M_2^4 - 4(M_1^2 \\
+ 4M_2^2) m_{s_1}^4 M_2^4 + (M_1^2 + M_2^2) m_{s_1}^2 M_2^4 \right\} \times g^2 (s s)^2 .
\]

4) Results for \( f_5 \):

\[
\dot{B}(f_5) = \dot{B}f_5^{\text{pert}} + \dot{B}f_5^{(3)} + \dot{B}f_5^{(5)} + \dot{B}f_5^{(6)} ,
\]

with

\[
\dot{B}f_5^{\text{pert}} = \int_{4m_1^2}^{s_1^2} ds_2 \int_{s_1^2}^{s_2^2} d\lambda \lambda \frac{-3e^{-s_1/2M_1^2 - s_2/2M_2^2}}{8M_1^2 M_2^2 \pi^2 \lambda^{3/2}} \left\{ 2(s_2 m_{s_1}^5 - 2s_2 m_{s_1} m_{s_3}^4 - 2s_2 (2m_{s_1}^2 + s_1 \\
- s_2 + q^2) m_{s_1}^3 + 2s_2 m_{s_1} (2M - s_2 + s_1 - s_2 + q^2) m_{s_1}^3 + [2s_2 m_{s_1}^4 \\
+ 2(-s_2^2 + s_1 s_2 + \lambda) m_{s_1}^2 - s_1^2 + q^2 (2s_2 m_{s_1}^2 + 2s_1 s_2 + \lambda)] \right\} ,
\]
\[
\hat{B}_s^{(3)} = \frac{e^{-m_s^2/(M_2^2 - m_s^2)}M_2^4}{12M_2^4M_2^2} \{M_2^4(M_2^2 + M_2^2)m_s^5 - M_2^2(M_2^2 + M_2^2)m_s^2
\}
[3M_2^4M_2 - (M_2^2 + 2M_2^2)m_s^2]m_s^4 + [3M_2^4m_\sigma M_2^6 - 9M_2^6(M_2^2 + M_2^2)
\]
\]
\[
m_2^2M_2 + (M_2^2 + 4M_2^2M_2^2 + 3M_2^4M_2^2)m_s^5 + M_2^4[-6M_2^4M_2^2 + (M_2^2
\]
\]
\[
+ 4M_2^2M_2^2 + 3M_2^4)m_s^6 - (6M_2^6 + 11M_2^2M_2^4 + 5M_2^4M_2^2)m_s^4
\]
\]
\[
+ 3(3M_2^4M_2^4 + 2M_2^4M_2^2)m_s^5 + M_2^4m_s[-15M_2^4M_2^2 + (2M_2^4
\]
\]
\[
+ 3M_2^2M_2^2 + 3M_2^4)m_s^6 - (2M_2^6 + 13M_2^2M_2^4 + 11M_2^4M_2^2)m_s^4
\]
\]
\[
+ (8M_2^2M_2^4 + 11M_2^4M_2^2)m_s^3 + \frac{4}{3}M_2^4M_2^2 - 4(M_2^2 + M_2^2)
\]
\]
\[
m_2^2M_2^2 + (M_2^2 + 2M_2^2)m_s^6 + (2M_2^6 + 5M_2^4M_2^2)m_s^4 + M_2^4M_2^2q^4m_s^2
\]
\]
\[
\]
\[
[(m_s^2 - 3M_2^2)M_2 + M_2^2m_\sigma m_\sigma + q^2(6M_2^2M_2^2 - (M_2^2 + 2M_2^2)m_s^6
\]
\]
\[
+ (7M_2^4 + 5M_2^2M_2^2)m_s^4 - 3(M_2^2 + 2M_2^2M_2^2)m_s^2]M_2^4 + M_2^2m_\sigma m_\sigma
\]
\]
\[
(3M_2^2(2M_2^2 + M_2^2) - (2M_2^2 + 3M_2^2)m_s^2)M_2^4 + M_2^2m_\sigma m_\sigma(-3M_2^2M_2^4
\]
\]
\[
- (3M_2^2 + 2M_2^2)m_s^4 + (2M_2^2 + 9M_2^2M_2^2)m_s^2]M_2^4 - M_2^2(2M_2^2 + M_2^2)
\]
\]
\[
[m_s^2m_\sigma^2]) \times \langle \bar{s}s \rangle ,
\]
\]
\[
\hat{B}_s^{(5)} = \frac{e^{-m_s^2/(M_2^2 - m_s^2)}M_2^4}{24M_2^4M_2^2} \{M_2^4(M_2^2 + M_2^2)m_s^5 - M_2^2(M_2^2 + M_2^2)[3M_2^4M_2^2
\]
\]
\[
- (M_2^2 + 2M_2^2)m_s^4 + M_2^2m_\sigma[(3M_2^4 + 4M_2^2M_2^2 + M_2^4)m_s^2
\]
\]
\[
- 2M_2^2M_2^2(5M_2^2 + 3M_2^2)]m_s^3 + [2M_2^4(M_2^2 + 3M_2^2)M_2^2 + (M_2^6
\]
\]
\[
+ 4M_2^4M_2^4 + 3M_2^4M_2^4)m_s^4 - 2(3M_2^2M_2^6 + 5M_2^4M_2^4)m_s^2]m_s^2 + M_2^4m_s
\]
\]
\[
[4M_2^2(M_2 + 3M_2^2)M_2^2 + 2(M_2^4 + 3M_2^2M_2^2 + M_2^4)m_s + (M_2^6
\]
\]
\[
- 8M_2^2M_2^4 - 13M_2^4M_2^2)m_s^3]m_s + M_2^4[4M_2^2M_2^2 + (M_2^2 + M_2^2)
\]
\]
\[
m_2^2(2M_2^2 + m_\sigma^2) - (M_2^2 + 5M_2^2M_2^2)m_s^4 + q^4[M_2^2(m_s^4
\]
\]
\[
- 3M_2^2)M_2^4 + M_2^4m_\sigma m_\sigma M_2^4 - q^2(2(M_2^2 + 3M_2^2)M_2^4 + (M_2^2 + 2M_2^2)m_s^4
\]
\]
\[
- 2(2M_2^2 + 3M_2^2M_2^2)m_s^2]M_2^4 + M_2^4m_\sigma m_\sigma(M_2^4 - 10M_2^2M_2^2
\]
\]
\[
+ (3M_2^2 + 2M_2^2)m_s^2)M_2^4 + M_2^2m_\sigma(2M_2^2 + 3M_2^2)m_s^2
\]
\]
\[
- 3(M_2^2 + 2M_2^2M_2^2)M_2^4 + M_2^4(2M_2^2 + M_2^2)m_s^3m_\sigma^2) \times \langle \bar{s}sTGs \rangle .
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