Sheaves of structures, Heyting-valued structures, and a generalization of Łoś’s theorem

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Sheaves of structures are useful to give constructions in universal algebra and model theory. We can describe their logical behavior in terms of Heyting-valued structures. In this paper, we first provide a systematic treatment of sheaves of structures and Heyting-valued structures from the viewpoint of categorical logic. We then prove a form of Łoś’s theorem for Heyting-valued structures. We also give a characterization of Heyting-valued structures for which Łoś’s theorem holds with respect to any maximal filter.

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1 Introduction

Sheaf-theoretic constructions have been used in universal algebra and model theory. In this context, sheaves of abelian groups or rings in geometry are generalized to sheaves of structures. We can obtain, e.g., the product (resp. an ultraproduct) of a family of structures from some sheaf by taking the set of global sections (resp. a stalk). This viewpoint originated from the early literature [19, 23, 43]. In combination with the theory of sheaf representations of algebras, Macintyre [43] succeeded in giving model-companions of some theories of commutative rings by transferring model-theoretic properties from stalks to global sections.

On the other hand, sheaves have another description as Heyting-valued sets. The notion of Heyting-valued sets originally arises from that of Boolean-valued models of set theory, which was introduced in relation to Cohen’s forcing. The development of topos theory in the early seventies, mainly due to Lawvere & Tierney, revealed profound relationships between toposes and models of set theory; objects in a topos can be regarded as “generalized sets” in a universe. Subsequently, Fourman & Scott [26] and Higgs1 independently established the categorical treatment of Heyting-valued sets (cf. Remark 2.6). The category $\text{Sh}(X)$ of sheaves of sets on a space $X$ and the category $\text{Set}(\mathcal{O}(X))$ of $\mathcal{O}(X)$-valued sets turned out to be categorically equivalent.

Some model-theorists of that era immediately applied Heyting-valued sets to concrete problems in sheaf-theoretic model theory. However, general methods of Heyting-valued model theory have not been explored enough, though Fourman & Scott mentioned such a direction in the preamble of [26]. Even worse, we are not aware of any clear explanation of the relationship between sheaves of structures and Heyting-valued structures. In this paper, employing well-established languages of categorical logic, we will give a gentle and coherent account of Heyting-valued semantics of first-order logic from the categorical point of view, and will apply that framework to obtain a generalization of Łoś’s theorem for Heyting-valued structures. We also provide a characterization of Heyting-valued structures for which Łoś’s theorem holds w.r.t. any maximal filter. Our theorems improve the works of Caicedo [15] and Pierobon & Viale [55]. While our principal examples of Heyting-valued sets are sheaves on topological spaces, other natural examples include sheaves on the complete Boolean algebra of regular open sets and Boolean-valued sets on the measure algebra. Therefore, we will develop our theory based on any complete Heyting algebra (a.k.a. a frame or a locale) rather than a topological space.

The intended audience for this paper is anyone who has interests both in model theory and in categorical logic. We assume the least amount of knowledge in topos theory and first-order categorical logic. Most categorical prerequisites are covered by [41]. In § 3.1, we will recall some elements of first-order categorical logic.

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1 Originally in his preprint written in 1973, part of which was later published as [29].
The areas related to this paper (and its sequels in the future) are diverse, including model theory, universal algebra, set theory, categorical logic, topos theory, and their applications to ordinary mathematics. The author did his best to ensure that the reader can follow the scattered literature (especially in model theory and topos theory) on each occasion.

The structure of this paper In §2, we will begin with preliminaries on sheaves and Heyting-valued sets. After we see basic properties of Heyting-valued sets and morphisms between them, we will give an outline of the equivalence of sheaves and Heyting-valued sets. We also provide some details on the topos $\text{Set}(\mathcal{O}(X))$ of $\mathcal{O}(X)$-valued sets. In §3, we will study structures in the toposes $\text{Sh}(X)$ and $\text{Set}(\mathcal{O}(X))$ and the relationship between them. We will also introduce forcing values of formulas categorically. In §4, observing that sheaves of structures generalize some model-theoretic constructions, we will introduce a further generalization of filter-quotients of sheaves to Heyting-valued structures and prove Łoś’s theorem and the characterization theorem. In §5, we will indicate some possible future directions with an expanded list of previous works.

Closing the introduction, we must mention Loullis’ work [40] on Boolean-valued model theory. The starting point of this research was trying to digest his work from a modern categorical viewpoint, though our work is still too immature to give the reader a full explanation of his contribution. If the author had not met his work, this paper would not have existed. The author regrets his untimely death, according to [12], in 1978.

2 Sheaves and Heyting-valued sets

Heyting-valued sets were introduced independently by Higgs [29] and by Fourman & Scott [26]. In this section, we will review the construction of the category $\text{Set}(\mathcal{O}(X))$ of $\mathcal{O}(X)$-valued sets for a locale $X$, its relation to sheaves on $X$, and its categorical structure as a topos. Most results are covered by [29], [26], [33, § C1.3] and [7, Chapter 2]. For the reader’s convenience, we will occasionally give brief sketches of proofs.

For aspects of Heyting-valued sets in intuitionistic logic, cf. [59, Chapters 13–14]. From a modern categorical viewpoint, it is worth noting that $\text{Set}(\mathcal{O}(X))$ is a prototypical example of the topos obtained from a tripos (cf. [31] and [52, Chapter 2]). Learning tripos logic will help the reader get intuition on Heyting-valued sets. Walters [62, 63] developed another direction of generalization of Heyting-valued sets.

2.1 Heyting-valued sets

**Definition 2.1** A frame is a complete lattice satisfying the infinitary distributive law:

$$a \wedge \bigvee_{i} b_{i} = \bigvee_{i} a \wedge b_{i}.$$ 

In particular, any frame has 0 and 1.

A frame is the same thing as a complete Heyting algebra: the infinitary distributive law for a frame $H$ says that each monotone map $a \wedge (-) : H \to H$ preserves arbitrary joins. This happens exactly when each map $a \wedge (-)$ has a right adjoint $a \to (-) : H \to H$, i.e., a monotone map satisfying

$$\forall b, c \in H, [a \wedge b \leq c \iff b \leq a \to c].$$

This fact follows either from category theory (the General Adjoint Functor Theorem), or from a direct construction

$$a \to c := \bigvee \{ b \in H ; a \wedge b \leq c \}.$$ 

On the other hand, frame homomorphisms differ from those for complete Heyting algebras (and even those for complete lattices):

**Definition 2.2** Let $H, H'$ be frames. A frame homomorphism $h : H \to H'$ is a map from $H$ to $H'$ preserving finite meets and arbitrary joins. Let $\text{Frm}$ denote the category of frames.
Similarly to the above, any frame homomorphism \( h : H \to H' \) has a right adjoint \( k : H' \to H \) given by
\[
k(b) = \bigvee \{ a \in H : h(a) \leq b \}.
\]

Any continuous map \( f : X \to Y \) of topological spaces gives rise to a frame homomorphism \( f^* : \mathcal{O}(Y) \to \mathcal{O}(X) \) given by \( f^*(V) = f^{-1}(V) \), where \( \mathcal{O}(X) \) (resp. \( \mathcal{O}(Y) \)) is the frame of open sets of \( X \) (resp. \( Y \)). The functor \( \mathcal{O}(-) : \text{Top} \to \text{ Frm}^\text{op} \) is full and faithful on sober spaces (i.e., spaces satisfying a suitable axiom between \( T_0 \) and \( T_2 \)). Therefore, it translates the language of spaces to that of frames, and we may consider frames as “point-free” spaces. This justifies the following definition:

**Definition 2.3** A frame considered as an object of \( \text{ Frm}^\text{op} \) is called a *locale*. We denote \( \text{ Frm}^\text{op} \) by \( \text{ Loc} \) and the frame corresponding to a locale \( X \in \text{ Loc} \) by \( \mathcal{O}(X) \). We will write \( U, V, \) etc. for elements of \( \mathcal{O}(X) \) and \( 0_X \) (resp. \( 1_X \)) for the smallest (resp. largest) element.

For a morphism \( f : X \to Y \) of locales, the corresponding frame homomorphism is denoted by \( f^* : \mathcal{O}(Y) \to \mathcal{O}(X) \). \( f^* \) has a right adjoint \( f_* : \mathcal{O}(X) \to \mathcal{O}(Y) \). Morphisms of locales are also called continuous maps of locales.

By writing \( X \) for the locale given by a topological space \( X \), i.e., \( \mathcal{O}(X) = \mathcal{O}(X) \), we now have a functor \( (-)_! : \text{ Top } \to \text{ Loc} \). It has a right adjoint \( pt : \text{ Loc } \to \text{ Top} \) sending a locale \( X \) to the space \( pt(X) \) of “points of \( X \)” (cf., e.g., [41, Chapter IX]). For more on frames and locales in point-free topology, cf. [32] and [53].

We are now ready to define Heyting-valued sets. In the remainder of this section, we fix a locale \( X \).

**Definition 2.4** An \( \mathcal{O}(X) \)-valued set \( (A, \alpha) \) is a pair of a set \( A \) and a map \( \alpha : A \times A \to \mathcal{O}(X) \) such that
1. \( \forall a, b \in A, \; \alpha(a, b) = \alpha(b, a) \);
2. \( \forall a, b, c \in A, \; \alpha(a, b) \land \alpha(b, c) \leq \alpha(a, c) \).

Instead of \( \alpha(a, b) \), the notation \([a = b]\) is frequently used in the literature. We introduce a few conventions:
1. \( \alpha(a) := \alpha(a, a) \) is called the extent of \( a \). Note that \( \alpha(a, b) \leq \alpha(a) \land \alpha(b) \).
2. If \( \alpha(a) = 1_X \), then \( a \) is called a *global element* of \( (A, \alpha) \).

**Definition 2.5** Let \( (A, \alpha), (B, \beta) \) be \( \mathcal{O}(X) \)-valued sets. A *morphism* \( \varphi : (A, \alpha) \to (B, \beta) \) of \( \mathcal{O}(X) \)-valued sets is a map \( A \times B \to \mathcal{O}(X) \) which satisfies
\[
\forall a, a' \in A, \forall b, b' \in B, \; \alpha(a, a') \land \varphi(a, b) \land \beta(b, b') \leq \varphi(a', b'),
\]
\[
\forall a \in A, \forall b, b' \in B, \; \varphi(a, b) \land \varphi(a, b') \leq \beta(b, b'),
\]
\[
\forall a \in A, \; \alpha(a) = \bigvee_{b \in B} \varphi(a, b).
\]

In particular, \( \varphi(a, b) \leq \alpha(a) \land \beta(b) \) always holds. If \( \psi : (B, \beta) \to (C, \gamma) \) is another morphism, we can define the composite \( \psi \circ \varphi \) by
\[
(\psi \circ \varphi)(a, c) = \bigvee_{b \in B} \varphi(a, b) \land \psi(b, c).
\]

We write \( \text{ Set}(\mathcal{O}(X)) \) for the category of \( \mathcal{O}(X) \)-valued sets and morphisms, where the identity \( \text{id}_{(A, \alpha)} \) is given by \( \alpha \) itself.

**Remark 2.6** In set theory, for a frame \( H \), we can construct a model \( V(H) \) of intuitionistic set theory (cf. [5, Chapter IV]). \( V(H) \) is called the *Heyting-valued universe*. The category \( \text{ Set}(H) \) is regarded as a categorical counterpart of \( V(H) \) (cf. [4, Appendix], [21]), and we can take arguments and examples from set theory to investigate Heyting-valued sets (cf. [55]).

We list useful facts on Heyting-valued sets, some of which will not be used in this paper.

**Lemma 2.7** Two morphisms \( \varphi, \psi : (A, \alpha) \Rightarrow (B, \beta) \) are identical if
\[
\forall a \in A, \; \forall b \in B, \; \varphi(a, b) = \psi(a, b).
\]

**Proposition 2.8** Let \( \varphi : (A, \alpha) \to (B, \beta) \) be a morphism in \( \text{ Set}(\mathcal{O}(X)) \).
(1) \( \varphi \) is a monomorphism if and only if
\[
\forall a, a' \in A, \forall b \in B, \quad \varphi(a, b) \land \varphi(a', b) \leq \alpha(a, a').
\]

(2) \( \varphi \) is an epimorphism if and only if
\[
\forall b \in B, \quad \beta(b) = \bigvee_{a \in A} \varphi(a, b).
\]

(3) \( \varphi \) is an isomorphism if and only if it is monic and epic. In other words, \( \text{Set}(\mathcal{O}(X)) \) is a balanced category. If \( \varphi \) is an isomorphism, \( \varphi^{-1} \) is given by \( \varphi^{-1}(b, a) = \varphi(a, b) \).

**Definition 2.9** We say that a morphism \( \varphi : (A, \alpha) \to (B, \beta) \) is represented by a map \( h : A \to B \) when
\[
\forall a \in A, \forall b \in B, \quad \varphi(a, b) = \alpha(a, \beta(ha, b)).
\]

**Proposition 2.10**

(1) A morphism \( \varphi : (A, \alpha) \to (B, \beta) \) is represented by a map \( h : A \to B \) if and only if
\[
\forall a \in A, \forall b \in B, \quad \varphi(a, b) \leq \beta(ha, b).
\]

(2) A map \( h : A \to B \) represents some morphism from \( (A, \alpha) \) to \( (B, \beta) \) if and only if
\[
\forall a, a' \in A, \quad \alpha(a, a') \leq \beta(ha, ha').
\]

Moreover, if \( h \) further satisfies \( \alpha(a) = \beta(ha) \) for all \( a \in A \), then the morphism \( \varphi \) represented by \( h \) is given simply by \( \varphi(a, b) = \beta(ha, b) \).

(3) Suppose two maps \( h, k : A \to B \) represent some morphisms. They represent the same morphism if and only if
\[
\forall a \in A, \quad \alpha(a) \leq \beta(ha, ka).
\]

(4) Let \( \varphi : (A, \alpha) \to (B, \beta) \) and \( \psi : (B, \beta) \to (C, \gamma) \) be morphisms. If \( \varphi \) and \( \psi \) are represented by maps \( h \) and \( k \) respectively, then \( \psi \varphi \) is represented by \( kh \).

**Proposition 2.11** Let \( \varphi : (A, \alpha) \to (B, \beta) \) be a morphism represented by \( h \).

1. \( \varphi \) is monic \( \iff \forall a, a' \in A, \alpha(a, a') = \alpha(a) \land \alpha(a') \land \beta(ha, ha') \).
2. \( \varphi \) is epic \( \iff \forall b \in B, \beta(b) = \bigvee_{a}[\alpha(a) \land \beta(ha, b)] \).

Further, if \( \alpha(a) = \beta(ha) \) for all \( a \in A \), these conditions reduce to

1. \( \varphi \) is monic \( \iff \forall a, a' \in A, \alpha(a, a') = \beta(ha, ha') \).
2. \( \varphi \) is epic \( \iff \forall b \in B, \beta(b) = \bigvee_{a} \beta(ha, b) \).

Combining the above facts, we obtain

**Corollary 2.12** If \( h \) satisfies \( \alpha(a) = \beta(ha) \) for all \( a \in A \), then \( h \) represents an isomorphism exactly when the following conditions hold:
\[
\forall a, a' \in A, \quad \alpha(a, a') = \beta(ha, ha'), \quad \text{and} \quad \forall b \in B, \quad \beta(b) = \bigvee_{a} \beta(ha, b).
\]

If \( h \) satisfies these conditions, the induced isomorphism \( \varphi(a, b) = \beta(ha, b) \) has the inverse \( \varphi^{-1}(b, a) = \beta(ha, b) \).

### 2.2 Sheaves on locales and complete Heyting-valued sets

Continuing from the previous section, we fix a locale \( X \). Let us discuss the relationship between sheaves on \( X \) and \( \mathcal{O}(X) \)-valued sets. Our presentation style here is largely due to [33, § C1.3].
Definition 2.13 (1) A presheaf on \(X\) is a functor \(\mathcal{O}(X)^{op} \to \textbf{Set}\). For a presheaf \(P\) and \(U \in \mathcal{O}(X)\), elements of \(PU\) (resp. \(PU_U\)) are called sections of \(P\) on \(U\) (resp. global sections of \(P\)). If \(a \in PU\) and \(W \subseteq U\), we will write \(a|_W\) for \(P(W \subseteq U)(a)\).

(2) A presheaf \(P\) is said to be a sheaf (resp. a separated presheaf) on \(X\) when it satisfies the following condition: For any covering \(\{U_i\}_i\) of \(U \in \mathcal{O}(X)\) (i.e., \(U = \bigvee_i U_i\)) and any family \(\{a_i\}_i\) of sections \(a_i \in PU_i\), if \(a_i|_{U_i \cap U_j} = a_j|_{U_i \cap U_j}\) for all \(i, j\), then there exists a unique (resp. at most one) \(a \in PU\) such that \(a|_{U_i} = a_i\) for all \(i\).

(3) Morphisms of presheaves are defined to be natural transformations. The functor category \(\textbf{Set}^{\mathcal{O}(X)^{op}}\) is also called the category of presheaves. Let \(\text{Sh}(X)\) denote its full subcategory spanned by sheaves.

We can associate a presheaf \(P\) with an \(\mathcal{O}(X)\)-valued set \(\Theta(P) := (\bigsqcup_U PU, \delta_P)\) as follows: for \((a, b) \in PU \times PV \subseteq \bigsqcup_U PU \times \bigsqcup_U PU\),

\[
\delta_P(a, b) := \bigvee \{W \leq U \land V ; a|_W = b|_W\}.
\]

Notice that

1. \(a \in PU\) if and only if \(\delta_P(a) = U\). Hence, global elements of \(\Theta(P)\) are exactly global sections of \(P\).
2. If \(P\) is a sheaf, then \(\delta_P(a, b)\) is the largest element on which the restrictions of \(a\) and \(b\) coincide. Moreover, if \(X\) is a topological space,

\[
\delta_P(a, b) = \{x \in X ; a_x = b_x\},
\]

where \(a_x, b_x\) are the germs of \(a, b\) over \(x\).

For a morphism \(\xi : P \to Q\) of presheaves on \(X\), the induced map \(h : \bigsqcup_U PU \to \bigsqcup_U QU\) satisfies

\[
\forall(a, b) \in \left(\bigsqcup_U PU\right)^2, \quad \delta_P(a, b) \leq \delta_Q(ha, hb) \quad \text{and} \quad \delta_P(a) = \delta_Q(ha).
\]

Therefore, by Proposition 2.10, \(h\) represents a morphism \(\Theta(\xi) : \Theta(P) \to \Theta(Q)\). This construction gives a functor \(\Theta : \textbf{Set}^{\mathcal{O}(X)^{op}} \to \text{Set}(\mathcal{O}(X))\).

We say a Heyting-valued set \((A, \alpha)\) is separated if \(\alpha(a) = \alpha(b) = \alpha(a, b)\) implies \(a = b\) for any \(a, b \in A\). A presheaf \(P\) is separated if and only if \(\Theta(P)\) is separated in this sense. To give a similar characterization of sheaves, we need a more involved definition.

Definition 2.14 For an \(\mathcal{O}(X)\)-valued set \((A, \alpha)\), define a preorder \(\sqsubseteq\) on \(A\) by

\[
b \sqsubseteq a \iff a(b) = \alpha(b).
\]

\((A, \alpha)\) is said to be complete if the following conditions hold:

1. \(\sqsubseteq\) is a partial order. (This is equivalent to separatedness.)
2. For any \(a \in A\) and \(U \leq \alpha(a)\), there exists \(b \in A\) such that \(b \sqsubseteq a\) and \(\alpha(b) = U\). (If \(\sqsubseteq\) is a partial order, such \(b\) is uniquely determined and denoted by \(a|_U\).)
3. If a family \(\{a_i\}_i\) of elements of \(A\) is pairwise compatible, i.e., \(\alpha(a_i, a_j) = \alpha(a_i) \land \alpha(a_j)\) for all \(i, j\), then it has a supremum w.r.t. \(\sqsubseteq\). (The supremum is called an amalgamation of \(\{a_i\}_i\).)

Let \(\text{CSet}(\mathcal{O}(X))\) denote the full subcategory of \(\textbf{Set}(\mathcal{O}(X))\) spanned by complete \(\mathcal{O}(X)\)-valued sets.

Proposition 2.15 A presheaf \(P\) on \(X\) is a sheaf if and only if \(\Theta(P)\) is complete as an \(\mathcal{O}(X)\)-valued set. Moreover, for any complete \(\mathcal{O}(X)\)-valued set \((A, \alpha)\), we can obtain a sheaf \(P\) on \(X\) determined by

\[
PU := \{a \in A ; \alpha(a) = U\},
\]

and \((A, \alpha)\) and \(\Theta(P)\) are isomorphic.

We can rephrase completeness in terms of singletons.

Definition 2.16 Let \((A, \alpha)\) be an \(\mathcal{O}(X)\)-valued set. A singleton on \((A, \alpha)\) is a function \(\sigma : A \to \mathcal{O}(X)\) such that

\[
\forall a, a' \in A, \quad \sigma(a) \land \alpha(a, a') \leq \sigma(a') \quad \text{and} \quad \sigma(a) \land \sigma(a') \leq \alpha(a, a').
\]
In particular, \( \sigma(a) \leq \alpha(a) \) always holds.

For each \( a \in A \), the map \( \sigma_a := \alpha(a, -) \) is a singleton of \( (A, \alpha) \).

**Lemma 2.17** For an \( \mathcal{O}(X) \)-valued set \( (A, \alpha) \), the following are equivalent:

(i) \( (A, \alpha) \) is complete.

(ii) Any singleton of \( (A, \alpha) \) is of the form \( \sigma_a \) for a uniquely determined \( a \).

**Proof.** (i) \( \implies \) (ii): Suppose \( (A, \alpha) \) is complete. Let \( \sigma \) be a singleton on \( (A, \alpha) \). Then the family

\[
\{ a_{\sigma(a)} ; a \in A \}
\]

is pairwise compatible, and its supremum \( s \in A \) satisfies \( \sigma = \sigma_s \).

(ii) \( \implies \) (i): Suppose the condition (ii) holds. If \( \alpha(a) = \alpha(a') = \alpha(a, a') \), then \( \sigma_a = \sigma_{a'} \) and hence \( a = a' \). Thus \( \subseteq \) is anti-symmetric. If \( a \in A \) and \( U \leq \alpha(a) \), the map \( \alpha(a, -) \cap U \) is a singleton, and we then have the restriction \( a_{\sigma(a)} \). If a family \( \{ a_i \} \) is pairwise compatible, the map \( \bigvee_i \alpha(a_i, -) \) is a singleton, and we then have the amalgamation. \( \square \)

**Lemma 2.18** Let \( (A, \alpha), (B, \beta) \) be \( \mathcal{O}(X) \)-valued sets with \( (B, \beta) \) complete. Each morphism \( \varphi : (A, \alpha) \to (B, \beta) \) is represented by a unique map \( h : A \to B \) which satisfies \( \alpha(a, a') \leq \beta(ha, ha') \) and \( \alpha(a) = \beta(ha) \) for all \( a, a' \in A \).

**Proof.** For any fixed \( a \in A \), the map \( \varphi(a, -) \) is a singleton of \( (B, \beta) \). By completeness, we can find a unique \( ha \in B \) such that \( \varphi(a, b) = \beta(ha, b) \) for every \( b \in B \). This defines a map \( h : A \to B \) representing \( \varphi \) and having the desired properties. \( \square \)

This lemma and Proposition 2.10(4) yield

**Proposition 2.19** \( \Theta \) induces a categorical equivalence between \( \text{Sh}(X) \) and \( \text{CSet}(\mathcal{O}(X)) \).

On the other hand, we can also show that \( \text{Set}(\mathcal{O}(X)) \) and \( \text{CSet}(\mathcal{O}(X)) \) are categorically equivalent.

**Proposition 2.20** Let \( \tilde{A} \) be the set of singletons on \( (A, \alpha) \). Define a valuation \( \tilde{\alpha} \) on \( \tilde{A} \) by

\[
\tilde{\alpha}(\sigma, \tau) := \bigvee_{a \in A} \sigma(a) \land \tau(a).
\]

Then \( (\tilde{A}, \tilde{\alpha}) \) is a complete \( \mathcal{O}(X) \)-valued set, and the map \( \tilde{A} \times A \ni (\sigma, a) \mapsto \sigma(a) \in \mathcal{O}(X) \) gives an isomorphism \( (A, \alpha) \simeq (\tilde{A}, \tilde{\alpha}) \). We call \( (\tilde{A}, \tilde{\alpha}) \) the completion of \( (A, \alpha) \).

For a morphism \( \varphi : (A, \alpha) \to (B, \beta) \), let \( \tilde{\varphi} \) be the composite of

\[
(\tilde{A}, \tilde{\alpha}) \xrightarrow{\sim} (A, \alpha) \xrightarrow{\varphi} (B, \beta) \xrightarrow{\sim} (\tilde{B}, \tilde{\beta}).
\]

Then \( (\tilde{-}) : \text{Set}(\mathcal{O}(X)) \to \text{CSet}(\mathcal{O}(X)) \) becomes a functor and also gives a quasi-inverse of the inclusion functor.

**Corollary 2.21** The categories \( \text{Sh}(X) \), \( \text{Set}(\mathcal{O}(X)) \) and \( \text{CSet}(\mathcal{O}(X)) \) are categorically equivalent. In particular, \( \text{Set}(\mathcal{O}(X)) \) is a topos.

We also remark that, for any morphism \( \varphi : (A, \alpha) \to (B, \beta) \), the composite of \( (A, \alpha) \xrightarrow{\varphi} (B, \beta) \xrightarrow{\sim} (\tilde{B}, \tilde{\beta}) \) is represented by the map \( A \ni a \mapsto \varphi(a, -) \in \tilde{B} \).

### 2.3 Topos structure of \( \text{Set}(\mathcal{O}(X)) \)

In the previous section, we saw that \( \text{Set}(\mathcal{O}(X)) \) is a topos. Here, we will give a concrete description of the topos structure of \( \text{Set}(\mathcal{O}(X)) \). Most results here (except for some details on the lattice \( \mathcal{P}(A, \alpha) \)) are borrowed from [29]. The constructions will be exploited later in this paper.
Proposition 2.22 (Finite limits in Set(\(\mathcal{O}(X)\))) (1) Let \(\{*, \top\}\) be the \(\mathcal{O}(X)\)-valued set with \(\top(*,*) = 1_X\). This yields a terminal object in Set(\(\mathcal{O}(X)\)).

(2) Let \(\{(A_i, \alpha_i)\}_{i \in I}\) be a finite family of \(\mathcal{O}(X)\)-valued sets. Define a valuation \(\delta\) on \(\prod_i A_i\) by \(\delta(a, a') = \bigwedge_i \alpha_i(a_i, a'_i)\) for \(a = \{a_i\}_{i \in I}\) and \(a' = \{a'_i\}_{i \in I}\). Then \((\prod_i A_i, \delta)\) equipped with the canonical projections \((\prod_i A_i, \delta) \rightarrow (A_i, \alpha_i)\) is a product of \(\{(A_i, \alpha_i)\}_{i \in I}\) in Set(\(\mathcal{O}(X)\)).

(3) Let \(\varphi, \psi : (A, \alpha) \rightarrow (B, \beta)\) be morphisms of \(\mathcal{O}(X)\)-valued sets. Define a valuation \(\delta\) on \(A\) by \(\delta(a, a') = \alpha(a, a') \land \bigvee_{b \in B} \psi(a, b) \land \psi(b, a)\). Then \((A, \delta)\) equipped with the canonical morphism \((A, \delta) \rightarrow (A, \alpha)\) is an equalizer of \(\varphi\) and \(\psi\) in Set(\(\mathcal{O}(X)\)).

(4) Let \((A, \alpha) \xrightarrow{\varphi} (C, \gamma) \xleftarrow{\psi} (B, \beta)\) be morphisms of \(\mathcal{O}(X)\)-valued sets. Define a valuation \(\delta\) on \(A \times B\) by

\[
\delta((a, b), (a', b')) = \alpha(a, a') \land \beta(b, b') \land \bigvee_{c \in C} \varphi(a, c) \land \psi(b, c).
\]

Then \((A \times B, \delta)\) equipped with the canonical projections is a pullback of that diagram in Set(\(\mathcal{O}(X)\)).

The following notion of strict relation is crucial in handling subobjects of an \(\mathcal{O}(X)\)-valued set. In the next section, it will enable us to define the “forcing values” of formulas.

Definition 2.23 Let \((A, \alpha)\) be an \(\mathcal{O}(X)\)-valued set. A strict relation on \((A, \alpha)\) is a function \(\sigma : A \rightarrow \mathcal{O}(X)\) such that

\[
\forall a, a' \in A, \quad \sigma(a) \land \alpha(a, a') \leq \sigma(a') \quad \text{and} \quad \sigma(a) \leq \alpha(a).
\]

Note that a singleton is a strict relation on the same \(\mathcal{O}(X)\)-valued set.

Proposition 2.24 Let \(\mathcal{P}(A, \alpha)\) be the set of strict relations on \((A, \alpha)\) ordered by

\[
\sigma \leq \tau \iff \forall a \in A, \quad \sigma(a) \leq \tau(a).
\]

Then, as ordered sets, \(\mathcal{P}(A, \alpha)\) is isomorphic to the poset Sub\((A, \alpha)\) of subobjects of \((A, \alpha)\).

Proof. For a strict relation \(\sigma\), we define a map \(\alpha_\sigma : A \times A \rightarrow \mathcal{O}(X)\) by

\[
\alpha_\sigma(a, a') := \sigma(a) \land \alpha(a, a') = \sigma(a') \land \alpha(a, a').
\]

Then \((A, \alpha_\sigma)\) is an \(\mathcal{O}(X)\)-valued set and the identity map on \(A\) represents a monomorphism \(\iota_\sigma : (A, \alpha_\sigma) \rightarrow (A, \alpha)\) by Proposition 2.11.

Conversely, for a monomorphism \(\varphi : (B, \beta) \rightarrow (A, \alpha)\), we define a strict relation \(\varphi_\sigma : A \rightarrow \mathcal{O}(X)\) by

\[
\varphi_\sigma(a) := \bigvee_{b \in B} \varphi(b, a).
\]

Then we can check

1. for any \(\sigma\), \(\iota_{\iota_\sigma} \circ \iota_\sigma = \sigma\);
2. for any \(\varphi\), \(\iota_{\varphi_\sigma} \simeq \varphi\) as subobjects of \((A, \alpha)\).

In fact, for an arbitrary morphism \(\varphi\), a strict relation \(\varphi_\sigma\) can be defined as above, and the image factorization of \(\varphi\) is given by

\[
\begin{array}{ccc}
(B, \beta) & \xrightarrow{\varphi} & (A, \alpha) \\
\downarrow \varphi & & \downarrow \iota_{\varphi_\sigma} \\
(A, \alpha_{\varphi_\sigma}) & \xrightarrow{\iota_{\varphi_\sigma}} & (A, \alpha)
\end{array}
\]

Set(\(\mathcal{O}(X)\)) is a topos and, in particular, a Heyting category (cf. [33, § A1.4]). The associated operations on subobject lattices are as follows:
Proposition 2.25 The operations on the frame $P(A, \alpha)$ are given by
\[
1_{P(A, \alpha)}(a) = \alpha(a), \quad 0_{P(A, \alpha)}(a) = 0_X,
\]
\[
(\sigma \land \tau)(a) = \sigma(a) \land \tau(a), \quad \bigvee_i \sigma_i(a) = \bigvee_i \sigma_i(a),
\]
\[
(\sigma \rightarrow \tau)(a) = \alpha(a) \land (\sigma(a) \rightarrow \tau(a)).
\]

Proposition 2.26 Let $\varphi : (B, \beta) \to (A, \alpha)$ be a morphism. Pulling back subobjects along $\varphi$ defines a frame homomorphism $\varphi^* : P(A, \alpha) \to P(B, \beta)$ such that, for $\sigma \in P(A, \alpha)$,
\[
(\varphi^* \sigma)(b) = \bigvee_{a \in A} \varphi(b, a) \land \sigma(a) = \beta(b) \land \bigwedge_{a \in A} [\varphi(b, a) \to \sigma(a)].
\]

Proposition 2.27 In the same notations as above, $\varphi^*$ has both a left adjoint $\exists_\varphi$ and a right adjoint $\forall_\varphi$; for $\tau \in P(B, \beta)$,
\[
(\exists_\varphi \tau)(a) = \bigvee_{b \in B} \varphi(b, a) \land \tau(b),
\]
\[
(\forall_\varphi \tau)(a) = \alpha(a) \land \bigwedge_{b \in B} [\varphi(b, a) \to \tau(b)].
\]
Finally, we describe the higher-order structure of $\text{Set}(O(X))$.

Proposition 2.28 Put $\delta(U, V) = (U \to V) \land (V \to U)$ for $U, V \in O(X)$. Then $(O(X), \delta)$ is an $O(X)$-valued set. Let $t : \{(\ast), \top\} \to (O(X), \delta)$ be the morphism defined by $t(\ast, U) = U$. This yields a subobject classifier of $\text{Set}(O(X))$.

Proof. Let $\chi : (A, \alpha) \to (O(X), \delta)$ be a morphism. Since $t$ corresponds to the strict relation $\text{id}_{O(X)}$ on $(O(X), \delta)$, the pullback of $t$ along $\chi$ is given by the strict relation $\sigma(a) = \bigvee_U \chi(a, U) \land U$. Conversely, given a strict relation $\sigma$ on $(A, \alpha)$, then $\sigma$ itself represents a morphism $\chi(a, U) = \alpha(a) \land (U \leftrightarrow \sigma(a))$. These correspondences yield a bijection between $P(A, \alpha)$ and $\text{Hom}((A, \alpha), (O(X), \delta))$.

\[
\begin{array}{ccc}
(A, \alpha) & \xrightarrow{\chi} & (O(X), \delta) \\
\downarrow j & & \downarrow t \\
\{\ast\} & \xrightarrow{1} & \{\ast\}, \top
\end{array}
\]

Similarly to $O(X)$, $P(A, \alpha)$ is not only a frame but also an $O(X)$-valued set.

Proposition 2.29 The power object of $(A, \alpha)$ is given by $P(A, \alpha)$ equipped with the valuation
\[
\tilde{a}(\sigma, \tau) := \bigwedge_{a \in A} \sigma(a) \leftrightarrow \tau(a).
\]

Proof. We would like to establish the following bijection
\[
\text{Hom}((B, \beta), P(A, \alpha)) \simeq P((B, \beta) \times (A, \alpha)).
\]
For a morphism $\varphi : (B, \beta) \to P(A, \alpha)$, we have a strict relation $\vartheta$ on $(B, \beta) \times (A, \alpha)$ such that
\[
\vartheta(b, a) = \bigvee_{\tau \in P(A, \alpha)} \varphi(b, \tau) \land \tau(a).
\]

\footnote{Note that $\tilde{a}$ does not necessarily coincide with $\hat{a}$ on $\hat{A}$ in Proposition 2.20. Indeed, while $\hat{a}(\sigma, \tau) \leq \tilde{a}(\sigma, \tau)$ holds for any $\sigma, \tau \in \hat{A}$, the converse inequality does not hold if $\sigma = \tau = a_0$ for a non-global element $a \in A$.}
On the other hand, for any strict relation \( \vartheta \), we have a morphism

\[
\varphi(b, \tau) = \beta(b) \land \bigwedge_{a \in A} \vartheta(b, a) \leftrightarrow \tau(a).
\]

These correspondences are mutual inverses. \( \square \)

## 3 Sheaves of structures and Heyting-valued structures

### 3.1 Structures in a topos

We will be concerned with categorical semantics in the toposes \( \text{Sh}(X) \) and \( \text{Set}(\mathcal{O}(X)) \). In this subsection, we take a glance at first-order categorical logic, which originates from [46]. The main reference is [33], in particular, Chapter D1 in volume 2. For an overview, we refer the reader to Caramello’s account [16], which is a preliminary version of the first two chapters of her book [17]. Although many fragments of (possibly infinitary) first-order logic are considered in the context of categorical logic, we restrict our attention to single-sorted intuitionistic logic. Examples of other fragments include Horn, cartesian, regular, coherent, classical, and geometric logics.

**Definition 3.1** A (first-order) language \( \mathcal{L} \) consists of the following data:

1. A set \( \mathcal{L}\text{-Func} \) of function symbols. Each function symbol \( f \in \mathcal{L}\text{-Func} \) is associated with a natural number \( n \) (the arity of \( f \)). If \( n = 0 \), \( f \) is called a constant (symbol).
2. A set \( \mathcal{L}\text{-Rel} \) of relation symbols. Each relation symbol \( R \in \mathcal{L}\text{-Rel} \) is associated with a natural number \( n \) (the arity of \( R \)). If \( n = 0 \), \( R \) is called an atomic proposition.

\( \mathcal{L} \)-terms and \( \mathcal{L} \)-formulas are defined as usual. We need some conventions.

**Definition 3.2**

1. A context is a finite list \( u \equiv u_1, \ldots, u_n \) of distinct variables. If \( n = 0 \), it is called the empty context and denoted by \( [\ ] \).
2. We say that a context \( u \) is suitable for an \( \mathcal{L} \)-formula \( \varphi \) when \( u \) contains all the free variables of \( \varphi \). A formula \( \varphi \) equipped with a suitable context \( u \) is called a formula-in-context and indicated by \( \varphi(u) \). Similarly, terms-in-context can be defined.

For an \( \mathcal{L} \)-formula-in-context \( \varphi(u, v) \), we abbreviate, e.g., \( \exists v_1 \cdots \exists v_n \varphi(u, v) \) as \( \exists v \varphi(u, v) \). We also abbreviate the \( \mathcal{L} \)-formula \( \bigwedge_{i} u_i = v_i \) as \( u = v \), where \( u, v \) are assumed to have the same length. A formula is closed if it contains no free variables.

We now give categorical semantics in an arbitrary elementary topos \( \mathcal{E} \).

**Definition 3.3** Let \( \mathcal{L} \) be a language and \( \mathcal{E} \) a topos. An \( \mathcal{L} \)-structure \( \mathcal{M} \) in \( \mathcal{E} \) is given by specifying the following data:

1. An underlying object \( |\mathcal{M}| \in \mathcal{E} \). We denote the \( n \)-ary product by \( |\mathcal{M}|^n \). In particular, \( |\mathcal{M}|^0 \) is the terminal object \( 1_{\mathcal{E}} \).
2. A morphism \( f^\mathcal{M} : |\mathcal{M}|^n \to |\mathcal{M}| \) for each \( n \)-ary function symbol \( f \).
3. A subobject \( R^\mathcal{M} \) of \( |\mathcal{M}|^n \) for each \( n \)-ary relation symbol \( R \).

As usual, we will not distinguish \( \mathcal{M} \) and its underlying object \( |\mathcal{M}| \) in notation.

Interpretations of \( \mathcal{L} \)-terms and \( \mathcal{L} \)-formulas are defined by using internal operations in \( \mathcal{E} \).

**Definition 3.4** (Interpretations of terms) Let \( \mathcal{M} \) be an \( \mathcal{L} \)-structure in a topos \( \mathcal{E} \). For an \( \mathcal{L} \)-term-in-context \( t(u) \), we define the interpretation \( t^\mathcal{M} : \mathcal{M}^n \to \mathcal{M} \) inductively.

1. If \( t \) is a variable \( u_i \), then \( t^\mathcal{M} \) is the \( i \)th product projection \( \pi_i : \mathcal{M}^n \to \mathcal{M} \).
2. If interpretations of \( L \)-terms \( t_i(u) \) and \( s(v) \) are given, the term \( s(t_1(u), \ldots, t_m(u)) \) is interpreted as the composite of the following morphisms:

\[
\mathcal{M}^n \xrightarrow{t_1^M, \ldots, t_m^M} \mathcal{M}^m \xrightarrow{s^M} \mathcal{M},
\]

where \( \langle t_1^M, \ldots, t_m^M \rangle \) is the morphism obtained from the morphisms \( t_i^M \) by using the universal property of the product \( \mathcal{M}^m \).

**Definition 3.5** (Interpretations of formulas) Let \( \mathcal{M} \) be an \( L \)-structure in a topos \( \mathcal{E} \). For an \( L \)-formula-in-context \( \varphi(u) \), we define the interpretation \( [u. \varphi]_\mathcal{M} \) as a subobject of \( \mathcal{M}^n \) inductively. (We drop the subscript \( \mathcal{M} \) if no confusion arises.)

1. If \( \varphi \equiv (s(u) = t(u)) \) where \( s, t \) are terms, then \( [u. \varphi]_\mathcal{M} \) is defined to be the equalizer of

\[
\mathcal{M}^n \xrightarrow{s^M} \mathcal{M} \xrightarrow{t^M} \mathcal{M}.
\]

2. If \( \varphi \equiv R(t_1(u), \ldots, t_m(u)) \), then \( [u. \varphi]_\mathcal{M} \) is the pullback

\[
\begin{array}{ccc}
\mathcal{M}^n & \xrightarrow{\langle t_1^M, \ldots, t_m^M \rangle} & \mathcal{M}^m \\
\downarrow & & \downarrow \\
\mathcal{M} & \rightarrow & \mathcal{M}^m
\end{array}
\]

3. If \( \varphi \equiv \top, \bot, \psi \land \vartheta, \psi \lor \vartheta, \psi \rightarrow \vartheta \) or \( \neg \varphi \), then \( [u. \varphi]_\mathcal{M} \) is defined as expected by using the Heyting operations on \( \text{Sub}(\mathcal{M}^n) \).

4. If \( \varphi \equiv \exists v \psi(u, v) \), then \( [u. \varphi]_\mathcal{M} \) is the image as in the following diagram:

\[
\begin{array}{ccc}
\mathcal{M}^n & \xrightarrow{[u. \exists v \psi]} & \mathcal{M}^n \times \mathcal{M} \\
\downarrow & & \downarrow \pi \\
\mathcal{M}^n & \rightarrow & \mathcal{M}^n
\end{array}
\]

where \( \pi \) is the projection onto \( \mathcal{M}^n \).

5. If \( \varphi \equiv \forall v \psi(u, v) \), then \( [u. \varphi]_\mathcal{M} := \forall \pi [u, v. \psi] \), where \( \forall \pi : \text{Sub}(\mathcal{M}^n \times \mathcal{M}) \rightarrow \text{Sub}(\mathcal{M}^n) \) is the right adjoint of \( \pi^* \).

In this paper, we will not consider the notions of models of a theory in a topos nor homomorphisms between structures.

### 3.2 Sheaves of structures and Heyting-valued structures

We now investigate the relationship between structures in \( \text{Sh}(X) \) and those in \( \text{Set}(\mathcal{O}(X)) \). We first consider the case of sheaves on a topological space \( X \). Let \( \text{LH} \) be the category of topological spaces and local homeomorphisms between them. Recall that the slice category \( \text{LH}/X \) is categorically equivalent to \( \text{Sh}(X) \). Comer [19], Ellerman [23], and Macintyre [43] used the following notion to obtain model-theoretic results:

**Definition 3.6** An \( \text{étalé bundle of } \mathcal{L} \text{-structures (on } X \text{)} \) is a tuple

\[
\langle X, E, \pi, \{f^E_x : x \in X, f \in \mathcal{L}-\text{Func}\}, \{R^E_x : x \in X, R \in \mathcal{L}-\text{Rel}\}\rangle
\]

such that

1. \( \pi : E \rightarrow X \) is a local homeomorphism of topological spaces,
2. each stalk \( E_x \) equipped with \( \{f^E_x\}_f \) and \( \{R^E_x\}_R \) is an \( \mathcal{L} \)-structure, and
   a. for each function symbol \( f \), the map \( \bigsqcup_x (E_x)^n \rightarrow E \) induced by \( \{f^E_x\}_x \) is continuous,
(b) for each relation symbol $R$, the subset $\bigsqcup_{i} R^{E_i} \subseteq \bigsqcup_{i} (E_i)^n$ is open, where $\bigsqcup_{i} (E_i)^n$ is seen as a subspace of the product space $E^n$ for $n > 0$, and $\bigsqcup_{i} (E_i)^0 \cong X$.

Sheaves of abelian groups or of rings in geometry are, of course, such examples for suitable languages. We will meet other examples which give model-theoretic constructions of (usual Set-valued) structures in the next section.

**Lemma 3.7** An étale bundle of $L$-structures is identified with an $L$-structure in $\text{LH}/X$.

**Proof.** Notice the following facts:
1. $\bigsqcup_i (E_i)^n$ is a fiber product $E \times_X \cdots \times_X E$, i.e., a product in $\text{LH}/X$.
2. Any monomorphism in $\text{LH}/X$ is an open embedding.

Hereafter, we fix a locale $X$. The previous lemma motivates the following definition:

**Definition 3.8** A sheaf of $L$-structures is an $L$-structure in $\text{Sh}(X)$.

When we mention a subsheaf $Q$ of a sheaf $P$, each $Q(U)$ is assumed to be a subset of $P(U)$.

Before we define Heyting-valued structures, let us introduce space-saving notations. If $(\mathcal{M}, \delta)$ is an $\mathcal{O}(X)$-valued set, then the $n$th power $\mathcal{M}^n$ is canonically equipped with the valuation as in Proposition 2.22(2). For tuples $a, a' \in \mathcal{M}^n$, we simply write $\delta(a, a')$ (resp. $\delta(a)$) for $\bigsqcup_i \delta(a_i, a'_i)$ (resp. $\bigsqcup_i \delta(a_i)$). These notations are useful, but, in the case $n = 2$, we will always write $\delta(a) \land \delta(b)$ for $\delta((a, b), (a, b))$ to avoid confusion between $\delta(a, b)$ and $\delta((a, b))$.

**Definition 3.9** An $\mathcal{O}(X)$-valued $L$-structure is an $L$-structure in the topos $\text{Set}(\mathcal{O}(X))$, i.e., it consists of the following data:
1. an $\mathcal{O}(X)$-valued set $(\mathcal{M}, \delta)$,
2. for each function symbol $f$, a morphism $f^\mathcal{M} : (\mathcal{M}^n, \delta) \to (\mathcal{M}, \delta)$,
3. for each relation symbol $R$, a strict relation $R^\mathcal{M} : \mathcal{M}^n \to \mathcal{O}(X)$.

The interpretation of equality is the diagonal $(\mathcal{M}, \delta) \mapsto (\mathcal{M}^2, \delta)$, which corresponds to the strict relation $(a, b) \mapsto \delta(a, b)$ on $(\mathcal{M}^2, \delta)$ under the bijection in Proposition 2.24.

Fourman & Scott [26, p. 365] defined Heyting-valued structures in a slightly less general form.

Recall the construction of $\Theta : \text{Set}(\mathcal{O}(X))^{\mathcal{P}^n} \to \text{Set}(\mathcal{O}(X))$ at the beginning of § 2.2. We can obtain $\mathcal{O}(X)$-valued structures from sheaves of structures on $X$ by applying $\Theta$.

**Lemma 3.10** Let $P$ be a presheaf on $X$. Then, the $n$-ary product $\Theta(P)^n$ is isomorphic to $\Theta(P^n)$ as $\mathcal{O}(X)$-valued sets. Indeed, the canonical map $h : \bigsqcup_U (PU)^n \to (\bigsqcup_U PU)^n$ represents an isomorphism $\iota : \Theta(P^n) \to \Theta(P)^n$ so that $\iota(b, a) = \iota^{-1}(a, b) = \delta_P(h(b), a)$.

Moreover, for a strict relation $\sigma$ on $\Theta(P^n)$, the corresponding strict relation $\tau$ on $\Theta(P)^n$ is given by

$$\tau(a) = \bigsqcup_{b \in \iota(\Theta(P^n))} \sigma(b) \land \delta_P(h(b), a).$$

**Proof.** For the case when $P$ is a sheaf, this lemma is an immediate consequence of the fact that $\Theta : \text{Sh}(X) \to \text{Set}(\mathcal{O}(X))$ is part of an equivalence of categories. We can also see directly that $\Theta : \text{Set}(\mathcal{O}(X))^{\mathcal{P}^n} \to \text{Set}(\mathcal{O}(X))$ preserves finite products by using Corollary 2.12.

For a given $\sigma$, by the proof of Proposition 2.24, the corresponding subobject of $\Theta(P^n)$ is $(\bigsqcup_U (PU)^n, (\delta_P)_o)$ with $(\delta_P)_o(b, b') = \sigma(b) \land \delta_P(b, b')$. Hence, $\tau$ is given by

$$\tau(a) = \bigsqcup_{b, b' \in \iota(\Theta(P^n))} (\delta_P)_o(b, b') \land \iota(b', a) = \bigsqcup_{b \in \iota(\Theta(P^n))} \sigma(b) \land \delta_P(h(b), a).$$

We remark that $\tau(h(b)) = \sigma(b)$ for any $b \in \Theta(P^n)$ and therefore $\tau$ is an extension of $\sigma$ along $h$. $\square$
Proposition 3.11 Let $P$ be a sheaf of $\mathcal{L}$-structures on $X$. The functor $\Theta$ induces an $\mathcal{O}(X)$-valued $\mathcal{L}$-structure $\mathcal{M} = \Theta(P)$, which is concretely described as follows:

\[
f^\mathcal{M}(a, a') = \delta_P(f^\mathcal{P}(a|_{\delta_P(a)}), a'),
\]

\[
R^\mathcal{M}(a) = \bigvee \{ W \leq \delta_P(a) : a|_W \in R^\mathcal{P}(W) \},
\]

where $a \in \Theta(P)^n$, $a' \in \Theta(P)$ and $a|_{\delta_P(a)} := (a_1|_{\delta_P(a)}, \ldots, a_n|_{\delta_P(a)})$.

Proof. For each function symbol $f$, we have a morphism $f^\mathcal{P} : P^n \to P$ of sheaves. This induces a morphism $\Theta(f^\mathcal{P}) : \Theta(P^n) \to \Theta(P)$. By the previous lemma, we obtain a morphism $f^\mathcal{M} : \Theta(P)^n \to \Theta(P)$, which can be computed as

\[
f^\mathcal{M}(a, a') = \bigvee_{b \in \Theta(P)^n} \iota^{-1}(a, b) \land \Theta(f^\mathcal{P})(b, a') = \delta_P(f^\mathcal{P}(a|_{\delta_P(a)}), a').
\]

In particular, $f^\mathcal{M}$ is represented by the map $k : \mathcal{M}^n \to \mathcal{M}$ with $k(a) = f^\mathcal{P}(a|_{\delta_P(a)})$.

For each relation symbol $R$, we have a subsheaf $R^\mathcal{P} \to P^n$. This induces a subobject $\Theta(R^\mathcal{P}) \to \Theta(P^n)$, which corresponds to the following strict relation $\sigma : \bigcup \Theta(\mathcal{P}U)^n \to \mathcal{O}(X)$: for $b \in (\mathcal{P}U)^n$,

\[
\sigma(b) = \bigvee_{b' \in \Theta(R^\mathcal{P})} \delta_P(b, b') = \bigvee \{ W \leq U : b|_W \in R^\mathcal{P}(W) \}.
\]

By the previous lemma, we obtain a subobject of $\Theta(P)^n$, which corresponds to the following strict relation $R^\mathcal{M} : \mathcal{M}^n \to \mathcal{O}(X)$: for $a \in PU_1 \times \cdots \times PU_n$,

\[
R^\mathcal{M}(a) = \bigvee_{b \in \Theta(P)^n} \iota^{-1}(a, b) \land \sigma(b)
\]

\[
= \bigvee_{b \in \Theta(P)^n} \left[ \delta_P(h(b), a) \land \bigvee \{ W \leq \delta_P(b) : b|_W \in R^\mathcal{P}(W) \} \right]
\]

\[
= \bigvee \{ W \leq U_1 \land \cdots \land U_n : a|_W \in R^\mathcal{P}(W) \}.
\]

Notice that the subobject $\Theta(P) \to \Theta(P)^2$ obtained from the diagonal $P \to P^2$ is the same as the one determined by the strict relation $\delta_P$ on $\Theta(P)$. \(\square\)

We could describe the converse construction (from Heyting-valued structures to sheaves of structures). This involves a complicated use of completion of Heyting-valued sets, and we do not find such details to be useful for the purpose of this paper. So we skip it at this point.

In the context of set theory, there are examples of Heyting-valued structures which do not come from sheaves (e.g. [55, Examples 2.9 & 2.13]).

### 3.3 Forcing values of formulas

Forcing values of formulas derive from Boolean-valued set theory. Here we first define them categorically and then observe that our definition is compatible with the usual one. The categorical description seems to be folklore but has not appeared in an explicit form elsewhere. For an $\mathcal{O}(X)$-valued $\mathcal{L}$-structure $(\mathcal{M}, \delta)$, we write $\mathcal{L}_\mathcal{M}$ for the language extending $\mathcal{L}$ by adding a new constant symbol for each element of $\mathcal{M}$.

**Definition 3.12** For an $\mathcal{L}$-formula-in-context $\varphi(u)$, the strict relation $\|\varphi(-)\|^\mathcal{M}$ on $(\mathcal{M}, \delta)^n$ is defined to be the one corresponding to the subobject $\lbrack u, \varphi \rbrack_{(\mathcal{M}, \delta)^n} \to (\mathcal{M}, \delta)^n$ via Proposition 2.24. For $a \in \mathcal{M}^n$, $\|\varphi(a)\|^\mathcal{M}$ is called the forcing value of the closed $\mathcal{L}_\mathcal{M}$-formula $\varphi(a)$. We drop the superscript $\mathcal{M}$ if no confusion arises.

Since the strict relation $a \mapsto \delta(a)$ is the greatest element in $\mathcal{P}(\mathcal{M}^n, \delta)$, $\|\varphi(a)\|^\mathcal{M} \leq \delta(a)$ always holds. Using the results in § 2.3, we can calculate the forcing values inductively.
**Proposition 3.13**

\[ \|R(t_1(a), \ldots, t_m(a))\|^M = \bigvee_{b \in M^n} (t_1^M, \ldots, t_m^M)(a, b) \land R^M(b), \]

\[ \|s(a) = t(a)\|^M = \bigvee_{b, c \in M} (s^M, t^M)(a, (b, c)) \land \delta(b, c), \]

\[ \|\varphi(a) \land \psi(a)\|^M = \|\varphi(a)\|^M \land \|\psi(a)\|^M, \]

\[ \|\varphi(a) \lor \psi(a)\|^M = \|\varphi(a)\|^M \lor \|\psi(a)\|^M, \]

\[ \|\varphi(a) \rightarrow \psi(a)\|^M = \delta(a) \land \left[ \|\varphi(a)\|^M \rightarrow \|\psi(a)\|^M \right], \]

\[ \|\exists v \varphi(a, v)\|^M = \bigvee_{b \in M} \|\varphi(a, b)\|^M, \]

\[ \|\forall v \varphi(a, v)\|^M = \delta(a) \land \bigwedge_{b \in M} \left[ \delta(b) \rightarrow \|\varphi(a, b)\|^M \right]. \]

**Remark 3.14** If a formula \( \varphi \) has a suitable context \( u \) and \( v \) is a variable distinct from \( u \), we have to distinguish the formulas-in-context \( \varphi(u) \) and \( \varphi(u, v) \). Indeed, the forcing values \( \|\varphi(a)\| \) and \( \|\varphi(a, b)\| \) can be different and

\[ \|\varphi(a, b)\| = \|\varphi(a)\| \land \delta(b). \]

This description of forcing values is compatible with those in [26, Definition 5.13] and [59, Definition 13.6.6]. The soundness and completeness theorems for Heyting-valued semantics are usually formulated with respect to intuitionistic predicate logic with existence predicate (for short, IQCE) as in [59, §2.2, §13.6]. However, we will only need soundness of the following form:

**Lemma 3.15** If the sentence \( \forall u [\varphi(u) \rightarrow \psi(u)] \) is intuitionistically valid, then \( \|\varphi(a)\|^M \leq \|\psi(a)\|^M \leq \delta(a) \) holds for any \( a \in M^n \).

**Proof.** The assumption implies \( \|u. \varphi\| \leq \|u. \psi\| \) as subobjects of \( (M, \delta)^n \). Therefore, the conclusion holds by the definition of forcing values.

Let \( P \) be a sheaf of \( L \)-structures and \( \Theta(P) = (M, \delta) \) be the \( O(X) \)-valued \( L \)-structure obtained from Proposition 3.11. We can see:

1. For any \( L \)-term \( t(u) \), the morphism \( t^M \) is represented by the map \( M^n \ni a \mapsto t^P(a|s(a)) \in M \) where \( t^P : P^n \rightarrow P \) is the interpretation of \( t \) by \( P \).
2. For any atomic \( L \)-formula \( R(t_1(a), \ldots, t_m(a)) \) and \( a \in M^n \),

\[ \|R(t_1(a), \ldots, t_m(a))\|^M = R^M(t_1^P(a|s(a)), \ldots, t_m^P(a|s(a))) \]

\[ = \bigvee \{ W \leq \delta(a) ; (t_1^P(a|W), \ldots, t_m^P(a|W)) \in R^P(W) \}. \]

Similarly for the formula \( s(a) = t(a) \).

More generally, the forcing value \( \|\varphi(-)\| \) for \( \Theta(P) \) can be described in terms of the subsheaf \( \|u. \varphi\| \) of \( P^n \). Let \( \Omega \) be the sheaf \( U \mapsto \Omega(U) = (U)^\downarrow \). This is a subobject classifier in \( \text{Sh}(X) \), and we thus obtain the characteristic morphism \( \chi : P^n \rightarrow \Omega \) by the universality of the subobject classifier:

\[ [u. \varphi] \xrightarrow{1} \frac{1}{\downarrow} \frac{\text{true}}{\chi U(a) = \bigvee \{ W \leq U ; a|W \in \|u. \varphi\| (W) \}}. \]

Using Proposition 2.28 and the fact that \( \Theta(\Omega) \) and \( (O(X), \delta) \) in that proposition are canonically isomorphic, we can verify the following:
Proposition 3.16 (Definable subsheaves and forcing values) In the above notation, \( \chi_U(a) = \| \varphi(a) \|_U \) for any \( a \in P^U \). We will denote \( \chi \) by \( \| \varphi(\cdot) \|_P \) and its component \( \chi_U \) by \( \| \varphi(\cdot) \|_U \).

Let \( \gamma : \mathcal{O}(X) \to \text{Set}^{\mathcal{O}(X)^{pp}} \) be the Yoneda embedding, and \( a : \text{Set}^{\mathcal{O}(X)^{pp}} \to \text{Sh}(X) \) the associated sheaf functor. We write \( a : ayU \to P^n \) for the morphism corresponding to \( a \in P^nU \) under the bijection
\[
P^nU \cong \text{Hom}_{\mathcal{O}(X)^{pp}}(yU, P^n) \cong \text{Hom}_{\text{Sh}(X)}(ayU, P^n).
\]
In terms of forcing values, the Kripke–Joyal semantics in \( \text{Sh}(X) \) (cf. [41, § VI.7]) has a simple description:
\[
U \models_P \varphi(a) \iff \text{the morphism } a : ayU \to P^n \text{ factors through the subsheaf } [a, \varphi] \to P^n
\]
\[
\iff \| \varphi(\cdot) \|_P \circ a = \text{true} \circ !
\]
\[
\iff \| \varphi(a) \|_U = U.
\]
This is the reason why we use the term “forcing values” similarly as in [23]. Using the above description, we can show the properties of forcing relation [41, Theorem VI.7.1] for the usual site on \( \mathcal{O}(X) \).

4 Filter-quotients of Heyting-valued structures and Łoś’s theorem

As we promised after Definition 3.6, we will observe that sheaves of structures give some constructions in model theory. These constructions can be generalized to constructions for Heyting-valued structures, and they provide an adequate setup to state our Łoś-type theorem.

4.1 Model-theoretic constructions via sheaves of structures

Definition 4.1 Let \( P \) be a sheaf of \( \mathcal{L} \)-structures on a locale \( X \).

1. We make the set \( P(U) \) for a fixed \( U \) into an \( \mathcal{L} \)-structure as follows:
\[
f^{P(U)}(a) := (f^P)_U(a), \quad P(U) \models R(a) \iff a \in R^P(U) \subseteq P(U)^m.
\]

2. For a filter \( f \) on \( \mathcal{O}(X) \), the colimit \( P/f := \lim_{U \in f} P(U) \) is the quotient of \( \coprod_{U \in f} P(U) \) by the following equivalence relation: for \( U, V \in f \) and \( a \in P(U), b \in P(V) \),
\[
(U, a) \sim (V, b) \iff \exists W \in f, W \subseteq U \cap V \text{ and } a|_W = b|_W.
\]
We often write \( [a]_f \) for a tuple \( (a_1, \ldots, a_n) \) of equivalence classes. Let \( \delta \) be the valuation of \( \Theta(P) \). We make \( P/f \) into an \( \mathcal{L} \)-structure as follows:
\[
f^{P/f}([a]_f) := [f^P(a_1|_{\delta(a_1)}, \ldots, a_n|_{\delta(a_n)})],
\]
\[
P/f \models R([a]_f) \iff \exists W \in f, a|_W \in R^P(W)
\]
\[
\iff \exists W \in f, P(W) \models R(a|_W).
\]
In particular, if \( X \) is a topological space and \( x \in X \), each stalk \( P_x \) is the quotient \( P/n_x \) by the filter \( n_x \) of open neighborhoods of \( x \).

Example 4.2 (Products) Let \( X \) be a set. Given an \( X \)-indexed family \( \{ M_x \}_{x \in X} \) of \( \mathcal{L} \)-structures, the product \( \mathcal{N} := \coprod_{x \in X} M_x \) is an \( \mathcal{L} \)-structure such that, for any elements \( a' = \{ a'_x \}_{x \in X} \),
\[
f^{\mathcal{N}}(a^1, \ldots, a^n) := \{ f^{M_x}(a^1_x, \ldots, a^n_x) \}_{x \in X},
\]
\[
\mathcal{N} \models R(a^1, \ldots, a^n) \iff \forall x \in X, M_x \models R(a^1_x, \ldots, a^n_x).
\]
This model-theoretic construction can be embodied by a sheaf of \( \mathcal{L} \)-structures on the discrete space \( X \). Let \( P \) be the sheaf of \( \mathcal{L} \)-structures corresponding to the étalé bundle of \( \mathcal{L} \)-structures \( \coprod_{x \in X} M_x \to X \) given by the canonical projection. Then, the \( \mathcal{L} \)-structure \( P(X) \) of global sections is the same as \( \mathcal{N} \).
Notice that, by induction based on Proposition 3.13,
\[ \| \varphi(a^1, \ldots, a^n) \|^{\mathcal{M}} = \{ x \in X : \mathcal{M} \models \varphi(a^1, \ldots, a^n) \} \]
holds for any formula \( \varphi \) and \( a^1, \ldots, a^n \in \mathcal{N} \).

**Example 4.3** (Ultraproducts) Let \( u \) be an ultrafilter over a set \( X \). In the same notation as the previous example, the ultraproduct \( \prod \mathcal{M}_x / u \) is the quotient of \( \prod \mathcal{M}_x \) by the equivalence relation
\[ a \sim b \quad \overset{def}{\iff} \quad \{ x \in X : a_x = b_x \} \in u \]
equipped with canonical interpretations of \( \mathcal{L} \), e.g.,
\[ \prod \mathcal{M}_x / u \models R([a_1]_u, \ldots, [a_n]_u) \; \overset{def}{\iff} \; \{ x \in X : \mathcal{M}_x \models R(a_1, \ldots, a_n) \} \in u. \]
If each \( \mathcal{M}_x \) is non-empty, \( \prod \mathcal{M}_x / u \) can be described as a filter-quotient of the sheaf \( P \) corresponding to \( \prod_{x \in X} \mathcal{M}_x \to X \). Since \( P(U) = \prod_{x \in U} \mathcal{M}_x \) and each local section can be extended to a global section by non-emptiness, we have
\[ \prod \mathcal{M}_x / u \cong \lim_{U \in u} P(U) = P / u. \]
Thus, it is reasonable to regard \( P / u \) as a “generalized” ultraproduct for any \( u \). Notice that we need the axiom of choice to extend local sections to global ones, but we do not need AC if \( \mathcal{L} \) contains a constant symbol.

**Example 4.4** (Bounded Boolean Powers) Let \( B \) be a Boolean algebra and \( \mathcal{M} \) be an \( \mathcal{L} \)-structure. We then have the sheaf \( P \) on the Stone space \( X \) dual to \( B \) determined by
\[ P(U) := \{ s : U \to \mathcal{M} : \text{locally constant map} \}. \]
This becomes a sheaf of \( \mathcal{L} \)-structures, and \( \mathcal{M}[B]_{lo} := P(X) \) is said to be the *bounded Boolean power* of \( \mathcal{M} \) (cf. [30, \$9.7]).

**Example 4.5** (Bounded Boolean Ultrapowers) In the same notation as the previous example, for any \( s, t \in \mathcal{M}[B]_{lo} \), the subsets
\[ \| R(s_1, \ldots, s_n) \| = \{ v \in X : \mathcal{M} \models R(s_1(v), \ldots, s_n(v)) \}, \]
\[ \| s = t \| = \{ v \in X : s(v) = t(v) \} \]
are clopen and identified with elements of \( B \). Let \( u \) be an ultrafilter on \( B \) (= a point of \( X \)). The *bounded Boolean ultrapower* \( \mathcal{M}[B]_{lo} / u \) is given by
\[ s \sim t \quad \overset{def}{\iff} \quad \| s = t \| \in u, \]
\[ \mathcal{M}[B]_{lo} / u \models R([s_1]_u, \ldots, [s_n]_u) \overset{def}{\iff} \| R(s_1, \ldots, s_n) \| \in u. \]
\( \mathcal{M}[B]_{lo} / u \) has a representation as a filter-quotient
\[ \mathcal{M}[B]_{lo} / u \cong \lim_{U \in u} P(D_U) = P_u, \]
where \( D_U = \{ v \in X : U \in v \} \) and \( P_u \) is the stalk over \( u \).

Bounded Boolean (ultra)powers are not direct generalizations of ordinary (ultra)powers. *Unbounded Boolean (ultra)powers* are such things, while they involve more complicated sheaf-theoretic constructions. Fish [25] gives a survey of bounded and unbounded Boolean (ultra)powers. These constructions can be further generalized to the notion of *Boolean product* (cf. [13, 14, 65]), which involves sheaves on Stone spaces (or, more precisely, étalé bundles over Stone spaces).
4.2 Filter-quotients of Heyting-valued structures

We will generalize the construction of \( P/\mathfrak{f} \) to Heyting-valued structures. We use filter-quotients of Heyting-valued sets (or structures), which appeared in, e.g., [55, Definition 2.6] and [50, Chapter 34]. Let \((\mathcal{M}, \delta)\) be an \( \mathcal{O}(X) \)-valued \( \mathcal{L} \)-structure. Given a filter \( \mathfrak{f} \) on \( \mathcal{O}(X) \), an \((\mathcal{O}(X)/\mathfrak{f})\)-valued \( \mathcal{L} \)-structure \( \mathcal{M}/\mathfrak{f} \) is defined as follows:\(^3\) we first observe the following fact:

**Claim** The following relation \( \sim_{\mathfrak{f}} \) on \( \mathcal{M} \) is an equivalence relation

\[
a \sim_{\mathfrak{f}} b \iff [\delta(a) \lor \delta(b) \rightarrow \delta(a, b)] \in \mathfrak{f}.
\]

**Proof.** For transitivity, observe

\[
(\delta(a) \lor \delta(b) \rightarrow \delta(a, b)) \land (\delta(b) \lor \delta(c) \rightarrow \delta(b, c)) \land (\delta(a) \lor \delta(c))
\]

\[
= [\delta(a) \land (\delta(a) \lor \delta(b) \rightarrow \delta(a, b)) \land (\delta(b) \lor \delta(c) \rightarrow \delta(b, c))] \\
\lor [\delta(c) \land (\delta(a) \lor \delta(b) \rightarrow \delta(a, b)) \land (\delta(b) \lor \delta(c) \rightarrow \delta(b, c))]
\]

(by using \( \delta(a, b) \leq \delta(a) \lor \delta(b) \) etc.)

\[
= \delta(a, b) \land \delta(b, c) \leq \delta(a, c).
\]

We then have

\[
(\delta(a) \lor \delta(b) \rightarrow \delta(a, b)) \lor (\delta(b) \lor \delta(c) \rightarrow \delta(b, c)) \leq \delta(a) \lor \delta(c) \rightarrow \delta(a, c).
\]

We denote the quotient \( \mathcal{M}/\sim_{\mathfrak{f}} \) by \( \mathcal{M}/\mathfrak{f} \) and the equivalence class of \( a \in \mathcal{M} \) by \( [a]_{\mathfrak{f}} \). In particular, by applying this to the \( \mathcal{O}(X) \)-valued set \( \mathcal{O}(X) \land \), for which \( U \sim_{\mathfrak{f}} V \) iff \( (U \leftrightarrow V) \in \mathfrak{f} \), we have the quotient Heyting algebra \( \mathcal{O}(X)/\mathfrak{f} \). By defining the valuation\(^4\)

\[
\delta_{\mathfrak{f}}([a]_{\mathfrak{f}}, [b]_{\mathfrak{f}}) := [\delta(a, b)]_{\mathfrak{f}},
\]

we can make \( \mathcal{M}/\mathfrak{f} \) into an \((\mathcal{O}(X)/\mathfrak{f})\)-valued set except that \( \mathcal{O}(X)/\mathfrak{f} \) is not necessarily complete. We may use the Dedekind–MacNeille completion of \( \mathcal{O}(X)/\mathfrak{f} \) (cf. [32, III.3.11]) to define forcing values as in [55, Definition 2.2], but such a complication will not be necessary for this paper because we will use forcing values \( \|\varphi([a]_{\mathfrak{f}})\|_{\mathcal{M}/\mathfrak{f}} \) only for atomic formulas \( \varphi \).

For each function \( f \) and each relation \( R \), the morphism \( f^{\mathcal{M}/\mathfrak{f}} : ((\mathcal{M}/\mathfrak{f})^n, \delta_{\mathfrak{f}}) \rightarrow (\mathcal{M}/\mathfrak{f}, \delta_{\mathfrak{f}}) \) and the strict relation \( R^{\mathcal{M}/\mathfrak{f}} : (\mathcal{M}/\mathfrak{f})^n \rightarrow \mathcal{O}(X)/\mathfrak{f} \) are defined canonically:

\[
R^{\mathcal{M}/\mathfrak{f}}([a_1]_{\mathfrak{f}}, \ldots, [a_n]_{\mathfrak{f}}) := [R^\mathcal{M}(a_1, \ldots, a_n)]_{\mathfrak{f}},
\]

We have finished the construction of the \((\mathcal{O}(X)/\mathfrak{f})\)-valued \( \mathcal{L} \)-structure \( \mathcal{M}/\mathfrak{f} \). We will call it the filter-quotient of \( \mathcal{M} \) by \( \mathfrak{f} \).

Next, we consider filter-quotients of \( \Theta(P) \) for a sheaf \( P \) of \( \mathcal{L} \)-structures. Recall that we already defined an \( \mathcal{L} \)-structure \( P/\mathfrak{f} \).

**Lemma 4.6** Let \( P \) be a sheaf of \( \mathcal{L} \)-structures and \( (\mathcal{M}, \delta) := \Theta(P) \). Then the canonical map \( P/\mathfrak{f} \rightarrow \mathcal{M}/\mathfrak{f} \) induces a bijection between \( P/\mathfrak{f} \) and the set of global elements of \( \mathcal{M}/\mathfrak{f} \).

**Proof.** Since \( \delta_{\mathfrak{f}}([a]_{\mathfrak{f}}) = [1_X]_{\mathfrak{f}} \) iff \( \delta(a) \in \mathfrak{f} \), it is obvious that the image of the canonical map \( P/\mathfrak{f} \rightarrow \mathcal{M}/\mathfrak{f} \) consists of global elements. We will show this map is injective.

For \( a, b \in \mathcal{M} \) with \( \delta(a), \delta(b) \in \mathfrak{f} \), they belong to the same equivalence class in \( P/\mathfrak{f} = \varprojlim_{U \in \mathfrak{f}} P(U) \) if and only if there exists \( U \in \mathfrak{f} \) such that \( U \leq \delta(a) \land \delta(b) \) and \( a|_U = b|_U \). On the other hand, by [26, Proposition 4.7(viii)]

---

\(^3\) We cannot consider a colimit \( \varinjlim_{a \in \mathcal{M}; \delta(a) = U} [a]_{\mathfrak{f}} \) as in the case of sheaves since restrictions do not necessarily exist.

\(^4\) Notice that we use the same notations \( \sim_{\mathfrak{f}} \) and \( [-]_{\mathfrak{f}} \) for two different equivalence relations on \( \mathcal{M} \) and \( \mathcal{O}(X) \).
and separatedness,
\[
a \sim b \iff \left[ \delta(a) \lor \delta(b) \rightarrow \bigvee \{ W \leq \delta(a) \land \delta(b) : a|_W = b|_W \} \right] \in \mathcal{F} \\
\iff W_0 := \bigvee \{ W \in O(X) : a|_{\delta(a) \land W} = b|_{\delta(b) \land W} \} \in \mathcal{F}.
\]
Again by separatedness, \( a|_{\delta(a) \land W_0} = b|_{\delta(b) \land W_0} \). Thus, the map \( P/\mathcal{F} \rightarrow \mathcal{M}/\mathcal{F} \) is injective.

Note that the map \( P/\mathcal{F} \rightarrow \mathcal{M}/\mathcal{F} \) is not surjective even if \( X \) is a discrete space.

To give a generalization of the construction of \( P/\mathcal{F} \) to Heyting-valued structures, we need to discuss how and when an ordinary structure can be obtained from some “local sections” of a Heyting-valued structure. The following construction is an analogue of Definition 4.1(1).

From an \( O(X) \)-valued \( L \)-structure \( (\mathcal{M}, \delta) \), we would like to construct an ordinary \( L \)-structure \( \Gamma(U, \mathcal{M}) \) as follows. Set \( \Gamma(U, \mathcal{M}) := \{ a \in \mathcal{M} : \delta(a) = U \} \) for \( U \in O(X) \). We would like to make \( \Gamma(U, \mathcal{M}) \) into an \( L \)-structure so that, for any relation \( R \) and any \( a \in \Gamma(U, \mathcal{M}) \),
\[
\Gamma(U, \mathcal{M}) \models R(a) \iff R^\mathcal{M}(a) = U.
\]
To define an interpretation \( f^\Gamma(U, \mathcal{M}) : \Gamma(U, \mathcal{M})^n \rightarrow \Gamma(U, \mathcal{M}) \) for each function symbol \( f \), we have to demand the following

**Assumption:**

For each function symbol \( f \), the morphism \( f^\mathcal{M} : (\mathcal{M}^n, \delta) \rightarrow (\mathcal{M}, \delta) \) is represented by some map \( h : \mathcal{M}^n \rightarrow \mathcal{M} \) satisfying \( \delta(a, a') \leq \delta(h(a), h(a')) \) and \( \delta(h(a)) = \delta(a) \) for any \( a, a' \).

For \( \mathcal{M} \) satisfying the Assumption, we can suitably define \( f^\Gamma(U, \mathcal{M}) \) to be the restriction of \( h \) to \( \Gamma(U, \mathcal{M}) \) and obtain an \( L \)-structure \( \Gamma(U, \mathcal{M}) \). The satisfaction relation \( \Gamma(U, \mathcal{M}) \models \phi(a) \) is defined as usual. The reader should notice that the relations \( \Gamma(U, \mathcal{M}) \models \varphi(a) \) and \( \| \varphi(a) \| = U \) do not coincide in general. Also, note that any relational Heyting-valued structure trivially satisfies the Assumption.

Given a filter \( \mathcal{F} \) on \( O(X) \), we write \( \Gamma(\mathcal{M}/\mathcal{F}) \) for the set of global elements of the \( (O(X)/\mathcal{F}) \)-valued \( L \)-structure \( \mathcal{M}/\mathcal{F} \). Note that \( [a]_1 \in \Gamma(\mathcal{M}/\mathcal{F}) \) iff \( \delta_1([a]_1) := [\delta(a)]_1 = [1_X]_1 \) iff \( \delta(a) \in \mathcal{F} \).

**Proposition 4.7** If \( \mathcal{M} \) satisfies the Assumption, then so does \( \mathcal{M}/\mathcal{F} \). The \( L \)-structure \( \Gamma(\mathcal{M}/\mathcal{F}) \) is determined by the following interpretations: for \( a_1, \ldots, a_n \in \mathcal{M} \) with \( \delta(a_i) \in \mathcal{F} \),
\[
\Gamma(\mathcal{M}/\mathcal{F}) \models R([a]_1) \iff R^\mathcal{M}/\mathcal{F}([a]_1) := [R^\mathcal{M}(a)]_1 = [1_X]_1
\]
\[
\iff R^\mathcal{M}(a) \in \mathcal{F}.
\]
**Proof.** If \( f^\mathcal{M} \) is represented by a map \( h : \mathcal{M}^n \rightarrow \mathcal{M} \),
\[
f^{\mathcal{M}/\mathcal{F}}([a]_1, [b]_1) := [f^\mathcal{M}(a, b)]_1 = [\delta(h(a), b)]_1 = [\delta(h(a))]_1, [b]_1.
\]
Therefore, \( f^{\mathcal{M}/\mathcal{F}} \) is represented by the map \( (\mathcal{M}/\mathcal{F})^n \ni ([a_1]_1, \ldots, [a_n]_1) \mapsto [h(a_1, \ldots, a_n)]_1 \in \mathcal{M}/\mathcal{F} \).

The resulting structure \( \Gamma(\mathcal{M}/\mathcal{F}) \) will play an essential role in describing our theorems.

We return to the case of \( (\mathcal{M}, \delta) = \Theta(P) \). As we observed in the proof of Proposition 3.11, the morphism \( f^{\Theta(P)} : \Theta(P)^n \rightarrow \Theta(P) \) is represented by the map \( \Theta(P)^n \ni a \mapsto f^\mathcal{M}(a)|_{\delta(a)} \in \Theta(P) \). Thus, \( \Theta(P) \) satisfies the Assumption, and we can consider the \( L \)-structure \( \Gamma(\Theta(P)/\mathcal{F}) \).

**Proposition 4.8** \( \Gamma(\Theta(P)/\mathcal{F}) \) is isomorphic to the \( L \)-structure \( P/\mathcal{F} \) under the bijection in Lemma 4.6.

**Proof.** By the above constructions and Definition 4.1(2),
\[
\Gamma(\Theta(P)/\mathcal{F}) \models R([a]_1) \iff \Theta(P)^\mathcal{F}(a) = \bigvee \{ W \leq \delta(a) : a|_W \in R^\mathcal{F}(W) \} \in \mathcal{F}
\]
\[
\iff \exists W \in \mathcal{F} \cap P(W) \models R(a|_W)
\]
\[
\iff P/\mathcal{F} \models R([a]_1).
\]
Thus, the construction of $\Gamma(\mathcal{M}/f)$ indeed generalizes that of $P/f$. In the remainder of this section, let $\mathcal{M}$ be an $\mathcal{O}(X)$-valued $\mathcal{L}$-structure satisfying the Assumption.

### 4.3 Łoś’s theorem

Łoś-type theorems for sheaves of structures appeared in [23, p. 179, Ultrastalk Theorem] (cf. Theorem 4.17), [8, Theorem 2.6 attributed to Miraglia], and [15, Teorema 5.2]. The first two of them restrict themselves to $\forall$-free formulas. Caicedo’s result is closer to ours, but no proof is given there. We give a generalization of Łoś’s theorem improving all these results, and also give a characterization of Heyting-valued structures for which Łoś’s theorem holds w.r.t. any maximal filter, which generalizes a similar theorem in [55, Theorem 2.8] for Boolean-valued structures consisting of global elements only.

We would like to connect the satisfaction relation $\Gamma(\mathcal{M}/f) \models \psi(\{a\}_f)$ with whether $\|\psi(a)\|_{\mathcal{M}}$ belongs to $\check{f}$. However, interpretations of formulas are inherently classical, while forcing values are of intuitionistic nature. To fill the gap, we exploit the Gödel translation:

**Definition 4.9** For each $\mathcal{L}$-formula $\psi$, the Gödel translation $\psi^G$ is defined inductively:

1. $\bot^G \equiv \bot$, and $\psi^G \equiv \neg \neg \psi$ if $\psi$ is atomic but not $\bot$.
2. $(\varphi \land \psi)^G \equiv \varphi^G \land \psi^G$, $(\varphi \lor \psi)^G \equiv \neg (\neg \varphi^G \land \neg \psi^G)$.
3. $(\varphi \to \psi)^G \equiv \varphi^G \to \psi^G$.
4. $(\forall \psi(v, u))^G \equiv \forall \psi^G(v, u)$, $(\exists \psi(v, u))^G \equiv \exists v \neg \psi^G(v, u)$.

We can reduce classical validity $(\vdash_c \varphi)$ of a sentence to intuitionistic validity $(\vdash_i \varphi^G)$ of its Gödel translation:

$$\vdash_c \varphi \iff \vdash_i \varphi^G.$$

For a proof, we refer the reader to [21, Theorem 6.2.8]. Observe that $\psi^G \iff \neg \neg \psi^G$ is intuitionistically valid since $\varphi \iff \neg \neg \varphi$ is classically valid and $(\varphi \iff \neg \neg \varphi)^G \equiv \psi^G \iff \neg \neg \psi^G$ holds.

**Definition 4.10** A filter $f$ on $\mathcal{O}(X)$ is $\mathcal{M}$-generic when it satisfies the following:

1. For each closed $\mathcal{L}_M$-formula $\varphi(a)$ with $\delta(a) \in f$, either $\|\varphi^G(a)\|_{\mathcal{M}} \in f$ or $\|\neg \varphi^G(a)\|_{\mathcal{M}} \in f$ holds.
2. For any $\mathcal{L}_M$-formula $\varphi(v, a)$ with $\delta(a) \in f$, if $\|\forall \psi^G(v, a)\|_{\mathcal{M}} \in f$, then there exists $b \in \mathcal{M}$ such that $\|\varphi^G(b, a)\|_{\mathcal{M}} \in f$.

**Theorem 4.11** (cf. [15, Teorema 5.2]) If $f$ is $\mathcal{M}$-generic, then, for any $\mathcal{L}$-formula $\varphi(v)$ and $a \in \mathcal{M}^a$ with $\delta(a) \in f$,

$$\Gamma(\mathcal{M}/f) \models \varphi(\{a\}_f) \iff \|\varphi^G(a)\|_{\mathcal{M}} \in f.$$
For the universal quantifier, by using the fact $\forall, \varphi^G \leftrightarrow \neg\neg\varphi^G$,
\[
\left\| \forall v \varphi^G(v, a) \right\| \in \mathcal{f} \iff \left\| \neg\forall v \neg\varphi^G(v, a) \right\| \notin \mathcal{f}
\iff \left\| \exists v \neg\varphi^G(v, a) \right\| \notin \mathcal{f}
\iff \forall b \in \mathcal{M}, \delta(b) \notin \mathcal{f} \implies \left\| \neg\varphi^G(b, a) \right\| \notin \mathcal{f}
\iff \forall b \in \mathcal{M}, \delta(b) \notin \mathcal{f} \implies \left\| \varphi^G(b, a) \right\| \in \mathcal{f}
\iff \forall b \in \mathcal{M}, \delta(b) \notin \mathcal{f} \implies \Gamma(M/f) \models \varphi([b], [a])
\iff \Gamma(M/f) \models \forall v \varphi(v, [a]).
\]

We say a formula is $\forall$-free if it is built up without $\forall$.

**Corollary 4.12** In the above notations, suppose that either of the following conditions holds:

1. $\mathcal{O}(X)$ is a complete Boolean algebra.
2. $\varphi$ is $\forall$-free. (In particular, $\varphi^G$ and $\neg\neg\varphi$ are intuitionistically equivalent.)

Then, for any $\mathcal{M}$-generic filter $\mathcal{f}$ and $a \in \mathcal{M}$ with $\delta(a) \notin \mathcal{f}$,
\[
\Gamma(M/f) \models \varphi([a]) \iff \left\| \neg\neg\varphi(a) \right\|^\mathcal{M} \in \mathcal{f}.
\]
A key to finding $\mathcal{M}$-generic filters is the following proposition. For proofs, the reader is guided to refer [49, Theorem 2.1] and [15, Teorema 3.3].

**Proposition 4.13** (Maximum Principle) If $\mathcal{M}$ is complete as an $\mathcal{O}(X)$-valued set, then, for any $\mathcal{L}_\mathcal{M}$-formula $\varphi(v, a)$, there exists $b \in \mathcal{M}$ such that
\[
\left\| \varphi(b, a) \right\|^\mathcal{M} \leq \left\| \exists v \varphi(v, a) \right\|^\mathcal{M} \leq \left\| \neg\varphi(b, a) \right\|^\mathcal{M} \quad \text{in} \quad \mathcal{O}(X).
\]

We say $\mathcal{M}$ satisfies the maximum principle if the conclusion holds.

In the topological case, the maximum principle means that we can find an open set $\left\| \exists v \varphi(v, a) \right\|$ dense in $\left\| \varphi(b, a) \right\|$.

**Remark 4.14** Volger [60, p. 4] pointed out that the maximum principle for Boolean-valued structures holds under a weaker assumption:

For any $\{a_i\}_{i \in I} \subseteq \mathcal{M}$ and any (strong) anti-chain $\{U_i\}_{i \in I} \subseteq \mathcal{O}(X)$ (i.e., a pairwise disjoint family) satisfying $U_i \leq \delta(a_i)$ for each $i \in I$, there exists $a \in \mathcal{M}$ such that $U_i \leq \delta(a, a_i)$ for each $i \in I$.

For a detailed proof, cf. [55, Proposition 2.11], where the authors call this the mixing property. This does not assume any existence of restrictions of elements, and we would like to remove such an assumption from the previous proposition. However, we cannot apply their argument to Heyting-valued structures because the anti-chain they consider may not cover $\exists v \varphi$ in general. Bell [5] assumes that the frame in consideration is refinable to ensure existence of an anti-chain refining $\exists v \varphi$ and to show that a specific Heyting-valued structure satisfies the maximum principle (he calls it the Existence Principle). We do not know whether the existence of restrictions and refinements can be removed from the previous proposition.

We also remark that all the results mentioned above on the maximum principle involve the use of the axiom of choice or its equivalents.

**Theorem 4.15** (Main Theorem) For any $\mathcal{O}(X)$-valued $\mathcal{L}$-structure $\mathcal{M}$ satisfying the Assumption, the following are equivalent:

(i) $\mathcal{M}$ satisfies the following variant of the maximum principle: for any $\mathcal{L}_\mathcal{M}$-formula $\varphi(v, a)$, there are finitely many $b_1, \ldots, b_r \in \mathcal{M}$ such that
\[
\bigvee_i \left\| \varphi^G(b_i, a) \right\|^\mathcal{M} \leq \left\| \exists v \varphi^G(v, a) \right\|^\mathcal{M} \leq \neg\neg \bigvee_i \left\| \varphi^G(b_i, a) \right\|^\mathcal{M}.
\]
(ii) Every maximal filter on \( O(X) \) is \( M \)-generic.

(iii) For any maximal filter \( m \) on \( O(X) \) and any closed \( L_M \)-formula \( \varphi(a) \) with \( \delta(a) \in m \),

\[
\Gamma(M/m) \models \varphi([a]_m) \iff \|\varphi_G(a)\|^M \in m.
\]

**Proof.** (i) \( \implies \) (ii): Let \( m \) be a maximal filter on \( O(X) \). For any \( U \in O(X) \), either \( U \in m \) or \( \neg U \in m \) holds. Moreover, if \( U \cup V \in m \), then \( U \in m \) or \( V \in m \). Thus, the maximum principle implies \( M \)-genericity of \( m \).

(ii) \( \implies \) (iii): By Theorem 4.11.

(iii) \( \implies \) (i): The following argument is a modification of the proof of [55, Theorem 2.8]. To simplify notations, we may assume \( \delta(a) = 1_X \) and suppress the parameter \( a \). For an arbitrary \( a \), we may use the frame \( O(\delta(a)) = (\delta(a)) \downarrow \) instead of \( O(X) \) in the following.

For any \( L_M \)-formula \( \varphi(v) \) with \( \|\exists v \varphi_G(v)\| \neq 0_X \), we can take a maximal filter \( m \supseteq \|\exists v \varphi_G(v)\| \). Since \( \exists v \varphi_G \to \neg \forall v \neg \varphi_G \) is intuitionistically valid, we have \( \|\exists v (\varphi_G(v))\|^M \in m \). By the assumption, \( \Gamma(M/m) \models \exists v \varphi(v) \). Then there exists \( b \in M \) such that \( \delta(b) \in m \) and \( \Gamma(M/m) \models \varphi([b]_m) \). Again by the assumption, there exists \( b \in M \) such that \( \|\varphi_G(b)\|^M \in m \).

We have just shown that any maximal filter containing \( \|\exists v (\varphi_G(v))\| \) also contains some \( \|\varphi_G(b)\| \). Notice that \( \|\varphi_G(b)\| \) is a regular element of \( O(\delta(b)) \) because \( \varphi_G \leftrightarrow \neg \neg \varphi_G \) is intuitionistically valid. We write \( \text{Reg}(O(X)) \) for the complete Boolean algebra of regular elements of \( O(X) \). Now we consider the spectrum \( \text{Spec}(\text{Reg}(O(X))) \) of \( \text{Reg}(O(X)) \), i.e., the Stone space of ultrafilters on \( \text{Reg}(O(X)) \) whose basic (closed) open sets are of the form

\[
D(U) := \{ u \in \text{Spec}(\text{Reg}(O(X))) \mid u \ni U \} \quad \text{for } U \in \text{Reg}(O(X)).
\]

Since maximal filters on \( O(X) \) correspond to ultrafilters on \( \text{Reg}(O(X)) \) (cf. [32, Exercise II.4.9], [57, Theorem 1.44]), the above observation yields

\[
D(\|\exists v \varphi_G(v)\|) \subseteq \bigcup_{b \in M} D(\neg \neg \|\varphi_G(b)\|).
\]

By compactness of \( D(\|\exists v \varphi_G(v)\|) \), we can find \( b_1, \ldots, b_r \) such that

\[
D(\|\exists v \varphi_G(v)\|) \subseteq \bigcup_i D(\neg \neg \|\varphi_G(b_i)\|) = D(\neg \neg \bigvee_i \|\varphi_G(b_i)\|).
\]

Hence, we have \( \|\exists v \varphi_G(v)\| \leq \|\exists v \varphi(v)^G\| \leq \neg \neg \bigvee_i \|\varphi_G(b_i)\| \).

Combining the results in this section, we obtain

**Corollary 4.16** (Classical Šoš’s theorem) Let \( X \) be a set, \( \{M_x\}_{x \in X} \) an \( X \)-indexed family of non-empty \( L \)-structures, and \( u \) an ultrafilter over \( X \). Then, for any \( L \)-formula \( \varphi(u_1, \ldots, u_n) \) and \( a^1, \ldots, a^n \in \biguplus_x M_x \),

\[
\prod_x M_x/u \models \varphi([a_1]_u, \ldots, [a^n]_u) \iff \{ x \in X \mid M_x \models \varphi(a^1, \ldots, a^n) \} \in u.
\]

**Proof.** Let \( P \) be the sheaf of \( L \)-structures corresponding to the \( \text{étalé} \) bundle of \( L \)-structures \( \biguplus_{x \in X} M_x \to X \) as in Example 4.2. The statement follows from the facts

\[
\prod_x M_x/u \simeq P/u \simeq \Gamma(\Theta(P)/u) \quad \text{and} \quad \|\varphi(a^1, \ldots, a^n)\|^{\Theta(P)} = \{ x \in X \mid M_x \models \varphi(a^1, \ldots, a^n) \}.
\]

We remark that Pierobon & Viale [55, Examples 2.9 and 2.13] give set-theoretic examples of Boolean-valued structures (not being sheaves).

1. satisfying the maximum principle but not having the mixing property (Remark 4.14), and
2. violating Šoš’s theorem (and the maximum principle).

The former example is given by the Boolean extension \( M^B \) of a countable transitive model \( M \) of ZFC by a Boolean algebra \( B \in M \) for which \( M \models \text{“} B \text{ is complete”} \). The latter is given by the set \( C^\infty(\mathbb{R}) \) of analytic functions \( \mathbb{R} \to \mathbb{R} \) with value \( \delta(f, g) \) in the measure algebra.

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5 While \( \|\varphi_G(b)\| \) is regular in \( O(\delta(b)) \), it is not necessarily regular in \( O(X) \). This is why we use \( \neg \neg \|\varphi_G(b)\| \) here.
Various Łoś-type theorems for specific sheaves of structures have been considered in the literature. Some of them are special cases of our theorem, but others are not. A notable one is Ellerman’s Ultrastalk Theorem. For simplicity, we treat ∀-free formulas only. Let $X$ be a topological space and $\text{Spec}(X)$ be the space of prime filters on the frame $\mathcal{O}(X)$. $\text{Spec}(X)$ has the basic open set $D_U = \{p : U \in p\}$ for each $U \in \mathcal{O}(X)$. We have a continuous map $\eta : X \to \text{Spec}(X)$ sending $x$ to $\mathfrak{n}_x$. For any sheaf $P$ of $\mathcal{L}$-structures, the direct image sheaf $\eta_* P$ on $\text{Spec}(X)$ is again a sheaf of $\mathcal{L}$-structures. Ellerman [23, p. 179] showed the following (cf. [51, 57]):

**Theorem 4.17 (Ultrastalk Theorem)** For any maximal filter $m \ni U$ and any closed $\mathcal{L}_{\Theta(P)}$-formula $\varphi(a)$ with $a \in P(U)^\gamma = (\eta_* P)(D_U)^\gamma$,

$$\eta_* P_m \models [\varphi[a]_m] \iff \|\varphi(a)\| \in m.$$ 

Our theorem subsumes the Ultrastalk Theorem since

$$\eta_* P_m = \lim_{D \ni m} (\eta_* P)(D) \simeq \lim_{D \ni m} (\eta_* P)(D_U) = \lim_{U \in m} P(U) = P/m.$$ 

Especially, Łoś theorem for unbounded Boolean ultrapowers [47, Theorem 1.5] is under our scope (cf. [45]). However, Ellerman’s approach suggests a significant viewpoint missing in ours: various model-theoretic constructions are realized by taking stalks of sheaves on the spectrum of a distributive lattice. For example, as we saw in Example 4.5, a bounded Boolean ultrapower is a stalk over an ultrafilter on a (possibly non-complete) Boolean algebra. There are Łoś-type theorems for such structures, e.g., [14, Lemma 7.1] for a family of Boolean products. The relationship between these theorems and our approach should be explored elsewhere (see the comments in the next section).

## 5 Related topics and future directions

Finally, we give an overview of various sheaf-theoretic methods in model theory with an expanded list of previous works, and indicate future directions from a topos-theoretic perspective.

**Forcing and generic models** We again assume all formulas are ∀-free. Let $P$ be a sheaf of $\mathcal{L}$-structures on a topological space $X$. As we noticed in § 3.3, forcing values give the sheaf semantics in $\text{Sh}(X)$. We can consider another forcing relation, for $x \in X$,

$$x \models_P \varphi(a) \overset{\text{def.}}{\iff} x \in \|\varphi(a)\|.$$ 

Caicedo [15] called “$U \models$” the local semantics and “$x \models$” the punctual semantics.

On the other hand, each stalk $P_\mathfrak{n}$ is an $\mathcal{L}$-structure, and we can also consider the relation $P_\mathfrak{n} \models \varphi(a_\mathfrak{n})$ for each closed $\mathcal{L}_M$-formula $\varphi(a)$ with $x \in \delta(\mathfrak{n})$. Define the discrete value of a formula:

$$|\varphi(a)| := \{x \in \delta(\mathfrak{n}) : P_\mathfrak{n} \models \varphi(a_\mathfrak{n})\}.$$ 

For any atomic relation $R$, by definition,

$$P_\mathfrak{n} \models R(a_\mathfrak{n}) \iff \exists V \ni x, P(V) \models R(a|_V),$$

i.e., $|R(a)| = \|R(a)\|$. However, in general, $|\varphi(a)| \neq \|\varphi(a)\|$. Some authors considered the relationship between them (cf. [48, §1] and [40, Theorem 4.3, Lemma 5.1]).

Kaiser [34] addressed the problem when the relations $P_\mathfrak{n} \models \varphi(a_\mathfrak{n})$ and $x \models_P \varphi(a)$ coincide for any formula. He called such $P_\mathfrak{n}$ a generic stalk. If the filter $\mathfrak{n}_x$ is $\Theta(P)$-generic, then $P_\mathfrak{n}$ is a generic stalk by our Łoś-type theorem. Kaiser used generic stalks to obtain omitting types and consistency results similar to those in [35] (cf. [15, §6], [9]).

From a topos-theoretic perspective, Blass & Scedrov [6] constructed the classifying toposes of existentially closed models and finite-generic models. Their work was apparently inspired by Keisler’s viewpoint [35] and might be related to ours.
Stalks, global sections, and induced geometric morphisms In addition to stalks of sheaves, the structure Γ(X, P) of global sections is of our future interest (see below). The Feferman–Vaught theorem works for global sections just like Łoś’s theorem does for stalks. Comer [19] gave a sheaf-theoretic interpretation of the original Feferman–Vaught theorem [24]. Feferman–Vaught type theorems and their applications to sentences preserved under taking global sections were pursued in [14, 38, 48, 58, 60] (cf. [61]).

From a topos-theoretic viewpoint, taking stalks and global sections can be seen as part of geometric morphisms. Any morphism f : X → Y of locales (Definition 2.3) or of topological spaces induces a geometric morphism (f∗, f∗) : Sh(X) → Sh(Y). Then,

1. The stalk Px is f∗P for the geometric morphism Set → Sh(X) induced by the point f = x : 1 → X.
2. The set P(X) is f∗P for the (essentially unique) geometric morphism Sh(X) → Set induced by f : X → 1.

Furthermore, we can construct a geometric morphism Set(𝒪(X)) → Set(𝒪(Y)), and it is canonically identified with (f∗, f∗) via the equivalence in Corollary 2.21. Therefore, we may investigate stalks and global sections in the more general framework of base change of Heyting-valued structures. This categorical approach has an advantage over the set-theoretic approach of [2] to base change of Heyting-valued universes since the construction of geometric morphisms is much simpler and the logical behavior under base change along them is well-understood for various classes of morphisms of locales [33, Chapter C3].

Sheaf representation and model theory for sheaves Algebraic structures often have representations as global sections of sheaves of structures. Knoebel’s monograph [36] includes a brief description of a history of sheaf representations of algebras (cf. [32, Chapter V]). Sheaf representations over Stone spaces, e.g., Pierce representation of commutative rings [54], play a special role in model theory. Following Lipshitz & Saracino [39], Macintyre [43] established a general method for obtaining model-companions of theories whose models have sheaf representations over Stone spaces with good stalks (cf. [18]). He exploited Comer’s version of the Feferman–Vaught theorem to transfer model-theoretic properties of stalks to global sections. This line of research was followed by [14, 20, 22, 64] (cf. [44, §6]). Later, Bunge & Reyes [12] gave a topos-theoretic unification (cf. [10]).

In this line of research, sheaves having good stalks are often sheaf models of well-behaved theories. For example, any (commutative) von Neumann regular ring R is represented by a sheaf of rings over a Stone space X(R) whose stalks are fields, and such a sheaf is a model of the theory of fields in the topos Sh(X(R)). The theory of von Neumann regular rings has the model-completion, whose models are represented by “algebraically closed fields” in sheaf toposes over Stone spaces. Thus, we may expect that developing model theory for sheaves will deepen our understanding of ordinary model theory. Model theory for sheaves has been studied intermittently by some authors. The pioneering work is [40], where Louliss had already pointed out the importance of the viewpoint we just mentioned. Our standpoint emphasizing Heyting-valued structures was greatly influenced by him too. Some other authors considered model-theoretic phenomena for models in various toposes (cf. [1, 3, 27, 49, 66]).

In fact, model theory for sheaves is part of what should be called topos-internal model theory or model theory in toposes. Topos-internal model theory concerns theories internal to toposes, and internal theories in a topos admit sheaves of function symbols and relation symbols (cf. [28]). It must be closer to doing model theory in a Heyting-valued universe (cf. [37]). The approach by Brunner & Miraglia [8], admitting a presheaf of constant symbols in place of a set of constants, is regarded as a restricted form of topos-internal model theory. In contrast to the scarcity of research on topos-internal model theory, there is much more on universal algebra in toposes and sheaf models for constructive mathematics.

Finally, we would like to mention a potential application of topos-internal model theory to algebraic geometry. At the end of [40], Louliss suggests that algebraic geometry over von Neumann regular rings [56] could be obtained by doing algebraic geometry in some topos. The works of Bunge [11] and her student MacCaull [42] reflect that idea, but no one followed them. We leave that direction as the ultimate goal of our research.

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