Asymptotics of Hankel determinants with a Laguerre-type or Jacobi-type potential and Fisher-Hartwig singularities

Christophe Charlier∗, Roozbeh Gharakhloo†

February 22, 2019

Abstract

We obtain large $n$ asymptotics of $n \times n$ Hankel determinants whose weight has a one-cut regular potential and Fisher-Hartwig singularities. We restrict our attention to the case where the associated equilibrium measure possesses either one soft edge and one hard edge (Laguerre-type) or two hard edges (Jacobi-type).

1 Introduction

Hankel determinants with Fisher-Hartwig (FH) singularities appear naturally in random matrix theory. Among others, they can express correlations of the characteristic polynomial of a random matrix, or gap probabilities in the point process of the thinned spectrum, see e.g. the introductions of [15, 8, 3] for more details. In these applications, the size $n$ of an $n \times n$ Hankel determinant is equal to the size of the underlying $n \times n$ random matrices. Large $n$ asymptotics for such determinants have already been widely studied, see e.g. [15, 14, 9, 1, 3]. Recent developments in the theory of Gaussian multiplicative chaos [1] provide a renewed interest in these asymptotics. For example, such asymptotics provide crucial estimates in the study of rigidity of eigenvalues of a random matrix [4].

In the present work, we restrict our attention on large $n$ asymptotics of Hankel determinants

$$
\det \left( \int_{\mathcal{I}} x^{j+k} w(x) dx \right)_{j,k=0,\ldots,n-1},
$$

(1.1)

whose weight $w$ is supported on an interval $\mathcal{I} \subset \mathbb{R}$, and is of the form

$$
w(x) = e^{-nV(x)} e^{W(x)} \omega(x).
$$

(1.2)

The function $W$ is continuous on $\mathcal{I}$ and $\omega$ contains the FH singularities (they will be described in more details below). The potential $V$ is real analytic on $\mathcal{I}$ and, in case $\mathcal{I}$ is unbounded, satisfies $\lim_{x \to \pm \infty, x \in \mathcal{I}} V(x) / \log |x| = +\infty$. Furthermore, we assume that $V$ is one-cut and regular. These properties are described in terms of the equilibrium measure $\mu_V$, which is the unique minimizer of the functional

$$
\iint \log |x-y|^{-1} \, d\mu(x) \, d\mu(y) + \int V(x) \, d\mu(x)
$$

(1.3)

∗Department of Mathematics, KTH Royal Institute of Technology, Lindstedtsvägen 25, SE-114 28 Stockholm, Sweden. e-mail: cchar@kth.se

†Department of Mathematical Sciences, Indiana University-Purdue University Indianapolis, 402 N. Blackford St., Indianapolis, IN 46202, USA. E-mail: rgharakh@iupui.edu
The function \( \omega \) where \( f \)ication \[10\]. Since these singularities are named after Fisher and Hartwig, due to their pioneering work in their identification \[10\]. It is known (see e.g. \[19\]) that \( \mu_V \) is completely characterized by the Euler-Lagrange variational conditions
\[
2 \int_{-1}^1 \log |x-s| d\mu_V(s) = V(x) - \ell, \quad \text{for } x \in [-1,1], \tag{1.4}
\]
\[
2 \int_{-1}^1 \log |x-s| d\mu_V(s) \leq V(x) - \ell, \quad \text{for } x \in I \setminus [-1,1], \tag{1.5}
\]
where \( \ell \in \mathbb{R} \) is a constant. Regular means that the Euler-Lagrange inequality \[1.5\] is strict on \( I \setminus [-1,1] \), and that the density of the equilibrium measure is positive on \((-1,1)\). The three canonical cases are the following:
1. \( I = \mathbb{R} \) and \( d\mu_V(x) = \psi(x) \sqrt{1-x^2} dx \),
2. \( I = [-1,\infty) \) and \( d\mu_V(x) = \psi(x) \sqrt{1-x^2} dx \),
3. \( I = [-1,1] \) and \( d\mu_V(x) = \psi(x) \sqrt{1-x^2} dx \),
where \( \psi \) is real analytic on \( I \), such that \( \psi(x) > 0 \) for all \( x \in [-1,1] \). We will refer to these three cases as Gaussian-type, Laguerre-type and Jacobi-type weights, respectively. Well-known examples for potentials of such weights are

1. \( V(x) = 2x^2 \) for Gaussian-type weight, with \( \ell = 1 + 2 \log 2 \) and \( \psi(x) = \frac{2}{\pi} \),
2. \( V(x) = 2(x+1) \) for Laguerre-type weight, with \( \ell = 2 + 2 \log 2 \) and \( \psi(x) = \frac{1}{x} \),
3. \( V(x) = 0 \) for Jacobi-type weight, with \( \ell = \log 2 \) and \( \psi(x) = \frac{1}{\pi} \).

In the language of random matrix theory, the interval \((-1,1)\) is called the bulk, and \( \pm 1 \) are the edges. An edge is said to be “soft” if there can be eigenvalues beyond it, and “hard” if this is impossible. On the level of the equilibrium measure, a soft edge translates into a square root vanishing of \( d\mu_V \), while a hard edge means that \( \frac{d\mu_V}{dx} \) blows up like an inverse square root. Thus, there are two soft edges at \( \pm 1 \) for Gaussian-type weights, one hard edge at \(-1\) and one soft edge at \( 1 \) for Laguerre-type weights, and two hard edges for Jacobi-type weights.

The function \( \omega \) that appears in \[12\] is defined by
\[
\omega(x) = \prod_{j=1}^m \omega_{\alpha_j}(x) \omega_{\beta_j}(x) \times \begin{cases} 1, & \text{for Gaussian-type weights}, \\ (x+1)^{\alpha_0}, & \text{for Laguerre-type weights}, \\ (x+1)^{\alpha_0}(1-x)^{\alpha_{m+1}}, & \text{for Jacobi-type weights}, \end{cases} \tag{1.6}
\]
where
\[
\omega_{\alpha_k}(x) = |x-t_k|^{\alpha_k}, \quad \omega_{\beta_k}(x) = \begin{cases} e^{-i\pi \beta_k}, & \text{if } x < t_k, \\ e^{i\pi \beta_k}, & \text{if } x > t_k, \end{cases} \tag{1.7}
\]
with
\[-1 < t_1 < \ldots < t_m < 1. \tag{1.8}\]

The functions \( \omega_{\alpha_k} \) and \( \omega_{\beta_k} \) represent the root-type and jump-type singularities at \( t_k \), respectively. These singularities are named after Fisher and Hartwig, due to their pioneering work in their identification \[10\]. Since \( \omega_{\beta_k+1} = -\omega_{\beta_k} \), we can assume without loss of generality that \( \Re \beta_k \in (-\frac{1}{2}, \frac{1}{2}] \) for all \( k \). Finally, to ensure integrability of the weight (at least for sufficiently large \( n \)), we require that \( \Re \alpha_k > -1 \) for all \( k \) and, in case \( I \) is unbounded, that \( W(x) = \mathcal{O}(V(x)) \) as \( x \to \pm \infty, x \in I \).
To summarise, the $n \times n$ Hankel determinant given by (1.1) depends on $n$, $m$, $V$, $W$, $\vec{t} = (t_1, \ldots, t_m)$, $\vec{\beta} = (\beta_1, \ldots, \beta_m)$ and $\vec{\alpha}$, where

$$\vec{\alpha} = \begin{cases} 
(\alpha_1, \ldots, \alpha_m) & \text{for Gaussian-type weight,} \\
(\alpha_0, \alpha_1, \ldots, \alpha_m) & \text{for Laguerre-type weight,} \\
(\alpha_0, \alpha_1, \ldots, \alpha_m, \alpha_{m+1}) & \text{for Jacobi-type weight.} 
\end{cases}$$

This determinant will be denoted by $G_n(\vec{\alpha}, \vec{\beta}, V, W)$, $L_n(\vec{\alpha}, \vec{\beta}, V, W)$ or $J_n(\vec{\alpha}, \vec{\beta}, V, W)$, depending on whether the weight is of Gaussian, Laguerre or Jacobi-type, respectively.

Many authors have contributed over the years to large $n$ asymptotics for $G_n(\vec{\alpha}, \vec{\beta}, V, W)$ in certain particular cases of the parameters $\vec{\alpha}$, $\vec{\beta}$, $V$ and $W$ (see the introduction of [3] for a global review). The most general result can be found in [3], see also Theorem 1.1 below for the precise statement. It is worth to note that these asymptotics are only valid for $\Re \beta_k \in (-\frac{1}{4}, \frac{1}{4})$ and not in the whole strip $\Re \beta_k \in (-\frac{1}{2}, \frac{1}{2})$. This is due to purely technical reasons, and we comment more on that in Remark 1.4 below.

Much less is known about large $n$ asymptotics for $L_n(\vec{\alpha}, \vec{\beta}, V, W)$ and $J_n(\vec{\alpha}, \vec{\beta}, V, W)$, and we briefly discuss this below.

The quantities $L_n(\vec{0}, \vec{0}, V, 0)$ and $J_n(\vec{0}, \vec{0}, V, 0)$ (i.e. no singularities and $W = 0$) represent partition functions of certain random matrix ensembles. In some very special cases of $V$ (like $V(x) = 2(x+1)$ for Laguerre-type weights and $V(x) = 0$ for Jacobi-type weights), these Hankel determinants reduce to Selberg integrals and are thus computable explicitly. Large $n$ asymptotics for $L_n(\vec{0}, \vec{0}, V, 0)$ and $J_n(\vec{0}, \vec{0}, V, 0)$ for a general $V$ were obtained in [2] (in fact the results of [2] are valid for more general ensembles than we consider). However, we believe our expansions, which are given by Theorem 1.2 and Theorem 1.3 below with $\vec{\alpha} = \vec{0}$, $\vec{\beta} = \vec{0}$ and $W = 0$, are more explicit (even though less general).

No results are available in the literature for Laguerre-type weight with FH singularities in the bulk (even in the case $V(x) = 2(x+1)$). There is more known about Jacobi-type weights. Asymptotics for $J_n((\alpha_0, 0, \ldots, 0, \alpha_{m+1}), \vec{0}, 0, W)$ (i.e. root-type singularities only at the edges) were computed in [10], however without the constant term. Major progress were achieved in [8, 9], in which the authors derived large $n$ asymptotics for $J_n(\vec{\alpha}, \vec{\beta}, 0, W)$ including the constant term (under very weak assumption on $W$, and for general value of $\vec{\beta}$ such that $\Re \beta_k \in (-\frac{1}{2}, \frac{1}{2})$).

The goal of the present paper is to fill a gap in the literature on large $n$ asymptotics of Hankel determinants with a one-cut potential and FH singularities. In Theorem 1.2 and Theorem 1.3 below, we find large $n$ asymptotics for $L_n(\vec{\alpha}, \vec{\beta}, V, W)$ and $J_n(\vec{\alpha}, \vec{\beta}, V, W)$ including to the constant term. First, we rewrite (in a slightly different way) the result of [3] in Theorem 1.1 for the reader’s convenience, in order to ease the comparison between the three canonical types of weights.

3
Theorem 1.1 (from [3] for Gaussian-type weight)

Let \( n \in \mathbb{N} \), and let \( t_j, \alpha_j \) and \( \beta_j \) be such that

\[-1 < t_1 < \ldots < t_m < 1, \quad \text{and} \quad \Re \alpha_j > -1, \quad \Re \beta_j \in (-\frac{1}{4}, \frac{1}{4}) \quad \text{for} \quad j = 1, \ldots, m.\]

Let \( V \) be a one-cut regular potential whose equilibrium measure is supported on \([-1, 1]\) with density \( \psi(x)\sqrt{1-x^2} \), and let \( W : \mathbb{R} \to \mathbb{R} \) be analytic in a neighbourhood of \([-1, 1]\), locally Hölder-continuous on \( \mathbb{R} \) and such that \( W(x) = O(V(x)) \), as \( |x| \to \infty \). As \( n \to \infty \), we have

\[ G_n(\alpha, \beta, V, W) = \exp \left( C_1 n^2 + C_2 n + C_3 \log n + C_4 + O\left( \frac{\log n}{n^{1-4\delta_{\text{max}}}} \right) \right), \quad (1.9) \]

with \( \delta_{\text{max}} = \max\{|\Re \beta_1|, \ldots, |\Re \beta_m|\} \) and

\[ C_1 = -\log 2 - \frac{3}{4} - \left( \frac{1}{2} \int_{-t_1}^{t_1} (V(x) - 2x^2) \left( \frac{2}{\pi} + \psi(x) \right) \sqrt{1-x^2} dx \right) \quad (1.10) \]

\[ C_2 = \log(2\pi) - A \log 2 - \frac{A}{2\pi} \int_{-t_1}^{t_1} \frac{V(x) - 2x^2}{\sqrt{1-x^2}} dx + \int_{-t_1}^{t_1} W(x) \psi(x) \sqrt{1-x^2} dx \]

\[ + \frac{1}{2} \sum_{j=1}^{m} \left( (V(t_j) - 1) + \sum_{k=1}^{m} \pi i \beta_j \left( 1 - 2 \int_{t_j}^{1} \psi(x) \sqrt{1-x^2} dx \right) \right), \quad (1.11) \]

\[ C_3 = -\frac{1}{12} + \frac{1}{24} \sum_{j=1}^{m} \left( \frac{\alpha_j^2}{4} - \beta_j^2 \right), \quad (1.12) \]

\[ C_4 = \zeta'(-1) - \frac{1}{24} \log\left( \frac{\pi}{2} \psi(-1) \right) - \frac{1}{24} \log\left( \frac{\pi}{2} \psi(1) \right) + \sum_{j=1}^{m} \left( \frac{\alpha_j^2}{4} - \beta_j^2 \right) \log\left( \frac{\pi}{2} \psi(t_j) \right) \]

\[ + \sum_{1 \leq j < k \leq m} \left[ \log\left( \frac{(1-t_j t_k - \sqrt{(1-t_j^2)(1-t_k^2)})^{2\beta_j \beta_k}}{2^{|t_j+t_k|} |t_j-t_k|^{|t_j|+|t_k|+2\beta_j \beta_k}} \right) + \frac{i\pi}{2} (\alpha_k \beta_j - \alpha_j \beta_k) \right] \]

\[ + \sum_{j=1}^{m} \left( \frac{\alpha_j^2}{4} \log(2\sqrt{1-t_j^2}) - \beta_j^2 \log\left( 8(1-t_j^2)^{3/2} \right) \right) + A \sum_{j=1}^{m} i \beta_j \arcsin t_j \]

\[ + \sum_{j=1}^{m} \log\left( \frac{G(1+2\beta_j \beta_j + \beta_j)G(1+2\beta_j - \beta_j)}{G(1+\alpha_j)} \right) \quad (1.13) \]

\[ + \frac{A}{2\pi} \int_{-t_1}^{t_1} \frac{W(x)}{\sqrt{1-x^2}} dx - \sum_{j=1}^{m} \frac{\alpha_j}{2} W(t_j) + \sum_{j=1}^{m} i \beta_j \pi \sqrt{1-t_j^2} \int_{-1}^{t_1} \frac{W(x)}{\sqrt{1-x^2}} dx \]

\[ - \frac{1}{4\pi^2} \int_{-1}^{t_1} W(y) \left( \int_{-1}^{1} \frac{W'(x) \sqrt{1-x^2}}{x-y} dx \right) dy, \quad (1.14) \]

where \( G \) is Barnes’ \( G \)-function, \( \zeta \) is Riemann’s zeta-function, where we use the notations \( J \) for the Cauchy principal value integral, and

\[ A = \sum_{j=1}^{m} \alpha_j. \quad (1.15) \]

Furthermore, the error term in (1.23) is uniform for all \( \alpha_k \) in compact subsets of \( \{ z \in \mathbb{C} : \Re z > -1 \} \), for all \( \beta_k \) in compact subsets of \( \{ z \in \mathbb{C} : \Re z \in \left( -\frac{1}{4}, \frac{1}{4} \right) \} \), and uniform in \( t_1, \ldots, t_m \), as long as there exists \( \delta > 0 \) independent of \( n \) such that

\[ \min_{j \neq k} \{ |t_j - t_k|, |t_j - 1|, |t_j + 1| \} \geq \delta. \quad (1.15) \]
Theorem 1.2 (for Laguerre-type weight)
Let \( m \in \mathbb{N} \), and let \( t_j, \alpha_j, \beta_j \) be such that

\[-1 = t_0 < t_1 < \ldots < t_m < 1, \quad \Re \alpha_j > -1, \quad \Re \beta_j \in (-\frac{1}{2}, \frac{1}{4}) \quad \text{for } j = 0, \ldots, m,\]

with \( \beta_0 = 0 \). Let \( V \) be a one-cut regular potential whose equilibrium measure is supported on \([-1, 1]\) with density \( \psi(x) \sqrt{1 + x^2} \), and let \( W : \mathbb{R}^+ \to \mathbb{R} \) be analytic in a neighbourhood of \([-1, 1]\), locally Hölder-continuous on \( \mathbb{R}^+ \) and such that \( W(x) = O(V(x)) \), as \( x \to +\infty \). As \( n \to \infty \), we have

\[
L_n(\bar{\alpha}, \bar{\beta}, V, W) = \exp \left( C_1 n^2 + C_2 n + C_3 \log n + C_4 + O\left( \frac{\log n}{n^{\frac{1-4\beta_{\max}}{2}}} \right) \right), \tag{1.16}
\]

with \( \beta_{\max} = \max\{|\Re \beta_1|, \ldots, |\Re \beta_m| \} \) and

\[
C_1 = -\log 2 - \frac{3}{2} - \frac{1}{2} \int_{-1}^{1} (V(x) - 2(x + 1)) \left( \frac{1}{\pi} + \psi(x) \right) \sqrt{1 - x} \frac{1}{1 + x} dx, \tag{1.17}
\]

\[
C_2 = \log(2\pi) - A \log 2 - A \log \left( \frac{1}{2\pi} \int_{-1}^{1} V(x) - 2(x + 1) dx + \int_{-1}^{1} W(x) \psi(x) \sqrt{1 - x} \frac{1}{1 + x} dx \right) \tag{1.18}
+ \sum_{j=1}^{m} \frac{\alpha_j^2}{2} (V(t_j) - 2) + \sum_{j=1}^{m} \frac{\pi i \beta_j}{2} \left( 1 - 2 \int_{t_j}^{1} \psi(x) \sqrt{1 - x} \frac{1}{1 + x} dx \right),
\]

\[
C_3 = -\frac{1}{6} + \frac{\alpha_0^2}{2} + \frac{1}{2} \sum_{j=1}^{m} \left( \frac{\alpha_j^2}{4} - \beta_j^2 \right), \tag{1.19}
\]

\[
C_4 = 2\zeta(-1) - \frac{1 - 4\pi^2}{8} \log(\pi \psi(-1)) - \frac{1}{24} \log(\pi \psi(1)) + \sum_{j=1}^{m} \left( \frac{\alpha_j^2}{4} - \beta_j^2 \right) \log(\pi \psi(t_j))
+ \alpha_0 \log(2\pi) + \sum_{0 \leq j < k \leq m} \left[ \log \left( \frac{1 - t_j t_k - \sqrt{(1 - t_j^2)(1 - t_k^2)}}{2\pi} \right) \right] + \frac{i \pi}{2} (\alpha_k \beta_j - \alpha_j \beta_k)
+ \sum_{j=1}^{m} \left( \frac{\alpha_j^2}{4} \log \left( \frac{1 - t_j}{1 + t_j} - \beta_j^2 \log \left( 4(1 - t_j)^{3/2}(1 + t_j)^{1/2} \right) \right) \right) + A \sum_{j=1}^{m} \beta_j \text{arcsin} t_j
- \log G(1 + \alpha_0) + \sum_{j=1}^{m} \log \frac{G(1 + \frac{\alpha_j}{2} + \beta_j)G(1 + \frac{\alpha_j}{2} - \beta_j)}{G(1 + \alpha_j)} \tag{1.20}
+ \frac{A}{2\pi} \int_{-1}^{1} W(x) \sqrt{1 - x^2} dx - \sum_{j=1}^{m} \frac{\alpha_j^2}{2} W(t_j) + \sum_{j=1}^{m} \frac{\pi i \beta_j}{2} \int_{-1}^{1} \frac{W(x)}{\sqrt{1 - x^2}} dx
- \frac{1}{4\pi^2} \int_{-1}^{1} \int_{-1}^{1} W(x) \sqrt{1 - x^2} \sqrt{1 - y^2} \sqrt{1 - x^2} \sqrt{1 - y^2} dx dy,
\]

where \( G \) is Barnes’ \( G \)-function, \( \zeta \) is Riemann’s zeta-function, where we use the notations \( \int \) for the Cauchy principal value integral, and

\[
A = \sum_{j=0}^{m} \alpha_j. \tag{1.21}
\]

Furthermore, the error term in (1.23) is uniform for all \( \alpha_k \) in compact subsets of \( \{z \in \mathbb{C} : \Re z > -1\} \), for all \( \beta_k \) in compact subsets of \( \{z \in \mathbb{C} : \Re z \in (\frac{1}{2}, \frac{1}{4})\} \), and uniform in \( t_1, \ldots, t_m \), as long as there exists \( \delta > 0 \) independent of \( n \) such that

\[
\min_{j \neq k} \{|t_j - t_k|, |t_j - 1|, |t_j + 1| \} \geq \delta. \tag{1.22}
\]
Theorem 1.3 (for Jacobi-type weight)
Let \( m \in \mathbb{N} \), and let \( t_j, \alpha_j \) and \( \beta_j \) be such that
\[-1 = t_0 < t_1 < \ldots < t_m < t_{m+1} = 1, \] and \( \Re \alpha_j > -1, \quad \Re \beta_j \in (-\frac{1}{2}, \frac{1}{4}) \) for \( j = 0, \ldots, m+1 \), with \( \beta_0 = 0 = \beta_{m+1} \). Let \( V \) be a one-cut regular potential whose equilibrium measure is supported on \([-1, 1]\) with density \( \frac{\psi(x)}{\sqrt{1-x^2}} \), and let \( W : [-1, 1] \to \mathbb{R} \) be analytic in a neighbourhood of \([-1, 1]\).

As \( n \to \infty \), we have
\[
J_n(\vec{a}, \vec{\beta}, V, W) = \exp \left( C_1 n^2 + C_2 n + C_3 \log n + C_4 + O\left( \frac{\log n}{n^{1-\beta_{\max}}} \right) \right), \tag{1.23}
\]
with \( \beta_{\max} = \max\{|\Re \beta_1|, \ldots, |\Re \beta_m|\} \) and
\[
C_1 = -\log 2 - \frac{1}{2} \int_{-1}^{1} V(x) \left( \frac{1}{\pi} + \psi(x) \right) \frac{dx}{\sqrt{1-x^2}}, \tag{1.24}
\]
\[
C_2 = \log(2\pi) - A \log 2 - \frac{A}{2\pi} \int_{-1}^{1} \frac{V(x)}{\sqrt{1-x^2}} dx + \int_{-1}^{1} W(x) \frac{\psi(x)}{\sqrt{1-x^2}} dx \tag{1.25}
+ \sum_{j=0}^{m+1} \frac{\alpha_j}{2} \int_{-1}^{1} V(t_j) + \sum_{j=1}^{m} \pi \Im \beta_j \left( 1 - 2 \int_{t_j}^{1} \frac{\psi(x)}{\sqrt{1-x^2}} dx \right),
\]
\[
C_3 = - \frac{1}{4} \sum_{j=1}^{m} \left( \frac{\alpha_j^2}{4} - \beta_j^2 \right), \tag{1.26}
\]
\[
C_4 = 3\zeta'(1) + \frac{\log 2}{12} - \frac{1-4\alpha_0^2}{8} \log (\pi \psi(-1)) - \frac{1-4\alpha_{m+1}^2}{8} \log (\pi \psi(1)) + \sum_{j=1}^{m} \left( \frac{\alpha_j^2}{4} - \beta_j^2 \right) \log (\pi \psi(t_j))
+ A \int_{-1}^{1} W(x) \frac{dx}{\sqrt{1-x^2}}
+ \sum_{j=1}^{m} \frac{\alpha_j}{2} \int_{-1}^{1} W(t_j) + \sum_{j=1}^{m} \Im \beta_j \left( 1 - t_j \right) \frac{W(x)}{\sqrt{1-x^2}} \int_{t_j}^{1} \frac{dx}{\sqrt{1-x^2}}
- \frac{1}{4\pi} \int_{-1}^{1} \frac{W(x) \sqrt{1-x^2}}{x-y} \left( \int_{-1}^{1} W''(x) \sqrt{1-x^2} \frac{dx}{x-y} \right) \frac{dy}{x-y}, \tag{1.27}
\]
where \( G \) is Barnes’ \( G \)-function, \( \zeta \) is Riemann’s zeta-function, where we use the notations \( \int \) for the Cauchy principal value integral, and
\[
A = \sum_{j=0}^{m+1} \alpha_j. \tag{1.28}
\]
Furthermore, the error term in (1.23) is uniform for all \( \alpha_k \) in compact subsets of \( \{ z \in \mathbb{C} : \Re z > -1 \} \), for all \( \beta_k \) in compact subsets of \( \{ z \in \mathbb{C} : \Re z \in (-\frac{1}{2}, \frac{1}{4}) \} \), and uniform in \( t_1, \ldots, t_m \), as long as there exists \( \delta > 0 \) independent of \( n \) such that
\[
\min_{j \neq k} \{|t_j - t_k|, |t_j - 1|, |t_j + 1|\} \geq \delta. \tag{1.29}
\]
Remark 1.4 The assumption \( \Re \beta_k \in (-\frac{1}{4}, \frac{1}{4}) \) comes from some technicalities in our analysis. Similar difficulties were encountered in \([14]\) for \( G_n(0, \beta, 2x^2, 0) \) with \( m = 1 \) (i.e. \( \beta = \beta_1 \)), and in \([19]\) for \( J_n(\alpha, \beta, 0, W) \). In \([9]\), the authors overcame these technicalities, and were able to extend their results from \( \Re \beta_k \in (-\frac{1}{4}, \frac{1}{4}) \) to \( \Re \beta_k \in (-\frac{1}{2}, \frac{1}{2}) \) by using Vitali’s theorem. Their argument relies crucially on \( w \) being independent of \( n \) (which is true only for Jacobi-type weights with \( V = 0 \)) and cannot be adapted straightforwardly to the situation of Theorem 1.1, 1.2 and 1.3. However, the method presented in this paper allows in principle, but with significant extra effort, to obtain asymptotics for the whole region \( \Re \beta_k \in (-\frac{1}{2}, \frac{1}{2}) \). Finally, extending the result from \( \Re \beta_k \in (-\frac{1}{2}, \frac{1}{2}) \) to \( \Re \beta_k \in (-\frac{1}{4}, \frac{1}{4}) \) would rely on so-called FH representations of the weight, see \([8]\) for more details.

Remark 1.5 Starting with a function \( f \) defined on the unit circle, the associated Toeplitz determinant is given by

\[
\det \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta}) e^{-i(j-k)\theta} d\theta \right)_{j,k=0,\ldots,n-1}.
\] (1.30)

Asymptotics of large Toeplitz determinants is another topic of high interest, which presents applications similar to those of Hankel determinants, but for point processes defined on the unit circle instead of the real line. In \([8]\), the authors obtained first large \( n \) asymptotics for certain Toeplitz determinants (with the zero potential), and deduced from them large \( n \) asymptotics for \( J_n(\alpha, \beta, 0, W) \). It is therefore natural to wonder if one can translate the results of Theorem 1.1, 1.2 and 1.3 into asymptotics for Toeplitz determinants with a one-cut regular potential. We explain here why we believe this is not obvious.

The main tool used in \([8]\) is a relation of Szegö \([27]\). If

\[
f(e^{i\theta}) = w(\cos \theta) |\sin \theta|,
\] (1.31)

we can express orthogonal polynomials on the unit circle associated to \( f \) in terms of orthogonal polynomials on the real line associated to \( w \). Note that this transformation can only work in all generality from Toeplitz to Hankel, and not the other way around. Indeed, the weight \( w \) can be arbitrary, but the function \( f \) is of a very particular type (in particular it satisfies \( f(e^{i\theta}) = f(e^{-i\theta}) \)).

We also believe that asymptotics for Toeplitz determinants with a one-cut regular potential and FH singularities would not imply Theorem 1.1, 1.2, and 1.3 (with the exception of \( V = 0 \) for Jacobi-type weights as done in \([8]\)). The main reason is that, as shown from the change of variables \( s = \cos \theta \) in \([14]\), the potential \( \hat{V} \) on the unit circle is related to the potential \( V \) on the interval \([-1,1]\) via the relation \( \hat{V}(e^{i\theta}) = V(\cos \theta) \), which means that at least one potential is not analytic (except if \( V \) is a constant as in \([8]\)). Finally, we also point out that regarding e.g. Gaussian-type weights, again the change of variables \( s = \cos \theta \) in \([14]\) shows that the associated equilibrium measure \( \mu_{\hat{V}} \) on the unit circle vanishes as a square at \( \theta = 0 \) and \( \theta = \pi \), which is not a “regular” weight. To avoid this problem, one could by a simple change of variables shrink the support of \( \mu_{\hat{V}} \) into \([-a,a] \) with \( 0 < a < 1 \), but then \( \mu_{\hat{V}} \) would be supported on two disjoint cuts.
Outline

The general strategy of our proof is close to the one done in [3], and can be schematized as

\[
L_n(0,0,2(x+1),0) \Rightarrow L_n(\vec{\alpha}, \vec{\beta}, 2(x+1),0) \Rightarrow L_n(\vec{\alpha}, \vec{\beta}, V, 0) \Rightarrow L_n(\vec{\alpha}, \vec{\beta}, V, W),
\]

\[
J_n(\vec{\alpha}, \vec{\beta}, 0,0) \Rightarrow J_n(\vec{\alpha}, \vec{\beta}, V, 0) \Rightarrow J_n(\vec{\alpha}, \vec{\beta}, V, W).
\]

(1.32)

In Section 2 we recall a well-known correspondence between Hankel determinants and orthogonal polynomials (OPs), and the characterization of these OPs in terms of a Riemann-Hilbert (RH) problem found by Fokas, Its and Kitaev [11], and whose solution is denoted by \(Y\). In Section 3 we derive suitable differential identities, which express the quantities

\[
\partial_\nu \log L_n(\vec{\alpha}, \vec{\beta}, 2(x+1),0), \quad \partial_\nu \log L_n(\vec{\alpha}, \vec{\beta}, V, 0), \quad \partial_\nu \log L_n(\vec{\alpha}, \vec{\beta}, V, W),
\]

\[
\partial_\nu J_n(\vec{\alpha}, \vec{\beta}, 0,0), \quad \partial_\nu J_n(\vec{\alpha}, \vec{\beta}, V, 0), \quad \partial_\nu J_n(\vec{\alpha}, \vec{\beta}, V, W),
\]

(1.33)

in terms of \(Y\), where \(\nu \in \{\alpha_0, \ldots, \alpha_m, \beta_1, \ldots, \beta_m\}\), and \(s \in [0,1]\) and \(t \in [0,1]\) are smooth deformation parameters (more details on these deformations are given in Subsection 2.1). In Section 4 we perform a Deift/Zhou steepest descent analysis of the RH problem to obtain large \(n\) asymptotics for \(Y\). We deduce from them asymptotics for the log derivatives given in (1.33), and we also proceed with their successive integrations (represented schematically by an arrow in (1.32)). These computations are rather long, and we organise them in several sections: Section 5 is devoted to integration in \(\vec{\alpha}\) and \(\vec{\beta}\), Section 6 to integration in \(s\) and Section 7 to integration in \(t\). Each integration only gives us asymptotics for a ratio of Hankel determinants. Therefore, it is important to choose carefully the starting point of integration in the set of parameters \((\vec{\alpha}, \vec{\beta}, V, W)\). For Laguerre-type weights, we chose this point to be \((0,0,2(x+1),0)\) and for Jacobi-type weights, we use the result of [8] and chose \((\vec{\alpha}, \vec{\beta}, 0,0)\). We recall large \(n\) asymptotics for \(L_n(0,0,2(x+1),0)\) and for \(J_n(\vec{\alpha}, \vec{\beta}, 0,0)\) in Section 5.

Notations. We will use repetitively through the paper the convention \(t_0 = -1, \ t_{m+1} = 1, \ \beta_0 = 0\) and \(\beta_{m+1} = 0\). Furthermore, for Laguerre-type weights, we define \(\alpha_{m+1} = 0\) and for Gaussian-type weights, we define \(\alpha_0 = 0\) and \(\alpha_{m+1} = 0\). This allows us for example to rewrite \(\omega\) given in (1.6) as

\[
\omega(x) = \prod_{j=0}^{m+1} \omega_{\alpha_j}(x)\omega_{\beta_j}(x).
\]

(1.34)

2 A Riemann-Hilbert problem for orthogonal polynomials

We consider the family of OPs associated to the weight \(w\) given in (1.2). The degree \(k\) polynomial \(p_k\) is characterized by the relations

\[
\int_{I} p_k(x)x^j w(x) dx = \kappa_k^{-1} \delta_{jk}, \quad j = 0, 1, 2, \ldots, k,
\]

(2.1)

where \(\kappa_k \neq 0\) is the leading order coefficient of \(p_k\). If \(\beta_j \in i\mathbb{R}\) and \(\Im \alpha_j > -1, \ j = 0, \ldots, m + 1,\) then \(w(x) > 0\) for almost all \(x \in I\). In this case, we can rewrite (2.1) as an inner product and it is a simple consequence of Gram-Schmidt that the OPs exist. However, for general values of \(\alpha_j\) and \(\beta_j\), the weight \(w\) is complex-valued and existence is no more guaranteed. This fact introduces some technicalities in the analysis that are briefly discussed in Section 6, Section 7 and Section 8.

We associate to these OPs a RH problem for a \(2 \times 2\) matrix-valued function \(Y\), due to [11]. As mentioned in the outline, it will play a crucial role in our proof.
**RH problem for Y**

(a) \( Y : \mathbb{C} \setminus \mathcal{I} \to \mathbb{C}^{2 \times 2} \) is analytic.

(b) The limits of \( Y(z) \) as \( z \) tends to \( x \in \mathcal{I} \setminus \{-1, t_1, \ldots, t_m, 1\} \) from the upper and lower half plane exist, and are denoted \( Y^\pm(x) \) respectively. Furthermore, the functions \( x \mapsto Y^\pm(x) \) are continuous on \( \mathcal{I} \setminus \{-1, t_1, \ldots, t_m, 1\} \) and are related by

\[
Y_+(x) = Y_-(x) \begin{pmatrix} 1 & w(x) \\ 0 & 1 \end{pmatrix}, \quad x \in \mathcal{I} \setminus \{-1, t_1, \ldots, t_m, 1\}. \tag{2.2}
\]

(c) As \( z \to \infty \),

\[
Y(z) = (I + O(z^{-1}))z^{n_{\sigma_3}}, \quad \text{where} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.3}
\]

(d) As \( z \to t_j \), for \( j = 0, 1, \ldots, m + 1 \) (with \( t_0 = -1 \) and \( t_{m+1} = 1 \)), we have

\[
Y(z) = \begin{cases} 
O(1) & \text{if } \Re \alpha_j \neq 0, \\
O(1) + O((z - t_j)^{\alpha_j}) & \text{if } \Re \alpha_j = 0.
\end{cases} \tag{2.4}
\]

The solution of the RH problem for \( Y \) is always unique, exists if and only if \( p_n \) and \( p_{n-1} \) exist, and is explicitly given by

\[
Y(z) = \begin{pmatrix} \kappa_n^{-1} p_n(z) & \kappa_n^{-1} \int_{\mathcal{I}} \frac{p_n(x)w(x)}{x - z} \, dx \\ -2\pi i \kappa_{n-1} p_{n-1}(z) & -\kappa_{n-1} \int_{\mathcal{I}} \frac{p_{n-1}(x)w(x)}{x - z} \, dx \end{pmatrix}. \tag{2.5}
\]

The fact that \( Y \) given by (2.5) satisfies the condition (b) of the RH problem for \( Y \) follows from the Sokhotski formula and relies on the assumption that \( W \) is locally Hölder continuous on \( \mathcal{I} \) (see e.g. [13]).

### 3 Differential identities

In this section, we express the logarithmic derivatives given in (1.33) in terms of \( Y \).

#### 3.1 Identity for \( \partial_x \log \nu L_n(\bar{\alpha}, \bar{\beta}, 2(x+1), 0) \) with \( \nu \in \{\alpha_0, \ldots, \alpha_m, \beta_1, \ldots, \beta_m\} \)

In this subsection, we specialize to the Laguerre-type weight \( w(x) = \omega(x) e^{-2\nu(x+1)} \).

Note that the second column of \( Y \) blows up as \( z \to t_k \), \( k = 0, 1, \ldots, m \) as shown in (2.4). The terms of order 1 in these asymptotics will contribute in our identity for \( \partial_x \log L_n(\bar{\alpha}, \bar{\beta}, 2(x+1), 0) \). To prepare ourselves for that matter, following [3 eq (3.6)], for each \( k \in \{1, \ldots, m\} \) we define a regularized integral by

\[
\text{Reg}_k(f) = \lim_{\epsilon \to 0^+} \left[ \alpha_k \int_{\mathcal{I} \setminus [t_k - \epsilon, t_k + \epsilon]} \frac{f(x)\omega(x)}{x - t_k} \, dx - f(t_k)\omega_k(t_k)(e^{\pi i \beta_k} - e^{-\pi i \beta_k})e^{\alpha_k} \right], \tag{3.1}
\]
where \( f \) is a smooth function on \( I = [-1, +\infty) \), and
\[
\omega_{t_k}(x) = \prod_{0 \leq j \leq m} \omega_{t_j}(x) \omega_{\beta_j}(x). \tag{3.2}
\]

For \( k = 0 \), we define the regularized integral as above, with \( e^{\pi i \beta_k} \) replaced by 0 and \( e^{-\pi i \beta_k} \) replaced by 1 (we also recall that \( t_0 = -1 \)), i.e. we have
\[
\text{Reg}_0(f) := \lim_{\varepsilon \to 0^+} \left[ \alpha_0 \int_{I \setminus [t_0, t_0+\varepsilon]} \frac{f(x)\omega(x)}{x-t_0} \, dx + f(t_0) \omega^{-1}(t_0) e^{\alpha_0} \right]. \tag{3.3}
\]

**Proposition 3.1** The regularized integrals (3.1) and (3.3) satisfy
\[
\text{Reg}_k(f) = \lim_{z \to t_k} \alpha_k \int_{I} \frac{f(x)\omega(x)}{x-z} \, dx - \mathcal{J}_k(z) \tag{3.4}
\]
where the limit is taken along a path in the upper-half plane which is non-tangential to the real line. For \( k = 1, \ldots, m \), \( \mathcal{J}_k(z) \) is given by
\[
\mathcal{J}_k(z) = \begin{cases} 
\pi \alpha_k \sin(\pi \alpha_k) f(t_k)\omega_{t_k}(t_k)(e^{\pi i \beta_k} - e^{-\pi i \beta_k})(z-t_k)^{\alpha_k}, & \text{if } \Re \alpha_k \leq 0, \alpha_k \neq 0, \\
f(t_k)\omega_{t_k}(t_k)(e^{\pi i \beta_k} - e^{-\pi i \beta_k}), & \text{if } \alpha_k = 0, \\
0, & \text{if } \Re \alpha_k > 0. 
\end{cases} \tag{3.5}
\]

For \( k = 0 \), we have
\[
\mathcal{J}_0(z) = \begin{cases} 
\pi \alpha_0 e^{-\pi i \alpha_0} \sin(\pi \alpha_0) f(t_0)\omega^{-1}(t_0)(z-t_0)^{\alpha_0}, & \text{if } \Re \alpha_0 \leq 0, \alpha_0 \neq 0, \\
f(t_0)\omega^{-1}(t_0), & \text{if } \alpha_0 = 0, \\
0, & \text{if } \Re \alpha_0 > 0. 
\end{cases} \tag{3.6}
\]

**Proof.** The proof for \( k = 1, \ldots, m \) can be found in [3, Proposition 3.1] (which is itself based on [15]). The proof for \( k = 0 \) can be proved similarly by a straightforward adaptation. It suffices to replace \( e^{\pi i \beta_k} \) by 0 and \( e^{-\pi i \beta_k} \) by 1 in the proof of [3, Proposition 3.1]. \( \square \)

Since the second column of \( Y(z) \) blows up as \( z \to t_j, j = 0, \ldots, m \), we regularize \( Y \) at these points using the definitions (3.1) and (3.3) as follows:
\[
\tilde{Y}(t_j) := \begin{pmatrix} Y_{11}(t_j) & \text{Reg}_j \left( \frac{1}{2\pi i} Y_{11}(x) e^{-2n(x+1)} \right) \\ Y_{21}(t_j) & \text{Reg}_j \left( \frac{1}{2\pi i} Y_{21}(x) e^{-2n(x+1)} \right) \end{pmatrix}. \tag{3.7}
\]

From Proposition 3.1 we have
\[
\tilde{Y}_{kj}(t_j) = \lim_{z \to t_j} \alpha_j Y_{kj}(z) - c_j Y_{kj}(t_j)(z-t_j)^{\alpha_j}, \quad k = 1, 2, \tag{3.8}
\]
where the limit is taken along a path in the upper half plane non-tangential to the real line. For \( j = 1, \ldots, m, \) \( c_j \) is given by
\[
c_j = \frac{\pi \alpha_j}{\sin(\pi \alpha_j)} e^{-2n(t_j+1)} \frac{1}{2\pi i} \omega_{t_j}(t_j)(e^{\pi i \beta_j} - e^{-\pi i \beta_j}), \tag{3.9}
\]

10
and for $j = 0$ we have
\begin{equation}
\alpha_0 = \frac{\pi \alpha_0}{\sin(\pi \alpha_0)} - e^{-\pi i \alpha_0} - \frac{1}{2 \pi i} \omega(-1).
\end{equation}
(3.10)

Note that $\det \tilde{\Upsilon}(t_j)$ is not equal to 1, but instead we have
\begin{equation}
\det \tilde{\Upsilon}(t_j) = \alpha_j, \quad j = 0, 1, \ldots, m.
\end{equation}
(3.11)

**Proposition 3.2** Let $p_0, p_1, \ldots$ be the family of OPs with respect to the weight $w(x) = \omega(x)e^{-2n(x+1)}$, whose leading coefficients are denoted by
\begin{equation}
p_k(x) = \kappa_k(x^k + \eta_k x^{k-1} + \ldots).
\end{equation}
(3.12)

Let $\nu \in \{\alpha_0, \alpha_1, \beta_1, \ldots, \alpha_m, \beta_m\}$ and let $n$, $\tilde{\alpha}$ and $\tilde{\beta}$ be such that $p_0, p_1, \ldots, p_n$ exist. We have the following identity:
\begin{equation}
\partial_\nu \log L_n(\tilde{\alpha}, \tilde{\beta}, 2(x+1), 0) = -(n + A) \partial_\nu \log (\kappa_n \kappa_{n-1}) + 2n \partial_\nu \eta_n + \sum_{j=0}^{m} \left( \tilde{\Upsilon}^2(t_j) \partial_{\nu} Y_{11}(t_j) - \tilde{\Upsilon}^2(t_j) \partial_{\nu} Y_{21}(t_j) + Y_{11}(t_j) \tilde{\Upsilon}^2(t_j) \partial_{\nu} \log (\kappa_n \kappa_{n-1}) \right),
\end{equation}
(3.13)

where $A = \sum_{j=0}^{m} \alpha_j$.

**Remark 3.3** We do not need an analogous formula for $\partial_\nu \log J_n(\tilde{\alpha}, \tilde{\beta}, 0, 0)$ as large $n$ asymptotics of $J_n(\tilde{\alpha}, \tilde{\beta}, 0, 0)$ are already known from [8], see the outline.

**Proof.** The proof is an adaptation of [3 Subsection 3.1] where the author obtained a differential identity for $\partial_\nu \log G_n(\tilde{\alpha}, \tilde{\beta}, 2x^2, 0)$ (this proof was itself a generalization of [15 [14]). Here, the proof is even slightly easier, due to the fact that the potential is a polynomial of degree 1 (and not of degree 2 as in [15 [14 [3]). Since we assume that $p_0, \ldots, p_n$ exist, we can use the following general identity, which was obtained in [15]
\begin{equation}
\partial_\nu \log L_n(\tilde{\alpha}, \tilde{\beta}, 2(x+1), 0) = -n \partial_\nu \log \kappa_{n-1} + \frac{\kappa_{n-1}}{\kappa_n} (I_1 - I_2),
\end{equation}
(3.14)

where
\begin{equation}
I_1 = \int_{\mathcal{I}_+} p_n(x) \partial_{\nu} p_n(x) w(x) dx, \quad \text{and} \quad I_2 = \int_{\mathcal{I}_+} p_n(x) \partial_{\nu} p_{n-1}(x) w(x) dx.
\end{equation}
(3.15)

Since $\Re \alpha_j > -1$ for all $j = 0, 1, \ldots, m$, we first note that
\begin{equation}
I_1 = \lim_{\epsilon \to 0^+} \int_{\mathcal{I}_\epsilon} p_n(x) \partial_{\nu} p_n(x) w(x) dx,
\end{equation}
(3.16)

where $\mathcal{I}_\epsilon$ is the union of $m + 1$ intervals given by
\begin{equation}
\mathcal{I}_\epsilon = [t_0 + \epsilon, t_1 - \epsilon] \cup [t_1 + \epsilon, t_2 - \epsilon] \cup \ldots \cup [t_{m-1} + \epsilon, t_m - \epsilon] \cup [t_m + \epsilon, \infty).
\end{equation}
\begin{equation}
\text{Along each of these $m + 1$ intervals, we integrate by parts (for each fixed and sufficiently small $\epsilon$), using}
\begin{equation}
w'(x) = \left( -2n + \sum_{j=0}^{m} \frac{\alpha_j}{x - t_j} \right) w(x), \quad x \in (-1, \infty) \setminus \{t_1, \ldots, t_m\}.
\end{equation}
(3.17)
Then, we simplify the expression by using the orthogonality relations (2.1). Finally, we substitute it in the limit (3.16) using (3.1) and (3.3), and we find

\[ I_1 = -(n + A) \frac{\partial \nu \kappa_n}{\kappa_n - 1} + 2n \frac{\kappa_n}{\kappa_n - 1} \partial \nu \eta_n - \sum_{j=0}^{m} \partial \nu p_n(t_j) \text{Reg}_j \left[ p_{n-1}(x) e^{-2n(x+1)} \right]. \] (3.18)

We proceed similarly to find the following expression for \( I_2 \) (the calculations are easier as several integrals can be identified as equal to 0 by using (2.1)):

\[ I_2 = -\sum_{j=0}^{m} \partial \nu p_{n-1}(t_j) \text{Reg}_j \left[ p_n(x) e^{-2n(x+1)} \right]. \] (3.19)

By rewriting first \( I_1 \) and \( I_2 \) in terms of \( Y \) and \( \tilde{Y} \), then by substituting these expressions into (3.14), and finally by using (3.11), we obtain the claim. \( \square \)

### 3.2 A general differential identity

We recall here a differential identity that is valid for all three types of weights. In Section 7 and Section 8, we will use Proposition 3.4 below with \( \nu = s \) or \( \nu = t \) to obtain identities for the quantities in (1.33) (save the case of \( \partial \nu L_n(\vec{\alpha}, \vec{\beta}, 2(x+1), 0) \) for which we will use Proposition 3.2).

**Proposition 3.4**

Let \( D_n \) be a Hankel determinant whose weight \( w \) depends smoothly on a parameter \( \nu \). Let us assume that the associated orthonormal polynomials \( p_0, \ldots, p_n \) exist. Then we have

\[ \partial \nu \log D_n = \frac{1}{2\pi i} \int_{\mathcal{I}} \left[ Y^{-1}(x) Y'(x) \right]_{21} \partial \nu w(x) dx, \] (3.20)

where \( \mathcal{I} \) is the support of \( w \), and \( Y \) is given by (2.5).

**Proof.** It suffices to start from the well-known [20] identity

\[ D_n = \prod_{j=0}^{n-1} \kappa_j^{-2}, \] (3.21)

take the log, differentiate with respect to \( \nu \), use the orthogonality relations and finally substitute \( Y \) in the expression. \( \square \)

### 4 Steepest descent analysis

In this section we will construct an asymptotic solution to the RH problem for \( Y \) through the Deift/Zhou steepest descent method, for Laguerre-type and Jacobi-type weights. The analysis goes via a series of transformations \( Y \leftrightarrow T \leftrightarrow S \leftrightarrow R \). The \( Y \leftrightarrow T \) transformation of Subsection 4.2 normalizes the RH problem at \( \infty \) by means of a so-called \( g \)-function (whose properties are presented in Subsection 4.1). We proceed with the opening of the lenses \( T \leftrightarrow S \) in Subsection 4.3. As a preliminary to the last step \( S \rightarrow R \), we first construct approximations (called “parametrices”) for \( S \) in different regions of the complex plane: a global parametrix in Subsection 4.4, local parametrices in the bulk around \( t_k \) in Subsection 4.5 and local parametrices at the edges \( \pm 1 \) in Subsection 4.6 and Subsection 4.7. These parametrices are rather standard: our global parametrix is close to the one done in [12] and local parametrices in the bulk are built out of confluent hypergeometric functions (as in [12] [14] [8]), local parametrices at soft edges in terms of Airy functions (as in [6]) and at a hard edge, in terms of Bessel functions (as in [16]). Finally, the last step \( S \rightarrow R \) is carried out in Subsection 4.8.
4.1 Equilibrium measure and \( g \)-function

It is convenient for us to introduce the notation \( \rho \) for the density of \( \mu_V \):

\[
d\mu_V(x) = \rho(x)dx = \begin{cases} \psi(x)\frac{\sqrt{-x}}{\sqrt{1+x}}dx, & \text{for Laguerre-type weight,} \\ \psi(x)\frac{1}{\sqrt{1-x^2}}dx, & \text{for Jacobi-type weight,} \end{cases}
\]

(4.1)

where we recall that by assumption \( \psi : I \to \mathbb{R} \) is analytic and positive on \([-1, 1]\). Let \( U_V \) be the maximal open neighbourhood of \( I \) in which \( V \) is analytic, and \( U_W \) be an open neighbourhood of \([-1, 1]\) in which \( W \) is analytic, sufficiently small such that \( U_W \subset U_V \). The so-called \( g \)-function is defined by

\[
g(z) = \int_{-1}^{1} \log(z - s)\rho(s)ds, \quad \text{for } z \in \mathbb{C} \setminus (-\infty, 1],
\]

(4.2)

where the principal branch is chosen for the logarithm. The \( g \)-function is analytic in \( \mathbb{C} \setminus (-\infty, 1] \) and has the following properties

\[
g_+(x) + g_-(x) = 2\int_{-1}^{1} \log|x - s|\rho(s)ds, \quad x \in \mathbb{R},
\]

(4.3)

\[
g_+(x) - g_-(x) = 2\pi i, \quad x \in (-\infty, -1),
\]

(4.4)

\[
g_+(x) - g_-(x) = 2\pi i \int_{x}^{1} \rho(s)ds, \quad x \in [-1, 1].
\]

(4.5)

The Euler-Lagrange conditions (4.1)-(4.5) can be rewritten in terms of the \( g \)-function as follows:

\[
g_+(x) + g_-(x) = V(x) - \ell, \quad x \in [-1, 1],
\]

(4.6)

\[
2g(x) < V(x) - \ell, \quad x \in I \setminus [-1, 1].
\]

(4.7)

The above inequality is relevant only for Laguerre-type weight (since for Jacobi-type weight \( I \setminus [-1, 1] = \emptyset \)), and is strict since we assume that \( V \) is regular.

For \( z \in U_V \setminus [-1, 1] \), we define

\[
\tilde{\rho}(z) = \begin{cases} -i\psi(z)\frac{\sqrt{z - 1}}{\sqrt{z + 1}}, & \text{for Laguerre-type weight,} \\ i\psi(z)\frac{1}{\sqrt{z^2 - 1}}, & \text{for Jacobi-type weight,} \end{cases}
\]

(4.8)

where the principal branches are chosen for \( \sqrt{z - 1} \) and \( \sqrt{z + 1} \). Note that for \( x \in (-1, 1) \) we have \( \tilde{\rho}_+(s) = -\tilde{\rho}_-(s) = \rho(s) \). Let us also define

\[
\xi(z) = -\pi i \int_{1}^{z} \tilde{\rho}(s)ds, \quad z \in U_V \setminus (-\infty, 1),
\]

(4.9)

where the path of integration lies in \( U_V \setminus (-\infty, 1) \). Since \( \xi_+(x) + \xi_-(x) = 0 \) for \( x \in (-1, 1) \), by (4.5) and (4.9), we have

\[
2\xi_\pm(x) = g_\pm(x) - g_+(x) = 2g_\pm(x) - V(x) + \ell.
\]

(4.10)

By analytic continuation, we have

\[
\xi(z) = g(z) + \frac{\ell}{2} - \frac{V(z)}{2}, \quad z \in U_V \setminus (-\infty, 1).
\]

(4.11)
Thus, the Euler-Lagrange inequality \( \xi(x) < 0 \) for \( x \in I \setminus [-1, 1] \). Furthermore, since \( g(z) \sim \log(z) \) as \( z \to \infty \), we have that \( (\xi_+(x) + \xi_-(x))/V(x) \to -1 \) as \( x \to +\infty \), \( x \in I \). Finally, by a standard and straightforward analysis of \( \xi \), we conclude that there exists a small enough neighbourhood of \((-1, 1)\) such that, for \( z \) in this neighbourhood with \( \Im z \neq 0 \), we have \( \Re \xi(z) > 0 \).

We will also need later large \( z \) asymptotics of \( e^{ng(z)} \) for the Laguerre-type potential \( V(x) = 2(x + 1) \). In this case, we recall that \( \psi(x) = \frac{1}{\pi} \), and after a straightforward calculation we obtain

\[
e^{ng(z)} = z^n \left( 1 + \frac{n}{2z} + \mathcal{O}(z^{-2}) \right), \quad \text{as } z \to \infty.
\]

### 4.2 First transformation: \( Y \mapsto T \)

We normalize the RH problem for \( Y \) at \( \infty \) by the standard transformation

\[
T(z) := e^{\frac{4\pi i}{3} Y(z)} e^{-n g(z) \sigma_3} e^{-\frac{4\pi i}{3} \sigma_3}.
\]

\( T \) satisfies the following RH problem.

**RH problem for \( T \)**

(a) \( T : \mathbb{C} \setminus I \to \mathbb{C}^{2 \times 2} \) is analytic.

(b) The jumps for \( T \) follows from (4.3), (4.6) and (4.11). We obtain

\[
T_+(x) = T_-(x) \begin{pmatrix} e^{-2n\xi_+(x)} & e^{W(x)} \omega(x) \\ 0 & e^{2n\xi_+(x)} \end{pmatrix}, \quad \text{if } x \in (-1, 1) \setminus \{t_1, \ldots, t_m\},
\]

\[
T_+(x) = T_-(x) \begin{pmatrix} 1 & 0 \\ e^{W(x)} \omega(x) e^{2n\xi(x)} & 1 \end{pmatrix}, \quad \text{if } x \in I \setminus [-1, 1].
\]

(c) As \( z \to \infty \), \( T(z) = I + \mathcal{O}(z^{-1}) \).

(d) As \( z \to t_j \), for \( j = 0, 1, \ldots, m + 1 \), we have

\[
T(z) = \begin{cases} 
\mathcal{O}(1) & \mathcal{O}(1) + \mathcal{O}((z-t_j)^{\alpha_{k_j}}) \\
\mathcal{O}(1) & \mathcal{O}(1) + \mathcal{O}((z-t_j)^{\alpha_{k_j}}) \\
\mathcal{O}(1) & \mathcal{O}(\log(z-t_j)) \\
\mathcal{O}(1) & \mathcal{O}(\log(z-t_j)) 
\end{cases}, \quad \text{if } \Re \alpha_j \neq 0,
\]

\[
T(z) = \begin{cases} 
\mathcal{O}(1) & \mathcal{O}(1) + \mathcal{O}((z-t_j)^{\alpha_{k_j}}) \\
\mathcal{O}(1) & \mathcal{O}(1) + \mathcal{O}((z-t_j)^{\alpha_{k_j}}) \\
\mathcal{O}(1) & \mathcal{O}(\log(z-t_j)) \\
\mathcal{O}(1) & \mathcal{O}(\log(z-t_j)) 
\end{cases}, \quad \text{if } \Re \alpha_j = 0.
\]

### 4.3 Second transformation: \( T \mapsto S \)

In this step, we will deform the contour of the RH problem. Therefore, we first consider the analytic continuations of the functions \( \omega_{\alpha_k} \) and \( \omega_{\beta_k} \) from \( \mathbb{R} \setminus \{t_k\} \) to \( \mathbb{C} \setminus \{z : \Re(z) = t_k\} \). They are given by

\[
\omega_{\alpha_k}(z) = \begin{cases} 
(t_k - z)^{\alpha_k}, \quad \text{if } \Re z < t_k, \\
(z - t_k)^{\alpha_k}, \quad \text{if } \Re z > t_k,
\end{cases}
\]

\[
\omega_{\beta_k}(z) = \begin{cases} 
e^{\pm i \pi \beta}, \quad \text{if } \Re z < t_k, \\
\end{cases}
\]

For \( k = 0, \ldots, m + 1 \), we also define

\[
\omega_{t_k}(z) = \prod_{0 \leq j < m \atop j \neq k} \omega_{\alpha_j}(z) \omega_{\beta_j}(z).
\]
Figure 1: The jump contour for the RH problem for $S$ with $m = 2$ and a Laguerre-type weight. For Jacobi-type weights, the jump contour for $S$ is of the same shape, except that there are no jumps on $(1, +\infty)$.

Note that for $x \in (-1, 1) \setminus \{t_1, \ldots, t_m\}$ we have the following factorization for $J_T(x)$:

$$
\begin{pmatrix}
\left(e^{-2n\xi_+(x)} & e^{W(x)\omega(x)}
\right)
&
\begin{pmatrix}
1
&
0
\end{pmatrix}

\end{pmatrix}
\begin{pmatrix}
\left(-e^{-W(x)\omega(x)} & e^{-2n\xi_-(x)}
\right)
&
\begin{pmatrix}
0
&
1
\end{pmatrix}

\end{pmatrix}
\begin{pmatrix}
\left(-e^{-W(x)\omega(x)} & e^{W(x)\omega(x)}
\right)^{-1}
&
\begin{pmatrix}
1
&
0
\end{pmatrix}

\end{pmatrix}
\begin{pmatrix}
\left(-e^{-W(x)\omega(x)} & e^{-2n\xi_+(x)}
\right)
&
\begin{pmatrix}
0
&
1
\end{pmatrix}

\end{pmatrix}

\times
\begin{pmatrix}
\left(-e^{-W(x)\omega(x)} & e^{W(x)\omega(x)}
\right)^{-1}
&
\begin{pmatrix}
1
&
0
\end{pmatrix}

\end{pmatrix}
\begin{pmatrix}
\left(-e^{-W(x)\omega(x)} & e^{-2n\xi_+(x)}
\right)
&
\begin{pmatrix}
0
&
1
\end{pmatrix}

\end{pmatrix}.

(4.19)

Let $\gamma_+$ and $\gamma_-$ be two curves (lying respectively in the upper and lower half plane) that join the points $-1, t_1, \ldots, t_m, 1$ as depicted in Figure 1. In order to be able to deform the contour of the RH problem, we choose them so that they both lie in $U_W$. In the constructions of the local parametries, they will be required to make angles of $\frac{\pi}{4}$ with $\mathbb{R}$ at the points $t_1, \ldots, t_m$, and angles of $\frac{\pi}{2}$ with $\mathbb{R}$ at the points $\pm 1$, and this is already shown in Figure 1. Also, we denote $\Omega_{\pm}$ for the open regions delimited by $\gamma_{\pm}$ and $\mathbb{R}$, see Figure 1. The next transformation is given by

$$
S(z) = T(z)
\begin{cases}
1 & \text{if } z \in \Omega_+,
-\left(e^{-W(z)\omega(z)}\right)^{-1}e^{-2n\xi(z)} & \text{if } z \in \Omega_-,
I, & \text{if } z \in \mathbb{C} \setminus (\Omega_+ \cup \Omega_- \cup (\mathbb{Z} \setminus S)).
\end{cases}

(4.20)

S satisfies the following RH problem.

**RH problem for $S$**

(a) $S : \mathbb{C} \setminus (\mathbb{I} \cup \gamma_+ \cup \gamma_-) \to \mathbb{C}^{2 \times 2}$ is analytic.

(b) The jumps for $S$ follows from those of $T$ and from (4.19). They are given by

$$
S_+(z) = S_-(z)
\begin{pmatrix}
1
&
0
\end{pmatrix}
\begin{pmatrix}
\left(-e^{-W(z)\omega(z)}\right)^{-1}e^{W(z)\omega(z)}
&
0
\end{pmatrix}

\begin{pmatrix}
1
&
0
\end{pmatrix},
\text{if } z \in (-1, 1) \setminus \{t_1, \ldots, t_m\},

(4.21)

$$

$$
S_+(z) = S_-(z)
\begin{pmatrix}
0
&
1
\end{pmatrix}
\begin{pmatrix}
\left(-e^{-W(z)\omega(z)}\right)^{-1}e^{W(z)\omega(z)}
&
0
\end{pmatrix}

\begin{pmatrix}
0
&
1
\end{pmatrix},
\text{if } z \in \mathbb{Z} \setminus [-1, 1],

(4.22)

$$

$$
S_+(z) = S_-(z)
\begin{pmatrix}
0
&
1
\end{pmatrix}
\begin{pmatrix}
\left(-e^{-W(z)\omega(z)}\right)^{-1}e^{-2n\xi(z)}
&
0
\end{pmatrix}

\begin{pmatrix}
0
&
1
\end{pmatrix},
\text{if } z \in \gamma_+ \cup \gamma_-.

(4.23)

(c) As $z \to \infty$, $S(z) = I + \mathcal{O}(z^{-1})$.  

15
are uniformly exponentially close to the identity matrix, except those on $S$ regions of the complex plane. If $z$ is away from neighbourhoods of $-1, t_1, \ldots, t_m, 1$, then the jumps for $P$ are uniformly exponentially close to the identity matrix, except those on $(-1, 1)$ (see the discussion at the end of Section 4.1). By ignoring the jumps that tend to the identity matrix, we are left with the following RH problem for $P$.

By disregarding the jump conditions on the lenses $\gamma_+ \cup \gamma_-$ and on $I \setminus [-1, 1]$, we are left with the RH problem for $P^{(\infty)}$ (condition (d) below ensures uniqueness of the RH problem and can not be seen from the RH problem for $S$).

**4.4 Global parametrix**

By disregarding the jump conditions on the lenses $\gamma_+ \cup \gamma_-$ and on $I \setminus [-1, 1]$, we are left with the following RH problem for $P^{(\infty)}$ (condition (d) below ensures uniqueness of the RH problem and can not be seen from the RH problem for $S$).

**RH problem for $P^{(\infty)}$**

(a) $P^{(\infty)} : \mathbb{C} \setminus [-1, 1] \to \mathbb{C}^{2 \times 2}$ is analytic.

(b) The jumps for $P^{(\infty)}$ are given by

$$P_{+}^{(\infty)}(z) = P_{-}^{(\infty)}(z) \begin{pmatrix} 0 & e^{W(z)\omega(z)} \\ -e^{-W(z)\omega(z)} & 0 \end{pmatrix}, \quad \text{if } z \in (-1, 1) \setminus \{t_1, \ldots, t_m\}. $$
Remark 4.1

Note that this RH problem is the same regardless of the weight, the only exception being that $\alpha_{m+1} = 0$ for Laguerre-type weight (and not necessarily for Jacobi-type weight).

This RH problem was solved first in \cite{6} with $W$ and $\omega$ \equiv 0. In \cite{16}, the authors explain how to construct the solution to the above RH problem for general $W$ and $\omega$ by using Szegő functions. Our RH problem for $P(\infty)$ is close to the one obtained in \cite{3} for Gaussian-type weights. The solution is given by

$$D(\infty) = D_\alpha(z)D_\beta(z)D_W(z),$$

where $D(\infty) = D_\alpha(z)D_\beta(z)D_W(z)$, where

$$D_W(z) = \exp \left( \frac{\sqrt{z^2 - 1}}{2\pi} \int_{-1}^{1} \frac{W(x)}{\sqrt{1 - x^2}} \frac{dx}{z - x} \right),$$

$$D_\alpha(z) = \prod_{j=0}^{m+1} \exp \left( \frac{\sqrt{z^2 - 1}}{2\pi} \int_{-1}^{1} \log \omega_\alpha(x) \frac{dx}{\sqrt{1 - x^2}} \frac{dx}{z - x} \right) = \left( z + \sqrt{z^2 - 1} \right)^{-\frac{m+1}{2}} \prod_{j=0}^{m+1} \frac{1}{(z - t_j)^{\alpha_j}},$$

$$D_\beta(z) = \prod_{j=1}^{m} \exp \left( \frac{\sqrt{z^2 - 1}}{2\pi} \int_{-1}^{1} \log \omega_\beta(x) \frac{dx}{\sqrt{1 - x^2}} \frac{dx}{z - x} \right) = e^{i\pi B} \prod_{j=1}^{m} \left( \frac{zt_j - 1 - i\sqrt{(z^2 - 1)(1 - t_j^2)}}{z - t_j} \right)^{\beta_j},$$

where $A = \sum_{j=0}^{m+1} \alpha_j$ and $B = \sum_{j=1}^{m} \beta_j$. The simplified forms of (4.31) and (4.32) were found in \cite{16} and \cite{4}, respectively. Also, $D(\infty) = \lim_{z \to \infty} D(z)$ appearing in (4.24) is given by

$$D(\infty) = 2^{-\frac{m}{2}} \exp \left( i \sum_{j=1}^{m} \beta_j \arcsin t_j \right) \exp \left( \frac{1}{2\pi} \int_{-1}^{1} \frac{W(x)}{\sqrt{1 - x^2}} dx \right).$$

The following asymptotic expressions were obtained in \cite{3} Section 4.4 with $\alpha_0 = \alpha_{m+1} = 0$. It is straightforward to adapt them for general $\alpha_0$ and $\alpha_{m+1}$. As $z \to t_k$, with $k \in \{1, \ldots, m\}$ and $3z > 0$, we have

$$D_\alpha(z) = e^{-i\frac{\pi}{2} \arccos t_k} \prod_{0 \leq j \neq k \leq m+1} \left( t_k - t_j \right)^{\alpha_j} \prod_{j=k+1}^{m} e^{i\pi \alpha_{j+1}} \left( z - t_k \right)^{\alpha_k} \left( 1 + O(\frac{z}{t_k}) \right),$$

$$D_\beta(z) = e^{-\frac{\pi}{2} (B_k + \beta_k)} \prod_{1 \leq j \neq k \leq m} T_{kj}^{3\beta_j} \left( 1 - t_k^2 \right)^{-\beta_k} \left( z - t_k \right)^{\beta_k} \left( 1 + O(\frac{z}{t_k}) \right).$$
where

\[ B_k = \sum_{j=1}^{k-1} \beta_j - \sum_{j=k+1}^{m} \beta_j, \quad T_{kj} = \frac{1 - t_k t_j - \sqrt{(1 - t_k^2)(1 - t_j^2)}}{|t_k - t_j|}. \quad (4.36) \]

Let us also define the following quantities:

\[ \beta_0 = \beta_{m+1} = 0. \]

As \( z \to 1 \), we have

\[ D_\alpha^2(z) \prod_{j=0}^{m+1} (z - t_j)^{-\alpha_j} = 1 - \sqrt{2} A \sqrt{z - 1} + A^2 (z - 1) + O((z - 1)^{3/2}), \quad (4.38) \]

\[ D_\beta^2(z) e^{i \pi B} = 1 + \sqrt{2 B_1} \sqrt{z - 1} + B_1^2 (z - 1) + O((z - 1)^{3/2}). \quad (4.39) \]

As \( z \to -1 \), \( \Im z > 0 \), we have

\[ D_\alpha^2(z) \prod_{j=0}^{m+1} (t_j - z)^{-\alpha_j} = 1 + \sqrt{2} A \sqrt{z + 1} - A^2 (z + 1) + O((z + 1)^{3/2}), \quad (4.40) \]

\[ D_\beta^2(z) e^{-i \pi B} = 1 + \sqrt{2 B_{-1}} \sqrt{z + 1} - B_{-1}^2 (z + 1) + O((z + 1)^{3/2}). \quad (4.41) \]

As \( z \to \infty \), with \( \Re z \equiv 0 \), we have

\[ P_1^{(\infty)} = \begin{pmatrix}
\frac{1}{2} \sum_{j=0}^{m+1} \left( \frac{\alpha_j t_j}{2} + i \sqrt{1 - t_j^2} \beta_j \right) \\
\frac{i}{2} D_\infty^{-2} - \sum_{j=0}^{m+1} \left( \frac{\alpha_j t_j}{2} + i \sqrt{1 - t_j^2} \beta_j \right)
\end{pmatrix}, \quad (4.42) \]

where we recall that \( \beta_0 = \beta_{m+1} = 0 \).

### 4.5 Local parametrix near \( t_k \), \( 1 \leq k \leq m \)

It is well-known [12, 14, 8] that \( P^{(t_k)} \) can be written in terms of hypergeometric functions. In [3], the local parametrix was obtained for Gaussian-type weights, and it is straightforward to adapt the construction for Laguerre-type and Jacobi-type weights, the only difference being in the definition of \( \xi \). Let us define the function \( f_{tk} \) by

\[ f_{tk}(z) = -2 \begin{cases} \xi(z) - \xi_+(t_k), & \exists z > 0, \\
-\xi(z) - \xi_-(t_k), & \exists z < 0,
\end{cases} = 2\pi i \int_{t_k}^{z} \rho(s) ds, \quad (4.43) \]

where in the above expression \( \rho \) is the analytic continuation on \( U_V \setminus ((-\infty, -1) \cup (1, +\infty)) \) of the density of the equilibrium measure (\( \rho \) was previously only defined on \([-1, 1]\)). This is a conformal map from \( D_{tk} \) to a neighbourhood of 0, and its expansion as \( z \to t_k \) is given by

\[ f_{tk}(z) = 2\pi i \rho(t_k)(z - t_k)(1 + O(z - t_k)), \quad \text{as } z \to t_k. \quad (4.44) \]

The lenses in a neighbourhood of \( t_k \) are chosen such that \( f_{tk}(\gamma_t \cap D_{tk}) \subset \Gamma_t \cup \Gamma_2 \) and \( f_{tk}(\gamma_\gamma \cap D_{tk}) \subset \Gamma_6 \cup \Gamma_8 \), see Figure [2]. Let us define \( Q_{\gamma,tk}^2 = f_{tk}^{-1}(\Pi) \cap D_{tk} \), that is, it is the subset of \( D_{tk} \) that lies
outside the lenses in the upper half plane and which is mapped by \( f_{t_k} \) into a subset of \( \Pi \). All we need is to find the expression of \( P^{(t_k)} \) in the region \( Q_{\Pi, t_k}^R \). This was done in \cite{[3]} equation (4.48) and below (5.2) for Gaussian-type weights. It is straightforward to adapt the construction in our situations, and we omit the details. For \( z \in Q_{\Pi, t_k}^R \), \( P^{(t_k)}(z) \) is given by

\[
P^{(t_k)}(z) = E_{t_k}(z) \times \left( \frac{\Gamma(1 + \frac{\alpha}{2})}{\Gamma(1 + \alpha_k)} G(1 + \frac{\alpha}{2} + \beta_k, \alpha; n f_{t_k}(z)) e^{-\frac{\alpha z}{2}} - \frac{\Gamma(1 + \frac{\alpha}{2} - \beta_k)}{\Gamma(1 + \alpha_k)} H(1 + \frac{\alpha}{2} - \beta_k, \alpha_k; n f_{t_k}(z) e^{-\pi i}) \right) \times (z - t_k) \frac{\alpha - \sigma_1 - \sigma_3 e^{-\pi i} e^{-\sqrt{z}}}{\sigma_1 - \sigma_3 e^{-\pi i} e^{-\sqrt{z}} - \frac{\alpha}{2}},
\]

where \( G \) and \( H \) are given in terms of the Whittaker functions (see \cite{[18], Chapter 13}):

\[
G(\alpha, \beta, z) = \frac{M_{\alpha, \beta}(z)}{\sqrt{z}}, \quad H(\alpha, \beta, z) = \frac{W_{\alpha, \beta}(z)}{\sqrt{z}}, \quad \mu = \frac{\alpha}{2}, \quad \kappa = \frac{1}{2}, \quad \frac{\alpha}{2} - a.
\]

The function \( E_{t_k} \) is analytic in \( D_{t_k} \) (see \cite{[3]} (4.49)-(4.51)) and its value at \( t_k \) is given by

\[
E_{t_k}(t_k) = \frac{D_{t_k}^{(2)}}{2 \sqrt{1 - t_k^2}} \left( i \left( e^{-\frac{\pi i}{4} \sqrt{1 + t_k} + e^{\frac{\pi i}{4} \sqrt{1 - t_k}} e^{-\frac{\pi i}{4} \sqrt{1 + t_k} + e^{\frac{\pi i}{4} \sqrt{1 - t_k}}} \right) \Lambda_k \right)\]

where

\[
\Lambda_k = e^{\frac{\pi i}{4} \sqrt{1 - t_k}} D_{W, t_k}(t_k)^{-1} e^{\frac{\pi i}{4} (4 \pi n(t_k)n(1 - t_k^2))^{\beta_k}} \prod_{1 \leq j \neq k \leq m} T_{k_j}^{-\beta_j},
\]

and

\[
\lambda_k = A \arccos t_k - \frac{\pi}{2} \alpha_k - \sum_{j=k+1}^{m+1} \pi \alpha_j + 2 \pi n \int_{t_k}^{1} \rho(s) ds.
\]

Also, we need a more detailed knowledge of the asymptotics \cite{[1, 26]}. By \cite{[3]} equation (4.52), we have

\[
P^{(t_k)}(z) P^{(\infty)}(z)^{-1} = I + \frac{v_k}{n f_{t_k}(z)} E_{t_k}(z) \left( -1 - \frac{1}{\tau(\alpha_k, -\beta_k)} \right) E_{t_k}(z)^{-1} + O(n^{-2 + 2|R_{s_k}|}),
\]

uniformly for \( z \in \partial D_{t_k} \) as \( n \to \infty \), where \( v_k = \beta_k^2 - \frac{\alpha_k^2}{4} \) and \( \tau(\alpha_k, \beta_k) = \frac{-\Gamma(\frac{\alpha_k}{2} - \beta_k)}{\Gamma(\frac{\alpha_k}{2} + \beta_k + 1)} \).

Figure 2: The neighborhood \( D_{t_k} \) and its image under the mapping \( f_{t_k} \).
4.6 Local parametrix near 1

The local parametrix near 1 cannot be treated for both Laguerre-type and Jacobi-type weights simultaneously, since 1 is a soft edge for Laguerre-type weights, and a hard edge for Jacobi-type weights. At a soft edge, the construction relies on the Airy model RH problem (whose solution is denoted $\Phi_{Ai}$), and at a hard edge on the Bessel model RH problem (whose solution is denoted $\Phi_{Be}$). For the reader’s convenience, we recall these model RH problems in the appendix.

**Laguerre-type weights**

Let us define $f_1(z) = (\frac{\pi}{2} \xi(z))^2/3$. This is a conformal map in $\mathcal{D}_1$ whose expansion as $z \to 1$ is given by

$$f_1(z) = \left( \frac{\pi \psi(1)}{\sqrt{2}} \right)^{2/3} (z - 1) \left( 1 - \frac{1}{10} \left( 1 - 4 \frac{\psi'(1)}{\psi(1)} \right) (z - 1) + O((z - 1)^2) \right).$$

(4.51)

The lenses $\gamma_+$ and $\gamma_-$ in a neighborhood of 1 are chosen such that $f_1(\gamma_+ \cap \mathcal{D}_1) \subset e^{\frac{2\pi i}{3}} \mathbb{R}^+$ and $f_1(\gamma_- \cap \mathcal{D}_1) \subset e^{-\frac{2\pi i}{3}} \mathbb{R}^+$. The local parametrix is given by

$$P^{(1)}(z) = E_1(z)\Phi_{Ai}(n^{2/3} f_1(z))\omega(z)^{-\frac{2\pi i}{3}} e^{-n\xi(z)\sigma_3} e^{-\frac{W(z)}{2} \sigma_3},$$

(4.52)

where $E_1$ is analytic in $\mathcal{D}_1$ and given by

$$E_1(z) = P^{(\infty)}(z) e^{\frac{W(z)}{2} \sigma_3} \omega(z)^{\frac{2\pi i}{3}} N^{-1} f_1(z)^{\frac{2\pi i}{3}} n^{\frac{2\pi i}{3}}, \quad N = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix},$$

(4.53)

and $\Phi_{Ai}(z)$ is the solution to the Airy model RH problem presented in the appendix (see Subsection 9.1). Using (9.2), we obtain a more detailed description of the matching condition (4.26):

$$P^{(1)}(z)P^{(\infty)}(z)^{-1} = I + \frac{P^{(\infty)}(z) e^{-\frac{W(z)}{2} \sigma_3} \omega(z)^{\frac{2\pi i}{3}}}{8nf_1(z)^{3/2}} \begin{pmatrix} i & i \\ -i & i \end{pmatrix} \omega(z)^{-\frac{2\pi i}{3}} e^{-\frac{W(z)}{2} \sigma_3} P^{(\infty)}(z)^{-1} + O(n^{-2})$$

(4.54)

uniformly for $z \in \partial \mathcal{D}_1$ as $n \to \infty$.

**Jacobi-type weights**

In this case we define $f_1(z) = \xi(z)/2$. This is a conformal map in $\mathcal{D}_1$ whose expansion as $z \to 1$ is given by

$$f_1(z) = \left( \frac{\pi}{\sqrt{2}} \psi(1) \right)^{2/3} (z - 1) \left( 1 + \frac{2}{3} \frac{\psi'(1)}{\psi(1)} \right) (z - 1) + O((z - 1)^2).$$

(4.55)

The lenses $\gamma_+$ and $\gamma_-$ in a neighborhood of 1 are again chosen such that $f_1(\gamma_+ \cap \mathcal{D}_1) \subset e^{\frac{2\pi i}{3}} \mathbb{R}^+$ and $f_1(\gamma_- \cap \mathcal{D}_1) \subset e^{-\frac{2\pi i}{3}} \mathbb{R}^+$. The local parametrix is given by

$$P^{(1)}(z) = E_1(z)\Phi_{Be}(n^{2/3} f_1(z); \alpha_{m+1})\omega_1(z)^{-\frac{2\pi i}{3}} (z - 1)^{-\frac{\alpha_{m+1}}{2} \sigma_3} e^{-n\xi(z)\sigma_3} e^{-\frac{W(z)}{2} \sigma_3},$$

(4.56)

where the principal branch is taken for $(z - 1)^{-\frac{\alpha_{m+1}}{2} \sigma_3}$, $\Phi_{Be}(z)$ is the solution to the Bessel model RH problem presented in Subsection 9.2 and $E_1$ is analytic in $\mathcal{D}_1$ and given by

$$E_1(z) = P^{(\infty)}(z) e^{\frac{W(z)}{2} \sigma_3} (z - 1)^{-\frac{\alpha_{m+1}}{2} \sigma_3} \omega_1(z)^{\frac{2\pi i}{3}} N^{-1} (2\pi n f(z)^{1/2})^{\frac{2\pi i}{3}}.$$

(4.57)
In this case, using (9.7), the matching condition (4.26) can be written as

\[ p^{(1)}(z)p^{(\infty)}(z)^{-1} = I + \frac{p^{(\infty)}(z)e^{\frac{W(z)}{2}w_1(z)}\frac{2}{R(z)}}{16nf_1(z)^{1/2}} \times \left( -\frac{(1 + 4\alpha_{m+1}^2)}{2i} \right) e^{-\frac{w_1(z)}{2}}p^{(\infty)}(z)^{-1} + O(n^{-2}), \quad (4.58) \]

uniformly for \( z \in \partial D \) as \( n \to \infty \).

### 4.7 Local parametrix near \(-1\)

Since Laguerre-type and Jacobi-type weights both have a hard edge at \(-1\), the construction of this local parametrix can be treated simultaneously for both cases, the only difference being in the conformal map. This map is defined by \( f_{-1}(z) = -\xi(z) - i\pi/2 \), and its expansion as \( z \to -1 \) is given by

\[ f_{-1}(z) = \begin{cases} \left( \sqrt{2}\pi\psi(-1)^2(z + 1) \right) (1 + \frac{2\psi(-1)}{3\psi(-1) - 1} + 0((z + 1)^2)), & \text{for Laguerre-type weights,} \\ \left( \frac{\pi}{2}\psi(-1)^2(z + 1) \right) (1 + \frac{2\psi(-1)}{3\psi(-1) + 1} + 0((z + 1)^2)), & \text{for Jacobi-type weights.} \end{cases} \quad (4.59) \]

The local parametrix is given by

\[ p^{(-1)}(z) = E_{-1}(z)\sigma_3\Phi_{\Omega}(z)\sigma_3w_{-1}(z)^{-\frac{\omega}{2}}(-z - 1)^{-\frac{\omega}{2}\sigma_3e^{-n\xi(z)\sigma_3}}, \quad (4.60) \]

where the principal branch is chosen for \(-z - 1)^{-\frac{\omega}{2}\sigma_3\), and \( E_{-1} \) is analytic in \( \mathcal{D}_{-1} \) and given by

\[ E_{-1}(z) = (-1)^n p^{(\infty)}(z)e^{\frac{W(z)}{2}w_1(z)}\frac{2}{R(z)}(-z - 1)^{-\frac{\omega}{2}\sigma_3}N(2\pi n(-f_{-1}(z))^{1/2})\frac{2}{R(z)}. \quad (4.61) \]

For Laguerre-type weights with \( W \equiv 0 \), by taking the limit \( z \to -1 \) in (4.61) (from e.g. the upper half plane) and using the asymptotics (4.40) - (4.41) we have

\[ E_{-1}(z) = (-1)^n D^n \left( N + \begin{pmatrix} 0 & n \\ 0 & \frac{\sqrt{2}(A + B_{-1})}{2} \end{pmatrix} \right) N(2\pi n(-f_{-1}(z))^{1/2})\frac{2}{R(z)}. \quad (4.62) \]

Furthermore, as \( n \to \infty \), we have

\[ p^{(-1)}(z)p^{(\infty)}(z)^{-1} = I + \frac{p^{(\infty)}(z)e^{\frac{W(z)}{2}w_1(z)}\frac{2}{R(z)}}{16nf_{-1}(z)^{1/2}} \times \left( -\frac{(1 + 4\alpha_0^2)}{2i} \right) e^{-\frac{w_1(z)}{2}}p^{(\infty)}(z)^{-1} + O(n^{-2}), \quad (4.63) \]

uniformly for \( z \in \partial \mathcal{D}_{-1} \).

### 4.8 Small norm RH problem

We are now in a position to do the last transformation. We recall that the disks are nonoverlapping. Using the parametrices, we define the matrix valued function \( R \) as

\[ R(z) = \begin{cases} S(z)p^{(\infty)}(z)^{-1}, & \text{if } z \in \mathbb{C} \setminus \bigcup_{j=0}^{m+1} \overline{\mathcal{D}_j}, \\ S(z)p^{(j)}(z)^{-1}, & \text{if } z \in \mathcal{D}_j, \quad j = 0, \ldots, m + 1. \end{cases} \quad (4.64) \]
We recall that the local parametrix has the same jumps as $S$ inside the disks and also that the global parametrix has the same jumps as $S$ on $(-1, 1)$, hence $R$ has jumps only on the contour $\Sigma_R$ depicted in Figure 3, where the orientation of the jump contour on $\partial D_j$ is chosen to be clockwise. Since $P(t_j)$ and $S$ have the same asymptotic behavior near $t_j$, $j = 0, \ldots, m + 1$, $R$ is bounded at these points. Therefore, it satisfies the following RH problem.

**RH problem for $R$**

(a) $R : \mathbb{C} \setminus \Sigma_R \to \mathbb{C}^{2 \times 2}$ is analytic.

(b) $R$ satisfies $R_+ (z) = R_- (z) J_R (z)$ for $z$ on $\Sigma_R \setminus \{\text{intersection points of } \Sigma_R\}$ with

$$J_R (z) = \begin{cases} 
    p(t_j)(z) p(\infty)(z)^{-1} & z \in \partial D_j, \\
    p(\infty)(z) J_S(z) p(\infty)(z)^{-1} & z \in \Sigma_R \setminus \cup_{j=0}^{m+1} \partial D_j,
\end{cases} \quad (4.65)$$

where $J_S(z) := S^{-1}(z) S_+(z)$ is given in (4.21)–(4.23).

(c) As $z \to \infty$, $R(z) = I + R_1 z^{-1} + \mathcal{O}(z^{-2})$ for a certain matrix $R_1$ independent of $z$. As $z \to z_\ast \in \{\text{intersection points of } \Sigma_R\}$, $R(z)$ is bounded.

We recall that outside fixed neighbourhoods of $t_j$, $j = 0, \ldots, m + 1$, the jumps for $S$ on $\gamma_+ \cup \gamma_-$ and on $\tilde{T} \setminus [-1, 1]$ are exponentially and uniformly close to the identity matrix (see the discussion at the end of Subsection 4.3). Therefore, from (4.50), (4.54), (4.58), (4.63) and (4.65), as $n \to \infty$ we have

$$J_R(z) = \begin{cases} 
    I + \mathcal{O}(e^{-cn}), & \text{uniformly for } z \in \Sigma_R \cap (\gamma^+ \cup \gamma^-), \\
    I + \mathcal{O}(n^{-1}), & \text{uniformly for } z \in \partial D_1 \cup \partial D_{-1}, \\
    I + \mathcal{O}(n^{-1+2|3\gamma|_k}), & \text{uniformly for } z \in \partial D_{kj}, k = 1, \ldots, m,
\end{cases} \quad (4.66)$$

for a positive constant $c$. By standard theory of small-norm RH problems (see e.g. [6, 7]), $R$ exists for sufficiently large $n$ (we also refer to [15] for very similar situations with more details provided). Furthermore, for any $r \in \mathbb{N}$, as $n \to \infty$, $R$ has an expansion given by

$$R(z) = I + \sum_{j=1}^{r} \frac{R^{(j)}(z)}{n^j} + R^{(r+1)}(z)n^{-r-1}, \quad (4.67)$$

$$R^{(j)}(z) = \mathcal{O}(n^{2\beta_{\max}}), \quad R^{(j)}(z)' = \mathcal{O}(n^{2\beta_{\max}}), \quad R^{(r+1)}(z) = \mathcal{O}(n^{2\beta_{\max}}), \quad R^{(r+1)}(z)' = \mathcal{O}(n^{2\beta_{\max}}),$$

uniformly for $z \in \mathbb{C} \setminus \Sigma_R$, uniformly for $(\bar{\alpha}, \bar{\beta})$ in any fixed compact set, and uniformly in $\tilde{T}$ if there exists $\delta > 0$, independent of $n$, such that

$$\min_{j \neq k}|t_j - t_k|, |t_j - 1|, |t_j + 1| \geq \delta. \quad (4.68)$$

Furthermore, in the way as done in [3], we show that

$$\partial_n R^{(j)}(z) = \mathcal{O}(n^{2\beta_{\max} \log n}), \quad \partial_n R^{(r+1)}(z) = \mathcal{O}(n^{2\beta_{\max} \log n}) \quad (4.69)$$

for $\nu \in \{\alpha_0, \alpha_1, \ldots, \alpha_{m+1}, \beta_1, \ldots, \beta_m\}$. From (4.50), (4.51), (4.58), (4.63), we show that $J_R$ admits an expansion as $n \to +\infty$ of the form

$$J_R(z) = I + \sum_{j=1}^{r} \frac{J^{(j)}_R(z)}{n^j} + \mathcal{O}(n^{-r-1+2\beta_{\max}}), \quad J^{(j)}_R(z) = \mathcal{O}(n^{2\beta_{\max}}), \quad (4.70)$$
uniformly for \( z \in \bigcup_{j=0}^{m+1} \partial D_j \). The matrices \( R^{(j)} \) are obtained in a recursive way via the Plemelj-Sokhotski formula (for instance see [16]), in particular one has

\[
R^{(1)}(z) = \sum_{j=0}^{m+1} \frac{1}{2\pi i} \int_{\partial D_j} \frac{J^{(1)}_R(s)}{s - z} ds,
\]

where we recall that the orientation on \( \partial D_j \) is clockwise. The goal for the rest of this section is to explicitly compute \( R^{(1)} \) in the case \( W \equiv 0 \) for Laguerre-type and Jacobi-type weights.

### Laguerre-type weights

From [4.50], [4.54], and [4.63] we easily show that \( J^{(1)}_R \) has a double pole at 1 and a simple pole at \( t_j \), \( j = 0, \ldots, m \). Therefore \( R^{(1)}(z) \) can be explicitly computed from (4.71) via a residue calculation.

For \( z \in \mathbb{C} \setminus \bigcup_{j=0}^{m+1} \partial D_j \), we have

\[
R^{(1)}(z) = \sum_{j=1}^{m} \frac{1}{z - t_j} \text{Res}(J^{(1)}_R(s), s = t_j) + \frac{1}{z + 1} \text{Res}(J^{(1)}_R(s), s = -1) + \frac{1}{z - 1} \text{Res}(J^{(1)}_R(s), s = 1) + \frac{1}{(z - 1)^2} \text{Res}((s - 1)J^{(1)}_R(s), s = 1).
\]

The residue at \( t_k \) can be computed from [4.50] (in the same way as in [3, eq (4.82)])

\[
\text{Res}(J^{(1)}_R(z), z = t_k) = \frac{v_k D^{\sigma}_2}{2\pi \rho(t_k) \sqrt{1 - t_k^2}} \left( t_k + \tilde{\Lambda}_{I,k} \quad -i - i\tilde{\Lambda}_{R,2,k} \right) D^{-\sigma}_1,
\]

where

\[
\tilde{\Lambda}_{I,k} = \frac{\tau(\alpha_k, \beta_k) \Lambda_k^2 - \tau(\alpha_k, -\beta_k) \Lambda_k^{-2}}{2i},
\]

\[
\tilde{\Lambda}_{R,1,k} = \frac{\tau(\alpha_k, \beta_k) \Lambda_k^2 e^{i \arcsin t_k} + \tau(\alpha_k, -\beta_k) \Lambda_k^{-2} e^{-i \arcsin t_k}}{2},
\]

\[
\tilde{\Lambda}_{R,2,k} = \frac{\tau(\alpha_k, \beta_k) \Lambda_k^2 e^{-i \arcsin t_k} + \tau(\alpha_k, -\beta_k) \Lambda_k^{-2} e^{i \arcsin t_k}}{2}.
\]

Furthermore, we note the following relation:

\[
\tilde{\Lambda}_{R,1,k} - \tilde{\Lambda}_{R,2,k} = -2t_k \tilde{\Lambda}_{I,k}.
\]
Now let us compute the other terms in (4.72). We compute the residue at \(-1\) from (4.29), (4.40), (4.41), (4.59) and (4.63), and we find
\[
\text{Res}(J_R^{(1)}(z), z = -1) = \frac{1 - 4a_3^2}{2^3\pi \psi(-1)} D_\infty^\sigma \begin{pmatrix} -i & 1 \\ 1 & -i \end{pmatrix} D_\infty^{-\sigma_3}.
\] (4.78)

Similarly, from (4.29), (4.38), (4.39), (4.51) and (4.54) we obtain
\[
\text{Res}((z - 1)J_R^{(1)}(z), z = 1) = \frac{5}{2^4 \pi \psi(1)} D_\infty^\sigma \begin{pmatrix} -1 & i \\ 1 & 1 \end{pmatrix} D_\infty^{-\sigma_3},
\] (4.79)

and
\[
\text{Res}(J_R^{(1)}(z), z = 1) = \frac{D_\infty^\sigma}{2^3\pi \psi(1)} \times
\begin{pmatrix}
-4(A - B_1)^2 + 1 + 2\psi'(1)
4i \left((A - B_1)^2 + 2(A - B_1) + \frac{11}{12} - \frac{1}{2} \psi'(1)\right)
4i \left((A - B_1)^2 - 2(A - B_1) - \frac{1}{2} \psi'(1)\right)
\end{pmatrix}
\] (4.80)

\[
D_\infty^{-\sigma_3}.
\]

The quantity \(R^{(1)}(-1)\) will also play an important role in Section 6. From another residue calculation, we obtain
\[
R^{(1)}(-1) = \sum_{j=1}^m \frac{-1}{1 + t_j} \text{Res}(J_R^{(1)}(s), s = t_j) - \text{Res}(J_R^{(1)}(s), s = -1) + \frac{1}{2} \text{Res}(j_R^{(1)}(s), s = 1) + \frac{1}{4} \text{Res}((s - 1)J_R^{(1)}(s), s = 1).
\] (4.81)

In (4.73), (4.79) and (4.80) we have already computed the above residues at \(t_1, \ldots, t_m\) and at 1, the other residue at \(-1\) can be computed from (4.29), (4.40), (4.41), (4.59) and (4.63), from which we obtain:
\[
\text{Res}(J_R^{(1)}(s)/s + 1, s = -1) = \frac{D_\infty^\sigma}{2^4 \pi \psi(-1)} \begin{pmatrix}
\frac{5}{2} (A + B_1)^2 - 2a_2^2 - 1 + \frac{1-4a_2^2}{4} \psi'(1)
4i \left(\frac{5}{2} (A + B_1)^2 - 3(A + B_1) + a_2^2 + \frac{9}{4} + \frac{1-4a_2^2}{4} \psi'(1)\right)
\cdots
\end{pmatrix}
\] (4.82)

Jacobi-type weights

In this case \(J_R^{(1)}(z)\) has simple poles at all \(t_j, j = 0, 1, \ldots, m + 1\) as can be seen from (4.60), (4.63), and (4.63). For \(z\) outside all of the disks \(D_{t_j}, j = 0, 1, \ldots, m + 1\), we have
\[
R^{(1)}(z) = \sum_{j=1}^m \frac{1}{z - t_j} \text{Res}(J_R^{(1)}(s), s = t_j) + \frac{1}{z + 1} \text{Res}(J_R^{(1)}(s), s = -1) + \frac{1}{z - 1} \text{Res}(J_R^{(1)}(s), s = 1).
\] (4.83)

Here the residue at \(t_k\) is again given by (4.73) (with \(\rho\) given by (4.11)). The residues at \(-1\) can be computed from (4.29), (4.40), (4.41), (4.59) and (4.63) and is given by
\[
\text{Res}(J_R^{(1)}(z), z = -1) = \frac{1 - 4a_3^2}{2^3 \pi \psi(-1)} D_\infty^\sigma \begin{pmatrix} -1 & -i \\ 1 & -i \end{pmatrix} D_\infty^{-\sigma_3}.
\] (4.84)
Similarly, from \((4.29), (4.38), (4.39), (4.55)\) and \((4.58)\) we obtain the residue at 1:

\[
\text{Res}(J_R^{(1)}(z), z = 1) = \frac{1 - 4\alpha_m^2 + 1}{24\pi \psi(1)} D^{\alpha_m^2}(1) D^{-\alpha_m^2}.
\]

(4.85)

5 Starting points of integration

Since we will find large \(n\) asymptotics only for the logarithmic derivative of Hankel determinant, we still face the classical problem of finding a good starting point for the integration. It turns out that in our case, it can be obtained by a direct computation, using some known results in the literature concerning standard Laguerre and Jacobi polynomials, and using the formula (3.21).

Lemma 5.1 As \(n \to \infty\), we have

\[
\log L_n((\alpha_0, 0, \ldots, 0), 0, 2(x + 1), 0) = \left(-\frac{3}{2} - \log 2\right) n^2 + (\log(2\pi) - \alpha_0(1 + \log 2)) n + \left(\frac{\alpha^2_0}{2} - \frac{1}{6}\right) \log n + \frac{\alpha_0}{2} \log(2\pi) + 2\zeta'(-1) \log G(1 + \alpha_0 + O(n^{-1}).
\]

(5.1)

As \(n \to \infty\), we have

\[
\log J_n((\alpha_0, 0, \ldots, 0, \alpha_m + 1), 0, 0, 0) = -n^2 \log 2 + [(1 - \alpha_0 - \alpha_m + 1) \log 2 + \log \pi] n + 2\alpha_m^2 + 2\alpha_m^2 + 1 \log n - \log(1 + \alpha_0(0 + \alpha_m + 1)) + 3\zeta'(-1) + \left(\frac{1}{12} \left(\frac{\alpha_0 + \alpha_m + 1}{2}\right)^2\right) \log 2 + \frac{\alpha_0 + \alpha_m + 1}{2} \log(2\pi) + O(n^{-1}).
\]

(5.2)

Proof. From [20] equations (5.1.1) and (5.1.8), the orthonormal polynomials of degree \(k\) with respect to the weight \(e^{-x}x^{\alpha_0}\) (supported on \((0, \infty)\)) has a leading coefficient given by

\[
\frac{(-1)^k}{\sqrt{k!} \Gamma(k + \alpha_0 + 1)}.
\]

Therefore, by a simple change of variables, the degree \(k\) orthonormal polynomials with respect to the weight \((x + 1)\alpha_0 e^{-2n(x+1)}\) (supported on \((-1, \infty)\)) has a leading coefficient given by

\[
\frac{(-1)^k (2n)^{k+1} \sqrt{\alpha_0}}{\sqrt{k!} \Gamma(k + \alpha_0 + 1)}.
\]

By applying formula \((3.21)\) for this weight, one obtains that

\[
L_n((\alpha_0, 0, \ldots, 0), 0, 2(x + 1), 0) = (2n)^{-n(n+\alpha_0)} \prod_{k=1}^n \Gamma(k + \alpha_0) \Gamma(k) = (2n)^{-n(n+\alpha_0)} \frac{G(n + 1) G(n + \alpha_0 + 1)}{G(1 + \alpha_0)},
\]

(5.3)

where we have used \(G(z+1) = \Gamma(z) G(z)\). The Barne’s \(G\)-function has a known asymptotics for large argument [see [18] eq (5.17.5)]. As \(z \to \infty\) with \(|\arg z| < \pi\), we have

\[
\log G(z+1) = z^2 + z \log \Gamma(z+1) - \left(\frac{z(z+1)}{2} + \frac{1}{12}\right) \log z - \frac{1}{12} + \zeta'(-1) + O(z^{-2}).
\]

(5.4)
The asymptotics of \( \log \Gamma(z) \) is given by

\[
\log \Gamma(z) = (z - \frac{1}{2}) \log z - z + \frac{1}{2} \log(2\pi) + \frac{1}{12z} + O(z^{-3}), \quad \text{as } z \to \infty, \quad |\arg z| < \pi, \quad (5.5)
\]

(see [18 eq (5.11.1)]). We obtain \((5.1)\) by using the above asymptotic formulas in \((5.3)\). Similarly, from [20 equations (4.3.3) and (4.21.6)], the degree \(k\) orthonormal polynomial with respect to the weight \((1 - x)^{n\alpha_0}(1 + x)^{\alpha_0}\) has a leading coefficient given by

\[
\frac{2^{-k} \sqrt{2k + \alpha_0 + \alpha_{m+1} + 1} \Gamma(2k + \alpha_0 + \alpha_{m+1} + 1)}{\sqrt{2\alpha_0 + \alpha_{m+1} + 1} \Gamma(k + 1) \Gamma(k + \alpha_0 + \alpha_{m+1} + 1) \Gamma(k + \alpha_0 + \alpha_{m+1} + 1)}
\]

By applying formula \((3.21)\) to this weight, one obtains

\[
J_n((\alpha_0, 0, \ldots, 0, \alpha_{m+1}), 0, 0, 0) = 2^{n^2 + n(\alpha_0 + \alpha_{m+1})} \prod_{k=0}^{n-1} \frac{\Gamma(k + 1) \Gamma(k + \alpha_0 + 1) \Gamma(k + \alpha_0 + \alpha_{m+1} + 1)}{\Gamma(2k + \alpha_0 + \alpha_{m+1} + 1) \Gamma(2k + \alpha_0 + \alpha_{m+1} + 2)}. \quad (5.6)
\]

Using the functional equation \(G(z + 1) = \Gamma(z)G(z)\) we can simplify the above product. We obtain

\[
J_n((\alpha_0, 0, \ldots, 0, \alpha_{m+1}), 0, 0, 0) = 2^{n^2 + n(\alpha_0 + \alpha_{m+1})} \frac{G(n + 1)G(n + \alpha_0 + 1)G(n + \alpha_0 + \alpha_{m+1} + 1)G(n + \alpha_0 + \alpha_{m+1} + 1)}{G(1 + \alpha_0)G(1 + \alpha_{m+1})G(2n + \alpha_0 + \alpha_{m+1} + 1)}. \quad (5.6)
\]

We obtain \((5.2)\) by expanding \((5.6)\) as \(n \to +\infty\), using the asymptotic formulas \((5.1)\) and \((5.5)\).

As mentioned in the outline, large \(n\) asymptotics for \(J_n(\vec{\alpha}, \vec{\beta}, 0, 0)\) are known in the literature, and we reproduce the precise statement here.

**Theorem 5.2** (Deift-Its-Krasovsky [8]). As \(n \to \infty\), we have

\[
\log J_n(\vec{\alpha}, \vec{\beta}, 0, 0) = \left[ 2n \sum_{j=1}^{m} \beta_j \arcsin t_j - A \log 2 \right] n + \left[ \frac{\alpha_0^2 + \alpha_{m+1}^2}{2} + \sum_{j=1}^{m} \left( \frac{\alpha_j^2}{4} - \beta_j^2 \right) \right] \log n
\]

\[
+ iA \sum_{j=1}^{m} \beta_j \arcsin t_j + i \frac{\pi}{2} \sum_{0 \leq j < k \leq m+1} \left( \alpha_k \beta_j - \alpha_j \beta_k \right) + \frac{\alpha_0 + \alpha_{m+1}}{2} \log(2\pi) - \frac{\alpha_0^2 + \alpha_{m+1}^2}{2} \log 2
\]

\[
+ \sum_{0 \leq j < k \leq m+1} \log \left( (1 - t_j t_k - \sqrt{(1 - t_j^2)(1 - t_k^2)})^{2\beta_j \beta_k} \right) \prod_{j=1}^{m} \log \left( G(1 + \frac{\alpha_0^2 + \beta_j^2}{4})G(1 + \frac{\alpha_0^2 + \beta_j^2}{4} - \beta_j) \right)
\]

\[
- \sum_{j=1}^{m} \left( \frac{\alpha_j^2}{4} + \beta_j^2 \right) \log(\sqrt{1 - t_j^2}) - \log(G(1 + \alpha_0)G(1 + \alpha_{m+1})) - \sum_{j=1}^{m} \beta_j^2 \log 2, \quad (5.7)
\]

where \(A = \sum_{j=0}^{m+1} \alpha_j\).

**Remark 5.3** The asymptotics \((5.7)\) with \(\vec{\beta} = \vec{0}\) and \(\alpha_1 = \ldots = \alpha_m = 0\) is consistent with \((5.2)\).

**Remark 5.4** Our notation differs slightly from the one used in [8]: \(\alpha_j\) and \(\beta_j\) in our paper corresponds to \(2\alpha_{m+1-j}\) and \(\beta_{m+1-j}\) in the paper [8].

The goal of the next section is to obtain a similar formula as \((5.7)\) for \(L_n(\vec{\alpha}, \vec{\beta}, 2(x+1), 0)\).
6 Integration in $\tilde{\alpha}$ and $\tilde{\beta}$ for the Laguerre weight

In this section, we specialize to the classical Laguerre weight with FH singularities

$$w(x) = e^{-2n(x+1)}\omega(x),$$ (6.1)

supported on $\mathcal{I} = [-1, +\infty)$. In this case, we recall that $\ell = 2 + 2 \log 2$ and $\psi(x) = \frac{1}{x}$. We will find large $n$ asymptotics for the differential identity (3.13), and then integrate in the parameters $\alpha_0, \ldots, \alpha_m, \beta_1, \ldots, \beta_m$. We first focus on finding large $n$ asymptotics for $\tilde{Y}(t_k)$, $k = 0, \ldots, m$.

**Proposition 6.1** For $k \in \{1, \ldots, m\}$, as $n \to +\infty$, we have

$$\tilde{Y}(t_k) = e^{-\frac{2}{n}\sigma_3}(I + O(n^{-1+2\beta_{max}}))E_{t_k}(k)\Phi_{k,11}\Phi_{k,12}\Phi_{k,21}\Phi_{k,22}e^{n(t_k+1)\sigma_3},$$ (6.2)

where

$$\Phi_{k,11} = \frac{\Gamma(1 + \frac{a}{\alpha_k} - \beta_k)}{\Gamma(1 + \alpha_k)}\left(2n\sqrt{1 - t_k}\right)^{\frac{\alpha}{\omega}}\omega^{-\frac{1}{4}}(t_k), \quad \Phi_{k,12} = \frac{-\alpha_k\Gamma(\alpha_k)}{\Gamma(\frac{a}{\omega} + \beta_k)}\left(2n\sqrt{1 - t_k}\right)^{-\frac{2}{\omega}}\omega^{-\frac{1}{4}}(t_k),$$

$$\Phi_{k,21} = \frac{\Gamma(1 + \frac{a}{\alpha_k} + \beta_k)}{\Gamma(1 + \alpha_k)}\left(2n\sqrt{1 - t_k}\right)^{\frac{\alpha}{\omega}}\omega^{-\frac{1}{4}}(t_k), \quad \Phi_{k,22} = \frac{\alpha_k\Gamma(\alpha_k)}{\Gamma(\frac{a}{\omega} - \beta_k)}\left(2n\sqrt{1 - t_k}\right)^{-\frac{2}{\omega}}\omega^{-\frac{1}{4}}(t_k).$$

As $n \to +\infty$, we have

$$\tilde{Y}(-1) = e^{-\frac{2}{n}\sigma_3}\left(I + \frac{R^{(1)}(-1)}{n}\right) + O(n^{-2+2\beta_{max}})\right)E_{-1}(-1)\left(\Phi_{0,11}\Phi_{0,12}\Phi_{0,21}\Phi_{0,22}\right),$$ (6.4)

where $R^{(1)}(-1)$ is given explicitly in (6.3) and

$$\Phi_{0,11} = \frac{1}{\Gamma(1 + \alpha_0)}\left(\sqrt{2n}\right)^{\alpha_0}\omega^{-\frac{1}{4}}(-1), \quad \Phi_{0,12} = \frac{-i\alpha_0\Gamma(\alpha_0)}{2\pi}\left(\sqrt{2n}\right)^{-\alpha_0}\omega^{-\frac{1}{4}}(-1),$$

$$\Phi_{0,21} = \frac{-i\pi\alpha_0}{\Gamma(1 + \alpha_0)}\left(\sqrt{2n}\right)^{\alpha_0}\omega^{-\frac{1}{4}}(-1), \quad \Phi_{0,22} = \frac{\alpha_0^2\Gamma(\alpha_0)}{2}\left(\sqrt{2n}\right)^{-\alpha_0}\omega^{-\frac{1}{4}}(-1).$$ (6.5)

**Proof.** For fixed $1 \leq k \leq m$, let $z \in D_{t_k} \cap Q^R_{-2, k}$ be outside the lenses. By inverting the RH transformations $Y \mapsto T \mapsto S \mapsto R$, we obtain

$$Y(z) = e^{-\frac{2}{n}\sigma_3}R(z)P^{(t_k)}(z)e^{n\sigma_3(z)}e^{\frac{2}{n}\sigma_3}$$ (6.6)

where $P^{(t_k)}(z)$ is given by (6.5). From [18, Section 13.14(iii)], we have

$$G(a, \alpha_k; z) = z^{\frac{a}{\omega}}(1 + O(z)), \quad z \to 0,$$ (6.7)

and, if $\alpha_k \neq 0$, and $a - \frac{a}{\omega} \pm \frac{a}{\omega} \neq 0$, then $z \to 0$ we have

$$H(a, \alpha_k; z) = \begin{cases} \frac{\Gamma(\alpha_k)}{\Gamma(a)}z^{\frac{a}{\omega}} + O(z^{1 - \frac{a}{\omega}}) + O(z^{\frac{a}{\omega}}) & \Re \alpha_k > 0, \\ \frac{\Gamma(-\alpha_k)}{\Gamma(a - \alpha_k)}z^{\frac{a}{\omega}} + \frac{\Gamma(\alpha_k)}{\Gamma(a)}z^{\frac{a}{\omega}} + O(z^{1 + \frac{a}{\omega}}) & -1 < \Re \alpha_k \leq 0. \end{cases}$$ (6.8)
From (2.5) and (3.12), we have

\[ Bessel functions \Rightarrow (3.9), (4.11), (4.67), (6.6) \text{ and } (6.9). \]

We extend it for general parameters \( \alpha \) given by \( P(1), \kappa \alpha \phi_k^2 + \frac{\pi}{2} \omega_k \). Recalling that \( 1 = \lim_{z \to 0} \Gamma(z) = 1 \) and \( \kappa \alpha \phi_k^2 + \frac{\pi}{2} \omega_k \geq 0 \) for the modified Bessel functions, we have

\[ \Phi_{k,11} \alpha_1 e^{-\frac{\pi}{2} \omega_k (t_k)} = e^{2n(t_k+1)\alpha_k} \]

where

\[ c_k = \frac{\Gamma(-\alpha_k) \Gamma(1 + \alpha_k) e^{-\frac{\pi \alpha_k}{2}} \omega_k(t_k)}{\Gamma(1 - \frac{\alpha_k}{2})} = \frac{\pi \alpha_k e^{i\pi \beta_k} e^{-i\pi \alpha_k} e^{-i\pi \beta_k}}{2\pi i} \omega_k(t_k) = e^{2n(t_k+1)\alpha_k} \]

and \( c_k \) is given by (6.1). The claim (6.2) for \( \alpha_k \neq 0, -\beta_k \pm \frac{\pi}{2} \neq 0, -1, -2, \ldots \) follows from (6.3). We extend it for general parameters \( \alpha_k \) and \( \beta_k \) (still subject to the constraint \( \Re \alpha_k > 0 \) and \( \Re \beta_k \in (-\frac{1}{2}, \frac{1}{2}) \)) by continuity of \( \bar{Y}(t_k) \) in \( \alpha_k \) and \( \beta_k \) (this can be shown by a simple contour deformation, see e.g. [15] eq (29) and below). Now we turn to the proof of (6.4). For \( z \in D_1 \setminus (\Omega + \Omega) \), from Section 4 we have

\[ Y(z) = e^{-\frac{\pi}{2} \omega z} R(z) P(-1)(z) e^{n\omega(z)\sigma_3} e^{\frac{i\pi \omega}{2}}. \]

In this region, by (4.10) and (6.6), \( P(-1)(z) \) is given by

\[ P(-1)(z) = E_{-1}(z) \sigma_3 \left( \frac{I_{\alpha_k}(2n(-f_{-1}(z)) \frac{1}{2})}{2\pi i n(-f_{-1}(z))} \right) - 2n(-f_{-1}(z)) \frac{1}{2} K_{\alpha_k}(2n(-f_{-1}(z)) \frac{1}{2}) \right) \]

\[ \times \sigma_3 \omega_{-1}(z) \frac{1}{2} \left( -z - 1 \right)^{\frac{\alpha_k}{2}} + \alpha_0 e^{\frac{i\pi \omega}{2}}. \]

From Section 10.30(i), we have the following asymptotic behaviors as \( z \to 0 \) for the modified Bessel functions

\[ I_{\alpha_0}(z) = \frac{1}{\Gamma(\alpha_0 + 1)} \left( \frac{z}{2} \right)^{\alpha_0} (1 + O(z^2)), \]

\[ K_{\alpha_0}(z) = \begin{cases} \frac{\Gamma(\alpha_0)}{\Gamma(\alpha_0 - \frac{1}{2})} z^{-\alpha_0} + \mathcal{O}(z^{1-R\alpha_0}) + \mathcal{O}(z^{R\alpha_0}), & \text{if } R\alpha_0 \geq 0, \alpha_0 \neq 0, \\ \frac{\Gamma(\alpha_0)}{\Gamma(\alpha_0 - \frac{1}{2})} z^{-\alpha_0} + \mathcal{O}(z^{2+R\alpha_0}), & \text{if } -1 < R\alpha_0 < 0. \end{cases} \]

Using (6.21), for \( \alpha_k \neq 0 \), we find that the leading terms of \( E_{-1}(z) P(-1)(z) e^{n\omega(z)\sigma_3} \) as \( z \to -1 \) are given by

\[ \left( \Phi_{0,11} \alpha_1 e^{-\frac{\pi}{2} \omega \alpha_1} \right) \left( \Phi_{0,21} \alpha_1 e^{-\frac{\pi}{2} \omega \alpha_1} \right) \]

where

\[ \tilde{c}_0 = \frac{\pi \alpha_0}{2 \sin(\pi \alpha_0)} \omega_{-1}(z) = e^{\pi \omega \alpha_0} \]

and where we recall that \( \tilde{c}_0 \) is defined in (6.4). This proves (6.4) for \( \alpha_0 \neq 0 \). The case \( \alpha_0 = 0 \) follows by continuity of \( \bar{Y}(-1) \).

6.1 Asymptotics for \( \partial_{\nu} \log L_n(\bar{\alpha}, \bar{\beta}, 2(x + 1), 0) \), \( \nu \in \{\alpha_0, \ldots, \alpha_m, \beta_1, \ldots, \beta_m\} \)

From (3.10) and (3.12), we have

\[ \kappa_n^{-1} = \lim_{z \to \infty} \frac{iY_{21}(z)}{2\pi z^{n-1}}, \quad \kappa_n^{-2} = -2\pi i \lim_{z \to \infty} z^{n+1} Y_{12}(z), \quad \eta_n = \lim_{z \to \infty} \frac{Y_{11}(z) - z^n}{z^{n-1}}. \]
Inverting the transformations $Y \mapsto T \mapsto S \mapsto R$ for $z \in \mathbb{C} \setminus \left( \Omega \cup \Omega^- \cup (T \setminus S) \cup \bigcup_{j=0}^{n-1} D_j \right)$ (i.e. outside the lenses and outside the disks) gives

$$Y(z) = e^{-\frac{2\pi i}{3} R(z)} P(\infty)(z) e^{\eta g(z) \sigma_3} e^{\frac{2\pi i}{3} \sigma_3}.$$  

(6.15)

From (6.12), (6.12), (6.167), (6.172) and (6.14), we find large $n$ asymptotic for $\kappa_{n-1}^2, \kappa_n^2$ and $\eta_n$. As $n \to +\infty$, we have

$$\kappa_{n-1}^2 = e^{2n^2(2(n-1)+A_{n-1})^{-1}} \exp \left( -2i \sum_{j=1}^{m} \beta_j \arcsin t_j \right) \left( 1 + \frac{R_{1,21}^{(1)}}{nP_{1,21}^{(\infty)}} \right) + O(n^{-2+2\beta_{\text{max}}}),$$  

(6.16)

where $A = \alpha_0 + \alpha_1 + \ldots + \alpha_m$ and

$$R_{1,21}^{(1)} = \sum_{j=1}^{m} v_j \left( 1 - \frac{\tilde{A}_{R,j}}{1 - t_j} \right) + \frac{1}{4} \left( (A - \tilde{B}_1)^2 + 2(A - \tilde{B}_1) + \frac{11}{12} \right).$$  

(6.17)

Similarly, for $\kappa_n^2$ we find

$$\kappa_n^2 = e^{2n^2(2n\alpha_{n-1})^{-1}} \exp \left( -2i \sum_{j=1}^{m} \beta_j \arcsin t_j \right) \left( 1 - \frac{R_{1,12}^{(1)}}{nP_{1,12}^{(\infty)}} + O(n^{-2+2\beta_{\text{max}}}) \right),$$  

(6.18)

as $n \to +\infty$, where by (6.172) we have

$$- \frac{R_{1,12}^{(1)}}{nP_{1,12}^{(\infty)}} = \sum_{j=1}^{m} v_j \left( 1 + \tilde{A}_{R,j} \right) \left( 1 - \frac{\tilde{A}_{R,j}}{1 - t_j} \right) + \frac{1}{4} \left( (A - \tilde{B}_1)^2 + 2(A - \tilde{B}_1) + \frac{11}{12} \right).$$  

(6.19)

Finally, for $\eta_n$ we obtain

$$\eta_n = \frac{n}{2} + P_{1,11}^{(\infty)} + \frac{R_{1,11}^{(1)}}{n} + O(n^{-2+2\beta_{\text{max}}}),$$  

as $n \to +\infty$, where $P_{1,11}^{(\infty)}$ is given by (6.124) and $R_{1,11}^{(1)}$ can be computed from (6.12) and is given by

$$R_{1,11}^{(1)} = \sum_{j=1}^{m} v_j \left( t_j + \tilde{A}_{I,j} \right) \left( 1 - \frac{\tilde{A}_{I,j}}{2(1 - t_j)} \right) - \frac{1}{32} + \frac{1}{32} \left( (A - \tilde{B}_1)^2 \right).$$  

(6.21)

Let $\nu \in \{ \alpha_0, \alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_m \}$. Then, from (6.169), (6.16), (6.18) and (6.20), we find that the large $n$ asymptotics of the first part of the differential identity (3.13) are given by

$$-(n + A) \partial_\nu \log(\kappa_n \kappa_{n-1}) + 2n \partial_\nu \eta_n = \partial_\nu \left( 2 \log D_{\infty} - \alpha_0 + \sum_{j=1}^{m} t_j \alpha_j + 2i \sum_{j=1}^{m} \sqrt{1 - t_j^2} \beta_j \right)n +$$

$$2A \partial_\nu \log D_{\infty} + \partial_\nu \left( \frac{\alpha_0^2}{2} \right) + \partial_\nu \sum_{j=1}^{m} v_j (\tilde{A}_{I,j} - 1) + O \left( \frac{n^{1-4\beta_{\text{max}}}}{n^{1-4\beta_{\text{max}}}} \right).$$  

(6.22)

Now we compute the second part of the differential identity (3.13). First, we compute the contributions from $t_j, j = 1, \ldots, m$ using (4.47), (6.69), (6.2) and (6.33). We obtain

$$\sum_{j=1}^{m} \left( \tilde{Y}_{22}(t_j) \partial_\nu Y_{11}(t_j) - \tilde{Y}_{12}(t_j) \partial_\nu Y_{21}(t_j) + Y_{11}(t_j) \tilde{Y}_{22}(t_j) \partial_\nu \log(\kappa_n \kappa_{n-1}) \right) =$$

$$-(A - \alpha_0) \partial_\nu \log D_{\infty} + \sum_{j=1}^{m} \left( \Phi_{j,22} \partial_\nu \Phi_{j,11} - \Phi_{j,12} \partial_\nu \Phi_{j,21} - 2 \beta_j \partial_\nu \log \Lambda_j \right) + O \left( \frac{n^{1-4\beta_{\text{max}}}}{n^{1-4\beta_{\text{max}}}} \right).$$  

(6.23)
Note that $E_{-1}(-1) = O(n^\frac{-2}{3})$ as $n \to +\infty$, while $E_{-k}(t_k) = O(n^{\frac{k-1}{2}})$, $k = 1, \ldots, m$. This makes the computations for the contribution from $-1$ more involved. From (6.22), (6.23) and (6.24), we obtain

$$
\tilde{Y}_{22}(-1)\partial_v Y_{11}(-1) - \tilde{Y}_{12}(-1)\partial_v Y_{21}(-1) + Y_{11}(-1)\tilde{Y}_{22}(-1)\partial_v \log(\kappa_n\kappa_{n-1}) = 
+ \partial_v \left( R^{(1)}_{11}(-1) - R^{(1)}_{22}(-1) + iD_{\infty}^{-2}R^{(1)}_{12}(-1) + iD_{\infty}^{-2}R^{(1)}_{21}(-1) + iD_{\infty}^{-2}R^{(1)}_{11} + iD_{\infty}^{-2}R^{(1)}_{21} \right) 
- \alpha_0 \partial_v \log D_{\infty} + \Phi_{0,22}\partial_v \Phi_{0,11} - \Phi_{0,12}\partial_v \Phi_{0,21} + O\left(\frac{\log n}{n^{1 - \frac{43}{15}}\max} \right). (6.24)
$$

We observe significant simplifications using (1.73), (1.74), (1.78), (1.80), (4.51), and (1.82):

$$
R^{(1)}_{11}(-1) - R^{(1)}_{22}(-1) + iD_{\infty}^{-2}R^{(1)}_{12}(-1) + iD_{\infty}^{-2}R^{(1)}_{21}(-1) + iD_{\infty}^{-2}R^{(1)}_{11} + iD_{\infty}^{-2}R^{(1)}_{21} = - \sum_{j=1}^{m} v_j \tilde{\Lambda}_{j,j}. (6.25)
$$

Adding (6.22), (6.23) and (6.24) yields

$$
\partial_v \log L_0(\alpha, \beta, 2(x + 1), 0) = \partial_v \left( 2\log D_{\infty} - \alpha_0 + \sum_{j=1}^{m} t_j \alpha_j + 2i\sum_{j=1}^{m} \sqrt{1 - t_j^2} \beta_j \right)n + A\partial_v \log D_{\infty} 
+ \partial_v \left( \frac{\alpha_0^2}{2} \right) + \sum_{j=0}^{m} \left( \Phi_{j,22}\partial_v \Phi_{j,11} - \Phi_{j,12}\partial_v \Phi_{j,21} \right) - \sum_{j=1}^{m} \partial_v v_j + 2\beta_j \partial_v \log \Lambda_j + O\left(\frac{\log n}{n^{1 - \frac{43}{15}}\max} \right), (6.26)
$$

as $n \to +\infty$. Now, we perform some computations to make the above asymptotic formula more explicit. From (6.5) and using the identity $z\Gamma(z) = \Gamma(z + 1)$ we have

$$
\Phi_{0,22}\partial_v \Phi_{0,11} - \Phi_{0,12}\partial_v \Phi_{0,21} = 
\frac{\alpha_0}{2} \partial_v \log \left( \frac{\alpha_0}{\Gamma(1 + \alpha_0)} \right) + \alpha_0 \log(\sqrt{2\pi}) \partial_v \alpha_0 - \frac{\alpha_0}{2} \partial_v \left( \sum_{\ell=1}^{m} \alpha_\ell \log(1 + t_\ell) + i\pi \sum_{\ell=1}^{m} \beta_\ell \right). (6.27)
$$

And from (6.3), after a long computation, for $1 \leq j \leq m$ we obtain

$$
\Phi_{j,22}\partial_v \Phi_{j,11} - \Phi_{j,12}\partial_v \Phi_{j,21} = \frac{\alpha_j}{2} \partial_v \log \left( \frac{\Gamma(1 + \frac{\alpha_j}{2} - \beta_j)\Gamma(1 + \frac{\alpha_j}{2} + \beta_j)}{\Gamma(1 + \alpha_j)^2} \right) + \frac{\alpha_j}{2} \log \left( 2\sqrt{\frac{1 - t_j}{1 + t_j}} \right) \partial_v \alpha_j 
+ \beta_j \partial_v \log \left( \frac{\Gamma(1 + \frac{\alpha_j}{2} + \beta_j)}{\Gamma(1 + \frac{\alpha_j}{2} - \beta_j)} \right) - \frac{\alpha_j}{2} \partial_v \left( \sum_{\ell=0}^{m} \alpha_\ell \log |t_\ell - t_j| - i\pi \sum_{\ell=1}^{m} \beta_\ell + i\pi \sum_{\ell=j+1}^{m} \beta_\ell \right). (6.28)
$$

Also, from (1.48) and (1.49), we have

$$
\partial_v \log \Lambda_j = \partial_v \left( \frac{iA}{2} \arccos t_j + \frac{\pi}{4} \alpha_j - \frac{\pi}{2} \sum_{\ell=j+1}^{m} \alpha_\ell + \beta_j \log(4\pi \rho(t_j)n(1 - t_j^2)) - \sum_{\ell=1}^{m} \beta_\ell \log T_{j,\ell} \right). (6.29)
$$
Substituting (6.21)–(6.29) into (6.26), and using the expression for \( D_\infty \) and \( v_j \) given by (4.33) and below (4.50), we obtain

\[
\partial_r \log L_n(\vec{\alpha}, \vec{\beta}, 2(x+1), 0) = \partial_r \left( \sum_{j=0}^{m} (t_j - \log 2)\alpha_j + 2i \sum_{j=1}^{m} \beta_j \left( \arcsin t_j + \sqrt{1 - t_j^2} \right) \right) n
\]

\[\]

\[
+ A \partial_r \left( i \sum_{j=1}^{m} \beta_j \arcsin t_j - \frac{A}{2} \log 2 \right) + \partial_r \left( \frac{\alpha_{0}^2}{2} \right) \partial_r \log \left( \frac{\alpha_{0}}{\Gamma(1 + \alpha_{0})^2} + \alpha_0 \log (\sqrt{2}n) \partial_r \alpha_0 \right)
\]

\[\]

\[
- \sum_{j=0}^{m} \frac{\alpha_j}{2} \partial_r \left( \sum_{\ell=0}^{j} \alpha_\ell \log |t_\ell - t_j| - i\pi \sum_{\ell=j+1}^{m} \beta_\ell + i\pi \right) \sum_{\ell=j+1}^{m} \beta_\ell + \frac{m}{2} \partial_r \left( 2n \sqrt{\frac{1 - t_j^2}{1 + t_j^2}} \right) \partial_r \alpha_j
\]

\[\]

\[
+ \sum_{j=1}^{m} \frac{\alpha_j}{2} \partial_r \log \left( \frac{\Gamma(1 + \frac{\alpha_{0}}{2} - \beta_j) \Gamma(1 + \frac{\alpha_{0}}{2} + \beta_j)}{\Gamma(1 + \alpha_j)^2} \right) + \sum_{j=1}^{m} \beta_j \partial_r \log \left( \frac{\Gamma(1 + \frac{\alpha_{0}}{2} + \beta_j)}{\Gamma(1 + \frac{\alpha_{0}}{2} - \beta_j)} \right) + \sum_{j=1}^{m} \partial_r \left( \frac{\alpha_j^2}{4} - \beta_j^2 \right)
\]

\[\]

\[
- \sum_{j=1}^{m} \frac{2\beta_j}{i} \partial_r \left( \frac{\pi i}{2} \alpha_j - \frac{\pi i}{2} \sum_{\ell=j+1}^{m} \alpha_\ell + \beta_j \log(4\pi \rho(t_j) n(1 - t_j^2)) - \sum_{\ell=1}^{m} \beta_\ell \log T_{j\ell} \right)
\]

\[\]

\[
+ O\left( \frac{\log n}{n^{1-4\delta_{\max}}} \right), \quad \text{as } n \to +\infty,
\]

where we recall that \( t_0 = -1 \). From the discussion in Subsection 4.8, the above error term is uniform for all \((\vec{\alpha}, \vec{\beta})\) in a given compact set \( \Omega \), and uniform in \( \vec{t} \) such that (4.63) holds. However, as stated in Proposition 7.2, the identity (6.30) itself is valid on the subset \( \Omega \setminus \Omega \) for which \( p_0, \ldots, p_n \) exist. From the determinantal representation of orthogonal polynomials, \( \Omega \) is locally finite and we can extend (6.30) for all \((\vec{\alpha}, \vec{\beta}) \in \Omega \) by continuity (for \( n \) large enough such that the r.h.s. exists). We refer to [15, 14, 8, 3] for very similar situations, with more details provided. Our goal for the rest of this section is to prove Proposition 6.2 below.

**Proposition 6.2** As \( n \to \infty \), we have

\[
\log \frac{L_n(\vec{\alpha}, \vec{\beta}, 2(x+1), 0)}{L_n(\vec{\alpha}_0, \vec{0}, 2(x+1), 0)} = \sum_{j=1}^{m} \beta_j \left( \arcsin t_j + \sqrt{1 - t_j^2} \right) + \sum_{j=1}^{m} (t_j - \log 2)\alpha_j n + \sum_{j=1}^{m} \frac{\alpha_j^2}{4} \log \left( \frac{n \sqrt{1 - t_j^2}}{\sqrt{1 + t_j^2}} \right)
\]

\[\]

\[
+ \sum_{j=1}^{m} \left( \frac{\alpha_j^2}{4} \log \left( \frac{n \sqrt{1 - t_j^2}}{\sqrt{1 + t_j^2}} \right) - \frac{\beta_j^2}{2} \log(4\pi \rho(t_j) n(1 - t_j^2)) \right) + \sum_{j=1}^{m} \log \left( \frac{G(1 + \frac{\alpha_{0}}{2} + \beta_j) G(1 + \frac{\alpha_{0}}{2} - \beta_j)}{G(1 + \alpha_j)^2} \right)
\]

\[\]

\[
- \frac{i}{2} \sum_{j=1}^{m} A_j \beta_j + i A \sum_{j=1}^{m} \beta_j \arcsin t_j + \sum_{1 \leq j < k \leq m} \beta_j \beta_k \log T_{jk} - \frac{2}{2} \sum_{0 \leq j < k \leq m} \alpha_j \alpha_k
\]

\[\]

\[
- \sum_{0 \leq j < k \leq m} \frac{\alpha_j \alpha_k}{2} \log |t_k - t_j| + \sum_{j=1}^{m} \log \left( \frac{G(1 + \frac{\alpha_{0}}{2})}{G(1 + \alpha_j)} \right) + O\left( \frac{\log n}{n^{1-4\delta_{\max}}} \right),
\]

where \( \log L_n(\vec{\alpha}_0, \vec{0}, 2(x+1), 0) \) is given by (5.1).

**6.2 Integration in \( \alpha_0 \)**

In this short subsection, we make a consistency check with (5.1). Let us set \( \alpha_1 = \ldots = \alpha_m = 0 = \beta_1 = \ldots = \beta_m \) and \( \nu = \alpha_0 \) in (6.30). With the notations \( \vec{\alpha}_0 = (\alpha_0, 0, \ldots, 0) \in \mathbb{C}^{m+1} \) and
\( \vec{0} = (0, \ldots, 0) \in \mathbb{C}^m \), this gives
\[
\partial_{\alpha_0} \log L_n(\vec{a}_0, \vec{0}, 2(x + 1), 0) = -(1 + \log 2)n - \frac{\log 2}{2} \alpha_0 + \alpha_0 + \frac{\alpha_0}{2} \partial_{\alpha_0} \log \frac{\alpha_0}{\Gamma(1 + \alpha_0)^2} + \alpha_0 \log(\sqrt{n}) + \mathcal{O}\left(\frac{\log n}{n}\right) \tag{6.32}
\]
as \( n \to +\infty \). Integrating (6.32) from \( \alpha_0 = 0 \) to an arbitrary \( \alpha_0 \), we obtain
\[
\log \frac{L_n(\vec{a}_0, \vec{0}, 2(x + 1), 0)}{L_n(\vec{0}, \vec{0}, 2(x + 1), 0)} = -(1 + \log 2)\alpha_0 n + \frac{\alpha_0^2}{2} \left(1 - \frac{\log 2}{2}\right) + \int_0^{\alpha_0} \frac{x}{2} \partial_x \log \frac{x}{\Gamma(1 + x)^2} dx + \frac{\alpha_0^2}{2} \log(\sqrt{n}) + \mathcal{O}\left(\frac{\log n}{n}\right) \tag{6.33}
\]
From [18, formula 5.17.4], we have
\[
\int_0^z \log \Gamma(1 + x) dx = \frac{z}{2} \log 2\pi - \frac{z(z + 1)}{2} + z \log \Gamma(z + 1) - \log G(z + 1), \tag{6.34}
\]
where \( G \) is Barnes’ \( G \)-function. Therefore, after an integration by parts, (6.33) can be rewritten as
\[
\log \frac{L_n(\vec{a}_0, \vec{0}, 2(x + 1), 0)}{L_n(\vec{0}, \vec{0}, 2(x + 1), 0)} = -(1 + \log 2)\alpha_0 n + \frac{\alpha_0^2}{2} \log n + \frac{\alpha_0}{2} \log 2\pi - \log G(1 + \alpha_0) + \mathcal{O}\left(\frac{\log n}{n}\right),
\]
which is consistent with (5.1).

### 6.3 Integration in \( \alpha_1, \ldots, \alpha_m \)

We set \( \alpha_2 = \ldots = \alpha_m = 0 = \beta_1 = \ldots = \beta_m \) and \( \nu = \alpha_1 \) in (6.30). With the notation \( \vec{\alpha}_1 = (\alpha_0, \alpha_1, 0, \ldots, 0) \in \mathbb{C}^{m+1} \), we obtain
\[
\partial_{\alpha_1} \log L_n(\vec{\alpha}_1, \vec{0}, 2(x + 1), 0) = (t_1 - \log 2)n - \frac{\log 2}{2} \alpha_0 - \frac{\alpha_0}{2} \log |t_1 - t_0| + \frac{\alpha_1}{2} \log \left(\frac{1}{\sqrt{1 + t_1}}\right) + \alpha_1 \partial_{\alpha_1} \log \frac{\Gamma(1 + \frac{\alpha_1}{2})}{\Gamma(1 + \alpha_1)} + \mathcal{O}\left(\frac{\log n}{n}\right), \tag{6.35}
\]
as \( n \to +\infty \). Using integration by parts and (6.34) we obtain the following relation
\[
\int_0^z x \partial_x \log \frac{\Gamma(1 + \frac{z}{2})}{\Gamma(1 + x)} dx = -\frac{z^2}{4} + \log \frac{G(1 + \frac{z}{2})^2}{G(1 + z)}. \tag{6.36}
\]
Using (6.36), we integrate (6.35) from \( \alpha_1 = 0 \) to an arbitrary \( \alpha_1 \). We get
\[
\log \frac{L_n(\vec{\alpha}_1, \vec{0}, 2(x + 1), 0)}{L_n(\vec{0}, \vec{0}, 2(x + 1), 0)} = (t_1 - \log 2)\alpha_1 n - \frac{\log 2}{2} \alpha_0 \alpha_1 - \frac{\alpha_0 \alpha_1}{2} \log |t_1 - t_0| + \frac{\alpha_1^2}{4} \log \left(\frac{1}{\sqrt{1 + t_1}}\right) + \log \frac{G(1 + \frac{\alpha_1}{2})^2}{G(1 + \alpha_1)} + \mathcal{O}\left(\frac{\log n}{n}\right), \quad \text{as } n \to +\infty. \tag{6.37}
\]
We proceed in a similar way for the other variables, by integrating successively in \(\alpha_2, \alpha_3, \ldots, \alpha_m\). At the last step, setting \(\beta_1 = \ldots = \beta_m = 0\) and \(\nu = \alpha_m\) in (6.30), we obtain

\[
\partial_{\alpha_m} \log L_n(\bar{\alpha}, \bar{\beta}, 2(x+1), 0) = (t_m - \log 2)n - \frac{\log 2}{2}(A - \alpha_m) - \sum_{j=0}^{m-1} \frac{\alpha_j}{2} \log |t_m - t_j| + \frac{\alpha_m}{2} \log \left( n \frac{\sqrt{1 - t_m}}{\sqrt{1 + t_m}} \right) + \alpha_m \partial_{\alpha_m} \log \frac{\Gamma(1 + \frac{t_m}{2})}{\Gamma(1 + \alpha_m)} + \frac{\alpha_m}{2} + O\left( \frac{\log n}{n} \right). \tag{6.38}
\]

Integrating (6.38) from \(\alpha_m = 0\) to an arbitrary \(\alpha_m\) using again (6.36), and with the notation \(\bar{\alpha}_{m-1} = (\alpha_0, \ldots, \alpha_{m-1}, 0)\), we obtain

\[
\log \frac{L_n(\bar{\alpha}, \bar{\beta}, 2(x+1), 0)}{L_n(\bar{\alpha}_{m-1}, 0, 2(x+1), 0)} = (t_m - \log 2)\alpha_m n - \frac{\log 2}{2} \sum_{j=0}^{m-1} \alpha_j \alpha_m - \sum_{j=0}^{m-1} \frac{\alpha_j \alpha_m}{2} \log |t_m - t_j| + \frac{\alpha_m^2}{4} \log \left( n \frac{\sqrt{1 - t_m}}{\sqrt{1 + t_m}} \right) + \log \frac{G(1 + \frac{2\alpha_m}{2})^2}{G(1 + \alpha_m)} + O\left( \frac{\log n}{n} \right), \tag{6.39}
\]

as \(n \to +\infty\). Summing the contributions of each step, we arrive at

\[
\log \frac{L_n(\bar{\alpha}, \bar{\beta}, 2(x+1), 0)}{L_n(\bar{\alpha}_0, 0, 2(x+1), 0)} = \sum_{j=1}^{m} (d_j - \log 2)\alpha_j n - \frac{\log 2}{2} \sum_{0 \leq j \leq k \leq m} \alpha_j \alpha_k - \sum_{0 \leq j < k \leq m} \frac{\alpha_j \alpha_k}{2} \log |t_k - t_j| + \sum_{j=1}^{m} \frac{\alpha_j^2}{4} \log \left( n \frac{\sqrt{1 - t_j}}{\sqrt{1 + t_j}} \right) + \sum_{j=1}^{m} \log \frac{G(1 + \frac{\alpha_j}{2})^2}{G(1 + \alpha_j)} + O\left( \frac{\log n}{n} \right), \tag{6.40}
\]

as \(n \to +\infty\).

### 6.4 Integration in \(\beta_1, \ldots, \beta_m\)

For convenience, we introduce the notation

\[
A_k = \sum_{j=0}^{k-1} \alpha_j - \sum_{j=k+1}^{m} \alpha_j, \quad k = 0, 1, \ldots, m. \tag{6.41}
\]

We set \(\beta_2 = \ldots = \beta_m = 0\) and \(\nu = \beta_1\) in (6.30). With the notation \(\bar{\beta}_1 = (\beta_1, 0, \ldots, 0)\), we have

\[
\partial_{\beta_1} \log L_n(\bar{\alpha}, \bar{\beta}_1, 2(x+1), 0) = 2i(\arcsin t_1 + \sqrt{1 - t_1^2})n + iA \arcsin t_1 - \frac{i\pi}{2} A_1 + \frac{\alpha_1}{2} \partial_{\beta_1} \log \Gamma(1 + \frac{\alpha_1}{2} - \beta_1) + \beta_1 \partial_{\beta_1} \log \frac{\Gamma(1 + \frac{\alpha_1}{2} + \beta_1)}{\Gamma(1 + \frac{\alpha_1}{2} - \beta_1)} - 2\beta_1
\]

\[
- 2\beta_1 \log \left( 4\pi \rho(t_1)n(1 - t_1^2) \right) + O\left( \frac{\log n}{n^{1-4\beta_{\max}}} \right). \tag{6.42}
\]

After some computations using (6.34), we obtain

\[
\int_0^{\beta_1} \left( \frac{\alpha_1}{2} \partial_x \log \Gamma(1 + \frac{\alpha_1}{2} - x) \Gamma(1 + \frac{\alpha_1}{2} + x) + x \partial_x \log \frac{\Gamma(1 + \frac{\alpha_1}{2} + x)}{\Gamma(1 + \frac{\alpha_1}{2} - x)} \right) dx
\]

\[
= \log \frac{G(1 + \frac{\alpha_1}{2} + \beta_1)G(1 + \frac{\alpha_1}{2} - \beta_1)}{G(1 + \frac{\alpha_1}{2})^2}. \tag{6.43}
\]
Integrating (6.42) from $\beta = 0$ to an arbitrary $\beta$, we obtain

\[
\log \frac{L_n(\vec{\alpha}, \vec{\beta}, 2(x+1), 0)}{L_n(\vec{\alpha}, 0, 2(x+1), 0)} = 2i\beta \left( \arcsin t_1 + \sqrt{1 - t_1^2} \right)n + iA \beta \arcsin t_1 - \frac{i\pi}{2} \mathcal{A}_1 \beta_1
+ \log \frac{G(1 + \frac{\alpha}{2} + \beta_1)G(1 + \frac{\alpha}{2} - \beta_1)}{G(1 + \frac{\alpha}{2})^2} - \beta_1^2 \log(4\pi\rho(t_1)(1 - t_1^2)) + \mathcal{O}\left(\frac{\log n}{n^{1-4\beta_{\text{max}}}}\right). \tag{6.44}
\]

We integrate successively in $\beta_2, \ldots, \beta_m$. At the last step, we set $\nu = \beta_m$ in (6.30), which gives

\[
\partial_{\beta_m} \log L_n(\vec{\alpha}, \vec{\beta}, 2(x+1), 0) = 2i\left( \arcsin t_m + \sqrt{1 - t_m^2} \right)n + iA \arcsin t_m - \frac{i\pi}{2} \mathcal{A}_m \beta_m
+ \frac{\alpha_m}{2} \partial_{\beta_m} \log \Gamma(1 + \frac{\alpha_m}{2} - \beta_m)\Gamma(1 + \frac{\alpha_m}{2} + \beta_m) + \beta_m \partial_{\beta_m} \log \frac{\Gamma(1 + \frac{\alpha_m}{2} + \beta_m)}{\Gamma(1 + \frac{\alpha_m}{2} - \beta_m)} - 2\beta_m
+ \sum_{j=1}^{m-1} 2\beta_j \log T_{jm} - 2\beta_m \log (4\pi\rho(t_m)(1 - t_m^2)) + \mathcal{O}\left(\frac{\log n}{n^{1-4\beta_{\text{max}}}}\right), \tag{6.45}
\]

as $n \to +\infty$. Integrating (6.45) from $\beta_m = 0$ to an arbitrary $\beta_m$, using the notation $\vec{\beta}_{m-1} = (\beta_1, \ldots, \beta_{m-1}, 0)$, we obtain

\[
\log \frac{L_n(\vec{\alpha}, \vec{\beta}, 2(x+1), 0)}{L_n(\vec{\alpha}, \vec{\beta}_{m-1}, 2(x+1), 0)} = 2i\beta_m \left( \arcsin t_m + \sqrt{1 - t_m^2} \right)n + iA \beta_m \arcsin t_m - \frac{i\pi}{2} \mathcal{A}_m \beta_m
+ \log \frac{G(1 + \frac{\alpha_m}{2} + \beta_m)G(1 + \frac{\alpha_m}{2} - \beta_m)}{G(1 + \frac{\alpha_m}{2})^2} - \beta_m^2 \log(4\pi\rho(t_m)(1 - t_m^2))
+ \sum_{j=1}^{m-1} 2\beta_j \beta_m \log T_{jm} + \mathcal{O}\left(\frac{\log n}{n^{1-4\beta_{\text{max}}}}\right). \tag{6.46}
\]

Summing all the contributions, as $n \to +\infty$ we obtain

\[
\log \frac{L_n(\vec{\alpha}, \vec{\beta}, 2(x+1), 0)}{L_n(\vec{\alpha}, 0, 2(x+1), 0)} = 2i n \sum_{j=1}^{m} \beta_j \left( \arcsin t_j + \sqrt{1 - t_j^2} \right) + iA \sum_{j=1}^{m} \beta_j \arcsin t_j
- \frac{i\pi}{2} \sum_{j=1}^{m} \mathcal{A}_j \beta_j + \sum_{j=1}^{m} \log \frac{G(1 + \frac{\alpha_j}{2} + \beta_j)G(1 + \frac{\alpha_j}{2} - \beta_j)}{G(1 + \frac{\alpha_j}{2})^2} - \sum_{j=1}^{m} \beta_j^2 \log(4\pi\rho(t_j)(1 - t_j^2))
+ 2 \sum_{1 \leq j < k \leq m} \beta_j \beta_k \log T_{jk} + \mathcal{O}\left(\frac{\log n}{n^{1-4\beta_{\text{max}}}}\right). \tag{6.47}
\]

The claim of Proposition 6.2 follows now by summing (6.40) and (6.47) using the definition of $\mathcal{A}_j$ given in (6.41).
7 Integration in $V$

In this section, we obtain asymptotics for general Laguerre-type and Jacobi-type weights by means of a deformation parameter $s$ and by using the analysis of Section 4 for the weight

$$w_s(x) = e^{-nV_s(x)} \omega(x),$$

(7.1)

where we emphasize in the notation the dependence in $s$. We specify in Subsection 7.1 the exact deformations we consider. In Subsection 7.2, we adapt several identities from [1] (that are valid for Gaussian-type weights) for our situations. Finally, we proceed with the integration in $s$ for Laguerre-type and Jacobi-type weights in Subsection 7.3 and Subsection 7.4, respectively.

### 7.1 Deformation parameters $s$

Inspired by [1, 3], for each $s \in [0, 1]$, we define

$$V_s(x) = (1-s)^2(x+1) + sV(x),$$

for Laguerre-type weights, (7.2)

$$V_s(x) = sV(x),$$

for Jacobi-type weights. (7.3)

If $s = 0$, we already know large $n$ asymptotics for the associated Hankel determinants (from Section 6 and the result of [8], see Proposition 6.2 and Theorem 5.2). It follows easily from (1.4)-(1.5) that $V_s$ is one-cut regular for each $s \in [0, 1]$, and the associated density $\psi_s$ and Euler-Lagrange constant $\ell_s$ are given by

$$\psi_s(x) = (1-s) \frac{1}{\pi} + s\psi(x), \quad \ell_s = (1-s)(2 + 2 \log 2) + s\ell,$$

(7.4)

$$\psi_s(x) = (1-s) \frac{1}{\pi} + s\psi(x), \quad \ell_s = (1-s) \log 2 + s\ell,$$

(7.5)

where the first and second lines read for Laguerre-type and Jacobi-type weights respectively. We will use the differential identities

$$\partial_s \log L_n(\vec{\alpha}, \vec{\beta}, V_s, 0) = \frac{1}{2\pi i} \int_{-1}^{+\infty} [Y^{-1}(x)Y'(x)]_{21} \partial_s w_s(x)dx,$$

(7.6)

$$\partial_s \log J_n(\vec{\alpha}, \vec{\beta}, V_s, 0) = \frac{1}{2\pi i} \int_{-1}^{1} [Y^{-1}(x)Y'(x)]_{21} \partial_s w_s(x)dx,$$

(7.7)

which were obtained in Proposition 3.4. Our objective in this section is to compute asymptotics of these differential identities, and finally integrate them in the parameter $s$ from 0 to 1.

### 7.2 Some identities

We generalize here several formulas of [1] (valid only for Gaussian-type potentials) for all three-types of canonical one-cut regular potentials. Most of the proofs are minor modifications of those done in [1].

**Lemma 7.1** For $t \in [-1, 1]$, we have

$$\int_{-1}^{1} \frac{V'(x)\sqrt{1-x^2}}{x-t} dx = -2\pi + 2\pi^2 \sqrt{1-t^2} \rho(t),$$

(7.8)

$$\int_{t}^{1} \rho(x)dx = \frac{\sqrt{1-t^2}}{2\pi^2} \int_{-1}^{1} \frac{V(x)}{t-x} dx + \frac{1}{\pi} \arccos t.$$

(7.9)
Proof. The proof goes as in [1] Lemma 5.8. Let $H : \mathbb{C} \setminus [-1,1] \to \mathbb{C}$ be defined by

$$H(z) = 2\pi\sqrt{z-1}\sqrt{z+1} \int_{-1}^{1} \frac{\rho(x)}{x-z} dx + \int_{-1}^{1} \frac{V'(x)\sqrt{1-x^2}}{x-z} dx$$

(7.10)

where the principal branches are chosen for $\sqrt{z-1}$ and $\sqrt{z+1}$. For $t \in (-1,1)$, one can check that $H_+(t) = H_-(t)$. Also $H$ is bounded at $\pm 1$ and $H(\infty) = -2\pi$; so Liouville’s theorem implies that $H(z) = -2\pi$. Considering $H_+(t) + H_-(t)$ for $t \in (-1,1)$ yields (7.8). Now, (7.9) follows from (7.8) and the following identity which is proved in [1] eq (5.18) and below

$$\sqrt{1-t^2} \int_{-1}^{1} \frac{V(x)}{t-x} \frac{dx}{\sqrt{1-x^2}} = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \left( \int_{1}^{1} \frac{V'(y)}{y-x} \frac{dy}{\sqrt{1-y^2}} \right) dx.$$  

(7.11)

Lemma 7.2 Let $C$ be a closed curve surrounding $[-1,1]$ in the clockwise direction, let $a(z) = \sqrt{\frac{z+1}{z-1}}$ be analytic on $\mathbb{C} \setminus [-1,1]$ such that $a(z) \sim 1$ as $z \to \infty$, and let $f$ be analytic in a neighbourhood of $[-1,1]$. We have

$$\frac{1}{2\pi i} \int_{C} \left[ \frac{a^2(z)}{a^2(t_j)} + \frac{a^2(t_j)}{a^2(z)} \right] \frac{f(z)}{(z-t_j)^2} dz = \frac{2}{\pi i} \int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}} dx,$$

(7.12)

$$\frac{1}{2\pi i} \int_{C} \left[ \frac{a^2(z)}{a^2(t_j)} - \frac{a^2(t_j)}{a^2(z)} \right] \frac{f(z)}{(z-t_j)^2} dz = \frac{2}{\pi i} \int_{1}^{1} \frac{f(x)}{(t_j-x)\sqrt{1-x^2}} dx.$$  

(7.13)

Proof. The proof is the same as in [1] equations (5.22)-(5.23) and above. □

Applying Lemma 7.2 to $f = \partial_s V_s$ (with $V_s$ given by (7.3)–(7.5)–(7.3)), and then simplifying using Lemma 7.1, we obtain

$$\int_{C} \left[ \frac{a^2(z)}{a^2(t_j)} + \frac{a^2(t_j)}{a^2(z)} \right] \frac{\partial_s V_s(z)}{(z-t_j)^2} dz = \begin{cases} 
8\pi^2 \left( \psi(t_j) - \frac{1}{\pi} \right) \sqrt{1-t_j} \sqrt{1+t_j} & \text{for Laguerre-type potentials} \\
8\pi^2 \left( \psi(t_j) - \frac{1}{\pi} \right) & \text{for Jacobi-type potentials}
\end{cases}$$

(7.14)

and

$$\int_{C} \left[ \frac{a^2(z)}{a^2(t_j)} - \frac{a^2(t_j)}{a^2(z)} \right] \frac{\partial_s V_s(z)}{(z-t_j)^2} dz = \begin{cases} 
8\pi^2 \int_{1}^{1} \left( \psi(x) - \frac{1}{\pi} \right) \sqrt{1-x} dx, & \text{for Laguerre-type potentials} \\
8\pi^2 \int_{1}^{1} \left( \psi(x) - \frac{1}{\pi} \right) \frac{1}{\sqrt{1-x}} dx, & \text{for Jacobi-type potentials}
\end{cases}$$

(7.15)

Lemma 7.3 Let $C$ be a closed curve surrounding $[-1,1]$ in the clockwise direction, let $a(z) = \sqrt{\frac{z+1}{z-1}}$ be analytic on $\mathbb{C} \setminus [-1,1]$ such that $a(z) \sim 1$ as $z \to \infty$, and let $f$ be analytic in a neighbourhood of
Proof. The proof of (7.16)–(7.18) is done in [1, Lemma 5.10], and the proof for (7.19) is similar.

7.3 Integration in $s$

Similarly, for Jacobi-type weights with $f(x) = \partial_s V_s = V(x) - 2(x + 1)$ with $V_s$ given by (7.2) for Laguerre-type potentials, and then simplifying using Lemma [14] we obtain

$$\int_{\mathbb{C}} \frac{a(z)^2}{(z-1)^2} \partial_s V_s(z) dz = 0,$$

(7.20)

$$\int_{\mathbb{C}} \frac{a(z)^2}{(z-1)^2} \partial_s V_s(z) dz = -\frac{4\pi^2 i}{3} \left( \psi(1) - \frac{1}{\pi} \right),$$

(7.21)

$$\int_{\mathbb{C}} \frac{a(z)^2}{(z-1)^2} \partial_s V_s(z) dz = -\frac{8\pi^2 i}{3} \left( \psi(1) - \frac{1}{\pi} \right),$$

(7.22)

$$\int_{\mathbb{C}} \frac{a(z)^2}{(z-1)^2} \partial_s V_s(z) dz = -\frac{8\pi^2 i}{3} \left( \psi(-1) - \frac{1}{\pi} \right).$$

(7.23)

Similarly, for Jacobi-type weights with $f(x) = \partial_s V_s = V(x)$ with $V_s$ given by (7.3) for Jacobi-type potentials, we obtain

$$\int_{\mathbb{C}} \frac{a(z)^2}{(z-1)^2} \partial_s V_s(z) dz = 4\pi^2 i \left( \psi(1) - \frac{1}{\pi} \right),$$

(7.24)

$$\int_{\mathbb{C}} \frac{a(z)^2}{(z+1)^2} \partial_s V_s(z) dz = -4\pi^2 i \left( \psi(-1) - \frac{1}{\pi} \right).$$

(7.25)

7.3 Integration in $s$ for Laguerre-type weights

In this subsection we prove Proposition [16] below.

Proposition 7.4 As $n \to +\infty$, we have

$$\log \frac{L_n(\alpha, \beta, V, 0)}{L_n(\alpha, \beta, 2(x+1), 0)} = -n^2 \frac{1}{2} \int_{-1}^{1} (V(x) - 2(x + 1)) \left( \frac{1}{\pi} + \psi(x) \right) \sqrt{\frac{1-x}{1+x}} dx$$

$$+ n \sum_{j=0}^{m} \frac{\alpha_j}{2} (V(t_j) - 2(1 + t_j)) + \frac{nA}{2\pi} \int_{-1}^{1} \frac{V(x) - 2(1+x)}{\sqrt{1-x^2}} dx - 2\pi n \sum_{j=1}^{m} \sum_{j=1}^{m} i\beta_j \int_{t_j}^{t_{j+1}} \left( \psi(x) - \frac{1}{\pi} \right) \sqrt{\frac{1-x}{1+x}} dx$$

$$+ \sum_{j=1}^{m} \left( \frac{\alpha_j^2}{4} - \beta_j^2 \right) \log(\pi \psi(t_j)) - \frac{1}{24} \log(\pi \psi(1)) - \frac{1}{8} n \sum_{j=1}^{m} \left( \frac{\alpha_j^2}{4} - \beta_j^2 \right) \log(\pi \psi(-1)) + O(n^{-1+\epsilon_{\text{max}}}).$$

(7.26)
Let \( C \) be a closed contour surrounding \([-1,1]\) and the lenses \( \gamma_+ \cup \gamma_- \), which is oriented clockwise and passes through \(-1-\varepsilon\) and \(1+\varepsilon\) for a certain \(\varepsilon > 0\). Using the jumps for \( Y \) given by (2.2), we rewrite the differential identity (7.28) as follows

\[
\partial_s \log L_n(\alpha, \beta, V, 0) = \int_{1+\varepsilon}^{1-\varepsilon} [Y^{-1}(x)Y'(x)]_{21} \partial_s w_s(x) \frac{dx}{2\pi i} = \frac{1}{2\pi i} \int_C [Y^{-1}(z)Y'(z)]_{11} \partial_s w_s(z) \frac{dz}{2\pi i}.
\]  

(7.27)

From (4.3), (4.7) and by inverting the transformations

\[
\begin{align*}
I_1,s &= \frac{-n}{2\pi i} \int_C g'(z) \partial_s \log w_s(z) dz, \\
I_2,s &= \frac{-1}{2\pi i} \int_C [p^{(\infty)}(z)^{-1} p^{(\infty)}(z)']_{11} \partial_s \log w_s(z) dz, \\
I_3,s &= \frac{-1}{2\pi i} \int_C [p^{(\infty)}(z)^{-1} R^{-1}(z) R'(z) p^{(\infty)}(z)]_{11} \partial_s \log w_s(z) dz.
\end{align*}
\]  

(7.28)

In exactly the same way as in [1, 3], we show from a detailed analysis of the Cauchy operator associated to \( R \) that the estimates in (4.67) hold uniformly for \((\alpha, \beta, s)\) in any fixed compact set \(\Omega\), and uniformly in \(s \in [0,1]\). However, from Proposition 3.4, the identity (7.28) itself is not valid for \(n \to +\infty\), for a positive constant \(c\), and that the integral over \( C \) can be decomposed into three integrals:

\[
\partial_s \log L_n(\alpha, \beta, V, 0) = I_{1,s} + I_{2,s} + I_{3,s} + O(e^{-cn}), \quad \text{as } n \to \infty,
\]

Note from (7.31) and (7.32) that \( \partial_s \log w_s(z) = -n \partial_s V_s(z) = -n(V(x) - 2(x+1)) \). Using the definition of \( g \) given by (4.2) and switching the order of integration, we get

\[
I_{1,s} = -n^2 \int_{-1}^{1} \rho(x) \partial_s V_s(x) dx = -n^2 \int_{-1}^{1} (V(x) - 2(x+1)){\left(1 - s \right)} \frac{1}{\pi} + s \psi(x) \sqrt{1 - x \over 1 + x} dx.
\]  

(7.29)

Therefore, we have

\[
\int_0^1 I_{1,s} ds = -n^2 \frac{2}{\pi} \int_{-1}^{1} (V(x) - 2(x+1)){\left(1 + \psi(x) \right)} \sqrt{1 - x \over 1 + x} dx.
\]  

(7.30)

From (4.29), (4.31), (4.32) and a contour deformation, we obtain the following expression for \( I_{2,s} \):

\[
I_{2,s} = n \sum_{j=0}^{m} {\alpha_j \over 2} (V(t_j) - 2(1 + t_j)) - nA \int_{-1}^{1} \frac{V(x) - 2(1 + x)}{\sqrt{1 - x^2}} dx
\]

\[
+ n \sum_{j=1}^{m} \frac{i\beta_j}{\pi} \sqrt{1 - t_j^2} \int_{-1}^{1} \frac{V(x) - 2(1 + x)}{\sqrt{1 - x^2}(x - t_j)} dx.
\]  

(7.31)

We simplify the last integral of (7.31) using (7.30):

\[
\sqrt{1 - t_j^2} \int_{-1}^{1} \frac{V(x) - 2(1 + x)}{\sqrt{1 - x^2}(x - t_j)} dx = -2\pi \int_{t_j}^{1} (\psi(x) - 1) \sqrt{1 - x \over 1 + x} dx.
\]  

(7.32)
Then, integrating in $s$ (note that $I_{2,s}$ is in fact independent of $s$), we obtain

$$
\int_0^1 I_{2,s} ds = n \sum_{j=0}^m \frac{\alpha_j}{2} \left( V(t_j) - 2(1 + t_j) \right) - \frac{nA}{2\pi} \int_{-1}^1 \frac{V(x) - 2(1 + x)}{\sqrt{1 - x^2}} dx
$$

\[ - 2\pi n \sum_{j=1}^m i\beta_j \int_{t_j}^1 \left( \psi(x) - \frac{1}{\pi} \right) \sqrt{\frac{1 - x}{1 + x}} dx. \tag{7.33} \]

Using the expansion of $R$ given by (4.61), we have

$$
I_{3,s} = \frac{1}{2\pi i} \int_C [P^{(\infty)}(z)^{-1} R^{(1)}(z) P^{(\infty)}(z)]_{11} \partial_s V_s(z) dz + O\left( n^{-1+4\beta_{\max}} \right), \quad \text{as } n \to \infty, \tag{7.34} \]

The leading term of $I_{3,s}$ can be written down more explicitly using the definition of $P^{(\infty)}$ given by (4.29), and we obtain

$$
I_{3,s} = \frac{1}{2\pi i} \int_C \left( \frac{a(z)^2 + a(z)^{-2}}{8} \left[ R^{(1)}(z)' - R^{(1)}(z) \right] + \frac{1}{2} \left[ R^{(1)}(z)' + R^{(1)}(z) \right] \right)
$$

\[ + \frac{1}{2} \left( 2 - \frac{a(z)^2 - a(z)^{-2}}{4} \right) \left[ R^{(1)}(z)'D^{-2} + R^{(1)}(z)'D^2 \right] \left( V(z) - 2(z + 1) \right) dz + O\left( n^{-1+4\beta_{\max}} \right). \tag{7.35} \]

From (4.72), (4.73), (4.78), (4.79) and (4.80) we have

$$
R^{(1)}(z) - R^{(1)}(z) = \sum_{j=1}^m \frac{1}{(z - t_j)^2} \frac{2\nu_j(t_j + \tilde{\Lambda}_{1,j})}{2\pi \rho_s(t_j) \sqrt{1 - t_j^2}} + \frac{1}{(z - 1)^2} \frac{223\pi \psi_s(1)}{2} \tag{7.36} \]

\[ + \frac{1}{(z - 1)^2} \frac{(A - \tilde{B}_j)^2 - \frac{1}{2} - \frac{\psi_s(1)}{2 \psi_s(1)}}{2^2 \pi \psi_s(1)} + \frac{1}{(z - 1)^2} \frac{1 - 4\alpha_0^2}{2^2 \pi \psi_s(1)}, \tag{7.37} \]

$$
R^{(1)}(z) + R^{(1)}(z) = 0, \tag{7.38} \]

\[ i[R^{(1)}(z)D^{-2} + R^{(1)}(z)D^2] = \sum_{j=1}^m \frac{1}{(z - t_j)^2} \frac{\nu_j(2 + \tilde{\Lambda}_{1,j} - \tilde{\Lambda}_{2,j})}{2\pi \rho_s(t_j) \sqrt{1 - t_j^2}} + \frac{1}{(z - 1)^2} \frac{223\pi \psi_s(1)}{2} \tag{7.39} \]

Therefore, from (7.35) - (7.38) and using the connection formula (4.77), we obtain

$$
I_{3,s} = \sum_{j=1}^m I_{3,s,t_j} + I_{3,s,1} + I_{3,s,-1} + O\left( n^{-1+4\beta_{\max}} \right), \quad \text{as } n \to \infty, \tag{7.39} \]

where

$$
I_{3,s,t_k} = \frac{-v_k}{8\pi^2 \rho_s(t_k)} \int_C \left[ \frac{a^2(z)}{a^2(z)} + \frac{a^2(z)}{a^2(z)} + \tilde{\Lambda}_{j,k} \left( \frac{a^2(z)}{a^2(z)} - \frac{a^2(z)}{a^2(z)} \right) \right] \partial_s V_s(z) (z - t_k)^2 dz, \tag{7.40} \]

$$
I_{3,s,1} = \int_C \left[ \frac{a^2(z)}{4\pi \psi_s(1)} \left( \frac{2(A - \tilde{B}_j)^2 + \frac{1}{2} - \frac{\psi_s(1)}{2 \psi_s(1)}}{2z^2} + \frac{5}{6} \frac{1}{(z - 1)^2} \right) + \frac{a^{-2}(z)}{4(z - 1)^2} \frac{7}{2^2 \pi \psi_s(1)} \right] \partial_s V_s(z) dz, \tag{7.41} \]

$$
I_{3,s,-1} = \int_C \left[ \frac{a^{-2}(z)}{4(z + 1)^2} \frac{1 - 4\alpha_0^2}{2^2 \psi_s(1)} \right] \partial_s V_s(z) dz. \tag{7.42} \]

39
Formulas (7.14) and (7.15) allow us to simplify $I_{3,s,t_k}$ as follows:

$$I_{3,s,t_k} = -\frac{v_k}{\psi_s(t_k)}(\psi(t_k) - \frac{1}{\pi}) - \frac{v_k \tilde{A}_{I,k}}{\rho_s(t_k)(1-t_k^2)} \int_{t_k}^{1} \left( \psi(x) - \frac{1}{\pi} \right) \sqrt{\frac{1-x}{1+x}} dx. \quad (7.40)$$

Integrating the above from $s = 0$ to $s = 1$, we have

$$\int_0^1 I_{3,s,t_k} ds = -v_k \log(\pi \psi(t_k)) - \frac{v_k}{1-t_k^2} \int_{t_k}^{1} \left( \psi(x) - \frac{1}{\pi} \right) \sqrt{\frac{1-x}{1+x}} dx \int_0^1 \frac{\tilde{A}_{I,k}}{\rho_s(t_k)} ds. \quad (7.41)$$

By the same argument as the one given in [3, equations (6.23) and (6.24)], the second term in the r.h.s of (7.41) is of order $O(n^{-1/2} \Re \alpha_{s,1})$ as $n \to +\infty$, that is,

$$\int_0^1 I_{3,s,t_k} ds = -v_k \log(\pi \psi(t_k)) + O(n^{-1/2} \Re \alpha_{s,1}). \quad (7.42)$$

We can also simplify the expression for $I_{3,s,1}$. Using the formulas (7.20)–(7.22), we obtain

$$I_{3,s,1} = -\frac{1}{24} \psi(1) - \frac{1}{\psi_s(1)}, \quad \text{and then} \quad \int_0^1 I_{3,s,1} ds = -\frac{1}{24} \log(\pi \psi(1)). \quad (7.43)$$

Similarly, using (7.23) we get

$$I_{3,s,-1} = -\frac{1-4\alpha_0^2 \psi(-1) - \frac{1}{\psi_s(-1)}}, \quad \text{and then} \quad \int_0^1 I_{3,s,-1} ds = -\frac{1-4\alpha_0^2}{8} \log(\pi \psi(-1)). \quad (7.44)$$

This finishes the proof of Proposition 7.4.

### 7.4 Jacobi-type weights

We prove here the analogue of Proposition 7.4 for Jacobi-type weights.

**Proposition 7.5** As $n \to \infty$, we have

$$\log J_n(\tilde{\alpha}, \tilde{\beta}, V, 0) = -\frac{n^2}{2} \int_{-1}^{1} \frac{V(x)}{\sqrt{1-x^2}} \left( \frac{1}{\pi} + \psi(x) \right) dx + n \sum_{j=0}^{m+1} \frac{\alpha_j}{2} V(t_j)$$

$$-\frac{nA}{2\pi} \int_{-1}^{1} \frac{V(x)}{\sqrt{1-x^2}} dx - 2\pi n \sum_{j=1}^{m} \left( \psi_j - \frac{1}{\pi} \right) \frac{dx}{\sqrt{1-x^2}} + \sum_{j=1}^{m} \left( \frac{\alpha_j^2}{4} - \beta_j^2 \right) \log(\pi \psi(t_j))$$

$$-\frac{1-4\alpha_0^2}{8} \log(\pi (1)) - \frac{1-4\alpha_0^2}{8} \log(\pi (-1)) + O(n^{-1+4\beta_{\max}}). \quad (7.45)$$

The computations of this subsection are organised similarly to those done in Subsection 7.3 and we provide less details. Let $C$ be a closed contour surrounding $[-1,1]$ and the lenses $\gamma_+ \cup \gamma_-$, which is oriented clockwise and passes through $-1 - \varepsilon$ and $1 + \varepsilon$ for a certain $\varepsilon > 0$. Using the jumps for $Y$, we rewrite the differential identity (7.46) as follows

$$\partial_s \log J_n(\tilde{\alpha}, \tilde{\beta}, V, 0) = -\frac{1}{2\pi i} \int_C [Y^{-1}(z)Y'(z)] \partial_s \log w_s(z) \frac{dz}{2\pi i}, \quad (7.46)$$

where from (7.11) and (7.20), we have $\partial_s \log w_s(z) = -n \partial_s V_s(z) = -nV'(z)$. In the same way as done in (7.28), by inverting the transformations $Y \mapsto T \mapsto S \mapsto R$ in the region outside the lenses and outside the disks, we have

$$\partial_s \log J_n(\tilde{\alpha}, \tilde{\beta}, V, 0) = I_{1,s} + I_{2,s} + I_{3,s}, \quad (7.47)$$

40
where $I_{1,s}$, $I_{2,s}$ and $I_{3,s}$ are given as in (7.28). For $I_{1,s}$, a simple calculation implies

$$I_{1,s} = -n^2 \int_{-1}^{1} \rho_s(x) \partial_s V_s(x) dx = -n^2 \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} V(x) \left( (1-s) \frac{1}{\pi} + s \psi(x) \right) dx, \quad (7.48)$$

which gives

$$\int_{0}^{1} I_{1,s} ds = -n^2 \int_{-1}^{1} \frac{V(x)}{\sqrt{1-x^2}} \left( \frac{1}{\pi} + \psi(x) \right) dx. \quad (7.49)$$

The computations of $I_{2,s}$ are similar to those done for $I_{1,s}$ and similar to (7.35) we get

$$\int_{0}^{1} I_{2,s} ds = n \sum_{j=0}^{m} \frac{\alpha_j}{2} V(t_j) - \frac{nA}{2n} \int_{-1}^{1} \frac{V(x)}{\sqrt{1-x^2}} dx - 2\pi n \sum_{j=1}^{m} i \beta_j \int_{t_j}^{1} \left( \psi(x) - \frac{1}{\pi} \right) \frac{1}{\sqrt{1-x^2}} dx. \quad (7.50)$$

For $I_{3,s}$, similar to (7.35) we get

$$I_{3,s} = \frac{1}{2\pi i} \int_{C} \left( \frac{a(z)^2 + a(z)^{-2}}{4} [R_{11}(z)' - R_{22}(z)'] + \frac{1}{2} [R_{11}(z)' + R_{22}(z)'] + i \frac{a(z)^2 - a(z)^{-2}}{4} [R_{12}(z)'D_{\infty}^{-2} + R_{21}(z)'D_{\infty}^{-2}] \right) V(z) dz + O(n^{-1+4\beta_{\max}}). \quad (7.51)$$

The quantities involving $R^{(1)}$ are made explicit using (4.83), we obtain

$$R_{11}(z)' - R_{22}(z)' = \sum_{j=1}^{m} \frac{1}{(z-t_j)^2} \frac{-2v_j(t_j + \tilde{\Lambda}_{1,j})}{2\pi \rho_s(t_j) \sqrt{1-t_j^2}} + \frac{1}{(z-1)^2} \frac{4\alpha_-^{m+1} - 1}{2^{3} \pi \psi_s(1)} + \frac{1}{(z+1)^2} \frac{1 - 4\alpha_0^2}{2^{3} \pi \psi_s(-1)},$$

$$R_{11}(z) + R_{22}(z) = 0,$$

$$i[R_{12}(z)'D_{\infty}^{-2} + R_{21}(z)'D_{\infty}^{-2}] = \sum_{j=1}^{m} \frac{1}{(z-t_j)^2} \frac{v_j(-2 + \tilde{\Lambda}_{R,1,j} - \tilde{\Lambda}_{R,2,j})}{2\pi \rho_s(t_j) \sqrt{1-t_j^2}} + \frac{1}{(z-1)^2} \frac{-(1-4\alpha_0^2)}{2^{4} \pi \psi_s(1)}$$

$$+ \frac{1}{(z+1)^2} \frac{1 - 4\alpha_0^2}{2^{4} \pi \psi_s(-1)}.$$

As in Subsection 7.3 we rewrite $I_{3,s}$ in the form

$$I_{3,s} = \sum_{j=1}^{m} I_{3,s,t_j} + I_{3,s,1} + I_{3,s,-1} + O(n^{-1+4\beta_{\max}}), \quad \text{as } n \to \infty, \quad (7.52)$$

where

$$I_{3,s,t_j} = \frac{-v_k}{8\pi^2 \rho_s(t_k)} \int_{C} \left[ \frac{a^2(z)}{a^2(t_k)} + \frac{a^2(t_k)}{a^2(z)} + \tilde{\Lambda}_{1,k} \left( \frac{a^2(z)}{a^2(t_k)} - \frac{a^2(t_k)}{a^2(z)} \right) \right] \frac{\partial_s V_s(z)}{(z-t_k)^2} dz,$$

$$I_{3,s,1} = \frac{4\alpha_-^{m+1} - 1}{2^{5} \pi^2 \psi_s(1)} \int_{C} \frac{a^2(z)}{(z-1)^2} \partial_s V_s(z) dz,$$

$$I_{3,s,-1} = \frac{1 - 4\alpha_0^2}{2^{5} \pi^2 \psi_s(-1)} \int_{C} \frac{a^{-2}(z)}{(z+1)^2} \partial_s V_s(z) dz.$$
From (7.14) and (7.15), \( I_{3,s,t} \) simplifies to
\[
I_{3,s,t} = -\frac{v_k}{\psi_s(t_k)} (\psi(t_k) - \frac{1}{\pi}) - \frac{v_k \bar{\Lambda}_{L,k}}{\rho_s(t_k)(1-t_k^2)} \int_{t_k}^1 (\psi(x) - \frac{1}{\pi}) \frac{dx}{\sqrt{1-x^2}}
\] (7.53)
and hence, similarly to (7.41)–(7.42), as \( n \to +\infty \) we have
\[
\int_0^1 I_{3,s,t} ds = -v_k \log(\pi \psi(t_k)) + O(n^{-1+2|\Re \beta_k|}).
\] (7.54)

Also, from (7.24)–(7.25), we have
\[
I_{3,s,1} = -\frac{1-4\alpha_0^2}{8\psi_s(1)} (\psi(1) - \frac{1}{\pi}) \quad \text{and} \quad I_{3,s,-1} = -\frac{1-4\alpha_0^2}{8\psi_s(-1)} (\psi(-1) - \frac{1}{\pi}),
\] (7.55)
and hence
\[
\int_0^1 I_{3,s,1} ds = -\frac{1-4\alpha_0^2}{8} \log(\pi \psi(1)) \quad \text{and} \quad \int_0^1 I_{3,s,-1} ds = -\frac{1-4\alpha_0^2}{8} \log(\pi \psi(-1)).
\] (7.56)

This concludes the proof of proposition 7.5.

8 Integration in \( W \)

The main result of this section is the following.

**Proposition 8.1** As \( n \to \infty \), we have
\[
\log \frac{D_n(\tilde{\alpha}, \tilde{\beta}, V, W)}{D_n(\tilde{\alpha}, \tilde{\beta}, V, 0)} = n \int_{-1}^1 W(x) \rho(x) dx - \frac{1}{4\pi^2} \int_{-1}^1 \frac{W(y)}{\sqrt{1-y^2}} \left( \int_{-1}^1 \frac{W'(x) \sqrt{1-x^2}}{x-y} dx \right) dy + \frac{A}{2\pi} \int_{-1}^1 \frac{W(x)}{\sqrt{1-x^2}} dx - \sum_{j=0}^{m+1} \frac{\alpha_j^2}{2} W(t_j) + \sum_{j=1}^{m} \frac{i\beta_j}{\pi} \sqrt{1-t_j^2} \int_{-1}^1 \frac{W(x)}{\sqrt{1-x^2} (t_j-x)} dx + O(n^{-1+2\beta_{\max}}).
\] (8.1)

where \( D_n \) stands for either \( L_n \) or \( J_n \).

**Remark 8.2** The difference between Laguerre-type and Jacobi-type weights in the r.h.s. of (8.1) is only reflected in the definitions of \( \rho \) and \( A \).

The proof of Proposition 8.1 goes in a similar way as in 8. The definition \( W_t(z) = \log \left( 1 - t + te^{W(z)} \right) \) (8.2)

where the principal branch is taken for the log. For every \( t \in [0,1] \), \( W_t \) is analytic on a neighbourhood of \([-1,1]\) (independent of \( t \) and is still H"older continuous on \( I \). This deformation is same as the one used in 8.3. Therefore, we can and do use the steepest descent analysis of Section 4 applied to the weight
\[
W_t(x) = e^{-nV(x)} e^{W_t(x)} \omega(x).
\] (8.3)
From Proposition 3.4, we have the following differential identities
\[
\partial_t \log L_n(\tilde{\alpha}, \tilde{\beta}, V, W_t) = \frac{1}{2\pi i} \int_{\gamma} [Y^{-1}(x)Y'(x)]_{21} \partial_t w_t(x) dx,
\]
(8.4)
\[
\partial_t \log J_n(\tilde{\alpha}, \tilde{\beta}, V, W_t) = \frac{1}{2\pi i} \int_{\gamma} [Y^{-1}(x)Y'(x)]_{21} \partial_t w_t(x) dx.
\]
(8.5)

The rest of the proof consists of inverting the transformations \(Y \mapsto T \mapsto S \mapsto R\) and evaluating certain integrals by contour deformations. These computations are identical to those done in Section 7] for Gaussian-type weights and we omit them here.

9 Appendix

We recall here some well-known model RH problems: the Airy model RH problem, whose solution is denoted \(\Phi_{Ai}\) and the Bessel model RH problem, whose solution is denoted \(\Phi_{Be}(\cdot) = \Phi_{Be}(\cdot; \alpha)\), where the parameter \(\alpha\) is such that \(R\alpha > -1\).

9.1 Airy model RH problem

(a) \(\Phi_{Ai} : \mathbb{C} \setminus \Sigma_A \to \mathbb{C}^{2 \times 2}\) is analytic, and \(\Sigma_A\) is shown in Figure 4.

(b) \(\Phi_{Ai}\) has the jump relations

\[
\Phi_{Ai,+}(z) = \Phi_{Ai,-}(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \text{ on } \mathbb{R}^-,
\]
\[
\Phi_{Ai,+}(z) = \Phi_{Ai,-}(z) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \text{ on } \mathbb{R}^+,
\]
\[
\Phi_{Ai,+}(z) = \Phi_{Ai,-}(z) \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \text{ on } e^{\frac{2\pi i}{3}} \mathbb{R}^+,
\]
\[
\Phi_{Ai,+}(z) = \Phi_{Ai,-}(z) \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \text{ on } e^{-\frac{2\pi i}{3}} \mathbb{R}^+.
\]
(9.1)

(c) As \(z \to \infty, z \notin \Sigma_A\), we have

\[
\Phi_{Ai}(z) = z^{-\frac{2\pi i}{3}} N \left( 1 + \sum_{k=1}^{\infty} \frac{\Phi_{Ai,k}}{z^{2k/3}} \right) e^{-\frac{2\pi i}{3} 2^{1/3}} z^{2/3} \sigma_3,
\]
(9.2)

where \(N = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}\) and \(\Phi_{Ai,1} = \frac{1}{8} \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}\).

As \(z \to 0\), we have

\[
\Phi_{Ai}(z) = O(1).
\]
(9.3)

The Airy model RH problem was introduced and solved in [5]. We have

\[
\Phi_{Ai}(z) := M_A \times \begin{cases}
\begin{pmatrix} \text{Ai}(z) & \text{Ai}(\omega^2 z) \\ \text{Ai}'(z) & \omega^2 \text{Ai}'(\omega^2 z) \end{pmatrix} e^{-\frac{2\pi i}{3} \sigma_3}, & \text{for } 0 < \arg z < \frac{2\pi}{3}, \\
\begin{pmatrix} \text{Ai}(z) & \text{Ai}(\omega^2 z) \\ \text{Ai}'(z) & \omega^2 \text{Ai}'(\omega^2 z) \end{pmatrix} e^{-\frac{2\pi i}{3} \sigma_3} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, & \text{for } \frac{2\pi}{3} < \arg z < \pi,
\end{cases}
\end{cases}
\]
\[
\begin{cases}
\begin{pmatrix} \text{Ai}(z) & \text{Ai}(\omega^2 z) \\ \text{Ai}'(z) & \omega^2 \text{Ai}'(\omega^2 z) \end{pmatrix} e^{-\frac{2\pi i}{3} \sigma_3} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, & \text{for } -\pi < \arg z < -\frac{2\pi}{3},
\end{cases}
\end{cases}
\]
\[
\begin{cases}
\begin{pmatrix} \text{Ai}(z) & \text{Ai}(\omega^2 z) \\ \text{Ai}'(z) & \omega^2 \text{Ai}'(\omega^2 z) \end{pmatrix} e^{-\frac{2\pi i}{3} \sigma_3}, & \text{for } -\frac{2\pi}{3} < \arg z < 0,
\end{cases}
\end{cases}
\]
(9.4)
with \( \omega = e^{\frac{2\pi i}{3}} \), Ai the Airy function and

\[
M_A = \sqrt{2\pi e^{\frac{\pi i}{6}}} \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}.
\] (9.5)

### 9.2 Bessel model RH problem

(a) \( \Phi_{Be} : \mathbb{C} \setminus \Sigma_{Be} \to \mathbb{C}^{2 \times 2} \) is analytic, where \( \Sigma_{Be} \) is shown in Figure 5.

(b) \( \Phi_{Be} \) satisfies the jump conditions

\[
\Phi_{Be,+}(z) = \Phi_{Be,-}(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad z \in \mathbb{R}^-,
\]

\[
\Phi_{Be,+}(z) = \Phi_{Be,-}(z) e^{\pi i \alpha} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad z \in e^{\frac{2\pi i}{3}} \mathbb{R}^+,
\] (9.6)

\[
\Phi_{Be,+}(z) = \Phi_{Be,-}(z) e^{-\pi i \alpha} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad z \in e^{-\frac{2\pi i}{3}} \mathbb{R}^+.
\]

(c) As \( z \to \infty \), \( z \notin \Sigma_{Be} \), we have

\[
\Phi_{Be}(z) = (2\pi z^\frac{1}{2})^{-\sigma_3} N \left( I + \sum_{k=1}^{\infty} \Phi_{Be,k} z^{-k/2} \right) e^{2z^\frac{1}{2} \sigma_3},
\] (9.7)

where \( \Phi_{Be,1} = \frac{1}{i\alpha} \begin{pmatrix} -1 + 4\alpha^2 & -2i \\ -2i & 1 + 4\alpha^2 \end{pmatrix} \).
(d) As $z$ tends to 0, the behaviour of $\Phi_{Be}(z)$ is

$$\Phi_{Be}(z) = \begin{cases} (O(1), O(\log z)), & |\arg z| < \frac{2\pi}{3}, \\ (O(\log z), O(\log z)), & \frac{2\pi}{3} < |\arg z| < \pi, \end{cases}$$

if $\Re \alpha = 0$.

$$\Phi_{Be}(z) = \begin{cases} (O(1), O(1)), & |\arg z| < \frac{2\pi}{3}, \\ (O(z^{-\frac{2}{3}}), O(z^{-\frac{2}{3}})), & \frac{2\pi}{3} < |\arg z| < \pi, \end{cases}$$

if $\Re \alpha > 0$.

$$\Phi_{Be}(z) = \begin{cases} (O(z^{\frac{2}{3}}), O(z^{\frac{2}{3}})), & |\arg z| < \frac{2\pi}{3}, \\ (O(z^{\frac{2}{3}}), O(z^{\frac{2}{3}})), & \frac{2\pi}{3} < |\arg z| < \pi, \end{cases}$$

if $\Re \alpha < 0$.

This RH problem was introduced and solved in [16]. Its unique solution is given by

$$\Phi_{Be}(z) = \begin{cases} I_\alpha(2z^{\frac{1}{3}}) + \frac{\pi}{4}K_\alpha(2z^{\frac{1}{3}}), & |\arg z| < \frac{2\pi}{3}, \\ \pi z^{\frac{2}{3}}H^{(1)}_\alpha(2(-z)^{\frac{1}{3}}) - 2\pi z^{\frac{2}{3}}K^{(1)}_\alpha(2(-z)^{\frac{1}{3}}) e^{\frac{2\pi}{3}\sigma_3}, & \frac{2\pi}{3} < |\arg z| < \pi, \\ \pi z^{\frac{2}{3}}H^{(2)}_\alpha(2(-z)^{\frac{1}{3}}) - \pi z^{\frac{2}{3}}K^{(2)}_\alpha(2(-z)^{\frac{1}{3}}) e^{-\frac{2\pi}{3}\sigma_3}, & -\pi < |\arg z| < -\frac{2\pi}{3}, \end{cases}$$

where $H^{(1)}_\alpha$ and $H^{(2)}_\alpha$ are the Hankel functions of the first and second kind, and $I_\alpha$ and $K_\alpha$ are the modified Bessel functions of the first and second kind.
Acknowledgements

C.C. acknowledges support from the Swedish Research Council, Grant No. 2015-05430, and the European Research Council, Grant Agreement No. 682537, and R.G. acknowledges support by NSF-grant DMS-1700261.

References

[1] N. Berestycki, C. Webb and M.D. Wong, Random Hermitian Matrices and Gaussian Multiplicative Chaos, *Probab. Theory Related Fields*, 172 (2018), 103–189.

[2] G. Borot and A. Guionnet, Asymptotic expansion of $\beta$ matrix models in the one-cut regime, *Comm. Math. Phys.* 317 (2013), 447–483.

[3] C. Charlier, Asymptotics of Hankel determinants with a one-cut regular potential and Fisher-Hartwig singularities, *Int. Math. Res. Not.* 2018 (2018), 62 pp. https://doi.org/10.1093/imrn/rny009.

[4] T. Claeys, B. Fahs, G. Lambert and C. Webb, How much can the eigenvalues of a random Hermitian matrix fluctuate ?, in preparation.

[5] P. Deift, Orthogonal polynomials and random matrices: A Riemann-Hilbert approach. Amer. Math. Soc. 3 (2000).

[6] P. Deift, T. Kriecherbauer, K. T-R. McLaughlin, S. Venakides, and X. Zhou, Strong asymptotics of orthogonal polynomials with respect to exponential weights, *Comm. Pure Appl. Math.* 52 (1999), 1491–1552.

[7] P. Deift, T. Kriecherbauer, K. T-R. McLaughlin, S. Venakides, and X. Zhou, Uniform asymptotics for polynomials orthogonal with respect to varying exponential weights and applications to universality questions in random matrix theory, *Comm. Pure Appl. Math.* 52 (1999), 1335–1425.

[8] P. Deift, A. Its and I. Krasovsky, Asymptotics of Toeplitz, Hankel, and Toeplitz+Hankel determinants with Fisher-Hartwig singularities, *Ann. Math.* 174 (2011), 1243–1299.

[9] P. Deift, A. Its and I. Krasovsky, On the asymptotics of a Toeplitz determinant with singularities, *MSRI Publications* 65 (2014), Cambridge University Press.

[10] M. E. Fisher and R. E. Hartwig, Toeplitz determinants: Some applications, theorems, and conjectures, *Adv. Chem. Phys.* 15 (1968), 333–353.

[11] A.S. Fokas, A.R. Its and A.V. Kitaev, The isomonodromy approach to matrix models in 2D quantum gravity, *Comm. Math. Phys.* 147 (1992), 395430.

[12] A. Foulquié Moreno, A. Martinez-Finkelshtein and V.L. Sousa, On a conjecture of A. Magnus concerning the asymptotic behavior of the recurrence coefficients of the generalized Jacobi polynomials, *J. Approx. Theory*, 162(2010), 807–831.

[13] F. Gakhov, *Boundary Value Problems*, Pergamon Press, Oxford (1966). Reprinted by Dover Publications, New York (1990).

[14] A. Its and I. Krasovsky, Hankel determinant and orthogonal polynomials for the Gaussian weight with a jump, *Contemporary Mathematics* 458 (2008), 215–248.
[15] I. Krasovsky, Correlations of the characteristic polynomials in the Gaussian unitary ensemble or a singular Hankel determinant, *Duke Math J.* **139** (2007), 581–619.

[16] A.B.J. Kuijlaars, K.T-R McLaughlin, W. Van Assche and M. Vanlessen, The Riemann–Hilbert approach to strong asymptotics for orthogonal polynomials on $[-1,1]$, *Adv. Math.* **188** (2004), 337–398.

[17] M.L. Mehta, *Random matrices, Third Edition*, Pure and Applied Mathematics Series **142** (2004), Elsevier Academic Press.

[18] F.W.J. Olver, D.W. Lozier, R.F. Boisvert and C.W. Clark, *NIST handbook of mathematical functions* (2010), Cambridge University Press.

[19] E. B. Saff and V. Totik, *Logarithmic Potentials with External Fields*, Springer-Verlag (1997).

[20] G. Szegő, *Orthogonal polynomials*, AMS Colloquium Publ. **23** (1959), New York: AMS.