Generalized symmetries and integrability conditions for hyperbolic type semi-discrete equations

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To the memory of A.B. Shabat and R.I. Yamilov

Abstract

In the article differential-difference (semi-discrete) lattices of hyperbolic type are investigated from the integrability viewpoint. More precisely we concentrate on a method for constructing generalized symmetries. This kind integrable lattices admit two hierarchies of generalized symmetries corresponding to the discrete and continuous independent variables \( n \) and \( x \). Symmetries corresponding to the direction of \( n \) are constructed in a more or less standard way while when constructing symmetries of the other form we meet a problem of solving a functional equation. We have shown that to handle with this equation one can effectively use the concept of characteristic Lie-Rinehart algebras of semi-discrete models. Based on this observation, we have proposed a classification method for integrable semi-discrete lattices. One of the interesting results of this work is a new example of an integrable equation, which is a semi-discrete analogue of the Tzizeica equation. Such examples were not previously known.

Keywords: generalized symmetry, characteristic vector field, integrability conditions, semi-discrete Tzizeica equation.

1 Introduction

We consider nonlinear differential-difference equations of the following form

\[ u_{n+1,x} = f(u_{n,x}, u_{n+1}, u_n, x) \] (1)

where the sought function \( u_n = u_n(x) \) depends on integer \( n \) and real \( x \) and \( u_{n,x} \) denotes the derivative of \( u_n(x) \) with respect to \( x \). Below we use also expressions \( u_{n,t} \) and \( u_{n,\tau} \) denoting the derivatives of \( u_n \) with respect to \( t \) and \( \tau \).

Lattice (1) can be regarded as a semi-discrete version of the hyperbolic type partial differential equation. To preserve the parity of the forward and backward directions, we lay down the following condition. We assume that equation (1) can be uniquely rewritten as

\[ u_{n-1,x} = \tilde{f}(u_{n,x}, u_{n-1}, u_n, x). \] (2)
By analogy with the hyperbolic PDE we introduce two sets of dynamical variables for the lattice (1) corresponding to two characteristic directions \( x = \text{const} \) and \( n = \text{const} \), first consists of the shifts of the variable \( u_n \): \( S_n := \{u_k\}_{k=-\infty}^{+\infty} \) and the second contains the derivatives of \( u_n(x) \) with respect to \( x \): \( S_x := \{\frac{d^k}{dx^k}u_n\}_{k=1}^{+\infty} \). As usual, dynamical variables can be viewed as independent. Unlike the class of evolutionary type semi-discrete equations, such as the well-known Volterra and Toda like chains, the class of lattices (1) received less attention. However it should be noted that there are many examples of integrable lattices from the class (1) (see for instance, [23]). Equations of this type, often referred to as dressing chains [20], usually arise as iterated applications of the Backlund transformations for the nonlinear partial differential equations. In such a case the corresponding PDE is interpreted as a symmetry for the lattice, and then the Lax pair and hierarchy of the higher symmetries are readily found.

Now we recall some necessary definitions (see cite ... for example). Let us given an evolutionary type PDE

\[
 u_t = g \left( x, u, u_x, u_{xx}, \ldots, \frac{\partial^N u}{\partial x^N} \right) \tag{3}
\]

of the order \( N \). We assume that solution of the lattice (1) depends on the variable \( t \) and similarly solution of the PDE (3) depends on \( n \) such that the latter is converted into the form

\[
 u_{n,t} = g \left( x, u_n, u_{n,x}, u_{n,xx}, \ldots, \frac{\partial^N u_n}{\partial x^N} \right). \tag{4}
\]

Equation (3) is called a symmetry of the lattice (1) in the direction of \( x \) if the flows defined by the equations (1) and (4) commute identically on the dynamical variables in \( S_x \cup S_n \). In other words the following relation holds

\[
 T D_x g - D_t f = 0 \quad \text{mod (1), (3)} \tag{5}
\]

for all dynamical variables. Here \( T \) is the shift operator of the discrete argument \( n \) and \( D_x, D_t \) – operators of the total derivative with respect to the variables \( x \) and \( t \).

We call the evolutionary type lattice

\[
 u_{n,\tau} = h(u_{n-M}, u_{n-M+1}, \ldots, u_{n+M-1}, u_{n+M}, x) \tag{6}
\]

a symmetry of the lattice (1) in the direction of \( n \) if the flows defined by the equations (1) and (6) commute with one another. More precisely we assume that equation

\[
 T D_x h - D_{\tau} f = 0 \quad \text{mod (1), (5)} \tag{7}
\]

is satisfied identically on the set \( S_x \cup S_n \).

The current state of symmetry approach in the integrability theory of the evolutionary type equations of the form (3) and (6) can be found in the works [5, 6, 12, 16, 18, 22, 24] and in the references therein.

Below in Section 2 we give a detailed explanation of the symmetries and illustrate the algorithm for finding them. Symmetries of the form (6) are found from an overdetermined system of differential equations, but the problem of constructing symmetries of the form (1) is rather complicated and requires non-standard techniques using the concept of the characteristic operators of a semi-discrete chain. In Section 3 we discuss all the stages of the algorithm for constructing symmetries using the example of a semi-discrete version the sine-Gordon equation. In section 4 we represent a new integrable semi-discrete model

\[
 u_{1,x} = u_{0,x} + (e^{-2u} + e^{-2u_1}) + \sqrt{e^{2u} + e^{2u_1}} \tag{8}
\]
that leads to the well-known Tzitzeika equation \[3, 13, 19, 25\] in the continuum limit. Here we show that lattice (8) does not admit any generalized symmetry of the order less than five in the direction of \(x\). Its fifth order symmetry (see below equation (61) with \(\lambda_1 = \lambda_2 = 1\)) coincides with an integrable equation from the list in [12]. The simplest generalized symmetry in the \(n\)-direction is a five-point evolutionary type lattice (62). To the best of our knowledge this equation is new.

An amazing fact is that the lattice (8) does not define the Backlund transformation either for the Tzizzeica equation or for its symmetry (89), but defines the Backlund transformation for some third equation of the form (4) connected with (89) by a very complex Miura-type transformation.

2 Evaluation of the symmetries

In this section, we discuss an algorithm for constructing generalized symmetries for the lattice (1). The problem is easily solved if a Lax pair or a lattice recursion operator is given. However, this method is not always available. For example, when solving the problem of integrable classification, we are dealing with equations of general form. It is curious that when we have only a lattice the problem of finding its generalized symmetry in the direction of \(x\) may turn out to be nontrivial since we arrive at a nonlinear functional equation and probably this is one of the reasons why up to now there is no any classification result for the soliton type lattices of the form (1).

Below we show how to reduce the over mentioned functional equation to a system of differential equations.

As it easily follows from the relations (3) and (7) the right hand sides \(g\) and \(h\) of the symmetries (3) and (6) should satisfy the linearization of the equations (1)

\[ V_{n+1,x} = \frac{\partial f}{\partial u_{n,x}} V_{n,x} + \frac{\partial f}{\partial u_{n+1}} V_{n+1} + \frac{\partial f}{\partial u_n} V_n. \]  

(9)

For the sake of convenience we will use the following abbreviated notations: \(u_k = u_{n+k}(x)\), \(u_{k,1} = \frac{du_{n+k}}{dx}\), \(u_{k,2} = \frac{d^2u_{n+k}}{dx^2}\), \ldots. Since the lattices are autonomous with respect to \(n\) these notations do not lead to confusion. Similarly we use expressions \(u_{k,t}\), \(u_{k,\tau}\) denoting the derivatives of the variable \(u_k\) with respect to \(t\) and \(\tau\).

In the abbreviated notations the symmetries (3), (6) and the dynamical variables take the form

\[ u_{0,t} = g(x, u_0, u_{0,1}, u_{0,2}, \ldots, u_{0,N}), \]  

(10)

\[ u_{0,\tau} = h(u_{M}, u_{M+1}, \ldots, u_{M-1}, u_M), \]  

(11)

\[ u_0, u_1, u_{-1}, u_{0,1}, u_2, u_{-2}, u_{0,2}, \ldots. \]  

(12)

Obviously, all other variables \(u_{k,m}\), \(m \in \mathbb{N}, k \in \mathbb{Z} - \{0\}\) can be expressed in terms of the dynamical variables (12) due to equation (11), its shifts and differential consequences. Completing the preparatory reasoning, we note that the equation (9) must be satisfied identically for all dynamical variables (12) that are considered as independent ones.

We first explain how to find a symmetry in the direction of \(x\). To this end we substitute \(V_n = g\), where \(g\) is from (11) into equation (9) and get a huge differential-difference functional
equation, which is to be solved:

\[
\frac{\partial}{\partial x} g(x, u_1, f, Df, \ldots, D^{N-1}f) + \sum_{i=0}^{N} D^{i}f \cdot \frac{\partial}{\partial u_{0,i}} g(x, u_1, f, Df, \ldots, D^{N-1}f) = \frac{\partial f}{\partial u_n} g(x, u, u_{0,1}, u_{0,2}, \ldots, u_{0,N}) + \frac{\partial f}{\partial u_{n+1}} g(x, u_1, f, Df, \ldots, D^{N-1}f) + \frac{\partial f}{\partial u_n} g.
\]

(13)

Then we split down equation (13) with respect to the highest order derivative \(u_{0,N+1}\). By collecting the coefficients in front of \(u_{0,N+1}\) we get a relation:

\[
\frac{\partial f}{\partial u_n,x} T \frac{\partial}{\partial u_{0,N}} g(x, u, u_{0,1}, u_{0,2}, \ldots, u_{0,N}) = \frac{\partial f}{\partial u_{n,x}} \frac{\partial}{\partial u_{0,N}} g(x, u_1, f, Df, \ldots, D^{N-1}f) + \frac{\partial f}{\partial u_n} g.
\]

(14)

that is simplified

\[
(T - 1) \frac{\partial}{\partial u_{0,N}} g(x, u_1, u_{0,1}, u_{0,2}, \ldots, u_{0,N}) = 0
\]

(15)

and easily solved

\[
\frac{\partial}{\partial u_{0,N}} g(x, u_1, u_{0,1}, u_{0,2}, \ldots, u_{0,N}) = C(x), \quad C(x) \neq 0.
\]

(16)

Thus the sought function \(g\) is partly specified and the symmetry (10) gets the form:

\[
\frac{\partial f}{\partial u_{n,x}} T \frac{\partial}{\partial u_{0,N}} g(x, u, u_{0,1}, u_{0,2}, \ldots, u_{0,N}) = \frac{\partial f}{\partial u_{n,x}} \frac{\partial}{\partial u_{0,N}} g(x, u_1, f, Df, \ldots, D^{N-1}f) + \frac{\partial f}{\partial u_n} g.
\]

(17)

Now we turn back to equation (13) where instead of \(g\) we substitute the rhs of the partly specified symmetry (17). The derivative of (13) with respect to \(u_{0,N}\) after some simplification due to the relation, which can be easily proved by induction

\[
D^k f = \frac{\partial f}{\partial u_{0,1}} u_{0,k+1} + \left(kD \left( \frac{\partial f}{\partial u_{0,1}} \right) + \frac{\partial f}{\partial u_{0,1}} \frac{\partial f}{\partial u_{1}} + \frac{\partial f}{\partial u_{0}} \right) u_{0,k} + \ldots, k \geq 3,
\]

reads as:

\[
ND \log \frac{\partial f}{\partial u_{0,1}} = (1 - T) \frac{\partial}{\partial u_{0,N-1}} g^{(1)}(x, u, u_{0,1}, u_{0,2}, \ldots, u_{0,N-1}), \quad N \geq 3.
\]

(19)

From (19) we obtain the first integrability condition for the lattice (11)

\[
D \log \frac{\partial f}{\partial u_{0,1}} \in \text{Im}(1 - T).
\]

(20)

We stress that this condition doesn’t depend on \(N\).

Now we concentrate on (19) that is in fact a functional equation, since it relates the values of the sought function \(g\) at two different points \((x, u_0, u_{0,1}, u_{0,2}, \ldots, u_{0,N-1})\) and \((x, u_1, u_{1,1}, u_{1,2}, \ldots, u_{1,N-1})\). For the shortness we denote \(z := \frac{\partial}{\partial u_{0,N-1}} g^{(1)}(x, u_0, u_{0,1}, u_{0,2}, \ldots, u_{0,N-1})\) and rewrite equation as follows

\[
T z - z = -ND \log \frac{\partial f}{\partial u_{0,1}}.
\]

(21)

Functional equations of such kind usually arise when constructing symmetries of hyperbolic type equations on quadrilateral graphs. In our work [4] we proposed a method for solving them
using the characteristic operators introduced earlier in [9]. Below, to solve equation (21), we use the idea of [4].

Since \( z \) does not depend on the variable \( u_1 \) then by applying the operator \( T^{-1} \frac{\partial}{\partial u_1} \) to both sides of (21) we reduce it to a differential equation:

\[
Y_1 z = -NT^{-1} \frac{\partial}{\partial u_1} (D \log \frac{\partial f}{\partial u_{0,1}}),
\]

where the vector field \( Y_1 \) called the characteristic operator is given by

\[
Y_1 = T^{-1} \frac{\partial}{\partial u_1} T = \frac{\partial}{\partial u_0} + \sum_{k=1}^{\infty} \left( T^{-1} \frac{\partial}{\partial u_1} D^k f \right) \frac{\partial}{\partial u_{0,k}}.
\]

In order to derive one more differential consequence of the functional equation (21) we apply to both sides of that the operator \( T^{-1} z - T^{-1} z = -NT^{-1} D \log \frac{\partial f}{\partial u_{0,1}} \) (24) and then apply to (24) the operator \( T \frac{\partial}{\partial u_{-1}} \). As a result we get

\[
Y_{-1} z = NY_{-1} (D \log \frac{\partial f}{\partial u_{0,1}})
\]

where \( Y_{-1} \) is another characteristic operator defined as

\[
Y_{-1} = T \frac{\partial}{\partial u_{-1}} T^{-1} = \frac{\partial}{\partial u_0} + \sum_{k=1}^{\infty} \left( T \frac{\partial}{\partial u_{-1}} D^k f \right) \frac{\partial}{\partial u_{0,k}}.
\]

Characteristic operators \( Y_1, Y_{-1} \) have been used earlier in [7], [8] for the classification of the Darboux integrable lattices of the form (1) in a particular case.

Let’s finalize the reasonings above. We have shown that function \( z \) satisfies a system of two first-order linear equations (22, 25) in partial derivatives with respect to the dynamical variables. However, \( z \) is a special solution, satisfying a very severe condition, indeed, by construction it does not depend on \( u_1 \) and \( u_{-1} \), despite the coefficients of the equations depend on these variables. To eliminate inappropriate solutions, we add to the system (22, 25) the following two extra equations

\[
X_1 z = \frac{\partial}{\partial u_1} z = 0, \quad X_{-1} z = \frac{\partial}{\partial u_{-1}} z = 0
\]

Moreover, according to general theory of the systems of the first order linear equations in partial derivatives, \( z \) must satisfy, in addition to these four equations, several other equations derived from these equations by taking cross applications of the operators such as \([X_1, Y_1] z = X_1 f^{(1)}\), \([Y_1, Y_{-1}] z = Y_1 f^{(-1)} - Y_{-1} f^{(1)}\) etc., where the bracket stands for the commutator of the operators, \([Y, Z] = YZ - ZY\) (Jacobi’s theorem). In this way we arrive at a closed system of linear equations for partial derivatives \( \frac{\partial z}{\partial u_{0,k}} \)

\[
\sum_{k=0}^{N-1} a_{s,k} \frac{\partial z}{\partial u_{0,k}} = b_s,
\]
such that the further manipulations with the cross applications produce only equations that are linearly expressed through the earlier obtained equations. Due to the well known Kronecker-Capelli theorem, system of linear equations (28) is compatible if and only if the rank of the coefficient matrix $A = (a_{s,i})$ is equal to that of the augmented matrix $B$ obtained from $A$ by adding the column of free terms $b_s$. Thus the condition $\text{rank}(A) = \text{rank}(B)$ is necessary for existence of a symmetry of the form (10).

By solving equation (28) we find an explicit expression for the function $z = \frac{\partial}{\partial u_0} g^{(1)}(x, u_0, u_{0,1}, u_{0,2}, \ldots, u_{0,N-1})$ in terms of the rhs of the lattice (1). This allows one to get a further specification of the formula (17)

$$u_t = u_{0,N} + p^{(1)}(x, u_0, u_{0,1}, u_{0,2}, \ldots, u_{0,N-1}) + q^{(2)}(x, u_0, u_{0,1}, u_{0,2}, \ldots, u_{0,N-2})$$

(29) for symmetry searched. Here $q^{(2)}$ is to be determined.

Applying several times the algorithm described above, we can completely determine the right side of the symmetry if it does exist. However in some cases we meet here additional problem related to the circumstance that equation (28) is a consequence of (21), but these two equations are not equivalent. Hence (28) might have solutions which do not satisfy (21). This problem is solved as follows. System (28) can be extended by adding some extra conditions obtained using the characteristic operators of a higher order (see [7], [8]).

By combining equation (21) and the following its consequence $T^2 z - Tz = -ND \log \frac{\partial f}{\partial u_{0,1}}$ we find

$$T^2 z = z - (1 + T)ND \log \frac{\partial f}{\partial u_{0,1}}.$$  

(30)

Now we apply to the latter the operator $T^{-2} \frac{\partial}{\partial u_1}$ and since $\frac{\partial z}{\partial u_1} = 0$ we obtain an equation

$$Y_2 z = -T^{-2} \frac{\partial}{\partial u_1}(1 + T)ND \log \frac{\partial f}{\partial u_{0,1}},$$

(31)

where $Y_2$ is a differential operator defined as

$$Y_2 = -T^{-2} \frac{\partial}{\partial u_1} T^2.$$  

(32)

By applying similar manipulations we can derive equation

$$Y_{-2} z = T^2 \frac{\partial}{\partial u_{-1}} (T^{-1} + T^{-2})ND \log \frac{\partial f}{\partial u_{0,1}},$$

(33)

where

$$Y_{-2} = -T^2 \frac{\partial}{\partial u_{-1}} T^{-2}.$$  

(34)

In the next step we derive linear equations similar to (31) and (33)

$$Y_3 z = F^{(3)} \quad \text{and} \quad Y_{-3} z = F^{(-3)}$$

where $Y_3 = -T^{-3} \frac{\partial}{\partial u_1} T^3$, $Y_{-3} = -T^3 \frac{\partial}{\partial u_{-1}} T^{-3}$ and $F^{(\pm 3)}$ are some functions depending on $f$, its derivatives and shifts. One can proceed with such kind reasonings and derive more equations. However, as it is shown by examples to find the desired $z$ it is enough, as a rule, to use (31) and (33), since the other equations do not produce additional conditions. Below in the next sections we illustrate application of the algorithm with some examples.
In the case of the lattice type symmetry \( (11) \) the usual scheme can be used since the defining equation is an overdetermined differential (not functional!) equation. Indeed in this case linearized equation \((9)\) takes the form:

\[
D \left( h(u_{-M+1}, u_{-M+2}, \ldots, u_M, u_{M+1}, x) \right) = \frac{\partial f}{\partial u_{n,M}} D(h(u_{-M}, u_{-M+1}, \ldots, u_{M-1}, u_M, x)) \\
+ \frac{\partial f}{\partial u_{n+1}} h(u_{-M+1}, u_{-M+2}, \ldots, u_M, u_{M+1}) + \frac{\partial f}{\partial u_n} h(u_{-M}, u_{-M+1}, \ldots, u_{M-1}, u_M, x).
\] \( (35) \)

At first we apply to both sides of the equation the operator \( T^{-1} - \frac{\partial}{\partial u_{M+1}} \) and get:

\[
D \log \left( \frac{\partial h}{\partial u_M} \right) = (T^{-1} - T^{M-1}) \frac{\partial f}{\partial u_1}.
\] \( (36) \)

and then we apply to the same equation the operator \( \frac{\partial}{\partial u_{-M}} \) that gives:

\[
D \log \left( \frac{\partial h}{\partial u_{-M}} \right) = (T - T^{-M+1}) \frac{\partial \tilde{f}}{\partial u_{-1}}.
\] \( (37) \)

The obtained relations \( (36) \) and \( (37) \) are linear partial differential equations for unknowns \( v = \log \frac{\partial h}{\partial u_M} \) and \( \tilde{v} = \log \frac{\partial h}{\partial u_{-M}} \), respectively. First of them \( (36) \) reads:

\[
\sum_{i=-M}^M D(u_i) \frac{\partial v}{\partial u_i} + \frac{\partial v}{\partial x} = (T^{-1} - T^{M-1}) \frac{\partial f}{\partial u_1}.
\] \( (38) \)

Here coefficients depend not only on \( u_i, -M \leq i \leq M \) but on \( u_{0,1} \) as well. So we can add equations of the form:

\[
\sum_{i=-M}^M \left( \frac{\partial^j}{\partial (u_{0,1})^j} D(u_i) \right) \frac{\partial v}{\partial u_i} = \frac{\partial^j}{\partial (u_{0,1})^j} (T^{-1} - T^{M-1}) \frac{\partial f}{\partial u_1}, j = 1, \ldots, \]

\( (39) \)

and all their differential consequences. From the system of equations \( (36) \), \( (39) \) we can find function \( v \) and then define the dependence of the function \( h \) on \( u_M \). In a similar way we can find dependence of \( h \) on \( u_{-M} \) from \( (37) \) and its consequences.

Evidently equations \( (36) \) and \( (37) \) imply the following necessary integrability conditions for the lattice \( (11) \):

\[
(T^{-1} - T^{M-1}) \frac{\partial f}{\partial u_1} \in \text{Im} D, \quad (T - T^{-M+1}) \frac{\partial \tilde{f}}{\partial u_{-1}} \in \text{Im} D.
\] \( (40) \)

We note that defining equations having the form of a functional equation similar to \( (13) \) often arise in the frame of the symmetry approach. Over the past two decades, a completely discrete analogue of the equation \( (11) \) has been actively studied. The symmetry approach to this class of equations is developed in \([4, 10, 11, 14]\). Since for \( (11) \) both characteristic directions are discrete then the problem of constructing symmetries is related with functional equations. In our work \([4]\) we suggested a method based on the characteristic operators, allowing to solve functional equation arising in the case \( (11) \). In the recent article \([21]\) by Xenitidis an alternative way to solve the functional equations for the symmetries of quad equations is developed.
3 An illustrative example

In this section we illustrate application of the algorithms outlined above by using the example of a semi-discrete version of the sine-Gordon equation [2]

\[ u_{1,x} = u_{0,x} + \sin 2u_1 + \sin 2u_0. \]  

(42)

For this lattice we derive generalized symmetries (17) with \( N = 3 \) and (11) with \( M = 1 \).

To look for a generalized symmetry (17) we use the defining equation (13) and its consequence (19) which for the lattice (42) take, respectively, the following form:

\[ (T - 1)(C'u_{0,3} + Cu_{0,4} + Dg^{(1)}(x, u, u_{0,1}, u_{0,2})) \]
\[ = 2\cos 2u_1(Cu_{1,3} + g^{(1)}(x, u, f, Df)) + 2\cos 2u(Cu_{0,3} + g(x, u, u_{0,1}, u_{0,2})) \]

(43)

and

\[ (1 - T)\frac{\partial}{\partial u_{0,2}}g^{(1)}(x, u, u_{0,1}, u_{0,2}) = 0. \]

(44)

Since equation (44) is easily solved explicitly, we do not need any characteristic operators at this stage. Indeed, obviously we have

\[ \frac{\partial}{\partial u_{0,2}}g^{(1)}(x, u, u_{0,1}, u_{0,2}) = C_1(x), \]

(45)

so formula (29) reads as

\[ g(x, u, u_{0,1}, u_{0,2}, u_{0,3}) = C(x)u_{0,3} + C_1(x)u_{0,2} + g^{(2)}(x, u, u_{0,1}). \]

(46)

Functions \( C_1(x) \) and \( g^{(2)}(x, u, u_{0,1}) \) will be defined in the next step. At this stage we need in the operators \( Y_{\pm 1} \). At first we evaluate the derivative of (43) with respect to \( u_{0,2} \):

\[ (T - 1)\frac{\partial}{\partial u_{0,1}}g^{(2)}(x, u, u_{0,1}) = 12C(u_{0,1} + \sin 2u_1)(\sin 2u + \sin 2u_1) - 2C'(\cos 2u + \cos 2u_1). \]

(47)

Let us apply the operators \( T^{-1}\partial u_1 \) and \( Y_{-1} \) to both sides of (47). After some simplification we obtain two equations:

\[ \frac{\partial^2 g^{(2)}}{\partial u \partial u_{0,1}} = 12C \sin 4u, \quad \frac{\partial^2 g}{\partial u_{0,1}^2} = 12Cu_{0,1} + 2C' \tan 2u_0. \]

(48)

The consistency condition of these equations implies \( C'(x) = 0 \). Without loss of generality we can put \( C(x) = 1 \) and define the following expression for the unknown function \( q^{(2)} \)

\[ g^{(2)} = 2u_{0,1}^3 - (3 \cos 4u + C_2(x))u_{0,1} + g^{(3)}(x, u). \]

(49)

We specify the formula (46) by virtue of the last relation and then substitute the found representation of the symmetry into equation (43). As a result we arrive at:

\[ -4u_{0,1}^2C_1(\sin 2u + \sin 2u_1) + u_{0,1}\left((T - 1)\frac{\partial}{\partial u_0}g^{(3)} + 2C'_1(\cos 2u + \cos 2u_1) \right. \]
\[ \left. -8\sin 2u_1C_1(\sin 2u + \sin 2u_1) \right) + \ldots = 0, \]

(50)
here the tail does not contain the variable \( u_{0,1} \). By comparing the coefficients in front of the powers of the variable \( u_{0,1} \) we find equations

\[
C_1(x) = 0, \quad g^{(3)} = C_3(x)u_0 + C_4(x)
\]

that allow to specify equation (50):

\[
(C'_{2} + C_3)(\sin 2u + \sin 2u_1) - 2C_3(u \cos 2u + u_1 \cos 2u_1) + C'_{3}(u_1 - u_0) - 2C_4(\cos 2u + \cos 2u_1) = 0.
\]

Since functions \( u, \sin 2u, \cos 2u \) and \( u \cos 2u \) are linearly independent we get

\[
C_3(x) = 0, \quad C_4(x) = 0, \quad C_2(x) = c = \text{const}.
\]

Thus, we have a complete solution of the defining equation (13) for the lattice (42)

\[
g = u_{0,3} + 2u_{0,1}^3 - 3u_{0,1} \cos 4u - cu_{0,1}.
\]  

(51)

The last term in \( g \) corresponds to the point symmetry of the lattice (42) and we can take \( c = 0 \). Finally, as expected, we obtain the well-known generalized symmetry of the lattice in the direction of \( x \):

\[
u_t = u_{xxx} + 2u_x^4 - 3u_x \cos 4u.
\]  

(52)

Now we turn to the symmetry in the direction of \( n \), which supposedly has the form

\[
u_n,\tau = h(u_{n-1}, u_n, u_{n+1}).
\]  

(53)

To look for the symmetry (53) we use the defining equation (35) with \( M = 1 \). In the case (42) the differential consequences (36) and (37) of the defining equation take the form

\[
(u_{0,1} - \sin 2u - \sin 2u_{-1}) \frac{\partial v}{\partial u_{-1}} + u_{0,1} \frac{\partial v}{\partial u_0} + (u_{0,1} + \sin 2u_1 + \sin 2u) \frac{\partial v}{\partial u_1} = 2 \cos 2u - 2 \cos 2u_1,
\]

\[
(u_{0,1} - \sin 2u - \sin 2u_{-1}) \frac{\partial \tilde{v}}{\partial u_{-1}} + u_{0,1} \frac{\partial \tilde{v}}{\partial u_0} + (u_{0,1} + \sin 2u_1 + \sin 2u) \frac{\partial \tilde{v}}{\partial u_1} = 2 \cos 2u_{-1} - 2 \cos 2u.
\]

From these equations the functions \( v = \log \frac{\partial h}{\partial u_1}, \tilde{v} = \log \frac{\partial h}{\partial u_{-1}} \) and \( h \) are easily found

\[
v = -2 \log \cos(u_1 - u_0) + \log C_1, \quad \tilde{v} = -2 \log \cos(u_0 - u_{-1}) + \log C_2.
\]  

(54)

\[
h = C_1 \tan(u_1 - u_0) + C_2 \tan(u_0 - u_{-1}) + h^{(1)}(u_0).
\]  

(55)

Now we get an equation for defining \( h^{(1)} \), \( C_1 \) and \( C_2 \):

\[
(T - 1)Dh^{(1)}(u) - 2(T + 1)(h^{(1)}(u) \cos 2u)
\]

\[
= 2(C_2 - C_1) \frac{(\cos 2u_1 - \cos 2u)(\sin 2u_1 - \sin 2u)}{\cos 2u_1 + \cos 2u}.
\]  

(56)

After differentiation of (56) with respect to \( u_{0,1} \) we get

\[
(T - 1) \frac{\partial}{\partial u_0} h^{(1)} = 0.
\]
Then we apply operator \( T^{-1} \frac{\partial}{\partial u_1} - \frac{\partial}{\partial u_0} \) to (56) and get a relation \( C_2 = C_1 \). Finally by applying the operator \( \frac{\partial}{\partial u_0} \) to (56) we obtain

\[ k^{(1)}(u_0) = 0. \]

We set \( C_1 = 1 \), and finally find:

\[ \partial_r u_0 = \tan(u_1 - u_0) + \tan(u_0 - u_{-1}) \quad (57) \]

After the Möbius transformation of the form

\[ u_n = \frac{i v_n - 1}{v_n + 1}, \quad I^2 = -1 \quad (58) \]

we arrive at the well-known equation (V2) in the Yamilov list (see [23], p. 596)

\[ \partial_r v_0 = -4v_0^2 \left( \frac{1}{v_1 + v_0} - \frac{1}{v + v_{-1}} \right). \quad (59) \]

4. A new example: integrable semi-discrete version of the Tzitzeika equation

The celebrated Tzitzeika equation (see [3, 13, 19, 25])

\[ u_{xy} = e^u + e^{-2u} \]

is the most complicated example of the integrable equations of the Klein-Gordon type. The problem of integrable discretization of this equation has not been solved for a long time. Its completely discrete integrable versions, defined on a quadrilateral graph, were found relatively recently (see [1, 15, 17]).

It was generally believed that a semi-discrete analog does not exist at all, since Tzitzeika equation does not admit a first-order Backlund transformation. However the following lattice can be regarded as the desired semi-discrete version

\[ u_{1,x} = u_{0,x} + \lambda_1(e^{-2u} + e^{-2u_1}) + \lambda_2\sqrt{e^{2u} + e^{2u_1}} \quad (60) \]

because it tends to Tzitzeika in the continuum limit, admits generalized symmetries on both directions of \( x \) and \( n \). Here \( \lambda_1 \) and \( \lambda_2 \) are arbitrary nonzero constants, which can be easily made equal to 1. However we keep them for some computational reasons. It can be shown that there is no any generalized symmetry of the form (17) for \( N = 3 \) as well as of the form (11) with \( M = 1 \). We succeeded to find generalized symmetries of the form (17) with \( N = 5 \):

\[ \partial_t u_0 = u_{0,5} + 5u_{0,3}(u_{0,2} - u_{0,1} - \lambda_2^2 e^{2u} - \lambda_1^2 e^{-4u}) - 5u_{0,2}u_{0,1} - 15u_{0,2}u_{0,1}(\lambda_1^2 e^{2u} - 4\lambda_1^2 e^{-4u}) + u_{0,1}^5 - 90\lambda_1^2 u_{0,1}e^{-4u} + 5u_{0,1}(\lambda_2^2 e^{2u} + \lambda_1^2 e^{-4u})^2, \quad (61) \]

and of the form (11) with \( M = 2 \):

\[ \partial_r u = \left( (v^2 - 1)^2 - 4v_{-1}^2 T^{-1} \right) \frac{(v_1^2 + 1)(v_{-1}^2 + 1)}{(v^2(v_1 + 1)^2 + (v_{-1} - 1)^2)(v_1(v + 1)^2 + (v_{-1} - 1)^2)}, \quad (62) \]

\[ v = \sqrt{1 + e^{2(u - u_1)} + e^{u - u_1}}. \]
Equation (61) for fixed $n$ is known to be integrable and can be found in [12, (3.8)]. Equation (62) and its modification (85) seem to be new.

Let’s get down to specific calculations. We look for the symmetry of the form (17) with $N = 5$

$$u_{0,1} = C(x)u_{0,5} + g(x, u, u_{0,1}, u_{0,2}, u_{0,3}, u_{0,4}).$$

Equations (13) and (19) take the form:

$$(T - 1)(C'u_{0,5} + Cu_{0,6} + Dg^{(1)}) = -2\lambda_1(T + 1)e^{2u}(Cu_{0,5} + g^{(1)}) + \frac{\lambda_2}{\sqrt{e^{2u} + e^{2u_1}}}(T + 1)e^{-2u}(Cu_{0,5} + g)$$

and

$$(1 - T)\frac{\partial}{\partial u_{0,2}}g^{(1)}(x, u, u_{0,1}, u_{0,2}, u_{0,3}, u_{0,4}) = 5D\log \frac{\partial f}{\partial u_{0,1}} = 0. \quad (64)$$

So, according to the general scheme (see also the previously studied example) we obtain right away

$$g(x, u, u_{0,1}, u_{0,2}, u_{0,3}, u_{0,4}, u_{0,5}) = C(x)u_{0,5} + C_1(x)u_{0,4} + g^{(2)}(x, u, u_{0,1}, u_{0,2}, u_{0,3}). \quad (65)$$

We substitute the obtained representation of $g^{(1)}$ into equation (63) and then differentiate the result with respect to $u_{0,4}$. That gives rise to:

$$(T - 1)\frac{\partial}{\partial u_{0,3}}g^{(2)} = 5C(u_{0,1}L^2f + (f - u_{0,1})\frac{d}{du_1}Lf) - C'Lf, \quad L = \frac{d}{du_1} + \frac{d}{du_0}. \quad (66)$$

By applying the operators $T^{-1}\partial_{u_1}$ and $Y_{-1}$ to both sides of (66) we find, respectively:

$$\sum_{k=0}^{3} Y_1(u_{0,k})\frac{\partial^2}{\partial u_{0,3}\partial u_{0,k}}g^{(2)} = 5C \left( 8\lambda_1^2e^{-4u} - \lambda_2^2e^{-2u} - 3\lambda_1\lambda_2 + \lambda_2 u_{0,1}e^{2u} \right) \frac{e^{2u}}{\sqrt{e^{2u_1} + e^{2u}}} + 8\lambda_1 u_{0,1}e^{-2u} \right)$$

$$-C' \left( 4\lambda_1 e^{-2u} + \lambda_2 \frac{e^{2u}}{\sqrt{e^{2u_1} + e^{2u}}} \right). \quad (67)$$

and

$$\sum_{k=0}^{3} Y_{-1}(u_{0,k})\frac{\partial^2}{\partial u_{0,3}\partial u_{0,k}}g^{(2)} = 5C \left( \lambda_2^2 e^{2u} - 8\lambda_1^2 e^{-4u} + 3\lambda_1\lambda_2 - \lambda_2 u_{0,1}e^{2u} \right) \frac{e^{2u}}{\sqrt{e^{2u_1} + e^{2u}}} + 8\lambda_1 u_{0,1}e^{-2u} \right)$$

$$-C' \left( 4\lambda_1 e^{-2u} + \lambda_2 \frac{e^{2u}}{\sqrt{e^{2u_1} + e^{2u}}} \right). \quad (68)$$

It can shown that these two equations are connected with each other by the transformation $\lambda_1 \rightarrow -\lambda_1, \lambda_2 \rightarrow -\lambda_2, u_{-1} \rightarrow u_1$. Therefore, below we provide a detailed study only of the first of them. We commute these equations with the operators $\frac{\partial}{\partial u_{-1}}$ and $\frac{\partial}{\partial u_{1}}$ respectively and obtain

$$(u_{0,1}^2 + u_{0,2} + \lambda_2^2 e^{2u} \pm 6\lambda_1 u_{0,1}e^{-2u} + 7\lambda_1^2 e^{-4u})\frac{\partial^2}{\partial u_{0,3}^2}g^{(2)} + (u_{0,1} \mp 3\lambda_1 e^{-2u})\frac{\partial^2}{\partial u_{0,3}\partial u_{0,1}}g^{(2)}$$

$$+ \frac{\partial^2}{\partial u_{0,3}\partial u_{0,1}}g^{(2)} = -5C(u_{0,1} \pm 3\lambda_1 e^{-2u}) - C'. \quad (69)$$
The consistency condition of the system (67)–(69) implies \( C'(x) = 0 \). So, without loss of generality we can put
\[ C(x) = 1. \]

Solution of the system (67)–(69) is
\[ g^{(2)} = -5(u_{0,1}^2 - u_{0,2} + \lambda_1^2 e^{-4u} + \lambda_2^2 e^{2u})u_{0,3} + C_2(x)u_{0,3} + g^{(3)}(x, u, u_{0,1}, u_{0,2}). \] (70)

Here \( C_2(x) \) and \( g^{(3)}(x, u, u_{0,1}, u_{0,2}) \) are functions to be defined in the next steps. Now we differentiate equation (63) with respect to \( u_{0,3} \) and get:
\[
(T - 1)\frac{\partial}{\partial u_{0,2}}g^{(3)} = -4C_1\left(Au_{0,1}(\lambda_2 + 4\lambda_1 A) + 4A^2\lambda_1^2 e^{-2u-4u_1} + \lambda_1\lambda_2(e^{-2u} + 4e^{-2u_1}) + \lambda_2^2 e^{2u_1}\right) \\
+ 10A^2\lambda_1(2u_{0,1}^2 - u_{0,2})e^{-2u-2u_1} - 10A\lambda_2(u_{0,1}^2 + u_{0,2}) \\
+ 5u_{0,1}(4A^2\lambda_1^2(5e^{-2u} - 2e^{-2u_1})e^{-4u-4u_1} + 6A\lambda_1\lambda_2 e^{-2u_1} + \lambda_2^2 e^{2u - 7e^{-2u_1}}) \\
+ 20A^2\lambda_1^3(e^{-4u-4u_1} + 4e^{-2u-6u_1}) - 10A\lambda_1\lambda_2(e^{-4u} - 3e^{-2u-2u_1} - 4e^{-4u_1}) \\
+ 5A^2\lambda_1^2(4e^{-2u_1} - 7e^{-2u}) - 25A\lambda_1^3 e^{2u_1} - C'_1 L f, \quad A = \sqrt{e^{2u_1} + e^{2u_0}}.
\]

Then we apply the operators \( T^{-1}\partial_u \) and \( Y_{-1} \) to both sides of the obtained relation. In the first case we arrive at an equation of the form:
\[
\left(4\lambda_2^2 e^{-4u} + 4\lambda_1 u_{0,1} e^{-2u} + \lambda_2 B(u_{0,1} e^{2u} - 3\lambda_1) + \lambda_2^2 e^{2u}\right) \frac{\partial^2}{\partial u_{0,2}^2} g^{(3)} \\
+ (\lambda_2 e^{2u} B - 2\lambda_1 e^{-2u}) \frac{\partial^2}{\partial u_{0,2} \partial u_{0,1}} g^{(3)} + \frac{\partial^2}{\partial u_{0,2} \partial u} g^{(3)} = -4C_1\lambda_2 B(3\lambda_1 + u_{0,1} e^{2u}) \\
+ 4C_1\left(8\lambda_2^2 e^{-4u} - \lambda_2^2 e^{2u} + 8\lambda_1 u_{0,1} e^{-2u}\right) + 40u_{0,1}(\lambda_2^2 e^{2u} + 7\lambda_1^2 e^{-4u}) \\
+ 20\lambda_1 e^{-2u}(u_{0,2} - 2u_{0,1}) + 30\lambda_1(\lambda_2^2 - 4\lambda_1^2 e^{-6u}) - 10\lambda_2 e^{2u} B(u_{0,2} + u_{0,1}) \\
+ 15\lambda_2 B(4\lambda_1^2 e^{-2u} - \lambda_2^2 e^{4u} + 2\lambda_1 u_{0,1}) \\
- C'_1\left(4\lambda_1 e^{-2u} + \lambda_2 e^{2u} B\right), \quad B = 1/\sqrt{e^{2u} + e^{2u_1}}.
\] (71)

The second one is the same up to the transformation \( \lambda_1 \rightarrow -\lambda_1, \lambda_2 \rightarrow -\lambda_2, u_{-1} \rightarrow u_1 \). We can compare in the equation (71) coefficients in front of the independent variable \( B \) and derive four equations. From the consistency condition of these equations we get
\[ C'_1(x) = 0, \quad C_1(x) = C_1. \]

Summarizing the reasoning above we find the solution:
\[
g^{(3)} = -2u_{0,2}(2\lambda_2^2 e^{2u} + 2\lambda_1^2 e^{-4u} + 2u_{0,1}^2 - u_{0,2})C_1 \\
- 5u_{0,2} u_{0,1}(3\lambda_2^2 e^{2u} - 12\lambda_1^2 e^{-4u} + u_{0,2}) + C_3(x) u_{0,2} + g^{(4)}(x, u, u_{0,1}).
\] (72)

In the next step we observe that equation (63) is quadratic in \( u_{0,2} \). Moreover, the second derivative with respect to \( u_{0,2} \) reads as:
\[-6C_1\lambda_2 \sqrt{e^{2u} + e^{2u_1}} = 0.\]

Hence we get
\[ C_1 = 0. \]
The first derivative with respect to \( u_{0,2} \) coincides with:

\[
(T - 1) \left( \frac{\partial}{\partial u_{0,1}} g^{(4)} - 5u_{0,1}^4 + 270\lambda_1^2 u_{0,1}^2 e^{-4u} - 5(\lambda_2^2 e^{2u} + \lambda_1^2 e^{-4u})^2 + C'_2(u_{0,1} - 3\lambda_1 e^{-2u}) + \frac{1}{2} C_2(3u_{0,1}^2 + 18\lambda_1 u_{0,1} e^{-2u} + 3\lambda_2^2 e^{2u} + 3\lambda_1^2 e^{-4u}) \right) + 6\lambda_1 e^{-2u}(3u_{0,1}C_2 - C'_2) = 0. \tag{73}
\]

Obviously equation \((73)\) implies

\[ C_2(x) = 0, \]
\[ g^{(4)} = u_{0,1}^5 - 90\lambda_1^2 u_{0,1}^3 e^{-4u} + 5(\lambda_2^2 e^{2u} + \lambda_1^2 e^{-4u})^2 u_{0,1} + C_4(x)u_{0,1} + g^{(5)}(x, u). \tag{74}\]

Now equation \((63)\) has the quadratic term in \( u_{0,1} \) with the coefficient \( C_3(x) \), so we should take

\[ C_3(x) = 0. \]

The remaining part in \((63)\) has the form:

\[
(T - 1) \left( u_{0,1} \frac{\partial}{\partial u_0} g^{(5)} + \frac{\partial}{\partial x} g^{(5)} + C'_4 u_{0,1} \right) = \frac{\partial f}{\partial u_0} g^{(5)} + \frac{\partial f}{\partial u_1} T g^{(5)}. \tag{75}\]

Let us apply the operator \( \frac{\partial^2}{\partial u_{0,1} \partial u_0} \) to \((75)\) and find

\[ \frac{\partial^2}{\partial u_0^2} g^{(5)} = 0, \quad g^{(5)} = C_5(x)u_0 + C_6(x). \]

Now equation \((75)\) does not depend on \( u_{0,1} \). This equation must be fulfilled identically for all values of \( u_0 \) and \( u_1 \), in particular, for \( u_1 = u_0 \) we get:

\[
\sqrt{2}\lambda_2(\lambda'_4 + C_5 - C_6)e^u + 2\lambda_1(\lambda'_4 + C_5 + 2C'_6)e^{-2u} + 4C_5\lambda_1 u e^{-2u} - \sqrt{2}u\lambda_2 C_5 e^{-u} = 0.
\]

Functions \( e^u, e^{-2u}, u e^u, u e^{-2u} \) are linearly independent, so we get

\[ C'_5 = 0, \quad C'_4 - C_6 = 0, \quad C'_4 + 2C'_6 = 0. \]

We find

\[ C_6 = 0, \quad C_4(x) = c = \text{const}. \]

Let us take \( c = 0 \) since this term is responsible for the classical symmetry. Finally, we get the generalized symmetry \((61)\).

We proceed to generalized symmetries in the direction of \( n \). It is easily proved that there is no generalized symmetry of the form \((11)\) with \( M = 1 \), therefore the simplest symmetry of this kind should be of the form

\[ \partial_k u_0 = h(u_{-2}, u_{-1}, u, u_1, u_2). \tag{76}\]

Upon closer examination of the equation \((35)\), adapted to our case, we can see that it is actually a linear expression with respect to the variable \( u_{0,1} \). Therefore the coefficient in front of \( u_{0,1} \) vanishes. This condition yields the following equation:

\[
(T - 1) \sum_{k=-2}^{2} \frac{\partial}{\partial u_k} h(u_{-2}, u_{-1}, u, u_1, u_2) = 0 \tag{77}
\]
that obviously implies a linear first order PDE for $h$

$$
\sum_{k=-2}^{2} \frac{\partial}{\partial u_k} h(u_{-2}, u_{-1}, u, u_1, u_2) = C_0,
$$

where $C_0$ is an arbitrary constant. General solution to the latter equation is easily found

$$
h(u_{-2}, u_{-1}, u, u_1, u_2) = h^{(1)}(u_{-2} - u_{-1} - u, u - u_1 - u_2) + C_0 u_0.
$$

(78)

Let us rewrite the original lattice (60) in terms of the variable $y_n = u_n - u_{n+1}$:

$$
\frac{dy}{dx} = \frac{d(u_0 - u_1)}{dx} = \lambda_1 e^{-2u}(1 + e^{2y}) + \lambda_2 e^{u-y} \sqrt{1 + e^{2y}}
$$

(79)

In order to bring the lattice to a rational form we make in addition a point transformation from $y$ to $v$ defined by formulas

$$
v = \sqrt{1 + e^{2y}} + e^y, \quad e^y = \frac{v^2 - 1}{2v}, \quad \sqrt{1 + e^{2y}} = \frac{1 + v^2}{2v}.
$$

In terms of the variable $v_k$ we have:

$$
\frac{dv_k}{dx} = \lambda_1 V_k e^{2u} + \lambda_2 U_k e^u,
$$

(80)

where $V_k$ and $U_k$ are functions depending on a finite set of the shifts $\{v_n\}$ only, and can be found recursively:

$$
V_0 = -\frac{v_0^4}{4v_0}, \quad U_0 = -v_0,
$$

$$
V_k = \frac{(v^2 - 1)^2}{4v^2} TV_{k-1}, \quad U_k = \frac{2v}{v^2 - 1} TU_{k-1}, \quad k \in \mathbb{N},
$$

$$
V_{-k} = T^{-1} \frac{4v}{(v^2 - 1)^2} V_{k+1}, \quad U_{-k} = T^{-1} \frac{2v}{v^2 - 1} U_{k+1}, \quad k \in \mathbb{N}.
$$

We look for the right hand side $h$ of the symmetry (76) in the form

$$
h(u_{-2}, u_{-1}, u, u_1, u_2) = H(v_{-2}, v_{-1}, v, v_1) + C_0 u_0.
$$

(81)

Note that now the set of functions $u := u_0, v_0, v_1, v_{-1}, \ldots$ defines a new set of dynamical variables. By passing in the equation (85) to this set of the variables we arrive at:

$$
\lambda_2 e^u M_1 + \lambda_1 e^{-2u} M_2 + C_0 u(v^2 + 1) \left( \frac{\lambda_1(v^2 + 1)}{2v^2} e^{-2u} - \frac{\lambda_2}{v^2 - 1} e^u \right) = 0,
$$

(82)

where

$$
M_1 = \sum_{k=-2}^{2} U_k \frac{\partial}{\partial v_k} (T - 1) H + \frac{v^2 - 1}{v^2 + 1} H + \frac{4v^2}{v^4 - 1} TH
$$

$$
- C_0 \left( \frac{v^2 + 1}{v(v^2 - 1)} + \frac{4v^2}{v^4 - 1} \log \frac{2v}{v^2 - 1} \right),
$$

$$
M_2 = \sum_{k=-2}^{2} V_k \frac{\partial}{\partial v_k} (T - 1) H - 2H - \frac{v^4 - 1}{2v^2} TH
$$

$$
- C_0 \left( \frac{v^2 + 1}{v(v^2 - 1)} - \frac{(v^2 - 1)^2}{2v^2} \log \frac{2v}{v^2 - 1} \right).
$$
Since functions $e^u, e^{-2u}, ue^u, ue^{-2u}$ are linearly independent we immediately obtain from [82]:

$$C_0 = 0, \quad M_1 = 0, \quad M_2 = 0.$$  

The last two equations are functional differential equations for function $H(v_{-1}, v_1)$. Dependence of $H$ on the variables $v_{-1}, v_1$ can be specified due to the system of partial differential equations

$$T^{-1} \frac{\partial}{\partial v_2} M_1 = 0, \quad T^{-1} \frac{\partial}{\partial v_2} M_2 = 0,$$

$$\frac{\partial}{\partial v_{-2}} M_1 = 0, \quad \frac{\partial}{\partial v_{-2}} M_2 = 0.$$  

(83)

Solution of this system is given by:

$$H = \frac{C_1 v (v_{-1}^2 + 1)}{(v_{-1}^2 + 1) (v^2 (v_{-1} + 1)^2 + (v_{-1} - 1)^2)} + \frac{C_2 (v^2 + 1) v_{-1}^2 (v_{-1} - 1)}{(v^2 (v_{-1} + 1)^2 + (v_{-1} - 1)^2) (v_{-1} + \frac{(v_{-2} - 1)^2}{(v_{-2} + 1)^2})} + H^{(1)}(v_{-1}, v).$$  

(84)

Any of the equations

$$\frac{\partial^2}{\partial v_{-1} \partial v_1} M_1 = 0, \quad \frac{\partial^2}{\partial v_{-1} \partial v_1} M_2 = 0$$

implies $C_2 = C_1/2$. Similarly from the system

$$\frac{\partial}{\partial v_{-1}} M_1 = 0, \quad \frac{\partial}{\partial v_{-1}} M_2 = 0, \quad T^{-1} \frac{\partial}{\partial v_1} M_1 = 0, \quad T^{-1} \frac{\partial}{\partial v_1} M_2 = 0$$

one can find

$$H^{(1)} = -\frac{C_1}{2} \frac{(v_{-1} + 1) v^2 - (v_{-1}^2 + 1) v - v_{-1} + 1}{v (v_{-1} + 1)^2 + (v_{-1} - 1)^2} + C_3.$$  

And now from $M_1 = 0$ or from $M_2 = 0$ we find $C_3 = -C_1/4$. For simplicity we put $C_1 = 4$ and finally get [62].

### 4.1 Equations in variable $v_n$

We can easily rewrite generalized symmetry [62] in terms of the variable $v$:

$$\partial_r v = v (v^2 - 1) \left( v^2 + 1 - \frac{4v_{-1}^2}{v^2 + 1} T^{-1} - \frac{(v_{-1}^2 - 1)^2}{v^2 + 1} \right)$$

$$\frac{(v_{-1}^2 + 1)(v_{-1} - 1)}{(v^2 (v_{-1} + 1)^2 + (v_{-1} - 1)^2)(v_1 (v + 1)^2 + (v - 1)^2)}$$

(85)

We can rewrite the semi-discrete equation [60] in terms of $v$ as well. However to this end we have to solve a cubic equation:

$$\lambda_2 v p^3 + v_0 x p^2 + \lambda_1 \frac{v^2 - 1}{4v} = 0.$$  

(86)

If $p(v_0, v_{0,x})$ is a solution of the cubic equation, then

$$p(v_0, v_{0,x}) \frac{2v}{1 - v^2} = p(v_1, v_{1,x})$$

is semi-discrete equation, defining the Backlund transformation for the equation [55].
4.2 Continuum limit

Here we compute the continuum limit of the semi-discrete Tzitzeika equation (60) and of its generalized symmetries. To do this we apply the transformation:

\[ u_n(x, t, \tau) = U \left( x, y - \frac{4}{9} \varepsilon \tau, t, \frac{2 \varepsilon^5}{135} \tau \right), \quad \lambda_1 = \varepsilon a/2, \quad \lambda_2 = \varepsilon b/2. \quad (87) \]

Then, as \( \varepsilon \to 0 \) from (60) we get:

\[ U_{x,y} = ae^{-2U} + be^U. \quad (88) \]

The same substitution (87) into (61) and (62) gives rise for \( \varepsilon \to 0 \) to equations

\[ \partial_t U = U_{xxxxx} + 5U_{xxx}(U_{xx} - U_x^2) - 5U_{xx}^2 U_x + U_x^5, \quad (89) \]

\[ \partial_y U = U_{yyyyy} + 5U_{yyy}(U_{yy} - U_y^2) - 5U_{yy}^2 U_y + U_y^5, \quad \theta = \frac{2 \varepsilon^5}{135} \tau. \]

The obtained equations are well known \( x- \) and \( y- \)symmetries of the Tzitzeika equation (88).

Conclusions

The article proposes a method for finding generalized symmetries for semi-discrete lattices of hyperbolic type. We hope that the method will find fruitful application in studying the problem of an integrable classification of lattices of this type. Despite the fact that a large number of integrable semi-discrete models of hyperbolic type have already been obtained within the framework of various discretization algorithms, including the Backlund transformations, this class requires further study: in our opinion, it may contain new integrable models. Our hypothesis is supported by the example of the Tziseica equation given in this article.

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