SLODOWY SLICES AND THE COMPLETE INTEGRABILITY OF Mishchenko-Fomenko SUBALGEBRAS ON REGULAR ADJOINT ORBITS

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Abstract. This work is concerned with Mishchenko-Fomenko subalgebras and their restrictions to the adjoint orbits in a finite-dimensional complex semisimple Lie algebra. In this setting, it is known that each Mishchenko-Fomenko subalgebra restricts to a completely integrable system on every orbit in general position. We improve upon this result, showing that each Mishchenko-Fomenko subalgebra yields a completely integrable system on all regular orbits (i.e., orbits of maximal dimension). Our approach incorporates the theory of regular $sl_2$-triples and associated Slodowy slices, as developed by Kostant.

1. Introduction

Mishchenko-Fomenko subalgebras are ubiquitous in both the classical and modern theories of integrable systems, and they give rise to a fruitful synergy of algebraic geometry, Lie theory, and symplectic geometry (see [1–5, 8–11], for instance). One studies them in the context of a finite-dimensional Lie algebra $g$, in which case $g^*$ carries a canonical Poisson structure. Each regular element $a \in g^*$ then determines a Mishchenko-Fomenko subalgebra $F_a$ of the polynomials on $g^*$, obtained by applying an argument-shifting procedure to invariants of the coadjoint representation. This Poisson-commutative subalgebra is often a completely integrable system on $g^*$, i.e., $F_a$ is often complete (see [4, Theorem 1.3]). The most classically studied cases arise when $g$ is complex semisimple, in which event every Mishchenko-Fomenko subalgebra is a completely integrable system.

We now describe the context of interest to us. Let $g$ be a complex semisimple Lie algebra of finite dimension $n$ and rank $r$. Note that $g$ is the Lie algebra of a connected, simply-connected complex semisimple linear algebraic group $G$. Note also that we have the adjoint representation

$$\text{ad} : g \rightarrow \text{gl}(g), \quad x \mapsto \text{ad}_x, \quad x \in g,$$

which satisfies $\text{ad}_x(y) = [x, y]$ for all $x, y \in g$. An element $x \in g$ is defined to be regular if $\text{dim} (\ker (\text{ad}_x)) = r$, and we shall let $g_{\text{reg}}$ denote the open, dense, $G$-invariant subvariety of all regular elements in $g$. An adjoint orbit $O \subseteq g$ is defined to be regular when $O \subseteq g_{\text{reg}}$, or equivalently $\text{dim}(O) = n - r$.

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Recall that the Killing form induces a $G$-equivariant identification of $\mathfrak{g}$ with $\mathfrak{g}^*$, through which we can transfer relevant algebraic and geometric structures (ex. the Poisson structure, Mishchenko-Fomenko subalgebras) from the latter space to the former. Note that $\mathbb{C}[\mathfrak{g}] := \text{Sym}(\mathfrak{g}^*)$ is then a Poisson algebra with Poisson centre equal to the subalgebra of all $G$-invariant polynomials, $\mathbb{C}[\mathfrak{g}]^G \subseteq \mathbb{C}[\mathfrak{g}]$. Now fix a regular element $a \in \mathfrak{g}_{\text{reg}}$, and associate to each $f \in \mathbb{C}[\mathfrak{g}]^G$ and $\lambda \in \mathbb{C}$ the polynomial $f_{\lambda,a} \in \mathbb{C}[\mathfrak{g}]$ defined by

$$f_{\lambda,a}(x) := f(x + \lambda a), \quad x \in \mathfrak{g}.$$ 

One then defines $\mathcal{F}_a$ to be the subalgebra of $\mathbb{C}[\mathfrak{g}]$ generated by all polynomials of the form $f_{\lambda,a}$, with $f \in \mathbb{C}[\mathfrak{g}]^G$ and $\lambda \in \mathbb{C}$.

We wish to study the restriction of a Mishchenko-Fomenko subalgebra $\mathcal{F}_a$ to the symplectic leaves of $\mathfrak{g}$, i.e. to the adjoint orbits of $G$. More precisely, let $\mathcal{F}_a|_O$ denote the algebra of all functions obtained by restricting elements of $\mathcal{F}_a$ to an adjoint orbit $O \subseteq \mathfrak{g}$. This algebra is Poisson-commutative with respect to the Poisson structure on $O$, so that it is natural to ask the following question: for which combinations of a regular element $a \in \mathfrak{g}$ and an adjoint orbit $O \subseteq \mathfrak{g}$ is $\mathcal{F}_a|_O$ a completely integrable system on $O$?

There are a few well-known results that address the question posed above. One such result is due to Mishchenko and Fomenko, who proved that $\mathcal{F}_a|_O$ is a completely integrable system whenever $a$ and $O$ are in general position (see [10, Theorem 4.2]). Their proof requires $a$ to be a regular semisimple element and $O$ to be a regular adjoint orbit, so that the meaning of “general position” is stronger than that of “regular” for both elements of $\mathfrak{g}$ and adjoint orbits. With this point in mind, there are two notable generalizations of the Mishchenko-Fomenko result. The first follows from Bolsinov’s work, and can be stated as follows: if $a \in \mathfrak{g}$ is any regular element, then $\mathcal{F}_a|_O$ is a completely integrable system on each regular adjoint orbit $O \subseteq \mathfrak{g}$ in general position (see [3]).

The second generalization derives from Kostant’s work, which effectively shows $\mathcal{F}_a|_O$ to be a completely integrable system whenever $a$ is a regular semisimple element and $O$ is a regular adjoint orbit (see [8, Proposition 4.7]).

This paper generalizes all of the above-mentioned results as follows.

**Theorem 1.1.** If $a \in \mathfrak{g}$ is any regular element, then $\mathcal{F}_a|_O$ is a completely integrable system on each regular adjoint orbit $O \subseteq \mathfrak{g}$.

2. Slodowy slices and $\mathfrak{sl}_2$-triples

Our proof of Theorem 1.1 draws from Kostant’s work on regular $\mathfrak{sl}_2$-triples and their Slodowy slices, the relevant parts of which we now review. Let all objects and notation be as described in Section 1, after the first paragraph. We recall that $(\xi, h, \eta) \in \mathfrak{g}^{\oplus 3}$ is called an $\mathfrak{sl}_2$-triple if the identities

$$[h, \xi] = 2\xi, \quad [h, \eta] = -2\eta, \quad [\xi, \eta] = h$$

hold in $\mathfrak{g}$. In this case, one may consider the associated Slodowy slice

$$S(\xi, h, \eta) := \xi + \ker(\text{ad}_\eta) := \{\xi + x : x \in \ker(\text{ad}_\eta)\} \subseteq \mathfrak{g}.$$
We will be particularly interested in those Slodowy slices that arise when $(\xi, h, \eta)$ is a regular $\mathfrak{sl}_2$-triple, i.e. when $\xi$ is regular.

**Theorem 2.1.** (cf. [7, Theorem 8]) Let $(\xi, h, \eta)$ be a regular $\mathfrak{sl}_2$-triple. If $O \subseteq g$ is any regular adjoint orbit, then $S(\xi, h, \eta)$ intersects $O$ in a unique point.

We will benefit from constructing a specific regular $\mathfrak{sl}_2$-triple. To this end, let $b_+, b_- \subseteq g$ be opposite Borel subalgebras. Note that $h := b_+ \cap b_-$ is then a Cartan subalgebra of $g$, and that we have roots $\Delta \subseteq h^*$. Given $\alpha \in \Delta$, let $g_\alpha$ denote the $\alpha$-root space in $g$, i.e.

$$g_\alpha := \{ x \in g : \text{ad}_y(x) = \alpha(y)x \text{ for all } y \in h\}.$$  

We may then define collections of positive roots $\Delta_+ \subseteq \Delta$ and negative roots $\Delta_- \subseteq \Delta$ by the condition that

$$b_\pm = h \oplus \bigoplus_{\alpha \in \Delta_\pm} g_\alpha$$

as $h$-modules. Let $\Pi \subseteq \Delta_+$ denote the resulting collection of simple roots. For each $\alpha \in \Pi$, let $h_\alpha \in h$ be the corresponding simple coroot and choose elements $e_\alpha \in g_\alpha$ and $e_{-\alpha} \in g_{-\alpha}$ such that $[e_{\alpha}, e_{-\alpha}] = h_\alpha$. Let us define $h$ to be the unique element of $h$ satisfying $\alpha(h) = -2$ for all $\alpha \in \Pi$. Noting that the simple coroots form a basis of $h$, we may write

$$-h = \sum_{\alpha \in \Pi} c_\alpha h_\alpha$$

for uniquely determined coefficients $c_\alpha \in \mathbb{C}$. Now consider the nilpotent elements of $g$ defined by

$$\xi := \sum_{\alpha \in \Pi} e_{-\alpha} \quad \text{and} \quad \eta := \sum_{\alpha \in \Pi} c_\alpha e_\alpha.$$  

It is then straightforward to verify that $(\xi, h, \eta)$ is an $\mathfrak{sl}_2$-triple. Since $\xi$ is known to be a regular element (see [6, Theorem 5.3]), we see that $(\xi, h, \eta)$ is regular.

The following lemma will be needed to help prove the main result of our paper.

**Lemma 2.2.** If $(\xi, h, \eta)$ is the $\mathfrak{sl}_2$-triple constructed above, then $\ker(\text{ad}_\eta) \subseteq b_+$.  

**Proof.** Since $\alpha(h) = -2$ for all $\alpha \in \Pi$, it follows that $b_+$ is the sum of those $\text{ad}_h$-eigenspaces corresponding to non-positive eigenvalues. At the same time, the representation theory of $\mathfrak{sl}_2$ implies that $h$ acts on $\ker(\text{ad}_\eta)$ with non-positive eigenvalues. We conclude that $\ker(\text{ad}_\eta)$ is contained in $b_+$. $\square$
3. Proof of Theorem 1.1

Let all objects be as described in the statement of Theorem 1.1 and the paragraphs preceding it. Our objective is to prove that the Poisson-commutative algebra $F_a|_{\mathcal{O}}$ is complete, i.e. that the subspace

$$d_x(F_a|_{\mathcal{O}}) := \{d_x f : f \in F_a|_{\mathcal{O}}\} \subseteq T_x^* \mathcal{O}$$

has dimension $\frac{1}{2} \dim(\mathcal{O}) = \frac{1}{2}(n - r)$ for all $x$ in an open dense subset of $\mathcal{O}$. To this end, consider the set

$$(1) \quad \text{Sing}_a(\mathcal{O}) := \left\{ x \in \mathcal{O} : \dim(d_x(F_a|_{\mathcal{O}})) < \frac{1}{2}(n - r) \right\}$$

of singularities of $F_a|_{\mathcal{O}}$. Note that $F_a|_{\mathcal{O}}$ is complete if and only if $\mathcal{O} \setminus \text{Sing}_a(\mathcal{O})$ is open and dense in $\mathcal{O}$. At the same time, $\mathcal{O}$ is irreducible and contains $\text{Sing}_a(\mathcal{O})$ as a Zariski-closed subset. This means that either $\mathcal{O} = \text{Sing}_a(\mathcal{O})$ or every irreducible component of $\text{Sing}_a(\mathcal{O})$ has strictly positive codimension in $\mathcal{O}$, in which case $\mathcal{O} \setminus \text{Sing}_a(\mathcal{O})$ is necessarily open and dense in $\mathcal{O}$. Accordingly, $\mathcal{O} \setminus \text{Sing}_a(\mathcal{O})$ is open and dense in $\mathcal{O}$ if and only if $\mathcal{O} \neq \text{Sing}_a(\mathcal{O})$.

In light of the above-mentioned equivalences, we are reduced to proving that $\mathcal{O} \neq \text{Sing}_a(\mathcal{O})$. Let us assume that $\mathcal{O} = \text{Sing}_a(\mathcal{O})$, for the sake of an argument by contradiction. Now recall that $\mathbb{C}^[\mathfrak{g}]^{\mathfrak{g}^*}$ is generated by $r$ homogeneous, algebraically independent polynomials $f_1, \ldots , f_r \in \mathbb{C}[\mathfrak{g}]^{\mathfrak{g}^*}$. Set $d_i := \text{degree}(f_i)$, $i \in \{1, \ldots , r\}$, and recall that $\sum_{i=1}^r d_i = \frac{1}{2}(n + r) =: \ell$ (see [12, Equation (1)]). We define new polynomials $f_{ij} \in \mathbb{C}[\mathfrak{g}]$, $i \in \{1, \ldots , r\}$, $j \in \{0, \ldots , d_i - 1\}$, by the following expansions:

$$f_i(x + \lambda a) = \sum_{j=0}^{d_i - 1} f_{ij}(x)\lambda^j + f_i(a)\lambda^{d_i}, \quad i \in \{1, \ldots , r\},$$

where $x \in \mathfrak{g}$ and $\lambda \in \mathbb{C}$. The $\ell$-many polynomials $f_{ij}$ turn out to form a list of algebraically independent generators of $F_a$ (see [11, Section 3], for instance). Note also that $f_{i0} = f_i$ for all $i \in \{1, \ldots , r\}$, so that we have introduced only $\ell - r = \frac{1}{2}(n - r)$ new polynomials. Let us record these new polynomials as $f_{r+1}, \ldots , f_{\ell}$.

Suppose that $x \in \mathcal{O}$. Since $x \in \text{Sing}_a(\mathcal{O})$, it follows that the $\frac{1}{2}(n - r)$ vectors $d_x(f_{r+1}|_{\mathcal{O}}), \ldots , d_x(f_{\ell}|_{\mathcal{O}})$ are linearly dependent in $T_x^* \mathcal{O}$. This is the statement that

$$\sum_{i=r+1}^{\ell} a_i(d_x(F_i|_{\mathcal{O}})) = 0$$

for some $a_{r+1}, \ldots , a_{\ell} \in \mathbb{C}$, not all equal to 0. Now note that $f_1, \ldots , f_r$ are constant on $\mathcal{O}$, so that the annihilator of $T_x \mathcal{O}$ in $T_x^* \mathfrak{g}$ must contain

$$U := \text{span}\{d_x f_1, \ldots , d_x f_r\}.$$

This annihilator is $r$-dimensional, since the regularity of $\mathcal{O}$ implies that $T_x \mathcal{O}$ has dimension $n - r$. At the same time, the fact that $x \in \mathfrak{g}_{\text{reg}}$ implies that
$U$ is $r$-dimensional (see [7, Theorem 7]). We conclude $U$ is precisely the annihilator of $T_x \mathcal{O}$ in $T_x^* \mathfrak{g}$, so that
\[ \sum_{i=r+1}^\ell a_i (d_x f_i) \in U. \]
Given how $U$ is defined, this means that $d_x f_1, \ldots, d_x f_\ell$ must be linearly dependent in $T_x^* \mathfrak{g}$. Now let $\text{Sing}(\mathcal{F}_a)$ denote the set of singularities of $\mathcal{F}_a$ in $\mathfrak{g}$, which one defines analogously to (1). Since $f_1, \ldots, f_\ell$ are algebraically independent generators of $\mathcal{F}_a$, we see that $\text{Sing}(\mathcal{F}_a)$ is the set of points at which $d f_1, \ldots, d f_\ell$ are linearly dependent. We also have that $\text{Sing}(\mathcal{F}_a) = \mathfrak{g}_{\text{sing}} + \mathbb{C} \cdot a$ (see [4, Proposition 3.1]), where $\mathfrak{g}_{\text{sing}} := \mathfrak{g} \setminus \mathfrak{g}_{\text{reg}}$ is the set of singular elements in $\mathfrak{g}$. Using these last two sentences, we conclude that $x \in \mathfrak{g}_{\text{sing}} + \mathbb{C} \cdot a$. In particular, this paragraph establishes the inclusion
\[ (2) \quad \mathcal{O} \subseteq \mathfrak{g}_{\text{sing}} + \mathbb{C} \cdot a. \]

Now choose opposite Borel subalgebras $b_+, b_- \subseteq \mathfrak{g}$ with the property that $a \in b_+$. Beginning with these opposite Borel subalgebras, we may repeat all of the constructions outlined between Theorem 2.1 and Lemma 2.2 to produce a specific $\mathfrak{sl}_2$-triple ($\xi, h, \eta$). Recall that this triple is regular, so that Theorem 2.1 implies that the Slodowy slice $S(\xi, h, \eta)$ must intersect $\mathcal{O}$. It now follows from (2) that $S(\xi, h, \eta)$ intersects $\mathfrak{g}_{\text{sing}} + \mathbb{C} \cdot a$. In other words, there exist $x \in \mathfrak{g}_{\text{sing}}$, $\lambda \in \mathbb{C}$, and $y \in \ker(\text{ad}_\eta)$ such that $x + \lambda a = \xi + y$, i.e.
\[ x = \xi + y - \lambda a. \]
Since $y \in b_+$ (by Lemma 2.2) and $a \in b_+$, we conclude that $x \in \xi + b_+$. At the same time, it is known that $\xi + b_+ \subseteq \mathfrak{g}_{\text{reg}}$ (see [7, Lemma 10]). The previous two sentences imply that $x \in \mathfrak{g}_{\text{reg}}$, contradicting the fact that $x \in \mathfrak{g}_{\text{sing}}$. In particular, $\mathcal{O} = \text{Sing}_a(\mathcal{O})$ cannot hold. This completes the proof.

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