Causal hydrodynamic fluctuations in non-static and inhomogeneous backgrounds

Koichi Murase

1 Department of Physics, Sophia University, 7-1 Kioicho, Chiyoda-ku, Tokyo 102-8554, Japan
2 Department of Physics, The University of Tokyo, 7-3-1 Hongo, Bunkyo-ku, Tokyo 113-0033, Japan

(Dated: October 29, 2019)

To integrate hydrodynamic fluctuations, namely thermal fluctuations of hydrodynamics, into dynamical models of high-energy nuclear collisions based on relativistic hydrodynamics, the property of the hydrodynamic fluctuations given by the fluctuation–dissipation relation should be carefully investigated. The fluctuation–dissipation relation for causal dissipative hydrodynamics with the finite relaxation time is naturally given in the integral form of the constitutive equation by the linear-response theory. While, the differential form of the constitutive equation is commonly used in analytic investigations and dynamical calculations for practical reasons. We give the fluctuation–dissipation relation for the general linear-response differential form and discuss the restrictions to the structure of the differential form, which comes from the causality and the positive semi-definiteness of the noise autocorrelation, and also the relation of those restrictions to the cutoff scale of the hydrodynamic fluctuations. We also give the fluctuation–dissipation relation for the integral form in non-static and inhomogeneous background by introducing new tensors, the pathline projectors. We find new modification terms to the fluctuation–dissipation relation for the differential form in non-static and inhomogeneous background which are particularly important in dynamical models to describe rapidly expanding systems.

Keywords: Relativistic hydrodynamics; hydrodynamic fluctuations; fluctuation–dissipation relation

CONTENTS

I. Introduction
II. Constitutive equation
   A. Differential form of constitutive equation
   B. Integral form of constitutive equation
III. Tensor structure of memory function
   A. Pathline projectors and their properties
   B. Memory function with tensor structures
   C. Memory function in simplified Israel–Stewart theory
IV. Causal hydrodynamic fluctuations in equilibrium
   A. Colored noise in integral form
   B. White noise in differential form in simplified Israel–Stewart theory
   C. General linear-response differential form
   D. Cutoff scale and structure of differential form
V. FDR in non-static and inhomogeneous backgrounds
   A. Integral form FDR
   B. Differential form FDR
VI. Summary

Acknowledgments

A. Properties of projectors
I. INTRODUCTION

Relativistic hydrodynamics has been widely used in various fields such as cosmology, astrophysics and nuclear physics to describe spacetime evolution of thermodynamic fields in a vast range of the scales from femtometer scale to astronomical scales. In this paper we argue general properties of thermal fluctuations of relativistic hydrodynamics which are originally needed to investigate the effects of the fluctuations on experimental observables of high-energy nuclear collisions but can also be applied to any relativistic systems with the situations the thermal fluctuations have relevant effects.

The aim of the high-energy nuclear collision experiments is to create and understand extremely hot and/or dense nuclear matter whose fundamental degrees of freedom are quarks and gluons. The dynamics of systems of quarks and gluons are ruled by quantum chromodynamics (QCD). While quarks and gluons are confined in hadrons in normal temperature and density, they are deconfined to form a novel state of the matter called quark gluon plasma (QGP) \cite{1-4} in extremely hot and/or dense circumstances such as in the early universe at microseconds after the big bang. The QGP can be experimentally created in high-energy nuclear collision experiments at Relativistic Heavy Ion Collider (RHIC) in Brookhaven National Laboratory and Large Hadron Collider (LHC) in CERN to study the properties of the created QGP.

The key to successfully extract the properties of the created QGP from experimental data is proper dynamical modeling of the collision reaction. The direct observable in the experiments is the distribution of final-state hadrons which formed after the non-equilibrium spacetime evolution of the created QGP. To reconstruct from the hadron distribution the information on the non-equilibrium dynamics of the created matter and to determine the transport properties of the QGP, sophisticated dynamical models, which describe the whole process of the high-energy nuclear collision reaction from the initial state and the QGP to the final state hadron gases, are needed. The standard structure of modern dynamical models is the combination of initial state models which describe the initial thermal-ization process, relativistic hydrodynamics which follows the spacetime evolution of the created matter, and hadronic transport models which describe the final-state interactions. Among them, the most important part is relativistic hydrodynamics reflecting the properties of the QGP. In the early 2000s, dynamical models based on relativistic ideal hydrodynamics \cite{5-8} has been successful to reproduce the experimental observables from RHIC such as the elliptic flows $v_2$ \cite{9-11}, and the new paradigm of the strongly-coupled QGP (sQGP) with tiny viscosity has been established \cite{12-14}. Since then, the hydrodynamic part of dynamical models has been updated to include dissipation such as shear viscosity and bulk viscosity and, along with sophisticated initial conditions and hadronic transport models, used to determine the quantitative values of such transport coefficients \cite{15-18}.

One of the important physics in determining such transport properties of the created matter is event-by-event fluctuations of collision reactions. For example, the flow harmonics $v_n$ reflects the anisotropy of the created matter caused by event-by-event fluctuations of collision reactions. The major part of the higher-order flow harmonics is determined by the initial-state fluctuations of the distributions of nucleons in colliding nuclei. Nevertheless other different kinds of fluctuations, such as thermal fluctuations and jets, have non-negligible contributions to the flow observables \cite{14}. Another example is the event-by-event fluctuations of the conserved charges which could be used as a signal of the critical point and the first-order phase transition of the QGP created in lower-energy collisions. For the search of the QCD critical point in high-baryon density domain of the phase diagram, a number of experiments such as the Beam Energy Scan programs at RHIC, NA61/SHINE experiment at SPS, CBM experiment at FAIR, MPD at NICA, CEE at HIAF and the heavy-ion program at J-PARC are ongoing or planned. To quantitatively determine the properties of the created matter, and to find signals of a critical point in the finite baryon region of the QCD phase.
Hydrodynamic fluctuations \cite{20, 21}, i.e., the thermal fluctuations of hydrodynamics, is one of the sources of event-by-event fluctuations of high-energy nuclear collisions. Also, the hydrodynamic fluctuations near the critical point play an important role in the dynamics near the critical point and the first-order phase transition, so the dynamical description of the hydrodynamic fluctuations is becoming increasingly important. Hydrodynamic fluctuations are introduced in hydrodynamic equations as noise fields, and the noise power spectrum is determined by viscosity and diffusion coefficients through the fluctuation–dissipation relation (FDR) \cite{22}. Now the hydrodynamic equations become stochastic partial differential equation, and such a framework is called fluctuating hydrodynamics \cite{20, 21}. Relativistic fluctuating hydrodynamics is first considered in the first-order theory \cite{19, 23}, and then extended for the second-order theory to apply it to the high-energy nuclear collisions \cite{24, 25}. The effects of hydrodynamic fluctuations to the observables in dynamical models are first investigated in linearized fluctuating hydrodynamics \cite{26} and then in fully non-linear relativistic fluctuating hydrodynamics \cite{27, 31}. For the critical point search, there are already several simulations solving the hydrodynamic fluctuations in the baryon current or other slow modes in simple setups \cite{32–40}. Instead of the Langevin type description (“event-by-event” description) of fluctuations, also the extension of the hydrodynamics (Hydro+) with slow modes and two-point functions is proposed for the critical dynamics \cite{41}. Another important topic of the hydrodynamic fluctuations is the renormalization of hydrodynamics due to the non-linear effects of the hydrodynamic fluctuations which are analyzed by various methods in various contexts \cite{12–50}. The transport coefficients and the equation of state should be renormalized depending on the cutoff scales of the hydrodynamic fluctuations, and also there arises the long-time tail of the two-point correlations, which cannot be renormalized into ordinary transport coefficients. The renormalization of the transport coefficients and the equation of state has already turned out to be important in dynamical models of the high-energy nuclear collisions \cite{25, 33}. The proper choice of the cutoff in dynamical models is also an important problem, for which the theoretical estimation of the bound of the cutoff has been given in the context of renormalization \cite{51, 52}. In existing analyses and calculations, the first-order hydrodynamics is usually assumed, or, even if the second-order causal hydrodynamics is assumed, a naive expression for the FDR in the global equilibrium has been used. However, the matter created in the high-energy nuclear collisions is rapidly expanding and highly inhomogeneous, so that the FDR should be carefully reconsidered.

In this paper we consider the properties of hydrodynamic fluctuations in causal dissipative hydrodynamics to clarify a proper treatment of the hydrodynamic fluctuations of non-static and inhomogeneous matter in dynamical models which is consistent with the causality. In the first-order dissipative hydrodynamics, which is also known as the Navier–Stokes theory, the hydrodynamic fluctuations are the white noise according to the FDR, i.e., the autocorrelation of the hydrodynamic fluctuations is a delta function in space and time. However, the Navier–Stokes theory has problems of acausal propagation of information and unstable modes \cite{53}. Instead, in dynamical models the second-order dissipative hydrodynamics is commonly used because the problem of the acausality is known to be resolved in the second-order hydrodynamics with an appropriate value of the relaxation time \cite{54, 55}. In the second-order causal dissipative hydrodynamics, the dissipative currents are treated as dynamical fields, and they respond to the thermodynamic forces with non-zero time of the relaxation time scale. To respect the causality, the relaxation time of the dissipative currents has a lower bound, which can be understood naively in the following way: the relaxation is caused by the interaction with many degrees of freedom in the system, so a non-zero time span is needed to achieve a sufficient interaction for the relaxation because the interaction is restricted within the light cone in relativistic systems. In such relativistic systems, the hydrodynamic fluctuations of the dissipative currents should also have finite relaxation time. The autocorrelation of the hydrodynamic fluctuations is no longer the delta function but has non-zero values within the time scale of the relaxation time \cite{24, 25}, which means that the FDR should be modified to a colored one from the white one of the Navier–Stokes theory. In the third–order theory, the noise has non-zero autocorrelations also in a finite spatial range \cite{56}.

In addition, another subtlety of the FDR arises with the non-zero relaxation time in non-static and inhomogeneous systems. Usually, the FDR is obtained by considering the linear response of the global equilibrium state to small perturbations, but it is non-trivial how the FDR is modified in non-static and inhomogeneous backgrounds. Nevertheless, in the case of the Navier–Stokes theory the FDR is a delta function so that it can be applied to solve the dynamics using the local temperature and chemical potentials even in non-static and inhomogeneous backgrounds. However, in the case of the second-order causal theory, the FDR is non-local so that a single thermal state cannot be assumed to determine the temperature and chemical potential appearing in the FDR of the global equilibrium. Here we need to explicitly consider the FDR in non-static and inhomogeneous systems. First, in Sec. III the integral form and the differential form of the constitutive equation are introduced to discuss the relation between the integral form in the linear-response regime and the differential form used in actual dynamical simulations. In Sec. IV to deal with the tensor structure of the integral form for the shear stress tensor and the diffusion currents in non-static and inhomogeneous backgrounds, new tensors, which are called the pathline projectors in this paper, are introduced. Next, in Sec. IV we obtain the FDR in the causal theories and discuss the restriction on the structure of the general linear-response
differential forms and its relation to the cutoff scales of the hydrodynamic fluctuations. Finally, in Sec. VI the FDR in non-static and inhomogeneous backgrounds are obtained for the integral and differential forms using the pathline projectors. Section VI is devoted to the summary of the conclusions and the discussions. Hereafter, we adopt the natural unit system with $c = k_B = 1$ and the metric tensor with the sign convention being $g_{\mu\nu} = \text{diag}(1, -1, -1, -1)$.

II. CONSTITUTIVE EQUATION

In this section we review the constitutive equation in relativistic hydrodynamics and introduce its two representations which are used to discuss the higher-order theory and the causality [57].

Hydrodynamics is based on the conservation laws:

$$\partial_{\mu}T^{\mu\nu} = 0, \quad \partial_{\mu}N_i^{\mu} = 0, \quad (i = 1, \ldots, n),$$

where the conserved currents, $T^{\mu\nu}$ and $N_i^{\mu}$, are the energy-momentum tensor and the Noether current for the $i$-th conserved charge, respectively. In the Landau frame in which the flow vector $u^{\mu}$ is defined to be the timelike eigenvector of $T^{\mu\nu}$, normalized as $u^{\mu}u_{\mu} = 1$, the conserved currents are decomposed into irreducible tensor components of SO(3) rotation in the local rest frame of the matter as

$$T^{\mu\nu} = eu^{\mu}u^{\nu} - (P + \Pi)\Delta^{\mu\nu} + \pi^{\mu\nu},$$

$$N_i^{\mu} = n_iu^{\mu} + \nu_i^{\mu},$$

where $\Delta^{\mu\nu} := g^{\mu\nu} - u^{\mu}u^{\nu}$ is the projector for vectors onto the spatial components in the local rest frame. The local conserved quantities, $(e, n_i)$, are the local energy density and the local $i$-th charge density, respectively. The symbol $P$ denotes the equilibrium pressure. The dissipative currents, $(\Pi, \pi^{\mu\nu}, \nu_i^{\mu})$, are the bulk pressure, the shear-stress tensor and the diffusion current for the $i$-th charge, respectively. The hydrodynamic equations can be written in terms of the flow velocity field $u^{\mu}(x)$ (for which the number of degrees of freedom is $\#d.o.f. = 3$), the thermodynamic fields, $(e(x), n_i(x), P(x))$ ($\#d.o.f. = 1 + n + 1$), and the dissipative currents $(\Pi(x), \pi^{\mu\nu}(x), \nu_i^{\mu}(x))$ ($\#d.o.f. = 1 + 5 + 3n$). The number of degrees of freedom is $11 + 4n$ in total. The conservation laws, (1) and (2), give $(4 + n)$ dynamical constraints. The equation of state, $P(x) = P(e(x), n_i(x))$, gives another constraint. The rest $(6 + 3n)$ degrees of freedom are constrained by the constitutive equations describing the dissipative currents $\Gamma(x) := (\Pi(x), \pi^{\mu\nu}(x), \nu_i^{\mu}(x))^T$. The dissipative currents, $\Gamma(x)$, can be in general written as the functional of the present $(x^0 = x^0)$ and past $(x^0 < x^0)$ dynamical fields, $X(x') := (u^\mu(x'), e(x'), n_i(x'), \Gamma(x'))$:

$$\Gamma(x) = \Gamma[\{X(x')\}_{x^0 \leq x^0}](x).$$

(5)

However, this form is too general to theoretically or experimentally determine the explicit behavior of the dissipative currents. Using various conditions and approximations, such as symmetry, the second law of thermodynamics, the gradient expansion or the linear-response theory, we shall obtain more restricted forms to express the behavior of the dissipative currents.

A. Differential form of constitutive equation

If one ignores treatment of the convergence and the discontinuity such as shock waves, the functional can be approximated to the function of a finite number of derivatives at the current position $x$ by substituting $X(x') \simeq X(x) + \sum_{i=1}^{k}(1/i!)[(x'^\mu - x^\mu) \cdot \partial_\mu]^iX(x)$:

$$\Gamma(x) = \Gamma(X(x), \partial_\mu X(x), \ldots, \partial_{\mu_1} \cdots \partial_{\mu_k} X(x)).$$

(6)

The differential form of the constitutive equation can be obtained by the gradient expansion of the right-hand side of Eq. (6) with respect to the derivatives. In hydrodynamics, this form of the constitutive equation is usually used to solve the dynamics. It should be noted here that $\Gamma$ itself is counted as the order 1 in the gradient expansion because the lowest order of the dissipative currents is given by the first-order terms from the symmetry consideration.

In the Navier–Stokes theory, using the second law of thermodynamics, the lowest order of the dissipative currents is constrained as

$$\Gamma = \kappa F,$$

(7)
where the Onsager coefficients, $\kappa$, and the first-order thermodynamic force, $F$, are defined as

$$\kappa := \text{diag}(\zeta, 2\eta \Delta^{\mu\alpha\beta}, -\kappa_{ij} \Delta^{\mu\alpha}),$$

$$F := \begin{pmatrix}
-\theta \\
\frac{\partial (\alpha u_{\beta})}{\partial \eta} \\
-T \nabla_{\alpha} \frac{\partial}{\partial \eta}
\end{pmatrix},$$

respectively. Here $\theta := \partial_{\mu} u^\mu$ and $\zeta(\mu) := \Delta^{\mu\alpha\beta} \sigma^{\alpha\beta}$ with $\Delta^{\mu\alpha\beta} := \frac{1}{2}(\Delta^{\mu\alpha} \Delta^{\nu\beta} + \Delta^{\mu\beta} \Delta^{\nu\alpha}) - \frac{1}{4} \Delta^{\mu\nu} \Delta^{\alpha\beta}$ being the projector for tensors onto the spatial symmetric traceless tensor components in the local rest frame. The symbol $\nabla_{\alpha} := \Delta_{\alpha\beta} \partial_{\beta}$ denotes the spatial derivative in the local rest frame. The Onsager coefficients $\kappa$ are thermodynamic quantities, i.e., functions of $(e, u^\mu, n_i)$. The Onsager coefficient matrix $\kappa$ is positive semi-definite due to the second law of thermodynamics and also symmetric due to the Onsager reciprocal relations\footnote{We do not consider time-reversal odd coefficients in this paper.}. It should noted that the factor $-\Delta_{\mu\nu}$ appearing in the diffusion component of $\kappa$ is positive semi-definite as $-\Delta_{\mu\nu} = \text{diag}(0, 1, 1, 1)_{LRF}$ in the local rest frame.

In relativistic hydrodynamics, the Navier–Stokes theory is known to have problems of acausal modes and numerical instabilities. To circumvent the problems, the minimal extension to the constitutive equation would be the simplified Israel–Stewart theory\footnote{We do not consider time-reversal odd coefficients in this paper.} which includes the term of the relaxation time as a second-order term:

$$\Gamma = \kappa F - \tau_{\Sigma} D \Gamma,$$

where $D$ denotes the substantial time derivative of the dissipative currents, and the transport coefficient matrix, $\tau_{\Sigma}$, is the relaxation time. They have the forms,

$$D = \Delta D,$$

$$\Delta = \text{diag}(1, \Delta^{\mu\alpha\beta}, \Delta^{\mu\alpha}),$$

$$\tau_{\Sigma} = \text{diag}(\tau_{\Pi}, \tau_{\pi}, \tau_{ij}),$$

where $D := u^\mu \partial_{\mu}$ is the time derivative in the local rest frame, and $\tau_{\Pi}$, $\tau_{\pi}$ and $\tau_{ij}$ are the relaxation times for the bulk pressure, the shear stress and the diffusion currents, respectively. The symbol $\Delta$ denotes the projectors onto the space of the dissipative current $\Gamma$, and the symbol $\tau_{\Sigma}$ denotes the matrix of the relaxation time. To make the maximal signal propagation speed smaller than the speed of light, the relaxation times should be chosen appropriately. Now we have the time derivative of the dissipative current itself on the right-hand side, so the constitutive equation is regarded as a dynamical equation. Also, one could include other second-order terms. This kind of second-order dissipative hydrodynamics which respects the causality is called causal dissipative hydrodynamics.

### B. Integral form of constitutive equation

To discuss the nature of the hydrodynamic fluctuations later in Sec.\textsuperscript{V} we here introduce another representation of the constitutive equation, i.e., the integral form of the constitutive equation which is for example obtained in the linear-response regime. It should be noted that this particular form of the constitutive equation is not usually used to solve the actual dynamics because of its analytical or numerical complexity. In this paper we consider the integral form obtained in the linear-response theory with the following structure:

$$\Gamma(x) = \int d^4x' G(x, x') \kappa(x') F(x'),$$

where the retarded kernel, $G(x, x')\kappa(x')$, is called the memory function. Here $\kappa$ was factored out for simplicity in the later discussion. In this paper, $G(x, x')$ is also called the memory function. Here physically required properties of the memory function $G(x, x')$ are summarized:

- The memory function should be retarded:

$$G(x, x') = 0, \quad \text{if} \quad x^0 < x'^0.$$
The memory function should vanish for spatially separated two points to respect the causality:
\[ G(x, x') = 0, \quad \text{if} \quad (x - x')^2 < 0. \] (16)

The memory function should vanish in the limit of large time:
\[ G(x, x') \to 0, \quad \text{as} \quad x^0 - x'^0 \to \infty. \] (17)

The memory function should not depend on the thermodynamic forces of infinite past because the system near the equilibrium should forget the information of the fluctuations from the local equilibrium in a finite time.

If the background fields are homogeneous and static, the memory function has a translational symmetry, i.e.,
\[ G(x, x') = G(x - x'). \]

Some class of the differential forms of the constitutive equation has corresponding equivalent integral forms. We here explicitly show the integral forms of the Navier–Stokes theory and the simplified Israel–Stewart theory by giving the expression for the memory functions. More general consideration on the class of the differential forms that has corresponding integral forms will be discussed in Sec. IV C. For the simplest example, in the Navier–Stokes theory, the memory function is identified to be the delta function,
\[ G(x, x') = \delta(4)(x - x'). \] (18)

For another example, the differential form of the simplified Israel–Stewart theory can be symbolically solved to obtain an explicit formula for the dissipative current as
\[
\Gamma(\tau) = \int_{-\infty}^{\tau} d\tau_1 G(\tau, \tau_1) \kappa(\tau_1) F(\tau_1),
\]
\[ G(\tau, \tau_1) \equiv T \exp \left( -\int_{\tau_1}^{\tau} d\tau_2 \tau_R^{-1}(\tau_2) \right) \tau_R^{-1}(\tau_1), \] (19)
\[ \text{where} \ \tau \ \text{is the proper time, and} \ T \ \text{exp is the} \ \tau\text{-ordered exponential for the matrix} \ \tau_R^{-1}(\tau_2). \] Here we used the fact that the derivative D corresponds to the derivative with respect to the proper time \( \tau \). In this section the symbol \( \equiv \) is used to express that the equality gives a symbolical solution rather than the exact solution. In solving the equation, we ignored the projectors appearing in \( D \) in Eq. (11), which have not been attacked directly in inhomogeneous and non-static backgrounds. For example, to avoid the complexity in inhomogeneous and non-static backgrounds, Ref. [57] introduced in the differential form a new term cancelling with the complicated effects of the projector, which is not present in the Israel–Stewart theory. In this paper, instead of modifying the differential form of the constitutive equations to match with the simple integral form, we will obtain the integral form corresponding to the Israel–Stewart theory in the linear-response regime, in Sec. III.

Finally we reconsider Eqs. (10) and (20) more carefully. In particular we give explicit definitions of \( \tau \) and the integration with respect to \( \tau \). The time \( \tau \) is defined to be the proper time of the fluid particle, which is a virtual particle that moves with the flow velocity. The proper time satisfies \( D\tau = 1 \), so solving the equation the proper time can be defined as
\[ \tau(t, x) := \tau(t_0, x) + \int_{t_0}^{t} dt' \left[ 1 - u' \partial_i \tau(t', x) \right], \] (21)

where the summation over spatial indices are taken for the upper and lower index pair of \( i \). Here the initial proper time distribution, \( \tau(t_0, x) \), can be freely chosen because the change of the initial proper time is just a time reparametrization. Fluid particles are specified with comoving spatial coordinates \( \sigma \), which is a counter part of the proper time \( \tau \). To satisfy \( D\sigma = 0 \) so that \( \sigma \) for the fluid particle does not change in time, \( \sigma \) is defined as
\[ \sigma(t, x) := \sigma(t_0, x) - \int_{t_0}^{t} dt' \left[ u' \partial_i \sigma(t', x) \right]. \] (22)

The initial coordinates \( \sigma(t_0, x) \) also have the freedom degrees of choice but should be chosen so that the spatial coordinates \( x \) and the coordinates \( \sigma \) are in one-to-one correspondence. In this way the spacetime point \( x^\mu \) and the comoving coordinates \( \sigma^\mu := (\tau, \sigma) \) becomes in one-to-one correspondence as a whole, so the comoving coordinates can be regarded as another coordinate system to specify spacetime points. It should be noted here that the time derivative \( D \) is reduced to the simple partial differential with respect to \( \tau \): \( D = \partial / \partial \tau \). The trajectory or the world
The above expression of the memory function can be recasted into a compatible form with Eq. (14) as follows:

\[
\Gamma(\tau, \sigma) = \int_{-\infty}^{\tau} d\tau_1 G(\tau, \tau_1; \sigma) \kappa(\tau_1, \sigma) F(\tau_1, \sigma),
\]

\[
G(\tau, \tau_1; \sigma) = \text{sym} \quad T \exp \left( - \int_{\tau_1}^{\tau} d\tau_2 \tau_2^{-1}(\tau_2, \sigma) \right) \tau_1^{-1}(\tau_1, \sigma).
\]

The above expression of the memory function can be recasted into a compatible form with Eq. (14) as follows:

\[
G(x, x') = \text{sym} \quad T \exp \left( - \int_{\tau(x')}^{\tau(x)} d\tau_2 \tau_2^{-1}(\tau_2, \sigma(x)) \right) \tau_1^{-1}(x', x),
\]

\[
\theta^{(4)}(\sigma, \sigma') := \Theta(\tau - \tau') \delta^{(3)}(\sigma - \sigma') \left. \frac{\partial \sigma^\mu}{\partial x^\alpha} \right|_\sigma,
\]

where \( \frac{\partial \sigma^\mu}{\partial x^\alpha} \) denotes the Jacobian, and \( \Theta(\tau) \) denotes the Heaviside function.

III. TENSOR STRUCTURE OF MEMORY FUNCTION

In Sec. [11] we ignored the projectors appearing in the time derivative \( \mathcal{D} \) to obtain Eq. (20) for simplicity. However, to obtain the correct memory function we need to consider the tensor structure of the memory function emerged from the projectors in \( \mathcal{D} \). The projectors in \( \mathcal{D} \) are related to the transversality of the dissipative currents to the flow velocity, i.e., \( u_\mu \pi^{\mu\nu} = u_\mu \nu^\mu = 0 \), which follows from the definition of the dissipative currents by tensor decomposition. If we do not consider the projectors in \( \mathcal{D} \) in solving Eq. (11), the resulting dissipative currents would break the transversality. For example, if we do not consider the projectors in \( \mathcal{D} \), the dynamical equation (11) for the shear-stress component would become

\[
\tau_\pi \mathcal{D} \pi^{\mu\nu} = -\pi^{\mu\nu} + 2\eta \theta^{(4)} u^{\mu\nu}.
\]

This constitutive equation would result in

\[
\mathcal{D}(u_\mu \pi^{\mu\nu}) = \pi^{\mu\nu} \mathcal{D} u_\mu + u_\mu \tau_\pi^{-1}(2\eta \theta^{(4)} u^{\mu\nu} - \pi^{\mu\nu}) = \pi^{\mu\nu} \mathcal{D} u_\nu.
\]

The right-hand side does not vanish if the velocity fields have time dependence \( \mathcal{D} u_\nu \neq 0 \), which means the transversality \( u_\mu \pi^{\mu\nu} = 0 \) would be broken by the time evolution even if the shear stress initially satisfies the transversality. If the projectors are correctly considered, the dynamical equation becomes

\[
\tau_\pi \Delta^{\alpha\beta} \mathcal{D} \pi^{\alpha\beta} = -\pi^{\mu\nu} + 2\eta \theta^{(4)} u^{\mu\nu},
\]

and the transversality \( u_\mu \pi^{\mu\nu} = 0 \) is preserved under this correct version of the constitutive equation. In fact the memory function (20), which is obtained without considering the projector, explicitly breaks the transversality of the non-vanishing shear stress in inhomogeneous and non-static background. In this section we obtain the correct memory function which preserves the transversality and also provides the solution to the original constitutive equation (11) with the projectors in \( \mathcal{D} \) considered. The well-known properties of the projectors are listed in Appendix A which will be referenced in Sec. [11].

A. Pathline projectors and their properties

In this section we introduce new tensors which we call the pathline projectors in this paper. First we give a definition of the pathline projectors and then discuss their properties. In this section all the projectors and flow velocities are evaluated on a single fixed pathline specified by \( \sigma = \sigma^* \), so we will omit the explicit dependence on \( \sigma^* \), i.e., \( \Delta(\tau)_{\alpha} := \Delta(\tau, \sigma^*)_{\alpha} \), \( \Delta(\tau)^{\mu\alpha} := \Delta(\tau, \sigma^*)^{\mu\alpha} \), and \( u(\tau)^{\mu} := u(\tau, \sigma^*)^{\mu} \).

First new projectors are introduced as follows:

\[
\Delta(\tau_1; \tau)_{\mu} := \lim_{N \to \infty} \Delta(\tau_1)^{\mu \alpha_0} \prod_{k=0}^{N-1} \Delta(\tau_1 + \frac{\tau - \tau_1}{N} k)_{\alpha_k \alpha_{k+1}} \Delta(\tau_1)^{\alpha_N \alpha},
\]

where \( \Delta(\tau_1)^{\mu \alpha} \equiv \int_{-\infty}^{\tau_1} d\tau_2 \tau_2^{-1}(\tau_2, \sigma) \).
\[ \Delta(\tau_i; \tau_j)^{\mu\nu}_{\alpha\beta} := \lim_{N \to \infty} \Delta(\tau_i)^{\mu\nu}_{\alpha\beta_0} \left[ \prod_{k=0}^{N-1} \Delta(\tau_i + \frac{k}{N})^{\alpha_k\beta_{k+1}} \right] \Delta(\tau_i)^{\alpha_N\beta_N}_{\alpha\beta}. \] (31)

With these newly introduced projectors, projections are performed at every moment on the pathline. It should be noted that the projected spaces by \( \Delta(\tau)^{\mu\nu}_{\alpha\beta} \) and \( \Delta(\tau)^{\nu\mu}_{\alpha\beta} \) are time dependent because the flow velocity \( u(\tau)^\mu \) which defines the local rest frame depends on the time.

When \( u^\mu \), \( Du^\mu \) and \( D^2 u^\mu \) are bounded and continuous in a considered domain, the limits in the definitions of the pathline projectors (30) and (31) are convergent, and the projectors become well-defined. Then the pathline projectors have the following properties:

\[ \Delta(\tau_i; \tau_j)^{\mu\nu}_{\alpha\beta} = \Delta(\tau_i; \tau_j)^{\mu\nu}_{\alpha\beta} \Delta(\tau_i)^{\kappa\lambda}_{\alpha\beta} = \Delta(\tau_i)^{\mu\nu}_{\kappa\lambda} \Delta(\tau_i)^{\kappa\lambda}_{\alpha\beta}, \] (32)

\[ \Delta(\tau_i; \tau_j)^{\mu\nu}_{\alpha\beta} = \Delta(\tau_i; \tau_j)^{\mu\nu}_{\alpha\beta} \Delta(\tau_i)^{\kappa\lambda}_{\alpha\beta} = \Delta(\tau_i)^{\mu\nu}_{\kappa\lambda} \Delta(\tau_i)^{\kappa\lambda}_{\alpha\beta}, \] (33)

\[ \Delta(\tau_i; \tau_j)^{\mu\nu}_{\alpha\beta} = \Delta(\tau_i; \tau_j)^{\mu\nu}_{\alpha\beta}, \] (34)

\[ \Delta(\tau_i; \tau_j)^{\mu\nu}_{\alpha\beta} = \Delta(\tau_i; \tau_j)^{\mu\nu}_{\alpha\beta}, \] (35)

\[ \Delta(\tau_i; \tau_j)^{\mu\nu}_{\alpha\beta} = \Delta(\tau_i; \tau_j)^{\mu\nu}_{\alpha\beta}, \] (36)

\[ \Delta(\tau_i; \tau_j)^{\mu\nu}_{\alpha\beta} = \Delta(\tau_i; \tau_j)^{\mu\nu}_{\alpha\beta}, \] (37)

\[ \Delta(\tau_i; \tau_j)^{\mu\nu}_{\alpha\beta} = \Delta(\tau_i; \tau_j)^{\mu\nu}_{\alpha\beta}, \] (38)

\[ \Delta(\tau_i; \tau_j)^{\mu\nu}_{\alpha\beta} = \Delta(\tau_i; \tau_j)^{\mu\nu}_{\alpha\beta}, \] (39)

\[ \Delta(\tau_i; \tau_j)^{\mu\nu}_{\alpha\beta} = \Delta(\tau_i; \tau_j)^{\mu\nu}_{\alpha\beta}, \] (40)

\[ \Delta(\tau_i; \tau_j)^{\mu\nu}_{\alpha\beta} = \Delta(\tau_i; \tau_j)^{\mu\nu}_{\alpha\beta}, \] (41)

\[ \Delta(\tau_i; \tau_j)^{\mu\nu}_{\alpha\beta} = \Delta(\tau_i; \tau_j)^{\mu\nu}_{\alpha\beta}, \] (42)

where \( D_1 := \partial / \partial \tau_1 \). For the details of the proof of the convergence of the pathline projectors and the above properties, see Appendix B. In the definition of the pathline projectors, projections are performed at discrete times and then the number of projections \( N \) are taken to be infinity. It is non trivial that such a limit exists and even the time derivative can be safely applied to the limit value. We here briefly give a sketch of the proof in Appendix B. First a function sequence \( P_N(\tau_i; \tau_j) \) indexed by \( N \), with which a pathline projector can be written as \( \lim_{N \to \infty} P_N(\tau_i; \tau_j) \), is introduced. Next \( P_N(\tau_i; \tau_j) \) is shown to be a Cauchy sequence using the fact that the time derivative of \( P_N(\tau_i; \tau_j) \) can be written by a regular part and the residual part scaling as \( 1/N \). Then we can define the pathline projector \( P(\tau_i; \tau_j) \) as the limit value and show several properties which do not include the time derivatives. Using the compact convergence of \( P_N(\tau_i; \tau_j) \), the time derivative \( D_1 P(\tau_i; \tau_j) \) is also obtained and shown to be convergent.

Eqs. (42) and (43) tell that the left (right) indices of the pathline projectors behave as spatial vector or spatial symmetric traceless tensor at the time \( \tau_i \) (\( \tau_j \)). As consequences the following properties follow:

\[ \Delta(\tau_i; \tau_j)^{\mu\nu}_{\alpha\beta} = \Delta(\tau_i; \tau_j)^{\mu\nu}_{\alpha\beta}, \] (43)

\[ \Delta(\tau_i; \tau_j)^{\mu\nu}_{\alpha\beta} = \Delta(\tau_i; \tau_j)^{\mu\nu}_{\alpha\beta}, \] (44)

\[ \Delta(\tau_i; \tau_j)^{\mu\nu}_{\alpha\beta} = \Delta(\tau_i; \tau_j)^{\mu\nu}_{\alpha\beta}, \] (45)

\[ \Delta(\tau_i; \tau_j)^{\mu\nu}_{\alpha\beta} = \Delta(\tau_i; \tau_j)^{\mu\nu}_{\alpha\beta}, \] (46)

\[ \Delta(\tau_i; \tau_j)^{\mu\nu}_{\alpha\beta} = \Delta(\tau_i; \tau_j)^{\mu\nu}_{\alpha\beta}, \] (47)

\[ \Delta(\tau_i; \tau_j)^{\mu\nu}_{\alpha\beta} = \Delta(\tau_i; \tau_j)^{\mu\nu}_{\alpha\beta}, \] (48)

\[ \Delta(\tau_i; \tau_j)^{\mu\nu}_{\alpha\beta} = \Delta(\tau_i; \tau_j)^{\mu\nu}_{\alpha\beta}, \] (49)

Using Eqs. (42)–(49), it can be shown that a pathline projection preserves the norms of a spatial vector and a spatial symmetric traceless tensor:

\[ \Delta(\tau_i; \tau_j)^{\mu\nu}_{\alpha\beta} A^\alpha B^\beta = A^\alpha B^\beta, \] (50)

\[ \Delta(\tau_i; \tau_j)^{\mu\nu}_{\alpha\beta} B^\alpha B^\beta = B^\alpha B^\beta, \] (51)

for any spatial vector \( A^\mu \) such that \( \Delta(\tau_i)^{\mu\nu}_{\alpha\beta} A^\alpha = A^\mu \) and any spatial symmetric traceless tensor \( B^{\mu\nu} \) such that \( \Delta(\tau_i)^{\mu\nu}_{\alpha\beta} B^\alpha B^\beta = B^{\mu\nu} \). This means that a pathline projection serves as a certain Lorentz transformation for spatial vectors and spatial symmetric traceless tensors.

By solving Eqs. (40) and (41) with the initial conditions (38) and (39), another representation of the projectors can be obtained as

\[ \Delta(\tau_i; \tau_j)^{\mu\nu}_{\alpha\beta} = \left[ T \exp \left( \int_{\tau_i}^{\tau_j} d\tau D \Delta(\tau)^{\mu\nu}_{\alpha\beta} \right) \right]^{\mu\nu}_{\alpha\beta} \Delta(\tau_i)^{\kappa\lambda}_{\alpha\beta}, \] (52)
\[ \Delta(\tau; \tau)_{\mu\nu\alpha\beta}^{\kappa\lambda} = \left[ T \exp \left( \int_{\tau}^{\tau_1} d\tau_1 D(\tau)_{\mu\nu\kappa\lambda} \right) \right]_{\kappa\lambda}^{\mu\nu} \Delta(\tau)_{\kappa\lambda\alpha\beta}, \]  

where \( T \exp \int d\tau \) is a \( \tau \)-ordered exponential, i.e., the contractions of the Lorentz indices of \( \Delta(\tau)_{\mu\nu} \) and \( \Delta(\tau)_{\mu\nu\kappa\lambda} \) are performed after the time ordering.

It is known from Eqs. (12) that the spatial traceless symmetric pathline projector, \( \Delta(\tau; \tau)_{\mu\nu\alpha\beta} \), can be simply written in terms of the spatial pathline projector, \( \Delta(\tau; \tau)_{\mu\nu} \), as

\[ \Delta(\tau; \tau)_{\mu\nu\alpha\beta} = \Delta(\tau; \tau)_{\mu\nu} \Delta(\tau; \tau)_{\alpha\beta}. \]  

This reflects the fact that the tracelessness and the symmetry of the tensor are independent of the local rest frame \( u^\mu \). Once the projection into the traceless symmetric components is made, there is no need of such projection in other times.

### B. Memory function with tensor structures

Now we are ready to give a solution to the second-order constitutive equation (10) using pathline projectors which replaces the symbolic solution (20) breaking the transversality. Using the properties (40)–(42), the following equations can be shown:

\[ \Delta(\tau)_{\mu\kappa} D(\tau; \tau)_{\nu\alpha} = 0, \]  
\[ \Delta(\tau)_{\mu\nu\kappa\lambda} D(\tau; \tau)_{\kappa\lambda\alpha\beta} = \Delta(\tau)_{\mu\nu} D(\tau; \tau)_{\kappa\lambda} \Delta(\tau; \tau)_{\kappa\lambda\alpha\beta} = 0. \]

These equations can be used to solve a certain class of the differential form of the constitutive equations.

For example, if a spatial vector \( A^\mu(\tau) \) and a spatial traceless symmetric tensor \( B_{\mu\nu}(\tau) \) obey the dynamical equations:

\[ \Delta_{\mu\kappa} D(\tau; \tau)_{\nu\alpha} = 0, \]  
\[ \Delta_{\mu\nu\kappa\lambda} D(\tau; \tau)_{\kappa\lambda\alpha\beta} = 0, \]

the solutions are written as

\[ A^\mu(\tau) = \Delta(\tau; \tau_0)^{\mu\nu} A^\nu(\tau_0), \]  
\[ B_{\mu\nu}(\tau) = \Delta(\tau; \tau_0)^{\mu\nu\kappa\lambda} B_{\kappa\lambda}(\tau_0), \]

with \( A^\kappa(\tau_0) \) and \( B_{\kappa\lambda}(\tau_0) \) being initial conditions.

For another example, let us consider the second-order constitutive equations of the form:

\[ \pi^{\mu\nu} + \tau_{\pi} \Delta_{\mu\nu\kappa\lambda} D\pi^{\kappa\lambda} = X^{\mu\nu}, \]  
\[ \nu^\mu_i + \sum_{j=1}^{n} \tau_{ij} \Delta_{\mu\kappa} D\nu^\kappa_j = X^\mu_i, \]

with \( X^{\mu\nu} \) and \( X^\mu_i \) being miscellaneous second-order terms which do not contain dissipative currents. The equations can be solved with respect to the dissipative currents using the pathline projectors as

\[ \pi^{\mu\nu}(\tau) = G^{\mu\nu\alpha\beta}(\tau, \tau_0) \tau_{\pi}(\tau_0) \pi^{\alpha\beta}(\tau_0) + \int_{\tau_0}^{\tau} d\tau_1 G^{\mu\nu\alpha\beta}(\tau, \tau_1) X^{\alpha\beta}(\tau_1), \]  
\[ \nu^{\mu}_i(\tau) = \sum_{j,k=1}^{n} G_{ijk}^{\mu\alpha}(\tau, \tau_0) \tau_{jk}(\tau_0) \nu^{\alpha}_j(\tau_0) + \sum_{j=1}^{n} \int_{\tau_0}^{\tau} d\tau_1 G_{ijk}^{\mu\alpha}(\tau, \tau_1) X^{\alpha}_j(\tau_1), \]  
\[ G^{\mu\nu\alpha\beta}(\tau, \tau_1) = \exp \left( -\int_{\tau_1}^{\tau} \frac{d\tau_2}{\tau_{\pi}(\tau_2)} \frac{1}{\tau_{\pi}(\tau_1)} \Delta(\tau; \tau_1)^{\mu\nu\alpha\beta}, \right) \]  
\[ G_{ijk}^{\mu\alpha}(\tau, \tau_1) = \sum_{j=1}^{n} \left[ T \exp \left( -\int_{\tau_0}^{\tau} d\tau_2 \tau_{ij}(\tau_2) \right) \tau_{ij}(\tau_1) \Delta(\tau; \tau_1)^{\mu\alpha}, \right] \]

where \( \pi^{\alpha\beta}(\tau_0) \) and \( \nu^{\alpha}(\tau_0) \) are initial conditions.
C. Memory function in simplified Israel–Stewart theory

To find the memory functions for the shear stress tensor and diffusion currents in the simplified Israel–Stewart theory [10], we can consider the case $X^{\alpha\beta} = 2\eta\partial^{(\mu}u^{\nu)}$ and $X^{\mu} = -T\nabla^{\mu}(\mu_i/T)$ and take the limit $\tau_0 \to -\infty$:

$$\pi^{\mu\nu}(\tau) = \int_{-\infty}^{\tau} d\tau_1 G^{\mu\nu}_{\alpha\beta}(\tau, \tau_1) 2\eta\partial^{(\alpha}u^{\beta)}(\tau_1),$$

$$\nu_i^{\mu}(\tau) = - \sum_{j,k=1}^{n} \int_{-\infty}^{\tau} d\tau_1 G_{ij}^{\mu\nu}(\tau, \tau_1) \kappa_{jk} T(\tau_1) \nabla_{\nu} \frac{\mu_k(\tau_1)}{T(\tau_1)}.$$

Thus the expression for the memory function [20] which correctly takes account of the tensor structure is

$$G(x, x') = \text{diag}(G_{\Pi}(x, x'), G_{\mu\nu\alpha\beta}(x, x'), G_{ik\mu\alpha}(x, x')),$$

where

$$G_{\Pi}(x, x') = \exp\left( -\int_{\tau(x')}^{\tau(x)} \frac{d\tau_2}{\tau_1(\tau_2, \sigma(x))} \right) \frac{1}{\tau_1(x')} \theta^{(4)}(\sigma(x), \sigma(x')),$$

$$G_{\mu\nu\alpha\beta}(x, x') = \exp\left( -\int_{\tau(x')}^{\tau(x)} \frac{d\tau_2}{\tau_1(\tau_2, \sigma(x))} \right) \frac{1}{\tau_1(x')} \Delta(\tau(x); \tau(x'), \sigma(x), \sigma(x'))^{\mu\nu\alpha\beta} \theta^{(4)}(\sigma(x), \sigma(x')),$$

$$G_{ik\mu\alpha}(x, x') = \sum_{j=1}^{n} \left[ \text{T} \exp\left( -\int_{\tau(x')}^{\tau(x)} d\tau_2 \tau_{ij}^{-1}(\tau_2, \sigma(x)) \right) \right] \tau_{jk}^{-1}(x') \Delta(\tau(x); \tau(x'), \sigma(x), \sigma(x'))^{\mu\alpha} \theta^{(4)}(\sigma(x), \sigma(x')).$$

By defining a pathline projector for the generic dissipative currents as

$$\Delta(\tau_1; \tau, \sigma) = \text{diag}(1, \Delta(\tau_1; \tau, \sigma)^{\mu\nu\alpha\beta}, \Delta(\tau_1; \tau, \sigma)^{\mu\alpha}),$$

these memory functions can be summarized in the following form:

$$G(x, x') = \text{T} \exp\left( -\int_{\tau(x')}^{\tau(x)} d\tau_2 \tau_{ij}^{-1}(\tau_2, \sigma(x)) \right) \tau_{jk}^{-1}(x') \Delta(\tau(x); \tau(x'), \sigma(x), \sigma(x')) \theta^{(4)}(\sigma(x), \sigma(x')).$$

Finally let us here give an expression of the memory function in the global equilibrium for later usage. In homogeneous and static backgrounds where $u^{\mu} = (1, 0, 0, 0)$ and $T = \mu_i = \text{const}$, the expression of the memory function simplifies as

$$G(x - x') = e^{-(x^0 - x'^0)} \tau_{R}^{-1} \Delta \Theta(x^0 - x'^0) \delta^{(3)}(x - x').$$

Components of the above memory function can be written down as

$$G_{\Pi}(x - x') = e^{-(x^0 - x'^0)} \tau_{R}^{-1} \Delta \Theta(x^0 - x'^0) \delta^{(3)}(x - x'),$$

$$G_{\mu\nu\alpha\beta}(x - x') = e^{-(x^0 - x'^0)} \tau_{R}^{-1} \Delta \Theta(x^0 - x'^0) \delta^{(3)}(x - x'),$$

$$G_{ik\mu\alpha}(x - x') = \sum_{j=1}^{n} e^{-(x^0 - x'^0)} \tau_{ij}^{-1} \Delta \Theta(x^0 - x'^0) \delta^{(3)}(x - x').$$

IV. CAUSAL HYDRODYNAMIC FLUCTUATIONS IN EQUILIBRIUM

A. Colored noise in integral form

So far we have considered the constitutive equations which determine the dissipative currents in terms of the present and past hydrodynamic fields. However such deterministic constitutive equations merely describe the “average” behavior of the dissipative currents. If the scale of interest is not sufficiently larger than the scale of the
microscopic dynamics, the deviation from the “average” affects the relevant dynamics. Such deviation is nothing but the hydrodynamic fluctuations, i.e., the thermal fluctuations of the dissipative currents:

\[
\delta \Gamma(x) = \begin{pmatrix} \delta \Pi(x) \\ \delta \pi^{\mu}(x) \\ \delta \nu^i(x) \end{pmatrix} := \Gamma(x) - \int d^4x' G(x, x') \kappa(x') F(x'). \tag{79}
\]

The hydrodynamic fluctuations originate from the dynamics of the microscopic degrees of freedom and therefore cannot be uniquely determined by the macroscopic information. Instead, the hydrodynamic fluctuations are treated as stochastic fields whose distribution is determined by the macroscopic information. The dissipative currents are calculated by the sum of the average part and the stochastic fields:

\[
\Gamma(x) = \int d^4x' G(x, x') \kappa(x') F(x') + \delta \Gamma(x). \tag{80}
\]

Now the hydrodynamic equations are stochastic partial differential equations (SPDE) which are similar to the Langevin equation for the Brownian motion. Such hydrodynamics with hydrodynamic fluctuations is called fluctuating hydrodynamics. The major difference to the Langevin equation is that the dynamical variable is fluid fields instead of a position of particle, so the noise terms become fields which have spatial dependence.

The distribution of the hydrodynamic fluctuations, \( \delta \Gamma(x) \), can be characterized by its moments such as average values and variances. The average values of the distribution are zero by definition: \( \langle \delta \Gamma(x) \rangle = 0 \). The variance–covariance matrix \( \delta \Gamma(x) \delta \Gamma(x'^T) \), determines the magnitude of the noises and is given in terms of macroscopic quantities by the FDR based on the linear-response theory \( \ref{FDR} \). The FDR can be obtained by the linear response theory by assuming Onsager’s regression hypothesis \( \ref{Onsager} \). The relaxation of spontaneous fluctuations on average behaves as the same as the relaxation of the deviation caused by external fields (see Ref. \( \ref{summary} \) for a summary of topics and discussions on the FDR).

In this section we consider the hydrodynamic fluctuations in an equilibrium state, i.e., in homogeneous and static backgrounds. In equilibrium, the FDR is given by

\[
\langle \delta \Gamma(x) \delta \Gamma(x'^T) \rangle \Theta(x^0 - x'^0) = G(x - x') \kappa T, \tag{81}
\]

or equivalently,

\[
\langle \delta \Gamma(x) \delta \Gamma(x'^T) \rangle = T[G(x - x') \kappa + \kappa G'(x' - x'^T)], \tag{82}
\]

where we used the Onsager reciprocal relation \( \kappa = \kappa^T \) to remove the transpose of the Onsager coefficient \( \kappa \).

Although we do not assume the Gaussian distribution of the hydrodynamic fluctuations in the following discussion, it should be noted here that the distribution of the hydrodynamic fluctuations is usually assumed to have the multivariate Gaussian distribution of the form,

\[
\text{Pr}[\delta \Gamma(x)] \propto \exp \left( -\frac{1}{2} \int d^4x \int d^4x' \delta \Gamma(x)^T C^{-1}(x, x') \delta \Gamma(x') \right), \tag{83}
\]

where \( \text{Pr}[\delta \Gamma(x)] \) is the probability density functional that a specific history of \( \delta \Gamma(x) \) is realized, and \( C^{-1}(x, x') \) is a matrix-valued function and is symmetric, i.e., \( C^{-1}(x, x') = C^{-1}(x', x) \). The reasoning of the Gaussian assumption is that, if the system is sufficiently large so that the dissipative currents are sum of many independent contributions from microscopic processes, the resulting distribution becomes Gaussian due to the central limit theorem. The freedom degrees of the multivariate Gaussian distribution, \( C^{-1}(x, x') \), are totally fixed by the autocorrelation given by the FDR through the relation:

\[
\int d^4x_2 \langle \delta \Gamma(x_1) \delta \Gamma(x_2)^T \rangle C^{-1}(x_2, x_3) = \delta^{(4)}(x_1 - x_3). \tag{84}
\]

Because in this way the FDR settles the basic statistical nature of the hydrodynamic fluctuations from the macroscopic information, it can be regarded as the most important relation in considering the hydrodynamic fluctuations.

\[2 \text{ Here } \langle \delta \Gamma(x) \delta \Gamma(x'^T) \rangle \text{ is regarded as a “matrix” whose indices include spatial indices } x \text{ and } x' \text{ as well as the usual indices that specifies the component of } \delta \Gamma. \]
Here let us demonstrate examples of the FDR for the Navier–Stokes theory and the simplified Israel–Stewart theory. The FDR in the Navier–Stokes theory is obtained by combining Eq. (18) and Eq. (82) as

$$\langle \delta \Gamma(x) \delta \Gamma(x') \rangle = 2T \kappa \delta^{(4)}(x - x').$$

(85)

From the above expression, the diagonal components read

$$\langle \delta \Pi(x) \delta \Pi(x') \rangle = 2T \zeta \delta^{(4)}(x - x'),$$

(86)

$$\langle \delta \pi^\mu\nu(x) \delta \pi^\alpha\beta(x') \rangle = 4T \eta \Delta^{\mu\nu,\alpha\beta} \delta^{(4)}(x - x'),$$

(87)

$$\langle \delta \nu^\mu_i(x) \delta \nu^\alpha_j(x') \rangle = -2T \kappa_i \Delta^{\mu\alpha} \delta^{(4)}(x - x').$$

(88)

(89)

The off-diagonal components vanish:

$$\langle \delta \Pi(x) \pi^\alpha\beta(x') \rangle = \langle \delta \pi^\mu\nu(x) \delta \nu^\alpha_j(x') \rangle = \langle \delta \nu^\mu_i(x) \delta \Pi(x') \rangle = 0.$$  

(90)

One can observe that the autocorrelations of the noise fields are delta functions which correspond to constant power spectra in the Fourier space. This type of noise fields is called white noise because modes of all the frequencies are contained in the spectrum with an equal strength.

In the simplified Israel–Stewart theory, the FDR is obtained from Eq. (74) and Eq. (82).

$$\langle \delta \Gamma(x) \delta \Gamma(x') \rangle = T \left\{ e^{-(x^0 - x'^0)\tau_R - 1} \kappa \Theta(x^0 - x'^0) \right\} \Delta \delta^{(3)}(x - x').$$

(91)

The diagonal components read

$$\langle \delta \Pi(x) \delta \Pi(x') \rangle = 2T \zeta \delta^{(3)}(x - x'),$$

(92)

$$\langle \delta \pi^\mu\nu(x) \delta \pi^\alpha\beta(x') \rangle = 4T \eta \Delta^{\mu\nu,\alpha\beta} \delta^{(3)}(x - x'),$$

(93)

$$\langle \delta \nu^\mu_i(x) \delta \nu^\alpha_j(x') \rangle = -T \sum_{k,l=1}^n \left\{ \left[ e^{-(x^0 - x'^0)\tau_R - 1} \kappa_i \Theta(x^0 - x'^0) \right] \kappa_j \delta^{(3)}(x - x') \right\} \Delta \delta^{(3)}(x - x').$$

(94)

The off-diagonal components vanish similarly to the case of the Navier–Stokes theory. One notices that, unlike in the Navier–Stokes theory, the memory function is no longer the delta function. The noise fields have finite time correlation with the time scale of the relaxation time $\tau_R$. This type of noise fields is called colored noise because the noise power spectrum in the Fourier space has a characteristic frequency corresponding to the inverse of the relaxation time. In general, causal dissipative hydrodynamics has non-vanishing relaxation times, so the noise fields appearing in the integral form of the constitutive equation are always colored in causal dissipative hydrodynamics.

### B. White noise in differential form in simplified Israel–Stewart theory

In Sec. [IV.A] we introduced noise terms $\delta \Gamma(x)$ in the integral form of the constitutive equation that gives an explicit form of the dissipative currents. While, in the actual analysis of causal dissipative hydrodynamics, we usually use the differential form of the constitutive equation that gives dynamical equations for the dissipative currents. We here consider how the noise term $\xi(x)$ enters the differential form of the constitutive equation and what is the nature of the fluctuations.

For a trivial example, in the case of the Navier–Stokes theory, the noise terms in the two forms are the same: $\xi(x) = \delta \Gamma(x)$ because the differential and integral forms are equivalent in this theory. Thus the autocorrelation of $\xi(x)$ is directly given by the FDR in the integral form [86]–[88].
In the differential forms of higher-order theories, the noise term, $\xi$, can be defined as the deviation of the right-hand side of the constitutive equation from the left-hand side similarly in the integral form. For example, in the case of the simplified Israel-Stewart theory (10), the noise term is defined as

$$\xi(x) = \left( \begin{array}{c} \xi^\mu(x) \\ \xi^\nu(x) \\ \xi_\alpha(x) \end{array} \right) := \Gamma(x) - [\kappa F(x) - \tau_R D \Gamma(x)]. \quad (95)$$

By substituting (94), one finds the following relation:

$$\xi(x) = (1 + \tau_R D) \delta \Gamma(x). \quad (96)$$

Because $\xi(x)$ is linear with respect to $\delta \Gamma(x)$, $\xi(x)$ is also a Gaussian noise if $\delta \Gamma(x)$ is a Gaussian noise. The autocorrelation of $\xi(x)$ can be calculated from the FDR of $\delta \Gamma(x)$ (52):

$$\langle \xi(x)\xi(x')^T \rangle = \langle [(1 + \tau_R D)\delta \Gamma(x)](1 + \tau_R D')\delta \Gamma(x')^T \rangle \quad (97)$$

$$= (1 + \tau_R D)\langle \delta \Gamma(x)\delta \Gamma(x')^T \rangle (1 + \tau_R D')^T \quad (98)$$

$$= (1 + \tau_R D)[G(x - x')\kappa T + T\kappa G(x' - x')^T](1 + \tau_R D')^T, \quad (99)$$

where $D$ and $D'$ are the time derivatives with respect to $x$ and $x'$, respectively. The symbol $\vec{D}$ denotes that the derivative operates on the left. Using the relation $(1 + \tau_R D)G(x - x') = \delta(4)(x - x')$, one obtains

$$\langle \xi(x)\xi(x')^T \rangle = 2\kappa \delta(4)(x - x') + 2\tau_R D\delta(4)(x - x'), \quad (100)$$

where $M^A := (1/2)(M - M^T)$ denotes the antisymmetric part of the matrix $M$. From the diagonal components one finds

$$\langle \xi^\mu(x)\xi^\mu(x') \rangle = 2\kappa \delta(4)(x - x'), \quad (102)$$

$$\langle \xi^\mu(x)\xi^\nu(x') \rangle = 4\tau_R \Delta^\mu\nu\delta(4)(x - x'), \quad (103)$$

$$\langle \xi^\mu(x)\xi_\alpha(x') \rangle = -2\tau_R \kappa_j\Delta^\mu\alpha\delta(4)(x - x') - T \sum_{k=1}^n (\tau_{ik}\kappa_{kj} - \tau_{jk}\kappa_{ki})\Delta^\mu\alpha\delta(4)(x - x'). \quad (104)$$

The non-diagonal components vanish so that there are no correlation between different types of noises: $\langle \xi^\mu(x)\xi^\alpha(x') \rangle = \langle \xi^\mu(x)\xi_\alpha(x') \rangle = \langle \xi^\alpha(x)\xi_\mu(x') \rangle = 0$.

One notices that the second term of Eq. (101), which is proportional to $[\tau_R \kappa]^A$, vanishes for the shear and bulk noise, $\xi^\mu(x)$ and $\xi^\nu(x)$, while it remains for the diffusion noise, $\xi_\alpha(x)$. In general the second term vanishes when we consider the single component diffusion current, or when the dissipative currents do not mix with one another due to some symmetry, i.e., $\tau_R$ and $\kappa$ are diagonal. The bulk pressure is a single-component dissipative current so that the second term trivially vanishes. Also, in the case of the shear stress, the second term vanishes because $\tau_R = \tau_\pi$ is scalar and $\kappa$ is symmetric due to the Onsager’s reciprocal relation so that $[\tau_R \kappa]^A = \tau_\pi \kappa^A = 0$. If the second term is present, it is difficult to interpret as the autocorrelation of $\xi(x)$ because the second term contains the derivatives on the delta function, $D\delta(4)(x - x')$. This implies that the differential form of the constitutive equation cannot be used to describe the hydrodynamic fluctuations when the dissipative currents mix with one another through the relaxation time $\tau_R$ and the Onsager coefficients $\kappa$. In such a case, we should consider solving the integral form instead of the differential form.

When the dissipative currents do not mix with one another, one notices that the differential form of the FDR of the simplified Israel–Stewart theory, Eqs. (112)–(114), match with the FDR of the Navier–Stokes theory, Eqs. (50)–(58). In particular, in spite of the fact that the noise in the integral form $\delta \Pi(x)$ is colored, the noise in the differential form $\xi(x)$ becomes white.

C. General linear-response differential form

In Sec. IV B we argued that in causal dissipative hydrodynamics, the noise in the integral form of the constitutive equations is colored reflecting the non-vanishing relaxation time required to maintain the causality. However, in the
simplified Israel–Stewart theory [11], we found that the noise in the differential form is still white despite of the colored noise in the integral form. In addition the FDR for the differential form is exactly the same with that of the Navier–Stokes theory [11]. This result appears to be non-trivial because the contribution from the relaxation time seems accidentally cancel with each other in Eq. (111) when the dissipative currents do not mix with one another. Here the question is whether there is a higher-order theory in which the noise of the differential form has a different colored noise in the integral form. In addition the FDR for the differential form is exactly the same with that of the simplified Israel–Stewart theory (10), we found that the noise in the differential form is still white despite of the colored noise in the integral form. In equilibrium the integral form (80) is written by the convolution of the memory function and the thermodynamic force:

\[ \Gamma(x) = \int d^4x' G(x - x') \kappa F(x') + \delta \Gamma(x). \]  

(105)

In this section we take the coordinates \((t, x)\) in the local rest frame where \(u^\mu = (1, 0, 0, 0)^T\). In the Fourier representation, the above expression is transformed to

\[ \Gamma_{\omega, k} = \hat{G}_{\omega, k} \kappa \hat{F}_{\omega, k} + \delta \Gamma_{\omega, k}. \]  

(106)

Here the Fourier transform is defined as

\[ f_{\omega, k} = \int d^4x e^{i k \cdot x} f(x) = \int d^4x e^{i \omega t - i k \cdot x} f(x), \]  

\[ f(x) = \frac{1}{(2\pi)^4} \int d^4k e^{-i k \cdot x} f_{\omega, k} = \frac{1}{(2\pi)^4} \int d^4k e^{-i \omega t + i k \cdot x} f_{\omega, k}, \]  

(107)

(108)

where \(k^\mu = (\omega, k)\).

The differential form in the linear-response (without noise) can generally be written in the following form [60, 61]:

\[ L_0(D, \nabla^\mu) \Gamma(x) = M_0(D, \nabla^\mu) \kappa F(x), \]  

(109)

where \(L_0(z, w)\) and \(M_0(z, w)\) are matrix-valued polynomials of \(z\) and \(w\). The time derivatives \(D\) in the polynomial \(M_0\) are acting on the thermodynamic forces \(F(x)\) which are written in terms of the thermodynamic fields \((u^\mu, e, n_i)\). Substituting the hydrodynamic equations to \((D u^\mu, D e, D n_i)\) in the right-hand side, the time derivatives \(D\) in \(M_0\) can be replaced by the spatial derivatives acting on dissipative currents or thermodynamic fields. If non-linear terms in the hydrodynamic equations are ignored in the linear-response regime, the above differential form can be reorganized into the following linear form:

\[ L(D, \nabla^\mu) \Gamma(x) = M(\nabla^\mu) \kappa F(x) + \xi(x), \]  

(110)

where the hydrodynamic fluctuations \(\xi(x)\) are introduced here. Here the polynomials \(L(z, w)\) and \(M(w)\) are required to have the following several properties:

• The polynomials \(L(z, w)\) and \(M(w)\) are normalized to be coprime, i.e., the two polynomials do not have common factors. If they have common factors, they can be eliminated by dividing both sides of Eq. (110).

• The coefficients of the polynomials \(L(z, w)\) and \(M(w)\) are the transport coefficients appearing in the constitutive equations, so all the coefficients of the polynomials should be real.

• The functions \(L(z, w)\) and \(M(w)\) are “polynomials”, i.e., the series are truncated to have a finite order of the derivatives so that the degrees of the polynomials \(L(z, w)\) and \(M(w)\) with respect to \(z\) and \(w\) are finite.

• The constant term of each polynomial is the identity in the space of the dissipative currents, i.e., \(\Delta\) defined in Eq. (12):

\[ L(z, w) = \Delta + O(z, w), \]  

(111)

\[ M(w) = \Delta + O(w). \]  

(112)

This is because the lowest-order terms are \(\Gamma = \kappa F(x)\).
• The polynomials \( L(z, w) \) and \( M(w) \) contain only even orders of \( w \):

\[
L(z, w) = L_2(z, w \otimes w), \\
M(w) = M_2(w \otimes w).
\]

This can be obtained by consideration on the parity inversion \( x \to x' = -x \). Because the constant terms of the polynomials are unity, the parity of the polynomials are \( L(z, w) \) and \( M(w) \) are +1. Note that, if the parity of the dissipative currents and the thermodynamic forces is different, the difference should be absorbed in the parity of the Onsager coefficients \( \kappa \), so that the polynomials are always parity +1. Therefore, the polynomials are even as a function of \( w \). It should be noted that the notation \( w \otimes w \) expresses that the even factors do not necessarily have the form of \( w \cdot w \) because the indices of \( w \) may be contracted with spatial indices of thermodynamic forces, e.g., \( M(\nabla^\mu)\kappa \Gamma \sim \nabla^\alpha \nabla^\beta (2\eta \partial_{(\alpha,\beta)}). \)

Then the Fourier representation of the differential form becomes

\[
L(-i\omega, -ik)\Gamma_{\omega, k} = M(-ik)\kappa F_{\omega, k} + \xi_{\omega, k}.
\]

(115)

Here \( D = \partial_0 \) was replaced by \(-i\omega \), and \( \nabla^\mu = (0, \partial^x) = (0, -\partial_t) \) was replaced by \(-ik \) since we now consider the coordinates in the rest frame.

Comparing the integral form (110) and the differential form (115), one finds the relations between the memory function and the polynomials and between the noise terms of the two forms:

\[
G_{\omega, k} = L(-i\omega, -ik)^{-1} M(-ik),
\]

(116)

\[
\xi_{\omega, k} = L(-i\omega, -ik) \delta \Gamma_{\omega, k}.
\]

(117)

The FDR in the integral form (82) is transformed into the following form in the Fourier representation:

\[
\langle \delta \Gamma_{\omega, k} \delta \Gamma_{\omega', k'} \rangle = T (G_{\omega, k} \kappa + \kappa G_{\omega, k}^\dagger) (2\pi)^4 \delta^{(4)}(k - k'),
\]

(118)

where \( \cdots ^\dagger \) denotes the Hermitian conjugate of a matrix or a vector. The delta function \( \delta^{(4)}(k - k') \) appears in the right-hand side due to the translational symmetry of the memory function \( G(x - x') \). The FDR in the differential form is obtained as

\[
\langle \xi_{\omega, k} \xi_{\omega', k'} \rangle = L(-i\omega, -ik) \langle \delta \Gamma_{\omega, k} \delta \Gamma_{\omega', k'} \rangle L(-i\omega', -ik)^\dagger
\]

(119)

\[
I_{\omega, k} = T [M(-ik) \kappa L(-i\omega, -ik)^\dagger + L(-i\omega, -ik) \kappa M(-ik)^\dagger].
\]

(120)

Here \( I_{\omega, k} \) represents the power spectrum of the noise field. The noise is white if the power spectrum is constant, and otherwise the noise is colored. The expression of \( I_{\omega, k} \) in general does not have the same spectrum with the Navier–Stokes theory \( I_{\omega, k} = 2T \kappa \). Also \( I_{\omega, k} \) is not white in general because it depends on the frequencies and the momentum \( (\omega, k) \) through the polynomials \( L(-i\omega, -ik) \) and \( M(-ik) \).

However, there are additional physical conditions which restrict the structure of the polynomials \( L(z, w) \) and \( M(w) \). For example, the properties of the memory function such as the causality (10), the retardation (15) and the relaxation (17) give constraints on the polynomials through the relation (110). Also the autocorrelation \( \langle \xi(x)\xi(x')^T \rangle \) is a variance–covariance matrix when its dependence on \( x \) and \( x' \) is seen as matrix indices, and therefore the autocorrelation should be positive semi-definite due to the general property of a variance–covariance matrix. This fact restricts the polynomials through the expression of power spectrum (121) written in terms of the polynomials.

When the dissipative currents do not mix with one another, all the matrices such as \( \kappa, G_{\omega, k}, L(-i\omega, -ik) \) and \( M(-ik) \) are diagonal, so that we can consider a single-component dissipative current for \( \Gamma(x) \) without loss of generality. In such a case, it can be shown that the restrictions on the polynomials and the power spectrum have the following forms (see Appendix D):

• The polynomial \( L(D, \nabla^\mu) \) should have the following form:

\[
L(D, \nabla^\mu) = 1 + \tau_R D,
\]

(122)

where the transport coefficient \( \tau_R \) is identified to be the relaxation time of the dissipative current.

• The polynomial \( M(-ik) \) is non-negative for all real \( k \):

\[
M(-i\kappa) \geq 0.
\]

(123)
The power spectrum of the noise in the differential form is independent of $\omega$, i.e., the noise is white in the frequency space:

\[ I_{\omega,k} = 2T\kappa M(-ik). \]  

(124)

From the power spectrum \((124)\), the autocorrelation in the real space is obtained as

\[ \langle \xi(x)\xi(x') \rangle = 2T\kappa M(\nabla^\mu)\delta^{(4)}(x-x'). \]  

(125)

Similarly to the case of Eq. \((\ref{eq:fourier-autocorr})\), it is subtle to interpret the derivatives on the delta function as the autocorrelation. Thus the polynomial $M$ is further restricted to $M(\nabla^\mu) = 1$. This means that the differential form of the constitutive equation is restricted to the simplified Israel–Stewart theory to be consistent with the noise terms obeying the FDR, and the noise is white as already shown in Sec. \(\text{IVB}\).

While a careful proof of these restrictions are given, in Appendix \(\text{D}\) we here briefly outline the essential part of the proof to use it in later discussions. First we consider the positive semi-definiteness which can be obtained from Eq. \((124)\) as follows:

\[ I_{\omega,k} = 2T\kappa \Re[M(-ik)L(-i\omega,-ik)] \geq 0. \]  

(126)

Here the polynomial $L(-i\omega,-ik)$ can be factorized with respect to $-i\omega$

\[ L(-i\omega,-ik) = (-1)^N A_k \prod_{p=1}^{N} i(\omega - \omega_p(k)), \]  

(127)

where $N = \deg_{\omega} L$ is the degree of $-i\omega$ in the polynomial $L(-i\omega,-ik)$. The zeroes $\{\omega_p(k)\}_{p=1}^{N}$ corresponding to the poles of the memory function must have negative imaginary part, i.e., $\Im\omega_p(k) < 0$, to keep the memory function retarded. For this reason, when $\omega$ moves from $-\infty$ to $\infty$, the complex argument of the factor $\arg(\omega - \omega_p(k))$ continuously decreases by the amount of $\pi$. Then, the argument of polynomial $L(-i\omega,-ik)$ decreases by $N\pi$ in total. Here, the change of the arguments is restricted within $\pi$ due to the inequality \((\ref{eq:inequality})\), which implies $N \leq 1$. With additional consideration on symmetries and the causality, the polynomials are constrained to the forms of Eqs. \((\ref{eq:form1})\)–\((\ref{eq:form2})\).

### D. Cutoff scale and structure of differential form

In Sec. \(\text{IVB}\) we discussed that it is difficult to interpret the noise autocorrelation \((\ref{eq:fourier-autocorr})\) which contains the derivatives of delta functions when the dissipative currents mix with one another through $[\tau_Rk]^A$. Also, in Sec. \(\text{IVC}\) we argued that even when the dissipative currents do not mix with one another, the structure of the differential form is restricted in Eqs. \((\ref{eq:form1})\) and \((\ref{eq:form2})\) due to the positive semi-definiteness of the autocorrelation, the causality of the memory function, etc. We also discussed that the polynomial $M(-ik)$ is further restricted to be unity because of the derivatives of the delta function in the autocorrelation \((\ref{eq:form1})\). The differential forms which have derivatives of delta functions in its noise autocorrelation or do not satisfy the restriction are not compatible with the naive FDR \((\ref{eq:FDR})\).

However, all of those discussions are based on the assumption that the FDR is applicable to the noise with arbitrarily short scales. For example, if an appropriate upper bound of the frequency $\omega$ or the wave numbers $k$ is introduced, the power spectrum of the noise is bounded even if derivatives (which are translated to $-i\omega$ and $ik$ in the spectrum) are present in the front of the delta function. For another example, when we considered the restriction to the polynomial $L(-i\omega,-ik)$ from the positive semi-definiteness of the autocorrelation, we took the limits $\omega \to \pm \infty$ in Eq. \((\ref{eq:form1})\). We also considered the limit of $|k| \to \infty$ in discussing the causality of the memory function represented in the derivative expansion. In the first place, the differential form is based on the derivative expansion in which the derivatives are assumed to be small enough with respect to the expansion coefficients, so it is subtle to discuss the short scale noise in the differential form. In this sense, even the problem of the acausal modes in the Navier–Stokes theory could be regarded as a kind of artifact of applying the constitutive equation of the Navier–Stokes theory to arbitrarily small scales.

In fact, in non-linear fluctuating hydrodynamics, one should introduce the cutoff length scale $\lambda_c$ and the cutoff time scale $\lambda_t$ of the hydrodynamic fluctuations. Because the autocorrelation of the noise terms is given in terms of delta functions, if infinitesimally small scales of the hydrodynamic fluctuations are considered, the fluctuations of the hydrodynamic fields diverge to have unphysical values. For example, the pressure can be negative due to infinitely large fluctuations. In actual physical systems, there should be some coarse-graining scale which gives a lower limit
of the scale where hydrodynamics is applicable. By introducing the hydrodynamic fluctuations, such a lower limit of the scale could be lowered so that the applicability of hydrodynamics is extended. However, there should be still a non-zero lower limit of the scales in which fluctuating hydrodynamics is applicable, such as \( \lambda_x \) and \( \lambda_t \). Therefore in fluctuating hydrodynamics only the hydrodynamic fluctuations of longer scales than \( \lambda_x \) and \( \lambda_t \) should be considered.

In such a framework, the power spectrum of the noise should be modified in the Fourier representation so that the noise with smaller scales are suppressed. While several ways of the regularization can be considered, we here employ a sharp momentum cutoffs by the Heaviside functions:

\[
I_{\omega, k} = T[M(-ik)\kappa L(-i\omega, -ik)^\dagger + L(-i\omega, -ik)\kappa M(-ik)^\dagger]\Theta(1 - \lambda_t\omega)\Theta(1 - \lambda_x|k|). \tag{128}
\]

With this regularized FDR, the power spectrum is bounded even when one takes the limit \(|\omega|/|k| \to \infty\). In the position space, the delta functions are replaced by the Bessel functions coming from the Heaviside functions:

\[
\langle \xi(x)\xi(x') \rangle = T[M(\nabla^\mu)\kappa L(-D, -\nabla^\mu)^T + L(D, \nabla^\mu)\kappa M(-\nabla^\mu)^T]\frac{j_0(|t - t'|/\lambda_t) j_1(|x - x'|/\lambda_x)}{2\pi^3\lambda_t \lambda_x^3} \frac{1}{|x - x'|/\lambda_x}. \tag{129}
\]

where \( j_0(x) \) and \( j_1(x) \) are the spherical Bessel functions of the order 0 and 1. The derivatives on the Bessel functions can be safely interpreted as the autocorrelations.

With the cutoff scales, the restriction from the the positive semi-definiteness is also loosened. The positive semi-definiteness could not be satisfied when the degree of the polynomial \( N = \deg_{\omega} L(-i\omega, -ik) \) is larger than one, because its complex argument changes too much when one takes the high-frequency limit in Eq. (128). However, when one introduces a cutoff time scale \( \lambda_t \), the positive semi-definiteness of \( \mathbb{R}[M(-ik)L(-i\omega, -ik)] \) is only required in the frequency region \(|\omega| < 1/\lambda_t \) so that the differential form with \( N \geq 2 \) can be valid with an appropriate cutoff scale. Conversely, this means that the structure of the differential form naturally introduces cutoff scales. For example, let us consider the following differential form for a single-component dissipative current:

\[
(1 + \tau_1D)(1 + \tau_2\nabla)\mathbb{G}(x) = \kappa \mathbb{F}(x) + \xi(x), \tag{130}
\]

where \( \tau_1 \) and \( \tau_2 \) are transport coefficients which can be identified to be two different relaxation times by calculating the memory function. The power spectrum of the noise is obtained as

\[
I_{\omega, k} = 2\kappa T \mathbb{R}[(1 - i\omega\tau_1)(1 - i\omega\tau_2)] = 2\kappa T(1 - \omega^2\tau_1\tau_2). \tag{131}
\]

If one takes the limit \( \omega \to \pm \infty \), one can see that the positive semi-definiteness is broken. For the positive semi-definiteness of the spectrum, the frequency should have an upper bound: \(|\omega| \leq (\tau_1\tau_2)^{-1/2} \). Therefore the cutoff scales should be introduced as

\[
I_{\omega, k} = 2\kappa T(1 - \omega^2\tau_1\tau_2)\Theta(1 - \lambda_t\omega)\Theta(1 - \lambda_x|k|), \tag{132}
\]

where the cutoff time scale satisfies \( \lambda_t \geq \sqrt{\tau_1\tau_2} \). In this way, the structure of the differential form of the constitutive equation naturally introduces the lower bound of the cutoff time or length scales. The exceptions are the Navier–Stokes theory and the simplified Israel–Stewart theory with non-mixing dissipative currents. In these exceptional theories, the positive semi-definiteness of the noise power spectrum is always satisfied without any cutoff scales.

V. FDR IN NON-STATIC AND INHOMOGENEOUS BACKGROUNDS

A. Integral form FDR

In Sec. IV, we discussed the hydrodynamic fluctuations in equilibrium, i.e., the nature of the hydrodynamic fluctuations in static and homogeneous backgrounds. While, in dynamical calculations of the high-energy nuclear collisions, the matter with highly inhomogeneous profile and short lifetimes of the order \( \sim c/\Lambda_{QCD} \) should be described in relativistic hydrodynamics. The matter created in high-energy nuclear collisions basically expands in the direction of beam axis with relativistic velocities, and also the matter is surrounded by the vacuum so that near its boundary the temperature gradient becomes very large. In such a situation, the FDR given in Sec. IV, Eq. (81), becomes subtle. For example, the temperature \( T \) can be in general different at two different points, \( x \) and \( x' \), so it is non-trivial to determine physically what temperature should be used to generate the hydrodynamic fluctuations.

The expression for the FDR in non-static and inhomogeneous backgrounds can be found in the generalized Green–Kubo formula for the Zubarev’s non-equilibrium statistical operator, in which the FDR for the integral form of the constitutive equation has the following form:

\[
\langle \delta \mathbb{G}(x)\delta \mathbb{G}(x')^T \rangle \Theta(x^0 - x'^0) = G(x, x')\kappa(x')T(x'), \tag{133}
\]
or equivalently,

\[ \langle \delta \Gamma(x) \delta \Gamma(x')^T \rangle = G(x, x') \kappa(x') T(x') + T(x) \kappa(x) G(x', x)^T. \] (134)

For the Navier–Stokes theory, the FDR is unmodified from the equilibrium case because the memory function is just a delta function:

\[ \langle \delta \Gamma(x) \delta \Gamma(x')^T \rangle = 2T(x) \kappa(x) \delta^{(4)}(x - x'). \] (135)

For the simplified Israel–Stewart theory, the FDR is written by the memory function in inhomogeneous backgrounds.

\[ \langle \Pi(x) \Pi(x') \Theta(x^0 - x'^0) \rangle = \exp \left( - \int_{\tau(x')}^{\tau(x)} \frac{d\tau_2}{\tau_1(\tau_2, \sigma(x))} \frac{\zeta(x') T(x') \theta^{(4)}(\sigma(x), \sigma(x'))}{\tau_1(x')} \right), \] (136)

\[ \langle \pi^{\mu\nu}(x) \pi^{\alpha\beta}(x') \Theta(x^0 - x'^0) \rangle = \exp \left( - \int_{\tau(x')}^{\tau(x)} \frac{d\tau_2}{\tau_1(\tau_2, \sigma(x))} 2\eta(x') T(x') \right) \times \Delta(\tau(x); \tau(x'), \sigma(x))^{\mu\nu\alpha\beta} \delta^{(4)}(\sigma(x), \sigma(x'))), \] (137)

\[ \langle \nu^\mu_i(x) \nu^\mu_j(x') \Theta(x^0 - x'^0) \rangle = \sum_{k_i=1}^n \left[ \text{exp} \left( - \int_{\tau(x')}^{\tau(x)} d\tau_2 \tau_{ik}^{-1}(\tau_2, \sigma(x)) \right) \right] \tau_{ik}^{-1}(x') \kappa_{ij}(x') T(x') \times \left[ -\Delta(\tau(x); \tau(x'), \sigma(x))^{\mu\alpha} \delta^{(4)}(\sigma(x), \sigma(x'))). \right] \] (138)

For the general linear-response differential form, due to the restriction [122], the constitutive equations have the form [61] and [62] with \( X^{\mu\nu} = M(\nabla^\mu)^{\alpha\beta} 2\eta^\alpha \delta^{[\alpha\beta]} \) and \( X^\mu = M(\nabla^\mu)^{\mu\alpha} T_{\kappa\alpha} \nabla^{\alpha} \frac{\partial}{\partial x^\alpha} \). Here, to apply the restriction, we assumed that the diffusion currents do not mix with one another. The components of the shear stress tensor essentially do not mix with one another due to the rotational symmetry. Therefore the corresponding integral forms can be written in terms of the pathline projectors as in Eqs. [63]–[66]. The FDR can be written down similarly by the memory function extracted from these integral forms. The FDR for the bulk pressure can also be written down similarly but more simply without a pathline projector.

### B. Differential form FDR

In dynamical calculations we apply the differential form of the constitutive equation to inhomogeneous and non-static fluid fields. Therefore we need to consider the FDR in the differential form in inhomogeneous backgrounds. Let us start from the general linear constitutive equation [110] in inhomogeneous backgrounds:

\[ L(D, \nabla^\mu; x) \Gamma(x) = M(\nabla^\mu; x) \kappa F(x) + \xi(x). \] (139)

Here it should be noted that the coefficients of polynomials, \( L \) and \( M \), now have spatial dependence on \( x \) unlike in the static and homogeneous backgrounds. Also, comparing the integral form [59] and Eq. [139], one finds relations of memory functions, polynomials and the noise terms, which correspond to the equilibrium version [116] and [117]:

\[ L(D, \nabla^\mu; x) G(x, x') = M(\nabla^\mu; x) \delta^{(4)}(x - x'), \] (140)

\[ \xi(x) = L(D, \nabla^\mu; x) \delta \Gamma(x). \] (141)

The autocorrelation of the noise in the differential form is calculated as

\[ \langle \xi(x) \xi(x')^T \rangle = L(D, \nabla^\mu; x) \delta \Gamma(x') L(D', \nabla^\mu; x')^T \]

\[ = L(D, \nabla^\mu; x)[G(x, x') \kappa(x') T(x') + T(x) \kappa(x) G(x', x)^T] L(D', \nabla^\mu; x')^T \]

\[ = M(\nabla^\mu; x) \delta^{(4)}(x - x') \kappa(x') T(x') L(D', \nabla^\mu; x')^T \]

\[ + L(D, \nabla^\mu; x) T(x) \kappa(x) \delta^{(4)}(x - x') M(\nabla^\mu; x')^T, \] (142)

where \( D' \) and \( \nabla' \) are the derivatives with respect to \( x' \) that operate on the left-hand side of the polynomials \( L \) and \( M \). Note that the derivatives do not operate on the coefficients in the polynomials.
For the Navier–Stokes theory in which \( L = M = \Delta \), the FDR is reduced to the normal one as is already shown in Eq. (135).

For the simplified Israel–Stewart theory in which \( L = \Delta + \tau_R D = \Delta (I + \tau_R D) \) and \( M = \Delta \), the FDR is calculated as follows:

\[
\langle \xi(x)\xi(x')^T \rangle = \Delta(x)\delta^{(4)}(x - x')\kappa(x')T(x') [1 + \tD'_{K}(x')]\Delta(x')^T \\
+ \Delta(x)[I + \tau_R(x)D][T(x)\kappa(x)\delta^{(4)}(x - x')\Delta(x')^T \\
= 2 \{ T(x)\kappa(x) + \Delta(x)[\tau_R(x)DT(x)\kappa(x)]^S \Delta(x)^T \} \delta^{(4)}(x - x') \\
+ \Delta(x)[K(x')^T D' \delta^{(4)}(x - x') + K(x)D \delta^{(4)}(x - x')] \Delta(x')^T ,
\]

(143)

where \( K(x) := \tau_R(x)T(x)\kappa(x) \) is defined, \( A^S := (A + A^T)/2 \) is the symmetric part of a matrix \( A \), and the relation \( \Delta(x)\kappa(x) = \kappa(x)\Delta(x)^T = \kappa(x) \) is used. The symmetric part of the last line can be transformed as follows:

\[
K(x')^S D' \delta^{(4)}(x - x') + K(x)^S D \delta^{(4)}(x - x') \\
= \partial_{\mu}'[u(x'^\mu)K(x'^S)\delta^{(4)}(x - x')] - [\partial_{\mu}'u(x'^\mu)K(x'^S)] \delta^{(4)}(x - x') + K(x)^S D \delta^{(4)}(x - x') \\
= -[K(x)^S D \delta^{(4)}(x - x') - DK(x)^S + \theta(x)K(x)^S \delta^{(4)}(x - x') + K(x)^S D \delta^{(4)}(x - x')] \\
= -[DK(x)^S + \theta(x)K(x)^S \delta^{(4)}(x - x')].
\]

(144)

where the relation \( \partial_{\mu}'[f(x)\delta(x - y)] = \partial_{\mu}'f(x)\delta(x - y) = f(x)\partial_{\mu}\delta(x - y) = -f(x)\partial_{\mu}\delta(x - y) \) is used to obtain the third line. Here it should be noted that the derivatives on the delta functions should be treated carefully. For example, the relation \( \partial_{\mu}\delta(x) = -\delta(x)\partial_{\mu} \) does not apply to \( \delta(x - y) \) when \( y \) is not fixed, i.e., \( \partial_{\mu}\delta(x - y) \neq -\delta(x - y)\partial_{\mu} \). This is because

\[
\int_D dx f(x)\partial_{\mu}\delta(x - y) = [f(x)\delta(x - y)]|_{\partial D} - \int_D dx \delta(x - y)\partial_{\mu}f(x),
\]

(145)

where the surface term does not vanish when \( y \) is on the boundary. Finally one obtains the following differential FDR:

\[
\langle \xi(x)\xi(x')^T \rangle = 2\Delta(x)\delta^{(4)}(x - x') \\
+ \Delta(x)[K(x)D \delta^{(4)}(x - x') - K(x)D' \delta^{(4)}(x - x')]^S \Delta(x)^T \\
+ \Delta[\tau_R DT \kappa - (D\tau_R)T \kappa - \theta \tau_R \kappa]^S \Delta^T \delta^{(4)}(x - x'),
\]

(146)

where the spatial dependence on \( x \) or \( x' \) of the coefficients of the first and third terms is omitted because it does not matter due to the delta function. Here the first and second term correspond to the terms in the FDR in equilibrium [111]. The second term is slightly modified but is easy to check that in equilibrium it exactly reduces to the second term of Eq. (111). As in the equilibrium case, this antisymmetric part vanishes when one considers the dissipative currents not mixing with one another. The third term is a newly obtained modification to the FDR originating from the time evolution of the background. The FDR for each dissipative current reads

\[
\langle \xi_{\Pi}(x)\xi_{\Pi}(x') \rangle = \left( 2 + \tau_{\Pi}D \ln \frac{T_{\Pi}}{\tau_{\Pi}} - \tau_{\Pi} \frac{T_{\Pi}}{\tau_{\Pi}} \right) T_{\Pi} \delta^{(4)}(x - x'),
\]

(147)

\[
\langle \xi_{\mu\nu}^{(4)}(x)\xi_{\mu\nu}^{(4)}(x') \rangle = \left[ 2 + \tau_{\nu}D \ln \frac{T_{\nu}}{\tau_{\nu}} - \tau_{\nu} \frac{T_{\nu}}{\tau_{\nu}} \right] \Delta^{\mu\nu\alpha\beta} + \tau_{\nu}D \Delta^{\mu\nu\alpha\beta} \right] T_{\nu} \delta^{(4)}(x - x'),
\]

(148)

\[
\langle \xi_{\mu}^{(4)}(x)\xi_{\nu}^{(4)}(x') \rangle = -2\Delta_{\mu\nu}[K_{ij}^{(4)}(x)D - K_{ij}^{(4)}(x')D'] \delta^{(4)}(x - x') \\
- \Delta_{\mu\nu}[\tau_{ik}DT_{kj} - (D\tau_{ik})T_{kj} - \tau_{ik}\theta T_{kj}]^S - K_{ij}^{(4)}D \Delta^{\mu\nu} \right] \delta^{(4)}(x - x'),
\]

(149)

where \( K_{ij}^{(4)}(x) = \sum_{k=1}^{n} T(x)(\tau_{ik}(x)\kappa_{kj}(x) \pm \tau_{jk}(x)\kappa_{ki}(x))/2, \) and \( [\omega_{ij}]^S = (\omega_{ij} + \omega_{ji})/2 \). The derivative \( D \) on the projectors is defined as

\[
D \Delta^{\mu\nu\alpha\beta} = \Delta^{\mu\nu\kappa\lambda} \delta^{\alpha\beta}_{\gamma\delta} D \Delta^{\kappa\lambda\gamma\delta},
\]

(150)

\[
D \Delta^{\mu\nu\alpha} = \Delta^{\mu\nu\kappa} \delta^{\alpha}_{\gamma} D \Delta^{\kappa\gamma}.
\]
As already seen in the bulk pressure case \[^{147}\], the typical structure of the FDR for a single-component dissipative
current is
\[
\langle \xi(x)\xi(x') \rangle = \left( 2 + \tau_R D \ln \frac{T\kappa}{\tau_R} - \tau_R \theta \right) T\kappa\delta^{(4)}(x - x'). \tag{152}
\]

Because the expansion rate \(\theta\) can be expressed in terms of the fluid element volume \(\Delta V\) as \(\theta = D \ln \Delta V\), the first factor can be regarded as \(2 + \tau_R D \ln (T\kappa/\tau_R \Delta V)\) intuitively. This factor becomes negative when \(T\kappa/\tau_R \Delta V\) rapidly decreases in a shorter time scale than \(\tau_R/2\), which occurs, e.g., when the expansion rate becomes extremely large or the temperature suddenly decreases. In such a situation, fluctuating hydrodynamics with the differential form of the constitutive equation breaks down because the noise with a negative norm is unphysical. Even if fluctuating hydrodynamics does not break down, it is still important in the dynamical description of the high-energy nuclear collisions because the matter created in the collision reaction expands with the relativistic velocity in the longitudinal direction so that \(\theta\) is expected to be large, and also because the lifetime of the matter is several fm/c so that the temperature decreases in a very small time scale.

\section{VI. SUMMARY}

To describe the rapid spacetime evolution of the highly inhomogeneous matter created in high-energy nuclear
collisions, non-linear equations of causal dissipative hydrodynamics with a non-vanishing relaxation time should be
solved in dynamical models. In causal dissipative hydrodynamics, the differential form of the constitutive equation,
in which the dissipative currents are treated as additional dynamical fields, is used for the analytic investigations
and numerical simulations because of its simplicity and smaller computational complexity compared to the integral
form. To introduce the hydrodynamic fluctuations in such a framework, we need to consider the non-trivial properties
of hydrodynamic fluctuations in non-static and inhomogeneous matter. The autocorrelations of the hydrodynamic
fluctuations are determined by the FDR, which is normally based on the linear-response theory of small perturba-
tions to global equilibrium. However, the background fields are no longer static and homogeneous in hydrodynamic
description of the created matter. Even when the background fields are non-static and inhomogeneous, one could
just apply the FDR of global equilibrium with the local thermal state given by the local temperature and chemical
potentials if the autocorrelations were local. However, because of the relaxation time which is needed to maintain
the causality, the hydrodynamic fluctuations have non-local correlations so that one cannot assume a single thermal
state for the FDR. Here we need to explicitly consider the FDR in the non-static and inhomogeneous background by
considering the linear-response constitutive equations.

In Sec. \(^{III}\) the differential and integral forms of the constitutive equation were explained. For practical purpose, we
usually use the differential form of the constitutive equations \(^{10}\), in which the dissipative current, \(\Gamma(x)\), is represented
by the gradient expansion truncated to a finite order. For examples, the differential forms in the Navier–Stokes
theory \(^{71}\) (which is first-order and acausal) and the simplified Israel–Stewart theory \(^{10}\) (which is second-order
and causal) are introduced. While, in the linear-response theory, the constitutive equation is naturally given by the
integral form \(^{[3]}\), in which the dissipative currents are given by past thermodynamic forces, \(\kappa F(x')\), convoluted
with the integration kernel, \(G(x, x')\), called the memory function. The FDR from the linear-response theory is naturally
given in this integral form.

The differential form of the constitutive equation usually contains the time derivatives of the dissipative current and
therefore is an implicit form with respect to the dissipative current. By solving the differential form with respect to
the dissipative current, the corresponding integral form shall be obtained for the later discussion of the FDR. In global
equilibrium where the translational symmetry can be utilized, such a solution can be easily obtained in the Fourier
representation. However, it is non-trivial to obtain the solution in non-static and inhomogeneous backgrounds. In
particular, the dissipative currents such as the shear stress and the diffusion currents have special constraints such as
the transversality to the flow velocity. To write down the explicit form of the integral form corresponding to a certain
class of the differential forms in non-static and inhomogeneous backgrounds, we newly introduced in Sec. \(^{III}\) the
pathline projectors, \(\Delta(\tau; \tau')^{\mu}_{\alpha}\) \(^{30}\) and \(\Delta(\tau; \tau')^{\mu}_{\nu \alpha \beta}\) \(^{31}\), which perform projection to the tensor components of the
dissipative currents at every moment along a pathline. Using the properties of the pathline projectors, Eqs. \(^{32} - ^{34}\),
the integral form corresponding to the differential form \(^{61} - ^{62}\) was obtained as in Eqs. \(^{63} - ^{66}\). In particular,
for the simplified Israel–Stewart theory, the explicit form of the memory function \(^{65} - ^{72}\) was obtained.

In Sec. \(^{IV}\), we next discussed the FDR in higher-order linear-response constitutive equations in equilibrium. First
we introduced the noise term, \(\delta\Gamma(x)\), in the integral form \(^{30}\) and gave its autocorrelation, \(\langle \delta\Gamma(x)\delta\Gamma(x') \rangle\), by the
FDR \(^{82}\). In the Navier–Stokes theory the noise is white as usual, \(i.e.,\) the noise autocorrelation is a delta function
and has no characteristic frequency (or “color”). While, in causal theories with a non-vanishing relaxation time,
\(\tau_R\), the noise in the integral form becomes colored, \(i.e.,\) the autocorrelation has a characteristic “color” of \(\sim 1/\tau_R\).
In the simplified Israel–Stewart theory, the corresponding noise term, $\xi(x)$, appears in the differential form, and its autocorrelation is given by Eqs. (101)–(104). However, when the components of the dissipative current mix with one another, the autocorrelation contains the derivative on the delta function which is naively unphysical. For the general linear-response differential form (110), the noise autocorrelation is given by Eqs. (120)–(121). When the dissipative currents do not mix with one another, it was shown that the differential form is restricted to a very simple form with Eqs. (122)–(123) using the positive semi-definiteness of the noise autocorrelations (D1) and the general properties of the memory function such as the relaxation (17) and the causality (16). Moreover, if one does not allow the derivative on the delta function, the only allowed causal differential form is the simplified Israel–Stewart theory. All of these restrictions come from the behavior of the hydrodynamics fluctuations in infinitesimally small length and time scales. However, in actual non-linear fluctuating hydrodynamics, the cutoff scale of the hydrodynamic fluctuations should be introduced. We discussed in Sec. [IV D] that those restrictions can be reinterpreted as the lower bound of the cutoff scales.

Finally, we discussed the FDR in inhomogeneous background given by Eq. (134). While the noise autocorrelation in the Navier–Stokes theory is not modified (135), the noise autocorrelation in the integral form of the simplified Israel–Stewart theory is given by Eqs. (136)–(138) using the pathline projectors. For the general linear-response differential form in inhomogeneous background (139), the noise autocorrelation is given by Eq. (142). For the simplified Israel–Stewart theory, we obtained the explicit form of the FDR (146)–(149) to find that new modification terms appear and are proportional to the relaxation time and the time derivatives of the background thermodynamic quantities.

The hydrodynamic fluctuations should be properly implemented in dynamical models for the high-energy nuclear collisions to satisfy the correct FDR corresponding to the actually used differential form. Also, the cutoff scales of the hydrodynamic fluctuations should be chosen so that the positive-semidefiniteness of the noise autocorrelation is satisfied when one uses the differential form other than the Israel–Stewart theory. The effects of the modification to the FDR in non-static and inhomogeneous backgrounds to the experimental observables have not yet been investigated. They should also be implemented in dynamical models and quantified by comparing the results to those without the modification by performing event-by-event simulations. Another future challenge would be to investigate the effects on the noise statistics coming from non-linear responses. The second-order constitutive equations actually used in the high-energy nuclear collisions contain the non-linear terms with respect to the thermodynamic forces. If one assumes that the non-linear terms are not too large compared to the linear terms near the equilibrium, one could approximate the noise autocorrelation by the FDR based on only the linear terms in the differential form. However, the non-linear terms can have the same order with the other second-order terms in actual dynamics. They may cause non-Gaussian statistics of the fluctuations and also may introduce additional modifications to the FDR.

### ACKNOWLEDGMENTS

The author thanks Tetsufumi Hirano and Yuji Hirono for useful discussions in the early stage of this study. This work was supported by JSPS KAKENHI Grant Number 12J08554 in the early stage.

### Appendix A: Properties of projectors

The properties of the projectors, which can be directly shown by the definition of the projectors, are listed here.

\begin{align}
\Delta^\mu{}_{\kappa} & = \Delta^\mu{}_{\alpha}, \\
\Delta^\mu{}_{\nu\kappa\lambda} & = \Delta^\mu{}_{\alpha\beta}, \\
\Delta^\mu{}_{\nu\kappa} & = \Delta^\nu{}_{\mu}, \\
\Delta^\mu{}_{\nu\alpha\beta} & = \Delta^\nu{}_{\mu\alpha\beta}, \\
\Delta^\mu{}_{\nu\alpha\beta} & = \Delta^\nu{}_{\mu\alpha\beta}, \\
\Delta^\mu{}_{\nu\alpha\beta} & = \Delta^\nu{}_{\mu\alpha\beta}, \\
\Delta^\mu{}_{\nu\alpha\beta} & = \Delta^\nu{}_{\mu\alpha\beta}, \\
\Delta^\mu{}_{\nu\alpha\beta} & = \Delta^\nu{}_{\mu\alpha\beta}. 
\end{align}
Appendix B: Proof of the convergence and the properties of the pathline projectors: Eqs. (32)–(42)

In later sections we will utilize the pathline projectors which have the useful properties, Eqs. (32)–(42). However these properties are non-trivial from the definition of the pathline projectors, Eqs. (30) and (31). In this section we aim to give proofs to these properties from the definition of the pathline projectors. We give a proof to the convergence of the limits in Eqs. (30) and (31), and then give proofs to the properties. The outline of the proof is as follows: First we introduce a general projector \( P(\tau) \) as a matrix-valued function of \( \tau \), and parametrize the projector by bases of \( \text{ker} P(\tau) \), such as \( \{ u_i(\tau) \} \) and \( \{ v_i(\tau) \} \), which are assumed to be continuously differentiable for two times with respect to \( \tau \), i.e., \( C^2 \) functions of \( \tau \). We also define a function sequence \( P_N(\tau_i; \tau) \) indexed by \( N \). Next we consider the derivative of \( P_N(\tau_i; \tau) \) to identify the residual contribution \( (1/N)R_N(\tau_i; \tau) \) to the derivative, and show the boundedness of \( R_N(\tau_i; \tau) \). Finally we show the compact convergence of \( P_N(\tau_i; \tau) \) and \( D_t P_N(\tau_i; \tau) \) to obtain the properties of the pathline projectors.

It should be noted in advance that the mean-value theorem and the Taylor’s theorem will be repeatedly used in the proof: for any \( C^2 \) function \( f(x) \), there exist \( \xi, \xi' \in I(x_0, x_1) \) such that
\[
\begin{align*}
  f(x_1) &= f(x_0) + (x_1 - x_0)f'(\xi), \\
  f(x_1) &= f(x_0) + (x_1 - x_0)f'(x_0) + \frac{(x_1 - x_0)^2}{2} f''(\xi'),
\end{align*}
\]
where the closed interval \( I(x, y) \) is defined as \( I(x, y) := \{ z \mid \min\{x, y\} \leq z \leq \max\{x, y\} \} \). Also, in this section dots are used to denote the time derivatives, i.e., \( \ddot{f}(\tau) = D\dot{f}(\tau), \dddot{f}(\tau) = D^2\dot{f}(\tau), \) etc.

1. Definition of \( P_N(\tau_i; \tau) \)

Let us consider an \( n \)-dimensional linear space, \( V \), and a time-dependent projector in \( V \), \( P(\tau) \), which satisfies \( P(\tau)^2 = P(\tau) \). Using the right and left eigenvectors belonging to the eigenspace \( \text{ker} P(\tau) \), \( \{ u_i(\tau) \} \) and \( \{ v_i(\tau)^T \} \) normalized as \( v_i^T u_j = \delta_{ij} \), the projector can be expressed as
\[
P(\tau) = 1 - \sum_{i=1}^m u_i(\tau)v_i(\tau)^T,
\]
where \( 1 \) denotes the identity matrix in \( V \), and \( m := \dim \text{ker} P(\tau) \). Here it is assumed that \( u_i(\tau) \) and \( v_i(\tau)^T \) are \( C^2 \) functions, and the derivatives are bounded in the considered domain \( D \):
\[
M_p := \sup_{i,a,\tau} \{ |D^p u_{ia}(\tau)|, |D^p v_{ia}(\tau)| \} < \infty, \quad (p = 0, 1, 2),
\]
where \( u_{ia}(\tau) := e_{ia}^T u_i(\tau) \) and \( v_{ia}(\tau) := v_i(\tau)^T e_{ia} \) (\( a = 1, \ldots, n \)) are the \( a \)-th components of \( u_i(\tau) \) and \( v_i(\tau)^T \), respectively, with \( \{ e_a \} \) being a (time-independent) orthonormal basis of \( V \). It should be noted that, for the projectors \( \Delta^\nu{}^a \) and \( \Delta^{\mu\nu}{}_{a\beta} \), one can explicitly construct \( u_i \) and \( v_i \) in terms of \( u^\mu \) (see Appendix [C]). Next a function sequence \( \{ P_N(\tau_i; \tau) \}_{N} \) is defined:
\[
P_N(\tau_i; \tau) := \begin{cases} 1 & \text{(N < 0)}, \\
P(\tau) & \text{(N = 0)}, \\
\prod_{k=0}^N P(\tau_k) & \text{(N > 0)}, \end{cases}
\]
where \( \tau_k := \tau_t - \frac{\Delta \tau}{N} k \) and \( \Delta \tau := \tau_t - \tau_i \). The following properties can be easily shown:
\[
\begin{align*}
P_N(\tau_i; \tau) &= P(\tau) P_N(\tau_i; \tau) = P_N(\tau_i; \tau) P(\tau), \\
P_N(\tau_i; \tau) &= P(\tau_i), \\
P_N(\tau_i; \tau) &= P_N(\tau_i; \tau)^T.
\end{align*}
\]

2. Derivative of \( P_N(\tau_i; \tau) \)

The derivative of the single projector, \( \dot{P}(\tau) \), can be calculated as
\[
\dot{P}(\tau) = -\sum_{i=1}^m [\dot{u}_i(\tau)v_i(\tau)^T + u_i(\tau)\dot{v}_i(\tau)^T],
\]
\[ \dot{P}(\tau) = D[P(\tau)^2] = P(\tau)\dot{P}(\tau) + \dot{P}(\tau)P(\tau) \]
\[ = -\sum_{i=1}^{m}[P(\tau)u_i(\tau)v_i(\tau)^T + u_i(\tau)\dot{v}_i(\tau)^TP(\tau)], \]
(B10)

where the relation \( P(\tau)u_i(\tau) = v_i(\tau)^TP(\tau) = 0 \) is used to obtain the last line. For \( N \geq 2 \), the derivative of a sequence term, \( P_N(\tau_1; \tau_1) \), with respect to \( \tau_1 \) is
\[ D_t P_N(\tau_1; \tau_1) = -\sum_{k=0}^{N-1}(1 - \frac{k}{N})P_{k-1}(\tau_1; \tau_k-1)\dot{P}(\tau_k)P_{N-k-1}(\tau_k+1; \tau_1) \]
\[ = -\sum_{k=0}^{N-1}(1 - \frac{k}{N})P_k(\tau_1; \tau_k)\sum_{i=1}^{m}u_i(\tau_k)v_i(\tau_k)^TP_{N-k-1}(\tau_k+1; \tau_1) \]
\[ - \sum_{k=0}^{N-1}(1 - \frac{k}{N})P_{k-1}(\tau_1; \tau_k-1)\sum_{i=1}^{m}u_i(\tau_k)v_i(\tau_k)^TP_{N-k}(\tau_k; \tau_1) \]
\[ = -\frac{1}{N}P_{N-1}(\tau_1; \tau_N-1)\sum_{i=1}^{m}u_i(\tau_{N-1})v_i(\tau_{N-1})^TP(\tau_1) - \sum_{i=1}^{m}u_i(\tau_1)v_i(\tau_1)^TP_N(\tau_1; \tau_1), \]
(B11)

where \( D_t := \partial/\partial \tau_1 \). In the first line, the relation \( D_t P(\tau_k) = (1 - \frac{k}{N})\dot{P}(\tau_k) \) is used. The index \( k \rightarrow k' = k + 1 \) of the first summation of Eq. (B11) was shifted to obtain the last line. From the Taylor’s theorem, there exist \( \xi^{(1)}_{kia}, \xi^{(2)}_{kia} \in \mathcal{I}(\tau_k, \tau_k-1) \) such that
\[ v_i(\tau_k-1)^T = v_i(\tau_k)^T + \frac{\Delta \tau}{N}v_i(\tau_k)^T + \frac{\Delta \tau^2}{2N^2}\sum_{a=1}^{n}e_a^Tv_i(\tau_k)(\xi^{(1)}_{kia}), \]
(B13)
\[ u_i(\tau_k) = u_i(\tau_k-1) - \frac{\Delta \tau}{N}u_i(\tau_k-1) + \frac{\Delta \tau^2}{2N^2}\sum_{a=1}^{n}e_a\dot{v}_i(\tau_k)(\xi^{(2)}_{kia}). \]
(B14)

One also notices that
\[ \dot{P}(\tau_1)P(\tau_1) = -\sum_{i=1}^{m}u_i(\tau_1)v_i(\tau_1)^TP(\tau_1). \]
(B15)

Substituting Eqs. (B13), (B14) into Eq. (B12), one obtains
\[ D_t P_N(\tau_1; \tau_1) = \dot{P}(\tau_1)P_N(\tau_1; \tau_1) - \frac{1}{N}R_N(\tau_1; \tau_1), \]
(B16)
\[ R_N(\tau_1; \tau_1) := \frac{\Delta \tau}{N}\sum_{k=1}^{N-1}P_{k-1}(\tau_1; \tau_k-1)\sum_{i=1}^{m}\left[u_i(\tau_k-1)v_i(\tau_k)^T \right. \]
\[ + \frac{\Delta \tau^2}{2}\sum_{a=1}^{n}\left[(1 - \frac{k}{N})u_i(\tau_k-1)e_a^Tv_i(\tau_k)(\xi^{(1)}_{kia}) + (1 - \frac{k}{N})e_a\dot{v}_i(\tau_k)(\xi^{(2)}_{kia})v_i(\tau_k)^T \right] \]
\[ \left. + P_{N-1}(\tau_1; \tau_N-1)\sum_{i=1}^{m}u_i(\tau_{N-1})\left[\frac{\Delta \tau}{N}v_i(\tau_1)^T + \frac{\Delta \tau^2}{2N^2}\sum_{a=1}^{n}e_a^Tv_i(\tau_1)(\xi^{(2)}_{Nia}) \right] P(\tau_1) \right) \right]. \]
(B17)

3. **Boundedness of** \( P_N(\tau_1; \tau_1) \) and \( R_N(\tau_1; \tau_1) \)

Here we will show the boundedness of the sequences \( \{P_N(\tau_1; \tau_1)\}_N \) and \( \{R_N(\tau_1; \tau_1)\}_N \) in the limit \( N \rightarrow \infty \). More specifically we here find upper bounds of those sequences which is independent of \( N \). First let us consider upper bounds of the single projector and its derivatives:
\[ |P_{ab}(\tau)| \leq 1 + mM_0^2, \]
(B18)
\[ |\tilde{P}_{ab}(\tau)| \leq 2mM_0M_1, \]  
\[ |\tilde{P}_{ab}(\tau)| \leq 2m(M_0M_2 + M_1^2), \]

where \( P_{ab}(\tau) := e_a^T P(\tau) e_b \) is the matrix element of the projector. Matrix elements for sequences are similarly defined: 
\[ P_{N,ab}(\tau_1; \tau_1) := e_a^T P_N(\tau_1; \tau_1) e_b, \]  
and 
\[ R_{N,ab}(\tau_1; \tau_1) := e_a^T R_N(\tau_1; \tau_1) e_b. \]

From the mean-value theorem, for any integers, \( k \) and \( l \), such that \( 0 \leq k < l \leq N \), there exists \( \xi_{kab}^{(3)} \in I(\tau_{k+1}, \tau_k) \) such that

\[ P_{l-k,ab}(\tau_k; \tau_1) = \sum_{c=1}^{n} P_{ac}(\tau_k) P_{l-k-1,cb}(\tau_{k+1}; \tau_1) \]
\[ = \sum_{c=1}^{n} \left[ P_{ac}(\tau_{k+1}) + \frac{\Delta \tau}{N} \tilde{P}_{ac}(\xi_{kac}^{(3)}) \right] P_{l-k-1,cb}(\tau_{k+1}; \tau_1) \]
\[ = P_{l-k-1,ab}(\tau_{k+1}; \tau_1) + \frac{\Delta \tau}{N} \sum_{c=1}^{n} \tilde{P}_{ac}(\xi_{kac}^{(3)}) P_{l-k-1,cb}(\tau_{k+1}; \tau_1). \]  

(B21)

Here one finds an upper bound as

\[ \sup_{a,b} |P_{l-k,ab}(\tau_k; \tau_1)| \leq \left( 1 + \frac{M'}{N} \right) \sup_{a,b} |P_{l-k-1,ab}(\tau_{k+1}; \tau_1)| \]
\[ \leq \cdots \leq \left( 1 + \frac{M'}{N} \right)^{l-k} \sup_{a,b} |P_{ab}(\tau_1)| \]
\[ \leq e^{M'(l-k)/N} (1 + mM_0^2), \]  

(B22)

where \( M' := 2|\Delta \tau|nmM_0M_3 \). In particular, when \( k = 0 \) and \( l = N \), one obtains \( |P_{N,ab}(\tau_1; \tau_1)| \leq e^{M'(1 + mM_0^2)} \).

Using Eq. (B22), one also obtain an upper bound of \( |R_{N,ab}(\tau_1; \tau_1)| \) from the definition \( (B17) \) as

\[ |R_{N,ab}(\tau_1; \tau_1)| \leq n^2 \frac{|\Delta \tau|}{N} \left[ \sum_{k=1}^{N-1} e^{M'(k-l)/N} (1 + mM_0^2)m \left( \frac{M_1^2}{2} \right) \right] \]
\[ + \frac{|\Delta \tau|}{2} \left[ (1 - \frac{k^2}{N})M_1M_2 + (1 - \frac{k}{N})M_1M_2 \right] e^{M'(N-k)/N} (1 + mM_0^2) \]
\[ + n^2 \frac{|\Delta \tau|}{N} e^{M'(N-1)/N} (1 + mM_0^2)mM_1 \left( M_1 + \frac{|\Delta \tau|}{2N}M_2 \right) (1 + mM_0^2) \]
\[ = |\Delta \tau| e^{M'(N-1)/N} n^2 m(1 + mM_0^2)^2 (M_1^2 + |\Delta \tau|M_1M_2/2) \]
\[ \leq M_R(\Delta \tau) := |\Delta \tau| e^{M'} n^2 m(1 + mM_0^2)^2 (M_1^2 + |\Delta \tau|M_1M_2/2). \]  

(B23)

4. **Compact convergence of** \( P_N(\tau_1; \tau_1) \)

Next we show that \( \{ P_N(\tau_1; \tau_1) \}_{N} \) is a Cauchy sequence. For \( N \geq 2 \), the difference between two consecutive sequence terms is

\[ P_{N,ab}(\tau_1; \tau_1) - P_{N-1,ab}(\tau_1; \tau_1) = [P_{N,ab}(\tau_1; \tau_1) - P_{N-1,ab}(\tau_1; \tau_1)] - [P_{N-1,ab}(\tau_1; \tau_1) - P_{N-1,ab}(\tau_1; \tau_1)] \]
\[ = \sum_{c=1}^{n} \left[ \frac{\Delta \tau}{N} \tilde{P}_{ac}(\tau_1) + \frac{\Delta \tau^2}{2N^2} \tilde{P}_{ac}(\xi_{\tau_1}^{(4)}) \right] P_{N-1,cb}(\tau_1; \tau_1) \]
\[ - \sum_{c=1}^{n} \frac{\Delta \tau}{N} \tilde{P}_{ac}(\xi_{\tau_1}^{(5)}) P_{N,cb}(\xi_{\tau_1}^{(5)}; \tau_1) + \frac{\Delta \tau}{N(N-1)} R_{N-1,ab}(\xi_{a}^{(5)}; \tau_1) \]  

(B24)

\[ = -\frac{\Delta \tau}{N} (\xi_{\tau_1}^{(5)} - \tau_1) \sum_{c=1}^{n} D[\tilde{P}_{ac}(\tau) P_{N-1,cb}(\tau; \tau_1)] \big|_{\tau=\xi_{\tau_1}^{(6)}} \]
\[ + \sum_{c=1}^{n} \frac{\Delta \tau^2}{2N^2} \tilde{P}_{ac}(\xi_{\tau_1}^{(4)}) P_{N-1,cb}(\tau_1; \tau_1) + \frac{\Delta \tau}{N(N-1)} R_{N-1,ab}(\xi_{a}^{(5)}; \tau_1). \]  

(B25)
Here, the Taylor’s theorem and the mean-value theorem were used: there exist \( \xi_{ab}^{(4)}, \xi_{ab}^{(5)} \in I(\tau_1, \tau_1) \) and \( \xi_{ab}^{(6)} \in I(\tau_1, \xi_{ab}^{(5)}) \) such that

\[
P_{ab}(\tau_1) = P_{ab}(\tau_1) + \frac{\Delta \tau}{N} \hat{P}_{ab}(\tau_1) + \frac{\Delta \tau^2}{2N^2} \hat{P}_{ab}(\xi_{ab}^{(4)}),
\]

\[
P_{N-1,ab}(\tau_1; \tau) = P_{N-1,ab}(\tau_1; \tau) + \frac{\Delta \tau}{N} D_{\tau}P_{N-1,ab}(\xi_{ab}^{(5)}; \tau),
\]

\[
\sum_{c=1}^{n} \hat{P}_{ac}(\xi_{ab}^{(5)}) P_{N-1,cb}(\xi_{ab}^{(5)}; \tau_1) = \sum_{c=1}^{n} \hat{P}_{ac}(\tau_1) P_{N-1,cb}(\tau_1; \tau_1)
\]

\[
+ (\xi_{ab}^{(5)} - \tau_1) \sum_{c=1}^{n} D[\hat{P}_{ac}(\tau) P_{N-1,cb}(\tau; \tau_1)]|_{\tau = \xi_{ab}^{(6)}}.
\]

Then one finds an upper bound of the difference of the two consecutive terms as

\[
\left| \sum_{c=1}^{n} D[\hat{P}_{ac}(\tau) P_{N-1,cb}(\tau; \tau_1)] \right|
\]

\[
= \left| \sum_{c=1}^{n} \left[ \hat{P}_{ac}(\tau) + \sum_{c=1}^{n} \hat{P}_{ac}(\tau) \hat{P}_{cd}(\tau) \right] P_{N-1,cb}(\tau; \tau_1) - \sum_{c=1}^{n} \frac{1}{N-1} \hat{P}_{ac}(\tau) R_{N-1,cb}(\tau; \tau_1) \right|
\]

\[
\leq n[2m(M_0M_2 + M_1^2) + n(2mM_0M_1)^2](1 + mM_0^2)e^{M'} + n \cdot \frac{1}{N-1} 2mM_0M_1M_R(\Delta \tau - \tau_1)
\]

\[
\leq 2mn[M_0M_2 + M_1^2 + 2mnM_0^2M_1^2](1 + mM_0^2)e^{M'} + 2mnM_0M_1M_R(\Delta \tau) =: M^{(1)}(\Delta \tau),
\]

\[
|P_{N,ab}(\tau_1; \tau_1) - P_{N-1,ab}(\tau_1; \tau_1)|
\]

\[
\leq \frac{|\Delta \tau(\xi_{ab}^{(5)} - \tau_1)|}{N} M^{(1)}(\Delta \tau) + n \cdot \frac{|\Delta \tau|^2}{2N^2} 2m(M_0M_2 + M_1^2)(1 + mM_0^2)e^{M'} + \frac{|\Delta \tau|}{N(N-1)} M_R(\xi_{ab}^{(5)} - \tau_1)
\]

\[
\leq \frac{1}{N(N-1)} \left[ |\Delta \tau|^2 M^{(1)}(\Delta \tau) + n \cdot |\Delta \tau|^2 \cdot M_0M_2 + M_1^2 \right](1 + mM_0^2)e^{M'} + |\Delta \tau|M_R(\Delta \tau)
\]

\[
=: \frac{1}{N(N-1)} M^{(2)}(\Delta \tau).
\]

To obtain the second inequality in Eq. \( \text{(B30)} \), we used \( |\xi_{ab}^{(5)} - \tau_1| \leq |\Delta \tau|/N, \ 1/N \leq 1/(N-1) \) and the fact that \( M_R(\Delta \tau) \) is monotonically increasing with respect to \( |\Delta \tau| \).

Now we are ready to show that the sequence is a Cauchy sequence. For integers \( N_1, N_2 \) such that \( N_2 > N_1 \geq 1 \),

\[
0 \leq |P_{N_2,ab}(\tau_1; \tau_1) - P_{N_1,ab}(\tau_1; \tau_1)|
\]

\[
\leq \sum_{N=N_1+1}^{N_2} |P_{N,ab}(\tau_1; \tau_1) - P_{N-1,ab}(\tau_1; \tau_1)|
\]

\[
\leq \frac{1}{N_2 - N_1} M^{(2)}(\Delta \tau) = \left( \frac{1}{N_1} - \frac{1}{N_2} \right) M^{(2)}(\Delta \tau).
\]

Therefore the sequence is a Cauchy sequence and pointwise convergent:

\[
\lim_{N_2 \to \infty \atop N_1 \to \infty} |P_{N_2,ab}(\tau_1; \tau_1) - P_{N_1,ab}(\tau_1; \tau_1)| = 0,
\]

Then a pathline projector can be defined as the limit value of the sequence:

\[
P(\tau_1; \tau_1) := \lim_{N \to \infty} P_N(\tau_1; \tau_1),
\]

which ensures that the limits in Eqs. \( \text{(33)} \) and \( \text{(34)} \) are convergent, and the original pathline projectors, \( \Delta(\tau_1; \tau_1)\mu_\alpha \) and \( \Delta(\tau_1; \tau_1)\mu_\alpha \), are well-defined. Taking the limit \( N \to \infty \) for Eqs. \( \text{(156)} \) and \( \text{(158)} \), one obtains the following relations for the pathline projectors:

\[
P(\tau_1; \tau_1) = P(\tau_1)P(\tau_1; \tau_1) = P(\tau_1; \tau_1)P(\tau_1),
\]
\begin{align}
P(\tau_1; \tau_i) &= P(\tau_i), \\
P(\tau_1; \tau_i) &= P(\tau_1; \tau_i)^T, \tag{B35} \tag{B36}
\end{align}

which are equivalent to the properties \[32\] of the pathline projectors, \(\Delta(\tau_1; \tau_i)^\mu\) and \(\Delta(\tau_1; \tau_i)^\mu\nu\). Also, taking the limit \(N_2 \to \infty\) for Eq. \([B31]\), it follows that

\[|P_{ab}(\tau_1; \tau_i) - P_{N,ab}(\tau_1; \tau_i)| \leq \frac{M^{(2)}(\Delta \tau)}{N_1}. \tag{B37}\]

It can even be shown that the sequence is compactly convergent. For any compact closed interval \([\tau_A, \tau_B]\) \(\subset D\), and for any \(\tau_i, \tau_j \in [\tau_A, \tau_B]\),

\[0 \leq |P_{N,ab}(\tau_i; \tau_j) - P_{ab}(\tau_i; \tau_j)| \leq \frac{M^{(2)}(\Delta \tau)}{N} \leq \frac{M^{(2)}(|\tau_B - \tau_A|)}{N}. \tag{B38}\]

Therefore the sequence is uniformly convergent in the compact interval \([\tau_A, \tau_B]\):

\[\lim_{N \to \infty} \sup \limits_{\tau_i, \tau_j \in [\tau_A, \tau_B]} |P_{N,ab}(\tau_i; \tau_j) - P_{ab}(\tau_i; \tau_j)| = 0, \tag{B39}\]

which means that the sequence is compactly convergent in \(D\).

5. Compact convergence of \(D_t P_N(\tau_1; \tau_i)\)

For any compact closed interval \([\tau_A, \tau_B]\) \(\subset D\), and for any \(\tau_i, \tau_j \in [\tau_A, \tau_B]\),

\[0 \leq \left| D_t P_{N,ab}(\tau_1; \tau_i) - \sum_{c=1}^{n} \dot{P}_{ac}(\tau_i) P_{cb}(\tau_1; \tau_i) \right| \leq \frac{1}{N} |R_{N,ab}(\tau_i; \tau_j)| + \sum_{c=1}^{n} |\dot{P}_{ac}(\tau_i)| \cdot |P_{N,cb}(\tau_i; \tau_j) - P_{cb}(\tau_i; \tau_j)| \leq \frac{1}{N} |M_R(\Delta \tau) + 2nm M_0 M_1 M^{(2)}(\Delta \tau)| \leq \frac{1}{N} |M_R(\tau_B - \tau_A) + 2nm M_0 M_1 M^{(3)}(\tau_B - \tau_A)| =: \frac{1}{N} M^{(3)}(\tau_B - \tau_A). \tag{B40}\]

Here one can use Eq. \([B30]\) to obtain the second line and Eq. \([B37]\) to obtain the third line. To obtain the fourth line, one can use the fact that \(M_R(\Delta \tau)\) and \(M^{(3)}(\Delta \tau)\) are monotonically increasing with \(|\Delta \tau|\). Therefore the derivative of the sequence is uniformly convergent in the closed interval \([\tau_A, \tau_B]\):

\[\lim_{N \to \infty} \sup \limits_{\tau_i, \tau_j \in [\tau_A, \tau_B]} \left| D_t P_{N,ab}(\tau_1; \tau_i) - \sum_{c=1}^{n} \dot{P}_{ac}(\tau_i) P_{cb}(\tau_1; \tau_i) \right| = 0, \tag{B41}\]

which means that the derivative \(D_t P_N(\tau_1; \tau_i)\) is compactly convergent in the domain \(D\). Thus the limit and the derivative commute:

\[D_t P(\tau_1; \tau_i) = D_t \lim_{N \to \infty} P_N(\tau_1; \tau_i) = \lim_{N \to \infty} D_t P_N(\tau_1; \tau_i) = D_t P(\tau_1) P(\tau_1; \tau_i). \tag{B42}\]

Here, the properties \([10]\) and \([11]\) are obtained.

Finally, let us consider the derivative of the product of two pathline projectors:

\[D[P(\tau_1; \tau) P(\tau; \tau_i)] = 2P(\tau_1; \tau) \dot{P}(\tau) P(\tau; \tau_i) \]
\[= - \sum_{i=1}^{m} 2P(\tau_1; \tau) \dot{u}_i(\tau) v_i(\tau)^T + u_i(\tau) \dot{v}_i(\tau)^T P(\tau_1; \tau_i) \]
\[= 0, \tag{B43}\]
where one can use Eq. \((33)\) to obtain the second line and \(P(\tau)u_i(\tau) = v_i(\tau)^TP(\tau) = 0\) to obtain the third line. Thus, the expression \(P(\tau_i; \tau)P(\tau; \tau_i)\) is unchanged for the change of \(\tau\). Choosing \(\tau_i\) as \(\tau\), one obtains

\[
P(\tau_i; \tau)P(\tau; \tau_i) = P(\tau_i; \tau_i)P(\tau_i; \tau_i)
\]

which is equivalent to the properties \((38)\) and \((39)\). Here one can use Eqs. \((B35)\) and \((B34)\) to obtain the second line.

The expression \(P\) where one can use Eq. \((B9)\) to obtain the second line and Eqs. \((A7)\) and \((A8)\) to obtain the second line and Eqs. \((A4)\) and \((A6)\) to obtain the third line.

It should be noted that Eq. \((42)\) is obtained from Eq. \((41)\) by a property of the spatial traceless symmetric projector \(\Delta^{\mu
\nu}_{\alpha\beta}\) as follows:

\[
\begin{align*}
\Delta^{\mu\nu}_{\kappa\lambda}\Delta^{\lambda\kappa}_{\alpha\beta} &= \frac{1}{2}(\Delta^{\mu\kappa}_{\nu\lambda} + \Delta^{\mu\lambda}_{\nu\kappa} + \Delta^{\nu\kappa}_{\mu\lambda} + \Delta^{\nu\lambda}_{\mu\kappa})\Delta^{\lambda\kappa}_{\alpha\beta}
\end{align*}
\]

where one can use Eqs. \((A7)\) and \((A8)\) to obtain the second line and Eqs. \((A4)\) and \((A6)\) to obtain the third line.

**Appendix C: Right and left eigenvectors of kernels of projectors**

Here we explicitly construct \(\{u_i\}_{i=1}^{m}\) and \(\{v^T_i\}_{i=1}^{m}\) in Appendix B for the projectors, \(\Delta^{\mu\alpha}\) and \(\Delta^{\mu\nu}_{\alpha\beta}\). For the spatial projector \(\Delta^{\mu\alpha}\), the dimension of the kernel is \(m = 1\), and the eigenvectors can be trivially defined as

\[
\begin{align*}
   u_{1,\alpha} &= u^\alpha, \\
   v_{1,\alpha} &= u_\alpha,
\end{align*}
\]

where \(u_{i,\alpha}\) and \(v_{i,\alpha}\) are the \(\alpha\)-th components of the \(u_i\) and \(v_i^T\).

For the spatial traceless symmetric projector \(\Delta^{\mu\nu}_{\alpha\beta}\), the dimension of the kernel is \(m = 11\). It is nontrivial to construct the set of the eigenvectors by \(u^\mu\) because there is freedom degrees which is not entirely fixed by \(u^\mu\), i.e., freedom degrees of the rotation in the local rest frame for spatial indices. We here fix the rotation degrees of freedom by introducing a specific choice of the basis \((a_0^\mu, a_1^\mu, a_2^\mu, a_3^\mu):\)

\[
(\begin{array}{c}
   a_0 \\
   a_1 \\
   a_2 \\
   a_3
\end{array}) := (\begin{array}{c}
   u^0 \\
   u^T \\
   u_1 \\
   u_{\mu+1}
\end{array}),
\]

where \((u^0, u) := u^\mu\). The right-hand side is actually the representation of the Lorentz transformation, \(\Lambda^{\mu\alpha}\), of the boost by \(u^\mu\). The vectors \(a_i^\mu\) \((i = 1, 2, 3)\), which are written in terms of \(u^\mu\), introduce specific directions of the spatial components. Corresponding covariant vectors \((b_{0\mu}, b_{1\mu}, b_{2\mu}, b_{3\mu})\) are defined as

\[
(\begin{array}{c}
   b_0^\mu \\
   b_1^\mu \\
   b_2^\mu \\
   b_3^\mu
\end{array}) := (\begin{array}{c}
   a_0 \\
   a_1 \\
   a_2 \\
   a_3
\end{array})^{-1} = (\begin{array}{c}
   u^0 \\
   -u^T \\
   -u_1 \\
   u_{\mu+1}
\end{array}),
\]

We note that \(a_0^\mu = u^\mu\) and \(b_{0\mu} = u_\mu\). With these bases, \(\{a_i^\mu\}\) and \(\{b_{i\mu}\}\), the right and left eigenvectors can be constructed as follows: The eigenvectors for the temporal–temporal tensor component are

\[
\begin{align*}
   u_{1,\mu\nu} &= u^\mu u^\nu, \\
   v_{1,\mu\nu} &= u_\mu u_\nu.
\end{align*}
\]

The eigenvectors for the temporal–spatial tensor components are

\[
\begin{align*}
   u_{2,\mu\nu} &= u^\mu a_1^\nu, \\
   v_{2,\mu\nu} &= u_\mu b_{1\nu}, \\
   u_{3,\mu\nu} &= u^\mu a_2^\nu, \\
   v_{3,\mu\nu} &= u_\mu b_{2\nu}, \\
   u_{4,\mu\nu} &= u^\mu a_3^\nu, \\
   v_{4,\mu\nu} &= u_\mu b_{3\nu}, \\
   u_{5,\mu\nu} &= a_1^\mu u^\nu, \\
   v_{5,\mu\nu} &= b_{1\mu} u_\nu,
\end{align*}
\]

\[
(\begin{array}{c}
   0 \\
   1 \\
   1 \\
   1
\end{array}) := (\begin{array}{c}
   a_0 \\
   a_1 \\
   a_2 \\
   a_3
\end{array})^{-1} = (\begin{array}{c}
   u^0 \\
   -u^T \\
   -u_1 \\
   u_{\mu+1}
\end{array}),
\]

We note that \(a_0^\mu = u^\mu\) and \(b_{0\mu} = u_\mu\). With these bases, \(\{a_i^\mu\}\) and \(\{b_{i\mu}\}\), the right and left eigenvectors can be constructed as follows: The eigenvectors for the temporal–temporal tensor component are

\[
\begin{align*}
   u_{1,\mu\nu} &= u^\mu u^\nu, \\
   v_{1,\mu\nu} &= u_\mu u_\nu.
\end{align*}
\]

The eigenvectors for the temporal–spatial tensor components are

\[
\begin{align*}
   u_{2,\mu\nu} &= u^\mu a_1^\nu, \\
   v_{2,\mu\nu} &= u_\mu b_{1\nu}, \\
   u_{3,\mu\nu} &= u^\mu a_2^\nu, \\
   v_{3,\mu\nu} &= u_\mu b_{2\nu}, \\
   u_{4,\mu\nu} &= u^\mu a_3^\nu, \\
   v_{4,\mu\nu} &= u_\mu b_{3\nu}, \\
   u_{5,\mu\nu} &= a_1^\mu u^\nu, \\
   v_{5,\mu\nu} &= b_{1\mu} u_\nu,
\end{align*}
\]
where we used the fact that $M$ diagonalized in the Fourier space as in Eq. (120). The condition for the positive semi-definiteness can be obtained is seen as matrix indices. Reflecting the translational symmetry of the equilibrium systems, the autocorrelation is $M$ with a zero of $\Delta_{\mu\nu}$ because $\omega, k$ are real-valued coefficients. Here a wave number $\omega, k$ is chosen so that $M(-ik)A_k \neq 0$ because the positive semi-definiteness is trivially fulfilled when $M(-ik)A_k = 0$. The zeroes $\omega_p(k)_{p=1}^N$ of the polynomial $L(-i\omega, -ik)$ correspond to the poles of the memory function $G_{\omega,k} = M(-ik)/L(-i\omega, -ik)$. Note that none of the zeroes $\omega_p(k)_{p=1}^N$ are canceled with a zero of $M(-ik)$ because $M(-ik)$ and $L(-i\omega, -ik)$ are defined to be coprime. To ensure the retardation and the relaxation of the memory function $G(x - x')$, the imaginary part of the poles should be negative:

$$\Im \omega_p(k) < 0, \quad (1 \leq p \leq N).$$

Here let us consider the complex argument of the polynomials:

$$\arg[M(-ik)L(-i\omega, -ik)] = \arg[M(-ik)A_k] + \sum_{p=1}^N \arg[\omega - \omega_p(k)] - \frac{N\pi}{2}. \quad (D4)$$

When the frequency $\omega$ goes from the negative infinity to the positive infinity, the argument of the factor $\omega - \omega_p(k)$ is continuously changed from $0$ to $-\pi$ because $\Im \omega_p(k) < 0$. Thus the whole argument continuously decreases by $N\pi$:

$$\lim_{\omega \to -\infty} \arg[M(-ik)L(-i\omega, -ik)] - \lim_{\omega \to +\infty} \arg[M(-ik)L(-i\omega, -ik)] = -N\pi.$$  

**Appendix D: Proof of restrictions on differential form: Eqs. (122)–(124)**

Here we show Eqs. (122)–(124) for a single-component dissipative current $\Gamma$.

1. Restriction by positive semi-definiteness of FDR

First we consider the positive semi-definiteness of the autocorrelation $\langle \xi(x)\xi(x') \rangle$ where its spatial dependence is seen as matrix indices. Reflecting the translational symmetry of the equilibrium systems, the autocorrelation is diagonalized in the Fourier space as in Eq. (120). The condition for the positive semi-definiteness can be obtained from Eq. (121):

$$L_{\omega, k} = 2T\kappa R[M(-ik)L(-i\omega, -ik)] \geq 0, \quad (D1)$$

where we used the fact that $M(-ik) = M(-ik)^*$ because $M(\omega)$ is an even polynomial with real-valued coefficients so that $M(-ik) = M_2((-ik) \otimes (ik)) = M_2(-k \otimes k) \in \mathbb{R}$. The polynomial $L(-i\omega, -ik)$ is factorized with respect to $-i\omega$ as

$$L(-i\omega, -ik) = (-1)^N A_k \prod_{p=1}^N (i(\omega - \omega_p(k))), \quad (D2)$$

where $N = \deg_L L$ is the degree of $-i\omega$ in the polynomial $L(-i\omega, -ik)$. The factor $A_k$ is the coefficient of the highest-order term $A_k(-i\omega)^N$. The factor $A_k$ is real because $L(-i\omega, -ik)$ is an even polynomial of $-ik$ with real-valued coefficients. Here a wave number $k$ is chosen so that $M(-ik)A_k \neq 0$ because the positive semi-definiteness is trivially fulfilled when $M(-ik)A_k = 0$. The zeroes $\omega_p(k)_{p=1}^N$ of the polynomial $L(-i\omega, -ik)$ correspond to the poles of the memory function $G_{\omega,k} = M(-ik)/L(-i\omega, -ik)$. Note that none of the zeroes $\omega_p(k)_{p=1}^N$ are canceled with a zero of $M(-ik)$ because $M(-ik)$ and $L(-i\omega, -ik)$ are defined to be coprime. To ensure the retardation and the relaxation of the memory function $G(x - x')$, the imaginary part of the poles should be negative:

$$\Im \omega_p(k) < 0, \quad (1 \leq p \leq N). \quad (D3)$$

Here let us consider the complex argument of the polynomials:

$$\arg[M(-ik)L(-i\omega, -ik)] = \arg[M(-ik)A_k] + \sum_{p=1}^N \arg[\omega - \omega_p(k)] - \frac{N\pi}{2}. \quad (D4)$$

When the frequency $\omega$ goes from the negative infinity to the positive infinity, the argument of the factor $\omega - \omega_p(k)$ is continuously changed from $0$ to $-\pi$ because $\Im \omega_p(k) < 0$. Thus the whole argument continuously decreases by $N\pi$:

$$\lim_{\omega \to -\infty} \arg[M(-ik)L(-i\omega, -ik)] - \lim_{\omega \to +\infty} \arg[M(-ik)L(-i\omega, -ik)] = -N\pi. \quad (D5)$$
The positive semi-definiteness condition (D1) is expressed in terms of the complex argument as
\[
\arg[M(-ik)L(-i\omega, -ik)] \in \left(2m\pi - \frac{\pi}{2}, 2m\pi + \frac{\pi}{2}\right),
\]
where \(m\) is an integer. Here the complex argument cannot be continuously changed by more than \(\pi\), and therefore the degree of the polynomial should be \(N = 0\) or \(1\). Now the polynomial \(L(-i\omega, -ik)\) has the form:
\[
L(-i\omega, -ik) = -i\omega A_k + B_k,
\]
where \(A_k\) and \(B_k\) are even polynomials of \(k\) and real due to the property of the polynomial \(L(-i\omega, -ik)\) (D13). The \(N = 0\) case corresponds to the case \(A_k = 0\). For the case \(N = 1\), \(B_k\) should have the same sign with \(A_k\) because the imaginary part of the pole \(\omega_1 = -iB_k/A_k\) should be negative (D8). Also, \(M(-ik)\) should have the same sign with \(A_k\) and \(B_k\) unless \(M(-ik) = 0\) so that the positive semi-definiteness is satisfied.

2. Restriction by causality in memory function

Another restriction comes from the causality of the memory function. The general conditions that the response function in the derivative expansion respects the causality are given in Ref. [64]:

- The coefficient of the highest order of \(\omega\) does not depend on \(k\).
- The poles \(\{\omega_p(k)\}_{p=1}^N\) of the memory function should satisfy the following equations:

\[
\lim_{|k|\to\infty} \frac{\Re \omega_p(k)}{|k|} < 1, \quad \lim_{|k|\to\infty} \frac{\Im \omega_p(k)}{|k|} < \infty.
\]

Let us apply these conditions to the polynomial \(L(-i\omega, -ik)\) (D7). Because the highest-order coefficients are constants, \(B_k = B\) for the case \(N = 0\), and \(A_k = A\) for the case \(N = 1\). For the case \(N = 1\), the pole \(\omega_1(k) = -iB_k/A\) is pure imaginary, and thus Eq. (D8) is already satisfied. To satisfy Eq. (D9), the degree of \(k\) in \(B_k\) should not be larger than one. Since \(B_k\) is an even polynomial of \(k\), \(B_k\) is a constant also in the case \(N = 1\). Because of the normalization (111), \(B = 1\) in both cases of \(N\). In the case \(N = 1\), \(A\) is positive because its sign is the same as \(B = 1\), and in fact \(A\) physically corresponds to the relaxation time \(\tau_R > 0\). Therefore one obtains the following form of the polynomial \(L\):
\[
L(-i\omega, -ik) = 1 - i\omega \tau_R,
\]
which is equivalent to (122). It should be noted that the \(N = 0\) case corresponds to the case \(\tau_R = 0\). For both cases of \(N\), also \(M(-ik)\) should be non-negative because its sign is the same as \(B = 1\). Also Eq. (124) is shown because \(M(-ik)\) should have the same sign with \(B = 1\) unless \(M(-ik) = 0\). Eq. (124) is immediately obtained by substituting (D10) into (D1).
[46] Y. Akamatsu, A. Mazeliauskas, and D. Teaney, *Proceedings, 26th International Conference on Ultra-relativistic Nucleus-Nucleus Collisions (Quark Matter 2017): Chicago, Illinois, USA, February 5-11, 2017*, Nucl. Phys. **A967**, 872 (2017), arXiv:1705.08199 [nucl-th].

[47] Y. Akamatsu, A. Mazeliauskas, and D. Teaney, Phys. Rev. **C97**, 024902 (2018), arXiv:1708.05657 [nucl-th].

[48] M. Hongo, N. Sogabe, and N. Yamamoto, JHEP **11**, 108 (2018), arXiv:1803.07267 [hep-ph].

[49] Martinez, M. and Schäfer, Thomas, Phys. Rev. **C99**, 054902 (2019), arXiv:1812.05279 [hep-th].

[50] X. An, G. Basar, M. Stephanov, and H.-U. Yee, Phys. Rev. **C100**, 024910 (2019), arXiv:1902.09517 [hep-th].

[51] P. Kovtun, J. Phys. **A48**, 265002 (2015), arXiv:1407.0690 [cond-mat.stat-mech].

[52] Martinez, Mauricio and Schäfer, Thomas, Phys. Rev. **A96**, 063607 (2017), arXiv:1708.01548 [cond-mat.quant-gas].

[53] W. A. Hiscock and L. Lindblom, Annals Phys. **151**, 466 (1983); Phys. Rev. **D31**, 725 (1985). Phys. Rev. **D35**, 3723 (1987).

[54] W. Israel, Annals Phys. **100**, 310 (1976).

[55] W. Israel and J. M. Stewart, Annals Phys. **118**, 341 (1979).

[56] J. I. Kapusta and C. Young, Phys. Rev. **C90**, 044902 (2014), arXiv:1404.4894 [nucl-th].

[57] T. Koide, G. S. Denicol, P. Mota, and T. Kodama, Phys. Rev. **C75**, 034909 (2007), arXiv:hep-ph/0609117 [hep-ph].

[58] L. Onsager, Phys. Rev. **38**, 22652279 (1931).

[59] U. M. B. Marconi, A. Puglisi, L. Rondoni, and A. Vulpiani, Physics Reports **461**, 111 (2008).

[60] G. S. Denicol, J. Noronha, H. Niemi, and D. H. Rischke, Phys. Rev. **D83**, 074019 (2011), arXiv:1102.4780 [hep-th].

[61] G. S. Denicol, H. Niemi, J. Noronha, and D. H. Rischke, in *Advances in Nuclear Physics in Our Time Goa, India, November 28-December 2, 2010* (2011) arXiv:1103.2476 [hep-th].

[62] D. N. Zubarev, *Nonequilibrium Statistical Thermodynamics* (Plenum, New York, 1974).

[63] A. Hosoya, M.-a. Sakagami, and M. Takao, Annals Phys. **154**, 229 (1984).

[64] Y. Minami and Y. Hidaka, (2013), arXiv:1401.0006 [hep-ph].