MATRIX FACTORIZATIONS AND REPRESENTATIONS OF QUIVERS I

ATSUSHI TAKAHASHI

Dedicated to Professor Kyoji Saito on the occasion of his 60th birthday

Abstract. This paper introduces a mathematical definition of the category of D-branes in Landau-Ginzburg orbifolds in terms of $A_{\infty}$-categories. Our categories coincide with the categories of (graded) matrix factorizations for quasi-homogeneous polynomials. After setting up the necessary definitions, we prove that our category for the polynomial $x^{n+1}$ is equivalent to the derived category of representations of the Dynkin quiver of type $A_n$. We also construct a special stability condition for the triangulated category in the sense of T. Bridgeland, which should be the "origin" of the space of stability conditions.

1. Introduction

This paper introduces new triangulated categories associated to quasi-homogeneous polynomials which define isolated singularities only at the origin and relates those categories with the derived categories of representations of quivers. Our motivation comes from K. Saito’s theory of primitive forms, especially from a problem in his study on regular weight systems and generalized root systems [Sa1]. We will explain the problem below.

Let $(a, b, c; h)$ be a quadruple of positive integers such that the function $\chi(T) := \frac{(T^{h-a} - 1)(T^{h-b} - 1)(T^{h-c} - 1)}{(T^a - 1)(T^b - 1)(T^c - 1)}$ has no poles. Such a quadruple $W := (a, b, c; h)$ is called a regular weight system. It is known that $W$ is a regular weight system if and only if we have at least one polynomial $f(x, y, z) \in \mathbb{C}[x, y, z]$ such that

$$ax \frac{\partial f}{\partial x} + by \frac{\partial f}{\partial y} + cz \frac{\partial f}{\partial z} = hf$$

and

$$X_0 := \{(x, y, z) \in \mathbb{C}^3 \mid f(x, y, z) = 0\}$$

has an isolated singularity only at the origin. Note that the (restricted) map

$$f : \mathbb{C}^3 \setminus f^{-1}(0) \to \mathbb{C} \setminus \{0\}$$

is a topologically locally trivial fiber bundle, and the general fiber $X_1 := f^{-1}(1)$ (called the Milnor fiber) is an open 2-dimensional complex manifold whose second homology group
$H_2(X_1, \mathbb{Z})$ is a free $\mathbb{Z}$-module of rank $\mu := (h-a)(h-b)(h-c)/abc = \lim_{t \to 1} \chi(T)$. Since $X_1$ is real 4-dimensional, $H_2(X_1, \mathbb{Z})$ has an intersection form

$$I_{H_2(X_1, \mathbb{Z})} : H_2(X_1, \mathbb{Z}) \times H_2(X_1, \mathbb{Z}) \to \mathbb{Z}.$$ 

If $f$ is a defining polynomial of a simple (ADE) singularity, then $(H_2(X_1, \mathbb{Z}), -I_{H_2(X_1, \mathbb{Z})})$ gives the root lattice of the finite root system corresponding to the singularity. In [Sa1], it is shown that $(H_2(X_1, \mathbb{Z}), -I_{H_2(X_1, \mathbb{Z})})$ with the set of vanishing cycles (which corresponds to the set of roots) and the Milnor monodromy (which corresponds to the Coxeter transformation) satisfies the axioms of the generalized root system which naturally extends the classical (finite) root systems. Since both weight systems and generalized root systems are combinatorial, it is natural to propose the following problem.

**Problem 1.1.** ([Sa1])

Construct directly from a regular weight system $W$, without passing through the homology group $H_2(X_1, \mathbb{Z})$ of the Milnor fiber, arithmetically or combinatorially, the generalized root system of the vanishing cycles.

The purpose of this paper is to develop a necessary tools in terms of $A_\infty$-categories and to give a partial answer to the above problem. Let $k$ be a field of characteristic zero. First, we introduce a notion of $\mathbb{Q}$-graded $A_\infty$-categories over $k$ (Definition 2.1) in order to consider the polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$ with a polynomial $f$ satisfying the quasi-homogeneous condition

$$\sum_{i=1}^{n} \frac{2a_i}{h} \cdot x_i \frac{\partial f}{\partial x_i} = 2f,$$

(1.1)

where $a_1, \ldots, a_n$ and $h$ are positive integers such that the greatest common divisor of them is 1, as a usual $\mathbb{Z}_2$-graded $A_\infty$-category with $m_0(1) = f$ and an "extra $\frac{2}{h} \cdot \mathbb{Z}$-grading". We shall denote the $\mathbb{Q}$-graded $A_\infty$-category defined by $f \in \mathbb{C}[x_1, \ldots, x_n]$ by $A_f$ (Example-Definition 2.2).

Next, we consider the category of twisted complexes over $\mathbb{Q}$-graded $A_\infty$-categories (Proposition 2.15) and the derived category of $\mathbb{Q}$-graded $A_\infty$-categories (Definition 2.17). The important fact is that the twisted complexes over $A_f$ coincide with matrix factorizations of $f$ introduced by Eisenbud [E] in his study of maximal Cohen-Macaulay modules. Since we consider the quasi-homogeneous polynomial $f$, we have a group action ($\mathbb{Z}$-action) on the category of matrix factorizations. Inspired by the work by Hori and Walcher [HW], we introduce the $\mathbb{Z}$-equivariant derived category of $A_f$ denoted by $D^b_Z(A_f)$ (Definition 2.23). We can now propose a conjecture to K. Saito’s problem.

**Conjecture 1.2.** Let $W$ be a regular weight system and $f$ be a quasi-homogeneous polynomial attached to $W$. Assume $W$ has a dual regular weight system $W^* = (a^*, b^*, c^*; h)$ in the sense
of $\text{S~\&~T}$ and let $f^*$ be a quasi-homogeneous polynomial attached to $W^*$. Then the following should hold.

(i) $\mathcal{D}^b_Z(A_{f^*})$ is generated as a triangulated category by objects $\{E_1, \ldots, E_\mu\}$ such that

$$\text{Hom}_{\mathcal{D}^b_Z(A_{f^*})}(E_i, E_j) = 0, \quad i > j, \quad \text{Hom}_{\mathcal{D}^b_Z(A_{f^*})}(E_i, E_j[k]) = 0, \quad k \neq 0, \forall i, j.$$ (1.2)

That is to say, $\mathcal{D}^b_Z(A_{f^*})$ is generated by a strongly exceptional collection.

(ii) $\mathcal{D}^b_Z(A_{f^*})$ has the Serre functor $S$ such that $S \cong [3h - 2a^* - 2b^* - 2c^*]$ where $[1]$ is the shift functor on $\mathcal{D}^b_Z(A_{f^*})$.

(iii) Let $a_{ij} := \chi(E_i, E_j) = \dim_k \text{Hom}_{\mathcal{D}^b_Z(A_{f^*})}(E_i, E_j)$. Put $A := (a_{ij})$ and $I_{K_0(\mathcal{D}^b_Z(A_{f^*}))} = A^{-1} + tA^{-1}$. Then $(K_0(\mathcal{D}^b_Z(A_{f^*})), I_{K_0(\mathcal{D}^b_Z(A_{f^*}))})$ is isomorphic to $(H_2(X_1, \mathbb{Z}), -I_{H_2(X_1, \mathbb{Z})})$ as a lattice.

This conjecture is based on the relation between the duality of regular weight systems and the mirror symmetry of Landau-Ginzburg orbifolds (see [T]). We do not discuss this background in detail here but we write the following diagram for reader’s convenience.

Quasi-homogeneous polynomial $f$ for $W^* \xrightarrow{\text{Milnor fiber}} \{\text{Vanishing cycles in } X_1 = f^{-1}(1)\} \xrightarrow{\text{Mirror Symmetry}} \{\text{A-branes in LG model for } W\}$

For ADE singularities, we know that $W \cong W^*$ and the generalized root systems for them are the classical finite root systems. Therefore, we may expect that the following conjecture should hold.

**Conjecture 1.3.** Let $W$ be a regular weight system corresponding to an ADE singularity and $f$ be a quasi-homogeneous polynomial attached to $W$. Then $\mathcal{D}^b_Z(A_f)$ is equivalent as a triangulated category to the bounded derived category of finite dimensional representations of Dynkin quiver corresponding to the type of singularity of $f$.

This is also expected from the homological mirror symmetry phenomena for ADE singularities studied by Seidel. See [Se2] for details.

In this paper, we will prove the conjecture for $A_n$-singularities (Theorem 3.1), where we reduce to the case $f := x^{n+1} \in \mathbb{C}[x]$ by Knörrer’s periodicity [K] (see also [O1]). We will give a proof of the above conjecture for general cases in a separate paper [KST].

Finally, we will construct a special stability condition for the triangulated category $\mathcal{D}^b_Z(A_f)$ in the sense of T. Bridgeland [B] for $f = x^h$. We can naturally introduce in our formulation the phase of objects (Definition 4.1) and the central charge $Z_\omega$ (Definition 4.3).
While our preparation of this paper, two papers related to our work appeared. One is the paper [W] by J. Walcher where he studies from physical point of view the similar categories and the stability conditions on them (his notion of ”R-stability”). Another is the paper [O2] by D. Orlov where he studies the triangulated category for singularities with a \(\mathbb{C}^*\)-action, which is equivalent to our category \(D^b_{\mathbb{Z}}(A_f)\).

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2. \(\mathbb{Q}\)-graded \(A_\infty\)-categories

In this section, we set up several definitions which we will use in the later sections. Let \(k\) be a field of characteristic zero.

**Definition 2.1.** Let \(h\) be a positive number. A \(\mathbb{Q}\)-graded \(A_\infty\)-category \(A\) of index \(h\) is a collection of the following data.

(i) A set of objects \(\text{Ob}(A)\),

(ii) A set of homomorphisms, a \(\mathbb{Q} \times \mathbb{Z}_2\) graded \(k\)-linear vector space for each \(a, b \in \text{Ob}(A)\)

\[
A(a, b) = \bigoplus_{q \in \mathbb{Q}} A^q(a, b)_+ \oplus A^q(a, b)_-,
\]

such that

\[
A^q(a, b)_+ = 0, \quad q \notin \frac{2}{h}\mathbb{Z}, \quad A^q(a, b)_- = 0, \quad q - 1 \notin \frac{2}{h}\mathbb{Z}.
\]

We call the subspaces

\[
A(a, b)_+ := \bigoplus_{q \in \frac{2}{h}\mathbb{Z}} A^q(a, b)_+, \quad A(a, b)_- := \bigoplus_{q - 1 \in \frac{2}{h}\mathbb{Z}} A^q(a, b)_-
\]

the even and the odd subspaces.

(iii) for \(n \geq 0\), \(k\)-multilinear maps

\[
m_n^A : A(a_{n-1}, a_n) \otimes \cdots \otimes A(a_0, a_1) \to A(a_0, a_n), \quad a_i \in \text{Ob}(A),
\]

of degree \(2 - n\) with respect to the \(\mathbb{Q}\)-grading which is even (odd) with respect to the \(\mathbb{Z}_2\)-grading when \(n\) is even (odd), where \(m_0^A\) is a map

\[
m_0^A : k \to A(a, a).
\]
The multilinear maps satisfy the following (\(A_\infty\)-relation). For fixed \(n\), we have

\[
\sum_{r+s+t=n} \sum_{r+1+t=u} \sum_{x_1,\ldots,x_{r+s+t}} (-1)^{|x_1|+\cdots+|x_r|+r} m^A_u(x_{r+s+t} \otimes \cdots \otimes x_{r+s+1} \otimes \cdots) = 0, \tag{2.1}
\]

where \(|x_i|\) is the parity of the morphism defined by

\[
|x_i| := \begin{cases} 
0, & x_i \in A(a_{i-1}, a_i)_+, \\
1, & x_i \in A(a_{i-1}, a_i)_-.
\end{cases} \tag{2.2}
\]

Remark 2.2. \(\mathbb{Q}\)-graded \(A_\infty\)-category of index 1 is nothing but an \(A_\infty\)-category with the usual \(\mathbb{Z}\)-grading, we call it a \(\mathbb{Z}\)-graded \(A_\infty\)-category or simply an \(A_\infty\)-category. See \cite{F} and \cite{Se1} for details of homological algebra of \(A_\infty\)-categories.

We write down explicitly the relation (2.1) when \(m^A_n = 0\) for \(n \geq 3\). For \(x, y, z \in \oplus_{a,b} A(a,b)\), we have

\[
m^A_1(m^A_0(1)) = 0,
\]

\[
m^A_1(m^A_1(x)) = (-1)^{|x|} m^A_2(m^A_0(1) \otimes x) - m^A_2(x \otimes m^A_0(1)),
\]

\[
m^A_1(m^A_2(x \otimes y)) = (-1)^{|y|} m^A_2(m^A_1(x) \otimes y) - m^A_2(x \otimes m^A_1(y)),
\]

\[
m^A_2(m^A_2(x \otimes y) \otimes z) = (-1)^{|z|} m^A_2(x \otimes m^A_2(y \otimes z)).
\]

Put \(u := -m^A_0(1)\), \(d(x) := (-1)^{|x|+1} m^A_1(x)\) and \(x \cdot y := (-1)^{|y|} m^A_2(x \otimes y)\). Then a triple \((u, d, \cdot)\) defines on \(\oplus_{a,b} A(a,b)\) a curved differential graded (CDG) algebra structure \cite{KL1}.

Example-Definition 2.3. Let \(f \in \mathbb{C}[x_1, \ldots, x_n]\) be a polynomial which satisfies the following quasi-homogeneous condition:

\[
\sum_{i=1}^n 2a_i \cdot \frac{2}{h} \cdot x_i \frac{\partial f}{\partial x_i} = 2f, \tag{2.3}
\]

where \(a_1, \ldots, a_n\) and \(h\) are positive integers such that the greatest common divisor of them is 1. We denote by \(\mathcal{A}_f\) the \(\mathbb{Q}\)-graded \(A_\infty\)-category of index \(h\) defined as follows:

\[
\text{Ob}(\mathcal{A}_f) = \{a\},
\]

\[
\mathcal{A}_f(a,a) := \mathbb{C}[x_1, \ldots, x_n], \quad \mathcal{A}_f(a,a)_- := 0,
\]

\[
m^A_0(1) := f \in \mathcal{A}_f^2(e,e)_+, \quad m^A_1 := 0,
\]

\[
m^A_2(\alpha \otimes \beta) := \alpha \cdot \beta, \quad \alpha, \beta \in \mathbb{C}[x_1, \ldots, x_n],
\]

where \(\cdot\) is the usual product on \(\mathbb{C}[x_1, \ldots, x_n]\).

In the above example, we have the special element \(1 \in \mathcal{A}_f^0(a,a)_+\) which defines a unit of the algebra \(\mathbb{C}[x_1, \ldots, x_n]\). It is well-known that the notion of units in \(A_\infty\)-categories can be introduced as follows:
Definition 2.4. Let \( a \in \text{Ob}(\mathcal{A}) \). \( e_a \in \mathcal{A}_0(a, a)_+ \) is called a unit if
\[
m^2_a(x, e_a) = x, \quad m^2_a(e_a, y) = (-1)^{|y|}y,
\]
for \( x \in \mathcal{A}(a, b) \) and \( y \in \mathcal{A}(b, a) \), and for \( n \neq 2 \)
\[
m^n_a(x_1, \ldots, x_n) = 0,
\]
if one of \( x_i \) coincides with \( e_a \).

Remark 2.5. It is easy to check that if a unit exists then it is unique.

Definition 2.6. \( \mathbb{Q} \)-graded \( \mathcal{A}_\infty \)-category is called unital if each object has a unit.

Let \( \mathcal{A} \) be a \( \mathbb{Q} \)-graded \( \mathcal{A}_\infty \)-category of index \( h \). We can construct another \( \mathbb{Q} \)-graded \( \mathcal{A}_\infty \)-category \( \overline{\mathcal{A}} \) of index \( h \) from \( \mathcal{A} \) as follows:

Definition 2.7. Let \( \mathcal{A} \) be a unital \( \mathbb{Q} \)-graded \( \mathcal{A}_\infty \)-category of index \( h \).

(i) a set of objects \( \text{Ob}(\overline{\mathcal{A}}) \) is given by
\[
\text{Ob}(\overline{\mathcal{A}}) := \left\{ a\{\frac{2k}{h}\}, a \in \text{Ob}(\mathcal{A}), k \in \mathbb{Z}, \quad b\{\frac{2l}{h}\}[-1], b \in \text{Ob}(\mathcal{A}), l \in \mathbb{Z} \right\},
\]
(ii) a set of homomorphisms is given by
\[
\bar{\mathcal{A}}^q\left(a\{\frac{2k}{h}\}, b\{\frac{2l}{h}\}\right)_\pm := \mathcal{A}^{q + \frac{2(l-k)}{h}}(a, b)_\pm,
\]
\[
\bar{\mathcal{A}}^q\left(a\{\frac{2k}{h}\}, b\{\frac{2l}{h}\}[-1]\right)_\pm := \mathcal{A}^{q + \frac{2(l-k)}{h} - 1}(a, b)_\mp,
\]
\[
\bar{\mathcal{A}}^q\left(a\{\frac{2k}{h}\}[-1], b\{\frac{2l}{h}\}\right)_\pm := \mathcal{A}^{q + \frac{2(l-k)}{h} + 1}(a, b)_\mp,
\]
\[
\bar{\mathcal{A}}^q\left(a\{\frac{2k}{h}\}[-1], b\{\frac{2l}{h}\}[-1]\right)_\pm := \mathcal{A}^{q + \frac{2(l-k)}{h}}(a, b)_\pm.
\]
(iii) \( k \)-multilinear maps \( \bar{m}_n^\mathcal{A} \) are defined using those on \( \mathcal{A} \) with additional signs as follows. For \( x_1 \in \overline{\mathcal{A}}(a_0, a_1), \ldots, x_n \in \overline{\mathcal{A}}(a_{n-1}, a_n) \),
\[
\bar{m}_n^\mathcal{A}(x_n \otimes \cdots \otimes x_1) := (-1)^{|a_0|}m_n^\mathcal{A}(x_n \otimes \cdots \otimes x_1),
\]
where we regard \( x_i \) in the right hand side as a homomorphism in \( \mathcal{A} \) by the above definition (ii) and \( |a_0| \) is the parity of \( a_0 \) defined by
\[
|a_0| := \begin{cases} 
0, & a_0 = a\{\frac{2k}{h}\}, \quad a \in \text{Ob}(\mathcal{A}), k \in \mathbb{Z} \\
1, & a_0 = a\{\frac{2l}{h}\}[-1], \quad a \in \text{Ob}(\mathcal{A}), l \in \mathbb{Z}.
\end{cases}
\]
A\textsubscript{\infty}-functors for \(\mathbb{Q}\)-graded \(A\textsubscript{\infty}\)-categories can be defined in an obvious way. There are the "translation" functor \(\{\frac{2}{h}\}\) and the shift functor \([1]\) on \(\overline{\mathcal{A}}\).

**Proposition 2.8.** The following functors \(\{\frac{2}{h}\}\) and \([1]\) define autoequivalences of \(\overline{\mathcal{A}}\):

\[
\{\frac{2}{h}\} \left( a \{\frac{2k}{h}\} \right) := a \{\frac{2(k + 1)}{h}\}, \quad \{\frac{2}{h}\} \left( a \{\frac{2k}{h}\}[-1] \right) := a \{\frac{2(k + 1)}{h}\}[-1], \quad (2.12)
\]

and

\[
[1] \left( a \{\frac{2k}{h}\} \right) := a \{\frac{2(k + h)}{h}\}[-1], \quad [1] \left( a \{\frac{2k}{h}\}[-1] \right) := a \{\frac{2k}{h}\}. \quad (2.13)
\]

Put \(\{\frac{2k}{h}\} := \{\frac{2}{h}\}^k\) and \([l] := [1]^l\) for \(k, l \in \mathbb{Z}\). We have the relation \(\{\frac{2k}{h}\} = [2]\).

Let \(\overline{\mathcal{A}}\) be an \(\mathbb{Q}\)-graded \(A\textsubscript{\infty}\)-category of index \(h\). Consider the \(\mathbb{Q}\)-graded \(A\textsubscript{\infty}\)-category \(\overline{\mathcal{A}}\) of index \(h\) whose set of objects is the set of finite (formal) direct sums of objects of \(\overline{\mathcal{A}}\),

\[
\text{Ob}(\overline{\mathcal{A}}) := \left\{ a = \bigoplus_i a_i \{\frac{2k_i}{h}\} \oplus \bigoplus_j a_j \{\frac{2l_j}{h}\}[-1], \quad a_i, a_j \in \text{Ob}(\overline{\mathcal{A}}), \quad k_i, l_j \in \mathbb{Z} \right\}, \quad (2.14)
\]

whose set of homomorphisms is

\[
\overline{\mathcal{A}}(a, b) := \bigoplus_{i_1, i_2} \overline{\mathcal{A}}(a_{i_1} \{\frac{2k_{i_1}}{h}\}, b_{i_2} \{\frac{2k_{i_2}}{h}\}) \oplus \bigoplus_{i_1, j_2} \overline{\mathcal{A}}(a_{i_1} \{\frac{2k_{i_1}}{h}\}, b_{j_2} \{\frac{2l_{j_2}}{h}\}[-1])
\]

\[
\oplus \bigoplus_{j_1, j_2} \overline{\mathcal{A}}(a_{j_1} \{\frac{2l_{j_1}}{h}\}[-1], b_{j_2} \{\frac{2k_{j_2}}{h}\}) \oplus \bigoplus_{j_1, j_2} \overline{\mathcal{A}}(a_{j_1} \{\frac{2l_{j_1}}{h}\}, b_{j_2} \{\frac{2l_{j_2}}{h}\}[-1]), \quad (2.15)
\]

and whose \(k\)-linear maps are defined by those on \(\overline{\mathcal{A}}\) using the natural "matrix multiplication" rule.

**Definition 2.9.** Take an object \(a \in \text{Ob}(\overline{\mathcal{A}})\) and \(Q \in \overline{\mathcal{A}}(a, a)_-\). \((a; Q)\) is called a twisted complex if \(Q\) satisfies the Maurer-Cartan equation

\[
\sum_{n \geq 0} m^A_n(Q ^{\otimes n}) = 0. \quad (2.16)
\]

The set of all twisted complexes is denoted by \(\text{Ob}(Tw(\mathcal{A}))\). If \(Q \in \overline{\mathcal{A}}^1(a, a)_-\) in addition, then \((a; Q)\) is called a graded twisted complex and we denote the set of all graded twisted complexes by \(\text{Ob}(Tw_{\mathbb{Z}}(\mathcal{A}))\).

An assumption is necessary for the equation (2.16) to make sense. If \(m^A_0 = 0\), then it is usually introduced that the notion of one-sided twisted complexes which makes the sum in the equation (2.16) finite. However we study in this paper the case when \(m^A_0 \neq 0\), we shall assume that our \(A\textsubscript{\infty}\)-categories have no higher product, in other words, \(m^A_n = 0\) for all \(n \geq 3\).
We often write $Q$ in the following form:

$$Q = \begin{pmatrix} Q_{++} & Q_{+-} \\ Q_{-+} & Q_{--} \end{pmatrix}$$

(2.17)

where

$$Q_{\pm\pm} \in \tilde{A}(a_\pm, a_\pm), Q_{\pm\mp} \in \tilde{A}(a_\pm, a_\mp),$$

(2.18)

and $a_\pm$ are given by the following decomposition

$$a = a_+ + a_-[-1], \quad a_+ = \bigoplus_i a_{+,i}\{\frac{2k_i}{h}\}, \quad a_- = \bigoplus_i a_{-,i}\{\frac{2l_i}{h}\}.$$  

(2.19)

Remark 2.10. If $\mathcal{A}$ is a unital $\mathbb{Q}$-graded $A_\infty$-category of index $h$ with $m_n^A = 0$, $n \geq 3$, then there exists at least one twisted complex for each object $a \in \text{Ob}(\tilde{\mathcal{A}})$. Indeed,

$$Q|_{ij} = \begin{cases} 0, & \text{for } i, j \text{ such that } a_{+,i} \neq a_{-,j} \text{ and } a_{+,i}\{\frac{2h}{n}\} \neq a_{-,j}, \\ e_{a_{+,i}} & \text{for } i, j \text{ such that } a_{+,i} = a_{-,j}, \\ e_{a_{+,i}} & \text{for } i, j \text{ such that } a_{+,i}\{\frac{2h}{n}\} = a_{-,j}, \\ m_0^A(1) & 0 \end{cases}$$

(2.20)

is a twisted complex.

Example 2.11. Since $\mathcal{A}_f$ has no odd homomorphisms, each twisted complex $(a = a_+ \oplus a_-[-1]; Q_a)$ has the following form

$$Q_a := \begin{pmatrix} 0 & Q_{+-} \\ Q_{-+} & 0 \end{pmatrix}, \quad Q_{+-} \in \tilde{A}(a_+, a_-), \quad Q_{-+} \in \tilde{A}(a_-, a_+).$$

The Maurer-Cartan equation (2.16) becomes

$$f \cdot \text{Id} - Q_a^2 = 0.$$  

(2.21)

This is exactly the same equation which first studied by Eisenbud in his work on maximal Cohen-Macaulay modules. $Q_a$ is called a matrix factorization of $f$.

Let $\mathcal{A}$ be a unital $\mathbb{Q}$-graded $A_\infty$-category of index $h$ with $m_n^A = 0$, $n \geq 3$.

Definition 2.12. Let $\alpha := (a; Q_a)$ and $\beta := (b; Q_b)$ be twisted complexes. We first put

$$Tw(\mathcal{A})(\alpha, \beta) := \tilde{A}(a, b)_+ \oplus \tilde{A}(a, b)_-.$$  

(2.22)

We define a $k$-multilinear maps $m_n^{Tw(\mathcal{A})}(n = 0, 1, 2)$ by

$$m_0^{Tw(\mathcal{A})}(1) := 0,$$

(2.23)

$$m_1^{Tw(\mathcal{A})}(\Phi) := m_1(\Phi) + m_2^A(Q_b \otimes \Phi) + m_2^A(\Phi \otimes Q_a),$$

(2.24)
where $\Phi \in Tw(A)(\alpha, \beta)$ and

$$m_2^{Tw(A)}(\Psi_2 \otimes \Psi_1) := m_2^A(\Psi_2 \otimes \Psi_1),$$

for $\Psi_1 \in Tw(A)(\alpha_0, \alpha_1) = \mathcal{A}(a_0, a_1)$ and $\Psi_2 \in Tw(A)(\alpha_1, \alpha_2) = \mathcal{A}(a_1, a_2)$.

We often write the spaces of morphisms in the matrix form:

$$Tw(A)(\alpha, \beta)_\pm = \begin{pmatrix} \mathcal{A}(a_+, b_+) & \mathcal{A}(a_-, b_+) \\ \mathcal{A}(a_+, b_-) & \mathcal{A}(a_-, b_-) \end{pmatrix}.$$

Lemma 2.13. $(m_1^{Tw(A)})^2 = 0$.

Proof. For $\Phi_\pm \in Tw(A)(\alpha, \beta)_\pm$, we have

$$(m_1^{Tw(A)})^2(\Phi_\pm) = (m_1^A)\Phi_\pm + m_1^A(m_2^A(Q_b \otimes \Phi_\pm)) + m_1^A(m_2^A(\Phi_\pm \otimes Q_a))$$

$$+ m_2^A(Q_b \otimes (m_1^A(\Phi_\pm) + m_1^A(Q_b \otimes \Phi_\pm) + m_2^A(\Phi_\pm \otimes Q_a)))$$

$$+ m_2^A((m_1^A(\Phi_\pm) + m_1^A(Q_b \otimes \Phi_\pm) + m_2^A(\Phi_\pm \otimes Q_a)) \otimes Q_a)$$

$$= m_2^A(m_1^A(1) \otimes \Phi_\pm) - m_2^A(\Phi_\pm \otimes m_1^A(Q_b)) \pm m_2^A(m_1^A(Q_b) \otimes \Phi_\pm)$$

$$- m_2^A(\Phi_\pm \otimes m_1^A(Q_a)) \pm m_2^A(m_1^A(Q_a) \otimes \Phi_\pm) - m_2^A(\Phi_\pm \otimes m_2^A(Q_a^{\otimes 2}))$$

$$= 0.$$

Lemma 2.14. For $\Phi \in Tw(A)(\alpha_0, \alpha_1)$ and $\Psi \in Tw(A)(\alpha_1, \alpha_2)$, we have

$$m_1^{Tw(A)}(m_2^{Tw(A)}(\Psi \otimes \Phi)) = (-1)^{\Phi}m_2^{Tw(A)}(m_1^{Tw(A)}(\Psi \otimes \Phi) - m_2^{Tw(A)}(\Psi \otimes m_1^{Tw(A)}(\Phi)).$$

Proof.

$$m_1^{Tw(A)}(m_2^{Tw(A)}(\Psi \otimes \Phi))$$

$$= m_1^A(m_2^A(\Psi \otimes \Phi)) + m_2^A(Q_{a_2} \otimes m_2^A(\Psi \otimes \Phi)) + m_2^A(m_1^A(\Psi \otimes \Phi) \otimes Q_{a_0})$$

$$= (-1)^{\Phi}m_2^A(m_1^A(\Psi \otimes \Phi) - m_2^A(\Psi \otimes m_1^A(\Phi))$$

$$= (-1)^{\Phi}m_2^A(m_1^A(Q_{a_2} \otimes \Psi) \otimes \Phi) - m_2^A(\Psi \otimes m_2^A(\Phi \otimes Q_{a_0}))$$

$$= (-1)^{\Phi}m_2^A(m_1^{Tw(A)}(\Psi \otimes \Phi) - (-1)^{\Phi}m_2^A(m_1^A(\Psi \otimes Q_{a_1}) \otimes \Phi)$$

$$- m_2^{Tw(A)}(\Psi \otimes m_1^{Tw(A)}(\Phi)) + m_2^A(\Psi \otimes m_2^A(Q_{a_2} \otimes \Phi))$$

$$= (-1)^{\Phi}m_2^{Tw(A)}(m_1^{Tw(A)}(\Psi \otimes \Phi) - m_2^{Tw(A)}(\Psi \otimes m_1^{Tw(A)}(\Phi)).$$
By the above two Lemmas, we have the following.

**Proposition 2.15.** Let $\mathcal{A}$ be a unital $\mathbb{Q}$-graded $A_\infty$-category of index $h$ with $m^n_\mathcal{A} = 0$, $n \geq 3$. A collection $\text{Ob}(Tw(\mathcal{A}))$, $Tw(\mathcal{A})(\alpha, \beta)$ and $(m^0_\mathcal{A}, m^1_\mathcal{A}, m^2_\mathcal{A})$ given by Definition 2.7 and Definition 2.12 determines a structure of a differential $(\mathbb{Z}_2)$-graded category. We denote it by $Tw(\mathcal{A})$. $\square$

**Remark 2.16.** Note that the condition that $\mathcal{A}$ is $\mathbb{Q}$-graded is not necessary for the above definition of the category $Tw(\mathcal{A})$ and the category $D^b(\mathcal{A})$ below. We need only the $\mathbb{Z}_2$-grading.

**Definition 2.17.** Let $\mathcal{A}$ be a unital $\mathbb{Q}$-graded $A_\infty$-category of index $h$ with $m^n_\mathcal{A} = 0$, $n \geq 3$. We construct the category $D^b(\mathcal{A})$ called the bounded derived category of $\mathcal{A}$ as follows. The set of objects is given by

$$\text{Ob}(D^b(\mathcal{A})) := \text{Ob}(Tw(\mathcal{A})), \quad (2.27)$$

and the set of homomorphisms is given by

$$\text{Hom}_{D^b(\mathcal{A})}(\alpha, \beta) := \text{Ker}(m^1_\mathcal{A} : Tw(\mathcal{A})(\alpha, \beta)_+ \to Tw(\mathcal{A})(\alpha, \beta)_-) \quad (2.28)$$

$$/ \text{Im}(m^1_\mathcal{A} : Tw(\mathcal{A})(\alpha, \beta)_- \to Tw(\mathcal{A})(\alpha, \beta)_+). \quad (2.29)$$

Let $T \in Tw(\mathcal{A})(\alpha, \beta)_+$ be a $m^1_\mathcal{A}$-closed homomorphism. We define a mapping cone $C(T)$ as an object

$$C(T) := (a[1] \oplus b; Q_{C(T)}), \quad Q_{C(T)} := \begin{pmatrix} Q_a[1] & 0 \\ T & Q_b \end{pmatrix}. \quad (2.30)$$

$C(T)$ is well-defined since the Maurer-Cartan equation (2.16) for $Q_{C(T)}$ is equivalent to the equation $m^1_\mathcal{A}(T) = 0$ and the Maurer-Cartan equation (2.13) for $Q_a$ and $Q_b$. Note also that there are natural closed morphisms

$$\beta \to C(T), \quad C(T) \to \alpha.$$

We define an exact triangle in the category $D^b(\mathcal{A})$ as a triangle of the form

$$\alpha \to \beta \to C(T) \to \alpha[1], \quad (2.31)$$

for some $T \in \text{Ker}(m^1_\mathcal{A} : Tw(\mathcal{A})(\alpha, \beta)_+ \to Tw(\mathcal{A})(\alpha, \beta)_-)$. $\square$

**Theorem 2.18.** The category $D^b(\mathcal{A})$ endowed with a shift functor [1] and the class of exact triangles defined above becomes a triangulated category.

**Proof.** The proof is essentially the same as the known results in the usual situation. See for example, [BK], [GM], [KS], [O1] and [Se1]. $\square$
Remark 2.19. The twice of the shift functor $[2] := [1]^\otimes 2$ is isomorphic to the identity functor in $D^b(A)$.

We shall add more objects to $D^b(A)$ following [Sel].

Definition 2.20. Consider the category $D^\pi(A)$ whose objects are pairs $(X, p)$ where $X \in \text{Ob}(D^b(A))$ and $p \in \text{Hom}_{D^b(A)}(X, X)$ an idempotent endomorphism, and whose spaces of homomorphisms are $\text{Hom}_{D^\pi(A)}((X_0, p_0), (X_1, p_1)) := p_1 \text{Hom}_{D^b(A)}(X_0, X_1)p_0$. The category $D^\pi(A)$ is called the split-closed derived category of $A$.

It is known that $D^\pi(A)$ is again a triangulated category (see [BS]).

Remark 2.21. Since any projective module over the polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$ is a free module, we have $D^b(A_f) \cong D^\pi(A_f)$. Therefore, we shall study in this paper only the category $D^b(A_f)$.

It is not difficult to see that our category $D^b(A_f)$ is equivalent to the category of matrix factorizations, or equivalently, the category of maximal Cohen-Macaulay modules over $\mathbb{C}[x_1, \ldots, x_n]/(f)$ without free summands studied by Eisenbud [E], Knörrer [K], Buchweitz-Greuel-Schreyer [BGS], Orlov [O1] and other people (see the book by Yoshino [Y] for the details on maximal Cohen-Macaulay modules). Indeed, we can construct a functor from the category of matrix factorizations to $D^b(A_f)$ once we choose a basis of the free module over the polynomial ring. Note that any object isomorphic to a direct sum of the following objects

\[
\begin{pmatrix}
0 & f \\
1 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 1 \\
f & 0
\end{pmatrix}
\]

becomes the zero object of the category $D^b(A_f)$.

Next, we shall define the $\mathbb{Z}$-equivariant bounded derived category $D^b_{\mathbb{Z}}(A)$ of $A$. Let $\alpha := (a; Q_a)$ and $\beta := (b; Q_b)$ be graded twisted complexes. We put

$$Tw_{\mathbb{Z}}(A)(\alpha, \beta) := \bigoplus_{q \in \mathbb{Z}} Tw^q_{\mathbb{Z}}(A)(\alpha, \beta),$$

where

$$Tw^q_{\mathbb{Z}}(A)(\alpha, \beta) := \left\{
\begin{array}{ll}
\tilde{A}^q(a, b)_+, & q \in 2\mathbb{Z}, \\
\tilde{A}^q(a, b)_-, & q - 1 \in 2\mathbb{Z}.
\end{array}
\right. \quad (2.32)$$

Since $Tw^q_{\mathbb{Z}}(A)(\alpha, \beta) \subset Tw(A)(\alpha, \beta)$, we can define a $k$-multilinear maps $m^{Tw_{\mathbb{Z}}(A)}_n$ by restricting $m^{Tw(A)}_n$ to the subspaces.

Proposition 2.22. Let $A$ be a unital $\mathbb{Q}$-graded $A_\infty$-category of index $h$ with $m^A_n = 0$, $n \geq 3$. A collection $\text{Ob}(Tw_{\mathbb{Z}}(A))$, $Tw_{\mathbb{Z}}(\alpha, \beta)$ and $m^{Tw_{\mathbb{Z}}(A)}_n$ given above determines a $\mathbb{Z}$-graded $A_\infty$-category with $m_n \neq 0$ only if $n = 1, 2$, i.e., a differential graded (DG) category in the usual sense. We denote it by $Tw_{\mathbb{Z}}(A)$. 

\[\square\]
Definition 2.23. Let $\mathcal{A}$ be a unital $\mathbb{Q}$-graded $A_\infty$-category of index $h$ with $m_n^A = 0$, $n \geq 3$. We call the cohomology category of $\text{Tw}_\mathbb{Z}(\mathcal{A})$ the $\mathbb{Z}$-equivariant bounded derived category of $\mathcal{A}$ and denote by $D^b_\mathbb{Z}(\mathcal{A})$. More precisely, the set of objects is given by

$$\text{Ob}(D^b_\mathbb{Z}(\mathcal{A})) := \text{Ob}(\text{Tw}_\mathbb{Z}(\mathcal{A})), \quad (2.33)$$

and the set of homomorphisms is given by

$$\text{Hom}_{D^b_\mathbb{Z}(\mathcal{A})}(\alpha, \beta) := \ker(m_1^{\text{Tw}_\mathbb{Z}(\mathcal{A})} : \text{Tw}_\mathbb{Z}(\mathcal{A})^0(\alpha, \beta) \to \text{Tw}_\mathbb{Z}(\mathcal{A})^1(\alpha, \beta)) \quad (2.34)$$

$$\big/ \text{Im}(m_1^{\text{Tw}_\mathbb{Z}(\mathcal{A})} : \text{Tw}_\mathbb{Z}(\mathcal{A})^{-1}(\alpha, \beta) \to \text{Tw}_\mathbb{Z}(\mathcal{A})^0(\alpha, \beta)). \quad (2.35)$$

Let $T \in \text{Tw}_\mathbb{Z}(\mathcal{A})^0(\alpha, \beta)$ be a $m_1^{\text{Tw}_\mathbb{Z}(\mathcal{A})}$-closed homomorphism. As in the case for $D^b(\mathcal{A})$, we define a mapping cone $C(T)$ as an object

$$C(T) := (a[1] \oplus b; Q_C(T)) = \left( \begin{array}{c} Q_{a[1]} \\ T \\ Q_b \end{array} \right), \quad (2.36)$$

We define an exact triangle in the category $D^b_\mathbb{Z}(\mathcal{A})$ as a triangle of the form

$$\alpha \to \beta \to C(T) \to \alpha[1], \quad (2.37)$$

for some $T \in \ker(m_1^{\text{Tw}_\mathbb{Z}(\mathcal{A})} : \text{Tw}_\mathbb{Z}(\mathcal{A})^0(\alpha, \beta) \to \text{Tw}_\mathbb{Z}(\mathcal{A})^1(\alpha, \beta))$.

Theorem 2.24. The category $D^b_\mathbb{Z}(\mathcal{A})$ endowed with a shift functor $[1]$ and the class of exact triangles defined above becomes a triangulated category.

Proof. As in the case for $D^b(\mathcal{A})$, the proof is essentially the same as the known results in the usual situation. \qed

Remark 2.25. The twice of the shift functor $[2]$ is not isomorphic to the identity functor in $D^b_\mathbb{Z}(\mathcal{A})$.

Consider the differential graded (DG) functor Tot as in [BK]

$$\text{Tot} : \text{Tw}_\mathbb{Z}(\text{Tw}_\mathbb{Z}(\mathcal{A})) \to \text{Tw}_\mathbb{Z}(\mathcal{A}), \quad (\bigoplus_{i=1}^k (a_i; Q_{a_i}); T) \mapsto (\bigoplus_{i=1}^k a_i; Q + T), \quad (2.38)$$

where

$$Q := \begin{pmatrix} Q_{a_1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & Q_{a_k} \end{pmatrix}, \quad (2.39)$$

and $T \in \text{Tw}_\mathbb{Z}(\mathcal{A})(\bigoplus_{i=1}^k (a_i; Q_{a_i}) \oplus_{i=1}^k (a_i; Q_{a_i}))$ satisfies

$m_1^{\text{Tw}_\mathbb{Z}(\mathcal{A})}(T) + m_2^{\text{Tw}_\mathbb{Z}(\mathcal{A})}(T \otimes 2) = 0.$
Example 2.28. Let \( T \) of \( \text{Corollary 2.27} \) be complexes. It is well-defined since it is a DG category, i.e., the case when \( m \) and the coboundary operator \( \Phi \). The following statement is easily shown as in [BK] where they consider the case when \( A \).

Proposition 2.26. \( \text{Tot} \) is an equivalence of DG categories. \( \square \)

Indeed, by definition of the twisted complexes and the differential on them, we see that \( (Tw_Z(Tw_Z(A)), m_1^{Tw_Z(Tw_Z(A))}) \) and \( (Tw_Z(A)(\text{Tot}(A), \text{Tot}(B), m_1^{Tw_Z(A)}) \) are the same as complexes.

Corollary 2.27. \( D^b_Z(A) \) is an enhanced triangulated category in the sense of Bondal-Kapranov [BK]. \( \square \)

Let us consider our category \( D^b_Z(A_f) \) a little bit in detail.

Example 2.28. Let \( \alpha := (a = a_+ \oplus a_- [-1]; Q_a) \) and \( \beta := (b = b_+ \oplus b_- [-1] : Q_b) \) be objects of \( Tw_Z(A_f) \). Then the space of homomorphisms is of the following form:

\[
\Phi \in Tw_Z^q(\alpha, \beta), \quad q \in 2\mathbb{Z} \iff \Phi = \begin{pmatrix} \Phi_{++} & 0 \\ 0 & \Phi_{--} \end{pmatrix}, \quad \Phi_{\pm \pm} \in \widehat{A}_f^{\pm q}(a_\pm, b_\pm)_+,
\]

\[
\Phi \in Tw_Z^q(\alpha, \beta), \quad q - 1 \in 2\mathbb{Z} \iff \Phi = \begin{pmatrix} 0 & \Phi_{-+} \\ \Phi_{+-} & 0 \end{pmatrix}, \quad \Phi_{\pm \mp} \in \widehat{A}_f^{q+1}(a_\pm, b_\mp)_+,
\]

and the coboundary operator \( m_1^{Tw(A_f)} \) becomes the differential of the usual form

\[
Q_b \Phi - (-1)^q \Phi Q_a, \quad \Phi \in Tw_Z^q(\alpha, \beta).
\]

Note that if \( \Phi \in \widehat{A}_f^q(\alpha, \beta) \), then

\[
E \Phi - R_\beta \Phi + \Phi R_\alpha = q \Phi,
\]

where we put

\[
R_\alpha := \text{diag}(\frac{2k_1}{h}, \ldots, \frac{2k_m}{h}, \frac{2l_1}{h} - 1, \ldots, \frac{2l_m}{h} - 1), \quad a = \bigoplus_{i=1}^m a\{\frac{2k_i}{h}\} \oplus \bigoplus_{i=1}^m a\{\frac{2l_i}{h}\}[-1],
\]
and

\[ R_\beta := \text{diag}(\frac{2k_1'}{h}, \ldots, \frac{2k_m'}{h}, \frac{2l_1'}{h}, \ldots, \frac{2l_m'}{h} - 1), \quad b = \bigoplus_{i=1}^{m'} a\{\frac{2k_i'}{h}\} \oplus \bigoplus_{i=1}^{m'} a\{\frac{2l_i'}{h}\}[-1]. \]

By integrating the equation (2.40), we get for \( \lambda \in \mathbb{C} \),

\[ e^{-\lambda R_\beta} \Phi(e^{\lambda \frac{2\pi i}{h}} x_1, \ldots, e^{\lambda \frac{2\pi i}{h}} x_n)e^{\lambda R_\alpha} = e^{q\lambda} \Phi(x_1, \ldots, x_n). \tag{2.41} \]

This is the analogue of the homogeneity condition discussed in [HW].

Consider the \( \mathbb{Z} \)-action defined by \( x_i \mapsto \exp(2\pi \sqrt{-1} p \cdot a_i/h) \cdot x_i, p \in \mathbb{Z} \). It is clear that \( f \) is invariant under this \( \mathbb{Z} \)-action. Note also that \( \{\frac{2k}{h}\}, k \in \mathbb{Z} \) can be considered as the irreducible representations of \( \mathbb{Z} \). For a graded twisted complex \( \alpha := (a; Q_\alpha) \), put

\[ S_\alpha := \text{diag}(\frac{2k_1}{h}, \ldots, \frac{2k_m}{h}, \frac{2l_1}{h}, \ldots, \frac{2l_m}{h}), \quad a = \bigoplus_{i=1}^{m} a\{\frac{2k_i}{h}\} \oplus \bigoplus_{i=1}^{m} a\{\frac{2l_i}{h}\}[-1]. \]

Since \( Q_\alpha \in \tilde{A}^1(a, a)_- \), the similar equation as (2.41) shows that \( Q_\alpha \) is equivariant with respect to the \( \mathbb{Z} \)-action, i.e., we have

\[ e^{-\pi \sqrt{-1} S_\alpha} Q_\alpha(e^{\pi \sqrt{-1} \frac{2\pi i}{h}} x_1, \ldots, e^{\pi \sqrt{-1} \frac{2\pi i}{h}} x_n)e^{\pi \sqrt{-1} S_\alpha} = Q_\alpha(x_1, \ldots, x_n). \tag{2.42} \]

One can show that there is also the \( \mathbb{Z} \)-action on the space of homomorphisms by (2.41). For \( \Phi_{\pm} \in \tilde{A}^1_f(q(\alpha, \beta)_{\pm}, \Phi_{\pm} \}
\]

\[ e^{-\pi \sqrt{-1} S_\alpha} \Phi_{\pm}(e^{\pi \sqrt{-1} \frac{2\pi i}{h}} x_1, \ldots, e^{\pi \sqrt{-1} \frac{2\pi i}{h}} x_n)e^{\pi \sqrt{-1} S_\alpha} = \pm e^{\phi \sqrt{-1} q} \Phi_{\pm}(x_1, \ldots, x_n). \tag{2.43} \]

Therefore, if \( \Phi \) is even (odd), then \( \Phi \) is \( \mathbb{Z} \)-invariant if and only if \( q \in 2\mathbb{Z} \) \((q-1) \in 2\mathbb{Z}\). These facts lead us to our definition of \( \mathbb{Z} \)-equivariant derived category \( D^b_{\mathbb{Z}}(A_f) \) of \( A_f \).

Note that the above \( \mathbb{Z} \)-action on \( A_f \) factors through \( \mathbb{Z}/h\mathbb{Z} \). The category whose set of objects is the set of \( \mathbb{Z}/h\mathbb{Z} \)-equivariant matrix factorizations and the space of morphisms is \( \mathbb{Z}/h\mathbb{Z} \)-invariant homomorphisms between matrix factorizations are called in physics the category of \( D \)-branes in Landau-Ginzburg (\( \mathbb{Z}/h\mathbb{Z} \))-orbifolds (see for example [HW]). We can construct it by considering the \( \mathbb{Z}/h\mathbb{Z} \)-equivariant version of \( D^b(A_f) \). Indeed, we can show that it is equivalent to \( D^b_{\mathbb{Z}}(A_f)/[2] \). In order to recover the \( \mathbb{Z} \)-grading by the shift functor, we introduced here the translation \( \{2/h\} \) and defined a new category \( D^b_{\mathbb{Z}}(A_f) \).

3. \( D^b_{\mathbb{Z}}(A_f) \) and representations of Dynkin quivers

The following is our main theorem in this paper.

Theorem 3.1. Let us put \( f(x) := x^h \in \mathbb{C}[x] \) for \( h \geq 2 \) and consider the unital \( \mathbb{Q} \)-graded \( A_\infty \)-category \( A_f \) of index \( h \). Then we have the following equivalence of triangulated categories

\[ D^b_{\mathbb{Z}}(A_f) \simeq D^b(\text{mod}-B), \tag{3.1} \]
where $B$ is the path algebra of the following Dynkin quiver of type $A_{h-1}$:

$$
\bullet_1 \rightarrow \bullet_2 \rightarrow \cdots \rightarrow \bullet_{h-2} \rightarrow \bullet_{h-1},
$$

(3.2)

(the algebra of upper triangular matrices over $k$), and $D^b(\text{mod-} B)$ is the bounded derived category of finitely generated right $B$-modules.

**Remark 3.2.** The above equivalence for $h = 2$, $D^b_Z(A_{x^2}) \simeq D^b(\text{mod-} C)$, gives the simplest example of Knörrer’s periodicity.

**Proof.** It is not difficult to see that our category $D^b_Z(A_f)$ is a Krull-Schmidt category, the spaces of homomorphisms are finite dimensional and the endomorphism rings of indecomposable objects are local rings. See, for example, section 5 of [KR] for the proof of the general $f$ which defines an isolated singularity. Therefore, we first study the set of isomorphism classes of indecomposable objects. We use the fact that the Auslander-Reiten quiver of the category $D^b(A_f)$ of matrix factorizations for $f$ is given by

$$
[Q_1] \equiv [Q_2] \equiv \cdots \equiv [Q_{h-2}] \equiv [Q_{h-1}],
$$

where

$$
Q_l = \begin{pmatrix}
0 & x^{h-l} \\
x^l & 0
\end{pmatrix}, \quad l = 1, \ldots, h-1, i \in \mathbb{Z},
$$

and each morphism corresponding to the arrow from left to right is given by $\text{diag}(1, x)$ and the one from right to left is given by $\text{diag}(x, 1)$. See [AR] and also [O1]. Hence we have the following.

**Lemma 3.3.** The set of isomorphism classes of all indecomposable objects of $D^b_Z(A_f)$ is given by

$$
\{[M_{l,i}], \quad l = 1, \ldots, h-1, i \in \mathbb{Z}\},
$$

where

$$
M_{l,i} := \left( a\left\{\frac{2i}{h}\right\} \oplus a\left\{\frac{2(l+i)}{h}\right\}\right)[{-1}]Q_l.
$$

We also have

$$
\begin{pmatrix}
1 & 0 \\
0 & x
\end{pmatrix} \in \text{Hom}_{D^b_Z(A_f)}(M_{l,i}, M_{l+1,i}), \quad \begin{pmatrix}
x & 0 \\
0 & 1
\end{pmatrix} \in \text{Hom}_{D^b_Z(A_f)}(M_{l,i}, M_{l-1,i+1}),
$$

and hence

$$
\text{Hom}_{D^b_Z(A_f)}(M_{k,i}, M_{l,j}) \neq 0, \quad \text{only if} \quad k + 2i \leq l + 2j.
$$

In particular,

$$
\text{Hom}_{D^b_Z(A_f)}(M_{k,0}, M_{l,0}) = \begin{cases} 
\mathbb{C}, & \text{if } k \leq l, \\
0, & \text{if } k > l.
\end{cases}
$$

(3.7)
Proof. One can easily show by direct computations. \hfill \square

Remark 3.4. Note that \( M_{l,i}[1] \simeq M_{h-l,i+l} \).

Serre duality holds in our category \( D^b_2(A_f) \).

Lemma 3.5. There are isomorphisms as \( \mathbb{C} \)-vector spaces

\[
\text{Hom}_{D^b_2(A_f)}(M_{k,i}, M_{l,j}) \simeq \text{Hom}_{D^b_2(A_f)}(M_{l,j}, M_{k,i-1}[1])^*, \quad \text{for all} \quad 1 \leq k, l \leq h-1, \quad i, j \in \mathbb{Z}.
\] (3.9)

Proof. There is a trace map \([KL2]\)

\[
\text{Tr}_k : \widetilde{A}_f^{1-\frac{2}{h}}(M_{k,i}, M_{k,i})_\rightarrow \mathbb{C}, \quad \Phi \mapsto \frac{1}{h-1} \text{Res} \left[ \frac{\text{Str}(dQ_k \cdot \Phi)}{dx} \right],
\] (3.10)

where

\[
\text{Str}(dQ_k \cdot \Phi) := [(h-k)x^{h-k-1}\Phi_+ - kx^{k-1}\Phi_-] dx, \quad \Phi = \begin{pmatrix} 0 & \Phi_+ \\ \Phi_- & 0 \end{pmatrix}.
\]

\( \text{Tr}_k(\Phi) = 0 \) if \( \Phi = Q_k \Psi + \Psi Q_k \) for some \( \Psi \), since \( dQ_k \cdot Q_k + Q_k \cdot dQ_k = df \cdot 1_{2 \times 2} \).

Note that

\[
\Phi_k := \begin{pmatrix} 0 & -x^{h-k-1} \\ x^{k-1} & 0 \end{pmatrix} \in \widetilde{A}_f^{1-\frac{2}{h}}(M_{k,i}, M_{k,i})_\rightarrow, \quad \text{Tr}_k(\Phi_k) = 1, \quad Q_k \Phi_k + \Phi_k Q_k = 0.
\]

Therefore, under the isomorphism given by

\[
\widetilde{A}_f^{1-\frac{2}{h}}(M_{k,i}, M_{k,i})_\rightarrow \simeq \widetilde{A}_f^0(M_{k,i}, M_{k,i-1}[1])_+, \quad \begin{pmatrix} 0 & \Phi_+ \\ \Phi_- & 0 \end{pmatrix} \mapsto \begin{pmatrix} \Phi_+ & 0 \\ 0 & -\Phi_- \end{pmatrix},
\]

\( \Phi_k \) determines an element of \( \text{Hom}_{D^b_2(A_f)}(M_{l,j}, M_{k,i-1}[1]) \). Moreover, from the knowledge of the Auslander-Reiten quiver \([3.3]\) of \( D^b(A_f) \), we see that \( \text{Hom}_{D^b_2(A_f)}(M_{l,j}, M_{k,i-1}[1]) \simeq \mathbb{C} \Phi_k \).

It is not difficult to see that the following pairings

\[
\widetilde{A}_f^{2m}(M_{k,i}, M_{l,j})_+ \otimes \widetilde{A}_f^{1-\frac{2}{h}-\frac{2m}{h}}(M_{l,j}, M_{k,i})_\rightarrow \rightarrow \widetilde{A}_f^{1-\frac{2}{h}-\frac{2m}{h}}(M_{k,i}, M_{k,i})_\rightarrow \text{Tr}_k \mathbb{C}, \quad m \in \mathbb{Z},
\]

induce the perfect pairings

\[
\text{Hom}_{D^b_2(A_f)}(M_{k,i}, M_{l,j}) \otimes \text{Hom}_{D^b_2(A_f)}(M_{l,j}, M_{k,i-1}[1]) \rightarrow \text{Hom}_{D^b_2(A_f)}(M_{k,i}, M_{k,i-1}[1]) \simeq \mathbb{C}.
\]

Remark 3.6. \( S := \left\{ \frac{2}{h} \right\} \circ [1] \) is the Serre functor on \( D^b_2(A_f) \). In particular, we have \( S^h = [h-2] \). Therefore \( D^b_2(A_f) \) is a fractional noncommutative Calabi-Yau manifold of dimension \( 1 - 2/h \) in the sense of \([56]\).
Combining the above Serre duality and the data of the Auslander-Reiten quiver \((3.3)\) of \(D^b(A_f)\), we see that there are no higher extensions among \(\{M_{l,0}\}\).

**Corollary 3.7.** For \(m \neq 0\), we have
\[
\text{Hom}_{D^b_Z(A_f)}(M_{k,0}, M_{l,0}[m]) = 0, \quad \text{for all } i, j = 1, \ldots, h - 1.
\]

**Corollary 3.8.** \(D^b(\text{mod}-B)\) is a full triangulated subcategory of \(D^b_Z(A_f)\).

**Proof.** Use the fact that \((M_{1,0}, \ldots, M_{h-1,0})\) is a strongly exceptional collection and
\[
B \cong \bigoplus_{i,j=1}^{h-1} \text{Hom}_{D^b_Z(A_f)}(M_{k,0}, M_{l,0}).
\]
Since \(D^b_Z(A_f)\) is an enhanced triangulated category, we can apply the theorem by Bondal-Kapranov (\[BK\] Theorem 1).

Note that the number of indecomposable objects of \(D^b_Z(A_f)/[2]\) is
\[
\# \{[M_{l,i}] \mid l = 1, \ldots, h-1, \quad i \in \mathbb{Z}/h\mathbb{Z}\} = (h-1) \cdot h,
\]
which is the number of roots for the root system \(A_{h-1}\). This number coincides with the number of indecomposable objects of \(D^b(\text{mod}-B)/[2]\) by Gabriel’s theorem \[G\]. Therefore, \(D^b(\text{mod}-B)/[2] \cong D^b_Z(A_f)/[2]\). This proves Theorem 3.7. \(\square\)

**Remark 3.9.** The similar proof can be applied for \(D_n\) and \(E_6, E_7, E_8\) cases since the heart of our proof is to use the Auslander-Reiten quivers of \(D^b(A_f)\), the fact that any matrix factorization over ADE singularities is gradable, the Serre duality and the theorem by Gabriel on the number of indecomposables. They are well-known or can be shown by direct calculations with explicit presentations of matrix factorizations. We shall discuss this in detail in the next paper \[KST\].

4. Stability condition on \(D^b_Z(A_f)\)

In this section, we will briefly discuss on a stability condition on \(D^b_Z(A_f)\).

**Definition 4.1.** Let \(\alpha := (\oplus_{i=1}^n a\{\frac{2k_i}{h}\} + \oplus_{i=1}^n a\{\frac{2l_i}{h}\}[-1]; Q_a)\) be an object of \(D^b_Z(A_f)\) such that \(Q_a\) is reduced, i.e., each matrix element of \(Q_a\) is in the maximal ideal generated by \((x_1, \ldots, x_n)\). Then we call the real number
\[
\phi_a := \frac{1}{2n} \text{Tr} S_a - \frac{1}{2}, \quad S_a := \text{diag}(\frac{2k_1}{h}, \ldots, \frac{2k_n}{h}, \frac{2l_1}{h}, \ldots, \frac{2l_n}{h})
\]
phase of the object \(\alpha\).
Example 4.2. Let \( f := x^{n+1} \) and consider the objects
\[
M_{l,i} := \left( \begin{array}{c} a\{\frac{2i}{h}\} \oplus a\{\frac{2(l+i)}{h}\}[-1]; \begin{pmatrix} 0 & x^{h-1} \\ x^l & 0 \end{pmatrix} \right). \quad (4.2)
\]
Then
\[
\phi_{M_{l,i}} = \frac{l + 2i}{h} - \frac{1}{2}.
\]

Definition 4.3. Let \( \omega := \exp 2\pi \sqrt{-1}/h \). For \( \alpha = (\oplus_{i=1}^{n} a\{\frac{2k_i}{h}\} \oplus \oplus_{i=1}^{n} a\{\frac{2l_i}{h}\}[-1]; Q_a) \), we define a \( \mathbb{C} \)-linear map \( Z_{\omega} : K_0(D^b_Z(A_f)) \to \mathbb{C} \) as follows:
\[
Z_{\omega}(\alpha) := \sum_{i=1}^{n} (\omega^{k_i} - \omega^{l_i}). \quad (4.3)
\]
By the above example, we see that \( \phi_{M_{l,i}} \) is the phase of \( Z_{\omega}([M_{l,i}]) \):

Proposition 4.4.
\[
Z_{\omega}([M_{l,i}]) = 2 \sin\left(\frac{l}{h} \pi \right) \cdot e^{\pi \sqrt{-1} \phi_{M_{l,i}}}. \quad (4.4)
\]
Since we know that all indecomposable objects in \( D^b_Z(A_f) \) for \( f = x^{n+1} \) have definite phases, we can define a stability condition on \( D^b_Z(A_f) \).

Theorem 4.5. Let \( f := x^{n+1} \) and \( P(\phi) \) be the full additive subcategory of \( D^b_Z(A_f) \) whose objects have phase \( \phi \in \mathbb{R} \). Then \( P(\phi) \) and \( Z_\omega \) define a stability condition on \( D^b_Z(A_f) \) in the sense of Bridgeland [B].

More precisely, \( P(\phi) \) and \( Z_\omega \) satisfy the following properties:
(i) if \( M \in P(\phi) \), then \( Z_\omega(M) = m(M) \exp(\sqrt{-1} \pi \phi) \) for some \( m(M) \in \mathbb{R}_{\geq 0} \),
(ii) for all \( \phi \in \mathbb{R} \), \( P(\phi + 1) = P(\phi)[1] \),
(iii) if \( \phi_1 > \phi_2 \) and \( M_i \in P(\phi_i) \), then \( \text{Hom}_{D^b_Z(A_f)}(M_1, M_2) = 0 \),
(iv) for each nonzero object \( M \in D^b_Z(A_f) \), there is a finite sequence of real numbers
\[
\phi_1 > \phi_2 > \cdots > \phi_n
\]
and a collection of exact triangles
\[
M_{i-1} \to M_i \to N_i \to M_{i-1}[1], \quad M_n := M, \quad M_0 := 0
\]
with \( N_j \in P(\phi_j) \) for all \( j \). □
The space of stability conditions for $D^b_Z(A_f)$ should be isomorphic to the base space of the universal unfolding of $f$ by the mirror symmetry. Therefore we expect that there exists a natural Frobenius (K. Saito’s flat) structure on the space of stability conditions and the stability condition constructed above should correspond to the origin of the base space of the universal unfolding. We shall study this in detail elsewhere.

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Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, Japan
E-mail address: atsushi@kurims.kyoto-u.ac.jp