Interacting Holographic Dark Energy Model as a Dynamical system and the Coincidence Problem

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Abstract

We examine the evolution of a holographic cosmological model with future event horizon as the infrared cut-off and dark matter and dark energy do not evolve independently — there is interaction between them. The basic evolution equations are reduced to an autonomous system and corresponding phase space is analyzed.

Keywords : Dynamical System, Phase plane, Holographic Dark energy.

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1 Introduction

Recent observational evidences particularly from Type Ia supernovae and Cosmic Microwave Background(CMB) speculate the existence of both gravitating and non-gravitating type of matter. There is a substantial amount of gravitating matter non-baryonic in nature and is termed as Dark Matter(DM) [1, 2, 3, 4, 5]. On the other hand, the non-gravitating matter, known as Dark Energy(DE) is the mysterious agent for the present phase of cosmic accelerated expansion. It is only known for certain that DE has huge negative pressure(comparable to its energy density) and there is sufficient reason to assume an even distribution of it over the space[for details see ref [6]]. Although DM energy density is expected to decrease at a faster rate than the density of DE throughout the evolution, interestingly they have comparable magnitude today.. This surprising matching is known as the 'coincident problem'. To resolve this problem, use of tracker fields [7] and oscillating DE models [8] are normally employed. But recently, there arises a third possibility [9, 10, 11, 12] by introducing DE and DM interact through an additional coupling term in the fluid equation. In the present work, we choose the third possibility as a solution of the problem.

To have some inside about the unknown and mysterious nature of DE, many people have suggested that DE should be compatible with Holographic principle, namely "the number of relevant degrees of freedom of a system dominated by gravity must vary along with the area of the surface bounding the system" [13]. Such a DE model is known as Holographic DE(HDE) model. Further the energy density of any given region should be bound by that ascribed to a Schwarzschild black hole(BH) that fills the same volume [14, 15]. Mathematically, we write

$$\rho_D \leq M_p^2 L^{-2},$$

where $\rho_D$ is the DE density, $L$ is the size of the region(or infrared cut off) and $M_p = (8\pi G)^{-\frac{1}{2}}$ is the reduced Planck mass. Usually, the DE density is written as

$$\rho_D = \frac{3M_p^2 c^2}{L^2} \quad (1)$$

Here the dimensionless parameter '$c^2$' takes care of the uncertainties of the theory and for mathematical convenience the factor 3 has been introduced. In HDE paradigm [14, 15, 16, 17, 18, 19] one determines an appropriate quantity to serve as an IR cut off for the theory and imposes the constraint that the total vacuum energy in the corresponding maximum value must not be greater than the mass of a BH of the same size. By saturating the

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inequality one identifies the acquired vacuum energy as HDE. Although the choice of the IR cut off has raised on discussion in the literature [15, 19, 20, 21, 22, 23], it has been shown, and it is generally accepted, that the radius of the event horizon of the universe \( R_E \) the most suitable choice for the IR cut off where \( R_E \) is defined as [18]:

\[
L = R_E = a \int_t^\infty \frac{dt}{a}.
\] (2)

Now for the interacting DM and DE to resolve the coincidence problem (as mentioned above), the interaction term is chosen in the present work in the following two ways: (a) usually, the interaction term is chosen as \( A \rho_m + B \rho_D \) where \( \rho_m \) and \( \rho_D \) are DM and DE densities and \( A, B \) are dimensionless constants. For convenience we shall choose \( A = B = 3b^2 \), i.e., interaction of the form \( 3b^2 H \rho \) where \( \rho = \rho_m + \rho_D \) is the total energy density. (b) a natural and physical variable interaction term is of the form \( \gamma \rho_m \rho_D \) with \( \gamma \) a dimension full \( (L^3 m t) \) constant. Note that this interaction term vanishes (as expected) if any one of the energy densities is zero while the interaction term grows with the increase of both the energy densities. Further, such an interaction term gives the best fit to observations [11] for HDE models. In the present work we choose DM in the form of pure dust while the HDE, as perfect fluid with equation of state \( p_D = \omega_D \rho_D \).

2 Basic Equations

We consider our universe to be homogeneous and isotropic flat FRW model and assume that it fills with DM in the form of dust (having energy density \( \rho_m \)) and HDE in the form of a perfect fluid having equation of state \( p_D = \omega_D \rho_D \) where \( \omega_D \) is variable.

The Einstein field equations for spatially flat model are

\[
3H^2 = \rho_m + \rho_D
\] (3)

and

\[
2 \dot{H} = -\rho_m - (1 + \omega_D) \rho_D
\] (4)

where for simplicity we choose \( 8\pi G = 1 = c \).

The conservation equations for the fluids are

\[
\dot{\rho}_m + 3H \rho_m = Q
\] (5)

and

\[
\dot{\rho}_D + 3H (1 + \omega_D) \rho_D = -Q
\] (6)

It is to be noted that for \( Q > 0 \), energy is transferred from DE to DM and opposite is the situation for \( Q < 0 \). As \( Q < 0 \) would worsen the coincidence problem so we choose \( Q > 0 \) throughout the work. Further, for validity of second law of thermodynamics and Lechatelier’s principle [12] one must take \( Q > 0 \). Also it should be mentioned that baryonic matter is not included in the interaction due to the constraints imposed by local gravity measurements [24, 25].

Using the field equations (3) and (4), the acceleration of the universe is given by

\[
\ddot{a} = a \left( \dot{H} + H^2 \right) = -\frac{a}{6} \left\{ \rho_m + (1 + \omega_D) \rho_D \right\}
\] (7)

which shows that for the present accelerating phase it is necessary (but not sufficient) to have \( \omega_D < -\frac{1}{3} \).

Using the density parameters

\[
\Omega_m = \frac{\rho_m}{3H^2}, \quad \Omega_D = \frac{\rho_D}{3H^2}
\] (8)

the Einstein equation (3) can be written as

\[
\Omega_m + \Omega_D = 1
\] (9)

Introducing \( u = \frac{\rho_m}{\rho_D} \) as the ratio of the energy densities we have

\[
\Omega_m = \frac{u}{1+u}, \quad \Omega_D = \frac{1}{1+u}
\] (10)
Detailed Calculations for $Q = 3b^2 H \rho$

Using the conservation equations (5) and (6) and the energy density of HDE from equation (1), we have the expression for the equation of state parameter as

$$\omega_D = -\frac{1}{3} - \frac{2\sqrt{\Omega_D}}{3c} - \frac{b^2}{\Omega_D} < -\frac{1}{3} \tag{11}$$

which shows that there will be always acceleration.

The evolution of the density parameter $\Omega_D$ is given by

$$\dot{\Omega}_D = H \Omega_D^2 (1 - \Omega_D) \left[ \frac{1}{\Omega_D} + \frac{2}{c\sqrt{\Omega_D}} - \frac{3b^2}{\Omega_D (1 - \Omega_D)} \right] \tag{12}$$

and hence the ratio of the energy densities evolves as

$$\dot{u} = H \left[ -u \left( 1 + \frac{2}{c\sqrt{1 + u}} \right) + 3b^2 (1 + u) \right] \tag{13}$$

The Friedmann equation (4) and the conservation equation (6) can be converted (after a bit simplification) into an autonomous system as

$$\dot{\rho}_D = 2\rho_D \left[ \frac{\sqrt{\rho_D}}{\sqrt{3c}} - H \right] \tag{14}$$

and

$$\dot{H} = \frac{1}{2} \left[ 3H^2 (1 - b^2) - \frac{\rho_D}{3} - \frac{2}{3\sqrt{3c}} \frac{\rho_D^{\frac{2}{3}}}{H} \right] \tag{15}$$

The dynamical system has a line of critical points along the parabola $\rho_D = 3c^2 H^2$ in the phase plane $(\rho_D, H)$ with the restriction $b^2 = 1 - c^2$. Then the linearized matrix $A$ has $\text{trace}(A) = H (1 - 4c^2)$ and $\text{determinant}(A) = 0$. So the phase paths form a family of parabolas [26]. The phase portrait for different choices of the parameter ‘$b$’ are shown in figures 1(a) and 1(b). Note that along the line of critical points $\Omega_D = c^2$ and $\omega_D = -\frac{1}{3} < -1$, $u = \frac{b^2}{2}$. Hence along the phase paths the ratio of the energy densities bears a constant value and the universe will be in the phantom era.

Further, fixed points corresponding to $\dot{u} = 0$ is essentially a cubic equation which has at least one real root say, $u_f$. Then the parameter ‘$b^2$’ can be estimated by the fixed point as

$$b^2 = \left( \frac{u_f}{1 + u_f} \right) \left( \frac{1}{3} + \frac{2}{3c\sqrt{1 + u_f}} \right) \tag{16}$$

Now, to analyse the stability of the fixed point we write

$$u' = \frac{du}{dx} = \frac{du}{dt} \frac{dt}{dx} = \frac{\dot{u}}{H} = -u \left( 1 + \frac{2}{c\sqrt{1 + u}} \right) + 3b^2 (1 + u) \tag{17}$$

where $x = \ln a$.

Then at the fixed point

$$\frac{du'}{du} = 3b^2 - 1 - \frac{(u + 2)}{3(1 + u)^2} \bigg|_{u=u_f} \tag{18}$$

$$= -\frac{1}{1 + u_f} - \frac{2 - u_f}{c (1 + u_f)^2} < 0.$$ 

Hence the fixed point $u_f$ is a stable one.

Moreover the conservation equations (5) and (6) can be written as

$$\dot{\rho}_m = \sqrt{3(\rho_m + \rho_D)} \left[ b^2 \rho_D - (1 - b^2) \rho_m \right] \tag{19}$$
Fig. 1(a)-1(b) represent the variation of $\rho_D - H$. Though in Fig 1(b) the whole region is not a physically valid region but for better understanding about the system we have drawn the whole region.

\[ \dot{\rho}_D = -\sqrt{3} (\rho_m + \rho_D) \left[ b^2 \rho_m + (1 + \omega_D + b^2) \rho_D \right] \]  

From equation (20) we see that $\dot{\rho}_D \leq 0$, i.e., DE density decreases at least in the quintessence era. Also from the equation (19), if we assume $\rho_D$ to be sufficiently large initially then matter density increases in the early phase and subsequently it decreases with $\dot{\rho}_m = 0$ along the straight line $\rho_m = \frac{b^2}{1 - \omega_D} \rho_D$ in the $(\rho_m, \rho_D)$-plane. Then in the phantom era ($\omega_D < -1$), $\rho_D$ may begin to increase and dominate over DM.

Thus the present model of the universe shows a DE domination initially and subsequently the universe evolve with DM domination. Then there may be DE dominated phase at late time as predicted by observation. Hence this scenario is favourable for the present universe.

Further, from the point of view of the coincidence problem we see that $u (\frac{\rho_m}{\rho_D})$ is less than unity in the early phase of the universe and then it gradually increases. $u \sim o(1)$ before $\rho_m = 0$ or after $\dot{\rho}_m = 0$ or along the straight line $\rho_m = \frac{b^2}{1 - \omega_D} \rho_D$ in the $(\rho_m, \rho_D)$-plane provided $b^2 > \omega_D < \omega_D = \frac{1}{2}$. Though the coincidence problem has partial solution around the straight line $\rho_m = \frac{b^2}{1 - \omega_D} \rho_D$, but it does not give any explanation for $u \sim o(1)$ in the present scenario.

4 Calculation details for $Q = \gamma \rho_m \rho_D$

Proceeding exactly as in the previous section the expression for equation of state parameter and the evolution of the density parameter are given by

\[ \omega_D = -\frac{1}{3} - \frac{2\sqrt{\Omega_D}}{3c} - \gamma H (1 - \Omega_D) < -\frac{1}{3} \]  

and

\[ \dot{\Omega}_D = H \Omega_D (1 - \Omega_D) \left[ 1 - 3\gamma H \Omega_D + \frac{2\sqrt{\Omega_D}}{c} \right] \]
Phase Portrait for $\gamma=0.232, c=0.9$

Phase Portrait for $\gamma=0.5, c=0.9$

Fig. 2(a) Fig. 2(b)

Fig. 2(a)-2(b) represent the variation of $\Omega_D-H$. Here also the negative coordinate of $H$ is not physically valid. But for completeness of the system we have drawn the whole figure.

So the evolution of the ratio of the energy densities is described as

$$\dot{u} = 3Hu \left[-\frac{1}{3} - \frac{2}{3c\sqrt{1+u}} + \frac{\gamma H}{1+u}\right]$$ (23)

Also the Friedmann equation (4) can be written as

$$\dot{H} = -\frac{3H^2}{2} \left[-\frac{\Omega_D}{3} - \frac{2\Omega^2_D}{3c} + 1 - \gamma H\Omega_D (1 - \Omega_D)\right]$$ (24)

Hence equations (22) and (24) constitute an autonomous system in the phase plane ($\Omega_D, H$). The possible critical points are

(i) $H = 0, \Omega_D$ is unrestricted,
(ii) $\Omega_D = 0, H = 0$,
(iii) $\Omega_D = 1, H = 0$,
(iv) $\Omega_D = 1, H$ is unspecified and
(v) $\Omega_D = c^2, H = \frac{1}{c^2}$.

The first three critical points correspond to static model of the universe. In the first one the universe may have any amount of DE while in the second one the universe does not have any DE. The third and fourth critical points correspond to universe in phantom era filled only with the DE. For the fifth critical point, the ratio of the matter densities $u = \frac{1-c^2}{c^2}$ and $\omega_D = -\frac{1}{c^2} < -1$. So the universe is again in the phantom region. A detailed investigation of this critical point will be done subsequently.

The conservation equations (5) and (6) can be written as

$$\dot{\rho}_m = \rho_m \left[\gamma \rho_D - \sqrt{3(\rho_m + \rho_D)}\rho_m\right]$$ (25)
\[ \rho_D = -\rho_D \left[ \gamma \rho_m + (1 + \omega_D) \sqrt{3(\rho_m + \rho_D)} \right] \]  

(26)

Apparently, we have similar situation as before, i.e., initially if we assume to have sufficient DE then \( \dot{\rho}_D < 0 \) and \( \dot{\rho}_m > 0 \) and subsequently \( \dot{\rho}_m < 0 \). Note that \( \dot{\rho}_m = 0 \) along the curve \( \gamma^2 \rho_D^2 - 3\rho_m^3 = 3\rho_D \rho_m^2 \). But due to \( \omega_D \) term in equation (26) DE density begins to increase in phantom era and we should have DE dominated universe as expected in the present scenario. So both the energy densities are of comparable magnitudes (i.e., \( u \sim o(1) \)) twice during the evolution and give a possible explanation to the coincidence problem.

To study the nature of the critical point \( \left( c^2, \frac{1}{\gamma c^2} \right) \) on the phase plane \( (\Omega_D, H) \) we start with the linearized system:

\[
\begin{align*}
\dot{x} &= \left( \frac{(1-c^2)}{c^2\gamma} \right) (2x + 3\gamma c^4 y) \\
\dot{y} &= \left( \frac{3(1-c^2)}{x^2\gamma^2} + \frac{3}{2c^2\gamma} \right) y
\end{align*}
\]

(27)

where \( x = \Omega_D - c^2 \), \( y = H - \frac{1}{\gamma c^2} \).

So the linearized matrix \( A \) has

\[ tr(A) = \frac{7}{2} \frac{(1-c^2)}{c^2\gamma} > 0 \]

\[ det(A) = \frac{3}{2} \frac{(1-c^2)}{c^4\gamma^2} < 0 \]

Thus for the linearized matrix \( A \) both the eigenvalues are real but of opposite sign and hence the critical point is of saddle type and unstable in nature. The phase portrait for different choices of \( \gamma \) and \( c^2 \) are presented in figures 2(a) and (b).

5 Discussion

In the present work, we study the cosmological evolution of interactive DM and DE in the background of a homogeneous and isotropic FRW model of the universe. The DM is chosen in the form of dust while for DE we choose holographic DE in the form of perfect fluid having variable equation of state. The interaction between DM and DE is chosen either as a linear combination of the energy densities or in their product form. For both choices of the interaction term, the evolution equations can be suitably converted into an autonomous system and the critical points are analyzed both analytically and graphically. Finally, the present model of the universe shows partial solution of the coincidence problem.

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