The Naïve versus the Adaptive Boston Mechanism*

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Abstract

The Boston mechanism is often criticized for its manipulability and the resulting negative implications for welfare and fairness. Nonetheless, it is one of the most popular school choice mechanisms used in practice. In this paper, we first study the traditional (naïve) Boston mechanism (NBM) in a setting with no priority structure and single uniform tie-breaking. We show that it imperfectly rank dominates the strategyproof Random Serial Dictatorship (RSD). We then formalize an adaptive variant of the Boston mechanism (ABM), which is also sometimes used in practice. We show that ABM has significantly better incentive properties than NBM (it is partially strategyproof), while it shares most of NBM’s efficiency advantages over RSD as markets get large. However, while a direct efficiency comparison of NBM and ABM via imperfect dominance is inconclusive, numerical simulations suggest that NBM is still somewhat more efficient than ABM, which can be interpreted as the cost of partial strategyproofness one pays when choosing ABM over NBM. Our results highlight the subtle trade-off between efficiency and strategyproofness a market designer must make when choosing between the two Boston mechanisms and RSD.

1. Introduction

Every year, millions of children enter a new public school. Frequently, they have to choose between a number of different schools, whose capacity is limited, such that not all wishes can be accommodated perfectly. A school choice mechanism is a procedure that collects the students’ (or the parents’) preferences over schools and determines a matching of students to schools. One such mechanism is the Boston mechanism, which is frequently used in practice, but has also been heavily criticized for its manipulability. In this paper, we consider a setting with no priority structure, in which we study two variants of the Boston mechanism in terms of incentives and efficiency.

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1.1. Strategyproofness versus Efficiency

By assuming no priority structure, the school choice problem becomes equivalent to the \textit{one-sided matching problem}, where indivisible objects must be allocated to self interested agents and monetary transfers are not permitted. As mechanism designers, we are interested in mechanisms that perform well with respect to incentives, efficiency, and fairness. Unfortunately, previous research on this problem has revealed that it is impossible to achieve the optimum on all dimensions simultaneously (Zhou, 1990).

Strategyproof mechanisms, such as Random Serial Dictatorship (RSD), are appealing because they make truthful reporting a dominant strategy for all agents. Participation in the mechanism becomes an easy task as there is no need for deliberation about the best response, thus reducing cognitive costs for the agents and (likely) endowing the mechanism with correct information about agents’ preferences. While strategyproofness is certainly a desirable property, it also imposes severe restrictions. In particular, strategyproofness is incompatible with ordinal efficiency and symmetry (Bogomolnaia and Moulin, 2001), and it is also incompatible with rank efficiency (Featherstone, 2011).

The Boston mechanism is an example of a manipulable mechanism that is frequently used in school choice settings (Abdulkadiroğlu and Sönmez, 2003; Kojima and Ünver, 2014). Under the Boston mechanism, agents submit their preferences in the form of rank ordered lists. The mechanism then lets agents “apply” to their reported first choice and the objects are distributed amongst applicants according to some (possibly random) priority ordering. If an agent does not obtain its first choice, it “applies” to its second choice in the second round, etc., until all agents have received an object or all objects are exhausted. Despite the manipulability of the Boston mechanism, advocates of this mechanism argue that it may lead to ex-ante welfare gains because agents may coordinate on equilibria that they find more desirable under their particular preference intensities (Abdulkadiroğlu, Che and Yasuda, 2010). On the other hand, the existence of manipulation opportunities may result in disadvantages for unsophisticated participants (Miralles, 2008) and inferior overall welfare (Ergin and Sönmez, 2006). Calsamiglia and Güell (2013) presented evidence from Barcelona, indicating that under a particular variant of the Boston mechanism, parents are so afraid of being matched to an undesirable school that the matching essentially degenerates to the matching determined purely by the neighborhood priority structure. In recent years, some school districts have adopted alternative procedures based on the Deferred Acceptance mechanism (which is strategyproof for students), e.g., Boston and Chicago (Pathak and Sönmez, 2013).

This paper remains agnostic with respect to this debate, i.e., we do not argue in favor or against the use of non-strategyproof mechanisms in school choice or other matching domains. Instead, we closely examine two variants of the Boston mechanism and compare them to each other and to the strategyproof RSD mechanism in terms of incentives and efficiency. Our analysis provides new insights into the subtle trade-offs between strategyproofness and efficiency that a choice between the two variants of the Boston mechanism and RSD entails.
1.2. An Adaptive Variant of the Boston Mechanism

Under the Boston mechanism as outlined above, an agent may apply to an object that has already been exhausted in a previous round. In that case the agent effectively “wastes” one round in which it cannot compete with other agents for other objects that are still available. This “naïve” variant of the Boston mechanism (NBM) is susceptible to a particular type of manipulation: suppose an agent knows that its 2nd choice will be exhausted by other agents in the first round; unless it ranks its second choice first the agent has no chance of obtaining that 2nd choice. Thus, by ranking its 2nd choice last instead of second, the agent can improve its chances of obtaining a better object either in round 2 or in any later round without foregoing any chances of the getting its 2nd choice. Obviously, NBM is highly manipulable.

A simple remedy for this particular problem is to let agents skip exhausted objects and instead let them apply to their best available choice in each round. This small change creates a slightly different Boston mechanism, which has been largely overlooked in the literature so far. We call this mechanism the adaptive Boston mechanisms (ABM). Like NBM, ABM is also used in practice, e.g., for the allocation of students to middle schools in Freiburg, Germany: parents apply to their preferred school; if the child is rejected, the parents receive a list of schools that still have capacity available; when the parents then apply to a school from this list, this procedure effectively implements ABM. In this paper, we formalize and study this mechanism in detail.

1.3. Understanding Trade-offs

The traditional approach to understanding the trade-off between strategyproofness and efficiency offered by various mechanisms is to verify (or falsify) that one mechanism satisfies a certain property while another mechanism does not. Such “binary” properties include strategyproofness, different forms of efficiency, and the existence of equilibria with desirable outcomes. However, as this paper will illustrate, this approach does not always work, i.e., a comparison of mechanisms via the traditional methods may yield inconclusive results.

For example, traditional methods do not reveal the efficiency advantages of NBM and ABM over RSD. All three mechanisms are ex-post efficient, but neither ordinally nor rank efficient. Regarding incentives, a comparison by vulnerability to manipulation (Pathak and Sönmez, 2013) does not differentiate between NBM and ABM in our setting. We therefore apply a set of new methods: first, regarding incentives, we successfully employ partial strategyproofness, a new, scalable concept to quantify the degree of strategyproofness of non-strategyproof mechanisms (Mennle and Seuken, 2014). Second, regarding efficiency, we consider existing notions of dominance for allocations and extend these notions towards the comparison of mechanisms. In combination with computational experiments, our methods shed new light on the trade-off between strategyproofness and efficiency when choosing between the two Boston mechanisms and RSD.
1.4. Overview of Contributions

In this paper, we study RSD, NBM, and ABM in a setting with no priority structure. We analyze their incentive and efficiency properties, and we obtain the following results:

1) **Price of Strategyproofness**: we show that NBM imperfectly rank dominates RSD, i.e., if the outcomes are comparable by rank dominance, then the outcome of NBM is at least weakly preferable to that of RSD. This efficiency advantage can be interpreted as the *cost of strategyproofness* when choosing RSD over NBM.

2) **Partial Strategyproofness of ABM**: we formalize the adaptive Boston mechanism and show that it is partially strategyproof, i.e., strategyproof on a domain of uniformly relatively bounded indifference. In contrast, NBM is not even weakly strategyproof. This clear distinction between NBM and ABM in terms of incentives is novel, since the comparison by vulnerability to manipulation (Pathak and Sönmez, 2013) does not differentiate between the two mechanisms in our setting.

3) **Imperfect Rank Dominance of ABM over RSD in Large Markets**: we show that ABM essentially imperfectly rank dominates RSD, i.e., while RSD may rank dominate ABM at certain type profiles, we find that this is almost never the case. We show that the share of type profiles at which RSD rank dominates ABM becomes arbitrarily small in large markets. Numerical evidence suggests that the efficiency advantages of ABM over RSD are similar in magnitude to those of NBM over RSD.

4) **Price of Partial Strategyproofness**: we show that a formal imperfect rank dominance comparison of NBM and ABM is inconclusive, i.e., NBM is not clearly preferable to ABM. However, numerical evidence suggests that, conditional on comparability, NBM is “usually” more efficient than ABM, which can be interpreted as the *cost of partial strategyproofness* when choosing ABM over NBM.

In summary, we find that choosing between NBM, ABM, and RSD remains a question of trade-offs between efficiency and incentives. ABM presents a viable alternative to NBM when partial strategyproofness is desirable and full strategyproofness is not a hard requirement.

2. Related Work

The Boston mechanism has received significant attention, because it is frequently used for the allocation of students to public schools in many school districts around the world. The mechanism has been heavily criticized for its manipulability and the resulting loss in welfare and fairness: for the case of strict priorities, Abdulkadiroğlu and Sönmez (2003) found that the Boston mechanism is neither strategyproof nor stable, and they suggested the Student Proposing Deferred Acceptance mechanism (Gale and Shapley, 1962) as an alternative that is strategyproof for students. Ergin and Sönmez (2006) showed that with full information, the Boston mechanism has undesirable equilibrium outcomes. Experimental studies by Chen
and Sönmez (2006) and Pais and Pintér (2008) revealed that the Boston mechanism is indeed manipulated more frequently than alternative, strategyproof mechanisms.

Kojima and Ünver (2014) provided an axiomatic characterization of this mechanism for the case of strict priorities. However, they also pointed out that the assumption of *strict* priorities is usually violated in school choice problems. Some recent work has considered coarse priorities, uncovering a number of surprising properties of the Boston mechanism: Abdulkadiroğlu, Che and Yasuda (2010) demonstrated that in a setting with *coarse* priorities and perfectly correlated preferences, the Boston mechanism can lead to higher ex-ante welfare than RSD. Similarly, simulations conducted by Miralles (2008) illustrate that for a setting without priorities, equilibria of the Boston mechanism can have desirable ex-ante welfare properties. In this paper, we consider a setting with no priority structure and give a systematic description of the efficiency advantages of the Boston mechanism over the strategyproof alternative RSD (which is equivalent to Student Proposing Deferred Acceptance if there is only one priority class and ties are broken using single uniform tie breaking).

Miralles (2008) also found that the manipulability of the Boston mechanism can lead to lower welfare for unsophisticated participants, and he noted that an adaptive adjustment could mitigate this effect. For the case of strict priorities, Dur (2013) characterized the adaptive Boston mechanism and showed that it is less vulnerable to manipulation than the traditional Boston mechanism. In this paper, we formalize the adaptive Boston mechanism in settings without priorities and study its properties. For coarse priorities, a characterization of either the traditional or the adaptive Boston mechanism remains an open research question.

In the absence of priorities, RSD is known to be strategyproof, ex-post efficient, and anonymous. Abdulkadiroğlu and Sönmez (1998) showed that it is equivalent to the Core from Random Endowments mechanism for house allocation, and Bade (2013) recently extended their result: she showed that any mechanism that randomly assigns agents to roles in some strategyproof, ex-post efficient, deterministic mechanism is equivalent to RSD. However, it remains an open conjecture whether RSD is the *unique* mechanism that is ex-post efficient, strategyproof, and anonymous.

Bogomolnaia and Moulin (2001) introduced the Probabilistic Serial mechanism and showed that it satisfies ordinal efficiency, a strict refinement of ex-post efficiency. However, they also showed that no mechanism can be ordinally efficient, strategyproof, and symmetric. Rank efficiency, introduced by Featherstone (2011), is an even stronger efficiency requirement, but it is incompatible with strategyproofness, independent of any fairness requirements. In this paper, we find that both variants of the Boston mechanism are ex-post efficient, but neither ordinally nor rank efficient. However, we show that both variants of the Boston mechanism provide significant efficiency gains over RSD.

Recently, Pathak and Sönmez (2013) introduced a general framework to compare different mechanisms by their vulnerability to manipulation. This framework is applicable for strict priority structures, but must be slightly extended for the study of mechanisms with coarse priorities in this paper. Unfortunately, even the weakest conceivable extension of the comparison concept does not differentiate between NBM and ABM in terms of manipulability. In (Mennle and Seuken, 2014) we have recently introduced *partial strategyproofness*, which yields a parametric measure for the degree of strategyproofness of non-strategyproof mechanisms. In
this paper, we use this new concept to compare the incentives of NBM and ABM. We find that ABM is partially strategyproof while NBM is not.

3. Model

3.1. Preferences

A setting \((N, M, q)\) consists of a set \(N\) of \(n\) agents, a set \(M\) of \(m\) objects, and a vector \(q = (q_1, \ldots, q_m)\) of capacities, i.e., there are \(q_j\) units of object \(j\) available. We assume that supply satisfies demand, i.e., \(n \leq \sum_{j \in M} q_j\), since we can always add dummy objects. Agents are endowed with von Neumann-Morgenstern (vNM) utilities \(u_i, i \in N\), over the objects. If \(u_i(a) > u_i(b)\), we say that agent \(i\) prefers object \(a\) over object \(b\), which we denote by \(a >_i b\). We exclude indifferences, i.e., \(u_i(a) = u_i(b)\) implies \(a = b\). A utility function \(u_i\) is consistent with preference ordering \(>_i\) if \(a >_i b\) whenever \(u_i(a) > u_i(b)\). Given a preference ordering \(>_i\), the corresponding type \(t_i\) is the set of all utilities that are consistent with \(>_i\), and \(T\) is the set of all types, called the type space. We use types and preference orderings synonymously.

3.2. Allocations

An allocation is a (possibly probabilistic) assignment of objects to agents. It is represented by an \(n \times m\)-matrix \(x = (x_{i,j})_{i \in N, j \in M}\) satisfying the fulfillment constraint \(\sum_{j \in M} x_{i,j} = 1\), the capacity constraint \(\sum_{i \in N} x_{i,j} \leq q_j\), and \(x_{i,j} \in [0, 1]\) for all \(i \in N, j \in M\). The entries of the matrix are interpreted as probabilities, where agent \(i\) gets object \(j\) with probability \(x_{i,j}\). An allocation is deterministic if \(x_{i,j} \in \{0, 1\}\) for all \(i \in N, j \in M\). The Birkhoff-von Neumann Theorem and its extensions (Budish et al., 2013) ensure that, given any allocation, we can always find a lottery over deterministic assignments that implements these marginal probabilities. Finally, let \(X\) denote the space of all allocations.

3.3. Mechanisms

A mechanism is a mapping \(f : T^n \to X\) that chooses an allocation based on a profile of reported types. We let \(f_i(t_i, t_{-i})\) denote the allocation that agent \(i\) receives if it reports type \(t_i\) and the other agents report \(t_{-i} = (t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n) \in T^{n-1}\). Note that mechanisms only receive type profiles as input. Thus, we consider ordinal mechanisms, where the allocation is independent of the actual vNM utilities. If \(i\) and \(t_{-i}\) are clear from the context, we may abbreviate \(f_i(t_i, t_{-i})\) by \(f(t_i)\). The expected utility for \(i\) is given by the dot product \(\langle u_i, f(t_i) \rangle\), i.e.,

\[
E_{f_i(t_i, t_{-i})}(u_i) = \sum_{j \in M} u_i(j) \cdot f_i(t_i, t_{-i})(j) = \langle u_i, f(t_i) \rangle. \tag{1}
\]

3.4. A Note on Priorities

The classical school choice problem can be considered a hybrid between the two-sided matching (or marriage) problem and the one-sided matching (or allocation) problem. In two-sided
matching, agents on both sides of the market have preferences over the agents on the other side, while in one-sided matching, objects are completely indifferent to whom they are allocated. In school choice, *priorities* take the role of the schools’ preferences, and they are usually not strict, i.e., many students may have the same priority, so that ties must be broken randomly.

In this paper, we do not model priorities explicitly. Instead, we can incorporate priorities implicitly in the formulation of the mechanism. The variants of the Boston mechanisms defined in Sections 5 and 6 choose a single priority ordering uniformly at random, i.e., all students are part of a single, large priority class, and ties are broken in the same way at all schools. This is consistent with (Miralles, 2008) and (Abdulkadiroglu, Che and Yasuda, 2010), and “orthogonal” to the case of strict priorities. Nonetheless, one of our main results (Theorem 1) generalizes to settings with some priority requirements. The definition of the Boston mechanisms for diverse priorities would not differ significantly from ours, but we leave the study of their properties in the more general environment to future research.

4. Preliminaries

In this section, we review common notions of strategyproofness, efficiency, and fairness, and we define a number of new concepts that we need for the comparison of mechanisms.

4.1. Strategyproofness Concepts

We begin with a revision of the standard strategyproofness requirement.

**Definition 1 (Strategyproofness).** A mechanism is strategyproof if for any agent $i \in N$, any type profile $t = (t_i, t_{-i}) \in T^n$, any misreport $t'_i \in T$, and any utility $u_i \in t_i$ we have

$$\langle u_i, f(t_i) - f(t'_i) \rangle \geq 0. \quad (2)$$

Strategyproofness is particularly attractive because it makes reporting truthfully a dominant strategy for all agents. This removes the need for agents to deliberate about potentially beneficial manipulations, and agents do not need to exert effort to coordinate on certain equilibria.

If agents are only interested in manipulations that improve the outcome (for them) in a first order-stochastic dominance sense (e.g., if they are not aware of their own preference intensities), then weak strategyproofness becomes useful. Weak strategyproofness was first used by Bogomolnaia and Moulin (2001) to describe the incentive properties of the Probabilistic Serial mechanism.

**Definition 2 (Weak Strategyproofness).** A mechanism is weakly strategyproof if for any type profile $t = (t_i, t_{-i}) \in T^n$, the outcome from truthful reporting is never strictly first order-stochastically dominated by the outcome from any misreport for agent $i$.

Weak strategyproofness is a very weak requirement for multiple reasons. In particular, it does not even guarantee that there exists a single utility profile for which truthful reporting is a best response (in terms of expected utility) (Mennle and Seuken, 2014).
In (Mennle and Seuken, 2014) we have introduced partial strategyproofness, a scalable requirement that bridges the gap between strategyproofness and weak strategyproofness. Intuitively, partially strategyproof mechanisms are strategyproof, but only on a particular domain restriction, i.e., the agents can still have any type, but their underlying utility functions are constrained. First we define this domain restriction.

**Definition 3** (Uniformly Relatively Bounded Indifference). A utility function \( u \) satisfies uniformly relatively bounded indifference with respect to bound \( r \in [0,1] \) (URBI(r)) if for any objects \( a, b \) with \( u(a) > u(b) \) we have

\[
r \cdot (u(a) - \min(u)) \geq u(b) - \min(u). \tag{3}
\]

Denote by URBI(r) the set of all utility functions that satisfy uniformly relatively bounded indifference with respect to bound \( r \).

A mechanism is partially strategyproof if it is strategyproofness in the domain restricted by the URBI(r) constraint. Formally:

**Definition 4** (r-partial Strategyproofness). Given a setting \((N, M, q)\) and a bound \( r \in [0,1] \), mechanism \( f \) is \( r \)-partially strategyproof in the setting \((N, M, q)\) if for any agent \( i \in N \), any type profile \( t = (t_i, t_{-i}) \in T^n \), any misreport \( t'_i \in T \), and any utility \( u_i \in \text{URBI}(r) \cap t_i \), we have

\[
\langle u_i, f(t_i) - f(t'_i) \rangle \geq 0. \tag{4}
\]

Sometimes, we simply want to state that a mechanism is \( r \)-partially strategyproof for some non-trivial \( r > 0 \) without explicitly naming the parameter \( r \). In this case, we simply say that the mechanism is partially strategyproof.

The partial strategyproofness concept has an axiomatic characterization. For details, please see Appendix A or (Mennle and Seuken, 2014). In the following we only briefly review the two axioms we need throughout this paper.

Each axiom restricts the way in which a misreport from the neighborhood of the agent’s true type, i.e., a misreport involving only a single swap, can affect the allocation of the reporting agent.

**Axiom 1** (Swap Monotonic). A mechanism \( f \) is swap monotonic if for any agent \( i \in N \), any type profile \( t = (t_i, t_{-i}) \in T^n \), and any type profile \( t'_i \in N_{t_i} \) (i.e., in the neighborhood of \( t_i \)) with \( a_k > a_{k+1} \) under \( t_i \) and \( a_{k+1} > a_k \) under \( t'_i \), one of the following holds:

1) \( i \)'s allocation is unaffected by the swap, i.e., \( f(t_i) = f(t'_i) \), or

2) \( i \)'s allocation for \( a_k \) strictly decreases and its allocation for \( a_{k+1} \) strictly increases, i.e.,

\[
f(t_i)(a_k) > f(t'_i)(a_k) \quad \text{and} \quad f(t_i)(a_{k+1}) < f(t'_i)(a_{k+1}). \tag{5}
\]

**Axiom 2** (Upper Invariance). A mechanism \( f \) is upper invariant if for any agent \( i \in N \), any type profile \( t = (t_i, t_{-i}) \in T^n \), and any type profile \( t'_i \in N_{t_i} \) with \( a_k > a_{k+1} \) under \( t_i \) and \( a_{k+1} > a_k \) under \( t'_i \), \( i \)'s allocation for the upper contour set \( U(a_k, t_i) \) is unaffected by the swap, i.e., \( f(t_i)(j) = f(t'_i)(j) \) for all \( j \in U(a_k, t_i) \).
Lower invariance is defined analogously by replacing the upper contour set by the lower contour set.

In (Mennle and Seuken, 2014), we have shown that a mechanism is strategyproof if and only if it is swap monotonic, upper invariant, and lower invariant. Furthermore, by dropping lower invariance, the class of partially strategyproof mechanisms arises, i.e., a mechanism is partially strategyproof if and only if it is swap monotonic and upper invariant.

For any partially strategyproof mechanism, we can consider the largest admissible indifference bound \( r \) as a single-parameter measure for “how strategyproof” the mechanism is.

**Definition 5.** For a setting \((N, M, q)\), the degree of strategyproofness (DOSP) of a mechanism \( f \) is given by

\[
\rho_{(N,M,q)}(f) = \max\{r \in [0, 1] : f \text{ is } r\text{-partially strategyproof in } (N, M, q)\}.
\] (6)

In (Mennle and Seuken, 2014) we have also shown that the DOSP is computable and consistent with the vulnerability to manipulation concept (Pathak and Sönmez, 2013).

### 4.2. Dominance and Efficiency

In matching markets, welfare usually cannot be expressed as the sum of the agents’ cardinal utilities, since the welfare of individual agents may not be comparable. Instead, we consider whether or not an allocation can be improved upon, i.e., whether it is *efficient*.

First, we review ex-post (Pareto) dominance and efficiency.

**Definition 6** (Ex-post Dominance). Given a type profile \( t = (t_i, t_{-i}) \in T^n \), a deterministic allocation \( x \) ex-post dominates another deterministic allocation \( y \) at \( t \) if all agents weakly prefer their allocation under \( x \) to their allocation under \( y \). The dominance is strict if at least one agent strictly prefers its allocation under \( x \).

**Definition 7** (Ex-post Efficiency). Given a type profile \( t = (t_i, t_{-i}) \in T^n \), a deterministic allocation \( x \) is ex-post efficient at \( t \) if it is not strictly ex-post dominated by any other deterministic allocation at \( t \). Finally, a probabilistic allocation is ex-post efficient if it has a lottery decomposition consisting only of ex-post efficient deterministic allocations.

Ex-post efficiency is ubiquitous in the matching literature and can be considered a baseline requirement for mechanism design. However, even when considering ex-post efficient probabilistic allocations, some unambiguous welfare gains may be possible: agents may wish to trade probability shares at some objects in such a way that all agents are weakly better off and some strictly (see (Bogomolnaia and Moulin, 2001) for an example). This kind of *ex-interim improvement*, i.e., before the lottery is implemented, is formalized by the following ordinal dominance and efficiency concepts.

**Definition 8** (Ordinal Dominance). For a type \( t : a_1 > \ldots > a_m \) and two allocation vectors \( v = v_{j \in M} \) and \( w = w_{j \in M} \), the allocation vector \( v \) first order-stochastically dominates \( w \) if for all ranks \( k \in \{1, \ldots, m\} \)

\[
\sum_{j \in M: j > a_k} v_j \geq \sum_{j \in M: j > a_k} w_j.
\] (7)
For a type profile \( \mathbf{t} \), an allocation \( \mathbf{x} \) ordinally dominates another allocation \( \mathbf{y} \) at \( \mathbf{t} \) if for all agents \( i \in N \) their respective allocation vector \( x_i \), first order-stochastically dominates \( y_i \) (for type \( t_i \)). \( \mathbf{x} \) strictly ordinally dominates \( \mathbf{y} \) if in addition, inequality (7) is strict for some agent \( i \in N \) and some rank \( k \in \{1, \ldots, m\} \). We let \( \succeq_O \) and \( >_O \) denote weak and strict ordinal dominance, respectively.

**Definition 9 (Ordinal Efficiency).** For a type profile \( \mathbf{t} \), an allocation \( \mathbf{x} \) is ordinally efficient at \( \mathbf{t} \) if it is not strictly ordinally dominated by any other allocation at \( \mathbf{t} \).

A refinement of ordinal efficiency is rank efficiency. Rank efficiency (Featherstone, 2011) captures the intuition that allocating two first choices and one second choice is preferable to allocating one first and two second choices.

**Definition 10 (Rank Dominance).** For a type \( t \in T \) and rank \( k \in \{1, \ldots, m\} \) let \( \text{ch}(k, t) \) denote the \( k \)th choice object of an agent of type \( t \). The rank distribution of an allocation \( \mathbf{x} \) at type profile \( \mathbf{t} \) is the vector \( \mathbf{d}_x = (d^x_1, \ldots, d^x_m) \) with

\[
d^x_k = \sum_{i \in N} x_{i, \text{ch}(k, t_i)}, \quad k \in \{1, \ldots, m\},
\]

i.e., \( d^x_k \) is the expected number of \( k \)th choices allocated under \( \mathbf{x} \) with respect to type profile \( \mathbf{t} \).

An allocation \( \mathbf{x} \) rank dominates another allocation \( \mathbf{y} \) at \( \mathbf{t} \) if the rank distribution \( \mathbf{d}_x \) first order-stochastically dominates \( \mathbf{d}_y \). \( \mathbf{x} \) strictly rank dominates \( \mathbf{y} \) if in addition, inequality (8) is strict for some rank \( r \in \{1, \ldots, m\} \). Let \( \succeq_R \) and \( >_R \) denote weak and strict rank dominance, respectively.

**Definition 11 (Rank Efficiency).** An allocation \( \mathbf{x} \) is rank efficient at \( \mathbf{t} \) if it is not strictly rank dominated by any other allocation at \( \mathbf{t} \).

Rank efficient allocations can be interpreted as maximizing some form of social value: suppose there is a value \( v_k \) associated with giving any agent its \( k \)th choice object. A vector \( \mathbf{v} = (v_1, \ldots, v_m) \) with \( v_1 > v_2 > \ldots > v_m \) is called rank valuation. An allocation \( \mathbf{x} \) is rank efficient if and only if there exists a rank valuation \( \mathbf{v} \) such that

\[
x \in \arg \max_{y \in \mathcal{X}} \sum_{k=1}^m \sum_{i \in N} v_k y_{i, \text{ch}(k, t_i)}.
\]

So far, we have only discussed dominance concepts for allocations. Dominance can easily be extended to mechanisms by requiring the allocations resulting from one mechanism to always dominate those from the other mechanism.

**Definition 12 (Dominance for Mechanisms).** Consider two mechanisms \( f \) and \( g \). Mechanism \( g \) weakly ordinally dominates \( f \) if \( g(t) \succeq_O f(t) \) at \( t \) for all type profiles \( t \in T^n \). \( g \) strictly ordinally dominates \( f \) if in addition \( g(t) >_O f(t) \) at \( t \) for some type profiles \( t \in T^n \). Weak and strict rank dominance are defined analogously. We denote weak and strict ordinal dominance for mechanisms by \( \succeq_O \) and \( >_O \), and weak and strict rank dominance for mechanisms by \( \succeq_R \) and \( >_R \), respectively, simply extending the notation from allocations.
To determine whether a mechanism can be improved upon with respect to efficiency, one must check whether there exists another mechanism that is unambiguously more efficient, but otherwise has the same desirable properties. This is captured by the efficient frontier.

**Definition 13** *(Efficient Frontier)*. For any set of properties $\mathcal{E}$ and a dominance concept $\succeq \in \{\succ_O, \succ_R\}$ (in the sense of Definition 12), a mechanism is on the efficient frontier (with respect to $\succ$, subject to $\mathcal{E}$), if it is not $\succ$-dominated by any mechanism that also satisfies $\mathcal{E}$.

Unfortunately, the dominance concept from Definition 12 may be overly restrictive for the comparison of many popular mechanisms, i.e., essentially all common mechanisms are not comparable in this way (e.g., Random Serial Dictatorship, Probabilistic Serial, variants of the Boston mechanism). Therefore, we introduce imperfect dominance for mechanisms, which only requires that one mechanism should produce dominant allocations whenever the resulting allocations are comparable.

**Definition 14** *(Imperfect Dominance for Mechanisms)*. Consider two mechanisms $f$ and $g$. $g$ weakly imperfectly ordinally dominates $f$ if $g(t) \succeq_O f(t)$ at $t$ for all type profiles $t \in T^n$ where $g(t)$ and $f(t)$ are comparable by the dominance relation. $g$ strictly imperfectly ordinally dominates $f$ if in addition $g(t) \succ_O f(t)$ at $t$ for some type profiles $t \in T^n$. Weak and strict imperfect rank dominance are defined analogously. We denote weak and strict imperfect ordinal dominance by $\succeq_{IO}$ and $\succ_{IO}$, respectively.

## 4.3. Fairness

In the absence of strict priorities, *anonymity* is a common fairness requirement, which all mechanisms we study in this paper satisfy.

**Definition 15** *(Anonymity)*. A mechanism is anonymous if the allocation only depends on the reports of the agents, but not their names, i.e., for any bijection $\phi : N \to N$ and all $i \in N$

$$f_{\phi(i)}(t_{\phi(1)}, \ldots, t_{\phi(n)}) = f_i(t_1, \ldots, t_n). \quad (10)$$

Anonymity implies symmetry (also called equal treatment of equals), which requires that agents with the same report also receive the same allocation. However, the inverse is not true.

Another important fairness requirement in school choice is the absence of justified envy: after the implementation of the lottery, agent $i$ envies another agent $i'$ if $i$ prefers the object that $i'$ received to its own object. The envy is justified if $i$ has a higher priority at that object than $i'$. Since in this paper, we consider mechanisms in the absence of strict priorities, an agent cannot have higher priority at any object ex-ante, and thus, justified envy is no concern.

Nonetheless, we can consider the absence of justified envy ex-post, i.e., after the implementation of the lottery and with respect to the particular tie-breaker. In this ex-post sense, Random Serial Dictatorship eliminates justified envy, while the Boston mechanisms may not. However, the Boston mechanisms do eliminate observable justified envy: if agents are not informed about the (randomly chosen) priority ordering, they only learn their relative priority with respect to
other agents against whom they have competed (and won or lost) at some objects. From the perspective of an agent \( i \), if any agent \( i' \) received an object that \( i \) prefers to its own allocation, then \( i' \) either applied to that object in an earlier round, in which case the relative priority of \( i \) and \( i' \) is unobservable, or \( i' \) applied to the object in the same round as \( i \) and \( i' \) had the higher priority. Thus, for any agent that \( i \) envies, that agent either has a higher priority or the priority is unobservable to \( i \).

4.4. The Random Serial Dictatorship Mechanism

We now define the Random Serial Dictatorship mechanism.

**Definition 16.** The Random Serial Dictatorship (RSD) mechanism works as follows:

1. Collect all type reports \( \hat{\mathbf{t}} = (\hat{t}_1, \ldots, \hat{t}_n) \in T^n \) from the agents.

2. Draw an ordering \( \pi \) of all agents uniformly at random from the space of all possible bijections \( \pi: N \to \{1, \ldots, n\} \). \( \pi(i) < \pi(i') \) means that \( i \) has priority over \( i' \).

3. Step 1: the agent with the highest priority gets a copy of the object it likes best and is removed together with that copy of the object.

4. Step \( k \): in each subsequent step, the agent with the highest priority (amongst the remaining agents) gets the object it likes best out of all objects that still have capacity left.

5. The process ends when all agents have been allocated an object.

Let \( \text{SD}_\pi(\hat{\mathbf{t}}) \) denote the allocation if a Serial Dictatorship (SD) is applied to reports \( \hat{\mathbf{t}} \) with fixed priority ordering \( \pi \). RSD can be viewed as producing a probabilistic allocation, because the priority ordering over the agents is unknown when the agents report their types. The probabilistic allocation is given by the convex combination over all possible deterministic allocations

\[
\text{RSD}(\hat{\mathbf{t}}) = \sum_{\pi: N \to \{1, \ldots, n\} \text{bijection}} \frac{1}{n!} \text{SD}_\pi(\hat{\mathbf{t}}).
\]

**Remark 1.** Any ex-post efficient deterministic allocation is supported by SD with respect to a particular priority ordering \( \pi \) of the agents (Featherstone, 2011). This implies that RSD performs a lottery over all ex-post efficient deterministic allocations, where each allocation is weighted by the number of different priority orderings \( \pi \) for which SD produces that allocation.

**Remark 2.** With a single, uniform tie-breaker, the Student Proposing Deferred Acceptance mechanism (Gale and Shapley, 1962) is equivalent to RSD.

5. The Naïve Boston Mechanism

In this section, we define and study the traditional (naïve) Boston mechanism (NBM) in the absence of priorities and with single uniform tie-breaking. It is well known that this mechanism
is ex-post efficient, but not even weakly strategyproof. We show that it is neither ordinally nor rank efficient, nor is it on the efficient frontier. Despite these negative findings, our first main result (Theorem 1) establishes that NBM does have efficiency advantages over RSD.

5.1. Formalization of NBM

**Definition 17.** The naïve Boston mechanism (NBM)\(^1\) works as follows:

1. Collect all type reports \(\hat{t} = (\hat{t}_1, \ldots, \hat{t}_n) \in T^n\) from the agents.

2. Draw an ordering \(\pi\) of all agents uniformly at random from the space of all possible bijections \(\pi : N \rightarrow \{1, \ldots, n\}\). \(\pi(i) < \pi(i')\) means that \(i\) has priority over \(i'\).

3. Round 1: all agents apply to their reported first choice object. At every object, if the object’s capacity is sufficient to satisfy all applications to this object, all the applicants get a copy of the object. Otherwise, the agents with the highest priority get copies of the object until it is exhausted. All remaining agents do not receive a copy and enter the next round.

4. Round \(k\): an agent who did not receive an object in any of the previous \(k - 1\) rounds applies to its reported \(k\)th choice object. Analogous to round 1, the agents with the highest priorities are allocated copies of the objects for which they applied. If an object has more applicants than remaining capacity, the remaining agents enter the next round.

5. The process ends when all agents have been allocated an object.

Let \(\text{NBM}_\pi(\hat{t})\) denote the allocation if NBM is applied to reports \(\hat{t}\) with priority ordering \(\pi\). Like RSD, NBM produces probabilistic allocations, because it randomizes over priority orderings. The probabilistic allocation is given by the convex combination

\[
\text{NBM}(\hat{t}) = \sum_{\pi : N \rightarrow \{1, \ldots, n\} \text{ bijection}} \frac{1}{n!} \text{NBM}_\pi(\hat{t}). \tag{12}
\]

5.2. Manipulability and Upper Invariance of NBM

NBM is known to be manipulable in a first order-stochastic dominance sense.

**Proposition 1.** NBM is not weakly strategyproof.

**Proof.** Consider a setting \((N, M, \mathbf{q})\) with \(N = \{1, \ldots, 4\}, M = \{a, b, c, d\}\), objects have unit capacities, and the type profile is

\[
t_1 : a > b > c > d, \\
t_2, t_3 : a > c > d > b, \\
t_4 : b > \ldots.
\]

\(^1\)The mechanism is called the Boston mechanism, despite the fact that it is no longer used in Boston. Some researchers argue that calling it the Immediate Acceptance mechanism would be more consistent with the naming of the Deferred Acceptance mechanism.
Agent 1’s allocation is $(1/3, 0, 0, 2/3)$ for the objects $a$ through $d$, respectively. If agent 1 swaps $b$ and $c$ in its report, its allocation will be $(1/3, 0, 1/3, 1/3)$, which first order-stochastically dominates the allocation under truthful reporting.

Despite this obvious manipulability, NBM does have one attractive incentive property: it satisfies upper invariance, i.e., changing the order of two adjacent objects in an agent’s type report ($t : a > b$ to $t' : b > a$, say) will not influence the allocation of any object that this agent strictly prefers to $a$.

**Proposition 2.** NBM is upper invariant.

*Proof outline (formal proof in Appendix B.1.1).* Consider each priority ordering $\pi$ separately. When swapping two objects $a$ and $b$, observe that if an object in the upper contour set is allocated, the swap has no effect, and if an object from the lower contour set is allocated, the swap will not lead to an allocation from the upper contour set. Therefore, the probabilities for the objects in the upper contour set remain unchanged.

**Proposition 3.** NBM is neither swap monotonic nor lower invariant.

*Proof.* The example in the proof of Proposition 1 shows that NBM has neither property.

### 5.3. Efficiency of NBM

In this section, we check whether NBM satisfies any existing efficiency concepts or lies on the efficient frontier, subject to upper invariance. First, we show that like RSD, NBM is ex-post efficient. However, also like RSD, NBM is neither ordinally nor rank efficient, nor does it lie on the efficient frontier. This motivates a direct comparison of NBM to RSD.

#### 5.3.1. Ex-post Efficiency of NBM

First, we state the well-known fact that NBM is ex-post efficient.

**Proposition 4.** NBM is ex-post efficient at all type profiles.

*Proof outline (formal proof in Appendix B.1.2).* For each type profile $t$ and each priority ordering $\pi$, we construct a priority order $\pi'$ such that $\text{NBM}_\pi(t) = \text{SD}_{\pi'}(t)$. The result follows, because SD is ex-post efficient for all priority orders $\pi$.

#### 5.3.2. Failure of Ordinal and Rank Efficiency

Since rank efficiency implies ordinal efficiency, ordinal inefficiency implies rank inefficiency. Thus, the following Proposition 5 shows ordinal inefficiency and rank inefficiency of NBM at the same time.

**Proposition 5.** NBM and RSD are neither ordinally efficient nor rank efficient.
Proof. Consider the setting with $N = \{1, \ldots, 4\}$, $M = \{a, b, c, d\}$, unit capacities, and the type profile

\[
\begin{align*}
t_1 : & \quad a > b > c > d, \\
t_2 : & \quad a > b > d > c, \\
t_3 : & \quad b > a > c > d, \\
t_4 : & \quad b > a > d > c.
\end{align*}
\]

The allocations are

\[
\text{RSD}(t) = \left( \begin{array}{cccc}
\frac{5}{12} & \frac{1}{12} & \frac{5}{12} & \frac{1}{12} \\
\frac{1}{12} & \frac{5}{12} & \frac{1}{12} & \frac{5}{12} \\
\frac{1}{12} & \frac{1}{12} & \frac{5}{12} & \frac{1}{12} \\
\frac{1}{12} & \frac{1}{12} & \frac{1}{12} & \frac{5}{12}
\end{array} \right) \quad \text{and} \quad \text{NBM}(t) = \left( \begin{array}{cccc}
\frac{1}{2} & 0 & \frac{3}{16} & \frac{1}{16} \\
\frac{1}{2} & 0 & \frac{1}{16} & \frac{3}{16} \\
0 & \frac{1}{2} & \frac{1}{16} & \frac{3}{16} \\
0 & \frac{1}{2} & \frac{1}{16} & \frac{3}{16}
\end{array} \right),
\]

which are both ordinally dominated by

\[
\text{PS}(t) = \left( \begin{array}{cccc}
\frac{1}{2} & 0 & \frac{1}{2} & 0 \\
0 & \frac{1}{2} & \frac{1}{2} & 0 \\
0 & \frac{1}{2} & 0 & \frac{1}{2} \\
0 & \frac{1}{2} & \frac{1}{2} & 0
\end{array} \right).
\]

Since ordinal dominance implies rank dominance, failure of ordinal efficiency implies failure of rank efficiency.

\[\square\]

5.3.3. Failure of Efficient Frontier

In the proof of Proposition 5, the Probabilistic Serial mechanism ordinally dominates NBM at a particular type profiles, i.e., it is more efficient. Since PS is ordinally efficient, it is a particularly interesting candidate mechanism which might ordinally dominate NBM at all type profiles. However, this is not the case, i.e., at some type profile, PS does not ordinally dominate NBM. Instead, NBM may even rank dominate PS at some type profile, as the following Proposition 6 shows.

Proposition 6. PS does not ordinally dominate NBM at all type profiles, and NBM rank dominates PS at some type profile.

Proof. Consider the setting with $N = \{1, 2, 3\}$, $M = \{a, b, c\}$, unit capacities, and the type profile

\[
\begin{align*}
t_1 : & \quad a > b > c, \\
t_2 : & \quad a > b > c, \\
t_3 : & \quad b > a > c.
\end{align*}
\]
The allocations for NBM is

\[ \text{NBM}(t) = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{pmatrix}, \]  
(15)

which has rank distribution \( d = (2, 0, 1) \). Probabilistic Serial on the other hand yields

\[ \text{PS}(t) = \text{RSD}(t) = \begin{pmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{2}{3} & \frac{1}{3} \end{pmatrix}, \]  
(16)

which has a strictly worse rank distribution \( d = (5/3, 1/3, 1) \). Thus, NBM strictly rank dominates PS at \( t \), and therefore PS does not ordinally dominate NBM even weakly at \( t \).

Since NBM is not ordinally dominated by the particularly interesting candidate PS, it could lie on the efficient frontier. However, we find that this is not the case, i.e., it is possible to construct a mechanism that is upper invariant and ordinally dominates NBM.

**Proposition 7.** NBM is not on the efficient frontier with respect to ordinal dominance, subject to upper invariance.

**Proof outline (formal proof in Appendix B.1.3).** We construct a mechanism NBM\(^+\) that is essentially equal to NBM, except for a certain set of type profiles. For these type profiles, NBM\(^+\) chooses the allocation selected by PS. It is easy to show that NBM\(^+\) ordinally dominates NBM, and the dominance may be strict. To show that NBM\(^+\) satisfies upper invariance, we argue that under any swap, the mechanism’s changes to the allocation are consistent with the axiom. In particular, this is true for transitions between the special type profiles where NBM\(^+\) and NBM choose different allocations and those type profiles where the allocations of both mechanisms are equal.

5.4. NBM versus RSD - The Price of Strategyproofness

So far we have not identified any efficiency advantage of NBM over RSD. In this section, we present our first main result, which shows in what sense NBM is in fact more efficient than RSD.

5.4.1. Imperfect Rank Dominance of NBM over RSD

**Theorem 1.** NBM strictly imperfectly rank dominates RSD, i.e., NBM strictly rank dominates RSD at some type profiles, but is never strictly rank dominated by RSD at any type profile.

**Proof outline (formal proof in Appendix B.1.4).** Proposition 6 has already established that NBM may strictly rank dominate RSD. For any fixed priority order \( \pi \) we show that if SD\(_{\pi}\) and NBM\(_{\pi}\) allocate the same number of first choices, they will in fact allocate these first choices to the same agents. We can remove these agents and the corresponding objects and proceed by induction, carefully handling the case when some object has capacity zero. Averaging across all priority orderings, we find that if RSD rank dominates NBM, the allocation from both mechanism must be the same, and therefore RSD never strictly rank dominates NBM.
Imperfect rank dominance of NBM over RSD by Theorem 1 can be viewed as a lower bound on the price of strategyproofness, i.e., by choosing RSD we forego some unambiguous improvements of the rank distribution that could be attained by using NBM instead.

**Remark 3.** Theorem 1 remains valid even if some priorities are imposed: let \( \Pi \) be the set of all possible priority orderings, i.e., bijections \( \pi : N \rightarrow \{1, \ldots, n\} \), and consider any probability distribution \( P \) on \( \Pi \). Mechanisms \( \text{RSD}_P \) and \( \text{NBM}_P \) can be defined analogously to Definitions 16 and 17, but each mechanism draws the priority ordering \( \pi \) from \( P \) instead of the uniform distribution. The proof of Theorem 1 considers every priority ordering separately, and thus, \( \text{RSD}_P \) never strictly ordinally dominates \( \text{NBM}_P \).

In this way it is possible to give priority to some agent \( i \) over another agent \( i' \), e.g., by setting \( P(\pi) = 0 \) for all \( \pi \) with \( \pi(i) > \pi(i') \). Thus, Theorem 1 remains valid if some groups of agents (e.g., minorities) are given preference at all objects. More complex priority structures where priorities can vary across different objects (e.g., walk zone priorities) lie outside the scope of this result.

We conclude that NBM is more efficient than the strategyproof baseline mechanism RSD. This result differs from prior attempts to compare Boston mechanisms to other mechanism, which relied on stylized assumptions about agents’ preference. Our results highlight that choosing between NBM and RSD involves a non-trivial decision about the trade-off between strategyproofness and efficiency.

### 5.4.2. Simulation Results

To determine how strongly NBM imperfectly rank dominates RSD, we conducted simulations: for settings with \( n = m \in \{3, \ldots, 10\} \) and \( q_j = 1 \) for all \( j \in M \), we sampled 100’000 type profiles uniformly at random for each value of \( n \). We then determined the rank dominance relation between the resulting allocations under NBM and RSD at the sampled type profiles. Figure 1 shows how often NBM strictly rank dominates (red bars) or has the same rank distribution as RSD (green bars), and how often the two mechanisms are incomparable (blue bars). We see that the share where NBM strictly rank dominates RSD shrinks for an increasing number of objects, but only slowly.

### 6. The Adaptive Boston Mechanism

This section contains our second main contribution: we define the adaptive Boston mechanism (ABM) and study its properties. The main take-away is that ABM has significantly better incentive properties than NBM, but has almost the same efficiency advantages over RSD.

Like NBM, ABM is manipulable. However, we show that, in contrast to NBM, it is partially strategyproof, i.e., strategyproof on the domain of uniformly bounded indifference. Like NBM, ABM is ex-post efficient, but neither ordinally nor rank efficient. We also show that it is not on the efficient frontier, subject to partial strategyproofness. Thus, we are in the same situation as with NBM, where traditional efficiency measures cannot be used to distinguish between...
NBM and ABM. Therefore, we compare ABM to RSD in terms of rank dominance. Rather surprisingly, we find that RSD may rank dominate ABM at certain type profiles. However, we also show that these cases are extremely rare, i.e., ABM imperfectly rank dominates RSD in the limit as markets become large. Numerically, we find evidence that the efficiency advantages of ABM over RSD are almost the same as those of NBM over RSD and that the theoretical possibility that RSD may rank dominate ABM is negligible.

6.1. Formalization of ABM

The adaptive Boston mechanism is designed to eliminate the obvious manipulations described in the introduction. Instead of applying to their \( k \)th choice object in the \( k \)th round, in each round agents apply to the object that they prefer most out of the objects that are still available. Formally:

**Definition 18.** The adaptive Boston mechanism (ABM) works as follows:

1. Collect all type reports \( \hat{t} = (\hat{t}_1, \ldots, \hat{t}_n) \in T^n \) from the agents.

2. Draw an ordering \( \pi \) of all agents uniformly at random from the space of all possible bijections \( \pi : N \to \{1, \ldots, n\} \). \( \pi(i) < \pi(i') \) means that \( i \) has priority over \( i' \).

3. Round 1: all agents apply to their reported first choice object. At every object, if the object’s capacity is sufficient to satisfy all applications to this object, all the applicants get a copy of the object. Otherwise, the agents with the highest priority get copies of the object until it is exhausted. All remaining agents do not receive a copy and enter the next round.

4. Round \( k \): an agent who did not receive an object in any of the previous \( k - 1 \) rounds applies to the object which it reported to prefer most out of the objects with non-zero remaining capacity. Analogous to round 1, the agents with the highest priorities are
allocated copies of the objects for which they applied. If an object has more applicants than capacity, the remaining agents enter the next round.

5. The process ends when all agents have been allocated an object.

Let \( \text{ABM}_\pi \) denote the adaptive Boston mechanism when the mechanism is used for a single, fixed priority order \( \pi \), in which case the resulting allocation is deterministic. We have

\[
\text{ABM}(t) = \sum_{\pi: N \to \{1, \ldots, n\} \text{ bijection}} \frac{1}{n!} \text{ABM}_\pi(t).
\] (17)

6.2. Partial Strategyproofness of ABM

In this section, we show that ABM is partially strategyproof and we asses its degree of strategyproofness.

6.2.1. Partial Strategyproofness Result for ABM

Like NBM, ABM it is upper invariant, but not strategyproof. However, we also show that ABM is swap monotonic. Thus, ABM is partially strategyproof, while NBM is not even weakly strategyproof.

Proposition 8. ABM is upper invariant.

The proof of Proposition 8 is essentially the same as for Proposition 2. ABM is also swap monotonic, as Theorem 2 shows.

Theorem 2. ABM is swap monotonic.

Proof outline (formal proof in Appendix B.2.1). We consider each priority ordering \( \pi \) separately. For each \( \pi \), we consider the reaction of the mechanism to a swap from \( t_i : a > b \) to \( t'_i : b > a \) by some agent \( i \). We show that

- if \( \pi \) leads to an allocation of \( b \) to \( i \) under \( t \), it will still lead to an allocation of \( b \) under \( t' \), i.e., the probability that \( b \) is allocated to \( i \) weakly increases,
- if \( \pi \) does not lead to an allocation of \( a \) to \( i \) under \( t \), it will not lead to an allocation of \( a \) to \( i \) under \( t' \) either, i.e., the probability that \( a \) is allocated to \( i \) weakly decreases,
- if the allocation changes at all under some \( \pi \), we can construct another ordering \( \pi' \), such that under \( \pi' \) a change of report from \( t \) to \( t' \) leads to a change in allocation from \( a \) to \( b \).

Thus, the probabilities for \( a \) and \( b \) change strictly if the allocation changes at all, which corresponds to the requirement of swap monotonicity. \( \square \)

Corollary 1. ABM is partially strategyproof.

Corollary 1 follows immediately if we consider that ABM is swap monotonic (Theorem 2) and upper invariant (Proposition 8) and hence partially strategyproof (Fact 2 in Appendix A).
| Number of objects $m$ | Capacity $q$ |
|-----------------------|--------------|
| 1                     | 0.5          |
| 2                     | 0.5          |
| 3                     | 0.5          |
| 3                     | 0.33         |
| 4                     | 0.33         |

Table 1: Degree of strategyproofness of ABM with balanced capacities and $n = mq$ agents.

6.2.2. Degree of Strategyproofness of ABM

Since ABM is partially strategyproof, the degree of strategyproofness measure

$$\rho_{(N,M,q)}(\text{ABM}) = \max\{r \in [0, 1] : \text{ABM is } r\text{-partially strategyproof in } (N, M, q)\} \quad (18)$$

is positive for any setting. We are interested in learning “how strategyproof” ABM is in different settings. Unfortunately, $\rho_{(N,M,q)}(\text{ABM})$ becomes small when the number of objects increases (assuming unit capacity and an equal number of objects and agents), which means intuitively that ABM becomes “less strategyproof.”

**Proposition 9.** For settings where $n = m, q_j = 1$ for all $j \in M$

$$\lim_{n \to \infty} \rho_{(N,M,q)}(\text{ABM}) = 0. \quad (19)$$

**Proof outline (formal proof in Appendix B.2.2).** We construct a particular sequence of type profiles. In each step, we consider a certain agent and show that this agent can exchange very little probability at its first choice for substantial probability at its second choice. This trade-off is beneficial for the agent, unless the relative difference in valuation between the first and second choice is very high.

Although the degree of strategyproofness becomes small for large numbers of objects and agents, numerical evidence suggests that for a fixed number of objects $m$ and evenly distributed capacity ($q_j = q$ for all $j \in M$), the degree of strategyproofness does not decrease as more agents (and more capacity) are added to the setting, as long as the capacity is evenly distributed. This can be seen in Table 1 and is formalized by the following conjecture.

**Conjecture 1.** For $q \neq q'$ and two settings $(N, M, q)$ and $(N', M', q')$ with

- $M = M'$,
- $q = (q, \ldots, q)$ and $q' = (q', \ldots, q')$,
- $n = \#N = qm$ and $n' = \#N' = q'm$,

we have

$$\rho_{(N,M,q)}(\text{ABM}) = \rho_{(N',M',q')}(\text{ABM}). \quad (20)$$
6.3. Efficiency of ABM

Analogous to our findings for NBM, the traditional methods fail to differentiate between ABM and RSD in terms of efficiency. ABM is ex-post efficient, but neither ordinally nor rank efficient, nor on the efficient frontier, even if we consider only partially strategyproof mechanisms.

**Proposition 10.** ABM is ex-post efficient at all type profiles.

The proof of Proposition 10 is analogous to the proof of Proposition 4.

**Proposition 11.** ABM is neither ordinally efficient nor rank efficient.

Proposition 11 follows from the same example as Proposition 5, because in the specific setting and type profile the results of ABM and NBM coincide.

**Proposition 12.** PS does not ordinally dominate ABM at all type profiles, and ABM rank dominates PS at some type profile.

Proposition 12 follows from the same example as Proposition 6.

**Proposition 13.** ABM is not on the efficient frontier with respect to ordinal dominance, subject to partial strategyproofness.

The proof is analogous to the proof of Proposition 7.

**Remark 4.** The mechanism ABM* constructed in the proof of Propositions 7 / 13 even satisfies partial strategyproofness. It remains an open question for future research whether compared to ABM, ABM* has the same, higher, or lower degree of strategyproofness.

### 6.4. ABM versus RSD - Imperfect Rank Dominance in Large Markets

In the previous section, we have shown that the efficiency of ABM is not differentiable from the efficiency of RSD by traditional methods. Thus, we again consider a direct comparison of ABM and RSD in terms of imperfect rank dominance. Surprisingly, ABM does not imperfectly rank dominate RSD, i.e., there exists at least one type profile at which RSD actually produces a rank dominant allocation. However, we recover imperfect rank dominance of ABM over RSD for large markets, and numerical evidence suggests that the violation of imperfect dominance is insignificant, even in finite markets.

#### 6.4.1. Failure of Imperfect Rank Dominance

The following Proposition yields the surprising result that RSD may rank dominate ABM.

**Proposition 14.** ABM does not imperfectly rank dominate RSD.
Proof. We present a counter example, i.e., a setting and type profile for which RSD strictly rank dominates ABM. Consider a setting with 6 agents, 6 objects in unit capacity, and the type profile

\[ t_1, \ldots, t_4 : a > b > c > d > e > f, \]
\[ t_5, t_6 : e > b > a > d > f > c. \]

Consider the ordering \( \pi = \text{id} \) of the agents. With this ordering, RSD will allocate objects \( a \) through \( d \) to agents 1 through 4. Agents 5 and 6 get objects \( e \) and \( f \). Thus, RSD allocates 2 first, 1 second, 1 third, 1 fourth, and 1 fifth choice. For the same ordering, ABM will allocate \( a, b, c \) to agents 1, 2, 3, respectively. Agents 5 and 6 will get \( e \) and \( d \), which leaves agent 4 with \( f \). Observe that with this ordering, ABM allocates 2 first, 1 second, 1 third, 1 fourth, no fifth, and 1 sixth choice. This is strictly rank dominated by the rank efficient allocation chosen by RSD.

The allocation under ABM is

\[
\text{ABM}(t) = \frac{1}{60} \begin{pmatrix}
15 & 12 & 15 & 3 & 0 & 15 \\
15 & 12 & 15 & 3 & 0 & 15 \\
15 & 12 & 15 & 3 & 0 & 15 \\
0 & 6 & 0 & 24 & 30 & 0 \\
0 & 6 & 0 & 24 & 30 & 0 \\
\end{pmatrix},
\]

and the allocation under RSD is

\[
\text{RSD}(t) = \frac{1}{60} \begin{pmatrix}
15 & 12 & 15 & 8 & 0 & 10 \\
15 & 12 & 15 & 8 & 0 & 10 \\
15 & 12 & 15 & 8 & 0 & 10 \\
0 & 6 & 0 & 14 & 30 & 10 \\
0 & 6 & 0 & 14 & 30 & 10 \\
\end{pmatrix}.
\]

The fact that all allocations chosen by ABM are rank inefficient, but the allocations chosen by RSD are sometimes rank efficient implies that RSD strictly rank dominates ABM for this type profile.

The example in the proof of Proposition 14 raises the question how frequently RSD rank dominates ABM. Interestingly, complete enumeration (using a computer) has revealed that RSD does not rank dominate ABM for any setting with less than 6 objects and unit capacities. In Section 6.4.3, we will show via simulation that even for more objects, such examples are pathological, i.e., out of 500’000 sampled type profiles (100’000 for each \( n \in \{6, \ldots, 10\} \)), there was not a single occurrence.

With this intuition in mind, we ask what the rank dominance relation looks like in large markets. We consider two independent notions of how settings “get large.” In Section 6.4.2 we employ the first notion, which is adopted from (Kojima and Manea, 2010), where the capacities
of a fixed number of objects increase. This approximates school choice settings, where the capacity of public schools frequently exceeds 100 seats. For the second notion, which we use in Section 6.4.3, we let the number of objects increase and they all have unit capacity. This is more adequate for house allocation problems, where every “house” is different and can be given to only one agent.

### 6.4.2. Limit Result for Large School Choice Markets

In this section, we present our first limit result, showing that as the capacity of the objects increases the share of type profiles where RSD rank dominates ABM becomes arbitrarily small.

Before we formulate our result, we introduce first-choice-maximizing allocations and provide an intuition of why the convergence holds. In words, an allocation \( x \) is first-choice-maximizing at type profile \( t \) if it can be represented as a lottery over deterministic allocations that give the maximum number of first choices, i.e., where

\[
d_1^x = \sum_{i \in N} x_{i, ch(1, t_i)} = \max_{y \in \mathcal{X}} d_1^y.
\]

**Remark 5.** We already observed that RSD puts positive probability on all ex-post efficient, deterministic allocations, while ABM assigns different probabilities, which may be zero. One key observation is that ABM maximizes the expected number of allocated first choices. Thus, ABM gives no weight to any allocation that does not yield the maximum possible number of first choices. In contrast, if at type profile \( t \) there exists any ex-post efficient deterministic allocation that is not first-choice-maximizing, then RSD does not allocate the maximum expected number of first choices, i.e., it is not first-choice-maximizing.

Using the observation from Remark 5, we can now prove the following Proposition 15, which in turn yields the limit result, Theorem 3.

**Proposition 15.** For any fixed number of objects \( m \geq 3 \) and any \( \epsilon > 0 \), there exists a \( q_{\min} \in \mathbb{N} \), such that for any capacity vector \( q = (q_1, \ldots, q_m) \) with \( q_j \geq q_{\min} \) for all \( j \in M \) and \( n = \sum_{j \in M} q_j \) agents, and for \( t \) chosen uniformly at random, the probability that RSD(\( t \)) is first-choice-maximizing is smaller than \( \epsilon \).

**Proof outline (formal proof in Appendix B.2.3).** Consider a fixed type profile \( t \). As discussed previously, RSD is first-choice-maximizing only if all ex-post efficient deterministic allocations with respect to \( t \) are first-choice-maximizing. To prove the proposition, we show that an ex-post efficient allocation that is not first-choice-maximizing can almost always be found (for sufficiently high capacities). Intuitively, we exploit the following: if objects \( j, j' \in M \) are the first choice of more than \( q_j \) and \( q_{j'} \) agents, respectively, then we can construct a priority ordering \( \pi \) that ranks the agents with first choice \( j \) first. Then under SD\( _\pi \) some \( q_j \) agents get their first choice \( j \), but it turns out to be “unlikely” that none of the remaining agents with first choice \( j \) have \( j' \) as their second choice.

Thus, RSD is almost never first-choice-maximizing as the capacities of the objects grow. Since ABM is first-choice-maximizing, it will allocate strictly more first choices than RSD at
almost all type profiles. Thus, even weak rank dominance of RSD over ABM almost always breaks already at the first choice.

**Theorem 3.** For any \( m \) and any sequence of settings \( (N^k, M^k, q^k)_{k \geq 0} \) with

- \( \#M^k = m \) (i.e., constant number of objects),
- \( \min_{j \in M^k} q^k_j \to \infty \) as \( k \to \infty \) (i.e., growing capacities),
- \( n_k = \#N^k = \sum_{j \in M^k} q^k_j \) (i.e., total supply matches total demand),

we have

\[
\frac{\#\{t \in T^n | RSD(t) \succeq_R ABM(t)\}}{\#\{t \in T^n\}} \to 0 \quad (k \to \infty). \quad (24)
\]

### 6.4.3. Limit Result for Large House Allocation Markets

In this section, we present our second limit result for large house allocation markets: we show that as the number of objects with unit capacity and agents increases, the share of type profiles where RSD rank dominates ABM becomes arbitrarily small. As in the previous section, we use the fact that RSD is almost never first-choice-maximizing to prove the convergence result.

**Proposition 16.** For any \( \epsilon > 0 \), there exists \( n \in \mathbb{N} \), such that for any setting \( (N, M, q) \) with \( \#M = \#N \geq n \) and \( q_j = 1 \) for all \( j \in M \), and for \( t \) chosen uniformly at random, the probability that \( RSD(t) \) is first-choice-maximizing is smaller than \( \epsilon \).

**Proof outline (formal proof in Appendix B.2.4).** The proof idea is similar to the proof of Proposition 15, but the formal analysis requires advanced combinatorics. We derive an upper bound for the probability that all agents with over-demanded first choices have second choices such that they do not exhaust some other agent’s first choice. We then show that this upper bound converges to zero.

Theorem 4 follows from Proposition 16 in the same way in which Theorem 3 follows from Proposition 15.

**Theorem 4.** For settings with \( n = m \) and \( q_j = 1 \), ABM is not rank-dominated by RSD in the limit, i.e.,

\[
\frac{\#\{t \in T^n | RSD(t) \succeq_R ABM(t)\}}{\#\{t \in T^n\}} \to 0 \quad (n \to \infty). \quad (25)
\]

### 6.4.4. Simulation Results

We conducted simulations similar to those in Section 5.4.2 to compare ABM and RSD: for settings with \( n = m \in \{3, \ldots, 10\} \) and \( q_j = 1 \) for all \( j \in M \), we sampled 100’000 type profiles uniformly at random for each value of \( n \). We then determined the rank dominance relation between the resulting allocations under ABM and RSD at the sampled type profiles. Figure 2
shows how often ABM strictly rank dominates (red bars) or has the same rank distribution (green bars) as RSD, and how often the two mechanisms are incomparable (blue bars). Note that the results are very similar to the findings for NBM and RSD (Figure 1), i.e., the share of profiles with comparability shrinks, but the share of those profiles where ABM strictly rank dominates RSD shrinks very slowly. Furthermore, the case where RSD rank dominates ABM never occurred in the entire sample. This suggests that such cases are extremely rare and should not matter for an evaluation of ABM against RSD. We conclude that ABM is “more efficient” than RSD.

7. NBM versus ABM - The Price of Partial Strategyproofness

In Section 5, we have established the price of strategyproofness in terms of efficiency: RSD is less efficient than NBM, a price we have to pay if we want to achieve full strategyproofness using RSD. We will now assess the price of partial strategyproofness by comparing NBM and ABM in terms of rank dominance. Contrary to intuition, this comparison is not unambiguous in favor of NBM: while NBM strictly rank dominates ABM at some type profiles, there also exist type profiles at which the opposite holds. Furthermore, we also present numerical evidence that the two mechanisms are usually not comparable by rank dominance, but if they are comparable, then dominance of NBM over ABM occurs more frequently than the opposite case. Therefore, when choosing ABM in favor of NBM, the fact that NBM rank dominates ABM more often than vice versa can be considered a price that we pay for partial strategyproofness. Of course, since neither mechanism is on the efficient frontier, there may be other, more desirable trade-offs that are not accessible via NBM and ABM.
7.1. Failure of Imperfect Rank Dominance

We first attempt a direct comparison between NBM and ABM via imperfect rank dominance. Interestingly, NBM is not always the more efficient mechanism, i.e., while there exists type profiles at which NBM strictly rank dominates ABM (Proposition 17), there also exist type profiles where the opposite holds (Proposition 18).

**Proposition 17.** NBM strictly rank dominates ABM at some type profiles.

*Proof.* Consider the setting with $N = \{1, \ldots, 4\}$, $M = \{a, b, c, d\}$, unit capacities, and the type profile

\[
t_1 : a > b > c > d,
\]

\[
t_{2,3} : a > c > d > b,
\]

\[
t_4 : b > \ldots,
\]

which is the same as Example 3. The allocations from NBM and ABM are

\[
\text{NBM}(t) = \begin{pmatrix}
\frac{1}{7} & 0 & 0 & \frac{2}{7} \\
\frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{6} \\
\frac{1}{3} & 0 & \frac{2}{3} & \frac{1}{6} \\
0 & 1 & 0 & 0
\end{pmatrix}
\]

and

\[
\text{ABM}(t) = \begin{pmatrix}
\frac{1}{7} & 0 & \frac{1}{3} & \frac{1}{6} \\
\frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{6} \\
\frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{6} \\
0 & 1 & 0 & 0
\end{pmatrix}.
\]  

The rank distribution under NBM is $d = (2, 1, 0, 1)$, which dominates $d' = (2, 2/3, 1/3, 1)$, the rank distribution under ABM.

The next result is more surprising: it shows that ABM may strictly rank dominate NBM. Note that the result holds for strict priorities as well.

**Proposition 18.** ABM strictly rank dominates NBM at some type profiles.

*Proof.* Consider the setting with $N = \{1, \ldots, 5\}$, $M = \{a, b, c, d, e\}$, unit capacities, and the type profile

\[
t_1, t_2 : a > b > c > d > e,
\]

\[
t_3, t_4 : a > d > c > e > b,
\]

\[
t_5 : b > \ldots.
\]

The allocations from NBM and ABM are

\[
\text{NBM}(t) = \frac{1}{60} \begin{pmatrix}
15 & 0 & 25 & 0 & 20 \\
15 & 0 & 25 & 0 & 20 \\
15 & 0 & 5 & 30 & 10 \\
15 & 0 & 5 & 30 & 10 \\
0 & 60 & 0 & 0 & 0
\end{pmatrix}
\]

and

\[
\text{ABM}(t) = \frac{1}{60} \begin{pmatrix}
15 & 0 & 30 & 0 & 15 \\
15 & 0 & 30 & 0 & 15 \\
15 & 0 & 0 & 30 & 15 \\
15 & 0 & 0 & 30 & 15 \\
0 & 60 & 0 & 0 & 0
\end{pmatrix}.
\]

The rank distribution under NBM is $d = (2, 1, 1, 1/3, 2/3)$, which is dominated by $d' = (2, 1, 1, 1/2, 1/2)$, the rank distribution under ABM.
Remark 6. Recall that a rank valuation is a vector that specifies the value of allocating first, second, third, etc. choices to agents. If an allocation $x$ rank dominates another allocation $y$, then the aggregate rank value is higher under $x$ than under $y$ for any rank valuation $v$, i.e.,

$$\sum_{i \in M} \sum_{k \in \{1, \ldots, m\}} v_k \cdot x_{i, ch(k,t_i)} \geq \sum_{i \in M} \sum_{k \in \{1, \ldots, m\}} v_k \cdot y_{i, ch(k,t_i)}.$$  

(Dur (2013) showed that for the case of strict priorities and for a particular rank valuation, ABM can lead to a higher aggregate rank value than NBM. Proposition 18 strengthens this result, since rank dominance of ABM over NBM implies that ABM has higher aggregate rank value for any rank valuation.

7.2. Simulation Results

So far, we have found that NBM and ABM are incomparable by imperfect rank dominance, because $\text{NBM} >_R \text{ABM}$ for some type profiles, but $\text{ABM} >_R \text{NBM}$ for others. This raises the question whether one of the cases is more common. Numerically we find that NBM is “usually” more efficient.

At each type profile $t$, one of the following cases can occur:

1. NBM$(t)$ strictly rank dominates ABM$(t)$,
2. NBM$(t)$ and ABM$(t)$ have the same rank distribution,
3. ABM$(t)$ strictly rank dominates NBM$(t)$,
4. NBM$(t)$ and ABM$(t)$ are incomparable by rank dominance.

Figure 3 shows how often each of these cases occurs when sampling type profiles uniformly at random in settings with unit capacities ($q_j = 1$ for all $j \in M$) and various $n(= m)$. The results suggest that the two mechanisms are usually not comparable (blue bars). However, conditioned on comparability, NBM rank dominates ABM more often than vice versa (red vs. yellow bars), and the share of profiles where the rank distributions coincide shrinks.

Remark 7. Even though NBM outperforms ABM in this comparison, the share of type profiles for which ABM dominates NBM is substantially larger than the share of profiles where RSD dominates ABM: conditional on comparability, ABM dominates NBM (yellow bars) in about 0.8% of the cases for $n \in \{7, 8, 9, 10\}$. Thus, while dominance of RSD over ABM is negligible, the out-performance is less pronounced for NBM and ABM.

8. Conclusion

In this paper, we have studied the traditional (naïve) and a new (adaptive) variant of the Boston mechanism in the absence of priorities. Our findings demonstrate the trade-off between efficiency and strategyproofness that is achievable by these mechanisms.
First, we have shown that the naïve Boston mechanism (NBM) strictly imperfectly rank dominates RSD, and we have presented numerical evidence suggesting that strict dominance occurs for a considerable share of type profiles. On the other hand, RSD is strategyproof, while NBM is not even weakly strategyproof. Thus, the efficiency advantages of NBM over RSD can be interpreted as a lower bound for the price of strategyproofness in terms of efficiency.

Second, we have introduced the adaptive Boston mechanism (ABM). We have shown that it satisfies partial strategyproofness, a relaxed notion of strategyproofness, which NBM fails to satisfy. Furthermore, ABM has almost the same efficiency advantages over RSD as NBM: the imperfect rank dominance of ABM over RSD has similar magnitude, and the share of type profiles where RSD rank dominants ABM converges to 0 as the markets get large.

Third, NBM and ABM are incomparable by imperfect rank dominance. Via simulations we have shown that - conditioned on comparability - NBM frequently rank dominates ABM (but not always). These efficiency advantages of NBM over ABM can be viewed as the price of partial strategyproofness one pays when choosing ABM over NBM.

Throughout the paper, it has become apparent that traditional analysis methods frequently fail to differentiate between two matching mechanisms. While a comparison by vulnerability to manipulation is inconclusive for NBM and ABM, except in the most basic case (see Appendix C), partial strategyproofness clearly captures the better incentive properties of ABM. Similarly, all mechanisms we considered are ex-post efficient, but neither ordinally nor rank efficient. Furthermore, NBM and ABM are not even on the efficient frontier. Nonetheless, a comparison by imperfect rank dominance, limit arguments, and numerical analysis revealed a hierarchy between NBM, ABM, and RSD: NBM has the most appeal in terms of rank dominance, but ABM is still more appealing than RSD; on the other hand, NBM exhibits the worst incentive properties, while ABM is at least partially strategyproof, and RSD is fully strategyproof.

In summary, we have found that a decision between NBM, ABM, and RSD requires a non-trivial trade-off between strategyproofness and efficiency. Our new methods facilitate the comparison in situations where traditional methods fail to differentiate, but where a decision is
nonetheless essential. Furthermore, ABM constitutes an attractive design alternative to NBM when incentive properties are a concern: ABM has almost the same efficiency advantages over the strategyproof alternative RSD, but it has significantly better incentive properties than NBM.

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Appendix

A. Review of Partial Strategyproofness

Partial strategyproofness is a scalable requirement that bridges the gap between strategyproofness and weak strategyproofness. Intuitively, partially strategyproof mechanisms are strategyproof, but only on a particular domain restriction, i.e., the agents can still have any type, but their underlying utility functions are constrained. First we define this domain restriction.

**Definition 19** (Uniformly Relatively Bounded Indifference). A utility function \( u \) satisfies uniformly relatively bounded indifference with respect to bound \( r \in [0, 1] \) (URBI\( (r) \)) if for any two objects \( a, b \) with \( u(a) > u(b) \) we have

\[
 r \cdot (u(a) - \min(u)) \geq u(b) - \min(u). 
\]

(29)

We denote by URBI\( (r) \) the set of all utility functions that satisfy uniformly relatively bounded indifference with respect to bound \( r \).

A mechanism is \( r \)partially strategyproof if it is strategyproofness in the domain restricted by the URBI\( (r) \) constraint. Formally:
Definition 20 (r-partial Strategyproofness). Given a setting \((N, M, q)\) and a bound \(r \in [0, 1]\), mechanism \(f\) is \(r\)-partially strategyproof in the setting \((N, M, q)\) if for any agent \(i \in N\), any type profile \(t = (t_i, t_{-i}) \in T^n\), any misreport \(t'_i \in T\), and any utility \(u_i \in \text{URBI}(r) \cap t_i\), we have
\[
\langle u_i, f(t_i) - f(t'_i) \rangle \geq 0.
\]

Sometimes, we simply want to state that a mechanism is \(r\)-partially strategyproof for some non-trivial \(r > 0\) without explicitly naming the parameter \(r\). In this case, we simply say that the mechanism is partially strategyproof. Partial strategyproofness has an axiomatic characterization, which gives a good intuition about the gain obtained from using a strategyproof (rather than partially strategyproof) mechanism. In the following, we define the axioms and give our characterization results, including the key auxiliary concepts, which we also require in our proofs. A more detailed presentation of the axioms and results for the partial strategyproofness concept can be found in (Mennle and Seuken, 2014).

Definition 21 (Neighborhood). The neighborhood of a type \(t\) is the set \(N_t\) of all types \(t'\) such that there exists \(k \in \{1, \ldots, m - 1\}\) with
\[
t : \ a_1 > \ldots > a_k > a_{k+1} > \ldots > a_m,
\]
\[
t' : \ a_1 > \ldots > a_k > a_{k+1} > a_k > \ldots > a_m,
\]
i.e., all the types \(t'\) where the corresponding preference order differs by just a swap of two adjacent objects.

Definition 22 (Contour Sets). For a type \(t : a_1 > \ldots > a_k > \ldots > a_m\), the upper contour set \(U(a_k, t)\) and lower contour set \(L(a_k, t)\) are given by
\[
U(a_k, t) = \{a_1, \ldots, a_{k-1}\} = \{j \in M | j > a_k\},
\]
\[
L(a_k, t) = \{a_{k+1}, \ldots, a_m\} = \{j \in M | a_k > j\},
\]
i.e., the sets of objects that an agent of type \(t\) strictly prefers to or likes strictly less than \(a_k\), respectively.

Next, we present three axioms, which in combination characterize strategyproofness. Each axiom restricts the way in which a misreport from the neighborhood of the agent’s true type, i.e., a misreport involving only a single swap, can affect the allocation of the reporting agent.

Axiom 1 (Swap Monotonic). A mechanism \(f\) is swap monotonic if for any agent \(i \in N\), any type profile \(t = (t_i, t_{-i}) \in T^n\), and any type \(t'_i \in N_{t_i}\) (i.e., in the neighborhood of \(t_i\)) with \(a_k > a_{k+1}\) under \(t_i\) and \(a_{k+1} > a_k\) under \(t'_i\), one of the following holds:

1) \(i\)'s allocation is unaffected by the swap, i.e., \(f(t_i) = f(t'_i)\), or

2) \(i\)'s allocation for \(a_k\) strictly decreases and its allocation for \(a_{k+1}\) strictly increases, i.e.,
\[
f(t_i)(a_k) > f(t'_i)(a_k) \quad \text{and} \quad f(t_i)(a_{k+1}) < f(t'_i)(a_{k+1}).
\]
**Axiom 2** (Upper Invariance). A mechanism $f$ is upper invariant if for any agent $i \in N$, any type profile $t = (t_i, -t_i) \in T^n$, and any type $t'_i \in N_t_i$ with $a_k > a_{k+1}$ under $t_i$ and $a_{k+1} > a_k$ under $t'_i$, $i$’s allocation for the upper contour set $U(a_k, t_i)$ is unaffected by the swap, i.e., $f(t_i)(j) = f(t'_i)(j)$ for all $j \in U(a_k, t_i)$.

**Axiom 3** (Lower Invariance). A mechanism $f$ is lower invariant if for any agent $i \in N$, any type profile $t = (t_i, -t_i) \in T^n$, and any type $t'_i \in N_t_i$ with $a_k > a_{k+1}$ under $t_i$ and $a_{k+1} > a_k$ under $t'_i$, $i$’s allocation for the lower contour set $L(a_{k+1}, t_i)$ is unaffected by the swap, i.e., $f(t_i)(j) = f(t'_i)(j)$ for all $j \in L(a_{k+1}, t_i)$.

The three axioms swap monotonicity, upper invariance, and lower invariance characterize strategyproof mechanisms.

**Fact 1** (Theorem 1 from (Mennle and Seuken, 2014)). A mechanism $f$ is strategyproof if and only if it is swap monotonic, upper invariant, and lower invariant.

Furthermore, relaxing the least intuitive of the axioms, lower invariance, the class of partially strategyproof mechanisms emerges.

**Fact 2** (Theorem 2 from (Mennle and Seuken, 2014)). Given a setting $(N, M, q)$, a mechanism $f$ is $r$-partially strategyproof for some $r \in (0, 1]$ if and only if $f$ is swap monotonic and upper invariant.

Finally, the URBI$(r)$ domain restriction is maximal, i.e., there is no systematically larger set of utilities for which a strategyproofness guarantee can also be provided (given $r$-partial strategyproofness).

**Fact 3** (Theorem 3 from (Mennle and Seuken, 2014)). Given a setting $(N, M, q)$, a bound $r \in (0, 1)$, and a utility function $\tilde{u} \in t$ that violates URBI$(r)$, there exists a mechanism $\tilde{f}$ such that

- $\tilde{f}$ is $r$-partially strategyproof,
- $\tilde{f}$ is not $\{\tilde{u}\}$-partially strategyproof, i.e., there exists a type $t' \neq t$ and reports $t_- \in T^{n-1}$ such that

$$\langle \tilde{u}, f(t, t_-) - f(t', t_-) \rangle < 0.$$  \hspace{1cm} (34)

For any partially strategyproof mechanism, we can now consider the largest admissible indifference bound as a single-parameter measure for how strategyproof the mechanism is.

**Definition 23.** For a setting $(N, M, q)$, the degree of strategyproofness (DOSP) of a mechanism $f$ is given by

$$\rho_{(N, M, q)}(f) = \max\{r \in [0, 1] : f \text{ is } r\text{-partially strategyproof in } (N, M, q)\}. \hspace{1cm} (35)$$

In (Mennle and Seuken, 2014), we have shown that the DOSP is computable and consistent with the vulnerability to manipulation comparison (Pathak and Sönmez, 2013).
B. Proofs

B.1. Proofs from Section 5

B.1.1. Proof of Propositions 2 and 8

Proof of Propositions 2 and 8. NBM and ABM are upper invariant.

Consider two adjacent objects $a, b, a > b$ in some agent’s preference order. Recall that NBM randomly determines an ordering of the agents. Some orderings lead to an allocation where the agent gets an object it strictly prefers to both $a$ and $b$, i.e., an object in the upper contour set of $a$. Let $\Pi^+$ be the set of all such orderings. All other orderings lead to an allocation of $a$, $b$, or an object in the lower contour set of $b$.

Suppose now that the agent swaps $a$ and $b$ in its report. It is obvious that for an ordering in $\Pi^+$ still leads to an allocation of the same object from the upper contour set, since all applications and rejections occur in the same way, independent of the relative ranking of $a$ and $b$. For an ordering from $\Pi^+ = \Pi \setminus \Pi^+$, the agent still does not get an object from the upper contour set, since until the agent applies to $a$ or $b$, the sequence of applications and rejections is not different under the two reports. Therefore, the probabilities of attaining any object in the upper contour set remain unaltered by the swap.

B.1.2. Proof of Propositions 4 and 10

Proof of Propositions 4 and 10. NBM and ABM are ex-post efficient at all type profiles.

Consider a fixed type profile $t$ and some ordering $\pi$ of the agents. Let $x_i$ be the object that agent $i$ receives under NBM and let $N_k$ be the agents who receive $x_i$ in the $k$th round of the mechanism. Construct an ordering $\pi'$ of the agents such that all agents in $N_k$ have higher priority than all agents in $N_k \setminus 1$ for any $k \in \{1, \ldots, m - 1\}$. Then if we use a serial dictatorship based on $\pi'$, the resulting allocation will be the same as from NBM, i.e., $\text{NBM}_\pi(t) = \text{SD}_{\pi'}(t)$.

An analogous argument proves ex-post efficiency of ABM.

B.1.3. Proof of Propositions 7 and 13

Proof of Propositions 7 and 13. NBM is not on the efficient frontier with respect to ordinal dominance, subject to upper invariance. ABM is not on the efficient frontier with respect to ordinal dominance, subject to partial strategyproofness.

In a setting with 4 agents and 4 objects in unit capacity, a type profile satisfies separable wants if the objects and agents can be renamed such that agents 1 and 2 have first choice $a$, agents 3 and 4 have first choice $b$, agents 1 and 3 prefer $c$ to $d$, and agents 2 and 4 prefer $d$ to $c$. Formally,

\begin{align*}
t_1 & : a > \{b, c, d\} \text{ and } c > d, \\
t_2 & : a > \{b, c, d\} \text{ and } d > c, \\
t_3 & : b > \{a, c, d\} \text{ and } c > d, \\
t_4 & : b > \{a, c, d\} \text{ and } d > c.
\end{align*}
The mechanism that improve on NBM arises by adjusting NBM at type profiles with separable wants. Let
\[
NBM^+(t) = \begin{cases} 
PS(t), & \text{if } t \text{ satisfies separable wants}, \\
NBM(t), & \text{else.} 
\end{cases} \tag{36}
\]

Similarly, the mechanism that dominates ABM is given by
\[
ABM^+(t) = \begin{cases} 
PS(t), & \text{if } t \text{ satisfies separable wants}, \\
ABM(t), & \text{else.} 
\end{cases} \tag{37}
\]

Consider a setting with 4 agents and 4 objects and type profile \( t \) that satisfies separable wants. The allocation under PS (after appropriately renaming of the agents and object) is
\[
PS(t) = \begin{pmatrix}
\frac{1}{2} & 0 & \frac{1}{2} & 0 \\
\frac{1}{2} & 0 & 0 & \frac{1}{2} \\
0 & \frac{1}{2} & 0 & \frac{1}{2} \\
0 & \frac{1}{2} & 0 & \frac{1}{2}
\end{pmatrix}. \tag{38}
\]

Furthermore, NBM and ABM split \( a \) equally between 1 and 2 and split \( b \) equally between 3 and 4. Consequently, agents 1 and 2 get no share of \( b \) and agents 3 and 4 get no share of \( a \), just as under PS. Amongst all allocations that split \( a \) and \( b \) in this way, agent 1 prefers the ones that give it maximum probability at \( c \), i.e., \( \frac{1}{2} \). Similarly, agents 2, 3, and 4 prefer their respective allocation under PS to any other allocation that split \( a \) and \( b \) in the same way as PS, NBM, and ABM. Therefore, \( NBM^+ \) and \( ABM^+ \) weakly ordinally dominate NBM and ABM, respectively, and the dominance is strict for type profiles \( t \) with separable wants.

It remains to be shown that \( NBM^+ \) is upper invariant and \( ABM^+ \) is swap monotonic and upper invariant. To validate these axioms, we only need to consider the change in allocation that the mechanism prescribes if some agent swaps two adjacent objects in its reported preference ordering. Starting with any type profile \( t \), the swap produces a new type profile \( t' \). If neither \( t \) nor \( t' \) satisfy separable wants, the mechanisms behave like NBM and ABM, which have the required properties.

For swaps where at least one of the type profiles satisfies separable wants, we can assume without loss of generality that this is \( t \). Such a swap will lead to a new type profile \( t' \) and one of the following three cases:

(i) The type profile still satisfies separable wants.

(ii) The composition of the first choices has changed.

(iii) The type profile no longer satisfies separable wants, but the composition of the first choices has not changed.

By symmetry, we can consider agent 1 whose preference order satisfies
\[
t_1: a > c > d \text{ and } a > b. \tag{39}
\]
Case (i) implies that \( t' \) still satisfies separable wants with respect to the same mappings \( \mu, \nu \). Thus, \( \text{NBM}^+ \) and \( \text{ABM}^+ \) will not change the allocation, and upper invariance (and swap monotonicity) are not violated.

In case (ii), the agent has a new first choice. If the new first choice is \( b \), agent 1 will receive \( b \) with probability \( \frac{1}{3} \) and \( a \) with probability 0 under both \( \text{NBM}(t') \) and \( \text{ABM}(t') \). If the new first choice is \( c \) or \( d \), the agent will receive that object with certainty under \( \text{NBM}(t') \) and \( \text{ABM}(t') \). Both changes are consistent with swap monotonicity or upper invariance.

Finally, in case (iii), the swap must involve \( c \) and \( d \). Since \( t' \) violates separable wants, we have \( \text{NBM}^+(t') = \text{NBM}(t') \) and \( \text{ABM}^+(t') = \text{ABM}(t') \). \( a \) will still be split equally between the agents who rank it first, and the same is true for \( b \). Thus, agent 1 will receive \( \frac{1}{2} \) of \( a \) and 0 of \( b \), which is the same as under \( \text{NBM}^+(t) = \text{ABM}^+(t) = \text{PS}(t) \). The only change can affect the allocation for the objects \( c \) and \( d \). This is consistent with upper invariance. Since under \( t \), agent 1 received a share of \( \frac{1}{2} \) of \( c \), the allocation for \( c \) can not increase under \( t' \). If the allocation for \( c \) remains constant, then the allocation does not change at all, which is consistent with swap monotonicity. If the allocation for \( c \) decreases, the allocation for \( d \) must increase by the same amount, which is also consistent with swap monotonicity. \( \square \)

B.1.4. Proof of Theorem 1

Proof of Theorem 1. \( \text{NBM} \) strictly imperfectly rank dominates \( \text{RSD} \), i.e., \( \text{NBM} \) strictly rank dominates \( \text{RSD} \) at some type profiles, but is never strictly rank dominated by \( \text{RSD} \) at any type profile.

Consider the rank distribution \( d^x = (d^x_1, \ldots, d^x_m) \) of an allocation \( x \) at a type profile \( t \), where \( d^x_k \) denotes the expected number of allocated \( k \)th choices. \( x \) strictly rank-dominates another allocation \( y \) if for all ranks \( K \in \{1, \ldots, m\} \),

\[
\sum_{k=1}^{K} d^x_k \geq \sum_{k=1}^{K} d^y_k, \tag{40}
\]

and the inequality is strict for some value of \( K \). For any lottery decomposition

\[
x = \sum_{l=1}^{L} p_l \cdot x^l \tag{41}
\]

of the allocation \( x \), the rank distribution of \( x \) is the convex combination of the rank distributions of the components \( x^l \), i.e.,

\[
d^x = \sum_{l=1}^{L} p_l \cdot d^{x^l}. \tag{42}
\]

Suppose allocations \( x \) and \( y \) have lottery decompositions

\[
x = \sum_{l=1}^{L} p_l \cdot x^l \text{ and } y = \sum_{l=1}^{L} p_l \cdot y^l. \tag{43}
\]

35
Suppose further that for each \( l \in \{1, \ldots, L\} \) there exists a \( K_l \) such that for all \( k \in \{1, \ldots, K_l - 1\} \)
\[
d^x_k = d^y_k, \tag{44}
\]
and
\[
d^x_{K_l} < d^y_{K_l}. \tag{45}
\]
In words, for each pair of components \((x^l, y^l)\) of the lottery decompositions, the rank-distribution is equal for a number of lower ranks, and under \( y^l \) more agents receive their \( K_l \)th choice than under \( x^l \). Let \( K := \min\{K_l | l \in \{1, \ldots, L\}\} \) be the lowest rank for which the rank distribution of some pair of components differs. Then for all \( k \leq K - 1 \) we have
\[
d^x_k = d^y_k, \text{ and } d^x_K < d^y_K. \tag{46}
\]
This means that \( x \) does not even weakly rank-dominate \( y \).

Keeping the above argument in mind, we will now proceed to show that for \( \text{RSD}(t) \) and \( \text{NBM}(t) \), the rank distributions at \( t \) are either the same, or we find lottery decompositions that satisfy (43), (44), and (45), implying that \( \text{RSD} \) never strictly rank-dominates \( \text{NBM} \).

Consider the pairs of components for the lottery decomposition that arise from selecting an ordering \( \pi \) of the agents uniformly at random and applying each of the mechanisms. Fix an ordering \( \pi \) and let \( x = \text{SD}_\pi(p) \) and \( y = \text{NBM}_\pi(p) \) denote the allocations from applying the mechanisms for the specific ordering \( \pi \). We now show by induction over the number of agents \( n \) that the rank distributions of the pair \((x, y)\) are either equal or satisfy (44) and (45). In order to do this, we consider a slight generalization of the mechanisms, where objects may have zero capacity even initially and agents rank all objects. In particular, under \( \text{NBM} \) agents will apply to empty objects in the first round, if they rank them first.

For \( n = 1 \), the allocations from both mechanisms are equal, hence the rank distributions of \( x \) and \( y \) coincide.

For a general number of agents \( n \), consider the set agents that receive their first choice under each mechanism \( R_x(1) \) and \( R_y(1) \). Suppose, more agents receive their first choice under \( x \) than under \( y \), i.e., there exists \( i \in R_x(1) \setminus R_y(1) \). Under \( \text{NBM}_\pi \), \( i \) applied to its first choice, say \( a \), in the first round. But if \( i \) did not receive \( a \), \( a \) must have been exhausted by higher ranking agents. Therefore, under \( \text{RSD}_\pi \), one of these higher-ranking agents, say \( i' \), must have received an object it prefers to \( a \). But since \( i' \) received \( a \) in the first round of \( \text{NBM}_\pi \), \( a \) must be its first choice - a contradiction. Thus, \( R_x(1) \subseteq R_y(1) \).

If \( R_x(1) \not\subseteq R_y(1) \), we are done, since in that case \( \text{RSD}_\pi \) allocates strictly less first choices than \( \text{NBM}_\pi \). If \( R_x(1) = R_y(1) \), we consider a new setting, where the allocated agents and their allocated objects are not present. The allocations for the remaining agents can be obtained by applying the generalized mechanisms to the reduced setting, but where all agents rank their (now exhausted) first choice last. The reduced setting contains at least one agent less, and hence we can invoke the induction hypothesis. This concludes the proof. \( \square \)
B.2. Proofs from Section 6

B.2.1. Proof of Theorem 2

Proof of Theorem 2. ABM is swap monotonic.

Suppose the agent is thinking about swapping $x$ and $y$ in its report, i.e., going from

$$\begin{align*}
t : a_1 > \ldots > a_K > x > y > b_1 > \ldots > b_L
\end{align*}$$

to

$$\begin{align*}
t' : a_1 > \ldots > a_K > y > x > b_1 > \ldots > b_L
\end{align*}$$

Let $\pi$ be any priority ordering of the agents. Consider the following cases:

| Case | Allocation from reporting $t$ | Allocation from reporting $t'$ | Possible |
|------|-------------------------------|-------------------------------|----------|
| I    | $a_k$                         | $a_k$                         | Yes      |
| II   | $a_k$                         | $j, j \neq a_k$               | ?        |
| III  | $x$                           | $x$                           | Yes      |
| IV   | $x$                           | $y$                           | Yes      |
| V    | $x$                           | $b_l$                         | Yes      |
| VI   | $y$                           | $y$                           | Yes      |
| VII  | $y$                           | $x$                           | No       |
| VIII | $y$                           | $b_l$                         | No       |
| IX   | $b_l$                         | $y$                           | Yes      |
| X    | $b_l$                         | $x$                           | No       |
| XI   | $b_l$                         | $b_l$                         | Yes      |
| XII  | $b_l$                         | $b_{l'}, l \neq l'$          | ?        |

- Cases I, III, VI, XI do not change the allocation.
- Cases VII, VIII, X are obviously impossible, due to the way the mechanism works.
- Case II is impossible. The argument is the same as that for upper invariance.
- Case XII is also impossible:
  - If $i$ applied to both $x$ and $y$, the agent would obtain $y$ with the swap.
  - If $i$ skipped $x$ (due to insufficient capacity), it would apply to $y$ in the same round after the swap, still not get it, and hence continue as under truthful preferences.
  - If $i$ applied to $x$ but skipped $y$, it may now apply to $y$, but fail to get it (otherwise it gets $y$ and we are not in a case XII). In the next round, $x$ is already exhausted, otherwise $i$ would have obtained $x$ under truthful reporting.
  - Finally, $i$ may skip both objects (due to insufficient capacities), in which case it will still skip both objects under the misreport.

Thus, we see that the allocation for $y$ can only increase (weakly) and the allocation for $x$ can only decrease (weakly). In case the ordering $\pi$ induces case IV, the mechanism will be swap...
consistent, since the changes for \( x \) and \( y \) become strict. We now show that if cases V or IX occur under realization \( \pi \), then case IV occurs at least once under some (possibly different) realization \( \pi' \). This is sufficient for swap monotonicity, because ABM chooses an ordering uniformly at random form the space of all orderings.

Suppose case V: \( i \) get \( x \), but after the swap, gets neither \( x \) nor \( y \). This means that \( y \) was not exhausted when \( i \) applied to \( x \) under truthful reporting. Otherwise, \( y \) would be skipped, in which case \( i \) would still get \( x \). Thus, let \( I_y \) be the set of agents who apply to \( y \) when \( i \) applies to \( x \) under truthful reporting. \( y \) is exhausted by the highest ranking of these agents \( I_y^+ \subseteq I_y \).

Consider a realization \( \pi' \) where all other agents \( I_x \) as well as the lowest ranking agent in \( I_y^+ \) are ranked below \( i \) (at \( y \)). Then:

- \( i \) does not get any of the objects \( a_k \), since this would violate upper invariance.
- \( i \) gets \( x \) under truthful reporting, but gets \( y \) under false reporting (case IV).

Suppose case IX: \( i \) did not get \( y \) before, but under the swap gets \( y \). This means that \( x \) is not exhausted in the round when \( i \) applies to \( x \) under truthful reporting (otherwise \( i \) would skip \( x \) and obtain \( y \)). We consider the set of successful and unsuccessful applicants at \( x \), \( I_x^- \), and find a realization \( \pi' \) that ranks all of \( I_x^- \) and the last agent in \( I_x^- \) below \( i \). As in case V, \( \pi' \) induces case IV.

\[ \square \]

### B.2.2. Proof of Proposition 9

**Proof of Proposition 9.** For settings where \( n = m, q_j = 1 \) for all \( j \in M \)

\[ \lim_{n \to \infty} \rho(N,M,q_j)(ABM) = 0. \]  
(47)

Consider the type profile

\[ t_1 : \quad a_1 > a_2 > a_3 > \ldots > a_n, \]  
(48)

\[ t_2, \ldots, t_{n-1} : \quad a_1 > a_2 > a_4 > a_5 > \ldots > a_n > a_3, \]  
(49)

\[ t_n : \quad a_2 > a_4 > \ldots > a_n > a_1 > a_3. \]  
(50)

Agent 1 has a chance to obtain object \( a_1 \) of \( \frac{1}{n-1} \) and receives \( a_3 \) otherwise. Thus, the expected utility from truthful reporting for agent 1 is

\[ \frac{1}{n-1} u_1(a_1) + \frac{n-2}{n-1} u_1(a_3). \]  
(51)

If agent 1 reports

\[ t'_1 : \quad a_2 > a_3 > \ldots \]  
(52)

instead, it will get object \( a_2 \) with probability \( \frac{1}{2} \) and \( a_3 \) otherwise. This yields utility of

\[ \frac{1}{2} u_1(a_2) + \frac{1}{2} u_1(a_3). \]  
(53)
But for any utility function \( u \) with \( u_1(a_1) > u_1(a_2) > \ldots \) the gain in expected utility from misreporting is lower bounded by

\[
\frac{1}{2} u_1(a_2) + \frac{1}{2} u_1(a_3) - \frac{1}{n-1} u_1(a_1) - \frac{n-2}{n-1} u_1(a_3) \geq \left( \frac{1}{2} u_1(a_2) - u_1(a_3) \right) - \frac{1}{n-1} u_1(a_1),
\]

which is positive for sufficiently large \( n \).

\[\square\]

**B.2.3. Proof of Proposition 15**

*Proof of Proposition 15.* For any fixed number of objects \( m \geq 3 \) and any \( \epsilon > 0 \), there exists a \( q_{\min} \in \mathbb{N} \), such that for any capacity vector \( \mathbf{q} = (q_1, \ldots, q_m) \) with \( q_j \geq q_{\min} \) for all \( j \in M \) and \( n = \sum_{j \in M} q_j \) agents, and for \( \mathbf{t} \) chosen uniformly at random, the probability that RSD(\( \mathbf{t} \)) is first-choice-maximizing is smaller than \( \epsilon \).

For a given type profile \( \mathbf{t} \in T^n \), the first choice profile \( \mathbf{k}^t = (k_j^t)_{j \in M} \) is the vector of non-negative integers, where \( k_j^t \) represents the number of agents who have first choice \( j \) under type profile \( \mathbf{t} \).

For a fixed setting \( (N, M, \mathbf{q}) \), we consider a uniform distribution on the space of type profiles \( T^n \). Under a given type profile \( \mathbf{t} \in T^n \), we consider the first choice profile \( \mathbf{k}^t = \mathbf{k} \), dropping the type profile in the notation. We say that an object \( j \in M \) is

- **un-demanded** if \( k_j = 0 \),
- **under-demanded** if \( k_j \in \{1, \ldots, q_j - 1\} \),
- **exhaustively demanded** if \( k_j = q_j \),
- and **over-demanded** if \( k_j > q_j \).

For any first choice profile \( \mathbf{k}^t \), one of the following cases holds:

(I) There is at least one un-demanded object.

(II) No object is over-demanded, i.e., all objects are exhaustively demanded.

(III) No object is un-demanded, there is at least one over-demanded object and at least one exhaustively demanded object.

(IV) No object is un-demanded, there is exactly one over-demanded object, and all other objects are under-demanded.

We will show that for fixed \( m \) and increasing minimum capacity, the probabilities for cases (I) and (II) become arbitrarily small. We will further show that in cases (III) and (IV), the probabilities that RSD allocates the maximum number of first choices become arbitrarily small.
(I) The probability that under a randomly chosen type profile at least one object is undemanded is upper-bounded by
\[ \frac{m}{m^n} (m-1)^n \leq m \left( \frac{m-1}{m} \right)^n, \]
which converges to 0 as \( n = \sum_{j \in M} q_j \geq mq_{\text{min}} \) becomes large (where \( m \) is fixed).

(II) Let \( \tilde{q} = \frac{n}{m} \). Without loss of generality, \( \tilde{q} \) can be chosen as a natural number (otherwise, we increase the capacity of the object with least capacity until \( n \) is divisible by \( m \)). The probability that under a randomly chosen type profile all objects are exhaustively demanded is
\[ \frac{\binom{n}{q_1, \ldots, q_m}}{m^n} \leq \frac{\binom{\tilde{q}m}{\tilde{q}m \tilde{q}m}}{(\tilde{q}m \tilde{q}m \tilde{q}m)} \leq \frac{m^{\tilde{q}}}{\tilde{q}^{m-1}}, \]
which converges to 0 as \( \tilde{q} \geq q_{\text{min}} \) becomes large (where \( m \) is fixed).

(III) If one object \( a \) is over-demanded and another object \( b \) is exhaustively demanded, than no agent with first choice \( a \) can have \( b \) as second choice. Otherwise, there exists an order of the agent such that an agent with first choice \( a \) will get \( b \). In that case, \( b \) is not allocated entirely to agents with first choice \( b \), and hence, the allocation cannot maximize the number of first choices. Thus, the probability that the \( k_{\tilde{q}} \) agents who have first choice \( a \) all have a second choice different from \( b \) (conditional on the first choice profile) is
\[ \left( \frac{m-2}{m-1} \right)^{k_{\tilde{q}}} \leq \left( \frac{m-2}{m-1} \right)^{q_{\text{min}}} \].

This becomes arbitrarily small for increasing \( q_{\text{min}} \). Thus, the probability that the maximum number of first choices is allocated by RSD, conditional on case (III) becomes small.

(IV) Suppose that for some type profile consistent with case (IV), RSD allocates the maximum number of first choices. Let 1 be the object that is over-demanded and let 2, \ldots, \( m \) be the under-demanded objects. Then the maximum number of first choices is allocated if and only if
- \( q_1 \) agents with first choice 1 receive 1, and
- all agents with first choices 2, \ldots, \( m \) receive their respective first choice.

If RSD maximizes the number of first choices, then for any ordering of the agents, the maximum number of first choices must be allocated, i.e., the two conditions are true. If the agents with first choice 1 get to pick before all other agents, then they exhaust 1 and get at most \( q_j - k_j \) of the objects \( j \); otherwise, if they got more than \( q_j - k_j \) of object \( j \), then some agent with first choice \( j \) would get a worse choice, which violates first choice maximization.
After any \( q_1 \) of the \( k_1 \) agents with first choice 1 consume object 1, there are \( k_1 - q_1 \) agents left, which will consume other objects. Since \( n = \sum_{j \in M} q_j = \sum_{j \in M} k_j \), we get that

\[
k_1 - q_1 = n - \sum_{j \neq 1} k_j - (n - \sum_{j \neq 1} q_j) = \sum_{j \neq 1} q_j - k_j.
\]

(59)

Therefore, the second choice profile of these \( k_1 - q_1 \) agents must be \((l_2, \ldots, l_m)\), where \( l_j = q_j - k_j \geq 1 \). In addition, some agent \( i' \) who consumed 1 has second choice \( j' \), and some agent \( i'' \) with first choice 1 gets its second choice \( j'' \neq j' \). If we exchange the place of \( i' \) and \( i'' \) in the ordering, \( i'' \) will get 1 and \( i' \) will get \( j' \). But then \( q_{j'} - k_{j'} + 1 \) agents with first choice 1 get their second choice \( j' \). Therefore, when the agents with first choice \( j' \) get to pick their objects, there are only \( k_{j'} - 1 \) copies of \( j' \) left, which is not sufficient. Thus, we have constructed an ordering of the agents under which the number of allocated first choices is not maximized. This implies that for any type profile with first choice profile satisfying case (IV), RSD never allocates the maximum number of first choices.

Combining the arguments for all cases, we can find \( q_{\min} \) sufficiently high, such that we can estimate the probability that RSD maximizes first choices (“RSD mfc.”) by

\[
P[RSD\ mfc.] = P[RSD\ mfc.|(I)]P[(I)] + P[RSD\ mfc.|(II)]P[(II)]
\]

\[+P[RSD\ mfc.|(III)]P[(III)] + P[RSD\ mfc.|(IV)]P[(IV)] \leq P[(I)] + P[(II)] + P[RSD\ mfc.|(III)] + P[RSD\ mfc.|(IV)] \leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} + 0 = \epsilon.
\]

(63)

\[P[RSD\ mfc.] \leq 3\epsilon.
\]

\[P[RSD\ mfc.] \leq \epsilon.
\]

B.2.4. Proof of Proposition 16

Proof of Proposition 16. For any \( \epsilon > 0 \), there exists \( n \in \mathbb{N} \), such that for any setting \((N, M, q)\) with \( \#M = \#N \geq n \) and \( q_j = 1 \) for all \( j \in M \), and for \( t \) chosen uniformly at random, the probability that RSD\( (t) \) is first-choice-maximizing is smaller than \( \epsilon \).

Given a preference profile \( t \), there exists a unique number \( d_{1}^{\max} \) of first choices that can be allocated to a set of agents with preference profile \( t \). For any ordering of the agents \( \pi \) let \( d_1(\text{SD}_\pi(t)) \) denote the number of allocated first choices under the serial dictatorship \( \text{SD}_\pi(t) \), i.e., the deterministic allocation chosen by RSD for the ordering \( \pi \). Obviously, \( d_{1}^{\max} \geq d_1(\text{SD}_\pi(t)) \), i.e., \( \text{SD}_\pi(t) \) allocates no more than the maximum number of first choices. The expected number of first choices allocated by RSD is then given by the average

\[
d_1(\text{RSD}(t)) = \frac{1}{n!} \sum_{\pi \text{ ordering of } N} d_1(\text{SD}_\pi(t)) \leq d_{1}^{\max}.
\]

(64)

If for any ordering \( \pi' \) we have \( d_1(\text{SD}_{\pi'}(t)) < d_{1}^{\max} \), it follows immediately from (64) that \( d_1(\text{RSD}(t)) < d_{1}^{\max} \). Put differently, RSD allocates the maximum expected number of first choices only if it allocates the maximum number of first choices for every ordering \( \pi \).
We now introduce no overlap, a necessary condition on the type profile that ensures that RSD allocates the maximum expected number of first choices. Conversely, if a type profile violates no overlap, RSD will not allocate the maximum expected number of first choices. Then we show that the share of type profiles that exhibit no overlap vanishes as $n$ becomes large.

Consider a setting where three agents have preferences $ąąą$, $ąąą$, and $ąąą$. The maximum number of first choices that can be allocated is 2, e.g. giving $a$ to 1 and $b$ to 3. But for the ordering $π = (1, 2, 3)$, agent 1 will get $a$ and agent 2 will get $c$. But then agent 3 cannot take $c$, and consequently RSD will not allocate the maximum expected number of first choices. Intuitively, no overlap means that this cannot happen.

The proof requires some more formal definitions: for convenience, we will enumerate the set $M$ of objects by the set $\{1, \ldots, n\}$. For a fixed type profile $t$ let $k^t = (k^t_1, \ldots, k^t_n)$ be a vector of non-negative natural numbers such that any object $j$ is the first choice of $k^t_j$ agents under the type profile $t$. $k^t$ is called the first choice profile of the type profile $t$.

For first choice profile $k$ and object $j$ we define

$$w_k(j) := \begin{cases} 1, & \text{if } k_j \geq 1 \\ 0, & \text{else} \end{cases}$$

and

$$o_k(j) := \begin{cases} 1, & \text{if } k_j \geq 2 \\ 0, & \text{else} \end{cases}.$$ 

(66)

$w$ indicates whether $j$ is not demanded as a first choice by at least one agent, and $o$ indicates whether $j$ is over-demanded, i.e., at least two agent rank $j$ as their first choice. Further, we define

$$W_k := \sum_{j \in M} w_k(j), \quad O_k := \sum_{j \in M} o_k(j), \quad C_k := \sum_{j \in M} k_j \cdot o_k(j).$$

$W_k$ is the number of objects that are demanded by at least one agent, $O_k$ is the number of over-demanded objects, and $C_k$ is the number of agents competing for over-demanded objects.

Finally, a type profile $t$ exhibits overlap if there exists an agent $i \in N$ with first choice $j_1 = \text{ch}(t, 1)$ and second choice $j_2 = \text{ch}(t, 2)$, such that $o_k(t)(j_1) = 1$ and $w_k(t)(j_2) = 1$, i.e., agent $i$’s first choice is over-demanded and its second choice object is demanded as a first choice by some other agent. If a type profile exhibits overlap, a situation as in (65) may arise, and therefore, RSD will not allocate the maximum expected number of first choices. Conversely, no overlap in $t$ is a necessary condition for RSD$(t)$ to allocate the maximum expected number of first choices. By showing that the share of type profiles exhibiting no overlap becomes small for increasing $n$, this implies the result.

Consider a uniform distribution on the type space, i.e., all agents draw their preference order independently and uniformly at random from the space of all possible preference orders. Then the statement that the share of type profiles exhibiting no overlap becomes small is equivalent to the statement that the probability of selecting a type profile with no overlap converges to 0. Using conditional probability, we can write the probability that a type profile is without overlap as

$$P[t \text{ no overlap}] = \sum_k P[k = k^t] \cdot P[t \text{ no overlap} | k = k^t].$$ 

(67)
The number of type profiles that have first choice profile \( \mathbf{k} = (k_1, \ldots, k_n) \) is proportional to the number of ways to distribute \( n \) unique balls (agents) across \( n \) urns (first choices), such that \( k_j \) balls end up in urn \( j \). Thus,

\[
\Pr[\mathbf{k} = \mathbf{k}^t] = \frac{(k_1 \ldots k_n)(n-1)!^n}{(n!)^n} = \frac{(k_1 \ldots k_n)}{n^n}.
\]  
(68)

In order to ensure no overlap, an agent with an over-demanded first choice cannot have as a second choice an object that is the first choice of any other agent. Agents whose first choice is not over-demanded can have any object (except for their first choice) as second choice. Thus, given a first choice profile \( \mathbf{k} \), the conditional probability of no overlap is

\[
\Pr[\text{no overlap} \mid \mathbf{k} = \mathbf{k}^t] = \prod_{j \in M} \left( 1 - o_k(j) + o_k(j) \left( \frac{n-W_k}{n-1} \right)^{k_j} \right) \right) \right) \]  
(69)

\[
= \left( \frac{n-W_k}{n-1} \right)^{\sum_{j \in M} k_j o_k(j)} = \left( \frac{n-W_k}{n-1} \right)^{C_k} \]  
(70)

\[
= \left( \frac{C_k - O_k}{n-1} \right)^{C_k}, \]  
(71)

where the last equality holds, since \( n - W_k = n - (C_k + O_k) = C_k - O_k \). Thus, the probability of no overlap can be determined as

\[
\Pr[\text{no overlap}] = \frac{1}{n^n} \sum_{\mathbf{k}} \left( \frac{n}{k_1, \ldots, k_n} \right) \left( \frac{C_k - O_k}{n-1} \right)^{C_k}.
\]  
(72)

\( C_k \) is either 0 or \( \geq 2 \), since a single agent cannot be in competition. If no agents compete \((C_k = 0)\), all must have different first choices. Thus, for \( \mathbf{k} = (1, \ldots, 1) \), the term in the sum is

\[
\left( \frac{n}{1, \ldots, 1} \right) \cdot 1 = n!.
\]  
(73)

Using this and sorting the terms for summation by \( c \) for \( C_k \) and \( o \) for \( O_k \), we get

\[
\Pr[\text{no overlap}] = \frac{1}{n^n} \left[ n! + \sum_{c=2}^n \sum_{o=1}^{\left\lfloor \frac{c}{2} \right\rfloor} \binom{c}{o} \sum_{k: C_k = c, O_k = o} \binom{n}{k_1, \ldots, k_n} \right].
\]  
(74)

Consider the sum

\[
\sum_{\mathbf{k}: C_k = c, O_k = o} \binom{n}{k_1, \ldots, k_n}
\]  
(75)

in (74): with a first choice profile \( \mathbf{k} \) that satisfies \( C_k = c \) and \( O_k = o \) there are exactly \( o \) over-demanded objects (i.e., objects \( j \) with \( k_j \geq 2 \)), \( n-c \) singly-demanded objects \( (k_j = 1) \),
\[ \sum_{k: C_k = c, O_k = o} \binom{n}{k_1, \ldots, k_n} = \binom{n}{c-o} \sum_{k': (k'_1, \ldots, k'_{n-c+o})} \binom{n}{k'_1, \ldots, k'_{n-c+o}} \]  

(76)

\[ = \binom{n}{c-o} \left( \frac{n-c+o}{n-c} \right) \frac{n!}{c!} \sum_{k'=(k''_1, \ldots, k''_o), k''_j \geq 2} \binom{c}{k''_1, \ldots, k''_o} \]  

(77)

The first equality holds, because we simply choose \( c-o \) of the \( n \) objects to be un-demanded, and

\[ \binom{n}{k_1, \ldots, k_{r-1}, 0, k_{r+1}, \ldots, k_m} = \binom{n}{k_1, \ldots, k_{r-1}, k_{r+1}, \ldots, k_m}. \]  

(78)

The second equality holds, because we select the \( n-c \) singly-demanded objects from the remaining \( n-c+o \) objects as well as the \( n-c \) agents to demand them. The sum (77) is equal to the number of ways to distribute \( c \) unique to \( o \) unique urns such that each urn contains at least 2 balls. This in turn is equal to

\[ o! \{ \{ c \} \} \{ \{ o \} \}, \]  

(79)

where \( \{ \{ \} \} \) denotes the 2-associated Stirling number of the second kind. This number represents the number of ways to partition \( c \) unique balls such that each partition contains at least 2 balls. The factor \( o! \) in (79) is included to make the partitions unique. \( \{ \{ \} \} \) is upper-bounded by \( \{ \} \), the Stirling number of the second kind, which represents the number of ways to partition \( c \) unique balls such that no partition is empty. Furthermore, the Stirling number of the second kind has the upper bound

\[ \left\{ \begin{array}{l} c \\ o \end{array} \right\} \leq \binom{c}{o} o^{c-o}. \]  

(80)

Thus, the sum in (77) can be upper-bounded by

\[ \sum_{k''=(k''_1, \ldots, k''_o), k''_j \geq 2} \binom{c}{k''_1, \ldots, k''_o} \leq o! \binom{c}{o} o^{c-o}. \]  

(81)

Combining all the previous observations, we can estimate the probability \( P[t \text{ no overlap}] \) from (74) by

\[ P[t \text{ no overlap}] = \frac{1}{n^n} \left[ n! + \sum_{c=2}^{n} \left( \frac{c-o}{n-1} \right)^c \binom{n}{c-o} \binom{n-c+o}{n-c} \right]. \]  

(82)

The Stirling approximation yields

\[ \sqrt{2\pi e^{\frac{1}{12n+1}}} \leq \frac{n!}{\sqrt{n} \left( \frac{n}{e} \right)^n} \leq \sqrt{2\pi e^{\frac{1}{12n}}}, \]  

(83)
and therefore $n! \approx \left(\frac{n}{e}\right)^n \sqrt{n}$ up to a constant factor. Using this, we observe that the first term in (82) converges to 0 as $n$ increases, i.e.,

$$\frac{n!}{n^n} \approx \frac{\sqrt{n}}{e^n} \to 0 \text{ for } n \to \infty.$$  \hfill (84)

Now we need to estimate the double sum in (82):

$$\frac{1}{n^n} \sum_{c=2}^{n} \sum_{o=1}^{\lfloor \frac{n}{c} \rfloor} \left( \frac{c-o}{n-1} \right)^c \left( \frac{n}{c} \right) \left( \frac{n-c+o}{n-c} \right) \left( \frac{c}{o} \right) \frac{n! o^{c-o}}{c!}$$

$$\left( \frac{c-o}{n-1} \right)^c \left( \frac{n}{c} \right) \left( \frac{n-c+o}{n-c} \right) \left( \frac{c}{o} \right) \frac{n! o^{c-o}}{c!}$$

$$\leq \left[ \sqrt{n} \left( \frac{n}{n-1} \right)^{n-1} \left( \frac{n-1}{n} \right) \right] \frac{1}{e^n} \sum_{c=2}^{n} \sum_{o=1}^{\lfloor \frac{n}{c} \rfloor} \left( \frac{n}{c} \right) \left( \frac{c}{o} \right) \frac{1}{n^c} \cdot \frac{(c-o)c^{c-o}}{(c-o)^{c-o}} e^{c-o}$$

$$\leq \left[ e\sqrt{n} \right] \frac{1}{e^n} \sum_{c=2}^{n} \sum_{o=1}^{\lfloor \frac{n}{c} \rfloor} \left( \frac{n}{c} \right) \left( \frac{c}{o} \right) \frac{1}{n^c} \cdot (c-o)^o o^{c-o} e^{c-o},$$

where we use that $(1 + \frac{z}{n})^n \leq e^z$. Using the binomial theorem and the fact that the function $o \mapsto (c-o)^o o^{c-o}$ is maximized by $o = \frac{c}{2}$, we can estimate (89) by

$$\left[ e\sqrt{n} \right] \frac{1}{e^n} \sum_{c=2}^{n} \left( \frac{n}{c} \right) \left( \frac{c}{o} \right) \frac{1}{n^c} \cdot (c-o)^o o^{c-o} e^{c-o},$$

$$\leq \left[ e\sqrt{n} \right] \frac{1}{e^n} \sum_{c=2}^{n} \left( \frac{n}{c} \right) \left( \frac{c}{o} \right) \frac{1}{n^c} \cdot (c-o)^o o^{c-o} e^{c-o},$$

with $\alpha = \frac{(1+e)}{2}$. To estimate the sum in (91), we first consider even $n$ and note the following:

- $\alpha = \frac{(1+e)}{2} \approx 1.85914 \ldots < e$, and therefore, the last term of the sum for $c = n$ can be ignored as $\left( \frac{n}{c} \right) \left( \frac{c}{o} \right) \frac{1}{n^c} \to 0$ for $n \to \infty$.

- $\left( \frac{n}{c} \right) = \left( \frac{n}{n-c} \right)$, and therefore, both terms $\left( \frac{n}{c} \right) \left( \frac{c}{o} \right)^c$ and $\left( \frac{n}{c} \right) \left( \frac{n-c}{n} \right)^n$ have the same binomial coefficient in the sum.

- The idea is to estimate the sum of both terms by an exponential function of the form $c \mapsto e^{mc+b}$, where $m$ and $b$ depend only on $n$ and $\alpha$. 

45
Indeed, the log of the sum, the function \( c \mapsto \log \left( \left( \frac{\alpha}{n} \right)^c + \left( \frac{n-c}{n} \right)^{n-c} \right) \), is strictly convex and on the interval \([1, \frac{n}{2}]\) it is upper-bounded by the linear function

\[
f(c) = \left( \frac{\log(4)}{n} - \log(2\alpha) \right) c + n \log(\alpha).
\]  

(92)

Thus,

\[
\left( \frac{n}{c} \right) \left( \frac{\alpha}{n} \right)^c + \left( \frac{n}{n-c} \right) \left( \frac{n-c}{n} \right)^{n-c} \leq \left( \frac{n}{c} \right) e^{f(c)}.
\]

(93)

We can bound (91) by

\[
\left[ e^{\sqrt{n}} \right] \frac{1}{e^n} \sum_{c=1}^{\frac{n}{2}} \left( \frac{n}{c} \right) e^{f(c)} = \left[ e^{\sqrt{n}} \right] \frac{1}{e^n} \sum_{c=1}^{\frac{n}{2}} \left( \frac{n}{c} \right) 4\pi \alpha^n \left( \frac{1}{2\alpha} \right)^c \\
\leq \left[ 4e^{\sqrt{n}} \right] \alpha^n \left( 1 + \frac{1}{2\alpha} \right)^n \\
= \left[ 4e^{\sqrt{n}} \right] \left( \frac{\frac{1}{2} + \alpha}{e} \right)^n \approx \left[ 4e^{\sqrt{n}} \right] \left( \frac{2.35914\ldots}{e} \right)^n. 
\]

(94)

(95)

(96)

Since 2.35914 < \( e \), the exponential convergence of the last term dominates the divergence of the first terms, which is of the order \( \sqrt{n} \), and the expression converges to 0.

For odd \( n \) the argument is essentially the same, except that we need to also consider the central term (for \( c = \frac{n}{2} + 1 \)) separately.

\[
\left( \frac{n}{2} + 1 \right) \left( \frac{\alpha}{\frac{n}{2} + 1} \right)^{\frac{n}{2} + 1} \leq 2^n \left( \sqrt{\alpha \left( \frac{1}{2} + \frac{1}{n} \right)} \right)^n \cdot \alpha \left( \frac{1}{2} + \frac{1}{n} \right) \\
= \left( \sqrt{\alpha \left( 2 + \frac{4}{n} \right)} \right)^n \cdot \alpha \left( \frac{1}{2} + \frac{1}{n} \right). 
\]

(97)

(98)

With \( \sqrt{\alpha \left( 2 + \frac{4}{n} \right)} \approx 1.92828\ldots < e \), the result follows for odd \( n \) as well.

C. Comparison by Vulnerability to Manipulation

In this section we compare NBM and ABM by their vulnerability to manipulation, considering the comparison “more manipulable” and “strongly more manipulable” in the case of strict and coarse priorities. We show that this comparison is inconclusive, except in the simplest case. Our findings do not diminish the value of the the vulnerability to manipulation concept, but they highlights the fact that there is no “one size fits all” solution, and we need a wide array of concepts. For NBM and ABM, the partial strategyproofness concept is able to clearly differentiate between the two mechanisms.
C.1. Formalization of Vulnerability to Manipulation

We first review notions for the comparison of mechanisms by their vulnerability to manipulation, introduced by Pathak and Sönmez (2013). Their definitions assume that the type reports can reflect all information about agents’ preferences. However, with underlying vNM utility functions, information about the preference intensities is hidden from ordinal mechanisms. In the case of deterministic mechanisms, this information is redundant, because determining which allocation is preferred by an agent is possible purely based on the ordinal preference order. To study probabilistic mechanisms, we define the concepts for comparison in such a way that they coincide with the definitions from (Pathak and Sönmez, 2013) in the deterministic case.

Definition 24 (Manipulable). \( f \) is manipulable by \( i \) at utility profile \( u = (u_1, \ldots, u_n) \in t_1 \times \ldots \times t_n \), \( t = (t_1, \ldots, t_n) \in T^n \) if there exits a type \( t'_i \in T \) such that \( i \) has higher expected utility from reporting \( t'_i \) than from reporting \( t_i \) truthfully, holding the reports from the other agents constant. Formally, 
\[
\langle u_i, f_{i}(t_i, t_{-i}) - f_i(t_i', t_{-i}) \rangle < 0.
\]

\( f \) is manipulable at \( u \) if there exits an agent \( i \) for which \( f \) is manipulable by \( i \) at \( u \).

Using this notion of manipulability, the weakest requirement that allows a weak or strict comparison of (probabilistic) mechanisms is the following.

Definition 25 (More Manipulable). \( g \) is as manipulable as \( f \) if for every utility profile \( u \) at which \( f \) is manipulable, \( g \) is also manipulable at \( u \). \( g \) is more manipulable than \( f \) if \( g \) is as manipulable as \( f \) and there exists a utility profile \( u \) at which \( g \) is manipulable, but \( f \) is not.

Note that the agents who can manipulate the mechanisms \( f \) and \( g \) in the previous definition do not necessarily have to be the same. The next concept includes this requirement.

Definition 26 (Strongly More Manipulable). \( g \) is as strongly manipulable as \( f \) if for every utility profile \( u \) at which \( f \) is manipulable by an \( i \), \( g \) is also manipulable at \( u \) by \( i \). \( g \) is strongly more manipulable than \( f \) if \( g \) is as strongly manipulable as \( f \) and there exists a utility profile \( u \) at which \( g \) is manipulable by \( i \), but \( f \) is not manipulable by \( i \).

The comparison concepts from Definitions 25 and 26 yield a best response notion of manipulability, i.e., under a more manipulable mechanism, if truthful reporting is a best response, then truthful reporting is also a best response under the less manipulable mechanism.

C.2. Failure of Strong Comparison for Strict Priorities

In the case of a single strict priority order \( \pi \), Dur (2013) showed that \( \text{NBM}_\pi \) is more manipulable than \( \text{ABM}_\pi \). The following examples show that this result cannot be strengthened using the strong distinction, i.e., \( \text{NBM}_\pi \) is not strongly more manipulable than \( \text{ABM}_\pi \). The following examples shows that there exist type profiles and priority orderings such that

- \( \text{NBM}_\pi \) is manipulable at \( t \) by \( i \), but \( \text{ABM}_\pi \) is not, and
• $ABM_{\pi'}$ is manipulable at $t'$ by $i$, but $NBM_{\pi'}$ is not.

**Example 1** ($ABM_{\pi}$ not as strongly manipulable as $NBM_{\pi}$). Consider a setting with $N = \{1, 2, 3, 4\}$, $M = \{a, b, c, d\}$, unit capacities, the type profile

$$
\begin{align*}
t_1 & : a > \ldots, \\
t_2 & : b > \ldots, \\
t_3 & : a > b > c > d, \\
t_4 & : a > c > \ldots, 
\end{align*}
$$

and the priority order $\pi = \text{id}$. Then agent 3 will get $d$ under $NBM_{\pi}$, but if agent 3 reports

$$
\begin{align*}
t_3' & : a > c > \ldots, 
\end{align*}
$$

it will get $c$ instead, a strict improvement. Under $ABM_{\pi}$ and truthful reporting, $b$ is exhausted by agent 2 in the first round, and therefore, both agents 3 and 4 apply for $c$ in the second round, where 3 gets $c$. It is clear that due to its low priority, agent 3 cannot get a better object than $c$ under $ABM_{\pi}$ with any misreport. Thus, truthful reporting is a best response for agent 3 under $ABM_{\pi}$, but not under $NBM_{\pi}$

**Example 2** ($NBM_{\pi}$ not as strongly manipulable as $ABM_{\pi}$). Consider a setting with $N = \{1, 2, 3, 4, 5\}$, $M = \{a, b, c, d, e\}$, unit capacities, the type profile

$$
\begin{align*}
t_1 & : a > \ldots, \\
t_2 & : b > \ldots, \\
t_3 & : d > \ldots, \\
t_4 & : a > b > d > c > e, \\
t_5 & : a > b > c > d > e, 
\end{align*}
$$

and the priority order $\pi = \text{id}$. Under $NBM_{\pi}$ and truthful reporting, agent 5 will get $c$, and there is no false report that will provide a better object, since $a$ and $b$ are exhausted in the first round. However, under $ABM_{\pi}$, agent 5 will get $e$. By reporting $c$ as its first choice instead, agent 5 can get $c$, which is better than $e$.

**C.3. Failure of Comparison for Coarse Priorities**

Recall that a probabilistic mechanism $g$ is as manipulable as $f$ if it is manipulable at least at the same utility profiles as $f$. Even though this is arguably the weakest way in which we can extend the weakest comparison concept from (Pathak and Sönmez, 2013) to probabilistic mechanisms, we find that $NBM$ and $ABM$ are incomparable in this sense, i.e., for some utility profile $NBM$ is manipulable, but $ABM$ is not, while for some other utility profile, $ABM$ is vulnerable, but $NBM$ is not. This is shown by the following examples.
**Example 3** (ABM not as manipulable as NBM). Consider the setting with $N = \{1, \ldots, 4\}$, $M = \{a, b, c, d\}$, unit capacities, and the type profile

\[
\begin{align*}
t_1 & : a > b > c > d, \\
t_2, t_3 & : a > c > b > d, \\
t_4 & : b > \ldots.
\end{align*}
\]

Agent 1’s allocation is $(1/3, 0, 0, 2/3)$ for the objects $a$ through $d$, respectively. If agent 1 swaps $b$ and $c$ in its report, the allocation will be $(1/3, 0, 1/3, 1/3)$. The outcome from this manipulation first order-stochastically dominates the outcome from truthful reporting. Thus, agent 1 will want to misreport under NBM (independent of its underlying utility).

Under ABM, the outcome for agent 1 is $(1/3, 0, 1/3, 1/3)$, independent of whether or not it swaps $b$ and $c$. Now suppose that all agents have utility $u = (9, 3, 1, 0)$ for their first, second, third, and fourth choice, respectively. Then no agent has an incentive to deviate from truthful reporting under ABM. Therefore, we found a utility profile at which NBM is vulnerable to manipulation, but ABM is not.

**Example 4** (NBM not as manipulable as ABM). Consider the setting with $N = \{1, \ldots, 6\}$, $M = \{a, b, c, d, e, f\}$, unit capacities, and the type profile

\[
\begin{align*}
t_1, t_2 & : a > e > c > d > f > b, \\
t_3, t_4 & : a > e > d > c > f > b, \\
t_5 & : b > c > \ldots, \\
t_6 & : b > d > \ldots.
\end{align*}
\]

Suppose that all agents have utility $u = (120, 30, 19, 2, 1, 0)$ for their first through sixth choice, respectively.

Consider the incentives of the agents under NBM: keeping the reports of all other agents constant, agent 5 has no incentive to deviate, and the same holds for agent 6. If agent 1 does not rank $a$ in first position, it looses all chances at $a$ and may at best get the second choice $e$ for sure, which is not an improvement under the particular utility chosen. Also, it is easy to check that changing the position of $f$ or $b$ will never be beneficial. Thus, any beneficial manipulation for agent 1 will involve only changes in the order of the objects $e, c, d$. However, none of these misreports are beneficial, which can be seen in Table 2 (middle column). Due to symmetry, none of the other agents 2, 3, 4 have an incentive to misreport either. Under ABM, however, agent 1 does have an incentive to misreport, which can also be seen in Table 2 (right column).

A natural next step to further understand the vulnerability of both mechanisms to manipulation is a quantitative analysis. This analysis should ask how often a mechanism is manipulable, i.e., given a probabilistic model of the way in which utility profiles are determined, how likely is each mechanism manipulable. In (Mennle and Seuken, 2013) we have studied NBM, ABM,
and PS in this way and find that under ABM truth-telling is a best response for all agents significantly more often than under NBM. This result is robust to changes in the size of the setting, the correlation of the preferences, and the underlying distributions in the utility model.

### D. Simulation Results

Tables 3, 4, and 5 contain the aggregate results of our simulation. The attributes are:

**Column 1:** The number of objects in the setting, where \( n = m \) and \( q_j = 1 \) for all \( j \in M \).

**Columns 2 through 5:** Rank dominance relation between NBM and ABM.

**Columns 6 through 7:** Rank dominance relation between NBM and RSD.

**Columns 8 through 11:** Rank dominance relation between ABM and RSD.

**Column 12:** Count of type profiles in the sample with the respective attributes.

| Misreport \( t'_1 \) | Gain from misreport \( t' \) under NBM | Gain from misreport \( t' \) under ABM |
|---------------------|----------------------------------------|---------------------------------------|
| \( a > e > d > c > f > b \) | -2.1 | 0.0 |
| \( a > c > e > d > f > b \) | -0.4 | 1.1 |
| \( a > c > d > e > f > b \) | -0.3 | 1.1 |
| \( a > d > c > e > f > b \) | -9.5 | -7.7 |
| \( a > d > c > e > f > b \) | -8.1 | -7.7 |

Table 2: Change in expected utility from misreports for agent 1 in Example 4.
| $n$ | NBM & ABM | NBM & RSD | ABM vs RSD | #   |
|-----|-----------|-----------|------------|-----|
| 3   | ✓         | ✓         | ✓          | 24874 |
| 3   | ✓         |           |            | 49969 |
| 4   | ✓         |            |            | 25157 |
| 4   | ✓         |            |            | 10176 |
| 4   |           |            |            | 152   |
| 4   |           |            | ✓          | 41182 |
| 4   | ✓         |            |            | 20074 |
| 4   | ✓         |            |            | 20821 |
| 4   | ✓         |            | ✓          | 5288  |
| 4   | ✓         |            |            | 2068  |
| 5   | ✓         |            | ✓          | 239   |
| 5   |           | ✓         |            | 130   |
| 5   |           | ✓         |            | 27    |
| 5   |           | ✓         |            | 64    |
| 5   | ✓         |            |            | 23945 |
| 5   | ✓         |            |            | 5     |
| 5   | ✓         |            |            | 999   |
| 5   |           | ✓         |            | 528   |
| 5   |           | ✓         |            | 347   |
| 5   |           |           | ✓          | 39705 |
| 5   |           | ✓         |            | 6937  |
| 5   |           |           | ✓          | 12283 |
| 5   |           |           | ✓          | 11876 |
| 5   | ✓         |           |            | 2301  |
| 5   |           | ✓         |            | 1     |
| 5   | ✓         |            |            | 852   |
| 6   | ✓         | ✓         |            | 237   |
| 6   |           | ✓         |            | 32    |
| 6   | ✓         |           |            | 80    |
| 6   |           |           | ✓          | 39359 |
| 6   |           |           | ✓          | 1496  |
| 6   |           |           | ✓          | 685   |
| 6   |           |           | ✓          | 585   |
| 6   |           |           |            | 29035 |
| 6   |           |           | ✓          | 2633  |
| 6   |           |           |            | 6813  |
| 6   |           |           |            | 15568 |
| 6   |           |           |            | 2347  |
| 6   |           |           |            | 1130  |

Table 3: Simulation results data cube ($n = 3, 4, 5, 6$)
| $n$ | NBM & ABM | NBM & RSD | ABM vs RSD | #  |
|-----|-----------|-----------|-------------|----|
| 7   | ✔         | ✔         | ✔           | 264|
| 7   | ✔         | ✔         | ✔           | 19 |
| 7   | ✔         | ✔         | ✔           | 67 |
| 7   | ✔         | ✔         | ✔           | 52804|
| 7   | ✔         | ✔         | ✔           | 1854|
| 7   | ✔         | ✔         | ✔           | 895|
| 7   | ✔         | ✔         | ✔           | 877|
| 7   | ✔         | ✔         | ✔           | 18798|
| 7   | ✔         | ✔         | ✔           | 1032|
| 7   | ✔         | ✔         | ✔           | 4078|
| 7   | ✔         | ✔         | ✔           | 15955|
| 7   | ✔         | ✔         | ✔           | 2290|
| 7   | ✔         | ✔         | ✔           | 1067|
| 8   | ✔         | ✔         | ✔           | 219|
| 8   | ✔         | ✔         | ✔           | 11 |
| 8   | ✔         | ✔         | ✔           | 54 |
| 8   | ✔         | ✔         | ✔           | 63354|
| 8   | ✔         | ✔         | ✔           | 1983|
| 8   | ✔         | ✔         | ✔           | 1103|
| 8   | ✔         | ✔         | ✔           | 980 |
| 8   | ✔         | ✔         | ✔           | 11373|
| 8   | ✔         | ✔         | ✔           | 398 |
| 8   | ✔         | ✔         | ✔           | 2523|
| 8   | ✔         | ✔         | ✔           | 14921|
| 8   | ✔         | ✔         | ✔           | 2113|
| 8   | ✔         | ✔         | ✔           | 968 |
| 9   | ✔         | ✔         | ✔           | 157|
| 9   | ✔         | ✔         | ✔           | 9  |
| 9   | ✔         | ✔         | ✔           | 38 |
| 9   | ✔         | ✔         | ✔           | 71904|
| 9   | ✔         | ✔         | ✔           | 2058|
| 9   | ✔         | ✔         | ✔           | 1212|
| 9   | ✔         | ✔         | ✔           | 1091|
| 9   | ✔         | ✔         | ✔           | 6664|
| 9   | ✔         | ✔         | ✔           | 175|
| 9   | ✔         | ✔         | ✔           | 1476|
| 9   | ✔         | ✔         | ✔           | 12617|
| 9   | ✔         | ✔         | ✔           | 1824|
| 9   | ✔         | ✔         | ✔           | 775|

Table 4: Simulation results data cube ($n = 7, 8, 9$)
| n  | $\neq_R$ | $>_R$ | $\sim_R$ | $<_R$ | $\neq_R$ | $>_R$ | $\sim_R$ | $<_R$ | $\neq_R$ | $>_R$ | $\sim_R$ | $<_R$ | #  |
|----|---------|-------|---------|-------|---------|-------|---------|-------|---------|-------|---------|-------|----|
| 10 | ✓       | ✓     | ✓       | ✓     | ✓       | ✓     | ✓       | ✓     | ✓       | ✓     | ✓       | ✓     | 96 |
| 10 | ✓       | ✓     |         | ✓     | ✓       | ✓     |         | ✓     | ✓       | ✓     |         | ✓     | 8  |
| 10 | ✓       |       | ✓       |       | ✓       | ✓     |         | ✓     |         | ✓     |       | ✓     | 33 |
| 10 | ✓       |       |         | ✓     |         | ✓     |         |       |         | ✓     |         | ✓     | 78154 |
| 10 | ✓       |       |         | ✓     |         |       |         | ✓     |         | ✓     |         |       | 1932 |
| 10 | ✓       |       |         |       |         | ✓     |         | ✓     |         |       |         | ✓     | 1261 |
| 10 | ✓       |       |         |       |         |       |         | ✓     |         |       |         | ✓     | 1109 |
| 10 | ✓       |       |         |       |         |       |         |       |         | ✓     |         | ✓     | 3694 |
| 10 | ✓       |       |         |       |         |       |         |       |         |       |         | ✓     | 72  |
| 10 | ✓       |       |         |       |         |       |         |       |         |       |         | ✓     | 833 |
| 10 | ✓       |       |         |       |         |       |         |       |         |       |         | ✓     | 10584 |
| 10 | ✓       |       |         |       |         |       |         |       |         |       |         | ✓     | 1609 |
| 10 | ✓       |       |         |       |         |       |         |       |         |       |         | ✓     | 615 |

Table 5: Simulation results data cube ($n = 10$)