On the fundamental group of an abelian cover ‡

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Résumé. Soient $X$ et $Y$ deux variétés complexes projectives et lisses de dimension $n \geq 2$ et soit $f : Y \to X$ un revêtement abelien totalement ramifié. Alors l’application $f_* : \pi_1(Y) \to \pi_1(X)$ est surjective et donne une extension centrale:

$$0 \to K \to \pi_1(Y) \to \pi_1(X) \to 1$$

(1)

où $K$ est un groupe fini.

Nous montrons comment le noyau $K$ et la classe de cohomologie $c(f) \in H^2(\pi_1(X), K)$ de (1) peuvent être calculés en termes de classes de Chern des composantes du diviseur critique de $f$ et des sous-faisceaux inversibles de $f_*\mathcal{O}_Y$ stables sous l’action du groupe de Galois.

Abstract. Let $X$, $Y$ be smooth complex projective varieties of dimension $n \geq 2$ and let $f : Y \to X$ be a totally ramified abelian cover. Then the map $f_* : \pi_1(Y) \to \pi_1(X)$ is surjective and gives rise to a central extension:

$$0 \to K \to \pi_1(Y) \to \pi_1(X) \to 1$$

(2)

where $K$ is a finite group.

Here we show how the kernel $K$ and the cohomology class $c(f) \in H^2(\pi_1(X), K)$ of (2) can be computed in terms of the Chern classes of the components of the branch divisor of $f$ and of the eigensheaves of $f_*\mathcal{O}_Y$ under the action of the Galois group.

1 Introduction.

This work generalizes a result of Catanese and the second author, who analyze in [CT] the fundamental group of a special type of covering $f : Y \to X$, with Galois group $(\mathbb{Z}/m\mathbb{Z})^2$, of a complex smooth projective surface $X$, the so-called "$m$-th root extraction" of a divisor $D$ on $X$.

By means of standard topological methods, the fundamental group $\pi_1(Y)$ can be described in that case as a central extension of the group $\pi_1(X)$, as follows:

$$0 \to \mathbb{Z}/r\mathbb{Z} \to \pi_1(Y) \to \pi_1(X) \to 1,$$

(3)

$r$ being a divisor of $m$ which depends only on the divisibility of $\pi^*(D)$ in $H^2(\tilde{X}, \mathbb{Z})$, where $\pi : \tilde{X} \to X$ is the universal covering of $X$.

The main result of [CT] (see Thm.2.16) is that the group cohomology class corresponding to the extension (3) can be explicitly computed in terms of the first Chern class of $D$.

This is an instance of a more general philosophy: in principle, it should be possible to recover all the information about an abelian cover $f : Y \to X$ from the "building data" of the cover,
i.e., from the Galois group $G$, the components of the branch locus, the inertia subgroups and the eigensheaves of $f_* \mathcal{O}_Y$ under the natural action of $G$ (see section 2 or [Pa] for more details).

Actually, the description of the general abelian cover given in [Pa] enables us to treat (under some mild assumptions on the components $D_1, \ldots, D_k$ of the branch locus) the case of any totally ramified abelian covering $f : Y \to X$, with $X$ a complex projective variety of dimension at least 2 (cf. section 2 for the definition of a totally ramified abelian cover).

Using the same methods as in [CT], we show that $\pi_1(Y)$ is a central extension as before:

$$0 \to K \to \pi_1(Y) \to \pi_1(X) \to 1,$$

where $K$ is a finite abelian group which is determined by the building data of the cover and the cohomology classes of $\pi^*(D_1), \ldots, \pi^*(D_k)$ (cf. Prop.3.2).

Consistently with the above "philosophy", a statement analogous to Thm.2.16 of [CT] actually holds in the general case: our main result (Thm.4.4, Rem.4.5) can be summarized by saying that the class of the extension (4) can be recovered from the Chern classes of the $D_j$’s and of the eigensheaves of $f_* \mathcal{O}_Y$; in some special case, this relation can be described in a particularly simple way (Thm.4.4, Cor.4.7, Rem.4.8). Moreover, one can construct examples of not homeomorphic varieties realized as covers of a projective variety $X$ with the same Galois group, branch locus and inertia subgroups (cf. Rem.4.5).

The idea of the proof is to exploit a natural representation of $\pi_1(Y)$ on a vector bundle on the universal covering $\tilde{X}$ of $X$ and the spectral sequence describing the cohomology of a quotient, in order to relate the group cohomology class of the extension (4) to the geometry of the covering. These are basically the same ingredients as in the proof of [CT], but we think that we have reached here a more conceptual and clearer understanding of the argument.

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2 A brief review of abelian covers.

In this section we set the notation and, for the reader’s convenience, we collect here the definitions and the notions concerning abelian covers that will be needed later. For further details and proofs, we refer to [Pa], sections 1 and 2.

Let $X, Y$ be complex algebraic varieties of dimension at least 2, smooth and projective, and let $f : Y \to X$ be a finite abelian cover, i.e. a Galois cover with finite abelian Galois group $G$.

The bundle $f_* \mathcal{O}_Y$ splits as a sum of one dimensional eigensheaves under the action of $G$, so that one has:

$$f_* \mathcal{O}_Y = \bigoplus_{\chi \in G^*} L^{-1}_\chi = \mathcal{O}_X \oplus \bigoplus_{\chi \in G^* \setminus \{1\}} L^{-1}_\chi \quad (5)$$

where $G^*$ denotes the group of characters of $G$ and $G$ acts on $L^{-1}_\chi$ via the character $\chi$.

We warn the reader that the notation here and in the next section is dual to the one adopted in [Pa]; however this does not affect the formulas quoted from there.

Under our assumptions, the ramification locus of $f$ is a divisor. Let $D_1, \ldots, D_k$ be the irreducible components of the branch locus $D$ and let $R_j = f^{-1}(D_j)$, $j = 1, \ldots, k$. For every $j = 1, \ldots, k$, one defines the inertia subgroup $G_j = \{g \in G \mid g(y) = y \text{ for each } y \in R_j\}$. Given any point $y_0 \in R_j$, one obtains a natural representation of $G_j$ on the normal space to $R_j$ at $y_0$ by taking differentials. The corresponding character, that we denote by $\psi_j$, is independent of the
choice of the point $y_0 \in R_j$. By standard results, the subgroup $G_j$ is cyclic and the character $\psi_j$ generates the group $G_j^*$ of the characters of $G_j$. We denote by $m_j$ the order of $G_j$, by $m$ the least common multiple of the $m_j$'s and by $g_j$ the generator of $G_j$ such that $\psi_j(g_j) = \exp(\frac{2\pi\sqrt{-1}}{m_j})$.

In what follows we will always assume that the cover $f : Y \to X$ is totally ramified, i.e., that the subgroups $G_j$ generate $G$; then the group of characters $G^*$ injects in $\bigoplus_{j=1}^k G_j^*$ and every $\chi \in G^*$ may be written uniquely as:

$$\chi = \sum_{j=1}^k a_{\chi,j} \psi_j, \quad 0 \leq a_{\chi,j} < m_j \text{ for every } j. \quad (6)$$

In particular, let $\chi_1, \ldots, \chi_n \in G^*$ be such that $G^*$ is the direct sum of the cyclic subgroups generated by the $\chi$'s, and let $d_i$ be the order of $\chi_i$, $i = 1, \ldots, n$. Write:

$$\chi_i = \sum_{j=1}^k a_{ij} \psi_j, \quad 0 \leq a_{ij} < m_j, \quad i = 1, \ldots, n. \quad (7)$$

Then one has ([Pa], Prop.2.1):

$$d_i L_{\chi_i} \equiv \sum_{j=1}^k \frac{d_i a_{ij}}{m_j} D_j \quad i = 1, \ldots n \quad (8)$$

the corresponding isomorphism of line bundles being induced by multiplication in the $\mathcal{O}_X$-algebra $f_* \mathcal{O}_Y$. More generally, if $\chi = \sum_{i=1}^n b_{\chi,i} \chi_i$, with $0 \leq b_{\chi,i} < d_i \forall i$, one has ([Pa], Prop.2.1):

$$L_\chi \equiv \sum_{i=1}^n b_{\chi,i} L_{\chi}, - \sum_{j=1}^k q^\chi_j D_j. \quad (9)$$

where $q^\chi_j$ is the integral part of the rational number $\sum_{i=1}^n b_{\chi,i} a_{ij} m_j$, $j = 1, \ldots k$.

Equations (8) are the characteristic relations of an abelian cover. Actually, since $X$ is complete, for assigned $G$, $D_j$, $G_j$, $\psi_j$, $j = 1, \ldots k$, to each set of line bundles $L_{\chi_i}$, $i = 1, \ldots n$, satisfying (8) there corresponds a unique, up to isomorphism, $G$-cover of $X$, branched on the $D_j$'s and such that $G_j$ is the inertia subgroup of $D_j$ and $\psi_j$ is the corresponding character ([Pa], Thm.2.1). Moreover, the cover is actually smooth under suitable assumptions on the building data.

### 3 The fundamental group and the universal covering of $Y$.

We keep the notation introduced in the previous section.

**Definition 3.1** ([MM], pag.218) A smooth divisor $\Delta$ on a variety $X$ is called flexible if there exists a smooth divisor $\Delta' \equiv \Delta$ such that $\Delta' \cap \Delta \neq \emptyset$ and $\Delta$ and $\Delta'$ meet transversely.

We recall that a flexible divisor on a projective surface is connected (see [Ca], Remark 1.5). Hence, by considering a general linear section, one deduces easily that a flexible divisor on a projective variety of dimension $\geq 2$ is connected.

**Proposition 3.2** Let $X$, $Y$ be smooth projective varieties over $\mathbb{C}$ of dimension $n$ at least 2. Let $f : Y \to X$ be a totally ramified abelian cover branched on irreducible, flexible and ample divisors $\{D_j\}_{j=1,\ldots,k}$. Then:
a) The natural map \( f_{\ast} : \pi_1(Y) \to \pi_1(X) \) is surjective.

b) Let \( K = \ker(f_{\ast}) \); then \( K \) is finite and
\[
0 \to K \to \pi_1(Y) \to \pi_1(X) \to 1.
\] (10)
is a central group extension.

c) Let \( \pi : \tilde{X} \to X \) be the universal covering of \( X \) and \( \tilde{D} = \pi^{-1}(D) \); then \( \tilde{D}_j = \pi^{-1}(D_j) \)
is connected, \( j = 1, \ldots, k \). Denote by \( H_c^2 \) the cohomology with compact supports and by \( \rho : H_c^2(X) \to H_c^2(\tilde{D}) \cong \bigoplus_{j=1}^k \mathbb{Z}D_j \) the restriction map. Finally, let \( \sigma \) be the map defined by:
\[
\sigma : H_c^2(\tilde{D}) \cong \bigoplus_{j=1}^k \mathbb{Z}D_j \to \bigoplus_{j=1}^k G_j \quad D_j \mapsto g_j.
\] (11)

Then \( N = \ker(\bigoplus G_j \to G) \) contains \( \text{Im}(\sigma \circ \rho) \) and \( K \) is isomorphic to the quotient group \( N/\text{Im}(\sigma \circ \rho) \).

Proof. a) and the fact that the extension \( \{\bigoplus\} \) is central can be proven exactly as in [Ca], Thm.1.6 and in [CT], Lemma 2.1.

For the proof of c) (that implies that \( K \) is finite), we refer the reader to [Ca], Prop.1.8 and to [CT], proof of Thm.2.16, Step I. One only has to notice that, by Lefschetz theorem (cf. [Bo], Remark 3.3)

\[ \text{a) From Prop.3.2, c), it follows in particular that the kernel } K \text{ of the surjection } f_{\ast} : \pi_1(Y) \to \pi_1(X) \text{ does not depend on the choice of the solution } L_X \text{ of } \{\bigoplus\}, \text{ once } G, \text{ the } g_j \text{'s and the class of the } \tilde{D}_j \text{'s in } H^2(\tilde{X}, \mathbb{Z}/m_j \mathbb{Z}), \text{ } j = 1, \ldots, k, \text{ are fixed.} \]

\[ \text{b) If } f : Y \to X \text{ is an abelian cover as in the hypotheses of Prop.3.2, then } H^1(Y, \mathcal{O}_Y) \cong H_1(X, \mathcal{O}_X) \text{ by } \{\bigoplus\} \text{ and the Kodaira Vanishing Theorem. Moreover, according to Prop.3.2, a) the map } f_{\ast} : H_1(Y, \mathbb{Z}) \to H_1(X, \mathbb{Z}) \text{ is surjective; thus the map } f_{\ast} : \text{alb}(Y) \to \text{alb}(X) \text{ between the Albanese varieties is an isomorphism.} \]

**Proposition 3.4** In the same hypotheses as in Prop.3.2, let \( q : \bar{Y} \to Y \) be the universal cover of \( Y \) and let \( \bar{f} : \bar{Y} \to \bar{X} \) be the map lifting \( f : Y \to X \). Then \( \bar{f} \) is a totally ramified abelian cover of \( \bar{X} \) with group \( \bar{G} = (\bigoplus_{j=1}^k G_j)/\text{Im}(\sigma \circ \rho) \), branched on \( \bar{D} \).

Proof. By diagram chasing, it is easy to show that \( \pi_1(\bar{X} \setminus \bar{D}) \) is isomorphic to the kernel \( V \) of the surjection \( \pi_1(X \setminus D) \to \pi_1(X) \) induced by the inclusion \( X \setminus D \subset X \). Since the \( D_i \)'s are flexible, one proves as in ([CT], Lemma 2.1) that \( V \) is an abelian group. It follows that \( \bar{f} \), being branched on \( \bar{D} \), is an abelian cover.

Consider now the fiber product \( Y' \) of \( f : Y \to X \) and \( \pi : \bar{X} \to X \), with the natural maps \( f' : Y' \to \bar{X} \) and \( q' : Y' \to Y \); \( f' \) is a \( G \)-cover ramified on \( \bar{D} \) and \( q' \) is unramified. According
to Prop. 3.2, b), the universal covering \( q : \tilde{Y} \to Y \) of \( Y \) factors as \( q = q' \circ q'' \), for a suitable unramified cover \( q'' : \tilde{Y} \to Y' \) with group \( K \), giving a commutative diagram as follows:

\[
\begin{array}{ccc}
\tilde{Y} & \xrightarrow{q'} & \tilde{X} \\
\downarrow & \nearrow \hat{f} \downarrow & \\
Y' & \xrightarrow{f'} & X \\
\downarrow & \nearrow \hat{f} \downarrow & \\
Y & \xrightarrow{f} & \hat{X} \\
\end{array}
\]

In particular, \( K \cong \pi_1(Y') \) and \( \tilde{f} = f' \circ q'' \). Hence, the Galois group \( \tilde{G} \) of \( \tilde{f} \) is given as an extension:

\[
0 \to K \to \tilde{G} \to G \to 0
\]

Moreover, if one denotes by \( \tilde{G}_j \) the inertia subgroup of \( \tilde{D}_j \) with respect to \( \tilde{f} \), then \( \tilde{G}_j \) maps isomorphically onto \( G_j \) for every \( j = 1, \ldots k \). The isomorphism \( \tilde{G} = (\bigoplus G_j)/\text{Im}(\sigma \circ \rho) \) can be obtained by computing the fundamental group of \( \tilde{Y} \) as in [CT], proof of Thm. 2.16.

The following lemma will be used in the next section.

**Lemma 3.5** Consider the subgroups \( \pi_1(Y) \) and \( \hat{G} \) of \( \text{Aut}(\tilde{Y}) \); then one has:

\[
\beta g = g\beta \quad \forall g \in \hat{G}, \forall \beta \in \pi_1(Y).
\]

**Proof.** Since the cover \( \tilde{f} \) is totally ramified, it is enough to show that all the elements of \( \pi_1(Y) \) commute with \( g_j, j = 1, \ldots k \).

Let \( \beta \in \pi_1(Y) \) and fix \( j = 1, \ldots k \). We remark firstly that \( \beta g_j \beta^{-1} \) is actually an element of \( \hat{G} \subset \text{Aut}(\tilde{Y}) \). In fact, consider the classes represented by \( \beta g_j \) and \( g_j \beta \) modulo \( K \): they do coincide as automorphisms of \( Y' \subseteq Y \times \hat{X} \), since the group \( G \times \pi_1(X) \) acts there via the natural action on the components. So, \( \beta g_j \beta^{-1} g_j^{-1} \in K \) and \( \beta g_j \beta^{-1} \in g_j K \subseteq \hat{G} \), as desired.

By diagram (12), we have \( \hat{R}_j = \tilde{f}^{-1}(D_j) = q^{-1}(R_j) \) \( \forall j = 1, \ldots k \). Since \( R_j = f^{-1}(D_j) \) is ample and connected, the same argument as in the proof of Lemma 3.2, c) shows that \( \hat{R}_j \) is connected. So, \( \beta \hat{R}_j = \hat{R}_j \) and \( \beta g_j \beta^{-1} \) fixes \( \hat{R}_j \) pointwise, namely \( \beta g_j \beta^{-1} \in \hat{G}_j \).

Finally, recalling the definition of the character \( \psi_j \in G_j^* \) introduced in section 2, one checks immediately that \( \psi_j(\beta g_j \beta^{-1}) = \psi_j(g_j) \). The conclusion now follows from the faithfulness of \( \psi_j \).

4 Computing the cohomology class of the central extension \( 0 \to K \to \pi_1(Y) \to \pi_1(X) \to 1 \).

We keep the notation and the assumptions introduced in the previous sections, unless the contrary is explicitly stated. We need two technical Lemmas in order to state the main result of this paper.

**Lemma 4.1** Let \( \mathcal{H} \) be a finite abelian group and \( \zeta_1, \ldots, \zeta_m \in \mathcal{H} \) be such that \( \mathcal{H} = \bigoplus_{j=1}^m <\zeta_j> \) is the direct sum of the cyclic subgroups generated by \( \zeta_j, j = 1, \ldots m \); denote by \( h_j \) the order of \( \zeta_j \), the typical element of \( <\zeta_j> \).
ζj. Let \( p \in \mathbb{Z} \) be a prime and \( \mathcal{H}_p \) be the \( p \)-torsion subgroup of \( \mathcal{H} \). Let \( \chi_1, \ldots, \chi_t \in \mathcal{H}_p \) such that \( < \chi_1, \ldots, \chi_t > = \bigoplus_{i=1}^t < \chi_i > \). Finally, let \( d_i \) be the order of \( \chi_i \) and write \( \chi_i = \sum_{j=1}^{h_i} a_{ij} \zeta_j \) with \( 0 \leq a_{ij} < h_j \).

Then, \( \forall x_1, \ldots, x_t \in \mathbb{Z} \) and \( \forall \gamma \geq 1 \), the system:

\[
\sum_{j=1}^{m} \frac{d_i a_{ij}}{h_j} s_j \equiv x_i \mod p^\gamma \quad i = 1, \ldots, t
\]  

(15)

admits a solution \( (s_1, \ldots, s_m) \in \mathbb{Z}^m \).

**Proof.** We set \( c_{ij} = \frac{d_i a_{ij}}{h_j} \) and, for \( x \in \mathbb{Z} \), we denote by \( \overline{x} \) the class of \( x \) in \( \mathbb{Z}/p\mathbb{Z} \). We proceed by induction on \( \gamma \).

Let \( \gamma = 1 \). We show that the matrix \((\overline{c}_{ij})\) has rank \( t \).

Let \( y_1, \ldots, y_m \in \mathbb{Z} \) and assume that:

\[
\sum_i \overline{c}_{ij} y_i = 0 \quad \forall j = 1, \ldots, m.
\]  

(16)

This implies that:

\[
\sum_{i=1}^{t} c_{ij} y_i \equiv 0 \mod p \quad \forall j = 1, \ldots, m
\]  

(17)

so that:

\[
\sum_{i=1}^{t} \frac{y_i d_i}{p} \frac{a_{ij}}{h_j} \in \mathbb{Z} \quad \forall j = 1, \ldots, m.
\]  

(18)

Recalling that \( p \) divides \( d_i \) \( \forall i \), we deduce that:

\[
\sum_{i=1}^{t} \left( \frac{y_i d_i}{p} \right) a_{ij} \equiv 0 \mod h_j \quad \forall j = 1, \ldots, m,
\]  

(19)

so that \( \sum_i \frac{y_i d_i}{p} \chi_i \) is the zero element in \( \mathcal{H} \). By the hypothesis on the \( \chi_i \)'s, it follows that:

\[
\frac{y_i d_i}{p} \equiv 0 \mod d_i
\]  

(20)

and finally:

\[
y_i \equiv 0 \mod p
\]  

(21)

showing, as desired, that the rows of the matrix \((\overline{c}_{ij})\) are linearly independent over \( \mathbb{Z}/p\mathbb{Z} \).

Let now \( \gamma > 1 \) and assume by inductive hypothesis that \( (s_1, \ldots, s_m) \in \mathbb{Z}^m \) is a solution of the system (15).

We set \( s'_j = s_j + \delta_j p^\gamma \) and we look for a suitable choice of the integers \( \delta_j \). We have:

\[
\sum_j c_{ij} s'_j = \sum_j c_{ij} s_j + p^\gamma \sum_j c_{ij} \delta_j = x_i + p^\gamma y_i + p^\gamma \sum_j c_{ij} \delta_j \quad \exists y_i \in \mathbb{Z},
\]  

(22)

so that:

\[
\sum_j c_{ij} s'_j \equiv x_i \mod p^{\gamma+1} \iff \sum_j c_{ij} \delta_j \equiv -y_i \mod p
\]  

(23)

and the latter system has a solution, by the case \( \gamma = 1 \). This conclude the proof. \( \blacksquare \)

We come back to the study of the cover \( f \):
Lemma 4.2. Let $A$ be the subgroup of $\text{Pic}(X)$ generated by $D_1, \ldots, D_k$ and $L_\chi$, $\chi \in G^*$. Then there exist $M_1, \ldots, M_q \in \text{Pic}(X)$ such that $A = \bigoplus_{l=1}^q <M_l>$ and

$$
\begin{pmatrix}
D_1 \\
\vdots \\
D_k
\end{pmatrix} = C
\begin{pmatrix}
M_1 \\
\vdots \\
M_q
\end{pmatrix}
$$

where $C = (c_{jl})$ is a matrix with integral coefficients such that each column $(c_{jl})_{j=1}^k$ represents an element of $N = \ker (\bigoplus_{j=1}^k G_j \to G)$.

**Proof.** $A$ is a finitely generated abelian group, so one can write $A = F \bigoplus T$, where $T$ is the torsion part of $A$ and $F$ is free.

Denote by $\{\xi_l\}_l$ a set of free generators of $F$ and by $\{\eta_l\}_l$ a set of generators of $T$ such that $T = \bigoplus <\eta_l>$ and the order $o(\eta_l)$ of $\eta_l$ is the power of a prime, $\forall l$. Let finally $\chi_i$ be generators of $G^*$ such that $G^* = \bigoplus_{i=1}^n <\chi_i>$ and the order $o(\chi_i)$ of $\chi_i$ is the power of a prime, $\forall i$.

One can write:

$$
L_\chi_i \equiv \sum_l \lambda_{il} \eta_l + \sum_l \lambda'_{il} \xi_l \quad \forall i = 1, \ldots, n,
$$

$$
D_j \equiv \sum_l c_{jl} \eta_l + \sum_l c'_{jl} \xi_l \quad \forall j = 1, \ldots, k,
$$

where the coefficients $\lambda'_{jl}$ and $c'_{jl}$ are uniquely determined, whereas $\lambda_{il}$ and $c_{jl}$ are determined only up to a multiple of $o(\eta_l)$.

We can apply the analysis of section 2 to the cover $f$. We write $\chi_i = \sum_{j=1}^k a_{ij} \psi_j$, with $0 \leq a_{ij} < m_j$, and we set $d_i = o(\chi_i)$ as in the previous Lemma; the equations (8) become here:

$$
d_i L_\chi_i \equiv \sum_{j=1}^k \frac{d_i a_{ij}}{m_j} D_j \quad i = 1, \ldots, n,
$$

so that we must have:

$$
d_i \lambda'_{il} \equiv \sum_{j=1}^k \frac{d_i a_{ij}}{m_j} c'_{jl} \quad i = 1, \ldots, n,
$$

showing that $(c'_{jl})_{j=1}^k$ represents an element of $N$, $\forall l$: in fact, by duality, $(t_1, \ldots, t_k) \in \mathbb{Z}^k$ represents an element of $N$ if and only if it satisfies the relations:

$$
\sum_{j=1}^k \frac{a_{ij}}{m_j} t_j \in \mathbb{Z} \quad \forall i = 1, \ldots, n.
$$

For the coefficients of the torsion part, we have:

$$
d_i \lambda_{il} \eta_l = \left( \sum_{j=1}^k \frac{d_i a_{ij}}{m_j} c_{jl} \right) \eta_l
$$

so that:

$$
d_i \lambda_{il} \equiv \sum_{j=1}^k \frac{d_i a_{ij}}{m_j} c_{jl} \mod o(\eta_l).
$$
We fix an index $l$. Let $p$ be a prime such that $o(\eta_l) = p^\alpha$. We want to show that, for a suitable choice of the $c_{jl}$, the following relation holds $\forall i = 1, \ldots n$:

$$d_i \lambda_{il} \equiv \sum_{j=1}^{k} \frac{d_i a_{ij}}{m_j} c_{jl} \mod d_i. \quad (32)$$

Let $\chi_i$ be a generator such that $d_i \equiv 0 \mod p$ and set $d_i = p^{\alpha_i}$. By (31), it is enough to consider the case in which $\alpha < \alpha_i$.

Setting $c_{jl}' = c_{jl} + p^\alpha s_j$ and recalling (31), one has:

$$\sum_{k=1}^{k} \frac{d_i a_{ij}}{m_j} c_{jl}' = \sum_{k=1}^{k} \frac{d_i a_{ij}}{m_j} c_{jl} + p^\alpha \sum_{k=1}^{k} \frac{d_i a_{ij}}{m_j} s_j$$

for a suitable choice of integers $x_i$. One concludes that the relation (32) holds if and only if:

$$\sum_{k=1}^{k} \frac{d_i a_{ij}}{m_j} s_j \equiv x_i \mod p^{\alpha_i - \alpha}. \quad (34)$$

Let $\beta = \max\{\alpha_i - \alpha\}_i$. The system of congruences:

$$\sum_{k=1}^{k} \frac{d_i a_{ij}}{m_j} s_j \equiv x_i \mod p^\beta \quad \forall i \text{ such that } d_i \equiv 0 \mod p \quad (35)$$

admits a solution by Lemma 4.1. So, we can assume that the coefficients $(c_{jl})_{j=1,\ldots,k}$ in (26) satisfy (32) for every $i$ such that $d_i \equiv 0 \mod p$.

To complete the proof, let $\gamma$ be an integer $\geqslant 0$; we can still modify the coefficients as $c_{jl}' = c_{jl} + p^\gamma t_j$. It is enough to notice that, setting $d = \lcm\{d_i \mid d_i \not\equiv 0 \mod p\}$, then $d$ and $p$ are coprime and the system of congruences:

$$c_{jl} + p^\gamma t_j \equiv 0 \mod d \quad \forall j \quad (36)$$

admits a solution. So we can assume that $c_{jl}' \equiv 0 \mod d$, and the proof is complete. 

To any decomposition (41) as in Lemma 4.2, we associate a cohomology class in $H^2(X, K)$:

**Definition 4.3** Given a decomposition (41) as in Lemma 4.2, consider the map:

$$\mathbb{Z}^q \rightarrow N$$

$$(x_1, \ldots, x_q) \mapsto \sum_{l=1}^{q} x_l [\xi] = (\sum_{l=1}^{q} x_l [c_{jl}])_{j=1,\ldots,k} \quad (37)$$

and denote by $\Theta : \mathbb{Z}^q \rightarrow K$ its composition with the projection $N \rightarrow K$ (cf. Prop.3.2, c)). Then, set:

$$\xi = \Theta_*([M_1], \ldots, [M_q]), \quad (38)$$

where $\Theta_* : H^2(X, \mathbb{Z}^q) \cong \bigoplus^q H^2(X, \mathbb{Z}) \rightarrow H^2(X, K)$ is the map induced in cohomology by $\Theta$ and $[M]$ is the Chern class of a divisor $M$ on $X$.
We briefly recall some facts about quotients by a properly discontinuous group action (see for instance [Mu], Appendix to section 1, [Gr], ch. 5).

Let \( \tilde{X} \) be a simply connected variety, let \( \Gamma \) be a group acting properly and discontinuously on \( \tilde{X} \) and let \( p: \tilde{X} \rightarrow X = \tilde{X}/\Gamma \) be the projection onto the quotient. Consider the following two functors:

\[
M \xrightarrow{F} M^\Gamma, \text{ for } M \text{ a } \Gamma\text{-module}
\]
\[
\mathcal{F} \xrightarrow{H} H^0(\tilde{X}, p^*\mathcal{F}), \text{ for } \mathcal{F} \text{ a locally constant sheaf on } X.
\]

The spectral sequence associated to the functor \( F \circ H \) yields in this case the exact sequence of cohomology group:

\[
0 \rightarrow H^2(\Gamma, H^0(\tilde{X}, p^*\mathcal{F})) \rightarrow H^2(X, \mathcal{F}) \rightarrow H^2(\tilde{X}, p^*\mathcal{F})^\Gamma
\]

that will be used several times in the following and it is natural with respect to the sheaf maps on \( X \).

**Theorem 4.4** Let \( X, Y \) be smooth projective varieties over \( \mathbb{C} \) of dimension at least 2. Let \( f: Y \rightarrow X \) be a totally ramified finite abelian cover branched on a divisor with flexible and ample components \( \{D_j\}_{j=1,...,k} \). According to Prop.3.2, b), the map \( f \) induces a central extension:

\[
0 \rightarrow K \rightarrow \pi_1(Y) \xrightarrow{f} \pi_1(X) \rightarrow 1
\]

Denote by \( c(f) \in H^2(\pi_1(X), K) \subseteq H^2(X, K) \) the cohomology class classifying the extension (40).

Let:

\[
\begin{pmatrix}
D_1 \\
\vdots \\
D_k
\end{pmatrix} \equiv C
\begin{pmatrix}
M_1 \\
\vdots \\
M_q
\end{pmatrix}
\]

be a decomposition as in Lemma 4.2 and let \( \xi \in H^2(X, K) \) be the class defined in Def.4.3.

In this notation, one has:

\[
c(f) = \xi;
\]

in particular, the class \( \xi \) does not depend on the chosen decomposition.

**Proof.** It is enough to show that \( \xi \) and \( c(f) \) admit cohomologous representatives. This can be done in three steps.

**Step I:** we compute a cocycle representing \( c(f) \in H^2(X, K) \).

We start by choosing suitable trivializations of the line bundles that appear in the computation.

Set \( \Gamma = \pi_1(X) \) and \( \tilde{\Gamma} = \pi_1(Y) \). Let \( \{U_r\} \) be a sufficiently fine cover of \( X \) such that \( \Gamma \) acts transitively on the set of connected components of \( \pi^{-1}(U_r), \forall \ r \). If we fix a component \( V_r \) of \( \pi^{-1}(U_r) \), then \( \pi^{-1}(U_r) = \bigcup_{\gamma \in \Gamma} \gamma(V_r) \); for every \( \gamma \in \Gamma \) we write:

\[
\gamma(V_r) = V_{(\gamma,r)}
\]

and, in particular: \( V_{(1,r)} = V_r \).

Such a covering has the following properties:
a) For every \((r, s)\) such that \(U_r \cap U_s = \emptyset\), there exists a unique element \(\beta(r, s) \in \Gamma\) such that:

\[
V_{(1, r)} \cap V_{(\beta(r, s), s)} \neq \emptyset. \tag{44}
\]

b) If \(U_r \cap U_s \neq \emptyset\), then \(V_{(\gamma, r)} \cap V_{(\gamma \beta(r, s), s)}\) have nonempty intersection.

c) Since \(\pi\) is a local homeomorphism, if \(U_r \cap U_s \cap U_t \neq \emptyset\), then:

\[
\emptyset \neq V_{(\beta(r, s), s)} \cap V_{(\beta(r, t), t)}. \tag{45}
\]

Hence the following relation is satisfied for every \(U_r \cap U_s \cap U_t \neq \emptyset\):

\[
\beta(r, t) = \beta(r, s) \beta(s, t). \tag{46}
\]

In particular: \(\beta(s, r) = \beta(r, s)^{-1}\).

For later use, we set:

\[
V_{(\alpha, r, s)} = \alpha(V_{(1, r)} \cap V_{(\beta(r, s), s)}) = V_{(\alpha, r)} \cap V_{(\alpha \beta(r, s), s)} \tag{47}
\]

for every \(\alpha \in \Gamma\) and for every \((r, s)\) such that \(U_r \cap U_s \neq \emptyset\).

For every \(r\) and for every \(j = 1, \ldots, k\), we choose a local generator \(w^j_r\) for \(\mathcal{O}_X(-D_j)\) on \(U_r\) (we ask that \(w^j_r\) is a local equation for \(D_j\)) and for every pair \((r, s)\) such that \(U_r \cap U_s \neq \emptyset\) we write:

\[
w^j_r = k^j_{(r, s)} w^j_s \text{ on } U_r \cap U_s. \tag{48}
\]

Now we apply to \(\tilde{f} : \tilde{Y} \to \tilde{X}\) the analysis of section 2, most of which can be easily extended to the case of analytic spaces. One has:

\[
\tilde{f}_* (\mathcal{O}_Y) = \bigoplus_{\chi \in \check{G}^*} L^{-1}_{\chi} \tag{49}
\]

Each element of the group \(\check{G}\) can be interpreted as an automorphism of the sheaf \(\tilde{f}_* (\mathcal{O}_Y)\). In particular, by duality, the elements of \(K \subseteq \check{G}\) are characterized by the property that they induce the identity on the subsheaf \(L^1_{\chi}\) for every \(\chi \in G^* \subseteq \check{G}^*\).

Let \(\tilde{\chi}_1, \ldots, \tilde{\chi}_h \in \check{G}^*\) be such that \(\check{G}^*\) is isomorphic to the direct sum of the cyclic subgroups generated by the \(\tilde{\chi}_i\)'s, and let \(\check{d}_i\) be the order of \(\tilde{\chi}_i\), \(i = 1, \ldots, h\). Let \(\check{D}_i\) be the inverse image of \(D_i\) via the universal covering map \(\pi : \tilde{X} \to X\), as before. If \(\tilde{\chi}_i = \sum_{j=1}^k \tilde{a}_{ij} \psi_j\), with \(0 \leq \tilde{a}_{ij} < m_j\), the system (8) yields in this case:

\[
\check{d}_i L_{\tilde{\chi}_i} \equiv \sum_{j=1}^k \frac{\check{d}_i \tilde{a}_{ij}}{m_j} \check{D}_j \quad i = 1, \ldots, h. \tag{50}
\]

So it is possible to choose local generators \(\tilde{z}^i_{(\alpha, r)}\) for \(L^{-1}_{\tilde{\chi}_i}\) on \(V_{(\alpha, r)}\) such that for every \(\alpha \in \Gamma\) and for every pair \((r, s)\) with \(U_r \cap U_s \neq \emptyset\) one has:

\[
\left(\tilde{z}^i_{(\alpha, r)}\right)^{\check{d}_i} = \prod_{j=1}^k \left(w^j_r\right)^{\frac{\check{d}_i \tilde{a}_{ij}}{m_j}}. \tag{51}
\]
Writing:

\[ z^i_{(\alpha,r)} = \tilde{h}^i_{(\alpha,r,s)} z^i_{(\alpha,\beta(r,s),s)} \quad \text{on } V_{(\alpha,r,s)} \]  

we have:

\[ (\tilde{h}^i_{(\alpha,r,s)})^{\bar{d}_i} = \prod_{j=1}^k (k^j_{(r,s)})^{d_{\alpha_{ij}}/m_j}. \]  

and the cocycle condition for \( \tilde{h}^i_{(\alpha,r,s)} \), that will be often used later on, yields the relation:

\[ 1 = \tilde{h}^i_{(\alpha,r,s)} \tilde{h}^i_{(\alpha,\beta(r,s),s,t)} \tilde{h}^i_{(\alpha,\beta(r,t),t,r)} \quad \forall \alpha \in \Gamma, \forall i, \forall r, s, t \text{ with } U_r \cap U_s \cap U_t \neq \emptyset. \]  

We observe that the generator \( \tilde{z}^i_{(\alpha,r)} \) is determined by (51) only up to a constant of the form \( \exp 2\pi \sqrt{-1} (u^i_{(\alpha,r)}/\bar{d}_i) \) with \( u^i_{(\alpha,r)} \in \mathbb{Z} \). Moreover, according to (9), every choice of local generators \( w^i_r \) for \( \mathcal{O}_X(-D_j) \) and \( \tilde{z}^i_{(\alpha,r)} \) for \( L_{\tilde{\chi}}^{-1} \) induces a choice of local generators for \( L_{\chi}^{-1} \), \( \forall \tilde{\chi} \in \hat{G} \ast \), by the rule:

\[ \tilde{z}^i_{(\alpha,r)} = \prod_{i=1}^n (z^i_{(\alpha,r)})^{b_{\tilde{\chi},i}} \prod_{j=1}^k (w^j_{r})^{-q_j^{\tilde{\chi}}} \quad \text{if } \tilde{\chi} = \sum_{i=1}^h b_{\tilde{\chi},i} \tilde{\chi}_i, \ 0 \leq b_{\tilde{\chi},i} < \bar{d}_i. \]  

where \( q_j^{\tilde{\chi}} \) denotes the integral part of the real number \( \sum_{i=1}^h b_{\tilde{\chi},i} \tilde{\chi}_i \).

Let now \( \chi_1, \ldots, \chi_n \) be a set of generators for \( G \ast \) such that \( G \ast \) is the direct sum of the cyclic subgroups generated by the \( \chi_v \)'s and the order \( d_v \) of \( \chi_v \) is a power of a prime number, \( v = 1, \ldots, n \).

We recall that \( G \ast \subseteq \hat{G} \ast \) and, \( \forall \chi \in G \ast \), the corresponding eigensheaf \( L_{\chi} \) is a pullback from \( X \).

We write \( \chi_v = \sum_{i=1}^n b_{v,i} \chi_i \in \hat{G} \ast \) (0 ≤ \( b_{v,i} < \bar{d}_i \)) and \( \tilde{q}_j^{\chi_v} = q_j^v \); the corresponding local generator for \( L_{\chi_v}^{-1} \) chosen in (52) is:

\[ z^v_{(\alpha,r)} = \prod_{i=1}^n (z^i_{(\alpha,r)})^{b_{v,i}} \prod_{j=1}^k (w^j_{v})^{-q_j^{v}} \]  

we show that, for a suitable choice of the \( \tilde{z}^i_{(\alpha,r)} \), we can assume that the expression in (56) is independent from \( \alpha \). In fact, using the characteristic equations of the cover \( f \), one can choose a local base \( y^v_r \) of \( L_{\chi_v}^{-1} \) on \( V_{(\alpha,r)} \) that does not depend on \( \alpha \) and satisfies the relation:

\[ (y^v_r)^{d_v} = \prod_{j=1}^k (w^j_{v})^{a_{v,j}d_v/m_j}. \]  

Since \( \sum_{i=1}^h b_{v,i} \tilde{\chi}_i = q_j^v m_j + a_{v,j} \forall j = 1, \ldots, k \), the two local generators \( y^v_r \) and \( z^v_{(\alpha,r)} \) on \( V_{(\alpha,r)} \) differ by a \( d_v \)-th root of unity, that we denote by \( \exp (2\pi \sqrt{-1} (x^v_{(\alpha,r)}/d_v)) \). If we multiply \( \tilde{z}^i_{(\alpha,r)} \) by \( \exp (2\pi \sqrt{-1} (u^i_{(\alpha,r)}/\bar{d}_i)) \), then \( (x^v_{(\alpha,r)}/d_v) \) becomes \( (x^v_{(\alpha,r)}/d_v) + \sum_{i=1}^h (b_{v,i}/\bar{d}_i) u^i_{(\alpha,r)} \). Hence, we only need to solve the linear system of congruences, \( \forall (\alpha, r) \):

\[ \sum_{i=1}^h \frac{d_{v,i} b_{v,i}}{\bar{d}_i} u^i_{(\alpha,r)} \equiv x^v_{(\alpha,r)} \mod d_v \quad v = 1, \ldots, n; \]  

since we assume that \( d_v \) is a power of a prime number, this system admits a solution according to the Chinese Remainder’s Theorem and Lemma 1.1.
So we can assume that the expression $z_{\alpha,r}^v$ in (55) does not depend on $\alpha$ and it is the pullback of a local generator of the corresponding eigensheaf on $X$: we write $z_{\alpha,r}^v = z_{\alpha}^v$. For later use, we define the corresponding cocycle $h_{(r,s)}^v$ ($v = 1, \ldots, n$) by the rule:

$$z_{\alpha}^v = h_{(r,s)}^v z_{\alpha}^v \quad \text{on } U_r \cap U_s$$

and we observe that, according to (57), the following relation holds:

$$(h_{(r,s)}^v)^{de} = \prod_{j=1}^{k} (k_{(r,s)}^{j})^{a_{ij}d_{uv}} m_{ij}$$

if $\chi_v = \sum_{j=1}^{k} a_{ij} \psi_j$, $0 \leq a_{ij} < m_{ij}$.

In order to compute the class of the extension (40), for every $\gamma \in \Gamma$ we choose a lifting $\tilde{\gamma} \in \tilde{\Gamma}$. By Lemma 3.5, the induced map $\tilde{\gamma} : \tilde{f} \ast O_{\tilde{Y}} \to \tilde{f} \ast O_{\tilde{Y}}$ is a $O_{\tilde{X}}$-algebra isomorphism lifting $\gamma : \tilde{X} \to \tilde{X}$; in terms of the chosen trivializations we may write:

$$\tilde{z}_{\alpha}^i : \tilde{\gamma}_{\alpha}^{i} \to \tilde{\sigma}_{\alpha}^{i,j} \tilde{z}_{\alpha}^j$$

for a suitable choice of a $\tilde{d}_{\alpha}$-th root of unity $\sigma_{\alpha}^{i,j}$, $i = 1, \ldots, h$.

For later use, we write down the transition relations for the constants $\sigma_{\alpha}^{i,j}$: Let $s, t$ be such that $U_s \cap U_t \neq \emptyset$ and let $\alpha, \gamma \in \Gamma$; then, for $i = 1, \ldots, h$:

$$\sigma_{\alpha,s,t}^{i,j} \tilde{h}_{(\gamma\alpha,s,t)}^{i} = \sigma_{\alpha,\beta(s,t),t}^{i,j} \tilde{h}_{(\alpha,s,t)}^{i} \circ \gamma^{-1} \quad \text{on } V_{\gamma\alpha,s,t}$$

We now exploit the action of the chosen elements in $\tilde{\Gamma}$ in order to compute the class $c(f) \in H^2(\Gamma, K)$ associated to the extension (40), and its image $c(f) \in H^2(X, K)$.

For any given $\delta, \gamma \in \tilde{\Gamma}$, the action of $(\delta \tilde{\gamma})^{-1}_{\alpha} \tilde{\delta}_{\alpha} \tilde{\gamma}_{\alpha}$ on $L_{\tilde{X}}^{-1}$, $i = 1, \ldots, h$, is described with respect to the chosen trivializations by:

$$\tilde{z}_{\alpha}^i \mapsto \left(\sigma_{\alpha}^{i,ij}(\delta_{\gamma})^{-1}\right)^{-1} \sigma_{\gamma\alpha}^{i,j} \tilde{z}_{\alpha}^j \quad \forall r, \forall \alpha \in \Gamma, \ i = 1, \ldots, h.$$ (63)

Since (63) represents a line bundle automorphism given by a root of the unity, the expression does not depend on $(\alpha, r)$ by the connectedness of $\tilde{X}$: therefore, we may set $\alpha = 1$. So, the class $c(f) \in H^2(\Gamma, K)$ is represented by the cocycle:

$$c(f)(\delta, \gamma) = \left(\sigma_{(1,r)}^{i,i((\delta\gamma))^{-1}} \sigma_{(\gamma\alpha)}^{i,j} \sigma_{(\alpha)}^{i,j}(\delta\gamma)\right)_{i=1,\ldots,h}$$

where an element of $K \subseteq \tilde{G}$ is represented by its coordinates with respect to the basis dual to $\{\tilde{x}_1, \ldots, \tilde{x}_h\}$.

According to ([Mu], page 23), the class $c(f) \in H^2(X, K)$ is represented on $V_{(1,r,s)} \cap V_{(1,r,t)}$ by the cocycle:

$$c(f)_{r,s,t} = c(f)(\beta(r, s), \beta(s, t)) = \left(\sigma_{(1,p)}^{i,\beta(r,t)} \sigma_{(\beta(s,t),p)}^{i,\beta(r,t)} \sigma_{(1,p)}^{i,\beta(s,t)}\right)_{i=1,\ldots,h}$$

for $r, s, t$ such that $U_r \cap U_s \cap U_t \neq \emptyset$. 

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We set \( p = t \) and, by the relation \( (\mathbb{Z}) \), we rewrite \( (\mathbb{Z}) \) as follows:

\[
c(f)_{r,s,t} = \left( \left( \sigma_{(1,p)}^{i,\beta(r,t)} \right)^{-1} \sigma_{(1,t)}^{i,\beta(r,t)} \sigma_{(t,l)}^{i,\beta(s,t)} \left( \tilde{h}_i^{(r,s),s,t} \beta(s,r) \right)^{-1} \right)_{i=1,\ldots,h} \tag{66}
\]

this shows that \( c(f)_{r,s,t} \) differs from the following cocycle (that we still denote by \( c(f)_{r,s,t} \) by abuse of notation):

\[
c(f)_{r,s,t} = \left( \tilde{h}_i^{(r,s),s,t} \left( \tilde{h}_i^{(1,s),t} \circ \beta(s,r) \right)^{-1} \right)_{i=1,\ldots,h} \tag{67}
\]

by the coboundary of the cochain:

\[
g_{r,t} = \left( \sigma_{(1,t)}^{i,\beta(r,t)} \right)_{i=1,\ldots,h} \tag{68}
\]

The cochain \( g_{r,t} \) actually takes values in \( K \): in fact, it is enough to check \( g_{r,t} \) acts trivially on the eigensheaves corresponding to the chosen generators \( \chi \) of \( G^* \). This follows easily by the previous choices since the action of \( \beta(r,t) \) on \( L_{\chi}^{-1} \) is given locally by \( \prod_i h(r,t)_{\beta(i,t)} \).

**Step II:** we compute a cocycle representing \( \xi \in H^2(X,K) \).

For every \( r \) and for every \( l = 1,\ldots,q \), we choose a local generator \( y_r^l \) for \( O_X(-M_l) \) on \( U_r \); if \( M_l \) has finite order \( e \), then we require:

\[
\left( y_r^l \right)^e = 1. \tag{69}
\]

We set \( m = \text{lcm}\{m_j\}_{j=1,\ldots,k} \). For every pair of indices \((r,s)\) such that \( U_r \cap U_s \neq \emptyset \) we write:

\[
y_r^l = \mu_{(r,s)}^l y_s^l \text{ on } U_r \cap U_s \tag{70}
\]

and we choose a \( m \)-th root \( \hat{\mu}_{(r,s)}^l \) of \( \mu_{(r,s)}^l \) in such a way that \( \hat{\mu}_{(s,r)}^l = \left( \hat{\mu}_{(r,s)}^l \right)^{-1} \). Then, as in \( (\mathbb{Z}) \), (2.45), one sees that the image of the class of \(-M_l \) in \( H^2(X,\mathbb{Z}/m_l \mathbb{Z}) \) is represented on \( U_r \cap U_s \cap U_l \) by the cocycle \( \left( \hat{\mu}_{(r,s)}^l \hat{\mu}_{(s,t)}^l \hat{\mu}_{(t,l)}^l \right)^{-\frac{m}{m_j} c_{ij} \tilde{a}_{ij}} \) \( i=1,\ldots,h \).

We conclude that the class \( \xi = \Theta_*([M_1],\ldots,[M_q]) \) is represented on \( U_r \cap U_s \cap U_l \) by:

\[
\xi_{r,s,t} = \left( \prod_{j=1}^k \prod_{i=1}^q \left( \hat{\mu}_{(r,s)}^l \hat{\mu}_{(s,t)}^l \hat{\mu}_{(t,l)}^l \right)^{-\frac{m}{m_j} c_{ij} \tilde{a}_{ij}} \right)_{i=1,\ldots,h} \tag{71}
\]

**Step III:** we show that \( \xi = c(f) \).

We remark that, according to \( (\mathbb{Z}) \), the cocycle \( k_{(r,s)}^j \) in \( (\mathbb{Z}) \) representing \( O_X(-D_j) \) \( (j = 1,\ldots,k) \) and the cocycles \( \mu_{(r,s)}^l \) representing \(-M_l \) \( (l = 1,\ldots,q) \) are related as follows:

\[
k_{(r,s)}^j = \prod_{j=1}^q \left( \hat{\mu}_{(r,s)}^l \right)^{c_{ji} \frac{f_{j,s}^i}{f_{s}^j}} \tag{72}
\]

for suitable nowhere vanishing holomorphic functions \( f_r \) on \( U_r \). For every \( j = 1,\ldots,k \) and every \( r \), we choose a \( m \)-th root \( \tilde{f}_{j,s}^i \) of \( f_{j,s}^i \) on \( U_r \); then, the expression:

\[
\hat{k}_{(r,s)}^j = \prod_{j=1}^q \left( \hat{\mu}_{(r,s)}^l \right)^{c_{ji} \frac{m}{m_j} \left( \frac{\tilde{f}_{j,s}^i}{f_{s}^j} \right)^{(m/m_j)}} \tag{73}
\]
is a $m_j$-th root of the cocycle $k^j_{(r,s)}$ and, as before, the product $\prod_{j=1}^g (\hat{\mu}^j_{(r,s)}\hat{\mu}^j_{(s,t)}\hat{\mu}^j_{(t,r)})^{c_{j}(m/m_j)}$ yields a cocycle representing the image of the class of $-D_j$ in $H^2(X,\mathbb{Z}/m_j\mathbb{Z})$. In this notation, by (34), we rewrite as follows the cocycle in (71) representing the class $\xi$:

$$\xi_{r,s,t} = \left( \prod_{j=1}^k \left( k^j_{(r,s)}k^j_{(s,t)}k^j_{(t,r)} \right)^{-\hat{a}_{ij}} \right)_{i=1,\ldots,h}.$$  \hspace{1cm} (74)

Let $\epsilon = \exp(\frac{2\pi \sqrt{-1}}{m})$. Then, by (33), one has:

$$\hat{h}^i_{(\alpha,r,s)} = \prod_{j=1}^k \left( \hat{k}^j_{(r,s)} \right)^{\hat{a}_{ij}} \epsilon^{-q^j_{(\alpha,r,s)}}.$$  \hspace{1cm} (75)

where $q^j_{(\alpha,r,s)}$ is an integer, multiple of $m/\hat{a}_i$, and (75) may be rewritten as:

$$c(f)_{r,s,t} = (\epsilon^{q^j_{(1,s,t)}-q^j_{(\beta(r,s),s,t)}})_{i=1,\ldots,h}.$$  \hspace{1cm} (76)

From the cocycle condition (54) for $\hat{h}^i_{(\alpha,r,s)}$, it follows:

$$\xi_{r,s,t} = (\epsilon^{-q^j_{(\alpha,r,s)}-q^j_{(\beta(r,s),s,t)}-q^j_{(\alpha\beta(r,t),t,r)}})_{i=1,\ldots,h} \hspace{0.5cm} \forall \hspace{0.1cm} r, s, t, \forall \alpha \in \Gamma.$$  \hspace{1cm} (77)

In particular, for $\alpha = 1$, one gets:

$$\xi_{r,s,t} = (\epsilon^{-q^j_{(1,r,s)}-q^j_{(\beta(r,s),s,t)}-q^j_{(\alpha\beta(r,t),t,r)}})_{i=1,\ldots,h}.$$  \hspace{1cm} (78)

So, one has:

$$c(f)_{r,s,t} = \xi_{r,s,t}(\epsilon^{q^j_{(1,r,s)}+q^j_{(1,s,t)}+q^j_{(\beta(r,t),t,r)}})_{i=1,\ldots,h}.$$  \hspace{1cm} (79)

By the definition of $q^j_{(\alpha,r,s)}$, this equality can be rewritten as follows:

$$c(f)_{r,s,t} = \xi_{r,s,t}(\epsilon^{q^j_{(1,r,s)}+q^j_{(1,s,t)}-q^j_{(1,t,r)}})_{i=1,\ldots,h}.$$  \hspace{1cm} (80)

To complete the proof of the theorem, we show that we can choose the $m$-th root $\hat{f}_j$ of $f_j$ ($j = 1, \ldots, k$) so that:

$$q_{(1,r,s)} = \left( \epsilon^{q^j_{(1,r,s)}} \right)_{i=1,\ldots,h}$$  \hspace{1cm} (81)

is an element of $K$, $\forall (r, s)$.

Let $\chi_v$ one of the chosen generators of $G^*$. According to (50), (51) and (75), the action of $q_{(1,r,s)}$ on $L^{-1}_{\chi_v}$ is given by a $d_v$-th root of unity, that we denote by $\exp(2\pi \sqrt{-1} \frac{x_v}{d_v})$ (for a suitable integer $x_v$). We want to show that we can assume that $x_v \equiv 0 \mod d_v$, $v = 1, \ldots n$.

We observe that:

$$\exp(2\pi \sqrt{-1} \frac{x_v}{d_v}) = \prod_{i=1}^h \epsilon^{q^i_{(1,r,s)}b_{vi}} = (\hat{h}^v_{(1,r,s)})^{-1} \prod_{j=1}^k (\hat{k}^j_{(r,s)})^{\epsilon_{v,j}}.$$  \hspace{1cm} (82)
and we compute the right-hand side of (82). By (73), one must have:

\[ h^v_{(1,r,s)} = \exp(-2\pi \sqrt{1 - \frac{z}{d_v}} \prod_{j=1}^k \hat{k}^j_{(r,s)} a_{w_j}) \]
\[ = \exp(-2\pi \sqrt{1 - \frac{z}{d_v}} \prod_{l=1}^q \hat{\mu}^l_{(r,s)} \sum_{j=1}^k \frac{c_{l,j} d_{w_j}}{m_j} \prod_{j=1}^k \left( \frac{f^j}{f^j_s} \right) a_{w_j} (m/m_j)) \] (83)

On the other hand, as in Lemma 4.2, we write

\[ L^\chi_v \equiv \sum_{l=1}^q \lambda_{vl} M^l \]

and we compute the following relation form of cocycles on \( V_{(1,r,s)} \):

\[ h^v_{(1,r,s)} = \prod_{l=1}^q (\mu^l_{(r,s)} \lambda_{vl}) \frac{\varphi^v_r}{\varphi^v_s} \] (84)

for suitable nowhere vanishing holomorphic functions \( \varphi^v_r \) on \( U_r \). According to Lemma 4.1 and to (69), we can then assume that in the previous equations one has:

\[ \prod_{l=1}^q (\hat{\mu}^l_{(r,s)} \sum_{j=1}^k \frac{c_{l,j} d_{w_j}}{m_j}) - d_v \lambda_{vl} = 1 \] (85)

so that one gets:

\[ \exp(2\pi \sqrt{1 - \frac{z}{d_v}} \prod_{j=1}^k \left( \frac{\hat{f}^j}{\hat{f}^j_s} \right) a_{w_j} (m/m_j)) = \frac{\varphi^v_r}{\varphi^v_s}. \] \( \text{(86)} \)

We observe that we may assume that:

\[ \varphi^v_r = \exp(2\pi \sqrt{1 - \frac{z}{d_v}} t^v \prod_{j=1}^k \left( \frac{\hat{f}^j}{\hat{f}^j_s} \right) a_{w_j} (m/m_j)) \] (87)

for suitable integers \( t^v_r \); hence the equation (86) gives:

\[ \frac{x^v}{d_v} - \frac{t^v_s}{d_v} + \frac{t^v_r}{d_v} \in \mathbb{Z}. \] \( \text{(88)} \)

If we replace \( \hat{f}^j_r \) by \( \exp(2\pi \sqrt{1 - \frac{z}{d_v}} f^j_r) \), then \( t^v_s + \sum_{j=1}^k a_{w_j s^j_r} \) is replaced by \( t^v_s + \sum_{j=1}^k a_{w_j s^j_r} \). Therefore, we need to solve the system:

\[ \sum_{j=1}^k a_{w_j s^j_r} \equiv t^v_r \mod d_v \quad v = 1, \ldots, n. \] \( \text{(89)} \)

Since this is possible according to Lemma 4.1 and the Chinese Remainder’s Theorem, the proof is complete.

**Remark 4.5** The cohomology class of the extension (40) of the fundamental groups depends on
the choice of the solution \( \{L^\chi\} \) of the characteristic relations (8) for the covering \( f \). Moreover, covers corresponding to different solutions \( \{L^\chi\} \) may not be homeomorphic.

This is shown, for instance, by the following class of examples. Denote by \( e_i \) the standard generators of the group \((\mathbb{Z}/4\mathbb{Z})^3\) and let \( G \) be the quotient of \((\mathbb{Z}/4\mathbb{Z})^3\) by the subgroup generated
by \(2e_1 + 2e_2 + 2e_3\). Let now \(X\) be a smooth projective surface such that \(H^2(\pi_1(X), \mathbb{Z}/2\mathbb{Z}) \neq 0\) and \(\text{Pic}(X)\) has a 2-torsion element \(\eta\) whose class in \(H^2(X, \mathbb{Z}/2\mathbb{Z})\) is non zero. Fix a very ample divisor \(H\) on \(X\) and choose suitable divisors \(D_i\) \((i = 1, 2, 3)\) such that \(D_i \equiv 4H\) and the \(D_i\)'s are in general position. Then there exists a smooth abelian \(G\)-cover \(f : Y \to X\) ramified on the \(D_i\)'s \((i = 1, 2, 3)\), with inertia subgroup \(G_i = \langle e_i \rangle = \mathbb{Z}/4\mathbb{Z}\) and character \(\psi_i\) dual to \(e_i\), respectively. In fact, taking the characters \(\chi_1 = \psi_1 + 3\psi_3, \chi_2 = \psi_2 + 3\psi_3, \chi_3 = 2\psi_3\) as generators of \(G^*\), the characteristic relations (8) of the cover \(f\) induce a central extension of the form:

\[
\begin{cases}
4L_1 & \equiv D_1 + 3D_3 \\
4L_2 & \equiv D_2 + 3D_3 \\
2L_3 & \equiv D_3
\end{cases}
\]  

and admit, in particular, the solution \(L_1 = L_2 = L_3 = 2H\). Under these hypotheses, \(L_3\) generates the subgroup \(\langle D_i, L_X \rangle (i = 1, 2, 3, \chi \in G^*)\) of \(\text{Pic}(X)\) and the decomposition \(D_i \equiv 2L_3\) has the properties requested in Prop.4.2. According to Prop.3.2 and Thm.1.4, since the pull back \(\tilde{D}_i\) of \(D_i\) under the universal cover \(\tilde{X}\) of \(X\) is 2-divisible, \(\forall i\), then the map \(f\) induces a central extension of the form:

\[
0 \to \mathbb{Z}/2\mathbb{Z} \to \pi_1(Y) \to \pi_1(X) \to 1
\]  

and the cohomology class of this extension in \(H^2(\pi_1(X), \mathbb{Z}/2\mathbb{Z}) \subseteq H^2(X, \mathbb{Z}/2\mathbb{Z})\) is the image \(\Psi_*([L_3])\) of the Chern class of \(L_3\) under the map induced in cohomology by the standard projection \(\Psi : \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}\): so, this class is trivial.

Let now \(\overline{Y}\) be the \(G\)-cover of \(X\) corresponding to the solution \(\overline{T}_i = 2H + \eta (i = 1, 2, 3)\) of (91); in this case the cohomology class describing \(\pi_1(\overline{Y})\) is given by \(\Psi_*([L_3 + \eta])\) and, by the hypotheses made, it is not trivial.

In particular, when \(X\) is a projective variety with \(\pi_1(X) = \mathbb{Z}/2\mathbb{Z}\), the previous construction yields two non homeomorphic \(G\)-covers \(Y, \overline{Y}\) of \(X\), branched on the same divisor, with the same inertia subgroups and characters, such that:

\[
\pi_1(Y) = (\mathbb{Z}/2\mathbb{Z})^2, \quad \pi_1(\overline{Y}) = \mathbb{Z}/4\mathbb{Z}.
\]  

The following theorem is an attempt to determine to what extent the class \(c(f)\) depends on the choice of the \(L_X\)'s, once the branch divisor and the covering structure are fixed.

**Theorem 4.6** Same hypotheses and notation as in the statement of Thm.4.4. Consider the class \(c(f) \in H^2(\pi_1(X), K)\) associated to the central extension (41) given by the fundamental groups and denote by \(i(c(f)) \in H^2(\pi_1(X), \tilde{G}) \subseteq H^2(X, \tilde{G})\) its image via the map induced in cohomology by the inclusion (13) \(K \subseteq \tilde{G}\).

Denote by \(\Phi\) the group homomorphism defined as follows:

\[
\Phi : \mathbb{Z}^k \to \tilde{G}, \quad (x_1, \ldots, x_k) \to g_1^{x_1} \cdots g_k^{x_k}
\]  

and by \(\Phi_* : H^2(X, \mathbb{Z}^k) \to H^2(X, \tilde{G})\) the map induced by \(\Phi\) in cohomology. Then:

\[
i(c(f)) = \Phi_*([D_1], \ldots, [D_k])
\]  

where \([\Delta]\) denotes the class of a divisor \(\Delta\) on \(X\) in \(H^2(X, \mathbb{Z})\).
Corollary 4.7 Same hypotheses and notation as in Thm 4.4. Assume moreover that the natural morphism $\text{Hom}(\pi_1(X), \hat{G}) \to \text{Hom}(\pi_1(X), G)$, induced by the surjection $\hat{G} \to G$, is surjective. Then the map $i : H^2(\pi_1(X), K) \to H^2(\pi_1(X), \hat{G})$ is injective and the class $\Phi_*([D_1], \ldots, [D_k])$ in $(\mathcal{D})$ determines uniquely the class $c(f) \in H^2(\pi_1(X), K)$ of the extension $(\mathcal{D})$ of the fundamental groups.

This happens, in particular, if $\pi_1(X)$ is torsion free or $\text{Hom}(\pi_1(X), G) = 0$ (e.g., if $\pi_1(X)$ is finite with order coprime to the order of $G$), or the sequence $0 \to K \to \hat{G} \to G \to 0$ splits.

Proof of Thm 4.4. We keep the notation and the results in Step I of the proof of Thm 4.4, noticing that the cocycle $c(f)_{r,s,t}$ in $(\mathcal{D})$ also represents the class $i(c(f))$ in $H^2(X, \hat{G})$.

We want to write down a cocycle representing the class $\Phi_*([D_1], \ldots, [D_k]) \in H^2(X, \hat{G})$ and to show that it represents the same cohomology class as the cocycle in $(\mathcal{D})$.

We consider as before the cocycle $k^j_{(r,s)}$ representing $\mathcal{O}_X(-D_j)$ in the choosen covering $U_r$ of $X$. For every pair of indices $r, s$ with $U_r \cap U_s \neq \emptyset$ and for every $j = 1, \ldots, k$, we choose a $m_j$-th root $\hat{k}^j_{(r,s)}$ of $k^j_{(r,s)}$ on $U_r \cap U_s$ in such a way that $\hat{k}^j_{(s,r)} = (\hat{k}^j_{(r,s)})^{-1}$. As before, the image of the class of $-D_j$ in $H^2(X, \mathbb{Z}/m_j \mathbb{Z})$ is represented on $U_r \cap U_s \cap U_t$ by the cocycle $\hat{k}^j_{(r,s)} \hat{k}^j_{(s,t)} \hat{k}^j_{(t,r)}$, $j = 1, \ldots, k$. Then the class $-\Phi_*([D_1], \ldots, [D_k]) \in H^2(X, \hat{G})$ is represented on $U_r \cap U_s \cap U_t$ by:

$$b_{r,s,t} = \left( \prod_{j=1}^k (\hat{k}^j_{(r,s)} \hat{k}^j_{(s,t)} \hat{k}^j_{(t,r)})^{a_{ij}} \right)_{i=1,\ldots,h} . \tag{95}$$

and we have shown in the equality $(\mathcal{F})$ in the proof of Thm 4.4, Step III, that this cocycle represents the same class then $c(f)_{r,s,t}$ in $H^2(X, \hat{G})$.

Remark 4.8 From Thm 4.4, it follows in particular that the class $i(c(f)) \in H^2(\Gamma, \hat{G})$ depends only on the class of the $D_j$’s in $H^2(X, \mathbb{Z}/m_j \mathbb{Z})$ ($j = 1, \ldots, k$), once $G$ and the $g_j$’s are fixed. In particular, if $D_j$ is $m_j$-divisible on $X$ ($\forall j = 1, \ldots, k$), then $i(c(f)) = 0$.

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