2-step nilpotent Lie groups and hyperbolic automorphisms

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Abstract

A connected Lie group admitting an expansive automorphism is known to be nilpotent but all nilpotent Lie groups do not admit expansive automorphisms. In this article, we find sufficient conditions for 2-step nilpotent Lie groups to admit expansive automorphisms.

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1 Introduction

Let $G$ be a locally compact (second countable) group and Aut($G$) be the group of (continuous) automorphisms of $G$. We consider automorphisms $\alpha$ of $G$ that have a specific property that there is a neighborhood $U$ of identity such that $\cap_{n \in \mathbb{Z}} \alpha^n(U) = \{e\}$ - such automorphisms are known as expansive. Since we are interested in Lie group, in the Lie theory context, expansive automorphisms are better known as hyperbolic automorphisms.

Structure of locally compact groups admitting expansive automorphisms have been studied for solvable groups [1], for compact totally disconnected groups [11], [18] and recently for locally compact totally disconnected groups [10]. It follows from Proposition 7.1 of [10] that any connected (real) Lie group admitting a hyperbolic automorphism is a nilpotent Lie group. By considering the one-dimensional torus, we observe that there are nilpotent Lie groups that doesn’t admit any hyperbolic automorphisms. We now attempt to classify which nilpotent groups admit hyperbolic automorphisms. In this note we consider the two-step nilpotent Lie groups.
Now let $N$ be a 2-step nilpotent Lie group and $\mathfrak{N}$ be its Lie algebra. Then we know that the exponential map $\exp : \mathfrak{N} \to N$ is surjective and the kernel of the map is a discrete subgroup $\Lambda$ of the center of $\mathfrak{N}$ (see [17] for details). We call the Lie algebra $\mathfrak{N}$ of type $(p, q)$ if $[\mathfrak{N}, \mathfrak{N}]$ has dimension $p$ and the complement of $[\mathfrak{N}, \mathfrak{N}]$ in $\mathfrak{N}$ has dimension $q$. Since $\mathfrak{N}$ is 2-step nilpotent, $[\mathfrak{N}, \mathfrak{N}]$ is contained in the center of $\mathfrak{N}$. In this article, we consider those 2-step nilpotent Lie groups for which $\Lambda$ is contained inside $[\mathfrak{N}, \mathfrak{N}]$. Being a discrete subgroup of the vector space $[\mathfrak{N}, \mathfrak{N}]$, $\Lambda$ is a vector space lattice inside $[\mathfrak{N}, \mathfrak{N}]$. For this article, we also assume that $\Lambda$ is a full lattice inside $[\mathfrak{N}, \mathfrak{N}]$ and call the Lie group $N$ of the form $[\mathfrak{N}, \mathfrak{N}]^c \oplus [\mathfrak{N}, \mathfrak{N}]/\Lambda$ a 2-step nilpotent Lie group of type $(p, q)$.

In the next section, we will see that any 2-step nilpotent Lie algebra is isomorphic to a Lie algebra of the form $\mathbb{R}^q \oplus W$ (see next section for the definition of the Lie bracket operation on this Lie algebra), where $W$ is a $p$-dimensional subspace of $\text{so}(q, \mathbb{R})$, the set of $q \times q$ skew symmetric real matrices. So, any 2-step nilpotent Lie group of type $(p, q)$ is isomorphic to $\mathbb{R}^q \oplus W/\Lambda$, $\Lambda$ being a vector space lattice of rank $p$ in $W$. $\text{GL}(q, \mathbb{R})$ acts on $\mathbb{R}^q \oplus \text{so}(q, \mathbb{R})$ by automorphism (see next section for details), where the action is given as follows: for $g \in \text{GL}(q, \mathbb{R})$ and $(v, w)$ in $\mathbb{R}^q \oplus \text{so}(q, \mathbb{R})$, $g(v, w) = (gv, gwg^t)$. Note that the action of $\text{GL}(q, \mathbb{R})$ on $\text{so}(q, \mathbb{R})$ given by $g(w) = gwg^t$ for $g \in \text{GL}(q, \mathbb{R})$ and $w \in \text{so}(q, \mathbb{R})$, gives rise to a representation $\rho$ (say) of $\text{GL}(q, \mathbb{R})$ inside $\text{GL}(\text{so}(q, \mathbb{R}))$.

For any $p$-dimensional subspace $W$ of $\text{so}(q, \mathbb{R})$, let $\text{GL}(q, \mathbb{R})_W$ be the stabilizer subgroup of $\text{GL}(q, \mathbb{R})$ corresponding to $W$, that is

$$\text{GL}(q, \mathbb{R})_W = \{g \in \text{GL}(q, \mathbb{R}) : gwg^t \in W, \text{ whenever } w \in W \}.$$ 

Then $\text{GL}(q, \mathbb{R})_W$ acts by automorphism on the Lie algebra $\mathbb{R}^q \oplus W$. The restriction of $\rho$ to $\text{GL}(q, \mathbb{R})_W$ is a representation of $\text{GL}(q, \mathbb{R})_W$ inside $\text{GL}(W)$ and we denote this representation by $\rho_W$. Let $\mathcal{O}$ be the subset of the Grassmann manifold $G(p, \text{so}(q, \mathbb{R}))$ for which the corresponding stabilizer subgroups have minimum dimension among the stabilizer subgroups corresponding to all the elements of $G(p, \text{so}(q, \mathbb{R})).$ In this article, we provide sufficient conditions for a large class of 2-step nilpotent Lie groups to admit hyperbolic automorphism.

**Theorem 1.1.** Let $N$ be a 2-step nilpotent Lie group of type $(p, q)$, i.e., $N$ is isomorphic to a Lie group of the form $\mathbb{R}^q \oplus W/\Lambda$, where $W$ is a $p$-dimensional subspace of $\text{so}(q, \mathbb{R})$, and $\Lambda$ is a lattice of rank $p$ inside $W$. If either $W$ is defined over the rational numbers or $W$ is in $\mathcal{O}$ and $\rho_W(\text{GL}(q, \mathbb{R})_W)$ is non-amenable, then $N$ admits a hyperbolic automorphism.
Remark 1.2. Note that in the above theorem, \( p \) is necessarily at least 2. Because if \( W \) is one-dimensional, then \( \rho_W(G_W) \) is abelian and hence amenable (See [12] for definition of amenability and other details). Consequently, \( q \) is at least 3. Some examples of both \( \rho_W(G_W) \) being amenable and non-amenable are discussed at the end of this article.

Remark 1.3. Dixmier and Lister [6] provides an example of a 3-step nilpotent Lie algebra \( g \) all whose derivations are nilpotent. Thus, if \( G \) is the simply connected nilpotent Lie group with Lie algebra \( g \), then the connected component of \( \text{Aut}(G) \) is a unipotent group. Since \( \text{Aut}(G) \) is an algebraic group, it has only finitely many connected components and hence no automorphism of \( G \) is hyperbolic. Thus, \( G \) is a 3-step simply connected nilpotent Lie group that does not admit any hyperbolic automorphism but simply connected 2-step nilpotent Lie groups admit expansive automorphisms (see section 2 and [10]).

2 Preliminaries

Let \( G \) be a locally compact (second countable) group and \( \text{Aut}(G) \) be the group of (continuous) automorphisms of \( G \). An automorphism \( \alpha \) of \( G \) is called expansive if there is a neighborhood \( U \) of the identity \( e \) in \( G \) such that \( \bigcap_{n \in \mathbb{Z}} \alpha^n(U) = \{e\} \).

Let \( G \) be a connected 2-step nilpotent Lie group and \( g \) be its Lie algebra. Then we know that the exponential map \( \exp: g \to G \) is surjective and as a Lie group \( G \) is isomorphic to \( g/\Gamma \), where \( \Gamma \) is a discrete subgroup of the center of \( g \). Given an \( \alpha \in \text{Aut}(G) \), let \( d\alpha \) denote the differential of \( \alpha \) which is a Lie algebra automorphism of \( g \). By Proposition 7.1 of [10], \( \alpha \in \text{Aut}(G) \) is hyperbolic if and only if \( d\alpha \) as a linear transformation is hyperbolic on \( g \). Therefore, we look for hyperbolic automorphism of a 2-step nilpotent Lie algebra preserving a given lattice inside its center.

We now recall general structure of 2-step nilpotent Lie groups, most of this materials are from [7] and [5]. Let the vector space \( F_2(q) = \mathbb{R}^q \oplus \text{so}(q, \mathbb{R}) \) be equipped with the inner product \( <,> \) such that the two direct summands are orthogonal to each other; when restricted to the first summand it is the Euclidean inner product on \( \mathbb{R}^q \) and for \( Z, Z' \) in \( \text{so}(q, \mathbb{R}) \) \( < Z, Z'> = -\text{trace}(ZZ') \). One can define a Lie bracket operation \([,]\) on \( F_2(q) \) as follows: for \( X, Y \in \mathbb{R}^q \),

\[
[X, Y] = -\frac{1}{2}\{XY^t - YX^t\},
\]

and declaring \( \text{so}(q, \mathbb{R}) \) to be the center of the Lie algebra. \( F_2(q) \) is called a standard metric free 2-step nilpotent Lie algebra on \( q \) generators.
A standard metric 2-step nilpotent Lie algebra of type \((p, q)\) with \(1 \leq p \leq \frac{1}{2}q(q-1)\) is of the form \(\mathbb{R}^q \oplus W\), where \(W\) is a \(p\)-dimensional subspace of \(\text{so}(q, \mathbb{R})\). The Lie bracket structure \([,]\) on \(\mathbb{R}^q \oplus W\) is defined as follows: if \(X, Y \in \mathbb{R}^q\) and \(Z \in W\), then
\[
< [X, Y], Z > = < Z(X), Y > .
\]
\(\mathbb{R}^q \oplus W\) is a quotient of \(F_2(q)\) by \(W^\perp\), where \(W^\perp\) is the orthogonal complement of \(W\) inside \(\text{so}(q, \mathbb{R})\) with respect to the inner product described above. Any 2-step nilpotent Lie algebra can be seen to be isomorphic to one of the standard metric 2-step nilpotent Lie algebra of type \((p, q)\) (see [7] for details). So from now on we will consider the Lie algebras of the form \(\mathbb{R}^q \oplus W\) as described above.

The group \(GL(q, \mathbb{R})\) acts on \(F_2(q)\) by automorphism as follows: for \(g\) in \(GL(q, \mathbb{R})\) and for \((v, w)\) in \(F_2(q)\),
\[
g(v, w) = (gv, gw^t)
\]
and \(GL(q, \mathbb{R})_W = \{ g \in GL(q, \mathbb{R}) : g(W) = W\}\) acts by automorphism on \(\mathbb{R}^q \oplus W\) (see [7] for details). It follows from [7] and [8] that a 2-step nilpotent Lie algebra \(\mathbb{R}^q \oplus W\) admits a hyperbolic automorphism if and only if an element of \(GL(q, \mathbb{R})_W\) acts hyperbolically on \(\mathbb{R}^q \oplus W\).

In case the 2-step nilpotent Lie group \(G\) is a simply connected nilpotent Lie group, then any Lie algebra automorphism of its Lie algebra \(\mathfrak{g}\) is the differential of an automorphism of \(G\). Since \(rI_q\) (\(I_q\) being the \(q \times q\) identity matrix) acts hyperbolically on \(\mathbb{R}^q \oplus W\), for any subspace \(W\) of \(\text{so}(q, \mathbb{R})\) and for \(r \neq \pm 1\), we get that any simply connected 2-step nilpotent Lie group admits hyperbolic automorphisms - in fact, it is known that simply connected 2-step nilpotent Lie groups are contractive (see [16] for more details and any unexplained notions).

As the simply connected case is dealt with rather easily, we now look at the non-simply connected cases. If \(G\) is a 2-step nilpotent Lie group, then there is a discrete subgroup \(\Gamma\) of the center of \(\mathfrak{g}\) such that any Lie algebra automorphism of \(\mathfrak{g}\) that preserves \(\Gamma\) is a differential of an automorphism of \(G\). This forces us to look at the action of \(SL(q, \mathbb{R})\) on \(F_2(q)\) and \(SL(q, \mathbb{R})_W = \{ g \in SL(q, \mathbb{R}) : g(W) = W\}\) action on \(\mathbb{R}^q \oplus W\).

Consider the action of \(SL(q, \mathbb{R})\) on \(\text{so}(q, \mathbb{R})\) given by
\[
g(w) = gwg^t
\]
for \(g \in SL(q, \mathbb{R})\) and \(w \in \text{so}(q, \mathbb{R})\), which is an algebraic action defined over \(\mathbb{Q}\). This gives rise to the representation \(\rho\) of \(SL(q, \mathbb{R})\) inside \(GL(\text{so}(q, \mathbb{R}))\), which
is an algebraic representation defined over \( \mathbb{Q} \). Let \( g \in \text{SL}(q, \mathbb{R}) \) be given by the matrix \((g)_{ij}\). We wish to write the matrix of \( \rho(g) \) in the standard orthonormal basis of \( \text{so}(q, \mathbb{R}) \). \( \rho(g) \) is an \( N \times N \) matrix where \( N = \frac{q(q-1)}{2} \).

Each row (column) of \( \rho(g) \) corresponds to a member of the double indexed set \( I = \{ij : 1 \leq i < j \leq q\} \) which has \( N \) number of elements because of the following: if \( \{e_1, \ldots, e_q\} \) denotes the standard basis of \( \mathbb{R}^q \), then

\[
e_{ij} := [e_i, e_j]
\]

is one of the elements of the standard orthonormal basis of \( \text{so}(q, \mathbb{R}) \) with respect to the inner product mentioned above and each element of the standard orthonormal basis of \( \text{so}(q, \mathbb{R}) \) is determined by (2). Now let \( \rho(g)_{ij} \) be an entry of the matrix \( \rho(g) \). Then \( i \) corresponds to two rows of the matrix \( g \) and we denote them by \( i_1 \) and \( i_2 \). Also, \( j \) corresponds to two columns of the matrix \( g \) which we denote by \( j_1 \) and \( j_2 \). Then a straightforward calculation shows that \( \rho(g)_{ij} = g_{i_1 j_2} g_{i_2 j_1} - g_{i_2 j_2} g_{i_1 j_1} \). That is \( \rho(g)_{ij} \) is the determinant of the \( 2 \times 2 \) minor of \( g \) formed by the \( i_1 \)th \( i_2 \)th rows and \( j_1 \)th, \( j_2 \)th columns of \( g \). This shows in particular that the representation \( \rho \) is in fact defined over \( \mathbb{Z} \). \( \text{SL}(q, \mathbb{R}) \) is an algebraic group defined over \( \mathbb{Q} \) if \( W \) itself is defined over \( Q \). We denote \( \text{SL}(q, \mathbb{R})_W \) by \( G_W \). If \( W \) is defined over \( \mathbb{Q} \), then it has a basis \( \{w_1, w_2, \ldots, w_p\} \) such that each of \( w_i \) is a \( \mathbb{Q} \)-linear combination of the standard orthonormal basis elements of \( \text{so}(q, \mathbb{R}) \).

Let \( V \) be a real vector space of dimension \( N \). For \( 1 \leq k < N \), Let \( G(k, V) \) denote the Grassmann manifold of \( k \)-dimensional subspaces of \( V \). Let \( W \in G(k, V) \). A local chart around \( W \) is given as follows. Choose a basis \( \{w_1, \ldots, w_k\} \) of \( W \) and form the matrix \( M_W \) consisting of \( w_1, \ldots, w_k \) as its column vectors. This matrix has rank \( k \) and therefore choosing \( k \) rows and naming them by \( i_1 \)th, \ldots, \( i_k \)th rows, construct a non-singular \( k \times k \) matrix \( M_W^k \). Now a local chart around \( W \) is given by \((U_W, F_W)\), where \( U_W \) consists of those \( k \)-dimensional subspaces of \( V \) such that if \( \{v_1, \ldots, v_k\} \) is a basis of \( V \), then the \( k \times k \) matrix \( M_V^k \) consisting of the \( i_1 \)th, \ldots, \( i_k \)th rows of \( M_V \), is nonsingular. The chart map \( F_W \) is such that \( F_W(V) \) is the \((n-k) \times k \) matrix \( M_V^{n-k} \) consisting of the remaining \( n-k \) rows of \( M_V \). We are also going to use another fact about the Grassmannian. If we equip \( V \) with an inner product, then

\[
d(W, W') = \|P_W - P_{W'}\|
\]

defines a metric on the Grassmannian, where \( P_W \) denotes the projection onto the subspace \( W \).

For the action of \( \text{SL}(q, \mathbb{R}) \) on \( \text{so}(q, \mathbb{R}) \) defined in (1), let \( d(W) \) be the dimension of the isotropy group \( \text{SL}(q, \mathbb{R})_W \). Let

\[
d(p, q) = \min\{d(W) : W \in G(p, \text{so}(q, \mathbb{R}))\}.
\]
It is well known that there exists an open Zariski dense subset \( \mathcal{O} \) of \( G(p, \text{so}(q, \mathbb{R})) \) (see \cite{7} for details) such that
\[
d(W) = d(p, q) = \min\{d(W) | W \in G(p, \text{so}(q, \mathbb{R}))\}
\]
for all \( W \) in \( \mathcal{O} \). In many cases \( d(p, q) = 0 \) and then \( \text{SL}(q, \mathbb{R}) \) is finite. See \cite{8} to get a complete list of \( (p, q) \) when \( d(p, q) \) is positive. Since in Theorem 1.1, non-amenability of \( \text{GL}(q, \mathbb{R}) \) is a part of the hypothesis and which is not the case if \( \text{SL}(q, \mathbb{R}) \) is finite, we will consider the cases when \( d(p, q) \neq 0 \) unless \( W \) is defined over \( \mathbb{Q} \).

3 On the space of closed subgroups

Given a locally compact (second countable) group \( G \), let \( \text{Sub}_G \) denote the space of closed subgroups of \( G \) equipped with Chabauty topology introduced by Claude Chabauty in \cite{5}. In this topology, a sequence \( H_n \in \text{Sub}_G \) converges to a limit \( H \) if and only if for any \( h \in H \) there is a sequence \( h_n \in H_n \) such that \( h_n \) converges to \( h \) and for any sequence \( h_{k_n} \in H_{k_n} \), with \( k_{n+1} > k_n \), which converges to a limit, the limit is in \( H \). With respect to the Chabauty topology, \( \text{Sub}_G \) is a compact metric space (see \cite{9} for more details).

Example 3.1. It is well-known that closed subgroups of \( \mathbb{R} \) are either discrete groups generated by a real number or the whole group \( \mathbb{R} \). This implies that \( \text{Sub}_\mathbb{R} \) is homeomorphic to \([0, \infty]\) as each positive real number is identified with the subgroup generated by it and \( \infty \) is identified with the whole group \( \mathbb{R} \): a sequence of subgroups that converge to \( \mathbb{R} \) is given by \( \frac{1}{n} \mathbb{Z} \).

In the case of closed subgroups of a finite-dimensional vector space \( V \), apart from the space \( \text{Sub}_V \), we also have the Grassmann \( G(k, V) \), the space of all \( k \)-dimensional subspaces of \( V \). Since subspaces of \( V \) are closed subgroups, \( G(k, V) \subset \text{Sub}_V \). We now prove that the inclusion is continuous when \( G(k, V) \) is equipped with its Grassmannian topology.

Lemma 3.2. The inclusion map \( \iota : G(k, V) \to \text{Sub}_V \) is a continuous map and \( G(k, V) \) is homeomorphic to \( \iota(G(k, V)) \). In particular, \( G(k, V) \) may be identified as a closed subset of \( \text{Sub}_V \).

Proof. Let \( W_n, W \in G(k, V) \) and \( W_n \to W \) in \( G(k, V) \), that is
\[
\|P_{W_n} - P_W\| \to 0 \text{ as } n \to \infty.
\]
Now let \( w \in W \), we want to show that there exist \( w_n \in W_n \) such that \( w_n \to w \). Let \( w_n = P_{W_n}(w) \). Then clearly \( w_n \in W_n \) and
\[
\|w_n - w\| = \|P_{W_n}(w) - P_W(w)\| \leq \|P_{W_n} - P_W\||w|| \to 0
\]
as $n \to \infty$. On the other hand assume that there exist $w_{k_n} \in W_{k_n}$ such that $w_{k_n} \to w$, we want to show that $w \in W$. It is enough to show that $P_{W}(w) = w$. Since $w_{k_n} \to w$ and $P_{W_{k_n}} \to P_{W}$, it follows that $P_{W_{k_n}}(w_{k_n}) \to P_{W}(w)$. But as $P_{W_{k_n}}(w_{k_n}) = w_{k_n}$, and $w_{k_n} \to w$, it follows that $P_{W}(w) = w$ and the proof is done.

We now look at the dimension function on $\text{Sub}_G$ when $G$ is a real Lie group. By Cartan’s Theorem closed subgroups of real Lie groups are real Lie groups (cf. Chapter V, Section 9 of [15]). Thus, for a Lie group $G$ and $H \in \text{Sub}_G$, dimension of $H$ is well-defined. Let $\dim : \text{Sub}_G \to \mathbb{N} \cup \{0\}$ be the dimension function where $\dim(H)$ is defined to be the dimension of $H$.

**Proposition 3.3.** Let $G$ be a real Lie group. Then $\dim$ is upper semi-continuous.

**Remark 3.4.** It may be noted that in general we can’t expect continuity of $\dim$ functions. Consider the $\text{Sub}_\mathbb{R}$ in Example 3.1, $\mathbb{Z} \to \mathbb{R}$ but $\dim(\mathbb{Z}) = 0$ and $\dim(\mathbb{R}) = 1$.

**Proof.** Let $H_n \to H$ in $\text{Sub}_G$. We first prove that $\dim(H) \geq r$ if $\dim(H_n) = r$ for some $r \in \mathbb{Z}$ and for all $n \in \mathbb{N}$.

Assume $\dim(H_n) = r$. Let $\mathfrak{H}_n$, $\mathfrak{H}$ denote the Lie algebras of $H_n$, $H$ respectively. Also, let $\mathfrak{G}$ denote the Lie algebra of $G$. Now $(\mathfrak{H}_n)$ being a sequence in the Grassmann manifold $G(r, \mathfrak{G})$ has a convergent subsequence. Therefore, passing to a subsequence, we may assume that $\mathfrak{H}_n$ converges to some $W \in G(r, \mathfrak{G})$. Then by Lemma [3.2], $\mathfrak{H}_n$ converges to $W$ in $\text{Sub}_\mathfrak{G}$ as well. Now for $w \in W$, there exist $w_n \in \mathfrak{H}_n$ such that $w_n \to w$. Consequently, $tw_n \to tw$ for all $t \in \mathbb{R}$. Then $\exp(tw_n) \to \exp(tw)$ for all $t \in \mathbb{R}$, where $\exp : \mathfrak{G} \to G$ is the exponential map. But as $H_n \to H$ in $\text{Sub}_G$, it follows that $\exp(tw) \in H$ for all $t \in \mathbb{R}$. Hence $W \subset \mathfrak{H}$. This shows that $\dim(H) \geq r$.

Now suppose $\dim(H_n) > \dim(H)$ for infinitely many $n$’s. Since $\dim(H_n) \leq \dim(G)$, we can extract a subsequence $(k_n)$ such that $\dim(H_{k_n}) > \dim(H)$ and $\dim(H_{k_n}) = r$ for some $r \in \mathbb{Z}$. But then by the above argument $\dim(H) \geq r$, which is a contradiction. Hence $\dim(H) \geq \dim(H_n)$ except for finitely many $n$’s. This proves the dim is upper semi-continuous.

We next look at algebraic actions: suppose a locally compact group $G$ acts on another locally compact group $X$ by automorphisms. In this case, for an $Y \in \text{Sub}_X$, there is a closed subgroup $G_Y = \{ \alpha \in G \mid \alpha(Y) = Y \}$ of $G$ - stabilizer map. The next two results discuss the continuity of this correspondence - the stabilizer map.
Lemma 3.5. If $Y_n \to Y$ in $\text{Sub}_X$, then for any sequence $(Z_n)$ with $Z_n \in \text{Sub}_{G_Y}$, limit points of $(Z_n)$ are in $\text{Sub}_{G_Y}$.

Remark 3.6. Stabilizer map $Y \mapsto G_Y$ need not be continuous. Consider the $\text{Sub}_R$ in example 3.1 and $R^+$ acts on $R$ by multiplication. Then stabilizer of any discrete group is $\{1\}$ but stabilizer of $R$ is $R^+$ but we have a sequence of discrete groups $\frac{1}{2}Z$ converging to $R$.

Proof. Let $Z_n \to Z$ with $Z_n \in \text{Sub}_{G_Y}$. Then for any $\alpha \in Z$ and $y \in Y$, there exist sequences $\alpha_n \in Z_n$ and $y_n \in Y_n$ such that $\alpha_n \to \alpha$ and $y_n \to y$. Therefore, $\alpha_n(y_n) \to \alpha(y)$. But $\alpha_n(y_n) \in Y_n$ for all $n \in \mathbb{N}$ and, therefore, it follows that $\alpha(y) \in Y$. Hence $\alpha \in G_Y$. Thus, it follows that $Z \subset G_Y$.

If we consider the linear action of $G = SL(q, \mathbb{R})$ on $so(q, \mathbb{R})$ described in (1), we get a continuity of the following stabilizer map on the open subset $\mathcal{O}$ defined in (3).

Corollary 3.7. Let $G = SL(q, \mathbb{R})$ and $\mathcal{O}$ be the open subset of $G(p, so(q, \mathbb{R}))$ as described in (3). Then the map $j : \mathcal{O} \to \text{Sub}_G$ defined by $j(W) = G_W^0$ for $W \in \mathcal{O}$, is continuous.

Proof. Suppose $W_n \to W$ in $G(p, so(q, \mathbb{R}))$. Since $\mathcal{O}$ is an open set, $W_n \in \mathcal{O}$ except for finitely many $n$‘s. Since $\text{Sub}_G$ is compact, there is a subsequence $G_{W_{n_k}}^0$ converging to a closed subgroup $H$. By Lemma 3.5, $H \subset G_W$. Then by Proposition 4.1, $\dim(H) \geq d(p, q)$, which implies that $H = G_W^0$.

It is easy to see that the set of all abelian subgroups in $\text{Sub}_G$ is a closed set. We next provide an affirmative answer to the interesting and useful question whether the set of all amenable closed subgroups is closed in $\text{Sub}_G$: the following result is proved in Corollary 4 of [4] but we include a proof which is based on Borel density theorem techniques of Furstenberg but doesn’t seem to have been noticed.

Proposition 3.8. Let $G$ be a linear group over a local field $\kappa$ of characteristic zero and $A_G$ be the set of all closed amenable subgroups of $G$. Then $A_G$ is a closed set in $\text{Sub}_G$.

Proof. Let $G$ be a closed subgroup of $GL_d$, the group of a invertible linear transformations for some $d \geq 1$ and $UT_d$ be the group of all upper triangular linear transformations in $GL_d$. By Corollary 2 of [14], a subgroup $H \in A_G$ if and only if $H$ has a fixed point in $\mathcal{P}(GL_d/UT_d)$. Let $(H_n)$ be a sequence in $A_G$ such that $H_n \to H$ in $\text{Sub}_G$. Then there is a sequence $(\mu_n)$ in $\mathcal{P}(GL_d/UT_d)$
such that each $\mu_n$ is fixed by $H_n$. Since $GL_d/UT_d$ is compact, $\mathcal{P}(GL_d/UT_d)$ is compact and hence, by passing to a subsequence we may assume that $(\mu_n)$ converges to a $\mu \in \mathcal{P}(GL_d/UT_d)$. We now claim that $\mu$ is fixed by $H$. Let $h \in H$. Since $H_n \to H$, there is a sequence $(h_n)$ such that $h_n \in H_n$ and $h_n \to h$. Since $\mu_n$ is fixed by $H_n$ and $\mu_n \to \mu$, we get that $\mu_n = h_n \mu_n \to h \mu$, hence $h \mu = \mu$. Now the amenability of $H$ follows from Corollary 2 of [14].

\[ \square \]

4 On hyperbolic automorphisms

Now we look for hyperbolic automorphisms on a standard metric 2-step nilpotent Lie algebra preserving a lattice of full rank inside the commutator ideal of the Lie algebra. We first deal with the case when the Lie algebra has a rational structure, i.e., the Lie algebra $\mathbb{R}^q \oplus W$ is such that $W$ is a rationally defined subspace of $so(q, \mathbb{R})$. We use several results from the theory of Lie groups, algebraic groups and lattices inside those groups to show the existence of a desired hyperbolic automorphism.

Proposition 4.1. Let $\mathcal{H} = \mathbb{R}^q \oplus W$ be a standard metric 2-step nilpotent Lie algebra of type $(p, q)$ with $q \geq 3$, $p \geq 2$ and $W$ being defined over $\mathbb{Q}$. Let $G_W$ be the corresponding group consisting of automorphisms of $\mathcal{H}$ as described above. Also let $\{w_1, \ldots, w_p\}$ be a basis of $W$ such that $w_i \in so(q, \mathbb{Q})$ for $1 \leq i \leq p$. Let $\Gamma$ be the lattice in $W$ of rank $p$ given by $\Gamma = \mathbb{Z}w_1 \oplus \mathbb{Z}w_2 \oplus \ldots \oplus \mathbb{Z}w_p$. If $\rho_W(G_W)$ ($\rho_W$ is as defined in the introduction) is non-amenable, then there exists an automorphism $\alpha$ of $\mathcal{H}$, which is also diagonalizable as an element of $SL(q, \mathbb{R})$ such that $\alpha$ is hyperbolic and satisfies $\alpha(\Gamma) = \Gamma$.

Proof. Because of the non-amenability assumption on $\rho_W(G_W)$, it contains a simple non-compact factor $S$ and $\rho^{-1}(S)$ is isogenous to an isomorphic copy of $S$. Therefore without loss of generality we may assume that $G_W$ contains a non-compact simple factor (in particular $G_W$ is non-amenable) $S_W$ which is again an algebraic group defined over $\mathbb{Q}$. The set of $\mathbb{Z}$-points of $S_W$, $S_W(\mathbb{Z})$ preserves the lattice $\Gamma$ since the representation $\rho$ is defined over $\mathbb{Z}$. So, the proposition will be proved if we can show that $S_W(\mathbb{Z})$ contains a hyperbolic automorphism of $\mathcal{H}$. By a result of Armand Borel ([3]) we know that the $\mathbb{Z}$-points of an algebraic group defined over $\mathbb{Q}$ is a lattice if and only if there is no polynomial homomorphism defined over $\mathbb{Q}$ from the algebraic group to the multiplicative group of non-zero complex numbers. Since $S_W$ is simple, we conclude that $S_W(\mathbb{Z})$ is a lattice in $S_W$.

It follows from Theorem 2.8 of [13] that there is a Cartan subgroup $C_W$ in $S_W$ such that $L_W = C_W \cap S_W(\mathbb{Z})$ is a uniform lattice in $C_W$. Now $C_W = \ldots$
$K_W D_W$, where $K_W$ is compact, $D_W$ is diagonalizable and they commute. Then the projection of $L_W$ to $D_W$, denoted by $\pi(L_W)$, is again a lattice in $D_W$. By Borel density theorem (see [2] for details) $L_W$ being a lattice, is Zariski dense in $D_W$. Let $D_W^h$ be the set of hyperbolic automorphisms of $\mathfrak{N}$ contained in $D_W$. The action of $D_W$ on $\mathfrak{N}$ has two parts, one is the linear action on $\mathbb{R}^q$ and another is the action on $so(q, \mathbb{R})$ as mentioned earlier. Since $D_W$ is diagonalizable, the set of hyperbolic elements in $D_W$ for both this action are Zariski open in $D_W$. $D_W^h$ being the intersection of these two Zariski open subsets is again Zariski open in $D_W$. Hence $\pi(L_W) \cap D_W^h \neq \emptyset$. This completes the proof. \hfill \Box

The above proposition deals with a particular lattice inside $W$, but it is desirable to have a more general result. The following lemma will help us to obtain the results for any lattice inside the rationally defined subspace $W$ as well as for any lattice inside $W \in \mathcal{O}$.

**Lemma 4.2.** Let $(W_n) \subset G(p, so(q, \mathbb{R}))$, $W \in G(p, so(q, \mathbb{R}))$ be such that $W_n \to W$. Let $(\Gamma_n)$ be a sequence of closed subgroups of $so(q, \mathbb{R})$ such that each $\Gamma_n$ is a lattice of rank $l$, $l \leq p$, in $W_n$ and $\Gamma_n \to \Gamma$, where $\Gamma$ is a lattice of rank $l$ in $W$. Suppose there exist $\alpha_n \in G_{W_n}$ with $\alpha_n(\Gamma_n) = \Gamma_n$ such that $\alpha_n$ is diagonalizable and the modulus of the eigenvalues of $\alpha_n$ are uniformly bounded away from 1 for all $n$. Then there exists $\alpha \in G_W$ such that $\alpha(\Gamma) = \Gamma$ and the modulus of the eigenvalues of $\alpha$ are different from 1.

**Proof.** Let $J_n = \{\alpha_n^k\}_{k \in \mathbb{Z}}$. Then $J_n$ has a convergent subsequence which converges to some closed subgroup of $G_W$ by Lemma 3.5. So, passing to a subsequence we may assume that $J_n \to J$ for some closed subgroup $J$ of $G_W$. Now let $\beta \in J$. Then there is a sequence $\{\beta_n\}$ with $\beta_n \in J_n$ such that $\beta_n \to \beta$. Let $\beta_n = Q_n R_n Q_n^t$ be the Schur decomposition of $\beta_n$ with $Q_n$ an orthogonal matrix and $R_n$ an upper triangular matrix with the diagonal entries of $R_n$ the eigenvalues of $\beta_n$. $\{Q_n\}$ being a sequence in the compact group $O(n, \mathbb{R})$, has a convergent subsequence, so passing to a subsequence we may assume that $Q_n \to Q$ for some orthogonal matrix $Q$ and consequently $Q_n^t \to Q^t$. These force $R_n$ to converge to some upper triangular matrix $R$. So $\beta = QRQ^t$ is a Schur decomposition of $\beta$. Since the eigenvalues of $\alpha_n$'s are uniformly bounded away from 1, it follows that $\beta$ does not have any eigenvalue of modulus 1.

Now we show that $\beta(\Gamma) = \Gamma$. Let

\[
\Gamma_n = \mathbb{Z}w_{1n} \oplus \ldots \oplus \mathbb{Z}w_{ln}
\]

and

\[
\Gamma = \mathbb{Z}w_1 \oplus \ldots \oplus \mathbb{Z}w_l
\]
such that $w_i \to w$ for $1 \leq i \leq l$. Let $\sum_{i=1}^{l} a_i w_i \in \Gamma$, $a_i \in \mathbb{Z}$. Then

$$\beta \left( \sum_{i=1}^{l} a_i w_i \right) = \sum_{i=1}^{l} a_i \beta(w_i).$$

Therefore, it is enough to show that $\beta(w_i) \in \Gamma$ for $i \in \{1, \ldots, l\}$. Since $w_i \to w$ and $\beta_n \to \beta$, we have $\beta_n(w_i) \to \beta(w_i)$. So, it is enough to show that given a sequence of elements $\sum_{i=1}^{l} a_n w_i \in \Gamma_n$ converging to $\sum_{i=1}^{l} a_i w_i$, we must have $a_i \in \mathbb{Z}$ for $i \in \{1, \ldots, l\}$. Now $\sum_{i=1}^{l} a_n w_i$ converges to $\sum_{i=1}^{l} a_i w_i$ means

$$M_{W_n} \begin{pmatrix} a_1 \\ \vdots \\ a_l \end{pmatrix} \to M_{W} \begin{pmatrix} a_1 \\ \vdots \\ a_l \end{pmatrix},$$

which implies that

$$M_{W_n}^{-1} M_{W} \begin{pmatrix} a_1 \\ \vdots \\ a_l \end{pmatrix} \to \begin{pmatrix} a_1 \\ \vdots \\ a_l \end{pmatrix}.$$

Since $M_{W_n}$ converges to $M_{W}$, $(M_{W_n})^{-1} M_{W}$ converges to the $l \times l$ identity matrix. Since $a_n \in \mathbb{Z}$ for all $n \geq 1$, $i \in \{1, \ldots, l\}$, it follows that each of the sequences $\{a_n\}$ stabilizes after some stage. Therefore, $a_i$ must be in $\mathbb{Z}$ for all $i \in \{1, \ldots, l\}$. This completes the proof. \qed

**Theorem 4.3.** Let $\mathfrak{N} = \mathbb{R}^q \oplus W$ be a standard metric 2-step nilpotent Lie algebra of type $(p, q)$ with $q \geq 3$, $p \geq 2$ and $W$ being defined over $\mathbb{Q}$. Let $G_W$ be the corresponding group consisting of automorphisms of $\mathfrak{N}$ as described before. Let $\Gamma$ be any lattice in $W$ of rank $p$. If $\rho_W(G_W)$ is non-amenable, then there exists an automorphism $\alpha$ of $\mathfrak{N}$, which is also diagonalizable as an element of $\text{SL}(q, \mathbb{R})$ such that $\alpha$ is hyperbolic and satisfies $\alpha(\Gamma) = \Gamma$.

**Proof.** Let $\Gamma = \mathbb{Z}v_1 \oplus \ldots \oplus \mathbb{Z}v_p$, $v_1, \ldots, v_p \in W$. Also let $w_1, \ldots, w_p \in \text{so}(q, \mathbb{Q})$ (since $W$ is defined over $\mathbb{Q}$) form a basis of $W$. Then

$$v_i = \sum_{j=1}^{p} v_{ij} w_j \text{ for } 1 \leq i \leq p.$$
Now choose \(v_{ij}^{(n)} \in \mathbb{Q}\) such that \(v_{ij}^{(n)} \to v_{ij}\). Let
\[
v_i^n = \sum_{j=1}^{p} v_{ij}^{(n)} w_j
\]
and
\[
\Gamma_n = \mathbb{Z}v_1^n \oplus \ldots \oplus \mathbb{Z}v_p^n.
\]
Then for sufficiently large \(n\), \(\Gamma_n\) is a lattice in \(W\) of rank \(p\) and \(\Gamma_n \to \Gamma\).

Buy applying Proposition 4.1, we see that for sufficiently large \(n\), there exists hyperbolic automorphism \(\alpha_n\) of \(\mathfrak{R}\) which is a diagonalizable element of \(\text{SL}(q, \mathbb{R})\) as well such that \(\alpha_n(\Gamma_n) = \Gamma_n\). Then the conclusion of the theorem follows from Lemma 4.2.

Now we turn to the case \(\mathfrak{R} = \mathbb{R}^q \oplus W\), when \(W\) is not necessarily defined over \(\mathbb{Q}\). We are going to use the fact that the set of subspaces define over \(\mathbb{Q}\) is dense in \(G(p, \text{so}(q, \mathbb{R}))\) to a good effect to obtain a similar statement as in Theorem 4.4 in the case when \(W\) is not necessarily defined over \(\mathbb{Q}\).

**Theorem 4.4.** Let \(\mathfrak{R} = \mathbb{R}^q \oplus W\) be a standard metric 2-step nilpotent Lie algebra of type \((p, q)\) with \(W \in \mathcal{O}\). Let \(\Gamma\) be a lattice of rank \(p\) in \(W\). Let \(G_W\) be the automorphism group of \(\mathfrak{R}\) as above. If \(\rho_W(G_W)\) is non-amenable, then there is a hyperbolic automorphism \(\alpha\) of \(\mathfrak{R}\) such that \(\alpha(\Gamma) = \Gamma\).

**Proof.** Since \(\Gamma\) is a lattice in \(W\), there is a basis \(\{w_1, ..., w_p\}\) of \(W\) such that
\[
\Gamma = \mathbb{Z}w_1 \oplus \ldots \oplus \mathbb{Z}w_p.
\]
Since the set of subspaces defined over \(\mathbb{Q}\) is dense in \(G(p, \text{so}(q, \mathbb{R}))\), it follows that there is a sequence \(\{W_n\}\) of subspaces with basis \(\{w_{1n}, ..., w_{pn}\} \subset \text{so}(q, \mathbb{Q})\) such that \(w_{in} \to w_i\) (consequently \(W_n \to W\)) as \(n \to \infty\). Also let
\[
\Gamma_n = \mathbb{Z}w_{1n} \oplus \ldots \oplus \mathbb{Z}w_{pn}.
\]
Since \(W \in \mathcal{O}\) and \(W_n\) converges to \(W\), from Lemma 3.7 it follows that a subsequence of \(G_{W_n}\) converges to \(G_W\), so passing to a subsequence we may assume that \(G_{W_n}\) converges to \(G_W\).

Since \(\rho_W(G_W)\) is non-amenable, \(G_W\) is non-amenable as well and then it follows from Lemma 3.8 that \(G_{W_n}\) is also non-amenable for sufficiently large \(n\). Now we wish to say that \(\rho_{W_n}(G_{W_n})\) is non-amenable as well for sufficiently large \(n\). Let \(T_n \in \text{GL}(\text{so}(q, \mathbb{R}))\) be such that \(T_n(W_n) = W\), and \(T_n\) tends to the identity element of \(\text{GL}(\text{so}(q, \mathbb{R}))\) as \(n \to \infty\), then \(W\) is invariant under \(T_n\rho(G_{W_n})T_n^{-1}\) for all \(n\). Let us denote the restriction of \(T_n\rho(G_{W_n})T_n^{-1}\) to \(W\)
by \((T_n \rho(G_{W_n})T_n^{-1})_W\). We show that \((T_n \rho(G_{W_n})T_n^{-1})_W\) is non-amenable for sufficiently large \(n\). The sequence \(( (T_n \rho(G_{W_n})T_n^{-1})_W \) in the compact space \(\text{Sub}_{\text{GL}(W)}\) has a convergent subsequence, so passing to a subsequence we may assume that \((T_n \rho(G_{W_n})T_n^{-1})_W\) converges to some closed subgroup \(H\) (say) of \(\text{GL}(W)\). We show that \(\rho_W(G_W) \subset H\) and consequently \(H\) is non-amenable. Then \(T_n \rho_W(G_{W_n})T_n^{-1}\) is non-amenable for sufficiently large \(n\), which will force \(\rho_W(G_{W_n})\) to be non-amenable. To show \(\rho_W(G_W) \subset H\), let \(\alpha \in G_W\), then there is \(\alpha_n \in G_{W_n}\) for all \(n\) such that \(\alpha_n \to \alpha\) (since \(G_{W_n} \to G_W\)). Then \(\rho(\alpha_n) \to \rho(\alpha)\) and, therefore, \((T_n \rho(\alpha_n)T_n^{-1})_W \to \rho_W(\alpha)\) as \(n \to \infty\), hence \(\rho_W(\alpha) \in H\).

Now, by Theorem 4.1, there exists \(\alpha_n \in G_{W_n}\) such that \(\alpha_n\) is hyperbolic and \(\alpha_n(\Gamma_n) = \Gamma_n\). Each \(\alpha_n\) is diagonalizable and if required, replacing \(\alpha_n\) by its large enough power we can assume that there is some \(\delta > 0\) such that the eigenvalues of \(\alpha_n\) are either greater than \(1 + \delta\) or less than \(\frac{1}{1 + \delta}\) for all \(n\). Clearly, \(\Gamma_n \to \Gamma\). Now the conclusion of the theorem follows from Lemma 4.2.

Proof of Theorem 4.1. Now the proof of Theorem 4.1 follows from Theorem 4.3 and Theorem 4.4.

5 Some examples

Let \(\{e_1, e_2, \ldots, e_q\}\) denote the standard orthonormal basis of \(\mathbb{R}^q\) and let \(\mathcal{B} = \{e_{st} = [e_s, e_t], 1 \leq s < t \leq q\}\) denote the orthonormal basis of \(\text{so}(q, \mathbb{R})\) with respect to the inner product described earlier. We call a \(p\)-dimensional subspace \(W \subset \text{so}(q, \mathbb{R})\) a standard subspace if \(W\) has a basis consisting of \(p\) elements from the set \(\mathcal{B}\). Now let \(p = 2\). Then it follows from [8], that if \(q \geq 3\), \(G_W\) is at least 6-dimensional. Let \(\{e_{st}, e_{su}\}\) be a basis of \(W\). Let \(g : \mathbb{R}^q \to \mathbb{R}^q\) be defined as follows: \(g(e_t) = ae_t + ce_u\), \(g(e_u) = be_t + de_u\), \(ad - bc = 1\), and \(g(e_r) = e_r\) for \(r \neq t, u\). Then it is easy to see that \(g \in \text{SL}(q, \mathbb{R})\). Also

\[
g(e_{st}) = g[e_s, e_t] = [e_s, ae_t + ce_u] = a e_{st} + ce_{su}.
\]

Similarly,

\[
g(e_{su}) = be_{st} + de_{su}.
\]

This shows that \(g \in G_W\) and moreover, \(\rho_W(G_W)\) contains a copy of \(\text{SL}(2, \mathbb{R})\) and hence non-amenable. More generally if \(W\) is a \(p\)-dimensional standard subspace of \(\text{so}(q, \mathbb{R})\) and \(W\) has a basis of the form \(\{e_{st_1}, e_{st_2}, \ldots, e_{st_p}\}\), then applying the same procedure one can find a copy of \(\text{SL}(p, \mathbb{R})\) inside \(\rho_W(G_W)\).
Note that if $q = 3$ and $W$ is a 2-dimensional standard subspace of $\text{so}(3, \mathbb{R})$, then $W$ always has a basis of the form $\{e_{st}, e_{su}\}$ and so $\rho_W(G_W)$ is non-amenable. But if $W$ is an 1-dimensional standard subspace then clearly $\rho_W(G_W)$ is amenable though $G_W$ is non-amenable as $G_W$ is isomorphic to $G_{W^\perp}$ (see [7] or [8] for details), where $W^\perp$ is the orthogonal complement of $W$ inside $\text{so}(3, \mathbb{R})$.

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