Critical Behavior in the Brans-Dicke Theory of Gravitation

Takeshi Chiba
Department of Physics, Kyoto University, Kyoto 606-01, Japan
Jiro Soda
Department of Fundamental Sciences, FIHS, Kyoto University, Kyoto 606, Japan

The collapse of a massless scalar field in the Brans-Dicke theory of gravitation is studied in the analysis of both analytical solution and numerical one. By conformally transforming the Roberts’s solution into the Brans-Dicke frame, we find for \( \omega > -3/2 \) that a continuous self-similarity continues and that the critical exponent does depend on \( \omega \). By conformally transforming the Choptuik’s solution into the Brans-Dicke frame, we find for \( \omega > -3/2 \) that at the critical solution shows discrete self-similarity, however, the critical exponent depends strongly on \( \omega \) while the echoing parameter weakly on it.

I. INTRODUCTION

Critical phenomena in black hole formation found by Choptuik [1] has renewed interest in the classical general relativistic black hole formation. Among the most interesting things in Choptuik’s results are

1. Power law: Black hole mass exhibits a power law

\[
M_{BH} \propto (p - p^*)^\beta, \quad (1.1)
\]

where \( p \) is a parameter which characterizes the strength of initial condition and \( p^* \) is the threshold value.

2. Echoing: Configurations sufficiently close to critical show a discrete homotheticity (or scale invariance)

\[
\phi(\rho - \Delta, \tau - \Delta) \simeq \phi(\rho, \tau), \quad (1.2)
\]

where \( \rho \) and \( \tau \) are logarithms of proper radius \( r \) and central proper time \( t \). Echoing means that the features of critical solution are repeated on ever decreasing time length scales.

3. Universality: The exponent \( \beta \simeq 0.37 \) and echoing parameter \( \Delta \) are independent of any choice of initial data \( p \).

It should be noted, however, that the universality mentioned there refers to the independence from the initial condition of the matter field considered. To investigate the dependence of the model of the matter, the collapse of gravitational wave [2] and radiation fluid collapse [3] were examined. For vacuum gravity, \( \beta \simeq 0.37 \) and \( \Delta \simeq 0.6 \), while for radiation fluid \( \beta \simeq 0.36 \) with \( \Delta \) being arbitrary (i.e., continuous self-similarity). These results excite the expectation that the critical exponent \( \beta \) may be universal among matter fields having massless property, although quantum effects will destroy the phenomena against expectation [4].

In this paper, we will investigate another model dependence: dependence on the theory of gravitation. To see this we take the Brans-Dicke theory of gravitation [5] for its simplicity. The theory contains a parameter \( \omega \) which controls the strength of gravity, and in the limit \( \omega \to \infty \) the theory reduces to the Einstein theory. We consider the Brans-Dicke theory in vacuum. The scalar field examined here is thus the Brans-Dicke scalar field.

The collapse of a massless scalar field in the Brans-Dicke theory is also interesting in the light of structural stability of Choptuik’s solution, which is our next problem.
Our paper is organized as follows. In sec. 2, Roberts’s self-similar solution in the Brans-Dicke theory is studied to introduce the notation and the detail of the conformal transformation. In sec. 3, basic equations and numerical procedures are given. We reproduce Choptuik solution. In sec. 4, numerical results for Brans-Dicke theory are given. Sec. 5 is devoted to summary.

II. ROBERTS’S SOLUTION IN BRANS-DICKE THEORY

Roberts [6] found solutions of the Einstein equations which describe continuous self-similar spherical collapse of a massless scalar field and he studied these solutions in the context of counter-examples to the cosmic censorship. Subsequently, the solutions were studied as an analytical example of critical behavior. We shall try to find the counterparts in the Brans-Dicke theory of Roberts’s solutions by conformal transformation.

The action of Brans-Dicke theory without matter term is

\[ S = \int d^4x \sqrt{-\tilde{g}} (\Phi \tilde{R} - \frac{\omega}{\Phi} (\nabla \Phi)^2). \]  (2.1)

By the conformal transformation

\[ \tilde{g}_{ab} = \Phi^{-1} g_{ab}, \]
\[ \Phi = \exp(\phi/\sqrt{2\omega + 3}), \]  (2.2)

the action Eq. (2.1) is reduced to that of a massless scalar field coupled to the Einstein gravity where the solutions are known:

\[ S = \int d^4x \sqrt{-g} (R - \frac{1}{2} (\nabla \phi)^2). \]  (2.3)

Therefore we simply replace the Roberts’s solutions by Eq. (2.2) to find the counterparts in the Brans-Dicke theory.

The Roberts’s solutions are written as

\[ ds^2 = -dudv + \frac{1}{4} (u - (1 - p)v)(u - (1 + p)v) d\Omega^2, \]  (2.4)
\[ \phi = \pm \left( \log \frac{u - (1 + p)v}{u - (1 - p)v} \right), \]  (2.5)

where \( p > 0 \) is an arbitrary constant. Then those in the Brans-Dicke theory are given by

\[ ds^2 = \pm^\alpha \left( \frac{u - (1 - p)v}{u - (1 + p)v} \right)^{\pm \alpha} dudv + \frac{1}{4} (u - (1 - p)v)^{1\pm \alpha} (u - (1 + p)v)^{1\mp \alpha} d\Omega^2, \]  (2.6)
\[ \Phi = \left( \frac{u - (1 - p)v}{u - (1 + p)v} \right)^{\mp \alpha}, \]  (2.7)

where \( \alpha = 1/\sqrt{2\omega + 3} \). Since the Ricci scalar takes the form

\[ \tilde{R} = \Phi R + 3 \nabla^2 \Phi - \frac{3}{2} (\nabla \Phi)^2, \]  (2.8)

and \( R = 8[(u - (1 - p)v)(u - (1 + p)v)]^{-2} p^2 uv \), we find that a singularity at \( u = (1 - p)v \) for \( v > 0 \) and \( u = (1 + p)v \) for \( v < 0 \). In the following we shall assume \( 1 \pm \alpha \geq 0 \) in order for the singularity to locate at the center. As in the case of the Einstein gravity, structure of the singularity depends on \( p \). It is easily found that for \( p > 1 \) it is timelike in the region \( v < 0 \) and spacelike \( v > 0 \), null for \( p = 1 \), timelike for \( p < 1 \).

By calculating the change of area with respect to derivative along advanced null coordinate, it is found that an apparent horizon is located at

\[ u = \frac{1 - p^2}{1 \pm \alpha p} v, \]  (2.9)

and that for \( p < 1 \) no apparent horizon exists. Hence the parameter \( p \) corresponds to the control parameter in this example of the critical behavior and \( p = 1 \) the point of the criticality.
Calculation of the black hole mass $M$ is also straightforward. It is done by calculating the square root of the proper area of an apparent horizon. After a little algebra it is found that $M$ is given by

$$M = \frac{\sqrt{(1 + \alpha)^{1/2}(1 - \alpha)^{1/2}pv(1 + p)^{1/2}2(1 - \alpha)^{1/2}}}{4(-1 \pm \alpha p)}.$$ \hfill (2.10)

Thus near the criticality $M$ behaves as $M \propto (p - 1)^{(1 \pm \alpha)/2}$. It is the power law of the critical behavior with the critical exponent

$$\beta = \frac{1}{2} \pm \frac{1}{2\sqrt{2\omega + 3}},$$ \hfill (2.11)

which clearly depends on $\omega$ and certainly reduces to the result of Einstein theory in the limit $\omega \to \infty$. This result strongly indicates that the critical phenomena depend on the parameter $\omega$. Indeed, it will be confirmed in the subsequent numerical analysis.

### III. COLLAPSE OF BRANS-DICKE SCALAR FIELD IN DOUBLE-NULL COORDINATES

Next we study numerically the collapse of the Brans-Dicke scalar field. We treat the problem with the action Eq. (2.3) in terms of null initial value formulation. Since in the null initial value formulation the grid points are tied to ingoing light rays, overall size of the grid becomes smaller as system evolves and the resolution improves. Adaptive mesh refinement algorithm used by Choptuik is not always necessary.

We take the following line element

$$ds^2 = -a(u, v)^2 du dv + r(u, v)^2 d\Omega^2.$$ \hfill (3.1)

Double-null coordinates are not unique. There remain two degrees of gauge freedom which correspond to redefining $u$ and $v$. We fix one of them by requiring $u = v$ on axis $r = 0$. The remaining gauge freedom will be fixed by the initial condition.

The equations of motion that derived from Eq. (2.3) are

$$rr_{uv} + r_u r_v + \frac{1}{4}a^2 = 0,$$ \hfill (3.2)

$$a^{-1}a_{uv} - a^{-2}a_u a_v - r^{-2}r_u r_v - \frac{1}{4}r^{-2}a^2 + \frac{1}{4}\phi_u \phi_v = 0,$$ \hfill (3.3)

And constraint equations are

$$r_{uu} - 2a^{-1}a_u r_u + \frac{1}{4}r \phi_u^2 = 0,$$ \hfill (3.5)

$$r_{vv} - 2a^{-1}a_v r_v + \frac{1}{4}r \phi_v^2 = 0,$$ \hfill (3.6)

where $r_u = \partial r / \partial u$, for example. These equations are solved numerically. Since the usual iteration scheme for solving these equations does not work because of their nonlinearity, we adopt the first order scheme developed by Hamade and Stewart \[7\].

We introduce the following new variables

$$d = \frac{a_v}{a}, f = r_u, g = r_v, p = \phi_u, q = \phi_v.$$ \hfill (3.7)

The system of equations (3.2-3.6) can then be converted to a first order system:

$$f_v + r^{-1}(fg + \frac{1}{4}a^2) = 0,$$ \hfill (3.8)

$$d_u - r^{-2}(fg + \frac{1}{4}a^2) + \frac{1}{4}pq = 0,$$ \hfill (3.9)

$$p_v + r^{-1}(fq + gp) = 0,$$ \hfill (3.10)

$$q_u + r^{-1}(fq + gp) = 0,$$ \hfill (3.11)

$$g_v - 2dg + \frac{1}{4}r q^2 = 0.$$ \hfill (3.12)
We then integrate outward from axis Eq.(3.13) to obtain critical parameter. The least square fit shows the exponent $\beta$ (or $\phi$) is the choice of $d$ on $u = 0$. This is arbitrary, and we choose $d = 0$. As for the initial condition of $q$ (or $\phi$) on $u = 0$, we typically take

$$\phi(u = 0, v) = 1 + \phi_0 v^2 \exp(- (v - v_0)^2 / v_1^2).$$

We then integrate outward from axis Eq.(3.13) to obtain $a$, Eq.(3.14) to obtain $r$, Eq.(3.12) to obtain $g$, Eq.(3.8) to obtain $f$ and Eq.(3.10) to obtain $p$. Thus we set initial conditions.

The integration in the $u$-direction is done using an explicit difference algorithm. The integration in the $v$-direction is done using an implicit algorithm, which is necessary to ensure stability. However, the integration can be made explicit. Details of algorithm are given in [7].

In Fig.1, we plot the scalar field at the center with marginally subcritical initial parameter as a function of logarithm of the central proper time $\tau = \log(t^* - t)$, where $t^*$ is the time when an infinitesimal black hole forms. We find that the scalar field oscillates with period $\Delta \simeq 3.43$ in $\tau$-coordinate, in agreement with Choptuik.

Fig.2 shows the black hole mass for marginally supercritical evolution as a function of $\phi_0 - \phi^*_0$ with $\phi^*_0$ being the critical parameter. The least square fit shows the exponent $\beta \simeq 0.38$, in agreement with Choptuik.

These results confirm the logical consistency and accuracy of our numerical code.

IV. CRITICAL PHENOMENA IN THE BRANS-DICKE THEORY OF GRAVITATION

Now we study the behavior of a scalar field for various values of $\omega$, $1000 \geq 2\omega + 3 > 0$ by conformally transforming the Choptuik’s solution reproduced in the previous section. This is done by the relation (2.2). The results are shown in Figures and Table. Fig.3 shows the scalar filed at the center as a function of logarithm of the central proper time $\tau = \log(t^* - t)$ for several values of $\omega$. The scalar field oscillates greatly for smaller $\omega$. Fig.4 shows the black hole mass for marginally supercritical evolution as a function of initial parameter for $2\omega + 3 = 2$. The least square fit gives the exponent $\beta \simeq 0.43$. Table summarizes the numerical results. As can be seen the exponent $\beta$ depends strongly on $\omega$, hence it is not universal quantity. This is the first numerical evidence of non-universality $\S$ for the collapse of single matter content. On the other hand the scaling parameter $\Delta$ depends weakly on $\omega$.

| $2\omega + 3$ | $\beta$ | $\Delta$ |
|--------------|--------|--------|
| 0.50         | 0.50   | 3.41   |
| 1.0          | 0.46   | 3.43   |
| 2.0          | 0.43   | 3.45   |
| 5.0          | 0.41   | 3.46   |
| 10           | 0.38   | 3.46   |
| 50           | 0.38   | 3.47   |
| 200          | 0.38   | 3.47   |
| 1000         | 0.38   | 3.47   |

$\S$ However, slight modification is necessary to ensure that the system has second-order accuracy; Eq.(3.5) in their paper should be replaced with $z_n = \frac{1}{2}(\hat{z}_n + \hat{z}_w + \frac{3}{2}(G(y_n, \hat{z}_n) + G(y_w, \hat{z}_w)))$. We thank T. Harada for pointing this out to us.
It is to be noted that critical phenomena in the Brans-Dicke theory does emerge in discrete self-similar manner. Thus stability analysis around a continuous self-similar solution gives no information for the stability of the actual dynamical solutions.

V. SUMMARY

We have studied the collapse of the Brans-Dicke scalar field to examine the dependence of the critical phenomena on the theory of gravitation. First, by conformally transforming the Roberts’s solution into the Brans-Dicke frame, we find for $\omega > -\frac{3}{2}$ that a continuous self-similarity continues and that the critical exponent does depend on $\omega$. Second, by conformally transforming the Choptuik’s solution into the Brans-Dicke frame, we find for $\omega > -\frac{3}{2}$ that at the critical solution shows discrete self-similarity. However, the critical exponent depends strongly on $\omega$, while the scaling parameter depends weakly on $\omega$. This weak dependence on the theory of gravitation may justify the approximation employed by Price and Pullin. They find that the critical solution can be well approximated by a flat spacetime scalar field solution. Numerical dynamical solutions show discrete self-similarity. These are another examples of discrete self-similar critical solutions.

Given these numerical examples, the critical phenomena are not universal in the original sense. The apparent analogy to statistical system may be just analogy not fundamental. Rather, the phenomena numerically observed can be regarded as a manifestation of the features of the infinite dimensional dynamical system. From this point of view, there are many interesting problems to be solved. For example, “Does the discrete self-similarity corresponds to the limit cycle?” or, more generally, “How can we classify the gravitational collapse phenomena from the point of the dynamical system?” “Are there any examples which give rise to bifurcation phenomena?”

We hope to report on the resolution of these problems in the future.

ACKNOWLEDGMENTS

T.C. would like to thank Dieter Mason for useful discussions and Misao Sasaki for useful comment on the conformal transformation. He is also grateful to H.Sato for continuous encouragement.

[1] M.W. Choptuik, *Phys. Rev. Lett.* 70 9 (1992).
[2] A.M. Abrahams and C.R. Evans, *Phys. Rev. Lett.* 70 2980 (1992).
[3] C.R. Evans and J.S. Coleman, *Phys. Rev. Lett.* 72 1782 (1993).
[4] T. Chiba and M. Siino, *Disappearance of Critical Behavior in Semiclassical General Relativity, KUNS-1384*, submitted to CQG.
[5] C.B. Brans and R.H. Dicke, *Phys. Rev.* 124 925 (1961).
[6] M.D.Roberts, *Gen. Rel. Grav.* 21 907 (1989); Y.Oshiro, K.Nakamura, and A.Tomimatsu, *Prog. Theor. Phys.* 91 1265 (1994); P.R.Brady, *Class. Quantum Grav.* 11 1255 (1994).
[7] R.S. Hamade and J.M. Stewart, *Class. Quantum Grav.* 13 497 (1996).
[8] D. Mason, *Phys. Lett.* B366 82 (1996).
[9] R.H.Price and J.Pullin, gr-qc/9601001.
FIGURE CAPTIONS

Fig.1. The scalar field on the center with a marginally subcritical initial parameter as a function of logarithm of the central proper time $\tau = -\log(t^*-t)$, where $t^*$ is the time when an infinitesimal black hole forms. In this time scale the scalar field oscillates with a periodicity $\Delta \simeq 3.43$.

Fig.2. The mass of a formed black hole for supercritical solutions as a function of $p - p^*$ with $p^*$ being the critical initial parameter. The least square fit gives the exponent $\beta \simeq 0.38$.

Fig.3. The scalar field on the center for $2\omega + 3 = 0.5, 1, 2$ with a marginally subcritical initial parameter as a function of logarithm of the central proper time $\tau = -\log(t^*-t)$.

Fig.4. The mass of a formed black hole for supercritical solutions for $2\omega + 3 = 2$ as a function of $p - p^*$. The least square fit gives the exponent $\beta \simeq 0.43$. 
