THREE-DIMENSIONAL CONSERVATIVE STAR FLOWS ARE ANOSOV

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Abstract. A divergence-free vector field satisfies the star property if any divergence-free vector field in some $C^1$-neighborhood has all the singularities and all closed orbits hyperbolic. In this article we prove that any divergence-free star vector field defined in a closed three-dimensional manifold is Anosov. Moreover, we prove that a $C^1$-structurally stable three-dimensional conservative flow is Anosov.

1. Introduction, basic definitions and statement of the results. Let $M$ be a three-dimensional closed and connected $C^\infty$ Riemannian manifold endowed with a volume-form and let $\mu$ denote the Lebesgue measure associated to it. We say that a vector field $X: M \to TM$ is divergence-free if its divergence is equal to zero or equivalently if the measure $\mu$ is invariant for the associated flow, $X^t$, $t \in \mathbb{R}$. In this case we say that the flow is conservative or volume-preserving. We denote by $X^r_\mu(M)$ ($r \geq 1$) the space of $C^r$ divergence-free vector fields on $M$ and we endow this set with the usual $C^1$ Whitney topology. Let also denote by $X^r(M) \supset X^r_\mu(M)$ ($r \geq 1$) the space of $C^r$ (dissipative) vector fields on $M$.

Given $X \in X^1(M)$ let $\text{Sing}(X)$ denote the set of singularities of $X$ and $R := M \setminus \text{Sing}(X)$ the set of regular points.

Given $x \in R$ we consider its normal bundle $N_x = X(x)^\perp \subset T_xM$ and define the linear Poincaré flow by $P^X_\ell(x) := \Pi_{X^\ell(x)} \circ DX^t_x$ where $\Pi_{X^\ell(x)} : T_{X^\ell(x)}M \to N_{X^\ell(x)}$ is the projection along the direction of $X(X^\ell(x))$. Let $\Lambda \subset R$ be an $X^1$-invariant set and $N = N^1 \oplus N^2$ be a $P^X_\ell$-invariant splitting over $\Lambda$; as $X$ is conservative these bundles are one-dimensional. We say that this splitting is an $\ell$-dominated splitting for the linear Poincaré flow if there exists an $\ell \in \mathbb{N}$ such that for all $x \in \Lambda$ we have:

$$\|P^X_\ell(x)|_{N^2_x}\| \cdot \|P^{-\ell}_X(X^\ell(x))|_{N^1_{X^\ell(x)}}\| \leq \frac{1}{2}.$$
This definition is weaker than hyperbolicity where it is required that
\[ \| P_x^t \|_{N_2^X} \leq \frac{1}{2} \] and also that \[ \| P_x^{-t} \|_{N_1^{X_t(x)}} \| \leq \frac{1}{2} \].

When \( \Lambda \) is compact this definition is equivalent to the usual definition of hyperbolic flow ([9, Proposition 1.1]).

The simplest examples of hyperbolic sets are singularities and closed orbits and it is well-know that these sets are stable by \( \mathcal{C}^1 \)-perturbations, that is, any other sufficiently \( \mathcal{C}^1 \)-close system has equivalent behavior or, in other words, it is possible to find a change of coordinates conjugating locally the two dynamics (for more details see [12]). Other classical examples are the Anosov ones where \( M \) is hyperbolic, and they form an open set of \( \mathcal{X}^1_\mu(M) \) (see e.g. [14]).

We say that a vector field is Axiom A if the closure of the union of the closed orbits and the singularities is the non-wandering set, denoted by \( \Omega(X) \), and this set is hyperbolic. Since, by Poincaré recurrence theorem, for conservative vector fields the non-wandering set is equal to \( M \), a conservative vector field that is Axiom A is actually an Anosov system. In the dissipative case, in order to obtain stability we must check if there exists no cycles. Recall that, by the spectral decomposition of an Axiom A flow, we have that \( \Omega(X) = \bigcup_{i=1}^k \Lambda_i \) where each \( \Lambda_i \) is a basic piece. We define an order relation by \( \Lambda_i \prec \Lambda_j \) if there exists \( x \in M \setminus (\Lambda_i \cup \Lambda_j) \) such that \( \alpha(x) \subset \Lambda_i \) and \( \omega(x) \subset \Lambda_j \). We say that \( X \) has a cycle if there exists a cycle with respect to \( \prec \) (see [14] for details).

We say that \( X \in \mathcal{X}^1(M) \) is a star flow if there exists a \( \mathcal{C}^1 \)-neighborhood \( V \ni X \) such that if \( Y \in V \), then all the closed orbits and all the singularities of \( Y \) are hyperbolic. Denote the set of star flows in \( M \) by \( \mathcal{G}^1(M) \).

Recently, Gan and Wen ([10]) proved a remarkable result about dissipative star flows defined in a \( d \)-dimensional manifold, where \( d \geq 3 \):

**Theorem 1.1.** If \( X \in \mathcal{G}^1(M^d) \) and \( \text{Sing}(X) = \emptyset \) then \( X \) is Axiom A without cycles.

In this paper we deal with these issues in the setting of three-dimensional divergence-free vector fields and our approach is of a completely different nature. We consider flows that are star flows restricted to the conservative setting, which we denote by \( \mathcal{G}^1_\mu(M) \). That is, \( X \in \mathcal{G}^1_\mu(M) \) if there exists a neighborhood \( V \) of \( X \) in \( \mathcal{X}^1_\mu(M) \) such that any \( Y \in V \), has all the closed orbits and all the singularities hyperbolic. Our main result states that such a flow has no singularities and is hyperbolic (Anosov). We note that Gan and Wen must consider non-singular flows due, in particular, to the fact that the Lorenz strange attractor is in \( \mathcal{G}^1(M) \). However, Arbieto and Matheus ([2, Corollary 4.1]) proved that, in the conservative setting, there are no geometrical Lorenz sets, which could indicate that it should be possible to remove the hypothesis of the non-existence of singularities.

Let us now state our main result.

**Theorem 1.2.** If \( X \in \mathcal{G}^1_\mu(M) \) then \( \text{Sing}(X) = \emptyset \) and actually \( X \) is Anosov.

We point out that the proof of this result is a consequence of several recent results on conservative three-dimensional flows. We believe that the previous result is also true in any dimension and its proof should be obtained by generalizing these recent results to any dimension\(^1\) and eventually following the strategy of the cited work.

\(^1\)In [7] the authors obtain a generalization of one of these results.
of Gan and Wen, namely by using the fact that vector fields in $G^{1}_\mu(M)$ cannot have heterodimensional cycles.

Let $A^3_\mu$ denote the open set of divergence-free Anosov vector fields on a three-dimensional manifold $M$.

It is clear that $G^{1}(M) \cap X^{1}_\mu(M) \subset G^{1}_\mu(M)$; Theorem 1.2 implies that

$$G^{1}(M) \cap X^{1}_\mu(M) = G^{1}_\mu(M) = A^3_\mu.$$  

As a consequence of Theorem 1.2 we also obtain the following result.

**Corollary 1.** The boundary of $A^3_\mu$ has no isolated points.

A vector field $X \in X^{1}_\mu(M)$ is said to be $C^1$-structurally stable in the conservative setting if there exists a $C^1$ neighborhood, $\mathcal{V}$, of $X$ in $X^{1}_\mu(M)$ such that every $Y \in \mathcal{V}$ is topological equivalent to $X$ (see, for example [12]).

Combining Theorem 1.2 with previous results of the first author with P. Duarte ([5]) and with V. Araújo ([1]) we are able to prove the stability conjecture for $C^1$ conservative 3-flows.

**Theorem 1.3.** If $X \in X^{1}_\mu(M)$ is a $C^1$-structurally stable three-dimensional flow then $X$ is Anosov.

2. Some main tools. If $p$ is a regular point of $X \in X^{1}_\mu(M)$, define the segment of orbit $\Gamma(p, \tau) = \{X^t(p); t \in [0, \tau]\}$. Now consider $V, V'$ sub-spaces of $N_p$, with $\text{dim}(V) = j$, for some $2 \leq j \leq n - 1$, and $N_p = V \oplus V'$. A one-parameter linear family $\{A_t\}_{t \in \mathbb{R}}$ associated to $\Gamma(p, \tau)$ and $V$ is defined as follows:

- $A_t: N_p \to N_p$ is a linear map, for all $t \in \mathbb{R}$,
- $A_t = Id$, for all $t \leq 0$, and $A_t = A_{\tau}$, for all $t \geq \tau$,
- $A_t|_V \in SL(j, \mathbb{R})$, and $A_t|_{V'} = Id$, $\forall t \in [0, \tau]$, in particular $\text{det}(A_t) = 1$, for all $t \in \mathbb{R}$, and
- the family $A_t$ is $C^\infty$ on the parameter $t$.

In this paper we will consider $n = 3$ and so $V = N_p$ and $\text{dim}(V) = 2$.

The following result is a kind of Franks’ Lemma for volume-preserving flows and was proved in [6].

**Theorem 2.1.** Given $\epsilon > 0$ and a vector field $X \in X^{1}_\mu(M)$ there exists $\xi_0 = \xi_0(\epsilon, X)$ such that $\forall \tau \in [1, 2]$, for any periodic point $p$ of period greater than 2, for any sufficient small flowbox $T$ of $\Gamma(p, \tau)$ and for any one-parameter linear family $\{A_t\}_{t \in [0, \tau]}$ such that $\|A'_tA^{-1}_t\| < \xi_0$, $\forall t \in [0, \tau]$, there exists $Y \in X^{1}_\mu(M)$ satisfying the following properties

1. $Y$ is $\epsilon$-$C^1$-close to $X$;
2. $Y^t(p) = X^t(p)$, for all $t \in \mathbb{R}$;
3. $P_Y(p) = P_X(p) \circ A_\tau$, and
4. $Y|_{\tau'} \equiv X|_{\tau'}$.

Let us state three crucial results that will help us to prove Theorem 1.2. They can be obtained, between other arguments (see [5]), by using Theorem 2.1, the fact that elliptic closed orbits are stable and also using a smoothness perturbation result due to Zuppa ([16]).

The first two lemmas give us different contexts where one can create a nearby elliptic closed orbit via a small perturbation, thus far from $G^{1}_\mu(M)$.
Lemma 2.2. ([5, Proposition 3.8]) Let $X \in \mathcal{X}^1_p(M)$ and $\epsilon > 0$ be given. There exists $\theta = \theta(\epsilon, X) > 0$ such that if a hyperbolic closed orbit $\mathcal{O}$ for $X$ has angle between its stable and unstable directions smaller than $\theta$, then we can find an $\epsilon$-$C^1$-close divergence-free vector field $Y$ such that $\mathcal{O}$ is an elliptic closed orbit for $Y_t$.

Lemma 2.3. ([5, Proposition 3.13]) Let $X \in \mathcal{X}^1_p(M)$ and $\epsilon, \theta > 0$ be given. There exist $m = m(\epsilon, \theta) \in \mathbb{N}$ and $T(m) > 0$ such that if $\mathcal{O}$ is a hyperbolic closed orbit for $X$ with

- angle between its stable and unstable directions bounded from below by $\theta$;
- period larger than $T(m)$, and

the linear Poincaré flow along $\mathcal{O}$ is not $m$-dominated,

then we can find an $\epsilon$-$C^1$-close divergence-free vector field $Y$ such that $\mathcal{O}$ is an elliptic closed orbit for $Y_t$.

Lemma 2.4. ([1, Lemma 2.6]) Let $X \in \mathcal{X}^1_p(M)$ and $\epsilon > 0$ be given and set $\theta = \theta(\epsilon, X)$, $m = m(\epsilon, \theta)$ and $T = T(m)$ given by Lemmas 2.2 and 2.3.

Assume that all divergence-free vector fields $Y$ which are $\epsilon$-$C^1$-close to $X$ do not admit elliptic closed orbits. Then, for every such $Y$, all closed orbits with period larger than $T$ are hyperbolic, $m$-dominated and with angle between its stable and unstable directions bounded from below by $\theta$.

We also recall the $C^1$-Closing Lemma adapted to the setting of volume-preserving flows by Pugh and Robinson ([13, Section 8(c)]) and also the contribution of Arnaud ([3]) for a more accurate version, which in particular assures a) and b) bellow.

The $X^t$-orbit of a recurrent point $x$ can be approximated for a very long time $T > 0$ by a closed orbit of a flow $Y$ which is $C^1$-close to $X$. In fact, given $r, T > 0$ we can find a $\epsilon$-$C^1$-neighborhood $\mathcal{U} \subset \mathcal{X}^1_p(M)$ of $X$, a closed orbit $p$ of $Y \in \mathcal{U}$ with period $\pi$, $\tilde{T} > T$ and a map $g : [0, T] \to [0, \pi]$ close to the identity such that

- $\text{dist}(X^t(x), Y^{\sigma(t)}(p)) < r$ for all $0 \leq t \leq \tilde{T}$;
- $Y = X$ over $M \setminus \bigcup_{0 \leq t \leq \tilde{T}} B(X^t(x), r)$.

Another ingredient of the proofs of our theorems is a generalization of Bochi’s dichotomy (see [8, Theorem A]) for the continuous-time class. This result was obtained recently by combining a Theorem of [4, Theorem 1], corresponding to the case when $X$ has no singularities, and a Theorem of [1, Theorem A], that corresponds to case when $X$ can have singularities. More precisely the following result was obtained.

**Theorem 2.5.** There exists a $C^1$-residual set $\mathcal{R} \subset \mathcal{X}^1_p(M)$ such that if $X \in \mathcal{R}$ then $X$ is Anosov or else almost every point in $M$ has zero Lyapunov exponents.

3. Proofs of the results. Let us recall that a singularity $\sigma$ of a vector field $X$ is linear hyperbolic if $\sigma$ is a hyperbolic singularity and there exists a smooth local change of coordinates around $\sigma$ that conjugates $X$ and $DX_\sigma$ (cf. [15, Definition 4.1]).

The proof of Theorem 1.2 is made in two steps. First we prove that if $X \in \mathcal{G}^1_p(M)$ then $X$ has no singularities and $P^t_X$ admits a dominated splitting over $M$ (Lemma 3.1) and then we prove that if $X \in \mathcal{G}^1_p(M)$ is such that $P^t_X$ admits a dominated splitting over $M$ then $X$ is Anosov (Lemma 3.2).
Lemma 3.1. If $X \in G^1_\mu(M)$ then $X$ has no singularities and $P_X^t$ admits a dominated splitting over $M$.

Proof. Let us first observe that $G^1_\mu(M)$ is $C^1$ open in $\mathcal{Y}^1_\mu(M)$.

To prove the lemma let us fix $X \in G^1_\mu(M)$ and a $C^1$-neighborhood $\mathcal{V}$ of $X$ in $G^1_\mu(M)$. Let us choose $Y \in \mathcal{V}$ such that all the singularities of $Y$ are linear hyperbolic. If $M \setminus \text{Sing}(Y)$ admits a dominated splitting for the linear Poincaré flow of $Y$ then [15, Proposition 4.1] implies that $\text{Sing}(Y) = \emptyset$. We observe that when $Y$ is robustly transitive then the previous conclusion follows directly from [11, Supplement].

It follows that there exists $\mathcal{U} \subset \mathcal{V}$, $Y \in \mathcal{U}$, whose elements do not have singularities and admit a dominated splitting for the associated linear Poincaré flow. So let us now assume that $M \setminus \text{Sing}(Y)$ does not admit a dominated splitting for the linear Poincaré flow of $Y$.

We claim that for all $m \in \mathbb{N}$, there exists a $Y^t$-invariant set $\Gamma_m \subset M \setminus \text{Sing}(Y)$ such that $\mu(\Gamma_m) > 0$ and $\Gamma_m$ do not have dominated splitting for $P_Y^t$. In fact, if this claim were false then there would exist $m$ such that $M \setminus \text{Sing}(Y)$ has an $m$-dominated splitting which contradicts our assumption.

Since $Y \in G^1_\mu(M)$, all divergence-free vector fields which are $\epsilon$-$C^1$-close to $Y$ do not admit elliptic closed orbits. Then, from Lemma 2.4, for every such a vector field $Y$ there are constants $\theta = \theta(\epsilon, Y)$, $m = m(\epsilon, \theta)$ and $T = T(m)$ such that, for each closed orbit with period greater then $T$, one has:

i) $m$-dominated splitting and
ii) angle between its stable and unstable directions bounded from below by $\theta$.

Observe that, since $Y \in G^1_\mu(M)$, these closed orbits are hyperbolic.

We will get a contradiction with the fact that there exists a positive measure set without domination.

Using the techniques involved in the proof of Theorem 2.5 (see [4]) it is straightforward to conclude that for $m$ sufficiently large and positive $\eta$ arbitrarily close to 0, there exist $T_0 > 0$ and $Y_1 \in \mathcal{V}$, $C^1$-close to $Y$, such that for a.e. $x \in \Gamma_m$, one has $e^{-t\eta} < \|P_{Y_1}^t(x)\| < e^{t\eta}$, for every $t > T_0$. Actually, let $\hat{U} \subset \Gamma_m$ be a measurable set with positive measure. Let $R \subset \hat{U}$ be the set given by the Poincaré recurrence theorem with respect to $Y_1$. Then, since the set of periodic points is a zero measure set, it follows that almost every $x \in R$ is not a periodic point and it returns to $\hat{U}$ infinitely many times under the flow $Y_1^t$. Let $Z_\eta$ denote the subset of points of $\Gamma_m$ having Lyapunov exponents, for $Y_1$, less than $\eta$.

Let us fix $\delta \in \left(0, \frac{\log(2)}{2m}\right)$ and $\eta < \delta$. Given $x \in Z_\eta \cap R$ there exists $T_x \in \mathbb{R}$ such that

$e^{-\delta t} < \|P_{Y_1}^t(x)\| < e^{\delta t}$ for every $t \geq T_x$.

Note that we can assume that $T_x \geq T$.

By the ergodic $C^1$-Closing Lemma ([3]) there exists a point $x \in Z_\eta \cap R$ such that the $Y_1^t$-orbit of $x$ can be approximated, by a closed orbit of a $C^1$-close flow $Y_2$: given $r, T > 0$ and a small $C^1$-neighborhood $\mathcal{U}$ of $Y_1$ in $\mathcal{X}^1_\mu(M)$, there exist a vector field $Y_2 \in \mathcal{U}$, $T > T$, a periodic orbit $p$ of $Y_2$ with period $\pi$ and a map $g: [0, T] \to [0, \pi]$ close to the identity such that

- $\text{dist}(Y_1^t(x), Y_2^{g(t)}(p)) < r$ for all $0 \leq t \leq T$;
- $Y_2 = Y_1$ over $M \setminus \bigcup_{0 \leq t \leq \tilde{T}} B(Y_1^t(x), r)$.

Thus, any point $x \in Z_\eta \cap R$ is $\epsilon$-close to a periodic point $p$ with period $\pi$ and returned with a probability greater than $\eta$.

Finally, observe that $\Gamma_m$ contains all periodic orbits, and it follows from the ergodic closing lemma that $\Gamma_m$ is dense in $M$. This implies that $\mu(\Gamma_m) > 0$. By the ergodic closing lemma we conclude that $Y_1^t$ is ergodic and admits a dominated splitting, which is a contradiction with our assumption. 

Therefore, we have proved that $M \setminus \text{Sing}(Y)$ admits a dominated splitting for the linear Poincaré flow of $Y$.
Letting \( r > 0 \) be small enough we obtain also that
\[
e^{-\delta \pi} < \| P_{Y_2}(p) \| < e^{\delta \pi},
\]
where \( \pi > T \).

Now, by construction, it follows that \( Y_2 \) is \( C^1 \)-close to \( Y \), so that the orbit of \( p \) under \( Y_2 \) satisfies the conclusion of Lemma 2.4. In particular we have that
\[
\| P_{Y_2}^m \| \leq \frac{1}{2} \| P_{Y_2}\| \text{ for all } x \in O_{Y_2}(p).
\]

Since the subbundles \( N^s \) and \( N^u \) are one-dimensional we write \( p_i := Y_2^{-in}(p) \) for \( i = 0, \ldots, \lfloor \pi/m \rfloor \) with \( \lfloor \cdot \rfloor := \max \{ k \in \mathbb{Z} : k \leq t \} \) and
\[
\| P_{Y_2}^\pi \| = \| P_{Y_2}^{-m \cdot \lfloor \pi/m \rfloor} \| \cdot \prod_{i=0}^{\lfloor \pi/m \rfloor - 1} \| P_{Y_2}^m \| \leq C(p, Y_2) \cdot \left(\frac{1}{2}\right)^{\lfloor \pi/m \rfloor},
\]
where \( C(p, Y_2) = \sup_{0 \leq t \leq m} \left( \| P_{Y_2}^\pi \| \cdot \| P_{Y_2}^{-m} \|^{-1} \right) \) depends continuously on \( Y_2 \) in the \( C^1 \) topology. Then, there exists a uniform bound for \( C(p, \cdot) \) for all vector fields which are \( C^1 \)-close to \( Y \).

Notice that we can take \( \pi > T \) arbitrarily large by letting \( r > 0 \) be small enough in the arguments described above. Therefore, the inequality in (2), ensures that
\[
\| P_{Y_2}^\pi \| = \| P_{Y_2}^{-m} \| \text{ and also }
\]
\[
\frac{1}{\pi} \log \| P_{Y_2}^\pi \| \leq \frac{1}{\pi} \log C(p, Y_2) + \frac{\lfloor \pi/m \rfloor}{\pi} \log \frac{1}{2} + \frac{1}{\pi} \log \| P_{Y_2}^\pi \|.
\]

Moreover, since \( P_{Y_2}^\pi \) is area-preserving we have that the sum of the Lyapunov exponents is zero, that is (recalling that \( \pi \) is the period of \( p \))
\[
\frac{1}{\pi} \log \| P_{Y_2}^\pi \| = -\frac{1}{\pi} \log \| P_{Y_2}^{-m} \|.
\]

The constants in inequality (2) do not depend on \( \pi \) so taking the period very large we can deduce that
\[
\frac{1}{\pi} \log \| P_{Y_2}(p) \| \geq \frac{1}{2m} \log 2 > \delta.
\]
This contradicts (1). Therefore \( \text{Sing}(Y) = \emptyset \) and \( P_X^\ell \) admits a dominated splitting over \( M \).

Let us now prove that \( X \) has no singularities and that \( P_X^\ell \) admits a dominated splitting over \( M \). In fact if \( X \) has a singularity then there exists \( Y_0 \in \mathcal{V} \) such that \( Y_0 \) has at least one linear hyperbolic singularity. Now we proceed as before to \( Y_\in \mathcal{V} \), arbitrarily close to \( Y_0 \), having all the singularities linear hyperbolic and with \( \text{Sing}(Y) \neq \emptyset \). Repeating the arguments above we get that \( \text{Sing}(Y) = \emptyset \), which is a contradiction. Therefore, in those arguments we can take \( Y = X \) and then conclude that \( X \) has a dominated splitting for the linear Poincaré flow.

\[\square\]

**Lemma 3.2.** If \( X \in G^1_{\mathrm{an}}(M) \) is such that \( P_X^{\ell} \) admits a dominated splitting over \( M \) then \( X \) is Anosov.

**Proof.** Since \( P_X^{\ell} \) admits a dominated splitting over \( M \) one gets that there exists \( \ell \in \mathbb{N} \) such that
\[
\Delta(x, \ell) = \| P_X^\ell(x) \| \cdot \| P_X^{-\ell}(X^\ell(x)) \| \leq \frac{1}{2} \quad \forall x \in M,
\]
where \( N = N^1 \oplus N^2 \), and these subbundles are \( P^1_X \)-invariant and are one-dimensional.

For any \( i \in \mathbb{N} \) we have \( \Delta(x, i\ell) \leq 1/2^i \). For every \( t \in \mathbb{R} \) we may write \( t = i\ell + r \) and since \( \|P^1_X\| \) is bounded, say by \( L \), take \( C = 2^7L^2 \) and \( \sigma = 2^{-1/8} \) to get \( \Delta(x, t) \leq C\sigma^i \), for every \( x \in M \) and \( t \in \mathbb{R} \). Denote by \( \alpha_t \) the angle \( \angle(N^1_{X^t(x)}, N^2_{X^t(x)}) \). We already know, by domination, that this angle is bounded below from zero, say by \( \beta \). Since we do not have singularities there exists \( K > 1 \) such that for all \( x \in M \), \( K^{-1} \leq \|X(x)\| \leq K \). Since the flow is conservative and the subbundles are both one dimensional we have that

\[
\sin(\alpha_0) = \|P^t(X(x))\| \|P^t(X(x))\| \sin(\alpha_t) \|X(X^t(x))\| \|X^t(x)\|.
\]

So,

\[
\|P^t_X(x)\|^2_{N^2} = \frac{\sin(\alpha_0)}{\sin(\alpha_t)} \|X(x)\| \|X(X^t(x))\| \Delta(x, t) \leq \Delta(x, i\ell + r) \sin(\beta)^{-1}K^2 \leq \sigma^C \sin(\beta)^{-1}K^2.
\]

Analogously we get

\[
\|P^{t'}_{X^t}(x)\|^2_{N^2} = \frac{\sin(\alpha_t)}{\sin(\alpha_0)} \|X(X^t(x))\| \|X^t(x)\| \Delta(x, t) \leq \Delta(x, i\ell + r) \sin(\beta)^{-1}K^2 \leq \sigma^C \sin(\beta)^{-1}K^2.
\]

These two inequalities show that \( M \) is hyperbolic for the linear Poincaré flow. Then by [9, Proposition 1.1] we obtain that \( M \) is a hyperbolic set, thus \( X \) is Anosov. This ends the proof of the lemma.

\[ \square \]

\textit{Proof.} (of Corollary 1) We claim that an isolated point \( X \) of the boundary of \( A^3_{\mu} \) do not have singularities. In fact if \( \text{Sing}(X) \neq \emptyset \) then, since Anosov vector fields do not have singularities, the singularities of \( X \) must be all nonhyperbolic. A nonhyperbolic singularity can be made hyperbolic by a small perturbation, thus there are vector fields arbitrarily close to \( X \) having (stably) hyperbolic singularities which is a contradiction because \( X \) is an isolated point of the boundary of \( A^3_{\mu} \).

Now we just have to follow the proof of Theorem 1.2, taking \( Y = X \) (where we don’t need to assume anymore that \( X \in \mathcal{G}^1_{\mu}(M) \)), concluding that the linear Poincaré flow of \( X \) admits a dominated splitting over \( M \). Now, as in the proof of the previous corollary, it follows that \( X \) is Anosov.

\[ \square \]

\textit{Proof.} (of Theorem 1.3) Let us fix a \( C^1 \)-structurally stable vector field in \( \mathcal{X}^1_{\mu}(M) \) and choose a neighborhood \( \mathcal{V} \) of \( X \) whose elements are topologically equivalent to \( X \). If \( X \notin A^3_{\mu} = \mathcal{G}^1_{\mu}(M) \) then it follows that \( \mathcal{V} \cap A^3_{\mu} = \emptyset \). Using [5, 1] one gets that there exists a residual subset \( \mathcal{R} \subset \mathcal{V} \) such that for every \( Y \in \mathcal{R} \) the set of elliptic closed orbits is dense in \( M \). Let us fix \( Y \in \mathcal{R} \) and choose a small neighborhood of \( Y \), \( \mathcal{W} \subset \mathcal{V} \).

Let \( x \) be an elliptic point of large period, say \( \pi \). Using Zuppà’s theorem ([16]) and the stability of elliptic points, we can approximated \( Y \), in the \( C^1 \) topology, by a \( C^2 \)-vector field \( Z \in \mathcal{W} \) such that the analytic continuation of \( x \) is also an elliptic
point with period close to \( \pi \). Now, if \( \pi \) is large enough, we apply Theorem 2.1 several times, by concatenating small rotations (the maps \( A_t \)), in order to obtain a new vector field \( W \in \mathcal{W} \) exhibiting a parabolic closed orbit. Since the existence of a parabolic point prevents structural stability and \( W \in \mathcal{W} \) we get a contradiction. Therefore \( X \in \mathcal{A}^3_{\mu} \), which ends the proof.

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