The tenfold way redux: Fermionic systems with $N$-body interactions

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We provide a systematic treatment of the tenfold way of classifying fermionic systems that naturally allows for the study of those with arbitrary $N$-body interactions. We identify four types of symmetries that such systems can possess, which consist of one ordinary type (usual unitary symmetries), and three non-ordinary symmetries (such as time reversal, charge conjugation and sublattice). Focusing on systems that possess no non-trivial ordinary symmetries, we demonstrate that the non-ordinary symmetries are strongly constrained. This approach not only leads very naturally to the tenfold classes, but also obtains the canonical representations of these symmetries in each of the ten classes. We also provide a group cohomological perspective of our results in terms of projective representations. We then use the canonical representations of the symmetries to obtain the structure of Hamiltonians with arbitrary $N$-body interactions in each of the ten classes. We show that the space of $N$-body Hamiltonians has an affine subspace (of a vector space) structure in classes which have either or both charge conjugation and sublattice symmetries. Our results can help address open questions on the topological classification of interacting fermionic systems.

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A. Introduction

Until very recently, our understanding of fermionic many body systems, for most part, could be traced to a handful of ground states and their excitations, e.g., the Fermi sea, the band insulator, the filled Landau level, and the BCS superconducting state. This picture has been drastically overhauled in the last decade initiated by the discovery of the two dimensional spin Hall insulators \[1-5\], and bolstered by its experimental realization [6]. It soon became clear \[7-9\] that systems with time reversal symmetry have new types of ground states – the topological insulator – in three dimensions as well (see \[10-15\] for a review). These developments naturally motivated the question of the classification of gapped states of fermionic many body systems which has now firmly established itself as a key research direction of condensed matter physics.

For systems of noninteracting fermions, a comprehensive classification has been achieved (see \[16-18\] for an overview), and indeed marks a milestone in condensed matter research. Central to this success is the symmetry classification of fermionic systems based on a set of “intrinsic” symmetries – the tenfold way of Altland and Zirnbauer [19] (see \[20-21\] for a more formal treatment) – which places any fermionic system into one of ten symmetry classes. From the point of view of fermionic physics, this work represents the culmination of a program of classification initiated by Dyson [22] via the threefold way. In each symmetry class, a gapped fermionic system may possess ground states which are topologically distinct. Early work in this direction \[23-24\] was developed into a complete picture \[25\], culminating in the “periodic table” of Kitaev [26]. The key result is that in any spatial dimension, there are only five symmetry classes that support nontrivial topological phases of noninteracting fermionic systems. The presence or absence of a topological phase is characterized by a nontrivial abelian group such as \(\mathbb{Z}, \mathbb{Z}_2\) or \(2\mathbb{Z}\). Even more remarkably, the pattern of nontrivial groups has a periodicity (in spatial dimensions) of 2 for the so called complex classes, and 8 in the real classes with a very specific relationship between nontrivial classes in a given dimension and ones just above and just below. These ideas have also been generalized to include defects [27], and have also been visited again from more formal perspectives [28-29] (see also [30]). Further developments in the physics of noninteracting systems came up with the study of the interplay of intrinsic symmetries with those of the environment (such as crystalline space groups) that have resulted in a more intricate classification [31-33].

A most intriguing story emerges up on the inclusion of interactions, i.e., for a system of fermions with “nonquadratic” terms in their Hamiltonian. [1-5] showed that the presence of two-body interactions results in a “collapse” of the noninteracting topological classification – for example, Kitaev’s Majorana chain [35], described by the group \(\mathbb{Z}\) collapses to \(\mathbb{Z}_8\) upon the inclusion of interactions. Following this, the natural question that arises is regarding the principles of classification of topological phases in the presence of interactions. Ideas based on group cohomology [36], and supercohomology for fermions [37] have been put forth. While these are important steps towards the final goal of topological classification of interacting systems, the problem remains at the very frontier of condensed matter research [38, 39].

A crucial aspect of the tenfold symmetry classification of noninteracting systems is the determination of the structure of the Hamiltonians allowed in each class. The knowledge of this structure then provides ways for viewing these systems from various perspectives e.g., classifying spaces [40], structure of the target manifolds of nonlinear \(\sigma\)-model descriptions [25] etc. To the best of our knowledge, a study of the structure of fermionic Hamiltonians in each of the ten symmetry classes with arbitrary $N$-body interactions is not available in the literature. An understanding of the structure of the interacting Hamiltonians will not only aid the analysis of these systems, but also help motivate models that could be crucial to

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build phenomenology and intuition (possibly through numerics) much like what the Kane-Mele and BHZ models did in the context of noninteracting systems. This paper aims to fill this lacuna. Enroute this pursuit, we build a transparent framework that allows for a systematic approach to these problems, which even provides further insights for the noninteracting case. Our framework is developed for a system comprising of \( L \) one-particle states (dubbed “orbitals”) which can be populated by fermions. Symmetries of such a system can be either linear or antilinear, and non-transposing (occupied orbitals mapped to occupied orbitals) or transposing (occupied orbitals mapped to unoccupied ones), i.e., four distinct types of symmetries. We call the linear non-transposing type of symmetries as \textit{ordinary}, and the remaining three types (antilinear non-transposing, linear transposing, antilinear transposing) as \textit{non-ordinary} symmetries. The focus of our work are those fermionic systems – which we dub as “grotesque” – which do not possess any nontrivial ordinary symmetries. We show that non-ordinary symmetries of a grotesque fermionic system (GFS) are solitary, i.e., a GFS can possess at most one from each type of non-ordinary symmetries. Identifying these solitary non-ordinary symmetries with time reversal (T), charge conjugation (C), and sublattice (S), leads us to the familiar Altland and Zirnbauer tenfold symmetry classes. We then address the key question of determining the structure of the Hamiltonian in these classes, by constructing “canonical” representations (by unitary matrices) for the symmetry operations in each class. This construction allows for a systematic and efficient determination of the structure of any arbitrary \( N \)-body interaction term in the Hamiltonian of any class. Not surprisingly we recover all known results of noninteracting systems and, more importantly, we provide physical insights that can, inter alia, aid model building thereby helping address the outstanding open problems vis a vis topological phases of interacting fermionic systems.

We begin the discussion with the setting of the problem in section \[I\]. This is followed in section \[C\] by a discussion of the four types of symmetry operations that a fermionic system can possess. Section \[D\] introduces and studies those fermionic systems that possess non trivial ordinary symmetries and obtains the constraints that the symmetries have to satisfy. The tenfold way is elucidated in section \[E\] which includes a treatment of the canonical representation of the symmetry in each of the symmetry classes, the result of which is displayed in table \[I\]. This is followed by section \[F\] that provides an understanding of the results shown in table \[I\] from the point of view of group cohomology. It is shown that every entry of table \[I\] is made up of copies of irreducible projective representations of the Klein group or its subgroups. Known results of noninteracting systems are reproduced in section \[C\] (see table \[I\]).

Section \[H\] lays down the framework for obtaining the structure of a Hamiltonian with \( N \)-body interactions by reducing the problem to the determination of certain vector subspaces and arbitrary \( K \)-body interaction term is shown to be an element of an affine subspace. Techniques needed for performing the analysis for determining the subspace structure are developed first for the two-body case in section \[I\] and later generalized to the arbitrary \( N \)-body case in section \[J\]. Tables \[II\] and \[IV\] contain the structures of the \( N \)-body interaction term in each class. The paper is concluded in section \[K\] where the significance and scope of our results are highlighted including their use in other problems of interest. For the convenience of the reader, all important symbols used in the paper are listed in appendix \[I\].

The paper is structured to be self contained and easy to use in that we develop the discussion from the very basic building blocks. This desideratum naturally results in some overlap with the results from the previous works. We shall specifically mention only those results that we have used explicitly in our discussion.

### B. The Setting

Our system consists of \( L \) one particle states \( |i\rangle \), \( i = 1, \ldots, L \) – which we call “orbitals” – that are orthonormal \( \langle i|j \rangle = \delta_{ij} \) (\( \delta_{ij} \) is the Kronecker delta symbol). Note that these states could denote a variety of situations; \( i \) could be orbitals at different sites of a lattice, or different atomic orbitals, or even flavor states of an elementary particle – our use of the term orbital denotes one particle states in any context including those just stated. Starting from the vacuum state \( |0\rangle \), we can create the one particle state

\[
|\tilde{i}\rangle \equiv \psi_i |0\rangle,
\]

where \( \psi_i (\psi_i^\dagger) \) is a fermion creation (annihilation) operator that creates (destroys) a particle in the one particle state \( |\tilde{i}\rangle \). These operators satisfy the well known fermion anticommutation relations

\[
\psi_i \psi_j^\dagger + \psi_j \psi_i^\dagger = \{\psi_i, \psi_j^\dagger\} = \delta_{ij},
\]

and

\[
\{\psi_i, \psi_j\} = 0.
\]

We collect these fermionic operators into convenient arrays

\[
\Psi = \begin{bmatrix}
\psi_1 \\
\vdots \\
\psi_L
\end{bmatrix}, \quad \Psi^\dagger = \begin{bmatrix}
\psi_1^\dagger & \ldots & \psi_L^\dagger
\end{bmatrix}.
\]

A different set of orthonormal one particle states \( |\tilde{j}\rangle \) \((i = 1, \ldots, L)\) can, of course, be used as effectively. The states \( |\tilde{j}\rangle \) are related to \( |i\rangle \) via \( |\tilde{j}\rangle = \sum_{j=1}^L R_{ij} |j\rangle \), where \( R_{ij} \) are the components of an \( L \times L \) unitary matrix \( \mathbf{R} \). In terms of the operators \( \Psi \), we have

\[
\Psi^\dagger = \mathbf{R}^{\dagger} \Psi \mathbf{R},
\]

along with other useful relations

\[
\Psi = \mathbf{R} \Psi^\dagger, \quad \Psi^{\dagger T} = \mathbf{R}^T \Psi^T, \quad \mathbf{R}^{\dagger T} = \mathbf{R}^T \mathbf{R}^*.
\]

where \( (\cdot)^T \) denotes the transpose.
The system can have any number of particles \( N_p \) ranging from 0 to \( L \). For each particle number \( N_p \), the set of allowed states is spanned by \( [\frac{L}{N_p}] \) states obtained by creating \( N_p \)-particle states from the vacuum using distinct combinations of the operators \( \phi_i^\dagger \). The vector space of \( N_p \)-particle states is denoted by \( \mathcal{V}_{N_p} \). The full Hilbert-Fock space of the system is given by

\[
\mathcal{V} = \bigoplus_{N_p=0}^{L} \mathcal{V}_{N_p},
\]

which is a vector space over complex numbers \( \mathbb{C} \), providing a complete kinematical description of the system. An important property of this vector space, which we will exploit, is that it is \( \text{“graded”} \) in a natural fashion by the sectors of different particle number. The dynamics of this fermionic system is determined by the Hamiltonian which contains up to \( N \)-body interactions where \( 0 \leq N \leq L \), and is formally written as

\[
\mathcal{H} = \sum_{K=0}^{N} (\Psi^\dagger)^{K} \mathcal{H}^{(K)} (\Psi)^{K}
\]

where

\[
(\Psi^\dagger)^{K} \mathcal{H}^{(K)} (\Psi)^{K} = \mathcal{H}^{(K)}_{i_{12...i_{K}; 1_2...j_{K}}}(\phi_{i_{1}}\phi_{j_{1}}\cdots\phi_{i_{K}}\psi_{j_{1}}\psi_{j_{2}}\cdots\psi_{j_{K}})
\]

with repeated indices \( i \) and \( j \) summed from 1, \ldots, \( L \). Note that here we depart slightly from the usual convention for the many-body Hamiltonian where \( \dagger \) operation is done on the complete string of \( \psi \); this notation will eventually prove to be useful in the later manipulations. Note also that \( (\phi_{i_{1}}\phi_{j_{1}}\cdots\phi_{i_{K}}) \) should be distinguished from the definition of the \( 1 \times L \) dimensional array \( \Psi \) in eqn. (4), \( \mathcal{H}^{(K)} \) is the matrix which describes the \( K \)-body interactions, and its components \( \mathcal{H}^{(K)}_{i_{12...i_{K}; 1_2...j_{K}}} \) have two properties. First, \( \mathcal{H}^{(K)}_{i_{12...i_{K}; 1_2...j_{K}}} \) is fully antisymmetric under permutations of the \( i \) indices among themselves, and also under the permutations of the \( j \) indices among themselves. Expressed in an equation

\[
\mathcal{H}^{(K)}_{i_{12...i_{K}; 1_2...j_{K}}} = \text{sgn}(X) \text{ sgn}(Y) \mathcal{H}^{(K)}_{j_{12...j_{K}; i_1i_2...i_{K}}}
\]

where \( X \) and \( Y \) are arbitrary permutations of \( K \) objects, and \( \text{sgn} \) denotes the sign of the permutation. Second, the Hermitian character of the Hamiltonian is reflected in the relation

\[
\mathcal{H}^{(K)}_{j_1j_2...j_{K}i_1i_2...i_{K}} = (\mathcal{H}^{(K)}_{i_1i_2...i_{K}; j_1j_2...j_{K}})^\ast.
\]

Each \( \mathcal{H}^{(K)} \) is an element of an \( (\frac{L}{K}) \)-dimensional vector space \( \mathcal{H}^{(K)} \) over the real numbers \( \mathbb{R} \). With no further restrictions other than eqn. (10) and eqn. (11), in fact, this vector space is endowed with a structure of a Lie algebra. This is achieved by constructing an isomorphic vector space \( i\mathcal{H}^{(K)} \) (multiplying every element of \( \mathcal{H}^{(K)} \) by \( i = \sqrt{-1} \)). It is evident that for any two matrices \( i\mathcal{H}_{a}^{(K)} \) and \( i\mathcal{H}_{b}^{(K)} \) of \( i\mathcal{H}^{(K)} \), the commutator

\[
[i\mathcal{H}_{a}^{(K)}, i\mathcal{H}_{b}^{(K)}] = i\mathcal{H}^{(K)}, \quad \text{and in fact,}
\]

\[
i\mathcal{H}^{(K)} \sim u \left( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right),
\]

i.e., \( i\mathcal{H}^{(K)} \) is isomorphic to the well-known Lie algebra \( u \left( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right) \) which generates the Lie group \( U \left( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right) \), and in fact,

\[
i\mathcal{H}^{(K)} \sim u \left( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right),
\]

where \( \mathcal{H} \) is the real vector space

\[
\mathcal{H} = \bigotimes_{K=0}^{N} \mathcal{H}^{(K)}
\]

The problem of classification of a fermionic system of \( L \)-orbitals with the Hilbert-Fock space eqn. (7) and upto \( N \)-body interactions can now be stated precisely. How many \( \text{“distinct”} \) spaces \( \mathcal{H} \) are possible? The symmetries of the system will determine the distinct structures of these spaces, placing them in different classes.

C. Symmetries

1. Symmetry Operations

A symmetry operation as defined by Wigner\(^{43}\) is a linear or antilinear (bijective) operator \( \mathcal{K} \) acting on the Hilbert space of the system that leaves the magnitude of the inner product invariant. Stated in the context of one particle physics of our \( L \)-orbital fermionic system, an operator \( \mathcal{K} \) is a symmetry operation if

\[
\langle \zeta' | \phi' \rangle = \langle \zeta | \phi \rangle, \quad \forall | \phi, \zeta \rangle \in \mathcal{V}
\]
with $|\phi\rangle = U|\phi\rangle$ and $|\zeta\rangle = U|\zeta\rangle$. A symmetry operation must preserve the graded structure of the Hilbert space eqn. (7), i.e., $N_p$ particle states must be mapped only to $N_p$ particle states in a bijective (invertible) fashion. We shall refer to such operations as usual symmetry operations; note that a usual symmetry operator can be either linear or antilinear. The above discussion can be summarized by the equation

$$\mathcal{U}_{\text{USL}}(V_{N_p}) = V_{N_p}, \forall N_p$$  \hspace{1cm} (16)

and the action of $\mathcal{U}_{\text{USL}}$ is implemented on the operators via

$$\mathcal{U}_{\text{USL}}|\psi\rangle = U_{\text{USL}}|\psi\rangle$$  \hspace{1cm} (17)

(in component form $\mathcal{U}_{\text{USL}}|\psi\rangle = U_{\text{USL}} = \sum_{j=0}^{L} \psi_j(U_{\text{USL}})_{ji}$) where $U_{\text{USL}}$ is an $L \times L$ unitary matrix that encodes the symmetry operation. If such a usual symmetry is linear then $\mathcal{U}_{\text{USL}}(\xi \mathcal{S}) = \xi \mathcal{S}$, and $\mathcal{U}_{\text{USL}}(\xi \mathcal{S}) = -\xi \mathcal{S}$ if it is antilinear where $\mathcal{S}$ is the identity of operator on $\mathcal{V}$, with the form eqn. (17) remaining the same for both linear and antilinear cases.

The key realization of Altland and Zirnbauer [19] is that many particle fermionic systems admit a larger classes of symmetry operations (see [21] for a discussion). The crucial point is that a linear or antilinear operation that maps $V_{N_p}$ to $V_{L-N_p}$, that preserves the magnitude of inner product and also preserves the graded nature of the Hilbert-Fock space eqn. (7) is also a legitimate symmetry operation. We call such operations transposing symmetry operations (see Fig. 2 for a schematic illustration) and they satisfy

$$\mathcal{U}_{\text{TRN}}(V_{N_p}) = V_{L-N_p}, \forall N_p$$  \hspace{1cm} (18)

with $\mathcal{U}_{\text{USL}}(\xi \mathcal{S}) = \pm \xi \mathcal{S}$ for linear (+) and antilinear (−) operations. Note that such operations are enabled by the fact that $\dim V_{N_p} = \dim V_{L-N_p}$. In fact, operations of the type eqn. (16) and eqn. (18) exhaust all possible (anti)linear automorphisms of $\mathcal{V}$ that preserve its graded structure. From a physical perspective, the transposing symmetry operation maps an $N_p$ “particle” state to an $N_p$ “hole” state. Holes are fermionic excitations obtained by starting from the “fully filled state” $|\Omega\rangle = (\psi_1 \psi_2 \ldots \psi_L)^{\dagger} |0\rangle$, and creating hole like excitations (such as $\psi_j |\Omega\rangle$, a 1-hole state). It is therefore natural to implement the action of a transposing symmetry operation $\mathcal{U}_{\text{TRN}}$ via

$$\mathcal{U}_{\text{TRN}}|\psi\rangle = U_{\text{TRN}}^\dagger |\psi\rangle$$  \hspace{1cm} (19)

where $^\dagger$ denotes complex conjugation, such that creation operators are mapped to annihilation operators, and the unitary matrix $U_{\text{TRN}}$ encodes “relabelling” of states in this transposing symmetry operation (which can, again, be linear or antilinear).

The main conclusion of the above discussion is that there are four distinct types of symmetry operations as illustrated in fig. 2. They are usual linear (UL), usual antilinear (UA), transposing linear (TL), and transposing antilinear (TA). We find it useful to introduce additional terminology – we call UL symmetry operations as ordinary symmetry operations, and all other types of symmetry operations (UA, TL and TA) as non-ordinary operations.

2. Symmetry conditions

A symmetry operation is a symmetry if it leaves the Hamiltonian of the system invariant. For usual symmetries, this is effected by the condition

$$\mathcal{U}_{\text{USL}} \mathcal{H} \mathcal{U}_{\text{USL}}^{-1} = \mathcal{H}$$  \hspace{1cm} (20)

while for the transposing symmetry operation the condition changes to

$$\mathcal{U}_{\text{TRN}} \mathcal{H} \mathcal{U}_{\text{TRN}}^{-1} = : \mathcal{H} :$$  \hspace{1cm} (21)

where $:\mathcal{H}:$ indicates that the expression $\mathcal{U}_{\text{TRN}} \mathcal{H} \mathcal{U}_{\text{TRN}}^{-1}$ has to be normal ordered (all creation operators to the left of annihilation operators) using the anticommutation relations eqn. (2). Both of these types of symmetries induce a mapping of $\mathcal{H}$ in eqn. (13) to $\tilde{\mathcal{H}}$ via

$$\mathcal{H} = (\mathcal{H}^{(0)}, \mathcal{H}^{(1)}, \ldots, \mathcal{H}^{(N)}) \mapsto \tilde{\mathcal{H}} = (\tilde{\mathcal{H}}^{(0)}, \tilde{\mathcal{H}}^{(1)}, \ldots, \tilde{\mathcal{H}}^{(N)})$$  \hspace{1cm} (22)

To obtain $\tilde{\mathcal{H}}^{(k)}$ for any $k$, we introduce an intermediate quantity $\mathcal{H}^{(k)}$ which is determined by the type of symmetry operation:

![Symmetry Operations](image-url)
\[
\tilde{H}_{i_1, \ldots, i_{k}; j_1, \ldots, j_{k}}^{(K)} = \begin{cases}
U_{i_1 \bar{i}_1} \cdots U_{i_k \bar{i}_k} H_{i_1, \ldots, i_{k}; j_1, \ldots, j_{k}}^{(K)} U^\dagger_{\bar{i}_1 j_1} \cdots U^\dagger_{\bar{i}_k j_k}, & \text{UL} \\
U_{i_1 \bar{i}_1} \cdots U_{i_k \bar{i}_k} \left( H_{i_1, \ldots, i_{k}; j_1, \ldots, j_{k}}^{(K)} \right)^* U^\dagger_{\bar{i}_1 j_1} \cdots U^\dagger_{\bar{i}_k j_k}, & \text{UA} \\
U^*_{i_1 \bar{i}_1} \cdots U^*_{i_k \bar{i}_k} H_{i_1, \ldots, i_{k}; j_1, \ldots, j_{k}}^{(K)} U^T_{j_1, \ldots, j_{k}; i_1, \ldots, i_{k}} U^T_{\bar{i}_1, \ldots, \bar{i}_k}, & \text{TL} \\
U^*_{i_1 \bar{i}_1} \cdots U^*_{i_k \bar{i}_k} \left( H_{i_1, \ldots, i_{k}; j_1, \ldots, j_{k}}^{(K)} \right)^* U^T_{j_1, \ldots, j_{k}; i_1, \ldots, i_{k}} U^T_{\bar{i}_1, \ldots, \bar{i}_k}, & \text{TA}
\end{cases}
\]

Here all the primed indices are summed from 1 to \( L \), and \( U \)s are the unitary matrices that encode the symmetry operations as defined in eqn. (17) and eqn. (19). Note that antilinear symmetry operations lead to a complex conjugation of the matrix elements. The transformation of the Hamiltonian eqn. (22) can now be specified completely. For usual symmetries (eqn. (17) and eqn. (20)), both linear and antilinear, we have

\[
\tilde{H}^{(K)} = H^{(K)}.
\] (24)

For transposing operation, the result is a bit more involved on account of the normal ordering operation of eqn. (21). We find

\[
\tilde{H}^{(K)} = \sum_{R=K}^{N} A_{R,K} \left[ tr_{R-K} \tilde{H}^{(K)} \right]^T,
\] (25)

The trace operation \( tr_{P} \tilde{H}^{(K)} = \tilde{P}_{i_1, \ldots, i_{k}; j_1, \ldots, j_{k}} \tilde{H}^{(K)} \) is defined as

\[
tr_{P} \tilde{H}^{(K)} = \tilde{P}_{i_1, \ldots, i_{k}; j_1, \ldots, j_{k}} \tilde{H}^{(K)}
\] (26)

accomplishing the tracing out (repeated \( k \) indices are summed) of \( P \) indices in \( \tilde{H}^{(K)} \), resulting in a \( (L - k) \times (L - k) \) matrix. The constants \( A_{R,K} \) in eqn. (25) are computed to be

\[
A_{R,K} = \begin{cases}
(-1)^{K} \frac{1}{(R - K)! \left( \frac{R!}{K!} \right)^2}, & 0 \leq K \leq R \\
0, & K > R
\end{cases}
\] (27)

Note that the RHS of eqn. (25) involves a matrix transpose, and this is a characteristic of the transposing symmetry operations justifying our terminology. Thus, eqns. (22), (23), (24) and (25) completely determine the mapping of \( \tilde{H} \) to \( \tilde{H} \). A symmetry operation (of any type) describes a symmetry if and only if

\[
\tilde{H} = H
\] (28)

which is a concise expression of eqn. (20) and eqn. (21).

3. Symmetry Group

The set of all symmetry operations \( \mathcal{U} \) of the system forms a group \( G_{\mathcal{V}} \). This group is a disjoint union of four types of symmetry operations (see fig. 2).

\[
G_{\mathcal{V}} = G^{UL}_{\mathcal{V}} \cup G^{UA}_{\mathcal{V}} \cup G^{TL}_{\mathcal{V}} \cup G^{TA}_{\mathcal{V}}.
\] (29)

The set of ordinary symmetries \( G^{UL}_{\mathcal{V}} \) form a normal subgroup of \( G_{\mathcal{V}} \). The non-ordinary symmetries have the following interesting properties. First, for any non-ordinary \( \mathcal{U} \) we have

\[
\mathcal{U} \mathcal{V} \mathcal{U} \in G^{UL}_{\mathcal{V}}, \quad \forall \mathcal{U} \in G^{UL}_{\mathcal{V}} \cup G^{TL}_{\mathcal{V}} \cup G^{TA}_{\mathcal{V}}.
\]

(30)

which can be paraphrased “the square of a non-ordinary symmetry operation is an ordinary symmetry operation”. Second, the product of two distinct types of non-ordinary symmetry operations is the third type – for example

\[
\mathcal{U}_{UA} \mathcal{V}_{TL} = \mathcal{V}_{TA};
\]

(31)

this is summarized in Fig. 3. The content of that figure can be restated as: the factor group \( G_{\mathcal{V}} / G^{UL}_{\mathcal{V}} \) is the Klein 4-group \( K_4 \).

The set of all the symmetry operations that satisfy either eqn. (20) or eqn. (21) makes up the symmetry group \( G \) of the system and is a subgroup of \( G_{\mathcal{V}} \).

D. Grotesque Fermionic Systems

We will now focus attention on a special class of \( L \)-orbital fermionic systems. Our systems of interest do not possess any nontrivial ordinary symmetries – we dub such systems as “grotesque fermionic systems (GFS)” to highlight this property. This would suggest that the only ordinary symmetry operation allowed is the trivial identity operation \( \mathcal{I} \). However, since we are working with the Hilbert-Fock vector space (not a projective or “ray” space of Wigner, see [44]), the operator

\[
\mathcal{I} = \hat{\phi}^0 \mathcal{N}
\]

(32)

with

\[
\mathcal{N} = \sum_{i=1}^{L} \psi_i^\dagger \psi_i
\]

(33)
is always an allowed symmetry operation, and indeed any Hamiltonian of the form eqn. \[34\] will be invariant under this operation. We will later discuss its relationship to projective representations (see sec. \[F\]). Thus, for the GFS, \(C^{UL} = I_\theta, \forall \theta \in [0, 2\pi] = U(1)\), i.e., the only ordinary symmetries are the trivial ones.

We will now demonstrate an important property of a GFS. A GFS can possess at most one each of UA, TL, and TA symmetries. In other words, non-ordinary symmetries of a GFS are solitary. To prepare to prove this statement, we adopt some useful notation. We will denote UA symmetry operations by \(\mathcal{I}\), TL by \(\mathcal{C}\), and TA by \(\mathcal{S}\) – this choice anticipates later discussion. Suppose, now, that we have two distinct UA symmetries of the GFS, say \(\mathcal{I}_1\) and \(\mathcal{I}_2\), then \(\mathcal{I}_1\mathcal{I}_2\) is also a symmetry of our system. But from figure \([3]\), we know that \(\mathcal{I}_1\mathcal{I}_2 = I_\theta\) or \(\mathcal{I}_1\mathcal{I}_2 = I_{-\theta}\mathcal{I}_1^{-1}\) and hence \(\mathcal{I}_2\) is not a distinct UA symmetry – it is simply a product of a trivial symmetry with \(\mathcal{I}_1^{-1}\). The same argument works for \(\mathcal{C}\) and \(\mathcal{I}\) symmetries and the solitariness of non-ordinary symmetries is proved (see also \([18]\)). In fact, we can conclude many more interesting facts about symmetries of a GFS.

Consider the symmetry operator \(\mathcal{J}^2\). From the previous paragraph we know \(\mathcal{J}^2 = I_\theta\) for some \(\theta\). We can determine \(\theta\) from \(\mathcal{J}^{-1}\mathcal{J}^2 = I = \mathcal{J}^2\mathcal{J}^{-1}\) which implies \(\mathcal{I}_0\mathcal{J}^{-1} = \mathcal{I} = \mathcal{I}_0\mathcal{J}^{-1}\). Applying this relation on \(\mathcal{V}_L\) immediately forces \(\mathcal{V}_L^2 = 1\) (due to the antilinearity of \(\mathcal{I}\)) or \(\mathcal{V}_L^0 = \pm 1\), and thus we immediately see that the action of \(\mathcal{J}^2\) on \(\mathcal{V}\) is

\[
\mathcal{J}^2 = (\pm 1)^N \mathcal{J}.
\]

We will mostly write eqn. \((34)\) simply as

\[
\mathcal{J}^2 = \pm \mathcal{J}.
\]

We turn now to \(\mathcal{C}^2\) which should also be equal to \(I_\theta\). We get, again, \(\mathcal{C}^{-1}\mathcal{J}_0 = \mathcal{J}_0\mathcal{C}^{-1}\). Applying this last relation on \(\mathcal{V}_{N_L}\), we get

\[
ed^{2N_L \theta} = \mathcal{C}^{(L-N_L)\theta}, \ \forall \ N_L = 0, \ldots, L.
\]

resulting in \(\mathcal{C}^2 = (\pm 1)^N \mathcal{J}\).

Note also that eqn. \((35)\) implies that \(L\) must be even when \(\theta = \pi\), i.e., a TL type of symmetry with the \((-\pi)\) signature can only be implemented in a GFS with even number of orbitals.

Finally, we discuss \(\mathcal{S}^2\). Noting that \(\mathcal{S}\) is an TA symmetry, we get

\[
ed^{2N_L \theta} = \mathcal{S}^{(L-N_L)\theta}, \ \forall \ N_L = 0, \ldots, L,
\]

and thus \(\mathcal{S}^2 = 1\), implies \(\theta = \frac{2\pi \ell}{L}\) where \(\ell\) is one of 0, 1, ..., \(L-1\). We will later show that \(\theta\) can always be chosen to be zero, leading to

\[
\mathcal{S}^2 = I.
\]

We conclude this discussion on the properties of GFS symmetries by noting that if a GFS has a \(\mathcal{I}\) and \(\mathcal{C}\) type symmetry, then \(\mathcal{S}\) is equal to \(\mathcal{I}\mathcal{C}\). If a GFS has a \(\mathcal{I}\) symmetry, we say that it possesses time reversal symmetry, \(\mathcal{C}\) implies the presence of charge conjugation symmetry, and \(\mathcal{S}\) endows a sublattice symmetry on the GFS.

### E. The Tenfold Way

In this section, we show how the ten symmetry classes of fermions arise and obtain the canonical representations of the symmetries in these classes.

#### I. Symmetry Classes

Based on the discussion of the previous section, we see that a GFS has to be of one of three types.

**Type 0:** Possesses no non-ordinary symmetries.

**Type 1:** Possesses one non-ordinary symmetry.

**Type 3:** Possesses all three non-ordinary symmetries.

The resulting symmetry classes and the class hierarchy is shown in fig. 4.

Whenever time reversal is present it be realized as \(\mathcal{J}^2 = \pm \mathcal{J}\), and we denote this by \(T = \pm 1\), similarly \(\mathcal{C}^2 = \pm \mathcal{J}\) is denoted by \(C = \pm 1\) and presence of \(\mathcal{S}\) is shown by \(S = 1\). Absence of these symmetries is denoted by \(T = 0\), \(C = 0\), or \(S = 0\) as the case may be. The “symmetry signature” of any class is denoted by a triple \((T, C, S)\) (see fig. 4). It is now immediately clear that there is only one class of type 0 GFS, called A with symmetry signature \((0, 0, 0)\). There are five classes of type 1 with a single non-ordinary symmetry: \(\text{All}(+1, 0, 0)\), \(\text{All}(0, 1, 0)\), \(\text{D}(0, +1, 0)\), \(\text{C}(0, -1, 0)\) and \(\text{All}(0, 0, 1)\). The type 3 systems come in four classes: \(\text{BDI}(+1, +1, 1)\), \(\text{Cl}(+1, -1, 1)\), \(\text{DII}(+1, +1, 1)\), and \(\text{CII}(-1, -1, 1)\).
2. Canonical Representation of Symmetries

An important step towards determining the structure of Hamiltonians (eqn. (13)) in each of these ten symmetry classes is the determination of the canonical representation of the symmetries of each class. This question is addressed in this section.

Each of the non-ordinary symmetry operations, \( T \), and \( \mathcal{S} \) is represented by a \( L \times L \) unitary matrix. Time reversal \( T \) is represented as

\[
\mathcal{T}\Psi^T = \Psi^T U_T \quad (40)
\]

using eqn. (17). Charge conjugation and sublattice, being transposing operations, are represented using eqn. (19) respectively as

\[
\mathcal{C}\Psi^T = \Psi^T U_C^* \quad (41)
\]

and

\[
\mathcal{J}\Psi^T = \Psi^T U_S^* \quad (42)
\]

3. Type 0

Class A: Class A has no non-ordinary symmetries and hence nothing to represent. There are, therefore, no restrictions on the \( L \)-orbital systems – any \( L \) orbital system can be in class A.

4. Type 1

Classes Al and All: Time reversal symmetry is the sole non-ordinary symmetry present in these classes with \( T = \pm 1 \). The condition eqn. (35) gives (using the antilinearity of \( \mathcal{T} \))

\[
\mathcal{T}^2\Psi^T = \Psi^T U_T U_T^* = T \Psi^T \quad (43)
\]

leading to

\[
U_T U_T^* = T \mathbf{1} \quad (44)
\]

where \( \mathbf{1} \) is an \( L \times L \) unit matrix. An immediate consequence of this is that \( |\det U_T|^2 = T^2 \), leading us to the conclusion that \( T = +1 \) can be realized in any \( L \)-orbital GFS, while \( T = -1 \) requires \( L \) to be an even number.

To construct a canonical \( U_T \) that satisfies eqn. (44), we consider a change of basis of the GFS as described by the matrix \( R \) defined in eqn. (5). The unitary \( \tilde{U}_T \) representing \( \mathcal{T} \) in this new basis can be obtained from

\[
R \tilde{U}_T R^T = U_T \quad (45)
\]

It is known that any unitary that satisfies eqn. (44) with \( T = +1 \) can be written as \( U_T = A A^T \) where \( A \) is a unitary matrix. This result from matrix theory, usually called the Takagi decomposition (see appendix D of [46]), allows us to conclude that we can always choose (\( R = A \)) a basis [47] where \( \tilde{U}_T = \mathbf{1} \) for \( T = +1 \). The outcome of this discussion is that \( T = +1 \) admits a canonical representation \( U_T = \mathbf{1} \).

In the case of \( T = -1 \), Takagi decomposition provides that any \( U_T \) satisfying eqn. (44) can decomposed as \( AJA^T = U_T \) where

\[
J = \begin{pmatrix} 0_{MM} & 1_{MM} \\ -1_{MM} & 0_{MM} \end{pmatrix}, \quad L = 2M, \quad JJ^* = -\mathbf{1}, \quad (46)
\]

and \( A \) is unitary. This is consistent with the fact that \( L \) must be necessarily even for \( T = -1 \) as concluded above. The subscripts on the matrices denote their sizes. Taken together with eqn. (45) allows us to conclude that \( T = -1 \) case is canonically represented by \( U_T = J \).

We can also obtain a “one particle” or “first quantized” representation of the time reversal operator. To see this consider a GFS with only a one-body Hamiltonian, form eqn. (24) and the symmetry condition eqn. (28), we obtain

\[
U_T \left[ H^{(1)} \right]^T U_T^* = H^{(1)} \quad (47)
\]

where \( T \) is

\[
T = U_T K \quad (48)
\]

where \( K \) is the complex conjugation “matrix”, recovering a well known result (see e.g., [18]).

Classes D and C: These classes respectively with symmetry signatures \((0, +1, 0)\) and \((0, -1, 0)\) possess the sole non-ordinary symmetry of charge conjugation. As noted just after eqn. (37) the latter symmetry can be realized only in GFSs with even number of orbitals.

As in the previous para, considering the action of \( \mathcal{C}^2 \) on the fermion operators, owing to eqn. (37), gives

\[
\mathcal{C}^2 \Psi^T = \mathcal{C} \Psi^T U_C^* \mathcal{C}^* \quad (49)
\]

resulting in

\[
U_C U_C^* = \mathbf{1}. \quad (50)
\]

Note that this relation, despite \( \mathcal{C} \) being a linear operation, looks very similar to eqn. (44) of an usual antilinear operator.

Under a change of basis, it can be shown that \( U_T \) transforms in exactly the same manner as \( U_T \), i.e., via eqn. (45). Precisely the same considerations using Takagi decomposition of the previous para then allow us to conclude that there is a basis for the GFS where \( U_C \) can be represented canonically as \( U_C = \mathbf{1} \) for \( C = +1 \) and \( U_C = J \) for \( C = -1 \). Finally, restricting again to a GFS described solely by one particle interactions, provides the first quantized version of the \( \mathcal{C} \) operator. Indeed, eqn. (24) and eqn. (28) provide that

\[
-CH^{(1)} C^{-1} = H^{(1)}. \quad (51)
\]
This is consistent with the first quantized form
\[ C = U_C K. \] (52)

It is easily seen that this feature is generic – any transposing linear symmetry operation has an antilinear first quantized representation.

Class All: The sole non-ordinary symmetry in this class with the symmetry signature \((0, 0, 1)\) is the sublattice symmetry.

Investigating the action of \(\mathcal{S}^2\), noting that \(\mathcal{S}\) is an antilinear operator, we obtain from eqn. (38)
\[ \mathcal{S}^2 \Psi^\dagger \mathcal{S}^{-2} = \mathcal{S} \Psi^T U_S^T \mathcal{S}^{-1} = \Psi^T U_S U_S = e^{i \frac{2\pi}{3}} \Psi^\dagger \] (53)
implying
\[ U_S U_S = e^{i \frac{2\pi}{3}} 1. \] (54)

It is evident that it can be redefined \((e^{i \frac{2\pi}{3}} U_S \mapsto U_S)\) as
\[ U_S U_S = 1. \] (55)

Since \(U_S\) is unitary, we obtain that \(U_S^T = U_S\), and thus, \(U_S\) is Hermitian. This condition implies that all eigenvalues of \(U_S\) are real and of unit magnitude. Quite interestingly, under a change of basis, \(U_S\) transforms as
\[ R U_S R^\dagger = U_S \] (56)
This implies that there is a basis in which \(U_S\) has the following canonical form
\[ U_S = 1_{p,q} \] (57)
where
\[ 1_{p,q} = \begin{pmatrix} 1_{pp} & 0_{pq} \\ 0_{qp} & -1_{qq} \end{pmatrix} \] (58)
where \(p + q = L\). This development makes the meaning of the sublattice symmetry clear. The orbitals of the system are divided into two groups – "sublattices \(A\) and \(B\)". The sublattice symmetry operation, a transposing antilinear operation, maps the "particle states" in the \(A\) orbitals to "hole states" on \(A\) orbitals, while \(B\)-particle states are mapped to negative \(B\)-hole states. In the first quantized language, the transposing antilinear operator \(\mathcal{S}\), is, therefore, represented by a linear matrix, i.e.,
\[ S = U_S \] (59)
and in a noninteracting system the symmetry is realized when
\[-SH^{(1)} S^{-1} = H^{(1)}.\]

5. Type 3

The remainder of the four classes are of type 3, i.e., they possess all three non-ordinary symmetries. As noted at the end of section [4], \(\mathcal{S} = \mathcal{S} C\) in these classes implying that
\[ U_S = U_T U_C. \] (60)

Our strategy in analyzing these classes is to choose a basis where \(U_S = 1_{p,q}\), and to determine the structure of \(U_T\) and \(U_C\). In this basis, we can write
\[ U_T = \begin{pmatrix} u_{pp} & u_{pq} \\ T_u u_{pq} & u_{qq} \end{pmatrix}, \quad u_T^{\dagger} = T u_{pp}, \quad u_T^{\dagger} = T u_{qq}, \] (61)
for \(T = \pm 1,\)
where \(u_{pq}\) are complex matrices of the dimension indicated by the suffix. This from automatically satisfies eqn. (44). Once, we fix \(U_T, \quad U_C\) is fixed for every class in type 3 via eqn. (60), as
\[ U_C = \begin{pmatrix} T u_{pp} & -T u_{pq} \\ T u_{pq} & -T u_{qq} \end{pmatrix}; \quad T = \pm 1. \] (62)

The conditions eqn. (44) and eqn. (60) constrain \(u_{pp}, \quad u_{pq}, \quad u_{qq}\) of eqn. (61) very strongly. Two possible cases arise.

\(T = C\): This case results in the condition \(U_T U_C = U_C U_T\) which provides the following constraints
\[ u_{pp} u_{pp}^* = T I_{pp}, \quad u_{pq} u_{pq}^* = T I_{pq}, \]
\[ u_{pq} u_{pq}^* = 0_{pp}, \quad u_{pp} u_{pq}^* = 0_{pq}, \quad u_{pq} u_{pq}^* = 0_{qp}, \quad u_{pp} u_{pq}^* = 0_{qp}, \quad u_{pq} u_{pq}^* = 0_{qp}, \] (63)
for \(T = C\).

\(T = -C\): Note that since either \(T\) or \(C\) is \(-1, \quad L\) is already even \(L = 2M\). The condition \(U_T U_C = -U_C U_T\) obtains the constraints
\[ u_{pp} u_{pp}^* = 0_{pp}, \quad u_{pq} u_{pq}^* = 0_{pq}, \quad u_{pp} u_{pq}^* = 0_{pq}, \quad u_{pq} u_{pq}^* = 0_{qp}, \] (64)
The second line of eqn. (64), forces \(p = q = M\) and \(u_{pq}\) to be a \(M \times M\) unitary matrix which we will call \(u_{MM}\). The other conditions provide \(u_{pp} = 0_{pp}\) and \(u_{pq} = 0_{pq}\).

Class BDI: The class has a symmetry signature \((1, 1, 1)\). Since \(T = C = +1\), we see from eqn. (63) that \(u_{pq} = 0_{pq}\), along with \(u_{pp} u_{pp}^* = +1_{pp}\) and \(u_{pq} u_{pq}^* = +1_{pq}\), providing
\[ U_T = \begin{pmatrix} u_{pp} & 0_{pq} \\ 0_{qp} & u_{qq} \end{pmatrix}, \quad U_C = \begin{pmatrix} u_{pp} & 0_{pq} \\ 0_{qp} & -u_{qq} \end{pmatrix} \] (65)
Apart from \(p + q = L\), there are no additional constraints on \(p\) and \(q\). The natural splitting of the orbitals into a \(p\)-subspace and \(q\)-subspace now allows us to find canonical forms of \(U_T\) and \(U_C\). Again, Takagi decomposition allows us to find a unitary matrix \(r_{pp}\) such that \(r_{pp} r_{pp}^* = u_{pp}\), another similar matrix \(r_{pq}\) that does \(r_{pq} r_{pq}^* = u_{pq}\). We can define a basis change matrix
\[ R = \begin{pmatrix} r_{pp} & 0_{pq} \\ 0_{qp} & r_{pq} \end{pmatrix} \] (66)
from which we immediately see that
\[ U_T = R1R^T, \quad U_C = R(1_{p,q})R^T. \] (67)

Moreover, we see that \( U_S \) transforms under this basis change via eqn. (56) as
\[ R^\dagger R_{p,q} = 1_{p,q} \] (68)
i.e., such a basis change does not alter the form of \( U_S \). The conclusion of this discussion is that when \( T = C = +1 \), we can always choose a basis in which
\[ U_T = 1, \quad U_C = 1_{p,q}, \quad U_S = 1_{p,q}. \] (69)

**Class CIII:** This symmetry class has signature \((-1,-1,1)\). From eqn. (63), we have \( u_{pq} = 0_{pq}, u_{pp}u_{pp}^\dagger = -1_{pp} \) and \( u_{pq}u_{pq}^\dagger = -1_{qq} \), resulting in
\[ U_T = \begin{pmatrix} u_{pp} & 0_{pq} \\ 0_{qp} & u_{qq} \end{pmatrix}, \quad U_C = \begin{pmatrix} -u_{pp} & 0_{pq} \\ 0_{qp} & u_{qq} \end{pmatrix}. \] (70)

Even as \( p + q = L = 2M \), there are additional constraints on \( p \) and \( q \). The conditions \( u_{pp}u_{pp}^\dagger = -1_{pp} \) and \( u_{pq}u_{pq}^\dagger = -1_{qq} \), force \( p = 2r \) and \( q = 2s \), i.e., both \( p \) and \( q \) have to be even numbers, \( 2(r + s) = 2M \).

We can find canonical forms for \( U_T \) and \( U_C \) much like eqn. (69). For this, noting that \( p \) and \( q \) are even, we know that there are unitary matrices \( r_{pp} \) and \( r_{qq} \) such that \( r_{pp}J_{pp}r_{pp}^\dagger = u_{pp} \) and \( r_{qq}J_{qq}r_{qq}^\dagger = u_{qq} \). Thus one can perform a basis change using a transformation similar to eqn. (66) to obtain natural forms
\[ U_T = \begin{pmatrix} J_{pp} & 0_{pq} \\ 0_{qp} & J_{qq} \end{pmatrix}, \quad U_C = \begin{pmatrix} -J_{pp} & 0_{pq} \\ 0_{qp} & -J_{qq} \end{pmatrix}, \quad U_S = 1_{p,q}, \] (71)

which provide the canonical representation of the symmetries in class CIII.

**Class CI:** This class with the symmetry signature \((+1,-1,1)\) can be analyzed using eqn. (64) which provides
\[ U_T = \begin{pmatrix} 0_{MM} & u_{MM} \\ u_{MM}^T & 0_{MM} \end{pmatrix}, \quad U_C = \begin{pmatrix} 0_{MM} & -u_{MM} \\ u_{MM}^T & 0_{MM} \end{pmatrix}. \] (72)

We can obtain canonical forms of \( U_T \) and \( U_C \) by the following manipulations. Consider a basis transformation of the type eqn. (66) of the from
\[ R = \begin{pmatrix} u_{MM} & 0_{MM} \\ 0_{MM} & 1_{MM} \end{pmatrix} \] (73)

This basis change effects the following changes (see eqn. (45), eqn. (56))
\[ U_T \leftrightarrow R^\dagger U_T R^\dagger = \begin{pmatrix} 0_{MM} & 1_{MM} \\ 1_{MM} & 0_{MM} \end{pmatrix} = F \]
\[ U_C \leftrightarrow R^\dagger U_C R^\dagger = -J \]
\[ U_S \leftrightarrow R^\dagger 1_{MM} R = 1_{MM} \] (74)

where the first line defines the \( L \times L \) matrix \( F \). Thus the canonical forms of class CII are
\[ U_T = F, \quad U_C = -J, \quad U_S = 1_{MM} \] (75)

**Class DIII:** The symmetry signature \((-1,+1,1)\) provides \( T = -1 = -C \). Form eqn. (64), we get
\[ U_T = \begin{pmatrix} 0_{MM} & u_{MM} \\ -u_{MM}^T & 0_{MM} \end{pmatrix}, \quad U_C = \begin{pmatrix} 0_{MM} & u_{MM} \\ u_{MM}^T & 0_{MM} \end{pmatrix}. \] (76)

Using the basis change eqn. (73), we can find canonical structure of the \( U \) matrices as
\[ U_T = J, \quad U_C = F, \quad U_S = 1_{MM}. \] (77)

In the classes CII and DIII, there is no further constraint on \( M \) which can be odd or even.

The determination of the canonical structure of the symmetry operation accomplishes a great deal in the determination of the structure of the Hamiltonians in each class. The summary of the findings of this section are given in table I.

\[ \begin{array}{c|c|c|c|c|c}
\mathcal{K}_4 & I & \Theta & \Xi & \Sigma \\
\hline
I & I & \Theta & \Xi & \Sigma \\
\hline
\Theta & \Theta & I & \Xi & \Sigma \\
\hline
\Xi & \Xi & \Sigma & I & \Theta \\
\hline
\Sigma & \Sigma & \Xi & \Theta & I \\
\end{array} \] (78)

**F. A Group Cohomology Perspective**

The results of the previous section can be obtained from a different perspective as we describe in this section. Readers interested in the structure of many body Hamiltonians in each class may proceed directly to sec. G. The main idea of the section is to demonstrate that the ten classes arise from different perspective as we describe in this section. Readers

The abstract symmetry group \( G \) of the GFS must be one of the (proper/improper) subgroups of \( \mathcal{K}_4 \). Thus, \( G \) is one of \( I = \{ I \} \), \( Z_2^1 = \{ I, \Theta \}, Z_2^2 = \{ I, \Xi \}, Z_2^3 = \{ I, \Sigma \} \) or \( \mathcal{K}_4 \).

We look for projective representations of the group \( G \) on a graded vector space \( \mathcal{W} = \mathcal{V}_1 \oplus \mathcal{V}_1 \), where \( \mathcal{V}_1 \) is an \( L \)-dimensional \( \mathbb{C} \) vector space. We are restricting here to the one particle sector \( \mathcal{V}_1 \) and its “transposed” hole space \( \mathcal{V}_1^{\dagger} \) (which is isomorphic to \( \mathcal{V}_L \), see eqn. (47)). The group elements are
represented by \( \mathbb{C} \) valued matrices of the form

\[
D(I) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad D(\Theta) = \begin{pmatrix} U_T & 0 \\ 0 & U_T \end{pmatrix}, \quad D(\Xi) = \begin{pmatrix} 0 & U_C \\ U_C & 0 \end{pmatrix}, \quad D(\Sigma) = \begin{pmatrix} 0 & U_S \\ U_S & 0 \end{pmatrix}.
\]

(79)

\( \mathbf{K} \) is the usual complex conjugation operator, \( U_T, U_C, \) and \( U_S \) are unitary matrices that represent the symmetries appearing in their subscripts. This choice of representations is motivated by discussions of the previous section.

Denoting the group elements of \( \mathcal{G} \) by \( g \), a projective representation reproduces the group multiplication table up to a \( U(1) \) phase factor, i.e.,

\[
D(g_1)D(g_2) = \omega(g_1, g_2)D(g_1g_2).
\]

(80)

With \( |\mathcal{G}| \) as the number of elements in \( \mathcal{G} \), the \( |\mathcal{G}|^2 \) numbers \( \omega(g_1, g_2) \in U(1) \), usually called “2-cocycles” or “Schur multipliers” must satisfy the following co-cycle condition

\[
\omega(g_1, g_2)\omega(g_1g_2g_3) = \omega(g_1, g_2g_3)\omega(g_2, g_3)\omega(g_1, g_2)\omega(g_2, g_3)\omega(g_1, g_2g_3)\omega(g_2, g_3).
\]

(81)

Here \( \omega^k \) is a short form for action of the group element \( g \) on an element of \( U(1) \), i.e., for every group element \( g \), there is an invertible function \( \varphi_g \) on \( U(1) \) which maps \( \omega \in U(1) \) to \( \varphi_g(\omega) = \omega^k \). The function must satisfy

\[
\varphi_{g_1}(\varphi_{g_2}(\omega)) = \varphi_{g_1g_2}(\omega).
\]

(82)

More technically, \( \varphi \) is a homomorphism from \( \mathcal{G} \) to \( \text{Aut}(U(1)) \), the group of automorphisms (invertible maps that preserve the group structure) of \( U(1) \). In our case, the homomorphism \( \varphi \) is defined as

\[
\varphi_I(\omega) = \omega, \quad \varphi_{\Theta}(\omega) = \omega^*, \quad \varphi_{\Xi}(\omega) = \omega^* \quad \varphi_{\Sigma}(\omega) = \omega.
\]

(83)

where (‘) denotes complex conjugation. The check of the homomorphic character of \( \varphi \) is routine. This particular choice of \( \varphi \) is made keeping in mind the physical aspects of the group elements, consistent with eqn. (79). Now two multiplier system \( \omega_1 \) and \( \omega_2 \) (both of which satisfy eqn. (81)) can be used to produce a third one \( \omega' \), in fact \( \omega'(g_1, g_2) = \omega_1(g_1, g_2)\omega_2(g_1, g_2) \). Note that \( \omega(g_1, g_2) = 1 \), the unit multiplier, satisfies the cocycle condition eqn. (81). Furthermore, any given \( \omega \) has an inverse \( \omega^{-1}(g_1, g_2) = \omega(g_1, g_2) \) such that \( \omega\omega^{-1} \) is the unit multiplier. Finally, two multipliers \( \omega_1 \) and \( \omega_2 \) are equivalent if there exists a \( U(1) \) valued function on \( \mathcal{G} \), \( a(g) \), such that

\[
\frac{\omega_1(g_1, g_2)}{\omega_2(g_1, g_2)} = \frac{a(g_1)a(g_2)}{a(g_1g_2)}.
\]

(84)

Suppose we denote the set of all multipliers equivalent to \( \omega \) by \(|\omega|\). It is easy to see that the set of equivalence classes of multipliers \(|\omega|\) forms an abelian group called the second co-homology group (see [49] and chapter 7 of [50]) \( H_2^C(U(1), \mathcal{G}) \). A key question of importance to us is how many elements (number of equivalence class of multipliers) does \( \mathcal{G} \) admit – this is answered by computing the group \( H_2^C(U(1), \mathcal{G}) \).

We can now make the central point: The number of symmetry classes corresponds to number of distinct equivalence classes of multiplier systems corresponding to the symmetry group \( \mathcal{G} \) which runs over of \( I, Z_2^I, Z_2^C, Z_2^2 \) and \( \mathcal{K}_3 \). To obtain the structure of each class, we find the multiplier system that labels the class, and find the irreducible representations, “smallest” vector space \( W \), and the matrices of eqn. (79). Again, we organize the classes via the types introduced in sec. [E1]

1. Type 0

Here the symmetry group \( \mathcal{G} = I \), and \( H_2^C(U(1), \mathcal{G}) \) is trivial containing a single element, and hence a trivial multiplier

---

**TABLE I.** Canonical representations of symmetry operators \( U_T, U_C \) and \( U_S \) are shown in the ten symmetry classes. \( \mathbf{I} \) is the \( L \times L \) identity matrix, \( \mathbf{J} \) defined in eqn. (66), \( \mathbf{J}_{pq} \) defined in eqn. (58), \( \mathbf{F} \) defined in eqn. (74).

| Class | T | C | S | L | \( U_T \) | \( U_C \) | \( U_S \) |
|-------|---|---|---|---|----------|----------|----------|
| A     | 0 | 0 | 0 | \( L \) | – | – | – |
| B     | +1| 0 | 0 | \( L \) | 1 | – | – |
| C     | 0 | –1| 0 | \( L = 2M \) | J | – | – |
| D     | 0 | +1| 0 | \( L \) | – | 1 | – |
| E     | 0 | 0 | 1 | \( L = p + q \) | – | – | \( \mathbf{1}_{pq} \) |
| F     | +1| +1| 1 | \( L = p + q \) | 1 | \( \mathbf{1}_{pq} \) | \( \mathbf{1}_{pq} \) |
| G     | –1| –1| 1 | \( L = p + q \) | \( \mathbf{J}_{pp} \) \( \mathbf{0}_{pq} \) \( \mathbf{0}_{qp} \) \( \mathbf{J}_{qq} \) | \( \mathbf{0}_{pp} \) \( \mathbf{J}_{pq} \) \( \mathbf{0}_{qp} \) \( \mathbf{J}_{qq} \) | \( \mathbf{1}_{pq} \) |
| H     | +1| –1| 1 | \( L = 2M \) | F | – | \( \mathbf{J} \) |
| I     | –1| +1| 1 | \( L = 2M \) | J | F | \( \mathbf{1}_{MM} \) |
system.

Class A: The sole irreducible representation corresponding to this trivial multiplier system is 1-dimensional. This labels class A, and we see that the L-orbital GFS in class A of table I is a reducible representation consisting of L copies of irreducible one obtained just now.

2. Type I

\[ G = \mathbb{Z}^T_2 \]

The second cohomology group \( H^2_0(U(1), \mathbb{Z}^T_2) \) consists two multiplier classes. The multipliers are

\[
\begin{pmatrix}
\omega(g_1 \downarrow g_2 \rightarrow) & I & \Theta \\
I & 1 & 1, \quad T = \pm 1 \\
\Theta & 1 & T
\end{pmatrix}
\]

Class AI: The multiplier of \( H^2_0(U(1), \mathbb{Z}^T_2) \) with \( T = +1 \) labels the class AI. To find the irreducible representation, we need to solve

\[ U_T U_T^* = 1 \]

and the solution with smallest dimension (in the sense of dimension of vector space \( \mathcal{V} \), that makes up \( \mathcal{W} \)) is 1, and \( U_T = \delta^a \). By the change of basis eqn. (45) \( \alpha \) can always chosen to the zero, and \( U_T = 1 \), and \( D(\Theta) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \). We see that the L-orbital GFS of table I is a reducible representation with L-copies of this irrep (irreducible representation).

Class All: The multiplier in eqn. (85) with \( T = -1 \) corresponds to class All. This leads to the equation \( U_T U_T^* = -1 \). The solution with the smallest dimension is 2, with

\[ U_T = J_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \]

We see that the All GFS of table I with \( L = 2M \) consists M copies of this this irrep corresponding to this multiplier system.

\[ G = \mathbb{Z}^C_2 \]

The second cohomology group \( H^2_0(U(1), \mathbb{Z}^C_2) \) consists \( \mathbb{Z}_2 \) – the mathematics of this group is same as \( \mathbb{Z}^T_2 \), and the multipliers are

\[
\begin{pmatrix}
\omega(g_1 \downarrow g_2 \rightarrow) & I & \Xi \\
I & 1 & 1 \\
\Xi & 1 & C
\end{pmatrix}
\]

The key physical difference is that the representation of \( \Xi \) is via a transposing antilinear matrix (see eqn. (79)). Very similar consideration as in the \( \mathbb{Z}^T_2 \) case, leads to two classes.

Class D: The multiplier with \( C = +1 \) gives this class. The irrep is 1-dimensional with \( U_C = 1 \) and \( D(\Xi) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \). L copies of which are present in the GFS shown in table I.

Class C: This class is labeled by the \( C = -1 \) multiplier of eqn. (88), with a 2-dimensional irrep such that \( D(\Xi) = \begin{pmatrix} 0 & J_2 \\ J_2 & 0 \end{pmatrix} \). The \( \Xi \) entry in table I consists of \( M \) copies of this irrep.

\[ G = \mathbb{Z}^S_2 \]

In the case of the sole non-ordinary sublattice symmetry, \( H^2_0(U(1), \mathbb{Z}^S_2) \) is trivial and has only the unit multiplier.

Class All: Being an abelian group of order 2, \( \mathbb{Z}^S_2 \) has two distinct 1-dimensional irreducible representations, which arise from the lowest dimensional solution to the equation \( U_S = 1 \). The first irrep has \( U_S = +1 \) and the second has \( U_S = -1 \) (note that these two irreps have the same multipliers). We immediately see that the L orbital All system in table I is made of the \( p \) copies of the +irrep and \( q \) copies of the –irrep.

3. Type 3

All the classes in type 3 arise from the multipliers of the symmetry group \( G = \mathcal{K}_4 \). Quite interestingly the second cohomology group of the Klein group is the Klein group itself \( \mathcal{K}_4 \). \( H^2_0(U(1), \mathcal{K}_4) = \mathcal{K}_4 \). This results in four multipliers labeled by \( T \) and \( C \)

\[ \omega(g_1 \downarrow g_2 \rightarrow) = \begin{pmatrix} I & \Theta & \Xi & \Sigma \\ I & 1 & 1 & 1 \\ \Theta & 1 & T & 1 & T \\ \Xi & 1 & TC & C & T \\ \Sigma & 1 & C & C & 1 \end{pmatrix} \]

To obtain the irreps, we seek smallest dimensions quantities \( U_T, U_C, U_S \) that simultaneously satisfy

\[ U_T U_T^* = T I, \quad U_C U_C^* = C I, \quad U_S U_S = 1, \quad U_T U_T^* = U_S \]

(90) (these are obtained by a straightforward application of eqn. (79)).

Class BDII: The multiplier eqn. (89) with \( T = C = +1 \) obtains this class. Eqn. (90) gives \( U_T U_T^* = 1, U_C U_C^* = 1, U_S U_S = 1, U_T U_T^* = U_S \). There are two 1-dimensional irreps, one with \( U_T = 1, U_C = 1 \) and \( U_S = 1 \), and the other \( U_T = 1, U_C = -1 \) and \( U_S = -1 \). We see that class BDII in table I contains \( p \) copies of the first irrep and \( q \) copies of the second one.

Class CII: This class is characterized by the multiplier in eqn. (89) with \( T = C = -1 \). Irreps are obtained from the solution of \( U_T U_T^* = -1, U_C U_C^* = -1, U_S U_S = 1, U_T U_T^* = U_S \).
These now provide two distinct 2-dimensional representations. First one has \( U_T = J_3, U_C = -J_3, U_S = -1 \), and the second has \( U_T = J_3, U_C = J_2, U_S = 1 \). The CI entry in table I has \( r \)-copies of the first 2-dimensional irrep, and \( s \)-copies of the second one.

**Class CI:** When \( T = -C = +1 \) in the multiplier eqn. (89), we get the CI class. The solution of eqn. (90) gives a single 2-dimensional representation with \( U_T = F_2, U_C = -J_2 \) and \( U_S = 1, F_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) and \( I_{1,1} \) is defined in eqn. (58). The GFS of the CI class in table I has \( M \) copies of this irrep.

**Class DII:** Class DII is obtained from the multiplier eqn. (89) when \( T = -C = -1 \). Eqn. (90), again, provides a single 2-dimensional representation with \( U_T = J_3, U_C = F_2 \) and \( U_S = 1, I_{1,1} \). DII class in table I is made of \( M \) copies of this irrep.

The discussion of this section provides a deeper insight of how the ten symmetry classes arise from projective representations on a graded vector space (this is where the fermionic nature of the states enters). Moreover, all the results of the canonical representations of the symmetry operations derived from elementary considerations can be better understood as copies of irreducible projective representations.

**G. Noninteracting Systems**

We devote this section to obtaining the structure of the Hamiltonians of each class when there are no interactions present. Here \( H \) of eqn. (13) has only two entries \( H = (H^{(0)}, H^{(1)}) \). Another important quantity of interest is Schrödinger time evolution operator

\[
U_{\text{Schrod}}(t) = e^{-iHt}
\]

where \( t \) is the time, \( H \) is the Hamiltonian matrix constructed out of \( H = (H^{(0)}, H^{(1)}) \). For example, the in this noninteracting setting the components of \( H_{ij} = H^{(0)} \delta_{ij} + H^{(1)}_{ij} \) for a fixed time, say \( t = 1 \), the time evolution operator spans out a “geometric structure” as \( H \) runs over all of the space \( \mathcal{H} \). As pointed out by [22] (generalizing the pioneering work of Dyson[22]), this geometric structure realized is a symmetric space (see [52] for a review symmetric spaces from a physicist’s perspective, [53] for a review of Lie groups and algebras). The ten classes realize the the ten different symmetric spaces classified by Cartan, and in fact the names of the classes borrow Cartan’s nomenclature. Our development of the ten fold symmetry classification not only allows us to recover these known results in a simple and direct manner but also provides a very clear structure of the Hamiltonian space that could be particularly useful for model building. Our results are recorded in table II.

To proceed with the discussion, we record the symmetry conditions eqn. (22) specifically for the noninteracting systems. Under the action of usual symmetries \( H \) transforms as

\[
H = (H^{(0)}, H^{(1)}) \implies \hat{H} = (H^{(0)}, H^{(1)})
\]

and for transposing symmetries we obtain

\[
H = (H^{(0)}, H^{(1)}) \implies \hat{H} = \left( H^{(0)} + \text{tr}_1 \hat{H}^{(1)} - [\hat{H}^{(1)}]^T \right)
\]

(see eqns. (23), (24), and (25)). The structure of \( H \) is obtained by imposing the symmetry condition eqn. (28) for all the appropriate symmetries. An immediate consequence is that \( H^{(0)} \), the “vacuum energy”, is allowed to have any real value \( H^{(0)} \) in all the ten classes. This is a feature that is always true, i.e., even when interactions are present. As the values of \( H^{(0)} \) spans the reals, the time evolution operator (at a fixed time) acquires a phase factor represented by the \( U(1) \) group (see discussion on class A in the next para).

**Class A:** There are no constraints on the Hamiltonian in this class. Any \( L \times L \) Hermitian matrix is in \( \mathcal{H}_A^{(1)} \). Consequently \( \mathcal{H}_A^{(1)} = \mathfrak{u}(L) \), and \( U_{\text{Schrod}}(t) \) acquires the structure of \( U(1) \times U(L) \). The \( U(1) \) factor arises from \( H^{(0)} \) which spans all of real numbers. As discussed in the previous para, this \( U(1) \) factor will appear in all cases (interacting and noninteracting). Henceforth, we suppress this factor (which is equivalent to setting \( H^{(0)} = 0 \) or any other fixed real number). The symmetric space of \( U_{\text{Schrod}}(t) \) in table II for this case, therefore is shown just as \( U(L) \).

**Class D:** Eqn. (28) appropriate for this class (with \( U_T = 1 \) provides that real symmetric matrices \( H^{(1)} = [H^{(1)}]^* \) are the allowed entries of \( \mathcal{H}_D^{(1)} \). This implies that the \( i \mathcal{H}_D^{(1)} = \mathfrak{u}(L) \oplus \mathfrak{o}(L) \), where \( \mathfrak{o}(L) \) is Lie algebra of the group of orthogonal matrices. The time evolution operator \( U_{\text{Schrod}}(t) \) spans the coset space \( U(L)/O(L) \).

**Class All:** Since \( U_T = J \) in this class with \( L = 2M \), we can split the orbitals into two types and write the Hamiltonian as

\[
H^{(1)} = \begin{pmatrix} h_{aa} & h_{ab} \\ h_{ab}^* & h_{bb} \end{pmatrix}
\]

(44)

The symmetry condition gives \( J[H^{(1)}]^*J^* = H^{(1)} \) resulting in \( h_{ab} = -h_{ab}^*, h_{bb} = h_{aa} \). Here, \( h_{aa}, h_{ab}, h_{bb} \) are \( M \times M \) matrices. The space \( \mathcal{H}_{All} \) is made of matrices of the type

\[
H^{(1)} = \begin{pmatrix} h_{aa} & h_{ab} \\ -h_{ab}^* & h_{aa} \end{pmatrix}
\]

(45)

It can be seen by explicit calculation that \( i\mathcal{H}_A^{(1)} = \mathfrak{u}(2M) \setminus \mathfrak{usp}(2M), \) leading to the time evolution \( U_{\text{Schrod}}(t) \) spanning the coset space \( U(2M)/USp(2M) \). Here \( USp(2M) \) is the symplectic group and \( \mathfrak{usp}(2M) \) is its associated Lie algebra.

**Class D:** This class with the charge conjugation described by \( U_C = 1 \) is made of Hamiltonians with vanishing trace \( \text{tr} H^{(1)} = 0 \) that satisfy, \( H^{(1)} = -[H^{(1)}]^* \). The crucial distinction between the time reversal class \( A(T = +1) \) is that the transposing nature of the charge conjugation give the negative sign in the just stated symmetry condition. The space \( i\mathcal{H}_D^{(1)} \) is made of real antisymmetric matrices which is the Lie algebra \( \mathfrak{o}(L) \). The time evolution operator spans \( O(L) \).
Table II: Structure of noninteracting Hamiltonians in the ten symmetry classes.

| Class | L | \( \mathbf{H}^{(1)} \) | \( \dim i\mathcal{H}^{(1)} \) | \( i\mathcal{H}^{(1)} \) | \( \mathbb{U}_{\text{Schrod}(t)} \) |
|-------|---|-----------------|-----------------|-----------------|-----------------|
| A(0, 0, 0) | \( L \) | \( \mathbf{H}^{(1)} = [\mathbf{H}^{(1)}]^\dagger \) | \( L^2 \) | \( \mathfrak{u}(L) \) | \( U(L) \) |
| A(±1, 0, 0) | \( L \) | \( \mathbf{H}^{(1)} = [\mathbf{H}^{(1)}]^\dagger \) | \( L(L + 1)/2 \) | \( \mathfrak{u}(L) \setminus \mathfrak{o}(L) \) | \( U(L)/O(L) \) |
| All(−1, 0, 0) | \( L = 2M \) | \( \begin{pmatrix} h_{aa} & h_{ab} \\ -h_{ab}^* & -h_{aa}^* \end{pmatrix} \) | \( M(2M - 1) \) | \( \mathfrak{u}(2M) \setminus \mathfrak{usp}(2M) \) | \( U(2M) / \mathfrak{usp}(2M) \) |
| D(0, ±1, 0) | \( L \) | \( \mathbf{H}^{(1)} = -[\mathbf{H}^{(1)}]^\dagger \) | \( L(L - 1)/2 \) | \( \mathfrak{o}(L) \) | \( \mathfrak{O}(L) \) |
| C(0, −1, 0) | \( L = 2M \) | \( \begin{pmatrix} h_{aa} & h_{ab} \\ -h_{ab}^* & -h_{aa}^* \end{pmatrix} \) | \( M(2M + 1) \) | \( \mathfrak{usp}(2M) \) | \( \mathfrak{USp}(2M) \) |
| All(0, 0, 1) | \( L = p + q \) | \( \begin{pmatrix} 0_{pp} & h_{pq} \\ h_{pq}^* & 0_{qq} \end{pmatrix} \) | \( 2pq \) | \( \mathfrak{u}(p + q) \setminus (\mathfrak{u}(p) \oplus \mathfrak{u}(q)) \) | \( U(p + q) / (U(p) \times U(q)) \) |
| BDI(±1, ±1, 0) | \( L = p + q \) | \( \begin{pmatrix} 0_{pp} & h_{pq} \\ h_{pq}^* & 0_{qq} \end{pmatrix} \) | \( \mathfrak{h}(p + q) \setminus (\mathfrak{h}(p) \oplus \mathfrak{h}(q)) \) | \( O(p + q) / (O(p) \times O(q)) \) |
| Cl(−1, −1, 1) | \( L = 2M \) | \( \begin{pmatrix} 0_{MM} & h_{MM} \\ h_{MM}^* & 0_{MM} \end{pmatrix} \) | \( 4rs \) | \( \mathfrak{usp}(p + q) \setminus (\mathfrak{usp}(p) \oplus \mathfrak{usp}(q)) \) | \( \mathfrak{USp}(2(r + s)) / (\mathfrak{USp}(2r) \times \mathfrak{USp}(2s)) \) |
| Cl(±1, −1, 0) | \( L = 2M \) | \( \begin{pmatrix} 0_{MM} & h_{MM} \\ h_{MM}^* & 0_{MM} \end{pmatrix} \) | \( \mathfrak{M}(M + 1) \) | \( \mathfrak{usp}(2M) \setminus \mathfrak{u}(M) \) | \( \mathfrak{USp}(2M) / U(M) \) |
| DIII(−1, 1, 1) | \( L = 2M \) | \( \begin{pmatrix} 0_{MM} & h_{MM} \\ h_{MM}^* & 0_{MM} \end{pmatrix} \) | \( \mathfrak{O}(2M) \setminus \mathfrak{u}(M) \) | \( \mathfrak{O}(2M) / U(M) \) |

Class C: The charge conjugation operation is described by \( \mathbf{U}_C = \mathbf{J} \), and the symmetry condition \( \mathbf{J}[\mathbf{H}^{(1)}][\mathbf{J}]^\dagger = -\mathbf{H}^{(1)} \), leads to matrices with \( \text{tr} \mathbf{H}^{(1)} = 0 \).

\[
\mathbf{H}^{(1)} = \begin{pmatrix} h_{aa} & h_{ab} \\ h_{ab}^* & -h_{aa}^* \end{pmatrix}
\]

i.e., \( i\mathcal{H}^{(1)}_C = \mathfrak{usp}(2M) \), and \( \mathbb{U}_{\text{Schrod}(t)} \) spans \( \mathfrak{USp}(2M) \).

Class All: The physics of this class is governed by the sublattice symmetry represented by the unitary \( \mathbf{U}_S = \mathbf{1}_{p,q} \) that naturally partitions the orbitals into two groups (sublattices). It is natural to write the Hamiltonian as

\[
\mathbf{H}^{(1)} = \begin{pmatrix} h_{pp} & h_{pq} \\ h_{pq}^* & h_{qq} \end{pmatrix}
\]

The symmetry condition leads immediately to \( h_{pp} = 0_{pp} \), \( h_{qq} = 0_{qq} \). Thus, \( \mathcal{H}_{\text{All}} \) is made of matrices of the form

\[
\mathbf{H}^{(1)} = \begin{pmatrix} 0_{pp} & h_{pq} \\ h_{pq}^* & 0_{qq} \end{pmatrix}
\]

Indeed, we have \( i\mathcal{H}^{(1)}_{\text{All}} = \mathfrak{u}(p + q) \setminus (\mathfrak{u}(p) \oplus \mathfrak{u}(q)) \), resulting in \( \mathbb{U}_{\text{Schrod}(t)} \) being in the coset space \( U(p + q) / (U(p) \times U(q)) \).

All of the type 3 classes can be viewed as descendants of class All, and thus all of these classes – often referred to as chiral classes – have Hamiltonians that imbibe the structure in eqn. (98).

Class BDI: The symmetry condition eqn. (92) with \( \mathbf{U}_T = \mathbf{1} \) gives \( h_{pq} = h_{pq}^* \), and thus elements of \( \mathcal{H}_{\text{BDI}} \) are of the form

\[
\mathbf{H}^{(1)} = \begin{pmatrix} 0_{pp} & h_{pq} \\ h_{pq}^* & 0_{qq} \end{pmatrix}, \quad h_{pq} = h_{pq}^*
\]

We obtain \( i\mathcal{H}^{(1)}_{\text{BDI}} = \mathfrak{o}(p + q) \setminus (\mathfrak{o}(p) \oplus \mathfrak{o}(q)) \), with \( \mathbb{U}_{\text{Schrod}(t)} \) spanning the coset space \( O(p + q) / (O(p) \times O(q)) \).

Class ClI: The symmetry condition eqn. (92) with \( \mathbf{U}_T \) as shown in table I and \( \mathbf{H}^{(1)} \) of the form eqn. (98) provides

\[
\mathbf{J}_{pq}^* h_{pq}^* = h_{pq} J_{pq}.
\]

The key point in this class is that the \( p \) orbitals themselves arise as \( 2r \) orbitals and \( q \) as \( 2s \) orbitals. Thus, \( h_{pq} \) can be written as

\[
h_{pq} = \begin{pmatrix} h_{rr} & h_{rs} \\ h_{rs}^* & h_{ss} \end{pmatrix}.
\]

resulting in \( h_{rr} = -h_{rs}^* \) and \( h_{ss} = h_{rr}^* \).

\[
h_{pq} = \begin{pmatrix} h_{rr} & h_{rs} \\ -h_{rs}^* & h_{ss}^* \end{pmatrix}.
\]
We see that $i\mathcal{H}^{(2)}_{\text{Cl}} = \mathfrak{usp}(2(r + s)) \setminus (\mathfrak{usp}(2r) \oplus \mathfrak{usp}(2s))$ and the symmetric space generated by the time evolution operator is $USp(2(r + s))/(USp(2r) \times USp(2s))$.

**Class Cl:** Here $L = 2M$, and thus eqn. (98) provides

$$
\mathbf{H}^{(1)} = \begin{pmatrix}
0_{MM} & \mathbf{h}_{MM} \\
\mathbf{h}_{MM}^t & 0_{MM}
\end{pmatrix}
$$

The symmetry condition eqn. (92) with $U_T = F$ gives $\mathbf{h}_{MM}^T = \mathbf{h}_{MM}$ with

$$
\mathbf{H}^{(1)} = \begin{pmatrix}
0_{MM} & \mathbf{h}_{MM} \\
\mathbf{h}_{MM}^t & 0_{MM}
\end{pmatrix}
$$

It is now clear that the $i\mathcal{H}^{(1)}_{\text{Cl}} = \mathfrak{usp}(2M) \setminus \mathfrak{u}(M)$, and the symmetric space corresponding to the time evolution is the coset space $USp(2M)/U(M)$.

**Class Dll:** In this $2M$ dimensional GFS with, the symmetry condition eqn. (92) leads to $\mathbf{h}_{MM}^T = -\mathbf{h}_{MM}$, and

$$
\mathbf{H}^{(1)} = \begin{pmatrix}
0_{MM} & \mathbf{h}_{MM} \\
-\mathbf{h}_{MM}^t & 0_{MM}
\end{pmatrix}
$$

Here $i\mathcal{H}^{(1)}_{\text{DII}}$ is isomorphic to $\mathfrak{o}(2M) \setminus \mathfrak{u}(M)$. The symmetric space spanned by the time evolution operator is $O(2M)/U(M)$.

The classes A and All are complex classes, while the remainder of the classes involve a “reality condition” of the form $\mathbf{H} = \mathbf{H}^*$ and are the real classes.

**H. Framework for Systems with Interactions**

In this section we establish ideas that allow for the determination of the structure of Hamiltonians with upto $N$-body interactions as in eqn. (3). For usual symmetries, the conditions that determine the spaces $i\mathcal{H}^{(K)}$ are straightforward. From, eqn. (25) we obtain

$$
\mathbf{H}^{(K)} = \tilde{\mathbf{H}}^{(K)}.
$$

These are a set of homogeneous equations in the matrix elements $H^{(K)}_{i_1, i_2, \ldots, i_{K}}$ that make $i\mathcal{H}^{(K)}$ of the appropriate class a vector subspace of $\mathfrak{u}\left(\left(\begin{smallmatrix} L \\ K \end{smallmatrix}\right)\right)$ following the discussion near eqn. (12). Eqn. (106) provides conditions to completely determine the structure of the admissible Hamiltonians.

Transposing symmetries have a more involved story. To make progress, we rewrite eqn. (28) using eqn. (25) to obtain

$$
\mathbf{H}^{(K)} = \sum_{R=K}^{N} A_{R,K} \left[\mathbf{tr}_{R-K} \mathbf{H}^{(R)}\right]^T
$$

$$
= (-1)^K \left[\mathbf{H}^{(K)}\right]^T + (-1)^K \sum_{R=K}^{N} \frac{1}{(R-K)!} \left(\begin{smallmatrix} R \\ K \end{smallmatrix}\right)^2 \left[\mathbf{tr}_{R-K} \mathbf{H}^{(R)}\right]^T
$$

We now define $\mathbf{H}^{(K)}_i$ and $\mathbf{H}^{(K)}_R$ as

$$
\mathbf{H}^{(K)}_i \equiv \mathbf{H}^{(K)} + (-1)^K \left[\mathbf{H}^{(K)}\right]^T
$$

$$
\mathbf{H}^{(K)}_R \equiv \mathbf{H}^{(K)} - (-1)^K \left[\mathbf{H}^{(K)}\right]^T
$$

with $\mathbf{H}^{(K)} = \frac{1}{2} \left(\mathbf{H}^{(K)}_i + \mathbf{H}^{(K)}_R\right)$ and $(-1)^K \left[\mathbf{H}^{(K)}\right]^T = \frac{1}{2} \left(\mathbf{H}^{(K)}_i - \mathbf{H}^{(K)}_R\right)$. With these definitions eqn. (25) becomes

$$
\mathbf{H}^{(K)} = \frac{(-1)^K}{2} \sum_{R=K}^{N} \frac{(-1)^R}{(R-K)!} \left(\begin{smallmatrix} R \\ K \end{smallmatrix}\right)^2 \left[\mathbf{tr}_{R-K} \mathbf{H}^{(R)}_i - \mathbf{tr}_{R-K} \mathbf{H}^{(R)}_R\right].
$$

Two points emerge from these considerations: (i) the transposing symmetry condition eqn. (107) puts no constraint on $\mathbf{H}^{(K)}_i$ for any $K$, and (ii) for every $K$, $\mathbf{H}^{(K)}_R$ is solely determined by $\mathbf{H}^{(K)}_i$ for $K$ from 0 to $N$. Further note that every $\mathbf{H}^{(K)}_i$ belongs to a vector subspace of $\mathfrak{u}\left(\left(\begin{smallmatrix} L \\ K \end{smallmatrix}\right)\right)$ defined by $\mathbf{H}^{(K)}_i = 0$, which we call $i\mathcal{H}^{(K)}_i$. Since, $i\mathcal{H}^{(K)}_i$ is made of objects of the type $\frac{1}{2} \left(\mathbf{H}^{(K)}_i + \mathbf{H}^{(K)}_R\right)$ where $\mathbf{H}^{(K)}_R$ is a “constant” determined by eqn. (109), $i\mathcal{H}^{(K)}_i = \frac{1}{2} \left(\mathbf{H}^{(K)}_i + \mathbf{H}^{(K)}_R\right)$, is no longer a vector subspace of $\mathfrak{u}\left(\left(\begin{smallmatrix} L \\ K \end{smallmatrix}\right)\right)$. In fact, $i\mathcal{H}^{(K)}_i$ is an affine subspace of $\mathfrak{u}\left(\left(\begin{smallmatrix} L \\ K \end{smallmatrix}\right)\right)$ whose dimension (as a manifold) is same as the subspace $i\mathcal{H}^{(K)}_i$.

These observations show that $\mathcal{H}$ in any class can be completely determined by starting from the $N$-body (highest multi-body interaction) term in $\mathbf{H}$ which satisfies $\mathbf{H}^{(N)} = 0$, and recursively using eqn. (109) to determine $\mathbf{H}^{(K)}_R$ for $K < N$. Again, for each $K$ the equation $\mathbf{H}^{(K)}_R = 0$ gives the subspace that defines $i\mathcal{H}^{(K)}_i$ which then makes up the affine space $i\mathcal{H}^{(K)}_i$. The problem of finding the structure of $\mathcal{H}$ is then reduced solely to finding the subspace $i\mathcal{H}^{(K)}_i$ for each $K$. In the subsequent sections we shall demonstrate the determination of this subspace $\mathbf{H}^{(K)} = 0$ for the highest multi-body ($N$-body) interaction in our system. Clearly, the same results will apply mutatis mutandis to all $K < N$. Finally, we note that $\mathbf{H}^{(K)}_i$ s satisfy the identity

$$
\frac{1}{2} \sum_{R=0}^{N} (-1)^R \left[\mathbf{tr}_{R} \mathbf{H}^{(R)}_i - \mathbf{tr}_{R} \mathbf{H}^{(R)}_R\right] = 0.
$$

In the next section we show how this is achieved for $N = 2$, and generalize this to higher $N$ in the subsequent sections. The main results of this exercise are summarized in tables [III] and [IV].
I. Structure of Two-body Hamiltonians

In this section we find the structure of two-body Hamiltonians in each class. This will serve as a precursor to our study of a generic even \(N\)-body Hamiltonian. We define a strictly two-body Hamiltonian as

\[ \mathcal{H} = \Psi \Psi^\dagger H^{(2)} \Psi \Psi, \quad (111) \]

where \(\Psi \Psi\) is a notational short-hand for a column vector comprised of \(\left(\frac{L}{2}\right)^2\) distinct product terms of two fermionic annihilation operators, written as

\[ \Psi \Psi \equiv \begin{pmatrix} \psi_1 \psi_2 \\ \vdots \\ \psi_{L-1} \psi_L \\ \psi_1 \psi_{L-1} \\ \vdots \end{pmatrix}. \quad (112) \]

Henceforth, we will call these product terms as states. The definition of \(\Psi \Psi\) also defines \(\Psi \Psi^\dagger \equiv \left(\left(\psi_1 \psi_2 \right) \ldots (\psi_1 \psi_{L}) \ldots (\psi_2 \psi_{L}) \ldots (\psi_{L-1} \psi_L) \right)^\dagger \). The preceding definitions are in the same spirit as our definition for the states in eqn. \((11)\).

We can also write \(H^{(2)}\) as (see eqn. \((9)\))

\[ (\Psi \Psi)^2 H^{(2)} (\Psi \Psi)^2 = \sum_{i_1, i_2, j_1, j_2} H^{(2)}_{i_1, i_2; j_1, j_2} \psi_{i_1} \psi_{i_2} \psi_{j_1} \psi_{j_2}. \quad (113) \]

Note here that although \(H^{(2)}\) is a four indexed object, fermionic anticommutation demands that it is a matrix of dimension \(\left(\frac{L}{2}\right) \times \left(\frac{L}{2}\right)\). As done in the case of one-body Hamiltonians we now construct the \(\left(\frac{L}{2}\right) \times \left(\frac{L}{2}\right)\) canonical representations of the symmetry operations \((U_{m_1}^{(2)}, U_{c_1}^{(2)}, U_{s_1}^{(2)})\) from their one-body counterparts (see table \(\|\)) and apply them to determine the structure of \(H^{(2)}\).

Class A: This class has no symmetries, therefore any hermitian matrix of dimension \(\left(\frac{L}{2}\right) \times \left(\frac{L}{2}\right)\) belongs to this class.

Class Al: For the Al class we have \(U_T = 1\). Hence for this case,

\[ \mathcal{T} \Psi^\dagger \mathcal{T}^{-1} = \Psi^\dagger \]

\[ \mathcal{T} \Psi \mathcal{T}^{-1} = U_T \Psi = \Psi, \quad (115) \]

and for the two-body states (see eqn. \((112)\) and eqn. \((113)\)) we have

\[ \mathcal{T} \Psi \Psi^\dagger \mathcal{T}^{-1} = \Psi \Psi^\dagger \]

\[ \mathcal{T} \Psi \Psi \mathcal{T}^{-1} = \Psi \Psi, \quad (116) \]

making \(U_T^{(2)} = 1\). Time reversal symmetry implies the following

\[ H^{(2)} = \left[H^{(2)}\right]^T, \quad (117) \]

which puts \(H^{(2)}\) in the class of real symmetric \(\left(\frac{L}{2}\right) \times \left(\frac{L}{2}\right)\) matrices. The dimension of this class is \(\frac{1}{2} \left(\frac{L}{2}\right) \left(\frac{L}{2} + 1\right)\).

Class All: For class All we have canonical \(U_T = J\) and

\[ \mathcal{T} \Psi^\dagger \mathcal{T}^{-1} = \Psi^\dagger U_T = \Psi^\dagger J \]

\[ \mathcal{T} \Psi \mathcal{T}^{-1} = U_T \Psi = -J \Psi. \quad (118) \]

Also we had seen that for this case to be realizable \(L = 2M\). We can divide the \(2M\) states as \(M\) states of flavor \(\alpha\) and \(M\) of flavor \(\beta\). Explicitly these transform in the following way

\[ \mathcal{T} \left( \psi_{1\alpha} \ldots \psi_{M\alpha} \psi^\dagger_{1\beta} \ldots \psi^\dagger_{M\beta} \right) \mathcal{T}^{-1} = \left( -\psi^\dagger_{1\beta} \ldots -\psi^\dagger_{M\beta} \psi_{1\alpha} \ldots \psi_{M\alpha} \right) \]

implying,

\[ \mathcal{T} \psi_{i\alpha} \mathcal{T}^{-1} = -\psi^\dagger_{i\beta} \]

\[ \mathcal{T} \psi^\dagger_{i\beta} \mathcal{T}^{-1} = \psi^\dagger_{i\alpha} \]

\[ \mathcal{T} \psi_{i\alpha} \mathcal{T}^{-1} = -\psi_{i\beta} \]

\[ \mathcal{T} \psi_{i\beta} \mathcal{T}^{-1} = \psi_{i\alpha}. \quad (120) \]

Since \(\Psi \Psi\) is formed by the product of these one-body operators, its elements can be divided into the following three kinds

\[ \Psi \Psi = \begin{pmatrix} \psi_{i\alpha} \psi_{j\alpha} \\ \psi_{i\beta} \psi_{j\beta} \\ \psi^\dagger_{i\alpha} \psi^\dagger_{j\beta} \end{pmatrix} \quad (121) \]

The number of states in each kind add up to \(\begin{pmatrix} L \end{pmatrix}\), i.e., \(2\begin{pmatrix} L \end{pmatrix}\) + \(M^2 = (M(M-1)) + 2(MM) = M(M-1) + M(M+1) = \frac{(2M-1)(2M+1)}{2} = \left(\frac{L}{2}\right)\). We find the effect of the \(\mathcal{T}\) operators on these states to be

\[ \mathcal{T} \Psi \Psi \mathcal{T}^{-1} = \begin{pmatrix} \psi_{i\alpha} \psi_{j\alpha} \\ \psi_{i\beta} \psi_{j\beta} \\ \psi^\dagger_{i\alpha} \psi^\dagger_{j\beta} \end{pmatrix} \mathcal{T}^{-1} = \begin{pmatrix} -\psi^\dagger_{i\beta} \psi_{j\beta} \\ \psi_{i\beta} \psi_{j\beta} \\ -\psi_{i\alpha} \psi_{j\alpha} \end{pmatrix} = \begin{pmatrix} \psi_{i\beta} \psi_{j\beta} \\ \psi_{i\beta} \psi_{j\beta} \\ \psi_{i\alpha} \psi_{j\alpha} \end{pmatrix} \quad (122) \]
| Class  | $L$  | $P$  | $Q$  | $\mathbf{H}^{(N)}$                                                                 | $\dim \mathcal{H}^{(N)}$ | $\mathcal{H}_N$ |
|--------|------|------|------|------------------------------------------------------------------------------------|---------------------------|---------------|
| A      | $L$  | $(\frac{L}{N})$ | –    | $\mathbf{H}^{(N)} = \left[\mathbf{H}^{(N)}\right]^T$                              | $p^2$                     | $\mathbf{u}(P)$ |
| AI     | $L$  | $(\frac{L}{N})$ | –    | $\mathbf{H}^{(N)} = \left[\mathbf{H}^{(N)}\right]^T$                              | $P(P + 1)/2$              | $\mathbf{u}(P) \setminus \mathbf{o}(P)$ |
| All    | $L = 2M$ | $\frac{1}{2} \left[ (\frac{L}{N}) + (\frac{M}{N/2}) \right]$ | $\frac{1}{2} \left[ (\frac{L}{N}) - (\frac{M}{N/2}) \right]$ | $\begin{bmatrix} \mathbf{h}^{(N)}_{PP} & \mathbf{h}^{(N)}_{PQ} \\ \mathbf{h}^{(N)}_{QP} & \mathbf{h}^{(N)}_{QQ} \end{bmatrix}$ | $\frac{P(P+1)}{2} + \frac{Q(Q+1)}{2} + PQ$ | $\mathbf{u}(P + Q) \setminus \mathbf{o}(P + Q)$ |
| D      | $L$  | $(\frac{L}{N})$ | –    | $\mathbf{H}^{(N)} = \left[\mathbf{H}^{(N)}\right]^T$                              | $P(P + 1)/2$              | $\mathbf{u}(P) \setminus \mathbf{o}(P)$ |
| C      | $L = 2M$ | $\frac{1}{2} \left[ (\frac{L}{N}) + (\frac{M}{N/2}) \right]$ | $\frac{1}{2} \left[ (\frac{L}{N}) - (\frac{M}{N/2}) \right]$ | $\begin{bmatrix} \mathbf{h}^{(N)}_{PP} & \mathbf{h}^{(N)}_{PQ} \\ \mathbf{h}^{(N)}_{QP} & \mathbf{h}^{(N)}_{QQ} \end{bmatrix}$ | $\frac{P(P+1)}{2} + \frac{Q(Q+1)}{2} + PQ$ | $\mathbf{u}(P + Q) \setminus \mathbf{o}(P + Q)$ |
| AllII  | $L = p + q$ | $\sum_{a=1,3,\ldots}^{N-1} (\frac{\partial}{\partial \rho})_{\rho=a}$ | $\sum_{a=0,2,\ldots}^{N-1} (\frac{\partial}{\partial \rho})_{\rho=a}$ | $\begin{bmatrix} \mathbf{h}^{(N)}_{PP} & 0_{QP} \\ 0_{QP} & \mathbf{h}^{(N)}_{QQ} \end{bmatrix}$ | $P^2 + Q^2$ | $\mathbf{u}(P) \oplus \mathbf{u}(Q)$ |
| BDI    | $L = p + q$ | $\sum_{a=1,3,\ldots}^{N-1} (\frac{\partial}{\partial \rho})_{\rho=a}$ | $\sum_{a=0,2,\ldots}^{N-1} (\frac{\partial}{\partial \rho})_{\rho=a}$ | $\begin{bmatrix} \mathbf{h}^{(N)}_{PP} & 0_{QP} \\ 0_{QP} & \mathbf{h}^{(N)}_{QQ} \end{bmatrix}$ | $\frac{P(P+1)}{2} + \frac{Q(Q+1)}{2}$ | $(\mathbf{u}(P) \setminus \mathbf{o}(P)) \oplus (\mathbf{u}(Q) \setminus \mathbf{o}(Q))$ |
| CII    | $L = p + q$ | $\sum_{a=1,3,\ldots}^{N-1} (\frac{\partial}{\partial \rho})_{\rho=a}$ | $\sum_{a=0,2,\ldots}^{N-1} (\frac{\partial}{\partial \rho})_{\rho=a}$ | $\begin{bmatrix} \mathbf{h}^{(N)}_{PP} & 0_{QP} \\ 0_{QP} & \mathbf{h}^{(N)}_{QQ} \end{bmatrix}$ | $\frac{P(P+1)}{2} + \frac{Q(Q+1)}{2}$ | $(\mathbf{u}(A + B) \setminus \mathbf{o}(A + B)) \oplus (\mathbf{u}(C + D) \setminus \mathbf{o}(C + D))$ |
| CI     | $L = 2M$ | $\left\{ \begin{array}{ll} P/2 & : N/2 \text{ even} \\ \frac{P}{2} \pm \frac{M}{N/2} & : N/2 \text{ odd} \end{array} \right.$ | $\left\{ \begin{array}{ll} Q/2 & : N/2 \text{ even} \\ Q/2 \pm \frac{M}{N/2} & : N/2 \text{ odd} \end{array} \right.$ | $\begin{bmatrix} \mathbf{h}^{(N)}_{AA} & \mathbf{h}^{(N)}_{AB} \\ \mathbf{h}^{(N)}_{BA} & \mathbf{h}^{(N)}_{BB} \end{bmatrix}$ | $A\left( A + B \right) + B\left( B + 1 \right) + AB$ | $(\mathbf{u}(A + B) \setminus \mathbf{o}(A + B)) \oplus (\mathbf{u}(C + D) \setminus \mathbf{o}(C + D))$ |
| DIII   | $L = 2M$ | $\left\{ \begin{array}{ll} P/2 & : N/2 \text{ even} \\ \frac{P}{2} \pm \frac{M}{N/2} & : N/2 \text{ odd} \end{array} \right.$ | $\left\{ \begin{array}{ll} Q/2 & : N/2 \text{ even} \\ Q/2 \pm \frac{M}{N/2} & : N/2 \text{ odd} \end{array} \right.$ | $\begin{bmatrix} \mathbf{h}^{(N)}_{AA} & \mathbf{h}^{(N)}_{AB} \\ \mathbf{h}^{(N)}_{BA} & \mathbf{h}^{(N)}_{BB} \end{bmatrix}$ | $A\left( A + B \right) + B\left( B + 1 \right) + AB$ | $(\mathbf{u}(A + B) \setminus \mathbf{o}(A + B)) \oplus (\mathbf{u}(C + D) \setminus \mathbf{o}(C + D))$ |

TABLE III. Structure of $N$-body interaction Hamiltonian $\mathbf{H}^{(N)}$ ($N$ even) in each symmetry class. The space of $K$-body ($K$ even) interaction Hamiltonians $\mathcal{H}^{(K)}$ in a class is an affine subspace $\mathcal{H}^{(K)} = \mathcal{H}^{(K)}_u + \mathbf{H}^{(K)}$ where $\mathcal{H}^{(K)}_u$ is to be read from the last column of this table and $\mathbf{H}^{(K)}$ is given in eqn. [109].
| Class | $L$ | $P$ | $Q$ | $H^{(N)}$ | $\dim i\mathcal{H}^{(N)}$ | $i\mathcal{H}_c^{(N)}$ |
|-------|-----|-----|-----|---------|----------------|----------------|
| A $(0,0,0)$ | $L$ | $f(L)$ | $-$ | $H^{(N)} = [H^{(N)}]^*$ | $P^2$ | $u(P)$ |
| Al $(+1,0,0)$ | $L$ | $f(L)$ | $-$ | $H^{(N)} = |H^{(N)}|^*$ | $P(P+1)/2$ | $u(P) \setminus o(P)$ |
| All $(-1,0,0)$ | $L = 2M$ | $\frac{1}{2} \binom{2M}{N}$ | $\frac{1}{2} \binom{2M}{N}$ | $\left[ \begin{array}{c} h_{PP}^{(N)} \\ h_{PQ}^{(N)} \\ -h_{QO}^{(N)} \\ h_{PP}^{(N)} \\ h_{QO}^{(N)} \end{array} \right]$ | $P^2 + 2 \times \frac{P(P+1)}{2}$ | $u(2P) \setminus \text{usp}(2P)$ |
| D $(0,+1,0)$ | $L$ | $f(L)$ | $-$ | $H^{(N)} = -|H^{(N)}|^*$ | $P(P-1)/2$ | $o(P)$ |
| C $(0,-1,0)$ | $L = 2M$ | $\frac{1}{2} \binom{2M}{N}$ | $\frac{1}{2} \binom{2M}{N}$ | $\left[ \begin{array}{c} h_{PP}^{(N)} \\ h_{PQ}^{(N)} \\ -h_{QO}^{(N)} \end{array} \right]$ | $P^2 + 2 \times \frac{P(P+1)}{2}$ | $\text{usp}(2P)$ |
| All $(0,0,1)$ | $L = p+q$ | $\frac{N}{\sum_{a=0}^{N-1}} \binom{a}{N-a}$ | $\frac{N}{\sum_{a=0}^{N-1}} \binom{a}{N-a}$ | $\left[ \begin{array}{c} 0_{PP} \\ h_{PQ}^{(N)} \\ 0_{QO} \end{array} \right]$ | $2PQ$ | $u(P+Q) \setminus (u(P) \oplus u(Q))$ |
| BDI $(+1,+1,1)$ | $L = p+q$ | $\frac{N}{\sum_{a=0}^{N-1}} \binom{a}{N-a}$ | $\frac{N}{\sum_{a=0}^{N-1}} \binom{a}{N-a}$ | $\left[ \begin{array}{c} 0_{PP} \\ h_{PQ}^{(N)} \\ 0_{QO} \end{array} \right]$ | $PQ$ | $o(P+Q) \setminus (o(P) \oplus o(Q))$ |
| CI $(-1,-1,1)$ | $L = p+q$ | $\frac{N}{\sum_{a=0}^{N-1}} \binom{a}{N-a}$ | $\frac{N}{\sum_{a=0}^{N-1}} \binom{a}{N-a}$ | $\left( \begin{array}{ccc} 0_{PP} \\ h_{PQ}^{(N)} \\ -h_{QO}^{(N)} \\ h_{PP}^{(N)} \\ h_{AD}^{(N)} \\ h_{AC}^{(N)} \end{array} \right)$ | $PQ$ | $\text{usp}(P+Q) \setminus (\text{usp}(P) \oplus \text{usp}(Q))$ |
| CI $(-1,+1,1)$ | $L = 2M$ | $\frac{1}{2} \binom{2M}{N}$ | $\frac{1}{2} \binom{2M}{N}$ | $\left[ \begin{array}{c} 0_{PP} \\ h_{PQ}^{(N)} \\ 0_{QO} \end{array} \right]$ | $P(P+1)$ | $\text{usp}(2P) \setminus u(P)$ |
| DII $(-1,+1,1)$ | $L = 2M$ | $\frac{1}{2} \binom{2M}{N}$ | $\frac{1}{2} \binom{2M}{N}$ | $\left[ \begin{array}{c} 0_{PP} \\ h_{PQ}^{(N)} \\ 0_{QO} \end{array} \right]$ | $P(P-1)$ | $o(2P) \setminus u(P)$ |

TABLE IV. The ten symmetry classes of fermions and the structure of many-body Hamiltonians $H^{(N)}$ when $N$ is odd. The space of $K$-body ($K$ odd) interaction Hamiltonians $i\mathcal{H}^{(K)}$ in a class is an affine subspace $i\mathcal{H}^{(K)} = i\mathcal{H}_c^{(K)} + H^{(K)}$, where $i\mathcal{H}_c^{(K)}$ is to be read from the last column of this table and $H^{(K)}$ is given in eqn. [109].
This suggests that we can re-organize the two-body basis into "symmetric" ($\sigma$) and "antisymmetric" ($\pi$) states as follows:

\[
\begin{pmatrix}
\psi_{ia} \psi_{ja} \\
M^2 \psi_{ia} \psi_{\beta j} \\
\psi_{\alpha i} \psi_{j\beta}
\end{pmatrix}
\rightarrow
\begin{pmatrix}
\sigma \\
M \\
\pi
\end{pmatrix}
\begin{pmatrix}
\psi_{ia} \psi_{ja} + \psi_{ib} \psi_{\beta j} \\
M \psi_{ia} \psi_{\beta j} \\
\psi_{\alpha i} \psi_{j\beta} - \psi_{\alpha j} \psi_{i\beta}
\end{pmatrix}.
\]

Conveniently, in this way of organizing our Hilbert space, we have

\[
\mathcal{F} \sigma \mathcal{F}^{-1} = \sigma \quad \mathcal{F} \pi \mathcal{F}^{-1} = -\pi.
\]

Also note that $M^2 = 2(M/2) + M$.

It will be useful to mention here that the transformation properties of these states under symmetry gives us the definitions of "symmetric" and "antisymmetric" states. This terminology will be used in later sections when the $N$-body Hamiltonians will be discussed. Note that the word "symmetric" may not necessarily imply a "+" sign in the linear combination of its constituent states and vice-versa.

Armed with this, we look for the structure of the $H^{(2)}$

\[
\mathcal{F} \psi^i \psi^j H^{(2)} \psi^i \psi^j \mathcal{F}^{-1} = \psi^i \psi^j H^{(2)} \psi^i \psi^j
\]

where $U_T$ in this new basis is obtained from eqn. (124) and is given by

\[
\left[U_T^{(2)}\right] = 1_{P,Q},
\]

where $P = 2(M/2) + M$ and $Q = 2(M/2)$. Assuming,

\[
H^{(2)} = \begin{pmatrix}
h_{pp}^{(2)} & h_{pq}^{(2)} \\
h_{qp}^{(2)} & h_{qq}^{(2)}
\end{pmatrix}
\]

and implementing the condition in eqn. (125), we get the following constraints,

\[
h_{pp}^{(2)} = \left[h_{pp}^{(2)}\right], \quad h_{qq}^{(2)} = \left[h_{qq}^{(2)}\right], \quad h_{pq}^{(2)} = -\left[h_{pq}^{(2)}\right].
\]

This leads to total $\frac{P(P+1)}{2} + \frac{Q(Q+1)}{2} + PQ$ independent parameters in $H^{(2)}$, which is equal to dim $H^{(2)}_{\text{All}}$.

**Class D:** We now look at transposing symmetries of the TL kind. The states transform according to eqn. (19) and since class D has canonical $U_C = 1$, we have

\[
\psi^i \psi^j \mathcal{F}^{-1} = \psi^i \psi^j
\]

which implies

\[
\psi^i \mathcal{F}^{-1} = \psi_i
\]

Thus even for the two-body Hamiltonian the symmetry condition (eqn. (21)) is

\[
H^{(2)} = \left[H^{(2)}\right],
\]

making $H^{(2)}$ a real symmetric matrix with $\frac{1}{2} \left(\frac{M}{2}\right) \left(\frac{M}{2} + 1\right)$ parameters.

**Class C:** Class C comes with canonical $U_C = J$ and requires $L = 2M$. The states transform as

\[
\psi \psi^i \mathcal{F}^{-1} = \psi^i J
\]

\[
\psi \psi^i \mathcal{F}^{-1} = -J(\psi^i)^T.
\]

Similar to class All, we label the states with $\alpha, \beta$ flavors and find there transformations to be

\[
\psi \psi^i \mathcal{F}^{-1} = -\psi_{\beta}
\]

\[
\psi \psi^i \mathcal{F}^{-1} = -\psi_{\alpha}
\]

\[
\psi \psi^i \mathcal{F}^{-1} = -\psi_{\beta}
\]

\[
\psi \psi^i \mathcal{F}^{-1} = -\psi_{\alpha}.
\]

However, unlike class All, one must remember that here we are dealing with a transposing symmetry. The two-body basis in this case can be rearranged in the following manner

\[
\begin{pmatrix}
\psi_{ia} \\
M \psi_{ia} \psi_{\beta j} \\
\psi_{\alpha i} \psi_{j\beta}
\end{pmatrix}
\rightarrow
\begin{pmatrix}
\sigma \\
M \\
\pi
\end{pmatrix}
\begin{pmatrix}
\psi_{ia} \psi_{ja} + \psi_{ib} \psi_{\beta j} \\
M \psi_{ia} \psi_{\beta j} \\
\psi_{\alpha i} \psi_{j\beta} - \psi_{\alpha j} \psi_{i\beta}
\end{pmatrix}.
\]

The action of $\mathcal{C}$ symmetry on these states are

\[
\mathcal{C} \sigma \mathcal{C}^{-1} = \sigma^T
\]

\[
\mathcal{C} \pi \mathcal{C}^{-1} = -\pi^T.
\]

From this we get $2(M/2)$ number of $\sigma$ states and $M^2$ number of $\pi$ states. Imposing $\mathcal{C}$ symmetry on the Hamiltonian gives

\[
\mathcal{C} \psi \psi^i \mathcal{F}^{-1} H^{(2)} \psi \psi^i \mathcal{F}^{-1} = \psi \psi^i \mathcal{F} \psi \psi^i H^{(2)} \psi \psi^i
\]

\[
U_C^{(2)} \left[H^{(2)}\right] U_C^{(2)} = H^{(2)}
\]

where

\[
U_C^{(2)} = -1_{P,Q}
\]

and $P = 2(M/2) + M$ and $Q = 2(M/2)$. Denoting the internal structure of $H^{(2)}$ with eqn. (127), the constraints turn out to be same as eqn. (128). This again leads to $\frac{P(P+1)}{2} + \frac{Q(Q+1)}{2} + PQ$ independent parameters.

**Class All:** For transposing symmetries of the TA kind, we know

\[
: (U \ H \ U^{-1}) : = H.
\]
The four classes will be of the form \( \mathcal{AIII} \) in contrast with structure of the one-body Hamiltonian in class \( \mathcal{BIIi} \) (see table II). The number of independent parameters are \( \frac{p(p+1)}{2} + \frac{q(q+1)}{2} \).

The presence of \( T = \pm 1 \) and \( C = \pm 1 \) symmetries will only put constraints on the non-zero sub-blocks of \( \mathbf{H}^{(2)} \).

**Class BDi:** This class has \( \mathbf{U}_S = \mathbf{1}_{p,q}, \mathbf{U}_C = \mathbf{1}_{p,q}, \mathbf{U}_T = \mathbf{1} \). Since time reversal symmetry operation has a canonical property, we now have additional conditions on the sub-blocks (see eqn. (147)).

\[
\mathbf{h}_{pp}^{(2)} = [\mathbf{h}_{pp}^{(2)}]^*, \quad \mathbf{h}_{QQ}^{(2)} = [\mathbf{h}_{QQ}^{(2)}]^*.
\] (148)

The two-body states look like \( L \). Before delving into the classes, we reorganize the states as given in eqn. (127) and imposing eqn. (147). Also note that \( \left( \frac{q}{2} \right) + pq + \left( \frac{q}{2} \right) = \left( \frac{q}{2} \right) \) and

\[
\mathcal{T} \mathbf{h} \mathcal{T}^{-1} = \mathbf{h}^{\star}, \quad \mathcal{T} \mathbf{h} \mathcal{T}^{-1} = \mathbf{h}^{\star},
\] (149)

Given sublattice symmetry, we have the structure of the Hamiltonian as shown in eqn. (147). We also have \( p = 2r \) and \( q = 2s \). We label the \( p \) states as \( \alpha_p \) and \( \beta_p \), both of which run from \( 1 \) to \( r \). Similarly the \( q \) states can be labeled with \( \alpha_q \) and \( \beta_q \), each running from \( 1 \) to \( s \). Transformations of these new flavors under symmetry are

\[
\begin{align*}
\mathcal{T} \mathbf{h} \mathcal{T}^{-1} &= \mathbf{h}, \\
\mathcal{T} \mathbf{h} \mathcal{T}^{-1} &= \mathbf{h}^\dagger, \\
\mathcal{T} \mathbf{h} \mathcal{T}^{-1} &= \mathbf{h}, \\
\mathcal{T} \mathbf{h} \mathcal{T}^{-1} &= \mathbf{h}^\dagger.
\end{align*}
\] (150)

The \( P = pq = 4rs \) states can be rearranged as

\[
\begin{pmatrix}
\psi_{ia_p} \psi_{ja_q} \\
\psi_{ia_q} \psi_{ja_p} \\
\psi_{ia_p} \psi_{ja_q} \\
\psi_{ia_q} \psi_{ja_p}
\end{pmatrix}
\rightarrow
\begin{pmatrix}
\psi_{ia_p} \psi_{ja_q} \\
\psi_{ia_q} \psi_{ja_p} \\
\psi_{ia_p} \psi_{ja_q} \\
\psi_{ia_q} \psi_{ja_p}
\end{pmatrix}
\] (151)

Remembering that \( Q = \left( \frac{q}{2} \right) + \left( \frac{q}{2} \right) \), both the \( \left( \frac{q}{2} \right) \) and \( \left( \frac{q}{2} \right) \) states can also be organized into symmetric and antisymmetric states. For example rearrangement of the \( \left( \frac{q}{2} \right) \) states leads to

\[
\begin{pmatrix}
\psi_{ia_p} \psi_{ja_q} \\
\psi_{ia_q} \psi_{ja_p} \\
\psi_{ia_p} \psi_{ja_q} \\
\psi_{ia_q} \psi_{ja_p}
\end{pmatrix}
\rightarrow
\begin{pmatrix}
\psi_{ia_p} \psi_{ja_q} + \psi_{ia_q} \psi_{ja_p} \\
\psi_{ia_p} \psi_{ja_q} - \psi_{ia_q} \psi_{ja_p} \\
\psi_{ia_p} \psi_{ja_q} + \psi_{ia_q} \psi_{ja_p} \\
\psi_{ia_p} \psi_{ja_q} - \psi_{ia_q} \psi_{ja_p}
\end{pmatrix}
\] (152)

which transform as

\[
\mathcal{T} \mathcal{T}^{-1} = \sigma, \quad \mathcal{T} \mathcal{T}^{-1} = -\pi.
\] (153)

It is interesting to note that this rearrangement of the states do not mix the \( P \) and \( Q \) blocks.
The two-body $U^{(2)}_T$ can now be written as

$$U^{(2)}_T = \begin{pmatrix} I_{A,B} & 0 \\ 0 & 1_{C,D} \end{pmatrix}$$  \hspace{1cm} (154)$$

where $A = 2r$ and $B = 2s$, $C = r^2 + s^2$ and $D = 2(\frac{r}{2}) + 2(\frac{s}{2})$. Imposing symmetry demands

$$\mathcal{U} \psi \Psi \mathcal{H}^{(2)} \Psi \mathcal{U}^{-1} = \Psi \mathcal{U} \psi \mathcal{H}^{(2)} \Psi$$

and $h^{(2)}_{pp}$, $h^{(2)}_{QQ}$ remain decoupled. Writing

$$h^{(2)}_{pp} = \begin{pmatrix} h^{(2)}_{AA} \\ h^{(2)}_{AB} \end{pmatrix}$$

and implementing the above conditions, we get the following constraints

$$h^{(2)}_{AA} = \begin{pmatrix} h^{(2)}_{BB} \\ h^{(2)}_{AB} \end{pmatrix}$$

This leads to total $\frac{A(B+1)}{2} + \frac{B(B+1)}{2} + AB$ independent parameters. Similarly setting

$$h^{(2)}_{QQ} = \begin{pmatrix} h^{(2)}_{CC} \\ h^{(2)}_{CD} \end{pmatrix}$$

and implementing the symmetry conditions, we get the following constraints

$$h^{(2)}_{CC} = \begin{pmatrix} h^{(2)}_{DD} \\ h^{(2)}_{CD} \end{pmatrix}$$

leading to $\frac{C(C+1)}{2} + \frac{D(D+1)}{2} + CD$ independent parameters for this part of the Hamiltonian.

Class Cl: This class arises with

$$U_T = F, \quad U_C = -J, \quad U_S = 1_{p,q}.$$  \hspace{1cm} (160)$$

Sublattice symmetry already implies a structure of Hamiltonian as given in eqn. (147). Also we have $P = M^2$ and $Q = 2^{(M)}$. States can be re-organized as

$$\begin{pmatrix} \psi_{\alpha} \psi_{\beta} \\ \psi_{\alpha} \psi_{\alpha} \\ \psi_{\beta} \psi_{\beta} \\ \psi_{\beta} \psi_{\alpha} \end{pmatrix} \rightarrow \begin{pmatrix} \chi \psi_{\alpha} \psi_{\beta} \psi_{\beta} \psi_{\alpha} \\ \chi \psi_{\alpha} \psi_{\alpha} \psi_{\alpha} \psi_{\alpha} \\ \chi \psi_{\alpha} \psi_{\alpha} \psi_{\alpha} \psi_{\alpha} \\ \chi \psi_{\alpha} \psi_{\alpha} \psi_{\alpha} \psi_{\alpha} \end{pmatrix}.$$  \hspace{1cm} (161)$$

Again both the sectors get decoupled and are symmetric and antisymmetric under time reversal. The remaining analysis follows closely the one for class Cl with the appropriate replacements of $A = (M/2)$, $B = (M/2) + M$, $C = (M/2)$ and $D = (M/2)$.

Class ClII: This one possesses

$$U_T = J, \quad U_C = F, \quad U_S = 1_{p,q}.$$  \hspace{1cm} (162)$$

and $L = 2M$. States can again be reorganized as symmetric and antisymmetric under time reversal as follows

$$\begin{pmatrix} \psi_{\alpha} \psi_{\beta} \\ \psi_{\alpha} \psi_{\alpha} \\ \psi_{\beta} \psi_{\beta} \\ \psi_{\beta} \psi_{\alpha} \end{pmatrix} \rightarrow \begin{pmatrix} \chi \psi_{\alpha} \psi_{\beta} \psi_{\beta} \psi_{\alpha} \\ \chi \psi_{\alpha} \psi_{\alpha} \psi_{\alpha} \psi_{\alpha} \\ \chi \psi_{\alpha} \psi_{\alpha} \psi_{\alpha} \psi_{\alpha} \\ \chi \psi_{\alpha} \psi_{\alpha} \psi_{\alpha} \psi_{\alpha} \end{pmatrix}.$$  \hspace{1cm} (163)$$

Yet again both the sectors get decoupled and are symmetric and antisymmetric under the time reversal. Rest of the analysis proceeds along the same lines as class ClII and Cl with $A = (M/2) + M$, $B = (M/2)$, $C = (M/2)$ and $D = (M/2)$.

J. $N$-body Interacting Hamiltonians

We now determine the structure of a $N$-body Hamiltonian of the generic form (see eqn. (23)),

$$\mathcal{H}^{(N)} = \sum_{i_1,i_2,...,i_N} (\psi_{i_1} \psi_{i_2} \cdots \psi_{i_N}) \mathcal{H}^{(N)}_{i_1,i_2,...,i_N} \psi_{j_1} \psi_{j_2} \cdots \psi_{j_N}.$$  \hspace{1cm} (164)$$

As discussed in section section 11 this is the key ingredient to determine the structure of $\mathcal{H}$ in eqn. (13). A basis of $\binom{N}{k}$ states is required to describe $\mathcal{H}^{(N)}$. We find that structure determination of $\mathcal{H}^{(N)}$ depends on whether $N$ is odd or even. In the next subsection we focus on the cases when $N$ is odd followed by the subsection for $N$ even. In both cases, the strategy is to use table 12 to find a canonical representation of a symmetry operation $U^{(N)}$, which then aids in finding the final structure of $\mathcal{H}^{(N)}$.

1. Structure of $\mathcal{H}^{(N)}$ for $N$ odd

Class A: This class has no symmetries and the only condition imposed on the Hamiltonian is

$$\mathcal{H}^{(N)} = [\mathcal{H}^{(N)}]^\dagger.$$  \hspace{1cm} (165)$$

The Hamiltonian has $\binom{N}{k}$ independent parameters.

Class Al: The implementation of the time reversal demands

$$\mathcal{H}^{(N)} = [\mathcal{H}^{(N)}]^\dagger.$$  \hspace{1cm} (166)$$

Hence the number of independent parameters are $\binom{N}{k} (\binom{N}{k} + 1)$. 

Class All: Given the conditions, as in (120), we know $L = 2M$. There are $M a$ and $M b$ states. A generic many-body state has the form

$$\alpha_{\alpha} \beta_{\beta_{N-a}} \equiv \psi_{\alpha_{1}} \cdots \psi_{\alpha_{a}} \psi_{\beta_{1}} \cdots \psi_{\beta_{N-a}}.$$  \hspace{1cm} (167)$$
This state transforms in the following way,
\[ \mathcal{T} \alpha \beta_{N-a} \mathcal{T}^{-1} = \beta \alpha_{N-a}(-1)^a = \alpha_{N-a} \beta_{N-a}(-1)^{(N-a+1)} \]
\[ = (-1)^a \alpha_{N-a} \beta_{N-a}. \] (168)
The last equality uses the fact that \( N \) is odd. Using this we form two kinds of states, where one is made of even number of \( a \)s(\( \equiv E_a \)) and the other with odd (\( \equiv O_a \)), having
\[
\dim O_a = \sum_{a=1, 3, \ldots}^{N} \binom{M}{a} \binom{M}{N-a} = \frac{1}{2} \binom{2M}{N} \]
\[
\dim E_a = \sum_{a=0, 2, \ldots}^{N-1} \binom{M}{a} \binom{M}{N-a} = \frac{1}{2} \binom{2M}{N}. \] (169)
These new states transform conveniently as
\[ \mathcal{T} O_a \mathcal{T}^{-1} = -E_a \]
\[ \mathcal{T} E_a \mathcal{T}^{-1} = O_a \] (170)
which overall gives the following transformation
\[ \mathcal{T} \Psi \cdots \Psi \mathcal{T}^{-1} = - \mathcal{J}^{(N)} \Psi \cdots \Psi \] (171)
where \( \mathcal{J}^{(N)} \) is the \( N \)-body version of \( \mathcal{J} \) (defined in 166), but now with dimensions \( \binom{N}{N} \). This tells us that the Hamiltonian transforms as the one-body case discussed near eqn. (95). However, the dimension now given is by, \( \dim \mathcal{H}^{(N)} = (2p+2)^{\frac{P(P-1)}{2}} \), where \( P = \frac{1}{2} \binom{2M}{N} \).

**Class D:** In this class the transformation of \( \psi \)'s are shown in eqn. (130). The Hamiltonian satisfies
\[ \mathcal{C} \mathcal{H}^{(N)} \mathcal{C}^{-1} = \mathcal{H}^{(N)}. \] (172)
This implies, \(-[\mathcal{H}^{(N)}]^{\dagger} = \mathcal{H}^{(N)} \), with the number of independent parameters being \( \frac{P(P+1)}{2} \), where \( P = \binom{M}{N} \).

**Class C:** Using the transformations for the fermionic operators given in eqn. (133), we find out that \( O_a \) and \( E_a \) transform as
\[ \mathcal{C} O_a \mathcal{C}^{-1} = (-1)^{\frac{N-a}{2}} (E_a)^{\dagger} \]
\[ \mathcal{C} E_a \mathcal{C}^{-1} = (-1)^{\frac{N-a}{2}} (O_a)^{\dagger}. \] (173)
Therefore as discussed near eqn. (96), we again have \( U^{(N)}_{s} \propto \mathcal{J}^{(N)} \). Hence the Hamiltonian satisfies the same condition as the single-body case, with the dimension \( \dim \mathcal{H}^{(N)} = (2p+2)^{\frac{P(P+1)}{2}} = P(2P+1) \) where \( P = \frac{1}{2} \binom{2M}{N} \).

**Class All:** Given the conditions for Class All (eqn. (140)), where \( p \) and \( q \) states are labeled by \( \alpha \) and \( \beta \), the states transform according to
\[ \mathcal{T} O_a \mathcal{T}^{-1} = (-1)^{\frac{N-a}{2}} (O_a)^{\dagger} \]
\[ \mathcal{T} E_a \mathcal{T}^{-1} = (-1)^{\frac{N-a}{2}} (E_a)^{\dagger}, \] (174)
with
\[
\dim O_a = \sum_{a=1, 3, \ldots}^{N} \binom{p}{a} \binom{q}{N-a} = P \]
\[
\dim E_a = \sum_{a=0, 2, \ldots}^{N-1} \binom{p}{a} \binom{q}{N-a} = Q. \] (175)
Therefore \( U^{(N)}_{s} \propto 1_{PQ} \) as we had seen in the one-body case (see near eqn. (97)). This implies that just like the one-body case, the diagonal blocks of \( H^{(N)} \) will be constrained to be zero (see eqn. (98)) forcing the structure of the Hamiltonian to be
\[ H^{(N)} = \begin{pmatrix}
0 & \hbar_{PQ} \\
\hbar_{PQ}^{\dagger} & 0_{QQ}
\end{pmatrix}. \] (176)
This gives the number of independent parameters as \( \dim \mathcal{H}^{(N)} = P \).

We again bring to attention of the reader that from here onwards and until the end of this subsection all the classes have sublattice symmetry and therefore their respective Hamiltonians will always possess the above form.

**Class BD:** The time reversal symmetry implementation on the class All, demands \( H^{(N)} \) to be real (see eqn. (176) and therefore, the \( \dim \mathcal{H}^{(BD)} = P \).

**Class CII:** The transformation of states is given by eqn. (150), and in this case we have \( p = 2r \), \( q = 2s \). We now show how these affect the \( P \) and \( Q \) states. The general structure of any state is of the form
\[ \psi_{\alpha_{p_1}} \cdots \psi_{\alpha_{p_r}} \psi_{\beta_{q_1}} \cdots \psi_{\beta_{q_s}} \psi_{\alpha_{p_{1}}} \cdots \psi_{\alpha_{p_{r}}} \psi_{\beta_{q_{1}}} \cdots \psi_{\beta_{q_{s}}} \]
where \( a \rightarrow a+c \) and \( d \rightarrow N-a-d \).

Given sublattice symmetry and the fact that \( \alpha_p \leftrightarrow \beta_p \), \( \alpha_q \leftrightarrow \beta_q \) under time reversal, the transformed states still live within their respective blocks. Therefore they do not change the constraints put on the Hamiltonian due to sublattice symmetry and hence preserve the structure of the Hamiltonian appearing in eqn. (176).

To see this more clearly, we write the above state schematically and find that it transforms under \( \mathcal{T} \)
\[ \alpha_{p_1} \beta_{p_{1}} \cdots \alpha_{q_{1}} \beta_{Q_{N-a-d}} \mathcal{T} \equiv \begin{cases}
(-1)^{r} \alpha_{p_{1}} \cdots \alpha_{p_{r}} \beta_{q_{1}} \cdots \beta_{q_{s}} &; a \text{ odd} \\
(-1)^{r} \alpha_{p_{1}} \cdots \alpha_{p_{r}} \beta_{q_{1}} \cdots \beta_{q_{s}} &; a \text{ even}
\end{cases} \] (178)
Therefore the odd-even structure of \( a \) is still preserved. On further substructuring of states into odd(even)-number of \( \alpha_p \) \( \equiv O_{a}(E_{a}) \) and odd(even)-number of \( \alpha_q \ \equiv O_{a}(E_{a}) \) within each \( P \) and \( Q \) block, we find that \( \mathcal{T} \) symmetry acts on them as
\[
\begin{pmatrix}
\alpha \rightarrow O_{a} \\
\beta \rightarrow E_{a}
\end{pmatrix} \rightarrow \begin{pmatrix}
\begin{pmatrix}
E_{a} & O_{a} \\
E_{a} & O_{a}
\end{pmatrix} & \begin{pmatrix}
O_{a} & E_{a} \\
O_{a} & E_{a}
\end{pmatrix} \\
\begin{pmatrix}
O_{a} & E_{a} \\
O_{a} & E_{a}
\end{pmatrix} & \begin{pmatrix}
O_{a} & E_{a} \\
O_{a} & E_{a}
\end{pmatrix}
\end{pmatrix} \mathcal{T} \equiv \begin{pmatrix}
\begin{pmatrix}
O_{a} & O_{a} \\
O_{a} & O_{a}
\end{pmatrix} & \begin{pmatrix}
E_{a} & E_{a} \\
E_{a} & E_{a}
\end{pmatrix} \\
\begin{pmatrix}
E_{a} & E_{a} \\
E_{a} & E_{a}
\end{pmatrix} & \begin{pmatrix}
O_{a} & E_{a} \\
O_{a} & E_{a}
\end{pmatrix}
\end{pmatrix} \] (179)
Hence \( U^{(N)}_T \) takes the same form as in the one-body case (see eqn. (71)) which is

\[
U^{(N)}_T = \begin{pmatrix}
J_{PP} & 0_{PQ} \\
0_{QP} & J_{QQ}
\end{pmatrix}.
\] (180)

The structure of Hamiltonian then follows from the one-body case discussed near eqn. (104) with the number of independent parameters given by \( \dim \mathcal{H}^{(N)}_{\text{Cl}} = PQ \).

Class CI: The presence of sublattice symmetry enforces that the Hamiltonian has a off-diagonal structure (see eqn. (176)) with dimension \( P = \frac{1}{2}(M+N) \). As we have seen, \( U_T \) in this case is \( F \) (see eqn. (75)), and converts \( \alpha \leftrightarrow \beta \) and vice-versa, therefore the transformation is

\[
\mathcal{T} O_\alpha \mathcal{T}^{-1} = E_\alpha \\
\mathcal{T} E_\alpha \mathcal{T}^{-1} = O_\alpha.
\] (181)

Hence \( U^{(N)}_T \) is \( F^{(N)} \) (a generalized version of \( F \) defined in (74)) but with dimension \( \left( \frac{L}{N} \right) \) and the constraint on \( H^{(N)} \) is same as that shown in eqn. (104) but with \( M \) replaced with \( P \). The dim of \( \mathcal{H}^{(N)}_{\text{Cl}} \) is \( P(P+1) \).

Class DII: For this class, \( U_T = J \) (see eqn. (76)) and states transform as

\[
\mathcal{T} \psi_\alpha \mathcal{T}^{-1} = -\psi_\beta \\
\mathcal{T} \psi_\beta \mathcal{T}^{-1} = \psi_\alpha.
\] (182)

The odd and even \( \alpha \) states transform in the following way

\[
\mathcal{T} O_\alpha \mathcal{T}^{-1} = -E_\alpha \\
\mathcal{T} E_\alpha \mathcal{T}^{-1} = O_\alpha.
\] (183)

Therefore \( U^{(N)}_T = J^{(N)} \) with matrix dimension \( P \). The constraints on the Hamiltonian is same as that shown in eqn. (105) but now with \( \dim \mathcal{H}^{(N)}_{\text{DII}} = P(P-1) \), where \( P = \frac{1}{2}(M+N) \). The generic structure of \( H^{(N)} \) for \( N \)-odd is summarized and tabulated in table IV.

2. Structure of \( H^{(N)} \) for \( N \) even

Class A: This class has no symmetries and the only condition that applies is \( H^{(N)} = \left[ H^{(N)} \right]^{\dagger} \). The Hamiltonian therefore has \( \left( \frac{N}{2} \right)^2 \) independent parameters.

Class AI: As is the case of \( N \)-odd, this class only enforces the condition \( H^{(N)} = \left[ H^{(N)} \right]^{\dagger} \) and the number of independent parameters are \( \frac{1}{4} \left( \frac{N}{2} \right) \left( \frac{N}{2} + 1 \right) \).

Class AII: Given the conditions in eqn. (120), we know \( L = 2M \). There are \( M \alpha \) and \( M \beta \) states. Note that given \( N \) is even, we can reorganize states into symmetric and antisymmetric states (like in two-body case, see eqn. (123)). The transformation on a generic state (see eqn. (167)) is given by

\[
\mathcal{T} \alpha_\beta \mathcal{T}^{-1} = \alpha_{N-\beta} \beta_\alpha.
\] (184)

Therefore symmetric and antisymmetric states can be made by linearly combining, \( \alpha_\beta \beta_\alpha \) and \( \alpha_{N-\beta} \beta_\alpha \) with a \( \pm \) sign. Note that \( \pi \) and \( \sigma \) don’t contain the same number of states. The state \( \alpha_\beta \pm \beta_\alpha \) (with same orbital labels) goes back to itself without any sign change under transformation and therefore is a symmetric state. Then dimensions of symmetric(\( \sigma \)) and antisymmetric(\( \pi \)) states are

\[
dim \sigma = \frac{1}{2} \left( \binom{L}{N} + \binom{M}{N/2} \right)
\]

\[
dim \pi = \frac{1}{2} \left( \binom{L}{N} - \binom{M}{N/2} \right).
\] (185)

and \( U^{(N)}_T = I_{PQ} \). The structure of the Hamiltonian in this basis is given by

\[
H^{(N)} = \begin{pmatrix}
h^{(N)}_{PP} & h^{(N)}_{PQ} \\
h^{(N)}_{QP} & h^{(N)}_{QQ}
\end{pmatrix}.
\] (186)

with symmetry conditions being

\[
\mathcal{T} \psi^{(N)} \mathcal{T}^{-1} = \psi^{(N)} H^{(N)} \psi
\]

\[
U^{(N)}_T \left[ H^{(N)} \right] U^{(N)}_T = H^{(N)}.
\] (187)

This imposes the following constraints,

\[
h^{(N)}_{PP} = \left[ h^{(N)}_{PP} \right]^{+}, \quad h^{(N)}_{PQ} = \left[ h^{(N)}_{PQ} \right]^{+}, \quad h^{(N)}_{QP} = -\left[ h^{(N)}_{PQ} \right]^{+}.
\] (188)

The total number of independent parameters are \( \frac{P(P+1)}{2} + \frac{Q(Q+1)}{2} + PQ \). This general formula reduces to the specific two-body case discussed above by substituting \( N = 2 \).

Class D: For this class the transformation of \( \psi \)s are determined by eqn. (130). The constraint on the Hamiltonian is then \( \left[ H^{(N)} \right]^{\dagger} = H^{(N)} \), which is same as that for Class AI.

Class C: The symmetric (\( \sigma \)) and antisymmetric states (\( \pi \)) are again linear combinations of \( \alpha_\beta \beta_\alpha \) and \( \alpha_{N-\beta} \beta_\alpha \) with a \( \pm \) sign. However depending on the value of \( N \), the + linear combination may transform under the \( \mathcal{C} \) symmetry with a \( - \) sign and therefore be an antisymmetric state by definition. In general a state will transform as

\[
\alpha_\beta \beta_\alpha \not\rightarrow (-1)^{N/2} \alpha_{N-\beta} \beta_\alpha.
\] (189)

With the definition

\[
P = \frac{1}{2} \left( \binom{L}{N} + \binom{M}{N/2} \right)
\]

\[
Q = \frac{1}{2} \left( \binom{L}{N} - \binom{M}{N/2} \right),
\] (190)

the dimensions of symmetric and antisymmetric states now are

\[
dim \sigma = \begin{cases}
P & : N/2 \text{ even} \\
Q & : N/2 \text{ odd}
\end{cases}
\]

\[
dim \pi = \begin{cases}
Q & : N/2 \text{ even} \\
P & : N/2 \text{ odd}
\end{cases}.
\] (191)
Therefore $U^{(N)}_S = (-1)^{N/2}I_{P,Q}$. The structure of the Hamiltonian is again constrained in the same way as we had seen in the eqn. (186) and eqn. (188).

Class All: Given the conditions for Class All (eqn. (140)) where $p$ and $q$ states are labeled by $\alpha$ and $\beta$, we look at the transformation of $E_\alpha$ and $O_\alpha$ states introduced near eqn. (167),

$$O_\alpha, O^{-1} = (-1)^{\frac{p}{2}}O_\alpha$$
$$E_\alpha, E^{-1} = (-1)^{\frac{q}{2}}E_\alpha,$$  \hspace{1cm} (192)

The dimensions are given by,

$$\dim O_\alpha = \sum_{a=1,3,...} \begin{pmatrix} p \cr N-a \end{pmatrix} = P$$
$$\dim E_\alpha = \sum_{a=0,2,...} \begin{pmatrix} q \cr N-a \end{pmatrix} = Q,$$

and $U^{(N)}_S = (-1)^{\frac{N}{2}}I_{P,Q}$. This imposes the condition that off-diagonal blocks in eqn. (186) are now zero and the independent parameters, which are $F^2 + Q^2$ in number, belong to the diagonal blocks.

The following classes discussed in this subsection all have sublattice symmetry and the form of the Hamiltonian in each class will satisfy

$$H^{(N)} = \begin{pmatrix} h^{(N)}_{PP} & 0_{PQ} \\ 0_{QP} & h^{(N)}_{QQ} \end{pmatrix}$$ \hspace{1cm} (193)

Class BDI: Apart from the implementation of the sublattice symmetry as in the previous section, time reversal operation for this class is just $U^{(N)}_T = 1$ and therefore just demands $h^{(N)}_{PP}$ and $h^{(N)}_{QQ}$ to be real. The number of independent parameters are $\frac{P(P+1)}{2} + \frac{Q(Q+1)}{2}$.

Class CII: This class requires that $p = 2r$ and $q = 2s$. Imposition of sublattice symmetry breaks the basis into blocks $O_\alpha$ and $E_\alpha$, of dimensions given in eqn. (193). Now the $\alpha$ states are made of two varieties $\alpha_\beta$ and $\beta_\beta$. While the $\beta$ states are made of $\alpha_\alpha$ and $\beta_\beta$. The time-reversal symmetry converts $\alpha_\beta \leftrightarrow \beta_\beta$, and $\alpha_\alpha \leftrightarrow \beta_\alpha$ states. Therefore one can make symmetric and antisymmetric combinations. The transformations under $\mathcal{F}$ are given in eqn. (150). Now let us discuss the $O_\alpha$ states. Dimension of $O_\alpha$ states is $P$ and is comprised of following kinds of states

$$\begin{pmatrix} 1 \\ E_{\alpha_\alpha}, E_{\alpha_\beta} \\ O_{\alpha_\alpha}, O_{\alpha_\beta} \\ O_{\beta_\beta} \end{pmatrix} \xrightarrow{\mathcal{F}} \begin{pmatrix} 1 \\ O_{\alpha_\alpha}, O_{\alpha_\beta} \\ E_{\alpha_\alpha}, E_{\alpha_\beta} \\ -E_{\alpha_\beta}, O_{\alpha_\beta} \end{pmatrix}.$$ \hspace{1cm} (194)

Hence the number of symmetric states(≡ $A$) and number of antisymmetric states(≡ $B$) equals $P/2$ within the $P$ block (see table III). A typical $E_\alpha$ state transforms under $\mathcal{F}$ as

$$\alpha_p, \beta_{p-a} \rightarrow \alpha_p, \beta_{p-a-a} \xrightarrow{\mathcal{F}} \alpha_p, \beta_{p-c} \alpha_{q-a} \rightarrow \alpha_p, \beta_{p-c} \alpha_{q-a-a} \beta_{q-a}.$$ \hspace{1cm} (195)

using which symmetric($\sigma$) and antisymmetric($\pi$) combinations can again be formed. For each $a$, the total number of states ($\sigma + \pi$) are,

$$\begin{pmatrix} p \cr a \end{pmatrix}, \begin{pmatrix} q \cr a \end{pmatrix} = \sum_i \begin{pmatrix} r \cr a - c \end{pmatrix}, \begin{pmatrix} s \cr d \end{pmatrix}, \begin{pmatrix} s \cr N - a - d \end{pmatrix}.$$ \hspace{1cm} (196)

The number of symmetric and antisymmetric states are equal except when $c = \frac{q}{2}$ and $d = \frac{N-a}{2}$ which gives individual counts of symmetric and antisymmetric states as

$$\sigma_\alpha = \frac{1}{2} \left( \begin{pmatrix} p \cr a \end{pmatrix}, \begin{pmatrix} q \cr a \end{pmatrix} + \begin{pmatrix} r \cr a - c \end{pmatrix}, \begin{pmatrix} s \cr d \end{pmatrix} \right)$$
$$\pi_\alpha = \frac{1}{2} \left( \begin{pmatrix} p \cr a \end{pmatrix}, \begin{pmatrix} q \cr a \end{pmatrix} - \begin{pmatrix} r \cr a - c \end{pmatrix}, \begin{pmatrix} s \cr d \end{pmatrix} \right).$$ \hspace{1cm} (197)

This brings the total symmetric($C$) and antisymmetric($D$) states in the $Q$ block to be

$$C = \sum_{a=0,2,...} \sigma_\alpha, D = \sum_{a=0,2,...} \pi_\alpha.$$ \hspace{1cm} (198)

The structure of the Hamiltonian for the $O_\alpha$ states is similar to the All case with $(P, Q)$ of All replaced with $(A, B)$ respectively. Likewise, replacing $(P, Q)$ with $(C, D)$ gives us the Hamiltonian structure for $E_\alpha$ states(see table III). The number of independent parameters are

$$A = \frac{P}{2} + \frac{B(P+1)}{2} + \frac{B(B+1)}{2} + \frac{AB}{2} + \frac{C(C+1)}{2} + \frac{D(D+1)}{2} + \frac{CD}{2}.$$ \hspace{1cm} (199)

Class CII: In this class, $L = 2M$. Imposition of sublattice symmetry again breaks the basis states into two blocks $P$ and $Q$ with dimensions as seen in eqn. (193). It can be seen that $\alpha_\beta \beta_{N-a} \rightarrow (-1)^{a-N-a} \alpha_{N-a} \beta_{a}$. A state having $a$ as odd(even) belongs to the $P(Q)$ block. Therefore symmetric and antisymmetric states can again be constructed. Given $\frac{q}{2}$ is an even integer, then the number of symmetric and antisymmetric states which can be formed using states of the $P(Q)$ block will be equal(unequal). This makes the $P, Q$ Hamiltonian blocks take the structure of the Hamiltonian for the case CII with

$$A = \frac{P}{2}, C = \frac{1}{2} \left( Q + \frac{M}{N/2} \right)$$
$$B = \frac{P}{2}, D = \frac{1}{2} \left( Q - \frac{M}{N/2} \right).$$ \hspace{1cm} (200)

Where as, for $N/2$ odd the number of symmetric and antisymmetric states become unequal(equal) leading to the same Hamiltonian structure(also see table III as before but with

$$A = \frac{1}{2} \left( P - \frac{M}{N/2} \right), C = \frac{Q}{2}$$
$$B = \frac{1}{2} \left( P + \frac{M}{N/2} \right), D = \frac{Q}{2}.$$ \hspace{1cm} (201)

The expression for total number of independent parameters is same as Class CII.
Class DIII: The reasoning for DIII class is similar to CI with the crucial distinction that $\alpha_2\beta_{N/2} \to \alpha_{N/2}\beta_2$. This distinction readjusts the number of symmetric and antisymmetric states in $P, Q$ blocks. The similarities between the two classes gives us the same cases depending on the oddness and evenness of $N/2$, however now with minor changes in the formulae for dimensions of the $A-D$ sub-blocks of $P, Q$, which now become

$$A = \begin{cases} \frac{P}{2} + \left(\frac{M}{N/2}\right) & N/2 \text{ even} \\ \frac{P}{2} - \left(\frac{M}{N/2}\right) & N/2 \text{ odd} \end{cases}$$

$$B = \begin{cases} \frac{P}{2} + \left(\frac{M}{N/2}\right) & N/2 \text{ even} \\ \frac{P}{2} - \left(\frac{M}{N/2}\right) & N/2 \text{ odd} \end{cases}$$

$$C = \begin{cases} \frac{Q}{2} + \left(\frac{M}{N/2}\right) & N/2 \text{ even} \\ \frac{Q}{2} - \left(\frac{M}{N/2}\right) & N/2 \text{ odd} \end{cases}$$

The structure of Hamiltonian remains same as discussed for class CI with the expression for total number of independent parameters given by eqn. (199). The generic structure of $H(N)$ for $N$-even is summarized and tabulated in table III.

K. Concluding Remarks

In this paper we have revisited the tenfold scheme of classification of fermions with the aim of studying interacting systems. We have endeavored to provide a simple and direct approach that makes clear the underlying physical content even while not being tied to the single particle picture. The canonical representation of symmetries in each of the ten classes (see table I) not only allows to obtain the structures of Hamiltonians in each class, but also provides some crucial physical insights into the nature of the symmetry operations. For example, our discussion of the symmetry operations in type 3 classes reveals a physical view of the interplay between the sublattice symmetry with other symmetries. For example, in the chiral classes with $T = C$ (BDI, CII) both time reversal and charge conjugation operation leaves the sublattice flavor unchanged, while for $T = -C$ (DIII, CI) the time reversal and charge conjugation operation flips the sublattice flavor. Furthermore, the results of our group cohomological considerations (section F) brings in crucial insights into the results of table I. From the perspective of interacting fermions, tables III and IV contain the key results on the structures of the $N$-body Hamiltonians in each class. We believe that the results will be useful to construct models for interacting systems in various classes, studies of which could be used to develop deeper understanding and phenomenology that can aid reaching the ultimate goal – the topological classification of interacting fermionic systems. Finally, not the least, our analysis provides a natural way to reveal the geometric structure of the time evolution operator (see eqns. (9)) in systems with $N$-body interactions. It is clear that Cartan’s symmetric spaces make their appearance again.

We conclude the paper by pointing out another very active area of condensed matter physics where our results would be of value – many body localization and thermalization. Our results can be used, again, to create models in any symmetry class with arbitrary disorder in the kinetic energy (quadratic Hamiltonian) or any $N$-body interaction. In fact, tables III and IV can be used to generate random matrix ensembles that can create models with different physical content which can be used to investigate outstanding issues in that area.

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Throughout the paper a bold roman symbol (e.g. $R$) is used to denote a matrix, and the light symbol with indices shown (e.g. $R_{ij}$) will denote its components.

This consideration can be easily generalized to superconducting Bogoliubov-de Gennes (BdG) Hamiltonians which are quadratic in fermion operators by expanding the operators eqn. (4) to the Nambu representation, and treating the problem as a quadratic fermion problem. In this case, $I_{\theta}$ is restricted to $\theta = 0, \pi$.

This new basis is not unique. In fact, existence of such a basis implies that any other basis related by a real orthogonal matrix will also be an equally valid one.

For the clarity of the equations, we will not explicitly put the factors of $\frac{1}{\sqrt{2}}$ in front of symmetric and antisymmetric states. It should be assumed that all the states are properly normalized.
1. Appendix: List of symbols

\(N\) Hamiltonian contains up to \(N\)-body interaction terms. First appearance in sec \(A\) page 1

\(L\) No. of orbitals. First appearance in sec \(A\) page 2

\(|i\rangle\) One-particle state. First appearance in sec \(B\) page 2

\(|0\rangle\) Vacuum state. First appearance in sec \(B\) page 2

\(\psi_i, \psi_i^\dagger\) Annihilation and creation operators for the \(i\)-th orbital. First appearance in sec \(B\) page 2

\(\Psi\) Row matrix of \(\psi_i\). First appearance in eqn. (4), page 2

\(\Psi^\dagger\) Column matrix of \(\psi_i^\dagger\). First appearance in eqn. (4), page 2

\(R\) Matrix for basis transformation of \(\Psi\). First appearance in sec \(B\) page 2

\(\Psi^B\) Basis transformed \(\Psi\). First appearance in eqn. (5), page 2

\(N_P\) Number of particles. First appearance in sec \(B\) page 3

\(\mathcal{V}_{N_P}\) Vector space of \(N_P\)-particle states. First appearance in sec \(B\) page 3

\(\mathcal{V}\) Hilbert-Fock space of the system. First appearance in eqn. (7), page 3

\(\mathcal{H}\) The Hamiltonian operator. First appearance in eqn. (8), page 3

\(\mathcal{H}^{(K)}\) Matrix for \(K\)-body interaction Hamiltonian. First appearance in eqn. (8), page 3

\(\mathcal{H}^{(K)}_{i_1j_1...i_kj_k}\) Matrix elements of \(\mathcal{H}^{(K)}\). First appearance in eqn. (9), page 3

\(\mathcal{H}^{(K)}\) Vector space of \(N\)-body interaction Hamiltonians. First appearance in sec \(B\) page 3

\(i\) \(i = \sqrt{-1}\). First appearance in sec \(B\) page 3

\(\mathfrak{u}(\cdot)\) Lie algebra of unitary group. First appearance in eqn. (12), page 3

\(U(\cdot)\) Unitary Lie group. First appearance in sec \(B\) page 3

\(\mathbf{H} = (\mathbf{H}^{(0)}, \ldots, \mathbf{H}^{(N)}) (N+1)\)-tuple Hamiltonian “vector”. First appearance in eqn. (13), page 3

\(U\) Symmetry operator. First appearance in sec \(C\) page 3

\(U_{\text{USL}}\) Symmetry operator for usual(USL) symmetries. First appearance in eqn. (16), page 4

\(U_{\text{USL}}\) Matrix associated with \(U_{\text{USL}}\). First appearance in eqn. (17), page 4

\(I\) Identity operator on \(\mathcal{V}\). First appearance in sec \(C\) page 4

\(U_{\text{TRN}}\) Symmetry operator for transposing(TRN) symmetries. First appearance in eqn. (18), page 4

\(|\Omega\rangle\) Fully filled state. First appearance in sec \(C\) page 4

\(U_{\text{TRN}}\) Matrix associated with \(U_{\text{TRN}}\). First appearance in eqn. (19), page 4

UL,UA,TL,TA Usual Linear, Usual Antilinear, Transposing Linear and Transposing Antilinear type of symmetry operations. First appearance in sec \(C\) page 4

\(:\) Normal ordering operation. First appearance in eqn. (21), page 4

\(\hat{H}\) Symmetry transformed \(H\). First appearance in eqn. (22), page 4

\(\hat{\mathbf{H}}^{(K)}\) Intermediate quantity generated when \(\mathbf{H}\) maps to \(\hat{\mathbf{H}}\) under symmetry operation. First appearance in sec \(C\) page 4
| Symbol | Description |
|--------|-------------|
| **tr** | Tracing out operation of \( P \) indices. Defined in eqn. (26), page 5 |
| **A_{R,K}** | Combinatorial factor generated when tracing over \( R − K \) indices of \( \hat{H}^{(R)} \). Defined in eqn. (27), page 5 |
| **G_{V}** | Group of all symmetry operations of \( V \). First appearance in sec C 3, page 5 |
| **G^{UL}_{V}, G^{UA}_{V}, G^{TL}_{V}, G^{TA}_{V}** | Set of all UL, UA, TL, TA type symmetry operations. First appearance in eqn. (29), page 5 |
| **K_{4}** | Klein 4-group. First appearance in sec C 3, page 5. Defined in eqn. (78), page 9 |
| **\(G_{V}^{UL}, G_{V}^{UA}, G_{V}^{TL}, G_{V}^{TA}\)** | Set of all UL, UA, TL, TA type symmetry operations. First appearance in eqn. (29), page 5 |
| **\(J_{0} e^{i \theta_{N}}\)** | Defined in eqn. (32), page 5 |
| **\(N\)** | Number operator. Defined in eqn. (33), page 5 |
| **\(T, C, S\)** | Time-reversal, charge-conjugation and sublattice operators. First appearance in sec D, page 6 |
| **\(U_{T}\)** | Unitary matrices associated with \( T \) operator. Defined in eqn. (40), page 7 |
| **\(U_{C}\)** | Unitary matrices associated with \( C \) operator. Defined in eqn. (41), page 7 |
| **\(U_{S}\)** | Unitary matrices associated with \( S \) operator. Defined in eqn. (42), page 7 |
| **\(1\)** | \( L \times L \) unit matrix. First appearance in eqn. (44), page 7 |
| **\(\tilde{U}_{T}\)** | Basis transformed \( U_{T} \). Defined in eqn. (44), page 7 |
| **\(M\)** | \( M = L/2 \), when \( L \) is even. First appearance in eqn. (46), page 7 |
| **\(J\)** | Matrix of dimension \( L \) of the form \[
\begin{pmatrix}
0_{MM} & 1_{MM} \\
-1_{MM} & 0_{MM}
\end{pmatrix}
\] Defined in eqn. (46), page 7 |
| **\(K\)** | Complex conjugation operation. Defined in eqn. (48), page 7 |
| **\(p, q\)** | Labels for indices of sub-blocks of \( U \) of various symmetry operators. Also indicates the label for states making up the blocks, together they form \( L \) states. First appearance in eqn. (64), page 8 |
| **\(r, s\)** | Half of \( p, q \) when they are even. First appearance in sec E 5, page 9 |
| **\(F\)** | Matrix of dimension \( L \) of the form \[
\begin{pmatrix}
0_{MM} & 1_{MM} \\
1_{MM} & 0_{MM}
\end{pmatrix}
\] Defined in eqn. (74), page 9 |
| **\(I, \Theta, \Xi, \Sigma\)** | Group elements of \( K_{4} \) denoting identity, time reversal, charge conjugation and sublattice symmetry operations. First appearance in eqn. (78), page 9 |
| **\(G\)** | Abstract symmetry group of GFS. First appearance in sec F, page 9 |
| **\(Z_{2}^{I}\)** | \( Z_{2}^{I} = \{I, \Theta\} \), sub-group of \( K_{4} \). First appearance in sec F, page 9 |
| **\(Z_{2}^{X}\)** | \( Z_{2}^{X} = \{I, \Xi\} \), sub-group of \( K_{4} \). First appearance in sec F, page 9 |
| **\(Z_{2}^{\Sigma}\)** | \( Z_{2}^{\Sigma} = \{I, \Sigma\} \), sub-group of \( K_{4} \). First appearance in sec F, page 9 |
| **\(D(\cdot)\)** | \( \mathbb{C} \) valued matrices representing elements of \( G \) group. Defined in eqn. (79), page 10 |
| **\(g_{\ell}\)** | Group element of \( G \). First appearance in sec F, page 10 |
| **\(\omega(g_{1}, g_{2})\)** | 2-cocycles or Schur multipliers. First appearance in sec F, page 10 |
| **\(\omega^{g}\)** | Short form for action of the group element \( g \) on an element of \( U(1) \). First appearance in eqn. (81), page 10 |
| **\(\{[\omega]\}\)** | Set of equivalence classes of Schur multipliers. First appearance in sec F, page 10 |
| **\(H_{2}^{G}(U(1), G)\)** | The second cohomology group. First appearance in sec F, page 10 |
$J_2$ 2 × 2 version of $J$. Defined in eqn. (87), page 11.

$F_2$ 2 × 2 version of $F$. First appearance in sec F 3, page 12.

$U_{\text{Schroed}}(t)$ Schödinger time evolution operator. Defined in eqn. (91), page 12.

$\mathfrak{o}(\cdot)$ Lie algebra of the group of orthogonal matrices. First appearance in sec G, page 12.

$O(\cdot)$ Group of orthogonal matrices. First appearance in sec G, page 12.

$\mathfrak{usp}(\cdot)$ Lie algebra of the group of symplectic matrices. First appearance in sec G, page 12.

$USp(\cdot)$ Symplectic group. First appearance in sec G, page 12.

$H^p_K, H^q_K$ Decompositions of $H^K$. Defined in eqn. (108), page 14.

$\alpha, \beta$ Labels the $p, q$ type of states respectively. First appearance in sec I, page 15.

$\sigma, \pi$ “symmetric” and “antisymmetric” states. Defined in eqn. (123), page 18.

$E_{\alpha}, E_{\alpha_p}, E_{\alpha_q}$ A state made of even number of $\alpha, \alpha_p, \alpha_q$ states respectively. First appearance in sec J 1, page 21.

$O_{\alpha}, O_{\alpha_p}, O_{\alpha_q}$ A state made of odd number of $\alpha, \alpha_p, \alpha_q$ states respectively. First appearance in sec J 1, page 21.

$P, Q, A, B, C, D$ Dimensions of various matrix blocks making up the parent many-body Hamiltonian matrix.

$J^{(N)}$ $N$-body version of $J$ defined in eqn. (46). First appearance in sec J 1, page 21.

$F^{(N)}$ $N$-body version of $F$ defined in eqn. (74). First appearance in sec J 1, page 22.