Self-consistent theory of compact QED$_3$ with relativistic fermions

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Abstract

We consider three dimensional quantum electrodynamics (cQED$_3$) with massless relativistic fermions coupled to a compact gauge field using a combined perturbative variational approach. Coupling to matter renders the bare interaction between magnetic monopoles logarithmic at large distances, suggesting the possibility of a confinement-deconfinement transition of the Berezinskii-Kosterlitz-Thouless type in the theory. Our self-consistent calculation suggests, however, that screening effects always destabilise the confined phase, in agreement with the previous renormalisation group study of the same model.
I. INTRODUCTION

Compact U(1) gauge theories in three \((d = 3)\) dimensions have long been of interest in high energy and condensed matter physics. In particle physics they serve as relatively simple models exhibiting non-perturbative phenomena such as chiral symmetry breaking and confinement \([1, 2, 3]\), believed to be crucial to our understanding of more realistic theories like quantum chromodynamics. In condensed matter physics the theories with compact U(1) gauge fields coupled to matter arise frequently in descriptions of strongly correlated electron systems \([4]\). In this case the three dimensional models are of direct significance to condensed matter systems in two \((d = 2)\) spatial dimensions and at zero temperature \((T = 0)\).

A crucial issue in all compact U(1) theories is the confinement of ‘charge’ due to the unbinding of magnetic monopoles, which are invariably introduced by the compact nature of the gauge field. In a pioneering work, Polyakov \([1]\) showed that in pure compact quantum electrodynamics without matter in \(d = 3\) confinement is permanent for all values of the gauge coupling. The situation where the gauge field is coupled to matter is more subtle, and a subject of current debate. It has been argued that coupling to relativistic massless fermions transforms the usual Coulombic interaction between monopoles into the much longer-ranged logarithmic interaction at large distances \([5, 6, 7, 8]\). When applied to a single monopole-antimonopole pair, this would suggest that monopoles may bind into dipoles, in analogy with the celebrated Berezinskii \([9]\), Kosterlitz and Thouless \([10]\) (BKT) transition in two dimensions. However, while the effects of a finite density of monopoles on the BKT transition in \(d = 2\) are well understood \([10, 11]\), the situation in \(d = 3\) appears less clear \([12]\). The difficulty lies in the fact that while the screening in the dipole phase in \(d = 2\) just amounts to renormalisation of the dielectric constant, in \(d = 3\) it changes the form of the interaction \([13, 14, 15, 16]\). In a recent paper, two of us \([15]\) presented an electrostatic argument and a renormalisation group calculation to show that the interaction between distant monopoles in the presence of other dipoles is screened back into the Coulomb potential. Together with the generalisation to the case of coupling to non-relativistic fermions \([16]\), this strongly suggests that the putative deconfined phase in \(d = 3\) is always unstable. Compact U(1) theories in \(d = 3\), with or without matter, would appear therefore generically to be permanently confining.

In the present article we study the issue of confinement in cQED\(_3\) using the variational
treatment of the anomalous sine-Gordon (ASG) theory, which is dual to the original cQED. By working to the second order in fugacity and including the screening effects we find that monopoles are free at any effective temperature in the ASG theory (i.e. for any number of fermion flavours in cQED). This suggests that fermions are permanently in the confined phase, and provides an additional support to the renormalisation group results of Refs. [15] and [16].

We introduce the cQED and its dual sine-Gordon version in Section II. In Section III we discuss the lowest order variational calculation that neglects screening and point to its limitations. We then propose a generalised self-consistent approach that includes higher orders in monopole fugacity and allows for the screening effects in Section IV. In Section V we present the calculations to the second order. A summary of our results is given in Section VI.

II. cQED AND THE ANOMALOUS SINE-GORDON THEORY

We will be interested in the phases of cQED, with the gauge field coupled to massless relativistic fermions on a lattice:

\[ S[\chi, a] = S_F[\chi, a] - \frac{1}{2e_0} \sum_{x,\mu,\nu} \cos (F_{\mu\nu}(x)). \]  

(1)

The sites of the three dimensional quadratic lattice are labeled by \( x_\mu = \{x_1, x_2, \tau\} \). Here, \( F_{\mu\nu} \) is the usual field-strength tensor \( F_{\mu\nu} = \Delta_\mu a_\nu - \Delta_\nu a_\mu \); the lattice derivative is defined by \( \Delta_\mu a_\nu(x) \equiv a_\nu(x + \hat{\mu}) - a_\nu(x) \). \( S_F \) is the lattice action of massless fermions coupled to the gauge field which reduces in the continuum limit to QED with \( N_f \) flavours of four-component Dirac spinors. Using staggered fermions, this takes the form

\[ S_F[\chi, a] = \frac{1}{2} \sum_{x,\mu,n=1}^{N_f/2} \eta_n(x) \left[ \bar{\chi}_n(x)e^{ia_\mu(x)}\chi_n(x + \hat{\mu}) - \bar{\chi}_n(x + \hat{\mu})e^{-ia_\mu(x)}\chi_n(x) \right] \]

(2)

where \( \eta_1 = 1 \), \( \eta_2 = (-1)^{x_1} \) and \( \eta_3 = (-1)^{x_1+x_2} \).

In the case of continuum QED, the fermion polarisation to one-loop order is

\[ \Pi_{\mu\nu}(p) = \frac{N_f}{16} p \left\{ \delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right\}. \]

(3)

Incorporating compactness of \( a_\mu \) in the spirit of Villain approximation, this suggests
that we consider a theory closely related to cQED$_3$
\[
S[a] = \frac{1}{2} \sum_{x,\mu,\nu} \left\{ (F_{\mu\nu}(x) - 2\pi n_{\mu\nu}(x)) \left( \frac{1}{2} e_0^2 + \frac{N_f}{16|\Delta|} \right) (F_{\mu\nu}(x) - 2\pi n_{\mu\nu}(x)) \right\},
\]
where the $n_{\mu\nu}$ are integers. The action (4) has the same continuum limit as cQED$_3$ to the leading order in large $N_f$ and may be understood as a compact quadratic approximation to it. In the remainder of the article we assume that the original cQED$_3$ and the theory (4) are in the same universality class.

In the presence of fermions, when $N_f \neq 0$, the original Maxwell term proportional to $1/e_0^2$ becomes irrelevant at large distances, and can be neglected with respect to the second term in Eqn. (4). This action can be then be put into the alternative form (see Appendix A)
\[
Z = \sum \exp \left\{ -\frac{\pi^2 N_f}{4} \sum_{a,b} q_a q_b V(x_a - x_b) \right\}.
\]
This is the partition function for a gas of monopoles of charge $q_a = \pm 1$, interacting with a potential $V(x)$. In our case, the potential has the form $V(k) = 1/|k|^3$ in Fourier space, which is the logarithmic interaction in three dimensions.

The problem now appears to be rather similar to the two dimensional Coulomb gas, where the logarithmic interaction may result in the BKT vortex-antivortex binding transition. The mechanism of such a transition stems from a simple energy-entropy competition, as both entropy and the interaction energy are proportional to the logarithm of linear dimension of the system: at low enough temperatures, it is energetically favourable for opposite vortex charges to form bound pairs, while as temperature is increased, entropy at some point takes over, and vortex-antivortex pairs unbind. Although the form of the partition function (5) resembles that of the two dimensional Coulomb gas, it is by no means guaranteed that such a scenario will still hold in three dimensions. In particular, the effect of screening of other dipoles on the potential felt by two widely-separated monopoles, which is neglected in this na"ive energy-entropy argument, can drastically affect the result.

To systematically address this issue, we first note that Eq. (5) is equivalent to the partition function with the anomalous sine-Gordon (ASG) action (see Appendix A)
\[
S_{\text{ASG}}[\phi] = \int d^3r \left\{ -\frac{T}{2} \phi \nabla^3 \phi - 2y \cos \phi \right\},
\]
where the fictitious temperature is $T \equiv 2/(\pi^2 N_f)$, and $y$ is the fugacity of the monopoles. The non-analytic gradient term proportional to $|q|^3$ is a consequence of the coupling of relativistic massless fermions to the gauge fields.
It is possible to construct an upper bound for $F_{\text{ASG}}$, the free energy associated with the action (6), using the Gibbs-Bogoliubov-Feynman (GBF) inequality, which is discussed in the next section. We will argue that this self-consistent mean-field approximation to the free energy of the system unfortunately misses the screening effects of the medium, and consequently incorrectly suggests the BKT transition. An improved calculation that incorporates such effects is then formulated in the following section.

III. VARIATIONAL APPROACH

The GBF inequality imposes a strict upper bound on the free energy $F_{\text{ASG}}$ through the relation

$$F_{\text{ASG}} \leq F_{\text{var}} \equiv F_0 + \langle S_{\text{ASG}} - S_0 \rangle_0,$$

(7)

where $S_{\text{ASG}}$ is defined in Eqn. (6) and $S_0$ is a trial action chosen to approximate $S_{\text{ASG}}$; $F_0$ is the free energy associated with $S_0$ and $\langle \ldots \rangle_0$ represents averaging within this ensemble. The trial action may be chosen to have the Gaussian form

$$S_0[\phi] = \frac{1}{V} \sum_q \frac{1}{2} \phi(q) G_0^{-1}(q) \phi(-q),$$

(8)

so it becomes particularly simple to calculate $F_{\text{var}}$:

$$\frac{F_{\text{var}}}{V} = -\frac{1}{2} \int \frac{d^3q}{(2\pi)^3} \ln (G_0(q)) + \frac{T}{2} \int \frac{d^3q}{(2\pi)^3} |q|^3 G_0(q) - 2y \exp \left\{ -\frac{1}{2} \int \frac{d^3q}{(2\pi)^3} G_0(q) \right\}. \quad (9)$$

Minimising $F_{\text{var}}$ with respect to $G_0(q)$ yields the optimal Gaussian theory that approximates $F_{\text{ASG}}$: \[ \frac{\delta F_{\text{var}}}{\delta G_0} = 0 \implies G_0^{-1}(q) = T|q|^3 + \sigma, \quad (10) \]

with the ‘mass’ $\sigma$ determined self-consistently through

$$\sigma = 2y \exp \left\{ -\frac{1}{2} \int \frac{d^3q}{(2\pi)^3} \frac{1}{T|q|^3 + \sigma} \right\} = 2y \left( 1 + \frac{T\Lambda^3}{\sigma} \right)^{-\frac{T}{\sigma}}. \quad (11)$$

$\Lambda$ is the ultraviolet cutoff and $T_c \equiv 1/(12\pi^2)$. Determining the solutions of Eqn. (11) amounts to identifying the roots of the function

$$f(\sigma) = \sigma - 2y \left( 1 + \frac{T\Lambda^3}{\sigma} \right)^{-\frac{T}{\sigma}}. \quad (12)$$
It is evident that $\sigma = 0$ is one such root for all values of $T$. We next demonstrate that a solution with finite $\sigma$ exists for $T > T_c$. In the limit of small $\sigma$, $f(\sigma)$ has the form

$$f(\sigma \ll \Lambda^3) = \begin{cases} \sigma, & T < T_c \\ -\sigma T_c/T, & T > T_c \end{cases}$$

while for large $\sigma$

$$f(\sigma \gg \Lambda^3) = \sigma, \quad \forall T.$$  \hspace{1cm} (14)

For $T > T_c$, $f(\sigma)$ changes sign and thus has a root with $\sigma > 0$, while only the $\sigma = 0$ solution exists for $T < T_c$ [21].

The stability of the $\sigma = 0$ solution for $T > T_c$ can be determined from the variational free energy (9) with the solution (10) for $G_{0}^{−1}$. Evaluating the free energy we get

$$\frac{F_{\text{var}}(\sigma)}{V} = T_c \Lambda^3 \ln \left(\sigma + T \Lambda^3\right) - 2y \left(1 + \frac{T \Lambda^3}{\sigma}\right)^{-T_c/T}.$$  \hspace{1cm} (15)

Then

$$\frac{1}{V} (F_{\text{var}}(\sigma) - F_{\text{var}}(0)) = \frac{\sigma (T_c - T)}{T} + O(\sigma^2),$$  \hspace{1cm} (16)

so that for $T > T_c$ any solution with $\sigma > 0$ is of lower free energy than with $\sigma = 0$. That is, the stable solution at $T > T_c$ has finite $\sigma$.

To understand the physical meaning of the non-trivial solution it is useful to calculate the monopole density from the variational free energy [9]:

$$\rho_M = \frac{1}{V} \frac{\partial F_{\text{var}}}{\partial \mu} = \frac{y}{V} \frac{\partial F_{\text{var}}}{\partial y} = \sigma,$$  \hspace{1cm} (17)

where we have used the definition of fugacity $y \equiv \exp \{\mu\}$. We see that $\sigma$ is exactly the monopole density $\rho_M$, so that $\sigma \neq 0$ may be identified with the plasma phase of free monopoles, while $\sigma = 0$ indicates the dipole phase. The simple variational calculation would therefore suggest that monopoles undergo a binding-unbinding transition at $T = T_c$ (i. e. at $N = N_c = 24$) in exact analogy with the equivalent calculation one can perform for the standard BKT transition. The value of $T_c$ also agrees with the simple energy-entropy argument that can be constructed for an isolated vortex [15].

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An obvious objection to this simple calculation is that minimisation of the variational free energy (9) by construction cannot yield any momentum dependence of the self-energy, but can only determine its constant part, the ‘mass’ $\sigma$. The renormalisation group \cite{15} treatment of the ASG theory suffers from the same problem to the lowest order in fugacity, and would likewise naïvely suggest the BKT transition. The same holds for the direct perturbative evaluation of the self-energy in the ASG. However, it is easy to check that the self-energy does become momentum dependent to the second order in fugacity, with the leading analytic term $\sim q^2$ at low momenta. This is just what one would expect based on the simple electrostatic analysis of the problem \cite{15}, where this term translates into the Coulombic interaction in real space (when $y = 0$). The presence of such a term would, however, drastically alter our present considerations. Indeed, if we add by hand the term $Qq^2$ with $Q \neq 0$ in the denominator of the integrand in the self-consistent equation (11), we find

$$f(\sigma) = \begin{cases} -2y \left(1 + \frac{\Lambda T}{Q}\right)^{-3T_c/T}, & \sigma \ll \Lambda^3 \\ \sigma, & \sigma \gg \Lambda^3 \end{cases}$$

for all $T$. Hence, the non-trivial solution would exist for all temperatures, exactly as in Polyakov’s original treatment of the pure gauge theory. This is natural since $Q \neq 0$ means that the original logarithmic interaction between monopoles is, even without free monopoles and only with a finite density of dipoles, screened into the Coulomb interaction for which the standard argument for the confined phase readily applies.

In the next section we propose a modified self-consistent calculation which provides a systematic perturbative approximation to the free energy and which reduces to the GBF method to the lowest order. As we will see in Section V such an approach has the advantage of including the screening effects in a self-consistent way, therefore overcoming the limitations of the purely variational theory discussed in this section.

IV. SELF-CONSISTENT PERTURBATIVE APPROACH

There are many ways in which one may generalise the variational method of the previous section. For instance, one may add a second-order term $-\frac{1}{2}(S_{\text{ASG}} - S_0)^2 + \frac{1}{2}(S_{\text{ASG}} - S_0)^2$ to $F_{\text{var}}$ and extremise the new energy functional \cite{19}. Such a second-order extension, however, has little variational justification. For a more systematic generalisation, we go back to the
GBF inequality (7) and exchange $S_{\text{ASG}}$ with $S_0$ to find

$$F_\prec \equiv F_0 + \langle S_{\text{ASG}} - S_0 \rangle \leq F_{\text{ASG}}.$$  \hspace{1cm} (19)

Extremising $F_\prec$ with respect to a quadratic action $S_0$ yields

$$\langle \phi(-q)\phi(q) \rangle_0 = \langle \phi(-q)\phi(q) \rangle,$$  \hspace{1cm} (20)

which is nothing but the equation for the exact propagator in the ASG theory. The right hand side (RHS) of the equation, on the other hand may be rewritten as

$$\langle \phi(-q)\phi(q) \rangle \equiv \frac{\langle \phi(-q)\phi(q)e^{-\Delta S} \rangle_0}{\langle e^{-\Delta S} \rangle_0},$$  \hspace{1cm} (21)

with $\Delta S \equiv S_{\text{ASG}} - S_0$. Eqn. (18) in this form may be understood as a self-consistent equation for the action $S_0$, which we may attempt to solve by expanding the RHS in powers of $\Delta S$, for example. To the first order in $\Delta S$ this becomes

$$\langle \phi(-q)\phi(q)\Delta S \rangle_0 - \langle \phi(-q)\phi(q) \rangle_0 \langle \Delta S \rangle_0 = 0,$$  \hspace{1cm} (22)

which is precisely the relation one would obtain from extremising $F_{\text{var}}$ with respect to $S_0$. That is, the first order approximation to Eqn. (20) reproduces the GBF result from the previous section.

Eqn. (20) forms the basis of our modified variational approximation to $F_{\text{ASG}}$. To the first order in $\Delta S$ it reduces to the GBF equation of the previous section, and when solved self-consistently to all orders gives the best variational lower bound to the free energy, provided by $F_\prec$ in (19). In addition, consider the expansion of Eqn. (20) to order $(\Delta S)^n$. One can show (see Appendix B2) that the resulting expression is the same as the one that would arise from extremising the function

$$F_{\text{var}}^{(n)} = \frac{F^{(1)} + F^{(2)} + \cdots + F^{(n)}}{n}.$$  \hspace{1cm} (23)

Here $F^{(n)}$ stands for the expansion of the true free energy of the system, $F_{\text{ASG}}$, in powers of $\Delta S$, truncated at $(\Delta S)^n$. Similarly denoting by $F_{\prec}^{(n)}$ the truncated expansion of $F_\prec$ in Eqn. (19), it is not difficult (see Appendix B1) to show

$$F_{\text{var}}^{(n)} = F^{(n)} + \frac{F^{(n)} - F_{\prec}^{(n)}}{n}.$$  \hspace{1cm} (24)
It is then clear that the sequence \( \{ F^{(n)}_{\text{var}} \} \) converges to \( F_{\text{ASG}} \) for any \( S_0 \). Therefore, the \( S_0 \) determined self-consistently from Eqn. (20) also yields the variational sequence that best approximates \( F_{\text{ASG}} \) within the family \( \{ F^{(n)}_{\text{var}}[S_0] \} \).

To the second order Eqn. (20) reads

\[
\langle \phi(-q)\phi(q)\Delta S \rangle_0^c - \frac{1}{2}\langle \phi(-q)\phi(q)(\Delta S)^2 \rangle_0^c = 0, \tag{25}
\]

where both terms are connected averages given by:

\[
\langle \phi(-q)\phi(q)\Delta S \rangle_0^c \equiv \langle \phi(-q)\phi(q)\Delta S \rangle_0 - \langle \phi(-q)\phi(q) \rangle_0 \langle \Delta S \rangle_0,
\]

\[
\langle \phi(-q)\phi(q)(\Delta S)^2 \rangle_0^c \equiv \langle \phi(-q)\phi(q)(\Delta S)^2 \rangle_0 - \langle \phi(-q)\phi(q) \rangle_0 \langle (\Delta S)^2 \rangle_0 - 2\langle \phi(-q)\phi(q)\Delta S \rangle_0 \langle \Delta S \rangle_0 + 2\langle \phi(-q)\phi(q) \rangle_0 \langle \Delta S \rangle_0^2. \tag{27}
\]

We discuss the results of the second-order self-consistent Eqn. (25) for the ASG model (6) in the next section. In particular, we will show that the density of free monopoles is finite at all \( T > 0 \), and that charge should consequently be permanently confined in cQED_3.

V. CONFINING SOLUTION FOR \( T > 0 \)

From the definitions of \( S_{\text{ASG}} \) and \( S_0 \) (Eqns. 6 and 8) it is straightforward to calculate the connected averages of Eqns. (26) and (27). Our second order equation (25) then yields the quadratic equation for \( G^{-1}_0(q) \)

\[
\left[ G^{-1}_0(q) \right]^2 - A[q, G_0]G^{-1}_0(q) + B[k, G_0] = 0, \tag{28}
\]

where

\[
A[q, G_0] = \frac{3}{2}T|q|^3 + 3a + ab - 2a^2 \left( c + \sum_{n=0}^{\infty} (-1)^n d_n q^{2n} \right), \tag{29}
\]

\[
B[q, G_0] = \frac{1}{2}T^2 q^6 + 2aT|q|^3. \tag{30}
\]

In Eqns. (29, 30), we have defined

\[
a = ye^{-\frac{1}{2}D_0(0)}, \tag{31}
\]

\[
b = \int \frac{d^3k}{(2\pi)^3} \left( \frac{T}{2}|k|^3 - \frac{1}{2}G^{-1}_0(k) \right) [G_0(k)]^2, \tag{32}
\]

\[
c = \int d^3R \left[ 1 - \cosh D_0(R) \right], \tag{33}
\]

\[
d_n = \int d^3R \frac{(R \cos \theta)^{2n}}{(2n)!} \sinh D_0(R), \tag{34}
\]
and the real-space propagator is \( D_0(R) = \int \frac{d^3k}{(2\pi)^3} G_0(k)e^{ikR} \).

We can solve the quadratic of Eqn. (28) and expand in powers of \(|q|^3/A_0\) to yield the result for \( G_0^{-1}(q) \):

\[
G_0^{-1}(q) = m + Q(m)q^2 + \tilde{T}(m)|q|^3 + \cdots
\]

(35)

where the coefficients are defined as

\[
m = \frac{1}{2} \{ A_0 \pm |A_0| \},
\]

(36)

\[
Q(m) = a^2 d_1 \left( 1 \pm \frac{|A_0|}{A_0} \right),
\]

(37)

\[
\tilde{T}(m) = \frac{3}{4} T \pm \frac{|A_0|}{A_0} \left( \frac{3}{4} T - \frac{2aT}{A_0} \right);
\]

(38)

and with \( A_0 \equiv A[q=0, G_0] \). For these equations, we should choose the solution corresponding to the upper sign in Eqns. (36–38) to ensure that \( m \geq 0 \). In what follows, we neglect terms higher order in \( q \) than \( q^3 \) as they should be irrelevant at low momenta.

As announced, the second order result includes additional renormalisation of the bare terms as well as the generation of new momentum dependent terms. Most importantly, the leading term proportional to \( q^2 \) has now appeared.

In the analysis in Section III we found that the bound phase of monopoles corresponded to low \( T \). In what follows we will restrict ourselves to low temperatures by assuming \( T\Lambda \ll Q \) and show that monopoles are unbound even for arbitrarily small temperatures. By continuity this would imply that they are free at all temperatures.

Let us start by examining \( a \):

\[
a = y \exp \left\{ -\frac{1}{2} D_0(0) \right\}
\]

\[
\approx y \exp \left\{ -\frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \frac{1}{(Q(m)k^2 + m)} \right\}
\]

\[
= y \exp \left\{ -\frac{1}{4\pi^2 Q(m)} \left( \Lambda - \sqrt{\frac{m}{Q(m)}} \arctan \left( \Lambda \sqrt{\frac{Q(m)}{m}} \right) \right) \right\},
\]

(39)

When \( m \to 0 \), we will assume \( m/Q(m) \to 0 \), and justify this assumption \textit{a posteriori}. The coefficient \( a \) now takes the form

\[
a = y \exp \left\{ -\frac{\Lambda}{4\pi^2 Q(m)} \right\}, \quad m \ll \Lambda^3 + O(T).
\]

(40)
Next, we examine the equation for $b$

$$b = \int \frac{d^3k}{(2\pi)^3} \left( \frac{T}{2} |k|^3 - \frac{1}{2} G_0^{-1}(k) \right) [G_0(k)]^2$$

$$= -\frac{1}{2} D_0(0) + O(T). \quad (41)$$

From this we find

$$b = -\frac{\Lambda}{4\pi^2 Q(m)}, \quad m \ll \Lambda^3 + O(T). \quad (42)$$

Next, as the terms $c$ and $d_0$ always appear together, we consider the combination

$$(c + d_0) \approx \int d^3R \left( 1 - \exp \left\{ -\int \frac{d^3k}{(2\pi)^3} \frac{e^{ik \cdot R}}{(Q(m)k^2 + m)} \right\} \right) + O(T)$$

$$= \int_0^\infty dR \frac{Re^{-\sqrt{Q(m)}R}}{Q(m)} + O(T). \quad (43)$$

Evaluating this yields

$$(c + d_0) = m^{-1}, \quad m \ll \Lambda^3 + O(T). \quad (44)$$

Similar analysis applies to the coefficient $d_1$:

$$d_1 = \frac{1}{6} \int_0^\infty dR \frac{\sqrt{2} R e^{-\sqrt{Q(m)}R}}{Q(m)} + O(T), \quad (45)$$

which gives

$$d_1 = \frac{Q(m)}{m^2}, \quad m \ll \Lambda^3 + O(T). \quad (46)$$

Evaluating the Eqn. (37) for $Q$ then we find

$$Q = 2a^2 d_1$$

$$= 2y^2 \exp \left\{ -\frac{\Lambda}{2\pi^2 Q} \right\} \frac{Q}{m^2} + O(T). \quad (47)$$

Solving this for $Q \neq 0$ yields

$$Q = \frac{\Lambda}{4\pi^2} \left( \ln \frac{\sqrt{2}y}{m} \right)^{-1} + O(T), \quad (48)$$

and we see that $m/Q(m)$ indeed approaches zero as $m \to 0$, thus justifying our earlier assumption. Substituting this solution for $Q(m)$ into our mass equation (36) gives

$$m = A_0$$

$$\approx \frac{m}{\sqrt{2}} \left[ 3 - \sqrt{2} - \ln \frac{\sqrt{2}y}{m} \right], \quad (49)$$
which can finally be solved for \( m \neq 0 \) to give the finite mass solution

\[
m^* = \sqrt{2}e^{2\sqrt{2}/3}y.
\]  

(50)

The corresponding finite value of \( Q \) is

\[
Q^* = \frac{\Lambda}{2\pi^2(3 - 2\sqrt{2})}.
\]  

(51)

Note that \( m^* \) is proportional to \( y \) so that small fugacity translates to small \( m^* \), in accord with our assumption that \( m \ll \Lambda^3 \).

To show that monopoles are free when \( m \neq 0 \), we calculate the monopole density as in Section III. From Eqn. (23) we see that the free energy associated with our second order equation (25) is

\[
F_{\text{var}}^{(2)} = F_0 + \langle \Delta S \rangle_0 - \frac{1}{4} \langle (\Delta S)^2 \rangle_0 + \frac{1}{4} \langle \Delta S \rangle_0^2.
\]  

(52)

From this, the monopole density can be calculated.

\[
\rho_{M}^{(2)} = -\frac{1}{V} \frac{\partial F_{\text{var}}^{(2)}}{\partial \mu}.
\]  

(53)

\[
= 2a + ab - 2a^2 c.
\]

For \( m = 0 \), the monopole density vanishes, while for our finite \( m \) solution

\[
\rho_{M}^{(2)} = \frac{m^*}{\sqrt{2}} \left( 2\sqrt{2} - 1 + \frac{1}{16\pi} \sqrt{\frac{2m^*}{(Q^*)^3}} \right)
\]

> 0.

(54)

From the free energy (52), it is also possible to show that the finite \( m \) solution is the stable solution for all temperatures. In fact, the free energy diverges as \( \log(1/m) \) as \( m \) approaches zero, but has a finite value for finite \( m \). It is then the free phase of monopoles which is favoured at all temperatures.

Thus we have demonstrated that for arbitrarily low \( T \) a finite mass solution always exists for the self-consistent equations (36 – 38). This implies that monopoles are always free at low temperatures, or, in terms of the original lattice model (1), that the electric charge is presumably confined for any number of fermion flavours.

VI. CONCLUSION

We have studied cQED\(_3\) where massless relativistic fermions coupled to the compact gauge field result in a logarithmic interaction between magnetic monopoles. One may suspect
that this could lead to a BKT-like transition where free monopoles bind into monopole-
antimonopole pairs at low enough effective temperatures. Although the simplest mean-field
approximation would predict such a transition, we argued that by design this treatment
misses the screening effects, argued to be crucial in this problem. To address this issue
we developed a combined variational-perturbative approach which allowed us to include
screening self-consistently. The modified theory then leads to the plasma phase of free
monopoles as being stable at all temperatures, in agreement with the renormalisation group
treatment of the problem [15].

cQED\textsubscript{3} has been studied numerically in [22] and [23]. Our calculation appears to be in
agreement with the numerical results of [23], where only a single phase was observed. We
hope that this and previous work on cQED\textsubscript{3} will motivate renewed efforts in this direction,
using bigger system sizes that have recently become available [24].

VII. ACKNOWLEDGMENT

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APPENDIX A: MONOPOLE GAS AND THE SINE-GORDON ACTION

In this appendix, we fill in the details in going from the action (1) to the Coulomb gas
partition function (5) and the sine-Gordon action (6). For this purpose, let us write (1) on
the lattice in a more general form as

\[ S[a,n] = \frac{1}{2} \sum_{x,y} (F_{\mu\nu}(x) - 2\pi n_{\mu\nu}(x)) \ u(x,y) \ (F_{\mu\nu}(y) - 2\pi n_{\mu\nu}(y)), \]

where \( u^{-1}(x,y) = (16/N_f)|\Delta_{x,y}|\delta_{x,y} \) and \( \Delta_{x,y} f \equiv f(x + \hat{\mu}) - f(x) \) denotes the lattice deriva-
tive. Introducing an antisymmetric Hubbard-Stratonovich field \( M_{\mu\nu} \) we find

\[ S \to \frac{1}{2} \sum_{x,y} M_{\mu\nu}(x) \ u^{-1}(x,y) \ M_{\mu\nu}(y) + i \sum_{x} M_{\mu\nu}(x)(F_{\mu\nu}(x) - 2\pi n_{\mu\nu}(x)) \]

\[ = \frac{1}{4} \sum_{x,y} b_{\mu}(x)u^{-1}(x,y)b_{\mu}(y) + i \sum_{x}(\nabla \times a - 2\pi n)(x) \cdot b(x), \]

with \( b_{\mu} \equiv \epsilon_{\mu\nu\lambda}M_{\nu\lambda} \) and \( n_{\mu} \equiv \epsilon_{\mu\nu\lambda}n_{\nu\lambda} \). Integrating over the gauge field constrains the \( b \)-field
to be curl-free, so we can take it to be a gradient on the lattice \( b = \Delta \varphi \). Performing the
lattice version of integration by parts and integrating over $\varphi$ yields

\[ S \rightarrow \frac{1}{4} \sum_{x,y} \Delta_\mu \varphi(x) \ u^{-1}(x,y) \Delta_\mu \varphi(y) + 2\pi i \sum_x \Delta \cdot n(x) \ \varphi(x) \] (A3)

\[ \rightarrow \frac{1}{2} \sum_{x,y} \rho(x) \ v(x,y) \ \rho(y), \] (A4)

where $v^{-1}(x,y) = -(1/8\pi^2)\Delta_{x,\mu} u^{-1}(x,y)\Delta_{y,\mu}$ is the (inverse of the) potential and $\rho = \Delta_\mu n_\mu$ is the density of magnetic monopoles. Using the expression for $u(x,y)$ we find, in the continuum limit,

\[ v(x,y) = \frac{\pi^2 N_f}{2} \int \frac{d^3 k}{(2\pi)^3} \frac{e^{i k \cdot (x-y)}}{k^3} \equiv \frac{\pi^2 N_f}{2} V(x - y). \] (A5)

Thus, for a system of $N$ monopoles with a density $\rho(x) = \sum_{a=1}^N q_a \delta(x - x_a)$, we obtain

\[ S_{\text{mon}} = \frac{\pi^2 N_f}{4} \sum_{a,b} q_a q_b V(x_a - x_b), \] (A6)

as in Eqn. (6).

We now proceed to show that (A6) with $q_a = \pm 1$ is equivalent to the sine-Gordon action (6). Higher charges are irrelevant for large enough $N_f$. To this end, let us introduce the bare action

\[ S_b = \frac{1}{2} \int d^3 x \ d^3 y \ \phi(x) v^{-1}(x - y) \phi(y), \] (A7)

so that $v(x - y) = \langle \phi(x) \phi(y) \rangle_b$. Showing the fugacity by $y$, we may write the monopole partition function in the grand-canonical ensemble as

\[ Z_{\text{mon}} = \sum_N \frac{y^N}{N!} \int \prod_i \frac{d^3 x_i}{\{q_a = \pm 1\}} \ \exp \left\{ -\frac{1}{2} \sum_{a,b} q_a q_b \langle \phi(x_a) \phi(x_b) \rangle_b \right\} \]

\[ = \sum_N \frac{y^N}{N!} \int \prod_i \frac{d^3 x_i}{\{q_a = \pm 1\}} \ \exp \left\{ i \sum_a q_a \phi(x_a) \right\} \]

\[ = \langle \exp \left\{ 2y \int d^3 x \ \cos \phi(x) \right\} \rangle_b \equiv Z_b^{-1} Z_{\text{ASG}}, \] (A8)

where $Z_b$ is independent of $\phi$ and

\[ Z_{\text{ASG}} = \int D\phi \ \exp \left\{ -\frac{1}{2} \int d^3 x \ d^3 y \ \phi(x) v^{-1}(x - y) \phi(y) + 2y \int d^3 x \cos \phi(x) \right\}. \] (A9)

Inserting the definition of $v^{-1}(x - y)$ into this last expression we immediately arrive at the anomalous sine-Gordon action (6).
APPENDIX B

1. $F^{(n)}_{<}$

Here we will show that $F^{(n)}_{<}$ indeed satisfies Eqn. (24). To this end, first let us define 
\[ \Delta F^{(n)} = F^{(n+1)} - F^{(n)} \]. Then we may equivalently show that 
\[ F^{(n)}_{<} = F_0 + \Delta F^{(1)} + 2\Delta F^{(2)} + \cdots + n\Delta F^{(n)}. \] (B1)

Let us also denote the path integral over the field $\phi(q)$ by $\text{Tr}$ and define, for a real variable $t$,
\[ F(t) \equiv -\ln \text{Tr} \left( e^{-S_0} e^{-t\Delta S} \right). \] (B2)

Then, $F(1) = -\ln \text{Tr} \exp(-S) = F_{\text{ASG}}$ and
\[ \frac{dF(t)}{dt} \bigg|_{t=1} = \frac{\text{Tr} \left( \Delta S e^{-S} \right)}{\text{Tr} \left( e^{-S} \right)} = \langle \Delta S \rangle. \] (B3)

On the other hand, we may expand the RHS of Eqn. (B2) in powers of $\Delta S$ as 
\[ F(t) = F_0 + \sum_{i=1}^{\infty} \Delta F^{(i)}(t) \] (B4)
where $\Delta F^{(i)}(t) = t^i \Delta F^{(i)}(1) = t^i \Delta F^{(i)}$. Thus
\[ \langle \Delta S \rangle = \frac{dF(t)}{dt} \bigg|_{t=1} = \sum_{i=1}^{\infty} i \Delta F^{(i)}. \] (B5)

Upon insertion of Eqn. (B5) into the definition of $F_{<}$ in Eqn. (19) and truncating the expansion at $i = n$ we find (B1).

2. $F^{(n)}_{\text{var}}$

In this appendix, we will give the proof for our claim that the extremum of $F^{(n)}_{\text{var}}$ as defined in (23) is given by the expansion of Eqn. (20) to order $(\Delta S)^n$, i.e.
\[ \frac{\delta F^{(n)}_{\text{var}}}{\delta G_0(k)} = 0 \iff \langle \phi(-k)\phi(k) \rangle_0 = \langle \phi(-k)\phi(k) \rangle^{(n)}. \] (B6)

The calculations are, for general $n$, cumbersome and not very instructive so we will first present the case for $n = 2$ which is also the one with which we are concerned in Section V.
Setting $n = 2$, we see that Eqn. (25) is readily found by an expansion of the RHS of Eqn. (21). To show that the same result arises from extremising $F^{(n)}_{\var}$, it is first useful to establish

$$\frac{\delta F_0}{\delta G_0(q)} = \left\langle \frac{\delta S_0}{\delta G_0(q)} \right\rangle_0 = \frac{-1}{2(2\pi)^d G_0(q)} \left\langle \phi(-q)\phi(q) \right\rangle_0,$$  \tag{B7}

$$\frac{\delta (g)_0}{\delta G_0(q)} = \frac{\delta F_0}{\delta G_0(q)} \langle g \rangle_0 + \left\langle \frac{\delta g}{\delta G_0(q)} - g \frac{\delta S_0}{\delta G_0(q)} \right\rangle_0,$$  \tag{B8}

where $g = g(S_0)$ is an arbitrary function of $S_0$. Thus choosing appropriate forms of $g$ for $F^{(1)} = F_0 + \langle \Delta S \rangle_0$ and $F^{(2)} = F^{(1)} - \frac{1}{2} \langle \langle \Delta S \rangle^2 \rangle_0 + \frac{1}{2} \langle \Delta S \rangle_0^2$ we find

$$\frac{\delta F^{(1)}}{\delta G_0(q)} = \frac{\delta F_0}{\delta G_0(q)} \langle \Delta S \rangle_0 - \left\langle \Delta S \frac{\delta S_0}{\delta G_0(q)} \right\rangle_0,$$  \tag{B9}

$$\frac{\delta F^{(2)}}{\delta G_0(q)} = \frac{\delta F_0}{\delta G_0(q)} \left[ -\frac{1}{2} \langle \Delta S^2 \rangle_0 + \langle \Delta S \rangle_0^2 \right]$$

$$+ \frac{1}{2} \left\langle \frac{\delta S_0}{\delta G_0(q)} \langle \Delta S \rangle_0^2 \right\rangle_0 - \left\langle \frac{\delta S_0}{\delta G_0(q)} \Delta S \right\rangle \langle \Delta S \rangle_0.$$  \tag{B10}

Inserting Eqn. (B7) into Eqns. (B9) (B10) and adding them we find that the restriction $\delta F^{(2)}_{\var}/\delta G_0(q) = 0$ leads to the same equation as Eqn. (25).

The proof for arbitrary $n$ goes along essentially the same steps as above. Various truncated expansions we have defined can be read off the Taylor expansion identity

$$-\ln \text{Tr} \left\{ e^{-S_0 - V} \right\} = F_b + \sum_{i=1}^{\infty} \sum_{l=1}^{i} \frac{(-1)^{i+l}}{l} \sum_{\{k_0\}}^{'} \frac{\langle V^k \rangle_b \cdots \langle V^k \rangle_b}{k_1! \cdots k_l!},$$  \tag{B11}

by setting $i$ to the desired order. In (B11) $S_b$ and $V$ give an arbitrary splitting of the action into a bare and potential part respectively and

$$\sum_{\{k_0\}}^{'} \equiv \sum_{k_1=1}^{i} \cdots \sum_{k_l=1}^{i} \delta_{k_1+\cdots+k_l,i}.$$  \tag{B12}

Notice that in Eqns. (B9) and (B10) all the terms are to the same order of $\Delta S$, which is also the largest in the corresponding expansion of the free energy. By choosing $S_0 = S_b$ and $V = \Delta S$ and setting $i = n$ in (B11) one can see, after some lengthy algebra, that the same is true for arbitrary $n$:

$$\frac{\delta F^{(n)}}{\delta G_0(q)} = \sum_{l=1}^{n} \frac{1}{(-1)^{n+l}} \sum_{\{k_0\}}^{'} \frac{\langle (\Delta S)^k \rangle_0 \cdots \langle (\Delta S)^k \rangle_0}{k_1! \cdots k_l!} \frac{\delta F_0}{\delta G_0(q)}$$

$$- \sum_{l=1}^{n} \frac{1}{(-1)^{n+l}} \sum_{\{k_0\}}^{'} \frac{\delta S_0}{\delta G_0(q)} \langle (\Delta S)^k \rangle_0 \frac{\langle (\Delta S)^k \rangle_0 \cdots \langle (\Delta S)^k \rangle_0}{k_1! k_2! \cdots k_l!}.$$  \tag{B13}
Let us now define, for a real variable \( t \),

\[
G(k, t) \equiv -\ln \text{Tr} \left\{ e^{-S_0 - \Delta S - t\phi(-k)\phi(k)} \right\},
\]

(B14)

so that \( \partial G(k, t)/\partial t |_{t=0} = \langle \phi(-k)\phi(k) \rangle \). Then, taking \( S_b = S_0 \) and \( V = \Delta S + t\phi(-k)\phi(k) \) in Eqn. (B11) to compute this derivative, it can be shown through additional tedious but straightforward algebra that

\[
\langle \phi(-q)\phi(q) \rangle^{(n)} - \langle \phi(-q)\phi(q) \rangle_0 = -2n(2\pi)^3[G_0(q)]^2 \frac{\delta F_{\text{var}}^{(n)}}{\delta G_0(q)},
\]

(B15)

where we have also made use of Eqn. (B13). Thus, the requirement that \( F_{\text{var}}^{(n)} \) be an extremum implies Eqn. (20) truncated at \( n \)th order, and \textit{vice versa}, proving our claim (B6).

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