In this paper we study arithmetic properties of a one-parameter family $H$ of Hénon maps over the affine line. Given a family of initial points $P$ satisfying a natural condition, we show the height function $h_P$ associated to $H$ and $P$ is the restriction of the height function associated to a semipositive adelically metrized line bundle on projective line. We then show various local properties of $h_P$. Next we consider the set $\Sigma(P)$ consisting of periodic parameter values, and study when $\Sigma(P)$ is an infinite set or not. We also study unlikely intersections of periodic parameter values.

**Introduction**

In this paper, we study arithmetic properties of families of Hénon maps. A Hénon map $H$ over a field $K$ is an automorphism of the affine plane of the form

$$H : \mathbb{A}^2 \to \mathbb{A}^2, \quad (x, y) \mapsto (\delta y + f(x), x),$$

where $\delta \in K \setminus \{0\}$ and $f(x) \in K[x]$ is a polynomial of degree $d \geq 2$. Up to conjugacy and a finite field extension, we may assume that $f(x)$ is a monic polynomial. Hénon maps constitute an important class of affine plane automorphisms. Indeed, Friedland–Milnor [FM89, Theorem 2.6] showed that any dynamically interesting affine plane automorphism (i.e. one with positive first dynamical degree) over $\mathbb{C}$ is, up to conjugacy, compositions of Hénon maps with monic $f(x)$. Properties of Hénon maps over $\mathbb{C}$ have been studied in e.g. [BS91, FS92, HOV94] from among the vast literature. Arithmetic properties of Hénon maps have also been studied in e.g. [Den95, DFT17, In14, Ka06, Ka13, Le13a, Le13b, Ma03, Si94c].

In this paper, we are interested in families of Hénon maps. To be precise, we consider a one-parameter family $H$ of Hénon maps over $\mathbb{A}^1 = \text{Spec}(K[t])$, so that we replace $K$ in (0.1) with $K[t]$. Note that, to have the inverse $H^{-1}$ over $\mathbb{A}^1$, $\delta \in K[t] \setminus \{0\}$ should belong to $K \setminus \{0\}$, and our family is thus of the form

$$H : \mathbb{A}^2 \to \mathbb{A}^2, \quad (x, y) \mapsto (\delta y + f_t(x), x),$$

where $\delta \in K \setminus \{0\}$ and $f_t(x) \in (K[t])[x]$ is a monic polynomial in $x$ of degree $d \geq 2$ (with respect to $x$). We regard $H$ as a family of Hénon maps $H_t$ over $\overline{K}$, where $t$ runs through elements of an algebraic closure $\overline{K}$ of $K$.

When $K$ is a field equipped with a set $M_K$ of inequivalent, non-trivial absolute values that satisfies the product formula, we have the canonical height function $h_{H_t}$ for $H_t$ for each $t \in \overline{K}$, as we now explain. For each $v \in M_K$, we denote by $K_v$ the completion of $K$ with respect to $v$, and by $\mathbb{K}_v$ the completion of an algebraic closure of $K_v$. Suppose that $v$ is archimedean, i.e., $\mathbb{K}_v = \mathbb{C}$. Then, as constructed and proven to be extremely useful in e.g.
we have the (complex) Green function \(G_{H, v}: \mathbb{A}^2(\mathbb{C}) \to \mathbb{R}_{\geq 0}\) defined by
\[
(0.3) \quad G_{H, v}(P) = \max \left\{ \lim_{n \to +\infty} \frac{1}{d^n} \log^+ |H^n_t(P)|, \lim_{n \to +\infty} \frac{1}{d^n} \log^+ |H^{-n}_t(P)| \right\},
\]
where \(P \in \mathbb{A}^2(\mathbb{C})\) and \(\log^+(r) := \log \max\{1, r\}\) for any \(r \in \mathbb{R}\). When \(v\) is nonarchimedean, we also have the \(v\)-adic Green function \(G_{H, v}: \mathbb{A}^2(\mathbb{K}_v) \to \mathbb{R}_{\geq 0}\), and we have the specialized canonical height function \(\tilde{h}_H: \mathbb{A}^2(\mathbb{K}) \to \mathbb{R}_{\geq 0}\) defined as the sum of these Green functions with some normalizing constants \(c_v\) (see §1 for details). Since \(K(t)\) is the function field of \(\mathbb{P}^1\) over a field \(K\), \(K(t)\) is also naturally equipped with a set of absolute values that satisfies the product formula (see Example 1.2), and we have the function field canonical height \(h: \mathbb{A}^2(\mathbb{K}(t)) \to \mathbb{R}_{\geq 0}\) as well.

Suppose we are given an initial point \(P \in \mathbb{A}^2(K[t])\), which we regard as a family of initial points \(P_t := P(t)\) with \(t \in \mathbb{K}\). Then both the specialized canonical heights \(\tilde{h}_H(P_t) \in \mathbb{R}_{\geq 0}\) for each \(t \in \mathbb{K}\) and the function field canonical height \(h(P) \in \mathbb{R}_{\geq 0}\) are defined. Assuming that \(\tilde{h}_H(P) \neq 0\), we put
\[
(0.4) \quad h_P(t) := \frac{\tilde{h}_H(P_t)}{\tilde{h}_H(P)}.
\]

One of the main results of this paper is the following theorem on variation of canonical heights for Hénon maps.

**Theorem A** (see Theorem 3.4). Let \(K\) be a number field or the function field of an integral projective variety that is regular in codimension one over another field \(F\). Let \(H\) and \(P\) be given as above such that \(\tilde{h}_H(P) \neq 0\). Then \(h_P\) is the restriction to \(A^1(\mathbb{K})\) of a height function on \(\mathbb{P}^1(\mathbb{K})\) associated to \(O_{\mathbb{P}^1}(1)\) with semipositive adelically metrized metrics. In other words, there exists a semipositive adelically metrized line bundle \(\mathcal{L}_P = (O_{\mathbb{P}^1}(1), \{\| \cdot \|_v\} \in M_K)\) on the parameter space \(\mathbb{P}^1\) such that, if \(h_{\mathcal{L}_P}\) denotes the height function associated to \(\mathcal{L}_P\), then we have
\[
\tilde{h}_H(P_t) = h_{\mathcal{L}_P}(t) \quad \text{for any } t \in A^1(\mathbb{K}).
\]
(For the definitions of a semipositive adelically metrized line bundle and the associated height function, see §1.6.)

Theorem A may best be compared with variation of canonical heights for families of elliptic curves. Let \(X\) (resp. \(C\)) be a smooth projective surface (resp. curve) over a number field \(K\), and let \(\pi: X \to C\) be an elliptic surface. For simplicity, we assume that \(C = \mathbb{P}^1\). Let \(U\) be a Zariski open subset of \(\mathbb{P}^1\) such that \(E_t := \pi^{-1}(t)\) is an elliptic curve for any \(t \in U(\mathbb{K})\). We denote by \(E\) the generic fiber of \(\pi\), which is an elliptic curve over \(K(\mathbb{P}^1)\). Let \(P: \mathbb{P}^1 \to X\) be a section. Then we have the Néron–Tate heights \(\hat{h}_E(P)\) and \(\hat{h}_{E_t}(P_t)\) for \(t \in U(\mathbb{K})\). Silverman [Si83] and Tate [Ta83] showed, among other things, that
\[
(0.5) \quad \hat{h}_{E_t}(P_t) = \hat{h}_E(P)h(t) + O(1) \quad \text{for any } t \in U(\mathbb{K}),
\]
where \(h(t)\) is the (usual) Weil height function on \(\mathbb{P}^1\). For the dynamical setting, Ingram [In13] showed a similar equality for a family of one-variable polynomial maps. In a subsequent paper, Ingram [In14] considered a family of Hénon maps and obtains similar results (see Theorem 1.13).

For applications to unlikely intersection problems and further generalizations (see §0.2 below), it has been noticed that it is important that there is a “nice” height function
without the $O(1)$ term in (0.5) or in its dynamical counterparts, just as it is important that Néron-Tate heights and canonical heights in dynamical systems are defined without the $O(1)$ term. In the setting of (0.5), assuming that $\hat{h}_E(P) \neq 0$ and putting $h_P(t) := \hat{h}_E(P_t)/\hat{h}_E(P)$ (for $t \in U(\overline{K})$), the question is whether or not $h_P$ is the restriction to $U(\overline{K})$ of a “nice” height function on $\mathbb{P}^1(\overline{K})$. This question has been answered in the affirmative in DeMarco–Wang–Ye [Dwy16, Theorem 1.5] and DeMarco–Mavraki [DM17+], which gives an alternate proof of Masser–Zannier’s unlikely intersections of simultaneous torsion sections of elliptic curves [MZ08, MZ10, MZ12] for the Legendre family. For various families of one-variable polynomial maps, this question has also been answered in the affirmative (see § 0.2–0.3 below). However, this question is still widely open for families of rational self-maps of higher dimensional varieties. Theorem A gives a refinement of Ingram’s result (= Theorem 1.13 with $\tilde{h}$ in place of $\hat{h}$), and answers this question in the affirmative for a family of Hénon maps.

In the following, we explain some properties and applications of $h_P$ in (0.4).

0.1. Local (i.e., $v$-adic) properties of $h_P$. Let $K$ be a field equipped with a set $M_K$ of inequivalent, non-trivial absolute values that satisfies the product formula. Then, for each $v \in M_K$, we have the $v$-adic Green function $G_{H_{t,v}}: A^2(K_v) \to \mathbb{R}_{\geq 0}$ as explained above (cf. (0.3)). We set

$$G_{P,v}(t) := G_{H_{t,v}}(P_t)$$

for any $t \in A^1(K_v)$. Let $| \cdot |_v$ denote the $v$-adic absolute value, and we write $\|a\|_v := \max\{|a_1|_v, \ldots, |a_k|_v\}$ for $a = (a_1, \ldots, a_k) \in K_v^k$. We define

$$W_{P,v} := \left\{ t \in A^1(K_v) \left| \lim_{n \to +\infty} \| (H^n_t(P_t), H^{-n}_t(P_t)) \|_v = +\infty \right. \right\},$$

$$K_{P,v} := \left\{ t \in A^1(K_v) \left| \text{the sequence} \{H^n_t(P_t)\}_{n \in \mathbb{Z}} \text{is bounded} \right. \right\}.$$

Using (the proof of) Theorem A we obtain the following.

**Theorem B** (see Proposition 4.2, Theorem 4.3, Corollary 4.5, Remark 4.6). (1) Set $c := \tilde{h}_H(P)$. Then $c$ is a rational number. The function $h_P$ decomposes into the sum of $v$-adic functions $G_{P,v}$ with coefficients $n_v/c$ (see (3.4) and (4.2)). Further, for any $v \in M_K$,

$$G_{P,v}(t) - c \log |t|_v$$

converges as $t \to \infty$.

(2) We have $A^1(K_v) = W_{P,v} \amalg K_{P,v}$, and $K_{P,v}$ is exactly the set of points where $G_{P,v}$ vanish:

$$K_{P,v} = \{ t \in A^1(K_v) \mid G_{P,v}(t) = 0 \}.$$

We note that, in the case of one-parameter families of elliptic curves, the continuity at the singular fibers was shown by Silverman [Si92, Si94a, Si94b], which provided a basis for [DM17+].

**Remark C.** The question of rationality of canonical heights associated to dynamical systems over function fields has been investigated in [DG18] for one-variable dynamics. In the case of Hénon maps, the canonical height $\tilde{h}_H(P)$ appearing in Theorem B provides an instance of rational canonical height for higher dimensional dynamics. It would be interesting to study whether or not the property of rationality of canonical heights holds for general Hénon maps $H$ and points $P$ over a function field.
As an illustration of $K_{P,v}$, Figure 1 below, drawn with Qfract (by Hiroyuki Inou), concerns the family of quadratic Hénon maps $H = (H_t)_{t \in \Q}$: $(x, y) \mapsto (y + x^2 + t, x)$ and the (constant) family of initial points $P = (0, 0)$ centered around $t = 0$. Here $K = \Q$ and $K_v = \C$. Shades depend on how fast the orbit $\{H_t^n((a, b))\}$ escapes as $|n|$ becomes large (the escaping rate is very small in the black region). Note that, at the center $t = 0$ of the images, the point $(0, 0)$ is fixed with respect to $H_0$. Note also that, for this example, there are infinitely many parameter values $t$ such that $(0, 0)$ is periodic respect to $H_t$ (see Theorem D below).

![Figure 1](image)

**Figure 1.** The region on the left is $\{t \in \C \mid |\Re(t)| \leq 1, |\Im(t)| \leq 1\}$, and that on the right is $\{t \in \C \mid |\Re(t)| \leq 0.01, |\Im(t)| \leq 0.01\}$.

0.2. **The set of periodic parameter values.** To study some global properties of $h_P$, we introduce the set $\Sigma(P)$ of periodic parameter values:

$$\Sigma(P) := \{t \in A^1(K) \mid P_t \text{ is periodic with respect to } H_t\}.$$  

We remark that, if $K$ is a number field, then the Northcott property implies that $\Sigma(P)$ is exactly the set of points where $h_P$ vanish (see Proposition 4.1):

$$\Sigma(P) = \{t \in A^1(K) \mid h_P(t) = 0\}.$$  

Now let $Q \in A^2(K[t])$ be another initial point with $\tilde{h}_H(Q) \neq 0$. We consider a question if there exist infinitely many $t \in K$ such that both $P_t$ and $Q_t$ are simultaneously periodic with respect to $H_t$, i.e., whether or not $\Sigma(P) \cap \Sigma(Q)$ is an infinite set.

This question is about the study of unlikely intersections in arithmetic dynamics, which was initiated by Baker–DeMarco [BD11] with motivation from Masser–Zannier’s study of unlikely intersections of simultaneous torsion sections of elliptic curves [MZ08, MZ10, MZ12] and the Pink–Zilber conjecture in arithmetic geometry (see [Za12]). Unlikely intersection in arithmetic dynamics has since been further and beautifully explored in e.g. [BD13, De16, DWY16, GHT13, GHT15, GHT16, FG16+].

For the present family of Hénon maps, however, we encounter a phenomenon that is not observed in families of one-variable maps, in which the set corresponding $\Sigma(P)$ is an infinite set. (This follows from Montel’s theorem on normal families in one-variable complex analysis.) In our case, as we now explain, $\Sigma(P)$ can be either an infinite set or a finite set.
First we explain our result toward infiniteness of $\Sigma(P)$. Inspired by the paper Dujardin–Favre [DF17] on dynamical Manin–Mumford conjecture for affine plane polynomial automorphisms, we suppose that our family $H$ has an involution $\iota: \mathbb{A}^2 \to \mathbb{A}^2$ over $K$ such that $\iota \circ H \circ \iota = H^{-1}$. Taking the Jacobian, we get $\delta \in \{1, -1\}$. It is straightforward to check that the involution $\iota_\delta: (x, y) \mapsto (-\delta y, -\delta x)$ reverses $H$ provided that $f_t(-\delta x) = f_t(x)$. Note that this condition gives non-trivial restriction on $f_t(x)$ only when $\delta = 1$. Let the line $C_\delta: \delta x + y = 0$ in $\mathbb{A}^2$ denote the set of $\iota_\delta$-fixed points for $\delta = 1, -1$. Then we have the following result.

**Theorem D** (= Theorem [5.1]). Let $K$ be a field (of any characteristic). Let $H$ be the family of Hénon maps in (0.2) such that $\delta \in \{1, -1\}$. When $\delta = 1$, we assume that $f_t(x)$ is an even polynomial in $x$. Let $P = (a(t), b(t)) \in \mathbb{A}^2(K[x])$. If $P$ lies in $C_\delta$, i.e., $\delta a(t) + b(t) = 0$ in $K[t]$, then $\Sigma(P)$ is an infinite set.

We remark that, for the quadratic Hénon maps we consider in Figure 1, Theorem D implies that $\Sigma((0,0))$ is an infinite set. It is natural to ask if $\Sigma((a,b))$ is a finite set when $a + b \neq 0$. In this opposite direction, we obtain, among other things, the following result on emptiness of $\Sigma(P)$.

**Proposition E** (= Proposition [7.4]). Let $H: (x, y) \mapsto (y + x + t^2, x)$ be the family of quadratic Hénon maps over $\mathbb{Q}$ (or any field of characteristic zero). Then, for any $b \notin \mathbb{Z}$ we have $\Sigma((0,b)) = \emptyset$ where $\mathbb{Z}$ denotes the ring of algebraic integers.

We say that a Hénon map $H$ is reversible if there exist an involution $\sigma$ of affine plane and a positive integer $m$ such that $\sigma \circ H^m \circ \sigma = H^{-m}$. Based on Theorem D, the dynamical Manin–Mumford conjecture for affine plane automorphisms in [DF17], we ask the following.

**Question F.** Let $K$ be an algebraically closed field of characteristic zero. Let $H: \mathbb{A}^2 \to \mathbb{A}^2$ be a Hénon map over $\mathbb{A}^1 = \text{Spec}(K[t])$, which we regard as a family $(H_t)_{t \in K}$ parametrized by $t$. Put $\text{Per}_{\mathbb{A}^2 \times \mathbb{A}^1}(H) := \{(P,t) \in \mathbb{A}^2(K) \times \mathbb{A}^1(K) \mid P \text{ is periodic with respect to } H_t\}$. Let $C$ be an integral curve in $\mathbb{A}^2 \times \mathbb{A}^1$, and we assume that $C(K) \cap \text{Per}_{\mathbb{A}^2 \times \mathbb{A}^1}(H)$ is infinite. Is it true that one of the following conditions must hold?

(i) $C \subseteq \mathbb{A}^2 = \mathbb{A}^2 \times \{t\}$ for some $t \in \mathbb{A}^1(K)$, and there exists an involution $\sigma_t$ of $\mathbb{A}^2$ over $K$ such that $\sigma_t \circ H_t^m \circ \sigma_t = H_t^{-m}$ for some $m \geq 1$. Furthermore, $C$ is contained in the set of fixed points of $\sigma_t$;

(ii) There exists a positive integer $m$ such that $C$ is an irreducible component of $\{(P,t) \in \mathbb{A}^2(K) \times \mathbb{A}^1(K) \mid H_t^m(P) = P\}$;

(iii) There exists an involution $\sigma$ of $\mathbb{A}^2$ over $K[t]$ such that $\sigma \circ H^m \circ \sigma = H^{-m}$ for some $m \geq 1$ and $C$ is contained in the set of fixed points of $\sigma$.

0.3. **Unlikely intersections of periodic parameter values.** By Theorem D, we have instances that $\Sigma(P)$ is infinite. Let $P, Q \in \mathbb{A}^2(K[t])$ be two initial points with $n_H(P) \neq 0$ and $\tilde{n}_H(Q) \neq 0$ such that both $\Sigma(P)$ and $\Sigma(Q)$ are infinite. Now we consider the problem of unlikely intersections, i.e., what happens if $\Sigma(P) \cap \Sigma(Q)$ is infinite.

**Theorem G** (= Theorem [6.2]). Let $K$ be a number field, and let $P, Q \in \mathbb{A}^2(K[t])$ be as above. Then the following is equivalent.
K¨ ottcher coordinates
proof relies on the B¨ ottcher coordinate in one-variable polynomial dynamics. Although the
points of the affine line, it is proved that, for two given polynomial families of initial points
to share infinite periodic parameter values, they must satisfy some orbital relations. The
Extended to the present situation. We would like to pose the following question (see also
literature for H´ enon maps (see [HOV94]), the techniques in the one-variable case seem not easily
extended to the present situation. We would like to pose the following question (see also Proposition 6.3).

Question H. (1) Let \( H : \mathbb{A}^2 \to \mathbb{A}^2 \) be a H´enon map over \( \mathbb{C} \), and let \( P, Q \in \mathbb{A}^2(\mathbb{C}) \). Suppose that \( \{ H^n(P) \}_{n \in \mathbb{Z}} \) and \( \{ H^n(Q) \}_{n \in \mathbb{Z}} \) are both unbounded, and that \( G(H^n(P)) = G(H^n(Q)) \) holds for any sufficiently large \( n \), where \( G \) is the complex Green function defined in (0.3). Is it true that there exists an invertible affine map \( \sigma : \mathbb{A}^2 \to \mathbb{A}^2 \) over \( \mathbb{C} \) such that \( \sigma^{-1} \circ H^m \circ \sigma = H^m \) or \( \sigma^{-1} \circ H^m \circ \sigma = H^{-m} \) for some \( m \geq 1 \) and \( P = \sigma(Q) \)?

(2) Let \( K \) be a number field. Let \( P, Q \in \mathbb{A}^2(K[t]) \) be two initial points with \( \tilde{h}_H(P) \neq 0 \) and \( \tilde{h}_H(Q) \neq 0 \). Suppose that \( h_P = h_Q \). Is it true that there exists an invertible affine map \( \sigma : \mathbb{A}^2 \to \mathbb{A}^2 \) over \( \mathbb{K}[t] \) with \( \sigma^{-1} \circ H^m \circ \sigma = H^m \) or \( \sigma^{-1} \circ H^m \circ \sigma = H^{-m} \) for some \( m \geq 1 \) and \( P = H^n(\sigma(Q)) \) for some \( n \in \mathbb{Z} \)?

In this paper, we aim to explore some fundamental arithmetic properties of families of H´enon maps. We believe that the methods and techniques used in this paper will be useful in studying further arithmetic properties, for example, in the case where the base parameter space is a punctured disk (in a suitable setting) or a higher dimensional space, or families of higher dimensional H´enon maps (i.e., affine space regular polynomial automorphisms). See also Remark 5.10 for a question about primitive prime divisors in a family of H´enon maps.

We briefly describe the organization of this paper. In Section 1 we set up the notation and recall some facts from the theory of canonical heights for H´enon maps that will be used in the paper. Section 2 and Section 3 are devoted to the construction of the height function \( h_P \) and the proof of Theorem A. In Section 2, we consider \( v \)-adic settings. As in [GHT15], we divide the parameter space \( \mathbb{A}^1(\mathbb{K}_v) \) into a bounded region and an unbounded region, and using various properties of H´enon maps, we show that certain \( v \)-adic metrics converge uniformly in each region. In Section 3 we prove Theorem A. In Section 4, we prove Theorem B by estimating multiplicities of some periodic points. Then in Section 5, we prove Theorem C on unlikely intersections of periodic parameter values. Here we use Yuan’s equidistribution theorem [Yu08], in hoping that our methods will be useful when one considers a family of H´enon maps over a higher-dimensional parameter space. In Section 7, we show some results towards emptiness of \( \Sigma(P) \), including Proposition 6.3.

Acknowledgement. We presented an earlier content of this paper at the BIRS Workshop on Arithmetic and Complex Dynamics in November, 2017. We thank the organizers of the workshop, and Eric Bedford, Patrick Ingram and Thomas Gautier for helpful comments at that occasion. We also thank Charles Favre for helpful discussions, and to Hiroyuki Inou for figures in Introduction and Example 4.4 drawn by Qfract. We thank Dragos Ghioca and Joe...
Silverman for helpful comments. The present paper grew out from our discussions at the Academia Scinica and Kyoto University, and we thank Julie Tzu-Yueh Wang and Kazuhiko Yamaki for warm hospitality. Part of this work was done during the second named author’s stay at the Mathematical Institute of Oxford, and he thanks the institute and Damian Rössler for warm hospitality.

1. Preliminaries

In this section, we recall some basic properties of Hénon maps that will be used in this paper.

1.1. Notation and terminology. We list some of the notation and terminology that we use throughout this paper.

- For a field $K$, we denote by $\overline{K}$ an algebraic closure of $K$.
- If $K$ is a number field, we denote by $O_K$ the ring of integers of $K$.
- If $K$ is equipped with a set $M_K$ of inequivalent, non-trivial absolute values (places), we denote by $M_K^\text{an}$ the subset of $M_K$ of all nonarchimedean absolute values, and by $M_K^\infty$ the subset of $M_K$ of all archimedean absolute values.
- For $v \in M_K$, the corresponding absolute value is denoted by $| \cdot |_v$. We let $K_v$ be the completion of $K$ with respect to $| \cdot |_v$, and let $\mathbb{K}_v$ be the completion of an algebraic closure of $K_v$.
- The (unique) extension of the absolute value $| \cdot |_v$ of $K_v$ to $\mathbb{K}_v$ will still be denoted by $| \cdot |_v$.
- Let $| \cdot |$ be an absolute value of a field $K$. For $(x_1, \ldots, x_n) \in K^n$, we write
  $$\|(x_1, \ldots, x_n)\| := \max\{|x_1|, \ldots, |x_n|\}.$$ 
- For $r \in \mathbb{R}$, we set $\log^+ r := \log \max\{r, 1\}$.
- Let $f : S \to S$ be a bijective self-map of a set $S$. We denote by $f^n$ the $n$-th iterate of $f$ under compositions of maps for $n \in \mathbb{Z}$. A point $P \in S$ is periodic for $f$ if there exists an integer $n \geq 1$ such that $f^n(P) = P$. The least positive integer with $f^n(P) = P$ is called the period of $P$.

1.2. Hénon maps. Let $K$ be a field. A Hénon map is a polynomial automorphism of the affine plane defined over $K$ of the form

\begin{equation}
H : \mathbb{A}^2 \to \mathbb{A}^2 \quad (x, y) \mapsto (\delta y + f(x), x),
\end{equation}

where $f(x) \in K[x]$ is a polynomial of degree $d \geq 2$ and $\delta \in K \setminus \{0\}$. The inverse $H^{-1}$ is given by

$$H^{-1} : \mathbb{A}^2 \to \mathbb{A}^2, \quad (x, y) \mapsto \left( y, \frac{1}{\delta} (x - f(y)) \right).$$

By taking a suitable finite extension $L$ of $K$ and conjugating by $(x, y) \mapsto (\varepsilon x, \varepsilon y)$ for suitable $\varepsilon \in L$, $f(x)$ becomes a monic polynomial. In this paper, we consider a Hénon map such that $f(x)$ is monic.

It is also common to consider a Hénon map $H'$ of the form

$$H' : \mathbb{A}^2 \to \mathbb{A}^2, \quad (x, y) \mapsto (y, \delta x + f(y)),$$

in place of $H : (x, y) \mapsto (\delta y + f(x), x)$. We remark that there is no real difference in choosing $H$ or $H'$ concerning their dynamical properties. Indeed, let $j : \mathbb{A}^2 \to \mathbb{A}^2$ be the involution given by $(x, y) \mapsto (y, x)$. Then we have

$$j \circ H \circ j(x, y) = (y, \delta x + f(y)) = H'(x, y),$$
so that $H^n = j \circ H^n \circ j$ for $n \in \mathbb{Z}$. Thus dynamical properties of $H'$ are deduced from those of $H$, and vice versa.

1.3. Canonical heights for Hénon maps. We say a field $K$ is a product formula field if it is equipped with a set $M_K$ of inequivalent absolute values (places) with the following property: The set $\{| \cdot |_v \in M_K \mid |\alpha|_v \neq 1\}$ is finite for any $\alpha \in K \setminus \{0\}$; Further, for each $v \in M_K$ there exists a positive integer $n_v$ such that the product formula $\prod_{v \in M_K} |\alpha|_v^{n_v} = 1$ holds for any $\alpha \in K \setminus \{0\}$. If $K$ is a product formula field, and $L$ is a finite extension field of $K$, then $L$ is naturally equipped with a set $M_L$ so that $L$ is also a product formula field (see [BG06, Proposition 1.4.2]). While some statements in this paper hold for any product formula field, we will assume that a product formula field is one of the following two types of fields (see for example, [La83, §2.3] and [BG06, §1.4.6] for more details).

Example 1.1 (number field). If $K$ is a number field, then we take $M_K$ as the set of all inequivalent absolute values of $K$. If $v \in M_K^{\text{fin}}$ lies over a prime $p \in \mathbb{Q}$, then $|p|_v = 1/p$. If $v \in M_K^{\text{fin}}$, then $| \cdot |_v$ is the extension of the usual absolute value of $\mathbb{Q}$. For $v \in M_K^{\text{fin}}$, the normalizing constant $n_v$ is defined to be $[K_v : \mathbb{Q}_p]/[K : \mathbb{Q}]$ if $v$ lies over $p$. For $v \in M_K^{\infty}$, the normalizing constant $n_v$ is defined to be $1/[K : \mathbb{Q}]$ if $v$ is a real place, and $2/[K : \mathbb{Q}]$ if $v$ is a complex place. Then $K$ is a product formula field.

Example 1.2 (function field). Let $K = F(B)$ be the function field of an integral projective variety $B$ over a field $F$ such that $B$ is regular in codimension one. In this case, we take $M_K$ as the set of all prime divisors of $B$. The valuation $v_Y$ associated to a prime divisor $Y$ is defined to be the vanishing order along $Y$, and the corresponding absolute value is defined by $| \cdot |_v := \exp(-\text{ord}_v(\cdot))$. In particular, $M_K = M_K^{\text{fin}}$. We fix an ample class $c \in \text{Pic}(B)$. The normalizing constant $n_v$ is defined to be $\text{deg}_c(Y)$. Then $K$ is a product formula field.

When we regard the rational function field $F(t)$ over a field $F$ as a product formula field, we always regard $F(t)$ as the function field of $F(\mathbb{P}^1)$, and we take an ample class $c \in \text{Pic}(\mathbb{P}^1)$ as a divisor class of degree 1 above.

Let $K$ be a product formula field and let $v \in M_K$. Let $H : \mathbb{A}^2 \to \mathbb{A}^2$ be a Hénon map of degree $d \geq 2$ defined over $\mathbb{K}_v$ as in (1.1). Let $P \in \mathbb{A}^2(K_v)$. The Green functions (also called local canonical height functions) for $H$ are defined by

\[
G_v^+(P) := \lim_{n \to \infty} \frac{1}{d^n} \log^+ \|H^n(P)\|_v \text{ and } G_v^-(P) := \lim_{n \to \infty} \frac{1}{d^n} \log^+ \|H^{-n}(P)\|_v.
\]

The limits exist for any $P \in \mathbb{A}^2(K_v)$, see, for example [HOV94, Proposition 5.5] and [BS91, §3] when $v$ is archimedean, and [Ka13, Theorem A] when $v$ is nonarchimedean.

Let $h : \mathbb{A}^2(\overline{K}) \to \mathbb{R}$ be the (usual) Weil height function on $\mathbb{A}^2$ defined by

\[
h(P) = \frac{1}{[K(P) : K]} \sum_{\sigma : K(P) \to \overline{K}} \sum_{v \in M_K} n_v \log^+ \|P^\sigma\|_v
\]

for $P \in \mathbb{A}^2(\overline{K})$. The height functions $\hat{h}^+_H$, $\hat{h}_H$ and $\hat{h}_H$ for $H$ are defined by

\[
\hat{h}_H^+(P) = \lim_{n \to \infty} \frac{1}{d^n} h(H^n(P)), \quad \hat{h}_H^-(P) = \lim_{n \to \infty} \frac{1}{d^n} h(H^{-n}(P)), \\
\hat{h}_H(P) = \hat{h}_H^+(P) + \hat{h}_H^-(P)
\]
for \( P \in \mathbb{A}^2(K) \). It follows from (1.2) and (1.3) that

\[
\hat{h}_H^\pm(P) = \frac{1}{[K(P) : K]} \sum_{\sigma : K(P) \to K} \sum_{\nu \in M_K} n_\nu G_\nu^\pm(P^\sigma)
\]

\[
= \sum_{\nu \in M_K} n_\nu \left( \frac{1}{[K(P) : K]} \sum_{\sigma : K(P) \to K} G_\nu^\pm(P^\sigma) \right).
\]

We consider a variant of \( \hat{h}_H^\pm \). Set

\[
G_v(P) = \max\{G_v^+(P), G_v^-(P)\}
\]

for \( P \in \mathbb{A}^1(K) \). As in [Ka13, Equation (6-5)], we define

\[
\tilde{h}_H(P) = \frac{1}{[K(P) : K]} \sum_{\sigma : K(P) \to K} \sum_{\nu \in M_K} n_\nu G_\nu(P^\sigma)
\]

\[
= \sum_{\nu \in M_K} n_\nu \left( \frac{1}{[K(P) : K]} \sum_{\sigma : K(P) \to K} G_\nu(P^\sigma) \right)
\]

for \( P \in \mathbb{A}^2(K) \).

In this paper, we mainly study \( \tilde{h}_H \) rather than \( \hat{h}_H \). For convenience, we call \( \tilde{h}_H \) the canonical height associated to the Hénon map \( H \). (Note the difference of terminologies used in [Ka06], where the “canonical height” refers to \( \hat{h}_H \).) As we see in Proposition 1.3 (1) below, \( \hat{h}_H \) and \( \tilde{h}_H \) are comparable, and one advantage of \( \tilde{h}_H \) is that \( \tilde{h}_H \) differs from the (usual) Weil height by a bounded function.

**Proposition 1.3.** (1) We have \((1/2)\hat{h}_H \leq \tilde{h}_H \leq \hat{h}_H\).

(2) We have \( \tilde{h}_H = h + O(1) \).

(3) Let \( K \) be a number field. Then, for \( P \in \mathbb{A}^2(K) \), \( \tilde{h}_H(P) = 0 \) if and only if \( P \) is periodic with respect \( H \).

**Proof.** The assertion (1) is obvious from the definitions of \( \hat{h}_H \) and \( \tilde{h}_H \). For (2), see [Ka13 Proposition 6.5], and for (3), see [Ka13 Theorem 6.3 (5) and Proposition 6.5]. \( \square \)

**Remark 1.4.** For Proposition 1.3 (2), Lee [Le13b, Theorem 6.5] proved a stronger result: If \( K \) is a number field, then the canonical height function \( \tilde{h}_H \) is a height function associated to \( \mathcal{O}_{\mathbb{P}^2}(1) \) with semipositive adelic metrics. (For a brief review of semipositive adelically metrized line bundles, see §1.6.)

**Remark 1.5.** Let \( K \) be a product formula field such that any \( \nu \in M_K \) is nonarchimedean. Assume that \( \delta = 1 \) in equation (1.1) of the Hénon map \( H \). Ingram [In14, Theorem 1.2] showed that, if \( H \) is not isotrivial over \( K \), then the conclusion of Proposition 1.3 (3) still holds: For \( P \in \mathbb{A}^2(K) \), \( \tilde{h}_H(P) = 0 \) if and only if \( P \) is periodic with respect \( H \). (For the definition of isotriviality, see [In14, p. 787].) It would be interesting to know if the same conclusion still holds when \( \delta \neq 1 \).
1.4. One-parameter families of Hénon maps: setting. Let $K$ be a field. We are interested in algebraic families of Hénon maps defined over $K$. We give our setting that will be fixed throughout this paper.

Let $f_t(x) \in K[t, x]$ be a polynomial such that, as a polynomial in $x$, $f_t(x)$ is monic and of degree $d \geq 2$. We write

\begin{equation}
(1.8) \quad f_t(x) = x^d + \sum_{i=1}^{d} c_i(t) x^{d-i} = x^d + c_1(t)x^{d-1} + \cdots + c_d(t),
\end{equation}

where $c_i(t) \in K[t]$ for $i = 1, \ldots, d$. Let $\delta \in K \setminus \{0\}$. We consider the one-parameter family of Hénon maps $H_t$ in (1.1) parametrized by $t$:

\begin{equation}
(1.9) \quad H = (H_t)_{t \in \mathbb{K}} : \mathbb{A}^2 \rightarrow \mathbb{A}^2, \quad (x, y) \mapsto (\delta y + f_t(x), x).
\end{equation}

We also fix an initial point $P = (a(t), b(t)) \in \mathbb{A}^2(K[t])$, which we regard as a family of initial points $(P_t)_{t \in \mathbb{K}}$.

We remark on our symbols $H$, $P$ and $(H_t)_t$, $(P_t)_t$. We use the symbols $H$ and $P$ when they are viewed as a Hénon map and a point defined over $K(t)$ (and also over $K[t]$). When we emphasize that they are algebraic families of Hénon maps and points parametrized by $t \in \mathbb{K}$, we use the symbols $(H_t)_t$ and $(P_t)_t$.

**Example 1.6.** Let $H_t : \mathbb{A}^2 \rightarrow \mathbb{A}^2$ be the Hénon map over a field $K$ defined by

\begin{equation}
H_t : \mathbb{A}^2 \rightarrow \mathbb{A}^2, \quad (x, y) \mapsto (y + x^2 + t, x).
\end{equation}

Then $H = (H_t)_{t \in \mathbb{K}}$ is an example of a one-parameter family of Hénon maps of (1.9) with $\delta = 1$ and $f_t(x) = x^2 + t$. Some properties of this family are discussed in Introduction, Example 1.6, Example 1.4, Example 5.2 and Section 7.

1.5. One-parameter families of Hénon maps: further setting. For $n \geq 1$, we set

\begin{equation}
(1.10) \quad \Phi_n : \mathbb{A}^1 \rightarrow \mathbb{A}^4, \quad t \mapsto (H_t^n(P_t), H_t^{-n}(P_t)).
\end{equation}

We embed $\mathbb{A}^n$ into $\mathbb{P}^n$ by $x \mapsto (x : 1)$. We denote the extension of $\Phi_n$ to projective spaces by

\begin{equation}
(1.11) \quad \Phi_n : \mathbb{P}^1 \rightarrow \mathbb{P}^4.
\end{equation}

Note that $\Phi_n$ is a morphism for any $n \geq 1$, because $\mathbb{P}^1$ is a smooth curve.

To fix the notation, we introduce the following polynomials $A_n(t)$ and $B_n(t)$ associated to the Hénon maps $H^n = (H_t^n)_t$ and the initial point $P = (P_t)_t$. Recall that the inverse $H_t^{-1}$ is given by

\begin{equation}
(1.12) \quad H_t^{-1} : \mathbb{A}^2 \rightarrow \mathbb{A}^2, \quad (x, y) \mapsto \left(y, \frac{1}{\delta}(x - f_t(y))\right).
\end{equation}

**Definition 1.7** ($A_n(t), B_n(t)$). We define $A_n(t), B_n(t) \in K[t]$ for $n \geq 0$ by the following recursive formulae

\begin{equation}
(1.13) \quad A_{n+1}(t) = \delta A_{n-1}(t) + f_t(A_n(t)) \quad \text{for} \quad n \geq 1, \quad A_1(t) = \delta b(t) + f_t(a(t)), \quad A_0(t) = a(t),
\end{equation}

\begin{equation}
(1.14) \quad B_{n+1}(t) = \frac{1}{\delta}(B_{n-1}(t) - f_t(B_n(t))) \quad \text{for} \quad n \geq 1, \quad B_1(t) = \frac{1}{\delta}(a(t) - f_t(b(t))), \quad B_0(t) = b(t).
\end{equation}
It follows from (1.9) and (1.12) that
\[(1.15) \quad H^{n}_{t}(P_{t}) = (A_{n}(t), A_{n-1}(t)) \quad \text{and} \quad H^{-n}_{t}(P_{t}) = (B_{n-1}(t), B_{n}(t)) \]
for any \(n \geq 1\). We set
\[(1.16) \quad \ell_{n} := \deg \Phi_{n} = \max \{ \deg A_{n}(t), \deg A_{n-1}(t), \deg B_{n}(t), \deg B_{n-1}(t) \}. \]

We consider the following assumption on \(P\).

**Assumption 1.8.** The sequence \((\deg A_{n}(t))_{n \geq 0}\) or \((\deg B_{n}(t))_{n \geq 0}\) is unbounded.

Note that Assumption 1.8 is mild. See Example 1.9 below, for example. As we see in Proposition 1.10, Assumption 1.8 is equivalent to \(\hat{h}_{H}(P) \neq 0\).

**Example 1.9.** Let \(H_{t}(x, y) = (y + x^{2} + t, x)\) as in Example 1.6. Then it is easy to see that, for any \(a, b \in K\), \(\deg(A_{t}(t)) = 2^{n-1}\) and \(\deg(B_{n}(t)) = 2^{n-1}\). Thus, for any constant initial point \(P = (a, b)\), Assumption 1.8 is satisfied.

**Proposition 1.10.** (1) If \((\deg A_{n}(t))_{n \geq 0}\) is unbounded, then there exists \(N \geq 1\) such that
\[(1.17) \quad \deg A_{N+j}(t) = d^{j} \deg A_{N}(t) \]
for all \(j \geq 0\). In particular, there exists a rational number \(\ell > 0\) such that \(\deg A_{n}(t) = d^{n}\ell\) for any sufficiently large \(n\).

(2) If \((\deg B_{n}(t))_{n \geq 0}\) is unbounded, then there exists \(N' \geq 1\) such that
\[(1.18) \quad \deg B_{N'+j}(t) = d^{j} \deg B_{N'}(t) \]
for all \(j \geq 0\). In particular, there exists a rational number \(\ell' > 0\) such that \(\deg B_{n}(t) = d^{n}\ell'\) for any sufficiently large \(n\).

(3) Assumption 1.8 is equivalent to \(\hat{h}_{H}(P) > 0\).

(4) Under Assumption 1.8, \(P\) is not periodic with respect \(H\). Further, there exists a rational number \(\ell \geq 1\) such that, for any sufficiently large \(n\), we have
\[(1.19) \quad \ell_{n} = d^{n}\ell. \]

In particular, \(\Phi_{n}\) is not a constant map for any sufficiently large \(n\).

**Proof.** (1) Fix a positive integer \(D\) such that \(D > \max\{\deg c_{1}(t), \ldots, \deg c_{d}(t), \deg A_{0}(t)\}\). Since \((\deg A_{n}(t))_{n \geq 0}\) is unbounded, there exists an \(n\) such that \(\deg A_{n}(t) \geq D\). Let \(N\) be the smallest positive integer satisfying this property. Thus \(\deg A_{n}(t) \geq D\) and \(\deg A_{i}(t) < D\) for \(0 \leq i \leq N-1\). We prove (1.17) by induction on \(n \geq N\). We write \(n = N + j\) with integer \(j \geq 0\).

Clearly, (1.17) is true for \(j = 0\). Let \(j \geq 0\) and assume that \(\deg A_{N+i}(t) = d^{i}\deg A_{N}(t)\) for integer \(0 \leq i \leq j\). From the recursive relation for polynomials \(A_{n}(t)\) we have
\[A_{N+j+1}(t) = \delta A_{N+j-1}(t) + f_{t}(A_{N+j}(t)) = \delta A_{N+j-1}(t) + A_{N+j}(t)^{d} + \sum_{i=1}^{d} c_{i}(t)A_{N+j}(t)^{d-i}. \]
Note that, by induction, if \(j \geq 1\), then \(\deg(\delta A_{N+j-1}(t)) = d^{j-1}\deg A_{N}(t) < d^{j+1}\deg A_{N}(t)\).

For \(j = 0\), we still have \(\deg(\delta A_{N+1}(t)) < D \leq \deg A_{N}(t) < d\deg A_{N}(t) = d^{j+1}\deg A_{N}(t)\).

Further, we have
\[\deg(c_{i}(t)A_{N+j}(t)^{d-i}) < D + (d-i)\deg A_{N+j}(t) = D + (d-i)d^{j}\deg A_{N}(t) < d^{j+1}\deg A_{N}(t). \]
Since \(\deg A_{N+j}(t)^{d} = d\deg A_{N+j}(t) = d^{j+1}\deg A_{N}(t)\), we have \(\deg A_{N+j+1}(t) = d^{j+1}\deg A_{N}(t)\).

This completes the induction steps and hence the proof of (1.17).
(2) One can prove (2) by the same argument as (1).

(3) Note that the Hénon map $H$ and the point $P$ are defined over the rational function field $K(t)$ of $\mathbb{P}^1$ over $K$. As in Example 1.2, we regard $K(t)$ as a product formula field equipped with the set of absolute values $M_{K(t)}$. We denote by $\infty$ the point of $\mathbb{P}^1(K)$ at which the function $t$ has a pole.

Since $A_n(t)$ and $B_n(t)$ belong to $K[t]$, we have, for any $v \in M_K(t)$ with $v \neq \infty$ and for any $n \geq 1$,

$$\|H^n(P)\|_v \leq 1 \quad \text{and} \quad \|H^{-n}(P)\|_v \leq 1.$$ 

It follows that $G^+_v(P) = 0$ if $v \neq \infty$. Then, the canonical height of $P$ is

$$\tilde{h}_H(P) = \sum_{v \in M_K(t)} n_v \max\{G^+_v(P), G^-_v(P)\} = \max\{G^+_\infty(P), G^-_\infty(P)\} = \lim_{n \to \infty} \frac{\deg \bar{F}_n}{d^n}.$$ 

Applying (1) and (2), we see that Assumption 1.8 is equivalent to $\tilde{h}_H(P) > 0$.

(4) The first assertion is obvious. The second assertion (1.18) follows from (1) and (2). The last assertion is obvious from the second assertion.

Remark 1.11. (1) For later reference, we note that in Proposition 1.10 (1), if we write $A_N(t) = \alpha_D t^D + \cdots + \alpha t + \alpha_0$ ($\alpha_0, \ldots, \alpha_D \in K$, $\alpha_D \neq 0$), then

$$(1.19) \quad A_{N+j}(t) = (\alpha_D t^D)^j + \text{(lower order terms)}.$$ 

(2) Similarly, in Proposition 1.10 (2), if we write $B_{N'}(t) = \beta_D t^{D'} + \cdots + \beta t + \beta_0$ ($\beta_0, \ldots, \beta_{D'} \in K$, $\beta_{D'} \neq 0$), then

$$(1.20) \quad B_{N'+j}(t) = (\beta_{D'} t^{D'})^j + \text{(lower order terms)}.$$ 

Remark 1.12. As we see in Proposition 1.10 (4), Assumption 1.8 implies that $P$ is not periodic with respect $H$. If $\delta = 1$ and $H$ is not isotrivial, then it follows from Proposition 1.10 (3) and Remark 1.5 that Assumption 1.8 is indeed equivalent to the assumption that $P$ is not periodic with respect $H$.

1.6. Heights associated to semipositive adelically metrized line bundles. In this subsection, following Zhang [Zh95], we recall the notion of semipositive adelically metrized line bundles over number fields and function fields. For details, see [Zh95, Gu08, Yu08].

Let $K$ be a number field (resp. a function field $F(B)$ of an integral projective variety $B$ over a field $F$ such that $B$ is regular in codimension one). In the following, we identify $M_K^\text{fin}$ with $\text{Spec}(O_K) \setminus \{(0)\}$ (resp. the $M_K$ with the set of all generic points of prime divisors of $B$).

Let $X$ be an integral projective variety. An adelically metrized line bundle on $X$, denoted by $\mathcal{L} := (L, \| \cdot \|_v)_{v \in M_K}$, is a pair consisting of a line bundle $L$ on $X$ and a collection of metrics $\| \cdot \|_v := \{\| \cdot \|_v | v \in M_K\}$ such that the following conditions are satisfied:

(i) Each $\| \cdot \|_v$ is a continuous and bounded metric of $L \otimes K_v$ that is $\text{Gal}(K_v/K_0)$-invariant;

(ii) There exist a non-empty Zariski open subset $U$ of $\text{Spec}(O_K)$ (resp. of $B$), a projective scheme $\mathcal{X}$ over $U$ with generic fiber $X$, a positive integer $e$, and a line bundle $\mathcal{L}$ on $\mathcal{X}$ with $L^e = \mathcal{L}|_X$ such that $\| \cdot \|_v^e$ is the metric induced from the model $\mathcal{L}_v := \mathcal{L} \otimes_U K_v$ at all $v \in M_K \cap U$. (When $K = F(B)$, the condition $v \in M_K \cap U$ means that $v \in M_K$ and the prime divisor corresponding to $v$ intersects with $U.$)
A sequence \( \{\| \cdot \|_n\}_{n \geq 1} \) of adelic metrics on \( L \) is said to converge to an adelic metric \( \| \cdot \| \) if there exists a non-empty Zariski open subset \( U \) of \( \text{Spec}(O_K) \) (resp. of \( B \)) such that \( \| \cdot \|_n, v = \| \cdot \|_v \) for all \( n \geq 1 \) and all \( v \in M_K \cap U \), and \( \log \left( \frac{\| \cdot \|_n, v}{\| \cdot \|_v} \right) \) converges to 0 uniformly on \( X(\mathbb{K}_v) \) for all places \( v \in M_K \setminus U \).

The adelically metrized line bundle \( \mathcal{L} = (L, \{\| \cdot \|_v\}_{v \in M_K}) \) is said to be semipositive if there is a family \( \mathcal{L}_n = (L, \{\| \cdot \|_{n,v}\}_{v \in M_K}) \) for \( n \geq 1 \) of adelically metrized line bundles satisfying the following conditions:

(a) Metrics \( \| \cdot \|_n := \{\| \cdot \|_{n,v}\} \) converge to \( \| \cdot \| := \{\| \cdot \|_v\} \).

(b) For any \( v \in M_K^\infty \), the metric \( \| \cdot \|_{n,v} \) is smooth and the curvature \( c_1(\mathcal{L}) \) of \( \| \cdot \|_{n,v} \) is semipositive for each \( n \);

(c) For each \( n \geq 1 \), there exist a projective scheme \( \mathcal{X}_n \) over \( \text{Spec}(O_K) \) (resp. of \( B \)) with generic fiber \( X \), a positive integer \( e_n \), and a line bundle \( \mathcal{L}_n \) on \( \mathcal{X}_n \) with \( L^{\otimes e_n} = \mathcal{L}_n|_X \) such that \( \mathcal{L}_n \) is relatively nef and that \( \| \cdot \|_{n,v} \) is induced from \( (\mathcal{X}_n, \mathcal{L}_n) \) for any \( v \in M_K^{\text{fin}} \).

For any given semipositive adelically metrized line bundle \( \mathcal{L} = (L, \{\| \cdot \|_v\}_{v \in M_K}) \) on \( X \), one can associate a height function \( h_{\mathcal{L}} \) on \( X(\overline{K}) \) as follows: For any \( P \in X(\overline{K}) \), we take a nonzero rational section \( \eta \) of \( L \) whose support is disjoint from the Galois conjugates of \( P \) over \( K \); then we set

\[
h_{\mathcal{L}}(P) = \frac{1}{[K(P) : K]} \sum_{\sigma : K(P) \to \overline{K}} \sum_{v \in M_K} -n_v \log \eta(P^\sigma) \|_{v}
\]

\[
= \sum_{v \in M_K} n_v \left( \frac{1}{[K(P) : K]} \sum_{\sigma : K(P) \to \mathbb{K}_v} -\log \eta(P^\sigma) \|_{v} \right) .
\]

Note that, since \( K \) satisfies the product formula, the value \( h_{\mathcal{L}}(P) \) does not depend on the choice of a nonzero rational section \( \eta \). We say that \( h_{\mathcal{L}} \) is a height associated to a semipositive adelically metrized line bundle.

### 1.7. Variation of canonical heights in families.

Let \( K \) be a number field, and let \( H = (H_t)_{t \in \mathbb{K}} \) be a Hénon map over \( K \) defined in (1.9).

Let \( P \in A^2(K[T]) \). In this setting, we have the (function field) height \( \hat{h}_H(P) \) defined in (1.4), and for each \( t \in \overline{K} \), we have the specialized height \( \hat{h}_{H_t}(P_t) \) defined in (1.4). Ingram [In14] studied variation of \( \hat{h}_{H_t}(P_t) \) and obtained the following theorem.

**Theorem 1.13 (In [In14] Theorem 1.1).** Let \( K \) be a number field. We have, for \( t \in \overline{K} \),

\[
(1.21) \quad \hat{h}_{H_t}(P_t) = \hat{h}_H(P) h(t) + O(1),
\]

where \( h(t) \) is the (usual) Weil height function on the parameter space \( t \in A^1(\overline{K}) \).

**Remark 1.14.** In fact, Ingram [In14] treated a more general case, where \( K(T) \) is replaced by the function field of a smooth projective curve \( C \) defined over a number field \( K \), and in this case, \( h \) is replaced by a height \( h_\eta \) associated to a degree one divisor \( \eta \in \text{Pic}(C) \) and \( O(1) \) is replaced by \( O(h_\eta(t)^{1/2}) \).

These types of results go back to the study of variation of canonical heights of abelian varieties in [Si83, Ta83]. See Introduction for more accounts. As we discuss in Introduction, just as a Néron-Tate height is important in the study of the arithmetic of abelian varieties, to study arithmetic properties of families of Hénon maps, a key ingredient is the existence
of a “nice” height function on the parameter space that encodes arithmetic information of the periodic parameters $\Sigma(P)$.

One of the main results in this paper is to show that for families of Hénon maps such a height function indeed exists on the parameter space (for $\tilde{h}$ in place of $h$). To be precise, Theorem A in Introduction asserts that, over a number field or a function field as in Example 1.2, if we put $h_P(t) := \tilde{h}_{H_t}(P_t)/\tilde{h}_H(P)$, then $h_P$ is the restriction of the height function associated to a semipositive adelically metrized line bundle $\mathcal{E}_P = (\mathcal{O}_{\mathbb{P}^1}(1), \{\| \cdot \|_v\}_{v \in M_K})$ on $\mathbb{P}^1$.

The proof of Theorem A will be given in Section 3 after establishing all technical results in the next section.

2. Construction of a metrized line bundle over a valued field

In this section, we construct a metrized line bundle on $\mathbb{P}^1$ over an algebraically closed valued field, which will be used to prove Theorem A.

Let $\Omega$ be an algebraically closed field that is complete with respect to an absolute value $| \cdot |$. Let $H = (H_t)_{t \in \Omega}$ and $P = (a(t), b(t)) = ((P_t)_{t \in \Omega}) \in \mathbb{A}^2(\Omega[t])$ be as in §1.4 such that $P$ satisfies Assumption 1.8, where the field $K$ in §1.4 is $\Omega$ in this section.

For later use, we set

\begin{equation}
    m := \max\{\deg(c_1(t)), \ldots, \deg(c_d(t)), \deg(a(t)), \deg(b(t))\}.
\end{equation}

We also put

\[ \Delta := \max(|\delta|, 1/|\delta|) \geq 1. \]

We use the following convenient notation: for a positive real number $r$, we set

\[ [r] := \begin{cases} r & \text{(if } | \cdot | \text{ is archimedean)}, \\ 1 & \text{(if } | \cdot | \text{ is nonarchimedean}). \end{cases} \]

2.1. Statement of the result. Let $P = (x_0 : x_1 : \ldots : x_4) \in \mathbb{P}^4(\Omega)$. Let $X_0, \ldots, X_4$ be homogeneous coordinates of $\mathbb{P}^4$. Then for any $\xi = c_0X_0 + \cdots + c_4X_4 \in H^0(\mathcal{O}_{\mathbb{P}^4}(1))$,

\[ \|\xi\|_{st}(P) := \frac{|c_0x_0 + \cdots + c_4x_4|}{\|(x_0, x_1, \ldots, x_4)\|} \]

is well-defined, thus defining a metric $\| \cdot \|_{st}$ (called the standard metric) on $\mathcal{O}_{\mathbb{P}^4}(1)$.

Recall that, for each positive integer $n$, we have defined a morphism

\begin{equation}
    \Phi_n : \mathbb{P}^1 \to \mathbb{P}^4,
    \quad (t : 1) \mapsto (H^n_t(P_t) : H^{-n}_t(P_t) : 1) = (A_n(t) : A_{n-1}(t) : B_{n-1}(t) : B_n(t) : 1)
\end{equation}

in (1.10) and (1.11). We write $\ell_n := \deg(\Phi_n)$ as before. We remark that by Proposition 1.10 (4), $\ell_n = d^n\ell > 0$ for sufficiently large $n$, so that $\Phi_n$ is not a constant morphism.

Now let $n$ be a large integer such that $\ell_n > 0$. We define a metric $\| \cdot \|_n$ on $\mathcal{O}_{\mathbb{P}^1}(1)$ by pulling back $\| \cdot \|_{st}$ via $\Phi_n$. To do this, we take a lift $A^2 \to A^5$ of $\Phi_n$ given by $(t, 1) \mapsto (H^n_t(P_t), H^{-n}_t(P_t), 1)$ which determines the isomorphism $\alpha_n : \Phi_n^*(\mathcal{O}_{\mathbb{P}^4}(1)) \cong \mathcal{O}_{\mathbb{P}^1}(1)^{\otimes \ell_n}$. Then we define

\[ \| \cdot \|_n := \left(\alpha_n(\Phi_n^*\| \cdot \|_{st})\right)^{1/\ell_n}. \]

Concretely, for a section $\eta = a_0X_0 + a_1X_1$ of $H^0(\mathcal{O}_{\mathbb{P}^4}(1))$, we have

\begin{equation}
    \|\eta\|_n(t : 1) = \frac{|a_0t + a_1|}{\|(A_n(t), A_{n-1}(t), B_{n-1}(t), B_n(t), 1)\|^{1/\ell_n}}.
\end{equation}
Theorem 2.1. The metrics \( \{ \log \| \cdot \|_n \} \) uniformly converge on \( \mathbb{P}^1(\Omega) \) as \( n \to \infty \).

The proof of Theorem 2.1 will be given in subsequent subsections. We first introduce a quantity related to the denominator of (2.3).

**Definition 2.2** (\( M_n(t) \)). Let \( n \geq 1 \). For \( t \in \mathbb{A}^1(\Omega) \), we set

\[
M_n(t) := \max \{|A_n(t)|, |A_{n-1}(t)|, |B_{n-1}(t)|, |B_n(t)|, 1\}.
\]

For a positive constant \( L \), we denote by \( \mathcal{B}_L \) the bounded region in the parameter space \( \mathbb{A}^1(\Omega) \) defined by

\[
\mathcal{B}_L = \{ t \in \Omega \mid |t| \leq L \}.
\]

If we take any sufficiently large \( L \), we then have, for any \( t \in \mathcal{B}_L \),

\[
|c_1(t)| \leq L^{m+1}, \ldots, |c_d(t)| \leq L^{m+1}, |a(t)| \leq L^{m+1}, |b(t)| \leq L^{m+1}, [3d] \Delta \leq L^{m+1}.
\]

In what follows, we fix \( L \geq 1 \) that satisfies (2.4).

### 2.2. Uniform convergence on a bounded region of \( \mathbb{A}^1 \).

In this subsection, we compare \( M_{n+1}(t) \) with \( M_n(t)^d \) on \( \mathcal{B}_L \).

**Lemma 2.3.** Let \( n \geq 1 \) be any integer. Then for any \( t \in \mathcal{B}_L \), we have

\[
M_{n+1}(t) \leq [d + 2] \Delta L^{m+1} M_n(t)^d.
\]

**Proof.** To simplify the notation, we write \( M_n, A_n, B_n \) for \( M_n(t), A_n(t), B_n(t) \). By (1.13), we have

\[
|A_{n+1}| = \left| \delta A_{n-1} + A_n^d + \sum_{i=1}^d c_i(t)A_n^{d-i} \right|
\]

\[
\leq [d + 2] \max \{|\delta A_{n-1}|, |A_n|^d, |c_1(t)A_n^{d-1}|, \ldots, |c_{d-1}(t)A_n|, |c_d(t)|\}
\]

\[
\leq [d + 2] \max \{\Delta |A_{n-1}|, |A_n|^d, L^{m+1}|A_n|^d-1, \ldots, L^{m+1}|A_n|, L^{m+1}\}
\]

\[
\leq [d + 2] \Delta L^{m+1} \max \{|A_{n-1}|, |A_n|^d, |A_n|^d-1, \ldots, 1\} \quad \text{(since } L \geq 1 \text{ and } \Delta \geq 1\}
\]

\[
\leq [d + 2] \Delta L^{m+1} M_n^d.
\]

Similarly, using (1.14) we have

\[
|B_{n+1}| = \left| \frac{1}{\delta} \left( B_{n-1} - B_n^d - \sum_{i=1}^d c_i(t)B_n^{d-i} \right) \right|
\]

\[
\leq \Delta \left| B_{n-1} - B_n^d - \sum_{i=1}^d c_i(t)B_n^{d-i} \right|
\]

\[
\leq [d + 2] \Delta \max \{|B_{n-1}|, |B_n|^d, L^{m+1}|B_n|^d-1, \ldots, L^{m+1}|B_n|, L^{m+1}\}
\]

\[
\leq [d + 2] \Delta L^{m+1} \max \{|B_{n-1}|, |B_n|^d, |B_n|^d-1, \ldots, 1\} \quad \text{(since } L \geq 1\}
\]

\[
\leq [d + 2] \Delta L^{m+1} M_n^d.
\]

We also have

\[
|A_n| \leq \max\{|A_n|, 1\} \leq [d + 2] \Delta L^{m+1} \max\{|A_n|, 1\}^d \leq [d + 2] \Delta L^{m+1} M_n^d,
\]

and

\[
|B_n| \leq [d + 2] \Delta L^{m+1} M_n^d \quad \text{as well. Combining these estimates, we get}
\]

\[
M_{n+1} = \max \{|A_{n+1}|, |A_n|, |B_{n+1}|, |B_{n+1}|, 1\} \leq [d + 2] \Delta L^{m+1} M_n^d.
\]
This completes the proof. 

For the other direction, we have the following estimate.

**Proposition 2.4.** Let $n \geq 1$ be any integer. Then for any $t \in \mathcal{B}_L$, we have

$$M_{n+1}(t) \geq \frac{M_n(t)^d}{([3d] \Delta L^{m+1})^d}.$$ 

To prove Proposition 2.4, we use some properties of Hénon maps. Set

$$V^+_L = \left\{ (x, y) \in \mathbb{A}^2(\Omega) \left| \| (x, y) \| \leq [3d] \Delta L^{m+1} \quad \text{or} \quad \| H_t(x, y) \| \geq \frac{1}{L^{m+1}} \| (x, y) \|^{d} \right. \right\},$$

$$V^-_L = \left\{ (x, y) \in \mathbb{A}^2(\Omega) \left| \| (x, y) \| \leq [3d] \Delta L^{m+1} \quad \text{or} \quad \| H_t^{-1}(x, y) \| \geq \frac{1}{L^{m+1}} \| (x, y) \|^{d} \right. \right\}.$$

Before we prove Proposition 2.4, we show the following lemma, which gives a filtration according to behaviour of the orbits of points under the action of the Hénon maps.

**Lemma 2.5.** Let $t \in \mathcal{B}_L$. Then the following hold.

1. $H_t(V^+_L) \subseteq V^+_L$ and $H_t^{-1}(V^-_L) \subseteq V^-_L$.
2. For any $(x, y) \in V^+_L$, we have

$$\max \{ \| H_t(x, y) \|, 1 \} \geq \frac{\max\{\| (x, y) \|, 1 \}^d}{([3d] \Delta L^{m+1})^d}.$$ 

3. For any $(x, y) \in V^-_L$, we have

$$\max \{ \| H_t^{-1}(x, y) \|, 1 \} \geq \frac{\max\{\| (x, y) \|, 1 \}^d}{([3d] \Delta L^{m+1})^d}.$$ 

4. $\mathbb{A}^2(\Omega) = V^+_L \cup V^-_L$.

**Proof.** This is essentially shown in [Ka13, Lemmas 2.6, 2.7, 2.9] with slightly different $V^+_L$ and $V^-_L$. Here we give an explicit proof. We first give useful estimates.

Let $(x, y) \in \mathbb{A}^2(\Omega)$ be any point. Noting that

$$H_t^{-1}(x, y) = \left( y, \frac{1}{\delta} \left( x - y^d - \sum_{i=1}^{d} c_i(t) y^{d-i} \right) \right),$$

we have

$$|y^d| = \left| \left( x - y^d - \sum_{i=1}^{d} c_i(t) y^{d-i} \right) + \left( -x + \sum_{i=1}^{d} c_i(t) y^{d-i} \right) \right|$$

$$\leq [3] \max \left\{ \delta \left| \frac{1}{\delta} (x - y^d - \sum_{i=1}^{d} c_i(t) y^{d-i}) \right|, \| (x, y) \|, \sum_{i=1}^{d} c_i(t) y^{d-i} \right\}$$

$$\leq [3d] \Delta \max \{ \| H_t^{-1}(x, y) \|, \| (x, y) \|, \| (x, y) \|^{d-1} \}$$

$$\leq [3] \Delta \max \{ \| H_t^{-1}(x, y) \|, L^{m+1} \| (x, y) \|^{d-1} \}$$

$$\leq [3d] \Delta \max \{ \| H_t^{-1}(x, y) \|, L^{m+1} \| (x, y) \|^{d-1}, L^{m+1} \}.$$
Similarly, since
\[ H_t(x, y) = \left( \delta y + x^d + \sum_{i=1}^d c_i(t)x^{d-i}, x \right) \]
and \( x^d = (\delta y + x^d + \sum_{i=1}^d c_i(t)x^{d-i}) - (\delta y + \sum_{i=1}^d c_i(t)x^{d-i}) \), we have
\[
| x^d | \leq [3d] \Delta \max \left\{ \| H_t(x, y) \|, L^{m+1} \| (x, y) \|^{d-1}, L^{m+1} \right\}.
\]
In conclusion, for any \((x, y) \in \mathbb{A}^2(\Omega)\), we obtain from (2.6) and (2.7) that
\[
\|(x, y)\|^{d} \leq [3d] \Delta \max \left\{ \| H_t^{-1}(x, y) \|, \| H_t(x, y) \|, L^{m+1} \| (x, y) \|^{d-1}, L^{m+1} \right\}.
\]
Now we prove the statements (1)–(4) of the lemma.

(1) We first show the first assertion.

Suppose that \((x, y) \notin V_L^+\). We will show that \(H_t^{-1}(x, y) \notin V_L^+\), which gives \(\mathbb{A}^2(\Omega) \setminus V_L^+ \supseteq H_t^{-1}(\mathbb{A}^2(\Omega) \setminus V_L^+)\), and thus \(H_t(V_L^+) \subseteq V_L^+\). In other words, we assume
\[
\|(x, y)\| > [3d] \Delta L^{m+1} \quad \text{and} \quad \| H_t(x, y) \| < \frac{1}{L^{m+1}} \| (x, y) \|^{d}.
\]
and we will show
\[
\| H_t^{-1}(x, y) \| > [3d] \Delta L^{m+1} \quad \text{and} \quad \| (x, y) \| < \frac{1}{L^{m+1}} \| H_t^{-1}(x, y) \|^{d}.
\]
Observe that (2.8), (2.9) and the choice of \(L\) give
\[
\|(x, y)\|^{d} \leq [3d] \Delta \| H_t^{-1}(x, y) \|.
\]
It follows from (2.11), (2.9) and \(d \geq 2\) that
\[
\| H_t^{-1}(x, y) \| \geq \| (x, y) \|^{d} \frac{[3d] \Delta L^{m+1})^{d}}{[3d] \Delta} \geq [3d] \Delta L^{m+1}.
\]
This gives the first part of (2.10).

(2) Let \((x, y) \in V_L^+\) be given. By the definition of \(V_L^+\) (see (2.5)), we have
\[
\| (x, y) \| \leq [3d] \Delta L^{m+1} \quad \text{or} \quad \| H_t(x, y) \| \geq \frac{1}{L^{m+1}} \| (x, y) \|^{d} > \frac{[3d] \Delta L^{m+1})^{d}}{[3d] \Delta L^{m+1})^{d}} \| (x, y) \| \geq \| (x, y) \|.
\]
This completes the proof of the first assertion of (1). Similar arguments also show that \(H_t^{-1}(V_L^-) \subseteq V_L^-\) which gives the second assertion of (1).

(2) Let \((x, y) \in V_L^+\) be given. By the definition of \(V_L^+\) (see (2.5)), we have
\[
\| (x, y) \| \leq [3d] \Delta L^{m+1} \quad \text{or} \quad \| H_t(x, y) \| \geq \frac{1}{L^{m+1}} \| (x, y) \|^{d} > \frac{[3d] \Delta L^{m+1})^{d}}{[3d] \Delta L^{m+1})^{d}} \| (x, y) \| \geq \| (x, y) \|.
\]
If the latter inequality holds, then noting that \(1 \geq 1/([3d] \Delta L^{m+1})^d\) we obtain the assertion.

Suppose that the former inequality holds. Then \(\| (x, y) \|^{d} \leq ([3d] \Delta L^{m+1})^d\). Consequently,
\[
\max \left\{ \| H_t(x, y) \|, 1 \right\} \geq 1 \geq \frac{1}{([3d] \Delta L^{m+1})^d} \max \left\{ \| (x, y) \|^{d}, 1 \right\},
\]
and we also obtain the assertion.

(3) By symmetry, one can show (3) by the same arguments as (2).
We claim that there does not exist \((x, y) \in \mathbb{A}^2(\Omega)\) such that
\[(2.12) \quad \|x, y\| > [3d] \Delta L^{m+1}, \quad \|H_t(x, y)\| < \frac{1}{L^{m+1}}\|x, y\|^d \quad \text{and} \quad \|H_t^{-1}(x, y)\| < \frac{1}{L^{m+1}}\|x, y\|^d.\]

Indeed, suppose that there exists \((x, y) \in \mathbb{A}^2(\Omega)\) satisfying (2.12). It follows from the first condition that
\[(2.13) \quad \|x, y\|^d > [3d] \Delta L^{m+1}\|x, y\|^{d-1} \quad \text{and} \quad \|x, y\|^d > [3d] \Delta L^{m+1}.\]
Since \(L^{m+1} > [3d] \Delta\) by our choice of \(L\) (see (2.4)), the second condition in (2.12) gives
\[(2.14) \quad \|x, y\|^d > [3d] \Delta \|H_t(x, y)\|.
Similarly, the third condition gives
\[(2.15) \quad \|x, y\|^d > [3d] \Delta \|H_t^{-1}(x, y)\|.
These estimates (2.13), (2.14), (2.15) contradict (2.8).

Thus we have shown the claim. Taking the converse, we obtain (4). \(\square\)

**Proof of Proposition 2.4.** As in the proof of Lemma 2.3, we write \(M_n, A_n, B_n\) for \(M_n(t), A_n(t), B_n(t)\).

Let \(t \in B_L\). Let \(P_t = (a(t), b(t))\) be the initial point. Since \(\|P_t\| \leq L^{m+1} \leq [3d] \Delta L^{m+1}\) (see (2.4)), we have \(P_t \in V_L^+\). It follows from Lemma 2.5 (1) that \(H_t^n(P_t) \in V_L^+\) for any \(n \geq 1\) and Lemma 2.5 (2) says that
\[
\max \left\{ \|H_t^{n+1}(P_t)\|, 1 \right\} \geq \left[ \frac{1}{[3d] \Delta L^{m+1}} \right]^d \max \left\{ \|H_t^n(P_t)\|^d, 1 \right\}.
\]
Since \(H_t^n(P_t) = (A_n, A_{n-1})\), we obtain
\[
\max \left\{ |A_{n+1}|, |A_n|, 1 \right\} \geq \left[ \frac{1}{[3d] \Delta L^{m+1}} \right]^d \max \left\{ |A_n|^d, |A_{n-1}|^d, 1 \right\}.
\]
Similarly, using Lemma 2.5 (1) (3), we have
\[
\max \left\{ |B_{n+1}|, |B_n|, 1 \right\} \geq \left[ \frac{1}{[3d] \Delta L^{m+1}} \right]^d \max \left\{ |B_n|^d, |B_{n-1}|^d, 1 \right\}.
\]
Thus
\[
\max \left\{ |A_{n+1}|, |A_n|, |B_{n+1}|, |B_n|, 1 \right\} \geq \left[ \frac{1}{[3d] \Delta L^{m+1}} \right]^d \max \left\{ |A_n|, |A_{n-1}|, |B_n|, |B_{n-1}|, 1 \right\}^d,
\]
which completes the proof of Proposition 2.4. \(\square\)

### 2.3. Uniform convergence on an unbounded region of \(\mathbb{A}^1\).

In this subsection we compare \(M_{n+1}(t)\) with \(M_n(t)\) on \(\mathbb{A}^1(\Omega) \setminus B_L\). We begin with the following lemma.

**Lemma 2.6.** Let \(s \geq 2\) be an integer. Let
\[
\varphi(x, y) = \gamma y + cx^s + \sum_{i=1}^s c_i(t)x^{s-i} \in (\Omega[t])[x, y],
\]
where \(\gamma, c \in \Omega \setminus \{0\}\) and \(c_i(t) \in \Omega[t]\) for \(i = 1, \ldots, s\). We take an integer \(m \geq 1\) with \(m \geq \max_i \{\deg c_i(t)\}\). Let \(x(t), y(t) \in \Omega[t]\) be polynomials such that there exists an \(L' \geq 1\) with the following properties:

(i) \(|x(t)| \geq |t|^{m+1}\) for any \(t \in \Omega\) with \(|t| > L'\);
(ii) \(|y(t)/x(t)^s| \leq 1/|t|^s\) for any \(t \in \Omega\) with \(|t| > L'\).
Then, for any \( \kappa > 1 \), there exists an \( L \geq 1 \) with the following properties:

(a) \( L^{m+1} \geq \sqrt{s}/\kappa t \); 
(b) \( \frac{|c| |x(t)|^s}{\kappa} \leq |\varphi(x(t), y(t))| \leq \kappa |c| |x(t)|^s \) for any \( t \in \Omega \) with \( |t| > L \); 
(c) \( |\varphi(x(t), y(t))| \geq |t|^{m+1} \) for any \( t \in \Omega \) with \( |t| > L \); 
(d) \( \left| \frac{x(t)}{\varphi(x(t), y(t))^s} \right| \leq \frac{1}{|t|^s} \) for any \( t \in \Omega \) with \( |t| > L \).

Proof. First, we note that if we take any sufficiently large \( L \geq L' \), then (a) holds. We write 
\[
\varphi(x, y) = c x^s \left( 1 + \frac{1}{c} \left( \sum_{i=1}^{s} \frac{c_i(t)}{x^i} + \frac{\gamma y}{x^s} \right) \right).
\]

For any \( t \in \Omega \) with \( |t| \geq L' \), conditions (i) and (ii) give
\[
\left| \sum_{i=1}^{s} \frac{c_i(t)}{x^i} + \frac{\gamma y(t)}{x^s} \right| \leq [s + 1] \max \left\{ \frac{|c_1(t)|}{|t|^{m+1}}, \ldots, \frac{|c_s(t)|}{|t|^{s(m+1)}}, \frac{|\gamma|}{|t|^s} \right\}.
\]
Since \( \deg c_i(t) \leq m \) for \( i = 1, \ldots, s \), we have
\[
[s + 1] \lim_{|t| \to \infty} \max \left\{ \frac{|c_1(t)|}{|t|^{m+1}}, \ldots, \frac{|c_s(t)|}{|t|^{s(m+1)}}, \frac{|\gamma|}{|t|^s} \right\} = 0,
\]
and hence
\[
\lim_{|t| \to \infty} \left| \sum_{i=1}^{s} \frac{c_i(t)}{x^i} + \frac{\gamma y(t)}{x^s} \right| = 0.
\]

We set \( \alpha := 1 - 1/\kappa \). Since \( \kappa > 1 \), we have \( 0 < \alpha < 1 \). We note that \( 1 + \alpha = 2 - 1/\kappa < \kappa \). If necessary, we replace \( L \) by a larger number, and we may assume that
\[
\left| \sum_{i=1}^{s} \frac{c_i(t)}{x^i} + \frac{\gamma y(t)}{x^s} \right| \leq |c| \alpha \quad \text{for all} \ t \in \Omega \ \text{with} \ |t| > L.
\]
Then, for any \( t \in \Omega \) with \( |t| > L \), we have
\[
(1 - \alpha) |c| |x(t)|^s \leq |\varphi(x(t), y(t))| \leq (1 + \alpha) |c| |x(t)|^s.
\]
By the definition of \( \alpha \), we conclude that \( |c| |x(t)|^s/\kappa \leq |\varphi(x(t), y(t))| \leq \kappa |c| |x(t)|^s \). This proves (b).

For (c), by condition (i) and \( L \geq L' \), we have \( |x(t)| \geq |t|^{m+1} \) for any \( t \in \Omega \) with \( |t| > L \). Then
\[
|\varphi(x(t), y(t))| \geq \frac{|c| |x(t)|^s}{\kappa} \geq \frac{|c|}{\kappa} |t|^{(s-1)(m+1)} |t|^{m+1} \geq \frac{L^{(s-1)(m+1)} |c|}{\kappa} |t|^{m+1} \geq |t|^{m+1},
\]
where we use the property (a) in the last inequality. For (d), observe that
\[
\left| \frac{x(t)}{\varphi(x(t), y(t))^s} \right| \leq \frac{(\kappa/|c|)^s |x(t)|}{|x(t)|^{s^2}} \leq \frac{(\kappa/|c|)^s}{|x(t)|^{s^2-1}} \leq \frac{(\kappa/|c|)^s}{|t|^{(s^2-1)(m+1)}} \leq \frac{(\kappa/|c|)^s}{L^{((s^2-1)(m+1)-s)} |t|^s}.
\]
Since $s \geq 2$ and $m \geq 1$, we have $(s^2 - 1)(m + 1) - s \geq s(s - 1)(m + 1)$. If follows from (a) that
\[ L^{(s^2-1)(m+1)-s} \geq L^{s(s-1)(m+1)} \geq \left( \frac{\kappa}{|\delta|} \right)^s \]
and thus $|x(t)/\varphi(x(t), y(t))| \leq 1/|t|^s$ for all $t \in \Omega$ with $|t| \geq L$. \hfill \Box

**Remark 2.7.** The proof of Lemma 2.6 shows that, if a number $L$ satisfies $L \geq L'$, $L^{m+1} \geq +\sqrt{\kappa}/|\delta|$, and
\[ \sum_{i=1}^{s} \frac{c_i(t)}{t^{(m+1)}} + \frac{\gamma}{t^s} \leq |\delta| \left( 1 - \frac{1}{\kappa} \right) \]
then we have (a)–(d) for such $L$.

Applying Lemma 2.6 to the family of Hénon maps $H_t$ with initial point $P_t = (a(t), b(t))$ we have the following.

**Proposition 2.8.** Let $H = (H_t)_{t \in \Omega}$ be the family of Hénon maps, and let $P = (P_t)_{t \in \Omega}$ be the family of initial points as in the beginning of § 3. Fix a constant $\kappa > 1$. Then, the following statements hold.

1. If $(\deg A_n(t))_{n \geq 0}$ is unbounded, then there exist an $L_A \geq 1$ and an integer $N_A \geq 1$ such that for all $n \geq N_A$
   \[ ||H_t^n(P_t)|| = |A_n(t)| \quad \text{and} \quad \frac{|A_n(t)|^d}{\kappa} \leq ||H_t^{n+1}(P_t)|| \leq \kappa |A_n(t)|^d \]
   for all $t \in \Omega$ with $|t| > L_A$.

2. If $(\deg B_n(t))_{n \geq 0}$ is unbounded, then there exist an $L_B \geq 1$ and an integer $N_B \geq 1$ such that for all $n \geq N_B$
   \[ ||H_t^{-n}(P_t)|| = |B_n(t)| \quad \text{and} \quad \frac{|B_n(t)|^d}{\delta \kappa} \leq ||H_t^{-n-1}(P_t)|| \leq \frac{\kappa |B_n(t)|^d}{|\delta|} \]
   for all $t \in \Omega$ with $|t| > L_B$.

**Proof.** We start with the proof of (1). Assume that $(\deg A_n(t))_{n \geq 0}$ is unbounded. By Proposition 1.10, there exists an integer $N$ such that $\deg A_{n+1}(t) = d \deg A_n(t)$ ($> 0$) for all $n \geq N$. Let $m$ be the integer defined in (2.1). Replacing $N$ by a larger number if necessary, we may assume that $\deg A_{N-1}(t) > m + 1$ and $\deg A_N(t) - \deg A_{N-1}(t) > d$.

We claim that there exists an $L \geq 1$ such that, for all $t \in \Omega$ with $|t| > L$ and for all $n \geq N$, the following inequalities hold:

\[ |A_n(t)| \geq |t|^{m+1} \quad \text{and} \quad ||H_t^n(P_t)|| = |A_n(t)|, \]
\[ \frac{|A_n(t)|^d}{\kappa} \leq |A_{n+1}(t)| \leq \kappa |A_n(t)|^d, \]
\[ \frac{|A_n(t)|}{A_{n+1}(t)^d} \leq \frac{1}{|t|^d} \]

We prove the claim by induction on $n \geq N$.

The recursive relation (1.13) of $A_n(t)$ gives
\[ A_{N+1}(t) = \delta A_N(t) + A_N(t)^d + \sum_{i=1}^{d} c_i(t) A_N(t)^{d-i}. \]
By our choice of $N$, we can find an $L' \geq 1$ such that, for all $|t| > L'$, (i) $|A_N(t)| \geq |t|^{m+1}$ and (ii) $|A_{N-1}(t)/A_N(t)| \leq 1/|t|^d$. In particular, $|A_{N-1}(t)/A_N(t)| \leq 1$. Hence, for any $t \in \Omega$ with $|t| > L'$, we have (2.17) with $n = N$.

Take $\varphi(x, y) := \delta y + f_t(x), \gamma := \delta, c := 1, s := \deg_x f_t(x) = d, x := A_N(t)$ and $y := A_{N-1}(t)$ in Lemma 2.6. We note that

$$|A_{N-1}(t)/A_N(t)^d| = |A_{N-1}(t)/A_N(t)| |1/A_N(t)^{d-1}| \leq |A_{N-1}(t)/A_N(t)| \leq 1/|t|^d.$$  

We take $L \geq L'$ such that $L^{m+1} \geq d^{-1} \sqrt{\kappa}$ and such that (2.16) holds. Then by Remark 2.7 and Lemma 2.6 (b) (d), we have (2.18) and (2.19) with $n = N$.

By induction on $n$, we assume that (2.17), (2.18) and (2.19) hold for all $t \in \Omega$ with $|t| > L$ for $n$, and we are going to deduce them for $n + 1$.

Let $t \in \Omega$ with $|t| > L$. Since $|A_n(t)| \geq |t|^{m+1} \geq L^{m+1}$ by (2.17), it follows from (2.18) that

$$|B_{n+1}(t)| \geq \left( \frac{|A_{n}(t)|}{\kappa} \right)^{d-1} |A_n(t)| \geq \left( \frac{(L^{m+1})^{d-1}}{\kappa} \right) |A_n(t)| \geq |A_n(t)| \geq |t|^{m+1}$$

and

$$\left| \frac{A_{n}(t)}{B_{n+1}(t)} \right| \leq \frac{\kappa |A_{n}(t)|}{|A_{n}(t)|^{d-1}} \leq \frac{\kappa}{L^{(d-1)(m+1)}} \leq 1.$$

Thus $\|H_t^{n+1}(P_t)\| = |A_{n+1}(t)|$ and (2.17) is satisfied for $n + 1$. Note that (2.19) and (2.20) show that conditions (i) and (ii) in Lemma 2.6 are satisfied for $s = d, x(t) := A_{n+1}(t)$ and $y(t) := A_n(t)$. Now (2.18) and (2.19) for $n + 1$ are simply the conclusions of Remark 2.7 and Lemma 2.6 (b) (d). Thus the claim is proved and by setting $N_A = N$ and $L_A = L$, we obtain (1).

The proof of (2) is similar to that of (1), so that we only give a sketch of it. We choose an integer $N_B$ such that $\deg B_{n+1}(t) = d \deg B_n(t) > 0$ for all $n \geq N_B$, $\deg B_{N_B-1}(t) > m + 1$ and $\deg B_{N_B}(t) - \deg B_{N_B-1}(t) > d$.

We take the polynomial $\varphi(x, y) := (1/\delta)y - (1/\delta)f_t(x), \gamma := 1/\delta, c = -1/\delta, s := d, x(t) = B_{n}(t)$ and $y(t) = B_{N-1}(t)$ in Lemma 2.6. We take $L_B \geq 1$ such that, for all $t \in \Omega$ with $|t| > L_B$, we have (i) $|B_{n}(t)| \geq |t|^{m+1}$ and (ii) $|B_{n-1}(t)/B_{n}(t)| \leq 1/|t|^d$ and such that $(L_B)^{m+1} \geq d^{-1} \sqrt{\kappa}$ and (2.16) holds. Then by a similar argument as in (1), one can show that for all $t \in \Omega$ with $|t| > L_B$ and for all $n \geq N_B$, the following equalities hold:

$$|B_{n}(t)| \geq |t|^{m+1} \text{ and } \|H_t^{-n}(P_t)\| = |B_{n}(t)|,$$

$$\left| \frac{B_{n}(t)}{B_{n+1}(t)} \right| \leq \frac{\kappa |B_{n}(t)|}{|\delta|} \leq \frac{\kappa |B_{n}(t)|}{|\delta|},$$

$$\left| \frac{B_{n}(t)}{B_{n+1}(t)} \right| \leq \frac{1}{|t|^d}.$$  

Then (2.21) and (2.22) give the assertion (2). \hfill \Box

As a corollary to Proposition 2.8, we have the main result of this subsection.

**Proposition 2.9.** Fix a constant $\kappa > 1$. Then, under Assumption 1.8, there exist an $L \geq 1$ and $N \in \mathbb{Z}_{> 0}$ such that

$$\frac{M_n^d(t)}{\Delta \kappa} \leq M_{n+1}(t) \leq \Delta \kappa M_n^d(t)$$

holds for all $t \in \Omega$ with $|t| > L$ and all $n \geq N$.  


Proof. Case 1. Suppose that both \((\deg A_n(t))_{n \geq 0}\) and \((\deg B_n(t))_{n \geq 0}\) are unbounded. Then we set \(L := \max \{L_A, L_B\}\) and \(N := \max \{N_A, N_B\}\) in Proposition 2.8 and then the estimates in (1) and (2) in Proposition 2.8 hold for \(A_n(t)\) and \(B_n(t)\) for all \(n \geq N\) and all \(|t| > L\). We note that \(\Delta := \max(\delta, 1/|\delta|) \geq 1\). It follows from Proposition 2.8(1) (2) that

\[
\frac{\|H_i^d(P)\|}{\Delta \kappa} \leq \|H_{i+1}^d(P)\| \leq \Delta \kappa \|H_i^d(P)\|
\]

\[
\frac{\|H_i^{-d}(P)\|}{\Delta \kappa} \leq \|H_{i-1}^{-d}(P)\| \leq \Delta \kappa \|H_i^{-d}(P)\|
\]

for all \(n \geq N\) and all \(|t| > L\). Our assertion now follows from the definition of \(M_n(t)\) (see Definition 2.2).

Case 2. Suppose that only one of the sequences \((\deg A_n(t))_{n \geq 0}\) and \((\deg B_n(t))_{n \geq 0}\) is unbounded. We will only give a proof for the case where \((\deg A_n(t))_{n \geq 0}\) is unbounded. The case where \((\deg B_n(t))_{n \geq 0}\) is unbounded can be proved similarly with the same choice of \(\varepsilon\) below.

We fix an integer \(D\) such that \(D > \deg B_n(t)\) for any \(n \geq 0\) and \(D > \deg c_i(t)\) for any \(i = 1, \ldots, d\) (cf. (1.8)). We fix an integer \(N\) with \(\deg A_N(t) \geq D\), whose existence is assured by the unbounded assumption of \((\deg A_n(t))_{n \geq 0}\), and such that the conclusion of Proposition 2.8(1) and equations (2.17), (2.18) and (2.19) hold with some \(L \geq 1\).

We fix any \(0 < \varepsilon < 1\) satisfying \((|d + 2| \kappa \Delta) \varepsilon^{d-1} \leq 1\). Note that, for such a \(\varepsilon\), we have \(\max\{\kappa \varepsilon^{-1}, (\kappa |d + 2|/|\delta|) \varepsilon^{d-1}\} \leq 1\). We replace \(L\) by a larger number if necessary, we may assume that, for all \(t \in \Omega\) with \(|t| > L\), we have

\[
\left| \frac{1}{A_N(t)} \right| \leq \varepsilon, \quad \left| \frac{B_{N-1}(t)}{A_N(t)} \right| \leq \varepsilon, \quad \left| \frac{B_N(t)}{A_N(t)} \right| \leq \varepsilon \quad \text{and} \quad \left| \frac{c_i(t)}{A_N(t)} \right| \leq \varepsilon \quad \text{for any} \quad i = 1, \ldots, d,
\]

where the \(c_i(t)\) are coefficients of \(f_i(x)\) (see (1.8)). With this setting, we claim the following.

Claim 2.1. For all \(t \in \Omega\) with \(|t| > L\) and all \(n \geq N\), the following inequalities hold:

\[
(2.23) \quad \left| \frac{1}{A_n(t)} \right| \leq \varepsilon, \quad \left| \frac{B_{n-1}(t)}{A_n(t)} \right| \leq \varepsilon, \quad \left| \frac{B_n(t)}{A_n(t)} \right| \leq \varepsilon \quad \text{and} \quad \left| \frac{c_i(t)}{A_n(t)} \right| \leq \varepsilon \quad \text{for} \quad i = 1, \ldots, d.
\]

We prove the claim by induction on integers \(n \geq N\). Write \(n = N + j, \ j \geq 0\) for \(n \geq N\). Then the claimed inequalities hold for the case \(j = 0\) by our choice of \(N\) and \(L\). Assume that (2.23) hold for integer \(j \geq 0\). It follows from the recursive relation (1.14) for \(B_n(t)\) and (2.18) that

\[
\left| \frac{B_{N+j+1}(t)}{A_{N+j+1}(t)} \right| \leq \kappa |d + 2| \max \left( \frac{|B_{N+j-1}(t)|}{|A_{N+j+1}(t)|^d}, \frac{|B_{N+j}(t)|^d}{|A_{N+j+1}(t)|^d}, \ldots, \frac{|c_i(t)B_{N+j-i}(t)|}{|A_{N+j+1}(t)|^d}, \ldots, \frac{|c_d(t)|}{|A_{N+j+1}(t)|^d} \right)
\]

\[
\leq \frac{\kappa |d + 2| \varepsilon^d}{|\delta|} \quad \text{by induction hypothesis},
\]

\[
\leq \varepsilon \quad \text{(since} \ (\kappa |d + 2|/|\delta|) \varepsilon^{d-1} \leq 1).\]
The induction hypothesis together with (2.18) also ensure that

\[
\begin{align*}
\frac{1}{A_{N+j+1}(t)} & \leq \frac{\kappa}{|A_{N+j}(t)|^d} \leq \kappa \varepsilon^d \leq (\kappa \varepsilon^{d-1}) \varepsilon \leq \varepsilon, \\
\frac{B_{N+j}(t)}{A_{N+j+1}(t)} & \leq \frac{\kappa |B_{N+j}(t)|}{|A_{N+j}(t)|^d} \leq (\kappa \varepsilon^{d-1}) \varepsilon \leq \varepsilon, \\
\frac{\ell d}{A_{N+j+1}(t)} & \leq \frac{\ell d}{A_{N+j}(t)} \leq (\kappa \varepsilon^{d-1}) \varepsilon \leq \varepsilon \quad \text{for } i = 1, \ldots, d.
\end{align*}
\]

This finishes the induction steps and hence the claim is proved.

Claim [2.1] and Proposition [2.8] (1) imply that \( M_n(t) = |A_n(t)| \) for all \( t \in \Omega \) with \( |t| > L \) and all \( n \geq N \). Noting that \( \Delta := \max\{|\delta|, |1/\delta|\} \geq 1 \) and applying Proposition [2.8] (1), we have

\[
\frac{M^d_n(t)}{\Delta \kappa} \leq M_{n+1}(t) \leq \Delta \kappa M^d_n(t).
\]

(We note that \( \Delta \) is needed when we treat the case where \((\deg B_n(t))_{n \geq 0}\) is unbounded.) \( \Box \)

2.4. Proof of Theorem [2.1] Combining the previous two subsections, we now give a proof for our main result in this section.

Proof of Theorem [2.1] Recall that

\[
\overline{\Theta}_n : \mathbb{P}^1 \to \mathbb{P}^4
\]

is given by \((t : 1) \mapsto (A_n(t) : A_{n-1}(t) : B_{n-1}(t) : B_n(t) : 1)\)

and that \( \ell_n = \deg(\overline{\Theta}_n) \). By Proposition [1.10], there exist a positive rational number \( \ell \) and a positive integer \( N \) such that \( \ell_n = d \ell \) for all \( n \geq N \). We fix such an \( N \).

For \( n \geq N \), we take the lift \( A^2 \to A^3 \) of \( \overline{\Theta}_n \) given by \((t, 1) \mapsto (A_n(t), A_{n-1}(t), B_{n-1}(t), B_n(t), 1)\).

Then the lift gives rise to an isomorphism \( \alpha_n : \overline{\Theta}_n^* \mathcal{O}_{\mathbb{P}^1}(1) \stackrel{\sim}{\to} \mathcal{O}_{\mathbb{P}^1}(\ell_n) \). Let \( \| \cdot \|_n \) be the metric on \( \mathcal{O}_{\mathbb{P}^1}(1) \) defined by \( \| \cdot \|_n := (\alpha_n(\| \cdot \|_n))^{1/\ell_n} \).

Our goal is to show that \( \log(\| \cdot \|_{n+1}/\| \cdot \|_n) \) converges uniformly on \( \mathbb{P}^1(\Omega) \) as \( n \) tends to \( \infty \).

For \((t' : t'') \in \mathbb{P}^1(\Omega)\), we have

\[
\log \left( \frac{\| \cdot \|_{n+1}(t' : t'')}{\| \cdot \|_n(t' : t'')} \right) = \log \left( \frac{\overline{\Theta}_n(t' : t'')^{1/\ell_n}}{\overline{\Theta}_{n+1}(t' : t'')^{1/\ell_{n+1}}} \right) = \begin{cases} \log \frac{M_n(t'/t'')^{1/\ell_n}}{M_{n+1}(t'/t'')^{1/\ell_{n+1}}} & \text{if } t'' \neq 0, \\ \log \frac{\| \Phi_n(1 : 0) \|^{1/\ell_n}}{\| \Phi_{n+1}(1 : 0) \|^{1/\ell_{n+1}}} & \text{if } t'' = 0, \end{cases}
\]

see (2.3) and Definition [2.2].

We set \( \kappa := \max\{|3d| \Delta, 2\} \). Let \( m \) be as in (2.1). Choose \( L \geq 1 \) such that \( L^{m+1} \geq d \sqrt{\kappa \Delta} \) and that \( \| P_t \| \leq L^{m+1} \) for \( t \in B_L \). We replace \( L \) and \( N \) by larger numbers if necessary, so that Lemma [2.3], Proposition [2.4] and Proposition [2.9] hold.

We claim that, for any \( n \geq N \) and for any \((t' : t'') \in \mathbb{P}^1(\Omega)\), we have

\[
\log \frac{\| \cdot \|_{n+1}(t' : t'')}{\| \cdot \|_n(t' : t'')} \leq \frac{1}{ld^n} \log(\kappa L^{m+1}).
\]
Case 1. Suppose that \( t'' \neq 0 \) and \( t := t'/t'' \) is in the bounded region \( B_L \). Then, for all \( n \geq N \), we have
\[
\frac{M_n(t)^d}{\kappa L^{m+1}} \leq \frac{M_n(t)^d}{([3d]\Delta L^{m+1})^d} \leq M_{n+1}(t) \quad (\text{by Proposition 2.4})
\]
\[
\leq [d + 2]\Delta L^{m+1} M_n(t)^d \quad (\text{by Lemma 2.3})
\]
\[
\leq [3d]\Delta L^{m+1} M_n(t)^d \leq (\kappa L^{m+1}) M_n(t)^d \leq (\kappa L^{m+1})^d M_n(t)^d.
\]
Thus
\[
\left| \log \frac{\| \cdot \|_{n+1}}{\| \cdot \|_n} (t : 1) \right| = \left| \log \frac{M_n(t)^{1/\ell_n}}{M_{n+1}(t)^{1/\ell_{n+1}}} \right|
\]
\[
= \frac{1}{\ell_{d^n+1}} \left| \log M_{n+1}(t) - \log M_n(t)^d \right| \leq \frac{1}{\ell d^n} \log (\kappa L^{m+1}).
\]
The claim now follows in this case.

Case 2. Suppose that \( t'' \neq 0 \) and \( t := t'/t'' \) is outside \( B_L \). By Proposition 2.9, for all \( n \geq N \), we then have
\[
\left| \log \frac{\| \cdot \|_{n+1}}{\| \cdot \|_n} (t : 1) \right| = \frac{1}{\ell d^n+1} \left| \log M_{n+1}(t) - \log M_n(t)^d \right|
\]
\[
\leq \frac{1}{\ell d^n+1} \log (\kappa \Delta) \leq \frac{1}{\ell d^n} \log (\kappa L^{m+1}),
\]
where in the last inequality we use \((\kappa L^{m+1})^d = (L^{m+1})^d - (\kappa L^{m+1})^d \geq (L^{m+1})^d - 1 \geq \kappa \Delta\). We obtain the claim in these case as well.

Case 3. Suppose that \( t'' = 0 \). In this case, \((t' : t'') = (1 : 0)\).

Subcase 3-1. Suppose that \((\deg(A_n(t)))_{n \geq 0}\) is unbounded, but \((\deg(B_n(t)))_{n \geq 0}\) is bounded. It follows from \((1.19)\) that for all \( n \geq N \),
\[
\log \| \Phi_n(1,0) \|^{|1/\ell_n} = \frac{1}{d^n \ell} \log |\alpha_D|^{d_n-N} = \frac{1}{d^N \ell} \log |\alpha_D| = \| \Phi_{n+1}(1,0) \|^{|1/\ell_{n+1}}.
\]
Thus, for all \( n \geq N \), we have
\[
(2.25) \quad \left| \log \frac{\| \cdot \|_{n+1}}{\| \cdot \|_n} (1 : 0) \right| = \log \frac{\| \Phi_n(1,0) \|^{|1/\ell_n}}{\| \Phi_{n+1}(1,0) \|^{|1/\ell_{n+1}}} = 0.
\]

Subcase 3-2. Suppose that \((\deg(B_n(t)))_{n \geq 0}\) is unbounded, but \((\deg(A_n(t)))_{n \geq 0}\) is bounded. Then using \((1.20)\), we similarly have \((2.25)\).

Subcase 3-3. Suppose that both \((\deg(A_n(t)))_{n \geq 0}\) and \((\deg(B_n(t)))_{n \geq 0}\) are unbounded. We may take \( N = N' \) in \((1.19)\) and \((1.20)\). If \( D > D' \), then as in Subcase 3-1, we have \((2.25)\). If \( D < D' \), then as in Subcase 3-2, we have \((2.25)\). If \( D = D' \), then we have, for any \( n \geq N \),
\[
\log \| \Phi_n(1,0) \|^{|1/\ell_n} = \frac{1}{d^n \ell} \log \max \{|\alpha_D|^{d_n-N}, |\beta_D|^{d_n-N}\}
\]
\[
= \frac{1}{d^N \ell} \log \max \{|\alpha_D|, |\beta_D|\} = \| \Phi_{n+1}(1,0) \|^{|1/\ell_{n+1}},
\]
and again we have \((2.25)\).

We have obtained the claim. By \((2.24)\), \{\log \| \cdot \|_n\}_{n \geq N}\) converges uniformly on the parameter space \( \mathbb{P}^1(\Omega) \). This completes the proof of Theorem 2.1. \(\square\)
Let $K$ be number field or a function field $K = F(B)$ of an integral projective variety $B$ over a field $F$ that is regular in codimension one. Let the family of Hénon maps $H = (H_t)_{t \in \mathbb{R}}$ and the family of initial points $P = (a(t), b(t)) = (P_t)_{t \in \mathbb{R}} \in \mathbb{A}^2(K[t])$ be as in §2.1. By Theorem 2.1, associated to $H$ and $P$, there exists a sequence of adelic metrics $\{\|\cdot\|_{n_v}\}_{v \in M_K}$ of $\mathcal{O}_{\mathbb{P}^1}(1)$ such that $\log \|\cdot\|_{n_v}$ converges uniformly to $\log \|\cdot\|_v$ as $n \to \infty$ for every $v \in M_K$.

**Proposition 3.1.** The pair $\mathcal{L}_P = (\mathcal{O}_{\mathbb{P}^1}(1), \{\|\cdot\|_v\}_{v \in M_K})$ is a semipositive adelicly metrized line bundle over $\mathbb{P}^1$ over $K$.

**Proof.** For brevity, in what follows, we will write $B$ for both $\text{Spec}(O_K)$ if $K$ is a number field and the underlying projective variety if $K = F(B)$ is a function field over another field $F$. As in §1.6, we identify $M_K^n$ with $\text{Spec}(O_K) \setminus \{(0)\}$ if $K$ is a number field, and with the set of all generic points of prime divisors of $B$ if $K = F(B)$.

Let $U$ be a non-empty Zariski open subset of $B$ with the following properties:

- We regard $U$ as a subset of $M_K^n$. (When $K = F(B)$, a place $v \in M_K^n$ belongs to $U$ if the prime divisor corresponding to $v$ intersects with $U$.) Then for each $v \in U$, all the coefficients of $c_i(t)$ $(i = 1, 2, \ldots, d)$ have absolute value 1 with respect to $|\cdot|_v$ (cf. (1.8));
- For each $v \in U$, all the coefficients of $a(t)$ and $b(t)$ have absolute value 1 with respect to $|\cdot|_v$;
- For each $v \in U$, if $(\deg A_n(t))_{n \geq 0}$ is unbounded, then the leading coefficients of $A_n(t)$ has absolute value 1 with respect to $|\cdot|_v$ for sufficiently large $n$. For each $v \in U$, if $(\deg B_n(t))_{n \geq 0}$ is unbounded, then the leading coefficients of $B_n(t)$ has absolute value 1 with respect to $|\cdot|_v$ for sufficiently large $n$. (See Remark 1.11.)
- For any sufficiently large $n$, the non-constant morphism $\Phi_n : \mathbb{P}^1 \to \mathbb{P}^4$ over $K$ extends to a morphism, denoted by $\Phi_{n,U}$, over $U$.

Then $M_K \setminus U$ is a finite set of places. In the following, we assume that $n$ is sufficiently large such that $\Phi_n : \mathbb{P}^1 \to \mathbb{P}^4$ is a non-constant morphism defined over $K$. Then the morphism $\Phi_n$ induces a rational map, still denoted by the same $\Phi_n$, over $B$.

$$\Phi_n : \mathbb{P}^1_B \dashrightarrow \mathbb{P}^4_B.$$  

Let $\mathbb{P}^1_B$ be the normalization of the map $\mathbb{P}^1_U \xrightarrow{\Phi_{n,U}} \mathbb{P}^4_U \hookrightarrow \mathbb{P}^4_B$ (cf. [Ha77, II. Ex. 3.8]). Then we have a morphism $\Phi_n : \mathbb{P}^1_B \to \mathbb{P}^4_B$. We set $\mathcal{L}_n := \Phi_n^*(\mathcal{O}_{\mathbb{P}^4}(1))$. Note that, since $\mathcal{O}_{\mathbb{P}^4}(1)$ is relatively ample and $\Phi_n$ is the normalization, $\mathcal{L}_n$ is relatively ample, and in particular relatively nef.

For $v \in M_K^n$, we endow $\mathcal{O}_{\mathbb{P}^1}(1) \otimes_B \mathbb{K}_v$ with the metric $\|\cdot\|_{n,v}$ induced from the model $(\mathbb{P}^1_B, \mathcal{L}_n)$. For $v \in M_K^n$, we endow $\mathcal{O}_{\mathbb{P}^1}(1) \otimes_B \mathbb{K}_v$ with the metric $\|\cdot\|_{n,v}$ by pulling back the Fubini-Study metric $\|\cdot\|_{FS,v}$ on $\mathcal{O}_{\mathbb{P}^1}(1)$.

We show that $(\mathcal{O}_{\mathbb{P}^1}(1), \{\|\cdot\|_{n,v}\})$ satisfies the conditions (a)(b)(c) in §1.6 and converges to $\mathcal{L}_P := (\mathcal{O}_{\mathbb{P}^1}(1), \{\|\cdot\|_{v}\}_{v \in M_K})$. Then by definition, $\mathcal{L}_P$ is a semipositive adelicly metrized line bundle.

We check (a). Let $v \in M_K^n$. Note that the standard metric $\|\cdot\|_{st,v}$ is the metric induced from the model $\mathcal{O}_{\mathbb{P}^1}(1)$. Then $\|\cdot\|_{n,v}$ is the pull-back of $\|\cdot\|_{st,v}$, and the convergence $\log(\|\cdot\|_{n,v}/\|\cdot\|_v)$ follows from Theorem 2.1. Let $v \in M_K^n$. Since

$$(1/5)\|\cdot\|_{st,v}^2 \leq \|\cdot\|_{FS,v}^2 \leq \|\cdot\|_{st,v}^2$$
and $5^{1/\ell_n} \to 1$ as $n \to +\infty$, the convergence of $\log(||-||_{n,v}/||\cdot||_v)$ also follows from Theorem 2.1.

Thus we have checked (a).

Condition (b) is obvious, because the Fubini-study metric is smooth and has positive curvature. Condition (c) follows from the definition of $(\mathcal{O}_{\mathbb{P}^1}(1), \{|| \cdot ||'_{n,v}\})$ where $e_n := \ell_n$ (see (1.16)).

It remains to show that $(\mathcal{O}_{\mathbb{P}^1}(1), \{|| \cdot ||'_{n,v}\})$ converges to $\mathcal{L}_P$. Since we have shown that $\log(|| \cdot ||'_{n,v}/||\cdot||_v)$ uniformly converges to 0, it suffices to show that $\| \cdot \|'_{n,v} = || \cdot ||_v$ for all sufficiently large $n$ and for all $v \in U$. Let $v \in U$, and we take any $(a : 1) \in \mathbb{P}^1(K_v)$. Note that all the coefficients of $A_n(t), B_n(t)$ have absolute value at most 1 with respect to $|| \cdot ||_v$, and that the leading coefficients of $A_n(t)$ (or $B_n(t)$) have absolute value 1 if the degrees of $A_n(t)$ (respectively $B_n(t)$) are unbounded as $n$ runs through all positive integers. Then, we have

$$
\| (A_n(a), A_{n-1}(a), B_{n-1}(a), B_n(a), 1) \|_v = \begin{cases} 1 & \text{if } |a|_v \leq 1, \\ |a\ell_n|_v & \text{if } |a|_v > 1. \end{cases}
$$

It follows that $\| \cdot \|'_{n,v}$ coincides with the standard metric $|| \cdot ||_{st,v}$ on $\mathcal{O}_{\mathbb{P}^1}(1)$, and thus $\| \cdot \|'_{n,v} = || \cdot ||_v$ ($= || \cdot ||_{st,v}$) for all $v \in U$ and any sufficiently large $n$. \hfill $\square$

**Definition 3.2.** We denote by $h_{\mathcal{L}_P} : \mathbb{P}^1(K) \to \mathbb{R}_{\geq 0}$ the height function associated to $\mathcal{L}_P$.

Before we prove Theorem A, we introduce the function $G_{P,v} : \mathbb{A}^1(K_v) \to \mathbb{R}_{\geq 0}$ defined by (3.2)

$$G_{P,v}(t) := G_v(P_t) = \max \{ G^+_v(P_t), G^-_v(P_t) \} \quad \text{for } v \in M_K,
$$

where $G_v(P_t)$ is the $v$-adic Green function as in (1.6). Recall from Definition 2.2 that $M_n(t) = ||(H_t^n(P_t), H_{t}^{-n}(P_t), 1)||_v$.

**Lemma 3.3.** We have $G_{P,v}(t) = \lim_{n \to \infty} \frac{1}{d^n} \log M_{n}(t)$ for all $t \in \mathbb{A}^1(K_v)$.

**Proof.** In general, suppose that $\{a_n\}_{n \geq 1}$ and $\{b_n\}_{n \geq 1}$ are convergent sequences of real numbers, and we write $\lim_{n \to \infty} a_n = \alpha$ and $\lim_{n \to \infty} b_n = \beta$. We set $c_n = \max\{a_n, b_n\}$. Then we see that $\{c_n\}_{n \geq 1}$ is also a convergent sequence, and that $\lim_{n \to \infty} c_n = \max\{\alpha, \beta\}$.

We apply this observation to $a_n = (\log^+ ||H_t^n(P_t)||)/d^n$ and $b_n = (\log^+ ||H_t^{-n}(P_t)||)/d^n$. Then we obtain the assertion. \hfill $\square$

Now we prove Theorem A by showing the following Theorem.

**Theorem 3.4.** With the notation and assumption as in Theorem A, we have $h_P = h_{\mathcal{L}_P}$.

**Proof.** For any $Q = (t : 1) \in \mathbb{P}^1(K)$ in the parameter space, we take a nonzero rational section $\eta$ of $\mathcal{O}_{\mathbb{P}^1}(1)$ whose support is disjoint from the Galois conjugates of $Q$ over $K$.

Let $v \in M_K$. Setting $\Omega = K_v$ in Section 2, we denote $M_n$ in Definition 2.2 by $M_{n,v}$. Further, for each place $v \in M_K$ we fix an embedding $K \hookrightarrow K_v$.

Then for any embedding $\sigma : K(Q) \to K$, it follows from (1.18), (2.3), Definition 2.2 and Lemma 3.3 that

$$
\log ||\eta(Q^\sigma)||_v = \log |\eta(Q^\sigma)|_v - \lim_{n \to \infty} \frac{1}{\ell_n} \log M_{n,v}(t^\sigma) = \log |\eta(Q^\sigma)|_v - \lim_{n \to \infty} \frac{1}{\ell d^n} \log M_{n,v}(t^\sigma) = \log |\eta(Q^\sigma)|_v - \frac{G_{P,v}(t^\sigma)}{\ell}.
$$
Then, we have
\begin{equation}
(3.3) \quad h_{\mathbb{Z}}(Q) = \frac{1}{[K(Q) : K]} \sum_{\sigma : K(\mathbb{Q}) \rightarrow \mathbb{K}} \sum_{v \in \mathcal{M}_K} -n_v \left( \log |\eta(Q^\sigma)|_v - \frac{G_{P^\sigma}(t^\sigma)}{\ell} \right)
\end{equation}
\begin{equation}
= \frac{1}{\ell} \left( \frac{1}{[K(t) : K]} \sum_{v \in \mathcal{M}_K} \sum_{\sigma : K(t) \rightarrow \mathbb{K}_v} n_v G_{P^\sigma}(t^\sigma) \right) = \frac{1}{\ell} \tilde{h}_{H_t}(P_t).
\end{equation}

Since \( P \in \mathbb{A}^2(K[t]) \), Proposition 1.10 (4) gives
\begin{equation}
(3.4) \quad \tilde{h}_H(P) = \lim_{n \to \infty} \frac{\deg \Phi_n}{d^n} = \ell.
\end{equation}

We obtain
\begin{equation}
(3.5) \quad h_P(t) := \frac{\tilde{h}_{H_t}(P_t)}{h_H(P)} = \frac{\tilde{h}_{H_t}(P_t)}{\ell}.
\end{equation}

Comparing (3.3) and (3.5), we obtain the assertion. \( \square \)

4. Local properties of parameter space height \( h_P \): A first application

Let \( K \) be a product formula field. Let \( H = (H_t)_{t \in \mathbb{K}} \) be a family of Hénon maps and let \( P = (a(t), b(t)) = (P_t)_{t \in \mathbb{K}} \) be a family of initial points satisfying Assumption 1.8 as in §1.4. Recall from Introduction that
\begin{equation}
(4.1) \quad \Sigma(P) := \{ t \in \mathbb{A}^1(\mathbb{K}) \mid \text{\( P_t \) is periodic with respect to \( H_t \)} \}.
\end{equation}

In this section, we establish some properties of \( h_P \) on the parameter space.

**Proposition 4.1.** Suppose that \( K \) is a number field. Then, for \( t \in \mathbb{A}^1(\mathbb{K}) \), \( P_t \) is periodic with respect to \( H_t \) if and only if \( h_P(t) = 0 \). In other words, we have
\[ \Sigma(P) = \{ t \in \mathbb{A}^1(\mathbb{K}) \mid h_P(t) = 0 \}. \]

**Proof.** Since \( h_P(t) := \tilde{h}_{H_t}(P_t)/h_H(P) \) by definition, \( h_P(t) = 0 \) is equivalent to \( \tilde{h}_{H_t}(P_t) = 0 \). The assertion follows from Proposition 1.3 (2). \( \square \)

Next, we decompose the parameter space height \( h_P \) into the sum of \( v \)-adic functions. Note that \( h_P(t) := \tilde{h}_{H_t}(P_t)/h_H(P) \) is defined over any product formula field \( K \) and that the function \( G_{P^\sigma} \) in (3.2) is defined for any place \( v \in M_K \).

**Proposition 4.2.** Let \( K \) be a product formula field. Then for any \( t \in \mathbb{A}^1(\mathbb{K}) \), we have
\begin{equation}
(4.2) \quad h_P(t) = \frac{1}{[K(t) : K]} \sum_{v \in M_K} \sum_{\sigma : K(t) \rightarrow \mathbb{K}_v} n_v G_{P^\sigma}(t^\sigma) \frac{1}{\ell}.
\end{equation}

**Proof.** As we saw in the proof of Theorem 3.4, the definition of \( \tilde{h}_{H_t}(P_t) \) gives
\begin{equation}
\tilde{h}_{H_t}(P_t) = \frac{1}{[K(t) : K]} \sum_{v \in M_K} \sum_{\sigma : K(t) \rightarrow \mathbb{K}_v} n_v G_{P^\sigma}(t^\sigma).
\end{equation}

Also we have \( h_P(t) = \tilde{h}_{H_t}(P_t)/\ell \) by (3.5), which holds for any product formula field. Thus we obtain the assertion. \( \square \)

As an application of \( h_P \), we show that the set of parameter values \( t \) where \( \{ \| H_t^a(P_t) \|_v \}_{a \in \mathbb{Z}} \) is bounded is characterized as the zero set of \( G_{P^\sigma} \).
Let $H$ and $P$ be as in Theorem $\text{[A]}$. In particular, we assume that $P$ satisfies Assumption $\text{[1.8]}$. Let $v \in M_K$. We set
\[
W_{P,v} := \{ t \in \mathbb{A}^1(K_v) \mid \| (H^n_t(P), H^{-n}_t(P)) \|_v \to +\infty \text{ as } n \to +\infty \},
\]
\[
K_{P,v} := \{ t \in \mathbb{A}^1(K_v) \mid \{ \| H^n_t(P) \|_v \}_{n \in \mathbb{Z}} \text{ is bounded} \}.
\]

**Theorem 4.3.** We have $\mathbb{A}^1(K_v) = W_{P,v} \amalg K_{P,v}$ (disjoint union), and $K_{P,v}$ is exactly the set of points where $G_{P,v}$ vanish:
\[
K_{P,v} = \{ t \in \mathbb{A}^1(K_v) \mid G_{P,v}(t) = 0 \}.
\]

**Proof.** For the first assertion, let $t \in \mathbb{A}^1(K_v)$ and suppose that $\{ \| H^n_t(P) \|_v \}_{n \in \mathbb{Z}}$ is not bounded. Then $\{ \| H^n_t(P) \|_v \}_{n \geq 0}$ is not bounded, or $\{ \| H^n_t(P) \|_v \}_{n \leq 0}$ is not bounded. In the former case, it follows from $\text{[Ka13, Theorem 3.1 (2)]}$ that $\lim_{n \to +\infty} \| H^n_t(P) \|_v = +\infty$. In the latter case, we similarly have $\lim_{n \to -\infty} \| H^n_t(P) \|_v = +\infty$ (see $\text{[Ka13, p. 1237]}$). Thus $\mathbb{A}^1(K_v) = W_{P,v} \amalg K_{P,v}$ (disjoint union).

We show the second assertion. Let $t \in \mathbb{A}^1(K_v)$. Suppose that $t \in K_{P,v}$. This means that $\{ \| A_n(t), A_{n-1}(t), B_{n-1}(t), B_n(t), 1 \|_v \}_{n \geq 1}$ is bounded, because $H^n_t(P) = (A_n(t), A_{n-1}(t))$ and $H^{-n}_t(P) = (B_{n-1}(t), B_n(t))$ for any $n \geq 1$. Then Lemma $\text{[3.3]}$ gives $G_{P,v}(t) = 0$.

Next suppose that $t \not\in K_{P,v}$. By $\text{[Ka13, Theorem 3.1 (1) and p. 1237]}$ and Lemma $\text{[3.3]}$, we then have $G_{P,v}(t) > 0$. This completes the proof. \(\square\)

**Example 4.4.** Let $H = \{ H_t \}$, with $H_t(x, y) = (y + x^2 + t, x)$ be as in Example $\text{[1.6]}$ where $H$ is regarded as a Hénon map defined over $\mathbb{Q}(t)$. We start from a constant initial point $P = (a, b)$ with $a, b \in \mathbb{Q}$, and we consider the archimedean place $v_\infty \in M_\mathbb{Q}$.

The following figures (Figure 2 and Figure 3), drawn with Qfract (by Hiroyuki Inou), are for $P = (0, 1/2)$ (centered around $t = 0.1$) and for $P = (1, -1)$ (centered around $t = -1$). Shades depend on how fast the orbit $\{ H^n_t((a, b)) \}$ escapes as $|n|$ becomes large (the escaping rate is very small in the black region). See also the figure in Introduction for $P = (0, 0)$.

---

**Figure 2.** Figures for $P = (0, 1/2)$. The region on the left is $\{ t \in \mathbb{C} \mid |\text{Re}(t) - 0.1| \leq 1, |\text{Im}(t)| \leq 1 \}$, and that on the right is $\{ t \in \mathbb{C} \mid |\text{Re}(t) - 0.1| \leq 0.01, |\text{Im}(t)| \leq 0.01 \}$. Proposition $\text{[E]}$ says that $\Sigma((0, 1/2)) = \emptyset$. 

**Figure 3.** Figures for $P = (1, -1)$. The region on the left is $\{ t \in \mathbb{C} \mid |\text{Re}(t) + 1| \leq 1, |\text{Im}(t)| \leq 1 \}$, and that on the right is $\{ t \in \mathbb{C} \mid |\text{Re}(t) + 1| \leq 0.01, |\text{Im}(t)| \leq 0.01 \}$. Proposition $\text{[E]}$ says that $\Sigma((1, -1)) = \emptyset$. 

Figures for $P = (-1,1)$. The region on the left is $\{ t \in \mathbb{C} \mid |\text{Re}(t) + 1| \leq 1, |\text{Im}(t)| \leq 1 \}$, and that on the right is $\{ t \in \mathbb{C} \mid |\text{Re}(t) + 1| \leq 0.01, |\text{Im}(t)| \leq 0.01 \}$. Note that, at the center $t = -1$ of the images, the point $(-1,1)$ is periodic with respect to $H_{-1}$ with period 2. Theorem $\overline{D}$ says that $\Sigma((-1,1))$ is an infinite set.

The following result, as a corollary of Theorem 2.1, gives the asymptotic behavior of $G_{P,v}(t)$ as $|t|_v \to \infty$.

**Corollary 4.5.** For any $v \in M_K$, $G_{P,v}(t) - \tilde{h}_H(P) \log |t|_v$ converges as $|t|_v \to \infty$.

**Proof.** Let $X_0, X_1$ be the homogeneous coordinates of $\mathbb{P}^1$, and we take the global section $X_0$ of $O_{\mathbb{P}^1}(1)$. It follows from Theorem 2.1 that

$$
\|X_0\|_v(t) = \lim_{n \to +\infty} \log \frac{|t|_v}{M_n(t)^{1/d_n}}
$$

uniformly converges around $t = \infty$. Since $G_P(t) = \lim_{n \to +\infty} \frac{1}{d_n} M_n(t)$ by Lemma 3.3 and $\ell = \tilde{h}_H(P)$ by (3.4), we obtain that $G_{P,v}(t) - \tilde{h}_H(P) \log |t|_v$ converges as $t \to \infty$. \qed

**Remark 4.6.** It follows from the proof of Theorem 2.1 that, with the notation therein, the limit in Corollary 4.5 is given by $-(1/d^n) \log \max\{|\alpha_D|_v, 1\}$, $-(1/d^n) \log \max\{|\beta_D|_v, 1\}$, or $-(1/d^n) \log \max\{|\alpha_D|_v, |\beta_D|_v, 1\}$, according to Subcases 3-1, 3-2, 3-3 of the proof of Theorem 2.1.

5. **Set of periodic parameter values: result on infiniteness**

Now we would like to consider another application of $h_P$, i.e., an application toward unlikely intersections of the sets of periodic parameter values of two different families of initial points.

As before, let $K$ be a field, and we let $H = (H_t)_{t \in K}$ be a family of Hénon map as in (1.9) and $P \in A^2(K[t])$ be a family of initial points. Let $Q \in A^2(K[t])$ be another family of initial points. We are interested in what will happen if the intersection of periodic parameters $\Sigma(P) \cap \Sigma(Q)$ is infinite. As we discussed in §0.2 that $\Sigma(P)$ (or $\Sigma(Q)$) may not be an infinite set for a family of Hénon maps.
The purpose of this section is to prove Theorem 5.1 which gives a sufficient condition for \( \Sigma(P) \) being infinite.

As we explain in Introduction, we assume that there exists an involution \( \iota: \mathbb{A}^2 \to \mathbb{A}^2 \) over \( K \) such that \( \iota \circ H \circ \iota = H^{-1} \). Then necessarily, \( \delta \in \{1,-1\} \). If \( \delta = 1 \), we assume that \( f_1(-x) = f(x) \) in \( K[t,x] \). Then one can check that the involution \( \iota_\delta: (x,y) \mapsto (-\delta y,-\delta x) \) reverses \( H \) (see §0.2 for more discussions). We note that the set of fixed points of \( \iota_\delta \) is given by the line

\[ C_\delta: \delta x + y = 0. \]

We recall Theorem 5.1 which we prove at the end of this section.

**Theorem 5.1.** Let \( K \) be a field (of any characteristic). Let \( H \) be the family of Hénon maps in (1.9) such that \( \delta \in \{1,-1\} \). If \( \delta = -1 \), we assume that \( f_1(x) \) is an even polynomial in \( x \). Let \( P = (a(t), b(t)) \in \mathbb{A}^2(K[t]) \) satisfy Assumption 1.8. Then, if \( P \) lies in \( C_\delta \), i.e., \( \delta a(t) + b(t) = 0 \) in \( K[t] \), then \( \Sigma(P) \) is an infinite set.

**Example 5.2.** Let \( H_1(x,y) = (y + x^2 + t, x) \) be as in Example 1.6, which is a family of Hénon maps that satisfies the assumption of Theorem 5.1 with \( \delta = 1 \) and \( f_1(x) = x^2 + t \). We take \( P = (a,b) \in K^2 \). As we see in Example 1.9, \( P \) satisfies Assumption 1.8. Theorem 5.1 says that, if \( a + b = 0 \), then \( \Sigma((a,b)) \) is an infinite set.

First, we prove several lemmas. For an ideal \( a \) of \( \overline{K}[t] \), let

\[ \sim: K[t] \to \overline{K}[t]/a \]

be composition of the inclusion map and the quotient map. We denote the base-changes by the morphism \( \sim \) of \( H \) (over \( \text{Spec}(K[t]) \)), \( \widehat{P} = (a(t), b(t)) \in \mathbb{A}^2(K[t]) \), \( H^n(P) = (A_n(t), A_{n-1}(t)) \)

by \( \widehat{H}, \widehat{P} = (a(t), b(t)) \), and \( \widehat{H}^n(P) = (A_n(t), A_{n-1}(t)) \).

**Lemma 5.3.** Assume that \( P \) lies in \( C_\delta \). Let \( m \geq 1 \) be a positive integer. Then the following are equivalent.

1. \( \delta A_m(t) + A_{m-1}(t) = 0 \) in \( \overline{K}[t]/a \).
2. \( \widehat{H}^m(P) = \widehat{P} \) in \( \mathbb{A}^2(\overline{K}[t]/a) \).

**Proof.** We show that (i) implies (ii). Indeed, condition (i) means that \( \widehat{H}^m(\widehat{P}) \in C_\delta(\overline{K}[t]/a) \).

Since \( P \) lies in \( C_\delta \), we have \( \widehat{P} \in C_\delta(\overline{K}[t]/a) \) as well. Thus \( \iota(\widehat{P}) = \widehat{P} \) and \( \iota(\widehat{H}^m(\widehat{P})) = \widehat{H}^m(\widehat{P}) \). We compute

\[ \widehat{H}^{-m}(\widehat{P}) = \iota \circ \widehat{H}^m \circ \iota(\widehat{P}) = \iota \circ \widehat{H}^m(\widehat{P}) = \widehat{H}^m(\widehat{P}), \]

so that \( \widehat{H}^{2m}(\widehat{P}) = \widehat{P} \).

Next we show that (ii) implies (i). Assume that \( \widehat{H}^{2m}(\widehat{P}) = \widehat{P} \), then

\[ \widehat{H}^m(\widehat{P}) = \widehat{H}^{-m}(\widehat{P}) = \iota \circ \widehat{H}^m \circ \iota(\widehat{P}) = \iota \circ \widehat{H}^m(\widehat{P}). \]

Thus \( \widehat{H}^m(\widehat{P}) \in C_\delta(\overline{K}[t]/a) \), which amounts to \( \delta A_m(t) + A_{m-1}(t) = 0 \) in \( \overline{K}[t]/a \).

Applying Lemma 5.3 to the ideal \( a = ((t-\alpha)^e) \) generated by the polynomial \((t-\alpha)^e\) for \( \alpha \in \overline{K} \) and \( e \geq 1 \), we have the following.

**Corollary 5.4.** Assume that \( P \) lies in \( C_\delta \). Let \( m \geq 1 \) be a positive integer. Then the following are equivalent.

1. \( \delta A_m(t) + A_{m-1}(t) \equiv 0 \pmod{(t-\alpha)^e}. \)
2. \( H^{2m}(P) \equiv P \pmod{(t-\alpha)^e}. \)
Remark 5.5. In the case where $\delta = -1$, the same arguments also lead to similar results for odd period $n = 2m + 1$ where one considers the polynomial $\delta A_{m+1}(t) + A_{m-1}(t)$ in place of $\delta A_m(t) + A_{m-1}(t)$ in Lemma 5.3 and Corollary 5.4.

Let $\alpha \in \overline{K}$. For $u(t) \in \overline{K}[t]$, we let $\text{ord}_\alpha(u(t))$ be the multiplicity of the polynomial $(t - \alpha)$ as a factor of $u(t)$. By convention, if $u(t)$ is the zero polynomial, then we set $\text{ord}_\alpha(u(t)) = +\infty$. More generally, if $r = (a_1(t), \ldots, a_n(t)) \in \overline{K}[t]^n$, then

$$\text{ord}_\alpha(r) = \min\{\text{ord}_\alpha(a_1(t)), \ldots, \text{ord}_\alpha(a_n(t))\}.$$ 

Thus, $\text{ord}_\alpha(H^{2m}(P) - P)$ is the largest non-negative integer $e$ such that Corollary 5.4(ii) holds. By definition, the condition that $P_\alpha$ is periodic with respect to $H_\alpha$ with period $n$ is equivalent to $\text{ord}_\alpha(H^n(P) - P) \geq 1$. Further, Corollary 5.4 says that

$$\text{ord}_\alpha(H^{2m}(P) - P) = \text{ord}_\alpha(\delta A_m(t) + A_{m-1}(t)).$$

Suppose that $P_\alpha$ is periodic with respect to $H_\alpha$ with period $n \geq 1$. Then $H^{nk}_\alpha(P_\alpha) = P_\alpha$ for any positive integer $k$. Further, one has

$$\text{ord}_\alpha(H^{nk}(P) - P) \geq \text{ord}_\alpha(H^n(P) - P)$$

for any $k \geq 1$.

Indeed, we put $e = \text{ord}_\alpha(H^n(P) - P)$. Then, we have $H^n(P) \equiv P \pmod{(t - \alpha)^e}$. Consequently, $H^{nk}(P) \equiv P \pmod{(t - \alpha)^e}$ for any positive integer $k$. This is equivalent to saying that $\text{ord}_\alpha(H^{nk}(P) - P) \geq e$.

The following result gives a control on the growth of $\text{ord}_\alpha(H^n(P) - P)$, when $n$ ranges over a certain subset of positive integers.

**Proposition 5.6.** Let $q$ be a positive integer. Then there exists an integer $D_q \geq 1$ such that, for any $\alpha \in \overline{K}$ with $\text{ord}_\alpha(H^n(P) - P) \geq 1$ and for any positive integer $k$ relatively prime to $D_q$, we have

$$\text{ord}_\alpha(H^{qk}(P) - P) = \text{ord}_\alpha(H^q(P) - P).$$

Before we start the proof of Proposition 5.6, we show two lemmas. Let $\alpha \in \overline{K}$. We set $e_q := \text{ord}_\alpha(H^q(P) - P)$, and we assume that $e_q \geq 1$. We write $H^q(P) - P = (\varepsilon(t), \zeta(t))$, so that $e_q = \min\{\text{ord}_\alpha(\varepsilon(t)), \text{ord}_\alpha(\zeta(t))\}$. Then

$$H^q(P) = (A_q(t), A_{q-1}(t)) = (a(t) + \varepsilon(t), b(t) + \zeta(t)).$$

Set the Jacobian matrix of $H^q$ at the point $P$ to be

$$\Psi_q(t) := J_{H^q}(P) = J_H(P) \cdots J_H(H^{q-2}(P))J_H(H^{q-1}(P))$$

$$= \begin{pmatrix} f'_1(a(t)) & 1 \\ \delta & 0 \end{pmatrix} \cdots \begin{pmatrix} f'_1(A_{q-2}(t)) & 1 \\ \delta & 0 \end{pmatrix} \begin{pmatrix} f'_1(A_{q-1}(t)) & 1 \\ \delta & 0 \end{pmatrix}.$$ 

Let $b \subseteq K[t]$ denote the ideal generated by $\varepsilon(t)^2, \varepsilon(t)\zeta(t), \zeta(t)^2$.

**Lemma 5.7.** Let $k \geq 1$. We define $r_k \in (K[t])^2$ by

$$H^{qk}(P) = P + (\varepsilon(t), \zeta(t)) \left( \sum_{i=0}^{k-1} \Psi_q(t)^i \right) + r_k.$$ 

Then $r_k \in (b)^2 \subseteq K[t]^2$. In particular, $\text{ord}_\alpha(r_k) \geq 2e_q$. 

where the last inequality is a consequence of the Taylor expansion and \( \xi \).

We continue the computation:

The right-hand side of (5.5)

\[
\begin{align*}
&= P + (\varepsilon(t), \zeta(t)) \left( \sum_{i=0}^{k-1} \Psi_q(t)^i \right) + r_k \\
&= P + (\varepsilon(t), \zeta(t)) \left( \sum_{i=0}^{k} \Psi_q(t)^i \right) + r_k \Psi_q(t) + s,
\end{align*}
\]

we complete the induction step, and obtain the lemma.

\[\square\]

**Lemma 5.8.** Let \( R \) be a discrete valuation ring equipped with valuation \( v \). Let \( r = (a_1, \ldots, a_n) \) be a non-zero element of \( R^n \). Set \( v(r) = \min \{ v(a_1), \ldots, v(a_n) \} \). Then, for every invertible matrix \( A \in \text{GL}(n, R) \), we have \( v(rA) = v(r) \).

**Proof.** It is an elementary exercise in algebra. Indeed, Let \( r \) and \( A \in \text{GL}(n, R) \) be as given. We note that coordinates of

\[ s := rA = (b_1, \ldots, b_n) \]

are \( R \)-linear combinations of the coordinates \( a_1, \ldots, a_n \). Thus \( v(b_i) \geq v(r) \) and we have \( v(s) \geq v(r) \). The same argument applies to the vector \( r = sA^{-1} \) and note that \( A^{-1} \) also has entries in \( R \). We can therefore conclude that \( v(r) \geq v(s) \) and \( v(rA) = v(s) = v(r) \). \[\square\]

We are ready to prove Proposition 5.6.

**Proof of Proposition 5.6.** For a positive integer \( k \), we set \( E_{q,k}(t) := \sum_{i=0}^{k-1} \Psi_q(t)^i \), where \( \Psi_q(t) \) is defined in (5.3). Then By Lemma 5.7 we have

\[
\H^q(P) = P + (\varepsilon(t), \zeta(t))E_{q,k}(t) + r_k.
\]

Let \( \alpha \in \overline{K} \) such that \( \text{ord}_q(\H^q(P) - P) \geq 1 \). Let \( \xi_1, \xi_2 \) be the eigenvalues of \( \Psi_q(\alpha) \). If the characteristic of \( K \) is zero, we define the integer \( D_{q,0} \) to be the least common multiple of the multiplicative orders of \( \xi_1 \) and \( \xi_2 \), where by convention, we define the multiplicative order of \( \xi_i (i = 1, 2) \) to be 1 if it is not a root of unity. If the characteristic of \( K \) is positive, then we define the integer \( D_{q,0} \) to be the least common multiple of the characteristic of \( K \) and the
multiplicative orders of $\xi_1$ and $\xi_2$. We claim that, for any positive integer $k$ relatively prime to $D_{q,\alpha}$, the equality
\[ \text{ord}_\alpha (H^{qk}(P) - P) = \text{ord}_\alpha (H^q(P) - P) \]
holds.

Since $\Psi_q(\alpha)$ is similar to the matrix \( \begin{pmatrix} \xi_1 & * \\ 0 & \xi_2 \end{pmatrix} \), $E_{q,k}(t)$ is similar to \( \begin{pmatrix} 1 + \cdots + \xi_1^{k-1} & * \\ 0 & 1 + \cdots + \xi_2^{k-1} \end{pmatrix} \).

As $k$ is relatively prime to $D_{q,\alpha}$, we have \( 1 + \cdots + \xi_1^{k-1} \neq 0 \) for $i = 1, 2$, and thus $\det E_{q,k}(\alpha) \neq 0$. It follows that $E_{q,k}(t) \in \text{GL}(2, R_\alpha)$ where $R_\alpha = \mathbb{K}[t](t - \alpha)$ is the localization of $\mathbb{K}[t]$ at the prime ideal $(t - \alpha)$.

By Lemma \ref{lem:ord}, we have $\text{ord}_\alpha (rE_{q,k}(t)) = \text{ord}_\alpha (r)$ for any non-zero row vector $r \in \mathbb{K}[t]^2$. Thus
\[ \text{ord}_\alpha ((\varepsilon(t), \zeta(t))E_{q,k}(t)) = \text{ord}_\alpha (\varepsilon(t), \zeta(t)) = e_q. \]

It follows from Lemma \ref{lem:ord} that $\text{ord}_\alpha r_k \geq 2e_q$. Thus
\[ \text{ord}_\alpha (H^{qk}(P) - P) = \text{ord}_\alpha ((\varepsilon(t), \zeta(t))E_{q,k}(t) + r_k) = e_q = \text{ord}_\alpha (H^q(P) - P). \]

Thus we obtain the claim.

If $H^q(P) = P$, then the assertion clearly holds because $\text{ord}_\alpha (H^{qk}(P) - P) = \text{ord}_\alpha (H^q(P) - P) = +\infty$. If $H^q(P) \neq P$, then there are only finitely many $\alpha \in \mathbb{K}$ with $\text{ord}_\alpha (H^q(P) - P) \geq 1$. We consider the product of $D_{q,\alpha}$ for all such $\alpha$, and put $D_q := \prod_\alpha D_{q,\alpha}$. Then $D_q$ gives the desired property that \( (5.2) \) holds for all positive integers $k$ which are relatively prime to $D_q$. \hfill \Box

Finally, we prove Theorem \ref{thm:main} For a positive integer $n$, we define the subset of $\Sigma(P)$ in \ref{eq:SigmaP} by
\[ \Sigma_n(P) := \{ t \in \mathbb{A}^1(\mathbb{K}) \mid H^n_t(P_t) = P_t \}. \]

\textbf{Proof of Theorem \ref{thm:main}.} Observe that $\Sigma (H^m(P)) = \Sigma (P)$ for any integer $m$. Since $t \circ H \circ t = H^{-1}$ and $t(P) = P$, it follows from Assumption \ref{ass:degBn} that both $(\deg A_n(t))_{n \geq 0}$ and $(\deg B_n(t))_{n \geq 0}$ are unbounded. We replace $P$ with $H^m(P)$ for sufficiently large $m$ if necessary, and we may assume that the degree $\deg a(t) > 0$ and that $\deg A_n(t) = d^n \deg a(t)$ for all $n \geq 0$ (see Proposition \ref{prop:deg}).

Let $D_2$ be the integer as given in Proposition \ref{prop:ord} for $q = 2$ and let $p$ be an odd prime number such that $p \nmid D_2$. We claim that there always exists a parameter $\beta \in \mathbb{K}$ such that $P_\beta$ has period $p$ or $2p$ under $H_\beta$. Indeed, by Lemma \ref{lem:ord2}
\[ \Sigma_{2p}(P) = \{ \alpha \in \mathbb{A}^1(\mathbb{K}) \mid \delta A_p(\alpha) + A_{p-1}(\alpha) = 0 \}. \]

By definition, if $\alpha \in \Sigma_{2p}(P)$ then $P_\alpha$ has period of $1, 2, p$ or $2p$ for $H_\alpha$.

Let $\alpha \in \mathbb{K}$ be a root of $\delta A_1(t) + A_0(t) = 0$. It follows from Corollary \ref{cor:ord} that $\text{ord}_\alpha (H^2(P) - P) \geq 1$. As a result, we have
\[ \text{ord}_\alpha (\delta A_p(t) + A_{p-1}(t)) = \text{ord}_\alpha (H^{2p}(P) - P) \]
\[ = \text{ord}_\alpha (H^2(P) - P) \]
\[ = \text{ord}_\alpha (\delta A_1(t) + A_0(t)) \]
by Corollary \ref{cor:ord}
by Proposition \ref{prop:ord}
by Corollary \ref{cor:ord}.

Thus there exists $W_{2p}(t) \in \mathbb{K}[t]$ such that
\[ \delta A_p(t) + A_{p-1}(t) = (\delta A_1(t) + A_0(t)) W_{2p}(t), \]
and $\delta A_1(t) + A_0(t)$ and $W_{2p}(t)$ are relatively prime. Since
\[ \deg (A_p(t) + A_{p-1}(t)) = \deg (A_p(t)) = d^p \ell > d \ell = \deg A_1(t) = \deg (A_1(t) + A_0(t)), \]
$W_{2p}(t)$ is a non-constant polynomial.

We take a root $\beta \in \overline{K}$ of $W_{2p}(t)$. Then $\beta \in \Sigma_{2p}(P)$. Since $\delta A_1(\beta) + A_0(\beta) \neq 0$, it follows that $H^2_\beta(P_\beta) \neq P_\beta$. In particular, we also have $H_\beta(P_\beta) \neq P_\beta$. Thus $P_\beta$ has period $p$ or $2p$.

We denote the above $\beta$ by $\beta_p$. Then $\beta_p \in \Sigma(P)$. Further, if $p$ and $p'$ are distinct odd prime numbers relatively prime to $D_2$, then the period of $P$ under $H_{\beta_p}$ and that under $H_{\beta_{p'}}$ are different, so that $\beta_p \neq \beta_{p'}$. We conclude that $\Sigma(P)$ is an infinite set as desired. \hfill $\square$

**Remark 5.9.** For $\delta = 1$ and $\delta = -1$, we actually show that the set $\Sigma_{\text{even}}(P) := \bigcup_{m \geq 1} \Sigma_{2m}(P)$ is an infinite set. In the case where $\delta = -1$, as indicated in Remark 5.5, by applying similar arguments one can prove that $\Sigma_{\text{odd}}(P) := \bigcup_{m \geq 1} \Sigma_{2m+1}(P)$ is also an infinite set.

**Remark 5.10 (Question about primitive prime divisors in a family of Hénon maps).** Related to Theorem 5.1 is the question of primitive prime divisors in arithmetic dynamics which has been considered in the case of one-variable rational maps in recent years (see for instance [FG11, GNT13, IS09]). For families of Hénon maps $H$ and initial points $P$, we say that $t - \alpha \in \overline{K}[t]$ is a primitive prime divisor for $H^n(P) - P$ if $H^n_\alpha(P_\alpha) = P_\alpha$ but $H^n_\delta(P_\delta) \neq P_\delta$ for any $1 \leq m \leq n - 1$. It would be interesting to investigate how often the sequence $H^n(P) - P$ has a primitive prime divisor as $n$ runs through all positive integers. Let $H$ and $P$ be given as in Theorem 5.1. We would like to ask the following question: Is it true that $H^{2m}(P) - P$ has a primitive prime divisor for all but finitely many positive integer $m$? In the case of $\delta = -1$ one can further ask whether or not $H^n(P) - P$ has a primitive prime divisor for all but finitely many positive integer $n$. An affirmative answer to these question strengthens Theorem 5.1.

### 6. Unlikely intersection for one-parameter families of Hénon maps

In this section, we prove Theorem 6.1 which addresses unlikely intersections for a one-parameter family Hénon maps. As we discuss in Introduction, a key element for the proof is the equidistribution theorem for points of small height associated to a semipositive adelically metrized line bundle.

Here we recall the following result as an application of Yuan’s equidistribution theorem [Yu08] which holds for varieties of any dimension. It is implicit in Section 3 of the arXiv version of [YZ17], and is stated in [GHT15]. We will apply it for $X = \mathbb{P}^1$. We note that, as our parameter space is $\mathbb{P}^1$, one could also apply the equidistribution theorem for curves as proven in [Am01, BR06, CL06, FRL04, FRL06, Th05].

**Proposition 6.1 ([GHT15 Corollary 4.3]).** Let $K$ be a number field. Let $X$ be an integral projective variety over $K$, and let $L$ be an ample line bundle on $X$. Let $\| \cdot \|_1 := \{ \| \cdot \|_{1,v} \}_{v \in M_K}$ and $\| \cdot \|_2 := \{ \| \cdot \|_{2,v} \}_{v \in M_K}$ be adelic metrics on $L$ such that $\overline{L}_1 := (L, \| \cdot \|_1)$ and $\overline{L}_2 := (L, \| \cdot \|_2)$ are semipositive adelically metrized line bundles. Assume that there exists a nonzero rational section $s$ and a point $P \in X(\overline{K})$ such that, for all but finitely many places $v \in M_K$, $\| s(P) \|_{1,v} = \| s(P) \|_{2,v}$. Let $\{x_m\}_{m \geq 1}$ be an infinite sequence of algebraic points in $X(\overline{K})$ that are Zariski dense in $X$. Suppose that

$$
\lim_{m \to \infty} h_{\overline{L}_1}(x_m) = \lim_{m \to \infty} h_{\overline{L}_2}(x_m) = h_{\overline{L}_1}(X) = h_{\overline{L}_2}(X) = 0.
$$

Then $h_{\overline{L}_1} = h_{\overline{L}_2}$ on $X(\overline{K})$.

For the convenience of the reader, we restate Theorem 6.1 as follows.

**Theorem 6.2.** Let $K$ be a number field. Let $H = (H_t)_{t \in \mathbb{K}}$ be the one-parameter family of Hénon maps as in § 1.4. Let $P, Q \in \mathbb{A}^2(\mathbb{K}[t])$ be initial points satisfying Assumption 1.5 such that $\Sigma(P)$ and $\Sigma(Q)$ are infinite. Then the followings are equivalent:
(i) $\Sigma(P) \cap \Sigma(Q)$ is infinite;
(ii) $\Sigma(P) = \Sigma(Q)$;
(iii) $G_{P,v} = G_{Q,v}$ on the parameter space $\mathbb{P}^1(K_v)$ for all $v \in M_K$.
(iv) $h_P = h_Q$ on the parameter space $\mathbb{P}^1(\overline{K})$.

**Proof.** We first show “(i) $\implies$ (iv).” Since $\Sigma(P) \cap \Sigma(Q)$ is infinite, we can take a sequence $\{t_m\}_{m \geq 1}$ of distinct points in $\Sigma(P) \cap \Sigma(Q)$. By the definition of $\Sigma(P)$ we have $h_{H_{t_m}}(P_{t_m}) = 0$ and $h_{H_{t_m}}(Q_{t_m}) = 0$. It follows from Theorem 3.4 (see also (3.3)) that

$$h_{\mathbb{Z}_P}(t_m) = h_{\mathbb{Z}_Q}(t_m) = 0. \tag{6.2}$$

Note that the underlying line bundle of the semipositive adelic metrics of $\mathbb{Z}_P$ is $\mathcal{O}_{\mathbb{P}^1}(1)$, and from the proof of Proposition 3.1 it is the uniform limit of $(\mathcal{L}_n, \{\|\cdot\|_{n,v}\})$, where $\mathcal{L}_n$ is relatively ample. Zhang’s fundamental inequality [Zh95, Theorem (1.10)] gives

$$\inf_{t \in \mathbb{P}^1(\mathcal{K})} h_{\mathbb{Z}_P}(t) \leq h_{\mathbb{Z}_P}(\mathbb{P}^1) \leq \sup_{\dim Z = 0} \inf_{t \in (\mathbb{P}^1 \setminus Z)(\mathcal{K})} h_{\mathbb{Z}_P}(t).$$

The existence of the infinite set $\{t_m\}_{m \geq 1}$ with $h_{\mathbb{Z}_P}(t_m) = 0$ implies that both the left-hand and right-hand sides are equal to 0, and we obtain $h_{\mathbb{Z}_P}(\mathbb{P}^1) = 0$. Similarly, we have $h_{\mathbb{Z}_Q}(\mathbb{P}^1) = 0$. By Proposition 6.1 and Theorem 3.4 we have $h_{\mathbb{Z}_P} = h_{\mathbb{Z}_Q}$, so that $h_P = h_Q$.

The implication “(iv) $\implies$ (ii)” follows from Proposition 4.1, and the implication “(ii) $\implies$ (i)” is obvious. The implication “(iii) $\implies$ (iv)” follows from Proposition 4.2.

Lastly we show “(i) $\implies$ (iii).” Let $v \in M_K$. The proof of [GHT15, Corollary 4.3] shows that there exists a constant $c_v \in \mathbb{R}$ such that $G_{P,v} = G_{Q,v} + c_v$. Take any $t \in \Sigma(P) \cap \Sigma(Q)$. Since $t \in \Sigma(P) \subseteq K_P$, we have $G_{P,v}(t) = 0$ by Theorem 4.3. Similarly, we have $G_{Q,v}(t) = 0$. Thus $c_v = 0$, and we obtain (iii).

Note that in the case of a family of one-variable polynomial dynamics over $K[t]$, using the Böttcher coordinate, it is proved that $P$ and $Q$ satisfy some orbital relations. For a family of Hénon maps, the techniques using the Böttcher coordinates seem not easily extended. In the following, we would like to point out that reversible Hénon maps play a role here. This is the easy direction, and we ask in Question 4 in Introduction if the converse also holds.

**Proposition 6.3.** (1) Let $\Omega$ be an algebraically closed field that is complete with respect to an absolute value. Let $H : \mathbb{A}^2 \to \mathbb{A}^2$ be a Hénon map over $\Omega$. We assume that there exists an invertible affine map $\sigma : \mathbb{A}^2 \to \mathbb{A}^2$ over $\Omega$ such that $\sigma^{-1} \circ H^m \circ \sigma = H^m$ or $\sigma^{-1} \circ H^m \circ \sigma = H^{-m}$ for some $m \geq 1$. Let $G$ be the Green function defined in (1.6). Then for any $P \in \mathbb{A}^2(\Omega)$, we have $G(\sigma(P)) = G(P)$.

(2) Let $K, \mathbf{H}, P$ be as in Theorem 6.2 such that $\Sigma(P)$ is infinite. Assume that there exists an invertible affine map $\sigma : \mathbb{A}^2 \to \mathbb{A}^2$ over $K[t]$ with $\sigma^{-1} \circ H^m \circ \sigma = H^m$ or $\sigma^{-1} \circ H^m \circ \sigma = H^{-m}$ for some $m \geq 1$. Let $n \in \mathbb{Z}$. Then $h_{\mathbf{H}^n(\sigma(P))} = h_P$.

**Proof.** (1) Since $\sigma$ is an invertible affine map of $\mathbb{A}^2(\Omega)$, there exists a constant $C \geq 1$ such that $C^{-1}\|Q\| \leq \|\sigma(Q)\| \leq C\|Q\|$ for any $Q \in \mathbb{A}^2(\Omega)$. In particular, we have $C^{-1}\|H^n(P)\| \leq \|\sigma(H^n(P))\| \leq C\|H^n(P)\|$ for any $n \in \mathbb{Z}$.
Suppose that $\sigma^{-1} \circ H^m \circ \sigma = H^m$. Then we have

$$G^+(\sigma(P)) = \lim_{n \to +\infty} \frac{1}{d^n} \log^+ \|H^n(\sigma(P))\| = \lim_{k \to +\infty} \frac{1}{dk^m} \log^+ \|H^{km}(\sigma(P))\|$$

$$= \lim_{k \to +\infty} \frac{1}{dk^m} \log^+ \|\sigma^{-1}H^{km}(\sigma(P))\| = \lim_{k \to +\infty} \frac{1}{dk^m} \log^+ \|H^{km}(P)\|$$

$$= \lim_{n \to +\infty} \frac{1}{d^n} \log^+ \|H^n(P)\| = G^+(P).$$

Similarly, we have $G^-(\sigma(P)) = G^-(P)$. Since $G = \max\{G^+, G^\prime\}$, we have $G(\sigma(P)) = G(P)$.

(2) Since $h_{\mathbf{P}}$ does not change under a finite extension of fields $K'/K$, replacing $K$ by a finite extension field, we may assume that $\sigma$ is defined over $K[t]$. It then follows from (1) that $h_{\sigma(\mathbf{P})} = h_{\mathbf{P}}$. Suppose that $t \in \Sigma(\mathbf{P})$, and we write $H^n_t(P_t) = P_t$. Then $H^{mn}_t(P_t) = P_t$, and we have $(\sigma^i \circ H^{m})_t(P_t) = \sigma_i(P_t)$, which implies $H^{mn}_t(\sigma_i(P_t)) = \sigma_i(P_t)$ if $\sigma^{-1} \circ H^m \circ \sigma = H^m$, and $H^{m-n}_t(\sigma_i(P_t)) = \sigma_i(P_t)$ if $\sigma^{-1} \circ H^m \circ \sigma = H^{-m}$. In either case, we have $t \in \Sigma(\sigma(\mathbf{P}))$. If follows that $\Sigma(\sigma(\mathbf{P}))$ is infinite. Since $\Sigma(\sigma(\mathbf{P})) = \Sigma(H^n(\sigma(\mathbf{P})))$, Theorem 6.2 implies that $h_{H^n(\sigma(\mathbf{P}))} = h_{\sigma(\mathbf{P})} = h_{\mathbf{P}}$.

7. Set of periodic parameter values: result on emptiness

This section is complementary to Section 5. Let $K$ be a field. We consider the family of quadratic Hénon maps in Example 1.6

$$H = (H_t)_{t \in \mathbb{K}}: \mathbb{K}^2 \to \mathbb{K}^2, \quad (x, y) \mapsto (y + x^2 + t, x).$$

We take $a, b \in \overline{K}$, and consider a constant family $\mathbf{P} = (a, b)$ as an initial point. In the following, we will simply denote the specialization of $\mathbf{P}$ at $t$ by $\mathbf{P}_t$ instead of $P_t$, because it is a constant family.

As we see in Example 5.2, if $a + b = 0$, then

$$\#\Sigma((a, b)) = \#\{t \in \mathbb{K} \mid (a, b) \text{ is periodic with respect to } H_t\} = \infty.$$
From Lemma 7.1, we are interested in the case where $K$ is the prime field $\mathbb{F}_p$, when the characteristic of $K$ is $p \geq 1$, and $\mathbb{Q}$ when the the characteristic of $K$ is zero.

For the positive characteristic case, we have the following.

**Lemma 7.3.** Let $K = \mathbb{F}_p$. Then for any $(a,b) \in \mathbb{F}_p$, we have $\Sigma((a,b)) = \mathbb{F}_p$.

**Proof.** We take any $t \in \mathbb{F}_p$. We take a finite subfield $F$ of $\mathbb{F}_p$ such that $a,b,t \in F$. Then the map $H_t(x, y) = (y + x^2 + t, x)$ gives an automorphism of $F^2$. Thus $(a,b) \in F^2$ is periodic with respect to $H_t$ (with period dividing $(\#F)^2)!$. We obtain that $t \in \Sigma((a,b))$. Since $t$ is arbitrary, we have $\Sigma((a,b)) = \mathbb{F}_p$. □

Thus the most interesting case would be the case $K = \mathbb{Q}$. Our main result in this section is the following proposition, giving many (constant) points $P$ with empty periodic parameter values. Let $\overline{\mathbb{Z}}$ denote the ring of algebraic integers.

**Proposition 7.4** (= Proposition 5). Let $K = \mathbb{Q}$ (or any field of characteristic zero). Then for any $(0,b)$ with $b \notin \mathbb{Z}$, we have $\Sigma((0,b)) = \emptyset$.

**Proof.** Let $P = (0,b)$ with $b \in \mathcal{K} \setminus \mathbb{Z}$. Since $b \neq 0$, we have $H_{t}(P) \neq P$ for any $t$. Suppose that $\Sigma(P) \neq \emptyset$. Then there exist $n \geq 2$ and $t_0 \in \mathbb{C}$ such that $H_{t_0}^n(P) = P$. Since the equations $A_n(t) = 0$ and $A_{n-1}(t) - b = 0$ have the common solution $t_0$, we have $\text{Res}_t(A_n(t), A_{n-1}(t)) = 0$. By Proposition 7.6 below, we then have $b \in \mathbb{Z}$, which contradicts with our assumption. □

**Example 7.5.** Let $K = \mathbb{Q}$. By Example 5.2, we have $\#(0,0)) = \infty$ and $\#((1,1)) = \infty$. By Proposition 7.4 we have $\Sigma((0,1/2)) = \emptyset$. On the other hand, Figure 1, 2, 3 depicting escaping rates of $P$ each looks complicated with fractal structures.

We finish this paper by showing the following proposition, which was used to prove Proposition 7.4.

**Proposition 7.6.** Let $n \geq 2$. As a polynomial in $b$, we have
\begin{equation}
\text{Res}_t(A_n(t), A_{n-1}(t) - b) = \pm b^{2^{n-1}} + c_1 b^{2^{n-1} - 1} + \cdots + c_{2^{n-1}}
\end{equation}
with $c_i \in \mathbb{Z}$ for $i = 1, \ldots, 2^{n-1}$.

**Proof.** We compute some Sylvester resultants to prove Proposition 7.6. Computations are not difficult, but lengthy. We only sketch a proof, leaving the details to the reader.

We observe that the condition $H_t^n(P) = P$ is equivalent to $H_{t}^{n/2}(P) = H_{-t}^{-n/2}(P)$ (resp. $H_{t}^{(n+1)/2}(P) = H_{-t}^{-(n-1)/2}(P)$) for even $n$ (resp. odd $n$). We divide the proof into two cases depending on whether $n$ is an even integer or an odd integer. We sketch a proof for the even case. (The odd case is shown similarly.) We set $n = 2m$.

In the following, we treat $b$ as another variable which is independent of $t$ and regard $A_n(t)$ and $B_n(t)$ as polynomials in variables $b$ and $t$ for all integers $n \geq 0$. By properties of resultants (cf. [CLO05 Chap. 3. Sect. 1]) and the recursive relations (1.13), (1.14), we have
\begin{equation}
\text{Res}_t(A_{2m}(t), A_{2m-1}(t) - b) = \pm \text{Res}_t(A_m(t) - B_{m-1}(t), A_{m-1}(t) - B_m(t)).
\end{equation}

We set $C_m(t) := A_m(t) - B_{m-1}(t)$ and $D_m(t) := A_{m-1}(t) - B_m(t)$. Then one can check that deg$_t(C_m(t)) = \text{deg}_t(D_m(t)) = 2^{m-1}$. We write
\begin{align*}
C_m(t) &= \gamma_m(b)t^{2^{m-1}} + \gamma_1(b)t^{2^{m-1} - 1} + \cdots + \gamma_{m-1}(b), \\
D_m(t) &= \delta_m(b)t^{2^{m-1}} + \delta_1(b)t^{2^{m-1} - 1} + \cdots + \delta_{m-1}(b)
\end{align*}
with $\gamma_{m_1}(b), \delta_{m_1}(b) \in \mathbb{Z}[b]$. 

For $C_m(t)$, one can check that $\gamma_{m_0}(b) = 1$, and $\deg(\gamma_{m_1}(b)) \leq i$ for any $0 \leq i \leq 2^m - 1$. For $D_m(t)$, one can check $\delta_{m_0}(b) = 1$ and $\deg(\delta_{m_1}(b)) \leq 2i$ for any $0 \leq i \leq 2^m - 1$. We also have 
\[
\delta_{m_2 2m-1}(b) = b^{2m} + \text{(lower terms in } b \text{ with integer coefficients)}.
\]

Then, by repeating elementary transformations of the Sylvester matrix of the resultant $\text{Res}_t(C_m(t), D_m(t))$, one can show that 
\[
\text{Res}_t(C_m(t), D_m(t)) = b^{2m-1} + c'_1 b^{2m-1} + \cdots + c'_{2^{2m-1}}
\]
with $c'_i \in \mathbb{Z}$.

By (7.3) and (7.4), we obtain (7.2).

\[\square\]

**Example 7.7.** We treat the case of $n = 4$ (i.e. $m = 2$) to illustrate how we repeat elementary transformations of the Sylvester matrix of the resultant $\text{Res}_t(C_m(t), D_m(t))$ in the proof of Proposition 7.6. Our goal is to verify that $\text{Res}_t(A_4(t), A_3(t) - b) = \pm b^8 + c_1 b^7 + \cdots + c_8$ with $c_i \in \mathbb{Z}$ in a way that generalizes to any even $n$.

It suffices to show that $\text{Res}_t(C_2(t), D_2(t)) = b^8 + c'_1 b^7 + \cdots + c'_4$ with $c'_i \in \mathbb{Z}$. We have 
\[
\begin{align*}
C_2(t) &= t^2 + \gamma_{21}(b) t + \gamma_{22}(b) \quad \text{with } (\gamma_{2i}(b) \in \mathbb{Z}[b], \deg(\gamma_{21}(b)) \leq 1, \deg(\gamma_{22}(b)) \leq 2), \\
D_2(t) &= t^2 + \delta_{21}(b) t + \delta_{22}(b) \quad \text{with } (\delta_{2i}(b) \in \mathbb{Z}[b], \deg(\delta_{21}(b)) \leq 2, \\
&\quad \delta_{22}(b) = b^4 + \text{(lower terms in } b \text{ with integer coefficients)}).
\end{align*}
\]

To simplify the notation, we denote $\gamma_{2i}(b)$ and $\delta_{2i}(b)$ by $\gamma_i$ and $\delta_i$ for $i = 1, 2$. Then the resultant is given by the determinant of the Sylvester matrix
\[
\text{Res}_t(C_2(t), D_2(t)) = \begin{vmatrix}
1 & \gamma_1 & \gamma_2 & 0 \\
0 & 1 & \gamma_1 & \gamma_2 \\
1 & \delta_1 & \delta_2 & 0 \\
0 & 1 & \delta_1 & \delta_2
\end{vmatrix} = \begin{vmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & \gamma_1 & \gamma_2 \\
1 & \delta_1 - \gamma_1 & \delta_2 - \gamma_2 & 0 \\
0 & 1 & \delta_1 & \delta_2
\end{vmatrix} = \begin{vmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
1 & \delta_1 - \gamma_1 & -\gamma_1(\delta_1 - \gamma_1) + (\delta_2 - \gamma_2) & -\gamma_2(\delta_1 - \gamma_1) \\
0 & 1 & \delta_1 - \gamma_1 & \delta_2 - \gamma_2
\end{vmatrix} = \begin{vmatrix}
-\gamma_1(\delta_1 - \gamma_1) + (\delta_2 - \gamma_2) & -\gamma_2(\delta_1 - \gamma_1) \\
\delta_1 - \gamma_1 & \delta_2 - \gamma_2
\end{vmatrix},
\]
where, for the second equality, the second (resp. third) column is subtracted by $\gamma_1$ (resp. $\gamma_2$) times the first column, and, for the third equality, the third (resp. fourth) column is subtracted by $\gamma_1$ (resp. $\gamma_2$) times the second column.

We write the right hand side of the above displayed equality as 
\[
\begin{vmatrix}
f_{11} & f_{12} \\
f_{21} & f_{22}
\end{vmatrix},
\]
so that, for example, $f_{11} = -\gamma_1(\delta_1 - \gamma_1) + (\delta_2 - \gamma_2)$. By (7.5) and (7.6), we have 
\[
f_{ii} = b^4 + \text{(lower terms in } b \text{ with integer coefficients)} \quad \text{for } i = 1, 2,
\]
and \( \deg(f_{21}) \leq 2 \) and \( \deg(f_{12}) \leq 4 \). Then

\[
\text{Res}_t(C_2(t), D_2(t)) = f_{11}f_{22} - f_{12}f_{21} = b^8 + (\text{lower terms in } b \text{ with integer coefficients}),
\]
as desired.

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