On the uniform distribution of zero ordinates of Epstein zeta-functions

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Abstract. Let \( \zeta(s; Q) \) be the Epstein zeta-function associated with an integral quadratic form \( Q[y] = y^t Q y \) for \( y \in \mathbb{Z}^n \) with nontrivial zeros \( \rho = \beta + i \gamma \). We prove that \( \sum_{|\gamma| < T} x_\rho = c_T T + O(\log T) \) for all Epstein zeta functions analogously to Landau’s explicit formula in the case of the Riemann zeta function. It follows that under the assumption \( \sum_{|\gamma| < T} |\beta - n/4| = o(N(T; Q)) \), where \( N(T; Q) \) counts the nontrivial zeros for \( |\gamma| < T \), the ordinates of Epstein zeta zeros are uniformly distributed modulo one.

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1 Introduction

In 1903, Epstein [5] introduced zeta functions associated with quadratic forms. If \( Q \) is a positive definite \( n \times n \) matrix with integer values and associated quadratic form \( Q[x] = x^t Q x \), then

\[
\zeta(s; Q) = \sum_{0 \neq x \in \mathbb{Z}^n} Q[x]^{-s}
\]

is called the associated Epstein zeta function. This Dirichlet series converges absolutely for \( \text{Re} \, s > n/2 \), and there exists an analytic continuation to the whole complex plane except for a simple pole at \( s = n/2 \). It is also well known (see, e.g., [16]) that it satisfies the functional equation

\[
\pi^{-s} \Gamma(s) \zeta(s; Q) = (\det Q)^{1/2} \pi^{s-n/2} \Gamma\left(\frac{n}{2} - s\right) \zeta\left(\frac{n}{2} - s; Q^{-1}\right)
\]

with inverse matrix \( Q^{-1} \) of \( Q \) and trivial zeros for \( s = -m, m \in \mathbb{N} \). We are interested in the remaining zeros, the nontrivial ones denoted by \( \rho = \beta + i \gamma \).
Let $N(T; Q)$ denote the number of nontrivial zeros with $|\gamma| < T$. There exists an analogue of the Riemann–von Mangoldt formula:

$$N(T; Q) = \frac{2T}{\pi} \log \frac{T}{\pi e \sqrt{m(Q) m(Q^{-1})}} + O(\log T);$$

(1.2)

see, for example, [15]. Here

$$m(Q) = \min\{ m \in \mathbb{N} \mid \exists x \in \mathbb{Z}^n: Q[x] = m \}$$

is the minimum value attained by the quadratic form, an important invariant of $Q$ frequently used in what follows.

In the case of the Riemann zeta-function, Steuding [17] showed more generally that the ordinates of the roots of $\zeta(s) = a$ are uniformly distributed modulo one. In the case of Epstein zeta-functions, Steuding [16] also proved that the mean value of the real parts of the nontrivial zeros is equal to $n/4$.

We use mainly methods from the latter two papers to arrive at the following:

**Theorem 1.** Let $x \in \mathbb{R}^+ \setminus \{1\}$. As $T \to \infty$,

$$\sum_{|\gamma| < T} x^\rho = \frac{T}{\pi} \left( \delta_{k(Q),x} b_k(Q) - x^{n/2} \delta_{k(Q^{-1}),1/x} b_k(Q^{-1}) \right) + O(\log T)$$

(1.3)

for all Epstein zeta-functions, where $l_k(Q)$ and $b_k(Q)$ are explicitly defined in Lemma 1.

Here $\delta_{a,b}$ is the Kronecker symbol (equal to 1 if $a = b$ and zero otherwise).

The theorem can be viewed as an analogue of Landau’s explicit formula (see [11]) in the case of the Riemann zeta-function, which states for $x > 1$ that

$$\sum_{0<\gamma<T} x^\rho = -A(x) \frac{T}{2\pi} + O(\log T).$$

Here the summation is again over all nontrivial zeros, and $A(x)$ is the well-known von Mangoldt function. Also, we note that the terms $o(\cdot)$ and $O(\cdot)$ are always allowed to depend on $x$. From Theorem 1 we will deduce the following:

**Corollary 1.** If $\sum_{|\gamma| < T} |\beta - n/4| = o(N(T; Q))$, then the ordinates of nontrivial zeros of the Epstein zeta-function $\zeta(s; Q)$ are uniformly distributed modulo one.

For the Riemann zeta-function as a particular case, barring in mind that $\zeta(s; 1_1) = 2\zeta(2s)$, where $1_1$ is the $1 \times 1$ matrix with entry 1, carries the critical line at real part $1/4$, this is obviously fulfilled since (more strongly)

$$\sum_{|\gamma| < T} \left| \beta - \frac{1}{4} \right| \ll \frac{T \log T}{\log \log T} = o(N(T; Q));$$

see [18] and ultimately [13].

We briefly discuss the (easiest) case of binary quadratic forms $Q = aX^2 + bXY + cY^2$, that is, $n = 2$; see, for example, [7]. If the class number equals one, then the associated Epstein zeta-function is a constant multiple of the Dedekind zeta-function to the related imaginary quadratic number field $Q(\sqrt{-d})$, where $-d = b^2 - 4ac$ (which is negative since $Q$ is supposed to be positive definite). In this case, $\zeta(s; Q)$ has an Euler product representation and is supposed to satisfy the analogue of the Riemann hypothesis, that is, all nonreal zeros lie on the critical line $\text{Re } s = 1/2$. Moreover, following [13], we can show that the condition in Corollary 1 is
fulfilled. If the class number is greater than one, \( \zeta(s, \mathbb{Q}) \) can be represented as a linear combination of Hecke \( L \)-functions with ideal class group characters of \( \mathbb{Q}(\sqrt{-d}) \), namely

\[
\zeta(s, \mathbb{Q}) = \epsilon \frac{1}{h} \sum_{1 \leq j \leq h} \chi_j(I) L(s, \chi_j),
\]

where \( h \) is the class number, \( \epsilon \) is the number of units in the ring of integers of \( \mathbb{Q}(\sqrt{-d}) \), the Hecke characters \( \chi_j \) are evaluated at the ideal \( I \) generated by \( a, b-\sqrt{-d}/2 \), and \( L(s, \chi_j) \) is the associated Hecke \( L \)-function. In this case, \( \zeta(s, \mathbb{Q}) \) has an infinitude of zeros in the half-plane \( \Re s > 1 \), as was shown by Davenport and Heilbronn [3,4]. Hejhal [8] and Bombieri and Hejhal [1] showed that almost all zeros of Epstein zeta-functions associated with binary quadratic forms lie on the critical line, provided that the generalized Riemann hypothesis and a widely accepted conjecture on the spacing of zeros of \( L \)-functions for ideal class characters is true. Recently, Rezyakova [14] showed that a positive proportion of the zeros of any \( \zeta(s, \mathbb{Q}) \) with a binary quadratic form \( Q \) independent of the class lie on the critical line. We illustrate these results by a few examples: the Gaussian number field \( \mathbb{Q}(\sqrt{-1}) \) has a factorial ring of integers, and hence the class number is one, and for the quadratic form \( Q = X^2 + Y^2 \) (which yields the norm of the elements \( X + Y \sqrt{-1} \)), the associated Epstein zeta-function has the representation

\[
\zeta(s; \mathbb{Q}) = 4 \zeta(\sqrt{-1}) L(s, \chi_{-4}) = 4 \zeta(s) L(s, \chi_{-4}),
\]

where \( \chi_d \) denotes the Kronecker symbol \( (d/\cdot) \). For the quadratic number field \( \mathbb{Q}(\sqrt{-5}) \), however, the ring of integers is not factorial; more precisely, the class number equals two, and the Epstein zeta-functions to the reduced inequivalent forms \( Q_1 = X^2 + 5Y^2 \) and \( Q_2 = 3X^2 + 4XY + 3Y^2 \) can be represented as

\[
\zeta(s; Q_1) = L(s, \chi_0) + L(s, \chi_1) \quad \text{and} \quad \zeta(s; Q_2) = L(s, \chi_0) - L(s, \chi_1),
\]

where \( L(s, \chi_0) = \zeta(\sqrt{-5}) L(s, \chi_{-20}) \) is the Dedekind zeta-function of this number field, and \( L(s, \chi_1) \) is the Hecke \( L \)-function to the nontrivial Hecke character.

It appears that here it is rather difficult to decide whether the condition of the corollary is fulfilled or not. Another example of this type is related to the Leech lattice of dimension \( n = 24 \). If \( L \) is the \( 24 \times 24 \) matrix related to the Leech lattice (explicitly given, e.g., in [2]) and \( Q = x^t L x \), then

\[
\zeta(s; \mathbb{Q}) = \frac{65520}{691} (\zeta(s) \zeta(s-11) - L(s, \Delta)),
\]

where \( L(s, \Delta) \) is the \( L \)-function associated with Ramanujan’s \( \tau \)-function (see [10]).

If the class number is one, we could derive the statement of Corollary 1 also from the uniform distribution of the ordinates of the zeros of the appearing factors in combination with the Weyl criterion (as described in [18, Sect. 6]).

The proofs of the theorem and corollary are given in Sections 3 and 4, respectively. We conclude with a paragraph containing a variant of a result of Landau, which is an essential tool for our reasoning; besides other preliminaries, this very result of Landau can be found in the following section.

\section{2 Preliminaries}

To facilitate the analysis, we first define the function

\[
Z(s; \mathbb{Q}) := \frac{m(\mathbb{Q})^s}{N(\mathbb{Q})^s} \zeta(s; \mathbb{Q}),
\]
where \( N(Q) \) is the number of \( y \) for which \( Q[y] = m(Q) \), the minimal nonzero value of the quadratic form. It is important that \( Z(s; Q) \) carries the same zeros as \( \zeta(s; Q) \), that is, \( Z(s; Q) = 0 \) is equivalent to \( \zeta(s; Q) = 0 \). In view of the functional equation (1.1),

\[
Z(s; Q) = \frac{m(Q)^s}{N(Q)} (\det Q)^{-1/2} \pi^{2s-n/2} \frac{\Gamma\left(\frac{n}{2} - s\right)}{\Gamma(s)} \frac{N(Q^{-1})}{m(Q^{-1})^{n/2-s}} Z\left(\frac{n}{2} - s; Q^{-1}\right). \tag{2.1}
\]

We also need to take a look at the Dirichlet series expansions of \( Z(s; Q) \) and at \( Z'(s; Q) \), as we will have to deal with the logarithmic derivative of \( Z(s; Q) \). With the definitions

\[
r(m; Q) := \# \{ x \in \mathbb{Z}^n : Q[x] = m \}
\]

and

\[
a(n) := \frac{r(n, Q)}{N(Q)},
\]

we observe that

\[
Z(s; Q) = \frac{m(Q)^s}{N(Q)} \zeta(s; Q) = 1 + \sum_{n>m(Q)} a(n) \left( \frac{m(Q)}{n} \right)^s \tag{2.2}
\]

and

\[
Z'(s; Q) = \sum_{n>m(Q)} a(n) \log \frac{m(Q)}{n} \left( \frac{m(Q)}{n} \right)^s. \tag{2.3}
\]

Before we further analyze the logarithmic derivative, we establish a zero-free region of \( Z(s; Q) \). Following Section 2 of Steuding [16], we denote by \( m_2(Q) \) the second minimum of \( Q \), that is, the least value \( > m(Q) \) of \( Q[y] \) for \( y \in \mathbb{Z}^n \). Setting

\[
\lambda := \frac{m_2(Q)}{m(Q)}
\]

for \( \sigma > n/2 \), we then have, as \( \sigma \to \infty \),

\[
Z(s; Q) = 1 + c\lambda^{-s} + o(\lambda^{-\sigma}), \tag{2.4}
\]

where \( c \) is a positive constant depending only on \( Q \). Therefore there exists a zero-free half-plane to the right. The same argument yields a zero-free region for \( Z(s; Q^{-1}) \), so from the functional equation (2.1) we deduce the existence of a positive constant \( B_1 \) such that neither \( Z(s; Q) \) nor \( Z(s; Q^{-1}) \) have nontrivial zeros outside the strip \(-B_1 < \sigma < B_1 \). It also follows from (2.4) that there exists a constant \( B_2 \) such that \( Z(\sigma + it; Q) > 1/2 \) for \( \sigma > B_2 \). Additionally, if we consider the supremum and infimum of the real parts \( \beta_i \) of the nontrivial zeros, then we can certainly find a constant \( B_3 \) such that both inequalities \( 1 \leq |B_3 - \sup \beta_i| \) and \( 1 \leq |B_3 - \inf \beta_i| \) hold. By defining \( B_3 := \max\{\sup \beta_i, \inf \beta_i\} + 1 \) we ensure that these inequalities also hold for all constants greater than \( B_3 \). The properties of the constants \( B_i \) are prerequisites for the proof of our main result.

The following construction is essentially that of Landau [12]. According to [6], we consider the unbounded sequence

\[
1 < \frac{m(Q) + 1}{m(Q)} < \frac{m(Q) + 2}{m(Q)} < \ldots \tag{2.5}
\]

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of real numbers belonging to the general Dirichlet series \( Z(s; Q) \) (see 2.2). We can define \( S \) as the set of all possible products of elements of this sequence and obtain an ordered sequence of all different numbers of \( S \) as

\[
l_n(Q) = \prod_{\nu=1}^{h} \frac{m(Q) + k_{\nu}}{m(Q)} \tag{2.6}
\]

with \( h \geq 0, k_{\nu} \in \mathbb{N} \).

Therefore Lemma 8 of [6] applied to the two general Dirichlet series \( Z(s; Q) \) and \( Z'(s; Q) \) ensures that the quotient \( Z'(s; Q)/Z(s; Q) \) is again a general Dirichlet series, which converges absolutely in a right half-plane and carries the numbers \( l_n(Q) \) (see 2.6) as its defining unbounded sequence. In particular, we can prove the following lemma giving an explicit representation.

**Lemma 1.** For \( Z(s; Q) \) and \( Z'(s; Q) \) defined as before, the quotient \( Z'(s; Q)/Z(s; Q) \) is again a general Dirichlet series, and there exists a constant \( B_4 \) such that \( Z'(s; Q)/Z(s; Q) \) converges absolutely for \( \sigma > B_4 \). We have

\[
\frac{Z'(s; Q)}{Z(s; Q)} = \sum_{n=1}^{\infty} \frac{b_n(Q)}{l_n(Q)^s},
\]

where the \( l_n(Q) \)'s are defined in (2.6) with \( l_n(Q) \to \infty \) as \( n \to \infty \). Moreover, the coefficients \( b_n(Q) \) are given explicitly as

\[
b_n(Q) = \sum_{\text{distinct } k_i} a(m(Q) + k_i) \log \frac{m(Q)}{m(Q) + k_i} \times \sum_{g_1, \ldots, g_r \in \mathbb{N}} \frac{(-1)^{h-1}}{(g_1, \ldots, g_j-1, g_j-1, g_{j+1}, \ldots, g_r)^{h-1}} \times a(m(Q) + k_1) \cdots a(m(Q) + k_{i-1}) a(m(Q) + k_{i+1}) \cdots a(m(Q) + k_h). \tag{2.7}
\]

Here \( g_1, \ldots, g_r \) are the multiplicities of the distinct values of \( k_1, \ldots, k_h \), and hence \( r \leq h \) and \( g_1 + \cdots + g_r = h \), where \( g_j \) is the multiplicity of \( k_i \).

**Proof.** Since \( Z(s; Q) \) starts with a constant term 1, there exists an inverse of the Dirichlet series

\[
\frac{1}{Z(s; Q)} = \frac{N(Q)}{m(Q)s \zeta(s; Q)}
\]

and we can apply the method of Landau [12], provided that \( \Re s \) is sufficiently large. More precisely, the inverse of \( Z(s; Q) \),

\[
\frac{1}{Z(s; Q)} = \sum_{n=1}^{\infty} \frac{c_n(Q)}{l_n(Q)^s},
\]

is a general Dirichlet series with \( l_n(Q) \) of the form (2.6) and some \( c_n(Q) \). This follows immediately from the first consideration in the Appendix of Landau [12], where in our case we set \( m = m(Q), f(s) = \zeta(s; Q)/N(Q) = \sum_{n=1}^{\infty} a(n)/n^s, \) and \( a(m(Q)) = 1, \) the first nonzero coefficient. Hence, the product \( Z'(s; Q) \cdot (1/Z(s; Q)) \) of two general Dirichlet series is again a Dirichlet series of the form

\[
\frac{Z'(s; Q)}{Z(s; Q)} = \sum_{n=1}^{\infty} \frac{b_n(Q)}{l_n(Q)^s}. \tag{2.8}
\]
Taking into account the uniqueness theorem for Dirichlet series, there exists a zero-free half-plane. Accordingly, we may find a positive number $B_4$ such that the abscissa of absolute convergence of the Dirichlet series (2.8) is less than or equal to $B_4$. Explicitly, we obtain

$$
\frac{1}{Z(s; Q)} = \sum_{h=0}^{\infty} (-1)^h \left( \sum_{n > m(Q)} a(n) \left( \frac{m(Q)}{n} \right)^h \right)
$$

$$
= \sum_{h=0}^{\infty} (-1)^h \sum_{\substack{k_1, \ldots, k_h \in \mathbb{N} \atop g_1 + \cdots + g_r = h}} \left( \sum_{i} \right) a(m(Q) + k_1) \cdots a(m(Q) + k_h)
$$

$$
\times \left( \frac{m(Q)^h}{(m(Q) + k_1) \cdots (m(Q) + k_h)} \right)^s
$$

with the tuple $g_1, \ldots, g_r \in \mathbb{N}$ running through all partitions $P(h)$ of $h$. Hence,

$$
\frac{Z'(s; Q)}{Z(s; Q)} = \sum_{n > m(Q)} a(n) \log \frac{m(Q)}{n} \left( \frac{m(Q)}{n} \right)^s
$$

$$
\times \sum_{h=0}^{\infty} (-1)^h \sum_{\substack{k_1, \ldots, k_h \in \mathbb{N} \atop g_1 + \cdots + g_r = h}} \left( \sum_{i} \right) a(m(Q) + k_1) \cdots a(m(Q) + k_h)
$$

$$
\times \left( \frac{m(Q)^h}{(m(Q) + k_1) \cdots (m(Q) + k_h)} \right)^s
$$

(2.9)

Now, for a given $l_n(Q)$, we see that the coefficient $b_n(Q)$ takes the form as stated before. This holds since in (2.9), which shows the factors of the product $Z'(s; Q) \cdot (1/Z(s; Q))$ separately, for a distinct $k_i$, always the coefficient of one summand of $Z'(s; Q)$ and the coefficient of

$$
\left( \frac{m(Q)^{h-1}}{(m(Q) + k_1) \cdots (m(Q) + k_{i-1})(m(Q) + k_{i+1}) \cdots (m(Q) + k_h)} \right)^s
$$

contribute to $b_n(Q)$. From this we obtain the representation of the lemma. □

Since the Dirichlet series representing the logarithmic derivative of $Z(s; Q)$ converges, we are now able to choose the constant

$$
B = \max(B_1, B_2, B_3, B_4)
$$

carrying all properties needed for the proof of our main result. We also recall Lemma 5 from [6], which will be very useful.

**Lemma 2.** If $f(s)$ is analytic and $f(s_0) \neq 0$ with

$$
\left| \frac{f(s)}{f(s_0)} \right| < e^M
$$

in $\{s: |s - s_0| \leq r\}$ with $M > 1$, then

$$
\left| \frac{f(s)}{f(s_0)} - \sum_{\rho} \frac{1}{s - \rho} \right| < C \frac{M}{r}
$$

for $|s - s_0| \leq r/4$, where $C$ is a constant, and $\rho$ runs through the zeros of $f(s)$ such that $|\rho - s_0| \leq r/2$. 

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Finally, in this section, we recall a well-known result from Section 3 of Steuding [15], which gives information about the growth of the Epstein zeta-function. It is based on an application of the Phragmén–Lindelöf principle: As $|t| \to \infty$, we have

$$\zeta(s; \mathcal{Q}) \ll |t|^\max\{n/2-\sigma, 0\}+\epsilon. \quad (2.10)$$

### 3 Proof of the theorem

We aim to prove the theorem by using an appropriate contour integral and residue calculus. By choosing $b > B$ and in view of the functional equation, we ensure that all nontrivial zeros of $\zeta(s; \mathcal{Q})$ lie inside $\text{Re} \, s > n/2 - b$. We further note that the logarithmic derivative of $\zeta(s; \mathcal{Q})$ and therefore also of $Z(s; \mathcal{Q})$ have simple poles at each zero with residue equal to the order. Therefore

$$\sum_{|\gamma|<T} x^\gamma = \frac{1}{2\pi i} \int_{\partial \mathcal{R}} x^s \frac{Z'(s; \mathcal{Q})}{Z(s; \mathcal{Q})} \, ds + O(1),$$

where the error term arises from the pole at $s = n/2$. Here the integration path is the counterclockwise oriented closed rectangle $\partial \mathcal{R}$ with points $B - iT, B + iT, n/2 - b + iT, \text{and} n/2 - b - iT$. We have

$$\int_{\partial \mathcal{R}} x^s \frac{Z'(s; \mathcal{Q})}{Z(s; \mathcal{Q})} \, ds = \left\{ \int_{B - iT} + \int_{B + iT} \right\} x^{n/2 - b + iT} + \left\{ \int_{B - iT} + \int_{B + iT} \right\} x^{n/2 - b - iT} =: \sum_{j=1}^{4} I_j.$$

The idea is now to estimate all integral paths piecewise and show that each integral is $O(T)$. We evaluate $I_1$ directly in the area of absolute convergence; for $I_3$, we achieve the same by using the functional equation, and $I_2$ and $I_4$ can be estimated by means of techniques from [6], essentially by using Lemma 2.

We begin with

$$I_1 = \int_{B - iT}^{B + iT} x^s \frac{Z'(s; \mathcal{Q})}{Z(s; \mathcal{Q})} \, ds.$$

Since summation and integration can be interchanged in the area of absolute convergence, we deduce from Lemma 1 that

$$I_1 = \int_{-T}^{+T} x^{B+iT} \frac{Z'(B+it; \mathcal{Q})}{Z(B+it; \mathcal{Q})} \, dt = \int_{-T}^{+T} x^{B+iT} \sum_{n=1}^{\infty} \frac{b_n(\mathcal{Q})}{l_n(\mathcal{Q})} \, dt$$

$$= i \sum_{n=1}^{\infty} b_n(\mathcal{Q}) \left( \frac{x}{l_n(\mathcal{Q})} \right)^B \int_{-T}^{+T} \left( \frac{x}{l_n(\mathcal{Q})} \right)^{it} \, dt.$$

The integral $\int_{-T}^{+T} (x/l_n(\mathcal{Q}))^{it} \, dt$ is unbounded if and only if $x/l_n(\mathcal{Q}) = 1$, that is, for $x = l_n(\mathcal{Q})$; otherwise, it is bounded, and in this case, we have

$$\int_{-T}^{+T} \left( \frac{x}{l_n(\mathcal{Q})} \right)^{it} \, dt = O(1). \quad (3.1)$$
Now we observe that the number of zeros with $|\sigma - (B + iT)| \leq r/2$ is bounded by $C \log T$. This is true since the number of zeros for $\zeta(s; \mathcal{Q})$ and accordingly $Z(s; \mathcal{Q})$ are given by (1.2), and hence we get

$$N\left(T + \frac{r}{2}; \mathcal{Q}\right) - N\left(T - \frac{r}{2}; \mathcal{Q}\right) = O(\log T).$$

Putting all this together, we get

$$I_1 = 2i \delta_{\text{inc},x} b_k(\mathcal{Q}) T + O(1)$$

with $b_k(\mathcal{Q})$ defined in Lemma 1.

Now we consider

$$I_2 = \int_{n/2-b+iT}^{B+iT} x^s \frac{Z'(s; \mathcal{Q})}{Z(s; \mathcal{Q})} \, ds,$$

To estimate this integral, we first apply Lemma 2. By setting the parameters

$$f(s) = Z(s; \mathcal{Q}), \quad s_0 = B + iT,$$

$$M = C \log T, \quad r = 4 \left( B - \left( \frac{n}{2} - b \right) \right)$$

with an appropriate constant $C$ we are able to meet the conditions of the Lemma since $B$ is large enough to ensure that $s_0 = B + iT$ is from the zero-free right half-plane essentially due to (2.4). Also, (2.10) ensures that $|f(s)/f(s_0)| < e^M$. We note that the constant $M$ may depend on $n$ and $\mathcal{Q}$. Hence we get

$$\frac{Z'(s; \mathcal{Q})}{Z(s; \mathcal{Q})} = \sum_{|\rho_0 - s_0| \leq r/2} \frac{1}{s - \rho_0} + O(\log T)$$

for $|s - s_0| \leq r/4 = B - (n/2 - b)$. With $\rho_0 = \beta_0 + i\gamma_0$, it follows that

$$I_2 \leq x^B \int_{n/2-b}^{B} \sum_{|\rho_0 - s_0| \leq r/2} \frac{1}{\sigma - \rho_0} \, d\sigma + O(\log T)$$

$$= x^B \sum_{|\rho_0 - s_0| \leq r/2} \int_{n/2-b}^{B} \frac{1}{\sqrt{(\sigma - \beta_0)^2 + (T - \gamma_0)^2}} \, d\sigma + O(\log T)$$

$$= x^B \sum_{|\rho_0 - s_0| \leq r/2} \left( \log \left( B - \beta_0 + \sqrt{(T - \gamma_0)^2 + (B - \beta_0)^2} \right) - \log \left( \frac{n}{2} - b - \beta_0 + \sqrt{(T - \gamma_0)^2 + \left( \frac{n}{2} - b - \beta_0 \right)^2} \right) \right) + O(\log T).$$

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Furthermore, all summands are bounded by a constant $C$ since
\[
\begin{align*}
\log (B - \beta_0 + \sqrt{(T - \gamma_0)^2 + (B - \beta_0)^2}) \\
- \log \left( \frac{n}{2} - b - \beta_0 + \sqrt{(T - \gamma_0)^2 + \left( \frac{n}{2} - b - \beta_0 \right)^2} \right) \\
= \log \frac{(B - \beta_0 + \sqrt{(T - \gamma_0)^2 + (B - \beta_0)^2})}{\left( \frac{n}{2} - b - \beta_0 + \sqrt{(T - \gamma_0)^2 + \left( \frac{n}{2} - b - \beta_0 \right)^2} \right)}.
\end{align*}
\]
Since we can choose $b, B_3$ such that $1 \leq |B - \sup \beta_0|$ and $1 \leq |B - \inf \beta_0|$ for $B > B_3$, we can also find constants $c_1, c_2$ with $c_1 |n/2 - b - \beta_0| \leq |B - \beta_0| \leq c_2 |n/2 - b - \beta_0|$ and hence $c_1 \leq |B - \beta_0|/|n/2 - b - \beta_0| \leq c_2$. Therefore there exists a constant $C$ such that
\[
\log \frac{(B - \beta_0 + \sqrt{(T - \gamma_0)^2 + (B - \beta_0)^2})}{\left( \frac{n}{2} - b - \beta_0 + \sqrt{(T - \gamma_0)^2 + \left( \frac{n}{2} - b - \beta_0 \right)^2} \right)} \leq C.
\]
With this, we have shown
\[
I_2 = \int_{n/2-b+iT}^{B+iT} x^s \frac{Z'(s; Q)}{Z(s; Q)} \, ds = O(\log T).
\]
Thus we can turn to
\[
I_3 = \int_{n/2-b-iT}^{n/2-b+iT} x^s \frac{Z'(s; Q)}{Z(s; Q)} \, ds.
\]
To get the logarithmic derivative of $Z(s; Q)$, we first take logarithms on both sides of (2.1), that is,
\[
\log Z(s; Q) = s \log m(Q) - \log N(Q) + \log(det\, Q)^{-1/2} \\
+ \left( 2s - \frac{n}{2} \right) \log \pi + \log \Gamma\left( \frac{n}{2} - s \right) - \log \Gamma(s) + \log N(Q^{-1}) \\
- \left( \frac{n}{2} - s \right) \log m(Q^{-1}) + \log Z\left( \frac{n}{2} - s; Q^{-1} \right),
\]
so that we obtain
\[
\frac{Z'(s; Q)}{Z(s; Q)} = \left[ \log m(Q) + 2 \log \pi + \log m(Q^{-1}) \right] + \frac{\Gamma'(\frac{n}{2} - s)}{\Gamma(\frac{n}{2} - s)} - \frac{\Gamma'(s)}{\Gamma(s)} + Z'(\frac{n}{2} - s; Q^{-1}).
\]
Therefore we have
\[
I_3 = \int_{n/2-b-iT}^{n/2-b+iT} x^s \left[ c_3 + \frac{\Gamma'(\frac{n}{2} - s)}{\Gamma(\frac{n}{2} - s)} - \frac{\Gamma'(s)}{\Gamma(s)} + \frac{Z'(\frac{n}{2} - s; Q^{-1})}{Z(\frac{n}{2} - s; Q^{-1})} \right] \, ds
\]
with
\[
c_3 := \log m(Q) + 2 \log \pi + \log m(Q^{-1}).
\]
We observe that $Z'(n/2 - s; Q^{-1})/Z(n/2 - s; Q^{-1})$ is again a Dirichlet series, and with another application of Lemma 1, we get the representation

$$
\frac{Z'(n/2 - s; Q^{-1})}{Z(n/2 - s; Q^{-1})} = \sum_{k=1}^{\infty} \frac{b_k(Q^{-1})}{l_k(Q^{-1})^{n/2-s}},
$$

which converges for $\Re s > n/2$. We can now divide the integral as follows:

$$
I_3 = \int_{n/2-b-iT}^{n/2-b+iT} x^s \left[ \frac{Z'(n/2 - s; Q^{-1})}{Z(n/2 - s; Q^{-1})} \right] ds + \int_{n/2-b-iT}^{n/2-b+iT} x^s \left( c_3 + \frac{\Gamma'(n/2 - s)}{\Gamma(n/2 - s)} - \frac{\Gamma'(s)}{\Gamma(s)} \right) ds
$$

For the first integral, in the area of absolute convergence, we get

$$
I_{3a} = \sum_{k=1}^{\infty} \frac{b_k(Q^{-1})}{l_k(Q^{-1})^{n/2-s}} ds = \sum_{k=1}^{\infty} \frac{b_k(Q^{-1})}{l_k(Q^{-1})^{n/2-s}} \int_{n/2-b-iT}^{n/2-b+iT} \frac{1}{x l_k(Q^{-1})^{-s}} ds
$$

Now repeating the argument of the calculation of $I_1$, we see that, for $x \neq 1/l_k(Q^{-1})$, the integral $I_{3a}$ is $O(1)$. Taking into account the contribution for $x = 1/l_k(Q^{-1})$, we obtain overall

$$
I_{3a} = -ix^{n/2} \delta_{l_k(Q^{-1}),1/x} b_k(Q^{-1}) 2T + O(1).
$$

It remains to estimate $I_{3b}$. Here we first apply Stirling’s formula for the gamma function and obtain

$$
|\Gamma(\sigma + it)| = \sqrt{2\pi} e^{-\pi|t|/2} |t|^{|\sigma|-1/2} \left( 1 + O\left( \frac{1}{|t|} \right) \right)
$$

and, for the logarithm,

$$
\log \Gamma(s) = \left( s - \frac{1}{2} \right) \log s - s + \frac{1}{2} \log 2\pi + O\left( \frac{1}{|s|} \right)
$$

for $|\arg s| < \pi$ and $|s| \geq 1$. Therefore, for the logarithmic derivative, we get the following asymptotics:

$$
\frac{\Gamma'(s)}{\Gamma(s)} = \log s - \frac{1}{2s} + O\left( \frac{1}{|s|^2} \right) = \log s + O\left( \frac{1}{|s|} \right);
$$

see, e.g., [9]. To handle the branch cut of the logarithm, we further separate the integral into

$$
I_{3b} = \left\{ \int_{n/2-b-iT}^{n/2-b+iT} x^s \left( c_3 + \frac{\Gamma'(n/2 - s)}{\Gamma(n/2 - s)} - \frac{\Gamma'(s)}{\Gamma(s)} \right) ds \right\}
$$

\[= L_1 + L_2 + L_3.\]
Clearly, we have \( L_1 = O(1) \). Next, we consider

\[
L_2 = \int_{n/2-b+iT}^{n/2-b+iT} x^s \left[ c_3 + \log s + O\left(\frac{1}{|s|}\right) + \log\left(\frac{n}{2} - s\right) + O\left(\frac{1}{|\frac{n}{2} - s|}\right) \right] ds.
\]

We see that

\[
\int_{n/2-b+iT}^{n/2-b+iT} x^s \left[ c_3 + O\left(\frac{1}{|s|}\right) + O\left(\frac{1}{|\frac{n}{2} - s|}\right) \right] ds
\]

\[
= i x^{n/2-b} \int_1^T x^{it} \left[ c_3 + O\left(\frac{1}{|\frac{n}{2} - b + it|}\right) + O\left(\frac{1}{|b - it|}\right) \right] dt = O(\log T).
\]

Moreover,

\[
\int_{n/2-b+iT}^{n/2-b+iT} x^s \left[ \log s + \log\left(\frac{n}{2} - s\right) \right] ds
\]

\[
= -i x^{n/2-b} \int_1^{+T} x^{it} \left[ \log\left(\frac{n}{2} - b + it\right) + \log(b - it) \right] dt.
\]

Now we apply partial integration to derive

\[
\int_1^{+T} x^{it} \left[ \log\left(\frac{n}{2} - b + it\right) t + \log(b - it) \right] dt
\]

\[
= \frac{x^{it}}{i \log x} \left[ \log\left(\frac{n}{2} - b + it\right) + \log(b - it) \right] \bigg|_1^{+T} - \int_1^{+T} \frac{x^{it}}{\log x} \left( \frac{1}{t} - \frac{1}{(\frac{n}{2} - t)} \right) dt.
\]

Both summands on the right-hand side are clearly \( O(\log T) \), therefore \( L_2 = O(\log T) \), and the same argument yields \( L_3 = O(\log T) \). Putting the pieces together, we get

\[
I_3 = I_{3a} + I_{3b} = I_{3a} + L_1 + L_2 + L_3
\]

\[
= -i x^{n/2} \delta_{k(Q^{-1}),1/x} b_k(Q^{-1}) 2T + O(\log T).
\]

Finnaly,

\[
I_4 = \int_{n/2-b-iT}^{B-iT} x^s \frac{Z'(s; Q)}{Z(s; Q)} ds
\]

can be estimated exactly like \( I_2 \), and hence \( I_4 = O(\log T) \). This concludes the proof of the theorem.
4 Proof of the corollary

Since

$$\exp y - 1 = \int_0^y \exp t \, dt \ll |y| \max\{1, \exp y\},$$

by setting $y = (\beta - n/4) \log x$ with positive $x \neq 1$ we get the estimate

$$\left| x^{(n/4+i\gamma)} - x^{(\beta+i\gamma)} \right| = \left| x^{i\gamma} \right| \left| x^{n/4} - x^\beta \right| = \left| x^\beta \left( x^{(n/4-\beta)} - 1 \right) \right| \leq x^\beta \exp \left( \left( \frac{n}{4} - \beta \right) \log x \right) - 1 \leq \left| \beta - \frac{n}{4} \right| \log x \max\{x^\beta, x^{n/4}\}.$$ 

In the following, we use Theorem 1 and Weyl’s criterion (see [19, 20]) to deduce the uniform distribution of the imaginary parts. We proceed like in [18] and obtain

$$\frac{1}{N(T; Q)} \sum_{|\gamma| < T} \left| x^{(n/4+i\gamma)} - x^{(\beta+i\gamma)} \right| \leq \frac{X}{N(T; Q)} \sum_{|\gamma| < T} \left| \beta - \frac{n}{4} \right|$$

with $X = \max\{x^D, 1\} |\log x|$ and $D$ as an upper bound for the real parts of the zeros. Such a positive number $D$ exists since $\zeta(s; Q)$ (having a Dirichlet series representation for $\Re s > n/2$) is nonvanishing in some right-half plane. Using the assumption $\sum_{|\gamma| < T} |\beta - n/4| = o(N(T; Q))$ of Corollary 1, we now get

$$\frac{1}{N(T; Q)} \sum_{|\gamma| < T} \left( x^{(n/4+i\gamma)} - x^{(\beta+i\gamma)} \right) = o(1),$$

where we dropped the dependency on $X$. Hence, applying Theorem 1, that is, $\sum_{|\gamma| < T} x^\rho = O(T)$, yields

$$\frac{1}{N(T; Q)} \sum_{|\gamma| < T} x^{(n/4+i\gamma)} = o(1).$$

Dividing this by $x^{n/4}$ and setting $x = z^m$ with positive real $z \neq 1$ and $m \in \mathbb{N}$, we get

$$\lim_{T \to \infty} \frac{1}{N(T; Q)} \sum_{|\gamma| < T} \exp(im\gamma \log z) = 0.$$ 

Now Weyl’s criteria implies the uniform distribution of the sequence $(1/(2\pi))\gamma \log z$ and finally the Epstein zeta ordinates.

5 Recursion formula for the logarithmic derivative

In this section, we derive a recursive representation for the logarithmic derivative $Z'(s; Q)/Z(s; Q)$. We also show that it agrees with Lemma 1. We split $Z'(s; Q)/Z(s; Q)$ into two summands depending on if the $l_n(Q)$ as defined in (2.6) consists of only one factor ($h = 1$) or is the product of at least two factors ($h \geq 2$). The first
summand of this representation equals \( Z'(s; Q) \), and for the second summand, we define quantities \( c_k \) and \( q_k \) recursively. The \( q_k \) will run through all possible products of the form (2.6) with at least two factors. The \( c_k \) will be the corresponding coefficients. In particular, we claim the following:

**Lemma 3.** For the logarithmic derivative of \( Z(s; Q) \) as defined in the previous sections, we have the following recursive relation:

\[
\frac{Z'(s; Q)}{Z(s; Q)} = \sum_{n > m(Q)} a(n) \log \frac{m(Q)}{n} \left( \frac{m(Q)}{n} \right)^s + \sum_{k=0}^\infty c_k q_k^{-s}.
\]

(5.1)

For \( k \geq 0 \), \( q_k \) are defined in increasing order as

\[
q_0 = \frac{l_1^2}{m(Q)^2}, \quad q_k = \min_{i \leq k-1} \left\{ \left( \frac{l_i}{m(Q)} \right), \left( \frac{l_3}{m(Q)} \frac{l_4}{m(Q)} \right) \right\}
\]

with \( q_k > q_{k-1} \), \( l_1 = m(Q) + 1 \), and in general all numbers \( l_i \in \mathbb{N} \) are greater than \( m(Q) \). For the corresponding \( c_k \), we have, with \( i_j < k \) and \( l_j > m(Q) \),

\[
c_k = \begin{cases} 
-a(l_2)c_{i_1} - \cdots - a(l_2)c_{i_n} & \text{if } q_k = l_2/m(Q) = \cdots = l_n/m(Q), \\
-a(l_3)^2 \log \frac{m(Q)}{l_3} & \text{if } q_k = \frac{l_3}{m(Q)} \frac{l_4}{m(Q)} \text{ and } l_3 = l_4, \\
-a(l_4)a(l_3) \log \frac{m(Q)}{l_3} - a(l_3)a(l_4) \log \frac{m(Q)}{l_4} & \text{if } q_k = \frac{l_3}{m(Q)} \frac{l_4}{m(Q)} \text{ and } l_3 \neq l_4.
\end{cases}
\]

**Proof.** First, we note that, obviously, \( Z'(s; Q)/Z(s; Q) \cdot Z(s; Q) = Z'(s; Q) \), and from this product representation we can derive the coefficients recursively by comparison of the series. Now the first sum of (5.1) is equal to \( Z'(s; Q) \) and stems from the fact that \( Z(s; Q) \) starts with constant term 1. In the second term, \( q_0 \) is the first base, which needs to occur since the lowest base of \( Z'(s; Q)/Z(s; Q) \) is \((l_1/m(Q))^{-s}\). Hence we need to define \( c_0 \) as to cancel out \( a(l_1) \log(m(Q)/l_1) \cdot a(l_1) \).

Now we consider the \( q_k \) defined in increasing order. The construction of the \( q_k \) ensures to represent all possible products with at least two factors, which are clearly of the form of the \( l_n(Q) \); see (2.6). In each step of the recursion, either the next larger \( q_k \) is a new product \( l_3/m(Q) \cdot l_4/m(Q) \), or it is a product of an already existing \( q_i \) with one new factor \( l_4/m(Q) \). The \( c_k \) are defined such that by multiplication with \( Z(s; Q) \) the coefficient of the lowest base is cancelled out. This means that if \( q_k \) is of the form \( q_i \cdot l_2/m(Q) \), then we have \( c_k = -a(l_2)c_{i_1} - \cdots - a(l_2)c_{i_n} \) since the \( q_k \) can equal several different products of this form and we need to cancel out all the contributions. If we have new products of the form \( q_k = l_3/m(Q) \cdot l_4/m(Q) \), then we again achieve the cancellation by the previous definitions. Here we note that the last case differentiation occurs if either \( l_3 = l_4 \) where there is only one term to be cancelled or if \( l_3 \neq l_4 \) where the two symmetric coefficients need to be wiped out.

To see that both representations from Lemmas 1 and 3 coincide, we first note that the number of weights \( l_i/m(Q) \) in the \( c_k \) grows according to the multinomial coefficient, which is the essential part of the \( b_n(Q) \) defined in (2.7). Further, the recursive identity \( c_k = -a(l_2)c_{i_1} - \cdots - a(l_2)c_{i_n} \) corresponds with that of the multinomial coefficient

\[
\binom{\nu}{k_1, \ldots, k_r} = \sum_{i=1}^{r} \binom{\nu - 1}{k_1, \ldots, k_{i-1}, k_i - 1, k_{i+1}, \ldots, k_r}.
\]

Finally, the factors \( a(m(Q) + k_i) \log(m(Q)/(m(Q) + k_i)) \) in \( b_n(Q) \) arise from the multiplication with \( Z'(s; Q) \). □
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