ON THE HYPERBOLIZING METRIC SPACES

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Abstract: In this paper, we prove that the metric space \((Z \setminus M, u_Z)\) defined by Z.Ibragimov is asymptotically PT\(_{-1}\) if the metric space \((Z, d)\) is PT\(_0\), where \(M\) is a nonempty closed proper subset of \(Z\). Secondly, based on the metric \(u_Z\), we define a new kind of metric \(k_z\) on the set \(Z \setminus M\) and show that the new metric space \((Z \setminus M, k_z)\) is also asymptotically PT\(_{-1}\) without the assumption of PT\(_0\) on the metric space \((Z, d)\).

Key Words: Gromov hyperbolic space, asymptotically PT\(_{-1}\) space, PT\(_0\) space.

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1. Introduction

Hyperbolization is an important process for converting a geometry object into a metric space with nonpositive curvature in the sense of Gromov. The process has been considered by many authors in the past several years. For examples, it was proved by M.Bonk, J.Heinonen and P.Koskela that the quasihyperbolic metric hyperbolizes uniform metric spaces in [1]. In Fractal geometry, Ka-Sing Lau and Xiang-Yang Wang proved that in [3], for an iterated function system \(\{S_j\}_{j=1}^N\) of similitudes that satisfies the open set condition, there is a natural graph structure in the representing symbolic space to make it a hyperbolic graph in the sense of Gromov, and the hyperbolic boundary is homeomorphic to the self-similar set generated by \(\{S_j\}_{j=1}^N\). The result of [3] has been generalized by Jun Jason Luo in [2] to the Moran set case. In [2], he proved that a Moran set is homeomorphic to the hyperbolic boundary of the representing symbolic space.

Let \((Z, d)\) be an arbitrary metric space. The distance from a point \(z\) to a set \(A\) is denoted by \(\text{dist}(z, A) = \inf\{d(z, y) : y \in A\}\). The diameter of a set \(A \subset Z\) is denoted by \(\text{diam}(A)\). Let \(M\) be a nonempty closed proper subset of \(Z\). For convenience we put \(d_M(z) = \text{dist}(z, M)\). In order to hyperbolize all locally compact noncomplete metric spaces, Z.Ibragimov introduced the following metric \(u_Z\) on \(Z \setminus M\) in the paper
For \(x, y \in Z \setminus M\), Z.Ibragimov defined
\[
u_Z(x, y) = 2 \log \frac{d(x, y) + d_M(x) \vee d_M(y)}{\sqrt{d_M(x)d_M(y)}},
\]
where \(a \vee b = \max\{a, b\}\) for \(a, b \in \mathbb{R}\) and obtained the following result about \(u_Z(x, y)\) in the paper [7].

**Theorem 1** (Z.Ibragimov). Let \((Z, d)\) be an arbitrary metric space and let \(M\) be a nonempty closed proper subset of \(Z\). Then

1. \(u_Z\) is a metric on \(Z \setminus M\);
2. the space \((Z \setminus M, u_Z)\) is \(\delta\)-hyperbolic with \(\delta \leq \log 4\);
3. the identity map between \((Z \setminus M, d)\) and \((Z \setminus M, u_Z)\) is \(5\)-quasiconformal;
4. if the space \((Z, d)\) is complete, then so is \((Z \setminus M, u_Z)\).

On the other hand, in order to generalize the well studied relation between the geometry of the classical hyperbolic space and the Möbius geometry of its boundary at infinity to \(\text{CAT}(-1)\) space case, R.Miao and V.Schroeder introduced the asymptotically \(\text{PT}_{-1}\) space and proved the asymptotically \(\text{PT}_{-1}\) space is a Gromov hyperbolic space in [5]. Since the metric space \((Z \setminus M, u_Z)\) is Gromov hyperbolic, thus a natural question is whether the metric space \((Z \setminus M, u_Z)\) introduced by Z.Ibragimov is an asymptotically \(\text{PT}_{-1}\). About this question, using the completely elementary methods, we obtain the following result in this paper.

**Theorem 2.** Let \((Z, d)\) be an arbitrary metric space and let \(M\) be a nonempty closed proper subset of \(Z\). If the metric space \((Z, d)\) is \(\text{PT}_0\), then the metric space \((Z \setminus M, u_Z)\) is asymptotically \(\text{PT}_{-1}\).

Secondly, based on the metric \(u_Z\) defined by Z.Ibragimov, we define the following metric on the set \(Z \setminus M\). Given \(\epsilon \in (0, \frac{1}{2}]\), let
\[
k_Z(x, y) = 2 \log \frac{d^\epsilon(x, y) + d_M(x) \vee d_M(y)}{\sqrt{d_M(x)d_M(y)}},
\]
for \(x, y \in Z \setminus M\), where \(d^\epsilon(x, y) = (d(x, y))^\epsilon\). We obtain the following result.

**Theorem 3.** Let \((Z, d)\) be an arbitrary metric space and let \(M\) be a nonempty closed proper subset of \(Z\). Then

1. \(k_Z\) is a metric on \(Z \setminus M\);
2. the space \((Z \setminus M, k_Z)\) is asymptotically \(\text{PT}_{-1}\);
3. the identity map between \((Z \setminus M, k_Z)\) and \((Z \setminus M, d)\) is \(2\)-quasiconformal;
4. if the space \((Z, d)\) is complete, then so is \((Z \setminus M, k_Z)\).
Obviously, the metric $k_Z$ can be defined on an arbitrary metric space and any a nonempty closed proper subset $M$ of $Z$. According to Theorem 3, it also preserves the quasiconformal geometry of the space. Since the asymptotically $PT_{-1}$ space is a Gromov hyperbolic space with more better properties, thus the metric $k_Z$ may be more appropriate to hyperbolize all locally compact noncomplete metric spaces.

2. Some remarks and definitions

Firstly, we begin with a brief discussion of Gromov hyperbolic spaces. Let $(X, d)$ denote a metric space. For $x, y, z \in X$, the Gromov product of $x$ and $y$ with respect to $z$ is defined by

$$(x|y)_z = \frac{1}{2}[d(x, z) + d(y, z) - d(x, y)].$$

**Definition 1.** A metric space $(X, d)$ is called Gromov $\delta$-hyperbolic if there is a $\delta \geq 0$ such that

$$(x|y)_o \geq \min \{(x|z)_o, (z|y)_o\} - \delta$$

for all $x, y, z, o \in X$.

The following are examples of Gromov $\delta$-hyperbolic space.

**Example 1.** (1) Every bounded metric space $(X, d)$ is $\text{diam}(X)$-hyperbolic space.

(2) The hyperbolic space $\mathbb{H}^n$ is $3\log 2$-hyperbolic space, refer to [4].

(3) Every tree with edges of arbitrary length is $0$-hyperbolic.

We refer the reader to [1] and [4] for further examples.

In general, although Gromov hyperbolicity has a simple definition, it is far from simple to verify for many of the metrics that arise in geometric function theory are Gromov hyperbolic. Some specific examples of results are that the quasihyperbolic [1] and the Klein-Hilbert metric [6] are Gromov hyperbolic (under particular conditions on the domain of definition).

To each Gromov hyperbolic space $X$, we associated a boundary at infinity, $\partial X$ (also called the Gromov boundary). Fix a base point $o \in X$. We say that a sequence $\{a_i\}$ of points in $X$ converges to infinity if

$$\lim_{i, j \to \infty} (a_i|a_j)_o = \infty.$$ 

It is easy to see that this definition does not depend on the choice of a base point. We say that two sequences $\{a_i\}$ and $\{b_i\}$ converging to infinity are equivalent and write $\{a_i\} \sim \{b_i\}$ if

$$\lim_{i \to \infty} (a_i|b_i)_o = \infty.$$
Once again, one can show that $\sim$ is an equivalence relation on the sequences converging to infinity and that the definition of the equivalence does not really depend on the choice of a base point $o \in X$.

**Definition 2.** Let $(X, d)$ be a Gromov $\delta$-hyperbolic space. The boundary at infinity $\partial X$ of $X$ is defined to be the equivalence classes of sequences converging to infinity.

The boundary at infinity supports a family of so-called visual metrics. A metric $d$ on $\partial X$ is called a visual metric if there is a $o \in X$, $C \geq 1$ and $\varepsilon > 0$ such that for all $x, y \in \partial X$,

$$\frac{1}{C} \rho_{o, \varepsilon}(x, y) \leq d(x, y) \leq C \rho_{o, \varepsilon}(x, y),$$

where $\rho_{o, \varepsilon}(x, y) = e^{-\varepsilon(x|y)_o}$. Here $(x|y)_o$ is the Gromov product on $\partial X$ defined by

$$(x|y)_o = \inf\{\lim\inf_{i \to \infty} (a_i|b_i)_o : \{a_i\} \in x, \{b_i\} \in y\}$$

and we set $e^{-\infty} = 0$. The boundary at infinity of any Gromov hyperbolic space endowed with a visual metric is bounded and complete.

A Gromov hyperbolic space is called boundary continuous, if the Gromov product extends continuously to the boundary in the following way: if $(x_i), (y_i)$ are sequences in $X$ which converge to points $\overline{x}, \overline{y}$ in $X$ or $\partial X$, then $(x_i|y_i)_o \to (\overline{x}|\overline{y})_o$ for all base points $o \in X$.

**Definition 3.** A homeomorphism $f : (X, d_X) \to (Y, d_Y)$ is called $K$-quasiconformal, $K \geq 1$, if for each $x \in X$, we have

$$\limsup_{r \to 0} \frac{\sup\{d_Y(f(x), f(y)) : d_X(x, y) \leq r\}}{\inf\{d_Y(f(x), f(y)) : d_X(x, y) \geq r\}} \leq K.$$  

**Definition 4.** A metric space $(X, d)$ is called $\text{PT}_0$, if the following inequality holds for all quadruples $x_1, x_2, x_3, x_4 \in X$.

$$d(x_1, x_2)d(x_3, x_4) \leq d(x_1, x_4)d(x_2, x_3) + d(x_1, x_3)d(x_2, x_4).$$

**Definition 5.** A metric space $(X, d)$ is called asymptotically $\text{PT}_{-1}$, if there exists some $\delta > 0$ such that for all quadruples $x_1, x_2, x_3, x_4 \in X$, we have

$$e^{\frac{1}{2}(\rho_{13} + \rho_{24})} \leq e^{\frac{1}{2}(\rho_{12} + \rho_{34})} + e^{\frac{1}{2}(\rho_{14} + \rho_{23})} + \delta e^{\frac{1}{2}\rho},$$

where $\rho_{ij} = d(x_i, x_j)$ and $\rho = \max_{i,j} \rho_{ij}$.

In the paper [5], the authors proved that an asymptotically $\text{PT}_{-1}$ space is a Gromov hyperbolic space and is boundary continuous and $\rho_o(x, y) = e^{-(x|y)_o}, x, y \in \partial X$ is a metric on $\partial X$ which is $\text{PT}_0$. These show that the asymptotically $\text{PT}_{-1}$ space has more better properties than the Gromov hyperbolic space.
3. Theorems and its proofs

Let \((Z, d)\) be an arbitrary metric space and \(M\) be a nonempty closed proper subset of \(Z\). In this section, firstly, using the elementary methods, we give the proof of Theorem 2. Secondly, based on the metric \(u_Z\), we introduce a new metric \(k_Z\) on \(Z \setminus M\) and prove Theorem 3 by a series propositions.

Proof of Theorem 2 Let \(x_1, x_2, x_3, x_4 \in Z \setminus M\), we introduce the following notation for convenience. \(b_{ij} = d(x_i, x_j), d_{ij} = \sqrt{d_M(x_i)d_M(x_j)}, \rho_{ij} = u_Z(x_i, x_j)\) and \(r_{ij} = d_M(x_i) \lor d_M(x_j)\). Let \(d_{ijkl} = d_{ij}d_{kl}\) for all \(i, j, k, l \in \{1, 2, 3, 4\}\).

Since 
\[
u_Z(x_i, x_j) = 2 \log \frac{d(x_i, x_j) + d_M(x_i) \lor d_M(x_j)}{\sqrt{d_M(x_i)d_M(x_j)}},
\]
using the above notation, we obtain
\[
\rho_{ij} = 2 \log \frac{b_{ij} + r_{ij}}{d_{ij}}.
\]
Thus
\[
b_{ij} = d_{ij}e^{\frac{1}{2}\rho_{ij}} - r_{ij}.
\]
Since the metric space \((Z, d)\) is \(PT_0\), thus
(1) \[b_{13}b_{24} \leq b_{12}b_{34} + b_{14}b_{23}\].
That is
(2) \[(d_{13}e^{\frac{1}{2}\rho_{13}} - r_{13})(d_{24}e^{\frac{1}{2}\rho_{24}} - r_{24}) \leq (d_{12}e^{\frac{1}{2}\rho_{12}} - r_{12})(d_{34}e^{\frac{1}{2}\rho_{34}} - r_{34}) + (d_{14}e^{\frac{1}{2}\rho_{14}} - r_{14})(d_{23}e^{\frac{1}{2}\rho_{23}} - r_{23})\].

By computation, we obtain
\[
d_{1234}(e^{\frac{1}{2}(\rho_{12} + \rho_{34})} + e^{\frac{1}{2}(\rho_{14} + \rho_{23})} - e^{\frac{1}{2}(\rho_{13} + \rho_{24})})
\geq \left[r_{34}d_{12}e^{\frac{\rho_{12}}{2}} + r_{12}d_{34}e^{\frac{\rho_{34}}{2}}\right] + \left[r_{23}d_{14}e^{\frac{\rho_{14}}{2}} + r_{14}d_{23}e^{\frac{\rho_{23}}{2}}\right]
- \left[r_{24}d_{13}e^{\frac{\rho_{13}}{2}} + r_{13}d_{24}e^{\frac{\rho_{24}}{2}}\right] - (r_{12}r_{34} + r_{14}r_{23} - r_{13}r_{24})
= \left[r_{34}b_{12} + r_{12}b_{34} + 2r_{34}r_{12}\right] + \left[r_{23}b_{14} + r_{14}b_{23} + 2r_{23}r_{14}\right]
- \left[r_{24}b_{13} + r_{13}b_{24} + 2r_{24}r_{13}\right] - (r_{12}r_{34} + r_{14}r_{23} - r_{13}r_{24})
= \left[r_{34}b_{12} + r_{12}b_{34}\right] + \left[r_{23}b_{14} + r_{14}b_{23}\right] - \left[r_{24}b_{13} + r_{13}b_{24}\right]
+ (r_{12}r_{34} + r_{14}r_{23} - r_{13}r_{24}).
\]
Because \((a \lor b)(c \lor d) = ac \lor ad \lor bc \lor bd\) for all \(a, b, c, d > 0\), we have
\[r_{12}r_{34} + r_{14}r_{23} \geq r_{13}r_{24}.\]
Thus, we have
\[ e^{\frac{1}{2}(\rho_{12} + \rho_{34})} + e^{\frac{1}{2}(\rho_{14} + \rho_{23})} - e^{\frac{1}{2}(\rho_{13} + \rho_{24})} \]
\[ \geq \frac{1}{d_{1234}} [r_{34}b_{12} + r_{12}b_{34}] + \frac{1}{d_{1234}} [r_{23}b_{14} + r_{14}b_{23}] \]
\[ - \frac{1}{d_{1234}} [r_{24}b_{13} + r_{13}b_{24}] \]
\[ = \frac{1}{d_{1234}} [b_{12}(r_{34} + d_{34}) + b_{34}(r_{12} + d_{12})] + \frac{1}{d_{1234}} [b_{14}(r_{23} + d_{23}) + b_{23}(r_{14} + d_{14})] \]
\[ - \frac{1}{d_{1234}} [r_{24}b_{13} + r_{13}b_{24}] \]
\[ \geq \frac{1}{d_{1234}} [(d_M(x_3) + d_M(x_4))b_{12} + (d_M(x_1) + d_M(x_2))b_{34}] \]
\[ + \frac{1}{d_{1234}} [(d_M(x_2) + d_M(x_3))b_{14} + (d_M(x_1) + d_M(x_4))b_{23}] \]
\[ - \frac{1}{d_{1234}} [r_{24}b_{13} + r_{13}b_{24}] \]
\[ \geq \frac{1}{d_{1234}} [(d_M(x_3) + d_M(x_1))b_{24} + (d_M(x_2) + d_M(x_4))b_{13}] - \frac{1}{d_{1234}} [r_{24}b_{13} + r_{13}b_{24}] \]
\[ \geq 0. \]
Thus, we obtain
\[ \frac{b_{ij}}{d_{ij}} \leq e^{\frac{1}{2}u_{max}} \]
for all \( i, j \in \{1, 2, 3, 4\} \) and
\[ e^{\frac{1}{2}(\rho_{12} + \rho_{34})} + e^{\frac{1}{2}(\rho_{14} + \rho_{23})} - e^{\frac{1}{2}(\rho_{13} + \rho_{24})} + 4e^{\frac{1}{2}u_{max}} \]
Since $\epsilon Z \setminus M$ is a nonempty closed proper subset of $Z$, we define the following metric on the set $Z \setminus M$. Given $\epsilon \in (0, \frac{1}{2}]$, we define
\[ k_Z(x, y) = 2\log \frac{d^k(x, y) + d_M(x) \vee d_M(y)}{\sqrt{d_M(x)d_M(y)}} \]
for $x, y \in Z \setminus M$, where $d^k(x, y) = (d(x, y))^{\epsilon}$.

Firstly, we show that $k_Z$ is a metric on the set $Z \setminus M$. We need the following Lemma.

**Lemma 1.** Let $(Z, d)$ be an arbitrary metric space and let $M$ be a nonempty closed proper subset of $Z$, then
\[ \log(1 + \frac{d^k(x, y)}{d_M(x)}) \leq \log(1 + \frac{d^k(x, y)}{d_M(x)})(1 + \frac{d^k(x, y)}{d_M(y)}) \leq k_Z(x, y). \]
for all $x, y \in Z \setminus M$.

**Proof.** The first inequality is obvious. Since
\[ (d_M(x) + d^k(x, y))(d_M(y) + d^k(x, y)) \leq (d^k(x, y) + d_M(x) \vee d_M(y))^2. \]
Dividing both sides by $d_M(x)d_M(y)$, we obtain that
\[ (1 + \frac{d^k(x, y)}{d_M(x)})(1 + \frac{d^k(x, y)}{d_M(y)}) \leq \frac{(d^k(x, y) + d_M(x) \vee d_M(y))^2}{d_M(x)d_M(y)}. \]
Thus,
\[ \log(1 + \frac{d^k(x, y)}{d_M(x)})(1 + \frac{d^k(x, y)}{d_M(y)}) \leq \log \frac{(d^k(x, y) + d_M(x) \vee d_M(y))^2}{d_M(x)d_M(y)} = k_Z(x, y). \]
Thus the second inequality holds. \[
\square
\]

**Proposition 1.** Let $(Z, d)$ be an arbitrary metric space and let $M$ be a nonempty closed proper subset of $Z$, then $(Z \setminus M, k_Z)$ is a metric space.

**Proof.** Firstly, because $d_M(x) \vee d_M(y) \geq \sqrt{d_M(x)d_M(y)}$, so $k_Z(x, y) \geq 0$.

Secondly, it is easy to see that when $x = y$, we have $k_Z(x, y) = 0$. According to Lemma 1, if $k_Z(x, y) = 0$, then $d^k(x, y) = 0$, so $x = y$.

Thirdly, we should proof the triangle inequality. Given $x, y, z \in Z \setminus M$, because $d$ is a metric, so we have $d(x, y) \leq d(x, z) + d(y, z)$. Since $\epsilon \in (0, \frac{1}{2}]$, we obtain that
\[ d^k(x, y) \leq (d(x, z) + d(y, z))^{\epsilon} \leq d^k(x, z) + d^k(y, z), \]
which implies Proposition 3. Let \( (X, d) \) be an arbitrary metric space. Then the proof of theorem 2, we have

\[
[M\{d(x, y) + d_M(x) \vee d_M(y)\}]^2 \leq \left[\frac{[d^*(x, z) + d_M(x) \vee d_M(z)]}{d_M(z)}\right]^2 \cdot \left[\frac{d^*(y, z) + d_M(y) \vee d_M(z)]}{d_M(z)}\right]^2,
\]

which implies \( k_Z(x, y) \leq k_Z(x, z) + k_Z(y, z) \). Thus the triangle inequality holds.

In order to prove Theorem 3, we need the following lemma of Z. Ibragimov in [8].

Lemma 2. [Z. Ibragimov] Let \( r_{ij} \geq 0 \) be real numbers such that \( r_{ij} = r_{ji} \) and \( r_{ij} \leq r_{ik} + r_{kj} \) for all \( i, j, k \in \{1, 2, 3, 4\} \). Then \( (r_1 r_3)^\epsilon \leq (r_2 r_4)^\epsilon \) for each \( \epsilon \in (0, \frac{1}{2}] \).

Obviously, Lemma 2 implies the following result.

Proposition 2. Let \( (X, d) \) be an arbitrary metric space. Then the metric space \( (X, \alpha d) \) is a \( PT_0 \) space for \( \alpha \in (0, \frac{1}{2}] \).

Secondly, we have the following result.

Proposition 3. The metric space \( (Z \setminus M, k_Z) \) is asymptotically \( PT_{-1} \).

Proof. For arbitrary four points \( x_1, x_2, x_3, x_4 \in (Z \setminus M, k_Z) \), we have

\[
e^{-\frac{k_z(x_i, x_j)}{2}} = \frac{d^*(x_i, x_j) + d_M(x_i) \vee d_M(x_j)}{\sqrt{d_M(x_i) d_M(x_j)}}.
\]

Let \( k_Z(x_i, x_j) = k_{ij} \) for \( i, j \in \{1, 2, 3, 4\} \). Using the same notation in the proof of theorem 2, we have

\[
e^{\frac{k_{ij}}{2}} = \frac{b_{ij} + r_{ij}}{d_{ij}},
\]

thus

\[
b_{ij} = d_{ij} e^{\frac{k_{ij}}{2}} - r_{ij},
\]
where \( b_{ij} = d(x_i, x_j) \). Because \( d \) is a metric on \( Z \), thus by Proposition 2, we have

\[
 b^\varepsilon_{13} b^\varepsilon_{24} \leq b^\varepsilon_{12} b^\varepsilon_{34} + b^\varepsilon_{14} b^\varepsilon_{23}.
\]

Taking the same argument in the proof of Theorem 2, we can obtain the result.

\[ \square \]

**Proposition 4.** If the metric space \((Z, d)\) is complete, then so is \((Z \setminus M, k_Z)\).

**Proof.** Firstly, notice that

\[
|d_M(x) - d_M(y)| \leq d(x, y)
\]

for all \( x, y \in Z \). Then, fixed \( x \in Z \setminus M \), we have

\[
k_Z(x, y) \leq \log \frac{[d^\varepsilon(x, y) + d(x, y) + d_M(x)]^2}{d_M(x)[d_M(x) - d(x, y)]}
\]

and

\[
|\log \frac{d_M(x)}{d_M(y)}| = 2 \log \frac{d_M(x) \vee d_M(y)}{\sqrt{d_M(x)d_M(y)}} \leq k_Z(x, y)
\]

for all \( y \in Z \setminus M \) with \( d(x, y) < d_M(x) \).

According to Lemma 1, we have

\[
\log(1 + \frac{d^\varepsilon(x, y)}{d_M(x)}) \leq \log(1 + \frac{d^\varepsilon(x, y)}{d_M(x)})(1 + \frac{d^\varepsilon(x, y)}{d_M(y)}) \leq k_Z(x, y).
\]

From the above two inequalities (9) and (11), we can find that \( k_Z(x, y) \to 0 \) if and only if \( d(x, y) \to 0 \), which show that the identity map between \((Z \setminus M, k_Z)\) and \((Z \setminus M, d)\) is a homeomorphism.

Let \( \{x_n\} \) be a Cauchy sequence in the metric space \((Z \setminus M, k_Z)\). Thus for any \( \varepsilon > 0 \), there is a \( N \in \mathbb{N} \) such that \( k_Z(x_n, x_m) < \varepsilon \) when \( n, m \geq N + 1 \). Fix \( m = N + 2 \). From (10) we get

\[
d_M(x_n) \leq \varepsilon^2 d_M(x_{N+2}).
\]

From (11), we get

\[
d_M(x_n) \geq (\varepsilon - 1)^{-1} d^\varepsilon(x_n, x_{N+2}).
\]

If \( \inf_n d^\varepsilon(x_n, x_{N+2}) = 0 \), we can take \( m = N + k \) for some \( k > 1 \) with \( x_{N+2} \neq x_{N+k} \) and \( \inf_n d^\varepsilon(x_n, x_{N+k}) > 0 \). Hence, we can assume that \( \inf_n d^\varepsilon(x_n, x_{N+2}) > 0 \) and obtain

\[
d_M(x_n) \geq (\varepsilon - 1)^{-1} d^\varepsilon(x_n, x_{N+2}) \geq \inf_n (\varepsilon - 1)^{-1} d^\varepsilon(x_n, x_{N+2}) > 0.
\]

Thus, we obtain

\[
0 < t = \inf_n d_M(x_n) \leq \sup_n d_M(x_n) = T < \infty
\]
and
\[ d^e(x_n, x_m) \leq d_M(x_n)(e^{k_Z(x_n, x_m)} - 1) \leq T(e^{k_Z(x_n, x_m)} - 1). \]

This shows that \( \{x_n\} \) is also a Cauchy sequence in \((Z, d)\). Since \((Z, d)\) is complete, thus \( \{x_n\} \) converges to some point \( x \in Z \) in metric space \((Z, d)\). Since \( d_M(x_n) \geq t > 0 \), from (8), we can see that \( d_M(x) > 0 \), which implies \( x \in Z \setminus M \). Since the identity map is a homeomorphism, so \( k_Z(x_n, x) \to 0 \) as \( n \to \infty \), thus \((Z \setminus M, k_Z)\) is complete. \( \square \)

**Proposition 5.** The identity map between \((Z \setminus M, k_Z)\) and \((Z \setminus M, d)\) is 2-quasiconformal.

**Proof.** Fix \( x \in Z \setminus M \) and let \( r < d_M(x) \). According to Lemma \[1\] we obtain
\[ \inf_{d(x,y) \geq r} k_Z(x,y) \geq \inf_{d(x,y) \geq r} \log \frac{d^e(x,y) + d_M(x)}{d_M(x)} \geq \log \frac{r^e + d_M(x)}{d_M(x)}. \]

Similarly, according to inequality \[9\], we obtain
\[ \sup_{d(x,y) \leq r} k_Z(x,y) \leq \sup_{d(x,y) \leq r} \log \frac{d^e(x,y) + d_M(x) + d_M(x)}{d_M(x)[d_M(x) - d(x,y)]} \leq \log \frac{[r^e + r + d_M(x)]^2}{d_M(x)[d_M(x) - r]^2}. \]

Thus, we obtain
\[ \limsup_{r \to 0} \frac{\sup \{k_Z(x,y) : d(x,y) \leq r\}}{\inf \{k_Z(x,y) : d(x,y) \geq r\}} \leq \limsup_{r \to 0} \frac{\log \frac{[r^e + r + d_M(x)]^2}{d_M(x)[d_M(x) - r]^2}}{\log \frac{r^e + d_M(x)}{d_M(x)}} = 2. \]

Thus the identity map is 2-quasiconformal. \( \square \)

**Proof of Theorem 3** According to Propositions \[1\][3][4][5] we obtain Theorem 3.

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