Some Remarks on the Braided Thompson Group $BV$

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Abstract

M. Brin and P. Dehornoy independently discovered a braided version $BV$ of R. Thompson’s group $V$. In this paper, we discuss some properties of $BV$ that might make the group interesting for group based cryptography. In particular, we show that $BV$ does not admit a non-trivial linear representation.

1 Introduction

One of the ways to visualize elements of R. Thompson’s group $F$ is to regard them as pairs of trees [9]. The trees forming such a pair, called the top tree and the bottom tree, are finite binary trees with the same number of leaves. We follow [3] in drawing the top tree with the root at the top and the bottom tree with the root at its bottom aligning their leaves to match. An element of Thompson’s group $V$ can be understood in a similar way: we still have a pair of trees, but now we wedge a permutation in between that decides which leaves are considered matching.

The braided version $BV$ of Thompson’s group $V$ was introduced independently by Brin in [6], [7] and Dehornoy in [11] and has been investigated further by several authors [5], [8]. Informally speaking, one obtains an element of the braided Thompson’s group $BV$ by using a braid instead of a permutation to connect the leaves of the top tree to the leaves of the bottom tree. In Section 3, we discuss complexity issues of computations in $BV$. In
particular, we show that multiplication of two elements of $BV$ given in tree-braid-tree form can be carried out in quadratic time on the input length. In Section 4, we analyze Brin’s presentation of $BV$ to prove the following:

**Theorem.** The group $BV$ does not admit non-trivial linear representations in any characteristic.

We note that relatives of $BV$, namely braid groups and Thompson’s group $F$, received some attention recently from a cryptographic point of view. Section 5 reflects on the possibility of using the group $BV$ as a platform group in cryptographic protocols.

## 2 The Group $BV$ and its Braided Band Diagrams

Recall that elements of Thompson’s group $F$ can be represented by band diagrams. A band diagram encodes splitting and merging of a band keeping track of the relative order of splits and merges. Pictorially, one can think of band diagrams as thickened tree diagrams. The following picture shows band diagrams for the canonical generators $x_0$ and $x_1$:

![Band Diagrams](image)

Two band diagrams are equivalent if one can pass from one to the other by means of a finite sequence of moves, where each move applies (forward or
backward) one of the following relations:

\[ \begin{align*}
\begin{array}{c}
D \\
K
\end{array} & \rightarrow \\
\begin{array}{c}
I \\
II
\end{array}
\end{align*} \quad \text{(first move: eye removal)}
\]

A band diagram is called **reduced** if neither of the above relations can be applied forward. It is well known that every band diagram can be reduced by a finite sequence of forward applications of the relations and that every equivalence class of band diagrams has a unique reduced representative.

Elements of Thompson’s group $F$ correspond to equivalence classes of band diagrams. Multiplication of elements of $F$ translates into stacking band diagrams.

Allowing bands to braid, one arrives at the notion of braided band diagrams. Those represent elements of the group $BV$. Note that bands are allowed to braid, but they are not allowed to twist, i.e., a twisted band segment like

\[ \begin{array}{c}
\text{\large$\downarrow$}
\end{array} \]

is not allowed in a braided band diagram.

Also note that we do not distinguish diagrams that just differ in the way the braiding is drawn (i.e., the diagrams themselves are supposed to live in 3-space and are regarded equal if they differ by an ambient homotopy not twisting bands). E.g., the following two pictures describe the same diagram:
Again, two diagrams are equivalent if there is a finite sequence of moves transforming one into the other; and we call a diagram reduced if it does not allow for a forward application of a relation.

M. Brin [7, Theorem 2] has shown that BV is generated by the following elements:

\[ \nu_0, \nu_1, \overline{\pi}_0, \overline{\pi}_1 \]

**Proposition 1.** Every equivalence class of braided band diagrams contains a unique reduced representative, and this representative can be obtained from any diagram in the equivalence class via a finite sequence of forward moves.

**Proof.** Let \( \Delta \) and \( \Theta \) be two braided band diagrams. We write \( \Delta \rightarrow \Theta \) if there is a forward move from \( \Delta \) to \( \Theta \). Since forward moves decrease the number of band-segments in a diagram, it follows that “\( \rightarrow \)” is a noetherian relation, i.e., there are no infinite \( \rightarrow \)-chains.

By Newman’s Lemma (a standard result on rewriting systems; see, e.g., [2, Corollary 4.76]), it suffices to show that the \( \rightarrow \)-relation is locally confluent, i.e., given a diagram \( \Delta \) and two forward moves \( \Delta \rightarrow \Theta_1 \) and \( \Delta \rightarrow \Theta_2 \), there exists a diagram \( \Lambda \) that can be obtained by forward move sequences from both \( \Theta_1 \) and \( \Theta_2 \).

The local confluence condition, however, is easily verified in our setting:

1. Any two forward moves removing eyes (joints) can be performed in any order since the two eyes (joints) do not interfere with each other.

2. Given two forward moves of different type, either they can be performed in any order, or they lead to equal diagrams (possibly after a suitable ambient homotopy). The latter happens when an eye meets a joint (removing either of them yields a tripod).
In the following example, we either delete the top-eye or the following joint and obtain identical diagrams (i.e., diagrams that are equal after a suitable ambient homotopy):

\[ \rightarrow \]

Remark 2. Note that, as a corollary, we recover the result of M. Brin [7, Lemma 4.3] that $BV$ contains a copy of $F$ realized as the set of reduced diagrams that do not exhibit braiding.

3 Complexity of the Word Problem

We want to devise an efficient method for computing products in $BV$. To do so, we have to establish a canonical method of representing elements of $BV$ in a way suitable for computations. Braided band diagrams will serve as our starting point.

Let us call a diagram semi-reduced if it does not admit joint-removal moves. Obviously, every reduced diagram is semi-reduced. Moreover, every semi-reduced diagram can be transformed into a reduced diagram via a (finite) sequence of eye-removal moves.

Observation 3. A diagram $\Delta$ is semi-reduced if and only if, along each route from top to bottom in $\Delta$, we never find a merge of bands followed by a split of the band.

Observation 4. Consider a semi-reduced braided band diagram $\Delta$. We can isotop the diagram so that all the splits precede any braiding and all the
merges occur after all the braiding is done:

Thus, a semi-reduced diagram always decomposes into three layers: the top-part that is a root-at-the-top tree where all the splits of the band occur; the middle part consisting of a braid of bands; the bottom part which is a root-at-the-bottom tree where the bands are merged back into a single ribbon.

Consequently, every element of $BV$ can be represented by a triple $(T^{\text{top}}, \beta, T^{\text{bot}})$, consisting of two planar trees $T^{\text{top}}$ and $T^{\text{bot}}$ and a braid $\beta$ interpolating between the leaves of the trees.

$q.e.d.$

Observation 5. Conversely, given a triple $(T^{\text{top}}, \beta, T^{\text{bot}})$ as above, we can form a braided band diagram by stacking the top tree on the top of the braid and appending an upside-down drawing of the bottom tree. Within such a diagram, along each ribbon we find no merge followed by a split, i.e., the diagram is semi-reduced.

$q.e.d.$

We shall now discuss how to detect removable eyes. Let the triple $(T^{\text{top}}, \beta, T^{\text{bot}})$ represent a semi-reduced diagram. Assuming that the braid $\beta$ is an element of the braid group $B_n$, where $n$ is the number of leaves of either tree, let

$$\pi_i : B_n \rightarrow B_{n-1}$$

be the map defined by deleting the $i$th strand (strands are indexed at the top of the braid); and let

$$\iota_i : B_{n-1} \rightarrow B_n$$

be the map defined by doubling the $i$th strand (i.e., splitting that strand into two all the way from the top to the bottom of the braid).

Observation 6. Let $\beta \in B_n$ be a braid. The $i$th and $(i+1)$st strands are parallel, i.e., can be united into a single strand without otherwise disrupting the braid $\beta$, if and only if $\iota_i(\pi_i(\beta)) = \beta$.

$q.e.d.$
Observation 7. A semi-reduced diagram represented as a triple \((T^{\text{top}}, \beta, T^{\text{bot}})\) can be further reduced if and only if there is a pair of parallel strands in \(\beta\) that connects a terminal caret in \(T^{\text{top}}\) to a terminal caret in \(T^{\text{bot}}\). Here, a terminal caret in \(T^{\text{top}}\) is a split of a band such that along both resulting bands there are no further splits. Symmetrically, a terminal caret in \(T^{\text{bot}}\) is a merge of two bands both of which had not previously been involved in merges. \(\text{q.e.d.}\)

We can use this to reduce diagrams algorithmically.

Algorithm 8. A triple \((T_{1}^{\text{top}}, \beta_{1}, T_{1}^{\text{bot}})\) can be reduced by applying a sequence of eye-removal moves according to Observation 7. The process can be organized as follows:

1. Find the left-most terminal caret of the top tree.
2. Check whether the strands issuing from this caret are parallel. If so, check whether they lead to a terminal caret in the bottom tree. If so, remove the eye and check if there is a terminal caret in the current position (in the top tree). Repeat this step, if there is one.
3. Move to the right and repeat the previous step on the next terminal caret in the top tree.
4. Repeat until all terminal carets of the top tree have been visited.

In this algorithm, we can proceed from the left to the right since an eye-removal cannot create terminal carets in the top tree to the left of the caret that is being removed.

Checking whether two triples represent the same group element in \(BV\) can be performed according to the following:

Algorithm 9. Given two triples \((T_{1}^{\text{top}}, \beta_{1}, T_{1}^{\text{bot}})\) and \((T_{2}^{\text{top}}, \beta_{2}, T_{2}^{\text{bot}})\), perform a sequence of eye-removal moves on either of them until both cannot be further reduced. The triples thus obtained represent the same group element if and only if they have the same top and bottom trees and the braids are equal as elements of the corresponding braid group.

Multiplication also has a natural interpretation in terms of diagrams:
Observation 10. If two elements \( g_1 \) and \( g_2 \) are represented by triples \((T_{1\text{top}}, \beta_1, T_{1\text{bot}})\) and \((T_{2\text{top}}, \beta_2, T_{2\text{bot}})\) where \( T_{2\text{top}} = T_{1\text{bot}} \), then the triple \((T_{1\text{top}}, \beta_1 \beta_2, T_{2\text{bot}})\) represents the product \( g_1 g_2 \). \(\text{q.e.d.}\)

Consequently, multiplication in \( BV \) can be carried out using the following:

Algorithm 11. Given two elements \( g_1, g_2 \in BV \), represented by semi-reduced triples \((T_{1\text{top}}, \beta_1, T_{1\text{bot}})\) and \((T_{2\text{top}}, \beta_2, T_{2\text{bot}})\), compute a semi-reduced triple for the product \( g_1 g_2 \) as follows: first unreduce both factors so that the bottom tree of the left-hand factor matches the top tree of the right-hand factor; then form a triple for the product using Observation 10. Note that the resulting triple is automatically semi-reduced.

So far, we have ignored complexity issues and we have taken operations on braids and trees for granted. Since braid operations dominate the time complexity of all algorithms, we will not discuss the complexity of operations on trees.

To meaningfully discuss the time complexity of the algorithms above, we need to settle on a representation of the braid component of a triple. The braid is an element of the braid group \( B_n \) where the number \( n \) of strands is determined by the tree components of the triple. A natural way to represent elements of \( B_n \) is as words over some fixed generating set. We will be using the set of non-repeating braids (also called the Garside generators). For this set of generators, W. Thurston has given a solution to the word problem in braid groups \([13, \text{Chapter 9}]\).

Recall that a braid \( \beta \in B_n \) is positive if it can be drawn so that all crossings are overcrossings (the down-right strand goes over the down-left strand). A positive braid is called non-repeating if any pair of strands crosses at most once. By \([13, \text{Lemma 9.1.10}]\), non-repeating braids of \( B_n \) are uniquely determined by the permutation they induce; and for each permutation, there is a non-repeating braid. Thus, non-repeating braids form a generating set for \( B_n \) whose elements can be represented by permutations on \( n \) letters.

The following observation makes the set of non-repeating braids convenient for our purposes:

Observation 12. Neither doubling a strand nor deleting a strand creates undercrossings out of nowhere. Also, both operations do not increase the number of crossings of any given pair of strands. Thus, if \( \beta \) is a non-repeating braid, then so are \( \pi_i(\beta) \) and \( \iota_i(\beta) \) for any \( i \).
It follows that the operations of deleting and doubling strands do not increase the word length with respect to the generating set of non-repeating braids.

We also note that non-repeating braids can be manipulated efficiently: the operations of doubling a strand or deleting a strand in a generator are linear in the length of the input and, therefore, take time $O(n \log(n))$ in the case of a non-repeating braid of $B_n$.

For the generating set of non-repeating braids, Thurston defines the right-greedy and the left-greedy normal forms, which are unique and can be efficiently computed:

**Lemma 13 ([13, Corollary 9.5.3]).** Let a braid $\beta \in B_n$ be a word of length $h$ with respect to the generating set of non-repeating braids. Then $\beta$ can be put in either normal form in time $O(h^2 n \log(n))$.

q.e.d.

For computations in $BV$, we use the right-greedy normal form.

**Definition 14.** The normal form of an element of $BV$ is a triple $(T_{\text{top}}, w, T_{\text{bot}})$, where $w$ is a word over the generating set of non-repeating braids in right-greedy normal form so that the diagram represented by the triple is reduced. (Of course, the way such a triple represents a diagram is by regarding the word as representing a braid.)

**Proposition 15.** Any triple $(T_{\text{top}}, w, T_{\text{bot}})$, where the trees have $n$ leaves and $w$ is of length $h$, can be put into normal form in time $O(h^2 n^2 \log(n))$.

**Proof.** Since a tree with $n$ leaves has at most $n$ carets, Algorithm 8 requires at most $n$ unsuccessful checks for eyes and at most $n$ successful checks. Each check can be carried out with complexity $O(h^2 n \log(n))$. Removing an eye that has been found is done by computing $\pi_i(w)$ for the corresponding $i$. This is done for each generator in the expression of $w$; and thus, it is linear in $h$. Thus, we can eliminate a single eye in $O(h n \log(n))$ time.

Eliminating an eye decreases the number of strands of the braid and therefore has to be done at most $n$ times. Note that during this process, the word length of the braid part in the triple does not increase by Observation 12.

Once the diagram is reduced, the braid part is put into right-greedy normal form in time $O(h^2 n \log(n))$. q.e.d.
Proposition 16. Let \((T_{1_{\text{top}}}, w_1, T_{1_{\text{bot}}})\) and \((T_{2_{\text{top}}}, w_2, T_{2_{\text{bot}}})\) be two triples in normal form representing the elements \(g_1\) and \(g_2\), respectively. Let \(n_1\) and \(n_2\) be their numbers of strands and let \(h_1\) and \(h_2\) be the word lengths of \(w_1\) and \(w_2\), respectively.

The normal form triple representing the product \(g_1g_2\) can be computed in time \(O \left( (h_1 + h_2)^2(n_1 + n_2)^2 \log(n_1 + n_2) \right)\).

Proof. Using Algorithm 11, we have to control how the number of strands and the word length of the braid grow in the unreducing step. For either factor, the number of strands grows at most to \(n_1 + n_2\) since \(T_{1_{\text{top}}}\) has at most \(n_2\) carets that need to be cloned in \(T_{1_{\text{bot}}}\) and \(T_{1_{\text{bot}}}\) has at most \(n_1\) carets that we might need to recreate in \(T_{2_{\text{top}}}\). Hence, we have to double at most \(n_2\) strands in \(w_1\), which can be done in time \(O \left( n_2h_1(n_1 + n_2) \log(n_1 + n_2) \right)\); and we have do double at most \(n_1\) strands in \(w_2\), which can be done in time \(O \left( n_1h_2(n_1 + n_2) \log(n_1 + n_2) \right)\). The total time for unreducing the diagrams is therefore \(O \left( (h_1 + h_2)(n_1 + n_2)^2 \log(n_1 + n_2) \right)\).

By Observation 12, unreducing does not increase the word length of the braids. Thus, it follows from Proposition 15 that we can reduce the triple that we obtain for the product \(g_1g_2\) to normal form in time \(O \left( (h_1 + h_2)^2(n_1 + n_2)^2 \log(n_1 + n_2) \right)\), which dominates all other bounds.

q.e.d.

Remark 17. On can save some computational effort by not putting all braids into normal form. Dropping the normalization steps from the algorithms above yields the following complexity bounds:

1. Any triple \((T_{\text{top}}, w, T_{\text{bot}})\), where the trees have \(n\) leaves and \(w\) has length \(h\), can be reduced in time \(O \left( hn^2 \log(n) \right)\).

2. Let \((T_{1_{\text{top}}}, w_1, T_{1_{\text{bot}}})\) and \((T_{2_{\text{top}}}, w_2, T_{2_{\text{bot}}})\) be two semi-reduced triples representing the elements \(g_1\) and \(g_2\), respectively. For \(i \in \{1, 2\}\), let \(n_i\) be the number of strands in \(w_i\) and let \(h_i\) be the word length of \(w_i\). A semi-reduced triple representing the product \(g_1g_2\) can be computed in time \(O \left( (h_1 + h_2)(n_1 + n_2) \log(n_1 + n_2) \right)\). This triple has trees with at most \(n_1 + n_2\) leaves and a braid that is represented as a word of length \(h_1 + h_2\).

From a practical point of view, it therefore pays off to put elements into normal form only when one needs to test for equality.
Remark 18. We note that the bit-length needed to encode a triple with \( n \) strands and the braid given as a word of length \( h \) is about \( hn \log(n) \). Thus, multiplication of elements in \( BV \) is actually quadratic in terms of total length of inputs, i.e., multiplication in \( BV \) is about as efficient as the elementary school algorithm for multiplying multi-digit integers.

4 Linear Representations

Lemma 19 ([6, Corollary 4.14]). The group \( BV \) is generated by three families of generators \( \nu_n, \pi_n, \) and \( \bar{\pi}_n \) (where \( n \geq 0 \)) subject to the following relations:

\[
\begin{align*}
\nu_q \nu_m &= \nu_m \nu_{q+1} & m < q \\
\pi_{m+1} \nu_m &= \nu_m \pi_{m+1} & m \geq 0, \varepsilon = \pm 1 \\
\pi_q \nu_m &= \nu_m \pi_q & m > q + 1 \\
\pi_q \nu_{q+1} &= \nu_{q+1} \pi_{q} & m < q \\
\pi_q \pi_{m+1} &= \pi_{m+1} \nu_{m+1} & m \geq 0 \\
\pi_q \pi_m &= \pi_{q} \pi_{m} & |m - q| \geq 2 \\
\pi_m \pi_{m+1} \pi_m &= \pi_{m+1} \pi_m \pi_{m+1} & m \geq 0 \\
\bar{\pi}_q \pi_m &= \pi_{q+1} \pi_{m} & q \geq m + 2 \\
\pi_m \bar{\pi}_{m+1} \pi_m &= \bar{\pi}_{m+1} \pi_m \bar{\pi}_{m+1} & m \geq 0 \\
\pi_n &= \bar{\pi}_n \nu_{n+1} \bar{\pi}_{n+1} & n \geq 0
\end{align*}
\]

Moreover,

1. The family \( \{\nu_n | n \geq 0\} \) generates a copy of \( F \) inside \( BV \).

2. Imposing the additional relations

\[
\bar{\pi}_n^2 = \pi_n^2 = 1, \quad n \geq 0
\]

turns the above into a presentation for \( V \).

In particular, \( V \) is a quotient of \( BV \). Thus, \( BV \) is not simple. We shall show, however, that it is not too far from being simple: the normal closure of \([F, F]\) (regarded as a subgroup of \( BV \)) is all of \( BV \):

Lemma 20. Consider \( F \) as a subgroup of \( BV \), generated by \( \{\nu_n | n \geq 0\} \). Then, \( BV \) does not have a proper normal subgroup containing \([F, F]\).
Proof. We first note that for $i \geq 1$,
\[ \nu_i \nu_i^{-1} = \nu_0 \nu_i \nu_0^{-1} \nu_i^{-1} = [\nu_0, \nu_i] \]
and
\[ \nu_i \nu_i^{-1} = \nu_0 \nu_i \nu_0^{-1} \nu_i^{-1} = [\nu_i, \nu_0] \]
are commutators. Telescoping products of such commutators shows that $\nu_i \nu_i^{-1} \in [F, F]$ for $i, j \geq 1$.

Let $N$ be the normal closure of $[F, F]$ in $BV$. For all $i \geq 1$,
\[ N \ni \pi_i \nu_i \nu_i^{-1} \pi_i \pi_i^{-1} \pi_i \pi_i^{-1} \nu_i^2. \]

Hence, $\pi_i \pi_i^{-1} \nu_i \nu_i^{-1} \pi_i \pi_i^{-1} \pi_i \pi_i^{-1} \nu_i^2 \in N$, and therefore $\pi_i \pi_i^{-1} \nu_i \nu_i^{-1} \pi_i \pi_i^{-1} \nu_i^2 \in N$. Thus, $\pi_i \pi_i^{-1} \nu_i \nu_i^{-1} \pi_i \pi_i^{-1} \nu_i^2 \in N$ for each $i \geq 1$.

Now, we show that all generators of $BV$ die in the quotient $BV/N$. We already know this for $\pi_i$ with $i \geq 2$. Using the braid relations between $\pi_1$ and $\pi_2$, we find that $\pi_1$ dies as well, and then, in view of the braid relation between $\pi_0$ and $\pi_1$, we find that $\pi_0$ dies as well.

The family of mixed braid relations (between $\pi_i$ and $\pi_i^{-1}$) now implies that $\pi_i = 1$ in $BV/N$ for $i \geq 1$. Now the relations $\pi_0 = \pi_0 \nu_0 \pi_0^{-1}$ and $\pi_0 = \pi_0 \nu_0 \pi_0^{-1}$ imply $\pi_0 = 1$ in $BV/N$.

Thus, the squares of all $\pi_i$ and all $\pi_i$ die in $BV/N$, whence $BV/N$ is a quotient of $V$. However, already too many generators are gone. So $BV/N$ is a proper quotient of $V$, and therefore trivial.

q.e.d.

Observation 21. Any linear representation of a simple group is either faithful or trivial.

q.e.d.

Corollary 22. Neither the commutator subgroup $[F, F]$ in Thompson’s group $F$ nor Thompson’s group $V$ do admit a non-trivial linear representation (in any characteristic).

Proof. First note that $F$ is not linear in any characteristic: it is finitely generated and not solvable. It it was linear, it would contain a non-abelian free subgroup by the Tits Alternative. But $F$ does not contain non-abelian free subgroups.

The commutator subgroup $[F, F]$ is also not linear in any characteristic since it contains a copy of $F$ as a subgroup. The claim for $[F, F]$ nor follows since $[F, F]$ is simple.

The same argument applies to Thompson’s group $V$, which is simple and also contains a copy of $F$.

q.e.d.
The main theorem now follows immediately:

**Theorem 23.** The group $BV$ does not admit non-trivial linear representations in any characteristic.

**Proof.** The subgroup $[F, F]$ lies within the kernel of any linear representation of $BV$. However, such a kernel is a normal subgroup and therefore exhausts $BV$ by Lemma 20.

q.e.d.

## 5 On the Cryptographic Use of $BV$

After the paper by Anshel, Anshel, and Goldfeld [1], group based cryptography got a huge boost and is rapidly developing since. An idea behind using groups in cryptography is that finding solutions of certain equations or systems of equations over a given group is computationally infeasible while generating equations with known or given solutions might be efficient since it only involves multiplication and computing normal forms.

We recall the key-exchange protocol proposed by Anshel, Anshel, and Goldfeld. Below, $m, k, n,$ and $l$ are integer parameters and $G$ is a group, called the platform group of the protocol. A key-exchange has the goal that Alice and Bob collaboratively create a secret that is shared between them. In this particular protocol, the shared secret will be an element of $G$. It is selected as follows:

1. Alice chooses randomly a public set $\{a_1, \ldots, a_m\} \subset G$ and a private key $a = a_{i_1}^{a_{i_1}} \cdots a_{i_k}^{a_{i_k}} \in \langle a_1, \ldots, a_m \rangle \subseteq G$.

2. Bob chooses randomly a public set $\{b_1, \ldots, b_n\} \subset G$ and a private key $b = b_{j_1}^{b_{j_1}} \cdots b_{j_l}^{b_{j_l}} \in \langle b_1, \ldots, b_n \rangle \subseteq G$.

3. Alice sends to Bob the $n$-tuple $\{ab_1a^{-1}, \ldots, ab_na^{-1}\}$.

4. Bob sends to Alice the $m$-tuple $\{ba_1b^{-1}, \ldots, ba_mb^{-1}\}$.

5. The shared secret is the commutator $[a, b] = a^{-1}b^{-1}ab$, which both of them can compute.
The security of this key-exchange protocol depends on how hard it is to solve the Simultaneous Conjugacy Search Problem in $G$: given elements $u_1, \ldots, u_t$ and $v_1, \ldots, v_t$ in $G$, find an element $c \in G$ such that $u_i = c^{-1}v_ic$ provided it is known that such a conjugating element exists.

Certain criteria on the choice of the platform group for a cryptosystem were given by Shpilrain [25]. We note that $BV$ satisfies those criteria. In Section 3, we have shown that computations in $BV$ can be performed in polynomial time and that the word problem can also be solved in polynomial time. The group $BV$ has a presentation with many short relations [7]. According to [25], this might make it harder to mount length based attacks on $BV$ (more on this below). Finally, both braid groups and Thompson’s groups $V$ and $F$ are widely known, which makes the braided version $BV$ “marketable”.

Both, braid groups and Thompson’s group $F$ were investigated in the context of cryptography, see [10], [20], [26] and references therein. In the remainder of this section, we shall compare $BV$ to $F$ and the braid groups from a cryptographic point of view.

The simultaneous conjugacy problem in $F$ was solved by Kassabov and Matucci [18] using the interpretation of elements of $F$ as piecewise linear functions. Such interpretation is not available for $BV$.

The conjugacy search problem seems to be harder for $BV$ than for braid groups. Efficient algorithms for solving the conjugacy problem in braid groups are based on associating a finite set (called summit set [15], super summit set, and ultra summit set [12], [16]) of conjugates to any braid $\beta \in B_n$. One should note that finiteness of the summit sets relies on the number of strands $n$ being fixed. Braids extracted from elements in $BV$ can have an arbitrary number of strands, which makes it impossible to directly transfer to $BV$ strategies successful for braid groups.

There are also known attacks on braid-group based crypto-systems using linear representations. Braid groups are known to be linear ([4], [19]), but more importantly, the Burau and colored Burau representations have small kernels and can be exploited. According to Theorem 23, such attacks on $BV$ will not work.

A very general approach, now known as length based attack, was described in [17] and further developed in [14]. It relies on the existence of a good length function on the platform group, and can be used to solve arbitrary systems of equations over the group. The main idea is to use the length function to turn the system of equations into a problem in combina-
torial optimization. We refer to [14], [24], and [22] for descriptions of length based attacks for the conjugacy search problem in braid groups and Thompson’s group $F$. Length based attacks are most successful if randomly chosen subgroups of the platform group are generically free (see [23] for a detailed analysis). This is the case for braid groups [21]. Both groups, $V$ and $BV$ are known to have free subgroups. It is not known whether random subgroups of $V$ and $BV$ are generically free. Thus, answers to the following questions will have an impact on the usability of $BV$ for cryptography:

**Question 24.** What are generic subgroups of $V$ and $BV$?

**Question 25.** Does $BV$ have a quotient with generically free subgroups?

### References

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