Quantum Caustics for Systems with Quadratic Lagrangians

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Abstract. We study caustics in classical and quantum mechanics for systems with quadratic Lagrangians of the form $L = \frac{1}{2} \dot{x}^2 - \frac{1}{2} \lambda(t)x^2 - \mu(t)x$. We derive a closed form of the transition amplitude on caustics and discuss their physical implications in the Gaussian slit (gedanken-)experiment. Application to the quantum mechanical rotor casts doubt on the validity of Jevicki’s correspondence hypothesis which states that in quantum mechanics, stationary points (instantons) arise as simple poles.

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1. Introduction

Semiclassical approximation is a powerful and perhaps the most commonly used approach to quantum mechanics for exploiting classical mechanics based on the idea that the former may be realized by supplementing the latter properly (see, e.g., [1]). In the path-integral language, it asserts that the transition amplitude between two arbitrarily given points may be approximated by summing up fluctuations around the classical path connecting the two points. Here, the existence of the classical path is assumed, not guaranteed, in the first place.

Caustics occur when this assumption breaks down. More precisely, when a family of classical trajectories focuses, the envelope of the trajectories forms a focal region called caustics. In one dimension, the region becomes a focal point, and hence a classical path connecting generic initial and final points \(a, b \in \mathbb{R}\) exists if and only if \(b\) is the focal point specified by \(a\). Given the action \(I\) of the system, this happens when the second variation of the action \(\delta^2 I\) along a classical path starting from \(a\) vanishes, and as such their analysis is purely classical and constitutes a branch of the Sturm-Liouville problem. In particular, if we confine ourselves to the action \(I[x] = \int_0^T dt L\) for a finite time interval \([0, T]\) with the quadratic Lagrangian\(^3\)

\[
L = \frac{1}{2} \dot{x}^2 - \frac{1}{2} \lambda(t) x^2 - \mu(t) x ,
\]

then \(\delta^2 I = 0\) is equivalent to the condition that a solution \(u\) obeying the homogeneous equation of motion \(\ddot{u}(t) + \lambda(t) u(t) = 0\) with \(u(0) = 0\) vanish at \(t = T\).

In quantum mechanics, the semiclassical approximation for the quadratic system is known to be exact, and in terms of the Morse index \(m(\lambda)\) and the action \(I[x_{cl}]\) evaluated for a classical path \(x_{cl}(t)\), the amplitude for the transition between the two points \(a, b\) reads [2, 3]

\[
K(b, T; a, 0) = \left( \frac{1}{2\pi |u(T)|} \right)^{\frac{1}{2}} e^{iI[x_{cl}]-\frac{i\pi}{4}m(\lambda)}.
\]

Thus the amplitude becomes singular when the harmonic potential \(\lambda\) admits the solution \(u\) to become \(u(T) = 0\), that is, when caustics occur. The appearance of the singularity suggests that, on caustics, a generalized prescription is required for semiclassical approximation to treat the cases where the classical path does not exist. Such a prescription has been devised in the path-integral framework [4, 5] (cf. section 3.1 of this paper). Strictly

\(^3\) Note that any Lagrangian at most quadratic in position and velocity can be brought into this form by partial integration. In this paper we use the dot to denote time derivative \(\dot{x} = dx/dt\), and put both the mass \(m\) of the particle and the Planck constant \(\hbar\) (except section 3.2) unity for convenience.
speaking, the singularity does not arise in actual physical processes, because there usually exist higher order terms (cubic, quartic etc. in \( x \)) in the Lagrangian, or even if the quadratic form (1.1) provides a reasonably accurate description of the system, the potential \( \lambda \) may not allow the solution \( u \) to vanish at \( t = T \), which is generically the case. Nonetheless, physical phenomena pertaining to caustics do arise, and the quantum analysis for such phenomena occurring in the presence of higher order terms has been carried out in the path-integral by Schulman [4] and DeWitt-Morette [6] (see also [5, 7, 8, 9]; for physical aspects of caustic phenomena in wave theory and the relation to Thom’s theorem, see, e.g., [10]).

In this paper, we wish to provide a study of quantum caustics characteristic to the quadratic Lagrangian (1.1) for a generic potential \( \lambda \) and an external force \( \mu \). The aim of the study is three-fold: (i) to present a basic but self-contained result on classical caustics which can be used for a fuller analysis of caustics for general Lagrangians (because caustics are characterized by the quadratic part of the Lagrangian), (ii) to obtain a formula for the transition kernel \( K(b,T; a,0) \) which covers the case of caustics and introduce the notion of a quantum Jacobi field to discuss the physical implications of caustics at the quantum level, and (iii) to apply the result to examine Jevicki’s correspondence hypothesis, which we now explain below.

In an attempt to resolve certain discrepancies between instantons in QCD and the \( 1/N \) expansion, it was argued by Witten [11] in the two dimensional \( \mathbb{C}P^N \) nonlinear sigma model (which is a prototype of QCD) that classical instantons are eliminated by quantum fluctuations and disappear at the quantum level. To this assertion Jevicki [12] contended — under a few but crucial assumptions — that instantons do not disappear but show up in the form of simple poles rather than stationary points at the quantum level. His point was illustrated by the example of the quantum mechanical rotor, \( \textit{i.e.} \), a free particle on the circle \( S^1 \), where the correspondence

\[
\text{stationary points } \iff \text{simple poles} \quad \quad (1.3)
\]

may become transparent. Embedded in the plane \( \mathbb{R}^2 \), the rotor can be transformed into a two dimensional harmonic oscillator with an arbitrary potential \( \lambda \), where our result for the quadratic Lagrangian is applicable. Our detailed analysis will show that, even in this toy model, the assumptions made in [12] cannot be justified and, therefore, the correspondence hypothesis (1.3) cannot be sustained. In short, there is no firm reason for rejecting Witten’s assertion.
The plan of the paper is as follows. In section 2 we provide a full account of classical caustics associated with the quadratic Lagrangian (1.1). To render the classical results most convenient for later use, we present them in a mathematical style. Readers who wish to see only the discussion of quantum caustics may skip this section. Section 3 is devoted to quantum caustics, where we first derive the kernel formula which admits the case caustics and then discuss the quantum effect associated with caustics by a simple Gaussian slit (gedanken-)experiment. Jevicki’s correspondence hypothesis is examined in section 4. Section 5 contains our conclusion and discussions. An appendix is provided at the end to supplement the argument of section 3.
2. Classical Caustics

In this section we review and summarize various aspects of the classical motion of a point particle governed by a quadratic Lagrangian. Our discussions focus especially on the occurrence of caustics and its consequences in the framework of classical physics. Part of the material presented is well-known, see e.g., [5], but is included here together with numerous other facets of the issue. In this way the following discussions illustrate the phenomenon of caustics in classical physics as opposed to that in quantum realm in later sections and further furnish a mathematical basis of the discussions in the following sections on quantum mechanics on $S^1$. First we consider the Jacobi equation, which in the case of quadratic Lagrangian is the homogeneous part of the equation of motion, and discuss some general properties of its solutions. Next we concentrate on the phenomenon of caustics and discuss two of its main characteristics, the one being the constant stretching factor between the final and the initial points, and the other being the Morse index.

2.1. Quadratic Lagrangian and Jacobi equation

Let us consider a point particle governed by the quadratic Lagrangian (1.1), whose equation of motion reads

$$A_\lambda x(t) = \mu(t), \quad \text{where} \quad A_\lambda := -\left[\frac{d^2}{dt^2} + \lambda(t)\right]. \quad (2.1)$$

For the discussion of caustics it is useful to consider the Jacobi field $J(p,t)$ (see e.g., [5]) which describes the spread of paths from an initial point when varying the momentum. Given a one-parameter family of classical paths $x(p,t)$ characterized by the initial momenta $\dot{x}(p,t) = p$ at $t = t_0$, it is defined by

$$J(p,t) := \frac{\partial x(p,t)}{\partial p} . \quad (2.2)$$

It satisfies $J(p,t_0) = 0$ by definition and solves the Jacobi equation, which in case of the quadratic Lagrangian (1.1) is the homogeneous part of (2.1). Hence, the Jacobi equation can itself be interpreted as the force-free equation of motion,

$$A_\lambda u(t) = 0 , \quad (2.3)$$

and as such contains a wider class of solutions (not necessarily zero at $t = 0$) which may be added to a solution curve of the full equation of motion (2.1) to yield other extremum paths. The investigation of the solution curves to (2.3) not only leads to a characterization
of caustics via the Jacobi field itself, but reveals further properties of the caustics, see in this context also [6]. In particular, the relationships between two linearly independent solutions of the homogeneous differential equation of second order (2.3) help us to clarify the notion of the stretching factor and the index which will be introduced later on.

As is well known, given initial data \( x(t_0) = x_0 \) and \( \dot{x}(t_0) = \dot{x}_0 \) at \( t_0 \), the differential equation (2.3) possesses a unique solution. If \( \lambda(t) \) is continuous, the solution is of class \( C^2 \), i.e., has continuous derivatives of order two, and when \( \lambda(t) \) is only piecewise continuous, the solution curve is at least \( C^1 \). In what follows, we shall use frequently the following

**Lemma 1.** Let \( v(t) \) and \( u(t) \) be solutions of the differential equation (2.3), and let \( t_1 \) and \( t_2 \) be two times. Then we have

\[
\left. v\dot{u} \right|_{t_1}^{t_2} = \left. \dot{v}u \right|_{t_1}^{t_2}. \tag{2.4}
\]

For the proof, we observe from (2.3) that

\[
\left. v\dot{u} \right|_{t_1}^{t_2} = \int_{t_1}^{t_2} dt \frac{d}{dt}(v\dot{u}) = \int_{t_1}^{t_2} dt \frac{d}{dt}(\dot{v}u) + \int_{t_1}^{t_2} dt(-\ddot{v}u + \ddot{v}) = \left. \dot{v}u \right|_{t_1}^{t_2}. \tag{2.5}
\]

There are two, linearly independent solutions to (2.3), which form the space of solutions associated with (2.3). In the case of constant positive \( \lambda := \omega^2 \), a basis for the 2-dimensional solution space of (2.3) may be given by \{\sin(\omega t), \cos(\omega t)\}. We shall remark here that, even for generic \( \lambda(t) \), any two linearly independent solution curves behave like sine and cosine if they are to possess zeros. To this end, first we state the following lemma, which is easily verified by direct differentiation:

**Lemma 2.** Let \( u(t) \) and \( v(t) \) be two linearly independent solutions of (2.3) and let \( u(t_0) = 0 \). Then \( v(t_0) \neq 0 \), and \( u \) may be expressed as

\[
u(t) = \hat{u}(t_0) \cdot v(t_0) \int_{t_0}^{t} \frac{ds}{(v(s))^2} \tag{2.6} \]

as long as \( v \) is non-zero.

Formula (2.6) can be used to create a linearly independent solution with prescribed zero position and initial velocity to a given solution curve. Before embarking on further relationships between \( v \) and \( u \), we show
Lemma 3. Let \( v(t) \) be a (non-trivial) solution of (2.3) and let \( \tau_1 \) be a time such that \( v(\tau_1) = 0 \). Then either \( \dot{v}(\tau_1) \neq 0 \) or else \( \lambda \) has a pole at \( \tau_1 \). In particular, if \( \lambda \) is piecewise continuous, then the zeros of \( v \) are separated.

Proof. Let \( t_0 \) be a time where \( v(t_0) \neq 0 \), and define \( u(t) \) according to Lemma 2 with \( \dot{u}(t_0) = 1 \). With the help of Lemma 1 applied to time \( t_0 \) and \( \tau_1 - \epsilon \) with \( \epsilon > 0 \), we have

\[
v(\tau_1 - \epsilon)\dot{u}(\tau_1 - \epsilon) - v(t_0) = \dot{v}(\tau_1 - \epsilon)u(\tau_1 - \epsilon).
\]  

(2.7)

Thus, by letting \( \epsilon \) go to zero, \( \dot{v}(\tau_1) = 0 \) is possible only if \( u(\tau_1) = \infty \) or \( \dot{u}(\tau_1) = \infty \), which both imply that \( u \) can not be \( C^1 \) at \( \tau_1 \). Therefore, \( \lambda \) is neither continuous nor even piecewise continuous at \( \tau_1 \), proving the assertion. Q.E.D.

Let us henceforth assume that \( \lambda \) be piecewise continuous or, if not, be at least such that whenever \( v(\tau_1) = 0 \), then \( \dot{v}(\tau_1) \neq 0 \). We then have the following

Lemma 4. In the situation of Lemma 2 let \( \tau_1 \) be the first zero position of \( v \) beginning with \( t_0 \). Then the formula (2.6) for \( u \) is well-defined at \( \tau_1 \) and its value is given by

\[
u(\tau_1 - \epsilon) = -\dot{u}(t_0)v(t_0)/\dot{v}(\tau_1).
\]

Proof. Since \( u \) is a solution of (2.3) and \( \lambda \) is piecewise continuous, any possible singularity of \( u \) stemming from its definition (2.6) can be restored, and the value of \( u \) at \( \tau_1 \) can be calculated with the help of Lemma 1. One can check this also directly from (2.6) in the following way. Let \( \epsilon_0 \) and \( \epsilon \) be positive numbers, such that \( 0 < \epsilon < \epsilon_0 \ll 1 \). Now if \( v(\tau_1) = 0 \), then from (2.3) it follows \( \ddot{v}(\tau_1) = 0 \). Thus in the interval \([\tau_1 - \epsilon_0, \tau_1]\) \( v \) can be approximated as \( v(t) = \dot{v}(\tau_1)(t - \tau_1) + O((t - \tau_1)^3) \). Keeping \( \epsilon_0 \) fixed, but letting \( \epsilon \) go to zero we obtain

\[
u(\tau_1 - \epsilon) = \dot{u}(t_0)\left(-\epsilon\dot{v}(\tau_1) + O(\epsilon^3)\right) v(t_0) \left[ \int_{t_0}^{\tau_1-\epsilon_0} \frac{ds}{(v(s))^2} + \int_{\tau_1-\epsilon_0}^{\tau_1-\epsilon} \frac{ds}{(v(s))^2} \right]
\]

\[
= O(\epsilon) + \dot{u}(t_0)\left(-\epsilon\dot{v}(\tau_1) + O(\epsilon^3)\right) v(t_0) \frac{1}{(\dot{v}(\tau_1))^2} \int_{\epsilon}^{\epsilon_0} dr \left\{ \frac{1}{r^2} (1 + O(\epsilon r^2)) \right\}
\]

\[
= O(\epsilon) - \frac{\dot{u}(t_0)v(t_0)}{\dot{v}(\tau_1)},
\]

proving our assertion. Q.E.D.

In the following proposition we now make precise in what sense two linearly independent solutions \( v \) and \( u \) behave like sine and cosine for the case when \( \lambda \) is a positive constant.
Figure 1. A schematic picture of how two linearly independent solutions, v and u, behave under a generic potential λ.

number. For the solution curves of the differential equation (2.3) to behave actually like sine or cosine, they must have zeros. This is certainly not the case for example for the constant $\lambda \equiv -c^2 < 0$ with one possible solution curve being $\exp(ct)$. On the other hand, as we shall see further below, for the case of critical $\lambda$, the solution curves necessarily have zeros. Thus for the following proposition we take it for granted that the solution curves indeed possess zeros, which are separated according to Lemma 3.

Proposition 5. Let $u$ and $v$ be two linearly independent solutions of (2.3). Let ..., $t_{-1}$, $t_0$, $t_1$, $t_2$, ... be the zero positions of $u$, and ..., $\tau_{-1}$, $\tau_0$, $\tau_1$, $\tau_2$, ... be those of $v$. Assume $t_0 \in (\tau_0, \tau_1)$, $v(t_0) > 0$ and $u(t_0) > 0$ for definiteness (other cases of signs may be treated analogously). Then $u$ is positive (negative) on $(t_i, t_{i+1})$ for $i$ even (odd), $v$ is positive (negative) on $(\tau_i, \tau_{i+1})$ for $i$ even (odd), and the zero positions of $v$ and of $u$ alternate such that

$$... < \tau_{-1} < t_{-1} < \tau_0 < t_0 < \tau_1 < t_1 < ...$$

(2.8)

Proof. First of all, the sign of $u$ (and likewise of $v$) must alternate as one passes the zeros, since at each zero position the slope of $u$ is non-zero by Lemma 3. We show first that the zero position $t_1$ following $t_0$ lies in $(\tau_1, \tau_2)$. Since $v$ is positive on $(\tau_0, \tau_1)$, $u$ may be defined by (2.6) and is therefore strictly positive on $(t_0, \tau_1)$. Moreover, since the derivative of $v$ at the zero position $\tau_1$ is non-zero (Lemma 3), and since $v$ becomes negative when it crosses $\tau_1$, we must have $\dot{v}(\tau_1) < 0$. Then, according to Lemma 4, $u$ is strictly positive also at $\tau_1$, which implies $\tau_1 < t_1$. By applying Lemma 1 to the interval $[\tau_1, \tau_2]$ one has

\[ u(\tau_2) = u(\tau_1) \dot{v}(\tau_1) = -\ddot{u}(\tau_0) v(\tau_0) < 0, \] the second equality following from Lemma 4. Since, by definition, \( v \) has a zero at \( \tau_2 \) and becomes positive thereafter (until the next zero at \( \tau_3 \)), its derivative \( \dot{v}(\tau_2) \) is positive, which implies that \( u(\tau_2) < 0 \), i.e., \( u \) must have a zero at some \( \tau_1 < \tau_2 \). Having proven \( \tau_1 < \tau_1 < \tau_2 \), we may proceed by re-defining \( u \) on the interval \([\tau_1, \tau_2]\) analogously to (2.6),

\[ u(t) = \ddot{u}(t_1) \cdot v(t_1) \int_{t_1}^{t} \frac{ds}{(v(s))^2}, \] (2.9)

The above reasoning for \( \tau_1 \) may be applied again to yield \( \tau_2 < \tau_2 < \tau_3 \) (see Fig.1). Proceeding similarly for other \( \tau_i \), one may prove (2.8), \( Q.E.D. \)

2.2. Critical potential and its characteristics

We now consider the full equation of motion (2.1) on a fixed time interval \([0, T]\), and examine the phenomenon of caustics. First we recall that, if \( u, v \) are the two linearly independent solutions of the homogeneous part (2.3) obeying the initial conditions,

\[ u(0) = 0, \quad \dot{u}(0) = 1, \quad v(0) = 1, \quad \dot{v}(0) = 0, \] (2.10)

then the general solution of the full equation (2.1) is given by

\[ x(t) = \alpha v(t) + \beta u(t) + s(t), \] (2.11)

where \( s \) is a special solution of (2.1) which we we take to be one satisfying \( s(0) = 0 \). (We could further specify by requiring, say, \( \dot{s}(0) = 0 \) but this is not important at the moment.) The constants \( \alpha, \beta \) are determined from the initial position \( x(0) \) and velocity \( \dot{x}(0) \), respectively. Note that the solution \( u \) may be obtained from the Jacobi field (2.2) by \( u(t) = J(p, t) / \dot{J}(p, 0) \).

Caustics occur when the solution \( u(t) \), or the Jacobi field \( J(p, t) \), vanishes at \( t = T \). If this happens then it follows that the final position \( x(T) = \alpha v(T) + s(T) \) does not depend on the initial velocity \( \dot{x}(0) \) (which is also obvious from the definition of the Jacobi field). More precisely, we have
Lemma 6. Given a time interval \([0, T]\), suppose that the Jacobi field \(J(p, t)\) beginning at \(t = 0\) vanishes at \(T\). Then there exists a solution of the full equation of motion (2.1) satisfying the Dirichlet boundary condition,

\[
x(0) = a, \quad x(T) = b
\]  

if and only if

\[
b = v(T) \cdot a + s(T),
\]

where \(v(T)\) and \(s(T)\) are the final values of the aforementioned solutions. Further, for the final point \(b\) given in (2.13) there are infinitely many solutions satisfying (2.12), and each solution is uniquely characterized by the initial velocity \(\dot{x}(0)\), which can take any value.

Proof. In the general solution (2.11) the Dirichlet boundary condition (2.12) implies that \(\alpha = a\) while \(\beta\) remains arbitrary since we now have \(u(T) = 0\). Thus whatever the initial velocity (which could determine \(\beta\)) may be, the final point \(x(T)\) turns out to be given by (2.13). Since \(\beta\) is left arbitrary, there are infinitely many solutions satisfying (2.12) with different initial velocities, \(\dot{x}(0) = \beta + \dot{s}(0)\). Q.E.D.

Thus, if \(J(p, T) = 0\), the different solution curves which spread out of the initial point \(a\) are all focused in one final point \(b\). This phenomenon can be considered as a special case of caustics in geometric optics, see e.g., [5]. In this paper we fix the interval \([0, T]\) and study the implications of caustics that arise under a generic potential \(\lambda\), and to this end we shall make the following

Definition 7. If the Jacobi field \(J(p, t)\) vanishes at \(t = T\), then the potential \(\lambda\) shall be called critical, otherwise non-critical.

In case of critical potentials, it is useful to introduce the constant

\[
k(\lambda) := \frac{v(T)}{v(0)},
\]

\(\dot{x}\) from a solution \(v\) satisfying (2.3) with \(v(0) \neq 0\), which gives the stretching factor during the period \([0, T]\). Clearly, this constant is independent of the choice of the solution \(v\) and hence determined solely by the critical potential \(\lambda\). Given an initial point \(x(0) = a\), the final point (2.13) to which all solution curves are focused on now reads \(x(T) = k(\lambda) a + s(T)\), and is called the conjugate point to \(a\) or focal point. The simplest example of caustics arises in the harmonic oscillator, where \(\lambda = \omega^2\) is a positive constant at one of the frequencies \(\omega = \frac{n\pi}{T}\) for \(n = 1, 2, \ldots\). The stretching factor associated with the potential is then
$k(\lambda) = (-1)^n$, and it is easy to see that there are infinitely many solutions $x(t)$ arriving at the conjugate point $(-1)^n a$.

The following Lemma shows how much the situation of a critical potential differs from that of non-critical one.

**Lemma 8.** If $\lambda$ is non-critical, there exists a unique solution of (2.1) for the Dirichlet boundary condition (2.12) with any $a$ and $b$.

**Proof.** Since $\lambda$ is non-critical we have $u(T) \neq 0$. Then, for any $a, b \in \mathbb{R}$, the Dirichlet boundary condition (2.12) determines uniquely the constants $\alpha, \beta$ in the general solution (2.11) to yield

$$x(t) = a v(t) + \frac{b - a v(T) - s(T)}{u(T)} u(t) + s(t).$$

(2.15)

Q.E.D.

We note in passing that a generic potential is non-critical, and this can be seen roughly as follows. Consider a non-critical potential and a solution curve of the force-free equation of motion (2.3) with initial value 0 and initial velocity 1, which thus does not vanish at the final time $T$. If the potential is varied in an infinitesimal manner, then this solution curve also varies very little, implying that the final value remains nonzero for such small variations. This implies that the set of non-critical potentials is open in an appropriate topology. Further, by the same reasoning, if the potential is critical, then in general a small variation suffices to make the potential non-critical. Thus, again in an appropriate topology, which we do not specify here but is easy to find, the set of critical potentials is nowhere dense.

The equation of motion (2.1) can be derived from the action functional $I[x] = \int_0^T dt L$ with the Lagrangian $L$ in (1.1), which now acts on the space of paths $x(t)$ with some given boundary condition $x(0) = a, x(T) = b$. If the potential is non-critical, then there exists a unique classical path from $a$ to $b$, denoted here by $x(t)$, at which the action becomes a minimum. Suppose we vary the potential $\lambda$ and with it the classical path $x(t)$ and its action $I[x]$ in a small neighbourhood of a critical potential $\bar{\lambda}$ with $a$ and $b$ fixed, wherein we only consider non-critical potentials. If $\lambda$ approaches $\bar{\lambda}$, two cases may occur: either $b$ happens to be conjugate to $a$ under $\bar{\lambda}$, i.e., $b = a k(\bar{\lambda}) + s(T)$, in which case there exists a classical path also for the critical potential, and the action of course remains finite, or else $b$ is not conjugate point to $a$, in which case there exists no classical path at the caustic, and the action becomes infinite, as the following lemma shows.
Lemma 9. In the situation of a non-critical potential \( \lambda \) approaching a critical potential \( \bar{\lambda} \), the minimum action \( I[x] \) of the classical path \( x(t) \) between two given fixed boundary points tends to infinity if the final point is not a conjugate one of the initial point for the critical potential \( \bar{\lambda} \).

Proof. Consider the classical path satisfying \( x(0) = a \) and \( x(T) = b \) given in the form (2.15), where we specify the initial velocity of the special solution \( s \) as \( \dot{s}(0) = 0 \) for definiteness. Denoting \( \tilde{x}(t) := a v(t) + s(t) \), and using the differential equations satisfied by \( s \) and \( u \), we obtain the identity,

\[
\int_0^T dt \mu(t) x(t) = \int_0^T dt \mu(t) \tilde{x}(t) - \frac{b - \tilde{x}(T)}{u(T)} \left( \dot{s}(T) u(T) - s(T) \dot{u}(T) \right). \tag{2.16}
\]

With the help of this identity and Lemma 1 applied to \( u \) and \( v \) of (2.15), we find that the action becomes

\[
I[x] = -\frac{1}{2} \int_0^T dt \mu(t) \tilde{x}(t) + \frac{1}{2} (2b - \tilde{x}(T)) \dot{x}(T) + \frac{1}{2} \frac{(b - \tilde{x}(T))^2}{u(T)} \dot{u}(T). \tag{2.17}
\]

As \( \lambda \) approaches \( \bar{\lambda} \), we observe that \( u(T) \to 0, \dot{u}(T) \to 1/k(\bar{\lambda}), \tilde{x}(T) \to k(\bar{\lambda}) a + s(T) \), while \( \dot{x}(T) \) remains finite since its initial velocity \( \dot{x}(0) = 0 \) is fixed for all \( \lambda \). The assertion then follows from the assumption, \( b \neq k(\bar{\lambda}) a + s(T) \). Q.E.D.

An important point to note is that the sign of the divergence \( I[x] \to \pm \infty \) depends on the sign of the product \( u(T) k(\lambda) \) in the limit \( \lambda \to \bar{\lambda} \), and this depends on how \( \lambda \) approaches \( \bar{\lambda} \), not just on \( \bar{\lambda} \) it is approaching.

If \( \lambda \) is critical, then besides the stretching factor \( k(\lambda) \), a further characteristics is the index of the differential operator \( A_\lambda \) viewed as a symmetric bilinear functional, which we now illustrate (see e.g., section 12 in [5]). Consider a generic potential \( \lambda \) with external force \( \mu \) and the action functional \( I[x] \) for paths \( x \) with fixed endpoints \( x(0) = a, x(T) = b \), and let \( x_{cl}(t) \) be a classical path (which in case of critical \( \lambda \) is assumed to exist) which obeys the boundary condition. To \( x_{cl}(t) \) we add a perturbation \( \eta(t) \) with \( \eta(0) = \eta(T) = 0 \) and expand the action \( I[x_{cl} + \eta] \) in terms of \( \eta \), the calculation of which is straightforward,

\[
I[x_{cl} + \eta] = I[x_{cl}] + \int_0^T dt \eta (A_\lambda x_{cl} - \mu) + \frac{1}{2} \int_0^T dt \eta A_\lambda \eta. \tag{2.18}
\]

Since \( x_{cl} \) is a classical solution, the second term linear in \( \eta \) on the right hand side vanishes. The last term on the right describes the action of a bilinear functional on the space of
paths defined on \([0, T]\) with vanishing boundary values. The index of this (non-degenerate) bilinear functional is then defined by the dimension of the space on which it is negative definite. It characterizes the type of the saddle point of the action functional \(I\) at the classical path \(x_{cl}\).

From the consideration of Sturm-Liouville problem the operator \(A_\lambda\) is known to possess a complete set of orthonormal eigenfunctions \(u_n(t)\) [13, 14]:

\[
A_\lambda u_n(t) = -\left[\frac{d^2}{dt^2} + \lambda(t)\right] u_n(t) = E_n u_n(t),
\]

with

\[
u_n(0) = u_n(T) = 0; \quad \int_0^T dt \, u_n(t) u_m(t) = \delta_{nm}.
\]

The index is then given by the number of negative eigenvalues of the eigenvalue problem (2.19) for non-degenerate \(A_\lambda\) for which no zero mode exists. It is known that for large \(n\) the eigenfunctions \(u_n\) approach trigonometric functions with the corresponding asymptotic eigenvalues \((\frac{n\pi}{T})^2\) [13, 14], which are thus unbounded from above. In the following, we assume the eigenvalues to be bounded from below, which is for example the case when \(\lambda\) is bounded from above.

The bilinear functional \(A_\lambda\) becomes degenerated if there arises a zero mode solution of (2.19), i.e., a \(u_m\) with \(E_m = 0\) for some \(m\). The index of \(A_\lambda\) may be extended even to this degenerate case by saying that it is given by the number of modes with \(E_n \leq 0\). By definition, the degeneracy occurs when the potential \(\lambda\) is critical under the given interval \([0, T]\). To illustrate this fact from another viewpoint (see e.g., pp79 in [5]) let us consider the initial value problem \(x(0) = x_0, \dot{x}(0) = \dot{x}_0\), where we no longer fix the final time \(T\), but let it be variable. Looking for each \(T\) the classical path, the corresponding expansion (2.18), and the eigenvalue problem (2.19), with gradually growing \(T\), we observe that the index of the operator \(A_\lambda\) changes only at focal points when an eigenvalue goes through zero, i.e., at those values of \(T\), where \(\lambda\) restricted on the interval \([0, T]\) becomes critical. The precise statement of this is the following

**Proposition 10.** Let \(u\) be a solution of (2.3) with initial condition \(u(0) = 0\). Then the index of the operator \(A_\lambda\) is given by the number of zeros (focal points) of \(u\) on the half-open interval \((0, T]\), which is called the Morse index. If \(\lambda\) is critical and \(v\) a solution of (2.3) with the initial condition \(v(0) \neq 0\), then \(v\) must have a zero on \((0, T]\). The number of zeros of \(v\) is given by the Morse index.
Proof. The first assertion on the index is a special case of the Morse theorem, cf. [15]. The second assertion follows directly from Proposition 5, since between two adjacent focal points of \(u\) there must lie a zero of \(v\). Q.E.D.

Note that, for \(\lambda\) critical, the Morse index, which is characterized by \(\lambda\) and is denoted by \(m(\lambda)\), gives the number of negative eigenvalues of (2.19) plus one. We now have the following consequence of the foregoing discussions on the stretching factor of a critical potential. Let \(\lambda\) be critical, and \(u\) be a zero mode solution of (2.3) with zeros at \(t_0 = 0, t_1, ..., t_m = T\), where \(m = m(\lambda)\) is the Morse index. Since a zero mode solution is unique up to a multiplicative factor, these zeros are fixed for a given potential \(\lambda\). Let \(\lambda_i := \lambda|_{[t_{i-1}, t_i]}\) be the restriction of \(\lambda\) on the \(i\)-th interval \([t_{i-1}, t_i], i = 1, ..., m\). Then the restriction of the zero mode \(u\) on each such interval is trivially a zero mode solution corresponding to \(\lambda_i\). Thus \(\lambda_i\) is critical with Morse index \(m(\lambda_i) = 1\). Then one has

**Lemma 11.** The stretching factor of a critical potential is the product of the stretching factors of the potentials restricted to each closed interval between two adjacent zeros of a zero mode solution, i.e.,

\[
k(\lambda) = k(\lambda_1) \cdots k(\lambda_m). \tag{2.21}
\]

Each \(k(\lambda_i)\) is negative, and thus the stretching factor is negative (positive) if the Morse index \(m(\lambda)\) is odd (even).

**Proof.** Let \(v\) be a solution of (2.3) linearly independent from \(u\). Then from the proof of Lemma 6 we have \(k(\lambda_i) = v(t_i)/v(t_{i-1})\) and

\[
k(\lambda) = v(T)/v(0) = \frac{v(t_m)}{v(t_{m-1})} \cdots \frac{v(t_1)}{v(t_0)}. \tag{2.22}
\]

Each stretching factor is negative, since by the proof of Proposition 5 between two adjacent zeros of the zero mode solution \(u\) there must lie a zero of \(v\), where its sign changes. Q.E.D.
3. Quantum Caustics

In the previous section we studied various dynamical aspects of a particle moving under the influence of time dependent harmonic potentials $\lambda(t)$ and external driving forces $\mu(t)$. When $\lambda(t)$ is critical, the dynamics of the system exhibits a number of singular characteristics at the classical level, notably in that transitions are allowed only between conjugate points specified by the stretching factor $k(\lambda)$ of the potential. The aim of this section is to study the quantum dynamics of the system by looking at the transition amplitude closely. In particular, for critical $\lambda(t)$, we shall confirm the known fact [5, 6] that classically forbidden transitions continue to be forbidden even at the quantum level. The novelty of our result is the closed form of the path-integral kernel for the transition amplitude expressed in terms of the stretching factor $k(\lambda)$ and the Morse index $m(\lambda)$. We further discuss the quantum effect in the momentum susceptibility of a Gaussian wave packet for non-critical $\lambda(t)$.

3.1. Transition amplitude and caustics

We begin by considering the transition amplitude for the system defined by the quadratic Lagrangian (1.1). Let $\hat{H}(t)$ be the Hamiltonian operator corresponding to the Lagrangian, and $\hat{U}(T, 0)$ the unitary operator for the time evolution obeying the Schrödinger equation, $(i\partial/\partial t)\hat{U}(t, 0) = \hat{H}(t)\hat{U}(t, 0)$. Then, in the path-integral formalism, the amplitude for the transition from $x = a$ at $t = 0$ to $x = b$ at $t = T$, where $a$ and $b$ are arbitrary two points on the line $\mathbb{R}$, is given by

$$K(b, T; a, 0) = \langle b | \hat{U}(T, 0) | a \rangle = \int_{x(0)=a}^{x(T)=b} Dx\, e^{iI[x]}.$$ (3.1)

In carrying out the path-integration, we need to take into account the fact that, when $\lambda(t)$ is critical, there may not exist a classical solution that respects the given boundary condition. The general case, including the critical one, may be handled by the following procedure [5, 4].

First, let $c$ be the endpoint at $t = T$ of a classical solution $\bar{x}_{cl}$ of the equation of motion (2.1) starting from $a$ at $t = 0$, i.e., $\bar{x}_{cl}(0) = a$ and $\bar{x}_{cl}(T) = c$. Here we put the bar to the solution in order to emphasize the fact that the endpoint value $c$ may not be equal to $b$, reserving the notation $x_{cl}$ without a bar for the actual solution, if any, having $x_{cl}(T) = b$. If $\lambda$ is non-critical, Lemma 8 states that $c$ can be chosen arbitrarily and hence the solution $x_{cl}(t)$ does exist, whereas if $\lambda(t)$ is critical, Lemma 6 states that the endpoint
is determined uniquely \( c = k(\lambda) a + s(T) \) by the stretching factor of the potential and the endpoint of the solution \( s(t) \). Now, let us choose a fixed, smooth function \( \rho(t) \) satisfying
\[
\rho(0) = 0, \quad \rho(T) = b - c. \tag{3.2}
\]
Then we may decompose any path \( x(t) \) connecting the endpoints, \( x(0) = a \) and \( x(T) = b \), as
\[
x(t) = \bar{x}_{\text{cl}}(t) + \rho(t) + \eta(t). \tag{3.3}
\]
The function \( \eta(t) \) in (3.3), which represents the fluctuations, vanishes at both of the ends \( \eta(0) = \eta(T) = 0 \) and may be expanded \( \eta(t) = \sum_n a_n u_n(t) \) in terms of the orthonormal eigenfunctions in (2.19) and (2.20).

Using the decomposition (3.3) we find
\[
I[\bar{x}_{\text{cl}} + \rho + \eta] = I[\bar{x}_{\text{cl}}] + I[\rho] + I[\eta] + \int_0^T dt \{ \dot{\bar{x}}_{\text{cl}} \dot{\rho} + \dot{\rho} \dot{\eta} + \dot{\eta} \dot{\bar{x}}_{\text{cl}} - \lambda(\bar{x}_{\text{cl}} \dot{\rho} + \rho \dot{\eta} + \eta \dot{\bar{x}}_{\text{cl}}) \}
= I[\bar{x}_{\text{cl}}] + \left. I[\rho] \right|_{\mu=0} + \frac{1}{2} \sum_n E_n a_n^2
+ \dot{\bar{x}}_{\text{cl}}(T) \rho(T) + \rho(T) \sum_n a_n \dot{u}_n(T) + \sum_n E_n a_n \int_0^T dt \rho(t) u_n(t). \tag{3.4}
\]
After the change of the integral variables \( Dx = D\eta = \mathcal{N} \prod_n da_n \) where \( \mathcal{N} \) is a Jacobian factor, the path-integral (3.1) becomes a product of Gaussian integrals over the modes \( a_n \). If \( \lambda \) is non-critical, there exists the solution \( x_{\text{cl}} \) and hence we are allowed to choose \( \rho(t) = 0 \) identically in (3.4). The Gaussian integrations then gives the standard result,
\[
K(b, T; a, 0) = \mathcal{N} \left[ \prod_n E_n \right]^{-\frac{1}{2}} e^{iI[x_{\text{cl}}]}, \tag{3.5}
\]
which can be shown [2, 3] to be equivalent to (1.2).

If, on the other hand, \( \lambda(t) \) is critical, then by definition we have \( E_m = 0 \) for some \( m \). It follows that the integration over \( a_m \), which is now non-Gaussian, yields the delta-function, \( \delta(\rho(T) \dot{u}_m(T)) \). Hence, for critical potentials the kernel formula (3.5) should be modified into
\[
K(b, T; a, 0) = \sqrt{\frac{2\pi}{i}} \mathcal{N} \left[ \prod_{n \neq m} E_n \right]^{-\frac{1}{2}} \delta(\rho(T) \dot{u}_m(T)) e^{iI[x_{\text{cl}}]}, \tag{3.6}
\]
where we have set \( \rho(t) = 0 \) in the phase part of the kernel (3.6), which is allowed in the presence of the delta-function and due to the fact \( \dot{u}_m(T) \neq 0 \). Since we have now
\[
\rho(T) = b - k(\lambda) a - s(T),
\]
the kernel (3.6) already shows that classically forbidden transitions to non-conjugate points, \( b \neq k(\lambda) a + s(T) \) from \( a \), are forbidden even quantum mechanically. We remark at this point that, strictly speaking, the action \( I[x_{cl}] \) in (3.6) should be written as \( I[\bar{x}_{cl}] \) since there is no solution \( x_{cl}(t) \) if \( \rho(T) \neq 0 \). However, the present form is still possible because the phase becomes insignificant if \( \rho(T) \neq 0 \) in which case the kernel vanishes identically.

The remarkable fact in the critical case is that one can express the transition kernel (3.6) in terms of the stretching factor \( k(\lambda) \) and the Morse index \( m(\lambda) \) of the potential \( \lambda(t) \). To see this, let us first write the kernel (3.6) in the form,
\[
K(b, T; a, 0) = R(T) \delta(\rho(T)) e^{i\Theta(T)},
\]
with
\[
R(T) = \sqrt{2\pi |\mathcal{N}|} \left[ \prod_{n \neq m} |E_n| \right]^{-\frac{1}{2}} |\dot{u}_m(T)|^{-1}.
\]
Then, the unitarity relation,
\[
\int db K^*(b, T; c, 0) K(b, T; a, 0) = \int db \langle c | \hat{U}^+(T, 0) | b \rangle \langle b | \hat{U}(T, 0) | a \rangle = \delta(a - c),
\]
determines the modulus \( R(T) \) to be
\[
R(T) = \sqrt{|k(\lambda)|}.
\]
On the other hand, since the Morse index gives the number of negative modes plus one, we have
\[
\left[ \prod_{n \neq m} E_n \right]^{-\frac{1}{2}} = \left[ \prod_{n \neq m} |E_n| \right]^{-\frac{1}{2}} e^{-\frac{i\pi}{4} (m(\lambda) - 1)}.
\]
Thus the phase part is given by
\[
\Theta(T) = I[x_{cl}] - \frac{\pi}{2} (m(\lambda) - 1).
\]
By combining these we find that the transition kernel (3.6) takes the simple form
\[
K(b, T; a, 0) = \sqrt{|k(\lambda)|} \delta(b - k(\lambda) a - s(T)) e^{iI[x_{cl}] - \frac{i\pi}{4} m(\lambda)},
\]
\footnote{An abstract form of the formula for the period before the first conjugate point has appeared in [6].}
up to an overall constant of unit modulus.

As application, take, for example, the forced harmonic oscillator given by \( \lambda(t) = \omega^2 \) and \( \mu(t) = -f(t) \). A special solution satisfying \( s(0) = 0 \) is then

\[
s(t) = \frac{1}{\omega} \int_0^t dt' \sin \omega(t - t') f(t').
\] (3.15)

As mentioned earlier for the pure harmonic oscillator case, caustics occur at \( \omega = \frac{n\pi}{T} \) with \( n = 1, 2, \ldots \), where we have the stretching factor \( k(\lambda) = (-1)^n \) and the Morse index \( m(\lambda) = n \). The endpoint of the solution is \( s(T) = -(-1)^n \int_0^T dt \sin \omega t f(t)/\omega \) and the classical action reads

\[
I[x_{cl}] = a \int_0^T dt \cos \omega t f(t) - \frac{1}{\omega} \int_0^T dt \int_0^t dt' \cos \omega t' \sin \omega t f(t) f(t').
\] (3.16)

In particular, for constant \( f \) the kernel (3.14) reduces to

\[
K(b, T; a, 0) = \delta \left( b - (-1)^n a - \left\{ 1 - (-1)^n \right\} f/\omega^2 \right) e^{i(f^2T/2\omega^2 - n\pi/2)}.
\] (3.17)

This agrees with the results obtained earlier [16, 17, 18] by other indirect means for the (forced) harmonic oscillator.

It is possible to combine the two kernel formulae, (3.5) and (3.6), into one expression. For this, we just introduce an infinitesimal \( \epsilon > 0 \) to replace \( E_n \) with \( E_n + i\epsilon \) in (3.4), and perform the integrations over \( a_n \) (which are now all Gaussian for any \( \lambda \)) to get

\[
K(b, T; a, 0) = N \left[ \prod_n (E_n + i\epsilon) \right]^{-\frac{1}{2}} e^{i\Phi(b, a; \lambda)},
\] (3.18)

where

\[
\Phi(b, a; \lambda) := I[\tilde{x}_{cl}] + I[\rho]_{\mu=0} + \tilde{x}_{cl}(T)\rho(T) - \frac{1}{2} \sum_n \frac{1}{(E_n + i\epsilon)} \left\{ \rho(T) \dot{u}_n(T) + (E_n + i\epsilon) \int_0^T dt \rho(t) u_n(t) \right\}^2.
\] (3.19)

Now for \( \lambda \) non-critical, the kernel (3.18) reduces to (3.5) in the limit \( \epsilon \to 0 \) by construction. (This can also be confirmed explicitly; see Appendix.) But it is also easy to see that, for \( \lambda \) critical, it reduces to (3.6) in the limit, since the mode \( m \) with \( E_m = 0 \) yields the required delta-function on account of the identity, \( \lim_{\epsilon \to 0} \frac{1}{\sqrt{2\pi\epsilon}} e^{-x^2/2\epsilon} = \delta(x) \). This expression

\[5\] The result contained in [18] for \( f(t) \neq 0 \) is marred by an error in the phase.
(3.18) with (3.19) will be useful later in section 4 when we examine the correspondence hypothesis.

### 3.2. Quantum effect in the Gaussian slit experiment

For $\lambda$ non-critical but close to a critical $\bar{\lambda}$, one expects that there will be a concentration in the transition amplitude around the focal point $b = k(\bar{\lambda})a - s(T)$ conjugate to a given initial point $a$. We now analyze how the concentration takes place quantum mechanically, based on the Gaussian slit experiment which is laid out in the textbook of Feynman-Hibbs [19].

To illustrate our point, let us first consider the harmonic oscillator $\lambda = \omega^2$ (with $\mu = 0$). In the Gaussian slit experiment we look at the evolution of a Gaussian wave packet prepared at $t = 0$, and for this we put an apparatus which emits a particle from the origin $x = 0$ at time $t = -\tau$. To get a Gaussian distribution at $t = 0$, we place a ‘Gaussian slit’ centered at $x = a$ with effective width (variance) $\sigma_0$. The slit is realized by means of the Gaussian transmission factor,

$$G(x - a; \sigma_0) = N \exp \left\{ \frac{-(x - a)^2}{4\sigma_0^2} \right\}. \tag{3.20}$$

The wave function $\psi(x, 0)$ of the particle at $t = 0$ is then furnished by the product of the free particle kernel$^6$ $K_0(x, 0; 0, -\tau) = \sqrt{1/(2\pi i\hbar \tau)} e^{ix^2/2\hbar \tau}$ and the transmission factor. From the normalization condition $\int dx |\psi(x, 0)|^2 = 1$ we determine the constant $N$ and obtain

$$\psi(x, 0) = G(x - a; \sigma_0) K_0(x, 0; 0, -\tau) = \frac{1}{(2\pi\sigma_0^2)^{1/4}} \exp \left\{ \frac{-(x - a)^2}{4\sigma_0^2} + \frac{ix^2}{2\hbar \tau} \right\}. \tag{3.21}$$

Note that the initial state (3.21) has the average momentum

$$p = \int dx \psi^*(x, 0) \left( -i\hbar \frac{d}{dx} \right) \psi(x, 0) = \frac{a}{\tau}. \tag{3.22}$$

At $t = T$, the wave function is given by

$$\psi(y, T) = \int dx K(y, T; x, 0) \psi(x, 0), \tag{3.23}$$

---

$^6$ To render the quantum effect manifest, we keep $\hbar$ throughout this subsection.
with the kernel for the harmonic oscillator [16],
\[
K(y, T; x, 0) = \left( \frac{\omega}{2\pi i \hbar |\sin \omega T|} \right)^{\frac{1}{2}} \exp \left( \frac{i\omega}{2\hbar \sin \omega T} \left\{ (x^2 + y^2) \cos \omega T - 2xy \right\} \right) e^{-\frac{i\pi}{4} \text{Int}[\frac{\omega T}{\pi}]},
\]
where \( \text{Int}[x] \) stands for the greatest integer below \( x \). Together with (3.21) the integration in (3.23) is Gaussian and hence we get
\[
\psi(y, T) = \frac{1}{(2\pi \sigma^2(T))^{\frac{1}{4}}} \exp \left\{ -\frac{(y - x_{\text{cl}}(T))^2}{4\sigma^2(T)} + i \text{(phase)} \right\}.
\]
Thus, as is well-known, the Gaussian distribution retains its shape for the harmonic oscillator, in which the center moves along the classical path having the initial momentum \( p \),
\[
x_{\text{cl}}(T) = a \cos \omega T + \frac{p}{\omega} \sin \omega T,
\]
while the variance pulsates as
\[
\sigma(T) = \sigma_0 \left\{ \left( \frac{x_{\text{cl}}(T)}{a} \right)^2 + \left( \frac{\hbar \sin \omega T}{2\sigma_0^2 \omega} \right)^2 \right\}^{\frac{1}{2}}.
\]
Clearly, the second term in (3.27) represents the quantum effect whereas the first term is just the classical variance, since the variation in the initial position \( \Delta x_{\text{cl}}(0) = \sigma_0 \) around the center \( x_{\text{cl}}(0) = a \) for the classical path results in the final variation \( \Delta x_{\text{cl}}(T) = \sigma_0 |x_{\text{cl}}(T)|/a \). (Notice that the initial momentum \( p \) is linearly dependent on the initial position where the path goes through.) Thus, as one expects, the quantum effect always enhances the spread of the Gaussian distribution.

Viewed as a function of the initial variance \( \sigma_0 \), the final variance \( \sigma(T) \) in (3.27) attains its minimum, \textit{i.e.}, the highest concentration of intensity,
\[
\sigma_{\text{min}}(T) = \left| \frac{\hbar x_{\text{cl}}(T) \sin \omega T}{a\omega} \right|^{\frac{1}{2}},
\]
at
\[
\sigma_0 = \left| \frac{\hbar a \sin \omega T}{2\omega x_{\text{cl}}(T)} \right|^{\frac{1}{2}},
\]
which is precisely the point where the quantum effect matches the classical variance. The infinite concentration \( \sigma_{\text{min}}(T) = 0 \) takes place at \( \omega = n\pi/T \), or at \( p = -a\omega/\tan \omega T \). The former case is the expected caustics, where the variance (3.27) becomes purely classical.
and reproduces the initial value $\sigma(T) = \sigma_0$. Thus the infinite concentration is obtained if we let $\sigma_0 \rightarrow 0$. By contrast, in the latter case the variance becomes purely quantum mechanical, as all paths passing through the slit coalesce toward the origin $x = 0$ at $t = T$. In fact, this case is again caustics occurring during the combined period $[-\tau, T]$ under the potential $\lambda(t)$ which vanishes for $t < 0$. Note that in this case the infinite concentration at the quantum level is achieved by letting $\sigma_0 \rightarrow \infty$. In practice, however, both of these concentrations in intensity are unstable, since a small fluctuation in the parameters $\omega$ or $\tau$ will generally bring the variance to a large value, as can be seen in Fig.2. Nevertheless, one could achieve a very high intensity by adjusting the initial variance along with the parameters according to (3.29).

An important quantity for characterizing the concentration is the susceptibility of the variance against initial momentum fluctuations. Exposing the momentum dependence of the variance explicit $\sigma(T) = \sigma(p, T)$, we use the following normalized quantity for the susceptibility,

$$S(p, T) := \frac{a}{\sigma_0 \partial_p} \sigma(p, T).$$

For the present harmonic oscillator (3.27) we find

$$S(p, T) = \left| \frac{\sin \omega T}{\omega} \right| \left\{ 1 + \left( \frac{\hbar \sin \omega T}{2\sigma_0^2 \omega x_{cl}(T)} \right)^2 \right\}^{-\frac{1}{2}}.$$  \hspace{1cm} (3.31)

In the classical limit $\hbar \rightarrow 0$, the susceptibility reduces to the (absolute value of the) Jacobi field, which in this case becomes $J(p, T) = \sin \omega T/\omega$. In the two cases mentioned above in which an infinite concentration can in principle be possible, the susceptibility vanishes $S(p, T) = 0$ and hence the Gaussian wave packet becomes free from momentum fluctuations. What is interesting in this result (3.31), however, is that the quantum effect suppresses the susceptibility of the variation against initial momentum fluctuations.

We now show that these features are universal for a generic potential $\lambda$ if $\mu = 0$. For this we need to note two crucial points. First, for systems with quadratic Lagrangians the semiclassical (WKB) approximation for the kernel gives the exact result,

$$K(y, T; x, 0) = \left( \frac{i}{2\pi \hbar} \left| \frac{\partial^2 I_{cl}}{\partial y \partial x} \right| \right)^{\frac{1}{2}} e^{\frac{i}{\hbar} I_{cl} - \frac{i\pi}{2} m(\lambda)},$$

where $I_{cl} = I_{cl}(y, T; x, 0) := I[x_{cl}]$ is the action evaluated for the classical path $x_{cl}(t)$ connecting the endpoints, $x_{cl}(0) = x$ and $x_{cl}(T) = y$.  \hspace{1cm} (3.32)
Figure 2. The final variance $\sigma(T) = \sigma(\sigma_0, p, T)$ as a function of the initial variance $\sigma_0$ and the momentum $p$. Fig.2a shows a generic case given by the parameters $a = 0.3$, $\omega = 2.5$ and $T = 1.0$ (we set $\hbar = 0.004$ here). The focal points due to the caustics associated with the period $[-\tau, T]$ appear at the bottom of the valley at $p = 1.0$. The minimum value $\sigma_{\min}(T)$ at the bottom tends to vanish as $\sigma_0 \to \infty$ but it is unstable under the fluctuation of $p$. When $\omega$ reaches the first critical value $\omega = \pi$ associated with the period $[0, T]$, the situation changes drastically to the one shown in Fig.2b. This time the minimum $\sigma_{\min}(T) = 0$ attained at $\sigma_0 = 0$ is unstable under the fluctuation of $\omega$.

Second, if $\mu = 0$ the classical action $I_{cl}(y, T; x, 0)$ is a quadratic polynomial homogeneous in $x$ and $y$, because we have

$$I_{cl}(cy, T; cx, 0) = I[cx_{cl}] = c^2I[x_{cl}] = c^2I_{cl}(y, T; x, 0),$$

(3.33)

for any constant $c$. Thus, with $A$, $B$ and $C$ being some functions of $T$, we may write

$$I_{cl}(y, T; x, 0) = Ax^2 + Bxy + Cy^2.$$

(3.34)

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Then it is straightforward to see that the Gaussian wave packet (3.21) evolves into the wave function (3.23) at time $t = T$ in exactly the same form (3.25) with

$$x_{cl}(T) = -\frac{1}{B}(2aA + p),$$

(3.35)

and

$$\sigma(T) = \sigma_0 \left\{ \left(\frac{x_{cl}(T)}{a}\right)^2 + \left(\frac{\hbar}{2\sigma_0^2 B}\right)^2 \right\}^{\frac{1}{2}}.$$  

(3.36)

The variance attains its minimum,

$$\sigma_{\text{min}}(T) = \left|\frac{\hbar x_{cl}(T)}{aB}\right|^{\frac{1}{2}} \quad \text{at} \quad \sigma_0 = \left|\frac{\hbar a}{2B x_{cl}(T)}\right|^{\frac{1}{2}}.$$  

(3.37)

The susceptibility of the variance (3.30) then reads

$$S(p,T) = |J(p,T)| \left\{ 1 + \left(\frac{\hbar a}{2\sigma_0^2 B x_{cl}(T)}\right)^2 \right\}^{-\frac{1}{2}},$$

(3.38)

where $J(p,T) = -(\partial^2 I_{cl}/\partial y \partial x)^{-1} = -1/B$ is the Jacobi field for the classical action (3.34). The generic behavior of the susceptibility (3.38) with respect to $p$ is analogous to what we have seen in the harmonic oscillator case.

The above features of the Gaussian wave packet persist even for $\mu \neq 0$, because then the classical action $I_{cl}(y,T;x,0)$ acquires only linear and constant terms in addition to the quadratic terms in (3.34) as can be explicitly seen from the form of the general classical solution (2.15). Accordingly, the time evolution by the integral (3.23) is essentially unchanged. Semiclassically, this may also be the case for more general systems, not only for those with quadratic Lagrangians we considered, in view of the earlier study [20] which suggests that these features are a norm for a generic system in the limit $\hbar \to 0$. 
4. The Role of Instantons in the Quantum Mechanical Rotor

In the previous section we obtained the path-integral kernel for the quadratic action with a generic time-dependent potential function $\lambda(t)$. The aim of this section is to apply it to the problem of quantum mechanical rotor on the circle $S^1$ and thereby examine Jevicki’s correspondence hypothesis [12] mentioned in the Introduction. We first furnish a general scheme to compute the path-integral kernel for the rotor based on the formula we just obtained. This will strengthen the basis on which Jevicki’s arguments for the hypothesis stand and highlight their loose ends at the same time. Our close examination will show that neither of the two crucial assumptions adopted in his arguments can be entirely justified for generic transition amplitudes.

4.1. Examination of the correspondence hypothesis

The system taken up by Jevicki to illustrate the correspondence (1.3) between stationary points and simple poles is the quantum mechanical rotor, namely, a free particle on the circle $S^1$. Embedded in the two dimensional plane $\mathbb{R}^2$, the system acquires a harmonic potential and develops a singularity in the transition kernel, and this has been used as a prototype to argue that the presence of instantons can be seen as simple poles (rather than stationary points) in the $\mathbb{C}P^N$ nonlinear sigma model. A salient feature of the $S^1$ system is that its non-trivial topology — $S^1$ is multiply-connected: $\pi_1(S^1) = \mathbb{Z}$ — admits multi-winding configurations called ‘instanton’ solutions labelled by the winding number $n \in \mathbb{Z}$. Here, our concern lies in the question whether the instantons can also be seen explicitly even if we embed the $S^1$ in the topologically trivial plane $\mathbb{R}^2$ using the Cartesian coordinates. Before enforcing this ‘Cartesian point of view’, we wish to recapitulate the known result of the path-integral on $S^1$ in the ‘intrinsic point of view’, that is, when the circle is regarded as intrinsic rather than being embedded in $\mathbb{R}^2$, where the instantons arise as sole contributors to the transition kernel.

For a free particle on $S^1$ coordinated by the angle $\varphi \in [0, 2\pi)$ we have the classical action $I_{S^1}^0[\varphi] = \int_0^T dt \frac{1}{2} (\ddot{\varphi})^2$. In the path-integral formalism, the transition amplitude going from $\alpha$ at $t = 0$ to $\beta$ at $t = T$ where $\alpha, \beta \in [0, 2\pi)$ is given by

$$K_{0}^{S^1}(\beta, T; \alpha, 0) = \int_{\varphi(0)=\alpha}^{\varphi(T)=\beta} \mathcal{D}\varphi \ e^{iI_{S^1}^0[\varphi]} \ . \quad (4.1)$$

This formal expression must be given a meaning such that it accommodates multi-winding paths allowed for the transition since $S^1$ is multiply-connected. The conventional method
to deal with such paths is that, instead of working on $S^1$, one considers the corresponding free particle system on the covering space of the circle, \textit{i.e.}, the line $\mathbb{R}$ governed by the action $I_0[x] = \int_0^T dt \frac{1}{2} (\dot{x})^2$, which is invariant under the translation by $2\pi$. Namely, to the amplitude on $S^1$ consisting of paths of winding number $n$, one assigns the free particle amplitude $K_0(\alpha + 2\pi n, T; \alpha, 0)$ on the line $\mathbb{R}$, and then sums it up over the integers $n \in \mathbb{Z}$ with a weight factor $w(n)$,

$$K_0^{S^1}(\beta, T; \alpha, 0) = \sum_{n=-\infty}^{\infty} w(n) K_0(\alpha + 2\pi n, T; \alpha, 0). \quad (4.2)$$

In short, this amounts to regarding $S^1$ as the coset space $\mathbb{R}/\mathbb{Z}$ by identifying those points on the line $\mathbb{R}$ which differ by $2\pi \times$ integer. The weight factor $w(n)$ appearing in (4.2) signals the ambiguity that can arise due to the multiply-connectedness of the space. It is furnished by the unitary representation $w(n) = e^{i n \theta}$ of the group $\mathbb{Z}$, where $\theta \in [0, 2\pi)$ is the angle parameter specifying the representation and, hence, the ambiguity.\footnote{For a general discussion of the path-integral quantization on a coset space $G/H$, see [21].}

Accordingly, the kernel (4.2) on the circle $S^1$ becomes

$$K_0^{S^1}(\beta, T; \alpha, 0) = \sqrt{\frac{1}{2\pi iT}} \sum_{n=-\infty}^{\infty} e^{i(\beta - \alpha + 2n\pi)^2/2T}, \quad (4.4)$$

which is just a sum of instanton contributions.

In the Cartesian coordinate point of view, on the other hand, one puts the particle on the two dimensional plane coordinated by $x = (x, y) \in \mathbb{R}^2$ and imposes the constraint $|x|^2 = 1$. The constraint can be implemented by introducing the Lagrange multiplier $\lambda(t)$ and writing (4.1) as

$$K_0^{S^1}(\beta, T; \alpha, 0) = \int \mathcal{D} \lambda e^{\frac{i}{2} \int_0^T dt \lambda \left( |\dot{x}|^2 - \lambda (|x|^2 - 1) \right)} \mathcal{D} x e^{iI_0[x]} \mathcal{D} y e^{iI[y]}, \quad (4.5)$$

\footnote{For a general discussion of the path-integral quantization on a coset space $G/H$, see [21].}
where $I[x]$ (and similarly $I[y]$) is the quadratic action with time-dependent harmonic potential $\lambda(t)$ under zero force $\mu(t) = 0$. In words, the kernel is given by the product of two kernels on the line sharing the potential $\lambda(t)$, one of which describing the transition from $\cos \alpha$ to $\cos \beta$ and the other from $\sin \alpha$ to $\sin \beta$, averaged over $\lambda(t)$ with the weight factor $e^{i\frac{1}{2} \int_0^T dt \lambda}$,

$$K_0^{S_1}(\beta, T; \alpha, 0) = \int D\lambda K(\cos \beta, T; \cos \alpha, 0) K(\sin \beta, T; \sin \alpha, 0) e^{i\frac{1}{2} \int_0^T dt \lambda}. \quad (4.6)$$

Using the expression (3.18) for $K(b, T; a, 0)$ obtained in section 3, the kernel (4.6) becomes

$$K_0^{S_1}(\beta, T; \alpha, 0) = \int D\lambda \left[ \prod_n (E_n + i\epsilon) \right]^{-1} e^{i\Phi(\cos \beta, \cos \alpha; \lambda) + i\Phi(\sin \beta, \sin \alpha; \lambda) + \frac{1}{2} \int_0^T dt \lambda}. \quad (4.7)$$

Now the generic formula for the phase (3.19) implies that the phase $\Phi(b, a; \lambda)$ reduces simply to the action $I_{cl}(b, T; a, 0) = I[x_{cl}]$ except for the cases where the potential $\lambda$ is critical and $b$ is not conjugate to $a$, i.e., $b \neq k(\lambda)a$. However, these exceptional cases may be neglected in the integration over $\lambda$ on the grounds that they form a set of measure zero in the entire space of potentials $\lambda$ and that the kernel $K(b, T; a, 0)$ vanishes at those $\lambda$ and hence cannot contribute to the integral (4.6). For this reason we replace the phase factor in the integration by the classical action with the prescription,

$$\Phi(b, a; \lambda) \rightarrow I_{cl}^{(\xi)}(b, T; a, 0) := (1 + i \text{sign}(I) \xi) I_{cl}(b, T; a, 0). \quad (4.8)$$

The prefactor $(1 + i \text{sign}(I) \xi)$, where $\text{sign}(I) := I/|I|$ gives the sign of the action with $\xi > 0$ being infinitesimal, is attached in order to guarantee that $e^{i(1 + i \text{sign}(I) \xi)I}$ vanishes for those critical potentials with $b \neq k(\lambda)a$ for which the action diverges (see Lemma 9).

To proceed further and make contact with Jevicki’s argument, let us consider the separation of the zero mode part $\lambda_0$ from the potential,

$$\lambda(t) = \lambda'(t) + \lambda_0, \quad \text{with} \quad \int_0^T dt \lambda'(t) = 0, \quad \lambda_0 = \frac{1}{T} \int_0^T dt \lambda(t). \quad (4.9)$$

As we did for the potential $\lambda(t)$ in (2.19) and (2.20), we can also consider a complete set of orthonormal eigenfunctions for the non-zero mode part $\lambda'(t)$. Let the eigenvalues of those eigenfunctions be $\varepsilon_n$. Obviously, both $\lambda(t)$ and $\lambda'(t)$ share the same set of eigenfunctions with eigenvalues related by

$$E_n = \varepsilon_n - \lambda_0. \quad (4.10)$$
Accordingly, a potential $\lambda(t)$ is critical if $\lambda_0 = \varepsilon_n(\lambda')$ for some $n$.

Substituting (4.10) in (4.7), we find

$$K_0^{Si}(\beta, T; \alpha, 0) = \int D\lambda' d\lambda_0 \left[ \prod_n (\varepsilon_n - \lambda_0 + i\varepsilon) \right]^{-1} \times \exp \left\{ iI_{cl}^{(\xi)}(\cos \beta, T; \cos \alpha, 0) + iI_{cl}^{(\xi)}(\sin \beta, T; \sin \alpha, 0) + \frac{i}{2} \lambda_0 T \right\} .$$

(4.11)

Consider now the integration over the zero mode $\lambda_0$. Observe that the integrand diverges (in the limit $\varepsilon \to 0$) for critical $\lambda$, because of the factor $[\prod_n (\varepsilon_n - \lambda_0 + i\varepsilon)]^{-1}$. Recall that the additional divergence that could arise in the actions $I_{cl}(\cos \beta, T; \cos \alpha, 0)$ and $I_{cl}(\sin \beta, T; \sin \alpha, 0)$ is taken care of by the prefactor $(1 + i \text{sign}(I) \xi)$. Thus we are naturally led to evaluate the integration over the zero mode by collecting the values at the critical potentials which admit the endpoint $b$ to be conjugate to $a$ in both $x$ and $y$-direction. This will be carried out by regarding the $\lambda_0$ integration as part of the contour integration in the complex $\lambda_0$ plane as shown in Fig.3. Since each critical potential which satisfies the above condition provides a simple pole in the integrand in (4.7), and since the contribution from the semicircle of the contour vanishes as the radius becomes large due to the weight factor $e^{\pm \lambda_0 T}$, the $\lambda_0$ integration boils down to the residue calculus at those simple poles. That the $\lambda_0$ integration in (4.7) can be carried out by the contour integration in the way described above, ignoring the contributions from those critical potentials which do not meet the condition $b = k(\lambda)a$, constitutes our first assumption.

Unfortunately, this assumption cannot be true in general. Indeed, the condition for a critical potential $\lambda(t)$ to create just a simple pole (not other type of potential singularities) is to have the stretching factor $k(\lambda)$ which matches the boundary conditions in both $x$ and $y$-direction simultaneously,

$$k(\lambda) = \frac{\cos \beta}{\cos \alpha} \quad \text{and} \quad k(\lambda) = \frac{\sin \beta}{\sin \alpha} ,$$

(4.12)

but, obviously, these are incompatible in general. It then follows from the absence of simple poles that the transition amplitude (4.7) on the circle must vanish for generic boundary conditions, a result in contradistinction with (4.4). Having no simple poles that could correspond to the stationary points in the intrinsic viewpoint, we find that the correspondence hypothesis (1.3) cannot be true in general. It is also worth mentioning that in the intrinsic viewpoint the stationary points contributing to (4.4) do not in general
Figure 3. The integration contour in the complex $\lambda_0$-plane. Among the potentially singular points $\lambda_0 = \varepsilon_n + i\varepsilon$, those critical potentials which meet the condition stated in the text give rise to simple poles shown by ‘•’. By contrast, those which do not meet the condition, shown here by ‘×’, will not create actual singularities for contributing to the integration.

correspond to the stationary points appearing under the critical potentials in the Cartesian viewpoint.

The condition (4.12) becomes compatible if $\tan \alpha = \tan \beta$, that is, if the two endpoints are the same or diametrically opposite with each other. To examine whether it is possible to sustain the correspondence hypothesis if restricted to this specific case, let us take $\alpha = \beta = 0$ from now on. Then we observe that, in this case, we have $I_{\text{cl}}(0, T; 0, 0) = 0$ for the classical action in $y$-direction, and the condition is fulfilled if $k(\lambda) = 1$. We thus arrive at

$$K_0^{S_1}(0, T; 0, 0) = F(T) \int D\lambda' \sum_n e^{i\Phi_n(\lambda')},$$

(4.13)

with the total phase

$$\Phi_n(\lambda') := \left( I_{\text{cl}}^{(1)}(1, T; 1, 0) - m(\lambda)\pi + \frac{1}{2}\lambda_0 T \right) \bigg|_{\lambda_0 = \varepsilon_n},$$

(4.14)

where we formally included all the contributions from critical potentials using (3.14) in view of the fact that those which correspond to $\lambda$ with $k(\lambda) \neq 1$ will drop out due to the prefactor attached in (4.8). In (4.13), the real function $F(T)$ accounts for the possible overall factor that arises in the change of integration measure and the residue computation, and $m(\lambda)$ is the index of the critical potential $\lambda(t)$. An important point to note is that, for any critical potential with $k(\lambda) = 1$ that can contribute to the kernel (4.13), the index
$m(\lambda)$ is always even on account of the alternate nature of the sign of the stretching factors $k(\lambda)$ (see Lemma 11). Thus we may drop the index term $m(\lambda)\pi$ in the total phase $\Phi_n(\lambda')$ without affecting the kernel (4.13).

According to Jevicki [12], we now pose the second assumption: the total phase factor $\Phi_n(\lambda')$ in (4.13) is independent of $\lambda'(t)$. This is a rather drastic assumption, allowing us to set $\lambda'(t) = 0$ and getting the eigenfunctions and the eigenvalues,

$$u_n(t) = \sqrt{\frac{2}{T}} \sin \left( \frac{n\pi t}{T} \right), \quad \varepsilon_n = \left( \frac{n\pi}{T} \right)^2,$$

(4.15)

for $n = 1, 2, 3, \ldots$. We then find that the critical potentials arise at $\lambda(t) = \lambda_0 = \varepsilon_n$, and they have the stretching factor $k(\lambda = \varepsilon_n) = (-1)^n$. Thus, among them those possessing $k(\lambda) = 1$ occur at even integers $n = 2l$. Since we have $I_{cl}(1, T; 1, 0) = 0$ for the classical solution in the (critical) harmonic oscillator, and since the index is given by $m(\lambda = \varepsilon_{2l}) = 2l$, the total phase becomes

$$\Phi_{2l}(0) = -2l\pi + \frac{2(l\pi)^2}{T}.$$  

(4.16)

Consequently, we obtain

$$K_0^{S^1}(0, T; 0, 0) = \frac{1}{2}F(T) \sum_{l \neq 0} e^{i2(l\pi)^2/T}.$$  

(4.17)

Thus, if we choose $F(T) = 2\sqrt{\frac{1}{2\pi iT}}$, the result (4.17) coincides with (4.4), up to the contribution from $l = 0$.

The foregoing argument is (a rigorous version of) the one used in [12] which asserts that, even in the Cartesian point of view, one can see the instantons contributing to the kernel by the form of simple poles, rather than stationary points, in the integrand. In fact, for the boundary condition $\alpha = \beta = 0$ the classical motions allowed under the critical potentials sitting at the simple poles correspond precisely to the instanton solutions (4.3) which are the stationary points in the intrinsic viewpoint. Thus we find that the correspondence hypothesis (1.3) holds if one restricts oneself to the specific transition process and ignores the lacking piece $l = 0$ in (4.17) which, if existed, would correspond to the trivial (0-instanton) solution $x_{cl}^{(0)}(t) = 0$ in the intrinsic viewpoint.

### 4.2. Validity of the second assumption: an example

Although we have seen that the first assumption cannot be sustained for generic transition processes, at the end of the computation the kernel (4.7) must recover the
integers (i.e., instanton numbers) appearing in (4.4) for any $\alpha$ and $\beta$. It is a matter of fact that the classical motions expressed by the instantons $x_{cl}^{(n)}(t)$ in (4.3) arise at $\lambda(t) = \text{constant}$, even though they do not necessarily appear as simple poles for generic $\alpha$ and $\beta$. In this respect the second assumption does lead to the correct result in effect, and one is curious whether there is an a priori reason for this. (Note that the two assumptions are independent of each other.) In the rest of this section we shall investigate this issue by looking at a simple but nontrivial example.

To this end, we return to the special case $\alpha = \beta = 0$ where the assertion of the first assumption is valid (up to the contribution at $l = 0$). The second assumption now turns out to be that, modulo $2\pi$, the total phase $\Phi_n(\lambda')$ in (4.14) is given by the phase $\Phi_n(\lambda' = 0)$ in (4.16) with $n = 2l$. To make the issue more transparent, let us fix some $\lambda(t)$ and consider the family of potentials $\lambda(t)$ which differ by constants $\lambda(t) = \lambda'(t) + \lambda_0$ as in the form (4.9). Within this family, the critical potentials which arise at $\lambda_0 = \varepsilon_n$ for $n \in \mathbb{Z}$ form a class and can be labelled by the integers $n$. In particular, those that have $k(\lambda) = 1$ form a subclass in the class and may further be labelled by another set of integers $l \in \mathbb{Z}$. The second assumption can then be justified if the total phase $\Phi_n$ admits the form (4.16) modulo $2\pi$ with the summation $l$ of $n = 2l$ being performed over the critical potentials in the subclass.

To see how this works, take the potential

$$\lambda'(t) = c \delta(t - t_0) - \frac{c}{T}, \quad (4.18)$$

where $c$ is a constant and the term $-\frac{c}{T}$ is inserted to ensure $\int dt \lambda' = 0$. This is a perturbation of the trivial case $\lambda'(t) = 0$ in the sense that (4.18) defines a one-parameter family of potentials containing the trivial case $c = 0$. The other parameter $t_0 \in (0, T)$ specifies the moment of impact inflicted on the particle and may be determined by requiring that $\lambda(t)$ admits cases where $k(\lambda) = 1$. From Lemma 6 we know that this requirement is met if there exists a classical solution $u(t)$ of the equation of motion vanishing at the endpoints $u(0) = u(T) = 0$ and further has the same velocity $\dot{u}(0) = \dot{u}(T)$. Below we shall seek for conditions under which such a solution exists, setting the initial velocity $\dot{u}(0) = 1$ for definiteness.

For this purpose, we first divide the interval $[0, T]$ into $[0, t_0]$ and $[t_0, T]$, and call them interval (I) and (II). Then, in each of the intervals, the solution which respects the boundary conditions takes, respectively, the form

$$u_I(t) = \frac{1}{\omega} \sin(\omega t), \quad u_{II}(t) = -\frac{1}{\omega} \sin(\omega(T - t)). \quad (4.19)$$
The continuity condition \( u_1(t_0) = u_{II}(t_0) \) at \( t = t_0 \) gives
\[
\frac{1}{\omega} \sin(\omega t_0) = -\frac{1}{\omega} \sin(\omega(T - t_0)) .
\] (4.20)

The velocity acquires a gap at \( t = t_0 \) prescribed by the delta-function interaction as \( \dot{u}_{II}(t_0) - \dot{u}_1(t_0) + c u_1(t_0) = 0 \), which amounts to
\[
\cos(\omega(T - t_0)) - \cos(\omega t_0) + \frac{c}{\omega} \sin(\omega t_0) = 0 .
\] (4.21)

The first condition (4.20) is fulfilled if
\[
\omega(T - t_0) = -\omega t_0 + 2\pi l , \quad \text{or} \quad \omega(T - t_0) = \omega t_0 + \pi(2l + 1) ,
\] (4.22)
for \( l \in \mathbb{Z} \). To each of these cases, the other condition (4.21) requires that (for \( c \neq 0 \))
\[
\omega = \frac{2\pi l}{T} , \quad c = \text{arbitrary} , \quad t_0 = \frac{m}{2l} T ,
\] (4.23)
where \( m \) can be any integer with \( 1 \leq m \leq 2l - 1 \), or
\[
\omega = \frac{(2l + 1)\pi}{T - 2t_0} , \quad c = \frac{2\omega \cos(\omega t_0)}{\sin(\omega t_0)} , \quad t_0 = \text{arbitrary} .
\] (4.24)

Note that the conditions (4.24) do not allow for a set of infinite number of solutions for fixed \( c \) and \( t_0 \) (specified by \( \lambda'(t) \) that defines the class). Hence in the latter case (4.24) the subclass with \( k(\lambda) = 1 \) consists of finite number (possibly one) of critical potentials. In the former case (4.23), on the other hand, if \( m = \frac{q}{p} \) with \( p \) and \( q \) being coprime positive integers satisfying \( \frac{q}{p} < 2 \), then the parameter \( t_0 \) lies in the interval \( 0 \leq t_0 \leq T \) with the fixed value,
\[
t_0 = \frac{q}{2p} T .
\] (4.25)

Thus the former case does provide the subclass consisting of infinite number of potentials, but for \( m \) to be integers the label \( l \) runs as \( l = p, 2p, 3p, \ldots \). We now examine whether our second assumption holds or not here.

To evaluate the total phase explicitly, we consider the classical solution \( x(t) \) for the above class of potentials satisfying the boundary condition,
\[
x(0) = x(T) = 1 , \quad \dot{x}(0) = 1 .
\] (4.26)
(Again, for definiteness we added the second condition in (4.26) because the action on caustics does not depend on the initial velocity.) As before, in each of the intervals (I) and (II) the solution obeying the boundary conditions (4.26) takes the form,

\[ x_I(t) = \frac{1}{\omega} \sin(\omega t) + \cos(\omega t) \]
\[ x_{II}(t) = -\frac{\gamma}{\omega} \sin(\omega(T - t)) + \cos(\omega(T - t)) , \]

where \( \gamma \) is a constant. From the continuity condition and the velocity gap equation at \( t = t_0 \) in (4.25) together with the \( \omega \) in (4.23), it is determined to be \( \gamma = 1 - c \). For this solution (4.27) the classical action becomes

\[ I_{cl}(1, T; 1, 0) = \frac{1}{2} x_I(t)\dot{x}_I(t)\bigg|_{t_0}^{t_0} + \frac{1}{2} x_{II}(t)\dot{x}_{II}(t)\bigg|_{t_0}^{T} - \frac{c}{2} x_I^2(t_0) = -\frac{c}{2} . \] (4.28)

On the other hand, the eigenvalues for the critical potentials in the subclass are

\[ \varepsilon_{2l} = \left( \frac{2\pi l}{T} \right)^2 + \frac{c}{T} , \] (4.29)

where we put the label \( 2l \) to match the notation (4.15). Consequently, the total phase (4.14) turns out to be

\[ \Phi_{2l} = -m(\lambda) \pi + \frac{2(l\pi)^2}{T} . \] (4.30)

Note that the final expression (4.30) is independent of the parameter \( c \). Since the index \( m(\lambda) \) is always even as remarked earlier, we see that, if \( p = 1 \) for which the label \( l \) takes all integers, the phase factor agrees with the original one (4.16) even if it is perturbed by the delta-function interaction. However, for \( p > 1 \), the total phase does not reproduce the unperturbed value (4.16).

We thus find that the assertion of the second assumption holds for the family of potentials furnished by (4.18) with the parameters given by (4.23), if one makes the special choice \( p = 1 \). It does not hold, however, for the case (4.24), where the final phase factor consists of a finite number of contributions. In conclusion, there is no a priori reason that the second assumption can lead to the correct transition amplitude, simply because there are cases where the assumption itself breaks down. Finally, we point out that the case \( p = 1 \) where the assumption does hold implies \( t_0 = T/2 \), and this is the only case in the type of potentials (4.18) that admits all the eigenstates to have a smooth transition to the unperturbed ones in the limit \( c \to 0 \). One may speculate that the cases where the non-zero mode \( \lambda' \)-independence is seen, such as the above case where the cancellation of the \( c \)-dependence occurs, may be found more generally, while those cases where it is not seen form a set of measure zero in the total space of the non-zero mode of the potential \( \lambda'(t) \).
5. Conclusion and Discussions

In this paper we investigated various aspects of classical and quantum caustics for quadratic Lagrangians in one dimension, and applied the results thereof to examine Jevicki's correspondence hypothesis (1.3) between stationary points and simple poles in the amplitude.

When caustics occur, classical paths spreading out from an initial point are focused into a unique conjugate point; in other words, only those paths connecting the conjugate points are allowed classically. The constant $k(\lambda)$, which is the stretching factor between the two conjugate points, is found to be a useful pointer, along with the Morse index $m(\lambda)$, for characterizing the intrinsic features of caustics. Correspondingly, in quantum mechanics the transition amplitude is nonvanishing only for the conjugate points (for which it diverges). We derived the path-integral kernel for the amplitude in a closed form, expressed solely in terms of the stretching factor, the Morse index, and the action of (any of the) solution paths.

Contrary to the situation on caustics, once one goes away from caustics the classical and quantum situations become different. Our study focused on how the typical feature of caustics, i.e., the concentration of intensity at the conjugate point, can be affected by quantum effect. The Gaussian slit (gedanken-)experiment using a Gaussian wave packet shows that, although the variance itself is enhanced at the quantum level, the susceptibility of the variance of the wave packet against initial momentum fluctuations is suppressed. High intensity is realized near caustics of two different types, the first being the caustics associated with the period $[0, T]$ while the second being those with $[-\tau, T]$ which is the entire period of our experiment. As a quantum analogue of the Jacobi field, we introduced the susceptibility which reduces to the ordinary Jacobi field in the classical limit $\hbar \to 0$. The susceptibility gives a measure of stability for the concentration, and our result shows that the intensity of the amplitude is stabilized by quantum effect. For practical purposes, however, we need to extend our analysis further to the effects of higher order terms in the Lagrangian near caustics and thereby supplement earlier works carried out on caustics. Another possible direction would be to consider these problems in higher dimensions allowing for settings more realistic for physical application.

The classical and quantum features of caustics studied here have then been employed to investigate in detail the role of instantons in quantum mechanics. Viewed as a constrained system on the plane $\mathbb{R}^2$, the quantum mechanical rotor over the circle $S^1$ becomes two harmonic oscillators sharing the same, time-dependent harmonic potential $\lambda$ given by
the Lagrange multiplier field. Singularities in the amplitude arise at those $\lambda$ under which caustics occur. We however learned that generically no simple pole can arise, simply because the two stretching factors of the harmonic oscillators do not coincide for generic boundary conditions on $S^1$, except when the two endpoints are the same or diametrically opposite with each other. But even then, the zero-instanton contribution is missing in the simple poles to complete the correspondence (1.3). Even if we are confined to the former specific cases, and ignoring the missing piece, we come across the problem that it is impossible to replace the path-integration over the potentials $\lambda$ by an ordinary integration over the constant (zero-mode) part $\lambda_0$ as originally assumed by Jevicki. We observed that, in the non-trivial example we looked at in section 4, the assumption of the non-zero mode independence of the phase factor does not hold in general, although there are cases where it holds thanks to a certain cancellation mechanism. From our result we find it hard to sustain the hypothesis even in the toy model, let alone in more general systems such as the $\mathbb{C}P^N$ model of the original concern. We, however, mention that there remains a possibility that, even though it is not strictly true, for some reason the assumption may still lead to a correct answer, validating the hypothesis in the end. We feel that this possibility deserves a fuller study in order to settle down the issue which is important in exploring non-perturbative methods in quantum field theory.

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Appendix

In this Appendix we give a direct argument to show that, for no n-critical $\lambda$, the phase (3.19) of the kernel (3.18) is independent of the choice of the function $\rho(t)$. More explicitly, we shall show that the phase is given by the action,

$$\Phi(b, a; \lambda) = I_{cl}(b, T; a, 0),$$

(A.1)

which is evaluated for the classical solution $x_{cl}$ under a non-critical $\lambda$ obeying the boundary condition, $x_{cl}(0) = a$ and $x_{cl}(T) = b$. Prior to our argument, we recall that the function $\rho(t)$ which satisfies $\rho(0) = 0$ and $\rho(T) = b - c$ is introduced to compensate the gap at $t = T$ between the given endpoint $b$ and the endpoint $c$ for which an actual classical solution $\bar{x}_{cl}$ exists. Such a gap is inevitable if $\lambda$ is critical and $k(\lambda) a + s(T) \neq b$. However, we stress that even for non-critical $\lambda$ it is perfectly legitimate to use the classical solution $\bar{x}_{cl}$ with $\bar{x}_{cl}(T) = c \neq b$ for computing the transition amplitude. Thus, for non-critical $\lambda$, it must be that the apparent $\rho(t)$-dependence in the result (3.19) disappear and the total phase be given by the classical action for the solution $x_{cl}(T) = b$, i.e., (A.1).

To confirm this, we first consider a solution $u(t)$ of the homogeneous equation (2.3) with $u(0) = 0$. We may expand it in terms of the orthonormal eigenfunctions $\{u_n\}$ in (2.19) as

$$u(t) = \sum_n c_n u_n(t) + u(T) \theta(t - T).$$

(A.2)

In (A.2) we introduced the step function $\theta(t - T)$, defined by $\theta(x) = 0$ for $x < 0$ and $\theta(x) = 1$ for $x \geq 0$, to account for the condition $u(T) \neq 0$. Differentiating (A.2) with respect to $t$ and setting $t = T$, we find

$$\dot{u}(T) = \sum_n c_n \dot{u}_n(T) + u(T) \delta(0).$$

(A.3)

Note that

$$c_n = \int_0^T dt u(t) u_n(t) = -\frac{1}{E_n} \int_0^T dt u(t) \left[ \frac{d^2}{dt^2} + \lambda(t) \right] u_n(t) = -\frac{u(T) \dot{u}_n(T)}{E_n},$$

(A.4)

where we used integration by parts and the equation of motion for $u$. Combining (A.4) with (A.3) we acquire the formal identity,

$$\frac{\dot{u}(T)}{u(T)} = \delta(0) - \sum_n \frac{\dot{u}_n^2(T)}{E_n}.$$  

(A.5)
It is true that the r.h.s. of this identity is ill-defined because of $\delta(0)$ and the sum of the infinite series, but it may be made sensible by demanding that it be valid for $\lambda = 0$. Indeed, since we have $u(t) = \alpha t$ with some constant $\alpha$ for $\lambda = 0$, the demand suggests

$$1 = T \delta(0) - 2 \sum_n 1. \quad (A.6)$$

This allows us to ‘renormalize’ (A.5) to get

$$\frac{\dot{u}(T)}{u(T)} = \frac{1}{T} - \sum_n \left( \frac{\dot{u}_n^2(T)}{E_n} - \frac{2}{T} \right). \quad (A.7)$$

This time the r.h.s. is well-defined, since each of the terms in the infinite series becomes small sufficiently fast when $n$ becomes large, as seen from the asymptotic behaviour of the eigenfunctions [13, 14].

Consider next the expansion of the function $\rho(t)$ appearing in (3.19) analogous to (A.2),

$$\rho(t) = \sum_n d_n u_n(t) + \rho(T) \theta(t - T), \quad (A.8)$$

where $d_n = \int_0^T dt \rho(t) u_n(t)$. Substituting (A.8) in the phase (3.19) we find

$$\Phi(b, a; \lambda) = I[\bar{x}_{cl}] + \dot{\bar{x}}_{cl}(T)\rho(T) + \frac{1}{2} \rho^2(T) \left\{ \delta(0) - \sum_n \frac{\dot{u}_n^2(T)}{E_n} \right\}. \quad (A.9)$$

Now, using the identity (A.5), and noticing that the solution $u$ may be given by $u = \beta(x_{cl} - \bar{x}_{cl})$ with $\beta$ a constant, we arrive at

$$\Phi(b, a; \lambda) = \frac{1}{2} \bar{x}_{cl} \dot{\bar{x}}_{cl} \bigg|_0^T + \frac{1}{2} \rho(T) \{ \dot{\bar{x}}_{cl}(T) + \dot{x}_{cl}(T) \}$$

$$= I_{cl}(b, T; a, 0) + \frac{1}{2} \left\{ a \dot{x}_{cl}(0) - a \dot{\bar{x}}_{cl}(0) + b \dot{x}_{cl}(T) - c \dot{\bar{x}}_{cl}(T) \right\}. \quad (A.10)$$

The relation (A.1) then follows by use of Lemma 1.
References

[1] V.P. Maslov and M.V. Fedoriuk, “Semiclassical Approximation in Quantum Mechanics”, D. Reidel Publ., London, 1981.

[2] I.M. Gel’fand and A.M. Yaglom, J. Math. Phys. 1 (1960) 48.

[3] S. Levit and U. Smilansky, Proc. Amer. Math. Soc. 21 (1977) 299; Ann. Phys. 103 (1977) 198.

[4] L.S. Schulman, in “Functional Integration and its Applications”, A.M. Arthurs, ed., Clarendon Press, Oxford, 1975.

[5] L.S. Schulman, “Techniques and Applications of Path Integration”, John Wiley & Sons, New York, 1981.

[6] C. DeWitt-Morette, Ann. Phys. 97 (1976) 367.

[7] C. DeWitt-Morette, A. Maheshwari and B. Nelson, Phys. Rep. 50C (1979) 256.

[8] C. DeWitt-Morette, B. Nelson and T.-R. Zhang, Phys. Rev. D28 (1983) 2526.

[9] G. Dangelmayr and W. Veit, Ann. Phys. 118 (1979) 108.

[10] M.V. Berry, Adv. Phys. 25 (1976) 1.

[11] E. Witten, Nucl. Phys. B149 (1979) 285.

[12] A. Jevicki, Phys. Rev. D20 (1979) 3331.

[13] R. Courant and D. Hilbert, “Methods of Mathematical Physics”, Interscience Publishers, New York, 1953.

[14] P.M. Morse and H. Feshbach, “Methods of Theoretical Physics”, McGraw-Hill Book Company, New York, 1953.

[15] J. Milnor, “Morse Theory”, Princeton University Press, Princeton, 1963.

[16] J.-M. Souriau, in “Group Theoretical Methods in Physics”, A. Janner, T. Janssen and M. Boon, eds., Lecture Notes in Physics, 50, Springer-Verlag, Berlin, 1976.

[17] P.A. Horváthy, Int. J. Theor. Phys. 13 (1979) 245.

[18] B.K. Cheng, Phys. Lett. 101A (1984) 464.

[19] R.P. Feynman and A.R. Hibbs, “Quantum Mechanics and Path Integrals”, McGraw-Hill, New York, 1965.

[20] G.A. Hagedorn, Commun. Math. Phys. 71 (1980) 77.

[21] S. Tanimura and I. Tsutsui, Ann. Phys. 258 (1997) 137.