DEEP SIGNATURE ALGORITHM FOR PATH-DEPENDENT AMERICAN OPTION PRICING

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Abstract. In this work, we study the deep signature algorithms for path-dependent FBSDEs with reflections. We follow the backward scheme in [Huré-Pham-Warin. Mathematics of Computation 89, no. 324 (2020)] for state-dependent FBSDEs with reflections, and combine it with the signature layer to solve American type option pricing problems while the payoff function depends on the whole paths of the underlying forward stock process. We prove the convergence analysis of our numerical algorithm and provide numerical example for Amerasian option under the Black-Scholes model.

1. Introduction

The recent development of deep learning and neural networks in optimal stopping [7,4,16,15], optimal control [31], and optimal switching [6] promote the developments and application of forward backward stochastic differential equations (FBSDEs) in various financial and economic fields. The idea of combining BSDE and deep learning is first introduced in [18]. There are several works further advance such an idea for coupled FBSDEs [19], path-dependent FBSDEs [13], which all relies on the forward Euler scheme of the backward process. On the other hand, the backward scheme of the backward process was recently studied in [30,22], which is then further developed to solve optimal stopping [41,5], and optimal switching [6] problems. In the current work, we focus on solving optimal stopping problems in the backward schemes in a path-dependent setting. The path-dependent American option pricing problems have been studied in [11,2] using diffusion operator integral methods, [20] by deriving analytical formulas, [3] using forward shooting grid, to list a few. The path-dependent FBSDE algorithm combined with the forward scheme has been introduced in [13] using signature layer and LSTM neural networks. We need to modify the signature layer in the current backward setting, as the LSTM structure does not fit into the backward scheme. Furthermore, we reduce the regularity assumption for the coefficients of the FBSDEs, which improves the signature layer idea introduced in [13]. We do not need to apply Taylor expansion to apply the signature approximation, which makes our algorithm more applicable to more general situations. We focus on solving path-dependent American
type option pricing problems (i.e., optimal stopping problems) by using signature layers, which naturally solve the difficulty from the path-dependent feature, improve training efficiency, and possibly enlarge the time horizon. It is reasonable to further apply our backward signature scheme together with obliquely reflected BSDEs to solve path-dependent optimal switching problems in the energy markets [9 6 8] and other related fields. This will be studied in future works. The paper is organized as below. In Section 2, we introduce the preliminary and the main algorithm. In Section 3, we show the convergence analysis for our path-dependent backward deep signature FBSDE algorithm. In Section 4, we present numerical examples for American type Asian option problems.

2. Main algorithm

In this paper we study numerical algorithms for the following reflected path-dependent decoupled FBSDE, for any $0 \leq t \leq T$,

\begin{align*}
X_t &= x + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s, \\
Y_t &= g(X_{\Lambda T}) + \int_t^T f(s, X_s, Y_s, Z_s)ds - \int_t^T Z_s dW_s + K_T - K_t, \\
Y_0 &= x, \quad 0 \leq t \leq T,
\end{align*}

(2.1)

where $K$ is an adapted non-decreasing process satisfying

$$
\int_0^T (Y_t - g(t, X_{\Lambda t})))dK_t = 0,
$$

(2.2)

and $W$ is a $d_1$-dimensional Brownian motion on a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with natural filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$. Such reflected FBSDE (2.1) is motivated by optimal stopping and American option pricing problems (e.g., [25 12]). The state-dependent and mean field version of (2.1) have been studied in [26 24 21] and the references therein. Following the idea from [22] for state-dependent reflected FBSDEs, we propose the following backward scheme in the current path-dependent setting. Firstly, we consider the following Euler scheme for the forward process $X_t$,

$$
X_{t_{i+1}}^n = X_{t_i}^n + b(t_i, X_{t_i}^n)\Delta t_i + \sigma(t_i, X_{t_i}^n)\Delta W_{t_i},
$$

for $i = 0, 1, \cdots, n-1$, $\Delta t_i = t_{i+1} - t_i = T/n$ and $X_0 = x_0$. We then apply the following signature layer to the discrete sequential points $\{X_{t_i}\}_{i=0}^n$.

**Definition 2.1 (Signature).** For a bounded variation path $x_t \in \mathbb{R}^d$, for $t \in [0, T]$, the signature of $x$ is defined as the iterated integrals of $x$. More precisely, for a word $J = (j_1, \cdots, j_k) \in \{1, \cdots, d\}^k$ with size $|J| = k$, we denote $\pi_m$ as the truncation of the signature up to degree $m$, which means

$$
\pi_m(\text{Sig}(x)_t) = \left(1, \sum_{j_1=1}^d \int_0^t dx_{t_1}^{j_1}, \cdots, \sum_{|J|=m} \int_{0\leq t_1<\cdots<t_N\leq t} dx_{t_1}^{j_1} \cdots dx_{t_N}^{j_m}\right).
$$

(2.3)

**Definition 2.2.** Consider a discrete $d$-dimensional time series $(x_{t_i})_{i=1}^n$ over time interval $[0, T]$. A signature layer of degree $m$ is a mapping from $\mathbb{R}^{d \times n}$ to $\mathbb{R}^d$, which computes $\text{Sig}_m^n$ as an output for any $x$, where $\text{Sig}_k^n$ is the truncated signature of $x$ over time interval $[0, t_k]$ of degree $m$ as follows:

$$
\text{Sig}_m^n(x)_{t_k} = \pi_m(\text{Sig}(x)_{[0,t_k]}),
$$

(2.4)
where $k \in \{1, \ldots, n\}$ and $\hat{d}$ is the dimension of the truncated signature.

Following the above definition, we denote $\tilde{n} = n/k$ as the number of segments (i.e. signature layers) for the sequence $\{X_t\}_{t=0}^n$ and denote $k \in \mathbb{N}^+$ as the number of data points in each segment. That said, we have $u_i = T/n*i*k$ for $i = 1, \ldots, \tilde{n}$. We then define
\[
X_{u_{i+1}}^n = X_{u_i}^n + b(u_i, X_{u_i}^n)\Delta u_i + \sigma(u_i, X_{u_i}^n)\Delta W_{u_i},
\]
for $i = 0, 1, \ldots, \tilde{n} - 1$. We are now ready to introduce the following backward scheme,
\[
\mathcal{U}_{u_{i+1}}^{\theta, \text{Sig}^m} = F^{\tilde{n}, \text{Sig}^m}(u_i, X_{\Lambda u_i}^n, \mathcal{U}_{u_i}^{\theta, \text{Sig}^m}, Z_{u_i}^{\theta, \text{Sig}^m}, \Delta u_i, \Delta W_{u_i}),
\]
where
\[
F^{\tilde{n}, \text{Sig}^m}(u_i, X_{\Lambda u_i}^n, \mathcal{U}_{u_i}^{\theta, \text{Sig}^m}, Z_{u_i}^{\theta, \text{Sig}^m}, \Delta u_i, \Delta W_{u_i})
= \mathcal{U}_{u_i}^{\theta, \text{Sig}^m} - f(u_i, X_{\Lambda u_i}^n, \mathcal{U}_{u_i}^{\theta, \text{Sig}^m}, Z_{u_i}^{\theta, \text{Sig}^m})\Delta u_i + Z_{u_i}^{\theta, \text{Sig}^m} \Delta W_{u_i},
\]
and we denote
\[
\mathcal{U}_{u_i}^{\theta, \text{Sig}^m} := \mathcal{R}^{\theta}(\text{Sig}^m(X^n)_{u_i}; \xi), \quad Z_{u_i}^{\theta, \text{Sig}^m} := \mathcal{R}^{\theta}(\text{Sig}^m(X^n)_{u_i}; n).
\]

Here, for $\theta = (\xi, n)$, we fix the neural network $\mathcal{R}^{\theta}$ at each time step $u_i$, for $i = 1, \ldots, \tilde{n} - 1$, with input dimension $d$ for the truncated signature $\text{Sig}^m(X^n)_{u_i}$, and output dimension $\hat{d} = 1$ for $\mathcal{U}_{u_i}$ and $\hat{d} = d_1$ for $Z_{u_i}$. The algorithm is given as below.

**Step 1.** Initialization : $\tilde{U}_N = g(X, X_T)$;

**Step 2.** For $i = \tilde{n} - 1, \ldots, 0$, given $\tilde{U}_{u_{i+1}}$, compute the minimizer of the loss function
\[
\tilde{L}_{u_i}(\theta) := \mathbb{E}\left|\tilde{U}_{u_i}(X_{\Lambda u_i}^{n+1}) - F^{\tilde{n}, \text{Sig}^m}(u_i, X_{\Lambda u_i}^n, \mathcal{U}_{u_i}^{\theta, \text{Sig}^m}, Z_{u_i}^{\theta, \text{Sig}^m}, \Delta u_i, \Delta W_{u_i})\right|^2,
\]
and set $\theta^* \in \arg\min_{\theta} \tilde{L}_{u_i}(\theta)$.

**Step 3.** Update : $\tilde{U}_{u_i} = \max\{\mathcal{U}_{u_i}^{\theta^*, \text{Sig}^m}, g(u_i, X, X_{\Lambda u_i})\}$, and set $\tilde{Z}_{u_i} = Z_{u_i}^{\theta^*, \text{Sig}^m}$.

### 3. Convergence Analysis

Throughout this section, we keep the following assumption.

**Assumption 3.1.** Let the following assumptions be in force.

- $b \in \mathbb{C}(\mathbb{R}^+ \times \mathbb{R}^d; \mathbb{R}^d), \sigma \in \mathbb{C}(\mathbb{R}^+ \times \mathbb{R}^d; \mathbb{R}^{d \times d_1}), f \in \mathbb{C}(\mathbb{R}^+ \times \mathbb{C}(\mathbb{R}^+; \mathbb{R}^d) \times \mathbb{R} \times \mathbb{R}^{d_1}; \mathbb{R}), g \in \mathbb{C}(\mathbb{R}^+ \times \mathbb{C}(\mathbb{R}^+; \mathbb{R}^d); \mathbb{R})$ are Lipschitz continuous functions with Lipschitz constant $L$ in all variables; $b(\cdot, 0), \sigma(\cdot, 0), f(\cdot, 0, 0, 0)$ and $g(\cdot, 0)$ are bounded.
- The terminal function $g$ satisfies a linear growth condition.

In this section, we show the convergence analysis for Algorithm 2.8. In particular, we first prove the convergence analysis for the backward scheme without reflection as below.

1. Initialization: $\tilde{U}_0 = g(X, X_T)$;

2. For $i = \tilde{n} - 1, \ldots, 0$, given $\tilde{U}_{u_{i+1}}$, compute the minimizer of the loss function
\[
L_{u_i}(\theta) := \mathbb{E}\left|\tilde{U}_{u_i}(X_{\Lambda u_i}^{n+1}) - F^{\tilde{n}, \text{Sig}^m}(u_i, X_{\Lambda u_i}^n, \mathcal{U}_{u_i}^{\theta, \text{Sig}^m}, Z_{u_i}^{\theta, \text{Sig}^m}, \Delta u_i, \Delta W_{u_i})\right|^2
\]
3 Update: $\hat{U}_{u_i} = U_{u_i}^{\theta^*, \text{Sig}^m}$, and set $\hat{Z}_{u_i} = Z_{u_i}^{\theta^*, \text{Sig}^m}$ with $\theta^* \in \arg \min_{\theta} L_{u_i}(\theta)$.

We first introduce the following error term $\varepsilon^Z$ as the $L^2$-regularity of $Z$,

$$
\varepsilon^Z := E\left[\sum_{i=0}^{n-1} \int_{u_i}^{u_{i+1}} |Z_t - \hat{Z}_{u_i}| dt\right],
$$

with $\hat{Z}_{u_i} := \frac{1}{\Delta u_i} E_{u_i}\left[\int_{u_i}^{u_{i+1}} Z_t dt\right],$

which means that $\varepsilon^Z = O(k\Delta t)$ if $g$ is Lipschitz (see e.g. [32]).

**Lemma 3.1.** Under Assumption [3.1] there exists a constant $C$ depending on $T$ and Lipschitz constant $L$, such that,

$$
\max_{1 \leq i \leq n-1} E[|Y_{u_i} - \hat{U}_{u_i}|] + \sum_{i=0}^{n-1} E\left[\int_{u_i}^{u_{i+1}} |Z_t - \hat{Z}_{u_i}|^2 dt\right] 
\leq C E|g(X, \leq T) - g(X^m_{\leq T})|^2 + Ckh + \varepsilon^Z + C \sum_{i=0}^{n-1} (\hat{\varepsilon}_{u_i}^N + \hat{\varepsilon}_{u_i}^Z).
$$

Here $\hat{\varepsilon}_{u_i}^N$ and $\hat{\varepsilon}_{u_i}^Z$ denotes the error introduced from the neural network (3.8) at time $u_i$.

**Proof** Following [22] Theorem 4.1 for the state-dependent backward implicit scheme, we consider the backward scheme in the current path-dependent setting as below,

$$
Y_{u_i} = E_{u_i}[Y_{u_{i+1}}] + E_{u_i}\left[\int_{u_i}^{u_{i+1}} f(t, X_{\leq t}, Y_{t}, Z_{t}) dt\right].
$$

Furthermore, we could define the implicit scheme, with $kh = k\Delta t_i = \Delta u_i = u_{i+1} - u_i$ and $\Delta W_{u_i} = W_{u_{i+1}} - W_{u_i}$,

$$
\hat{\nu}_{u_i} = E_{u_i}\left[\hat{U}_{u_{i+1}}(\text{Sig}^m(X^n)_{u_{i+1}}]\right] + f(u_i, X^n_{\leq u_i}, \hat{\nu}_{u_i}, \bar{Z}_{u_i})kh,
$$

$$
\bar{Z}_{u_i} = \frac{1}{kh} E_{u_i}\left[\hat{U}_{u_{i+1}}(\text{Sig}^m(X^n)_{u_{i+1}})\Delta W_{u_i}\right].
$$

We investigate the convergence analysis for the following quantity,

$$
\max_{i=0,1,\ldots,n} E[|Y_{u_i} - \hat{U}_{u_i}(\text{Sig}^m(X^n)_{u_i})|^2] + E\left[\sum_{i=0}^{n-1} \int_{u_i}^{u_{i+1}} |Z_t - \hat{Z}_{u_i}(\text{Sig}^m(X^n)_{u_i})| dt\right],
$$

where $\hat{U}_{u_i}(\text{Sig}^m(X^n)_{u_i}) = U_{u_i}(\text{Sig}^m(X^n)_{u_i}; \theta^*_u)$ and $\hat{Z}(\text{Sig}^m(X^n)_{u_i}) = Z(\text{Sig}^m(X^n)_{u_i}; \theta^*_u)$. Due to the Markovian property of the discretized forward process $X^n_{\leq t}$, there exists deterministic continuous functions $\nu_{u_i}, z_{u_i} : C([0, T], \mathbb{R}^d) \rightarrow \mathbb{R}$ such that,

$$
\hat{\nu}_{u_i} = \nu_{u_i}(X^n_{\leq u_i}), \quad \bar{Z}_{u_i} = z_{u_i}(X^n_{\leq u_i}).
$$

By the martingale representation theorem, there exists an $\mathbb{R}^d$-valued square integrable process $\hat{Z}_t$, such that,

$$
\hat{U}_{u_{i+1}}(\text{Sig}^m(X^n)_{u_{i+1}}) := \hat{U}_{u_{i+1}}(X^n_{\leq u_{i+1}})
= \hat{\nu}_{u_i} - f(u_i, X^n_{\leq u_i}, \hat{\nu}_{u_i}, \bar{Z}_{u_i})\Delta u_i + \int_{u_i}^{u_{i+1}} \hat{Z}_t dW_t,
$$

(3.3)
and we have,

$$
\overline{Z}_{u_i} = \frac{1}{kh} E_{u_i} \left[ \int_{u_i}^{u_{i+1}} \overline{Z}_s ds \right], \text{ for } i = 1, \ldots, n.
$$

According to the implicit scheme above, we have

$$
Y_{u_i} - \overline{V}_{u_i} = E_{u_i} \left[ Y_{u_{i+1}} - \overline{U}_{u_{i+1}} (\text{Sig}^m(X^n)_{u_{i+1}}) \right] + E_{u_i} \left[ \int_{u_i}^{u_{i+1}} f(t, X_{\wedge t}, Y_t, Z_t) - f(u_i, X^n_{\wedge u_i}, \overline{V}_{u_i}, \overline{Z}_{u_i}) dt \right].
$$

By using the Young inequality, Cauchy-Schwartz inequality, and Assumption 3.1, we get

$$
E|Y_{u_i} - \overline{V}_{u_i}|^2 \leq (1 + \gamma |\Delta u_i|) E \left[ |Y_{u_{i+1}} - \overline{U}_{u_{i+1}} (\text{Sig}^m(X^n)_{u_{i+1}})|^2 \right] + \frac{4L^2 (1 + \gamma |\Delta u_i|)}{\gamma}
$$

$$
\times \left\{ \left( \sup_{t \in [u_i, u_{i+1}]} E|X_{\wedge t} - X^n_{\wedge u_i}|^2 \right)^2 + 2\Delta u_i E|Y_{u_i} - \overline{V}_{u_i}|^2 + E \left[ \int_{u_i}^{u_{i+1}} |Z_t - \overline{Z}_{u_i}|^2 dt \right] \right\}. \tag{3.4}
$$

Here the parameter $\gamma$ is a constant to be chosen later. Following [23] [Theorem 9.6.2], for some nonnegative function $c(\Delta u_i)$ with $\lim_{\Delta u_i \to 0} c(\Delta u_i) = 0$, we obtain

$$
\sup_{t \in [u_i, u_{i+1}]} E|X_{\wedge t} - X^n_{\wedge u_i}|^2 \leq C(\Delta u_i + c(\Delta u_i)).
$$

The estimate of $(\sup_{t \in [u_i, u_{i+1}]} E|X_{\wedge t} - X^n_{\wedge u_i}|^2)^2$ is the first difference here compared to the state-dependent FBSDE setting. For $\Delta u_i$ small enough, choosing $\gamma = 8dL^2$, following Step 1 in the proof of [22] [Theorem 4.1] for the backward implicit Euler scheme,

$$
E|Y_{u_i} - \overline{V}_{u_i}|^2 \leq (1 + C|\Delta u_i|) E \left[ |Y_{u_{i+1}} - \overline{U}_{u_{i+1}} (\text{Sig}^m(X^n)_{u_{i+1}})|^2 \right] + (\Delta u_i + c(\Delta u_i))^2
$$

$$
+ CE \left[ \int_{u_i}^{u_{i+1}} |Z_t - \overline{Z}_{u_i}|^2 dt \right] + C|\Delta u_i| E \left[ \int_{u_i}^{u_{i+1}} |f(t, X_t, Y_t, Z_t)|^2 dt \right]. \tag{3.5}
$$

Applying the Young inequality, i.e.

$$(a+b)^2 \geq (1-|\Delta u_i|)a^2 + (1-|\Delta u_i|)b^2 \geq (1-|\Delta u_i|)a^2 - \frac{1}{|\Delta u_i|}b^2,$$

we have

$$
E|Y_{u_i} - \overline{V}_{u_i}|^2 = E|Y_{u_i} - \overline{U}_{u_i} (\text{Sig}^m(X^n)_{u_i}) + \overline{U}_{u_i} (\text{Sig}^m(X^n)_{u_i}) - \overline{V}_{u_i}|^2
$$

$$
\geq (1 - |\Delta u_i|) E|Y_{u_i} - \overline{U}_{u_i} (\text{Sig}^m(X^n)_{u_i})|^2 - \frac{1}{|\Delta u_i|} E|\overline{U}_{u_i} (\text{Sig}^m(X^n)_{u_i}) - \overline{V}_{u_i}|^2. \tag{3.6}
$$

Plugging estimate (3.6) into estimate (3.5), for $|\Delta u_i| = kh$ small enough, we have

$$
E|Y_{u_i} - \overline{U}_{u_i} (\text{Sig}^m(X^n)_{u_i})|^2
\leq (1 + Ckh) E|Y_{u_{i+1}} - \overline{U}_{u_{i+1}} (\text{Sig}^m(X^n)_{u_{i+1}})|^2 + (Ckh + c(kh))^2
$$

$$
+ CE \left[ \int_{u_i}^{u_{i+1}} |Z_t - \overline{Z}_{u_i}|^2 dt \right] + Ckh E \left[ \int_{u_i}^{u_{i+1}} |f(t, X_t, Y_t, Z_t)|^2 dt \right]
$$

$$
+ C\tilde{\gamma} E|\overline{U}_{u_i} - \overline{V}_{u_i} (\text{Sig}^m(X^n)_{u_i})|^2. \tag{3.7}
$$

Applying the Gronwall’s inequality, and using the error $\varepsilon Z$ for the $L^2$-regularity of $Z$ and the fact

$E \left[ \int_0^T f(t, X_{\wedge t}, Y_t, Z_t)^2 dt \right] < \infty$, and recalling the fact $Y_{u_0} = g(X_{\wedge T})$, and $\overline{U}_{u_0} (\text{Sig}^m(X^n)_{T})$ =
where we use the estimate \( C((kh) + c(kh))^2 < Ck\) when \( kh \) is small enough. Now, we are left to estimate \( \tilde{\eta} \sum_{i=0}^{\tilde{n}-1} E[\tilde{\eta}_{u_i} - \tilde{\eta}_{u_i}(X^n_{\lambda u_i})]^2 \), which is the main difference compared to Step 3 and Step 4 in [22][Theorem 4.1]. Following (3.2), we first define the following errors at each time step \( u_i \), for \( i = 1, \ldots, \tilde{n} \).

\[
\tilde{\eta}_{u_i} := \inf_k E[\tilde{\eta}_{u_i}(X^n_{\lambda u_i}) - \tilde{\eta}_{u_i}(X^n_{\lambda u_i})] \leq \tilde{\eta}_{u_i}(X^n_{\lambda u_i}),
\]

We then decompose the error in (3.1) as \( L_{u_i}(\theta) := \tilde{L}_{u_i} + E[\int_{u_i}^{u_{i+1}} |\tilde{Z}_t - \bar{Z}|^2 dt] \) for \( \theta = (\xi, \eta) \), where we define

\[
\tilde{L}_{u_i}(\theta) := \Delta u_i E[|\tilde{Z}_{u_i} - Z_{u_i}(X^n_{\lambda u_i})]^2]
\]

In the following, we show that the errors defined above are indeed small at each time \( u_i \). We first observe that

\[
\tilde{L}_{u_i}(\theta) \leq (1 + C\Delta u_i)E[\tilde{\eta}_{u_i} - \tilde{\eta}_{u_i}(X^n_{\lambda u_i})]^2 + C\Delta u_i E[|\tilde{Z}_{u_i} - Z_{u_i}(X^n_{\lambda u_i})|^2].
\]  

(3.10)

Furthermore, applying the Young inequality: \((a + b)^2 \geq (1 - \gamma\Delta u_i)a^2 + (1 - \frac{1}{\gamma\Delta u_i})b^2 \geq (1 - \gamma\Delta u_i)a^2 - \frac{1}{\gamma\Delta u_i}b^2\), we have

\[
\tilde{L}_{u_i}(\theta) \geq \Delta u_i E[|\tilde{Z}_{u_i} - Z_{u_i}(X^n_{\lambda u_i})|^2] + (1 - \gamma\Delta u_i)E[\tilde{\eta}_{u_i} - \tilde{\eta}_{u_i}(X^n_{\lambda u_i})^2] - \frac{2\Delta u_i L^2}{\gamma} (E[\tilde{\eta}_{u_i} - \tilde{\eta}_{u_i}(X^n_{\lambda u_i})]^2 + \Delta u_i E[|\tilde{Z}_{u_i} - Z_{u_i}(X^n_{\lambda u_i})|^2]).
\]

Let \( \gamma = 4L^2 \), we get

\[
\tilde{L}_{u_i}(\theta) \geq (1 - C\Delta u_i)E[\tilde{\eta}_{u_i} - \tilde{\eta}_{u_i}(X^n_{\lambda u_i})]^2 + \frac{\Delta u_i}{2} E[|\tilde{Z}_{u_i} - Z_{u_i}(X^n_{\lambda u_i})|^2].
\]

Combining with (3.10), we observe that

\[
(1 - C\Delta u_i)E[\tilde{\eta}_{u_i} - \tilde{\eta}_{u_i}(X^n_{\lambda u_i})]^2 + \frac{\Delta u_i}{2} E[|\tilde{Z}_{u_i} - Z_{u_i}(X^n_{\lambda u_i})|^2] \leq \tilde{L}_{u_i}(\theta^*)
\]

\[
\tilde{L}_{u_i}(\theta) \leq (1 + C\Delta u_i)E[\tilde{\eta}_{u_i} - \tilde{\eta}_{u_i}(X^n_{\lambda u_i})]^2 + C\Delta u_i E[|\tilde{Z}_{u_i} - Z_{u_i}(X^n_{\lambda u_i})|^2].
\]
Next, we apply the following interpolation inequality for $I$,
\[
|v_{u_i}(X_{\wedge u_i}^n) - U_{u_i}(\pi_m(\text{Sig}^m(X^n)_{u_i}; \xi))| \leq |v_{u_i}(X_{\wedge u_i}^n) - \mathcal{L}(\text{Sig}(X^n)_{u_i})| \ldots I_1
+ |\mathcal{L}(\text{Sig}(X^n)_{u_i}) - \mathcal{L}(\text{Sig}^m(X^n)_{u_i})| \ldots I_2
+ |\mathcal{L}(\text{Sig}^m(X^n)_{u_i}) - U(\text{Sig}^m(X^n)_{u_i}; \xi)| \ldots I_3,
\]
where we denote $\mathcal{L}$ as a linear functional for $v_{u_i}$. According to the universal nonlinearity proposition [e.g., [1], see also [23] Proposition A.6], for continuous function $v_{u_i}$, and continuous paths $X_{\wedge u_i}$, there exists a linear functional $\mathcal{L}$, such that
\[
I_1 = |v_{u_i}(X_{\wedge u_i}^n) - \mathcal{L}(\text{Sig}(X^n)_{u_i})| \leq \varepsilon_{u_i}^{N,v,1}.
\]
Since $\mathcal{L}$ is a linear functional, applying the remainder term estimates for the signature of the linear path generated by $\{X_{u_1}^n, \ldots, X_{u_3}^n\}$ (see e.g. [28] Lemma 4.1), we have
\[
I_2 = |\mathcal{L}(\text{Sig}(X^n)_{u_i}) - \mathcal{L}(\text{Sig}^m(X^n)_{u_i})| \leq \varepsilon_{u_i}^{N,v,2},
\]
where the error $\varepsilon_{u_i}^{N,v,2}$ goes to zero as the projection order $m$ of the signature goes to infinity. Next, applying the universality property [14], for the linear functional $\mathcal{L}$, there exists a neural network $U$, such that
\[
I_3 = |\mathcal{L}(\text{Sig}^m(X^n)_{u_i}) - U(\text{Sig}^m(X^n)_{u_i}; \xi)| \leq \varepsilon_{u_i}^{N,v,3}.
\]
Combining the above estimates, for a fixed neural network at time $u_i$, we conclude that
\[
\varepsilon_{u_i}^{N,v} \leq \varepsilon_{u_i}^{N,v,1} + \varepsilon_{u_i}^{N,v,2} + \varepsilon_{u_i}^{N,v,3}.
(3.11)
\]
Similarly, we obtain the same estimates for $\varepsilon_{u_i}^{N,z}$, namely,
\[
\varepsilon_{u_i}^{N,z} \leq \varepsilon_{u_i}^{N,z,1} + \varepsilon_{u_i}^{N,z,2} + \varepsilon_{u_i}^{N,z,3}.
(3.12)
\]
From now on, we will refer to (3.8) as the overall error coming from the estimates (3.11). For $\Delta u_i$ small enough, we end up with
\[
\mathbb{E}|\hat{Y}_{u_i} - U_{u_i}(\text{Sig}^m(X^n)_{u_i}; \xi)|^2 + \Delta u_i \mathbb{E}|\hat{Z}_{u_i} - Z_{u_i}(\text{Sig}^m(X^n)_{u_i}; \eta)|^2 \leq C(\varepsilon_{u_i}^{N,v} + \Delta u_i \varepsilon_{u_i}^{N,z}).
(3.13)
\]
Applying the error defined in (3.8) and Step 3, Step 4 in [22] Theorem 4.1], we have
\[
\max_{t=0,1,\ldots,n-1} \mathbb{E}|Y_{u_i} - \hat{U}_{u_i}(\text{Sig}(X)_{u_i})|
\leq C\mathbb{E}|g(X_{\wedge T}) - g(X_{\wedge T})|^2 + Ck\triangle + Cn \sum_{i=0}^{n-1} (\tilde{n}_i \varepsilon_{u_i}^{N,v} + \varepsilon_{u_i}^{N,z}).
\]
The estimates for $\mathbb{E}[\int_{u_i}^{u_{i+1}} |Z_t - \hat{Z}_{u_i}|^2 dt]$ follows from the following observation,
\[
\mathbb{E}[\int_{u_i}^{u_{i+1}} |Z_t - \hat{Z}_{u_i}|^2 dt] \leq 2\mathbb{E}[\int_{u_i}^{u_{i+1}} |Z_t - \hat{Z}_{u_i}|^2 dt] + 2\Delta u_i \mathbb{E}|\hat{Z}_{u_i} - \hat{Z}_{u_i}(\text{Sig}^m(X)_{u_i})|^2
\leq \mathcal{J}_1 + \mathcal{J}_2.
\]
The estimate of $J_2$ follows from error (3.8), the estimate of $J_1$ is similar to Step 5 in [22][Theorem 4.1] by using our neural network error defined in 3.8:

$$
\sum_{i=0}^{n-1} \mathbb{E}\left[|Z_t - \tilde{Z}_{u_i}|^2 dt\right] \leq C \mathbb{E}|g(X_{\cdot T}) - g(X_{\cdot T}^n)|^2 + C \varepsilon^2 + C \sum_{i=0}^{n-1} (\bar{n} \varepsilon_{u_i}^{N_1} + \varepsilon_{u_i}^{N_2}).
$$

Combining the above two estimates, we complete the proof.

3.1. Convergence Analysis: variational inequality. We present the convergence analysis for Algorithm 2.8 which produces the following scheme,

$$
\begin{cases}
\bar{Y}_{u_i} = \mathbb{E}_{u_i}\left[\tilde{U}_{u_{i+1}}(\text{Sig}^m(X^n_{u_{i+1}}))\right] + f(u_i, X^n_{\cdot T}, \bar{V}_{u_i}, \tilde{Z}_{u_i}) \Delta u_i \\
\bar{Z}_{u_i} = \frac{1}{kh} \mathbb{E}_{u_i}\left[\tilde{U}_{u_{i+1}}(\text{Sig}^m(X^n_{u_{i+1}})) \Delta W_{u_i}\right] \\
\bar{V}_{u_i} := \max\{\bar{V}_{u_i}, g(u_i, X^n_{\cdot T})\}
\end{cases}
$$

(3.14)

Similar to (3.8) we consider the error below,

$$
\begin{align*}
\varepsilon_{u_i}^{N_{\tilde{U}}} &:= \inf_\xi \mathbb{E}[\tilde{U}_{u_i}(X^n_{\cdot T}, \xi)]^2; \\
\varepsilon_{u_i}^{N_{\tilde{Z}}} &:= \inf_\eta \mathbb{E}[\tilde{Z}_{u_i}(X^n_{\cdot T}, \eta)]^2,
\end{align*}
$$

(3.15)

which could be bounded as shown in (3.8).

**Theorem 3.2.** Let Assumption 3.7 hold. There exists a constant $C > 0$, such that,

$$
\max_{0 \leq i \leq \bar{n} - 1} \mathbb{E}[|Y_{u_i} - \tilde{U}_{u_i}(\text{Sig}^m(X^n_{u_i})|] + \sum_{i=0}^{n-1} \mathbb{E}\left[|Z_t - \tilde{Z}_{u_i}|^2 dt\right] \leq C (k + \sum_{i=0}^{n-1} (\bar{n} \varepsilon_{u_i}^{N_{\tilde{U}}} + \varepsilon_{u_i}^{N_{\tilde{Z}}})).
$$

**Proof** We first introduce the discrete-time approximation of the path-dependent reflected BSDE, according to the well-posedness for a general oblique reflected BSDE in [17][Section 2.2], in our current situation, $Y_t$ is one-dimensional and the path-dependence of the generator $f$ on $X_{\cdot T}$ does not affect the measurability assumption in [17][Section 2.2], the existence of the RBSDE follows directly. The rate of convergence follows similar to the ones in [10], where they derive the convergence for relaxed condition on the coefficient in the state-dependent case. The dependence on the path of $X$ will not introduce extra difficulty. To be precise, adding the path-dependence of the generator $f$ and terminal condition $g$ on $(X_{\cdot T})_{0 \leq t \leq T}$, we can update the discretized scheme in [17][equation (3.2)] (or [10][equation (1.5)] and [27]), as long as $f$ and $g$ are Lipschitz continuous (Assumption 3.1) for all the variables and $X$ itself is Markovian, we get the similar estimates as below,

$$
\max_{i=0, \ldots, \bar{n} - 1} \mathbb{E}[|Y_{u_i} - Y^n_{u_i}|^2 = O(kh), \quad \sum_{i=0}^{n-1} \int_{u_i}^{u_{i+1}} |Z_t - Z^n_{u_i}|^2 dt = O(\sqrt{k}h).
$$

(3.17)
The rest of the proof follows closely to [22] Theorem 4.4] after the following adjustment,
\[ \hat{U}_{u_i}(\text{Sig}^m(X^n)_{u_i}) = \max \{ \mathcal{U}(\text{Sig}^m(X^n)_{u_i}, \xi^*), g(u_i, X_{\wedge u_i}) \}. \] (3.18)

First, we observe that
\[
\bar{Y}^{n} - \bar{V}_{u_i} = E_n[\bar{Y}^{n} - \hat{U}_{u_i}(\text{Sig}^m(X^{n})_{u_i+1})] + \Delta u_i(f(u_i, X_{\wedge u_i}, \bar{Y}^{n}, Z_{u_i}) - f(u_i, X_{\wedge u_i}^{n}, \bar{V}_{u_i}, \bar{Z}_{u_i}))
\]
for \( i = 0, 1, \ldots, \bar{n} - 1 \). Similar to (3.4), we further get
\[
E|\bar{Y}^{n} - \bar{V}_{u_i}|^2 \leq (1 + \gamma \Delta u_i) E[|Y_{u_i+1} - \hat{U}_{u_i+1}(\text{Sig}^m(X^n)_{u_i+1})|^2]
\]
\[ + 2 \frac{L^2}{\gamma} (1 + \gamma \Delta u_i) [\Delta u_i E|\bar{Y}^{n} - \bar{V}_{u_i}|^2 + \Delta u_i E|Z_{u_i} - \bar{Z}_{u_i}|^2]. \]

Notice that (3.14) and (3.16) share the same \( X_{\wedge u_i} \) dependence for \( f \), we get the following similar estimates as in [22] Theorem 4.4,
\[
\Delta u_i E|Z_{u_i} - \bar{Z}_{u_i}|^2 \leq 2d [E|Y_{u_i+1} - \hat{U}_{u_i+1}(\text{Sig}^m(X^n)_{u_i+1})|^2
\]
\[ - E|E_n[Y_{u_i+1} - \hat{U}_{u_i+1}(\text{Sig}^m(X^n)_{u_i+1})]|^2]. \]

Let \( \gamma = 4dL^2 \), for \( \Delta u_i \) small enough, we get
\[
E|\bar{Y}^{n} - \bar{V}_{u_i}|^2 \leq (1 + Ch) E|Y_{u_i+1} - \hat{U}_{u_i+1}(\text{Sig}^m(X^n)_{u_i+1})|^2.
\]

Following the steps for deriving the estimates in (3.7), we have
\[
E|\bar{Y}^{n} - \bar{U}_{u_i}(\text{Sig}^m(X^n)_{u_i}; \xi)|^2 \leq (1 + Ch) E|Y_{u_i+1} - \hat{U}_{u_i+1}(\text{Sig}^m(X^n)_{u_i+1})|^2
\]
\[ + Cn E|\bar{Y}_{u_i} - \bar{U}_{u_i}(\text{Sig}^m(X^n)_{u_i}; \xi)|^2. \] (3.20)

Similar to (3.2) and (3.3), applying the Martingale representation theorem for \( \hat{U}_{u_i+1}(\text{Sig}^m(X^n)_{u_i+1}) \), there exists an \( \mathbb{R}^d \)-valued square integrable process \( \bar{Z} \) such that
\[
\hat{U}_{u_i+1}(\text{Sig}^m(X^n)_{u_i+1}) = \bar{V}_{u_i} - f(u_i, X^n_{\wedge u_i}, \bar{V}_{u_i}, \bar{Z}_{u_i}) \Delta u_i + \int_{u_i}^{u_{i+1}} \bar{Z}_t^T dW_t.
\]

Following (3.9) in the previous case, in order to estimate the second term on the right hand side of (3.20), we denote \( \tilde{L}_{u_i} = \bar{L}_{u_i} + E[\int_{u_i}^{u_{i+1}} |\bar{Z}_t - \bar{Z}_{u_i}|^2 dt] \), and
\[
\tilde{L}_{u_i}(\theta) := \Delta u_i E |\bar{Z}_{u_i} - Z_{u_i}(\text{Sig}^m(X^n)_{u_i}; \eta)|^2 + E|\bar{V}_{u_i} - \bar{U}_{u_i}(\text{Sig}^m(X^n)_{u_i}; \xi)
\]
\[ + [f(u_i, X^n_{\wedge u_i}, \bar{U}_{u_i}(\text{Sig}^m(X^n)_{u_i}; \xi), Z_{u_i}(\text{Sig}^m(X^n)_{u_i}; \eta)) - f(u_i, X^n_{\wedge u_i}, \bar{V}_{u_i}, \bar{Z}_{u_i})] \Delta u_i |^2. \]

Following the steps in (3.10), (3.11), (3.13) and applying the new error in (3.15), recalling the fact \( \hat{U}(\text{Sig}^m(X^n)_{u_i}) = \max \{ \mathcal{U}(\text{Sig}^m(X^n)_{u_i}; \xi^*); g(X^n_{\wedge u_i}, u_i) \}, Y^n_{u_i} = \max \{ Y^n_{u_i}; g(X_{\wedge u_i}, u_i) \} \), together with (3.20) and \( |\max(a, c) - \max(b, c)| \leq |a - b| \), we have
\[
E|Y_{u_i} - \hat{U}_{u_i}(\text{Sig}^m(X^n)_{u_i})|^2 \leq (1 + Ch) E|Y_{u_i+1} - \hat{U}_{u_i+1}(\text{Sig}^m(X^n)_{u_i+1})|^2 + Cn (\varepsilon_{u_i}^N, \Delta u_i \varepsilon_{u_i}^N).
\]

By induction, we conclude
\[
\max_{i=0, \ldots, \bar{n}} E|Y_{u_i} - \hat{U}_{u_i}(\text{Sig}^m(X^n)_{u_i})|^2 \leq C \sum_{i=0}^{\bar{n}-1} (\bar{n} \varepsilon_{u_i}^N, \Delta u_i \varepsilon_{u_i}^N).
\] (3.21)

Combining (3.17) and (3.21), we finish the proof.
We consider the following Amerasian option under the Black-Scholes model with $d$ stocks $X_1, \ldots, X_d$. The risk neural dynamics are given by
\begin{equation}
    dX^i_t = rX^i_t dt + \sigma_i X^i_t dW^i_t, \quad X_0^i = x_0^i, \quad i = 1, \ldots, d, \tag{4.1}
\end{equation}
where $r$ is the risk free rate, $\sigma_i$ is the volatility, and $W^1, \ldots, W^d$ are independent standard Brownian motions. Given a vector of weights $(w_i)_{i=1,\ldots,d}$, the payoff of the basket Amerasian call option at strike price $K$ is
\begin{equation}
    g(X, T) = \left( \sum_{i=1}^{d} w_i \frac{1}{T} \int_0^T X^i_t dt - K \right)^+ \tag{4.2}
\end{equation}
Experiment results for Bermudan options are summarized in Table 1 where the parameters are chosen as:
\begin{align*}
    X_0^i &= 100, \quad r = 5\%, \quad \sigma_i = 0.15, \quad w_i = \frac{1}{d}, \quad T = 1, \quad K = 100, \quad n = 1000, \quad \forall i \in \{1, \ldots, d\}.
\end{align*}
Recall from Section 2, we denote $n$ as the number of Euler scheme steps, and $\tilde{n}$ as the number of segments for the signature layers (this also corresponds to the number of exercise times in the Bermudan approximation, which explains the monotonicity in the table with respect to this variable). Bermuda options are a restricted form of the American option that allows for early exercise but only at set dates. In addition, we use a fully connected feedforward network with five hidden layers with 16 neurons on each hidden layer. We choose tanh as the activation function for the hidden layers, and identity function as the activation function for the output layer. We use Adam Optimizer, implemented in TensorFlow and mini-batch with 100 trajectories for the stochastic gradient descent.\footnote{The code can be found in the following URL link: \url{https://github.com/zhaoyu-zhang/sig_american}. The desktop we used in this study is equipped with an i7-8700 CPU. For all the examples in this paper, we generated in a total of 10,000 paths for the forward processes. 100 paths were used to test, and the rest were used to train the neural network.}

The most related results that we could compare are from \cite{Pham}, which proposed a series expansion formula for the price of the Amerasian option. However, the approach is not able to price multi-dimensional Amerasian option. While choosing the same parameters but $\tilde{n} = 2024$, the American option price from \cite{Pham} is 4.6643, and the 95\% confidence interval is (4.5506, 4.7780). As shown in Table 1, in one dimensional case, the option price approaches to the result in \cite{Pham} as the number of segments $\tilde{n}$ increases. Moreover, our 95\% confidence interval is tighter, which implies the price obtained from our price is more accurate. Our scheme can price high dimensional path-dependent American type options. Applying the Jensen’s inequality for $E[\cdot]$ and plugging in the parameters, we observe that $E[e^{-rT} \left( \sum_{i=1}^{d} \frac{1}{d} \int_0^T X^i_t dt - K \right)^+] \geq (e^{-rT} E[\sum_{i=1}^{d} \frac{1}{d} \int_0^T X^i_t dt - K])^+ + 100e^{-r} \left( \frac{1}{d} \sum_{i=1}^{d} \int_0^1 E[e^{-rT} \left( \int_0^T X^i_t dt - K \right)^+] \right)$ for the European option price, which also provides a lower bound for the American option. We can find an upper bound from the following observation: For any stopping time $\tau \in [0, T]$, applying Jensen’s inequality for the summation, we have $e^{-r\tau} \left( \sum_{i=1}^{d} \frac{1}{d} \int_0^\tau X^i_t dt - K \right)^+ \leq e^{-r\tau} \frac{1}{d} \sum_{i=1}^{d} \left( \frac{1}{d} \int_0^\tau X^i_t dt - K \right)^+$. After taking expectations and maximizing over $\tau$ and using the fact that the stocks are identically distributed we see that $d = 1$ case provides an upper bound. Moreover, the value of the option is monotonically decreasing with respect $d$, which can be seen again using Jensen’s inequality: for example

4. Numerical Example
the value for the option 2d dimensions can be bounded by the d dimensional counterpart using
\[ e^{-r\tau}(\sum_{i=1}^{2d} \frac{1}{\tau(2d)} \int_0^{\tau} X_i^d dt - K)^+ \leq \frac{1}{2} e^{-r\tau}(\sum_{i=1}^{d} \frac{1}{\tau} \int_0^{\tau} X_i^d dt - K)^+ + \frac{1}{2} e^{-r\tau}(\sum_{i=d+1}^{2d} \frac{1}{\tau} \int_0^{\tau} X_i^d dt - K)^+ \]. Furthermore, since the option price is decreasing in d, we can take the limit in d first and use Law of Large Numbers and Merton’s no-early exercise theorem to conclude that the option price, in fact, converges as \( d \to \infty \) to \( X_0 e^{-r\tau}(e^{r\tau}/r - 1/r - 1)^+ = 2.42 \). Our calculations in Table (1) are in line with these theoretical observations.

The running time is also reported in Table (1), which shows that the computation time is not exponential in dimension. The limitation of our signature algorithm is due to the fact that the dimension of the signature input grows exponentially in dimension. To reduce the limitation from the dimension of the signature is interesting on its own, we leave this for future studies. Indeed, assuming that we have enough storage to save the signature for the sample paths \( \{X_{t_i}\}_{i=1,\ldots,n} \) (offline), the training time of neural network for our algorithm is similar to (even shorter compared to) the state-dependent deep learning algorithms (e.g., [22]). To be more precise, our algorithm is tuned for \( \tilde{n} \) independent neural networks while the deep learning backward algorithms ((e.g., [22])) is tuned for \( n \) (i.e. number of steps in Euler scheme) independent neural networks.

| d   | Price CI            | Time  | d   | Price CI            | Time  |
|-----|--------------------|-------|-----|--------------------|-------|
| 1   | 4.2177 [4.1990, 4.2363] | 322.40s | 10  | 2.2994 [2.2837, 2.3152] | 620.53s |
| 1   | 4.4471 [4.4286, 4.4656] | 604.36s | 20  | 2.5006 [2.4688, 2.5323] | 1001.80s |
| 1   | 4.6172 [4.5786, 4.6558] | 1391.24s | 50  | 2.6712 [2.6042, 2.740] | 2108.54s |
| 5   | 2.6120 [2.5954, 2.6287] | 514.98s | 10  | 2.1850 [2.1608, 2.2091] | 904.01s |
| 5   | 2.8058 [2.7478, 2.8639] | 849.39s | 20  | 2.3313 [2.3007, 2.3619] | 1467.11s |
| 5   | 3.0350 [2.9626, 3.1075] | 1789.29s | 50  | 2.3481 [2.3549, 2.6134] | 3153.25s |

Table 1. Estimate of Bermudan option price \( Y_0 \) from the average over 100 independent runs, and 95% confidence interval (CI) of \( Y_0 \) are reported.

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