BRAUER-SEVERI MOTIVES AND DONALDSON-THOMAS INVARIANTS OF QUANTIZED THREEFOLDS

LIEVEN LE BRUYN

Abstract. Motives of Brauer-Severi schemes of Cayley-smooth algebras associated to homogeneous superpotentials are used to compute inductively the motivic Donaldson-Thomas invariants of the corresponding Jacobian algebras. This approach can be used to test the conjectural exponential expressions for these invariants, proposed in [3]. As an example we confirm the second term of the conjectured expression for the motivic series of the homogenized Weyl algebra.

1. Introduction

We fix a homogeneous degree $d$ superpotential $W$ in $m$ non-commuting variables $X_1, \ldots, X_m$. For every dimension $n \geq 1$, $W$ defines a regular functions, sometimes called the Chern-Simons functional

$$\text{Tr}(W) : \mathbb{M}_{m,n} = \bigoplus_{1 \leq i \leq m} \mathbb{M}_n(C) \to \mathbb{C}$$

obtained by replacing in $W$ each occurrence of $X_i$ by the $n \times n$ matrix $n$ the $i$-th component, and taking traces.

We are interested in the (naive, equivariant) motives of the fibers of this functional which we denote by

$$\mathbb{M}^W_{m,n}(\lambda) = \text{Tr}(W)^{-1}(\lambda).$$

Recall that to each isomorphism class of a complex variety $X$ (equipped with a good action of a finite group of roots of unity) we associate its naive equivariant motive $[X]$ which is an element in the ring $K^0(\text{Var}_C)[L^{-1/2}]$ (see [4] or [3]) and is subject to the scissor- and product-relations

$$[X] - [Z] = [X - Z] \text{ and } [X].[Y] = [X \times Y]$$

whenever $Z$ is a Zariski closed subvariety of $X$. A special element is the Lefschetz motive $L_1 = [\mathbb{A}_1, id]$ and we recall from [12] Lemma 4.1 that $[GL_n] = \prod_{k=0}^{n-1}(L^n - L^k)$ and from [3] 2.2 that $[\mathbb{A}^n, \mu_k] = L^n$ for a linear action of $\mu_k$ on $\mathbb{A}^n$. This ring is equipped with a plethystic exponential $\text{Exp}$, see for example [2] and [4].

The representation theoretic interest of the degeneracy locus $Z = \{d\text{Tr}(W) = 0\}$ of the Chern-Simons functional is that it coincides with the scheme of $n$-dimensional representations

$$Z = \text{rep}_n(R_W) \text{ where } R_W = \frac{\mathbb{C}[X_1, \ldots, X_m]}{(\partial_{X_i}(W) : 1 \leq i \leq m)}$$

of the corresponding Jacobi algebra $R_W$ where $\partial_{X_i}$ is the cyclic derivative with respect to $X_i$. As $W$ is homogeneous it follows from [4] Thm. 1.3 (or [1]) if the
superpotential allows 'a cut') that its virtual motive is equal to

\[ \left[ \text{rep}_n(R_W) \right]_{\text{virt}} = \mathbb{L} \cdot \frac{e^{m^2}}{\mu_d} \left( [M^{W}_{m,n}(0)] - [M^{W}_{m,n}(1)] \right) \]

where \( \mu \) acts via \( \mu_d \) on \( m^{W}_{m,n}(1) \) and trivially on \( m^{W}_{m,n}(0) \). These virtual motives can be packaged together into the motivic Donaldson-Thomas series

\[ U_W(t) = \sum_{n=0}^{\infty} \mathbb{L} \cdot \frac{e^{m^2}}{\mu_d - \mathbb{L}^{1/2} - \mathbb{L}^{-1/2} - 1} t^n \]

In [3] A. Cazzaniga, A. Morrison, B. Pym and B. Szendrői conjecture that this generating series has an exponential expression involving simple rational functions of virtual motives determined by representation theoretic information of the Jacobi algebra \( R_W \)

\[ U_W(t) = \exp \left( - \sum_{i=1}^{k} \frac{M_i}{L^{1/2} - L^{-1/2} - 1 - t^{m_i}} \right) \]

where \( m_1, \ldots, m_k \) are the dimensions of simple representations of \( R_W \) and \( M_i \in M_C \) are motivic expressions without denominators, with \( M_1 \) the virtual motive of the scheme parametrizing (simple) 1-dimensional representations. Evidence for this conjecture comes from cases where the superpotential admits a cut and hence one can use dimensional reduction, introduced by A. Morrison in [12], as in the case of quantum affine three-space [3].

The purpose of this paper is to introduce an inductive procedure to test the conjectural exponential expressions given in [3] in other interesting cases such as the homogenized Weyl algebra and elliptic Sklyanin algebras. To this end we introduce the following quotient of the free necklace algebra on \( m \) variables

\[ T^{W}_m(\lambda) = \frac{\mathbb{C}(X_1, \ldots, X_m) \otimes \text{Sym}(V_m)}{(W - \lambda)}, \quad \text{where} \quad V_m = \frac{\mathbb{C}(X_1, \ldots, X_m)}{[\mathbb{C}(X_1, \ldots, X_m), \mathbb{C}(X_1, \ldots, X_m)]_{\text{rect}}} \]

is the vectorspace space having as a basis all cyclic words in \( X_1, \ldots, X_m \). Note that any superpotential is an element of \( \text{Sym}(V_m) \). Substituting each \( X_k \) by a generic \( n \times n \) matrix and each cyclic word by the corresponding trace we obtain a quotient of the trace ring of \( m \) generic \( n \times n \) matrices

\[ T^{W}_{m,n}(\lambda) = \frac{T^{W}_m(\lambda)}{(Tr(W) - \lambda)} \quad \text{with} \quad M^{W}_{m,n}(\lambda) = \text{rep}_n(T^{W}_{m,n}) \]

such that its scheme of trace preserving \( n \)-dimensional representations is isomorphic to the fiber \( M^{W}_{m,n}(\lambda) \). We will see that if \( \lambda \neq 0 \) the algebra \( T^{W}_{m,n}(\lambda) \) shares many ringtheoretic properties of trace rings of generic matrices, in particular it is a Cayley-smooth algebra, see [10]. As such one might hope to describe \( M^{W}_{m,n}(\lambda) \) using the Luna stratification of the quotient and its fibers in terms of marked quiver settings given in [10]. However, all this is with respect to the étale topology and hence useless in computing motives.

For this reason we consider the Brauer-Severi scheme of \( T^{W}_{m,n}(\lambda) \), as introduced by M. Van den Bergh in [17] and further investigated by M. Reineke in [10], which are quotients of a principal \( GL_n \)-bundles and hence behave well with respect to motives. More precisely, the Brauer-Severi scheme of \( T^{W}_{m,n}(\lambda) \) is defined as

\[ BS^{W}_{m,n}(\lambda) = \{ (v, \phi) \in C^n \times \text{rep}_n(T^{W}_{m,n}(\lambda)) \mid \phi(T^{W}_{m,n}(\lambda))v = C^n \}/GL_n \]
and their motives determine inductively the motives in the Donaldson-Thomas series. In Proposition 5 we will show that
\[(
\mathbb{L}^n - 1
\frac{[M_{m,n}^W(0)] - [M_{m,n}^W(1)]}{[GL_n]}
\]
is equal to
\[BS_{m,n}^W(0) - BS_{m,n}^W(1) + \sum_{k=1}^{n-1} \frac{L^{(m-1)k(n-k)}}{[GL_{n-k}]} ([BS_{m,k}^W(0)] - [BS_{m,k}^W(1)]([M_{m,k}^W(0)] - [M_{m,k}^W(1)])
\]

In section 4 we will compute the first two terms of $U_W(t)$ in the case of the quantized 3-space in a variety of ways. In the final section we repeat the computation for the homogenized Weyl algebra and show that it coincides with the conjectured expression of [3]. In a forthcoming paper [11] we will compute the first two terms of the series for elliptic Sklyanin algebras both in the generic case and the case of 2-torsion points.

Acknowledgement: I would like to thank Brent Pym for stimulating conversations concerning the results of [3] and Balazs Szendrői for explaining the importance of the monodromy action (which was lacking in a previous version) and for sharing his calculations on the Exp-expressions of [3]. I am grateful to Ben Davison for pointing out a computational error in summing up the terms in the homogenized Weyl algebra case and explaining the equality with the conjectured motive.

2. Brauer-Severi motives

With $T_{m,n}$ we will denote the trace ring of $m$ generic $n \times n$ matrices. That is, $T_{m,n}$ is the $C$-subalgebra of the full matrix-algebra $M_n(C[x_{ij}(k) \mid 1 \leq i, j \leq n, 1 \leq k \leq m])$ generated by the $m$ generic matrices
\[
X_k = \begin{bmatrix}
x_{11}(k) & \cdots & x_{1n}(k) \\
\vdots & \ddots & \vdots \\
x_{n1}(k) & \cdots & x_{nn}(k)
\end{bmatrix}
\]

Together with all elements of the form $Tr(M)1_n$, where $M$ runs over all monomials in the $X_i$. These algebras have been studied extensively by ringtheorists in the 80ties and some of the results are summarized in the following result.

**Proposition 1.** Let $T_{m,n}$ be the trace ring of $m$ generic $n \times n$ matrices, then

1. $T_{m,n}$ is an affine Noetherian domain with center $Z(T_{m,n})$ of dimension $(m-1)n^2 + 1$ and generated as $C$-algebra by the $Tr(M)$ where $M$ runs over all monomials in the generic matrices $X_k$.
2. $T_{m,n}$ is a maximal order and a noncommutative UFD, that is all twosided prime ideals of height one are generated by a central element and $Z(T_{m,n})$ is a commutative UFD which is a complete intersection if and only if $n = 1$ or $(m,n) = (2,2), (2,3)$ or $(3,2)$.
3. $T_{m,n}$ is a reflexive Azumaya algebra unless $(m,n) = (2,2)$, that is, every localization at a central height one prime ideal is an Azumaya algebra.

**Proof.** For (1) see for example [13] or [15]. For (2) see for example [8], for (3) for example [7].
A Cayley-Hamilton algebra of degree \( n \) is a \( \mathbb{C} \)-algebra \( A \), equipped with a linear trace map \( tr : A \rightarrow A \) satisfying the following properties:

1. \( tr(a).b = b.tr(a) \)
2. \( tr(a.b) = tr(b.a) \)
3. \( tr(tr(a).b) = tr(a).tr(b) \)
4. \( tr(a) = n \)
5. \( \chi_n^{(1)}(a) = 0 \) where \( \chi_n^{(1)}(t) \) is the formal Cayley-Hamilton polynomial of degree \( n \), see [14].

For a Cayley-Hamilton algebra \( A \) of degree \( n \) it is natural to look at the scheme \( \text{trep}_n(A) \) of all trace preserving \( n \)-dimensional representations of \( A \), that is, all trace preserving algebra maps \( A \rightarrow M_n(\mathbb{C}) \). A Cayley-Hamilton algebra \( A \) of degree \( n \) is said to be a smooth Cayley-Hamilton algebra if \( \text{trep}_n(A) \) is a smooth variety. Procesi has shown that these are precisely the algebras having the smoothness property of allowing lifts modulo nilpotent ideals in the category of all Cayley-Hamilton algebras of degree \( n \), see [14]. The étale local structure of smooth Cayley-Hamilton algebras and their centers have been extensively studied in [10].

**Proposition 2.** Let \( W \) be a homogeneous superpotential in \( m \) variables and define the algebra

\[
T_{m,n}^W(\lambda) = \frac{T_{m,n}}{(Tr(W) - \lambda)} \quad \text{then} \quad M_{m,n}^W(\lambda) = \text{trep}_n(T_{m,n}^W(\lambda))
\]

If \( Tr(W) - \lambda \) is irreducible in the UFD \( \mathbb{Z}(T_{m,n}) \), then for \( \lambda \neq 0 \)

1. \( T_{m,n}^W(\lambda) \) is a reflexive Azumaya algebra.
2. \( T_{m,n}^W(\lambda) \) is a smooth Cayley-Hamilton algebra of degree \( n \) and of Krull dimension \( (m - 1)n^2 \).
3. \( T_{m,n}^W(\lambda) \) is a domain.
4. The central singular locus is the non-Azumaya locus of \( T_{m,n}^W(\lambda) \) unless \( (m,n) = (2,2) \).

**Proof.** (1) : As \( M_{m,n}^W(\lambda) = \text{trep}_n(T_{m,n}^W(\lambda)) \) is a smooth affine variety for \( \lambda \neq 0 \) (due to homogeneity of \( W \)) on which \( GL_n \) acts by automorphisms, we know that the ring of invariants,

\[
\mathbb{C}[\text{trep}_n(T_{m,n}^W(\lambda))]^{GL_n} = Z(T_{m,n}^W(\lambda))
\]

which coincides with the center of \( T_{m,n}^W(\lambda) \) by e.g. [10] Prop. 2.12, is a normal domain. Because the non-Azumaya locus of \( T_{m,n} \) has codimension at least 3 (if \( (m,n) \neq (2,2) \)) by [7], it follows that all localizations of \( T_{m,n}^W(\lambda) \) at height one prime ideals are Azumaya algebras. Alternatively, using (2) one can use the theory of local quivers as in [10].

(2) : That the Cayley-Hamilton degree of the quotient \( T_{m,n}^W(\lambda) \) remains \( n \) follows from the fact that \( T_{m,n} \) is a reflexive Azumaya algebra and irreducibility of \( Tr(W) - \lambda \). Because \( M_{m,n}^W(\lambda) = \text{trep}_n(T_{m,n}^W(\lambda)) \) is a smooth affine variety, \( T_{m,n}^W(\lambda) \) is a smooth Cayley-Hamilton algebra. The statement on Krull dimension follows from the fact that the Krull dimension of \( T_{m,n} \) is known to be \( (m - 1)n^2 + 1 \).

(3) : After taking determinants, this follows from factoriality of \( Z(T_{m,n}) \) and irreducibility of \( Tr(W) - \lambda \).

(4) : This follows from the theory of local quivers as in [10]. The most general non-simple representations are of representation type \((1,a;1,b)\) with the dimensions
of the two simple representations \( a, b \) adding up to \( n \). The corresponding local quiver is

\[
(m-1)a^2 + \underbrace{\cdots} + (m-1)ab + (m-1)b^2
\]

and as \((m - 1)ab \geq 2\) under the assumptions, it follows that the corresponding singular point is singular. \( \Box \)

Let us define for all \( k \leq n \) and all \( \lambda \in \mathbb{C} \) the locally closed subscheme of \( \mathbb{C}^n \times \text{trep}_n(T^W_{m,n}(\lambda)) \)

\[
X_{k,n,\lambda} = \{(v, \phi) \in \mathbb{C}^n \times \text{trep}_n(T^W_{m,n}(\lambda)) \mid \text{dim}_\mathbb{C}(\phi(T^W_{m,n}(\lambda)).v) = k\}
\]

Sending a point \((v, \phi)\) to the point in the Grassmannian \( \text{Gr}(k, n) \) determined by the \( k \)-dimensional subspace \( V = \phi(T^W_{m,n}(\lambda)).v \subset \mathbb{C}^n \) we get a Zariskian fibration as in [12]

\[
X_{k,n,\lambda} \longrightarrow \text{Gr}(k, n)
\]

To compute the fiber over \( V \) we choose a basis of \( \mathbb{C}^n \) such that the first \( k \) base vectors span \( V = \phi(T^W_{m,n}(\lambda)).v \). With respect to this basis, the images of the generic matrices \( X_i \) all are of the following block-form

\[
\phi(X_i) = \begin{bmatrix}
\phi_k(X_i) & \sigma(X_i) \\
0 & \phi_{n-k}(X_i)
\end{bmatrix}
\]

with

\[
\begin{align*}
\phi_k(X_i) & \in M_k(\mathbb{C}) \\
\phi_{n-k}(X_i) & \in M_{n-k}(\mathbb{C}) \\
\sigma(X_i) & \in M_{n-k \times k}(\mathbb{C})
\end{align*}
\]

Using these matrix-form it is easy to see that

\[
\text{Tr}(\phi(W(X_1, \ldots, X_m))) = \text{Tr}(\phi_k(W(X_1, \ldots, X_m))) + \text{Tr}(\phi_{n-k}(W(X_1, \ldots, X_m)))
\]

That is, if \( \phi_k \in \text{trep}_k(T^W_{m,n}(\mu)) \) then \( \phi_{n-k} \in \text{trep}(T^W_{m,n-k}(\lambda - \mu)) \) and moreover we have that \((v, \phi_k) \in X_{k,n,\mu} \). Further, the \( m \) matrices \( \sigma(X_i) \in M_{n-k \times k}(\mathbb{C}) \) can be taken arbitrary. Rephrasing this in motives we get

\[
[X_{k,n,\lambda}] = L^{mk(n-k)}[\text{Gr}(k, n)] \sum_{\mu \in \mathbb{C}} [X_{k,k,\mu}][\text{trep}_{n-k}(T^W_{m,n-k}(\lambda - \mu))]
\]

Here the summation \( \sum_{\mu \in \mathbb{C}} \) is shorthand for distinguishing between zero and non-zero values of \( \mu \) and \( \lambda - \mu \). For example, with \( \sum_{\mu \in \mathbb{C}} [X_{k,k,\mu}][\text{trep}_{n-k}(T^W_{m,n-k}(\lambda - \mu))] \) we mean for \( \lambda \neq 0 \)

\[
(L-2)[X_{k,k,1}][\text{trep}_{n-k}(T^W_{m,n-k}(1))] + [X_{k,k,0}][\text{trep}_{n-k}(T^W_{m,n-k}(\lambda))] + [X_{k,k,1}][\text{trep}_{n-k}(T^W_{m,n-k}(0))]
\]

and when \( \lambda = 0 \)

\[
(L-1)[X_{k,k,1}][\text{trep}_{n-k}(T^W_{m,n-k}(1))] + [X_{k,k,0}][\text{trep}_{n-k}(T^W_{m,n-k}(0))].
\]

Further, we have

\[
[\text{Gr}(k, n)] = \frac{[GL_n]}{[GL_k][GL_{n-k}]L^{k(n-k)}} \quad \text{and} \quad [X_{k,k,\mu}] = [GL_k][BS^W_{m,n-k}(\mu)]
\]

and substituting this in the above, and recalling that \( M^W_{m,1}(\alpha) = \text{trep}_1(T^W_{m,1}(\alpha)) \), we get
Proposition 3. With notations as before we have for all $0 < k < n$ and all $\lambda \in \mathbb{C}$ that

$$[X_{k,n,\lambda}] = [GL_n]L^{(m-1)(k(n-k))} \sum_{\mu \in \mathbb{C}} [BS_{m,k}^{W}(\mu)] \frac{[M_{m,n,k}^{W}(\lambda-\mu)]}{[GL_{n-k}]}$$

Further, we have

$$[X_{0,n,\lambda}] = [M_{m,n}^{W}(\lambda)] \quad \text{and} \quad [X_{n,n,\lambda}] = [GL_n][BS_{m,n}^{W}(\lambda)]$$

We can also express this in terms of generating series. Equip the commutative ring $\mathcal{M}_{\mathbb{C}}[[t]]$ with the modified product

$$t^a * t^b = L^{(m-1)ab} t^{a+b}$$

and consider the following two generating series for all $\frac{1}{2} \neq \lambda \in \mathbb{C}$

$$B_{\lambda}(t) = \sum_{n=1}^{\infty} [BS_{m,n}^{W}(\lambda)] t^n \quad \text{and} \quad R_{\lambda}(t) = \sum_{n=1}^{\infty} \frac{[M_{m,n}^{W}(\lambda)]}{[GL_n]} t^n$$

$$B_{\frac{1}{2}}(t) = \sum_{n=0}^{\infty} [BS_{m,n}^{W}(\frac{1}{2}})] t^n \quad \text{and} \quad R_{\frac{1}{2}}(t) = \sum_{n=0}^{\infty} \frac{[M_{m,n}^{W}(\frac{1}{2})]}{[GL_n]} t^n$$

Proposition 4. With notations as before we have the functional equation

$$1 + R_1(Lt) = \sum_{\mu} B_{\mu}(t) * R_{1-\mu}(t)$$

Proof. The disjoint union of the strata of the dimension function on $\mathbb{C}^n \times \text{trep}_n(T_{m,n}^{W}(\lambda))$ gives

$$\mathbb{C}^n \times M_{m,n}^{W}(\lambda) = X_{0,n,\lambda} \sqcup X_{1,n,\lambda} \sqcup \ldots \sqcup X_{n,n,\lambda}$$

Rephrasing this in terms of motives gives

$$L^n[M_{m,n}^{W}(\lambda)] = [M_{m,n}^{W}(\lambda)] + \sum_{k=1}^{n-1} [X_{k,n,\lambda}] + [GL_n][BS_{m,n}^{W}(\lambda)]$$

and substituting the formula of proposition $\mathbb{B}$ into this we get

$$\frac{[M_{m,n}^{W}(\lambda)]}{[GL_n]} L^n t^n = \frac{[M_{m,n}^{W}(\lambda)]}{[GL_n]} t^n + \sum_{k=1}^{n-1} \sum_{\mu \in \mathbb{C}} ([BS_{m,k}^{W}(\mu)] t^k) * \left( \frac{[M_{m,n,k}^{W}(\lambda-\mu)]}{[GL_{n-k}]} t^{n-k} \right)$$

Now, take $\lambda = 1$ then on the left hand side we have the $n$-th term of the series $1 + R_1(Lt)$ and on the right hand side we have the $n$-th factor of the series $\sum_{\mu} B_{\mu}(t) * R_{1-\mu}(t)$. The outer two terms arise from the product $B_{\frac{1}{2}}(t) * R_{\frac{1}{2}}(t)$, using that $W$ is homogeneous whence for all $\lambda \neq 0$

$$BS_{m,n}^{W}(\lambda) \simeq BS_{m,n}^{W}(1) \quad \text{and} \quad M_{m,n}^{W}(\lambda) \simeq M_{m,n}^{W}(1)$$

This finishes the proof. □

These formulas allow us to determine the motive $[M_{m,n}^{W}(\lambda)]$ inductively from lower dimensional contributions and from the knowledge of the motive of the Brauer-Severi scheme $[BS_{m,n}^{W}(\lambda)]$. 


Proposition 5. For all $n$ we have the following inductive description of the motives in the Donalson-Thomas series
\[
(\mathbb{L}^n - 1) \frac{[M^W_{m,n}(0)] - [M^W_{m,n}(1)]}{[GL_n]}
\]
is equal to
\[
[BS^W_{m,n}(0)] - [BS^W_{m,n}(1)] + \sum_{k=1}^{n-1} \frac{L(m-1)k(n-k)}{[GL_{n-k}]} ([BS^W_{m,k}(0)] - [BS^W_{m,k}(1)])([M^W_{m,k}(0)] - [M^W_{m,k}(1)])
\]

Proof. Follows from Proposition 3 and the fact that for all $\mu \neq 0$ we have that $[M^W_{m,k}(\mu)] = [M^W_{m,k}(1)]$ and $[BS^W_{m,k}(\mu)] = [BS^W_{m,k}(1)]$. $\square$

3. Deformations of affine 3-space

The commutative polynomial ring $\mathbb{C}[x, y, z]$ is the Jacobi algebra associated with the superpotential $W = XYZ - XZY$. For this reason we restrict in the rest of this paper to cases where the superpotential $W$ is a cubic necklace in three non-commuting variables $X, Y$ and $Z$, that is $m = 3$ from now on. As even in this case the calculations become quickly unmanageable we restrict to $n \leq 2$, that is we only will compute the coefficients of $t$ and $t^2$ in $U_W(t)$. We will have to compute the motives of fibers of the Chern-Simons functional

\[
M_2(\mathbb{C}) \oplus M_2(\mathbb{C}) \oplus M_2(\mathbb{C}) \xrightarrow{Tr(W)} \mathbb{C}
\]
so we want to express $Tr(W)$ as a function in the variables of the three generic $2 \times 2$ matrices

\[
X = \begin{bmatrix} n & p \\ q & r \end{bmatrix}, \quad Y = \begin{bmatrix} s & t \\ u & v \end{bmatrix}, \quad Z = \begin{bmatrix} w & x \\ y & z \end{bmatrix}.
\]

We will call $\{n, r, s, v, w, x\}$ (resp. $\{p, t, x\}$ and $\{q, u, y\}$) the diagonal- (resp. upper- and lower-) variables. We claim that

\[
Tr(W) = C + Q_{q,q}q + Q_{u,u}u + Q_{y,y}y
\]

where $C$ is a cubic in the diagonal variables and $Q_{q,q}, Q_{u,u}$ and $Q_{y,y}$ are bilinear in the diagonal and upper variables, that is, there are linear terms $L_{ab}$ in the diagonal variables such that

\[
\begin{cases}
Q_{q,q} = L_{qq}p + L_{qt}t + L_{qy}x \\
Q_{u,u} = L_{up}p + L_{ut}t + L_{ux}x \\
Q_{y,y} = L_{yp}p + L_{yt}t + L_{yx}x
\end{cases}
\]

This follows from considering the two diagonal entries of a $2 \times 2$ matrix as the vertices of a quiver and the variables as arrows connecting these vertices as follows

and observing that only an oriented path of length 3 starting and ending in the same vertex can contribute something non-zero to $Tr(W)$. Clearly these linear and cubic terms are fully determined by $W$. If we take

\[
W = \alpha X^3 + \beta Y^3 + \gamma Z^3 + \delta XYZ + \epsilon XZY
\]
then we have $C = W(n, s, w) + W(r, v, z)$ and

$$\begin{aligned}
\left\{
\begin{array}{ll}
L_{qp} &= 3\alpha(n + r) \\
L_{qt} &= ew + \delta z \\
L_{qx} &= \delta s + \epsilon v
\end{array}
\right.
\begin{array}{ll}
L_{up} &= \delta w + \epsilon z \\
L_{ut} &= 3\beta(s + v) \\
L_{ux} &= \epsilon n + \delta r
\end{array}
\begin{array}{ll}
L_{yp} &= \epsilon s + \delta v \\
L_{yt} &= \delta n + \epsilon r \\
L_{yx} &= 3\gamma(w + z)
\end{array}
\end{aligned}$$

By using the cellular decomposition of the Brauer-Severi scheme of $T_{3,2}$ one can simplify the computations further by specializing certain variables. From [16] we deduce that $BS_2(T_{3,2})$ has a cellular decomposition as $\mathbb{A}^{10} \sqcup \mathbb{A}^8 \sqcup \mathbb{A}^8$ where the three cells have representatives

$$\begin{aligned}
\text{cell}_1 : v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, X = \begin{bmatrix} 0 & p \\ 1 & r \end{bmatrix}, Y = \begin{bmatrix} s & t \\ u & v \end{bmatrix}, Z = \begin{bmatrix} w & x \\ y & z \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\text{cell}_2 : v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, X = \begin{bmatrix} n & p \\ 0 & r \end{bmatrix}, Y = \begin{bmatrix} 0 & t \\ 1 & v \end{bmatrix}, Z = \begin{bmatrix} w & x \\ y & z \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\text{cell}_3 : v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, X = \begin{bmatrix} n & p \\ 0 & r \end{bmatrix}, Y = \begin{bmatrix} s & t \\ 0 & v \end{bmatrix}, Z = \begin{bmatrix} 0 & x \\ 1 & z \end{bmatrix}
\end{aligned}$$

It follows that $BS_2^W(1)$ decomposes as $S_1 \sqcup S_2 \sqcup S_3$ where the subschemes $S_i$ of $\mathbb{A}^{11-i}$ have defining equations

$$\begin{aligned}
S_1 : (C + Q_{u,u} + Q_{y,y} + Q_{x})_{|n=0} &= 1 \\
S_2 : (C + Q_{y,y} + Q_{u})_{|s=0} &= 1 \\
S_3 : (C + Q_{y})_{|w=0} &= 1
\end{aligned}$$

Note that in using the cellular decomposition, we set a variable equal to 1. So, in order to retain a homogeneous form we let $\mathbb{G}_m$ act on $n, s, w, r, v, z$ with weight one, on $q, u, y$ with weight two and on $x, t, p$ with weight zero. Thus, we need a slight extension of [4 Thm. 1.3] as to allow $\mathbb{G}_m$ to act with weight two on certain variables.

From now on we will assume that $W$ is as above with $\delta = 1$ and $\epsilon \neq 0$. In this generality we can prove:

**Proposition 6.** With assumptions as above

$$[S_3] = \begin{cases} 
L^7 - L^4 + L^3[W(n, s, 0) + W(-\epsilon^{-1}n, -\epsilon s, 0)]_{|A^2} = 1 & \text{if } \gamma \neq 0 \\
L^7 - L^5 + L^3[W(n, s, 0) + W(-\epsilon^{-1}n, -\epsilon s, z)]_{|A^3} = 1 & \text{if } \gamma = 0
\end{cases}$$

**Proof.** $S_3$ : The defining equation in $\mathbb{A}^8$ is equal to

$$W(n, s, 0) + W(r, v, z) + (\epsilon s + v)p + (n + er)t + 3\gamma(z)x = 1$$

If $\epsilon s + v \neq 0$ we can eliminate $p$ and get a contribution $L^5(L^2 - L)$. If $v = -\epsilon s$ but $n + er \neq 0$ we can eliminate $t$ and get a term $L^4(L^2 - L)$. From now on we may assume that $v = -\epsilon s$ and $r = -\epsilon^{-1}n$.

$\gamma \neq 0$ : Assume first that $z \neq 0$ then we can eliminate $x$ and get a contribution $L^4(L - 1)$. If $z = 0$ then we get a term

$$L^3[W(n, s, 0) + W(-\epsilon^{-1}n, -\epsilon s, 0)]_{|A^2}$$
\[\gamma = 0: \text{Then we have a remaining contribution}\]

\[L^3 [W(n, s, 0) + W(-\epsilon^{-1} n, -\epsilon s, z) = 1]_{\mathbb{A}^3}\]

Summing up all contributions gives the result. \(\square\)

Calculating the motives of \(S_2\) and \(S_1\) in this generality quickly leads to a myriad of subcases to consider. For this reason we will defer the calculations in the cases of interest to the next sections. Specializing Proposition 5 to the case of \(n = 2\) we get

**Proposition 7.** For \(n = 2\) we have that

\[\{L^2 - 1\} \frac{[\mathcal{M}_{3,2}^W(0)] - [\mathcal{M}_{3,2}^W(1)]}{[GL_2]}\]

is equal to

\[[\mathcal{B}_S^W_{3,2}(0)] - [\mathcal{B}_S^W_{3,2}(1)] + \frac{L^2}{(L-1)} ([\mathcal{M}_{3,1}^W(0)] - [\mathcal{M}_{3,1}^W(1)])^2\]

**Proof.** The result follows from Proposition 5 and from the fact that \(\mathcal{B}_S^W_{3,1}(1) = \mathcal{M}_{3,1}^W(1)\) and \(\mathcal{B}_S^W(0) = \mathcal{M}_{3,1}^W(0)\). \(\square\)

4. **Quantum affine three-space**

For \(q \in \mathbb{C}^\ast\) consider the superpotential \(W_q = XYZ - qXZY\), then the associated algebra \(R_{W_q}\) is the quantum affine 3-space

\[R_{W_q} = \frac{\mathbb{C}\langle X, Y, Z \rangle}{(XY - qYX, ZX - qXZ, YZ - qZY)}\]

It is well-known that \(R_{W_q}\) has finite dimensional simple representations of dimension \(n\) if and only if \(q\) is a primitive \(n\)-th root of unity. For other values of \(q\) the only finite dimensional simples are 1-dimensional and parametrized by \(XYZ = 0\) in \(\mathbb{A}^3\). In this case we have

\[\begin{align*}
\{[\mathcal{M}_{3,1}^W(1)] &= [(q - 1)XYZ = 1]_{\mathbb{A}^3} = (L-1)^2 \\
\{[\mathcal{M}_{3,1}^W(0)] &= [(1 - q)XYZ = 0]_{\mathbb{A}^3} = 3L^2 - 3L + 1 \\
\end{align*}\]

That is, the coefficient of \(t\) in \(U_{W_q}(t)\) is equal to

\[L^{-1} \frac{[\mathcal{M}_{3,1}^W(0)] - [\mathcal{M}_{3,1}^W(1)]}{[GL_1]} = L^{-1} \frac{2L^2 - L}{L - 1} = \frac{2L - 1}{L - 1}\]

In \([3\text{, Thm. 3.1}]\) it is shown that in case \(q\) is not a root of unity, then

\[U_{W_q}(t) = \exp\left(\frac{2L - 1}{L - 1} \frac{t}{1 - t}\right)\]

and if \(q\) is a primitive \(n\)-th root of unity then

\[U_{W_q}(t) = \exp\left(\frac{2L - 1}{L - 1} \frac{t}{1 - t}\right) + (L - 1) \frac{t^n}{1 - t^n}\]

In \([3\text{, 3.4.1}]\) a rather complicated attempt is made to explain the term \(L - 1\) in case \(q\) is an \(n\)-th root of unity in terms of certain simple \(n\)-dimensional representations of \(R_{W_q}\). Note that the geometry of finite dimensional representations of the algebra \(R_{W_q}\) is studied extensively in \([5]\) and note that there are additional simple \(n\)-dimensional representations not taken into account in \([3\text{, 3.4.1}]\).
Perhaps a more conceptual explanation of the two terms in the exponential expression of $U_W(t)$ in case $q$ is an $n$-th root of unity is as follows. As $W_q$ admits a cut $W_q = X(YZ - qZY)$ it follows from \cite{[12]} that for all dimensions $m$ we have

$$[M_{3,m}^{W_q}(0)] - [M_{3,m}^{W_q}(1)] = \mathbb{L}^m [\text{rep}_m(C_q[Y, Z])].$$

where $C_q[Y, Z] = \mathbb{C}[Y, Z]/(YZ - qZY)$ is the quantum plane. If $q$ is an $n$-th root of unity the only finite dimensional simple representations of $C_q[Y, Z]$ are of dimension 1 or $n$. The 1-dimensional simples are parametrized by $YZ = 0$ in $\mathbb{A}^2$ having as motive $2\mathbb{L} - 1$ and as all have $GL_1$ as stabilizer group, this explains the term $(2\mathbb{L} - 1)/(\mathbb{L} - 1)$.

The center of $C_q[Y, Z]$ is equal to $\mathbb{C}[Y^n, Z^n]$ and the corresponding variety $\mathbb{A}^2 = \text{Max}(\mathbb{C}[Y^n, Z^n])$ parametrizes $n$-dimensional semi-simple representations. The $n$-dimensional simples correspond to the Zariski open set $\mathbb{A}^2 - (Y^n Z^n = 0)$ which has as motive $(\mathbb{L} - 1)^2$. Again, as all these have as $GL_2$-stabilizer subgroup $GL_1$, this explains the term

$$L - 1 = \frac{(L - 1)^2}{[GL_1]}.$$

As the superpotential allows a cut in this case we can use the full strength of \cite{[1]} and can obtain $[M_{3,2}^{W_q}(0)]$ from $[M_{3,2}^{W_q}(1)]$ from the equality

$$\mathbb{L}^{12} = [M_{3,2}^{W_q}(0)] + (\mathbb{L} - 1)[M_{3,2}^{W_q}(1)].$$

To illustrate the inductive procedure using Brauer-Severi motives we will consider the case $n = 2$, that is $q = -1$ with superpotential $W = XYZ + XZY$. In this case we have from \cite{[3]} Thm. 3.1] that

$$U_W(t) = \exp\left(\frac{2\mathbb{L} - 1}{\mathbb{L} - 1} \frac{t}{1 - t} + (\mathbb{L} - 1) \frac{t^2}{1 - t^2}\right).$$

The basic rules of the plethystic exponential on $\mathcal{M}_{\mathbb{C}}[[t]]$ are

$$\exp\left(\sum_{n \geq 1} [A_n] t^n\right) = \prod_{n \geq 1} (1 - t^n)^{-[A_n]} \quad \text{where} \quad (1 - t)^{-L^m} = (1 - L^m t)^{-1}$$

and one has to extend all infinite products in $t$ and $\mathbb{L}^{-1}$. One starts by rewriting $U_W(t)$ as a product

$$U_W(t) = \exp\left(\frac{t}{1 - t}\right) \exp\left(\frac{\mathbb{L}}{\mathbb{L} - 1} \frac{t}{1 - t}\right) \exp\left(\frac{\mathbb{L} t^2}{1 - t^2}\right) \exp\left(\frac{t^2}{1 - t^2}\right)^{-1}.$$

where each of the four terms is an infinite product

$$\exp\left(\frac{t}{1 - t}\right) = \prod_{m \geq 1} (1 - t^m)^{-1}, \quad \exp\left(\frac{\mathbb{L}}{\mathbb{L} - 1} \frac{t}{1 - t}\right) = \prod_{m \geq 1} \prod_{j \geq 0} (1 - \mathbb{L}^{-j} t^m)^{-1},$$

$$\exp\left(\frac{\mathbb{L} t^2}{1 - t^2}\right) = \prod_{m \geq 1} (1 - \mathbb{L} t^2m)^{-1}, \quad \exp\left(\frac{t^2}{1 - t^2}\right)^{-1} = \prod_{m \geq 1} (1 - t^{2m})^{-1}.$$

That is, we have to work out the infinite product

$$\prod_{m \geq 1} ((1 - t^{2m-1})^{-1} (1 - \mathbb{L} t^{2m})^{-1}) \prod_{m \geq 1} \prod_{j \geq 0} (1 - \mathbb{L}^{-j} t^m)^{-1}$$

as a power series in $t$, at least up to quadratic terms. One obtains

$$U_W(t) = 1 + \frac{2\mathbb{L} - 1}{\mathbb{L} - 1} t + \frac{\mathbb{L}^4 + 3\mathbb{L}^3 - 2\mathbb{L}^2 - 2\mathbb{L} + 1}{(\mathbb{L}^2 - 1)(\mathbb{L} - 1)} t^2 + \ldots.$$
That is, if \( W = XYZ + XZY \) one must have the relation:
\[
[M^{W}_{3,2}(0)] - [M^{W}_{3,2}(1)] = L^5(L^4 + 3L^3 - 2L^2 - 2L + 1)
\]

4.1. Dimensional reduction. It follows from the dimensional reduction argument of [12] that
\[
[M^{W}_{3,2}(0)] - [M^{W}_{3,2}(1)] = L^5[\text{rep}_2 \mathbb{C}_{-1}[X,Y]]
\]
where \( \mathbb{C}_{-1}[X,Y] \) is the quantum plane at \( q = -1 \), that is, \( \mathbb{C}(X,Y)/(XY + YX) \).

The matrix equation
\[
\begin{bmatrix}
a & b \\
c & d
\end{bmatrix}
\begin{bmatrix}
e & f \\
g & h
\end{bmatrix}
+ \begin{bmatrix}
e & f \\
g & h
\end{bmatrix}
\begin{bmatrix}
a & b \\
c & d
\end{bmatrix}
= \begin{bmatrix}0 & 0 \\
0 & 0
\end{bmatrix}
\]
gives us the following system of equations
\[
\begin{align*}
2ae + bg + fc &= 0 \\
2hd + bg + fc &= 0 \\
f(a + d) + b(e + h) &= 0 \\
c(h + e) + g(a + d) &= 0
\end{align*}
\]
where the two first are equivalent to \( ae = hd \) and \( 2ae + bg + fc = 0 \). Changing variables
\[
x = \frac{1}{2}(a + d), \quad y = \frac{1}{2}(a - d), \quad u = \frac{1}{2}(e + h), \quad v = \frac{1}{2}(e - h)
\]
the equivalent system then becomes (in the variables \( b, c, f, u, v, x, y \))
\[
\begin{align*}
xv + yu &= 0 \\
xu + yv + bg + fc &= 0 \\
f(x) + bu &= 0 \\
cu + gx &= 0
\end{align*}
\]

**Proposition 8.** The motive of \( R_2 = \text{rep}_2 \mathbb{C}_{-1}[x,y] \) is equal to
\[
[R_2] = L^5 + 3L^4 - 2L^3 - 2L^2 + L
\]

**Proof.** If \( x \neq 0 \) we obtain
\[
v = \frac{yu}{x}, \quad f = \frac{bu}{x}, \quad g = \frac{cu}{x}
\]
and substituting these in the remaining second equation we get the equation(s)
\[
u(y^2 - x^2 + 2bc) = 0 \quad \text{and} \quad x \neq 0
\]
If \( u \neq 0 \) then \( y^2 - x^2 + 2bc = 0 \). If in addition \( b \neq 0 \) then \( c = \frac{x^2 - y^2}{2b} \) and \( y \) is free.
As \( x,u,b \) are non-zero this gives a contribution \((L - 1)^3\mathbb{L})\). If \( b = 0 \) then \( c \) is free and \( x^2 - y^2 = 0 \), so \( y = \pm x \). This together with \( x \neq 0 \neq u \) leads to a contribution of \( 2\mathbb{L}(L - 1)^2 \). If \( u = 0 \) then \( y,b \) and \( c \) are free variables, and together with \( x \neq 0 \) this gives \((L - 1)\mathbb{L}^3\).

Remains the case that \( x = 0 \). Then the system reduces to
\[
\begin{align*}
yu &= 0 \\
yv + bg + fc &= 0 \\
bu &= 0 \\
cu &= 0
\end{align*}
\]
If \( u \neq 0 \) then \( y = 0, b = 0 \) and \( c = 0 \) leaving \( c, g, v \) free. This gives \((L - 1)L^3\). If \( u = 0 \) then the only remaining equation is \( yv + bg + fc = 0 \). That is, we get the cone in \( A^6 \) of the Grassmannian \( Gr(2, 4) \) in \( \mathbb{P}^5 \). As the motive of \( Gr(2, 4) \) is

\[
[Gr(2, 4)] = (L^2 + 1)(L^2 + L + 1)
\]

we get a contribution of

\[
(L - 1)(L^2 + 1)(L^2 + L + 1) + 1
\]

Summing up all contributions gives the desired result. \( \square \)

4.2. **Brauer-Severi motives.** In the three cells of the Brauer-Severi scheme of \( T_{3,2} \) of dimensions resp. 10, 9 and 8 the superpotential \( Tr(XYZ + XZY) \) induces the equations:

\[
\begin{align*}
S_1 : & \ 2rvz + puz + pvy + rty + psy + rux + pw + tz + vx + sx + tw = 1 \\
S_2 : & \ 2rvz + pvy + rty + nty + pz + rx + nx + pw = 1 \\
S_3 : & \ 2rvz + pv + rt + nt + ps = 1
\end{align*}
\]

**Proposition 9.** With notations as above, the Brauer-Severi scheme of \( T^W_{3,2}(1) \) has a decomposition

\[
BS^W_{3,2}(1) = S_1 \sqcup S_2 \sqcup S_3
\]

where the schemes \( S_i \) have motives

\[
\begin{align*}
[S_1] & = L^9 - L^6 - 2L^5 + 3L^4 - L^3 \\
[S_2] & = L^8 - 2L^5 + L^4 \\
[S_3] & = L^7 - 2L^4 + L^3
\end{align*}
\]

Therefore, the Brauer-Severi scheme has motive

\[
[BS^W_{3,2}(1)] = L^9 + L^8 + L^7 - L^6 - 4L^5 + 2L^4
\]

**Proof.** \( S_1 \) : From Proposition 8 we obtain

\[
[S_3] = L^7 - L^6 + L^3[W(n, s, 0) + W(-n, -s, z) = 1]_{A^3}
\]

and as \( W(n, s, 0) + W(-n, -s, z) = 2nsz \) we get \( L^7 - L^6 + L^3(L - 1)^2 \).

\( S_2 \) : The defining equation is

\[
2rvz + y(pv + (r + n)t) + p(z + w) + x(r + n) = 1
\]

If \( r + n \neq 0 \) we can eliminate \( x \) and have a contribution \( L^6(L^2 - L) \). If \( r + n = 0 \) we get the equation

\[
2rvz + p(yv + z + w) = 1
\]

If \( yv + z + w \neq 0 \) we can eliminate \( p \) and get a term \( L^3(L^4 - L^3) \). If \( r + n = 0 \) and \( yv + z + w = 0 \) we have \( 2rvz = 1 \) so a term \( L^4(L - 1)^2 \). Summing up gives us

\[
[S_2] = L^4(L - 1)(L^3 + L^2 + L - 1) = L^8 - 2L^5 + L^4
\]

\( S_1 \) : The defining equation is

\[
2rvz + p(u(z + w) + y(v + s)) + t(z + w + ry) + x(v + s + ru) = 1
\]
If $v + s + ru \neq 0$ we can eliminate $x$ and get $L^5(L^4 - L^3)$. If $v + s + ru = 0$ and $z + w + ry \neq 0$ we can eliminate $t$ and have a term $L^4(L^4 - L^3)$. If $v + s + ru = 0$ and $z + w + ry = 0$, the equation becomes (in $\mathbb{A}^3$, with $t$, $x$ free variables)

$$2r(vz - puy) = 1$$

giving a term $L^2(L^5 - [vz = puy])$. To compute $[vz = puy]_{\mathbb{A}^5}$ assume first that $v \neq 0$, then this gives $L^3(L - 1)$ and if $v = 0$ we get $L(3L^2 - 3L + 1)$. That is, $[vz = puy]_{\mathbb{A}^5} = L^2 + 2L^3 - 3L^2 + L$. In total this gives us

$$[S_1] = L^3(L - 1)(L^5 + L^4 + L^3 - 2L + 1) = L^9 - L^6 - 2L^5 + 3L^4 - L^3$$

finishing the proof. \qed

**Proposition 10.** From the Brauer-Severi motive we obtain

$$\left\{ \begin{array}{l}
M_3W^W(1) = L^{11} - L^8 - 3L^7 + 2L^6 + 2L^5 - L^4 \\
M_3W^W(0) = L^{11} + L^9 + 2L^8 - 5L^7 + 3L^5 - L^4
\end{array} \right.$$  

As a consequence we have,

$$[M_3W^W(0)] - [M_3W^W(1)] = L^4(L^5 + 3L^4 - 2L^3 - 2L^2 + L)$$

**Proof.** We have already seen that $M_3W^W(1) = \{(x, y, z) \mid 2xyz = 1\}$ and $M_3W^W(0) = \{(x, y, z) \mid xyz = 0\}$ whence

$$[M_3W^W(1)] = (L - 1)^2 \quad \text{and} \quad [M_3W^W(0)] = 3L^2 - 3L + 1$$

Plugging this and the obtained Brauer-Severi motive into Proposition 5 gives $[M_3W^W(1)]$. From this $[M_3W^W(0)]$ follows from the equation $L^{12} = (L - 1)[M_3W^W(1)] + [M_3W^W(0)]$. \qed

### 5. The homogenized Weyl algebra

If we consider the superpotential $W = XYZ - XZY - \frac{1}{3} X^3$ then the associated algebra $R_W$ is the homogenized Weyl algebra

$$R_W = \frac{C(X, Y, Z)}{(XZ - ZX, XY - YX, YZ - ZY - X^2)}$$

In this case we have $M_3W^W(1) = \{x^3 = -3\}$ and $M_3W^W(0) = \{x^3 = 0\}$, whence

$$[M_3W^W(1)] = L^2[\mu_3], \quad \text{and} \quad [M_3W^W(0)] = L^2$$

where, as in [3.1.3] we denote by $[\mu_3]$ the equivariant motivic class of $\{x^3 = 1\} \subset \mathbb{A}^1$ carrying the canonical action of $\mu_3$. Therefore, the coefficient of $t$ in $U_W(t)$ is equal to

$$[GL_1^{-1}] \frac{[M_3W^W(1)] - [M_3W^W(0)]}{L(1 - [\mu_3])} = \frac{L(1 - [\mu_3])}{L - 1}$$

As all finite dimensional simple representations of $R_W$ are of dimension one, this leads to the conjectural expression [3. Conjecture 3.3]

$$U_W(t) \overset{?}{=} \text{Exp} \left( \frac{L(1 - [\mu_3])}{L - 1} \frac{t}{1 - t} \right)$$

Balazs Szendrogi kindly provided the calculation of the first two terms of this series. Denote with $\tilde{M} = 1 - [\mu_3]$, then

$$U_W(t) \overset{?}{=} 1 + \frac{LM}{L - 1} + \frac{L^2M^2 + L(L^2 - 1)M + L^2(L - 1)\tilde{M}}{(L^2 - 1)(L - 1)}t^2 + \ldots$$
As was pointed out by B. Pym and B. Davison it follows from \cite{[4, Defn 4.4 and Prop 4.5 (4)]} that \( \sigma_2(M) = L \), so the second term is equal to 
\[
\frac{LL^3(L-1) + ML(L^2 - 1) + MLL^2}{(L^2 - 1)(L-1)}
\]

We will now compute the this second term using Brauer-Severi motives.

Recall that \( BS^W_{3,2}(i) \), for \( i = 0, 1 \), decomposes as \( S_1 \sqcup S_2 \sqcup S_3 \) where the subschemes \( S_i \) of \( A^{11-i} \) have defining equations

\[
\begin{align*}
S_1 & : -\frac{1}{3}n^3 + (w - z)p + rx)u + ((v - s)p - rt)y - rp + (z - w)t + (s - v)x = \delta_1 \\
S_2 & : -\frac{1}{3}n^3 - \frac{1}{3}v^3 + (vp + (n - r)t)y + (w - z)p + (r - n)x = \delta_1 \\
S_3 & : -\frac{1}{3}n^3 - \frac{1}{3}v^3 + (v - s)p + (n - r)t = \delta_1
\end{align*}
\]

If we let the generator of \( \mu_3 \) act with weight one on the variables \( n, s, w, r, v, z \), with weight two on \( x, t, p \) and with weight zero on \( q, u, y \) we see that the schemes \( S_j \) for \( i = 1 \) are indeed \( \mu_3 \)-varieties. We will now compute their equivariant motives:

**Proposition 11.** With notations as above, the Brauer-Severi scheme of \( T^W_{3,2}(1) \) has a decomposition

\[
BS^W_{3,2}(1) = S_1 \sqcup S_2 \sqcup S_3
\]

where the schemes \( S_i \) have equivariant motives

\[
\begin{align*}
[S_1] & = L^9 - L^6 \\
[S_2] & = L^8 + (\mu_3 - 1)L^6 = L^8 - ML^6 \\
[S_3] & = L^7 + (\mu_3 - 1)L^5 = L^7 - ML^5
\end{align*}
\]

Therefore, the Brauer-Severi scheme \( BS^W_{3,2}(1) \) has equivariant motive

\[
[BS^W_{3,2}(1)] = L^9 + L^8 + L^7 + (\mu_3 - 2)L^6 + (\mu_3 - 1)L^5
\]

**Proof.** \( S_3 \) : If \( v - s \neq 0 \) we can eliminate \( p \) and obtain a contribution \( L^4(L^2 - L) \).

If \( v = s \) and \( n - r \neq 0 \) we can eliminate \( t \) and obtain a term \( L^4(L^2 - L) \).

Finally, if \( v = s \) and \( n = r \) we have the identity \( -\frac{2}{3}n^3 = 1 \) and a contribution \( L^5[\mu_3] \).

\( S_2 \) : If \( r - n \neq 0 \) we can eliminate \( x \) and get a term \( L^6(L^2 - L) \).

If \( r - n = 0 \) we get the equation in \( A^8 \)

\[
-\frac{2}{3}n^3 + p(vy + w - z) = 1
\]

If \( vy + w - z \neq 0 \) we can eliminate \( p \) and get a contribution \( L^3(L^4 - L^3) \).

Finally, if \( vy + w - z = 0 \) we get the equation \( -\frac{2}{3}n^3 = 1 \) and hence a term \( L^3L^5[\mu_3] \).

\( S_1 \) : If \( (w - z)p + rx \neq 0 \) then we can eliminate \( u \) and get a contribution

\[
L^4(L^5 - [(w - z)p + rx = 0]_{A^5}) = L^5(L - 1)(L^2 - 1)
\]

If \( (w - z)p + rx = 0 \) but \( (v - s)p - rt \neq 0 \) we can eliminate \( y \) and get a term

\[
L_i[(w - z)p + rx = 0, (v - s)p - rt \neq 0]_{A^5}
\]

To compute the equivariant motive in \( A^8 \) assume first that \( r \neq 0 \) then we can eliminate \( x \) from the equation and obtain

\[
L^2[r \neq 0, (v - s)p - rt \neq 0]_{A^5} = L^2(L^4(L - 1) - [r \neq 0, (v - s)p - rt = 0]_{A^5}) = L^5(L - 1)^2
\]
If $r = 0$ we have to compute $[(w - z)p = 0, (v - s)p \neq 0]_{A^7} = \mathbb{L}^2(\mathbb{L} - 1)(\mathbb{L}^2 - \mathbb{L}) = \mathbb{L}^4(\mathbb{L} - 1)^2$. So, in total this case gives a contribution
\[
\mathbb{L}[(w - z)p + rx = 0, (v - s)p - rt \neq 0]_{A^8} = \mathbb{L}^5(\mathbb{L} - 1)(\mathbb{L}^2 - 1)
\]
If $(w - z)p + rx = 0$, $(v - s)p - rt = 0$ and $r \neq 0$ we can eliminate $x = \frac{v - w}{r}p$ and $t = \frac{w}{r}p$ and substituting in the defining equation of $S_1$ we get
\[
-\frac{1}{3}r^3 - rp = 1
\]
so we can eliminate $p$ and obtain a contribution $\mathbb{L}^6(\mathbb{L} - 1)$. Finally, if $(w - z)p + rx = 0$, $(v - s)p - rt = 0$ and $r = 0$ we get the system of equations
\[
\begin{cases}
(w - z)p = 0 \\
(v - s)p = 0 \\
(z - w)t + (s - v)x = 1
\end{cases}
\]
If $p \neq 0$ we must have $w - z = 0$ and $v - s = 0$ which is impossible, so we must have $p = 0$ and the remaining equation is $(z - w)t + (s - v)x = 1$ giving a contribution $\mathbb{L}^5(\mathbb{L}^2 - 1)$. Summing up these contributions gives the claimed motive. \(\square\)

**Proposition 12.** With notations as above, the Brauer-Severi scheme of $\mathbb{T}^W_{3,2}(0)$ has a decomposition

\[
BS^W_{3,2}(0) = S_1 \sqcup S_2 \sqcup S_3
\]
where the schemes $S_i$ have (equivariant) motives

\[
\begin{align*}
[S_1] &= \mathbb{L}^9 + \mathbb{L}^7 - \mathbb{L}^6 \\
[S_2] &= \mathbb{L}^8 \\
[S_3] &= \mathbb{L}^7
\end{align*}
\]

Therefore, the Brauer-Severi scheme $BS^W_{3,2}(0)$ has (equivariant) motive

\[
[BS^W_{3,2}(0)] = \mathbb{L}^9 + \mathbb{L}^8 + 2\mathbb{L}^7 - \mathbb{L}^6
\]

**Proof.** $S_3$: If $v - s \neq 0$ we can eliminate $p$ and obtain a contribution $\mathbb{L}^5(\mathbb{L}^2 - \mathbb{L})$. If $v = s$ and $n - r \neq 0$ we can eliminate $t$ and obtain a term $\mathbb{L}^4(\mathbb{L}^2 - \mathbb{L})$. Finally, if $v = s$ and $n = r$ we have the identity $n^3 = 0$ and a contribution $\mathbb{L}^5$.

$S_2$: If $r - n \neq 0$ we can eliminate $x$ and get a term $\mathbb{L}^6(\mathbb{L}^2 - \mathbb{L})$. If $r - n = 0$ we get the equation in $A^8$
\[
-\frac{2}{3}n^3 + p(vy + w - z) = 1
\]
If $vy + w - z \neq 0$ we can eliminate $p$ and get a contribution $\mathbb{L}^3(\mathbb{L}^4 - \mathbb{L}^3)$. Finally, if $vy + w - z = 0$ we get the equation $n^3 = 0$ and hence a term $\mathbb{L}^6$.

$S_1$: If $(w - z)p + rx \neq 0$ we can eliminate $u$ and obtain a term
\[
\mathbb{L}^4(\mathbb{L}^5 - [(w - z)p + rx = 0]_{A^5}) = \mathbb{L}^6(\mathbb{L} - 1)(\mathbb{L}^2 - 1)
\]
If $(w - z)p + rx = 0$ but $(v - s)p - rt \neq 0$ then we can eliminate $y$ and obtain a contribution
\[
\mathbb{L}[(w - z)p + rx = 0, (v - s)p - rt \neq 0]_{A^8} = \mathbb{L}^5(\mathbb{L} - 1)(\mathbb{L}^2 - 1)
\]
Now, assume that \((w - z)p + rx = 0\) and \((v - s)p - rt = 0\). If \(r \neq 0\) then we can eliminate \(p, t\) as before and substituting them in the defining equation of \(S_1\) we get

\[-\frac{1}{3}p^3 - rp = 0\]

and we can eliminate \(p\) giving a contribution \(\mathbb{L}^6(\mathbb{L} - 1)\). Finally, if \((w - z)p + rx = 0\) and \((v - s)p - rt = 0\) and \(r = 0\) we have the system of equations

\[
\begin{cases}
(w - z)p = 0 \\
(v - s)p = 0 \\
(z - w)t + (s - v)x = 0
\end{cases}
\]

If \(p \neq 0\) we get \(w - z = 0\) and \(v - s = 0\) giving a contribution \(\mathbb{L}^6(\mathbb{L} - 1)\). If \(p = 0\) the only remaining equation is \((z - w)t + (s - v)x = 0\) which gives a contribution \(\mathbb{L}^2(\mathbb{L}^2 + \mathbb{L} - 1)\). Summing up all terms gives the claimed motive. \(\square\)

Now, we have all the information to compute the second term of the motivic Donaldson-Thomas series. We have

\[
\begin{align*}
&[BSW_{3,2}^W(0)] - [BSW_{3,2}^W(1)] = \mathbb{L}^7 + \mathcal{M}L^6 + \mathcal{M}L^5 \\
&[M^W_{3,1}(0)] - [M^W_{3,1}(1)] = \mathcal{M}L^2
\end{align*}
\]

By Proposition \(\ref{proposition:virtual-motive}\) this implies that

\[
(\mathbb{L}^2 - 1)\frac{[M^W_{3,2}^W(0)] - [M^W_{3,2}^W(1)]}{[GL_2]} = \mathbb{L}^7 + \mathcal{M}L^6 + \mathcal{M}L^5 + \mathcal{M}^2L^2 - \mathcal{M}L^6(\mathbb{L} - 1)
\]

Therefore the virtual motive is equal to

\[
\mathbb{L}^{-4}\frac{[M^W_{3,2}^W(0)] - [M^W_{3,2}^W(1)]}{[GL_2]} = \mathbb{L}^3(\mathbb{L} - 1) + \mathcal{M}L(\mathbb{L}^2 - 1) + \mathcal{M}^2L^2
\]

which coincides with the conjectured term in \(\cite{cazzaniga2015geometry} Conjecture 3.3\).

\section*{References}

[1] K. Behrend, J. Bruan and B. Szendroi, \textit{Motivic degree zero Donaldson-Thomas invariants}, \texttt{arXiv:0909.5088} (2009)

[2] J. Bryan and A. Morrison, \textit{Motivic classes of commuting varieties via power structures}, J. Algebraic Geom. \textbf{24} (2015) 183-199

[3] Alberto Cazzaniga, Andrew Morrison, Brent Pym and Balazs Szendroi, \textit{Motivic Donaldson-Thomas invariants for some quantized threefolds}, \texttt{arXiv:1510.08116} (2015)

[4] Ben Davison and Sven Meinhardt, \textit{Motivic DT-invariants for the one loop quiver with potential}, \texttt{arXiv:1103.5966} (2011)

[5] Kevin De Laet, \textit{Geometry of representations of quantum spaces, \texttt{arXiv: 1405.1938}} (2014)

[6] Kevin De Laet and Lieven Le Bruyn, \textit{The geometry of representations of 3-dimensional Sklyanin algebras}, Algebras and Representation Theory, \textbf{18} , 761-776 (2015)

[7] Lieven Le Bruyn, \textit{Quiver concomitants are often reflexive Azumaya}, Proc. AMS, \textbf{105} (1989) 10-16

[8] Lieven Le Bruyn, \textit{Trace rings of generic matrices are unique factorization domains}, Glasgow Math. J. \textbf{28} (1986) 11-13

[9] Lieven Le Bruyn and Michel Van den Bergh, \textit{An explicit description of }T(3,2), In "Ring Theory, Proceedings Antwerp 1985", 109-113, Lecture Notes in Mathematics 1197, (1986).

[10] Lieven Le Bruyn, \textit{Noncommutative geometry and Cayley-smooth orders}, Pure and Appl. Math. \textbf{290}, Chapman & Hall (2008)

[11] Lieven Le Bruyn, \textit{The superpotential }XY^2Z + XZ^2Y - \frac{1}{6}(X^3 + Y^3 + Z^3), to appear.

[12] Andrew Morrison, \textit{Motivic invariants of quivers via dimensional reduction, \texttt{arXiv:1103.3819}} (2011)
[13] Claudio Procesi, *The invariant theory of $n \times n$ matrices*, Adv. in Math. **19** (1976) 306-381
[14] Claudio Procesi, *A formal inverse to the Cayley-Hamilton theorem*, J. Alg. **107** (1987) 63-74
[15] Y. P. Razmyslov, *Trace identities of full matrix algebras over a field of characteristic zero*, Math. USSR-Izv. **8** (1974) 727-760
[16] Markus Reineke, *Cohomology of non-commutative Hilbert schemes*, Alg. Repr. Theory **8** (2005) 541-561
[17] Michel Van den Bergh, *The Brauer-Severi scheme of the trace ring of generic matrices*, Perspectives in Ring Theory (Antwerp 1987), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol 233, Kluwer (1988)
[18] Chelsea Walton, *Representation theory of three-dimensional Sklyanin algebras*, Nuclear Phys. B, **860**, 167-185 (2012)

Department of Mathematics, University of Antwerp, Middelheimlaan 1, B-2020 Antwerp (Belgium), lieven.lebruyn@uantwerpen.be