Maximal Entanglement and Teleportation using an Arthurs-Kelly type Interaction for Qubits

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We study entanglement generation between a system qubit and three apparatus qubits using an exactly soluble Arthurs-Kelly type model. We demonstrate the possibility of generating an EPR-like maximally entangled system-apparatus state, in which the second qubit of the usual EPR state is replaced by a three qubit state. We design a very simple teleportation protocol to transfer the unknown state of the system onto one of the apparatus qubits which can then be teleported via a quantum channel.

PACS numbers: 13.85.Dz, 13.85.Lg, 13.85.Hd, 11.55.Jy, 12.40.Nn

Introduction. The idea of quantum entanglement was introduced by Schrödinger [1]. The truly non-classical features of entanglement drew wide attention after the work of Einstein, Podolsky and Rosen (EPR) [2] and John Bell [3]. In the last two decades, quantum entanglement has emerged as an important resource for quantum information processing tasks, such as Quantum Teleportation [4-7], Quantum Key Distribution [8], Quantum Computing [9] and Quantum Metrology [10].

We study entanglement generation between a system qubit and three apparatus qubits using an interaction that maximally entangles the qubit with three apparatus qubits.

Entangling interactions have been used previously in quantum measurement theory. Von Neumann introduced the idea of tracking of a system observable by using an apparatus observable [11]: the system interacts with the apparatus for some time such that the apparatus observable has the same expectation value in the final state as the system observable in the initial state. The idea was extended by Arthurs and Kelly [12] to the joint tracking of two canonically conjugate observables by two commuting apparatus observables. The tracking cannot be noiseless and hence, only approximate joint measurement of non-commuting system observables is possible. This yields a joint measurement uncertainty relation [13, 14] for conjugate observables. It has also been shown that the Arthurs Kelly (AK) interaction can be utilised for remote quantum tomography of continuous variable systems [15]. Extensions of Arthurs-Kelly type measurements have been made to joint measurement of different components of spin observables [16-18].

Here we consider the Levine et al [16] AK-type measurement interaction between a system qubit and three apparatus qubits such that the three (mutually non-commuting) spin components of the system qubit are tracked by mutually commuting spin components of the apparatus qubits. We (i) derive a joint measurement uncertainty relation, (ii) show that the interaction can give rise to maximal entanglement generation between an unknown system qubit and the apparatus, and (iii) utilise the maximal entanglement to devise a new protocol to teleport the unknown state of the system qubit.

Joint Measurement Uncertainty Relation in an Arthurs Kelly Type Measurement of Spin Components. In the usual Arthurs-Kelly model for simultaneous approximate measurement of conjugate variables $q, p$ of a system particle $P$, an interaction proportional to $q_{P1} + q_{P2}$ with mutually commuting apparatus variables $P_1, P_2$ is assumed. There is extensive literature on its experimental implementation in quantum optics where $q, p$ denote conjugate quadratures of photons [12, 14].

Consider now possible generalizations to spin measurements. In the quantum model of Levine et al [16] the three non-commuting spin components of a spin half particle $P$ are coupled with three meter qubits $(A_1, A_2$ and $A_3$) via an Arthurs-Kelly type interaction,

$$H = K(\sigma^P_z \sigma^{A_1}_z + \sigma^P_y \sigma^{A_2}_z + \sigma^P_x \sigma^{A_3}_z) = K \sum_{i=1}^3 \sigma^P_i \sigma^{A_i}_z,$$ (1)

where $\sigma^Q_i$ is the $i$th Pauli Operator for the particle $Q$, where $Q = P, A_1, A_2, or A_3$, and $\sigma^P_1 = \sigma^Q_y, \sigma^P_2 = \sigma^Q_x, \sigma^P_3 = \sigma^Q_z$. This is reminiscent of a system particle $P$ of magnetic moment $M$ interacting classically with apparatus particles $A_1, A_2$ and $A_3$ with respective magnetic moments $M^1, M^2$ and $M^3$ via magnetic moment interactions with a Hamiltonian proportional to $M.(M^1 + M^2 + M^3)$.

Neglecting other interactions during the short interaction time $T$, the unitary evolution of the four qubit initial state $|0\rangle$ to the final state $|T\rangle$ is given by,

$$|T\rangle = \hat{U}|0\rangle, \hat{U} = \exp[-iHT].$$ (2)

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The unitary evolution operator can be simplified to give
\[ \hat{U} = \cos \theta \mathbf{1} - \frac{i}{\sqrt{3}} \sin(\theta) \sum_{i=1}^{3} \sigma_i^x \sigma_i^z. \]  
(3)

where \( \mathbf{1} \) denotes the identity operator and \( \theta = \sqrt{3} KT \).

The time evolved meter operators after time \( T \) in the Heisenberg picture can be written as
\[ \sigma_i^A(T) = U \sigma_i^A U^\dagger = \cos \theta \sigma_i^A - \sin(2\theta) \sigma_i^y \sigma_i^A / \sqrt{3} \]
\[ + \frac{1}{3} \sin^2 \theta \left( \sigma_i^A + 2 \sigma_i^y \sum_{j,k=1}^{3} \epsilon_{ijk} \sigma_j^x \sigma_k^z \right). \]  
(4)

where \( \epsilon_{ijk} \) is the totally antisymmetric symbol with \( \epsilon_{123} = 1 \), and \( \sigma_i^1 = \sigma_i^x, \sigma_i^2 = \sigma_i^y, \sigma_i^3 = \sigma_i^z \); this yields, for example,
\[ \langle T | \sigma_i^A | T \rangle = \langle 0 | \cos \theta \sigma_i^A - \sin(2\theta) \sigma_i^y \sigma_i^A / \sqrt{3} \]
\[ + \frac{1}{3} \sin^2 \theta \left( \sigma_i^A + 2 \sigma_i^y \sum_{j,k=1}^{3} \epsilon_{ijk} \sigma_j^x \sigma_k^z \right) \langle 0 |. \]  
(5)

If we start with an initial state,
\[ |0\rangle = |\psi\rangle|A_1\rangle|A_2\rangle|A_3\rangle \equiv |\psi,+,+,+\rangle \]
(6)

where \( |\psi\rangle \) is the unknown initial state of particle \( P \), \( |\pm\rangle \) are eigenstates of the Pauli Matrix \( \sigma_y \) with eigenvalues \( \pm 1 \),
\[ \sigma_y|\pm\rangle = \pm|\pm\rangle, \sigma_y|+\rangle = |-\rangle, \]
(7)

we obtain,
\[ \langle 0 | \Sigma_i | 0 \rangle = \langle \psi | \sigma_i^p | \psi \rangle, \]
(8)

where,
\[ \Sigma_i = -\frac{\sqrt{3}}{\sin(2\theta)} \sigma_i^A(T). \]
(9)

For the variances,
\[ (\Delta \sigma_i^p)^2 = \langle \psi (\sigma_i^p)^2 \psi \rangle - \langle \psi \sigma_i^p | \psi \rangle^2 \]
\[ (\Delta \Sigma_i)^2 = \langle 0 | \Sigma_i^2 | 0 \rangle - \langle 0 | \Sigma_i | 0 \rangle^2 \]
(10)

we have the uncertainty relations,
\[ (\Delta \Sigma_i)^2 - (\Delta \sigma_i^p)^2 = \frac{3}{\sin^2(2\theta)} - 1 \geq 2 \]
\[ \sum_{i=1}^{3} (\Delta \sigma_i^p)^2 = 2, \]
\[ \sum_{i=1}^{3} (\Delta \Sigma_i)^2 = \frac{9}{\sin^2(2\theta)} - 1 \geq 8. \]
(11)

The uncertainty relations in Eqs. (10) and (11) in the case of spin measurements are the analogues of the measurement uncertainty relations in the standard Arthurs-Kelly case of \( q, p \) measurements. The tracking of \( \sigma_i^p \) by \( \Sigma_i \) is not noiseless; the minimum noise is achieved for \( \theta = \pi/4 \).

**Entanglement generation.** The time evolved state after time \( T \) is,
\[ |T\rangle = \cos \theta |\psi,+,+,+\rangle - \frac{i \sin \theta}{\sqrt{3}} \left( \sigma_y^+ |\psi,+,+,+\rangle \right. \]
\[ + \sigma_y^- |\psi,+,+,+\rangle + \sigma_x^+ |\psi,+,+,+\rangle \].  
(12)

If the system qubit is denoted by
\[ |\psi\rangle = \begin{pmatrix} a \\ b \end{pmatrix} = a|0\rangle + b|1\rangle, \]
(13)

the above state can be expressed in the following form:
\[ |T\rangle = |0\rangle \left( a \cos \theta |++\rangle - i \sin \theta / \sqrt{3} \right) \]
\[ \times (b|---\rangle - ib|--\rangle + a|++\rangle + |++\rangle)) \]
\[ + |1\rangle \left( b \cos \theta |++\rangle - i \sin \theta / \sqrt{3} \right) \]
\[ \times (a|--\rangle + ia|--\rangle - b|--\rangle + |++\rangle) \].  
(14)

Note that the apparatus states multiplying the system states \( |0\rangle \) and \( |1\rangle \) are mutually orthogonal if and only if \( \cos^2 \theta = 1/4 \); in that case the above state is expressed in the Schmidt biorthogonal form [19].

The final reduced density matrices for the system qubit \( P \) and the three-qubit apparatus \( A_1, A_2, A_3 \) are,
\[ \rho^P = Tr(A_1, A_2, A_3) |T\rangle \langle T|; \rho^A = Tr(P) |T\rangle \langle T|. \]
(15)

This yields,
\[ \rho^P_{00} = \cos^2 \theta |a|^2 + \frac{1}{3} \sin^2 \theta (1 + |b|^2) \]
\[ \rho^P_{11} = (\rho^P_{10})^* = (\cos^2 \theta - \frac{1}{3} \sin^2 \theta) ab^* \]
\[ \rho^P_{01} = \cos^2 \theta |b|^2 + \frac{1}{3} \sin^2 \theta (1 + |a|^2) \]

The system-apparatus entanglement \( E \) is given by the von Neumann entropy of the reduced density matrix of system \( P \) or equivalently of the apparatus \( A \), [20]
\[ E = -Tr \rho^P \ln \rho^P = -Tr \rho^A \ln \rho^A \]
\[ = -\lambda \ln \lambda - (1 - \lambda) \ln (1 - \lambda), \]
(16)

where \( \lambda \) and \( 1 - \lambda \) are eigenvalues of \( \rho^P \) which obey
\[ \lambda(1 - \lambda) = det \rho^P = \frac{2}{9} \sin^2 \theta (1 + 2 \cos^2 \theta). \]

The entanglement is maximum when \( \lambda = 1/2 \), i.e. \( \cos^2 \theta = 1/4 \),
\[ E \leq \ln 2; \ E = \ln 2 \text{ for } \cos^2 \theta = 1/4. \]  
(17)
For $\cos\theta = 1/2$, $\sin\theta = \pm\sqrt{3}/2$, the corresponding maximally entangled final states $|T\pm\rangle$ assume the Schmidt biorthogonal forms,

$$|T\pm\rangle = \frac{|0\rangle}{2} \left(a|++\rangle \mp i|b|--\rangle - i|b|--\rangle + a|---\rangle\right)$$

These states, are analogous to the two qubit EPR states, with one of the qubits replaced by three qubits.

Since $\theta = \sqrt{3}KT$, it is seen that by varying the product of the strength and duration of interaction, such that $\cos\theta = 1/2, \sin\theta = \pm\sqrt{3}/2$, maximal entanglement between the system and the apparatus can be achieved.

**Teleportation.**

Suppose we wish to transfer the unknown state of the particle $P$ on to one of the apparatus particles, say $A_2$, which can then be teleported. This might be useful e.g. if $A_2$ is more easily transported over a quantum channel than $P$ or if it has a longer lifetime than $P$. If we expand the state $|T\rangle$ for general $\theta$ in the basis of the orthogonal states $|\pm\rangle$ for the $A_1, A_3$ particles, we obtain,

$$|T\rangle = |+\rangle^{A_1} |+\rangle^{A_3}$$

$$|0\rangle(a \cos\theta|+\rangle^{A_2} - b(\sin\theta/\sqrt{3})|−\rangle^{A_2})$$

$$+|1\rangle(b \cos\theta|+\rangle^{A_2} + a(\sin\theta/\sqrt{3})|−\rangle^{A_2})$$

$$+|+\rangle^{A_1}|−\rangle^{A_1} |+\rangle^{A_2}(-i \sin\theta/\sqrt{3})(a|0\rangle - b|1\rangle)$$

$$+|−\rangle^{A_1} |+\rangle^{A_3} |+\rangle^{A_2}(-i \sin\theta/\sqrt{3})(b|0\rangle + a|1\rangle).$$

(19)

Strikingly, we see a connection between maximal entanglement and perfect teleportation: the coefficients of $|+\rangle^{A_1} |+\rangle^{A_3}|0\rangle$ and $|+\rangle^{A_1} |+\rangle^{A_3}|1\rangle$ are states of particle $A_2$ which are unitary transforms of the original unknown state of particle $P$ if and only if $\sin\theta = \pm\sqrt{3}/2$ (maximal entanglement between the system and the apparatus). For these special values of $\theta$, Eqn. (19) immediately suggests the following teleportation protocol. If we make measurements on $|T\pm\rangle$ to project it on to the sub-spaces $|+\rangle^{A_1} |+\rangle^{A_3}|0\rangle$ and $|+\rangle^{A_1} |+\rangle^{A_3}|1\rangle$, we obtain respectively the following normalized states of the qubit $A_2$,

$$2\langle 0| A_1 A_3 | T\pm\rangle = a|+\rangle^{A_2} - b|−\rangle^{A_2}$$

$$= -it_{U_1}^T (a|0\rangle \pm b|1\rangle)$$

$$2\langle 1| A_1 A_3 | T\pm\rangle = b|+\rangle^{A_2} \pm a|−\rangle^{A_2}$$

$$= \pm iU_{U_2}^T (a|0\rangle \pm b|1\rangle)$$

(20)

where $U_1$ and $U_2$ are the unitary transformations

$$U_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -1 & i \end{pmatrix} ; U_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$$

(21)

For general $\theta$, we can read off the corresponding non-unitary transformations $U_1, U_2$, and see that,

$$(U_1^T U_1 + U_2^T U_2)/2 - 1 = (1 - (4/3) \sin^2\theta)1,$$  

(22)

where 1 denotes the unit matrix, and the right-hand side gives a quantitative measure of the imperfection of teleportation when $\sin^2\theta \neq 3/4$.

If $\sin^2\theta = 3/4$, we recover $|\psi\rangle^{A_2}$ by applying the appropriate unitary transforms $U_1, U_2$ if $\cos\theta = 1/2, \sin\theta = \pm\sqrt{3}/2$, and the unitary transforms $\sigma_z U_1, \sigma_z U_2$ if $\cos\theta = 1/2, \sin\theta = -\sqrt{3}/2$. On the other hand coefficients of $|+\rangle^{A_1} |−\rangle^{A_1} |+\rangle^{A_2}$ and $|−\rangle^{A_1} |+\rangle^{A_1} |−\rangle^{A_2}$ are proportional to $U_3 |\psi\rangle^P$ and $U_4 |\psi\rangle^P$, where,

$$U_3 = \sigma_z : U_4 = \sigma_x.$$  

(23)

A flowchart of the teleportation protocol is provided in Fig.1 for the case $\cos\theta = 1/2, \sin\theta = \pm\sqrt{3}/2$. A measurement of $\sigma_z^{A_1} = Y(A_1)$ and $\sigma_x^{A_3} = Y(A_3)$ separates the emerging particles into three beams with probabilities 1/2, 1/4 and 1/4.

The first beam with $Y(A_1) = +1, Y(A_3) = +1$ (probability 1/2) is subjected to a measurement of $\sigma_z^P = Z(P)$: this yields two beams of $A_2$ particles with $Z(P) = +1$ and $Z(P) = -1$ which are then subjected to the unitary transformations $U_1$ and $U_2$ respectively (see Eq. (21)) to yield the state $|\psi\rangle^{A_2}$ which is teleported through a quantum channel.

The second beam with $Y(A_1) = +1, Y(A_3) = -1, Y(A_2) = +1$ (probability 1/4) is subjected to a unitary transformation $U_3 = \sigma_z$ on the particle $P$, and the third beam $Y(A_1) = -1, Y(A_3) = +1, Y(A_2) = +1$ (probability 1/4) is subjected to a unitary transformation $U_4 = \sigma_x$ on the particle $P$ in both cases the original state $|\psi\rangle^P$ is recovered and recycled to get a fresh sequence of Arthurs-Kelly interaction and measurements.

For $\cos\theta = 1/2, \sin\theta = -\sqrt{3}/2$, the only change in the flow chart is that

$$U_1 \to U'_1 = \sigma_z U_1; U_2 \to U'_2 = \sigma_z U_2.$$  

(24)

The probability of getting the teleported state $|\psi\rangle^{A_2}$ after 0,1,2,...recyclings are $\frac{1}{2}, \frac{1}{4}, \frac{1}{8},...$ which add up to 1.

**Entanglement Swapping.** Suppose now that the particle $P$ sent for Arthurs-Kelly type interaction, instead of being in a state $|\psi\rangle^P$ is actually entangled with another particle $R$ in Alice’s laboratory, and their joint state is,

$$|\phi_1\rangle^R |\psi_1\rangle^P + |\phi_2\rangle^R |\psi_2\rangle^P.$$  

(25)

Then, using the linearity of Schrödinger Eqn., after the state of particle $P$ is teleported to that of $A_2$ in Bob’s laboratory, the particle $R$ in Alice’s lab. becomes entangled with $A_2$ in Bob’s lab. and their joint state is,

$$|\phi_1\rangle^R |\psi_1\rangle^{A_2} + |\phi_2\rangle^R |\psi_2\rangle^{A_2}.$$  

(26)
FIG. 1: A schematic diagram showing the new teleportation protocol for the case $\cos \theta = 1/2$, $\sin \theta = +\sqrt{3}/2$. The qubit in black is the system qubit $P$ in an unknown state, and the qubits in green, blue and red are suitably prepared apparatus qubits $A_1$, $A_2$ and $A_3$ respectively. After an Arthurs-Kelly type interaction, a measurement of $\sigma_{A_1}^y = Y(A_1)$ and $\sigma_{A_3}^y = Y(A_3)$ separates the emerging particles into three beams with probabilities $1/2$, $1/4$ and $1/4$. The first beam with $Y(A_1) = +1$, $Y(A_3) = +1$ (probability $1/2$) is subjected to a measurement of $\sigma_P^z = Z(P)$: this yields two beams of $A_2$ particles with $Z(P) = +1$ and $Z(P) = -1$ which are then subjected to the unitary transformations $U_1$ and $U_2$ respectively (see Eq. (21)) to yield the state $|\psi\rangle_{A_2}$ which is teleported through a quantum channel. The second beam with $Y(A_1) = +1$, $Y(A_3) = -1$, $Y(A_2) = +1$ (probability $1/4$) is subjected to a unitary transformation $U_3 = \sigma_z$ on the particle $P$, and the third beam $Y(A_1) = -1$, $Y(A_3) = +1$, $Y(A_2) = +1$ (probability $1/4$) is subjected to a unitary transformation $U_4 = \sigma_x$ on the particle $P$. In both cases the original state $|\psi\rangle_P$ is recovered and recycled to get a fresh sequence of Arthurs-Kelly interaction and measurements. For $\cos \theta = 1/2$, $\sin \theta = -\sqrt{3}/2$, the only change necessary in the flow chart is that $U_1 \rightarrow \sigma_x U_1$; $U_2 \rightarrow \sigma_x U_2$. The probability of getting the teleported state $|\psi\rangle_{A_2}$ after 0, 1, 2, ... recyclings are $1/2$, $1/4$, $1/8$, ... which add up to 1.

The main difference from the usual protocol is that the particle $A_2$ is taken not from a previously prepared EPR pair, but from the final state of the Arthurs-Kelly type interaction.

Comparison with usual teleportation protocols. A conventional Quantum Teleportation scheme has 4 main steps: (i) An EPR pair $E_1, E_2$ is shared by Alice and Bob at distant locations. (ii) The system particle $P$ with unknown state is received by Alice and she makes a Bell-state measurement on the joint state of that particle and $E_1$ and (iii) communicates the result via a classical channel to Bob; (iv) Bob then makes a unitary transformation on $E_2$ depending on the classical information to replicate the unknown system state. In the alternative Teleportation scheme reported here, the steps of EPR-pair sharing, Bell projection and classical communication are not necessary; instead, the Arthurs-Kelly entangling interaction and single particle spin measurements are used to “transfer” an unitary transform of the unknown state to an apparatus qubit. The unknown state can be recovered from the apparatus qubit by applying the inverse unitary transform, either before or after teleportation of that qubit. One advantage of the present scheme, apart from not needing EPR-sharing, is that single particle spin measurements are much easier than Bell state measurements.

Conclusions. We have shown that the Levine et al Arthurs Kelly type interaction can generate maximal entanglement between a system qubit and three apparatus qubits. We utilise this to introduce a novel scheme
of teleportation which has some advantages over the conventional methods. The main new task is the realization of the Arthurs-Kelly interaction. The technology and experimental realizations of the Arthurs-Kelly interaction for optical quadratures (such as $q,p$) are widely known in the context of optical Homodyne and Heterodyne Tomography (see e.g. [21], [22], [23], [24]). The present work provides a concrete motivation to extend the technology to realize the Levine et al [16] proposal of an Arthurs Kelly type interaction between qubits. Possible qubit systems are solid-state nuclear and electronic qubits in diamond as in the “Unconditional quantum teleportation between distant solid-state quantum bits” [25], in

ion-traps as in ‘Quantum teleportation between distant qubits’ [26], and polarized photon qubits as in tests of Bell inequalities [27]. Once the teleported state $|\psi\rangle_A^Z$ is realized, applications to long distance quantum communications via quantum memory and quantum repeaters might be possible [28].

Acknowledgments. S.S. thanks the NIUS program in HBCSE, TIFR and KVPY scholarship program; S. M. R thanks the Indian National Science Academy for an INSA senior scientist grant. We thank the referees for many suggestions on improvements of the manuscript and important references to experimental work.