DONUT CHOIRS AND SCHIEMANN’S SYMPHONY
AN IMAGINATIVE INVESTIGATION OF THE ISOSPECTRAL
PROBLEM FOR FLAT TORI.

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Abstract. Flat tori are among the only types of Riemannian manifolds for
which the Laplace eigenvalues can be explicitly computed. In 1964, Milnor
used a construction of Witt to find an example of isospectral non-isometric
Riemannian manifolds, a striking and concise result that occupied one page in
the Proceedings of the National Academy of Science of the USA. Milnor’s ex-
ample is a pair of 16-dimensional flat tori, whose set of Laplace eigenvalues are
identical, in spite of the fact that these tori are not isometric. A natural ques-
tion is: what is the lowest dimension in which such isospectral non-isometric
pairs exist? Do you know the answer to this question? The isospectral ques-
tion for flat tori can be equivalently formulated in analytic, geometric, and
number theoretic language. We take this opportunity to explore this question
in all three formulations and describe its resolution by Schiemann in the 1990s.
We explain the different facets of this area; the number theory, the analysis,
and the geometry that lie at the core of it and invite readers from all back-
grounds to learn through exercises. Moreover, there are still a wide array of
open problems that we share here. In the spirit of Mark Kac and John Horton
Conway, we introduce a playful description of the mathematical objects, not
only to convey the concepts but also to inspire the reader’s imagination, as
Kac and Conway have inspired us.

1. AN APPETIZING INTRODUCTION

The Laplace eigenvalue problem is broadly appealing because it connects physics,
number theory, analysis and geometry. At the same time, it is a challenging and
frustrating problem because in general, one cannot solve it analytically. Wielding
heavy tools from functional analysis, we can prove that solutions exist, but this is
not nearly as satisfying as being able to stare down the solution in the face and say,
gotcha! There is, however, a notable exception: flat tori. One can visualize a two
dimensional flat torus as a donut in Figure 1a. However, this is not quite correct,
because as a Riemannian surface, such a donut has a curved surface whereas the
curvature of a flat torus is identically zero. Although there is no smooth isometric
embedding of a flat $n$ dimensional torus into $n + 1$ dimensional Euclidean space,
there is a $C^1$ embedding discovered by Nash [36] and Kuiper [33]. This embedding
remained mysterious, eluding visualization until 2012 by Borrelli, Jabrane, Lazarus

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This visualization is much more complicated than Figure 1a, because the surface of the torus exhibits a fractal behavior in the normal vector, resulting in a corrugated structure that is reminiscent of ridges of piped frosting as shown Figure 1b. Consequently, rather than dismissing the association of donuts and flat tori, we imagine that their complicated surface structure reflects the wealth of possible frostings, sprinkles, and further decorations on a donut’s surface, and we will throughout use the term donut for a flat torus.

(a) A two dimensional flat torus is reminiscent of a tasty donut - with one caveat - whereas this donut is curved a flat torus is flat.

(b) A $C^1$ isometric embedding of a two-dimensional flat torus into three dimensional Euclidean space has a rough, corrugated looking surface as depicted in [3].

Figure 1. Combining these two observations, we identify flat tori with donuts keeping in mind that the surface of the donut may be quite complicated, perhaps due to frosting, sprinkles, and further toppings. Images sources and licenses: openclipart.org, CC0 1.0; HEVEA Project (V. Borrelli, S. Jabrane, F. Lazarus, D. Rohmer, B. Thibert.), CC BY-SA 2.0.

Mathematically, a donut is the quotient of $\mathbb{R}^n$ by a full-rank lattice with Riemannian metric induced by the standard Euclidean metric on $\mathbb{R}^n$. It is a smooth and compact Riemannian manifold, whose Riemannian curvature tensor is identically zero. For the sake of completeness and inclusivity, we recall

Definition 1.1 (Lattices and donuts). An $n$-dimensional (full rank) lattice $\Gamma \subset \mathbb{R}^n$ is a set which can be expressed as $\Gamma := AZ^n$ for an invertible $n \times n$ matrix $A$ with real coefficients. The matrix $A$ is called a basis matrix of $\Gamma$. The lattice defines the flat torus $T_\Gamma = \mathbb{R}^n/\Gamma$, with Riemannian metric induced by the Euclidean metric on $\mathbb{R}^n$. We equivalently refer to $T_\Gamma$ as a donut, or as the donut belonging to the lattice $\Gamma$.

The Laplace eigenvalue problem in this context is to find all functions defined on $\mathbb{R}^n$ for which there exists $\lambda \in \mathbb{C}$ such that

$$\Delta f(x) = \lambda f(x), \quad f(x + \ell) = f(x) \quad \text{and}$$

$$\nabla f(x + \ell) = \nabla f(x) \quad \forall \ell \in \Gamma \text{ and } x \in \mathbb{R}^n.$$

(1.1)
Our sign convention for the Euclidean Laplace operator is
\[ \Delta = -\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}. \]

If \( f \) is a solution to the Laplace eigenvalue problem, it is an \textit{eigenfunction}. With a bit of functional analysis [13] one can prove that the corresponding eigenvalue is non-negative. Let’s keep things light and simple and start with a one-dimensional donut. In that case, a one-dimensional lattice consists of all integer multiples of a number, \( \ell \). The Laplace eigenvalue problem on the donut \( \mathbb{R}/\ell\mathbb{Z} \) is to find all functions which satisfy
\[ -f''(x) = \lambda f(x), \quad f(x + k\ell) = f(x), \quad f'(x + k\ell) = f'(x), \quad \forall k \in \mathbb{Z}. \]

We can solve this using elementary methods. The functions
\[ f_n(x) := e^{2\pi inx/\ell}, \quad n \in \mathbb{Z} \]
are eigenfunctions, with corresponding eigenvalues
\[ \lambda_n = \frac{4\pi^2 n^2}{\ell^2}. \]

To prove that the functions \( \{f_n\}_{n \in \mathbb{Z}} \) are all of the eigenfunctions up to scaling, it suffices to prove that they constitute an orthogonal basis for the Hilbert space \( L^2 \), as demonstrated in [18].

The language in our title is inspired by Kac’s famous paper [27]. Imagine a vibrating donut. According to the laws of physics, the sound this vibrating donut creates is described by its Laplace eigenvalues, which are in bijection with its resonant frequencies. For a particular donut, the set of all of its Laplace eigenvalues with multiplicities is known as its \textit{spectrum}. Consequently, if you happen to have perfect hearing, then you could hear all of these eigenvalues. You would therefore also be able to hear all of the \textit{spectral invariants}, which are all quantities that are entirely determined by the spectrum. In this spirit, we imagine that the vibrating donut is in fact singing, and the song that it sings is identified with its spectrum. We refer to a quantity as \textit{audible} if it is a spectral invariant, so in that sense it can be heard. If several donuts are isospectral (meaning they possess identical spectrum) then they sing exactly the same song, and so we can further imagine that they belong to a choir that sings in unison.

If two donuts belong to the same choir, then what can we say about their shapes? Are they necessarily the same? As Riemannian manifolds, one would say that two flat tori are geometrically identical if they are isometric. Are isospectral flat tori necessarily also isometric, or, in our playful language, if donuts sing the same song then are they the same shape? If not, are there certain geometric characteristics that \textit{are} identical, and if so, what are those characteristics? In other words, what geometric features can be heard from the songs the donuts sing?

1.1. \textbf{Three perspectives.} We are interested in the question: if two flat tori are isospectral, then are they isometric? In our playful language inspired by Kac and Conway, we may rephrase this as: if two donuts sing the same song, are they the same shape? It turns out that this question has an equivalent formulation in both number theoretic terms as well as purely geometric terms. This observation is crucial to obtain a thorough investigation of the question. These three equivalent questions are:
Does the spectrum of the Laplace operator determine the geometry of flat tori?

Are positive definite quadratic forms determined by their values over the integers counting multiplicity?

Do the lengths of points of a lattice with multiplicity determine the lattice itself up to congruency?

The first perspective gives the subject physical motivation. The second perspective is in some ways more convenient for practical purposes of investigating this problem as demonstrated in for example Schiemann’s work [44]. The third perspective is a more intuitive and purely geometric question about lattices. It is unfortunate that the mathematical language is quite different when the problem is investigated from these different viewpoints in the sense that disparate fields do not cross-reference each other. Consequently, we will take this opportunity to connect some of the different terminologies. We aim to provide interested readers with a more thorough understanding of the question by studying it from all three perspectives.

1.2. Organization: the menu. This note introduces spectral geometry in general, and the spectral theory of donuts and their associated lattices in particular. We further take the opportunity to highlight the connections between geometry, analysis, and number theory. To ease readers into the field, instead of providing proofs for the ‘basics,’ we invite you to learn by doing, that is to work out these proofs. To support your efforts, we also provide references just in case anyone gets stuck; see FR’s homepage, that contains hints and solutions.

In §2 we collect the essential ingredients required to investigate donuts, their songs and choirs, and we present a case study, referred to as an amuse-bouche. To manipulate these ingredients, in §3 we introduce useful techniques, or key kitchenware, for investigating the spectrum and geometry of donuts. We explore in §4 famous examples of different donuts (non-isometric) that nonetheless sing the same song (are isospectral). Section §5 is dedicated to popularizing the fundamental yet not widely known theorem of Schiemann. We conclude in §6 with a collection of conjectures and open problems. This includes a discussion of the open question: how many donuts can sing the same song in a given dimension? We hope that this work will inspire, engage, and entice mathematicians from all backgrounds and career stages.

2. Essential ingredients and an amuse-bouche

We begin with the key ingredients in the mathematics of donuts and the songs they sing.

2.1. Lattices, congruency, isometry, and identifying donuts of the same shape. Donuts are defined in terms of lattices that in turn are based on matrices, so we begin by recalling the fundamental matrix groups. The set of all invertible $n \times n$ matrices with real coefficients is denoted $GL_n(\mathbb{R})$. This set enjoys a group structure with the group operation matrix multiplication and is known as the general linear group. The subgroup $GL_n(\mathbb{Z}) \subset GL_n(\mathbb{R})$ consists of those matrices $M \in GL_n(\mathbb{R})$ such that both $M$ and $M^{-1}$ have integer coefficients. The elements of $GL_n(\mathbb{Z})$ are known as unimodular matrices, and $GL_n(\mathbb{Z})$ is the unimodular group. The subgroup
$O_n(\mathbb{R}) \subset GL_n(\mathbb{R})$ consists of those matrices whose inverse matrix and transpose matrix are equal, and is known as the orthogonal group. It may be equivalently described as the group of rotations and reflections of $\mathbb{R}^n$, with the group operation composition.

Lemma 2.1. A matrix is an element of $O_n(\mathbb{R})$ if and only if its column vectors form an orthonormal basis of $\mathbb{R}^n$. Further, $A \in O_n(\mathbb{R})$ if and only if $Ax \cdot Ay = x \cdot y$ for any $x, y \in \mathbb{R}^n$, where $\cdot$ denotes the Euclidean inner product.

Exercise 2.2. This proof is an exercise!

Exercise 2.3. Show that $B \in GL_n(\mathbb{Z})$ if and only if $B \in \mathbb{Z}^{n \times n}$ and $\det(B) = \pm 1$. Moreover, $C \in O_n(\mathbb{R})$ if and only if the linear transformation $C : \mathbb{R}^n \to \mathbb{R}^n$ defined by $C(v) = Cv$ maps any orthonormal basis of $\mathbb{R}^n$ to another orthonormal basis of $\mathbb{R}^n$.

Proposition 2.4. Two matrices $A_1$ and $A_2$ in $GL_n(\mathbb{R})$ are both bases for the same (full-rank) lattice, meaning $A_1\mathbb{Z}^n = A_2\mathbb{Z}^n$ if and only if there is a matrix $B \in GL_n(\mathbb{Z})$ such that $A_2 = A_1B$.

Exercise 2.5. This proof is an exercise.

One way of constructing full-rank lattices is to build them from lower rank lattices, and we therefore define these.

Definition 2.6 (Lattices of arbitrary rank and trivial lattices). Let $v_1, \ldots, v_k \in \mathbb{R}^n$ be a set of linearly independent vectors. They define a $k$-rank lattice in the following way,

$$\Gamma := v_1\mathbb{Z} + \cdots + v_k\mathbb{Z} = \left\{ \sum_{j=1}^{k} z_jv_j : z_j \in \mathbb{Z} \ \forall j \right\}.$$

The matrix whose column vectors are equal to $v_1, \ldots, v_k$ is a basis matrix. The dimension, or equivalently, rank of the lattice is $k$. Consequently, full-rank lattices are $n$-rank lattices in $\mathbb{R}^n$. A trivial lattice is a set whose only element is the additive identity, 0.

The following result from [38, p. 24] shows that all non-trivial discrete additive subgroups in $\mathbb{R}^n$ are lattices as defined here.

Proposition 2.7. A non-trivial additive subgroup $\Gamma \subset \mathbb{R}^n$ is discrete if and only if it is a $k$-rank lattice for some $1 \leq k \leq n$.

A sublattice $\Lambda$ of a lattice $\Gamma$ is a lattice such that $\Lambda \subseteq \Gamma$. The following characterization will be useful in §3.

Proposition 2.8. Let $\Lambda$ and $\Gamma$ be full-rank lattices and let $A_\Lambda, A_\Gamma$ be corresponding bases.

1. $\Lambda$ is a sublattice of $\Gamma$ if and only if $A_\Lambda = A_\Gamma V$ for some $V \in \mathbb{Z}^{n \times n}$.
2. If $\Lambda \subseteq \Gamma$, then $\det(A_\Lambda)/\det(A_\Gamma) \in \mathbb{Z} \setminus \{0\}$ and $(\det(A_\Lambda)/\det(A_\Gamma))\Gamma \subseteq \Lambda$.
3. If $\Lambda \subseteq \Gamma$, then the index of $\Lambda$ in $\Gamma$ as a subgroup, denoted $[\Gamma : \Lambda]$, is equal to $|\det(A_\Lambda)/\det(A_\Gamma)|$. 
Proof. Statements (1), (2) are direct observations. A proof of (3) can be found in [17, §1.2.4].

2.1.1. Congruence and isometry. Some donuts are indistinguishable because they are exactly the same shape; mathematically the flat tori are said to be isometric. This is the case if and only if the lattices that define these tori are congruent, meaning that they can be mapped to each other by a finite sequence of rotations and reflections.

Definition 2.9 (Congruent lattices). Let $\Gamma_1 \subset \mathbb{R}^n$ and $\Gamma_2 \subset \mathbb{R}^m$ be lattices. If $n = m$, then $\Gamma_1$ and $\Gamma_2$ are congruent if there is a $C \in O_n(\mathbb{R})$ such that $\Gamma_2 = C\Gamma_1$. If $n > m$, then $\Gamma_1$ is congruent to $\Gamma_2$ if $\Gamma_2 \times \{0\} = C\Gamma_1$ for an orthogonal matrix $C \in O_n(\mathbb{R})$ and a trivial lattice in $\mathbb{R}^{n-m}$. Congruency is denoted by $\cong$ and is an equivalence relation.

Lemma 2.10. Let $\Gamma_1 = A_1\mathbb{Z}^n$ and $\Gamma_2 = A_2\mathbb{Z}^n$ be two full-rank lattices. Then

$$\Gamma_1 \cong \Gamma_2 \iff CA_1 = A_2B$$

for some $B \in \text{GL}_n(\mathbb{Z}), C \in O_n(\mathbb{R})$.

Exercise 2.11. Prove the lemma.

Definition 2.12 (Isometric donuts). We say that two flat tori are isometric if they are isometric in the Riemannian sense, viewing the flat tori as Riemannian manifolds. We write $T_1 \cong T_2$ to denote that $T_1$ and $T_2$ are isometric.

We use the same notation $\cong$ for both isometry of flat tori as well as congruency of lattices because of the following

Theorem 2.13 (see p. 5 of [1]). Two flat tori are isometric in the Riemannian sense if and only if their associated lattices are congruent.

2.2. The spectrum of a flat torus is the song the donut sings. To investigate the sound of donuts, we need to define their spectra, which shall in turn be defined via the dual lattice.

Definition 2.14. For a lattice $\Gamma \subset \mathbb{R}^n$, its dual lattice is defined to be

$$\Gamma^\ast := \{\ell \in \mathbb{R}^n : \ell \cdot \gamma \in \mathbb{Z}, \forall \gamma \in \Gamma\}.$$

Exercise 2.15. Prove that the dual lattice of a full-rank lattice $\Gamma$ is itself a lattice, and if $A$ is a basis for $\Gamma$, then $A^{-T}$ is a basis for $\Gamma^\ast$. Notice that there is a natural bijection between $\Gamma^\ast$ and $\text{Hom}(\Gamma, \mathbb{Z})$, the set of homomorphisms from $(\Gamma, +)$ to $(\mathbb{Z}, +)$, justifying the name "dual" lattice.

Theorem 2.16 (The Laplace spectrum of a flat torus). The eigenvalues of a flat torus $\mathbb{R}^n/\Gamma$ are precisely $4\pi^2||\ell||^2$ such that $\ell$ is an element of the dual lattice, $\Gamma^\ast$. The multiplicity of such an eigenvalue is the number of distinct elements of $\Gamma^\ast$ that have the same length as $\ell$. The eigenspace is spanned by the functions

$$\{u_\ell(x) = e^{2\pi ix \cdot \ell}\}_{\ell \in \Gamma^\ast}.$$

The collection of eigenvalues, counted with multiplicity, is the spectrum of the flat torus.

The proof of this theorem follows from Exercise 2.17 below.

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1The symbol used here to indicate the end of a proof is reminiscent of a donut.
Exercise 2.17. Verify that for any $\ell$ in the dual lattice, the function $u_\ell(x) = e^{2\pi i x \cdot \ell}$ satisfies both the Laplace eigenvalue equation on $\mathbb{R}^n$ as well as $u_\ell(x + \gamma) = u_\ell(x)$, and $\nabla u_\ell(x + \gamma) = \nabla u_\ell(x)$ for all $\gamma \in \Gamma$. For an extra challenge, prove that the functions $u_\ell$ are an orthogonal basis for the Hilbert space $L^2$ on the torus. Hint: start in one dimension using the pointwise convergence of Fourier series [18].

The eigenvalues determine the resonant frequencies of solutions to the wave equation on the torus. For this reason we identify the eigenvalues with sound.

Definition 2.18 (Isospectrality). Two flat tori $\mathbb{R}^n/\Gamma$ and $\mathbb{R}^n/\Lambda$ are isospectral if they have the same Laplace spectrum.

We say that two donuts sing the same song if they are isospectral; to inspire your imagination see Figure 2.

Exercise 2.19. Show that donuts that are geometrically identical always sing the same song; that is, if two flat tori are isometric, then they are isospectral.

Is the converse true? We explore this question in §5.

2.2.1. Poisson’s summation formula elucidates the songs donuts sing. Poisson’s summation formula is a powerful tool because it equates a purely analytical object with a purely geometric one. We say that Poisson’s summation formula elucidates the songs donuts sing, because we use it to prove that the dimension, volume, and length spectrum are spectral invariants, meaning we can hear them.

Definition 2.20. For a flat torus $\mathbb{T}_\Gamma = \mathbb{R}^n/\Gamma$, with lattice $\Gamma = AZ^n$, the volume of $\mathbb{T}_\Gamma$ (with respect to the flat Riemannian metric induced by the Euclidean metric on $\mathbb{R}^n$) is equal to

$$\text{vol}(\mathbb{T}_\Gamma) := |\det(A)|.$$

We define the volume of a non-trivial, not necessarily full-rank lattice $\Gamma$ to be $\text{vol}(\Gamma) := |\det(A')|$, where $A'Z^k$ is a full-rank lattice such that $\Gamma \cong A'Z^k$.

Exercise 2.21. It may not be immediately apparent that the right side above is well defined, because the lattice may be equivalently expressed by infinitely many different basis matrices. Prove that the volume is indeed well defined because it is independent of the choice of basis matrix.
Exercise 2.22. An integer lattice $\Gamma$ lies in $\mathbb{Z}^n$. Prove that if $\Gamma$ is an integer lattice, then $\text{vol}(\Gamma)\mathbb{Z}^n \subseteq \Gamma$. Further, for any full-rank lattices $\Lambda \subseteq \Lambda'$, we have $\Lambda = \Lambda'$ if and only if $\text{vol}(\Lambda) = \text{vol}(\Lambda')$.

Definition 2.23 (The length spectrum). For a flat torus $T_{\Gamma} = \mathbb{R}^n / \Gamma$, with lattice $\Gamma = A\mathbb{Z}^n$, the length spectrum of $T_{\Gamma}$ (with respect to the flat Riemannian metric induced by the Euclidean metric on $\mathbb{R}^n$) is equal to the collection of lengths of closed geodesics, counted with multiplicity, and denoted by $L_\Gamma$. This length spectrum is also equal to the collection of lengths of lattice vectors, $||\gamma||$ for $\gamma \in \Gamma$, counted with multiplicity, which is how we define the length spectrum of the lattice $\Gamma$.

The Poisson summation formula [9, p. 125] equates a sum over the Laplace spectrum with a sum over the length spectrum, thereby equating a purely analytical object with a purely geometric quantity.

Theorem 2.24 (Poisson summation formula). For an $n$-rank lattice $\Gamma$ the following series converge for any $t \in (0, \infty)$ and satisfy

\begin{align}
\sum_{\gamma^* \in \Gamma^{*}} e^{-4\pi^2 ||\gamma^*||^2 t} &= \frac{\text{vol}(\Gamma)}{(4\pi t)^{n/2}} \sum_{\gamma \in \Gamma} e^{-||\gamma||^2 / 4t} \\
\sum_{\gamma \in \Gamma} e^{-||\gamma||^2 / 4t} &= (4\pi t)^{n/2} \frac{\text{vol}(\Gamma)}{\sum_{\gamma^* \in \Gamma^{*}} e^{-4\pi^2 ||\gamma^*||^2 t}}
\end{align}

The first series in Poisson’s summation formula is a spectral invariant known as the heat trace.

Definition 2.25. The heat trace of a flat torus $T_{\Gamma}$ is defined for $t > 0$ by

$$
\sum_{\gamma^* \in \Gamma^{*}} e^{-4\pi^2 ||\gamma^*||^2 t} = \sum_{k \geq 0} e^{-\lambda_k t},
$$

where the eigenvalues $\lambda_k$ are ordered as $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \uparrow \infty$ and counting multiplicities.

A closely related spectral invariant is the theta series.

Definition 2.26. Let $\Gamma$ be a lattice. Then we define the theta series of the lattice (and its donut) as

$$
\theta_{\Gamma}(z) := \sum_{\gamma \in \Gamma} e^{i\pi z ||\gamma||^2}, \quad z \in \mathbb{C} \text{ with } \text{Im } z > 0.
$$

Exercise 2.27. Use Poisson’s summation formula to prove that the lattices $\Gamma, \Lambda$ are isospectral if and only if $\Gamma^{*}, \Lambda^{*}$ are.

The Poisson summation formula helps us to hear the dimension, volume, and length spectrum of a donut.

Corollary 2.28. Let $T^n_{\Gamma}$ be a flat torus. Then the heat trace of $T^n_{\Gamma}$ determines the spectrum of $T^n_{\Gamma}$, and therefore the flat tori $T_{\Gamma}$ and $T_{\Lambda}$ are isospectral if and only if they have identical heat traces. Two flat tori have the same length spectra if and only if they are isospectral. Two flat tori are isospectral if and only if they have identical theta series. If two flat tori are isospectral, then they have identical dimension and volume.

\[\text{We note that there is a related question of length-equivalence of lattices [39], in which one considers the set of lengths of lattice vectors, ignoring their multiplicity.}\]
Proof. We provide an outline and leave the details to the reader. To prove that the heat trace determines the spectrum, proceed inductively to prove that the heat trace determines all of the eigenvalues and their multiplicities by analyzing the asymptotic behavior of the heat trace as $t \to \infty$. This shows that if two flat tori have identical heat traces, then they are isospectral. If two flat tori are isospectral, then by definition, they have identical heat traces. Next, use Poisson’s summation formula to prove that two flat tori are isospectral if and only if they have the same length spectra. To prove the statement regarding the theta series, note that the series converges locally uniformly in the upper half space, and is therefore a holomorphic function of $z$ on this half space. The heat trace is obtained by evaluating the theta series of the dual lattice at $z = 4\pi it$ for $t > 0$. Use the identity theorem in complex analysis to prove that the heat traces of the dual lattices are identical if and only if their theta series are identical. Then use exercise 2.27 to prove that flat tori are isospectral if and only if they have the same theta series. To conclude that isospectral tori have identical dimension and volume, investigate the asymptotic behavior for $t$ approaching 0 and for $t$ approaching infinity in Poisson’s formula.

Remark 2.29. This corollary, facilitated by Poisson’s powerful summation formula, shows that the analytical and geometrical formulations in §1.1 are equivalent.

2.3. Quadratic forms are the key to our number theoretic formulation. We have seen that the analytical and geometrical formulations in §1.1 are equivalent with help of Poisson’s powerful summation formula. To equate these formulations in number theoretic language, we collect several facts about quadratic forms. In §2.3.1, we will use these facts to associate a quadratic form to a flat torus and identify its representation numbers with the donut’s song. Quadratic forms will also be essential in §5.

Definition 2.30. A quadratic form, $q$, of $n$ variables is a homogeneous polynomial of degree 2. If $q(x) \geq 0$ for all nonzero $x \in \mathbb{R}^n$, it is positive semi-definite, and if the inequality is strict, then the form is positive definite. We may equivalently refer to quadratic forms of $n$ variables as $n$-dimensional quadratic forms.

There is a well-known natural bijection between positive definite quadratic forms and symmetric matrices.

Exercise 2.31. Prove that for any quadratic form $q$ of $n$ variables, there is a unique symmetric $n \times n$ matrix $Q$, known as the associated matrix, such that $q(x) = x^T Q x$, $\forall x \in \mathbb{R}^n$.

The matrix $Q$ is positive (semi-)definite if and only if the quadratic form $q$ is.

Although it is a slight abuse of notation, we hope the reader will pardon our identification of quadratic forms with their associated matrices in the following

Definition 2.32. The notation $S^+_n$, respectively $S^0_n$, is the set of $n \times n$ positive definite, respectively semi-definite, matrices and is also identified with the set of positive definite, respectively semi-definite, quadratic forms of $n$ variables. A quadratic form is rational if the entries of its associated matrix are all rational. A quadratic form is even if the entries of its associated matrix are integers, and the
diagonal consists of even numbers. If a quadratic form \( q \) is positive definite, then its dual form, \( q^* \), is defined by
\[
q^*(x) := x^T Q^{-1} x, \quad \forall x \in \mathbb{R}^n.
\]

The matrix \( Q \) associated with a positive definite quadratic form admits a Cholesky factorization.

**Theorem 2.33** (Theorem 11.2 in [47]). Assume that \( Q \) is a symmetric \( n \times n \) real matrix such that \( x^T Q x > 0 \) for all non-zero \( x \in \mathbb{R}^n \). Then there is an invertible matrix \( A \) with \( Q = A^T A \). This is known as a Cholesky factorization.

Motivated by the Cholesky factorization, we define underlying lattices of a quadratic form.

**Definition 2.34.** For a positive definite \( n \times n \) matrix \( Q \) with Cholesky factorization \( Q = A^T A \), we say that \( A\mathbb{Z}^n \) is an underlying lattice of both \( Q \) and the associated quadratic form.

The following exercise shows that Cholesky factorization is unique up to left-multiplication with an orthogonal matrix, and that all underlying lattices of a given positive definite quadratic form are congruent.

**Exercise 2.35.** Let \( A_1, A_2 \) be invertible real matrices. Show that both \( A_1^T A_1 \) are symmetric and positive definite. Show that \( A_1^T A_1 = A_2^T A_2 \) if and only if \( A_2 = CA_1 \) for some \( C \in O_n(\mathbb{R}) \). Use this to conclude that all underlying lattices of a given positive definite quadratic form are congruent.

**Definition 2.36.** Two quadratic forms \( q \) and \( p \) on \( \mathbb{R}^n \) are integrally equivalent if their associated matrices \( Q \) and \( P \) satisfy \( B^T QB = P \), for a unimodular matrix \( B \).

The following proposition collects several useful facts about quadratic forms.

**Proposition 2.37.** Let \( Q \) be a real symmetric \( n \times n \) matrix. Let \( \lambda_{\min} \) be its smallest eigenvalue and \( \lambda_{\max} \) its biggest.

1. \( x^T Q x > 0 \) for all \( x \in \mathbb{Z}^n \setminus \{0\} \) if and only if \( x^T Q x > 0 \) for all \( x \in Q^n \setminus \{0\} \).
2. \( x^T Q x > 0 \) for all \( x \in \mathbb{Z}^n \setminus \{0\} \) implies \( Q \in S_{\geq 0}^n \).
3. \( Q \in S_{\geq 0}^n \) if and only if \( Q \) has only non-negative real eigenvalues.
4. \( Q \in S_{\geq 0}^n \) if and only if \( Q \) has only positive real eigenvalues.
5. \( \lambda_{\min} \|x\|^2 \leq x^T Q x \leq \lambda_{\max} \|x\|^2 \) for any \( x \in \mathbb{R}^n \).
6. \( Q \in S_{\geq 0}^n \) implies \( Q = E^T E \) for some \( n \times n \) real matrix \( E \).
7. \( Q \in S_{\geq 0}^n \) if and only if \( Q \in S_{\geq 0}^n \) and \( Q \) is of full rank.
8. Viewed as a quadratic form, the image of \( Q \in S_{\geq 0}^n \) over \( \mathbb{Z}^n \) is discrete, and all multiplicities are finite.

**Exercise 2.38.** Prove the proposition. Hint: use the spectral theorem for symmetric matrices.

**2.3.1. Representation numbers of quadratic forms echo the song the donut sings.**

The connection between the spectra of flat tori and quadratic forms is obtained using the representation numbers of quadratic forms. The representation numbers are the image of \( \mathbb{Z}^n \) under the quadratic form, taking into account multiplicities.

**Definition 2.39** (Representation Numbers). If \( q \) is an \( n \)-dimensional positive definite quadratic form, its representation numbers are defined as follows for \( t \in \mathbb{R}_{\geq 0} \)
\[
\mathcal{R}(q, t) := \# \{ x \in \mathbb{Z}^n : q(x) = t \}.
\]
We may also consider a subset \( X \subset \mathbb{Z}^n \) and define 
\[
R_X(q,t) := \# \{ x \in X : q(x) = t \}.
\]

**Exercise 2.40.** Let \( \mathbb{Z}_n^* := \{ x \in \mathbb{Z}^n \setminus \{0\} : \text{GCD}(x_1, \ldots, x_n) = 1, \text{and the last non-zero coordinate is positive} \} \).

Show that for two quadratic forms \( q_1 \) and \( q_2 \),
\[
R(q_1,t) = R(q_2,t) \iff R_X(q_1,t) = R_X(q_2,t) \text{ for } X = \mathbb{Z}_n^*, \quad \forall t \geq 0.
\]

A positive definite quadratic form \( q \) has a collection of underlying lattices, all of which are congruent. Consequently, the associated flat tori are all isometric and therefore also isospectral. On the one hand, for a full-rank lattice \( AZ^n \), for any unimodular matrix \( G \in \text{GL}_n(\mathbb{Z}) \), \( AGZ^n \) and \( AZ^n \) are the same lattice. On the other hand, the quadratic form with matrix \( (AG)^T(AG) \) is not necessarily the same as the quadratic form with matrix \( A^T A \). These two quadratic forms are, however, integrally equivalent, and the following exercise shows that they have identical representation numbers.

**Exercise 2.41.** Prove that if two \( n \)-dimensional positive definite quadratic forms are integrally equivalent, then they have identical representation numbers for all \( t \geq 0 \).

For a donut \( \mathbb{R}^n/\Gamma \) with \( \Gamma = AZ^n \) we associate the equivalence class of quadratic forms that are integrally equivalent to \( A^T A \). There is a natural bijection between the length spectrum of the donut and the representation numbers of this equivalence class, taking for some \( x \in \mathbb{Z}^n \), \( \|Ax\| \) to \( q(x) = x^T A^T Ax = \|Ax\|^2 \). It then follows from Corollary 2.28 that two donuts are isospectral if and only if their representation numbers associated in this way are identical. They are isometric if and only if their equivalence classes of quadratic forms are in fact identical. We therefore define isospectrality for donuts, lattices, and quadratic forms.

**Definition 2.42.** We say that two lattices \( \Gamma_i \subset \mathbb{R}^n \), \( i = 1, 2 \), are isospectral if and only if the flat tori \( \mathbb{R}^n/\Gamma_i \) have identical Laplace spectra, or equivalently, they have identical length spectra. Two quadratic forms are isospectral if and only if they have identical representation numbers.

The precise number theoretic formulation in §1.1 is then: is a quadratic form uniquely determined by its representation numbers, up to integral equivalence? We will see that the answer to the question depends on the dimension.

### 2.4. Composition and decomposition of lattices with implications for isospectrality.

One way of building higher dimensional donuts is to take products of lower dimensional ones, or similarly, to build a full-rank lattice from lower rank lattices. This technique has been historically important, which will be apparent in Section 4. We will also use it to give an elegant proof of the lower bound for the number of isospectral but non-isometric donuts in each dimension in §6.1. If a donut has been built as a product of lower dimensional donuts it is reducible, and if not, it is irreducible.\(^3\) We use the same terminology for the lattice that defines the donut. Reducibility of lattices have connections to root systems and Dynkin

\(^3\)Irreducible may also be termed indecomposable.
diagrams that are studied in the theory of Lie groups; see for example [23, p. 217]. A key ingredient in the definition of reducibility is the Minkowski sum.

Let $A$ and $B$ be two non-empty sets in $\mathbb{R}^n$. We define their Minkowski sum, denoted $A + B$, and their product, denoted $A \cdot B$ by the sets
\[
A + B := \{a + b : a \in A, b \in B\}, \quad A \cdot B := \{a \cdot b : a \in A, \quad b \in B\},
\]
where $a \cdot b$ is the scalar product in $\mathbb{R}^n$. If $A \cdot B = \{0\}$, we may write $A \oplus B$ to denote the Minkowski sum, which in this case is a direct sum.

**Definition 2.43** (Reducible & irreducible lattices and their donuts). Let $\Gamma \subseteq \mathbb{R}^n$ be a non-trivial lattice. Then $\Gamma$ is reducible if $\Gamma = \Gamma_1 \oplus \Gamma_2$ for two non-trivial lattices $\Gamma_1, i = 1, 2$. Otherwise $\Gamma$ is irreducible. The associated flat torus (donut) $\mathbb{R}^n/\Gamma$ is reducible or irreducible if $\Gamma$ is reducible or irreducible, respectively.

By the definition of irreducibility, a full-rank lattice $\Gamma \subset \mathbb{R}^n$ can be decomposed as a sum of irreducible sublattices $\Gamma_i$, so that
\[
\Gamma = \Gamma_1 \oplus \ldots \oplus \Gamma_k,
\]
for some $k \geq 1$. This is known as the irreducible decomposition, and is unique up to re-ordering. Kneser investigated a more general setup in [30]. The following lemma shows that an equivalent way to identify reducible or irreducible donuts is to investigate whether the associated lattice is congruent to a product of two (or more) lower rank lattices.

**Lemma 2.44.** A lattice $\Gamma \neq \{0\}$ is reducible if and only if it is congruent to a lattice of the form $\Gamma_1 \times \Gamma_2$ where $\Gamma_1, \Gamma_2$ are of dimensions at least 1. Conversely, if there are no such $\Gamma_1$ and $\Gamma_2$, then $\Gamma$ is irreducible.

**Proof.** Without loss of generality, we may assume that $\Gamma$ is a full-rank lattice in $\mathbb{R}^n$. Then, $\Gamma$ is reducible if and only if there are two non-trivial sublattices $G_1$ and $G_2$ such that $\Gamma = G_1 \oplus G_2$. These induce an orthogonal decomposition of $\mathbb{R}^n$ into two subspaces of dimensions $k_1$ and $k_2$, the ranks of $G_1$ and $G_2$, respectively, with $k_1 + k_2 = n$. There is an isometry, denoted $\Phi_i$, between each of these subspaces and $\mathbb{R}^{k_i}$, respectively, for $i = 1, 2$. Consequently, $G_i \cong \Phi_i(G_i) =: \Gamma_i \subset \mathbb{R}^{k_i}$, for each $i = 1, 2$, and $\Gamma = G_1 \oplus G_2 \cong \Phi_1(G_1) \times \Phi_2(G_2) = \Gamma_1 \times \Gamma_2$. \( \Box \)

**Exercise 2.45.** Prove that a lattice is irreducible if and only if its dual lattice is irreducible.

The following lemma shows that re-arranging the constituents in a product of lattices results in a congruent lattice. Essentially, we could think of re-arranging a collection of donuts in a box; as long as we don’t eat any of the donuts, it’s still the same collection of donuts as depicted in Figure 3. In the context of re-arranging donuts, we recall the symmetric group $S_n$, that is defined to be the group of permutations on $n$ elements.

**Lemma 2.46.** The product of lattices $\Gamma_1 \times \cdots \times \Gamma_m$ is congruent to the product $\Gamma_{\sigma(1)} \times \cdots \times \Gamma_{\sigma(m)}$ for any permutation $\sigma \in S_m$.

**Proof.** We outline the key ideas; details are left to the reader. Let $A = [a_{ij}]$ be an $n \times n$ matrix. The elements of $\text{GL}_n(\mathbb{Z})$ can by right multiplication change the order of the columns of $A$ in any way, and the elements of $O_n(\mathbb{R})$ can by left multiplication change the order of the rows of $A$ in any way. \( \Box \)
Proposition 2.47. Assume that a lattice $\Gamma$ in $\mathbb{R}^n$ can be decomposed into an orthogonal sum of sublattices

$$\Gamma = \Gamma_1 \oplus \cdots \oplus \Gamma_k.$$ 

Then $\Lambda \cong \Gamma$ if and only if $\Lambda$ is a direct sum of $k$ sublattices $\Lambda_i \cong \Gamma_i$.

Proof. Without loss of generality we assume that $\Lambda \subset \mathbb{R}^n$, and that $\Gamma$ is full-rank. If $\Lambda \cong \Gamma$, there is $C \in O_n(\mathbb{R})$ such that $\Lambda = C\Gamma = C(\Gamma_1 \oplus \cdots \oplus \Gamma_k)$. Since $C$ preserves orthogonality, $\Lambda = C\Gamma_1 \oplus \cdots \oplus C\Gamma_k$. Thus, defining $\Lambda_i := C\Gamma_i \cong \Gamma_i$ completes the proof in this direction. For the other direction, assume that $\Lambda_i \cong \Gamma_i$ for each $i = 1, \ldots, k$, with $\Lambda := \Lambda_1 \oplus \cdots \oplus \Lambda_k$. Then, there exist orthogonal transformations $C_i \in O_n(\mathbb{R})$ such that $C_i\Gamma_i = \Lambda_i$. By orthogonality, $\mathbb{R}^n$ admits an orthogonal decomposition into $k$ subspaces, the $i$th subspace containing $\Gamma_i$. Let $\Pi_i$ be orthogonal projection onto the $i$th subspace. We therefore define

$$C := \sum_{i=1}^k C_i \circ \Pi_i \in O_n(\mathbb{R}), \quad C(\Gamma) = \Lambda_1 \oplus \cdots \oplus \Lambda_k = \Lambda \implies \Gamma \cong \Lambda.$$ 

We leave the proof of the following corollary to the reader.

Corollary 2.48. Two products of irreducible lattices are congruent:

$$\Gamma_1 \times \cdots \times \Gamma_k \cong \Lambda_1 \times \cdots \times \Lambda_k'$$

if and only if $k = k'$, and up to reordering $\Gamma_i \cong \Lambda_i$.

Proposition 2.49. Two lattices $\Gamma, \Lambda$ are congruent if and only if $\Gamma^n, \Lambda^n$ are congruent for some $n \in \mathbb{Z}_{\geq 2}$.

Proof. If $\Gamma$ and $\Lambda$ are congruent, we leave it to the reader to show that $\Gamma^n$ and $\Lambda^n$ are congruent. On the other hand, we may up to congruence consider irreducible decompositions

$$\Gamma_1^n \times \cdots \times \Gamma_k^n \quad \& \quad \Lambda_1^n \times \cdots \times \Lambda_k^n.$$ 

By Corollary 2.48, $nk = nk'$, and therefore $k = k'$. By possibly re-ordering and re-naming, without loss of generality, $\Lambda_i \cong \Gamma_i$ for each $i = 1, \ldots, k$.

In the following theorem, we show that we can ‘slice off’ part of a product, if it is congruent on both sides.
Theorem 2.50. Fix two congruent lattices $\Lambda$ and $\Lambda'$. Two lattices $\Gamma$ and $\Gamma'$ are congruent if and only if $\Gamma \times \Lambda$ is congruent to $\Gamma' \times \Lambda'$.

Proof. We leave the proof that if $\Gamma \cong \Gamma'$, then $\Gamma \times \Lambda \cong \Gamma' \times \Lambda'$ to the reader. Conversely, since it is equivalent, we express the products as the orthogonal Minkowski sums $G \oplus H \cong G' \oplus H'$, with the assumption that $G \cong \Gamma$, $G' \cong \Gamma'$ and $H \cong H'$. Consider the decomposition into irreducible sublattices, $G = G_1 \oplus \cdots \oplus G_k$, and $G' = G'_1 \oplus \cdots \oplus G'_j$. By Proposition 2.47, since $H \cong H'$, they satisfy $H = \Lambda_1 \oplus \cdots \oplus \Lambda_m$, $H' = \Lambda'_1 \oplus \cdots \oplus \Lambda'_m$, and can be ordered so that $\Lambda_i \cong \Lambda'_i$ for all $i$. It therefore follows from Proposition 2.47 that we also have $k = j$, and by possibly re-arranging, $\Gamma_i \cong \Gamma'_i$ for all $i$ and $G \cong G'$.

With help from the theta series we obtain a similar method for slicing off an identical term in a product to obtain that what remains is isospectral if the original products were isospectral.

Lemma 2.51. If $\Gamma_1$ and $\Gamma_2$ are lattices and $\Gamma = \Gamma_1 \times \Gamma_2$, then $\theta_{\Gamma} = \theta_{\Gamma_1} \theta_{\Gamma_2}$. As a consequence, for an arbitrary lattice $\Lambda$ we have that $\Lambda \times \Gamma_1$ is isospectral to $\Lambda \times \Gamma_2$ if and only if $\Gamma_1$ and $\Gamma_2$ are isospectral.

Proof. The first statement is left to the reader; it can be demonstrated using the Pythagorean theorem. Consider $n$-dimensional lattices $\Gamma_1, \Gamma_2$ and an $m$-dimensional lattice $\Gamma$. Now $T\Gamma_1 \times T\Gamma$ and $T\Gamma_2 \times T\Gamma$ are isospectral if and only if $\theta_{T\Gamma_1 \times \Gamma} = \theta_{T\Gamma_2 \times \Gamma}$. This is equivalent to $\theta_{\Gamma_1}, \theta_{\Gamma} = \theta_{\Gamma_2}, \theta_{\Gamma}$. Since theta functions are non-zero where they are defined, this means exactly $\theta_{\Gamma_1} = \theta_{\Gamma_2}$, which is true if and only if $T\Gamma_1$ and $T\Gamma_2$ are isospectral.

The next lemma can be proven by combining the ingredients we have collected thus far to show that once we have different donuts singing the same song in dimension $n$, we obtain different donuts singing the same song in all higher dimensions.

Lemma 2.52 (Movin’ on up). If there exist $k$ mutually isospectral and pairwise non-isometric flat tori in dimension $n$, then there exist $k$ mutually isospectral and pairwise non-isometric flat tori in all higher dimensions.

Exercise 2.53. Prove the lemma by (1) using Theorem 2.50 and Lemma 2.51, and (2) via a more elementary method, not requiring Theorem 2.50.

2.5. An amuse bouche consisting of rectangular donuts and their fundamental domains. We combine our ingredients to study a special type of donuts that are particularly simple because they can be completely reduced to a product of one-dimensional donuts. These are rectangular flat tori.

Definition 2.54 (Rectangular Lattices and Donuts). A rectangular lattice $\Gamma$ is a lattice that has a diagonal basis matrix. The associated flat torus $\mathbb{R}^n/\Gamma$ is a rectangular donut.

If an $n$-rank lattice $\Gamma$ has a diagonal basis matrix, then there are scalars $\{c_j\}_{j=1}^n$ such that a basis for the lattice is $\{c_je_j\}_{j=1}^n$, where $e_j$ are the standard orthonormal basis vectors of $\mathbb{R}^n$. Consequently, the rectangular lattice $\Gamma \cong \Gamma_1 \times \ldots \times \Gamma_n$, for the one-dimensional lattices $\Gamma_j = \mathbb{Z}c_j \cong \mathbb{Z}c_je_j$.

Theorem 2.55. If two rectangular flat tori are isospectral then they are isometric.
Proof. The proof is by induction on the dimension. If two rectangular flat tori are
isospectral, then they are the same dimension, so they are both defined by full-rank
lattices in \( \mathbb{R}^n \) that have diagonal basis matrices. The case \( n = 1 \) is an exercise for
the reader. Assume the theorem has been proven for dimensions up to some \( n \geq 1 \).
Now assume that two \( n+1 \) dimensional rectangular flat tori are isospectral. The
rectangular tori are each defined by the products of the one-dimensional lattices

\[
\Gamma = \Gamma_1 \times \ldots \times \Gamma_{n+1}, \quad \text{and} \quad \Lambda = \Lambda_1 \times \ldots \times \Lambda_{n+1},
\]

with each \( \Gamma_j = \mathbb{Z}c_j \cong \mathbb{Z}c_je_j \), for the standard unit vector \( e_j \) and for some non-zero
\( c_j \). The length of the shortest non-zero vector in \( \Gamma \) is therefore the minimal \( |c_j| \).
The length of the shortest non-zero vectors of \( \Gamma \) and \( \Lambda \) are the same by exercise
2.28. By Lemma 2.46 we can without loss of generality re-arrange these products
to assume that \( \Gamma_1 = \mathbb{Z}c_1e_1 \), and \( \Lambda_1 = \mathbb{Z}c_1e_1 \). Consequently, these are congruent
and isospectral. By Lemma 2.51, we therefore have that \( \Gamma_2 \times \ldots \times \Gamma_{n+1} \) and
\( \Lambda_2 \times \ldots \times \Lambda_{n+1} \) are isospectral, and they are \( n \) dimensional. Consequently, by the
induction assumption, they are congruent. We therefore have by Theorem 2.50
that \( \Gamma \) and \( \Lambda \) are also congruent, and consequently the flat tori they define are
isometric.

Can we listen to the song of a donut and distinguish whether or not its shape
is rectangular? Or, is there a rectangular flat torus that is isospectral to a non-
rectangular flat torus? The answers to these questions will be revealed in Section
3.3.1.

2.5.1. One can hear the shape of a Euclidean box. A fundamental domain of a
rectangular flat torus is a Euclidean box, that is a product of bounded intervals.
The Dirichlet boundary condition for the Laplace eigenvalue equation demands that
eigenfunctions vanish on the boundary of this set, whereas the Neumann boundary
condition demands that on the interior of the boundary faces the normal derivative
of the eigenfunctions vanish. Although we would expect the following result to be
known, we are unaware of a reference in the literature and therefore include it here.

**Theorem 2.56.** Assume that two Euclidean boxes are isospectral with respect to
the Dirichlet boundary condition or the Neumann boundary condition. Then the
two boxes are isometric.

We first note that if two boxes are isospectral, then the same boundary condition
must be taken on both boxes, because in the Neumann case, 0 is an eigenvalue,
whereas in the Dirichlet case, the spectrum is strictly positive. Although the proof
is similar to the one for rectangular flat tori, the flavor is a bit different. For flat
tori, one has the Poisson summation formula, which we do not have for Euclidean
boxes. We replace this ingredient with the explicit calculation of the eigenvalues.

**Exercise 2.57.** Calculate the eigenfunctions and eigenvalues in the Dirichlet and
Neumann case for boxes. Show that if one orders them from smallest to largest,
then the \( k^{th} \) eigenvalue grows with \( k \to \infty \) on the order of \( k^{2/n} \).

**Proof.** To begin, we invite the reader to complete Exercise 2.57. As a consequence,
if two boxes are isospectral, then they are the same dimension. We now continue
the proof by induction on the dimension in the Dirichlet case; the Neumann case
is completely analogous.
In one dimension, the spectrum consists of \( n^2 \pi^2 / \ell^2 \) for \( n \in \mathbb{N} \), with \( \ell \) the length of the one-dimensional box (interval). The base case is therefore clearly true; if one dimensional boxes are isospectral then they are isometric. Now assume the statement is true for all dimensions from 1 to \( n \) for some \( n \geq 1 \). We consider an \( n + 1 \) dimensional box. Using separation of variables, the first two eigenvalues are

\[
\lambda_1 = \sum_{k=1}^{n+1} \frac{\pi^2}{\ell_k^2}, \quad \lambda_2 = \frac{4\pi^2}{\ell_{\max}^2} + \sum_{\ell_k \neq \ell_{\max}} \frac{\pi^2}{\ell_k^2}.
\]

Here, \( \ell_{\max} \) is the length of the longest side of the box. If two boxes are isospectral, they have the same first two eigenvalues, as well as the same difference between these first eigenvalues, that is \( 3\pi^2 / \ell_{\max}^2 \). Consequently, \( \ell_{\max} \) is the same for both boxes. The two boxes are therefore each respectively isometric to

\[
B \times [0, \ell_{\max}], \quad B' \times [0, \ell_{\max}].
\]

Here, \( B \) and \( B' \) are boxes of dimension \( n \).

For simplicity, set \( \ell := \ell_{\max} \). Since the eigenvalues of \( B \times [0, \ell] \) are equal to the sum of the eigenvalues of \( B \) and the eigenvalues of \( [0, \ell] \), the heat trace satisfies \( H_{B \times [0, \ell]}(t) = H_B(t)H_{[0, \ell]}(t) \) and similarly \( H_{B' \times [0, \ell]}(t) = H_{B'}(t)H_{[0, \ell]}(t) \). The two heat traces are equal by isospectrality, and therefore \( H_B = H_{B'} \), from which it follows that \( B \) and \( B' \) are isospectral. Since \( B \) and \( B' \) are dimension \( n \), by induction they are also isometric. By Theorem 2.50, the two boxes are respectively isometric to \( B \times [0, \ell] \cong B' \times [0, \ell] \).

3. Key kitchenware for computation and construction

In the preceding section we introduced the essential ingredients to understand the geometry and spectra of flat tori. Here we present some essential tools for making use of and combining these ingredients. In this section, all lattices are full-rank.

3.1. Techniques for determining congruency. Given two lattices \( \Gamma_1 = A_1 \mathbb{Z}^n \) and \( \Gamma_2 = A_2 \mathbb{Z}^n \) with two explicit bases, we can easily check if they are the same lattice, since this is the case if and only if \( A_1^{-1}A_2 \) is a unimodular matrix. Checking whether or not the two lattices are congruent is much more of a challenge. The lattices are congruent if and only if there is an orthogonal matrix \( C \) and a unimodular matrix \( B \) such that \( CA_1B = A_2 \). There are infinitely many orthogonal matrices and infinitely many unimodular matrices, so checking congruency is a seemingly infinite task. One way to conclude that lattices are not congruent is furnished by Proposition 2.47, from which we immediately obtain

Corollary 3.1. If \( \Gamma \) is reducible and \( \Lambda \) is not, then \( \Gamma \) is not congruent to \( \Lambda \).

A second method uses the fact that orthogonal transformations preserve the scalar product.

Lemma 3.2. Let \( \Gamma_1, \Gamma_2 \) be two lattices. For \( s > 0 \), let \( S_i(s) \) be the sets of all vectors in \( \Gamma_i \) of length \( s \). If there exists an \( s > 0 \) such that the number of elements in \( S_i(s) \) are not equal, then \( \Gamma_i \) are not congruent. Let \( P_i(s) = \{a \cdot b : a, b \in S_i(s)\} \). If there exists an \( s > 0 \) such that \( P_i(s) \) are not equal, then \( \Gamma_i \) are not congruent.

Exercise 3.3. Prove the preceding corollary and lemma.
The third and perhaps most robust method uses the equivalence classes of quadratic forms we associate to lattices, with the help of the following Lemma that is a consequence of Proposition 2.37.

**Lemma 3.4.** Let $Q_1$ and $Q_2$ be positive definite $n \times n$ matrices. Let $\lambda_{\text{min}}$ be the smallest eigenvalue of $Q_1$. If $B = [b_j]$ is a matrix with column vectors $b_j$, and $B^T Q_1 B = Q_2$, then

$$b_j^T Q_1 b_j = (Q_2)_{jj}, \quad j = 1, \ldots, n.$$  

Moreover, $\|b_j\|^2 \leq (Q_2)_{jj}/\lambda_{\text{min}}$ for each $j = 1, \ldots, n$.

To lattices $A_i \mathbb{Z}^n$ for $i = 1, 2$ we associate the class of quadratic forms that are integrally equivalent to $Q_i = A_i^T A_i$ for each $i$. The lattices are congruent if and only if these equivalence classes are identical. In turn, these equivalence classes are identical if and only if there is a unimodular matrix such that $B^T Q_1 B = Q_2$. Since the entries of unimodular matrices are integers, there are only finitely many unimodular matrices whose column vectors satisfy the conditions of Lemma 3.4. Consequently, one can write a computer program to check congruency.

If we have a parametrized infinite family of pairs, and we want to say something about their congruence as a whole, it might not be reasonable to use the above methods because they rely on explicit bases and computer searches. In 2011, Cervino and Hein developed a systematic method to analyze isometry and non-isometry of infinite families of pairs using modular forms [25]. An entirely different approach based on nodal counts was given one year later in [5].

### 3.2. Modular forms in connection to isospectrality

Verifying with complete confidence that two doughnuts sing the same song requires checking that all of their eigenvalues are identical counting multiplicity, yet again an apparently infinite task. It turns out that the theory of modular forms give a useful criterion for determining precisely when two doughnuts sing the same song. This is one of several reasons why the theory and language of modular forms appear in many of the articles related to the spectral theory of flat tori.

To state the definition, recall that $\text{SL}_2(\mathbb{Z})$ is the *special linear group* of $2 \times 2$ integer matrices with determinant 1. The *congruence groups* are defined as

$$\Gamma_0(N) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{GL}_2(\mathbb{Z}) : c \equiv 0 \mod N \right\}.$$  

**Definition 3.5** (Modular Forms). Let $\mathbb{H}$ denote the complex upper half-plane. Given a subgroup $G \subseteq \text{SL}_2(\mathbb{Z})$ of finite index, known as an *arithmetic group*, a *modular form* of level $G$ and weight $k$ is a function $f : \mathbb{H} \to \mathbb{C}$ such that:

1. $f$ is holomorphic.
2. For any $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G$, $f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$.
3. For any $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{SL}_2(\mathbb{Z})$, $(cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right)$ is bounded as $\text{Im}(z) \to \infty$.

For a congruence group $\Gamma_0(N)$, $f : \mathbb{H} \to \mathbb{C}$ is a modular form with weight $k$, Dirichlet character $\chi \mod N$ of level $\Gamma_0(N)$ if it satisfies conditions (1) and (3) above, together with:

$$f\left(\frac{az+b}{cz+d}\right) = \chi(d)(cz+d)^k f(z).$$

We may write $f \in M_k(\Gamma_0(N), \chi)$ for such a form, following the notation of [32, p. 127].
Exercise 3.6. Show that it is enough to check conditions (2) and (3) for the generators of $G$ and $\text{SL}_2(\mathbb{Z})$ respectively. (Note that the matrices $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ generate $\text{SL}_2(\mathbb{Z})$).

Let us now restrict to rational donuts (respectively rational lattices and rational quadratic forms), meaning donuts defined by a lattice with a basis matrix whose entries are rational numbers. We are justified in making this restriction: it turns out that the relation between isospectrality and congruence can be reduced to only looking at rational donuts, in the sense of Proposition 6.6. Rational numbers also enjoy an amenability towards computers. Up to a constant, any rational quadratic form is an even quadratic form, as in Definition 2.32. One connection between modular forms and the spectrum of flat tori is through the following theorem that shows that the theta series of an even-dimensional rational donut is a modular form.

**Theorem 3.7** (see p. 295 of [16], or Corollary 4.9.5 (iii) of [35]). Let $Q$ be an even positive definite quadratic form of $2k$ variables, and $N_Q$ be the smallest positive integer such that $N_Q Q^{-1}$ is even. If $\Gamma$ is an underlying lattice, then $\theta_{\Gamma}(z) \in M_k(\Gamma_0(N_Q), \chi)$ for a real Dirichlet character $\chi$ that is determined by $\det(Q)$.

Exercise 3.8 (⋆). Show that $N_Q$ in the theorem above satisfies $1 \leq N_Q \leq 2 \det Q$. If $\Gamma$ is an underlying lattice of $Q$, show that $N_Q$ is determined by the spectrum of the flat torus $\mathbb{R}^n/\Gamma$, so in this sense, $N_Q$ is audible. Hint: let $Q_1$ and $Q_2$ be even isospectral quadratic forms. Use the fact that their dual forms are also isospectral.

**Theorem 3.9** (Hecke’s identity theorem for modular forms [24, p. 811]). Let $f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z} \in M_k(\Gamma_0(N), \chi)$, with real-valued $\chi$. If $\mu_0(N) := N \prod_{p | N, \text{prime}} \left( 1 + \frac{1}{p} \right)$, then

$$a_n = 0 \text{ for all } 0 \leq n \leq \frac{\mu_0(N)k}{12} + 1 \text{ implies } f = 0.$$

We first apply the identity theorem to modular forms in even dimensions. It is straightforward to generalize the result to odd dimensions as shown in Exercise 3.11.

**Corollary 3.10** (Isospectrality certificate). Let $P$ and $Q$ be two even positive definite quadratic forms of $2k$ variables. They are isospectral if and only if $\det(P) = \det(Q)$, $N_P = N_Q$, and their multiplicities over the integers of values less than or equal to $\mu_0(N_P)k/12 + 1$ coincide.

**Proof.** We leave it to the reader to verify that if $P$ and $Q$ are isospectral, then the statements of the corollary hold. For the converse, note that for any underlying lattice of $Q$, denoted $AZ^n$, the theta series of this lattice is

$$\sum_{\gamma \in AZ^n} e^{i\pi z ||\gamma||^2} = \sum_{x \in \mathbb{Z}^n} e^{i\pi z (Ax)^T (Ax)} = \sum_{x \in \mathbb{Z}^n} e^{i\pi z x^T Q x}.$$

---

4By ⋆ we mean that the exercise is more difficult than others.

5By $p | N$ we mean that $p$ divides $N$. 
Consequently, the theta series is identical for all underlying lattices, and we can therefore define the theta series associated to \( P \) and \( Q \)

\[
\theta_P(z) := \sum_{x \in \mathbb{Z}^n} e^{i\pi x^T P x}, \quad \theta_Q(z) := \sum_{x \in \mathbb{Z}^n} e^{i\pi x^T Q x}.
\]

If \( N_P = N_Q \), and \( \det(P) = \det(Q) \), then their corresponding theta series \( \theta_P, \theta_Q \) lie in \( M_k(\Gamma_0(N_P), \chi) \) for some \( \chi \) determined by \( \det(P) = \det(Q) \), by Theorem 3.7. The quadratic forms are isospectral if and only if their theta series are identical. So, consider \( f(z) := \theta_P(z) - \theta_Q(z) \). We have

\[
M_k(\Gamma_0(N_P), \chi) \ni f(z) = \sum_{x \in \mathbb{Z}^n} e^{i\pi x^T P x} - \sum_{x \in \mathbb{Z}^n} e^{i\pi x^T Q x} = \sum_{n \in \mathbb{N}} (m_P(n) - m_Q(n)) e^{i\pi zn},
\]

where \( m_P(n), m_Q(n) \) respectively denote the multiplicities of the value \( n \) of \( P \) and \( Q \) as quadratic forms over \( \mathbb{Z}^n \). By Theorem 3.9, if \( m_P(n) - m_Q(n) = 0 \) for all \( 0 \leq n \leq \mu_0(N_P)k/12 + 1 \), then \( f(z) = 0 \) for all \( z \), and \( \theta_P = \theta_Q \).

**Exercise 3.11.** Let \( P, Q \) be positive definite \( k \times k \) matrices. Show that \( P \) and \( Q \) are isospectral if and only if the \( k + 1 \)-dimensional forms

\[
P' = \begin{bmatrix} 2 & 0 \\ 0 & P \end{bmatrix} \quad \& \quad Q' = \begin{bmatrix} 2 & 0 \\ 0 & Q \end{bmatrix},
\]

are isospectral. Further, \( P', Q' \) are integrally equivalent if and only if \( P, Q \) are.

### 3.3. Building donuts from linear codes.

Codes are used in numerous everyday circumstances including data compression, cryptography, error detection and correction, data transmission, and data storage. Linear codes are useful for studying lattices because they allow one to translate questions for lattices, infinite discrete groups, into questions for finite groups, known as linear codes. For a general treatment we refer to [15] and [37].

**Definition 3.12.** A linear \( q \)-nary code \( C \) of length \( n \) is a \( \mathbb{Z} \)-linear subspace (and a subgroup) of the vector space \((\mathbb{Z}/q\mathbb{Z})^n\), where \( \mathbb{Z}/q\mathbb{Z} \) is the ring of integers modulo \( q \). Its elements are codewords. The linear space \((\mathbb{Z}/q\mathbb{Z})^n\) is equipped with the inner product \( x \cdot y := \sum_{i=1}^n x_i y_i \mod q \).

In the literature, \( q \) is often assumed to be prime, but we don’t need this assumption for our purposes. To construct a lattice from a linear code consider the projection

\[
\pi : \mathbb{Z}^n \to (\mathbb{Z}/q\mathbb{Z})^n \quad \text{by} \quad z \mapsto z \mod q,
\]

where \( \mod q \) acts coordinate-wise. Importantly, \( \pi \) is a group homomorphism. For a code \( C \) in \((\mathbb{Z}/q\mathbb{Z})^n\) its pre-image under \( \pi_q \) is a lattice,

\[
\pi^{-1}_q(C) = \{ \ell \in \mathbb{Z}^n : \ell \mod q \in C \}.
\]

In [11], building a lattice in this way is called construction \( A \).

**Exercise 3.13.** Let \( C \) be a linear code in \((\mathbb{Z}/q\mathbb{Z})^n\). Check that the codewords \( c_i \in C \) partition \( \pi^{-1}_q(C) \) into the subsets \( \pi^{-1}_q(c_i) \). In particular, if \( C_1, C_2 \subseteq (\mathbb{Z}/q\mathbb{Z})^n \)
are different codes, then \( \pi_q^{-1}(C_1) \neq \pi_q^{-1}(C_2) \). Further, show that if \( c_1, \ldots, c_k \) are generators of \( C \), then

\[
\pi_q^{-1}(C) = [c_1 \cdots c_k qI] \mathbb{Z}^{k+n} \subseteq \mathbb{Z}^n.
\]

Above \( I \) is the \( n \times n \) identity matrix.

In particular, \( \pi_q^{-1}(C) \) is always a full-rank lattice which follows from [28, Prop. 16.2]. In the following theorem, we prove that all integer lattices can be obtained by construction \( A \), indicating the usefulness of this approach. As a consequence of this characterization, there are only finitely many distinct integer lattices of a given determinant \( q \), because the number of linear codes in \( (\mathbb{Z}/q\mathbb{Z})^n \) is finite.

**Theorem 3.14.** Any integer lattice \( L \) is the pre-image of the \( \text{vol}(L) \)-nary code \( \pi_{\text{vol}(L)}(L) \).

**Proof.** Assume that \( L \) is an integer lattice. For any \( q \),

\[
L' := \pi_q^{-1}(\pi_q(L)) = \{ x \in \mathbb{Z}^n : \pi_q(x) \in \pi_q(L) \}.
\]

For \( x \in \mathbb{Z}^n \), \( \pi_q(x) \in \pi_q(L) \) if and only if there is some \( \gamma \in L \) such that \( \pi_q(x) = \pi_q(\gamma) \). This is equivalent to \( \pi_q(x - \gamma) = 0 \), which holds if and only if \( x = \gamma + qz \) for some \( z \in \mathbb{Z}^n \). This proves \( L' = L + q\mathbb{Z}^n \). By Exercise 2.22, if \( q = \text{vol}(\Gamma) \), then \( q\mathbb{Z}^n \subseteq L \), and therefore \( L' = L \).

**Corollary 3.15.** Let \( C \subseteq (\mathbb{Z}/q\mathbb{Z})^n \) be a linear code and \( L = AQ^n \) be a lattice, where \( A = [a_{ij}] \). If the codewords \( \pi_q(a_{ij}) \) generate \( C \), then \( L \subseteq \pi_q^{-1}(C) \). If in addition, \( q\mathbb{Z}^n \subseteq L \), then \( L = \pi_q^{-1}(C) \).

**Proof.** If \( \pi_q(a_{ij}) \) generate \( C \), then \( \pi_q(L) = C \). By the previous proof, \( L \subseteq \pi_q^{-1}(C) \), and if \( q\mathbb{Z}^n \subseteq L \), then inclusion becomes an equality.

We next relate linear codes to isospectrality. For this we need

**Definition 3.16.** Let \( C_1, C_2 \subseteq (\mathbb{Z}/q\mathbb{Z})^n \) be two linear codes of equal cardinality and list their respective elements as \( c_1^{(1)}, \ldots, c_k^{(1)} \) for \( i = 1, 2 \). The codes have the same weight distribution if for each pair \( (c_j^{(1)}, c_j^{(2)}) \), there is a permutation \( \sigma \in S_n \) such that \( (c_j^{(2)})_k = (c_j^{(1)})_{\sigma(k)} \) for each coordinate \( k \). The codes constitute an absolute pairing if for each \( (c_j^{(1)}, c_j^{(2)}) \) we have \( (c_j^{(2)})_k = \pm (c_j^{(1)})_k \) in \( \mathbb{Z}/q\mathbb{Z} \) for each \( k \). Both relations are equivalence relations.

**Proposition 3.17.** Let \( C_1, C_2 \) be \( q \)-nary linear codes, and let \( L_i = \pi_q^{-1}(C_i) \). If the weight distributions of \( C_1 \) and \( C_2 \) are the same, then \( L_1 \) and \( L_2 \) are isospectral. The converse does not hold.

**Proof.** We prove the first part. Let \( c_1^{(1)}, \ldots, c_m^{(1)} \) be lists of the codewords as in the definition of equal weight distribution. We give length-preserving bijections from \( \pi_q^{-1}(c_j^{(1)}) \) to \( \pi_q^{-1}(c_j^{(2)}) \) for each \( j \), which is enough by Exercise 3.13. Let \( \sigma \) be the permutation corresponding to the pair \( (c_j^{(1)}, c_j^{(2)}) \). It can be realized as a permutation matrix \( \Sigma \), which is orthogonal. Then \( x \mapsto \Sigma x \) is length-preserving and by construction a bijection from \( \pi_q^{-1}(c_j^{(1)}) \) to \( \pi_q^{-1}(c_j^{(2)}) \). Consequently, \( L_i \) have identical length spectra and are therefore isospectral.
Exercise 3.18. Finish the proof by finding a counterexample, namely a pair of isospectral integer lattices $L_i = \pi_q^{-1}(C_i)$ such that the weight distributions of $C_1$ and $C_2$ are not the same.

**Theorem 3.19.** Let $C_1, C_2 \subseteq (\mathbb{Z}/q\mathbb{Z})^n$ be linear codes with

$$L_i = A_i\mathbb{Z}^n = \pi_q^{-1}(C_i).$$

There is a bijection $\phi : L_1 = A_1\mathbb{Z}^n \rightarrow L_2 = A_2\mathbb{Z}^n$ preserving the absolute value of each coordinate if and only if $C_1, C_2$ make an absolute pairing. If either is true, then $DL_1$ and $DL_2$ are isospectral for any diagonal invertible matrix $D$.

**Exercise 3.20 (⋆).** Prove direction $\Rightarrow$ of the theorem.

Proof of Theorem 3.19. $\Leftarrow$ Let $c_1^{(i)}, \ldots, c_m^{(i)}$ be lists of codewords as in the definition of an absolute pairing, and view them as elements of $\{0, 1, \ldots, q-1\}^n \subseteq \mathbb{Z}^n$. We again give a length-preserving bijection $\phi : \pi_q^{-1}(c_j^{(1)}) \rightarrow \pi_q^{-1}(c_j^{(2)})$ for a fixed $j$. Write $x \in \pi_q^{-1}(c_j^{(1)})$ uniquely as $c_j^{(1)} + qt$, where $t \in \mathbb{Z}^n$. Let $\phi$ act coordinate-wise bijectively as follows:

$$x_k = (c_j^{(1)})_k + qt_k \mapsto \begin{cases} (c_j^{(2)})_k + qt_k, & \text{if } (c_j^{(2)})_k = (c_j^{(1)})_k \\ (c_j^{(2)})_k - q(t_k + 1), & \text{otherwise if } (c_j^{(2)})_k = q - (c_j^{(1)})_k. \end{cases}$$

We therefore obtain that the lattices are isospectral. For the last part, consider $\phi : L_1 \rightarrow L_2$, a bijection between lattices and a diagonal matrix $D$ as in the statement above. By assumption, if $x \in L_1$, then $|x_i| = |\phi(x)_i|$ for each coordinate. Now define $\varphi : DL_1 \rightarrow DL_2$ as $\varphi(Dx) := D\phi(x)$ for $x \in L_1$. We have for $x \in L_1$,

$$|\langle Dx \rangle| = |d_i x_i| = |d_i \phi(x)_i| = |\langle D\phi(x) \rangle| = |\varphi(Dx)|.$$ 

Therefore $\varphi$ is again a bijection that preserves the absolute values of coordinates, from whence it follows that $DL_1$ and $DL_2$ are isospectral.

**Exercise 3.21.** Prove that isomorphic codes (in the sense of groups) do not in general correspond to either isospectral or congruent lattices.

3.3.1. *Can one hear cubicity?* In §2.5, we proved that if a pair of rectangular donuts sing the same song, then they have the same shape. Suppose we only know a priori that one donut out of the pair singing the same song is rectangular, then surely the other must be as well? As an application of linear codes, we show that the answer depends on the dimension. Following Conway [10, p. 40–42], we define *cubic lattices* as those that are congruent up to scaling to $\mathbb{Z}^n$ and say that *cubicity* is the property of being cubic.

**Proposition 3.22.** The two lattices

$$\Lambda = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \mathbb{Z}^6 \quad \Omega = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \mathbb{Z}^6,$$

are isospectral and non-congruent. In particular, $\Lambda$ is cubic, but $\Omega$ is not. Cubicity is audible in dimensions five and lower; it is not audible in dimensions 6 and higher.

Proof. We prove that the pair is isospectral and non-congruent. As a consequence, Lemma 2.52 shows that cubicity is not audible for $n > 6$. The fact that cubicity is audible when $n < 6$ is shown in [10, p. 60]. Consider the following two binary linear
codes of length 6. The rows are the codewords, and the codewords of the same row differ by permutation, showing that the codes are of the same weight distribution:

\[
C_1 : \begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 \\
1 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1
\end{array}
\]

\[
C_2 : \begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 & 0 & 1 \\
0 & 1 & 1 & 1 & 0 & 1 \\
1 & 1 & 1 & 1 & 1 & 1
\end{array}
\]

We have that, modulo 2, the vectors of the given bases of \( \Lambda \), \( \Omega \) generate the codes \( C_1 \), \( C_2 \) respectively. It is straightforward to check that, \( 2\mathbb{Z}^6 \subseteq \Lambda, \Omega \). So, the preimages of the codes are equal to \( \Lambda, \Omega \) respectively by Corollary 3.15. They are isospectral by Proposition 3.17 and non-isometric since the vectors of length \( \sqrt{2} \) in \( \Lambda \) that are non-parallel are orthogonal, while this is not the case for \( \Omega \).

4. Donuts of different shapes singing the same song

In 1964, a paper of one single page was published in Proceedings of the National Academy of Science that is famous to this day [34], Eigenvalues of the Laplace operator on certain manifolds. In it Milnor described an example of two sixteen dimensional donuts that sing exactly the same song (are isospectral) but are not of the same shape (not isometric). One can imagine this as a duet, a pair of perfectly attuned yet differently shaped donuts singing exactly the same song. Milnor’s paper inspired Kac’s acclaimed work [27] titled Can one hear the shape of a drum?

4.1. The race to find duets. Milnor’s paper [34] referred to a construction of two lattices by Witt [53] that begins with the root lattice \( D_n \), also called the checkerboard lattice,

\[
D_n := \left\{ z = (z_1, \ldots, z_n) \in \mathbb{Z}^n : \sum_{i=1}^{n} z_i \in 2\mathbb{Z} \right\}.
\]

A basis for \( D_n \) is given by the vectors \( \{e_1 + e_2, e_{j-1} - e_j\}_{j=2}^{n} \). The root lattice of the \( E_n \) root system, also denoted by \( E_n \), for \( n \) divisible by 4 is

\[
E_n := \left\{ x \in \mathbb{Z}^n \cup \left( \frac{1}{2} \mathbb{I} + \mathbb{Z}^n \right) : \sum_{i=1}^{n} x_i \in 2\mathbb{Z} \right\}, \quad \mathbb{I} := \sum_{j=1}^{n} e_j.
\]

We note that \( E_8 \) is sometimes known as the Gosset lattice after [21].

When \( n \) is divisible by 4, a basis for \( E_n \) is given by \( \{e_1 + e_2, e_{j-1} - e_j, \frac{1}{2} \mathbb{I} \}_{j=2}^{n} \). The classical theory of root lattices tells us that \( D_n \) with \( n > 2 \) and \( E_8 \) are irreducible lattices; see [15, §1.4] and [12, §4.7-4.8.1]. We give an alternative method for checking irreducibility that may be of independent interest as we have not seen this elsewhere in the literature.

Lemma 4.1. Let \( A = [a_j] \) be a basis of a lattice, \( \Gamma \subseteq \mathbb{R}^n \) with irreducible components \( \Gamma_i \). For a scalar \( s \neq 0 \), and a vector \( v \in \mathbb{R}^n \), the lattice

\[
\Lambda := \begin{bmatrix} A & v \\ 0 & s \end{bmatrix} \mathbb{Z}^{n+1}
\]

is irreducible if:

(a) each \((a_j, 0)\) is of shortest non-zero length in \( \Lambda \),

(b) \((v + \gamma) \cdot \Gamma_i \neq \{0\}\) for each \( \gamma \in \Gamma \) and \( i \).
Proof. By contradiction, assume that we can write $\Lambda = \Lambda_1 \oplus \Lambda_2$, where $\Lambda_i$ are non-trivial, and $\Lambda_1$ is irreducible. Without loss of generality, since $s \neq 0$, we may assume that there is some $\omega \in \Lambda_1$ with $\omega = (x,t)$ for some $t \neq 0$ (for simplicity we use row vector notation in this proof). Then, $\{\omega, (a_j,0)\}_{j=1}^n$ are all contained in $\Lambda$, and they are all linearly independent. Each $(a_j,0)$ lies in precisely one of $\Lambda_i$, by virtue of (a) and the Pythagorean theorem. Consequently, the number of $(a_j,0)$ that lie in $\Lambda_2$ is equal to the rank of $\Lambda_2$. In particular, the last coordinate of every element of $\Lambda_2$ vanishes. Therefore, the projection of $\Lambda_2$ onto the first $n$ coordinates is an orthogonal sum of some $\Gamma_i$. We fix an element $\Gamma_i$ in this sum.

Exercise 4.2. Show that $D_n$ for $n > 2$ and $E_{4n}$ for $n > 0$ are irreducible lattices.

Milnor argued that $E_{16}$ and $E_8 \times E_8$ are isospectral and not congruent. As we have seen, one dimensional isospectral flat tori are always isometric, so a natural question is, what is the lowest dimension in which there are isospectral non-isometric flat tori? Following Milnor, the search for isospectral and non-isometric duets of flat tori, became a race towards the lowest possible dimension. Kneser found a 12 dimensional example [31] in 1967. Ten years later, Kitaoka [29] reduced this to 8. In 1986, Conway and Sloane [11] found 5 and 6 dimensional examples. In 1990, Schiemann [43] constructed a 4 dimensional example. Independently, and using a different approach, Shiota [48] found another example one year later in 1991. The same year, Earnest and Nipp [14] contributed with one more pair.

Exercise 4.3. Use the previous exercise to show that the two pairs $E_{16}$ and $E_8 \times E_8$, and $D_{12}$ and $E_8 \times D_4$ are non-isometric.

Kneser’s 12-dimensional pair is $D_{12}$ and $E_8 \times D_4$ [31]. Kitaoka also made use of $D_4$ in his construction [29]. We refer interested readers to the literature for the aforementioned constructions and recall here Schiemann’s four dimensional pair. Consider the positive definite matrices

\[
\begin{bmatrix}
4 & 2 & 0 & 1 \\
2 & 8 & 3 & 1 \\
0 & 3 & 10 & 5 \\
1 & 1 & 5 & 10 \\
\end{bmatrix}
&
\begin{bmatrix}
4 & 0 & 1 & 1 \\
0 & 8 & 1 & -4 \\
1 & 1 & 8 & 2 \\
1 & -4 & 2 & 10 \\
\end{bmatrix}
\]

Schiemann proved that the quadratic forms they define have identical representation numbers using Corollary 3.10. He showed that these forms are not integrally equivalent using the theory of Minkowski reduction; see 5.2. This can also be seen by Lemma 3.4. These two quadratic forms were systematically found in the sense that they are the integral forms that satisfy these conditions and have the smallest determinant.

4.2. Conway and Sloane’s isospectral family. One way to obtain a family of infinitely many pairs of isospectral non-isometric 4-dimensional donuts is to start with Schiemann’s pair, denoted by $(S_1, S_2)$ and take the one parameter family $(cS_1, cS_2)_{c \in \mathbb{R}}$. Can one obtain an infinite family of isospectral non-isometric pairs that are not simply obtained by scaling? Conway and Sloane were the first to present an infinite family of 4-dimensional pairs via
Theorem 4.4 (Conway-Sloane [11], and Cervino-Hein [25]). Let \(a, b, c, d > 0\).
Consider the two matrices
\[
A_\pm = \frac{1}{\sqrt{12}} \begin{bmatrix}
\sqrt{\pi} & 0 & 0 & 0 \\
0 & \sqrt{5} & 0 & 0 \\
0 & 0 & \sqrt{\pi} & 0 \\
0 & 0 & 0 & \sqrt{\pi}
\end{bmatrix}
\begin{bmatrix}
\pm 3 & 1 & 1 & 1 \\
1 & \pm 3 & -1 & 1 \\
-1 & -1 & \pm 3 & -1 \\
-1 & -1 & -1 & \pm 3
\end{bmatrix}
\]

There is a bijection \(T_+ \mathbb{Z}^4 \to T_- \mathbb{Z}^4\) preserving absolute values of the coordinates, and therefore the lattices \(A_+ \mathbb{Z}^4, A_- \mathbb{Z}^4\) are isospectral for any \(a, b, c, d > 0\). They are non-isometric if and only if \(a, b, c, d\) are all distinct.

Both \(A_\pm\) have determinant \(\sqrt{abcd}\). The observant reader may see a connection to Theorem 3.19. Indeed, \(T_\pm \mathbb{Z}^4\) correspond to linear codes that constitute an absolute pairing. However, since \(\det(T_\pm) = 144\), this is not easy to check directly.

This theorem encompasses, up integral equivalence, Schiemann’s pair \((S_1, S_2)\) of quadratic forms by letting \(a = 1, b = 7, c = 13, d = 19\), which can be seen using Lemma 3.4. Conway and Sloane were only able to verify non-isometry for integers \(a < b < c < d\) with \(abcd < 10,000\) [11]. Note that the proof for the isospectrality part given by Conway and Sloane is short and quite elementary, in contrast to the isometry part, which was proven in full by Cervino and Hein [25] in 2011. This motivates us to demonstrate a simple method to obtain an infinite family of isospectral non-isometric four dimensional pairs of flat tori that do not differ by scaling, assuming only the isospectrality part of the theorem. We have not seen this method in the literature, but it is so simple, we expect it is known to experts.

Lemma 4.5. Let \(\Gamma_i = A_i \mathbb{Z}^n\) and \(\Lambda_i = F_i \mathbb{Z}^n\) be full-rank lattices for each \(i \in \mathbb{N}\). Assume also that \(A_i \to A\) and \(F_i \to F\) in the usual sense of entrywise convergence, and that \(A\) and \(F\) are invertible. If we write \(\Gamma = AZ^n, \Lambda = FZ^n\), then the following hold:

1. If \(\Gamma_i\) and \(\Lambda_i\) are congruent for each \(i\), then so are \(\Gamma\) and \(\Lambda\).
2. If \(\Gamma_i\) and \(\Lambda_i\) are isospectral for each \(i\), then so are \(\Gamma\) and \(\Lambda\).

Exercise 4.6 (⋆). Prove Lemma 4.5!

Proposition 4.7. There are infinitely many pairs of isospectral non-isometric pairs of four-dimensional flat tori, that do not differ by scaling.

Proof. As discussed above, letting \(a = 1, b = 7, c = 13, d = 19\) we get a pair of isospectral non-congruent pair of lattices, say \(\Gamma_1, \Gamma_2\). By Theorem 4.4, for any diagonal invertible matrix \(D, DF_1\) and \(DF_2\) are isospectral. If there were a sequence of distinct diagonal matrices \(D_\epsilon\) such that \(D_\epsilon \to I\) and \(D_\epsilon \Gamma_1, D_\epsilon \Gamma_2\) were congruent for each \(\epsilon\), then we would find a contradiction by Lemma 4.5 (1).

5. The race ends to the sound of Schiemann’s symphony

The race to find isospectral non-isometric donuts in successively lower dimensions finally ended in 1994, over three decades after it began, when Alexander Schiemann determined the final answer to the three equivalent questions in §1.1. The aim of this section is to describe Schiemann’s symphony, the elaborate algorithm that gave a final answer to this question, and that was performed with a comprehensive computer search [44], [45].
5.1. Schiemann’s theorem. The following result is well known, and the reader has already demonstrated it in the one dimensional case.

**Theorem 5.1.** If two donuts of dimension at most two sing the same song, then they are isometric.

**Exercise 5.2.** Show that in dimension 2, a full-rank lattice always has a basis consisting of a shortest non-zero vector, and a shortest vector that is linearly independent of the first one. Interestingly, it is *not* always possible to find such a basis for higher dimensional lattices; see §5.2.

**Proof of Theorem 5.1.** We prove the two-dimensional case and equivalently work with lattices. Let \( A, A' \) be bases of two isospectral lattices \( \Gamma, \Gamma' \) in \( \mathbb{R}^2 \) as in the exercise above. By left-multiplication with an orthogonal matrix, we may assume that

\[
A = \begin{bmatrix} a & c \\ 0 & b \end{bmatrix} \quad \text{and} \quad A' = \begin{bmatrix} a' & c' \\ 0 & b' \end{bmatrix}.
\]

Here, \( a, a', b, b' \) are positive numbers. Exercise 2.28 tells us that \( a = a' \), and by Corollary 2.28, \( b = b' \). If \( |c| < |c'| \), then the closed ball \( D(0, \|(c,b)\|) \) contains more points from \( \Gamma \) than from \( \Gamma' \), implying that \( \Gamma, \Gamma' \) are not isospectral. If \( c = -c' \), then by letting \( B = C = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \), we see that \( CA' = AB \) and \( \Gamma \) is congruent to \( \Gamma' \).

To quantify the isospectral question for flat tori, we introduce the *choir numbers*.

**Definition 5.3.** In each dimension \( n \in \mathbb{N} \), we define the *choir number* \( \flat_n \) to be the maximal number \( k \) such that the three following equivalent conditions hold:

1. There is a sequence of \( k \) \( n \)-dimensional mutually isospectral (1) flat tori that are non-isometric,
2. Positive definite quadratic forms that are non-integrally equivalent,
3. Lattices that are non-congruent.

The choir numbers tell us the biggest choir of donuts in each dimension, such that all members sing in perfect unison and are geometrically different. In 1990–1994, Schiemann proved the next remarkable theorem [43], [44]. Although talented mathematicians have tried, no one has been able to prove his theorem without the use of a computer.

**Theorem 5.4** (Schiemann’s theorem). The choir numbers are equal to 1 for \( n = 1, 2, 3 \) and greater than 1 otherwise.

The proof in full detail is only available in German [43]. Consequently, we take this opportunity to present the main ideas, general structure, and strategy of the proof. This strategy is independently interesting, and we further suspect that Schiemann’s methods can be generalized to higher dimensions in order to determine, say \( \flat_4 \), which is unknown. Continuing the musical analogy, we will describe the proof of Theorem 5.4 as Schiemann’s symphony. This piece is performed using positive definite quadratic forms. Up to integral equivalence, these forms can be geometrically represented as a polyhedral cone, which is a convenient structure for our computer algorithms. The proof is rather technical, so we employ musical analogies as a mnemonic technique to keep track of the different elements in the proof and the roles they play.

---

\(^6\) We use the musical symbol \( \flat \), “flat” since we are working with flat tori.
5.2. Minkowski reduction. There are infinitely many representatives in each integral equivalence class of positive definite quadratic forms. A Minkowski reduced form is a particularly natural representative which has been of historical interest as described in Schürman’s survey [46].

Definition 5.5. A positive definite quadratic form \( q \) is Minkowski reduced if for all \( k = 1, \ldots, n \) and for all \( x \in \mathbb{Z}^n \) with \( \gcd(x_k, \ldots, x_n) = 1 \), we have \( q(x) \geq q_{kk} \). Moreover, a lattice basis \( A = [a_j] \in \text{GL}_n(\mathbb{R}) \) is Minkowski reduced if for each \( j \),

\[
a_j \in AZ^n \setminus \{0, a_1, \ldots, a_{j-1}\},
\]

is a shortest choice of vector such that \( a_1, \ldots, a_j \) is part of some basis of \( \Gamma \).

There is a correspondence between these notions of Minkowski reduction. In fact, \( A \) is a Minkowski reduced lattice basis if and only if \( A^T A \) is a Minkowski reduced positive definite form [7, Cor. 4, p. 14] (this is non-trivial). The lattice formulation gives the geometric intuition that makes it a natural choice for a basis. A representative of a quadratic form \( Q \) is any element of the equivalence class of integrally equivalent quadratic forms containing \( Q \).

Exercise 5.6 (⋆). A quadratic form \( q \) is positive definite and Minkowski reduced if and only if \( q_{11} > 0 \) and \( q(x) \geq q_{kk} \) for all \( k = 1, \ldots, n \) as long as \( x \in \mathbb{Z}^n \) with \( \gcd(x_k, \ldots, x_n) = 1 \).

Theorem 5.7 (see [7, p. 27-28], or Section 4.4.2 of [51]). To each positive definite quadratic form, the number of Minkowski reduced representatives is non-zero and finite.

Is it possible to find an even more intuitive reduction? For example, is there always, given a lattice \( \Gamma \), a basis matrix \( A = [a_j] \) such that each

\[
a_j \in \Gamma \setminus \text{Span}\{0, a_1, \ldots, a_{j-1}\}
\]

is any shortest choice of vector? In four dimensions, the following example constructed by van der Waerden [52, p. 286] shows that this is not always possible. Consider the basis matrix

\[
A_4 = \begin{bmatrix}
1 & 0 & 0 & 1/2 \\
0 & 1 & 0 & 1/2 \\
0 & 0 & 1 & 1/2 \\
0 & 0 & 0 & 1/2 \\
\end{bmatrix}
\]

of the lattice \( \Gamma = A_4 \mathbb{Z}^4 \), with the column vector notation \( A_4 = [a_j] \). We see that \( e_4 = -a_1 - a_2 - a_3 + 2a_4 \) is of the same length as \( a_4 \) and is linearly independent of \( a_1, a_2, a_3 \), however the vectors \( a_1, a_2, a_3, e_4 \) do not make a basis for \( \Gamma \). In general for \( n \geq 5 \), let \( A_n = [e_1, \ldots, e_{n-1}, \frac{1}{2} \mathbb{1}] \), where \( \mathbb{1} = e_1 + \cdots + e_n \). Then \( e_n \in A_n \mathbb{Z}^n \) is shorter than \( \frac{1}{2} \mathbb{1} \), but \( e_1, \ldots, e_n \) is not a basis for \( A_n \mathbb{Z}^n \).

Theorem 5.8 (see [52, p. 278]). As long as \( n \leq 3 \), we can find a basis matrix \( A = [a_j] \) for any \( n \)-dimensional lattice \( \Gamma \) such that \( a_1 \) is a shortest non-zero vector of \( \Gamma \) and each \( a_j \) is any shortest choice such that \( a_1, \ldots, a_j \) are linearly independent. If \( n = 4 \), then we have the equivalent statement if we replace any with some.

Definition 5.9 (Minkowski Domain). We define \( \mathcal{M}_n \) to be the set of \( n \)-dimensional symmetric positive definite quadratic forms that are Minkowski reduced.
5.2.1. Polyhedral cones. In order to describe the Minkowski domain geometrically we define polyhedral cones. Properties of these cones are used throughout Schiemann’s symphony. First note that we can naturally embed $n$-dimensional symmetric quadratic forms $q(x) = x^T Q x$ where $Q = (q_{ij})_{ij}$ in $\mathbb{R}^{n(n+1)/2}$. There are many ways to perform such an embedding. One of the most common in three dimensions is the following,

$$q \mapsto (q_{11}, q_{22}, q_{33}, q_{12}, q_{13}, q_{23}).$$

Definition 5.10. Let $A, B$ be (possibly empty) sets of non-zero $n$-dimensional vectors. Then a set of the form

$$P(A, B) := \{x \in \mathbb{R}^n : a \cdot x \geq 0, b \cdot x > 0 \text{ for each } a \in A, b \in B\}.$$

is a polyhedral cone. For $a \in A$, a set of the form $\{x \in \mathbb{R}^n : a \cdot x = 0\}$ is a supporting hyperplane, similarly for $b \in B$. The dimension of $P$ is the dimension of the smallest vector space containing it. A polyhedral cone is pointed if it does not contain any lines.

In terms of computer calculations, rational polyhedral cones (i.e., the elements of $A$ and $B$ are vectors with rational entries) are suitable to work with because they are easily stored and a computer can do exact calculations. Another fundamental concept for Schiemann’s symphony is:

Definition 5.11. Let $P$ be $k$-dimensional polyhedral cone, where $\overline{P}$ is its closure. A facet of $P$ is a $k-1$-dimensional intersection of $\overline{P}$ and some of its supporting hyperplanes. A $j$-face is similarly a $j$-dimensional intersection. A 1-face is an edge and a 0-face is a vertex.

Exercise 5.12. Let $P(A, B) \subset \mathbb{R}^n$ and $P(C, D) \subset \mathbb{R}^m$ be polyhedral cones. Their Cartesian product is also a polyhedral cone. More specifically, show:

1. $P(A, B) \times P(C, D) = P \left(A \times \{0\} \cup \{0\} \times C, B \times \{0\} \cup \{0\} \times D\right),$

2. If $n = m$, then $P(A, B) \cap P(C, D) = P \left(A \cup C, B \cup D\right)$.

Exercise 5.13. Let $P(A, B)$ be a non-empty polyhedral cone and $U$ a closed set in $\mathbb{R}^n$. Then the closure of $P(A, B)$ is equal to $P \left(A \cup B, \emptyset\right)$, and further, $P(A, B) \subseteq U$ if and only if $P \left(A \cup B, \emptyset\right) \subseteq U$.

Exercise 5.14. If a polyhedral cone $P(A, B) \subset \mathbb{R}^n$ is pointed, show that each edge is a set of the form $k \mathbb{R}_{\geq 0}$ for some $k \in \mathbb{R}^n$. Let $k_i \mathbb{R}_{\geq 0}$ be the edges of $P(A, \emptyset)$ and $k'_j \mathbb{R}_{\geq 0}$ the edges of $P(C, \emptyset)$. Show that the edges of $P(A, \emptyset) \times P(C, \emptyset)$ are $(k_i, 0) \mathbb{R}_{\geq 0}$ and $(0, k'_j) \mathbb{R}_{\geq 0}$.

For any pointed polyhedral cone we have the Minkowski sum

$$P(A, B) = \sum_{i=1}^{r} k_i \mathbb{R}_{\geq 0},$$

where $k_i \mathbb{R}_{\geq 0}$ are the finite number of edges of $P(A, B)$. This is a consequence of the fact that a polytope is the convex hull of its vertices; see Theorem 3.10 of [26]. If no confusion shall arise, we may simply write the directions $k_i$ to denote edges. To prove that every element of a pointed polyhedral cone has a certain property, it is often enough to check the property for only its edges, as exemplified in the next exercise.
Exercise 5.15. Let \( P(A, B) \) be a pointed polyhedral cone, and let \( C \) be a convex set. Check that \( P(A, B) \subseteq C \) if and only if the edges \( k_i \) of \( P(A, B) \) lie in \( C \). Further, observe that the dimension of \( P(A, B) \) is the number of linearly independent vectors among \( k_i \).

5.2.2. Minkowski’s domain. The next lemma describes \( \mathcal{M}_n \) for \( n \leq 4 \) as a pointed polyhedral cone. We refer to a proof that is quite elegant, but technical [6, p. 257-258].

Lemma 5.16. As long as \( n \leq 4 \), a quadratic form \( q \) is a Minkowski reduced positive definite form if and only if the following hold,

i) \( 0 < q_{11} \leq q_{22} \leq \cdots \leq q_{nn} \),

ii) \( q(x) \geq q_{kk} \) for \( x \in \{-1, 0, 1\}^n \) with \( x_k = 1 \) and \( x_{k+1} = \cdots = x_n = 0 \).

We can directly write out \( \mathcal{M}_3 \) as a pointed polyhedral cone, because we now only have a finite number of inequalities to consider.

Theorem 5.17 (\( \mathcal{M}_3 \) as a polyhedral cone). The set of symmetric positive definite Minkowski reduced forms \( q(x) = x^T Q x \) in 3-dimensions is given by \( 0 < q_{11} \) and the following systems of inequalities,

\[
\begin{align*}
0 & \leq q_{22} - q_{11}, \\
0 & \leq q_{33} - q_{22}, \\
0 & \leq q_{11} - 2q_{12}, \\
0 & \leq q_{11} + 2q_{12}, \\
0 & \leq q_{11} + q_{22} + 2q_{12} - 2q_{13} - 2q_{23}, \\
0 & \leq q_{11} + q_{22} - 2q_{12} - 2q_{13} + 2q_{23}, \\
0 & \leq q_{11} + q_{22} - 2q_{12} - 2q_{13} - 2q_{23}, \\
0 & \leq q_{11} + q_{22} - 2q_{12} + 2q_{13} + 2q_{23}, \\
0 & \leq q_{11} + q_{22} + 2q_{12} - 2q_{13} + 2q_{23}, \\
0 & \leq q_{11} + q_{22} - 2q_{12} - 2q_{13} - 2q_{23}.
\end{align*}
\]

\( \& \)

\[
\begin{align*}
0 & \leq q_{11} + q_{22} - 2q_{12} + 2q_{13} - 2q_{23}, \\
0 & \leq q_{11} + q_{22} + 2q_{12} + 2q_{13} + 2q_{23}, \\
0 & \leq q_{11} + q_{22} - 2q_{12} - 2q_{13} - 2q_{23}, \\
0 & \leq q_{11} + q_{22} - 2q_{12} + 2q_{13} + 2q_{23}, \\
0 & \leq q_{11} + q_{22} - 2q_{12} - 2q_{13} - 2q_{23}.
\end{align*}
\]

Proof. This follows from Lemma 5.16. \( \square \)

The Minkowski domain is always a pointed polyhedral cone. Unfortunately, in higher dimensions the number of inequalities explode, increasing quickly; see Tammel’s list [46, p. 20], [50].

Theorem 5.18 (\( \mathcal{M}_n \) is a polyhedral cone [6, p. 256-257]). Out of the infinitely many conditions for a quadratic form \( q \) to be in \( \mathcal{M}_n \) as given by Lemma 5.6, all but finitely many are non-redundant. In other words, \( \mathcal{M}_n \) is a pointed polyhedral cone.

Although the set of Minkowski reduced forms may contain more than one representative for each equivalence class, the representatives are almost unique in the following sense.

Theorem 5.19. Assume that \( q \) is in the interior of \( \mathcal{M}_n \) and has associated matrix \( Q \). Then for any unimodular \( B \), the quadratic form with associated matrix \( B^T Q B \) is in \( \mathcal{M}_n \) if and only if \( B \) is diagonal.

Proof. Write \( B = [b_j] \), where \( b_j \) are column vectors. Since the determinant of a unimodular matrix is \( \pm 1 \), the entries of the first column, \( b_1 \), satisfy \( \text{GCD}((b_1)_1, \ldots, (b_n)_1) = 1 \). Then, \( q(b_1) > q_1 \) unless \( b_1 = \pm e_1 \). However, we know that \( q(b_1) = q_{11} \), because by Minkowski reduction \( q(B e_1) \) is a smallest non-zero value. This means that the lower right \( n - 1 \times n - 1 \) matrix of \( B \), say \( B' \), has determinant \( \pm 1 \) and is also unimodular. For this reason, the second vector \( b_2 \) has
GCD((b_2)_2, \ldots, (b_n)_n) = 1. Therefore q(b_2) > q_{22} unless b_2 = \pm e_2. The equality q(b_2) = q_{22} follows from the second formulation of Minkowski reduction in terms of lattices; write Q = A^T A, A = \{a_j\} and note that AB = [\pm a_1 A_2 \cdots A_n]. Since a_1, A_2 are part of a basis of AZ^n, the length of A_2 must be equal to the length of a_2. We can repeat this process until each b_i = \pm e_i.

**Corollary 5.20.** The set \( \mathcal{M}_n^+ := \mathcal{M}_n \cap \{q : q_{1j} \geq 0 \text{ for each } 1 \leq j \leq n\} \) contains a representative of each positive definite quadratic form. In the interior, the representatives are unique.

This corollary is a direct consequence of Theorem 5.19, and we leave the details to the reader.

**5.2.3. Successive minima.** To understand one of the most striking results about Minkowski reduction, we require successive minima.

**Definition 5.21.** Assume that Q is a positive definite symmetric n \times n matrix. The i:th successive minimum of Q is

\[
\lambda_i(Q) := \min \left\{ q(x^{(i)}) : x^{(1)}, \ldots, x^{(i)} \in \mathbb{Z}^n \text{ are linearly independent and } q(x^{(j)}) \leq q(x^{(j+1)}) \text{ for each } 1 \leq j \leq i-1 \right\}.
\]

**Theorem 5.22** (van der Waerden, Satz 7 [52]). For a Minkowski reduced positive definite form q we have

\[
q_{ii} \leq \Delta_i \lambda_i(Q) \quad \text{with} \quad \Delta_i := \max \left\{ 1, \left( \frac{5}{4} \right)^{i-4} \right\}.
\]

This theorem helps us to understand the Minkowski domain in higher dimensions. Moreover, it is a generalization of Theorem 5.8 that tells us how much this reduction could fail to be “intuitive.” It is conjectured in [46] that there is an even tighter bound: \( q_{ii} \leq i \lambda_i(Q)/4 \) for \( i > 5 \). The next two exercises are not necessary to understand Schiemann’s symphony but are included due to their relevance to the spectral geometry of flat tori.

**Exercise 5.23.** Prove that if two Minkowski reduced forms \( q^{(1)}, q^{(2)} \) are isospectral, then \( q_{22}^{(1)} = q_{22}^{(2)} \) using Theorem 5.22. (There is no guarantee however that \( q_{33}^{(1)} = q_{33}^{(2)} \), which we saw in §4).

**Exercise 5.24.** Can you write an algorithm that inputs a positive definite form q and outputs all Minkowski reduced representatives of q? Hint: Find first the successive minima and then use Theorem 5.22.

**5.3. Schiemann’s choir.** In his thesis [44], Schiemann introduced the following reduction, which he called the Vorzeichennormalform, and which we call Schiemann reduction.

**Definition 5.25** (Schiemann reduction). A ternary positive definite form f is said to be Schiemann reduced if

1a) f is Minkowski reduced,
1b) \( f_{12} \geq 0, f_{13} \geq 0, \)
1c) 2f_{23} > -f_{22},

and the following facet conditions hold:

2a) \( f_{12} = 0 \implies f_{23} \geq 0, \)
2b) \( f_{13} = 0 \implies f_{23} \geq 0, \)
3a) \( f_{11} = f_{22} \implies |f_{23}| \leq f_{13}, \)
3b) \( f_{22} = f_{33} \implies f_{13} \leq f_{12}, \)
4a) \( f_{11} + f_{22} - 2f_{12} - 2f_{13} + 2f_{23} = 0 \implies f_{11} - 2f_{13} - f_{12} \leq 0, \)
4b) \( 2f_{12} = f_{11} \implies f_{13} \leq 2f_{23}, \)
4c) \( 2f_{13} = f_{11} \implies f_{12} \leq 2f_{23}, \)
4d) \( 2f_{23} = f_{22} \implies f_{12} \leq 2f_{13}. \)

The first three conditions define a non-closed pointed polyhedral cone that properly contains all Schiemann reduced forms embedded in \( \mathbb{R}^6 \). This cone has several facets. The facet conditions \( 2 - 4 \) exclude the forms that lie on these facets but do not satisfy the corresponding inequalities. Note that the facet conditions cannot be expressed according to the definition of polyhedral cone. The more well-known set of Eisenstein reduced forms (see [40, p. 142-143]) contains precisely one representative of each positive definite ternary quadratic form, and in fact, Schiemann used this theory to prove

**Theorem 5.26.** Any positive definite quadratic form in three variables has a unique representative that is Schiemann reduced.

The proof is not very interesting, it is long and technical [44]. What is interesting is that the Schiemann reduction of a positive definite quadratic form is unique. We define Schiemann’s choir, \( C \), to be the set of Schiemann reduced forms embedded in \( \mathbb{R}^6 \). Henceforth we may simply refer to this as the choir.

**Lemma 5.27.** The closure of the choir \( C \) is a pointed polyhedral cone equal to the closure of \( M_3^+ \). Alternatively, it is given by the following system

\[
\begin{align*}
0 &\leq q_{11} \leq q_{22} \leq q_{33} \\
0 &\leq 2q_{12} \leq q_{11} \& 0 \leq 2q_{13} \leq q_{11} \\
-q_{22} &\leq 2q_{23} \leq q_{22} \\
q_{11} + q_{22} &- 2q_{12} - 2q_{13} + 2q_{23} \geq 0
\end{align*}
\]

**Proof.** The fact that these inequalities give the closure of \( M_3^+ \) is a consequence of Theorem 5.17. Now it is straight-forward to see that \( M_3^+ \subseteq C \subseteq \overline{M}_3^+ \), which suffices.

One can calculate the edges of \( \overline{C} \), for instance via a package like Polymake [19] or by applying Section 5.5.1, here written as the set \( M \) which we decompose into \( M = M_1 \cup M_2 \cup M_3 \) accordingly:

\[
\begin{align*}
M_1 &:= \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}, \quad M_2 := \left\{ \begin{pmatrix} 0 & 0 & \pm 1 \\ 0 & \pm 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\} \& \\
M_3 &:= \left\{ \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & \pm 2 \\ 0 & \pm 1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 & 1 \\ 0 & \pm 2 & 1 \\ 0 & \pm 1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 1 & 0 \\ 1 & \pm 2 & 0 \\ 1 & \pm 1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 1 & 1 \\ 1 & \pm 2 & 1 \\ 1 & \pm 1 & 2 \end{pmatrix} \right\}.
\end{align*}
\]

These are edges in the sense that any element of \( \overline{C} \) can be expressed as an \( \mathbb{R}_{\geq 0} \)-linear combination of elements in \( M \). Schiemann gives a brief theoretical motivation in [44].

By definition of \( C \), we can for three sets of vectors \( \mathcal{A}, \mathcal{B}, \) and \( \mathcal{C} \) write

\[
\mathcal{C} = P(\mathcal{A}, \mathcal{B}) \cap \{ f \in \mathbb{R}^6 : \forall (c, d) \in \mathcal{C} : (c \cdot f = 0 \implies d \cdot f \geq 0) \}.
\]

The sets \( \mathcal{A}, \mathcal{B} \) consist of vectors in \( \mathbb{R}^6 \) that correspond to the conditions 1a), b), c) in the definition of Schiemann reduced forms. The set of vectors in \( \mathcal{C} \subseteq \mathbb{R}^6 \times \mathbb{R}^6 \)
produce the facet conditions. The elements of these three sets can be written out explicitly using Theorem 5.17 and Definition 5.25. There are elements in $C$ that have representatives in $C^{\prime} \setminus C$. Since they are integrally equivalent, the underlying lattices are congruent, so we can imagine that the forms look similar. We therefore call the representatives in $C^{\prime} \setminus C$ choir imposters. The following algorithm is a method to expel these imposters to speed up the symphony. To state the algorithm, for $v \in \mathbb{R}^n$ we define the sets

\begin{equation}
  v^0 := \{ x \in \mathbb{R}^n : v \cdot x \geq 0 \}, \quad v^\bot := \{ x \in \mathbb{R}^n : v \cdot x = 0 \}.
\end{equation}

The goal of the following lemma is to eliminate the imposters. This is done by checking if the first or second factor lies in a facet of $C$ which has facet conditions, and then removing imposter forms according to the corresponding condition; imposters lurking in $C^{\prime} \setminus C$ are depicted in Figure 4.

**Lemma 5.28 (Imposter Eliminator).** Assume that the polyhedral cone $T \subseteq C^{\prime} \times C$. We define a sequence of polyhedral cones $T_i$ for $i \in \mathbb{N}_0$ by

\[
  T_0 := \left( P(\mathfrak{A}, \mathfrak{B}) \times P(\mathfrak{A}, \mathfrak{B}) \right) \cap T,
\]

\[
  T_i := T_{i-1} \cap \bigcap_{(c,d) \in C : T_{i-1} \subseteq (c,0)^+} (d,0)^\geq \cap \bigcap_{(c,d) \in C : T_{i-1} \subseteq (0,c)^+} (0,d)^\geq.
\]

The sequence $T_i$ becomes stationary at some $i_0$. We define $T_{\mathcal{C} \times \mathcal{C}} := T_{i_0}$. Further, $T \cap (\mathcal{C} \times \mathcal{C}) = T_{\mathcal{C} \times \mathcal{C}}$ and $T \cap (\mathcal{C} \times \mathcal{C}) \subseteq T_{\mathcal{C} \times \mathcal{C}} \subseteq T$. 

**Figure 4.** The choir of Schiemann reduced forms is properly contained in its closure, $\overline{C}$, which is a pointed polyhedral cone. The facet conditions exclude certain regions on the facets of this cone so that Schiemann’s choir contains precisely one representative of each integral equivalence class of quadratic forms. Those forms on the facets that are not in Schiemann’s choir are described as imposters, because they are integrally equivalent to a choir member, hence the underlying lattices are congruent, so the imposters resemble legitimate choir members.
Proof. Clearly we have $T_{i+1} \subseteq T_i$ for each $i$. Let $C_1^i$, respectively $C_2^i$, be the set of $c \in \pi_1(\mathcal{C})$ such that $T_i \subseteq (c, 0)^\perp$, respectively $T_i \subseteq (0, c)^\perp$. Since $T_i$ monotonically decreases, $C_1^i$ and $C_2^i$ monotonically increase. However $\mathcal{C}$ is finite meaning that $C_1^i$ and $C_2^i$ converge and become stationary. It follows that $T_i$ must become stationary.

We are left to show $T \cap (\mathcal{C} \times \mathcal{C}) \subseteq T_{\mathcal{C} \times \mathcal{C}}$. Take any $(f, g) \in T \cap (\mathcal{C} \times \mathcal{C})$. It is clear that $(f, g) \in T_0$. Now say $(f, g) \in T_i$. When calculating $T_{i+1}$, note that $T_i \subseteq (c, 0)^\perp$ only if $f \cdot c = 0$ at which point we already know $f \cdot d \geq 0$ for $(c, d) \in \mathcal{C}$, since $(f, g) \in \mathcal{C} \times \mathcal{C}$. We have the analogous situation for $T_i \subseteq (0, c)^\perp$. In either case, this implies $(f, g) \in T_{i+1}$.

Note that we always have $(T_{\mathcal{C} \times \mathcal{C}})_{\mathcal{C} \times \mathcal{C}} = T_{\mathcal{C} \times \mathcal{C}}$.

5.4. Auditioning choir sections. We imagine isospectral quadratic forms as elements of the choir who sing exactly the same. To distinguish whether or not two or more forms sing exactly the same, we begin by sorting forms into choir sections, in which the forms sing similarly, but not necessarily identically. Mathematically, at least some of the representation numbers of the forms coincide, but not necessarily all. One can imagine that we are auditioning the forms to sort them into bass, tenor, alto, and soprano. We will later refine these rough auditions to locate those forms that create a perfect duet by singing exactly identically; mathematically identifying those forms with identical representation numbers. The minimal set induces a transitive relation on quadratic forms which is helpful for sorting them.

Definition 5.29. Let $\mathcal{Q}$ be a finite set of symmetric quadratic forms in $n$-dimensions. First define the transitive relation

$$x \preceq_{\mathcal{Q}} y \iff q(x) \leq q(y) \quad \text{for each } q \in \mathcal{Q}.$$ 

The set of minimal vectors with respect to a set $X \subseteq \mathbb{Z}^n$ is defined

$$\text{MIN}_{\mathcal{Q}}(X) := \{x \in X : y \not\preceq_{\mathcal{Q}} x \text{ for all } y \in X \setminus \{x\}\}.$$ 

Note that $\text{MIN}_{\mathcal{Q}}(X)$ consists of all $x \in X$ for which no other element in $X$ is smaller with respect to $\preceq_{\mathcal{Q}}$. We are naturally interested in the case when $n = 3$, and $\mathcal{Q}$ is the set of edges of $\mathcal{C}$, listed in (5.3) and (5.4). When we simply write $\text{MIN}$ and $\preceq$, this is assumed to be the case, but for the sake of generality we shall continue to keep $n$ and $\mathcal{Q}$ as variables.

Definition 5.30. For a set $\mathcal{Q}$ of $n$-dimensional quadratic forms, recall its positive hull,

$$\text{pos}(\mathcal{Q}) := \left\{ \sum_{i=1}^N \lambda_i q_i : \lambda_i \geq 0, \quad q_i \in \mathcal{Q} \right\}.$$ 

Above, the sum may contain any finite number of elements of $\mathcal{Q}$, that is to say $N$ may assume any value in $\mathbb{N}$. $\mathcal{Q}$ is regular if it is finite, all its elements are rational and positive semi-definite, the identity matrix is in the positive hull of $\mathcal{Q}$, and the embedding of $\mathcal{Q}$ into $\mathbb{R}^{n(n+1)/2}$ spans $\mathbb{R}^{n(n+1)/2}$.

One can check that the edges of $\mathcal{C}$ form a regular set. Next we characterize $\text{MIN}_{\mathcal{Q}}$ and describe its most important properties.

Lemma 5.31. We have $x \preceq_{\mathcal{Q}} y$ if and only if $q(x) \leq q(y)$ for each $q \in \text{pos}(\mathcal{Q})$.

Exercise 5.32. Prove this lemma.
Lemma 5.33. Assume that $Q$ is a regular set of quadratic forms, and that $\text{MIN}_Q(X) \neq \emptyset$. The minimal set $\text{MIN}_Q(X)$ is the unique smallest set in $X$ such that for any $z \in X$ there is a $y \in \text{MIN}_Q(X)$ with $y \preceq_z z$.

Proof. By contradiction, assume that for some $z \in X$, each $y \in \text{MIN}_Q(X)$ has $y \not\preceq_Q z$. Then $z \notin \text{MIN}_Q(X)$, meaning that there is an element $z^{(1)} \in X \setminus \{z\}$ with $z^{(1)} \preceq_z z$. If $z_1 \in \text{MIN}_Q(X)$, then we arrive at a contradiction. Otherwise, we construct a sequence $z^{(i)}$ inductively by letting $z^{(i+1)}$ be an element in $X \setminus \{z, z^{(1)}, \ldots, z^{(i)}\}$ such that $z^{(i+1)} \preceq_Q z^{(i)}$. By transitivity of $\preceq_Q$, we would arrive at a contradiction if at any point $z^{(i)} \in \text{MIN}_Q(X)$. Since $I \in \text{pos}(Q)$, by Lemma 5.31 we know that $\|z^{(i)}\| \leq \|z^{(i-1)}\| \leq \cdots \leq \|z\|$. Each $z^{(i)} \in \mathbb{Z}^n$ is distinct, therefore this sequence of norms must become stationary. This will eventually contradict $z^{(i+1)} \in X \setminus \{z, z^{(1)}, \ldots, z^{(i)}\}$, because there are only finitely many vectors in $\mathbb{Z}^n$ of any given fixed length.

To see that $\text{MIN}_Q(X)$ is the unique smallest set with this property, note that if $z \in \text{MIN}_Q(X)$, then the only element $y \in X$ with $y \preceq_Q z$ is $z$ itself.

Exercise 5.34. Show that $\text{MIN}_M(\mathbb{Z}^3) = \{(1, 0, 0)\}$, where $M$ is as in (5.3) and (5.4).

Exercise 5.35. Find a set $\emptyset \neq X \subseteq \mathbb{Z}^3$ such that $\text{MIN}_Q(X) = \emptyset$ regardless of $Q$. If $Q$ is regular and $0 \in X$, show that $\text{MIN}_Q(X) = \{0\}$.

Exercise 5.36. If $q$ is rational and semi-definite, show that $q(\mathbb{Z}^n)$ is a discrete set. Find a positive semi-definite $q$ such that $q(\mathbb{Z}^n)$ is not discrete.

Proposition 5.37. Let $Q$ be a regular set. We have the following:

(i) $x \preceq_Q y$ implies $\|x\| \leq \|y\|$.
(ii) If $x \preceq_Q y \preceq_Q x$, then $x = \pm y$.
(iii) $\text{MIN}_Q(X)$ is finite.
(iv) If $0 \notin X \neq \emptyset$, and for any shortest vector $u$ of $X$ we have $-u \notin X$, then $\text{MIN}_Q(X) \neq \emptyset$.
(v) $\emptyset \neq X \subseteq \mathbb{Z}^n$ implies $\text{MIN}_Q(X) \neq \emptyset$.

Proof.

(i) Follows from Lemma 5.31 and the fact that $I \in \text{pos}Q$.
(ii) We have $q(x) = q(y)$ for all $q \in Q$. This can be written:

$$0 = q(x) - q(y) = \sum_i q_{ii}(x_i^2 - y_i^2) + \sum_{i<j} 2q_{ij}(x_ix_j - y_iy_j).$$

The fact that $Q$ spans $\mathbb{R}^{(n+1)/2}$ implies that $x_i x_j = y_i y_j$ for each $i, j$. Then $y_i = \pm x_i$ for each $i$. Assume that $x_1 \neq 0$ and $y_1 = \pm x_1$. If there is some $y_j \neq 0$ with $y_j = \mp x_j$, then it follows that $y_1 y_j = -x_1 x_j \neq x_1 x_j$.

(iii) If $\text{MIN}_Q(X)$ were infinite, then there is a sequence $x^{(i)} \in \text{MIN}_Q(X)$ of distinct elements such that $x^{(1)} \not\preceq_Q x^{(2)} \not\preceq_Q x^{(3)} \ldots$. We can find a subsequence $x^{(i_j)}$ such that there is a $q \in Q$ with $x^{(i_1)} > q(x^{(i_2)}) > q(x^{(i_3)}) \ldots$. The elements of $Q$ are semi-positive and rational, implying by Exercise 5.36 that this sequence becomes stationary, which is a contradiction.

(iv) Denote the distinct vectors in $X$ that share the shortest length in $X$ by $u^{(1)}, \ldots, u^{(k)} \in X$. Assume by contradiction that no $u^{(i)}$ is in $\text{MIN}_Q(X)$. Then for each $u^{(i)}$ there is an $x \in X \setminus \{u^{(i)}\}$ such that $x \preceq_Q u^{(i)}$. By (i), $x = u^{(j)}$ for
some \( j \neq i \). If \( k = 1 \), we directly have a contradiction. If \( k > 1 \), then we could find two vectors \( u^{(i)} \neq u^{(j)} \) such that \( u^{(j)} \preceq u^{(i)} \preceq u^{(j)} \) as a consequence of the set of \( u^{(i)} \) being finite. By (ii) this implies \( u^{(j)} = \pm u^{(i)} \) which is a contradiction to the assumption in (iv).

(v) Follows directly from (iv) and the way we have defined \( Z^n_0 \) in Exercise 2.40.

\[ \square \]

5.4.1. \textit{Calculating} \( \text{MIN}_Q(X) \). It helps to know that any minimal set is finite given a regular set \( Q \), but how do we find its elements? The following lemma was used by Schiemann to reduce the calculation of MIN under certain conditions.

\textbf{Lemma 5.38.} Let \( \emptyset \neq Y \subseteq X \subseteq \mathbb{Z}^n_0 \), and let \( Q \) be regular. If

\[
W \supseteq \{ x \in X : x \not\preceq_Q y \ \forall y \in Y \} \cup Y,
\]

then \( \text{MIN}_Q(X) = \text{MIN}_Q(X \cap W) \).

\textbf{Proof.} We first show \( \text{MIN}_Q(X) \subseteq \text{MIN}_Q(X \cap W) \), and it’s not hard to see that it suffices to prove \( \text{MIN}_Q(X) \subseteq X \cap W \). Take \( z \in \text{MIN}_Q(X) \subseteq X \). If \( z \not\preceq_Q y \) for all \( y \in Y \), then by the assumption on \( W, z \in W \). If for some \( y \in Y \) we have \( z \geq_Q y \), then by construction of \( \text{MIN}_Q(X) \), \( y = z \in Y \), meaning that also in this case we have \( z \in W \).

Secondly we prove \( \text{MIN}_Q(X) \supseteq \text{MIN}_Q(X \cap W) \). Let \( z \in \text{MIN}_Q(X \cap W) \) and take any \( x \in X \) such that \( x \preceq_Q z \). We show that \( x \) is necessarily equal to \( z \), since then \( z \in \text{MIN}_Q(X) \) by definition. If \( x \in X \cap W \), then since \( z \in \text{MIN}_Q(X \cap W) \), we have \( z = x \). If \( x \in X \setminus W \), then it follows that there is a \( y \in Y \) such that \( y \preceq_Q x \). By transitivity, \( y \preceq_Q z \). Now since \( Y \subseteq X \cap W \), this implies \( y = z \). As a consequence, \( z \preceq_Q x \preceq_Q z \) which by Proposition 5.37 means \( z = x \).

To determine \( \text{MIN}_Q(X) \) we want to find appropriate, finite \( Y \) and \( W \) as above. We present a new result for this purpose. First, let \( \pi : \mathbb{R}^n \to \mathbb{R}^{n-1} \) denote the projection onto the last \( n - 1 \) coordinates.

\textbf{Theorem 5.39.} Assume that \( \mathcal{E} \) is a regular set of \( n \)-dimensional quadratic forms. Let \( \mathcal{E}' \) be the set of non-invertible matrices of \( \mathcal{E} \), with first row and column deleted. Let \( \mathcal{E}^x \) be the invertible matrices of \( \mathcal{E} \). If \( \mathcal{E}' \) is regular, then for \( a \in \mathbb{Z} \) and \( X \subseteq \mathbb{Z}^n \) such that \( \text{MIN}_{\mathcal{E}'}(\pi(X)) \neq \emptyset \), we define \( Y(a) := \{(a, z) : z \in \text{MIN}_{\mathcal{E}'}(\pi(X))\} \). Then,

\[
W(a) := \{ x \in X : t \not\preceq_{\mathcal{E}} x \text{ for each } t \in Y(a) \} \cup Y(a)
\]

\[
\subseteq \{ x \in \mathbb{Z}^n : \lambda_{\min} \|x\|^2 < \lambda_{\max} \max_{t \in Y(a)} \|t\|^2 \},
\]

where \( \lambda_{\min}, \lambda_{\max} \) are respectively the smallest and largest eigenvalues among elements of \( \mathcal{E}^x \). In particular, \( W(a) \) is a finite set.

\textbf{Proof.} Take any \( x \in W(a) \) and note that if \( f \in \mathcal{E}' \), then \( f(x) = f(\pi(x)) \). By assumption and Proposition 5.33, there is an element \( z \) of \( \text{MIN}_{\mathcal{E}'}(\pi(X)) \) such that \( z \preceq_{\mathcal{E}'} \pi(x) \). Therefore, if \( (a, z) \not\preceq_{\mathcal{E}} x \) we would require \( g(a, z) > g(x) \) for some \( g \in \mathcal{E}^x \). It follows that \( \lambda_{\min} \|x\|^2 < g(a, z) \leq \max \{ g(t) \} \leq \lambda_{\max} \max \{ \|t\|^2 \} \), where the maximum is taken over \( t \in Y(a) \). This proves the inclusion, since \( \lambda_{\min} > 0 \), \( \lambda_{\max}/\lambda_{\min} \geq 1 \), and we end by noting that both \( \mathcal{E}^x \) and \( Y(a) \) are finite.

\[ \square \]

If \( \mathcal{E} = M \) is the set of edges of \( \overline{\mathcal{G}} \), then the conditions of the theorem are satisfied. To apply the theorem in Schiemann’s symphony, let

\[
X_0 := \mathbb{Z}_3^3 \setminus e_1 \mathbb{Z}, \quad X_1 := \mathbb{Z}_3^3 \setminus (e_1 \mathbb{Z} + e_2 \mathbb{Z}), \quad X_2 := \mathbb{Z}_3^3 \setminus ((e_1 \mathbb{Z} + e_2 \mathbb{Z}) \cup (e_1 \mathbb{Z} + e_3 \mathbb{Z})).
\]
In Theorem 5.39, we get the corresponding

\[(5.6) \quad Y_0(a) = (a, 1, 0), \quad Y_1(a) = (a, 0, 1), \quad Y_2(a) = (a, \pm 1, 1), \]
\[(5.7) \quad W_i(a) \subseteq \{ x : \|x\| < 2\sqrt{2(a^2 + 2)} \}. \]

Schiemann showed similar, sharper bounds through less general means. We could find sharper bounds by determining \(\max\{g(t)\}\) as in the proof above, instead of estimating using \(\lambda_{\text{max}}\). In Schiemann’s symphony, we calculate the minimal sets of \(X_1 \setminus Z\) for finite sets \(Z\). Since \(Z\) is finite, we can always find an appropriate \(a\) such that \(Y_i(a) \subseteq X \setminus Z\) and in this way apply Lemma 5.38.

**Exercise 5.40 (⋆).** With \(E, X_0, X_1\) and \(X_2\) and as above, prove (5.6) and (5.7). Note that in this case, \(\lambda_{\text{min}} \approx 0.59\), and \(\lambda_{\text{max}} = 4\).

As mentioned above, the minimal sets will be useful for understanding the *choir sections.* Although we work in 3 dimensions here, the statements can be generalized to any dimension without problems, essentially by replacing \(\overline{\mathbf{e}}\) by the closure of \(\mathcal{M}_n^*\).

**Definition 5.41.** Let \(x^{(1)}, \ldots, x^{(k)} \in X \subseteq \mathbb{Z}_+^3\) be distinct elements. We define the choir section of \(X\) and the sequence \(x^{(i)}\) to be

\[\mathcal{S}(X, \{x^{(i)}\}_{i=1}^k) := \{ f \in \mathbf{e} : f(x^{(j)}) = \min(f(X \setminus \{x^{(1)}, \ldots, x^{(j-1)}\})) \forall j = 1, \ldots, k \}. \]

We set \(\mathcal{S}(X, \emptyset) := \overline{\mathbf{e}}\).

The choir section of \(X\) and \(x^{(i)}\) is the set of quadratic forms in \(\overline{\mathbf{e}}\) for which \(x^{(1)}, \ldots, x^{(k)}\) yield the successively smallest values among elements of \(X\). A connection to the representation numbers is that if \(f \in \mathcal{S}(X, \{x^{(i)}\}_{i=1}^k)\), then

\[(5.8) \quad f(x) = \min f(X \setminus \{x^{(1)}, \ldots, x^{(k)}\}) = \min \{ t_0 \in \mathbb{R}_{\geq 0} : \sum_{0 \leq t \leq t_0} R_X(f, t) \geq k + 1 \}, \]
\[(5.9) \quad f(x) = \min \{ t_0 \in \mathbb{R}_{\geq 0} : \sum_{0 \leq t \leq t_0} R_X(f, t) \geq k + 1 \}, \]

where we by \(\{x^{(i)}\}_{i=1}^k\) refer to the sequence \(x^{(1)}, \ldots, x^{(k)}, x\) in this order. We describe these sets of quadratic forms as choir sections because imagining the representation numbers as the notes they sing, the elements of a choir section have a similar vocal range.

**Lemma 5.42.** Let \(X \subseteq \mathbb{Z}_+^3\) and \(x^{(1)}, \ldots, x^{(k)} \in X\) be distinct. Assume that \(\text{MIN}(X \setminus \{x^{(i)}\}_{i=1}^k) \neq \emptyset\). Then the choir section \(\mathcal{S}(X, \{x^{(i)}\}_{i=1}^k)\) is a pointed polyhedral cone. More precisely,

\[\mathcal{S}(X, \{x^{(i)}\}_{i=1}^k) = \left\{ f \in \mathbf{e} : f(x^{(1)}) \leq \cdots \leq f(x^{(k)}) \text{ and } f(x^{(k)}) \leq f(x) \forall x \in \text{MIN}(X \setminus \{x^{(i)}\}_{i=1}^k) \right\}. \]

**Exercise 5.43.** Prove this lemma. Hint: apply Lemma 5.33 for the “≥” direction.

To see how the choir section is realized as a polyhedral cone in the computer, we proceed as follows. For fixed \(x, y \in \mathbb{R}^n\) and any quadratic form \(f\) we have

\[f(y) \geq f(x) \iff \sum_i f_{ii}(y_i^2 - x_i^2) + \sum_{i<j} f_{ij}(2y_iy_j - 2x_ix_j) \geq 0. \]

The right hand side is a linear inequality for the values of \(f_{ij}\). Then, given \(x^{(i)}\) and \(\text{MIN}(X \setminus \{x^{(i)}\}_{i=1}^k)\), which we calculated in the last section, we note that
Lemma 5.44. Let all the linear inequalities defining the choir section.

Proof.

"$\subseteq$" This inclusion is immediate since $S(X, \{x^{(i)}\}_{i=1}^k) \subseteq S(X, \{x^{(i)}\}_{i=1}^k)$ by definition.

"$\supseteq$": Let $f \in S(X, \{x^{(i)}\}_{i=1}^k)$ and define

$$Y_f := \left\{ y \in X \setminus \{x^{(i)}\}_{i=1}^k : f(y) = \min f(X \setminus \{x^{(i)}\}_{i=1}^k) \right\}.$$ 

The set $Y_f$ is non-empty by the discreteness of $f(X \setminus \{x^{(i)}\}_{i=1}^k)$ given by Exercise 5.36. By Proposition 5.37 (v), $\text{MIN}(Y_f) \neq \emptyset$. Fix some $y \in \text{MIN}(Y_f)$. We proceed to show that $y \in \text{MIN}(X \setminus \{x^{(i)}\}_{i=1}^k)$. In that case, we are done since by definition we have $f \in S(X, \{x^{(i)}\}_{i=1}^k)$. We consider the following decomposition,

$$X \setminus \{x^{(i)}\}_{i=1}^k = Y_f \cup \left( X \setminus \{x^{(i)}\}_{i=1}^k \setminus Y_f \right).$$

To see that $y \in \text{MIN}(X \setminus \{x^{(i)}\}_{i=1}^k)$, it suffices to show that $x \not\preceq y$ for any $x \in X \setminus \{x^{(i)}\}_{i=1}^k$. Since $y \in \text{MIN}(Y_f)$, we know that $x \not\preceq y$ for $x \in Y_f \setminus \{y\}$. Now consider $x \in (X \setminus \{x^{(i)}\}_{i=1}^k) \setminus Y_f$ and note that $x \not\preceq y$ (if $(X \setminus \{x^{(i)}\}_{i=1}^k) \setminus Y_f = \emptyset$, then we are already done). For such an $x$, we get $f(x) > f(y)$, because $y \in Y_f$ which implies $x \not\preceq y$ and we are done.

5.5. Schiemann’s symphony. We are almost ready to perform Schiemann’s symphony, the algorithm that proves $b_3 = 1$; three-dimensional flat tori are determined by their spectra. For this we define the duets,

$$\mathcal{D} := \{(f, g) \in \mathcal{C} \times \mathcal{C} : f \text{ and } g \text{ are isospectral}\}.$$ 

The goal of Schiemann’s symphony is to prove that all duets are in fact solos, that is $\mathcal{D} \subseteq \mathcal{S}$, where

$$\mathcal{S} := \{(f, f) : f \in \mathbb{R}^6\},$$

meaning that choir members that sing the same songs are identical.

Definition 5.45. We say that $P$, respectively $P'$, is a covering of the set $A \subseteq \mathbb{R}^n$, respectively refinement of $P$, if

$$A \subseteq \bigcup_{U \in P} U \quad \& \quad A \subseteq \bigcup_{V \in P'} V \subseteq \bigcup_{U \in P} U.$$

To prove that $\mathcal{D} \subseteq \mathcal{S}$, from which it immediately follows that $b_3 = 1$, we use a computer algorithm to calculate polyhedral cones that cover $\mathcal{D}$ and iteratively refine these coverings to prove that eventually they are contained in $\mathcal{S}$. We could imagine that the symphony is performed by pairs of quadratic forms that could include both members of Schiemann’s choir as well as imposters. The refinement procedure systematically ejects the imposters until all that remains are duets comprised of members of Schiemann’s choir, simultaneously showing that isospectral Schiemann reduced forms are necessarily identical, so the duets are actually solos.
Definition 5.46. A polyhedral cone $T$ is in tune with respect to $\Lambda(T), k = k(T)$ and sequences $x^{(1)}, \ldots, x^{(k)}, y^{(1)}, \ldots, y^{(k)}$ if the following properties hold

**P1:** $T \subseteq \mathcal{C} \times \mathcal{C}$ is a polyhedral cone and $T_{\mathcal{C} \times \mathcal{C}} = T$ as in Lemma 5.28.

**P2:** $\Lambda := \Lambda(T)$ is the largest of the three nested subsets of $\mathbb{R}^3$,

$$e_1 \mathbb{Z} \subseteq e_1 \mathbb{Z} + e_2 \mathbb{Z} \subseteq (e_1 \mathbb{Z} + e_2 \mathbb{Z}) \cup (e_1 \mathbb{Z} + e_3 \mathbb{Z})$$

such that $f|_{\Lambda(T)} = g|_{\Lambda(T)}$ for all $(f, g) \in T$.

**P3:** We have $x^{(i)}, y^{(i)} \in \mathbb{Z}_+^3 \setminus \Lambda$ and

$$T \subseteq S(\mathbb{Z}_+^3 \setminus \Lambda, \{x^{(i)}\}_{i=1}^k) \times S(\mathbb{Z}_+^3 \setminus \Lambda, \{y^{(i)}\}_{i=1}^k),$$

$$T \subseteq \{(f, g) \in \mathcal{C} \times \mathcal{C} : f(x^{(i)}) = g(y^{(i)}) \quad \forall i = 1, \ldots, k\}.$$ 

A covering $T$ of $\mathcal{D}$ is in tune if each $T \in T$ is in tune.

Aiming to define a sequence of coverings of $\mathcal{D}$, we start with

$$T_6 := \{(f, g) \in \mathcal{C} \times \mathcal{C} : f_{11} = g_{11} \} \mathcal{C} \times \mathcal{C}.$$ 

Denote by $T$ the single element of $T_6$. Observe that $\mathcal{D} \subseteq T$, by Minkowski reduction and Corollary 2.28. The set $T$ satisfies P1, P2 and P3 by letting $k = 1$, $x^{(1)} = y^{(1)} = e_1$ and $\Lambda = e_1 \mathbb{Z}$. For an in tune covering $\overline{T}$ of $\mathcal{D}$, we may define a refinement as follows. As the symphony is being performed, this refinement process removes the imposters and retains only the best performers.

Definition 5.47. Let $T$ be an in tune polyhedral cone with corresponding $\Lambda, k, x^{(i)}, y^{(i)}$ as in Definition 5.46. We define its refinement $\mathcal{T}'$ according to the following cases.

**Case 1.** $T \subseteq \mathcal{D}$: Let $\mathcal{T} := \{T\}$. The set $T \in \mathcal{T}$ is in tune with $\Lambda, k, x^{(i)}, y^{(i)}$.

**Case 2.** $T \not\subseteq \mathcal{D}$: Write $\mathfrak{L}(\Lambda, \{x^{(i)}\}) := \text{MIN}(\mathbb{R}(\mathbb{Z}_+^3 \setminus \Lambda, \{x^{(i)}\}_{i=1}^k)$ for a sequence $z^{(i)}$. Let for each $x \in \mathfrak{L}(\Lambda, \{x^{(i)}\})$ and $y \in \mathfrak{L}(\Lambda, \{y^{(i)}\})$,

$$S_{xy} := T \cap \left(S(\mathbb{Z}_+^3 \setminus \Lambda, \{x^{(i)}, x\}_{i=1}^k) \times S(\mathbb{Z}_+^3 \setminus \Lambda, \{y^{(i)}, y\}_{i=1}^k)\right),$$

$$T_{xy} := [S_{xy} \cap \{(f, g) \in \overline{\mathcal{C}} \times \overline{\mathcal{C}} : f(x) = g(y)\}] \mathcal{C} \times \mathcal{C}.$$ 

We define

$$\mathcal{T}' := \bigcup_{x \in \mathfrak{L}(\Lambda, \{x^{(i)}\}), \ y \in \mathfrak{L}(\Lambda, \{y^{(i)}\})} \{T_{xy}\}.$$ 

Each $T_{xy}$ is in tune with variables as follows. Let $\Lambda_{xy} = \Lambda(T_{xy})$ be maximal with $f|_{\Lambda_{xy}} = g|_{\Lambda_{xy}}$ for all $(f, g) \in T_{xy}$. Let $k(T_{xy}) = k(T) + 1$ and $x^{(k+1)} = x, y^{(k+1)} = y$.

The values in the sets $\mathfrak{L}$ are vectors at which the quadratic forms in the coverings are evaluated, for this reason we use the musical notation and imagine the quadratic forms are singing musical notes. Observe that $T_{xy} \subseteq S_{xy}$, and $(f, g) \in T_{xy}$ implies $(f, g) \in T$ and $f(x) = g(y)$. The sets $S_{xy}$ decompose $T$ into possibly overlapping subsets whose union is equal to $T$. Passing from $S_{xy}$ to $T_{xy}$, multiple representatives of the same equivalence class are removed; the imposters are expelled according to Lemma 5.28. We are now ready to define the sequence $T_i$ of coverings of $\mathcal{D}$.

Definition 5.48. If $T$ is a covering of $\mathcal{D}$, then we define its refinement as

$$T' := \bigcup_{T \in T} \mathcal{T}_T.$$ 

With $T_0$ as in Equation 5.10, we define the sequence $T_i$ by $T_{i+1} = T'_i$ for each $i \geq 0$. 
We must now argue that $T'$ is actually a refinement of a given covering $T$. This is done via the following proposition.

**Proposition 5.49.** $T_i$ is a sequence of coverings of $\mathcal{D}$, and each iteration is a refinement of the previous one.

**Proof.** Assume that $T = T_i$ for some $i$ is a covering of $\mathcal{D}$. We show that $T' = T_{i+1}$ is a refinement. Fix some arbitrary $T \in T$. Let us first note that by Lemma 5.44,

$$\bigcup_{x \in \mathcal{Z}(\Lambda, \{x^{(i)}\}), y \in \mathcal{Z}(\Lambda, \{y^{(i)}\})} S(Z^3_s \setminus \Lambda, \{x^{(i)}, x\}^{k}_{i=1}) \times S(Z^3_s \setminus \Lambda, \{y^{(i)}, y\}^{k}_{i=1})$$

is equal to $S(Z^3_s \setminus \Lambda, \{x^{(i)}\}^{k}_{i=1}) \times S(Z^3_s \setminus \Lambda, \{y^{(i)}\}^{k}_{i=1})$. By property P3 of $T$, we have

$$\bigcup_{x \in \mathcal{Z}(\Lambda, \{x^{(i)}\}), y \in \mathcal{Z}(\Lambda, \{y^{(i)}\})} S_{xy} = T \cap S(Z^3_s \setminus \Lambda, \{x^{(i)}\}^{k}_{i=1}) \times S(Z^3_s \setminus \Lambda, \{y^{(i)}\}^{k}_{i=1})$$

$$= T.$$

Consider $(f, g) \in S_{xy} \cap \mathcal{D}$. We have $\mathcal{R}(f, t) = \mathcal{R}(g, t)$ for all $t \in \mathbb{R}^+_0$. Since $f|_X = g|_X$, $\mathcal{R}(Z^3_s \setminus \Lambda)(f, t) = \mathcal{R}(Z^3_s \setminus \Lambda)(g, t)$ for all $t \in \mathbb{R}^+_0$. By Equation 5.9 and the definition of $S_{xy}$, we have

$$f(x) = \min \left\{ t_0 \in \mathbb{R}^+_0 : \sum_{0 \leq t \leq t_0} \mathcal{R}(Z^3_s \setminus \Lambda)(f, t) \geq k + 1 \right\} = \min \left\{ t_0 \in \mathbb{R}^+_0 : \sum_{0 \leq t \leq t_0} \mathcal{R}(Z^3_s \setminus \Lambda)(g, t) \geq k + 1 \right\} = g(y),$$

This implies $(f, g) \in T_{xy}$ due to Lemma 5.28, and $S_{xy} \cap \mathcal{D} \subseteq T_{xy}$. We get

$$T \supseteq \bigcup_{x \in \mathcal{Z}(\Lambda, \{x^{(i)}\})}, y \in \mathcal{Z}(\Lambda, \{y^{(i)}\}) \bigcup_{T \cap \mathcal{D}} \bigcup_{T \cap \mathcal{D}} (S_{xy} \cap \mathcal{D}) = T \cap \mathcal{D}.$$

We are done since by the fact that $T$ is a covering of $\mathcal{D}$, we have

$$\bigcup_{T \cap \mathcal{D}} \bigcup_{T \cap \mathcal{D}} (S_{xy} \cap \mathcal{D}) = \mathcal{D} \cap \bigcup_{T \cap \mathcal{D}} T = \mathcal{D}.$$

If $T_i$ ever becomes stationary, then by Definitions 5.47 and 5.48 each $T \in T_i$ lies in $\mathcal{J}$, and we have $\mathcal{D} \subseteq \mathcal{J}$. We cannot a priori conclude that this will occur, but we can show that the pairs of the coverings become more and more isospectral in the following sense.

**Lemma 5.50.** We have

$$\bigcap_{i \in \mathbb{N}} \bigcup_{T \in T_i} T \cap (\mathcal{C} \times \mathcal{C}) = \mathcal{D}.$$

**Sketch of proof.** By definition, $\mathcal{D}$ is contained in the left side of (5.11). So, assume that $(f, g)$ is in the left side of (5.11), we wish to show that $(f, g) \in \mathcal{D}$. There is a sequence of $T_i \in T_i$ such that $(f, g) \in T_i$ and $T_i \subseteq T_{i+1}$ for each $i$ by construction. Since $\Lambda(T_i)$ increases monotonically and takes three values, it becomes stationary and equals, say $\Lambda$. Clearly, for $x \in \Lambda$, we have $f(x) = g(x)$. Next we construct a bijection $\phi$ from $Z^3_s \setminus \Lambda$ to itself, such that $f(x) = g(\phi(x))$ for each $x \in Z^3_s \setminus \Lambda$. The existence of $\phi$ proves that $f$ and $g$ are isospectral. There are sequences $x^{(i)}, y^{(i)} \in Z^3_s \setminus \Lambda$ for which $T_i$ are in tune with, meaning $f(x^{(i)}) = \min f((Z^3_s \setminus \Lambda) \setminus \{x^{(i)}\}^{k-1}_{i=1})$ for
each $k$, and the analogous for $g(y^{(k)})$. Note that $\mathbb{Z}^n_k \setminus \Lambda = \{x^{(i)}\}_{i=1}^\infty = \{y^{(i)}\}_{i=1}^\infty$, or we’d arrive at a contradiction by Proposition 2.37 (5). In addition, $f(x^{(k)}) = g(y^{(k)})$ for each $k$, and so defining $\phi(x^{(k)}) = y^{(k)}$ finalizes the proof.

5.5.1. Calculating edges. Schiemann’s symphony is performed by calculating $T_i$ in Definition 5.48 using a computer. An algorithm calculates the edges of polyhedral cones in order to find $\Lambda$ in Definition 5.46 and to check the termination criterion $T \subseteq S$ of Definition 5.47 using Exercise 5.15. The statements here follow from the classic literature of [22, chapters 2 and 3]. As it would take to much space to write down rigorous proofs, we refer to the proofs that are contained in Schiemann’s thesis [44]. To start with, consider linearly independent vectors $a_1, \ldots, a_n \in \mathbb{R}^n$. To find the edges of $P((a_i)_{i=1}^n, \emptyset)$ we calculate all the kernels of the $n$ different $n - 1 \times n$ matrices we obtain from the set of $a_i$ by removing one those vectors.

This approach is very computationally expensive for a larger set of constraints $a_i$, and therefore we proceed as follows. Assume that we know the edges $k_1, \ldots, k_r$ of some polyhedral cone $P(A, B)$. What can we say about the edges of a polyhedral cone $P(A \cup \{v\}, B) \subseteq \mathbb{R}^n$ for some vector $v$? Since $P(A \cup B, B) = P(A, B)$, we can without loss of generality assume that $B \subseteq A$. In this case, Exercise 5.13 says that $P(A, B) = P(A, \emptyset)$.

**Theorem 5.51.** Let $P(A, B) \neq \emptyset$ be a pointed polyhedral cone such that $B \subseteq A$. Let $k_1, \ldots, k_r$ be the edges of $P(A, B)$. For a non-zero vector $v$ and the set $P(A \cup \{v\}, B)$, we have

Case 1: If $k_i \in v^{\geq 0}$ for each $i$, then the edges of $P(A \cup \{v\}, B)$ are $k_1, \ldots, k_r$.

Case 2: If $k_i \notin v^{\geq 0}$ for some $i$ and each $k_j$ has $k_j \cdot v \leq 0$, then proceed as follows. Let $k_1', \ldots, k_r'$ be those edges among $k_1, \ldots, k_r$ that lie in $v^\perp$. If there are no such $k_i'$, then $P(A \cup \{v\}, B)$ is either empty or equal to $\{0\}$. The set $P(A \cup \{v\}, B)$ is empty if and only if $k := \sum k_i'$ has $k \cdot b = 0$ for some $b \in B$. If it is non-empty and non-zero, then its edges are $k_1', \ldots, k_r'$.

Case 3: If $k_i \notin v^{\geq 0}$ for some $i$ and some $k_j$ has $k_j \cdot v > 0$, then proceed as follows. The set $P(A \cup \{v\}, B)$ is non-empty and its edges are calculated as those edges among $k_1, \ldots, k_r$ such that $k_i \cdot v \geq 0$ and the elements of the set

$$\left\{ F \cap v^\perp : F = k_1\mathbb{R}_{\geq 0} + k_2\mathbb{R}_{\geq 0} \text{ is a 2-face of } P(A, \emptyset) \text{ with } k_1 \cdot v > 0, k_2 \cdot v < 0 \right\}.$$ 

In our algorithm, we calculate a great number of edges. Theorem 5.51 reduces the computing time since it allows us to do it cumulatively, by keeping track of all the edges at all times. Next we give a computable criterion for finding the 2-faces for Case 3 above.

**Lemma 5.52.** Let $k_1 \neq k_2$ represent different edges of a polyhedral cone $P(A, \emptyset)$. Let $\{a_1, \ldots, a_r\} = \{a \in A : k_1 \subseteq a^\perp\}$ and $\{a_1', \ldots, a_s'\} = \{a \in A : k_2 \subseteq a^\perp\}$. Then $k_1\mathbb{R}_0^+ + k_2\mathbb{R}_0^+$ is a 2-face of $P_c(A, \emptyset)$ if and only if

$$\dim_{a \in \{a_1, \ldots, a_r\} \cap \{a_1', \ldots, a_s'\}} a^\perp = 2.$$ 

At each step, we want the number of elements of $A$ (and $B$) to be as few as possible, since it makes the program faster. The next lemma gives a simple condition with which we can remove some of the redundant constraints.
Table 1. This table shows our computational results for Schiemann’s symphony. Here, HH:MM correspond to the time in hours and minutes after the $i$:th iteration, and the number of cones is to the number of elements of $T_i$.

| $i$ | HH:MM | # Cones |
|-----|--------|---------|
| 0   | 00:00  | 1       |
| 1   | 00:00  | 1       |
| 2   | 00:00  | 4       |
| 3   | 00:00  | 42      |
| 4   | 00:02  | 500     |
| 5   | 00:05  | 3,311   |
| 6   | 00:13  | 11,164  |
| 7   | 00:28  | 31,334  |

| $i$ | HH:MM | # Cones |
|-----|--------|---------|
| 8   | 00:59  | 59,970  |
| 9   | 01:48  | 34,658  |
| 10  | 02:22  | 4,452   |
| 11  | 02:42  | 1,283   |
| 12  | 02:53  | 702     |
| 13  | 03:00  | 18      |
| 14  | 03:01  | 0       |

**Lemma 5.53.** Let $P(A, \emptyset)$ be a pointed polyhedral cone of dimension $d$ and with edges $k_1, \ldots, k_r$. We have $P(A, \emptyset) = P(A', \emptyset)$ for

$$A' := \{c \in A : \#\{k_i : k_i \in c^\perp\} \geq d - 1\}.$$ 

5.5.2. **Results from the algorithm.** The finale of Schiemann’s symphony is the following result.

**Theorem 5.54.** The sequence $T_i$ becomes stationary for $i \geq 14$. Further, we have that each set $T \in T_{14}$ lies in $\mathcal{S}$.

As we have previously noted, this completes the symphony. In other words, we have shown $\flat_3 = 1$. As documented in Table 1, with one processor it took about 3 hours to finish, and at least 147,442 polyhedral cones were computed. With 50 processors the algorithm took 19 minutes. We wrote the code in Julia [2] with the following packages and specifications of our computer: CPU: Intel(R) Xeon(R) Platinum 8180 CPU @ 2.50 Ghz; OS: Fedora 32; packages: Abstract Algebra v.0.9.0, Nemo v.0.17.0 & Hecke v.0.8.0. We found that the calculation of the minimal sets was not very time consuming in comparison to calculating edges. One way to make this program faster would be to minimize the amount of edges that need to be calculated, since the calculations of the minimal sets are relatively quick. Here, we perform the symphony at a comfortable andante pace. In his thesis, Schiemann performed the symphony at a quicker vivace tempo by incorporating clever tricks to speed up the algorithm which we have not included here so as to keep the focus on the main arguments of the proof.

6. **Open problems and food for thought**

To our best knowledge, it is not very well understood exactly what kind of tuples of isospectral non-isometric donuts can exists in any given dimension. We take this opportunity to highlight some very interesting related problems.
6.1. **Asymptotics of the choir numbers.** The celebrated geometer Wolpert, one of the three authors who proved *one cannot hear the shape of a drum* [20], studied the moduli space of flat tori in [54]. This space consists of the set of equivalence classes of donuts, where all members of the same equivalence class are isometric. He gave a geometric description of this moduli space and proved that it is in a certain sense an unusual phenomenon that donuts of different shapes sing the same song; the isometry class of a *generic* (which essentially means random) donut is determined by its spectrum. In his article, he showed that if each donut in a *continuous family* of donuts, meaning a one-parameter family of flat tori that has a continuous family of basis matrices, is isospectral to any other donut in the family, then all donuts are isometric. We obtain Wolpert’s result as a consequence of the following

**Lemma 6.1.** Consider for \( k \in \mathbb{N} \) the sequence of full-rank lattices \( \Gamma_k = A_k \mathbb{Z}^n \).

Assume that all \( T_{\Gamma_k} \) are mutually isospectral, and that \( A_k \to A \). At some point the sequence \( \Gamma_k \) becomes stationary up to congruency.

**Proof.** The positive definite quadratic forms \( A_k^T A_k \) all have the same image over \( \mathbb{Z}^n \). By Corollary 2.28 and continuity of the determinant, \( \det(A) = \det(A_k) \) for each \( k \). Therefore, \( \Gamma = AZ^n \) is a full-rank lattice, and its length spectrum is a discrete set. Fix any \( x \in \mathbb{Z}^n \) and consider the triangle inequality \( \|A_k x\| - \|Ax\| \leq \|(A_k - A)x\| \). For sufficiently large \( k \), \( \|A_k x\| = \|Ax\| \) since the spectra are discrete and identical for all \( k \), and \( A_k \to A \). There is now a \( k \) big enough such that for all \( x = e_i + e_j \), where \( 1 \leq i, j \leq n \), we have

\[
x^T (A_k^T A_k - A^T A) x = 0.
\]

We leave it to the reader to check that \( A_k^T A_k = A^T A \) for all \( k \) sufficiently large. By Exercise 2.35, \( A_k = C_k A \) for \( C_k \in O_n(\mathbb{R}) \).

As a direct consequence: let \( \Gamma = AZ^n \) be any full-rank lattice. There is a constant \( r(A) > 0 \) depending only on \( A \) such that if an \( n \times n \) matrix \( A' \) satisfies \( \|A - A'\| < r(A) \) and if \( \Gamma' = A'Z^n \) is isospectral to \( \Gamma \), then \( \Gamma' \) and \( \Gamma \) are congruent.

**Exercise 6.2.** Can you find such an explicit function \( r(A) \)?

When studying limits of donuts or lattices, Mahler’s Compactness Theorem is crucial, without which this survey would not be complete. According to Cassels, this theorem “may be said to have completely transformed the subject” in the context of lattice theory. See [7, p. 136-139] for this quote and a proof.

**Theorem 6.3** (Mahler’s Compactness Theorem). Let \( \Lambda_i \) be an infinite sequence of lattices of the same dimension, satisfying the following two conditions:

1. there exists a number \( K > 0 \) such that \( \text{vol}(\Lambda_i) \leq K \) for all \( i \);
2. there exists a number \( r > 0 \) such that \( \inf_{0 \neq v \in \Lambda_i} \|v\| \geq r \) for all \( i \).

There is then a subsequence \( \Lambda_{i_k} \) that converges to some lattice \( \Lambda \).

As an application of the results collected above, one can prove that only finitely many donuts can sing the same song.

**Theorem 6.4** (Finiteness Theorem, [54]). The total number of distinct donuts that sing the same song is finite; that is, the number of non-isometric flat tori with a given Laplace spectrum is finite.
Exercise 6.5. Prove the theorem by deriving a contradiction. We note that this result was first demonstrated by Kneser in an unpublished work.

In the spirit of honoring Wolpert’s work, we generalize his lemma in [54] to tuples.

**Proposition 6.6.** If \( \varphi_n \geq k \), then there exist \( k \) isospectral non-isometric integral \( n \)-dimensional quadratic forms.

**Proof.** We follow the ideas of Wolpert. Consider the \( n \)-variable positive definite forms \( Q_1, \ldots, Q_k \) that are isospectral and non-isometric. Then there are bijections \( \phi_i : \mathbb{Z}^n \to \mathbb{Z}^n \) such that

\[
Q_1(\phi_1(z)) = \cdots = Q_k(\phi_k(z))
\]

for all \( z \in \mathbb{Z}^n \). In particular, this tuple \( Q_i \) lies in the set

\[
U := \bigcap_{z \in \mathbb{Z}^n} \left\{ (P_1, \ldots, P_k) \in (S^n)^k : P_1(\phi_1(z)) = \cdots = P_k(\phi_k(z)) \right\},
\]

where \( S^n \) is the set of real symmetric \( n \times n \) matrices, and it is not hard to see that \( U \) is a linear space. Since it is defined by integral constraints, we can find a basis \((f_1^{(i)}, \ldots, f_k^{(i)})\) of integer matrices for \( U \). This means that \( Q_1 \) can be written as a linear combination of the \( f_i^{(i)} \) with real coefficients \( \lambda_i \). Since the set of positive definite forms is open, we can approximate \( Q_1 \) by a rational matrix \( \tilde{Q}_1 \), by choosing rational \( \tilde{\lambda}_i \approx \lambda_i \), such that it is still positive definite and lies in the first factor of \( U \). Then

\[
(\tilde{Q}_1, \ldots, \tilde{Q}_k) := \sum_i \tilde{\lambda}_i (f_1^{(i)}, \ldots, f_k^{(i)}),
\]

is a tuple of isospectral rational positive definite forms in \( U \), and up to a constant \( \tilde{Q}_i \) are integral.

In 1984, Suwa-Bier, a student of Kneser, was able to show the following powerful result [49]:

**Theorem 6.7.** The choir numbers are finite.

With this in mind, we ask about the asymptotic behaviour of the choir numbers. We have already seen in Lemma 2.52 that the choir numbers grow monotonically. In light of the four dimensional examples of isospectral but non-isometric lattices we obtain the following lower bound that we have not been able to find in the literature.

**Theorem 6.8.** If \( \varphi_m \geq k \), then

\[
\varphi_n \geq \left( \frac{|n/m| + k - 1}{k - 1} \right).
\]

In particular, \( \varphi_n \geq |n/4| + 1 \) and \( \varphi_n \) tends towards infinity.

**Proof.** In dimension \( mn \), for some positive integer \( n \), we construct \( \binom{n+k-1}{k-1} \) pairwise isospectral non-isometric flat tori. This would prove the statement by Lemma 2.52. In dimension \( m \) we have \( k \) flat tori \( T_{\Gamma_1}, \ldots, T_{\Gamma_k} \) that are isospectral and non-isometric. Consider the sequence of \( mn \)-dimensional lattices

\[
\Omega_{i_1, \ldots, i_k} := \Gamma_1^{i_1} \times \cdots \times \Gamma_k^{i_k},
\]
with non-negative $i_j$ such that $i_1 + \cdots + i_k = n$. There are $\binom{n+k-1}{k-1}$ different choices of the sequence $i_1, \ldots, i_k$. As a direct consequence of the Theorem 2.50 and Proposition 2.49, the flat tori $T_{i_1, \ldots, i_k}$ all share a common Laplace spectrum, but are pairwise non-isometric.

The exact values of the choir numbers remains an open problem. Do they have polynomial or exponential growth?

6.2. The fourth choir number. Since the third choir number has already been determined, the next step is to consider the fourth one. We could start by trying to show that $\mathfrak{b}_4 = 2$, by looking at triplets of quadratic forms instead of pairs. Let $\mathcal{C}_n$ be a set containing a unique representative of each $n$-dimensional positive definite form. Let $k, n$ be positive integers. We define

$$D_{n,k} := \{(f_1, \ldots, f_k) \in (\mathcal{C}_n)^k : f_i \text{ are isospectral for all } i = 1, \ldots, k\},$$

so that for instance $D_{4,3}$ consists of triplets $(f, g, h) \in \mathcal{C}_4 \times \mathcal{C}_4 \times \mathcal{C}_4$. It’s possible to modify Schiemann’s algorithm to deal with triplets instead of pairs. Our termination criterion in this case would be that for a polyhedral cone $T$, each triplet $(f, g, h) \in T$ should have either $f = g, f = h$ or $g = h$, since if the elements of a covering would have this property, then there are no triplets of different forms that all share representation numbers. The first main difficulty in working with triplets is that each iteration would go through three MIN sets instead of two. For this reason, the algorithm might be too slow, but considering the massive increase in modern computation power since Schiemann’s thesis approximately thirty years ago, this might not pose too much difficulty. However, it could be that the algorithm never terminates. In that case, we would propose a computer search to obtain a triplet of isospectral non-isometric donuts and perform the analogous algorithm with $D_{4,4}$ to try to prove $\mathfrak{b}_4 = 3$. This process could be repeated iteratively. In this way, using $D_{n,k}$, we could write a theoretically functioning program in the spirit of Schiemann’s symphony for determining $\mathfrak{b}_n$ for each $n$.

6.3. $k$-Spectra. In 2011, Jahan Claes wrote a report\textsuperscript{7} on the spectral determination of flat tori [8]. Reading it, it is apparent that he was not aware of Schiemann’s work, even though it certainly would have been relevant. In any case, Claes introduced the following definition which we have not seen elsewhere.

**Definition 6.9.** Let $\Gamma$ be a lattice and $\Lambda \subseteq \Gamma$ a sublattice. Write $[\Lambda]$ for the equivalence class of sublattices in $\Gamma$ with respect to congruence. We define the $k$-spectrum of $\Gamma$ to be the set

$$L^k(\Gamma) := \{([\Lambda], m_\Lambda) : \Lambda \text{ is a } k\text{-rank sublattice of } \Gamma\}.$$

Here, $m_\Lambda = \#[\Lambda]$ is the number of sublattices of $\Gamma$ that are congruent to $\Lambda$.

The $k$-spectrum is a generalization of the length spectrum, and it gives rise to new problems, the most natural of which is the following. For triplets of positive integers $(n, k, m)$ with $n \geq k, m$ we ask: given any $n$-dimensional lattices $\Gamma_1, \Gamma_2$, does $L^k(\Gamma_1) = L^k(\Gamma_2)$ imply $L^m(\Gamma_1) = L^m(\Gamma_2)$?

**Exercise 6.10.** Check that two full-rank lattices in $\mathbb{R}^n$ are congruent if and only if their 1-spectra agree. Check that two lattices are isospectral if and only if their 1-spectra agree.

---

\textsuperscript{7}The report has not been published to the best of our knowledge.
It follows from our work here that the answer is positive for some triplets, for example $(3,1,3)$ as in §5, and negative for others, like $(4,1,4)$ as in §4. Claes showed that the answer is yes if $n = m = 3$ and $k = 2$ by hand. The proposition below effectively gives an alternative demonstration of Claes’s claim using Schiemann’s theorem.

**Proposition 6.11.** Let $k > m$. If the $k$-spectra of $\Gamma_1$ and $\Gamma_2$ agree, then their $m$-spectra also agree.

Observe that as a consequence, the number of $n$-dimensional non-congruent lattices whose $k$-spectra all agree is bounded by $\flat_n$, for any $k$. One could also phrase the $k$-spectrum as a property of flat tori or quadratic forms. Indeed, in terms of quadratic forms, there is a clear connection to Siegel modular forms, motivated by the proof below. For more information, see Chapter 1 of [41]. For a certificate similar to Corollary 3.10 in terms of $k$-spectra, see [42].

**Proof.** The $k$-spectra agree if and only if there is a bijection

$$\phi : \Gamma_1^k \to \Gamma_2^k,$$

such that $\phi$ maps any set of linearly independent vectors $u_1, \ldots, u_k \in \Gamma_1$ to a set of linearly independent $v_1, \ldots, v_k \in \Gamma_2$, where the parallelopiped spanned by $u_i$ is congruent to that which is spanned by $v_i$, meaning that we can order $v_i$ such that $(\langle u_i, u_j \rangle)_{ij} = (\langle v_i, v_j \rangle)_{ij}$. Now consider the function $\phi' : \Gamma_1^{k-1} \to \Gamma_2^{k-1}$, sending the sets $u_1, \ldots, u_{k-1} \in \Gamma_1$ to $v_1, \ldots, v_{k-1} \in \Gamma_2$ corresponding to $\phi$. Since the upper left $k - 1 \times k - 1$ submatrices of the above $k \times k$ Gram matrices are equal, we get that $\phi'$ satisfies precisely the condition that the $k - 1$-spectra of $\Gamma_1$ and $\Gamma_2$ agree. □

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