KLEINIAN GEOMETRY AND
THE N=2 SUPERSTRING.

J. Barrett*, G.W. Gibbons*, M.J. Perry*, C.N.Pope† and P.Ruback‡

* Department of Mathematics
University of Nottingham
University Park
Nottingham
NG7 2RD
England

* Department of Applied Mathematics and Theoretical Physics
University of Cambridge
Silver Street
Cambridge
CB3 9EW
England

† Department of Physics
Texas A and M University
College Station
Texas 77843
USA

‡ Airports Policy Division
Department of Transport
2 Marsham Street
London
SW1P 3EB
England

Typeset by AMSTeX
Abstract. This paper is devoted to the exploration of some of the geometrical issues raised by the $N = 2$ superstring. We begin by reviewing the reasons that $\beta$-functions for the $N = 2$ superstring require it to live in a four-dimensional self-dual spacetime of signature $(- - + +)$, together with some of the arguments as to why the only degree of freedom in the theory is that described by the gravitational field. We then move on to describe at length the geometry of flat space, and how a real version of twistor theory is relevant to it. We then describe some of the more complicated spacetimes that satisfy the $\beta$-function equations. Finally we speculate on the deeper significance of some of these spacetimes.
§1 Introduction

This paper is aimed towards providing a description of some of the physics of the $N=2$ superstring, together with some of the geometrical consequences of such theories. It follows on from some seminal work of Ooguri and Vafa [1] who showed how to make sense of this type of string theory. They showed that the critical dimension of such strings is four real dimensions, and then computed some scattering amplitudes. The amplitudes indicate that the bosonic part of the $N=2$ theory corresponds to self-dual metrics of a spacetime of ultrahyperbolic signature $(+ - - -)$. We call metrics of such signature Kleinian. The situation should be contrasted with the bosonic string where we expect 26-dimensional metrics of Lorentz signature to play an important role, and the $N=1$ superstring where 10-dimensional metrics of Lorentz signature are the important objects. If we try to extend the number of supersymmetries beyond two superconformal supersymmetries, then we find that the critical dimensions are negative. It is unclear how one should think about such types of string theory.

The present paper is divided into six sections. In section two, we discuss some of the basic properties of the $N=2$ superstring, and explain how self-dual spaces of dimension four arise. In section three, we discuss the properties of flat four-dimensional Kleinian spacetime, and see how a real version of twistor theory is relevant. In section four we discuss some curved metrics that arise in this type of geometry. In section five, we discuss some cosmic string metrics which are in some sense related to the planar strings, and in section six we discuss some metrics that are related to string of higher genus. We conclude with some speculations. Readers who are more interested in Kleinian geometry than superstrings may skip section two and proceed directly to section three.

§2 The N=2 String

The traditional view of string theory is to regard the string worldsheet as an object embedded in a classical spacetime background. If we have no other structure then we would be thinking about the bosonic string. In perturbation theory, the closed bosonic string describes gravity with the Einstein action, together with various other spacetime fields. Such a theory is both conformally and Lorentz invariant in 26-dimensional flat spacetime of Lorentz signature. The bosonic string however is believed to be inconsistent because its spectrum contains a tachyon [2].

A different string theory can be found by supposing that the worldsheet has some additional structure. If it has $N=1$ superconformal supersymmetry, we find that the string is quantum mechanically consistent in a 10-dimensional spacetime of Lorentz signature. The closed superstring describes gravity with the Einstein action coupled to various other spacetime fields, the massless fields being those found in either of the two $N=2$ supergravity theories. There is no longer a tachyon in its spectrum and it is believed that this theory is a self-consistent finite theory. Nevertheless the description in terms of the worldsheet is incomplete because, like the bosonic string, the density of states function $\rho(E)dE$, the number of physical spacetime states between $E$ and $E+dE$, grows exponentially. As a consequence, there is a maximum temperature, the Hagedorn temperature, at which there is a phase transition to some other “exotic” phase of string theory [3]. For the superstring this temperature is $T_H \sim (8\pi^2\alpha')^{-1}$ where $\alpha'$ is the inverse string tension. The nature of the new phase is not currently understood - although much has been made of the analogy between hadronic physics and QCD. It may be extremely difficult to resolve the issue problem for the superstring precisely because of the complexity of the theory.

What is needed is a much simpler model theory in which some of the issues can be investigated without the complications of the infinite number of degrees of freedom of the
\[ N = 1 \text{ string. The model we propose to investigate here is the } N = 2 \text{ superstring, which is in some sense a “topological” string theory. The } N = 2 \text{ superstring is a theory with two superconformal supersymmetries. The action for such a theory can be conveniently written so that the two supersymmetries can be combined into a single complex supersymmetry. If we let } Z^i \text{ be the } i\text{-th component of a complex bosonic field and } \psi^i \text{ be the } i\text{-th component of a complex world-sheet spinor field (i.e. a Dirac spinor) then the action is invariant under an } N = 2 \text{ rigid supersymmetry. In this expression } \alpha' \text{ is the inverse string tension, and the string world-sheet is spanned by coordinates } \sigma^\mu, \text{ and has metric tensor } \gamma_{\mu\nu}. \]

\[ \text{Gamma matrices are determined with respect to the metric } \gamma_{\mu\nu}. \text{ The spacetime metric is given by } \eta_{ij} \text{ and has not, as yet, been specified. The rigid supersymmetry is generated by a complex (Dirac) spinor } \epsilon, \text{ and the transformations of } Z^i \text{ and } \psi^i \text{ are given by} \]

\[ \delta Z^i = \epsilon \psi^i, \quad \delta \psi^i = -i(\partial_\mu Z^i)\gamma^\mu \epsilon. \]

\[ \text{The global supersymmetry can be promoted to a local supersymmetry by coupling the matter multiplet to } N = 2 \text{ world-sheet supergravity. The gravitino multiplet contains a complex gravitino } \chi_\mu \text{ and an } U(1) \text{ gauge field } A_\mu, \text{ besides the world-sheet metric } \gamma_{\mu\nu}. \text{ The action is now invariant under world-sheet reparametrizations and world-sheet Weyl rescalings, and can be written explicitly as [4]} \]

\[ I = -\frac{1}{2\pi\alpha'} \int \gamma^{1/2}d^2\sigma \left\{ \gamma^{\mu\nu} \partial_\mu Z^i \partial_\nu Z^j - i\bar{\psi}^i\gamma^\mu \partial_\mu \psi^j + i(\partial_\mu \bar{\psi}^i)\gamma^\mu \psi^j \right\}\eta_{ij} \]

\[ + A_\mu \bar{\psi}^i\gamma^\mu \psi^j + (\partial_\mu Z^i - \frac{1}{2} \bar{\psi}^i\chi_\mu \gamma^\mu)\bar{\psi}^j\gamma^\nu \gamma^\mu \chi_\nu \quad \text{(2.4)} \]

\[ + (\partial_\mu \bar{Z}^i - \frac{1}{2} \bar{\psi}^i \gamma^\mu \gamma^\nu \psi^j)\eta_{ij}. \]

\[ \text{There are no kinetic terms for any components of the } N=2 \text{ supergravity multiplet, and in fact these fields can be gauged away by the introduction of the appropriate ghost fields. As it stands the action (2.4) describes a string propagating in a complex } \frac{1}{2}d\text{-dimensional background. However, it can be reinterpreted as a } d\text{-dimensional real background, where the signature of the metric } \eta_{ij} \text{ has an even number of both positive and negative directions so as to be consistent with the complex structure inherent with } N = 2 \text{ supersymmetry.} \]

\[ \text{We must now examine the consistency of our string theory. Suppose that the string is propagating in a flat real metric } \eta_{ij}. \text{ First, there is the issue of the Weyl anomaly. It is easily computed, since the contribution to the anomaly from a ghost-antighost pair in which the ghost field has conformal weight } \frac{1}{2}(1 - j) \text{ and statistics } \epsilon, \text{ is } \epsilon(1 - 3j^2). \text{ The gravitino field has } \epsilon = +1 \text{ and } j = 3 \text{ thus contributes } -26 \text{ to the anomaly. There are two sets of gravitino ghosts, one for each gravitino both with } \epsilon = -1 \text{ and } j = 2 \text{ giving a contribution of 22. The ghost for the } U(1) \text{ gauge field has } \epsilon = 1 \text{ and } j = 1 \text{ and so contributes } -2. \text{ The total ghost contributions are therefore } -6. \text{ The matter fields contribute } d \text{ from the bosonic fields and } d/2 \text{ from the fermionic fields and thus the Weyl anomaly will cancel provided that } d = 4. \text{ Thus we conclude that the } N = 2 \text{ string is anomaly-free in a flat spacetime of four dimensions. The choice of metric signature is however not completely arbitrary. The target space must have a complex structure and so we can choose the target space metric to have} \]

\[ \text{[98x172]} \]
signature 4 or 0. However, if the worldsheet is to be Lorentzian, then the metric signature is \((- - + +\)). We will briefly comment about the Euclidean signature case later.

The next step is to determine the physical states of the theory. These are the states of positive norm modulo gauge equivalence. For signature \((- - + +\)) one expects twice as many negative norm states as for the bosonic string, and all of these must be removed by the various ghost contributions. In fact, there are two sets of fermionic ghosts, those that cancel the graviton, and those that cancel the gauge field. These are sufficient to cancel the negative norm states. On the face of it, all the states of the theory are cancelled, since each ghost field absorbs both a timelike and a spacelike degree of freedom. It is in this sense that we mean the \(N = 2\) superstring is a topological string theory.

We must however be more precise, and look at the question of degrees of freedom more carefully. There are two equivalent ways of proceeding. One is to examine the cohomology of the BRS operator defined on a cylindrical worldsheet. Physical states are then those that are \(Q\)-closed but not \(Q\)-exact \([5]\). Alternatively one can look at the partition function evaluated on the torus. This will exhibit contributions from each physical state. Unitarity considerations indicate that the two approaches yield the same conclusions. In going through such a process, it is clear that there is considerable ambiguity in how one deals with the global issues, and it appears that there are in fact many consistent string theories that might eventually emerge.

To see what the physical states are in the BRS formulation, we must introduce sets of operators with which to describe the excited states of the string. We are interested in closed strings propagating in the flat metric \(\eta_{ij}\) on \(\mathbb{R}^4\). The string worldsheet is cylindrical with spacelike coordinate \(\sigma\) identified with period \(2\pi\), and the timelike coordinate \(\tau\) being unbounded. We work, for convenience, with complex fields, \(X^i\) and \(\psi^i\). It therefore follows that \(X^i\) is periodic

\[
X^i(0, \tau) = X^i(2\pi, \tau),
\]

and \(\psi^i\) is either periodic or antiperiodic

\[
\psi^i(0, \tau) = \pm \psi^i(2\pi, \tau).
\]

Introducing mode expansions for these fields

\[
X^i = x^i_0 + p^i_\tau + \sum_{n \neq 0} \frac{1}{n} (\alpha_n^i e^{-in(\tau + \sigma)} + \tilde{\alpha}_n^i e^{-in(\tau - \sigma)}),
\]

and

\[
\psi^i = \sum_r d^i_r e^{-in(\tau + \sigma)} + \tilde{d}^i_r e^{-in(\tau - \sigma)},
\]

where \(r \in \mathbb{Z}\) (R sector) or \(r \in \mathbb{Z} + 1/2\) (NS sector). The canonical (anti)-commutation rules then lead to

\[
[\alpha_n^i, \tilde{\alpha}_m^j] = n\eta^{ij} \delta_{n+m,0}
\]

and

\[
\{d^i_r, \tilde{d}^j_s\} = \eta^{ij} \delta_{r+s,0},
\]

together with similar expressions for the left-handed modes. All other (anti)-commutators vanish.

As a result of \(N = 2\) superconformal symmetry the energy-momentum tensor \(T_{\mu\nu}\), the (complex) supercurrent, \(S_\mu\) and the \(U(1)\) gauge current \(T_\mu\) must all vanish identically.

\[
T_{\mu\nu} = [\partial_\mu X^i \partial_\nu X^j + \bar{\psi}^i \gamma_{(\mu} \partial_{\nu)} \psi^j - \text{(trace)} \] \(\eta_{ij}.\)
\[ S_\mu = \gamma^\nu \gamma_\mu \bar{\psi}^j \partial_\nu X^j \eta_{ij}. \]  
\[ T_\mu = \bar{\psi}^j \gamma_\mu \psi^j \eta_{ij}. \]  
(2.12)  
(2.13)

The Fourier components of these are given by the following operator expressions, each of which should be taken to be normal ordered in the quantum theory:

\[ L_n = \sum_m \alpha_m \bar{\alpha}_{n+m} + \sum_r (r + \frac{n}{2}) d_{-r} \bar{d}_{n+r}, \]  
(2.14)

\[ G_r = \sum_s d_s \alpha_{r-s}, \]  
(2.15)

\[ \bar{G}_r = \sum_s \bar{d}_s \bar{\alpha}_{r-s}, \]  
(2.16)

\[ T_n = \sum_r d_r \bar{d}_{n-r}. \]  
(2.17)

The \( N = 2 \) superconformal algebra can then be constructed in the standard way. It is, taking care of all anomalous (central) terms:

\[ [L_n, L_m] = (n-m)L_{n+m} + \frac{1}{2}n(n^2 - 1)\delta_{n+m,0}, \]  
(2.18)

\[ [L_n, T_m] = -mT_{n+m}, \]  
(2.19)

\[ [L_n, G_r] = (\frac{1}{2}n - r)G_{n+r}, \]  
(2.20)

\[ [L_n, \bar{G}_r] = (\frac{1}{2}n - r)\bar{G}_{n+r}, \]  
(2.21)

\[ [T_n, G_r] = G_{n+r}, \]  
(2.22)

\[ [T_n, \bar{G}_r] = -\bar{G}_{n+r}, \]  
(2.23)

\[ [T_n, T_m] = 2m\delta_{n+m,0}, \]  
(2.24)

\[ \{G_r, \bar{G}_s\} = L_{r+s} + 1/2(r-s)T_{r+s} + (r^2 - 1/4)\delta_{r+s,0}. \]  
(2.25)

The remaining (anti)-commutators all vanish.

The physical states conditions are then:

\[ L_n | \phi \rangle = T_n | \phi \rangle = G_r | \phi \rangle = \bar{G}_r | \phi \rangle = 0, \quad n, r > 0, \]  
(2.26)

\[ T_0 | \phi \rangle = 0, \]  
(2.27)

\[ L_0 | \phi \rangle = h | \phi \rangle, \]  
(2.28)

for a physical state \( | \phi \rangle \). The conditions (2.26-8) are precisely equivalent to the requirement that \( | \phi \rangle \) be described by the cohomology of the BRS operator \( Q \). It should be noted that these conditions imply, from equation (2.25), that all physical states have vanishing \( U(1) \) charge, whereas the currents \( G \) and \( \bar{G} \) have a \( U(1) \) charge of 1 and \(-1\) respectively.

The highest weight representations of the \( N = 2 \) super-Virasoro algebra have been investigated by Boucher, Friedan and Kent [6]. We can apply their results directly to the string problem. For the NS-sector, the only possible choice of \( h \) consistent with the requirement of neutrality with respect to the \( U(1) \) charge for the ground state, and central charge corresponding to the physical dimension in which conformal invariance can be maintained is...
Furthermore, beyond the ground state, all excited states formed by acting with the \( \alpha_n \)’s or \( \bar{\alpha}_n \)’s on the vacuum state will not be neutral with respect to the \( U(1) \) charge, or spurious states. The conclusion is therefore that the only state that represents a spacetime boson is the ground state of the string, which is a massless. This agrees with the results of Mathur and Mukhi [7] who examined the first few excited states of this superstring.

In the R-sector of the string, their results can be similarly applied, and we find no physical states whatsoever. Thus we conclude that the only physical state of the string is a single massless scalar boson. This string therefore does not suffer from a Hagedorn transition together with its associated complications.

If instead we wished to derive the same results from considering the cohomology of \( Q \), then we would have to supplement the definitions of \( L_n, G_n, \bar{G}_n \) and \( T_n \) by the inclusion of extra ghost terms. We would then discover that the (2.18) to (2.25) would still hold, but now all the central terms would cancel, in both the NS and R sectors, as a consequence of choosing the critical dimension to be four, and of having zero shift of the vacuum energy in both sectors. We note that the above situation is rather different to the \( N = 1 \) superstring where there are different shifts of the vacuum energies in different sectors. Having carried out this procedure, we would be guaranteed that \( Q^2 = 0 \), and then its cohomology could be investigated. Thus the analysis of the previous paragraph would be repeated.

The interpretation of this scalar particle has been explored by Vafa and Ooguri [1] who have computed the scattering amplitudes at the tree level. They find that the amplitudes obtained show that the scalar is the Kähler potential for a self-dual gravitational field. Thus it seems reasonable to conclude that this string theory is just self-dual gravity, for the case of spacetime signature \((- - + +)\).

One can also support their conclusion by two other calculations. Firstly, we can compute the string partition function directly from the path integral which requires the explicit introduction of the ghost and antighost fields for each of the constraints, given by equations (2.14-17). The ghosts are just minimal b-c systems with statistics \( \epsilon \) and conformal weight \( j \) referred to earlier. We can now simply evaluate the partition function by standard methods. Consider states at level \( n \) so that the mass squared of these states is proportional to \( n \), and define the partition function to be

\[
G(w) = Tr \ w^N = \sum p(n)w^n
\]

(2.29)

where \( N \) is the number operator which counts the level to which the string has been excited, and thus \( p(n) \) is the number of states at the \( N \)th level. Since we are dealing with a closed string theory, we must take account of both left-moving excitations and right-moving excitations, imposing the constraint that the level of excitation of the left-movers is the same as that of the right-movers. Consider first of all the NS-sector (for just right-movers), then

\[
G_{NS}(w) = \prod_{n>0} (1 - w^n)^{-4}(1 + w^{n+1/2})^4(1 + w^n)^4(1 + w^{n+1/2})^{-4}
\]

(2.30)

where each term in the product arises from the \( X^i \)’s, \( \psi^i \)’s, graviton and gauge field ghosts, and gravitino ghosts respectively. We thus find that \( G_{NS}(w) = 1 \) as was expected from previous considerations. Now we shall consider the R-sector. There are no states that have vanishing charge, and so we can conclude immediately that \( G_R(w) = 0 \). Hence, we believe the only physical state is the massless scalar. The fact that \( G(w) = 1 \) is the reason that the theory can be termed topological.

Of course, we have chosen one particular GSO projection [8] in the preceding treatment, and it is the one in which all the fermions have the same choice of periodicity. We need not have done that, and Ooguri and Vafa [1] exhibit other possible choices. We note here
however that if we make a choice that results in spacetime fermions, then there will be at least two spacetime supersymmetries that are never broken. That is because in a self-dual space, we are guaranteed to have at least two covariantly constant spinors which can be used to generate two independent spacetime supersymmetries.

We have so far examined the theory in flat space only. Suppose that instead we started from the $\sigma$-model type action and asked what would happen if the model were immersed in some curved space. Then, the would need to compute the renormalization group equations for the curved space metric $\eta_{ij}$. At the one-loop level, we would find that the $\beta$-function equation for the renormalization of the metric would just be

$$R_{ij} = 0$$

Thus to lowest order, we have discovered that the metric must be Ricci flat. Stringy considerations lead us to believe that this metric must be Kähler because of the behaviour of the scattering amplitudes alluded to earlier. Thus, in perturbation theory we can conclude that there are no other terms. Suppose that we have a Ricci flat Kähler metric, since we are four dimensions we can conclude that the metric must be self-dual. It is therefore impossible to find any non-trivial higher order symmetric tensors built from only the Riemann tensor and so any such target space will solve the string $\beta$-function equations to all orders. Consequently in most of the remainder of our paper, we will describe some of the geometry of such target spaces.

We conclude this section by contemplating the string propagating in a space of signature $(+ + + +)$ once more. We have discovered that the only physical excitations of the string were the ground state. If we were in flat $\mathbb{R}^4$ then with this signature, there would be no string theory as the ground state would always correspond to the string having collapsed down to a point. However, if the target space, $\mathcal{M}$ had some non-trivial $\pi^2$, then the string at genus zero could shrink down onto the non-contractible $S^2$ and give some extra contribution to the partition function. We can extrapolate our present results to conclude that

$$G_{NS}(0) = \dim(\pi^2(\mathcal{M}))$$

in general. Furthermore, similar considerations at genus $g$ would seem to indicate that all of $H_2(\mathcal{M})$ can be probed by considering all possible genera of the string worldsheet. It may even be possible to extract further topological information from $N = 2$ string theory. This is a matter that is currently under investigation.

§3 Flat Ultra-hyperbolic 4-Geometry

The properties of four-dimensional metrics of signature $(2,2)$ are comparatively unknown amongst most of the physics community. The purpose of this section therefore is to provide a brief review of their basic properties. We shall begin with the flat space $\mathbb{R}^{2,2}$, and discuss curved spaces in the next section.

We will start with some purely terminological remarks. In an arbitrary dimension, an ultrahyperbolic metric is a metric with signature $(s,t)$ or $(t,s)$ with $s \geq t$ and for which $t \neq 0,1$. The cases of $t = 0,1$ are called Riemannian and Lorentzian (or hyperbolic) metrics respectively. In dimension four, there remains only the case $(2,2)$. In dimensions greater than four, the term “ultrahyperbolic,” as it is usually used, is ambiguous. Another case that is especially interesting for us is that of $(3,3)$, since $GL(4, \mathbb{R})$ is a 2-fold cover of $SO(3,3; \mathbb{R})$ which itself is a 2-fold cover of the conformal group of $\mathbb{R}^{2,2}$. We propose calling metrics of signature $(r,r)$ “Kleinian.” We shall also refer to a flat space of signature $(r,r)$ as “Plücker” space by analogy with flat Euclidean space and flat Minkowski spacetime. Not the least
interesting feature of Kleinian spacetimes is the complete symmetry between space and time which corresponds to the freedom to multiply the metric $g_{\mu\nu}$ by $-1$ and obtain a new metric of the same signature. Only in dimension 2 is it possible for a Lorentzian spacetime because in that dimension Lorentzian and Kleinian are the same thing. In string theory the above symmetry is associated with what is usually referred to as “crossing symmetry.” Since the analogy is not precise and since the word “dual” is used in many unrelated situations, we prefer to use the term “chronal-chiral symmetry” for the symmetry under $g_{\mu\nu} \to -g_{\mu\nu}$.

The reason for the introduction of the names of Klein [9] and Plücker [10] is of course because of the well-known Plücker correspondence between unoriented lines in Euclidean 3-space $\mathbb{R}^3$ and null rays through the origin in $\mathbb{R}^{3,3}$, via the corresponding relation to simple 2-forms in $\mathbb{R}^{1,-1}$ for all values of $t$, and its exploitation and popularization by Klein. The Plücker correspondence has numerous important applications to the geometry and physics of Euclidean 3-space. The extension of the Plücker correspondence to 4-dimensional Minkowski space is what Penrose [11] calls twistor theory [12]. The Plücker correspondence works as follows: Let $X^\alpha, Y^\beta$ be homogeneous coordinates for two points $x$ and $y$ in $\mathbb{R}^3$, or more precisely $P_3(\mathbb{R})$. The line $l$ through $x$ and $y$ corresponds to the simple bi-vector

$$P^{\alpha\beta} = 2X^{[\alpha}Y^{\beta]},$$

up to a scale factor. The condition that the bi-vector be simple is that

$$\epsilon_{\alpha\beta\gamma\delta} P^{\alpha\beta} P^{\gamma\delta} = 0,$$

where the left hand side of (3.2) defines a Kleinian metric on the six-dimensional space of all bivectors in $\mathbb{R}^4$. Equation (3.2) says that lines in $P_3(\mathbb{R})$ correspond to null rays through the origin in $\mathbb{R}^{3,3}$. The six numbers $P^{\alpha\beta}$ subject to the constraint (3.2) are referred to as the Plücker coordinates for the straight line $l$.

If $X^\alpha = (1, x^i)$ etc. then

$$P^{i0} = x^i - y^i,$$

$$P^{ij} = x^i y^j - x^j y^i.$$  \hspace{1cm} (3.3a)

(3.3b)

Using a suitable numbering system, we may introduce coordinates $p^a$ in $\mathbb{R}^{3,3}$ and equation (3.2) becomes

$$p^a p^b k_{ab} = 0,$$

where $k_{ab}$ are the components of the Plücker metric.

Thus if

$$\epsilon_{0123} = 1,$$

$$p^i \equiv P^{0i},$$

$$p^{3+i} = \frac{1}{2} \epsilon^{ijk} P^{jk},$$

then

$$k_{ab} = \delta_{3,|a-b|}. \hspace{1cm} (3.5)$$

The action of the projective group of $\mathbb{R}^3$, $PSL(4,\mathbb{R}) \cong SL(4;\mathbb{R})/\pm 1$, may be lifted to a linear action on $\mathbb{R}^4$ preserving the alternating tensor $\epsilon_{\alpha\beta\gamma\delta}$. This provides the isomorphism $SO_0(3,3;\mathbb{R}) \cong SL(4,\mathbb{R})/\pm 1$, the elements of $\mathbb{R}^4$ corresponding to what are called in the physics literature Majorana-Weyl spinors for $SO(3,3;\mathbb{R})$.

The space of lines in $P_3(\mathbb{R})$, $Gr_2(\mathbb{R}^4)$ carries a natural conformal structure: two lines $l_1$ and $l_2$ are null separated iff they intersect. Equivalently, their respective bivectors satisfy

$$\epsilon_{\alpha\beta\gamma\delta} P_1^{\alpha\beta} P_2^{\gamma\delta} = 0,$$

(3.6)
that is to say that
\[ p^a_i p^b_j k_{ab} = 0. \quad (3.7) \]
In other words the Kleinian 6-metric \( k_{ab} \) on \( \mathbb{R}^{3,3} \) induces on the 4-dimensional space of lines \( \text{Gr}_2(\mathbb{R}^4) \) this conformal structure. Topologically \( \text{Gr}_2(\mathbb{R}^4) \) is \((S^2 \times S^2)/\pm 1 \) where the action of \( \pm 1 \) is the antipodal map on each factor. If we set
\[ a_i = p^i + p^{i+3} = P^{0i} + \frac{1}{2} \epsilon^{ijk} p^{jk}, \quad (3.8a) \]
\[ b_i = p^i - p^{i+3} = P^{0i} - \frac{1}{2} \epsilon^{ijk} p^{jk}, \quad (3.8b) \]
the light cone condition (3.2), or equivalently (3.4), is
\[ a^2 = b^2, \quad (3.9) \]
whilst the pairs \((a, b)\) and \((\lambda a, \lambda b), \lambda \neq 0 \) must be identified. Choosing \( \lambda \) to be positive we may set, without any loss of generality, \( a^2 = b^2 = 1 \) and the remaining freedom is \((a, b) \rightarrow (-a, -b)\). This metric is conformally flat, but not Einstein, and shows in fact that we may regard \( S^2 \times S^2/\pm 1 \) as the conformal compactification of flat Plücker 4-spacetime \( \mathbb{R}^{2,2} \), just as \( S^3 \times S^1/\pm 1 \) and \( S^4/\pm 1 \) with their standard product metrics provide the conformal compactifications of Minkowski spacetime and Euclidean space respectively. To obtain flat Plückert 4-spacetime itself, we consider not all null rays in Plücker 6-spacetime \( \mathbb{R}^{3,3} \), but those which intersect a null hypersurface. The rays intersecting a timelike or a spacelike hyperplane correspond to the conformally flat space with constant curvature and signature \((2, 2)\) whose isometry group is \( SO(3, 2; \mathbb{R}) \), that is to the quadric in \( \mathbb{R}^{3,2} \) given by
\[ (x^1)^2 + (x^2)^2 + (x^3)^2 - (x^4)^2 - (x^5)^2 = \frac{3}{|\Lambda|}. \quad (3.10) \]
The induced metric satisfies
\[ R_{\alpha\beta} = \Lambda g_{\alpha\beta}, \quad (3.11) \]
and plays the rôle of de Sitter spacetime for \( SO(2, 2; \mathbb{R}) \). Note that as a consequence of chiral-chronal symmetry there is only one de Sitter spacetime for \( SO(2, 2; \mathbb{R}) \). The anti-de Sitter version with the opposite sign of \( \Lambda \) corresponds to changing \( g_{\alpha\beta} \) to \(-g_{\alpha\beta}\).

Suppressing one timelike and one spacelike direction exhibits \( SO(2, 2; \mathbb{R})/\pm 1 \) as the conformal group of \( \mathbb{R}^{1,1} \), 2-dimensional Minkowski spacetime whose conformal compactification is the 2-torus, \( S^1 \times S^1/\pm 1 \). If we regard \( \mathbb{R}^{1,1} \) as the intersection of a null hyperplane in \( \mathbb{R}^{2,2} \) with the light cone of the origin, we can also consider the further intersection with spacelike, timelike or null hyperplanes in \( \mathbb{R}^{2,2} \). These intersect \( \mathbb{R}^{1,1} \) in hyperbolic circles; these are timelike, spacelike or null curves of constant acceleration. Thus \( SO(2, 2; \mathbb{R}) \) plays the same rôle in hyperbolic circle geometry as the Lorentz group \( SO(3, 1; \mathbb{R}) \) in the circle geometry of the 2-dimensional Euclidean plane. In other words the set of hyperbolae corresponds to two copies of \((AdS)_3/J\), where \( J \) is the antipodal map, and \((AdS)_3\) refers to three dimensional Anti-de Sitter spacetime.

At this point we should emphasize that in all dimensions greater than two, the conformal group \( \text{Conf}(s, t; \mathbb{R}) \) of \( \mathbb{R}^{s,t} \) is locally isomorphic to \( SO_{s+t}(s + 1, t + 1; \mathbb{R}) \) and the conformal compactifications of \( \mathbb{R}^{s,t} \) correspond to a set of null rays in \( \mathbb{R}^{s,t} \). What is special about \( \mathbb{R}^{2,2} \) is that we may identify \( \mathbb{R}^{3,3} \) with its Kleinian metric as the space of two forms in \( \mathbb{R}^4 \) and hence with the Lie algebra of \( SO(4 - t, t; \mathbb{R}) \). The Kleinian metric is invariant under the adjoint action of \( SO(4 - t, t; \mathbb{R}) \) on \( so(4 - t, t; \mathbb{R}) \) and so corresponds to one of the two quadratic Casimir operators.
Thought of as $so(4-t,t;\mathbb{R})$, the space of all two-forms in $\mathbb{R}^4$, has another natural metric, the negative of the $Ad_{SO(4-t,t;\mathbb{R})}$ invariant Killing metric. If $\omega_{\mu\nu} = -\omega_{\nu\mu}$ is a two form, then this metric is given by

$$G(\omega,\omega) = \frac{1}{2} \omega^a_{\mu\nu} \omega^b_{\lambda\rho} g^{\mu\lambda} g^{\nu\rho} = g_{ab} \omega^a \omega^b,$$  \hspace{1cm} (3.12)

where $g_{\mu\nu}$ is the flat metric with signature $(4-t,t)$ and $\omega^a$ with $a = 1,2,\ldots$ are the components of $\omega_{\mu\nu}$ in a suitable basis.

The (negative) Killing metric has signature $(6,0), (3,3)$ or $(2,4)$ if $t = 0, 1, 2$ respectively. Moreover, since we have two metrics $k_{ab}$ and $g_{ab}$ on $\Lambda^2(\mathbb{R}^4)$, we can form the Hodge star operator

$$(*a)_{\mu}\nu = k^{ac} g_{c\nu},$$ \hspace{1cm} (3.13)

from them. Thus

$$(*\omega)_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\tau} g^{\rho\lambda} g^{\tau\sigma} \omega^\lambda_{\sigma}.$$ \hspace{1cm} (3.14)

Now it is easily seen that

$$** = (-)^t,$$ \hspace{1cm} (3.15)

or equivalently

$$k^{ab} g_{bc} = (-)^t g^{ab} k_{bc}.$$ \hspace{1cm} (3.16)

Thus, if $t = 2$ (or $t = 0$) the two metrics can be simultaneously diagonalized to yield the orthogonal direct sum decomposition

$$SO(2,2;\mathbb{R}) = SL(2;\mathbb{R})_L \oplus SL(2;\mathbb{R})_R,$$ \hspace{1cm} (3.17)

where the left and right subspaces $SL(2,\mathbb{R})_L$ and $SL(2,\mathbb{R})_R$ are the eigenspaces of the Hodge * operator with eigenvalues 1 and $-1$ respectively. The eigenspaces are generally referred to as the self-dual and anti-selfdual spaces. Restricting each factor to the (negative) Killing metric, they each have signature $(2,1)$ while the Klein metric has the same or opposite signature respectively. It follows that for self-dual, or anti-self-dual two forms, the property of being simple coincides with that of being null. Such self-dual (or anti-self-dual) simple two forms are called $\alpha$-(or $\beta$-)forms. Moreover the simultaneous requirements that $\omega_{\alpha\beta}$ be simple

$$\omega_{\alpha\beta} = u_{[\alpha} v_{\beta]},$$ \hspace{1cm} (3.18)

and self-dual (or anti-self-dual)

$$\omega_{\alpha\beta} = \pm \frac{1}{2} \epsilon_{\alpha\beta}^{\gamma\delta} \omega_{\gamma\delta},$$ \hspace{1cm} (3.19)

which is to say that

$$u_{[\alpha} v_{\beta]} = \pm \frac{1}{2} \epsilon_{\alpha\beta}^{\gamma\delta} u_{\gamma} v_{\delta},$$ \hspace{1cm} (3.20)

are easily seen to imply that the two one-forms $u_\alpha$ and $v_\beta$ are themselves both null and orthogonal to each other

$$g^{\alpha\beta} u_\alpha u_\beta = g^{\alpha\beta} v_\alpha v_\beta = g^{\alpha\beta} u_\alpha v_\beta = 0.$$ \hspace{1cm} (3.21)

It follows from the above that the 2-planes annihilated by $\omega_{\alpha\beta}$ are themselves totally null, that is spanned by two vectors $a^\alpha = g^{\alpha\beta} u_\alpha$ and $b^\alpha = g^{\alpha\beta} v_\beta$ such that

$$a^\alpha a^\beta g_{\alpha\beta} = b^\alpha b^\beta g_{\alpha\beta} = a^\alpha b^\beta g_{\alpha\beta} = 0,$$ \hspace{1cm} (3.22)
with
\[ \omega_{\alpha \beta} a^\alpha = \omega_{\alpha \beta} b^\alpha = 0. \quad (3.23) \]

Since \( \alpha \) and \( \beta \) planes play an important role in what follows, we shall pause to describe them in more detail. To do so we reconsider the light cone in \( \mathbb{R}^{2,2} \). If the metric is given by
\[ ds^2 = dx^+ dx^- - dy^+ dy^- , \quad (3.24) \]
then a general null 1-form \( k(\omega, \alpha, \beta) \) may be parametrized as
\[ k_\mu = k^\mu \frac{\partial k_\mu}{\partial \alpha} = k^\mu \frac{\partial k_\mu}{\partial \beta} = 0. \quad (3.26) \]
Moreover
\[ \frac{\partial k^\mu}{\partial \alpha} \frac{\partial k_\mu}{\partial \alpha} = \frac{\partial k^\mu}{\partial \beta} \frac{\partial k_\mu}{\partial \beta} = 0, \quad (3.27) \]
and
\[ \frac{\partial k^\mu}{\partial \alpha} \frac{\partial k_\mu}{\partial \beta} = \omega^2. \quad (3.28) \]
Thus the parameters \( \alpha \) and \( \beta \) label two null directions on the set of null rays, and the curves \( \alpha = \text{constant} \ 0 \leq \beta \leq 2\pi \) or \( \beta = \text{constant} \ 0 \leq \alpha \leq 2\pi \) are circles which wind around each of the two fundamental generators of the torus. Since
\[ k_\alpha \frac{\partial k}{\partial \alpha} = -\omega^2(-\sin \beta dx^+ + \cos \beta dy^+) \wedge (\cos \beta dx^- - \sin \beta dy^-) , \quad (3.29) \]
and
\[ k_\beta \frac{\partial k}{\partial \beta} = \omega^2(\sin \alpha dx^+ + \cos \alpha dx^- \wedge (\sin \alpha dy^+ + \cos \alpha dy^-) , \quad (3.30) \]
these two families of null vectors lie in two totally null 2-planes which are called the \( \alpha \)-planes or \( \beta \)-planes respectively. Moreover as \( \alpha \) varies, we obtain a circle of \( \alpha \)-planes, and as \( \beta \) varies, we obtain a circle of \( \beta \)-planes.

The structure of \( \alpha \) and \( \beta \) planes has important consequences for the behavior of massless fields in Pl"ucker 4-space. It means, for example, that a massless particle with 4-momentum \( k_\mu \) can decay into two other massless particles with 4-momenta \( k_2^\mu \) and \( k_3^\mu \) conserving momentum, so that
\[ k_1^\mu = k_2^\mu + k_3^\mu , \quad (3.31) \]
provided that both \( k_2^\mu \) and \( k_3^\mu \) lie either in the \( \alpha \)-plane or the \( \beta \)-plane passing through \( k_1^\mu \). In Minkowski spacetime this would only be possible if \( k_1^\mu \), \( k_2^\mu \) and \( k_3^\mu \) were mutually parallel. More seriously, particles can decay into products that are more massive than their progenitors.

Another interesting feature of massless fields and totally null 2-planes is that an arbitrary function of two coordinates spanning such a 2-plane is automatically a solution of the wave equation. Take, for example, the totally null 2-plane spanned by \( x^+ \) and \( y^+ \). Since the wave equation is
\[ \frac{\partial^2 \phi}{\partial x^+ \partial x^-} - \frac{\partial^2 \phi}{\partial y^+ \partial y^-} = 0, \quad (3.32) \]
then

\[ \phi = f(x^+, y^+) \]  

is automatically a solution of (3.32). Since such solutions are analogous to the left-movers in \( \mathbb{R}^{1,1} \), we propose calling such solutions left-handed or right-handed if they are constant on \( \alpha \)-planes or \( \beta \)-planes respectively. Thus, with our conventions, \( f(x^+, y^+) \) is a left-handed because it is constant on the \( \alpha \)-plane spanned by \( x^- \) and \( y^- \). Note that since both \( dx^+ \wedge dy^+ \) and \( dx^- \wedge dy^- \) are self-dual 2-forms, it is the case that an arbitrary function \( g(x^-, y^-) \) is also left-handed. Examples of right-handed solutions would be \( h(x^+, y^-) \) or \( k(x^-, y^+) \).

How now do we treat a general solution of the wave equation? We can express \( \phi \) as a Fourier integral, so that

\[ \phi = \int_0^\infty \int_0^{2\pi} \int_0^{2\pi} \omega d\omega d\alpha d\beta e^{ik_\mu(\omega, \alpha, \beta) x^\mu} \tilde{f}(\omega, \alpha, \beta), \]  

(3.34)

where \( \omega d\omega d\alpha d\beta \) is the invariant measure on the null-cone. If we perform the \( \omega \) integral, we get a Whittaker-type formula

\[ \phi = \int_0^{2\pi} d\alpha \int_0^{2\pi} d\beta f(-\sin \alpha \sin \beta x^+ + \sin \alpha \cos \beta y^+ + \cos \alpha \cos \beta x^- - \cos \alpha \sin \beta y^-, \alpha, \beta), \]  

(3.35)

where

\[ f(\lambda, \alpha, \beta) = \int_0^\infty d\omega \tilde{f}(\omega, \alpha, \beta)e^{i\omega \lambda}, \]  

(3.36)

and \( \tilde{f}(\omega, \alpha, \beta) \), or equivalently \( f(\lambda, \alpha, \beta) \) is an arbitrary function. If we do the \( \beta \) integration first, we find that

\[ \phi = \int_0^{2\pi} d\alpha F(\alpha), \]  

(3.37)

where

\[ F(\alpha) = \int_0^{2\pi} d\beta f(-\sin \alpha \sin \beta x^+ + \sin \alpha \cos \beta y^+ + \cos \alpha \cos \beta x^- - \cos \alpha \sin \beta y^-, \alpha, \beta). \]  

(3.38)

\( F(\alpha) \) is a superposition of solutions each of which is constant along the null direction \( k^\mu(\omega, \alpha, \beta) \). As \( \beta \) varies, the null vectors span an \( \alpha \)-plane specified by the fixed value of \( \alpha \), that is to say we have a left-handed solution associated with the \( \alpha \)-plane spanned by \( k^\mu \) and \( \frac{\partial k^\mu}{\partial \beta} \). Performing the \( \alpha \) integration, we see that the general solution may be expressed as a superposition of left-handed solutions or, by interchanging the \( \alpha \) and \( \beta \) integrations, as a superposition of right-handed solutions. Note that unlike the situation in \( \mathbb{R}^{1,1} \) where one needs both the left and right movers, for \( \mathbb{R}^{2,2} \) one only needs either the left-handed solutions or the right-handed solutions. The reason is presumably because the space of unoriented null directions in \( \mathbb{R}^{2,2} \) is connected, whereas in the case of \( \mathbb{R}^{1,1} \) it consists of two disconnected points. It would appear that the ability to write general solutions of the wave equation in terms of solutions of a single handedness is rather similar to the idea of duality found in string theory, where a single string scattering process must be represented by a number of distinct Feynman diagrams.

A more group theoretic description of the \( \alpha \)-planes and \( \beta \)-planes is provided by exhibiting the isomorphism \( SO(2,2; \mathbb{R}) \cong SL(2, \mathbb{R})_L \otimes SL(2, \mathbb{R})_R \pm 1 \) by regarding flat Plücker 4-spacetime \( \mathbb{R}^{2,2} \) as the space of \( 2 \times 2 \) real matrices:

\[ \mathbf{x} = \begin{pmatrix} x^+ & y^+ \\ y^- & x^- \end{pmatrix}, \]  

(3.39)
with metric
\[ ds^2 = \det d\bf{x} = dx^+ dx^- - dy^+ dy^- . \] (3.40)

Left multiplication of \( \bf{x} \) by an element \( L \) of \( SL(2, \mathbb{R}) \), and right multiplication by the transpose of another element \( L' \) of \( SL(2, \mathbb{R}) \) preserves the determinant, and hence the Kleinian metric on \( \mathbb{R}^2 \). The spin space of \( SO(2, 2; \mathbb{R}) \) similarly splits into the direct sum of two real 2-dimensional spin spaces \( S \oplus S' \) of unprimed and primed spinors \( \alpha^A, \beta^A' \) respectively.

Elements of \( S \) (or \( S' \)) are Majorana-Weyl spinors for \( SO(2, 2; \mathbb{R}) \). A spinor dyad for \( S \) (or \( S' \)) is given by two unprimed (primed) spinors \( o^A, \iota^A \) (\( \tilde{o}^A, \tilde{\iota}^A \)) such that
\[ o^A \iota^B - o^B \iota^A = -\epsilon^{AB}, \] (3.41)
\[ \tilde{o}^A \tilde{\iota}^{B'} - \tilde{o}^{B'} \tilde{\iota}^A = -\tilde{\epsilon}^{A'B'}, \] (3.42)
where \( \epsilon^{AB} \) and \( \tilde{\epsilon}^{A'B'} \) are the symplectic 2-forms on \( S \) and \( S' \) defining \( SL(2, \mathbb{R}) \). If we parametrize elements of \( SL(2, \mathbb{R}) \) by four coordinates subject to one constraint
\[ L = \begin{pmatrix} a_+ & b_+ \\
                    b_- & a_- \end{pmatrix}, \] (3.43)
with
\[ \det L = 1, \] (3.44)
we see that we may identify \( SL(2, \mathbb{R}) \) with the bi-invariant Killing metric with three dimensional anti-de Sitter spacetime \((AdS)_3\). To get \( SO(2, 1; \mathbb{R}) \) we must identify \( L \) and \(-L\) which corresponds to factoring \((AdS)_3\) by the antipodal map. Thus \( SO_+(2, 2; \mathbb{R}) \) with its Killing metric may be identified with two copies of \((AdS)_3\) quotiented by the simultaneous action of the antipodal map on each factor. The Lie algebra of \( SO(2, 2; \mathbb{R}) \) may be identified with the tangent space at the origin and splits as a direct sum as was mentioned earlier. The adjoint action of \( SO(2, 2; \mathbb{R}) \) decomposes into the adjoint action of the two \( SL(2, \mathbb{R}) \) factors on their Lie algebras. This action may be described as follows: we may identify \( SL(2, \mathbb{R}) \) with its (negative) Killing metric with three dimensional Minkowski spacetime \((dS)_2\). The adjoint action of \( SL(2, \mathbb{R}) \) decomposes into the adjoint action of the two \( SL(2, \mathbb{R}) \) factors on their Lie algebras. This action may be described as follows: we may identify \( SL(2, \mathbb{R}) \) with its (negative) Killing metric with three dimensional Minkowski spacetime, the adjoint action being equivalent to the usual action of the Lorentz group. The non-trivial orbits under the Lorentz group action then decompose into five strata according to how their tangent vectors are classified: future timelike, past timelike, future null, past null and spacelike. Each timelike orbit may be identified with 2-dimensional hyperbolic space \( H^2 \) and the spacelike orbits with 2-dimensional de Sitter spacetime \((dS)_2\).

One may choose a basis \((T, I, S)\) in \( SL(2, \mathbb{R}) \) normalized with respect to the Killing metric such that
\[ -I^2 = T^2 = S^2 = 1, \] (3.45)
and
\[ TIS = 1. \] (3.46)
It follows that \( T, I \) and \( S \) mutually anticommute. A convenient representation is
\[ I = \begin{pmatrix} 0 & 1 \\
                        -1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 \\
                        0 & -1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 1 \\
                        1 & 0 \end{pmatrix}. \] (3.47)
The relations (3.45) and (3.46) generate \( R(2) \), the algebra of all real \( 2 \times 2 \) real matrices, or what are sometimes called the “pseudoquaternions.” The relations continue to hold when
$I, T$ and $S$ act on $\mathbb{R}^2$ where $I$ provides a complex structure and $T$ and $S$ are a pair of real structures. Thus, for example, if $I^2$ acts on the right of the matrix $x$, we find that

$$(x^+, x^-, y^+, y^-) \rightarrow (y^+, -y^-, -x^+, x^-),$$

which shows that if we introduce complex coordinates

$$z^\pm = x^\pm \pm iy^\pm,$$

the effect of $I$ is to multiply the complex coordinates $z^\pm$ by $i$. In terms of complex coordinates, the flat Plücker metric becomes

$$ds^2 = \frac{1}{2}(dz^+d\bar{z}^- + d\bar{z}^+dz^-),$$

which is manifestly pseudo-Kähler. The Kähler form is antiselfdual and equal to to

$$\frac{1}{2}I_{\alpha\beta}dx^\alpha \wedge dx^\beta = \frac{1}{2}(dy^- \wedge dx^+ + dx^- \wedge dy^+),$$

$$= \frac{i}{4}(dz^+ \wedge d\bar{z}^- + d\bar{z}^+ \wedge dz^-).$$

$I_{\alpha\beta}$ is an isometry of the flat metric, and because $I^2 = -1$ it leaves no vector fixed. However it may rotate a two-planes worth of vectors into themselves. Such two-planes are called holomorphic with respect to the complex structure $I$. These holomorphic two-planes may, or may not, be null.

The situation with regard to the real structures $T$ and $S$ has some similarities and some differences. Since $S^2 = 1$ and antisymmetric $S_{\alpha\beta} = -S_{\alpha\beta}$ we see that

$$g(SX, SY) = g(X, Y).$$

Thus $S$ is not an isometry of $g$. Moreover, the positive and negative eigenspaces of $S$ are totally null. In other words, associated with $S$ are a pair of $\beta$-planes, which correspond in the Lie algebra to the intersection of the timelike two-plane orthogonal to $S$ with the light cone (see figure 3.3). In this basis, such $\beta$-planes correspond to

$$B_\pm = \frac{1}{2}(I \pm S),$$

and satisfy

$$B_\pm^2 = 0.$$  

$B_\pm$ is thus a nilpotent element of $SL(2, \mathbb{R})$.

We may bring out the analogies rather than the differences between real structures and complex structures by introducing “double numbers” in place of complex numbers. Just as the complex numbers $\mathbb{C}$ may be considered as an algebra over $\mathbb{R}$ generated by unity and another element $i$ such that $i^2 = -1$, we can consider the commutative and associative algebra of double numbers $\mathbb{E}$ generated by a unit element $I$, and another generator $e$ such that $e^2 = 1$. We introduce double number-valued coordinates by

$$w^+ = x^+ + ey^+,$$
$$w^- = x^- - ey^-,$$
together with their conjugates
\[ \bar{w}^+ = x^+ - ey^+, \]
\[ \bar{w}^- = x^- - ey^-, \]
(3.55)

If \( ST \) acts on the right of the matrix \( x \), it is equivalent to multiplication by \( e \), just as right multiplication by \( IT \) was equivalent to multiplication by \( i \). In these coordinates the metric takes the form
\[ ds^2 = \frac{1}{2}(dw^+ d\bar{w}^- + d\bar{w}^+ dw^-), \]
where now the analogue of the Kähler form is
\[ S_{\alpha \beta} dx^\alpha dx^\beta = \frac{1}{2}(dx^+ \wedge dy^- + dx^- \wedge dy^+) \]
\[ = -\frac{e}{4}(dw^+ \wedge d\bar{w}^- + d\bar{w}^+ \wedge dw^-). \]
(3.57)

Further analogies between complex numbers and double numbers are revealed by recalling that with respect to the complex structure \( I_{\alpha \beta} \), the complex valued 2-form \( T_{\alpha \beta} + iS_{\alpha \beta} \) is closed and holomorphic. In fact
\[ dz^+ \wedge dz^- = -2(T + iS). \]
(3.58)

Analogously for double numbers
\[ dw^+ \wedge dw^- = -2(T + eS). \]
(3.59)

The double numbers \( E \) do not usually receive a great deal of attention because they are not a division algebra, the modulus \( z^\dagger z' = (x^-)^2 - (y^+)^2 \) being non-positive. Moreover as an algebra \( E \) splits as a direct sum of two copies of \( \mathbb{R} \), \( E = \mathbb{R} \oplus \mathbb{R} \), the two copies being generated by the idempotents \( 2^{-1/2}(1 \pm e) \) whose product vanishes. Nevertheless, the great formal similarity between them and the complex numbers, including the ability to introduce an analytic function theory with an obvious analogue of the Cauchy-Riemann equations makes them very useful as a calculational tool for ultrahyperbolic geometry since one can interpret most of the usual formulae of complex and Kähler geometry in terms of double numbers. Moreover, it allows a simple passage from the general analysis of Plebanski et. al[13] in terms of complex metrics, with two independent sets of complex coordinates to the restriction to the ultrahyperbolic case.

An alternative basis for \( SL(2, \mathbb{R}) \) is obtained in terms of a non-compact generator \( T \) and the two nilpotent generators \( A_{\pm} \) given by
\[ A_{\pm} = \frac{1}{2}(S \pm I) \]
(3.60)

which satisfy
\[ A^2_+ = A^2_- = 0, \]
\[ [A_+, A_-] = T, \]
\[ [T, A_\pm] = \pm 2A_\pm. \]
(3.61)

Thus \( A_+ \) and \( A_- \) are null vectors in the Killing metric. Let us consider \( A_- \) which has a two-by-two matrix representation
\[ A_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \]
with
\[
\exp(tA_-) = \begin{pmatrix} 1 & 0 \\ t & 0 \end{pmatrix},
\]
acting on the left of the matrix \(x\)
\[
\exp(tA_-) : (x^+, x^-, y^+, y^-) \rightarrow (x^+, x^- + ty^+, y^+ + tx^+),
\]
which stabilizes the self-dual totally null 2-plane (or \(\alpha\)-plane), \(x^+ = y^+ = 0\). The action of \(A_-\) may be reproduced by introducing coordinates taking values in the algebra \(D\) of “dual numbers,” that is the algebra generated by unity and another element \(\epsilon\) satisfying \(\epsilon^2 = 0\). If
\[
z'' = y^+ + \epsilon x^-, \quad w'' = x^+ + \epsilon y^-,
\]
then
\[
A_- : z'' \rightarrow \epsilon z'', \quad w'' \rightarrow \epsilon w''.
\]
Since \(\det A_- = 0\), it also stabilizes a primed spinor, \(o_A'\), say. The converse is also true, every spinor determines a nilpotent generator \(A_-\) and an associated totally null 2-plane, or \(\alpha\) plane corresponding to the simple antiselfdual 2-form whose spinor expression is \(\epsilon_{AB} o_A' o_B'\).

To illustrate the utility of complex, dual and double numbers for ultra hyperbolic geometry, we return to \(\mathbb{R}^6\) thought of as the Lie algebra of \(SO(4-t, t; \mathbb{R})\), is equivalent to \(\Lambda^2(\mathbb{R}^4)\). Recall that this space has two distinct metrics on it, the Kleinian metric \(k_{ab}\) given by equation 3.4 and the Killing metric, \(g_{ab}\) given by equation 3.12. Additionally, there is the Hodge star operator, *, which since
\[
** = (-1)^t,
\]
may be regarded as providing a complex structure on \(SO(3,1; \mathbb{R})\), and a double structure on \(SO(4, \mathbb{R})\) and \(SO(2,2; \mathbb{R})\). If the combinations
\[
a + ib \quad \text{or} \quad a + eb
\]
are used where \(a\) and \(b\) are vectors in \(so(3; \mathbb{R})\) or \(so(2,1; \mathbb{R})\), we may exhibit the following isomorphisms:
\[
SO(4,1; \mathbb{R}) \cong SO(3; \mathbb{C}),
\]
\[
SO(4; \mathbb{R}) \cong SO(3; \mathbb{E}),
\]
\[
SO(2,2; \mathbb{R}) \cong SO(2,1; \mathbb{E}).
\]
If instead we had considered the dual vectors in the combination
\[
a + eb,
\]
we would have found an isomorphism between the Euclidean group and rotations
\[
E(3; \mathbb{R}) \cong SO(3; \mathbb{D}).
\]

If we identify \(SO(3, \mathbb{R})\) with the real quaternion algebra and \(SO(2,1; \mathbb{R})\) with the real pseudo-quaternion algebra, then we obtain complex quaternions, “bi-quaternions” and what are sometimes called “motors” respectively from the isomorphisms given in equation (3.68).

These were introduced originally by Clifford, Ball, Study and others to discuss rigid body motion in \(H^3, S^3\) and \(\mathbb{R}^3\). The last case is of considerable technological importance. The relation of all of these concepts to the geometry of ultra hyperbolic manifolds and of Euclidean
three space is quite remarkable, and is worthy of study even though the physical significance of Kleinian geometry seems rather tenuous.

The last case is of considerable physical and practical importance, as well as being mathematically interesting. The Lie algebra of the Euclidean group \( e(3) \) (i.e. the space of “motors”) is 6-dimensional. It consists of pairs of 3-vectors \((\mathbf{v}, \omega)\) where \( \mathbf{v} \) is an infinitesimal velocity and \( \omega \) an infinitesimal angular velocity with respect to some origin \( O \) in \( \mathbb{R}^3 \). Under a change of origin (i.e. under the adjoint action of a translation \( \mathbf{a} \)), we find

\[
\mathbf{v} \rightarrow \mathbf{v} + \omega \times \mathbf{a},
\]

\[
\omega \rightarrow \omega,
\]

which leaves invariant both the negative Killing metric

\[
\omega \cdot \omega,
\]

and the extra quadratic Casimir

\[
2\omega \cdot \mathbf{v}.
\]

The latter provides the space of motors with a Kleinian metric of signature \((3, 3)\) which coincides with \( g^{ab} \) if we regard the components \((\mathbf{v}, \omega)\) of a motor as giving the six components \( v_a \) of a co-vector in \( \mathbb{R}^3 \). The motors \((\mathbf{v}, \omega)\) and \((\lambda \mathbf{v}, \lambda \omega)\) define the same one-parameter subgroups of \( E(3) \) and have as orbits on \( \mathbb{R}^3 \) helical curves with an axis (straight line) and pitch \( p = \omega \cdot \mathbf{v}/\omega^2 \). For this reason elements of the projective space \( \mathbb{P}(E(3)) = \mathbb{P}_6(\mathbb{R}) \) are referred to as “screws”. Note that every Euclidean motor is a twist about a screw.

The rate of doing work \( \frac{dW}{dt} \) by a system of forces equivalent to a net force \( \mathbf{F} \) and a couple \( \mathbf{G} \) is

\[
\frac{dW}{dt} = \mathbf{v} \cdot \mathbf{F} + \omega \cdot \mathbf{G}.
\]

Moreover under a change of origin we have

\[
\mathbf{F} \rightarrow \mathbf{F},
\]

\[
\mathbf{G} \rightarrow \mathbf{G} + \mathbf{F} \times \mathbf{a},
\]

which leaves invariant both the right hand side of equation (3.74) as well as both

\[
\mathbf{F} \cdot \mathbf{F},
\]

and

\[
\mathbf{F} \cdot \mathbf{G}.
\]

The rate of doing work \( \frac{dW}{dt} \), that is the right hand side of (3.74) is invariant under a change of origin, and so we can identify the pair \((\mathbf{F}, \mathbf{G})\), called a “wrench” as an element of the dual space \( e(3)^* \). The transformations 3.75 and 3.76 are just the coadjoint actions of the Euclidean group \( E(3) \) on the dual of its Lie algebra \( e(3)^* \). In an appropriate basis the six components of the wrench \((\mathbf{F}, \mathbf{G})\) may be written as \( g^a \) so that (3.74) becomes

\[
\frac{dW}{dt} = v_0 g^a
\]

Note that the components \((\mathbf{y} - \mathbf{x}, \mathbf{x} \times \mathbf{y})\) of the Plücker coordinates of the line through \( \mathbf{x} \) and \( \mathbf{y} \) transform under a change of origin as

\[
\mathbf{y} - \mathbf{x} \rightarrow \mathbf{y} - \mathbf{x}
\]
\[ x \times y \rightarrow x \times y + (y - x) \times a \]  
(3.81)

which is the same way as 3.77 and 3.78 transform, that is under the coadjoint action. A comparison with 3.3 and 3.5 reveals why the components of a motor should be thought of as comprising a covariant vector rather than contravariant vector. We could also have adopted a two-form notation with

\[ G^{ij} = \epsilon^{ijk} G^k; \]  
(3.82)

\[ G^0_i = F^i; \]  
(3.83)

\[ V_{ij} = \epsilon_{ijk} \omega^k, \]  
(3.84)

\[ V_{0i} = v_i. \]  
(3.85)

Now equation (3.79) can be rewritten as

\[ \frac{dW}{dt} = \frac{1}{2} V_{\alpha\beta} G^{\alpha\beta}. \]  
(3.86)

There is obviously a reciprocity or duality between wrenches and motors in that the pairs \( \omega \) and \( F \) and \( G \) play identical roles in all formulae. This reciprocity is just the isomorphism between \( \mathbb{R}^{3,3} \) and its dual space induced by the Kleinian metric \( k_{ab} \). In physicists language this corresponds to the lowering and raising of indices. Thus we may set

\[ G_{\alpha\beta} = \frac{1}{2} \epsilon_{\alpha\beta\mu\nu} G^{\mu\nu}; \]  
(3.87)

which then has components

\[ G_{0i} = G_i, \]  
(3.88)

\[ G_{ij} = \epsilon_{ijk} F^k. \]  
(3.89)

Correspondingly \( V^{\alpha\beta} \) has components

\[ V^{0i} = \omega^i, \]  
(3.90)

\[ V^{ij} = \epsilon^{ijk} V^k. \]  
(3.91)

The isomorphism between \( e(3) \) and \( e(3)^* \) is an essentially a three dimensional phenomenon. It is not true for \( e(n) \) and \( e(n)^* \), \( n \neq 3 \). It arises solely because of the extra Casimir, equation (3.73). It means, among other things, that all statements in statics (i.e. about wrenches) have analog statements in the theory of infinitesimal kinematics (i.e. about motors and screws). Moreover there is also a connection to line geometry, which in turn has important application in optics and symplectic geometry, all resulting from the ultrahyperbolic metric structure.

It is illuminating to place the geometry of Kleinian 6-space in a more general context by looking at it from a slightly different point of view. Consider a general Lie group \( G \) with Lie algebra \( g \). Let \( g^* \) be the dual of the Lie algebra \( g \) so that if \( \omega^i \in g \) and \( v_i \in g^* \), \( i = 1, 2 \ldots \dim G \), there is a natural product which we write as

\[ \langle v, \omega \rangle = v_i \omega^i. \]  
(3.92)

The cotangent bundle \( T^*(G) \) may, since any Lie group may be parallelized by, for example, left translating a basis for \( g^* = T^*_0(G) \) over \( G \) be identified topologically with \( G \times g^* \). We may also endow \( T^*(G) \) with a group structure by considering the semi-direct product of \( G \ltimes g^* \) where \( g^* \) is thought of as an additive abelian subgroup, and \( G \) acts on \( g^* \) by the
co-adjoint action. The Lie algebra of \( G \ltimes g^* \) may be identified with the direct sum \( g \oplus g^* \). The cotangent bundle \( T^*(G) \) is naturally a symplectic manifold with a symplectic form, at the origin, given by

\[
\Omega = dv_i \wedge d\omega^i. \tag{3.93}
\]

One can also endow \( g \oplus g^* \) with a Kleinian metric, \( g \), using the product given by equation (3.92)

\[
g = 2v_i \omega^i. \tag{3.94}
\]

Using left translation we can extend this metric over all of \( T^*(G) \). Moreover, as the metric and symplectic structures are compatible we can view \( T^*(G) \) as a pseudo-Kähler manifold with complex structure at the origin given by

\[
J(\omega) = \omega, \tag{3.95}
\]

and

\[
J(V) = -V. \tag{3.96}
\]

Thus in the obvious basis for \( g \oplus g^* \)

\[
g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{3.97}
\]

\[
J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{3.98}
\]

\[
\omega = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{3.99}
\]

The theory given above is quite general and may well be useful in, for example, quantizing a particle moving on a group manifold. The particular case of interest is when \( G = SO(3) \) in which case \( T^*(G) = SO(3) \ltimes \mathbb{R}^3 = E(3) \), the Euclidean group, and we are back to the case of the theory of “motors,” \( \omega \) being an infinitesimal angular velocity and \( v \) and infinitesimal velocity.

§4 Curved Kleinian Einstein Metrics

In this section we wish to make a few remarks about curved Kleinian 4-metrics. We commence by noticing that on topological grounds not every 4-manifold \( \mathcal{M} \) can admit such a metric. Every paracompact manifold \( \mathcal{M} \) admits a (not unique) Riemannian metric, \( g_{ab} \) say, and thus if \( \mathcal{M} \) admits a Kleinian metric \( k_{ab} \), then we may diagonalize \( k_{ab} \) relative to \( g_{ab} \). It follows that \( \mathcal{M} \) must admit an everywhere non-vanishing 2-plane distribution. According to Atiyah and Dupont [14] a necessary condition for this, assuming that \( \mathcal{M} \) is closed, \( \partial \mathcal{M} = \emptyset \), and orientable, and the 2-plane distribution to be oriented, is that the Euler character \( \chi \) and the signature \( \tau \) obey certain conditions, namely

\[
\chi = 0 \ mod \ 2, \tag{4.1}
\]

and

\[
\tau = \chi \ mod \ 4. \tag{4.2}
\]

Hence only if (4.1) and (4.2) are satisfied can a manifold admit a Kleinian metric.
Let us now turn to metrics that obey the Einstein condition
\[ R_{ab} = \Lambda g_{ab}. \]  
(4.3)

Whilst this is not directly relevant to string theory, it is useful to examine such metrics as they illustrate some of the peculiarities of Kleinian geometry. The first obvious non-flat example is the constant curvature space, which is a generalization of de Sitter spacetime. Consider the quadric in \( \mathbb{R}^{2,3} \) given by
\[ (X^1)^2 + (X^2)^2 + (X^3)^2 - (X^4)^2 - (X^5)^2 = \frac{3}{\Lambda}, \]  
(4.4)
with \( \Lambda > 0 \). The isometry group on the surface is \( SO(3,2;\mathbb{R}) \). This is same construction as for Lorentzian anti-de Sitter space except that the cosmological constant has been chosen to be positive rather than negative. We should however note that in contrast to the Lorentzian case there is just a single metric of constant curvature. This is because changing the sign of \( \Lambda \) is the same as changing the overall sign of the metric, which is a discrete isometry in the present case. One could regard (4.4) as a generalized Friedmann-Robertson-Walker metric in a variety of ways, rather as one does for de Sitter spacetime, but we will not explore this here.

We now pass on to the analog of the Schwarzschild vacuum solution. For Lorentzian signature, the Schwarzschild metric is usually regarded as the field of a particle with a timelike worldline. Thus it has as its isometry group \( SO(3) \otimes \mathbb{R} \), the \( SO(3) \) having two-dimensional orbits, and the isometry group being the stabilizer of a timelike line in Minkowski spacetime. This property completely characterizes the metric up to the sign of the mass parameter \( M \). Both possible signs give rise to an incomplete singular metric. If \( M > 0 \), the singularity is hidden inside the event horizon, whereas if \( M < 0 \) there would be a naked singularity. If instead we had wished to consider the gravitational field of a tachyon, that is a particle with a spacelike worldline, we would replace \( SO(3) \) by \( SO(2,1) \). We then have a choice as to whether the orbit of \( SO(2,1) \) is spacelike, giving the two-dimensional hyperboloid \( H^2 \), or timelike, giving two-dimensional de Sitter spacetime \( (dS)_2 \). We also have a choice of whether the mass is positive or negative. Since these metrics are warped products of a two-dimensional Lorentzian spacetime and a Riemannian two-surface, we may illustrate the possibilities by he self-explanatory diagrams below which include the analogous possibilities for Kleinian and Riemannian choices of signature.

It is noteworthy that among the list are three complete non-singular metrics, one for each signature. In each of these cases, the \((r,t)\) plane has a positive definite metric, \( t \) being periodic with period \( 8\pi M \), and \( M > 0 \).

We can construct these solutions in a way that is analogous to the de Sitter space example. They can be regarded as algebraic varieties in a seven dimensional flat space of appropriate signature. Lets start with the Schwarzschild solution of Lorentz signature. It can be regarded as an algebraic variety in \( \mathbb{R}^{5,1} \). Thus consider the space
\[ ds^2 = (dX^1)^2 + (dX^2)^2 + (dX^3)^2 + (dX^4)^2 + (dX^5)^2 + (dX^6)^2 - (dX^7)^2 \]  
(4.5)
and construct the four-dimensional submanifold formed by the intersection of the following three hypersurfaces [15]
\[ (X^6)^2 - (X^7)^2 + 4/3(X^5)^2 = 16M^2, \]
\[ ((X^1)^2 + (X^2)^2 + (X^3)^2)(X^5)^4 = 576M^6, \]
\[ \sqrt{3}X^4 X^5 + (X^5)^2 = 24M^2. \]  
(4.6)
We can relate this to the usual exterior Schwarzschild solution by means of the following substitutions:

\[
X^1 = r \sin \theta \cos \phi, \\
X^2 = r \sin \theta \sin \phi, \\
X^3 = r \cos \theta, \\
X^4 = -2M(2M/r)^{1/2} + 4M(r/2M)^{1/2}, \\
X^5 = (24M^3/r)^{1/2}, \\
X^6 = 4M(1 - 2M/r)^{1/2} \cosh(t/4M), \\
X^7 = 4M(1 - 2M/r)^{1/2} \sinh(t/4M).
\] (4.7)

In fact, the algebraic variety described here does in fact cover the entire Kruskal manifold. Since this construction works in a flat seven dimensional space, it is clear that we can construct metrics of other signatures in a straightforward way by complexifying the entire construction, and taking real sections in an appropriate way. By this technique, we can find all of the metrics alluded to earlier. We will not do this explicitly here as the method is fairly transparent, but rather tedious to record.

§5 Self-dual metrics

The requirement that the curvature of a 4-metric $g_{ab}$ of signature $(2,2)$ be self-dual,

\[
R_{ab}^{\ cd} = \frac{1}{2} \varepsilon_{ab}^{\ ef} R_{cdef},
\]

is easily seen to imply that the metric is Ricci flat and has holonomy $SL(2,\mathbb{R})$ [16]. Such metrics are also said to be half-flat. In $4n$-dimensions, metrics with Kleinian signature and holonomy $Sp(2n,\mathbb{R})$ have been termed hypersymplectic by Hitchin [17]. Hence self-dual Kleinian metrics are hypersymplectic. In two component notation, the only non-vanishing components of the Weyl curvature spinor is $\Psi_{ABCD}$. Thus there exist a pair of covariantly constant spinor fields $\iota^A$ and $\sigma^A$, which may be normalized conveniently so that

\[
\sigma^A \iota_A = 1 \Leftrightarrow \delta^A_{B'} = -\iota^A \sigma_{B'} + \sigma^A \iota_{B'}.
\] (5.2)

The bundle of anti-self-dual 2-forms is flat and one may choose a basis such that

\[
A_+ = \iota_A \iota_{B'} = \frac{1}{2}(S + I),
\]

\[
A_- = -\sigma_A \sigma_{B'} = \frac{1}{2}(S - I),
\]

\[
T = -\sigma_A \iota_{B'} - \iota_A \sigma_{B'}.
\] (5.5)

Hence

\[
I = \sigma_A \sigma_{B'} - \iota_A \iota_{B'},
\]

\[
S = -\sigma_A \sigma_{B'} - \iota_A \iota_{B'}.
\] (5.7)

Thus $I, S$ and $T$ are covariantly constant. It follows that $I$ endows $\mathcal{M}$ with the structure of a complex Kähler manifold, as indeed does every other 2-form lying in the 2-sheeted hyperbola.
in $\Lambda^2(\mathcal{M})$. The same statements can be made but with “complex” changed to “double” and “double-sheeted” to “single-sheeted.” Moreover, $A_+$ and $A_-$ belong to two circles worth of “dual” solutions, i.e. to a single circles worth of covariantly constant $\beta$-planes.

Let us consider now a self-dual Kleinian four manifold admitting an isometry group $G$ generated by Killing vector fields $K^A$. If $I, S$ and $T$ are invariant under the action $G$ then $G$ is said to act (locally) hypersymplectically. The action of $G$ is holomorphic with respect to every complex structure on the double sheeted hyperboloid, and similar statements apply in terms of double and dual structures. The necessary and sufficient conditions that the action is (locally) hypersymplectic is that

$$\mathcal{L}_K\{I, S, T\} = 0$$  \hspace{1cm} (5.8)

for every Killing vector filed $K \in \mathfrak{g}$, the Lie algebra of $G$. since $I, S$ and $T$ are covariantly constant we obtain as a necessary and sufficient condition for hypersymplecticity is the the two-form $\nabla_a K_b$ be self-dual. One then refers to the Killing vector being self-dual. If the action of $G$ does not leave $I, S$ and $T$ invariant, it must act on them preserving the Killing metric, in other words the action of $G$ provides a homomorphism from $G$ to a (possibly improper) subgroup of $SO(2,1;\mathbb{R})$. For example, if $G$ is one-dimensional, its action on the self-dual two form may either be a spacelike rotation leaving invariant a privileged complex structure, a Lorentz boost leaving invariant a privileged double structure, or a null rotation leaving invariant a privileged dual structure. In the spacelike case it is clear that $G$ will not leave invariant any dual or double structure, while in the second case it will leave invariant two privileged dual structures - or equivalently two privileged foliations by $\beta$-planes, while in the last case there will be no privileged double or complex structures.

The classification of isometries just given is very similar to that in the positive definite case. Moreover it is obvious that by means of analytic continuation (or “Wick” rotation) of known examples of Hyper-Kähler manifolds one may obtain numerous local forms of hypersymplectic 4-manifolds. Since all known Hyper-Kähler 4-manifolds admit isometries, the corresponding hypersymplectic forms will also admit isometries. The analytically continued Killing fields $K^a$ will always have, however, non-vanishing length $g_{ab}K^aK^b$. It follows that various constructions and uniqueness arguments in the Riemannian case will pass over to the Kleinian case without much change. However, there is a special case which cannot occur in Riemannian geometry which is when the Killing vector field is null,

$$g_{ab}K^aK^b = 0.$$  \hspace{1cm} (5.9)

This specific case will be studied in the next section. Now let us turn our attention to the case of a self-dual Killing vector in a self-dual spacetime where $K$ is not null. The results of Tod and Ward show that it may be cast in the form

$$ds^2 = V^{-1}(d\tau + \omega_i dx^i)^2 - V\eta_{ij}dx^i dx^j$$  \hspace{1cm} (5.10)

where

$$\text{curl } \omega = \text{grad } V$$  \hspace{1cm} (5.11)

with $\text{grad}$ and $\text{curl}$ being defined with respect to the flat Lorentzian three metric $\eta_{ij}$. This of course implies that flat space Laplacian acting on $V$ must vanish. Similar remarks apply in the case of a non-self-dual Killing vector in a self-dual space as has been considered by Boyer and Finley [18], and Gegenberg and Das [19], and also Park [20].

If instead we had considered the Riemannian form of these metrics,

$$ds^2 = V^{-1}(d\tau + \omega_i dx^i)^2 + V\delta_{ij}dx^i dx^j,$$  \hspace{1cm} (5.12)
we would have found a family of Hyper-kähler metrics that have a self-dual Killing vector and self-dual curvature, and that are Asymptotically Locally Euclidean (ALE). These are the "N-center" metrics where

\[ V = \sum_{i}^{N} \frac{1}{|x - x_i|}, \]  

(5.13)

with \(x_i\) is a Euclidean three vector, and \(\omega\) is determined by equation (5.11). If \(N = 1\), then the metric is that of flat space. If \(N = 2\), the space is the Eguchi-Hanson instanton metric. In general the N-center metrics are the unique self-dual metrics that ALE and whose boundary at infinity is \(S^3/Z_2\). One can analytically continue any of these metrics to find hypersymplectic Kleinian metrics, however there is a considerable arbitrariness in how one goes about this. We now give an example which appears to have some stringy significance.

Consider the Eguchi-Hanson metric [21] in the form

\[ ds^2 = \frac{dr^2}{f^2} + 1/4r^2(\sigma_1^2 + \sigma_2^2 + f^2\sigma_3^2), \]  

(5.14)

where

\[ f^2 = (1 - \frac{a^4}{r^4}), \]  

(5.15)

and \(\sigma_i\) are a set of left-invariant one forms on \(S^3\). Thus

\[ d\sigma_i = -\frac{1}{2}\epsilon_{ijk}\sigma_j \wedge \sigma_k. \]  

(5.16)

Explicitly, in terms of Euler angles \((\theta, \phi, \psi)\),

\[ \sigma_1 + i\sigma_2 = e^{-i\psi}(d\theta + i\sin \theta d\phi), \]  

(5.17)

\[ \sigma_3 = d\psi + \cos \theta d\phi. \]  

(5.18)

One way in which one can analytically continue this metric is by the coordinate transformation \(\theta \rightarrow i\theta\), so that the metric becomes

\[ ds^2 = \frac{dr^2}{f^2} + \frac{1}{4}r^2(-d\theta^2 - \sin^2 \theta d\phi^2 + f^2(d\psi + \cosh \theta d\phi)^2). \]  

(5.19)

This can be rewritten in terms of a pseudo-orthonormal basis of one-forms

\[ e^0 = \frac{dr}{f}, \quad e^1 = \frac{1}{2}r\sigma_1, \quad e^2 = \frac{1}{2}r\sigma_2, \quad e^3 = \frac{1}{2}rf\sigma_3, \]  

(5.20)

where now \(\sigma_i\) are a basis of \(SO(2,1;\mathbb{R})\) invariant one-forms:

\[ \sigma_1 = \cos \psi d\theta + \sin \psi \sinh \theta d\phi, \]  

(5.21)

\[ \sigma_2 = -\sin \psi d\theta + \cos \psi \sinh \theta d\phi, \]  

(5.22)

\[ \sigma_3 = d\psi + \cosh \theta d\phi. \]  

(5.23)

Hence

\[ d\sigma_i = -\frac{1}{2}\epsilon_{ijk}\sigma_j \wedge \sigma_k. \]  

(5.24)
The $c_{ijk}$ being the structure constants of $so(2,1)$ The above metric is hypersymplectic as can be seen by constructing the basis of self-dual 2-forms described earlier. Explicitly

$$I = e^0 \wedge e^3 - e^1 \wedge e^2,$$

$$S = e^0 \wedge e^2 + e^1 \wedge e^3,$$

$$T = e^0 \wedge e^1 + e^2 \wedge e^3.$$ (5.25, 5.26, 5.27)

This metric has the following stringy interpretation. Consider the submanifold spanned by $e^1$ and $e^2$. This two surface is Riemannian and of constant curvature. It therefore is conformal to any Riemann surface of genus $g > 1$, after appropriate identification to find the unit cell. Let us suppose that this is the string worldsheet $\Sigma$, appropriate for string loop diagrams. It has Riemannian signature indicating that it corresponds to a classically forbidden process, but nevertheless one that is quantum mechanically allowed. The four-manifold is then a self-dual metric on $T^*(\Sigma)$. This is precisely the situation envisaged by Ooguri and Vafa. This lends some support to the conjecture that $N=2$ string theory is in fact the same thing as self-dual gravity.

A second rather different continuation can be found by making the analytic continuation $\phi \rightarrow i\phi$, $\psi \rightarrow i\psi$. Were it the case that $f = 1$, then this continuation would yield a space diffeomorphic to the previous manifold. However, we instead find that

$$ds^2 = \frac{dr^2}{f^2} + \frac{1}{4} \psi^2 (d\theta^2 - \sin^2 \theta d\phi^2 - f^2 (d\psi + \cos \theta d\phi)^2).$$ (5.28)

This metric is also hypersymplectic. Now however the analog of the subspace spanned by $e^1$ and $e^2$ is a Lorentz manifold rather than a Riemannian one, again of constant negative curvature, therefore it represents a physical string worldsheet $\Sigma_\ast$, corresponding to a classically allowed process. Despite this difference, the four manifold still has a Kleinian self-dual metric and is of the form $T^*(\Sigma)$. It would therefore seem that Kleinian self-dual four metrics provide an arena for discussing both quantum and classical processes on a democratic footing. This lends further support to the idea that string theory is in some, as yet ill-understood way, related to self-dual gravity.

§6 Cosmic Strings

There has been some considerable interest recently in the relation between on the one hand cosmic strings and solitons, and on the other hand fundamental strings and $p$-branes. In particular [22], a fundamental string was modelled by a solution of the equations of motion of the zero-slope limit of the superstring, with a distributional source with support on the worldsheet of the string. On the other hand, there is a dual formulation in which the ten-dimensional superstring written in terms of seven-form field strengths, there are source-free soliton like solutions of the zero-slope equations of motion representing a 5-brane. In the present section, we shall show how to construct non-singular, source-free solutions of the self-dual Einstein equations with signature $(2,2)$ which may be interpreted as a “thickening” or “de-singularization” of a distributional cosmic string with curvature supported on a totally null 2-plane. We shall begin by describing the distributional model, and then its thickening. The idea of the distributional model is due to Mason [23], but our specific construction is a little different.

A distributional “Regge Calculus” model of a cosmic string may be obtained by taking a flat spacetime $\mathbb{R}^{t,s}$, with $s+t = 4$, and an isometry, following the method of Ellis and Schmidt
[24]. A suitable isometry $\Gamma$ is one with an 2-dimensional fixed point set $M^2$ and which lies in a one-parameter subgroup. For the construction, it is necessary to fix a particular parameter $c$, so that

$$\Gamma: x^\alpha \rightarrow (\exp cM^2_\beta)x^\beta,$$  \hspace{1cm} (6.1)

For example, in Euclidean space this fixes the rotation angle, which is otherwise ambiguous by the addition of multiples of $2\pi$.

The element $M^\alpha_\beta$ of the Lie algebra is determined up to scale by the fixed point set, and in fact $M^2$ is given by points $x^\alpha$ such that

$$M^\alpha_\beta x^\beta = 0.$$  \hspace{1cm} (6.2)

The spacetime is obtained by an identification of points on the universal covering space of $R_{t,s}$ with the fixed point set $M^2$ removed. This covering space is denoted $C$.

The action of $\Gamma$ lifts uniquely to the covering space, due to the choice of parameter in (6.1). The fundamental group of $(R_{t,s} - M^2)$ also acts freely on the covering space. Let $e$ be a generator of the fundamental group, and identify points under the action of $\Gamma$ composed with $e$, so that the exterior of the cosmic string is given by $M^4 = C / (\Gamma \circ e)$.

The resulting spacetime is homeomorphic to $R_{t,s} - M^2$, but has a non-standard metric, the holonomy group for curves encircling the string worldsheet $M^2$ being generated by $\Gamma$. The metric varies continuously with the parameter $c$, the case $c = 0$ being flat space. The points $M^2$ can be reinserted if desired.

Now return to our construction, which includes the action of $\pi_1$. Suppose the string worldsheet $M^2$ lies on $x_1 = x_2 = 0$, and that $M^\alpha_\beta$ is scaled so that $M_{12} = 1$. The distributional Riemann curvature tensor has support on the string worldsheet $M^2$ and is given by

$$R^\alpha_\beta\gamma\delta = cM^\alpha_\beta M_\gamma^\delta \delta(x_1)\delta(x_2),$$  \hspace{1cm} (6.3)

as follows from the definition of the Riemann tensor in [26].

The distributional Ricci tensor associated with the cosmic string is given by

$$R_{\alpha\beta} = cM^\alpha_\sigma M_{\sigma\beta} \delta(x_1)\delta(x_2).$$  \hspace{1cm} (6.4)

Clearly the string is therefore entirely specified by the simple 2-form $M^\alpha_\beta$ and the strength of the source $c$.

For conventional cosmic strings, $t = 1$ and $s = 3$, and $\Gamma$ is a spacelike rotation through $c$ about an axis lying in the string, and $c$ is the deficit angle. Thus

$$\exp(cM) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos c & -\sin c & 0 \\ 0 & \sin c & \cos c & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$  \hspace{1cm} (6.5)
The string worldsheet is the timelike 2-surface $x_1 = x_2 = 0$. The Ricci tensor is non-zero. The case of null strings in Minkowski spacetime has been discussed by Bruno, Shapley and Ellis [27]. They take

\[
M_{\alpha\beta} = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
1 & -1 & 0 & 0
\end{pmatrix}.
\]

(6.6)

The null string worldsheet satisfies

\[
x^0 - x^1 = 0, \quad x^3 = 0.
\]

(6.7) (6.8)

Since

\[
M_{\alpha\beta}M_{\beta\gamma} = \begin{pmatrix}
1 & -1 & 0 & 0 \\
1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
\]

(6.9)

the distributional Ricci curvature is not zero. Thus in Minkowski spacetime neither timelike or null strings are distributional source-free solutions of the Einstein vacuum equations, and one does not expect to find non-singular thickenings which are Ricci-flat either.

By contrast in $\mathbb{R}^{2,2}$, Mason pointed out that a simple self-dual (or anti-self-dual) 2-form $M_{\alpha\beta}$ will satisfy

\[
M_{\beta\gamma}M_{\beta\gamma} = 0.
\]

(6.10)

and therefore the associated cosmic string has distributional curvature with support on an $\alpha$-plane (or $\beta$-plane if anti-self-dual). An explicit example is provided by introducing coordinates such that

\[
ds^2 = dx^+dx^- - dy^+dy^-.
\]

(6.11)

The null self-dual $O(2, 2)$ transformations, $\Gamma$, are

\[
x^+ \to x^+,
\]

\[
y^+ \to y^+,
\]

\[
x^- \to x^- + cx^+,
\]

\[
y^- \to y^- + cy^+,
\]

(6.12)

and these fix the null self-dual two-plane given by

\[
x^+ = 0,
\]

\[
y^+ = 0.
\]

(6.13)

In this (non-orthonormal) basis

\[
M_{\alpha\beta} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}
\]

(6.14)

where

\[
M_{\alpha\beta}M_{\beta\gamma} = 0.
\]

(6.15)

This establishes that the distributional Ricci tensor is indeed zero. The $O(2, 2)$ transformations $\Gamma$ commute with

1. anti-self-dual rotations which act transitively on the quadrics $x^+x^- - y^+y^- = constant$
2. two null and covariantly constant translations generated by $\frac{\partial}{\partial x^-}$ and $\frac{\partial}{\partial y^-}$. 

Since \( \Gamma \) is obtained by exponentiating the null and self-dual Killing vector field \( K = x^+ \frac{\partial}{\partial y} + y^+ \frac{\partial}{\partial x} \), the manifold 
\[
\mathcal{M}^4 = \mathcal{C}/(\Gamma \circ e)
\] (6.16)

admits a six parameter group isometries which act transitively. It is perhaps surprising that removing the worldsheet from \( \mathbb{R}^{2,2} \) leaves a spacetime that is still homogeneous.

In searching for a thickening of the cosmic string, it is natural to insist that the resulting spacetime should continue to admit some of these symmetries. In particular we shall demand that \( \mathcal{M}^4 \) admits a null self-dual Killing vector field.

We therefore have the following:

**Proposition.** Every \((2,2)\) self-dual spacetime admitting a null self-dual Killing vector field \( K \) is automatically Ricci flat, and locally be cast into the following form:

\[
ds^2 = dpdt - \frac{1}{2}p^2 du((dv + H(p,u)du)
\]

where \( H(p,u) \) is an arbitrary \( C^2 \) function of its arguments and \( K^a \partial_a = \partial_v \).

Suppose that the one-form associated with the Killing vector is \( \mathcal{K} \), then \( d\mathcal{K} \) is self-dual,

\[
\mathcal{K} \wedge d\mathcal{K} = *d(g(\mathcal{K},\mathcal{K}))
\] (6.17)

and so since \( \mathcal{K} \) is null, it follows that \( \mathcal{K} \) is hypersurface orthogonal. Hence we can put \( K^a \partial_a = \partial_v \) and \( \mathcal{K} = K_\alpha dx^\alpha = -2\omega du \), with \( v \) being a null coordinate. The \( K \)-invariant metric then takes the form

\[
ds^2 = P^{-2}(x,y,u)(dx^2 - dy^2) - 2du(\omega dv + m_\alpha dx^\alpha + Hdu). \quad (6.18)
\]

There is a considerable amount of coordinate freedom that preserves this metric form. Firstly however, we must impose the self-duality of \( d\mathcal{K} \), and hence obtain

\[
\omega = \omega(x-y,u)
\] (6.19)

Suppose that \( \omega \) is not only a function of \( u \), then setting \( s = \omega \), and \( t = x + y \) we obtain

\[
ds^2 = P^{-2}(s,t,u)dsdt - 2sdudv + m_\alpha ds + m_\alpha dt + Hdu). \quad (6.20)
\]

By changing the \( v \) coordinate by \( v \rightarrow v + g(s,t,u) \) we can set \( m_\alpha = 0 \). Now imposing the self-duality of the curvature form we discover that the metric must be of the form

\[
ds^2 = s^{-1/2}dsdt - 2sdudv + H(s,u)), \quad (6.21)
\]

where \( H(s,u) \) is an arbitrary function. Now by rescaling \( s = \frac{1}{4}p^2 \) we obtain the metric form given. The condition that \( H(p,u) \) be \( C^2 \) emerges from the requirement that the curvature form be well defined. A special case not covered by this treatment is when \( \omega \) is a function of \( u \) only. Then \( d\mathcal{K} \) vanishes, and the Killing vector \( K \) is covariantly constant. This class of metrics is included in the form (6.20). □

We can now proceed with our cosmic string calculation. The null self-dual Killing vector field is

\[
K = \frac{\partial}{\partial v}. \quad (6.22)
\]
If one uses the frame basis of 1-forms
\[ e^1 = \frac{1}{2} dp, \quad e^2 = dt, \quad e^3 = \frac{1}{4} p^2 du, \quad e^4 = -(dv + H du), \quad (6.23) \]
in
\[ ds^2 = e^1 \otimes e^2 + e^2 \otimes e^1 + e^3 \otimes e^4 + e^4 \otimes e^3, \quad (6.24) \]
then the only non-vanishing component of the curvature 2-form is
\[ \Omega_{13} = \frac{16}{p^2} \frac{\partial}{\partial p} \left( p^2 \frac{\partial H}{\partial p} \right). \quad (6.25) \]

Thus the holonomy is not only self-dual, but null. Clearly the metric will be flat away from \( p = 0 \) if
\[ H = \frac{f''}{p}, \quad (6.26) \]
where prime denotes the derivative with respect to \( u \), and \( f(u) \) is an arbitrary function of \( u \). One can recover the usual flat form of the metric by setting
\[ x^- = \frac{p}{2} u (v + \frac{f(u)}{p}) + t - f(u), \]
\[ x^+ = p, \]
\[ y^- = \frac{p}{2} \left( v + (\frac{f'(u)}{p}) \right), \]
\[ y^+ = pu. \quad (6.27) \]

It now follows that the null Killing vector \( \frac{\partial}{\partial v} \) generates the null self-dual rotation \( v \to v + c \) or
\[ x^- \to x^- + cy^+, \]
\[ y^- \to y^- + cx^+, \quad (6.28) \]
and the covariantly constant null Killing vector \( \frac{\partial}{\partial t} \) the null translation \( t \to t + e \) or
\[ y^- \to y^- + e. \quad (6.29) \]

Thus \( \frac{\partial}{\partial v} \) corresponds to \( y^+ \frac{\partial}{\partial p} + x^+ \frac{\partial}{\partial \frac{1}{p}} \) and \( \frac{\partial}{\partial t} \) to \( \frac{\partial}{\partial u} \). It is interesting that the assumption of a null self-dual Killing vector always implies the existence of an additional null covariantly constant Killing vector \( \frac{\partial}{\partial t} \). These solutions are thus members of Plebanski’s class of complex solutions [13] to the vacuum Einstein equations that are analogous to pp-waves. They have also previously been described by Ward [28] in the context of twistor theory.

In order to obtain the cosmic string, one should note that
\[ \frac{d}{dp} \left( p^2 \frac{d}{dp} \left( \frac{1}{p} \right) \right) = \frac{-1}{4\pi} \delta(p). \quad (6.30) \]
Therefore choosing \( H = \frac{c}{|p|}, \ c \) constant, we find that the coordinate change to the flat coordinates \( x^+, x^-, y^+, y^- \) becomes
\[ x^- = \frac{puv}{2} + \frac{c}{2} \text{sgn}(p) + t - \frac{cu^2}{2}, \]
\[ x^+ = p, \quad (6.31) \]
\[ y^- = pv + \frac{c}{2} \text{sgn}(p), \]
\[ y^+ = pu. \]
These coordinate transformations are discontinuous across the totally null 2-surface \( p = 0 \)
i.e.

\[
\begin{align*}
x^+ &= 0, \\
y^+ &= 0.
\end{align*}
\]  

(6.32)

Points with the same \( u, v, \) and \( t \) values but with \( p \) very small and positive, and points with \( p \) very small and negative are mapped to points with identical \( x^+ \) and \( y^+ \) but with \( x^- \) and \( y^- \) which differ by \( c \). The metric (6.21) with \( H = \frac{c}{|p|} \) corresponds to the distributional cosmic string. To obtain a thickened non-singular string we need only to replace \( H \) by a function which is everywhere bounded, but which tends to \( \frac{c}{|p|} \) for large values of \( p \).

Thus we can construct an everywhere non-singular self-dual spacetime representing a cosmic string. This contrasts sharply with the work of Bruno, Shapley and Ellis [27] who found that in signature (1,3) that they could not eliminate the singularities. In fact the situation is strongly resembles that for self-dual metrics with signature (0,4). The “orbifold” construction is obtained by quotienting \( \mathbb{R}^4 \), thought of as \( \mathbb{C}^2 \) by the action of a discrete subgroup \( \Gamma \subset SU(2)_L \subset SU(2)_L \times SU(2)_R \equiv SO(4) \). The subgroup \( \Gamma \) can be thought of as generating a self-dual rotation acting on \( \mathbb{R}^4 \) which leaves fixed the origin. The resulting orbifold has a Kleinian singularity at the origin. Deletion of the origin gives a metric with a flat connection but with self-dual holonomy. Since the holonomy is self-dual any distributional curvature will be Ricci-flat. Thus it is not surprising that the Kleinian singularity can be blown up to obtain a family of complete non-singular self-dual metrics which are ALE (Asymptotically Locally Euclidean). These Riemannian self-dual metrics, or gravitational instantons as they are sometimes called, depend upon a finite number of moduli. In our case, with signature (2,2) the moduli space is, by contrast, infinite dimensional. Another difference is that because the support of the curvature is zero dimensional from the orbifold construction of the self-dual instantons, the usual Regge calculus approach to distributional metrics, where curvature has support only on two-dimensional sets, does not apply. What is needed is a formalism in which the quadratic functions of the Riemann tensor that gives rise to the Euler density and the Hirzebruch density have support at the origin.

Let us return to our solution. It is easy to see that the metric admits two covariantly constant spinors, since the curvature is self-dual. In addition there is a further covariantly constant spinor of opposite chirality. Let us call these spinors \( o_A, \iota_A \) and \( \tilde{o}_A \) respectively. It follows immediately from this that the metric must admit two covariantly constant null vectors given by \( o_A \tilde{o}_A \) and \( \iota_A \tilde{o}_A \). These vectors automatically obey Killings equation, and span the covariantly constant null 2-surface with 2-form \( \tilde{o}_A \tilde{o}_B \) which is self-dual. This corresponds to our privileged family of \( \alpha \)-planes.

It is tempting to identify these spacetimes with the physical spacetime in which an \( N = 2 \) superstring moves, and the cosmic string with the 2-surface itself. The existence of the covariantly constant spinors is promising because it shows that the spacetime can admit unbroken supersymmetry as one might expect.

REFERENCES

1. H. Ooguri and C. Vafa, Self-Duality and \( N=2 \) String Magic, Int. J. Mod. Phys. A5 (1990), 1389-1398; H. Ooguri, Geometry of \( N=2 \) String Theory, Proceedings of the Trieste Spring School 23 April-1 May 1990, eds M. Green, R. Iengo, S. Randjbar-Daemi, E. Sezgin and H. Verlinde., World Scientific, Singapore, 1991.
2. M.B. Green, J.H. Schwarz and E. Witten, Superstring Theory, Cambridge University Press, Cambridge, 1987.
3. K. Huang and S. Weinberg, Ultimate Temperature and the Early Universe, Phys. Rev. Lett. 25 (1970), 895-7; R. Hagedorn, Hadronic Matter near the Boiling Point, Nuovo Cimento 56A (1968), 1027-57; J. Atick and E. Witten, The Hagedorn Transition and the Number of Degrees of Freedom of String Theory, Nucl. Phys. B310 (1988), 291-334.
4. M. Ademollo, L. Brink, A. D’Adda, R. D’Auria, E. Napolitani, S. Sciuto, E. del Guidice, P. di Vecchia, S. Ferrara, F. Gliozzi, R. Musto, R. Pettorin and J.H. Schwarz, Dual String with U(1) Colour Symmetry, Nucl. Phys. B111 (1976), 77-110; L. Brink and J.H. Schwarz, Local complex supersymmetry in two dimensions, Nucl. Phys. B121 (1977), 285-.

5. M. Kato and K. Ogawa, Covariant Quantization of String based on BRS Invariance, Nucl. Phys. B111 (1976), 77-110; L. Brink and J.H. Schwarz, Local complex supersymmetry in two dimensions, Nucl. Phys. B121 (1977), 285-.

6. W. Boucher, D. Friedan and A. Kent, Determinant Formulas and Unitarity for N=2 superconformal algebras in two dimensions or exact results on string compactifications, Phys. Lett. 172B (1986), 316-322.

7. M. Kato and K. Ogawa, Covariant Quantization of String based on BRS Invariance, Nucl. Phys. B111 (1976), 77-110; L. Brink and J.H. Schwarz, Local complex supersymmetry in two dimensions, Nucl. Phys. B121 (1977), 285-.

8. M. Ferraris and M. Francavigilia, An Algebraic Isometric Embedding of Kruskal Spacetime, Gen. Rel. and Grav. 10 (1979), 283-296.

9. N. Hitchin, Hypersymplectic Quotients, Acta Academiae Scientiarum Taurinensis, Suplemento al numero 124 degli Atti della Accademia delle Scienze di Torino Classe di Scienze Fisiche, Matematiche e Naturali, (1990).

10. J.D. Gegenberg and A. Das, Stationary Riemannian Spacetimes with Self-Dual Curvature, J. Gen. Rel. and Grav. 16 (1984), 817-29.

11. R. Penrose and W. Rindler, Spinors and Spacetime, Cambridge University Press, Cambridge, 1984.

12. R.O. Wells and R.S. Ward, Twistor Geometry and Field Theory, Cambridge University Press, Cambridge, 1990.

13. C.W. Misner, Taub-NUT spacetime as a counterexample to almost anything, Relativity Theory and Astrophysics I: Relativity and Cosmology (J. Ehlers, ed.), Lectures in Applied Mathematics Volume 8, American Mathematical Society, Rhode Island, 1967, pp. 160 -169.

14. J.D. Gegenberg and A. Das, Stationary Riemannian Spacetimes with Self-Dual Curvature, J. Gen. Rel. and Grav. 16 (1984), 817-29.

15. J. Plebanski, Some solutions of the complex Einstein equations, J. Math. Phys. 16 (1975), 2395-2402.

16. C.P. Boyer and J.D. Finley, Killing Vectors in self-dual Euclidean Einstein spaces, J. Math. Phys. 23 (1982), 1126-30.

17. B.G. Schmidt and G.F.R. Ellis, Quasi-Regular Singularities based on null planes, J. Gen. Rel. and Grav. 19 (1987), 203-206.

18. L.J. Mason, Twistor and the Regge Calculus, Twistor Newsletter 28 (1989), 42.

19. B.G. Schmidt and G.F.R. Ellis, Singular Spacetimes, J. Gen. Rel. and Grav. 8 (1977), 915-933.

20. C.W. Misner, Taub-NUT spacetime as a counterexample to almost anything, Relativity Theory and Astrophysics I: Relativity and Cosmology (J. Ehlers, ed.), Lectures in Applied Mathematics Volume 8, American Mathematical Society, Rhode Island, 1967, pp. 160 -169.

21. A. Dabolkar and J.A. Harvey, Nonrenormalization of the superstring tension, Phys. Rev. Lett. 63 (1989), 478-481; A. Dabolkar, G.W. Gibbons, J.A. Harvey and F. Ruiz-Ruiz, Superstrings and Solitons, Nucl. Phys. B340 (1990), 33.