Deterministic Abelian Sandpile and square-triangle tilings
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Abstract: The Abelian Sandpile Model, seen as a deterministic lattice automaton, on two-dimensional
periodic graphs generates complex regular patterns displaying (fractal) self-similarity. In particular, on a
variety of lattices and initial conditions, at all sizes, there appears what we call an exact Sierpinski structure:
the volume is filled with periodic patterns, glued together along straight lines, with the topology of a tri-
angular Sierpinski gasket. Various lattices (square, hexagonal, kagome,...) initial conditions, and toppling
rules show Sierpinski structures which are apparently unrelated and involve different mechanisms. As will
be shown elsewhere, all these structures fall under one roof, and are in fact different projections of a unique
mechanism pertinent to a family of deterministic surfaces in a 4-dimensional lattice. This short note gives a
description of this surface, and of the combinatorics associated to its construction.

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Introduction. Let Λ be the lattice in dimension 4, tensor product of two copies of the triangular lattice,
Λ = (e1, e2, e3, e4, e5, e6 | \sum_{1 \leq i \leq 3} e_i = \sum_{4 \leq i \leq 6} e_i = 0)/2. Consider the two-dimensional cell complex containing
all the vertices and edges of Λ, and, as (oriented)
faces, the triangles of the two lattices and the parallelograms spanned by pairs (e1, e4), (e2, e5) and
(e3, e6). We choose the orientation such that the cy-
cles (e1, e2, e3), (−e1, −e2, −e3) and (e1, −e4, −e1, e4)
are upward faces, and similarly with (123) → (456)
and (14) → (25), (36). We call a surface a connected
and simply-connected collection of faces in the cell
complex above, with all upward faces.

Embeddings of e1, . . . , e6 in R2 satisfying the fore-
mentioned orientation constraints correspond to proj-
ections of the 4-dimensional cell complex on a 2-di-
nensional real space, such that surfaces are mapped
injectively. An example of such an embedding is
(e1, . . . , e6) = (ω3, ω11, ω7, ω9, ω8, ω4), where ωk =
(\cos \frac{2\pi k}{6}, \sin \frac{2\pi k}{6}). In this case, surfaces correspond to
tilings of regions of the plane, composed only of squares
and triangles of unit sides, and along directions mul-
tiple of π/6. These tilings are called square-triangle
tilings in the literature.

Any other projection is topologically equivalent, pro-
vided that e1 + e2 + e3 = e4 + e5 + e6 = 0 and the orien-
tation of the faces is preserved. We call valid such
a projection. The set of valid projections is an open
portion of an algebraic projective variety. We call de-
genenerate projections those on the boundary of this open
set. Under degenerate projections, the image of some
faces is a segment or a point.

A seminal work of de Bruijn for Penrose–Ammann
lozenge tilings [BD81] has first illustrated the possi-
bility that projections of deterministic surfaces from a
high-dimensional periodic cell-complex could ex-
plain features of two-dimensional aperiodic incommen-
surable tilings. The square-triangle case discussed here
shows a similar phenomenon.

Square-triangle tilings have also distinguished prop-
erties, among which is a relation with Algebraic Ge-
ometry, generalising the well-known connection be-
 tween lozenge tilings and Schur functions (see e.g.
[BP13]). The algebra of Schur functions has ubiquitous
three-index structure constants cλ,µ,ν called Littlewood–
Richardson (LR) coefficients [LR34]. When the Young
diagrams λ, µ, ν are boxed in a rectangle (d − n) × n
(as is the case, e.g., when they label cells of the Schu-
bert variety), there exists a relation (Poincaré dual-
ity) which acts as complementation at the level of

\[ \lambda \leftrightarrow \mu \leftrightarrow \nu \]

\[ c_{\lambda,\mu,\nu} = 0 \]

As shown by P. Zinn-Justin [Z109], and Purbhoo [P109], the LR coefficients cor-
respond to the enumerations of square-triangle tilings
over triangoloids whose three sides are built from λ,
µ, and ν, respectively. Two degenerate projections of
these surfaces reduce to portions of the square and of
the triangular lattice. As degenerate projections trans-
form some faces into segments or points, the bijective
correspondence is preserved only if extra integer label-
ings, encoding the disappeared faces, are added to the
resulting structures. These limiting tilings, together
with the auxiliary labelings, correspond to the origi-
nal Littlewood–Richardson rule [LR34] in the square
case, and to the Knutson–Tao (discrete) honeycombs
[KT199] [KT200] in the triangular case.
ASM and square-triangle tilings. The purpose of this paper is to illustrate another unsuspected feature specific of square-triangle tilings, namely of encoding the exact Sierpinski structures that arise in the Deterministic Abelian Sandpile Model. These structures have been identified on various regular two-dimensional lattices, under various abelian toppling rules, initial conditions and deterministic evolution protocols, and square-triangle tilings describe them in a unified way.

The first occurrences of such structures have been presented, by the authors, in [Pao13; CPS10], while the observation of approximated versions of these structures (reproduced at a coarse-grained scale, but locally deformed by some 1-dimensional defects) is much older [Ost03], and has first been made, only on the square lattice, for the two most natural deterministic protocols: the evaluation of the identity configuration in simple geometries [DM94; LKG90; LR02], and the relaxation of a large amount of sand put at the origin, in the (elsewhere empty) infinite lattice [DSC09; Hal08; LPS12; LPS13].

The ‘universal role’ of the square-triangle tiling, in different ASM realisations, should sound surprising, as the generic projection gives incommensurable parallelogram-triangle tilings and does not live on a discrete two-dimensional lattice, as is instead the case for the sandpile models we consider. What comes out is that, in a remarkable analogy with the mechanism discussed above for the combinatorics of the Littlewood–Richardson rule and Knutson–Tao honeycombs, different lattice ASM realisations occur at different “rational” points in the set of valid projections (and its boundary, of degenerate projections).

As this short paper is within a series, we do not give here an introduction to the Abelian Sandpile Model. The interested reader can consult the beautiful review by Deepak Dhar [Dim99], who first established a large part of the theory. For aspects of the model more strictly related to the features discussed here, the reader can refer to the PhD thesis of one of the authors [Pao13], or the shorter papers [CPS10] and [CPS12]. Here we will only concentrate on the aspects concerning the surfaces in the square-triangle tiling corresponding to the exact Sierpinski structures in the ASM.

The sandpile configurations are height vectors \( \vec{z} = \{z_i\} \), with variables \( z_i \in \mathbb{N} \) associated to vertices \( i \) of a graph \( G = (V,E) \). There exists a notion of stable configuration, and a more restrictive notion of recurrent one. Transient is a synonym for non-recurrent. There exists a notion of forbidden sub-configuration (FSC), and a stable configuration is recurrent if it has no FSC. More generally, a configuration is recurrent over \( W \subseteq V(G) \) if it has no FSC contained within \( W \), thus making recurrence a local notion (like instabiliy). Local recurrence and instability are dual notions, if we set in the wider frame of multitoppling ASM, as first shown in [CPS12]. The toppling matrix \( \Delta \) encodes the dynamics of the sandpile, and determines a subdivision of \( \mathbb{Z}^V(G) \) into equivalence classes. There exists exactly one stable recurrent configuration within each class. Unstable configurations \( \vec{z} \) can be relaxed to stable ones, \( \vec{w} = R \vec{z} \). Stable transient configurations can be projected to the unique recurrent representative in the class, \( \vec{w} = P \vec{z} \). The operators \( R \) and \( P \) correspond to find the fixed point of iterated maps, \( R_0 \) and \( P_0 \), corresponding to “rounds” of the procedure.

A number of structures and operations on square-triangle tilings can be introduced, that will reproduce, under the various projection procedures, the aforementioned counterparts in the various ASM realisations. We dub all these features of the square-triangle setting with the “axiomatic” attribute, as the reason for their names emerges only when the projection procedure is explicitted. Note that we are not able to reproduce all the relevant features of the sandpile model. In particular, we are not able to reproduce the \( a_i \) operators (nor their counterparts \( a_i^1 \) defined in [CPS12]). The main things we are able to reproduce are summarised by the following list:

- The notion of (ASM-)equivalence of configurations is trivialised at the axiomatic level: two tilings are equivalent if they have the same boundary.
- The axiomatic notion of FSC correspond to cycles in the tiling satisfying certain local rules.
- We have an axiomatic notion of \( P_0 \), consisting in a local deformation along the cycles of maximal FSC’s (w.r.t. inclusion).
- Similarly, we can certify that regions encircled by certain cycles will undergo a round of relaxation. This gives an axiomatic local notion of unstable subconfiguration (USC).\(^1\)
- We have an axiomatic notion of \( R_0 \), consisting in a local deformation along the cycles of maximal USC’s (w.r.t. inclusion).
- We have a recursive description of the Sierpinski structures at the axiomatic level. As these structures in the ASM determine the classification of patches and propagators in certain backgrounds [Pao13], this induces a corresponding classification of axiomatic patches and propagators.
- A choice of vectors \( e_1,\ldots,e_6 \in \mathbb{R}^2 \), and of “masses” \( \{m_{123},m_{456},m_{14},m_{25},m_{36}\} \) for the five types of tiles, induces a notion of density for the patches. This allows to state an axiomatic version of the Dhar–Sadhu–Chandra incidence formula, first introduced, for the ASM, in [DSC09].

\(^1\)In \( R_0 \) one can perform at most one toppling per site, in \( P_0 \) one adds a single frame identity, and then relax.

\(^2\)Corresponding to the waves of topplings [KPS92; IPS9].
Sierpinski structures. Let \( s = (s_k, \ldots, s_1, s_0) \) be a finite string of positive integers, and \( n(s) = \sum_j 3^j s_j \). A Sierpinski structure is labeled by a string \( s \), and \( n(s) \) is its size. Structures of the same size are equivalent.

An abstract Sierpinski gasket of index \( k \) is defined as follows. At index 0, it is just a dark upward triangle. At index \( k + 1 \), it is obtained from the gasket at index \( k \) by subdividing all dark upward triangles into three dark upward and one light downward triangles, all of half the side. Light triangles which are there at index \( k \), will remain unchanged at all \( k' > k \). A light triangle has index \( k \) if it first appeared in a gasket at index \( k + 1 \). A gasket of index \( k \) has \( 3^k \) dark triangles, and \( 3^k \) light triangles of index \( h \), for \( 0 \leq h < k \). See Fig. 1.

In the sandpile setting, the triangles of the gasket will determine polygonal regions filled with a biperiodic patterns, called patches [Ost10]. Patches may be recurrent, transient or marginal, depending on their behaviour under the burning test (see [CPS10]).

In a Sierpinski structure identified by \( s \), all the dark triangles correspond to transient patches, of triangular shape, with a side of \( s_k \) unit tiles. Light triangles of index \( h \) correspond to polygonal regions filled with recurrent patches. These regions have the aspect of triangoloids with concave sides, the sides being polygonal lines composed of \( 2k^{-h} - 1 \) segments. The packing of unit tiles depends in a certain fixed way on the integer \( k - h \) and the variables \( s_{h'} \) for \( h' > h \), and has no extra freedom, with an exception: starting at the vertices of the triangoloids, we can have a band of a patch with marginal density, of width \( s_h - 1 \). The three bands meet at a triangular transient patch.

A transient patch contains a FSC only if “sufficiently large”, namely if it contains at least 7 unit tiles, packed in a shape \( \boxdot \). Thus, a triangle of side up to 3 units filled with a transient patch, i.e. the shape \( \boxdot \), may still be part of an overall recurrent configuration. This has a consequence on our Sierpinski structures: a structure with label \( s \) is recurrent if and only if \( 1 \leq s_h \leq 3 \) for all \( h \leq k \). These are the structures ultimately appearing in sandpile protocols.

Each region of the Sierpinski structure is filled with a periodic pattern. The geometry of every region, including the number and location of the unit tiles, is determined through a recursive procedure. Also the shape of the unit tiles, and their content in terms of elementary squares and triangles, are determined recursively. At this aim it is useful to introduce a labeling of the regions of the Sierpinski gasket. We label the dark upward triangles with words in the alphabet \( \{a, b, c\} \), and the light downward triangles with the same word as the dark triangle that originated them. When a triangle of label \( w \) is split, the three new triangles, in the three directions, have labels \( wa, wb \) and \( wc \). We also give labels to the three external regions of the triangles, as \( a^{-1}, b^{-1} \) and \( c^{-1} \). See Fig. 1.

A triangle with label \( w \) has three larger adjacent light triangles, in the three directions, that have labels \( \alpha(w), \beta(w) \) and \( \gamma(w) \). These three functions can be defined as follows. Let \( \alpha_w, \beta_w \) and \( \gamma_w \) the rightmost position along \( w \) such that, at its right, there are no more \( a, b \) or \( c \), respectively; let us call \( w[1] \) the truncation of \( w \) to its first \( l \) letters; let us understand that \( aa^{-1} = bb^{-1} = cc^{-1} = 1 \). Then \( \alpha(w) = w[1]a^{-1} \), and so on.

Complex tiles arise from the superposition of more elementary ones. Only three tiles are indecomposable, and must be given as input. These tiles correspond to the three square orientations in our square-triangle tilings. The corresponding tilings appear outside the triangle, at the three sides. The unit tile of label \( w \) is composed of the superposition of two copies of the tiles of labels \( \alpha(w), \beta(w) \) and \( \gamma(w) \). Unless \( w = 1 \), one of these three words has higher degree than the other (say \( \alpha(w) \)). In this case, the six tiles do not overlap, with the only exception that the two \( \alpha(w) \) tiles do overlap exactly on a \( \alpha(\alpha(w)) \) tile. If \( w = 1 \), no tiles overlap. Each tile has 12 special positions along its boundary, which determine the translation vectors of the recurrent, transient and marginal tilings involving it, and the new tile inherits its own positions from those of the three subtiles.

\[ \text{This corresponds to } s_k - 1 \text{ parallel type-I propagators, w.r.t. the definitions in [CPS10, Pao13].} \]
sequence of the form $\{1, 1 \}$ by a cyclic sequence in central symmetry $(\text{some square-triangle tiling})$. A polygon

\[
\begin{align*}
\alpha & = (1 \bar{6} 1 \bar{5} 1 \bar{4} 1 \bar{3} 1 \bar{2} 1 \bar{1} 1 \bar{0}), \\
\beta & = (2 \bar{6} 2 \bar{5} 2 \bar{4} 2 \bar{3} 2 \bar{2} 2 \bar{1} 2 \bar{0}), \\
\gamma & = (3 \bar{6} 3 \bar{5} 3 \bar{4} 3 \bar{3} 3 \bar{2} 3 \bar{1} 3 \bar{0}),
\end{align*}
\]

Each $P$ is neighbour to $4$ $P$'s and $4$ $Q$'s, alternating. The fundamental triangles are at the triple points of the square-octagon topology. Each $P$ and $Q$ tile is adjacent to $8$ and $4$ triangles, respectively, alternating dark / light, and, within dark and light ones, of opposite orientations.

For each $w$, the pairs of tiles $(P(\alpha(w)), P(w))$, $(P(\beta(w)), P(w))$ and $(P(\gamma(w)), P(w))$ are dual pairs. For example, the dodecagon and any of the fundamental parallelograms form a dual pair.

Exceptionally, and analogously to what happens for hex tilings, also all pairs of fundamental parallelograms are dual pairs, although with a different topology, and with no ordering.

Each tile $P = P(w)$ appears in two hex tilings, three sq-oct tilings as ‘octagon’, and infinitely many sq-oct tilings as ‘square’. The union of the positions of triple points among all these tilings has cardinality $12$. These $12$ special positions break the perimeter of the tile into open paths, related by the central symmetry. Thus, a list of $6$ paths, $u(P) = (u_1, \ldots, u_6)$, determines simultaneously the perimeter and the special positions, and $P = (u_1, u_2, u_3, u_4, u_5, u_6)$.

The recursive construction, at the level of these paths, leads to the formulas (completed by $C_3$-covariance)

\[
\begin{align*}
(u_1 u_2) & = (u_1, u_2)_{\alpha(w)} 6 (u_1 u_2)_{\gamma(w)} \\
(u_1) & = (u_1)_{\alpha(w)} \quad |\alpha(w)| = |\beta(w)| \\
(u_2) & = (u_2)_{\beta(w)} \quad |\alpha(w)| < |\beta(w)| \\
(u_1) & = (u_2)_{\gamma(w)} \quad \alpha(w) = a^{-1}, \beta(w) = b^{-1}
\end{align*}
\]

The geometry of these paths is such that:

- The sq-oct patches based on a $(P, Q)$ dual pair may be adjacent to both recurrent and transient hex patches, based both on $P$ and on $Q$, although with a restriction on the direction of the (straight) boundary.
- The hex transient tiling based on $P(w)$ can be adjacent to the hex recurrent tiling based on $P(w')$, if $w'$ is a prefix of $w$.

This ultimately leads to the consistency of the construction of the Sierpinski structures (see Fig. 3).

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4This fixes who's who among $P$ and $Q$. 

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Fig. 2. The tile associated to $w = cba$. The interior orange lines describe the decomposition into $\alpha(w)$, $\beta(w)$ and $\gamma(w)$ tiles. The overlap, composed of a $\alpha(\alpha(w))$ tile and two light triangles, is in the middle. The triangles outside the tile denote the $12$ special positions. We have $u(P) = ((61), (2616), (43426162), (34), (5343), (16153435))$. 

**Dual tiles.** Our construction in terms of the vectors $e_1, \ldots, e_6$ has a number of covariances that allow to shorten our description.

\[C_3\text{-covariance} \quad (2\pi/3 \text{ rotations):} \]

\[
\begin{align*}
(e_1, e_2, e_3, e_4, e_5, e_6) & \rightarrow (e_2, e_3, e_4, e_5, e_6, e_4); \\
\text{exchange} \ (123) \leftrightarrow (456) \ (\pi/2 \text{ rotations}); \\
\text{central symmetry} \ (\pi \text{ rotations}): \\
(e_1, e_2, \ldots, e_6) & \rightarrow (-e_1, -e_2, \ldots, -e_6).
\end{align*}
\]

We call **polygon** a closed curve that is the boundary of some square-triangle tiling. A polygon $P$ is determined by a cyclic sequence in $\{1, \ldots, 6, 1, \ldots, 6\}$, where $1, 1$ stand for $+e_1$, $-e_1$, and so on. We use the shortcuts $\triangledown, \nabla, \triangleright$ and $\triangleleft$ for the polygons $(123), (456)$ and $(456)$, respectively.

A centrally symmetric polygon $P$ is determined by a sequence of the form $P = (i_1 i_2 \ldots i_k i_1 i_2 \ldots i_k)$, where $i_1 = i$. We use the shortcut $(i_1 i_2 \ldots i_k \bar{i}_k)$ in such a case.

A polygon $P$ is a **dual tile** if both the triple of polygons $(P, \triangledown, \nabla)$ and the triple $(P, \triangleright, \triangleleft)$ (in these proportions) tile periodically the plane. We call a **transient/recurrent hex tiling** a tiling of the two forms above, respectively.

The three fundamental parallelogram tiles are dual tiles. The dodecagon, $(162435\bar{1})$, is another example. All the tilings associated to dual tiles, except those deriving from the fundamental parallelograms, have the topology of a **hexagonal tiling**: each polygon $P$ is neighbour to other $6$ $P$’s. The fundamental triangles are at the $6$ triple points, with alternating orientations cyclically along each $P$.

To each word $w$ as in the previous section can be associated a dual tile $P(w)$, which is centrally symmetric. The three fundamental parallelograms are $P(1)$. The dodecagon is $(162435\bar{1}) = P(1)$.

A pair of polygons $(P, Q)$ is a **dual pair** if the sextuplet $(P, Q, \triangledown, \nabla, \triangleright, \triangleleft)$ (in these proportions) tiles the plane. We call a **sq-oct tiling** a tiling obtained as above.

Neglecting triangles (e.g., replacing them with $Y$-shapes), the tiling has the square-octagon topology: any $Q$ tile is neighbour to $4$ $P$ ones, and any $P$ tile is neighbour to $4$ $P$’s and $4$ $Q$’s, alternating. The fundamental triangles are at the triple points of the square-octagon topology. Each $P$ and $Q$ tile is adjacent to $8$ and $4$ triangles, respectively, alternating dark / light, and, within dark and light ones, of opposite orientations.

For each $w$, the pairs of tiles $(P(\alpha(w)), P(w))$, $(P(\beta(w)), P(w))$ and $(P(\gamma(w)), P(w))$ are dual pairs. For example, the dodecagon and any of the fundamental parallelograms form a dual pair.

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This fixes who’s who among $P$ and $Q$. 

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Fig. 3. Four classes of equivalent configurations: Left: three deterministic configurations, of size $n = 6$. The two on top are stable but transient, and the one on the bottom is recurrent but unstable. Applying $P$ and $R$, respectively, we obtain our axiomatic Sierpinski structure, (on the right at $s = (1, 2, 2)$, thus $n(s) = 17$). The patch structure is highlighted by the orange construction lines, showing the same topology of the Sierpinski gasket in Figure 1.

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