Theoretical proposal for the experimental realisation of a monochromatic electromagnetic knot

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Abstract

We propose an antenna designed to generate monochromatic electromagnetic knots and other ‘unusual electromagnetic disturbances’ in the microwave domain. Our antenna is a spherical array of radiating dipolar elements configured to approximate the desired electromagnetic field near its centre. We show numerically that a specific embodiment of the antenna with a radius of 61.2 cm and only 20 element pairs driven at a frequency of 2.45 GHz can yield linked and torus-knotted electric and magnetic field lines approximating those of an ‘electromagnetic tangle’: a monochromatic electromagnetic knot closely related to the well-known Rañada–Hopf type electromagnetic knots but simpler in its construction. The antenna could be used to locally excite plasmas.

Keywords: electromagnetic knots, microwaves, antenna theory

Some figures may appear in colour only in the online journal

1. Introduction

Electromagnetic knots were first described by Rañada and collaborators in a series of works \cite{1–4} based on earlier studies by Wolfer \cite{5} and Moffatt \cite{6}, making use of the Bateman construction \cite{7}. Several properties of these Rañada–Hopf electromagnetic knots have been worked out in theory, including their orbital angular momentum \cite{8} and helicity \cite{9–14}, and similar solutions of Maxwell’s equations have been found more recently in the form of propagating light beams \cite{15–17}. Although analogous structures have been observed experimentally in liquid crystals \cite{18} and in fluid dynamics \cite{19}, no feasible proposal has been given to date for the realisation of a Rañada–Hopf type electromagnetic knot, the closest work being a theoretical proposal involving plasma physics and self-organisation \cite{20, 21}. A key difficulty is that Rañada–Hopf type electromagnetic knots are \textit{polychromatic}; they have a broad frequency spectrum, which is difficult to realise experimentally.

One of us recently described a collection of ‘unusual electromagnetic disturbances’ \cite{22}, which are essentially electromagnetic standing waves \cite{23} that appear to be well localised in 3D, even in complete vacuum. Included among these are various \textit{monochromatic} electromagnetic knots, closely related to the well-known Rañada–Hopf type electromagnetic knots but simpler in their construction.

In this paper we propose an antenna designed to generate monochromatic electromagnetic knots and other unusual electromagnetic disturbances in the microwave domain. We work in an inertial frame of reference with time $t$ and position vector $\mathbf{r} = x\mathbf{x} + y\mathbf{y} + z\mathbf{z} = \mathbf{s}\mathbf{s} + \mathbf{z}\mathbf{z} = \mathbf{r}\mathbf{r}$, where $x$, $y$ and $z$ are right-handed Cartesian coordinates with associated unit vectors $\mathbf{x}$,
\[\mathbf{E} = \mathbb{R} \left\{ E_0 \left[ \tilde{k}_0 (8g + \tilde{z}h) + i\tilde{B}_0 \phi \right] e^{-i\omega_0 t} \right\} \]
\begin{align}
\mathbf{B} &= \mathbb{R} \left\{ \frac{E_0}{c} \left[ i\tilde{A}_0 \phi - \tilde{B}_0' (8g + \tilde{z}h) \right] e^{-i\omega_0 t} \right\}
\end{align}

with
\begin{align}
f &= \int_0^\pi \! J_1 (k_0 \sin \vartheta) \cos (k_0 \cos \varphi \varphi) \sin \vartheta d\vartheta, \\
g &= -\int_0^\pi \! J_1 (k_0 \sin \vartheta) \sin (k_0 \cos \varphi \varphi) \cos \vartheta d\vartheta \\
h &= -\int_0^\pi \! J_0 (k_0 \sin \vartheta) \cos (k_0 \cos \varphi \varphi) \sin^2 \vartheta d\vartheta,
\end{align}

where \(E_0\) is an electric-field strength, \(\tilde{k}_0\) and \(\tilde{B}_0\) are constants that derive from the polarisation state of the waves and \(\omega_0 = c k_0\) is the angular frequency of the disturbance, \(k_0\) being the angular wavenumber. A particular unusual electromagnetic disturbance of the first kind is determined by specifying \(\tilde{k}_0\) and \(\tilde{B}_0\). For example: taking \(\tilde{k}_0 = i\) and \(\tilde{B}_0 = 0\) corresponds to each wave being linearly polarised along the polar direction and gives an ‘electric globule’; taking \(\tilde{k}_0 = 0\) and \(\tilde{B}_0 = -i\) corresponds to each wave being linearly polarised along the azimuthal direction and gives an ‘electric ring’; taking \(\tilde{k}_0 = 1/\sqrt{2}\) and \(\tilde{B}_0 = i\pi/\sqrt{2}\) with \(\pi = \pm 1\) corresponds to each wave having left- or right-handed circular polarisation and gives an ‘electromagnetic tangle’, which can be regarded as a (monochromatic) electromagnetic knot in that it has linked and torus-knotted electric and magnetic field lines.

Unusual electromagnetic disturbances of the first kind can be superposed in various ways to create more exotic structures, referred to as ‘unusual electromagnetic disturbances of the second kind’.

We consider the generation of an (approximate) electromagnetic tangle in section 4.
The electric field $\mathbf{E} = \mathbf{E}(r,t)$ and magnetic field $\mathbf{B} = \mathbf{B}(r,t)$ generated by the antenna as a whole are superpositions of the fields generated by the individual elements:

$$
\mathbf{E} = \sum_{N} \sum_{k,b,N=1}^{N} \mathbf{E}_{kn}
$$

$$
\mathbf{B} = \sum_{N} \sum_{k,b,N=1}^{N} \mathbf{B}_{kn}.
$$

The integrals in (8) and (9) can be calculated numerically.

We will now elucidate the conditions under which our antenna functions in the desired manner, specializing to vacuum ($\mu = \mu_0$ and $k = k_0$) for a direct comparison with the results presented in [22] and summarised in section 2. The antenna is designed to generate an approximation to an unusual electromagnetic disturbance internally, near the origin ($r = 0$). Let us focus, therefore, on a spherical region of radius $r_{\text{max}}$ inside the antenna ($0 < r_{\text{max}} < R$), centred on the origin ($0 \leq r \lesssim r_{\text{max}}$). Assuming that the antenna is many wavelengths in radius ($k_0R \gg 1$) and that we are in the far-field regime with respect to each of the elements ($k_0(R-r_{\text{max}}) \gg 1$), we neglect near-field terms and thus see (8)–(11) reduce to

$$
\mathbf{E} \approx \Re \left\{ \sum_{k,b,N=1}^{N} \int_{-L/2}^{L/2} \frac{i j \mu_0 \omega_0 \mathbf{A}_{kn} \mathbf{w} \cdot (\mathbf{r} - \mathbf{r}_n - \mathbf{u}_{kn} \times \omega_0 + \varphi_{\text{max}})}{4 \pi |\mathbf{r} - \mathbf{r}_n - \mathbf{u}_{kn}|} \right\}
$$

$$
\mathbf{B} \approx \Re \left\{ \sum_{k,b,N=1}^{N} \int_{-L/2}^{L/2} \frac{i j \mu_0 \omega_0 \mathbf{A}_{kn} \mathbf{w} \cdot (\mathbf{r} - \mathbf{r}_n - \mathbf{u}_{kn} \times \omega_0 + \varphi_{\text{max}})}{4 \pi c |\mathbf{r} - \mathbf{r}_n - \mathbf{u}_{kn}|} \right\}
$$

Assuming that the spherical region lies sufficiently well within the antenna ($r_{\text{max}} \ll R$) that the electromagnetic wave produced by each element appears planar, we take

$$
\frac{1}{|\mathbf{r} - \mathbf{r}_n - \mathbf{u}_{kn}|} \approx \frac{1}{R},
$$

$$
e^{i k_0 |\mathbf{r} - \mathbf{r}_n - \mathbf{u}_{kn}|} \approx e^{i k_0 (R - \mathbf{r}_n)},
$$

$$
\mathbf{u}_{kn} \times (\mathbf{r} - \mathbf{r}_n - \mathbf{u}_{kn} \times \mathbf{r}_n) \approx \mathbf{u}_{kn}
$$

and thus see (12) and (13) reduce further still to

$$
\mathbf{E} \approx \Re \left\{ \sum_{x=k,b,N=1}^{N} \frac{i j \mu_0 \omega_0 \mathbf{A}_{kn} \mathbf{w} \cdot (\mathbf{r} - \mathbf{r}_n - \mathbf{u}_{kn})}{4 \pi R} \right\}
$$

$$
\mathbf{B} \approx \Re \left\{ \sum_{x=k,b,N=1}^{N} \frac{i j \mu_0 \omega_0 \mathbf{A}_{kn} \mathbf{w} \cdot (\mathbf{r} - \mathbf{r}_n - \mathbf{u}_{kn})}{4 \pi c R} \right\}
$$

with

$$
W = \int_{-L/2}^{L/2} \mathbf{w} \cdot d\mathbf{l},
$$

where we have used $L = \lambda_0/2 \ll R$. Assuming that there are a large number of element pairs ($N \gtrsim 16 \pi r_{\text{max}}^2/\lambda_0^2$) and that these are indeed uniformly distributed, we replace the discrete summation over $N$ with a continuous integral over $4\pi r_0$ and thus see (18) and (19) reduce finally to

$$
\mathbf{E} \approx \Re \left\{ \int_{0}^{2\pi} \int_{0}^{\pi} \frac{i j \mu_0 \omega_0 \mathbf{A}_{kn} \mathbf{w} \cdot (\mathbf{r} - \mathbf{r}_n - \mathbf{u}_{kn})}{16 \pi^2} \sin \theta d\theta d\phi \right\}
$$

$$
\mathbf{B} \approx \Re \left\{ \int_{0}^{2\pi} \int_{0}^{\pi} \frac{i j \mu_0 \omega_0 \mathbf{A}_{kn} \mathbf{w} \cdot (\mathbf{r} - \mathbf{r}_n - \mathbf{u}_{kn})}{16 \pi^2 c} \sin \theta d\theta d\phi \right\}
$$

where we have introduced continuous peak electric current functions $I_{\text{max}} = I_{\text{max}}(\theta, \phi)$ as well as continuous phase constant functions $\varphi_{\text{max}} = \varphi_{\text{max}}(\theta, \phi)$ defined such that $I_{\text{max}} = I_{\text{max}}(\mathbf{r}_n)$ and $\varphi_{\text{max}} = \varphi_{\text{max}}(\mathbf{r}_n)$ ($x \in \{A, B\}, n \in \{1, \ldots, N\}$). To generate an (approximate) unusual electromagnetic disturbance of the first kind we take the $I_{\text{max}}$ and the $\varphi_{\text{max}}$ to be constants, corresponding to each $A$-type element having the same (possibly zero) peak electric current and phase constant and similarly for each $B$-type element; (21) and (22) become

$$
\mathbf{E} \approx \Re \left\{ \frac{E_0}{c} \left[ k_0 \mu_0 \mathbf{A}_{kn} \mathbf{w} \cdot (\mathbf{r} - \mathbf{r}_n - \mathbf{u}_{kn}) \right] e^{-i k_0 |\mathbf{r} - \mathbf{r}_n - \mathbf{u}_{kn}|} \right\}
$$

$$
\mathbf{B} \approx \Re \left\{ \frac{E_0}{c} \left[ k_0 c \mathbf{A}_{kn} \mathbf{w} \cdot (\mathbf{r} - \mathbf{r}_n - \mathbf{u}_{kn}) \right] e^{-i k_0 |\mathbf{r} - \mathbf{r}_n - \mathbf{u}_{kn}|} \right\}
$$
where and shown in figure of five wavelengths (on the vertices of a dodecahedron [4], half-wave dipoles (the electric field.

In this section we consider a specific embodiment of our antenna configured to generate an approximation of an electromagnetic tangle with antenna (configured to generate an approximation of an electromagnetic disturbance (‘... a continuous distribution associated with each of its constituent unusual electromagnetic disturbances of the first kind.

In summary, our antenna is a spherical array of radiating dipolar elements configured to approximate the desired electromagnetic field near its centre. This design is arguably the simplest possible, given the reciprocal-space interpretation of the unusual electromagnetic disturbances (‘... a continuous spherical superposition of plane electromagnetic waves...’).

4. A specific embodiment

In this section we consider a specific embodiment of our antenna configured to generate an approximation of an electromagnetic tangle with \( \hat{E}_0' = 1/\sqrt{2} \) and \( \hat{B}_0' = i/\sqrt{2} \) at a frequency of \( f_0 = 2.45 \text{ GHz} \). We work in vacuum and focus on the electric field.

Our embodiment consists of \( N = 40 \) elements in the form of half-wave dipoles (\( L = \lambda_0/2 = 6.12 \text{ cm} \)), arranged in pairs on the vertices of a dodecahedron [26] with circumradius equal to five wavelengths (\( R = 5\lambda_0 = 61.2 \text{ cm} \)) as indicated in table 1 and shown in figure 1. We take

\[
I_0 e^{i \omega t} = I_0 e^{i(-k_0 R - \pi/2)} \tag{27}
\]

\[
I_0 e^{i \omega t} = I_0 e^{i(-k_0 R + \pi)} \tag{28}
\]

where \( I_0 \) is a peak electric current. This corresponds to \( \hat{E}_0' \propto 1/\sqrt{2} \) and \( \hat{B}_0' \propto i/\sqrt{2} \), as can be seen by comparing (27) and (28) with (25) and (26).

Figure 1. The specific embodiment of our antenna considered in section 4. Included is a 5 × magnified view of the \( n = 1 \) element pair.

We only expect our antenna to generate a good approximation of the exact electromagnetic tangle within a spherical region of radius

\[
r_{\text{max}} = \lambda_0 \sqrt{\frac{N}{16 \pi}}
\]

\[
= 0.63 \lambda_0, \tag{29}
\]

centred on the origin (\( r = 0 \)). This conclusion can be reached by considering the surface area of the sphere to be divisible into \( N \) patches of area equal to a half wavelength squared (\( \lambda_0^2/4 \)); within the spherical region the (discrete) array of elements that comprise the antenna is essentially indistinguishable from the continuous distribution required in the exact case. Outwith the spherical region the discrete nature of the
giving the trajectory (approximate) electromagnetic tangle at time localised, as desired. The (approximate) electromagnetic tangle is reasonably well local maximum value of shown in figure 4. We note for completeness that our antenna also generates linked and torus-knotted magnetic field lines with an overall structure similar to that of the electric field lines only shifted in time by a quarter cycle, as is the case for the exact electromagnetic tangle. A selection of the latter electric field lines are plotted individually in figure 4 for \( \tau = 30\lambda_0 \), where their torus-knotted forms can be seen clearly. Note that these electric field lines co-exist and are linked with each other.

We note for completeness that our antenna also generates linked and torus-knotted magnetic field lines with an overall structure similar to that of the electric field lines only shifted in time by a quarter cycle, as is the case for the exact electromagnetic tangle. In summary, a specific embodiment of our antenna with a radius of \( R = 5\lambda_0 = 61.2 \text{ cm} \) and only \( N = 20 \) element pairs driven at a frequency of \( f_0 = 2.45 \text{ GHz} \) can yield linked and torus-knotted electric and magnetic field lines approximating those of an exact electromagnetic tangle, as desired.

5. Outlook

There is much still to be done, most notably the experimental realisation of our antenna.

We have taken our antenna to be embedded in a simple medium; a transparent, isotropic, homogeneous, linear dielectric. It might prove fruitful to consider other, more exotic media such as microwave metamaterials, for example. We thank an anonymous referee for this suggestion.

Our antenna could be used to locally excite plasma. One motivation for doing so is the possible connection hypothesised in [27] between unusual electromagnetic disturbances and the as-yet unexplained natural phenomenon of ball lightning [28]; a modern take, perhaps, on some old ideas [4, 29].

Although our focus in this paper has been on the microwave domain, we recognise that analogous ideas might be pursued in other frequency domains. For example: a monochromatic electromagnetic knot might be generated in the visible domain by placing the antenna in a dielectric medium such as microwave metamaterials, for example. We note that torus knots (including the trivial ‘knot’) qualitatively similar to those found in the exact electromagnetic tangle, as desired. We have taken our antenna to be embedded in a simple medium; a transparent, isotropic, homogeneous, linear dielectric. It might prove fruitful to consider other, more exotic media such as microwave metamaterials, for example. We thank an anonymous referee for this suggestion.

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Figure 3. Streamline plots of electric field lines inside the specific embodiment of our antenna considered in section 4. A total of 11 electric field lines have been seeded at 11 positions equally spaced along the positive x axis in the range \((0, \lambda_0/2]\). Segments of electric field lines that lie outwith a spherical region of radius \(r_{\text{max}} = 0.63\lambda_0\) centred on the origin \((r = 0)\) are coloured red; these segments are not expected to behave in the desired manner. The streamlines were calculated numerically using the fourth-order Runge–Kutta method with a step size equal to \(\lambda_0/100\).

Figure 4. Streamline plots of 5 individual torus-knotted electric field lines and a trivially ‘knotted’ electric field line inside the specific embodiment of our antenna considered in section 4. The seed position \(e(0)\) of each field line has been chosen carefully such that the line closes on itself within the integration length. The streamlines were calculated numerically using the fourth-order Runge–Kutta method with a step size equal to \(\lambda_0/100\).
superposing multiple laser beams or perhaps more elegantly by exciting the appropriate modes in an optical cavity.

Data availability statement

No new data were created or analysed in this study.

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