On the integrability of codimension 1 invariant subbundles of partially hyperbolic skew-products

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Abstract

We prove there is a class of maps $\gamma : T^2 \to S^1$ such that a conservative dynamically coherent partially hyperbolic skew-product on $T^2 \times S^1$ with fixed hyperbolic dynamics on the base and rotation by angle $\gamma$ acting on the fibers have integrable hyperbolic structure which also implies in particular that they are not contact diffeomorphisms.

1 Introduction

The goal of this work is to contribute in the study of partially hyperbolic contact diffeomorphisms. Most of the works related to contact dynamics is about Anosov contact flows, which are Anosov flows defined in contact manifolds such that its hyperbolic structure $E^s \oplus E^u$ is the contact structure of the manifold [1, 2, 10, 11].

In general, when the invariant center bundle of a partially hyperbolic diffeomorphism has dimension 1, the bundles $E^s$ and $E^u$ are not jointly integrable, that is, $E^s \oplus E^u$ is not integrable. However, nonintegrability of $E^s \oplus E^u$ does not mean that it qualifies to be a contact structure. Indeed, a contact structure is defined to be a codimension 1 subbundle of an odd-dimensional

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manifold that is as far as possible from being integrable or maximally nonintegrable. The geometric meaning of maximal nonintegrability is that there are no hypersurfaces tangent to the given subbundle, not even locally, whereas an ordinary nonintegrable subbundle may have some form of local integrability. This suggests that we can establish different types of nonintegrability.

A way to establish the different types of nonintegrability is through the Frobenius' Integrability Theorem. For codimension 1 subbundles, the Frobenius' Integrability Theorem states that such subbundles are integrable if and only if \( \alpha \wedge d\alpha = 0 \), where \( \alpha \) is local defining 1-form for the subbundle. This theorem implies that one can achieve nonintegrability by providing a subbundle such that \( (\alpha \wedge d\alpha)(p) \neq 0 \), for some \( p \in M \). On the other hand, the contact condition for \( \xi \) requires that \( \alpha \wedge (d\alpha)^n \neq 0 \) at every point in \( M \), that is, \( \alpha \wedge (d\alpha)^n \) is required to be a volume form in \( M \). When \( \dim(M) = 3 \) it becomes clear what “as far as possible from being integrable” means: \( \xi \) is a contact structure if \( \alpha \wedge d\alpha \neq 0 \) at every point in \( M \), which is the complete opposite of it being integrable.

For the intermediate case when \( \xi \) is nonintegrable but not a contact structure, Eliashberg and Thurston defined the concept of confoliations and studied stability properties of such objects [4]. Their work is done mainly in dimension 3 since all 3-dimensional manifolds admit a contact structure due to Martinet’s Theorem in [6] and they study methods to perturb (co)foliations into contact structures.

For dynamically coherent partially hyperbolic diffeomorphisms, a natural possibility for the invariant contact structure is \( E^s \oplus E^u \) and in dimension 3 this is actually the unique choice, i.e., \( E^s \oplus E^u \) is a contact structure for any dynamically coherent partially hyperbolic contact diffeomorphisms in a 3-dimensional manifold. However, there still remains the question if such diffeomorphisms exist. For instance, in the Heisenberg manifold, which is the quotient of the Heisenberg group by a discrete subgroup and can also be viewed as a \( S^1 \)-bundle of over \( T^2 \), all bundle automorphisms over Anosov automorphisms of \( T^2 \) are contact diffeomorphisms with \( E^s \oplus E^u \) being the invariant contact structures in Heisenberg manifold, see [9].

We consider conservative partially hyperbolic skew-products \( F : T^2 \times S^1 \to T^2 \times S^1 \) of the form

\[
F(x, t) = (f(x), t + \gamma(x)),
\]

where \( f \) is an Anosov diffeomorphism and \( \gamma \in C^r(T^2, S^1) \). The motivation for this choice is due to the following conjecture posed by Pujals, see details in Bonatti-Wilkinson [8], concerning the classification of partially hyperbolic diffeomorphisms:
Conjecture 1.1 (Pujals). In dimension 3 any partially hyperbolic diffeomorphism is leaf conjugate to one of the following models:

(i) An Anosov diffeomorphism of $\mathbb{T}^3$;

(ii) A partially hyperbolic skew-product over an Anosov diffeomorphism of $\mathbb{T}^3$ (in this case the skew-product is defined in either $\mathbb{T}^3$ or a 3-dimensional nilmanifold);

(iii) The time-1 map of an Anosov flow.

We show that for a certain class of maps $\gamma : \mathbb{T}^2 \to \mathbb{S}^1$ the skew-product (1) fails to be a contact diffeomorphism.

Existence of dynamically coherent partially hyperbolic contact diffeomorphisms in $\mathbb{T}^3$ is an interesting matter since the contact structure must be $E^s \oplus E^u$. In particular, if there are any contact partially hyperbolic diffeomorphisms in $\mathbb{T}^3$ then all of them must have the accessibility property due to Chow’s Theorem, see [8, Theorem 3.3].

The main result of this work follows.

Theorem A. There are no dynamically coherent partially hyperbolic contact skew-products $F : \mathbb{T}^2 \times \mathbb{S}^1 \to \mathbb{T}^2 \times \mathbb{S}^1$ of the form $F(p, t) = (f(p), t + \gamma(p))$, where $f : \mathbb{T}^2 \to \mathbb{T}^2$ is an Anosov diffeomorphism and $\gamma \in C^r(\mathbb{T}^2; \mathbb{S}^1)$ is a coboundary.

We give two proofs of Theorem A. The first proof is based on local considerations about the defining 1-form of the bundle $E^s \oplus E^u$ and the existence of a surface locally tangent to it, which is enough to guarantee that the skew-product is not a contact diffeomorphism. The second one is based on the notion of characteristic foliations and Lemma 3.1 that gives a necessary and sufficient condition for a bundle to be a contact structure in a neighborhood of a surface.

When $E^s \oplus E^u$ is at least of class $C^1$ we are actually proving that the bundle is a confoliation.

2 Basic definitions

A diffeomorphism $f : M \to M$ is said to be partially hyperbolic if there exists a $Df$-invariant splitting $TM = E^s \oplus E^c \oplus E^u$ of the tangent bundle satisfying the following conditions:

(i) $Df(E^\delta) = E^\delta$, para $\delta = s, c, u$;
(ii) there are constants $C > 0$ and $0 < \lambda < 1$ such that
\[
||Df^n v_s|| \leq C \lambda^n ||v_s|| \quad \text{e} \quad ||Df^{-n} v_u|| \leq C \lambda^n ||v_u||,
\]
para todo $p \in M$, $v_s \in E_p^s$ e $v_u \in E_p^u$;

(iii) there are real numbers $0 < \mu_1 < 1 < \mu_2$, usually dependent on the point $p$, such that
\[
||Df_p|E_p|| < \mu_1 < ||Df_p|E^\delta_p|| < \mu_2 < ||Df_p|E^\delta_{f(p)}||,
\]
where $||Df_p|E^\delta_p||$ is the norm of the linear transformation $Df_p|E^\delta_p : E^\delta_p \rightarrow E^\delta_{f(p)}$. When $\mu_1$ and $\mu_2$ are independent of $p$ we say that $f$ is absolutely partially hyperbolic.

A contact manifold $(M, \xi)$ is an odd dimensional manifold with a codimension 1 subbundle $\xi$ that is as far as possible from being integrable in the sense of Frobenius. This means that if $\alpha$ is a local defining 1-form for $\xi$, i.e., $\xi|_U = \ker(\alpha)|_U$ is some open neighborhood $U \subset M$, then $\alpha \wedge (d\alpha)^n \neq 0$ at every point. Such a bundle is called a contact structure in $M$. When a globally defined 1-form $\alpha$ satisfies the condition $\alpha \wedge (d\alpha)^n \neq 0$ at every point in $M$, we say that $\alpha$ is contact form in $M$ and $\xi = \ker(\alpha)$ is the associated contact structure.

A diffeomorphism $f : M \rightarrow M$ is a contact diffeomorphism if there exist a $C^\infty$ function $\lambda : M \rightarrow \mathbb{R}^+$ such that $f^*\alpha = \lambda \alpha$. This condition implies that $\xi$ is a $Df$-invariant subbundle.

Let $S \subset M^3$ be a surface in a 3-dimensional manifold and $\xi \subset TM$ be codimension 1 subbundle. The characteristic foliation of $\xi$ in $S$ is the foliation of $S$ generated by the field of directions $\xi \cap TS$. A field of directions can be thought as an equivalence class of vector fields modulo multiplication by $C^\infty$ strictly positive functions and a representative of this equivalence class is what we call a characteristic vector field. In general, the characteristic foliation has singular points which are precisely the point where $\xi$ is tangent to $S$.

Let $(M, \omega)$ be a symplectic manifold and $g$ a riemannian metric in $M$. An almost complex structure in $M$ is a $(1, 1)$-tensor $J : TM \rightarrow TM$ satisfying $J^2 = -I_{2n}$. The almost complex structure $J$ is said to be compatible with the symplectic and the riemannian structures of $M$ if the following conditions holds:

(i) $\omega(u, Ju) = g(u, v)$;

(ii) $g(Ju, v) = \omega(u, v)$,
(iii) $Ju = S^{-1} \mu(u),$

where $S : TM \to TM$ and $\mu : TM \to TM$ are the isomorphisms induced by $g$ and $\omega$, respectively. In this case we also say that $(\omega, g, J)$ is a compatible triple. All symplectic manifolds admits a compatible almost complex structure (see Proposition 4.1 in [7]).

Let $H : M \to \mathbb{R}$ be a $C^k$ function on a symplectic almost complex manifold $(M, \omega, g, J)$ with a compatible triple of structures. A hamiltonian vector field $X_H$ in $M$ is the vector field defined by

$$X_H = -J \text{grad}(H),$$

where grad is the gradient of the function $H$ with respect to the metric $g$.

### 3 Proof of Theorem A

#### 3.1 First proof

The main ingredient in the first proof is the following theorem.

**Theorem B.** Let $M^n$ be a compact $n$-dimensional manifold and $F : M \times \mathbb{S}^1 \to M \times \mathbb{S}^1$ be a skew-product of the form $F(p, t) = (f(x), t + \gamma(p))$, where $f$ is an Anosov $C^r$ diffeomorphism and $\gamma \in C^r(M; \mathbb{S}^1)$, $r \geq 1$. If $\gamma$ is a coboundary relative to $f$ with transfer map $\mu \in C^1(T^2, \mathbb{S}^1)$ then there exists an $F$-invariant codimension 1 subbundle $\xi \subset T(M \times \mathbb{S}^1)$ transversal to $T\mathbb{S}^1$ such that $\xi$ is tangent to the graph of $\mu$.

For a partially hyperbolic diffeomorphism on a 3-dimensional manifold the only codimension 1 invariant subbundles are $E^s \oplus E^u$, $E^s \oplus E^c$ or $E^c \oplus E^u$. Since the skew-product in Theorem A is dynamically coherent it follows that the only possibility for an invariant contact structure is $E^s \oplus E^u$.

The $DF$-invariant central direction is the direction of the fibers, i.e., $E^c = T\mathbb{S}^1$ and since $\gamma$ is a coboundary, Theorem B implies that there exist a codimension 1 subbundle $\xi$ transversal to $T\mathbb{S}^1 = E^c$ such that $\xi$ is tangent to the graph of the transfer map $\mu \in C^1(T^2, \mathbb{S}^1)$ of $\gamma$. However, the only $DF$-invariant subbundle transversal to $T\mathbb{S}^1$ is $E^s \oplus E^u$. Therefore, $E^s \oplus E^u$ is tangent to the graph of $\mu$ and Theorem A follows.

#### 3.2 Second proof

In this proof we use the following result.
Theorem C. Let $M^2$ be a compact surface without boundary, $F: M \times S^1 \to M \times S^1$ be a skew-product of the form $F(p, t) = (f(x), t + \gamma(p))$, where $f$ is an Anosov $C^r$ diffeomorphism and $\gamma \in C^r(M; S^1)$, $r \geq 1$. If $\gamma$ is a coboundary relative to $f$ with transfer map $\mu \in C^1(M; S^1)$ then there exists a $DF$-invariant codimension 1 subbundle $\xi \subset T(M \times S^1)$ transversal to $T S^1$ such that the vector field $X_\xi$ representing the characteristic foliation of $\xi$ in any hypersurface $S \subset M \times S^1$ transversal to $T S^1$ is locally hamiltonian.

Since $\gamma$ is a coboundary, Theorem C implies that there exist a codimension 1 subbundle $\xi$ such that for any surface $S$ transversal to $S^1$ the characteristic foliation can represented by a locally hamiltonian vector field $X_\xi = -J \operatorname{grad}(H)$, for some differentiable function $H: M \to S^1$.

The following lemma gives a necessary and sufficient condition for a codimension 1 subbundle to be a contact structure near an embedded surface in a 3-dimensional manifold.

Lemma 3.1. A vector field $X$ on an embedded surface $S$ in a 3-dimensional manifold represents the characteristic foliation of a contact structure if and only if the following condition holds: $\operatorname{div}_\omega(X)(p) \neq 0$ on all points singular points $p \in S$ of $X$, where $\operatorname{div}_\omega(X)$ is the divergence of the vector field $X$ with respect an area form $\omega$ in $S$.

For a proof of Lemma 3.1 see Lemma 4.6.3 in [5].

Recall that the divergence of vector field $X$ with respect to some area form $\omega$ in $S$ is uniquely defined the function $\operatorname{div}_\omega(X): S \to \mathbb{R}$ such that

$$
\mathcal{L}_X \omega = \operatorname{div}_\omega(X) \omega,
$$

where $\mathcal{L}_X \omega$ is the Lie derivative of $\omega$ with respect to $X$. By Cartan’s Formula,

$$
\mathcal{L}_X \omega = \iota_X d\omega + d(\iota_X \omega),
$$

and the fact $\omega$ is a symplectic form in $S$, we have

$$
\operatorname{div}_\omega(X) \omega = d(\iota_X \omega).
$$

Now, choosing Darboux local (symplectic) coordinates on $S$ we write $\omega = dx \wedge dy$ and

$$
X = -\frac{\partial H}{\partial y} \frac{\partial}{\partial x} + \frac{\partial H}{\partial x} \frac{\partial}{\partial y},
$$

in these coordinates. It follows that $\iota_X \omega = dH$ and since $\omega$ is nondegenerate, we conclude that $\operatorname{div}_\omega(X)(p) \omega = 0$ at any point. This implies that the $DF$-invariant codimension 1 subbundle $\xi$ of Theorem C cannot be a contact structure.

Again, since the only $DF$-invariant codimension 1 subbundle transversal to $TS^1$ is $E^s \oplus E^u$ it follows that $F$ cannot be a contact diffeomorphism and Theorem A follows.
4 Proof of Theorem B

Let $\alpha$ be a 1-form in $M \times S^1$, transversal to $S^1$ in the sense that the tangent direction $\frac{\partial}{\partial t}$ to $S^1$ is transversal to $\ker(\alpha)$, for all $(p, t) \in M \times S^1$.

**Lemma 4.1.** Let $\alpha = v dt + \beta_t$ be a 1-form in $M \times S^1$, where $\beta_t \in \Omega^1_t(M)$, and suppose that $\ker(\alpha)$ is $F$-invariant and transversal to $S^1$. Then,

$$F^* \left( \frac{\beta_t}{v_t} \right) - \phi'_t \frac{\beta_t}{v_t} = -d\phi_t,$$

where $\phi'_t$ is the derivative of $\phi_t$ with respect to $t$ and $F^*: \Omega^1_t(M \times S^1) \to \Omega^1_t(M \times S^1)$ is the pull-back transformation induced by $F$. Conversely, if (2) holds, with $v_t \neq 0$ at every point, then $\ker(\alpha)$ $DF$-invariant.

**Proof.** Consider the splitting $T(M \times S^1) \cong TM \oplus TS^1$ induced by the global trivialization of the trivial bundle $M \times S^1$. This splitting induces a module isomorphism between the $C^* (M \times S^1, \mathbb{R})$-modules $\Omega^1_t(M \times S^1)$ and $\Omega^1_t(M) \oplus \Omega^1_t(S^1)$, which is realized by taking the natural projections $\pi_1: TM \oplus TS^1 \to TM$ and $\pi_2: TM \oplus TS^1 \to TS^1$ and noting that the induced pullback maps $\pi^*_1: \Omega^1_t(M) \to \Omega^1_t(M \times S^1)$ and $\pi^*_2: \Omega^1_t(S^1) \to \Omega^1_t(M \times S^1)$ satisfies $\Omega_t^1(M \times S^1) = \pi^*_1 \Omega^1_t(M) \oplus \pi^*_2 \Omega^1_t(S^1)$. Hence any 1-form in $\Omega^1_t(M)$ or $\Omega^1_t(S^1)$ can be considered as a 1-form in $\Omega^1_t(M \times S^1)$ via this identification.

Since $\ker(\alpha)$ is $DF$-invariant there exists $\lambda: M \times S^1 \to \mathbb{R}$ such that $F^* \alpha = \lambda \alpha$. Then

$$F^* \alpha = \lambda \alpha \Rightarrow F^*(v dt + \beta_t) = \lambda (v dt + \beta_t) \Rightarrow$$

$$F^*(v dt) + F^* \beta_t = \lambda v dt + \lambda \beta_t \quad (3)$$

Note that

$$F^*(v dt) = (v_{\phi_t} \circ f) d\phi_t = (v_{\phi_t} \circ f) d\phi_t + (v_{\phi_t} \circ f) \phi'_t dt,$$

where $d\phi_t$ is the partial derivative of $\phi_t$ in the direction of $TM$ with respect to the splitting $T(M \times S^1) \cong TM \oplus TS^1$. For simplicity, we will write $v_{\phi_t}$ instead of $v_{\phi_t} \circ f$ so that $F^*(v dt) = v_{\phi_t} d\phi_t + v_{\phi_t} \phi'_t dt$. Then, from this and (3) we have

$$(v_{\phi_t} \phi'_t - \lambda v_t) dt = F^* \beta_t - \lambda \beta_t + v_{\phi_t} d\phi_t.$$

Note that the left side of this equation is a multiple of $dt$ while that right side is a linear combination of $dx$ and $dy$, since $\beta_t \in \Omega^1_t(M)$. Now, since $\{dx, dy, dt\}$ is a local basis for $\Omega^1_t(M \times S^1)$, it follows that
\[ \phi'_tv_{\phi_t} - \lambda v_t = 0 \]  

(4)

and

\[ F^*\beta_t - \lambda \beta_t + v_{\phi_t} d\phi_t = 0 \]  

(5)

by linear independency of \( dx, dy \) and \( dt \). The transversality condition of \( \ker(\alpha) \) implies that either \( v_t > 0 \) or \( v_t < 0 \) at every point, otherwise there would be point \((p, t) \in M \times S^1\) such that \( T_p S \subset \ker(\alpha)(p, t) \). Thus, \( \frac{\beta_t}{v_t} \) is defined in all of \( M \times S^1 \) and by (5) we obtain

\[ -d\phi_t = \frac{F^*\beta_t}{v_{\phi_t}} - \frac{\lambda \beta_t}{v_{\phi_t}} \]

Recall that \( F^*v_t = v_{\phi_t} \), which implies \( F^*(v_t)^{-1} = v_{\phi_t}^{-1} \). Also, by (4) we have \( \frac{\lambda}{v_{\phi_t}} = \frac{\phi'_t}{v_t} \). Then, by the previous equation we obtain

\[ -d\phi_t = F^* \left( \frac{\beta_t}{v_t} \right) - \phi'_t \frac{\beta_t}{v_t} \]

Conversely, suppose (2) holds. Then, \( F^* \left( \frac{\beta_t}{v_t} \right) = -d\phi_t + \phi'_t \frac{\beta_t}{v_t} \) and

\[ F^* \left( \frac{\alpha}{v_t} \right) = F^*(dt) + F^* \left( \frac{\beta_t}{v_t} \right) = \]

\[ = d\phi_t + \phi'_t dt + \phi'_t \frac{\beta_t}{v_t} - d\phi_t = \phi'_t dt + \phi'_t \frac{\beta_t}{v_t} \Rightarrow \]

\[ F^*(v_t^{-1}) F^*\alpha = \frac{\phi'_t}{v_t} (v_t dt + \beta_t) \Rightarrow \]

\[ (v_{\phi_t} \circ f)^{-1} F^*\alpha = \frac{\phi'_t}{v_t} (v_t dt + \beta_t) \Rightarrow \]

\[ F^*\alpha = \frac{\phi'_t v_{\phi_t} \circ f}{v_t} (v_t dt + \beta_t) = \lambda \alpha \]

where \( \lambda = v_t^{-1} \cdot \phi'_t \cdot (v_{\phi_t} \circ f) \). Therefore, \( \ker(\alpha) \) is \( F \)-invariant.

The previous lemma assumes a differential 1-form with a specific form, which at first seems to be very restrictive. The next lemma however, shows that this is not really the case.
Lemma 4.2. Let $M^n$ be a compact riemannian manifold and $S \subset M$ be a codimension 1 closed submanifold of $M$. Then, every $C^r$ differential 1-form $\alpha$ can be written as $\alpha = v_t dt + \beta_t$ in a neighborhood $V_\varepsilon$ of $S$, where $v_t \in C^r(S; \mathbb{R})$ and $\beta_t \in \Omega^1_1(S)$ are families of $C^r$ maps and families of $C^r$ differential 1-forms of $S$, respectively.

Proof. Consider a tubular neighborhood $V$ of $S$ in $M$. Since $M$ is compact we can find $\varepsilon > 0$ such that $S \times (-\varepsilon, \varepsilon)$ is embedded into $V$, where we identify $S$ with $S \times \{0\}$.

Let $\varphi : S \times (-\varepsilon, \varepsilon) \to V$ be this embedding and $\widetilde{V}_\varepsilon = \varphi(S \times (-\varepsilon, \varepsilon))$. Consider the projection $\pi : \widetilde{V}_\varepsilon \to S$ induced by the canonical projection $\pi_0 : S \times (-\varepsilon, \varepsilon) \to S$ defined by $\pi = \pi_0 \circ \varphi^{-1}$. For $\varepsilon > 0$ sufficiently small we can assume that $\widetilde{V}_\varepsilon$ belongs to a cover of $S$ by slice charts of $M$, i.e.,

$$S \subset \bigcup_{i \in I} W_i,$$

where $(W_i, x_1, \ldots, x_{n-1}, t)$ are local coordinates with the property that $S \cap W_i = \{p \in W_i; t(p) = 0\}$. Note that we can always find such an $\varepsilon$ by passing to a finite subcover of $S$ and take $\varepsilon > 0$ to be the smallest $\varepsilon_i > 0$ such that $\widetilde{V}_\varepsilon_i = \varphi((S \cap W_i) \times (-\varepsilon_i, \varepsilon_i)) \subset W_i$. Then, if $\alpha$ is a 1-form in $M$ its restriction to the local charts $\widetilde{V}_\varepsilon_i$ is written as

$$i_{\widetilde{V}_\varepsilon_i}^* \alpha = v_t dt + \sum_{j=1}^{n-1} u_j dx_j = v_t dt + \beta_t,$$

where $i_{\widetilde{V}_\varepsilon_i} : \widetilde{V}_\varepsilon_i \to M$ is the inclusion, $v_t \in C^k(S, \mathbb{R})$ and $\beta_t = \sum_{j=1}^{n-1} u_j(\cdot, t)dx_j \in \Omega^1_1(S)$ is the local representation of $\beta_t$ with coordinate functions $u_j(\cdot, t) \in C^k(\widetilde{V}_\varepsilon \cap S; \mathbb{R})$.

Proof of Theorem \[\Box\] Let $\xi$ be an $DF$-invariant codimension 1 subbundle transversal to $TS^1$ and consider a local defining 1-form $\alpha$ for $\xi$. Let $U \subset M \times S^1$ be such that $\xi|_U = \ker(i_U^* \alpha)$, for some 1-form $\alpha$, where $i_U : U \to M \times S^1$ is the inclusion.

By Lemma 4.2 there exist a sufficiently small open neighborhood $V_{\varepsilon, t_0}$ of $M \times \{t_0\}$ such that

$$i_{\widetilde{V}_{\varepsilon, t_0}}^* \alpha = v_t dt + \beta_t,$$
where \( v_t : U \cap V_{\epsilon,t_0} \to \mathbb{R} \) and \( \beta_t \in \Omega^1(M) \) are \( C^\ell \) families with parameters in \( \mathbb{S}^1 \), with \( \ell \leq r \).

Since \( \xi \) is transversal to \( T\mathbb{S}^1 \) we have either \( v_t > 0 \) or \( v_t < 0 \). Recall that
\[
\phi_t = t + \gamma \pmod{1},
\]
for some function \( \gamma \in C^k(M; \mathbb{R}) \). From the hypothesis of \( F \)-invariance of \( \xi \) and by Lemma 4.1 we have
\[
F^* \left( \frac{\beta_t}{v_t} \right) - \frac{\beta_t}{v_t} = -d\gamma,
\]
with \( v_t \) satisfying \( \lambda v_t = v_{t+\gamma} \circ f \), for some \( C^\infty \) function \( \lambda : M \times \mathbb{S}^1 \to \mathbb{R}^+ \) due to \([4]\) and the fact that \( \phi'_t = 1 \). In particular, \( d\gamma \) is coboundary relative to \( F \).

It follows that the 1-form \( \eta = dt + \tilde{\beta}_t \) is \( F \)-invariant, where \( \tilde{\beta}_t = v_t^{-1} \beta_t \), since
\[
F^* \eta = d(t + \gamma) + F^* \tilde{\beta}_t = dt + d\gamma + \tilde{\beta}_t - d\gamma = dt + \tilde{\beta}_t = \eta,
\]
as consequence of \([4]\).

Now suppose that \( \gamma \) is coboundary relative to \( f \) with transfer function \( \mu \in C^1(M; \mathbb{S}^1) \). Then
\[
d\gamma = f^*(d\mu) - d\mu = F^*(d\tilde{\mu}) - d\tilde{\mu},
\]
where \( \tilde{\mu} : M \times \mathbb{S}^1 \to \mathbb{S}^1 \) is defined as \( \tilde{\mu}(x,t) = \mu(x) \).

Therefore, \( \tilde{\alpha} = dt - d\tilde{\mu} \) is also an \( F \)-invariant 1-form since
\[
F^* \tilde{\alpha} = d(t + \gamma) - F^*(d\tilde{\mu}) = dt + F^*(d\tilde{\mu}) - d\tilde{\mu} - F^*(d\tilde{\mu}) = dt - d\tilde{\mu} = \tilde{\alpha}.
\]
We also note that \( \tilde{\alpha} \) is also transversal to \( \mathbb{S}^1 \) since \( \tilde{\alpha} \left( \frac{\partial}{\partial t} \right) = 1 \) and consequently \( \frac{\partial}{\partial t} \not\in \text{ker}(\tilde{\alpha}) \). Now \( F \) being partially hyperbolic with one dimensional central direction there is only one \( DF \)-invariant codimension 1 sub-bundle transversal to \( E^c = TM \mathbb{S}^1 \) and hence \( \xi|_U = \text{ker}(\eta|_U) = \text{ker}(i^*_\gamma \tilde{\alpha}) \), where \( U \subset U \cap V_{\epsilon,t_0} \) is an open set.

Define \( \mu_{t_0} : M \to \mathbb{S}^1 \) as
\[
\mu_{t_0}(p) = t_0 + \mu(p),
\]
for some \( t_0 \in \mathbb{S}^1 \), and \( \tilde{\mu}_{t_0} : M \times \mathbb{S}^1 \to \mathbb{S}^1 \) as \( \tilde{\mu}_{t_0}(p,t) = \mu_{t_0}(p) \). Let \( S_{\mu_t} = \text{Graph}(\mu_t) \subset M \times \mathbb{S}^1 \) be the graph of \( \mu_t \) and \( i_{S_{\mu_t}} : S_{\mu_t} \to M \times \mathbb{S}^1 \) e the inclusion. Then, for fixed \( t \), we have
\[
\tilde{i}^*_{S_{\mu_t} \cap \tilde{U}} \tilde{\alpha} = d\mu_t - d\tilde{\mu} = d\tilde{\mu} - d\tilde{\mu} = 0.
\]
Hence, \( T_{(p,t)}S_{\mu_t} = \ker(i_\tilde{U}^*\tilde{\alpha})_{(p,t)} \), for all \((p, t) \in S_{\mu_t} \cap \tilde{U}\). That is, \( \xi|_\tilde{U} = \ker(i_\tilde{U}^*\tilde{\alpha}) \) is integrable and the family \( \mathcal{F} = (S_{\mu_t})_{t \in S^1} \) is a local \( F \)-invariant codimension 1 foliation.

\[ \square \]

## 5 Proof of Theorem C

To prove Theorem C we first give a useful characterization of the characteristic vector fields near the hypersurface \( S \subset M \) where they are defined and the local defining 1-forms of the bundle \( \xi \) the characteristic vector fields originate from.

**Lemma 5.1.** Let \( S \subset M \) be a hypersurface and \( U \subset M^3 \) be an open set such that \( U \cap S \neq \emptyset \) and \( \xi|_U = \ker(\alpha) \), for some 1-form \( \alpha \). Then, a vector field \( X_\xi \in TS \) is a generator of the characteristic foliation if and only if \( i_{X_\xi}\omega = i_\xi^*\alpha \) for some area form \( \omega \) in \( S \).

**Proof.** Let \( X_\xi \in TS \) be a vector field satisfying \( i_{X_\xi}\omega = i_\xi^*\alpha \) for some area form \( \omega \) in \( S \). It suffices to show that \( X_\xi \in \ker(\alpha) \) since this implies that \( X_\xi \in TS \cap \ker(\alpha) \) and hence is a generator of the characteristic foliation. But this is true since

\[ i_\xi^*\alpha(X_\xi) = i_{X_\xi}\omega(X_\xi) = \omega(X_\xi, X_\xi) = 0. \]

Conversely, let \( X_\xi \) be a vector field generating the characteristic foliation and \( \Omega \) be an arbitrary area form in \( S \). Then, the vector field \( Y \) satifying \( i_Y\Omega = i_\xi^*\alpha \) is also a generator of the characteristic foliation and thus, there exists a positive \( C^\infty \) function \( \lambda : M \to \mathbb{R}^+ \) such that \( Y = \lambda X_\xi \).

Let \( \omega = \lambda\Omega. \) Since \( \lambda > 0 \), the 2-form \( \omega \) is also an area form \( S \) and

\[ i_{X_\xi}\omega = \lambda i_{X_\xi}\Omega = i_{\lambda X_\xi}\Omega = i_Y\Omega = i_\xi^*\alpha. \]

\[ \square \]

Let \((M^2, \omega)\) be a symplectic manifold, \( F : M \times S^1 \to M \times S^1 \) be a skew-product as in Theorem A, \( M_{t_0} = M \times \{t_0\} \) and \( \xi \subset TM \) a codimension 1 subbundle and consider the slice chart formed by local symplectic coordinates \( x, y \) in \( M \) and the global coordinate \( t \) in \( S^1 \). Then, by Lemma 12 there exists an open set \( U \subset M \times S^1 \) such that \( U \cap M_t \neq \emptyset \) \( e \xi|_U = \ker \alpha, \) with \( \alpha = v_t dt + \beta_t \in \Omega_1(M_t) = \Omega_1^H(M) \).

\[ \square \]

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Lemma 5.2. If \( X_\xi = X^1_\xi \frac{\partial}{\partial x} + X^2_\xi \frac{\partial}{\partial y} \) is the characteristic vector field of \( \xi \) in local symplectic coordinates of \( M_t \) associated to the symplectic form \( \omega \), then \( \beta_t = -X^2_\xi dx + X^1_\xi dy \).

Proof. We first observe that \( M_t \) is diffeomorphic to \( M \) through \( \pi_t : M_t \to M \) obtained by restricting the projection along the fibers \( \pi : M \times S^1 \to M \) to \( M_t \). Indeed, the projection along the fibers of the trivial bundle \( M \times S^1 \) is the natural projection on the first component and the graph of any smooth map is diffeomorphic to its domain through the natural projection. Since \( M \) is a symplectic manifold with symplectic form \( \omega \) we pull back this symplectic form to \( M_t \) with \( \pi_t \) obtaining a symplectic form \( \omega_t = \pi_t^* \omega \) on each \( M_t \).

Choosing symplectic local coordinates \( x, y \) on \( M \) and the global coordinate \( t \) on \( S^1 \) we have \( \omega = dx \wedge dy \) and \( \pi_t(x, y, t) = (x, y) \). Then, we trivially have \( \omega_t = \pi_t^* \omega = \omega \).

The symplectic form \( \omega = dx \wedge dy \) is an area form in \( M_t \) and by Lemma 5.1, the characteristic vector field of \( \xi \) in \( M_t \) satisfies \( \iota_{X_\xi} \omega = i_{\pi_t^* \alpha} M_t \).

But, fixing \( t_0 \in S^1 \), we have

\[
\iota_{M_{t_0}}^* \alpha = d(t_0) + \beta_{t_0} = \beta_{t_0},
\]

for any \( t_0 \in S^1 \). Then,

\[
(\beta_t)_p(v) = (\iota_{M_t}^* \alpha)_p(v) = (\iota_{X_\xi} \omega)_p(v) = \det \left( \begin{array}{cc} X^1_\xi & X^2_\xi \\ v_1 & v_2 \end{array} \right) = -X^2_\xi v_1 + X^1_\xi v_2,
\]

for any \( p \in M \) and \( v = v_1 \frac{\partial}{\partial x} \bigg|_p + v_2 \frac{\partial}{\partial y} \bigg|_p \in T_p M \). Therefore \( \beta_t = -X^2_\xi dx + X^1_\xi dy \).

Thus, in dimension 3 we have that every subbundle \( \xi \subset T(M \times S^1) \) transversal to \( S^1 \) can be locally written as the kernel of a 1-form given by

\[
\alpha = v_1 dt - X_2 dx + X_1 dy,
\]

where \( X_1 \) and \( X_2 \) are the components of the characteristic vector field of \( \xi \) in \( M_t \). If \( v_t \neq 0 \) at every point, then we can rescale the characteristic vector field \( X_\xi \) by multiplying it by \( v_t \), thus obtaining \( \alpha = v_t \tilde{\alpha} \), where

\[
\tilde{\alpha} = dt - X_2 dx + X_1 dy
\]

and \( \xi = \ker(\alpha) = \ker(\tilde{\alpha}) \). In this case, we can work \((8)\) instead of \((7)\).
Proof of Theorem C. Let $S \subset M \times S^1$ be a surface transversal to $S^1$. By the Implicit Function Theorem there exist an open set $U \subset M$ such that $S \cap \pi^{-1}(U)$ is the graph of a map $h : U \to S^1$ and since $\dim(S) = \dim(M)$, it follows that the restriction of the bundle projection $\pi : M \times S^1 \to M$ to $S$ is a local diffeomorphism.

Shrinking $U$ if necessary, let $(U, x, y)$ be a local symplectic chart in $M$ and define local symplectic coordinates and area form in $S \cap \pi^{-1}(U)$ as the ones induced by the pullback of the coordinates $x, y$ and area form $\omega = dx \wedge dy$ by the bundle projection $\pi$. We shall use the same notation for both coordinates and area forms in $S$ and $M$. By a simple calculation, we see that $\pi^*\omega = \omega$.

Since $\gamma$ is a coboundary with transfer map $\mu$, we know by a part of the proof of Theorem B that $\xi$ is locally the kernel of the 1-form $\alpha = dt - d\mu$. From the defining equation $\iota_{X_\xi}\omega = \iota_{X_\xi}^*\alpha$ of a vector field representing the characteristic foliation of $\xi$ in $S$, we have

$$-X_2 dx + X_1 dy = \iota_{X_\xi} = \iota_{X_\xi}^*\alpha = \left(\frac{\partial h}{\partial x} - \frac{\partial \mu}{\partial x}\right) dx + \left(\frac{\partial h}{\partial y} - \frac{\partial \mu}{\partial y}\right) dy.$$ 

and so $X_1 = -\frac{\partial (h - \mu)}{\partial y}$ and $X_2 = \frac{\partial (h - \mu)}{\partial x}$.

Let $g$ be the riemannian metric in $S$ compatible with the symplectic form $\omega$, $J$ be the almost complex structure in $S$ and $\text{grad}_0(h - \mu) = \frac{\partial (h - \mu)}{\partial x} + \frac{\partial (h - \mu)}{\partial y} \frac{\partial}{\partial y}$ in the local symplectic coordinates $x, y$, where $\text{grad}_0(h - \mu)$ is the jacobian matrix of $h - \mu$. Then, recalling that $\text{grad}_0(h) = G \text{grad}(h)$, we have

$$X_\xi|_{S \cap \pi^{-1}(U)} = -J_0 \text{grad}_0(h - \mu) = J_0^{-1} G \text{grad}(h - \mu) = J^{-1} \text{grad}(h - \mu) = -J \text{grad}(h - \mu),$$

where $J_0$ and $G$ are the tensors associated to $\omega$ and $g$, respectively, and we used the fact that if $(\omega, g, J)$ is a compatible triple in $M$ then $G^{-1} J_0 = J$.

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