C*-ALGEBRAS GENERATED BY PARTIAL ISOMETRIES

ILWOO CHO AND PALLE JORGENSEN

Abstract. We prove a structure theorem for a finite set \( G \) of partial isometries in a fixed countably infinite dimensional complex Hilbert space \( H \). Our result is stated in terms of the C*-algebra generated by \( G \). The result is new even in the case of a single partial isometry which is not an isometry or a co-isometry; and in this case, it extends the Wold decomposition for isometries. We give applications to groupoid C*-algebras generated by graph groupoids, and to partial isometries which have finite defect indices and which parametrize the extensions of a fixed Hermitian symmetric operator with dense domain on the Hilbert space \( H \). Our classification parameters for the “Wold decomposed” set \( G_W \) of our finite set \( G \) of partial isometries involve infinite and explicit Cartesian product sets, and they are computationally attractive. Moreover, our classification labels generalize the notion of defect indices in the special case of the family \( G_W \) of partial isometries from the Cayley transform theory and Hermitian extensions of unbounded Hermitian operators with dense domain.

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1. Introduction

In this paper, all Hilbert spaces are separable infinite dimensional. We will consider $C^*$-subalgebras of $B(H)$ generated by finitely many partial isometries on $H$, where $B(H)$ is the operator algebra consisting of all (bounded linear) operators. First, we characterize the $C^*$-subalgebras generated by a single partial isometry characterized by a $*$-isomorphism. We show that they are characterized by a $*$-isomorphism index in $(\mathbb{N}_0^\infty)^4$, where $\mathbb{N}_0^\infty = \mathbb{N} \cup \{0, \infty\}$, up to finite numbers. This shows that the $C^*$-subalgebras of $B(H)$ generated by a single partial isometry are characterized by quadruples of numbers in $\mathbb{N}_0^\infty$. More generally, we consider $C^*$-subalgebras of $B(H)$ generated by finitely many partial isometries. We prove that they are determined by certain combinatorial objects, called the conditional iterated glued graphs and their corresponding graph groupoids. By observing the block structures of the graph groupoids, we can have the block structures of a $C^*$-algebra generated by finitely many partial isometries in $B(H)$.

Let $H$ be a Hilbert space and let $a \in B(H)$ be an operator, where $B(H)$ is the operator algebra consisting of all bounded linear operators on $H$. We say that this operator $a$ is a partial isometry if the operator $a^*a$ is a projection on $H$, where $a^*$ is the adjoint of $a$. Recall that an operator $p \in B(H)$ is a projection if $p$ satisfies that $p^2 = p = p^*$. We study $C^*$-subalgebras generated by finitely many partial isometries in $B(H)$. We will extend the classification results of [13].

Directed graphs, both finite and infinite, play a role in a myriad of areas of applications; electrical network of resistors, the internet, and random walk in probability theory, to mention only a few. Some of the results in the subject stress combinatorial aspects of the graphs, while others have a more analytic slant. Applications to quantum theory fall in the latter group, and that is where Hilbert space and non-commuting operators play a role. To see this, consider a given graph $G$, and some matrix operation which transfers data encoded in one site (or vertex) of $G$ onto another. If there is a notion of energy, or some other conserved quantity for the entire graph, then there is also a naturally associated Hilbert space, and with the energy now representing the norm-squared. Under these conditions, the matrix transfer between vertices will take the form of partial isometries: As a result, we will then be assigning partial isometries to the directed edges in $G$. In this paper we will derive some of the $C^*$-algebraic invariants associated with such an approach. Hence, the analytic approach asks for representations of the directed graphs, but each application motivates different representations.
Before turning to our main theorem, we recall some basic definitions and results in the literature which are natural precursors to our present model. A partial isometry $a$ in a Hilbert space $H$ is a linear mapping $a : H \to H$, which restricts to an isometry between two closed subspaces in $H$, the initial space $H^a_{\text{init}}$ and the final space $H^a_{\text{fin}}$. In addition, it is assumed that $a$ maps the orthogonal complement of $H^a_{\text{init}}$ into $\{0\}$. The dimensions of the ortho-complements of the two spaces are called the defect indices of $a$. If the first defect index is 0, we say that $a$ is an isometry, and if the second is 0, a co-isometry. While there are good classification theorems for isometries, e.g., [27]; the case of partial isometries when both the defect indices are non-zero have so far resisted classification. See however [28]. From the pioneering work of Akhiezer and Glazman [26], we know that the Cayley transform is a partial isometry with finite defect indices. In Section 3.1, we generalize such indices.

The simplest and most primitive case, of course would be asking for a universal $C^*$-algebra which is generated by a single partial isometry. Using a certain grading on this algebra, the author of [33] constructs an associated analog of the Cuntz algebras [31]. Recall that the Cuntz algebra $O_d$ is the $C^*$-algebra on $d$-isometries on $H$ which have mutually orthogonal final spaces adding up to $H$.

A popular invariant is the $KK$-group, and with his universal algebra, Kandelaki gives a homotopical interpretation of $KK$-groups (See [33] and [37]). He proves that his universal $C^*$-algebra is homotopically equivalent to $M_2(\mathbb{C})$ up to stabilization by $(2 \times 2)$-matrices. Therefore, these Kandelaki algebras are $KK$-isomorphic.

This paper provides not only a characterization of $C^*$-subalgebras of $B(H)$ generated by a single partial isometry (See Section 3.1) but also a topological reduced free block structures for the analysis of $C^*$-subalgebras of $B(H)$ generated by finitely many partial isometries on $H$. In particular, if $a$ is a partial isometry, then we have the so-called Wold decomposition $u + s$ of $a$, where $u$ is the unitary part of $a$ and $s$ is the shift part of $a$, and hence the $C^*$-subalgebra $C^*(\{u\})$ of $B(H)$ is the $C^*$-algebra $C^*(\{u, s\})$, which is $*$-isomorphic to $C^*(\{u\}) \oplus C^*(\{s\})$ inside $B(H)$, generated by $u$ and $s$. We realize that $C^*(\{u\})$ and $C^*(\{s\})$ are characterized by certain graph groupoids $\mathcal{G}_u$ and $\mathcal{G}_s$, induced by the corresponding graphs $G_u$ and $G_s$ of $u$ and $s$, respectively.

In conclusion, we show that if $\mathcal{G} = \{a_1, ..., a_N\}$ is a family of partial isometries in $B(H)$, and $\mathcal{G}_W = \{x_1, ..., x_n\}$, the corresponding Wold decomposed family of $\mathcal{G}$, and if $\mathcal{G}_G = \{G_{x_1}, ..., G_{x_n}\}$ is the family of corresponding graphs of $\mathcal{G}_W$, then the $C^*$-subalgebra $C^*(\mathcal{G})$ generated by $\mathcal{G}$ is $*$-isomorphic to the groupoid algebra $\mathcal{A} = C^*_\alpha(\mathcal{G})$, as embedded $C^*$-subalgebras of $B(H)$. Furthermore, the $C^*$-algebra $\mathcal{A}$ has its building blocks determined by the topological reduced free product.

Our motivations come from both pure and applied mathematics, including mathematical physics.
On the applied side, we notice that the internet offers graphs of very large size, hence to an approximation, infinite. This is a context where algebraic models have been useful (e.g., [55] and the references cited in it).

Features associated with infinite models are detected especially nicely with the geometric tools from operators on a Hilbert space. A case in point is the kind of Transfer Operator Theory or Spectral Theory which goes into the mathematics of internet search engines (e.g., [55]). A second instance is the use of graph models in the study of spin models in Quantum Statistical Mechanics (e.g., [30], [32], [49] and [50]). In both of these classes of “infinite” models mentioned above, what happens is there is an essential distinction between conclusions from the use of infinite dimensional models, as compared to the finite dimensional counterparts.

And there is the same distinction between the mathematics of a finite number of degrees of freedom vs. the statistics of infinite models: For a finite number of degrees of freedom, we have the Stone-von Neumann uniqueness theorem, but not so in the infinite case. Nonetheless, isomorphism classes of $C^*$-algebras are essential for the study of fields and statistics in Physics. In fact, these math physics models lie at the foundations of Operator Algebra Theory.

In Statistical Mechanics, we know that phase-transition may happen in certain infinite models, but of course is excluded for the corresponding finite models.

On the pure side, there are several as well: First, in the literature, there is a variety of classification theorems for $C^*$-algebras generated by systems of isometries, starting with the case of a single isometry; Coburn’s Theorem (See [54]). Second, in Free Probability (e.g., [16] and [18]), $C^*$-algebras built on generators and relations are the context for “con-commutative” random variables.

2. Preliminaries

In this paper, we will consider a $C^*$-algebra $C^*(\mathcal{G})$ generated by a family $\mathcal{G}$ of finitely many partial isometries in $B(H)$. We first introduce the main objects we will need throughout the paper.

2.1. Partial Isometries on a Hilbert Space.

We say that an operator $a \in B(H)$ is a partial isometry, if the operators $a^*a \in B(H)$ is a projection. The various equivalent characterizations of partial isometries are well-known: the operator $a$ is a partial isometry if and only if $a = aa^*a$, if and only if its adjoint $a^*$ is a partial isometry, too, in $B(H)$. Recall that an operator $p$
in $B(H)$ is a projection if $p$ is a self-adjoint idempotent, i.e., $p^2 = p = p^*$ in $B(H)$. Every partial isometry $a$ has its initial space $H^a_{\text{init}}$ and its final space $H^a_{\text{fin}}$ which are closed subspaces of $H$. Notice that a partial isometry $a$ is a unitary from $H^a_{\text{init}}$ onto $H^a_{\text{fin}}$, i.e., $a \in B(H^a_{\text{init}}, H^a_{\text{fin}})$ is a unitary satisfying that $a^*a = 1_{H^a_{\text{init}}}$ and $aa^* = 1_{H^a_{\text{fin}}}$, where $1_K$ means the identity operator on a Hilbert space $K$. Therefore, as Hilbert spaces, $H^a_{\text{init}}$ and $H^a_{\text{fin}}$ are Hilbert-space isomorphic.

2.2. Directed Graphs and Graph Groupoids.

Countable directed graphs and their applications have been studied extensively in Pure and Applied Mathematics (e.g., [4], [5], [6], [9], [10], [17], [19], [24], [35] and [36]). Not only are they connected with certain noncommutative structures but they also let us visualize such structures. Furthermore, the visualization has a nice matricial expressions.

A graph is a set of objects called vertices (or points or nodes) connected by links called edges (or lines). In a directed graph, the two directions are counted as being distinct directed edges (or arcs). A graph is depicted in a diagrammatic form as a set of dots (for vertices), jointed by curves (for edges). Similarly, a directed graph is depicted in a diagrammatic form as a set of dots jointed by arrowed curves, where the arrows point the direction of the directed edges.

Our global results and theorems have applications to Operator Theory as well as to a class of finite and infinite electrical networks used in Analysis on fractals (e.g., [53]), and Physics (e.g., [49] and [50]). We have given our results an axiomatic formulation, making use here of a general Operator Theoretic framework. The operator-theoretic approach is new, especially in its general scope. When stated in the context of Hilbert-space operators, we believe that our results are also of independent interest; quite apart from their other applications. Our results have a variety of applications, e.g., to fractals with affine selfsimilarity (See [53]), to models in Quantum Statistical Mechanics (See [51]), to the analysis of energy forms, interplaying between selfsimilar measures and associated energy forms; and to random walks on graphs (See [53]).

By a graph $G$, we will mean a system of vertices $V(G)$ and edges $E(G)$ (See the previous axioms below). Intuitively, the points in $E(G)$ will be “lines” connecting prescribed pairs of points in $V(G)$. Each vertex $x$ will have a finite set $g(x)$ of edges connecting it to other points in $V(G)$. (For models in Physics, each set $g(x)$ will correspond to a set of nearest neighbors for the vertex $x$, with the vertices typically arranged in a lattice (See [49], [50] and [51]).)

But our present graphs will be more general. Starting with $G$, we specify a function $g$ from $V(G)$ into the set of all finite subsets of $V(G)$, subject to certain axioms; see below for details.
An important difference between those graphs and those that are more traditional in Geometric Analysis is that we do not “a priori” fix an orientation. Nonetheless, for electrical network models, we will be using orientations, but they will be introduced only indirectly, via a prescribed function $R$, positive, real valued and defined on $E(G)$, i.e., on the set of edges on $G$. (The letter $R$ is for resistance.) By analogy to models in Electrical Engineering, we think of such a function $R$ as a system of resistors, and we will be interested in the pair $(G, R)$. Given a system $R$ of resistors, then potential theoretic problems for $(G, R)$ will induce currents in the global system, the associated current functions will be denoted by $I$, and each admissible (subject to Kirchhoff’s laws) current function induces an orientation.

This is a second class of functions $I$ ($I$, for electrical current) again functions from $E(G)$ into the real numbers. It is only when such a function $I$ is prescribed, that our graph $G$ will acquire an orientation: Specially we say that an edge $e$ is positively oriented (relative to $I$), if $I(e) > 0$. And a different choice of current function $I$ will induce a different orientation on the graph $G$.

Our results will be stated in terms of precise mathematical axioms, and we present our theorems in the framework of Operator Theory / Operator Algebra, which happens to have an element of motivation from the theory of infinite systems of resistors. The main tools in our proofs will be operators on a Hilbert space. In particular, we will solve problems in Discrete Potential Theory with the use of a family of operators and their adjoints. Good references to the fundamentals for operators are [44] and [48]. The second author of this paper recalls conversations with Raul Bott with a remainder about an analogous Hilbert-space Operator Theoretic approach to Electrical Networks; apparently attempted in the 1950’s in the engineering literature, but we did not find details in journals. The closest we could come is the fascinating paper [45], by Bott et al.

A motivation for our present study is a series of papers in the 1970’s by Powers which introduced infinite systems of resistors into the resolution of an important question from Quantum Statistical Mechanics (e.g., [49], [50], [51] and [52]). This is coupled with the emergence of these same techniques, or their close vintage. By “Analysis on Fractals” we refer to a set of Potential Theoretic tools developed recently by Kigami, Strichartz and others (e.g., [44] and [46]).

In this paper, we observe how an algebraic structure, a graph groupoid induced by a finite directed graph, acts on a Hilbert space $H$. In [9] and [10], we showed that every graph groupoid maybe realized with a system of Hilbert space operators on a so-called graph Hilbert space.

Let $G$ be a countable directed graph with its vertex set $V(G)$ and its edge set $E(G)$. Denote the set of all finite paths of $G$ by $FP(G)$. Clearly, the edge set $E(G)$ is contained in $FP(G)$. Let $w$ be a finite path in $FP(G)$. Then it is represented as a word in $E(G)$. If $e_1, \ldots, e_n$ are connected directed edges in the order $e_1 \rightarrow e_2 \rightarrow \cdots \rightarrow e_n$, for $n \in \mathbb{N}$, then we can express $w$ by $e_1 \ldots e_n$ in $FP(G)$. If there exists a finite path $w = e_1 \ldots e_n$ in $FP(G)$, where $n \in \mathbb{N} \setminus \{1\}$, we say that the directed edges $e_1, \ldots, e_n$ are admissible. The length $|w|$ of $w$ is defined to be $n$, which is the
cardinality of the admissible edges generating \( w \). Also, we say that finite paths \( w_1 = e_{11} \ldots e_{1k_1} \) and \( w_2 = e_{21} \ldots e_{2k_2} \) are admissible, if \( w_1 w_2 = e_{11} \ldots e_{1k_1} e_{21} \ldots e_{2k_2} \) is again in \( FP(G) \), where \( e_{11}, \ldots, e_{1k_1}, e_{21}, \ldots, e_{2k_2} \in E(G) \). Otherwise, we say that \( w_1 \) and \( w_2 \) are not admissible. Suppose that \( w \) is a finite path in \( FP(G) \) with its initial vertex \( v_1 \) and its terminal vertex \( v_2 \). Then we write \( w = v_1 w \) or \( w = w v_2 \) or \( w = v_1 w v_2 \), for emphasizing the initial vertex of \( w \), respectively the terminal vertex of \( w \), respectively both the initial vertex and the terminal vertex of \( w \). Suppose \( w = v_1 w v_2 \) in \( FP(G) \) with \( v_1, v_2 \in V(G) \). Then we also say that \([v_1 \ ) and \( w \) are admissible] and \([w \ ) and \( v_2 \) are admissible]. Notice that even though the elements \( w_1 \) and \( w_2 \) in \( V(G) \cup FP(G) \) are admissible, \( w_2 \) and \( w_1 \) are not admissible, in general. For instance, if \( e_1 = v_1 e_1 v_2 \) is an edge with \( v_1, v_2 \in V(G) \) and \( e_2 = v_2 e_2 v_1 \) is an edge with \( v_3 \in V(G) \) such that \( v_3 \neq v_1 \), then there is a finite path \( e_1 e_2 \) in \( FP(G) \), but there is no finite path \( e_2 e_1 \).

The free semigroupoid \( F^+(G) \) of \( G \) is defined by a set

\[
F^+(G) = \{\emptyset\} \cup V(G) \cup FP(G),
\]

with its binary operation \((\cdot)\) on \( F^+(G) \), defined by

\[
(w_1, w_2) \mapsto w_1 \cdot w_2 = \begin{cases} 
  w_1 & \text{if } w_1 = w_2 \in V(G) \\
  w_1 & \text{if } w_1 \in FP(G), w_2 \in V(G) \text{ and } w_1 = w_1 w_2 \\
  w_2 & \text{if } w_1 \in V(G), w_2 \in FP(G) \text{ and } w_2 = w_1 w_2 \\
  w_1 w_2 & \text{if } w_1, w_2 \in FP(G) \text{ and } w_1 w_2 \in FP(G) \\
  \emptyset & \text{otherwise,}
\end{cases}
\]

where \( \emptyset \) is the empty word in \( V(G) \cup E(G) \). (Sometimes, the free semigroupoid \( F^+(G) \) of a certain graph \( G \) does not contain the empty word \( \emptyset \). For instance, the free semigroupoid of the one-vertex–multi-loop-edge graph does not have the empty word. But, in general, the empty word \( \emptyset \) is contained in the free semigroupoid, whenever \(|V(G)| > 1 \). So, if there is no confusion, then we usually assume that the empty word is contained in free semigroupoids.) This binary operation \((\cdot)\) on \( F^+(G) \) is called the admissibility. i.e., the algebraic structure \((F^+(G), \cdot)\) is the free semigroupoid of \( G \). For convenience, we denote \((F^+(G), \cdot)\) simply by \( F^+(G) \).

For the given countable directed graph \( G \), we can define a new countable directed graph \( G^{-1} \) which is the opposite directed graph of \( G \), with

\[
V(G^{-1}) = V(G) \text{ and } E(G^{-1}) = \{e^{-1} : e \in E(G)\},
\]

where \( e^{-1} \in E(G^{-1}) \) is the opposite directed edge of \( e \in E(G) \), called the shadow of \( e \in E(G) \). i.e., if \( e = v_1 e_1 v_2 \) in \( E(G) \) with \( v_1, v_2 \in V(G) \), then \( e^{-1} = v_2 e^{-1} v_1 \) in \( E(G^{-1}) \) with \( v_2, v_1 \in V(G^{-1}) = V(G) \). This new directed graph \( G^{-1} \) is said to be the shadow of \( G \). It is trivial that \((G^{-1})^{-1} = G \). This relation shows that the admissibility on the shadow \( G^{-1} \) is oppositely preserved by that on \( G \).
A new countable directed graph $\hat{G}$ is called the shadowed graph of $G$ if it is a directed graph with

$$V(\hat{G}) = V(G) = V(G^{-1})$$

and

$$E(\hat{G}) = E(G) \cup E(G^{-1}).$$

**Definition 2.1.** Let $G$ be a countable directed graph and $\hat{G}$, the shadowed graph of $G$, and let $\mathbb{F}^+(\hat{G})$ be the free semigroupoid of $\hat{G}$. Define the reduction (RR) on $\mathbb{F}^+(\hat{G})$ by

$$(RR) \quad ww^{-1} = v \quad \text{and} \quad w^{-1}w = v',$$

whenever $w = vuv'$ in $FP(\hat{G})$, with $v, v' \in V(\hat{G})$. The set $\mathbb{F}^+(\hat{G})$ with this reduction (RR) is denoted by $\mathbb{F}^+_r(\hat{G})$. And this set $\mathbb{F}^+_r(\hat{G})$ with the inherited admissibility $(\cdot)$ from $\mathbb{F}^+(\hat{G})$ is called the graph groupoid of $G$. Denote $(\mathbb{F}^+_r(\hat{G}), (\cdot))$ of $G$ by $G$. Define the reduced finite path set $FP_r(\hat{G})$ of $G$ by

$$FP_r(\hat{G}) \overset{\text{def}}{=} G \setminus (V(\hat{G}) \cup \{\emptyset\}).$$

All elements of $FP_r(\hat{G})$ are said to be reduced finite paths of $\hat{G}$.

Let $w_1$ and $w_2$ be reduced finite paths in $FP_r(\hat{G}) \subset G$. We will use the same notation $w_1 w_2$ for the (reduced) product of $w_1$ and $w_2$ in $G$. But we have to keep in mind that the product $w_1 w_2$ in the graph groupoid $G$ is different from the (non-reduced) product $w_1 w_2$ in the free semigroupoid $\mathbb{F}^+(\hat{G})$. Suppose $e_1$ and $e_2$ are edges in $E(\hat{G})$ and assume that they are admissible, and hence $e_1 e_2$ is a finite path in $FP(\hat{G})$. Then the product $e_1 e_2 e_2^{-1}$ of $e_1$, $e_2$, and $e_2^{-1}$ is a length-3 finite path in $FP(\hat{G}) \subset \mathbb{F}^+(\hat{G})$, but the product $e_1 e_2 e_2^{-1}$ of them is $e_1 (e_2 e_2^{-1}) = e_1$, which is a length-1 reduced finite path, in $FP_r(\hat{G}) \subset G$.

### 2.3. $C^*$-Algebras Induced by Graphs.

In [13], we observed the $C^*$-algebras generated by certain family of partial isometries on a fixed Hilbert space $H$. Suppose a family $\mathcal{G} = \{a_1, ..., a_N\}$ in $B(H)$ is a collection of finite number of partial isometries, where $N \in \mathbb{N}$. We say that such family $\mathcal{G}$ construct a finite directed graph $G$ if there exists a finite directed graph $\hat{G}$ such that

(i) $|E(G)| = |\mathcal{G}|$, and $|V(G)| = |\mathcal{G}_{\text{pro}}|$, where
\[ \mathcal{G}_{pro} = \{ a^* : a \in \mathcal{G} \} \cup \{ a a^* : a \in \mathcal{G} \}, \]

(ii) the edges \( e_1 \) and \( e_2 \) create a nonempty finite path \( e_1 e_2 \) on \( G \) if and only if \( H_{init}^{e_1} = H_{fin}^{e_2} \), where “=” means “being identically same in \( H \).”

In this case, the family \( \mathcal{G} \) is called a \( G \)-family of partial isometries in \( B(H) \). The family \( \hat{\mathcal{G}} = \mathcal{G} \cup \mathcal{G}^* \) generates a \( C^* \)-subalgebra \( C^*(\hat{\mathcal{G}}) \) of \( B(H) \), where \( \mathcal{G}^* = \{ a_j^* : j = 1, ..., N \} \).

Let \( \mathcal{G} \) be the graph groupoid of a finite directed graph \( G \), with \( |V(\hat{G})| = n \) and \( |E(\hat{G})| = 2N \) (i.e., \( |E(G)| = N \)). We give a indexing on \( V(G) \) by \( \{1, ..., n\} \). i.e., we will let \( V(G) = \{ v_1, ..., v_n \} \). By doing that, we can also index the elements \( E(G) \) and \( E(\hat{G}) \) as follows:

\[
E(G) = \left\{ e_{m;ij} \mid m = 1, ..., k_{ij}, k_{ij} \neq 0 \right\}
\]

and

\[
E(\hat{G}) = \left\{ x_{m;ij} \mid x_{m;ij} = e_{m;ij} \text{ if } x_{m;ij} \in E(G), \right. \\
\left. x_{m;ij} = e_{m;ij}^{-1} \text{ if } x_{m;ij} \in E(G^{-1}) \right\},
\]

where \( k_{ij} \) means the cardinality of edges connecting the vertex \( v_i \) to \( v_j \). By the finiteness of \( G \), \( k_{ij} < \infty \), whenever \( k_{ij} \neq 0 \). Clearly, “\( k_{ij} = 0 \)” means that “there is no edge connecting \( v_i \) to \( v_j \).” In \([13]\), we showed that such indexing process on \( V(G) \) \cup \( E(\hat{G}) \) is uniquely determined up to graph-isomorphisms. This means that if we fix an indexing, then this indexing contains the full (admissibility) combinatorial data of \( G \).

We will say that a finite directed graph \( G \) is connected, if its graph groupoid \( \mathcal{G} \) satisfies that: for any \( (v_1, v_2) \in V(G) \times V(G) \), where \( v_1 \neq v_2 \), there exists an element \( w \in \mathcal{G} \) such that \( w = v_1 w v_2 \) and \( w^{-1} = v_2 w^{-1} v_1 \). Otherwise, we say that the graph \( G \) is not connected or is disconnected.

Under this setting, if \( G \) is a connected finite directed graph, then the graph groupoid \( \mathcal{G} \) has its matricial representation \( (\mathcal{H}_G, \pi) \), where \( \mathcal{H}_G \) is a Hilbert space defined by \( \bigoplus_{n=1}^{p} (\mathbb{C} \xi_{v_n}) \), which is Hilbert-space isomorphic to \( \mathbb{C}^{\oplus n} \), and where \( \pi : \mathcal{G} \rightarrow B(\mathcal{H}_G) \) is a groupoid action satisfying that

\[ \pi(v_j) = P_j \]

and

\[ \pi(x_{m;ij}) = \begin{cases} E_{m;ij} & \text{if } x_{m;ij} = e_{m;ij} \\
E_{m;ji}^* & \text{if } x_{m;ij} = e_{m;ji}^{-1} \end{cases} \]
where $P_j$ is the diagonal matrix in $M_n(\mathbb{C})$, having its only nonzero $(j, j)$-entry 1,

\[
\begin{pmatrix}
0 & 0 \\
\ddots & \ddots \\
0 & 0 & 1 & 0 \\
\end{pmatrix}
\]

and $E_{m;j}$ is the matrix in $M_n(\mathbb{C})$, having its only nonzero $(i, j)$-entry $\omega^m$, where $\omega$ is the root of unity of the polynomial $z^{k_{ij}}$, whenever $i \neq j$,

\[
\begin{pmatrix}
0 & 0 \\
0 & 0 \\
\omega^m & \ddots \\
0 & \ddots & 0 \\
\end{pmatrix}
\]

or it is the diagonal matrix in $M_n(\mathbb{C})$, having its only nonzero $(j, j)$-entry $e^{i\theta_{m;j}}$, where $\theta_{m;j} \in \mathbb{R} \setminus \{0\}$, satisfying that $\theta_{m_1;j} \neq \theta_{m_2;j}$, whenever $m_1 \neq m_2$.

\[
\begin{pmatrix}
0 & 0 \\
\ddots & \ddots \\
e^{i\theta_{j;j}} & \ddots \\
0 & 0 & 0 \\
\end{pmatrix}
\]

Then we can see that the graph groupoid $G$ generates the $C^*$- algebra $M_G$ which is a $C^*$-subalgebra of $B(\mathcal{H}_G)$ $\ast$-isomorphic $M_n(\mathbb{C})$. This $C^*$-subalgebra $M_G$ induced by $G$ in $M_n(\mathbb{C})$ is called the matricial graph $C^*$-algebra of a connected finite directed graph $G$.

If the family $G$ of partial isometries in $B(H)$ constructs a connected finite directed graph $G$, then, since $G$ generates a groupoid $G_G$, under the operator multiplication on $B(H)$, and it is groupoid-isomorphic to a certain graph groupoid $G$, we can check that the $C^*$-subalgebra $C^*(G)$ of $B(H)$ is $\ast$-isomorphic to the affiliated matricial graph $C^*$- algebra

\[
M_G(H_0) \overset{def}{=} (G \cdot 1_{H_0}) \otimes \mathcal{C} M_G \subseteq M_n(B(H_0)),
\]
for certain Hilbert space $H_0$ which is embedded in $H$, where

$$M_n(B(H_0)) = \left\{ \begin{pmatrix} T_{11} & \cdots & T_{1n} \\ \vdots & \ddots & \vdots \\ T_{n1} & \cdots & T_{nn} \end{pmatrix} \middle| T_{ij} \in B(H_0) \right\}.$$

We conclude this section with two results which will be extended later.

**Theorem 2.1.** (See [13]) Let $\mathcal{G}$ be a finite family of partial isometries in $B(H)$, and assume that $\mathcal{G}$ constructs a connected finite directed graph $G$. Then the $C^*$-subalgebra $C^*(\mathcal{G})$ of $B(H)$ is $*$-isomorphic to the affiliated matricial algebra $M_G(H_0)$, where $H_0$ is a Hilbert space which is Hilbert-space isomorphic to $pH$, for all $p \in \mathcal{G}_{\text{pro}}$. □

**Corollary 2.2.** (Also See [13]) Let $\mathcal{G}$ be a finite family of partial isometries in $B(H)$, and assume that $\mathcal{G}$ constructs a finite directed graph $G$, having its connected components $G_1, \ldots, G_t$, for $t \in \mathbb{N}$. Then the $C^*$-subalgebra $C^*(\mathcal{G})$ of $B(H)$ is $*$-isomorphic to the direct product algebra $\bigoplus_{j=1}^t M_G(H_j)$ of the affiliated matricial graph $C^*$-algebras $M_G(H_j) \overset{\text{def}}{=} (\mathbb{C} : 1_{H_j}) \otimes_{\mathbb{C}} M_G$, where $M_G$’s are the matricial $C^*$-algebras induced by $G_j$, for $j = 1, \ldots, N$. □

The above theorem and corollary are the main motivation of this paper. They provide a certain connection between partial isometries and graph groupoids. We will extend the above results. We realize that, in general, we have much more complicated structures than the above special cases.

### 2.4. Groupoids and Groupoid Actions.

As we observed in [9], [10] and [12], every graph groupoid is indeed a (categorical) groupoid. This means that graph groupoids have rough but rich algebraic structures. The following definition is inspired by [19] and [25].

**Definition 2.2.** We say an algebraic structure $(X, Y, s, r)$ is a (categorical) groupoid if it satisfies that (i) $Y \subset X$, (ii) for all $x_1, x_2 \in X$, there exists a partially-defined binary operation $(x_1, x_2) \mapsto x_1 x_2$, for all $x_1, x_2 \in X$, depending on the source map $s$ and the range map $r$ satisfying that:

1. **(ii-1)** the operation $x_1 x_2$ is well-determined, whenever $r(x_1) s(x_2) \in Y$,

2. **(ii-2)** the operation $(x_1 x_2) x_3 = x_1 (x_2 x_3)$ is defined, if the constituents are well-determined in the sense of (ii-1), for $x_1, x_2, x_3 \in X$. 


(ii-3) if \( x \in \mathcal{X} \), then there exist \( y, y' \in \mathcal{Y} \) such that \( s(x) = y \) and \( r(x) = y' \), satisfying \( x = y \times x' \).

(ii-4) if \( x \in \mathcal{X} \), then there exists a unique groupoid-inverse \( x^{-1} \) satisfying \( x x^{-1} = s(x) \) and \( x^{-1} x = r(x) \), in \( \mathcal{Y} \).

By definition, we can conclude that every group is a groupoid \((\mathcal{X}, \mathcal{Y}, s, r)\) with \(|\mathcal{Y}| = 1\) (and hence \( s = r \) on \( \mathcal{X} \)). This subset \( \mathcal{Y} \) of \( \mathcal{X} \) is said to be the base of \( \mathcal{X} \). Remark that we can naturally assume that there exists the empty element \( \emptyset \) in a groupoid \( \mathcal{X} \). Roughly speaking, the empty element \( \emptyset \) means that there are products \( x_1 \times x_2 \) which are not well-defined, for \( x_1, x_2 \in \mathcal{X} \). Notice that if \(|\mathcal{Y}| = 1\) (equivalently, if \( \mathcal{X} \) is a group), then the empty word \( \emptyset \) is not contained in the groupoid \( \mathcal{X} \). However, in general, whenever \(|\mathcal{Y}| \geq 2\), a groupoid \( \mathcal{X} \) always contain the empty word. So, if there is no confusion, we will automatically assume that the empty element \( \emptyset \) is contained in \( \mathcal{X} \).

It is easily checked that our graph groupoid \( G \) of a finite directed graph \( G \) is indeed a groupoid with its base \( \hat{V} \), i.e., every graph groupoid \( G \) of a countable directed graph \( G \) is a groupoid \((G, \hat{V}, s, r)\), where \( s(w) = s(v w) = v \) and \( r(w) = r(w v') = v' \), for all \( w = v w v' \in G \) with \( v, v' \in \hat{V} \).

Let \( \mathcal{X}_k = (\mathcal{X}_k, \mathcal{Y}_k, s_k, r_k) \) be groupoids, for \( k = 1, 2 \). We say that a map \( f : \mathcal{X}_1 \rightarrow \mathcal{X}_2 \) is a groupoid morphism if (i) \( f(\mathcal{Y}_1) \subseteq \mathcal{Y}_2 \), (ii) \( s_2(f(x)) = f(s_1(x)) \) in \( \mathcal{X}_2 \), for all \( x \in \mathcal{X}_1 \), and (iii) \( r_2(f(x)) = f(r_1(x)) \) in \( \mathcal{X}_2 \), for all \( x \in \mathcal{X}_1 \). If a groupoid morphism \( f \) is bijective, then we say that \( f \) is a groupoid-isomorphism, and that the groupoids \( \mathcal{X}_1 \) and \( \mathcal{X}_2 \) are groupoid-isomorphic.

Notice that, if two countable directed graphs \( G_1 \) and \( G_2 \) are graph-isomorphic, via a graph-isomorphism \( g : G_1 \rightarrow G_2 \), in the sense that (i) \( g \) is bijective from \( V(G_1) \) onto \( V(G_2) \), (ii) \( g \) is bijective from \( E(G_1) \) onto \( E(G_2) \), (iii) \( g(e) = g(v_1) e v_2 = g(v_1) g(e) g(v_2) \) in \( E(G_2) \), for all \( e = v_1 e v_2 \in E(G_1) \), with \( v_1, v_2 \in V(G_1) \), then the graph groupoids \( G_1 \) and \( G_2 \) are groupoid-isomorphic. More generally, if \( G_1 \) and \( G_2 \) have graph-isomorphic shadowed graphs \( \hat{G} \) and \( \hat{G}_2 \), then \( \hat{G}_1 \) and \( \hat{G}_2 \) are groupoid-isomorphic (See Section 3, in detail).

Let \( \mathcal{X} = (\mathcal{X}, \mathcal{Y}, s, r) \) be a groupoid. We say that this groupoid \( \mathcal{X} \) acts on a set \( X \) if there exists a groupoid action \( \pi \) of \( \mathcal{X} \) such that \( \pi(x) : X \rightarrow X \) is a well-determined function. Sometimes, we call the set \( X \), a \( \mathcal{X} \)-set. We are interested in the case where a \( \mathcal{X} \)-set \( X \) is a Hilbert space. A nice example of groupoid actions acting on a Hilbert space are graph-representations defined and observed in [9]. In this paper, we will consider a new kind of graph-groupoid action on certain Hilbert spaces.
The section has two parts, one centered around our Theorem 3.3. This result generalizes the more traditional notion of deficiency indices (See Section 5). The second part shows that our $C^*$-algebras can be realized in representations of groupoids.

In [13], we considered the case where the collection of finite partial isometries are graph-families. How about the general cases where the family of finite partial isometries on $H$ is not a graph-family in the sense of [13]? We provide an answer of this question. As before, we will fix a Hilbert space $H$ and the corresponding operator algebra $B(H)$. Let $G = \{a_1, ..., a_N\}$ be a family of partial isometries $a_1, ..., a_N$ of $B(H)$, where $N \in \mathbb{N}$. Compared with [13], we will have much more complicated results.

3.1. $\ast$-Isomorphic Indices of Partial Isometries.

In this section, we will classify the set of all partial isometries of $B(H)$. For convenience, we denote the subset of all partial isometries of $B(H)$ by $PI(H)$. Let $a \in PI(H)$ with its initial space $H^a_{init}$ and its final space $H^a_{fin}$. Recall that $H^a_{init} = (a^*a)H$ and $H^a_{fin} = (aa^*)H$. Then we can find the subspaces

$$(H^a_{init})^\perp = H \ominus H^a_{init} = (1_H - a^*a)H = \ker a$$

and

$$(H^a_{fin})^\perp = H \ominus H^a_{fin} = (1_H - a a^*)H = \ker a^*.$$ 

Assume that

$$\varepsilon_+ \overset{def}{=} \dim \left( (H^a_{init})^\perp \right) \quad \text{and} \quad \varepsilon_- \overset{def}{=} \dim \left( (H^a_{fin})^\perp \right),$$

where $\varepsilon_+, \varepsilon_- \in \mathbb{N}_0^\infty = \mathbb{N} \cup \{0, \infty\}$. Then we can determine a subset $PI_H(\varepsilon_+, \varepsilon_-)$ of $PI(H)$ by

$$PI_H(\varepsilon_+, \varepsilon_-) \overset{def}{=} \left\{ a \in PI(H) \mid \begin{array}{l} \dim \left( (H^a_{init})^\perp \right) = \varepsilon_+, \\ \dim \left( (H^a_{fin})^\perp \right) = \varepsilon_- \end{array} \right\}.$$ 

Note that

$$PI(H) = \bigcup_{(\varepsilon_+, \varepsilon_-) \in (\mathbb{N}_0^\infty)^2} (PI_H(\varepsilon_+, \varepsilon_-)).$$
set-theoretically, in \( B(H) \), where \( \mathbb{N}_0^\infty = \mathbb{N} \cup \{0, \infty\} \). For instance, if \( U \) is the well-known unilateral shift (which is an isometry) on the Hilbert space \( l^2(\mathbb{N}_0) \), sending \((\xi_0, \xi_1, \xi_2, \ldots) \in l^2(\mathbb{N}_0)\) to

\[
U ((\xi_0, \xi_1, \xi_2, \ldots)) \overset{def}{=} (0, \xi_0, \xi_1, \xi_2, \ldots) \in l^2(\mathbb{N}_0),
\]

where \( \mathbb{N}_0 \overset{def}{=} \mathbb{N} \cup \{0\} \), then it is contained in \( PI_{l^2(\mathbb{N}_0)}(0, 1) \), since \( H^U_{in\text{it}} = l^2(\mathbb{N}_0) \) and \( H^U_{in\text{m}} = l^2(\mathbb{N}_0) \ominus \mathbb{C} \). Notice that every operator \( U^\varepsilon \) on \( l^2(\mathbb{N}_0) \) can be regarded as elements in \( PI_{l^2(\mathbb{N}_0)}(0, \varepsilon) \), for all \( \varepsilon \in \mathbb{N} \), and every unitaries on an arbitrary Hilbert space \( K \) can be regarded as elements in \( PI_K(0, 0) \).

Let \( u_1 \neq u_2 \in PI_H(0, 0) \) be unitaries, satisfying that \( \text{spec}(u_1) = \text{spec}(u_2) \) in \( T \subseteq \mathbb{C} \), where \( T \) is the unit circle in \( \mathbb{C} \) and \( \text{spec}(x) \) means the spectrum of \( x \), for all \( x \in B(H) \). Then we can get that:

**Lemma 3.1.** If the subspaces \( H_{u_1} \) and \( H_{u_2} \) of \( H \) are Hilbert-space isomorphic, where \( H_{u_k} = (u_k^* u_k) H \), for \( k = 1, 2 \), and if the spectrums \( \text{spec}(u_1) \) and \( \text{spec}(u_2) \) are identical in \( T \subseteq \mathbb{C} \), then the \( C^*(\{u_1\}) \) and \( C^*(\{u_2\}) \) of \( B(H) \) are \( * \)-isomorphic.

\[ \text{Proof.} \] By [13], we have that the \( C^* \)-algebras \( C^*(\{u_k\}) \) are \( * \)-isomorphic to \( (\mathbb{C} \cdot 1_{H_{u_k}}) \otimes_{\mathbb{C}} C(\text{spec}(u_k)) \), as embedded \( C^* \)-subalgebra of \( B(H) \), for \( k = 1, 2 \), where \( C(X) \) means the \( C^* \)-algebra consisting of all continuous functions on a compact subset \( X \) of \( \mathbb{C} \). Since the subspaces \( H_{u_1} \) and \( H_{u_2} \) are Hilbert-space isomorphic in \( H \) and since the spectrums \( \text{spec}(u_1) \) and \( \text{spec}(u_2) \) are same in \( T \), the \( C^* \)-algebras \( (\mathbb{C} \cdot 1_{H_k}) \otimes_{\mathbb{C}} C(\text{spec}(u_k)) \)'s are \( * \)-isomorphic \( C^* \)-subalgebras of \( B(H) \).

If \( a \in PI(H) \) is a partial isometry on \( H \), then we can regard \( a \) as an isometry on \( H \ominus \ker a \). Recall that if \( x \in B(K) \) is an isometry on a Hilbert space \( K \), then \( x \) has its Wold decomposition \( x = u_x + s_x \), where \( u_x \) is the unitary part of \( x \) and \( s_x \) is the shift part of \( x \). Futhermore, the Hilbert space \( K \) is decomposed by \( K = K_{u_x} \oplus K_{s_x} \), where \( K_{u_x} \) (resp., \( K_{s_x} \)) is the subspace of \( K \) where \( u_x \) (resp., \( s_x \)) is acting on, as a unitary (resp., as a shift).

**Definition 3.1.** Suppose \( a \in PI(H) \) is a partial isometry. We say that \( u_a + s_a \in B(H) \) is the Wold decomposition of \( a \), if \( u_a \) is the unitary part of \( a \) on \( H_a \) and \( s_a \) is the shift part of \( a \) on \( H_a \), by regarding \( a \) as an isometry on \( H_a \), where \( H_a = (a^* a) H = H \ominus \ker a \). Equivalently, the Wold decomposition of a \( |H_a \) of the isometry \( a \) \( |H_a \) is said to be the Wold decomposition of the partial isometry \( a \).

Let \( a \in PI_H(\varepsilon_+, \varepsilon_-) \). Then, by the Wold decomposition, we can decompose \( a \)

\[
a = u_a + s_a,
\]
where $u_a$ is the unitary part on $H_{u_a} = (u_a^* u_a) H$ of $a$; and $s_a$ is the shift part on $H_{s_a} = (s_a^* s_a) H$ of $a$. Note that the subspaces $H_{u_a}$ and $H_{s_a}$ of $H$ are also the subspaces of the initial space $H_{\text{init}}^a$ of $a$, and they decompose $H_{\text{init}}^a = H_{u_a} \oplus H_{s_a}$. We will call the subspaces $H_{u_a}$ and $H_{s_a}$ of $H$, the unitary part of $H$ and the shift part of $H$, in terms of $a$.

By the Wold decomposition of $a$, we then get the following diagram:

\[
\begin{array}{c}
H \\
\downarrow a \\
H_{u_a} \oplus H_{s_a} \oplus \ker a
\end{array}
\]

\[
\begin{array}{c}
H \downarrow a \\
H_{u_a} \oplus (\ker s_a^* \oplus H_{s_a}) \oplus \ker a^* \oplus \ker s_a^* \\
H_{u_a} \oplus H_{s_a} \oplus \ker a
\end{array}
\]

i.e., the partial isometry $a$ has the following block operator-matricial form,

\[
a = \begin{pmatrix} u_a & s_a \\ s_a & 0 \end{pmatrix},
\]

on $H_{u_a} \oplus H_{s_a} \oplus \ker a$.

Let $a \in PI_H(\varepsilon_+, \varepsilon_-)$, with its Wold decomposition $a = u_a + s_a$, and assume that the shift part $s_a$ is contained in $PI_{H_{s_a}}(0, \varepsilon_-)$. Then we have that

\[
\varepsilon_- = \varepsilon^- + \varepsilon_-^-, \text{ for some } \varepsilon_-^- \in \mathbb{N}_0^\infty,
\]

where

\[
\varepsilon^- = \dim (\ker s_a^*) \text{ and } \varepsilon_-^- = \varepsilon_- - \varepsilon^-.
\]

**Definition 3.2.** Let $a \in PI_H(\varepsilon_+, \varepsilon_-)$ with its Wold decomposition $u_a + s_a$, and assume that $s_a \in PI_{H_{s_a}}(0, \varepsilon_-)$. Then we will denote the collection of such partial isometries $a$ by $PI_H(\varepsilon_+, \varepsilon^-, \varepsilon_-^-)$.

Let $a \in PI_H(\varepsilon_+, \varepsilon^-, \varepsilon_-^-)$ be a partial isometry. Then we can have the following properties.

**Proposition 3.2.** Let $a \in PI_H(\varepsilon_+, \varepsilon^-, \varepsilon_-^-)$ be a nonzero partial isometry with its Wold decomposition $a = u + s$, where $u$ and $s$ are the unitary part and the shift part of $a$, respectively.

1. We can have that

\[
C^*(\{a\}) \text{-isomorphic } C^*(\{u, s\}) = C^*(\{u\}) \oplus C^*(\{s\}) \subseteq B(H).
\]

2. If $s = 0$, then
\[
C^*(\{a\}) = C^*(\{u\}) \ast\text{-isomorphic} (\mathbb{C} \cdot 1_{H_a}) \otimes_{\mathbb{C}} C(\text{spec}(u)).
\]

(3) If \(u = 0\) and if \(s \in PI_{H_s}(0, \varepsilon^-)\) with \(\varepsilon^- \in \mathbb{N}\), then
\[
C^*(\{a\}) = C^*(\{s\}) \ast\text{-isomorphic} T(H_s),
\]
where \(T(H_s)\) means the classical Toeplitz algebra defined on \(H_s\).

(4) If \(u = 0\) and if \(s \in PI_{H_s}(0, \infty)\), then
\[
C^*(\{a\}) = C^*(\{s\}) \ast\text{-isomorphic} (\mathbb{C} \cdot 1_{H_s}) \otimes_{\mathbb{C}} M_2(\mathbb{C}).
\]

Proof. (1) Let \(a \in PI_H(\varepsilon_+, \varepsilon_-, \varepsilon^-)\) be a partial isometry on \(H\) having its Wold decomposition \(a = u + s\), where \(u\) is the unitary part of \(a\) on \(H_u\) and \(s\) is the shift part of \(a\) on \(H_s\), where the subspaces \(H_u\) and \(H_s\) are the unitary part and the shift part of \(H\) in terms of \(a\). Recall the diagram
\[
\begin{array}{ccc}
H & = & H_u \oplus H_s \oplus \ker a \\
\downarrow a & & \downarrow u \quad \downarrow s \\
H & = & H_u \oplus \ker s^* \oplus H_s \oplus (\ker a^* \ominus \ker s^*).
\end{array}
\]

Notice that \(\ker s^* \oplus H_s\) is Hilbert-space isomorphic to \(H_s\), since \(s\) is a shift. So, the \(C^*\)-subalgebra \(C^*(\{a\})\) generated by \(a\) is \(\ast\)-isomorphic to the \(C^*\)-algebra \(C^*(\{u, s\})\) generated by \(u\) and \(s\). Moreover, since \(H_u\) and \(H_s\) are orthogonal (as subspaces) in \(H\), the \(C^*\)-subalgebra \(C^*(\{u, s\})\) is \(\ast\)-isomorphic to \(C^*(\{u\}) \oplus C^*(\{s\})\) as an embedded \(C^*\)-subalgebra of \(B(H)\).

(2) Suppose the shift part \(s\) of a partial isometry \(a\) is the zero operator. Then \(a = u\), and hence the \(C^*\)-subalgebra \(C^*(\{a\})\) is \(\ast\)-isomorphic to \(C^*(\{u\})\), as an embedded \(C^*\)-subalgebra of \(B(H_u) \subseteq B(H)\). By [13], for the given unitary part \(u\), we can create a one-vertex-one-edge graph \(G\), i.e., \(\{u\}\) is a connected \(G\)-family in the sense of Section 2.3. Therefore,
\[
C^*(\{u\}) \ast\text{-isomorphic} (\mathbb{C} \cdot 1_{H_u}) \otimes_{\mathbb{C}} C(T).
\]

(3) Assume now that the unitary part \(a\) is the zero operator. Then \(a = s\), and hence the \(C^*\)-subalgebra \(C^*(\{a\})\) is \(\ast\)-isomorphic to \(C^*(\{s\})\), as an embedded \(C^*\)-subalgebra of \(B(H_s)\). Suppose that \(\varepsilon^- = k_s \in \mathbb{N}\). For example, let \(k_s = 1\), and hence let \(s \in PI_{H_s}(0, 1)\). Then this operator \(s\) is unitarily equivalent to the unilateral shift \(U\) on \(l^2(\mathbb{N}_0)\) which is Hilbert-space isomorphic to \(H_s\). It is well-known that
the unilateral shift $U$ generates the Toeplitz algebra $T(l^2(\mathbb{N}_0))$. So, the shift part $s$ generates $C^*$-algebra $*$-isomorphic to the classical Toeplitz algebra $T(H)$.

Suppose $k_s > 1$ in $\mathbb{N}$. Then the shift part $s$ on $H_s$ is unitarily equivalent to the operator $U^k_s$, where $U$ is the unilateral shift on $l^2(\mathbb{N}_0)$. Note that the operator $U^k_s$ satisfies that

$$U^k_s \left( (\xi_0, \xi_1, \xi_2, \ldots) \right) = \left( 0, \ldots, 0, \xi_0, \xi_1, \ldots \right)_{k_s \text{-times}},$$

for all $(\xi_0, \xi_1, \xi_2, \ldots) \in l^2(\mathbb{N}_0)$, with dim$(U^k_s)^* = k_s$. So, the $C^*$-algebra $C^*(\{U^k_s\})$ is also $*$-isomorphic to the Toeplitz algebra $T(l(\mathbb{N}_0))$. Therefore, if $k_s < \infty$ in $\mathbb{N}$, then the $C^*$-subalgebra $C^*(\{s\})$ is $*$-isomorphic to the classical Toeplitz algebra $T(H)$, as an embedded $C^*$-subalgebra of $B(H)$.

(4) Suppose $s \in PI_H(0, \infty)$. Then the operator $s$ is unitarily equivalent to the block operator matrix

$$\begin{pmatrix} 0_{H_s} & 0_{H_s} \\ 1_{H_s} & 0_{H_s} \end{pmatrix},$$

and hence the $C^*$-algebra $C^*(\{s\})$ is $*$-isomorphic to $(C \cdot 1_{H_s} \otimes \mathbb{C} M_2(\mathbb{C}))$, by [13]. Notice that, on the previous case, the Hilbert space $H_s$ is Hilbert-space isomorphic to $H_1 \oplus H_2$, where both $H_1$ and $H_2$ are Hilbert-space isomorphic to $H$.

Motivated by the statement (2) of the previous proposition, we now define a new number $\varepsilon_u \in \mathbb{N}_0^\infty$.

**Definition 3.3.** Let $a \in PI_H(\varepsilon_+, \varepsilon^-, \varepsilon^-)$, with its Wold decomposition $a = u + s$, where $u$ and $s$ are the unitary part and the shift part of $a$, respectively, and let $H_u$ be the unitary part of $H$, in terms of $a$. Define the number $\varepsilon_u$ by

$$\varepsilon_u \stackrel{def}{=} \dim H_u \in \mathbb{N}_0^\infty.$$

And define the subclass $PI_H(\varepsilon_0, \varepsilon_+, \varepsilon^-, \varepsilon^-)$ of $PI_H(\varepsilon_+, \varepsilon^-, \varepsilon^-)$ by

$$PI_H(\varepsilon_0, \varepsilon_+, \varepsilon^-, \varepsilon^-) \stackrel{def}{=} \{ a \in PI_H(\varepsilon_+, \varepsilon^-, \varepsilon^-) : \dim H_{ua} = \varepsilon_0 \},$$

where $H_{ua}$ is the unitary part of $H$, in terms of $a$. 


Since \( PI(H) = \bigcup_{(\varepsilon_+, \varepsilon_-) \in \mathbb{N}_0^\infty} PI_H(\varepsilon_+, \varepsilon_-) \), we can get that

\[
PI(H) = \bigcup_{(\varepsilon_0, \varepsilon_+, \varepsilon_-, \varepsilon_-) \in (\mathbb{N}_0^\infty)^4} (PI_H(\varepsilon_0, \varepsilon_+, \varepsilon_-, \varepsilon_-)).
\]

Define an equivalence relation \( R_* \) on \( PI(H) \) by

\[
(3.1) \quad a_1 R_* a_2 \iff C^*(\{a_1\}) \ast\text{-isomorphic} = C^*(\{a_2\}) \text{ in } B(H).
\]

By the previous proposition, if \( s_1 \in PI_H(0, k_1) \) and \( s_2 \in PI_H(0, k_2) \) are shifts (which are isometries on \( H \)), where \( k_1, k_2 < \infty \) in \( \mathbb{N} \), then we can get that

\[
C^*(\{s_1\}) \ast\text{-isomorphic} = C^*(\{U_{k_1}\}) = T(H) = C^*(\{U_{k_2}\}) \ast\text{-isomorphic} = C^*(\{s_2\})
\]

where \( T(H) \) is the classical Toeplitz algebra.

**Definition 3.4.** Let \( a \in PI_H(\varepsilon_0, \varepsilon_+, \varepsilon_-, \varepsilon_-) \in (\mathbb{N}_0^\infty)^4 \) is called the \( \ast\text{-isomorphic index of } a \). The \( \ast\text{-isomorphic index of } a \) is denoted by \( i_*(a) \), i.e., \( i_*(a) = (\varepsilon_0, \varepsilon_+, \varepsilon_-, \varepsilon_-) \).

Consider the set

\[
(\mathbb{N}_0^\infty)^4 = \mathbb{N}_0^\infty \times \mathbb{N}_0^\infty \times \mathbb{N}_0^\infty \times \mathbb{N}_0^\infty,
\]

in detail. We will define a binary operation \(( - )\) on \((\mathbb{N}_0^\infty)^4\) by

\[
(i_1, j_1, k_1, l_1) - (i_2, j_2, k_2, l_2) \defeq (|i_1 - i_2|, |j_1 - j_2|, |k_1 - k_2|, |l_1 - l_2|)
\]

under the additional rule:

\[
\infty - \infty \defeq 0 \text{ in } \mathbb{N}_0^\infty,
\]

for all \((i_t, j_t, k_t, l_t) \in (\mathbb{N}_0^\infty)^4, \) for \( t = 1, 2 \). From now, if we denote \((\mathbb{N}_0^\infty)^4\), then it means the algebraic pair \(((\mathbb{N}_0^\infty)^4, -)\). Then, we can get the following theorem.
Theorem 3.3. Let $a_j \in PI(H)$ be partial isometries, and suppose the spectra $\text{spec}(u_j)$ of the unitary parts $u_j$ of $a_j$ are identical (if they are nonzero), for $j = 1, 2$. Then we can have that

\[(3.2) \quad i_*(a_1) - i_*(a_2) = (0, k_1, k_2, 0)\]

for some $k_1, k_2 < \infty$ in $\mathbb{N}_0$, if and only if $a_1 R_* a_2$, equivalently, the $C^*$-algebras $C^*(\{a_1\})$ and $C^*(\{a_2\})$ are $*$-isomorphic, as embedded $C^*$-subalgebras of $B(H)$.

Proof. Suppose the $*$-isomorphic indices $i_*(a_k)$ are $(t_1^{(k)}, t_2^{(k)}, t_3^{(k)}, t_4^{(k)})$ in $(\mathbb{N}_0^\infty)^4$, for $k = 1, 2$. Equivalently, assume that $a_k \in PI_H(t_1^{(k)}, t_2^{(k)}, t_3^{(k)}, t_4^{(k)})$, for $k = 1, 2$. Suppose $a_k$ have their Wold decomposition $u_k + s_k$, where $u_k$ are the unitary parts of $a_k$ and $s_k$ are the shift parts of $a_k$, for $k = 1, 2$. Also, let $H_{u_k}$ and $H_{s_k}$ be the unitary parts of $H$ and the shift parts of $H$, in terms of $a_k$, for $k = 1, 2$, respectively.

$(\Rightarrow)$ Suppose the $*$-isomorphic indices $i_*(a_1)$ and $i_*(a_2)$ satisfies (3.2). Then the condition (3.2) says that

$t_1^{(1)} = \dim H_{u_1} = t_1 = \dim H_{u_2} = t_1^{(2)},$

$t_2^{(1)} = \dim(\ker a_1) = t_2 = \dim(\ker a_2) = t_2^{(2)},$

$t_4^{(1)} = \dim(\ker a_1^* \ominus \ker s_1^*) = t_4 = \dim(\ker a_2^* \ominus \ker s_2^*) = t_4^{(2)}$

and

$\left|t_2^{(1)} - t_2^{(2)}\right| = k_1, \left|t_4^{(1)} - t_4^{(2)}\right| = k_2$ in $\mathbb{N}_0$.

Assume now that $k_1 < \infty$ and $k_2 < \infty$. Then

$a_1 \in PI_H(t_1, t_2^{(1)}, t_3^{(1)}, t_4)$ and $a_2 \in (t_1, t_2^{(1)}, t_3^{(2)}, t_4)$.

Since $C^*(\{a_j\})$ $*$-isomorphic $C^*(\{a_j\}) \oplus C^*(\{s_j\})$, for $j = 1, 2$, it suffices to show that $C^*(\{s_1\})$ and $C^*(\{s_2\})$ are $*$-isomorphic. By the previous proposition, $C^*(\{s_k\})$ are $*$-isomorphic either $(C \cdot 1_{H_k}) \otimes_{\mathbb{C}} M_2(\mathbb{C})$ or $T(H_{s_k})$, where $T(H_{s_k})$ is the Toeplitz algebra on $H_{s_k}$, for $k = 1, 2$. By the assumption that

$k_1 = \left|t_2^{(1)} - t_2^{(2)}\right| < \infty$ and $k_2 = \left|t_4^{(1)} - t_4^{(2)}\right| < 0,$
either \([t_3^{(1)} < \infty\) and \(t_3^{(2)} < \infty\) or \([t_3^{(1)} = \infty = t_3^{(2)}\). If \(t_3^{(1)} < \infty\) and \(t_3^{(2)} < \infty\), then

\[
C^*\{\{s_1\}\} \ast\text{-isomorphic} \; T(\ell^2(N_0)) \ast\text{-isomorphic} \; C^*\{\{s_2\}\},
\]

by the previous proposition. If \(t_3^{(1)} = t_3^{(2)}\), then

\[
C^*\{\{s_1\}\} \ast\text{-isomorphic} \; (\mathbb{C} \cdot 1_{H_0}) \otimes_{\mathbb{C}} M_2(\mathbb{C}) \ast\text{-isomorphic} \; C^*\{\{s_2\}\},
\]

again by the previous proposition. Therefore, if (3.2) holds true, then \(C^*\{\{s_1\}\}\) and \(C^*\{\{s_2\}\}\) are \(*\)-isomorphic, and hence, \(C^*\{\{a_1\}\}\) and \(C^*\{\{a_2\}\}\) are \(*\)-isomorphic, as embedded \(C^*\)-subalgebras of \(B(H)\). i.e., \(a_1 \mathcal{R}_* a_2\).

\((\Leftarrow)\) Assume now that two partial isometries \(a_1, a_2 \in PI(H)\) satisfy \(a_1 \mathcal{R}_* a_2\), equivalently, the \(C^*\)-subalgebras \(C^*\{\{a_1\}\}\) and \(C^*\{\{a_2\}\}\) are \(*\)-isomorphic in \(B(H)\). This means that \(C^*\{\{u_1\}\}\) and \(C^*\{\{u_2\}\}\) (resp., \(C^*\{\{s_1\}\}\) and \(C^*\{\{s_2\}\}\)) are \(*\)-isomorphic, by the previous proposition. It is clear that \(C^*\{\{u_1\}\}\) and \(C^*\{\{u_2\}\}\) are \(*\)-isomorphic, as embedded \(C^*\)-subalgebras of \(B(H)\), if and only if \(H_{u_1}\) is Hilbert-space isomorphic to \(H_{u_2}\), if and only if \(t_1^{(1)} = t_1^{(2)},\) whenever \(\text{spec}(u_1) = \text{spec}(u_2)\) in \(T \subset \mathbb{C}\). Clearly, as partial isometries, if \(a_1 \mathcal{R}_* a_2\), then \(t_4^{(1)} = t_4^{(2)}\).

Assume that both \(C^*\{\{s_1\}\}\) and \(C^*\{\{s_2\}\}\) are \(*\)-isomorphic to the classical Toeplitz algebra \(T(\ell^2(N_0))\). Then \(t_2^{(1)}, t_2^{(2), t_3^{(1)}, t_3^{(2)} < \infty}\) in \(N_0^\infty\). Thus

\[
\left| t_2^{(1)} - t_2^{(2)} \right| = k_1 < \infty \text{ and } \left| t_3^{(1)} - t_3^{(2)} \right| = k_2 < \infty.
\]

Assume now that both \(C^*\{\{s_1\}\}\) and \(C^*\{\{s_2\}\}\) are \(*\)-isomorphic to the \(C^*\)-algebra \((\mathbb{C} \cdot 1_{K}) \otimes_{\mathbb{C}} M_2(\mathbb{C})\). Then \(t_3^{(1)} = \infty = t_3^{(2)}\) in \(N_0^\infty\). Thus

\[
\left| t_3^{(1)} - t_3^{(2)} \right| = 0 = \left| t_2^{(1)} - t_2^{(2)} \right|,
\]

by the rule \(\infty - \infty \overset{\text{def}}{=} 0\) in \(N_0^\infty\). This shows that if \(a_1 \mathcal{R}_* a_2\) and if \(i_*(a_k) = (t_1^{(k)}, t_2^{(k)}, t_3^{(k)}, t_4^{(k)})\) in \((N_0^\infty)^4\), then

\[
i_*(a_1) - i_*(a_2) = (0, k_1, k_2, 0), \text{ with } k_1, k_2 < \infty.
\]
The above theorem shows that the $\ast$-isomorphic indexing

$$i_\ast(\cdot) : PI(H) \to (N_0^{\infty})^4$$

is a kind of invariant on $\ast$-isomorphic $C^*$-subalgebras generated by a single partial isometry in $B(H)$, up to the spectrums of unitary parts and the “finite” difference of the second and the third entries of $i_\ast(\cdot)$.

Example 3.1. Let $H = l^2(N_0)$, and let $a \in B(H)$ be an operator which is unitarily equivalent to the following matrix

$$\begin{pmatrix}
\begin{pmatrix}
\theta \\
\vdots \\
\theta
\end{pmatrix}
\begin{pmatrix}
0 \\
1 \\
1
\end{pmatrix}
\begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}
\end{pmatrix},$$

where $\theta \in \mathbb{T}$ in $\mathbb{C}$. Without loss of generality, let $a$ be the operator having above matrix form on $H$. Then it is a partial isometry on $H$. Assume that the left cornered block matrix

$$\begin{pmatrix}
\begin{pmatrix}
\theta \\
\vdots \\
\theta
\end{pmatrix}
\end{pmatrix} \overset{\text{denote}}{=} a_u$$

of $a$ is contained in the matricial algebra $M_n(\mathbb{C})$, with $\text{spec}(a_u) = \{\theta\}$, and the middle block matrix

$$\begin{pmatrix}
0 \\
1 \\
1 \\
\vdots
\end{pmatrix} \overset{\text{denote}}{=} a_s$$
is the unilateral shift. Then this partial isometry $a$ has its Wold decomposition $a = u + s$, and the unitary part $H_{u}$ of $H$ is Hilbert-space isomorphic to $\mathbb{C}^\oplus n$ and the shift part $H_{s}$ of $H$ is Hilbert-space isomorphic to $l^2(\mathbb{N}_0)$. So, we can get that this partial isometry $a$ is contained in $PI_H(n, \infty, 1, \infty)$, and hence the $*$-isomorphic index $i_*(a)$ is

$$i_*(a) = (n, \infty, 1, \infty).$$

Also, the $C^*$-subalgebra $C^*(\{a\})$ generated by $a$ is $*$-isomorphic to

$$\mathbb{C}^\oplus n \oplus T(l^2(\mathbb{N}_0)) \subseteq B(H_{u}) \oplus B(H_{s})$$

$$\subseteq B(H).$$

### 3.2. Graph Groupoids Induced by Partial Isometries.

Throughout this section, we will use the same notations we used in the previous sections. In Section 3.1, we characterized $C^*$-subalgebras of $B(H)$ generated by a single partial isometry. In the rest of this Section, we will observe $C^*$-subalgebras of $B(H)$ generated by multi-partial isometries on $H$. So, let $G = \{a_1, ..., a_N\}$ be a family of partial isometries $a_j$'s on $H$, for all $j = 1, ..., N$. And let’s assume that $N > 1$. Also, let

$$i_*(a_j) = (k_1^{(j)}, k_2^{(j)}, k_3^{(j)}, k_4^{(j)}), \text{ for all } j = 1, ..., N.$$ 
i.e.,

$$k_1^{(j)} = \dim H_{u_j}, \text{ where } u_j \text{ is the unitary part of } a_j,$$

$$k_2^{(j)} = \dim (\ker a_j^*), \quad k_3^{(j)} = \dim (\ker s_j^*)$$

and

$$k_4^{(j)} = \dim (\ker a_j^* \ominus \ker s_j^*),$$

for all $j = 1, ..., N$, where $a_j$ have their Wold decomposition $a_j = u_j + s_j$, for $j = 1, ..., N$. Then we can construct a family $G_W$ induced by $G$:

$$G_W = G^{(u)} \cup G^{(s)},$$

where

$$G^{(u)} \overset{\text{def}}{=} \{u_j : j = 1, ..., N\}.$$
and
\[ G^{(s)} \overset{\text{def}}{=} \{ s_j : j = 1, \ldots, N \}. \]

Also, for our purpose, we decompose \( G^{(s)} \) by
\[ G^{(s)} = G^{(s)}_1 \cup G^{(s)}_\infty, \]
where
\[ G^{(s)}_f \overset{\text{def}}{=} \left\{ x \in G^{(s)} \bigg| i_*(x) = (0, \varepsilon_+, \varepsilon_-), \right. \]
\[ \left. \varepsilon_+, \varepsilon_- \in \mathbb{N}_0^\infty, \text{ and } \varepsilon^- < \infty \text{ in } \mathbb{N}_0 \right\} \]
and
\[ G^{(s)}_\infty \overset{\text{def}}{=} \left\{ x \in G^{(s)} \bigg| i_*(x) = (0, \varepsilon_+, \varepsilon_-), \right. \]
\[ \left. \varepsilon_+, \varepsilon_- \in \mathbb{N}_0^\infty, \text{ and } \varepsilon^- = \infty \right\}. \]

Define a subset \( \overline{G_W} \) of \( B(H) \) by \( G_W \cup G^*_W \). Also, define the set \( G_{pro} \) induced by \( G_W \) by
\[ G_{pro} \overset{\text{def}}{=} \{ x^* x, x x^* : x \in G_W \} \subset B(H)_{pro}, \]
where \( B(H)_{pro} \) is the subset of \( B(H) \) consisting of all projections on \( H \).

Consider now the subset \( G_0^0 \subset B(H) \), consisting of words in \( \{ x^k x^k : x \in \overline{G_W}, k \in \mathbb{N} \} \), i.e.,
\[ G_0^0 \overset{\text{def}}{=} \{ 0 \} \cup G_{pro} \cup \left( \bigcup_{k=1}^{\infty} \{(x^k)(x^k)^*,(x^k)^*(x^k) : x \in G_W \} \right) \]
\[ = \{ 0 \} \cup \left( \bigcup_{k=1}^{\infty} \{(x^k)(x^k)^*,(x^k)^*(x^k) : x \in G_W \} \right) \]

Notice that, if \( x \in \overline{G^{(u)}} \subset \overline{G_W} \), where \( \overline{G^{(u)}} = G^{(u)} \cup G^{(u)*} \), then
\[ x^* x = x x^* = 1_{H_u} = (x^n)(x^n)^* = (x^n)^*(x^n), \text{ for all } n \in \mathbb{N}. \]

If \( x \in \overline{G^{(s)}} = G^{(s)}_\infty \cup G^{(s)}_{\infty*} \), then
\[ x^n = 0 = (x^n)^* = (x^*)^n, \text{ for all } n \in \mathbb{N} \setminus \{1\}. \]

Suppose now that \( x \in \overline{G^{(s)}_f} = G^{(s)}_f \cup G^{(s)}_{f*} \). Then we can have that
\[ (x^{n_1})(x^{n_1})^* \neq (x^{n_2})(x^{n_2})^* \text{ in } B(H)_{pro}, \text{ for all } n_1 \neq n_2 \in \mathbb{N}. \]
Remark 3.1. Suppose $x \in \mathcal{G}_{f}^{(s)}$. Then, $x^{n} \in B(H)$ is a shift, too. Indeed, if $i_{*}(x) = (0, \varepsilon_{+}, k, \varepsilon_{-})$, with $k < \infty$ in $\mathbb{N}_{0}$, then $i_{*}(x^{n}) = (0, \varepsilon_{+}, nk, \varepsilon_{-})$ in $(\mathbb{N}_{0}^{\infty})^{4}$. This shows that the operators $(x^{n}) (x^{n})^{*}$ are projections on $H$, too, since $x^{n}$ are partial isometries on $H$, for all $n \in \mathbb{N}$.

By the previous remark, we can conclude that:

Lemma 3.4. Let $p \in \mathcal{G}_{0}^{0}$. Then $p$ is a projection on $H$.

Proof. Let $p \in \mathcal{G}_{0}^{0}$. Then there exists $x \in \mathcal{G}_{W}^{\mathcal{W}}$ and $m \in \mathbb{N}$, such that $p = (x^{m}) (x^{m})^{*}$. Assume that $x \in \mathcal{G}_{W}^{(s)}$. Then

$$(x^{m}) (x^{m})^{*} = 1_{H_{x}} = x x^{*} = x^{*} x,$$

for all $m \in \mathbb{N}$, where $H_{x} = (x x^{*}) H$ is the subspace of $H$, which is both the initial and the final spaces of $x$. Suppose that $x \in \mathcal{G}_{W}^{(s)}$. Then

$$(x^{m})(x^{m})^{*} = \begin{cases} x x^{*} & \text{if } m = 1 \\ 0 & \text{otherwise,} \end{cases}$$

since $x^{m} = 0 = x^{* m}$, whenever $m > 1$. Therefore, it is a projection on $H$. Assume now that $x \in \mathcal{G}_{W}^{(s)}$. Then, by the previous remark, the operator $(x^{m}) (x^{m})^{*}$ is a projection on $H$, too. This shows that $\mathcal{G}_{0}^{0} \subset B(H)_{\text{pro}}$. □

Now, define the partial ordering $\leq$ on $\mathcal{G}_{0}^{0}$ by the rule:

$$p \leq q \iff p H \text{ Subspace } q H, \text{ for all } p, q \in \mathcal{G}_{0}^{0}.$$

The above partial ordering $\leq$ is well-defined, since all elements of $\mathcal{G}_{0}^{0}$ are projections on $H$.

Notation Without loss of generality, we will denote the partially ordered set $(\mathcal{G}_{0}^{0}, \leq)$ in $B(H)_{\text{pro}}$ simply by $\mathcal{G}_{0}^{0}$. □
3.2.1. Construction of Corresponding Graphs of Partial Isometries.

Let \( x \in \mathcal{G}_W \). Then we can construct a corresponding directed graph \( G_x \) as follows:

(Case I) Suppose \( u \in \mathcal{G}^{(u)} \). Then construct the one-vertex-one-loop-edge graph \( G_u \) by a directed graph with

\[
V(G_u) = \{ u^* u = uu^* \} \quad \text{and} \quad E(G_u) = \{ u \}.
\]

i.e., we regard \( u \) as a loop-edge and we regard the projections \( u^* u \) and \( uu^* \) as the initial and the terminal vertices of \( u \):

\[ G_u = \begin{array}{c}
\bullet \\
\overset{u^*u=uu^*}{\longrightarrow} \\
\end{array} \]

(Case II) Suppose \( s \in \mathcal{G}_d^{(s)} \). Then construct the one-edge graph \( G_s \) by a directed graph with

\[
V(G_s) = \{ s^* s, ss^* \} \quad \text{and} \quad E(G_s) = \{ s \}.
\]

i.e., we regard \( s, s^* s \) and \( ss^* \) as the non-loop edge and the corresponding initial and terminal vertices of the edge, respectively:

\[ G_s = \begin{array}{c}
\bullet \\
\overset{s}{\longrightarrow} \\
\bullet \\
\overset{ss^*}{\longrightarrow} \\
\end{array} \]

(Case III) Assume now that \( x \in \mathcal{G}_f^{(x)} \). Then construct the directed graph \( G_x \) by the countable directed linear graph with

\[
V(G_x) = \{ x^* x \} \cup \{ (x^n)^* : n \in \mathbb{N} \}
\]

and

\[
E(G_x) = \{ x^{(1)} = x \} \cup \{ x^{(k)} : k \in \mathbb{N} \setminus \{ 1 \} \}.
\]

i.e., we can have the following graph \( G_x \):

\[ G_x = \begin{array}{c}
\bullet \\
\overset{x^*x}{\longrightarrow} \\
\overset{x_{x^*}}{\longrightarrow} \\
\overset{x_{x^*}}{\longrightarrow} \\
\overset{x_{x^*}}{\longrightarrow} \\
\overset{x_{x^*}}{\longrightarrow} \\
\overset{x_{x^*}}{\longrightarrow} \\
\overset{x_{x^*}}{\longrightarrow} \\
\end{array} \]

Notice that \( x^{(n)} \)'s satisfy that

\[
x^{(1)} = x \quad \text{and} \quad x^{(n+1)} = x = x \mid_{(x^nx^*)H}, \quad \text{for all} \ n \in \mathbb{N}
\]

and

\[
x^{(1)} x^{(2)} \ldots x^{(m)} = x^m \in B(H), \quad \text{for all} \ m \in \mathbb{N} \setminus \{ 1 \},
\]
which are shifts on $H$. Moreover, each vertex $(x^n)$ $(x^n)^*$ are the projection onto the final space of the shift $x^n$, acting as the terminal vertex of the edge $x^{(n)}$, for all $n \in \mathbb{N} \setminus \{1\}$.

**Definition 3.5.** Let $G_x$ be the directed graphs induced by $x \in \mathcal{G}_W$, by the above rules. Denote the collection of countable directed graphs $G_x$’s induced by $x$’s in $\mathcal{G}_W$ by $\mathcal{G}_G$. The collection $\mathcal{G}_G$ is called the (collection of) corresponding graphs induced by $\mathcal{G}_W$.

3.2.2. The Glued Graph $G_1 \overset{v_1}{\#} v_2 \ G_2$ of $G_1$ and $G_2$.

Let $G_k$ be countable directed graphs and let $v_k \in V(G_k)$ be the fixed vertices, for $k = 1, 2$. Then, by identifying the fixed two vertices $v_1$ and $v_2$, we can construct a new countable directed graph $G = G_1 \overset{v_1}{\#} v_2 \ G_2$, by a countable directed graph

$$V(G) = (V(G_1) \setminus \{v_1\}) \cup \{v_{12}\} \cup (V(G_2) \setminus \{v_2\})$$

and

$$E(G) = E(G_1) \cup E(G_2),$$

where $v_{12}$ is the identified vertex of the vertices $v_1$ and $v_2$ (after identifying $v_1$ and $v_2$). If $e_k \in E(G_k)$ and if $e_k = v_k e_k$ or $e_k = e_k v_k$, then this edge $e_k$ is regarded as $e_k = v_{12} e_k$, respectively, $e_k = e_k v_{12}$ in $E(G)$. For instance, let

$$G_1 = \bullet \rightarrow^{v_1} \bullet \rightarrow \bullet$$

and

$$G_2 = \bullet \rightarrow^{v_2} \cdot.$$

Then the graph $G = G_1 \overset{v_1}{\#} v_2 \ G_2$ is the graph

$$G = \bullet \rightarrow^{v_{12}} \bullet \rightarrow \bullet \uparrow \bullet.$$

**Definition 3.6.** The graph $G = G_1 \overset{v_1}{\#} v_2 \ G_2$ is called the glued graph of $G_1$ and $G_2$ by gluing (or identifying) $v_1$ and $v_2$. The vertex $v_{12}$ of $v_1$ and $v_2$ in $V(G)$ is said to be the identified (or glued) vertex.

Inductively, we can construct the iterated glued graph $G$ of the countable directed graphs $K$ and $T$ by
\[ G = (K v_1 \# v_2 T) v_3 \# v_4 T v_5 \# v_6 T \ldots \]

e tc.

As usual, we can create the shadowed graph \( \hat{G} \) and the corresponding graph groupoid \( \mathbb{G} \) of the glued graph \( G \). Notice that

\[ G_1 v_1 \# v_2 G_2 = G_2 v_2 \# v_1 G_1. \]

3.2.3. Conditional Iterated Gluing on \( \mathcal{G}_G \).

Let \( \mathcal{G}_G \) be the corresponding graphs induced by the Wold decomposed family \( \mathcal{G}_W \) of \( G \) in \( \mathcal{P} \mathcal{I}(H) \). Define now the subset \( \mathbb{G}_0 \) of \( B(H) \) by

\[ \mathbb{G}_0 \overset{\text{def}}{=} \{ \text{all reduced words in } \mathbb{G}_0^0 \} \subset B(H). \]

i.e., if \( y \in \mathbb{G}_0 \) is nonzero, then there exist \( n \in \mathbb{N} \) and \( p_1, \ldots, p_n \in \mathbb{G}_0^0 \) such that

\[ y = \text{the operator formed by the product } p_1 \ldots p_n. \]

Remark that, even though \( p_1, \ldots, p_n \) are projections on \( H \), we cannot guarantee that the operator \( y \) is a projection on \( H \). Recall that the operator product \( p q \) of projections \( p \) and \( q \) is a projection if and only if \( p q = q p \) on \( H \).

First, we will determine the connecting relation on \( \hat{\mathcal{G}}_W = \mathcal{G}_W \cup \mathcal{G}_W^* \) on \( H \), by defining the map

\[ \pi : \hat{\mathcal{G}}_W \times \hat{\mathcal{G}}_W \to \mathbb{G}_0 \]

by

\[ \pi(x, y) \overset{\text{def}}{=} (x^* x) (y y^*), \]

for all \( x, y \in \hat{\mathcal{G}}_W \). Clearly, we can get that

\[ \pi(x, y) = \begin{cases} 
  x^* x & \text{if } x^* x \leq y y^* \text{ in } \mathbb{G}_0^0 \\
  y y^* & \text{if } x^* x \geq y y^* \text{ in } \mathbb{G}_0^0 \\
  (x^* x) (y y^*) & \text{if } (x^* x) (y y^*) \neq 0 \text{ on } H \\
  0 & \text{otherwise}, 
\end{cases} \]

for all \( x, y \in \hat{\mathcal{G}}_W \). Consider that if \( \pi(x, y) \neq 0 \) in \( \mathbb{G}_0 \), then

\[ xy = x (x^* x) (y y^*) y = x \pi(x, y) y \neq 0 \text{ on } H. \]
Definition 3.7. The map $\pi: \hat{G}_W \times \hat{G}_W \to \mathcal{G}_0$ is said to be the $G_W$-admissibility map.

Now, define a new subset $\mathcal{G}$ of $B(H)$ by the collection of all reduced words in $\hat{G}_W$. i.e.,

\[ \mathcal{G} \overset{\text{def}}{=} \{ \text{all reduced words in } \hat{G}_W \} \subset B(H). \]

The reduction on $\mathcal{G}$ is determined by the operator multiplication on $B(H)$, and this reduction is totally explained by the $G_W$-admissibility map $\pi$:

\[
x_1, \ldots, x_n \in \hat{G}_W \text{ and } w = x_1 \ldots x_n \in \mathcal{G} \setminus \{0\} \iff \pi(x_1, x_{i+1}) \neq 0 \text{ in } \mathcal{G}_0, \text{ for all } i = 1, \ldots, n-1.
\]

Notice that every element $y \in \mathcal{G}_0$ is contained in $\mathcal{G}$, too. i.e.,

\[ \mathcal{G}_0 \subseteq \mathcal{G}. \]

Thus we can extend the $G_W$-admissibility map $\pi$ on $\hat{G}_W$ to that on $\mathcal{G}$, also denoted by $\pi$:

\[ \pi: \mathcal{G} \times \mathcal{G} \to \mathcal{G}_0 \]

defined by

\[
\pi(w_1, w_2) = \begin{cases} 
\pi(x_1, x_n) &, \text{if } w_1 \in \mathcal{G}_0 \text{ and } w_2 \notin \mathcal{G}_0 \text{ and } w_2 = x_1 \ldots x_n, \text{ } x_j \in \hat{G}_W \\
(x_n^* x_n) w_2 &, \text{if } w_1 \notin \mathcal{G}_0 \text{ and } w_2 \in \mathcal{G}_0 \text{ and } w_1 = x_1 \ldots x_n, \text{ } x_j \in \hat{G}_W \\
\pi(x_n, y_1) &, \text{if } w_1, w_2 \notin \mathcal{G}_0 \text{ in } \mathcal{G}, \text{ and } w_1 = x_1 \ldots x_n \text{ and } w_2 = y_1 \ldots y_m, \text{ where } x_j, y_i \in \hat{G}_W \\
\end{cases}
\]

for all $w_1, w_2 \in \mathcal{G}$.

Denote now the collection of the shadowed graphs $\hat{G}_x$'s of $G_x \in \mathcal{G}_G$ by $\hat{G}_G$. i.e.,

\[ \hat{G}_G \overset{\text{def}}{=} \{ \hat{G}_x : G_x \in \mathcal{G}_G \}. \]

Take $G_1$ and $G_2$ in $\mathcal{G}_G$ and choose $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$. For the chosen vertices $v_1$ and $v_2$, we can compute
\[ \pi(v_1, v_2) = v_1 v_2 \in G_0 \text{ in } B(H), \]

since \( V(G_x) \in G_0 \subset G_0 \), for all \( G_x \in \mathcal{G}_G \).

Observe now the partial ordering on \( \mathcal{G}_G \). Define

\[ G_1 \leq G_2 \text{ in } \mathcal{G}_G \iff G_1 \text{ is a full-subgraph of } G_2. \]

Recall that we say that \( K_1 \) is a full-subgraph of \( K_2 \), where \( K_1 \) and \( K_2 \) are countable directed graphs, if

\[ E(K_1) \subseteq E(K_2), \]

and

\[ V(K_1) = \{ v, v' \in V(K_2) : e = v \to v', \forall e \in E(K_1) \}. \]

Remark the difference between subgraphs and full-subgraphs. We say that \( K_1 \) is a subgraph of \( K_2 \) if

\[ V(K_1) \subseteq V(K_2) \]

and

\[ E(K_1) = \{ e \in E(K_2) : e = v \to v', \forall v, v' \in V(K_1) \}. \]

Our partial ordering \( \leq \) on \( \mathcal{G}_G \) is determined by the concept, “full”-subgraphs.

**Notation** From now, if we denote \( \mathcal{G}_G \), then it means the partial ordered set \( (\mathcal{G}_G, \leq) \).

We now can construct the (\( \pi \)-depending) conditional glued graph \( G_1 \ ^{v_1 \#_\pi} v_2 \ G_2 \), by a directed graph,

\[
G_1 \ ^{v_1 \#_\pi} v_2 \ G_2 \overset{def}{=} \begin{cases} 
G_1 \ ^{v_1 \#_\pi} v_2 \ G_2 & \text{if } \pi(v_1, v_2) \neq 0 \\
G_2 & \text{if } G_1 \leq G_2 \\
G_1 & \text{if } G_1 \geq G_2 \\
G_1 \cup G_2 & \text{otherwise,}
\end{cases}
\]

where \( G_1 \cup G_2 \) is the directed graph with

\[ V(G_1 \cup G_2) = V(G_1) \sqcup V(G_2) \]

and

\[ E(G_1 \cup G_2) = E(G_1) \sqcup E(G_2), \]

where the symbol \( \sqcup \) on the right-hand sides means the disjoint union.
Now, consider the conditional gluing \( \#_\pi \) on \( G \) as follows: if \( G_1, G_2 \in G \) and if
\[
V(G_1) = \{ p_1, p_2, p_3, \ldots \} \quad \text{and} \quad V(G_2) = \{ q_1, q_2, q_3, \ldots \},
\]
in \( G_0^0 \), then the conditional glued graph \( G_1 \#_\pi G_2 \) of \( G_1 \) and \( G_2 \) is defined by
\[
G_1 p_1 \#_\pi q_1 \quad G_2 p_1 \#_\pi q_2 \quad G_2 p_1 \#_\pi q_3 \quad G_2 \ldots
\]
\[
... \quad G_1 p_2 \#_\pi q_1 \quad G_2 p_2 \#_\pi q_2 \quad G_2 p_2 \#_\pi q_3 \quad G_2 \ldots
\]
\[
... \quad G_1 p_n \#_\pi q_1 \quad G_2 p_n \#_\pi q_2 \quad G_2 p_n \#_\pi q_3 \quad G_2 \ldots
\]

**Definition 3.8.** The graph (3.3) of \( G_1 \) and \( G_2 \) in \( G \) is denoted by \( G_1 \#_\pi G_2 \). And we call it the conditional (or the \( \pi \)-dependent) glued graph of \( G_1 \) and \( G_2 \). And the symbol “\( \#_\pi \)” is called the conditional (or the \( \pi \)-dependent) gluing.

By the conditional iterated gluing on \( G \), we can decide the countable directed graph \( G_{GW} \) as a directed graph with
\[
G_{GW} \overset{\text{def}}{=} \#_{x \in G_W} G_x.
\]

**Definition 3.9.** The conditional iterated glued graph \( G_{GW} \) defined in (3.4) is called the \( G \)-graph. And the corresponding graph groupoid \( G_{GW} \) of \( G_{GW} \) is called the \( G \)-groupoid.

Notice that every element in \( G_{GW} \) is the reduced words in \( \hat{G}_W \), under the operator multiplication on \( B(H) \). Thus we can have the following lemma.

**Lemma 3.5.** \( G_{GW} = G \), where \( G \overset{\text{def}}{=} \{ \text{all reduced words in } \hat{G}_W \} \) defined at the beginning of this subsection. \( \square \)

**Example 3.2.** Suppose that \( G_W = G^{(u)} \cup G^{(s)}_\infty \) and assume that
\[
\pi(x, y) \neq 0, \quad \text{for } (x, y) \in G_W \times G_W \iff x^* x = y y^* \text{ in } G_0^0.
\]

Then this finite family \( G_W \) of partial isometries in \( B(H) \) is a \( G_{GW} \)-family in the sense of [13] (Also See Section 2.3).
Example 3.3. Let $\mathcal{G} = \{a\}$ and assume that $\mathcal{G}_W = \{u, s\}$, where $a$ has its Wold decomposition $a = u + s$, with $u \neq 0$ and $s \neq 0$ on $H$, equivalently, the $*$-isomorphic index $\iota_*(a)$ is $(k_1, k_2, k_3, k_4)$ in $\mathbb{N}_0^\infty$, satisfying that $k_1 \neq 0$, $k_3 = \infty$ in $\mathbb{N}_0^\infty$. Then we can create a family $\mathcal{G}_G = \{G_u, G_s\}$ of directed graphs. In particular,

$$V(G_u) = \{u^* u = u u^*\} \text{ and } E(G_u) = \{u\}$$

and

$$V(G_s) = \{s^* s, s s^*\} \text{ and } E(G_s) = \{s\}.$$ 

Assume that

$$\pi(u, s) = (u^* u) (s s^*) = (u u^*) (s s^*) = \pi(u^*, s) \neq 0,$$

and

$$\pi(u, s^*) = (u^* u) (s^* s) = (u u^*) (s^* s) = \pi(u^*, s^*) = 0.$$ 

Then we can construct the conditional iterated glued graph $G = G_u \#_\pi G_s$,

$$G = G_u \#_\pi G_s = (G_u \#_{uu^*} \#_{ss^*} G_s),$$

Then this graph $G$ is graph-isomorphic to the graph $K$,

$$K = \bullet \leftarrow \bullet \circ.$$ 

Example 3.4. Let $\mathcal{G} = \{u_1, u_2\} = \mathcal{G}^{(u)}$ be a finite family of partial isometries in $B(H)$. Then we can have $\mathcal{G}_G = \{G_{u_1}, G_{u_2}\}$, where $G_k$ are the graph with

$$V(G_k) = \{u_k^* u_k = u_k u_k^*\} \text{ and } E(G_k) = \{u_k\},$$

for $k = 1, 2$. Suppose $\pi(u_1, u_2) = \pi(u_1^*, u_2^*) = \pi(u_1, u_2^*) = \pi(u_1^*, u_2) \neq 0$ in $\mathbb{G}_0^0$. Then we can construct the iterated glued graph $G = G_{u_1} \#_\pi G_{u_2}$ which is identified with $G_{u_1} \#_{u_1^* u_1} \#_{u_2^* u_2} G_{u_2}$. Then this graph $G$ is graph-isomorphic to the graph $K$ with

$$V(K) = \{v\} \text{ and } E(K) = \{e_1 = v e_1, v, e_2 = v e_2 v\}.$$ 

Assume now that $\pi(u_1, u_2) = 0$ in $\mathbb{G}_0^0$. Then the iterated glued graph $G = G_{u_1} \#_\pi G_{u_2}$ is identified with the graph $G_{u_1} \sqcup G_{u_2}$. This graph is graph-isomorphic to the graph $H$ with

$$V(H) = \{v_1, v_2\} \text{ and } E(H) = \{e_1 = v_1 e_1 v_1, e_2 = v_2 e_2 v_2\}.$$
Example 3.5. Let $\mathcal{G} = \{s_1, s_2\} = \mathcal{G}^{(s)}$. Then we have the corresponding family $\mathcal{G}_G = \{G_{s_1}, G_{s_2}\}$ of directed graphs. Suppose $\pi(s_1, s_2) \neq 0$ in $\mathcal{G}$. Then the conditional iterated glued graph $\Delta = G_{s_1} \# \pi G_{s_2}$ is graph-isomorphic to the graph $\Delta$:

$$\Delta = \bullet \xrightarrow{} \bullet \downarrow \bullet.$$ 

Here, we can understand the horizontal part

$$\bullet \xrightarrow{} \bullet$$

of $\Delta$ are graph-isomorphic to $G_{s_2}$, and the vertical part

$$\bullet \downarrow \bullet$$

of $\Delta$ are graph-isomorphic to $G_{s_1}$.

The more examples would be provided in Section 5. In the following section, we will observe how the $\mathcal{G}$-groupoid $\mathcal{G}_W$ induced by the $\mathcal{G}$-graph $G_{W}$ works on the fixed Hilbert space $H$.

3.2.4. The Vertex Set of the $\mathcal{G}$-Graph.

In this subsection, we observe the vertex set $V(G)$ of the $\mathcal{G}$-graph $G$, which is the conditional iterated glued graph of $\mathcal{G}_G = \{G_x : x \in \mathcal{G}_W\}$. Suppose $x, y \in \mathcal{G}_W$, and assume that $\pi(x, y) \neq 0$ in $\mathcal{G}_0$. Then we can identify the vertices $x^* x \in V(G_x)$ and $y y^* \in V(G_y)$, and we can create the glued graph,

$$G_{x,y} = G_x \xleftarrow{x^* x} \pi \# y y^* G_y,$$

which is a full-subgraph of the $\mathcal{G}$-graph $G$. Then, inside $G_{x,y}$, the vertices $x^* x \in V(G_x)$ and $y y^* \in V(G_y)$ are identified. Denote the identified vertex by $p_0$. Then, how we can understand this vertex $p_0$? Combinatorially (in the sense of Subsection 3.2.1), we can simply understand $p_0$ is the identified vertex which is a pure combinatorial object. However, operator-theoretically, it is not clear to see how this vertex $p_0$ works as an operator on $H$. In fact, this vertex $p_0$ is not determined uniquely as an operator. Operator-theoretically, we can understand $p_0$ as:

$$p_0 = x^* x \text{ or } p_0 = y y^* \text{ or } p_0 = \pi(x, y)$$
or
\[ p_0 = \pi(x, y)^* = \pi(y^*, x^*), \]
case by case. Notice that
\[ \pi(x, y) \neq 0 \text{ in } G_0 \text{ if and only if } \pi(y^*, x^*) \neq 0 \text{ in } G_0, \]
since \( \pi(x, y)^* = \pi(y^*, x^*) \). Anyway, the vertex \( p_0 \) just represents the connection on \( G_{x,y} \) (or on \( G \)), in terms of the \( G_W \)-admissibility (map \( \pi \)).

The best way to choose \( p_0 \in V(G_{x,y}) \subset V(G) \), as an operator (if we have to: but we have not to), is that: \( p_0 \) may be the projection from \( H \) onto the subspace \( H_{init}^x \cap H_{fin}^y \). So, by gluing \( x^* \) \( x \) and \( y \) \( y^* \) to \( p_0 \) as the projection onto \( H_{init}^x \cap H_{fin}^y \) (operator-theoretically), we may construct the conditional glued graph \( G_{x,y} \). However, we will not fix \( p_0 \), as an operator. We regard \( p_0 \) as a pure combinatorial object representing the connection of the operators \( x \) and \( y \). In other words, the identified vertex \( p_0 \) can be explained case by case, differently in Operator Theory point of view.

**Observation** Combinatorially, the identified vertices of the \( G \)-graph represent the \( G_W \)-admissibility. Operator-theoretically, these vertices are understood differently case by case. For instance, the identified vertex \( p_0 \) in the text represents \( x^* \) \( x \) or \( y \) \( y^* \) or \( \pi(x, y) \) or \( \pi(y^*, x^*) \), or more depending on the \( G_W \)-admissibility, operator-theoretically. □

### 3.3. A Representation of the \( G \)-Groupoid.

Throughout this section, we will use the same notations we used in the previous sections. We will consider a certain representation of the \( G \)-groupoid \( G \), which is the graph groupoid induced by the conditional iterated glued graph, the \( G \)-graph \( G \). Recall that, the operator \( w \in B(H) \) is contained in \( G \) if and only if \( w = x_1 \ldots x_n \), for \( n \in \mathbb{N} \) and \( x_1, \ldots, x_n \in E(\hat{G}) = \hat{G}_W \).

Let \( x \in E(\hat{G}) \). Then we can construct the subspace \( H_x = (x^* \ x) \ H \) of \( H \), where \( H_x \) is a Hilbert space where \( x \in \hat{G}_W \) is acting on, as a unitary or a shift. Then we can have the class \( \mathcal{H}_{\hat{G}_W \rightarrow} \) of subspaces of \( H \),

\[ \mathcal{H}_{\hat{G}_W \rightarrow} \overset{def}{=} \{ H_x = (x^* \ x) \ H : x \in E(\hat{G}) \}. \]

Let \( H_{x_1}, H_{x_2} \in \mathcal{H}_{\hat{G}_W \rightarrow} \). Then we can define subspaces of \( H \),

\[ H_{x_1} \wedge \pi H_{x_2} \text{ and } H_{x_1} \vee \pi H_{x_2}, \]
by

\[ H_{x_1} \cap \pi H_{x_2} \overset{\text{def}}{=} \begin{cases} H_{x_1} \cap H_{x_2} & \text{if } \pi(x_1, x_2) \neq 0 \\ \{0_H\} & \text{otherwise} \end{cases} \]

and

\[ H_{x_1} \lor \pi H_{x_2} \overset{\text{def}}{=} \begin{cases} \text{span}(H_{x_1} \cup H_{x_2})^H & \text{if } \pi(x_1, x_2) \neq 0 \\ H_{x_1} \oplus H_{x_2} & \text{otherwise} \end{cases} \]

Then we can define the Hilbert space \( H_G \) by

\[ (3.5) \quad H_G \overset{\text{def}}{=} \lor_{w \in \text{FP}_r(\hat{G})} \pi H_w \]

with

\[ H_w \overset{\text{def}}{=} \bigwedge_{j=1}^n H_{x_j} , \]

whenever \( w = x_1 \ldots x_n \in \text{FP}_r(\hat{G}) \), with \( x_1, \ldots, x_n \in E(\hat{G}) \),

where

\[ \text{FP}_r(\hat{G}) \overset{\text{def}}{=} \mathbb{G} \setminus \left( V(\hat{G}) \cup \{0\} \right) . \]

Clearly, the Hilbert space \( H_G \) is a well-determined subspace of \( H \), and every operator \( w \in G \subset B(H) \) acts on \( H_G \). Suppose \( v \in V(\hat{G}) \). Then there always exists \( e \in E(\hat{G}) \) such that \( (e^* e) H \supseteq v H \) or \( (e e^*) H \supseteq v H \). So, all operators in \( G \) acts on the subspace \( H_G \) of \( H \).

We now define the groupoid action \( \alpha \) of \( G \) acting on \( H_G \),

\[ \alpha : G \rightarrow B(H_G) \]

sending \( w \) to \( \alpha_w \in B(H_G) \), by

\[ (3.6) \quad \alpha_w = w, \text{ for all } w \in G . \]

Then, clearly, the action \( \alpha \) is a groupoid action of \( G \) on \( H_G \). Indeed,

\[ \alpha_{w_1 w_2} = w_1 w_2 = \alpha_{w_1} \alpha_{w_2} , \text{ for all } w_1, w_2 \in G . \]

We can easily check that \( \alpha_{w_1 w_2} \neq 0 \) if and only if \( \pi(w_1, w_2) \neq 0 \), for all \( w_1, w_2 \in G \). Now, we can determine a representation of the \( G \)-groupoid \( G \) by the pair \( (H_G, \alpha) \).
Definition 3.10. The representation $(H_G, \alpha)$ of the $G$-groupoid $G$ is called the $G$-representation in $B(H)$.

In the following theorem, we observe the equivalence of $G$-representations.

Theorem 3.6. Let $G_k = \{a_1^{(k)}, \ldots, a_{N_k}^{(k)}\} \subset PI(H)$ be the finite families of partial isometries in $B(H)$, and let $G_k$ be the $G_k$-graphs having the $G_k$-groupoids $G_k$, respectively, for $k = 1, 2$. Assume that (i) the shadowed graphs $\hat{G}_k$ of $G_k$ are graph-isomorphic, via the graph-isomorphism $\phi : \hat{G}_1 \rightarrow \hat{G}_2$, and (ii) $i_* (x) - i_* (g(x)) = (0, 0, 0, 0)$ in $\mathbb{N}^\infty$, for all $x \in E(\hat{G}_1)$. Then the $G_k$-representations $(H_{G_k}, \alpha^{(k)})$ are equivalent, for $k = 1, 2$.

Proof. Suppose $G_1$ and $G_2$ are the finite families of partial isometries in $B(H)$, and let $G_k = G_k^{(u)} \cup G_k^{(s)}$, where $G_k^{(u)}$ and $G_k^{(s)}$ are the collection of all unitary parts and all shift parts of $G_k$, respectively, for $k = 1, 2$.

Also, by the condition (i), we can conclude that the $G_k$-groupoids $G_k$ are groupoid-isomorphic, for $k = 1, 2$. Indeed, we can have the groupoid-isomorphism $\varphi : G_1 \rightarrow G_2$:

$$\varphi(w) \overset{def}{=} \begin{cases} 0 & \text{if } w = 0 \\ g(w) & \text{if } w \in V(\hat{G}) \cup E(\hat{G}) \\ g(x_1) \ldots g(x_n) & \text{if } w = e_1 \ldots e_n \in FP_r(\hat{G}), \ n > 1, \end{cases}$$

in $G_2$, for all $w \in G_1$ (Also, see [9] and [10]). This means that algebraically the groupoidal structures of $G_1$ and $G_2$ are same.

By the condition (ii), the subspaces $H_x = (x^* x) H$ and $H_{g(x)} = (g(x)^* g(x)) H$ are Hilbert-space isomorphic in $H$, for all $x \in E(G_1)$. Therefore, the Hilbert spaces $H_{G_1}$ and $H_{G_2}$ are Hilbert-space isomorphic, as embedded subspaces in $H$. i.e., we can have the Hilbert-space isomorphism $\Phi : H_{G_1} \rightarrow H_{G_2}$ induced by the groupoid-isomorphism $\varphi : G_1 \rightarrow G_2$:

$$\Phi (H_w) = H_{\varphi(w)}, \text{ in } H_{G_2}, \text{ for all } H_w \subset H_{G_1}, \text{ for all } w \in FP_r(\hat{G}).$$

Therefore, the Hilbert spaces $H_{G_1}$ and $H_{G_2}$ are Hilbert-space isomorphic, via $\Phi$.

Since we have the following commuting diagram,

$$
\begin{array}{ccc}
H_{G_1} & \overset{\Phi}{\longrightarrow} & H_{G_2} \\
\downarrow_{\alpha^{(1)}} & & \downarrow_{\alpha^{(2)}} \\
H_{G_1} & \overset{\Phi}{\longrightarrow} & H_{G_2}
\end{array}
$$
for all $w \in G_1$, the groupoid actions $\alpha^{(k)}$ satisfy that
\[
\alpha^{(2)}_{\varphi(w)} = \varphi(w) = \Phi w \Phi^{-1} = \Phi \alpha_w \Phi^{-1} = \alpha^{(1)}_{\Phi w \Phi^{-1}},
\]
for all $w \in G_1$. Therefore, the actions $\alpha^{(1)}$ and $\alpha^{(2)}$ are equivalent. Thus the $G_k$-representations $(H_{G_k}, \alpha^{(k)})$ are equivalent, for $k = 1, 2$.

The above theorem shows that the equivalence of the $G_k$-representations $(H_{G_k}, \alpha^{(k)})$, for $k = 1, 2$, is determined by the combinatorial conditions of $G_k$: $W$, for $k = 1, 2$.

3.4. $C^*$-Subalgebras Generated by Partial Isometries.

As usual, throughout this section, we will use the same notations we used in the previous sections. Let $G$ be a finite family of partial isometries in $B(H)$, with its Wold decomposed family $G_W$. Let $x \in G_W$. We already observed the $C^*$-subalgebra $A_x = C^*(\{x\})$ of $B(H)$, in Section 3.1. Notice that, by the very definition, $A_x = A_x^*$, for all $x \in G_W$. Here, the symbol “$=$” means “being identically same in $B(H)$”.

Proposition 3.7. (See Section 3.1) Let $x \in \hat{G}_W$ and $A_x = C^*(\{x\})$. Then

\[
A_x \cong \begin{cases} 
(C \cdot 1_{H_x}) \otimes \mathbb{C} \ C(\text{spec}(x)) & \text{if } i^*(x) = (k_1, k_2, 0, k_4), \\
T(H_x) & \text{if } i^*(x) = (0, k_2, k_3, k_4), \\
(C \cdot 1_{H_x}) \otimes \mathbb{C} M_2(\mathbb{C}) & \text{if } i_*(x) = (0, k_2, \infty, k_4)
\end{cases}
\]

where $i_*(\cdot)$ means the $*$-isomorphic index.

The following theorem is the main result of this paper. This shows that a $C^*$-subalgebra of $B(H)$ generated by finitely many partial isometries is $*$-isomorphic to a groupoid $C^*$-algebra.

Theorem 3.8. Let $G$ be a finite family of partial isometries in $B(H)$ and assume that it has its $G$-groupoid $G$. Then the $C^*$-algebra $C^*(G)$ generated by $G$ is
*-isomorphic to the groupoid C*-algebra C* α(G), as embedded C*-subalgebras of B(H G) ⊆ B(H), where (H G, α) is the G-representation of G.

Proof. Observe that

\[ C^*(G) = C^*(G_W) = C^* \left( \hat{G}_W \right) \]

\[ \cong \left[ C^* \alpha(G) \right] \]

\[ = C^*_\alpha(G) \subseteq B(H_G) \subseteq B(H). \]

The first identity of the above formulae holds true because of the Wold decomposition. And the *-isomorphic relation and the first *-subalgebra inclusion hold true, by the definition of the G-representation (H G, α) of G.

The above theorem shows that C*-subalgebras of B(H) generated by finitely many partial isometries on H are *-isomorphic to certain groupoid C*-algebras.

**Corollary 3.9.** Let G_1 and G_2 be the collection of finitely many partial isometries in B(H) having their G-graphs G_1 and G_2. Assume that (i) the shadowed graphs \( \hat{G}_1 \) and \( \hat{G}_2 \) are graph-isomorphic, via a graph-isomorphism \( g : \hat{G}_1 \to \hat{G}_2 \), (ii) \( i_*(x) - i_*(g(x)) = (0, 0, 0, 0) \) in \( \mathbb{N}_0^4 \), for all \( x \in E(\hat{G}_1) \), and (iii) For any unitary parts \( u \in E(G_1) \), \( \text{spec}(u) = \text{spec}(g(u)) \) in \( \mathbb{T} \). Then the C*-algebras \( C^*(G_1) \) and \( C^*(G_2) \) are *-isomorphic, as embedded C*-subalgebras of B(H).

Proof. Observe that

\[ C^*(G_1) \cong C^*_\alpha(1)(G_1) \]

in \( B(H_{G_1}) \), by the previous theorem, where \( G_k \) are the \( G_k \)-groupoids and where \( (H_{G_k}, \alpha^{(k)}) \) are the \( G_k \)-representations of \( G_k \), for \( k = 1, 2 \)

\[ \cong C^*_\alpha(2)(G_2) \cong C^*(G_2), \]

of \( B(H_{G_2}) \), by (ii) and (iii), since \( (H_{G_k}, \alpha^{(k)})'s \) are equivalent.

4. **Block Structures of C*(G)**
The main result in this section is a two-part structure theorem (Theorem 4.3 and 4.13) for the $C^*$-algebras introduced above. In proving them, we will need some facts and lemmas from topological (reduced) free product algebras, introduced in Section 4.1.

Throughout this Section, we will keep using the same notations we used before. Let $G$ be the given family of finitely many partial isometries in $B(H)$ and let $G_W$ be the corresponding Wold decomposed family of $G$. We showed that the $C^*$-algebra $C^*(G)$ is $*$-isomorphic to the groupoid $C^*$-algebra $C^*_{\alpha}(G)$, where $G$ is the $G$-groupoid induced by the $G$-graph $G$ in the operator algebra $C^*$-subalgebra $\subseteq B(H)$, where $(H_G, \alpha)$ is the $G$-representation.

4.1. Topological Reduced Free Product Algebras.

Let $X_1$ and $X_2$ be arbitrary sets. Then we can define the algebraic free product $X_1 \ast X_2$ of $X_1$ and $X_2$ by the set of all words in $X_1 \cup X_2$, i.e.,

$$(4.1) \quad X_1 \ast X_2 \overset{\text{def}}{=} \text{all words in } X_1 \cup X_2.$$

i.e., $x \in X_1 \ast X_2$ if and only if there exist $n \in \mathbb{N}$ and $(i_1, \ldots, i_n) \in \{1, 2\}^n$ and $x_{i_j} \in X_{i_j}$, for $j = 1, \ldots, n$, such that

$$i_1 \neq i_2, i_2 \neq i_3, \ldots, i_{n-1} \neq i_n$$

and

$$x = x_{i_1} x_{i_2} \ldots x_{i_n}.$$

Recall now that we say a topological space $A$ is a topological algebra over a scalar field $F$, if it is an algebra over $F$ and all operations on $A$ are continuous under the topology for $A$, i.e., $A$ is a topological algebra over a scalar field $F$, if a topological space $A$, containing $F$, satisfies

(i) $A$ is a vector space over $F$ with its vector addition $(+): (a_1, a_2) \mapsto a_1 + a_2$ and the scalar multiplication $(\times): (t, a_1) \mapsto ta_1$, for all $t \in F$ and $a_1, a_2 \in A$,

(ii) $A$ has a vector multiplication $(\cdot): (a_1, a_2) \mapsto a_1 a_2$, for all $a_1, a_2 \in A$, and it is associative,

(iii) the vector addition $(+)$ and the vector multiplication $(\cdot)$ on $A$ are left and right distributive, and

(iv) the operations $(+), (\cdot)$ and $(\times)$ are continuous, under the topology for $A$. 
Recall also that we say \( A \) is an (algebraic) algebra over \( F \), if \( A \) satisfies (i), (ii) and (iii). For example, \( C^* \)-algebras and von Neumann algebras are topological algebras.

**Definition 4.1.** Let \( A_1 \) and \( A_2 \) be algebraic algebras over the same scalar field \( F \). Then we can define their algebraic free product algebra \( A_1 \ast_{a_{lg}} A_2 \) of \( A_1 \) and \( A_2 \) by the algebra generated by all words in \( A_1 \cup A_2 \), in the algebra \( Alg_{a_{lg}}(A_1, A_2) = F[A_1 \cup A_2] \), generated by \( A_1 \) and \( A_2 \). Notice that \( A_1 \ast_{a_{lg}} A_2 \) is not a topological space (even though \( A_1 \) and \( A_2 \) are topological spaces).

By the definition of algebraic free product and algebraic free product algebras, we can get the following proposition.

**Proposition 4.1.** Let \( A_1 \) and \( A_2 \) be algebraic algebras over a scalar field \( F \), and let \( A_1 \ast_{a_{lg}} A_2 \) be the algebraic free product algebra of \( A_1 \) and \( A_2 \). Assume now that \( A_k = F[X_k] \), where \( X_k \) are the generator set of \( A_k \), for \( k = 1, 2 \). Then

\[
A_1 \ast_{a_{lg}} A_2 \overset{Algebra}{=} F[X_1 \ast X_2],
\]

where \( \overset{Algebra}{=} \) means “being algebra-isomorphic”.

**Proof.** Suppose \( A_k = F[X_k] \) are the algebraic algebras generated by \( X_k \), for \( k = 1, 2 \). Then we have that

\[
A_1 \ast_{a_{lg}} A_2 = F[A_1 \ast A_2]
\]

by the definition of \( \ast_{a_{lg}} \)

\[
= F[F[X_1] \ast F[X_2]] \overset{Algebra}{=} F[F[X_1 \ast X_2]]
\]

by the definition of \( \ast \)

\[
= F[X_1 \ast X_2].
\]

i.e., the above proposition shows that

\[
(4.2) \quad F[X_1 \ast X_2] \overset{Algebra}{=} F[X_1] \ast_{a_{lg}} F[X_2],
\]

where \( X_1 \) and \( X_2 \) are arbitrary sets.

Similar to the previous construction, we can determine algebraic reduced free structures. Let \( (X, \cdot) \) be an arbitrary algebraic pair. i.e., \( (\cdot) : X \times X \to X \) is
a binary operation on $X$. (Notice that $(X, \cdot)$ is not necessarily be a well-known algebraic structures, for instance, a semigroup or a group, or a groupoid, etc.) Let $\mathcal{X}$ be the collection of all reduced words in $X$, where the reduction is totally depending on the binary operation $(\cdot)$ on $X$.

**Definition 4.2.** Let $(X, \cdot)$ and $\mathcal{X}$ be given as in the previous paragraph. Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be subsets of $\mathcal{X}$. Define the algebraic reduced free product set $\mathcal{X}_1 \ast^r \mathcal{X}_2$ of $\mathcal{X}_1$ and $\mathcal{X}_2$, by the subset of $\mathcal{X}$ consisting of all reduced words in $\mathcal{X}_1 \cup \mathcal{X}_2$.

Similarly, we can determine the algebraic reduced free product algebra of algebraic algebras $A_1$ and $A_2$:

**Definition 4.3.** Let $A$ be an algebraic algebra over its scalar field $\mathbb{F}$, and let $A_1$ and $A_2$ be the algebraic subalgebras of $A$. Define the algebraic reduced free product algebra $A_1 \ast^r_{a_{\text{lg}}} A_2$ of $A_1$ and $A_2$ in $A$, by the algebra generated by all reduced words in $A_1 \cup A_2$. i.e.,

$$A_1 \ast^r_{a_{\text{lg}}} A_2 \overset{\text{def}}{=} \mathbb{F}\left[ \{ \text{all reduced words in } A_1 \cup A_2 \} \right].$$

Remark that the reduction is dependent upon the vector multiplication on $A$.

Then, similar to the previous proposition or (4.2), we can get the following proposition.

**Proposition 4.2.** Let $A$ be an algebraic algebra over a scalar field $\mathbb{F}$ and assume that $A = \mathbb{F}[X]$, i.e., $A$ is generated by a set $X$. Let $A_1$ and $A_2$ be algebraic subalgebras of $A$ and suppose $A_k = \mathbb{F}[X_k]$ where $X_k$’s are the subset of $X$. Then the algebraic reduced free product algebra $A_1 \ast^r_{a_{\text{lg}}} A_2$ in $A$ is $*$-isomorphic to the algebraic algebra $\mathbb{F}[X_1 \ast^r X_2]$, where $X_1 \ast^r X_2$ is the reduced free product of $X_1$ and $X_2$, under the vector multiplication on $A$.

**Proof.** First, notice that we can construct an algebraic pair $(X, \cdot)$, where $(\cdot)$ is the restricted vector multiplication on $A$, and we can understand $X$ as $(X, \cdot)$. (So, the algebraic reduced free product set $X_1 \ast^r X_2$ is well-defined.) Similar to the proof of the previous proposition, we can get that

$$A_1 \ast^r_{a_{\text{lg}}} A_2 = \mathbb{F}[A_1 \ast^r A_2]$$

by the definition of “$\ast^r_{a_{\text{lg}}}$”,

$$= \mathbb{F}[\mathbb{F}[X_1] \ast^r \mathbb{F}[X_2]]^{\text{Algebra}} \mathbb{F}[\mathbb{F}[X_1 \ast^r X_2]]$$

by the definition of “$\ast^r$” and by the vector addition and multiplication on $A$.
\[ = \mathbb{F}[X_1 * r X_2]. \]

So, the above proposition shows the relation between \( * \) and \( *_{\text{alg}} \):

\[ F[X_1 * r X_2]_{\text{Algebra}} \equiv F[X_1] *_{\text{alg}}^r F[X_2], \text{ inside } F[X], \]

whenever \( X_1 \) and \( X_2 \) are subsets of \( X \).

Now, we define the topological reduced free product algebras.

**Definition 4.4.** Let \( A \) be a topological algebra over its scalar field \( \mathbb{F} \), and let \( A_1 \) and \( A_2 \) be topological subalgebras of \( A \). Define the topological reduced free product algebra \( A_1 *_{\text{top}}^r A_2 \) of \( A_1 \) and \( A_2 \) in \( A \) by the \( \tau \)-closure of \( A_1 *_{\text{alg}}^r A_2 \), where \( \tau \) is the topology for \( A \), i.e.,

\[ A_1 *_{\text{top}}^r A_2 \overset{\text{def}}{=} \overline{A_1 *_{\text{alg}}^r A_2}_\tau \text{ inside } A. \]

By definition, we can get the following theorem.

**Theorem 4.3.** Let \( A \) be a topological algebra over its scalar field \( \mathbb{F} \), and let \( A_1 \) and \( A_2 \) be topological subalgebras of \( A \). Assume that \( A = F[X] \) and \( A_k = F[X_k] \), where \( X \) is the generator set of \( A \) and the \( X_k 's \) are subsets of \( X \), for \( k = 1, 2 \). Then

\[ A_1 *_{\text{top}}^r A_2 \overset{\text{Top-Algebra}}{=} F[X_1 * r X_2]^\tau, \]

where \( "\text{Top-Algebra}" \) means "being topological algebra isomorphic".

**Proof.** Observe that

\[ A_1 *_{\text{top}}^r A_2 = \overline{F[A_1 *_{\text{alg}}^r A_2]}^\tau = \overline{F[F[X_1] *_{\text{alg}}^r F[X_2]]}^\tau = \overline{F[X_1 * r X_2]}^\tau = F[X_1 * r X_2]^\tau, \]

by (4.3).}

i.e., by the previous theorem, we have

\[ F[X_1]^\tau *_{\text{top}}^r F[X_2]^\tau = F[X_1 * r X_2]^\tau, \text{ in } F[X], \]

whenever \( X_1, X_2 \) are subsets of \( X \).
The following corollary is the direct consequence of the previous theorem.

**Corollary 4.4.** Let \( \mathcal{G} \) be a finite family of partial isometries in \( B(H) \) and let \( \mathcal{G} \) be the \( \mathcal{G} \)-groupoid. Let \( \{ \mathcal{G}_x : x \in \mathcal{G}_W \} \) be the subgroupoids of the \( \mathcal{G} \)-groupoid \( \mathcal{G} \), where \( \mathcal{G}_x \) are the graph groupoids induced by the corresponding graphs \( G_x \) of \( x \) in \( \mathcal{G} \), for all \( x \in \mathcal{G}_W \). Then

\[
(4.6) \quad C^*(\mathcal{G}) \overset{\ast}\cong \bigast_{x \in \mathcal{G}_W} (C^*_\alpha(\mathcal{G}_x)) \text{ in } B(H),
\]

as embedded \( C^* \)-subalgebras of \( B(H) \), where \((H_G, \alpha)\) is the \( \mathcal{G} \)-representation of \( \mathcal{G} \).

**Proof.** Denote \( C^*_\alpha(G) \) and \( C^*_\alpha(G_x) \)'s by \( A \) and \( A_x \)'s, respectively. By Section 4, we know that the \( C^* \)-subalgebra \( C^*(\mathcal{G}) \) of \( B(H) \) is \( \ast \)-isomorphic to the groupoid \( C^* \)-algebra \( A \) in \( B(H_G) \subseteq B(H) \), where \((H_G, \alpha)\) is the \( \mathcal{G} \)-representation of \( \mathcal{G} \). It is easily checked that

\[
\mathcal{G} = \bigast_{x \in \mathcal{G}_W} \mathcal{G}_x,
\]

Thus, we can have that

\[
C^*(\mathcal{G}) \overset{\ast}\cong A = \mathcal{C}[\alpha(\mathcal{G})] = \mathcal{C}[\mathcal{G}]
\]

\[
= \mathcal{C} \left[ \bigast_{x \in \mathcal{G}} \mathcal{G}_x \right] \overset{\ast}\cong \bigast_{x \in \mathcal{G}_W} \left( \mathcal{C}[\mathcal{G}_x] \right)
\]

by \((4.5)\)

\[
= \bigast_{x \in \mathcal{G}_W} A_x.
\]

The above corollary shows that the \( C^* \)-algebra \( C^*(\mathcal{G}) \) generated by a finite family \( \mathcal{G} \) of partial isometries in \( B(H) \) has its block structure determined by the topological reduced free product, and the blocks are \( A_x = C^*_\alpha(\mathcal{G}_x) \)'s, for all \( x \in \mathcal{G}_W \). Since \( A_x \)'s are characterized in Section 3, we have the topological reduced free block structures and characterized blocks.

**Remark 4.1.** Our topological reduced free (product) structure is basically different from the (amalgamated reduced) free structures observed in [9], [10], [11] and [13]. (Amalgamated) Reduced freeness on those papers are determined by the free probabilistic settings of Voiculescu (See [16]). However, there are some connections between them (See [13]).
In the following section, we will consider this topological reduced free block structures of $C^*(G)$ more in detail.

4.2. Topological Free Block Structures on $C^*(G)$.

Let $G$ be the given finite family of partial isometries in $B(H)$, and let $G_G = \{G_x : x \in G_W\}$ be the family of corresponding graphs induced by $G_W$. Also, let $G$ be the $G$-graph induced by $G_W$, which is the conditional iterated glued graph of $G_G$, and let $G$ be the $G$-groupoid, the graph groupoid of $G$. In the previous section, we showed that $G$ is the topological reduced free product of $G_x$’s, where $G_x$ are the $G_x$-groupoids of $G$ for all $x \in G_W$, and

$$C^*(G) \overset{\text{*-isomorphic}}{=} C^*_a(G) \overset{\text{*-isomorphic}}{=} \underset{x \in G_W}{\underset{\text{top}}{\ast}} C^*_a(G_x).$$

This means that the $C^*$-subalgebra $C^*(G)$ of $B(H)$ has rough but characterized block structures $\{A_x : x \in G_W\}$, by Section 3.1. In this section, we will consider this block structure more in detail, inside $C^*(G)$.

Suppose $u \in G^{(u)}$ and $G_u \in G_G$. Then, by Subsection 3.2.1, the graph $G_u$ is the one-vertex-one-loop-edge graph. Therefore, the graph groupoid $G_u$ of $G_u$, which is a subgroupoid of $G$-groupoid $G$, is a group. Also, if $s \in G^{(s)}$, then the corresponding graph $G_s \in G_G$ is a one-edge graph having distinct initial and terminal vertices. So, the graph groupoid $G_s$ of $G_s$ is a finite (sub)groupoid consisting of the elements $0$, $s$, $s^*$, $s^* s$ and $s s^*$. Therefore, if $x \in G^{(u)} \cup G^{(s)}$, then we can handle the corresponding subgroupoid induced by $x$ relatively easy.

Lemma 4.5. Let $u_1$, $u_2$, $u \in G^{(u)}$ and $s_1$, $s_2$, $s \in G^{(s)}$.

(1) If $\pi(u_1, u_2) \neq 0$ in $G_0$, then

$$C^*(\{u_1, u_2\}) \overset{\text{*-isomorphic}}{=} C^*(\{u_1\}) \ast_{\text{alg}} C^*(\{u_2\}).$$

Thus, $C^*(\{u_1, u_2\})$ is $\ast$-isomorphic to

$$\left(\mathbb{C} \cdot 1_{H(u_1, u_2)}\right) \otimes \left(\mathbb{C} \cdot \text{spec}(u_1) \ast_{\text{alg}} \mathbb{C} \cdot \text{spec}(u_2)\right),$$

where $H(u_1, u_2)$ is the $\{u_1, u_2\}$-Hilbert space in the sense of Section 3.

(2) If $\pi(s_1, s_2) \neq 0$ in $G_0$, then
Thus, \( C^* (\{s_1, s_2\}) \) is \(*\)-isomorphic to

\[
(\mathbb{C} \cdot 1_{H(s_1, s_2)}) \otimes_{\mathbb{C}} (M_2(\mathbb{C}) \otimes_{\mathbb{C}} M_2(\mathbb{C})) = (\mathbb{C} \cdot 1_{H(s_1, s_2)}) \otimes_{\mathbb{C}} M_4(\mathbb{C}),
\]

where \( H(s_1, s_2) \) is the \( \{s_1, s_2\} \)-Hilbert space in the sense of Section 3.

(3) if \( \pi(u, s) \neq 0 \) in \( \mathbb{G}_0 \), then

\[
C^* (\{u, s\}) \text{ is } *\text{-isomorphic to } \left( \mathbb{C} \cdot 1_{H(u, s)} \right) \otimes (C (\text{spec}(u)) \ast_{a lg} M_2(\mathbb{C})).
\]

Proof. By Section 4.1, we have that if \( \mathcal{G}_W = \{x, y\} \), then

\[
C^* (\mathcal{G}_W) \text{ is } *\text{-isomorphic to } C^*_a (\mathcal{G}) = C^*_a (\mathcal{G}_x \ast^r \mathcal{G}_y)
\]

\[
\text{def } \to C^*_a (\mathcal{G}_x) \ast_{top} C^*_a (\mathcal{G}_y)
\]

\[
\text{def } \to \left( C^*_a (\mathcal{G}_x) \ast_{a lg} C^*_a (\mathcal{G}_y) \right)
\]

where “\( \ast_{a lg} \)” means the “algebraic reduced algebra free product”.

(1) Suppose \( u_1 \) and \( u_2 \) are unitary parts of certain partial isometries in \( B(H) \), and let \( G_{u_1} \) and \( G_{u_2} \) be the corresponding one-vertex-one-loop-edge graphs in \( \mathcal{G}_G \). Then we can construct the conditional glued graph \( G = G_{u_1} \# \pi G_{u_2} \) of \( G_{u_1} \) and \( G_{u_2} \), which is the \( \mathcal{G} \)-graph. It is graph-isomorphic to the one-vertex-two-loop-edge graph. Then \( \mathcal{G} \)-groupoid \( \mathcal{G} \) satisfies that \( \mathcal{G} = \mathcal{G}_{u_1} \ast^r \mathcal{G}_{u_2} \), where \( \mathcal{G}_{u_k} \) are the graph groupoids of \( G_{u_k} \), for \( k = 1, 2 \). However, by the conditional gluing on the \( \mathcal{G} \)-graph \( G \), we can realize that \( \mathcal{G}_{u_1} \ast^r \mathcal{G}_{u_2} \) is identified with \( \mathcal{G}_{u_1} \ast \mathcal{G}_{u_2} \), where “\( \ast \)” means the “algebraic (non-reduced) free product”. Therefore,

\[
C^* (\{u_1, u_2\}) \text{ is } *\text{-isomorphic to } \left( C^* (\{u_1\}) \ast_{a lg} C^* (\{u_2\}) \right)
\]

\[
\text{def } \to \left( C^* (\{u_1\}) \ast_{a lg} C^* (\{u_2\}) \right).
\]
where “$*_{a.lg}$” means the “algebraic non-reduced algebra free product”.

(2) Assume now that $s_1$ and $s_2$ are shift parts in $G_{s_k}(s)$ of certain partial isometries in $B(H)$, and let $G_{s_k}$ be the corresponding graphs with their graph groupoids $G_{s_k}$, for $k = 1, 2$. Suppose $\pi(s_1, s_2) \neq 0$. Then we can create the conditional glued graph $G = G_{s_1} \#_\pi G_{s_2}$ which is graph-isomorphic to

\[ \bullet \longrightarrow \bullet \longrightarrow \bullet. \]

Then we can easily check that the graph groupoid $G$ of $G$ is $\{0, s_1, s_2, s_1^*, s_2^*, s_1 s_2, s_2^* s_1^*\}$, set-theoretically. So,

$$G = G_{s_1} * G_{s_2} = G_{s_1} \ast G_{s_2}.$$

Therefore, we have that

$$C^*\{s_1, s_2\} \ast\text{-isomorphic} = \frac{C^*\{s_1\} \ast_{a.lg} C^*\{s_2\}}{\frac{C^*\{s_1\} \ast_{a.lg} C^*\{s_2\}}{C^*\{s_1\} \ast_{a.lg} C^*\{s_2\}}}.$$

(3) Let $u \in G^{(u)}$ and $s \in G^{(s)}_{\infty}$ with their corresponding graphs $G_u, G_s \in G_G$, respectively. Then similar to the previous two cases, we can get the desired result.

The above lemma shows that it is relatively easy to deal with the groupoid $C^*$-algebraic structures in terms of their topological reduced free blocks of them, since the topological reduced free product of blocks is determined by the algebraic “non-reduced” free product of generator sets of blocks. This means that we do not need to consider the reduction on blocks induced by the elements in $G^{(u)} \cup G^{(s)}_{\infty}$, inside $C^*(G)$.

However, if $x$ is contained in $G^{(s)}_f$, then the corresponding graph $G_x$ is graph-isomorphic to the infinite linear graph,

\[ \bullet \xrightarrow{x} \bullet \xrightarrow{x_{(2)}} \bullet \xrightarrow{x_{(3)}} \bullet \xrightarrow{x_{(4)}} \cdots, \]

and hence it is hard to handle the graph groupoid $G_x$ of $G_x$, where $x^{(k)} = x$ on $H$, for all $k \in \mathbb{N}$, with $x^{(1)} = x$, satisfying

$$x \ x^{(2)} \ldots \ x^{(m)} = x^m \text{ on } H, \text{ for all } m \in \mathbb{N}.$$
We will concentrate on the case where we have a fixed shift part $x$ in $G_{f}^{(s)}$. Let $x \in G_{f}^{(s)}$ and $G_{x} \in G_{G}$, and let $y \in G_{W}$ with its corresponding graph $G_{y} \in G_{G}$. First, assume that $y \in G^{(u)}$. If $\pi(x, y) = 0$ in $G_{0}$, then we can have the disconnected graph $G_{x,y} = G_{x} \#_{\pi} G_{y} = G_{x} \cup G_{y},$

$$\bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \ldots$$

If $\pi(x, y) \neq 0$ in $G_{0}$, then $\pi(x^{n}, y) \neq 0$, for all $n \in \mathbb{N}$, since

$$x^{*} x \geq x x^{*} \geq x^{2} x^{2*} \geq x^{3} x^{3*} \geq x^{4} x^{4*} \geq \ldots.$$ 

Thus, the conditional iterated glued graph $G_{x,y}$ is

$$\bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \ldots$$

If we consider the graph groupoid $G_{x,y}$ of the conditional glued graph $G_{x,y}$ of $G_{x}$ and $G_{y}$, then we can conclude that

$$G_{x,y} = G_{x} * G_{y} = G_{x} \star G_{y},$$

where "$\star$" means the reduced algebraic free product and "$\ast$" means the non-reduced algebraic free product.

**Lemma 4.6.** Let $x \in G_{f}^{(s)}$ and $u \in G^{(u)}$, assume that $\pi(x, u) \neq 0$ in $G_{0}$. Then the $C^{*}$-subalgebra $C^{*}(\{x, u\})$ of $B(H)$ is $\ast$-isomorphic to $C^{*}(\{\pi(x, u)\}) *_{alg} C^{*}(\{\pi(u)\})$. Thus, $C^{*}(\{x, u\})$ is $\ast$-isomorphic to

$$(\mathbb{C} \cdot 1_{H(x, u)}) \otimes_{\mathbb{C}} (T(l^{2}(\mathbb{N}_{0})) *_{alg} C(spec(u))).$$

**Proof.** It suffices to show that the graph groupoid $G$ of the conditional glued graph $G_{x,y}$ of $G_{x}$ and $G_{y}$ satisfies $G = G_{x} \star G_{y}$, where $G_{x}$ and $G_{y}$ are the graph groupoids of the corresponding graphs $G_{x}$ and $G_{y}$ of $x$ and $y$. By (4.7), it is done. \[\blacksquare\]

Assume now that $x \in G_{f}^{(s)}$ and $s \in G_{\infty}^{(s)}$, and let $\pi(x, s) \neq 0$ in $G_{0}$. Then, $\pi(x^{n}, s) \neq 0$ in $G_{0}$, too, for all $n \in \mathbb{N}$, because

$$x^{*} x \geq x x^{*} \geq x^{2} x^{2*} \geq x^{3} x^{3*} \geq \ldots,$$
as projections on $H$. Since $\pi(x, s)^* = \pi(s^*, x^*)$, for any $x, s \in \mathcal{G}_W$, Therefore, we can get the conditional glued graph $G = G_x \#_\pi G_s$, graph-isomorphic to the following graph,

\[
\begin{array}{ccccccc}
\bullet & \rightarrow & \bullet & \rightarrow & \bullet & \rightarrow & \bullet \rightarrow \cdots \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \cdots \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \cdots 
\end{array}
\]

The upper row of the above graph can be regarded as the graph $G_x$, and the columns are regarded as the graphs $G_s$’s. This shows that, similar to the previous lemma, we can conclude that

\[
C^*(\{x, s\}) \ast-isomorphic = C^*(\{x\}) \ast_{alg} C^*(\{s\}).
\]

**Lemma 4.7.** Let $x \in \mathcal{G}_f^{(s)}$, and $s \in \mathcal{G}_s^{(s)}$. Then the $C^*$-algebra $C^*(\{x, s\})$ is $\ast$-isomorphic to the $C^*$-closure $C^*(\{x\}) \ast_{alg} C^*(\{s\})$ of the algebraic free product algebra of $C^*(\{x\})$ and $C^*(\{s\})$. Thus, $C^*(\{x, s\})$ is $\ast$-isomorphic to

\[
(\mathbb{C} \cdot 1_{H_{\{x, s\}}}) \otimes_{\mathbb{C}} (T(H_x) \otimes_{\mathbb{C}} M_2(\mathbb{C})).
\]

□

Finally, let’s take $x, y \in \mathcal{G}_f^{(s)}$, and assume that $\pi(x, y) \neq 0$, in $\mathbb{G}_0$. Let $H_{\{x, y\}}$ be the $\{x, y\}$-Hilbert space in the sense of Section 3, induced by the subspaces $H_x$ and $H_y$ of $H$, and let $\alpha = \alpha|_{\mathbb{G}_{x, y}}$ be the restriction of the $\mathbb{G}$-groupoid action $\alpha$, where $\mathbb{G}_{x, y}$ is the graph groupoid, which is a subgroupoid of the $\mathbb{G}$-groupoid $\mathbb{G}$, induced by the conditional glued graph $G_{x, y} = G_x \#_\pi G_y$ of $G_x$ and $G_y$ of $\mathcal{G}_G$. We know that

\[
C^*(\{x, y\}) \ast-isomorphic C^*_{\alpha}(\mathbb{G}_{x, y}),
\]

\[
\ast-isomorphic = C^*_{\alpha}(\mathbb{G}_x) \ast_{top} C^*_{\alpha}(\mathbb{G}_y)
\]

by the definition of $\mathbb{G}_{x, y}$ (or $G_{x, y}$) and by Section 4.2

\[
\ast-isomorphic = C^*_{\alpha}(\mathbb{G}_x) \ast_{top} C^*_{\alpha}(\mathbb{G}_y)
\]

(4.8)

\[
\ast-isomorphic = T(H_x) \ast_{top} T(H_y),
\]
as a $C^*$-subalgebra of $B(H)$. By the reduction on "*top" (and on "*$alg" and on "*$"), it is somewhat hard to deal with it, compared with the previous cases. In the previous cases, we only need to observe the non-reduced free products, moreover simple algebraic (non-reduced) free products. However, in the above case, we need to observe the reduction on the free product.

In fact, it is also interesting to observe the *-isomorphic $C^*$-subalgebra of $T(H_x)$ $*$top $T(H_y)$, in (4.8), inside $B(H)$. Clearly, if the corresponding graphs $G_x$ and $G_y$ in $G_G$ satisfies either $G_x > G_y$ or $G_x < G_y$, or equivalently, if either

$$\dim(\text{ker} y^*) | \dim(\text{ker} x^*) \text{ or } \dim(\text{ker} x^*) | \dim(\text{ker} y^*),$$

then $G_{x,y} = G_y$, respectively, $G_{x,y} = G_x$. Therefore, we can get that:

**Lemma 4.8.** Let $x, y \in G^{(x)}_f$, and assume that the corresponding graphs $G_x$ and $G_y$ in $G_G$ satisfy $G_x > G_y$. Then $C^*$-algebra $C^*\{x, y\}$ generated by $\{x, y\}$ is *-isomorphic to the Toeplitz algebra $T(H_y)$, i.e.,

$$C^*\{x, y\} \text{ *-isomorphic } C^*_G(G_y) \text{ *-isomorphic } T(H_y).$$

Assume now that $\pi(x, y) \neq 0$ in $G_0$, and also assume that neither $G_x > G_y$ nor $G_x < G_y$. Then the conditional glued graph $G_{x,y}$ of $G_x$ and $G_y$ is graph-isomorphic to the following graph,

```
   :   :   :   :   :   ...  
  ↓  ↓  ↓  ↓  ↓  
• —• —• —• —• —• —• —...  
  ↓  ↓  ↓  ↓  ↓  
• —• —• —• —• —• —• —...  
  ↓  ↓  ↓  ↓  ↓  
• —• —• —• —• —• —• —...  
  ↓  ↓  ↓  ↓  ↓  
• —• —• —• —• —• —• —...  
  ↓  ↓  ↓  ↓  ↓  
• —• —• —• —• —• —• —...  
  ↓  ↓  ↓  ↓  ↓  
• —• —• —• —• —• —• —...  
```

We can regard all edges of the previous graph as shifts $x$’s or $y$’s. Thus all finite paths of the above graph are regarded as shifts, too. And hence, all reduced finite paths in the graph groupoid $G_{x,y}$ of $G_{x,y}$ are regarded as finite dimensional shifts on $H$. So, it is easy to check that

$$T(H_x) *_{\text{top}} T(H_y) \subseteq T(H_{\{x,y\}}),$$

in $B(H)$. Conversely, assume that $T \in T(H_{\{x,y\}})$. Since $T(H_{\{x,y\}})$ is *-isomorphic to
Lemma 4.9. Let \( x, y \in \mathcal{G}_f^{(s)} \) and assume that \( \pi(x, y) \neq 0 \) in \( \mathcal{G}_0 \), and neither \( G_x < G_y \) nor \( G_y < G_x \) in \( \mathcal{G}_G \). Then the \( C^* \)-algebra \( C^* \{ x, y \} \) is \(*\)-isomorphic to the Toeplitz algebra \( T(H_{\{x,y\}}) \), where \( H_{\{x,y\}} \) is the \( \{x,y\} \)-Hilbert space in the sense of Section 3. \( \Box \)

Readers can realize the difference between the above lemma and the previous lemmas. The above lemma shows that, inductively, if we have a subfamily \( \mathcal{G}_f \) of \( \mathcal{G}_f^{(s)} \), satisfying that (i) \( \pi(x_1, x_2) \neq 0 \) in \( \mathcal{G}_0 \), and (ii) neither \( G_{x_1} < G_{x_2} \) nor \( G_{x_2} < G_{x_1} \), in \( \mathcal{G}_G \), for all pair \((x_1, x_2) \in \mathcal{G}_f \times \mathcal{G}_f \), then the \( C^* \)-subalgebra \( C^*(\mathcal{G}_f) \) of \( B(H) \), generated by \( \mathcal{G}_f \), is \(*\)-isomorphic to the Toeplitz algebra \( T(H_{\mathcal{G}_f}) \), where \( H_{\mathcal{G}_f} \) is the \( \mathcal{G}_f \)-Hilbert space in the sense of Section 3, which is a subspace of \( H \).

Proposition 4.10. Let \( \mathcal{G}_f \) be a subfamily of \( \mathcal{G}_f^{(s)} \subseteq \mathcal{G}_W \) satisfying (i) \( \pi(x_1, x_2) \neq 0 \) in \( \mathcal{G}_0 \), and (ii) neither \( G_{x_1} < G_{x_2} \) nor \( G_{x_2} > G_{x_1} \) in \( \mathcal{G}_G \), for all \((x_1, x_2) \in \mathcal{G}_f \times \mathcal{G}_f \), then the \( C^* \)-subalgebra \( C^*(\mathcal{G}_f) \) of \( B(H) \) is \(*\)-isomorphic to \( T(H_{\mathcal{G}_f}) \), where \( H_{\mathcal{G}_f} \) is the \( \mathcal{G}_f \)-Hilbert space, which is a subspace of \( H \). \( \Box \)

Let \( \mathcal{G}_f^{(s)} \subseteq \mathcal{G}_W \) be nonempty. Define the corresponding subset \( \mathcal{G}_G^{(s,f)} \) of \( \mathcal{G}_G \) by

\[
\mathcal{G}_G^{(s,f)} \overset{\text{def}}{=} \{ G_x \in \mathcal{G}_G : x \in \mathcal{G}_f^{(s)} \} \subseteq \mathcal{G}_G.
\]

Recall that the family \( \mathcal{G}_G \) is in fact a partially ordered set \((\mathcal{G}_G, \leq)\), where \( \leq \) is the full-subgraph inclusion on directed graphs. Then, without loss of generality, we can regard \( \mathcal{G}_G^{(s,f)} \) as a partially ordered set \((\mathcal{G}_G^{(s,f)}, \leq)\), under the inherited partial
ordering \( \leq \). By this partial ordering, we can take the chains inside \( G_G^{(s;f)} \), and, for each chain, we can take the minimal elements in \( G_G^{(s;f)} \).

**Definition 4.5.** Let \( G_G^{(s;f)} \subseteq G_G \) be given as before. Define the subset \( G_G^{(s;\text{min})} \) of \( G_G^{(s;f)} \subseteq G_G \) by

\[
G_G^{(s;\text{min})} \overset{\text{def}}{=} \left\{ G \in G_G^{(s;f)} : G \text{ is minimal in } G_G^{(s;f)} \right\}.
\]

By the collection \( G_G^{(s;\text{min})} \), we can take the corresponding subset \( G_f^{(s;\text{min})} \) of \( G_f^{(s)} \) by

\[
G_f^{(s;\text{min})} \overset{\text{def}}{=} \left\{ x \in G_f^{(s)} : G_x \in G_G^{(s;\text{min})} \right\}.
\]

By the previous lemmas, we can realize that:

**Proposition 4.11.** The \( C^* \)-subalgebra \( C^* \left( G_f^{(s)} \right) \) of \( C^*(G) \) is \( * \)-isomorphic to

\[
\left( \mathbb{C} \cdot 1_{G_f^{(s)}} \right) \otimes C^*_\alpha \left( G_f^{(s;\text{min})} \right),
\]

where \( G_f^{(s;\text{min})} \) is the subgroupoid of the \( G \)-groupoid \( G \), generated by the graph

\[
G_f^{(s;\text{min})} \overset{\text{def}}{=} \#_{\pi} \ G_x.
\]

Observe the graph \( G_f^{(s;\text{min})} \overset{\text{def}}{=} \#_{\pi} \ G_x \). By the previous lemmas, we can conclude that if \( x, y \in G_f^{(s;\text{min})} \) having their corresponding graphs \( G_x, G_y \in G_G^{(s;\text{min})} \), then there are two cases where \( \pi(x, y) \neq 0 \) or \( \pi(x, y) = 0 \), in \( G_0 \). In particular, if \( \pi(x, y) \neq 0 \), the full-subgraph \( G_{x,y} = G_x \#_{\pi} G_y \) of \( G_f^{(s;\text{min})} \) induces the graph groupoid \( G_{x,y} = G_x \ast \pi G_y \), and \( C^*_\alpha(G_{x,y}) \) is \( * \)-isomorphic to \( T(H_{x,y}) \). Clearly, if \( \pi(x, y) = 0 \), then

\[
C^*_\alpha(G_{x,y}) \overset{* \text{-isomorphic}}{=} C^*_\alpha(G_x) \oplus C^*_\alpha(G_y) \overset{* \text{-isomorphic}}{=} T(H_x) \oplus T(H_y).
\]

Therefore, we can get that:

**Proposition 4.12.** Let \( G_f^{(s;\text{min})} \) of \( G_W \) be given as above. Then there exists \( k \in \mathbb{N} \) such that (i) \( k \leq \left| G_f^{(s)} \right| \), and (ii) we can decompose \( G_f^{(s;\text{min})} \) by \( G_f^{(s;\text{min})} = \bigsqcup_{m=1}^{k} G_{f,m}^{(s;\text{min})} \), where \( \bigsqcup \) means the disjoint union and where
So, the $C^*$-algebra $C^*(\mathcal{G}_{f}^{(s:\min)})$ is $\ast$-isomorphic to $\bigoplus_{m=1}^{k} T \left( H_{\mathcal{G}_{f,m}^{(s:\min)}} \right)$, as $C^*$-subalgebras of $B(H_{\mathcal{G}}) \subseteq B(H)$. □

The proof of the previous proposition is done by the previous lemmas and proposition, by induction.

**Notation** In the rest of this paper, we will keep using the same notations used in the previous lemmas and propositions. □

Now, we will combine all previous observations. The following theorem provides the topological free block structures in $C^*(\mathcal{G})$.

**Theorem 4.13.** Let $\mathcal{G}$ be a finite family of partial isometries in $B(H)$. Define the set $\mathcal{G}_{W}^{\min}$ of partial isometries induced by $\mathcal{G}_{W}$ of $\mathcal{G}$ by

$$\mathcal{G}_{W}^{\min} = \mathcal{G}^{(u)} \cup \mathcal{G}_{(s)}^{(\infty)} \cup \mathcal{G}_{f}^{(s:\min)} = \mathcal{G}^{(u)} \cup \mathcal{G}_{(s)}^{(\infty)} \cup \left( \bigcup_{m=1}^{k} \mathcal{G}_{f;m}^{(s:\min)} \right),$$

Then the $C^*$-subalgebra $C^*(\mathcal{G})$ of $B(H)$ is $\ast$-isomorphic to

$$(\mathbb{C} \cdot 1_{H_{\mathcal{G}}}) \otimes_{\mathbb{C}} C^* \left( \mathcal{G}_{W}^{\min} \right),$$

and hence it is $\ast$-isomorphic to

$$\mathcal{A}_{\ast \log} \left( \mathbb{C} \cdot 1_{H_{\mathcal{G}}} \otimes_{\mathbb{C}} C(\text{spec}(u)) \right) \mathcal{A}_{\ast \log} \left( \mathbb{C} \cdot 1_{H_{\mathcal{G}}} \otimes_{\mathbb{C}} M_{2}(\mathbb{C}) \right) \mathcal{A}_{\ast \log} \left( \bigoplus_{m=1}^{k} T_{m} \right),$$

where $T_{m} = T \left( H_{\mathcal{G}_{f,m}^{(s:\min)}} \right)$ are the Toeplitz algebras, for all $m = 1, \ldots, k$.

**Proof.** Let $\mathcal{G}$ be a finite family of partial isometries in $B(H)$ and let $\mathcal{G}_{W}^{\min}$ be given as above. Then

$$C^* \left( \mathcal{G} \right) \ast\text{-isomorphic} C^* \left( \mathcal{G} \right) \ast\text{-isomorphic} (\mathbb{C} \cdot 1_{H_{\mathcal{G}}}) \otimes_{\mathbb{C}} C^* \left( \mathcal{G}_{W}^{\min} \right)$$

by the previous lemmas and propositions.
\[\text{*-isomorphic} = (C \cdot 1_{H_u}) \otimes C \left( C^* \left( \mathcal{G}^{(s)} \right) \right)\]

\[\text{*-isomorphic} = \left( \mathcal{A}_u \right)_{u \in \mathcal{G}^{(u)}} \right) \left( \mathcal{A}_s \right)_{s \in \mathcal{G}^{(s)}} \right) \left( \mathcal{T}_m \right)_{m=1}^k\]

where \( \mathcal{A}_x = C^*\left( \{x\} \right) \), for all \( x \in \mathcal{G}_W \), and \( \mathcal{T}_m = T \left( H_{\mathcal{G}_f^{(s)}} \right) \), by the previous propositions, for all \( m = 1, \ldots, k \)

\[\text{*-isomorphic} = \left( \mathcal{A}_u \right)_{u \in \mathcal{G}^{(u)}} \right) \left( \mathcal{A}_s \right)_{s \in \mathcal{G}^{(s)}} \right) \left( \mathcal{T}_m \right)_{m=1}^k\]

by the previous lemmas. Recall that

\[\mathcal{A}_u \text{ *-isomorphic } (C \cdot 1_{H_u}) \otimes C \left( \text{spec}(u) \right), \text{ for all } u \in \mathcal{G}^{(u)} , \]

and

\[\mathcal{A}_s \text{ *-isomorphic } (C \cdot 1_{H_s}) \otimes C \left( \text{spec}(s) \right), \text{ for all } s \in \mathcal{G}^{(s)} .\]

The above theorem provides the characterization of the block structures of \( C^*\)-subalgebras generated by finitely many partial isometries in terms of the algebraic (non-reduced) free product and the characterization of \( C^*\)-subalgebras generated by a single partial isometry.

4.3. Examples.

In this section, we will consider some examples.

Example 4.1. In this first example, we will consider the graph-families we observed in [13]. Let \( \mathcal{G} = \{a_1, \ldots, a_N\} \) be a finite family of partial isometries in \( B(H) \). And assume that \( \mathcal{G} \) constructs a finite directed graph \( G \) in the sense that

(i) \( |\mathcal{G}| = |E(G)| \) and \( |\mathcal{G}_p| = |V(G)| \), where

\[\mathcal{G}_p \overset{def}{=} \{a_j^* a_j : j = 1, \ldots, N\} \cup \{a_j a_j^* : j = 1, \ldots, N\},\]
with the corresponding bijection $h_E : E(G) \to \mathcal{G}$.

(ii) the edges $e_1$ and $e_2$ generate nonempty finite path $e_1 e_2$ if and only if $H_{\text{init}}^{h_E(e_1)} = H_{\text{fin}}^{h_E(e_2)}$, where the symbol “$H_1 = H_2$” means that $H_1$ and $H_2$ are identically same as subspaces of $H$. So, “$H_1 \neq H_2$” means that $H_1 \cap H_2 = \{0_H\}$.

The above graph-family-setting makes us understand the elements $a_j$ satisfies that either $a_j = u_j$ or $a_j = s_j$, where $u_j$ are the unitary parts of $a_j$ and $s_j$ are the shift parts of $a_j$. Moreover, by the condition (ii), if $a_j = s_j$, for some $j \in \{1, \ldots, N\}$, then its $*$-isomorphism index $\iota_e(a_j) = (0, \varepsilon, \infty, 0)$, for some $\varepsilon \in \mathbb{N}_0^\infty$. Thus we can get that the $C^*$-subalgebra $C^*(\{a_j\})$ generated by a single partial isometry $a_j \in \mathcal{G}$ is $*$-isomorphic to either

$$(C \cdot 1_{H_{\text{fin}}}) \otimes_C C(T)$$

(if $h_E^{-1}(a_j)$ is a loop-edge in $E(G)$) or

$$(C \cdot 1_{H_{\text{init}}}) \otimes_C M_2(\mathbb{C})$$

(if $h_E^{-1}(a_j)$ is a non-loop edge in $E(G)$). i.e., the $\mathcal{G}_W$-admissibility is completely determined by (ii), and hence, we have either,

$$\pi(a_i, a_j) = 0$$

or

$$\pi(a_i, a_j) = (a_i^* a_i) (a_j a_j^*) = a_i^* a_i = a_j a_j^*,$$

for all $i, j \in \{1, \ldots, N\}$. In other words, in this case, we have that

$$\mathcal{G} = \mathcal{G}_W = \mathcal{G}_W^{(u)} \cup \mathcal{G}_W^{(s)}, \text{ with } \mathcal{G}_W^{(f)} = \emptyset.$$  

Therefore, indeed, the $C^*$-algebras $C^*(\{a_j\})$ are $*$-isomorphic to either

$$(C \cdot 1_{H_0}) \otimes_C C(T) \text{ or } (C \cdot 1_{H_0}) \otimes_C M_2(\mathbb{C}),$$

by Section 3.1. Moreover, if $G$ is connected in the sense that, for any $(v_1, v_2) \in V(\hat{G}) \times V(\hat{G})$, there always exists at least one reduced finite path $w \in FP_1(\hat{G})$ such that $w = v_1 w v_2$ and $w^{-1} = v_2 w^{-1} v_1$, then the $C^*$-subalgebra $C^*(\mathcal{G})$ generated by $\mathcal{G}$ is $*$-isomorphic to the affiliated matricial graph $C^*$-algebra $M_G(H_0)$ which is $*$-isomorphic to

$$(C \cdot 1_{H_0}) \otimes_C M_G,$$
where $\mathcal{M}_G$ is the matricial graph $C^*$-algebra which is the $C^*$-subalgebra of $M_n(\mathbb{C})$ (See [13]), and where $H_0$ is the Hilbert space that is Hilbert-space isomorphic to the initial and the final subspaces of all elements in $\mathcal{G}$. Notice that, indeed, $H_0 = H_G$ in $H$.

So, in general, if the family $\mathcal{G}$ constructs a finite directed graph $G$ in the above sense and if $G$ has its connected components $G_1$, ..., $G_t$, for some $t \in \mathbb{N}$, where $G_1$, ..., $G_t$ are the connected full-subgraphs of $G$. Then we can decompose $\mathcal{G}$ into the disjoint union of $\mathcal{G}_1$, ..., $\mathcal{G}_t$, constructing $G_1$, ..., $G_t$, respectively. Then the $C^*$-algebra $C^*(\mathcal{G})$ is $\bigoplus_{j=1}^t C^*(\mathcal{G}_j)$ which is *-isomorphic to the direct sum of the affiliated matricial graph $C^*$-algebras:

$$C^*(\mathcal{G}) = \bigoplus_{j=1}^t C^*(\mathcal{G}_j) \quad \text{*-isomorphic to} \quad \bigoplus_{j=1}^t \mathcal{M}_G(H_j),$$

where $H_j$’s are the embedded subspaces of $H$ which are Hilbert-space isomorphic to $p_j H$, for all $p_j \in C^*(\mathcal{G}^{(j)}_p)$, for all $j = 1, \ldots, t$.

**Example 4.2.** Let $U$ be the unilateral shift on the Hilbert space $H = l^2(\mathbb{N}_0)$, and let $k_1 > k_2 \in \mathbb{N}$. We can have the shift operators $U^{k_1}$ and $U^{k_2}$ on $H$. Then they generate the $C^*$-subalgebras $C^*(\{U^{k_1}\})$ and $C^*(\{U^{k_2}\})$ of $B(H)$, which are *-isomorphic to the classical Toeplitz algebra $T(H)$. By hypothesis, $\pi(U^{k_1}, U^{k_2}) \neq 0$ on $H$. Therefore, $C^*(\{U^{k_1}, U^{k_2}\})$ is *-isomorphic to the classical Toeplitz algebra $T(H)$, by Section 4.2, since $H_{\{U^{k_1}, U^{k_2}\}}$ is Hilbert-space isomorphic to $H = l^2(\mathbb{N}_0)$.

Indeed, if $\mathcal{G}_W = \{U^{k_1}, U^{k_2}\}$, then $\mathcal{G}_W^{(\text{max})} = \{U^{k_1}\}$. Therefore, we can get the desired result.

**Example 4.3.** Let $k \in \mathbb{N}$ and $U$, the unilateral shift on

$$H = l^2(\mathbb{N}_0) \oplus l^2(\mathbb{N}_0) \quad \text{denote} \quad H_1 \oplus H_2.$$

Define the operator $V$ on $H$ by

$$V \overset{\text{def}}{=} \begin{pmatrix} 0_H & \ast_H \\ 1_H & 0_H \end{pmatrix} \quad \text{on} \quad H_1 \oplus H_2.$$

Consider the family $\mathcal{G} = \{x_1, x_2\}$, where $x_1 = U^k$ and $x_2 = V$. We can check that

$$i_*(x_1) = (0, 0, k, 0) \quad \text{and} \quad i_*(0, 0, \infty, 0) \quad \text{in} \quad (\mathbb{N}_0^\infty)^4.$$

Therefore, the $C^*$-subalgebras $\mathcal{A}_{x_k} = C^*(\{x_k\})$ of $B(H)$, for $k = 1, 2$, satisfy that
$A_{x_1} \ast$-isomorphic $= T(H),$

and

$A_{x_2} \ast$-isomorphic $= (\mathbb{C} \cdot 1_{H_1}) \otimes_\mathbb{C} M_2(\mathbb{C}),$

by Section 3.1. Also, we can have the family $G_G = \{G_{x_1}, G_{x_2}\}$ of the corresponding graphs $G_{x_k}$ of $x_k$, for $k = 1, 2$. The graphs are

$$G_{x_1} = \begin{array}{c}
\bullet \xrightarrow{x_1} \bullet \xrightarrow{x_1^*} \bullet \xrightarrow{x_1^{(2)}} \bullet \xrightarrow{x_1^{*}} \bullet \xrightarrow{x_1^{(4)}} \bullet \xrightarrow{x_1^{*}} \bullet \xrightarrow{x_1^{(5)}} \bullet
\end{array}$$

and

$$G_{x_2} = \begin{array}{c}
\bullet \xrightarrow{x_2} \bullet
\end{array}$$

We can check that

$$\pi(x_1^n, x_2^m) = (x_1^{n*} x_1^n) (x_2^m x_2^{m*}) = \begin{cases}
x_2^m x_2^{m*} 
eq 0 & \text{if } m = 1 \\
0 & \text{if } m \neq 1
\end{cases},$$

since $x_2^k = 0$, for all $k \in \mathbb{N} \setminus \{1\}$, and

$$\pi(x_1^{n*}, x_2^{m*}) = (x_1^{n*} x_1^n) (x_2^{m*} x_2^m) = \begin{cases}
(1_H)(1_H \oplus 0) = 1_H & \text{if } m = 1 \\
0 & \text{if } m \neq 1
\end{cases},$$

since $x_2^k = 0$, for all $k \in \mathbb{N} \setminus \{1\}$, for all $n, m \in \mathbb{N}$. Thus we can get the conditional iterated glued graph $G = G_{x_1} \# = G_{x_2}$ as the graph which is graph-isomorphic to the following graph:

$$G = \begin{array}{c}
\bullet \xrightarrow{\vdots} \bullet \xrightarrow{\vdots} \bullet \xrightarrow{\vdots} \bullet \xrightarrow{\vdots} \bullet \xrightarrow{\vdots} \bullet \xrightarrow{\vdots} \bullet
\end{array}$$

By Section 4.2, the $C^*$-algebra $A = C^*_r(G)$ is $\ast$-isomorphic to

$$A \ast$-isomorphic $= \overline{T(H) \ast_{alg} ((\mathbb{C} \cdot 1_{H_1}) \otimes_\mathbb{C} M_2(\mathbb{C}))}.$$

**Example 4.4.** Let $H_{\text{Hilbert-Space}} = l^2(\mathbb{N}_0)$ having its Hilbert basis $B_H = \{\xi_n : n \in \mathbb{N}_0\}$, where

$$\xi_0 = (1, 0, 0, 0, \ldots)$$

and
\[ \xi_n = \begin{pmatrix} 0, \ldots, 0, 1, 0, \ldots \end{pmatrix}, \text{ for all } n \in \mathbb{N}. \]

Define an operator \( x \in B(H) \) by the operator having its infinite matricial form,

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & \cdots \\
\vdots & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix},
\]

i.e., the operator \( x \) satisfies that

\[
(t_0, t_1, t_2, \ldots, \ldots) \xrightarrow{x} (0, t_0, 0, t_1, 0, t_2, 0, t_3, 0, \ldots),
\]

for all \((t_0, t_1, t_2, \ldots) \in H\). The adjoint \( x^* \) of \( x \) has its infinite matricial form,

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix},
\]

so, it satisfies that

\[
(t_0, t_1, t_2, t_3, t_4, \ldots) \xrightarrow{x^*} (t_1, t_3, t_5, \ldots),
\]

for all \((t_0, t_1, t_2, \ldots) \in H\).

Instead of observing the Wold decomposition of \( x \), we will consider \( x \) by the operator determined by the (finite dimensional) shifts \((y_n)_{n=0}^{\infty}\). First, we define the following subspaces \( K_n \)'s of \( H \):

\[
K_n \overset{def}{=} \mathbb{C} : \xi_n \quad \text{Hilbert Space} = \mathbb{C} \quad \text{Subspace} \subset H, \text{ for all } n \in \mathbb{N}_0.
\]

Now, define the operators \( y_n \) by
\[ y_0 : (t, 0, 0, 0, \ldots) \mapsto (0, t, 0, 0, 0, \ldots) \]

and

\[ y_n : \begin{pmatrix} 0, \ldots, 0, t, 0, 0, \ldots \end{pmatrix}_{n\text{-times}} \mapsto \begin{pmatrix} 0, \ldots, 0, t, 0, 0, \ldots \end{pmatrix}_{(2n+1)\text{-times}} \]

on \( H \), for all \( n \in \mathbb{N} \) and \( t \in \mathbb{C} \). Then each operator \( y_n \) is a partial isometry on \( H \) having its initial space and its final space as follows:

\[ H_{\text{init}}^{y_0} = K_0 \quad \text{and} \quad H_{\text{fin}}^{y_0} = K_1 \]

and

\[ H_{\text{init}}^{y_n} = K_n \quad \text{and} \quad H_{\text{fin}}^{y_n} = K_{2n+1}, \quad \text{for all} \quad n \in \mathbb{N}. \]

Then the given operator \( x \) is defined by

\[ x \xi_k \overset{\text{def}}{=} y_k \xi_k, \quad \text{for all} \quad k \in \mathbb{N}_0, \quad \xi_k \in B_H. \]

i.e., we can define

\[ x \left( \sum_{k=0}^{\infty} t_k \xi_k \right) \overset{\text{def}}{=} \sum_{k=0}^{\infty} t_k y_k(\xi_k) = \sum_{k=0}^{\infty} t_k \xi_{2k+1}, \]

for all \( \sum_{k=0}^{\infty} t_k \xi_k \in H \), where \( t_k \in \mathbb{C} \). Now, construct the subspaces \( (H_n)_{n=0}^{\infty} \) of \( H \) by

\[ H_n = K_n \oplus K_{2n+1}, \quad \text{for all} \quad n \in \mathbb{N}_0. \]

Note that \( H_n^{\text{Hilbert Space}} = \mathbb{C}^{\oplus 2} \), for all \( n \in \mathbb{N}_0 \). Then the operators \( y_n \) can be understood as shifts on \( H_n \). (Notice that they are regarded as shifts on finite dimensional space \( \mathbb{C}^{\oplus 2} \).) Moreover, the operators \( y_n \) on \( H_n \) have their matricial forms,

\[ y_n = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \text{on} \quad H_n = \mathbb{C}^{\oplus 2} = K_n \oplus K_{2n+1} \]

for all \( n \in \mathbb{N}_0 \). Therefore, the \( C^* \)-algebras \( C^*(\{y_n\}) \) generated by the shifts \( y_n \) are \(*\)-isomorphic to \( M_2(\mathbb{C}) = B(H_n) \), as embedded \( C^* \)-subalgebras of \( B(H) \), for all \( n \in \mathbb{N}_0 \). Similar to Section 3.3, if we let \( G = \{y_n \in B(H) : n \in \mathbb{N}_0\} \) (by regarding \( y_n \)'s as operators on \( H \) ), then we get the \( G \)-space \( H_G = H \), and we can have the \( G \)-action \( \alpha \) of the \( G \)-groupoid \( G \), where

\[ G_{y_n} = \bullet \xrightarrow{y_n} y_n \quad \text{for all} \quad n \in \mathbb{N}_0. \]
Define a map \( f : \mathbb{N}_0 \to \mathbb{N}_0 \) by
\[
f(m) = 2m + 1, \text{ for all } m \in \mathbb{N}_0.
\]
We will denote the iterated compositions \( f \circ \ldots \circ f \) by \( f^{(k)} \), for all \( k \in \mathbb{N} \).
Then we can have the subset \( X_{(n)} \) of \( \mathbb{N} \) by
\[
X_{(n)} \overset{\text{def}}{=} \{n\} \cup \{f^{(k)}(n) : k \in \mathbb{N}\} \subset \mathbb{N}_0,
\]
for all \( n \in \mathbb{N}_0 \). It is easy to check that if \( n \in \mathbb{N}_0 \), then, for any \( k \in \mathbb{N} \), the sets \( X_{(f^{(k)}(n))} \) are contained in \( X_{(n)} \), i.e.,
\[
X_{(f^{(k)}(n))} \subset X_{(n)}, \text{ for all } n \in \mathbb{N}_0 \text{ and } k \in \mathbb{N}.
\]
For instance, \( X_{(7)} \subset X_{(3)} \subset X_{(1)} \subset X_{(0)} \), etc. This means that we can take chains under the usual set-inclusion on the collection \( \mathcal{Y} = \{X_{(n)} : n \in \mathbb{N}_0\} \) of \( X_{(n)} \)'s. Denote \( \mathcal{X} \) by the collection of all maximal elements of \( \mathcal{Y} \), under the partial ordering \( \subset \). For instance, \( X_{(0)}, X_{(2)}, X_{(4)}, X_{(6)} \) are the first five elements of \( \mathcal{X} \).
Let \( I \) be the subset of \( \mathbb{N}_0 \) defined by
\[
I = \{n \in \mathbb{N}_0 : X_{(n)} \in \mathcal{X}\} \subset \mathbb{N}_0.
\]
Let \( \mathcal{G} = \{y_0, y_1, y_2, y_3, \ldots\} \) and let \( \mathcal{G}_I = \{G_{(n)} : n \in I\} \). Then we can define the conditional iterated glued graph
\[
G = \#_{n \in I} G_{(n)},
\]
where
\[
G_{(n)} = G_n \, \#_{\pi} \left( \# _{k=1}^{ \infty} G_{y_{2k+1}} \right)
\]
\[
= y_n \xrightarrow{y_{n+1}} y_n y_{n+1} y_{n+1} \xrightarrow{y_{2n+1}} y_{2n+1} y_{2n+1} y_{2n+1} \xrightarrow{y_{2(2n+1)+1}} \ldots,
\]
for all \( n \in I \). Clearly, we can create the corresponding graph groupoid \( \mathbb{G} \) of \( G \). Then we can determine \( H_{\mathbb{G}} = H \) and the groupoid action \( \alpha \) of \( \mathbb{G} \) on \( H \), like Section 3.3. We can check that
\[
C^*(\mathbb{G}) \overset{\text{*-isomorphic}}{=} C_\alpha^*(\mathbb{G}),
\]
by Section 3.4. By Section 4, we can check that the graph groupoid \( \mathbb{G}_{(n)} \) of \( G_{(n)} \) generates the \( C^* \)-algebra \( C_\alpha^*(\mathbb{G}_{(n)}) \) satisfies that
\[
C^*_\alpha(\mathbb{G}(n)) \overset{\ast\text{-isomorphic}}{\overset{\cong}{\ast}} \bigotimes_{k=1}^{\infty} \mathcal{M}_k, \text{ with } \mathcal{M}_k = B(\mathbb{C}\xi_k \oplus \mathbb{C}\xi_{2k+1}),
\]

for all \( k \in \mathbb{N} \), and \( n \in I \). Notice that \( \mathcal{M}_k \overset{\ast\text{-isomorphic}}{\overset{\cong}{=}} M_2(\mathbb{C}) \), for all \( k \in \mathbb{N} \).

### 5. Extensions of Unbounded Operators

In this Section, we will consider an application of our results to Unbounded Operator Theory, in particular, the Cayley Transform Theory. Let \( H \) be a countably infinite dimensional Hilbert space and let \( D \) be a dense subspace of \( H \). A linear unbounded operator \( S \) defined on \( D \) is said to be Hermitian if

\[
< S \xi_1, S \xi_2 > = < S \xi_2, S \xi_1 >
\]

holds for all \( \xi_1, \xi_2 \in D \). Notice that, the relation (6.1) means that the dense subspace \( D \) is contained in the domain of the adjoint operator \( S^* \) of \( S \), and that

\[
S \xi = S^* \xi, \text{ for all } \xi \in D.
\]

In other words, the graph \( \mathcal{G}(S) \) of \( S \) is contained in \( \mathcal{G}(S^*) \) of \( S^* \). Here, the graph \( \mathcal{G}(S) \) of \( S \) means that the subset \( \{ (\xi, S \xi) : \xi \in D \} \) of \( H \times H \). We denote this relation by

\[
S \subseteq S^*.
\]

We are interested in finding Hermitian elements \( T \) of \( S \); i.e., Hermitian operators \( T \) such that

\[
\mathcal{G}(S) \subseteq \mathcal{G}(T).
\]

**Definition 5.1.** The Cayley transform \( V \) of \( S \) is a unbounded operator

\[
V \overset{\text{def}}{=} (S + i)(S - i)^{-1}, \quad \text{i.e.,}
\]

\[
V(S \xi - i\xi) = S \xi + i\xi, \text{ for all } \xi \in D,
\]

where \( i \) of (6.3) means \( i1_H \), with the imaginary number \( i \) satisfying \( i^2 = -1 \) in \( \mathbb{C} \), where \( 1_H \) means the identity operator on \( H \).
It is immediate that the given unbounded operator $S$ is Hermitian if and only if the Cayley transform $V$ is a partial isometry. Moreover, the orthogonal compliment $E_+$ of the operator $V^* V$ satisfies
\begin{equation}
E_+ = \ker (S^* - i) = \{ \xi_+ \in \text{dom}(S^*) | S^* \xi_+ = i \xi_+ \},
\end{equation}
where $\text{dom}(T)$ means the domain of an operator $T$. Also, the orthogonal compliment $E_-$ of $V^* V$ satisfies
\begin{equation}
E_- = \ker (S^* + i) = \{ \xi_- \in \text{dom}(S^*) | S^* \xi_- = -i \xi_- \}.
\end{equation}

**Definition 5.2.** The subspaces $E_+$ and $E_-$ are called the defect spaces or the deficiency space of $V$. And the numbers
\begin{equation}
n_+ \overset{\text{def}}{=} \dim E_+ \text{ and } n_- \overset{\text{def}}{=} \dim E_-
\end{equation}
are called the deficiency indices of $V$.

**Lemma 5.1.** (See [48]) A Hermitian symmetric operator $S$ has its self-adjoint extensions $T$ if and only if $n_+ = n_-$. In this case, all self-adjoint extensions $T$ satisfy that
\begin{equation}
S \subseteq T \subseteq S^*.
\end{equation}
Moreover, there is an one-to-one correspondence between all the unitary extensions $U$ of the Cayley transform $V$ and all the self-adjoint extensions $T$ of $S$:
\begin{enumerate}
  \item If $U$ is a unitary extension of $V$, i.e., $\mathcal{G}(V) \subseteq \mathcal{G}(U)$, then
  \begin{equation}
  T \overset{\text{def}}{=} i(U + 1)H(U - 1H)^{-1}
  \end{equation}
is a self-adjoint extension of $S$.
  \item If conversely $T$ is a self-adjoint extension of $S$, then
  \begin{equation}
  U = (T + i)(T - i)^{-1}
  \end{equation}
is a unitary extension of $V$.
\end{enumerate}
(iii) In general, the correspondence between (i) and (ii) holds for the family of all Hermitian extensions $V_T$ of $V$, and all the Hermitian extensions $T$ of $S$; and the correspondence is decided by

\begin{align}
T &= i (V_T + 1_H) (V_T - 1_H)^{-1} \\
V_T &= (T + i) (T - i)^{-1},
\end{align}

respectively. □

We provide several examples to illustrate our purpose.

**Example 5.1.** (1) Let $H = L^2(0, 1)$ and let

\begin{align}
S &\stackrel{\text{def}}{=} \frac{1}{i} \frac{d}{dx} \\
\mathcal{D} &= \text{dom}(S) = C^1_0(0, 1) = \text{all } C^1\text{-functions } \varphi \text{ on } [0, 1] \\
&\text{such that} \\
\varphi(0) &= \varphi(1) = 0.
\end{align}

Then the operator $S$ has its deficiency indices

\[ n_+ = 1 = n_- \]

and the defect spaces are

\begin{align}
\mathcal{E}_+ &= \mathbb{C} \cdot e^{-x} \\
\mathcal{E}_- &= \mathbb{C} \cdot e^x.
\end{align}

(2) Every $\theta \in [0, 2\pi)$ determines a unique self-adjoint extension $T_\theta$ of $S$ as follows. Set

\[ \mathcal{B}_\theta \stackrel{\text{def}}{=} \{ e^{i(\theta + 2\pi n)x} : n \in \mathbb{Z} \} . \]

Then

\[ \mathcal{B}_\theta \subseteq \text{dom}(T_\theta) \]

and

\[ T_\theta e^{i(\theta + 2\pi n)x} = (\theta + 2\pi n) e^{i(\theta + 2\pi n)x}, \text{ for } n \in \mathbb{Z}, \]

i.e.,

\[ \text{spec}(T_\theta) = \theta + 2\pi \mathbb{Z} \text{ in } \mathbb{C}, \]

where $\text{spec}(T_\theta)$ means the spectrum of $T_\theta$.

The following proposition is the main result of this Section.
Proposition 5.2. Let $S$ be a Hermitian symmetric operator with its dense domain in a Hilbert space $H$, and assume that $S$ has the deficiency indices $(1, 1)$. Then all the self-adjoint extensions of $S$ have Cayley transform

$$U = V \oplus W,$$

where $V$ is the Cayley transform of $S$ determined in (6.3), and where $W$ has $W^* W$ and $W W^*$ of rank 1. Moreover, there exists $\alpha \in \mathbb{C}$, $|\alpha| < 1$, such that

$$(6.18) \quad W^2 = \alpha W.$$ 

Proof. The defect spaces $\mathcal{E}_+$ and $\mathcal{E}_-$ in (6.5) and (6.6) are the initial and final spaces of the partial isometries $W$ from the previous lemma which extend the Cayley transform $V$ of $S$. So,

$$W^* W = \mathcal{E}_+ \quad \text{and} \quad W W^* = \mathcal{E}_-.$$ 

Since $\dim \mathcal{E}_{\pm 1} = 1$, there are vectors $e_{\pm 1} \in \mathcal{D}$ such that $\|e_{\pm 1}\| = 1$ and

$$\mathcal{E}_+ = \mathbb{C} \cdot e_+ \quad \text{and} \quad \mathcal{E}_- = \mathbb{C} \cdot e_-.$$ 

It follows that

$$(6.19) \quad W = |e_-><e_+|,$$ 

where we use the Dirac notation,

$$(6.20) \quad |\xi><\eta| \zeta \defeq <\eta, \zeta > \xi, \quad \text{for all} \; \zeta, \eta, \xi \in H.$$ 

Hence we have that

$$W^2 = <e_+, e_-> W$$

and

$$\alpha = <e_+, e_->$$

works in (6.18).

If $|\alpha| < 1$, then, by Schurz, it would follow that $\mathcal{E}_+ = \mathcal{E}_-$, which contradicts

$$S^* e_{\pm 1} = \pm i e_{\pm 1},$$
from (6.5) to (6.6).

By the previous proposition, we can get the following corollary.

**Corollary 5.3.** Let \( W \) be as in the previous proposition. Then \( W^n \to 0 \), as \( n \to \infty \).

**Proof.** By (6.18) and by Induction, we have that

\[
W^{n+1} = \alpha^n W, \text{ for all } n \in \mathbb{N}.
\]

Thus \( W^n \to 0 \), as \( n \to \infty \).

---

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