PERFECT CRYSTALS AND $q$-DEFORMED FOCK SPACES

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Abstract. In [S, KMS] the semi-infinite wedge construction of level 1 $U_q(A^{(1)}_n)$ Fock spaces and their decomposition into the tensor product of an irreducible $U_q(A^{(1)}_n)$-module and a bosonic Fock space was given. Here a general scheme for the wedge construction of $q$-deformed Fock spaces using the theory of perfect crystals is presented.

Let $U_q(g)$ be a quantum affine algebra. Let $V$ be a finite-dimensional $U'_q(g)$-module with a perfect crystal base of level $l$. Let $V_{\text{aff}} \simeq V \otimes \mathbb{C}[z, z^{-1}]$ be the affinization of $V$, with crystal base $(L_{\text{aff}}, B_{\text{aff}})$. The wedge space $V_{\text{aff}} \wedge V_{\text{aff}}$ is defined as the quotient of $V_{\text{aff}} \otimes V_{\text{aff}}$ by the subspace generated by the action of $U_q(g)[z^a \otimes z^b + z^b \otimes z^a]_{a,b \in \mathbb{Z}}$ on $v \otimes v$ ($v$ an extremal vector). The wedge space $\wedge^r V_{\text{aff}}$ ($r \in \mathbb{N}$) is defined similarly. Normally ordered wedges are defined by using the energy function $H : B_{\text{aff}} \otimes B_{\text{aff}} \rightarrow \mathbb{Z}$. Under certain assumptions, it is proved that normally ordered wedges form a base of $\wedge^r V_{\text{aff}}$.

A $q$-deformed Fock space is defined as the inductive limit of $\wedge^r V_{\text{aff}}$ as $r \rightarrow \infty$, taken along the semi-infinite wedge associated to a ground state sequence. It is proved that normally ordered wedges form a base of the Fock space and that the Fock space has the structure of an integrable $U_q(g)$-module. An action of the bosons, which commute with the $U'_q(g)$-action, is given on the Fock space. It induces the decomposition of the $q$-deformed Fock space into the tensor product of an irreducible $U_q(g)$-module and a bosonic Fock space.

As examples, Fock spaces for types $A^{(2)}_{2n}$, $B^{(1)}_n$, $A^{(2)}_{2n-1}$, $D^{(1)}_n$ and $D^{(2)}_{n+1}$ at level 1 and $A^{(1)}_1$ at level $k$ are constructed. The commutation relations of the bosons in each of these cases are calculated, using two point functions of vertex operators.

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1. Introduction

Let \( g \) be an affine Lie algebra. The construction of integrable highest weight modules for \( g \) has been studied extensively for more than 15 years, with applications to problems in mathematical physics like soliton equations and conformal field theories. More recently, a further item was added to the list of interactions between representation theory and integrable systems: the link between quantum affine algebras, \( U_q(\mathfrak{g}) \), and solvable lattice models (see [JM] and references therein).

The link is twofold: (a) the R-matrices, which appear as the Boltzmann weights of solvable lattice models, are intertwiners of level 0 \( U_q(\mathfrak{g}) \)-modules, and (b) the irreducible integrable highest weight modules for \( U_q(\mathfrak{g}) \) appear as the spaces of the eigenvectors of the corner transfer matrices. This suggests a construction of integrable highest weight modules by means of semi-infinite tensor products of level 0 modules. In fact, in the crystal limit, such a construction was given for a large class of representations known as the representations with perfect crystals [KMN1].

The idea of using Fock spaces of bosons or fermions goes back to earlier works before the above link was found. In fact, the literature is vast. Let us mention some of the works that are closely related to the present work. In [LW, KKLW], bosonic Fock spaces were used to construct some level 1 highest weight modules of affine Lie algebras using the fact that the actions of the principal Heisenberg subalgebras are irreducible. In [DJKM] the level 1 highest weight modules of \( \mathfrak{gl}_\infty \) were constructed in the fermionic Fock space. By the boson-fermion correspondence one has the action of bosons on the Fock space. The action of affine Lie algebras such as \( \mathfrak{sl}_n \), as subalgebras of \( \mathfrak{gl}_\infty \), was then realized as the commutant of bosons of degree divisible by \( n \). Likewise, level 1 highest weight modules of other affine Lie algebras \( \mathfrak{g} \) were constructed by realizing \( \mathfrak{g} \) as a subalgebra of \( \mathfrak{go}_\infty \) (see also [JY]) or \( \mathfrak{go}_{2\infty} \).

Under the influence of quantum groups, several developments were made further in this direction. A \( q \)-deformed construction of the fermion Fock space was achieved in [H]. In [MM], this was connected to the crystal base theory of Kashiwara [K1]. These works and the developments in solvable lattice models led to the semi-infinite construction of affine crystals mentioned above.

Very recently, in [S], Stern gave a semi-infinite construction of the level 1 Fock spaces for \( U_q(\mathfrak{g}) \) when \( \mathfrak{g} = \mathfrak{sl}_n \). Subsequently, in [KMS], the decomposition of the Fock spaces into the level 1 irreducible highest weight modules and the bosonic Fock space, was given. In the present paper, we give a similar construction of Fock spaces and their decomposition, for various cases in the class of representations with perfect crystals. The case in [S, KMS] corresponds to the perfect crystal of level 1 for \( A^{(1)}_n \). Here we treat

\[
\text{level 1 } A^{(2)}_{2n}, B^{(1)}_n, A^{(2)}_{2n-1}, D^{(1)}_n, D^{(2)}_{n+1} \quad \text{and} \quad \text{level } k \ A^{(1)}_1.
\]

In order to handle these cases, we not only follow the basic strategy in [S, KMS], but also develop some new machinery, where the R-matrix and crystal bases play an important role.

In the following we recall the basic construction in [KMS] and compare it with the newer version developed in this paper, by taking the examples of level \( l \ A^{(1)}_1 \),
Let $V$ be a finite-dimensional $U'_q(g)$-module, and $V_{\text{aff}} = V \otimes \mathbb{C}[z, z^{-1}]$ its affinization. The $r$-th $q$-wedge space is given by

$$\bigwedge^r V_{\text{aff}} = V_{\text{aff}}^\otimes r / N_r,$$

where

$$N_r = \sum_{i=0}^{r-2} V_{\text{aff}}^\otimes i \otimes N \otimes V_{\text{aff}}^\otimes (r-2-i)$$

and the space $N$ is a certain subspace of $V_{\text{aff}} \otimes V_{\text{aff}}$. Namely, the $q$-wedge space is defined as a quotient of the tensor product of $V_{\text{aff}}$ modulo certain relations of nearest neighbour type.

For the level 1 $A^{(1)}_1$ case, the space $V$ is the 2-dimensional representation of $U'_q(\widehat{\mathfrak{sl}}_2)$, $V = \mathbb{Q}v_0 \oplus \mathbb{Q}v_1$. In [S, KMS], the action of the Hecke algebra generator $T$ was given on $V_{\text{aff}} \otimes V_{\text{aff}}$, and the space $N$ was defined by

$$N = \text{Ker}(T + 1).$$

It was also noted that $N = U'_q(\widehat{\mathfrak{sl}}_2) \cdot v_0 \otimes v_0$. In this paper, we define, in general,

$$N = U'_q(g)[z \otimes z, z^{-1} \otimes z^{-1}, z \otimes 1 + 1 \otimes z] \cdot v \otimes v,$$

(1.1.1)

where $v$ is an extremal vector in $V_{\text{aff}}$ (see §3.1 for the definition). For $l = 1$, any $z^n v_i$ ($n \in \mathbb{Z}, i = 0, 1$) is extremal. For $l = 2$, we take

$$V = \mathbb{Q}v_0 \oplus \mathbb{Q}v_1 \oplus \mathbb{Q}v_2.$$

The extremal vectors are $z^n v_0$ and $z^n v_2$ ($n \in \mathbb{Z}$). For $l = 1$, in the $q = 1$ limit, the construction gives rise to ordinary wedges with anti-commutation relations

$$z^m v_i \wedge z^n v_j + z^n v_j \wedge z^m v_i = 0.$$

For $l = 2$, this is not the case, e.g. $v_1 \wedge v_1 \neq 0$, even in the $q = 1$ limit.

The definition (1.1.1) is appropriate for computational use. For theoretical use, we have the following equivalent definition

$$N = \text{Ker}(R - 1).$$

Here $R$ is the $R$-matrix acting on $V_{\text{aff}} \otimes V_{\text{aff}}$ (strictly speaking, the image of $R$ belongs to a certain completion of $V_{\text{aff}} \otimes V_{\text{aff}}$).

The $R$-matrix satisfies the Yang-Baxter equation

$$R_{12}R_{23}R_{12} = R_{23}R_{12}R_{23},$$

commutes with the $U_q(g)$-action on $V_{\text{aff}} \otimes V_{\text{aff}}$, satisfies

$$R(z \otimes 1) = (1 \otimes z)R, \quad R(1 \otimes z) = (z \otimes 1)R,$$

and is normalized as

$$R(v \otimes v) = v \otimes v,$$

where $v$ is an extremal vector.
1.2. Energy function and the normal ordering rules. In [KMS], it was shown that the $q$-wedge relations give a normal ordering rule of products of vectors. Define $u_m (m \in \mathbb{Z})$ by

$$z^n v_i = u_{2n-i}. \quad (1.2.1)$$

It was shown that the vectors $u_{m_1} \wedge \cdots \wedge u_{m_r} \quad (m_1 < \cdots < m_r)$

form a base of $\wedge^r V_{\text{aff}}$.

To describe the normal ordering rules in the general case, we use the energy function

$$H : B_{\text{aff}} \otimes B_{\text{aff}} \longrightarrow \mathbb{Z}.$$ 

The set $B_{\text{aff}}$ is the crystal of $V_{\text{aff}}$. For each element $b$ in $B_{\text{aff}}$, we have a corresponding vector $G(b)$ in $V_{\text{aff}}$. In this section we use the same symbol for $b$ and $G(b)$: e.g. a general element of $B_{\text{aff}}$ for the level 1 $A_1^{(1)}$ case and that of $V_{\text{aff}}$ are denoted by $z^n v_i$. The energy function $H$ is such that

$$R(G(b_1) \otimes G(b_2)) = z^{H(b_1 \otimes b_2)} G(b_1) \otimes z^{-H(b_1 \otimes b_2)} G(b_2) \mod qL(V_{\text{aff}}) \otimes L(V_{\text{aff}}),$$

where $L(V_{\text{aff}})$ is the free module generated by $G(b)$ ($b \in B_{\text{aff}}$) over $A \overset{\text{def}}{=} \{ f \in \mathbb{Q}(q); \ f \text{ is regular at } q = 0 \}$. For the level 2 $A_1^{(1)}$ case,

$$B_{\text{aff}} = \{ z^m v_i; m \in \mathbb{Z}, i = 0, 1, 2 \}$$

and

$$H(z^m v_i \otimes z^n v_j) = -m + n + h_{ij}$$

where the $(h_{ij})_{i,j=0,1,2}$ are given by

$$\begin{bmatrix} j = 0 & 1 & 2 \\ i = 0 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 2 & 0 \end{bmatrix}.$$ 

We show that the set of vectors

$$G(b_1) \wedge \cdots \wedge G(b_r)$$

such that

$$H(b_i \otimes b_{i+1}) > 0 \quad (i = 1, \ldots, r - 1) \quad (1.2.2)$$

is a base of $\wedge^r V_{\text{aff}}$.

The vectors satisfying (1.2.2) are called normally ordered wedges. To show that the normally ordered wedges span the $q$-wedge space, we need to write down the basic $q$-wedge relations explicitly. This part of the work is technically much involved. We do it case by case. The generality in handling examples in this paper is narrower than that of [KMN2] because of this limitation.
In [KMS] the linear independence of the normally ordered wedges is proved by reduction to the \( q = 1 \) limit. Since the \( q = 1 \) result is not known for the general case, we prove the linear independence directly by using the Yang-Baxter equation for \( R \) and the crystal base theory.

1.3. Fock representations. In [KMS], the Fock spaces are constructed by means of an inductive limit of \( \Lambda^r V_{\text{aff}} \). In the case of level 1 \( A_1^{(1)} \), we take the sequence \( (u_m)_{m \in \mathbb{Z}} \) as in (1.2.1). The Fock space \( F_m \) is defined as the space spanned by the semi-infinite wedges

\[
 u_{j_1} \wedge u_{j_2} \wedge u_{j_3} \wedge \cdots
\]

such that \( j_k = m + k - 1 \) for sufficiently large \( k \). The action of \( U_q(\widehat{\mathfrak{sl}}_2) \) on \( F_m \) is defined by using the semi-infinite coproduct. It was shown that \( F_m \) is the tensor product \( V(\lambda_m) \otimes \mathbb{C}[H_-] \).

Here \( V(\lambda_m) \) is the irreducible highest weight representation with the highest weight \( \lambda_m \), where

\[
 \lambda_m = \begin{cases} 
 \Lambda_1 & \text{if } m \equiv 0 \mod 2; \\
 \Lambda_0 & \text{if } m \equiv 1 \mod 2,
\end{cases}
\]

and \( \mathbb{C}[H_-] \) is the Fock space of the Heisenberg algebra generated by \( B_n \) (\( n \in \mathbb{Z} \setminus \{0\} \)) that acts on \( F_m \) by

\[
 B_n = \sum_{k=1}^{\infty} 1 \otimes \cdots \otimes 1 \otimes z^k \otimes 1 \otimes \cdots.
\]

To construct the Fock spaces in the general case, we use the construction of affine crystals developed in [KMN1]. We assume that \( V \) has a perfect crystal \( B \) of level \( l \). Then we can choose a sequence \( b_m^o \) in \( B_{\text{aff}} \) such that

\[
 \langle c, \varepsilon(b_m^o) \rangle = l, \\
 \varepsilon(b_m^o) = \varphi(b_{m+1}^o), \\
 H(b_m^o \otimes b_{m+1}^o) = 1
\]

(see subsection 3.1 for the definition of \( \varepsilon(b) \) and \( \varphi(b) \)). In the case of level 2 \( A_1^{(1)} \), we have

\[
 b_m^o = \begin{cases} 
 z^k v_j & \text{if } m \text{ is odd}; \\
 z^{k+1-j} v_{2-j} & \text{if } m \text{ is even},
\end{cases}
\]

(1.3.1)

for some \( k \in \mathbb{Z} \) and \( j \in \{0, 1, 2\} \) independent of \( m \). Then we shall define the Fock space \( F_m \) as a certain quotient of the space spanned by the semi-infinite wedges

\[
 G(b_1^o) \wedge G(b_2^o) \wedge G(b_3^o) \wedge \cdots
\]

such that \( b_n = b_{m+n-1}^o \) for sufficiently large \( n \). In particular, the Fock space contains the highest weight vector

\[
 |m\rangle = G(b_m^o) \wedge G(b_{m+1}^o) \wedge G(b_{m+2}^o) \wedge \cdots
\]
with the highest weight

\[ \lambda_m = \begin{cases} 
  j\Lambda_1 + (2 - j)\Lambda_0 & \text{if } m \text{ is odd;} \\
  (2 - j)\Lambda_1 + j\Lambda_0 & \text{if } m \text{ is even.}
\end{cases} \]

The quotient is such that if

\[ H(b \otimes b^*_m) \leq 0 \]

we require that

\[ G(b) \wedge |m\rangle = 0. \]

Here is a significant difference between level 1 $A_n^{(1)}$ and other cases. For the former if $H(b \otimes b^*_m) \leq 0$ then

\[ G(b) \wedge G(b^*_m) \wedge \cdots \wedge G(b^*_m') = 0 \]

for sufficiently large $m'$. But, this is not true in general. The correct statement is that for any $n$ we can find $m'$ such that the $q$-wedge $G(b) \wedge G(b^*_m) \wedge \cdots \wedge G(b^*_m')$ is a linear combination of normally ordered wedges whose coefficients are $O(q^n)$ at $q = 0$. Therefore, we need to impose the separability of the $q$-adic topology, taking the quotient by the closure of $\{0\}$.

It is necessary to check that the action of $U_q(g)$ given by the semi-infinite co-product, is well-defined. A careful study of the $q$-wedges shows that

\[ \Delta^{(\infty/2)}(f_i)|m\rangle = G(\tilde{f_i}b^*_m) \wedge |m + 1\rangle, \tag{1.3.2} \]

where

\[ \Delta^{(\infty/2)}(f_i) = \sum_{n=1}^{\infty} 1 \otimes \cdots \otimes 1 \otimes \tilde{f_i} \otimes t_i \otimes t_i \otimes \cdots. \]

In the case in [KMS], the action of $\Delta^{(\infty/2)}(f_i)$ on each vector in $F_m$ is such that only finitely many terms in the sum are different from 0. This is not true in general. For example, consider the case $k = 1$ and $j = 1$ in (1.3.1). We have $f_1v_1 = |2\rangle v_2$ ($|2\rangle = q + q^{-1}$) and $t_1|m\rangle = q|m\rangle$. Therefore, we have

\[ \Delta^{(\infty/2)}(f_1)(v_1 \wedge v_1 \wedge v_1 \wedge \cdots) = q[2](v_2 \wedge v_1 \wedge v_1 \wedge \cdots) + q[2](v_1 \wedge v_2 \wedge v_1 \wedge \cdots) + \cdots. \]

On the other hand, we have

\[ v_1 \wedge v_2 + q^2 v_2 \wedge v_1 = 0, \]

and hence

\[ \Delta^{(\infty/2)}(f_1)(v_1 \wedge v_1 \wedge \cdots) = v_2 \wedge v_1 \wedge v_1 \wedge \cdots, \]

by summing up

\[ 1 + (-q^2) + (-q^2)^2 + \cdots = \frac{1}{1 + q^2} \]

in the $q$-adic topology.

In general, based on (1.3.2) we can show the well-definedness of the $U_q(g)$-action. The decomposition of the $q$-Fock spaces into the irreducible $U_q(g)$-modules and the bosonic Fock space goes the same as the level 1 $A_n^{(1)}$ case. We carry out the
computation of the exact commutation relations of the bosons in each case, by reducing it to the commutation relations of vertex operators.

The plan of this paper is as follows. We list the notations in section 2. We define the finite $q$-wedges in section 3 and prove that the normally ordered wedges form a base. In section 4, we define the $q$-Fock space and the actions of $U_q(\mathfrak{g})$ and the Heisenberg algebra. We give level 1 examples in section 5 for which we check the conditions assumed in section 3. We compute the level 1 two point functions in section 6 in order to find the commutation relations of the bosons. Section 7 is devoted to a higher level example. We add four appendices. In Appendix A we prove a proposition on crystal base which is necessary in this paper but was not proved in [KMN1]. Appendix B is a proof that the Serre relations follow from the integrability of representations. Appendix C is the computation of the two-point correlation functions of the $q$-vertex operators in the $D_{n+1}^{(2)}$ case. In Appendix D we consider the $q \to 1$ limit for the $A_{2n}^{(2)}$ case and compare it to the result in [JY].

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2. Preliminary

2.1. Notations. In this paper we use the following notations.

\[
\delta(P) = \begin{cases} 
1 & \text{if a statement } P \text{ is true} \\
0 & \text{if } P \text{ is false.}
\end{cases}
\]

$\mathfrak{g}$ : an affine Lie algebra.

$\mathfrak{h}$ : its Cartan subalgebra with dimension $\text{rank}(\mathfrak{g}) + 1$.

$I$ : the index set for simple roots.

$\alpha_i$ : a simple root $\in \mathfrak{h}^*$ corresponding to $i \in I$.

$h_i$ : a simple coroot $\in \mathfrak{h}$ corresponding to $i \in I$.

We assume that the simple roots and the simple coroots are linearly independent.

$W$ : the Weyl group of $\mathfrak{g}$.

$\langle \ , \ \rangle$ : a $W$-invariant non-degenerate bilinear symmetric form on $\mathfrak{h}^*$ such that $\langle \alpha_i, \alpha_i \rangle \in 2\mathbb{Z}_{>0}$.

$\langle \ , \ \rangle$ : the coupling $\mathfrak{h} \times \mathfrak{h}^* \to \mathbb{C}$.

$P$ : a weight lattice $\subset \mathfrak{h}^*$.

$Q = \sum_i \mathbb{Z}\alpha_i$ the root lattice.

$Q_{\pm} = \pm \sum_i \mathbb{Z}_{\geq 0}\alpha_i$.

$\delta$ : an element of $Q_+$ such that $\mathbb{Z}\delta = \{ \lambda \in Q; \langle h_i, \lambda \rangle = 0 \}$.

$c$ : an element of $\sum_i \mathbb{Z}_{>0}h_i$ such that $\mathbb{Z}c = \{ h \in \sum_i \mathbb{Z}h_i; \langle h, \alpha_i \rangle = 0 \}$.

We write
\[ \delta = \sum_i a_i \alpha_i \quad \text{and} \]
\[ c = \sum_i a'_i h_i. \]
\[ P_{cl} = P/\mathbb{Z}\delta. \]
\[ \text{cl : } P \rightarrow P_{cl}. \]

We assume for the sake of simplicity
\[ P_{cl} \cong \text{Hom}_\mathbb{Z}(\bigoplus_{i \in I} \mathbb{Z} h_i, \mathbb{Z}). \]

This implies \( \{ \lambda \in P : \langle h_i, \lambda \rangle = 0 \text{ for any } i \in I \} = \mathbb{Z}\delta. \)

\[ \Lambda_i : \text{a fundamental weight in } P, \]
i.e. an element of \( P \) such that \( \langle h_j, \Lambda_i \rangle = \delta_{ij}. \)

\[ \Lambda_{\text{cl}}^i = \text{cl}(\Lambda_i), \text{the fundamental weight in } P_{\text{cl}}. \]

Note that \( \Lambda_i \) is determined modulo \( \mathbb{Z}\delta. \)

\[ P^0 : \text{the level 0 part of } P, \text{ i.e. } \{ \lambda \in P : \langle c, \lambda \rangle = 0 \}. \]

\[ P^0_{\text{cl}} : \text{the level 0 part of } P_{\text{cl}}, \text{ i.e. } \text{cl}(P^0). \]

\[ U_q(g) : \text{the quantized universal enveloping algebra} \]
with \( \{ q^h; h \in P^* \} \) as its Cartan part.

\[ U'_q(g) : \text{the quantized universal enveloping algebra} \]
with \( \{ q^h; h \in P_{\text{cl}}^* \} \) as its Cartan part.

Hence \( U'_q(g) \) is a subalgebra of \( U_q(g). \)

\[ K = \mathbb{Q}(q). \]

We consider \( U_q(g) \) and \( U'_q(g) \) over \( K. \)

\[ A = \{ f \in K; f \text{ has no pole at } q = 0 \}. \]

\[ U'_q(g)_\mathbb{Z} : \text{the } \mathbb{Z}[q, q^{-1}]-\text{subalgebra of } U'_q(g) \text{ generated by the divided powers} \]
e\[ \{ e^{(n)}_i, f^{(n)}_i, t_i \} \text{ and } \{ t_n \}. \]

\[ U_q(g)_\mathbb{Z} : \text{the } \mathbb{Z}[q, q^{-1}]-\text{subalgebra of } U_q(g) \text{ generated by } U'_q(g)_\mathbb{Z} \]
and \( \{ q^h \} (h \in P^*). \)

The quantized affine algebra \( U_q(g) \) is a \( K \)-algebra generated by \( e_i, f_i (i \in I) \) and \( q^h (h \in P^*) \) with the commutation relations

\[ q^h = 1 \text{ for } h = 0, \]
\[ q^h q^{h'} = q^{h+h'} \text{ for } h, h' \in P^*, \]
\[ q^h e_i q^{-h} = q^{(h,\alpha_i)} e_i \quad \text{and} \quad q^h f_i q^{-h} = q^{-(h,\alpha_i)} f_i, \]

\[ [e_i, f_j] = \delta_{ij} \frac{t_i - t_i^{-1}}{q_i - q_i^{-1}}, \]
for \( i \neq j \in I \)

\[ \sum_k (-1)^k e_i^{(k)} e_j e_i^{(-h_i,\alpha_j) - k} = 0, \]
\[ \sum_k (-1)^k f_i^{(k)} f_j f_i^{-(h_i,\alpha_j) - k} = 0. \]

Here
\[ q_i = q^{\frac{(\alpha_i,\alpha_i)}{2}} \quad \text{and} \quad t_i = q^{\frac{(\alpha_i,\alpha_i)}{2} h_i}. \]

2.2. Coproducts. There are several coproducts of \( U_q(g) \) used in the literature.

In this paper, we use a coproduct different from the ones used in \[ \text{[DJO, JM, K1, K2,} \]
In this subsection, we shall explain the relations among four coproducts:\n\[\Delta_+ : \begin{cases} q^h \mapsto q^h \otimes q^h \\
e_i \mapsto e_i \otimes 1 + t_i \otimes e_i \\
f_i \mapsto f_i \otimes t_i^{-1} + 1 \otimes f_i \end{cases} \quad (2.2.1)\]
\[\Delta_- : \begin{cases} q^h \mapsto q^h \otimes q^h \\
e_i \mapsto e_i \otimes t_i^{-1} + 1 \otimes e_i \\
f_i \mapsto f_i \otimes 1 + t_i \otimes f_i \end{cases} \quad (2.2.2)\]
\[\bar{\Delta}_+ : \begin{cases} q^h \mapsto q^h \otimes q^h \\
e_i \mapsto e_i \otimes 1 + t_i \otimes e_i \\
f_i \mapsto f_i \otimes t_i^{-1} + 1 \otimes f_i \end{cases} \quad (2.2.3)\]
\[\bar{\Delta}_- : \begin{cases} q^h \mapsto q^h \otimes q^h \\
e_i \mapsto e_i \otimes 1 + t_i \otimes e_i \\
f_i \mapsto f_i \otimes 1 + t_i^{-1} \otimes f_i \end{cases} \quad (2.2.4)\]

Their antipodes are given by\n\[a_+ : \begin{cases} q^h \mapsto q^{-h} \\
e_i \mapsto -t_i^{-1}e_i \\
f_i \mapsto -f_i t_i \end{cases} \quad (2.2.5)\]
\[a_- : \begin{cases} q^h \mapsto q^{-h} \\
e_i \mapsto -e_i t_i \\
f_i \mapsto -t_i^{-1} f_i \end{cases} \quad (2.2.6)\]
\[\bar{a}_+ : \begin{cases} q^h \mapsto q^{-h} \\
e_i \mapsto -t_i e_i \\
f_i \mapsto -f_i t_i^{-1} \end{cases} \quad (2.2.7)\]
\[\bar{a}_- : \begin{cases} q^h \mapsto q^{-h} \\
e_i \mapsto -e_i t_i^{-1} \\
f_i \mapsto -t_i f_i \end{cases} \quad (2.2.8)\]

For two \(U_q(g)\)-modules \(M_1\) and \(M_2\), let us denote by \(M_1 \otimes_+ M_2\), \(M_1 \otimes_- M_2\), \(M_1 \bar{\otimes}_+ M_2\) and \(M_1 \bar{\otimes}_- M_2\) the vector space \(M_1 \otimes_K M_2\) endowed with the \(U_q(g)\)-module structure via the coproduct \(\Delta_+, \Delta_-, \bar{\Delta}_+, \text{ and } \bar{\Delta}_-\), respectively.

We have functorial isomorphisms of \(U_q(g)\)-modules\n\[M_1 \otimes_+ M_2 \xrightarrow{\sim} M_2 \bar{\otimes}_- M_1 \quad (2.2.9)\]
\[M_1 \otimes_- M_2 \xrightarrow{\sim} M_2 \bar{\otimes}_+ M_1 \quad (2.2.10)\]
by $u_1 \otimes u_2 \mapsto u_2 \otimes u_1$.

We have functorial isomorphisms of $U_q(\mathfrak{g})$-modules

$$q^{-} : M_1 \otimes^+ M_2 \cong M_1 \otimes M_2 \quad (2.2.11)$$
$$q^{-} : M_1 \otimes^+ M_2 \cong M_1 \otimes M_2 \quad (2.2.12)$$

Here $q^{-}$ sends $u_1 \otimes^+ u_2$ to $q^{-}(\text{wt}(u_1), \text{wt}(u_2))u_1 \otimes u_2$ and $q^{-}$ sends $u_1 \otimes^+ u_2$ to $q^{-}(\text{wt}(u_1), \text{wt}(u_2))u_1 \otimes u_2$.

The tensor products $\otimes^+$ and $\otimes^-$ behave well under upper crystal bases and $\otimes^-$ and $\otimes^+$ behave well under lower crystal bases. Namely, if $(L_j, B_j)$ is an upper crystal base of an integrable $U_q(\mathfrak{g})$-module $M_j$ ($j = 1, 2$), then $(L_1 \otimes^+ A L_2, B_1 \otimes B_2)$ is an upper crystal base of $M_1 \otimes^+ M_2$ and $M_1 \otimes^+ M_2$. Similarly, if $(L_j, B_j)$ is a lower crystal base of $M_j$, then $(L_1 \otimes A L_2, B_1 \otimes B_2)$ is a lower crystal base of $M_1 \otimes^+ M_2$ and $M_1 \otimes^+ M_2$. If we use $\otimes^+$ or $\otimes^-$, the tensor product of crystal base is described as follows. For two crystals $B_1, B_2$ and $b_1 \in B_1, b_2 \in B_2$,

$$\text{wt} (b_1 \otimes b_2) = \text{wt} (b_1) + \text{wt} (b_2),$$
$$\varepsilon_i(b_1 \otimes b_2) = \max(\varepsilon_i(b_1), \varepsilon_i(b_2) - \langle h_i, \text{wt} (b_1) \rangle),$$
$$\varphi_i(b_1 \otimes b_2) = \max(\varphi_i(b_1) + \langle h_i, \text{wt} (b_2) \rangle, \varphi_i(b_2)), $$
$$\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} \tilde{e}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2), \\ b_1 \otimes \tilde{e}_i b_2 & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2), \end{cases}\quad (2.2.12)$$
$$\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2), \\ b_1 \otimes \tilde{f}_i b_2 & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2). \end{cases}\quad (2.2.13)$$

If we use the other tensor products $\otimes^+$ or $\otimes^-$, we have to exchange the first and the second factors in the formulas above. Namely the tensor product of crystals is given as follows.

$$\text{wt} (b_1 \otimes b_2) = \text{wt} (b_1) + \text{wt} (b_2),$$
$$\varepsilon_i(b_1 \otimes b_2) = \max(\varepsilon_i(b_1) - \langle h_i, \text{wt} (b_2) \rangle, \varepsilon_i(b_2)), $$
$$\varphi_i(b_1 \otimes b_2) = \max(\varphi_i(b_1), \varphi_i(b_2) + \langle h_i, \text{wt} (b_1) \rangle),$$
$$\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} \tilde{e}_i b_1 \otimes b_2 & \text{if } \varepsilon_i(b_1) > \varphi_i(b_2), \\ b_1 \otimes \tilde{e}_i b_2 & \text{if } \varepsilon_i(b_1) \leq \varphi_i(b_2), \end{cases}$$
$$\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i b_1 \otimes b_2 & \text{if } \varepsilon_i(b_1) \geq \varphi_i(b_2), \\ b_1 \otimes \tilde{f}_i b_2 & \text{if } \varepsilon_i(b_1) < \varphi_i(b_2).\quad (2.2.13)\end{cases}$$

In this article, we mainly use the tensor product $\otimes^+$ and lower crystal bases. The rule of the tensor product of crystals is therefore by $(2.2.13)$. Note that $\otimes^+$ is used in [DJO, JM] and $\otimes^-$ in [K2, KMN1].
3. Wedge products

3.1. Perfect crystal. Let us take an integrable finite-dimensional representation $V$ of $U'_q(\mathfrak{g})$. Let $V = \oplus_{\lambda \in P} V_\lambda$ be its weight space decomposition. Its affinization is defined by

$$V_{\text{aff}} = \bigoplus_{\lambda \in P} (V_{\text{aff}})_\lambda$$

where $(V_{\text{aff}})_\lambda = V_{\text{cl}(\lambda)}$ for $\lambda \in P$. Let $\text{cl} : (V_{\text{aff}})_\lambda \to V_{\text{cl}(\lambda)}$ denote the canonical isomorphism. Then $V_{\text{aff}}$ has a natural structure of a $U'_q(\mathfrak{g})$-module such that $\text{cl} : V_{\text{aff}} \to V$ is $U'_q(\mathfrak{g})$-linear (see [KMN1]).

Let $z : V_{\text{aff}} \to V_{\text{aff}}$ be the endomorphism of weight $\delta$ given by

$$z : (V_{\text{aff}})_\lambda \to (V_{\text{aff}})_{\lambda + \delta}$$

The endomorphism $z$ is $U'_q(\mathfrak{g})$-linear.

Taking a section of $\text{cl} : P \to P_{\text{cl}}$, $V_{\text{aff}}$ may be identified with $V \otimes \mathbb{C}[z, z^{-1}]$ (see section 5.1).

We assume that

(P): $V$ has a perfect crystal base $(L, B)$.

Let us recall its definition in [KMN1]. A crystal base $(L, B)$ is called perfect of level $l \in \mathbb{Z}_{>0}$ if it satisfies the following axioms (P1)–(P3).

(P1): There is a weight $\lambda^o \in P^0_{\text{cl}}$ such that the weights of $V$ are contained in the convex hull of $W\lambda^o$ and that $\dim V_{w\lambda^o} = 1$ for any $w$ in the Weyl group $W$.

We call a vector in $V_{w\lambda^o}$ an extremal vector with extremal weight $w\lambda^o$.

(P2): $B \otimes B$ is connected.

(P3): There is a positive integer $l$ satisfying the following conditions.

(i) For every $b \in B$, $\langle c, \varepsilon(b) \rangle = \langle c, \varphi(b) \rangle \geq l$. Here we set

$$\varepsilon(b) = \sum_{i \in I} \varepsilon_i(b) \Lambda_i^{\text{cl}} \in P_{\text{cl}}$$

$$\varphi(b) = \sum_{i \in I} \varphi_i(b) \Lambda_i^{\text{cl}} \in P_{\text{cl}}$$

with the fundamental weights $\Lambda_i^{\text{cl}} \in P_{\text{cl}}$.

(ii) Set $B_{\text{min}} = \{ b \in B ; \langle c, \varepsilon(b) \rangle = l \}$ and $(P_{\text{cl}}^+_l) = \{ \lambda \in P_{\text{cl}} ; \langle c, \lambda \rangle = l \}$ and $\langle h_i, \lambda \rangle \geq 0$ for every $i \in I$. Then

$$\varepsilon, \varphi : B_{\text{min}} \to (P_{\text{cl}}^+_l)$$

are bijective.

Note that (P1) is equivalent to the irreducibility of $V$ (see [CP]).

Note that the equality $\langle c, \varepsilon(b) \rangle = \langle c, \varphi(b) \rangle$ in (P3) (i) follows from

$$\varphi(b) = \text{wt}(b) + \varepsilon(b)$$
and the fact that $V$ is a $U'_q(\mathfrak{g})$-module of level 0.

Remark. The map $\varepsilon(b) \mapsto \varphi(b)$ ($b \in B_{\text{min}}$) defines an automorphism of $(P_{\text{cl}}^+)$. In all the examples of perfect crystals that we know, this automorphism is induced by a Dynkin diagram automorphism.

We have constructed $V_{\text{aff}}$ out of $V$. Similarly we construct the crystal base $(L_{\text{aff}}, B_{\text{aff}})$ of $V_{\text{aff}}$ out of $(L, B)$. We define similarly $\text{cl} : B_{\text{aff}} \to B$ and $z : B_{\text{aff}} \to B_{\text{aff}}$.

We assume further that $V$ has a good base $\{G(b)\}_{b \in B}$ in the following sense.

\begin{equation}
\text{(G): } V \text{ has a lower global base } \{G(b)\}_{b \in B}.
\end{equation}

This means that the base $\{G(b)\}_{b \in B}$ satisfies the following conditions (cf. [K2]).

(i) $\bigoplus_{b \in B} \mathbb{Z}[q, q^{-1}]G(b)$ is a $U'_q(\mathfrak{g})_{\mathbb{Z}}$-submodule of $V$.

(ii) $b \equiv G(b) \mod L/qL$.

(iii) $e_iG(b) = [\varphi_i(b) + 1], G(e_i) + \sum E_{b,b'}^i G(b')$,

(iv) $f_iG(b) = [\varepsilon_i(b) + 1], G(f_i) + \sum F_{b,b'}^i G(b')$.

In both cases, the sum ranges over $b'$ that belongs to an $i$-string strictly longer than that of $b$ ($\Leftrightarrow \varepsilon_i(b') \geq \varepsilon_i(b)$ or $\varphi_i(b') \geq \varphi_i(b)$ according to (iii) or (iv)). Moreover the coefficients satisfy

\begin{equation}
E_{b,b'}^i \in q q_i^{-\varphi_i(b')} \mathbb{Z}[q] \cup q^{-1} q_i^{\varphi_i(b')} \mathbb{Z}[q^{-1}]
\end{equation}

\begin{equation}
F_{b,b'}^i \in q q_i^{-\varepsilon_i(b')} \mathbb{Z}[q] \cup q^{-1} q_i^{\varepsilon_i(b')} \mathbb{Z}[q^{-1}].
\end{equation}

Remark. The reason why we choose a lower global base is explained in Theorem 4.2.5 and the remark after Proposition 4.2.8.

We define the base $\{G(b)\}_{b \in B_{\text{aff}}}$ of $V_{\text{aff}}$ by $\text{cl}(G(b)) = G(\text{cl}(b))$. We have $G(z^n b) = z^n G(b)$ for $n \in \mathbb{Z}$ and $b \in B_{\text{aff}}$.

3.2. Energy function. Let $H$ be an energy function (see [KMN1]). Namely $H : B_{\text{aff}} \otimes B_{\text{aff}} \to \mathbb{Z}$ satisfies

\begin{itemize}
    \item[(E1):] $H(z b_1 \otimes b_2) = H(b_1 \otimes b_2) - 1.$
    \item[(E2):] $H(b_1 \otimes z b_2) = H(b_1 \otimes b_2) + 1.$
    \item[(E3):] $H$ is constant on every connected component of the crystal graph $B_{\text{aff}} \otimes B_{\text{aff}}$.
\end{itemize}

By (E1–3), $H$ is uniquely determined up to a constant. We normalize $H$ by

\begin{itemize}
    \item[(E4):] $H(b \otimes b) = 0$ for any (or equivalently some) extremal $b \in B_{\text{aff}}$ (i.e. $\text{cl}(\text{wt}(b)) \in W \lambda^\vee$).
\end{itemize}

We know already its existence and uniqueness ([KMN1]). The existence is in fact proved by using $R$-matrix. Let us explain their relation. There is a $U'_q(\mathfrak{g})$-linear endomorphism ($R$-matrix) $R$ of $V_{\text{aff}} \otimes V_{\text{aff}}$ such that

\begin{equation}
R \circ (z \otimes 1) = (1 \otimes z) \circ R
\end{equation}

\begin{equation}
R \circ (1 \otimes z) = (z \otimes 1) \circ R
\end{equation}
and normalized by
\[ R(u \otimes u) = u \otimes u \quad \text{for every extremal } u \in V_{\text{aff}}. \] (3.2.3)

Strictly speaking, \( R \) is a homomorphism from \( V_{\text{aff}} \otimes V_{\text{aff}} \) to its completion \( V_{\text{aff}} \hat{\otimes} V_{\text{aff}} \). It is proved in [KMN1] that \( R \) sends \( L_{\text{aff}} \otimes L_{\text{aff}} \) to \( L_{\text{aff}} \hat{\otimes} L_{\text{aff}} \) and
\[ R(G(b_1) \otimes G(b_2)) = G(z^{H(b_1 \otimes b_2)}b_1) \otimes G(z^{-H(b_1 \otimes b_2)}b_2) \mod qL_{\text{aff}} \hat{\otimes} L_{\text{aff}} \] (3.2.4)
for every \( b_1, b_2 \in B_{\text{aff}} \).

We know that \( R \) has finitely many poles. It means that there is a non-zero \( \psi \in K[z \otimes z^{-1}, z^{-1} \otimes z] \) such that \( \psi R \) sends \( V_{\text{aff}} \otimes V_{\text{aff}} \) into itself. We assume that the denominator \( \psi \) of \( R \) satisfies the following property.

\[ \text{(D): } \psi \in A[z \otimes z^{-1}] \text{ and } \psi = 1 \text{ at } q = 0. \]

We take a linear form \( s : P \to \mathbb{Q} \) such that \( s(\alpha_i) = 1 \) for every \( i \in I \), and define \( l : B_{\text{aff}} \to \mathbb{Z} \) by \( l(b) = s(\wt(b)) + c \) for some constant \( c \). With a suitable choice of \( c \), \( l \) is \( \mathbb{Z} \)-valued. It satisfies
\[ \begin{align*}
\text{(i) } & l(zb) = l(b) + a \text{ for any } b \in B_{\text{aff}}. \text{ Here } a \text{ is a positive integer independent of } b. \\
\text{(ii) } & l(\tilde{e}_i b) = l(b) + 1 \text{ if } i \in I \text{ and } b \in B_{\text{aff}} \text{ satisfy } \tilde{e}_i b \neq 0. \\
\end{align*} \]

We assume that it satisfies
\[ \text{(L): If } H(b_1 \otimes b_2) \leq 0, \text{ then } l(b_1) \geq l(b_2). \]

### 3.3. Wedge products.
We define \( L(V_{\text{aff}}^{\otimes 2}) \) by \( L_{\text{aff}} \otimes_A L_{\text{aff}} \). Let us set \( \tilde{R} = \psi(z \otimes 1, 1 \otimes z) R = R \psi(1 \otimes z, z \otimes 1) \). Then it is an endomorphism of \( V_{\text{aff}}^{\otimes 2} \) and \( L(V_{\text{aff}}^{\otimes 2}) \) is stable by \( \tilde{R} \). We shall denote by the same letter \( \tilde{R} \) the endomorphism of \( L(V_{\text{aff}}^{\otimes 2})/qL(V_{\text{aff}}^{\otimes 2}) \) induced by \( \tilde{R} \). Then by (D) and (3.2.4) we have the equality in \( L(V_{\text{aff}}^{\otimes 2})/qL(V_{\text{aff}}^{\otimes 2}) \)
\[ \tilde{R}(b_1 \otimes b_2) = z^{H(b_1 \otimes b_2)}b_1 \otimes z^{-H(b_1 \otimes b_2)}b_2 \quad \text{for every } b_1, b_2 \in B_{\text{aff}}. \] (3.3.1)

Since \( R^2 = 1 \), we have
\[ (\tilde{R} - \psi(z \otimes 1, 1 \otimes z)) \circ (\tilde{R} + \psi(1 \otimes z, z \otimes 1)) = 0. \] (3.3.2)

Let us choose an extremal vector \( u \in V_{\text{aff}} \). Then we define
\[ N = U_q(g)[z \otimes z, z^{-1} \otimes z^{-1}, z \otimes 1 + 1 \otimes z](u \otimes u). \]

This definition does not depend on the choice of \( u \), because an extremal vector \( u \) of weight \( \lambda \) satisfies
\[ \begin{align*}
(f_i^{(n)} u) \otimes (f_i^{(n)} u) & = f_i^{(2n)}(u \otimes u) \quad \text{if } \langle h_i, \lambda \rangle = n \geq 0, \\
(e_i^{(n)} u) \otimes (e_i^{(n)} u) & = e_i^{(2n)}(u \otimes u) \quad \text{if } \langle h_i, \lambda \rangle = -n \leq 0.
\end{align*} \]
By the definition, we have

\[ f(z \otimes 1, 1 \otimes z)N \subset N \]

for any symmetric Laurent polynomial \( f(z_1, z_2) \). \hspace{1cm} (3.3.3)

We put the following postulate.

\[ \text{(R): For every pair } (b_1, b_2) \text{ in } B_{\text{aff}} \text{ with } H(b_1 \otimes b_2) = 0, \text{ there exists } C_{b_1, b_2} \in N \text{ which has the form} \]

\[ C_{b_1, b_2} = G(b_1) \otimes G(b_2) - \sum_{b_1', b_2'} a_{b_1', b_2'} G(b_1') \otimes G(b_2') . \]

\[ \text{Here the sum ranges over } (b_1', b_2') \text{ such that} \]

\[ H(b_1' \otimes b_2') > 0, \]
\[ l(b_2) \leq l(b_1') < l(b_1), \]
\[ l(b_2) < l(b_1') \leq l(b_1), \]

\[ \text{and the coefficients } a_{b_1', b_2'} \text{ belong to } \mathbb{Z}[q, q^{-1}]. \]

Later in Lemma 3.3.2, we see that \( a_{b_1', b_2'} \) belong to \( q \mathbb{Z}[q] \).

Since we have normalized the R-matrix by \( R(u \otimes u) = u \otimes u \), we have

\[ \tilde{R}(v) = \psi(z \otimes 1, 1 \otimes z)v \text{ for every } v \in N. \hspace{1cm} (3.3.4) \]

Hence \( \tilde{R} \) sends \( N \) to itself.

We set

\[ L(N) = N \cap L(V_{\text{aff}}^\otimes 2). \]

Then by (D) and (3.3.4), we have the equality in \( L(V_{\text{aff}}^\otimes 2)/qL(V_{\text{aff}}^\otimes 2) \)

\[ \tilde{R}(b) = b \text{ for every } b \in L(N)/qL(N). \hspace{1cm} (3.3.5) \]

We define the wedge product by

\[ \wedge^2 V_{\text{aff}} = V_{\text{aff}}^\otimes 2/N. \]

For \( v_1, v_2 \in V \), let us denote by \( v_1 \wedge v_2 \) the element of \( \wedge^2 V_{\text{aff}} \) corresponding to \( v_1 \otimes v_2 \). We set

\[ L(\wedge^2 V_{\text{aff}}) = L(V_{\text{aff}}^\otimes 2)/L(N) \subset \wedge^2 V_{\text{aff}}. \]

Now we shall study the properties of \( \wedge^2 V_{\text{aff}} \) under conditions (P), (G), (D), (L) and (R).

We conjecture that (P) and (G) imply the other conditions (D), (L) and (R).

**Lemma 3.3.1.** If \( \sum_{H(b_1 \otimes b_2) > 0} a_{b_1, b_2} G(b_1) \otimes G(b_2) \) belongs to Ker \( (\tilde{R} - \psi(z \otimes 1, 1 \otimes z)) \), then all \( a_{b_1, b_2} \) vanish.
Proof. It is enough to show that for \( n \in \mathbb{Z} \)
if \( a_{b_1,b_2} \in q^n A \) for all \( b_1, b_2 \), then \( a_{b_1,b_2} \in q^{n+1} A. \) (3.3.6)

By (D), (3.3.4) and (3.3.1), we obtain the identity in \( L(V_{\text{aff}}^2) \),
\[
\sum_{H(b_1 \otimes b_2) > 0} (q^{-n}a_{b_1,b_2})b_1 \otimes b_2 = \sum_{H(b_1 \otimes b_2) > 0} (q^{-n}a_{b_1,b_2})z^{H(b_1 \otimes b_2)}b_1 \otimes z^{-H(b_1 \otimes b_2)}b_2.
\]

Since \( H(z^{H(b_1 \otimes b_2)}b_1 \otimes z^{-H(b_1 \otimes b_2)}b_2) = -H(b_1 \otimes b_2) < 0 \), we obtain the desired assertion (3.3.6).

A similar argument leads to the following result.

**Lemma 3.3.2.** If \( H(b_1 \otimes b_2) = 0 \) and \( G(b_1) \otimes G(b_2) - \sum_{H(b'_1 \otimes b'_2) > 0} a_{b'_1,b'_2} G(b'_1) \otimes G(b'_2) \)
belongs to \( N \), then \( a_{b'_1,b'_2} \in qA. \)

We shall call a pair \((b_1, b_2)\) of elements in \( B_{\text{aff}} \) normally ordered and \( G(b_1) \wedge G(b_2) \) a normally ordered wedge if \( H(b_1 \otimes b_2) > 0 \). The axiom (R) may be considered as a rule to write \( G(b_1) \wedge G(b_2) \) as a linear combination of normally ordered wedges when \( H(b_1 \otimes b_2) = 0 \). In order to treat the case \( H(b_1 \otimes b_2) = -c < 0 \), we introduce an element of \( N \) (see (3.3.3))
\[
C'_{b_1,b_2} = (1 \otimes z^{-c} + z^{-c} \otimes 1)C_{b_1,z^{-c}b_2}
\]
(3.3.7)

Note that \( H(b_1 \otimes z^{-c}b_2) = H(z^{-c}b_1 \otimes b_2) = 0. \)

**Lemma 3.3.3.** If \( H(b_1 \otimes b_2) \leq 0 \), then \( C'_{b_1,b_2} \) has the form
\[
G(b_1) \otimes G(b_2) - \sum_{b'_1,b'_2} a_{b'_1,b'_2} G(b'_1) \otimes G(b'_2).
\]
Here the sum ranges over \((b'_1, b'_2)\) such that
\[
H(b'_1 \otimes b'_2) > H(b_1 \otimes b_2), \quad l(b_2) \leq l(b'_1) < l(b_1), \quad l(b'_2) < l(b_1).
\]
Moreover \( a_{b'_1,b'_2} \) belongs to \( \mathbb{Z}[q]. \)

**Proof.** Assume \( H(b_1 \otimes b_2) = -c < 0 \). Set
\[
C'_{z^{-c}b_1,b_2} = G(z^{-c}b_1) \otimes G(b_2) - \sum_{H(b'_1 \otimes b'_2) > 0} a_{b'_1,b'_2} G(b'_1) \otimes G(b'_2).
\]
Here the sum ranges over
\[
l(b_2) \leq l(b'_1) < l(z^{-c}b_1), \quad l(b'_2) < l(b'_1) \leq l(z^{-c}b_1).
\]
Then
\[
C'_{b_1,b_2} = G(b_1) \otimes G(b_2) + G(z^{-c}b_1) \otimes G(z^c b_2).
\]
Proof. Lemma 3.3.1 implies the linear independence of the normally ordered wedges and Corollary 3.3.4 implies that they generate \( N \). The desired properties can be easily checked.

By the repeated use of the proposition above, we obtain the following result.

**Corollary 3.3.4.** If \( H(b_1 \otimes b_2) \leq 0 \) then \( N \) contains an element \( C_{b_1,b_2} \), which has the form

\[
G(b_1) \otimes G(b_2) - \sum_{b_1', b_2'} a_{b_1', b_2'} G(b_1') \otimes G(b_2').
\]

Here the sum ranges over \((b_1', b_2')\) such that

\[
H(b_1' \otimes b_2') > 0, \\
l(b_2') \leq l(b_1') < l(b_1), \\
l(b_2') < l(b_2') \leq l(b_1).
\]

and \( a_{b_1', b_2'} \in \mathbb{Z}[q] \).

By Lemma 3.3.1, \( C_{b_1,b_2} \) is uniquely determined. Note that we shall see

\[
a_{b_1', b_2'}(0) = -\delta(b_1' \otimes b_2') = z^{H(b_1 \otimes b_2)} b_1 \otimes z^{-H(b_1 \otimes b_2)} b_2
\]

(see Lemma 3.3.8).

The following corollary is a consequence of the corollary above and Lemma 3.3.1.

**Lemma 3.3.5.** \( L(N) \) is a free \( A \)-module with \( \{C_{b_1,b_2}\}_{H(b_1 \otimes b_2) \leq 0} \) as its basis.

**Proposition 3.3.6.**

(i) The normally ordered wedges form a base of \( \wedge^2 V_{\text{aff}} \).

(ii) \( L(\wedge^2 V_{\text{aff}}) \) is a free \( A \)-module with the normally ordered wedges as a base.

**Proof.** Lemma 3.3.1 implies the linear independence of the normally ordered wedges and Corollary 3.3.4 implies that they generate \( \wedge^2 V_{\text{aff}} \).

(ii) follows from (i) and Corollary 3.3.4.

**Corollary 3.3.7.** \( N = \text{Ker} \left( \tilde{R} - \psi(z \otimes 1, 1 \otimes z) \right) \).

**Proof.** We know already that \( N \) is contained in \( \text{Ker} \left( \tilde{R} - \psi(z \otimes 1, 1 \otimes z) \right) \). Since the normally ordered wedges are linearly independent in \( V_{\text{aff}}^\otimes / \text{Ker} \left( \tilde{R} - \psi(z \otimes 1, 1 \otimes z) \right) \) by Lemma 3.3.1, \( \wedge^2 V_{\text{aff}} \to V_{\text{aff}}^\otimes / \text{Ker} \left( \tilde{R} - \psi(z \otimes 1, 1 \otimes z) \right) \) is injective.

We define for \( n > 0 \)

\[
N_n = \sum_{k=0}^{n-2} \left( V_{\text{aff}}^\otimes \otimes N \otimes V_{\text{aff}}^\otimes(n-k-2) \right) \subset V_{\text{aff}}^\otimes
\]

and then

\[
\wedge^n V_{\text{aff}} = V_{\text{aff}}^\otimes / N_n.
\]

For \( u_1, u_2, \ldots, u_n \in V_{\text{aff}} \), we denote by \( u_1 \wedge u_2 \wedge \cdots \wedge u_n \) the image of \( u_1 \otimes u_2 \otimes \cdots \otimes u_n \) in \( \wedge^n V_{\text{aff}} \).
There is a $U_q(g)$-linear homomorphism

$$\wedge : \Lambda^n V_{\text{aff}} \otimes \Lambda^m V_{\text{aff}} \rightarrow \Lambda^{n+m} V_{\text{aff}}.$$ 

Let us set $L(V_{\text{aff}}^\otimes n) = L_{\text{aff}}^\otimes n$ and let $L(\Lambda^n V_{\text{aff}})$ be the image of $L(V_{\text{aff}}^\otimes n)$ in $\Lambda^n V_{\text{aff}}$. We call a sequence $(b_1, b_2, \ldots, b_n)$ normally ordered if its every consecutive pair is normally ordered, i.e. if $H(b_j \otimes b_{j+1}) > 0$ for $j = 1, \ldots, n - 1$. In this case we call $G(b_1) \wedge \cdots \wedge G(b_n)$ a normally ordered wedge. Set

$$L(N_n) = \sum_{k=0}^{n-2} L(V_{\text{aff}})^{\otimes k} \otimes_A L(N) \otimes_A L(V_{\text{aff}})^{\otimes (n-2-k)} \subset L(V_{\text{aff}}^\otimes n).$$

Note that we have not yet seen $L(N_n) \supset N_n \cap L(V_{\text{aff}}^\otimes n)$, which will follow from Lemma 3.3.11. In the formulae below, we have to pay attention to a difference between modulo $ql(N_n)$ and modulo $ql(V_{\text{aff}}^\otimes n)$.

**Lemma 3.3.8.**

(i) If $H(b_1 \otimes b_2) = 0$ then

$$G(z^a b_1) \wedge G(z^b b_2) \equiv -G(z^b b_1) \wedge G(z^a b_2) \mod ql(\Lambda^2 V_{\text{aff}}).$$

(ii) If $H(b_1 \otimes b_2) < 0$ then

$$C_{b_1, b_2} \equiv b_1 \otimes b_2 + \delta(H(b_1 \otimes b_2) < 0)z^{H(b_1 \otimes b_2)}b_1 \otimes z^{-H(b_1 \otimes b_2)}b_2 \mod ql(V_{\text{aff}}^\otimes 2).$$

(iii) If $H(b_j \otimes b_{j+1}) = 0$ for $j = 1, \ldots, n - 1$, then for any $\sigma \in S_n$, 

$$G(z^{a_1} b_1) \wedge G(z^{a_2} b_2) \wedge \cdots \wedge G(z^{a_n} b_n) \equiv \text{sgn}(\sigma)G(z^{\sigma(a_1)} b_1) \wedge G(z^{\sigma(a_2)} b_2) \wedge \cdots \wedge G(z^{\sigma(a_n)} b_n) \mod ql(\Lambda^n V_{\text{aff}}).$$

**Proof.** By Lemma 3.3.3, (i) holds for $a = b = 0$. The general case is obtained by operating $z^a \otimes z^b + z^b \otimes z^a$ on $G(b_1) \otimes G(b_2) \equiv 0$.

The other assertions follow from (i). \hfill \Box

**Proposition 3.3.9.** Let $a, c \in \mathbb{Z}$ and $n \in \mathbb{Z}_{\geq 0}$. Then for $b_1, \ldots, b_n \in B_{\text{aff}}$ with $a \leq l(b_j) \leq c$, we have

$$G(b_1) \otimes \cdots \otimes G(b_n) \in \mathbb{Z}[q]G(b'_1) \otimes \cdots \otimes G(b'_n) + L(N_n)$$

where the sum ranges over normally ordered sequences $(b'_1, \ldots, b'_n)$ with $a \leq l(b'_j) \leq c$ and $l(b'_1) \leq l(b_1)$.

**Proof.** We shall prove this by induction on $n$ and $l(b_1)$. By the induction hypothesis on $n$, we may assume that $(b_2, \ldots, b_n)$ is normally ordered. If $H(b_1 \otimes b_2) > 0$, then we are done. Assume that $H(b_1 \otimes b_2) \leq 0$. Then by Corollary 3.3.4, we can write

$$G(b_1) \otimes G(b_2) \equiv \sum_{b'_1, b'_2} a_{b'_1, b'_2} G(b'_1) \otimes G(b'_2) \mod L(N)$$

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with \( H(b'_1 \otimes b'_2) > 0 \) and \( l(b_2) \leq l(b'_1) < l(b_1) \) and \( l(b_2) < l(b'_2) \leq l(b_1) \). Then we have

\[
G(b_1) \otimes G(b_2) \otimes \cdots \otimes G(b_n)
\equiv \sum a_{b_1,b_2}G(b'_1) \otimes G(b_2) \otimes G(b_3) \otimes \cdots \otimes G(b_n) \mod L(N_n).
\]

Since \( a \leq l(b_2) \leq l(b'_1) < l(b_1) \), the induction proceeds.

This proposition says in particular that \( \wedge^n V_{\text{aff}} \) is generated by the normally ordered wedges. In order to see their linear independence, we need the compatibility of the relations, which follow from the Yang–Baxter equation for \( R \).

**Lemma 3.3.10.** Assume \( H(b_1 \otimes b_2) = H(b_2 \otimes b_3) = 0 \). Then for \( a \geq b \geq c \), we have

\[
(1 + \delta_{a,b})C_{z^a b_1, z^b b_2} \otimes G(z^c b_3)
\equiv (1 + \delta_{b,c})C_{z^b b_1, z^c b_2} \otimes G(z^a b_3)
\equiv (1 + \delta_{a,c})C_{z^a b_2, z^c b_3} \otimes G(z^b b_1)
\equiv (1 + \delta_{a,b})C_{z^a b_1, z^b b_2} \otimes G(z^c b_3)
\mod qL(N).\]

**Proof.** We have the Yang-Baxter equation

\[
\tilde{R}_{12} \circ \tilde{R}_{23} \circ \tilde{R}_{12} = \tilde{R}_{23} \circ \tilde{R}_{12} \circ \tilde{R}_{23}.
\]

Here \( \tilde{R}_{ij} \) is the action of \( \tilde{R} \) on the \( i, j \)-th components on \( V_{\text{aff}}^\otimes 3 \). Set \( \psi_{21} = \psi(1 \otimes z \otimes 1, z \otimes 1 \otimes 1) \), etc. Since \( \tilde{R} + \psi(1 \otimes z, z \otimes 1) \) sends \( L(V_{\text{aff}}^\otimes 2) \) to \( L(N) \), \( \tilde{R}_{ij} + \psi_{ji} \) sends \( L(V_{\text{aff}}^\otimes 2) \) to \( L(N) \). Also we have

\[
(\tilde{R} + \psi(1 \otimes z, z \otimes 1))(G(z^a b_1) \otimes G(z^b b_2))
\equiv G(z^b b_1) \otimes G(z^a b_2) + G(z^a b_1) \otimes G(z^b b_2)
\equiv (1 + \delta_{a,b})C_{z^a b_1, z^b b_2} \mod qL(V_{\text{aff}}^\otimes 2).
\]

Since \( L(N) = N \cap L(V_{\text{aff}}^\otimes 2) \), the above congruence is also true modulo \( qL(N) \). Since we have

\[
\tilde{R}_{23} \circ \tilde{R}_{12} \left(G(z^a b_1) \otimes G(z^b b_2) \otimes G(z^c b_3)\right)
\equiv G(z^b b_1) \otimes G(z^c b_2) \otimes G(z^a b_3) \mod qL(V_{\text{aff}}^\otimes 3),
\]

etc., we have

\[
(\tilde{R}_{12} + \psi_{21}) \circ \tilde{R}_{23} \circ \tilde{R}_{12} \left(G(z^a b_1) \otimes G(z^b b_2) \otimes G(z^c b_3)\right)
\equiv (\tilde{R}_{12} + \psi_{21}) \left(G(z^b b_1) \otimes G(z^c b_2) \otimes G(z^a b_3)\right)
\equiv (1 + \delta_{b,c})C_{z^b b_1, z^c b_2} \otimes G(z^a b_3) \mod qL(N),
\]

and similarly

\[
(\tilde{R}_{23} + \psi_{23}) \circ \tilde{R}_{12} \left(G(z^a b_1) \otimes G(z^b b_2) \otimes G(z^c b_3)\right)
\equiv (1 + \delta_{a,c})G(z^b b_1) \otimes C_{z^a b_2, z^c b_3} \mod qL(N_3).
\]
They imply
\[ \tilde{R}_{12} \circ \tilde{R}_{23} \circ \tilde{R}_{12} \left( G(z^a b_1) \otimes G(z^b b_2) \otimes G(z^c b_3) \right) \]
\[ \equiv (1 + \delta_{b,c})C_{z^a b_1, z^b b_2} \otimes G(z^a b_3) \]
\[ - \psi_{21} \tilde{R}_{23} \circ \tilde{R}_{12} \left( G(z^a b_1) \otimes G(z^b b_2) \otimes G(z^c b_3) \right) \]
\[ \equiv (1 + \delta_{b,c})C_{z^a b_1, z^b b_2} \otimes G(z^a b_3) - (1 + \delta_{a,c})\psi_{21} G(z^b b_1) \otimes C_{z^b b_2, z^c b_3} \]
\[ + \psi_{21} \psi_{32} \tilde{R}_{12} \left( G(z^a b_1) \otimes G(z^b b_2) \otimes G(z^c b_3) \right) \]
\[ \equiv (1 + \delta_{b,c})C_{z^a b_1, z^b b_2} \otimes G(z^a b_3) - (1 + \delta_{a,c})G(z^b b_1) \otimes C_{z^b b_2, z^c b_3} \]
\[ + (1 + \delta_{a,b})C_{z^b b_2, z^c b_3} \otimes G(z^c b_3) \]
\[ - \psi_{21} \psi_{32} \psi_{31} G(z^a b_1) \otimes G(z^b b_2) \otimes G(z^c b_3). \]

Here \( \equiv \) is taken modulo \( qL(N_3) \). Similarly we have

\[ \tilde{R}_{23} \circ \tilde{R}_{12} \circ \tilde{R}_{23} \left( G(z^a b_1) \otimes G(z^b b_2) \otimes G(z^c b_3) \right) \]
\[ \equiv (1 + \delta_{a,b})G(z^b b_1) \otimes C_{z^a b_2, z^c b_3} - (1 + \delta_{a,c})C_{z^a b_1, z^b b_2} \otimes G(z^b b_3) \]
\[ + (1 + \delta_{b,c})G(z^a b_1) \otimes C_{z^b b_2, z^c b_3} - \psi_{31} \psi_{32} \psi_{21} G(z^a b_1) \otimes G(z^b b_2) \otimes G(z^c b_3). \]

Comparing these two identities, we obtain the desired result. \( \square \)

**Lemma 3.3.11.** The \( \mathbb{Q} \)-vector space \( L(N_n)/qL(N_n) \) is generated by \( G(b_1) \otimes \cdots \otimes G(b_{i-1}) \otimes C_{b_i, b_{i+1}} \otimes G(b_{i+2}) \otimes \cdots \otimes G(b_n) \) where \( (b_1, \ldots, b_n) \) ranges over the elements in \( B^n_{\text{aff}} \) such that \( (b_{i+1}, \ldots, b_n) \) is normally ordered and \( H(b_i \otimes b_{i+1}) \leq 0 \).

**Proof.** \( L(N_n) \) is generated by \( G(b_1) \otimes \cdots \otimes G(b_{i-1}) \otimes C_{b_i, b_{i+1}} \otimes G(b_{i+2}) \otimes \cdots \otimes G(b_n) \). Here \( H(b_i \otimes b_{i+1}) \leq 0 \) but \( (b_{i+1}, \ldots, b_n) \) is not necessarily normally ordered. We shall prove that such a vector can be written as a \( \mathbb{Q} \)-linear combination of vectors satisfying the conditions as in the lemma, by induction on \( n \) and descending induction on \( i \). Arguing by induction on \( n \), we may assume \( i = 1 \). Write \( b_k = z^{a_k} \tilde{b}_k \) with \( H(\tilde{b}_k \otimes \tilde{b}_{k+1}) = 0 \). Then \( a_1 \geq a_2 \). By Lemma 3.3.8 (iii), we may assume that \( a_3 < a_4 < \cdots < a_n \). If \( a_2 < a_3 \), there is nothing to prove. Assume \( a_2 \geq a_3 \). Then the preceding lemma implies

\[ (1 + \delta_{a_1, a_2})C_{z^{a_1} \tilde{b}_1, z^{a_2} \tilde{b}_2} \otimes G(z^{a_3} \tilde{b}_3) \]
\[ \equiv -(1 + \delta_{a_2, a_3})C_{z^{a_2} \tilde{b}_1, z^{a_3} \tilde{b}_2} \otimes G(z^{a_1} \tilde{b}_3) - (1 + \delta_{a_1, a_3})C_{z^{a_1} \tilde{b}_1, z^{a_3} \tilde{b}_2} \otimes G(z^{a_2} \tilde{b}_3) \]
\[ + (1 + \delta_{a_2, a_3})G(z^{a_1} \tilde{b}_1) \otimes C_{z^{a_2} \tilde{b}_2, z^{a_3} \tilde{b}_3} + (1 + \delta_{a_1, a_3})G(z^{a_2} \tilde{b}_1) \otimes C_{z^{a_1} \tilde{b}_2, z^{a_3} \tilde{b}_3} \]
\[ + (1 + \delta_{a_1, a_2})G(z^{a_3} \tilde{b}_1) \otimes C_{z^{a_1} \tilde{b}_2, z^{a_2} \tilde{b}_3} \mod qL(N_3). \]

Note that \( a_3 \) is the smallest among \( (a_1, \ldots, a_n) \). After tensoring \( G(z^{a_1} \tilde{b}_1) \otimes \cdots \otimes G(z^{a_n} \tilde{b}_n) \), the first two terms can be written in the desired form by Lemma 3.3.8 (iii), and the last three terms can be written in the desired form by the hypothesis of induction on \( i \). \( \square \)

**Theorem 3.3.12.** The normally ordered wedges form a base of \( \wedge^n V_{\text{aff}} \).
Proof. The normally ordered wedges generate $\wedge^n V_{aff}$ by Proposition 3.3.9. We shall show that any linear combination of normally ordered tensors in $N_n$ are linearly independent by the theorem above. Let $C$ be such a linear combination. Since $\cap_k q^k L(N_n) \subset \cap_k q^k L(\wedge^n V_{aff}) = 0$, it is enough to show that $C \in L(N_n)$ implies $C \in qL(N_n)$. By the preceding lemma, we can write

$$C = \sum_{i=1}^{n-1} \sum_{(b_1, \ldots, b_n) \in K_i} a_i(b_1, \ldots, b_n) \ G(b_1) \otimes \cdots \otimes G(b_{i-1}) \otimes C_{b_i, b_{i+1}} \otimes G(b_{i+2}) \otimes \cdots \otimes G(b_n) \mod qL(N_n).$$

Here the coefficients $a_i(b_1, \ldots, b_n)$ belong to $\mathbb{Q}$ and $(b_{i+1}, \ldots, b_n)$ is normally ordered for $(b_1, \ldots, b_n) \in K_i$. In order to show the vanishing of $a_i(b_1, \ldots, b_n)$, let us calculate $C$ modulo $qL(V_{aff}^\otimes n)$.

$$C = \sum_{i=1}^{n-1} \sum_{(b_1, \ldots, b_n) \in K_i} a_i(b_1, \ldots, b_n) \ b_1 \otimes \cdots \otimes b_{i-1} \otimes C_{b_i, b_{i+1}} \otimes b_{i+2} \otimes \cdots \otimes b_n \mod qL(V_{aff}^\otimes n).$$

Since Lemma 3.3.8 (ii) implies

$$C_{b_i, b_{i+1}} \equiv b_i \otimes b_{i+1} + \delta(H(b_i \otimes b_{i+1}) < 0)z^{H(b_i \otimes b_{i+1})} b_i \otimes z^{-H(b_i \otimes b_{i+1})} b_{i+1},$$

we have

$$C \equiv \sum_{i=1}^{n-1} \sum_{(b_1, \ldots, b_n) \in K_i} a_i(b_1, \ldots, b_n) \ b_1 \otimes \cdots \otimes b_{i-1} \otimes \left(b_i \otimes b_{i+1} + \delta(H(b_i \otimes b_{i+1}) < 0)z^{H(b_i \otimes b_{i+1})} b_i \otimes z^{-H(b_i \otimes b_{i+1})} b_{i+1}\right) \otimes b_{i+2} \otimes \cdots \otimes b_n \mod qL(V_{aff}^\otimes n). \quad (3.3.8)$$

We shall show $a_i(b_1, \ldots, b_n) = 0$ by the descending induction on $i$. Assume that $a_k(b_1, \ldots, b_n) = 0$ for $k > i$. Note that $H(b_i \otimes b_{i+1}) \leq 0$, and $H(z^{H(b_i \otimes b_{i+1})} b_i \otimes z^{-H(b_i \otimes b_{i+1})} b_{i+1}) > 0$ when $H(b_i \otimes b_{i+1}) < 0$. We also note that $(b_1, \ldots, b_n)$ is not normally ordered for $(b_1, \ldots, b_n) \in K_i$ but it is normally ordered for $(b_1, \ldots, b_n) \in K_k$ with $k < i$. By these observations, for $(b_1, \ldots, b_n) \in K_i$, the coefficient of $b_1 \otimes b_2 \otimes \cdots \otimes b_n$ in the right hand side of (3.3.8) is $a_i(b_1, \ldots, b_n)$ and $b_1 \otimes b_2 \otimes \cdots \otimes b_n$ does not appear in $C$. Hence $a_i(b_1, \ldots, b_n)$ must vanish.

**Corollary 3.3.13.** $L(\wedge^n V_{aff})$ is a free $A$-module with the normally ordered wedges as a base.

In fact, the normally ordered wedges generate $L(\wedge^n V_{aff})$ by Proposition 3.3.9 and are linearly independent by the theorem above.

Let $B(\wedge^n V_{aff})$ be the set of normally ordered sequences. Let us regard $B(\wedge^n V_{aff})$ as a subset of $B_{aff}^\otimes n$. Since it is invariant by $\tilde{e}_i$ and $\tilde{f}_i$, we can endow $B(\wedge^n V_{aff})$ with the structure of crystal induced by $B_{aff}^\otimes n$. We regard $B(\wedge^n V_{aff})$ as a basis of $L(\wedge^n V_{aff})/qL(\wedge^n V_{aff})$. Then we have
**Proposition 3.3.14.** \( \left( L(\wedge^n V_{\text{aff}}), B(\wedge^n V_{\text{aff}}) \right) \) is a crystal base of \( \wedge^n V_{\text{aff}} \).

The following lemma follows immediately from (3.3.3).

**Lemma 3.3.15.** Let \( f(z_1, \ldots, z_n) \) be a symmetric Laurent polynomial. Then \( f(z \otimes 1 \otimes \cdots \otimes 1, 1 \otimes z \otimes 1 \otimes \cdots \otimes 1, \ldots, 1 \otimes \cdots \otimes 1 \otimes z) \) induces an endomorphism of \( \wedge^n V_{\text{aff}} \).

4. Fock space

4.1. Ground state sequence. In this section we shall introduce a \( q \)-deformed Fock space in a similar way to the \( A_n^{(1)} \)-case ([KMS]).

We continue the discussion on the perfect crystal \( B \) of level \( l \). Let us take a sequence \( \{ b_m \}_{m \in \mathbb{Z}} \) in \( B_{\text{aff}} \) such that

\[
\langle c, \varepsilon(b_m^\circ) \rangle = l, \quad \varepsilon(b_m^\circ) = \varphi(b_{m+1}^\circ) \quad \text{and} \quad H(b_m^\circ \otimes b_{m+1}^\circ) = 1.
\]

We call \( (\cdots, b_{-1}^\circ, b_0^\circ, b_1^\circ, \ldots) \) a ground state sequence. If we give one of \( b_m^\circ \), then the other members of a ground state sequence are uniquely determined.

Since \( B \) is a finite set, there exist a positive integer \( N \) and an integer \( c \) such that

\[
b_{k+N}^\circ = \varepsilon^\circ b_k^\circ \quad \text{for every } k.
\]

Take weights \( \lambda_m \in P \) of level \( l \) satisfying

\[
\lambda_m = \text{wt} (b_m^\circ) + \lambda_{m+1} \quad \text{and} \quad \text{cl}(\lambda_m) = \varphi(b_m^\circ) = \varepsilon(b_{m-1}^\circ).
\]

Set \( v_m^\circ = G(b_m^\circ) \in V_{\text{aff}} \).

4.2. Definition of Fock space. For \( m \in \mathbb{Z} \), let us define first a (fake) \( q \)-deformed Fock space \( \mathcal{F}_m \) as the inductive limit \( (k \to \infty) \) of \( \wedge^{k-m} V_{\text{aff}} \), where \( \wedge^{k-m} V_{\text{aff}} \to \wedge^{k+1-m} V_{\text{aff}} \) is given by \( u \mapsto u \wedge v_k^\circ \). Intuitively \( \mathcal{F}_m \) is the subspace of \( \wedge^\infty V_{\text{aff}} \) generated by the vectors of the form \( u_m \wedge u_{m+1} \wedge \cdots \) with \( u_k = v_k^\circ \) for \( k \gg m \).

Similarly we define \( L(\mathcal{F}_m) \) as the inductive limit of \( L(\wedge^{k-m} V_{\text{aff}}) \). We define the vacuum vector \( |m\rangle = v_m^\circ \wedge v_{m+1}^\circ \wedge \cdots \in \mathcal{F}_m \). Then any vector can be written as \( v \wedge |m+r\rangle \) for some positive integer \( r \) and \( v \in \wedge^r V_{\text{aff}} \). Note that \( v \wedge |m+r\rangle = 0 \) if and only if \( v \wedge v_{m+r}^\circ \wedge \cdots \wedge v_{m+s}^\circ = 0 \) for some \( s > r \).

Then we introduce the true (\( q \)-deformed) Fock space by

\[
\mathcal{F}_m = \mathcal{F}_m / \left( \bigcap_{n>0} q^n L(\mathcal{F}_m) \right).
\]

Let \( L(\mathcal{F}_m) \subset \mathcal{F}_m \) be the image of \( L(\mathcal{F}_m) \), and \( |m\rangle \) the image of \( |m\rangle \).

We have the homomorphism

\[
\wedge : \wedge^r V_{\text{aff}} \otimes \mathcal{F}_{m+r} \to \mathcal{F}_m.
\]
For a normally ordered sequence \((b_m, b_{m+1}, \ldots)\) in \(B_{aff}\) such that \(b_k = b_k^\circ\) for \(k \gg m\), we call \(G(b_m) \land G(b_{m+1}) \land \cdots \in \mathcal{F}_m\) a normally ordered wedge.

**Theorem 4.2.1.** The normally ordered wedges form a base of \(\mathcal{F}_m\).

In order to prove this theorem, we need some preparations.

**Lemma 4.2.2.** If \(l(b) > l(b_{m+1}^\circ)\), then \(H(b \otimes b_{m+1}^\circ) \leq 0\).

**Proof.** If \(l(b) \gg 0\), then the assertion holds. Let us prove it by descending induction on \(l(b)\). Assume that there is \(i \in I\) such that \(e_i(b \otimes b_{m+1}^\circ) = (e_i b) \otimes b_{m+1}^\circ \neq 0\). Then \(l(b) < l(e_i b)\) and hence \(H(b \otimes b_{m+1}^\circ) = H(e_i b \otimes b_{m+1}^\circ) \leq 0\) by the hypothesis of induction. Hence we may assume that there is no such \(i\). Then \(e_i(b) \leq \varphi_i(b_{m+1}^\circ)\) for any \(i\), and hence \(b = z(ab_{m+1}^\circ)\) for some \(a \in \mathbb{Z}\). Since \(l(b) > l(b_{m+1}^\circ)\), we have \(a > 0\). Therefore \(H(b \otimes b_{m+1}^\circ) = 1 - a \leq 0\). \(\Box\)

**Proposition 4.2.3.** Assume \(H(b \otimes b_{m+1}^\circ) \leq 0\). Then for every \(n\) we can find \(m_1 \geq m\) such that

\[ G(b) \land v_m^\circ \land \cdots \land v_{m_1}^\circ \in q^nL(\Lambda^{m_1-m+2}V_{aff}). \]

**Proof.** We shall prove this by induction on \(n\) and \(H(b \otimes b_{m+1}^\circ)\). Set \(H(b \otimes b_{m}^\circ) = -c\) and

\[ G(b) \land v_m^\circ = \sum a(b_1, b_2)G(b_1) \land G(b_2). \]

Here the sum ranges over normally ordered pairs \((b_1, b_2)\) such that

\[ l(b_{m_1}^\circ) \leq l(b_1) < l(b), \]
\[ l(b_{m_1}^\circ) < l(b_2) \leq l(b). \] \(4.2.1\)

By the preceding lemma \(H(b_2 \otimes b_{m_1}^\circ) \leq 0\). Lemma 3.3.8 (i) implies

\[ a(b_1, b_2) \equiv -\delta(c < 0 \text{ and } (b_1, b_2) = (z^{-c}b, z^{c}b_{m_1}^\circ)) \mod qA. \]

We have

\[ G(b) \land v_m^\circ \land \cdots \land v_{m_1}^\circ = \sum a(b_1, b_2)G(b_1) \land G(b_2) \land v_{m_1}^\circ \land \cdots \land v_{m_1}^\circ. \]

Since \(l(b_2) > l(b_{m_1}^\circ)\), we have \(G(b_2) \land v_{m_1}^\circ \land \cdots \land v_{m_1}^\circ \in q^{n-1}L(\Lambda V_{aff})\). Hence \(a(b_1, b_2)G(b_1) \land G(b_2) \land v_{m_1}^\circ \land \cdots \land v_{m_1}^\circ\) belongs to \(q^nL(\Lambda V_{aff})\) except \(c < 0\) and \((b_1, b_2) = (z^{-c}b, z^{c}b_{m_1}^\circ)\).

Assume that \(c < 0\) and \((b_1, b_2) = (z^{-c}b, z^{c}b_{m_1}^\circ)\). Then we have \(0 \geq H(z^{c}b_{m_1}^\circ \otimes b_{m+1}^\circ) = 1 - c > H(b \otimes b_{m}^\circ)\). Hence \(a(b_1, b_2)G(b_1) \land G(b_2) \land v_{m_1}^\circ \land \cdots \land v_{m_1}^\circ\) belongs to \(q^nL(\Lambda^{m_1-m+2}V_{aff})\) by the hypothesis of induction on \(H(b \otimes b_{m}^\circ)\). \(\Box\)

**Remark.** Assume that \(c\) in (4.1.1) is positive (or equivalently, \(l(b_{m_1}^\circ)\) tends to infinity as \(m\) tends to infinity). Then \(H(b \otimes b_{m}^\circ) \leq 0\) implies \(G(b) \land v_m^\circ \land \cdots \land v_{m_1}^\circ = 0\) for \(m_1 \gg m\). In fact by the same argument as above we have \(G(b) \land v_{m_1}^\circ \land \cdots \land v_{m_1}^\circ \in \sum_{b'} \Lambda^{m_1-m+1}V_{aff} \land G(b')\) where \(b'\) satisfies \(l(b_{m_1}^\circ) < l(b') \leq l(b)\). Note that, under the condition of the proposition, \(G(b) \land v_m^\circ \land v_{m_1}^\circ \land \cdots \land v_k^\circ = 0\) for \(k \gg m\) is false in general.
A similar argument shows the following dual statement.

**Proposition 4.2.4.** Assume \( H(b_m \otimes b) \leq 0 \). Then for every \( n \) we can find \( m_1 \leq m \) such that

\[
v_{m_1}^\circ \wedge \cdots \wedge v_m^\circ \wedge G(b) \in q^n L(\wedge^{m-m_1+2} V_{\text{aff}}).
\]

As an immediate consequence of Proposition 4.2.3, we obtain the following result.

**Theorem 4.2.5.** For any vector \( b \in B_{\text{aff}} \) such that \( H(b \otimes b_m^\circ) \leq 0 \), we have the equality in \( F_m \)

\[
G(b) \wedge |m\rangle = 0.
\]

**Proof of Theorem 4.2.1.** Any vector in \( F_m \) can be written in the form \( v \wedge |m+r\rangle \) with \( v \in \wedge^r V_{\text{aff}} \). We may assume that \( v \) is a normally ordered wedge \( G(b_m) \wedge \cdots \wedge G(b_{m+r-1}) \). If \( H(b_{m+r} \otimes b_m^\circ) > 0 \), then \( v \wedge |m+r\rangle \) is a normally ordered wedge and otherwise \( v \wedge |m+r\rangle = 0 \) by Proposition 4.2.3.

The linear independence follows immediately from the corresponding statement for the wedge space (Corollary 3.3.13).

By a similar argument, we have

**Proposition 4.2.6.** \( L(F_m) \) is a free \( A \)-submodule of \( F_m \) generated by the normally ordered wedges.

**Proposition 4.2.7.**

\[
\bigcap_{n>0} q^n L(F_m) = \sum_{H(b_m \otimes b_m^\circ) < 0} \wedge^{r-1} V_{\text{aff}} \wedge G(b) \wedge |m+r\rangle
\]

\[
= \sum_{l(b) > l(b_m^\circ)} \wedge^{r-1} V_{\text{aff}} \wedge G(b) \wedge |m+r\rangle.
\]

**Proof.** The first equality follows from Theorems 4.2.1 and 4.2.5 and the last follows from Lemma 4.2.2 and (4.2.1).

As a corollary of Theorem 4.2.5 we have the following result concerning vertex operators.

**Proposition 4.2.8.** Let \( V(\lambda_m) \) be the irreducible \( U_q(\mathfrak{g}) \)-module with highest weight \( \lambda_m \) and \( u_{\lambda_m} \) its highest weight vector. Let \( \Phi : V_{\text{aff}} \otimes V(\lambda_m) \to V(\lambda_{m-1}) \) be an intertwiner. Then for any vector \( b \in B_{\text{aff}} \) such that \( H(b \otimes b_m^\circ) \leq 0 \), \( \Phi(G(b) \otimes u_{\lambda_m}) = 0 \).

**Proof.** As proved in [DJO], the intertwiner is unique up to a constant. As seen in the next two subsections, \( F_m \) has a \( U_q(\mathfrak{g}) \)-module structure and contains \( V(\lambda_m) \) as a direct summand. By this embedding, the highest vector \( u_{\lambda_m} \) of \( V(\lambda_m) \) corresponds to \( |m\rangle \). Therefore \( \Phi \) is given as the composition:

\[
V_{\text{aff}} \otimes V(\lambda_m) \to V_{\text{aff}} \otimes F_m \to F_{m-1} \to V(\lambda_{m-1}).
\]

Now the result follows from Theorem 4.2.5.
Remark. It is known (see e.g. [DJO]) that \( \Phi(v \otimes u_{\lambda_m}) = 0 \) for \( v \in (V_{\text{aff}})_{\lambda_{m-1} - \lambda_m} \) such that \( v \in \sum_i e_i^{1+(h_i, \lambda_{m-1})} V_{\text{aff}} \). On the other hand, by the property of the lower global base ([K2]), \( G(b) \) belongs to \( \sum_i e_i^{1+(h_i, \lambda_{m-1})} V_{\text{aff}} \) if and only if \( \varphi_i(b) > (h_i, \lambda_{m-1}) \) for some \( i \). Therefore, \( \Phi(G(b) \otimes u_{\lambda_m}) = 0 \) for \( b \in (B_{\text{aff}})_{\lambda_{m-1} - \lambda_m} \) other than \( b_{m-1}^b \).

This observation shows that we have to take a lower global base in order to have Theorem 4.2.5. Theorem 4.2.5, as well as Proposition 4.2.8, does not hold for an arbitrary choice of base other than the lower global base. In the course of our construction of the Fock space, we have not used explicitly the property of the lower global base. This is hidden in postulate (R). This postulate fails for an arbitrary choice of base.

### 4.3. \( U_q(\mathfrak{g}) \)-module structure on the Fock space.

Let us define the action of \( U_q(\mathfrak{g}) \) on \( F_m \). We define first the action of the Cartan part of \( U_q(\mathfrak{g}) \) by assigning weights. We set \( \text{wt}(\{m\}) = \lambda_m \) and \( \text{wt}(v \wedge |m+r\rangle) = \text{wt}(v) + \text{wt}(|m+r\rangle) \) for \( v \in \Lambda^r V_{\text{aff}} \). This defines the weight decomposition of the Fock space.

Let \( B(F_m) \) denote the set of normally ordered sequences \( (b_m, b_{m+1}, \ldots) \) in \( B_{\text{aff}} \) such that \( b_k = b_k^b \) for \( k \gg m \). Then it has a crystal structure as in [KMN1]. Moreover \( B(F_m) \) may be considered as a base of \( L(F_m)/qL(F_m) \) by Proposition 4.2.6. We write \( b_m \wedge b_{m+1} \wedge \cdots \) for \( (b_m, b_{m+1}, \ldots) \).

**Proposition 4.3.1.**

(i) \( \text{ch}(F_m) = \text{ch}(V(\lambda_m)) \prod_{k>0} (1 - e^{-k\delta})^{-1} \).

(ii) The weights of \( F_m \) appear as weights of \( V(\lambda_m) \). In particular, any weight \( \mu \) of \( F_m \) satisfies \( s(\mu) \leq s(\lambda_m) \) (see the end of §3.2 for \( s : P \to \mathbb{Q} \)). Moreover, \( s(\mu) = s(\lambda_m) \) implies \( \mu = \lambda_m \).

(iii) For any \( \mu \in P \), \( \dim(F_m)_\mu < \infty \).

(iv) \( (F_m)_{\lambda_m - n\alpha_i} = \begin{cases} KG(f_i^n b_m^\circ) \wedge |m+1\rangle & \text{if } 0 \leq n \leq \langle h_i, \lambda_m \rangle, \\ 0 & \text{otherwise}. \end{cases} \)

(v) If \( b \in B_{\text{aff}} \) satisfies \( \text{wt}(b) = \text{wt}(b^\circ_m) - n\alpha_i \), then \( G(b) \wedge |m+1\rangle = 0 \) unless \( 0 \leq n \leq \langle h_i, \lambda_m \rangle \) and \( b = f_i^n b^\circ_m \).

(vi) Any highest weight element of \( B(F_m) \) has the form \( z^a_m b^\circ_m \wedge z^a_{m+1} b^\circ_{m+1} \wedge \cdots \) with \( a_m \leq a_{m+1} \leq \cdots \) and \( a_k = 0 \) for \( k \gg m \).

(vii) For \( b_m \wedge b_{m+1} \wedge \cdots \in B(F_m) \), \( b_m = b^\circ_m \) implies \( b_k = b^\circ_k \) for any \( k \geq m \).

**Proof.** By Proposition 4.6.4 in [KMN1] (see also Appendix A), we have

\[
\text{ch}(V(\lambda_m)) = e^{\lambda_m} \sum_n e^{\sum_{n \geq m} (\text{wt}(b_n) - \text{wt}(b^\circ_n))}
\]

where the sum ranges over the family \( B_0 \) of sequences \( b_m, b_{m+1}, \ldots \) in \( B_{\text{aff}} \) such that \( b_n = b^\circ_n \) for \( n \gg m \) and \( H(b_n \otimes b_{n+1}) = 1 \) for any \( n \geq m \). On the other hand, we have

\[
\text{ch}(F_m) = e^{\lambda_m} \sum_n e^{\sum_{n \geq m} (\text{wt}(b_n) - \text{wt}(b^\circ_n))}
\]

where the sum ranges over the family \( B \) of normally ordered \( b_m, b_{m+1}, \ldots \) such that \( b_n = b^\circ_n \) for \( n \gg m \). We have

\[
B = \left\{ (z^{-a_m} b_m, z^{-a_{m+1}} b_{m+1}, \ldots) ; (b_m, b_{m+1}, \ldots) \in B_0, a_m \geq a_{m+1} \geq \cdots \text{ and } a_n = 0 \text{ for } n \gg m \right\}.
\]
To obtain (i), it is enough to remark that \( z \) has weight \( \delta \).

The assertions (ii)–(vi) follow from (i) and Theorem 4.2.5. The assertion (vii) follows from (vi) and

\[
\tilde{f}_i(z^{a_m} b_m^\circ \wedge z^{a_{m+1}} b_{m+1}^\circ \wedge \cdots) = z^{a_m} \tilde{f}_i b_m^\circ \wedge z^{a_{m+1}} b_{m+1}^\circ \wedge \cdots.
\]

\( \square \)

Now we shall define the action of \( e_i \) and \( f_i \) on \( \mathcal{F}_m \).

Taking \( \{q^n L(\mathcal{F}_m)\}_n \) as a neighborhood system of 0, \( \mathcal{F}_m \) is endowed with a so called \( q \)-adic topology. Since \( \cap_n q^n L(\mathcal{F}_m) = 0 \) by construction, the \( q \)-adic topology is separated. Since we use \( K = \mathbb{Q}(q) \) as a base field, \( \mathcal{F}_m \) is not complete with respect to this topology. For any \( \mu \in P \), the completion of \( (\mathcal{F}_m)_\mu \) is \( \mathbb{Q}((q)) \otimes_K (\mathcal{F}_m)_\mu \).

**Proposition 4.3.2.** For any vectors \( u_m, u_{m+1}, \cdots \in V_{\text{aff}} \) such that \( u_k = v_k^\circ \) for \( k \gg m \),

\[
\sum_{k \geq m} t_i^{-1}(u_m \wedge \cdots \wedge u_{k-1}) \wedge e_i u_k \wedge u_{k+1} \wedge \cdots \tag{4.3.1}
\]

and

\[
\sum_{k \geq m} u_m \wedge \cdots \wedge u_{k-1} \wedge f_i u_k \wedge t_i(u_{k+1} \wedge \cdots) \tag{4.3.2}
\]

converge in the \( q \)-adic topology to elements of \( \mathbb{Q}((q)) \otimes_K \mathcal{F}_m \).

**Proof.** First note that \( (e_i v_k^\circ) \wedge |k+1| = 0 \) because \( \lambda_k + \alpha_i \) is not a weight of \( \mathcal{F}_k \).

Hence, only finitely many terms survive in (4.3.1).

In order to prove the convergence of (4.3.2), we may assume that \( u_k = v_k^\circ \) for every \( k \geq m \). Then

\[
v_m^\circ \wedge \cdots \wedge v_{k-1}^\circ \wedge f_i v_k^\circ \wedge t_i(v_{k+1}^\circ \wedge \cdots) = q_i^{(h_i, \lambda_{k+1})} v_m^\circ \wedge \cdots \wedge v_{k-1}^\circ \wedge f_i v_k^\circ \wedge |k+1|.
\]

Since \( (h_i, \lambda_{k+1}) \) takes only finitely many values, it is enough to show that \( v_m^\circ \wedge \cdots \wedge v_{k-1}^\circ \wedge f_i v_k^\circ \wedge |k+1| \) converges in the \( q \)-adic topology. This follows from the following lemma. \( \square \)

**Lemma 4.3.3.** Let \( C \) be an endomorphism of the \( K \)-vector space \( V_{\text{aff}} \) of weight \( \mu \neq 0 \). Assume that \( Cz = zC \). Then for any \( m \), \( v_m^\circ \wedge \cdots \wedge v_{k-1}^\circ \wedge C v_k^\circ \wedge |k+1| \) converges to 0 in the \( q \)-adic topology when \( k \) tends to infinity.

**Proof.** Write

\[
C v_k^\circ = \sum_{\nu} c_{k,\nu} G(b_{k,\nu}).
\]

Take \( N \) and \( c \) as in (4.1.1). Then we have also the periodicity \( b_{k+N,\nu} = z^c b_{k,\nu} \) and \( c_{k+N,\nu} = c_{k,\nu} \). Hence \( c_{k,\nu} \) is bounded with respect to the \( q \)-adic topology. Therefore it is enough to show that \( v_m^\circ \wedge \cdots \wedge v_{k-1}^\circ \wedge G(b_{k,\nu}) \wedge |k+1| \) converges to 0.

By Proposition 4.3.1 (vi), \( (b_m^\circ, \ldots, b_{k-1}^\circ, b_{k+1}^\circ, \ldots) \) is not normally ordered. It means that either \( H(b_{k,\nu} \otimes b_{k+1}^\circ) \leq 0 \) or \( H(b_{k-1}^\circ \otimes b_{k,\nu}) \leq 0 \). If \( H(b_{k,\nu} \otimes b_{k+1}^\circ) \leq 0 \) then \( G(b_{k,\nu}) \otimes |k+1| \) vanishes. If \( H(b_{k-1}^\circ \otimes b_{k,\nu}) \leq 0 \), then \( v_m^\circ \wedge \cdots \wedge v_{k-1}^\circ \wedge G(b_{k,\nu}) \otimes |k+1| \) converges to 0 when \( s \) tends to infinity by Proposition 4.2.4. By shifting the indices, \( v_m^\circ \wedge \cdots \wedge v_{k-1}^\circ \wedge G(b_{k,\nu}) \otimes |k+1| \) converges to 0. \( \square \)
Let us set
\[ f_i\overline{m} = f_i v_m \wedge t_i|m + 1| + v_m \wedge f_i v_{m+1} \wedge t_i|m + 2| + \cdots \quad (4.3.3) \]
Then it is an element of \( \mathbb{Q}((q)) \otimes_K F_m \).

**Lemma 4.3.4.** \( f_i\overline{m} \) belongs to \( F_m \).

**Proof.** Let us take \( c \) and \( N \) as in (4.1.1). We define the isomorphism \( \psi_m : F_m \to F_{m+N} \) by \( u_m \wedge u_{m+1} \wedge \cdots \mapsto z^c u_m \wedge z^c u_{m+1} \wedge \cdots \). Then \( f_i\overline{m} \) satisfies the recurrence relation
\[
\begin{align*}
    f_i\overline{m} - v_m^o \wedge v_{m+1}^o \wedge \cdots \wedge v_{m+N-1}^o \wedge \psi_m(f_i\overline{m}) \\
    = f_i(v_m \wedge v_{m+1} \wedge \cdots \wedge v_{m+N-1}^o) \wedge t_i|m + N| \in F_m.
\end{align*}
\]
Hence the result follows from the following lemma. \( \square \)

**Lemma 4.3.5.** For \( \mu \in P \setminus \{ \lambda_m \} \), the endomorphism of \( (F_m)_\mu \) given by \( w \mapsto w - v_m^o \wedge v_{m+1}^o \wedge \cdots \wedge v_{m+N-1}^o \wedge \psi_m(w) \) is an isomorphism.

**Proof.** It is enough to show its injectivity. We show that \( w = v_m^o \wedge v_{m+1}^o \wedge \cdots \wedge v_{m+N-1}^o \wedge \psi_m(w) \) implies \( w = 0 \).

For \( b_m \wedge b_{m+1} \wedge \cdots \in B(F_m)_\mu \) is not normally ordered by Proposition 4.3.1 (vii), and hence \( H(b_m^o \otimes b_m) \leq 0 \). Proposition 4.2.4 implies that \( v_m^{o-kN} \wedge \cdots \wedge v_{m-1}^o \wedge G(b_m) \wedge G(b_m+1) \wedge \cdots \) belongs to \( qL(F_m-kN) \) for \( k \gg 0 \). Shifting the indices, we conclude that
\[
v_m^o \wedge \cdots \wedge v_{m+kN-1}^o \wedge \psi_{m+(k-1)N} \cdots \psi_{m+N} \psi_m \bigl( G(b_m) \wedge G(b_m+1) \wedge \cdots \bigr)
\]
belongs to \( qL(F_m) \) for \( k \gg 0 \). Therefore the homomorphism
\[
C : w \mapsto v_m^o \wedge \cdots \wedge v_{m+kN-1}^o \wedge \psi_{m+(k-1)N} \cdots \psi_{m+N} \psi_m(w)
\]
sends \( L(F_m)_\mu \) to \( qL(F_m)_\mu \) for \( k \gg 0 \). This shows the injectivity of the endomorphism \( \text{id}_{(F_m)_\mu} - C \). \( \square \)

Now we define
\[
\begin{align*}
e_i(v \wedge \overline{m+r}) &= e_i v \wedge \overline{m+r}, \\
f_i(v \wedge \overline{m+r}) &= f_i v \wedge t_i|\overline{m+r}| + v \wedge f_i\overline{m+r}
\end{align*}
\]
for \( v \in \Lambda^r V_{\text{aff}} \). Then \( e_i \) and \( f_i \) are well-defined homomorphisms from \( F_m \) to \( F_m \).

They satisfy
\[
\begin{align*}
e_i(v \wedge u) &= e_i v \wedge u + t_i^{-1} v \wedge e_i u \\
f_i(v \wedge u) &= f_i v \wedge t_i u + v \wedge f_i u
\end{align*}
\]
for \( v \in \Lambda^r V_{\text{aff}} \) and \( u \in F_{m+r} \). In order to see that they define endomorphisms of \( F_m \), we need to show the following proposition (see Proposition 4.2.7).
Proposition 4.3.6. Assume that \( b \in B_{\text{aff}} \) satisfies \( l(b) > l(b^0_m) \). Then we have the equalities in \( F_m \).

\[
e_i\left(G(b) \wedge \overline{m+1}\right) = 0, \quad \text{(4.3.6)}
\]
\[
f_i\left(G(b) \wedge \overline{m+1}\right) = 0. \quad \text{(4.3.7)}
\]

The first equality (4.3.6) follows from the fact that \( \text{wt} \ G(b) + \alpha_i + \lambda_{m+1} \) is not a weight of \( F_m \) (following Proposition 4.3.1 (ii)).

Let us prove (4.3.7). First note that the same consideration on the weight implies that

\[
\text{if } l(b) > l(b^0_m) \text{ and } \text{wt} \ (b) \neq \text{wt} \ (b^0_m) + \alpha_i, \text{ then (4.3.7) holds.} \quad \text{(4.3.8)}
\]

Hence in order to prove (4.3.7), we may assume that

\[
\text{wt} \ (b) = \text{wt} \ (b^0_m) + \alpha_i. \quad \text{(4.3.9)}
\]

Sublemma 4.3.7. Under the condition (4.3.9), we can write

\[
G(b) \wedge v^0_{m+1} \wedge \cdots \wedge v^0_{m+r} = u + \sum a(b_0, \ldots, b_r)G(b_0) \wedge \cdots \wedge G(b_r). \quad \text{(4.3.10)}
\]

Here \( u \) satisfies \( f_i(u \wedge \overline{m+r+1}) = 0 \), the coefficients \( a(b_0, \ldots, b_r) \) belong to \( \mathfrak{q}'A \), and the sum ranges over \( (b_0, \ldots, b_r) \) such that

\[
\text{wt} \ (b_j) = \begin{cases} 
\text{wt} \ (b^0_{m+j}) & \text{for } 0 \leq j < r \\
\text{wt} \ (b^0_{m+r}) + \alpha_i & \text{for } j = r
\end{cases}. \quad \text{(4.3.11)}
\]

Proof. We shall prove this by induction on \( r \). Assuming (4.3.10) for \( r \), let us prove (4.3.10) for \( r + 1 \). Since \( H(b_r \otimes b^0_{m+r+1}) \leq 0 \) by Lemma 4.2.2, we can write

\[
G(b_r) \wedge v^0_{m+r+1} = \sum a_{b',b''}G(b') \wedge G(b''). \quad \text{(4.3.12)}
\]

Here \( (b',b'') \) ranges over normally ordered pairs such that

\[
l(b^0_{m+r+1}) \leq l(b') < l(b_r),
\]
\[
l(b^0_{m+r+1}) < l(b'') \leq l(b_r). \quad \text{(4.3.13)}
\]

If \( \text{wt} \ (b'') \neq \text{wt} \ (b^0_{m+r+1}) + \alpha_i \), then we have \( G(b'') \wedge |m + r + 2| = 0 \) and \( f_i(G(b'') \wedge |m + r + 2|) = 0 \) by (4.3.13) and (4.3.8). Therefore \( f_i(G(b_0) \wedge \cdots \wedge G(b_{r-1}) \wedge G(b') \wedge G(b'') \wedge |m + r + 2|) = 0 \). If \( \text{wt} \ (b'') = \text{wt} \ (b^0_{m+r+1}) + \alpha_i \), then \( \text{wt} \ (b') = \text{wt} \ (b^0_{m+r}) \).

Moreover Lemma 3.3.8 (i) implies \( a_{b',b''} \in \mathfrak{q}A \). Thus the induction proceeds. \( \square \)

We resume the proof of Proposition 4.3.6. We have

\[
f_i(G(b) \wedge \overline{m+1}) = \sum a(b_0, \ldots, b_r)
\]
\[
\left( G(b_0) \wedge \cdots \wedge G(b_r) \wedge f_i|\overline{m+r+1}|ight.
\]
\[
+ \sum_{0 \leq j \leq r} G(b_0) \wedge \cdots \wedge G(b_{j-1}) \wedge f_iG(b_j) \wedge t_iG(b_{j+1}) \wedge \cdots \wedge t_iG(b_r) \wedge t_i|m + r + 1). \]

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There is a constant $s$ such that $f_i L_{\text{aff}} \in q^s L_{\text{aff}}$, and $f_i m + r + 1]$ is bounded with respect to the $q$-adic topology. Moreover, $t_i G(b_{j+1}) \land \cdots \land t_i G(b_r) \land t_i m + r + 1) = q_{i}^{(n_i,\lambda_{m+j+1})+28(j<r)} G(b_{j+1}) \land \cdots \land G(b_r) \land m + r + 1)$ and $\langle h_i, \lambda_{m+j} \rangle$ is bounded from below. Hence there is a constant $d$ independent of $r$ such that
\[
 f_i \left( G(b) \land m + 1) \right) \in q^{r+d} L(F_m) \quad \text{for every } r.
\]
This implies the equality (4.3.7) in $F_m$. This completes the proof of Proposition 4.3.6.

Thus we have defined the action of $e_i$ and $f_i$ on $F_m$. Now we shall show the commutation relations between them.

**Proposition 4.3.8.** On $F_m$ we have
\[
 [e_i, f_j] = \delta_{ij}(t_i - t_i^{-1})/(q_i - q_i^{-1}). \tag{4.3.14}
\]

**Proof.** First note that (4.3.5) implies
\[
 [e_i, f_j](v \land u) = [e_i, f_j]v \land t_i u + t_i^{-1} v \land [e_i, f_j]u. \tag{4.3.15}
\]
for $v \in \land^r V_{\text{aff}}$ and $u \in F_m + 1]$. Hence, it is enough to prove that the equality (4.3.14) holds when it is applied to the vacuum vector. If $i \neq j$ then $[e_i, f_j]|m \rangle = 0$ because $\lambda_m + \alpha_i - \alpha_j$ is not a weight of $F_m$.

Now we shall show $[e_i, f_i] = \{t_i\}_i$. Here $\{x\}_i = (x - x^{-1})/(q_i - q_i^{-1})$.

Since $[e_i, f_i]|m \rangle$ has weight $\lambda_m$, there is $c_m \in K$ such that $[e_i, f_i]|m \rangle = c_m|m \rangle$. Then by (4.3.15), we have
\[
 [e_i, f_i]|m \rangle = [e_i, f_i]|v^o_m \land t_i|m + 1 \rangle + t_i^{-1} v^o_m \land [e_i, f_i]|m + 1 \rangle
 = \left( q_i^{(h_i,\lambda_{m+1})}\langle h_i, \lambda_m - \lambda_{m+1} \rangle \right) + q_i^{(h_i,\lambda_{m+1} - \lambda_m)} c_{m+1} |m \rangle.
\]
Hence we have a recurrence relation
\[
 c_m = q_i^{(h_i,\lambda_{m+1})} \langle h_i, \lambda_m - \lambda_{m+1} \rangle + q_i^{(h_i,\lambda_{m+1} - \lambda_m)} c_{m+1}.
\]
Solving this, there is a constant $a \in K$ such that
\[
 c_m = \langle h_i, \lambda_m \rangle + q_i^{-(h_i,\lambda_m)} a \quad \text{for every } m. \tag{4.3.16}
\]
Namely we have $[e_i, f_i]|(m \rangle = \{t_i\}_i + at_i^{-1})|m \rangle$. Hence for $v \in \land^r V_{\text{aff}}$
\[
 [e_i, f_i](v \land m + r) = [e_i, f_i]v \land t_i|m + r \rangle + t_i^{-1} v \land [e_i, f_i]|m + r \rangle
 = \{t_i\}_i v \land t_i|m + r \rangle + t_i^{-1} v \land \{t_i\}_i + at_i^{-1})|m + r \rangle
 = \{t_i\}_i + at_i^{-1} \langle v \land m + r \rangle.
\]
Thus we obtain
\[
 [e_i, f_i] = \{t_i\}_i + at_i^{-1}. \tag{4.3.17}
\]
Let us show the vanishing of $a$. 

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By induction on $n$ we can see the following commutaiton relation
\[
e_i^{(n)} f_i^{(n)} = \sum_{k=0}^{n} \frac{f_i^{(n-k)} e_i^{(n-k)} \prod_{\nu=0}^{k-1} \{q_i^{-\nu} t_i\} i + a q_i^{\nu} t_i^{-1}}{[k]_i!}.
\] (4.3.18)

Setting $c = \langle h_i, \lambda_m \rangle$, we have $f_i^{(c+1)} \langle m \rangle = 0$ by Proposition 4.3.1 (iv). Hence
\[
0 = e_i^{(c+1)} f_i^{(c+1)} \langle m \rangle = \frac{\prod_{\nu=0}^{c+1} \{c - \nu\} i + a q_i^{\nu - c}}{[c + 1]_i!} \langle m \rangle.
\]

Therefore there is an integer $s$ such that $a = -q_i^s [s]_i$. Then the commutation relation (4.3.17) can be rewritten as
\[
[e_i, q_i^{-s} f_i] = \{q_i^{-s} t_i\}_i.
\]

Hence $e_i$, $q_i^{-s} f_i$ and $q_i^{-s} t_i$ form $U_q(\mathfrak{sl}_2)$. Then the representation theory of $U_q(\mathfrak{sl}_2)$ and Proposition 4.3.1 (iv) implies $s = 0$. In fact, the string containing the weight of $\langle m \rangle$ (with respect to $q_i^{-s} t_i$) is $\{c - s - 2n; 0 \leq n \leq c\}$, and hence the symmetry of a string under the simple reflection implies $c - s = -(c - s)$.

Thus the actions of $e_i$ and $f_i$ satisfy the commutation relations. By Proposition 4.3.1 (ii), for any $i \in I$ and $\mu \in P$, $\mu + n \alpha_i$ is a weight of $F_m$ only for a finitely many integers $n$. Therefore $F_m$ is integrable over the $U_q(\mathfrak{sl}_2)_i = \langle e_i, f_i, t_i, t_i^{-1} \rangle$.

This implies the Serre relations (see Appendix B).

Thus we obtain

**Proposition 4.3.9.** $F_m$ has the structure of an integrable $U_q(\mathfrak{g})$-module.

By Proposition 4.3.1 (i), $F_m$ is a direct sum of $V(\lambda_m - k \delta)$’s. This decomposition is studied in the next subsection through bosons.

Note that
\[
\wedge : \wedge^r V_{\text{aff}} \otimes F_{m+r} \rightarrow F_m
\]
is $U_q(\mathfrak{g})$-linear.

**Lemma 4.3.10.**
\[
f_i^{(k)} \langle m \rangle = G(f_i^{k b_m^o}) \wedge \langle m + 1 \rangle.
\]

**Proof.** If $k > \langle h_i, \lambda_m \rangle$, then the both side vanish. Assume that $0 \leq k \leq \langle h_i, \lambda_m \rangle$.

By Proposition 4.3.1 (iv), there is $c \in K$ such that
\[
f_i^{(k)} \langle m \rangle = c G(f_i^{k b_m^o}) \wedge \langle m + 1 \rangle.
\]

We have
\[
e_i^{(k)} f_i^{(k)} \langle m \rangle = \left[ \frac{\langle h_i, \lambda_m \rangle}{k} \right]_i \langle m \rangle.
\]

On the other hand, by the repeated use of (iii) in (G), we have
\[
e_i^{(k)} G(f_i^{k b_m^o}) = \left[ \frac{\langle h_i, \lambda_m \rangle}{k} \right]_i v_m^o + \ldots.
\]

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Here, \( \cdots \) is a linear combination of global bases other than \( v^0_m \), which is annihilated after being wedged with \( |m + 1\rangle \) by Proposition 4.3.1 (v). Hence we have

\[
e^{(k)}_i \left( G(j_i b^0_m) \wedge |m + 1\rangle \right) = \left( e^{(k)}_i G(j_i b^0_m) \right) \wedge |m + 1\rangle
\]

\[
= \left[ \langle h_i, \lambda_m \rangle \atop k \right] v^0_m \wedge |m + 1\rangle.
\]

Comparing these two identities, we obtain \( c = 1 \).

Let \( \mathcal{F}^\mathbb{Z}_m \) be the \( \mathbb{Z}[q, q^{-1}] \)-submodule of \( \mathcal{F}_m \) generated by the normally ordered wedges. Then \( \mathcal{F}^\mathbb{Z}_m \) is a module over \( U_q(\mathfrak{g})_\mathbb{Z} \) by Lemma 4.3.10. Hence by specializing at \( q = 1 \), we obtain a Fock representation of \( U(\mathfrak{g}) \).

However, the action of the bosons on \( \mathcal{F}_m \) introduced in the next subsection may have a pole at \( q = 1 \) and it cannot be specialized at \( q = 1 \) in a naive way.

**4.4. The action of Bosons.** We shall define the action of the bosons \( B_n \) (\( n \neq 0 \)) on \( \mathcal{F}_m \).

**Proposition 4.4.1.** For \( n \neq 0 \) and any \( u_m, u_{m+1}, \cdots \in V_{\text{aff}} \) such that \( u_k = v^0_k \) for \( k \gg m \),

\[
\left( z^n u_m \wedge u_{m+1} \wedge u_{m+2} \wedge \cdots \right)
+ \left( u_m \wedge z^n u_{m+1} \wedge u_{m+2} \wedge \cdots \right)
+ \left( u_m \wedge u_{m+1} \wedge z^n u_{m+2} \wedge \cdots \right)
+ \cdots
\]

(4.4.1)

converges in the \( q \)-adic topology.

**Proof.** Reducing to the case \( u_k = v^0_k \) for every \( k \geq m \), apply Lemma 4.3.3. \( \square \)

**Lemma 4.4.2.** \( z^n v^0_m \wedge |m + 1\rangle + v^0_m \wedge z^n v^0_{m+1} \wedge |m + 2\rangle + \cdots \) belongs to \( \mathcal{F}_m \)

The proof is similar to the one for Lemma 4.3.4.

By these lemmas and Lemma 3.3.15, (4.4.1) defines a homomorphism from \( \mathcal{F}^\mathbb{Z}_m \) to \( \mathcal{F}_m \). Since \( L(\mathcal{F}_m) \) is stable by the correspondence (4.4.1), it induces an endomorphism of \( \mathcal{F}_m \). We denote it by \( B_n \). It is clear that \( B_n \) is a \( U'_q(\mathfrak{g}) \)-linear endomorphism of \( \mathcal{F}_m \) with weight \( n\delta \).

By the definition, we have

\[
B_n(v \wedge u) = z^n v \wedge u + v \wedge B_n(u) \quad \text{for } v \in V_{\text{aff}} \text{ and } u \in \mathcal{F}_m.
\]

(4.4.2)

**Proposition 4.4.3.** There is \( \gamma_n \in K \) (independent of \( m \)) such that

\[
[B_n, B_n'] = \delta_{n+n',0} \gamma_n.
\]

**Proof.** (4.4.2) implies

\[
[B_n, B_n'](v \wedge u) = v \wedge [B_n, B_n']u.
\]

Since \([B_n, B_n']|m\) has weight \( \lambda_m + (n+n')\delta \) and hence it must vanish when \( n+n' > 0 \). Therefore \([B_n, B_n'] = 0\) in this case.
Assume $n + n' < 0$. Write $[B_n, B_{n'}]|m\rangle$ as a linear combination of normally ordered wedges:

$$[B_n, B_{n'}]|m\rangle = \sum_{\nu} c_{\nu} G(b_{1,\nu}) \wedge \cdots .$$

Then $b_{1,\nu} \neq b_{m}^\circ$. Take $N$ and $c$ as in (4.1.1). Then we have

$$[B_n, B_{n'}]|m + jN\rangle = \sum_{\nu} c_{\nu} G(z^j b_{1,\nu}) \wedge \cdots .$$

We have also $H(b_{m+jN-1}^\circ \otimes z^j b_{1,\nu}) = H(b_{m-1}^\circ \otimes b_{1,\nu}) \leq 0$. Hence by Proposition 4.2.4, $v_m^\circ \wedge \cdots \wedge v_{m+jN-1}^\circ \wedge G(z^j b_{1,\nu}) \wedge |m + jN + 1\rangle$ converges to 0 when $j$ tends to infinity. Hence

$$[B_n, B_{n'}]|m\rangle = v_m^\circ \wedge \cdots \wedge v_{m+jN-1}^\circ \wedge [B_n, B_{n'}]|m + jN\rangle$$

$$= \sum_{\nu} c_{\nu} v_m^\circ \wedge \cdots \wedge v_{m+jN-1}^\circ \wedge G(z^j b_{1,\nu}) \wedge \cdots$$

converges to 0. Therefore $[B_n, B_{n'}]|m\rangle$ must vanish.

Now assume that $n + n' = 0$. Since $[B_n, B_{-n}]|m\rangle$ has the same weight as $|m\rangle$, there is $\gamma_{m,n}$ such that $[B_n, B_{-n}]|m\rangle = \gamma_{m,n}|m\rangle$. Since

$$[B_n, B_{-n}]|m\rangle = v_m^\circ \wedge [B_n, B_{-n}]|m + 1\rangle = \gamma_{m+1,n} v_m^\circ \wedge |m + 1\rangle = \gamma_{m+1,n}|m\rangle,$$

$\gamma_{m,n}$ does not depend on $m$. Write $\gamma_n$ for $\gamma_{m,n}$. Now we have $[B_n, B_{-n}](v \wedge |m\rangle) = v \wedge [B_n, B_{-n}]|m\rangle = \gamma_n v \wedge |m\rangle$.

Now we shall show that $\gamma_n$ does not vanish.

**Lemma 4.4.4.** Let $n$ be a positive integer.

(i) $z^n v_k^\circ \wedge |k + 1\rangle = 0$.

(ii) $v_m^\circ \wedge v_{m+1}^\circ \wedge \cdots \wedge v_{k-1}^\circ \wedge z^{-n} v_k^\circ \wedge |k + 1\rangle \equiv 0$ for $k \geq m + n$.

(iii) $z^n v_m^\circ \wedge v_{m+1}^\circ \wedge \cdots \wedge v_{k-1}^\circ \wedge z^{-n} v_k^\circ \wedge |k + 1\rangle \equiv 0$ for $m < k < m + n$.

Here $\equiv$ is modulo $qL(\mathcal{F}_m)$.

**Proof.** (i) follows from Theorem 4.2.5.

In order to prove the other statements, write $b_k = z^{-k} b_k^\circ$. Then $H(b_k \otimes b_{k+1}) = 0$.

We have

$$v_m^\circ \wedge v_{m+1}^\circ \wedge \cdots \wedge v_{k-1}^\circ \wedge z^{-n} v_k^\circ \equiv z^m b_m \wedge z^{1+m} b_{m+1} \wedge \cdots \wedge z^{k-1} b_{k-1} \wedge z^{k-n} b_k.$$

Since $m \leq k - n \leq k - 1$, it is zero modulo $qL(\wedge V_{\text{aff}})$ by Lemma 3.3.8 (iii).

The proof of (iii) is similar to that of (ii). We have

$$z^n v_m^\circ \wedge v_{m+1}^\circ \wedge \cdots \wedge v_{k-1}^\circ \wedge z^{-n} v_k^\circ \wedge v_{k+1}^\circ \wedge \cdots \wedge v_{n+m}^\circ \equiv$$

$$z^{n+m} b_m \wedge z^{n+1} b_{m+1} \wedge z^{k-1} b_{k-1} \wedge z^{k-n} b_k \wedge z^{k+1} b_{k+1} \wedge \cdots \wedge z^{n+m} b_{n+m}.$$

Then it is zero modulo $qL(\wedge V_{\text{aff}})$ again by Lemma 3.3.8 (iii).

**Proposition 4.4.5.** For $n \neq 0$, $\gamma_n \in K$ has no pole at $q = 0$ and $\gamma_n(0) = n$. 

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Proof. We may assume \( n > 0 \). Noting that \( B_{\pm n} \) sends \( L(F_m) \) to itself, let us calculate the commutator modulo \( qL(F_m) \). We have \([B_n, B_{-n}]|m\rangle = B_n B_{-n}|m\rangle\). By Lemma 4.4.4 (ii), we have
\[
B_{-n}|m\rangle = v^o_m \wedge |m + 1\rangle + v^o_m \wedge z^{-n} v^o_{m+1} \wedge |m + 2\rangle + \cdots
\]
Here \( \equiv \) is taken modulo \( qL(F_m) \). Hence we have by Lemma 4.4.4 (i) and (iii)
\[
B_{-n} B_{-n}|m\rangle = \sum_{m \leq k < m+n} \left( \sum_{m \leq j < k} v^o_m \wedge v^o_{m+1} \wedge \cdots \wedge z^n v^o_j \wedge \cdots \wedge v^o_{k-1} \wedge z^{-n} v^o_k \wedge |k + 1\rangle
\]
\[
+ \sum_{j > k} v^o_m \wedge v^o_{m+1} \wedge \cdots \wedge v^o_{k-1} \wedge z^{-n} v^o_k \wedge v^o_{k+1} \wedge \cdots \wedge z^n v^o_j \wedge |j + 1\rangle \right)
\]
\[
\equiv n|m\rangle.
\]

Let \( H \) be the Heisenberg algebra generated by \( \{B_n\}_{n \in \mathbb{Z} \setminus \{0\}} \) with the defining relations \([B_n, B_{n'}] = \delta_{n+n', 0} \gamma_n\). Then \( H \) acts on the Fock space \( F_m \) commuting with the action of \( U'_q(g) \). Let \( \mathbb{Q}[H_-] \) be the Fock space for \( H \). Namely, \( \mathbb{Q}[H_-] \) is the \( H \)-module generated by the vacuum vector \( 1 \) with the defining relation \( B_n 1 = 0 \) for \( n > 0 \). Since \( |m\rangle \) is annihilated by the \( e_i \) and the \( B_n \) with \( n > 0 \), we have an injective \( U'_q(g) \otimes H \)-linear homomorphism
\[
\iota_m : V(\lambda_m) \otimes \mathbb{Q}[H_-] \to F_m
\]
sending \( u_{\lambda_m} \otimes 1 \) to \( |m\rangle \). Comparing their characters (see Proposition 4.3.1 (i)), we obtain

**Theorem 4.4.6.** \( \iota_m : V(\lambda_m) \otimes \mathbb{Q}[H_-] \to F_m \) is an isomorphism.

### 4.5. Vertex operator.

Similarly to the case \( A_n^{(1)} \) in [KMS], the intertwiner
\[
\Omega_m : V_{\text{aff}} \otimes F_{m+1} \to F_m,
\]
\[
v \otimes u \mapsto v \wedge u \quad (v \in V_{\text{aff}}, u \in F_{m+1}),
\]
induced by the wedge product is related with vertex operators. Let us describe it briefly. The proof is similar to [KMS].

Take an intertwiner
\[
\Phi_m : V_{\text{aff}} \otimes V(\lambda_{m+1}) \to V(\lambda_m)
\]
and normalize it by
\[
\Phi_m(v^o_m \otimes u_{\lambda_{m+1}}) = u_{\lambda_m}
\]
(cf. Appendix A).

Let
\[
\iota_m : V(\lambda_m) \otimes \mathbb{Q}[H_-] \cong F_m
\]
be the isomorphism in (4.4.3).

We define

$$\Omega'_m : V_{\text{aff}} \otimes V(\lambda_{m+1}) \otimes \mathbb{Q}[H_-] \to V(\lambda_m) \otimes \mathbb{Q}[H_-]$$

by requiring the commutativity of the following diagram

$$
\begin{array}{ccc}
V_{\text{aff}} \otimes V(\lambda_{m+1}) \otimes \mathbb{Q}[H_-] & \sim & V_{\text{aff}} \otimes \mathcal{F}_{m+1} \\
\downarrow \Omega'_m & & \downarrow \Omega_m \\
V(\lambda_m) \otimes \mathbb{Q}[H_-] & \sim & \mathcal{F}_m
\end{array}
$$

(4.5.2)

We shall write the intertwiners in the form of generating functions. Namely, introducing an indeterminate $w$ (of weight $\delta$), we set for $v \in V_{\text{aff}}$

$$v(w) = \sum_n z^n v \otimes w^{-n},$$

$$\Phi_m(w)(v \otimes u) = \Phi_m(v(w) \otimes u) = \sum_n \Phi_m(z^n v \otimes u) w^{-n},$$

$$\Omega_m(w)(v \otimes u) = \Omega_m(v(w) \otimes u) = \sum_n \Omega_m(z^n v \otimes u) w^{-n},$$

$$\Omega'_m(w)(v \otimes u) = \Omega'_m(v(w) \otimes u) = \sum_n \Omega'_m(z^n v \otimes u) w^{-n}.$$

Here $u \in V(\lambda_{m+1}), \mathcal{F}_{m+1}$ or $V(\lambda_m) \otimes \mathbb{Q}[H_-].$

We define the vertex operator for the bosons by

$$\Theta(w) = \exp \left( \sum_{n \geq 1} \frac{B_n w^n}{\gamma_n} \right) \exp \left( - \sum_{n \geq 1} \frac{B_n w^{-n}}{\gamma_n} \right).$$

(4.5.3)

**Theorem 4.5.1.** $\Omega'_m(w) = \Phi_m(w) \otimes \Theta(w).$

As a corollary of this theorem, we have the relations of the two-point functions of the vertex operators and $\gamma_n$ as in [KMS].

Set

$$\Phi'_m(w)(u) = \Phi_m(w)(v \otimes u)$$

for $u \in \mathcal{F}_{m+1}.$

For $v, v' \in V_{\text{aff}},$ we define $\langle \Phi'_{m-1}(w_1) \Phi'_m(w_2) \rangle$ to be the coefficient of $u_{\lambda_{m-1}}$ in $\Phi'_{m-1}(w_1) \Phi'_m(w_2) w_{\lambda_{m+1}} \in V(\lambda_{m-1}).$ We introduce functions by

$$\omega_{v,v'}(w_2/w_1) = \langle m-1 | v(w_1) \wedge v'(w_2) \wedge | m+1 \rangle$$

(4.5.4)

$$\phi_{v,v'}(w_2/w_1) = \langle \Phi'_{m-1}(w_1) \Phi'_m(w_2) \rangle$$

(4.5.5)

and

$$\theta(w_2/w_1) = \exp \left( - \sum_{n > 0} \frac{(w_2/w_1)^n}{\gamma_n} \right).$$

(4.5.7)

Here for $u \in \mathcal{F}_{m-1}, \langle m-1 | u \rangle$ means the coefficient of $| m-1 \rangle$ in $u.$ Then the theorem above implies

**Proposition 4.5.2.** For $v, v' \in V_{\text{aff}},$ we have

$$\omega_{v,v'}(w_2/w_1) = \phi_{v,v'}(w_2/w_1) \theta(w_2/w_1).$$

(4.5.8)
This formula will be used later to calculate $\gamma_n$.

5. Examples of level 1 Fock spaces

In this section we give some examples of the theory developed in the earlier sections. The case of level 1 type $A_n^{(1)}$ described in [S, KMS] is first reviewed in the perfect crystal language. Then we present results for types $A_{2n}^{(2)}$, $B_n^{(1)}$, $A_{2n-1}^{(2)}$, $D_n^{(1)}$ and $D_{n+1}^{(2)}$ at level 1, corresponding to the perfect crystals of [KMN1] Table 2.

5.1. Preliminaries. Define $[m, n] := \{i \in \mathbb{Z} \mid m \leq i \leq n\}$. We label the simple roots by $I = \{0, n\}$. We choose $0 \in I$ so that $W_{cl}$ is generated by $\{s_i\}_{i \in I \setminus \{0\}}$ and $a_0 = 1$.

We take fundamental weights $\{\Lambda_i\}_{i \in I}$ such that

$$\alpha_i = \sum_{j \in I} \langle h_j, \alpha_i \rangle \Lambda_j + \delta_{i,0}\delta.$$ 

Let $s_0 : P_{cl}^0 \to P^0$ be a section of $cl : P^0 \to P_{cl}^0$ such that

$$s_0(P_{cl}^0) \subset \sum_{i \in I \setminus \{0\}} \mathbb{Q}\alpha_i = \sum_{i \in I \setminus \{0\}} \mathbb{Q}(a_i^\vee \Lambda_i - a_i^\vee \Lambda_0).$$

Then we have

$$s_0(\lambda + cl(\alpha_i)) = \begin{cases} 
    s_0(\lambda) + \alpha_i & \text{for } i \in I \setminus \{0\}, \\
    s_0(\lambda) + \alpha_0 - \delta & \text{for } i = 0.
\end{cases}$$

We regard $V$ as a subspace of $V_{aff}$ by $V \supset V_\Lambda \simeq (V_{aff})_{s_0(\lambda)} \subset V_{aff}$. Then $V_{aff}$ is identified with $\mathbb{Q}[z, z^{-1}] \otimes V$. With this identification, the action of $U_q(\mathfrak{g})$ on $\mathbb{Q}[z, z^{-1}] \otimes V$ is given by

$$e_i(a \otimes v) = z^{\delta_{i,0}}a \otimes e_i v,$$

$$f_i(a \otimes v) = z^{-\delta_{i,0}}a \otimes f_i v.$$ 

Similarly we identify $B$ as a subset of $B_{aff}$.

In the examples that we treat in this paper, the action of $U_q(\mathfrak{g})$ on the lower global base of $V_{aff}$ (respectively $V$) is completely determined by its crystal structure as we have

$$e_i G(b) = [1 + \varphi_i(b)] G(e_i b),$$

$$f_i G(b) = [1 + \varepsilon_i(b)] G(f_i b),$$

$$q^h G(b) = q^{(h, \text{wt}(b))} G(b),$$

for $b \in B_{aff}$ ($b \in B$), $i \in I$ and $h \in P^*$ ($h \in P_{cl}^*$).

5.2. Level 1 $A_n^{(1)}$.
5.2.1. *Cartan datum.* The Dynkin diagram for $A_n^{(1)}$ ($n \geq 1$) is

```
0 −−− 1 −−− 2
|     |     |
n 3
|     |
n 4
n−1 −−− ··· −−−
```

For $A_n^{(1)}$ we have

\[
\delta = \sum_{i \in I} \alpha_i, \\
c = \sum_{i \in I} h_i, \\
(\alpha_i, \alpha_i) = 2 \quad (i \in I).
\]

5.2.2. *Perfect crystal.* Let $J := [0, n]$. Let $V$ be the $(n + 1)$-dimensional $U'_q(A_n^{(1)})$-module with the level 1 perfect crystal $B := \{b_i\}_{i \in J}$ with crystal graph:

```
  b_0 ↑ 1 −→ b_1 2 −→ b_2 ↓ 3
  |     |
  b_n \uparrow \vdots \downarrow \vdots
  |     
  b_{n-1} \leftarrow b_{n-2} \leftarrow \cdots \leftarrow b_{n-3}
```

The elements of $B$ have the following weights

\[
\text{wt}(b_i) = \Lambda_{i+1} - \Lambda_i \quad (i \in J).
\]

Let $v_j := G(b_j)$ ($j \in J$). The action of $U'_q(A_n^{(1)})$ on $v_j \in V$ obeys (5.1.1).

5.2.3. *Energy function.* The energy function $H$ takes the following values on $B \otimes B$

\[
H(b_i \otimes b_j) = \begin{cases} 
1 & \text{for } i > j, \\
0 & \text{for } i \leq j.
\end{cases}
\]

Write $H(i, j)$ for $H(b_i \otimes b_j)$ $(i, j \in J)$.

The Coxeter number of $A_n^{(1)}$ is $h = n + 1 = \dim V$. We take $l : B_{\text{aff}} \to \mathbb{Z}$ to be

\[
l(z^m b_j) = mh - j \quad (m \in \mathbb{Z}, j \in J).
\]

The functions $H$ and $l$ satisfy condition (L) (see end of §3.2). The map $l$ gives a total ordering of $B_{\text{aff}}$.

5.2.4. *Wedge relations.* We have

\[
N := U_q(A_n^{(1)})[z \otimes z, z^{-1} \otimes z^{-1}, z \otimes 1 + 1 \otimes z] \cdot v_0 \otimes v_0 \subset V_{\text{aff}} \otimes V_{\text{aff}}.
\]

The following elements are contained in $U_q(A_n^{(1)}) \cdot v_0 \otimes v_0 \subset N$:

\[
C_{i,i} = v_i \otimes v_i \quad (i \in J), \\
C_{i,j} = v_i \otimes z^{-H(i,j)} v_j + qz^{-H(i,j)} v_j \otimes v_i \quad ((i, j) \in J^2 \setminus \{(k, k)\}_{k \in J}).
\]

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Proposition 5.2.1. Identify $C_{i,j}$ with $C_{b_{i-1},z^{-n(i,j)}b_{j}}$. Then \{\(z^m \otimes z^m \cdot C_{i,j}\)\}_{m \in \mathbb{Z}, i,j \in J}$ with the function \(l\) satisfy condition \((R)\) of subsection 3.3.

5.2.5. Fock space. For \(U_q(A^{(1)}_n)\) we have

\[
B_{\text{min}} = B, \\
\left(P^+_{\text{cl}}\right)_1 = \{A_i^{\text{cl}}\}_{i \in I},
\]

with

\[
\varepsilon(b_j) = \Lambda_j^{\text{cl}}, \quad \varphi(b_j) = \Lambda_{j+1}^{\text{mod h}} \quad (j \in J).
\]

Since \(H(b_j \otimes b_{j-1}) = 1\) \((j \in [1, n])\) and \(H(b_0 \otimes z b_n) = 1\) there is a unique ground state sequence given as follows: every \(m \in \mathbb{Z}\) fixes uniquely \(a \in \mathbb{Z}\) and \(j \in J\) such that \(m = ah - j\), then

\[
b_m^o = z^o b_j \quad (m \in \mathbb{Z}),
\]

\[
\text{cl}(\lambda_m) = \Lambda_{j+1}^{\text{mod h}} \quad (m \in \mathbb{Z}).
\]

With \(v_m^o = G(b_m^o)\), the vacuum vector of \(F_m\) is then given by

\[
|m\rangle = v_m^o \land v_{m+1}^o \land v_{m+2}^o \land \cdots.
\]

with highest weight \(\lambda_m\).

5.3. Level 1 \(A^{(2)}_{2n}\).

5.3.1. Cartan datum. The Dynkin diagram for \(A^{(2)}_{2n}\) \((n \geq 1)\) is

\[
0 \implies 1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow n-2 \longrightarrow n-1 \implies n.
\]

For \(A^{(2)}_{2n}\) we have

\[
\delta = \alpha_0 + \sum_{i=1}^n 2\alpha_i, \\
c = (\sum_{i=0}^{n-1} 2h_i) + h_n,
\]

\[
(\alpha_i, \alpha_i) = \begin{cases} 
8 & \text{for } i = 0, \\
4 & \text{for } i \in [1, n-1], \\
2 & \text{for } i = n.
\end{cases}
\]

5.3.2. Perfect crystal. Let \(J := [-n, n]\). Let \(V\) be the \((2n + 1)\)-dimensional \(U'_q(A^{(2)}_{2n})\)-module with the level 1 perfect crystal \(B := \{b_i\}_{i \in J}\) and crystal graph:

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 b_1 \overset{1}{\longrightarrow} b_2 \overset{2}{\longrightarrow} \cdots \overset{n-2}{\longrightarrow} b_{n-1} \overset{n-1}{\longrightarrow} b_n \\
 0 \\
b_{-1} \overset{1}{\leftarrow} b_{-2} \overset{2}{\leftarrow} \cdots \overset{n-2}{\leftarrow} b_{-n-1} \overset{n-1}{\leftarrow} b_{-n}
\end{array}
\end{array}
\end{array}
\end{array}
\]
Let \( v_j := G(b_j) \) \((j \in J)\). The action of \( U_q^\prime(A^{(2)}_{2n}) \) on \( v_j \in V \) obeys (5.1.1).

5.3.3. **Energy function.** Define the following ordering of \( J \)

\[ 1 \succ 2 \succ \cdots \succ n \succ 0 \succ -n \succ 1 \succ \cdots \succ -1. \]

The energy function \( H \) takes the following values on \( B \otimes B \)

\[
H(b_i \otimes v_j) = \begin{cases} 
1 & \text{for } (i, j) \in \{(i', j') \in J^2 \mid i' \prec j'\} \cup \{(0,0)\}, \\
0 & \text{for } (i, j) \in \{(i', j') \in J^2 \mid i' \succ j'\} \cup \{(k,k)\}_{k \in J \setminus \{0\}}.
\end{cases}
\]

Write \( H(i, j) \) for \( H(b_i \otimes b_j) \) \((i, j \in J)\).

The Coxeter number of \( A^{(2)}_{2n} \) is \( h = 2n + 1 = \dim V \). We take \( l : B_{\text{aff}} \to \mathbb{Z} \) to be

\[
l(z^m b_j) = \begin{cases} 
hm + n + 1 - j & \text{for } j \in [1,n], \\
hm & \text{for } j = 0, \\
hm - (n + 1 + j) & \text{for } j \in [-n,-1].
\end{cases}
\]

The functions \( H \) and \( l \) satisfy condition (L) (see end of §3.2). The map \( l \) gives a total ordering of \( B_{\text{aff}} \).

5.3.4. **Wedge relations.** In \( V_{\text{aff}} \otimes V_{\text{aff}} \) we have

\[ N := U_q(A^{(2)}_{2n}) \cdot [z \otimes z, z^{-1} \otimes z^{-1}, z \otimes 1 \oplus 1 \otimes z] \cdot v_1 \otimes v_1. \]

The following elements are contained in \( U_q(A^{(2)}_{2n}) \cdot v_1 \otimes v_1 \subset N: \)

\[
\begin{align*}
\tilde{C}_{i,i} & = v_i \otimes v_i \quad (i \in J \setminus \{0\}), \\
\tilde{C}_{i,-i} & = v_i \otimes z^{-H(i,-i)}v_{-i} + q^2 v_{i+1} \otimes z^{-H(i,-i)}v_{-i-1} + q^2 z^{-H(i,-i)}v_{-i-1} \otimes v_{i+1} + q^4 z^{-H(i,-i)}v_{-i} \otimes v_i \quad (i \in J \setminus \{-1,0,n\}), \\
\tilde{C}_{i,j} & = v_i \otimes z^{-H(i,j)}v_j + q^2 z^{-H(i,j)}v_j \otimes v_i \quad ((i,j) \in J^2 \setminus \{(k,k),(k,-k)\}_{k \in J}), \\
\tilde{C}_{0,0} & = v_0 \otimes z^{-1}v_0 + q^2[2]v_n \otimes z^{-1}v_n + q^4 z^{-1}v_n \otimes v_0, \\
\tilde{C}_{n,-n} & = v_n \otimes v_n + qv_0 \otimes v_0 + q^4 v_n \otimes v_n, \\
\tilde{C}_{-1,1} & = v_{-1} \otimes z^{-1}v_1 + q^4 z^{-1}v_1 \otimes v_{-1}.
\end{align*}
\]

Notice that each \( \tilde{C}_{i,j} \) has \( v_i \otimes z^{-H(i,j)}v_j \) as its first term and a term in \( z^{-H(i,j)}v_j \otimes v_i \).
Define the following elements in $N$.

$$C_{i,j} := \begin{cases} C_{i,j} & \text{for } (i,j) \in J^2 \setminus \{(k,-k)\}_{k \in J}, \\ \sum_{k=i}^{n} (-q^2)^{k-i} C_{k,-k} & \text{for } (i,j) \in \{(k,-k)\}_{k \in [1,n]}, \\ \sum_{k=1}^{n} (-q^2)^{j-k} C_{-k,k} & \text{for } (i,j) \in \{(-k,k)\}_{k \in [1,n]}, \\ C_{0,0} - q^2[2]C_{-n,n} & \text{for } (i,j) = (0,0). \end{cases}$$

Explicitly for $(i,j) \in \{(k,-k)\}_{k \in J}$, we have

$$C_{j,-j} = v_j \otimes v_{-j} + q^4 v_{-j} \otimes v_j + q(-q^2)^{n-j} v_0 \otimes v_0$$

$$- (1 - q^4)^n \sum_{k=j+1}^{n} (-q^2)^{k-j} v_{-k} \otimes v_k \quad (j \in [1,n]),$$

$$C_{-j,j} = v_{-j} \otimes z^{-1} v_j + q^4 z^{-1} v_j \otimes v_{-j}$$

$$- (1 - q^4)^{j-1} \sum_{k=1}^{j-1} (-q^2)^{j-k} z^{-1} v_k \otimes v_{-k} \quad (j \in [1,n]),$$

$$C_{0,0} = v_0 \otimes z^{-1} v_0 + q^2 z^{-1} v_0 \otimes v_0$$

$$+ q^2[2](1 - q^4)^n \sum_{k=1}^{n} (-q^2)^{n-k} z^{-1} v_k \otimes v_{-k}.$$

**Proposition 5.3.1.** Identify $C_{i,j}$ with $C_{b_i,z^{-n(i,j)}b_j}$. Then $\{z^m \otimes z^m \cdot C_{i,j}\}_{m \in \mathbb{Z}, i,j \in J}$ with the function $l$ satisfy condition (R) of subsection 3.3.

5.3.5. *Fock space.* We have $B_{\min} = \{b_0\}$ and $(P^+_1)_1 = \{\Lambda^d_{n}\}$. Since $H(b_0, b_0) = 1$ we have a unique ground state sequence (up to an overall shift by $z^k$ $(k \in \mathbb{Z})$): $b_m = b_0$ and $\lambda_m = \Lambda_n$ $(m \in \mathbb{Z})$. Therefore the vacuum vector of the Fock space $\mathcal{F}_m$ is

$$|m\rangle := v_0 \wedge v_0 \wedge v_0 \wedge v_0 \wedge \cdots \quad (m \in \mathbb{Z}).$$

We set $\text{wt} (|m\rangle) = \Lambda_n$.

As an illustration of the use of the $q$-adic topology, let us check Proposition 4.3.8 on $|m\rangle$: i.e. that $[e_i, f_i] \cdot |m\rangle = \frac{t_i - t^{-1}_i}{q_i - q^{-1}_i} \cdot |m\rangle$. The case $i \in I \setminus \{n\}$ is trivial. For $e_n|m\rangle$, consider first $v_n \wedge |m + 1\rangle = (-q^2)^j(v_0)^{\wedge j} \wedge v_n \wedge |m + j + 1\rangle$ $(j \in \mathbb{N})$. As $j \to \infty$, the vector vanishes by the $q$-adic topology on $\mathcal{F}_m$. Hence

$$e_n \cdot |m\rangle = \sum_{j=0}^{\infty} (v_0)^{\wedge j} \wedge (e_n \cdot v_0) \wedge |m + j + 1\rangle$$

$$= [2] \sum_{j=0}^{\infty} (v_0)^{\wedge j} \wedge v_n \wedge |m + j + 1\rangle$$

$$= 0.$$
For $f_n$ we have

\[
    f_n \cdot |m\rangle = \sum_{j=0}^{\infty} (v_0)^\wedge j \wedge (f_n \cdot v_0) \wedge t_n |m + j + 1\rangle
\]

\[
    = q[2] \sum_{j=0}^{\infty} (-q^2)^j v_0 \wedge |m + 1\rangle
\]

\[
    = q[2](1 + q^2)^{-1} v_0 \wedge |m + 1\rangle
\]

\[
    = v_0 \wedge |m + 1\rangle.
\]

Then since

\[
    e_n \cdot f_n \cdot |m\rangle = e_n \cdot v_0 \wedge |m + 1\rangle = v_0 \wedge |m + 1\rangle = |m\rangle,
\]

and $[\langle h_n, \Lambda_n \rangle]_n = 1$, this completes the check.

5.4. Level 1 $B_n^{(1)}$.

5.4.1. Cartan datum. The Dynkin diagram for $B_n^{(1)}$ ($n \geq 3$) is

\[
    \begin{array}{ccccccc}
    0 & 2 & 3 & \cdots & n-2 & n-1 & n \\
    \end{array}
\]

For $B_n^{(1)}$ we have

\[
    \delta = \alpha_0 + \alpha_1 + \sum_{i=2}^{n} 2\alpha_i,
\]

\[
    c = h_0 + h_1 + \sum_{i=2}^{n-1} 2h_i + h_n,
\]

\[
    (\alpha_i, \alpha_i) = \begin{cases} 4 & \text{for } i \in [0, n-1], \\ 2 & \text{for } i = n. \end{cases}
\]

5.4.2. Perfect crystal. Let $J := [-n, n]$. Let $V$ be the $(2n + 1)$-dimensional $U_q'(B_n^{(1)})$-module with the level 1 perfect crystal $B := \{b_i\}_{i \in J}$ with crystal graph:

\[
    b_2 \xrightarrow{2} b_3 \xrightarrow{3} \cdots \xrightarrow{n-2} b_{n-1} \xrightarrow{n-1} b_n
\]

\[
    b_0 \xleftarrow{1} b_1 \xleftarrow{b_{-1}} \cdots \xleftarrow{b_{n-2}} b_{n-1} \xleftarrow{b_{n-3}} b_{n}.
\]
The elements of $B$ have the following weights:

$$\text{wt} (b_i) = \sum_{k=1}^{n} \alpha_k = (1 + \delta_{i,n}) \Lambda_i - \Lambda_{i-1} - \delta_{i,2} \Lambda_0 \quad (i \in [1, n]),$$

$$\text{wt} (b_0) = 0,$$

$$\text{wt} (b_{-i}) = -\text{wt} (b_i) \quad (i \in [1, n]).$$

Let $v_j := G(b_j) \ (j \in J)$. The action of $U_q(B_n^{(1)})$ on $v_j \in V$ obeys (5.1.1).

5.4.3. Energy function. Define the following ordering of $J$:

$$1 \succ 2 \succ \cdots \succ n \succ 0 \succ -n \succ 1 - n \succ \cdots \succ -1.$$

The energy function $H$ takes the following values on $B \otimes B$

$$H(b_i \otimes b_j) = \begin{cases} 2 & \text{for } (i, j) = (-1, 1), \\ 1 & \text{for } (i, j) \in \{(i', j') \in J^2 \mid \{(-1, 1) \mid i' < j'\} \cup \{(0, 0)\}, \\ 0 & \text{for } (i, j) \in \{(i', j') \in J^2 \mid i' > j'\} \cup \{(k, k)\}_{k \in J \setminus \{0\}}. \end{cases}$$

Write $H(i, j)$ for $H(b_i \otimes b_j) \ (i, j \in J)$.

The Coxeter number of $B_n^{(1)}$ is $h = 2n = \dim V - 1$. We take $l$ to be

$$l(z^m b_j) = \begin{cases} hm + n + 1 - j & \text{for } j \in [1, n], \\ hm & \text{for } j = 0, \\ hm - (n + 1 + j) & \text{for } j \in [-n, -1]. \end{cases}$$

The functions $H$ and $l$ satisfy condition (L). Note that $l(z^m b_1) = l(z^{m+1} b_{-1}) \ (m \in \mathbb{Z})$, so the map $l$ gives a partial ordering of $B_{aff}$.

5.4.4. Wedge relations. We have

$$N := U_q(B_n^{(1)})[z \otimes z, z^{-1} \otimes z^{-1}, z \otimes 1 + 1 \otimes z] \cdot v_1 \otimes v_1 \subset V_{aff} \otimes V_{aff}.$$

The following elements are contained in $U_q(B_n^{(1)}) \cdot v_1 \otimes v_1 \subset N$:

$$\tilde{C}_{i,i} = v_i \otimes v_i \quad (i \in J \setminus \{0\}),$$

$$\tilde{C}_{i,-i} = v_i \otimes z^{-H(i,-i)} v_{-i} + q^2 v_{i+1} \otimes z^{-H(i,-i)} v_{-i-1} + q^2 z^{-H(i,-i)} v_{-i-1} \otimes v_{i+1} + q^4 z^{-H(i,-i)} v_{-i} \otimes v_i \quad (i \in J \setminus \{-1, 0, n\}),$$

$$\tilde{C}_{i,j} = v_i \otimes z^{-H(i,j)} v_j + q^2 z^{-H(i,j)} v_j \otimes v_i \quad ((i, j) \in J^2 \setminus \{(k, k), (k, -k)\}_{k \in J}),$$

$$\tilde{C}_{0,0} = v_0 \otimes z^{-1} v_0 + q^2 [2] v_{-n} \otimes z^{-1} v_n + q^2 [2] z^{-1} v_n \otimes v_0,$$

$$\tilde{C}_{n,-n} = v_n \otimes v_{-n} + q v_0 \otimes v_0 + q^4 v_{-n} \otimes v_n,$$

$$\tilde{C}_{-1,1} = v_{-1} \otimes z^{-2} v_1 + q^2 z^{-1} v_{-2} \otimes z^{-1} v_2 + q^2 z^{-1} v_2 \otimes z^{-1} v_{-2} + q^4 z^{-2} v_1 \otimes v_{-1}.$$
Each $\tilde{C}_{i,j}$ has $v_i \otimes z^{-H(i,j)}v_j$ as its first term and a term in $z^{-H(i,j)}v_j \otimes v_i$.

Define the following elements in $N$.

$$
C_{i,j} := \begin{cases}
    \tilde{C}_{i,j} & \text{for } (i,j) \in J^2 \setminus \{(k,-k)\}_{k \in J}, \\
    \sum_{k=1}^{n}(q^2)^{k-i}\tilde{C}_{k,-k} & \text{for } (i,j) \in \{(k,-k)\}_{k \in [1,n]}, \\
    \sum_{k=2}^{n}(q^2)^{j-k}\tilde{C}_{-k,k} & \text{for } (i,j) \in \{(-k,k)\}_{k \in [2,n]}, \\
    \tilde{C}_{0,0} - q^2[2]C_{-n,n} & \text{for } (i,j) = (0,0), \\
    \tilde{C}_{-1,1} - q^2(z^{-1} \otimes z^{-1})C_{2,-2} & \text{for } (i,j) = (-1,1).
\end{cases}
$$

Explicitly for $(i,j) \in \{(k,-k)\}_{k \in J}$, we have

$$
C_{j,-j} = v_j \otimes v_{-j} + q^4v_{-j} \otimes v_j + q(-q^2)^{n-j}v_0 \otimes v_0
- (1 - q^4) \sum_{k=1}^{n}(-q^2)^{k-j}v_k \otimes v_k \quad (j \in [1,n]),
$$

$$
C_{-j,j} = v_{-j} \otimes z^{-1}v_j + q^4z^{-1}v_j \otimes v_{-j}
- (1 - q^4) \sum_{k=2}^{j-1}(-q^2)^{j-k}z^{-1}v_k \otimes v_k
- (-q^2)^{j-1}(z^{-1}v_1 \otimes v_{-1} + v_{-1} \otimes z^{-1}v_1) \quad (j \in [2,n]),
$$

$$
C_{0,0} = v_0 \otimes z^{-1}v_0 + q^2z^{-1}v_0 \otimes v_0
+ q^2[2](1 - q^4) \sum_{k=2}^{n}(-q^2)^{n-k}z^{-1}v_k \otimes v_k
+ q^2[2][(-q^2)^n-1](z^{-1}v_1 \otimes v_{-1} + v_{-1} \otimes z^{-1}v_1),
$$

$$
C_{-1,1} = v_{-1} \otimes z^{-2}v_1 + q^4z^{-2}v_1 \otimes v_{-1}
+ q(-q^2)^{n-1}z^{-1}v_0 \otimes v_0 - (1 - q^4) \sum_{k=2}^{n}(-q^2)^{k-1}z^{-1}v_{-k} \otimes v_k.
$$

**Proposition 5.4.1.** Identify $C_{i,j}$ with $C_{b_{i,z^{-H(i,j)}b_j}}$. Then \{$z^m \otimes z^m \cdot C_{i,j}$\}_{m \in \mathbb{Z}, j \in J} with the function $l$ satisfy condition (R) of subsection 3.3.

5.4.5. **Fock space.** For $U_q(B_n^{(1)})$ we have

$$
B_{\min} = \{b_1, b_0, b_{-1}\},
(P_{cl}^+)_1 = \{\Lambda^0_1, \Lambda^1_1, \Lambda^0_0\},
$$

with

$$
\varepsilon(b_1) = \Lambda^0_0, \quad \varepsilon(b_0) = \Lambda^1_0, \quad \varepsilon(b_{-1}) = \Lambda^0_1,
\varphi(b_1) = \Lambda^1_0, \quad \varphi(b_0) = \Lambda^0_n, \quad \varphi(b_{-1}) = \Lambda^0_0.
$$

Since $H(b_0 \otimes b_0) = 1$, $H(b_1 \otimes zb_{-1}) = 1$ and $H(zb_{-1} \otimes b_1) = 1$, there are two ground state sequences (up to overall shifts by $z^k$ $(k \in \mathbb{Z})$):

$$
b_m^0 = b_m \quad (m \in \mathbb{Z}),
\lambda_m = \Lambda_n \quad (m \in \mathbb{Z}).$$
and

\[
b^\circ_m = \begin{cases} 
b_1 & \text{for } m \in 2\mathbb{Z}, \\
z b_{-1} & \text{for } m \in 2\mathbb{Z} + 1, \end{cases}
\]

\[
\lambda_m = \begin{cases} 
\Lambda_1 - \frac{m}{2} \delta & \text{for } m \in 2\mathbb{Z}, \\
\Lambda_0 - \frac{m-1}{2} \delta & \text{for } m \in 2\mathbb{Z} + 1. \end{cases}
\]

The vacuum vectors are respectively

\[
|m\rangle := v_0 \wedge v_0 \wedge v_0 \wedge v_0 \wedge \cdots \quad (m \in \mathbb{Z}),
\]

with \(\text{wt} (|m\rangle) = \Lambda_n\), and

\[
|m\rangle := \begin{cases} 
v_1 \wedge z v_{-1} \wedge v_1 \wedge \cdots & \text{for } m \in 2\mathbb{Z}, \\
z v_{-1} \wedge v_1 \wedge z v_{-1} \wedge \cdots & \text{for } m \in 2\mathbb{Z} + 1, \end{cases}
\]

with \(\text{wt} (|m\rangle) = \Lambda_1 - \frac{m}{2} \delta \) (\(m : \text{even}\)), \(\Lambda_0 - \frac{m-1}{2} \delta \) (\(m : \text{odd}\)).

5.5. Level 1 \(A^{(2)}_{2n-1}\).

5.5.1. Cartan datum. The Dynkin diagram for \(A^{(2)}_{2n-1}\) \((n \geq 3)\) is

\[
\begin{array}{c}
0 \\
\downarrow
\end{array} \quad \begin{array}{c}
2 \\
\downarrow
\end{array} \quad \begin{array}{c}
3 \\
\downarrow
\end{array} \quad \cdots \quad \begin{array}{c}
n-2 \\
\downarrow
\end{array} \quad \begin{array}{c}
n-1 \\
\leftarrow
\end{array} n.
\end{array}
\]

For \(A^{(2)}_{2n-1}\) we have

\[
\delta = \alpha_0 + \alpha_1 + \left( \sum_{i=2}^{n-1} 2\alpha_i \right) + \alpha_n,
\]

\[
c = h_0 + h_1 + \left( \sum_{i=2}^{n} 2h_i \right),
\]

\[
(\alpha_i, \alpha_i) = \begin{cases} 
2 & \text{for } i \in [0, n-1], \\
4 & \text{for } i = n. \end{cases}
\]

5.5.2. Perfect crystal. Let \(J := [-n, -1] \cup [1, n]\). Let \(V\) be the \((2n)\)-dimensional \(U_q(\mathfrak{sl}(2n))\)-module with the level 1 perfect crystal \(B := \{b_i\}_{i \in J}\) with crystal graph:

\[
\begin{array}{cccccccc}
b_2 & \xrightarrow{2} & b_3 & \xrightarrow{3} & \cdots & \xrightarrow{n-2} & b_{n-1} & \xrightarrow{n-1} b_n \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
1 & \xrightarrow{1} & b_{-1} & \xrightarrow{1} & \cdots & \xrightarrow{n-2} & b_{1-n} & \xrightarrow{n-1} b_{-n} \\
\end{array}
\]

The elements of \(B\) have the following weights

\[
\text{wt} (b_i) = \frac{\alpha_n}{2} + \sum_{k \in [i, n-1]} \alpha_k = \Lambda_i - \Lambda_{i-1} - \delta_{i,2} \Lambda_0 \quad (i \in [1, n]),
\]

\[
\text{wt} (b_{-i}) = -\text{wt} (b_i) \quad (i \in [1, n]).
\]
Let $v_j := G(b_j)$ ($j \in J$). The action of $U_q'(A_{2n-1}^{(2)})$ on $v_j \in V$ obeys (5.1.1).

### 5.5.3. Energy function

Define the following ordering of $J$
\[
1 \succ 2 \succ \cdots \succ n \succ -n \succ 1 - n \succ \cdots \succ -1.
\]

The energy function $H$ takes the following values on $B \otimes B$
\[
H(b_i \otimes b_j) = \begin{cases} 
2 & \text{for } (i, j) = (-1, 1), \\
1 & \text{for } (i, j) \in \{(i', j') \in J^2 \setminus \{(-1, 1) \} | i' \prec j'\}, \\
0 & \text{for } (i, j) \in \{(i', j') \in J^2 | i' \succ j'\} \cup \{(k, k)\}_{k \in J}.
\end{cases}
\]

Write $H(i, j)$ for $H(b_i \otimes b_j)$ $(i, j \in J)$.

The Coxeter number of $A_{2n-1}^{(2)}$ is $h = 2n - 1 = \dim V - 1$. We take $l$ to be
\[
l(z^mb_j) = \begin{cases} 
hm + n - j & \text{for } j \in [1, n], \\
hm - (n + 1 + j) & \text{for } j \in [-n, -1].
\end{cases}
\]

The functions $H$ and $l$ satisfy condition (L). Note that $l(z^mb_1) = l(z^{m+1}b_{-1})$ ($m \in \mathbb{Z}$), so $l$ gives a partial ordering of $B_{\text{aff}}$.

### 5.5.4. Wedge relations

We have
\[
N := U_q(A_{2n-1}^{(2)})[z \otimes z, z^{-1} \otimes z^{-1}, z \otimes 1 + 1 \otimes z] \cdot v_1 \otimes v_1 \subset V_{\text{aff}} \otimes V_{\text{aff}}.
\]

The following elements are contained in $U_q(A_{2n-1}^{(2)}) \cdot v_1 \otimes v_1 \subset N$:
\[
\begin{align*}
\tilde{C}_{i,i} &= v_i \otimes v_i \quad (i \in J), \\
\tilde{C}_{i,-i} &= v_i \otimes z^{-H(i, -i)}v_{-i} + qv_{i+1} \otimes z^{-H(i, -i)}v_{-i-1} \\
&\quad + qz^{-H(i, -i)}v_{-i-1} \otimes v_{i+1} \\
&\quad + q^2z^{-H(i, -i)}v_{i-1} \otimes v_i \quad (i \in J \setminus \{-1, n\}), \\
\tilde{C}_{i,j} &= v_i \otimes z^{-H(i, j)}v_j \\
&\quad + qz^{-H(i, j)}v_j \otimes v_i \quad ((i, j) \in J^2 \setminus \{(k, k), (k, -k)\}_{k \in J}), \\
\tilde{C}_{n,-n} &= v_n \otimes v_{-n} + q^2v_{-n} \otimes v_n, \\
\tilde{C}_{-1,1} &= v_{-1} \otimes z^{-2}v_1 + qz^{-1}v_{-2} \otimes z^{-1}v_2 \\
&\quad + qz^{-1}v_2 \otimes z^{-1}v_{-2} + q^2z^{-2}v_1 \otimes v_{-1}.
\end{align*}
\]

Each $\tilde{C}_{i,j}$ has $v_i \otimes z^{-H(i, j)}v_j$ as its first term and a term in $z^{-H(i, j)}v_j \otimes v_i$.

Define the following elements in $N$.
\[
C_{i,j} := \begin{cases} 
\tilde{C}_{i,j} & \text{for } (i, j) \in J^2 \setminus \{(k, -k)\}_{k \in J}, \\
\sum_{k=1}^n (-q)^{k-i} \tilde{C}_{k,-k} & \text{for } (i, j) \in \{(k, -k)\}_{k \in [1,n]}, \\
\sum_{k=2}^j (-q)^{j-k} \tilde{C}_{-k,k} & \text{for } (i, j) \in \{(-k, k)\}_{k \in [2,n]}, \\
\tilde{C}_{-1,1} - q(z^{-1} \otimes z^{-1})C_{2,-2} & \text{for } (i, j) = (-1, 1).
\end{cases}
\]
Explicitly for \((i, j) \in \{(k, -k)\}_{k \in J}\), we have
\[
C_{j,-j} = v_j \otimes v_{-j} + q^2 v_{-j} \otimes v_j \\
- (1 - q^2) \sum_{k=j+1}^{n} (-q)^{j-k} v_{-k} \otimes v_k \quad (j \in [1, n]),
\]
\[
C_{-j,j} = v_{-j} \otimes z^{-1} v_{j} + q^2 z^{-1} v_{j} \otimes v_{-j} \\
- (1 - q^2) \sum_{k=2}^{j-1} (-q)^{j-k} z^{-1} v_k \otimes v_{-k} \\
- (-q)^{j-1} (z^{-1} v_1 \otimes v_{-1} + v_{-1} \otimes z^{-1} v_1) \quad (j \in [2, n]),
\]
\[
C_{-1,1} = v_{-1} \otimes z^{-2} v_{1} + q^2 z^{-2} v_{1} \otimes v_{-1} \\
- (1 - q^2) \sum_{k=2}^{n} (-q)^{k-1} z^{-1} v_{-k} \otimes z^{-1} v_k.
\]

Proposition 5.5.1. Identify \(C_{i,j}\) with \(C_{b_i,z^{-m(i,j)}b_j}\). Then \(\{z^m \otimes z^m \cdot C_{i,j}\}_{m \in \mathbb{Z}, i,j \in J}\) with the function \(l\) satisfy condition \((R)\) of subsection 3.3.

5.5.5. Fock space. For \(U_q(A^{(2)}_{2n-1})\) we have
\[
B_{\text{min}} = \{b_1, b_{-1}\}, \\
(P^+_{cl})_1 = \{\Lambda^1_{0}, \Lambda^1_{1}\},
\]
with
\[
\varepsilon(b_1) = \Lambda^1_{0}, \quad \varepsilon(b_{-1}) = \Lambda^1_{1}, \\
\varphi(b_1) = \Lambda^1_{1}, \quad \varphi(b_{-1}) = \Lambda^1_{0}.
\]
Since \(H(b_1 \otimes z b_{-1}) = 1\) and \(H(z b_{-1} \otimes b_1) = 1\) there is one ground state sequence:
\[
b_m^0 = \begin{cases} 
    b_1 & \text{for } m \in 2\mathbb{Z}, \\
    z b_{-1} & \text{for } m \in 2\mathbb{Z} + 1,
\end{cases}
\]
\[
\lambda_m = \begin{cases} 
    \Lambda_1 - \frac{m}{2} \delta & \text{for } m \in 2\mathbb{Z}, \\
    \Lambda_0 - \frac{m-1}{2} \delta & \text{for } m \in 2\mathbb{Z} + 1.
\end{cases}
\]
The vacuum vector of \(\mathcal{F}_m\) is
\[
|m\rangle := \begin{cases} 
    v_1 \wedge z v_{-1} \wedge v_1 \wedge \cdots & \text{for } m \in 2\mathbb{Z}, \\
    z v_{-1} \wedge v_1 \wedge z v_{-1} \wedge \cdots & \text{for } m \in 2\mathbb{Z} + 1,
\end{cases}
\]
with \(\text{wt} (| m \rangle) = \Lambda_1 - \frac{m}{2} \delta \ (m : \text{even}), \ \Lambda_0 - \frac{m-1}{2} \delta \ (m : \text{odd}).\)

5.6. Level 1 \(D^{(1)}_n\)
5.6.1. **Cartan datum.** The Dynkin diagram for $D_n^{(1)}$ $(n \geq 4)$ is

```
0   2   3   ···   n-3   n-2   n
|    |    |    |    |    |
1    |    |    |    |    |
```

For $D_n^{(1)}$ we have

\[
\begin{align*}
\delta &= \alpha_0 + \alpha_1 + \sum_{i=2}^{n-2} 2\alpha_i + \alpha_{n-1} + \alpha_n, \\
c &= h_0 + h_1 + \sum_{i=2}^{n-2} 2h_i + h_{n-1} + h_n, \\
(\alpha_i, \alpha_i) &= 2 \quad (i \in I).
\end{align*}
\]

5.6.2. **Perfect crystal.** Let $J := [-n, -1] \cup [1, n]$. Let $V$ be the $(2n)$-dimensional $U'_q(D_n^{(1)})$-module with the level 1 perfect crystal $B := \{b_i\}_{i \in J}$ with crystal graph:

```
\begin{array}{c}
b_2 \rightarrow b_3 \rightarrow \cdots \rightarrow b_{n-2} \rightarrow b_{n-1} \\
1 \leftarrow b_1 \leftarrow b_{-1} \\
0 \leftarrow b_{-2} \leftarrow \cdots \leftarrow b_{-n+2} \leftarrow b_{-n+1} \\
\end{array}
```

The elements of $B$ have the following weights

\[
\begin{align*}
\text{wt} (b_i) &= \left(\sum_{k=i}^{n-2} \alpha_k\right) + (\alpha_{n-1} + \alpha_n)/2 \\
&= \Lambda_i - \Lambda_{i-1} + \delta_{i,n-1} \Lambda_n - \delta_{i,2} \Lambda_0 \quad (i \in [1, n]), \\
\text{wt} (b_{-i}) &= -\text{wt} (b_i) \quad (i \in [1, n]).
\end{align*}
\]

Let $v_j := G(b_j)$ $(j \in J)$. The action of $U'_q(D_n^{(1)})$ on $v_j \in V$ obeys (5.1.1).

5.6.3. **Energy function.** Define the following ordering of $J$

\[1 \succ 2 \succ \cdots \succ n \succ -n \succ 1-n \succ \cdots \succ -1.
\]

The energy function $H$ takes the following values on $B \otimes B$

\[
H(b_i \otimes b_j) = \begin{cases} 
2 & \text{for } (i, j) = (-1, 1), \\
1 & \text{for } (i, j) \in \{(i', j') \in J^2 \setminus \{(-1, 1)\} \mid i' \prec j'\} \cup \{(n, -n)\}, \\
0 & \text{for } (i, j) \in \{(i', j') \in J^2 \setminus \{(n, -n)\} \mid i' \succ j'\} \cup \{(k, k)\}_{k \in J}.
\end{cases}
\]

Write $H(i, j)$ for $H(b_i \otimes b_j)$ $(i, j \in J)$.

The Coxeter number of $D_n^{(1)}$ is $h = n + 1 = \dim V - 2$. We take $l$ to be

\[
l(z^m b_j) = \begin{cases} 
hm + n - j & \text{for } j \in [1, n], \\
hm - (n + j) & \text{for } j \in [-n, -1].
\end{cases}
\]

The functions $H$ and $l$ satisfy condition (L). Note that $l(z^m b_1) = l(z^{m+1} b_{-1})$ and $l(z^m b_n) = l(z^{m} b_{-n})$ $(m \in \mathbb{Z})$, so $l$ gives a partial ordering of $B_{aff}$. 

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5.6.4. **Wedge relations.** We have

\[ N := U_q(D^{(1)}_n)[z \otimes z, z^{-1} \otimes z^{-1}, z \otimes 1 + 1 \otimes z] \cdot v_1 \otimes v_1 \subset V_{aff} \otimes V_{aff}. \]

The following elements are contained in \( U_q(D^{(1)}_n) \cdot v_1 \otimes v_1 \subset N \):

\[
\begin{align*}
\tilde{C}_{i,i} &= v_i \otimes v_i & (i & \in J), \\
\tilde{C}_{i,-i} &= v_i \otimes z^{-H(i,i)}v_{-i} + qv_{i+1} \otimes z^{-H(i,i)}v_{-i-1} & (i & \in J \setminus \{-1, n\}), \\
\tilde{C}_{i,j} &= v_i \otimes z^{-H(i,j)}v_{j} & (i, j) \in J^2 \setminus \{(k, k), (k, -k)\}_{k \in J}, \\
\tilde{C}_{n,-n} &= v_n \otimes z^{-1}v_{-n} + qv_{1-n} \otimes z^{-1}v_{n-1} + qz^{-1}v_{n-1} \otimes v_{1-n} + q^2z^{-1}v_{n-n} \otimes v_{1}, \\
\tilde{C}_{-1,1} &= v_{-1} \otimes z^{-2}v_{1} + qz^{-1}v_{-2} \otimes z^{-1}v_{2} + qz^{-2}v_{1} \otimes z^{-2}v_{1} \otimes v_{-1}.
\end{align*}
\]

Each \( \tilde{C}_{i,j} \) has \( v_i \otimes z^{-H(i,j)}v_{j} \) as its first term and a term in \( z^{-H(i,j)}v_{j} \otimes v_{i} \).

Define the following elements in \( N \):

\[
C_{i,j} := \begin{cases}
\tilde{C}_{i,j} & \text{for } (i, j) \in J^2 \setminus \{(k, k)\}_{k \in J}, \\
\sum_{k=i}^{n-1} (-q)^{k-i} \tilde{C}_{k,-k} & \text{for } (i, j) \in \{(k, -k)\}_{k \in [1, n-1]}, \\
\sum_{k=2}^{j} (-q)^{j-k} \tilde{C}_{-k,k} & \text{for } (i, j) \in \{(-k, k)\}_{k \in [2, n]}, \\
C_{n,-n} - qC_{1-n,n-1} & \text{for } (i, j) = (n, -n), \\
\tilde{C}_{-1,1} - q(z^{-1} \otimes z^{-1})C_{2,-2} & \text{for } (i, j) = (-1, 1).
\end{cases}
\]

Explicitly for \( (i, j) \in \{(k, -k)\}_{k \in J} \), we have

\[
\begin{align*}
C_{j,-j} &= v_j \otimes v_{-j} + q^2v_{-j} \otimes v_{j} \\
&\quad - (1 - q^2) \sum_{k=j+1}^{n-1} (-q)^{k-j}v_{-k} \otimes v_k \\
&\quad - (-q)^{n-j}(v_n \otimes v_{-n} + v_{-n} \otimes v_n) \quad (j \in [1, n]), \\
C_{-j,j} &= v_{-j} \otimes z^{-1}v_{j} + q^2z^{-1}v_{j} \otimes v_{-j} \\
&\quad - (1 - q^2) \sum_{k=2}^{j-1} (-q)^{j-k}z^{-1}v_{k} \otimes v_{-k} \\
&\quad - (-q)^{j-1}(z^{-1}v_{1} \otimes v_{-1} + v_{-1} \otimes z^{-1}v_{1}) \quad (j \in [2, n]), \\
C_{n,-n} &= v_n \otimes z^{-1}v_{-n} + q^2z^{-1}v_{-n} \otimes v_n \\
&\quad - (1 - q^2) \sum_{k=2}^{n-1} (-q)^{n-k}z^{-1}v_{k} \otimes v_{-k} \\
&\quad - (-q)^{n-1}(z^{-1}v_{1} \otimes v_{-1} + v_{-1} \otimes z^{-1}v_{1}) \\
C_{-1,1} &= v_{-1} \otimes z^{-2}v_{1} + q^2z^{-2}v_{1} \otimes v_{-1}.
\end{align*}
\]
\[-(1-q^2)\sum_{k=2}^{n-1}(-q)^{k-1}v_{-k} \otimes z^{-1}v_k\]
\[-(-q)^{n-1}(z^{-1}v_n \otimes z^{-1}v_{-n} + z^{-1}v_{-n} \otimes z^{-1}v_n).\]

**Proposition 5.6.1.** Identify $C_{i,j}$ with $C_{b_1,z^{-\eta(u,j)}b_j}$. Then $\{z^m \otimes z^n : C_{i,j}\}_{m \in \mathbb{Z}, i,j \in J}$ with the function $l$ satisfy condition (R) of subsection 3.3.

5.6.5. **Fock space.** For $U_q(D_n^{(1)})$ we have
\[
\begin{align*}
B_{\min} & = \{b_1, b_{-1}, b_n, b_{-n}\}, \\
(P_{cl}^\dagger)_1 & = \{\Lambda^\dagger_1, \Lambda^\dagger_0, \Lambda^\dagger_{n-1}, \Lambda^\dagger_n\},
\end{align*}
\]
with
\[
\begin{align*}
\varepsilon(b_1) & = \Lambda^\dagger_0, & \varphi(b_1) & = \Lambda^\dagger_1, \\
\varepsilon(b_{-1}) & = \Lambda^\dagger_1, & \varphi(b_{-1}) & = \Lambda^\dagger_0, \\
\varepsilon(b_n) & = \Lambda^\dagger_n, & \varphi(b_n) & = \Lambda^\dagger_{n-1}, \\
\varepsilon(b_{-n}) & = \Lambda^\dagger_{n-1}, & \varphi(b_{-n}) & = \Lambda^\dagger_n.
\end{align*}
\]

Since $H(b_1 \otimes z b_{-1}) = 1$, $H(z b_{-1} \otimes b_1) = 1$, $H(b_n \otimes b_{-n}) = 1$ and $H(b_{-n} \otimes b_n) = 1$, there are two ground state sequences:
\[
\begin{align*}
b^\circ_m & = \begin{cases} 
  b_1 & \text{for } m \in 2\mathbb{Z}, \\
  z b_{-1} & \text{for } m \in 2\mathbb{Z} + 1,
\end{cases} \\
\lambda_m & = \begin{cases} 
  \Lambda_1 - \frac{m}{2} \delta & \text{for } m \in 2\mathbb{Z}, \\
  \Lambda_0 - \frac{m-1}{2} \delta & \text{for } m \in 2\mathbb{Z} + 1,
\end{cases}
\end{align*}
\]
and
\[
\begin{align*}
b_m^\circ & = \begin{cases} 
  b_n & \text{for } m \in 2\mathbb{Z}, \\
  b_{-n} & \text{for } m \in 2\mathbb{Z} + 1,
\end{cases} \\
\lambda_m & = \begin{cases} 
  \Lambda_{n-1} & \text{for } m \in 2\mathbb{Z}, \\
  \Lambda_n & \text{for } m \in 2\mathbb{Z} + 1.
\end{cases}
\end{align*}
\]

The vacuum vector of $\mathcal{F}_m$ are respectively
\[
|m\rangle := \begin{cases} 
  v_1 \wedge z v_{-1} \wedge v_1 \wedge \cdots & \text{for } m \in 2\mathbb{Z}, \\
  z v_{-1} \wedge v_1 \wedge z v_{-1} \wedge \cdots & \text{for } m \in 2\mathbb{Z} + 1,
\end{cases}
\]
with $\text{wt} (|m\rangle) = \Lambda_1 - \frac{m}{2} \delta$ ($m : \text{even}$), $\Lambda_0 - \frac{m-1}{2} \delta$ ($m : \text{odd}$), and
\[
|m\rangle := \begin{cases} 
  v_n \wedge v_{-n} \wedge v_n \wedge \cdots & \text{for } m \in 2\mathbb{Z}, \\
  v_{-n} \wedge v_n \wedge v_{-n} \wedge \cdots & \text{for } m \in 2\mathbb{Z} + 1,
\end{cases}
\]
with $\text{wt} (|m\rangle) = \Lambda_{n-1}$ ($m : \text{even}$), $\Lambda_n$ ($m : \text{odd}$).

5.7. **Level 1 $D_n^{(2)}$.**
5.7.1. Cartan datum. The Dynkin diagram for $D^{(2)}_{n+1}$ ($n \geq 4$) is

\[ 0 \leftarrow 1 \leftarrow 2 \leftarrow \cdots \leftarrow n-2 \leftarrow n-1 \leftarrow n. \]

For $D^{(2)}_{n+1}$ we have

\[ \delta = \sum_{i \in I} \alpha_i, \]
\[ c = h_0 + (\sum_{i=1}^{n-1} 2h_i) + h_n, \]
\[ (\alpha_i, \alpha_i) \begin{cases} 2 & \text{for } i \in \{0, n\}, \\ 4 & \text{for } i \in I \setminus \{0, n\}. \end{cases} \]

5.7.2. Perfect crystal. Let $J := [-n, n] \cup \{\phi\}$. Let $V$ be the $(2n + 2)$-dimensional $U'_q(D^{(2)}_{n+1})$-module with the level 1 perfect crystal $B := \{b_i\}_{i \in J}$ and crystal graph:

The elements of $B$ have the following weights

\[ \text{wt} (b_i) = \sum_{k=i}^{n} \alpha_k = (1 + \delta_{i,n}) \Lambda_i - (1 + \delta_{i,1}) \Lambda_i - 1 \quad (i \in [1, n]) \]
\[ \text{wt} (b_0) = 0 \]
\[ \text{wt} (b_\phi) = 0 \]
\[ \text{wt} (b_{-i}) = - \text{wt} (b_i) \quad (i \in [1, n]). \]

Let $v_j := G(b_j)$ ($j \in J$). The action of $U'_q(D^{(2)}_{n+1})$ on $v_j \in V$ obeys (5.1.1).

Let $J_0 := J \setminus \{\phi\}$. Let $V_0$ denote that subspace of $V$ spanned by $\{v_j\}_{j \in J_0}$. Then, $V_{aff}$ decomposes into two $U'_q(D^{(2)}_{n+1})$-modules:

\[ V_{aff} = \left( V_0 \otimes \mathbb{C}[z^2, z^{-2}] + v_\phi \otimes \mathbb{C}[z^2, z^{-2}] \right) \]
\[ \oplus \left( V_0 \otimes \mathbb{C}[z^2, z^{-2}] + v_\phi \otimes \mathbb{C}[z^2, z^{-2}] \right). \]

5.7.3. Energy function. Define the following ordering of $J$

\[ 1 \succ 2 \succ \cdots \succ n \succ 0 \succ -n \succ 1 - n \succ \cdots \succ -1 \succ \phi. \quad (5.7.1) \]

The energy function $H$ takes the following values on $B \otimes B$

\[ H(b_i \otimes b_j) = \begin{cases} 2 & \text{for } (i, j) \in \{(i', j') \in J_0^2 \mid i' \prec j'\} \cup \{(0, 0), (\phi, \phi)\}, \\ 1 & \text{for } (i, j) \in \{(k, \phi), (\phi, k) \in J^2 \mid k \in J \setminus \{\phi\}\}, \\ 0 & \text{for } (i, j) \in \{(i', j') \in J_0^2 \mid i' \succ j'\} \cup \{(k, k)\}_{k \in J \setminus \{0, \phi\}}. \end{cases} \]

Write $H(i, j)$ for $H(b_i \otimes b_j)$ ($i, j \in J$).
The Coxeter number of $D_{n+1}^{(2)}$ is $h = n + 1 = \frac{1}{2} \dim V$. We take $l$ to be

$$l(z^m b_j) = \begin{cases} hm + n + 1 - j & \text{for } j \in [1, n], \\ hm & \text{for } j \in \{0, \phi\}, \\ hm - (n + 1 + j) & \text{for } j \in [-n, -1]. \end{cases}$$

The functions $H$ and $l$ satisfy condition (L). Note that $l(z^m b_0) = l(z^m b_\phi)$ and $l(z^m b_i) = l(z^{m+1} b_{i-h})$ ($m \in \mathbb{Z}$ and $i \in [1, n]$), so $l$ gives a partial ordering of $B_{aff}$. ($l$ gives a total ordering of each of the crystals of the two irreducible submodules.)

5.7.4. Wedge relations. In $V_{aff} \otimes V_{aff}$ we have

$$N := U_q(D_{n+1}^{(2)})[z \otimes z, z^{-1} \otimes z^{-1}, z \otimes 1 + 1 \otimes z] \cdot v_1 \otimes v_1.$$ The following elements are contained in $U_q(D_{n+1}^{(2)})[z \otimes z, z^{-1} \otimes z^{-1}] \cdot v_1 \otimes v_1 \subset N$:

$$\tilde{C}_{i,i} = v_i \otimes v_i \quad (i \in J \setminus \{0, \phi\}),$$

$$\tilde{C}_{i,-i} = v_i \otimes z^{-H(i,-i)} v_{-i} + q^2 v_{i+1} \otimes z^{-H(i,-i)} v_{-i-1}$$

$$+ q^2 z^{-H(i,-i)} v_{-i-1} \otimes v_{i+1}$$

$$+ q^4 z^{-H(i,-i)} v_{-i} \otimes v_i \quad (i \in J \setminus \{-1, 0, \phi, n\}),$$

$$\tilde{C}_{i,j} = v_i \otimes z^{-H(i,j)} v_j + q^2 z^{-H(i,j)} v_j \otimes v_i$$

$$\quad \left((i, j) \in J^2 \setminus \{(k, k), (k, -k)\}_{k \in J}\right),$$

$$\tilde{C}_{0,0} = v_0 \otimes z^{-2} v_0 + q^2 [2] v_{-n} \otimes z^{-2} v_n$$

$$+ q^2 [2] z^{-2} v_n \otimes v_{-n} + q^2 z^{-2} v_0 \otimes v_0,$$

$$\tilde{C}_{n-n} = v_n \otimes v_n + q v_0 \otimes v_0 + q^4 v_{-n} \otimes v_n,$$

$$\tilde{C}_{-1,1} = v_{-1} \otimes z^{-2} v_1 + q z^{-1} v_{\phi} \otimes z^{-1} v_{\phi} + q^4 z^{-2} v_1 \otimes v_{-1},$$

$$\tilde{C}_{\phi, \phi} = v_{\phi} \otimes z^{-2} v_{\phi} + q^2 [2] z^{-1} v_1 \otimes z^{-1} v_{-1}$$

$$+ q^2 [2] z^{-1} v_{-1} \otimes z^{-1} v_1 + q^2 z^{-2} v_{\phi} \otimes v_{\phi}.$$

Notice that each $\tilde{C}_{i,j}$ has $v_i \otimes z^{-H(i,j)} v_j$ as its first term and a term in $z^{-H(i,j)} v_j \otimes v_i$.

Define the following elements in $N$:

$$C_{i,j} := \begin{cases} \tilde{C}_{i,j} & \text{for } (i, j) \in J^2 \setminus \{(k, -k)\}_{k \in J}, \\ \sum_{i=1}^n (-q^2)^{k-i} \tilde{C}_{k,-k} & \text{for } (i, j) \in \{(k, -k)\}_{k \in [1, n]}, \\ \sum_{i=1}^n (-q^2)^{j-k} \tilde{C}_{-k,k} & \text{for } (i, j) \in \{(-k, k)\}_{k \in [1, n]}, \\ \tilde{C}_{0,0} - q^2 [2] C_{n-n} & \text{for } (i, j) = (0, 0), \\ \tilde{C}_{\phi, \phi} - q^2 [2] (z^{-1} \otimes z^{-1}) C_{1,1} & \text{for } (i, j) = (\phi, \phi). \end{cases}$$

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Explicitly for \((i, j) \in \{(k, -k)\}_{k \in J} \cup \{\phi\}\), we have
\[
C_{j, -j} = v_j \otimes v_{-j} + q^2 v_{-j} \otimes v_j + q(-q^2)^{n-j} v_0 \otimes v_0 \\
-(1 - q^4) \sum_{k=j+1}^{n} (-q^2)^{k-j} v_{-k} \otimes v_k \quad (j \in [1, n]),
\]
\[
C_{-j, j} = v_{-j} \otimes z^{-2} v_j + q^4 z^{-2} v_j \otimes v_{-j} + q(-q^2)^{j-1} z^{-1} v_0 \otimes z^{-1} v_{\phi} \\
-(1 - q^4) \sum_{k=1}^{j-1} (-q^2)^{j-k} z^{-2} v_k \otimes v_{-k} \quad (j \in [1, n]),
\]
\[
C_{0, 0} = v_0 \otimes z^{-2} v_0 + q^2 z^{-2} v_0 \otimes v_0 + q[2](-q^2)^n z^{-1} v_0 \otimes z^{-1} v_{\phi} \\
-2(1 - q^4) \sum_{k=1}^{n} (-q^2)^{n+1-k} z^{-2} v_k \otimes v_{-k},
\]
\[
C_{\phi, \phi} = v_{\phi} \otimes z^{-2} v_{\phi} + q^2 z^{-2} v_{\phi} \otimes v_{\phi} + q[2](-q^2)^n z^{-1} v_0 \otimes z^{-1} v_0 \\
-2(1 - q^4) \sum_{k=1}^{n} (-q^2)^{j} z^{-1} v_{-j} \otimes z^{-1} v_j.
\]

**Proposition 5.7.1.** Identify \(C_{i, j}\) with \(C_{b_i z^{-m(i,j)} b_j}\). Then \(\{z^m \otimes z^m \cdot C_{i, j}\}_{m \in \mathbb{Z}, i, j \in J}\) with the function \(l\) satisfy condition \((R)\) of subsection 3.3.

5.7.5. **Fock space.** For \(U_q(D_{n+1}^{(2)})\) we have
\[
B_{\min} = \{b_0, b_{\phi}\}, \\
(P_{cl}^+) = \{\Lambda_n^{cl}, \Lambda_0^{cl}\},
\]
with
\[
\varepsilon(b_0) = \Lambda_n^{cl}, \quad \varepsilon(b_{\phi}) = \Lambda_0^{cl}, \\
\varphi(b_0) = \Lambda_n^{cl}, \quad \varphi(b_{\phi}) = \Lambda_0^{cl}.
\]

Since \(H(b_0 \otimes z^{-1} b_0) = 1\) and \(H(b_{\phi} \otimes z^{-1} b_{\phi}) = 1\), there are two ground state sequences (see also the remark at the end of §6.6):
\[
b_{m}^o = z^{-m} b_0 \quad (m \in \mathbb{Z}), \\
cl(\lambda_m) = \Lambda_n \quad (m \in \mathbb{Z}),
\]
and
\[
b_{m}^o = z^{-m} b_{\phi} \quad (m \in \mathbb{Z}), \\
cl(\lambda_m) = \Lambda_0 \quad (m \in \mathbb{Z}).
\]
The vacuum vectors of \(\mathcal{F}_m\) are respectively
\[
|m\rangle := z^{-m} v_0 \wedge z^{-m-1} v_0 \wedge z^{-m-2} v_0 \wedge \ldots \quad (m \in \mathbb{Z}),
\]
with \(\text{wt} \ (|m\rangle) = \Lambda_n\), and
\[
|m\rangle := z^{-m} v_0 \wedge z^{-m-1} v_0 \wedge z^{-m-2} v_0 \wedge \ldots \quad (m \in \mathbb{Z}),
\]
with \(\text{wt} \ (|m\rangle) = \Lambda_0\).
6. LEVEL 1 TWO POINT FUNCTIONS

In this section we calculate the boson commutation relations using the decomposition of the Fock space vertex operator into a product of a $U_{q}(g)$-vertex operator and a bosonic vertex operator (Theorem 4.5.1), for level 1 $A_{2n}^{(2)}$, $B_{n}^{(1)}$, $A_{2n-1}^{(2)}$, $D_{n}^{(2)}$, and $D_{n+1}^{(2)}$. The two point functions of the level 1 $U_{q}(g)$-vertex operators that we use are due to Date and Okado [DO] (except for type $D_{n+1}^{(2)}$ which is given in Appendix C).

6.1. Summary. In the following table we list the dual Coxeter number $h_{\vee} := \sum_{i \in I} a_{i}^{\vee}$, $p := q^{\alpha_{0,\alpha_{0}}}/(2a_{0}^{\vee})$ and $\xi := (-1)^{-1}p^{h_{\vee}}$ for $g = X^{(r)}_{n}$ of types $A_{n}^{(1)}$, $A_{2n}^{(2)}$, $B_{n}^{(1)}$, $A_{2n-1}^{(2)}$ and $D_{n}^{(1)}$.

| $g$      | $A_{n}^{(1)}$ | $A_{2n}^{(2)}$ | $B_{n}^{(1)}$ | $A_{2n-1}^{(2)}$ | $D_{n}^{(1)}$ |
|----------|---------------|----------------|---------------|-------------------|---------------|
| $h_{\vee}$ | $n + 1$       | $2n + 1$       | $2n - 1$      | $2n$              | $2n - 2$      |
| $p$      | $q$           | $q^{r}$        | $q^{r}$       | $q$               | $q$           |
| $\xi$   | $q^{n+1}$     | $-q^{2(2n+1)}$ | $q^{2(2n-1)}$ | $-q^{2n}$         | $q^{2n-2}$    |

Proposition 6.1.1 ([KMS]). For $A_{n}^{(1)}$ at level 1, we have

$$\gamma_{m} = m \frac{1 - \xi^{2m}}{1 - q^{2m}}.$$  

See [KMS] §2 for the proof.

Let $g = X^{(r)}_{n}$ be of type $A_{2n}^{(2)}$, $B_{n}^{(1)}$, $A_{2n-1}^{(2)}$ or $D_{n}^{(1)}$. Let $|g\rangle$ be one of the vacuum vectors of the $U_{q}(g)$-Fock modules described in the previous section. For each type, direct calculations of $B_{m} \cdot |g\rangle$ for small $m$, suggest the following result.

Theorem 6.1.2. For $g = X^{(r)}_{n} \in \{A_{2n}^{(2)}, B_{n}^{(1)}, A_{2n-1}^{(2)}, D_{n}^{(1)}\}$ at level 1, we have

$$\gamma_{m} = m \frac{1 + \xi^{m}}{1 - p^{2m}}.$$  

In this section we prove this theorem case by case using Proposition 4.5.2. We also give a corresponding result for level 1 $D_{n+1}^{(2)}$.

For this boson commutation relation, the boson two point function (4.5.7) is

$$\theta(w_{2}/w_{1}) = \frac{(w_{2}/w_{1}; \xi^{2})_{\infty}(p^{2}w_{2}/w_{1}; \xi^{2})_{\infty}}{(p^{2}w_{2}/w_{1}; \xi^{2})_{\infty}(\xi w_{2}/w_{1}; \xi^{2})_{\infty}}. \quad (6.1.1)$$

Let us introduce the operator $Z(t, d) \in \text{End}(V_{\text{aff}} \otimes V_{\text{aff}})$ defined by:

$$Z(t, d) := z^{t} \otimes z^{d-t} + \delta(2t > d)z^{d-t} \otimes z^{t} - \delta(2t < d)z^{t} \otimes z^{d-t} \quad (t, d \in \mathbb{Z}).$$

Note that $Z(t, d)$ is a symmetric Laurent polynomial in $z \otimes 1$ and $1 \otimes z$, so we have

Lemma 6.1.3. $Z(t, d) \cdot N \subset N \ (t, d \in \mathbb{Z})$. 

6.2. Type $A^{(2)}_{2n}$. Recall we have $\lambda_m = \Lambda_n$ and $b_m^0 = b_0$ ($m \in \mathbb{Z}$). So the level 1 intertwiner maps

$$\Phi_m : V_{\text{aff}} \otimes V(\Lambda_n) \rightarrow V(\Lambda_n) \quad (m \in \mathbb{Z}).$$

From [DO], (up to a factor of a constant power in $w_2/w_1$) we have in our notation (4.5.5)

$$\phi_{v_0,v_0}(w_2/w_1) = \left(1 - p^{h^\vee + 1}w_2/w_1\right) \times \frac{\left(p^{2h^\vee + 2}w_2/w_1; p^{2h^\vee}\right)_\infty \left(-p^{3h^\vee}w_2/w_1; p^{2h^\vee}\right)_\infty}{\left(-p^{h^\vee + 2}w_2/w_1; p^{2h^\vee}\right)_\infty \left(p^{2h^\vee}w_2/w_1; p^{2h^\vee}\right)_\infty}.$$  

Define

$$g_j(t) := \langle m - 1 | z^t v_j \wedge z^{-t} v_{-j} \wedge |m + 1 \rangle \quad (j \in J; t \in \mathbb{Z}).$$

Note that $g_0(0) = 1$, $g_{-j}(0) = 0$ ($j \in [1, n]$) and $g_j(t) = 0$ ($j \in J; t \in \mathbb{Z}_{<0}$).

**Proposition 6.2.1.** $g_0(t)$ satisfies the following recurrence relation

$$g_0(t) - (p^2 - p^{h^\vee})g_0(t - 1) - p^{h^\vee + 2}g_0(t - 2) = \delta_{t,0} - (1 + p^{h^\vee + 1})\delta_{t,1} + p^{h^\vee + 1}\delta_{t,2}. \quad (6.2.1)$$

**Proof.** The proof for $n > 1$ goes as follows (the exceptional case $n = 1$ is similar). First note that any element in $N$, that is generated by $C_{j,k}$ ($k \in [1, n]$) gives rise to a linear relation of some $g_k(t)$ ($k \in J, t \in \mathbb{Z}$). For example $(z \otimes 1 + 1 \otimes z) \cdot C_{0,0}$ gives

$$g_0(1) + q^2g_0(0) + q^2[2](1 - q^4) \sum_{k=1}^{n} (-q^2)^{n-k}g_k(0) + g_0(0) = 0.$$  

From $C_{k,k}$ ($k \in [1, n]$) we get

$$g_k(0) + q(-q^2)^{n-k}g_0(0) = 0.$$  

Combining these two relations, we get

$$g_0(1) + (1 - q^4 + q^{4n+2} + q^{4n+4})g_0(0) = 0,$$

which is (6.2.1) with $t = 1$.

The recurrence relation in the general case ($t \in \mathbb{N}$) comes from

$$\mathcal{A}_t = \left(Z(t, 1) - p^{h^\vee + 1}Z(t - 1, 1)\right) \cdot C_{0,0}$$

$$+ [2](1 - p^2)(-p)^{n+1} \sum_{j=1}^{n} (-p)^{-j}Z(t - 1, 0) \cdot C_{j,k}$$

$$- (-p)^{j}Z(t - 1, 1) \cdot C_{-j,k}.$$
Let $\mathcal{A}_t^\wedge$ denote the image of $\mathcal{A}_t$ in $V_{\text{aff}} \wedge V_{\text{aff}}$. We have:

$$
\langle m - 1 | \mathcal{A}_t^\wedge \wedge | m + 1 \rangle =

\begin{align*}
g_0(t) - (p^{h\gamma+1} - p)g_0(t - 1) - p^{h\gamma+2}g_0(t - 2) \\
+ [2](1 - p^2) \left\{ \sum_{j=1}^{n} (-p)^{n+1-j} \left[ g_j(t - 1) - (-p)^{h\gamma+1}g_j(t - 2) \right] \\
+ \sum_{j=1}^{n} (-p)^{n+1-j} \left[ g_j(t - 1) + p^2 g_{-j}(t - 1) \right] \\
- q \frac{(-p)^{h\gamma} - (-p)}{1 - p^2} g_0(t - 1) \\
- (1 - p^2)p^{n+1} \sum_{j=1}^{n} (-j)[j - 1]_p g_{-j}(t - 1) \\
- \sum_{j=1}^{n} (-p)^{n+1+j} \left[ g_{-j}(t - 1) + p^2 g_{j}(t - 2) \right] \\
- (1 - p^2)p^{h\gamma+1} \sum_{j=1}^{n} (-)^{n-j}[n - j]_p g_{j}(t - 2) \right\} \\
+ \delta(2t > 1)g_0(1 - t) - \delta(2t < 1)g_0(t) \\
- p^{h\gamma+1}\left( \delta(2t > 3)g_0(2 - t) - \delta(2t < 3)g_0(t - 1) \right) = 0.
\end{align*}
$$

All terms in $g_k$ ($k \in J \setminus \{0\}$) cancel and the proposition follows. \hfill \Box

The two point function of Fock intertwiners (4.5.4) is given by

**Corollary 6.2.2.**

$$
\omega_{\nu_0,\nu_0}(w_2/w_1) = \frac{(1 - w_2/w_1)(1 - p^{h\gamma+1}w_2/w_1)}{(1 - p^2 w_2/w_1)(1 + p^{h\gamma}w_2/w_1)}.
$$

**Proof.** Note that $\omega_{\nu_0,\nu_0}(w) = \sum_{t \in \mathbb{N}} w^{t}g_0(t)$. Multiplying both sides of (6.2.1) by $w^{t} (w := w_2/w_1)$ and summing over non-negative $t$ we get

$$
\left( 1 - (p^2 - p^{h\gamma})w - p^{h\gamma+2}w^2 \right) \sum_{t \in \mathbb{N}} w^{t}g_0(t) =

1 - (1 + p^{h\gamma+1})w + p^{h\gamma+1}w^2,
$$

from which the result follows. \hfill \Box

Hence

$$
\omega_{\nu_0,\nu_0}(w_2/w_1) = \frac{(-p^{h\gamma+2}w_2/w_1;p^{2h\gamma})_{\infty}(w_2/w_1;p^{2h\gamma})_{\infty}}{(p^{2}w_2/w_1;p^{2h\gamma})_{\infty}(-p^{h\gamma}w_2/w_1;p^{2h\gamma})_{\infty}},
$$

in agreement with (6.1.1).
6.3. Type $B_n^{(1)}$. Recall that we have two ground state sequences ($\kappa = 0, 1$)

$$b_m^\kappa = \begin{cases} 
  b_\kappa & \text{for } m \in 2\mathbb{Z}, \\
  z^\kappa b_{-\kappa} & \text{for } m \in 2\mathbb{Z} + 1,
\end{cases}$$

and

$$\lambda_m = \begin{cases} 
  (1 - \kappa)\Lambda_n + \kappa(\Lambda_1 - \frac{m}{2}\delta) & \text{for } m \in 2\mathbb{Z}, \\
  (1 - \kappa)\Lambda_n + \kappa(\Lambda_0 - \frac{m-1}{2}\delta) & \text{for } m \in 2\mathbb{Z} + 1.
\end{cases}$$

From [DO], (up to a factor of a constant power in $w_2/w_1$) we have in our notation (4.5.5)

$$\phi_{\nu_m^{-1},v_m^n}(w_2/w_1) = (1 + p^{h^\vee + 1}w_2/w_1)^{\delta_{s,0}} (p^{2h^\vee + 2}w_2/w_1; p^{2h^\vee})_\infty (p^{3h^\vee}w_2/w_1; p^{2h^\vee})_\infty,$$

$$\times (p^{2h^\vee + 2}w_2/w_1; p^{2h^\vee})_\infty (p^{2h^\vee}w_2/w_1; p^{2h^\vee})_\infty.$$

By a diagram automorphism, it is sufficient just to consider the case when $m$ is even. Let $m \in 2\mathbb{Z}$. Define

$$g_j(t) := \langle m - 1|z^{t+\kappa}v_j \wedge z^{-t}v_{-j} \wedge |m + 1 \rangle \quad (j \in J; t \in \mathbb{Z}).$$

Note that $g_{-\kappa}(0) = 1$, $g_j(-\kappa) = 0$ ($j \in [1, n]$) and $g_j(t) = 0$ ($j \in J; t \in \mathbb{Z}_{<0}$).

**Proposition 6.3.1.** $g_{-\kappa}(t)$ satisfies the following recurrence relation

$$g_{-\kappa}(t) = (p^2 - p^{h^\vee})g_{-\kappa}(t - 1) - p^{h^\vee}g_{-\kappa}(t - 2) = \delta_{t,0} - (1 - \delta_{s,0}p^{h^\vee + 1})\delta_{t,1} - \delta_{s,0}p^{h^\vee + 1}\delta_{t,2}. \quad (6.3.1)$$

**Proof.** The proof is like the proof of Proposition 6.2.1 for type $A_{2n}^{(1)}$, but using

$$A_t := \left( Z(t, 1 + \kappa) + p^{1 + h^\vee \delta_{s,0}} Z(t + \kappa - 1, 1 + \kappa) \right) C_{0,0},$$

$$+ [2]Z(t - 1, \kappa) \left( (-p)^n C_{1,-1} + (1 - p^2) \sum_{j=2}^{n} (-p)^{n+1-j}C_{j,-j} \right),$$

$$+ [2]p^{-h^\vee \delta_{s,1}} \left( (1 - p^2) Z(t + \kappa - 1, 1 + \kappa) \sum_{j=2}^{n} (-p)^{n+j-1}C_{j,j} \right.$$ 

$$\left. + (-p)^n Z(t + \kappa, 2 + \kappa) C_{-1,1} \right).$$

The two point function of Fock intertwiners (4.5.4) is given by

**Corollary 6.3.2.** Let $m \in 2\mathbb{Z}$.

$$\omega_{\nu_{m^{-1}},v_m}(w_2/w_1) = \frac{(1 - w_2/w_1)(1 + p^{h^\vee + 1}w_2/w_1)^{\delta_{s,0}}}{(1 - p^2w_2/w_1)(1 - p^{h^\vee}w_2/w_1)}.$$
Proof. Note that \( \omega_{\varphi_m^{-1},\varphi_m}(w) = \sum_{t \in \mathbb{N}} w^t g_{-\kappa}(t) \). Multiplying both sides of (6.3.1) by \( w^t \) \((w := w_2/w_1)\) and summing over non-negative \( t \) we get

\[
(1 - (p^2 - p^{h^\vee})w - p^{h^\vee}w^2) \sum_{t \in \mathbb{N}} w^t g_{-\kappa}(t) = 1 - (1 - \delta_{\kappa,0}p^{h^\vee+1})w - \delta_{\kappa,0}p^{h^\vee+1}w^2,
\]

from which the result follows.

Hence

\[
\frac{\omega_{\varphi_m^{-1},\varphi_m}(w_2/w_1)}{\phi_{\varphi_m^{-1},\varphi_m}(w_2/w_1)} = \frac{(p^{h^\vee+2}w_2/w_1;p^{2h^\vee})_\infty(w_2/w_1;p^{2h^\vee})_\infty}{(p^{2h^\vee}w_2/w_1;p^{2h^\vee})_\infty(p^{h^\vee}w_2/w_1;p^{2h^\vee})_\infty},
\]

in agreement with (6.1.1).

### 6.4. Type \( A_{2n-1}^{(2)} \)

Recall that we have

\[
b_m^\delta = \begin{cases} 
 b_1 & \text{for } m \in 2\mathbb{Z}, \\
 zb_{-1} & \text{for } m \in 2\mathbb{Z} + 1,
\end{cases}
\]

and \( \lambda_m = \begin{cases} 
 \Lambda_1 - \frac{m}{2} \delta & \text{for } m \in 2\mathbb{Z}, \\
 \Lambda_0 - \frac{m-1}{2} \delta & \text{for } m \in 2\mathbb{Z} + 1.
\end{cases} \)

From [DO], (up to a factor of a constant power in \( w_2/w_1 \)) we have in our notation (4.5.5)

\[
\phi_{\varphi_m^{-1},\varphi_m}(w_2/w_1) = \frac{(q^{2h^\vee+2}w_2/w_1;q^{2h^\vee})_\infty(-q^{3h^\vee}w_2/w_1;q^{2h^\vee})_\infty}{(-q^{h^\vee+2}w_2/w_1;q^{2h^\vee})_\infty(q^{2h^\vee}w_2/w_1;q^{2h^\vee})_\infty},
\]

By a diagram automorphism it is sufficient just to consider the case when \( m \) is even. Let \( m \in 2\mathbb{Z} \). Define

\[
g_j(t) := (m-1|z^{t+1}v_j \wedge z^{-t}v_{-j} \wedge |m+1) \quad (j \in J; t \in \mathbb{Z}).
\]

Note that \( g_{-1}(0) = 1 \) and \( g_j(t) = 0 \) \((j \in J; t \in \mathbb{Z}_{<0})\).

**Proposition 6.4.1.** \( g_{-1}(t) \) satisfies the following recurrence relation

\[
g_{-1}(t) - (q^2 - q^{h^\vee})g_{-1}(t-1) - q^{h^\vee+2}g_{-1}(t-2) = \delta_{t,0} - \delta_{t,1}. \quad (6.4.1)
\]

**Proof.** The proof is like the proof of Proposition 6.2.1, but using

\[
\mathcal{A}_t := Z(t,1)(z \otimes z)C_{-1,1} + Z(t,2)(1 - q^2)\sum_{j=2}^{n}(-q)^{j-1}C_{-j,j} - Z(t-1,1)q^{h^\vee}(C_{1,-1} + (1 - q^2)\sum_{j=2}^{n}(-q)^{1-j}C_{j,-j}).
\]

The two point function of Fock intertwiners (4.5.4) is given by
Corollary 6.4.2. Let \( m \in 2\mathbb{Z} \).

\[
\omega_{v_m^\circ, v_n^\circ}(w_2/w_1) = (1 - w_2/w_1)/(1 - q^2w_2/w_1)(1 + q^{h^\vee}w_2/w_1).
\]

Proof. Note that \( \omega_{v_m^\circ, v_n^\circ}(w) = \sum_{t \in \mathbb{N}} w^t g_{-1}(t) \). Multiplying both sides of (6.4.1) by \( w^t \) \( (w := w_2/w_1) \) and summing over non-negative \( t \) we get

\[
\left(1 - (q^2 - q^{h^\vee})w - q^{h^\vee+2}w^2\right) \sum_{t \in \mathbb{N}} w^t g_{-1}(t) = 1 - w,
\]

from which the result follows. \( \square \)

Hence

\[
\frac{\omega_{v_m^\circ, v_n^\circ}(w_2/w_1)}{\phi_{v_m^\circ, v_n^\circ}(w_2/w_1)} = \frac{(-q^{h^\vee+2}w_2/w_1; q^{2h^\vee})_\infty(w_2/w_1; q^{2h^\vee})_\infty}{(q^{h^\vee+2}w_2/w_1; q^{2h^\vee})_\infty(-q^{h^\vee}w_2/w_1; q^{2h^\vee})_\infty},
\]

in agreement with (6.1.1).

6.5. Type \( D^{(1)}_n \). Recall that we have two ground state sequences \( (\kappa = 0, 1) \)

\[
b_m^\circ = \begin{cases} b_{n\delta_{k,0} + 1\delta_{k,1}} & \text{for } m \in 2\mathbb{Z}, \\ z^\kappa b_{n\delta_{k,0} - 1\delta_{k,1}} & \text{for } m \in 2\mathbb{Z} + 1, \end{cases}
\]

From [DO], (up to a factor of a constant power in \( w_2/w_1 \)) we have in our notation (4.5.5)

\[
\phi_{v_m^\circ, v_m^\circ}(w_2/w_1) = \frac{(q^{2h^\vee+2}w_2/w_1; q^{2h^\vee})_\infty(q^{3h^\vee}w_2/w_1; q^{2h^\vee})_\infty}{(q^{h^\vee+2}w_2/w_1; q^{2h^\vee})_\infty(q^{2h^\vee}w_2/w_1; q^{2h^\vee})_\infty}.
\]

By diagram automorphisms, it is sufficient just to consider the case when \( \kappa = 0 \) and \( m \) is odd. Let \( \kappa = 0 \) and \( m \in 2\mathbb{Z} + 1 \). Define

\[
g_j(t) := (m - 1)z^t v_j \wedge z^{-t} v_{-j} \wedge |m + 1| \quad (j \in J; t \in \mathbb{Z}).
\]

Note that \( g_n(0) = 1, g_{-j}(0) = 0 \) \( (j \in [1, n]) \) and \( g_j(t) = 0 \) \( (j \in J; t \in \mathbb{Z}_{<0}) \).

Proposition 6.5.1. \( g_n(t) \) satisfies the following recurrence relation

\[
g_n(t) - (q^2 - q^{h^\vee})g_n(t-1) - q^{h^\vee+2}g_n(t-2) = \delta_{t,0} - \delta_{t,1}. \quad (6.5.1)
\]

Proof. The proof is like the proof of Proposition 6.2.1, but using

\[
\mathcal{A}_t := Z(t, 1)C_{n,-n} - q^{h^\vee}Z(t-1, 1)C_{-n,n} + Z(t-1, 0)((-q)^{n-1}C_{1,-1} + (1 - q^2) \sum_{j=2}^{n-1} (-q)^{n-j}C_{j,-j}) - (-q)^{n-1}Z(t-1, 0)(z \otimes z)C_{-1,1} + (1 - q^2)Z(t-1, 1) \sum_{j=2}^{n-1} (-q)^{j-1}C_{-j,j}. \]

\( \square \)
The two point function of Fock intertwiners (4.5.4) is given by

Corollary 6.5.2. Let \( m \in 2\mathbb{Z} + 1 \).

\[
\omega_{\phi_{m-1}^w,m_0}^w(w_2/w_1) = \frac{(1 - w_2/w_1)}{(1 - q^2w_2/w_1)(1 - q^{h^V}w_2/w_1)}.
\]

Proof. Note that \( \omega_{\phi_{m-1}^w,m_0}^w(w) = \sum_{t \in \mathbb{N}} w^t g_n(t) \). Multiplying both sides of (6.5.1) by \( w^t \) (\( w := w_2/w_1 \)) and summing over non-negative \( t \) we get

\[
(1 - (q^2 - q^{h^V})w - q^{h^V+2}w^2) \sum_{t \in \mathbb{N}} w^t g_n(t) = 1 - w,
\]

from which the result follows. \( \square \)

Hence

\[
\frac{\omega_{\phi_{m-1}^w,m_0}^w(w_2/w_1)}{\phi_{\phi_{m-1}^w,m_0}^w(w_2/w_1)} = \frac{(q^{h^V+2}w_2/w_1; q^{2h^V})_\infty (w_2/w_1; q^{2h^V})_\infty}{(q^{2w_2/w_1}; q^{2h^V})_\infty (q^{h^V}w_2/w_1; q^{2h^V})_\infty},
\]

in agreement with (6.1.1).

6.6. Type \( D_{n+1}^{(2)} \). This type is somewhat special because of the fact that \( V_{aff} \) is not irreducible. The dual Coxeter number is \( h^V = 2n \). We define \( p = q^2 \) and \( \xi^2 = p^{h^V} \).

Recall that we have two ground state sequences (\( \kappa = 0, \phi \) )

\[
b_0^\phi = z^{-m}b_\kappa.
\]

By a diagram automorphism, it is sufficient just to consider one of the two cases \( \kappa \in \{0, \phi\} \). We choose \( \kappa = 0 \).

The boson commutator \( \gamma_m = [B_m, B_{-m}] \) is given as follows:

Proposition 6.6.1.

\[
\gamma_m = \begin{cases}
\frac{(1+\xi^m)}{(1-2p^m-\xi^m)} & \text{for } m \in 2\mathbb{Z}, \\
m & \text{for } m \in 2\mathbb{Z} + 1.
\end{cases}
\]

(6.6.1)

This corresponds to the following boson two point function

\[
\theta(w) = (1 - w) \frac{(\xi^4w^2; \xi^4)_\infty (p^2\xi^2w^2; \xi^4)_\infty}{(\xi^2w^2; \xi^4)_\infty (p^2w^2; \xi^4)_\infty}.
\]

(6.6.2)

From Appendix C we have

Lemma 6.6.2. Let \( w = w_2/w_1 \).

\[
\phi_{z_1^{-m_0},z^{-m_0}}(w_2/w_1) = (1 + p\xi^2w^2) \frac{(\xi^6w^2; \xi^4)_\infty (p^2\xi^4w^2; \xi^4)_\infty}{(\xi^4w^2; \xi^4)_\infty (p^2\xi^2w^2; \xi^4)_\infty}.
\]

It is sufficient just to consider the case \( m = 0 \). Define

\[
g_j(t) := \langle -1 \rangle z^{j+1}v_j \wedge z^{-t}v_{-j} \wedge \langle 1 \rangle \quad (j \in J; t \in \mathbb{Z}).
\]

Note that \( g_0(0) = 1, g_{-j}(0) = 0 \ (j \in [1, n]) \) and \( g_j(t) = 0 \ (j \in J; t \in \mathbb{Z}_{\leq 0}) \).
Proposition 6.6.3. \( g_0(t) \) satisfies the following recurrence relation
\[
g_0(t) - (p^2 + \xi^2)g_0(t - 2) + p^2 \xi^2 g_0(t - 4) = \\
\delta_{t,0} - \delta_{t,1} + p\xi^2 \delta_{t,2} - p\xi^2 \delta_{t,3}. 
\] (6.6.3)

Proof. The proof is like the proof of Proposition 6.2.1, but using
\[
A_t := (Z(t,1)(z \otimes z) + p^{h+1}Z(t - 1,3))C_{0,0} \\
+ Z(t - 1,1)[2](-q(-p)^n(z \otimes z)C_{q,q} \\
+ (1 - p^2) \sum_{j=1}^{n}(-p)^{n+1-j}C_{j,-j}) \\
- Z(t - 1,3)[2](1 - p^2) \sum_{j=1}^{n}(-p)^{n+j}C_{-j,j}.
\]

The two point function of Fock intertwiners (4.5.4) is given by

Corollary 6.6.4.
\[
\omega_{\varphi_0,\varphi_0}(w_2/w_1) = (1 - w_2/w_1)(1 + p\xi^2(w_2/w_1)^2) \\
(1 - p^2(w_2/w_1)^2)(1 - \xi^2(w_2/w_1)^2).
\]

Proof. Note that \( \omega_{\varphi_0,\varphi_0}(w) = \sum_{t \in \mathbb{N}} w^t g_0(t) \). Multiplying both sides of (6.6.3) by \( w^t \) (\( w := w_2/w_1 \)) and summing over non-negative \( t \) we get
\[
(1 - (p^2 + \xi^2)w^2 - p^2 \xi^2 w^4) \sum_{t \in \mathbb{N}} w^t g_0(t) = 1 - w + p\xi^2 w^2 - p\xi^2 w^3,
\]
from which the result follows.

Finally we have \( \omega_{\varphi_0,\varphi_0}(w) = (6.6.2) \), which proves Proposition 6.6.1.

Remark. It is possible to work in an irreducible component of \( V_{\text{aff}} \), say \( V_{\text{aff}}^{\text{even}} = V_{\text{aff}} \otimes \mathbb{C}[z^2, z^{-2}] + v_0 \otimes \mathbb{C}[z^2, z^{-2}] \). On \( V_{\text{aff}}^{\text{even}} \otimes V_{\text{aff}}^{\text{even}} \) the energy function takes only even values. The condition \( H(b_m^o \otimes b_{m+1}^o) = 1 \) for a ground state sequence \( \{b_m^o\}_{m \in \mathbb{Z}} \) should then be replaced by \( H(b_m^o \otimes b_{m+1}^o) = 2 \) for all \( m \in \mathbb{Z} \).

The ground state sequence in \( B_{\text{aff}}^{\text{even}} \) is given by \( b_m^o = b_0 \) for all \( m \in \mathbb{Z} \). The Fock two-point function can be shown to be given by
\[
\omega_{\varphi_0,\varphi_0}(w) = (1 - w^2)(1 + p\xi^2 w^2) \\
(1 - \xi^2 w^2)(1 - p^2 w^2),
\] (6.6.4)
where \( w = w_1/w_2 \). Comparing with Lemma 6.6.2, we find that \( \gamma_m \) is now given by the same formula as in Theorem 6.1.2
\[
\gamma_m = \frac{m + \xi^2 m}{1 - p^{2m}}.
\] (6.6.5)
Higher level examples: level $k A^{(1)}_1$

7.1. Cartan datum. $I = \{0, 1\}$. The Dynkin diagram for $A^{(1)}_1$ is

\[ 0 = \cdashline{1-1}^1 = \]

We have

\[ \delta = \alpha_0 + \alpha_1, \]
\[ c = h_0 + h_1, \]
\[ (\alpha_i, \alpha_i) = 2 \quad (i \in I). \]

7.2. Perfect crystal. Fix $k \in \mathbb{Z}_{>0}$. Let $J := [0, k]$. Let $V$ be the $(k + 1)$-dimensional $U'_q(A^{(1)}_1)$-module with the level $k$ perfect crystal $B := \{b_j\}_{j \in J}$ with crystal graph:

\[ b_0 \rightleftharpoons b_1 \rightleftharpoons b_2 \rightleftharpoons \cdots \rightleftharpoons 0 \rightleftharpoons b_k. \]

The elements of $B$ have the following weights

\[ \text{wt}(b_j) = (k/2 - j)\alpha_1 \]
\[ = (k - 2j)(\Lambda_1 - \Lambda_0) \quad (j \in J). \]

Let $v_j := G_b(j) \ (j \in J)$. The action of $U'_q(A^{(1)}_1)$ on $v_j \in V$ obeys (5.1.1).

7.3. Energy function. The energy function $H$ has the following values on $B \otimes B$

\[ H(b_i \otimes b_j) = \min(i, k - j) \quad (i, j \in J). \]

Write $H(i, j)$ for $H(b_i \otimes b_j) \ (i, j \in J)$.

The Coxeter number of $A^{(1)}_1$ is $h = 2$. We take

\[ l(z^m b_j) = 2m - j \quad (m \in \mathbb{Z}, j \in J). \]

$H$ and $l$ satisfy condition (L). Note that $l(z^m b_j) = l(z^{m+1} b_{j+2}) \ (m \in \mathbb{Z} \text{ and } j \in [0, k - 2])$, so $l$ gives a partial ordering of $B_{aff}$ for $k > 1$.

7.4. $q$-binomials. Define the $q$-binomial coefficient $\binom{m}{n} (m, n \in \mathbb{Z})$ by

\[ \binom{m}{n} = \begin{cases} 
\prod_{i=0}^{\lfloor (m-n)/m-1 \rfloor} (q^{n-2i} - \eta) : m \geq n \geq 0 \\
0 : \text{otherwise.} 
\end{cases} \]

We will often write sums involving $q$-binomial coefficients as sums over all integers. The advantage is that we can then freely change variables without worrying about the range of summation. The following result is widely used in the sequel:

**Lemma 7.4.1.**

(i) For any $\eta \in \mathbb{C}(q)$ and $n \in \mathbb{Z}_{>0}$, we have

\[ \sum_{j \in \mathbb{Z}} (-\eta)^j \binom{n}{j} \left( \frac{[m][m-1] \cdots [m-n+1]}{[n][n-1] \cdots [1]} \right) \left( q^n - 1 \right) \eta = \prod_{i=0}^{n-1} (q^{n-2i} - \eta). \]

(ii) The sum in (i) vanishes if $\eta = q^m$ with $m$ an integer lying in the range $[-n + 1, n - 1]$. Here $[a, a + 2b]_2$ means $\{a + 2i; 0 \leq i \leq b\}$. 

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7.5. Wedge relations. Define the vectors $C_{i,j} \in N \; (i, j \in J)$ by
\[
C_{i,j} = \begin{cases} 
(z \otimes z)^{-i} e_0^{(i)} f_1^{(j)} (v_0 \otimes v_0) & : i + j \leq k, \\
(1-k+i) e_0^{(k-j)} f_0^{(j)} (v_k \otimes v_k) & : i + j > k.
\end{cases}
\]
Explicitly, we have
\[
C_{i,j} = \begin{cases} 
\sum_{i', a, b} q^{(k-j')(i' - b) - (k-i') a} \left[ \begin{array}{c} j' \\ a \\ b \end{array} \right] z^{-a} v_{i'} \otimes z^{-b} v_{j'} & : i' + j' \leq k, \\
\sum_{i', a, b} q^{(k-j' - b) + j' a} \left[ \begin{array}{c} k-j' \\ a \\ b \end{array} \right] z^{-a} v_{i'} \otimes z^{-b} v_{j'} & : i' + j' > k.
\end{cases}
\] (7.5.1)
The summation in both cases is over $i', j' \in J$ and $a, b \geq 0$ with
\[
i' + j' = i + j, \\
a + b = H(i, j).
\]

**Proposition 7.5.1.** Identify $C_{i,j}$ with $C_{b_i, z^{-H(i,j)} b_j}$. Then $\{z^m \otimes z^m : C_{i,j}\}_{m \in \mathbb{Z}, i,j \in J}$ with the function $l$ satisfy condition (R) of subsection 3.3.

7.6. Fock space. We have
\[
\mathcal{F}_m = B, \\
(P_{cl})_k = \{j \Lambda_0^d + (k-j) \Lambda_1^d \}_{j \in J},
\]
with the bijections
\[
\varepsilon(b_j) = (k-j) \Lambda_0^d + j \Lambda_1^d, \\
\varphi(b_j) = j \Lambda_0^d + (k-j) \Lambda_1^d.
\]
Fix $\kappa \in J$ and let $\kappa' = k - \kappa$. We have $H(z^{-\ell(k-2)} b_\kappa \otimes z^{-\ell(k-2) - \kappa + 1} b_{\kappa'}) = H(b_\kappa \otimes b_{\kappa'}) - 1 = 1$. $H(z^{-\ell(k-2) - \kappa + 1} b_{\kappa'} \otimes z^{-(\ell+1)(k-2)} b_\kappa) = 1$ and so the following is a ground state sequence ($\ell \in \mathbb{Z}$)
\[
b_{2\ell-1} = z^{-\ell(k-2)} b_\kappa, \\
b_{2\ell} = z^{-\ell(k-2) - \kappa + 1} b_{\kappa'}.
\] (7.6.1)
with
\[
\text{cl}(\lambda_m) = \begin{cases} 
\kappa \Lambda_0 + \kappa' \Lambda_1 & : \text{if } m \text{ is odd}, \\
\kappa' \Lambda_0 + \kappa \Lambda_1 & : \text{if } m \text{ is even}.
\end{cases}
\]
With $v_m^o = G(b_m^o)$, the vacuum vector of $\mathcal{F}_m$ is
\[
|m\rangle = v_m^o \wedge v_{m+1}^o \wedge v_{m+2}^o \wedge \cdots
\]
with weight $\lambda_m$. 

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7.7. Two point functions. A priori \( \gamma_n = [B_n, B_{-n}] \) may depend on the choice of \( \kappa \). However, we find that it is independent of \( \kappa \).

**Theorem 7.7.1.**

\[
\gamma_n = n \frac{1 - q^{4n}}{1 - q^{2n} - q^{4n} + q^{2(k+1)n}}.
\]

The theorem follows by applying Proposition 4.5.2 to Proposition 7.7.2 and Corollary 7.7.4 below.

From [IIJMNT] we have

**Proposition 7.7.2.**

\[
\phi_{v_{2r-1};v_{2r}}(w) = \frac{(q^{2(2r+2)}w; q^4)_\infty}{(q^{4}w; q^4)_\infty} \sum_{p=0}^{\infty} (q^{k+2}w)^p \begin{bmatrix} \kappa \\ p \end{bmatrix} \begin{bmatrix} \kappa' \\ p \end{bmatrix},
\]

where \( w = w_2/w_1 \).

Without loss of generality, we can choose \( \ell = 0 \). Define

\[
g_{\kappa}(t) := \langle -1 | z^t v_\kappa \wedge z^{-t+1} v_{\kappa'} \wedge | +1 \rangle \quad (t \in \mathbb{Z}; j \in J).
\]

Note that \( g_{\kappa}(t) = \delta_{t,0} \) for \( t \in \mathbb{Z}_{\leq 0} \) by Theorem 4.2.5.

**Proposition 7.7.3.** \( g_{\kappa}(t) \) satisfies the following recurrence relation

\[
\sum_{\alpha \in \mathbb{Z}} (-q^{k+1})^\alpha \begin{bmatrix} k \\ \alpha \end{bmatrix} g_{\kappa}(t - \alpha) - (q^{k+2})^t \begin{bmatrix} \kappa \\ t \end{bmatrix} \begin{bmatrix} \kappa' \\ t \end{bmatrix} + (q^{k+2})^{t-1} \begin{bmatrix} \kappa \\ t-1 \end{bmatrix} \begin{bmatrix} \kappa' \\ t-1 \end{bmatrix} = 0 \quad (7.7.1)
\]

**Corollary 7.7.4.**

\[
\omega_{v_{2r-1};v_{2r}}(w) = \frac{(1 - w)}{\prod_{j=1}^{\infty} (1 - q^{2j}w)} \sum_{p=0}^{\infty} (q^{k+2}w)^p \begin{bmatrix} \kappa \\ p \end{bmatrix} \begin{bmatrix} \kappa' \\ p \end{bmatrix},
\]

where \( w = w_2/w_1 \).

**Proof.** We have

\[
\omega_{v_{2r-1};v_{2r}}(w) = \sum_{j \in \mathbb{Z}} \left( \frac{w_2}{w_1} \right)^j g_{\kappa}(j). \quad (7.7.2)
\]

Multiply both sides of (7.7.1) by \( w^t \) and sum over all \( t \geq 0 \). After relabelling of \( t \) and using (7.7.2) we obtain

\[
\omega_{v_{2r-1};v_{2r}}(w) \sum_{\alpha \in \mathbb{Z}} (-q^{k+1}w)^\alpha \begin{bmatrix} k \\ \alpha \end{bmatrix} = (1 - w) \sum_{t=0}^{\infty} (q^{k+2}w)^t \begin{bmatrix} \kappa \\ t \end{bmatrix} \begin{bmatrix} \kappa' \\ t \end{bmatrix}. \quad (7.7.3)
\]

From Lemma 7.4.1 (i) we have

\[
\sum_{\alpha \in \mathbb{Z}} (-q^{k+1}w)^\alpha \begin{bmatrix} k \\ \alpha \end{bmatrix} = \prod_{j=1}^{k} (1 - q^{2j}w),
\]

thereby proving the result. \( \square \)
Only Proposition 7.7.3 remains to be proved.

7.8. Proof of recurrence relation. Let $Z(t, d)$ be the operator defined in §6.1:

$$Z(t, d) = z^t \otimes z^{d-t} + \delta(2t > d)z^{d-t} \otimes z^t - \delta(2t < d)z^t \otimes z^{d-t} \quad (t, d \in \mathbb{Z}).$$

For $t \in \mathbb{Z}$, define

$$A_t := \sum_{i \in J, \gamma \in \mathbb{Z}} (-q^{k+1})^{k-i-\kappa} q^{\gamma(k+2)} \binom{i}{\gamma} \binom{k-i}{\kappa - \gamma} Z(t - \gamma, -i + \kappa' + 1)C_{k-i,i}.$$  

We split the proof into three parts. Define

$$Z^{(1)}(t, d) = z^t \otimes z^{d-t},$$
$$Z^{(2)}(t, d) = -z^t \otimes z^{d-t} \delta(2t < d),$$
$$Z^{(3)}(t, d) = z^{d-t} \otimes z^t \delta(2t > d).$$

Then $Z(t, d) = Z^{(1)}(t, d) + Z^{(2)}(t, d) + Z^{(3)}(t, d)$. Let Define $A^{(i)}_t$ ($i \in \{1, 2, 3\}$) by replacing $Z$ by $Z^{(i)}$ in the definition of $A_t$. Then $A_t = A^{(1)}_t + A^{(2)}_t + A^{(3)}_t$. We will deal with each $A^{(i)}_t$ separately. Note that $A^{(i)}_t \notin N$ ($i \in \{1, 2, 3\}$), only $A_t \in N$.

7.8.1. $A^{(1)}_t$. From (7.5.1) we obtain

$$C_{k-i,i} = \sum_{j \in J, \gamma \in \mathbb{Z}} q^{j-\gamma+2(k+1)b+2(k+b)} \left[ \begin{array}{c} j \\ k-i-b \end{array} \right] \left[ \begin{array}{c} k-j \\ b \end{array} \right] z^{b+i-k}v_{k-j} \otimes z^{-b}v_j.$$  

Substituting into (7.8) and performing a change of variable $b \rightarrow \gamma + k - i - \alpha$, followed by $i \rightarrow i + \gamma$ we obtain

$$A^{(1)}_t = \sum_{i, \gamma, k \in \mathbb{Z}} (-q^{k+1})^{k-i-\kappa-\gamma} q^{2\gamma + k\alpha + (i-j+\gamma)(k-j)} \binom{i+\gamma}{\gamma} \binom{k-i-\gamma}{\kappa-\gamma}$$
$$\times \left[ \begin{array}{c} j \\ \alpha-\gamma \end{array} \right] \left[ \begin{array}{c} k-j \\ k-i-\alpha \end{array} \right] z^{t-\alpha}v_{k-j} \otimes z^{-t+\alpha-\kappa+1}v_j. \quad (7.8.2)$$

Let us now argue that only the $j = k - \kappa$ terms contribute in the above sum. Recall that our convention for $q$-binomial coefficients implicitly defines for us the upper and lower limits of summation in formulae like $A^{(1)}_t$. For instance, the constraints on $i$ are

$$\max(0, j - \alpha) \leq i \leq \min(k - \kappa, k - \alpha). \quad (7.8.3)$$

Let us assume first that $j \leq k - \kappa - 1$. The strategy is to recast the sum over $i$ in (7.8.2), more specifically,

$$I_j = \left[ \begin{array}{c} j \\ \alpha-\gamma \end{array} \right] \sum_{i \in \mathbb{Z}} (-q^{k-j-\kappa-1})^{i} \binom{i+\gamma}{\gamma} \binom{k-i-\gamma}{\kappa-\gamma} \binom{k-j}{k-i-\alpha}. \quad (7.8.4)$$  

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into a form such that Lemma 7.4.1 applies. Consider the case \( j \leq \alpha \leq \kappa \), so that according to (7.8.3) we have \( 0 \leq i \leq \kappa - \alpha \). By manipulating the \( q \)-binomial coefficients we obtain

\[
I_j = \frac{[k-j][j]!}{[k-\kappa]!\left([\kappa-1]-\gamma\right)!} \sum_{i=0}^{k-\kappa} (-q^{k-j-\kappa-1})^i \left[\begin{array}{c} k-\kappa \\ i \end{array}\right] \left[\begin{array}{c} i+\gamma \\ j-\alpha+\gamma \end{array}\right] \left[\begin{array}{c} k-i-\gamma \\ \alpha-\gamma \end{array}\right].
\]

(7.8.5)

Now treat the product of the last two \( q \)-binomial coefficients in \( I_j \) together with \((-q^{k-j-\kappa-1})^i\) as a polynomial in \( q^i \); the powers of \( q^i \) which appear can be seen to lie in the range \([k-\kappa-1-2j, k-\kappa-1]_2\). In fact, due to the assumption on \( j \) the range is \([1-k+\kappa, k-\kappa-1]_2\). Therefore \( I_j \) is a finite sum of sums for which Lemma 7.4.1 (ii) applies and thus vanishes.

For the other three remaining cases (a) \( j, \kappa \leq \alpha \), (b) \( j, \kappa \geq \alpha \) and (c) \( j \geq \alpha \geq \kappa \) we use, respectively, the identities for \( I_j \):

\[
I_j = \frac{[k-j][j]!}{[k-\alpha]!\left([\alpha-1]-\gamma\right)!} \sum_{i=0}^{k-\alpha} (-q^{k-j-\alpha-1})^i \left[\begin{array}{c} k-\alpha \\ i \end{array}\right] \left[\begin{array}{c} i+\gamma \\ j-\alpha+\gamma \end{array}\right] \times \left[\begin{array}{c} k-i-\gamma \\ \kappa-\gamma \end{array}\right].
\]

\[
I_j = \frac{[k-j][j]!}{[j-\alpha+\gamma]!\left([\kappa-\gamma]-\alpha\right)!} \sum_{i=j-\alpha}^{k-\kappa} (-q^{k-j-\alpha-1})^i \left[\begin{array}{c} k-\kappa-j+\alpha \\ i \end{array}\right] \left[\begin{array}{c} i+\gamma \\ \gamma \end{array}\right] \left[\begin{array}{c} k-i-\gamma \\ \alpha-\gamma \end{array}\right] \times \left[\begin{array}{c} k-j \\ i-j+\alpha \end{array}\right].
\]

\[
I_j = \sum_{i=j-\alpha}^{k-\alpha} (-q^{k-j-\alpha-1})^i \left[\begin{array}{c} k-j \\ i-j+\alpha \end{array}\right] \left[\begin{array}{c} i+\gamma \\ \gamma \end{array}\right] \left[\begin{array}{c} k-i-\gamma \\ \kappa-\gamma \end{array}\right].
\]

In each case \( I_j \) vanishes by application of Lemma 7.4.1 (ii).

We have proved that the sum over \( j < k-\kappa \) in (7.8.2) vanishes. The sum over \( j > k-\kappa \) vanishes for similar reasons. Keeping only the \( j = k-\kappa \) term we arrive at

\[
A^{(1)}_t = \sum_{\gamma,\alpha \in \mathbb{Z}} (-q)^{k-\kappa-\gamma} q^{2\gamma+k\kappa} I_{k-\kappa} z^{t-\alpha} v_\kappa \otimes z^{-t+\alpha-\kappa+1} v_{\kappa'}
\]

(7.8.6)

where \( I_{k-\kappa} \) is given by (7.8.4). Once again, we have the constraint (7.8.3) and have to treat the four cases separately. We consider in detail only the case \( k-\kappa \leq \alpha \leq \kappa \), using the form (7.8.5) for \( I_{k-\kappa} \). The other three cases are similar. We proceed as before but now find that the powers of \( q^i \) lie in the range \([-1-k+\kappa, -1+k-\kappa]_2\). By Lemma 7.4.1 (ii) only the term whose power of \( q^i \) is \(-1-k+\kappa \) survives. In other words,

\[
I_{k-\kappa} = \left[\begin{array}{c} \kappa \\ \gamma \end{array}\right] \sum_{i=0}^{k-\kappa} (-q^{-1})^i \left[\begin{array}{c} k-\kappa \\ i \end{array}\right] \frac{q^{-(i+\gamma)} q^{-(i+\gamma-1)} \cdots q^{k-i-\gamma} q^{k-i-\gamma-1} \cdots (-1)^{k-\kappa-\alpha+\gamma}}{[k-\kappa-\alpha+\gamma]! \left[\begin{array}{c} \alpha-\gamma \\
q^{-1} \right]!} \left(\frac{q-q^{-1}}{q-q^{-1}}\right)^{k-\kappa}.
\]

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Applying Lemma 7.4.1 (i) and simplifying we find
\[ I_{k-\kappa} = \left[ \frac{\kappa}{\gamma} \right] \left[ \frac{k - \kappa}{\alpha - \gamma} \right] (-q^{\kappa+1})^{\alpha} (-q^{k+1})^{-\gamma} (-q)^{\kappa-k}. \]

Substituting into (7.8.6) we obtain
\[ \mathcal{A}_t^{(1)} = \sum_{\gamma, \alpha \in \mathbb{Z}} (-q^{k+1})^\alpha (-q^{\kappa+1})^{-\gamma} \left[ \frac{\kappa}{\gamma} \right] \left[ \frac{k - \kappa}{\alpha - \gamma} \right] z^{t-\alpha} v_\kappa \otimes z^{-t+\alpha-\kappa+1} v_\kappa'. \] (7.8.7)

We now note the identity
\[ \left( \sum_{\beta \in \mathbb{Z}} (-q^{k+1})^\beta \left[ \frac{k - \kappa}{\beta} \right] \right) \left( \sum_{\gamma \in \mathbb{Z}} (-q^{\kappa+1})^\gamma \left[ \frac{\kappa}{\gamma} \right] \right) = \prod_{i=1}^{k} (1 - q^{2i} x) = \sum_{\alpha \in \mathbb{Z}} (-q^{k+1})^\alpha \left[ \frac{k}{\alpha} \right], \]
which follows from the ubiquitous Lemma 7.4.1, to perform the \( \gamma \)-sum in (7.8.7) with the result
\[ \mathcal{A}_t^{(1)} = \sum_{\alpha \in \mathbb{Z}} (-q^{k+1})^\alpha \left[ \frac{k}{\alpha} \right] z^{t-\alpha} v_\kappa \otimes z^{-t+\alpha+1-\kappa} v_\kappa'. \]

**7.8.2. \( \mathcal{A}_t^{(2)} \).** The only difference between \( \mathcal{A}_t^{(1)} \) and \( \mathcal{A}_t^{(2)} \) is that the latter has a negative sign and an additional constraint
\[ i < \gamma - 2t + k - \kappa + 1 \] (7.8.8)
on the sum (denoted by prime) due to the definition of \( Z^{(2)} \):
\[ \mathcal{A}_t^{(2)} = - \sum_{i, \gamma, \alpha \in \mathbb{Z}} (-q^{\kappa+1})^{k-i-\kappa-\gamma} q^{2\gamma+(i-j+\gamma)(k-j)} \left[ \frac{k - i - \gamma}{\gamma} \right] \left[ \frac{\gamma}{\kappa - \gamma} \right] \]
\[ \times \left[ \frac{j}{\alpha - \gamma} \right] \left[ \frac{k - j}{k - i - \alpha} \right] z^{t-\alpha} v_{k-j} \otimes z^{-t+\alpha-\kappa+1} v_j. \] (7.8.9)

Furthermore we are now interested in dropping terms that annihilate the vacuum. Using Theorem 3.5 this means that we require
\[ H(z^{-t+\alpha-m(k-2)} b_j \otimes z^{-(m+1)(k-2)} b_\kappa) = t - \alpha + k - \kappa + 1 + \min(j, k - \kappa) > 0. \] (7.8.10)

Let us assume first that \( j > k - \kappa \). From (7.8.10) we need \( \alpha - t < 0 \). Now from the last \( q \)-binomial in (7.8.9) and (7.8.8) we have
\[ j \leq i + \alpha < (\alpha - t) + (\gamma - t) + k - \kappa + 1. \] (7.8.11)
Thus we have \( k - \kappa < j < \gamma - t + k - \kappa + 1 \) and so \( \gamma - t > 0 \). But this means \( \gamma > t \geq \alpha \) which contradicts the requirement \( \gamma \leq \alpha \) coming from the third \( q \)-binomial in (7.8.9).
Next assume that $j < k - \kappa$. From (7.8.10) we now need $j \geq k - \kappa + \alpha - t$ and thus $\alpha - t < 0$. But again we have (7.8.11), and so

$$j \leq (\alpha - t) + (\gamma - t) + k - \kappa < (\gamma - t) + k - \kappa.$$  

Thus we have $k - \kappa + \alpha - t \leq j < \gamma - t + k - \kappa$ and so $\alpha < \gamma$ which again contradicts $\gamma \leq \alpha$.

Hence we must have

$$j = k - \kappa.$$  

(7.8.12)

According to (7.8.10) we need $\alpha - t < 0$. But again (7.8.11) is required, which leads to $0 \leq (\alpha - t) + (\gamma - t) \leq \gamma - t$. Therefore $\alpha \leq t \leq \gamma$ which together with $\gamma \leq \alpha$ from the third $q$-binomial in (7.8.9) makes mandatory

$$\alpha = \gamma = t.$$  

(7.8.13)

This means that (7.8.8) can be rephrased as $i \leq -t + k - \kappa$. But from the last $q$-binomial in (7.8.9), together with (7.8.12) and (7.8.13) we must have also

$$i = -t + k - \kappa.$$  

(7.8.14)

Substituting (7.8.12), (7.8.13) and (7.8.14) into (7.8.9) we arrive at

$$\mathcal{A}_t^{(2)} = - (q^{k+2})^t \left[ \begin{array}{c} \kappa \\ t \end{array} \right] \left[ \begin{array}{c} \kappa' \\ t \end{array} \right] v_\kappa \otimes z^{-\kappa+1} v_{k-\kappa} + \cdots.$$ 

7.8.3. $\mathcal{A}_t^{(3)}$. One argues in the same way that

$$(q^{k+2})^{t-1} \left[ \begin{array}{c} \kappa \\ t-1 \end{array} \right] \left[ \begin{array}{c} \kappa' \\ t-1 \end{array} \right] v_\kappa \otimes z^{-\kappa+1} v_{k-\kappa} + \cdots.$$ 

Let $\mathcal{A}_t^\wedge$ denote the image of $\mathcal{A}_t$ in $V_{\text{aff}}^\wedge$. Adding the three parts together, the relation $\langle -1 \mid \mathcal{A}_t^\wedge \wedge \mid 1 \rangle = 0$ gives us Proposition 7.7.3.

APPENDIX A. PERFECT CRYSTAL

Let $V$ be an integrable finite-dimensional $U'_q(\mathfrak{g})$-module with a perfect crystal base $(L, B)$ of level $l$. We assume that it has a lower global base (i.e. satisfies (G)). In [KMN1], we proved that the “semi-infinite tensor product” $B \otimes B \otimes \cdots$ is isomorphic to the crystal base $B(\lambda)$ of the highest irreducible module, provided that the rank of $\mathfrak{g}$ is greater than 2. In this appendix, we prove the same statement for any rank. In [KMN1], the proof is combinatorial, and here it is by the use of a vertex operator. Let us take a ground state sequence $(\cdots, b_m^\circ, b_{m+1}^\circ, \cdots)$ in $B_{\text{aff}}$. Set $v_k^\circ = G(b_k^\circ)$. For an integral dominant weight $\lambda$, we denote by $V(\lambda)$ the irreducible $U'_q(\mathfrak{g})$-module with highest weight $\lambda$ and highest weight vector $u_\lambda$, and by $(L(\lambda), B(\lambda))$ its crystal base.

Proposition A.1. $B \otimes B(\lambda_m) \cong B(\lambda_{m-1})$.

This proposition implies the following result.
Lemma A.3. \[ B(\lambda_m) \cong \{(b_m, b_{m+1}, \cdots); b_k \in B_{\text{aff}}, H(b_k \otimes b_{k+1}) = 1 \text{ for any } k \geq m \] and \[ b_k = b_k^\circ \text{ for } k \gg m \} \].

The following lemma is proved in [DJO].

Lemma A.4. \[ \text{Hom}_{U_q(\mathfrak{g})} \left( V_{\text{aff}} \otimes V(\lambda_m), V(\lambda_{m-1}) \right) = K. \]

Let \( \Phi : V_{\text{aff}} \otimes V(\lambda_m) \to V(\lambda_{m-1}) \) be a \( U_q(\mathfrak{g}) \)-linear homomorphism. We normalize it by

\[ \Phi(u_{m-1}^\circ \otimes u_{\lambda_m}) = u_{\lambda_{m-1}}. \]

Then the following lemma is also proved in [DJO] in the dual form.

Lemma A.5. \[ \text{Hom}_{A} (L(\lambda_m)) \subset L(\lambda_{m-1}). \]

Let \( \tilde{\Phi} : (L(\lambda_m) \otimes A) / q(L_{\text{aff}} \otimes A) (\lambda_m) \to L(\lambda_{m-1}) / qL(\lambda_{m-1}) \) be the induced homomorphism.

The following two lemmas are easily proved.

Lemma A.6. Let \( \lambda \) be a dominant integral weight. Let \( B \) be a semi-regular crystal (i.e. \( \varepsilon_i(b) = \max\{n \in \mathbb{Z}_{\geq 0}|e_i^n b \neq 0\} \) and \( \varphi_i(b) = \max\{n \in \mathbb{Z}_{\geq 0}|f_i^n b \neq 0\} \)). We assume further that \( B \) is connected.

(i) If \( \psi : B(\lambda) \to B \) is a strict morphism such that \( \psi(B(\lambda)) \subset B \), then \( \psi \) is an isomorphism.

(ii) If \( \psi : B \to B(\lambda) \) is a strict morphism such that \( \psi(B) \subset B(\lambda) \), then \( \psi \) is an isomorphism.

Let \( B' \) be the connected component of \( B_{\text{aff}} \otimes B(\lambda_m) \) containing \( b_{m-1}^\circ \otimes u_{\lambda_m} \). Then \( \Phi \) sends \( B' \) to \( B(\lambda_{m-1}) \). Hence \( B' \) is a subcrystal of \( B_{\text{aff}} \otimes B(\lambda_m) \), and Lemma A.5 implies \( B' \to B_{\text{aff}} \otimes B(\lambda_m) \) and \( B' \to B(\lambda_{m-1}) \) are strict morphisms. Moreover any \( b \in B' \) is not mapped to 0 by the morphism \( B' \to B(\lambda_{m-1}) \). Hence by Lemma A.6, \( B' \) is isomorphic to \( B(\lambda_{m-1}) \). Hence we obtain a strict morphism \( B(\lambda_{m-1}) \to B_{\text{aff}} \otimes B(\lambda_m) \). Composing it with \( B_{\text{aff}} \to B \), we obtain a strict morphism \( B(\lambda_{m-1}) \to B \otimes B(\lambda_m) \).

The following lemma is proved in [KMN1].

Lemma A.7. \( B \otimes B(\lambda_m) \) is connected.

Thus \( B(\lambda_{m-1}) \to B \otimes B(\lambda_m) \) is an isomorphism by Lemma A.6.
Appendix B. Serre relations

Let $U_q(g)$ be the algebra associated to a symmetrizable Kac-Moody algebra with the same generators and the defining relations as the quantized universal enveloping algebra except the Serre relations. Let $U_q(g)_i$ be its subalgebra generated by $e_i$, $f_i$ and $t_i^{-1}$. In this appendix, we prove the following proposition.

**Proposition B.1.** Let $M$ be a $U_q(g)$-module. Assume that $M$ is an integrable $U_q(g)_i$-module for every $i$. Then the action of $U_q(g)$ on $M$ satisfies the Serre relations.

Hence $M$ has the structure of a $U_q(g)$-module.

Let $M$ and $N$ be integrable $U_q(g)_i$-modules. We endow the structure of $U_q(g)_i$-module on $\text{Hom}(M,N)$ such that $\text{Hom}(M,N) \otimes M \to N$ is $U_q(g)_i$-linear. Namely for $x \in U_q(g)_i$ with $\Delta(x) = \sum x(1) \otimes x(2)$, $x$ acts on $\psi \in \text{Hom}(M,N)$ by $x(1)\psi a(x(2))$.

Recall that an element $\psi$ of $\text{Hom}(M,N)$ is called locally $U_q(g)_i$-finite, if it is contained in an integrable $U_q(g)_i$-submodule.

**Lemma B.2.** Let $M$ and $N$ be integrable $U_q(g)_i$-modules. Assume that a weight vector $\psi \in \text{Hom}(M,N)$ satisfies $f_i^{n+1}\psi = 0$ for some $n \geq 0$.

Then $\psi$ is locally $U_q(g)_i$-finite.

**Proof.** Assume $t_i^s \psi = q_i^m \psi$. It is enough to show

$$e_i^s \psi = 0. \quad (B.1)$$

Here $s = \max(n - m + 1, 0)$. In order to see this, we may assume that $M$ is finite-dimensional. Replacing $N$ with the $U_q(g)_i$-module generated by $\psi(M)$, we may assume that $N$ is also finite-dimensional. Hence $\text{Hom}(M,N)$ is finite-dimensional and hence integrable. In this case it is a well-known fact that $f_i^{n+1}\psi = 0$ implies (B.1).

**Proof of Proposition B.1.** Let $\text{ad} : U_q(g) \to \text{End}(U_q(g))$ be a $U_q(g)$-module structure on $U_q(g)$ such that the multiplication $U_q(g) \otimes M \to M$ is $U_q(g)_q$-linear. We have

$$\text{ad}(t_i)(a) = t_i a t_i^{-1} \quad (B.2)$$
$$\text{ad}(e_i)(a) = e_i a - t_i^{-1} a t_i e_i \quad (B.3)$$
$$\text{ad}(f_i)(a) = [f_i, a] t_i^{-1} \quad (B.4)$$

for $a \in U_q(g)$. Let $X : U_q(g) \to \text{End}(M)$ be the homomorphism given by the $U_q(g)$-module structure on $M$. Let $i \neq j$. Since $[f_i, e_j] = 0$, $f_i X(e_j) = 0$. Since $X(e_j)$ has weight $\langle h_i, \alpha_j \rangle$ with respect to $U_q(g)_i$, the preceding lemma implies

$$e_i^b X(e_j) = 0, \quad (B.5)$$

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where $b = 1 - \langle h_i, \alpha_j \rangle$. On the other hand

$$e_i^b X(e_j) = X(\text{ad}(e_i^b) e_j)$$

$$= X \left( \sum_{k=0}^b (-1)^k q_i^{-k(b-k)} \binom{b}{k} e_i^{b-k} t_i^{k} e_j (t_i e_i)^k \right)$$

$$= X \left( \sum_{k=0}^b (-1)^k \binom{b}{k} e_i^{b-k} e_j e_i^k \right).$$

This along with (B.5) gives the Serre relation

$$X \left( \sum_{k=0}^b (-1)^k \binom{b}{k} e_i^{b-k} e_j e_i^k \right) = 0.$$

By applying the automorphism $e_i \mapsto f_i, f_i \mapsto e_i, q^h \mapsto q^{-h} (h \in P^*)$ of $U_q(\mathfrak{g})$, we obtain the other Serre relations

$$X \left( \sum_{k=0}^b (-1)^k \binom{b}{k} f_i^{b-k} f_j f_i^k \right) = 0.$$

**Appendix C. Two-point function for $D^{(2)}_{n+1}$**

In this appendix we will sketch the calculation for level 1 $D^{(2)}_{n+1}$, of the two-point function $\Psi(z_1/z_2) = \langle \Lambda_n | \Phi_{\lambda_n V_2}^V(z_2) \Phi_{\lambda_n V_1}^V(z_1) | \Lambda_n \rangle$, for the intertwiner $\Phi_{\lambda_n V}(z) : V(\lambda) \rightarrow V(\mu) \otimes V_z$, by solving the corresponding $q$-KZ equation it must satisfy. The corresponding calculations for the other cases in this paper have been done in [IIJMNT] and [DO]. For the theoretical background the appendix in [IIJMNT] should be consulted. To conform with their conventions, we will use here the upper global base and corresponding coproduct $\Delta_+$, in contrast to the main text of this paper.

Recall the total order $\succ$ on the index set $J$ defined in (5.7.1). Extend the natural definition of minus on $J \setminus \{\phi\}$ to all of $J$ by defining $-\phi = \phi$. Let

$\overline{j} = j, \quad \overline{-j} = 2n + 1 - j, \quad j = 1, \ldots, n$

$\overline{0} = n, \quad \overline{-0} = 2n.$

Denote, as usual, by $E_{jk}$ the matrix acting on $\{v_j\}_{j \in J}$ as $E_{jk}v_i = \delta_{ki}v_j$. The R-matrix $\mathcal{R}(z)$ with normalization $\mathcal{R}(z)v_1 \otimes v_1 = v_1 \otimes v_1$ is then given by

$$\mathcal{R}(z) = \sum_{i \neq 0, \phi} E_{ii} \otimes E_{ii} + q^2 \frac{(1-z^2)}{(1-q^2 z^2)} \sum_{i \neq j, -j} E_{ii} \otimes E_{jj}$$

$$+ \frac{(1-q^4)}{(1-q^4 z^2)} \left( \sum_{i > j, i \neq -j} z^{\alpha_{ij}} E_{ij} \otimes E_{ji} + z^2 \sum_{i > j, i \neq -j} z^{-\alpha_{ij}} E_{ij} \otimes E_{ji} \right)$$

$$+ \sum a_{ij}(z) E_{ij} \otimes E_{-i, -j}$$

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where
\[ \alpha_{ij} = \begin{cases} 1 & : i = \phi \text{ or } j = \phi, \\ 0 & : \text{otherwise}, \end{cases} \]
\[ a_{ij}(z) = \begin{cases} (1 - z^2)(q^4 - \xi^2z^2) + \delta_{i,-i}(1 - q^2)(q^2 + z^2)(1 - \xi^2z^2) & : i = j, \\ (1 - q^4)\left\{ z^{\alpha_{ij}}(z^2 - 1) s_{\alpha_{ij}}(-q^2)^{-i-j} + \delta_{i,-j}(1 - \xi^2z^2) \right\} & : i > j, \\ (1 - q^4)\xi^2z^{-\alpha_{ij}}(z^2 - 1) s_{\alpha_{ij}}(-q^2)^{-i-j} + \delta_{i,-j}(1 - \xi^2z^2) \right\} & : i < j, \end{cases} \]
and
\[ s_{i0} = \frac{1}{[2]} \sgn(i) \quad (i \neq \phi), \quad s_{0j} = -[2] \sgn(j) \quad (j \neq \phi), \]
\[ s_{i\phi} = \frac{1}{[2]} \sgn(i) \quad (i \neq 0), \quad s_{\phi j} = [2] \sgn(j) \quad (j \neq 0), \]
\[ s_{0\phi} = s_{\phi 0} = -1, \quad s_{ij} = \sgn(i) \sgn(j) \quad (i, j \neq 0, \phi). \]

Also we have \( \xi^2 = q^{4n}. \) The expression for \( \overline{\mathcal{R}}(z) \) is given in [J] in a different basis.

Let \( \{ v_j^* \}_{j \in J} \) be the canonical dual basis of the upper global base. The following isomorphism of \( U_q(\mathfrak{g}) \)-modules
\[ C : V_{\xi^{-1}} \to (V_\xi)^* \]
\[ v_j \mapsto \sgn(j) (-q^2)^{-j-1} v_j^* \quad (j \in J/\{0, \phi\}) \]
\[ v_0 \mapsto -\frac{1}{[2]} (-q^2)^{-1} v_0^* \]
\[ v_\phi \mapsto \frac{1}{[2]} \xi^{-1} (-q^2)^{-1} v_\phi^* \]
gives rise to crossing-symmetry for the R-matrix
\[ \left( R^{-1}(z) \right)^{t_1} = \beta(z)(C \otimes 1)R(z \xi^{-1})(C \otimes 1)^{-1}, \quad (C.1) \]
with
\[ \beta(z) = q^{-4} \frac{(1 - z^2)(1 - q^{-4n+4}z^2)}{(1 - q^{-4n}z^2)(1 - q^{-4}z^2)}. \quad (C.2) \]

The image \( R^+(z_1/z_2) = \pi_{V_{z_1}} \otimes \pi_{V_{z_2}}(\mathcal{R}') \) of the modified universal R-matrix \( \mathcal{R}' \) also satisfies (C.1) with \( z = z_1/z_2 \) and \( \beta(z) = 1. \) Therefore we have
\[ R^+(z) = q^{-2} \frac{(q^4 z^2; \xi_\infty^4 z^2; \xi_\infty^4 z^2; \xi^4)_\infty(q^{-4} \xi^4 z^2; \xi^4)_\infty(q^4 \xi^4 z^2; \xi^4)_\infty(q^4 \xi^4 z^2; \xi^4)_\infty}{(z^2; \xi^4)_\infty(q^{-4} \xi^4 z^2; \xi^4)_\infty(q^4 \xi^4 z^2; \xi^4)_\infty(q^4 \xi^4 z^2; \xi^4)_\infty} \overline{\mathcal{R}}(z). \]
The two-point function satisfies the \( q \)-KZ equation
\[ \Psi(q^{2(h^\vee + k)} z) = R^+(q^{2(h^\vee + k)} z)(q^{-\phi} \otimes 1) \Psi(z), \quad (C.3) \]
where \( k = 1 \) is the level and \( \phi = 2\Lambda^c_n + 2\rho^c \) and, as a consequence, also
\[ (\pi_{V_{z_1}} \otimes \pi_{V_{z_2}}) \Delta'(e_i)^{(h_i, \Lambda^c_n) + 1} \Psi(z) = 0, \quad \text{wt} \ \Psi(z) = 0. \quad (C.4) \]
It can be shown that

\[ w(z) = (1 + z^2 q^2 \xi^2) v_0 \otimes v_0 + q[2] (-q^2)^n z v_\phi \otimes v_\phi - \]

\[ q \sum_{i=1}^{n} (-q^2)^{n-i} (v_0 \otimes v_{-i} + z^2 q^{4i-2} v_{-i} \otimes v_i) \]  

(C.5)

solves (C.4) and satisfies

\[ \overline{R}(q^2 \xi^2 z)(q^{-\phi} \otimes 1) w(z) = q^2 \frac{(1 - q^4 \xi^2 z^2)(1 - \xi^4 z^2)}{(1 - q^8 \xi^4 z^2)(1 - q^4 \xi^6 z^2)} w(q^2 \xi^2 z). \]  

(C.6)

Letting \( \Psi(z) = \phi(z) w(z) \) and substituting (C.6) into (C.3) one gets a scalar \( q \)-difference equation for \( \phi(z) \) which can be solved to obtain the final result

\[ \Psi(z) = \frac{\langle q^4 \xi^4 z^2; \xi_4 \rangle_\infty (\xi^6 z^2; \xi_4 \rangle_\infty}{\langle q^4 \xi^2 z^2; \xi_4 \rangle_\infty (\xi^4 z^2; \xi_4 \rangle_\infty} w(z). \]  

(C.7)

**Appendix D. The limit \( q \to 1 \) for the \( U_q(A_n^{(2)}) \) Fock space**

In this appendix we will show how to recover the known classical (i.e. at \( q = 1 \)) Fock space \( \mathcal{F}_{\text{class}} \) for \( g = A_n^{(2)} \) at level 1. This involves reduction of the Fock space \( \mathcal{F} \) defined for generic \( q \) by means of an invariant inner product on \( \mathcal{F} \). To facilitate the discussion we shall make a transcription from the semi-infinite wedge description of \( \mathcal{F} \) to one involving Young diagrams or, synonymously, partitions (the so-called “combinatorial description”).

Define the following subspace of \( V_{\text{aff}}^+ \):

\[ V_{\text{aff}}^+ = z^{-1} Q[z^{-1}] \otimes V \oplus Q\langle v_{-1}, \ldots, v_{-n}, v_0 \rangle. \]  

(D.1)

In any normally ordered pure wedge in \( \mathcal{F} \) it is clear that only bases in \( V_{\text{aff}}^+ \) appear as components. Recall the single-valued function \( l \) on \( B_{\text{aff}} \) in (5.3.3). To the normally ordered pure wedge \( u = G(u_1) \land G(u_2) \land \cdots \land G(u_k) \land v_0 \land v_0 \land \cdots \) let us associate the sequence \( Y(u) = [-l(u_1), -l(u_2), \ldots, -l(u_k), 0, 0, \ldots] \), whose tail of zeroes we shall ignore. Now, \(-l\) takes non-negative values on \( V_{\text{aff}}^+ \). Also, the sequence \( Y(u) \) is non-increasing because of the normal-ordering rules. Furthermore, the only integers allowed to repeat belong to \( h \mathbb{N} \), where \( h = 2n + 1 \), because of the rule \( v_i \land v_i = 0 \) if \( i \neq 0 \). Thereby we have the identification

\[ \mathcal{F} \simeq Q(q) \langle Y \rangle_{Y \in \text{DP}_h}, \]  

(D.2)

where \( \text{DP}_k \) is the set of Young diagrams whose rows are allowed to repeat only if their length is \( 0 \mod k \). In this notation, \( \text{DP}_\infty \) is the set of Young diagrams with no repeating rows, i.e., the set of Distinct Partitions.

The action of \( U_q(g) \) on \( \mathcal{F} \) can be transcribed to the Young diagram setting. The generators \( t_i \) act diagonally, of course, while \( f_i \) (respectively \( e_i \)) act by adding (removing) one box in the following manner. Let the Young diagram \( Y \) be denoted
by \([y_1, \ldots, y_m]\). For \(y \in \mathbb{Z}_{>0}\), let \(\alpha_Y(y)\) denote the number of occurrences of \(y\) in \(Y\). Define the functions \(\beta_i\) for \(i = 0, 1, \ldots, n\) by

\[
\begin{align*}
\beta_0(y) & = \begin{cases} 
\pm 4 & : y \in h\mathbb{Z} + n \\
0 & : \text{otherwise}
\end{cases} \\
\beta_i(y) & = \begin{cases} 
\pm 2 & : y \in h\mathbb{Z} \mp n \pm (i - 1) \\
\mp 2 & : y \in h\mathbb{Z} \mp n \pm i \\
0 & : \text{otherwise}
\end{cases} \\
\beta_n(y) & = \begin{cases} 
\pm 2 & : y \in h\mathbb{Z} \mp 1 \\
0 & : \text{otherwise}
\end{cases}
\end{align*}
\]

Then the action of \(U_q(\mathfrak{g})\) on \(Y\) is given explicitly by

\[
\begin{align*}
t_i \cdot Y & = q^{\sum_j \beta_i(y_j) + \delta_{i,n}} Y \\
f_i \cdot Y & = \sum_{y_k \in \mathbb{N}^{n+1} \atop y_k - 1 \neq y_k + 1} q^{\sum_{j=k}^n \beta_i(y_j)} [y_1, \ldots, y_k + 1, \ldots, y_m] \\
& \quad + \sum_{y_k \in \mathbb{N}^{n-1} \atop y_k - 1 \neq y_k + 1} q^{\sum_{j=k}^n \beta_i(y_j) + 1} \left(1 - (-q^2)^{\alpha_Y(y_k)}\right) [y_1, \ldots, y_k + 1, \ldots, y_m] \\
& \quad + \delta(y_m \neq 1)[y_1, \ldots, y_m, 1] \\
e_i \cdot Y & = \sum_{y_k \in \mathbb{N}^{n+1} \atop y_k - 1 \neq y_k - 1} q^{-\sum_{j=k}^n \beta_i(y_j)} [y_1, \ldots, y_k - 1, \ldots, y_m] \\
e_n \cdot Y & = \sum_{y_k \in \mathbb{N}^{n+1} \atop y_k - 1 \neq y_k - 1} q^{-\sum_{j=k}^n \beta_i(y_j)} [y_1, \ldots, y_k - 1, \ldots, y_m] \\
& \quad + \sum_{y_k \in \mathbb{N}^n \atop y_k - 1 \neq y_k - 1} q^{-\sum_{j=k}^n \beta_i(y_j) - 1} \left(1 - (-q^2)^{\alpha_Y(y_k)}\right) [y_1, \ldots, y_k - 1, \ldots, y_m].
\end{align*}
\]

Note that all Young diagrams appearing on the right-hand side belong to \(\text{DP}_h\). In other words, the corresponding pure wedges are already normally ordered. The factors \(1 - (-q^2)^{\alpha_Y(y_k)}\) come from normal ordering and summing up Young diagrams which arise when \(Y\) has repeated rows. Note also that the vacuum vector is the empty Young diagram \(\emptyset\) and \(f_n \cdot \emptyset = [1]\). This combinatorial description is in the same spirit as that for \(U_q(\mathfrak{B}^{(1)}_n)\) given in [MM].

Let us now introduce an inner product \((\ , \ )\) on \(\mathcal{F}\). We shall require that the normally ordered pure wedges, or equivalently Young diagrams in \(\text{DP}_h\), form an orthogonal basis with respect to \((\ , \ )\). We shall also require that with respect to
The adjoints of the generators satisfy
\[ f_i^\dagger = q_i e_i t_i, \]
\[ e_i^\dagger = q_i f_i t_i^{-1}, \]
\[ t_i^\dagger = t_i. \]

These conditions are natural for a \( U_q(\mathfrak{g}) \)-module \( V \) because then on the module \( V \otimes V \) with induced inner product given by \((v_1 \otimes v_2, u_1 \otimes u_2) = (v_1, u_1)(v_2, u_2)\) we have \( \Delta(f_i)^\dagger = q_i \Delta(e_i) \Delta(t_i) \), etc. Using the explicit description of the \( U_q(\mathfrak{g}) \) action on \( \mathcal{F} \) one can show that the squared norm of an arbitrary Young diagram \( Y \) in \( \text{DP}_h \) is given by
\[ ||Y||^2 = (Y, Y) = \prod_{y \in h \mathbb{Z}_{\geq 0}} \alpha_y(y) \prod_{i=1} \left( 1 - (-q^2)^i \right). \] (D.3)

From calculations for small \( k \) we conjecture that the boson operators satisfy
\[ B_{-k}^\dagger = B_k. \]

As at the end of §4.3, we denote by \( \mathcal{F}_Q \) the \( \mathbb{Q}[q, q^{-1}] \)-vector space generated by the pure wedges. Set \( \mathcal{F}_1 = \mathcal{F}_Q/(q - 1)\mathcal{F}_Q \). Then the action of \( U_q(\mathfrak{g}) \) on \( \mathcal{F}_1 \) induces an action of \( U_q(\mathfrak{g}) \) on \( \mathcal{F}_1 \). The inner product \((, )\) on \( \mathcal{F}_Q \) induces a \( \mathbb{Q} \)-valued inner product on \( \mathcal{F}_1 \), which we also denote by \((, )\). The adjoint of operators in \( \mathfrak{g} \) is then given by \( e_i^\dagger = f_i, f_i^\dagger = e_i \) and \( h_i^\dagger = h_i \). Define the subspace \( \mathcal{F}_0 = \{ u \in \mathcal{F}_1 : (u, \mathcal{F}_1) = 0 \} \). The reduced Fock space \( \mathcal{F}_{\text{red}} = \mathcal{F}_1/\mathcal{F}_0 \) is a \( U(\mathfrak{g}) \)-module. From (D.3) we note that \( \mathcal{F}_0 \) is the \( \mathbb{Q} \)-span of Young diagrams with some repeated rows. It follows that \( \mathcal{F}_{\text{red}} \) is the \( \mathbb{Q} \)-span of Young diagrams in \( \text{DP}_\infty \). This is isomorphic to the well-known classical Fock space \( \mathcal{F}_{\text{class}} \simeq \mathbb{Q}[x_k]_{k \in \mathbb{N}_{\text{odd}}} \) [KKLW, DJKM]. In fact, the action of the generators on \( \mathcal{F}_{\text{red}} \) and at \( q = 1 \) reduces to a known classical action [JY]. Furthermore we recover the known decomposition of \( \mathcal{F}_{\text{class}} \simeq \mathbb{Q}[x_k]_{k \in \mathbb{N}_{\text{odd}}} \otimes \mathbb{Q}[x_{hk}]_{k \in \mathbb{N}_{\text{odd}}} \otimes \mathbb{Q}^{\otimes H_-} \mathbb{Q}^{\otimes H_-} \) as a \( U(\mathfrak{g}) \otimes \mathbb{Q}[H_-] \)-module. Here we identify bosons \( x_{hk} \sim B_{-k} \) for \( k \in \mathbb{N}_{\text{odd}} \). The even boson commutators \( \gamma_k \) for \( k \in \mathbb{N}_{\text{even}} \) have a pole at \( q = 1 \). After appropriately rescaling we find that such \( B_{-k} \) act as 0 on \( \mathcal{F}_{\text{red}} \) at \( q = 1 \).

In most of the cases considered in this paper, the boson commutator \( \gamma_k \) has a pole at \( q = 1 \) for some \( k \). We take this to indicate that similar Fock space reductions to the one considered in this Appendix are necessary to recover any known classical Fock spaces.
References

[CP] V. Chari and A. Pressley, A Guide to Quantum Groups, Cambridge University Press, 1994.

[DJKM] E. Date, M. Jimbo, M. Kashiwara, and T. Miwa, Transformation Groups for soliton equations — Euclidean Lie algebras and reduction of the KP hierarchy, Publ. R.I.M.S., Kyoto Univ. 18 (1982), 1077–1110.

[DJO] E. Date, M. Jimbo, and M. Okado, Crystal bases and q-vertex operators, Commun. Math. Phys. 155 (1993), 47–69.

[DO] E. Date and M. Okado, Calculation of excitation spectra of the spin model related with the vector representations of the quantized affine algebra of type $A^{(1)}_n$, Int. J. Mod. Phys. A 9 (1994), no. 3, 399–417.

[H] T. Hayashi, $q$ analogues of Clifford and Weyl algebras — spinor and oscillator representations of quantum enveloping algebras, Commun. Math. Phys. 127 (1990), 129–144.

[IJMNT] M. Idzumi, K. Iohara, M. Jimbo, T. Miwa, T. Nakashima, and T. Tokihiro, Quantum affine symmetry in vertex models, Int. J. Mod. Phys. A 8 (1993), 1479–1511, (hep-th/9208066).

[J] M. Jimbo, Quantum R-matrix for the generalised Toda system, Commun. Math. Phys. 102 (1986), 537–547.

[JL] M. Jimbo and T. Miwa, Algebraic analysis of solvable lattice models, CBMS Regional conference series in mathematics, vol. 85, A.M.S., Providence, Rhode Island, 1995.

[JY] P. D. Jarvis and C. M. Yung, Combinatorial description of the Fock representation of the affine Lie algebra $g(o)(x)$, Lett. Math. Phys. 30 (1994), 45–52.

[K1] M. Kashiwara, Crystallising the $q$-analogue of universal enveloping algebras, Commun. Math. Phys. 133 (1990), 249–260.

[K2] M. Kashiwara, Global crystal bases of quantum groups, Duke Math. J. 69 (1993), 455–485.

[KKLW] V. Kac, D. A. Kazhdan, J. Lepowsky, and R. L. Wilson, Realization of the basic representations of the Euclidean Lie algebras, Adv. Math. 42 (1981), 83–112.

[KMN1] S.-J. Kang, M. Kashiwara, K. C. Misra, T. Miwa, T. Nakashima, and A. Nakayashiki, Affine crystals and vertex models, Int. J. Mod. Phys. A 7, Suppl. 1A (1992), 449–484, Proceedings of the RIMS Project 1991 “Infinite Analysis”.

[KMN2] S.-J. Kang, M. Kashiwara, K. C. Misra, T. Miwa, T. Nakashima, and A. Nakayashiki, Perfect crystals of quantum affine Lie algebras, Duke Math. J. 68 (1992), 499–607.

[KMS] M. Kashiwara, T. Miwa, and E. Stern, Decomposition of $q$-deformed Fock spaces, preprint 1025, R.I.M.S., August 1995, (q-alg/9508006), to appear in Selecta Mathematica.

[LW] J. Lepowsky and R. L. Wilson, Construction of the affine Lie algebra $A^{(1)}_1$, Commun. Math. Phys. 62 (1978), 43–53.

[MM] K. C. Misra and T. Miwa, Crystal base for the basic representation of $U_q(\hat{sl}(n))$, Commun. Math. Phys. 134 (1990), 79–88.

[S] E. Stern, Semi-infinite wedges and vertex operators, Internat. Math. Res. Notices 4 (1995), 201–220, (q-alg/9505030).

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