Vacuum misalignment corrections to tri-bimaximal mixing and form dominance

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Abstract

Tri-bimaximal neutrino mixing may arise from see-saw models based on family symmetry which is spontaneously broken by flavons with particular vacuum alignments. In this paper we derive approximate analytic results which express the deviations from tri-bimaximal neutrino mixing due to vacuum misalignment. We also relate vacuum misalignment to departures from form dominance, corresponding to complex deviations from the real orthogonal $R$ matrix, where such corrections are necessary to allow for successful leptogenesis. The analytic results show that the corrections to tri-bimaximal mixing and form dominance depend on the pattern of the vacuum misalignment, with the two effects being uncorrelated.

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1 Introduction

It is well known that the solar and atmospheric data are consistent with so-called tri-bimaximal (TB) lepton mixing [1],

\[ U_{TB} = \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 1 \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}. \] (1)

In the flavour basis (i.e. diagonal charged lepton mass basis), it has been shown that the TB neutrino mass matrix is invariant under \( S, U \) transformations [3, 4]

\[ M_{TB}^{\nu} = S M_{TB}^{\nu} S^T = U M_{TB}^{\nu} U^T. \] (2)

A very straightforward argument [5] shows that this neutrino flavour symmetry group has only four elements corresponding to Klein’s four-group \( Z_2^S \times Z_2^U \) corresponding to the two generators \( S, U \). By contrast the diagonal charged lepton mass matrix (in this basis) satisfies a diagonal phase symmetry corresponding to the generator \( T \). The matrices \( S, T, U \) form the three generators of the group \( S_4 \) in the triplet representation, while the \( A_4 \) subgroup is generated by \( S, T \). This suggests that TB lepton mixing matrix calls for a discrete non-Abelian family symmetry in nature. There has been a considerable amount of theoretical work in this direction [6–19].

As discussed in [5], the Klein symmetry of the neutrino mass matrix may originate either directly as above or accidentally as an indirect effect of the family symmetry \( G_f \). In such indirect models the flavons responsible for the neutrino masses break \( G_f \) completely so that none of the generators of \( G_f \) survive. The Klein symmetry \( Z_2^S \times Z_2^U \) emerges as an accidental symmetry due to the appearance of quadratic combinations of flavons in the neutrino sector, with particular vacuum alignments along the columns of the TB matrix. This is essentially the approach followed in many existing models in the literature [7–12]. In such models there is no compelling reason why the vacuum alignments should take this form, and it is quite possible to have alternative vacuum alignments which would lead to alternative types of mixing which violate the Klein symmetry.

Recently it has been argued [20] that TB mixing may not originate from a family symmetry at all, discrete or otherwise, but may be a pure accident. The authors of [20] explored the experimentally allowed violations of the TB symmetry relations present in the effective neutrino mass matrix \( M^{\nu} \) and found that very strong deviations of the neutrino mass matrix element relations arising from from TB mixing were allowed within current experimental errors on the mixing parameters in \( U \). We point out that \( M^{\nu} \) is

\[^1\text{Note that the position of the minus signs in the TB mixing matrix are phase convention dependent. We have adopted the phase conventions consistent with the standard PDG parametrisation [2].}\]
comprised of sums of component matrices $C_i$, weighted by neutrino mass eigenvalues $m_i$, where the $C_i$ are directly linked to the underlying symmetry. We shall trace the origin of the observation in [20] to leading order zeroes in the $C_3$ matrix which in indirect or accidental models originates from a flavon aligned along the third column of the TB matrix which has a zero in the first entry. The observation of [20] that large violations of the TB symmetry relations are allowed then translates into the observation that this zero entry will have large (formally infinite) fractional corrections due to any finite correction to the vacuum alignment. However it has already previously been pointed out that these zeroes can be filled in at the leading order without disturbing the tri-bimaximal predictions for the atmospheric and solar angles [19]. We conclude that these results [19,20] do not disfavour family symmetry models but do show that tri-bimaximal mixing may be insensitive to certain corrections to vacuum alignment. This provides a motivation for the present study.

In the remainder of the paper we focus on models based on the type I see-saw mechanism [21] which explain TB mixing as a consequence of spontaneously broken family symmetry. In such models the Dirac neutrino mass matrix in the diagonal right-handed neutrino mass basis satisfies the conditions of form dominance (FD) [14] at leading order. FD means that, in this basis, the columns of the Dirac mass matrix are proportional to the columns of the TB matrix. In practice this is achieved by vacuum alignment of flavon fields. In the most natural models [7–12] there is a separate flavon $\phi_i$ contributing to each of the component matrices $C_i$, and each of the neutrino masses $m_i$ arises from a separate flavon vacuum expectation value (VEV), and the TB mixing cannot depend on cancellations involving neutrino masses. By contrast, in the direct models [13–15], the component matrices originate from linear combinations of flavon VEVs [14].

The way the vacuum alignment is achieved is quite model dependent, but in general the mechanisms may be classified as being due to D-terms [10,22], F-terms [13] or extra dimensional orbifold boundary conditions [23]. Although in principle the desired vacuum alignment of the flavon fields originates from some high energy family symmetry such as $A_4$, albeit in a model dependent way, in all cases there will be corrections to the leading order vacuum alignment of flavons. For example, such corrections can fill in the zeroes, or violate the equality between different components of a flavon VEV. In addition the effects of higher order operators can allow flavons with a particular alignment to pollute the sector containing at leading order only flavons of a different alignment. All such effects, which in general lead to a violation of the Klein symmetry, will referred to here as “vacuum misalignment” since in all cases we are perturbing away from the forms of vacuum alignment which are known to reproduce TB mixing exactly. It is worth emphasising that the term “vacuum misalignment” as used here could either refer to a leading order vacuum alignment (in the case where the original form of vacuum alignment contains a zero) or a correction to the non-zero components of the leading order vacuum alignment, and the results in this paper apply to both situations.
One of the main motivations for this paper is to derive analytic formulae which relate vacuum misalignments to the deviation from TB mixing. This paper contains the first analytic results in the literature which relate the effect of general vacuum misalignments to the deviations from TB mixing. The value of the analytic approach is that it enables simple physical insights to be obtained which are not possible with a purely numerical approach, and we illustrate this with some simple examples. These examples include (admittedly rather arbitrary) special cases where the vacuum misalignment does not lead to any corrections to TB mixing. However, the main value of this paper lies not in the special cases we consider but in the general analytic results which relate vacuum misalignment to deviations from TB mixing, and the only purpose of the examples is to provide simple illustrations of the general results.

It is important to differentiate between two distinct consequences of vacuum misalignment, namely (i) deviations to TB mixing, and (ii) departures from FD, where the two effects are in principle independent of each other. Apart from presenting analytic formulae describing the first effect (i), we also present a formalism which enables the second effect (ii) to be discussed, which is also the first time that such effects have been studied analytically in the literature. Note that nearly all models which give TB mixing using a family symmetry also satisfy the conditions of FD at the leading order, so the FD approximation for the unperturbed alignment is not restrictive at all in practice and applies to all of the models in [6–19], for example, which describe TB mixing.

We stress that both sets of analytic results, i.e. which relate vacuum misalignment to both (i) TB mixing corrections and (ii) FD corrections, are original results which have not appeared before in the literature. Furthermore, the analytic results are both useful and physically relevant. Firstly, the analytic results are useful since in practice some degree of vacuum misalignment is always present in realistic models which attempt to describe TB mixing as the result of a family symmetry. Secondly, such vacuum misalignment will have important physical implications regarding neutrino oscillation experiments and leptogenesis. The physical relevance of the results to precision neutrino oscillation experiments is clear since future experiments will be sensitive to deviations from TB mixing [24], and the analytic results enable such deviations to be related to vacuum misalignment in realistic models, which facilitates theoretical insights which complement the numerical studies. The physical relevance of the results to leptogenesis is also clear, since the lepton asymmetries vanish exactly in the FD limit where it would correspond to a real $R$ matrix [25] for which leptogenesis vanishes [26], as previously observed in particular family symmetry models [27]. The analytical expressions we derive for the complex corrections to the real $R$ matrix in terms of the vacuum misalignment are therefore physically relevant since they allow for non-zero leptogenesis.

In this paper, then, we derive approximate analytic formulae which relate general vacuum misalignment, as defined above, to the deviations from TB mixing. We also relate vacuum misalignment to violations of FD via a small complex angle expansion of the orthogonal $R$ matrix. Conventional wisdom says that vacuum misalignment always
leads to violation of TB mixing and FD, however the resulting analytic formulae show that vacuum misalignment may or may not lead to deviations from TB mixing and does not necessarily imply violation of FD either, with the two effects being uncorrelated. Also, as already discussed above, the recent analyses hint that TB mixing may be insensitive to vacuum misalignment [19, 20], and it is interesting to apply our analytic results to study this question here, although this is not the main motivation for the paper. We emphasise that the results here have very general applicability and may be applied to all direct or indirect family symmetry models based on spontaneously broken family symmetry in order to estimate the deviations from TB neutrino mixing due to vacuum alignment corrections. However, to use the results here, the Dirac mass matrix must be rotated to the basis in which the charged lepton and right-handed neutrino mass matrices are both diagonal, which is automatically the case for the indirect models, at least approximately. However, in the case of direct models, the Dirac mass matrix needs to be rotated to the diagonal right-handed neutrino mass basis before the results can be applied [14]. Such models should also be formulated in the diagonal charged lepton mass basis, corresponding to the choice of diagonal $T$ generator basis [13]. We stress that it is not the goal of this paper to study the dynamics of vacuum misalignment via some potential or superpotential of a particular model, as was done for example in [18]. Instead we are only interested in the effects of vacuum misalignment on TB mixing and FD, and for this purpose it is sufficient to simply parameterise the misalignment in a particular basis where the dynamical origin of the misalignment can have a general origin as discussed above.

We remark that TB deviations due to vacuum alignment corrections have previously only been studied numerically in the framework of the direct $A_4$ models [28]. We also note that the results in this paper are complementary and more general than the analytic results in [29] which were confined to sequential dominance (SD) [30], and were derived in a completely different way based on a perturbative diagonalization of the neutrino mass matrix in powers of small neutrino mass ratios assuming a hierarchical mass spectrum. By contrast, our results here are applicable to any pattern of neutrino masses. Finally we recall that vacuum alignment corrections, though important and in some cases dominant, are only one of a number of corrections to TB mixing which may arise in realistic models, the other ones being renormalisation group corrections, canonical normalisation corrections and charged lepton corrections, but since these have all been studied elsewhere [31] they will not be revisited here.

The layout of the remainder of the paper is as follows. In section 2 we show that the neutrino mass matrix is comprised of a sum of component matrices $C_i$ which are closely related to the Klein symmetry. We show that the observation in [20] that large violations in the neutrino mass matrix may be allowed consistently with current limits on tri-bimaximal deviations is due to the presence of leading order zeroes in the $C_3$ matrix. In section 3 we show how TB mixing arises naturally from the type I see-saw mechanism if the conditions of FD are satisfied. Working in the diagonal right-handed
neutrino mass basis, perturbations of the Dirac mass matrix (identified with vacuum misalignment) are related to deviations from TB mixing via the effective neutrino mass matrix. We also relate vacuum misalignment to the complex corrections to the real $R$ matrix predicted by FD. Section 4 summarises and concludes the paper.

2 The effective neutrino mass matrix

2.1 Symmetry and the component matrices

Let us begin by considering the general case of leptonic mixing. In the neutrino flavour basis, in which the charged lepton mass matrix is diagonal and mixing arises from the neutrino sector, the effective neutrino mass matrix $M_ν$, a complex symmetric matrix containing six phases, may be diagonalised as,

$$U^\dagger P_E M_ν P_E U^* = \text{diag}(m_1, m_2, m_3),$$

where we find it convenient to work with three complex neutrino masses $m_i$ and a mixing matrix $U$ containing only one Dirac phase, where $P_E$ is a diagonal phase matrix. The usual MNS matrix is written in terms of three real and positive neutrino masses $|m_i|$ as $U_{MNS} = U P_{Maj}$, where $P_{Maj}$ contains two Majorana phases, after absorbing the unphysical phases $P_E$ and an overall phase in the diagonal charged lepton sector. Thus $U$ is the analogue of the CKM mixing matrix for quarks, involving three mixing angles $\theta_{ij}$ and one phase $\delta$, in the standard convention. Given any such mixing matrix $U$, this enables the neutrino mass matrix $\tilde{M}_ν = P_E M_ν P_E$ to be determined in terms of the three complex neutrino masses,

$$\tilde{M}_ν = U \text{diag}(m_1, m_2, m_3) U^T = m_1 \Phi_1 \Phi_1^T + m_2 \Phi_2 \Phi_2^T + m_3 \Phi_3 \Phi_3^T,$$

(4)

corresponding to the orthonormal column vectors $\Phi_i$ which are just equal to the columns of $U$,

$$U = (\Phi_1, \Phi_2, \Phi_3)$$

(5)

with the orthonormality relations,

$$\Phi_i^\dagger \Phi_j = \delta_{ij}.$$  

(6)

It is convenient to define the component matrices

$$C_i = \Phi_i \Phi_i^T,$$

(7)

$\tilde{M}_ν$ is loosely referred to as the neutrino mass matrix in the literature even though the true effective neutrino mass matrix, as determined for example by the see-saw mechanism, is $M_ν$. 

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in terms of which the neutrino mass matrix $\tilde{M}^\nu$ is simply written as a sum, weighted by neutrino masses,

$$\tilde{M}^\nu = m_1 C_1 + m_2 C_2 + m_3 C_3.$$  

(8)

Let us now consider the special case of TB mixing, where the columns of $U = U^{TB}$ have particularly simple forms which may be written in the standard parametrisation (the PDG convention with mixing angles given by $\sin \theta_{12} = 1/\sqrt{3}$, $\sin \theta_{23} = 1/\sqrt{2}$) as $[32]$,

$$\Phi^{TB}_1 = \frac{1}{\sqrt{6}} \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \quad \Phi^{TB}_2 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \quad \Phi^{TB}_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix},$$

(9)

where $U^{TB}_{MNS} = U^{TB}.P_{Maj}$. In the TB example, $S,U$ in Eq$[2]$ take the particularly simple forms,

$$S = \frac{1}{3} \begin{pmatrix} -1 & 2 & -2 \\ 2 & -1 & -2 \\ -2 & -2 & -1 \end{pmatrix}, \quad U = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

(10)

such that the elements $S,U,T$ generate the closed finite group $G_f = S_4$ $[3]$ which may be used as a family symmetry capable of enforcing TB mixing.

The important point is that the TB symmetry transformations $G$, contained in $G_f$ leave invariant not only the effective neutrino mass matrix invariant but also the component matrices $C_i^{TB}$ from which the effective neutrino mass matrix is formed. In natural models it is these component matrices $C_i^{TB}$ which result from the family symmetry $G_f$, with the effective neutrino matrix emerging as a sum of such matrices weighted by neutrino masses $m_i$ which are not predicted by the symmetry. It may happen that a particular element of $\tilde{M}^\nu$ is small due to an accidental cancellation in the sum of terms in Eq$[3]$ but since the masses are not predicted by symmetry the models have nothing to say about this special point. It is precisely such special points that give the largest deviations from the TB mixing relations studied in $[20]$. It is clear that such special accidental points are irrelevant from the perspective of symmetry models. What is relevant for a discussion of the robustness of the symmetry approach is the deviation of the matrix elements of the component matrices $C_i^{TB}$ from the TB form due to deviations in the mixing parameters from their TB values, as we discuss later.

### 2.2 Direct vs Indirect Models

As discussed in $[5]$, the flavour symmetry of the neutrino mass matrix may originate from two quite distinct classes of models. The first class of models, which we call direct models, are based on a family symmetry $G_f = S_4$, or a closely related family symmetry as discussed below, some of whose generators are directly preserved in the lepton sector and
are manifested as part of the observed flavour symmetry. The second class of models, which we call indirect models, are based on some more general family symmetry $G_f$ which is completely broken in the neutrino sector, while the observed neutrino flavour symmetry $Z_2^S \times Z_2^U$ in the neutrino flavour basis emerges as an accidental symmetry which is an indirect effect of the family symmetry $G_f$. In such indirect models the flavons responsible for the neutrino masses break $G_f$ completely so that none of the generators of $G_f$ survive in the observed flavour symmetry $Z_2^S \times Z_2^U$.

In the direct models, the symmetry of the neutrino mass matrix in the neutrino flavour basis (which we are calling the neutrino mass matrix for brevity) is a remnant of the $G_f = S_4$ symmetry of the Lagrangian, where the generators $S,U$ are preserved in the neutrino sector, while the diagonal generator $T$ is preserved in the charged lepton sector. For direct models, a larger family symmetry $G_f$ which contains $S_4$ as a subgroup is also possible e.g. $G_f = PSL(2, 7)$ [17]. Typically direct models require flavon F-term vacuum alignment and may include an $SU(5)$ type unification [13]. Such minimal $A_4$ models lead to neutrino mass sum rules between the three masses $m_i$, resulting in/from a simplified neutrino mass matrix. $A_4 \times SU(5)$ SUSY GUT models are typically constructed in extra dimensions [13], where such models in 8D enables vacuum alignment to be elegantly achieved by boundary conditions [23].

In the indirect models [5] the idea is that the three columns of $U_{TB}, \Phi_i^{TB}$, are promoted to new Higgs fields called “flavons”, with the particular vacuum alignments along the directions $\Phi_i^{TB}$ in Eq.9 breaking the family symmetry. In the indirect models the underlying family symmetry of the Lagrangian $G_f$ is completely broken, and the flavour symmetry of the neutrino mass matrix $Z_2^S \times Z_2^U$ emerges entirely as an accidental symmetry, due to the quadratic appearance of such flavons in effective Majorana Lagrangian which results in a neutrino mass matrix of the desired form in Eq.4 [5]. Such vacuum alignments can be elegantly achieved using D-term vacuum alignment, which allows the large classes of discrete family symmetry $G_f$, namely the $\Delta(3n^2)$ and $\Delta(6n^2)$ groups [5]. We shall discuss an explicit example of an indirect model in subsection 3.3.

2.3 Deviations from tri-bimaximal mixing

In general, not assuming TB mixing, we can write the neutrino mass matrix as a sum of the component matrices weighted by the neutrino masses:

$$\tilde{M}\nu = m_1 \Phi_1 \Phi_1^T + m_2 \Phi_2 \Phi_2^T + m_3 \Phi_3 \Phi_3^T,$$

where $\Phi_i$ are the orthonormal columns of the mixing matrix $U = (\Phi_1, \Phi_2, \Phi_3)$. If we are close to the TB case, as current data tells us that we must be, then we can expand the columns of $U$ to lowest order as:

$$\Phi_i = \Phi_i^{TB} + \Delta \Phi_i.$$
Expanding Eqs. 11 to lowest order in 12,
\[
\tilde{M}^\nu \approx \begin{bmatrix}
    m_1 & \Phi^T_1 B \Delta \Phi_1^T + \Delta \Phi_1^T_1 \\
    m_2 & \Phi^T_2 B \Delta \Phi_2^T + \Delta \Phi_2^T_2 \\
    m_3 & \Phi^T_3 B \Delta \Phi_3^T + \Delta \Phi_3^T_3 \\
\end{bmatrix},
\]
(13)

In the following discussion it is convenient use the expansion about TB mixing introduced in [34],
\[
s_{13} = \frac{r}{\sqrt{2}}, \quad s_{12} = \frac{1}{\sqrt{3}}(1 + s), \quad s_{23} = \frac{1}{\sqrt{2}}(1 + a),
\]
(14)
where the three real parameters \(r, s, a\) describe the deviations of the (r)eactor, (s)olar and (a)tmospheric angles from their tri-bimaximal values.

The global fits of the conventional mixing angles [33] can be translated into the 1σ ranges:
\[0.07 < r < 0.21, \quad -0.05 < s < 0.003, \quad -0.09 < a < 0.04\].
(15)

To first order in \(r, s, a\) the lepton mixing matrix \(U\) (where as usual \(U_{MNS} = U_P\)) can be written as [34, 3]
\[
U = \begin{pmatrix}
    \frac{1}{\sqrt{3}}(1 - \frac{1}{2}s) & \frac{1}{\sqrt{3}}(1 + s) & \frac{1}{\sqrt{2}} r e^{-i\delta} \\
    -\frac{1}{\sqrt{6}}(1 + s - a + re^{i\delta}) & \frac{1}{\sqrt{6}}(1 - \frac{1}{2}s - a - \frac{1}{2}re^{i\delta}) & \frac{1}{\sqrt{2}}(1 + a) \\
    \frac{1}{\sqrt{6}}(1 + s + a - re^{i\delta}) & \frac{1}{\sqrt{6}}(1 - \frac{1}{2}s + a + \frac{1}{2}re^{i\delta}) & \frac{1}{\sqrt{2}}(1 - a)
\end{pmatrix},
\]
(16)
from which the deviations of the columns of \(U = (\Phi_1, \Phi_2, \Phi_3)\) in Eq.12 namely \(\Delta \Phi_i\), may be read off as follows,
\[
\Delta \Phi_1 = \frac{1}{\sqrt{6}} \begin{pmatrix}
    -s \\
    -s + a - re^{i\delta} \\
    s + a - re^{i\delta}
\end{pmatrix}, \quad \Delta \Phi_2 = \frac{1}{\sqrt{3}} \begin{pmatrix}
    \frac{1}{2}s - a - \frac{1}{2}re^{i\delta} \\
    \frac{1}{2}s - a - \frac{1}{2}re^{i\delta} \\
    \frac{1}{2}s - a - \frac{1}{2}re^{i\delta}
\end{pmatrix}, \quad \Delta \Phi_3 = \frac{1}{\sqrt{2}} \begin{pmatrix}
    re^{-i\delta} \\
    a \\
    -a
\end{pmatrix}.
\]
(17)

It is manifest from Eq.13 that the deviations in the component matrices \(C_{Ti}^{TB} = \Phi_i^{TB} \Phi_i^{TB T}\) from the TB form due to deviations in the mixing parameters from their TB values are proportional to \(\Delta \Phi_i\) which are, from Eq.17 proportional to the mixing deviation parameters \(r, s, a\). Thus there is a linear relationship between the deviation from the elements of the component matrices and the deviation between the mixing parameters.

\(^3\)Other related proposals to parametrize the lepton mixing matrix have been considered in [35].
2.4 Example: deviations due to the reactor angle

Let us now consider as an example the case where TB mixing is only corrected by the presence of a non-zero reactor angle parameterised by the deviation parameter $r$ (with $s = a = 0$ in this example) [19]. This example is interesting since the experimental limit on $r$ is weaker than on $s, a$ according to Eq.[15] and it is also sufficient to understand the observations in [20]. In this case, to first order in $r$, from Eqs.13,16 we find that the component matrices $C_i$ which comprise the neutrino mass matrix can be written as a sum of the TB matrices $C_i^{\text{TB}}$ plus a correction proportional to the reactor parameter $r$,

\[
C_1 = \Phi_1\Phi_1^T = \frac{1}{6} \begin{pmatrix}
4 & -2 & 2 \\
-2 & 1 & -1 \\
2 & -1 & 1 \\
\end{pmatrix} - \frac{1}{3} r e^{i\delta} \begin{pmatrix}
0 & 1 & 1 \\
1 & -1 & 0 \\
1 & 0 & 1 \\
\end{pmatrix},
\]

\[
C_2 = \Phi_2\Phi_2^T = \frac{1}{3} \begin{pmatrix}
1 & 1 & -1 \\
1 & 1 & -1 \\
-1 & -1 & 1 \\
\end{pmatrix} - \frac{1}{6} r e^{i\delta} \begin{pmatrix}
0 & 1 & 1 \\
1 & 2 & 0 \\
1 & 0 & -2 \\
\end{pmatrix},
\]

\[
C_3 = \Phi_3\Phi_3^T = \frac{1}{2} \begin{pmatrix}
0 & 0 & 0 \\
0 & 1 & 1 \\
0 & 1 & 1 \\
\end{pmatrix} + \frac{1}{2} r e^{-i\delta} \begin{pmatrix}
0 & 1 & 1 \\
1 & 0 & 0 \\
1 & 0 & 0 \\
\end{pmatrix}.
\]

From Eq.11 and the component matrices in Eq.18 we may write $\tilde{M}^\nu$ as the symmetric matrix,

\[
\tilde{M}^\nu = \begin{pmatrix}
m_{ee} & m_{e\mu} & m_{e\tau} \\
m_{e\mu} & m_{\mu\mu} & m_{\mu\tau} \\
m_{e\tau} & m_{\mu\tau} & m_{\tau\tau} \\
\end{pmatrix},
\]

where,

\[
m_{ee} = \frac{2}{3} m_1 + \frac{1}{3} m_2,
\]

\[
m_{e\mu} = -\frac{1}{3} m_1 + \frac{1}{3} m_2 - r e^{i\delta} \left(\frac{1}{3} m_1 + \frac{1}{6} m_2\right) + r e^{-i\delta} \left(\frac{1}{2} m_3\right),
\]

\[
m_{e\tau} = \frac{1}{3} m_1 - \frac{1}{3} m_2 - r e^{i\delta} \left(\frac{1}{3} m_1 + \frac{1}{6} m_2\right) + r e^{-i\delta} \left(\frac{1}{2} m_3\right),
\]

\[
m_{\mu\mu} = \frac{1}{6} m_1 + \frac{1}{3} m_2 + \frac{1}{2} m_3 + r e^{i\delta} \left(\frac{1}{3} m_1 - \frac{1}{3} m_2\right),
\]

\[
m_{\tau\tau} = \frac{1}{6} m_1 + \frac{1}{3} m_2 + \frac{1}{2} m_3 + r e^{i\delta} \left(\frac{1}{3} m_1 + \frac{1}{3} m_2\right),
\]

\[
m_{\mu\tau} = -\frac{1}{6} m_1 - \frac{1}{3} m_2 + \frac{1}{2} m_3.
\]

In the limit that $r = 0$, $\tilde{M}^\nu$ reduces to the TB neutrino mass matrix $\tilde{M}_{TB}^\nu$, and the relations $m_{e\mu} = -m_{e\tau}$ and $m_{\mu\mu} = m_{\tau\tau}$ and $-m_{\mu\tau} = m_{ee} + m_{e\mu} - m_{\mu\mu}$ emerge as
the characteristic signatures of the TB neutrino mass matrix in the flavour basis, in
the convention for the TB matrix in Eq. 1. This implies that the origin of the reactor
parameter $r$ is due to a violation of the family symmetry that would lead to TB mixing.
Following [20] we may consider the parameters which signal a violation of the TB matrix
element relations. For example, in our convention, we may consider,

$$\Delta_e = \frac{m_{e\mu} + m_{e\tau}}{m_{e\mu}}.$$  \hspace{1cm} (21)

In [20] the parameter $\Delta_e$ was shown to suffer very large discrepancies from zero due to a
pole at $m_{e\mu} = 0$. The origin of this pole is apparent from the second line of Eq. 20 where
it is clear that cancellations can occur in the case of a normal hierarchy, for example,
where the correction term of order $rm_3$ can compete with the TB term of order $m_2$ for
$r$ of order $m_2/m_3$. From Eq. 18 it is clear that the component matrix $C_3$ is responsible
for this effect since $C_3^{TB}$ has zeroes in the first row and column, and thus technically
any non-zero value of $r$ will provide an infinite correction to the symmetry prediction of
these elements of the $C_3$ matrix. Thus, as emphasised in [20], this may open the door
to alternative approaches to neutrino mixing which violate Klein symmetry, especially
if $r$ is not much smaller than unity.

We shall see later that the origin of these zeroes in indirect or accidental models is
due to a flavon aligned along the third column of the TB matrix which has a zero in
the first entry. If this zero is filled, corresponding to a violation of Klein symmetry,
then this switches on a reactor angle, while preserving the tri-bimaximal predictions
for the solar and atmospheric angles, corresponding to the example discussed in this
subsection. This was first discussed in [19] where it was referred to as tri-bimaximal
reactor mixing. The symmetry approach should not be abandoned since it provides an
excellent approximation to and understanding of the observed near TB mixing which is
so far lacking in alternative approaches. On the contrary, the analyses [19, 20] seem to
motivate indirect family symmetry models with the accidental emergence of the Klein
symmetry, and indicate that tri-bimaximal mixing may be insensitive to relatively large
vacuum misalignment.

3 The see-saw mechanism

3.1 Form Dominance

We now show how TB mixing can arise at leading order from see-saw models based
on form dominance (FD) [14] which includes most symmetry based models. To set the
notation, recall that, in the type I see-saw mechanism [21], the starting point is a heavy
right-handed Majorana neutrino mass matrix $M_{RR}$ and a Dirac neutrino mass matrix
(in the left-right convention) $M_D$, with the light effective left-handed Majorana neutrino
mass matrix $M'\nu$ given by the type I see-saw formula \[21\],
\[
M'\nu = M_D M^{-1}_{RR} M_D^T.
\]

In a basis in which $M_{RR}$ is diagonal with real and positive eigenvalues $M_i$, we may write,
\[
M_{RR} = \text{diag}(M_1, M_2, M_3)
\]
and $M_D$ may be written in terms of three general column vectors $m_{D1}, m_{D2}, m_{D3}$,
\[
M_D = (m_{D1}, m_{D2}, m_{D3}).
\]

The see-saw formula then gives,
\[
M'\nu = \frac{m_{D1} m_{D1}^T}{M_1} + \frac{m_{D2} m_{D2}^T}{M_2} + \frac{m_{D3} m_{D3}^T}{M_3}.
\]

As first observed in \[7,14\] $M'_{TB}$ may be achieved if the columns of the Dirac mass matrix are aligned along the columns of the TB mixing matrix, $U^{TB} = (\Phi^{TB}_1, \Phi^{TB}_2, \Phi^{TB}_3)$,
\[
m_{D1}^{TB} = a_1 \Phi^{TB}_1, \quad m_{D2}^{TB} = a_2 \Phi^{TB}_2, \quad m_{D3}^{TB} = a_3 \Phi^{TB}_3,
\]
where $a_i$ are three complex constants.

Using Eq.26 we see that this leads to,
\[
M'_{TB} = \frac{a_1^2}{M_1} \Phi^{TB}_1 \Phi^{TB}_1^T + \frac{a_2^2}{M_2} \Phi^{TB}_2 \Phi^{TB}_2^T + \frac{a_3^2}{M_3} \Phi^{TB}_3 \Phi^{TB}_3^T,
\]
diagonalized using Eq.3 with $U = U^{TB}$ and $P_E = I$ (i.e. zero phases) leading to complex neutrino mass eigenvalues given by $m_1 = a_1^2/M_1$, $m_2 = a_2^2/M_2$, $m_3 = a_3^2/M_3$. This mechanism allows a completely general neutrino mass spectrum and, since $M'_{TB}$ is form diagonalizable (i.e. the mixing angles are independent of the neutrino masses), it is referred to as form dominance (FD) \[14\]. It is interesting to compare FD to Constrained Sequential Dominance (CSD) defined in \[7\]. In CSD a strong hierarchy $|m_1| \ll |m_2| < |m_3|$ is assumed which enables $m_1$ to be effectively ignored (typically this is achieved by taking $M_A$ to be very heavy leading to a very light $m_1$) then CSD is defined by only assuming the second and third conditions in Eq.26 \[7\]. Thus CSD is seen to be just a special case of FD corresponding to a strong neutrino mass hierarchy. FD on the other hand is more general and allows any choice of neutrino masses including a mild hierarchy, an inverted hierarchy or a quasi-degenerate mass pattern.

In the case of direct symmetry models, for example those in \[13\], in the diagonal right-handed neutrino mass basis, each column vector in Eq.26 corresponds to a linear combination of flavon VEVs, which requires some mild tuning in order to achieve a mild neutrino mass hierarchy. To eliminate such tuning one may consider the case that each
column vector in Eq.26 arises from a separate flavon VEV, and this possibility, called natural FD \[14\], is realised in the classes of indirect symmetry models. For example, if \( m_1 \ll m_2 < m_3 \) then the precise form of \( m_{D1} \) becomes irrelevant, and in this case FD reduces to constrained sequential dominance (CSD) \[7\]. The CSD mechanism has been applied in this case to the class of indirect models with Natural FD based on the family symmetries \( SO(3) \ [7,9] \) and \( SU(3) \ [8] \), and their discrete subgroups \[10\]. The results here will be most useful for the indirect models which are naturally expressed in the diagonal right-handed neutrino mass basis, although the direct models may also be rotated to this basis \[14\].

### 3.2 Deviations from tri-bimaximal mixing on the see-saw

In models based on family symmetry, we have seen that the Dirac mass matrix takes a very special form in the diagonal right-handed neutrino (and charged lepton) mass basis, namely its columns are proportional to the columns of the TB mixing matrix \( U_{TB} \), as in Eq.26. This observation is known as FD, since it implies a form diagonalizable neutrino mass matrix. Now we want to consider the effect of deviations \( \Delta m_{Di} \), given by,

\[
m_{Di} = m_{Di}^{TB} + \Delta m_{Di},
\]

and study the resulting deviations from TB mixing corresponding to the mixing matrix being changed to \( U = (\Phi_1, \Phi_2, \Phi_3) \), where as in Eq.12

\[
\Phi_i = \Phi_i^{TB} + \Delta \Phi_i.
\]

In this subsection we need determine the linear relation between \( \Delta m_{Di} \) and \( \Delta \Phi_i \). From the symmetry model building point of view, the \( \Delta m_{Di} \) may arise from corrections to vacuum alignment. From this perspective the results in this subsection provide useful relations between TB deviations and vacuum alignment corrections.

Expanding Eqs.25 to lowest order in the Dirac mass matrix perturbations in Eq.28

\[
M' \approx \sum_i \left[ \frac{1}{M_i} [m_{Di}^{TB} + \Delta m_{Di}] \right].
\]

The first observation is that any deviations \( \Delta m_{Di} \propto \Phi_i^{TB} \) will not result in any mixing angle deviations, i.e. \( \Delta \Phi_i = 0 \) since FD is maintained in this case. This suggests expanding \( \Delta m_{Di} \) in the TB basis \( \Phi_i^{TB} \),

\[
\Delta m_{Di} = \sum_j \alpha_{ij} \Phi_j^{TB},
\]

where \( \alpha_{ij} \) are small complex mass parameters, \( |\alpha_{ij}| \ll |a_i| \), for all \( i, j \), where \( a_i \) are defined by Eq.26. Using Eq.26 and Eq.31 in Eq.30

\[
M' \approx \sum_{i,j} \left[ \frac{1}{M_i} [a_i^2 \Phi_i^{TB} + a_i \alpha_{ij} \Phi_j^{TB}] \right].
\]
where $\Phi^T_{ij} B \equiv \Phi^T_{i} B_{j} T_B$. 

In order to extract the TB deviation parameters we compare the perturbed neutrino mass matrix in Eq. (32) to the perturbed neutrino mass matrix in Eq. (13) repeated below, 

$$M' \approx \sum_i m_i [\Phi^T_{i} B_{i} T_B + \Phi^T_{i} \Delta\Phi_{i} T + \Delta\Phi_{i} \Phi^T_{i} T_B].$$ \hspace{1cm} (33)

The general results for complex leading order neutrino masses $m^0_i = a_i^2 / M_i$ are derived in Appendix A with the MNS parameters given in Eq. (68) and Eq. (72) and the neutrino masses in Eq. (69).

In the special case that the leading order neutrino masses are real (due for example to a real vacuum alignment with $a_i$ real) but allowing arbitrary complex vacuum alignment corrections we find from Eq. (68) of Appendix A rather compact expressions:

$$s \approx \sqrt{2} \text{Re} \left( \frac{\delta m^+_2}{m_{21}} \right) \Rightarrow \frac{\sqrt{3}}{m_{32}} = \frac{1}{\sqrt{3}} \text{Re} \left( \frac{\delta m^+_3}{m_{31}} \right)$$

$$a \approx \sqrt{2} \text{Re} \left( \frac{\delta m^+_3}{m_{32}} \right) - \frac{1}{\sqrt{3}} \text{Re} \left( \frac{\delta m^+_3}{m_{31}} \right)$$

$$r e^{i\delta} \approx \sqrt{2} \text{Re} \left( \frac{\delta m^+_3}{m_{32}} \right) + \frac{2}{\sqrt{3}} \text{Re} \left( \frac{\delta m^+_3}{m_{31}} \right)$$ \hspace{1cm} (34)

where we have written,

$$m^0_i = a_i^2 / M_i,$$

$$m^\pm_{ij} = m^0_i \pm m^0_j,$$

$$\delta m^+_{ij} = m^0_i a_{ij} / a_i + m^0_j a_{ji} / a_j.$$ \hspace{1cm} (35)

From Eq (36) the magnitude of the corrected neutrino masses are:

$$|m_i| \approx |m_i^0| \left[ 1 + 2 \text{Re} \left( \frac{\alpha_{ii}}{a_i} \right) \right].$$ \hspace{1cm} (36)

### 3.3 Vacuum misalignment and deviations from TB mixing

In this subsection we shall discuss the application of the results of subsection 3.2 to models based on a family symmetry $G_f$. We shall consider here only an extremely simple example of an indirect model expressed in the diagonal right-handed neutrino mass basis. We emphasise that this example is for illustrative purposes only, and that the results in this paper apply to all models in which TB mixing results from a family...
symmetry. For example the results also apply to the direct family symmetry models based on $A_4$ when rotated to this basis [14]. Consider the see-saw Lagrangian in the diagonal charged lepton basis,

$$\mathcal{L}_{Yuk}^N \sim L_i(y_1\phi_1^Nc_i + y_2\phi_2^Nc_i + y_3\phi_3^Nc_i)H,$$  \hspace{1cm} (37)

$$\mathcal{L}_{Maj}^N \sim M_1N_1^cN_1^c + M_2N_2^cN_2^c + M_3N_3^cN_3^c,$$  \hspace{1cm} (38)

where $y_i$ are Yukawa couplings and these diagonal forms are enforced by additional symmetries. Since the (CP conjugated) right-handed neutrinos are family singlets $N_i^c \sim \mathbf{1}$, the combination of family triplet left-handed leptons $L_i \sim \mathbf{3}$ and flavons $\phi_i \sim \mathbf{3}$ (or $\phi_i \sim \mathbf{3}$ if the representations are complex) must yield a singlet of $G_f$. After the see-saw mechanism takes place, this results in an effective Lagrangian of the form,

$$\mathcal{L}_{Maj} \sim L\left(\frac{\phi_1^T\phi_1}{M_1} + \frac{\phi_2^T\phi_2}{M_2} + \frac{\phi_3^T\phi_3}{M_3}\right)LHH.$$  \hspace{1cm} (39)

Thus we see the appearance of the quadratic combinations of flavons which serve to preserve an accidental neutrino flavour symmetry of the neutrino mass matrix, in the effective Lagrangian after the see-saw mechanism has taken place. This is also an example of “natural FD” since a separate flavon VEV is responsible for each physical neutrino mass.

In matrix notation, when the flavons get their VEVs in the three columns of the Dirac mass matrix $M_D$ are proportional to the VEVs of the three flavons,

$$M_D = (y_1\langle\phi_1\rangle, y_2\langle\phi_2\rangle, y_3\langle\phi_3\rangle) \equiv (m_{D1}, m_{D2}, m_{D3}).$$  \hspace{1cm} (40)

Thus in the indirect family symmetry models each column of the Dirac mass matrix $m_{Di}$ is identified with the VEV of a separate flavon field $\phi_i$, where

$$\langle\phi_i\rangle \propto m_{Di} = m_{Di}^{TB} + \Delta m_{Di}.$$  \hspace{1cm} (41)

Note that, although we have taken a very specific model here for illustrative purposes, a similar procedure may be followed for any model in which a general family symmetry $G_f$ leads to TB mixing. Namely, the general model will have some aligned flavon VEVs which will lead to some Dirac mass matrix and some heavy Majorana mass matrix in the diagonal charged lepton mass basis. The Dirac mass matrix of the model in question must then be rotated to the basis in which the heavy Majorana mass matrix is diagonal. Then the columns of the Dirac mass matrix in that basis may be identified with the columns given in Eq.40. The only difference will be that, in a general model, the

\footnote{Note that these models are formulated in a basis where the family indices are trivially summed over. We emphasise again that this particular model with this choice of matter and representation content is chosen purely for illustrative purposes.}
columns of the Dirac mass matrix will not correspond to a unique flavon, but in general will correspond to a linear combination of flavons. This makes the analysis of vacuum misalignment more complicated to interpret than in the simple example considered here, but notwithstanding this complication, the results may be applied to any such model. The main point to note is that all such models satisfy FD at the leading order \([14]\), which is the crucial requirement for this procedure to be followed.

The leading order vacuum alignment discussed in \([7–12]\) respects FD with,

\[
m_{D1}^{TB} = \frac{a_1}{\sqrt{6}} \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \quad m_{D2}^{TB} = \frac{a_2}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \quad m_{D3}^{TB} = \frac{a_3}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.
\]

(42)

The corrections to the leading order vacuum alignment can be expressed as in Eq.31,

\[
\Delta m_{D1} \approx \frac{\alpha_{11}}{\sqrt{6}} \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + \frac{\alpha_{12}}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + \frac{\alpha_{13}}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}
\]

\[
\Delta m_{D2} \approx \frac{\alpha_{21}}{\sqrt{6}} \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + \frac{\alpha_{22}}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + \frac{\alpha_{23}}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}
\]

\[
\Delta m_{D3} \approx \frac{\alpha_{31}}{\sqrt{6}} \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + \frac{\alpha_{32}}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + \frac{\alpha_{33}}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.
\]

(43)

The above discussion shows how indirect family symmetry models lead to natural FD at leading order, since each neutrino mass eigenvalue \(m_i\) is associated with a particular flavon field \(\phi_i\), so no cancellations of flavon VEVs are required to generate a particular neutrino mass. In such models the neutrino masses are free parameters and not predicted by the theory. In the following we consider the case of a hierarchical neutrino mass spectrum \(|m_1| \ll |m_2| < |m_3|\) where, since the flavon \(\phi_i\) associated with the neutrino mass \(m_i\), it is clear that the flavon \(\phi_1\) is irrelevant and may be ignored. This then reduces to the example of leading order CSD where the dominant flavon \(\phi_3\) is responsible for the atmospheric neutrino mass and mixing angle and the subdominant flavon \(\phi_2\) is responsible for the solar neutrino mass and mixing angle. Including vacuum misalignment, these flavons have VEVs from Eqs.31,32,33 as follows,

\[
\langle \phi_2 \rangle \propto a_2 \sqrt{3} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + \frac{\alpha_{21}}{\sqrt{6}} \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + \frac{\alpha_{22}}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + \frac{\alpha_{23}}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}
\]

\[
\langle \phi_3 \rangle \propto a_3 \sqrt{2} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \frac{\alpha_{31}}{\sqrt{6}} \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + \frac{\alpha_{32}}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + \frac{\alpha_{33}}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix},
\]

(44)
where $\alpha_{ij}$ and $a_i$ are complex in general with $|\alpha_{ij}| \ll |a_i|$ so the leading order vacuum alignments are given by the first term of the right-hand sides, familiar from CSD models \[7-12\]. The remaining terms parametrize the vacuum misalignment.

For the case of hierarchical neutrino masses (allowing complex $\alpha_{ij}$ and $a_i$) the TB mixing deviations parameters are given from Eq.75 of Appendix A:

$$s \approx \sqrt{2} \text{Re} \left( \frac{\alpha_{21}}{a_2} \right),$$

$$a \approx \sqrt{\frac{2}{3}} \text{Re} \left( \frac{\alpha_{32}}{a_3} - \frac{1}{\sqrt{2}} \frac{\alpha_{31}}{a_3} \right),$$

$$r e^{-i\delta} \approx \sqrt{\frac{2}{3}} \left( \frac{\alpha_{32}}{a_3} + \sqrt{2} \frac{\alpha_{31}}{a_3} \right).$$

\[45\]

### 3.3.1 Preserving the TB solar prediction

The first observation is that the solar angle deviation parameter $s$ in Eq.15 is only sensitive to $\phi_2$ vacuum misalignments in Eq.44 and in particular only those corrections proportional to $\alpha_{21}$. The solar angle does not care about any $\phi_3$ vacuum misalignments. Thus the prediction tri-maximal prediction $\sin \theta_{12} = 1/\sqrt{3}$ corresponding to $s = 0$ can be maintained in the presence of any vacuum alignment corrections such that $\alpha_{21} = 0$. Thus any vacuum alignment correction orthogonal to $\Phi_T^1$ will preserve the TB prediction for the solar angle ($s = 0$). An example of such an alignment is:

$$\langle \phi_2^{s=0} \rangle \propto \frac{a_2}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + \frac{\alpha_{23}}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix},$$

\[46\]

corresponding to $\alpha_{21} = 0$ and in addition the optional condition $\alpha_{22} = 0$, chosen to make the misalignment have a simple form. In these models the tri-bimaximal prediction for the solar angle is therefore relatively robust in the presence of vacuum alignment corrections, and indeed there is some experimental support for this observation in Eq.15. If Eq.46 is the only vacuum alignment correction then the atmospheric and reactor angles are also unchanged and so TB mixing will be preserved with $r = s = a = 0$.

### 3.3.2 Preserving the TB atmospheric and reactor predictions

The second observation is that the atmospheric and reactor tri-bimaximal deviation parameters in Eq.15 are only sensitive to $\phi_3$ vacuum misalignments in Eq.44 and do not care about $\phi_2$ vacuum misalignments. Note that the atmospheric and reactor tri-bimaximal deviation parameters in Eq.15 do not depend on the parameter $\alpha_{33}$ and hence they are insensitive to $\phi_3$ corrections proportional to the leading order alignment $\Phi_T^3$,
as expected. Since the atmospheric and reactor tri-bimaximal deviation parameters in Eq.45 depend on different linear combinations of $\alpha_{32}$ and $\alpha_{31}$ one can envisage corrections for which either $a = 0$ or $r = 0$, as we now discuss.

(i) The case $a = 0$ can be achieved for $\alpha_{32} = \alpha_{31}/\sqrt{2}$, corresponding to the vacuum misalignment,

$$\langle \phi_3^{a=0} \rangle \propto \frac{a_2 + \alpha_{33}}{\sqrt{2}} \begin{pmatrix} r e^{-i\delta} \\ 1 \\ 1 \end{pmatrix}, \quad (47)$$

where we have used $r e^{-i\delta} = \sqrt{6}\alpha_{32}/a_3$ from Eq.45. If in addition $\alpha_{21} = 0$, then only the reactor angle and CP phase $r e^{-i\delta}$ are non-zero, and the tri-bimaximal predictions for the solar and atmospheric angles are both preserved ($s = a = 0$). This was called tri-bimaximal reactor (TBR) mixing in [19], where it was assumed that the only vacuum misalignment was due to Eq.47. Here we see that additional misalignments such as in Eq.46 are also consistent with TBR mixing, which is a new result. We emphasise that the vacuum misalignment in Eq.47 corresponds to a violation of Klein symmetry at the leading order, since the first component of $\langle \phi_3^{TBR} \rangle$ is zero. This example corresponds to a leading order vacuum misalignment rather than a correction to a vacuum alignment, as discussed in [19]. This is related to the observations in [20], as discussed earlier.

(ii) The case of zero reactor angle $r = 0$ can be achieved for $\alpha_{32} = -\sqrt{2}\alpha_{31}$, corresponding to the vacuum misalignment,

$$\langle \phi_3^{r=0} \rangle \propto \frac{a_3 + \alpha_{33}}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \sqrt{\frac{3}{2}} \alpha_{32} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}. \quad (48)$$

If in addition $\alpha_{21} = 0$, then only the atmospheric angle will deviate and the tri-bimaximal predictions for the solar and reactor angles are both preserved ($s = r = 0$).

### 3.3.3 Tri-maximal mixing

One may arrange for the vacuum alignment corrections to lead to the MNS matrix taking the special forms as proposed in the literature (see e.g. [36] are references therein). For example, tri-maximal mixing [37], in which the second column of the TB mixing matrix is preserved, corresponds to $s = 0, a = -\frac{1}{2}r \cos \delta$. From Eq.45 this be achieved for misalignments with $\alpha_{21} = 0$ (giving $s = 0$) and $\alpha_{32} = 0$ (giving $a = -\frac{1}{2}r \cos \delta$). An example of a $\phi_3$ misalignment with $\alpha_{32} = 0$ is,

$$\langle \phi_3^{\text{trimax}} \rangle \propto \frac{a_3}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \sqrt{\frac{2}{3}} \alpha_{31} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad (49)$$

where we have set $\alpha_{33}/\sqrt{2} = -\alpha_{31}/\sqrt{2}$ in order to lead to a simple looking misalignment.
3.4 Vacuum misalignment and departures from FD

We have already remarked that violation of FD is welcome since, in the exact FD limit, corresponding to a real $R$ matrix, leptogenesis asymmetries vanish identically [26]. From this perspective, vacuum misalignment is to be welcomed. However it is not clear that vacuum misalignment will lead to violation of FD, even though it leads to deviations from TB mixing. As emphasised in [14], FD corresponds to the columns of the Dirac mass matrix $M_D = (m_{D1}, m_{D2}, m_{D3})$ being proportional to the columns of the general MNS matrix $U = (\Phi_1, \Phi_2, \Phi_3)$. In family symmetry models the MNS matrix is identified with the TB mixing matrix and the columns of the Dirac mass matrix then take simple TB forms which are identified with simple flavon vacuum alignments as discussed in the previous subsection. Vacuum misalignment will induce departures from the simple Dirac TB forms, resulting in $U$ deviating from $U^{TB}$. However vacuum misalignment will not necessarily induce departures from FD. The point is that the columns of the corrected Dirac mass matrix may still in principle be proportional to the columns of the corrected mixing matrix, in which case FD would not be violated and the leptogenesis asymmetries would remain zero even in the presence of vacuum misalignment.

To investigate this question we recall that FD may be expressed in the language of the orthogonal $R$ matrix [25] where for exact FD the $R$ matrix is a real matrix. Departures from FD are then signalled by departures of the $R$ matrix from the real matrix. In a suitable convention, we can expand the $R$ matrix in a small angle approximation about the real matrix, and these small angles will be related to vacuum misalignment. For many phenomenological applications it is convenient to perform numerical scans over the Dirac mass matrix parametrized in terms of the orthogonal $R$ matrix, thus it is useful in any case to be able to have a dictionary between vacuum misalignment and the $R$ matrix, expanded to leading order in terms of small $R$ matrix angles.

We begin by recalling the derivation of the $R$ matrix in the diagonal charged lepton and right-handed neutrino mass basis [25]. From Eqs.322, one obtains,

$$P_{\text{Maj}}^*P_E M_D D_{\text{M}}^{-1} M^T_D P_E U^* P_{\text{Maj}}^* = D_k,$$

(50)

where $D_k, D_{\text{M}}$ are diagonal matrices of positive neutrino mass eigenvalues,

$$D_k = \text{diag}(|m_1|, |m_2|, |m_3|), \quad D_{\text{M}} = \text{diag}(M_1, M_2, M_3).$$

(51)

The $R$ matrix is then defined as,

$$R = D_{\sqrt{\text{M}}}^{-1} M_D^T P_E U^* P_{\text{Maj}}^* D_{\sqrt{\text{M}}}^{-1},$$

(52)

where from Eq.50 we see that $R$ is a complex orthogonal matrix $R^T R = I$.

From Eq. (52) we can write,

$$P_E M_D D_{\sqrt{\text{M}}}^{-1} = U P_{\text{Maj}} D_{\sqrt{\text{M}}} R^T,$$

(53)
which shows that the $R$ matrix serves to parametrize $P_E M_D$, for fixed values of $U_{MNS} = U P_{Maj}, D_k$ and $D_M$. It is instructive to expand this equation in terms of the columns of $M_D$ and $U$,

$$P_E ((M_D)_{i1} M_1^{-1/2}, (M_D)_{i2} M_2^{-1/2}, (M_D)_{i3} M_3^{-1/2}) = (U_{11} m_{1}^{1/2}, U_{12} m_{2}^{1/2}, U_{13} m_{3}^{1/2}) R^T, \quad (54)$$

reverting again to complex neutrino masses $m_i$. In the case of FD, where the columns of $M_D$ are proportional to the columns of $U$, it is apparent that the orthogonal $R$ matrix is equal to permutations of the unit matrix with $P_E = I$. In the convention where the right-handed neutrino of mass $M_i$ is associated with the physical neutrino of mass $m_i$ in the FD limit we can write $R = I$ [20]. In this convention, deviations from FD are then parametrized by a small $R$ matrix angle expansion. In the standard convention where the right-handed neutrinos are ordered according to mass $M_1 < M_2 < M_3$ then this may require a trivial re-ordering of the columns of the Dirac mass matrix and hence the rows of the $R$ matrix, as is clear from Eq.(54) (see also [38]).

Taking the transpose of Eq.(54) we can rewrite this equation in the column vector notation, where $M_D = (m_D1, m_D2, m_D3)$ and $U = (\Phi_1, \Phi_2, \Phi_3),

$$m_{Di} M_i^{-1/2} = \sum_k R_{ik} m_k^{1/2} \Phi_k P_E^\dagger. \quad (55)$$

It is clear from Eq.(55) that the $R$ matrix parametrizes an expansion of the columns of the Dirac mass matrix in the basis of the columns of the *perturbed* mixing matrix $U$. This is different from our previous approach which was based on an expansion of the columns of the Dirac mass matrix in the basis of the columns of the *unperturbed* TB mixing matrix $U_{TB}$. Eq.(55) shows that violations of FD are related to the non-orthogonality of the Dirac columns, since from this equation,

$$(m_{Dj}^\dagger m_{Di}) M_i^{-1/2} M_j^{-1/2} = \sum_k R_{ik} |m_k| R_{jk}^*, \quad (56)$$

where the mixing matrices vanish by unitarity. Eq.(56) shows that the Dirac columns are orthogonal when the $R$ matrix is diagonal, and so off-diagonal elements of $R_{ij}$ are associated with non-orthogonality of the Dirac columns. The since Dirac columns are orthogonal in the FD limit, we again see that the violations of FD may thus be parameterised in terms of a small angle expansion of the $R$ matrix, where Eq.(56) may be used as the starting point for such an expansion.

The $R$ matrix is a complex orthogonal $3 \times 3$ matrix which can be parameterized in terms of three complex angles $\alpha_{ij}$ as $R = R_1 R_2 R_3$ where $R_i$ take the form

$$R_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix}, \quad R_2 = \begin{pmatrix} c_{13} & 0 & s_{13} \\ 0 & 1 & 0 \\ -s_{13} & 0 & c_{13} \end{pmatrix}, \quad R_3 = \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (57)$$
where \( s_{ij} = \sin z_{ij} \approx z_{ij} \), \( c_{ij} = \cos z_{ij} \approx 1 \) in the small complex angle approximation.

In the small angle approximation, we find the following elements of Eq.\( \ref{eq:56} \)

\[
(m_{D2}^* m_{D1}) M_2^{-1/2} M_1^{-1/2} \approx |m_2| z_{12} - |m_1| z_{12}^* + |m_3| z_{23}^* z_{13} \\
(m_{D3}^* m_{D1}) M_3^{-1/2} M_1^{-1/2} \approx |m_3| z_{13} - |m_1| z_{13}^* - |m_2| z_{23}^* z_{12} \\
(m_{D3}^* m_{D2}) M_3^{-1/2} M_2^{-1/2} \approx |m_3| z_{23} - |m_2| z_{23}^* + |m_1| z_{13}^* z_{12}. \tag{58}
\]

Expanding the Dirac columns in the TB basis, as in Eqs.\( \ref{eq:41} \) to first order, we may evaluate the Dirac matrix elements which appear in Eq.\( \ref{eq:58} \) to first order,

\[
(m_{Dj}^* m_{Di}) \approx a_j^* a_i \delta_{ji} + \alpha_j^* a_i + a_j^* \alpha_{ij}. \tag{59}
\]

Eq.\( \ref{eq:58} \) can then be solved to find \( R \) matrix complex angles \( z_{ij} \). For example, in the case of a hierarchical neutrino mass spectrum \(|m_1| \ll |m_2| < |m_3|\), Eq.\( \ref{eq:58} \) may be solved to leading order in the \( R \) matrix angles,

\[
z_{12} \approx (\alpha_{21} a_1 + a_2^* \alpha_{12}) M_2^{-1/2} M_1^{-1/2} |m_2|^{-1} \\
z_{13} \approx (\alpha_{31} a_1 + a_3^* \alpha_{13}) M_2^{-1/2} M_1^{-1/2} |m_3|^{-1} \\
z_{23} \approx (\alpha_{32} a_2 + a_3^* \alpha_{23}) M_3^{-1/2} M_2^{-1/2} |m_3|^{-1}, \tag{60}
\]

where the neutrino masses \(|m_i|\) are given in Eq.\( \ref{eq:56} \).

Eq.\( \ref{eq:56} \) shows, as expected, that only the off-diagonal vacuum alignment corrections \( \alpha_{ij} \) with \( i \neq j \) will lead to non-zero \( R \) matrix angles and hence violation of FD. From Eq.\( \ref{eq:45} \), it is seen that, to first order, only \( \alpha_{21} \) affects the solar angle deviation from TB mixing, and only \( \alpha_{31} \) and \( \alpha_{32} \) affect the atmospheric and reactor deviations from TB mixing. Thus it is possible to have vacuum misalignments which maintain TB mixing, as discussed in the previous subsection, but which lead to violations of FD due for example to \( \alpha_{12}, \alpha_{13} \) and \( \alpha_{23} \) being non-zero, allowing successful leptogenesis. Alternatively, vacuum misalignment can in principle lead to deviations from TB mixing with \( r, s, a \neq 0 \) while maintaining FD with \( z_{ij} \approx 0 \) due to approximate cancellations in Eq.\( \ref{eq:56} \). \( \alpha_{ij}^* a_i + a_{ij}^* \alpha_{ij} \approx 0 \). Clearly having a vacuum misalignment which gives deviations from TB mixing is not sufficient to guarantee violation of FD and hence successful leptogenesis.

Finally note that, as seen in Eq.\( \ref{eq:11} \), the flavon \( \phi_1 \), associated with the right-handed neutrino of mass \( M_1 \), decouples from the seesaw mechanism in the limit \( m_1 \to 0 \), meaning that the TB deviations are independent of the alignment of this flavon. However, since \( m_1 \approx a_1^2/M_1 \), this decoupling may be due to either \( a_1 \to 0 \) or \( M_1 \to \infty \). If \( M_1 \to \infty \) then Eq.\( \ref{eq:56} \) shows that \( z_{12}, z_{13} \to 0 \), which is the two right-handed neutrino limit. However, if \( a_1 \to 0 \), with \( M_1 \) fixed, then Eq.\( \ref{eq:56} \) shows that \( z_{12}, z_{13} \) remain non-zero in addition to \( z_{23} \). In this case we can have violations of FD due to vacuum misalignment of the flavon \( \phi_1 \) which is irrelevant for the seesaw mechanism. This can lead to the \( R \) matrix angles \( z_{12}, z_{13} \) being significantly different from zero, while maintaining accurate TB mixing, allowing successful leptogenesis in the framework of CSD models [26].
4 Summary and Conclusions

TB neutrino mixing may arise from see-saw models based on family symmetry which is spontaneously broken by flavons with particular vacuum alignments. However in practice some degree of vacuum misalignment is always present in realistic models. In this paper we have derived analytic results which relate such general vacuum misalignment to deviations in TB mixing and FD. Since the method here only involves inspecting the Dirac mass matrix, the results have very general applicability and may be applied to all direct or indirect family symmetry models, including the effects of higher order operators. However, while the results are readily applicable for indirect models, the Dirac mass matrix in the direct models needs to be rotated to the diagonal charged lepton and right-handed neutrino mass basis. For example, even if the direct $A_4$ models are formulated in the diagonal charged lepton basis, the right-handed neutrino mass matrix still needs to be diagonalised and the Dirac mass matrix correctly identified in this basis before the results in this paper can be applied.

The results have important physical implications regarding neutrino oscillation experiments and leptogenesis. Future precision neutrino oscillation experiments will be sensitive to deviations from TB mixing and the analytic results presented here enable such deviations to be related to vacuum misalignment in realistic models. In simple cases we show that certain patterns of vacuum misalignment can preserve TB mixing in full or in part with one or more of the TB deviation parameters $r, s, a$ being zero, or can lead to tri-maximal mixing where the second columns of the TB matrix is preserved. The physical relevance of the results to leptogenesis is also clear, since the lepton asymmetries vanish exactly in the FD limit where it would correspond to a real $R$ matrix. The analytical expressions in Eq.60 for the complex corrections to the real $R$ matrix in terms of the vacuum misalignment are therefore physically relevant since they allow for non-zero leptogenesis.

In conclusion, for the classes of family symmetry models studied in the stated approximations, the analytic results in this paper provide useful insight into the effects of vacuum misalignment on deviations from TB mixing and FD. The analytic results clearly show how the corrections to TB mixing and FD depend on the pattern of the vacuum misalignment, with the two effects being uncorrelated.

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Appendix

A  Effect of vacuum misalignment on TB mixing including phases

In this appendix we shall give a full derivation of the results relating the deviations from TB mixing due to vacuum misalignment, including a careful treatment of the phases. The starting point of the derivation is the comparison of the perturbed neutrino mass matrix in Eq. 32 to the one in Eq. 33. However, before they can be compared, one must take account of the extra phases present in the most general effective neutrino mass matrix $\tilde{M}^{\nu}$ (which contains six independent phases) as compared to $\tilde{M}^{\nu}$ (which only contains four phases). They are related by $\tilde{M}^{\nu} = P_{E} M^{\nu} P_{E}$. In the FD limit we saw that $P_{E} = I$, and close to this limit the phases will be small so that the diagonal phase matrix is approximately equal to the unit matrix. This implies that, to leading order, Eq. 32 can be written as

$$\tilde{M}^{\nu} \approx \sum_{i,j} \frac{1}{M_{i}} [a_{i}^{2} P_{E} \Phi_{ii}^{TB} P_{E} + a_{i} \alpha_{ij} (\Phi_{ij}^{TB} + \Phi_{ji}^{TB})],$$  

(61)

where the small phases in $P_{E}$ only modify the leading order terms. Expanding the phase matrix to first order in the small phases,

$$P_{E} = \begin{pmatrix} e^{i \delta_{1}} & 0 & 0 \\ 0 & e^{i \delta_{2}} & 0 \\ 0 & 0 & e^{i \delta_{3}} \end{pmatrix} \approx I + i \begin{pmatrix} \delta_{1} & 0 & 0 \\ 0 & \delta_{2} & 0 \\ 0 & 0 & \delta_{3} \end{pmatrix}.$$  

(62)

Eq. 61 can be written to first order as,

$$\tilde{M}^{\nu} \approx \frac{1}{M_{1}} \left[ \left( a_{1}^{2} [1 + \frac{i}{3} (4 \delta_{1} + \delta_{2} + \delta_{3})] + 2a_{1} \alpha_{11} \right) \Phi_{11}^{TB} \right] + \frac{1}{M_{1}} \left[ \left( a_{1} \alpha_{12} + a_{1}^{2} \frac{i}{\sqrt{18}} (2 \delta_{1} - \delta_{2} - \delta_{3}) \right) (\Phi_{12}^{TB} + \Phi_{21}^{TB}) \right] + \frac{1}{M_{1}} \left[ \left( a_{1} \alpha_{13} + a_{1}^{2} \frac{i}{\sqrt{12}} (\delta_{3} - \delta_{2}) \right) (\Phi_{13}^{TB} + \Phi_{31}^{TB}) \right] + \frac{1}{M_{2}} \left[ \left( a_{2}^{2} [1 + \frac{2i}{3} (\delta_{1} + \delta_{2} + \delta_{3})] + 2a_{2} \alpha_{22} \right) \Phi_{22}^{TB} \right] + \frac{1}{M_{2}} \left[ \left( a_{2} \alpha_{21} + a_{2}^{2} \frac{i}{\sqrt{18}} (2 \delta_{1} - \delta_{2} - \delta_{3}) \right) (\Phi_{12}^{TB} + \Phi_{21}^{TB}) \right] + \frac{1}{M_{2}} \left[ \left( a_{2} \alpha_{23} + a_{2}^{2} \frac{i}{\sqrt{6}} (\delta_{2} - \delta_{3}) \right) (\Phi_{23}^{TB} + \Phi_{32}^{TB}) \right]$$

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\[ + \frac{1}{M_3} \left[ \left( a_3 \alpha_{31} + a_3^2 \frac{i}{\sqrt{12}}(\delta_3 - \delta_2) \right) (\Phi_{13}^{TB} + \Phi_{31}^{TB}) \right] \]

\[ + \frac{1}{M_3} \left[ \left( a_3 \alpha_{32} + a_3^2 \frac{i}{\sqrt{6}}(\delta_2 - \delta_3) \right) (\Phi_{23}^{TB} + \Phi_{32}^{TB}) \right], \tag{63} \]

where \( \Phi_{ij}^{TB} \equiv \Phi_i^{TB} \Phi_j^{TB}^T \).

The TB deviation columns \( \Delta \Phi \) in Eq.17 may also be expanded in the TB basis \( \Phi_i^{TB} \),

\[ \Delta \Phi_1 = -\frac{s}{\sqrt{2}} \Phi_2^{TB} + \frac{1}{\sqrt{3}}(a - re^{i\delta})\Phi_3^{TB}, \]

\[ \Delta \Phi_2 = \frac{s}{\sqrt{2}} \Phi_1^{TB} - \sqrt{\frac{2}{3}}(a + \frac{1}{2}re^{i\delta})\Phi_3^{TB}, \]

\[ \Delta \Phi_3 = \frac{1}{\sqrt{6}}(a + \frac{1}{2}re^{-i\delta})\Phi_2^{TB} - \frac{1}{\sqrt{3}}(a - re^{-i\delta})\Phi_1^{TB}. \tag{64} \]

Inserting Eq.64 in Eq.33

\[ \tilde{M}^{\nu} \approx m_1[\Phi_{11}^{TB} - \frac{s}{\sqrt{2}}(\Phi_{12}^{TB} + \Phi_{21}^{TB}) + \frac{1}{\sqrt{3}}(a - re^{i\delta})(\Phi_{13}^{TB} + \Phi_{31}^{TB})] \]

\[ + m_2[\Phi_{22}^{TB} + \frac{s}{\sqrt{2}}(\Phi_{12}^{TB} + \Phi_{21}^{TB}) - \sqrt{\frac{2}{3}}(a + \frac{1}{2}re^{i\delta})(\Phi_{23}^{TB} + \Phi_{32}^{TB})] \tag{65} \]

\[ + m_3[\Phi_{33}^{TB} + \sqrt{\frac{2}{3}}(a + \frac{1}{2}re^{-i\delta})(\Phi_{23}^{TB} + \Phi_{32}^{TB}) - \frac{1}{\sqrt{3}}(a - re^{-i\delta})(\Phi_{13}^{TB} + \Phi_{31}^{TB})], \]

where \( \Phi_{ij}^{TB} \equiv \Phi_i^{TB} \Phi_j^{TB}^T \).

Comparing the coefficients of \( \Phi_{ij}^{TB} \equiv \Phi_i^{TB} \Phi_j^{TB}^T \) in Eq.63 to those in Eq.65, we find the following relations to first order in the small dimensionless quantities \( r, s, a, \alpha_{ij}/a, \delta_i, \)

\[ m_1 \approx m_1^0 \left[ 1 + 2\frac{\alpha_{11}}{a_1} + \frac{i}{3}(4\delta_1 + \delta_2 + \delta_3) \right], \]

\[ m_2 \approx m_2^0 \left[ 1 + 2\frac{\alpha_{22}}{a_2} + \frac{2i}{3}(\delta_1 + \delta_2 + \delta_3) \right], \]

\[ m_3 \approx m_3^0 \left[ 1 + 2\frac{\alpha_{33}}{a_3} + i(\delta_2 + \delta_3) \right], \]

\[ \frac{s}{\sqrt{2}}m_{21} \approx \frac{im_{12}^+}{\sqrt{18}}(2\delta_1 - \delta_2 - \delta_3) + \delta m_{12}^+, \]

\[ \frac{m_{21}^0}{\sqrt{3}}(a - re^{i\delta}) - \frac{m_{21}^0}{\sqrt{3}}(a - re^{-i\delta}) \approx \frac{im_{13}^+}{\sqrt{12}}(3\delta_3 - \delta_2) + \delta m_{13}^+, \]

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\[ m^0_3 \sqrt{\frac{2}{3}}(a + \frac{1}{2}e^{-i\delta}) - m^0_2 \sqrt{\frac{2}{3}}(a + \frac{1}{2}e^{i\delta}) \approx \frac{im^+_{21}}{\sqrt{6}}(\delta_2 - \delta_3) + \delta m^+_{23}, \]  

(66)

where we have written,

\[
\begin{align*}
m^0_i &= \frac{a_i^2}{M_i}, \\
m^\pm_{ij} &= m^0_i \pm m^0_j, \\
\delta m^+_{ij} &= \frac{m^0_i \alpha_{ij}}{a_i} + \frac{m^0_j \alpha_{ji}}{a_j}. 
\end{align*}
\]

(67)

Eqs. 66 may be solved for the three complex neutrino mass eigenvalues \( m_i = |m_i|e^{i\phi} \), together with the three real mixing angle deviations \( r, s, a \) plus the Dirac oscillation phase \( \delta \), in terms of the underlying see-saw parameters consisting of the three real positive heavy right-handed Majorana masses \( M_i \), the three complex leading order Dirac masses \( a_i \), and the nine small complex Dirac masses \( \alpha_{ij} \). The unphysical phases \( \delta_i \) are fixed by the conditions that \( r, s, a \) are real. From Eq. 66 we find the results:

\[
\begin{align*}
s &\approx \sqrt{2} \text{Re} \left( \frac{\delta m^+_{21}}{m^0_{21}} \right) + \sqrt{2} \text{Im} \left( \frac{\delta m^+_{21}}{m^0_{21}} \right) \tan \text{arg} \left( \frac{m^0_{21}}{m^0_{21}} \right), \\
a &\approx \sqrt{\frac{2}{3}} \text{Re} \left( \frac{\delta m^+_{32}}{m^0_{32}} \right) - \sqrt{\frac{1}{3}} \text{Re} \left( \frac{\delta m^+_{31}}{m^0_{31}} \right), \\
r \cos \delta &\approx \sqrt{\frac{2}{3}} \text{Re} \left( \frac{\delta m^+_{32}}{m^0_{32}} \right) + \sqrt{\frac{2}{3}} \text{Re} \left( \frac{\delta m^+_{31}}{m^0_{31}} \right), \\
r \sin \delta &\approx -\sqrt{\frac{2}{3}} \text{Im} \left( \frac{\delta m^+_{32}}{m^0_{32}} \right) + \sqrt{\frac{2}{3}} \text{Re} \left( \frac{\delta m^+_{32}}{m^0_{32}} \right) \tan \text{arg} \left( \frac{m^0_{32}}{m^0_{32}} \right) \\
&- \frac{2}{\sqrt{3}} \text{Im} \left( \frac{\delta m^+_{31}}{m^0_{31}} \right) + \frac{2}{\sqrt{3}} \text{Re} \left( \frac{\delta m^+_{31}}{m^0_{31}} \right) \tan \text{arg} \left( \frac{m^0_{31}}{m^0_{31}} \right). 
\end{align*}
\]

(68)

We write the complex neutrino masses as \( m_i = |m_i|e^{i\phi} \), and the lowest order complex masses as \( m^0_i = |m^0_i|e^{i\phi_i} \). The magnitude of the neutrino masses are:

\[
|m_i| \approx |m^0_i| \left[ 1 + 2 \text{Re} \left( \frac{\alpha_{ii}}{a_i} \right) \right].
\]

(69)
and the phases of the neutrino masses are given by:

\[ \phi_1 \approx \phi_1^0 + 2Im \left( \frac{\alpha_{11}}{a_1} \right) + \frac{1}{3} (4\delta_1 + \delta_2 + \delta_3) \]

\[ \phi_2 \approx \phi_2^0 + 2Im \left( \frac{\alpha_{22}}{a_2} \right) + \frac{2}{3} (\delta_1 + \delta_2 + \delta_3) \]

\[ \phi_3 \approx \phi_3^0 + 2Im \left( \frac{\alpha_{33}}{a_3} \right) + (\delta_2 + \delta_3). \] (70)

However only the relative neutrino mass phases \( \phi_i - \phi_j \) are physical (these are the Majorana phases). Only one phase combination appears in the Majorana phases, and this is fixed by the requirement that \( s \) is real, which gives,

\[ - \frac{1}{3} (2\delta_1 - \delta_2 - \delta_3) \approx \sqrt{2} \frac{Im \left( \frac{\delta m_{21}}{m_{21}} \right)}{Re \left( \frac{m_{21}}{m_{21}} \right)}. \] (71)

For example we find the following Majorana phases,

\[ \phi_2 - \phi_1 \approx \phi_2^0 - \phi_1^0 + 2Im \left( \frac{\alpha_{22}}{a_2} \right) - 2Im \left( \frac{\alpha_{11}}{a_1} \right) + \sqrt{2} \frac{Im \left( \frac{\delta m_{21}}{m_{21}} \right)}{Re \left( \frac{m_{21}}{m_{21}} \right)} \]

\[ \phi_3 - \phi_1 \approx \phi_3^0 - \phi_1^0 + 2Im \left( \frac{\alpha_{33}}{a_3} \right) - 2Im \left( \frac{\alpha_{11}}{a_1} \right) + 2\sqrt{2} \frac{Im \left( \frac{\delta m_{21}}{m_{21}} \right)}{Re \left( \frac{m_{21}}{m_{21}} \right)}. \] (72)

The results greatly simplify for the case of hierarchical neutrinos,

\[ |m_1| \ll |m_2| < |m_3|. \] (73)

In this case Eq.66 simplifies to,

\[ s \approx \frac{i}{3} (2\delta_1 - \delta_2 - \delta_3) + \sqrt{2} \frac{\alpha_{21}}{a_2}, \]

\[ a - re^{-i\delta} \approx \frac{i}{2} (\delta_2 - \delta_3) - \sqrt{3} \frac{\alpha_{31}}{a_3}, \]

\[ a + \frac{r}{2} e^{-i\delta} \approx \frac{i}{2} (\delta_2 - \delta_3) - \frac{\sqrt{3}}{2} \frac{\alpha_{32}}{a_3}. \] (74)
From Eq.\text{[74]} we find,
\begin{align*}
s & \approx \sqrt{2} \text{Re}\left(\frac{\alpha_{21}}{a_2}\right), \\
a & \approx \sqrt{\frac{2}{3}} \text{Re}\left(\frac{\alpha_{32}}{a_3} - \frac{1}{\sqrt{2}} \frac{\alpha_{31}}{a_3}\right), \\
re^{-is} & \approx \sqrt{\frac{2}{3}} \left(\frac{\alpha_{32}}{a_3} + \sqrt{2} \frac{\alpha_{31}}{a_3}\right),
\end{align*}
(75)
where the phase combination fixed by the requirement of real \(s\) is,
\begin{equation}
-\frac{1}{3}(2\delta_1 - \delta_2 - \delta_3) \approx \sqrt{2} \text{Im}\left(\frac{\alpha_{21}}{a_2}\right).
\end{equation}
(76)

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