SPACE-TIME FOAM FROM
NON–COMMUTATIVE INSTANTONS

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ABSTRACT

We show that a $U(1)$ instanton on non-commutative $\mathbb{R}^4$ corresponds to a non-singular $U(1)$ gauge field on a commutative Kähler manifold $X$ which is a blowup of $\mathbb{C}^2$ at a finite number of points. This gauge field on $X$ obeys Maxwell’s equations in addition to the susy constraint $F^{0,2} = 0$. For instanton charge $k$ the manifold $X$ can be viewed as a space-time foam with $b_2 \sim k$. A direct connection with integrable systems of Calogero-Moser type is established. We also make some comments on the non-abelian case.
1. Introduction

The moduli space $\mathcal{M}_{k,N}$ of the charge $k$ instantons in the gauge group $U(N)$ shows up in many problems in mathematical physics and more recently in string theory. This space is non-compact, due to the well-known phenomenon of instantons being able to shrink, and there are several celebrated ways of (partially) compactifying this space. One option, motivated by the Uhlenbeck’s theorem concerning the extension of finite action gauge fields to an isolated point, is to add to the space $\mathcal{M}_{k,N}$ the space $\mathcal{M}_{k-1,N} \times X$ which corresponds to a single point-like instanton in the background of the smooth charge $k-1$ instanton. One then further adds the space $\mathcal{M}_{k-2,N} \times \text{Sym}^2 X$ corresponding to the pairs of the point-like instantons, and so on. In this way one obtains the Donaldson compactification:

$$\mathcal{M}^D_{k,N} = \mathcal{M}_{k,N} \cup \mathcal{M}_{k-1,N} \times X \cup \mathcal{M}_{k-2,N} \times \text{Sym}^2 X \ldots \cup \text{Sym}^k X.$$ (1.1)

If the space-time $X$ is a projective surface $S$ with the Kähler form $\omega$ then there is a finer compactification, the space of the torsion free sheaves. This compactification $\mathcal{M}^G_{k,N}$ is the space of all $\omega$-stable torsion free sheaves of rank $N$ and second Chern class $c_2 = k$. In the case $S = \mathbb{CP}^2$ one can study the sheaves which are trivial when restricted to the projective line $\mathbb{CP}^1$ at infinity. This space $\mathcal{M}^G_{k,N}$ has an ADHM-like description. It was shown in [1] that this space parameterises instantons on the non-commutative space $\mathbb{R}^4$ where the degree of the non-commutativity is related to the metric on the space $\mathcal{M}^G_{k,N}$. This deformation of the ADHM equations also arises in the study of integrable systems of Calogero-Moser type [2][3][4]; these same models have appeared in connection with supersymmetric gauge theories [5][6][7][8][9][10][11] and admit a brane description [12][13][14].

An outline of our paper is as follows. In section two we review the physical motivation for our work. Next we will review the deformed ADHM equations we are interested in, paralleling the usual ADHM construction. For a particular choice of complex structure we find the resulting equations describe appropriate holomorphic data. Our aim is to show this actually describes holomorphic bundles on a “blown up” spacetime. We begin (section 4) by focusing attention on the abelian setting which is rather illustrative, and an unexpected richness is found for sufficiently large charge. A direct correspondence with the Calogero-Moser integrable system is established. Section 5 continues with the nonabelian situation. We conclude with a brief discussion.
2. Physical motivation

Consider the theory on a stack of $N$ D3 branes in the Type IIB string theory. Add a collection of $k$ D-instantons and switch on a constant, self-dual $B$-field along the D3-brane worldvolume. The D-instantons cannot escape the D3-branes without breaking supersymmetry [14]. From the point of view of the gauge theory living on the D3-branes, the D-instantons are represented by field configurations with non-trivial instanton charge [15]. Those instantons which shrink to zero size become D-instantons, and such can escape from the D3-brane worldvolume. Therefore, in the presence of the $B$-field, one cannot make the instanton shrink. One realization of this scenario was suggested in [1] where it was proposed to view the D-instantons within the D3-brane with $B$-field as the instantons of a gauge theory on a non-commutative space-time. However, the non-commutative gauge theory as arising in the zero slope limit of the open string theory in a particular regularization can be mapped to the ordinary commutative gauge theory, as shown in [14]. Therefore one is led to the following puzzle in the $N = 1$ case: how is it possible for the $U(1)$ gauge field on $\mathbb{R}^4$ to have a non-trivial instanton charge? It is easy to show that a non-trivial charge is incompatible with the vanishing of $F$ at infinity.

At the same time, one can look at what is happening from the point of view of the D-instantons. Equally, by T-duality one can study the D0-D4 system, and look at the quantum mechanics of D0-branes. The latter has a low-energy target space which coincides with the resolution of the singularities $\overline{M}_{k,N}$ of the instanton moduli space.

One can imagine probing the instanton gauge field as in [16] (perhaps employing further T-dualities). When the $B$-field is turned on the probed gauge field is given by the deformed ADHM construction described below. As we shall see, the resulting gauge fields are singular unless one changes the topology of the space-time.

We suggest that this is what indeed happens. In this way we resolve the paradox with the $U(1)$ gauge fields, since if the space-time contains non-contractible two-spheres (and this is precisely what we shall get) then the $U(1)$ gauge field can have a non-trivial instanton charge. As far as the concrete mechanism for such a topology change within string theory is concerned this will be left to future work.
3. The deformed ADHM construction

From now on we make the change of notation: \( k = v, N = w \). Let \( V \) and \( W \) be hermitian complex vector spaces of dimensions \( v \) and \( w \) respectively. Let \( B_1 \) and \( B_2 \) be the maps from \( V \) to itself, \( I \) be the map from \( W \) to \( V \) and finally let \( J \) be the map from \( V \) to \( W \). We can form a sequence of linear maps

\[
V \xrightarrow{\sigma} V \otimes \mathbb{C}^2 \oplus W \xrightarrow{\tau} V
\]  

(3.1)

where

\[
\sigma = \begin{pmatrix} -B_2 \\ B_1 \\ J \end{pmatrix}, \quad \tau = \begin{pmatrix} B_1 \\ B_2 \\ I \end{pmatrix}.
\]

(3.2)

We will also use

\[
\sigma_z = \begin{pmatrix} -B_2 + z_2 \\ B_1 - z_1 \\ J \end{pmatrix}, \quad \tau_z = \begin{pmatrix} B_1 - z_1 \\ B_2 - z_2 \\ I \end{pmatrix}.
\]

Suppose now that the matrices \((B_1, B_2, I, J)\) obey the following equations:

\[
\tau \sigma = \zeta_c \mathbf{1}_V, \\
\tau \tau^\dagger = \Delta + \zeta_r \mathbf{1}_V, \\
\sigma^\dagger \sigma = \Delta - \zeta_r \mathbf{1}_V.
\]

(3.3)

Let us collect the numbers \((\zeta_r, \text{Re} \zeta_c, \text{Im} \zeta_c)\) into a three-vector \( \vec{\zeta} \in \mathbb{R}^3 \). When \( \vec{\zeta} = 0 \) these equations, together with the injectivity and surjectivity of \( \sigma_z \) and \( \tau_z \) respectively, yield the standard ADHM construction. If one relaxes the injectivity condition then one gets the Donaldson compactification of the instanton moduli space. In the nomenclature of Corrigan and Goddard [17] describing charge \( v \) \( SU(w) \) instantons,

\[
\Delta = \begin{pmatrix} -B_2 & B_1^\dagger \\ B_1 & B_2^\dagger \\ J & I^\dagger \end{pmatrix},
\]

and \( \Delta^\dagger \Delta = \Delta \otimes \mathbf{1}_2 \) corresponds to the equations (3.3) when \( \vec{\zeta} = 0 \). We are considering a deformation of the standard ADHM equations.

The space of all matrices \((B_1, B_2, I, J)\) is a hyperkähler vector space and the equations (3.3) may be interpreted as \( U(k) \) hyperkähler moment maps [18]. In particular by performing an \( SU(2) \) transformation

\[
\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \mapsto \begin{pmatrix} \alpha B_1 - \beta B_2^\dagger \\ \alpha B_2 + \beta B_1^\dagger \end{pmatrix}, \quad \begin{pmatrix} I \\ J \end{pmatrix} \mapsto \begin{pmatrix} \alpha I - \beta J^\dagger \\ \alpha J + \beta I^\dagger \end{pmatrix},
\]

(3.4)
with $|\alpha|^2 + |\beta|^2 = 1$, we can always rotate $\zeta$ into a vector $(\zeta_r, 0, 0)$. Such a transformation corresponds to singling out a particular complex structure on our data, for which $z = (z_1, z_2)$ are the holomorphic coordinates on the Euclidean space-time. Further we may choose the complex structure such that $\zeta_r > 0$.

The moduli space $\tilde{M}_{v,w}$ is the space of solutions to the equations (3.3) up to a symmetry transformation

$$(B_1, B_2, I, J) \mapsto (g^{-1}B_1g, g^{-1}B_2g, g^{-1}I, Jg)$$ (3.5)

for $g \in U(w)$. It is the space of freckled instantons on $\mathbb{R}^4$ in the sense of [19], a “freckle” simply being a point at which $\sigma_z$ fails to be injective. Observe that for $\zeta_r > 0$ (3.3) shows that $\tau_z \tau^+_z$ is invertible and $\tau_z$ is surjective.

One can learn from [20] that the deformed ADHM data parameterise the (semistable) torsion free sheaves on $\mathbb{C} \mathbb{P}^2$ whose restriction on the projective line $\ell_\infty$ at infinity is trivial. Each torsion free sheaf $E$ is included into the exact sequence of sheaves

$$0 \longrightarrow E \longrightarrow F \longrightarrow S_Z \longrightarrow 0$$ (3.6)

where $F$ is a holomorphic bundle $E^{**}$ and $S_Z$ is a skyscraper sheaf supported at points, the set $Z$ of freckles [13]. From this exact sequence one learns that

$$\text{ch}_i(E) = \text{ch}_i(F) - \#Z \delta_{i,2}.$$ (3.7)

### 3.1. Constructing the gauge field

The fundamental object is the solution of

$$\mathcal{D}_z^\dagger \Psi_z = 0, \quad \Psi_z : W \rightarrow V \otimes \mathcal{O}^2 \oplus W$$ (3.8)

where

$$\mathcal{D}_z^\dagger = \begin{pmatrix} \tau_z \\ \sigma^+_z \end{pmatrix}.$$ 

We shall need the components

$$\Psi_z = \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \chi \end{pmatrix} = \begin{pmatrix} \varphi \\ \chi \end{pmatrix}, \quad \Psi_{1,2} \in V, \varphi \in V \otimes \mathcal{O}^2, \chi \in W.$$ (3.9)

\footnote{Of course the algebra of functions at such points carries interesting information: it is a finite-dimensional commutative associative algebra which still may have nilpotents. In this sense the freckle is a “fat point” (or “zero dimensional subscheme”).}
The solution of (3.8) is not uniquely defined and one is free to perform a $GL(w, \mathbb{C})$ gauge transformation, 

$$\Psi_z \rightarrow \Psi_z g(z, \bar{z}), \quad g(z, \bar{z}) \in GL(w, \mathbb{C}).$$

This gauge freedom can be partially fixed by normalising the vector $\Psi_z$ as follows:

$$\Psi_z^\dagger \Psi_z = 1_W. \quad (3.10)$$

With this normalisation the $U(w)$ gauge field is given by

$$A = \Psi_z^\dagger d\Psi_z, \quad (3.11)$$

and its curvature is given by

$$F = \Psi_z^\dagger dD_z \frac{1}{D_z^\dagger D_z} dD_z^\dagger \Psi_z. \quad (3.12)$$

More explicitly,

$$D_z^\dagger D_z = \Delta_z \otimes 1 + 1_V \otimes \zeta^a \sigma_a,$$

hence

$$\frac{1}{D_z^\dagger D_z} = \frac{1}{\Delta_z^2 - \zeta^2}(\Delta_z \otimes 1 - 1_V \otimes \zeta^a \sigma_a).$$

Formula (3.12) makes sense for $z \in X^\circ \equiv \mathbb{R}^4 \setminus Z$, where $X^\circ$ is the complement in $\mathbb{R}^4$ to the set $Z$ of points (freckles) at which

$$\text{Det} \left( \Delta_z^2 - \zeta^2 \right) = 0. \quad (3.13)$$

Now it is a straightforward exercise to show that on $X^\circ$

$$F^+ = \frac{i}{2} (F + \ast F) = \varphi^\dagger \frac{1}{\Delta_z^2 - \zeta^2} \varphi \hat{\zeta}, \quad (3.14)$$

where $\hat{\zeta} = \zeta_r \varpi_r + \zeta_c \varpi_c + \bar{\zeta}_c \bar{\varpi}_c$, $\ast$ is flat space Hodge star, and

$$\varpi_r = \frac{i}{2} (dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2), \quad \varpi_c = dz_1 \wedge dz_2. \quad (3.15)$$

If $\zeta^c = 0$ then (3.14) implies that $F^{0,2} = 0$, i.e. the $A_{\bar{z}_1}, A_{\bar{z}_2}$ define a holomorphic structure on the bundle $\mathcal{E}_z = \ker D_z^\dagger$ over $X^\circ$. As we have a unitary connection, $F^{2,0} = F^{0,2} = 0$.

From (3.6) the holomorphic bundle $\mathcal{E}$ extends to a holomorphic bundle $\tilde{\mathcal{F}}$ on the whole of $\mathbb{R}^4$. We will now construct a compactification $X$ of $X^\circ$ with a holomorphic bundle $\tilde{\mathcal{F}}$ over $X$ such that $\tilde{\mathcal{E}}|_{X^\circ} \approx \mathcal{E}$, and whose connection $\tilde{A}$ is a smooth continuation of the connection $A$ over $X^\circ$. This compactification $X$ projects down to $\mathbb{C}^2$ via a map $p : X \rightarrow \mathbb{C}^2$. The pull-back $p^* \mathcal{F}$ is a holomorphic bundle over $X$ which differs from $\tilde{\mathcal{E}}$. This difference is localised at the exceptional variety, which is the preimage $p^{-1}(Z)$ of the set of freckles.
4. Abelian case in detail

Let us rotate $\vec{\zeta}$ so that $\zeta_c = 0, \zeta_r = \zeta > 0$ and consider the case $w = 1$. Then \[20\] shows that $J = 0$. Hence, $I^\dagger I = 2v\zeta$ and $[B_1, B_2] = 0$.

We can now solve the equations (3.8) rather explicitly:

\[
\begin{pmatrix}
\Psi_1 \\
\Psi_2
\end{pmatrix} = -\begin{pmatrix} B_1^\dagger - \bar{z}_1 \\ B_2^\dagger - \bar{z}_2 \end{pmatrix} GI\chi,
\]

where

\[
G^{-1} = (B_1 - z_1)(B_1^\dagger - \bar{z}_1) + (B_2 - z_2)(B_2^\dagger - \bar{z}_2)
\]

and

\[
\chi = \frac{1}{\sqrt{1 + I^\dagger GI}}
\]

Let $P(z) = \text{Det}G^{-1}$. It is a polynomial in $z, \bar{z}$ of degree $v$. Clearly (4.3) implies that:

\[
\chi^2 = \frac{P(z)}{Q(z)}
\]

where $Q(z) = P(z) + I^\dagger \tilde{G}^{-1}I$ is another degree $v$ polynomial in $z, \bar{z}$, $\tilde{G}^{-1}$ being the matrix of minors of $G^{-1}$.

The gauge field (3.11) is calculated to be

\[
A = (\partial - \bar{\partial})\log\chi,
\]

and its curvature is

\[
F = \partial\bar{\partial}\log\chi^2.
\]

The formula (4.4) provides a well-defined one-form on the complement $X^\circ$ in $\mathbb{R}^4$ to the set $Z$ of zeroes of $P(z)$. This is just where $B_1 - z_1$ and $B_2 - z_2$ fail to be invertible (and so $\sigma_z$ fails to be injective), that is a “freckle”. We start with the study of one such point and then generalize.

4.1. Charge one instantons.

To see what happens at such a point let us first look at the case $v = 1$. Then (after shifting $\bar{z}_1$ by $B_1^\dagger$, etc.)

\[
\Psi_z = \frac{1}{r\sqrt{r^2 + 2\zeta}} \begin{pmatrix} \bar{z}_1\sqrt{2\zeta} \\ \bar{z}_2\sqrt{2\zeta} \end{pmatrix}, \quad \chi = \frac{r}{\sqrt{r^2 + 2\zeta}},
\]

(4.6)
where \( r^2 = |z_1|^2 + |z_2|^2 \). Thus in this case

\[
P(z) = z_1\bar{z}_1 + z_2\bar{z}_2, \quad Q(z) = z_1\bar{z}_1 + z_2\bar{z}_2 + 2\zeta
\]

The gauge field is given by (setting \( 2\zeta = 1 \)):

\[
A = \frac{1}{2r^2(1 + r^2)}(z_1d\bar{z}_1 - \bar{z}_1dz_1 + z_2d\bar{z}_2 - \bar{z}_2dz_2), \quad (4.7)
\]

and

\[
F = \frac{dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2}{r^2(1 + r^2)} - \frac{1 + 2r^2}{r^4(1 + r^2)^2} \sum_{i,j} z_i\bar{z}_j dz_j \wedge d\bar{z}_i. \quad (4.8)
\]

4.2. Comparison with the non-commutative instanton

Notice the similarity of the solution (4.7) to the formulae (4.56), (4.61) of the paper [14]. It has the same asymptotics both in the \( r^2 \to 0 \) and \( r^2 \to \infty \) limits. Of course the formulae in [14] were meant to hold only for slowly varying fields and that is why we don’t get precise agreement. Nevertheless, we conjecture that all our gauge fields are the transforms of the non-commutative instantons from [1] under the field redefinition described in [14]. From our analysis below, it follows that one has to modify the topology of space-time in order to make non-singular the corresponding gauge fields of the ordinary gauge theory.

4.3. The first blowup

To examine (4.7) further let us rewrite \( A \) as follows:

\[
A = A_0 - A_\infty,
\]

\[
A_0 = \frac{1}{2r^2} (z_1d\bar{z}_1 - \bar{z}_1dz_1 + z_2d\bar{z}_2 - \bar{z}_2dz_2),
\]

\[
A_\infty = \frac{1}{2(1 + r^2)} (z_1d\bar{z}_1 - \bar{z}_1dz_1 + z_2d\bar{z}_2 - \bar{z}_2dz_2).
\]

The form \( A_\infty \) is regular everywhere in \( \mathbb{R}^4 \). The form \( A_0 \) has a singularity at \( r = 0 \). Nevertheless, as we now show, this becomes a well-defined gauge field on \( \mathbb{R}^4 \) blown up at one point \( z = 0 \).

Let us describe the blowup in some details. We start with \( \mathbb{C}^2 \) with coordinates \((z_1, z_2)\). The space blown up at the point \( 0 = (0, 0) \) is simply the space \( X \) of pairs \((z, \ell)\), where \( z \in \mathbb{C}^2 \), and \( \ell \) is a complex line which passes through \( z \) and the point 0. \( X \) projects to \(\mathbb{C}^2 \).
via the map $p(z, \ell) = z$. The fiber over each point $z \neq 0$ consists of a single point while the fiber over the point 0 is the space $\mathbb{CP}^1$ of complex lines passing through the point 0.

In our applications we shall need a coordinatization of the blowup. The total space of the blowup is a union $X = U \cup U_0 \cup U_\infty$ of three coordinate patches. The local coordinates in the patch $U_0$ are $(t, \lambda)$ such that

$$z_1 = t, \quad z_2 = \lambda t. \quad (4.9)$$

In this patch $\lambda$ parameterises the complex lines passing through the point 0, which are not parallel to the $z_1 = 0$ line. In the patch $U_\infty$ the coordinates are $(s, \mu)$, such that

$$z_1 = \mu s, \quad z_2 = s. \quad (4.10)$$

There is also a third patch $U$, where $(z_1, z_2) \neq 0$. This projects down to $\mathbb{C}^2$ such that over each point $(z_1, z_2) \neq 0$ the fiber consists of just one point. The fiber over the point $(z_1, z_2) = 0$ is the projective line $\mathbb{CP}^1 = \{ \lambda \} \cup \infty$. We now show that on this blown up space our gauge field is well defined.

On $U \cap U_0$ we may write

$$A_0 = \frac{t \bar{d}t - \bar{t} dt}{2|t|^2} + \frac{\lambda d\bar{\lambda} - \bar{\lambda} d\lambda}{2(1 + |\lambda|^2)}. \quad (4.11)$$

Define $A_{U_0, \infty}$ as

$$A_{U_0} = \frac{\lambda d\bar{\lambda} - \bar{\lambda} d\lambda}{2(1 + |\lambda|^2)}, \quad A_{U_\infty} = \frac{\mu d\bar{\mu} - \bar{\mu} d\mu}{2(1 + |\mu|^2)}. \quad (4.12)$$

Now $A_0$ is a well-defined one-form on $U$. On the intersections $U \cap U_0$ the one-forms $A_0$ and $A_{U_0}$ are related via a gauge transformation

$$i d \text{arg} t.$$  

On the intersection $U_0 \cap U_\infty$ the one-forms $A_{U_0}$ and $A_{U_\infty}$ are related via

$$i d \text{arg} \lambda = -i d \text{arg} \mu$$

gauge transformations. Finally on $U \cap U_\infty$ the one-forms $A_0$ and $A_{U_\infty}$ are related via the gauge transformation

$$i d \text{arg} s.$$
We have shown therefore that $A_0$ is a well-defined gauge field on $X$. Observe also that at infinity $A \to 0$ as $o(r^{-3})$, which yields a finite action. In fact the gauge field (4.7) has a non-trivial Chern class $\text{ch}_2$:

$$F \wedge F = -\frac{2}{r^2(1 + r^2)^3} dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2$$  \hspace{1cm} (4.13)

so that

$$\frac{1}{4\pi^2} \int F \wedge F = 1.$$  

Finally, the restriction of $A$ on the exceptional divisor $E$, defined by the equation $t = 0$ in $U_0$ and $s = 0$ in $U_{\infty}$, has non-trivial first Chern class:

$$\frac{1}{2\pi i} \int_E F = -1.$$  

4.4. Charge two

In the case $v > 1$ the formulae (4.4), (4.3) are rather intricate. Nevertheless we show that by a sequence of blowups we are able to construct a space $X$ on which the formula (4.4) defines a well-defined gauge field.

For $v = 2$ the matrices $(B_1, B_2)$ and the vector $I$ can be brought to the following normal form by a complexified gauge transformation (3.5) with $g \in \text{GL}_2(\mathbb{C})$:  

$$B_1 = \begin{pmatrix} 0 & p_1 \cr 0 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & p_2 \cr 0 & 0 \end{pmatrix}, \quad I = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$  \hspace{1cm} (4.14)

where the only modulus is $p = (p_1 : p_2)$ which is a point in $\mathbb{C} \mathbb{P}^1$. This data parameterises the torsion free ideal sheaves $\mathcal{I}$ on $\mathbb{C}^2$ which become locally free on the manifold $X$ which is a blowup of $\mathbb{C}^2$ at the point 0 subsequently blown up at the point $p$ on the exceptional divisor.

The sheaf and its liberation. The ideal $\mathcal{I}_p$ corresponding to $p \in \mathbb{P}^1$ is spanned by the functions $f(z_1, z_2)$ on $\mathbb{C}^2$ such that:

$$f \in \mathcal{I}_p \Leftrightarrow f(0,0) = 0, \quad p_1 \partial_1 f|_0 + p_2 \partial_2 f|_0 = 0,$$

i.e. $\mathcal{I}_p = \langle p_1 z_2 - p_2 z_1, z_1^2, z_1 z_2, z_2^2 \rangle$. This sheaf becomes locally free on the manifold $X$. Indeed, consider first the manifold $Y$ which is $\mathbb{C}^2$ blown up at the point $(0, 0)$. Suppose $p_1 \neq 0$, hence we may set $p_1 = 1, p_2 = p$. In the chart $U_0$ where the good coordinates are: $(z_1, \lambda = z_2/z_1)$ the ideal is spanned by:

$$z_1(p - \lambda), z_1^2$$
which is the sheaf $\mathcal{O}(-E)$, where $E = \{z_1 = 0\}$ is the exceptional divisor, tensored with the ideal sheaf of the point $z_1 = 0$, $\lambda = p$. Upon the further blow up $X \to Y$ at the point $\lambda = p, z_1 = 0$ we get the locally free sheaf, whose sections vanish at the exceptional variety.

The gauge field construction. It turns out that the solution of the equations (3.3) up to the (3.5) group action does not differ much from (4.14). In fact,

$$
B_1 = \begin{pmatrix} 0 & p_1 \\ 0 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & p_2 \\ 0 & 0 \end{pmatrix}, \quad I = \sqrt{4\zeta} \begin{pmatrix} 0 \\ 1 \end{pmatrix}
$$

is the solution, provided

$$
|p_1|^2 + |p_2|^2 = 2\zeta.
$$

Then formula (4.4) still holds with

$$
\chi^2 = \frac{r^2(r^2 + 2\zeta) - |\eta|^2}{(r^2 + 2\zeta)(r^2 + 4\zeta) - |\eta|^2},
$$

where $\eta = z_1 p_1 + z_2 p_2$. This expression naively leads to a singular gauge field when $r^2 = 0$. (The denominator is nonvanishing for $\zeta > 0$.) We deal with that by blowing up the point $r = 0$. In this way we can write

$$
r^2 = |z_1|^2(1 + |\lambda|^2), \quad \eta = z_1 (p_1 + \lambda p_2)
$$

(in the patch $U_0$) to get a factor $|z_1|^2$ from the numerator of $\chi^2$. This factor is then removed by a gauge transformation which enters the gluing function. Then, for $z_1 = 0$, we find a singularity at the point $\lambda = p_2/p_1$ on the exceptional divisor in $Y$ which is removed in a similar way by the next blowup $X \to Y$.

4.5. The higher charge case

The nice feature of the cases $v \leq 2$ is that all information about the sheaves is encoded in the geometry of the manifolds $X, X_p$. This property may seem to be lost once $v > 2$. Take for example the ideal $\mathcal{I}^3 = \langle z_1^2, z_1 z_2, z_2^2 \rangle$. The quotient $\mathcal{C}[z_1, z_2]/\mathcal{I}^3 = \langle [1], [z_1], [z_2] \rangle$ is three-dimensional. Similarly the ideal $\mathcal{I}^4 = \langle z_1^2, z_2^2 \rangle$ produces a four dimensional quotient $\mathcal{C}[z_1, z_2]/\mathcal{I}^4 = \langle [1], [z_1], [z_2], [z_1 z_2] \rangle$. Clearly these ideal sheaves are different. Now consider the first blowup $X$. Obviously these sheaves then lift to the same sheaf of ideals, since on $X$: $z_1 z_2 = z_1^2 \lambda \in \mathcal{I}^3$. Thus we find two different sheaves (for charges 3 and 4) which lift to the same holomorphic bundle after a single blowup. The puzzle is what extra data is needed to distinguish them? Therefore one needs a more refined way of extracting the
properties of the sheaf from the properties of the manifold $X$. Perhaps the metrics on the blown up space differ for different charges. We now show that the gauge fields constructed out of the deformed ADHM data describing these two ideals are different, so distinguishing them!

**Charge three gauge fields.** Consider the ideal $I$ spanned by $z_1^2, z_1z_2, z_2^2$. Let us choose the basis

$$e_1 = 1, \quad e_2 = z_1, \quad e_3 = z_2$$

in the quotient $V = \mathbb{C}[z_1, z_2]/I$. The matrices $B_1, B_2$ act in $V$ as follows:

$$B_1 e_1 = e_2, \quad B_1 e_2 = B_1 e_3 = 0,$$

$$B_2 e_1 = e_2, \quad B_2 e_2 = B_2 e_3 = 0.$$  

It turns out that a very simple modification makes them a solution of the equations (3.3). We find

$$B_1 = \sqrt{2\zeta} e_2 e_1^\dagger, \quad B_2 = \sqrt{2\zeta} e_3 e_1^\dagger, \quad I = \sqrt{6\zeta} e_1,$$

and consequently

$$G^{-1} = \begin{pmatrix} \frac{r^2}{r^2 + 2\zeta} & -z_1 \sqrt{2\zeta} & -z_2 \sqrt{2\zeta} \\ -\bar{z}_1 \sqrt{2\zeta} & \frac{r^2}{r^2 + 2\zeta} & 0 \\ -\bar{z}_2 \sqrt{2\zeta} & 0 & \frac{r^2}{r^2 + 2\zeta} \end{pmatrix}.$$  

One finds that

$$G_{11} = \frac{(r^2 + 2\zeta)^2}{r^4(r^2 + 2\zeta)}$$

and so

$$\chi^2 = \frac{1}{1 + 6\zeta G_{11}} = \frac{r^4}{(r^2 + 3\zeta)^2 + 4\zeta^2}. \quad (4.17)$$

**Charge four.** Now let us take the ideal $I = \langle z_1^2, z_2^2 \rangle$. The quotient $V = \mathbb{C}[z_1, z_2]/I$ is four dimensional with the basis

$$e_1 = 1, \quad e_2 = z_1, \quad e_3 = z_2, \quad e_4 = z_1 z_2.$$  

The corresponding solution to the real moment map equations turns out to be

$$B_1 = \sqrt{3\zeta} e_2 e_1^\dagger + \sqrt{\zeta} e_4 e_3^\dagger, \quad B_2 = \sqrt{3\zeta} e_3 e_1^\dagger + \sqrt{\zeta} e_4 e_2^\dagger, \quad I = \sqrt{8\zeta} e_1. \quad (4.18)$$

We were advised by R. Thomas that at the sheaf level the distinction is captured by the torsion groups $Tor$ one can construct in the course of lifting the sheaf to the blowup
Then
\[
G^{-1} = \begin{pmatrix}
r^2 & -z_1\sqrt{3}\zeta & -z_2\sqrt{3}\zeta & 0 \\
-z_1\sqrt{3}\zeta & r^2 + 3\zeta & 0 & -z_2\sqrt{3}\zeta \\
-z_2\sqrt{3}\zeta & 0 & r^2 + 3\zeta & -z_1\sqrt{3}\zeta \\
0 & -z_2\sqrt{3}\zeta & -z_1\sqrt{3}\zeta & r^2 + 2\zeta
\end{pmatrix}
\]
and
\[
\chi^2 = \frac{1}{1 + 8G_{11}} = \frac{r^4((r^2 + 2\zeta)^2 + 2\zeta^2) - 12\zeta^2|z_1z_2|^2}{((r^2 + 4\zeta)^2 + 8\zeta^2)((r^2 + 2\zeta)^2 + 2\zeta^2) - 12\zeta^2|z_1z_2|^2}.
\] (4.19)

Clearly this expression is quite different from (4.17) and so the gauge fields do somehow distinguish the different ideals. Both (4.17) and (4.19) require a single blowup to make the gauge field non-singular. In the charge three case the gauge field restricted onto the exceptional divisor looks like \((\partial - \bar{\partial})\log\chi'\) with \(\chi'_3 = 1 + |\lambda|^2\) in the \(U_0\) chart and \(1 + |\mu|^2\) in \(U_\infty\) chart. In contrast the charge four gauge field on the exceptional divisor has \(\chi'_4 = \sqrt{1 + |\lambda|^4}\) on \(U_0\) and \(\sqrt{1 + |\mu|^4}\) on \(U_\infty\). One may say that the exceptional divisor in the charge three case is more “rounded” than the one in the charge four case.

**Elongated instantons: the general case.** These are special solutions to the deformed ADHM equations that describe \(v \geq 1\) points which sit along a complex line. The ideal \(\mathcal{I}\) corresponding to this configuration is (in an appropriately rotated coordinate system) generated by \(\langle P(z_1), z_2\rangle\), where \(P(z)\) is an arbitrary degree \(v\) polynomial. In other words the space of elongated torsion free sheaves of rank one is isomorphic to the space of degree \(v\) polynomials \(P\). If \(P(z) = z^v\) then we get \(v\) points on top of each other. We shall study this case in some detail a little later after first presenting the case of general \(P(z)\). Since \(z_2 \in \mathcal{I}\) we immediately conclude that \(B_2 = 0\). Then the moment map equation \(\mu_r = 2\zeta\) coincides precisely with the moment map leading to the Calogero-Moser integrable system \([21]\). In the light of \([22]\) its solutions form a part of the phase space of the Sutherland model. For us the most convenient presentation is in terms of the dual \([23]\) - rational Ruijsenaars system \([24]\). Consider the polar decomposition \(B_1 - z_1 = U^\dagger H^{1/2}, B_2 = 0\), with \(H\) hermitian and positive definite, and \(U\) unitary. We may take \(B_1\) traceless by appropriately shifting \(z_1\). Then equation (3.3) becomes
\[
U^\dagger HU - H + II^\dagger = 2\zeta.
\]
This can be solved (as in \([23]\)) by first diagonalising \(H\)
\[
H = 2\zeta \text{diag} (r_1^2, \ldots, r_v^2),
\] (4.20)
(with \(r_i^2 \geq r_{i-1}^2 + 1\)) and then solving for \(U\) and \(I\). Let \(P(z) = \prod_{i=1}^{v} (z - r_i^2)\). Then with

\[
I_i = \sqrt{2\zeta y_i}, \quad x_i = (Uy)_i,
\]

we find

\[
U_{ij} = \frac{x_i\bar{y}_j}{r_j^2 - r_i^2 + 1}, \quad z_1 = -\frac{1}{v} \sum_{i=1}^{v} \bar{x}_iy_ir_i,
\]

(4.21)

where (employing manipulations familiar in Lagrange interpolation)

\[
|x_i|^2 = \frac{P(r_i^2 - 1)}{-P'(r_i^2)}, \quad |y_i|^2 = \frac{P(r_i^2 + 1)}{P'(r_i^2)}.
\]

The remaining gauge invariance allows us to make \(y_i\) real and non-negative. The phases of \(x_i\) for \(r_i^2 > r_{i-1}^2 + 1\) are arbitrary, and given by

\[
x_i = \sqrt{\frac{P(r_i^2 - 1)}{-P'(r_i^2)}} e^{-i\theta_i}.
\]

The polynomial \(P(z)\) which corresponds to the solution (4.20), (4.21) is given by

\[
P(z_1) = \text{Det}(B_1 - z_1) = \prod_{i} r_i e^{i\theta_i}.
\]

Finally, a short calculation shows that for these solutions

\[
\chi = \sqrt{\frac{P\left(-\frac{|z_2|^2}{2\zeta}\right)}{P\left(-\frac{|z_2|^2}{2\zeta} - 1\right)}}.
\]

(4.22)

For the case \(P(z) = z^v\) the solution (4.22) can be made more explicit. By a change of basis in \(V\), the solution to the hyperkähler moment map equations (3.3) acquires a simpler form, viz.

\[
B_1 = \sum_{i=1}^{v-1} \sqrt{2(v-i)\zeta} e_{i+1}^\dagger, \quad B_2 = 0, \quad I = \sqrt{2v\zeta} e_1.
\]

(4.23)

Then \(P(z) = \text{Det}(B_1 - z) = z^v\) and

\[
G^{-1} = r^2 + \sum_{i=1}^{v} \left(2(v-i)\zeta e_{i+1}^\dagger e_i^\dagger - \sqrt{2(v-i)\zeta} \left(z_1 e_i^\dagger e_{i+1}^\dagger + \bar{z}_1 e_{i+1} e_i^\dagger\right)\right).
\]
Observe that $G^{-1}$ is a tridiagonal matrix. Now in order to find $\chi$ we again need $e_1^\dagger Ge_1$. This is easily done, the tridiagonal nature of $G^{-1}$ reducing the problem to one of a three term recursion. Suppose $u_k$ satisfies

$$-\sqrt{2\zeta(k+1)}z_1 u_{k+1} + (r^2 + 2\zeta(k+1)) u_k - \sqrt{2\zeta(k)} z_1 u_{k-1} = 0. \quad (4.24)$$

Then

$$G^{-1}(u_{v-1}, u_{v-2}, \ldots u_0)^t = (r^2 u_{v-1} - z_1 \sqrt{2\zeta(v-1)} u_{v-2}, 0, \ldots 0)^t$$

and consequently

$$G_{11} = u_{v-1}/(r^2 u_{v-1} - z_1 \sqrt{2\zeta(v-1)} u_{v-2}).$$

The normalisation of the $u_k$’s is irrelevant. Now the substitution

$$u_k = \frac{z_1^k}{\sqrt{(2\zeta)^k k!}} w_k$$

simplifies (4.24) to give

$$xw_{k+1} - (x + y + 1 + k)w_k + kw_{k-1} = 0, \quad (4.25)$$

where we have set $|z_1|^2 = 2\zeta x$, $|z_2|^2 = 2\zeta y$. Together with the normalization $w_0 = 1$, we recognize the recursion for the Charlier polynomials $w_k = C_k(-1-y; x) = _2F_0(-k,-1-y; -1/x)$.

These may also be expressed in terms of the Laguerre polynomials as $(-x)^k C_k(a; x)/k! = L_k^{(a-k)}(x)$. They have the generating function

$$e^t \left(1 - \frac{t}{x}\right) = \sum_{k=0}^{\infty} \frac{C_k(a; x)}{k!} t^k.$$ 

In terms of $w_k$, we find

$$G_{11} = \frac{1}{2\zeta} \frac{w_{v-1}}{x^2 w_{v-1} - (v-1)w_{v-2}}, \quad 1 + I^\dagger GI = \frac{w_v}{2\zeta^2 w_{v-1} - (v-1)w_{v-2}}.$$ 

Differentiation of the generating function shows that

$$(x + y)C_k(-1-y; x) - kC_{k-1}(-1-y; x) = xC_{k+1}(-y; x)$$

from which it follows that

$$\chi = \sqrt{\frac{C_v(-y; x)}{C_v(-y-1; x)}}. \quad (4.26)$$

One may show that the polynomial $\mathcal{P}$ corresponding to this special solution of the ADHM equations is simply

$$\mathcal{P}(z) = C_v(z; x).$$
5. Non-abelian charge one freckled instantons

We now proceed with the investigations of the non-abelian case, considering the example of charge one instantons. The deformed ADHM construction in the case \( v = 1 \) gives the space \( M_{1,w} \approx \mathbb{R}^4 \times T^*\mathbb{C}P^{w-1} \). The first factor is the space of pairs \((B_1, B_2)\) which parameterize the center of the instanton. The second factor is responsible for its size and orientation.

Specifically, the second factor emerges as a quotient of the space of pairs \((I, J)\), \(I \in W^*, J \in W\), such that
\[
IJ = 0, \quad II^\dagger - J^\dagger J = 2\zeta > 0
\]
by the action of the group \( U(1) \)
\[
(I, J) \mapsto (Ie^{i\theta}, Je^{-i\theta}).
\]

Let us introduce two projectors
\[
P_1 = II^\dagger, \quad P_2 = JJ^\dagger,
\]
and two numbers
\[
\rho_1^2 = II^\dagger, \quad \rho_2^2 = J^\dagger J.
\]

Then \(\rho_1^2 - \rho_2^2 = 2\zeta\). In particular \(\rho_1 > \rho_2 \geq 0\), and if \(\rho_2 > 0\) we can write
\[
I^\dagger = \rho_1 e_1, J = \rho_2 e_2,
\]
where \(e_1, e_2\) form an orthonormal pair of vectors in \(W\). We shall distinguish between the \(\rho_2 = 0\) and \(\rho_2 > 0\) cases in what follows.

Let us proceed with the ADHM construction. Without any loss of generality we may assume that \(B_1 = B_2 = 0\), by shifting \(z_1, z_2\). The vector \(\Psi_z : W \to \mathbb{C}^2 \oplus W\) is found to be
\[
\Psi_z = \frac{1}{r^2} \left( \frac{z_1}{\sqrt{r^2 + P_1 + P_2}} + \frac{z_2}{\sqrt{r^2 + P_1 + P_2}} \right) \chi, \quad \chi = \frac{r}{\sqrt{r^2 + P_1 + P_2}} =
\]
\[
= 1 + \frac{P_1}{\rho_1^2} \left( \frac{r}{\sqrt{r^2 + \rho_1^2}} - 1 \right) + \frac{P_2}{\rho_2^2} \left( \frac{r}{\sqrt{r^2 + \rho_2^2}} - 1 \right),
\]
where in the process of solving for \(\Psi_z\) we used the gauge \(\chi^\dagger = \chi\).
Notice that in order to write the explicit formula (5.5) for the vector $\Psi_z$ we had to make a choice of the vectors $I, J$ in the orbit (5.2). When working on flat $\mathbb{R}^4$ this choice can be made globally, i.e. in a $z$ independent way. If we are to replace $\mathbb{R}^4$ by a manifold $X$ over which non-trivial line bundles exist, then this choice may well become a subtle matter, i.e. the solution to the equations (5.1) may depend on $z$ while staying in the orbit of the gauge group (3.5). In other words $\theta$ may depend on $z$. In a moment we shall see that this indeed happens.

Using the relations $I\chi = \frac{r}{\sqrt{r^2 + \rho_1^2}} I$, and $J^\dagger \chi = \frac{r}{\sqrt{r^2 + \rho_2^2}} J^\dagger$, we may write

$$
\Psi = \begin{pmatrix}
\bar{z}_1 r \sqrt{r^2 + \rho_1^2} I - \bar{z}_2 r \sqrt{r^2 + \rho_2^2} J^\dagger \\
\bar{z}_2 r \sqrt{r^2 + \rho_1^2} I + \bar{z}_1 r \sqrt{r^2 + \rho_2^2} J^\dagger \\
\chi
\end{pmatrix}.
$$

(5.6)

This expression is well-defined for $r \neq 0$. Moreover $\chi$ is well-defined everywhere, while $\Psi_1, \Psi_2$ have singularities at $r = 0$. Let us perform a sigma process at $(z_1, z_2) = 0$. Introduce the coordinates $(t, \lambda)$ and $(s, \mu)$ by the formulae (4.9), (4.10).

The locally free sheaves: $\rho_2 > 0$. In this case we may write

$$
\chi = \chi^\perp + \chi^\parallel, \quad \chi^\parallel = \frac{r}{\sqrt{r^2 + \rho_1^2}} e_1 e_1^\dagger + \frac{r}{\sqrt{r^2 + \rho_2^2}} e_2 e_2^\dagger.
$$

(5.7)

The component $\chi^\perp = (1 - e_1 e_1^\dagger - e_2 e_2^\dagger) \chi$ decouples. In this sense, it is sufficient to study the case $w = 2$ only. We are free to perform a gauge transformation on $\chi^\parallel$ and $\varphi$ not affecting the $\chi^\perp$ part.

In the patch $U_0$ we may write

$$
\varphi_0 = \frac{1}{\sqrt{1 + |\lambda|^2}} \left( \frac{\lambda J^\dagger}{\sqrt{r^2 + \rho_1^2}} - \frac{\lambda J^\dagger}{\sqrt{r^2 + \rho_2^2}} \right),
$$

(5.8)

while in the patch $U_\infty$ we similarly have

$$
\varphi_\infty = \frac{1}{\sqrt{1 + |\mu|^2}} \left( \frac{\mu J^\dagger}{\sqrt{r^2 + \rho_1^2}} - \frac{\mu J^\dagger}{\sqrt{r^2 + \rho_2^2}} \right).
$$

(5.9)

The gluing across the intersection $U_0 \cap U$ is achieved with the help of a $U(w)$ gauge transformation which acts on the vectors $e_1, e_2$ only, leaving $\chi^\perp, \chi^\parallel$ unchanged,

$$
g_{U_0} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} t e_1 \\ \frac{\bar{t}}{|t|} e_2 \end{pmatrix},
$$

(5.10)
so that $\varphi = \varphi_0 g_{\mathcal{U}\mathcal{U}_0}^\dagger$. Analogously,

$$g_{\mathcal{U}\mathcal{U}_0} \left( \begin{array}{c} e_1 \\ e_2 \end{array} \right) = \left( \begin{array}{c} \frac{s}{\bar{s}} e_1 \\ \frac{s}{|s|} e_2 \end{array} \right), \quad g_{\mathcal{U}_0} = g_{\mathcal{U}\mathcal{U}_0}^{-1}. \quad (5.11)$$

Finally, $g_{\mathcal{U}_0\mathcal{U}_\infty} = g_{\mathcal{U}\mathcal{U}_\infty}^2$.

In the patch $\mathcal{U}$ the gauge field is given by

$$A = e_1 e_1^\dagger \left( \bar{\partial} - \partial \right) \log \frac{r}{\sqrt{r^2 + \rho_1^2}} - e_2 e_2^\dagger \left( \bar{\partial} - \partial \right) \log \frac{r}{\sqrt{r^2 + \rho_2^2}}$$

$$+ \frac{\rho_1 \rho_2}{\sqrt{(r^2 + \rho_1^2)(r^2 + \rho_2^2)}} \left( e_2 e_1^\dagger \frac{\bar{z}_1 d\bar{z}_2 - \bar{z}_2 d\bar{z}_1}{r^2} + e_1 e_2^\dagger \frac{z_2 d\bar{z}_1 - z_1 d\bar{z}_2}{r^2} \right) \quad (5.12)$$

and its field strength

$$F = \sum_{i,j} e_i e_j^\dagger F_{ij},$$

$$F_{11} = \partial \bar{\partial} \log \frac{r^2}{r^2 + \rho_1^2} - \frac{\rho_1 \rho_2}{(r^2 + \rho_1^2)(r^2 + \rho_2^2)} \partial \bar{\partial} \log r^2,$$

$$F_{22} = -\partial \bar{\partial} \log \frac{r^2}{r^2 + \rho_2^2} + \frac{\rho_1 \rho_2}{(r^2 + \rho_1^2)(r^2 + \rho_2^2)} \partial \bar{\partial} \log r^2,$$

$$F_{12} = -2 \frac{(2 \rho^2 + \rho_1^2 + \rho_2^2) \rho_1 \rho_2}{r^4(r^2 + \rho_1^2)^{3/2}(r^2 + \rho_2^2)^{3/2}} \left( z_1 d\bar{z}_2 - z_2 d\bar{z}_1 \right) \wedge \bar{\partial} r^2,$$

$$F_{21} = -F_{12}. \quad (5.13)$$

In the patch $\mathcal{U}_0$ the expression for the gauge field is modified to

$$A = e_1 e_1^\dagger \left( \bar{\partial} - \partial \right) \log \frac{1 + |\lambda|^2}{\sqrt{r^2 + \rho_1^2}} - e_2 e_2^\dagger \left( \bar{\partial} - \partial \right) \log \frac{1 + |\lambda|^2}{\sqrt{r^2 + \rho_2^2}}$$

$$+ \frac{\rho_1 \rho_2}{\sqrt{(r^2 + \rho_1^2)(r^2 + \rho_2^2)}} \left( e_2 e_1^\dagger \frac{d\bar{\lambda}}{1 + |\lambda|^2} + e_1 e_2^\dagger \frac{-d\lambda}{1 + |\lambda|^2} \right). \quad (5.14)$$

The freckles: $\rho_2 = 0$. This value of the parameter $\rho_2$ corresponds to the torsion free sheaves which are not locally free. For $\rho_2 = 0$ the vector $J$ vanishes, $\rho_1^2 = 2 \zeta$, $I = \sqrt{2 \zeta} e^\dagger$, $e \in W$ and the vector $\Psi_z$ simplifies to (removing the $\chi^\perp$ part):

$$\Psi_z = \frac{1}{r \sqrt{r^2 + 2 \zeta}} \left( \begin{array}{c} \sqrt{2 \zeta} \bar{z}_1 e^\dagger \\ \sqrt{2 \zeta} \bar{z}_2 e^\dagger \end{array} \right). \quad (5.15)$$

We easily recognize the vector $\Psi_z$ from the abelian section. So in this case the torsion free sheaf of rank $w$ splits as a direct sum of the trivial holomorphic bundle of rank $w - 1$ and a standard charge one torsion free sheaf of rank 1 which lifts to a line bundle on the blowup.
6. Discussion

Our paper has been concerned with the deformed ADHM equations. These equations may be viewed as giving instantons on a noncommutative space-time. Equally, and this is the focus of our paper, they may be interpreted as gauge fields on an ordinary commutative space-time manifold $\mathbb{C}^2$ or $\mathbb{CP}^2$ blown up at a finite number of points. The deformation singles out a particular complex structure. In terms of this complex structure, the deformed ADHM construction yields a holomorphic bundle $\mathcal{E}_z = \ker D_z^\dagger$ outside of a set of points where $\sigma_z$ fails to be surjective. The constraint $F_{0,2}^0 = 0$ yielding this holomorphic structure may be viewed as a restriction for supersymmetry. In addition ordinary instantons obey the equation $F_{1,1}^{1,1} = 0$, which is usually viewed as fixing the “non-compact” part of the complexified gauge group. Our solutions apparently obey another equation $Z(F_{1,1}^1) = 0$ which also serves as a gauge fixing condition. At the points for which $\sigma_z$ fails to be surjective (elsewhere called “freckles”) the ADHM gauge fields look singular, and we have shown that by suitably blowing up such points the gauge fields may be extended in a regular manner.

Our construction is consistent with the work of Seiberg and Witten who show that (to any finite order) there is a mapping from ordinary gauge fields to non-commutative gauge fields that respects gauge equivalence. Presumably the equation $\hat{F}_{1,1}^{1,1} = 0$ is mapped into our equation $Z(F_{1,1}^1)$ which we admittedly weren’t able to identify in full generality (perhaps the results of [26] may help to solve this problem).

Our blowups will only be seen at short wavelengths and regulate the divergences encountered by Seiberg and Witten (cf. Section 4 of [14]). We believe the modifications to the topology of space-time we have described are necessary in order to make the corresponding gauge fields of the ordinary gauge theory non-singular. For large instanton number we interpret our results as producing space-time foam.

Although our study has focused on the $U(1)$ situation, we have shown how one may extend to the non-abelian situation. Several of the $U(1)$ constructs reappeared in that case. As well as being rather concrete, the $U(1)$ situation has revealed a rather rich structure. Our low order computations show how the gauge fields can distinguish between two different sheaves that lift to the same holomorphic bundle after blowup. We have also described a general class of instantons that we called elongated. We were able to associate to this class of solution precisely the moment map equation of the Calogero-Moser integrable system and used the machinery of integrable systems to describe this case in some detail. This appearance of the Calogero-Moser system is somewhat different to that of Wilson
This appearance of integrable systems here, in Seiberg-Witten theory more generally, and in the various brane descriptions of these same phenomena, still awaits a complete explanation.

6.1. Notes added in a year

After this paper was posted in the archives, a few papers appeared which addressed the issue of the non-singularity of the noncommutative instantons more thoroughly. It was shown \[27\] \[28\] that there is indeed a memory of the blowup of the commutative space in the noncommutative description (through the appearance of the shift operators $S$ and $S^\dagger$ \[28\]). However, the very noncommutative space over which the instantons are defined, is not altered in any way. In this way the noncommutative description is simpler, though a physical mechanism for the topology change in the commutative description we have encountered may be forthcoming.

On the other hand, there was some progress in the search for the equations $Z(F^{1,1}) = 0$, replacing the ordinary $F^{1,1+} = 0$. We found that the charge one $U(1)$ gauge field given by (4.7) is in fact anti-self-dual in Burns\[29\] metric on the blowup of $\mathbb{R}^4$. The higher charge case however remains open.

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