We consider a possibility of capture of a heavy charged massive particle $\chi^-$ by the nucleus leading to appearance of a bound state. A simple analytic formula allowing to calculate binding energies of the $N\chi^-$ bound state for different nuclei is derived. If the binding energy is sufficiently large the particle $\chi^-$ is stable inside the nucleus. The probabilities of the nuclear fissions for such states are calculated. It is shown that the bound states are more stable to a possible fission in comparison to the bare nucleus. This makes an observation of this hypothetical charged massive particle and the superheavy nuclei more probable.

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I. INTRODUCTION

The problem what the dark matter consists of is one of a hottest topics of the modern physics. In particular, a possibility of nonbaryonic dark matter is widely discussed (see, e.g., [1, 2]). One of the possible models of the existence of nonbaryonic dark matter was discussed by Pospelov and Ritz in their recent paper [3]. They considered the direct and indirect detection signatures of weakly interacting massive particles (WIMP) $\chi^0$ with the mass $m_{\chi^0}$ with a heavier, but nearly degenerate, charged state $\chi^\pm$. The WIMP-nucleus interaction may be dominated by inelastic recombination process leading to the formation of $N\chi^-$ bound state.

Following this scenario and assuming that there is a heavy particle with the mass $m_{\chi^-}$ and the charge equal to the electron charge $e$, we derive a simple analytic formula allowing to calculate binding energies of the $N\chi^-$ bound state for different nuclei. This formula can be applied for any $Z$ and especially useful for heavy and superheavy nuclei, whose properties are less studied so far.

In particular, we can consider the following inelastic scattering of $\chi^0$ with heavy nucleus leading to the capture of the particle $\chi^-$ by the nucleus and the appearance of the bound state $N\chi^-$ [3]:

$$\chi^0 + N \rightarrow (N\chi^-) + e^+ + \nu$$  \hfill (1)

Knowing the binding energy $E_0$ we can predict how stable is the bound state that occurred due to this $\beta^+\text{-type process.}$ The condition of stability of the particle $\chi^-$ inside the nucleus is $|E_0| > m_{\chi^-} - m_{\chi^0}$.

The appearance of the bound state leads, in turn, to that the probability of the tunneling through the fission barrier is changed. From general considerations we can expect that the $N\chi^-$ bound state will be more stable to a possible fission in comparison with the bare nucleus. Our calculations show that for heavy nuclei the tunneling probability of the bound state is reduced by a few orders of magnitude compared to the bare nuclei. Since the bound states are more stable to a possible fission in comparison to the bare nucleus, it opens new possibilities for the observations of such systems.

The paper is organized as follows. In Sec. II we derive the analytic formula and calculate the binding energies for different nuclei. In Sec. III we provide the expression for the tunneling probability and describe the Coulomb interaction potential for a deformed nucleus. Section IV is devoted to the discussion of the obtained results and contains concluding remarks. The units ($\hbar = c = 1$ and $e^2 \equiv \alpha \approx 1/137$) are used throughout.

II. BINDING ENERGY

In this section we will find the binding energies of the $\chi^-$ particle to a nucleus. Pospelov and Ritz [3] numerically found the binding energies of the state $(N\chi^-)$ assuming a Gaussian and steplike nuclear charge distribution for several elements and solving the Schrödinger equation with a given charge distribution inside the nucleus. We derive analytical formulas which reproduce the numerical results obtained in Ref. [3] to reasonable accuracy and allow us to calculate the binding energies for heavy and superheavy nuclei.

Let us model a nucleus containing $Z$ protons as a uniformly charged sphere of radius $R$. The electric potential is

$$V(r) = \begin{cases} -\frac{Ze}{r} \left( \frac{3}{2} - \frac{1}{2} \frac{r^2}{R^2} \right), & 0 < r \leq R, \\ -\frac{Ze}{r}, & r \geq R, \end{cases}$$  \hfill (2)

where $r$ is the distance from the center of the nucleus.

Let us show that for heavy nuclei the wave function of the bound state $N\chi^-$ will be localized well within the nucleus. In other words, the characteristic distances $r$ of the localization of the wave function are smaller than the nuclear radius $R$ even for light nuclei. For heavy nuclei $r \ll R$.

As follows from Eq. (2) (upper line) the second term of $V(r)$ is the potential of a three-dimensional harmonic
oscillator. The potential energy can be written as

$$U_{osc}(r) = \frac{1}{2} Z\alpha r^2 \frac{1}{R^3} \equiv \frac{1}{2} M\omega^2 r^2,$$

(3)

where we define

$$\omega = \sqrt{\frac{Z\alpha}{MR^3}},$$

(4)

and $M \equiv m_{\chi} - m_N/(m_{\chi} + m_N)$ is the reduced mass of the particle $\chi^-$ and the nucleus. We assume that the mass of the particle $m_{\chi}^-$ is comparable to or greater than the mass of the nucleus $m_N$. If $m_{\chi}^- \gg m_N$ then $M \approx m_N$.

For the harmonic oscillator, as follows from the virial theorem, the average potential and kinetic energies are equal and for the total energy of the ground state we obtain

$$E_{osc} = \frac{3}{2} \omega = M\omega^2 \langle r^2 \rangle,$$

(5)

and, respectively,

$$\langle r^2 \rangle = \frac{3}{2M\omega}.$$  

(6)

Here and in the following it is sufficient for our purposes to use the following approximations

$$R \approx r_0 A^{1/3} \quad \text{and} \quad M \approx A m_p,$$

(7)

where $r_0 \approx 1.2$ fm, $A$ is the nucleon number, and $m_p$ is the proton mass.

Using Eqs. (4), (6), and (7) and denoting $r_{av} \equiv \sqrt{\langle r^2 \rangle}$, we arrive at the following expression for the ratio $r_{av}/R$:

$$\frac{r_{av}}{R} \approx \frac{2.7}{A^{1/3} Z^{1/3}}.$$  

(8)

In Table I we present the results of calculation of $r_{av}/R$ for different elements. As seen from the table, for the heavy nuclei and approximate the potential energy using the electric potential inside the nucleus only. This is the potential energy of a three-dimensional harmonic oscillator with the additional constant term $-\frac{3}{2} Z\alpha r_0$. It is given by

$$U(r) = -\frac{3}{2} \frac{Z\alpha}{R} + \frac{1}{2} M\omega^2 r^2.$$  

(9)

The eigen-values of the Schrödinger equation for the potential of a three-dimensional harmonic oscillator are well-known. In our case there is the additional constant term, so that the energies of this system are

$$E = -\frac{3}{2} \frac{Z\alpha}{R} + \omega (3/2 + n_x + n_y + n_z).$$  

(10)

Let us apply this model to the case of a nucleus with atomic weight $A$. Using Eqs. (4) and (7) and writing the mass $M$ in units of proton mass $m_p$ we obtain for the frequency:

$$\omega = 7.73 \sqrt{\frac{Z m_p}{A M r_0^3}} \text{ MeV.}$$  

(11)

The energy of the ground state is given by

$$E_0 = \left( -2.16 \frac{Z}{A^{1/3}} + 11.6 \sqrt{\frac{Z}{A M r_0^3}} \right) \text{ MeV.}$$  

(12)

As we have already mentioned the characteristic distance $r$ is much smaller than $R$ for heavy nuclei. For this reason the contribution of the second term in Eq. (12) is small for heavy nuclei.

Taking into account Eq. (7) we arrive at the following approximate formulas for the frequency $\omega$ and the ground state energy:

$$\omega \approx 5.9 \frac{\sqrt{Z}}{A} \text{ MeV,}$$  

(13)

$$E_0 \approx \left( -1.8 \frac{Z}{A^{1/3}} + 8.8 \sqrt{Z} \frac{\sqrt{Z}}{A} \right) \text{ MeV.}$$  

(14)

In Table II we list the frequencies and the binding energies calculated for several elements using Eqs. (13) and (14). As seen from the table there is a good agreement between the energy values obtained in our simple approach and by Pospelov and Ritz. The heavier the atom the better agreement. A reason is that in our approach we assumed that $r \ll R$ and, as a result, neglected the part of the Coulomb potential for $r > R$. This approximation is good for the heavy elements while for the light elements (like $^{11}B$) it can be used only for rough estimates.
TABLE II: The frequencies \( \omega \) (in MeV) and the binding energies (in MeV) of \( \chi^- \) for different nuclei. The results are compared (where available) with those obtained in Ref. [3]. If \( |E_0| \) exceeds the mass difference \( m_{\chi^-} - m_{\chi^0} \), the particle \( \chi^- \) is stable inside the nucleus.

| \((^A\chi^-)\) | \(Z\) | \(\omega\) (MeV) | \(-E_0\) (MeV) |
|-----------------|-----|----------------|----------------|
| \((^{11}\chi^-)\) | 5   | 1.2            | 2.3            |
| \((^{13}\chi^-)\) | 6   | 1.2            | 2.9            |
| \((^{14}\chi^-)\) | 7   | 1.1            | 3.6            |
| \((^{16}\chi^-)\) | 8   | 1.0            | 4.2            |
| \((^{40}\chi^-)\) | 18  | 0.63           | 8.5            |
| \((^{74}\chi^-)\) | 32  | 0.45           | 13.0           |
| \((^{132}\chi^-)\) | 54  | 0.33           | 18.6           |
| \((^{208}\chi^-)\) | 80  | 0.26           | 24.2           |
| \((^{252}\chi^-)\) | 90  | 0.24           | 26.0           |
| \((^{282}\chi^-)\) | 100 | 0.23           | 28.0           |

III. TUNNELING PROBABILITY

In the work of Dzuba and Flambaum [4] the effect of atomic electrons on nuclear fission was considered. It was shown that atomic electrons influence on the nuclear fission very insignificantly. The probability of the fission of the nuclei with \( Z \sim 100 \) is changed only at the level of 0.2%.

In this paper we consider effect of a heavy charged particle, which forms the bound state, on the probability of the nuclear fission. Note that a real shape of the fission barrier is complicated and accurate calculation requires the knowledge of the nuclear structure. At the same time the qualitative features in the structure of the fission barrier can be described by a simple parabolic barrier model [3, 6]. Since our goal is to make an estimate of the nuclear fission probability, the parabolic barrier model is sufficient for our purposes.

The probability of the tunneling through the barrier can be written as [4, 6]

\[
P = \left[ 1 + \exp \left( 2\pi \frac{|U_B - E|}{\omega_B} \right) \right]^{-1}, \tag{15}
\]

where \( U_B \) is the maximum of the potential energy and the potential barrier width \( \omega_B \sim 0.5 - 1 \text{ MeV} \).

In a case of spontaneous fission we can estimate the difference \( |U_B - E| \) for typical energies \( E \) as \( |U_B - E| \sim 5 \text{ Mev} \) and therefore \( 2\pi |U_B - E|/\omega_B \gg 1 \). It is convenient to determine the probability of the spontaneous fission \( P_0 \) as

\[
P_0 = \exp \left( -2\pi \frac{|U_B - E|}{\omega_B} \right). \tag{16}
\]

Following Ref. [4] we present the tunneling probability \( P \) in the form

\[
P = P_0 \exp \left( -2\pi \frac{\delta E}{\omega_B} \right), \tag{17}
\]

where \( \delta E \equiv E_{\text{max}}^C - E_{\text{min}}^C \) is the difference between Coulomb energies of the particle \( \chi^- \) in the points of maximum and minimum of the nuclear energy \( U \) as a function of the deformation parameter. To find these Coulomb energies we have to specify the form of the nucleus. It will be discussed in detail in the next section.

A. Prolate ellipsoid

Our subsequent calculations are based on the following experimental fact. The minimum and maximum of the Coulomb energy correspond to spheroidal deformations of the nucleus. Let us consider, for example, prolate spheroid with the minor semiaxis \( a \) and the major semiaxis \( c \). If this spheroid was obtained as a result of deformation of the sphere with the radius \( R \), then the condition of volume conservation reads as

\[
a^2 c = R^3. \tag{18}
\]

If the eccentricity \( \varepsilon \) is defined by

\[
\varepsilon^2 = 1 - \frac{a^2}{c^2}, \tag{19}
\]

then for prolate ellipsoids

\[
a = R (1 - \varepsilon^2)^{1/6}, \tag{20}
\]

\[
c = R (1 - \varepsilon^2)^{-1/3}. \tag{21}
\]

In Ref. [7] (see also [8]) it was introduced the parameter deformation of the sphere \( \eta \) connected with the minor and major semiaxes of the prolate spheroid by a simple relation

\[
\frac{a}{c} = \frac{1 - \frac{2\pi}{3} \eta}{1 + \frac{4\pi}{3} \eta}. \tag{22}
\]

Using Eq. (19) we obtain for the eccentricity

\[
\varepsilon^2 = \frac{3\eta(6 - \eta)}{(3 + \eta)^2}. \tag{23}
\]

B. Coulomb potential

The Coulomb potential of the spheroid can be written as [9]

\[
U_C(r, \theta) = -\frac{3}{2} \frac{Z\alpha}{R} \left[ K(1 - \varepsilon^2)^{1/3} + \left\{ \frac{K - 1}{\varepsilon^2} (\varepsilon^2 - 1) \cos^2 \theta + \frac{1}{2} \frac{K - 1}{r^2} \sin^2 \theta \right\} \frac{r^2}{R^2} \right], \tag{24}
\]

where \( K \) is the maximum of the potential energy and \( \omega_B \sim 0.5 - 1 \text{ MeV} \).
where the coefficient $K$ is determined as

$$K \equiv \frac{1}{2\varepsilon} \ln \frac{1+\varepsilon}{1-\varepsilon} \quad (25)$$

and $\theta$ is the angle between $r$ and the axis $z$.

It is easy to show that if the eccentricity $\varepsilon \to 0$, i.e. $a = c$, then

$$K \to 1 \quad \text{and} \quad \frac{K-1}{\varepsilon^2} \to \frac{1}{3}$$

and the potential $U_C$ goes over to the potential of the uniformly charged sphere $U_C^{sh}$

$$U_C^{sh}(r, \theta) = -\frac{Z\alpha}{R} \left( \frac{3}{2} - \frac{1}{2} \frac{r^2}{R^2} \right). \quad (26)$$

C. Deformation parameter

According to experimental investigations (see, e.g., [8], in light actinides (like $^{236}$U) the deformation $\eta$ changes from 0.2 in the minimum to about 0.8 in the last maximum of the fission barrier. In superheavy elements $\eta \approx 0.4$ in the last maximum.

Using Eq. (24) and supposing that in the minimum $\eta = 0.2$ and in the maximum $\eta = 0.4$ we can find $\delta E = E_C^{\text{max}} - E_C^{\text{min}}$. Taking into account that for heavy nuclei the wave function of the bound state is localized at the distances $r \ll R$, for an estimate we can neglect the term $\sim r^2/R^2$ in Eq. (24). Then

$$U_C(r, \theta) \approx -\frac{3}{2} \frac{Z\alpha}{R} K(1-\varepsilon^2)^{1/3} \quad (27)$$

The change of the parameter $\eta$ from 0.2 to 0.4 corresponds to change of the eccentricity $\varepsilon$ from 0.58 to 0.76. Then we obtain

$$\frac{3}{2} K(1-\varepsilon^2)^{1/3} = \frac{3}{4\varepsilon} \ln \frac{1+\varepsilon}{1-\varepsilon} \quad (28)$$

and, respectively,

$$E_C^{\text{min}}(\min) \approx -1.494 \frac{Z\alpha}{R} \quad E_C^{\text{max}}(\max) \approx -1.475 \frac{Z\alpha}{R} \Rightarrow \delta E \approx 0.019 \frac{Z\alpha}{R}. \quad (29)$$

IV. RESULTS AND DISCUSSION

Using the derived expressions for the minimal and maximal Coulomb energies given by Eq. (29) we can find the tunneling probabilities for the heavy and superheavy elements with $Z$ ranging from 80 to 160. For calculation we again approximate the nuclear radius by $R \approx 1.2 A^{1/3} \text{fm} \approx 0.006 A^{1/3} \text{MeV}^{-1}$. For the superheavy nuclei with $Z \geq 120$ whose nucleon numbers $A$ are not determined yet we put $A \approx 2.5 Z$. For an estimate we assume that $\omega_B \approx 0.5 \text{MeV}$ [6].

The results of calculation of $\delta E$ and $P/P_0$ are listed in Table III. Our calculation confirmed that the $(N\chi^-)$ bound state is more stable to a possible fission in comparison with the bare nucleus. For the potential barrier $\omega_B \approx 0.5 \text{MeV}$ the tunneling probability of the bound state are 2–3 orders of magnitude smaller than that of the bare nucleus. The greater nuclear charge $Z$ the larger difference between the probabilities $P$ and $P_0$. When $Z$ changes from 80 to 160 the ratio $P/P_0$ changes from $1.9 \times 10^{-2}$ to $1.8 \times 10^{-3}$, i.e. it becomes 10 times smaller. It is not surprisingly because $P \sim \exp(-\delta E)$ and $\delta E \sim Z/R$. Since $R \sim A^{1/3} \sim Z^{1/3}$, $\delta E$ increases as $Z^{2/3}$ with increasing $Z$.

To conclude, we have considered the process of capture of the heavy charged massive particle $\chi^-$ by the nucleus leading to appearance of a bound state. We derived a simple analytic formula allowing to calculate binding energies of the $N\chi^-$ bound state for different nuclei. These energies can be calculated rather accurately for heavy and superheavy nuclei while for light elements the derived formula can be used for the estimates of the binding energies.

We have calculated the tunneling probabilities for a number of the $N\chi^-$ bound states for the heavy and superheavy nuclei and showed that these states are more stable to a possible fission in comparison to the bare nucleus. Their tunneling probabilities are 2–3 orders of magnitude smaller than the tunneling probabilities of the bare nuclei. This result is important because it opens new perspectives to observe such bound states and get a new information about the hypothetical particle $\chi^-$ and the superheavy nuclei which were not observed so far due to their instability.

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| $Z$ | $A$ | $\delta E$ (MeV) | $P/P_0$ |
|-----|-----|-----------------|---------|
| 80  | 202 | 0.315           | 1.9[-2] |
| 90  | 232 | 0.339           | 1.4[-2] |
| 100 | 257 | 0.364           | 1.0[-2] |
| 110 | 269 | 0.394           | 7.1[-3] |
| 120 | 300 | 0.414           | 5.5[-3] |
| 130 | 325 | 0.437           | 4.1[-3] |
| 140 | 350 | 0.459           | 3.1[-3] |
| 150 | 375 | 0.481           | 2.4[-3] |
| 160 | 400 | 0.501           | 1.8[-3] |
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[1] G. Jungman, M. Kamionkowski, and K. Griest, Phys. Rep. 267, 195 (1996).
[2] G. Bertone, D. Hooper, and J. Silk, Phys. Rep. 405, 279 (2005).
[3] M. Pospelov and A. Ritz, Phys. Rev. D 78, 055003 (2008).
[4] V. A. Dzuba and V. V. Flambaum, Europhys. Lett. 84, 22001 (2008).
[5] E. Segrè, *Nuclei and Particles* (The Benjamin / Cummings Publishing Company, Inc., Reading, Massachusetts, 1977), 2nd ed.
[6] A. Bohr and B. Mottelson, *Nuclear Structure* (W. A. Benjamin Inc., New York, 1974).
[7] S. G. Nilsson, C. F. Tsang, A. Sobiczewski, Z. Szymański, S. Wycech, C. Gustafson, I. Lamm, P. Möller, and B. Nilsson, Nucl. Phys. A 131, 1 (1969).
[8] G. Leander and P. Möller, Phys. Lett. B 57, 245 (1975).
[9] R. W. Hasse and W. D. Myers, *Geometrical Relationships of Macroscopic Nuclear Physics* (Springer-Verlag, Heidelberg, 1988).