Integration with respect to Euler characteristic over the projectivization of the space of functions and the Alexander polynomial of a plane curve singularity.

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For a reduced plane curve singularity \(C = \bigcup_{i=1}^{r} C_i\) (\(C_i\) are its irreducible components), let \(\Delta^C(t_1,\ldots,t_r)\) be the Alexander polynomial of the link \(C \cap S^2_{\epsilon} \subset S^3_{\epsilon}\) for \(\epsilon > 0\) small enough (see, e.g., [4]). We fix the Alexander polynomial \(\Delta^C(t_1,\ldots,t_r)\) (in general it is defined only up to multiplication by a monomial \(\pm t_1^{n_1} \cdots t_r^{n_r}\)) assuming that it is really a polynomial (i.e., it does not contain variables in negative powers) and that \(\Delta^C(0,\ldots,0) = 1\). Let \(\zeta_C(t)\) be the zeta-function of the classical monodromy transformation of the singularity \(C\), i.e., of the function germ \(f : (\mathbb{C}^2,0) \to (\mathbb{C},0)\) such that \(C = \{f = 0\}\) (see, e.g., [1]). For \(r > 1\), one has \(\zeta_C(t) = \Delta^C(t,\ldots,t)\) (for \(r = 1\), \(\zeta_C(t) = \Delta^C(t)/(1-t)\)).

It was shown that all the coefficients of the Alexander polynomial \(\Delta^C(\underline{t})\) (\(\underline{t} = (t_1,\ldots,t_r)\)) can be described as Euler characteristics of some spaces – complements to arrangements of projective hyperplanes in projective spaces ([3]). For a hypersurface singularity of any dimension, the Lefschetz numbers of iterates of the classical monodromy transformation have been described as Euler characteristics of some subspaces in the space of (truncated) arcs ([3]). The last result is connected with the theory of integration with

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respect to the Euler characteristic in the space of arcs. Here we discuss a
similar notion (integration with respect to the Euler characteristic) in the
projectivization \( \mathbb{P}\mathcal{O}_{\mathbb{C}^n,0} \) of the ring \( \mathcal{O}_{\mathbb{C}^n,0} \) of germs of functions on \( \mathbb{C}^n \) at the
origin (here we consider it as a linear space) and show that the Alexan-
der polynomial and the zeta-function of a plane curve singularity can be
expressed as certain integrals over \( \mathbb{P}\mathcal{O}_{\mathbb{C}^2,0} \) with respect to the Euler charac-
teristic.

Let \( \mathcal{J}_k^{n,0} \) be the space of \( k \)-jets of functions at the origin in \( (\mathbb{C}^n,0) \)
(\( \mathcal{J}_k^{n,0} = \mathcal{O}_{\mathbb{C}^n,0}/m^{k+1} \cong \mathbb{C}^{(n+k)} \)), where \( m \) is the maximal ideal in \( \mathcal{O}_{\mathbb{C}^n,0} \). For a
complex linear space \( L \) (finite or infinite dimensional) let \( \mathbb{P}L = (L \setminus \{0\})/\mathbb{C}^{*} \)
be its projectivization, let \( \mathbb{P}^*L \) be the disjoint union of \( \mathbb{P}L \) with a point (in
some sense \( \mathbb{P}^*L = L/\mathbb{C}^{*} \)). One has natural maps \( \pi_{k,\ell} : \mathbb{P}\mathcal{O}_{\mathbb{C}^n,0} \to \mathbb{P}^*\mathcal{J}_k^{n,0} \)
and \( \pi_{k,\ell} : \mathbb{P}^*\mathcal{J}_k^{n,0} \to \mathbb{P}^*\mathcal{J}_\ell^{n,0} \) for \( k \geq \ell \). Over \( \mathbb{P}\mathcal{J}_\ell^{n,0} \subset \mathbb{P}^*\mathcal{J}_\ell^{n,0} \) the map \( \pi_{k,\ell} \) is a
locally trivial (and in fact trivial) fibration, the fibre of which is a complex
linear space of some dimension.

**Definition:** A subset \( X \subset \mathbb{P}\mathcal{O}_{\mathbb{C}^n,0} \) is said to be cylindric if \( X = \pi_{k}^{-1}(Y) \) for
a semi-algebraic subset \( Y \subset \mathbb{P}\mathcal{J}_k^{n,0} \subset \mathbb{P}^*\mathcal{J}_k^{n,0} \).

**Definition:** For a cylinder subset \( X \subset \mathbb{P}\mathcal{O}_{\mathbb{C}^n,0} \) (\( X = \pi_{k}^{-1}(Y), Y \subset \mathbb{P}\mathcal{J}_k^{n,0} \))
its Euler characteristic \( \chi(X) \) is defined as the Euler characteristic \( \chi(Y) \) of
the set \( Y \).

**Remark.** A semi-algebraic subset of a finite dimensional projective space
(e.g., the set \( Y \) above) can be represented as the union of a finite number of
cells which do not intersect each other. The Euler characteristic of such a set
is defined as the alternative sum of numbers of cells of different dimensions.
Defined this way, the Euler characteristic satisfies the additivity property:

\[
\chi(Y_1 \cup Y_2) = \chi(Y_1) + \chi(Y_2) - \chi(Y_1 \cap Y_2),
\]

and therefore can be considered as a generalized (non-positive) measure on
the algebra of semi-algebraic subsets.

Let \( \psi : \mathbb{P}\mathcal{O}_{\mathbb{C}^n,0} \to A \) be a function with values in an Abelian group \( G \).

**Definition:** We say that the function \( \psi \) is cylindric if, for each \( a \neq 0 \) the
set \( \psi^{-1}(a) \subset \mathbb{P}\mathcal{O}_{\mathbb{C}^n,0} \) is cylindric.

**Definition:** The integral of a cylindric function \( \psi \) over \( \mathbb{P}\mathcal{O}_{\mathbb{C}^n,0} \) with respect
to the Euler characteristic is

\[
\int_{\mathbb{P}\mathcal{O}_{\mathbb{C}^n,0}} \psi d\chi = \sum_{a \in A, a \neq 0} \chi(\psi^{-1}(a)) \cdot a
\]
if this sum has sense in $A$. If the integral exists (has sense) the function $\psi$ is said to be integrable.

**Remark.** In a similar way one can define a generalized Euler characteristic $[X]$ of a cylindric subset of $\mathbb{P}O_{\mathbb{C}^n,0}$ (or of $O_{\mathbb{C}^n,0}$) with values in the Grothendieck ring of complex algebraic varieties localized by the the class $L$ of the complex line and thus the corresponding notion of integration (see, e.g., [3]). For that one can define $[X]$ as $[Y] \cdot \mathbb{L}^{-\binom{n+k}{k}}$.

For a plane curve singularity $C = \bigcup_{i=1}^{r} C_i$, let $\varphi_i : (\mathbb{C},0) \to (\mathbb{C}^2,0)$ be parameterizations (uniformizations) of the branches $C_i$ of the curve $C$ (i.e., $\text{Im} \varphi_i = C_i$ and $\varphi_i$ is an isomorphism between $\mathbb{C}$ and $C_i$ outside of the origin). For a germ $g \in O_{\mathbb{C}^2,0}$, let $v_i(g)$ be the power of the leading term in the power series decomposition of the germ $g \circ \varphi_i : (\mathbb{C},0) \to \mathbb{C}$: $g \circ \varphi_i(\tau_i) = c_i \cdot \tau_i^n + \text{terms of higher degree}$, where $c_i \neq 0$. If $g \circ \varphi_i(t) \equiv 0$, $v_i(g)$ is assumed to be equal to $\infty$. Let $u(g) = (v_1(g), \ldots, v_r(g)) \in \mathbb{Z}_{\geq 0}^r$, $v(g) = \|u(g)\| = v_1(g) + \ldots + v_r(g)$. Let $\mathbb{Z}[t]$ (respectively $\mathbb{Z}[t_1, \ldots, t_r]$) be the group (with respect to addition) of formal power series in the variable $t$ (respectively in $t_1, \ldots, t_r$). For $u = (v_1, \ldots, v_r) \in \mathbb{Z}_{\geq 0}^r$, let $t^u = t_1^{v_1} \cdot \ldots \cdot t_r^{v_r}$; we assume $t^\infty = 0$.

**Theorem 1** For each $u \in \mathbb{Z}_{\geq 0}^r$, the set $\{ g \in \mathbb{P}O_{\mathbb{C}^2,0} : u(g) = u \}$ is cylindric. Therefore the functions $t^u(g)$ and $t^v(g)$ on $\mathbb{P}O_{\mathbb{C}^2,0}$ with values in $\mathbb{Z}[t_1, \ldots, t_r]$ and $\mathbb{Z}[t]$ respectively are cylindric.

**Proof** follows from the fact that, for $g \in m^s$, $v_i(g) \geq s$, i.e., the Taylor series of $g \circ \varphi_i(\tau_i)$ starts from terms of degree at least $s$. Therefore the functions $t^u(g)$ and $t^v(g)$ on $\mathbb{P}O_{\mathbb{C}^2,0}$ are integrable (since $\sum_{\mathbb{Z}^+} \ell(u \cdot u) t^u \in \mathbb{Z}[t_1, \ldots, t_r]$)

for any integers $\ell(u \cdot u)$.

**Theorem 2** For $r > 1$,

$$\int_{\mathbb{P}O_{\mathbb{C}^2,0}} t^u(g) d\chi = \Delta^C(t_1, \ldots, t_r);$$

for $r \geq 1$,

$$\int_{\mathbb{P}O_{\mathbb{C}^2,0}} t^v(g) d\chi = \zeta^C(t).$$

**Proof** follows from the results of [3].
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