Learning Graph Partitions

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December 16, 2021

Abstract

Given a partition of a graph into connected components, the membership oracle asserts whether any two vertices of the graph lie in the same component or not. We prove that for \( n \geq k \geq 2 \), learning the components of an \( n \)-vertex hidden graph with \( k \) components requires at least \( \frac{1}{2}(n-k)(k-1) \) membership queries. This proves the optimality of the \( O(nk) \) algorithm proposed by Reyzin and Srivastava [22] for this problem, improving on the best known information-theoretic bound of \( \Omega(n \log k) \) queries. Further, we construct an oracle that can learn the number of components of \( G \) in asymptotically fewer queries than learning the full partition, thus answering another question posed by the same authors. Lastly, we introduce a more applicable version of this oracle, and prove asymptotically tight bounds of \( \tilde{\Theta}(m) \) queries for both learning and verifying an \( m \)-edge hidden graph \( G \) using this oracle.

1 Introduction

1.1 Background and Applications

A graph \( G = (V, E) \) consists of a set of vertices \( V \) and edges \( E \subseteq \binom{V}{2} \). The field of graph learning deals with learning a hidden graph using queries to black-box oracles that reveal partial information about the graph. In several real world scenarios, learning a full graph by checking only pairwise adjacency is inefficient, and several oracles can speed up the process of learning by encoding more information per query. Different oracles could be useful or easier to implement in different scenarios. For example, in the context of trying to learn a hidden network graph, a so-called traceroute query between a pair of vertices can give information about a shortest path between the vertices and their distance in the graph. In the context of bioinformatics, one can model a graph with vertices corresponding to chemicals, and two vertices are joined by an edge if they react when mixed together. In such a situation, oracles such as edge-detection and edge-counting can be implemented by mixing different sets of chemicals and measuring the intensity of reaction. As calling an oracle incurs cost, researchers try to estimate the query complexity: the least number of queries to the oracle required to learn a specific graph or graph property.

A separate type of problem that is also widely studied is the problem of graph verification. In this setting, we have a hidden graph \( G = (V, E) \) and a known graph \( \hat{G} = (V, \hat{E}) \), and an oracle that reveals information about \( G \). The main task in this area is to verify whether \( G = \hat{G} \) using as few oracle queries as possible. Verification tasks are prevalent in real networks where it is important to make sure a recent snapshot of a network is accurate.

Due to their practical and theoretical importance, both graph learning and verification have garnered a lot of interest in recent years. Perhaps the first problem considered in the literature was the problem of learning a

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degree-bounded tree using the shortest path oracle \cite{12,16}. The shortest path oracle and the distance oracle were extensively studied in \cite{22,15,23}. The current best upper bound on learning a connected bounded-degree graph on \(n\) vertices is due to Mathieu and Zhou \cite{18}, where they prove that \(\tilde{O}(n^{3/2})\) distance queries suffice to learn such a graph. Abrahamsen et al. \cite{1} prove the exact bound on the query complexity using a weaker oracle called the betweenness oracle, thus suggesting that the bound of \(\tilde{O}(n^{3/2})\) is not tight. Parallel results in graph verification have also been obtained for the distance oracle \cite{14} and betweenness oracle \cite{13}. Although there is still a gap between the lower bound of \(\Omega(n)\) and upper bound of \(\tilde{O}(n^{3/2})\) for learning degree-bounded graphs using distance oracle, the gap was closed recently for random degree-bounded regular graphs by Mathieu and Zhou \cite{19}.

Another well-studied oracle is the edge-detection oracle, which, given a set of vertices of the hidden graph \(G\), tells if it is an independent set or not. Results on learning restricted classes of graphs using this oracle such as matchings \cite{3}, stars and cliques \cite{2}, Hamiltonian cycles \cite{11} were succeeded by a very general treatment by Angluin and Chen \cite{5}. Using a recursive coloring argument, they show that \(O(m \log n)\) edge-detection queries are sufficient to learn an arbitrary hidden graph. Reyzin and Srivastava \cite{22} consider and compare the shortest path, edge-detection and edge-counting queries and prove a variety of lower and upper bounds for learning partitions, trees and arbitrary graphs. In other related work, Beerliova et. al. \cite{6} consider an oracle called the layered-graph oracle.

1.2 Our Results

One of the problems studied by Reyzin and Srivastava \cite{22} entails learning the components of an \(n\)-vertex hidden graph \(G\) with \(k\) components. More precisely, if \(\alpha\) denotes the membership query given by

\[
\alpha(u, v) = \begin{cases}
1, & \text{if } u \text{ and } v \text{ belong to the same component}, \\
0, & \text{otherwise};
\end{cases}
\]

then they demonstrate an algorithm that uses \(O(nk)\) many \(\alpha\)-queries to learn the components of \(G\). However, the best known lower bound was an information-theoretic bound of \(\Omega(n \log k)\) queries. Although a bound of \(\Omega(nk)\) seems very intuitive, to the knowledge of the author, no proof of this result has been published till date. Our first result is the first tight lower bound of \(\Omega(nk)\) to this problem (when \(k \leq n/2\)), which proves that their algorithm is asymptotically optimal.

**Theorem 1.1.** Given any algorithm \(A\) that makes membership queries on a hidden graph \(G\) with \(n\) vertices and \(k\) components, there is an adversary that can force \(A\) to make at least \(\frac{1}{2}(n-k)(k-1)\) queries to learn the partition of \(G\).

**Remark.** We make a note here that Theorem 1.1 and the membership oracle is applicable to a much more general discrete setting: **learning partitions of any finite \(n\)-element set into \(k\) parts.** However, the problem of learning partitions is not well-represented in the literature and hence we use the more familiar and equivalent language of graph learning for presenting Theorem 1.1.

Reyzin and Srivastava \cite{22} also posed a question on whether there is an oracle that can learn the number of components in an \(n\)-vertex hidden graph in fewer queries than learning the components. For a vertex \(v\) and a set \(S\) of vertices not containing \(v\), we define the multiple-membership query

\[
\alpha_m(u, S) = \begin{cases}
1, & \text{if } u \text{ and } v \text{ belong to the same component for some } v \in S, \\
0, & \text{otherwise}.
\end{cases}
\]

Our second result gives a positive answer to their question:

**Theorem 1.2.** For an \(n\)-vertex hidden graph \(G\), learning the number of components of \(G\) can be done using \(O(n)\) \(\alpha_m\)-queries. However, learning all the components requires \(\Theta(n \log k)\) \(\alpha_m\)-queries.
Although $\alpha_m$ is an oracle which is theoretically interesting in its own right, it may be difficult to efficiently implement it in practical scenarios. Therefore, we modify $\alpha_m$ into a new type of query, which we call the vertex-neighborhood detection query $\beta$ given by:

$$\beta(u, S) = \begin{cases} 1, & \text{if } u \text{ and } v \text{ are adjacent for some } v \in S, \\ 0, & \text{otherwise.} \end{cases}$$

$\beta$ can be useful in the setting of biochemistry where it is easy to detect a reaction between a fixed reagent and a set of other chemicals, and could be applicable to genome sequencing using polymerase chain reaction (PCR) tests [7, 8].

We analyze the problems of graph learning and graph verification using $\beta$-queries, and prove tight bounds for these problems (upto logarithmic factors):

**Theorem 1.3.** Learning or verifying an $n$-vertex hidden graph on $m$ edges requires $\Omega(m)$ $\beta$-queries. Conversely, learning such a graph can be done in $O(m \log n)$ queries, whereas verifying can be done in $O(m + n)$ queries.

**Remark.** As an immediate consequence to Theorem 1.3, note that all sparse families of graphs which satisfy $m = O(n)$, such as trees, planar graphs and minor-free graphs, can be both learned and verified using $\tilde{\Theta}(n)$ queries to $\beta$.

This paper is organized as follows. In Section 2, we prove Theorems 1.1 and 1.2. Section 3 presents our results on the $\beta$ oracle, and proves Theorem 1.3. Finally, we make some concluding remarks in Section 4.

## 2 Graph partitions and the membership query

In this section, we consider the membership query given by $\alpha(u, v) = 1$ iff $u$ and $v$ belong to the same connected component. Our goal in this section is to prove Theorem 1.1.

### 2.1 Preliminaries

We briefly state some preliminary results before diving into the proof. Given a graph $G$, we say that it is $k$-colorable if there is a labelling $\chi : V(G) \to \{1, \ldots, k\}$ such that the two endpoints of any edge receives distinct labels. Such a $\chi$ is also called a proper $k$-coloring. $G$ is said to be uniquely $k$-colorable if it has a unique proper $k$-coloring $\chi$.

It is known (see, for e.g. [21, 23]) that graphs which are uniquely $k$-colorable have at least $\Omega(nk)$ edges:

**Lemma 2.1.** A uniquely $k$-colorable graph on $n$ vertices must have at least $(k - 1)n - \binom{k}{2}$ edges.

For the sake of completeness, we give an outline of the proof. We urge the reader to refer to [21, 25] for further information on uniquely $k$-colorable graphs.

**Proof Sketch for Lemma 2.1.** Suppose $G$ is a uniquely $k$-colorable graph on $n$ vertices, and its color classes are $I_1, \ldots, I_k$. Each $I_i$ is an independent set, and the edges of $G$ go across these color classes. Let us fix any pair $i \neq j$, and consider the bipartite graph $G[I_i \cup I_j]$. If $G[I_i \cup I_j]$ was disconnected, we could switch around these connected components to obtain a new proper $k$-coloring of $G$, a contradiction. Thus, $G[I_i \cup I_j]$ is a connected bipartite graph, implying there are at least $|I_i| + |I_j| - 1$ edges between components $I_i$ and $I_j$. It
can then be seen that $G$ has at least

$$
\sum_{i<j} |I_i| + |I_j| - 1 = (k-1)(|I_1| + \cdots + |I_k|) - \binom{k}{2}
$$

$$
= (k-1)n - \binom{k}{2}
$$

many edges.

We now have all the machinery required for proving Theorem 1.1, it will be presented in the following subsection. Recall that $\alpha(u, v) = 1$ iff $u$ and $v$ belong to the same component in an $n$-vertex hidden graph with $k$ components, and we are aiming to prove a lower bound of $\frac{1}{2}(n - k)(k - 1)$ queries on learning all its edges and non-edges.

### 2.2 Proof of Theorem 1.1

As mentioned earlier, our proof is adversarial. We (the adversary) start with initializing an empty auxiliary (simple) graph $H$ with $|V(H)| = n$, and pick any $k$-coloring $\chi : V(H) \rightarrow \{1, \ldots, k\}$ of $H$. Let $G$ be the hidden graph corresponding to $H$, where each color class corresponds to a partition. For the sake of simplicity, we shall call vertices of $H$ with degree at least $k - 1$ big, and those with degree less than $k - 1$ small.

Suppose now that $A$ makes a query $\alpha(x, y)$. If $\chi(x) \neq \chi(y)$, we add $xy$ to $E(H)$ and reply “no” to the algorithm. If $\chi(x) = \chi(y)$ and either $x$ or $y$ is small, say $x$, then we note that $x$ has at most $k - 2$ neighbors
and hence two admissible colors. We then modify \( \chi(x) \) to its other admissible color different from \( \chi(y) \), add the edge \( xy \) to \( E(H) \), and reply “no”. Finally, if \( \chi(x) = \chi(y) \) and both \( x \) and \( y \) are big, we identify (i.e., contract) vertices \( x \) and \( y \). In terms of \( G \), this would mean \( x \) and \( y \) belong to the same component. We illustrate some intermediate steps in the evolution of \( H \) and \( \chi \) against a sample algorithm in Figure 2.1.

Note that by construction, every edge of \( H \) corresponds to a non-coincidence in \( G \’s \) components. Vertices of \( H \) that have been identified together must lie in the same component. Further, \( \chi \) is always a proper \( k \)-coloring of \( H \), and hence induces a partition of \( G \) into \( k \) components.

Now, suppose that \( A \) learns a unique partition of \( G \) into \( k \) components. There are two cases to consider:

- **Case 1. The adversary contracts at least \( \frac{n}{2} \) vertices:** In this case, as each contracted vertex has degree at least \( k-1 \), \( A \) must have made at least \( \frac{n}{2}(k-1) \) queries.

- **Case 2. The adversary contracts less than \( \frac{n}{2} \) vertices:** In this case, \( A \) can learn a unique partition of \( G \) iff \( \chi \) is a unique \( k \)-coloring of \( H \). Since \( H \) started with \( n \) vertices, \( H \) is a graph on at least \( \frac{n}{2} \) vertices which is uniquely \( k \)-colorable. By Lemma 2.1, \( H \) has at least \( \frac{n}{2}(k-1) - \binom{k}{2} \) edges.

In either case, \( A \) makes at least \( \frac{n}{2}(k-1) - \binom{k}{2} = \frac{1}{2}(n-k)(k-1) \) queries, as desired. \( \square \)

Now we turn our attention to the problem of learning both \( k \) and individual components using multiple-membership queries.

### 2.3 Proof of Theorem [1,2]

Recall that \( \alpha_m(u,S) = 1 \) if and only if there is a \( v \in S \) such that \( u \) and \( v \) belong to the same component. There are three assertions in Theorem [1,2] and we prove each of them below.

- **Part 1. Learning \( k \):** First, we demonstrate an algorithm that learns the number of components of an \( n \)-vertex hidden graph \( G \) using \( O(n) \) queries to \( \alpha_m \). Start with \( S = V(G) \). While there is a vertex \( v \in S \) such that \( \alpha_m(v,S \setminus \{v\}) = 1 \), we delete \( v \) from \( S \). When this algorithm terminates, we end up with an independent set \( S \) whose each vertex lies in a different component. Conversely, as every other vertex \( v \in V(G) \) lies in the same component as some vertex in \( G \), \( |S| \) will equal the number of components of \( G \). Therefore, the above algorithm learns \( k \) using \( O(n) \) many \( \alpha_m \)-queries.

- **Part 2. Lower bound on learning components:** Next, we claim that learning all components of \( G \) requires at least \( \Omega(n \log k) \) queries. This proof is information-theoretic. Encoding the component for each vertex takes \( \log k \) bits of information, and each \( \alpha_m \)-query gives one bit of information.

- **Part 3. Upper bound on learning components:** Note that Algorithm [1] learns the components of the hidden graph \( G \).

The binary search in Step [7] works because \( \alpha_m \) can query \( v_i \) with a union of several \( C_j \’s \) at once. This step requires \( O(\log k) \) queries, and therefore our algorithm has query complexity of \( O(n \log k) \). \( \square \)

This completes the proof of Theorem [1,2] and hence \( \alpha_m \) is an oracle that can learn the number of components \( k \) in a hidden graph with fewer queries than learning the components. \( \square \)

### 3 The vertex-neighborhood detection query

For the remainder of the paper, we consider the vertex-neighborhood detection query given by \( \beta(v,S) = 1 \) iff there is some edge from \( v \) to some vertex in \( S \). We now prove tight bounds of \( \Theta(m) \) for both learning and verifying graphs (with \( m \) edges) using \( \beta \)-queries.
Algorithm 1: Algorithm for learning components using $\alpha_m$.

**Input**: Vertex set $V(G) = \{v_1, \ldots, v_n\}$, Oracle $\alpha_m$.

**Output**: Partition $V(G) = \bigcup_{i=1}^k C_i$ such that each $C_i$ is a connected component.

1. Initialize $C_1 = \{v_1\}$, $k = 1$;
2. \textbf{for} $i = 2$ to $n$ \textbf{do}
3. \hspace{1em} if $\alpha_m(v_i, C_1 \cup \cdots \cup C_k) = 0$ then
4. \hspace{2em} $k += 1$; \hspace{1em} // add $v_i$ in its own component
5. \hspace{2em} Add $v_i$ to $C_k$;
6. \hspace{1em} else
7. \hspace{2em} Figure out which $C_j$ satisfies $\alpha_m(v_i, C_j) = 1$ using binary search among $\{C_1, \ldots, C_k\}$;
8. \hspace{2em} Add $v_i$ to its corresponding $C_j$;
9. \textbf{return} $\{C_1, \ldots, C_k\}$

### 3.1 Proof of Theorem 1.3

Now we move onto analyzing the problems of graph learning and verification using the oracle $\beta$. Let us take a hidden graph $G$ on $n$ vertices and $m$ edges. It is clear that $\beta$ can only detect one edge at a time, and hence both learning and verifying a hidden graph using $\beta$ would trivially require at least $\Omega(m)$ queries. Hence, it suffices to prove upper bounds of $O(m + n)$ for verification and $O(m \log n)$ for learning $G$.

#### 3.1.1 Verification using $\beta$

For the verification problem, we have a graph $\hat{G}$ with $V(\hat{G}) = V(G) = V$ that is known to us. We verify each edge $uv \in E(\hat{G})$ individually by checking $\beta(u, \{v\}) = 1$, and this requires $m$ queries.

Next, we verify the non-edges of $\hat{G}$. Fix a vertex $v$ and compute its neighborhood $N_{\hat{G}}(v) = \{u \in V : uv \in E(\hat{G})\}$. Note that if $\hat{G}$ was the same as $G$, we would have

$$\beta(v, V \setminus N_{\hat{G}}(v)) = 0.$$ 

Since checking all non-edges through a vertex takes a single $\beta$-query, we can verify all non-edges of $\hat{G}$ using $n$ queries. Hence, verifying all edges and non-edges of $G$ using can be done using $O(m + n)$ $\beta$-queries.

#### 3.1.2 Learning using $\beta$

Although slightly more involved, our algorithm for learning a hidden graph $G$ is also based on divide-and-conquer. The main step consists of devising a recursive method (which we call findNeighbors) that, for a fixed vertex $v \in V$, learns all neighbors (and non-neighbors) of $v$.

Let us first analyze the query complexity of findNeighbors.

**Claim 3.1.** Suppose $\ell$ is the number of neighbors of $v$ in $S$. Then, findNeighbors$(v, S)$ takes at most $O(\ell \log |S|)$ queries to mark all neighbors of $v$ in $S$ red and non-neighbors blue.

**Proof of Claim 3.1** We take a look at the recursion tree $T$ for findNeighbors$(v, S)$, and color its nodes red or blue as follows. The root node corresponds to the computation of findNeighbors$(v, S)$, and so we keep the set $S$ in it. If $\beta(v, S) = 1$, we color the root node blue, otherwise we color it red. In the next level, the node containing $S$ has two children: $S_1$ and $S_2$ corresponding to the equipartition of $S$. Let us color $S_1$ blue if
**Function** findNeighbors($v, S$).

**Input:** Vertex $v$, Set $S$, Hidden graph $G$, Oracle $\beta$.

**Result:** Mark all neighbors of $v$ in $S$ blue, and non-neighbors red.

1. **if** $|S| = 1$ **then**
2.  **if** $\beta(v, S) = 1$ **then**
3.  Mark the vertex of $S$ blue and terminate;
4.  **else**
5.  Mark the vertex of $S$ red and terminate;
6. **else**
7.  Divide $S$ into two (approximately) equal parts $S_1 \sqcup S_2$;
8. **if** $\beta(v, S_1) = 1$ **then**
9.  findNeighbors($v, S_1$);
10. **else**
11.  Mark all vertices of $S_1$ red and terminate;
12. **if** $\beta(v, S_2) = 1$ **then**
13.  findNeighbors($v, S_2$);
14. **else**
15.  Mark all vertices of $S_2$ red and terminate;

$\beta(v, S_1) = 1$, and red otherwise. Similarly, we color $S_2$, and continue this coloring scheme down the entire recursion tree. A sample $T$ and its coloring is depicted in Figure 3.1.

![Figure 3.1: A sample recursion tree $T$ corresponding to findNeighbors($v, S$)](image)

Let $T_b$ be the subtree with blue vertices. It is clear that each leaf of $T_b$ corresponds to a vertex marked blue by findNeighbors, implying that $T_b$ is a tree with $\ell$ leaves and depth at most $\log |S|$. Therefore,

$$|V(T_b)| \leq O(\ell \log |S|).$$

Now we take a closer look at the red vertices of $T$. By construction, each red vertex is a leaf, i.e. the computation stops at these vertices. This means that the parents of red vertices are always blue. Further, two red nodes cannot be siblings, as if $\beta(v, A) = \beta(v, B) = 0$, then $\beta(v, A \cup B) = 0$, implying that the computation would have stopped at the parent of $A$ and $B$. Thus, every red vertex has a *unique* blue parent. This implies that the number of red vertices in $T$ is bounded above by $|V(T_b)|$, leading to

$$|V(T)| \leq 2|V(T_b)| \leq O(\ell \log |S|),$$

completing the proof of Claim 3.1.
Therefore, to learn $G$, we can run $\text{findNeighbors}(v, V \setminus \{v\})$ over all vertices $v$. Observe that the total number of queries made is at most

$$\sum_{v \in V(G)} O(\deg(v) \log n) = O(m \log n),$$

thus proving the required upper bound.

\section{Conclusion and future work}

In this paper, we demonstrated a fundamental lower bound on the problem of learning partitions using membership queries, filling a long lasting gap in the literature. We generalized the membership oracle to take subsets of vertices as one of its input, and demonstrated its ability to learn the number of components in fewer queries than the components themselves. In the second section, we also demonstrated a powerful oracle that is very efficient at learning and verifying sparse graphs.

It would be very interesting to see other oracles which can exploit structural properties of sparse graphs, such as the existence of small separators. Graph families that admit the existence of small separators include a vast array of graphs such as planar graphs \[\text{[17, 4]},\text{ bounded genus graphs}\[\text{[10]}\], minor-free graphs \[\text{[21, 26]}\], etc. Further, it should be possible to extend these results to learning and verification of graphs with polynomially bounded expansion \[\text{[20, 9]}\], and we leave this as a future avenue of investigation.

\textbf{Acknowledgments.} The author is immensely thankful to Mano Vikash Janardhanan for bringing the membership oracle and the paper \[\text{[22]}\] to his attention, and for several fruitful discussions on graph learning theory.

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