Inhomogeneities in 3 dimensional oscillatory media

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Abstract

We consider localized perturbations to spatially homogeneous oscillations in dimension 3 using the complex Ginzburg-Landau equation as a prototype. In particular, we will focus on heterogeneities that locally change the phase of the oscillations. In the usual translation invariant spaces and at $\varepsilon = 0$ the linearization about these spatially homogeneous solutions result in an operator with zero eigenvalue embedded in the essential spectrum. In contrast, we show that when considered as an operator between Kondratiev spaces, the linearization is a Fredholm operator. These spaces consist of functions with algbraical localization that increases with each derivative. We use this result to construct solutions close to the equilibrium via the Implicit Function Theorem and derive asymptotics for wavenumbers in the far field.

1 Introduction

This paper is concerned with the effect of inhomogeneities in oscillatory media. As a prototype we study the complex Ginzburg-Landau equation,

$$A_t = (1 + i\alpha)\Delta A + A - (1 + i\gamma)A|A|^2,$$

which is known to approximate the phase and amplitude of modulation patterns in reaction diffusion systems near a supercritical Hopf bifurcation \cite{1}. Stationary in time inhomogeneities which produce a localized change in the phase of oscillations in such a system can be well modeled by the inclusion of a term $i\varepsilon g(x)A$. The effects of such inhomogeneities can vary dramatically depending on the sign of $\varepsilon$ and the space dimension. This has been explored formally in the phase-diffusion approximation in \cite{16} and for radially symmetric inhomogeneities in \cite{6}. Most notably, inhomogeneities can create wave sources in space dimension 1 and 2. In dimension 3 and radial geometry it was shown in \cite{6} that sources are weak, that is, wavenumbers decay in the far field. In this note, we establish a similar result without the assumption of radial symmetry and without relying on spatial dynamics. In addition, we relax the assumption of spatial decay of $g(x)$.

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One would hope to use the Implicit Function Theorem to find approximations near a spatially homogeneous solution to the complex Ginzburg-Landau equation, but as we will see the linearization results in an operator which is not Fredholm in the usual translation invariant Sobolev spaces. This is a consequence of zero belonging to the essential spectrum, which in some instances can be taken care of by working in exponentially localized spaces. However, since we will be considering algebraically localized inhomogeneities these spaces do not provide the appropriate framework, the use of exponential weights would turn the linearization into a semi-Fredholm operator with infinite dimensional cokernel. Instead, we will use a functional analysis approach and try to recover Fredholm properties of the linearization using Kondratiev spaces and the results from [11], where it was shown that the Laplacian is a Fredholm operator.

The algebraic localization of the heterogeneity will force us to consider weights \((1 + |x|^2)^{\delta/2}\) with \(\delta > 0\). As a consequence, the linearized operator has a nonzero cokernel and we need to add variables to our system. The result are expressions for the far field corrections, which unfortunately prove problematic in the critical case of dimension 2 since the nonlinearity produces terms which are not well posed, i.e. they do no belong to the correct weighted space. We hope to address these issues in the future and restrict ourselves in the present paper to the 3 dimensional case. This will provide a straightforward example where the advantage of viewing the linearization in the setting of Kondratiev spaces can be appreciated without the extra complications coming from the nonlinearity.

We begin the analysis by considering the spatially homogeneous solution \(A_*(t) = e^{-i\gamma t}\) of equation (3.1) and looking for approximations of the form \(A(x, t) = (1 - s(x))e^{-i(\gamma t - \phi(x))}\). We will show in the last section, using Lyapunov-Schmidt reduction, that in dimension 3 it is possible to find solutions near \(A_*\).

The asymptotics for the function \(\phi(x)\) will show that in the far field the wavenumber \(k \sim \nabla \phi\) decays to zero and hence target patterns will not form. We state this result in the following Theorem:

**Theorem 1** Suppose \(\delta \in (0, 1/2)\), \(g \in L^2_{\delta + 2}\) and \(1 + a\gamma > 0\). Then, there exist \(\epsilon_0 > 0\) and smooth functions \(S(x, \epsilon)\) and \(\Phi(x, t; \epsilon)\) such that

\[
A(x, t; \epsilon) = S(x, \epsilon)e^{-i\Phi(x, t; \epsilon)}
\]

is a family of solutions to (3.1) near \(A = e^{-i\gamma t}\) for all \(\epsilon \in (-\epsilon_0, \epsilon_0)\). Furthermore, for fixed \(\epsilon \in (-\epsilon_0, \epsilon_0)\) and \(t\), the functions \(S(x; \epsilon)\) and \(\Phi(x, t; \epsilon)\) satisfy the following asymptotics in \(x\),

\[
S(x, \epsilon) = 1 + O\left(\frac{1}{|x|^{\delta+2}}\right),
\]

\[
\Phi(x, t; \epsilon) = -i\gamma t + i\frac{c(\epsilon)}{|x|} \left(1 + o_1(1/|x|)\right),
\]

as \(|x| \to \infty\), where \(c(\epsilon)\) is a smooth function satisfying the expansion \(c(\epsilon) = \epsilon c_1 + O(\epsilon^2)\). In particular,

\[
c_1 = \frac{-1}{4\pi(1 + a\gamma)} \int g \, dx.
\]

Again we point out that the above agrees with the results found in [6] where the authors show that in dimensions 3 and higher, there exists only contact defects (the wave number \(k \sim \nabla \phi = 0\)) and obtain asymptotics for the wavenumber \(k\),

\[
k(r, \epsilon) = \frac{M\epsilon}{r^{n-1}}(c + O_1/r(1))
\]
Figure 1.1: Snapshots of $|A|$ (see Section 3.1) at the intersection of the planes $y = 10$ and $z = 10$ for times $T = 500$, $T = 750$, $T = 1000$

where the notation $O_y(1)$ means that these terms go to zero as $y \to 0$. This implies that for large values of $|x|$ and fixed $\epsilon$ we do not see a pace maker effect. Nonetheless, if we fix $|x|$ large we can approximate the group velocity, $c_g$, for the family of solutions $A(x, t; \epsilon)$ in terms of $\epsilon$:

$$c_g(\epsilon) = 2(\alpha - \gamma)k \sim -2(\alpha - \gamma)\frac{\epsilon c_1}{|x|^3}.$$  

In particular, if $\epsilon(\gamma - \alpha) \int g > 0$ then $c_g > 0$ and we obtain weak wave sources. These results were confirmed in numerical simulations with a cubic domain of length $l = 20$ and with parameter values $\alpha = 1$, $\gamma = 5$, see figure 1.1. Furthermore, our numerical results confirm that as $|x| \to \infty$ the amplitude, $|A|$ decays to 1, see figure 1.2 where the inhomogeneity was given by

$$g(x, y, z) = \frac{1}{(1 + 1/4(x - 10)^2 + 2(y - 10)^2 + (z - 10)^2)^3}.$$  

Notice as well that the color scale in the case of positive group velocity, i.e. $\epsilon = -2$, is slightly larger indicating that the effects of the inhomogeneity are stronger. All simulations were done with an exponential time differencing algorithm (ETDRK4) following the methods found in [5, 4].

Figure 1.2: Snapshots of $|A|$ (see Section 3.1) at cross section $z = 10$ for time $T = 250$. 

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This paper is organized as follows: In Section 2, we define weighted Sobolev spaces and Kondratiev spaces and state Fredholm properties for the Laplacian. Finally in Section 3 we give a proof of our main result.

2 Weighted and Kondratiev spaces

2.1 Weighted spaces

Weighted Sobolev spaces refer to a large class of Banach spaces defined with the same norm as regular Sobolev spaces, but with the difference that the Lebesgue measure is replaced by a measure that comes from a density function, i.e. the weight \( w(x) \). These spaces have been considered when showing existence and uniqueness of solutions to degenerate elliptic PDE’s \(^2\). In this paper we consider the weight \( \langle x \rangle = (1 + |x|^2)^{1/2} \) and define the spaces \( W^{k,p}_{\delta} \) as the completion of \( C^\infty_0(\mathbb{R}^n) \) under the norm

\[
\|u\|_{W^{k,p}_{\delta}} = \left( \sum_{|\alpha| \leq k} \|D^\alpha u \cdot \langle x \rangle^\delta \|_{L^p}^p \right)^{1/p},
\]

with \( 1 < p < \infty, \delta \in \mathbb{R} \) and \( k \in \mathbb{N} \). Notice that we have inclusions of the form \( W^{k,p}_{\beta} \subset W^{k,p}_{\alpha} \) for any real numbers \( \alpha, \beta \) such that \( \alpha < \beta \). Furthermore, we have the following proposition which was proven in \(^3\)

**Proposition 2.1** The operator \( \Delta - a : W^{2,p}_{\delta} \to L^p_{\delta} \) is invertible for all real numbers \( a > 0 \) and \( p \in (1, \infty) \).

The above proposition also shows us why the Laplace operator does not have closed range when considered in the setting of weighted Sobolev spaces: just as in the case of \( \Delta : H^2 \to L^2 \), we can construct Weyl’s sequences for the Laplace operator proving that zero is in the essential spectrum. We summarize this results as a lemma:

**Lemma 2.2** The operator \( \Delta_{\delta} : W^{2,p}_{\delta} \to L^p_{\delta} \) is not a Fredholm operator for \( p \in (1, \infty) \).

2.2 Kondratiev spaces

A slight variation of the above spaces are Kondratiev spaces, where the exponent in the weight \( \langle x \rangle \) is increased by one every time we take a derivative. We denote them here by \( M^{k,p}_{\delta} \) and defined them as the completion of \( C^\infty_0(\mathbb{R}^n) \) under the norm

\[
\|u\|_{M^{k,p}_{\delta}} = \left( \sum_{|\alpha| \leq k} \|D^\alpha u \cdot \langle x \rangle^\delta + |\alpha| \|_{L^p}^p \right)^{1/p},
\]

again with \( 1 < p < \infty, \delta \in \mathbb{R} \) and \( k \in \mathbb{N} \).

Kondratiev spaces were introduced in connection with boundary value problems for elliptic equations in domains with critical points \(^7\). They also appear in the setting of unbounded domains, for example, Nirenberg and Walker showed in \(^14\) that a class of elliptic operators with coefficients that decay sufficiently fast at infinity have finite dimensional kernel. These results were later used by McOwen...
and Lockhart to study Fredholm properties of elliptic operators and systems of elliptic operators in non-compact manifolds \[8, 9, 10\]. More recently a variant of these spaces was used in \[13\] to study Poisson’s equation in a one-periodic infinite strip \(Z = [0, 1] \times \mathbb{R}\).

These spaces have also been used in the description of far field asymptotics for fluid problems, in particular when studying the flow past obstacles, since they lend themselves to the study of problems in exterior domains, see \[15\] for the case of \(\mathbb{R}_3\) and \[12\] for an application towards bifurcation theory.

The main advantage for us is that in Kondratiev spaces the Laplace operator is a Fredholm operator. These results are shown in McOwen’s paper \[11\] and are summarized in the following theorem.

**Theorem 2** Let \(1 < p = \frac{q}{q-1} < \infty, n \geq 2,\) and \(\delta \neq -2+n/q+m\ or \delta \neq -n/p-m\, for\ some \ m \in \mathbb{N}\). Then

\[\Delta : M^{2,p}_\delta \to L^{p}_{\delta+2},\]

is a Fredholm operator and

(i) for \(-n/p < \delta < -2+n/q\ the map is an isomorphism;

(ii) for \(-2+n/q+m < \delta < -2+n/q+m+1\, m \in \mathbb{N},\) the map is injective with closed range equal to

\[R_m = \left\{ f \in L^p_{\delta+2} : \int f(y)H(y) = 0 \text{ for all } H \in \bigcup_{j=0}^m \mathcal{H}_j \right\};\]

(iii) for \(-n/p-m-1 < \delta < -n/p-m\, m \in \mathbb{N},\) the map is surjective with kernel equal to

\[N_m = \bigcup_{j=0}^m \mathcal{H}_j.\]

Here, \(\mathcal{H}_j\) denote the harmonic homogeneous polynomials of degree \(j\).

On the other hand, if \(\delta = -n/p-m\ or \delta = -2+n/q+m\ for\ some \ m \in \mathbb{N},\) then \(\Delta\) does not have closed range.

3 Proof of Theorem

To facilitate the analysis we will split this section into four parts. In Subsection 3.1 we describe how we set up the problem and how we obtain a linearization which is easier to work with. Next, in Subsection 3.2 we state conditions that allow us to use the Implicit Function Theorem (IFT) and derive expansions for the amplitude and phase, effectively proving the results of Theorem. Finally, in the last two subsections we show that the linearization is invertible and the nonlinear operator associated to our problem is well defined.

3.1 Set up

We recall here our main equation, the complex Ginzburg-Landau equation in dimension 3,

\[A_t = (1 + ia)\Delta A + A - (1 + iy)A|A|^2 + i\varepsilon g(x)A,\quad (3.1)\]
where $g(x)$ is a localized real valued function and $\varepsilon$ is small. In what follows we describe how we arrive at our linearization.

We pass to a corotating frame $A = e^{-i\Omega t} \tilde{A}$, so that $\tilde{A}$ satisfies the following equation,

$$\tilde{A}_t = (1 + i\alpha)\Delta \tilde{A} + (1 + i\Omega)\tilde{A} - (1 + i\gamma)\tilde{A}|\tilde{A}|^2 + i\varepsilon g(x)\tilde{A}. \quad (3.2)$$

At parameter values $\Omega = \gamma$ and $\varepsilon = 0$ the function $\tilde{A}_s = 1$ is a solution of (3.2). Substituting $\tilde{A} = (1 + s)e^{i\phi}$ into equation (3.2) and separating the real and imaginary parts leads to the following system

$$s_t = \Delta s - 2s - \alpha\Delta \phi - (1 + s)|\nabla \phi|^2 - 2\alpha \nabla s \cdot \nabla \phi - 3s^2 - s^3 - \alpha s \Delta \phi, \quad (3.3)$$

$$(1 + s)\phi_t = \alpha\Delta s + \Delta \phi(1 + s) - \alpha(1 + s)|\nabla \phi|^2 + 2\nabla s \cdot \nabla \phi + \Omega(1 + s) - \gamma(1 + s)^3 + \varepsilon g(x)(1 + s). \quad (3.4)$$

The right hand side of the above equations will help us define the nonlinear operator $F : \mathcal{X} \rightarrow \mathcal{Y}$, where $\mathcal{X}, \mathcal{Y}$ are Banach spaces that will be defined later. Our aim is to find steady solutions near $\tilde{A}_s = 1$ via the Implicit Function Theorem, and so we are interested in the linearization at $\Omega = \gamma, \varepsilon = 0$:

$$L \begin{bmatrix} s \\ \phi \end{bmatrix} = \begin{bmatrix} \Delta - 2 & -\alpha \Delta \\ \alpha \Delta - 2\gamma & \Delta \phi \end{bmatrix} \begin{bmatrix} s \\ \phi \end{bmatrix}.$$ 

In Fourier space the linearization, $L$, can be represented by a matrix, $\mathcal{F}(L)(k)$, which at $k = 0$ has eigenvalues $\lambda_1 = -2$, and $\lambda_2 = 0$. This suggest that in order to simplify future computations we use the following change of coordinates

$$\hat{s} = \gamma s, \quad \hat{\phi} = -\gamma s + \phi,$$

so as to diagonalize $\mathcal{F}(L)(0)$. The resulting operator that comes from the right hand side of the equations for $\hat{s}_t$ and $\hat{\phi}_t$, and which we will label again as $F$, is given by the following two components

$$F_1(s, \phi) = (1 - \alpha \gamma)\Delta s - 2s - \gamma \alpha \Delta \phi - (\gamma + s)|\nabla s|^2 + 2\nabla s \cdot \nabla \phi + |\nabla \phi|^2 - 2\alpha |\nabla s|^2 - 2\alpha \nabla s \cdot \nabla \phi \quad (3.5)$$

$$- \alpha s(\Delta s + \Delta \phi) - \frac{3}{\gamma} s^2 - \frac{1}{\gamma} s^3,$$

$$F_2(s, \phi) = \left(\frac{\alpha}{\gamma} + \alpha \gamma\right)\Delta s + (1 + \alpha \gamma)\Delta \phi + (\gamma - \alpha s)|\nabla \phi|^2 + 2\nabla s \cdot \nabla \phi + |\nabla \phi|^2 + \alpha s(\Delta s + \Delta \phi) \quad (3.6)$$

$$+ 2\alpha |\nabla s|^2 + 2\alpha \nabla s \cdot \nabla \phi + \frac{3 s^2}{\gamma} + \frac{s^3}{\gamma^2} + (\gamma + s)^{-1} \left[2|\nabla s|^2 + 2\nabla s \cdot \nabla \phi - s^2 - \frac{s^3}{\gamma} - \alpha s \Delta s\right] + \varepsilon g(x)$$

where we have suppressed the “hats” from the variables. We will show in the last section that this operator is well defined and smooth.

Next, we introduce the following Ansatz for equation (3.1)

$$\tilde{A}(x, t, \varepsilon) = S(x, \varepsilon)e^{i\phi(x, t, \varepsilon)}, \quad (3.7)$$

where

$$S(x, \varepsilon) = 1 + s(x, \varepsilon), \quad \Phi(x, \varepsilon) = -i(\gamma t - \phi(x, \varepsilon)), \quad \phi(x, \varepsilon) = \phi(x, \varepsilon) + \varepsilon(\varepsilon \frac{\chi(|x|)}{|x|},$$

$$\frac{p}{\varepsilon}.$$
and $\chi \in C^\infty(\mathbb{R})$ is a cut-off function equal to zero near the origin and equal to 1 for $|x| > 2$. This amounts to letting $\phi(x, \varepsilon) = \hat{\phi}(x, \varepsilon) + c(\varepsilon)P(x)$ in the definition for $\hat{A}$ and so the linearization about the origin, $L : \mathcal{X} \times \mathbb{R} \to \mathcal{Y}$, of the nonlinear operator $F$ defined by the right hand sides of (3.5) and (3.6) is given by

$$L \begin{bmatrix} s \\ \phi \\ c \end{bmatrix} = \begin{bmatrix} (1 - \alpha \gamma)\Delta - 2 & -\alpha \gamma \Delta & -\alpha \gamma \Delta P \\ (\alpha \gamma + \hat{\gamma})\Delta & (1 + \alpha \gamma)\Delta & (1 + \alpha \gamma)\Delta P \end{bmatrix} \begin{bmatrix} s \\ \phi \\ c \end{bmatrix}. $$

This is the linear operator that we analyze in Subsection 3.3.

### 3.2 Main results: Expansions for phase $\phi$ and amplitude $s$

The following proposition, whose proof is done in the last two sections, states conditions for $L : \mathcal{X} \times \mathbb{R} \to \mathcal{Y}$ to be invertible. The choice of Banach spaces $\mathcal{X} = W^{2,2}_{\delta+2} \times M^{2,2}_\delta$ and $\mathcal{Y} = L^2_{\delta+2} \times L^2_{\delta+2}$ is done so that the Laplace operator is Fredholm and to guarantee that the nonlinear terms are in the correct weighted space.

**Proposition 3.1** Let $\delta \in (0, 1/2)$ and let $g \in L^2_{\delta+2}$. Then the operator $F : W^{2,2}_{\delta+2} \times M^{2,2}_\delta \times \mathbb{R} \to L^2_{\delta+2} \times L^2_{\delta+2}$ defined by (3.5) and (3.6) and the Ansatz (3.7) is smooth and its Fréchet derivative $DF$ evaluated at the origin is invertible.

This result and the Implicit Function Theorem show the existence of $\varepsilon_0 > 0$ and steady solutions to (3.5), (3.6) of the form $\xi(x; \varepsilon) = (s(x, \varepsilon), \hat{\phi}(x, \varepsilon), c(\varepsilon))$. Furthermore, the functions $s, \hat{\phi}$ and $c$ are smooth in $\varepsilon$ for all $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ and are equal to zero at $\varepsilon = 0$.

Now, the decay rates for the functions $S(x, \varepsilon)$ and $\Phi(x, t, \varepsilon)$ as $|x| \to \infty$ stated in Theorem I follow from our choice of weighted spaces. Since we can write $s(x, \varepsilon)(x)^{\delta+2} \in W^{2,2}(\mathbb{R}^3)$ then the inclusion $W^{2,2}(\mathbb{R}^3) \subset L^\infty(\mathbb{R}^3)$ implies that $S(x, \varepsilon) \to 1$. To show the decay of the phase $\Phi(x, t, \varepsilon)$ we need a similar inclusion of the space $M^{2,2}_\delta$ into $L^\infty$. This result follows from the following lemma whose proof can be found in [3].

**Lemma 3.2** If $f \in M^{2,2}_\gamma$ then $|f(x)| \leq C\|f\|_{M^{2,2}_\gamma(x)}^{-\gamma-1}$ as $|x| \to \infty$.

It is then clear that in the far field the function $\hat{\phi}(x, \varepsilon)$ decays to zero faster than $1/|t|$ and consequently we obtain the desired decay rate for $\Phi(x, t, \varepsilon)$.

Lastly, we prove the expansion for the function $c(\varepsilon)$ stated in Theorem I. This is obtained by using Lyapunov-Schmidt reduction and the results of the next subsection where we show that the vector $(0, 1)^T$, spans the cokernel of the operator $\hat{L} : W^{2,2}_\delta \times M^{2,2}_{\delta+2} \to L^2_\delta \times L^2_\delta$ defined by the first two columns of $L$. Then, taking the projection of $F(s(x, \varepsilon), \hat{\phi}(x, \varepsilon), c(\varepsilon))$ onto the cokernel and assuming expansions of the form $\xi(x; \varepsilon) = \varepsilon\xi_1 + O(\varepsilon^2)$, we obtain at order $O(\varepsilon)$ an expression for the coefficient $c_1$ corresponding to the far field correction $\Delta P$:

$$\int g \, dx = \int (\alpha \gamma + \frac{\alpha}{\gamma})\Delta s_1 + (1 + \alpha \gamma)\Delta \hat{\phi}_1 + c_1(1 + \alpha \gamma)\Delta P \, dx$$

$$= - 4\pi(1 + \alpha \gamma)c_1$$

$$c_1 = -\frac{\int g \, dx}{4\pi(1 + \alpha \gamma)}.$$
where the last two equalities follow from Theorem [2] and the fact that
\[ \int \Delta \left( \frac{\chi(|x|)}{|x|} \right) \, dx = -4\pi. \]

### 3.3 The Linear operator

In this subsection we prove the results from Proposition [3.1]. First, we will use the results from Section 2 to show that the linearization, \( \hat{L} : X \to Y \), of the full operator \( F \) defined by (3.5) and (3.6) is Fredholm with index \(-1\). Next, we show that the Ansatz (3.7) adds good far field corrections so that the resulting linearization, \( L : X \times \mathbb{R} \to Y \) is an invertible operator.

Here we define \( \hat{L} \) explicitly for future reference,
\[
\hat{L} \begin{bmatrix} s \\ \phi \end{bmatrix} = \begin{bmatrix} (1 - \alpha \gamma) \Delta - 2 & -\gamma \alpha \Delta \\ \gamma \alpha + \frac{\phi}{\gamma} & (1 + \gamma \alpha) \Delta \end{bmatrix} \begin{bmatrix} s \\ \phi \end{bmatrix}, \tag{3.8}
\]
and set \( X = W^{2,p}_{\delta+2} \times M^{2,p}_\delta \) and \( Y = L^p_{\delta+2} \times L^p_{\delta+2} \).

**Lemma 3.3** Let \( \delta \in (0,1/2) \), and \( 1 + \gamma \alpha > 0 \). The linear operator \( \hat{L} : W^{2,2}_{\delta+2} \times M^{2,2}_\delta \to L^2_{\delta+2} \times L^2_{\delta+2} \).

defined by (3.8) is a Fredholm operator with index \( i = -1 \) and cokernel spanned by the vector \((0,1)^T\).

**Proof.** From the second component we obtain
\[
\Delta \phi = \frac{g}{1 + \alpha \gamma} - \frac{\alpha \gamma + \alpha / \gamma}{1 + \alpha \gamma} \Delta s,
\]
which can then be plugged into the first component in order to solve for \( s \), provided that \( 1 + \alpha \gamma > 0 \):
\[
s = [(1 + \alpha^2) \Delta - 2(1 + \alpha \gamma)]^{-1}(1 + \gamma \alpha)f + [(1 + \alpha^2) \Delta - 2(1 + \alpha \gamma)]^{-1} \alpha \gamma g.
\]
Then, using this result in the equation for \( \Delta \phi \) and exploiting the linearity of the operators we obtain the following formula:
\[
\Delta \phi = [(1 + \alpha^2) \Delta - 2(1 + \alpha \gamma)]^{-1}[(1 - \alpha \gamma) \Delta - 2]g + \Delta[(1 + \alpha^2) \Delta - 2(1 + \alpha \gamma)]^{-1}(1 + \alpha \gamma)f.
\]
In order to solve for \( \phi \) we need the right hand side of the above equation to be in the range of the Laplacian. It is clear that the second component satisfies this requirement for any \( f \in L^2_{\delta+2} \) given that it involves the Laplacian and that the operator \([(1 + \alpha^2) \Delta - 2(1 + \alpha \gamma)]^{-1} : L^2_{\delta+2} \to W^{2,2}_{\delta+2} \) is bounded.

We will show that if \( g \) satisfies \( \int g = 0 \) then the first component is also in the range of the Laplacian. The result follows if we can show that the operator, \( A : L^2_{\delta+2} \to L^2_{\delta+2} \) defined by
\[
A = [(1 + \alpha^2) \Delta - 2(1 + \alpha \gamma)]^{-1}[(1 - \alpha \gamma) \Delta - 2]
\]

preserves this condition. To see this observe that the condition \( \int g = 0 \) is equivalent to the Fourier transform of \( g \) satisfying \( \hat{g}(0) = 0 \). So that if we consider the Fourier symbol of \( A \).
\[
\hat{A}(k) = \frac{(1 - \alpha \gamma)|k|^2 + 2}{(1 + \alpha^2)|k|^2 + 2(1 + \alpha \gamma)}
\]
we see that \( \mathcal{F}(Ag)(0) = 0 \) if and only if \( g(0) = 0 \), again because \( 1 + \alpha \gamma > 0 \). This proves the Lemma.
Remark 3.4 Observe that the condition $1 + a\gamma > 0$ is also required for spectral stability, an indication that these methods are consistent with previous results.

Next, consider the Ansatz:

$$\phi = \tilde{\phi} + c \frac{\chi(|x|)}{|x|},$$

where $\chi \in C^\infty(\mathbb{R})$ is defined as in the introduction. With this Ansatz the linearization of the system (3.5), (3.6) about the origin is given by the following operator, $L : W^{2,2}_{\delta+2} \times M^{2,2}_{\delta} \times \mathbb{R} \to L^2_{\delta+2} \times L^2_{\delta+2}$,

$$L \begin{bmatrix} s \\ \phi \\ c \end{bmatrix} = \begin{bmatrix} (1 - a\gamma)\Delta - 2 & -a\gamma\Delta & -a\gamma\Delta P \\ (a\gamma + \frac{2}{\gamma})\Delta & (1 + a\gamma)\Delta & (1 + a\gamma)\Delta P \end{bmatrix} \begin{bmatrix} s \\ \phi \\ c \end{bmatrix},$$  \hspace{1cm} (3.9)

which admits the following decomposition,

$$L = \begin{bmatrix} \hat{L} & M \end{bmatrix}.$$

Here, $\hat{L}$ is the same as (3.8) and $M : \mathbb{R} \to L^p_{\delta+2} \times L^p_{\delta+2}$ is given by

$$Mc = \begin{bmatrix} -a\gamma\Delta P \\ (1 + a\gamma)\Delta P \end{bmatrix} c.$$

It is clear that the operator $M$ is well defined since $\Delta P = \Delta \frac{\chi(|x|)}{|x|}$ has compact support. Notice as well that

$$\int_{\mathbb{R}^3} \Delta \frac{\chi(|x|)}{|x|} \, dx = -4\pi,$$

so that the range of $M$ and the cokernel of $L$ intersect. The Bordering lemma for Fredholm operators shows that the operator $L : W^{2,2}_{\delta+2} \times M^{2,2}_{\delta} \times \mathbb{R} \to L^2_{\delta+2} \times L^2_{\delta+2}$ is invertible. This proves the following lemma

**Lemma 3.5** Let $\delta \in (0, 1/2)$ and $1 + a\gamma > 0$. Then the operator $L : W^{2,2}_{\delta+2} \times M^{2,2}_{\delta} \times \mathbb{R} \to L^2_{\delta+2} \times L^2_{\delta+2}$, defined by (3.9) is an invertible operator.

In order to finish the proof of Proposition 3.1 we just need to show that the full operator $F : W^{2,2}_{\delta+2} \times M^{2,2}_{\delta} \times \mathbb{R} \to L^2_{\delta+2} \times L^2_{\delta+2}$ is well defined and smooth, justifying our assertion that $DF(0) = L$. This will be done in the following section.
3.4 Nonlinear terms

We now consider the full non-linear operator $F : M^2_{δ} \times W^2_{δ+2} \times \mathbb{R} \to L^2_{δ+2} \times L^2_{δ+2}$, and we recall here the expressions for its two components,

$$F_1(s, φ, c) = (1 - αγ)Δs - 2s - γαΔφ - (γ + s)[|∇s|^2 + 2∇s · ∇φ + |∇φ|^2] - 2α|∇s|^2 - 2α∇s · ∇φ$$

$$- αs(Δs + Δφ) - \frac{3}{γ^2}α + \frac{3}{γ^2}/3,$$

$$F_2(s, φ, c) = (\frac{α}{γ} + αγ)Δs + (1 + αγ)Δφ + (γ - α + s)[|∇s|^2 + 2∇s · ∇φ + |∇φ|^2] + αs(Δs + Δφ)$$

$$+ 2α|∇s|^2 + 2α∇s · ∇φ + \frac{α}{γ} + \frac{s}{γ^2} + (γ + s)^{-1}(2|∇s|^2 + 2∇s · ∇φ - s^2 - \frac{s^3}{γ} - \frac{α}{γ}sΔs) + ϵg(x)$$

where again we use $φ = ̂φ + cP$, with $P = \frac{χ(|x|)}{|x|}$. With the help of the next lemma we show that $F$ is well defined in the sense that all non-linear terms are in the space $L^p_{δ+2}$.

**Lemma 3.6** Let $f, g ∈ W^{1, p}_{δ+1}$, with $δ > 0$ and $p ∈ (1, ∞)$. Then the product $fg ∈ L^p_{δ+2}$.

**Proof.** This lemma is a consequence of Hölder’s inequality and the Sobolev embeddings. ■

Notice also that if $δ > 0$ then $W^{2, p}_{δ+2} ⊂ W^{2, p}$. Furthermore if $p = 2$ we have $W^{2, 2}_{δ+2} ⊂ W^{2, 2} ↔ BC(\mathbb{R}^3)$.

**Proposition 3.7** Let $δ ∈ (0, 1/2)$, and $g ∈ L^2_{δ+2}$. Then the linear operator $F : W^{2, 2}_{δ+2} \times M^2_{δ} \times \mathbb{R} \to L^2_{δ+2} \times L^2_{δ+2}$ defined by (3.5) and (3.6), is well defined and smooth.

**Proof.** The results form Lemma 3.6 and the above remark suggest that all non-linear terms which do not involve the parameter $c$ are in the space $L^2_{δ+2}$. Since all derivatives of $\frac{χ(|x|)}{|x|}$ are bounded the only terms we need to worry about come from the expression $|∇φ|^2$. Recall here that $φ = ̂φ + cP$, with $P = \frac{χ(|x|)}{|x|}$ and $̂φ ∈ M^{2, 2}_{δ}$, so that

$$|∇φ|^2 = |∇̂φ|^2 + 2c∇̂φ · ∇P + c^2|∇P|^2.$$

It is clear from Lemma 3.6 that the expression $|∇̂φ|^2 ∈ L^2_{δ+2}$. Also, because $∇P$ is bounded in compact sets and behaves like $(x)^{-2}$ for large $|x|$, a straightforward calculation shows that $∇̂φ · ∇P$ is in the desired space. Finally, since $δ ∈ (0, 1/2)$ the following integral converges

$$\int_{\mathbb{R}^3} |∇P|^{4}(x)^{2(δ+2)} dx ≤ \int_{0}^{∞} r^{2(δ+2)-8} r^2 dr$$

Given that all non-linear terms are defined via superposition operators of algebraic functions, they are smooth once well defined. ■
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