Integrals of monomials over the orthogonal group

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Abstract

A recursion formula is derived which allows to evaluate invariant integrals over the orthogonal group $O(N)$, where the integrand is an arbitrary finite monomial in the matrix elements of the group. The value of such an integral is expressible as a finite sum of partial fractions in $N$. The recursion formula largely extends presently available integration formulas for the orthogonal group.

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1 Introduction

Integrals over the classical compact groups\textsuperscript{1,2} are of interest in various fields, such as harmonic analysis\textsuperscript{3} or random matrix theory\textsuperscript{4}. In these applications the integrand is often a polynomial in the matrix elements of the group itself, (i.e. of the true matrix representation of the group). Thus we have to integrate an arbitrary monomial of these matrix elements. Closed formulas are available only for very special cases\textsuperscript{3,5−7} and even a new method by Prosen et al.\textsuperscript{8} using computer algebras is practically limited to low degrees. Though note that, for arbitrary monomials there is strong evidence, that the results are exact at least up to the next leading order in $N^{-1}$ with respect to the approximation of the group integral by independent Gaussian distributed matrix elements.\textsuperscript{9}

In the present paper we shall address the case of the orthogonal group $O(N)$. First we rederive the well known one-vector formula\textsuperscript{4,6}. In this context, the terms “$R$-vector formula” or “$R$-vector integral” refer to the case where the monomial in question contains only powers of matrix elements from $R$ rows or $R$ columns respectively. Next we derive a recursion formula that relates an $R$-vector integral to a linear combination of $(R-1)$-vector integrals. This is the central result of the present paper. Together with the one-vector formula, it allows to calculate any integral over a monomial of finite degree in a finite number of steps. This result is then used, to obtain a closed expression for general two-vector integrals that is much simpler than the one known before.\textsuperscript{6} Besides, the older formula contains mistakes which (to the best of my knowledge) had never been corrected in the literature.

The paper is organized as follows: In Sec.\textsuperscript{2} we describe the current approach to the problem. In addition, we introduce some compact non-standard notations, which help to keep the mathematical expressions manageable. Then the one-vector result of Ullah\textsuperscript{6} is rederived, as it is the base for the recursion formula developed later on. In passing we obtain an equally simple formula for the corresponding one-vector integral over the unitary group. In Sec.\textsuperscript{3} we derive the general recursion formula. In Sec.\textsuperscript{4} some applications are presented. As an immediate consequence, we obtain a closed expression for the two-vector integral, which is then compared to the corrected old result.\textsuperscript{6}

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We also illustrate the use of our general formula for $R > 2$, calculating a particular three-vector integral. Sec. 5 contains the conclusions.

2 General considerations

To be specific, let us consider the orthogonal matrix $w \in O(N)$ as a point in Euclidean $N^2$-dimensional space. Then we are interested in integrals of monomials in the coordinates of $w$. These are denoted by:

$$\langle M \rangle = \int d\sigma(w) \prod_{i,\xi=1}^{N,R} w_{i\xi}^M \,.$$

(1)

Here $\sigma$ is the normalized Haar measure of $O(N)$, i.e. $\int d\sigma(w) = 1$, and $M$ is a $N \times R$ matrix of non-negative integers, with $R \leq N$. $M$ is called the power matrix. In the recursion formula to be developed, $R$ is used as the recursion parameter. Hence it is important, that $R$, the number of columns of $M$, is as small as possible.

The integral over the orthogonal group is invariant under any permutation of columns or rows of the integration variable $w \in O(N)$. It is also invariant under taking the transpose. Therefore it is sufficient to consider such monomials which contain matrix elements from the first $R \leq N$ columns of $w$ only. According to Ullah one may then write:

$$\langle M \rangle = \frac{\mathcal{N}(M)}{\mathcal{N}(o)} \,,$$

$$\mathcal{N}(M) = \int \prod_{\xi=1}^{R} \left\{ d\Omega(\vec{w}_\xi) \prod_{i=1}^{N} w_{i\xi}^{M_{i\xi}} \right\} \prod_{\mu<\nu} \delta(\langle \vec{w}_\mu | \vec{w}_\nu \rangle) \,,$$

(2)

where $o$ is a $N \times R$ matrix of zeros. The integration region is the Cartesian product of $R$ unit spheres with constant measures $d\Omega(\vec{w}_\xi)$, and $\{ \vec{w}_1, \ldots, \vec{w}_R \}$ are the corresponding unit vectors. The orthogonality of the unit vectors is implemented with the help of appropriately chosen $\delta$-functions. $\langle \vec{w}_\mu | \vec{w}_\nu \rangle$ denotes the scalar product between the two vectors: $\vec{w}_\mu$ and $\vec{w}_\nu$.

2.1 Compact notations for certain products of multinomials

In the calculations to follow, we will frequently deal with certain products of binomial and multinomial coefficients. For better legibility we use two special notations: In what follows, $\vec{x}$ and $\vec{y}$ are $N$-dimensional real vectors, and $\vec{m}$ and $\vec{n}$ are $N$-dimensional vector-indices of non-negative integers. There are two typical cases in which products of multinomials appear:

1) Consider the expression $E = \prod_{i=1}^{N} (x_i + y_i)^{n_i}$. Its expansion gives:

$$E = \sum_{\vec{k}} \prod_{i=1}^{N} \binom{n_i}{k_i} x_i^{k_i} y_i^{n_i-k_i} \,,$$

(3)

where the sum runs over all $\vec{k}$ for which $\forall i : k_i \leq n_i$. In this case, the product of binomials is denoted by the following symbol:

$$\prod_{i=1}^{N} \binom{n_i}{k_i} = \binom{\vec{n}}{\vec{k}} \,.$$

(4)

2) In other occasions, we encounter expressions of the type: $E' = \prod_{i=1}^{N} \langle \vec{\tau} | \vec{w}_i' \rangle^{n_i}$, where $\langle \vec{\tau} | \vec{w}_i' \rangle$ is the scalar product of the two $(R-1)$-dimensional vectors $\vec{\tau}$ and $\vec{w}_i'$. In this case the expansion reads:

$$E' = \sum_{\vec{K}} \left\{ \prod_{i=1}^{N} \binom{n_i}{K_{i1}, \ldots, K_{i,R-1}} \right\} \prod_{\xi=1}^{R-1} \frac{\vec{K}_\xi}{\vec{k}_\xi} \prod_{i=1}^{N} w_{i\xi}^{K_{i\xi}} \,.$$

(5)
Here we have to use the $N \times (R-1)$ matrix $K$ as an index. The elements of $K$ are non-negative integers. $\vec{k}_\xi$ is the $\xi$th column vector of $K$, and $\vec{k}_\xi$ is the sum of its components: $\vec{k}_\xi = \sum_{\xi=1}^{N} K_{i\xi}$. The sum in Eq. (3) runs over all $K$ for which $\forall i : \sum_{\xi=1}^{R-1} K_{i\xi} = n_i$. In this case, we use the following notation:

$$\prod_{i=1}^{N} (n_i | K_{i1}, \ldots, K_{i,R-1}) = (\vec{m} | K).$$

### 2.2 The one-vector integral

In the one-vector case, $R = 1$, there are no orthogonality relations to respect. The power matrix $M$ consists of one single column vector, here denoted by $\vec{m}$. According to Eq. (2) we may write:

$$\langle \vec{m} \rangle = \frac{\mathcal{N}(\vec{m})}{\mathcal{N}(\vec{0})}, \quad \mathcal{N}(\vec{m}) = \int d\Omega(\vec{w}) \prod_{i=1}^{N} u_i^{m_i},$$

where $\vec{0}$ is a $N$-dimensional vector of zeros. Following the original calculation of Ullah,\textsuperscript{4,6} we integrate over the full vector space $\mathbb{R}^N$ and implement the normalization of the column vector with the help of a $\delta$-function. This introduces the integration constant $c_1(N)$:

$$\mathcal{N}(\vec{m}) = c_1(N)^{-1} \left\{ \prod_{i=1}^{N} \int_{-\infty}^{\infty} dx_i x_i^{m_i} \right\} \delta (||\vec{x}||^2 - 1).$$

The next step is to remove the $\delta$-function. Setting $x_i = u_i/\sqrt{r}$, we obtain:

$$\mathcal{N}(\vec{m}) \Gamma \left( \frac{N+\vec{m}}{2} \right) c_1(N) = \prod_{i=1}^{N} \int_{-\infty}^{\infty} du_i u_i^{m_i} e^{-u_i^2} = \prod_{i=1}^{N} \Gamma \left( \frac{1+m_i}{2} \right).$$

Solving this equation for $\mathcal{N}(\vec{m})$, the ratio $\mathcal{N}(\vec{m})/\mathcal{N}(\vec{0})$ can be calculated, which leads to the desired result:

$$\langle \vec{m} \rangle = \left( \frac{N}{2} \right)^{-1} \prod_{i=1}^{N} \left( \frac{1}{2} \right)_{m_i/2}.$$

Here it is convenient to use the Pochhammer symbol $(z)_n = \Gamma(z+n)/\Gamma(z)$,\textsuperscript{11} Note that Eq. (11) implies, that the integral $\langle \vec{m} \rangle$ vanishes if at least one component of $\vec{m}$ is odd.

### 2.3 The one-vector integral over the unitary group

It is natural to consider also integrals over the unitary group $U(N)$. This is in general much more complicated because usually the monomials to integrate contain powers of the matrix elements and their complex conjugated counterparts. However in the one-vector case, the integral over the unitary group can be mapped on a corresponding integral over the orthogonal group, which leads again to a simple result (it seems that such a formula has never been published elsewhere). If more vectors are involved, $R > 1$, the orthogonality conditions destroy this simple correspondence.

To obtain the desired expression for one-vector integrals, it is convenient to consider monomials in the real and imaginary parts of the complex unit vector $\vec{w}$. They can be identified with the
coordinates in a 2N-dimensional Euclidean space, where the Haar measure reduces to the constant measure \( \Omega \) on the unit hypersphere. Denoting the one-vector integral of an arbitrary monomial by \( \langle \vec{m} : \vec{n} \rangle = \prod_{i=1}^{N} x_i^{m_i} y_i^{n_i} \), where \( w_i = x_i + iy_i \), we may write:

\[
\langle \vec{m} : \vec{n} \rangle = \mathcal{M}(\vec{m}, \vec{n}) = \int d\Omega_2(\vec{w}) \prod_{i=1}^{N} x_i^{m_i} y_i^{n_i}, \quad w_i = x_i + iy_i . \tag{12}
\]

Note the different notations: \( \langle \vec{m} : \vec{n} \rangle \) stands for the one-vector integral over the unitary group, while \( \langle \vec{m}, \vec{n} \rangle \) is used in Sec. 4 for the two-vector integral over the orthogonal group.

Equation (12) shows that we may express \( \langle \vec{m} : \vec{n} \rangle \) as a one-vector integral over the orthogonal group \( O(2N) \): \( \langle \vec{m} : \vec{n} \rangle = \langle \vec{p} \rangle \), where \( \vec{p} \) is the \( 2N \)-dimensional concatenation of \( \vec{m} \) and \( \vec{n} \). Then we may apply Eq. (11) to this integral:

\[
\langle \vec{m} : \vec{n} \rangle = (N)^{-1/2} \prod_{i=1}^{N} \left( \frac{1}{2} \right) m_{i/2} \left( \frac{1}{2} \right) n_{i/2} . \tag{13}
\]

Again the integral \( \langle \vec{m} : \vec{n} \rangle \) vanishes, if at least one component of \( \vec{m} \) or \( \vec{n} \) is odd.

### 3 The recursion formula

The desired recursion formula shall express an arbitrary integral \( \langle M \rangle \), where \( M \) is a power matrix with \( R \) columns, as a linear combination of simpler integrals \( \langle M' \rangle \), where \( M' \) has only \( R-1 \) columns. Starting from Eq. (12) one may attack this problem head on, and separate the integration on the last unit vector \( \vec{u}_R \) from the remaining integral:

\[
\mathcal{N}(M) = \int \left\{ \prod_{\xi=1}^{R-1} d\Omega(\vec{u}_\xi) \right\} \left\{ \prod_{i=1}^{N} \mathcal{M}_{i\xi}^M \right\} \left\{ \prod_{\mu<\nu} \delta(\langle \vec{w}_\mu | \vec{w}_\nu \rangle) \right\} J(\vec{w}_1, \ldots, \vec{w}_{R-1}; \vec{m}_R) . \tag{14}
\]

Here \( \vec{m}_R \) is the last column vector of the power matrix \( M \), and

\[
J(\vec{w}_1, \ldots, \vec{w}_{R-1}; \vec{m}_R) = \int d\Omega(\vec{u}) \left\{ \prod_{i=1}^{R} \mathcal{M}_{iR} \right\} \prod_{\xi=1}^{R-1} \delta(\langle \vec{u}_\xi | \vec{u} \rangle) . \tag{15}
\]

As shown below, the value of this integral can be expressed as a linear combination of monomials in the integration variables \( \{ w_{i\xi} | \xi \leq R - 1 \} \). If this is inserted back into Eq. (14), it obviously leads to the desired recursion formula.

To evaluate the integral (15), the integration over the unit sphere is replaced by an integration over the full space \( \mathbb{R}^N \), implementing the normalization condition with the help of a \( \delta \)-function. This introduces again the normalization constant \( c_1(N) \) [cf. Eq. (8)]. The \( \delta \)-function is then removed again, using the same trick as in Sec. 2. After that, the remaining \( \delta \)-functions (responsible for the orthogonality relations) are replaced by their respective Fourier representations. In this way one obtains:

\[
J(\vec{w}_1, \ldots, \vec{w}_{R-1}; \vec{m}_R) = c_1(N)^{-1} \left\{ \prod_{i=1}^{N} dx_i x_i^{M_{iR}} \right\} \delta(\|\vec{x}\|^2 - 1) \prod_{\xi=1}^{R-1} \delta(\langle \vec{w}_\xi | \vec{x} \rangle)
\]

\[
= c_1(N)^{-1} \Gamma \left( \frac{N-R+\vec{m}_R+1}{2} \right)^{-1} \left\{ \prod_{i=1}^{N} dx_i x_i^{M_{iR}} e^{-x_i^2} \right\} \prod_{\xi=1}^{R-1} \delta(\langle \vec{w}_\xi | \vec{x} \rangle)
\]

\[
= c_1(N)^{-1} \Gamma \left( \frac{N-R+\vec{m}_R+1}{2} \right)^{-1} \frac{4^{R-1}}{\pi^{R-1}} \prod_{i=1}^{N} dx_i x_i^{M_{iR}} e^{-x_i^2 + 2i\vec{x}^\prime \vec{w}_i} , \tag{16}
\]
where \( \vec{w}' \) stands for the row-vector \((w_1, \ldots, w_{i,R-1})^T\). The integrals on \(x_i\) are easily evaluated, leading to:

\[
J(\ldots) = \frac{1}{\sqrt{2\pi \Gamma(N) c_1(N) \Gamma\left(\frac{N-R+\bar{m}+1}{2}\right)}} \left\{ \begin{array}{l}
(\bar{m}_R) \\
(N-R+\bar{m}) \end{array} \right\} \int d\tau^R \prod_{i=1}^N e^{-\frac{1}{2} \sum_{i=1}^N (\vec{w}'_i)^2 \Gamma M_i A R} e^{-\frac{1}{2} \sum_{i=1}^N (\vec{w}'_i)^2 \Gamma M_i R - \vec{K}} \right]
\]

Expanding the \(N\)-fold product into a sum over the vector-index \(K = (\kappa_1, \ldots, \kappa_N)\), we obtain for the l.h.s.:

\[
l.h.s. = \sum_K (\bar{m}_R) R^{-\bar{m} - \bar{K}} \left\{ \begin{array}{l}
R-1 \end{array} \right\} \int d\tau^R \prod_{i=1}^N \left\{ \begin{array}{l}
\sum_{\kappa} w_i^{\bar{m}_R - \bar{K}} \end{array} \right\} \int d\tau^R \prod_{i=1}^N (\vec{w}'_i)^2 \prod_{i=1}^N w_i^\kappa \right]
\]

where \(K\) is a matrix index with \(R-1\) columns, as introduced in Eq. \(\ref{eq:4}\) together with the abbreviation for the product of binomials the abbreviation from Eq. \(\ref{eq:5}\) is used. A bar over vector quantities such as \(\bar{m}_R\) and \(\bar{K}\) denotes the sum of all their components. The quadratic matrix \(A\), with elements \(A_{\mu\nu} = \langle \vec{w}'_\mu | \vec{w}'_\nu \rangle\), has dimension \(R-1\).

Now, the key observation is the following: The orthogonality conditions implemented in the form of \(\delta\)-functions in Eq. \(\ref{eq:14}\) select from the total integration region only a sub-manifold. There it holds that \(\langle \vec{w}'_\mu | \vec{w}'_\nu \rangle = \delta_{\mu\nu}\), so that the matrix \(A\) may be replaced by the unit matrix. Then it is possible to integrate the \(\vec{w}'\)-integral. The expansion of the product of scalar products leads to:

\[
l.h.s. = \sum_K (\bar{m}_R) R^{-\bar{m} - \bar{K}} \left\{ \begin{array}{l}
R-1 \end{array} \right\} \int d\tau^R \prod_{i=1}^N \left\{ \begin{array}{l}
\sum_{\kappa} w_i^{\bar{m}_R - \bar{K}} \end{array} \right\} \int d\tau^R \prod_{i=1}^N (\vec{w}'_i)^2 \prod_{i=1}^N w_i^\kappa \right]
\]

Net that, as a consequence of the \(\tau^R\)-integrals, the sum in the second line runs over such \(K\) only, for which all \(k_1, \ldots, k_{R-1}\) are even. Inserting this expression into the initial Eq. \(\ref{eq:14}\) we obtain:

\[
N(M) = \frac{\pi^{1-R}}{c_1(N) \Gamma(N-R+\bar{m})} \sum_K (\bar{m}_R) R^{-\bar{m} - \bar{K}} \left\{ \begin{array}{l}
R-1 \end{array} \right\} \int d\Omega(\vec{w}'_i) \prod_{i=1}^N \left\{ \begin{array}{l}
\sum_{\kappa} w_i^{\bar{m}_R - \bar{K}} \end{array} \right\} \int d\Omega(\vec{w}'_i) \prod_{i=1}^N w_i^\kappa \right]
\]
The integral over the normalized vectors \( \vec{w}_1, \ldots, \vec{w}_{R-1} \) can be identified with \( \mathcal{N}(M^{(R-1)} + K) \) which is a \( (R-1) \)-vector integral. In this way, we obtain a recursion formula for \( \mathcal{N}(M) \). For the normalization constant, we find:

\[
\mathcal{N}(o) = \frac{\pi^{1-R}}{c_1(N) \Gamma \left( \frac{N-R+1}{2} \right)} \left\{ \prod_{i=1}^N \Gamma \left( \frac{1}{2} \right) \right\} \left\{ \prod_{\xi=1}^{R-1} \Gamma \left( \frac{1}{2} \right) \right\} \mathcal{N}(o^{(R-1)}) .
\]  

(22)

Thus we obtain for the \( R \)-vector integral \( \langle M \rangle \), defined in Eq. (2):

\[
\langle M \rangle = \left( \frac{N-R+1}{2} \right)^{-1} \bar{m}_{R/2} \sum_{\vec{\kappa}} \left( \frac{\bar{\kappa}}{\vec{\kappa}} \right) (-1)^{(\bar{\kappa}_R - \bar{\kappa})/2} \left\{ \prod_{i=1}^N \left( \frac{1}{2} \right) \kappa_i/2 \right\} \times \sum_{K} (\bar{m}_R - \bar{\kappa} | K) \left\{ \prod_{\xi=1}^{R-1} \left( \frac{1}{2} \right) \bar{\kappa}_\xi/2 \right\} \langle M^{(R-1)} + K \rangle .
\]  

(23)

This is the desired recursion formula and the main result of the present paper. As mentioned before it is understood, that the first sum runs over such \( \vec{\kappa} \) only for which all components are even, while the second runs over such \( K \) only for which all \( \bar{\kappa}_\xi = \sum_{i=1}^N K_{i\xi} \) are even. Furthermore \( \bar{m}_R = \sum_1^N M_{1i}, \bar{\kappa} = \sum_1^N \kappa_i, \) and \( M^{(R-1)} \) stands for the matrix consisting of the first \( R-1 \) columns of the matrix \( M \).

In principle Eq. (23) allows to evaluate integrals of arbitrary monomials of finite degree. The result is always expressible as a rational function of the dimension \( N \), because the repeated expansion of Eq. (23) leads to nested sums of partial fractions in \( N \). In this context it is useful to note, that only the prefactor of the r.h.s. depends explicitly on \( N \). The formula (23) can be used conveniently if either \( R \) or the degree of the monomial to be integrated are small. Otherwise Eq. (23) may lead to very lengthy expressions. However, such expressions should still be manageable with an appropriate computer algebra system. This would allow for further systematic studies of this class of integrals.

In contrast to what one might expect, the integral \( \langle M \rangle \) does not necessarily vanish if the power matrix \( M \) has odd elements. It rather holds the following: If any sum over a row or column of \( M \) is odd, then \( \langle M \rangle = 0 \). Though this is in fact well known,\(^{12} \) it is still instructive to see that it follows almost immediately from the recursion relation (23).

To this end, permute columns and rows, and take the transpose if necessary, to transform \( M \) in such a way that its last column contains the row or column whose sum of components is odd. Then apply Eq. (23): The sum over \( \vec{\kappa} \) is restricted to such \( \vec{\kappa} \) which have only even components. Hence \( \bar{\kappa} \) is even. As \( \bar{m}_R \) is odd, and \( \sum_{\xi=1}^{R-1} \bar{k}_\xi = \bar{m}_R - \bar{\kappa} \), at least one of the sums \( \bar{k}_1, \ldots, \bar{k}_{R-1} \) must be odd as well. Such a term vanishes, because of what is said below Eq. (23). This implies that all terms of the sum over \( K \) vanish likewise, so that the complete integral gives zero.

4 Applications

In random matrix theory, many matrix ensembles are based on the concept of orthogonal invariance. Physically this corresponds to a situation where the Hamiltonian for a spin-less quantum particle possesses an anti-unitary symmetry, e.g. time reversal invariance. The Gaussian and circular orthogonal ensembles\(^{4,13,14} \) are well known examples based on this concept. Other examples are the Poisson orthogonal ensemble,\(^{15} \) or the recently introduced matrix ensembles for semi-separable systems.\(^8 \) In those cases where the orthogonal invariance applies directly to the Hamiltonian, the statistical properties of the eigenvectors are uniquely determined by the orthogonal group and its invariant measure. Therefore any correlators between the eigenvectors can be expressed and calculated in terms of \( R \)-vector integrals.
In what follows, we first apply our integration formula (23) to the two-vector case. In this way we obtain a closed expression for arbitrary two-vector integrals. Then we compare this result with the corrected formula of Ullah. For illustration, we finally compute a simple three-vector integral, which can be evaluated with an independent method also. As it should be, we find the same answer with both methods.

4.1 The two-vector integral

Consider the arbitrary two-vector integral \( \langle M \rangle = \langle \vec{m}, \vec{n} \rangle \), where the first column vector of \( M \) is denoted by \( \vec{m} \) and the second by \( \vec{n} \). In this case, Eq. (23) leads directly to the following expression:

\[
\langle \vec{m}, \vec{n} \rangle = \left( \frac{N-1}{2} \right)^{-1} \bar{n}/2 \sum_{\vec{r}} \left( \begin{array}{c} m \\ n \end{array} \right) \left( -1 \right)^{(n-r)/2} \left( \prod_{i=1}^{N} \left( \frac{1}{2} \right) \kappa_{i}/2 \left( \frac{1}{2} \right) \right) \left( \bar{m} + \bar{n} - \bar{r} \right). \tag{24}
\]

The sum runs over such \( \vec{r} \) only, for which all components are even. A bar over a vector quantity denotes, as before, the sum of all its components. This formula has already been used in Ref. 16 to calculate various two-vector integrals. The numerical tests performed in parallel confirm its validity.

For later purpose we use Eq. (24) to evaluate the following simple integral:

\[
\left\langle \begin{array}{c} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ \vdots \\ \vdots \\ \end{array} \right\rangle = \frac{2}{N-1} \left( -1 \right)^{1/2} \left\langle \begin{array}{c} 2 \\ 0 \\ \vdots \\ \vdots \\ \end{array} \right\rangle = -\frac{1}{(N-1)N(N+2)}. \tag{25}
\]

Note that the same result can be obtained by an indirect method12 also.

In principle, an integration formula for general two-vector integrals has already been published some time ago.6 After the correction of two misprints, it reads:

\[
\prod_{i} u_{m_{i}, n_{i}}^{2m_{i}, 2n_{i}} = \pi^{N+1} 2^{-2N+4-2\sum (m_{i}+n_{i})} \frac{\Gamma(N-1) \Gamma[N-1+\sum (m_{i}+n_{i})]}{\Gamma[\sum m_{i}+(N-1)/2] \Gamma[\sum n_{i}+(N-1)/2]} \times \sum_{k_{1}, \ldots, k_{N}, l_{1}, \ldots, l_{N}=0}^{2m_{1}, \ldots, 2m_{N}, 2n_{1}, \ldots, 2n_{N}} (-1)^{\sum_{i} l_{i}} \times \frac{\prod_{i} \left( \begin{array}{c} 2m_{i} \\ k_{i} \end{array} \right) \left( \begin{array}{c} 2n_{i} \\ l_{i} \end{array} \right) \Gamma[k_{i}+l_{i}+1/2] \Gamma[m_{i}+n_{i}-(k_{i}+l_{i}-1)/2]}{\Gamma[N/2+\sum_{i}(k_{i}+l_{i})/2] \Gamma[N/2+\sum_{i}(m_{i}+n_{i})-\sum_{i}(k_{i}+l_{i})/2]}, \tag{26}
\]

where \( \forall i : k_{i}+l_{i} \) must be even. Here the original notation of Ref. 6 is used. The corrections concern the first line, where the nominator has been multiplied with \( \Gamma[N-1+\sum (m_{i}+n_{i})] \), and the sum over \( k_{1}, \ldots, k_{N}, l_{1}, \ldots, l_{N} \), where the vector-indices must start with zeros instead of ones. Finally the notation is quite unfortunate, as it seems to prohibit monomials with odd powers, though there is no reason for it. Indeed, Eq. (26) holds in those cases as well. This can be checked, for instance, by computing the integral (25) with the help of Eq. (26), setting \( m_{1} = m_{2} = n_{1} = n_{2} = 1/2 \). Using the notation adopted in the present work, Eq. (25) reads:

\[
\langle \vec{m}, \vec{n} \rangle = \frac{(N-1)\langle \vec{m}+\vec{n} \rangle /2}{2^{n+m} \left( \frac{N-1}{2} \right)^{n/2} \left( \frac{N-1}{2} \right)^{m/2}} \sum_{\vec{k}, \vec{l}} \left( \begin{array}{c} \vec{m} \\ \vec{k} \end{array} \right) \left( \begin{array}{c} \vec{n} \\ \vec{l} \end{array} \right) (-1)^{\vec{j} \cdot \vec{k} + \vec{l}} \langle \vec{m} - \vec{k} + \vec{n} - \vec{l} \rangle. \tag{27}
\]

If we compare the integration formulas (24) and (27), they differ considerably. It seems rather difficult to prove their equivalence directly. Note moreover, that our result is much simpler, because there the sum runs over a single vector-index only.
4.2 A simple three-vector integral

The three-vector integral considered here, is chosen because of its simplicity and because it may be evaluated using an indirect method, which allows to crosscheck the result. We will compute the integral $\langle M \rangle$ with

$$M = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \end{pmatrix}. \quad (28)$$

Henceforth we will skip those parts of the column vectors which are zero anyway. Using our recursion formula (23) the three-vector integral $\langle M \rangle$ can be reduced to a linear combination of two-vector integrals, for which we already have a closed expression, namely Eq. (24). Thus, the evaluation of $\langle M \rangle$ needs only a few steps:

$$\begin{align*}
\left\langle \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \right\rangle &= \frac{2}{N-2} \left\{ \sum_K \left( \begin{pmatrix} 0 \\ 0 \\ K \end{pmatrix} \right) \left( \begin{pmatrix} 1 \\ 0 \\ K \end{pmatrix} \right) + \sum_K \left( \begin{pmatrix} 0 \\ 0 \\ K \end{pmatrix} \right) \left( \begin{pmatrix} 1 \\ 0 \\ K \end{pmatrix} \right) \right\} + \frac{1}{2} \left( \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \right) \left( \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} \right) \\
&= \frac{2}{N-2} \left\{ \sum_K \left( \begin{pmatrix} 0 \\ 0 \\ K \end{pmatrix} \right) \left( \begin{pmatrix} 1 \end{pmatrix} \right) + \sum_K \left( \begin{pmatrix} 0 \\ 0 \\ K \end{pmatrix} \right) \left( \begin{pmatrix} 1 \end{pmatrix} \right) \right\} + \frac{1}{2} \left( \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \right) \left( \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} \right) \\
&= \frac{1}{N-2} \left\{ \frac{N+1}{(N-1)N(N+2)} - 2 \frac{N+3}{(N-1)N(N+2)(N+4)} \right\} \\
&= \frac{N^2 + 3N - 2}{(N-2)(N-1)N(N+2)(N+4)}. \quad (29)
\end{align*}$$

The result is expressed as a rational function, where care has been taken, that nominator and denominator have no more common factors.

Alternatively we may compute $\langle M \rangle$, starting from the following identity:

$$\left( \sum_i w_{i1}^2 \right) \left( \sum_j w_{j2}^2 \right) \left( \sum_k w_{k3}^2 \right) = 1, \quad (30)$$

which holds for an arbitrary $w \in O(N)$. Now we expand the products on the l.h.s. and integrate on both sides over the group. This gives:

$$N(N-1)(N-2) \left\langle \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \right\rangle + 3N(N-1) \left\langle \begin{pmatrix} 2 & 0 \\ 0 & 2 \\ 0 & 0 \end{pmatrix} \right\rangle + N \left\langle \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} \right\rangle = 1. \quad (31)$$

It allows to express the three-vector integral $\langle M \rangle$ as a linear combination of a one-vector and a two-vector integral. According to the Eqs. (11) and (24), these integrals are given by:

$$\left\langle \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} \right\rangle = \frac{1}{N(N+2)(N+4)}, \quad \left\langle \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} \right\rangle = \frac{N+3}{(N-1)N(N+2)(N+4)}. \quad (32)$$
Thus we finally obtain:

\[
\begin{pmatrix}
2 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 2
\end{pmatrix} = \frac{(N + 2)(N + 4) - 3(N + 3) + 1}{(N - 2)(N - 1)N(N + 2)(N + 4)}
\]

\[
= \frac{N^2 + 3N - 2}{(N - 2)(N - 1)N(N + 2)(N + 4)}. \tag{33}
\]

As expected, the result coincides with the one above, i.e., Eq. (29). Here the indirect method worked so well because we first wrote down the identities (30) and (31), and then chose our particular example \(M\). However, if the value of a certain integral is needed, one would have to guess useful identities which allow to express the integral by a linear combination of simpler ones, a procedure which is certainly very difficult. In contrast to that, the recursion formula (23) always provides a well defined finite procedure, for the computation of any integral.

5 Conclusions

To summarize, we have derived a recursion formula, which expresses an arbitrary \(R\)-vector integral over the orthogonal group as a linear combination of \((R - 1)\)-vector integrals. It allows to successively evaluate the group integral of any finite monomial in the matrix elements of the group. The simplicity of the result depends primarily on \(R\), the number of column or row vectors involved, and only secondarily on the degree of the monomial in question. The result is always given as a finite sum of partial fractions in \(N\).

As an immediate consequence of the general result, we obtained a closed integration formula for arbitrary two-vector integrals, which is quite different and much simpler than the corrected, previously known result.

In principle a similar recursion formula can be obtained for integrals over the unitary group also. To that end one should consider monomials in the real and imaginary parts of the matrix elements. Though the derivation along the lines of the orthogonal case is rather straightforward, the resulting expressions are much more involved. It seem that the simple result for the case \(R = 1\) is only an exception. More work is clearly necessary to clarify the situation in this case.

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References

1. H. Weyl. *The classical groups*. Princeton: Princeton U.P., 1939.
2. E. Cartan. *Abh. Math. Sem. Univ. Hamburg*, 11:116, 1935.
3. L. K. Hua. *Harmonic analysis of functions of several complex variables*. Providence, RI: American Mathematical Society, 1963.
4. M. L. Mehta. *Random Matrices and the statistical theory of energy levels*. Academic Press, Boston, 1991.
5. N. Ullah and C. E. Porter. *Phys. Rev.*, 132(2):948, 1963.
6. N. Ullah. *Nucl. Phys.*, 58:65, 1964.
7. P. A. Mello and T. H. Seligman. *Nucl. Phys. A*, 344:489, 1980.
8. T. Prosen, T. H. Seligman, and H. A. Weidenmüller. *Europhys. Lett.*, 55(1):12, 2001.
9. T. H. Seligman. private communication.
10. A. Haar. *Ann. Math.*, 34:147–169, 1933.
11. M. Abramowitz and I. A. Stegun, editors. *Handbook of mathematical functions*. Dover, New York, 1964.
12 T. A. Brody, J. Flores, J. B. French, P. A. Mello, A. Pandey, and S. S. M. Wong. 
*Rev. Mod. Phys.*, 53(3):385, July 1981.

13 F. J. Dyson and M. L. Metha. *J. Math. Phys.*, 4:701, 1963.

14 C. E. Porter, editor. *Statistical theories of spectra: Fluctuations*. New York: Academic Press, 1965.

15 F.-M. Dittes, I. Rotter, and T. H. Seligman. *Phys. Lett. A*, 158:14, 1991.

16 T. Gorin and T. H. Seligman. *nlin.CD* /0101018, January 2001.