Atom canonicity, Dedekind-MacNeille completions, neat embeddings and omitting types

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Abstract

Let $n$ be finite $> 2$. We show that any class between $S\forall\forall_n CA_n + 3$ and $RCA_n$ is not atom canonical, and any class containing the class of completely representable algebras and contained in $S\forall\forall_n CA_n + 3$ is not elementary. We show that there is no finite variable universal axiomatization of many diagonal free reducts of representable cylindric algebras of dimension $n$, like the varieties of representable diagonal-free cylindric algebras and Halmos’ polyadic algebras (without equality). We apply our hitherto obtained algebraic results to show that the omitting types theorem fails for finite variable fragments of first order logic with and without equality, having $n$ variables, even if we count in severely relativized models as candidates for omitting single non-principle types. Finally, we show that for many cylindric-like algebras, like diagonal free cylindric algebras and Halmos’ polyadic algebras with and without equality the class of strongly representable atom structures of finite dimension $> 2$ is not elementary.

1 Introduction

We follow the notation of [2] which is in conformity with that of [6]. Assume that we have a class of Boolean algebras with completely additive operators for which we have a semantical notion of representability (like Boolean set algebras or cylindric set algebras). A weakly representable atom structure is an atom structure such that at least one atomic algebra based on it is representable. It is strongly representable if all atomic algebras having this atom structure are representable. The former is equivalent to that the term algebra, that is, the algebra generated by the atoms, in the complex algebra is representable, while the latter is equivalent to that the complex algebra is representable.
Could an atom structure be possibly weakly representable but not strongly representable? Ian Hodkinson [14], showed that this can indeed happen for both cylindric algebras of finite dimension $\geq 3$, and relation algebras, in the context of showing that the class of representable algebras, in both cases, is not closed under Dedekind-MacNeille completions. In fact, he showed that this can be witnessed on an atomic algebras, so that the variety of representable relation algebras algebras and cylindric algebras of finite dimension $> 2$ are not atom-canonical. (The complex algebra of an atom structure is the completion of the term algebra.) This construction is somewhat complicated using a rainbow atom structure. It has the striking consequence that there are two atomic algebras sharing the same atom structure, one is representable the other is not.

This construction was simplified and streamlined, by many authors, including the author [19], but Hodkinson’s construction, as we indicate below, has the supreme advantage that it has a huge potential to prove analogous theorems on Dedekind-MacNeille completions, and atom-canonicity for several varieties of algebras including properly the variety of representable cylindric-like algebras, whose members have a neat embedding property, such as polyadic algebras with and without equality and Pinter’s substitution algebras. In fact, in such cases atomic representable countable algebras will be constructed so that their Dedekind-MacNeille completions are outside such varieties.

Restricted to the cylindric algebra case, we show, that for $n > 2$ finite, there is representable atomic algebra $\mathfrak{A} \in \mathcal{C}A_n$ such that $\mathfrak{C}m\mathfrak{A} \notin \mathfrak{S}\mathfrak{M}_n\mathcal{C}A_{n+3}$ inferring that the varieties $\mathfrak{S}\mathfrak{M}_n\mathcal{C}A_{n+k}$, for $n > 2$ finite and for any $k \geq 3$, are not closed under Dedekind-MacNeille completions. Such results, as illustrated below will have non-trivial (to say the least) repercussions on omitting types for finite variable fragments of first order logic with and without equality.

Our construction presented herein model-theoretically, in the spirit of Hodkinson’s rainbow construction, gives a polyadic atomic representable equality algebra of finite dimension $n > 2$ such that the diagonal free reduct of its completion is not representable.

Now that we have two distinct classes, namely, the class of weakly representable atom structures and that of the strongly representable atom structures; the most pressing need is to try to classify them. Venema proved (in a more general setting) that the former is elementary, while Hirsch and Hodkinson show that the latter is not elementary. Their proof is amazing depending on an ultraproduct of Erdos probabilistic graphs [11].

We know that there is a sequence of strongly representable atom structures whose ultraproduct is only weakly representable, it is not strongly representable. This gives that the class $K = \{ \mathfrak{A} \in \mathcal{C}A_n : \mathfrak{A} \text{ is atomic and } \mathfrak{C}m\mathfrak{A} \in \mathfrak{RCA}_n \}$ is not elementary, as well.

Here we extend Hirsch and Hodkinson’s result to many cylindric-like alge-
bras, answering a question of Hodkinson’s for PA and PEA, that was answered also by Bulian and Hodkinson [16, Theorem 9.7]. The proof presented here is based on algebras that are simpler than those used in [16], which are tailored for a different purpose, namely, proving that, any first order axiomatization of the classes of representable and cylindric algebras of finite dimension > 2 must contain infinitely many non-canonical sentences.

The algebras we use are based on atom structures of cylindric algebras constructed in [11] by noting that these algebras can be endowed with polyadic operations (in an obvious way) and that they are generated by elements whose dimension sets do not exhaust the dimension. The latter implies that an algebra is representable if and only if its diagonal free reduct is representable.

Lately, it has become fashionable in algebraic logic to study abstract algebras that posses complete representations, witness [22] for an extensive overview. A representation of \( A \) is roughly an injective homomorphism from \( f: A \to \wp(V) \) where \( V \) is a set of \( n \)-ary sequences; \( n \) is the dimension of \( A \), and the operations on \( \wp(V) \) are concrete and set theoretically defined, like the Boolean intersection and cylindrifiers or projections. A complete representation is one that preserves arbitrary disjuncts carrying them to set theoretic unions. If \( f: A \to \wp(V) \) is such a representation, then \( A \) is necessarily atomic and \( \bigcup_{x \in \text{At} A} f(x) = V \).

Let us focus on cylindric algebras for some time to come. It is known that there are countable atomic \( RCA_n \)s when \( n > 2 \), that have no complete representations; in fact, the class of completely representable \( CA_n \)s when \( n > 2 \), is not even elementary [12, Corollary 3.7.1].

Such a phenomena is also closely related to the algebraic notion of \textit{atom-canonicity}, as indicated, which is an important persistence property in modal logic and to the metalogical property of omitting types in finite variable fragments of first order logic [22] Theorems 3.1.1-2, p.211, Theorems 3.2.8, 9, 10]. Recall that a variety \( V \) of Boolean algebras with completely additive operators is atom-canonical, if whenever \( A \in V \), and \( A \) is atomic, then the complex algebra of its atom structure, \( \text{CmAt}A \) for short, is also in \( V \).

If \( A \) is a weakly representable but not strongly representable, then \( \text{CmAt}A \) is not representable; this gives that \( RCA_n \) for \( n > 2 \) finite, is \textit{not} atom-canonical. Also \( \text{CmAt}A \) is the Dedekind-MacNeille completion of \( A \), and so obviously \( RCA_n \) is not closed under Dedekind-MacNeille completions.

On the other hand, \( A \) cannot be completely representable for, it can be shown without much ado, that a complete representation of \( A \) induces a representation of \( \text{CmAt}A \) [12 Definition 3.5.1, and p.74].

Finally, if \( A \) is countable, atomic and has no complete representation then the set of co-atoms (a co-atom is the complement of an atom), viewed in the corresponding Tarski-Lindenbaum algebra, \( \wp_T \), as a set of formulas, is a non principal-type that cannot be omitted in any model of \( T \); here \( T \) is consistent.
This last connection was first established by the author leading up to [4] and more, see e.g. [12]. The reference [22] contains an extensive discussion of such notions.

In the context of cylindric algebras, closure under complete neat embeddings and complete representability was proved equivalent for countable atomic algebras by the author [23, 22]. The characterization also works for relation algebras, using the same method, which is a Baire category argument at heart [2]; later re-proved by Robin Hirsch using games [8]. It was also proved that all three conditions cannot be omitted; atomicity, countability and complete embeddability. There are examples, that show that such conditions cannot be dispensed with. Hirsch and Hodkinson [10] prove that the class of completely representable CA_nS is not elementary, for any n ≥ 3. Below we shall strengthen such a result considerably.

We use techniques similar to that of Hirsch’s in [8] that deal with relation algebras, and those of Hirsch and Hodkinson in [10] on complete representations. The results in the latter had to do with investigating the existence of complete representations for both relation and cylindric algebras and for this purpose, an infinite (atomic) game that tests complete representability was devised, and such a game was used on so-called rainbow algebras. More sophisticated games were also played on rainbow relation algebras in [8]; to get sharper result on relation algebra reducts of cylindric algebras and complete representations. We will use similar games addressing cylindric-like rainbow atom structures of finite dimension > 2.

In [10] one game is used to test complete representability (the usual atomic game, a winning strategy for ∃ in k rounds can be coded in a first order sentence referred to as the kth Lyndon condition). In [8] three games were devised testing different neat embeddability properties.

Here we use only two games adapted to the cylindric paradigm. This suffices for our purposes. The main result in [10], namely, that the class of completely representable algebras of finite dimension n ≥ 3, is non elementary, follows from the fact that ∃ cannot win the infinite length game, but she can win all the finite rounded ones.

To obtain a stronger result on neat embeddings, namely, that any class containing the class of completely representable algebras and further contained in ScRt_nCA_{n+3} with n ≥ 3, we use two distinct games, both having ω rounds, but one has a limited number of pebbles, namely, n + 3, that ∀ can use, but he has the option to re-use them. Here Sc denotes the operation of forming complete subalgebras. That is by ScK, where K is a class having a Boolean reduct, we understand the class of complete subalgebras of K, that is A ∈ ScK if there exists B ∈ K such that A ⊆ B and for all X ⊆ A whenever ∑_A X = 1, then ∑_B X = 1.

Summarizing, our main results are:
(1) For any finite $n > 2$, any class $K$ such that $\text{RCA}_n \subseteq K \subseteq S_c \text{N} \text{r}_n \text{CA}_{n+3}$, $K$ is not atom-canonical, hence not closed under Dedekind-MacNeille completions. This result is also proved for many cylindric-like algebras like Pinter’s substitution algebras and Halmos polyadic algebras with and without equality, theorem 3.11. This solves an open problem first announced by Hirsch and Hodkinson in [9] and re-appearing in the late [2]. In the first reference the question is attributed to Németi and the present author, and in the second reference, it appears among the open problems in [22].

(2) For any finite $n > 2$, any class $K$ such that $K$ contains the class of completely representable $\text{CA}_n$s and is contained in $S_c \text{N} \text{r}_n \text{CA}_{n+3}$, $K$ is not elementary; it is not closed under ultraroots. This result is also proved for the relatives of CAs mentioned in the previous item, theorem 4.3.

(3) Call an atomic $\text{CA}_n$ strongly representable if its Dedekind-MacNeille completion is representable. We show that, for finite $n > 2$, there are uncountable atomic algebras in $S_c \text{N} \text{r}_n \text{CA}_\omega$ that are not completely representable, but any algebra in $S_c \text{N} \text{r}_n \text{CA}_\omega$ is strongly representable, theorem 5.8.

(4) There is no finite variable universal axiomatization for several classes of representable diagonal free reducts of cylindric algebras of finite dimension $> 2$, like diagonal free cylindric algebras and quasi-polyadic algebras, theorem 5.3. This solves an old open problem that dates back to the eighties of the last century, formulated partially in [17].

(5) Applying the algebraic results in the first two items to show that the omitting types theorem fails for first order logic with finitely many variables, as long as the number of variables available are $> 2$, even if we considerably broaden the permissible class of models omitting single non-principal types, dealing with clique guarded semantics, theorem 5.10.

(6) Introducing, for each $n > m > 2$, $m$ finite, a new class of $m$ dimensional cylindric algebras denoted by $\text{CAB}_{m,n}$. This class is a strict approximation to the class $\text{RCA}_m$, when $n$ is finite, and it the CA analogue of relational algebras embedding into complete and atomic relation algebras having $n$ dimensional relational bases introduced by Maddux. In particular, $\bigcap_k \text{CB}_{m,m+k} = \text{RCA}_m$. We show that $\text{CB}_{m,n}$ is a canonical variety for all $n > m$ that is not atom-canonical when $n \geq m + 3$ and $m$ finite $> 3$, theorems 5.15 5.17.

(7) For $2 < n < \omega$ and $\mathcal{T}$ be any signature between $\text{Df}_n$ and $\text{PEA}_n$, the class of strongly representable atom structures of type $\mathcal{T}$ is not elementary,
This proves a result of Bu and Hodkinson’s [10], except that we believe our proof is simpler. Though atomicity and complete additivity of an operator in a Boolean algebra with operators are first order notions, we show that for any finite \( n > 2 \) and any \( K \) between \( Df \) and \( PEA \), the class of strongly representable \( K_n \)’s is not elementary. More precisely, we show that there is a sequence of completely additive \( K_n \)’s \( (\mathcal{A}_i : i \in \omega) \) such that \( \mathcal{C} \text{Mat}\mathcal{A}_i \in RK_n \) for each \( i \in \omega \), but for any non-principal ultrafilter on \( \omega \), \( \mathcal{C} \text{Mat}(\Pi_{i \in \omega} \mathcal{A}_i) \) is not representable, theorem [6.14]

2 Notation and Preliminaries

We follow, as stated above, the notation in [2]. But, for the reader’s convenience, we include the following list of notation that will be used throughout the paper. Other than that our notation is fairly standard or self explanatory. Unusual notation will be explained at its first occurrence in the text.

An ordinal \( \alpha \) is transitive set (i.e., any member of \( \alpha \) is also a subset of \( \alpha \)) that is well-ordered by \( \in \). Every well-ordered set is order isomorphic to a unique ordinal. For ordinals \( \alpha, \beta \), \( \alpha < \beta \) we means \( \alpha \in \beta \). An ordinal is therefore the set of all smaller ordinals, so for a finite ordinal \( n \) we have \( n = \{0, 1, \ldots, n - 1\} \) and the least infinite ordinal is \( \omega = \{0, 1, 2, \ldots\} \).

A cardinal is an ordinal not in bijection with any smaller ordinal, briefly an initial ordinal and the cardinality \( |X| \) of a set \( X \) is the unique cardinal in bijection with \( X \). Cardinals are ordinals and are therefore ordered by \( < \) (i.e., \( \in \)). The first few cardinals are \( 0 = \emptyset, 1, 2, \ldots, \omega \) (the first infinite ordinal), \( \omega_1 \) (the first uncountable cardinal). A set will be said to be countable if it has cardinality \( \leq \omega \), uncountable otherwise, and countable infinite if it has cardinality \( \omega \). \( 2^\omega \) denotes the power of the continuum.

For a set \( X \), \( \wp(X) \) denotes the set of all subsets of \( X \), i.e. the powerset of \( X \). Ordinals will be identified with the set of smaller ordinals. In particular, for finite \( n \) we have \( n = \{0, \ldots, n - 1\} \). \( \omega \) denotes the least infinite ordinal which is the set of all finite ordinals. For two given sets \( A \) and \( B \), \( A^B \) denotes the set of functions from \( A \) to \( B \), and \( A \sim B = \{x \in A : x \notin B\} \). If \( f \in A^B \) and \( X \subseteq A \) then \( f \upharpoonright X \) denotes the restriction of \( f \) to \( X \). We denote by \( \text{dom}f \) and \( \text{rng}f \) the domain and range of a given function \( f \), respectively. We frequently identify \( f \) with the sequence \( \langle f_x : x \in \text{dom} f \rangle \). We write \( fx \) or \( f_x \) or \( f(x) \) to denote the value of \( f \) at \( x \). We define composition so that the right-hand function acts first, thus for given functions \( f, g \), \( f \circ g(x) = f(g(x)) \), whenever the left hand side is defined, i.e when \( g(x) \in \text{rng} f \).

For a non-empty set \( X \), \( f(X) \) denotes the image of \( X \) under \( f \), i.e \( f(X) = \{f(x) : x \in X\} \). \( |X| \) denotes the cardinality of \( X \) and \( \text{Id}_X \), or simply \( \text{Id} \) when \( X \) is clear from context, denotes the identity function on \( X \). A set \( X \) is
countable if $|X| \leq \omega$; if $X$ and $Y$ are sets then $X \subseteq Y$ denotes that $X$ is a finite subset of $Y$.

Algebras will be denoted by Gothic letters, and when we write $\mathfrak{A}$ then we will be tacitly assuming that $A$ will denote the universe of $\mathfrak{A}$. However, in some occasions we will identify (notationally) an algebra and its universe.

Fix some ordinal $n \geq 2$. For $i, j < n$ the replacement $[i/j]$ is the map that is like the identity on $n$, except that $i$ is mapped to $j$ and the transposition $[i, j]$ is the like the identity on $n$, except that $i$ is swapped with $j$. A map $\tau : n \to n$ is finitary if the set $\{i < n : \tau(i) \neq i\}$ is finite, so if $n$ is finite then all maps $n \to n$ are finitary. It is known, and indeed not hard to show, that any finitary permutation is a product of transpositions and any finitary non-injective map is a product of replacements.

The standard reference for all the classes of algebras to be dealt with is [7]. Each class in $\{\text{Df}_n, \text{Sc}_n, \text{CA}_n, \text{PA}_n, \text{PEA}_n, \text{QEA}_n, \text{QEA}_n\}$ consists of Boolean algebras with extra operators, as shown in figure 1, where $d_{ij}$ is a nullary operator (constant), $c_i, s_r, s_j^i$ and $s_{[i,j]}$ are unary operators, for $i, j < n$, $\tau : n \to n$.

The algebras considered can be conceived as a generalization from Boolean algebras to algebras of relations of higher rank and algebraic versions of different variants of first order logic. We start by recalling the concrete versions of such algebras. Such algebras consist of sets of sequences, and the operations are set-theoretic operations on such sets. Let $\alpha$ be an ordinal. Let $U$ be a set. For $t, s \in ^\alpha U$ and $i < \alpha$, write $t \equiv_i s$ if $t(j) = s(j)$ for all $i \neq j$. Then we define for $i, j < \alpha$, $\tau : \alpha \to \alpha$ and $X \subseteq ^\alpha U$:

$$c_i X = \{s \in ^\alpha U : \exists t \in X, t \equiv_i s\},$$

$$s_\tau X = \{s \in ^\alpha U : s \circ \tau \in X\},$$

$$d_{ij} = \{s \in ^\alpha U : s_i = s_j\}.$$ 

$[i/j]$ is the replacement on $\alpha$ that takes $i$ to $j$ and leaves every other thing fixed, while $[i, j]$ is the transposition interchanging $i$ and $j$.

The extra concrete non-Boolean operations we deal with are as specified above. For a set $X$, let $\mathfrak{B}(X) = (\mathfrak{B}(X), \cup, \cap, \sim, \emptyset, X)$ be the full Boolean set algebra with universe $\mathfrak{B}(X)$. Let $S$ be the operation of forming subalgebras, and $P$ be that of forming products.

$$\text{RDF}_\alpha = \text{SP}\{(\mathfrak{B}(^\alpha U), c_i)_{i<\alpha} : U \text{ is a set}\}.$$

$$\text{RSC}_\alpha = \text{SP}\{(\mathfrak{B}(^\alpha U), c_i, s_{[i,j]})_{i,j<\alpha} : U \text{ is a set}\}.$$

$$\text{RQA}_\alpha = \text{SP}\{(\mathfrak{B}(^\alpha U), c_i, s_{[i,j]}, s_{[i,j]})_{i,j<\alpha} : U \text{ is a set}\}.$$

$$\text{RCA}_\alpha = \text{SP}\{(\mathfrak{B}(^\alpha U), c_i, s_{[i,j]}, s_{[i,j]})_{i,j<\alpha} : U \text{ is a set}\}.$$

$$\text{RQEA}_\alpha = \text{SP}\{(\mathfrak{B}(^\alpha U), c_i, s_{[i,j]}, s_{[i,j]})_{i,j<\alpha} : U \text{ is a set}\}.$$
### Table: Extra Operators for Various Classes

| Class          | Extra Operators          |
|----------------|--------------------------|
| $Df_n$         | $c_i : i < n$            |
| $Sc_n$         | $c_i, s_{[i/j]} : i, j < n$ |
| $CA_n$         | $c_i, d_{ij} : i, j < n$  |
| $PA_n$         | $c_i, s_{\tau} : i < n, \tau \in \mathbb{n}^n$ |
| $PEA_n$        | $c_i, d_{ij}, s_{\tau} : i, j < n, \tau \in \mathbb{n}^n$ |
| $QA_n$         | $c_i, s_{[i/j]}, s_{[i,j]} : i, j < n$ |
| $QEA_n$        | $c_i, d_{ij}, s_{[i/j]}, s_{[i,j]} : i, j < n$ |

**Figure 1:** Non-Boolean operators for the classes

All such classes are varieties, so that they are closed under forming homomorphic images, too. For finite $n$, polyadic algebras are the same as quasi-polyadic algebra and for the infinite dimensional case we restrict our attention to quasi-polyadic algebras in $QA_n, QEA_n$. Each class is defined by a finite set of equation schema. Existing in a somewhat scattered form in the literature, finite equational axiomatizations defining $Sc_n, QA_n$ and $QEA_n$ are given in the appendix of [13]. For $CA_n$ we follow the standard axiomatization given in definition 1.1.1 in [6].

For any operator $o$ of any of these signatures, we write $\text{dim}(o) (\subseteq n)$ for the set of dimension ordinals used by $o$, e.g. $\text{dim}(c_i) = \{i\}$, $\text{dim}(s_{ij}) = \{i, j\}$. An algebra $\mathfrak{A}$ in $QEA_n$ has operators that can define any operator of $QA_n, CA_n, Sc_n$ and $Df_n$. Thus we may obtain the reducts $\mathfrak{R}_K \mathfrak{A}$ for $K \in \{QEA_n, QA_n, CA_n, Sc_n, Df_n\}$ and it turns out that the reduct always satisfies the equations defining the relevant class so $\mathfrak{R}_K \mathfrak{A} \in K$. Similarly from any algebra $\mathfrak{A}$ in any of the classes $QEA_n, QA_n, CA_n, Sc_n$ we may obtain the reduct $\mathfrak{R}_K \mathfrak{A} \in Sc_n$, which we write as $\mathfrak{R}_Sc \mathfrak{A}$. We also write $\mathfrak{R}_df \mathfrak{A}$ for the diagonal free reduct of $\mathfrak{A}$, so that $\mathfrak{R}_df \mathfrak{A} \in Df_n$.

Let $K \in \{QEA, QA, CA, Sc, Df\}$. Then for $n > 2$ any ordinal, $\mathfrak{R}_K \mathfrak{A}$ is not axiomatizable by a finite schema; in particular, $\mathfrak{R}_K \mathfrak{A} \neq K_n$. Let $\mathfrak{A} \in K_n$ and let $2 \leq m \leq n$ (possibly infinite ordinals). The **reduct to $m$ dimensions** $\mathfrak{R}_m \mathfrak{A} \in K_m$ is obtained from $\mathfrak{A}$ by discarding all operators with indices $m \leq i < n$. The **neat reduct to $m$ dimensions** is the algebra $\mathfrak{N}_m \mathfrak{A} \in K_m$ with universe $\{a \in \mathfrak{A} : m \leq i < n \rightarrow c_i a = a\}$ where all the operators are induced from $\mathfrak{A}$ (see [6] definition 2.6.28) for the CA case).

Let $\mathfrak{A} \in K_m$, $\mathfrak{B} \in K_n$. An injective homomorphism $f : \mathfrak{A} \rightarrow \mathfrak{B}$ is a **neat embedding** if the range of $f$ is a subalgebra of $\mathfrak{N}_m \mathfrak{B}$. The notions of neat reducts and neat embeddings have proved useful in analyzing the number of variables needed in proofs, as well as for proving representability results, via the so-called neat embedding theorems.

A generalized space of dimension $n$ is a set of the form $\bigcup_{i \in I} n^i U_i$ where $I$ is a non-empty indexing set, $U_i \neq \emptyset$ for each $i \in I$ and $U_i \cap U_j = \emptyset$ for dis-
tinct \( i, j \in I \). For any such space \( V \) of dimension \( n \), \( \wp(V) \) with the Boolean operations of union, intersection and complementation, and cylindrifiers, substitution operators and diagonal elements (if in the signature) defined like set algebras is a \( K_n \); in fact an \( RK_n \). Conversely, for any \( \mathfrak{A} \in RK_n \) there is a generalized space \( V \) and an embedding from \( f: \mathfrak{A} \to \wp(V) \). We call \( f \) a representation of \( \mathfrak{A} \).

Let \( \mathfrak{A} \in K_n \) and \( f: \mathfrak{A} \to \wp(V) \) be a representation of \( \mathfrak{A} \), where \( V \) is a generalized space of dimension \( n \).

A complete representation of \( \mathfrak{A} \) is a representation \( f \) satisfying

\[
f(\prod X) = \bigcap f[X]
\]

whenever \( X \subseteq \mathfrak{A} \) and \( \prod X \) is defined.

The action of the non-Boolean operators in a completely additive atomic Boolean algebra with operators is determined by their behavior over the atoms, and this in turn is encoded by the atom structure of the algebra. We use \( \text{BAO}(s) \) short for Boolean algebra(s) with operators; operators are operations that distribute over the Boolean join in every component.

**Definition 2.1. (Atom Structure)** Let \( \mathfrak{A} = \langle A,+,-,0,1,\Omega_i : i \in I \rangle \) be an atomic Boolean algebra with operators \( \Omega_i : i \in I \). Let the rank of \( \Omega_i \) be \( \rho_i \).

The atom structure \( \text{At}\mathfrak{A} \) of \( \mathfrak{A} \) is a relational structure

\[
\langle \text{At}\mathfrak{A}, R_{\Omega_i} : i \in I \rangle
\]

where \( \text{At}\mathfrak{A} \) is the set of atoms of \( \mathfrak{A} \), and \( R_{\Omega_i} \) is a \((\rho(i)+1)\)-ary relation over \( \text{At}\mathfrak{A} \) defined by

\[
R_{\Omega_i}(a_0, \ldots, a_{\rho(i)}) \iff \Omega_i(a_1, \ldots, a_{\rho(i)}) \geq a_0.
\]

Similar ‘dual’ structure arise in other ways, too. For any not necessarily atomic \( \text{BAO} \mathfrak{A} \), its ultrafilter frame is the structure

\[
\mathfrak{A}^+ = \langle \text{Uf}(\mathfrak{A}), R_{\Omega_i} : i \in I \rangle,
\]

where \( \text{Uf}(\mathfrak{A}) \) is the set of all ultrafilters of (the Boolean reduct of) \( \mathfrak{A} \), and for \( \mu_0, \ldots, \mu_{\rho(i)} \in \text{Uf}(\mathfrak{A}) \), we put \( R_{\Omega_i}(\mu_0, \ldots, \mu_{\rho(i)}) \) iff \( \{ \Omega(a_1, \ldots, a_{\rho(i)}) : a_j \in \mu_j \text{ for } 0 \leq j \leq \rho(i) \} \subseteq \mu_0 \).

**Definition 2.2. (Complex algebra)** Conversely, if we are given an arbitrary structure \( S = \langle S, r_i : i \in I \rangle \) where \( r_i \) is a \((\rho(i)+1)\)-ary relation over \( S \), we can define its complex algebra

\[
\mathfrak{C}m(S) = \langle \wp(S), \cup, \setminus, \phi, S, \Omega_i \rangle_{i \in I},
\]

where \( \wp(S) \) is the power set of \( S \), and \( \Omega_i \) is the \( \rho(i) \)-ary operator defined by

\[
\Omega_i(X_1, \ldots, X_{\rho(i)}) = \{ s \in S : \exists s_1 \in X_1 \cdots \exists s_{\rho(i)} \in X_{\rho(i)}, r_i(s, s_1, \ldots, s_{\rho(i)}) \},
\]

for each \( X_1, \ldots, X_{\rho(i)} \in \wp(S) \).
It is easy to check that, up to isomorphism, $\text{At}(Cm(S)) \cong S$ always, and $\mathfrak{A} \subseteq Cm(\text{At}\mathfrak{A})$ for any completely additive atomic Boolean algebra with operators $\mathfrak{A}$. If $\mathfrak{A}$ is finite then of course $\mathfrak{A} \cong Cm(\text{At}\mathfrak{A})$. For an atomic algebra $\mathfrak{A}$, $\text{TmAt}\mathfrak{A}$ denotes the term algebra; it is the subalgebra of $Cm\text{At}\mathfrak{A}$ generated by the atoms, the two algebras share the same atom structure. If $\mathfrak{A}$ is completely additive then $\text{TmAt}\mathfrak{A} \subseteq \mathfrak{A} \subseteq Cm\text{At}\mathfrak{A}$.

A variety $V$ of BAOs is atom-canonical if whenever $\mathfrak{A} \in V$, and $\mathfrak{A}$ is atomic, then $Cm\text{At}\mathfrak{A} \in V$. If $V$ is completely additive, then $Cm\text{At}\mathfrak{A}$ is the Dedekind-MacNeille completion of $\mathfrak{A}$. $\mathfrak{A}$ is always dense in its Dedekind-MacNeille completion and so the Dedekind-MacNeille completion of $\mathfrak{A}$ is atomic iff $\mathfrak{A}$ is atomic.

Not all varieties we will encounter are completely additive; the varieties $\text{Sc}_n$ and $\text{PA}_n$ for $n > 1$, are not, but Dedekind-MacNeille completions of atomic completely additive algebras remain in the variety, which is not the case with the variety of representable algebras as shown below.

The canonical extension of $\mathfrak{A}$, namely, $Cm(Uf(\mathfrak{A}))$ will be denoted by $\mathfrak{A}^+$. This algebra is always complete and atomic. $\mathfrak{A}$ embeds into $\mathfrak{A}^+$ under the usual ‘Stone representability’ function mapping $a \in \mathfrak{A}$ to the set of all Boolean ultrafilters of $\mathfrak{A}$ containing $a$. The algebra $\mathfrak{A}^+$ is the Dedekind-MacNeille completion of $\mathfrak{A}$ if and only if $\mathfrak{A}$ is finite.

For other operators on classes of algebras, $H$ denotes closure under forming homomorphic images and $Up$ under ultraproducts.

For a class $K$, and an algebra $\mathfrak{A}$ of the same signature as $K$, $\mathfrak{A} \in \text{Ur}K$ iff an ultrapower of $\mathfrak{A}$ is in $K$. $\text{UpUr}K$ is the elementary closure of $K$ that is the least elementary class containing $K$; $K$ is elementary or synonymously first order definable, if it is closed under ultraproducts and ultraroots. More succinctly $K$ is elementary if $\text{UpUr}K = K$. $K$ is a quasi-variety if it is closed under $\text{SPUp}$ and a variety if it is closed under $\text{HSP}$. If $K$ is a variety then it can be axiomatized by a set of equations; if its only a quasi-variety, then it can be axiomatized by a set quasi-equations. $K$ is elementary if it is definable by a set of first order sentences.

### 3 Rainbows, atom-canonicity

We will show using the so called blow up and blur construction, a very indicative name suggested in [4], that for any finite $n > 2$, any $K \in \{\text{Sc}, \text{CA}, \text{PA}, \text{PEA}\}$, and any $k \geq 3$, $\text{Snr}_nK_{n+k}$ is not atom canonical. We will blow up and blur a finite rainbow algebra.

We give the general idea for cylindric algebras, though the idea is much more universal as we will see. Assume that $\text{RCA}_n \subseteq K$, and $K$ is closed under forming subalgebras. Start with a finite algebra $\mathfrak{C}$ outside $K$. Blow up and
blur $\mathcal{C}$, by splitting each atom to infinitely many, getting a new atom structure $\mathbf{At}$. In this process a (finite) set of blurs are used.

They do not blur the complex algebra, in the sense that $\mathcal{C}$ is there on this global level. The algebra $\mathcal{CmAt}$ will not be in $K$ because $\mathcal{C} \not\in K$ and $\mathcal{C}$ embeds into $\mathcal{CmAt}$. Here the completeness of the complex algebra will play a major role, because every element of $\mathcal{C}$, is mapped, roughly, to the join of its splitted copies which exist in $\mathcal{CmAt}$ because it is complete.

These precarious joins prohibiting membership in $K$ do not exist in the term algebra, only finite-cofinite joins do, so that the blurs blur $\mathcal{C}$ on this level; $\mathcal{C}$ does not embed in $\mathcal{TmAt}$.

In fact, the the term algebra will not only be in $K$, but actually it will be in the possibly smaller $\mathbf{RCA}_n$. This is where the blurs play their other role. Basically non-principal ultrafilters, the blurs are used as colours to represent $\mathcal{TmAt}$.

In the process of representation we cannot use only principal ultrafilters, because $\mathcal{TmAt}$ cannot be completely representable, that is, it cannot have a representation that preserves all (possibly infinitary) meets carrying them to set theoretic intersections, for otherwise this would give that $\mathcal{CmAt}$ is representable.

But the blurs will actually provide a complete representation of the canonical extension of $\mathcal{TmAt}$, in symbols $\mathcal{TmAt}^+$; the algebra whose underlying set consists of all ultrafilters of $\mathcal{TmAt}$. The atoms of $\mathcal{TmAt}$ are coded in the principal ones, and the remaining non-principal ultrafilters, or the blurs, will be finite, used as colours to completely represent $\mathcal{TmAt}^+$, in the process representing $\mathcal{TmAt}$.

We start off with a conditional theorem: giving a concrete instance of a blow up and blur construction for relation algebras due to Hirsch and Hodkinson. The proof is terse highlighting only the main ideas.

**Theorem 3.1.** Let $m \geq 3$. Assume that for any simple atomic relation algebra with atom structure $S$, there is a cylindric atom structure $H$, constructed effectively from $S$, such that:

1. If $\mathcal{TmS} \in \mathbf{RRA}$, then $\mathcal{TmH} \in \mathbf{RCA}_m$,

2. If $S$ is finite, then $H$ is finite,

3. $\mathcal{CmS}$ is embeddable in $\mathbf{Ra}$ reduct of $\mathcal{CmH}$.

Then for all $k \geq 3$, $S\mathfrak{m}_m \mathbf{CA}_{m+k}$ is not closed under completions,

**Proof.** Let $S$ be a relation atom structure such that $\mathcal{TmS}$ is representable while $\mathcal{CmS} \not\in \mathbf{RA}_\infty$. Such an atom structure exists [[9, Lemmas 17.34-35-36-37]].
We give a brief sketch at how such algebras are constructed by allowing complete irreflexive graphs having an arbitrary finite set nodes, slightly generalizing the proof in *op.cit*, though the proof idea is essentially the same.

Another change is that we refer to non-principal ultrafilters (intentionally) by *blurs* to emphasize the connection with the blow up and blur construction in [4] as well as with the blow up and blur construction outlined above, to be encountered in full detail in a little while, witness theorem 3.11.

In all cases a finite algebra is blown up and blurred to give a representable algebra (the term algebra on the blown up and blurred finite atom structure) whose Dedekind-MacNeille completion does not have a neat embedding property.

We use the notation of the above cited lemmas in [12] without warning, and our proof will be very brief just stressing the main ideas. $G^n_r$ denotes the usual atomic $r$ rounded game played on atomic networks having $n$ nodes of an atomic relation algebra, where $n, r \leq \omega$, and $K_r (r \in \omega)$ denotes the complete irreflexive graph with $r$ nodes.

Let $R$ be the rainbow algebra $\mathfrak{A}_{K_m,K_n}$, $m > n > 2$. Let $T$ be the term algebra obtained by splitting the reds. Then $T$ has exactly two blurs $\delta$ and $\rho$. $\rho$ is a flexible non-principal ultrafilter consisting of reds with distinct indices and $\delta$ is the reds with common indices. Furthermore, $T$ is representable, but $\mathfrak{C}mAtT \notin S\mathfrak{RaCA}_{m+2}$, in particular, it is not representable [12, Lemma 17.32].

Now we use the Ehrenfeucht–Fraïssé forth pebble game $EF^k_r(A, B)$ where $A$ and $B$ are relational structures. This game has $r$ rounds and $k$ pebbles. The rainbow theorem [9, Theorem 16.5] says that $\exists$ has a winning strategy in the game $G^n_{1+r}(\mathfrak{A}_{A,B})$ if and only if she has a winning strategy in $EF^k_r(A, B)$.

Using this theorem it is obvious that $\exists$ has a winning strategy over $\mathfrak{A}R$ in $m + 2$ rounds, hence $R \notin \mathfrak{RA}_{m+2}$, hence is not in $S\mathfrak{RaCA}_{m+2}$. $\mathfrak{C}mAtT$ is also not in the latter class for $R$ embeds into it, by mapping every red to the join of its copies. Let $D = \{r^n_{il} : n < \omega, l \in n\}$, and $\mathcal{R} = \{r^n_{im}, l, m \in n, l \neq m\}$. If $X \subseteq R$, then $X \subseteq T$ if and only if $X$ is finite or cofinite in $R$ and same for subsets of $D$ [9 lemma 17.35]. Let $\delta = \{X \subseteq T : X \cap D$ is cofinite in $D\}$, and $\rho = \{X \subseteq T : X \cap R$ is cofinite in $R\}$. Then these are the non principal ultrafilters, they are the blurs and they are enough to (to be used as colours), together with the principal ones, to represent $T$ as follows [12, bottom of p. 533]. Let $\Delta$ be the graph $n \times \omega \cup m \times \{\omega\}$. Let $\mathfrak{B}$ be the full rainbow algebras over $\mathfrak{At}\mathfrak{A}_{K_m,\Delta}$ by deleting all red atoms $r_{ij}$ where $i, j$ are in different connected components of $\Delta$.

Obviously $\exists$ has a winning strategy in $EF^\omega_r(K_m, K_m)$, and so it has a winning strategy in $G^\omega_r(\mathfrak{A}_{K_m,K_m})$. But $\mathfrak{At}\mathfrak{A}_{K_m,K_m} \subseteq \mathfrak{At}\mathfrak{B} \subseteq \mathfrak{At}_{K_m,\Delta}$, and so $\mathfrak{B}$ is representable.

One then defines a bounded morphism from $\mathfrak{At}\mathfrak{B}$ to the the canonical extension of $T$, which we denote by $T^+$, consisting of all ultrafilters of $T$. The
blurs are images of elements from $K_m \times \{\omega\}$, by mapping the red with equal double index, to $\delta$, for distinct indices to $\rho$. The first copy is reserved to define the rest of the red atoms the obvious way. (The underlying idea is that this graph codes the principal ultrafilters in the first component, and the non principal ones in the second.) The other atoms are the same in both structures. Let $S = \mathcal{C}mAtT$, then $\mathcal{C}mS \notin S\mathcal{R}a\mathcal{C}A_{m+2}$ [9 lemma 17.36].

Note here that the Dedekind-MacNeille completion of $T$ is not representable while its canonical extension is completely representable, via the representation defined above. However, $T$ itself is not completely representable, for a complete representation of $T$ induces a representation of its Dedekind-MacNeille completion, namely, $\mathcal{C}mAtA$.

Now let $H$ be the $\mathcal{C}A_m$ atom structure obtained from $AtT$ provided by the hypothesis of the theorem. Then $\mathcal{C}mH \in R\mathcal{C}A_m$. We claim that $\mathcal{C}mH \notin S\mathcal{M}t_m\mathcal{C}A_{m+k}$, $k \geq 3$. For assume not, i.e. assume that $\mathcal{C}mH \in S\mathcal{M}t_m\mathcal{C}A_{m+k}$, $k \geq 3$. We have $\mathcal{C}mS$ is embeddable in $\mathcal{R}a\mathcal{C}mH$. But then the latter is in $S\mathcal{R}a\mathcal{C}A_6$ and so is $\mathcal{C}mS$, which is not the case.

Hodkinson constructs atom structures of cylindric and polyadic algebras of any pre-assigned finite dimension $> 2$ from atom structures of relation algebras [15]. One could well be tempted to use such a construction with the above proof to obtain an analogous result for cylindric and polyadic algebras. However, we emphasize that the next result cannot be obtained by lifting the relation algebra case [9 lemmas 17.32, 17.34, 17.35, 17.36] to cylindric algebras using Hodkinson’s construction in [15] as it stands. It is true that Hodkinson constructs from every atomic relation algebra an atomic cylindric algebra of dimension $n$, for any $n \geq 3$, but the relation algebras does not embed into the $R\mathcal{A}$ reduct of the constructed cylindric algebra when $n \geq 6$. If it did, then the $R\mathcal{A}$ result would lift as indeed is the case with $n = 3$ [20]. Now we are faced with two options. Either modify Hodkinson’s construction, implying that the embeddability of the given relation algebra in the $R\mathcal{A}$ reduct of the constructed cylindric algebras, or avoid completely the route via relation algebras. We tend to think that it is impossible to adapt Hodkinson’s construction the way needed, because if $\mathcal{A}$ is a non representable relation algebra, and $\mathcal{C}mAt(\mathcal{A})$ embeds into the $R\mathcal{A}$ reduct of a cylindric algebra of every dimension $> 2$, then $\mathcal{A}$ will be representable, which is a contradiction.

Therefore we choose the second option. We instead start from scratch. We blow up and blur a finite rainbow cylindric algebra.

In [12] the rainbow cylindric algebra of dimension $n$ on a graph $\Gamma$ is denoted by $R(\Gamma)$. We consider $R(\Gamma)$ to be in $\mathcal{P}E\mathcal{A}_n$ by expanding it with the polyadic operations defined the obvious way (see below). In what follows we consider $\Gamma$ to be the indices of the reds, and for a complete irreflexive graph $\mathcal{G}$, by $\mathcal{P}E\mathcal{A}_{\mathcal{G},\Gamma}$ we mean the rainbow cylindric algebra $R(\Gamma)$ of dimension $n$, where $G = \{g_i : 1 \leq i < n - 1\} \cup \{g_0 : i \in \mathcal{G}\}$. 

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More generally, we consider a rainbow polyadic algebra based on relational structures $A, B$, to be the rainbow algebra with signature the binary colours (binary relation symbols) $\{r_{ij} : i, j \in B\} \cup \{w_i : i < n - 1\} \cup \{g_i : 1 \leq i < n - 1\} \cup \{g_0 : i \in A\}$ and $n - 1$ shades of yellow ($n - 1$ ary relation symbols) $\{y_S : S \subseteq A, or S = A\}$.

We look at models of the rainbow theorem as coloured graphs [10]. This class is denoted by $\text{CRG}_{A,B}$ or simply $\text{CRG}$ when $A$ and $B$ are clear from context.

A coloured graph is a graph such that each of its edges is labelled by one of the first three colours mentioned above, namely, greens, whites or reds, and some $n - 1$ hyperedges are also labelled by the shades of yellow. Certain coloured graphs will deserve special attention.

**Definition 3.2.** Let $i \in A$, and let $M$ be a coloured graph consisting of $n$ nodes $x_0, \ldots, x_{n-2}, z$. We call $M$ an $i$-cone if $M(x_0, z) = g_0^i$ and for every $1 \leq j \leq n - 2$, $M(x_j, z) = g_j$, and no other edge of $M$ is coloured green. $(x_0, \ldots, x_{n-2})$ is called the center of the cone, $z$ the apex of the cone and $i$ the tint of the cone.

**Definition 3.3.** The class of coloured graphs $\text{CRG}$ are

- $M$ is a complete graph.
- $M$ contains no triangles (called forbidden triples) of the following types:

  \[(g, g', g^*), (g_i, g_0, w_i) \quad \text{any } 1 \leq i < n - 1 \quad (1)\]
  \[(g_0^i, g_k^j, w_0) \quad \text{any } j, k \in A \quad (2)\]
  \[(r_{ij}, r_{j'k'}, r_{i'k'^*}) \quad i, j, j', k', i^*, k^* \in B \quad (3)\]

  unless $i = i^*$, $j = j'$ and $k' = k^*$

and no other triple of atoms is forbidden.

- If $a_0, \ldots, a_{n-2} \in M$ are distinct, and no edge $(a_i, a_j) i < j < n$ is coloured green, then the sequence $(a_0, \ldots, a_{n-2})$ is coloured a unique shade of yellow. No other $(n - 1)$ tuples are coloured shades of yellow.

- If $D = \{d_0, \ldots, d_{n-2}, \delta\} \subseteq M$ and $M \upharpoonright D$ is an $i$ cone with apex $\delta$, inducing the order $d_0, \ldots, d_{n-2}$ on its base, and the tuple $(d_0, \ldots, d_{n-2})$ is coloured by a unique shade $y_S$ then $i \in S$.

One then can define a polyadic equality atom structure of dimension $n$ from the class $\text{CRG}$. It is a rainbow atom structure. Rainbow atom structures are what Hirsch and Hodkinson call atom structures built from a class of models.
Our models are, according to the original more traditional view \cite{10} coloured graphs. So let \textbf{CRG} be the class of coloured graphs as defined above. Let

\[ \text{At} = \{ a : n \to M, M \in \text{CRG} : a \text{ is surjective} \}. \]

We write \( M_a \) for the element of \text{At} for which \( a : n \to M \) is a surjection. Let \( a, b \in \text{At} \) define the following equivalence relation:

1. \( a(i) = a(j) \iff b(i) = b(j) \),
2. \( M_a(a(i), a(j)) = M_b(b(i), b(j)) \) whenever defined,
3. \( M_a(a(k_0), \ldots, a(k_{n-2})) = M_b(b(k_0), \ldots, b(k_{n-2})) \) whenever defined.

Let \text{At} be the set of equivalences classes. Then define

\[ [a] \in E_{ij} \iff a(i) = a(j). \]

\[ [a]T_i[b] \iff a \upharpoonright n \setminus \{i\} = b \upharpoonright n \setminus \{i\}. \]

Define accessibility relations corresponding to the polyadic (transpositions) operations as follows:

\[ [a]S_{ij}[b] \iff a \circ [i, j] = b. \]

This, as easily checked, defines a \( \text{PEA}_n \) atom structure. The complex algebra of this atom structure is denoted by \( \text{PEA}_{A,B} \) where \( A \) is the greens and \( B \) is the reds.

Consider the following two games on coloured graphs, each with \( \omega \) rounds, and limited number of pebbles \( m > n \). They are translations of \( \omega \) atomic games played on atomic networks of a rainbow algebra using a limited number of nodes \( m \). Both games offer \( \forall \) only one move, namely, a cylindrifier move.

From the graph game perspective both games \cite{10} p.27-29 build a nested sequence \( M_0 \subseteq M_1 \subseteq \ldots \) of coloured graphs.

First game \( G^m. \forall \) picks a graph \( M_0 \in \text{CRG} \) with \( M_0 \subseteq m \) and \( \exists \) makes no response to this move. In a subsequent round, let the last graph built be \( M_i \). \forall \) picks

1. a graph \( \Phi \in \text{CRG} \) with \( |\Phi| = n \),
2. a single node \( k \in \Phi \),
3. a coloured graph embedding \( \theta : \Phi \setminus \{k\} \to M_i \). Let \( F = \phi \setminus \{k\} \). Then \( F \) is called a face. \( \exists \) must respond by amalgamating \( M_i \) and \( \Phi \) with the embedding \( \theta \). In other words she has to define a graph \( M_{i+1} \in C \) and embeddings \( \lambda : M_i \to M_{i+1} \mu : \phi \to M_{i+1} \), such that \( \lambda \circ \theta = \mu \upharpoonright F \).
\(F^m\) is like \(G^m\), but \(\forall\) is allowed to reuse nodes.

\(F^m\) has an equivalent formulation on atomic networks of atomic algebras.

Let \(\delta\) be a map. Then \(\delta[i \to d]\) is defined as follows. \(\delta[i \to d](x) = \delta(x)\) if \(x \neq i\) and \(\delta[i \to d](i) = d\). We write \(\delta_i^j\) for \(\delta[i \to \delta j]\).

**Definition 3.4.** Let \(2 < n < \omega\). Let \(\mathcal{C}\) be an atomic PEA\(_n\). An atomic network over \(\mathcal{C}\) is a map

\[
N : \Delta \to \text{At}\mathcal{C},
\]

where \(\Delta\) is a non-empty set called a set of nodes, such that the following hold for each \(i, j < n\), \(\delta \in \Delta\) and \(d \in \Delta\):

- \(N(\delta^j_i) \leq d_{ij}\),
- \(N(\delta[i \to d]) \leq c_i N(\delta)\),
- \(N(\bar{x} \circ [i, j]) = s_{[i,j]} N(\bar{x})\) for all \(i, j < n\).

**Definition 3.5.** Let \(2 \leq n < \omega\). For any \(\mathcal{Sc}_n\) atom structure \(\alpha\) and \(n < m \leq \omega\), we define a two-player game \(F^m(\alpha)\), each with \(\omega\) rounds.

Let \(m \leq \omega\). In a play of \(F^m(\alpha)\) the two players construct a sequence of networks \(N_0, N_1, \ldots\) where \(\text{nodes}(N_i)\) is a finite subset of \(m = \{j : j < m\}\), for each \(i\).

In the initial round of this game \(\forall\) picks any atom \(a \in \alpha\) and \(\exists\) must play a finite network \(N_0\) with \(\text{nodes}(N_0) \subseteq m\), such that \(N_0(\bar{d}) = a\) for some \(\bar{d} \in \text{nodes}(N_0)\).

In a subsequent round of a play of \(F^m(\alpha)\), \(\forall\) can pick a previously played network \(N\) an index \(l < n\), a face \(F = \langle f_0, \ldots, f_{n-2} \rangle \in n^{-2}\text{nodes}(N), k \in m \sim \{f_0, \ldots, f_{n-2}\}\), and an atom \(b \in \alpha\) such that

\[
b \leq c_i N(f_0, \ldots, f_i, x, \ldots, f_{n-2}).
\]

The choice of \(x\) here is arbitrary, as the second part of the definition of an atomic network together with the fact that \(c_i(c_i x) = c_i x\) ensures that the right hand side does not depend on \(x\).

This move is called a cylindrifier move and is denoted

\[(N, \langle f_0, \ldots, f_{n-2} \rangle, k, b, l)\]

or simply by \((N, F, k, b, l)\). In order to make a legal response, \(\exists\) must play a network \(M \supseteq N\) such that \(M(f_0, \ldots, f_{i-1}, k, f_{i+1}, \ldots, f_{n-2}) = b\) and \(\text{nodes}(M) = \text{nodes}(N) \cup \{k\}\).

\(\exists\) wins \(F^m(\alpha)\) if she responds with a legal move in each of the \(\omega\) rounds. If she fails to make a legal response in any round then \(\forall\) wins.
Next we adapt certain notions worked out for relation algebras in [8] to the CA context culminating in theorem 3.9 that will be used several times, to show that certain constructed algebras do not have a neat embedding property.

**Definition 3.6.** Let $n$ be an ordinal. An $s$ word is a finite string of substitutions ($s^i_j$), a $c$ word is a finite string of cylindrifications ($c_k$). An $sc$ word is a finite string of substitutions and cylindrifications. Any $sc$ word $w$ induces a partial map $\hat{w} : n \to n$ by

- $\hat{e} = Id,$
- $\hat{w}_j^i = \hat{w} \circ [i|j],$
- $\hat{w}\bar{c}_j = \hat{w} \mid (n \setminus \{i\}).$

If $\bar{a} \in {{}^{n-1}}n$, we write $s_{\bar{a}}$, or more frequently $s_{0\ldots a_{k-1}}$, where $k = |\bar{a}|$, for an arbitrary chosen $sc$ word $w$ such that $\hat{w} = \bar{a}$. $w$ exists and does not depend on $w$ by [9] definition 5.23 lemma 13.29. We can, and will assume [9] Lemma 13.29 that $w = sc_{n_1-c_0}$. [In the notation of [9] definition 5.23, lemma 13.29], $\hat{s}_{ijk}$ for example is the function $n \to n$ taking 0 to $i$, 1 to $j$ and 2 to $k$, and fixing all $l \in n \setminus \{i, j, k\}$.

We need some more technical lemmas which are cylindric analogues of lemmas formulated for relation algebras in [8].

The next definition is formulated for the least reduct $Scs$ of PEAs, so that it applies to all its expansions studied here.

**Definition 3.7.** For $m \geq 5$ and $\mathcal{C} \in Sc_m$, if $\mathfrak{A} \subseteq \mathfrak{A} \in Sc_m$, and $\mathcal{N}$ is an $\mathfrak{A}$-network with $\text{nodes}(\mathcal{N}) \subseteq m$, then we define $\hat{N} \in \mathcal{C}$ by

$$\hat{N} = \prod_{i_0\ldots,i_{n-1} \in \text{nodes}(\mathcal{N})} s_{i_0\ldots,i_{n-1}} N(i_0, \ldots, i_{n-1})$$

$\hat{N} \in \mathcal{C}$ depends implicitly on $\mathcal{C}$.

In what follows we write $\mathfrak{A} \subseteq_c \mathfrak{B}$ if $\mathfrak{A}$ is a complete subalgebra of $\mathfrak{B}$, that is, if $\mathfrak{A} \subseteq \mathfrak{B}$ and whenever $X \subseteq \mathfrak{A}$ is such that $\sum^\mathfrak{A} X = 1$, then $\sum^\mathfrak{B} X = 1$.

**Lemma 3.8.** Let $n < m$ and let $\mathfrak{A}$ be an atomic $Sc_n$, $\mathfrak{A} \subseteq_c \mathfrak{A} \subseteq_n \mathfrak{C}$ for some $\mathfrak{C} \in Sc_m$. For all $x \in \mathcal{C} \setminus \{0\}$ and all $i_0, \ldots, i_{n-1} < m$ there is $a \in \text{At}(\mathfrak{A})$ such that $s_{i_0\ldots,i_{n-1}} a$. $x \neq 0$.

**Proof.** We can assume, see definition 3.6 that $s_{i_0\ldots,i_{n-1}}$ consists only of substitutions, since $c_m c_{m-1} \ldots c_n x = x$ for every $x \in \mathfrak{A}$. We have $s^i_j$ is a completely additive operator (any $i, j$), hence $s_{i_0\ldots,i_{n-1}}$ is too (see definition 3.6). So $\sum \{s_{i_0\ldots,i_{n-1}} a : a \in \text{At}(\mathfrak{A})\} = s_{i_0\ldots,i_{n-1}} \sum \text{At}(\mathfrak{A}) = s_{i_0\ldots,i_{n-1}} 1 = 1$, for any $i_0, \ldots, i_{n-1} < n$. Let $x \in \mathcal{C} \setminus \{0\}$. It is impossible that $s_{i_0\ldots,i_{n-1}} x = 0$ for all $a \in \text{At}(\mathfrak{A})$ because this would imply that $1 - x$ was an upper bound for $\{s_{i_0\ldots,i_{n-1}} a : a \in \text{At}(\mathfrak{A})\}$, contradicting $\sum \{s_{i_0\ldots,i_{n-1}} a : a \in \text{At}(\mathfrak{A})\} = 1$. \hfill $\Box$
For networks $M, N$ and any set $S$, we write $M \equiv^S N$ if $N|_S = M|_S$, and we write $M \equiv_S N$ if the symmetric difference $\Delta(\text{nodes}(M), \text{nodes}(N)) \subseteq S$ and $M \equiv^{(\text{nodes}(M) \cup \text{nodes}(N)) \setminus S} N$. We write $M \equiv_k N$ for $M \equiv_{\{k\}} N$.

We write $\text{Id}_{i \mapsto j}$ for the function $\{(k, k) : k \in n \setminus \{i\}\}$. For a network $N$ and a partial map $\theta$ from $n$ to $n$, that is $\text{dom}\theta \subseteq n$, then $N\theta$ is the network whose labelling is defined by $N\theta(x) = N(\tau(x))$ where for $i \in n$, $\tau(i) = \theta(i)$ for $i \in \text{dom}\theta$ and $\tau(i) = i$ otherwise. Recall that $F^m$ is the usual atomic game on networks, except that the nodes are $m$ and $\forall$ can re use nodes. Then:

**Theorem 3.9.** Let $K$ be any class between $\text{Sc}$ and PEA. Let $n < m$, and let $\mathfrak{A}$ be an atomic $K_n$. If $\mathfrak{A} \in S_\text{Amn}K_m$, then $\exists \mathfrak{A}$ has a winning strategy in $F^m(\text{At}\mathfrak{A})$ (the latter involves only cylindrifier moves so it applies to algebras in $K$). In particular, if $\mathfrak{A}$ is completely representable, then $\exists \mathfrak{A}$ has a winning strategy in $F^\omega(\text{At}\mathfrak{A})$.

**Proof.** The proof of the first part is based on repeated use of lemma 3.8. We first show (*):

1. For any $x \in \mathcal{C} \setminus \{0\}$ and any finite set $I \subseteq m$ there is a network $N$ such that $\text{nodes}(N) = I$ and $x \cdot \hat{N} \neq 0$.

2. For any networks $M, N$ if $\hat{M} \cdot \hat{N} \neq 0$ then $M \equiv^{\text{nodes}(M) \cap \text{nodes}(N)} N$.

We define the edge labelling of $N$ one edge at a time. Initially no hyperedges are labelled. Suppose $E \subseteq \text{nodes}(N) \times \text{nodes}(N) \ldots \times \text{nodes}(N)$ is the set of labelled hyper edges of $N$ (initially $E = \emptyset$) and $x \cdot \prod_{\bar{c} \in E} s_{\bar{c}} N(\bar{c}) \neq 0$. Pick $\bar{d}$ such that $\bar{d} \not\in E$. Then there is $\bar{a} \in \text{At}(\mathfrak{A})$ such that $x \cdot \prod_{\bar{c} \in E} s_{\bar{c}} N(\bar{c}) \cdot s_{\bar{d}} \theta \neq 0$.

Include the edge $\bar{d}$ in $E$. Eventually, all edges will be labelled, so we obtain a completely labelled graph $N$ with $\hat{N} \neq 0$. It is easily checked that $N$ is a network.

For the second part, if it is not true that $M \equiv^{\text{nodes}(M) \cap \text{nodes}(N)} N$, then there are $\bar{c} \in \text{nodes}(M) \cap \text{nodes}(N)$ such that $M(\bar{c}) \neq N(\bar{c})$. Since edges are labelled by atoms we have $M(\bar{c}) \cdot N(\bar{c}) = 0$, so $0 = s_{\bar{c}} 0 = s_{\bar{c}} M(\bar{c}) \cdot s_{\bar{c}} N(\bar{c}) \geq \hat{M} \cdot \hat{N}$.

Next, we show that (**):

1. If $i \not\in \text{nodes}(N)$ then $c_i \hat{N} = \hat{N}$.

2. $\hat{N} Id_{i \mapsto j} \geq \hat{N}$.

3. If $i \not\in \text{nodes}(N)$ and $j \in \text{nodes}(N)$ then $\hat{N} \neq 0 \rightarrow \hat{N}[i/j] \neq 0$, where $N[i/j] = N \circ [i/j]$.

4. If $\theta$ is any partial, finite map $n \rightarrow n$ and if $\text{nodes}(N)$ is a proper subset of $n$, then $\hat{N} \neq 0 \rightarrow \hat{N} \theta \neq 0$. 

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The first part is easy. The second part is by definition of $\hat{\cap}$. For the third part, suppose $\hat{N} \neq 0$. Since $i \notin \text{nodes}(N)$, by part [1] we have $c_i \hat{N} = \hat{N}$. By cylindric algebra axioms it follows that $\hat{N} \cdot d_{ij} \neq 0$. From the above there is a network $M$ where $\text{nodes}(M) = \text{nodes}(N) \cup \{i\}$ such that $\hat{M} \cdot d_{ij} \neq 0$. From the first part, we have $M \supseteq N$ and $M = N[i/j]$. Hence $\hat{N}[i/j] \neq 0$.

For the final part (cf. [9, lemma 13.29]), since there is $k \in n \setminus \text{nodes}(N)$, $\theta$ can be expressed as a product $\sigma_0 \sigma_1 \ldots \sigma_t$ of maps such that, for $s \leq t$, we have either $\sigma_s = \text{Id}_i$ for some $i < n$ or $\sigma_s = [i/j]$ for some $i, j < n$ and where $i \notin \text{nodes}(N \sigma_0 \ldots \sigma_{s-1})$. Now apply parts [2] and [3].

Now we prove the required. Suppose that $\mathfrak{A} \subseteq_c \mathfrak{Nt}_n \mathfrak{C}$ for some $\mathfrak{C} \in K_m$, then $\exists$ always plays networks $N$ with $\text{nodes}(N) \subseteq n$ such that $\hat{N} \neq 0$. In more detail, in the initial round, let $\forall$ play $a \in \text{At}\mathfrak{A}$. $\exists$ plays a network $N$ with $N(0, \ldots, n - 1) = a$. Then $\hat{N} = a \neq 0$. At a later stage suppose $\forall$ plays the cylindrifier move $(N, \langle f_0, \ldots, f_{n-2} \rangle, k, b, l)$ by picking a previously played network $N$ and $f_i \in \text{nodes}(N)$, $l < n, k \notin \{f_i : i < n - 2\}$, and $b \leq c_k \hat{N}(f_0, \ldots, f_{i-1}, x, f_{i+1}, \ldots, f_{n-2})$.

Let $\bar{a} = \langle f_0 \ldots f_{i-1}, k, f_{i+1}, \ldots f_{n-2} \rangle$. Then by (*), we have that $c_k \hat{N} \cdot s_a b \neq 0$ and so by item 1 in (**), there is a network $M$ such that $\hat{M} \cdot c_k \hat{N} \cdot s_a b \neq 0$. Hence

$$M \langle f_0, \ldots, f_{i-1}, k, f_{i+1}, \ldots, f_{n-2} \rangle = b,$$

and $M$ is the required response.

The last part follows from the fact that if $\mathfrak{A}$ is completely representable, then $\mathfrak{A} \in S_c \mathfrak{Nt}_n \mathfrak{QEA}_c$, witness theorem [5,8] below.

We shall also need:

**Theorem 3.10.** Let $2 < n < \omega$. Let $K \in \{\text{Df, Sc, CA, PA, PEA}\}$. If $\mathfrak{A} \in K_n$ is generated by $\{a \in \mathfrak{A} : \Delta a \neq n\}$, then $\mathfrak{A}$ is completely representable if and only if $\mathfrak{Ro}_d \mathfrak{A}$ is completely representable.

**Proof.** [15, Proposition 4.10].

The idea of the proof of our next theorem is summarized in the following:

1. We construct a labelled hypergraph $M$ that can be viewed as an $n$ homogeneous model of a certain first order rainbow theory. This model gets its labels from a rainbow signature. By $n$ homogeneous we mean that every partial isomorphism of $M$ of size $\leq n$ can be extended to an automorphism of $M$.

2. We have a shade of red $\rho$, outside the signature; this is basically the blur; a non-principal ultrafilter, in the term algebra which consists of finite and cofinite sets on a countable set, but $\rho$ can be used as a label.
(3) We build a relativized set algebra based on $M$, by discarding all assignments whose edges are labelled by the shade of red getting $W \subseteq n^M$.

(4) $W$ is definable in $n^M$ be an $L_{\infty,\omega}$ formula hence the semantics with respect to $W$ coincides with classical Tarskian semantics (when assignments are in $n^M$). This is proved using certain back and forth systems.

(5) The set algebra $\mathfrak{A}$ based on $W$ (consisting of sets of sequences (without shades of reds labelling edges) satisfying formulas in $L^n$ in the given signature) will be an atomic simple representable algebra such that its completion is not representable; the atoms will be coloured graphs. The completion $\mathfrak{C}$, which is the complex algebra of the rainbow atom structure, will consist of interpretations of $L^n_{\infty,\omega}$ formulas; though represented over $W$, it will not be, and cannot be, representable in the classical sense. In fact, its $\mathfrak{Sc}$ reduct will be outside $S\mathfrak{Mr}_n\mathfrak{Sc}_{n+3}$.

(6) The last part in the previous item is proved by embedding a finite rainbow polyadic equality algebra whose $\mathfrak{Sc}$ reduct is not in $S\mathfrak{Mr}_n\mathfrak{Sc}_{n+3}$ in $\mathfrak{C}$; so that the term algebra which is obtained by blowing up and blurring this finite rainbow algebra is representable, while the $\mathfrak{Sc}$ reduct of $\mathfrak{C}$, the Dedekind-MacNeille completion of $\mathfrak{A}$, is outside $S\mathfrak{Mr}_n\mathfrak{Sc}_{n+3}$, too.

In the next proof some parts overlap with parts in [14]; we include them to make the proof self contained as much as possible referring to [14] every time we do this.

**Theorem 3.11.** Let $n$ be finite > 2. Then there exists a countable representable atomic $\mathfrak{A} \in \mathfrak{PEA}_n$ such that $\mathfrak{Rd}\mathfrak{At}\mathfrak{A}$ is not in $S\mathfrak{Mr}_n\mathfrak{Sc}_{n+3}$, and $\mathfrak{Rd}_d\mathfrak{A}$ is not completely representable. In particular, for any finite $n > 2$, any $K \in \{\mathfrak{Sc}, \mathfrak{PA}, \mathfrak{PEA}, \mathfrak{CA}\}$ any class between $S\mathfrak{Mr}_nK_{n+3}$ and $\mathfrak{RK}_n$ is not atom-canonical.

**Proof.** We blow up and blur a finite rainbow polyadic equality algebra, namely, $R(\Gamma)$ where $\Gamma$ is the complete irreflexive graph $n$, and the greens are $G = \{g_i : 1 \leq i < n - 1\} \cup \{g^i_0 : 1 \leq i \leq n + 1\}$, we denote this finite rainbow algebra by $\mathfrak{PEA}_{n+1,n}$.

Let $\mathfrak{At}$ be the rainbow atom structure similar to that in [14] except that we have $n + 1$ greens and $n$ indices for reds, so that the rainbow signature $L$ now consists of $g_i : 1 \leq i < n - 1$, $g^i_0 : 1 \leq i \leq n + 1$, $w_t : i < n - 1$, $r^t_{kl} : k < l \in n$, $t \in \omega$, binary relations, and $y_S, S \subseteq n + 1, n - 1$ ary relations. We also have a shade of red $\rho$; the latter is a binary relation but is outside the rainbow signature, though it is used to label coloured graphs during a certain game devised to prove representability of the term algebra [14], and in fact $\exists$ can win the $\omega$ rounded game and build the $n$ homogeneous model $M$ by using $\rho$ whenever she is forced a red, as will be shown in a while.
So $At$ is obtained from the rainbow atom structure of the algebra $\mathfrak{A}$ defined in [14, section 4.2 starting p. 25] truncating the greens to be finite (exactly $n+1$ greens) and everything else is the same. In [14] it shown that the complex algebra $\mathfrak{CmAt}\mathfrak{A}$ is not representable; the result to be obtained now, because the greens are finite but still outfit the red, is sharper; it will imply that $\mathfrak{RmSc}\mathfrak{CmAt} \notin \mathfrak{SR}_{n+3}$.

Now $\mathfrak{ImAt} \in \mathfrak{RPEA}_n$; this can be proved exactly like in [14]. Strictly speaking the cylindric reduct of $\mathfrak{ImAt}$ can be proved representable like in [14]; representing the polyadic operations is straightforward, by swapping variables in representing formulas.

Let us spell out more details. The first three items are very similar to Hodkinson’s arguments. The only difference between our atom structure and his is that we use finitely many greens, while he uses infinitely many. The greens do not contribute to this part of the proof; all the other colours do. The reds are the most important in this part of the proof.

(1) **Constructing an $n$ homogeneous model**

We define a new class of coloured graphs $\mathcal{G}$: they are obtained from $\mathcal{CGR}$ by adding the shade of red $\rho$, and new forbidden triples of reds involving the shade of red $\rho$, namely, $(r_{jk}, r_{jk'}, \rho)$ for any $i, j, k, i', j', k' \in n$ and $(r_{jk}, \rho, \rho)$ for any $i, j, k \in n+1$. Strictly speaking the reds here are different from the reds specified in the rainbow colours defined in 3.3, for they have superscripts coming from $\omega$. To deal with such reds we stipulate that $(r_{ij}, r_{ij'}^*, r_{ij''})$ for $i, j, i', j' \in n$ is forbidden unless $l = l' = l''$ and $i = i^*, j = j'$ and $k = k^*$.

On the other hand $\mathfrak{PEA}_{n+1,n}$ is the standard rainbow algebra as defined in 3.3.

Now one can view the complete undirected coloured graphs in $\mathcal{G}$ as first order models for the rainbow signature together with $\rho$ viewed as a binary relation. (In case we have infinitely many greens like in [14] the rainbow theory [12] is an $L_{\omega_1,\omega}$ theory.)

Using the standard rainbow argument adopted in [14], one shows that there is a countable $n$ homogeneous model $M \in \mathcal{G}$ with the following property:

- If $\triangle \subseteq \triangle' \in \mathcal{G}$, $|\triangle'| \leq n$, and $\theta : \triangle \to M$ is an embedding, then $\theta$ extends to an embedding $\theta' : \triangle' \to M$.

To prove this we use, like Hodkinson, a simple game. Two players, $\forall$ and $\exists$, play a game to build a labelled graph $M$. They play by choosing a chain $\Gamma_0 \subseteq \Gamma_1 \subseteq \ldots$ of finite graphs in $\mathcal{G}$; the union of the chain will be the graph $M$. There are $\omega$ rounds. In each round, $\forall$ and $\exists$ do the following. Let $\Gamma \in \mathcal{G}$ be the graph constructed up to this point in
the game. ∀ chooses \( \Delta \in \mathcal{G} \) of size \(< n \), and an embedding \( \theta : \Delta \to \Gamma \). He then chooses an extension \( \Delta \subseteq \Delta^+ \in \mathcal{G} \), where \(|\Delta^+ \setminus \Delta| \leq 1\). These choices, \((\Delta, \theta, \Delta^+)\), constitute his move. ∃ must respond with an extension \( \Gamma \subseteq \Gamma^+ \in \mathcal{G} \) such that \( \theta \) extends to an embedding \( \theta^+ : \Delta^+ \to \Gamma^+ \). Her response ends the round. The starting graph \( \Gamma_0 \in \mathcal{G} \) is arbitrary. We claim that ∃ can always find a suitable extension \( \Gamma^+ \in \mathcal{G} \).

Let \( \Gamma \in \mathcal{G} \) be the graph built at some stage, and suppose that ∀ choose the graphs \( \Delta \subseteq \Delta^+ \in \mathcal{G} \) and the embedding \( \theta : \Delta \to \Gamma \). Thus, his move is \((\Delta, \theta, \Delta^+)\). We may assume with no loss of generality that ∀ actually played \((\Gamma \upharpoonright F, \text{Id}_F, \Delta^+)\), where \( \Gamma \upharpoonright F \subseteq \Delta^+ \in \mathcal{G} \), \( \Delta^+ \setminus F = \{\delta\} \), and \( \delta \notin \Gamma \). Then ∀ has to build a labelled graph \( \Gamma^* \supseteq \Gamma \), whose nodes are those of \( \Gamma \) together with \( \delta \), and whose edges are the edges of \( \Gamma \) together with edges from \( \delta \) to every node of \( F \). The labelled graph structure on \( \Gamma^* \) is given by

- \( \Gamma \) is an induced subgraph of \( \Gamma^* \) (i.e., \( \Gamma \subseteq \Gamma^* \))
- \( \Gamma^* \upharpoonright (F \cup \{\delta\}) = \Delta^+ \). Now ∃ must extend \( \Gamma^* \) to a complete graph on the same node and complete the colouring yielding a graph \( \Gamma^+ \in \mathcal{G} \). Thus, she has to define the colour \( \Gamma^+(\beta, \delta) \) for all nodes \( \beta \in \Gamma \setminus F \), in such a way as to meet the required conditions. The strategy of ∃ is as follows:

1. If there is no \( f \in F \), such that \( \Gamma^*(\beta, f), \Gamma^*(\delta, f) \) are coloured \( g_t \) and \( g_u \) for some \( t, u \), then ∃ defined \( \Gamma^+(\beta, \delta) \) to be \( w_0 \).
2. Otherwise, if for some \( i \) with \( 0 < i < n - 1 \), there is no \( f \in F \) such that \( \Gamma^*(\beta, f), \Gamma^*(\delta, f) \) are both coloured \( g_i \), then ∃ defines the colour \( \Gamma^+(\beta, \delta) \) to be \( w_i \) say the least such.
3. Otherwise \( \delta \) and \( \beta \) are both the apexes on \( F \) in \( \Gamma^* \) that induce the same linear ordering on (there are no green edges in \( F \) because \( \Delta^+ \in \mathcal{G} \), so it has no green triangles). Now ∃ has no choice but to pick a red. The colour she chooses is \( \rho \).

This defines the colour of edges. Now for hyperedges, for each tuple of distinct elements \( \bar{a} = (a_0, \ldots, a_{n-2}) \in n^{-1}(\Gamma^+) \) such that \( \bar{a} \notin n^{-1} \Gamma \cup n^{-1} \Delta \) and with no edge \((a_i, a)\) coloured green in \( \Gamma^+ \), ∃ colours \( \bar{a} \) by \( y_S \) where \( S = \{i < \omega : \text{there is a } i \text{ cone with base } \bar{a}\} \). Notice that \(|S| \leq F \). This strategy works \([14, \text{lemma 2.7}]\).

Now there are only countably many finite graphs in \( \mathcal{G} \) up to isomorphism, and each of the graphs built during the game is finite. Hence ∀ can play every possible \((\Delta, \theta, \Delta^+)\) (up to isomorphism) at some round in the game. Suppose he does this, and let \( M \) be the union of the graphs played in the game. Then \( M \) is as required \([14]\).
Relativization, back and forth systems ensuring that relativized semantics coincide with the classical semantics

Let $W = \{ \bar{a} \in {}^n M : M \models (\bigwedge_{i<j<n,l<n} \neg \varphi(x_i,x_j))(\bar{a}) \}$. Here assignments who have a $\rho$ labelled edge are discarded.

For an $L_{\infty \omega}^n$-formula $\varphi$, define $\varphi^W$ to be the set $\{ \bar{a} \in W : M \models W \varphi(\bar{a}) \}$. Then set $A$ to be the relativised set algebra with domain $\{ \varphi^W : \varphi$ a first-order $L^n$-formula $\}$ and unit $W$, endowed with the algebraic operations $d_{ij}, c_i$, etc., in the standard way.

Let $S$ be set algebra with domain $\varphi^n M$ and unit $^n M$. Then the map $h : A \longrightarrow S$ given by $h : \varphi^W \longmapsto \{ \bar{a} \in ^n M : M \models \varphi(\bar{a}) \}$ can be checked to be well-defined and one-one; this is a representation of $A$ [14 Proposition 3.13, Proposition 4.2]. This follows from the fact that classical semantics and relativized semantic with respect to $W$ coincide for first order formulas of the rainbow signature which follows from the $n$-homogeneity built into $M$, that implies that the set of all partial isomorphisms of $M$ of cardinality at most $n$ forms an $n$-back-and-forth system.

Let us elaborate some more. Let $\chi$ be a permutation of the set $\omega \cup \{ \rho \}$. Let $\Gamma, \Delta \in \mathcal{G}$ have the same size, and let $\theta : \Gamma \rightarrow \Delta$ be a bijection. We say that $\theta$ is a $\chi$-isomorphism from $\Gamma$ to $\Delta$ if for each distinct $x, y \in \Gamma$,

- If $\Gamma(x, y) = r_{jk}^i$, 
  \[ \Delta(\theta(x), \theta(y)) = \begin{cases} r_{jk}^{\chi(i)}, & \text{if } \chi(i) \neq \rho \\ \rho, & \text{otherwise}. \end{cases} \]

- If $\Gamma(x, y) = \rho$, then 
  \[ \Delta(\theta(x), \theta(y)) = \begin{cases} r_{jk}^{\chi(\rho)}, & \text{if } \chi(\rho) \neq \rho \\ \rho, & \text{otherwise}. \end{cases} \]

For any permutation $\chi$ of $\omega \cup \{ \rho \}$, $\Theta^\chi$ is the set of partial one-to-one maps from $M$ to $M$ of size at most $n$ that are $\chi$-isomorphisms on their domains. We write $\Theta$ for $\Theta^{Id_{\omega \cup \{ \rho \}}}$.

Then like the proof of [14 Lemma 3.10], for any permutation $\chi$ of $\omega \cup \{ \rho \}$, $\Theta^\chi$ is an $n$-back-and-forth system on $M$.

Using this we now derive a connection between classical and relativized semantics in $M$, over the set $W$:
Recall that $W$ is simply the set of tuples $\bar{a}$ in $\mathfrak{A}$ such that the edges between the elements of $\bar{a}$ don’t have a label involving the red shade $\rho$. Their labels come only from the rainbow signature. We can replace $\rho$-labels by suitable red rainbow labels within an $n$-back-and-forth system. Thus, it can be arranged that the system maps a tuple $\bar{b} \in \mathfrak{A}$ to a tuple $\bar{c} \in W$ and this will preserve any formula containing no red rainbow relation symbols moved by the system.

Indeed, we can show that the classical and $W$-relativized semantics agree. $M \models W \varphi(\bar{a})$ iff $M \models \varphi(\bar{a})$, for all $\bar{a} \in W$ and all $L^n$-formulas $\varphi$, hence as claimed $\mathfrak{A}$ defined above is representable as a set algebra.

The proof is by induction on $\varphi$ [14, Proposition 3.13]. If $\varphi$ is atomic, the result is clear; and the Boolean cases are simple. Let $i < n$ and consider $\exists_i \varphi$. If $M \models W \exists_i \varphi(\bar{a})$, then there is $\bar{b} \in W$ with $\bar{b} =_i \bar{a}$ and $M \models W \varphi(\bar{b})$. Inductively, $M \models \varphi(\bar{b})$, so clearly, $M \models W \exists_i \varphi(\bar{a})$. For the (more interesting) converse, suppose that $M \models W \exists_i \varphi(\bar{a})$. Then there is $\bar{b} \in \mathfrak{A}$ with $\bar{b} =_i \bar{a}$ and $M \models \varphi(\bar{b})$. Take $L_{\varphi,\bar{b}}$ to be any finite subsignature of $L$ containing all the symbols from $\mathfrak{B}$ that occur in $\varphi$ or as a label in $M \upharpoonright \text{rng}(\bar{b})$. Choose a permutation $\chi$ of $\omega \cup \{\rho\}$ fixing any $i'$ such that some $r_{j'k}'$ occurs in $L_{\varphi,\bar{b}}$ and moving $\rho$. Let $\theta = \text{Id}_{\{a_m:m \neq i\}}$. Take any distinct $l, m \in n \setminus \{i\}$. If $M(a_l, a_m) = r_{j'k}'$, then $M(b_l, b_m) = r_{j'k}'$ because $\bar{a} =_i \bar{b}$, so $r_{j'k}' \in L_{\varphi,\bar{b}}$ by definition of $L_{\varphi,\bar{b}}$. So, $\chi(i') = i'$ by definition of $\chi$. Also, $M(a_l, a_m) \neq \rho$ because $\bar{a} \in W$. It now follows that $\theta$ is a $\chi$-isomorphism on its domain, so that $\theta \in \Theta^x$. Extend $\theta$ to $\theta' \in \Theta^x$ defined on $b_i$, using the “forth” property of $\Theta^x$. Let $\bar{c} = \theta'(\bar{b})$. Now by choice of $\chi$, no labels on edges of the subgraph of $M$ with domain $\text{rng}(\bar{c})$ involve $\rho$. Hence, $\bar{c} \in W$. Moreover, each map in $\Theta^x$ is evidently a partial isomorphism of the reduct of $M$ to the signature $L_{\varphi,\bar{b}}$. Now $\varphi$ is an $L_{\varphi,\bar{b}}$-formula. We have $M \models \varphi(\bar{a})$ iff $M \models \varphi(\bar{c})$. So $M \models \varphi(\bar{c})$. Inductively, $M \models W \varphi(\bar{c})$. Since $\bar{c} =_i \bar{a}$, we have $M \models W \exists_i \varphi(\bar{a})$ by definition of the relativized semantics. This completes the induction, and proves that $h : \mathfrak{A} \rightarrow \mathcal{S}$ above is indeed a representation of $\mathfrak{A}$. However, it is not a complete representation. This will be clear from our subsequent discussion, when we explicitly describe the atoms of $\mathfrak{A}$ and show that their union is not $\mathfrak{A}$; it is $W$. In fact $\mathfrak{A}$ has no complete classical representation; even more its $\text{Df}$ reduct does not have such a representation as stated in the theorem to be proved in a while.

(3) The atoms and the complex algebra; the Dedekind-MacNeille completion

Recall that $L$ denotes the rainbow signature (without $\rho$). The logics $L^n$ and $L^n_{\omega\omega}$ are taken in this signature.

We show that $\mathfrak{A}$ is atomic and we give the atoms, following Hodkinson,
a syntactical description in terms of special formulas in the rainbow signature specified above taken in $L^n$; this will enable us to transparently define the embedding of $\text{PEA}_{n+1,n}$ into the hitherto constructed complex algebra.

Now every atom in the (representable) relativized set algebra $A$ is uniquely defined by an MCA formula $[13]$. A formula $\alpha$ of $L^n$ is said to be MCA (‘maximal conjunction of atomic formulas’) $[13$, Definition 4.3] if (i) $M \models \exists x_0 \ldots, x_{n-1} \alpha$ and (ii) $\alpha$ is of the form

$$\bigwedge_{i \neq j < n} \alpha_{ij}(x_i, x_j) \land \bigwedge \eta_\mu(x_0, \ldots, x_{n-1}),$$

where for each $i, j, \alpha_{ij}$ is either $x_i = x_j$ or $R(x_i, x_j)$ a binary relation symbol in the rainbow signature, and for each $\mu : (\omega \cup \{\rho\}) \to n$, $\eta_\mu$ is either $y_S(x_{\mu(0)}, \ldots x_{\mu(n-2)})$ for some $y_S$ in the signature, if for all distinct $i, j < n$, $\alpha_{\mu(i), \mu(j)}$ is not equality nor green, otherwise it is $x_0 = x_0$.

A formula $\alpha$ being MCA says that the set it defines in $^nM$ is nonempty, and that if $M \models \alpha(\bar{a})$ then the graph $M \upharpoonright \text{rng}(\bar{a})$ is determined up to isomorphism and has no edge whose label is of the form $\rho$. Now since we have for any permutation $\chi$ of $\omega \cup \{\rho\}$, $\Theta^\chi$ is an $n$-back-and-forth system on $M$, any two tuples (graphs) satisfying $\alpha$ are isomorphic and one is mapped to the other by the $n$-back-and-forth system $\Theta$ of partial isomorphisms from $M$ to $M$; they are the same coloured graph.

No $L^n_{\omega}$-formula can distinguish any two graphs satisfying an MCA formula $\alpha$. So $\alpha$ defines an atom of $\mathfrak{A}$. Since the MCA-formulas clearly cover $W$, the atoms defined by them are dense in $\mathfrak{A}$, hence $\mathfrak{A}$ is atomic. The coloured graphs whose edges are not labelled by the shade of red $\rho$ (up to isomorphism) determined by MCA formulas are the atoms of $\mathfrak{A}$.

In more detail, let $\varphi$ be any $L^n_{\omega}$-formula, and $\alpha$ any MCA-formula. If $\varphi^W \cap \alpha^W \neq \emptyset$, then $\alpha^W \subseteq \varphi^W$. Indeed, take $\bar{a} \in \varphi^W \cap \alpha^W$. Let $\bar{b} \in \alpha^W$ be arbitrary. Clearly, the map $(\bar{a} \mapsto \bar{b})$ is in $\Theta$. Also, $W$ is $L^n_{\omega}$-definable in $M$, since we have

$$W = \{\bar{a} \in ^nM : M \models (\bigwedge_{i < j < n} (x_i = x_j \lor \bigvee_{R \in L} R(x_i, x_j)))((\bar{a}))\}.$$  

We have $M \models_W \varphi(\bar{a})$ iff $M \models_W \varphi(\bar{b})$. Since $M \models_W \varphi(\bar{a})$, we have $M \models_W \varphi(\bar{b})$. Since $\bar{b}$ was arbitrary, we see that $\alpha^W \subseteq \varphi^W$. Let

$$F = \{\alpha^W : \alpha \text{ an MCA, } L^n \text{ - formula} \} \subseteq \mathfrak{A}.$$  

Evidently, $W = \bigcup F$. We claim that $\mathfrak{A}$ is an atomic algebra, with $F$ as its set of atoms. First, we show that any non-empty element $\varphi^W$ of $\mathfrak{A}$
contains an element of $F$. Take $\bar{a} \in W$ with $M \models_W \varphi(\bar{a})$. Since $\bar{a} \in W$, there is an MCA-formula $\alpha$ such that $M \models_W \alpha(\bar{a})$. Then $\alpha^W \subseteq \varphi^W$. By definition, if $\alpha$ is an MCA formula then $\alpha^W$ is non-empty. If $\varphi$ is an $L^n$-formula and $\emptyset \neq \varphi^W \subseteq \alpha^W$, then $\varphi^W = \alpha^W$. It follows that each $\alpha^W$ (for MCA $\alpha$) is an atom of $\mathfrak{A}$. We can also conclude from this that $\mathfrak{A}$ is simple, for if $\varphi^W \in \mathfrak{A}$ is non zero then there is an MCA formula $\alpha$ such that $\alpha^W \leq \varphi^W$, and $M \models \exists x_0 \ldots x_{n-1} \alpha$, hence $c_{(n)} \varphi^W = nM$. Also $\mathfrak{A}$ is not completely representable; this will follow from the fact that its completion is not representable.

Let $\mathfrak{C}$ be the the relativized set algebra with domain

$$\{\varphi^W : \varphi \text{ an } L_{\infty, \omega}^n \text{ formula}\},$$

and operations defined like for $\mathfrak{A}$. Then $\mathfrak{C}$ is the Dedekind-MacNeille completion of $\mathfrak{A}$, that is $\mathfrak{C} \cong \mathfrak{CmAtA}$ [14, Proposition 4.6].

(4) Embedding $\text{PEA}_{n+1,n}$ in the complex algebra

Hodkinson proves the non representability of the completion of his algebra $\mathfrak{A}$ [14] using the greens, which are in his case infinite. In case of the existence of a representation, he reaches a contradiction, by using infinitely many cones having a common base, to force an inconsistent triple of reds. So the greens play an essential role in this part of the proof. Here we show that if we truncate the greens to be finite, but as long as they outfit the red, then we still can force such an inconsistent triple using only finitely many cones, getting a sharper result. Roughly this finite number determines the number of dimensions $> n$, which $\mathfrak{CmAtA}$ cannot neatly embed into an algebra having this number as its dimension.

We have $\text{At} = \text{AtA} = \text{AtC}$, and $\text{TmAt} \subseteq \mathfrak{A} \subseteq \mathfrak{CmAt} \cong \mathfrak{C}$; the former two are representable. We can proceed like [14] and show that though $\mathfrak{NDO_{ca}C}$ is representable in a relativized sense it is not classically representable. But we can do better.

We show that the $\text{Sc}$ reduct of the last is not in $S\mathfrak{N}_{n+1}^{\text{Sc}}$. This will be done by showing that the $\text{Sc}$ reduct of $\text{PEA}_{n+1,n}$ is not in $S\mathfrak{N}_{n+1}^{\text{Sc}}$ and that $\text{PEA}_{n+1,n}$ embeds into $\mathfrak{C}$ as polyadic equality algebras. Recall that the colours used for coloured graphs involved in building the finite atom structure of the algebra $\text{PEA}_{n+1,n}$ are:

- greens: $g_i$, $1 \leq i \leq n - 2$, $g^i_0$, $1 \leq i \leq n + 1$,
- whites: $w_i$, $i \leq n - 2$,
- reds: $r_{ij}$, $i < j \leq n$,
- shades of yellow: $y_S$, $S \subseteq n + 1$. 

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with forbidden triples

\[(g_i^g, g^g_i), (g_i, g_i, w_i)\quad \text{any } 1 \leq i \leq n - 2\]

\[(g_i^k, g_0^k, w_0)\quad \text{any } 1 \leq j, k \leq n + 1\]

\[(r_{ij}, r_{ij}^{*'}, r_{ij}^{*''})\quad i, j, i', j', i^*, j^* \in n,\]

unless \(i = i^*, j = j'\) and \(k' = k^*\).

and no other triple is forbidden. One can say metaphorically that \(At\) is isomorphic to the rainbow atom structure obtained from the finite atom structure of \(\text{PEA}_{n+1,n}\) if each red \(r_{ij}\), \(i < j < n\) is ‘split’ into \(\omega\) many copies \(r_{ij}^l\), \(l \in \omega\), and adding the consistency condition stated above, namely, \((r_{ij}^l, r_{ij}^{l'}, r_{ij}^{l''})\) for \(i, j, j', k', i^*, k^* \in n\) is forbidden unless \(l = l' = l''\) and \(i = i^*, j = j'\) and \(k' = k^*\). But splitting has to do with splitting atoms, in our context coloured graphs, not colours; so this ‘image’, which, all the same, we find useful to highlight at this stage, will be made precise in a while.

A coloured graph is red if at least one of its edges is labelled red. Here we do not have \(\rho\), it simply does not exist in the set of availablereds. Now the algebra \(\text{PEA}_{n+1,n}\) embeds into \(\mathfrak{c}\) as polyadic equality algebras. The map is defined on the atoms then extended the obvious way to the whole algebra. Every \([a] : n \rightarrow \Gamma\), where \(\Gamma\) a red graph is mapped to the join of its copies, which exists because \(\mathfrak{c}\) is complete. A copy of a red graph is one that is isomorphic to this graph, modulo removing superscripts of reds. Every other atom (graph) is mapped to itself.

More precisely, for brevity write \(r\) for \(r_{jk}(j < k < n)\). If \(\Gamma\) is a coloured graph using the colours in \(\text{AtPEA}_{n+1,n}\), and \(a : n \rightarrow \Gamma\) is in \(\text{AtPEA}_{n+1,n}\), then \(a' : n \rightarrow \Gamma'\) with \(\Gamma' \in \text{CGR}\) is a copy of \(a : n \rightarrow \Gamma\), if for any non red binary colour \(c\) and any red \(r\) we have for any \(i, j, k_0, \ldots, k_{n-1} < n\):

- \(a(i) = a(j) \iff a'(i) = a'(j)\),
- \((a(i), a(j)) \in c \iff (a'(i), a'(j)) \in c\),
- \((a(i), a(j)) \in r \iff (a'(i), a'(j)) \in r'\) for some \(l \in \omega\),
- \(M_a(a(k_0), \ldots, a(k_{n-2})) = M'_a(a'(k_0), \ldots, a'(k_{n-2}))\) whenever defined.

In other words, all non red edges and \(n - 1\) tuples have the same colour (whenever defined) and for all \(i < j < n\), for every red \(r\), if \((a(i), a(j)) \in r\), then there exits \(l \in \omega\) such that \((a'(i), a'(j)) \in r'\). Here we implicitly require that for distinct \(i, j, k < n\), if \((a(i), a(j)) \in r\), \((a(j), a(k)) \in r'\), \((a(i), a(k)) \in r''\), and \((a'(i), a'(j)) \in r'_1\), \((a'(j), a'(k)) \in r'_2\), and \((a'(i), a'(k)) \in r''_3\), then \(l_1 = l_2 = l_3 = l\), say, so that \((r', [r']^l, [r'']^l)\) is a consistent triangle in \(\Gamma'\).
Then every $a : n \to \Gamma$, where $\Gamma$ is red in $\text{PEA}_{n+1,n}$ is mapped to the join of $\phi^W$, where $\phi$ is an MCA formula, corresponding to $a' : n \to \Gamma'$ in $\mathfrak{C}$, such that $a'$ is a copy of $\Gamma$.

These joins exist because $\mathfrak{C}$ is complete.

Hence we have a map $\Psi : \text{At}(\text{PEA}_{n+1,n}) \to \mathfrak{C}$. This induces another map, which we denote also by $\Psi$ from the finite algebra $\text{PEA}_{n+1,n}$ to $\mathfrak{C}$ taking finite (unions) sums of atoms to the corresponding sum in $\mathfrak{C}$.

We now show that $\Psi$ is an injective homomorphism. It is injective since distinct atoms are mapped to distinct elements. We proceed to show that it preserves the operations; in the process we clarify the fact that we can look at $\text{At}$ as begotten from $\text{PEA}_{n+1,n}$ by splitting every red graph into $\omega$ many copies, in the following sense.

If $a' : n \to \Gamma'$ and $\Gamma'$ is a red graph using the colours of the rainbow signature of $\text{At}$, then there is a unique $a : n \to \Gamma$, $\Gamma$ a red graph using the red colours in the rainbow signature of $\text{PEA}_{n+1,n}$, such that $a'$ is a copy of $a$. We denote $a$ by $o(a')$, $o$ short for original; $a$ is the original of its copy $a'$.

For $i < n$, let $T_i$ be the accessibility relation corresponding to the $i$th cylindrifier in $\text{At}$. Let $T_i^s$, be that corresponding to the $i$th cylindrifier in $\text{PEA}_{n+1,n}$. Then if $c : n \to \Gamma$ and $d : n \to \Gamma'$ are surjective maps $\Gamma, \Gamma'$ are coloured graphs for $\text{PEA}_{n+1,n}$, that are not red, then for any $i < n$, we have

\[(c, d) \in T_i \iff (c, d) \in T_i^s.\]

If $\Gamma$ is red using the colours for the rainbow signature of $\text{At}$ (without $\rho$) and $a' : n \to \Gamma$, then for any $b : n \to \Gamma'$ where $\Gamma'$ is not red and any $i < n$, we have

\[(a', b) \in T_i \iff (o(a'), b) \in T_i^s.\]

Extending the notation, for $a : n \to \Gamma$ a graph that is not red in $\text{At}$, set $o(a) = a$. Then for any $a : n \to \Gamma$, $b : n \to \Gamma'$, where $\Gamma, \Gamma'$ are coloured graphs at least one of which is not red in $\text{At}$ and any $i < n$, we have

\[[a]T_i[b] \iff [o(a)]T_i^s[o(b)].\]

Now we deal with the last case, when the two graphs involved are red. Now assume that $a' : n \to \Gamma$ is as above, that is $\Gamma \in \text{CGR}$ is red, $b : n \to \Gamma'$ and $\Gamma'$ is red too, using the colours in the rainbow signature of $\text{At}$.

Say that two maps $a : n \to \Gamma$, $b : n \to \Gamma'$, with $\Gamma$ and $\Gamma' \in \text{CGR}$ having the same size are $r$ related if all non red edges and $n - 1$ tuples have
the same colours (whenever defined), and for all red r, whenever $i < j < n$, $l \in \omega$, and $(a(i), a(j)) \in r'$, then there exists $k \in \omega$ such that $(b(i), b(j)) \in r^k$. In more detail, for any red $r$ we have for any $i, j, k_0, \ldots, k_{n-1} < n$:

- $a(i) = a(j) \iff a'(i) = a'(j)$,
- $(a(i), a(j)) \in c \iff (a'(i), a'(j)) \in c$,
- $(a(i), a(j)) \in r^k \iff (a'(i), a'(j)) \in r^l$ for some $l, k \in \omega$,
- $M_a(a(k_0), \ldots, a(k_{n-2})) = M_{a'}(a'(k_0), \ldots, a'(k_{n-2}))$ whenever defined.

Let $i < n$. Assume that $([o(a')], [o(b')]) \in T_i$. Then there exists $c : n \to \Gamma$ that is red related to $a'$ such that $[c]T_i[b']$. Conversely, if $[c]T_i[b']$, then $[o(c)]T_i[o(b')]$.

Hence, by complete additivity of cylindrifiers, the map $\Theta : \text{At}(\text{PEA}_{n+1, n}) \to \text{CmAt}$ defined via

$$\Theta([a]) = \begin{cases} \{a' : a' \text{ copy of } a\} & \text{if } a \text{ is red}, \\ \{[a]\} & \text{otherwise}. \end{cases}$$

induces an embedding from $\text{PEA}_{n+1, n}$ to $\text{CmAt}$, which we denote also by $\Theta$.

We first check preservation of diagonal elements. If $a'$ is a copy of $a$, $i, j < n$, and $a(i) = a(j)$, then $a'(i) = a'(j)$.

We next check cylindrifiers. We show that for all $i < n$ and $[a] \in \text{At}(\text{PEA}_{n+1, n})$ we have:

$$\Theta(c_i[a]) = \bigcup \{\Theta([b]) : [b] \in \text{AtPEA}_{n+1, n}, [b] \leq c_i[a]\} = c_i\Theta([a]).$$

Let $i < n$. If $[b] \in \text{AtPEA}_{n+1}, [b] \leq c_i[a]$, and $b' : n \to \Gamma$, $\Gamma \in \text{CGR}$, is a copy of $b$, then there exists $a' : n \to \Gamma'$, $\Gamma' \in \text{CGR}$, a copy of $a$ such that $b' \upharpoonright n \setminus \{i\} = a' \upharpoonright n \setminus \{i\}$. Thus $\Theta([b]) \leq c_i\Theta([a])$.

Conversely, if $d : n \to \Gamma$, $\Gamma \in \text{CGR}$ and $[d] \in c_i\Theta([a])$, then there exist $a'$ a copy of $a$ such that $d \upharpoonright n \setminus \{i\} = a' \upharpoonright n \setminus \{i\}$. Hence $o(d) \upharpoonright n \setminus \{i\} = a \upharpoonright n \setminus \{i\}$, and so $[d] \in \Theta(c_i[a])$, and we are done.

Now we also have $\text{CmAt} \cong \mathcal{C}$, via the map $g : X \mapsto \bigcup X$. The proof is similar to Hodkinson’s [?], Hodkinson or his Dedekind-MacNeille completion which is different than ours. The map is clearly injective. It is surjective, since

$$\phi^W = \bigcup \{\alpha^W : \alpha \text{ an MCA-formula, } \alpha^W \subseteq \phi^W\}$$
for any $L^\omega_{\omega \omega}$ formula $\phi$. Preservation of the Boolean operations and diagonals is clear. We check cylindrifications. We require that for any $X \subseteq \text{At} \mathfrak{A}$, we have $\bigcup c_i^X X = c_i^{\bigcup X}$ that is

$$\bigcup \{ S \in \text{At} \mathfrak{A} : S \subseteq c_i^S S' \text{ for some } S' \in X \} = \{ \bar{a} \in W : \bar{a} \equiv_i \bar{a}' \text{ for some } \bar{a}' \in \bigcup X \}.$$ 

Let $\bar{a} \in S \subseteq c_i S'$, where $S' \in X$. So there is $\bar{a}' \equiv_i \bar{a}$ with $\bar{a}' \in S'$, and so $\bar{a}' \in \bigcup X$.

Conversely, let $\bar{a} \in W$ with $\bar{a} \equiv_i \bar{a}'$ for some $\bar{a}' \in \bigcup X$. Let $S \in \text{At} \mathfrak{A}$, $S' \in X$ with $\bar{a} \in S$ and $\bar{a}' \in S'$. Choose MCA formulas $\alpha$ and $\alpha'$ with $S = \alpha^W$ and $S' = \alpha'^W$, then $\bar{a} \in \alpha^W \cap (\exists x_1 \alpha')^W$ so $\alpha^W \subseteq (\exists x_1 \alpha')^W$, or $S \subseteq c_i^\mathfrak{A}(S')$. The required now follows. We leave the checking of substitutions to the reader. Thus

$$g \circ \Theta = \Psi,$$

and so $\Psi : \text{PEA}_{n+1,n} \to \mathfrak{C}$ is an embedding.

A subset of $\mathfrak{C}$ is red if it consists only of red graphs. Notice that for every red atom $a$, we have $|\Theta(a)| \geq \omega$, and it is not co-finite, that is its complement is also infinite. These sets do not exist in the term algebra, which contains only finite or co-finite red sets. Here $\rho$ functions in a sense as a non standard red colour, corresponding to the non-principal ultrafilter of $\mathfrak{A}$ generated by all co-finite sets of red atoms.

(5) $\forall$ winning $F^{n+3}$ on $\text{At}(\text{PEA}_{n+1,n})$

But now we can show that $\forall$ can win the game $F^{n+3}$ on $\text{At}(\text{PEA}_{n+1,n})$ in only $n + 2$ rounds as follows. Viewed as an Ehrenfeucht–Fraïssé forth game pebble game, with finitely many rounds and pairs of pebbles, played on the two complete irreflexive graphs $n+1$ and $n$, in each round $0, 1, \ldots, n$, $\forall$ places a new pebble on an element of $n + 1$. The edge relation in $n$ is irreflexive so to avoid losing $\exists$ must respond by placing the other pebble of the pair on an unused element of $n$. After $n$ rounds there will be no such element, and she loses in the next round. Hence $\forall$ can win the graph game on $\text{At}(\text{PEA}_{n+1,n})$ in $n + 2$ rounds using $n + 3$ nodes.

In the game $F^{n+3}$ $\forall$ forces a win on a red clique using his excess of greens by bombarding $\exists$ with $\alpha$ cones having the same base ($1 \leq \alpha \leq n + 2$).

In his zeroth move, $\forall$ plays a graph $\Gamma$ with nodes $0, 1, \ldots, n - 1$ and such that $\Gamma(i, j) = w_0(i < j < n - 1), \Gamma(i, n - 1) = g_i(i = 1, \ldots, n - 2), \Gamma(0, n - 1) = g_0$, and $\Gamma(0, 1, \ldots, n - 2) = y_{n+2}$. This is a 0-cone with base $\{0, \ldots, n - 2\}$. In the following moves, $\forall$ repeatedly chooses the face
(0, 1, . . . , n − 2) and demands a node α with Φ(i, α) = g_i, (i = 1, . . . , n − 2) and Φ(0, α) = g_0^α, in the graph notation – i.e., an α-cone, without loss n − 1 < α ≤ n + 1, on the same base. ∃ among other things, has to colour all the edges connecting new nodes α, β created by ∀ as apexes of cones based on the face (0, 1, . . . , n − 2), that is α, β ≥ n − 2. By the rules of the game the only permissible colours would be red. Using this, ∀ can force a win in n + 2 rounds, using n + 3 nodes without needing to re-use them, thus forcing ∃ to deliver an inconsistent triple of reds.

Let B = PEA_{n+1,n}. Then Rd sc B is outside Sm_{n}Sc_{n+3} for if it was in Sm_{n}Sc_{n+3}, then being finite it would be in Sc_{n}Sm_{n}Sc_{n+3} because Rd sc B is the same as its canonical extension D, say, and D ∈ Sc_{n}Sm_{n}Sc_{n+3}. But then by theorem 3.9 ∃ would have won.

Hence Rd sc CmAt CmAt Soc_{n}Sm_{n+3} is not completely representable, because if it were then A, generated by elements whose dimension sets < n, as a PEA_n would be completely representable by theorem [3.10] and this induces a representation of its Dedekind-MacNeille completion CmAt A.

□

4 Complete representability

Now we approach the notion of complete representations for any K between Sc and PEA. Rainbows will offer solace here as well. Throughout this subsection n will be finite and > 1. We identify notationally set algebras with their universes.

Our next couple of theorems are formulated and proved for CAs but the proof works for any K as specified above.

Let A ∈ CA_n and f : A → φ(V) be a representation of A, where V is a generalized space of dimension n. If s ∈ V we let

\[ f^{-1}(s) = \{ a ∈ A : s ∈ f(a) \}. \]

An atomic representation f : A → φ(V) is a representation such that for each s ∈ V, the ultrafilter \( f^{-1}(s) \) is principal.

Theorem 4.1. Let A ∈ CA_n. Let f : A → φ(V) be a representation of A. Then f is a complete representation iff f is an atomic one. Furthermore, if A is completely representable, then A is atomic and A ∈ Sc Sm_n CA_n.
Proof. Witnesss [23, Theorems 5.3.4, 5.3.6], [12, Theorem 3.1.1] for the first three parts. It remains to show that if $\mathcal{A}$ is completely representable, then $\mathcal{A} \in S_c\mathfrak{N}\iota \omega \mathcal{C}A_\omega$. Assume that $M$ is the base of a complete representation of $\mathcal{A}$, whose unit is a generalized space, that is, $1^M = \bigcup_{i \in I}^\omega U_i$, where $U_i \cap U_j = \emptyset$ for distinct $i$ and $j$ in $I$ where $I$ is an index set $I$. Let $t : \mathcal{A} \to \wp(1^M)$ be the complete representation. For $i \in I$, let $E_i = {}^\omega U_i$, pick $f_i \in {}^\omega U_i$, let $W_i = \{f \in {}^\omega U_i : |\{k \in \omega : f(k) \neq f_i(k)\}| < \omega\}$, and let $\mathcal{C}_i$ be the $\mathcal{C}A_n$ with universe $\wp(W_i)$, with the $\mathcal{C}A$ operations defined the usual way on weak set algebras. Then $\mathcal{C}_i$ is atomic; indeed the atoms are the singletons.

Let $x \in \mathfrak{N}\iota_n \mathcal{C}_i$, that is $c_jx = x$ for all $n \leq j < \omega$. Now if $f \in x$ and $g \in W_i$ satisfy $g(k) = f(k)$ for all $k < n$, then $g \in x$. Hence $\mathfrak{N}\iota_n \mathcal{C}_i$ is atomic; its atoms are $\{g \in W_i : g(0), \ldots g(n-1) \subseteq U_i\}$. Define $h_i : \mathcal{A} \to \mathfrak{N}\iota_n \mathcal{C}_i$ by

$$h_i(a) = \{f \in W_i : \exists a \in \mathfrak{A}^\omega : (f(0) \ldots f(n-1)) \in t(a)\}.$$ 

Let $\mathcal{C} = \prod_i \mathcal{C}_i$. Let $\pi_i : \mathcal{C} \to \mathcal{C}_i$ be the $i$th projection map. Now clearly $\mathcal{C}$ is atomic, because it is a product of atomic algebras, and its atoms are $(\pi_i(\beta) : \beta \in \mathfrak{A}(\mathcal{C}_i))$. Now $\mathcal{A}$ embeds into $\mathfrak{N}\iota_n \mathcal{C}$ via $I : a \mapsto (\pi_i(a) : i \in I)$, and we may assume that the map is surjective.

If $a \in \mathfrak{N}\iota_n \mathcal{C}$, then for each $i$, we have $\pi_i(x) \in \mathfrak{N}\iota_n \mathcal{C}_i$, and if $x$ is non zero, then $\pi_i(x) \neq 0$. By atomicity of $\mathcal{C}_i$, there is a tuple $\bar{m}$ such that $\{g \in W_i : g(k) = [\bar{m}]_k\} \subseteq \pi_i(x)$. Hence there is an atom $a$ of $\mathcal{A}$, such that $\bar{m} \in t(a)$, so $x \cdot I(a) \neq 0$, and so the embedding is complete and we are done. Note that in this argument no cardinality condition is required. (The reverse inclusion does not hold in general for uncountable algebras, as will be shown in theorem 5.8, though it holds for atomic algebras with countably many atoms as shown in our next theorem). Hence $\mathcal{A} \in S_c\mathfrak{N}\iota \omega \mathcal{C}A_\omega$.

Conversely, we have [23, Theorem 5.3.6]:

**Theorem 4.2.** If $\mathcal{A}$ is countable and atomic and $\mathcal{A} \in S_c\mathfrak{N}\iota \omega \mathcal{C}A_\omega$, then $\mathcal{A}$ is completely representable.

Now we use a rainbow construction. Coloured graphs and rainbow algebras are defined like above. The algebra constructed now is very similar to $\mathcal{P}E\mathcal{A}_\omega \omega \omega$ but is not identical; for in coloured graphs we add a new triple of forbidden colours involving two greens and one red synchronized by an order preserving function. In particular, we consider the underlying set of $\omega$ endowed with two orders, the usual order and its converse.

**Theorem 4.3.** Let $3 \leq n < \omega$. Then there exists an atomic $\mathcal{C} \in \mathcal{P}E\mathcal{A}_n$ with countably many atoms such that $\mathfrak{N}\iota_n \omega \mathcal{C} \notin S_c\mathfrak{N}\iota_n \mathfrak{S}_c_{n+3}$, and there exists a countable $\mathcal{B} \in S_c\mathfrak{N}\iota_n \mathcal{Q}E\mathcal{A}_\omega$ such that $\mathcal{C} \equiv \mathcal{B}$ (hence $\mathcal{B}$ is also atomic). In
particular, for any class \( L \), such that \( S \circ \mathfrak{R}_n K_\omega \subseteq L \subseteq S \circ \mathfrak{R}_n K_{n+3} \), \( L \) is not elementary, and the class of completely representable \( K \) algebras of dimension \( n \) is not elementary.

Proof. Let \( \mathbb{N}^{-1} \) denote \( \mathbb{N} \) with reverse order, let \( f : \mathbb{N} \to \mathbb{N}^{-1} \) be the identity map, and denote \( f(a) \) by \(-a\), so that for \( n, m \in \mathbb{N} \), we have \( n < m \) iff \(-m < -n\). We assume that 0 belongs to \( \mathbb{N} \) and we denote the domain of \( \mathbb{N}^{-1} \) (which is \( \mathbb{N} \)) by \( \mathbb{N}^{-1} \). We alter slightly the standard rainbow construction. The colours we use are the same colours used in rainbow constructions:

- **greens**: \( g_i \) \((1 \leq i \leq n - 2)\), \( g_i^0, i \in \mathbb{N}^{-1} \),
- **whites**: \( w_i : i \leq n - 2 \),
- **reds**: \( r_{ij} (i, j \in \mathbb{N}) \),
- **shades of yellow**: \( y_S : S \subseteq \omega \mathbb{N}^{-1} \) or \( S = \mathbb{N}^{-1} \).

And the class \( \mathcal{G} \) consists of all coloured graphs \( M \) such that

1. \( M \) is a complete graph.
2. \( M \) contains no triangles (called forbidden triples) of the following types:

   \[ (g, g', g^*), (g_i, g_i, w_i) \quad \text{any } 1 \leq i \leq n - 2 \]  
   \[ (g_i^j, g_i^k) \quad \text{any } j, k \in \mathbb{N} \]  
   \[ (g_i^j, g_0^j, r_{kl}) \quad \text{unless } \{(i, k), (j, l)\} \text{ is an order-} \]  
   \[ \text{preserving partial function } \mathbb{N}^{-1} \to \mathbb{N} \]  
   \[ (r_{ij}, r_{ij'}, r_{i+1}) \quad \text{unless } i = i^*, j = j' \text{ and } k' = k^*. \]

   and no other triple of atoms is forbidden.

3. The last two items concerning shades of yellow are as before.

   But the forbidden triple \((g_0^i, g_0^j, r_{kl})\) is not present in standard rainbow constructions, adopted example in [10] and in a more general form in [12]. Therefore, we cannot use the usual rainbow argument adopted in [10]; we have to be selective for the choice of the indices of reds if we are labelling the apexes of two cones having green tints; *not any red* will do. Inspite of such a restriction (that makes it harder for \( \exists \) to win), we will show that \( \exists \) will always succeed to choose a suitable red in the finite rounded atomic games. On the other hand this bonus for \( \forall \) will enable him to win the game \( F_n^{n+3} \) by forcing \( \exists \)
to play a decreasing sequence in $\mathbb{N}$. Using and re-using $n + 3$ nodes will suffice for this purpose.

One then can define (what we continue to call) a rainbow atom structure of dimension $n$ from the class $\mathcal{G}$. Let

$$\text{At} = \{ a : n \to M, M \in \mathcal{G} : a \text{ is surjective} \}.$$  

We write $M_a$ for the element of $\text{At}$ for which $a : n \to M$ is a surjection. Let $a, b \in \text{At}$ define the following equivalence relation:

- $a(i) = a(j) \iff b(i) = b(j)$,
- $M_a(a(i), a(j)) = M_b(b(i), b(j))$ whenever defined,
- $M_a(a(k_0), \ldots, a(k_{n-2})) = M_b(b(k_0), \ldots, b(k_{n-2}))$ whenever defined.

Let $\text{At}$ be the set of equivalences classes. Then define

$$[a] \in E_{ij} \text{ iff } a(i) = a(j).$$

$$[a][T_i][b] \text{ iff } a \upharpoonright n \setminus \{ i \} = b \upharpoonright n \setminus \{ i \}.$$  

Define accessibility relations corresponding to the polyadic (transpositions) operations as follows:

$$[a]S_{ij}[b] \text{ iff } a \circ [i, j] = b.$$  

This, as easily checked, defines a PEA$_n$ atom structure. Let $\mathcal{C}$ be the complex algebra. Let $k > 0$ be given. We show that $\exists$ has a winning strategy in the usual graph game in $k$ rounds (now there is no restriction here on the size of the graphs) on $\text{AtC}$. We recall the ‘usual atomic’ $k$ rounded game $G_k$ played on coloured graphs. $\forall$ picks a graph $M_0 \in \mathcal{G}$ with $M_0 \subseteq n$ and $\exists$ makes no response to this move. In a subsequent round, let the last graph built be $M_i$. $\forall$ picks

- a graph $\Phi \in \mathcal{G}$ with $|\Phi| = n$,
- a single node $m \in \Phi$,
- a coloured graph embedding $\theta : \Phi \setminus \{ m \} \to M_i$. Let $F = \phi \setminus \{ m \}$. Then $F$ is called a face. $\exists$ must respond by amalgamating $M_i$ and $\Phi$ with the embedding $\theta$. In other words she has to define a graph $M_{i+1} \in C$ and embeddings $\lambda : M_i \to M_{i+1}$, $\mu : \phi \to M_{i+1}$, such that $\lambda \circ \theta = \mu \upharpoonright F$.

We define $\exists$’s strategy for choosing labels for edges and $n - 1$ tuples in response to $\forall$’s moves. Assume that we are at round $r + 1$. Our arguments are similar to the arguments in [8, Lemmas, 41-43].
Let $M_0, M_1, \ldots, M_r, r < k$ be the coloured graphs at the start of a play of $G_k(\alpha)$ just before round $r + 1$. Assume inductively that $\exists$ computes a partial function $\rho_s : \mathbb{N}^{-1} \rightarrow \mathbb{N}$, for $s \leq r$, that will help her choose the suffixes of the chosen red in the critical case. In our previous rainbow construction we had the additional shade of red $\rho$ that did the job. Now we do not have it, so we proceed differently. Inductively for $s \leq r$ we assume:

1. If $M_s(x, y)$ is green then $(x, y)$ belongs $\forall$ in $M_s$ (meaning he coloured it),
2. $\rho_0 \subseteq \ldots \rho_r \subseteq \ldots$,
3. $\text{dom}(\rho_s) = \{i \in \mathbb{N}^{-1} : \exists t \leq s, x, x_0, x_1, \ldots, x_{n-2} \in \text{nodes}(M_t)$ where the $x_i$’s form the base of a cone, $x$ is its apex and $i$ its tint \}$.
   The domain consists of the tints of cones created at an earlier stage,
4. $\rho_s$ is order preserving: if $i < j$ then $\rho_s(i) < \rho_s(j)$. The range of $\rho_s$ is widely spaced: if $i < j \in \text{dom}\rho_s$ then $\rho_s(i) - \rho_s(j) \geq 3^{m-r}$, where $m - r$ is the number of rounds remaining in the game,
5. For $u, v, x_0 \in \text{nodes}(M_s)$, if $M_s(u, v) = \iota \mu, \delta, M_s(x_0, u) = g_i^0, M_s(x_0, v) = g_j^0$, where $i, j$ are tints of two cones, with base $F$ such that $x_0$ is the first element in $F$ under the induced linear order, then $\rho_s(i) = \mu$ and $\rho_s(j) = \delta$,
6. $M_s$ is a a coloured graph,
7. If the base of a cone $\Delta \subseteq M_s$ with tint $i$ is coloured $y_S$, then $i \in S$.

To start with if $\forall$ plays $a$ in the initial round then $\text{nodes}(M_0) = \{0, 1, \ldots, n-1\}$, the hyperedge labelling is defined by $M_0(0, 1, \ldots, n) = a$.

In response to a cylindrifier move for some $s \leq r$, involving a $p$ cone, $p \in \mathbb{N}^{-1}$, $\exists$ must extend $\rho_r$ to $\rho_{r+1}$ so that $p \in \text{dom}(\rho_{r+1})$ and the gap between elements of its range is at least $3^{m-r-1}$. Properties (3) and (4) are easily maintained in round $r + 1$. Inductively, $\rho_r$ is order preserving and the gap between its elements is at least $3^{m-r}$, so this can be maintained in a further round. If $\forall$ chooses a green colour, or green colour whose suffix already belong to $\rho_r$, there would be fewer elements to add to the domain of $\rho_{r+1}$, which makes it easy for $\exists$ to define $\rho_{r+1}$.

Now assume that at round $r + 1$, the current coloured graph is $M_r$ and that $\forall$ chose the graph $\Phi$, $|\Phi| = n$ with distinct nodes $F \cup \{\delta\}$, $\delta \notin M_r$, and $F \subseteq M_r$ has size $n - 1$. We can view $\exists$ s move as building a coloured graph $M^*$ extending $M_r$ whose nodes are those of $M_r$ together with the new node $\delta$ and whose edges are edges of $M_r$ together with edges from $\delta$ to every node of $F$. 

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Now $\exists$ must extend $M^*$ to a complete graph $M^+$ on the same nodes and complete the colouring giving a graph $M_{r+1} = M^+$ in $\mathcal{G}$ (the latter is the class of coloured graphs). In particular, she has to define $M^+(\beta, \delta)$ for all nodes $\beta \in M_r \sim F$, such that all of the above properties are maintained.

(1) If $\beta$ and $\delta$ are both apexes of two cones on $F$.

Assume that the tint of the cone determined by $\beta$ is $a \in \mathbb{N}^{-1}$, and the two cones induce the same linear ordering on $F$. Recall that we have $\beta \notin F$, but it is in $M_r$, while $\delta$ is not in $M_r$, and that $|F| = n - 1$. By the rules of the game $\exists$ has no choice but to pick a red colour. $\exists$ uses her auxiliary function $\rho_{r+1}$ to determine the suffices, she lets $\mu = \rho_{r+1}(p)$, $b = \rho_{r+1}(q)$ where $p$ and $q$ are the tints of the two cones based on $F$, whose apexes are $\beta$ and $\delta$. Notice that $\mu, b \in \mathbb{N}$; then she sets $N_s(\beta, \delta) = r_{\mu, b}$ maintaining property (5), and so $\delta \in \text{dom}(\rho_{r+1})$ maintaining property (4). We check consistency to maintain property (6).

Consider a triangle of nodes $(\beta, y, \delta)$ in the graph $M_{r+1} = M^+$. The only possible potential problem is that the edges $M^+(\gamma, \beta)$ and $M^+(\gamma, \delta)$ are coloured green with distinct superscripts $p, q$ but this does not contradict forbidden triangles of the form involving $(g^p_0, g^q_0, r_{kl})$, because $\rho_{r+1}$ is constructed to be order preserving. Now assume that $M_r(\beta, y)$ and $M_{r+1}(y, \delta)$ are both red (some $y \in \text{nodes}(M_r)$). Then $\exists$ chose the red label $N_{r+1}(y, \delta)$, for $\delta$ is a new node. We can assume that $y$ is the apex of a $t$ cone with base $F$ in $M_r$. If not then $N_{r+1}(y, \delta)$ would be coloured $w$ by $\exists$ and there will be no problem. All properties will be maintained. Now $y, \beta \in M$, so by by property (5) we have $M_{r+1}(\beta, y) = r_{\rho+1(p), \rho+1(t)}$. But $\delta \notin M$, so by her strategy, we have $M_{r+1}(y, \delta) = r_{\rho+1(t), \rho+1(q)}$. But $M_{r+1}(\beta, \delta) = r_{\rho+1(p), \rho+1(q)}$, and we are done. This is consistent triple, and so have shown that forbidden triples of reds are avoided.

(2) If there is no $f \in F$ such that $M^*(\beta, f), M^*(\delta, f)$ are coloured $g^p_0, g^q_0$ for some $t, u$ respectively, then $\exists$ defines $M^+(\beta, \delta)$ to be $w_0$.

(3) If this is not the case, and for some $0 < i < n - 1$ there is no $f \in F$ such that $M^*(\beta, f), M^*(\delta, f)$ are both coloured $g_i$, she chooses $w_i$ for $M^+(\beta, \delta)$.

It is clear that these choices in the last two items avoid all forbidden triangles (involving greens and whites).

She has not chosen green maintaining property (1). Now we turn to colouring of $n - 1$ tuples, to make sure that $M^+$ is a coloured graph maintaining property (7).
Let $\Phi$ be the graph chosen by $\forall$ it has set of node $F \cup \{\delta\}$. For each tuple $\bar{a} = a_0, \ldots, a_{n-2} \in M^{n-1}$, $\bar{a} \notin M^{n-1} \cup \Phi^{n-1}$, with no edge $(a_i, a_j)$ coloured green (we already have all edges coloured), then $\exists$ colours $\bar{a}$ by $y_S$, where

$$S = \{i \in A : \text{ there is an } i \text{ cone in } M^* \text{ with base } \bar{a}\}.$$  

We need to check that such labeling works, namely that last property is maintained.

Recall that $M$ is the current coloured graph, $M^* = M \cup \{\delta\}$ is built by $\forall$ s move and $M^+$ is the complete labelled graph by $\exists$ whose nodes are labelled by $\exists$ in response to $\forall$ s moves. We need to show that $M^+$ is labelled according to the rules of the game, namely, that it is in $\mathfrak{B}$. It can be checked $(n - 1)$ tuples are labelled correctly, by yellow colours using the same argument in [14, p.16] [10, p.844] and [12].

We show that $\forall$ has a winning strategy in $F^{n+3}$, the argument used is the $\mathbf{CA}$ analogue of [8, Theorem 33, Lemma 41]. The difference is that in the relation algebra case, the game is played on atomic networks, but now it is translated to playing on coloured graphs, [10, lemma 30].

In the initial round $\forall$ plays a graph $\Gamma$ with nodes $0, 1, \ldots, n - 1$ such that $\Gamma(i, j) = w_0$ for $i < j < n - 1$ and $\Gamma(i, n - 1) = g_i$ ($i = 1, \ldots, n - 2$), $\Gamma(0, n - 1) = g_0^0$ and $\Gamma(0, 1, \ldots, n - 2) = y_B$.

In the following move $\forall$ chooses the face $(0, \ldots, n - 2)$ and demands a node $n$ with $\Gamma_2(i, n) = g_i$ ($i = 1, \ldots, n - 2$), and $\Gamma_2(0, n) = g_0^{-1}$. $\exists$ must choose a label for the edge $(n + 1, n)$ of $\Gamma_2$. It must be a red atom $r_{mn}$. Since $-1 < 0$ we have $m < n$. In the next move $\forall$ plays the face $(0, \ldots, n - 2)$ and demands a node $n + 1$, with $\Gamma_3(i, n) = g_i$ ($i = 1, \ldots, n - 2$), such that $\Gamma_3(0, n + 2) = g_0^{-2}$. Then $\Gamma_3(n + 1, n)$ and $\Gamma_3(n + 1, n - 1)$ both being red, the indices must match. $\Gamma_3(n + 1, n) = r_{mn}$ and $\Gamma_3(n + 1, n - 1) = r_{mn}$ with $l < m$. In the next round $\forall$ plays $(0, 1, \ldots, n - 2)$ and reuses the node 2 such that $\Gamma_4(0, 2) = g_0^{-3}$. This time we have $\Gamma_4(n, n - 1) = r_{jl}$ for some $j < l \in \mathbb{N}$. Continuing in this manner leads to a decreasing sequence in $\mathbb{N}$.

Now that $\forall$ has a winning strategy in $F^{n+3}$, it follows by theorem [3.9] that $\mathfrak{Ro}_wC \notin S_\mathfrak{Me}_nS \mathfrak{Sc}_{n+3}$, but it is elementary equivalent to a countable completely representable algebra. Indeed, using ultrapowers and an elementary chain argument, we obtain $\mathfrak{B}$ such $\mathfrak{C} \equiv \mathfrak{B}$ [8, lemma 44], and $\exists$ has a winning strategy in $G_\omega$ (the usual $\omega$ rounded atomic game) on $\mathfrak{B}$ so by [12, theorem 3.3.3], $\mathfrak{B}$ is completely representable.

In more detail we have $\exists$ has a winning strategy $\sigma_k$ in $G_k$. We can assume that $\sigma_k$ is deterministic. Let $\mathfrak{D}$ be a non-principal ultrapower of $\mathfrak{C}$. One can show that $\exists$ has a winning strategy $\sigma$ in $G(At\mathfrak{D})$ — essentially she uses $\sigma_k$ in the $k$'th component of the ultraproduct so that at each round of $G(At\mathfrak{D})$, $\exists$ is still winning in co-finitely many components, this suffices to show she has still not lost.
We can also assume that $\mathcal{C}$ is countable. If not then replace it by its subalgebra generated by the countably many atoms (the term algebra); winning strategy $s$ that depended only on the atom structure persist for both players.

Now one can use an elementary chain argument to construct a chain of countable elementary subalgebras $\mathcal{C} = A_0 \preceq A_1 \preceq \ldots \preceq \mathcal{D}$ inductively in this manner. One defines $A_{i+1}$ be a countable elementary subalgebra of $\mathcal{D}$ containing $A_i$ and all elements of $\mathcal{D}$ that $\sigma$ selects in a play of $G(\text{At}\mathcal{D})$ in which $\forall$ only chooses elements from $A_i$. Now let $\mathcal{B} = \bigcup_{i<\omega} A_i$. This is a countable elementary subalgebra of $\mathcal{D}$, hence necessarily atomic, and $\exists$ has a winning strategy in $G(\text{At}\mathcal{B})$, so by [12, Theorem 3.3.3], noting that $\mathcal{B}$ is countable, then $\mathcal{B}$ is completely representable; furthermore $\mathcal{B} \equiv \mathcal{C}$. 

In [8] more sophisticated games are devised for relation algebras and Robin Hirsch deduces from the fact that $\exists$ can win the $k$ rounded game on a certain atomic relation algebra $A$ for every finite $k$, then $A$ is elementary equivalent to $\mathcal{B} \in \text{RaCA}_\omega$. This is a mistake. All we can deduce from $\exists$’s winning strategy is that $\mathcal{B} \in S_{c \text{RaCA}}_\omega$ but $\text{At}\mathcal{B} \cong \text{At}\mathcal{D}$ with $\mathcal{D} \in \text{RaCA}_\omega$; this does not mean that $\mathcal{B}$ itself is in $\text{RaCA}_\omega$.

Indeed we show that we may well have algebras $A, B \in \text{CA}_n (n > 1)$, and even more in RCA$_n$ such that $\text{At}\mathcal{A} = \text{At}\mathcal{B}$, $A \in \text{Nr}_n \text{CA}_\omega$ but $B \notin \text{Nr}_n \text{CA}_{\omega+1}$, a fortiori $B \notin \text{Nr}_n \text{CA}_\omega$.

Example 4.4. $FT_\alpha$ denotes the set of all finite transformations on $\alpha$. Let $\alpha$ be an ordinal $> 1$; could be infinite. Let $\mathfrak{F}$ is field of characteristic 0.

$$V = \{s \in ^\alpha \mathfrak{F} : |\{i \in \alpha : s_i \neq 0\}| < \omega\},$$

$$\mathcal{C} = (\varphi(V), \cup, \cap, \sim, \emptyset, V, c_i, d_{i,j}, s_\tau)_{i,j \in \alpha, \tau \in FT_\alpha}.$$

Then clearly $\varphi(V) \in \text{Nr}_\alpha \text{QEA}_{\alpha+\omega}$. Indeed let $W = {^\alpha + \omega} \mathfrak{F}^{(0)}$. Then $\psi : \varphi(V) \rightarrow \text{Nr}_\alpha \varphi(W)$ defined via

$$X \mapsto \{s \in W : s \upharpoonright \alpha \in X\}$$

is an isomorphism from $\varphi(V)$ to $\text{Nr}_\alpha \varphi(W)$. We shall construct an algebra $\mathfrak{A}$, $\mathfrak{A} \notin \text{Nr}_\alpha \text{QPEA}_{\alpha+1}$. Let $y$ denote the following $\alpha$-ary relation:

$$y = \{s \in V : s_0 + 1 = \sum_{i>0} s_i\}.$$

Let $y_s$ be the singleton containing $s$, i.e. $y_s = \{s\}$. Define $\mathfrak{A} \in \text{QEA}_\alpha$ as follows:

$$\mathfrak{A} = \mathfrak{G}_{\mathfrak{F}}^\mathcal{C}\{y, y_s : s \in y\}.$$

Now clearly $\mathfrak{A}$ and $\varphi(V)$ share the same atom structure, namely, the singletons. Then we claim that $\mathfrak{A} \notin \text{Nr}_\alpha \text{QEA}_\beta$ for any $\beta > \alpha$. The first order
sentence that codes the idea of the proof says that \( \mathfrak{A} \) is neither an elementary nor complete subalgebra of \( \varphi(V) \). We use \( \land \) and \( \rightarrow \) in the meta language with their usual meaning. Let \( \text{At}(x) \) be the first order formula asserting that \( x \) is an atom. Let

\[
\tau(x,y) = c_1(c_0x \cdot s_0^1c_1y) \cdot c_1x \cdot c_0y.
\]

Let

\[
\text{Rc}(x) := c_0x \cap c_1x = x,
\]

\[
\phi := \forall x(x \neq 0 \rightarrow \exists y(\text{At}(y) \land y \leq x)) \land \forall x(\text{At}(x) \rightarrow \text{Rc}(x)),
\]

\[
\alpha(x,y) := \text{At}(x) \land x \leq y,
\]

and \( \psi(y_0,y_1) \) be the following first order formula

\[
\forall z(\forall x(\alpha(x,y_0) \rightarrow x \leq z) \rightarrow y_0 \leq z) \land \forall x(\text{At}(x) \rightarrow \text{At}(c_0x \cap y_0) \land \text{At}(c_1x \cap y_0)) \rightarrow [\forall x_1\forall x_2(\alpha(x_1,y_0) \land \alpha(x_2,y_0) \rightarrow \tau(x_1,x_2) \leq y_1) \land \forall z(\forall x_1\forall x_2(\alpha(x_1,y_0) \land \alpha(x_2,y_0) \rightarrow \tau(x_1,x_2) \leq z) \rightarrow y_1 \leq z)].
\]

Then

\[
\mathfrak{N}_{\alpha \text{QEA}_\beta} \models \phi \rightarrow \forall y_0 \exists y_1 \psi(y_0,y_1).
\]

But this formula does not hold in \( \mathfrak{A} \). We have \( \mathfrak{A} \models \phi \) and not \( \mathfrak{A} \models \forall y_0 \exists y_1 \psi(y_0,y_1) \).

In words: we have a set \( X = \{y_s : s \in V\} \) of atoms such that \( \sum S X = y \), and \( \mathfrak{A} \) models \( \phi \) in the sense that below any non zero element there is a rectangular atom, namely a singleton.

Let \( Y = \{\tau(y_r,x),r,s \in V\} \), then \( Y \subseteq \mathfrak{A} \), but it has no supremum in \( \mathfrak{A} \), but it does have one in any full neat reduct \( \mathfrak{B} \) containing \( \mathfrak{A} \), and this is \( \tau_B^\alpha(y,y) \), where \( \tau_B^\alpha(x,y) = c_\alpha(s_0^1c_\alpha x \cdot s_0^1c_\alpha y) \).

In \( \varphi(V) \) this last is \( w = \{s \in S(0)^0 : s_0 + 2 = s_1 + 2 \sum_{i>1}s_i\} \), and \( w \notin \mathfrak{A} \). The proof of this can be easily distilled from [24] main theorem. For \( y_0 = y \), there is no \( y_1 \in \mathfrak{A} \) satisfying \( \psi(y_0,y_1) \). Actually the above proof proves more. It proves that there is a \( \mathfrak{C} \in \mathfrak{N}_{\alpha \text{QEA}_\beta} \) for every \( \beta > \alpha \) (equivalently \( \mathfrak{C} \in \mathfrak{N}_{\alpha \text{QEA}_{\omega}} \)), and \( \mathfrak{A} \subseteq \mathfrak{C} \), such that \( \mathfrak{N} \mathfrak{S} \alpha \mathfrak{A} \notin \mathfrak{N} \mathfrak{S} \alpha + 1 \).

See [23] Theorems 5.1.4-5.1.5] for an entirely different example.

### 4.1 Neat embeddings in connection to complete and strong representations

Here we approach the notion of complete representations and strong representations using neat embeddings. In this subsection \( K \) is any class between Sc and PEA and \( n \) denotes a finite ordinal. Lifting from atom structures, we set:

**Definition 4.5.** An atomic completely additive algebra \( \mathfrak{A} \in K_n \) is strongly representable if its Dedekind-MacNeille completion, namely, \( \text{cmAt} \mathfrak{A} \) is representable.
It is not hard to see that a completely representable algebra is completely additive and also strongly representable. Indeed if $\mathfrak{A}$ is completely representable then it is isomorphic to a generalized set algebra where joins, even infinite ones, are unions, and it is easy to see that the extra non-Boolean operators of substitutions and cylindrifiers distribute over arbitrary joins. Also if $\mathfrak{A}$ is completely representable, then it satisfies all Lyndon conditions hence will be strongly representable. The converse is false as will be shown in our next example. On the other hand, not every atomic completely additive representable algebra is strongly representable; our algebra $\mathfrak{A}$ in theorem 3.11 is an example; in fact we shall see that unlike the class of representable algebras which is a variety, the class of strongly representable $K$ algebras of dimension $> 2$ is not elementary.

Let $\text{CRK}_n$ denote the class of completely representable $K$ algebras of dimension $n$. In [23] it is shown that $S_c \text{Mr}_n \mathcal{CA}_\omega$ and $\text{CRCA}_\omega$ coincide on countable atomic algebras. This characterization works for any $K$ between $S_c$ and $\text{PEA}$, the argument used is an omitting types argument implemented via the Baire category theorem. The result can be slightly generalized to allow algebras with countably many atoms, that may not be countable. In our next theorem we show that this does not generalize any further as far as cardinalities are concerned.

**Theorem 4.6.** For any $n > 2$ we have, $\text{Mr}_n K_\omega \not\subseteq \text{CRK}_n$, while for $n > 1$, $\text{CRK}_n \not\subseteq \text{UpUr} \text{Mr}_n K_\omega$. In particular, there are completely representable, hence strongly representable algebras that are not in $\text{Mr}_n K_\omega$. Furthermore, such algebras can be countable. However, for any $n \in \omega$, if $\mathfrak{A} \in S_c \text{Mr}_n K_\omega$ is atomic and completely additive, then $\mathfrak{A}$ is strongly representable.

**Proof.** (1) We first show that $\text{Mr}_n K_\omega \not\subseteq \text{CRK}_n$. This cannot be witnessed on countable algebras, so our constructed neat reduct that is not completely representable, must be uncountable.

In [8] a sketch of constructing an uncountable relation algebra $R \in \mathcal{RA} \mathcal{CA}_\omega$ (having an $\omega$ dimensional cylindric basis) with no complete representation is given. It has a precursor in [15] which is the special case of this example when $\kappa = \omega$ but the idea in all three proofs are very similar using a variant of the rainbow relation algebra $R_{\omega_1, \omega}$.

Now we give the details of the construction in [8, Remark 31].

Using the terminology of rainbow constructions, we allow the greens to be of cardinality $2^\kappa$ for any infinite cardinal $\kappa$, and the reds to be of cardinality $\kappa$. Here a winning strategy for $\forall$ witnesses that the algebra has no complete representation. But this is not enough because we want our algebra to be in $\mathcal{RA} \mathcal{CA}_\omega$; we will show that it will be.
As usual we specify the atoms and forbidden triples. The atoms are $\text{Id}, g^i_0 : i < 2^\kappa$ and $r_j : 1 \leq j < \kappa$, all symmetric. The forbidden triples of atoms are all permutations of $(\text{Id}, x, y)$ for $x \neq y$, $(r_j, r_j, r_j)$ for $1 \leq j < \kappa$ and $(g^i_0, g^j_0, g^k_0)$ for $i, j, k < 2^\kappa$. In other words, we forbid all the monochromatic triangles.

Write $g_0$ for $\{ g^i_0 : i < 2^\kappa \}$ and $r_+$ for $\{ r_j : 1 \leq j < \kappa \}$. Call this atom structure $\alpha$.

Let $\mathfrak{A}$ be the term algebra on this atom structure; the subalgebra of $\mathfrak{C}ma$ generated by the atoms. $\mathfrak{A}$ is a dense subalgebra of the complex algebra $\mathfrak{C}ma$. We claim that $\mathfrak{A}$, as a relation algebra, has no complete representation.

Indeed, suppose $\mathfrak{A}$ has a complete representation $M$. Let $x, y$ be points in the representation with $M \models r_1(x, y)$. For each $i < 2^\kappa$, there is a point $z_i \in M$ such that $M \models g^i_0(x, z_i) \land r_1(z_i, y)$.

Let $Z = \{ z_i : i < 2^\kappa \}$. Within $Z$ there can be no edges labeled by $r_0$ so each edge is labelled by one of the $\kappa$ atoms in $r_+$. The Erdos-Rado theorem forces the existence of three points $z^1, z^2, z^3 \in Z$ such that $M \models r_j(z^1, z^2) \land r_j(z^2, z^3) \land r_j(z^3, z_1)$, for some single $j < \kappa$. This contradicts the definition of composition in $\mathfrak{A}$ (since we avoided monochromatic triangles).

Let $S$ be the set of all atomic $\mathfrak{A}$-networks $N$ with nodes $\omega$ such that $\{ r_i : 1 \leq i < \kappa : r_i$ is the label of an edge in $N \}$ is finite. Then it is straightforward to show $S$ is an amalgamation class, that is for all $M, N \in S$ if $M \equiv_{ij} N$ then there is $L \in S$ with $M \equiv_i L \equiv_j N$. Hence the complex cylindric algebra $\mathfrak{C}a(S) \in \mathcal{QE}A_\omega$.

Now let $X$ be the set of finite $\mathfrak{A}$-networks $N$ with nodes $\subseteq \omega$ such that

1. each edge of $N$ is either (a) an atom of $\mathfrak{A}$ or (b) a cofinite subset of $r_+ = \{ r_j : 1 \leq j < \kappa \}$ or (c) a cofinite subset of $g_0 = \{ g^i_0 : i < 2^\kappa \}$ and
2. $N$ is ‘triangle-closed’, i.e. for all $l, m, n \in \text{nodes}(N)$ we have $N(l, n) \leq N(l, m); N(m, n)$. That means if an edge $(l, m)$ is labeled by $\text{Id}$ then $N(l, n) = N(mn)$ and if $N(l, m), N(m, n) \leq g_0$ then $N(l, n) \cdot g_0 = 0$ and if $N(l, m) = N(m, n) = r_j$ (some $1 \leq j < \omega$) then $N(l, n) \cdot r_j = 0$.

For $N \in X$ let $N' \in \mathfrak{C}a(S)$ be defined by

$$\{ L \in S : L(m, n) \leq N(m, n) \text{ for } m, n \in \text{nodes}(N) \}$$
For \( i \in \omega \), let \( N \upharpoonright_{-i} \) be the subgraph of \( N \) obtained by deleting the node \( i \). Then if \( N \in X \), \( i < \omega \) then \( c_i N' = (N \upharpoonright_{-i})' \). The inclusion \( c_i N' \subseteq (N \upharpoonright_{-i})' \) is clear.

Conversely, let \( L \in (N \upharpoonright_{-i})' \). We seek \( M \equiv_i L \) with \( M \in N' \). This will prove that \( L \in c_i N' \), as required. Since \( L \in S \) the set \( X = \{ r_i \notin L \} \) is infinite. Let \( X \) be the disjoint union of two infinite sets \( Y \cup Y' \), say. To define the \( \omega \)-network \( M \) we must define the labels of all edges involving the node \( i \) (other labels are given by \( M \equiv_i L \)). We define these labels by enumerating the edges and labeling them one at a time. So let \( j \neq i < \omega \).

Suppose \( j \in \mathrm{nodes}(N) \). We must choose \( M(i, j) \leq N(i, j) \). If \( N(i, j) \) is an atom then of course \( M(i, j) = N(i, j) \). Since \( N \) is finite, this defines only finitely many labels of \( M \). If \( N(i, j) \) is a cofinite subset of \( a_0 \) then we let \( M(i, j) \) be an arbitrary atom in \( N(i, j) \). If \( N(i, j) \) is a cofinite subset of \( r_i \) then let \( M(i, j) \) be an element of \( N(i, j) \cap Y \) which has not been used as the label of any edge of \( M \) which has already been chosen (possible, since at each stage only finitely many have been chosen so far).

If \( j \notin \mathrm{nodes}(N) \) then we can let \( M(i, j) = r_k \in Y \) some \( 1 \leq k < \kappa \) such that no edge of \( M \) has already been labeled by \( r_k \). It is not hard to check that each triangle of \( M \) is consistent (we have avoided all monochromatic triangles) and clearly \( M \in N' \) and \( M \equiv_i L \). The labeling avoided all but finitely many elements of \( Y' \), so \( M \in S \). So \( (N \upharpoonright_{-i})' \subseteq c_i N' \).

Now let \( X' = \{ N' : N \in X \} \subseteq \mathfrak{C} a(S) \). Then the subalgebra of \( \mathfrak{C} a(S) \) generated by \( X' \) is obtained from \( X' \) by closing under finite unions. Clearly all these finite unions are generated by \( X' \). We must show that the set of finite unions of \( X' \) is closed under all cylindric operations. Closure under unions is given. For \( N' \in X \) we have \( -N' = \bigcup_{m,n \in \mathrm{nodes}(N)} N_{mn}' \) where \( N_{mn} \) is a network with nodes \( \{ m, n \} \) and labeling \( N_{mn}(m, n) = -N(m, n) \). \( N_{mn} \) may not belong to \( X \) but it is equivalent to a union of at most finitely many members of \( X \). The diagonal \( d_{ij} \in \mathfrak{C} a(S) \) is equal to \( N' \) where \( N \) is a network with nodes \( \{ i, j \} \) and labeling \( N(i, j) = \mathbf{1}_d \). Let \( \mathfrak{C} \) be the subalgebra of \( \mathfrak{C} a(S) \) generated by \( X' \). Then we claim that \( \mathfrak{A} = \mathfrak{Ra}(\mathfrak{C}) \). Each element of \( \mathfrak{A} \) is a union of a finite number of atoms and possibly a co-finite subset of \( a_0 \) and possibly a co-finite subset of \( a_+ \). Clearly \( \mathfrak{A} \subseteq \mathfrak{Ra}(\mathfrak{C}) \). Conversely, each element \( z \in \mathfrak{Ra}(\mathfrak{C}) \) is a finite union \( \bigcup_{N \in F} N' \), for some finite subset \( F \) of \( X \), satisfying \( c_i z = z \), for \( i > 1 \). Let \( i_0, \ldots, i_k \) be an enumeration of all the nodes, other than 0 and 1, that occur as nodes of networks in \( F \). Then, \( c_{i_0} \cdots c_{i_k} z = \bigcup_{N \in F} c_{i_0} \cdots c_{i_k} N' = \bigcup_{N \in F} (N \upharpoonright_{\{0,1\}})' \in \mathfrak{A} \). So \( \mathfrak{Ra}(\mathfrak{C}) \subseteq \mathfrak{A} \). We have shown that \( \mathfrak{A} \) is relation algebra reduct of \( \mathfrak{C} \in \mathrm{QEA}_{\omega} \) but has no complete representation.

Let \( n > 2 \). Let \( \mathfrak{B} = \mathfrak{R}_n \mathfrak{C} \). Then \( \mathfrak{B} \in \mathfrak{R}_n \mathrm{QEA}_{\omega} \), is atomic, but has
that \( \text{CRTA}_n \not\\subseteq \text{UpUr}\mathfrak{Nr}_n\text{QEA}_\omega \) for \( n > 1 \) follows from example 4.4.

(3) Now for the last part, namely, that complete subneat reducts are strongly representable. Let \( n > 2 \). Let \( \mathfrak{A} \in \mathfrak{S}_c\mathfrak{Nr}_n\mathfrak{CA}_\omega \) be atomic and completely additive. Then \( \exists \) has a winning strategy in \( F^\omega \) by theorem 3.9, hence it has a winning strategy in \( G \) (the usual \( \omega \) rounded atomic game) and so it has a winning strategy for \( G_k \) for all finite \( k \) (\( G \) truncated to \( k \) rounds.) Thus \( \mathfrak{A} \models \sigma_k \) which is the \( k \) th Lyndon sentence coding that \( \exists \) has a winning strategy in \( G_k \), called the \( k \)th Lyndon condition. Since \( \mathfrak{A} \) satisfies the \( k \)th Lyndon conditions for each \( k \), then any algebra on its atom structure is representable, so that \( \mathfrak{CmAt}\mathfrak{A} \) is representable, hence it is strongly representable, and we are done.

\( \Box \)

## 5 Non-finite axiomatizability

We now prove a non-finite axiomatizability result, addressing diagonal free reducts of cylindric and polyadic algebras answering an open problem formulated in [17]. Here the methods of ‘Andréka’s splitting’, adopted in [1] to prove the analogous result for cylindric algebras depend essentially on the presence of diagonal elements and it does not work when the signature lacks diagonal elements, so accordingly we use a rainbow construction instead. The rainbow construction we use is a cylindric version of that used in [9, Definition 17.8]. Throughout this subsection \( n \) is a finite ordinal \( > 2 \).

We alter the used colours a little bit. The reds will have only a single index. We deal with finite rainbow algebras. Let \( \kappa, \mu \) be finite ordinals \( > 0 \). The colours we use are:

- **greens**: \( g_i \ (1 \leq i < n - 2) \cup \{g^0_i : i \in \mu\} \),
- **whites**: \( w_{i,i} : i < n - 1 \),
- **reds**: \( r_i, i \in \kappa \),
- **shades of yellow**: \( y_S : S \subseteq \mu \).

And coloured graphs are:

(1) \( M \) is a complete graph.
(2) $M$ contains no triangles (called forbidden triples) of the following types:

\[
\begin{align*}
(g, g', g^*), & \quad (g_i, g_i, w_i) & \text{any } 1 \leq i \leq n - 1 \\
(g_j^k, g_0^k, w_0) & \quad \text{any } j, k \in \mu \\
(r_i, r_i, r_j) & \quad i, j \in \kappa.
\end{align*}
\]

and no other triple of atoms is forbidden. The items concerning cones and $n - 1$ tuples are the same as before.

We denote the resulting rainbow polyadic equality (finite) algebra defined as before from the new coloured graphs by $\mathfrak{A}_{\mu, \kappa}$.

Let $m > 1$. Let $\alpha = n.2^m$ and $\beta = (\alpha + 1)(\alpha + 2)/2$. Let $\mathfrak{A} = \mathfrak{A}_{\alpha+2, \beta}$ and $\mathfrak{B} = \mathfrak{A}_{\alpha+2, \alpha}$, here $\alpha + 2$ is the number of greens. We consider the usual atomic $k$ rounded game $G_k$ played on coloured graphs. Recall that $G$ denotes the $\omega$ rounded atomic game.

**Theorem 5.1.**

(1) $\forall$ has a winning strategy for $\mathfrak{B}$ in $G_{\alpha+2}$; hence $\mathfrak{M}_{df}\mathfrak{B} \notin \mathcal{RDf}_n$.

(2) $\exists$ has a winning strategy for $G$ on $\mathfrak{A}$, hence $\mathfrak{A} \in \mathcal{RPEA}_n$.

**Proof.** We first show that $\forall$ has a winning strategy for $\mathfrak{B}$ in $\alpha + 2$ rounds; hence it is not representable, and by [7, Theorem 5.4.26], its diagonal free reduct $\mathfrak{M}_{df}\mathfrak{B} \notin \mathcal{RDf}_n$.

$\forall$ plays a coloured graph $M$ with nodes $0, 1, \ldots, n - 1$ and such that $M(i, j) = w_0(i < j < n - 1), M(i, n - 1) = g_i(i = 1, \ldots, n), M(0, n - 1) = g_0^\alpha,$ and $M(0, 1, \ldots, n - 2) = y_{\alpha+2}$. This is a $0$-cone with base $\{0, \ldots, n - 2\}$. In the following moves, $\forall$ repeatedly chooses the face $(0, 1, \ldots, n - 2)$ and demands a node $t < \alpha + 2$ with $\Phi(i, \alpha) = g_i(i = 1, \ldots, n - 2)$ and $\Phi(0, t) = g_0^t$, in the graph notation – i.e., a $t$-cone on the same base. $\exists$ among other things, has to colour all the edges connecting nodes. By the rules of the game the only permissible colours would be red. Using this, $\forall$ can force a win in $\alpha + 2$ rounds eventually using her enough supply of greens, which $\exists$ cannot match using his $< \alpha + 2$ number of reds. The conclusion now follows since $\mathfrak{B}$ is generated by elements whose dimension sets are $< n$.

But we claim that $\mathfrak{A} \in \mathcal{RPEA}_n$. If $\forall$ plays like before, now $\exists$ has more reds, so $\forall$ cannot force a win. In fact $\forall$ can only force a red clique of size $\alpha + 2$, not bigger. So $\exists$’s strategy within red cliques is to choose a label for each edge using a red colour and to ensure that each edge within the clique has a label unique to this edge (within the clique). Since there are $\beta$ many reds she can do that.
Let $M$ be a coloured graph built at some stage, and let $\forall$ choose the graph $\Phi$, $|\Phi| = n$, then $\Phi = F \cup \{\delta\}$, where $F \subseteq M$ and $\delta \notin M$. So we may view $\forall$'s move as building a coloured graph $M^*$ extending $M$ whose nodes are those of $\Gamma$ together with $\delta$ and whose edges are edges of $\Gamma$ together with edges from $\delta$ to every node of $F$.

Colours of edges and $n-1$ tuples in $M^*$ but not in $M$ are determined by $\forall$ moves. No $n-1$ tuple containing both $\delta$ and elements of $M \sim F$ has a colour in $M^*$.

Now $\exists$ must extend $M^*$ to a complete the graph on the same nodes and complete the colouring giving a complete coloured graph $M$. In particular, she has to define $M^+(\beta, \delta)$ for all nodes $\beta \in M \sim F$.

(1) if $\beta$ and $\delta$ are both apexes of cones on $F$, that induces the same linear ordering on $F$, the $\exists$ has no choice but to pick a red atom, and as we described above, she can choose one avoiding inconsistencies.

(2) Other wise, this is not the case, so for some $i < n - 1$ there is no $f \in F$ such that $M^*(\beta, f), M^*(f, \delta)$ are both coloured $g_i$ or if $i = 0$, they are coloured $g_0$ and $g'_0$ for some $l$ and $l'$.

In the second case $\exists$ uses the normal strategy in rainbow constructions. She chooses $w_0$, for $M^+(\beta, \delta)$.

Now we turn to colouring of $n-1$ tuples. For each tuple $\bar{a} = a_0, \ldots a_{n-2} \in M^{n-1}$ with no edge $(a_i, a_j)$ coloured green, then $\exists$ colours $\bar{a}$ by $y_{S}$, where

$$S = \{i \in \alpha + 2 : \text{there is an } i \text{ cone in } M^* \text{ with base } \bar{a}\}.$$ 

This works exactly as in the previous proofs of theorems 3.11 and 4.3.

Recall that a coloured graph is red, if at least one of its edges are labelled red. We write $r$ for $a : n \rightarrow \Gamma$, where $\Gamma$ is a red graph, and we call it a red atom. (Here we identify an atom with its representative, but no harm will follow).

**Theorem 5.2.** Let $m > 1$. Then for any $m$ variable equation in the signature of $\text{Df}_n$ the two algebras $\text{Rd}_q\mathfrak{A}$ and $\text{Rd}_q\mathfrak{B}$, as defined above, falsify it together or satisfy it together.

**Proof.** Cf. [9, Lemma 17.10]. We consider the $\text{Df}$ reducts of $\mathfrak{A}$ and $\mathfrak{B}$, which we continue to denote, with a slight abuse of notation, $\mathfrak{A}$ and $\mathfrak{B}$. Let $R$ be the set of red atoms of $\mathfrak{A}$, and $R'$ be the set of red atoms in $\mathfrak{B}$. Then $|R| \geq |R'| \geq n.2^n$. Assume that the equation $s = t$, using $m$ variables does not hold in $\mathfrak{A}$. Then there is an assignment $h : \{x_0, \ldots, x_{m-1}\} \rightarrow \mathfrak{A}$, such that $\mathfrak{A}, h \models s \neq t$. We construct an assignment $h'$ into $\mathfrak{B}$ that also falsifies $s = t$. Now $\mathfrak{A}$ has more red atoms, but $\mathfrak{A}$ and $\mathfrak{B}$ have identical non-red atoms. So for any non red
atom $a$ of $\mathcal{B}$, and for any $i < m$, let $a \leq h'(x_i)$ iff $a \leq h(x_i)$. To complete the
definition of $h'$ it remains to identify which red atoms of $\mathcal{B}$ (which are only
a part of those of $\mathcal{A}$) lie below $h'(x_i)$ ($i < m$). The assignment $h$ induces a
partition of $R$ into $2^m$ parts $R_S$, $S \subseteq m = \{0, 1 \ldots, m - 1\}$, by

$$R_S = \{r : r \leq h(x_i), i \in S, r \cdot h(x_i) = 0, i \in m \sim S\}. \quad (\star)$$

Partition $R'$ into $2^m$ parts $R'_S$ for $S \subseteq m$ such that $|R'_S| = |R_S|$ if $|R_S| < n$, and
$|R'_S| \geq n$ iff $|R_S| \geq n$. This possible because $|R| \geq n.2^m$. Now for each
$i < m$ and each red atom $r'$ in $R'$, we complete the definition of $h'(x_i) \in \mathcal{B}$
by $r' \leq h'(x_i)$ iff $r' \in R'_S$ for some $S$ such that $i \in S$. It can be easily shown
inductively that for any term $\tau$ using only the first $m$ variables and any $S \subseteq m$, we have

$$R_S \subseteq h(\tau) \iff R'_S \subseteq h'(\tau),$$

$$h(\tau) \sim R \sim h'(\tau),$$

$$|h(\tau) \cap R| = |h'(\tau) \cap R'| \text{ iff } |h(\tau) \cap R| < n,$$

$$|h(\tau) \cap R| \geq n \iff |h'(\tau) \cap R'| \geq n.$$

Hence $\mathcal{B}$ does not model $s = t$. The converse is entirely analogous.  \hfill \Box

**Corollary 5.3.** For any $K$ such that $\text{Df}_n \subseteq K \subseteq \text{PEA}_n$, there is no finite vari-
able universal prenex axiomatization of $\mathcal{R}K_n$, namely, the class of representable
algebras of dimension $n$.

**Proof.** If $\Sigma$ is any $m$ variable equational theory then the appropriate reduct
of $\mathcal{A}$ and $\mathcal{B}$ either both validate $\Sigma$ or neither do. Since one algebra is in $V$
while the other is not, it follows that $\Sigma$ does not axiomatize $V$. But $\mathcal{A}$ and $\mathcal{B}$
are simple (in any of the considered signatures), then they are subdirectly irre-
ducible. In a discriminator variety every universal prenex formula is equivalent
in subdirectly irreducible members to an equation using the same number of
variables. Hence the desired.  \hfill \Box

### 5.1 Omitting types for clique guarded semantics

Throughout this section, unless otherwise indicated, $m$ will denote the dimen-
sion of algebras considered. We consider only cylindric algebras but all results
extend to all of its relatives dealt with above; that is for any class between $\mathcal{S}c$
and $\text{PEA}$. Unless otherwise indicated $m$ will be always finite and $> 2$.

**Definition 5.4.** A non-empty set $M$ is a *relativized representation* of an abstract algebra $\mathcal{A}$ in $\text{CA}_m$, if there exists $V \subseteq ^mM$, and an injective homomorphism $f : \mathcal{A} \to \wp(V)$, where cylindric operations are the usual concrete ones
relativized to $V$ and $M = \bigcup_{s \in V} \text{rng}(s)$. We write $M \models 1(\bar{s})$, if $\bar{s} \in V$. More generally, we write for $a \in A$, $M \models a(\bar{s})$ if $\bar{s} \in f(a)$. We say that $M$ is a relativized representation of $\mathfrak{A}$ with representing function $f$, if we want to highlight the role of $f$, or if $f$ is relevant to the situation at hand.

Let $L(A)$ be the $\mathfrak{L}_m$ signature with an $m$ predicate symbol for every element of $a \in A$ and $L(A)_n$ denotes the set of $n$ variable formulas in this signature.

Next we specify certain relativized representations. The idea, borrowed from Hirsch and Hodkinson who introduced such notions for relation algebras [9] Section 1.3.1., p.400-404], is that the properties of genuine representations manifest itself only locally, that is on $m$-cliques so we have a notion of clique guarded semantics, where witnesses of cylindrifiers are only found if we zoom in adequately on the representation by a ‘movable window’. We may also require that cylindrifiers commute on this localized level; this is the case with $n$ flat representations.

**Definition 5.5.** Let $M$ be a relativized representation of a $\mathfrak{CA}_m$ and $n > m$. An $m$ clique in $M$, or simply a clique in $M$, is a subset $C$ of $M$ such that $M \models 1(\bar{s})$ for all $\bar{s} \in mC$, that is it can be viewed as a hypergraph such that every $m$ tuple is labelled by the top element. Let $C^n(M) = \{ \bar{a} \in {}^n M : \text{rng}(\bar{a}) \text{ is a clique in } M \}$.

**Definition 5.6.** Let $M$ be a relativized representation of $\mathfrak{A}$ with representing function $f$. We define the $n$ dimensional clique guarded semantics $M \models_C \phi[\bar{a}]$ for $\phi \in L(A)_n$ and $\bar{a} \in C^n(M)$ as follows:

- If $\phi$ is $r(x_{i_0} \ldots x_{i_{m-1}})$, $i : m \to n$, then $M \models \phi[\bar{a}]$ iff $(a_{i_0}, \ldots a_{i_{m-1}}) \in f(r)$.

- Boolean clauses are as expected.

- For $i < n$, $M \models \exists x_i \phi[\bar{a}]$ iff $M \models \phi[\bar{b}]$ for some $b \in C^n(M)$ with $\bar{b} \equiv_i \bar{a}$.

**Definition 5.7.**

1. Let $\mathfrak{A} \in \mathfrak{CA}_m$, and $M$ be a relativized representation of $\mathfrak{A}$ with representing function $f$. $M$ is said to be $n$ square, if whenever $\bar{s} \in C^n(M)$, $a \in A$, $i < m$, and $l : m \to n$ if $M \models c_i a(s_{l(0)} \ldots s_{l(m-1)})$, then there is a $t \in C^n(M)$ with $\bar{t} \equiv_i \bar{s}$, and $M \models a(t_{l(0)}, \ldots t_{l(m-1)})$, that is, $(t_{l(0)}, \ldots t_{l(m-1)}) \in f(a)$.

2. Let $M$ be a relativized $n$ square representation of a $\mathfrak{CA}_m$ and $n > m$. $M$ is $n$ flat if for all $\phi \in L(A)^n$, for all $\bar{a} \in C^n(M)$, for all $i, j < n$, we have $M \models_C [\exists x_i \exists x_j \phi \iff \exists x_j \exists x_i \phi](\bar{a})$.

Complete relativized representations are defined exactly like the classical case. One can easily show that if $\mathfrak{A}$ has a complete relativized representation
then it is atomic. The following theorem says that a relativized complete representation of the term algebra induces a relativized representation of its Dedekind-MacNeille completion.

**Lemma 5.8.** If $\mathcal{A} \in \mathcal{CA}_m$ is countable and atomic and $M$ is an $n(n > m)$ flat complete representation of $\mathcal{A}$ via $f$, then $M$ gives an $n$ flat representation of $\mathcal{CMAT}\mathcal{A}$.

**Proof.** Let $f : \mathcal{A} \rightarrow \varphi(V)$ be a representing function for $\mathcal{A}$ where $V \subseteq \n M$. For $a \in \mathcal{CMAT}\mathcal{A}$, $a = \sum X$, say, with $X \subseteq \mathcal{A}$, define $g(a) = \bigcup_{x \in X} f(x)$. Then $g : \mathcal{CMAT}\mathcal{A} \rightarrow \varphi(V)$ is a complete $n$ flat representation of $\mathcal{A}$.  

**Lemma 5.9.** If $\mathcal{A} \in \mathcal{CA}_m$ has an $n$ flat representation, then $\mathcal{A} \in S\mathcal{MR}_m \mathcal{CA}_n$. If $\mathcal{A}$ has an $n$ complete flat representation, then $\mathcal{A} \in S\mathcal{MR}_m \mathcal{CA}_n$. For any $n \geq m + 3$, the class of algebras having $n$ complete flat representations is not elementary.

**Proof.** Let $L(A)$ denote the $\mathcal{L}_m$ signature that contains an $m$ ary predicate for every $a \in A$. Let $M$ be an $n$ flat representation. For $\phi \in L(A)^n$, let $\phi^M = \{ \bar{a} \in C^n(M) : M \models_C \phi(\bar{a}) \}$. Here $\models_C$ is the $n$ dimensional clique guarded semantics. Let $\mathcal{D}$ be the algebra with universe $\{ \phi^M : \phi \in L(A)^n \}$ with usual Boolean operations, cylindrifiers and diagonal elements [9, Theorem 13.20].

Certainly $\mathcal{D}$ is a subalgebra of the $\mathcal{CR}_n$ (the class of algebras whose units are arbitrary sets of $n$ ary sequences) with domain $\varphi(C^n(M))$ so $\mathcal{D} \in \mathcal{CR}_n$. The unit $C^n(M)$ of $\mathcal{D}$ is symmetric, closed under substitutions, so $\mathcal{D} \in \mathcal{G}_n$ (these are relativized set algebras whose units are locally cube, they are closed under substitutions.) Because the representation is $n$ flat, quantifier commute, so cylindrifiers commute, hence we have $\mathcal{D} \in \mathcal{CA}_n$. Now define the map $\theta : \mathcal{A} \rightarrow \mathcal{D}$ by $\theta(r) = r(\bar{x})^M$. Preservation of operations is straightforward. We show that $\theta$ is injective. Let $r \in A$ be non-zero. But $M$ is a relativized representation, so there $\bar{a} \in M$ with $r(\bar{a})$ hence $\bar{a}$ is a clique in $M$, and so $M \models r(\bar{x})(\bar{a})$, and $\bar{a} \in \theta(r)$ proving the required.

Let $L(A)^n_{\infty, \omega}$ be the expansion of $L(A)$ by infinite conjunctions. Let $\mathcal{D} = \{ \phi^M : \phi \in L(A)^n_{\infty, \omega} \}$, that is $\mathcal{D}$ consists of all sets definable by infinitary $n$ variable formulas in the clique relativized semantics (we allow infinite disjunctions). Then as before $\mathcal{D} \in \mathcal{CA}_n$, but in this case we claim that $\mathcal{D}$ is also atomic. Let $\phi^M \in \mathcal{D} \sim \{ 0 \}$. Let $\bar{a} \in \phi^C$ and let $\tau = \bigwedge \{ \psi \in L(A)^n_{\infty, \omega} : M \models \psi(\bar{a}) \}$. Then $\tau^M$ is an atom. Since $\mathcal{A}$ has a relativized complete representation then $\mathcal{A}$ is atomic. We show that $\theta$ is a complete embedding. Let $\phi \in L(A)^n_{\infty, \omega}$ be such that $\phi^M \neq 0$. Let $\bar{a} \in \phi^M$. Since $M$ is a complete relativized representation and $a \in C^n(M)$, then there is an atom $\alpha$ of $\mathcal{A}$ such that $M \models \alpha(\bar{a})$, hence $\theta(\alpha) \cdot \phi^M \neq 0$ and we are done.
The last part follows from theorem 4.3 since \( A = R_{o, A} \) is elementary equivalent to a completely representable algebra \( \mathcal{B} \), and so \( R_{o, A} \in S_{c, \mathfrak{M}_m CA_{m+3}} \).

Results in algebraic logic are most attractive when they have direct new impact on variants of first order logic. We show that our previously proved algebraic results, theorems 3.11, 4.3 are such.

We consider omiting types theorems for \( \mathfrak{L}_m \) a task implemented also in [4], but our approach is different because we move away from classical Tarskian semantics. We show that when we broaden considerably the class of allowed models, permitting the so called \( m+3 \) flat ones, there still might not be countable models omitting a single non-principal type. Furthermore, this single-non principal type can be chosen to be the set of co-atoms in an atomic theory. In the next theorem by \( T \models \phi \) we mean that \( \phi \) is valid in any (classical) model of \( T \).

Let \( T \) be a consistent \( L_m \) theory, and \( m > n \). A model \( M \) of \( T \) is \( n \) flat if it is the base of an \( n \) flat relativized representation of the Tarski-Lindenbaum algebra \( \mathfrak{z}m_T \), that is there exists \( V \subseteq m M \), where \( M \) is the smallest such set, and an injective homomorphism \( f : \mathfrak{z}m_T \rightarrow \wp(V) \).

If \( T \) is countable then an \( \omega \) flat model is an ordinary model, and if it is consistent, then it has an \( n \) flat model for every finite \( n \geq m \). A type \( \Gamma \) is isolated if there exists \( \phi \in \mathfrak{L}_m \) such that \( T \models \phi \rightarrow \alpha \) for all \( \alpha \in \Gamma \); otherwise it is non-principal.

**Theorem 5.10.** There is a countable consistent \( \mathfrak{L}_m \) theory \( T \) with a non-principal type \( \Gamma \) that cannot be omitted in any \( m+3 \) flat model of \( T \).

**Proof.** We give two different proofs relying on theorems 3.11 and 4.3 respectively.

1. Let \( \mathfrak{A} \) be the term algebra as in theorem 3.11. Then \( \mathfrak{A} \) is countable and atomic. Let \( \Gamma' \) be the set of co-atoms. We can assume that \( \mathfrak{A} \) is simple \([3]\). Then \( \mathfrak{A} = \mathfrak{z}m_T \) for some countable consistent \( \mathfrak{L}_m \) complete theory \( T \). Then \( \Gamma = \{ \phi : \phi \models T \} \) is not principal. If it can be omitted in an \( m+3 \) flat model \( \mathfrak{M} \), then this gives an \( m+3 \) complete flat representation of \( \mathfrak{A} = \mathfrak{z}m_T \), which gives an \( m+3 \) flat representation of \( \mathfrak{M} \text{At}\mathfrak{A} \), which we know, by theorems 3.11 and 5.9, does not exist.

2. Now we use theorem 4.3. Let \( \mathfrak{A} \) be the term algebra of the atom structure of that \( CA_m \mathfrak{C} \) constructed in theorem 4.3. Then \( \mathfrak{A} \notin S_{c, \mathfrak{M}_m CA_{m+3}} \). Let \( T \) be such that \( \mathfrak{A} \models \mathfrak{z}m_T \) and let \( \Gamma \) be the set of co-atoms. Then \( \Gamma \) cannot be omitted in an \( m+3 \) flat \( \mathfrak{M} \), for else, this gives a complete \( m+3 \) flat representation of \( \mathfrak{A} \), which means that \( \mathfrak{A} \in S_{c, \mathfrak{M}_m CA_{m+3}} \) by theorem 5.9.

\[ \square \]
5.2 A stronger result on omitting types

For finite $m > 2$ and $k \geq m$, we introduce a new variety of cylindric algebras $CAB_{m,k}$ of dimension $m$ satisfying $S\mathfrak{r}_m CA_{m+k} \subseteq CAB_{m,k} \subseteq RCA_m$ and $\bigcap_{k \geq m} CAB_{m,k} = RCA_m$. Every $\mathfrak{A} \in CAB_{k,m}$ has a $k$ dimensional basis that is determined by an $\omega$ rounded game using $k$ pebbles where $\forall$ has only a cylindrifier move. We show that for any finite $m > 2$ and any $L$ such that $CAB_{m,m+3} \subseteq L \subseteq RCA_m$, $L$ is not atom-canonical. We use this result to show that $OTT$ fails for $\mathfrak{L}_m$ ($m$ finite $> 2$) even if we dispense with the condition of $m$ flatness and require only $m$ squareness. It can be also proved that $n \geq m+3$, $S\mathfrak{r}_m CA_n$ is not finitely axiomatizable over $CAB_{m,n}$ and that $CAB_{m,n+1}$ is not finitely axiomatizable over $CAB_{m,n}$, but we defer this to another paper. The results in this subsection apply equally well to any $K$ between $Sc$ and $PEA$. We formulate and prove our results just for $CA$s.

We recall the definition of networks for $CA$s obtained by deleting the clause that has to do with substitutions corresponding to transpositions in $PEA$s.

Definition 5.11. Let $2 < m < \omega$. Let $\mathfrak{C}$ be an atomic $CA_m$. An atomic network over $\mathfrak{C}$ is a map

$$N : ^m \Delta \to \text{At} \mathfrak{C},$$

where $\Delta$ is a non-empty set called a set of nodes, such that the following hold for each $i, j < n, \delta \in ^m \Delta$ and $d \in \Delta$:

- $N(\delta_j^i) \leq d_{ij}$
- $N(\delta[i \rightarrow d]) \leq c_i N(\delta)$

Definition 5.12. An $n$ dimensional basis of an algebra in $\mathfrak{A} \in CA_m$ is a set of $n$ dimensional $CA$ networks that satisfy only

1. For all $a \in \text{At} \mathfrak{A}$, there is an $N \in H$ such that $N(0, 1, \ldots, m-1) = a$.
2. For all $N \in H$ all $\bar{x} \in ^n \text{nodes}(N)$, for all $i < m$ for all $a \in \text{At} \mathfrak{A}$ such that $N(\bar{x}) \leq c_i a$, there exists $\bar{y} \equiv_i \bar{x}$ such that $N(\bar{y}) = a$.

We define a game on an atomic $\mathfrak{A} \in CA_m$s that characterizes algebras with $n$ dimensional basis. Let $2 < m < n$. Let $r \leq \omega$, the game $G^m_r(\mathfrak{A})$ is played over atomic $\mathfrak{A}$ networks between $\forall$ and $\exists$ and has $r$ rounds, and $n$ pebbles, where $r, n \leq \omega$.

1. In round 0, $\forall$ plays an atom $a \in \text{At} \mathfrak{A}$. $\exists$ must respond with a network $N_a$ with set of nodes $\bar{x}$ such that $N_a(\bar{x}) = a$. 

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2. In any round $t$ with $0 < t < r$, assume that the current network is $N_{t-1}$. Then $\forall$ plays as follows. First if $|N_{t-1}| = n$ then he deletes a node $z \in N_{t-1}$ and defines $N_t$ to be the resulting network. Otherwise, he lets $N = N_{t-1}$; now $\forall$ chooses $\bar{x} \in \mathcal{N}_i$, $i < m$ and an atom $a \in \mathcal{A}$, such that $N(\bar{x}) \leq c_i a$. $\exists$ must respond to this cylindrifier move by providing a network $N_{t} \supseteq N$, with $|N_t| \leq n$, having a node $z$ such and $N_t(\bar{y}) = a$, where $\bar{y} \equiv_i \bar{x}$ and $\bar{y}(i) = z$.

The number of pebbles $n$ measures the squareness, flatness of the relativized representations; these are significantly distinct notions in relativized semantics, though the distinction between the first two diffuses at the limit of genuine representations.

Crudely, the classical case then becomes a limiting case, when $n$ goes to infinity. However, this is not completely accurate because $\omega$ complete relativized representations may not coincide with classical ones on uncountable algebras. Indeed the uncountable algebra proved in theorem 5.8 not have a complete representation can be easily proved to have an $\omega$ relativized complete representation using that $\exists$ can win the $\omega$ rounded atomic game.

**Theorem 5.13.** Let $2 < m < n$. Then following are equivalent for an atomic $\mathfrak{A} \in \mathcal{C}A_m$

1. $\mathfrak{A}$ has an $n$ dimensional bases.

2. $\exists$ has a winning strategy in $G^\omega_n(\mathfrak{A})$.

**Proof.** The proof is similar to [9, proposition 12.25].

**Definition 5.14.** Let $2 < m < n$ and suppose that $m$ is finite. We define $\mathfrak{A} \in \mathcal{C}AB_n$ if $\mathfrak{A}$ is a subalgebra of a complete atomic $\mathfrak{B} \in \mathcal{C}A_m$ that has an $n$ dimensional basis.

**Theorem 5.15.** Fix a finite dimension $m > 2$, and let $n \in \omega$ with $n \geq m$.

1. $\mathcal{C}AB_{m,n}$ is a canonical variety,

2. If $\mathfrak{A} \in \mathcal{C}A_m$ then $\mathfrak{A} \in \mathcal{C}AB_{m,n}$ iff $\mathfrak{A}^+$ has an $n$ dimensional basis,

3. If $\mathfrak{A} \in \mathcal{C}A_m$, then $\mathfrak{A} \in \mathcal{C}AB_{m,n}$ iff $\mathfrak{A}$ has an $n$ square relativized representation,

4. $\bigcap_{k \in \omega, k \geq m} \mathcal{C}AB_{m,k} = \mathcal{R}CA_m$.

**Proof.**

1. The proof is similar to that of [12, proposition 12.31]

2. One side is trivial. The other side is similar to the proof of [9, Proposition 13.37].
(3) One side is easy. Let $M$ be the given representation. For a network $N$ and $v : N \to M$, we say that $v$ embeds $N$ in $M$ if for all $r \in \text{At} \mathfrak{M}$, we have $N(\bar{x}) = r \iff M \models r(v(\bar{x}))$.

All such networks that embed in $M$ are the desired basis. The other more involved part is also similar to [9 Proposition 13.37].

(4) Since a classical representation is $\omega$ square, then $\bigcap_{k \geq m} \text{CAB}_{m,k}$ and $\text{RCA}_m$ coincide on simple countable algebras. But each is a discriminator variety and so they are equal.

The next theorem (that we state without proof) says that the class of algebras having $n$ flat relativized representations is not finitely axiomatizable over that having $n$ square representations. This means that commutativity of cylindrifiers adds a lot.

**Theorem 5.16.**

1. For $m \geq 3$ and $k \in \omega \sim \{0\}$, $\mathfrak{M} \mathfrak{t}_m \text{CA}_{m+k} \subseteq \text{CAB}_{m,m+k}$

2. For $3 \leq m \leq n < \omega$, $\mathfrak{M} \mathfrak{t}_m \text{CA}_{n+1}$ is not finitely axiomatizable over $\text{CAB}_{m,n+1}$

**Theorem 5.17.**

1. The atom structure $\text{At}$ constructed in theorem 3.11 satisfies that $\mathfrak{Rd}_\text{ca} \mathfrak{S} \text{mAt} \in \text{RCA}_m$, but $\mathfrak{Rd}_\text{ca} \mathfrak{C} \text{mAt} \not\in \text{CAB}_{m,m+3}$.

2. The omitting types theorem fails if we allow $k + 3$ square models.

**Proof.** Like the proof of theorem 3.11 by noting that $\forall$ can win $G^m_{m+3}$ on the finite rainbow algebra $m$ dimensional algebra $\mathfrak{C} \text{A}_{m+1,m}$.  

Our next corollary formulated only for $\text{CAs}$ is true for any $K$ between $\text{Sc}$ and $\text{PEA}$. It also holds for $\text{CAB}_{n,n+k}$ in place of $\mathfrak{M} \mathfrak{t}_n \text{CA}_{n+k}$, by replacing ‘flat’ by ‘square’ in the last item.

**Corollary 5.18.** Let $n$ be finite with $n > 2$ and $k \geq 3$. Then the following hold:

1. There exist two atomic cylindric algebras of dimension $n$ with the same atom structure, one representable and the other is not in $\mathfrak{M} \mathfrak{t}_n \text{CA}_{n+k}$.

2. The variety $\mathfrak{M} \mathfrak{t}_n \text{CA}_{n+k}$ is not closed under Dedekind-MacNeille completions and is not atom-canonical. In particular, $\text{RCA}_n$ is not atom-canonical.
(3) There exists an algebra outside $\mathcal{SNrnCAn+k}$ with a dense representable subalgebra.

(4) The variety $\mathcal{SNrnCAn+k}$ is not Sahlqvist axiomatizable. In particular, $\text{RCA}_n$ is not Sahlqvist axiomatizable.

(5) There exists an atomic representable $\text{CA}_n$ with no $n+k$ flat complete representation; in particular it has no complete representation.

Proof. Throughout $\mathfrak{A}$ is the algebra constructed in theorem 3.11.

(1) $\mathfrak{A}$ and $\mathcal{CmAt}\mathfrak{A}$ are such.

(2) $\mathcal{CmAt}\mathfrak{A}$ is the Dedekind-MacNeille completion of $\mathfrak{A}$ (even in the $\text{PA}$ and $\text{Sc}$ cases, because $\mathfrak{A}$, hence its $\text{Sc}$ and $\text{PA}$ reducts are completely additive), hence $\mathcal{SNrnCAn+k}$ is not atom canonical [9, Proposition 2.88, Theorem, 2.96].

(3) $\mathfrak{A}$ is dense in $\mathcal{CmAt}\mathfrak{A}$.

(4) Completely additive varieties defined by Sahlqvist equations are closed under Dedekind-MacNeille completions [9, Theorem 2.96].

(5) Like the proof of theorem 5.9.

$\square$

6 Strongly representable atom structures

Here we extend Hirsch and Hodkinson’s celebrated result that the class of strongly representable atom structures of cylindric algebras of finite dimension $>2$ is not elementary, to any class of algebras with signature between $\text{Df}$ and $\text{PEA}$, reproving a result by Bu and Hodkinson [16]. We believe that our proof is simpler. Throughout this section, $n$ is a finite ordinal $>2$.

Definition 6.1. Let $\Gamma = (G, E)$ be a graph.

1. A set $X \subseteq G$ is said to be independent if $E \cap (X \times X) = \phi$.

2. The chromatic number $\chi(\Gamma)$ of $\Gamma$ is the smallest $\kappa < \omega$ such that $G$ can be partitioned into $\kappa$ independent sets, and $\infty$ if there is no such $\kappa$.

Definition 6.2.

1. For an equivalence relation $\sim$ on a set $X$, and $Y \subseteq X$, we write $\sim \upharpoonright Y$ for $\sim \cap (Y \times Y)$. For a partial map $K : n \to \Gamma \times n$ and $i, j < n$, we write $K(i) = K(j)$ to mean that either $K(i), K(j)$ are both undefined, or they are both defined and are equal.
2. For any two relations \( \sim \) and \( \approx \). The composition of \( \sim \) and \( \approx \) is the set
\[
\sim \circ \approx = \{(a, b) : \exists c(a \sim c \land c \approx b)\}.
\]

**Definition 6.3.** Let \( \Gamma \) be a graph. We define an atom structure \( \eta(\Gamma) = \langle H, D_{ij}, \equiv_i, \equiv_{ij} : i, j < n \rangle \) as follows:

1. \( H \) is the set of all pairs \( (K, \sim) \) where \( K : n \to \Gamma \times n \) is a partial map and \( \sim \) is an equivalence relation on \( n \) satisfying the following conditions
   
   (a) If \(|n/\sim| = n\), then \text{dom}(K) = n\) and \text{rng}(K) is not an independent subset of \( n\)
   
   (b) If \(|n/\sim| = n - 1\), then \( K \) is defined only on the unique \( \sim \) class \( \{i, j\} \) say of size 2 and \( K(i) = K(j)\)
   
   (c) If \(|n/\sim| \leq n - 2\), then \( K \) is nowhere defined.

2. \( D_{ij} = \{(K, \sim) \in H : i \sim j\}\)

3. \((K, \sim) \equiv_i (K', \sim')\) iff \( K(i) = K'(i) \) and \( \sim \upharpoonright (n \setminus \{i\}) = \sim' \upharpoonright (n \setminus \{i\})\)

4. \((K, \sim) \equiv_{ij} (K', \sim')\) iff \( K(i) = K'(j), K(j) = K'(i)\), and \( K(\kappa) = K'(\kappa) (\forall \kappa \in n \setminus \{i, j\})\) and if \( i \sim j \) then \( \sim = \sim'\), if not, then \( \sim = \sim \circ [i, j]\).

It may help to think of \( K(i) \) as assigning the nodes \( K(i) \) of \( \Gamma \times n \) not to \( i \) but to the set \( n \setminus \{i\} \), so long as its elements are pairwise non-equivalent via \( \sim \).

For a set \( X \), \( \mathcal{B}(X) \) denotes the Boolean algebra \( \langle \wp(X), \cap, \cup, \sim, \emptyset, X \rangle \). We denote \( s_{i/j} \) by \( s^j_i \).

**Definition 6.4.** Let \( \mathfrak{B}(\Gamma) = (\mathfrak{B}(\eta(\Gamma)), c_i, s^i_j, s^i_{[i,j]}(D_{ij}))_{i,j < n} \) be the algebra, with extra non-Boolean operations defined for all \( X \subseteq \eta(\Gamma) \) as follows:

\[
\begin{align*}
d_{ij} &= D_{ij}, \\
c^i_i X &= \{c : \exists a \in X, a \equiv_i c\}, \\
s^i_{[i,j]} X &= \{c : \exists a \in X, a \equiv_{ij} c\}, \\
s^j_i X &= \begin{cases} 
  c(X \cap D_{ij}), & \text{if } i \neq j, \\
  X, & \text{if } i = j.
\end{cases}
\end{align*}
\]

**Definition 6.5.** For any \( \tau \in \{\pi \in n^n : \pi \text{ is a bijection}\} \), and any \( (K, \sim) \in \eta(\Gamma) \), we define \( \tau(K, \sim) = (K \circ \tau, \sim \circ \tau)\).

The proof of the following two Lemmas is straightforward.

**Lemma 6.6.** For any \( \tau \in \{\pi \in n^n : \pi \text{ is a bijection}\} \), and any \( (K, \sim) \in \eta(\Gamma) \), \( \tau(K, \sim) \in \eta(\Gamma) \).
Theorem 6.8. For any graph $\Gamma$, $(K, \sim), (K', \sim'),$ and $(K'', \sim'') \in \eta(\Gamma)$, and $i, j \in n$:

1. $(K, \sim) \equiv_{ii} (K', \sim') \iff (K, \sim) = (K', \sim').$

2. $(K, \sim) \equiv_{ij} (K', \sim') \iff (K, \sim) \equiv_{ji} (K', \sim').$

3. If $(K, \sim) \equiv_{ij} (K', \sim')$, and $(K, \sim) \equiv_{ij} (K'', \sim'')$, then it follows that $(K', \sim') = (K'', \sim'').$

4. If $(K, \sim) \in D_{ij}$, then we have $(K, \sim) \equiv_i (K', \sim') \iff \exists(K_1, \sim_1) \in \eta(\Gamma), (K, \sim) \equiv_j (K_1, \sim_1) \land (K', \sim') \equiv_{ij} (K_1, \sim_1)$.

5. $s_{[i,j]}(\eta(\Gamma)) = \eta(\Gamma)$.

The proof of the next lemma is tedious but not too hard.

Theorem 6.8. For any graph $\Gamma$, $B(\Gamma)$ is a simple $\text{PEA}_n$.

Proof. We follow the axiomatization in [17] except renaming the items by $Q_i$. Let $X \subseteq \eta(\Gamma)$, and $i, j, k \in n$:

1. $s^i = Id$ by definition [6.4]. $s_{[i,j]}X = \{c : \exists a \in X, a \equiv_{ii} c\} = \{c : \exists a \in X, a = c\} = X$ (by Lemma 6.7 (1));

$s_{[i,j]}X = \{c : \exists a \in X, a \equiv_{ij} c\} = \{c : \exists a \in X, a \equiv_{ji} c\} = s_{[j,i]}X$ (by Lemma 6.7 (2)).

2. Axioms $Q_1, Q_2$ follow directly from the fact that the reduct $\mathfrak{M}_{ca}B(\Gamma) = \langle B(\eta(\Gamma)), c_i, d_{ij} \rangle_{i,j<n}$ is a cylindric algebra which is proved in [11].

3. Axioms $Q_3, Q_4, Q_5$ follow from the fact that the reduct $\mathfrak{M}_{ca}B(\Gamma)$ is a cylindric algebra, and from [7] (Theorem 1.5.8(i), Theorem 1.5.9(ii), Theorem 1.5.8(ii)).

4. $s^i$ is a Boolean endomorphism by [7] (Theorem 1.5.3).

$s_{[i,j]}(X \cup Y) = \{c : \exists a \in (X \cup Y), a \equiv_{ij} c\}$

$= \{c : (\exists a \in X \lor \exists a \in Y), a \equiv_{ij} c\}$

$= \{c : \exists a \in X, a \equiv_{ij} c\} \cup \{c : \exists a \in Y, a \equiv_{ij} c\}$

$= s_{[i,j]}X \cup s_{[j,i]}Y$.

$s_{[i,j]}(-X) = \{c : \exists a \in (-X), a \equiv_{ij} c\}$, and $s_{[i,j]}X = \{c : \exists a \in X, a \equiv_{ij} c\}$ are disjoint. For, let $c \in (s_{[i,j]}(X) \cap s_{[i,j]}(-X))$, then $\exists a \in X \land b \in (-X)$, such that $a \equiv_{ij} c$, and $b \equiv_{ij} c$. Then $a = b$, (by Lemma 6.7 (3)), which is a contradiction. Also,

$s_{[i,j]}X \cup s_{[i,j]}(-X) = \{c : \exists a \in X, a \equiv_{ij} c\} \cup \{c : \exists a \in (-X), a \equiv_{ij} c\}$

$= \{c : \exists a \in (X \cup -X), a \equiv_{ij} c\}$

$= s_{[i,j]}\eta(\Gamma)$

$= \eta(\Gamma)$. (by Lemma 6.7 (5))
therefore, \( s_{[i,j]} \) is a Boolean endomorphism.

5. 
\[
\begin{align*}
  s_{[i,j]}s_{[i,j]}X &= s_{[i,j]} \{ c : \exists a \in X, a \equiv_{ij} c \} \\
  &= \{ b : (\exists a \in X \land c \in \eta(\Gamma)), a \equiv_{ij} c, \text{ and } c \equiv_{ij} b \} \\
  &= \{ b : \exists a \in X, a = b \} \\
  &= X.
\end{align*}
\]

6. 
\[
\begin{align*}
  s_{[i,j]}s^{i}X &= \{ c : \exists a \in s^{i}X, a \equiv_{ij} c \} \\
  &= \{ c : \exists b \in (X \cap d)_{ij}, a \equiv_{i} b \land a \equiv_{ij} c \} \\
  &= \{ c : \exists b \in (X \cap d)_{ij}, c \equiv_{j} b \} \text{ (by Lemma 6.7 (4))} \\
  &= s^{j}X.
\end{align*}
\]

7. We need to prove that \( s_{[i,j]}s_{[i,\kappa]}X = s_{[j,\kappa]}s_{[i,j]}X \) if \(|\{i,j,\kappa\}| = 3\).

Let \((K, \sim) \in s_{[i,j]}s_{[i,\kappa]}X\) then \(\exists(K', \sim') \in \eta(\Gamma)\), and \(\exists(K'', \sim'') \in X\) such that \((K'', \sim'') \equiv_{\kappa} (K', \sim')\) and \((K', \sim') \equiv_{ij} (K, \sim)\).

Define \(\tau : n \rightarrow n\) as follows:
\[
\begin{align*}
  \tau(i) &= j \\
  \tau(j) &= \kappa \\
  \tau(\kappa) &= i, \text{ and} \\
  \tau(l) &= l \text{ for every } l \in (n \setminus \{i, j, \kappa\}).
\end{align*}
\]

Now, it is easy to verify that \(\tau(K', \sim') \equiv_{ij} (K'', \sim'')\), and \(\tau(K', \sim') \equiv_{j\kappa} (K, \sim)\). Therefore, \((K, \sim) \in s_{[j,\kappa]}s_{[i,j]}X\), i.e., \(s_{[i,j]}s_{[j,\kappa]}X \subseteq s_{[j,\kappa]}s_{[i,j]}X\).

Similarly, we can show that \(s_{[j,\kappa]}s_{[i,j]}X \subseteq s_{[i,j]}s_{[j,\kappa]}X\).

8. Axiom \(Q_{10}\) follows from [7] (Theorem 1.5.7)

9. Axiom \(Q_{11}\) follows from axiom 2, and the definition of \(s^{i}\).

Since \(\mathfrak{R}_{\text{ca}}B\) is a simple \(\mathcal{CA}_{n}\), by [11], then \(B\) is a simple \(\mathcal{PEA}_{n}\). This follows from the fact that ideals \(I\) is an ideal in \(\mathfrak{R}_{\text{ca}}B\) if and only if it is an ideal in \(B\).

**Theorem 6.9.** \(B(\Gamma)\) is a simple \(\mathcal{PEA}_{n}\) generated using infinite unions by the set of the \(n-1\) dimensional elements.

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Proof. \( \mathcal{B}(\Gamma) \) is a simple \( \mathsf{PEA}_n \) from Theorem 6.8. It remains to show, since \( \mathcal{B} \) is atomic, so that every element is an infinite union of atoms that \( \{(K, \sim)\} = \prod \{c_i((K, \sim)) : i < n\} \) for any \((K, \sim) \in H\). Let \((K, \sim) \in H\), clearly \(\{(K, \sim)\} \leq \prod \{c_i((K, \sim)) : i < n\}\). For the other direction assume that \((K', \sim') \in H\) and \((K, \sim) \neq (K', \sim')\). We show that \((K', \sim') \notin \prod \{c_i((K, \sim)) : i < n\}\). Assume toward a contradiction that \((K', \sim') \in \prod \{c_i((K, \sim)) : i < n\}\) for all \(i < n\), i.e., \(K'(i) = K(i)\) and \(\sim'\mid (n \setminus \{i\}) = \sim\mid (n \setminus \{i\})\) for all \(i < n\). Therefore, \((K, \sim) = (K', \sim')\) which makes a contradiction, and hence we get the other direction. \(\square\)

**Theorem 6.10.** Let \( \Gamma \) be a graph.

1. Suppose that \( \chi(\Gamma) = \infty \). Then \( \mathcal{B}(\Gamma) \) is representable.

2. If \( \Gamma \) is infinite and \( \chi(\Gamma) < \infty \) then \( \mathcal{R}_{df} \mathcal{B}(\Gamma) \) is not representable.

**Proof.** 1. For the sake of brevity, we denote \( \mathcal{B}(\Gamma) \) by \( \mathcal{B} \). We have \( \mathcal{R}_{ca} \mathcal{B} \) is representable (c.f., [11]). Let \( X = \{x \in \mathcal{B} : \Delta x \neq n\} \). Call \( J \subseteq \mathcal{B} \) inductive if \( X \subseteq J \) and \( J \) is closed under infinite unions and complementation. Then \( B \) is the smallest inductive subset of \( \mathcal{B} \). Let \( f \) be an isomorphism of \( \mathcal{R}_{ca} \mathcal{B} \) onto a cylindric set algebra with base \( U \). Clearly, by definition, \( f \) preserves \( s_i^{ij} \) for each \( i, j < n \). It remains to show that \( f \) preserves \( s_{[i,j]} \) for every \( i < j < n \). Let \( i < j < n \), since \( s_{[i,j]} \) is a Boolean endomorphism and completely additive, it suffices to show that \( f(s_{[i,j]}x) = s_{[i,j]}f(x) \) for all \( x \in \mathsf{At} \mathcal{B} \). Let \( x \in \mathsf{At} \mathcal{B} \) and \( \mu \in n \setminus \Delta x \). If \( \kappa = \mu \) or \( l = \mu \), say \( \kappa = \mu \), then

\[
 f(s_{[\kappa,l]}^\mu) = f(s_{[\kappa,l]}^\kappa x) \\
 = f(s_i^{\kappa} x) \\
 = s_i^{\kappa} f(x) \\
 = s_{[\kappa,l]} f(x).
\]

If \( \mu \notin \{ \kappa, l \} \) then

\[
 f(s_{[\kappa,l]}^\mu) = f(s_i^\mu s_i^\kappa s_{[\kappa,l]}^\mu x) \\
 = s_i^\mu s_i^\kappa c_\mu f(x) \\
 = s_{[\kappa,l]} f(x).
\]

2. Assume for contradiction that \( \mathcal{R}_{df} \mathcal{B} \) is representable. Since \( \mathcal{R}_{ca} \mathcal{B} \) is generated by \( n - 1 \) dimensional elements then \( \mathcal{R}_{ca} \mathcal{B} \) is representable. But this contradicts [11, Proposition 5.4]. \(\square\)
Theorem 6.11. Let $2 < n < \omega$ and $T$ be any signature between $Df_n$ and $PEA_n$. Then the class of strongly representable atom structures of type $T$ is not elementary.

Proof. By Erdős's famous 1959 Theorem [5], for each finite $\kappa$ there is a finite graph $G_\kappa$ with $\chi(G_\kappa) > \kappa$ and with no cycles of length $< \kappa$. Let $\Gamma_\kappa$ be the disjoint union of the $G_l$ for $l > \kappa$. Clearly, $\chi(\Gamma_\kappa) = \infty$. So by Theorem 6.10 (1), $\mathfrak{B}(\Gamma_\kappa) = \mathfrak{B}(\Gamma_\kappa)^+$ is representable.

Now let $\Gamma$ be a non-principal ultraproduct $\prod_D \Gamma_\kappa$ for the $\Gamma_\kappa$. It is certainly infinite. For $\kappa < \omega$, let $\sigma_\kappa$ be a first-order sentence of the signature of the graphs stating that there are no cycles of length less than $\kappa$. Then $\Gamma_l \models \sigma_\kappa$ for all $l \geq \kappa$. By Łoś's Theorem, $\Gamma \models \sigma_\kappa$ for all $\kappa$. So $\Gamma$ has no cycles, and hence by, [11] Lemma 3.2, $\chi(\Gamma) \leq 2$. By Theorem 6.10 (2), $\mathfrak{R}_{df} \mathfrak{B}$ is not representable. It is easy to show (e.g., because $\mathfrak{B}(\Gamma)$ is first-order interpretable in $\Gamma$, for any $\Gamma$) that

$$\prod_D \mathfrak{B}(\Gamma_\kappa) \cong \mathfrak{B}(\prod_D \Gamma_\kappa).$$

Combining this with the fact that: for any $n$-dimensional atom structure $S$

$S$ is strongly representable $\iff \mathfrak{C}mS$ is representable, the desired follows. $\square$

Again like in the case of rainbow algebras, define the polyadic operations corresponding to transpositions using the notation in [12, Definition 3.6.6], in the context of defining atom structures from classes of models by $R_{df}^{[ij]} = \{(\omega, \omega) : f \circ [i, j] = g\}$. This is well defined. In particular, we can (and will) consider the Monk-like algebra $\mathfrak{M}(\Gamma)$ as defined in [12, top of p. 78] as a polyadic equality algebra.

Theorem 6.12. Let $\mathfrak{M}(\Gamma)$ be the polyadic equality algebra defined above. If $\chi(\Gamma) = \infty$, then $\mathfrak{M}(\Gamma)$ is representable as a polyadic equality algebra. If $\chi(\Gamma) < \infty$, then $\mathfrak{R}_{df} \mathfrak{M}(\Gamma)$ is not representable.

Proof. Only networks are changed but the atomic game is the same, so clearly $\exists$ has a winning strategy infinite (possibly transfinite) game over $\mathfrak{M}(\Gamma)^\sigma$ which is the canonical extension of $\mathfrak{M}(\Gamma)$ [12, Lemma 3.6.4, Lemma 3.6.7]

In using the latter lemma the proof is unaltered, in the former lemma $\exists$ strategy to win the $G([\ell(\mathfrak{M}(\Gamma))^\sigma])$ game is exactly the same so that $\mathfrak{M}(\Gamma)^\sigma$ is completely representable, hence $\mathfrak{M}(\Gamma)$ is representable as a polyadic equality algebra.

The second part follows from the fact that $\mathfrak{M}(\Gamma)$ is generated by elements whose dimension sets $< n$, [7, Theorem 5.4.29]. $\square$

Definition 6.13. (1) A Monks algebra $\mathfrak{M}(\Gamma)$ is good if $\chi(\Gamma) = \infty$. 

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A Monk’s algebra $\mathfrak{M}(\Gamma)$ is bad if $\chi(\Gamma) < \infty$.

It is easy to construct a good Monk’s algebra as an ultraproduct (limit) of bad Monk algebras. Monk’s original algebras can be viewed this way. The converse is, as illustrated above, is much harder. It took Erdős probabilistic graphs, to get a sequence of good graphs converging to a bad one. Recall that we defined a strongly representable $\mathfrak{A} \in \text{CA}_n$ to be atomic, such that $C^m \mathfrak{A} \in \text{RCA}_n$. Extending the definition to any $K$ between $\text{Df}$ and $\text{PEA}$, we stipulate that $\mathfrak{A} \in K$ is strongly representable if it is atomic, completely additive and $C^m \mathfrak{A} \in \text{RK}_n$. Atomicity and complete additivity on an atomic BAO are first order definable. Let $\text{SRK}_n$ denote the class of strongly representable $K$ algebras of dimension $n$. Using our modified Monk-like algebras one can obtain the result formulated in theorem 6.11 using the same graphs. Also answering a question of Hodkinson’s in [15, p. 284] we get:

**Corollary 6.14.** For any finite $n > 2$, and any $K$ between $\text{Df}$ and $\text{PEA}$ the class $\text{SRK}_n$ is not elementary. In fact, for any class $K$ between $\text{Df}$ and $\text{PEA}$ and any finite $n > 2$, the class $K_n = \{ \mathfrak{A} \in K_n : C^m \mathfrak{A} \in \text{RK}_n \}$, is not elementary. Furthermore, for $K$ between $\text{Sc}$ and $\text{PEA}$, $\mathfrak{N}_n K_\omega \subseteq S\mathfrak{N}_n K_\omega \subseteq \text{SRK}_n$ and all inclusions are proper.

**Proof.** For the first part and second parts, we proceed like in the proof of theorem 6.11 using the Monk algebras $\mathfrak{M}(\Gamma)$, $\Gamma$ a graph. By Erdős’s celebrated Theorem, for each finite $\kappa$ there is a finite graph $\mathfrak{G}_\kappa$ with $\chi(\mathfrak{G}_\kappa) > \kappa$ and with no cycles of length $< \kappa$. Let $\Gamma_\kappa$ be the disjoint union of the $\mathfrak{G}_l$ for $l > \kappa$. Clearly, $\chi(\Gamma_\kappa) = \infty$ hence $\mathfrak{M}(\Gamma_\kappa)$ is representable.

Now let $\Gamma$ be a non-principal ultraproduct $\Pi_D \Gamma_\kappa$ of the $\Gamma_\kappa$’s. It is certainly infinite. For $\kappa < \omega$, let $\sigma_\kappa$ be a first-order sentence of the signature of the graphs stating that there are no cycles of length less than $\kappa$. Then $\Gamma_l \models \sigma_\kappa$ for all $l \geq \kappa$. By Lös’s Theorem, $\Gamma \models \sigma_\kappa$ for all $\kappa$. So $\Gamma$ has no cycles, and hence $\chi(\Gamma) \leq 2$ hence $\text{Rd}_d \mathfrak{M}(\Gamma)$ is not representable. But

$$\Pi_D \mathfrak{M}(\Gamma_\kappa) \cong \mathfrak{M}(\Pi_D \Gamma_\kappa), = \mathfrak{M}(\Gamma)$$

and we are done.

So $\mathfrak{M}(\Gamma_i)$ is a family of algebras in $K_s$ whose ultraproduct is not in $K_s$.

We prove the last part. The first inclusion follows from example 4.4 and the last follows from the construction in theorem 4.3. Taking for example cylindric algebras we have $\mathfrak{A} = \text{Rd}_c \text{PEA}_{N-1,N} \not\in S\mathfrak{N}_n \text{CA}_{n+3} \supseteq S\mathfrak{N}_n \text{CA}_\omega$, but $\mathfrak{A}$ satisfies the Lyndon conditions since $\exists \text{ can win } G_k$ for all finite $k$ hence $\mathfrak{A}$ is strongly representable.

Henceforth we carry out our discussions and formulate our theorems for cylindric algebras. Everything said about cylindric algebras carries over to...
the other cylindric-like algebras approached in our previous investigations like $Sc$, $PA$ and $PEA$.

However, $Dfs$ does not count here, for the notion of neat reducts for such algebras is trivial. Lifting from atom structures, we define several classes of atomic representable algebras. For a class $K$ with a Boolean reduct, $K \cap \text{At}$ denotes the class of atomic algebras in $K$; the former is elementary iff the latter is.

Fix $n > 2$ finite. $\text{CRA}_n$ denotes the class of completely representable algebras of dimension $n$. Let $\text{SRCA}_n$ be the class of strongly representable atomic algebras of dimension $n$, $A \in \text{SRCA}_n$ iff $\mathcal{C}m\text{At}A \in \text{RCA}_n$. $\text{WRCA}_n$ denotes the class of weakly representable algebras of dimension $n$, and this is just $\text{RCA}_n \cap \text{At}$. We have the following strict inclusions lifting them up from atom structures $^{12}$ (*):

$$\text{CRA}_n \subset \text{LCA}_n \subset \text{SRCA}_n \subset \text{WCRA}_n$$

The second and fourth classes are elementary but not finitely axiomatizable, bad Monk algebras converging to a good one (Monk’s original algebras are like that), can witness this, while $\text{SRCA}_n$ is not closed under both ultraroots and ultraproducts, good Monks algebras converging to a bad one witnesses this. The algebra $\mathcal{R}o_{ca} \text{PEA}_{n+1,n}$ witness that $\text{CRA}_n$ is not elementary, since it is not completely representable (in fact it is not in $S_n \mathcal{N}t_n \text{CA}_{n+3}$), but is elementary equivalent to (countable atomic) algebra that is. From this we readily conclude that $\text{CRCA}_n$ is properly contained in $\text{LCA}_n$.

For a cylindric algebra atom structure $\mathfrak{F}$ the first order algebra over $\mathfrak{F}$ is the subalgebra of $C_m \mathfrak{F}$ consisting of all sets of atoms that are first order definable with parameters from $S$. $\text{FOCA}_n$ denotes the class of atomic such algebras of dimension $n$.

This class is strictly bigger than $\text{SRCA}_n$. Indeed, let $\mathfrak{A}$ be any the rainbow term algebra obtained by blowing up and blurring the finite rainbow algebra $\text{CA}_{n+1,n}$ proving that $S_n \mathcal{N}t_n \text{CA}_{n+3}$ is not atom canonical. This algebra was defined using first order formulas in the rainbow signature (the latter is first order since we had only finitely many greens). Though the usual semantics was perturbed, first order logic did not see the relativization, only infinitary formulas saw it, and that’s why the complex algebra could not be represented. These examples all show that $\text{SRCA}_n$ is properly contained in $\text{FOCA}_n$. Another way to view this is to notice that $\text{FOCA}_n$ is elementary, that $\text{SRCA}_n \subseteq \text{FOCA}_n$, but $\text{SRCA}_n$ is not elementary.

Last inclusion follows from the following RA to CA adaptation of an example of Hirsch and Hodkinson which we use to show that $\text{FOCA}_n \subset \text{WRCA}_n$. This is not at all obvious because they are both elementary.

**Example 6.15.** Take an $\omega$ copy of the $n$ element graph with nodes $\{1, 2, \ldots, n\}$ and edges $1 \rightarrow 2 \rightarrow \ldots \rightarrow n$. Then of course $\chi(\Gamma) < \infty$. Now $\Gamma$ has an $n$
first order definable colouring. Since $\mathfrak{M}(\Gamma)$ as defined above and in \cite{12} top of p. 78] is not representable, then the algebra of first order definable sets is also not representable because $\Gamma$ is first order interpretable in $\rho(\Gamma)$, the atom structure constructed from $\Gamma$ as defined in \cite{12}. However, it can be shown that the term algebra is representable. (This is not so easy to prove).

**Theorem 6.16.** Let $n > 2$ be finite. Then we have the following inclusions (note that $At$ commutes with $UpUr$):

\[
\mathfrak{M}_n CA_\omega \cap At \subset UpUr\mathfrak{M}_n CA_\omega \cap At \\
\subset UpUrS_c \mathfrak{M}_n CA_\omega \cap At = UpUrCRA_n = LCA_n \subset SRCA_n \\
\subset UpSRCA_n = UrSRCA_n = UpUrSRCA_n \subseteq FOCA_n \\
\subset S\mathfrak{M}_n CA_\omega \cap At = WRCA_n = RCA_n \cap At.
\]

**Proof.** The majority of inclusions, and indeed their strictness, can be distilled without much difficulty from our previous work. It is known that $UpSRCA_n = UrSRCA_n$ \cite{11}. To show that the elementary closure of $CRA_n$ is $LCA_n$, let $A \in LCA_n$, then $\exists$ can win the $k$ rounded atomic game $G_k$ for all $k \in \omega$ on $AtA$. Using ultrapowers and an elementary chain argument one gets a countable (atomic) $C$ such that $C \equiv A$, and $\exists$ can win $G$, the $\omega$ rounded atomic game, on $AtC$, hence $C$ being countable is completely representable. It follows that $A \in URUp\{C\} \subseteq URUpCRA_n$, and we are done.

The first inclusion is witnessed by a slight modification of the algebra $B$ used in the proof of \cite{23} Theorem 5.1.4], showing that for any pair of ordinals $1 < n < m \cap \omega$, the class $\mathfrak{M}_n CA_m$ is not elementary.

In the constructed model $M$ \cite{23} lemma 5.1.3] on which (using the notation in op.cit), the two algebras $A$ and $B$ are based, one requires (the stronger) that the interpretation of the 3 ary relations symbols in the signature in $M$ are disjoint not only distinct as above. Atomicity of $B$ follows immediately, since its Boolean reduct is now a product of atomic algebras. For $n = 3$ these are denoted by $A_u$ except for one countable component $B_{id}$, $u \in ^33 \sim \{Id\}$, cf. \cite{23} p.113-114. Second inclusion follows from example 4.4. Third inclusion follows from theorem 3.11 and fourth inclusion follows from theorem 6.11. Last one follows from example 6.15.

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