Tumbling of asymmetric microrods in a microchannel flow

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We describe results of measurements of the orientational motion of glass microrods in a microchannel flow, following the orientational motion of particles with different shapes. We determine how the orientational dynamics depends on the shape of the particle and on its initial orientation. We find that the dynamics depends so sensitively on the degree to which axisymmetry is broken that it is difficult to find particles that are sufficiently axisymmetric so that they exhibit periodic tumbling (‘Jeffery orbits’).

I. INTRODUCTION

We study experimentally the orientational dynamics of neutrally buoyant non-axisymmetric particles suspended in a viscous shear flow. The orientational dynamics of a particle suspended in a viscous flow is determined by resistance tensors that relate the local flow velocity and its gradients to the torque acting on the particle 1,2. In the absence of inertial effects the equation of motion is given by the condition that the torque vanishes at every instant 3. For particles that possess three orthogonal mirror planes 4, particle shape enters the orientational equation of motion through two parameters, Λ and K. For an ellipsoid with half-axes a, b and c, for example, Λ = (λ^2 - 1) / (λ^2 + 1) and K = (κ^2 - 1) / (κ^2 + 1) with aspect ratios λ = a/c and κ = b/c.

The case K = 0 corresponds to an axisymmetric particle. For axisymmetric particles in a steady shear flow the equation of motion has been tested by several experiments (we give a brief account below). The case K ≠ 0 of non-axisymmetric particles, by contrast, has received little attention experimentally. This is surprising because it is known 5,6 that the orientational dynamics depends crucially on the parameter K, even for very small deviations from the axisymmetric limit. In this paper we present experimental observations of single-particle dynamics for particles that are axisymmetric, slightly non-axisymmetric, and substantially non-axisymmetric.

For K = 0 the orientational dynamics is exactly solvable 3. When |Λ| < 1 (so that 0 < λ < ∞) there are infinitely many degenerate periodic orbits, the so-called ‘Jeffery orbits’. This is a consequence of the fact that the dynamical system has a conserved quantity, usually called the ‘orbit constant’. The value of the orbit constant is determined by the initial orientation of the particle. The orientational motion of an axisymmetric particle in a simple shear is sometimes referred to as ‘tumbling’, the particle spends a long time aligned with the flow direction, and it periodically changes orientation by 180 degrees. Different Jeffery orbits differ in the functional form of these periodic ‘flips’.

The degeneracy for K = 0 is particular to the simple shear flow, and it means that small perturbations can have a large effect. Inertial forces, for example, are neglected in Jeffery’s theory. These forces induce ‘orbit drift’ into a final stable orbit. This was already suggested by Jeffery 3, and was discussed in many experimental papers starting with Taylor 7. See for instance Ref. 8.

Small particles may be affected by thermal noise so that the orbit constant performs a random walk giving rise to a statistical distribution of orientations. This mechanism forms the theoretical basis for understanding the rheology of dilute suspensions 9,10.

A third possibility, the topic of this work, is that the particle is not perfectly axisymmetric. This leads to a more complicated orientational equation of motion, also derived by Jeffery 3. Some numerical examples of its solutions were reported by Gierszewski and Chaffey 11 who found that the motion of a non-axisymmetric particle in a simple shear flow is qualitatively different from that of an axisymmetric particle. Hinch and Leal 5 analysed the structure of the solutions to the equation of motion. They found that for short times...

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a nearly axisymmetric ellipsoidal particle approximately follows a Jeffery orbit, but on longer time scales
the ‘orbit constant’ does not remain constant. It oscillates, giving rise to ‘doubly periodic’ tumbling: time
series of the components of the unit vector aligned with the major axis of the particle show two distinct
periods. Subsequently Yarin et al. inferred from numerical experiments and analytical calculations that
ellipsoidal particles may tumble periodically, quasi-periodically, or in a chaotic fashion – depending on
the particle shape and on its initial orientation. The term ‘quasi-periodic’ refers to doubly periodic motion
with incommensurable periods.

In this paper we describe and analyse experimental results obtained by measuring the orientational motion
of a micron-sized particle suspended in a pressure-driven microchannel flow. Our analysis demonstrates that
the tumbling may indeed be periodic, doubly periodic, or possibly chaotic, depending on particle shape and
initial orientation. We exclude effects of inertia or thermal noise by observing the invariance of the dynamics
under pressure and time inversion. An optical trap allows us to manipulate the same particle into different
initial orientations. Our results are in good qualitative agreement with theoretical predictions.

In the wider context of this work that considers the orientational motion of small particles in a viscous shear
flow. This is a special but important case. It is of theoretical interest because of its degeneracy and
sensitivity to small perturbations, and it is of practical interest because it fundamentally relates to theories
and experiments concerning the rheology of dilute suspensions. Theories are commonly formulated in terms
of Jeffery’s equation. Recently there has been a surge of interest in describing the tumbling of non-
spherical particles in turbulent and other complex flows. Since it is difficult to solve the coupled
particle-flow problem most theoretical and numerical studies rely on Jeffery’s equation as an approximation
to the orientational dynamics. Some exceptions are described in Refs. 22–24.

This article is organised as follows. In Section II we enumerate previous experimental efforts to validate
Jeffery’s equations. In Section III we describe the experimental setup. Section IV contains our experimental
results. These results are discussed in Section V, and we conclude in Section VI.

II. PREVIOUS EXPERIMENTAL WORK

In this section we give a brief account of earlier experiments observing the orientational dynamics of single
particles in shear flows.

Taylor immersed millimeter-sized aluminum spheroids in sodium silicate between two concentric rotating
cylinders approximately 10 mm apart. In his brief report he asserts that the tumbling of the spheroids is
in qualitative agreement with Jeffery’s predictions, but that the orientational dynamics drifts after many
(order of 100) particle rotations.

In a related study Eirich et al. observed the orientations of glass rods and silk fibres in a Taylor-Couette
device. The particles had diameters between 10 and 50 µm and aspect ratios between 5 and 100. No
quantitative data on the orientational dynamics was measured, but they observed that the particles tend to
align with the flow direction or with the vorticity direction.

Binder studied fibers of many different aspect ratios suspended in glycerine. Binder employed a similar
device with two concentric cylinders and found, as Taylor, that the orientational dynamics slowly drifts.

Mason and co-workers have studied the orientational dynamics of small particles in shear flows during
two decades. Initially Trevelyan and Mason used a setup of two concentric cylinders rotating in
opposite directions, making it possible to study a single particle over an extended period of time. The
gap between the cylinders was approximately 10 mm, the suspending liquid was white corn syrup, and the
particles were 9.5 µm diameter glass fibers cut to different lengths. By observing the particle orientation
in a plane orthogonal to vorticity, Trevelyan and Mason found fairly good quantitative agreement of
their experimental results with Jeffery’s theory for one particle rotation (Fig. 7 in Ref. 27). However,
for longer time series (up to 30 revolutions) their results were inconclusive: sometimes orbit drift was
observed, sometimes not, and sometimes the change of orbit appeared seemingly erratic. In order to compare
quantitatively with Jeffery’s equations, valid for spheroidal particles, Trevelyan and Mason fitted the value
of λ to measurements of the tumbling period, yielding in their words an ‘effective aspect ratio’ for cylinders.
Bretherton later showed that this procedure is consistent. Mason and Manley extended the experiment to
hundreds of particle rotations (Fig. 2 in Ref. 28), but the observed orbit drift was apparently erratic. Mason
and Manley mention convective currents as a possible cause for the observed drift, but no conclusions could
be drawn concerning the single-particle dynamics. Bartok and Mason used a similar device consisting of
concentric cylinders, with a camera-equipped microscope observing along the vorticity axis, allowing to
very precisely measure the tumbling behavior of high aspect ratio (λ ≈ 45) acrylic (‘Orlon’) fibres. The
experimental results were found to agree quantitatively with Jeffery’s predictions for one particle rotation (Fig. 5 in Ref. 29). However, no data on the orbit drift, if any, was presented. Goldsmith and Mason performed the first quantitative measurements of the rotations of disks. They used the same coaxial-cylinder setup described above, with silicone oil for the suspending liquid. The disks were fabricated by heating and compressing polystyrene spheres. The disk diameters were 400-850 µm, and their aspect ratios ranged from λ = 1/20 to λ = 1/4. Goldsmith and Mason showed that Jeffery’s theory quantitatively predicts the orientational motion of axisymmetric disks, and that the orbit remained constant over 120 particle rotations (Table II in Ref. 30). In a sequel Goldsmith and Mason described measurements of the motion of particles suspended in a liquid flowing through a circular tube. The tube diameters were 2-8 mm, the flow was pressure-driven by a syringe pump, and observations were recorded through a microscope traveling along the tube. Measurements were made on an assortment of particles of sizes ~0.1 mm with aspect ratios ranging from λ = 1/20 (disks) to λ = 100 (fibres). Goldsmith and Mason concluded that the dynamics along Jeffery orbits compares well with theory for short times (Fig. 5 in Ref. 31). However, they found measurable orbit drift after a single particle rotation, for both a rod and a disk, see Fig. 7 in the same paper. The drift was attributed to particle asymmetry. Anczurowski and Mason fabricated prolate spheroidal particles by polymerising an electrostatically deformed droplet, and showed that Jeffery’s theory holds quantitatively for one particle rotation given the true aspect ratio λ. They used the same concentric-cylinder device described above.

Harris et al. performed experimental measurements on non-axisymmetric particles (K ≠ 0). They used an apparatus with counter-rotating cylinders with a gap of 27 mm, which was filled with a glucose solution. The particles were machined from a composite material (‘Tufnol’) into cuboids of 1.75 mm×1.28 mm cross-section and 2.5-9.5 mm in length. They measured the unknown elements of the resistance tensors by observing simple rotations around each of the principal axes. With these numerical values of the resistance tensors they compared orientational trajectories of the cuboids to numerical solutions of Jeffery’s equations and found reasonable quantitative agreement for two particle revolutions (Fig. 9 in their paper).

Stover and Cohen investigated the effect of a wall on fibre motion using a pressure-driven flow of corn syrup through a rectangular channel. The fibres had cross-sectional diameters of 50 µm and lengths 600 µm, resulting in an aspect ratio of 12. They found that the orientational dynamics are in quantitative agreement with Jeffery’s theory for two particle rotations when the particle is at least one particle length away from the wall.

Kaya and Kosser observed E. coli bacteria advected in a microfluidic channel of rectangular cross section. They found that the orientational motion of the bacteria approximately follows Jeffery orbits.

Einarsson et al. described examples of orientational trajectories of polymer microrods in a microchannel. They found that the trajectories of some particles were periodic admitting comparison to Jeffery’s theory. Other trajectories were seen to be aperiodic, perhaps chaotic. It was argued that the aperiodicity is due the fact that the particles are not axisymmetric.

III. METHODS

Fig. 1(a) shows a schematic drawing of the experimental setup to observe the orientational motion of small particles advected in a microchannel flow. A dilute suspension of microrods in a density-matched fluid is introduced into the microchannel. A syringe pump (a standard Harvard Apparatus infuse/withdraw model) is used to drive the flow of the particles. The system is placed under an inverted microscope equipped with a motorised translation stage. A CCD camera is used to register the orientation of a given particle. The initial orientation and position of the particle in the channel is set using an optical trap.

Fig. 1(b) shows the coordinate system that is used in this paper. The x-axis is parallel to the channel length along the flow direction. The y-axis is directed along the depth of the channel. The z-axis is also the optical axis of the microscope objective. The z-axis points along the channel width. The orientation of the particle is determined by the unit vector $\mathbf{n}$ along the major axis of the particle.

The experiment is performed with cylindrical glass rods with diameters 3 µm ± 0.01 µm (PF-30S, Nippon Electric Glass Co., Ltd). The microrods were manufactured as spacers in liquid-crystal devices (PF-30S, Nippon Electric Glass Co., Ltd). This requires precise diameters. The lengths of the rods vary between approximately 10 µm and 30 µm. An electron-microscope image of the particles is shown in Fig. 2(a). This Figure shows that the end surfaces of the cylindrical rods are randomly inclined and uneven, indicating that the particles were obtained by breaking longer glass rods. While this is irrelevant for the intended industrial use as spacers, it is important for our application. Random inclinations of the end surfaces break
FIG. 1. (a) Schematic picture of the experimental setup, elements are not drawn to scale. $L_{1,2}$: lenses in a Keplerian telescope configuration. $M_{1,2,3}$: mirrors. $D_{1,2}$: diaphragms. DM: dichroic mirror reflecting the laser beam vertically towards the microfluidic system. The microscopic objective focuses the beam on the sample plane. Illumination is provided from the top. (b) Coordinate system in the lab frame spanned by orthogonal unit vectors $\hat{x}$, $\hat{y}$, and $\hat{z}$. The $x$-axis (flow direction) is directed along the channel length, the $y$-axis along the depth of the channel, and the $z$-axis along the channel width. The unit vector $\mathbf{n}$ points along the major axis of the particle. The polar angle between $\mathbf{n}$ and the $z$-axis is denoted by $\theta$. The particles are kept roughly equally far away from the side walls of the channel. Since the channel is much wider than deep this means that the $\hat{y}$-direction is the shear direction, so that the local linearisation of the flow-velocity obeys $u_x = sy$ in the frame co-moving with the centre-of-mass of the particle, where $s$ is the shear rate.

FIG. 2. (a) Electron-microscope image of the glass particles used in the experiment. Taken by Stefan Gustafsson, Chalmers. (b) Optical-microscope image of double particles.

axisymmetry: sometimes only very slightly [lower left particle in Fig. 2(a)], sometimes more [c.f. particle in the centre of Fig. 2(a)]. We investigate the orientational dynamics of highly asymmetrical particles by studying aggregates of glass rods. Following Lewandowski et al.\textsuperscript{36} a dilute suspension of microrods is left to evaporate in order to produce double particles, Fig. 2(b). The glass particles have an index of refraction of 1.56 and a density of $\rho_p = 2.56 \text{ g cm}^{-3}$.

To achieve neutral buoyancy the fluid must have the same density as the particles. This is achieved by mixing 22.2\%wt water, 4.4\%wt glycerol and 73.4\%wt sodium metatungstate monohydrate (Alfa Aesar GmbH). Density matching usually requires further titration until the particle is neutrally buoyant. The suspension is highly diluted in order to avoid particle-particle interactions. The mixture has a viscosity of $\mu = 25 \text{ mPa s}$ at $20^\circ\text{C}$. We estimate the shear Reynolds number $\text{Re}_s = \rho_f s a^2 / \mu$ as follows. The particle length $a$ is of the order of 20$\mu$m. The shear rate $s$ is determined by the flow speed $u_x$ which is in turn given by the flow rate 8 $\mu$l/min and the cross section of the channel, $2.5 \text{ mm} \times 200 \mu$m. This results in $u \approx 0.3\text{mm s}^{-1}$ and a shear rate of $s \approx 3s^{-1}$ at 60$\mu$m depth assuming a parabolic profile in the $y$-direction, and that the channel is much wider than deep. The density of the fluid $\rho_f$ equals the particle density, $\rho_p = 2.56 \text{ g cm}^{-3}$. This gives $\text{Re}_s \approx 10^{-4}$. Inertial effects are thus negligible on the time-scale of the experiment.

The microchannel is produced using standard soft lithography. The process begins by milling a rectangular moulding form in aluminium. The surfaces of the mould are mechanically polished. A 10:1 mixture of polydimethylsiloxane (PDMS) and sylgard 184 Dow Corning (Sigma Aldrich) is poured on the moulding form and allowed to cure for several hours. The PDMS replica obtained after peeling off the mould is sealed to a glass slide (thickness 0.17 mm) by oxygen plasma bonding. This results in a rectangular channel that is 40 mm long, 2.5 mm wide, and 150 $\mu$m deep. For some measurements, a channel with a depth of 200 $\mu$m was used, see Table I. The suspension is injected into the channel using thin tubing connected to a syringe pump.
PTFE tubes (Cole-Parmer) with outer diameters of 0.76 or 1.07 mm are used, the former in connection with a plastic syringe (1 ml, Terumo), the latter in connection with a glass syringe (500 µl, Hamilton).

The optical system is built around a Nikon X60 microscope objective (NA 1.0, WD 2 mm). The particle motion is recorded with a CCD video camera (Leica). The channel is mounted on a translation stage that moves the microchannel over the fixed observation microscopic objective. The stage is driven by a stepper motor that records the position of the stage. By moving the channel a given particle is kept within the field of view of the objective, despite the fact that the particle is advected by the fluid through the microchannel.

A single-beam optical trap is used to set the initial orientation and position of a given particle. The optical trap makes use of the microscope objective (Fig. 1), it provides sufficient magnification to not only visualise the particle, but also to trap it with a continuous infrared laser of wavelength 1075 nm (10 W, IPG-Laser GmbH). The most efficient way of trapping a glass rod with this setup is to direct the laser beam towards one of the two ends of the particle. Different orientations can be imposed on the particle by moving the channel sufficiently quickly to cause one end of the particle to leave the trap, yet sufficiently slowly so that the other end of the particle is kept trapped.

We use the image-analysis algorithm employed by Einarsson et al.\textsuperscript{23}. Images are acquired at a rate of 100 frames per second. Each frame is stored as 8-bit gray-scale bitmap with 692 × 520 pixels. The pixel size is 0.21µm. For a given frame the image analysis proceeds in three basic steps. First static noise is reduced by subtracting the time-averaged intensity from each frame. Then the boundary of the projection of the particle into the image plane is detected, and finally an ellipse is fitted to the boundary. Details are given in Ref. 23. The output defines the position and the orientation of the particle in the image. The centre-of-mass coordinates of the particle in the laboratory frame are determined using the output from the stepper motor recording the position of the stage.

A typical experiment starts by capturing a particle with the optical trap. The particle is brought into the desired location in the $x$-$z$-plane by moving the channel. All particles are started close to one of the inlets at approximately equal $z$-distances to both side walls. We verified that the $z$-position remains centred, with an error typically at most one particle length. This implies that the shear in the $z$-direction remains very small (the channel is much wider than deep). The $y$-coordinate thus corresponds to the shear direction, and the $z$-coordinate is the vorticity direction. We strive to always position the centre-of-mass of the particle at the same initial depth, of about 60 µm. But it is difficult to estimate the depth precisely because it is not directly observed. We obtain a rough estimate by changing the focal depth of and determining when the particle appears to be in focus. The particle is brought to the desired initial orientation as described above and then released to follow the flow in the microchannel. We then invert the pressure gradient so that the particle is advected back in the opposite $x$-direction. For each orientational trajectory we record both forward and backward dynamics. Since Stokes’ equation is invariant under simultaneous pressure inversion and time reversal, the backward dynamics must exactly retrace the forward dynamics unless irreversible effects such as inertia or thermal noise affect the dynamics. Examples for the resulting video-microscopy recordings of the orientational dynamics in the $x$-$z$-plane are given in the Supplementary Online Material.

For a given particle this procedure is repeated many times to obtain orientational trajectories with different initial orientations. We record the length of the projection of the particle into the $x$-$z$-plane as a function of time. We estimate the particle length using the procedure described in Ref. 23. Once the particle length is known we can extract the components of the unit vector $\mathbf{n}$ determining the orientation of the particle, as a function of time.

We plot the orientation not as a function of time but as a function of distance that the centre-of-mass of the particle has traveled through the channel, advected by the flow:

$$x(t) = \int_0^t dt' u_x(t'). \tag{1}$$

Here $u_x(t)$ is the instantaneous flow velocity at time $t$. This transformation simplifies the analysis because it accounts for the fact that the shear-rate is time-dependent: the flow velocity changes when the pressure is reversed, and in order to avoid inertial effects these reversals must be performed slowly. The invariance of Stokes equation under time and pressure reversal implies $x(t) = -x(-t)$. We overlay forward and backwards dynamics by plotting the orientation as a function of centre-of-mass position.
FIG. 3. Orientational dynamics of particle 1. (a) Dynamics of $n_z$. Here $n_z^{(i)}$ denote the values of $n_z$ at subsequent zero crossings of $n_z$, $i = 1, 2, 3, \ldots$. The data are taken from panels (b-f). Red ◦ data from panel (b); blue □ data from (c); green △ data from (d); black + data from (e); magenta ⋆ data from (f). Panels (b-f) show orientational dynamics as a function of c.o.m.-position $x$ in the channel, Eq. (1). Data in different panels correspond to different initial orientations. Solid blue and dashed red lines represent forward and backward trajectories, respectively. The flow direction is reversed at $x = 0$. The horizontal arrow in panel (b) indicates the period $X_p[\text{mm}]$ of the trajectory.

FIG. 4. Orientational dynamics of particle 2. See caption of Fig. 3 for details.
IV. EXPERIMENTAL RESULTS

Figures 3, 4, and 5 show orientational dynamics of three different particles. For each particle five different orientational trajectories are shown, corresponding to different initial orientations [panels (b) to (f)]. In all three Figures we show trajectories of the $x$- and $z$-components of the unit vector $\mathbf{n}$ that points along the major axis of the particle. Here $n_x$ is the component in the flow direction, and $n_z$ is the component in the vorticity direction. The third component of $\mathbf{n}$ is determined by normalisation, $|\mathbf{n}| = 1$.

In panel (a) of each Figure we summarise the orientational dynamics by recording the values of $n_z$ whenever $n_x = 0$. We denote the resulting sequence of consecutive $n_z$-values by $n_z^{(i)}$, $i = 1, 2, 3, \ldots$. For an axisymmetric particle Jeffery’s equation predicts that $n_z^{(i+1)} = n_z^{(i)}$, shown as the solid line along the diagonal in panel (a).

As mentioned above, the particle is first advected along a stream line of the pressure-driven flow in the channel. We then invert the pressure gradient so that the centre-of-mass of the particle is advected back to where it came from. For each orientational trajectory we show the ‘forward dynamics’ (blue solid line), going from right-to-left in the Figure. After the reversal follows the ‘backward dynamics’ (red dashed line). Since Stokes’ equation is invariant under simultaneous pressure inversion and time reversal, the backward orientational dynamics must exactly retrace the forward dynamics unless irreversible effects due to inertia or thermal noise affect the dynamics noticeably on the time-scale of the experiment.

Consider first the trajectories shown in Fig. 3, corresponding to particle 1. Panels (b) to (f) show orientational trajectories of $n_x$ and $n_z$ for different initial orientations. In all cases the backward dynamics retraces the forward dynamics very well. This shows that neither inertial forces nor rotational diffusion affect the orientational dynamics. We attribute the small dephasing visible in each panel to a small density mismatch causing the particle to sink (or float), changing the shear rate it experiences. Apart from this slight dephasing all orientational trajectories are fairly periodic. For a given $n_x$-trajectory the relative variation of the centre-of-mass distance between two consecutive $n_x = 0$-events (‘half-period’) is of the order of 10%. Between different panels we observe substantial variations in the period $X_p$, though, between 2.1 mm and 2.6 mm. This is probably due to the fact that the initial depth is difficult to reproduce precisely after changing the initial orientation. Panel (a) indicates that $n_z^{(i+1)}$ is approximately equal to $n_z^{(i)}$.

Fig. 4 shows the orientational dynamics of particle 2 for different initial orientations [panels (b) to (f)]. In all cases the backward dynamics retraces the forward dynamics well, at least for a few millimetres. As for particle 1 the trajectories show a slight dephasing within each panel. Different panels show quite different periods $X_p$, ranging between 1.7 mm and 5.9 mm. In panels (d) and (f) we see that the amplitude of $n_z$...
Table I. Description of particles and other experimental parameters. *Single or double particle. \( ^b \)Particle length as extracted from image-analysis algorithm, see Section III. We estimate the error to be of the order of 1\( \mu m \). \( ^c \)These Figures are in the Supplementary Online Material.

V. DISCUSSION

The results summarised in Figs. 3 to 5 show the orientational dynamics of single and double glass rods. In general the particles are not perfectly axisymmetric (as seen in Fig. 2), and therefore do not follow perfect Jeffery orbits. In this section we relate our experimental results to the theoretical predictions valid for ellipsoidal particles\(^3,5 \). The particles in our experiment do not satisfy the mirror symmetries assumed in this theory, but it is plausible that the effects of breaking axisymmetry predicted by this theory apply at least qualitatively to our glass rods.

The orientational dynamics of an ellipsoidal particle\(^3,5 \) can be cast in the form\(^{37} \)

\[
\dot{n} = \mathcal{D} n + \Lambda (S n - (n \cdot S n) n) + \frac{K(1 - \Lambda^2)}{K \Lambda - 1} (n \cdot S p) p ,
\]

\(2a\)

\[
\dot{p} = \mathcal{D} p + K (S p - (p \cdot S p) p) + \frac{\Lambda(1 - K^2)}{K \Lambda - 1} (n \cdot S p) n.
\]

\(2b\)

Following the convention outlined in Section III, \( n \) is a unit vector that points along the major axis of the ellipsoidal particle. The unit vector \( p \) is orthogonal to \( n \), directed along the particle axis corresponding to the length \( b \) used in the definition of the aspect ratio \( \kappa \) (defined in the Introduction). The geometry of the ellipsoid is characterised by the two shape parameters \( \Lambda \) and \( K \) that are defined in the Introduction, Section I. The matrices \( S \) and \( \mathcal{D} \) are the symmetric and anti-symmetric parts of \( A \), the matrix of fluid-velocity gradients. In our case this matrix takes the form

\[
A = \begin{bmatrix}
0 & s & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]
where $s$ is the shear strength (Fig. 1).

Eqs. (2) are symmetric under the simultaneous exchange of $n$ and $p$ as well as $\Lambda$ and $K$: describing the motion of the same particle using a different coordinate system within the particle must result in the same dynamics. Note also that the non-linear coupling between $n$ and $p$ involves the strain $S$ only. This coupling maintains orthogonality of the two vectors $n$ and $p$. The anti-symmetric part $\mathbb{O}$ just causes a solid-body rotation. For axisymmetric particles, $K = 0$, so that the tumbling of $n$ becomes independent of the dynamics of $p$ (but not vice versa). The resulting equation for $n$ has infinitely many degenerate periodic solutions, the Jeffery orbits. The dynamics of $p$ describes how the particle spins around its symmetry axis.

When $K \neq 0$ (and $\Lambda \neq 0$), no general closed-form solutions of Eqs. (2) are known. It is convenient to represent the numerical solutions of Eq. (2) in terms of a Poincaré surface-of-section, recording the locations at which the dynamics intersects a surface in the phase space of Eq. (2). This section is constructed as follows. Hinch and Leal have shown that the vector $n$ rotates around the vorticity at a positive angular velocity so that one can reduce the dimensionality of the problem by parametrising the orientational dynamics in terms of the corresponding angle. A suitable condition defining the surface-of-section is that $n$ is perpendicular to the flow direction, $n_x = 0$. Following Yarin et al. we take the coordinates in the surface-of-section to be $\psi$ and $n_z$, where $\psi$ is the Euler angle parametrising the spin of the particle around the axis $n$, and $n_z$ is the $z$-component of $n$, the cosine of the angle $\theta$ between $n$ and vorticity (Fig. 1). When $n_x = 0$ we record the coordinates $(\psi, n_z)$. To define the Euler angles $(\theta, \phi, \psi)$ we use the convention of Goldstein and express $n$ and $p$ as

$$
n = \begin{bmatrix}
\sin \theta \sin \phi \\
-\sin \theta \cos \phi \\
\cos \theta
\end{bmatrix}, \quad \text{and} \quad p = \begin{bmatrix}
-\sin \psi \cos \phi - \cos \theta \sin \phi \cos \psi \\
-\sin \psi \sin \phi + \cos \theta \cos \phi \cos \psi \\
\cos \psi \sin \theta
\end{bmatrix}.
$$

(4)

So $\theta$ is the polar angle depicted in Fig. 1. The angle $\phi$ is referred to as the ‘precession angle’. This angle measures the direction of the projection of $n$ into the flow-shear plane. Eqs. (2) correspond to Eqs. (2.1) and (2.2) in Ref. 6, setting $s = -1$ and defining the aspect ratios $\lambda$ and $\kappa$ in terms of the particle axes as follows: $a_x = \lambda a_z$ and $a_y = \kappa a_z$.

In the experimental time series shown in Figs. 3 to 5, instances where $n_x = 0$ correspond to peaks in the oscillation in $n_z$. The $n_z$-coordinate in the surface-of-section is therefore easily read off from the experimentally observed time series. The angle $\psi$, by contrast, cannot be measured in our experiment because we cannot track how the particles spin around $n$.

Four different surfaces-of-section are shown in Fig. 6, obtained by numerical integration of Eqs. (2) for a large number of different initial orientations, and plotting the sequence $[\psi^{(i)}, n_z^{(i)}]$ of $(\psi, n_z)$ evaluated at consecutive zero crossings of $n_z$, labeled by $i = 1, 2, 3, \ldots$. Similar sections are found in Ref. 6. The map that gives $[\psi^{(i+1)}, n_z^{(i+1)}]$ in terms of $[\psi^{(i)}, n_z^{(i)}]$ is called the Poincaré map.

Fig. 6(a) depicts the orientational dynamics of an axisymmetric particle, $K = 0$. The coordinate $n_z$ is a conserved quantity on the surface-of-section, Jeffery orbits appear as horizontal lines in Fig. 6(a), one-parameter families parametrised by $\psi$. In the literature Jeffery orbits are commonly identified by their orbit constant $C$. It is given by the value of $\tan \theta$ at $n_x = 0$ (see for example Eq. (3) in Ref. 5). In this paper we characterise Jeffery orbits by $n_z = \cos \theta = 1/(1 + C^2)$ on the surface-of-section ($n_x = 0$). Fig. 6(a) illustrates that the orientational dynamics depends on the initial orientation, determining the value of $n_z$. We remark that the periods of $n$ and $\psi$ are not in general commensurate for $K = 0$. But the tumbling of $n$ is independent of that of the angle $\psi$ and thus periodic for axisymmetric particles.

Fig. 6(b) shows the orientational dynamics of a weakly asymmetric ellipsoidal particle ($K \approx 0.095$ and $\Lambda = 12/13$). We see that Jeffery orbits with $n_z \approx 1$ remain almost unchanged. But there are substantial changes for smaller values of $|n_z|$, compared with the surface-of-section for $K = 0$. We see that $n_z$ ceases to be a constant of motion, doubly-periodic orientational dynamics results. The most substantial changes occur near $n_z = 0$. The $n_z = 0$-Jeffery orbit is replaced by two fixed points at $(0, 0)$ and $(\pm \pi/2, 0)$ on the surface-of-section.

This fact and the surface-of-section patterns in the vicinity of these points follow from the time-reversal symmetry of Eqs. (2). The general principle is explained in Section 6.6 of Ref. 38. See also Ref. 40. The invariance of Stokes equation referred to in Section 3.1 implies that Eqs. (2) are invariant under

$$
t \to -t, \quad n_x \to -n_x, \quad \text{and} \quad p_x \to -p_x.
$$

(5)

The fixed point $(0, 0)$ is mapped onto itself under this transformation. It follows that the dynamics in its immediate neighbourhood can neither be expanding nor contracting. In other words the determinant
describing the linearised motion in the vicinity of this fixed point,

\[
\text{det} \left[ \frac{\partial \psi^{(i+1)}}{\partial n^{(i+1)}} \frac{\partial \psi^{(i+1)}}{\partial n^{(i+1)}} \right],
\]

must be of unit modulus despite the fact that the dynamics (2) is dissipative. We find numerically that the eigenvalues are \(\exp(\pm i \omega)\). The point \((0,0)\) is thus an elliptic fixed point, surrounded by a one-parameter family of closed orbits that appear as concentric closed curves, much like so-called ‘tori’ in so-called ‘Hamiltonian’ systems with area-preserving phase-space dynamics\(^{41}\). For near-axisymmetric particles Hinch and Leaf\(^5\) analysed the corresponding orbits by multiple-scales analysis. As these orbits rotate around the elliptic point, the value of \(n_z\) changes sign. This doubly-periodic motion may be quasi-periodic or periodic, depending on whether the two frequencies are incommensurate or not (corresponding to irrational or rational winding numbers, respectively).

The point \((\pm \pi/2,0)\) is a hyperbolic fixed point (real and opposite eigenvalues, \(\gamma\) and \(\gamma^{-1}\)). The fact that the \(n_z = 0\)-orbit is destroyed upon infinitesimal perturbation and replaced by a discrete set of fixed points of alternating stability is typical for Hamiltonian systems\(^{41}\). The mechanism in our dissipative system is analogous, a consequence of the symmetry (5).

For larger asymmetries chaotic orientational dynamics occurs, seen as a region with a stochastic scatter of points in Fig. 6(c). Almost all Jeffery orbits are modified, only those with \(n_z\) close to \(\pm 1\) remain. Whether the orientational dynamics is periodic, quasi-periodic, or chaotic depends upon the initial condition on the surface-of-section.

Fig. 6(d) shows the orientational dynamics for ellipsoidal particles with \(K = 3/5\) and \(\Lambda = 12/13\). This value of \(K\) is similar the corresponding value for the double particle 3 (Figs. 2(b) and 5), but this particle has \(\Lambda \approx 40/41\). We do not seek quantitative correspondence between these parameters because the particles used in our experiment are not ellipsoidal. This prevents us from drawing quantitative conclusions, but still allows for a qualitative comparison between theory and experiment. The orientational dynamics displayed in Fig. 6(d) is either on tori or chaotic. Motion on tori can occur with large amplitudes, so that \(n_z\) changes from \(n_z \approx -1\) to \(n_z \approx 1\).

How are these observations reflected in the experimental time series shown in Figs. 3 to 5? Since we cannot measure the angle \(\psi\) in our experiments, we concentrate on the dynamics of \(n_z\). Fig. 7 shows the range of changes of \(n_z\) in one iteration of the Poincaré map for particles with different degrees of asymmetry, determined by numerically recording the changes \(n_z^{(i+1)} - n_z^{(i)}\) along orientational trajectories. The diagonal corresponds to symmetric particles, where \(n_z^{(i+1)} = n_z^{(i)}\). The larger the asymmetry, the larger is the range of \(n_z^{(i+1)} - n_z^{(i)}\) that may occur, reflecting the properties of the surfaces-of-section shown in Fig. 6.

Fig. 7 can be directly compared with Figs. 3(a) to 5(a). Our experimental results show that the Poincaré map may scatter significantly around the diagonal. The above discussion explains that this is a consequence of doubly-periodic and possibly chaotic orientational dynamics of asymmetric particles. The range of scatter differs between different particles, a consequence of different degrees of asymmetry. We see that the scatter is largest for the double particle 3, with \(K \approx 3/5\) [Fig. 5(a)]. Particle 1, by contrast, shows only negligible scatter around the diagonal [Fig. 3(a)]. We infer that this particle is highly symmetric, \(K\) very small. The data shown in Fig. 3 are consistent with the conclusion that particle 1 follows Jeffery orbits, \(n_z\) remains approximately constant in the surface-of-section. But we remark that the shape of the \(n_z \approx 0\)-orbit does not look like a Jeffery orbit for an axisymmetric particle. We cannot exclude that this is due to possible non-ellipsoidal deviations from axisymmetry. The surface-of-section dynamics is most sensitive to such shape perturbations near \(n_z = 0\). A more likely explanation is that the shape of the trajectory is a consequence of systematic (and reproducible) tracking errors due to diffraction and finite pixel size.

Particle 2 is also a single glass rod, but it shows a somewhat larger range of scatter around the diagonal [Fig. 4(a)]. We attribute this to a more substantial breaking of axisymmetry at the tips of the particle [as seen for instance in Fig. 2(a)]. Particle 2 shows fairly periodic motion for \(n_z \approx 1\) and distinctly doubly-periodic orientational dynamics for small values of \(n_z\). This confirms that the initial orientation determines whether the particle tumbles periodically or aperiodically. Note also that \(n_z\) changes sign along the trajectories that remain near \(n_z = 0\) [Fig. 4(b), (d), and (f)]. All of these observations are qualitatively consistent with the surfaces-of-section shown in Fig. 6(b) and 6(c).

For particle 3 the value of \(n_z\) changes sign for all orientational trajectories shown. This is consistent with the fact that the Fig. 6(d) shows predominantly this type of motion. Fig. 5(e) is consistent with chaotic orientational motion in the stochastic layer around the elliptic island. The surfaces-of-section in Fig. 6 show that there are two types of doubly-periodic motion: either \(n_z\) has the same sign, or its sign...
FIG. 6. Poincaré surfaces-of-section for $\Lambda = 12/13$ and different values of $\kappa$ [$K = (\kappa^2 - 1)/(\kappa^2 + 1)$]. The angle $\psi$ is defined up to $(\psi + \pi/2) \mod \pi$.

FIG. 7. Range of Poincaré map for $\Lambda = 12/13$ and different values of $\kappa$ corresponding to the values used in Fig. 6, $K = (\kappa^2 - 1)/(\kappa^2 + 1)$. The data shown was obtained by sampling $\psi$ uniformly over the surface-of-section for a given initial value of $n_z$. 
changes periodically. The trajectory in Fig. 5(e) exhibits both behaviours, indicating chaotic dynamics. Now consider the trajectory shown in Fig. 5(c). It is not periodic or doubly-periodic, but the reversal works well (at least initially). But we cannot conclude that the dynamics is consistent with surfaces-of-section discussed above. The \( n_z \)-values at \( n_z = 0 \) in Fig. 5(c) change from approximately 0.25 to 0.75 in modulus. This could be the result of uncontrolled external factors. Explaining this behaviour in terms of chaotic dynamics on the surfaces-of-section requires a large stochastic region, larger than the one shown in Fig. 6(d). This might mean that the trajectory shown in panel (d) is a doubly-periodic piece of a chaotic trajectory that may show different behaviours at larger times. But to determine whether these behaviours can be explained by chaotic dynamics would require to derive and numerically integrate the orientational equations of motion for the precise shape of the particle, and to experimentally record the angle \( \psi \). Since the current experimental setup does not allow to reliably extract how the angle \( \psi \) changes, we plan to perform experiments with small triangular platelets that will allow to record the angle \( \psi \).

The winding number of the trajectory shown in Fig. 5(d) is roughly 7 corresponding to seven \( n_x = 0 \)-crossings while the trajectory winds approximately once round the elliptic fixed point in the centre of the surface-of-section. The winding number of the trajectory in Fig. 6(e) is roughly 4 corresponding to four \( n_x = 0 \)-crossings while the surface winds once around the elliptic fixed point. These observations are in qualitative agreement with the fact that the winding numbers of tori winding around the elliptic fixed points increase as the distance from that point increases (corresponding to larger maximal values of \( n_z \)).

Additional results for seven more particles are shown in the Supplementary Online Material, Figs. S1 to S7. In general the results shown in these Figures support the observations and qualitative conclusions summarised above. But the trajectory shown in Fig. S7 is difficult to reconcile with the surfaces-of-section shown in Fig. 6. Fig. S7(e) shows an orbit near \( n_z = 0 \) where \( n_z \) appears not to change sign. But panels (d) and (f) in Fig. S7 show distinct sign changes, not consistent with the surfaces-of-section shown in Fig. 6.

It was pointed out in Section IV that the periods \( X_p \) observed in the orientational trajectories can differ substantially between different trajectories of the same particle. Consider for instance particle 1 (Fig. 3). For an axisymmetric ellipsoidal particle \( (K = 0) \) the Jeffery period (time units) is given by \( 3 \)

\[
T_p = \pi \frac{\lambda^2 + 1}{s\lambda}.
\]

For \( K = 0 \) this is twice the return time to the surface-of-section. For \( K \neq 0 \) the return time depends upon the starting position on the surface-of-section, but for small values of \( K \) the deviations from (7) are small, of the order of \( K \). For the nearly axisymmetric particle 1 we can use Eq. (7) to estimate the period \( X_p \) in Fig. 3. Using the parameters from Table I and assuming that the particle was located at a depth of 60 \( \mu \)m, Eq. (7) gives \( X_p \approx 2.3 \) mm. This is consistent with the range of periods observed in Fig. 3(b) to (f), namely 2.1 mm to 2.6 mm. We infer that the precision in determining the depth at which particle 1 moves through the channel is of the order of one particle length. The variations in periods observed in Figs. 4 and 5 indicate that the actual depths vary more between different panels than in Fig. 3.

VI. CONCLUSIONS

Theory and numerical simulations predict that the orientational dynamics of small neutrally buoyant particles in a shear flow is very sensitive to breaking of axisymmetry.\(^ {5,6,11} \) Axisymmetric particles tumble periodically on Jeffery orbits, but when the symmetry is broken, the Jeffery orbits are modified. Depending on the initial orientation, periodic, doubly-periodic, or chaotic tumbling may result.

In this paper we have reported on measurements of the orientational motion of small glass rods suspended in a micro-channel shear flow. Reverting the pressure-driven flow shows that the orientation retraces its trajectory. This means that neither inertial effects nor rotational diffusion matter. The glass rods have highly symmetric circular cross sections. We find that slight imperfections at the ends have substantial effects upon the tumbling dynamics. Our measurements show periodic and doubly-periodic tumbling, and that the nature of the orientational dynamics depends upon the initial orientation and on the degree to which axisymmetry is broken. We have also found orientational trajectories that are neither periodic nor quasi-periodic, that may correspond to chaotic dynamics. These results are in qualitative agreement with theory for small ellipsoidal particles, and demonstrate how sensitively the orientational dynamics depends on the breaking of axisymmetry.

In the future we plan to experimentally map out Poincaré surfaces-of-section (Fig. 6). With the present particles this is not possible because we cannot resolve how the particle spins around its major axis. We
therefore plan to perform corresponding experiments using micron-sized triangular platelets. We expect that it will be possible to resolve the tumbling and spinning dynamics of these particles using the methods described in this paper.

Finally we note that the shear flow is a special (yet important) case. It remains to be seen whether tri-axiality has significant effects on the orientational dynamics in other flows.

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Supplementaray online material

Tumbling of asymmetric microrods in a microchannel flow

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FIG. S1. Orientational dynamics of particle 4. (a) Dynamics of $n_z$. Here $n_z(i)$ denote the values of $n_z$ at subsequent zero crossings of $n_x$, $i = 1, 2, 3, \ldots$. The data are taken from panels (b-f). Red ○ data from panel (b); blue □ data from (c); green △ data from (d); black + data from (e); magenta ⋆ data from (f). Panels (b-f) show orientational dynamics as a function of c.o.m.-position $x$ in the channel, Eq. (1) in the main text. Data in different panels correspond to different initial orientations. Solid blue and dashed red lines represent forward and backward trajectories, respectively. The flow direction is reversed at $x = 0$.

FIG. S2. Orientational dynamics of particle 5.
FIG. S3. Orientational dynamics of particle 6.

FIG. S4. Orientational dynamics of particle 7.
FIG. S5. Orientational dynamics of particle 8.

FIG. S6. Orientational dynamics of particle 9.
FIG. S7. Orientational dynamics of particle 10.