The population dynamics of a trapped Bose-Einstein condensate, subject to the action of an oscillatory field, is studied. This field produces a modulation of the trapping potential with the frequency close to the transition frequency between the ground state and an excited energy level. Unusual critical effects are found exhibiting sharp qualitative changes in the population dynamics. An effective averaged system is constructed explicitly illustrating the occurrence of critical phenomena. The related critical indices are calculated.

Bose-Einstein condensates of trapped atoms present an interesting example of essentially nonlinear statistical systems with rich properties that are being intensively studied both experimentally and theoretically [1,2]. Although experiments deal with rather dilute Bose gases, their interaction, nevertheless, cannot be considered as a small perturbation. This is because in a system of condensed atoms coherence develops, resulting in an effective interaction proportional to the number of particles. The fact that Bose-Einstein condensation and coherence appear simultaneously can be naturally understood remembering that both these phenomena arise when the thermal wavelength of atoms exceeds the mean interatomic distance. All the measurements done so far show the evidence for coherence of Bose-Einstein condensates [1,2]. Essential nonlinearity of the latter makes their behaviour quite different from that of ensembles of weakly interacting gases.

In the present article we show that the nonlinearity in the Hamiltonian describing Bose condensates results in very unusual dynamical effects resembling critical phenomena in statistical systems. We consider the dynamics of population levels of a Bose system, when all atoms are condensed and the system can be described by the nonlinear Schrödinger equation, often called the Gross-Pitaevskii equation [1,2]. Assume that, in addition to a stationary trapping potential, there is a time-dependent potential, which we call the resonance field and whose meaning will be specified below. Thus, we consider the equation

\[ i\hbar \frac{\partial \phi}{\partial t} = \left[ \hat{H}(\phi) + V_{res} \right] \phi, \quad \hat{H}(\phi) = -\frac{\hbar^2}{2m_0} \nabla^2 + U(\vec{r}) + A|\phi|^2, \]

(1)

with the nonlinear Hamiltonian in which \( U \) is a trapping potential; \( A \equiv 4\pi\hbar^2a_sN/m_0; \) \( a_s \) is a scattering length; \( m_0 \) is mass; and \( N \) is the number of atoms. The resonance field has the form \( V_{res} = V(\vec{r}) \cos\omega t \). This can be treated as the modulation of the trapping potential.

Tuning the frequency \( \omega \) of the resonant field close to the transition frequency \( \omega_{mn} \equiv (E_m - E_n)/\hbar \) between the energy levels defined by the stationary problem \( \hat{H}(\phi_n)\phi_n = E_n\phi_n \), we may realize the corresponding interlevel transitions [3]. Since the nonlinear Schrödinger equation is an exact equation for the wave function of a coherent state [4], the stationary solutions \( \phi_n \) can be called \textit{coherent modes}. These should not be confused with collective excitations defined as small deviations from the ground-state and described by a linearized equation [1,2], although resonant transitions between different branches of collective excitations are also possible [5]. But the equation for coherent modes is principally nonlinear.
Suppose that at the initial time \( t = 0 \) the Bose gas was condensed, with all atoms being in the ground state \( \varphi(\vec{r}, 0) = \varphi_0(\vec{r}) \). After this, the resonance field is switched on, with a frequency \( \omega \) being in quasiresonance with the transition frequency \( \omega_j \) between the ground-state level and a chosen energy level \( j \). The quasiresonance condition is \( |\Delta \omega| \ll \omega_j \), where the detuning \( \Delta \omega \equiv \omega - \omega_j \). The solution of Eq. (1) can be presented as a sum \( \varphi(\vec{r}, t) = \sum_n c_n(t) \varphi_n(\vec{r}, t) \) over coherent modes \( \varphi_n(\vec{r}, t) = \varphi_n(\vec{r}) \exp(-iE_n t / \hbar) \). Substituting this expansion in Eq. (1), one gets a system of equations for the coefficients \( c_n(t) \). These equations can be simplified employing the quasiresonance condition. To this end, it is convenient to introduce the notation for the fractional level population \( n_i(t) \equiv |c_i(t)|^2 \), and for the interaction intensity \( \alpha \) and the transition amplitude \( \beta \), respectively,

\[
\alpha \equiv \frac{A}{\hbar} \int |\varphi_0(\vec{r})|^2 (2|\varphi_j(\vec{r})|^2 - |\varphi_0(\vec{r})|^2) \, d\vec{r}, \quad \beta \equiv \frac{1}{\hbar} \int \varphi_0^*(\vec{r}) V(\vec{r}) \varphi_j(\vec{r}) \, d\vec{r}.
\]

Then, in the quasiresonance approximation [3], we have the system of equations

\[
\frac{dc_0}{dt} = -i \alpha n_j c_0 - \frac{i}{2} \beta e^{i \Delta \omega t} c_j, \quad \frac{dc_j}{dt} = -i \alpha n_0 c_j - \frac{i}{2} \beta^* e^{-i \Delta \omega t} c_0
\]

for the dynamics of the ground-state coefficient \( c_0(t) \) and of the coefficient \( c_j(t) \) related to a chosen excited mode \( j \). According to the assumption that at the initial time all atoms are condensed in the ground state \( \varphi_0(\vec{r}) \), the initial conditions to Eqs. (3) are \( c_0(0) = 1 \) and \( c_j(0) = 0 \). The derivation of Eqs. (3) has been explained in detail in Ref. [3]. However, it is worth emphasizing here a couple of important points. Looking for a solution of Eq. (1) in the form of the sum \( \sum_n c_n \varphi_n \) does not require that the modes \( \varphi_n \) compose a complete set. Recall that the completeness of a basis is the possibility of expanding over it an arbitrary function from the considered space. Such an excessively restrictive property is of no need for us. What we need is just to present the sole function \( \varphi \) as a sum over modes, with the coefficients \( c_n \) to be defined from Eq. (1). In the theory of differential equations, this way of looking for a solution is called the method of substitution or the method of variation of parameters. This method is often used for solving nonlinear differential equations, e.g. for nonlinear optical waveguide equations [6,7] and for the nonlinear Schrödinger equation [8,9]. An exact orthogonality of the nonlinear modes is also not necessary, but, in general, it is sufficient that the modes \( \varphi_n \) be approximately orthogonal in the sense of the smallness of the scalar product \( \langle \varphi_m, \varphi_n \rangle \) for \( m \neq n \). In the case of the nonlinear Schrödinger equation, taking for coherent modes the variational wavefunctions of Ref. [3], it is easy to check, by direct calculations, that the maximal values of \( \langle \varphi_m, \varphi_n \rangle \) for different \( m \neq n \) are of order 0.1. Moreover, even this approximate orthogonality is not compulsory in the frame of the quasiresonance picture employed in Ref. [3]. For deriving Eqs. (3), it is sufficient that \( c_n(t) \) be considered as a slow function as compared to \( \exp(-iE_n t / \hbar) \). Then, integrating Eq. (1), one can make use of the exact orthonormality on average of functions \( \varphi_n \) in the sense of the equality \( \lim_{T \to \infty} \frac{1}{T} \int_0^T \left[ \int \varphi^*_m(\vec{r}, t) \varphi_n(\vec{r}, t) \, d\vec{r} \right] dt = \delta_{mn} \). Calculations of Ref. [3] show that \( c_n(t) \) can, really, be treated as slow, compared to \( \exp(-iE_n t / \hbar) \) since the variation rate of \( c_n \) is of order \( \alpha \) which is an order smaller than \( E_n \), that is, \( |dc_n / dt| \ll E_n \).

We solved the system of nonlinear equations (3) numerically, carefully analyzing the behaviour of the solutions for different parameters. This behaviour turned out to be surprisingly rich exhibiting unexpected critical effects. Analyzing Eqs. (3), it is convenient to make there a scaling, measuring time in units of \( \alpha^{-1} \) and introducing the dimensionless parameters \( b \equiv |\beta| / \alpha \) and \( \delta \equiv \Delta \omega / \alpha \). The dimensionless detuning is assumed to be always small, \( \delta \ll 1 \). When the dimensionless transition amplitude \( b \) is also small, the fractional populations oscillate according to the sine-squared law. When \( b \) increases, the amplitude of oscillations also increases. The overall behaviour continues to be normal unless we reach a critical value \( b_c \), when the dynamics of the system changes drastically. More generally, there exists a critical line connecting the parameters \( b \) and \( \delta \), so that \( b_c \equiv 0.5 - \delta \). In the vicinity of this line, the system dynamics experiences sharp changes, while the parameters are varied just a little. We illustrate this in Figs. 1–4, slightly varying the detuning and keeping \( b = 0.4999 \), so that we are close to the critical line. In Fig. 1, the detuning is zero, and the oscillations of the fractional populations are yet normal. Shifting the detuning to \( \delta = 0.0001 \) transforms the picture to that in Fig. 2, where the top of \( n_j(t) \) and the bottom of \( n_0(t) \) become flat, while the oscillation period is approximately doubled. Changing the detuning to \( \delta = 0.0001001 \) results in Fig. 3, where the period is again doubled and there appear the upward cusps of \( n_j(t) \) and the downward cusps of \( n_0(t) \). Increasing further the detuning to \( \delta = 0.00011 \) squeezes the oscillation period twice, as is shown in Fig. 4. The same phenomena occur when we cross the critical line \( b + \delta \equiv 0.5 \) at other values of parameters.

This unusual behaviour of the fractional populations is certainly due to the nonlinearity of the evolution equations (3). Systems of nonlinear differential equations, as is known, can possess qualitatively different solutions for parameters differing by infinitesimally small values. The transfer from one type of solutions to another type, in the theory of dynamical systems, is termed bifurcation. At a bifurcation point, dynamical system is structurally unstable. Bifurcations in dynamical systems are somewhat analogous to phase transitions and critical phenomena in equilibrium statistical systems. To elucidate this analogy for the present case, we have to consider the time-averaged behaviour
of the system, which can be done as follows. First, we need to define an effective Hamiltonian generating the evolution equations (3). To this end, we notice that the latter equations can be presented in the Hamiltonian form
\[ i\hbar \frac{d\rho}{dt} = \frac{\partial H_{\text{eff}}}{\partial \rho^*}, \]
with the effective Hamiltonian
\[ H_{\text{eff}} = -\frac{i \hbar}{2} \left( \beta e^{i\Delta \omega t} c_j^* c_j + \beta^* e^{-i\Delta \omega t} c_j c_j^* \right). \]

Then, using the averaging method, we solve the evolution equations (3) finding
\[ c_0 = \left[ \cos \frac{\Omega t}{2} + i \frac{\alpha (n_0 - n_j) - \Delta \omega}{\Omega} \sin \frac{\Omega t}{2} \right] \exp \left\{ -\frac{i}{2} (\alpha - \Delta \omega) t \right\}, \]
\[ c_j = -i \beta^* \frac{\Omega}{\Omega} \sin \frac{\Omega t}{2} \exp \left\{ -\frac{i}{2} (\alpha + \Delta \omega) t \right\}, \]
\[ \Omega^2 = [\alpha (n_0 - n_j) - \Delta \omega]^2 + |\beta|^2. \]

The usage of the averaging method here is accomplished in the same way as it has been done for solving nonlinear evolution equations describing superradiant spin relaxation in magnets [10,11], or nonadiabatic dynamics of atoms in magnetic traps [12,13]. Technical details can be found in the latter references. Employing the found solutions (5), with the normalization \( n_0 + n_j = 1 \), we obtain for the effective Hamiltonian
\[ H_{\text{eff}} = \alpha n_0 n_j + \frac{1}{2} \left( \beta e^{i\Delta \omega t} c_j^* c_j + \beta^* e^{-i\Delta \omega t} c_j c_j^* \right). \]

Finally, let us analyse the optimal conditions when the considered effects could be separately, taking into account stability conditions as well as the lifetime of the condensate as such, which is defined of the system to store the energy pumped in by the resonant field is characterized by the derivative
\[ \dot{\chi}_s = \frac{\partial H_{\text{eff}}}{\partial \chi_s}. \]

The pumping capacity grows from zero at
\[ b > b_c, \]
becoming divergent at \( b = b_c \). The same concerns the susceptibility. All this suggests that \( b_c \) is a critical point and the relation
\[ b_c + \delta = 0.5 \]
defines the critical line, which is in agreement with our numerical solution of the evolution equations (3). The asymptotic behaviour of the characteristic quantities in the vicinity of the critical line defines the corresponding critical indices. Thus, for the small relative deviation \( \tau = |b - b_c|/b_c \to 0 \) from the critical point \( b_c \), we have
\[ \chi_s \simeq \chi_s^0 \left( 1 - 2\delta \right)^{\nu_1/2}, \quad \nu_1 \approx \frac{\chi_s^0}{\chi_s^0} \left( 1 - 2\delta \right)^{-1/2}, \quad \chi_s \simeq \frac{\chi_s^0}{\chi_s^0} \left( 1 - 2\delta \right)^{-1/2}. \]

Hence, the related critical indices are 1/2 for all characteristics. It is interesting that the critical indices of \( \eta, \chi_s \), and \( \chi_s \) satisfy the known scaling relation:
\[ \text{ind}(\chi_s) = 2 + \text{ind}(\eta) + \text{ind}(\chi_s) = 2, \]
where \( \text{ind} \) is the evident abbreviation for index.

If, after exciting a coherent mode, one switches off the pumping, then the following dynamics is to be considered separately, taking into account stability conditions as well as the lifetime of the condensate as such, which is defined by depolarizing collisions [14]. Finally, let us analyse the optimal conditions when the considered effects could be achieved experimentally. To reach the bifurcation line one needs to invoke the transition amplitude \( \beta \approx \alpha/2 \). At the same time, the interaction parameter \( \alpha \), at least in the strong-coupling limit [3], can be of the order of the transition frequency \( \omega \). When the magnitude of the transition amplitude \( \beta \) becomes close to the transition frequency \( \omega \), the effect of power broadening is expected. This effect is well known in optics [15], where the role of \( \beta \) is played by the Rabi frequency. In the regime of power broadening, one would expect the ground state to be coupled to more than one excited mode, so that the quasi-resonant two-mode picture could become not a good approximation. This picture remains a good approximation provided that the probability of nonresonant excitation of neighboring levels is small. The probability of an induced transition between levels \( i \) and \( j \), due to a monochromatic field with frequency \( \omega \), can be estimated [15] as
\[ P_{ij} \approx \frac{\beta_i^2}{2} \left( |\omega - \omega_{ij}|^2 + \beta_j^2 \right), \]
where \( \beta_i^2 \) and \( \omega_{ij} \) are the related transition amplitude and frequency, respectively. To estimate \( P_{ij} \), we may use the calculations of Ref. [3], slightly modifying them to take into account the cylindrical symmetry of a trap with radial and axial frequencies \( \omega_r \) and \( \omega_z \), respectively. Dealing with a cylindrical trap gives an additional explicit parameter \( \nu \equiv \omega_{z}/\omega_r \), hence, more possibilities for varying conditions. In the case of nonresonant excitation, when \( \omega - \omega_{ij} \approx \omega \), the magnitude of \( P_{ij} \) essentially depends on the value of the transition amplitude \( \beta_{ij} \). The maximal value of the latter, occurring when approaching the bifurcation line, is close to \( \alpha/2 \). The value of \( \alpha \) depends on the parameter \( g \nu \), with \( g \equiv 4\pi a_s N/l_r \), where \( a_s \) is a scattering length, \( N \) is
the number of atoms, and \( l_r \equiv \sqrt{\hbar / m_0 \omega_r} \). If \( g \nu \) is not large, then \( \alpha \ll \omega \). Thence the maximal \( \beta_{ij} \approx \alpha / 2 \ll \omega \) and \( P_{ij} \ll 1 \). Therefore the nonresonant excitation can be neglected, and the two-level picture is a good approximation. When \( g \nu \gg 1 \), then \( \alpha \sim \omega \), so that the maximal \( \beta_{ij} \approx \omega / 2 \), from where \( P_{ij} \sim 0.1 \). This tells that, although power broadening can influence quantitative results, the two-level picture still serves as a reasonable first approximation. To reduce the influence of power broadening, one can proceed as follows. In the case of an almost spherical trap, with \( \nu \approx 1 \), the number of atoms is to be restricted in order not to make \( g \) too high. When the number of atoms is large, so that \( g \gg 1 \), this can be compensated by taking a long cigar-shape trap, with \( \nu \ll 1 \), which would result in \( g \nu \sim 1 \). In the present-day experiments there is such a wide variety of possibilities for changing characteristic parameters [1,2] by choosing an appropriate shape of a trap, by changing the number or the type of trapped atoms, by varying the scattering length using Feshbach resonances, and so on, that it looks rather feasible to observe the critical phenomena described.

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Figure captions

**Fig. 1.** The time dependence of the fractional populations \(n_0(t)\) and \(n_j(t)\) for \(b = 0.4999\) and \(\delta = 0\). Here and in all following pictures the dashed line corresponds to the ground-state population \(n_0(t)\) and the solid line, to the excited-level population \(n_j(t)\).

**Fig. 2.** Flattening of the fractional populations, with their oscillation period being doubled, at \(b = 0.4999\) and \(\delta = 0.0001\).

**Fig. 3.** The appearance of the upward cusps of \(n_j(t)\) and of the downward cusps of \(n_0(t)\) for \(b = 0.4999\) and \(\delta = 0.0001001\).

**Fig. 4.** Fractional populations versus time for \(b = 0.4999\) and \(\delta = 0.00011\).