On Normality of the Wijsman Topology

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Outline

1. Introduction
2. Normality of the Wijsman Topology
3. Cardinal invariants of the Wijsman Topology
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What is a hyperspace?

A **hyperspace** is a space \((CL(X), \tau)\), where \(CL(X)\) is the set of all nonempty closed subsets of a Hausdorff space \(X\). \(\tau\) is called a **hypertopology**.

**Definition**

A topology generated by all sets of the form 
\[ U^- = \{ A \in CL(X); A \subset U \}, \quad V^+ = \{ A \in CL(X); A \cap V \neq \emptyset \}, \]
where \(U, V\) are open subsets of \(X\), is called the **Vietoris topology**. If \(V\) runs only through complements of compact sets, the corresponding topology is called the **Fell topology**.

**Definition**

Let \((X, \rho)\) be a metric space. The topology on \(CL(X)\) generated by all functions of the form \(\rho(x, \cdot): CL(X) \to R\) is called the **Wijsman topology**.
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Question from [Di Maio and Meccariello, 1998, Problem I]:

It is known that \((X, \rho)\) is a separable metric space, iff \((CL(X), W_\rho)\) is metrizable; and then it is normal.

Is the opposite true?

Is \((CL(X), W_\rho)\) normal if and only if it is metrizable?
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**Theorem**

*Suppose GCH holds. If $(CL(X), W_\rho)$ is normal then $d(CL(X)) = d(X)$.***

**Theorem**

*Let $(X, \rho)$ be a discrete metric space with 0-1 metric. If $(CL(X), W_\rho)$ is normal then $X$ is countable.*
Simple results

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Theorem

Let \((X, \rho)\) be a metric space. If for every metric \(\delta\) uniformly equivalent to \(\rho\) the space \((\text{CL}(X), W_\delta)\) is normal, then \(X\) is separable.

Idea of the Proof

If \(X\) is not separable, then there is an \(\epsilon\)-discrete set \(Y \subset X\) with \(|Y| > \aleph_0\). Put \(\delta(x, y) = \min(\rho(x, y), \epsilon)\), which is uniformly equivalent to \(\rho\). One can prove that \((\text{CL}(Y), W_{\delta|Y})\) can be embedded as a closed subset of \((\text{CL}(X), W_\delta)\), so it has to be normal. Since \(\delta|Y\) is \(0 - \epsilon\) metric, then by previous result we have that \(Y\) is countable, a contradiction.
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Normality of the Wijsman Topology

More general results II

**Theorem**

Let $(X, \rho)$ be a linear metric space. $(CL(X), W_\rho)$ is normal iff $X$ is separable.

**Idea of the Proof**

If $X$ is not separable, then no point has a compact neighborhood. Then one can prove that $\omega^{\aleph_1}$ can be embedded as a closed subset of $(CL(X), W_\rho)$. Since $\omega^{\aleph_1}$ is not normal, we have a contradiction.

**Theorem (J. Cao, H.J. K. Junnila, W.B. Moors)**

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Cardinal Invariants of the Wijsman Topology

Theorem

\[ d(X) = f(CL(X)) = hf(CL(X)), \] where \( f \) is any function from: spread \( (s) \), extent \( (e) \), netweight \( (nw) \), weight \( (w) \), pseudo weight \( (\psi w) \), \( \pi \)-weight \( (\pi) \), character \( (\chi) \), pseudocharacter \( (\psi) \), \( \pi \)-character \( (\pi \chi) \), tightness \( (t) \), Lindelöf number \( (L) \), diagonal degree \( (\Delta) \), weak weight \( (ww) \), uniform weight \( (u) \) and hereditary density \( (hd) \).
Cardinal invariants of a Tychonoff space

- $w(\text{CL}(X)) \leq d(X)$
- $d(X) \leq \psi(\text{CL}(X))$
- $d(X) \leq t(\text{CL}(X))$
- $d(X) \leq \pi \chi(\text{CL}(X))$
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**Theorem**

\[ \log(d(X)) \leq d(CL(X)) \leq d(X), \quad \aleph_0 \leq c(CL(X)) \leq d(CL(X)) \]

\[ \log(n) = \min\{m; n \leq 2^m\} \]

**Example**

Let \( X \) be a discrete metric space with the \( 0-1 \) metric, then
\[ d(CL(X)) = \log(d(X)) \] and \[ c(CL(X)) = \aleph_0. \]
- \( CL(X) \) with Wijsman topology is homeomorphic to \( 2^X \setminus \{1\} \)
- Hewitt-Marczewski-Pondiczery theorem

Let \( M \subset X \) fulfill \( |M| = m \geq \aleph_0 \), let \( \rho \) be a metric such that for every \( x \in M \) there is \( x' \in M \) such that \( (x')' = x \) and \( \rho(x, x') = 2; \rho(x, x) = 0 \) and \( \rho(x, y) = 1 \) otherwise.
Then \( d(CL(X)) = \log(d(X)) + m \) and \( c(CL(X)) = m. \)
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Then \( d(CL(X)) = \log(d(X)) + m \) and \( c(CL(X)) = m \).
Density and cellularity

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Let \( X \) be a discrete metric space with the \( 0-1 \) metric, then \( d(CL(X)) = \log(d(X)) \) and \( c(CL(X)) = \aleph_0 \).

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For some classes of metric spaces, the normality of the Wijsman topology (on the corresponding hyperspace) is equivalent to its metrizability;

General question is still open. Is the normality of the Wijsman topology always equivalent to its metrizability?

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Summary

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- General question is still open. Is the normality of the Wijsman topology always equivalent to its metrizability?
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For Further Reading

- G. Di Maio and E. Meccariello. Wijsman topology. *Quaderni di Matematica, 3:55–92, 1998.*

- J. Cao, H.J. K. Junnila, W.B. Moors. Wijsman hyperspaces: subspaces and embeddings *preprint*

- G. Beer. *Topologies on Closed and Closed Convex Sets.* Kluwer Academic Publishers, Dodrecht, 1993.
Thank You for Your Attention