Class numbers in the imaginary quadratic field and the 1/f noise of an electron gas

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Abstract: - Partition functions $Z(x)$ of statistical mechanics are generally approximated by integrals. The approximation fails in small cavities or at very low temperature, when the ratio $x$ between the energy quantum and thermal energy is larger or equal to unity. In addition, the exact calculation, which is based on number theoretical concepts, shows excess low frequency noise in thermodynamical quantities, that the continuous approximation fails to predict. It is first shown that Riemann zeta function is essentially the Mellin transform of the partition function $Z(x)$ of the non degenerate (one dimensional) perfect gas. Inverting the transform leads to the conventional perfect gas law. The degeneracy has two aspects. One is related to the wave nature of particles: this is accounted for from quantum statistics, when the de Broglie wavelength exceeds the mean distance between particles. We emphasize here the second aspect which is related to the degeneracy of energy levels. It is given by the number of solutions $r_3(p)$ of the three squares diophantine equation, a highly discontinuous arithmetical function. In the conventional approach the density of states is proportional to the square root of energy, that is $r_3(p) \simeq 2\pi p^{1/2}$. We found that the exact density of states relates to the class number in the quadratic field $Q(\sqrt{-p})$. One finds $1/f$ noise around the mean value.

Key-Words: - Electronic circuits, number theory, quantum statistical physics, $1/f$ noise

1 Introduction to Noise in Electrical Circuits

Noise in electrical circuits is found in many forms, some of them have been well explained: thermal noise, shot noise, partition noise, burst noise... and one which is still subject to much debate due to its universality and the lack of a general accepted model: $1/f$ noise.

Let us first review briefly our understanding of electrical thermal noise. Due to thermal agitation free electrons in a metallic conductor are moving around continuously causing collisions with the atoms and a continuous exchange of energy between the modes. This was first investigated experimentally by Johnson [1] and theoretically explained by Nyquist [2]. The noise in any circuit kept at uniform temperature $T$ can be described by a noise voltage $(v^2)^{1/2}$ in series with a resistor $R$ of the circuit such that for a small frequency interval $df$

$$\overline{v^2} = 4kRTp(f)df,$$

(1)

where $p(f) = \frac{hf}{kT}(e^{hf/kT} - 1)^{-1}$ is the Planck factor, $k = 1.38 \times 10^{-23}$ J/K is Boltzmann’s constant and $h = 6.62 \times 10^{-34}$ J.s is Planck’s constant. For room temperature and not too high frequency $hf/kT \ll 1$, so that $p(f)$ is equal to unity. These equations are verified from experiments. For example with a resistor of $R = 1$ KΩ, at room temperature $T = 300$ K, if one uses a filter of bandwith $df = 1$ MHz, the expected voltage fluctuation is of the order $4 \mu V$ easily detectable on an oscilloscope.

Our second example is shot noise in a diode. Such a noise is produced by random emission of electrons from the cathode to the anode. Schottky theorem [3] states that the mean current $I$ undergoes a fluctuation

$$\overline{i^2} = 2eI \, df,$$

(2)

where $e = 1.6 \times 10^{-19}$ C is the charge of one electron. Using a typical value $I = 1$ mA and the same bandwith as above the expected current fluctuation is on the order $18$ nA.

Our third exemple is partition noise which occurs any time a current is distributed randomly between two electrodes. Its net effect is an extra multiplicative factor in the relation for the shot noise current [4]. A modern variant of shot noise and partition noise is quantum partition noise. Eq. [2] was used as an extra proof that the electrical charge is indivisible and the measurement of current fluctuations allowed the determination of the unit $e$. Recently it
has been argued that low dimensional systems like quantum wires or quantum dots may produce a fractional charge $e^q/p$ ($p$ and $q$ integers) associated to a quasi particle tunneling state. The corresponding fractional shot noise current has been observed [5].

In most small size transistors a noise of the low frequency type is observed: burst noise also called random telegraph signal (or RTS noise). The noise typically consists of random discrete-switching events in the current flowing through a transistor. It was first seen in reverse-biased p-n junctions and bipolar transistors [6]. It was then related to random single electron capture events into localized defects of very small transistors such as MOSFET’s [7]. Assuming that to the probability of an electron making a transition from one state of amplitude 0 to another state of amplitude 1 there is associated a switching time $\tau_1$. The power spectral density is found to be of the Lorentzian type

$$S_1(f) = \frac{2 I^2 \tau_1}{4 + (2\pi f \tau_1)^2},$$

that is a flat spectrum up to the turn over frequency and a $1/f^2$ dependance above.

We now turn to $1/f$ or flicker noise. Examples above refer to a constant power spectral density $S_V(f)$ for thermal noise and $S_1(f) = 2eI$ for shot noise. Flicker effect is the large ammount of noise generated in any solid state device (vacuum tube, diode, transistor...) at low frequencies which was discovered by Johnson [8]. It is generally described phenomenologically as

$$S_V(f) = KV^{a} f^{b},$$

with $a$ close to 2 and $b$ close to 1. By integrating between the low frequency cut-off $1/\tau$ and the high frequency cut-off $f_c$, we get

$$S_y^2(\tau) = \frac{1}{2} \delta y(\tau)^2,$$

where $\delta y(\tau) = y_{i+1}(\tau) - y_i(\tau)$ is the deviation between two consecutive samples of index $i$ and $i + 1$ counted over an averaging time $\tau$ and $y_i = f_i/f_0$ refers to the ratio between the instantaneous frequency and the mean frequency of the oscillator. It can be shown [15] that $1/f$ noise of the power spectral density $S_y(f) = C/f$, with $C$ a constant related to the physical set-up, is associated to a constant value of the Allan deviation $\sigma_y(\tau) \sim 2 \ln 2 C$. This so-called flicker floor is the limit to the stability of oscillators. Values in the range $10^{-12}$ for quartz oscillators to $10^{-15}$ for Hydrogen maser based oscillators have been obtained [16].

We recently discovered a possible relationship between $1/f$ frequency noise in oscillator measurements and prime number theory [17], [18]. In this paper we pursue this quest by relating exact statistical mechanics of electrons of mass $m$ in a box of
size \( L \) to the quadratic field \( Q(\sqrt{-p}) \) of negative discriminant, with \( p \) the number of energy quanta in units of \( \delta E = h^2/8mL^2 \). We find numerically a \( 1/f \) noise of the fractional density of states about the average classical value. Sec. 2 shows that analytical number theory (and the Riemann zeta function) underlies the statistical mechanics of the perfect non degenerate gas. Sec. 3 shows that the degeneracy of energy levels needs the extended limits of number theory. For a perfect classical gas the partition function becomes

\[
\psi = \frac{2}{L} \left( \sqrt{\pi n} \right) \sin \left( \frac{n \pi x}{L} \right),
\]

with \( E_n = \delta E n^2 \), \( n = 1, 2, \cdots \) so that the partition function becomes

\[
Z_0 = \sum_{n=1}^{n_{\text{max}}} \exp \left( -\frac{E_n}{kT} \right).
\]

The energy levels are obtained by solving Schrödinger equation with free boundary conditions

\[
\frac{d^2 \psi}{dx^2} + \frac{2mE}{\hbar^2} \psi = 0,
\]

\[
\psi = 0 \text{ at } x = 0 \text{ and } x = L.
\]

The solution is

\[
\psi = \left( \frac{2}{L} \right)^{1/2} \sin \left( \frac{n \pi x}{L} \right),
\]

with \( E_n = \delta E n^2 \). Using the first order Mac-Laurin expansion and if \( x \) is small

\[
Z_0(x) \approx \sqrt{2\pi x} \text{erfc} \left( \frac{n_{\text{max}} \sqrt{\pi n}}{x} \right) \frac{1}{2} \approx \frac{1}{2 \sqrt{x}}.
\]

For the three dimensional gas of \( N \) particles we need the indistinguishability parameter \( N! \) so that

\[
Z = \frac{Z_0^N}{N!}.
\]

The gas pressure is defined from

\[
P(V, T) = 3NkT \left( \frac{\partial \ln Z}{\partial V} \right)_{T, N} \quad \text{with } V = L^3,
\]

and the free energy \( F \) and the entropy \( S \) as

\[
S = -\frac{\partial F}{\partial T}, \quad F = -kT \ln Z.
\]

The Sackur-Tetrode equation follows

\[
S = S(E, V, N) = Nk \left[ \frac{5}{2} + \ln \left( \frac{V}{N} \right) + \frac{3}{2} \ln \left( \frac{m}{3\pi \hbar^2} \frac{E}{N} \right) \right].
\]

Thanks to the inclusion of \( N! \) term in the definition of partition function the entropy equation satisfies the property of extensivity.

\section{The Non Degenerate Perfect Gas}

\subsection{The classical approach to the perfect gas}

In statistical thermodynamics the partition function allows to establish all thermodynamical properties \[19\]. It is called generating function in the context of number theory. For a perfect classical gas the partition function per degree of freedom and per particle is

\[
Z_0 = \sum_{n=1}^{n_{\text{max}}} \exp \left( -\frac{E_n}{kT} \right).
\]

The energy levels are obtained by solving Schrödinger equation with free boundary conditions

\[
\frac{d^2 \psi}{dx^2} + \frac{2mE}{\hbar^2} \psi = 0,
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The solution is

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\psi = \left( \frac{2}{L} \right)^{1/2} \sin \left( \frac{n \pi x}{L} \right),
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with \( E_n = \delta E n^2 \). Using the first order Mac-Laurin expansion and if \( x \) is small

\[
Z_0(x) \approx \sqrt{2\pi x} \text{erfc} \left( \frac{n_{\text{max}} \sqrt{\pi n}}{x} \right) \frac{1}{2} \approx \frac{1}{2 \sqrt{x}}.
\]

Substituting \( \pi n^2 x \) to \( x \) in the integral \[19\] and summing for all integers \( n \) one gets

\[
\pi^{-s} \Gamma(s) \zeta(2s) = \int_0^\infty Z_0(x) x^{s-1} dx,
\]

where \( N_A \) is the Avogadro constant.

Others useful state equations are derived such as the mean energy per mole of the gas

\[
E = -\frac{\partial \ln Z}{\partial \beta} = 3 \frac{RT}{2} \quad \text{with } \beta = \frac{1}{kT},
\]

and the free energy \( F \) and the entropy \( S \) as

\[
S = -\frac{\partial F}{\partial T}, \quad F = -kT \ln Z.
\]
which is the desired relation between \( \zeta(s) \) and the Mellin transform of \( Z_0(x) \). Here we assumed \( n_{\text{max}} \to \infty \) in the definition of the partition function. The expression
\[
\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s),
\] is the so-called completed Riemann zeta function. This is because \( \xi(s) \) admits an analytic continuation to the whole complex plane \( s \) except at the simple poles \( s = 0 \) and \( s = 1 \) of residues \(-1\) and \( 1 \) respectively. It satisfies the functional relation \( \xi(s) = \xi(s-1) \).

The approximation of \( Z_0(x) \) may be obtained by taking the inverse Mellin transform and by applying the residue theorem to the two poles of \( \xi(2s) \) at \( s = 0 \) and \( s = 1/2 \) so that
\[
Z_0(x) \approx \frac{1}{2\sqrt{x}} - \frac{1}{2},
\] which is similar to the previous Mac-Laurin approximation \([12]\). The step from the discrete to the continuous is condensed into the contribution of poles of the Riemann zeta function.

The remaining poles at \( s = -m \) (integer \( \geq 1 \)) of \( \Gamma(s) \) are absorbed by the trivial zeros at \( s = -l \) ( \( l \) a positive integer) of \( \xi(2s) \). The unproven Riemann hypothesis \([17]\) is that all remaining zeros are located on the critical line \( s = 1/2 \). The modulating action of the zeros is expected to be significant within a box at the nanoscopic scale \( x = \frac{\delta E}{kT} \geq 1 \).

### 3 The Degenerate Perfect Gas

#### 3.1 First form of quantum degeneracy due to wave-particle aspects

In the quantum gas, the first form of degeneracy is due to the indiscernability of (wave)particles of size \( \lambda_{\text{th}} \) (the de Broglie wavelength) larger than the typical spacing \( d \). The degeneracy parameter is \([19]\)
\[
\frac{\lambda_{\text{th}}}{d} = \frac{h/(mv)}{V^{1/3}}, \quad \text{with} \quad v = \left( \frac{2\pi kT}{m} \right)^{1/2}.
\] (For conduction electrons in silver at room temperature the ratio is about 40).

The partition function for electrons is taken to satisfy Fermi-Dirac statistics with chemical potential \( \alpha \)
\[
\ln Z = \sum_s \ln [1 + \exp(-\alpha - \beta E_s)] + N\alpha.
\] In the conventional approach of statistical mechanics, the discreteness of quantum states is neglected and one goes from the discrete to the continuous by taking the number of states \( \Omega_0(E) \) as \( \frac{1}{2} \), times the volume of the Fermi sphere of radius \( R_F \), times 2 (for the two spin states of the electron). The density of states \( \omega_0(E) \) per unit volume \( V = L^3 \) follows from
\[
\omega_0(E) = \frac{1}{V} \frac{d\Omega_0(E)}{dE} = 2\pi \left( \frac{2m}{\hbar^2} \right)^{3/2} E^{1/2}.
\]

#### 3.2 Second quantum aspect due to the degeneracy of energy levels

Here the number of possible quantum states is calculated exactly from number theory.
\[
\Omega(E) = \sum_{n_1^2 + n_2^2 + n_3^2 \leq R_F^2} 1 \quad \text{with} \quad R_F^2 = \frac{E_F}{\delta E},
\]
where \( n_1, n_2 \) and \( n_3 \) are positive integers, \( E_F \) is the Fermi energy and \( \delta E \) is the energy quantum as above. In quantum dots the energy is quantized in units of \( \delta E \): this may be observed when the Coulomb blockade energy \( E_B = \frac{e^2}{C} \) is such that \( E_B \gg \delta E \gg kT \), that is at very low temperatures or for small samples with large capacity \( C \). The partition function is
\[
Z = \sum_{p=1}^{p_{\text{max}}} g_p \exp(-\frac{E_p}{kT}) \quad \text{with} \quad E_p = \delta E p,
\]
where the degeneracy parameter \( g_p \) is the number of states of energy \( E_p \).

![Figure 1: Relative fluctuations \( \epsilon(p) \) of the density of states versus energy \( p \) (We took \( \delta p = 100 \)).](image-url)

The number of quantum states \( g_p \) also equals \( \frac{1}{3} \) times 2, times the number \( r_3(p) \) of solutions of the three squares diophantine equation
\[
n_1^2 + n_2^2 + n_3^2 = p \quad \text{with} \quad p \leq R_F^2,
\]
and \( n_1, n_2, n_3, p \) integers. This is a central problem of number theory. Gauss proved that for \( p > 4 \) and squarefree
\[
    r_3(p) = \begin{cases} 
    24h(-p) & \text{for } p \equiv 3 \pmod{8}, \\
    12h(-4p) & \text{for } p \equiv 1, 2, 5 \text{ or } 6 \pmod{8}, \\
    0 & \text{for } p \equiv 7 \pmod{8}, 
    \end{cases}
\]
where \( h(-p) \) is the class number in the field \( Q(\sqrt{-p}) \) (that is the number of primitive quadratic forms with given negative discriminant \(-p\)). The general case is \( r_3(p) = \sum_{q \mid p} R_3(\frac{p}{q^2}) \), where \( R_3(p) \) holds for square free values of \( p \). More explicit solutions can be found [21]. They are available in the Mathematica package [22].

The class number \( h(-p) \), and thus \( r_3(p) \) is a highly discontinuous function. We defined the (exact) density of states as
\[
    \omega(p) = \frac{1}{m} \sum_{n=p}^{n=p+\delta p} r_3(n) 
    \approx \omega_0(p) = 2\pi p^{1/2} \quad \text{with} \quad \delta p \ll p. \tag{31}
\]

Relative fluctuations of the density of states around the mean value \( \omega_0(p) \) were calculated as
\[
    \epsilon(p) = \frac{\omega(p)-\omega_0(p)}{\omega_0(p)} \quad \text{(see Fig. 1).}
\]
As a result the power spectral density of low frequency fluctuations versus Fourier frequency \( f \) was found as
\[
    F F T(f) \sim \frac{\delta p}{\sqrt{p}}, \quad \text{with } \alpha \text{ close to } 1 \text{ and } A \sim 0.1, \text{ that is a } 1/f \text{ spectrum as found in Fig. 2.}
\]

Physically one can take \( \delta p = [\frac{kT}{\delta E}] \), with \([\ ]\) the integer part and \( \delta E \) the granularity parameter, since \( kT \) is the width of the Fermi surface at finite temperature \( T \). One can conclude that the \( 1/f \) spectrum arising in the arithmetical approach connects to the different regimes one gets for the electron gas in quantum dots, versus the typical size \( L \) of the dot, the temperature \( T \) and the effective mass \( m \).

Due to the low frequency noise in the density of states all thermodynamical quantities are perturbed, as well as the conductivity of electronic devices. This will be developed in a future paper.

### 4 Concluding Remarks

Thermodynamical quantities attached to the quantum description of an electronic gas in a cubic box are not smooth versus energy or time, contrary to the classical viewpoint: this is because the summatory function of an arithmetical function such as \( r_3(n) \) is discontinuous. The fractional deviation with respect to the asymptotic average very often leads to a low frequency power spectrum of the fractal type \( 1/f^b \), \( b \) close to 1. Number theory thus is a good candidate to explain many low frequency noises encountered in solid state physics and engineering as well as in others fields.

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### References:

[1] J. B. Johnson. Thermal agitation of electricity in conductors. Phys. Rev.: 32, 97–109, 1928.

[2] H. Nyquist. Thermal agitation of electrical charge in conductors. Phys. Rev.: 32, 110–113, 1928.

[3] W. Schottky. Small shot effect and flicker effect. Phys. Rev.: 28, 74–103, 1926.

[4] Van der Ziel. Noise, Sources, Characterization, Measurement. Prentice Hall, Englewood Cliffs, New Jersey, 1970.

[5] L. Saminadayar L., D. C. Glattli, Y. Jin, B. Etienne. Observation of the \( e/3 \) fractionally charged Laughlin quasiparticle. Phys. Rev. Lett.: 79, 2526–2529, 1997.

[6] M. J. Buckingam. Noise in Electronic Devices and Systems. Chichester, Ellis Horwood, 1983.

[7] M. J. Kirton and M. J. Uren. Noise in solidstate microstructures. A new perspective on individual defects, interface states and low frequency \((1/f)\) noise. Advances in Physics: 38, 367–468, 1989.
[8] T. J. M. Kleinpenning. Relation between variance and sample duration of $1/f$ noise signals. J. Appl. Phys.: 63, 43–45, 1988.

[9] M. N. Mihaila. Phonon fine structure in the $1/f$ noise of metals, semiconductors and semiconductor devices, in Noise, oscillators and algebraic randomness, ed. M. Planat. Lect. Notes in Phys., Springer Verlag, Berlin, 216–231, 2000.

[10] F. N. Hooge. $1/f$ noise is no surface effect. Phys. Lett.: 29A, 139–140, 1969.

[11] R. F. Voss and J. Clarke. $1/f$ noise from systems in thermal equilibrium. Phys. Rev. Lett.: 36, 42–45, 1976.

[12] P. H. Handel. Noise, low frequency, in Wiley Encyclopedia of Electrical and Electronic Engineering, ed. John G. Webster, John Wiley and Sons, N. Y.: 14, 428–449, 1999.

[13] L. B. Kiss. An exact proof of the invalidity of Handel’s quantum $1/f$ noise model based on quantum electrodynamics. J. Phys. C: 19, L631, 1986.

[14] M. B. Weissman. $1/f$ noise and other slow, non exponential kinetics in condensed matter. Rev. Mod. Phys.: 60, 537–571, 1988.

[15] D. W. Allan. Time and Frequency (Time Domain) Characterization, Estimation and Prediction of Precision Clocks and Oscillators. IEEE Trans. Ultrason. Ferroelect. Freq. Contr.: 34, 647–654, 1987.

[16] C. Audoin and B. Guinot. Les fondements de la mesure du temps, Masson, Paris, 1998.

[17] M. Planat. $1/F$ noise, the measurement of time and number theory. Fluc. and Noise Lett.: 1, R65–R79, 2001.

[18] M. Planat, H. Rosu and S. Perrine. Ramanujan sums for signal processing of low frequency noise. Phys. Rev. E66, 56128, 2002; arXiv:math-ph/0209002.

[19] J. Kestin J. and J. R. Dorfman. A course in statistical thermodynamics. Acad. Press, N. Y., 1971.

[20] H. M. Edwards. Riemann’s zeta function. Academic Press, New York, 1974.

[21] E. Grosswald. Representations of integers as a sum of squares. Springer, Berlin, 54–65, 1984.

[22] Wolfram Research. Standard Add-on Packages. Wolfram Media 317-323, 1999.