On Regional Boundary Gradient Strategic Sensors In Diffusion Systems

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Abstract:
This paper is aimed at investigating and introducing the main results regarding the concept of Regional Boundary Gradient Strategic Sensors (RBGS-sensors) in Diffusion Distributed Parameter Systems (DDP-Systems). Hence, such a method is characterized by Parabolic Differential Equations (PDEs) in which the behavior of the dynamic is created by a Semigroup \( S(\tau) \) \( \tau \geq 0 \) of Strongly Continuous type (SCSG) in a Hilbert Space (HS). Additionally, the grantee conditions which ensure the description for such sensors are given respectively to together with the Regional Boundary Gradient Observability (RBG-Observability) can be studied and achieved. Finally, the results gotten are applied to different situations with altered sensors positions are undertaken and examined.

Keywords: WRBG-Observability, ERBG-Observability, RBGS-Sensors, DDP-Systems.

1 Introduction

The Observation Problem [1−3] is one of the most important notion in the analysis of DDP-Systems was attracted the attention of many researchers [4−7]. In various cases, one may interest in the cognition of the state of a PDEs system on a sub region \( \Omega \) of internal and boundary the domain \( \Omega \) in an unbounded interval [8−18] or bounded time [19−23].

The investigation of this notion is incited by specific Physical Problem, in Thermic, Mechanic, Environment, for example some physical problems concern the determination of laminar flux conditions, developed in steady state by vertical uniformly heated plate [24−27].

This approach can be applied to find the unknown boundary convective condition on the front face of the active plate, as in [26]. The reconstruction is based on knowledge of the dynamical system via measurement information given by internal sensors type pointwise \((b_1, b_2)\) (that means by the thermocouples for instance see (Figure 1).

![Figure 1: Real heated plate diffusion.](image)

Thence, this study designed at giving the required conditions of the RBGS-Sensors in this region, that builds RBG-State. Thus, the main reasons for presenting this notion are: Firstly it makes cognition for the usual observer concept closer to actual world quandaries, Secondly it can be introduced and explore the main results concerned to the DDP-Systems [24−26] in connection with RBGS−Sensors. This job is arranged in the following:
Certain definitions with identification of the ERBG – Observability for exactly case and WRBG-Obervability for weakly case, are given in the next section. Section three introduces most for the required ailments to RBGS-Sensors and a reformation process is developed to come across the internal state region to the boundary. Later, several applications for sensors positions in regions of rectangular types are presented and illustrated.

2 RBG-Observability in DDP-Systems

The current section invests to study the notion of RBG- Observability in DDP-Systems. It makes certain important outcomes concerning this notion.

2.1. Preliminaries Considerations Of The System

The following assumptions are to be given

- \( \mathbb{U} \) stay Open and Bounded in \( \mathbb{R}^n \), is the space domain with smooth boundary \( \partial \mathbb{U} \).
- \( \mathbb{G} \) remains a sub-boundary on \( \partial \mathbb{U} \).
- \([0,T], T > 0\) stand to a space-time interval cylinder.
- The \( HSs \) with \( \mathbb{W}, \mathbb{U} \) and \( \mathbb{Y} \) are separable where \( \mathbb{W} \) is the space of the state \( \omega, \ \mathbb{U} = L^2(0, T, \mathbb{R}^p) \) is the space of the input \( u \) and \( \mathbb{Y} = L^2(0, T, \mathbb{R}^q) \) is the space of output \( \mathbb{Y} \) [16].

- Reflected DDP – System described by the following PDEs

\[
\begin{align*}
\frac{\partial \omega}{\partial \tau}(\zeta, \tau) &= \Delta \omega(\zeta, \tau) + Bu(\tau) \quad \Pi_{\tau} \\
\omega(\zeta, 0) &= \omega_0(\zeta) \quad \mathbb{U} \\
\frac{\partial \omega}{\partial \nu}(\mu, \tau) &= 0 \quad \mathbb{G}_\tau
\end{align*}
\]  

(1)

where \( \Pi_{\tau} = \mathbb{U} \times ] 0, \tau [ \ , \mathbb{G}_\tau = \partial \mathbb{U} \times ] 0, \tau [ , \mathbb{Z}_\tau = \mathbb{U} \times [0, \tau [ \), and \( (\zeta, \tau) \in \mathbb{U} \times ] 0, \tau [ , (\mu, \tau) \in \partial \mathbb{U} \times ] 0, \tau [ , (\zeta, 0) \in \mathbb{G} \). where \( \mathbb{G} \) represents \( \mathbb{U} \) closure and \( \frac{\partial \omega}{\partial \nu} \) indicates the derivative of normal vector \( \nu \) on \( \partial \mathbb{U} \).

Then DDP – System remains augmented with the measurement function

\[
\mathbb{Y}(., \tau) = C \omega(., \tau)
\]  

(2)

where,

- \( \Delta \) stays an operator, linear and differential of second order type, in which is produced a \( SCs \) – group \( (S_\Delta(t))_{t \geq 0} \) on \( HS \) may be symbolized by \( \mathbb{W} = H^1(\mathbb{U}) \) such that it is self adjoint through resolvent of compact type.

- So the operators \( B \in L(\mathbb{R}^p, \mathbb{W}) \) and \( C \in L(\mathbb{W}, \mathbb{R}^q) \) be dependent on the of sensors (actuators) construction [6]. Thus the reflected DDP – Systems (1) possesses a solution of unique kind [1-3] illustrated in the subsequent form

\[
\omega(\zeta, \tau) = S_\Delta(t)\omega_0(\zeta) + \int_0^\tau S_\Delta(t - \tau)Bu(\tau)d\tau
\]  

(3)

- The problematic underlies, in what way to realize satisfactory conditions of RBGS – Sensors on specified sub-boundary \( \mathbb{G} \).

- Thence, the operator is defined

\[
\mathcal{K}: \omega \in \mathbb{W} \rightarrow \mathcal{K}\omega = CS_\Delta(.)\omega \in \mathbb{Y}
\]

And, the adjoint operator of \( \mathcal{K} \) indicates by \( \mathcal{K}^* \) identified by

\[
\mathcal{K}^*\mathbb{Y}^* = \int_0^\tau S_\Delta^*(s)C^*\mathbb{Y}^*(s)ds
\]

- Consider the gradient operator

\[
\begin{cases}
\nabla: H^1(\mathbb{U}) \rightarrow (H^1(\mathbb{U}))^n \\
\omega \rightarrow \nabla \omega = (\frac{\partial \omega}{\partial \zeta_1}, ..., \frac{\partial \omega}{\partial \zeta_n})
\end{cases}
\]
and the adjoint of $\nabla$ indicated by $\nabla^*$ is given as
\[
\begin{align*}
\nabla^* : (H^1(\Omega))^n &\to H^1(\Omega) \\
\omega &\to \nabla^* \omega = \nu
\end{align*}
\]
whereas $\nu$ is a solution of the Dirichlet problem
\[
\begin{align*}
\Delta \nu = -\text{div}(\omega) &\quad \text{in } \Omega \\
\nu/\partial \nu = 0 &\quad \text{in } \partial \Omega
\end{align*}
\]
• Then Operator of Trace type of zero-order is offered by
\[
\gamma^*_\Omega : H^1(\Omega) \to H^{1/2}(\partial \Omega)
\]
Therefore, the propagation of the trace operator where is described via
\[
\gamma : (H^1(\Omega))^n \to (H^{1/2}(\partial \Omega))^n
\]
with the related Adjoint Operators $\gamma^*_\Omega$ and $\gamma^*$.  
• On behalf of a sub-boundary $\mathcal{B} \subset \partial \Omega$, we take into account a gradient restriction operator
\[
\mathcal{X}_{\mathcal{B}} : (H^{1/2}(\partial \Omega))^n \to (H^{1/2}(\mathcal{B}))^n
\]
and
\[
\tilde{\mathcal{X}}_{\mathcal{B}} : H^{1/2}(\partial \Omega) \to H^{1/2}(\mathcal{B})
\]
where the adjoints are correspondingly presented by $\mathcal{X}^*_{\mathcal{B}}, \tilde{\mathcal{X}}^*_{\mathcal{B}}$.
• If $\omega$ remains a subregion of $\Omega$, then $\mathcal{X}_\omega$ is an operator specified by
\[
\mathcal{X}_\omega : \begin{cases}
(H^1(\Omega))^n \to (H^1(\omega))^n \\
\omega \to x_\omega \omega = \omega \mid_\omega
\end{cases}
\]
where $\omega \mid_\omega$ represented the restriction of the state $\omega$ to $\omega$ [28]. These adjoints are respectively denoted by $\mathcal{X}^*_\omega$ are defined by
\[
\begin{align*}
\mathcal{X}^*_{\omega} : \begin{cases}
(H^1(\omega))^n \to (H^1(\Omega))^n \\
\omega \to \mathcal{X}^*_{\omega} \omega = \{ \omega \mid_\omega &\quad \text{in } \omega \\
0 &\quad \text{in } \Omega \setminus \omega
\end{cases}
\end{align*}
\]
• Finally, we introduced the operator $H_{\mathcal{B}} = \mathcal{X}_{\mathcal{B}} \nabla \mathcal{X}^* \ni \mathcal{V}$ from $\mathcal{V}$ into $(H^{1/2}(\mathcal{B}))^n$.

2.2 Definitions and Descriptions

This section part presents necessary results about the RBG – Observability notion devoted to a particular devoted sensors. On behalf of this objective, one can deliberate the ADDP – Systems characterizes (1) in the autonomous case via next form.
\[
\begin{align*}
\frac{\partial \omega}{\partial t}(\xi, t) &= \Delta \omega(\xi, t) &\quad \text{in } \Omega \\
\omega(\xi, 0) &= \omega_0(\xi) &\quad \text{in } \bar{\Omega} \\
\frac{\partial \omega}{\partial \nu}(\mu, t) &= 0 &\quad \text{in } \partial \Omega \\
\end{align*}
\]
(4)

The Problem Solution of ADDP – Systems (4) is obtainable in the following form
\[
\omega(\xi, t) = S_\omega(t) \omega_0(\xi) \quad \text{for all } t \in [0, T]
\]
(5)

Definition2.1: ADDP – Systems (4) is increased with the measurement function (2) is so-called to be an ERG – Observable in a region $\omega \subset \Omega$, if
\[
\text{Im } H_{\omega} = (H^1(\omega))^n
\]
and ADDP – Systems (4) is increased with the measurement function (2) is so-called to be an WRG – Observablie if
\[
\text{Im } H_{\omega} = (H^1(\omega))^n
\].
**Definition 2.2:** ADDP – Systems (4) increased with measurement function (2) is so-called to be an ERBG – Observabile in a boundary region $\overline{\mathbb{G}} \subset \partial \mathbb{U}$, if

$$\text{Im} H_\partial = (H^{1/2}(\overline{\mathbb{G}}))^n$$

and ADDP – Systems (4) increased with measurement function (2) is so-called to be an WRBG – Observabile if

$$\text{Im} H_\partial = (H^{1/2}(\overline{\mathbb{G}}))^n.$$

**Remark 2.3:** We conclude that, this equation

$$\text{Im} H_\partial = (H^{1/2}(\overline{\mathbb{G}}))^n \iff \ker H_\partial = \{0\}.$$

**Proposition 2.4:** ADDP – Systems (4) increased with measurement function (2) are ERBG – Observabile if and only if $\exists \nu > 0$, such that for all $\omega^\ast \in (H^{1/2}(\overline{\mathbb{G}}))^n$, then,

$$\|X_\partial \omega^\ast\|_{(H^{1/2}(\overline{\mathbb{G}}))^n} \leq \nu \|\mathcal{K} \nabla \gamma^\ast X_\partial \omega^\ast\|_\gamma.$$  (6)

**Proof:**

The proof of Proposition 2.4 can be deducted via the next overall conclusions [1]. Supposing $E, F$ as well as $G$ be a reflexive Banach spaces and $f \in L(E, G)$, $g \in L(F, G)$, then the following properties are analogous

1. $\text{Im} f \subset \text{Im} g.$
2. $\exists \nu > 0$, such that

$$\|f \ast \omega^\ast\|_E \leq \nu \|g \ast \omega^\ast\|_F,$$ for all $\omega^\ast \in G^\ast$.

If we apply this outcome, considered $E = G = (H^{1/2}(\overline{\mathbb{G}}))^n$, $F = \mathcal{Y}$, $f = \text{Id}_{(H^{1/2}(\overline{\mathbb{G}}))^n}$ and $g = X_\partial \mathcal{V} \mathcal{X}^\ast$. Therefore, we obtain the inequality

$$\|X_\partial \omega^\ast\|_{(H^{1/2}(\overline{\mathbb{G}}))^n} \leq \nu \|\mathcal{K} \nabla \gamma^\ast X_\partial \omega^\ast\|_\gamma.$$  ■

Now, the following proposition can be arrived at:

**Proposition 2.5:** If the ADDP – System is ERB – Observable then it is ERBG – Observabile.

**Proof:** The ADDP – System is an ERB – Observable. Therefore $\exists \gamma_\partial > 0$, such that for all $\omega_0 \in H^{1/2}(\overline{\mathbb{G}})$, we have

$$\|\omega_0\|_{H^{1/2}(\overline{\mathbb{G}})} \leq \gamma_\partial \|\mathcal{K} \nabla \gamma \omega_0\|_{L^2(0, T, \mathcal{Y})},$$ for all $\gamma_\partial > 0$

Since $(H^{1/2}(\overline{\mathbb{G}}))^n \subset H^{1/2}(\overline{\mathbb{G}})$, then

$$\|\mathcal{V} \omega_0\|_{(H^{1/2}(\overline{\mathbb{G}}))^n} = \|\omega_0\|_{(H^{1/2}(\overline{\mathbb{G}}))^n} \leq \|\omega_0\|_{H^{1/2}(\overline{\mathbb{G}})},$$ for all $\omega_0 \in H^{1/2}(\overline{\mathbb{G}})$,

where,

$$H^{1/2}(\overline{\mathbb{G}}) = \{\omega_0: \int_\partial |\omega_0|^2 < \infty\}$$

and,

$$(H^{1/2}(\overline{\mathbb{G}}))^n = \{\|\mathcal{V} \omega_0 = g_i: \int_\partial |g_i|^2 < \infty, g_i = \frac{\partial \omega_0}{\partial n_i}, \text{for all } i = 1, 2, ...\}.$$  (7)

So to demonstrate $\|\omega_0\|_{(H^{1/2}(\overline{\mathbb{G}}))^n} \leq v \|\mathcal{K} \nabla \gamma^\ast X_\partial \omega_0\|_{L^2(0, T, \mathcal{Y})},$ we have, from (7) and since the ADDP – System is ERB – Observable, then there exists $\gamma_\partial > 0$ and $v > 0$, such that $\gamma_\partial = \frac{1}{v}$, by setting

$$v = \frac{\|\mathcal{K} \nabla \gamma \omega_0\|_{L^2(0, T, \mathcal{Y})}}{\|\mathcal{K} \nabla \gamma^\ast X_\partial \omega_0\|_{L^2(0, T, \mathcal{Y})}}.$$  (8)

consequently we can get
\[\|u_0\|_{(H^{1/2}((g)))^n} \leq \|\mathcal{K} \mathcal{V}^* X_0 u_0\|_Y.\]  
(9)

Therefore, ADPD - System is ERBG - Observable with \( \gamma = 1.\n\]

3. Sufficient Conditions For RBGS - Sensors

For accomplishing the RBG - Observability, we must grant the appropriate condition for the characterization of in a specified region \( g.\)

Remark 3.2:

1- Sensor \((D, f)\) may be pointwise if \( D = \{ b \}, \) with \( b \in \Omega \) and \( f = \delta (\cdot - b), \) whereas \( \delta \) is the mass of Dirac focused in \( b. \) Then the measurement output function (2) formulated by [1-3],

\[\mathcal{Y}(t) = \int_{\Omega} w(\zeta, t) \delta_b(\zeta - b) d\zeta = w( b, t)\]

2- So, in the zone circumstance, \( D \subset \Omega \) as well as \( f \in L^2(D). \) Hence the measurement function

\[\mathcal{Y}(t) = \int_{D} w(\zeta, t) f(\zeta) d\zeta.\]

Definition 3.2: The couple \((D, f)\) is RBGS - Sensor, if the linked ADPD - System is WRBG - Observable.

Definition 3.3: Sensors \((D_i, f_i)_1 \leq i \leq q\) are RBGS - Sensor, if one of them symbolized by \((D_i, f_i)\) is RBGS - Sensor.

Proposition 3.4: The couple \((D, f)\) is RBGS - Sensor if and only \( N_\delta = H H^* \) represent a positive definite operator.

Proof: As \((D, f)\) is RBGS - Sensor means that the linked ADPD - System is WRBG - Observable. Thus, if \( w^* \in (H^{1/2}(\overline{\Omega}))^n, \) achieves the subsequent

\[\langle N_\delta w^*, w^* \rangle_{(H^{1/2}(\overline{\Omega}))^n} = 0, \text{ then } \|H^* w^*\|_Y = 0\]

Henceforth \( w^* \in \ker H^*, \) thus \( w^* = 0, \) i.e., \( N_\delta \) is positive definite.

Conversely, let \( w^* \in (H^{1/2}(\overline{\Omega}))^n, \) such that

\[H^* w^* = 0, \text{ then } \langle H^* w^*, H^* w^* \rangle_Y = 0\]

and thus,

\[\langle N_\delta w^*, w^* \rangle_{(H^{1/2}(\overline{\Omega}))^n} = 0.\]

Thus \( w^* = 0, \) therefore, the linked ADPD - System is WRBG - Observable.

and then, \((D, f)\) is RBGS - Sensor. □

Proposition 3.5: The couple \((D, f)\) is RBGS - Sensor, if the linked ADPD - System is ERBG - Observable.

Proof: As the ADPD - System is ERBG - Observable. Thus,

\[\text{Im } H_\delta = (H^{1/2}(\overline{\Omega}))^n\]

As well known \((H^{1/2}(\partial \Omega)) \) is HS. So that leads to the form

\[\ker X_\delta + \text{Im } X_\delta^* X_\delta \mathcal{V} \mathcal{K}^* = (H^{1/2}(\partial \Omega)) \]

we obtain that,

\[\ker \mathcal{K}(t) \mathcal{V} \mathcal{Y} X_\delta^* = \{ 0 \}\]

and this is equivalent to

\[\text{Im } X_\delta^* \mathcal{V} \mathcal{K}^* = (H^{1/2}(\overline{\Omega}))^n.\]

Later, the connected ADPD - System is WRBG - Observable. Consequently, \((D, f)\) stays RBGS - Sensor. □

Remark 3.6: As of the preceding outcomes, we can realized the next:
(I) An ADPD – System is EBG – Observable, then the ADPD – System is WRGB – Observable.

(II) If a couple (𝒟, 𝑓) is RBGS – Sensor in 𝓂₁ for an ADPD – System , then it is RBGS – Sensor in 𝓂₂ subregion of 𝓂₁.

3.2. The Main Results

This part concerns with developing the consequences to the concept of RBGS – Sensors in the corresponding ADPD – System , and presents the enough conditions for such sensor. So that, it is assumed that there is (𝜑ₙ)ₙ≠0, 𝑛∈ℕ, of 𝐄 in 𝐇¹(Ω) denotes a set of eigenfunctions [10], associated with eigenvalue 𝛾ₙ of multiplicity 𝑚ₙ and 𝑚ₙ = supₙ∈ℕ 𝑚ₙ is finite. For 𝑏 = (𝑤₁,...,𝑤ₙ−1) and 𝑛 = (𝑛₁,...,𝑛ₙ−1). Suppose that the function 𝜓_{𝑖𝑗}(𝑏) = 𝑥_{𝑖𝑗}∀ 𝜑ₙ(𝑤), 𝑛 ∈ 𝐼, is a complete set in (𝐻¹(Ω))ⁿ. If the reflected DDP – System (1) where 𝑆 satisfies instability property. Thus, the succeeding outcome can be obtained.

Theorem 3.7: Suppose that sup 𝑚ₙ = 𝑚 < ∞, then the couples (𝒟, 𝑓)₁≤𝑖≤𝑞 are RBGS – Sensors iff

1. 𝑞 ≥ 𝑚,
2. rank 𝑔ₙ = 𝑚ₙ, for all 𝑛 ≥ 1, where 𝑔ₙ = (𝑔ₙ)₁≤𝑖≤𝑞 with 1 ≤ 𝑖 ≤ 𝑞, 1 ≤ 𝑗 ≤ 𝑚ₙ, and

   (𝑔ₙ)₁≤𝑖≤𝑞 = \left\{ \begin{array}{ll}
   \sum_{ℎ=1}^{𝑛} \frac{∂\phi}{∂𝑤} (ℎ₁) & \text{point wise sensor} \\
   \sum_{ℎ=1}^{𝑛} \frac{∂\phi}{∂𝑤} (ℎ₁) & \text{zone sensor}
   \end{array} \right.

Proof: First, we evoke that the ADPD – System is WRGB – Observable, this means:

[∀ 𝜒 \in (𝐻¹(Ω))ⁿ | ∫Ω 𝑥_1 𝜒 = 0 ⇒ 𝜒 = 0],

which allows to state that the couples (𝒟, 𝑓)₁≤𝑖≤𝑞 are RBGS – Sensors iff

\{ 𝜒 \in (𝐻¹(Ω))ⁿ | ∫Ω 𝑥_1 𝜒 = 0 \} = \{ 0 \} [12].

By supposing that the couples (𝒟, 𝑓)₁≤𝑖≤𝑞 are RBGS – Sensors, but for a certain 𝑛 ∈ 𝑁, then, rank 𝑔ₙ ≠ 𝑚ₙ, i.e.:

∀ 𝑤ₙ = (𝑤₁, 𝑤₂, ..., 𝑤ₙ) \neq 0,

such that

𝑔ₙ𝑤ₙ = 0, 𝑤₀ = ∑₁≤𝑖≤𝑛 𝑤_{𝑛𝑖} 𝜓_{𝑖𝑛} ∈ 𝐇¹(Ω) ≠ 0

So, we can rebuild a non-zero 𝑤₀ ∈ 𝐇¹(Ω) in considering

\{ 𝑤₀, 𝜓_{𝑖𝑛}\}ᵦ¹(Ω) = 0.

If 𝑝 ≠ 𝑛, and \{ 𝑤₀, 𝜓_{𝑖𝑛}\}ᵦ¹(Ω) = 𝜓_{𝑖𝑛}, 1 ≤ 𝑖 ≤ 𝑚ₙ, then

\{ 𝐻𝐿, 𝑤₀\}ᵦ¹(Ω) = ∑₀≤𝑛=₁ (𝑥₀, 𝑥₀, 𝑥₀, 𝑤₀) = 0,

where 𝜓 maybe signifies as a solution of the ensuing form

\begin{align}
\frac{∂\ddot{x}}{∂𝑡} (\zeta, 𝑡) &= Δ^∗ 𝜓(\zeta, 𝑡) + \sum_{i=1}^{𝑞} 𝑦_{𝑖}(𝑇 − 𝑡) \quad \text{Π}_T \\
\frac{∂\ddot{x}}{∂𝑡} (\zeta, 0) &= 0 \quad \text{Π}_T \\
\frac{∂\ddot{x}}{∂𝑡} (\mu, 𝑡) &= 0 \quad \text{Π}_T
\end{align}

(10)

Now, consider the system

\begin{align}
\frac{∂\psi}{∂𝑡} (\zeta, 𝑡) &= −Δ\phi(\zeta, 𝑡) \quad \text{Π}_T \\
\frac{∂\psi}{∂𝑡} (\zeta, 0) &= 𝑦_{𝑖} 𝑥_{𝑖} 𝑤₀ \quad \text{Π}_T \\
\frac{∂\psi}{∂𝑡} (\mu, 𝑡) &= 0 \quad \text{Π}_T
\end{align}

(11)
multiply (10) by $\frac{\partial \psi}{\partial \xi}$ and integrate on $\Pi_{\tau}$, we get that

$$
\int_{\Pi_{\tau}} \frac{\partial \psi}{\partial \xi}(\zeta, t) \frac{\partial \varphi}{\partial t}(\zeta, t) \, d\zeta \, dt = \int_{\Pi_{\tau}} \Delta^* \bar{\omega}(\zeta, t) \frac{\partial \psi}{\partial \xi}(\zeta, t) \, d\zeta \, dt
$$

But, we have

$$
\int_{\Pi_{\tau}} \frac{\partial \psi}{\partial \xi}(\zeta, t) \frac{\partial \varphi}{\partial t}(\zeta, t) \, d\zeta \, dt = \int_{\partial \Omega} \frac{\partial \varphi}{\partial \xi}(\zeta, t) \bar{\omega}(\zeta, t) \, d\zeta + \int_{\Pi_{\tau}} \Delta \bar{\omega}(\zeta, t) \frac{\partial \psi}{\partial \xi}(\zeta, t) \, d\zeta \, dt.
$$

Integrating by parts, we obtain

$$
\int_{\partial \Omega} \frac{\partial \varphi}{\partial \xi}(\zeta, t) \bar{\omega}(\zeta, t) \, d\zeta = -\int_{\Pi_{\tau}} \frac{\partial \varphi}{\partial \xi}(\zeta, t) \bar{\omega}(\zeta, t) \, d\zeta + \frac{\partial}{\partial t} \int_{\Pi_{\tau}} \frac{\partial \psi}{\partial \xi}(\zeta, t) \bar{\omega}(\zeta, t) \, d\zeta dt.
$$

the boundary conditions give

$$
\int_{\partial \Omega} \frac{\partial \varphi}{\partial \xi}(\zeta, t) \bar{\omega}(\zeta, t) \, d\zeta = \int_{\Pi_{\tau}} \frac{\partial \psi}{\partial \xi}(\zeta, t) \bar{\omega}(\zeta, t) \, d\zeta dt.
$$

Thus,

$$
\int_{\partial \Omega} \psi(\zeta, t) \frac{\partial \varphi}{\partial \xi}(\zeta, t) \bar{\omega}(\zeta, t) \, d\zeta = -\sum_{i=1}^{q} \int_{0}^{T} \frac{\partial \psi}{\partial \xi}(b_i, t) \bar{\omega}(T - t) \, dt.
$$

and, we have

$$
(X_3)^{\gamma} \nabla^\gamma (\omega)_{\nu(\gamma)} = \sum_{K=1}^{n} \int_{0}^{T} \frac{\partial \varphi}{\partial \xi}(\zeta, t) \psi(\zeta, t) \, d\zeta = -\sum_{K=1}^{n} \int_{0}^{T} \int_{K=1}^{n} \frac{\partial \psi}{\partial \xi}(b_i, t) \bar{\omega}(T - t) \, dt.
$$

but,

$$
\psi(\zeta, t) = \sum_{p=1}^{\infty} e^{-\lambda_p(T-t)} \sum_{j=1}^{m_p} \omega_{\nu(p, p, \nu)} \mathcal{L}^2(\omega) \psi_{p, j}
$$

Then,

$$
\sum_{K=1}^{n} \frac{\partial \psi}{\partial \xi}(b_i, t) = \sum_{p=1}^{\infty} e^{-\lambda_p(T-t)} \sum_{j=1}^{m_p} \omega_{\nu(p, p, \nu)} \mathcal{L}^2(\omega) \sum_{K=1}^{n} \frac{\partial \psi}{\partial \xi}(b_i) = \sum_{p=1}^{\infty} e^{-\lambda_p(T-t)} (G_p \omega_p)_i.
$$

therefore,

$$
(X_3)^{\gamma} \nabla^\gamma (\omega)_{\nu(\gamma)} = -\sum_{K=1}^{n} \int_{0}^{T} \sum_{p=1}^{\infty} e^{-\lambda_p(T-t)} (G_p \omega_p) \mathcal{L}^2(\omega) \psi_{p, j} \, dt.
$$

thus,

$$
(X_3)^{\gamma} \nabla^\gamma (\omega)_{\nu(\gamma)} = -\sum_{i=1}^{q} \int_{0}^{T} e^{-\lambda_n(T-t)} (G_n \omega_n)_i \bar{\omega}(T - t) \, dt = 0, \forall \, \bar{\omega} \in \mathcal{L}^2(0, T, R^q).
$$
Subsequently, \( \omega_0 \in \ker H_0^* \) this is conflicted to the hypotheses. Thus, \((D_i f_i)_{1 \leq i \leq q}\) are RBGS – Sensors for the ADPD – System (4).

### 3.3. Internal and RBG-Reconstruction via Internal Region

RBGS – Sensors problem for ADPD – System may be seen as internal RGS – Sensors, if we deliberate \( \overline{\omega}_r \subset \overline{U} \) [27].

- Let \( \mathcal{R} \) is an linear operator of extension continuous type which is represented via

\[
\mathcal{R}: (H^{1/2}(\partial U))^n \to (H^1(\Omega))^n, \quad \text{such that} \quad y \mathcal{R} h(\zeta, t) = h(\zeta, t), \quad \text{for all} \quad h(\zeta, t) \in (H^{1/2}(\partial U))^n \tag{12}
\]

- For \( r > 0 \) any real number such that satisfactorily small we can define

\[
E = \bigcup_{\omega \in \mathbb{R}} B(\omega, r), \quad \overline{\omega}_r = E \cap \overline{U} \quad \text{and} \quad \overline{\mathcal{R}}_r = \overline{\omega}_r \cap \partial U,
\]

where \( B(\omega, r) \) is a ball radius \( r \) focused in \( \omega(\zeta, r) \), so \( \overline{\mathcal{R}}_r \) is a subregion of \( \overline{\omega}_r \) (Figure 2).

![Fig.2: Internal region \( \omega_r \) and boundary \( \overline{\omega}_r \).](image)

In the next consequences, we demonstrate that the link between the RBGS – Sensors problem and \( \overline{\omega}_r \) GS – Sensors.

**Proposition 3.8:**

(i) If the couples \((D_i f_i)_{1 \leq i \leq q}\) are \( \overline{\omega}_r \) GS – Sensors in ADPD – System, then, there are RBGS – Sensors.

(ii) If the ADPD – System is \( E\overline{\omega}_r G – Observable \) then, the couples \((D_i f_i)_{1 \leq i \leq q}\) then, there are RBGS – Sensors.

**Remark 3.9:** As of the preceding outcomes, then, we have:

(i) If the ADPD – System is \( E\overline{\omega}_r G – Observable \) then, it is ERBG – Observable, i.e., \( \exists X_{\overline{\omega}_r} Y: \mathcal{Y} \to (H^{1}(\omega_r))^n \) an operator given by

\[
H_{\overline{\omega}_r} Y(\zeta, t) = X_{\overline{\omega}_r} Y(\zeta, t) = X_{\overline{\omega}_r} \mathcal{R} \zeta, t). \]

Hence,

\[
X_{\zeta} \big( Y X_{\overline{\omega}_r} Y(\zeta, t) \big) = \omega (\zeta, t).
\]

where \( \omega (\zeta, t) \in (H^{1/2}(\overline{\mathcal{R}}))^n \) and \( \overline{\omega}(\zeta, t) \) be an extension to \((H^{1/2}(\partial U))^n \).

(iii) If the ADPD – System is \( W\overline{\omega}_r G – Observable \) then, it is WRGB – Observable.

(ii) An development of the outcomes can be employed for diverse issues of RG–Observability [5, 29], and to the RBG – Observability of asymptotic reduced case in ADPD – Systems [7].

### 4. Applications Of Some Sensor Locations

This part is devoted to the application of these outcomes for ADPD – System described in \( U = [0, a_1] \times [0, a_2] \), via the form
he supports $m\pi - \pi - \pi + n\pi \notin \mathcal{W}$

\begin{align}
\frac{\partial^2 w}{\partial t^2} (\zeta_1, \zeta_2, t) &= \frac{\partial^2 w}{\partial \zeta_1^2} (\zeta_1, \zeta_2, t) + \frac{\partial^2 w}{\partial \zeta_2^2} (\zeta_1, \zeta_2, t) + \nu (\zeta_1, \zeta_2, t) \quad \Pi_T \\
\omega (\zeta_1, \zeta_2, 0) &= \omega_0 (\zeta_1, \zeta_2) \quad \bar{\Omega} \\
\frac{\partial w}{\partial \nu} (\mu_1, \mu_2, t) &= 0 \quad \bar{\zeta}_T
\end{align}

(13)

where $\bar{\Omega} = \{0, a_2 \times \{a_2\} \text{ or } \bar{\Omega} = \{a_1\} \times \{0, a_2\}$, the eigenfunctions of the system (13) is given by

$$
\psi_{nm} (\zeta_1, \zeta_2) = \frac{2}{\sqrt{a_1 a_2}} \cos n\pi \frac{\zeta_1}{a_1} \cos m\pi \frac{\zeta_2}{a_2}
$$

(14)

associated with eigenvalues

$$
\lambda_{nm} = \frac{-n^2 \pi^2}{a_1^2} - \frac{m^2 \pi^2}{a_2^2}, \quad n, m \geq 1
$$

(15)

If we assume that $a_1^2 / a_2^2 \notin \mathcal{Q}$, and hence $\lambda_{nm}$ is the multiplicity of $r_{nm} = 1$. Consequently the couple $(\mathcal{D}, f)$ may be enough to realize $RBG - \text{Observability}$ of the observed ADPD – System as in [3-6]. Now, in the following outcomes give information on the location of (pointwise and zone) $RBGS - \text{Sensors}$.

4.1 Sensor of Zone Type

This sub-section will be devoted to study the subsequent cases.

4.1.1 Case Figure3

Take into consideration the ADPD – System (13) with the measurement equation (2) which is formulated via

$$
\mathcal{Y}(t) = \int_{\mathcal{D}} \omega (\zeta_1, \zeta_2, t) f(\zeta_1, \zeta_2) d\zeta_1 d\zeta_2
$$

(16)

with the couple $(\mathcal{D}, f)$ sensor of type zone is placed in the domain $\bar{\Omega}$, done the supports $\mathcal{D} = [\zeta_1 - \ell_1, \zeta_1 + \ell_1] \times [\zeta_2 - \ell_2, \zeta_2 + \ell_2] \in \bar{\Omega}$ as in (Figure3).

Fig 3: Internal zone sensor $\mathcal{D}$.

Then, we have the subsequent consequence.

Proposition 4.1: If $f$ satisfies symmetry property with around to $\zeta = (\zeta_1, \zeta_2)$, so the couple $(\mathcal{D}, f)$ is $RBGS - \text{Sensor}$ in $\bar{\Omega} = \{0, a_2 \times \{a_2\}$ for the ADPD – System (13 – 16) , if

$$
\frac{n_\varphi \ell_1}{a_1} \text{ and } \frac{m_\varphi \ell_2}{a_2} \in \mathcal{Q}, \text{ for all } n_\varphi, m_\varphi = \{1, \ldots, f\}.
$$

4.1.2 Boundary Zone Case

We discuss this case as the follows:

1. Case of Figure3

In this case, where $\bar{\Omega}_0 = [\mu_0 - \ell_1, \mu_0 + \ell_1] \times \{a_2\}$ is the support of the boundary sensor and $f \in L^2(\bar{\Omega}_0)$ as in (Figure4).
The measurements are shown by the output function
\[ Y(t) = \int_{\delta_0} \frac{\partial \omega}{\partial v}(\mu_1, \mu_2, t) f(\mu_1, \mu_2) d\mu_1 d\mu_2 \]  

(17)

Then, we arrive to the result:

**Proposition 4.2:**
Assume that the sensors \((\delta_0, f)\) are located on \(\delta_0 \subset \partial \Omega\) and \(f\) is symmetric with respect to \(\mu_1 = \mu_1^0\), then the couple \((\delta_0, f)\) is RBGS \(-\) Sensor in \(\delta = \{a_1\} \times ]0, a_2[ \subset \partial \Omega\) for the ADPD \(-\) System (13 \(-\) 17), if

\[ \frac{n \mu_1^0}{a_1} \in Q, \text{ for all } n = \{1, \ldots, J\}. \]

**2. Figure 5 case**

In this case, where \(\delta \subset \partial \Omega\) is the support of the boundary sensor and \(f \in L^2(\delta)\) as in (Figure 5).

Now, \(\delta = \{a_1\} \times ]0, a_2[ \subset \partial \Omega\) is the observed region and the measurements are shown by the output

\[ Y(t) = \int_{\delta} \frac{\partial \omega}{\partial v}(\mu_1, \mu_2, t) f(\mu_1, \mu_2) d\mu_1 d\mu_2 \]  

(18)

Then, we reach to the subsequent consequence.

**Proposition 4.3:**
Suppose that \((\delta, f)\) to be the situated sensors on \(\delta = \delta_1 \cup \delta_2 = [0, \bar{\mu}_1 + \ell_1] \times \{0\} \cup \{0\} \times [0, \bar{\mu}_2 + \ell_2] \subset \partial \Omega\) and \(f|_{\delta_1}\) is symmetric around to \(\bar{\mu}_1 = \bar{\mu}_1^0\) and \(f|_{\delta_2}\) is symmetric around to \(\bar{\mu}_2 = \bar{\mu}_2^0\), then the couple \((\delta, f)\) is RBGS \(-\) Sensor on \(\delta\) for the ADPD \(-\) System (13 \(-\) 18), if

\[ \frac{n \bar{\mu}_1^0}{a_1} \text{ and } \frac{m \bar{\mu}_2^0}{a_2} \in Q, \text{ for all } n, m = \{1, \ldots, J\}. \]

This indicates that the (RBG \(-\) observability) relies on the sensors support shape and measurements equation.

**4.2 Sensor of Pointwise type**

This sub-section is devoted for discussing and describing the RBGS \(-\) Sensor on \(\delta\) for the ADPD \(-\) System indifferent situations.

**4.2.1 Internal Pointwise Sensor**

In this situation, we have two cases:
**I Pointwise case:**

The output equation described by

\[ Y(t) = \int_0 w(\zeta_1, \zeta_2, t) \delta(\zeta_1 - b_1, \zeta_2 - b_2) d\zeta_1 d\zeta_2 \]  

where \( b = (b_1, b_2) \) is the sensor pointwise position in \( U = [0, a_1] \times [0, a_2] \) as defined in (Figure 6).

---

**Proposition 4.4:** If \( n b_1/a_1 \) and \( m b_2/a_2 \notin Q \), for all \( n, m = \{1, \ldots, J\} \), then the couple \( b, \delta_b \) is RBGS - Sensor on \( \Omega = [0, a_2] \times \{a_2\} \) for the ADPD - System (13 - 19).

**II Filament case:**

Deliberate the case where the measurement information is given via the curve \( \beta = Im(\rho) \) such that \( \rho \in \mathbb{C}^1(0,1) \) (Figure 7).

---

**Proposition 4.5:** Assume that the \( \beta \) satisfy symmetry property around line filament \( b = (b_1, b_2) \), if \( n b_1/a_1 \) and \( m b_2/a_2 \notin Q \), for all \( n, m = \{1, \ldots, J\} \), then the couple \( (\beta, \delta_\beta) \) is RBGS - Sensor on \( \Omega = [0, a_2] \times \{a_2\} \) for the ADPD - System (13 - 19).

**4.2.2 Boundary Pointwise Sensor**

Assume that the sensor \( (b, \delta_b) \) is placed on \( b \), where \( b = (b_1, b_2) \in \partial \Omega \) such that \( b = (0, b_2) \) by way of (Figure 8).

The output function is got by

\[ Y(t) = \int_{\partial \Omega} w(\mu_1, \mu_2, t) \delta(0, \mu_2 - b_2) d\mu_1 d\mu_2 \]  

Therefore, we acquire the subsequent outcomes.
Proposition 4.6:
The couple \((\beta, \delta_p)\) is RBGS – Sensor on \(\mathcal{X} \times \mathbb{R}^n\) for the ADPD – System if \(\frac{m_2}{a_2} \in \mathcal{Q}\), for all \(m \in \{1, \ldots, J\}\).

Conclusions
This work has been tackled RBGS- sensors concept for the ADPD-System under which situation accomplishes the unknown gradient of the initial state. Additionally the associations of WRBG- observability and ERBG-observability notions have been deliberated and examined in a region \(\mathcal{X}\). So, for DDP-Systems in HS, many remarkable consequences concerning the choice of sensor constructing which are demonstrated in discreet results. Finally, we have specified that there is a linking between the RBGS- sensor with number, sensors characters and related domains. Many complications are not treated, the likelihood to develop these outcomes to the case of HS in quasi forms.

Conflict of interest
The authors declare that they have no conflict of interest.

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