DEFORMATION QUANTIZATION OF MODULI SPACES OF HIGGS BUNDLES ON A RIEMANN SURFACE WITH TRANSLATION STRUCTURE

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Abstract. Let $X$ be a compact connected Riemann surface of genus $g \geq 1$ equipped with a nonzero holomorphic 1-form. Let $\mathcal{M}_X(r)$ denote the moduli space of semistable Higgs bundles on $X$ of rank $r$ and degree $r(g-1)+1$; it is a complex symplectic manifold. Using the translation structure on the open subset of $X$ where the 1-form does not vanish, we construct a natural deformation quantization of a certain nonempty Zariski open subset of $\mathcal{M}_X(r)$.

1. Introduction

Let $X$ be a compact connected Riemann surface of genus $g$, with $g \geq 1$. Take a holomorphic 1-form

$$\beta \in H^0(X, K_X) \setminus \{0\}.$$ 

So $\beta$ defines a translation structure on the Zariski open subset

$$X_0 := \{ x \in X \mid \beta(x) \neq 0 \} \subset X.$$ 

This means that $X_0$ is covered by a distinguished class of holomorphic coordinate functions such that all the transition functions are translations of $\mathbb{C}$. Consider the total space of the holomorphic cotangent bundle $K_{X_0}$ of $X_0$ equipped with the Liouville symplectic form. Using the translation structure on $X_0$, we construct a deformation quantization of this symplectic manifold (see Proposition 3.2); the definition of a deformation quantization is recalled in Section 2.1 (more details on deformation quantization can be found in [BFPLS], [DWL] and references therein).

It may be mentioned that if $X$ is equipped with a projective structure, then the complement $K_X \setminus \{0_X\}$ of the zero section of $K_X$, equipped with the Liouville symplectic form, has a natural deformation quantization [BB].

For any integer $n \geq 1$, consider

$$K^n_0 := \{(x_1, \cdots, x_n) \in (K_X)^n \mid x_i \neq x_j \ \forall \ i \neq j\} ;$$

the Liouville symplectic form on $K_X$ produces a holomorphic symplectic form on $K^n_0$, which is denoted by $\Omega'_{K^n}$. The above mentioned deformation quantization of $K_{X_0}$ produces

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a deformation quantization of the Zariski open subset $K_0^n \cap (K_{X_0})^n \subset (K_X)^n$ equipped with the symplectic form $\Omega'_K|_{K_0^n \cap (K_{X_0})^n}$.

The symplectic form $\Omega'_K$ on $K_0^n$ is preserved under the natural action of the symmetric group $\mathbb{S}_n$ of permutations of \{1, \cdots, n\}. Since the action of $\mathbb{S}_n$ on $K_0^n$ is free, we get a holomorphic symplectic form on the manifold $K_0^n/\mathbb{S}_n$ given by $\Omega''_K$, which is denoted by $\Omega''_{K_0^n}$. The deformation quantization of $K_0^n \cap (K_{X_0})^n$ is also invariant under the natural action of $\mathbb{S}_n$ on $K_0^n \cap (K_{X_0})^n$. Consequently, it produces a deformation quantization of the symplectic structure $\Omega''_{K_0^n}$ on $(K_0^n \cap (K_{X_0})^n)/\mathbb{S}_n$.

For any integer $r \geq 1$, let $\mathcal{M}_X(r)$ denote the moduli space of semistable Higgs bundles on $X$ of rank $r$ and degree $r(g-1)+1$; it is a smooth complex quasiprojective variety of dimension $2(r^2(g-1)+1)$ equipped with an algebraic symplectic form. We will denote the symplectic form on $\mathcal{M}_X(r)$ by $\Omega_M$.

Set $\delta = r^2(g-1)+1$. The moduli space $\mathcal{M}_X(r)$ is birational to $K_0^\delta/\mathbb{S}_\delta$ [GNR], [ER]. More precisely, there is a nonempty Zariski open subset $\tilde{U}_M \subset \mathcal{M}_X(r)$ and a nonempty Zariski open subset $\tilde{U}_S \subset (K_0^n \cap (K_{X_0})^n)/\mathbb{S}_n$ together with a natural algebraic isomorphism

$$\tilde{\Phi} : \tilde{U}_M \rightarrow \tilde{U}_S$$

such that

$$\tilde{\Omega}_M := \Omega_M|_{\tilde{U}_M} = \tilde{\Phi}^*\tilde{\Omega}_S,$$

where $\tilde{\Omega}_S := \Omega''_{K_0^n}|_{\tilde{U}_S}$.

The above mentioned deformation quantization of the symplectic manifold

$$\left((K_0^\delta \cap (K_{X_0})^\delta)/\mathbb{S}_\delta, \Omega''_{K}((K_0^\delta \cap (K_{X_0})^\delta)/\mathbb{S}_\delta)\right)$$

produces a deformation quantization of the symplectic manifold $(\tilde{U}_S, \tilde{\Omega}_S)$. Using the above isomorphism $\tilde{\Phi}$, this produces a deformation quantization of the symplectic manifold $(\tilde{U}_M, \tilde{\Omega}_M)$; see Theorem 4.1.

2. Constant symplectic form and its quantization

2.1. Deformation quantization. Let $Y$ be a connected complex manifold. Its holomorphic tangent and cotangent bundles will be denoted by $TY$ and $T^*Y$ respectively. Assume that $Y$ is equipped with a holomorphic symplectic form $\Theta$. The closed 2–form $\Theta$ defines a holomorphic homomorphism

$$\eta : TY \rightarrow T^*Y$$
that sends any \( v \in T_xY \) to \( -i_v(\Theta(x)) \in T_xY \), so \( \eta(x)(w)(v) = \Theta(v, w) \) for all \( v, w \in T_xY \) and \( x \in Y \). This homomorphism \( \eta \) is an isomorphism, because \( \Theta \) is nondegenerate.

Define

\[
\tau := \eta^{-1} : T^*Y \rightarrow TY.
\]

For any two holomorphic functions \( f_1, f_2 \) defined on some open subset \( U \subset Y \), define the holomorphic function on \( U \)

\[
\{f_1, f_2\} := \Theta(\tau(df_1), \tau(df_2)).
\] (2.1)

We have

- \( \{f_1, f_2\} = -\{f_2, f_1\} \),
- \( \{f_1, f_2, f_3\} = \{f_1, f_2\} \cdot f_3 + \{f_1, f_3\} \cdot f_2 \), and
- \( \{f_1, \{f_2, f_3\}\} + \{f_2, \{f_3, f_1\}\} + \{f_3, \{f_1, f_2\}\} = 0. \)

So the operation \( \{-, -\} \) defines a Poisson structure on \( Y \).

The algebra of locally defined holomorphic functions on \( Y \) will be denoted by \( \mathcal{H}(Y) \).

Let

\[
\mathcal{A}(Y) := \mathcal{H}(Y)[[h]] := \left\{ \sum_{i=0}^{\infty} h^i f_i \bigg| f_i \in \mathcal{H}(Y) \right\}
\]

be the space of all formal power series.

Consider an associative algebra operation

\[
\mathcal{A}(Y) \times \mathcal{A}(Y) \rightarrow \mathcal{A}(Y).
\]

The image of any pair \((f, g)\), where \( f = \sum_{i=0}^{\infty} h^i f_i \) and \( g = \sum_{i=0}^{\infty} h^i g_i \), is an element

\[
f \star g = \sum_{i=0}^{\infty} h^i \alpha_i \in \mathcal{A}(Y).
\] (2.2)

A deformation quantization of the Poisson manifold \((Y, \{-, -\})\) is an associative algebra operation “\( \star \)” as in (2.2) satisfying the following conditions:

1. each \( \alpha_i \) in (2.2) is some polynomial in the derivatives, of arbitrary order, of \( \{f_i\}_{i \geq 0} \) and \( \{g_i\}_{i \geq 0} \) (it should be emphasized that the polynomial \( \alpha_i \) itself is independent of \( f \) and \( g \)),
2. \( \alpha_0 = f_0 g_0 \),
3. \( 1 \star t = t = t \star 1 \) for all \( t \in \mathcal{H}(Y) \), and
4. \( f \star g - g \star f = \sqrt{-1}h\{f_0, g_0\} + h^2 \gamma \), where \( \gamma \in \mathcal{A}(Y) \) (it depends on \( f, g \)).

(See [BFFLS], [DWL], [Fe], [We] for more details.)

2.2. Moyal–Weyl deformation quantization. We will now recall an explicit deformation quantization of a constant symplectic form on a vector space.

Let \( V \) be a complex vector space of even dimension, say \( 2n \). Fix a constant symplectic form

\[
\Theta_0 \in \bigwedge^2 V^*.
\]
on \( V \); in other words, \( \Theta_0 \) is nondegenerate. Following the notation of Section 2.1, the space of all locally defined holomorphic functions on \( V \) (respectively, \( V \times V \)) is denoted by \( \mathcal{H}(V) \) (respectively, \( \mathcal{H}(V \times V) \)). The form \( \Theta_0 \) defines a Poisson structure on \( \mathcal{H}(V) \); see (2.1). As before, the Poisson structure will be denoted by \{−, −\}. Define as before

\[
\mathcal{A}(V) := \mathcal{H}(V)[[h]] := \left\{ \sum_{i=0}^{\infty} h^i f_i \in \mathcal{H}(V) \right\}.
\]

For any \( f_1, f_2 \in \mathcal{H}(V) \), the element of \( \mathcal{H}(V \times V) \) defined by \( (v, w) \mapsto f_1(v) \cdot f_2(w) \), where \( v, w \in V \), will be denoted by \( f_1 \otimes f_2 \). Let

\[
\Delta : V \to V \times V, \quad v \mapsto (v, v)
\]

be the diagonal embedding. It defines a homomorphism

\[
\Delta^* : \mathcal{H}(V \times V) \to \mathcal{H}(V), \quad \Delta^*(f)(v) = f(v, v).
\]

There is a unique differential operator with constant coefficients

\[
D : \mathcal{H}(V \times V) \to \mathcal{H}(V \times V)
\]

that satisfies the following condition: for any \( f_1, f_2 \in \mathcal{H}(V) \),

\[
\{f_1, f_2\} = \Delta^* D(f_1 \otimes f_2)
\]

(see [Fe], [We]).

The **Moyal–Weyl deformation quantization** of the Poisson manifold \((\mathcal{H}(V), \{-, -\})\) is defined by the following conditions:

1. For all \( f_1, f_2 \in \mathcal{H}(V) \),

\[
f_1 \star f_2 = \Delta^* \exp(\sqrt{-1} hD/2)(f_1 \otimes f_2) \in \mathcal{A}(Y),
\]

where \( D \) is defined in (2.3), and

2. Extend the above multiplication operation \( \star \) to \( \mathcal{A}(V) \) using the bilinearity condition with respect to the formal parameter \( h \). In other words, for \( f, g \in \mathcal{A}(V) \) with \( f = \sum_{i=0}^{\infty} h^i f_i \) and \( g = \sum_{i=0}^{\infty} h^i g_i \), define

\[
f \star g = \sum_{i=0}^{\infty} h^i \left( \sum_{j=0}^{i} f_j \star g_{i-j} \right) \in \mathcal{A}(V).
\]

This multiplication operation \( \star \) is a deformation quantization of the Poisson manifold \((\mathcal{H}(V), \{-, -\})\) (see [We], [Fe] and references therein). It is known as the **Moyal–Weyl deformation quantization**.

Take any \( v_0 \in V \). Define the translation map

\[
\mathcal{T}_0 : V \to V, \quad v \mapsto v + v_0.
\]

Let

\[
\mathcal{T} : \mathcal{A}(V) \to \mathcal{A}(V)
\]

be the automorphism that sends any \( f = \sum_{i=0}^{\infty} h^i f_i \) to \( \sum_{i=0}^{\infty} h^i (f_i \circ \mathcal{T}_0) \in \mathcal{A}(V) \).
Lemma 2.1. For any $f, g \in \mathcal{A}(V)$,
\[
\mathcal{T}(f \ast g) = \mathcal{T}(f) \ast \mathcal{T}(g).
\]

Proof. Recall that the differential operator $D$ in (2.3) has constant coefficients. From this it follows immediately that
\[
(f_1 \circ \mathcal{T}_0) \ast (f_2 \circ \mathcal{T}_0) = \mathcal{T}(f_1 \ast f_2)
\]
for all $f_1, f_2 \in \mathcal{H}(V)$ (see (2.4)). The lemma follows from this. \hfill \square

It may be mentioned that the Moyal–Weyl deformation quantization has the important property that it is invariant under the action of the symplectic group $\text{Sp}(V, \Theta_0) \subset \text{GL}(V)$ on $V$, however, we won’t need it here.

3. One-form on Riemann surfaces

Let $X$ be a compact connected Riemann surface of genus $g$, with $g \geq 1$. The holomorphic cotangent bundle of $X$ will be denoted by $K_X$. Fix a nonzero holomorphic 1-form
\[
\beta \in H^0(X, K_X) \setminus \{0\}
\]
Consider the zero-set
\[
Z_\beta := \{x \in X \mid \beta(x) = 0\} \subset X.
\]
Its cardinality is at most $\text{degree}(K_X) = 2g - 2$, and $Z_\beta = \emptyset$ if $g = 1$. Let
\[
X_0 := X \setminus Z_\beta
\]
be the complement.

Take any simply connected open subset $U \subset X_0$. There is a holomorphic function $f_U$ on $U$ such that $df_U = \beta|_U$. This condition uniquely determines $f_U$ up to an additive constant, meaning for any $\tilde{f}_U$ with $d\tilde{f}_U = \beta|_U$, there is a constant $c \in \mathbb{C}$ such that
\[
\tilde{f}_U = f_U + c.
\]
Since $\beta$ does not vanish on $U$, the above function $f_U$ is an immersion. So for every $x \in U$, there is an open neighborhood $x \in U_x \subset U$ such that the restriction $\tilde{f}_U|_{U_x}$ is an embedding. Hence $\tilde{f}_U$ is a holomorphic coordinate function on $U_x$. As $x$ runs over points on $X_0$, we get a holomorphic coordinate atlas on $X_0$ satisfying the condition that all the transition functions are translations of $\mathbb{C}$. Such a holomorphic coordinate atlas defines a translation structure on $X_0$. So $\beta$ defines a translation structure on $X_0$ (see [EMZ] and references therein for translation structure). In fact, $\beta$ defines a branched translation structure on entire $X$ (see [LF], [Ka], [BJJP], [De]); see [BD] for general branched structures.

On $\mathbb{C}^2$ we have the standard constant symplectic form
\[
\omega_0 := (dz_2) \wedge (dz_1).
\]
Identify $\mathbb{C}^2$ with the total space of the holomorphic cotangent bundle $T^*\mathbb{C}$ by sending $u \cdot dz \in T^*_c \mathbb{C}$ to $(c, u) \in \mathbb{C}^2$. Under this identification of $\mathbb{C}^2$ with $T^*\mathbb{C}$, the form $\omega_0$ in (3.4) gets identified with the Liouville symplectic form on $T^*\mathbb{C}$.

Take a pair $(U, f_U)$, where $U$ is a connected open subset of $X_0$ and $f_U : U \longrightarrow \mathbb{C}$ is a holomorphic embedding with $df_U = \beta|_U$. Let $F : K_U = K_{X|_U} = T^*U \longrightarrow T^*\mathbb{C} = \mathbb{C}^2$ (3.5) be the holomorphic embedding given by the differential of $f_U$, more precisely,

$$F(u, \lambda \cdot \beta(u)) = (f_U(u), \lambda)$$

for all $u \in U$ and $\lambda \in \mathbb{C}$.

**Lemma 3.1.** The pullback $F^*\omega_0$ of the form $\omega_0$ in (3.4) coincides with the Liouville symplectic form on $K_U = T^*U$.

**Proof.** This follows from the fact that the earlier mentioned identification of $\mathbb{C}^2$ with $T^*\mathbb{C}$ takes the form $\omega_0$ to the Liouville symplectic form on $T^*\mathbb{C}$. $\Box$

Consider the Moyal–Weyl deformation quantization of the symplectic manifold

$$(T^*\mathbb{C}, \omega_0) = (\mathbb{C}^2, \omega_0),$$

where $\omega_0$ is the constant form in (3.4). In view of Lemma 3.1 using the holomorphic embedding $F$ in (3.5), the Moyal–Weyl deformation quantization of $(\mathbb{C}^2, \omega_0)$ produces a deformation quantization of the symplectic manifold $T^*U = K_U$ equipped with the Liouville symplectic form. To describe this deformation quantization explicitly, let

$$F^{-1} : F(K_U) \longrightarrow K_U$$

be the inverse of $F$ on the image of $F$. For $f, g \in A(K_U)$ with $f = \sum_{i=0}^\infty h^i f_i$ and $g = \sum_{i=0}^\infty h^i g_i$, where $f_i, g_i \in \mathcal{H}(K_U)$, if

$$\left(\sum_{i=0}^\infty h^i (f_i \circ F^{-1})\right) \star \left(\sum_{i=0}^\infty h^i (g_i \circ F^{-1})\right) = \sum_{i=0}^\infty h^i b_i,$$

where $\star$ is the Moyal–Weyl deformation quantization and $b_i \in \mathcal{H}(F(K_U))$, then define

$$f \star g = \sum_{i=0}^\infty h^i (b_i \circ F).$$

(3.6)

Using Lemma 3.1 we conclude that this defines a deformation quantization of the symplectic manifold $T^*U$ equipped with the Liouville symplectic form.

The holomorphic coordinate function $f_U$ is not uniquely determined by $\beta$. However, any two choices of the holomorphic coordinate function are related by (3.3). Using Lemma 2.1 and (3.3) we conclude that the deformation quantization in (3.6), of the symplectic manifold $K_U$, equipped with the Liouville symplectic form, is actually independent of the choice the function $f_U$. Consequently, the locally defined deformation quantizations of the symplectic manifold $T^*X_0 = K_{X_0}$ equipped with the Liouville symplectic form patch.
together compatibly to define a deformation quantization of the symplectic manifold $K_{X_0}$ equipped with the Liouville symplectic form.

we summarize the above construction in the following proposition:

**Proposition 3.2.** Given a nonzero holomorphic 1-form $\beta$ on $X$, the symplectic manifold $T^*X_0 = K_{X_0}$ equipped with the Liouville symplectic form has a natural deformation quantization. It is locally given by the Moyal–Weyl deformation quantization of the symplectic manifold $(T^*\mathbb{C}, \omega_0)$, where $\omega_0$ is the constant form in (3.4). These locally defined deformation quantizations patch together compatibly to define a global deformation quantization.

### 4. Moduli space of Higgs bundles and deformation quantization

#### 4.1. A description of moduli spaces of Higgs bundles

A Higgs bundle on $X$ is a holomorphic vector bundle $E$ on $X$ together with a holomorphic section $\theta \in H^0(X, \text{End}(E) \otimes K_X)$.

A Higgs bundle $(E, \theta)$ is called semistable if

$$\frac{\text{degree}(F)}{\text{rank}(F)} \leq \frac{\text{degree}(E)}{\text{rank}(E)}$$

for all holomorphic subbundles $0 \neq F \subset E$ with $\theta(F) \subset F \otimes K_X$. If

$$\frac{\text{degree}(F)}{\text{rank}(F)} < \frac{\text{degree}(E)}{\text{rank}(E)}$$

for all holomorphic subbundles $0 \neq F \subsetneq E$ with $\theta(F) \subset F \otimes K_X$, then $(E, \theta)$ is called stable.

Fix an integer $r \geq 1$. Let $\mathcal{M}_X(r)$ denote the moduli space of semistable Higgs bundles on $X$ of rank $r$ and degree $d = r(g - 1) + 1$, where $g$ as before is the genus of $X$ (see [Hi], [N1], [Si2] for the construction of this moduli space). Since $r$ and $d$ are coprime, any $(E, \theta) \in \mathcal{M}_X(r)$ is in fact stable. Consequently, $\mathcal{M}_X(r)$ is an irreducible smooth quasiprojective variety defined over $\mathbb{C}$. Its dimension is

$$2\delta := 2(r^2(g - 1) + 1). \quad (4.1)$$

Moreover, it has a natural algebraic symplectic structure [Hi].

Consider the Zariski open subset of the Cartesian product

$$K_0^\delta := \{(x_1, \cdots, x_\delta) \in (K_X)^\delta \mid x_i \neq x_j \forall i \neq j\} \subset (K_X)^\delta \quad (4.2)$$

given by the locus of all distinct $\delta$ points, where $\delta$ is the integer in (4.1). The symmetric group $S_\delta$ of permutations of $\{1, \cdots, \delta\}$ acts freely on $K_0^\delta$. So the quotient $K_0^\delta / S_\delta$ is an irreducible smooth complex quasiprojective variety of dimension $2\delta$.

Let

$$p : K_X \longrightarrow X \quad (4.3)$$

be the natural projection.
There is a natural nonempty Zariski open subset

$$U_M \subset \mathcal{M}_X(r)$$

(4.4)

and a natural nonempty Zariski open subset

$$U_S \subset K_0^X/S_0$$

(4.5)

such that there is a canonical algebraic isomorphism

$$\Phi : U_M \sim U_S$$

(4.6)

[GNR], [ER], [BM], [Hu]. We will briefly recall it below.

Take any \((E, \theta) \in \mathcal{M}_X(r)\). Since degree\((E) = d = \text{rank}(E)(g-1)+1\), from Riemann–Roch theorem we know that \(\dim H^0(X, E) - \dim H^1(X, E) = 1\). There is a nonempty Zariski open subset

$$U_0 \subset \mathcal{M}_X(r)$$

(4.7)

such that \(\dim H^0(X, F) = 1\) for all \((F, \theta) \in U_0\). Assume that \((E, \theta) \in U_0\).

For \((E, \theta)\), we have

- a 1-dimensional closed subscheme \(C \subset K_X\) such that the projection

$$\phi := p|_C : C \rightarrow X$$

(see (1.3) for the map \(p\)) is a finite map, and a

- a torsionfree coherent sheaf \(\mathbb{L} \rightarrow C\) of rank one,

satisfying the condition that \(E = \phi_*\mathbb{L}\) [Hi]. The Higgs field \(\theta\) on \(E\) is constructed from \((C, \mathbb{L})\) as follows. Let

$$\sigma \in H^0(K_X, p^*K_X)$$

be the tautological section, where \(p\) is the projection in [1.3]. Tensoring with it produces a homomorphism

$$\mathbb{L} \rightarrow \mathbb{L} \otimes p^*K_X$$

Taking the direct image and invoking the projection formula, we get a homomorphism

$$E = \phi_*\mathbb{L} \xrightarrow{\phi_*(-\otimes\sigma)} \phi_*(\mathbb{L} \otimes p^*K_X) = (\phi_*\mathbb{L}) \otimes K_X = E \otimes K_X.$$  

This homomorphism coincides with \(\theta\) [Hi], [Si2].

Since \(E = \phi_*\mathbb{L}\), and \(\phi\) is a finite map, we have

$$H^i(X, E) = H^i(C, \mathbb{L})$$

for all \(i\). In particular, we have \(H^0(X, E) = H^0(C, \mathbb{L})\). So

$$\dim H^0(C, \mathbb{L}) = \dim H^0(X, E) = 1.$$  

Take a nonzero section \(s \in H^0(C, \mathbb{L}) \setminus \{0\}\). Let \(D_s = \text{div}(s)\) denote its divisor; since \(\dim H^0(C, \mathbb{L}) = 1\), we know that \(D_s\) is actually independent of the choice of \(s\).

There is a nonempty Zariski open subset

$$U_M \subset U_0$$
such that for all \((E, \theta) \in U_M\), we have
\[ D_s \in K_0^\delta / S_\delta \]
(see (4.2)). The isomorphism \(\Phi\) in (4.6) sends any \((E, \theta) \in U_M\) to \(D_s \in K_0^\delta / S_\delta\). See [GRN], [ER] for more details.

The Liouville symplectic form on \(K_X\) produces an algebraic symplectic form on the Cartesian product \((K_X)^\delta\); let
\[ \tilde{\Omega}_K' \in H^0((K_X)^\delta, \Omega_{(K_X)^\delta}^2) \quad (4.8) \]
be this symplectic form on \((K_X)^\delta\). Let
\[ \Omega'_K \in H^0(K_0^\delta, \Omega_{K_0^\delta}^2) \quad (4.9) \]
be the restriction of \(\tilde{\Omega}_K'\) to the Zariski open subset \(K_0^\delta\) in (4.2). This 2-form \(\Omega'_K\) is evidently preserved by the action of the symmetric group \(S_\delta\) on \(K_0^\delta\). So \(\Omega'_K\) produces an algebraic symplectic form \(\Omega''_K\) on the quotient space \(K_0^\delta / S_\delta\). Let
\[ \Omega_S \in H^0(U_S, \Omega_{U_S}^2) \quad (4.10) \]
be the restriction of \(\Omega''_K\) to the open subset \(U_S\) in (4.5).

Let \(\Omega'_M\) denote the algebraic symplectic form on the moduli space \(M_X(r)\). Let
\[ \Omega_M \in H^0(U_M, \Omega_{U_M}^2) \quad (4.11) \]
be its restriction to the open subset \(U_M\) in (4.4). For the isomorphism \(\Phi\) in (4.6), we have
\[ \Phi^* \Omega_S = \Omega_M, \quad (4.12) \]
where \(\Omega_S\) and \(\Omega_M\) are defined in (4.10) and (4.11) respectively (see [GRN], [ER], [BM], [Hu]).

### 4.2. Deformation quantization of the moduli space

As in (3.1), fix a nonzero 1-form
\[ \beta \in H^0(X, K_X) \setminus \{0\}. \]

In Proposition 3.2 we constructed a deformation quantization of the symplectic manifold \(T^*X_0 = K_{X_0}\) equipped with the Liouville symplectic form. Restrict the symplectic form \(\tilde{\Omega}'_K\) in (4.8) to the open subset \((K_{X_0})^\delta \subset (K_X)^\delta\).

The deformation quantization of \(K_{X_0}\) produces a deformation quantization of this symplectic manifold \(((K_{X_0})^\delta, \tilde{\Omega}'_K|_{(K_{X_0})^\delta})\). From the construction of this deformation quantization of the symplectic manifold \(((K_{X_0})^\delta, \tilde{\Omega}'_K|_{(K_{X_0})^\delta})\) it is evident that the permutation action of the symmetric group \(S_\delta\) on \((K_{X_0})^\delta\) preserves this deformation quantization.

The deformation quantization of the symplectic manifold \(((K_{X_0})^\delta, \tilde{\Omega}'_K|_{(K_{X_0})^\delta})\) in turn restricts to a deformation quantization of the Zariski open subset \((K_{X_0})^\delta \cap K_0^\delta \subset K_0^\delta\).
equipped with the symplectic form $\Omega'_K|_{(K_{X_0})^\delta \cap K_0^\delta}$, where $K_0^\delta$ is constructed in (4.2) and $\Omega'_K$ is the symplectic form in (4.9). Let

$$K_0^\delta / S_\delta \subset K_0^\delta / S^\delta$$

be the Zariski open subset given by the image of $(K_{X_0})^\delta \cap K_0^\delta$. From the above observation, that the permutation action of the symmetric group $S_\delta$ on $(K_{X_0})^\delta$ preserves the deformation quantization of the symplectic manifold $((K_{X_0})^\delta, \tilde{\Omega}'_K|_{(K_{X_0})^\delta})$, it follows immediately that the above deformation quantization of the symplectic manifold

$$((K_{X_0})^\delta \cap K_0^\delta, \Omega'_K|_{(K_{X_0})^\delta \cap K_0^\delta})$$

produces a deformation quantization of the above symplectic manifold $K_0^\delta / S_\delta$ equipped with the symplectic form constructed using $\Omega'_K|_{(K_{X_0})^\delta \cap K_0^\delta}$. Consequently, we obtain a deformation quantization of the Zariski open subset

$$\hat{U}_S := U_S \cap K_0^\delta / S_\delta \subset U_S$$

equipped with the algebraic symplectic form $\hat{\Omega}_S := \Omega_S|_{\hat{U}_S}$, where $\Omega_S$ and $U_S$ are as in (4.10) and (4.5) respectively.

Define

$$\hat{U}_M := \Phi^{-1}(\hat{U}_S) \subset U_M,$$

where $\Phi$ is the isomorphism in (4.6). Let

$$\hat{\Phi} := \Phi|_{\hat{U}_M} : \hat{U}_M \sim \hat{U}_S$$

be the isomorphism obtained by restricting $\Phi$.

The restriction of the symplectic form $\Omega_M$ in (4.11) to the above Zariski open subset $\hat{U}_M \subset U_M$ will be denoted by $\hat{\Omega}_M$. From (4.12) it follows immediately that

$$\hat{\Phi}^* \hat{\Omega}_S = \hat{\Omega}_M. \quad (4.14)$$

Using $\beta$, we constructed above a deformation quantization of the symplectic manifold $\left(\hat{U}_S, \hat{\Omega}_S\right)$. In view of (4.14) this produces a deformation quantization of the symplectic manifold $\left(\hat{U}_M, \hat{\Omega}_M\right)$. To describe this deformation quantization explicitly, let

$$\Psi : \mathcal{A}(\hat{U}_M) \longrightarrow \mathcal{A}(\hat{U}_S)$$

be the isomorphism that sends any $\sum_{i=0}^{\infty} h^i f_i \in \mathcal{A}(\hat{U}_M)$, where $f_i \in \mathcal{H}(\hat{U}_M)$, to

$$\sum_{i=0}^{\infty} h^i (f_i \circ \hat{\Phi}^{-1}) \in \mathcal{A}(\hat{U}_S).$$

Now for any $f_1, f_2 \in \mathcal{A}(\hat{U}_M)$, define

$$f_1 \ast f_2 := \Psi^{-1}(\Psi(f_1) \ast \Psi(f_2)).$$

This evidently defines a deformation quantization of the symplectic manifold $\left(\hat{U}_M, \hat{\Omega}_M\right)$.

The above construction is summarized in the following theorem:
Theorem 4.1. Take a nonzero 1-form $\beta \in H^0(X, K_X) \setminus \{0\}$. It produces a natural deformation quantization of the Zariski open subset $\tilde{U}_M$ of the moduli space $\mathcal{M}_X(r)$ of semistable Higgs bundles on $X$ of rank $r$ and degree $r(g - 1) + 1$.

Data Availability

Data sharing not applicable — no new data generated.

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