Distinguishing general relativity from Chern-Simons gravity using gravitational wave polarizations

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Abstract

Quasi-normal modes (QNMs) uniquely characterize the final black-hole. Till now, only the QNM frequency and damping time are used to test General relativity. In this work, we show explicitly that another property of the QNMs — their polarization — can be a reliable tool for probing gravity. We provide a consistent test for General relativity by considering Chern-Simons gravity. Distinguishing Chern-Simons gravity from General relativity using only template matching is highly challenging. Thus a parameter that can differentiate between Chern-Simons gravity and GR will be a suitable candidate for any modified theories of gravity. We discuss the implications of our result for the future gravitational wave detectors.

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I. INTRODUCTION

The direct detection of gravitational waves (GW) and subsequent detections, from compact binary objects, have energized the search for deviations from the general theory of relativity (GR) [1]. Model-independent studies for deviations from GR and constraints have been placed on the graviton mass [2]. Currently, tests of GR is done by matching templates from a numerical simulation with the observed data, and introducing new parameters corresponding to extended gravity theories [3–5]. However, as the accuracy of the current and upcoming detectors (including LISA) increases, the sensitivities of the template matching technique will saturate, and there is an urgent need to find alternative strategies to test for deviations from GR.

Broadly, modified theories of gravity can be classified into two categories [6]: First, modify the Einstein–Hilbert action while leaving the coupling of all matter fields to the metric unchanged, like \( f(R, R, R_{\mu\nu}, R_{\mu\alpha\nu\beta}) \) [7, 8]. Second, modify the coupling to the matter action like Brans-Dicke theory. There are numerous possible modifications [7]. Even if we can obtain the waveforms for many of these models, it is imperative to obtain a handful of parameters that can be used as a consistency test of General relativity. More specifically, it is essential to find a dimensionless parameter which vanishes for GR and is finite for modified gravity theories. Such parameters have been constructed to distinguish between dark energy models and modified gravity theories (see, for instance, [9]).

Quasi-normal modes (QNMs) — superposition of damped sinusoids emitted when blackholes settles to an equilibrium configuration — are the fingerprints of the final black-hole as they depend only on the parameters characterizing the BH (like Mass, Charge and angular momentum) and independent of the initial configuration that caused the excitation [10–13]. Thus, extracting the QNM frequency and damping time allows one to test General relativity [14–16]. However, another property of the QNMs — their polarization — can be a reliable tool for probing gravity. Recently, the current authors used the inequality between polar and axial gravitational perturbations between GR and \( f(R) \) theories to obtain a parameter that vanishes for GR while it is finite for \( f(R) \) theories [17, 18]. In this study, we propose a parameter to distinguish GR from (dynamical) Chern Simons (CS) Gravity and show that the inequality between polar and axial perturbations is, in general, valid for any modification to GR.
CS Gravity is indistinguishable from GR for all conformally flat space-times and space-times that possess a maximally symmetric 2-dimensional subspace [19]. Thus a parameter that can distinguish between CS gravity and GR will be a suitable candidate for any modified theories of gravity. Naturally, of late, there is a lot of interest in studying the perturbations about Schwarzschild and slowly-rotating black-holes in dynamical CS gravity [20–25] and, more recently, in Ref. [26] as the authors were preparing the current article for submission.

In this work, we show that the isospectral equality [27] between odd and even parity perturbations is broken for a perturbed Schwarzschild and slowly rotating black holes in dynamical Chern-Simons (dCS) gravity in a gauge invariant manner [28]. Consequently, the odd-even parity modes carry different amounts of gravitational energy to asymptotic infinity. We quantify the relative difference between the two modes by constructing an energy-momentum pseudotensor of perturbation and show that the modification to gravitational radiation is more significant around the black hole region compared to flat space-times. We discuss the implications of the result for future detectors.

For easy comparison, we use the notations/conventions of Ref. [28] with a few changes. We use (−, +, +, +) signature, Greek for the 4-D space-time, upper case Latin for the (θ, φ) and lower case Latin for (t, r), c = G = 1 such that κ² = 8π. ∇ and subscript semicolon are covariant derivatives of the full space-time, D is the covariant derivative for (t, r), ˆD is the covariant derivative on a 2-sphere, and Ω ≡ Ω(θ, φ) denote the coordinates on a 2-sphere. Overbarred quantities are background and A(i) denotes the ith order perturbation of the object A.

II. GAUGE INVARIANT PERTURBATION IN GR AND DCS

A. Perturbations in GR

To keep things transparent, we will consider Schwarzschild space-time as our background \( \bar{g}_{\mu\nu} \). Later will discuss the slowly-rotating space-time.

\[
\begin{align*}
    ds^2 &= \bar{g}_{ab}dx^a dx^b + \bar{g}_{AB}dz^A dz^B \\
    &= -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2 \\
    f(r) &\equiv f = 1 - \frac{2M}{r}
\end{align*}
\]
Here, the background space-time is split into an orbit space $x^a \equiv (t,r)$ and a 2-sphere $z^A \equiv (\theta, \phi)$. The metric perturbations ($h_{\mu\nu}$)

$$g_{\mu\nu} = \tilde{g}_{\mu\nu} + h_{\mu\nu}; g^{\mu\nu} = \tilde{g}^{\mu\nu} - h^{\mu\nu}$$

(4)
can be separated using spherical harmonics. This leads to metric perturbations of two kinds of parity: odd parity (expanded using vector spherical harmonics) and even parity (expanded using scalar spherical harmonics) [28, 29]. Gravitational or scalar field perturbations thus reduce to a one dimensional scattering problem of the form [27, 29–32]

$$\frac{d^2 \Phi_i}{dr_+^2} + (\omega^2 - V_i) \Phi_i = S_i \quad i = \text{scalar, odd/even gravitational},$$

(5)

where $r_+$ is the Tortoise coordinate, and $V_i$ are effective potentials induced by the background space-time curvature, depending on the type of perturbation (gravitational or scalar). An inhomogeneous source term $S_i$ in the RHS of (5) can arise if the background has some non-zero energy-momentum tensor. For theories other than general relativity, $S_i$ can also be an effective source comprising of terms coming due to modifications to the Einstein-Hilbert action. The individual profiles of $V_i$ determine the amount of gravitational or scalar radiation that escapes to infinity due to perturbation. For the gravitational perturbations in GR, the profiles of $V_i$ for both odd and even parity perturbations are the same owing to an isospectral relationship that exists between them [27]. This leads to the conclusion that the gravitational energy flux radiated through each of the parity modes to infinity is equal [17] — a feature that has important consequences on non-Einsteinian theories of gravity.

**B. Perturbations in Dynamical Chern-Simons**

One of the motivations for the CS gravity is to write a field theory of gravity as a gauge theory. In GR, the space-time is a dynamical object and is governed by dynamical equations, namely the Einstein field equations. Therefore, the construction of a gauge theory of gravity requires an action that does not consider a fixed space-time background [20, 33–35].

The action for the dynamical Chern-Simons (dCS) is given by [20]

$$S = \int d^4x \sqrt{-g} \left[ \frac{R}{2\kappa^2} + \frac{\alpha}{4} \varphi^* RR - \frac{\beta}{2} (\nabla \varphi)^2 - \frac{\beta}{2} V(\varphi) \right]$$

(6)
where $\vartheta$ is the dynamical pseudo-scalar field. We have chosen $\vartheta$ to be dimensionless, which leads to $[\alpha] = L^2$ and $[\beta]$ is dimensionless and $*RR$ is

$$ *RR = \frac{1}{2} R_{\mu\nu\rho\sigma} \epsilon^{\mu\nu\alpha\beta} R^\rho_{\alpha\beta} $$

referred to as Pontryagin density that quantifies the extent to which local Lorentz invariance is violated in a region. For spherically symmetric space-times, the Pontryagin density vanishes, making the Schwarzschild space-time a solution of the dCS modified field equations. Only recently, non-slow rotation black-hole space-times have been constructed \[36\]. For slow-rotating perturbative solutions, see \[37, 38\].

To simplify the problem, usually one sets $V(\vartheta) = 0$ \[21, 22\]. For the vacuum case, we obtain the following equations of motion:

$$ R_{\mu\nu} = -2\kappa^2 \alpha C_{\mu\nu} + \kappa^2 \beta \partial_{\mu} \vartheta \partial_{\nu} $$

(8)

$$ \Box \vartheta = -\frac{\alpha}{4\beta} *RR $$

(9)

$$ C_{\mu\nu} = \frac{1}{2} \left[ \partial_{\rho} \left( \epsilon^{\rho\sigma\alpha\beta} R_{\nu\beta;\alpha} + \epsilon^{\sigma\alpha\beta} R_{\mu\beta;\alpha} \right) + \partial_{\rho\tau} \left( *R^\rho_{\mu\nu} \sigma + *R^\rho_{\nu\sigma} \mu \right) \right] $$

(10)

$$ *R^\tau_{\sigma\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} R_{\sigma\alpha\beta} $$

(11)

where $*R^\tau_{\sigma\mu\nu}$ is referred as the dual of the Riemann tensor.

Since $C_{\mu\nu}$ vanishes on a spherically symmetric background, (8) becomes a system whose only static and the spherically symmetric solution is Schwarzschild, with a vanishing background scalar field \[39\]. Thus perturbing about the Schwarzschild background, the perturbed equations of motion become

$$ R_{\mu\nu}^{(1)} = -2\kappa^2 \alpha C_{\mu\nu}^{(1)} $$

(12)

$$ \Box \vartheta^{(1)} = -\frac{\alpha}{4\beta} \delta (*RR) $$

(13)

where, we have used the notation $\vartheta^{(1)} \equiv \vartheta$, and

$$ C_{\mu\nu}^{(1)} = \frac{1}{2} \partial_{\rho\tau} \left( *\tilde{R}^\rho_{\mu\nu} \sigma + *\tilde{R}^\rho_{\nu\sigma} \mu \right) $$

(14)

Using the $2+2$ decomposition by expanding the metric perturbation and the pseudo-scalar using spherical harmonics \[28\], Eqs. (12, 13) leads to two coupled equations for the odd parity mode, while the even parity is the same as in GR. The odd parity sector becomes:

$$ \frac{d^2 \Phi_O}{dr_s^2} + \left( \omega^2 - \tilde{V}_O \right) \Phi_O = \mathcal{S}_\varphi \varphi $$

(15)

$$ \frac{d^2 \varphi}{dr_s^2} + \left( \omega^2 - \tilde{V}_\varphi \right) \varphi = \mathcal{S}_\Phi \Phi_O $$

(16)
where the exact forms of $\bar{V}_O$, $\bar{V}_\varphi$, $\mathcal{I}_\varphi$, and $\mathcal{I}_\Phi$ have been given in Appendix A.3, $\varphi$ is related to $\vartheta$ as \cite{21, 22}

$$
\vartheta(t, r, \Omega) = \frac{\varphi(r)}{r}S(\Omega)e^{i\omega t},
$$

and $S(\Omega)$ is a scalar spherical harmonic function.

This is the first result of this work and shows that the isospectral relation between even and odd parity modes break in CS gravity. The reason for this breaking is the appearance of inhomogeneous terms in the RHS of (15) and an absence of any such term in the even parity sector. Thus, the QNM frequencies of the even and odd parity modes will be different, implying that the energy radiated through these two modes.

Besides the breaking of the isospectral relation breaking, the above relations also show a preferential interaction between the dCS field and odd parity. To see this, take the asymptotic limit of the functions $\mathcal{I}_\Phi, \mathcal{I}_\varphi$ in Eqs. (A13) and (A10). In the limit of $r \to \infty$, $\mathcal{I}_\Phi \sim r^{-3}$ while $\mathcal{I}_\varphi \sim r^3$. Thus, $\mathcal{I}_\Phi$ can be neglected in (16) while $\mathcal{I}_\varphi$ can not be neglected in (15). Thus, the dCS field has a long range effect on the odd parity perturbation, while the odd parity perturbation has only short-range effect on the dCS field. This is expected since the dCS field, in effect, is a light scalar field and is easier to excite compared to the massive extra scalar gravitational degree of freedom of \textit{f}(\textit{R}) theories \cite{18, 40}. As mentioned above, since the even parity sector does not interact with the dCS field, a wave scattering process leads to different amounts of scattered gravitational energies in the even and odd sectors. In the next section, we define a dimensionless number which quantifies the relative difference between radiated gravitational energies of even and odd sectors.

\section{Difference in Energy Flux}

To quantify the relative difference in the energies of odd and even parity waves, we calculate the effective energy-momentum pseudotensor of perturbation in the curved background (for earlier work, see \cite{41}). We use the Isaacson’s shortwave approximation \cite{42, 43}. In this approximation, we average over the rapidly fluctuating spatial components of the metric perturbation compared to the length scales over which the background significantly changes and obtain the \textit{back-reaction} effect on the background metric $\bar{g}_{\mu\nu}$.
The back-reaction effect on the background metric is given by

$$\bar{\Theta}_{\mu\nu} = -t_{\mu\nu} \equiv 2e^2\kappa^2\alpha \langle G^{(2)}_{\mu\nu} \rangle - e^2 \langle G^{(2)}_{\mu\nu} \rangle$$  \hspace{1cm} (18)$$

where $G_{\mu\nu}$ is the Einstein tensor. It is useful to obtain $t_{\mu\nu}$ in the TT gauge and in-terms of the traced-reversed perturbed tensor $\psi_{\mu\nu} = h_{\mu\nu} - \frac{\kappa}{2}g_{\mu\nu}$, where $h$ is the trace of $h_{\mu\nu}$. The first order perturbation of Eqs. (8) and (9), we get:

$$\Box \psi_{\mu\nu} + 2\bar{R}_{\mu\alpha\nu\beta} \psi^{\alpha\beta} = 2\kappa^2\alpha\partial^i\partial^\sigma (\ast \bar{R}^i_\mu^\sigma \nu + \ast \bar{R}^i_\nu^\sigma \mu)$$  \hspace{1cm} (19)

$$\Box \vartheta = -\frac{\alpha}{4\beta} \left[ 2\psi^{\mu\nu} \ast \bar{R}_{\mu\alpha\nu\beta} + \ast \bar{R}_{\mu\beta\alpha} \right] + \bar{R}^{\alpha\beta\gamma\mu} (\ast \bar{R}_{\alpha}^{\gamma\mu} \psi_{\beta\nu}^{\ast} + \ast \bar{R}_{\alpha}^{\beta\sigma\mu} \psi^{\sigma}_{\gamma})$$  \hspace{1cm} (20)

It is important to note that, like Eqs. (15, 16), in the asymptotic limit, Eqs. (19, (20) decouple, and $\vartheta$ is a light field.

Thus, the perturbed energy-momentum pseudotensor $t_{\mu\nu}$ is given by:

$$t_{\mu\nu} = -\frac{e^2}{4r_H^2} \left\langle \left( \nabla^2 \psi \right)^2 \right\rangle_{\mu\nu} - e^2\kappa^2\alpha^2 \left\langle \left( \nabla^2 \psi \nabla^2 \psi \right) \right\rangle_{\mu\nu} + \left\langle \nabla^2 \psi \nabla \psi \right\rangle_{\mu\nu} + \left\langle \psi \nabla^2 \psi \right\rangle_{\mu\nu}$$  \hspace{1cm} (21)$$

where $\psi_{\mu\nu}$ and its derivatives have been scaled with respect to the horizon radius $r_H$ such that quantities inside the angular brackets are dimensionless, and the averaging is over the short wavelength modes. (See Appendix B for details.)

This is the second significant result, regarding which we would like to stress the following points: First, the first term in the RHS of Eq. (21) is the energy-momentum pseudotensor of perturbation for GR [42]. The second term is the correction term that arises from the modifications to gravity. In this case, these are the corrections from the dCS gravity. Second, the DCS field $\vartheta$ does not appear in the expression, and hence its perturbation does not propagate to asymptotic infinity. As seen from their exact forms in Appendix B, the terms quadratic and cubic in the Riemann tensor in the square bracketed terms of (21) ensure that the contribution from those terms vanishes at a large distance from the black hole. The leading order contribution at large distances from the black hole will come from terms with a single Riemann as a product, i.e., the first term in the square brackets of the form $\left\langle \nabla^2 \psi \nabla^2 \psi \right\rangle_{\mu\nu}$, as can be seen in (B9). Hence, the effect of the terms in square brackets is strongest around the black hole, and any scattering process happening around that region will have signatures of $\Delta_{\text{CS}}$ in GW signals appearing at asymptotic infinity - specifically in the relative ratio of the energy fluxes of odd (vector) and even (scalar) parity polarizations.
Third, the quantities that are averaged in both the terms, including derivatives of the perturbation higher than one, coming due to the corrections, are of the same order. This allows us to define a dimensionless parameter $\Delta_{CS}$ given by

$$\Delta_{CS} = \frac{\kappa^2 \alpha^2}{\beta r_H^4}$$

which can serve as a consistent test for GR. We will discuss more on this in the next section.

Lastly, an attentive reader would have noticed that the energy-momentum pseudotensor analysis we have obtained is independent of the background metric and holds for any black hole space-time with characteristic length scale ($r_H$). In particular, the analysis applies to slowly rotating black-holes. Like in the spherically symmetric black-holes, in the slowly-rotating case, the odd and even perturbations decouple at linear order (see, for instance, \([44-47]\)). Since the decoupling holds at the linear order for a slowly rotating Kerr solution, the broken energetic equality between odd and even parities will hold for those cases \([48]\). Specifically, for a slowly rotating background space-time, the 1D scattering problem for odd and even parity perturbations become

$$\frac{d^2 \Phi_{O/E}^{\ell,m}}{dr_*^2} + \left[\omega^2 - \frac{4maM^2\omega}{r^3} - \tilde{V}_{O/E}^{\ell,m}(r; a, \omega)\right] \Phi_{O/E}^{\ell,m} = 0 \quad (23)$$

where $\tilde{V}_{O/E} \sim O(a)$ and unlike spherically symmetric cases, the scattering potential now depends on the frequency $\omega$ and orbital number $m$. Thus, the analysis in this section and the definition (22) holds for the slowly rotating black hole.

IV. IMPLICATIONS FOR FUTURE GW DETECTORS

In the rest of this article, we will discuss the implications of our work for future observations to distinguish between general relativity and modified theories of gravity, in general.

Using gauge-invariant formalism \([29]\) for dCS gravity, we found that the parity-violating scalar field coupled only to the odd parity perturbations of a spherically symmetric space-time, keeping the even parity unchanged from its GR counterpart. Similarly, the dynamics of the dCS scalar field $\vartheta$ is also influenced by the odd parity master function, leading to the coupled system of equations (15) and (16). The odd parity scattering potential $\tilde{V}_O$ and the dCS field potential $\tilde{V}_\varphi$ in the coupled case does not asymptote to zero at large distances.
from the black hole but saturate to the value

\[ \lim_{r \to \infty} \tilde{V}_\phi = \lim_{r \to \infty} \tilde{V}_O = \frac{6\kappa^2 \alpha^2 k^2 \omega^2}{\beta M^4 (k+1)} \]  

which is at the leading order of the factor \((\kappa^2 \alpha^2)/\beta\). Importantly, Eq. (24) implies that a noticeable change in the asymptotic value of \(\tilde{V}_O\) and \(\tilde{V}_\phi\) occurs for small black-hole and at high frequencies. LIGO-VIRGO observations have shown that stellar mass black-holes heavier than 25 \(M_\odot\) exists. This is the third significant result, and our analysis points that large mass black-holes are natural laboratories to test GR.

Eqs. (24) and (16) also imply that at infinity the dCS field decouples from the odd parity perturbation and attains a mass given by

\[ m_\phi = \sqrt{\frac{6\kappa^2 \alpha^2 k^2 \omega}{\beta (k+1) M^2}}. \]  

which is again significant only for large frequencies.

This naturally leads to the following question: Can the future detectors measure the difference between the two modes? To answer this question, we go back to the pseudo-energy-momentum tensor (21). The direct coupling of the dynamics of \(\vartheta\) with the gravitational perturbation \(\Phi_O\) allows one to write the energy-momentum tensor solely in terms of the trace-reversed perturbation tensor \(\psi_{\mu\nu}\), as seen in Eq. (21). Thus, the parity-violating terms in the energy-momentum tensor of gravitational radiation can be represented as a correction to the usual graviton-graviton interaction term in GR. The appearance of the background Riemann curvature in the stress tensor leads to the conclusion that the terms appearing as a correction to GR does no propagate to asymptotic infinity, implying that the dCS field also does not propagate to asymptotic infinity. However, the effect of the dCS term to the gravitational radiation is strongest close to the black hole.

A wave scattering process occurring near a black hole will thus leak energy from the odd parity mode to the dCS field. The decoupled nature of the odd and even parity modes in the linear regime will ensure no energy exchange takes place between them, leading to a difference in the radiated energy between the two modes. The gravitational wave detectors only see the two polarization modes, and it is possible to observe this difference directly with more detectors being planned/commissioned.

The even and odd parity modes manifest as the usual plus and cross polarizations at detectors at asymptotic infinity. The relations between the odd/even wavefunctions and the
plus/cross amplitudes are given by [18, 49, 50]:

\[
\frac{d\Psi_{E}^{\ell,m} (t, r)}{d\Omega} = h_+ (t, r, \Omega) S_{\theta\theta}^{\ell,m} (\Omega) + \sin^2 \theta h_\times (t, r, \Omega) S_{\theta\phi}^{\ell,m} (\Omega)
\]

(26)

\[
\frac{d\Psi_{O}^{\ell,m} (t, r)}{d\Omega} = h_+ (t, r, \Omega) V_{\theta\theta}^{\ell,m} (\Omega) + \sin^2 \theta h_\times (t, r, \Omega) V_{\theta\phi}^{\ell,m} (\Omega)
\]

(27)

where $\Psi_{E/O} = \frac{\Psi_{E/O}}{r}$, while $S_{AB}^{\ell,m}$ and $V_{AB}^{\ell,m}$ are even and odd parity traceless spherical harmonic tensors. The quantities $h_{+/\times}$ defined at distances far from the source space-time have the spherical wave form $\frac{f(t-r)}{r}$ and have the angular dependence of -2 spin weighted spherical harmonics $-2Y_{\ell,m}$ [51]. Specifically, for a black hole of mass $M$ located at distance $D$

\[
h_+ (t, D, \Omega) = \frac{M}{D} \sum_{\ell,m} h_{\ell,m} (t) Y_{+}^{\ell,m} (\Omega)
\]

(28)

\[
h_\times (t, D, \Omega) = \frac{M}{D} \sum_{\ell,m} h_{\ell,m} (t) Y_{\times}^{\ell,m} (\Omega)
\]

(29)

where $h_{\ell,m} (t)$ are the mode amplitudes at time $t$, while $Y_{+/\times}^{\ell,m} (\Omega)$ are related to $-2Y_{\ell,m}$. Thus, given the polarization amplitudes $h_{\ell,m}$, the relative ratio of the intensities of the two quantities in the RHS would help us constrain the amount to which general relativity is violated. Specifically, a dimensionless parameter quantifying the relative energy ratio and hence, a quantifier for deviation from general relativity can be defined from (26) and (27) as

\[
\Delta_{\ell,m} = \left| \frac{d\Omega \Psi_{E}^{\ell,m}}{d\Omega \Psi_{O}^{\ell,m}} \right|^2 - \left| \frac{d\Omega \Psi_{E}^{\ell,m}}{d\Omega \Psi_{O}^{\ell,m}} \right|^2
\]

(30)

where the overdot denotes time derivative and in the present case $O (\Delta_{\ell,m}) \sim O (\Delta_{CS})$. In the future gravitational wave detectors (for instance, Cosmic Explorer [52]) the signal-to-noise ratio in the QNM regime could be 50 [53]. These observations will help us to put a stringent bound on the factor $(\alpha^2/\beta)$ and constrain any deviation from GR in general much better than from the currently used template matching techniques.

The earlier analyses [17, 18] and the current work strongly establishes the fact that any modification to GR will lead to parity preferences of the odd and even modes and hence, the quantity $\Delta_{\ell,m}$ will be nonzero. The general nature of isospectrality breaking and its relations to modified theories of gravity have been discussed [54, 55]. However, to our knowledge, our work is the first to evaluate the difference and obtain a quantifying tool.

It is also possible that the isospectrality is broken due to the environmental contaminants around a black hole [56]. However, environmental contaminants around a black hole will be

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different in each detection and would show up as different $\Delta_{\ell,m}$ in different cases. However, modified theories of gravity will lead to a consistent nonzero value of $\Delta_{\ell,m}$ in all observations.

We note again that the calculation for the energy-momentum pseudotensor in Sec. (III) is independent of the background. For the slowly rotating black-holes, the odd and even metric perturbations remain decoupled and hence, our current analysis holds. One possible future work will be to obtain the dominant mode $(\ell, m) = (2, 2)$ calculation of (30) using numerical relativity data of non-spinning or slowly spinning final states of binary black hole mergers and precisely evaluate the changes between GR and modified theories of gravity.

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Appendix A: Various first order perturbed 2+2 decomposed quantities

1. Perturbed Pontryagin density

The perturbed metric tensor can be 2+2 decomposed following [28]

$$h_{\mu\nu} \equiv \begin{pmatrix} f_{ab}S & r(f^E_aS_A + f^O_aV_A) \\ Sym r^2 \left(2H^E_T S_{AB} + 2H^O_T V_{AB} + 2H^E_L \gamma_{AB} S\right) \end{pmatrix}$$

(A1)

where $f_{ab}, f^E_a, f^O_a, H^E_T, H^O_T, \text{and } H^E_L$ are a set of ten scalars only dependent on $(t, r)$ (subscript T and L imply transverse and longitudinal components respectively). An implicit summation over $\ell, m$ was assumed in (A1). $S, S_A, \text{and } S_{AB}$ are even parity spherical harmonic scalar, vector, and tensor respectively. $V_A$ and $V_{AB}$ are odd parity spherical harmonic vector and tensor respectively. The odd parity spherical harmonic vector on a 2-sphere is related to the even parity spherical harmonic scalar as

$$V_A = \frac{1}{k^2} \epsilon_{AB} \hat{D}^B S; \quad k^2 = \ell (\ell + 1)$$

(A2)
where $\epsilon_{AB}$ is the covariant Levi-Civita density and $\hat{D}$ is the covariant derivative defined on a 2-sphere. $k^{-1}$ appears as a factor to make sure orthonormality of $V_A$ and $S_A$ are the same, i.e.

$$
\int_{\Omega} \sqrt{-\gamma} V_A V^B d\Omega = \int_{\Omega} \sqrt{-\gamma} S_A S^B d\Omega \tag{A3}
$$

where $\sqrt{-\gamma}$ is the determinant of the metric on a 2-sphere.

A covariant Levi-Civita on a 2-sphere can be constructed by projecting out of a Levi-Civita in the full space-time as

$$
\epsilon_{AB} = \frac{r^2}{\sqrt{2}} \epsilon_{AaBb} \epsilon^{ab} \tag{A4}
$$

where $\epsilon_{ab}$ is the covariant Levi-Civita on the $(t,r)$ space. (A4) satisfies all the properties of the antisymmetric 2-form in the 2-sphere. Using (A1), (A2), and (A4), the perturbed Pontryagin density for a background Schwarzschild space-time in terms of the odd parity master variable becomes

$$
\delta (^*RR) = \frac{12 Mk}{r^4 f} \frac{d^2 \Phi_\Omega}{dr_*^2} + k \Phi_\Omega \left( -\frac{32 M}{r^6} + \frac{96 M^2}{r^7} + \frac{12 M^3}{r^8} \right) \tag{A5}
$$

where it’s seen that only the odd parity master function contributes to the perturbed Pontryagin density.

2. Perturbed Cotton tensor as an effective source.

Any tensor that can be written as an effective energy-momentum tensor sourcing the Einstein tensor can be decomposed as (in first order of perturbation)

$$
\delta T_{\mu\nu} \equiv \begin{pmatrix} 0 & r \tau_a V_A \\ Sym \tau_T V_{AB} \end{pmatrix} \tag{A6}
$$

where only the odd parity part have been used since only the odd parity contribution is non-vanishing under a linear perturbation of the Cotton tensor in a spherically symmetric space-time. Also, $\tau_a$ and $\tau_T$ are gauge invariant. Comparing $-\alpha \delta C_{\mu\nu}$ with $\delta T_{\mu\nu}$ we obtain the effective source term $S_\phi$ appearing in the RHS of the odd parity master equation using the prescription of [32]

$$
S_\phi = \sqrt{2} \kappa^2 \alpha kr f \left[ \frac{2r^2 k}{M^6 f} \frac{d^2 \phi}{dr_*^2} + \frac{k}{M^6} \left( -4 + \frac{2r^2 \omega^2}{f} \right) \phi \right] \tag{A7}
$$
where \( \varphi \) has been defined in (17). The above result do not match with the form of \( S_\varphi \) as obtained in [22, 24] using the Regge-Wheeler gauge choice. Our source terms are purely real unlike theirs which are purely imaginary. We are confident with our calculations and have verified them with CADABRA.

3. 2+2 decomposed odd parity master equations

The perturbed odd parity sector equations of motion were found to be (12) and (13). Using (A5), (A7), and reducing double spatial derivatives in the source side of the master equations one obtains (15) and (16). The various coefficients are given by

\[
\tilde{V}_O = V_O + \frac{2\sqrt{2}\kappa^2 \alpha kr^3}{M^6 (k^2 - 1)} \mathcal{S}_\varphi \tag{A8}
\]

\[
V_O = \frac{f}{r^2} \left( k^2 - \frac{6M}{r} \right) \tag{A9}
\]

\[
\mathcal{S}_\varphi = \frac{\sqrt{2}\kappa^2 \alpha kr}{k^2 - 1} \left[ -\frac{2r^2}{M^6} (\omega^2 - \tilde{V}_\varphi) + af \right] \tag{A10}
\]

\[
\tilde{V}_\varphi = V_\varphi - \frac{3\kappa^2 \alpha^2 k^2 M}{\beta r^2 (k^2 - 1)} \left[ -\frac{2r^2}{M^6} (\omega^2 - V_\varphi) + af \right] \left[ 1 + \frac{6\kappa^2 \alpha^2 k^2}{\beta M^5 (k^2 - 1)} \right]^{-1} \tag{A11}
\]

\[
V_\psi = \frac{f}{r^2} \left( k^2 + \frac{2M}{r} \right) \tag{A12}
\]

\[
\mathcal{S}_\Phi = \frac{\alpha kr}{4\sqrt{2}\beta} \left[ \frac{12M}{r^4} \left\{ \omega^2 - V_O + \frac{\kappa^2 \alpha^2 k^2 r^4 b}{2\beta M^6 f (k^2 - 1)} \right\} \left[ 1 + \frac{6\kappa^2 \alpha^2 k^2}{\beta M^5 (k^2 - 1)} \right]^{-1} - \frac{b}{f} \right] \tag{A13}
\]

\[
a = \frac{k}{M^6} \left( 4 - \frac{2r^2\omega^2}{f} \right) \tag{A14}
\]

\[
b = k \left( -\frac{32M}{r^6} + \frac{96M^2}{r^7} + \frac{12M\omega^2}{r^4 f} \right) \tag{A15}
\]

Appendix B: Gravitational radiation in the shortwave limit

A vanishing background \( \vartheta \) and using the transverse-traceless gauge by defining trace reversed field \( \psi_{\mu\nu} = h_{\mu\nu} - \frac{k}{2} \bar{g}_{\mu\nu} \) leads to (19) and (20). For a metric and CS perturbation

\[
g_{\mu\nu} = \bar{g}_{\mu\nu} + eh_{\mu\nu} \tag{B1}
\]

\[
\vartheta = e \vartheta, \tag{B2}
\]

the modified field tensor \( \mathcal{G} \) can be expanded in powers of \( e \) as

\[
\bar{\mathcal{G}}_{\mu\nu} + e \mathcal{G}^{(1)}_{\mu\nu} + e^2 \mathcal{G}^{(2)}_{\mu\nu} = 0 \tag{B3}
\]
Solving for $\mathcal{E}_{\mu\nu}^{(1)} = 0$ gives the dynamics of the perturbation. While the radiated energy and momentum flux due to perturbation can be found from an energy-momentum pseudotensor due to perturbation. From (B3) we then get

$$\bar{\mathcal{E}}_{\mu\nu} = \kappa^2 t_{\mu\nu}$$

(B4)

$$t_{\mu\nu} = -e^2 \langle \mathcal{G}_{\mu\nu}^{(2)} \rangle$$

(B5)

$$\mathcal{G}_{\mu\nu}^{(2)} = -G_{\mu\nu}^{(2)} + 4\kappa^2 \alpha \left[ \nabla^{(1)}_{\sigma} \nabla_{\tau} \vartheta \ast \bar{R}_{\tau}^{(\mu \sigma \nu)} + \nabla_{\sigma} \nabla_{\tau} \vartheta \ast \bar{R}^{(1)\tau}_{\mu \sigma \nu} \right]$$

(B7)

$\langle \ldots \rangle$ was defined in [43] and consists of the following effective operations

- Total derivative terms are put to zero.
- $\langle A_{\mu} B_{\nu} \rangle = - \langle A_{\nu} B_{\mu} \rangle$
- Covariant derivatives commute.
- Average of a product of two different fields are put to zero.

From which $\langle \mathcal{G}_{\mu\nu}^{(2)} \rangle$ was found to be

$$\langle \mathcal{G}_{\mu\nu}^{(2)} \rangle = \frac{1}{4} \langle \psi_{\mu}^{\rho \tau} \psi_{\nu}^{\rho \tau} \rangle - \frac{\kappa^2 \alpha^2}{2\beta} \langle \mathcal{P}_{\mu\nu} \rangle$$

(B8)

$$\langle \mathcal{P}_{\mu\nu} \rangle = -2 \langle \psi^{\beta \gamma \delta \lambda} \psi^{\alpha \sigma \tau \rho} \rangle \epsilon_{\mu \nu}^{\sigma \alpha} (\ast \bar{R}_{\lambda \beta \delta \gamma} + \ast \bar{R}_{\delta \beta \lambda \gamma}) - 2 \langle \psi^{\beta \gamma \delta \lambda} \psi^{\mu \alpha \sigma \tau} \rangle \epsilon_{\nu \rho}^{\sigma \alpha} (\ast \bar{R}_{\lambda \beta \delta \gamma} + \ast \bar{R}_{\delta \beta \lambda \gamma})$$

$$-2 \langle \psi^{\rho \sigma \delta \lambda} \psi^{\alpha \beta \gamma \lambda} \rangle \left[ \ast \bar{R}_{\gamma \alpha \beta \delta} (\ast \bar{R}_{\mu \sigma \nu \rho} + \ast \bar{R}_{\nu \sigma \mu \rho}) + \ast \bar{R}_{\delta \gamma \alpha \beta} (\ast \bar{R}_{\mu \sigma \nu \rho} + \ast \bar{R}_{\nu \sigma \mu \rho}) \right]$$

$$+ \bar{R}^{\sigma \rho \delta \lambda} \left[ \epsilon_{\mu \gamma}^{\delta \lambda} \left( \langle \psi^{\eta \gamma \delta \lambda} \psi^{\mu \lambda \delta} \rangle \ast \bar{R}_{\rho \sigma \alpha \beta} + \langle \psi^{\eta \gamma \delta \lambda} \psi^{\nu \lambda \delta} \rangle \ast \bar{R}_{\rho \sigma \eta \beta} \right) + \epsilon_{\nu \lambda}^{\delta \gamma} \left( \langle \psi^{\eta \gamma \delta \lambda} \psi^{\mu \lambda \delta} \rangle \ast \bar{R}_{\rho \eta \alpha \beta} \right. \right.$$  

$$+ \langle \psi^{\eta \gamma \delta \lambda} \psi^{\mu \lambda \delta} \rangle \ast \bar{R}_{\rho \eta \sigma \beta} \rangle \bigg] + \bar{R}^{\rho \sigma \alpha \beta} \left[ \langle \psi^{\gamma \delta \lambda} \psi^{\mu \lambda \delta} \rangle \ast \bar{R}_{\rho \sigma \lambda \beta} (\ast \bar{R}_{\mu \delta \nu \gamma} + \ast \bar{R}_{\nu \delta \mu \gamma}) \right. \bigg]$$

$$+ \langle \psi^{\gamma \delta \lambda} \psi^{\mu \lambda \delta} \rangle \ast \bar{R}_{\rho \lambda \alpha \beta} (\ast \bar{R}_{\mu \delta \nu \gamma} + \ast \bar{R}_{\nu \delta \mu \gamma}) \bigg]$$

(B9)

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