Integration by Parts formula for SPDEs with Multiplicative Noise and its Applications*

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Abstract

By using the Malliavin calculus, the Driver-type integration by parts formula is established for the semigroup associated to to SPDEs with Multiplicative Noise. Moreover, estimates on the density of heat kernel w.r.t. Lebesgue measure are obtained in finite dimension case.

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1 Introduction

A significant application of the Malliavin calculus ([5, 6]) is to describe the density of a Wiener functional using the integration by parts formula. In 1997, Driver [1] established the following integration by parts formula for the heat semigroup $P_t$ on a compact Riemannian manifold $M$:

$$(1.1) \quad P_t(\nabla Z f) = \mathbb{E}(f(X_t)N_t), \quad f \in C^1(M), Z \in \mathcal{X},$$

where $\mathcal{X}$ is is the set of all smooth vector fields on $M$, and $N_t$ is a random variable depending on $Z$ and the curvature tensor. This formula has many applications. For example, we are

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able to characterize the derivative w.r.t. the second variable $y$ of the heat kernel $p_t(x,y)$, see [7] for a study on integration by parts formulas and applications for SDEs and SPDEs driven by Wiener processes. Moreover, if $N_t$ is exponential integrable, (1.1) implies the shift Harnack inequality, see also [7] for details.

So far, there are many results on the Driver-type integration by parts formula for SDEs or SPDEs. The backward coupling method developed in [7] has been used in [2, 11] for SDEs driven by fractional Brownian motions and SPDEs driven by Wiener processes. Recently, using a finite many jumps approximation and Malliavin calculus, Wang obtains integration by parts formula for SDEs and SPDEs with additive noise driven by subordinated Brownian motion, see [8, 9].

However, all the above results are considered in additive noise case. The aim of this paper is to derive the integration by parts formula for SPDEs with multiplicative noise by Malliavin calculus and to derive estimates on the derivatives of heat kernel.

The main difficulty in obtaining the integration by parts formula is to give a representation of $D_h \kappa^T$ (see the proof of Theorem 2.2). Unfortunately, in multiplicative noise case, the equations for $D_h \kappa^T$ is so sophisticated that the Duhamel’s formula used in the additive noise case is unavailable. Instead, we applying Lemma 3.1 which is crucial in the proof of the main results.

Consider the following SPDE on a separable Hilbert spaces $(\mathbb{H}, \langle \cdot, \cdot \rangle, | \cdot |)$:

\begin{equation}
(1.2) \quad dX_t = AX_t dt + b_t(X_t) dt + \sigma_t(X_t) dW_t, \quad X_0 = x \in \mathbb{H},
\end{equation}

where $b : [0, \infty) \times \mathbb{H} \to \mathbb{H}$ are measurable locally bounded (i.e. bounded on bounded sets), and $\sigma : [0, \infty) \times \mathbb{H} \to \mathcal{L}(\mathbb{H})$ is measurable, where $\mathcal{L}(\mathbb{H})$ is the space of bounded linear operators on $\mathbb{H}$ equipped with the operator norm $\| \cdot \|$. Moreover,

(i) $W$ is a cylindrical Brownian motion on $\mathbb{H}$ with respect to a complete filtration probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$. More precisely, $W = \sum_{n=1}^{\infty} w^n e_n$ for a sequence of independent one dimensional Brownian motions $\{w^n\}_{n \geq 1}$ with respect to $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$, where $\{e_n\}_{n \geq 1}$ is an orthonormal basis on $\mathbb{H}$.

(ii) $(A, \mathcal{D}(A))$ is a linear operator generating a $C_0$-contraction semigroup $e^{At}$ such that

\begin{equation}
(1.3) \quad \int_0^T \|e^{At}\|_{\text{HS}}^2 dt < \infty, \quad T > 0,
\end{equation}

where $\| \cdot \|_{\text{HS}}$ is the Hilbert-Schmidt norm.

(iii) There exists a non-negative function $K \in C([0, \infty), [0, \infty))$ such that

\begin{equation}
(1.4) \quad \|\nabla_v b_s(x)\| \vee \|\nabla_v \sigma_s(x)\|_{\text{HS}} \leq K(s)|v|, \quad s \geq 0, x, v \in \mathbb{H}.
\end{equation}

Then the equation (1.2) has a unique mild solution $X_t(x)$, and the associated Markov semigroup $P_t$ is defined as follows:

$$P_t f(x) := \mathbb{E} f(X_t(x)), \quad f \in \mathcal{B}_b(\mathbb{H}), t \geq 0, x \in \mathbb{H}.$$
Since for any \( t \geq 0 \), \( \text{Ker}(e^{At}) = 0 \), the inverse operator \( e^{-At} : \text{Im}(e^{At}) \to \mathbb{H} \) is well defined.

To establish the integration by parts formula, we need the following assumptions:

\((H1)\) For any \((t, x) \in [0, \infty) \times \mathbb{H}, b_t, \sigma_t \in C^2(\mathbb{H})\), and there holds \( \nabla b_t(x) : \text{Im}(e^{At}) \to \text{Im}(e^{At}), \nabla \sigma_t(x) : \text{Im}(e^{At}) \to \mathcal{L}(\mathbb{H}, \text{Im}(e^{At})) \). Let

\[
B_t(x) := e^{-At} \nabla b_t(x) e^{At}, \quad \Sigma_t(x) := e^{-At} \nabla \sigma_t(x) e^{At},
\]

\(\sigma_t^{(k)}(x) := \sigma_t(x) e_k, \quad \Sigma_t^{(k)}(x) := e^{-At} \nabla \sigma_t^{(k)}(x) e^{At}, \quad k \geq 1, (t, x) \in [0, \infty) \times \mathbb{H}.\)

\(B_t(x)\) and \(\Sigma_t^{(k)}(x)\) are the corresponding eigenbasis. \((H1)\) holds if

\[
\sum_{k=1}^{\infty} \left\| \Sigma_t^{(k)}(x) \right\|^2 \leq \lambda_1(t), \quad t \geq 0, x \in \mathbb{H},
\]

Assume

\[
\left\| B_t(x) \right\| \vee \left\{ \sum_{k=1}^{\infty} \left\| \Sigma_t^{(k)}(x) \right\|^2 \right\}^{\frac{1}{2}} \leq K_1(t), \quad t \geq 0, x \in \mathbb{H},
\]

and

\[
\left\| \nabla B_t(x) \right\| \vee \left\{ \sum_{k=1}^{\infty} \left\| \nabla \Sigma_t^{(k)}(x) \right\|^2 \right\}^{\frac{1}{2}} \leq K_2(t), \quad t \geq 0, x \in \mathbb{H}
\]

for two increasing functions \(K_1, K_2 \in C([0, \infty), [0, \infty))\).

\((H2)\) \(\sigma\) is invertible. Moreover, there exist a strictly positive increasing function \(\lambda, \lambda \in C([0, \infty), (0, \infty))\) such that

\[
\left\| \sigma_t^{-1}(x) \right\| \leq \lambda(t), \quad t \geq 0, x \in \mathbb{H}.
\]

\((H2)\) is a standard non-degenerate assumption, while \((H1)\) comes from \([9]\), which means that the interaction between far away directions are weak enough. For example, let \(0 < \lambda_1 \leq \lambda_2 \leq \cdots \) be all eigenvalues of \(-A\) counting multiplicities, and \(\{e_k\}_{k \geq 1}\) are the corresponding eigenbasis. \((H1)\) holds if

\[
\left| \langle \nabla e_i b_t, e_j \rangle \right| \leq K_1(t) e^{-t|\lambda_i - \lambda_j|},
\]

\[
\left| \langle \nabla \nabla e_i b_t, e_j \rangle \right| \leq K_2(t) e^{-t|\lambda_i - \lambda_j|},
\]

\[
\sum_{k=1}^{\infty} \left| \langle \nabla e_i \sigma_t^{(k)}, e_j \rangle \right|^2 \leq K_1^2(t) e^{-2t|\lambda_i - \lambda_j|},
\]

\[
\sum_{k=1}^{\infty} \left| \langle \nabla \nabla e_i \sigma_t^{(k)}, e_j \rangle \right|^2 \leq K_2^2(t) e^{-2t|\lambda_i - \lambda_j|}, \quad t \geq 0, i, j \geq 1.
\]

In addition, for simplicity, set

\[
(\nabla B_t)(x)(u, v) := \nabla_v (B_t(\cdot)u)(x),
\]

\[
(\nabla \Sigma_t^{(k)})(x)(u, v) := \nabla_v \left( \Sigma_t^{(k)}(\cdot)u \right)(x), \quad k \geq 1, (t, x, u, v) \in [0, \infty) \times \mathbb{H}^3.
\]
2 Main results

To state our main results, for any \( s \geq 0 \), we introduce \( \mathcal{L}(\mathbb{H}) \)-valued processes \((J_{s,t})_{t \geq s}\) and \((J^{A}_{s,t})_{t \geq s}\), which solve the following SDEs:

\[
dJ_{s,t} = B_t(X_t)J_{s,t}dt + \sum_{k=1}^{\infty} \Sigma_{t}^{(k)}(X_t)J_{s,t}dw^k_t, \quad J_{s,s} = I
\]

and

\[
dJ^{A}_{s,t} = (A + \nabla b_t(X_t))J^{A}_{s,t}dt + \sum_{k=1}^{\infty} \Sigma_{t}^{(k)}(X_t)J^{A}_{s,t}dw^k_t, \quad J^{A}_{s,s} = I.
\]

From (1.7), (2.1) and (2.2) are well defined. Set \( J_t = J_{0,t} \) and \( J^{A}_{t} = J^{A}_{0,t} \), then \( J^{A}_{t} = e^{At}J_t \).

Firstly, we use Itô’s formula to derive the equation for \( \{J^{-1}_t\}_{t \geq 0} \).

Assume

\[
dJ^{-1}_t = G_t dt + \sum_{k=1}^{\infty} H^{(k)}_t dw^k_t, \quad J^{-1}_0 = I,
\]

where \( G \) and \( H \) are to be determined.

Applying Itô’s formula (see [3, section 23]), combining with (2.1), we have

\[
dJ_tJ^{-1}_t = B_t(X_t)J_tJ^{-1}_tdt + \sum_{k=1}^{\infty} \Sigma^{(k)}_t(X_t)J_tJ^{-1}_tdw^k_t
\]

\[+ J_tG_t dt + \sum_{k=1}^{\infty} J_tH^{(k)}_t dw^k_t + \sum_{k=1}^{\infty} \Sigma^{(k)}_t(X_t)J_tH^{(k)}_t dt.
\]

Since \( dJ_tJ^{-1}_t = 0 \), it holds that

\[
G_t = -J^{-1}_t \left\{ B_t(X_t) - \sum_{k=1}^{\infty} \left( \Sigma^{(k)}_t(X_t) \right)^2 \right\}, \quad H^{(k)}_t = -J^{-1}_t \Sigma^{(k)}_t(X_t), \quad k \geq 1.
\]

Thus, we obtain

\[
dJ^{-1}_t = -J^{-1}_t \left\{ B_t(X_t) - \sum_{k=1}^{\infty} \left( \Sigma^{(k)}_t(X_t) \right)^2 \right\} dt - \sum_{k=1}^{\infty} J^{-1}_t \Sigma^{(k)}_t(X_t)dw^k_t, \quad J^{-1}_0 = I.
\]

**Remark 2.1** Since the inverse of \( J^{A}_t \) does not exist in infinite dimension, we will use \( J^{-1}_t \) to construct \( h \) in (3.5) in stead of \((J^{A}_t)^{-1}\) in finite dimension, seeing details in the proof of Theorem 2.1. Moreover, to ensure the existence of \( J^{-1}_t \), we assume (H1).

We have the following estimates for \( J_t \) and \( J^{-1}_t \).
Lemma 2.1. Assume (H1) and (H2). Then for any $x \in \mathbb{H}$, $t \geq 0$, $p \geq 2$, it holds that

\begin{equation}
(2.4) \quad \sup_{s \in [0,t]} \mathbb{E} \| J_s \|^p \leq 3^{p-1} \exp \left\{ 3^{p-1} \left( t^{p-1} + t^{\frac{p}{2}-1} \right) K_1^p(t) \right\}
\end{equation}

and

\begin{equation}
(2.5) \quad \sup_{s \in [0,t]} \mathbb{E} \| J_s^{-1} \|^p \leq 3^{p-1} \exp \left\{ 3^{p-1} \left[ t^{p-1} (K_1(t) + K_1^2(t))^p + t^{\frac{p}{2}-1} K_1^p(t) \right] \right\}.
\end{equation}

Proof. By Burkholder-Davis-Gundy inequality, it follows from (2.1) that

\begin{align*}
\mathbb{E} \| J_t \|^p & \leq 3^{p-1} + 3^{p-1} \mathbb{E} \left\| \int_0^t B_s(X_s) J_s ds \right\|^p + 3^{p-1} \mathbb{E} \left\| \int_0^t \sum_{k=1}^{\infty} \Sigma_s^{(k)}(X_s) J_s dw_s^k \right\|^p \\
& \leq 3^{p-1} + 3^{p-1} t^{p-1} \mathbb{E} \int_0^t \| B_s(X_s) \|^p \| J_s \|^p ds \\
& \quad + 3^{p-1} t^{\frac{p}{2}-1} \mathbb{E} \int_0^t \left( \sum_{k=1}^{\infty} \| \Sigma_s^{(k)}(X_s) \|^2 \right)^{\frac{p}{2}} \| J_s \|^p ds \\
& \leq 3^{p-1} + 3^{p-1} \left[ t^{p-1} K_1^p(t) + t^{\frac{p}{2}-1} K_1^p(t) \right] \int_0^t \mathbb{E} \| J_s \|^p ds.
\end{align*}

Applying Gronwall inequality, we have

\begin{equation}
(2.6) \quad \sup_{s \in [0,t]} \mathbb{E} \| J_s \|^p \leq 3^{p-1} \exp \left\{ 3^{p-1} \left( t^{p-1} + t^{\frac{p}{2}-1} \right) K_1^p(t) \right\}.
\end{equation}

Similarly, (2.3) yields that

\begin{equation}
(2.7) \quad \sup_{s \in [0,t]} \mathbb{E} \| J_s^{-1} \|^p \leq 3^{p-1} \exp \left\{ 3^{p-1} \left[ t^{p-1} (K_1(t) + K_1^2(t))^p + t^{\frac{p}{2}-1} K_1^p(t) \right] \right\}.
\end{equation}

Thus, we finish the proof.

The main result is the following.

Theorem 2.2. Assume (H1) and (H2), then the integration formula by parts holds, i.e.

\begin{equation}
(2.8) \quad P_T(\nabla_{e^{AT}v} f) = \frac{1}{T} \mathbb{E} \{ f(X_T) M^n_T \}, \quad v \in \mathbb{H}, f \in C_b(\mathbb{H})
\end{equation}
Then the following assertions hold.

\[
M_T^p := \left\langle \int_0^T \left[ \sigma_t^{-1}(X_t) J_t^A \right]^* dW_t, J_T^{-1}v \right\rangle 
+ \int_0^T \text{Tr} \left\{ tJ_t^{-1} \left[ (\nabla B_t)(X_t) (J_tJ_T^{-1}v, J_t^A) \right] \right\} dt
+ \sum_{k=1}^\infty \left[ \int_0^T J_t^{-1} \sum_{j=1}^\infty \left( \nabla_t e_k e_k \Sigma_t^{(j)} \right) (X_t) J_t d w_t^j \right] J_T^{-1}v, e_k \right\}
(2.9)
\]

Then the following assertions hold.

\[
\sum_{k=1}^\infty \left[ \int_0^T J_t^{-1} \sum_{j=1}^\infty \left( \nabla_t e_k e_k \Sigma_t^{(j)} \right) (X_t) J_t d w_t^j \right] J_T^{-1}v, e_k \right\}
\]

\[
\int_0^T \text{Tr} \left\{ J_t^{-1} [\Sigma_t(X_t) J_t J_T^{-1}v] (\sigma_t^* (\sigma_t^*)^{-1}) (X_t) J_t^A \right\} dt
- \int_0^T \text{Tr} \left\{ tJ_t^{-1} \sum_{j=1}^\infty \Sigma_t^{(j)} (X_t) \left[ (\nabla \Sigma_t^{(j)}) (X_t) (J_t J_T^{-1}v, J_t^A) \right] \right\} dt.
\]

**Remark 2.2** Every term in (2.9) is well defined by (1.3), (H1), (H2), Lemma 2.1.

This result extends [7, Theorem 5.1] where \( \sigma \) only depends on time, see also [10, Theorem 3.2.4(1)]. When \( \mathbb{H} = \mathbb{R}^d \) is finite-dimensional, we may take \( A = 0 \) and so that Theorem 2.2 with \( J^A = J \) covers the result in [7, Theorem 2.1]. In this case, according to [7], the integration by parts formula implies that \( P_T \) has a density \( p_T(x,y) \) with respect to the Lebesgue measure, which is differentiable in \( y \) with

\[
(2.10) \quad \nabla_y \log p_T(x,\cdot)(y) = -\mathbb{E} (M_T^p | X_T(x) = y), \quad x, v \in \mathbb{R}^d.
\]

The next corollary is an application of Theorem 2.2 for finite dimension case.

**Corollary 2.3.** Assume (H1) and (H2), \( \mathbb{H} = \mathbb{R}^d, A = 0 \). Let

\[
\beta_1(p,t) := 3^{p-1} \exp \left\{ 3^{p-1} \left( t^{p-1} + t^{-\frac{d}{2}} \right) K_1^p(t) \right\},
\beta_2(p,t) := 3^{p-1} \exp \left\{ 3^{p-1} \left[ t^{p-1} (K_1(t) + K_2(t))^p + t^{-\frac{d}{2}} K_1^p(t) \right] \right\}.
\]

Then the following assertions hold.

1. For any \( T > 0, v \in \mathbb{R}^d \),

\[
\| P_T(\nabla_v f) \|_\infty \leq \| v \|_\infty \| f \|_\infty \frac{\Gamma_T}{T}, \quad f \in C^1_b(\mathbb{R}^d),
\]

\[
\int_{\mathbb{R}^d} \left| \nabla_y \log p_T(x,\cdot)(y) p_T(x,y) \right| dy \leq |v| \frac{\Gamma_T}{T}, \quad x \in \mathbb{R}^d,
\]

where

\[
\Gamma_T = \lambda(T) \sqrt{dT} (\beta_1(2,T) \beta_2(2,T))^\frac{1}{2}
+ dT^2 K_2(T) (\beta_2(4,T) \beta_1(8,T) \beta_2^3(2,T))^\frac{1}{2}
+ (dT^2 K_2(T) + dT \lambda_2(T) K_1(T) + dT^2 K_1(T) K_2(T)) (\beta_1(4,T) \beta_2(4,T))^\frac{1}{2}.
\]
(2) For any \( p > 1, T > 0, \) it holds that

\[
|P_T(\nabla_v f)| \leq \frac{|v|}{T} (P_T|f|^p)^{\frac{1}{p}} \left\{ 5 \left( \frac{p-2}{p-1} \right)^{\frac{1}{2}} \Gamma_{T, \frac{p-1}{p-2}} \right\}^{\frac{1}{2}},
\]

\[
\int_{\mathbb{R}^d} |\nabla_v \log p_T(x, \cdot)| \frac{|v|}{T} (y)p_T(x, y)dy \leq \frac{|v|}{T} \left\{ 5 \left( \frac{p-2}{p-1} \right)^{\frac{1}{2}} \Gamma_{T, \frac{p-1}{p-2}} \right\}^{\frac{1}{2}}, \quad x \in \mathbb{R}^d,
\]

where

\[
\Gamma_{T,q} = C(q)\lambda^q(T)d^2T\frac{\beta_1(2q,T)}{2} \frac{\beta_2(2q,T)}{2} \left\{ \beta_1(2q,T) \beta_2(2q,T) \right\}^{\frac{1}{2}}
\]

\[
+ C(q)d^2T^{2q+1}K^q_2(T) \left( \beta_1(4q,T) \beta_2(4q,T) \right)^{\frac{1}{2}}
\]

\[
+ \left( d^qT^{2q}K^q_2(T) + d^qT^q\lambda^q(T)K^q_2(T) + d^qT^{2q}K^q_2(T) \right) \left( \beta_1(4q,T) \beta_2(4q,T) \right)^{\frac{1}{2}},
\]

and \( C(q) \) is a nonnegative constant only depending on \( q \geq 2. \)

**Remark 2.3** From (2.1) and (2.3), it is easy to see that \( \mathbb{E}(\exp(\delta |M^T_t|)) = \infty \) for any \( \delta > 0, \)

for \( M^T_t \) has the form like \( \exp(\exp(X)) \), where \( X \) is a Gaussian random variable. Thus, it can not yield the shift Harnack inequality by Young’s inequality from (2.8) as in the additive noise case.

The remainder of the paper is organized as follows. In Section 3, we give a proof of Theorem 2.2 in Section 4, we prove Corollary 2.3.

## 3 Proof of Theorem 2.2

Firstly, we introduce a formula for the solution of a class of semi-linear SDEs on \( \mathcal{L}(\mathbb{H}, \mathbb{H}) \).

**Lemma 3.1.** Let \( \{Y_t\}_{t \geq 0} \) solves the following SDE on \( \mathcal{L}(\mathbb{H}, \mathbb{H}) \):

\[
dY_t = a_t dt + b_t Y_t dt + \sum_{k=1}^{\infty} c^k_t Y_t dw^k_t + \sum_{k=1}^{\infty} f^k_t dw^k_t,
\]

where \( a, b, \{c^k\}_{k \geq 1}, \{f^k\}_{k \geq 1} \) are \( \mathcal{L}(\mathbb{H}, \mathbb{H}) \)-valued progressive measurable processes satisfying

\[
\int_0^t \mathbb{E} \left( \|a_s\|^2 + \|b_s\|^2 \right) ds + \sum_{k=1}^{\infty} \int_0^t \mathbb{E} \left( \|c^k_s\|^2 + \|f^k_s\|^2 \right) ds < \infty, \quad t \geq 0.
\]

Then \( Y_t \) satisfies

\[
(3.1) \quad Y_t = G_t \left\{ Y_0 + \int_0^t G^{-1}_s a_s ds + \int_0^t G^{-1}_s \sum_{k=1}^{\infty} f^k_s dw^k_s - \int_0^t G^{-1}_s \sum_{k=1}^{\infty} c^k_s f^k_s ds \right\}, \quad t \geq 0,
\]
where \( \{G_t\}_{t \geq 0} \) and \( \{G_t^{-1}\}_{t \geq 0} \) satisfy
\[
dG_t = b_t G_t dt + \sum_{k=1}^{\infty} c^k_t G_t dw^k_t, \quad G_0 = I;
\]
\[
dG_t^{-1} = -G_t^{-1} \left( b_t - \sum_{k=1}^{\infty} (c^k_t)^2 \right) dt - G_t^{-1} \sum_{k=1}^{\infty} c^k_t dw^k_t, \quad G_0^{-1} = I.
\]

Proof. Let \( \{F_t\}_{t \geq 0} \) (called as integrating factor) solves the following SDE:
\[
dF_t = -F_t b_t dt - F_t \sum_{k=1}^{\infty} c^k_t dw^k_t + F_t \sum_{k=1}^{\infty} (c^k_t)^2 dt, \quad F_0 = I.
\]
Then Itô formula yields
\[
dF_t Y_t = (dF_t) Y_t + F_t dY_t + (dF_t)(dY_t)
\]
\[
= F_t \left[ -b_t dt - \sum_{k=1}^{\infty} c^k_t dw^k_t + \sum_{k=1}^{\infty} (c^k_t)^2 dt \right] Y_t
+ F_t \left[ a_t dt + b_t Y_t dt + \sum_{k=1}^{\infty} c^k_t Y_t dw^k_t + \sum_{k=1}^{\infty} f^k_t dw^k_t \right]
- F_t \sum_{k=1}^{\infty} (c^k_t)^2 Y_t dt - F_t \sum_{k=1}^{\infty} c^k_t f^k_t dt
= F_t a_t dt + F_t \sum_{k=1}^{\infty} f^k_t dw^k_t - F_t \sum_{k=1}^{\infty} c^k_t f^k_t dt.
\]
So we obtain
\[
(3.2) \quad Y_t = F_t^{-1} \left\{ F_0 Y_0 + \int_0^t F_s a_s ds + \int_0^t F_s \sum_{k=1}^{\infty} f^k_s dw_s^k - \int_0^t F_s \sum_{k=1}^{\infty} c^k_s f^k_s ds \right\},
\]
where \( F_t^{-1} \) satisfies
\[
(3.3) \quad dF_t^{-1} = b_t F_t^{-1} dt + \sum_{k=1}^{\infty} c^k_t F_t^{-1} dw^k_t, \quad F_0^{-1} = I.
\]
Let \( G_t := F_t^{-1} \), we obtain (3.1).

Proof of Theorem 1.1. We will use Malliavin calculus to derive the integration by parts formula, see for instance [3] [7] [8]. For \( (W_t)_{t \in [0,T]} \), let \( (D, \mathcal{D}(D)) \) be the Malliavin gradient, and let \( (D^*, \mathcal{D}(D^*)) \) be its adjoint operator (i.e. the Malliavin divergence). Since \((J_{s,t}, J_{s,t}^A)_{t \geq s}\) satisfy linear SDEs, from (2.1), it is easy to see that
\[
(3.4) \quad J_T = J_{t,T} J_t, \quad J_T^A = J_{t,T}^A J_t^A, \quad J_t^A = e^{At} J_t, \quad T \geq t \geq 0.
\]
Take

\begin{equation}
(3.5) \quad h(t) = \int_{0}^{t} \sigma_{s}^{-1}(X_{s})J_{s}^{A}J_{T}^{-1}vds, \quad t \in [0, T].
\end{equation}

From (H1), we see that \( J_{t} \) and \( J_{t}^{-1} \) are Malliavin differentiable for every \( t \in [0, T] \) such that \( h \in \mathcal{D}(D^{*}) \), so that \(1.2\) yields

\begin{equation}
(3.6) \quad \frac{d}{dt}D_{h}X_{t} = (A + \nabla b_{t}(X_{t}))D_{h}X_{t}dt + \sum_{k=1}^{\infty} \nabla \sigma_{t}^{(k)}(X_{t})D_{h}X_{t}dw_{t}^{k} + \sigma_{t}(X_{t})dh(t), \quad D_{h}X_{0} = 0.
\end{equation}

Then by Duhamel’s formula, we obtain

\begin{equation}
D_{h}X_{T} = \int_{0}^{T} J_{t,T}^{A} \sigma_{t}(X_{t})dh(t) = \int_{0}^{T} J_{t,T}^{A}J_{t}^{-1}vdt = Te^{AT}v.
\end{equation}

Therefore,

\begin{equation}
E(\nabla e^{AT}v)(X_{T}) = \frac{1}{T}E(\nabla D_{h}X_{T}v)(X_{T}) = \frac{1}{T}E\{D_{h}f(X_{T})\} = \frac{1}{T}E\{f(X_{T})D^{*}(h)\}.
\end{equation}

Since \( h \) is not adapted, to calculate \( D^{*}(h) \), let

\begin{equation}
(3.7) \quad h_{k}(t) = \int_{0}^{t} \sigma_{s}^{-1}(X_{s})J_{s}^{A}e_{k}ds, \quad F_{k} = \langle J_{T}^{-1}v, e_{k} \rangle, \quad k \geq 1, t \in [0, T].
\end{equation}

Then \( h(t) \) can be written as

\begin{equation}
(3.8) \quad h(t) = \sum_{k=1}^{\infty} h_{k}(t)F_{k}, \quad t \in [0, T].
\end{equation}

Noting that \( h_{k} \) is adapted with \( E \int_{0}^{T} |h_{k}'(t)|^{2}dt < \infty \), we have

\begin{equation}
D^{*}(h_{k}) = \int_{0}^{T} \langle h_{k}'(t), dW_{t} \rangle = \int_{0}^{T} \langle (\sigma_{s}^{*}(\sigma_{s}^{*}J_{s}^{A})^{-1})(X_{s})J_{s}^{A}e_{k}, dW_{s} \rangle, \quad k \geq 1.
\end{equation}

Thus, using the formula \( D^{*}(F_{k}h_{k}) = F_{k}D^{*}(h_{k}) - D_{h_{k}}F_{k} \), we obtain

\begin{equation}
D^{*}(h) = \sum_{k=1}^{\infty} \{F_{k}D^{*}(h_{k}) - D_{h_{k}}F_{k}\}
\end{equation}

\begin{equation}
(3.8) \quad = \left\langle \int_{0}^{T} \left[ (\sigma_{s}^{*}(\sigma_{s}^{*}J_{s}^{A})^{-1})(X_{s})J_{s}^{A} \right]^{*} dW_{s}, J_{T}^{-1}v \right\rangle - \sum_{k=1}^{\infty} \langle D_{h_{k}}J_{T}^{-1}v, e_{k} \rangle.
\end{equation}
From (2.1) for $J_t := J_{0,t}$, we have

$$
dD_{hk} J_t = B_t (X_t) D_{hk} J_t dt + \sum_{j=1}^{\infty} \Sigma_t^{(j)} (X_t) D_{hk} J_t dw_t^j
$$

$$
+ \left( \nabla_{D_{hk} X_t} B_t \right) (X_t) J_t dt + \sum_{j=1}^{\infty} \left( \nabla_{D_{hk} X_t} \Sigma_t^{(j)} \right) (X_t) J_t dw_t^j
$$

$$
+ \sum_{j=1}^{\infty} \Sigma_t^{(j)} (X_t) J_t dh_k^j(t), \quad D_{hk} J_0 = 0,
$$

where $h_k^j := \langle h_k, e_j \rangle$, $j \geq 1$. By Lemma 3.1, we obtain

$$
D_{hk} J_T = J_T \int_0^T J_t^{-1} \left( \nabla_{D_{hk} X_t} B_t \right) (X_t) J_t dt
$$

$$
+ J_T \int_0^T J_t^{-1} \sum_{j=1}^{\infty} \left( \nabla_{D_{hk} X_t} \Sigma_t^{(j)} \right) (X_t) J_t dw_t^j
$$

$$
+ J_T \int_0^T J_t^{-1} \sum_{j=1}^{\infty} \left\langle \left( \sigma_t^* (\sigma_t \sigma_t^*)^{-1} \right) (X_t) J_t^4 e_k, e_j \right\rangle \Sigma_t^{(j)} (X_t) J_t dt
$$

$$
- J_T \int_0^T J_t^{-1} \sum_{j=1}^{\infty} \Sigma_t^{(j)} (X_t) \left( \nabla_{D_{hk} X_t} \Sigma_t^{(j)} \right) (X_t) J_t dt.
$$

(3.9)

In addition, it follows from (3.6) that

$$
D_{hk} X_t = t J_t^4 e_k.
$$

(3.10)

Since $0 = D_{hk} (J_T J_T^{-1}) = J_T D_{hk} J_T^{-1} + (D_{hk} J_T) J_T^{-1}$, it means that

$$
D_{hk} J_T^{-1} = - J_T^{-1} (D_{hk} J_T) J_T^{-1}.
$$

(3.11)
Combining (3.8), (3.9), (3.10) and (3.11), we get

\[
D^*(h) = \left\langle \int_0^T \left[ \sigma_s^{-1}(X_s)J_s^A \right]^* dW_s, J_T^{-1}v \right\rangle \\
+ \sum_{k=1}^{\infty} \int_0^T \left\langle J_t^{-1} \left( \nabla_{tJ_t^A e_k} B_t \right)(X_t)J_tJ_T^{-1}v dt, e_k \right\rangle \\
+ \sum_{k=1}^{\infty} \left\langle \left[ \int_0^T J_t^{-1} \sum_{j=1}^{\infty} \left( \nabla_{tJ_t^A e_k} \Sigma_t^{(j)} \right)(X_t)J_t dW_t \right] J_T^{-1}v, e_k \right\rangle \\
+ \sum_{k=1}^{\infty} \int_0^T \sum_{j=1}^{\infty} \left\langle \sigma_t^{-1}(X_t)J_t^A e_k, e_j \right\rangle \left\langle J_t^{-1} \Sigma_t^{(j)}(X_t)J_tJ_T^{-1}v dt, e_k \right\rangle \\
- \sum_{k=1}^{\infty} \int_0^T \sum_{j=1}^{\infty} \left\langle \Sigma_t^{(j)}(X_t) \left( \nabla_{tJ_t^A e_k} \Sigma_t^{(j)} \right)(X_t)J_tJ_T^{-1}v dt, e_k \right\rangle \\
= \left\langle \int_0^T \left[ \sigma_t^{-1}(X_t)J_t^A \right]^* dW_t, J_T^{-1}v \right\rangle \\
+ \int_0^T \text{Tr} \left\{ tJ_t^{-1} \left[ \left( \nabla B_t \right)(X_t) \left( J_tJ_T^{-1}v, J_t^A \right) \right] \right\} dt \\
+ \sum_{k=1}^{\infty} \left\langle \left[ \int_0^T J_t^{-1} \sum_{j=1}^{\infty} \left( \nabla_{tJ_t^A e_k} \Sigma_t^{(j)} \right)(X_t)J_t dW_t \right] J_T^{-1}v, e_k \right\rangle \\
+ \int_0^T \text{Tr} \left\{ J_t^{-1} \left[ \Sigma_t(X_t)J_tJ_T^{-1}v \right] \left( \sigma_t^*(\sigma_t^{-1}) \right)(X_t)J_t^A \right\} dt \\
- \int_0^T \text{Tr} \left\{ tJ_t^{-1} \sum_{j=1}^{\infty} \Sigma_t^{(j)}(X_t) \left[ \left( \nabla \Sigma_t^{(j)} \right)(X_t) \left( J_tJ_T^{-1}v, J_t^A \right) \right] \right\} dt
\]

Substituting this into (3.7), we obtain (2.8). \qed
4 Proof of Corollary 2.3

Proof of Corollary 2.3. (1) For simplicity, letting

\[ \Theta_1 = \left\langle \int_0^T \left[ \sigma_t^{-1}(X_t) J_t \right]^* dW_t, J_T^{-1} v \right\rangle; \]

\[ \Theta_2 = \int_0^T \text{Tr} \left\{ t J_t^{-1} \left[ (\nabla B_t)(X_t) \left( J_t J_T^{-1} v, J_t \right) \right] \right\} dt; \]

\[ \Theta_3 = \sum_{k=1}^d \left\langle \left[ \int_0^T J_t^{-1} \sum_{j=1}^d \left( \nabla_t e_k, \Sigma_t^{(j)} \right) (X_t) J_t dW_t \right] J_T^{-1} v, e_k \right\rangle; \]

\[ \Theta_4 = \int_0^T \text{Tr} \left\{ J_t^{-1} \left[ \Sigma_t(X_t) J_t J_T^{-1} v \right] \sigma_t^{-1}(X_t) J_t \right\} dt; \]

\[ \Theta_5 = -\int_0^T \text{Tr} \left\{ t J_t^{-1} \sum_{j=1}^\infty \Sigma_t^{(j)}(X_t) \left[ (\nabla \Sigma_t^{(j)}) (X_t) \left( J_t J_T^{-1} v, J_t \right) \right] \right\} dt, \]

then \( M_T^X = \sum_{i=1}^5 \Theta_i \).

Firstly, by Hölder inequality, Itô isometric formula, (H2), Lemma 2.1 we have

\[
E|\Theta_1| \leq \left\{ \int_0^T E \| \sigma_t^{-1}(X_t) J_t \|_{\text{HS}}^2 dt E \| J_T^{-1} v \|^2 \right\}^{\frac{1}{2}} \\
\leq \lambda(T) \sqrt{dT} |v| (\beta_1(2, T) \beta_2(2, T))^\frac{1}{4}.
\]

Next, Hölder inequality, (H1), Lemma 2.1 yield that

\[
E|\Theta_2| \leq dTK_2(T)|v| \int_0^T \left\{ E \| J_t \|_4 \right\} \left\{ E \| J_t^{-1} \|^2 \| J_T^{-1} \|^2 \right\} \left\{ E \| J_T^{-1} \|^4 \right\}^{\frac{1}{4}} dt \\
\leq dTK_2(T)|v| \int_0^T \left\{ E \| J_t \|_4 \right\} \left\{ E \| J_t^{-1} \|^4 \right\} \left\{ E \| J_T^{-1} \|^4 \right\}^{\frac{1}{4}} dt \\
\leq dT^2 K_2(T)|v| (\beta_1(4, T) \beta_2(4, T))^\frac{1}{4}.
\]

Again by Hölder inequality, Itô isometric formula, (H1), Lemma 2.1 it holds that

\[
E|\Theta_3| \leq dTK_2(T)|v| \int_0^T \left\{ E \left( \| J_t^{-1} \|^2 \| J_t \|^4 \right) \right\} \left\{ E \| J_T^{-1} \|^2 \right\} \left\{ E \| J_T^{-1} \|^4 \right\} \left\{ E \| J_T^{-1} \|^4 \right\}^{\frac{1}{4}} dt \\
\leq dTK_2(T)|v| \int_0^T \left\{ E \left( \| J_t^{-1} \|^4 \right) \right\} \left\{ E \| J_t \|^8 \right\} \left\{ E \| J_T^{-1} \|^2 \right\} \left\{ E \| J_T^{-1} \|^4 \right\} \left\{ E \| J_T^{-1} \|^4 \right\}^{\frac{1}{4}} dt \\
\leq dT^2 K_2(T)|v| (\beta_2(4, T) \beta_1(8, T) \beta_2^2(2, T))^\frac{1}{4}.
\]

Similarly as (4.2), it is easy to see that

\[
E|\Theta_4| \leq d\lambda_2(T) K_1(T)|v| \int_0^T \left\{ E \| J_t \|^4 \right\} \left\{ E \| J_t^{-1} \|^2 \| J_T^{-1} \|^2 \right\} \left\{ E \| J_T^{-1} \|^4 \right\} \left\{ E \| J_T^{-1} \|^4 \right\}^{\frac{1}{4}} dt \\
\leq dT\lambda_2(T) K_1(T)|v| (\beta_1(4, T) \beta_2(4, T))^\frac{1}{2},
\]

\[\text{Proof of Corollary 2.3.}(1)\text{For simplicity, letting}\]

\[\Theta_1 = \left\langle \int_0^T \left[ \sigma_t^{-1}(X_t) J_t \right]^* dW_t, J_T^{-1} v \right\rangle;\]

\[\Theta_2 = \int_0^T \text{Tr} \left\{ t J_t^{-1} \left[ (\nabla B_t)(X_t) \left( J_t J_T^{-1} v, J_t \right) \right] \right\} dt;\]

\[\Theta_3 = \sum_{k=1}^d \left\langle \left[ \int_0^T J_t^{-1} \sum_{j=1}^d \left( \nabla_t e_k, \Sigma_t^{(j)} \right) (X_t) J_t dW_t \right] J_T^{-1} v, e_k \right\rangle;\]

\[\Theta_4 = \int_0^T \text{Tr} \left\{ J_t^{-1} \left[ \Sigma_t(X_t) J_t J_T^{-1} v \right] \sigma_t^{-1}(X_t) J_t \right\} dt;\]

\[\Theta_5 = -\int_0^T \text{Tr} \left\{ t J_t^{-1} \sum_{j=1}^\infty \Sigma_t^{(j)}(X_t) \left[ (\nabla \Sigma_t^{(j)}) (X_t) \left( J_t J_T^{-1} v, J_t \right) \right] \right\} dt,\]

then \( M_T^X = \sum_{i=1}^5 \Theta_i \).

Firstly, by Hölder inequality, Itô isometric formula, (H2), Lemma 2.1 we have

\[E|\Theta_1| \leq \left\{ \int_0^T E \| \sigma_t^{-1}(X_t) J_t \|_{\text{HS}}^2 dt E \| J_T^{-1} v \|^2 \right\}^{\frac{1}{2}} \]

\[\leq \lambda(T) \sqrt{dT} |v| (\beta_1(2, T) \beta_2(2, T))^\frac{1}{4}.\]

Next, Hölder inequality, (H1), Lemma 2.1 yield that

\[E|\Theta_2| \leq dTK_2(T)|v| \int_0^T \left\{ E \| J_t \|_4 \right\} \left\{ E \| J_t^{-1} \|^2 \| J_T^{-1} \|^2 \right\} \left\{ E \| J_T^{-1} \|^4 \right\} \left\{ E \| J_T^{-1} \|^4 \right\} \left\{ E \| J_T^{-1} \|^4 \right\} \left\{ E \| J_T^{-1} \|^4 \right\}^{\frac{1}{4}} dt \]

\[\leq dTK_2(T)|v| \int_0^T \left\{ E \| J_t \|_4 \right\} \left\{ E \| J_t^{-1} \|^4 \right\} \left\{ E \| J_T^{-1} \|^4 \right\} \left\{ E \| J_T^{-1} \|^4 \right\} \left\{ E \| J_T^{-1} \|^4 \right\} \left\{ E \| J_T^{-1} \|^4 \right\}^{\frac{1}{4}} dt \]

\[\leq dT^2 K_2(T)|v| (\beta_1(4, T) \beta_2(4, T))^\frac{1}{4}.\]

Again by Hölder inequality, Itô isometric formula, (H1), Lemma 2.1 it holds that

\[E|\Theta_3| \leq dTK_2(T)|v| \int_0^T \left\{ E \left( \| J_t^{-1} \|^2 \| J_t \|^4 \right) \right\} \left\{ E \| J_T^{-1} \|^2 \right\} \left\{ E \| J_T^{-1} \|^4 \right\} \left\{ E \| J_T^{-1} \|^4 \right\} \left\{ E \| J_T^{-1} \|^4 \right\}^{\frac{1}{4}} dt \]

\[\leq dTK_2(T)|v| \int_0^T \left\{ E \left( \| J_t^{-1} \|^4 \right) \right\} \left\{ E \| J_t \|^8 \right\} \left\{ E \| J_T^{-1} \|^2 \right\} \left\{ E \| J_T^{-1} \|^4 \right\} \left\{ E \| J_T^{-1} \|^4 \right\} \left\{ E \| J_T^{-1} \|^4 \right\} \left\{ E \| J_T^{-1} \|^4 \right\}^{\frac{1}{4}} dt \]

\[\leq dT^2 K_2(T)|v| (\beta_2(4, T) \beta_1(8, T) \beta_2^2(2, T))^\frac{1}{4}.\]

Similarly as (4.2), it is easy to see that

\[E|\Theta_4| \leq d\lambda_2(T) K_1(T)|v| \int_0^T \left\{ E \| J_t \|^4 \right\} \left\{ E \| J_t^{-1} \|^2 \| J_T^{-1} \|^2 \right\} \left\{ E \| J_T^{-1} \|^4 \right\} \left\{ E \| J_T^{-1} \|^4 \right\} \left\{ E \| J_T^{-1} \|^4 \right\} \left\{ E \| J_T^{-1} \|^4 \right\}^{\frac{1}{4}} dt \]

\[\leq dT\lambda_2(T) K_1(T)|v| (\beta_1(4, T) \beta_2(4, T))^\frac{1}{2},\]
and
\[
\mathbb{E} |\Theta_3| \leq dT K_1(T) K_2(T) |v| \int_0^T \left\{ \mathbb{E} \left| J_t \right|^4 \right\} \frac{q}{2} \left\{ \mathbb{E} \left| J_t^{-1} \right|^2 \left| J_t^{-1} \right|^2 \right\} \frac{1}{2} dt \\
\leq dT^2 K_1(T) K_2(T) |v| (\beta_1(4,T) \beta_2(4,T))^{\frac{q}{2}}.
\]
Combining (4.4)-(4.5), we have
\[
\mathbb{E} |M_T^v| \leq |v| \Gamma_T,
\]
here, $\Gamma_T$ is in Corollary 2.3. According to (2.8), it is easy to see that
\[
|P_T(\nabla_v f)| \leq \frac{\|f\|_\infty}{T} \mathbb{E} |M_T^v| \leq |v| \|f\|_\infty \frac{\Gamma_T}{T}, \quad f \in C^1_b \mathbb{R}^d.
\]
By [7] Theorem 2.4(1)] with $H(r) = r$, (2.10) yields that
\[
\int_{\mathbb{R}^d} |\nabla_v \log p_T(x,\cdot)|(y) p_T(x,y) dy \leq |v| \frac{\Gamma_T}{T}, \quad x \in \mathbb{R}^d,
\]
(2) For any $q \geq 2$, by Burkholder-Davis-Gundy inequality, Hölder inequality, (H2), Lemma 2.1 we have
\[
\mathbb{E} |\Theta_1|^q \leq C(q) \lambda^q(T) T^{q \frac{q - 1}{2}} |v|^q \left\{ \int_0^T \mathbb{E} \| J_t \|_{H^q}^2 dt \right\} \frac{q}{2} \left\{ \mathbb{E} \left| J_t^{-1} \right|^2 \right\} \frac{1}{2} \\
\leq C(q) \lambda^q(T) d^{q} T^q |v|^q \left\{ \beta_1(2q,T) \beta_2(2q,T) \right\}^{\frac{q}{2}}.
\]
Next, Hölder inequality, (H1), Lemma 2.1 yield that
\[
\mathbb{E} |\Theta_3| \leq d^{q - 1} T^{q - 1} dT^q K_2(T) |v|^q \int_0^T \left\{ \mathbb{E} \| J_t \|_{L^q}^4 \right\} \frac{q}{2} \left\{ \mathbb{E} \left| J_t^{-1} \right|^2 \left| J_t^{-1} \right|^2 \right\} \frac{1}{2} dt \\
\leq d^{q - 1} T^{q - 1} dT^q K_2(T) |v|^q \int_0^T \left\{ \mathbb{E} \| J_t \|_{L^q}^4 \right\} \frac{q}{2} \left\{ \mathbb{E} \left| J_t^{-1} \right|^2 \left| J_t^{-1} \right|^2 \right\} \frac{1}{2} dt \\
\leq d^{q - 1} T^{q - 1} K_2(T) |v|^q \left( \beta_1(4q,T) \beta_2(4q,T) \right)^{\frac{q}{2}}.
\]
Again by Burkholder-Davis-Gundy inequality, Hölder inequality, (H1), Lemma 2.1 it holds that
\[
\mathbb{E} |\Theta_3|^q \leq C(q) d^q T^{\frac{3q - 1}{2}} K_2(T) |v|^q \int_0^T \left\{ \mathbb{E} \left( \left| J_t^{-1} \right|^2 \left| J_t \right| \right) \right\} \frac{q}{2} \left\{ \mathbb{E} \left| J_t^{-1} \right|^2 \right\} \frac{1}{2} dt \\
\leq C(q) d^q T^{\frac{3q - 1}{2}} K_2(T) |v|^q \int_0^T \left\{ \mathbb{E} \left| J_t^{-1} \right|^4 \right\} \frac{q}{2} \left\{ \mathbb{E} \left| J_t \right|^4 \right\} \frac{1}{2} \left\{ \mathbb{E} \left| J_t^{-1} \right|^2 \right\} \frac{1}{2} dt \\
\leq C(q) d^q T^{\frac{3q - 1}{2}} K_2(T) |v|^q \left( \beta_2(4q,T) \beta_1(8q,T) \beta_2^2(2q,T) \right)^{\frac{q}{2}}.
\]
Similarly as (4.7), it is easy to see that
\begin{equation}
\mathbb{E}|\Theta_1|^q \leq d^{q-1}T^{q-1}d\lambda_2^q(T)K_1^q(T)|v|^q \int_0^T \left\{ \mathbb{E} \|J_t\|^{4q}\right\}^{\frac{1}{2}} \left\{ \mathbb{E} \|J_t^{-1}\|^{2q}\|J_t^{-1}\|^{2q}\right\}^{\frac{1}{2}} dt \\
\leq d^qT^q\lambda_2^q(T)K_1^q(T)|v|^q (\beta_1(4q,T)\beta_2(4q,T))^{\frac{1}{2}},
\end{equation}
and
\begin{equation}
\mathbb{E}|\Theta_5|^q \leq d^{q-1}T^{q-1}dK_1^q(T)T^qK_2^q(T)|v|^q \int_0^T \left\{ \mathbb{E} \|J_t\|^{4q}\right\}^{\frac{1}{2}} \left\{ \mathbb{E} \|J_t^{-1}\|^{2q}\|J_t^{-1}\|^{2q}\right\}^{\frac{1}{2}} dt \\
\leq d^qT^qK_1^q(T)K_2^q(T)|v|^q (\beta_1(4q,T)\beta_2(4q,T))^{\frac{1}{2}}.
\end{equation}
Combining (4.6)-(4.10), for any \( q \geq 2 \), it holds that
\begin{equation}
\mathbb{E}|M_T^q|^q \leq \left\{ \frac{5^{q-1}}{q} \sum_{i=1}^5 \mathbb{E}|\Theta_i|^q \right\}^{\frac{1}{q}} |v| \leq \left\{ 5^{q-1}T_{\Gamma,q} \right\}^{\frac{1}{q}} |v|,
\end{equation}
here, \( \Gamma_{T,q} \) is in Corollary 2.3. On the other hand, Jensen inequality yields that
\begin{equation}
\mathbb{E}|M_T^q|^q \leq \left( \mathbb{E}|M_T|^2 \right)^{\frac{q}{2}}
\end{equation}
for any \( 1 < q < 2 \). Combining (4.11) and (4.12), we obtain for any \( q > 1 \),
\begin{equation}
\mathbb{E}|M_T^q|^q \leq \left\{ \frac{5^{q+2}}{q} \Gamma_{T,q} \right\}^{\frac{1}{q}} |v|.
\end{equation}
It follows from (2.8), Hölder inequality and (4.13) that for any \( p > 1 \),
\begin{equation}
|P_T(\nabla_v f)| \leq \frac{1}{T} (P_T |f|^p)^{\frac{1}{p}} \left( \mathbb{E}|M_T^p|^{\frac{p}{p-1}} \right)^{\frac{p-1}{p}} \\
\leq \frac{|v|}{T} (P_T |f|^p)^{\frac{1}{p}} \left\{ 5^{\frac{q-1}{p}} |2-1| \Gamma_{T,\|J_T\|^{\frac{p}{2}}} \right\}^{\frac{p-1}{p}}^{\frac{1}{2}}, \ f \in C_b^1 \mathbb{R}^d.
\end{equation}
Finally, applying [7] Theorem 2.4(1) with \( H(r) = r^{\frac{p}{p-1}} \), (2.10) yields that
\begin{equation}
\int_{\mathbb{R}^d} \nabla_v \log p_T(x,\cdot)|\nabla_r f(y)p_T(x,y) dy \leq \frac{|v|}{T} \left\{ 5^{\frac{q-1}{p}} |2-1| \Gamma_{T,\|J_T\|^{\frac{p}{2}}} \right\}^{\frac{p-1}{p}}^{\frac{1}{2}}, \ x \in \mathbb{R}^d.
\end{equation}
Thus, the proof is completed.

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