WELL-POSEDNESS, BLOW-UP CRITERIA AND GEVREY REGULARITY FOR A ROTATION-TWO-COMPONENT CAMASSA-HOLM SYSTEM

LEI ZHANG* AND BIN LIU
School of Mathematics and Statistics, Hubei Key Laboratory of Engineering Modeling and Scientific Computing
Huazhong University of Science and Technology, Wuhan 430074, Hubei, China

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ABSTRACT. In this paper, we are concerned with the Cauchy problem for a new two-component Camassa-Holm system with the effect of the Coriolis force in the rotating fluid. We first investigate the local well-posedness of the system in $B^{s}_{p,r} \times B^{s-1}_{p,r}$ with $s > \max\{1 + \frac{1}{p}, \frac{3}{2}, 2 - \frac{1}{p}\}$, $p, r \in [1, \infty]$ by using the transport theory in Besov space. Then by means of the logarithmic interpolation inequality and the Osgood's lemma, we establish the local well-posedness in the critical Besov space $B^{3/2}_{2,1} \times B^{1/2}_{2,1}$, and we present a blow-up result with the initial data in critical Besov space by virtue of the conservation law. Finally, we study the Gevrey regularity and analyticity of solutions to the system in a range of Gevrey-Sobolev spaces in the sense of Hardamard. Moreover, a precise lower bound of the lifespan is obtained.

1. Introduction. This paper considers a rotation-two-component Camassa-Holm system (R2CH), which is derived to model the equatorial water waves with the effect of the Coriolis force in the rotating fluid [22]:

\[
\begin{align*}
    u_t - u_{xxt} - Au_x + 3uu_x &= \sigma(2uu_{xx} + uu_{xxx}) - \mu u_{xxx} - (1 - 2\Omega A)\rho x + 2\Omega \rho (\rho u)_x, \\
    \rho_t + (\rho u)_x &= 0, \\
    u(0, x) &= u_0(x), \quad \rho(0, x) = \rho_0(x).
\end{align*}
\]

Here, the function $u(t, x)$ stands for the fluid velocity in $x$ direction and $\rho(t, x)$ describes the free surface elevation from equilibrium. In (1), the real non-dimensional parameter $\sigma$ is a parameter which provides the competition, or balance, in fluid convection between nonlinear steepening and amplification due to stretching, the parameter $A$ is related to a linear underlying shear flow, $\mu \in \mathbb{R}$ is a parameter and $\Omega$ characterizes the constant rotational speed of the Earth.

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* Corresponding author.
It is worth pointing out that the system (1) is closely related to several models describing the motion of waves at the free surface of shallow water under the influence of gravity.

For example, without considering the effect of the Earth’s rotation (i.e., $\Omega = 0$), the system (1) reduces to the generalized two-component integrable Dullin-Gottwald-Holm (gDGH) system, see [32] for more details. If we further take $\sigma = 1$ in (1), it then recovers the standard two-component Dullin-Gottwald-Holm (2DGH) system in the form of

$$\begin{cases}
u_t - u_{xxt} - Au_x + 3u u_x = 2u_x u_{xx} + uu_{xxx} - \mu u_{xxx} - \rho u_x, \\ \rho_t + (\rho u)_x = 0,
\end{cases}$$

which was equipped with the boundary assumptions $u \to 0$, $\rho \to 1$ as $|x| \to \infty$ [10, 29, 44]. It is shown that the 2DGH system is completely integrable, and it can be expressed as a compatibility condition of two linear systems (Lax pair) with a spectral parameter $\xi$ [30, 48]:

$$\Psi_t = \left( -\frac{\xi}{2} - u + \gamma \right) \Psi_x + \frac{1}{2} u_x \Psi,$$

$$\Psi_{xx} = \left( -\xi^2 \rho^2 + \xi \left( m - \frac{4}{2} + \frac{3}{2} \right) + \frac{1}{4} \right) \psi,$$

where $m = u - u_{xx}$. After its first appearance, lots of papers are devoted to studying the Cauchy problem of the 2DGH system. Here we only review some relevant results on our topic. By applying the Kato’s theorem for quasilinear hyperbolic systems, Guo, etc. [30] obtain the local well-posedness of the system in Sobolev space $H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$ with $s \geq 2$, and they showed that the solutions can only have singularities that correspond to wave breaking. Moreover, a sufficient condition was provided to ensure the existence of global solutions. In [6], by virtue of the bi-linear estimate technique to the approximate solutions, Chen, etc. established the local as well as global well-posedness in $H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$ with $s \geq \frac{3}{2}$, which improves the results in [30]. Later, in [36], Liu and Yin proved that the 2DGH is local well-posed in the nonhomogeneous Besov spaces $B^p_r(\mathbb{R}) \times B^{s-1}_r(\mathbb{R})$, $s > \max\{1 + \frac{1}{p}, 2 - \frac{1}{p}\}$ and $1 \leq p, r \leq \infty$, which extends the results obtained in [6, 29]. On the other hand, the authors investigated the orbital stability of the smooth solitary wave solutions in the energy space $H^1(\mathbb{R}) \times L^2(\mathbb{R})$. In [29], Guo and Wang considered the persistence property of solutions to the 2DGH system, which indicates that the corresponding solutions enjoy the same exponential decay property as the initial datum. Moreover, they proved that the strong solutions must identically equal to zero if the solutions and their spacial derivatives decay exponentially initially and at a later time. The wave breaking phenomena of solutions to the 2DGH system in the period case $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ was discussed by Zhu and Xu [48].

By taking $\Omega = 0$, $\sigma = 1$ and $\mu = 0$, the system (1) becomes the celebrated two-component integrable Camassa-Holm (2CH) system:

$$\begin{cases}
m_t + um_x + 2um_x - Au_x + \rho u_x = 0, \\ \rho_t + (\rho u)_x = 0,
\end{cases}$$

where $m = u - u_{xx}$. The 2CH system was originally introduced by Olver and Rosenau [40] as a bi-Hamiltonian model, which is complete integrable and has a Lax pair formulation. It is worth mentioning that the 2CH system is closely related to the first negative flow of the AKNS hierarchy via a reciprocal transformation [5, 21], in which the authors proved that the 2CH system is integrable under the
assumption that it has Lax pair, and it also has peakons and multi-kink solitons. The great interest in the 2CH system lies in the fact that Constantin and Ivanov \[10\] provided the hydrodynamical derivation of the system as a valid approximation to the governing equations for water waves in the shallow water regime without vorticity (i.e., \(A = 0\)). In \[19, 25\], the authors studied the well-posedness and wave breaking phenomenon for 2CH system. In \[31\], Constantin and Escher studied the wave breaking phenomena for the 2CH system. The local well-posedness of the 2CH system in a range of the Besov spaces was studied in \[27\], wherein the wave-breaking mechanism for strong solutions and the exact blow-up rate of solutions were also investigated. With the initial data satisfying some certain conditions, it was shown that the 2CH system admits global solutions \[26, 28\]. Furthermore, for the Cauchy problem for some high-order 2CH systems, see for example \[20, 35, 47\] and the references therein.

It is worth pointing out that the system (1) (or (5)) can be regarded as a generalization of the famous Camassa-Holm (CH) equation in the case of \(\rho = 0\):

\[
m_t + um_x + 2um_x = 0, \quad m = u - u_{xx},
\]

which was first introduced as a bi-Hamiltonian system by Fokas and Fuchssteiner in \[24\], and then Camassa and Holm \[4\] independently re-derived it by approximating directly in the Hamiltonian for the Euler equations in the shallow water regime. The CH equation possesses the bi-Hamiltonian structures and is completely integrable \[4, 7, 9, 24\]. One of the remarkable property for Equ.(6) is that it possesses the peaked solitons (called peakons) in the form of \(u(x, t) = ce^{-|x-ct|}(c \neq 0)\) which is orbitally stable \[15, 16\], and in the periodic case \(u(x, t) = \frac{c \cosh(x - ct - [x - ct] - 1/2)}{\sinh(1/2)}\).

Since the peakons capture an essential feature that is characteristic for the traveling waves of largest amplitude, the CH equation has attracted a lot of attention in the literature over the past decades. For instance, the local well-posedness and blow-up phenomena for (6) in the Sobolev as well as the Besov spaces were investigated in \[8, 11, 12, 13, 17, 44, 46\]. For the global existence of the weak and strong solutions, we refer the readers to \[8, 11, 12, 14, 45\]. Moreover, the global conservative and dissipative solutions to the CH equation are investigated in \[2, 3, 33\].

Recently, Fan, etc. \[22\] derived the system (1) in the spirit of Ivanov’s asymptotic perturbation analysis for the governing equations of two-dimensional rotational gravity water waves \[34\]. The authors mainly established the precise blow-up mechanism, and it is shown that the model (1) can only have singularities which corresponding to wave breaking. Moreover, some initial conditions are provided to guarantee that the wave breaking phenomena occurs in finite time.

To the best known of our knowledge, the Cauchy problem of (1) in Besov spaces and the Gevrey regularity and analyticity of the solutions have not been studied yet. In this paper, based on the Littlewood-Paley theory and the transport theory, we shall show that the system (1) is locally well-posed in \(B^s_{p,r}(\mathbb{R}) \times B^s_{p,r}(-1)(\mathbb{R})\) with \(s > \max\{1 + \frac{1}{p}, \frac{3}{2}, 2 - \frac{1}{p}\}\). Then we establish the local well-posedness in the critical space \(B^2_{2,1}(\mathbb{R}) \times B^2_{2,-1}(\mathbb{R})\). The major difficulties in our discussion are as follows. First, due to the appearance of the Earth’s rotation, the system (1) involves two cubic-order nonlinear terms, which makes the proof of several required nonlinear estimates in Besov spaces more delicate than the standard 2CH system and the 2DGH system. Second, it seems that one can not obtain the convergence of the iterated sequence \((u^{(n)}, v^{(n)}))_{n \in \mathbb{N}}\) in \(C([0, T]; B^2_{2,1}(\mathbb{R}) \times B^{-2}_{2,1}(\mathbb{R}))\) directly, which is
caused by the low regularity of $B^{-\frac{1}{2}}_{2,1}(\mathbb{R})$. Instead, we need to prove the convergence in $C([0,T];B^{-\frac{1}{2}}_{2,\infty}(\mathbb{R})\times B^{\frac{1}{2}}_{2,\infty}(\mathbb{R}))$ by using an endpoint Moser-type estimate and the Log-type interpolation inequality. Then, we give a blow-up result for the solution to (1) with data in $B^{-\frac{3}{2}}_{2,1}(\mathbb{R})\times B^{\frac{3}{2}}_{2,1}(\mathbb{R})$ with the help of the conservation law. Finally, we study the Gevrey regularity of the system (1) by using a generalized Cauchy-Kovalevsky theorem, which implies that the system (1) admits analytical solutions locally in time and globally in space. Moreover, we obtain a precise lower bound of the lifespan and the continuity of the solution mapping in the sense of this paper.

The structure of the paper is organized as follows. In section 2, we recall some necessary results on the Besov space. We refer the readers to [1, 18] devoted to the local well-posedness of solutions to (1) in Besov spaces, and a blow-up result is provided. In section 5, we investigate the Gevrey regularity and analyticity of the system (1).

**Notation.** All function spaces are considered in $\mathbb{R}$, and we shall drop them in our notation if there is no ambiguity. We denote by $C$ the estimates that hold up to some universal constant which may change from line to line but whose meaning is clear throughout the context.

2. Preliminaries. The investigation of the well-posedness of the system (1) strongly depends on the transport theory in Besov spaces. To begin with, let us recall some necessary results on the Besov space. We refer the readers to [1, 18] for more details of the proof.

**Definition 2.1.** Let $1 \leq p, r \leq \infty$, $s \in \mathbb{R}$, the 1-D nonhomogeneous Besov space $B^s_{p,r}$ is defined by

$$B^s_{p,r} := \left\{ u \in \mathcal{S}'; \| u \|_{B^s_{p,r}} = \left( \sum_{j \in \mathbb{N}} 2^{js} \| \Delta_j u \|_{L^p}^r \right)^{\frac{1}{r}} < \infty \right\},$$

where $\{ \Delta_j \}_{j \geq -1}$ are the nonhomogeneous Littlewood-Paley decomposition operators defined through an almost-orthogonality partition of unity (see, e.g., [1]).

**Lemma 2.2.** Let $s \in \mathbb{R}$, $1 \leq p, r, p_1, r_1 \leq \infty$, $i = 1, 2$, then

1. Fatou's lemma: if $\{ u_n \}_{n \in \mathbb{N}^+}$ is bounded in $B^s_{p,r}$ and $u_n \rightharpoonup u$ in $\mathcal{S}'$, then $u \in B^s_{p,r}$ and

$$\| u \|_{B^s_{p,r}} \leq \liminf_{n \to \infty} \| u_n \|_{B^s_{p,r}}.$$  

2. If $s_1 \leq \frac{1}{p} < s_2$ ($s_2 \geq \frac{1}{p}$ if $r = 1$) and $s_1 + s_2 > 0$, then we have

$$\| uv \|_{B^{s_1}_{p,r}} \leq C \| u \|_{B^{s_1}_{p,r}} \| v \|_{B^{s_2}_{p,r}}.$$  

3. For any $f \in B^{-\frac{1}{2}}_{2,\infty}$ and $g \in B^{\frac{1}{2}}_{2,\infty} \cap L^\infty$, there exists a constant $C > 0$ such that

$$\| fg \|_{B^{-\frac{1}{2}}_{2,\infty}} \leq C \| f \|_{B^{-\frac{1}{2}}_{2,\infty}} \| g \|_{B^{\frac{1}{2}}_{2,\infty} \cap L^\infty}.$$  

4. A smooth function $f : \mathbb{R}^d \to \mathbb{R}$ is said to be an $S^m$-multiplier: if $\forall \alpha \in \mathbb{N}^n$, there exists a constant $C_\alpha > 0$ such that $|\partial^\alpha f(\xi)| \leq C_\alpha (1 + |\xi|)^{m-|\alpha|}$, for all $\xi \in \mathbb{R}^d$. The operator $f(D)$ is continuous from $B^s_{p,r}$ to $B^{s-m}_{p,r}$, for all $s \in \mathbb{R}$ and $1 \leq p, r \leq \infty$.

**Lemma 2.3.** (Interpolation inequality) (1) Let $s_2 > s_1$, $\theta \in (0,1)$, we have

$$\| u \|_{B^{s_2+(1-\theta)s_2}_{p,1}} \leq \frac{C}{s_2 - s_1} \left( \frac{1}{\theta} + \frac{1}{1-\theta} \right) \| u \|_{B^{s_1}_{p,\infty}} \| u \|_{B^{s_2}_{p,\infty} \cap L^\infty}^{1-\theta}.$$
(2) For $\forall s \in \mathbb{R}$, $\epsilon > 0$ and $1 \leq p \leq \infty$, there exists a constant $C > 0$ such that

$$
\|u\|_{B^s_{p,1}} \leq C \frac{\epsilon + 1}{\epsilon} \|u\|_{B^s_{p,\infty}} \left(1 + \log \frac{\|u\|_{B^{s+\epsilon+1}_{p,\infty}}}{\|u\|_{B^s_{p,\infty}}} \right).
$$

Lemma 2.4. (Commutator estimate) Let $\sigma > 0$, $1 \leq r \leq \infty$ and $1 \leq p \leq p_1 \leq \infty$. Let $v$ be a vector field over $\mathbb{R}$. Then the following estimates hold,

$$
\|(2^{j\sigma}\|v \partial_x, \Delta_j f\|_{L^p})\|_{j \in \mathbb{N}} \|r \leq C(\|v_x\|_{L^\infty} \|f\|_{B^s_{p,r}} + \|f\|_{L^p} \|\nu\|_{B^{s-\frac{1}{p}}_{p,1}}),
$$

where $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. In addition, if $\sigma < 1$, we have

$$
\|(2^{j\sigma}\|v \partial_x, \Delta_j f\|_{L^p})\|_{j \in \mathbb{N}} \|r \leq C(\|v_x\|_{L^\infty} \|f\|_{B^s_{p,r}}).
$$

Lemma 2.5. (Osgood’s Lemma) Let $\rho \geq 0$ be a measurable function, $\gamma > 0$ be a locally integrable function and $\mu$ be a continuous and increasing function. For some $a \geq 0$, if

$$
\rho(t) \leq a + \int_0^t \gamma(s)\mu(\rho(s))ds.
$$

(1) If $a > 0$, then $-\mathcal{M}(\rho(t)) + \mathcal{M}(a) \leq \int_0^t \gamma(s)(s)ds$, where $\mathcal{M}(x) := \int_x^1 \frac{1}{\mu(r)}dr$.

(2) If $a = 0$ and $\mu$ satisfies the condition $\int_0^1 \frac{dr}{\mu(r)}dr = +\infty$, then $\rho \equiv 0$.

Lemma 2.6. Let $v$ be the smooth vector field with bounded first order space derivatives. Let $\psi_t(x)$ be the flow generalized by the vector filed $v(t, x)$, i.e.,

$$
\psi_t(x) = x + \int_0^t v(t, \psi_s(x))dt.
$$

Then for all $t \geq 0$, $\psi_t(\cdot) \in C^1$ is a diffeomorphism over $\mathbb{R}$, and the following estimates hold:

$$
\|D\psi_t^{\pm1}(\cdot)\|_{L^\infty} \leq e^{J_0^t \|Dv(t)\|_{L^\infty} dt},
$$

$$
\|D\psi_t^{\pm1}(\cdot)\|_{L^\infty} \leq e^{2J_0^t \|Dv(t)\|_{L^\infty} dt} - 1,
$$

$$
\|D^2\psi_t^{\pm1}(\cdot)\|_{L^\infty} \leq e^{J_0^t \|Dv(t)\|_{L^\infty} dt} \int_0^t \|D^2v(t)\|_{L^\infty} e^{2J_0^t \|Dv(\tau)\|_{L^\infty} d\tau} d\tau.
$$

Lemma 2.7. (A Priori estimates) Let $1 \leq p, r \leq \infty$, $s \geq -\min\left(\frac{1}{p}, \frac{1}{p_1} - \frac{1}{r}\right)$, and assume that $f_0 \in B^s_{p,r}$ and $g \in L^1([0, T]; B^s_{p,r})$. For any solution $f \in L^\infty([0, T]; B^s_{p,r})$ of the linear transport equation

$$
\begin{align*}
\partial_t f + v \partial_x f &= g, \\
f(x, 0) &= f_0(x),
\end{align*}
$$

(7)

with $v_x \in L^1([0, T]; B_{p,r}^{s-\frac{1}{p}})$ if $s > 1 + \frac{1}{p}$ or $v_x \in L^1([0, T]; B_{p,r}^{\frac{1}{r}} \cap L^\infty)$ otherwise.

(1) If $r = 1$ or $s \neq 1 + \frac{1}{p}$, there exists $C > 0$ depending only on $s, p$ and $r$ such that

$$
\|f\|_{B^s_{p,r}} \leq e^{VC_P(t)}\|f_0\|_{B^s_{p,r}} + \int_0^t e^{VC_P(t) - VC_P(\tau)}\|g(\tau)\|_{B^s_{p,r}} d\tau,
$$

where

(8)

$$
V_P(t) := \begin{cases}
\int_0^t \|v_x(\tau)\|_{B^{s-\frac{1}{p}}_{p,\infty} \cap L^\infty} d\tau, & \text{if } s < 1 + \frac{1}{p}; \\
\int_0^t \|v_x(\tau)\|_{B^{s-\frac{1}{p}}_{p,r}} d\tau, & \text{if } s > 1 + \frac{1}{p} \text{ or } s = 1 + \frac{1}{p}, r = 1.
\end{cases}
$$

(9)
(2) If \( r < \infty \), then \( f \in C([0, T]; B^s_{p,r}). \) If \( r = \infty \), then \( f \in C([0, T]; B^s_{p,1}) \) for all \( s' < s \).

(3) If \( v = f, s > 0 \), the inequality (8) holds true with \( V_p(t) := \int_0^t \| v_x(s) \|_{L^p} \, ds \).

**Lemma 2.8.** (Well-posedness) Let \( p, r, s, f_0 \) and \( g \) be as in Lemma 2.7. If \( v \in L^p([0, T]; B^{-M}\infty) \) for some \( \rho > 1 \), \( M > 0 \) and \( v_x \in L^1([0, T]; B^{s-1}_{p,r}) \) if \( s > 1 + \frac{1}{p} \) or \( s = 1 + \frac{1}{p} \) and \( r = 1 \), and \( v_x \in L^1([0, T]; B^\frac{1}{r}\infty \cap L^\infty) \) if \( s < 1 + \frac{1}{p} \). Then the transport equation (7) admits a unique solution \( u \) in the space \( C([0, T]; B^s_{p,r}) \) if \( r < \infty \) or \( L^\infty([0, T]; B^s_{p,r}) \cap (\cap s < C([0, T]; B^s_{p,1}) \) if \( r < \infty \). Moreover, the inequalities of Lemma 2.7 hold true.

### 3. Well-posedness in Besov space.

This section is devoted to studying the local well-posedness of the system (1) in Besov spaces. To this end, let us first introduce the following notation: For \( \forall T > 0, s \in \mathbb{R} \) and \( 1 \leq p \leq +\infty \), define

\[
E^s_{p,r}(T) := C([0, T]; B^s_{p,r} \times B^{s-1}_{p,r}) \cap C^1([0, T]; B^{s-1}_{p,r} \times B^{s-2}_{p,r}), \quad \text{if } r < \infty,
\]
\[
E^\infty_{p,r}(T) := L^\infty([0, T]; B^s\infty \times B^{s-1}\infty) \cap \text{Lip}([0, T]; B^{s-1}\infty \times B^{s-2}\infty).
\]

Apparently, for all \( f \in L^2(\mathbb{R}), (1 - \partial_x^2)^{-1}f = \ast f \), where \( \ast f \) is the Gere’s function \( \frac{1}{2} e^{-|x|} \), so one can rewrite the system (1) in the following form:

\[
\begin{aligned}
&u_t + (\sigma u - \mu)u_x = -\partial_x (1 - \partial_x^2)^{-1}((\mu - A)u + \frac{3-\sigma}{2}u^2 + \frac{\sigma}{2}u_x^2 + \frac{1-2\sigma \lambda}{2}\rho^2), \\
&-\Omega^\rho u + \Omega (1 - \partial_x^2)^{-1}(\rho^2 u_x), \\
&\rho_t + u\rho_x = -\rho u_x, \\
u(0, x) = u_0(x), \quad \rho(0, x) = \rho_0(x).
\end{aligned}
\]

(10)

We are now ready to state the first main result of the paper.

**Theorem 3.1.** Let \((u_0, \rho_0) \in B^s_{p,r} \times B^{s-1}_{p,r} \) with \( s > \max\{1 + \frac{1}{p}, \frac{3}{2} - \frac{1}{p}, 2 - \frac{1}{p}\}, \) \( p, r \in [1, \infty] \), there exists a time \( T := T(||(u_0, \rho_0)||_{B^s_{p,r} \times B^{s-1}_{p,r}}) > 0 \) such that the system (10) admits a unique solution \((u, \rho) \in E^s_{p,r}(T)\). Moreover, the data-to-solution mapping

\[
(u_0, \rho_0) \mapsto (u, \rho) : B^s_{p,r} \times B^{s-1}_{p,r} \to C([0, T]; B^{s'}_{p,r} \cap C^1([0, T]; B^{s'}_{p,r-1}) \times S'(i = 1, 2)) \]

is continuous for all \( s' < s \) if \( r = \infty \), and \( s' = s \) if \( 1 \leq r < \infty \).

The uniqueness and continuity with respect to the initial data in some sense can be guaranteed by the following priori estimates.

**Lemma 3.2.** Assume that \((u^{(i)}, \rho^{(i)}) \in L^\infty(0, T; B^s_{p,r} \times B^{s-1}_{p,r}) \cap C([0, T]; S' \times S'(i = 1, 2)) \) are two solutions of the system (10) corresponding to the initial datum \((u^{(i)}, \rho^{(i)}) \in B^s_{p,r} \times B^{s-1}_{p,r} \) \(i = 1, 2\), where \( 1 \leq p, r \leq \infty \) and \( s > \max\{\frac{3}{2}, 2 - \frac{1}{p}, 1 + \frac{1}{p}\} \). Denoting \( u^{(12)} = u^{(1)} - u^{(2)} \) and \( \rho^{(12)} = \rho^{(1)} - \rho^{(2)} \), then the following estimates hold:

(1) for \( s > \max\{\frac{3}{2}, 2 - \frac{1}{p}, 1 + \frac{1}{p}\} \) but \( s \neq 2 + \frac{1}{p} \) and \( s \neq 3 + \frac{1}{p} \), we have

\[
\| u^{(12)}(t) \|_{B^{s-1}_{p,r}} + \| \rho^{(12)}(t) \|_{B^{s-1}_{p,r}} \leq C(\| u_0^{(12)} \|_{B^s_{p,r}} + \| \rho_0^{(12)} \|_{B^s_{p,r}}) \mathcal{E}^{s,s-1}(t),
\]

where

\[
\mathcal{E}^{s,s-1}(t) = C \int_0^t \left( \sum_{i=1,2} (\| u^{(i)}(\tau) \|_{B^s_{p,r}} + \| \rho^{(i)}(\tau) \|_{B^s_{p,r}}) \right) d\tau.
\]
Proof. It is easy to verify that the function \( (u^{(12)}, \rho^{(12)}) \in L^\infty(0, T; B_{p,r}^s \times B_{p,r}^{s-1}) \cap C([0,T]; S' \times S') \) solves the following transport system:

\[
\begin{cases}
  u^{(12)}(t) + (\sigma u^{(2)} - \mu) u^{(12)} = F(t,x), \\
  \rho^{(12)}(t) + u^{(2)} \rho^{(12)} = G(t,x), \\
  u^{(12)}(0,x) = u^{(12)}_0 := u^{(1)}_0(x) - u^{(2)}_0(x), \\
  \rho^{(12)}(0,x) = \rho^{(12)}_0 := \rho^{(1)}_0(x) - \rho^{(2)}_0(x).
\end{cases}
\]

(14)

where \( F(t,x) := -\sigma u^{(12)} u^{(1)}_x - \partial_x (1 - \partial_x^2)^{-1} \left( (\mu - A) u^{(12)} + \frac{3-\gamma}{2} (u^{(1)} + u^{(2)}) u^{(12)} + \frac{\sigma}{2} (u^{(1)} + u^{(2)}) u^{(12)} + \frac{1-2\alpha}{2} \rho^{(12)} + \Omega(u^{(1)} + \rho^{(2)}) \rho^{(12)} u^{(12)} - \Omega(u^{(1)} + \rho^{(2)}) \rho^{(12)} u^{(1)} - \Omega(u^{(12)} + \rho^{(2)} \rho^{(12)} - \Omega(u^{(12)} + \rho^{(2)} \rho^{(12)} - \Omega(u^{(12)} + \rho^{(2)}) u^{(12)} + \rho^{(2)}) u^{(12)} + u^{(12)} \rho^{(2)})^2 \Omega(t,x) := -\rho^{(12)} u^{(2)} (\rho^{(1)} u^{(12)} - \rho^{(12)} u^{(12)} x).

By applying the prior estimate for the transport equation (Lemma 2.7) to system (14) with respect to \( u^{(12)} \), if \( s > \max \left\{ \frac{2}{p}, 2 - \frac{1}{p}, 1 + \frac{1}{p} \right\} \) but \( s \neq \frac{2}{p} \) and \( s \neq 3 + \frac{1}{p} \), we obtain

\[
\| u^{(12)}(t) \|_{B_{p,r}^{s-1}} \leq \| u^{(12)}_0 \|_{B_{p,r}^{s-1}} + \int_0^t \| F(s,\cdot) \|_{B_{p,r}^{s-1}} ds
\]

(15)

where \( \| \partial_x (\sigma u^{(2)} - \mu) \|_{B_{p,r}^{s-1}} \leq C \| u^{(2)} \|_{B_{p,r}^s} \) for some positive constant \( C \).

In order to get the estimation for \( \| u^{(12)}(t) \|_{B_{p,r}^{s-1}} \), we are now in a position to estimate the nonlinear term \( \| F(s,\cdot) \|_{B_{p,r}^{s-1}} \). Noting that the operator \( \partial_x (1 - \partial_x^2)^{-1} \) is a \( S^{-1} \) multiplier, for \( \max \left\{ \frac{2}{p}, 2 - \frac{1}{p}, 1 + \frac{1}{p} \right\} < s \leq 2 + \frac{1}{p} \), it follows from Lemma...
2.2 that
\[
\|\partial_x(1 - \partial_x^2)^{-1} \left( \frac{3 - \sigma}{2}(u^{(1)} + u^{(2)})u^{(12)} + \frac{\sigma}{2}(u_x^{(1)} + u_x^{(2)})u_x^{(12)} \right) \|_{B_{p,r}^{-1}}
\leq \|\partial_x(1 - \partial_x^2)^{-1} \left( \frac{3 - \sigma}{2}(u^{(1)} + u^{(2)})u^{(12)} + \frac{\sigma}{2}(u_x^{(1)} + u_x^{(2)})u_x^{(12)} \right) \|_{B_{p,r}^{-1}}
\leq C\|u^{(1)}\|_{B_{p,r}^{-2}} \|u^{(12)}\|_{B_{p,r}^{-1}} + \|u_x^{(1)}\|_{B_{p,r}^{-1}} \|u_x^{(12)}\|_{B_{p,r}^{-1}}
\leq C\|\|u^{(1)}\|_{B_{p,r}^{-1}} + \|u^{(2)}\|_{B_{p,r}^{-1}}\|u^{(12)}\|_{B_{p,r}^{-1}}
\]
(16)
\[
\|\partial_x(1 - \partial_x^2)^{-1}(\rho^{(1)} + \rho^{(2)})\|_{B_{p,r}^{-2}} \leq C(\|\rho^{(1)}\|_{B_{p,r}^{-1}} + \|\rho^{(2)}\|_{B_{p,r}^{-1}})\|\rho^{(12)}\|_{B_{p,r}^{-2}}
\]
(17)

For the cubic nonlinearities, by means of the algebra property in Besov spaces, one can deduce that
\[
\|\partial_x(1 - \partial_x^2)^{-1}\Omega(\rho^{(1)} + \rho^{(2)})\rho^{(12)}u^{(1)} + u^{(12)}(\rho^{(2)})^2\|_{B_{p,r}^{-1}}
\leq C\|\|\rho^{(1)}\|_{B_{p,r}^{-1}} + \|\rho^{(2)}\|_{B_{p,r}^{-1}}\|u^{(1)}\|_{B_{p,r}^{-1}} + \|u^{(12)}\|_{B_{p,r}^{-1}}\|\rho^{(12)}\|_{B_{p,r}^{-2}}
\leq C\|\|\rho^{(1)}\|_{B_{p,r}^{-1}} + \|\rho^{(2)}\|_{B_{p,r}^{-1}}\|u^{(1)}\|_{B_{p,r}^{-1}} + \|u^{(12)}\|_{B_{p,r}^{-1}}\|\rho^{(12)}\|_{B_{p,r}^{-2}}
\]
\times(\|u^{(12)}\|_{B_{p,r}^{-1}} + \|\rho^{(12)}\|_{B_{p,r}^{-2}})
\]
(18)

and
\[
\|\Omega(1 - \partial_x^2)^{-1}(\rho^{(1)} + \rho^{(2)})^2u^{(1)} + u^{(12)}(\rho^{(2)})^2\|_{B_{p,r}^{-1}}
\leq C\|\|\rho^{(1)}\|_{B_{p,r}^{-1}} + \|\rho^{(2)}\|_{B_{p,r}^{-1}}\|u^{(1)}\|_{B_{p,r}^{-1}} + \|u^{(12)}\|_{B_{p,r}^{-1}}\|\rho^{(2)}\|_{B_{p,r}^{-2}}
\leq C\|\|\rho^{(1)}\|_{B_{p,r}^{-1}} + \|\rho^{(2)}\|_{B_{p,r}^{-1}}\|u^{(1)}\|_{B_{p,r}^{-1}} + \|u^{(12)}\|_{B_{p,r}^{-1}}\|\rho^{(2)}\|_{B_{p,r}^{-2}}
\]
\times(\|u^{(12)}\|_{B_{p,r}^{-1}} + \|\rho^{(12)}\|_{B_{p,r}^{-2}})
\]
(19)

Moreover, one observe that
\[
\|\sigma u^{(12)}u_x^{(1)} + \partial_x(1 - \partial_x^2)^{-1}(\mu - A)u^{(12)}\|_{B_{p,r}^{-2}} \leq C\|\|u^{(1)}\|_{B_{p,r}^{-1}} + 1\|u^{(12)}\|_{B_{p,r}^{-2}}.
\]
(20)

By putting (16)-(20) together and using the Cauchy-Schwarz inequality, we get
\[
\|F(s, \cdot)\|_{B_{p,r}^{-2}} \leq C\|\|u^{(1)}\|_{B_{p,r}^{-1}} + \|u^{(2)}\|_{B_{p,r}^{-2}}^2 + \|\rho^{(1)}\|_{B_{p,r}^{-1}}^2 + \|\rho^{(2)}\|_{B_{p,r}^{-2}}^2 + 1\|\|u^{(12)}\|_{B_{p,r}^{-1}} + \|\rho^{(12)}\|_{B_{p,r}^{-2}}\|
\]
(21)

On the other hand, due to the fact that $B_{p,r}^{-2}$ is a Banach algebra for $s > 2 + \frac{1}{p}$, it is easier to derive the similar estimation for $\|F(s, \cdot)\|_{B_{p,r}^{-2}}$ for $s > 2 + \frac{1}{p}$, we omit the details here. Therefore, combining (15) and (21), we obtain
\[
\|u^{(12)}(t)\|_{B_{p,r}^{-1}} \leq \|u_0^{(12)}\|_{B_{p,r}^{-1}} + C\int_0^t\|u^{(2)}\|_{B_{p,r}^{-2}}\|u^{(12)}\|_{B_{p,r}^{-1}}ds
\]
\[
+ C\int_0^t\|\|u^{(1)}\|_{B_{p,r}^{-1}}^2 + \|u^{(2)}\|_{B_{p,r}^{-2}}^2 + \|\rho^{(1)}\|_{B_{p,r}^{-1}}^2 + \|\rho^{(2)}\|_{B_{p,r}^{-2}}^2 + 1\|\|u^{(12)}\|_{B_{p,r}^{-1}} + \|\rho^{(12)}\|_{B_{p,r}^{-2}}\|
\]
\begin{align}
	\times &\left(\|u^{(12)}\|_{B_{p,r}^{s-1}} + \|\rho^{(12)}\|_{B_{p,r}^{s-2}}\right)ds \\
\leq &\ |u_0^{(12)}|_{B_{p,r}^{s-1}} + C \int_0^t (\|u^{(1)}\|_{B_{p,r}^{s-1}} + \|u^{(2)}\|_{B_{p,r}^{s-1}} + \|\rho^{(1)}\|_{B_{p,r}^{s-1}} \\
&\ + \|\rho^{(2)}\|_{B_{p,r}^{s-1}} + 1)^2 (\|u^{(12)}\|_{B_{p,r}^{s-1}} + \|\rho^{(12)}\|_{B_{p,r}^{s-2}})ds.
\end{align}

(22)

Since \( s \neq 3 + \frac{1}{p} \), by applying the Lemma 2.7 to the second equation in (10), we get

\begin{align}
\|\rho^{(12)}(t)\|_{B_{p,r}^{s-2}} \leq &\ |\rho_0^{(12)}|_{B_{p,r}^{s-2}} + \int_0^t |G(s, \cdot)|_{B_{p,r}^{s-2}}ds \\
&\ + C \int_0^t \|u^{(2)}(s)\|_{B_{p,r}^{s-2}} \|\rho^{(12)}(s)\|_{B_{p,r}^{s-2}}ds.
\end{align}

(23)

Let us estimate the term \( |G(t, x)|_{B_{p,r}^{s-2}} \) for \( 1 + \frac{1}{p} < s \leq 2 + \frac{1}{p} \), otherwise one can easily show these inequalities hold true in view of the fact that \( B_{p,r}^{s-2} \) is a Banach algebra as \( s > 2 + \frac{1}{p} \). Indeed, using Lemma 2.2, we have

\begin{align}
\|G(t, \cdot)\|_{B_{p,r}^{s-2}} &= \|\rho^{(12)}(t, \cdot)\|_{B_{p,r}^{s-2}} + (\|\rho^{(12)}(t, \cdot)\|_{B_{p,r}^{s-2}}) \\
&\leq C(\|\rho^{(12)}(t, \cdot)\|_{B_{p,r}^{s-2}} \|u^{(2)}\|_{B_{p,r}^{s-1}} + \|\rho^{(1)}(t, \cdot)\|_{B_{p,r}^{s-2}} \|u^{(12)}\|_{B_{p,r}^{s-1}} + \|\rho^{(2)}(t, \cdot)\|_{B_{p,r}^{s-1}}) \\
&\leq C(\|\rho^{(12)}(t, \cdot)\|_{B_{p,r}^{s-2}} \|u^{(12)}\|_{B_{p,r}^{s-1}} + \|\rho^{(1)}(t, \cdot)\|_{B_{p,r}^{s-2}} \|u^{(12)}\|_{B_{p,r}^{s-1}} + \|\rho^{(2)}(t, \cdot)\|_{B_{p,r}^{s-1}}),
\end{align}

(24)

which combined with (23) yields that

\begin{align}
\|\rho^{(12)}(t)\|_{B_{p,r}^{s-2}} \leq &\ |\rho_0^{(12)}|_{B_{p,r}^{s-2}} + C\int_0^t (\|u^{(2)}(s)\|_{B_{p,r}^{s-2}} + \|\rho^{(1)}(s)\|_{B_{p,r}^{s-1}} + 1)^2 \\
&\times (\|\rho^{(12)}(s)\|_{B_{p,r}^{s-2}} + \|u^{(12)}(s)\|_{B_{p,r}^{s-1}})ds.
\end{align}

(25)

From the inequalities (22) and (25), we deduce that

\begin{align}
\|u^{(12)}(t)\|_{B_{p,r}^{s-1}} + \|\rho^{(12)}(t)\|_{B_{p,r}^{s-2}} \\
\leq &\ |u_0^{(12)}|_{B_{p,r}^{s-1}} + |\rho_0^{(12)}|_{B_{p,r}^{s-2}} + C\int_0^t (\|u^{(1)}\|_{B_{p,r}^{s-1}} + \|u^{(2)}\|_{B_{p,r}^{s-2}} + \|\rho^{(1)}\|_{B_{p,r}^{s-1}} \\
&\ + \|\rho^{(2)}\|_{B_{p,r}^{s-1}} + 1)^2 (\|u^{(12)}\|_{B_{p,r}^{s-1}} + \|\rho^{(12)}\|_{B_{p,r}^{s-2}})ds.
\end{align}

(26)

Then the desired estimation (11) can be obtained by taking advantage of the Gronwall inequality.

We are now in a position to deal with the critical case \( s = 2 + \frac{1}{p} \), which can be done by virtue of the interpolation argument. Indeed, let us choose

\begin{align}
\theta = \frac{1 + \frac{1}{p} - s_2}{s_1 - s_2}, \quad s_1 \in (\max\{\frac{1}{2} + 1 - \frac{1}{p}, 1 + \frac{1}{p}\}, 1 + \frac{1}{p}) \text{ and } s_2 \in (1 + \frac{1}{p}, 2 + \frac{1}{p}).
\end{align}

According to the complex interpolation (see (1) of Lemma 2.3) and the estimation (26), we have

\begin{align}
\|u^{(12)}(t)\|_{B_{p,r}^{s-1}}^{1+\frac{1}{p}} \leq &\ |u^{(12)}(t)|_{B_{p,r}^{s-1}}^{1+\frac{1}{p}} + \|u^{(12)}(t)\|_{B_{p,r}^{s-1}}^{1-\theta} \\
&\times C(\|u_0^{(12)}\|_{B_{p,r}^{s-1}}^{1+\frac{1}{p}} + \|\rho_0^{(12)}\|_{B_{p,r}^{s-1}}^{1+\frac{1}{p}})^{\theta} \\
&\times \left(\|u^{(1)}(t)\|_{B_{p,r}^{s-1}}^{2+\frac{1}{p}} + \|u^{(2)}(t)\|_{B_{p,r}^{s-1}}^{2+\frac{1}{p}}\right)^{1-\theta}e^{\theta\|u^{(12)}(t)\|_{B_{p,r}^{s-1}}^{1+\frac{1}{p}}},
\end{align}

(27)
where the function $\mathcal{N}^{2+\frac{1}{p},1+\frac{1}{p}}(t)$ is the same as in (1). On the other hand, the fact of $s - 2 = \frac{1}{p} \neq 1 + \frac{1}{p}$ implies that the estimation in (1) also holds true, i.e.,

$$\left\| \rho^{(12)}(t) \right\|_{B_{p,r}^{s}} \leq C(\left\| u_{0}^{(12)} \right\|_{B_{p,r}^{s+1}} + \left\| \rho_{0}^{(12)} \right\|_{B_{p,r}^{s}}) \mathcal{N}^{2+\frac{1}{p},1+\frac{1}{p}}(t),$$

which combined with (27) yields the desired inequality.

For the critical case $s = 3 + \frac{1}{p}$, since $s - 1 = 2 + \frac{1}{p} > 1 + \frac{1}{p}$, the estimation for $\left\| u(t) \right\|_{B_{p,r}^{s}}$ in case (1) also holds true. Moreover, by taking the similar argument as that in case (2), one can obtain the estimation for $\left\| \rho(t) \right\|_{B_{p,r}^{s}}$, which leads to the inequality (13). Therefore, the proof of Lemma 3.2 is completed.

Next we focus on the existence of the solution to the system (10) by using the classical iterated method, which is ensured by the following result.

**Lemma 3.3.** Let $p$, $r$ and $s$ be as in the statement of Lemma 3.2. Assume that $(u^{(0)}, \rho^{(0)}) = (0, 0)$. There exists a sequence of smooth functions $(u^{(n)}, \rho^{(n)}) \in C([0, \infty); B_{p,r}^{\infty} \times B_{p,r}^{\infty})$ solving

$$(T^{(n)}): \left\{ \begin{aligned} &\partial_{t}u^{(n+1)} + (\sigma u^{(n)} - \mu)\partial_{x}u^{(n+1)} = F_{1}^{(n)}(t, x) + F_{2}^{(n)}(t, x), \\
&\partial_{t}\rho^{(n+1)} + u^{(n)}\partial_{x}\rho^{(n+1)} = -\rho^{(n)}u^{(n)}x, \\
u^{(n+1)}(0, x) := u_{0}^{(n+1)}(x) = S_{n+1}u_{0}, \\
\rho^{(n+1)}(0, x) := \rho_{0}^{(n+1)}(x) = S_{n+1}\rho_{0}, \\
\end{aligned} \right.$$  

where $F_{1}^{(n)}(t, x) = -\partial_{x}(1 - \partial_{x}^{2})^{-1}\left( (\mu - A)u^{(n)} + \frac{3 - \sigma}{2}(u^{(n)})^{2} + \frac{\sigma}{2}(u_{x}^{(n)})^{2} + \frac{1 - 2\Omega}{2}A(\rho^{(n)})^{2} - \Omega(\rho^{(n)})^{2}u^{(n)} \right)$, $F_{2}^{(n)}(t, x) = \Omega(1 - \partial_{x}^{2})^{-1}((\rho^{(n)})^{2}u_{x}^{(n)})$, and $S_{n+1} = \sum_{q \geq -1}D_{q}$ is the low frequency cut-off operator.

Moreover, there is a positive time $T = T(||(u_{0}, \rho_{0})||_{B_{p,r}^{1}} B_{p,r}^{1}) > 0$ such that the solutions satisfying the following properties:

(1) $(u^{(n)}, \rho^{(n)})_{n \in \mathbb{N}}$ is uniformly bounded in $E_{p,r}^{s}(T)$.

(2) $(u^{(n)}, \rho^{(n)})_{n \in \mathbb{N}}$ is a Cauchy sequence in $C([0, T]; B_{p,r}^{s+1} \times B_{p,r}^{s+1})$.

**Proof.** Due to the fact that the initial data $(S_{n+1}u_{0}, S_{n+1}\rho_{0}) \in B_{p,r}^{\infty} \times B_{p,r}^{\infty}$, it thus follows from the Lemma 2.8 and by induction with respect to the index $n$ that problem $(T^{(n)})$ admits a global solution. It remains to prove (1) and (2).

Indeed, since $\partial_{x}(1 - \partial_{x}^{2})^{-1}$ is a Fourier multiplier of degree $-1$, and $B_{p,r}^{s}$ is a Banach algebra with $s - 1 > \frac{1}{p}$, one can obtain the estimation for the nonlinear term $\|F^{(n)}(s, \cdot)\|_{B_{p,r}^{s}}$ as follows:

$$\|F_{1}^{(n)}(s, \cdot)\|_{B_{p,r}^{s}} \leq \|\mu - A\|_{u^{(n)}} + \frac{3 - \sigma}{2}(u^{(n)})^{2} + \frac{\sigma}{2}(u_{x}^{(n)})^{2} + \frac{1 - 2\Omega}{2}A(\rho^{(n)})^{2} - \Omega(\rho^{(n)})^{2}u^{(n)}\|_{B_{p,r}^{s}},$$

which completes the proof of Lemma 3.3.
For convenience, we denote

$$A_n(t) := \|u^{(n)}(t)\|_{B_{p,r}^s} + \|\rho^{(n)}(t)\|_{B_{p,r}^{s-1}}, \quad A_0 := \|u_0\|_{B_{p,r}^s} + \|\rho_0\|_{B_{p,r}^{s-1}}.$$ 

Noting that \((1-\partial^2_x)^{-1}\in S^{-2}\) and \(B_{p,r}^s \hookrightarrow B_{p,r}^{s-1}\), if \(\max\{\frac{3}{2}, 2-\frac{1}{p}, 1+\frac{1}{p}\} < s \leq 2+\frac{1}{p}\), we have

$$\|F_2^{(n)}(t, \cdot)\|_{B_{p,r}^s} \leq C\|\rho^{(n)}\|^2_{B_{p,r}^{s-1}} \|u_x^{(n)}\|_{B_{p,r}^{s-2}} \leq C\|\rho^{(n)}\|^2_{B_{p,r}^{s-1}} \|u^{(n)}\|_{B_{p,r}^s} \leq CA_n(t)^3.$$  

Otherwise, the \(B_{p,r}^{s-2}\) is a Banach algebra for \(s > 2 + \frac{1}{p}\), in which case is easier to deal with. By applying the a priori estimates for the transport equation to the first equation of \((T^{(n)})\), and using the inequalities (28) and (29), we get

$$\|u^{(n+1)}(t)\|_{B_{p,r}^s} \leq \|S_{n+1}u_0\|_{B_{p,r}^s} + \int_0^t \|F(\tau, \cdot)\|_{B_{p,r}^s} d\tau + C \int_0^t \|u^{(n)}(\tau)\|_{B_{p,r}^s} \|u^{(n+1)}(\tau)\|_{B_{p,r}^s} d\tau \leq C \left(\|u_0\|_{B_{p,r}^s} + \int_0^t (A_n^3 + A_n^2 + A_n) d\tau + \int_0^t A_n(\tau) \|u^{(n+1)}(\tau)\|_{B_{p,r}^s} d\tau\right),$$

where we used the fact that \(\|S_{n+1}u_0\|_{B_{p,r}^s} \leq C\|u_0\|_{B_{p,r}^s}\) for some positive \(C\) independent of \(n\). By taking the Gronwall’s inequality, we deduce

$$\|u^{(n+1)}(t)\|_{B_{p,r}^s} \leq Ce^{C \int_0^t A_n(\tau) d\tau} \left(\|u_0\|_{B_{p,r}^s} + \int_0^t e^{-C \int_0^\tau A_n(\tau') d\tau'} (A_n^3 + A_n^2 + A_n) d\tau\right).$$

By taking the similar argument for the second equation in \((T^{(n)})\), we can also obtain

$$\|\rho^{(n+1)}(t)\|_{B_{p,r}^{s-1}} \leq Ce^{C \int_0^t A_n(\tau) d\tau} \left(\|\rho_0\|_{B_{p,r}^{s-1}} + \int_0^t e^{-C \int_0^\tau A_n(\tau') d\tau'} A_n(\tau) d\tau\right).$$

Putting the estimates (30) and (31) together leads to

$$A_{n+1}(t) \leq Ce^{C \int_0^t A_n(\tau') d\tau'} \left(A_0 + \int_0^t e^{-C \int_0^\tau A_n(\tau') d\tau'} (A_n^3 + A_n^2 + A_n) d\tau\right).$$

If \(A_n(t) < 1\), it follows from (32) that \(A_{n+1}(t) \leq Ce^t(A_0 + t)\), which shows that the sequence \((u^{(n)}, \rho^{(n)})_{n\in\mathbb{N}}\) is uniformly bounded in \(C([0,T]; B_{p,r}^s \times B_{p,r}^{s-1})\). Otherwise, if \(A_n(t) > 1\), we deduce from inequality (32) that

$$A_{n+1}(t) \leq C \left(e^t \int_0^t A_n(\tau') d\tau' A_0 + \int_0^t e^{C \int_0^\tau A_n(\tau') d\tau'} A_n^3(\tau) d\tau\right).$$

To obtain the uniformly boundedness, let us choose \(T > 0\) satisfying \(T < 1/8CA_0^3\), and suppose by induction that

$$A_n(t) \leq \frac{\sqrt{2}A_0}{(1-SCA_0^2t)^{\frac{1}{2}}} \leq \frac{\sqrt{2}A_0}{(1-SCA_0^2T)^{\frac{1}{2}}} = \mathcal{K}, \quad \forall t \in [0, T].$$
After some direct calculation, we get
\[
C \int_t^\tau A_n(\tau') d\tau' \leq \int_t^\tau C A_n(\tau') + C(A_n(\tau'))^2 d\tau' \\
\leq \int_t^\tau \sqrt{2} C A_0 \left( 1 - 8C A_0^2 \tau' \right)^{\frac{1}{2}} d\tau' + \int_t^\tau \frac{2 C A_0^2}{1 - 8 C A_0^2 \tau'} d\tau' \\
= \frac{\sqrt{2}}{4} \frac{C A_0}{(1 - 8 C A_0^2 \tau)} \left( 1 - 8 C A_0^2 \tau - \sqrt{1 - 8 C A_0^2 \tau} \right) + \frac{1}{4} \ln(1 - 8 C A_0^2 \tau) - \frac{1}{4} \ln(1 - 8 C A_0^2 \).
\]
Therefore, using the fact that \( \sqrt{1 - 8 C A_0^2 \tau} < 1 \) for \( \forall \tau \in [0, T] \), we obtain
\[
e^{C \int_t^\tau A_n(\tau') d\tau'} \leq e^{\frac{\sqrt{2}}{4} C A_0} \left( \frac{1 - 8 C A_0^2 \tau}{1 - 8 C A_0^2} \right)^{\frac{1}{2}}.
\]
And then inserting the above inequality and (34) into (33) yields that
\[
A_{n+1}(t) \leq \frac{C A_0 e^{\frac{\sqrt{2}}{4} C A_0}}{(1 - 8 C A_0^2 \tau)^{\frac{1}{2}}} + C \int_0^t e^{\frac{\sqrt{2}}{4} C A_0} \left( \frac{1 - 8 C A_0^2 \tau}{1 - 8 C A_0^2} \right)^{\frac{1}{2}} \frac{2 \sqrt{2} C A_0^3}{(1 - 8 C A_0^2 \tau)^{\frac{3}{2}}} d\tau \\
\leq \frac{C A_0 e^{\frac{\sqrt{2}}{4} C A_0}}{(1 - 8 C A_0^2 \tau)^{\frac{1}{2}}} + \frac{2 \sqrt{2} C A_0^3}{(1 - 8 C A_0^2 \tau)^{\frac{3}{2}}} \int_0^t \frac{1}{(1 - 8 C A_0^2 \tau)^{\frac{3}{2}}} d\tau \\
= \frac{C A_0 e^{\frac{\sqrt{2}}{4} C A_0}}{(1 - 8 C A_0^2 \tau)^{\frac{1}{2}}} + \frac{2 \sqrt{2} C A_0^3}{(1 - 8 C A_0^2 \tau)^{\frac{3}{2}}} \left( \frac{1}{(1 - 8 C A_0^2 \tau)^{\frac{3}{2}}} - 1 \right) \\
\leq \frac{\sqrt{2} C A_0 e^{\frac{\sqrt{2}}{4} C A_0}}{(1 - 8 C A_0^2 \tau)^{\frac{1}{2}}},
\]
where we chosen the constant \( C > 1 \), and it implies that the sequence \( (u^{n}, \rho^{n})_{n \in \mathbb{N}} \) is uniformly bounded in \( C([0, T]; B_{p, r}^{s}) \times C([0, T]; B_{p, r}^{s-2}) \). Using the system \( (T^{(n)}) \) and the similar argument as above, one can prove that \( (\partial_t u^{n}, \partial_t \rho^{n})_{n \in \mathbb{N}} \) is uniformly bounded in \( C([0, T]; B_{p, r}^{s-1}) \times B_{p, r}^{s-2} \). Therefore, \( (u^{n}, \rho^{n})_{n \in \mathbb{N}} \) is uniformly bounded in \( E_{p, r}^{s}(T) \).

Now it suffices to show that \( (u^{n}, \rho^{n})_{n \in \mathbb{N}} \) is a Cauchy sequence in \( C([0, T]; B_{p, r}^{s-1}) \times B_{p, r}^{s-2} \). Indeed, for all \( n, m \in \mathbb{N} \), it follows from \( (T^{(n)}) \) that
\[
\begin{align*}
\partial_t (u^{n+m+1} - u^{m+1}) + (\sigma u^{m} - \mu) \partial_x (u^{n+m+1} - u^{m+1}) \\
= -\sigma u^{(n,m)} u_x^{(n+m+1)} - \partial_x (1 - \partial_x^{(n,m)})^{-1} \left( \frac{3}{2} (u^{n+m} + u^{(n,m)}) u_x^{(n,m)} \\
+ (\mu - A) u^{(n,m)} + \frac{\sigma}{2} (u_x^{(n+m)} + u_x^{(m)}) u_x^{(n,m)} + 2 \Omega A (\rho^{(n,m)} + \rho^{(m)}) u^{(n,m)} \\
- \Omega (\rho^{(n+m)} + \rho^{(m)}) u^{(n+m)} - \Omega u^{(n,m)} (\rho^{(m)})^2 \\
+ \Omega (1 - \partial_x^{(n,m)})^{-1} (\rho^{(n+m)} + \rho^{(m)}) u_x^{(n+m)} + u_x^{(n,m)} (\rho^{(m)})^2 \\
:= F_{1, n}^{m}(t, x),
\end{align*}
\]
and
\[ \partial_t (\rho^{n+m+1} - \rho^{m+1}) + u^{(n+m+1)} \partial_x (\rho^{n+m+1} - \rho^{m+1}) = -\rho^{n,m} \partial_x \rho^{(n+m+1)} - \partial_t \rho^{(n+m+1)} \]
where we denote \( u^{(n,m)} := u^{(n+m)} - u^{(m)} \) and \( \rho^{(n,m)} := \rho^{(n+m)} - \rho^{(m)} \) for convenience.

Similar to the proof of Lemma 3.2, for \( s > \max \left( \frac{3}{2}, 2 - \frac{1}{p}, 1 + \frac{1}{p} \right) \) but \( s \neq 2 + \frac{1}{p}, 3 + \frac{1}{p} \), one can deduce from the Lemma 2.7 that
\[ H^{n,m+1}(t) \leq C e^{C U^{m}(t)} \left( H^{n,m+1}(0) + C \int_0^t e^{-C U^{m}(\tau)} f^{n,m}(\tau) d\tau \right), \]
where \( H^{n,m}(t) := \|u^{(n+m+1)}(t) - u^{(m+1)}(t)\|_{B_{p,r}^{s+1}} + \|\rho^{(n+m+1)}(t) - \rho^{(m+1)}(t)\|_{B_{p,r}^{s+2}}, \)
\( H^{n,m}(0) := \|u_0^{(n+m+1)} - u_0^{(m+1)}\|_{B_{p,r}^{s+1}} + \|u_0^{(n+m+1)} - u_0^{(m+1)}\|_{B_{p,r}^{s+2}}, \)
\( f^{n,m}(t) := \{||u^{(n+m)}(t)||_{B_{p,r}^{s+1}} + ||u^{(m)}(t)||_{B_{p,r}^{s+1}} + ||\rho^{(n+m)}(t)||_{B_{p,r}^{s+2}} + ||\rho^{(m)}(t)||_{B_{p,r}^{s+2}} + 1\}^2, \)
\( U(t) := \int_0^t \|u^{(m)}(\tau)\|_{B_{p,r}^{s+2}} d\tau. \)

However, notice that \( (u^{(n)}, \rho^{(n)})_{n \in \mathbb{N}} \) is uniformly bounded in \( C([0,T]; B_{p,r}^s \times B_{p,r}^{s-1}) \), it thus follows (38) that
\[ H^{n,m+1}(t) \leq C 2^{-m} + C \int_0^t H^{n,m}(\tau) d\tau, \forall t \in [0,T]. \]

Here by the definition of the operator \( S_q \), we have used the following fact:
\[ \|u_0^{(n+m+1)} - u_0^{(m+1)}\|_{B_{p,r}^{s+1}} = \left\| \sum_{q=m+1}^{n+m} \Delta_q u_0 \right\|_{B_{p,r}^{s+1}} \]
\[ = \left\{ \sum_{k=-2}^{n-m} 2^k r^{2-k} \left\| \Delta_k \sum_{q=m+1}^{n+m} \Delta_q u_0 \right\|_{L_p}^{\frac{1}{r}} \right\} \leq C 2^{-m} \|u_0\|_{B_{p,r}^s}. \]

Likewise we also have \( \|\rho_0^{(n+m+1)} - \rho_0^{(m+1)}\|_{B_{p,r}^{s-2}} \leq C 2^{-m} \|\rho_0\|_{B_{p,r}^{s-1}}. \) Hence, we get
\[ H^{n,m+1}(0) \leq C 2^{-m} \|(u_0, \rho_0)\|_{B_{p,r}^s \times B_{p,r}^{s-1}}. \] By induction with respect to the index \( m \) in (39) and then taking the limit as \( m \to +\infty \), we arrive at
\[ H^{n,m+1}(t) \leq C 2^{-m} \sum_{k=0}^{m} \frac{(2TC)^k}{k!} + \frac{(CT)^{m+2}}{2T} \leq \frac{2^{-m} e^{2TC}}{2T} + \frac{(CT)^{m+2}}{2T} \to 0, \]
which implies that \( (u^{(n)}, \rho^{(n)})_{n \in \mathbb{N}} \) is a Cauchy sequence in \( C([0,T]; B_{p,r}^s \times B_{p,r}^{s-2}) \).

By using the interpolation argument as we did in the proof of Lemma 3.2, one can also prove the results for the critical cases \( s = 2 + \frac{1}{p} \) and \( s = 3 + \frac{1}{p} \), we omit the details here. This completes the proof of Lemma 3.3.

Now we give the proof of the Theorem 3.1.

Proof. (Proof of Theorem 3.1) By (2) of Lemma 3.3, there exists a function \((u, \rho)\) such that \((u^{(n)}, \rho^{(n)}) \to (u, \rho)\) in \( C([0,T]; B_{p,r}^s \times B_{p,r}^{s-2}) \) as \( n \to \infty \). By the Fatou's Lemma in Besov spaces, we get \((u, \rho) \in L^\infty([0,T]; B_{p,r}^s \times B_{p,r}^{s-1}) \). Therefore, an interpolation argument ensures that the convergence holds true in \( C([0,T]; B_{p,r}^s \times B_{p,r}^{s-2}) \).
$B_{p,r}^{s'-1}$, for $\forall s'<s$. Taking limit in $(T^{(n)})$ as $n \to \infty$, we obtain that $(u, \rho)$ is a solution to the system (10).

Since $(u, \rho) \in L^\infty([0,T]; B_{p,r}^s \times B_{p,r}^{s'-1})$, the right hand side of the (10) belong to $L^\infty([0,T]; B_{p,r}^s \times B_{p,r}^{s'-1})$. Using the system (10) again, we see that $(\partial_t u, \partial_t \rho) \in C([0,T]; B_{p,r}^{s'-1} \times B_{p,r}^{s'-2})$ for $r < \infty$, and $(\partial_t u, \partial_t \rho) \in L^\infty([0,T]; B_{p,r}^{s'-1} \times B_{p,r}^{s'-2})$ otherwise, hence we get that $(u, \rho) \in E_{p,r}^s(T)$.

Finally, the continuity of the solution mapping can be proved through using a sequence of viscosity approximation of solutions $\{u, \rho\}_{t>0}$ to the system (10), which uniformly converges in $C([0,T]; B_{p,r}^s \times B_{p,r}^{s'-2}) \cap C^1([0,T]; B_{p,r}^{s'-1} \times B_{p,r}^{s'-2})$. For $s'<s$, the continuity of solution with respect to the initial data in $C([0,T]; B_{p,r}^{s'} \times B_{p,r}^{s'-2}) \cap C^1([0,T]; B_{p,r}^{s'-1} \times B_{p,r}^{s'-2})$ can be shown by Lemma 3.3 and a simple interpolation argument. This finishes the proof of Theorem 3.1. 

\[ \square \]

**Remark 3.4.** Noting that $B_{2,2}^s \approx H^s$, one can conclude from Theorem 3.1 that the solution to the system (10) is local well-posed in $C([0,T]; H^s) \cap C^1([0,T]; H^{s-1})(s > \frac{3}{2})$ with some $T > 0$ depends only on the initial data, and the solution mapping is Hölder continuous. Therefore, the Theorem 3.1 covers the corresponding result provided in [22].

4. **The critical case and blow up phenomena.** This section aims at investigating the local well-posedness of the system (10) in the critical Besov space $B_{2,1}^{\frac{3}{2}} \times B_{2,1}^{\frac{1}{2}}$, and it is shown that the data-to-solution mapping is Hölder continuous. Moreover, we give a blow up result with the initial data in the critical Besov space.

4.1. **The well-posedness in the critical space.** The uniqueness and existence of the solution is guaranteed by the following result.

**Theorem 4.1.** Let $(u_0, \rho_0) \in B_{2,1}^{\frac{3}{2}} \times B_{2,1}^{\frac{1}{2}}$, the system (10) admits a unique solution $(u, \rho) \in E_{2,1}^{\frac{3}{2}}(T)$ for some positive $T$ depends only on the initial data, and there exists a constant $C$ such that if $T = 1/16C(||u_0||_{B_{2,1}^{\frac{3}{2}}} + ||\rho_0||_{B_{2,1}^{\frac{1}{2}}})^2$, then
\[
||u||_{L^\infty(0,T; B_{2,1}^{\frac{3}{2}})} + ||\rho||_{L^\infty(0,T; B_{2,1}^{\frac{1}{2}})} \leq 2(||u_0||_{B_{2,1}^{\frac{3}{2}}} + ||\rho_0||_{B_{2,1}^{\frac{1}{2}}}).
\]

Moreover, the data-to-solution mapping $(u_0, \rho_0) \mapsto (u, \rho): B_{2,1}^{\frac{3}{2}} \times B_{2,1}^{\frac{1}{2}} \mapsto B_{2,1}^{\frac{3}{2}} \times B_{2,1}^{\frac{1}{2}}$ is Hölder continuous for any $s \in (\frac{1}{2}, \frac{3}{2})$. Namely, let $(u^{(i)}, \rho^{(i)})$ be solutions to the system (10) with respect to $(u_0^{(i)}, \rho_0^{(i)})$, $i = 1, 2$, then we have
\[
\sup_{t \in [0,T]} ||u^{(1)}(t) - u^{(2)}(t)||_{B_{2,1}^{\frac{3}{2}}} + \sup_{t \in [0,T]} ||\rho^{(1)}(t) - \rho^{(2)}(t)||_{B_{2,1}^{\frac{1}{2}}}
\leq C \left( ||u_0^{(1)} - u_0^{(2)}||_{B_{2,1}^{\frac{3}{2}}} + ||\rho_0^{(1)} - \rho_0^{(2)}||_{B_{2,1}^{\frac{1}{2}}} \right)^{\frac{1}{2}} \exp(-Ct),
\]

for $\forall \theta = \frac{3}{2} - s \in (0,1)$.

**Proof.** The proof will be divided into several steps.

**Step 1.** Consider the system $(T^{(n)})$ in Section 3 and assume that $(u^{(n)}, \rho^{(n)}) \in L^\infty(0,T; B_{2,1}^{\frac{3}{2}} \times B_{2,1}^{\frac{1}{2}})$. Since $B_{2,1}^{\frac{3}{2}} \hookrightarrow B_{2,1}^{\frac{1}{2}}$ and $B_{2,1}^{\frac{1}{2}}$ is a Banach algebra, it is easy to verify that the right hand side of the system $(T^{(n)})$ belongs to $L^\infty_{loc}(\mathbb{R}^+; B_{2,1}^{\frac{3}{2}})$.
and $L^\infty_{loc}(\mathbb{R}^+; B^{\frac{3}{2}}_{2,1})$, respectively. Taking advantage of the Lemma 2.8 ensures that $(T^{(n)})$ has a global solution in $E^{\frac{3}{2}}_{2,1}(T)$ for any given $T > 0$.

**Step 2.** Similar to the proof of Theorem 3.1, one can find $T > 0$ such that $T < 1/8CA_0^2$ and

$$A_n(t) \leq \frac{\sqrt{2}A_0}{(1 - 8CA_0^2T)^\frac{3}{2}} \leq \frac{\sqrt{2}A_0}{(1 - 8CA_0^T)^\frac{3}{2}} := \mathcal{W}, \ \forall t \in [0, T].$$

(42)

where $A_n(t) := \|u^{(n)}(t)\|_{B^{\frac{3}{2}}_{2,1}} + \|\rho^{(n)}(t)\|_{B^{\frac{3}{2}}_{2,1}}$, $A_0 := \|u_0\|_{B^{\frac{3}{2}}_{2,1}} + \|\rho_0\|_{B^{\frac{3}{2}}_{2,1}}$. This shows that the sequence $(u^{(n)}, \rho^{(n)})_{n \in \mathbb{N}}$ is uniformly bounded in $L^\infty(0, T; B^{\frac{3}{2}}_{2,1} \times B^{\frac{3}{2}}_{2,1})$. By using the system $(T^{(n)})$, we can verify that the sequence $(u^{(n)}, \rho^{(n)})_{n \in \mathbb{N}}$ is uniformly bounded in $E^{\frac{3}{2}}_{2,1}(T)$. Especially, the estimate (40) can be easily obtained by taking $T = 1/16CA_0^2$.

**Step 3.** We are going to show that $(u^{(n)}, \rho^{(n)})_{n \in \mathbb{N}}$ is a Cauchy sequence in $C([0, T]; B^{\frac{3}{2}}_{2,\infty} \times B^{-\frac{3}{2}}_{2,\infty})$. Applying the Lemma 2.7 to (36), it follows from the uniformly boundedness of $(u^{(n)}, \rho^{(n)})_{n \in \mathbb{N}}$ that

$$\|u^{n+m+1}(t) - u^{m+1}(t)\|_{B^{\frac{3}{2}}_{2,\infty}} \leq C\left(\|S_{n+m+1}u_0 - S_{m+1}u_0\|_{B^{\frac{3}{2}}_{2,\infty}} + C\int_0^t \|F^{n,m}_1(\tau, \cdot)\|_{B^{\frac{3}{2}}_{2,\infty}} d\tau\right).$$

(43)

Noting that $\partial_x(1-\partial^2)^{-1} \in S^{-1}$ and $(1-\partial^2)^{-1} \in S^{-2}$, by virtue of the embedding $B^{\frac{3}{2}}_{2,1} \hookrightarrow B^{\frac{3}{2}}_{2,\infty} \hookrightarrow B^{\frac{3}{2}}_{2,\infty} \cap L^\infty$, the second term on the right hand side of (43) can be estimated as

$$\|F^{n,m}_1(\tau, \cdot)\|_{B^{\frac{3}{2}}_{2,\infty}} \leq C\left(\|u^{(n,m)}u^{(n+m+1)}\|_{B^{\frac{3}{2}}_{2,1}} + \|u^{(n+m)}u^{(m)}u^{(n,m)}\|_{B^{\frac{3}{2}}_{2,1}} + \|u^{(n,m)}\|_{B^{\frac{3}{2}}_{2,1}} + \|u^{(n,m)}\|_{B^{\frac{3}{2}}_{2,1}} + \|\rho^{(n,m)}\|_{B^{\frac{3}{2}}_{2,\infty}} + \|\rho^{(n,m)}\|_{B^{\frac{3}{2}}_{2,\infty}} + \|\rho^{(m)}\|_{B^{\frac{3}{2}}_{2,\infty}} + \|\rho^{(m)}\|_{B^{\frac{3}{2}}_{2,\infty}} \right).$$

(44)

where $u^{(n,m)} := u^{(n+m)} - u^{(m)}$ and $\rho^{(n,m)} := \rho^{(n+m)} - \rho^{(m)}$ are the same as in (36) and (37). By using the Banach algebra property of $B^{\frac{3}{2}}_{2,1}$ and the embedding $B^{\frac{3}{2}}_{2,1} \hookrightarrow B^{\frac{3}{2}}_{2,\infty} \cap L^\infty$, one can estimate the terms on the right hand side of (44) by

$$\|u^{(n,m)}u^{(n+m+1)}\|_{B^{\frac{3}{2}}_{2,1}} + \|u^{(n+m)}u^{(m)}u^{(n,m)}\|_{B^{\frac{3}{2}}_{2,1}} + \|u^{(n,m)}\|_{B^{\frac{3}{2}}_{2,1}} + \|u^{(n,m)}\|_{B^{\frac{3}{2}}_{2,1}}$$

$$\leq C\left(\|u^{(n,m)}\|_{B^{\frac{3}{2}}_{2,1}} \|u^{(n+m+1)}\|_{B^{\frac{3}{2}}_{2,1}} + \|u^{(n+m)} + u^{(m)}\|_{B^{\frac{3}{2}}_{2,1}} \|u^{(n,m)}\|_{B^{\frac{3}{2}}_{2,1}} + \|u^{(n,m)}\|_{B^{\frac{3}{2}}_{2,1}} \right).$$

(45)

and
since \( \|u^{(n)}(t)\|_{L^\infty(0,T;B^{3/2}_{2,1})} \leq \overline{W} \) is uniformly bounded.

Moreover, by using the Moser-type estimate (see (3) of Lemma 2.2), we have

\[
\begin{align*}
&\|(u_x^{(n+m)} + u_x^{(m)})u_x^{(n,m)}\|_{B^{3/2}_{2,\infty}} + \|(\rho^{(n+m)} + \rho^{(m)})\rho^{(n,m)}\|_{B^{3/2}_{2,\infty}} \\
&\leq C\left(\|u_x^{(n+m)} + u_x^{(m)}\|_{B^{1/2}_{2,1}} \|u^{(n,m)}\|_{B^{3/2}_{2,\infty}} + \|\rho^{(n,m)}\|_{B^{1/2}_{2,1}}\right) \\
&+ \rho^{(m)}\|\rho^{(n,m)}\|_{B^{1/2}_{2,1}} \\
&\leq C\left(\|u^{(n+m)} - u^{(m)}\|_{B^{1/2}_{2,1}} + \|\rho^{(n+m)} - u^{(m)}\|_{B^{3/2}_{2,\infty}}\right), \\
&\|(\rho^{(n+m)} + \rho^{(m)})\rho^{(n,m)}u^{(n+m)}\|_{B^{3/2}_{2,\infty}} + \|u^{(n,m)}(\rho^{(m)})^2\|_{B^{3/2}_{2,\infty}} \\
&\leq \|\rho^{(n+m)} + \rho^{(m)}\|_{B^{1/2}_{2,1}} \|\rho^{(n,m)}\|_{B^{3/2}_{2,\infty}} \|u^{(n,m)}\|_{B^{3/2}_{2,1}} + \|u^{(n,m)}\|_{B^{3/2}_{2,1}} \|\rho^{(m)}\|_{B^{3/2}_{2,1}} \\
&\leq C\left(\|\rho^{(n+m)} - u^{(m)}\|_{B^{1/2}_{2,1}} + \|u^{(n,m)} - u^{(m)}\|_{B^{1/2}_{2,1}}\right), \\
&\|(\rho^{(n+m)} + \rho^{(m)})\rho^{(n,m)}u^{(n,m)}\|_{B^{3/2}_{2,\infty}} + \|u^{(n,m)}(\rho^{(m)})^2\|_{B^{3/2}_{2,\infty}} \\
&\leq \|\rho^{(n+m)} + \rho^{(m)}\|_{B^{1/2}_{2,1}} \|\rho^{(n,m)}\|_{B^{3/2}_{2,\infty}} \|u^{(n,m)}\|_{B^{3/2}_{2,1}} + \|u^{(n,m)}\|_{B^{3/2}_{2,1}} \|\rho^{(m)}\|_{B^{3/2}_{2,1}} \\
&\leq C\left(\|\rho^{(n+m)} - u^{(m)}\|_{B^{1/2}_{2,1}} + \|u^{(n,m)} - u^{(m)}\|_{B^{1/2}_{2,1}}\right).
\end{align*}
\]

It is not difficult to verify that

\[
\begin{align*}
&\|u^{(n+m+1)}_0 - u^{(m+1)}_0\|_{B^{1/2}_{2,\infty}} = \|S^{n+m+1}_0 u_0 - S^{m+1}_0 u_0\|_{B^{1/2}_{2,\infty}} \leq C2^{-m}\|u_0\|_{B^{1/2}_{2,1}}, \\
\end{align*}
\]

and

\[
\|\rho^{(n+m+1)}_0 - \rho^{(m+1)}_0\|_{B^{1/2}_{2,\infty}} = \|S^{n+m+1}_0 \rho_0 - S^{m+1}_0 \rho_0\|_{B^{1/2}_{2,\infty}} \leq C2^{-m}\|\rho_0\|_{B^{1/2}_{2,1}}.
\]

Plugging the above estimates into (44), one can conclude from (43) that

\[
\begin{align*}
\|u^{n,m+1}(t) - u^{m+1}(t)\|_{B^{1/2}_{2,\infty}} &\leq C\left(2^{-m}\|u_0\|_{B^{1/2}_{2,1}} \right. \\
&\left. + \int_0^t \|\rho^{(n+m)} - u^{(m)}\|_{B^{1/2}_{2,\infty}} + \|u^{(n,m)} - u^{(m)}\|_{B^{1/2}_{2,1}} \, d\tau\right). \\
\end{align*}
\]  

(45)

On the other hand, by the same taken for the second equation in (T^{(n)}), one can deduce that

\[
\begin{align*}
\|\rho^{n,m+1}(t) - \rho^{m+1}(t)\|_{B^{1/2}_{2,\infty}} &\leq C\left(2^{-m}\|\rho_0\|_{B^{1/2}_{2,1}} + C\int_0^t \|\rho^{(n+m)} - \rho^{(m)}\|_{B^{1/2}_{2,\infty}} \\
&\left. + \|u^{(n,m)} - u^{(m)}\|_{B^{1/2}_{2,1}} \, d\tau\right). \\
\end{align*}
\]  

(46)

Setting

\[
P^{n,m}(t) := \|u^{n,m}(t) - u^{m}(t)\|_{B^{1/2}_{2,\infty}} + \|\rho^{n,m}(t) - \rho^{m}(t)\|_{B^{1/2}_{2,\infty}}.
\]

It follows from (45) and (46) that

\[
P^{n,m+1}(t) \leq C\left(2^{-m} + C\int_0^t P^{n,m}(t) + \|u^{(n,m)} - u^{(m)}\|_{B^{1/2}_{2,1}} \, d\tau\right). \\
\]  

(47)
However, by means of the Log-type interpolation inequality (see Lemma 2.3), we have

\[
\|u^{n+m} - u^m\|_{B^{\frac{1}{2}}_{2,1}} \leq C\|u^{n+m} - u^m\|_{B^{\frac{1}{2}}_{2,\infty}} \ln \left( e + \frac{\|u^{n+m} - u^m\|_{B^{\frac{1}{2}}_{2,\infty}}}{2W} \right).
\]

Since \( f(x) = x \ln(e + \frac{C}{x})(C > 0) \) is a nondecreasing function, which combined with (47) yields that

\[
P^{n,m+1}(t) \leq C2^{-m} + C\int_0^t P^{n,m}(t) \ln \left( e + \frac{2W}{P^{n,m}(t)} \right) d\tau.
\]

Define \( P^m(t) = \sup_{\tau \in [0,t],n \in \mathbb{N}} P^{n,m}(\tau) \), the previous inequality implies that

\[
P^{n+1}(t) \leq C2^{-m} + C\int_0^t P^m(t) \ln \left( e + \frac{2W}{P(t)} \right) d\tau. \tag{48}
\]

Let \( \overline{P} = \limsup_{m \to \infty} P^m(t) \). By taking the superior limit on both sides of the inequality (48) with respect to \( m \), and using the Fatou’s Lemma and again the monotone property of function \( f(x) = x \ln(e + \frac{C}{x}) \), we obtain

\[
\overline{P}(t) \leq C\int_0^t \limsup_{m \to \infty} P^m(t) \ln \left( e + \frac{2W}{\limsup_{m \to \infty} P(t)} \right) d\tau = C\int_0^t \overline{P}(t) \ln \left( e + \frac{2W}{\overline{P}(t)} \right) d\tau.
\]

An application of the Osgood’s Lemma to the above inequality implies that \( \overline{P}(t) \equiv 0 \), which shows that the sequence \( (u^{(n)}, \rho^{(n)})_{n \in \mathbb{N}} \) is a Cauchy sequence in \( C([0, T]; B_{2,\infty}^{\frac{1}{2}} \times B_{2,\infty}^{-\frac{1}{2}}) \). Hence there exits a function \( (u, \rho) \) such that \((u^{(n)}, \rho^{(n)}) \to (u, \rho) \) in \( C([0, T]; B_{2,\infty}^{\frac{1}{2}} \times B_{2,\infty}^{-\frac{1}{2}}) \) as \( n \to \infty \). Then by taking the similar argument as those in the proof of Theorem 3.1, one can conclude that \((u, \rho) \in E_{2,1}^{3}(T)\) is indeed a solution for the system (10).

**Step 4.** In order to prove the uniqueness of the solution, let \((u^{(i)}, \rho^{(i)})\) be two solutions corresponding to the initial data \((u_0^{(i)}, \rho_0^{(i)})\), \(i = 1, 2\). Consider the system (14) with respect to the difference between two solutions. Using the Lemma 2.7 and following the similar computations as we did in the Step 3, we obtain

\[
\|u^{(12)}(t)\|_{B_{2,\infty}^{\frac{1}{2}}} + \|\rho^{(12)}(t)\|_{B_{2,\infty}^{-\frac{1}{2}}} \leq C \left( \|u_0^{(12)}\|_{B_{2,1}^{\frac{3}{2}}} + \|\rho_0^{(12)}\|_{B_{2,1}^{\frac{1}{2}}} + \int_0^t \|\rho^{(12)}\|_{B_{2,\infty}^{-\frac{1}{2}}} + \|u^{(12)}\|_{B_{2,1}^{\frac{1}{2}}} d\tau \right). \tag{49}
\]
Applying the (2) of Lemma 2.3, we have
\[ \|u^{(12)}(t)\|_{B^{\frac{1}{2}}_{2,1}} \leq \|u^{(12)}(t)\|_{B^{\frac{1}{2}}_{2,\infty}} \ln \left( e + \frac{2W}{\|u^{(12)}(t)\|_{B^{\frac{1}{2}}_{2,\infty}}} \right). \]

Inserting the above inequality into (49), and noting that \( \ln(e + \frac{2}{x}) \leq \ln(e + a)(1 - \ln x) \) with \( x > 0 \) and \( a > 0 \), we get
\[ \frac{\mathcal{H}(t)}{2W} \leq \frac{\mathcal{H}(0)}{2W} + C \int_0^{\tau} \frac{\mathcal{H}(\tau)}{2W} \ln \left(e + \frac{2W}{\mathcal{H}(\tau)}\right) d\tau \]
\[ \leq \frac{\mathcal{H}(0)}{2W} + C \int_0^{\tau} \frac{\mathcal{H}(\tau)}{2W} \left(1 - \ln \frac{\mathcal{H}(\tau)}{2W}\right) d\tau, \]

where \( \mathcal{H}(t) = \|u^{(12)}(t)\|_{B^{\frac{1}{2}}_{2,\infty}} + \|\rho^{(12)}(t)\|_{B^{-\frac{1}{2}}_{2,\infty}} \).

By taking \( c = C\frac{\mathcal{H}(0)}{2W} \), \( \mu(x) = x(1 - \ln x) \) and \( \gamma(t) = C \), it then follows from the Osgood’s Lemma that \( \mathcal{H}(t) \leq \left(C\frac{\mathcal{H}(0)}{2W}\right)\exp(-Ct) \), which is equivalent to
\[ \|u^{(12)}(t)\|_{B^{\frac{1}{2}}_{2,\infty}} + \|\rho^{(12)}(t)\|_{B^{-\frac{1}{2}}_{2,\infty}} \leq C\left(\|u_0^{(12)}\|_{B^{\frac{3}{2}}_{2,1}} + \|\rho_0^{(12)}\|_{B^{-\frac{1}{2}}_{2,1}}\right)\exp(-Ct). \]

By using the interpolation inequality and the uniformly boundedness of the solutions, we have
\[ \|u^{(12)}(t)\|_{B^{\frac{1}{2}}_{2,1}} + \|\rho^{(12)}(t)\|_{B^{-\frac{1}{2}}_{2,1}} \leq C\|u^{(12)}(t)\|_{B^{\frac{1}{2}}_{2,\infty}}^{1-\theta} \|u^{(12)}(t)\|_{B^{\frac{1}{2}}_{2,\infty}}^\theta + \|\rho^{(12)}(t)\|_{B^{-\frac{1}{2}}_{2,\infty}}^{1-\theta} \|\rho^{(12)}(t)\|_{B^{-\frac{1}{2}}_{2,\infty}}^\theta \]
\[ \leq C\left(\|u_0^{(12)}\|_{B^{\frac{3}{2}}_{2,1}} + \|\rho_0^{(12)}\|_{B^{-\frac{1}{2}}_{2,1}}\right)^\theta \exp(-Ct), \]

for any \( \theta = \frac{3}{2} - s \in (0, 1) \), which implies that the data-to-solution mapping is H"older continuous from \( B^{\frac{3}{2}}_{2,1} \times B^{-\frac{1}{2}}_{2,1} \) into \( B^{s}_{2,1} \times B^{s\,-1}_{2,1} \) for all \( s \in (\frac{1}{2}, \frac{3}{2}) \). This completes the proof of the Theorem 4.1.

4.2. The blow-up result. Thanks to the conservation law for the system (10) and the transport equation theory, we can deduce the following blow-up phenomena.

**Theorem 4.2.** Assume that \((u_0, \rho_0) \in B^{\frac{3}{2}}_{2,1} \times B^{-\frac{1}{2}}_{2,1} \) and \( 1 - 2\Omega A > 0 \). Let \( T^* \) be the maximum existence time of the solution \((u, \rho)\) to the system (10). Then there exists a positive constant \( C \) such that
\[ T^* \geq \frac{C}{\left(\|u_0\|_{B^{3/2}_{2,1}} + \|\rho_0\|_{B^{-1/2}_{2,1}}\right)^2}. \]

Moreover, the solution will blow up if \( T^* \) is finite, namely,
\[ T^* < \infty \implies \int_0^{T^*} \|\partial_x u(t)\|_{L^\infty}^2 dt = +\infty. \]
Proof. The first part is a direct conclusion of Step 2 in the proof of Theorem 4.1, and it remains to prove the blow-up result. To this end, by applying the Littlewood-Paley decomposition operator $\Delta_j$ to the system (10), we get

$$\partial_t (\Delta_j u) + (\sigma u - \mu) \partial_x (\Delta_j u) = \left[ (\sigma u - \mu) \partial_x, \Delta_j \right] u + \Delta_j F(t, x),$$

$$\partial_t (\Delta_j \rho) + u \partial_x (\Delta_j \rho) = [u \partial_x, \Delta_j] \rho - \Delta_j (\rho u_x),$$

with the initial data $\Delta_j u|_{t=0} = \Delta_j u_0$, $\Delta_j \rho|_{t=0} = \Delta_j \rho_0$, where $F(t, x) = -\partial_x (1 - \partial_x^2)^{-1} \left( (\mu - A) u + \frac{3 - \sigma}{2} u^2 + \frac{\sigma}{2} u_x^2 + \frac{1 - 2\Omega A}{2} \rho^2 - \Omega \rho^2 u \right) + \Omega (1 - \partial_x^2)^{-1} (\rho^2 u_x)$.

Multiplying both sides of the (52) by $\Delta_j \mu$, and then integrating by parts on $\mathbb{R}$ with respect to $x$, it follows from the Cauchy-Schwarz inequality that

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} |\Delta_j u|^2 \, dx = \frac{\sigma}{2} \int_{\mathbb{R}} u_x |\Delta_j u|^2 \, dx + \int_{\mathbb{R}} \Delta_j u [(\sigma u - \mu) \partial_x, \Delta_j] u \, dx + \int_{\mathbb{R}} \Delta_j u \Delta_j F(t, x) \, dx \leq \frac{\sigma}{2} \|u_x\|_{L^\infty} \|\Delta_j u\|_{L^2}^2 + \|\Delta_j u\|_{L^2} \|(\sigma u - \mu) \partial_x, \Delta_j\|_{L^2} + \|\Delta_j F(t, \cdot)\|_{L^2}.$$  

It thus transpires that

$$\frac{d}{dt} \|\Delta_j u\|_{L^2} \leq \frac{\sigma}{2} \|u_x\|_{L^\infty} \|\Delta_j u\|_{L^2} + \|(\sigma u - \mu) \partial_x, \Delta_j\|_{L^2} \|\Delta_j F(t, \cdot)\|_{L^2},$$

Multiplying both sides of (54) by $2^{\frac{j}{2}}$ and taking $L^1$-norm with respect to $j$, we deduce that

$$\frac{d}{dt} \|u(t)\|_{B^\frac{3}{2}} \leq \frac{\sigma}{2} \|u_x\|_{L^\infty} \|u(t)\|_{B^\frac{3}{2}} + 2^{\frac{j}{2}} \|(\sigma u - \mu) \partial_x, \Delta_j\|_{L^2} \|\Delta_j F(t, \cdot)\|_{L^2}.$$  

By virtue of the commutator estimate (see Lemma 2.4), we have

$$\|2^{\frac{j}{2}} [(\sigma u - \mu) \partial_x, \Delta_j] u\|_{B^\frac{1}{2}} \leq C \|u_x\|_{L^\infty} \|u\|_{B^\frac{3}{2}} + \|u_x\|_{L^\infty} \|u\|_{B^\frac{1}{2}} \leq C \|u_x\|_{L^\infty} \|u\|_{B^\frac{3}{2}}.$$  

Moreover, since $\partial_x (1 - \partial_x^2)^{-1} \in S^{-1}$ and $(1 - \partial_x^2)^{-1} \in S^{-2}$, we deduce that

$$\|\Delta_j F(t, \cdot)\|_{B^\frac{3}{2}} \leq C \|\mu - A\| u + 3 - \sigma \|u^2\|_{B^\frac{1}{2}} + \|u_x^2\|_{B^\frac{1}{2}} + 2 \Omega A \|\rho^2 u\|_{B^\frac{1}{2}} + 2 \Omega \|\rho^2 u_x\|_{B^\frac{1}{2}} + \|\rho\|_{L^\infty} \|\rho\|_{B^\frac{1}{2}} + \|\rho\|_{L^\infty} \|\rho\|_{B^\frac{1}{2}} \leq C \left( \|u\|_{B^\frac{3}{2}} + \|u^2\|_{B^\frac{1}{2}} + \|u_x^2\|_{B^\frac{1}{2}} + \|\rho\|_{L^\infty} \|\rho\|_{B^\frac{1}{2}} + \|\rho\|_{L^\infty} \|\rho\|_{B^\frac{1}{2}} \right).$$

where we used the fact that

$$\|\rho^2 u_x\|_{B^\frac{1}{2}} \leq \|\rho^2 u\|_{B^\frac{1}{2}} \leq \|\rho\|_{L^\infty} \|\rho\|_{B^\frac{1}{2}} \leq \|\rho\|_{L^\infty} \|\rho\|_{B^\frac{1}{2}} + \|\rho\|_{L^\infty} \|\rho\|_{B^\frac{1}{2}}.$$
Plugging (56) and (57) into (55), we obtain
\[
\frac{d}{dt} \|u(t)\|_{B^2_{2,1}} \leq C \left( 1 + \|u\|_{L^\infty}^2 + \|u_x\|_{L^\infty}^2 + \|\rho\|_{L^\infty}^2 \right) \left( \|u(t)\|_{B^2_{2,1}} + \|\rho(t)\|_{B^2_{2,1}} \right).
\]

By taking the similar argument to (53), we can get
\[
\frac{d}{dt} \|\rho(t)\|_{B^\frac{1}{2}_{2,1}} \leq C(\|u_x\|_{L^\infty} + \|\rho\|_{L^\infty}) \left( \|u(t)\|_{B^2_{2,1}} + \|\rho(t)\|_{B^2_{2,1}} \right).
\]

Therefore, we obtain
\[
\frac{d}{dt} \left( \|u(t)\|_{B^2_{2,1}} + \|\rho(t)\|_{B^\frac{1}{2}_{2,1}} \right) \leq C \left( 1 + \|u\|_{L^\infty}^2 + \|u_x\|_{L^\infty}^2 + \|\rho\|_{L^\infty}^2 \right) \left( \|u(t)\|_{B^2_{2,1}} + \|\rho(t)\|_{B^2_{2,1}} \right),
\]
which combined with the Gronwall inequality yields that, for \(\forall t \in [0, T^*)\),
\[
\|u(t)\|_{B^2_{2,1}} + \|\rho(t)\|_{B^\frac{1}{2}_{2,1}} \leq \left( \|u_0\|_{B^2_{2,1}} + \|\rho_0\|_{B^\frac{1}{2}_{2,1}} \right) e^{\int_0^t (1 + \|u(s)\|_{L^\infty}^2 + \|u_x(s)\|_{L^\infty}^2 + \|\rho(s)\|_{L^\infty}^2)ds}. \tag{58}
\]

Thanks to the conservation law associated to the system (10) \(E(u, \rho) = \int_\mathbb{R} u^2 + u_x^2 + (1 - 2\Omega A)(\rho - 1)^2 dx \) [22]. Using the Sobolev embedding theorem and the assumption \(1 - 2\Omega A > 0\), it follows that
\[
\|u\|_{L^\infty} \leq C \|u\|_{H^1} \leq CE(u, \rho)^\frac{1}{2} = CE(u_0, \rho_0)^\frac{1}{2} < \infty.
\]

Therefore, we deduce from (58) that, for \(\forall t \in [0, T^*)\),
\[
\|u(t)\|_{B^2_{2,1}} + \|\rho(t)\|_{B^\frac{1}{2}_{2,1}} \leq \left( \|u_0\|_{B^2_{2,1}} + \|\rho_0\|_{B^\frac{1}{2}_{2,1}} \right) e^{\int_0^t (1 + \|u_x(s)\|_{L^\infty}^2 + \|\rho(s)\|_{L^\infty}^2)ds}. \tag{59}
\]

Moreover, by using the embedding \(C([0, T^*], B^2_{2,1}) \hookrightarrow C([0, T^*], C^{0,1})\), an application of the Lemma 2.6 to the equation \(\rho_x + \rho u_x = -\rho_x u\) ensures that the flow \(\psi_t(x) = \psi(t, x)\) is smooth enough, and the corresponding solution can be expressed by
\[
\rho(t, x) = \rho_0(\psi_t^{-1}(x)) + \int_0^t (u_x(\tau, \psi_\tau(x)))(\psi_\tau^{-1}(x))d\tau.
\]

Taking \(L^\infty\)-norm to the both sides of the above equality and using the Gronwall inequality, we get
\[
\|\rho(t)\|_{L^\infty} \leq \|\rho_0\|_{L^\infty} e^{C \int_0^t \|u_x(\tau)\|_{L^\infty}d\tau} \leq C \|\rho_0\|_{B^\frac{1}{2}_{2,1}} e^{C \int_0^t \|u_x(\tau)\|_{L^\infty}d\tau}, \tag{60}
\]
for \(\forall t \in [0, T^*)\).

Assume that the maximum existence time \(T^*\) is finite and \(\int_0^{T^*} \|u_x(\tau)\|_{L^\infty}d\tau < \infty\), it then follows from (59) and (60) that
\[
\lim_{t \to T^*} \sup (\|u(t)\|_{B^2_{2,1}} + \|\rho(t)\|_{B^\frac{1}{2}_{2,1}}) < \infty,
\]
which indicates that we are able to extend the solution \((u, \rho)\) beyond \(T^*\), and this contradicts to the fact that \(T^*\) is the maximum existence time. Hence the proof of Theorem 4.2 is completed.

The Sobolev embedding theorem and Theorem 4.2 imply the following result.
**Corollary 4.3.** Let \((u_0, \rho_0) \in B^{\frac{3}{2}}_{2,1} \times B^{\frac{3}{2}}_{2,1}\), \(1 - 2\Omega A > 0\) and \(T^*\) be the maximum existence time of the solution \((u, \rho)\) to the system (10). Then the solution will blow up in finite time if and only if 

\[
\limsup_{t \to T^*} \|\partial_x u(t)\|_{L^\infty} = +\infty.
\]

5. **Gevrey regularity and analyticity.** In this section, we show that the system (10) is locally well-posed in the Gevrey-Sobolev spaces in the sense of Hardamard. Especially, the system (10) admits unique analytic solutions locally in time and globally in space. Moreover, we obtain a lower bound of the lifespan.

To begin with, we recall the classical definition of the Gevrey class.

**Definition 5.1.** A function \(g \in C^\infty(\mathbb{R}^n)\) is said to be in the Gevrey class \(G^\tau(\mathbb{R}^n)\) for some \(\tau > 0\), if there exist positive constants \(C\) and \(R\) such that

\[
|D^\alpha g(x)| \leq C \frac{(k!)^\tau}{R^k}, \quad |\alpha| = k, \quad \alpha \in \mathbb{N}^n, \quad x \in \mathbb{T}^n,
\]

where \(R\) represents the radius of Gevrey-class regularity of the function \(f\).

**Remark 5.2.** The class \(G^1(\mathbb{R}^n)\) is equal to the space of real-analytic functions on \(\mathbb{R}^n\). If \(0 < \tau < 1\), the \(f\) is called an ultra-analytic function. Moreover, the functions in \(G^\tau(\mathbb{R}^n)\) with \(\tau > 1\) is a smooth but not analytic function.

The following definition, which was introduced by Foias and Temam [23] to study the analyticity of solution to the Navier-Stokes equation, is an equivalent characterization of the Gevrey-class.

**Definition 5.3.** [23] For all \(\tau \geq 1\) and \(s \geq 0\), the Gevrey-class \(G^\tau(\mathbb{R}^n)\) is given by

\[
\bigcup_{\delta > 0} D((-\Delta)^s e^{\delta(-\Delta)^{\frac{1}{2}}}),
\]

where

\[
\|(-\Delta)^s e^{\delta(-\Delta)^{\frac{1}{2}}} f\|_{L^2}^2 = (2\pi)^n \sum_{k \in \mathbb{Z}^n} |k|^{2s} e^{2\delta |k|^{1/2}} |\hat{f}(k)|^2,
\]

where \(\hat{f}\) is the Fourier coefficient of \(f\) in \(\mathbb{T}^n\).

Combining the above two definitions, we now introduce the framework of the Gevrey class spaces \(G^\tau_{s,\delta}(\mathbb{R}^n)\), which is the working space in the present paper.

**Definition 5.4.** Given \(\tau, \delta > 0\) and \(s \in \mathbb{R}\), the Sobolev-Gevrey spaces \(G^\delta_{\tau,\delta}(\mathbb{R}^n)\) is defined by

\[
G^\delta_{\tau,\delta}(\mathbb{R}^n) = \{f \in C^\infty(\mathbb{R}^n); \quad \|f\|_{G^\delta_{\tau,\delta}} = \left(\int_{\mathbb{R}^n} (1 + |\xi|^2)^s e^{2\delta |\xi|^{1/2}} |\hat{f}(\xi)|^2 d\xi \right)^{1/2} < \infty\},
\]

**Remark 5.5.** In the period case \(\mathbb{T}^n\), the Gevrey-Sobolev norm can be stated as

\[
\|f\|_{G^\delta_{\tau,\delta}(\mathbb{T}^n)} = \left(\sum_{k \in \mathbb{Z}^n} (1 + |k|^2)^s e^{2\delta |k|^{1/\tau}} |\hat{f}(k)|^2 \right)^{1/2}.
\]

In the following paper, we mainly focus on the solutions in the Gevrey-Sobolev spaces defined on \(\mathbb{R}^n\), the similar results still hold true for the period case, and we omit the details here.

The following two lemmas on the property of \(G^\delta_{\tau,\delta}(\mathbb{R})\) can be found in [37].

**Lemma 5.6.** Let \(s\) be a real number and \(\tau > 0\). Assume that \(0 < \delta' < \delta\). Then

\[
\|\partial_x f\|_{G^\delta_{\tau,\delta}(\mathbb{R})} \leq \frac{e^{-\delta' \tau}}{\delta - \delta'} \|f\|_{G^\delta_{\tau,\delta}(\mathbb{R})},
\]
Moreover, for \( s \in \mathbb{R}, \tau, \delta > 0 \) and \( f \in G^\delta_{\tau,s-1} \), we have
\[
\| (1 - \partial_x^2)^{-1} f \|_{G^\delta_{\tau,s}(\mathbb{R})} = \| f \|_{G^\delta_{\tau,s-2}(\mathbb{R})} \leq \| f \|_{G^\delta_{\tau,s}(\mathbb{R})},
\]
\[
\| \partial_x (1 - \partial_x^2)^{-1} f \|_{G^\delta_{\tau,s}(\mathbb{R})} \leq \frac{1}{2} \| f \|_{G^\delta_{\tau,s}(\mathbb{R})},
\]
\[
\| \partial_x (1 - \partial_x^2)^{-1} f \|_{G^\delta_{\tau,s}(\mathbb{R})} \leq \| f \|_{G^\delta_{\tau,s-1}(\mathbb{R})},
\]
\[
\| \partial_x^k f \|_{G^\delta_{\tau,s}(\mathbb{R})} \leq \| f \|_{G^\delta_{\tau,s}(\mathbb{R})}, \quad \forall k \in \mathbb{N}^+.
\]

Proof. We just need to prove (61), and the other results can be found in [37]. Indeed, by the Fourier transform implies that \( \hat{\partial_x^k f} = (i\xi)^k \hat{f} \), it follows that
\[
\| \partial_x^k f \|_{G^\delta_{\tau,s-k}(\mathbb{R})}^2 = \int_{\mathbb{R}} (1 + |\xi|^2)^{s-k} e^{2\delta |\xi|^2} |\xi|^k |\hat{f}(\xi)| d\xi
\]
\[
\leq \frac{1}{2^k} \int_{\mathbb{R}} (1 + |\xi|^2)^{s-k} e^{2\delta |\xi|^2} (1 + |\xi|^2)^k |\hat{f}(\xi)| d\xi
\]
\[
\leq \frac{1}{2^k} \| f \|^2_{G^\delta_{\tau,s}(\mathbb{R})} \leq \| f \|^2_{G^\delta_{\tau,s}(\mathbb{R})}.
\]

This finish the proof of Lemma Theorem 5.6. \( \square \)

**Lemma 5.7.** (1) Let \( s > \frac{1}{2}, \tau \geq 1 \) and \( \delta > 0 \). The space \( G^\delta_{\tau,s} \) is a Banach algebra, and there exist constants \( C_s, C'_s > 0 \) such that for \( \forall u, v \in G^\delta_{\tau,s} \),
\[
\| uv \|_{G^\delta_{\tau,s}} \leq C_s \| u \|_{G^\delta_{\tau,s}} \| v \|_{G^\delta_{\tau,s}} \quad \text{and} \quad \| uv \|_{G^\delta_{\tau,s-1}} \leq C'_s \| u \|_{G^\delta_{\tau,s-1}} \| v \|_{G^\delta_{\tau,s}}.
\]

(2) Let \( 0 < \delta' < \delta, 0 < \tau' < \tau \) and \( s' < s \), then we have
\[
G^\delta_{\tau,s} \hookrightarrow G^{\delta'}_{\tau',s'}, \quad G^\delta_{\tau,s} \hookrightarrow G^{\delta'}_{\tau',s}, \quad G^\delta_{\tau,s} \hookrightarrow G^{\delta'}_{\tau,s'}.\]

Let \( \{ X_\delta \}_{0 < \delta < 1} \) be a scale of decreasing Banach spaces with the norm denoted by \( \| \cdot \|_{X_\delta} \), we consider the following Cauchy problem
\[
(IP) \quad \frac{du}{dt} = F(t, u(t)), \quad u(0) = u_0,
\]
where the function \( F : t \mapsto F(t, u(t)) \) satisfying the following conditions: (i) If \( 0 < \delta' < \delta < 1 \) and \( u(t) \) is holomorphic in \( |t| < T \) and continuous on \( |t| \leq T \) with values in \( X_\delta \) and \( \sup_{|t| < T} \| u(t) \|_{X_\delta} < R \), then the mapping \( t \mapsto F(t, u(t)) \) is holomorphic function \( |t| < T \) with values in \( X_{\delta'} \). Moreover, there exists \( M > 0 \) only depending on \( u_0 \) and \( R \) such that
\[
\sup_{|t| < T} \| F(t, u_0) \|_{X_\delta} \leq \frac{M}{(1 - \delta^2)}, \quad \forall \delta \in (0, 1).
\]

(ii) For \( 0 < \delta' < \delta < 1 \) and any \( u, v \in X_\delta \) with \( \| u - u_0 \|_{X_\delta} < R, \| v - v_0 \|_{X_\delta} < R \), there exists \( K > 0 \) only depending on \( u_0 \) and \( R \) such that
\[
\sup_{|t| < T} \| F(t, u) - F(t, v) \|_{X_{\delta'}} \leq \frac{K}{(\delta - \delta')} \| u - v \|_{X_\delta}.
\]

The abstract Cauchy-Kovalevsky theorem was studied by many authors, for example [38, 39, 41, 42, 43]. Very recently, Luo and Yin proved a generalized Ovsyannikov theorem (see Theorem 5.8), which contains the classical Cauchy-Kovalevsky theorem as a special case.
Theorem 5.8. (Ovsyannikov’s theorem) Let \( \{X_\delta\}_{0<\delta \leq 1} \) and the function \( F(\cdot) \) satisfy the conditions (i) and (ii) stated as above, then there exists a time \( T = \min\{\frac{1}{R(K_{R0})}, \frac{(2^{2s-1})^R}{(2^{2s-1})^R + K_{R0}}\} \), such that the problem (IP) admits a unique function \( u(t) \) in \( X_\delta \) for every \( \delta \in (0, 1) \), and it is a holomorphic in \( |t| < \frac{T(1-\delta)^\tau}{2^{2s-1}} \).

For our needs, let us first rewrite the system (10) in the form of (IP), i.e.,

\[
\frac{d}{dt}\begin{pmatrix} u \\ \rho \end{pmatrix} = \begin{pmatrix} H_1(u, \rho) \\ H_2(u, \rho) \end{pmatrix} \cdot \begin{pmatrix} u(0) \\ \rho(0) \end{pmatrix} = \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix},
\]

(62)

where the autonomous terms \( H_i(u, \rho)(i = 1, 2) \) are given by

\[
H_1(u, \rho) = -\left(\sigma u - \mu\right)u_x - \partial_x(1 - \partial^2_x)^{-1}\left(\mu - A\right)u + \frac{3 - \sigma}{2}u^2 + \frac{\sigma}{2}u^2_x
+ \frac{1 - 2\Omega A}{2}\rho^2 - \Omega\rho^2u_x
+ \Omega(1 - \partial^2_x)^{-1}(\rho^2u_x),
\]

\[
H_2(u, \rho) = -\mu \rho_x - \rho u_x.
\]

For the sake of discussion, we denote \( \overrightarrow{u} = (u, \rho)^t, \overrightarrow{H}(u, \rho) = (H_1(u, \rho), H_2(u, \rho))^t \), where notation \( ^t \) means the transposition of a vector. In the sequel, for any \( \tau \geq 1 \) and \( s > \frac{3}{2} \), we shall choose the space \( X_\delta = G^s_{\tau, s} \times G^s_{\tau, s-1} \) to investigate the Gevrey regularity of the solutions to system (10), which is equipped with the norm

\[
\|\overrightarrow{u}\|_{X_\delta} = \|(u, \rho)^t\|_{X_\delta} = \|u\|_{G^s_{\tau, s}} + \|\rho\|_{G^s_{\tau, s-1}}, \forall u \in G^s_{\tau, s}, \rho \in G^s_{\tau, s-1}.
\]

(63)

By using the embedding property of the Gevrey-Sobolev spaces (see Lemma 5.6), it is easy to see that the \( \{X_\delta\}_{0<\delta \leq 1} \) is a decreasing scale of Banach spaces, namely, for any \( \delta' < \delta \) we have

\[
X_\delta \subset X_{\delta'}, \| \cdot \|_{X_{\delta'}} \leq \| \cdot \|_{X_\delta}.
\]

The following two lemmas are crucial in proving the main result.

Lemma 5.9. Let \( \tau \geq 1, s > \frac{3}{2} \) and the initial data \( \overrightarrow{u_0} = (u_0, \rho_0)^t \in X_\delta \). Then there exists a positive constant \( M \), which depends only on the initial data \( \overrightarrow{u_0} \), such that

\[
\|H(\overrightarrow{u_0})\|_{X_\delta} \leq \frac{M}{(1 - \delta)^{\tau}}, \forall \delta \in (0, 1).
\]

Proof. To obtain the result, let us first prove that under the conditions of the Lemma 5.9, there exists a constant \( M > 0 \) such that

\[
\|H(\overrightarrow{u})\|_{X_{\delta'}} \leq \frac{M}{(\delta - \delta')^{\tau}} \|\overrightarrow{u}\|_{X_\delta}, \text{ for any } 0 < \delta' < \delta \leq 1.
\]

(64)

Let \( C_s > 0 \) be the same constant as that given in (1) of Lemma 5.7, and using the fact that \( G^s_{\tau, s-1} \) is a Banach algebra for \( s > \frac{3}{2} \), then we can estimate that

\[
\left\|\partial_x(1 - \partial^2_x)^{-1}\left(\mu - A\right)u + \frac{3 - \sigma}{2}u^2 + \frac{\sigma}{2}u^2_x + \frac{1 - 2\Omega A}{2}\rho^2 - \Omega\rho^2u_x + \Omega(1 - \partial^2_x)^{-1}(\rho^2u_x)\right\|_{G^s_{\tau, s}}
\]

\[
\leq \left\|\left(\mu - A\right)u + \frac{3 - \sigma}{2}u^2 + \frac{\sigma}{2}u^2_x + \frac{1 - 2\Omega A}{2}\rho^2 - \Omega\rho^2u_x\right\|_{G^s_{\tau, s-1}}
\]

\[
\leq |\mu - A|\|u\|_{G^s_{\tau, s}} + \frac{3 - \sigma|C_s|}{2}\|u\|_{G^s_{\tau, s}}^2 + \frac{\sigma C_s}{2}\|u_x\|_{G^s_{\tau, s-1}}^2.
\]
\begin{align}
+ \Omega C_s^2 \| \rho \|_{G_{\tau,s}^\prime}^2 \| u \|_{G_{\tau,s}^\prime} + \frac{1 - 2 \Omega A | C_s |}{2} \| \rho \|_{G_{\tau,s}^\prime}^2 \\
\leq | \mu - A | \| u \|_{G_{\tau,s}^\prime} + \frac{3 - \sigma | C_s |}{2} \| u \|_{G_{\tau,s}^\prime} + \frac{\sigma e^{-\tau} C_s}{2 (\delta - \delta')^r} \| u \|_{G_{\tau,s}^\prime}^2 \\
\frac{2 \Omega C_s^2}{3} \| \rho \|_{G_{\tau,s}^\prime}^3 + \frac{\Omega C_s^2}{3} \| u \|_{G_{\tau,s}^\prime}^3 + \frac{1 - 2 \Omega A | C_s |}{2} \| \rho \|_{G_{\tau,s}^\prime}^2 \\
\leq \frac{2 \Omega C_s^2}{3} \| \overline{u} \|_{X_\delta}^3 + \left( \frac{3 - \sigma | C_s |}{2} + \frac{\sigma e^{-\tau} C_s}{2 (\delta - \delta')^r} + \frac{1 - 2 \Omega A | C_s |}{2} \right) \| \overline{u} \|_{X_\delta}^3 \\
+ | \mu - A | \| \overline{u} \|_{X_\delta}.
\end{align}

Similarly, we have
\begin{align}
\| (\sigma u - \mu) u_x + \Omega (1 - \partial_x^2)^{-1} (\rho^2 u_x) \|_{G_{\tau,s}^\prime} \leq \frac{\sigma e^{-\tau} C_s}{2 (\delta - \delta')^r} \| u \|_{G_{\tau,s}^\prime}^2 \\
\frac{\Omega e^{-\tau} C_s^2}{(\delta - \delta')^r} \| u \|_{G_{\tau,s}^\prime}^3 + \frac{\Omega e^{-\tau} C_s^2}{(\delta - \delta')^r} (\| \rho \|_{G_{\tau,s}^\prime}^3 + \| u \|_{G_{\tau,s}^\prime}^3) \\
\leq \frac{\Omega e^{-\tau} C_s^2}{(\delta - \delta')^r} \| \overline{u} \|_{X_\delta}^3 + \frac{\sigma e^{-\tau} C_s}{2 (\delta - \delta')^r} \| \overline{u} \|_{X_\delta}^2 + \frac{1 - 2 \Omega A | C_s |}{2} \| \overline{u} \|_{X_\delta}.
\end{align}

Therefore, combining (65) and (66), we get the estimate for $H_1(\overline{u})$:
\begin{align}
\| H_1(\overline{u}) \|_{G_{\tau,s}^\prime} \\
\leq \left( \frac{2 \Omega}{3} C_s^2 + \frac{\Omega e^{-\tau} C_s^2}{(\delta - \delta')^r} \right) \| \overline{u} \|_{X_\delta}^3 + \left( | mu - A | + \frac{\mu e^{-\tau}}{(\delta - \delta')^r} \right) \| \overline{u} \|_{X_\delta} \\
+ \left( \frac{3 - \sigma | C_s |}{2} + \frac{e^{-\tau} \sigma C_s}{2 (\delta - \delta')^r} + \frac{1 - 2 \Omega A | C_s |}{2} \right) \| \overline{u} \|_{X_\delta}^2.
\end{align}

Next, by using Lemma 5.6, the functional $H_2(\overline{u})$ can also be estimated as follows:
\begin{align}
\| H_2(\overline{u}) \|_{G_{\tau,s}^\prime} \\
\leq \| u \rho_x \|_{G_{\tau,s}^\prime} + \| \rho u_x \|_{G_{\tau,s}^\prime} \\
\leq C_s \| u \|_{G_{\tau,s}^\prime} \| \rho_x \|_{G_{\tau,s}^\prime} + C_s \| \rho \|_{G_{\tau,s}^\prime} \| u_x \|_{G_{\tau,s}^\prime} \\
\leq \frac{e^{-\tau} C_s}{2 (\delta - \delta')^r} (\| u \|_{G_{\tau,s}^\prime}^2 + \| \rho \|_{G_{\tau,s}^\prime}^2) \leq \frac{e^{-\tau} C_s}{2 (\delta - \delta')^r} \| \overline{u} \|_{X_\delta}^2.
\end{align}

It thus follows from (67)-(68) that
\begin{align}
\| H(\overline{u}) \|_{X_\delta} = \| H_1(\overline{u}) \|_{G_{\tau,s}^\prime} + \| H_2(\overline{u}) \|_{G_{\tau,s}^\prime} \\
\leq \left( \frac{2 \Omega}{3} C_s^2 + \frac{\Omega e^{-\tau} C_s^2}{(\delta - \delta')^r} \right) \| \overline{u} \|_{X_\delta}^3 + \left( | mu - A | + \frac{\mu e^{-\tau}}{(\delta - \delta')^r} \right) \| \overline{u} \|_{X_\delta} \\
+ \left( \frac{3 - \sigma | C_s |}{2} + \frac{e^{-\tau} (1 + \sigma) C_s}{(\delta - \delta')^r} + \frac{1 - 2 \Omega A | C_s |}{2} \right) \| \overline{u} \|_{X_\delta}^2 \\
\leq \frac{1}{(\delta - \delta')^r} (M_1 \| \overline{u} \|_{X_\delta}^3 + M_2 \| \overline{u} \|_{X_\delta}^2 + M_3 \| \overline{u} \|_{X_\delta}),
\end{align}

where the positive constants $M_i$ are given by $M_1 = \frac{2 \Omega}{3} C_s^2 + \Omega e^{-\tau} C_s^2$, $M_2 = \frac{1}{2} (3 - \sigma | C_s | + 2 e^{-\tau} (1 + \sigma) C_s + | 1 - 2 \Omega A | C_s |)$ and $M_3 = | \mu - A | + | \mu e^{-\tau} |$.

By taking the similar argument, it is not difficult to obtain that, for any $\delta \in (0, 1)$,
\begin{align}
\| H(\overline{u}_0) \|_{X_\delta} \leq \frac{1}{(1 - \delta)^r} (M_1 \| \overline{u}_0 \|_{X_\delta}^3 + M_2 \| \overline{u}_0 \|_{X_\delta}^2 + M_3 \| \overline{u}_0 \|_{X_\delta}),
\end{align}
where $M_i$ are defined in (69). Thus the proof of Lemma 5.9 is finished. \[\square\]

**Lemma 5.10.** Let $\tau \geq 1$, $s > \frac{3}{2}$ and the initial data $\vec{u}_0 \in X_1$. Assume that

$$
\|\vec{u}_1 - \vec{u}_0\|_{X_s} \leq R, \quad \|\vec{u}_2 - \vec{u}_0\|_{X_s} \leq R, \quad 0 < \delta \leq 1
$$

with $\vec{u}_i = (u_i, \rho_i)^t \in X_s$, $i = 1, 2$ and $R > 0$. Then for any $0 < \delta' < \delta \leq 1$, there exists a positive constant $K$, which depends only on $R$ and $\vec{u}_0$, such that

$$
\|H(\vec{u}_1) - H(\vec{u}_2)\|_{X_s'} \leq \frac{K}{(\delta - \delta')^2} \|\vec{u}_1 - \vec{u}_2\|_{X_s}.
$$

**Proof.** Taking advantage of the Lemma 5.6 and the fact that the space $G_{r,s-1}^2$ is a Banach algebra for $s > \frac{3}{2}$, we deduce that

$$
\begin{align*}
\|H(\vec{u}_1) - H(\vec{u}_2)\|_{G_{r,s'}^r} & \leq \|(\sigma u_1 - \mu)\partial_x u_1 - (\sigma u_2 - \mu)\partial_x u_2\|_{G_{r,s'}^r} \\
& + \left\|\frac{1}{2} \frac{\rho_1^2 - \rho_2^2}{2} - \Omega (\rho_1^2 u_1 - \rho_2^2 u_2)\right\|_{G_{r,s'}^r} \\
& + \Omega \|\rho_1^2 \partial_x u_1 - \rho_2^2 \partial_x u_2\|_{G_{r,s'}^r} := A_1 + A_2 + A_3 + A_4.
\end{align*}
$$

(70)

To finish the proof, we are now in a position to estimate the right hand side in the above inequality.

Estimation for $A_1$.

$$
A_1 \leq \frac{\sigma}{2} \|\partial_x (u_1^2 - u_2^2)\|_{G_{r,s'}^r} + |\mu| \|\partial_x (u_1 - u_2)\|_{G_{r,s'}^r} \\
\leq \frac{e^{-\tau \tau^r}}{(\delta - \delta')^\tau} \left( \frac{\sigma}{2} \|u_1^2 - u_2^2\|_{G_{r,s'}^r} + |\mu| \|u_1 - u_2\|_{G_{r,s'}^r} \right) \\
\leq \frac{e^{-\tau \tau^r}}{(\delta - \delta')^\tau} \left( \sigma C_{\tau,\sigma} \|u_0\|_{G_{r,s'}^r} + R \right) + |\mu| \|u_1 - u_2\|_{G_{r,s'}^r}.
$$

(71)

Estimation for $A_2$.

$$
A_2 \leq \left\|\frac{\mu - A}{2} (u_1^2 - u_2^2) + \frac{3 - \sigma}{2} (u_1^2 - u_2^2) + \frac{\sigma}{2} \left( (\partial_x u_1)^2 - (\partial_x u_2)^2 \right) \right\|_{G_{r,s'}^r} \\
\leq |\mu - A| \|u_1 - u_2\|_{G_{r,s'}^r} + \frac{3 - \sigma}{2} |C_{\tau,\sigma}| \|u_1 + u_2\|_{G_{r,s'}^r} \|u_1 - u_2\|_{G_{r,s'}^r} \\
+ \frac{\sigma C_{\tau,\sigma}}{2} \|\partial_x u_1 + \partial_x u_2\|_{G_{r,s'}^r} \|\partial_x u_1 - \partial_x u_2\|_{G_{r,s'}^r} \\
\leq |\mu - A| \|u_1 - u_2\|_{G_{r,s'}^r} + \frac{3 - \sigma}{2} |C_{\tau,\sigma}| \left( \|u_0\|_{G_{r,s'}^r} + R \right) \|u_1 - u_2\|_{G_{r,s'}^r} \\
+ \frac{e^{-\tau \tau^r \sigma C_{\tau,\sigma}}}{(\delta - \delta')^\tau} \left( \|u_0\|_{G_{r,s'}^r} + R \right) \|u_1 - u_2\|_{G_{r,s'}^r}.
$$

(72)
Estimation for $A_3$.

\[
A_3 \leq \frac{1 - 2\Omega A}{2} ||\delta^2_1 - \delta^2_2||_{G^\prime_{r,s-1}} + \Omega(||\delta^1_1\ ||_{G^\prime_{r,s-1}} + \rho_1^2 - \rho_2^2||_{G^\prime_{r,s-1}})
\]

\[
\leq ||1 - 2\Omega A||C_s||\rho_0||_{G^\prime_{r,s-1}} + R||\rho_1 - \rho_2||_{G^\prime_{r,s-1}} + \Omega C^2_s||\rho_1 - \rho_2||_{G^\prime_{r,s-1}}
\times ||\rho_1 - \rho_2||_{G^\prime_{r,s-1}} + R\||u_1||_{G^\prime_{r,s-1}} + u_2||_{G^\prime_{r,s-1}}
\]

\[
\leq \left( ||1 - 2\Omega A||C_s + 2\Omega C^2_s(||u_0||_{G^\prime_{r,s-1}} + R) \right)(||\rho_0||_{G^\prime_{r,s-1}} + R)||\rho_1 - \rho_2||_{G^\prime_{r,s-1}}
\]

\[
+ \Omega C^2_s(||\rho_0||_{G^\prime_{r,s-1}} + R)^2||u_1||_{G^\prime_{r,s-1}} + u_2||_{G^\prime_{r,s-1}}.
\]

(73)

Estimation for $A_4$.

\[
A_4 \leq \Omega(||\delta^2_1 - \delta^2_2||_{G^\prime_{r,s-2}} + \Omega(||\delta^1_1\ ||_{G^\prime_{r,s-2}} + \rho_1^2 - \rho_2^2||_{G^\prime_{r,s-2}})
\]

\[
\leq 2\Omega C^2_s(||\rho_0||_{G^\prime_{r,s-1}} + R)||\rho_1 - \rho_2||_{G^\prime_{r,s-1}} + \Omega C^2_s||\rho_0||_{G^\prime_{r,s-1}} + R\||u_1||_{G^\prime_{r,s-1}} + \rho_2||_{G^\prime_{r,s-1}}
\]

\[
\leq \left( ||1 - 2\Omega A||C_s + 2\Omega C^2_s(||u_0||_{G^\prime_{r,s-1}} + R) \right)(||\rho_0||_{G^\prime_{r,s-1}} + R)||\rho_1 - \rho_2||_{G^\prime_{r,s-1}}
\]

\[
+ \frac{\Omega C^2_s(||\rho_0||_{G^\prime_{r,s-1}} + R)^2||u_1||_{G^\prime_{r,s-1}} + u_2||_{G^\prime_{r,s-1}}.}
\]

(74)

Combining the estimation (71)-(74), we get

\[
\frac{1}{(\delta - \delta')^\tau} \left( e^{-\tau^\tau} ||\mu + 2\sigma C_s|| + |\mu - A| + |3 - \sigma| C_s ||u_0||_{G^\prime_{r,s}} + R \right)
\]

\[
+ \Omega C^2_s(e^{-\tau^\tau} + 1)(||\rho_0||_{G^\prime_{r,s-1}} + R^2)||u_1||_{G^\prime_{r,s}}
\]

\[
+ \left( ||1 - 2\Omega A||C_s ||\rho_0||_{G^\prime_{r,s-1}} + R \right)(||\rho_0||_{G^\prime_{r,s-1}} + R)||\rho_1 - \rho_2||_{G^\prime_{r,s-1}}
\]

\[
\leq \frac{K_1}{(\delta - \delta')^\tau} ||u_1 - u_2||_{G^\prime_{r,s-1}} + \frac{K_2}{(\delta - \delta')^\tau} ||\rho_1 - \rho_2||_{G^\prime_{r,s-1}}.
\]

(75)

where

\[
K_1 = e^{-\tau^\tau} ||\mu + 2\sigma C_s|| + |\mu - A| + |3 - \sigma| C_s ||u_0||_{G^\prime_{r,s}} + R
\]

\[
+ \Omega C^2_s(e^{-\tau^\tau} + 1)(||\rho_0||_{G^\prime_{r,s-1}} + R^2),
\]

\[
K_2 = ||1 - 2\Omega A||C_s ||\rho_0||_{G^\prime_{r,s}} + R + 4\Omega C^2_s(||\rho_0||_{G^\prime_{r,s-1}} + R^2).
\]

Similarly, for the term $H_2(u)$, we have

\[
\frac{1}{(\delta - \delta')^\tau} \left( e^{-\tau^\tau} ||\rho_1 - \rho_2||_{G^\prime_{r,s-1}} + ||\rho_1 - \rho_2||_{G^\prime_{r,s-1}} \right)
\]

\[
\leq \frac{e^{-\tau^\tau} C^2_s}{(\delta - \delta')^\tau} (||u_1 - u_2||_{G^\prime_{r,s-1}} + ||u_2||_{G^\prime_{r,s-1}} ||\rho_1 - \rho_2||_{G^\prime_{r,s-1}})
\]

\[
\leq \frac{K_3}{(\delta - \delta')^\tau} ||\rho_1 - \rho_2||_{G^\prime_{r,s-1}}
\]

(76)
where \( K_3 = 2e^{-\tau^2}C_s(\|u_0\|_{X_1} + R) \). It follows that from (75) and (76) that
\[
\|H(u_1^t) - H(u_2^t)\|_{X_{\delta'}} = \|H_1(u_1^t) - H_1(u_2^t)\|_{G^\prime_{\tau,\varepsilon}} + \|H_2(u_1^t) - H_2(u_2^t)\|_{G^\prime_{\tau,\varepsilon-1}} \leq \frac{1}{(\delta - \delta')^2}(K_1 + K_2 + K_3)\|u_1^t - u_2^t\|_{X_{\delta}},
\]
where the constants \( K_i \) (\( i = 1, 2, 3 \)) only depend on the radius \( R \) and the initial data \( u_0^\tau \). This completes the proof of Lemma 5.10 by taking \( K = K_1 + K_2 + K_3 \).

Based on the above two lemmas, we are now able to give the first main result.

**Theorem 5.11.** Let \( \tau \geq 1 \), \( s > \frac{3}{2} \) and the initial data \( u_0^\tau \in X_1 \). For all \( \delta \in (0, 1) \), there exists a time \( T > 0 \) such that the system (10) admits a unique solution \( \vec{u}^\tau \) which is holomorphic in \( X_\delta \) for \( |t| < \frac{T(1-\delta)^\tau}{2^{\tau-1}} \). Moreover, the lifespan \( T \) is given by
\[
T = \frac{1}{2^{2\tau+4}K} = O\left(\frac{1}{\|u_0\|_{X_1}^3}\right),
\]
where \( K = e^{-\tau^2}(\|\mu\| + 2\sigma C_s) + |\mu - A| + 4\Omega C_s^2(e^{-\tau^2} + 5)\|u_0\|_{X_1}^2 + 2C_s(1 - 2\Omega A \geq |3 - \sigma| + 2\tau)\|u_0\|_{X_1} \), and the \( C_s \) is provided in Lemma 5.7.

**Proof.** By the definition of \( H(\vec{u}(t)) \), if the mapping \( t \mapsto \vec{u}(t) \) is holomorphic, so is the mapping \( t \mapsto H(\vec{u}(t)) \). Moreover, by Lemma 5.9 and Lemma 5.10, one concludes that the conditions (i) and (ii) are satisfied. It thus follows from the Ovsyannikov theorem that there exists \( T \in (0, T) \) such that the Cauchy problem (62) admits a unique function \( \vec{u} = (u, \rho)^t \in X_\delta \). Moreover, for every \( \delta \in (0, 1) \), it is a holomorphic in \( |t| < \frac{T(1-\delta)^\tau}{2^{\tau-1}} \), and the time \( T > 0 \) is given by
\[
\bar{T} = \min \left\{ \frac{1}{2^{2\tau+4}K}, \frac{(2^\tau - 1)R}{(2^\tau - 1)2^{2\tau+3}KR + M} \right\},
\]
where the constants \( K, M \) are defined as that in Lemma 5.9 and Lemma 5.10. Especially, by taking \( R = \|u_0\|_{X_\delta} \), it follows that
\[
K = e^{-\tau^2}(\|\mu\| + 2\sigma C_s) + |\mu - A| + 4\Omega C_s^2(e^{-\tau^2} + 5)\|u_0\|_{X_1}^2 + 2C_s(1 - 2\Omega A \geq |3 - \sigma| + 2\tau)\|u_0\|_{X_1}.
\]

On the other hand, from the definition of \( M \), it is easy to verify that \( M \leq 2^{2\tau+3}K\|u_0\|_{X_1} \), and then we obtain
\[
\frac{(2^\tau - 1)R}{(2^\tau - 1)2^{2\tau+3}KR + M} \geq \frac{2^{\tau} - 1}{2^{2\tau+3}K} \geq \frac{1}{2^{2\tau+4}K},
\]
which implies that \( \bar{T} = \frac{1}{2^{2\tau+4}K} \). It is easy to see that \( \bar{T} = O(1/\|u_0\|_{X_1}^3) \). Therefore, the proof of Theorem 5.11 is completed.

To prove the well-posedness of the system in the sense of Hadamard, following the idea in [37], we shall first introduce a new Banach space \( E_a \) with \( a > 0 \).

**Definition 5.12.** Let \( \tau \geq 1 \) and \( a > 0 \). We denote by \( E_a \) the Banach space of all functions \( \vec{u}(t) \) such that for every \( 0 < \delta < 1 \) and \( |t| < \frac{(1-\delta)^\tau}{a(1-\delta)^\tau} \), the functions \( \vec{u}(t) \) are holomorphic and continuous functions of \( t \) with values in \( X_\delta \). The space \( E_a \) is equipped with the norm
\[
\|\vec{u}\|_{E_a} := \sup_{|t| < \frac{(1-\delta)^\tau}{a(1-\delta)^\tau}, 0 < \delta < 1} \left\{ \|\vec{u}(t)\|_{X_\delta}(1-\delta)^\tau \left(1 - \frac{|t|}{a(1-\delta)^\tau}\right)^\frac{3}{2} \right\},
\]
where the norm in space $X_\delta$ is defined by (63).

**Lemma 5.13.** [37] Let $\tau \geq 1$. For every $a > 0$, $\vec{u} \in E_a$, $0 < \delta < 1$ and $0 \leq t < a(1-\delta)^\tau$, we have

$$\int_0^t \|u(\eta)\|_{X_{a(\eta)}} d\eta \leq \frac{a 2^{2\tau+3}}{(1-\delta)^\tau} \left( \frac{a}{a(1-\delta)^\tau} \right)^\tau \left[ (1-\delta)^\tau - \frac{t}{a} \right] \leq \|\vec{u}\|_{X_\delta} \left( \frac{a}{a(1-\delta)^\tau} \right)^\tau \left[ (1-\delta)^\tau - \frac{t}{a} \right].$$

where

$$\delta_a(\eta) = \frac{1}{2} (1+\delta) + \frac{1}{2} \left( \frac{1}{2} \right) \left( 2\delta - \frac{t}{a} \right) \leq \delta_a(\eta) = \frac{1}{2} (1+\delta) + \frac{1}{2} \left( \frac{1}{2} \right) \left( 2\delta - \frac{t}{a} \right) \leq \delta_a(\eta) = \frac{1}{2} (1+\delta) + \frac{1}{2} \left( \frac{1}{2} \right) \left( 2\delta - \frac{t}{a} \right).$$

Inspired by [37], we give the definition of what means the data-to-solution mapping is continuous.

**Definition 5.14.** Let $\sigma \geq 1$ and $s > \frac{3}{2}$. We say that the solution mapping for the system (10) is continuous, if for any $\vec{u}_0^\infty = (u_0^\infty, \rho_0^\infty) \in X_1$, there exists $T > 0$ such that for any $\vec{u}_0^n = (u_0^n, \rho_0^n) \in X_1$ satisfying $\|\vec{u}_0^n - \vec{u}_0^\infty\|_{X_1} \to 0$, the corresponding solutions $\vec{u}^n$ satisfying $\|\vec{u}^n - \vec{u}^\infty\|_{E_T} \to 0$, $n \to \infty$, where the norm $E_T$ is defined in (5.17) with $a = T$.

**Theorem 5.15.** Let $\tau \geq 1$ and $s > \frac{3}{2}$. Given initial data $\vec{u}_0^n \in X_1$, there exits a time $T$ such that the data-to-solution mapping $\vec{u}_0^n \mapsto \vec{u}^n : X_1 \mapsto E_T$ is continuous.

**Proof.** The existence and uniqueness of the solution has been proved in Theorem 5.11, and it follows that the lifespan for the solutions with data $\vec{u}_0^n$ and $\vec{u}_0^\infty$ are given by $T^n = \frac{1}{2^\tau+1} \tau$ and $T^\infty = \frac{1}{2^\tau+1} \tau$ respectively, where

$$K^n = e^{-\tau \tau}(|\mu| + 2\sigma C_s) + |\mu - A| + 4\Omega C_s^2(e^{-\tau \tau} + 5) ||\vec{u}_0^n||_{X_1},$$

and

$$K^\infty = e^{-\tau \tau}(|\mu| + 2\sigma C_s) + |\mu - A| + 4\Omega C_s^2(e^{-\tau \tau} + 5) ||\vec{u}_0^\infty||_{X_1}.$$ 

Since $\vec{u}_0^n \to \vec{u}_0^\infty$ in $X_1$ as $n \to \infty$, there must be an integer $N > 0$ such that $\|\vec{u}_0^n\|_{X_1} \leq \|\vec{u}_0^\infty\|_{X_1}$, $\forall n > N$. (78)

For each $n > N$, there is a constant $\vec{T} < \min\{\vec{T}^\infty, \vec{T}^n\}$ such that, for any $|t| < \frac{\vec{T}(1-\delta)^\tau}{2^\tau-1}$, the solutions to the Cauchy problem (62) satisfying

$$\vec{u}^n(t) = \vec{u}_0^n + \int_0^t \vec{H}(\vec{u}^n(s)) ds, \text{ and } \vec{u}^\infty(t) = \vec{u}_0^\infty + \int_0^t \vec{H}(\vec{u}^\infty(s)) ds. \quad (79)$$

where the functional $\vec{H}(\vec{u}(t))$ is the same as (72). For instance, one can take $T = e^{-\tau \tau}(|\mu| + 2\sigma C_s) + |\mu - A| + 4\Omega C_s^2(e^{-\tau \tau} + 5)(||\vec{u}_0^\infty||_{X_1} + 1)^2 + 2C_s(|1 - 2\Omega A| + |3 - \sigma| + 2e^{-\tau \tau})(||\vec{u}_0^\infty||_{X_1} + 1)$.

It follows from (79) that for each integer $n > N$ and $|t| < \frac{\vec{T}(1-\delta)^\tau}{2^\tau-1}$, we have

$$\|\vec{u}^n(t) - \vec{u}^\infty(t)\|_{X_\delta} \leq \|\vec{u}_0^n - \vec{u}_0^\infty\|_{X_\delta} + \int_0^t \|\vec{H}(\vec{u}^n(s)) - \vec{H}(\vec{u}^\infty(s))\|_{X_\delta} ds. \quad (80)$$
By applying the Lemma 5.13, for $0 \leq t < \frac{T(1-\delta)^\tau}{2\tau-1}$, we have $\delta \leq \delta_T(\eta) < 1$. Taking the similar argument in the Theorem 5.11, one can obtain

$$\| \widetilde{H}(u^n) - \widetilde{H}(u^\infty) \|_{X_s} \leq \frac{K}{(\delta_T(\eta) - \delta)^\tau} \| u^n - u^\infty \|_{X_{s,T}(\eta)}, \quad (81)$$

where $K$ is a constant similar to that in the Lemma 5.10, i.e., $K = e^{-\tau(\mu + 2\sigma C_s + |\mu - A| + 4\Omega C_s^2(e^{-\tau(\sigma + 5)}) \| u_0^\infty \|_{X_1}^2 + 2C_s(1-2\Omega A + |3-\sigma| + 2e^{-\tau(\sigma)}) \| u_1^\infty \|_{X_1}$.

On the other hand, by virtue of the Lemma 5.13 with $a = \widetilde{T}$, we have $\delta_T(\eta) \in (\delta, 1)$ and

$$\int_0^t \| \| \| u^n(\eta) - u^\infty(\eta) \|_{X_{s,T}(\eta)} \|_{X_s} \leq \frac{K\widetilde{T}2^{2\tau+3} \| u^n(t) - u^\infty(t) \|_{E_{s,T}} (\frac{\widetilde{T}(1-\delta)^\tau}{\widetilde{T}(1-\delta)^\tau - t})^{\frac{1}{2}}. \quad (82)$$

Then it follows from (80)-(82) that

$$\| \| u^n(t) - u^\infty(t) \|_{X_s} \leq \frac{K\widetilde{T}2^{2\tau+3} \| u^n(t) - u^\infty(t) \|_{E_{s,T}} (\frac{\widetilde{T}(1-\delta)^\tau}{\widetilde{T}(1-\delta)^\tau - t})^{\frac{1}{2}}. \quad (83)$$

Simple calculation yields that $K\widetilde{T}2^{2\tau+3} < \frac{1}{2}$, we get from the above inequality that

$$(1-\delta)^\tau \left( \frac{\widetilde{T}(1-\delta)^\tau - t}{\widetilde{T}(1-\delta)^\tau} \right)^{\frac{1}{2}} || u^n(t) - u^\infty(t) ||_{X_s} \leq \frac{1}{2} || u^n(t) - u^\infty(t) ||_{E_{s,T}} + (1-\delta)^\tau \left( \frac{\widetilde{T}(1-\delta)^\tau - t}{\widetilde{T}(1-\delta)^\tau} \right)^{\frac{1}{2}} || u_0^n - u_0^\infty ||_{X_1}.$$

By taking the supremum both sides over $0 < \delta < 1$, $0 < t < \frac{T(1-\delta)^\tau}{2\tau-1}$, we deduce from the definition of the space $E_{s,T}$ that

$$\| u^n(t) - u^\infty(t) \|_{E_{s,T}} \leq \frac{1}{2} \| u^n(t) - u^\infty(t) \|_{E_{s,T}} + (1-\delta)^\tau \| u_0^n - u_0^\infty \|_{X_1}, \quad \forall n > N.$$

Hence, we have

$$\| u^n(t) - u^\infty(t) \|_{E_{s,T}} \leq 2(1-\delta)^\tau \| u_0^n - u_0^\infty \|_{X_1} \to 0, \quad \text{as} \quad n \to \infty.$$

Therefore, the proof of Theorem 5.15 is completed. \qed

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**REFERENCES**

[1] H. Bahouri, J. Y. Chemin and R. Danchin, *Fourier Analysis and Nonlinear Partial Differential Equations*, Springer, Heidelberg, 2011.  
[2] A. Bressan and A. Constantin, *Global conservative solutions of the Camassa-Holm equation*, *Arch. Ration. Mech. Anal.*, **183** (2007), 215–239.  
[3] A. Bressan and A. Constantin, *Global dissipative solutions of the Camassa-Holm equation*, *Anal. Appl.*, **5** (2007), 1–27.  
[4] R. Camassa and D. Holm, *An integrable shallow water equation with peaked solitons*, *Phys. Rev. Lett.*, **71** (1993), 1661–1664.
[5] M. Chen, S. Liu and Y. Zhang, A 2-component generalization of the Camassa-Holm equation and its solutions, Lett. Math. Phys., 75 (2006), 1–15.

[6] Y. Chen, H. Gao and Y. Liu, On the cauchy problem for the two-component Dullin-Gottwald-Holm system, Discrete Contin. Dyn. Syst., 33 (2013), 3407–3441.

[7] A. Constantin, The Hamiltonian structure of the Camassa-Holm equation, Expo. Math., 15 (1997), 53–85.

[8] A. Constantin, Global existence of solutions and breaking waves for a shallow water equation: A geometric approach, Ann. Inst. Fourier (Grenoble), 50 (2000), 321–362.

[9] A. Constantin and R. I. Ivanov, On an integrable two-component Camassa-Holm shallow water system, Phys. Lett. A., 372 (2008), 7129–7132.

[10] A. Constantin and J. Escher, Global existence and blow-up for a shallow water equation, Ann. Sc. Norm. Super. Pisa Cl. Sci., 26 (1998), 303–328.

[11] A. Constantin and J. Escher, Well-posedness, global existence, and blowup phenomena for a periodic quasi-linear hyperbolic equation, Comm. Pure Appl. Math., 51 (1998), 475–504.

[12] A. Constantin and J. Escher, Wave breaking for nonlinear nonlocal shallow water equations, Acta Math., 181 (1998), 229–243.

[13] A. Constantin and L. Molinet, Global weak solutions for a shallow water equation, Comm. Math. Phys., 211 (2000), 45–61.

[14] A. Constantin and W. Strauss, Stability of solitons, Commun. Pure Appl. Math., 53 (2000), 603–610.

[15] A. Constantin and W. Strauss, Stability of the Camassa-Holm solitons, J. Nonlinear Sci., 12 (2002), 415–422.

[16] R. Danchin, A few remarks on the Camassa-Holm equation, Differ. Integral Equ., 14 (2001), 953–988.

[17] R. Danchin, Fourier analysis methods for PDEs, Lecture notes, 14 (2005).

[18] J. Escher, O. Lechtenfeld and Z. Yin, Well-posedness and blow-up phenomena for the 2-component Camassa-Holm equation, Discrete Contin. Dyn. Syst., 19 (2007), 493–513.

[19] J. Escher and T. Lyons, Two-component higher order Camassa-Holm systems with fractional inertia operator: A geometric approach, J. Geom. Mech., 7 (2015), 281–293.

[20] G. Falqui, On a Camassa-Holm type equation with two dependent variables, J. Phys. A: Math. Gen., 39 (2006), 327–342.

[21] L. Fan, H. Gao and Y. Liu, On the rotation-two-component Camassa-Holm system modelling the equatorial water waves, Adv. Math., 291 (2016), 59–89.

[22] C. Foias and R. Temam, Gevrey class regularity for the solutions of the Navier-Stokes equations, J. Funct. Anal., 87 (1989), 359–369.

[23] A. Fokas and B. Fuchssteiner, Symplectic structures, their Bäcklund transformations and hereditary symmetries, Phy. D., 4 (1981), 47–66.

[24] C. Guan and Z. Yin, Global existence and blow-up phenomena for an integrable two-component Camassa-Holm shallow water system, J. Differential Equations., 248 (2010), 2003–2014.

[25] C. Guan and Z. Yin, Global weak solutions for a two-component Camassa-Holm shallow water system, J. Funct. Anal., 260 (2011), 1132–1154.

[26] G. Gui and Y. Liu, On the Cauchy problem for the two-component Camassa-Holm system, Math. Z., 268 (2011), 45–66.

[27] G. Gui and Y. Liu, On the global existence and wave-breaking criteria for the two-component Camassa-Holm system, J. Funct. Anal., 258 (2010), 4251–4278.

[28] F. Guo and R. Wang, On the persistence and unique continuation properties for an integrable two-component Dullin-Gottwald-Holm system, Nonlinear Anal., 96 (2014), 38–46.

[29] F. Guo, H. Gao and Y. Liu, On the wave-breaking phenomena for the two-component Dullin-Gottwald-Holm system, J. Lond. Math. Soc., 86 (2012), 810–834.

[30] Z. Guo and M. Zhu, Wave breaking for a modified two-component Camassa-Holm system, J. Differential Equations., 252 (2012), 2759–2770.

[31] Y. Han, F. Guo and H. Gao, On solitary waves and wave-breaking phenomena for a generalized two-component integrable Dullin-Gottwald-Holm system, J. Nonlinear. Sci., 23 (2013), 617–656.

[32] H. Holden and X. Raynaud, Global conservative solutions of the Camassa-Holm equations-A Lagrangian point of view, Comm. Partial Differential Equations., 32 (2007), 1511–1549.
[34] R. Ivanov, Two-component integrable systems modelling shallow water waves: the constant vorticity case, Wave Motion, 46 (2009), 389–396.
[35] X. Li and L. Zhang, The Cauchy problem and blow-up phenomena for a new integrable two-component peakon system with cubic nonlinearities, Discrete Contin. Dyn. Syst., 37 (2017), 3301–3325.
[36] X. Liu and Z. Yin, Local well-posedness and stability of solitary waves for the two-component Dullin-Gottwald-Holm system, Nonlinear Anal., 88 (2013), 1–15.
[37] W. Luo and Z. Yin, Gevrey regularity and analyticity for Camassa-Holm type systems, arXiv preprint, arXiv:1507.05250, 2015.
[38] L. Nirenberg, An abstract form of the nonlinear Cauchy-Kowalevski theorem, J. Differ Geom., 6 (1972), 561–576.
[39] T. Nishida, A note on a theorem of Nirenberg, J. Differ Geom., 12 (1977), 629–633.
[40] P. J. Olver and P. Rosenau, Tri-Hamiltonian duality between solitons and solitary-wave solutions having compact support, Physical Review E., 53 (1996), 1900–1906.
[41] L. V. Ovsyannikov, Singular operators in Banach spaces scales, Doklady Akademii Nauk SSSR.
[42] L. V. Ovsyannikov, Non-local Cauchy problems in fluid dynamics, Actes du Congrès International des Mathématiciens, 3 (1971), 137–142.
[43] L. V. Ovsyannikov, A nonlinear Cauchy problem in a scale of Banach spaces, Doklady Akademii Nauk SSSR., 200 (1971), 789–792.
[44] G. Rodríguez-Blanco, On the Cauchy problem for the Camassa-Holm equation, Nonlinear Anal., 46 (2001), 309–327.
[45] Z. Xin and P. Zhang, On the weak solutions to a shallow water equation, Comm. Pure Appl. Math., 53 (2000), 1411–1433.
[46] L. Zhang and B. Liu, On the Cauchy problem for a class of shallow water wave equations with (k+1)-order nonlinearities, J. Math. Anal. Appl., 445 (2017), 151–185.
[47] L. Zhang and X. Li, The local well-posedness, blow-up criteria and Gevrey regularity of solutions for a two-component high-order Camassa-Holm system, Nonlinear Anal. RWA., 35 (2017), 414–440.
[48] M. Zhu and J. Xu, On the wave-breaking phenomena for the periodic two-component Dullin-Gottwald-Holm system, J. Math. Anal. Appl., 391 (2012), 415–428.

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E-mail address: lzhang@907012163.com (Lei Zhang)
E-mail address: binliu@mail.hust.edu.cn (Bin Liu)