On power integral bases of certain pure number fields defined by $x^{3^r} - m$

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Abstract
Let $K = \mathbb{Q}(\alpha)$ be a pure number field generated by a root $\alpha$ of a monic irreducible polynomial $F(x) = x^{3^r} - m$, with $m \neq \pm 1$ is a square-free rational integer and $r$ is a positive integer. In this paper, we study the monogenity of $K$. We prove that if $m \not\equiv \pm 1 \pmod{9}$, then $K$ is monogenic. We give also sufficient conditions on $r$ and $m$ for $K$ to be not monogenic. Some illustrating examples are given too.

Keywords  Power integral basis · Theorem of Ore · Prime ideal factorization · Common index divisor

Mathematics Subject Classification  11R04 · 11R16 · 11R21

1 Introduction

Let $K$ be a number field defined by a monic irreducible polynomial $f(x) \in \mathbb{Z}[x]$ and $\mathbb{Z}_K$ its ring of integers. It is well known that the ring $\mathbb{Z}_K$ is a free $\mathbb{Z}$-module of rank $n = [K : \mathbb{Q}]$. Thus, the abelian group $\mathbb{Z}_K/\mathbb{Z}[\alpha]$ is finite. Its cardinal order is called the index of $\mathbb{Z}[\alpha]$, and denoted $(\mathbb{Z}_K : \mathbb{Z}[\alpha])$. The ring $\mathbb{Z}_K$ is said to have a power integral basis if it has a $\mathbb{Z}$-basis $(1, \theta, \ldots, \theta^{n-1})$ for some $\theta \in \mathbb{Z}_K$, that means $\mathbb{Z}_K = \mathbb{Z}[\theta]$ or $\mathbb{Z}_K$ is mono-generated as a ring, with a single generator $\theta$. In such a case, the field $K$ is said to be monogenic and not monogenic otherwise. The field index of $K$ is $i(K) = \gcd \{(\mathbb{Z}_K : \mathbb{Z}[\theta]), \theta \in \mathbb{Z}_K \text{ generates } K\}$. A rational prime $p$ dividing $i(K)$ is called a prime common index divisor of $K$. If $\mathbb{Z}_K$ has a power integral basis, then

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$i(K) = 1$. Thus, when a field possessing a prime common index divisor is not monogenic. The problem of checking the monogenity of the field $K$ is called a problem of Hasse [10, 11, 16, 17]. The problem of testing the monogenity of number fields and constructing power integral bases have been intensively studied, mainly by teams of Gaál, Nakahara and Pethő in the past century (see for instance [1, 2, 11, 13, 20]). In [22], it is shown that there exist infinitely many cubic fields with a power integral basis such that its splitting field contains a given quadratic field. In [21], it is proved that every cubefree integer occurs as the minimal index of infinitely many pure cubic fields. In [5], El Fadil gave conditions for the existence of power integral bases of pure cubic fields in terms of the index form equation. In [10], Funakura, proved that for a pure quartic fields $K$, $i(K) = 1$ if $m \not\equiv 1 \pmod{16}$ and $i(K) = 2$ otherwise. In [2], Ahmad, Nakahara and Husnine proved that if $m \equiv 2, 3 \pmod{4}$ and $m \not\equiv \mp 1 \pmod{9}$, then the sextic number field generated by $m^{1/6}$ is monogenic. They also showed in [1], that if $m \equiv 1 \pmod{4}$ and $m \not\equiv \mp 1 \pmod{9}$, then the sextic number field generated by $m^{1/6}$ is not monogenic. Also, in [7], based on prime ideal factorization, El Fadil showed that if $m \equiv 1 \pmod{4}$ or $m \not\equiv \mp 1 \pmod{9}$, then the sextic number field generated by $m^{1/6}$ is not monogenic. Also, Hameed and Nakahara [14], proved that if $m \equiv 1 \pmod{4}$, then the octic number field generated by $m^{1/8}$ is not monogenic, but if $m \equiv 2, 3 \pmod{4}$, then it is monogenic. In [12], Gaál and Remete, by applying the explicit form of the index forms, they obtained new results on monogenity of the number fields generated by $m^{1/8}$, where $3 \leq n \leq 9$. The pure number fields defined by $x^n - m$ have been considered in [15]. While Gaál’s and Pethő’s techniques are based on the index calculation, Nakahara’s methods are based on the existence of power relative integral bases of some special sub-fields. The goal of this paper is to study the monogeneity of pure number fields $K = \mathbb{Q}(\alpha)$ generated by a root $\alpha$ of a monic irreducible polynomial $F(x) = x^r - m$, where $m \not\in \pm 1$ is a square-free rational integer and $r$ is a positive integer. As in [3, 6, 7], our method is based on prime ideal factorization.

2 Main results

Let $K = \mathbb{Q}(\alpha)$ be a pure number field generated by a root $\alpha$ of a monic irreducible polynomial $F(x) = x^r - m$, with $m \not\in \pm 1$ is a square free rational integer and $r$ is a positive integer.

**Theorem 1** If $m \not\equiv \pm 1 \pmod{9}$, then $\mathbb{Z}[\alpha]$ is the ring of integers of $K$; $K$ is monogenic and $\alpha$ generates a power integral basis of $\mathbb{Z}_K$.

**Theorem 2** If $r \geq 3$ and $m \equiv \pm 1 \pmod{81}$, then $K$ is not monogenic; $\mathbb{Z}_K$ has no power integral basis.
3 Preliminaries

As our method is based on prime ideal factorization, we begin by recalling some fundamental notions on Newton polygon techniques. For more details, we refer to [8, 9, 19]. Let \( p \) be a rational prime integer and \( \phi \in \mathbb{Z}[x] \) be a monic polynomial whose reduction modulo \( p \) is irreducible. Any monic irreducible polynomial \( f(x) \in \mathbb{Z}[x] \) admits a unique \( p \)-adic development

\[
f(x) = a_0(x) + a_1(x)\phi(x) + \cdots + a_n(x)\phi(x)^n,
\]

with \( \deg (a_i(x)) < \deg (\phi(x)) \). For every \( 0 \leq i \leq n \), let \( u_i = v_p(a_i(x)) \). The \( p \)-Newton polygon of \( f(x) \) is the lower boundary convex envelope of the set of points \( \{(i, u_i), 0 \leq i \leq n, a_i(x) \neq 0\} \) in the Euclidean plane, which we denote by \( N_p(f) \). The polygon \( N_p(f) \) is the union of different adjacent sides \( S_1, S_2, \ldots, S_s \) with increasing slopes \( \lambda_1 < \lambda_2 < \cdots < \lambda_s \). We shall write \( N_p(f) = S_1 + S_2 + \cdots + S_s \). The polygon determined by the sides of negative slopes of \( N_p(f) \) is called the \( p \)-principal Newton polygon of \( f(x) \) and well denoted by \( N_p^+(f) \).

The length of \( N_p^+(f) \) is \( \ell(N_p^+(f)) = v_p(f(x)) \); the highest power of \( \phi \) dividing \( f(x) \) modulo \( p \).

Let \( \mathbb{F}_\phi \) be the finite field \( \mathbb{Z}[x]/(p, \phi(x)) \simeq \mathbb{F}_p[x](\overline{\phi}) \) (note that if \( \deg(\phi) = 1 \), then \( \mathbb{F}_\phi \simeq \mathbb{F}_p \)). We attach to any abscissa 0 \( \leq i \leq \ell(N_p^+(f)) \) the following residual coefficient \( c_i \in \mathbb{F}_\phi \) as follows:

\[
c_i = \begin{cases} 
0, & \text{if } (i, u_i) \text{ lies strictly above } N_p^+(f), \\
\left( \frac{a_i(x)}{p^{u_i}} \right) \pmod{p, \phi(x)}, & \text{if } (i, u_i) \text{ lies on } N_p^+(f).
\end{cases}
\]

Now, let \( S \) one of the sides of \( N_p^+(f) \) and \( \lambda = -\frac{h}{e} \) be its slope, where \( e \) and \( h \) are two positive coprime integers. The length of \( S \), denoted \( l(S) \), is the length of its projection to the horizontal axis. The degree of \( S \) is \( d = d(S) = \frac{l(S)}{e} \), is equal to the number of segments into which the integral lattice divides \( S \). More precisely, if \( (s, u_s) \) is the initial point of \( S \), then the points with integer coordinates lying in \( S \) are exactly \((s, u_s), (s + e, u_s - h), \ldots, (s + de, u_s - dh)\). We attach to \( S \) the following residual polynomial defined by \( f_S(y) = c_s + c_{s+e}y + \cdots + c_{s+(d-1)e}y^{d-1} + c_{s+de}y^d \in \mathbb{F}_\phi[y] \). As defined in [9, Def. 1.3], the \( \phi \)-index of \( f(x) \), denoted by \( \text{ind}_\phi(f) \), is \( \deg(\phi) \) times the number of points with natural integer coordinates that lie below or on the polygon \( N_p^+(f) \), strictly above the horizontal axis and strictly beyond the vertical axis (see Fig. 1). We say that the polynomial \( f(x) \) is \( \phi \)-regular with respect to \( p \) if for each side \( S \) of \( N_p^+(f) \), the associated residual polynomial \( f_S(y) \) is separable in \( \mathbb{F}_\phi[y] \). The polynomial \( f(x) \) is said to be \( p \)-regular if \( \overline{f}(x) = \phi \overline{f}(x) \)-regular for every \( 1 \leq i \leq t \), where \( f(x) = \prod_{i=1}^t \overline{f}_i \) is the factorization of \( f(x) \) into a product of powers of distinct irreducible polynomials over \( \mathbb{F}_p[x] \). For every \( i = 1, \ldots, t \), let \( N_{f_i}(x) = S_{i_1} + \cdots + S_{i_r} \), and for every \( j = 1, \ldots, r_i \), let \( f_{S_{ij}}(y) = \prod_{i=1}^{a_{ij}} \psi_{ij}^{a_{ij}}(y) \) be the factorization of \( f_{S_{ij}}(y) \) in \( \mathbb{F}_\phi[y] \). Then we have the following theorem of Ore, which plays a significant role in the proof of our theorems (see [9, Theorem 1.7 and Theorem 1.9], [8, Theorem 3.9] and [19]):

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Theorem 3 (Theorem of Ore)

1. \( \nu_2(\text{ind}(f)) = \nu_2((\mathbb{Z}_K : \mathbb{Z}[\alpha])) \geq \sum_{i=1}^{t'} \text{ind}_{\psi_i}(f) \) and equality holds if \( f(x) \) is \( p \)-regular; every \( n_{ijs} = 1 \).
2. If every \( n_{ijs} = 1 \), then \( p\mathbb{Z}_K = \prod_{i=1}^{t} \prod_{j=1}^{r_i} \prod_{s=1}^{s_j} p_{ijs}^{e_{ij}} \)

where \( e_{ij} \) is the ramification index of the side \( S_{ij} \) and \( f_{ijs} = \text{deg}(\phi_i) \times \text{deg}(\psi_{ijs}) \) is the residue degree of \( p_{ijs} \) over \( p \).

Example

Consider the monic irreducible polynomial \( f(x) = x^8 + 544x - 17 \), which factors in \( \mathbb{F}_2[x] \) into \( f(x) = \overline{\phi^8} \), where \( \phi = x - 1 \), then the \( \phi \)-adic development of \( f(x) \) is

\[
    f(x) = 528 + 552\phi + 28\phi^2 + 56\phi^3 + 70\phi^4 + 56\phi^5 + 28\phi^6 + 8\phi^7 + \phi^8.
\]

Thus \( N^+_{\phi}(f) = S_1 + S_2 + S_3 \) with respect to \( \nu_2 \) has three sides, with \( d(S_1) = 2 \) and \( d(S_2) = d(S_3) = 1 \) (see Fig. 1). The residual polynomials attached to the sides of \( N^+_{\phi}(f) \) are \( f_{S_1}(y) = 1 + y + y^2 \) and \( f_{S_2}(y) = f_{S_3}(y) = 1 + y \), which are irreducible polynomials in \( \mathbb{F}_2[y] \cong \mathbb{F}_2[y] \). Thus \( f(x) \) is \( \phi \)-regular, hence it is 2-regular. By Theorem 3, \( \nu_2(\text{ind}(f)) = \nu_2((\mathbb{Z}_K : \mathbb{Z}[\alpha])) = \text{ind}_{\phi}(f) = 7 \) and \( 2\mathbb{Z}_K = p_1^2p_2^3p_3^4 \), with respective residue degrees \( f_1 = 2 \) and \( f_2 = f_3 = 1 \).

4 Proofs

In order to prove our theorems, we need the following lemma which allows to evaluate the \( p \)-adic valuation of the binomial coefficient \( \binom{p^r}{j} \).

Lemma 1 Let \( p \) be a rational prime integer and \( r \) be a positive integer. Then
\[ v_p\left(\binom{p^r}{j}\right) = r - v_p(j) \]
for any integer \( j = 1, \ldots, p^r - 1 \).

**Proof** Since for any natural number \( m \), 
\[ v_p(m!) = \sum_{t=1}^{m} v_p(t) \], we have
\[
v_p\left(\binom{p^r}{j}\right) = v_p(p^r! - v_p((p^r - j)!)) = v_p(p^r!) - \sum_{t=1}^{p^r} v_p(t) - \sum_{t=1}^{p^r-j} v_p(t) = \sum_{t=p^r-j+1}^{p^r} v_p(t) - \sum_{t=1}^{j} v_p(t) = v_p(p^r) + \sum_{t=1}^{j-1} v_p(p^r - t) - \sum_{t=1}^{j} v_p(t).
\]

As for every \( t = 1, \ldots, j-1 < p^r \), then \( v_p(p^r - t) = v_p(t) \). Hence
\[ v_p\left(\binom{p^r}{j}\right) = r - v_p(j). \]

**Remark 1** For the simplicity of the calculations and as remarked in [3], we note that when a prime \( p \) does not divide a non-zero integer \( m \), then for every positive integer \( k \), 
\[ v_p(m^{p^k} - m) = v_p(m^{p^k-1} - 1) = v_p(m^{p^k} - 1). \]

**Proof of Theorem 1** Let \( D(\alpha) \) the discriminant of the algebraic integer \( \alpha \) and \( d_K \) the field discriminant of \( K \). Since \( F(x) \) is the minimal polynomial of \( \alpha \) over \( \mathbb{Q} \), then by [18, Propositions 2.9 and 2.13], one has
\[
D(\alpha) = D(1, \alpha, \ldots, \alpha^{3^r-1}) = (-1)^{\frac{3^r(3^r-1)}{2}}N_{K/\mathbb{Q}}(F'(\alpha)) = \pm N_{K/\mathbb{Q}}(3^r \alpha^{3^r-1})
\]
\[ = \pm (3^r)^{3r-1}N_{K/\mathbb{Q}}(\alpha)^{3^r-1} = \pm (3^r)^{3^r-1}m^{3^r-1} = (\mathbb{Z}_K : \mathbb{Z}[\alpha])^2 \cdot d_K, \]
then \( \mathbb{Z}[\alpha] \) is integrally closed if and only if \( p \) does not divide \( (\mathbb{Z}_K : \mathbb{Z}[\alpha]) \) for every rational prime \( p \) dividing \( 3 \cdot m \). Let \( p \) be a rational prime dividing \( m \), then \( F(x) \equiv \phi^{3^r} \pmod{p} \), where \( \phi = x \). As \( m \) is a square free integer, the \( \phi \)-principal Newton polygon with respect to \( v_p \), \( N_{\phi}^+(F) = S \) has a single side of degree 1; it is the side joining the points \((0, 1)\) and \((3^r, 0)\). Thus, the residual polynomial \( F_3(y) \) is irreducible over \( \mathbb{F}_3 \), where \( \mathbb{F}_3 \) is a field of characteristic 3. By Theorem 3, we get \( v_p((\mathbb{Z}_K : \mathbb{Z}[\alpha])) = \text{ind}_\phi(F) = 0 \); \( p \) does not divide \( (\mathbb{Z}_K : \mathbb{Z}[\alpha]) \). For \( p = 3 \) and \( 3^3 \) does not divide \( m \), then by little Fermat’s theorem and Lemma 1, \( F(x) = (x - m) \) in \( \mathbb{F}_3[x] \). Set \( \phi = x - m \) and applying Binomial theorem, we see that
\[
F(x) = x^{3^r} - m = (\phi + m)^{3^r} - m = m^{3^r} - m + \sum_{j=1}^{3^r-1} \binom{3^r}{j} m^{3^r-j} \phi^j + \phi^{3^r}.
\]

Since \( m \not\equiv \pm 1 \pmod{9} \); \( v_3(m^{3^r} - m) = v_3(m^2 - 1) = 1 \), then by (1) and Lemma 1, \( N_{\phi}^+(F) = S \) has a single side of degree 1, thus \( F_3(y) \) is irreducible over \( \mathbb{F}_3 \), \( S \) follows that the polynomial \( F(x) \) is 3-regular. By Theorem 3, we get \( v_3(\text{ind}(F)) = v_3((\mathbb{Z}_K : \mathbb{Z}[\alpha])) = \text{ind}_\phi(F) = 0 \); the rational prime integer 3 does not divide \( (\mathbb{Z}_K : \mathbb{Z}[\alpha]) \). This completes the proof of the theorem. \( \square \)
For the proof of Theorem 2, we use the following Lemma, which gives a sufficient condition for a prime $p$ to be a prime common index divisor of $K$; it is a consequence of the theorem of Dedekind (see [18, Theorems 4.33 and 4.34] and [4]).

**Lemma 2** Let $p$ be a rational prime integer and $K$ be a number field. For every positive integer $f$, let $P_f$ be the number of distinct prime ideals of $\mathbb{Z}_K$ lying above $p$ with residue degree $f$ and $N_f$ be the number of monic irreducible polynomials of $\mathbb{F}_p[x]$ of degree $f$. If $P_f > N_f$, then $p$ is a common index divisor of $K$.

**Proof of Theorem 2** Let $v = v_3(m^3 - m) = v_3(m^2 - 1)$. Since $m \equiv \pm 1 \pmod{81}$, then $v \geq 4$. Recall that $F(x) \equiv \phi^{m^3} \pmod{3}$, where $\phi = x - m$. By the above $\phi$-adic development (1) of $F(x)$ and Lemma 1, the $\phi$-principal Newton polygon of the polynomial $F(x)$ is the lower boundary convex envelope of the set of points $\{(0, v), (3^j, f - j), 0 \leq j \leq r\}$ in the Euclidean plane. More precisely, if $v > r$, then $N_\phi^{+}(F)$ is the polygon joining the points $\{(0, v), (1, r), (3, r - 1), \ldots, (3^r, 0)\}$ (see Fig. 2 as example) and if $r \geq v$, then $N_\phi^{-}(F)$ is the polygon joining the points $\{(0, v), (3^{v+1} - 1, v - 1), \ldots, (3^r, 0)\}$ (see Fig. 3). Thus $N_\phi^{+}(F) = S_1 + S_2 + \cdots + S_g$ has $g$ sides of degree 1 each one with $g \geq \min(v, r + 1) \geq 4$. So, the residual polynomial $F_{S_1}(y)$ is irreducible over $\mathbb{F}_3$ for every $1 \leq k \leq g$. It follows that $F(x)$ is $\phi$-regular with respect to 3, hence it is 3-regular. By Theorem 3, there are at least 4 distinct prime ideals of $\mathbb{Z}_K$ lying above 3 with residue degree 1 each one. More explicitly,
As there are only three linear monic irreducible polynomials over the Galois field \( \mathbb{F}_3 \), namely \( x \), \( x + 1 \), and \( x + 2 \). Therefore, by Lemma 2, 3 is a common index divisor of \( K \). Hence, \( K \) is not monogenic; because \( i(K) > 1 \).

\[ 3 \mathbb{Z}_K = \left\{ \begin{aligned} p_0 p_1^2 p_2^3 \ldots p_i^{2.3^{i+1}} \ldots p_r^{2.3^{r-1}}, & \quad f_0 = f_1 = \ldots = f_r = 1, & \text{if } v > r, \\
 p_1^{2.3^{i+1}} p_2^{2.3^{i+1}} \ldots p_i^{2.3^{i+1}} \ldots p_v^{2.3^{v-1}}, & \quad f_1 = \ldots = f_v = 1, & \text{if } r \geq v. \end{aligned} \right. \]

\[ \mathbb{Z}_K = \mathbb{Z}/(3^2 \cdot 2^{i_v} \cdot 2^{i_{v+1}} \ldots 2^{i_{r-1}} \cdot 5^{i_0} \cdot 5^{i_1} \ldots 5^{i_{r-1}}). \]

\[ f_0 = f_1 = \ldots = f_r = 1, \quad \text{if } v > r, \]

\[ f_1 = \ldots = f_v = 1, \quad \text{if } r \geq v. \]

5 Examples

Let \( F(x) \in \mathbb{Z}[x] \) a monic irreducible polynomial and \( K \) the number field defined by a root of \( F(x) \).

1. If \( F(x) = x^{27} - 22; \) \( F(x) \) is irreducible, because it is 2-Eisenstein polynomial. Since \( m = 22 \), then \( m \equiv 4 \pmod{9} \). By Theorem 1, \( K \) is monogenic.
2. If \( F(x) = x^{314} - 82; \) \( F(x) \) is irreducible, because it is 41-Eisenstein polynomial. Since \( m = 82 \), then \( m \equiv 1 \pmod{81} \). By Theorem 2, \( K \) is not monogenic.
3. Let \( F(x) = x^{243} - 161; \) \( F(x) \) is irreducible, because it is 7-Eisenstein polynomial. Since \( m = 161 \), then \( m \equiv -1 \pmod{81} \). By Theorem 2, \( K \) is not monogenic.
4. If \( F(x) = (x - 5)^{30} - 5^2 \), then \( F(x) \equiv x^{36} \pmod{5} \). By applying Binomial theorem, write

\[ F(x) = -(5^2 + 3^6) + \sum_{j=1}^{36}(-1)^{j+1} \cdot \binom{36}{j} \cdot 5^{36-j}x^j. \]

It follows that \( N_N^+(F) \) at \( p = 5 \) has a single side of degree 1; it is the side joining the points \((0, 2)\) and \((3^6, 0)\). Thus \( F(x) \) is irreducible over \( \mathbb{Q} \). Let \( \alpha \) be a root of \( F(x) \) and \( \beta = \frac{\alpha - 5}{5} \), where \( u \) and \( v \) are two positive integers such that \( 2u - 3^6v = 1 \). Then \( \beta \) is a root of the monic irreducible polynomial \( G(x) = x^{36} - 5 \) and \( K = \mathbb{Q}(\alpha) = \mathbb{Q}(\beta) \). Since \( 5 \not\equiv \pm 1 \pmod{9} \), then by Theorem 1, \( K \) is monogenic and \( \beta \) generates a power integral basis of \( \mathbb{Z}_K \).

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