Abstract

Ideas and results of the generalized wave operator theory for dynamical and stationary cases are developed further and exact expressions for generalized scattering operators are obtained for wide classes of differential equations. New results on the structure of the generalized scattering operators are derived. Interesting interrelations between dynamical and stationary cases are found for radial Schrödinger and Dirac equations, and for Dirac-type equations as well. For some important examples we explain why the well-known divergences in the higher order approximations of the scattering matrices do not appear in the "generalized wave operator" approach.

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1 Introduction

Wave and scattering operators belong to the basic notions of mathematical physics and spectral theory. Three books studying this topic (in the mentioned above domains) [3,12,31] as well as hundreds of papers were published.
in 2015 only. Generalized wave operators were introduced in the 1960s [11,26], whereas interesting nontrivial examples appeared already in [11] and much wider classes were considered in [26]. The theory of generalized wave operators was actively developed at the end of the 1960s and at the beginning of the 1970s (see, e.g., [7,10,16,20,22,23,27]). Using generalized wave operators one can consider scattering theory for wider classes of important operators and in greater detail than using the classical wave operators. More precisely, in many important cases the initial and final states of the system cannot be regarded as free (i.e., no free states at \( t = \pm \infty \)). In these cases the theory of generalized wave and generalized scattering operators proves to be effective.

Several interesting papers (see, e.g., [32,34]) as well as an important book by L.D. Faddeev and S.P. Merkuriev [13] on generalized wave operators appeared in the 1980s. However, numerous open problems remain and various further important results and applications may be obtained. In the recent years, the ideas developed in the theory of generalized wave operators are starting to attract attention again (see [15,19,21,33] and references therein). In particular, some ideas are used in the theory of modified wave operators (the case of nonlinear differential equations). At the same time various other results and publications on the spectral and scattering theory of radial Dirac and Schrödinger equations and equations with singularities appeared last years (see, e.g., [3,4,18,25] and references therein). In the present paper we plan to demonstrate new important applications of the generalized wave operators to scattering theory.

Ideas and results of the generalized wave operator theory for dynamical and stationary cases are developed further and exact expressions for scattering operators are obtained for wide classes of differential equations. New results on the structure of the generalized scattering operators are derived. Interesting interrelations between dynamical and stationary cases are found for radial Schrödinger and Dirac equations, and for Dirac-type equations as well. A "generalized wave operator" approach to the well-known divergences in the higher order approximations of the scattering matrices is discussed, and for some important examples we explain why these divergences do not appear in our approach.

In Section 2 we introduce the notions of the generalized wave and gen-
eralized scattering operators and of the deviation factors (see [7], [16], [26] and [27]). The deviation factors describe the deviation of the initial and final states from the free state. In Section 3 we consider the case of Schrödinger equation with Coulomb potential.

Section 4 is dedicated to the radial Schrödinger equation. The generalized scattering matrix $S_{dyn}$ describes the behavior of a system when the time tends to infinity (dynamical case). In the present paper we introduce also the generalized scattering matrix $S_{st}$ which describes the behavior of a system when the space coordinate tends to infinity (stationary case). We discuss the connections between $S_{dyn}$ and $S_{st}$ and derive the corresponding ergodic-type theorem in Section 4. Similar results for the radial Dirac equation are obtained in Section 5 (and for Dirac-type equation in Section 6).

Dirac equation in $\mathbb{R}^4$ is studied in Section 7. Under some natural conditions we prove that the scattering matrix of the Dirac operator in $\mathbb{R}^4$ (momentum representation) has a special structure (see (7.16)) which was not known before. This structure follows from the fact that the generalized scattering matrix commutes with the unperturbed operator. We note that the unperturbed Dirac operator (in the momentum representation) is the operator of multiplication by the matrix function $H(q)$ given by (7.8).

In the last part of the paper (Sections 8-10), we investigate the fundamental equation of quantum electrodynamics. The structure of the scattering operator for the fundamental equation of quantum electrodynamics is given in Theorem 8.1. In the electrodynamic theory, the higher order approximations of matrix elements of the scattering matrix contain integrals which diverge. We think that these divergences appear because the corresponding scattering matrix is introduced in the form of the small parameter series. We produce exact formulas for the scattering matrices and for some important examples we prove (see Section 8 and, especially, Sections 9 and 10) that the divergences do not exist in the case of ”generalized wave operator” approach. In our approach we essentially use the results from Section 2.

The last section of the paper is Conclusion.

As usual, $\mathbb{R}$ stands for the real axis and $\mathbb{C}$ stands for the complex plane, $\overline{\lambda}$ stands for the complex number which is complex conjugate to $\lambda$, $I$ stands for the identity operator, $e$ is the base of the natural logarithm, $i$ is the
imaginary unit and Span stands for linear span.

2 Preliminaries: generalized wave operators

Consider linear (not necessarily bounded) operators $A$ and $A_0$ acting in some Hilbert space $H$ and assume that the operator $A_0$ is self-adjoint. The absolutely continuous subspace of the operator $A_0$ (i.e., the subspace corresponding to the absolutely continuous spectrum) is denoted by $G_0$, and $P_0$ is the orthogonal projector on $G_0$. Generalized wave operators $W_+(A, A_0)$ and $W_-(A, A_0)$ are introduced by the equality

$$W_{\pm}(A, A_0) = \lim_{t \to \pm \infty}[e^{itA}e^{-itA_0}W_0(t)^{-1}]P_0, \quad (2.1)$$

where $i$ is the imaginary unit and $W_0$ is an operator function taking operator values $W_0(t)$ acting in $G_0$ in the domain $|t| > R$ ($t \in \mathbb{R}$) for some $R \geq 0$. More precisely, we have the following definition (see [26,27]) of the generalized wave operators $W_{\pm}(A, A_0)$ and deviation factor $W_0$.

Definition 2.1 An operator function $W_0(t)$ is called a deviation factor and operators $W_{\pm}(A, A_0)$ are called generalized wave operators if the following conditions are fulfilled:

1. The operators $W_0(t)$ and $W_0(t)^{-1}$ acting in $G_0$, are bounded for all $t$ ($|t| > R$), and

$$\lim_{t \to \pm \infty} W_0(t + \tau)W_0(t)^{-1}P_0 = P_0, \quad \tau = \mp. \quad (2.2)$$

2. The following commutation relations hold for arbitrary values $t$ and $\tau$:

$$W_0(t)A_0P_0 = A_0W_0(t)P_0, \quad W_0(t)W_0(t+\tau)P_0 = W_0(t+\tau)W_0(t)P_0. \quad (2.3)$$

3. The limits $W_{\pm}(A, A_0)$ in (2.1) exist in the sense of strong convergence.

If $W_0(t) \equiv I$ in $G_0$, then the operators $W_{\pm}(A, A_0)$ are usual wave operators.
Definition 2.2  The operators $A$ and $A_0$ are called comparable in the generalized sense if the generalized wave operators $W_\pm(A, A_0)$ and $W_\pm(A_0, A)$ exist.

Although the notion of the generalized wave operator was introduced in [7, 11, 16, 26], its description in the form of Definition 2.1 was given several years later in [10, 27].

Proposition 2.3  Let conditions (2.1)–(2.3) be fulfilled. Then

$$W_\pm(A, A_0)e^{iA_0 t}P_0 = e^{iAt}W_\pm(A, A_0)P_0$$  \hspace{1cm} (2.4)

Proof. Rewrite (2.1) in the form

$$W_\pm(A, A_0) = \lim_{t \to \pm \infty} [e^{iA(t+s)}e^{-iA_0(t+s)}W_0(t+s)^{-1}P_0].$$  \hspace{1cm} (2.5)

Now, formula (2.2), the second equality in (2.3) and equality (2.5) imply that

$$W_\pm(A, A_0)P_0 = e^{iAs}W_\pm(A, A_0)e^{-iA_0 s}P_0.$$  \hspace{1cm} (2.6)

This proves the proposition. ■

Definition 2.4  Let conditions (2.1)–(2.3) be fulfilled. Then the generalized wave operators $W_\pm(A, A_0)$ are called complete if

$$W_\pm(A, A_0)G_0 = G_A,$$  \hspace{1cm} (2.7)

where $G_A$ is the absolutely continuous subspace of the operator $A$.

Using (2.6), we obtain the following assertion (see [2 Ch. 7]).

Proposition 2.5  Let conditions (2.1)–(2.3) be fulfilled. If the operator $W_+(A, A_0)$ is complete, then

$$A_a W_+(A, A_0) f = W_+(A, A_0) A_{0,a} f, \quad f \in G_0,$$  \hspace{1cm} (2.8)

where $A_a$ and $A_{0,a}$ are the operators induced by $A$ and $A_0$ in the spaces $G_A$ and $G_0$, respectively.
Theorem 2.6 Let self-adjoint operators $A_0$, $A_1$ and $A$ be given. If the operator $A_1 - A$ belongs to the trace class $\sigma_1$ and the generalized wave operators

$$W_\pm(A, A_0) = \lim_{t \to \pm \infty} [e^{iAt}e^{-iA_0t}W_0(t)^{-1}]P_0$$

exist and are complete, then the generalized wave operators

$$W_\pm(A_1, A_0) = \lim_{t \to \pm \infty} [e^{iA_1t}e^{-iA_0t}W_0(t)^{-1}]P_0$$

exist and are complete as well.

Proof. We need the well-known Rosenblum–Kato theorem [17, 24]: If the operators $A$ and $A_1$ are self-adjoint and the operator $A_1 - A$ belongs to the trace class $\sigma_1$, then the wave operators

$$W_\pm(A_1, A) = \lim_{t \to \pm \infty} [e^{iA_1t}e^{-iA_0t}]P_A,$$

where $P_A$ is the orthogonal projector on $G_A$, exist and are complete. Using Rosenblum–Kato approach and theorem, it is easy to show that the operators $W_\pm(A, A_0)$ map the subspace $G_0$ onto $G_A$. Hence,

$$W_\pm(A_1, A_0) = W_\pm(A_1, A)W_\pm(A, A_0).$$

The theorem is proved. ■

Clearly, the choice of the deviation factor is not unique.

Remark 2.7 Let unitary operators $C_-$ and $C_+$ satisfy commutation conditions $A_0C_{\pm} = C_{\pm}A_0$. If $W_0(t)$ is a deviation factor, then the operator function given (for $t > 0$ and $t < 0$, respectively) by the equalities $W_+(t) = C_+W_0(t)$ ($t > 0$), and $W_-(t) = C_-W_0(t)$ ($t < 0$) is the deviation factor as well.

The choice of the operators $C_{\pm}$ is very important and is determined by specific physical problems. The definition below shows that generalized scattering operators also depend on the choice of $C_{\pm}$. 
**Definition 2.8**  The generalized scattering operator \( S(A, A_0) \) has the form

\[
S(A, A_0) = W_+^*(A, A_0)W_-(A, A_0),
\]

(2.13)

where

\[
W_\pm(A, A_0) = \lim_{t \to \pm \infty} \left( e^{iAt} e^{-iA_0t} W_\pm(t)^{-1} \right) P_0.
\]

(2.14)

In fact, operator functions \( W_\pm(t) \) are uniquely determined up to some factors \( C_\pm(t) \) tending to \( C_\pm \) when \( t \) tends to \( \infty \) or \( -\infty \), respectively. This means that \( S(A, A_0) \) is uniquely determined by the choice of \( C_\pm \).

It is not difficult to prove that the operator \( S(A, A_0) \) unitarily maps \( G_0 \) onto itself and that

\[
A_0 S(A, A_0) P_0 = S(A, A_0) A_0 P_0.
\]

(2.15)

3  **Coulomb potential**

1. We introduce the operators

\[
\mathcal{L}_1 f = -\frac{d^2}{dx^2} f + \left( \frac{\ell(\ell + 1)}{r^2} - \frac{2z}{r} \right) f, \quad \mathcal{L}_0 f = -\frac{d^2}{dx^2} + \frac{\ell(\ell + 1)}{r^2} f,
\]

(3.1)

where \( z > 0, \ell \) is some nonnegative integer (i.e., \( \ell \in \mathbb{N}_0 \)), and the boundary condition is given by

\[
f(0) = 0.
\]

(3.2)

In momentum representation, \( \mathcal{L}_0 \) and \( \mathcal{L}_1 \) have the following form [8, Sec. 8]:

\[
\tilde{\mathcal{L}}_0 f = k^2 f(k), \quad f \in L^2(0, \infty),
\]

(3.3)

\[
\tilde{\mathcal{L}}_1 f = k^2 f(k) + \int_0^\infty f(p) R_\ell(k, p) dp.
\]

(3.4)

Here \( R_\ell(k, p) \) is given by the equality

\[
R_\ell(k, p) = -\frac{2z}{\pi} Q_\ell \left( \frac{k^2 + p^2}{2pk} \right).
\]

(3.5)
and $Q_{\ell}(x)$ is Legendre function of the second kind. The deviation factors in the momentum representations are operators of multiplication by functions of the form (see [27]):

$$\tilde{W}_\pm(t, k) = |t|^{\mp(iz/k)},$$

which we sometimes call deviation factors as well. Thus, the generalized scattering operator exists and can be written in the form

$$S(\tilde{L}_1, \tilde{L}_0) = \lim_{\tau \to -\infty, t \to +\infty} |t|^{-iz/k} S(t, \tau) |\tau|^{-iz/k},$$

where

$$S(t, \tau) = e^{it\tilde{L}_0} e^{-it\tilde{L}_1} e^{ir\tilde{L}_1} e^{-ir\tilde{L}_0}. \quad (3.8)$$

We have shown in [27] that the scattering operator $S(\tilde{L}_1, \tilde{L}_0)$ has the form

$$S(\tilde{L}_1, \tilde{L}_0)f(k) = S(k)f(k), \quad k > 0,$$

where $S(k)$ is introduced by the equality

$$S(k) = (2k)^{4iz/k} \frac{\Gamma(\ell + 1 - iz/k)}{\Gamma(\ell + 1 + iz/k)}. \quad (3.10)$$

Here $\Gamma(x)$ is the Euler gamma function.

Formulas (3.9) and (3.10) present an expression for the dynamical scattering operator. On the other hand, we note that the stationary scattering problem is very well known for the case of Coulomb potential [9]. In particular, for the stationary scattering operator $S_{st}$ the following equality holds [9 Ch. 1]:

$$S_{st}(k) = \frac{\Gamma(\ell + 1 - iz/k)}{\Gamma(\ell + 1 + iz/k)}. \quad (3.11)$$

The difference between the functions $S(k)$ and $S_{st}(k)$ consists only in the factor $(2k)^{4iz/k}$, which does not depend on $\ell$. Such a difference is insignificant and is caused by the choice of normalization. This fact confirms that the introduced definitions are correct from the physical point of view. In Section[4] and further we discuss some other important stationary scattering problems (and in greater detail).
2. Our further results in this section are useful for understanding interconnections between removal of divergence problem and generalized scattering matrix. Expanding the right-hand side of (3.7) in powers of $z$ we obtain the normalized perturbation series

$$\tilde{S}(\tilde{L}_1, \tilde{L}_0) = I + \sum_{i=1}^{\infty} z^i \tilde{S}_k(\infty, -\infty).$$

(3.12)

The first term of the normalized series is defined by the relation

$$\left(\tilde{S}_1(\infty, -\infty)f\right)(k) = \lim_{\substack{\tau \to -\infty \\ t \to +\infty}} \left( -\frac{i}{k} \left( \ln |t\tau| \right) f(k) \right) + \frac{2i}{\pi} \int_{t}^{\infty} \int_{0}^{\infty} f(p) Q_\ell \left( \frac{k^2 + p^2}{2pk} \right) e^{i(k^2 - p^2)t_1} dp dt_1. \quad (3.13)$$

It follows from (3.10) that

$$\left(\tilde{S}_1(\infty, -\infty)f\right)(k) = -\frac{2iz}{k} \left( \frac{\Gamma'\left(\ell + 1\right)}{\Gamma\left(\ell + 1\right)} - 2 \ln (2k) \right) f(k). \quad (3.14)$$

3. Now, let us expand (in powers of $z$) the right-hand side of (3.9):

$$S(t, \tau) = I + \sum_{i=1}^{\infty} z^i S_k(t, \tau).$$

(3.15)

It is easy to see that

$$S_1(t, \tau)f = \frac{2i}{\pi} \int_{\tau}^{t} \int_{0}^{\infty} f(p) Q_\ell \left( \frac{k^2 + p^2}{2pk} \right) e^{i(k^2 - p^2)t_1} dp dt_1. \quad (3.16)$$

Relations (3.13), (3.14) and (3.16) imply that

$$S_1(t, \tau)f = \left( \frac{i}{k} \left( \ln |t\tau| \right) + O(1) \right) f(k), \quad k > 0, \ t \to +\infty, \ \tau \to -\infty. \quad (3.17)$$

**Remark 3.1** The operator $S_1(t, \tau)$ has a logarithmic type divergence. Using generalized scattering operator we obtain the results (see (3.10) and (3.14)) without divergences.
4 Classical and generalized stationary scattering problems (Schrödinger equation)

1 Classical and Coulomb cases. Stationary scattering results are well-known in the classical and Coulomb cases [9]. The classical radial Schrödinger equation has the form

\[
\left( \frac{d^2}{dr^2} - \frac{\ell(\ell + 1)}{r^2} + U(r) + k^2 \right) y_\ell(r) = 0 \quad (U(r) = U(r)), \quad (4.1)
\]

where

\[
\int_0^\infty r^p|U(r)|dr < \infty \quad (p = 1, 2). \quad (4.2)
\]

There exists a solution of (4.1) which satisfies the following boundary conditions

\[
y_\ell(r) \sim N r^{\ell+1}, \quad r \to 0, \quad (4.3)
\]

\[
y_\ell(r) \sim \exp(-i\theta_\ell) - \exp(i\theta_\ell) S_\ell(k), \quad r \to \infty, \quad (4.4)
\]

where \(N\) is a constant, \(\theta_\ell = kr - \frac{1}{2}\ell\pi\) and \(S_\ell(k)\) is the scattering function.

The radial Schrödinger equation with Coulomb potential has the form

\[
\left( \frac{d^2}{dr^2} - \frac{\ell(\ell + 1)}{r^2} + U_c(r) + k^2 \right) y_\ell(r) = 0, \quad (4.5)
\]

where

\[
U_c(r) = \frac{2z}{r}, \quad z > 0. \quad (4.6)
\]

Again, there exists (see [9] Ch. 1)) a solution of (4.5) satisfying the boundary conditions

\[
y_\ell(r) \sim N r^{\ell+1}, \quad r \to 0, \quad (4.7)
\]

\[
y_\ell(r) \sim \exp(-i\theta_\ell^c) - \exp(i\theta_\ell^c) S_\ell^c(k), \quad r \to \infty, \quad (4.8)
\]

where \(N\) is a constant and \(\theta_\ell^c = kr - \frac{1}{2}\ell\pi - (z/k) \ln(2kr)\). The Coulomb stationary scattering function was already considered in Section 3 and is given by the formula (3.11). Below we consider a more general case.
2 Generalized stationary scattering function. Let us consider the radial Schrödinger equation

\[
\left( \frac{d^2}{dr^2} - \frac{\ell(\ell + 1)}{r^2} + \varphi(r) + k^2 \right) y_\ell(r) = 0 \quad (\varphi(r) = \overline{\varphi(r)}),
\]

where

\[
\int_a^\infty |\varphi'(r)|dr + \int_a^\infty |\varphi^2(r)|dr + \int_0^a |\varphi(r)|rdr < \infty, \quad 0 < a < \infty.
\]

It is easy to see that the Coulomb potential satisfies (4.10). In the present (more general) case, a solution of (4.9) satisfying the boundary conditions (4.7) exists as well. We use the following result [5, Ch. II, Theorem 8]:

Under conditions (4.10), equation (4.9) has two linear independent solutions

\[
Z_1(r, k, \ell) \sim \exp (-i\theta)V_0^{-1}(r, k), \quad r\to\infty, \quad (4.11)
\]

\[
Z_2(r, k, \ell) \sim \exp (i\theta)V_0(r, k), \quad r\to\infty, \quad (4.12)
\]

where \( \theta = kr - \frac{1}{2}\ell\pi \) and

\[
V_0(r, k) = \exp \left( \frac{i}{2k} \int_a^r \varphi(u)du \right). \quad (4.13)
\]

Hence, we obtain our main assertion in this section.

**Proposition 4.1** Let (4.10) be fulfilled. Then the solution \( y_\ell \) of (4.9), which satisfies (4.7), has the form

\[
y_\ell(r) \sim \exp (-i\theta)V_0^{-1}(r, k) - \exp (i\theta)V_0(r, k)S_\ell(k), \quad r\to\infty, \quad (4.14)
\]

where \( V_0(r, k) \) is the deviation factor given by (4.12).

**Definition 4.2** The functions \( S_\ell(k) \) and \( V_0(r, k) \) in (4.14) are called the stationary generalized scattering function and deviation factor (in the case of radial Schrödinger equation), respectively.

**Remark 4.3** The deviation factor \( V_0(r, k) \) characterizes the deviation of the wave \( y_\ell(r) \) from the free wave. It is important that \( V_0(r, k) \) does not depend on \( \ell \).
3. Let us consider the operators

\[ Hf = \left( -\frac{d^2}{dr^2} + \frac{\ell(\ell + 1)}{r^2} - \varphi(r) \right) f, \quad f(0) = 0 \]  
(4.15)

and

\[ H_0f = -\frac{d^2}{dr^2} f, \quad f(0) = 0. \]  
(4.16)

According to Buslaev–Matveev results [10], the following assertion is valid.

**Proposition 4.4** Let the potential \( \varphi(x) \) and its derivatives satisfy the conditions

\[ \left| \varphi^{(\kappa)}(x) \right| \leq C(1 + x)^{-\alpha-\kappa}, \quad \alpha > 1/2, \quad 0 \leq \kappa \leq 2. \]  
(4.17)

Then there exists the generalized wave operator \( W_+(H, H_0) \), and the corresponding deviation factor (in momentum representation) is given by the formula

\[ \tilde{W}_0(t, k) = \exp \left( \frac{i}{2k} \int_a^{tk} \varphi(u) du \right). \]  
(4.18)

Thus, we derived that the stationary deviation factor \( V_0(r) \), when \( r \to \infty \), and the dynamical deviation factor, when \( t \to \infty \), are connected by the important *ergodic* equality

\[ V_0(tk, k) = \tilde{W}_0(t, k). \]  
(4.19)

The following remark is an anologue (for the stationary case) of Remark 2.7.

**Remark 4.5** Let a unitary operator \( C \) be such that \( A_0C = CA_0 \). If \( V_0(r) \) is a deviation factor, then \( V(r) = CV_0(r) \) is a deviation factor as well.

The choice of the operator \( C \) depends on the specific physical problem.
5 Generalized stationary scattering problems (radial Dirac and Dirac-type systems)

1. Radial Dirac system has the form

\[
\left( \frac{d}{dr} + \frac{k}{r} \right) f - (\lambda + m - v(r)) g = 0, \quad (5.1)
\]

\[
\left( \frac{d}{dr} - \frac{k}{r} \right) g + (\lambda - m - v(r)) f = 0 \quad (k > 0, \ m > 0), \quad (5.2)
\]

where \( \lambda = \overline{\lambda} \) and

\[ v(r) = -\frac{A}{r} + \varphi(r) \quad (A > 0, \ |k| > A, \ \varphi(r) = \overline{\varphi(r)}). \quad (5.3) \]

We assume that the following inequality

\[
\int_a^\infty |\varphi'(r)|dr + \int_a^\infty |\varphi^2(r)|dr + \int_0^a |\varphi(r)|dr < \infty, \quad 0 < a < \infty \quad (5.4)
\]

is fulfilled. Then, there exists (see [29]) a solution of (5.1), (5.2) satisfying the boundary condition

\[ \text{col}[f, g] \sim r^\alpha \text{col}[1, b_0], \quad r \to 0, \quad (5.5) \]

where \( \alpha = \sqrt{k^2 - A^2} \), \( b_0 = (\alpha + k)/A \), \( \Re \alpha > 0. \quad (5.6) \)

Let us rewrite the system (5.1), (5.2) in the matrix form:

\[
\frac{dZ}{dr} = \mathcal{A}(r)Z, \quad \mathcal{A}(r) = \begin{bmatrix} -k/r & m + \lambda - v(r) \\ m - \lambda + v(r) & k/r \end{bmatrix}, \quad Z = \begin{bmatrix} f \\ g \end{bmatrix}. \quad (5.7)
\]

We use the following result (see [5, Ch. II, Theorem 8]).

The system (5.7) has two linearly independent solutions:

\[
Z_1(r, \lambda, k) \sim \exp(-i\theta)V_0(r, \lambda)^{-1}C_1(\lambda, k), \quad r \to \infty, \quad (5.8)
\]

\[
Z_2(r, \lambda, k) \sim \exp(i\theta)V_0(r, \lambda)C_2(\lambda, k), \quad r \to \infty, \quad (5.9)
\]
where $\theta = \eta r$, $\eta = \sqrt{\lambda^2 - m^2}$, $|\lambda| > m$, $C_1(\lambda, k)$ and $C_2(\lambda, k)$ are $2 \times 1$ vectors, $C_1(\lambda, k) = C_2(\lambda, k)$, and

$$V_0(r, \lambda) = \exp \left( \frac{i}{\eta} \int_a^r v(u) du \right). \quad (5.10)$$

Hence, we obtain the following assertion.

**Proposition 5.1** Let (5.4) hold and let $\text{col}[f, g]$ be the solution of (5.7) satisfying (5.5). Then, for $r \to \infty$, we have

$$\text{col}[f, g] \sim \exp (-i \theta)V_0(r, \lambda)^{-1}C_1(\lambda, k) + \exp (i \theta)V_0(r, \lambda)C_2(\lambda, k), \quad (5.11)$$

where $\theta = \eta r$, $|\lambda| > m$.

Consider the entries of $C_1(\lambda, k) = \text{col}[c_{11}(\lambda, k), c_{21}(\lambda, k)]$. Taking into account (5.1), (5.2) and (5.8) we have $\frac{c_{11}}{c_{21}} = i \sqrt{\frac{\lambda + m}{\lambda - m}}$, and so

$$\frac{c_{11}}{c_{11}} = -\frac{c_{21}}{c_{21}} \quad (|\lambda| > m). \quad (5.12)$$

**Definition 5.2** The scattering matrix function $S(\lambda, k)$ of system (5.1), (5.2) (where (5.4) holds) is defined by the relation

$$S(\lambda, k) = \begin{bmatrix} -\frac{c_{11}(\lambda, k)}{c_{11}(\lambda, k)} & 0 \\ 0 & \frac{c_{11}(\lambda, k)}{c_{11}(\lambda, k)} \end{bmatrix} \quad (5.13)$$

Recall that $C_1(\lambda, k) = C_2(\lambda, k)$. Hence, equalities (5.12) and (5.13) yield

$$S(\lambda, k)C_1(\lambda, k) = -C_2(\lambda, k). \quad (5.14)$$

In view of (5.14), equality (5.11) (for $|\lambda| > m$ and $r \to \infty$) can be rewritten in the form

$$\text{col}[f, g] \sim \left( \exp (-i \theta)V_0(r, \lambda)^{-1} - \exp (i \theta)V_0(r, \lambda)S(\lambda, k) \right)C_1(\lambda, k). \quad (5.15)$$

**Remark 5.3** We emphasize that the deviation factor $V_0(r, \lambda)$ does not depend on $k$. 

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Remark 5.4 Comparing expression (5.10) for $V_0(r, \lambda)$ with the corresponding formula for $\tilde{W}_0(t, p)$ from [28, Theorem 2.2], we obtain our next ergodic equality

$$V_0(ts, \lambda) = \tilde{W}_0(t, p), \quad (5.16)$$

where $\lambda = \sqrt{p^2 + m^2}$ and $s = p/\lambda$.

2. Next, consider Dirac-type system:

$$\left(\frac{d}{dr} + a(r)\right)f - \left(\lambda + m - b(r)\right)g = 0, \quad (5.17)$$

$$\left(\frac{d}{dr} - a(r)\right)g + \left(\lambda - m - b(r)\right)f = 0 \quad (m > 0), \quad (5.18)$$

where $a(r) = \overline{a(r)}$ and $b(r) = \overline{b(r)}$. We assume that $a$ and $b$ satisfy the inequalities

$$\int_1^\infty |b'(r)|dr + \int_1^\infty |b^2(r)|dr + \int_0^1 |b(r)|dr < \infty, \quad (5.19)$$

$$\int_0^\infty |a'(r)|dr + \int_1^\infty |a^2(r)|dr + \int_0^\infty |a(r)|dr < \infty. \quad (5.20)$$

Then, there exists a solution $\text{col}[f, g]$ of the system (5.17), (5.18) (written down in matrix form), which satisfies the boundary condition

$$\text{col}[f(0), g(0)] = \text{col}[0, 1]. \quad (5.21)$$

If inequalities (5.19) and (5.20) hold, then there are linearly independent solutions $Z_1$ and $Z_2$ of the Dirac-type system, which admit representations (5.8) and (5.9), respectively. The corresponding deviation factor $V_0(r, \lambda)$, in this case, has the form

$$V_0(r, \lambda) = \exp \left(\frac{i\lambda}{\eta} \int_1^r b(u)du\right)I_2, \quad (5.22)$$

where (similar to the equality (5.10)) $\eta = \sqrt{\lambda^2 - m^2}$. The next proposition follows.

Proposition 5.5 Let inequalities (5.19) and (5.20) hold. Then formulas (5.13) and (5.15) are valid for the Dirac-type case as well.
6 Generalized dynamical scattering problem (Dirac-type system)

1. Introduce the Dirac operator $\mathcal{L}$ acting in $L^2_2(0, \infty)$:

$$\left( \mathcal{L}\Psi \right)(r) = \left( J \frac{d}{dr} + B(r) \right) \Psi(r), \quad (6.1)$$

where

$$J := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad B(r) := \begin{bmatrix} m + b(r) & a(r) \\ a(r) & -m + b(r) \end{bmatrix}. \quad (6.2)$$

Then, we can rewrite Dirac-type system (5.17), (5.18) in the form

$$\mathcal{L}\Psi = \lambda\Psi, \quad \Psi = \begin{bmatrix} f \\ g \end{bmatrix}. \quad (6.3)$$

The operator $\mathcal{L}_0$ is introduced by the equality

$$\left( \mathcal{L}_0\Psi \right)(r) = \left( J \frac{d}{dr} + B_0(r) \right) \Psi(r), \quad B_0 := \begin{bmatrix} m & 0 \\ 0 & -m \end{bmatrix}. \quad (6.4)$$

The boundary condition for the operators $\mathcal{L}_0$ and $\mathcal{L}$ has the form

$$f(0) = 0. \quad (6.5)$$

Next, we determine $a$ and $b$ on the semiaxis $(-\infty, 0)$ by the equalities $a(-r) = a(r)$ and $b(-r) = b(r)$ ($r > 0$), and consider operators $\mathcal{L}$ and $\mathcal{L}_0$ given by (6.1), (6.2) and by (6.4), respectively, and acting in $L^2_2(-\infty, \infty)$. The operator $\mathcal{L}_0$ in the momentum representation has the form

$$\left( \tilde{\mathcal{L}}_0\tilde{\Psi} \right)(p) = H_0(p)\tilde{\Psi}(p), \quad H_0(p) := \begin{bmatrix} -m & -ip \\ ip & m \end{bmatrix}. \quad (6.6)$$

where $\tilde{\Psi}(p) \in L^2_2(-\infty, \infty)$. The matrix $H_0(p)$ admits representation

$$H_0(p) = U(p)D(p)U(p)^{-1}, \quad (6.7)$$

where

$$U(p) = c \begin{bmatrix} is & 1 \\ 1 & is \end{bmatrix}, \quad D(p) = \begin{bmatrix} \mu & 0 \\ 0 & -\mu \end{bmatrix}, \quad (6.8)$$

$$\mu = \sqrt{p^2 + m^2}, \quad s = p/\mu, \quad c = (1 + s^2)^{-1/2}. \quad (6.9)$$
Applying to the radial Dirac equation the approach, which V.S. Buslaev and V.B. Matveev used for the Schrödinger (see [10]), we obtain the following important theorem.

**Theorem 6.1** Let \( a(r) \) and \( b(r) \) satisfy (for \( r > 0 \), fixed values \( \alpha \) and \( \gamma \), and for \( \nu \) taking values \( 0, 1 \) and \( 2 \)) the inequalities

\[
\left( |a^{(\nu)}(r)| + |b^{(\nu)}(r)| \right) \leq C(1 + r)^{-\alpha - \nu}, \quad 1 > \alpha > 3/4; \tag{6.10}
\]

\[
|a(r)| < C(1 + r)^{-1 - \alpha + \gamma}, \quad 1 - \alpha < \gamma < \alpha - 1/2. \tag{6.11}
\]

Then the generalized wave operators \( W_{\pm}(L, L_0) \) exist and the corresponding deviation factor \( W_0(t) \) in the momentum representation has the form

\[
\tilde{W}_0(t, p) = \exp \left( i \text{sgn}(t) \int_1^{\frac{|t|}{b(rs)}} b(rs)dr \right) I_2. \tag{6.12}
\]

**Proof.** First introduce the operator function

\[
\Theta(t) = \exp(itL) \exp(-itL_0)W_0(t). \tag{6.13}
\]

In order to prove the theorem, it suffices to show that the equality

\[
\left\| \frac{d\Theta(t)}{dt} \Psi \right\| = O(t^{1+\varepsilon}) \quad (\varepsilon > 0, \ t \to \infty) \tag{6.14}
\]

holds (for \( t > 0 \)) on some set which is dense in \( L^2_2(-\infty, \infty) \). Introduce the set \( S \) of vector functions \( \Psi \in L^2_2(-\infty, \infty) \) such that their Fourier transforms (i.e., images in the momentum space)

\[
\tilde{\Psi}(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ipr} \Psi(r)dr \tag{6.15}
\]

belong to the class \( C^\infty \) and

\[
\text{supp}(\tilde{\Psi}) \subset \{ p : 0 < c_1(\Psi) < |p| < c_2(\Psi) < \infty \}. \tag{6.16}
\]

Further we consider \( \Psi \in S \) of the forms \( \Psi = \text{col}[h, 0] \) and \( \Psi = \text{col}[0, h] \) with Fourier transforms \( \tilde{\Psi} = \text{col}[\tilde{h}, 0] \) and \( \tilde{\Psi} = \text{col}[0, \tilde{h}] \), respectively. Using momentum representation of \( \frac{d\Psi}{dt} \) (for \( \Psi \) mentioned above) we obtain

\[
\left\| \left( \frac{d\Theta}{dt} \Psi \right)(r, t) \right\| \leq \|J_1(r, t)\| + \|J_2(r, t)\|, \tag{6.17}
\]

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where

\[ J_1(r, t) = \int_{-\infty}^{+\infty} (b(r) - b(zt)) \exp\{itF(p, r, t)\} \tilde{h}(p) dp, \quad (6.18) \]

\[ J_2(r, t) = \int_{-\infty}^{+\infty} a(r) \exp\{itF(p, r, t)\} \tilde{h}(p) dp. \quad (6.19) \]

Here \( s \) is given in (6.9) and

\[ F(p, r, t) = pr/t - \mu - \left(1/t\right) \int_1^t b(us) du. \quad (6.20) \]

Integrating the integrals in (6.18) and (6.19) by parts, we obtain (in the same way it was done in [10]) the inequalities

\[ \left( \int_{|r|\leq \varepsilon t} |J_k^2(r, t)| dr \right)^{1/2} \leq C_1 t^{-(\alpha+1/2)} \quad (k = 1, 2), \quad (6.21) \]

which hold for some \( C_1 > 0 \). Now, let us estimate the function \( J_1(r, t) \) in the domain \( |r| \geq \varepsilon t \). The stationary-phase point \( p_0(r, t) \) is the solution of the equation

\[ \frac{\partial}{\partial p} F(p, r, t) = 0. \quad (6.22) \]

Thus, we have

\[ r - ts - \frac{ds}{dp} \int_1^t ub'(us) du = 0 \quad \left( b'(z) = \frac{db}{dz} \right), \quad (6.23) \]

where \( s(p) \) is given in (6.9). Using (6.10), we rewrite (6.23) in the form

\[ r - ts + O(t^{1-\alpha}) = 0. \quad (6.24) \]

Hence, the stationary-phase point \( p_0(r, t) \) is such that

\[ ts_0(r, t) = r + O(t^{1-\alpha}) \quad \left( s_0 = p_0/\mu_0 = p_0/\sqrt{p_0^2 + m^2} \right), \quad (6.25) \]

Let us consider the case when \( p > 0 \) and \( t > 0 \). We say that \( s \) belongs to the domain \( \Delta(r, t) \) if

\[ |r - st| \leq C_2 t^\gamma \quad (6.26) \]
for some fixed $\gamma$ satisfying (6.11) (and for some fixed $C_2 > 0$). In particular, the point $s_0 = p_0/\sqrt{p_0^2 + m^2}$ belongs to $\Delta(r, t)$ for all sufficiently large values of $t$. Using (6.10) and the Lagrange’s mean value equality

$$b(r) - b(st) = b'(\xi)(r - st),$$

we obtain

$$|b(r) - b(st)| = O(t^{-1-\alpha+\gamma}), \quad s \in \Delta(r, t).$$

When $s \in \Delta(r, t)$, relations (6.11) and (6.28) can be written in the form

$$|a(r)| + |b(r) - b(st)| = O\left(\left((r + 1)t\right)^{-1/2-\alpha/2+\gamma/2}\right), \quad s \in \Delta(r, t).$$

Similar to (6.18) and (6.19), we introduce the integrals $J_k$ on $\Delta$:

$$J_1(\Delta) = \int_{\Delta} (b(r) - b(st)) \exp\{itF(p, r, t)\}\tilde{h}(p)dp,$$

$$J_2(\Delta) = \int_{\Delta} a(r) \exp\{itF(p, r, t)\}\tilde{h}(p)dp.$$

It follows from (6.29) that

$$\left(\int_{r \geq \varepsilon t} |J_k^2(\Delta(r, t))| dr\right)^{1/2} \leq C_3 t^{\gamma - \alpha - 1/2} \quad (k = 1, 2),$$

where, according to (6.11), we have

$$\gamma - \alpha - 1/2 < -1.$$  

Now, consider domain $\tilde{\Delta}(r, t)$ of values of $s$ such that

$$s \notin \Delta(r, t), \quad |r| \geq \varepsilon t.$$  

We write down $J_1(\tilde{\Delta}(r, t))$ in the form

$$J_1(\tilde{\Delta}(r, t)) = J_{11}(\tilde{\Delta}(r, t)) - J_{12}(\tilde{\Delta}(r, t)),$$

$$J_{11}(\tilde{\Delta}(r, t)) := \int_{\tilde{\Delta}(r, t)} b(r) \exp\{itF(p, r, t)\}\tilde{h}(p)dp,$$

$$J_{12}(\tilde{\Delta}(r, t)) := \int_{\tilde{\Delta}(r, t)} b(st) \exp\{itF(p, r, t)\}\tilde{h}(p)dp.$$
Integrating $J_{11}$ and $J_2$ by parts, we obtain the inequalities

$$|J_{11}(\tilde{\Delta}(r,t))| \leq |b(r)|t^{-1-\gamma}, \quad |J_2(\tilde{\Delta}(r,t))| \leq |a(r)|t^{-1}. \quad (6.38)$$

It follows from (6.10) and (6.38) that

$$\left( \int_{r \geq \epsilon t} |J_{11}^2(\tilde{\Delta}(r,t))| \, dr \right)^{1/2} + \left( \int_{r \geq \epsilon t} |J_2^2(\tilde{\Delta}(r,t))| \, dr \right)^{1/2} \leq C_4 t^{-1/2-\alpha}. \quad (6.39)$$

Next, integrating by parts the expression $J_{12}$, we derive

$$J_{12}(\tilde{\Delta}(r,t)) = \frac{1}{t} \int_{\tilde{\Delta}(r,t)} B(s,r,t) \exp\{ipr\} \, dp, \quad (6.40)$$

where $B(s,r,t) = \frac{\partial}{\partial p} \left( b(st)\tilde{\eta}(p) / \frac{\partial}{\partial p} F(p,r,t) \right)$, and so

$$|B(s,r,t)| \leq C_5 t^{-\alpha}(t/r). \quad (6.41)$$

Hence, we have

$$\left( \int_{r \geq \epsilon t} |J_{12}^2(\tilde{\Delta}(r,t))| \, dr \right)^{1/2} \leq C_6 t^{-1/2-\alpha}. \quad (6.42)$$

Finally, by virtue of (6.17), (6.21), (6.32), (6.33), (6.39) and (6.42), we easily obtain (6.14). The theorem is proved.

Comparing (5.22) and (6.12), we see that again (similar to the non-relativistic case) the stationary deviation factor $V_0(r,\lambda)$ and the dynamic deviation factor $\tilde{W}_0(t,p)$ are connected by a simple equality

$$V_0(ts,\sqrt{p^2+m^2}) = \mathcal{C}(p)\tilde{W}_0(t,p) \quad (t > 0), \quad (6.43)$$

where $|\mathcal{C}(p)| = 1$. Equality (6.43) shows that the class of deviation factors for a Dirac-type stationary problem coincides with the class of deviation factors for the corresponding dynamical problem.
7 Dirac equation in $\mathbb{R}^4$

1. The classical Dirac equation in the space of four variables has the form

$$\left( -e\varphi(x,t)I_4 + mc^2\beta + \sum_{n=1}^{3} \alpha_n(cp_n + eA_n(x,t)) \right) \psi(x,t) = i\hbar \frac{\partial \psi(x,t)}{\partial t},$$

(7.1)

where $\psi(x,t)$ is the wave $4 \times 1$ vector function for the particle of the rest mass $m$, $c$ is the speed of light, $\hbar$ is the Planck constant divided by $2\pi$, and $t$ are the space-time coordinates, $p_k = -i\hbar \frac{\partial}{\partial x_k}$, $e$ is the charge of the particle, $\varphi$ is a scalar potential, $A = \text{col}[A_1, A_2, A_3]$ is a vector potential. The Hermitian $4 \times 4$ matrices $\alpha_k$ and $\beta$ satisfy the relations

$$\alpha_k^2 = \beta^2 = I_4; \quad \alpha_k \beta + \beta \alpha_k = 0; \quad \alpha_k \alpha_\ell + \alpha_\ell \alpha_k = 0 \quad (k \neq \ell).$$

(7.2)

Without loss of generality we put

$$\begin{align*}
\beta &= \begin{bmatrix} I_2 & 0 \\ 0 & -I_2 \end{bmatrix}, & \alpha_1 &= \begin{bmatrix} 0 & I_2 \\ I_2 & 0 \end{bmatrix}, & \alpha_k &= \begin{bmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{bmatrix} \quad (k = 2, 3), \\
\sigma_2 &= \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, & \sigma_3 &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},
\end{align*}$$

(7.3)

where $\sigma_k$ are Pauli matrices. It is convenient to rewrite (7.1) in the form

$$\left( mc^2\beta + c \sum_{n=1}^{3} \alpha_n p_n + V(x,t) \right) \psi(x,t) = i\hbar \frac{\partial \psi(x,t)}{\partial t},$$

(7.5)

where the potential $V$ is given by

$$V(x,t) = -e\varphi(x,t)I_4 + e \sum_{n=1}^{3} \alpha_n A_n(x,t).$$

(7.6)

2. Dirac equation, momentum representation. Further we use the natural units measure, that is, $c = \hbar = 1$. In the momentum space, Dirac equation takes the form

$$H(q)\Phi(t,q) + \int_{\mathbb{R}^3} U(q - r)\Phi(t,r)dr = i \frac{\partial \Phi(t,q)}{\partial t},$$

(7.7)
where \( x = (x_1, x_2, x_3) \), \( q = (q_1, q_2, q_3) \), \( r = (r_1, r_2, r_3) \).

\[
H(q) = \begin{bmatrix}
  m & 0 & q_3 & q_1 - iq_2 \\
  0 & m & q_1 + iq_2 & -q_3 \\
  q_3 & q_1 - iq_2 & -m & 0 \\
  q_1 + iq_2 & -q_3 & 0 & -m
\end{bmatrix}.
\] (7.8)

\[
\Phi(t, q) = \int_{\mathbb{R}^3} e^{ix \cdot \psi(x, t)} dx, \quad U(q) = \int_{\mathbb{R}^3} e^{ix \cdot V(x)} dx.
\] (7.9)

The eigenvalues \( \lambda_k \) and the corresponding eigenvectors \( g_k \) of \( H(q) \) are important, and we find them below:

\[
\lambda_{1,2} = -\sqrt{m^2 + |q|^2}, \quad \lambda_{3,4} = \sqrt{m^2 + |q|^2} \quad (|q|^2 := q_1^2 + q_2^2 + q_3^2); \quad (7.10)
\]

\[
g_1 = \begin{bmatrix}
  (-q_1 + iq_2)/(m + \lambda_3) \\
  q_3/(m + \lambda_3) \\
  0 \\
  1
\end{bmatrix}, \quad
g_2 = \begin{bmatrix}
  -q_3/(m + \lambda_3) \\
  (-q_1 - iq_2)/(m + \lambda_3) \\
  1 \\
  0
\end{bmatrix}, \quad (7.11)
\]

\[
g_3 = \begin{bmatrix}
  (-q_1 + iq_2)/(m - \lambda_3) \\
  q_3/(m - \lambda_3) \\
  0 \\
  1
\end{bmatrix}, \quad
g_4 = \begin{bmatrix}
  -q_3/(m - \lambda_3) \\
  (-q_1 - iq_2)/(m - \lambda_3) \\
  1 \\
  0
\end{bmatrix}. \quad (7.12)
\]

We need the linear spans of some sets \( \{g_k\} \) of vectors \( g_k \) and we introduce the following notations:

\[
M_1(q) = \text{Span}\{g_k(q) : k = 1, 2\}, \quad M_2(q) = \text{Span}\{g_k(q) : k = 3, 4\}. \quad (7.13)
\]

We shall use the fact that the eigenvectors \( g_1(q) \) and \( g_2(q) \) of \( H(q) \) have the same eigenvalues \( \lambda_{1,2} = -\sqrt{m^2 + |q|^2} \), and the eigenvectors \( g_3(q) \) and \( g_4(q) \) have the same eigenvalues \( \lambda_{3,4} = \sqrt{m^2 + |q|^2} \).

3. Let us introduce the self-adjoint operators

\[
A_0 \Phi(q) = H(q) \Phi(q), \quad A \Phi(q) = H(q) \Phi(q) + \int_{\mathbb{R}^3} U(q, r) \Phi(r) dr. \quad (7.14)
\]
Equations (7.7) form a subclass of the class of equations
\[ i \frac{\partial \Phi(t, q)}{\partial t} = A \Phi(t, q). \]  
(7.15)

**Theorem 7.1**  
Let the generalized wave operators \( W_\pm(A, A_0) \) exist. Then the corresponding scattering operator \( S(q) \) and the deviation factor \( \tilde{W}_0(t, q) \) have the following structure:

\[ S(q) = \sum_{n=1}^{2} s_n(q), \quad \tilde{W}_0(t, q) = \sum_{n=1}^{2} w_n(t, q), \]  
(7.16)

where

\[ s_n(q) = P_n(q)S(q)P_n(q), \quad w_n(t, q) = P_n(q)\tilde{W}_0(t, q)P_n(q), \]

\( P_n(q) \) is the orthogonal projector of the space \( \mathbb{C}^4 \) onto \( M_n(q) \), and \( M_n(q) \) \( (n = 1, 2) \) are introduced in (7.13).

We note that \( S(q) \) and \( \tilde{W}(t, q) \) are unitary transformations. Hence, the transformations \( s_n(q) \) and \( w_n(t, q) \) are unitary transformations in the spaces \( M_n(q) \) \( (n = 1, 2) \).

### 8 Scattering operator in quantum electrodynamics

1. The fundamental equation of quantum electrodynamics (often called interaction picture \[1\] p. 273, see also \[30\]) has the form

\[ i \frac{\partial \Phi(t)}{\partial t} = \varepsilon V(t)\Phi(t), \]  
(8.1)

where \( \Phi(t) \) is the wave operator function which describes the state of the field at the time \( t \) and \( \varepsilon \) is the small parameter. Rewrite (8.1) in the following way:

\[ \Phi(t) = S(t, t_0)\Phi(t_0); \quad i \frac{\partial S(t, t_0)}{\partial t} = \varepsilon V(t)S(t, t_0), \quad S(t_0, t_0) = I. \]  
(8.2)
Now, we consider the series expansion of $S(t, t_0)$ and write down the recursive formulas for coefficients:

$$ S(t, t_0) = \sum_{k=0}^{\infty} \varepsilon^k S_k(t, t_0); \quad S_0(t, t_0) = I, \quad (8.3) $$

$$ S_k(t, t_0) = -i\varepsilon \int_{t_0}^{t} V(u) S_{k-1}(u, t_0) du, \quad k > 0. \quad (8.4) $$

2. Let $\Phi(-\infty)$ and $\Phi(+\infty)$ be the operators which describe the states of the field at the time $t = -\infty$ and $t = +\infty$, respectively. In view of (8.2) we have

$$ \Phi(+\infty) = S(+\infty, -\infty) \Phi(-\infty), \quad (8.5) $$

where $S(+\infty, -\infty)$ is the scattering operator. It is often assumed that the initial and final states of the system are free, that is, $V(\pm\infty) = 0$. However, in many important cases the initial and final states are not free, and we shall show that in these cases the theory of generalized wave and generalized scattering operators is useful. In particular, the deviation factors $W_-(t)$ and $W_+(t)$ describe the deviation of the initial and final states from the free state.

3. Dirac operators in the presence of the electromagnetic fields (and in momentum representations) have the form [1, Ch. IV]:

$$ A_0 \Phi(q) = \hat{H}(q) \Phi(q), \quad A \Phi(q) = \hat{H}(q) \Phi(q) + \int U(q, r) \Phi(r) dr, \quad (8.6) $$

where

$$ \hat{H}(q) = \begin{bmatrix} H(q) & 0 \\ 0 & H(q) \end{bmatrix}, \quad (8.7) $$

and $H(q)$ is given by (7.8). The fundamental equation (Schrödinger picture) in the momentum space has the form

$$ i \frac{\partial \Phi(q, t)}{\partial t} = A \Phi(q, t). \quad (8.8) $$

Further we consider the scattering matrix $S(+\infty, -\infty)$ in the momentum representation. In order to consider $S(q)$ we recall the results on eigenvectors
of $H(q)$ in Section 7 and easily write down the eigenvectors $G_k \in \mathbb{C}^8$ of $\hat{H}(q)$:

$$G_k = \begin{bmatrix} g_k \\ 0 \end{bmatrix} \quad \text{for } 1 \leq k \leq 4; \quad G_k = \begin{bmatrix} 0 \\ g_{k-4} \end{bmatrix} \quad \text{for } 5 \leq k \leq 8.$$ (8.9)

where $g_k \in \mathbb{C}^4$ are given by (7.11) and (7.12). We set

$$N_1(q) = \text{Span}\{G_k : k = 1, 2, 5, 6\},$$ (8.10)

$$N_2(q) = \text{Span}\{G_k : k = 3, 4, 7, 8\}.$$ (8.11)

Clearly, the eigenvectors $G_1(q), G_2(q), G_5(q)$ and $G_6(q)$ have equal eigenvalues $\lambda_{1,2,5,6} = -\sqrt{m^2 + |q|^2}$, and the eigenvectors $G_3(q), G_4(q), G_7(q)$ and $G_8(q)$ have equal eigenvalues $\lambda_{3,4,7,8} = \sqrt{m^2 + |q|^2}$.

**Theorem 8.1** Let the generalized wave operators $W_{\pm}(A, A_0)$, where $A$ and $A_0$ are given by (8.6), exist. Then the corresponding scattering operator $S(q)$ and the deviation factor $\tilde{W}_0(t, q)$ have the following structure:

$$S(q) = s_1(q) + s_2(q),$$ (8.12)

$$\tilde{W}_0(t, q) = w_1(t, q) + w_2(t, q),$$ (8.13)

where

$s_n(q) = \hat{P}_n(q)S(q)\hat{P}_n(q), \quad w_n(t, q) = \hat{P}_n(q)\tilde{W}_0(t, q)\hat{P}_n(q),$

$\hat{P}_n(q)$ is the orthogonal projector of the space $\mathbb{C}^8$ onto $N_n(q)$, and $N_n(q)$ ($n = 1, 2$) are introduced in (8.10) and (8.11).

We note that $S(q)$ and $\tilde{W}(t, q)$ are unitary transformations. Hence, the transformations $s_n(q)$ and $w_n(t, q)$ are unitary transformations in the spaces $N_n(q)$. Further we shall use the next corollary.

**Corollary 8.2** Assume that there are such $k$ and $\ell$ ($1 \leq k, \ell \leq 8$) that all the entries of the $k$-th row and of the $\ell$-th column of the generalized scattering matrix $S(q)$ (excluding $s_{k\ell}(q)$) equal zero. Then $s_{k\ell}(q)$ satisfies the equality

$$|s_{k\ell}(q)| = 1.$$ (8.14)
9 Example, logarithmic type singularity

Consider the series expansion of the scattering function:

\[ S(t, \tau, q) = \sum_{k=0}^{\infty} \varepsilon^k S_k(t, \tau, q). \]

(9.1)

The following conditions are fulfilled for a number of problems in quantum electrodynamics (see [1] and [6]):

1) The function \( S_1(t, \tau, q) \) is uniformly bounded when both inequalities \( t > 1 \) and \( \tau < -1 \) hold.

2) The function \( S_2(t, \tau, q) \) has the form

\[ S_2(t, \tau, q) = \varepsilon^2 \left( \varphi(q) \ln |t\tau| + O(1) \right), \quad t \to \infty, \quad \tau \to -\infty. \]

(9.2)

In other words, the function \( S_2(t, \tau, q) \) has a logarithmic type singularity.

Let the conditions 1) and 2) above be fulfilled, and introduce the functions

\[ \tilde{W}_+(t, q) = t^{i\varphi(q)} \text{ (} t > 0 \text{)}, \quad \tilde{W}_-(\tau, q) = |\tau|^{-i\varphi(q)} \text{ (} \tau < 0 \text{)}, \]

(9.3)

Similar to (9.1), the generalized scattering function

\[ \tilde{S}(t, \tau, q) = \tilde{W}_+(t, q)^* S(t, \tau, q) \tilde{W}_-(\tau, q) \]

(9.4)

admits a series expansion:

\[ \tilde{S}(t, \tau, q) = \sum_{k=0}^{\infty} \varepsilon^k \tilde{S}_k(t, \tau, q). \]

(9.5)

It is easy to see that the following assertion is valid.

**Proposition 9.1** The function \( \tilde{S}_2(t, \tau, q) \) is uniformly bounded when both inequalities \( t > 1 \) and \( \tau < -1 \) hold.
Remark 9.2 Earlier we proved the absence of divergences in some important cases of Schrödinger and Dirac equations with Coulomb potentials (see [27, 28] as well as Remark 3.1 here). Moreover, the existence of the generalized wave operators implies the existence of the generalized scattering operator and absence of divergences in the cases considered in Proposition 4.4 and in Theorem 8.1.

Below we consider one of the examples where the coefficient $S_2(t, \tau, q)$ (of the non-generalized scattering function) has a logarithmic type singularity and the condition (9.2) is fulfilled.

Example 9.3 Introduce operators

$$L_0 f = x^2 f, \quad f \in L^2(0, \infty),$$

$$L f = x^2 f + \int_0^\infty f(y)R(x, y)dy,$$

where

$$R(x, y) = \varepsilon^2 p(x)p(y) \ln |x^2 - y^2|, \quad p(x) > 0.$$  (9.8)

Proposition 9.4 Let $p(x)$ satisfy the condition

$$|p(x_2) - p(x_1)| \leq C|x_2 - x_1|^\alpha, \quad \alpha > 0.$$  (9.9)

Then the equality (9.2) is valid for $S_2(t, \tau, q)$ corresponding to the operators $L_0$ and $L$ above and for $\varphi(q) = -\pi p^2(q)/(2q)$.

Proof. It is easy to see that

$$S_2(t, \tau)f = -i \int_\tau^t \int_0^\infty f(y)R(x, y)e^{i(x^2 - y^2)t_1} dy dt_1.$$  (9.10)

Let the function $f(x)$ be differentiable and such that $f(x) = 0$ for all $x \notin [0, M]$ (with some fixed $M > 0$). Introduce new variables $v = x^2$ and $u = y^2$. Using (9.9) and (9.10) we obtain

$$S_2(t, \tau)f \sim f(x)\frac{p^2(x)}{2x} \int_0^{\sqrt{M}} \left( \ln |u - v| \right) \frac{e^{i(\tau - v)t} - e^{i(\tau - v)t_1}}{u - v} dv.$$  (9.11)
We have
\[
\int_0^{\sqrt{M}} (\ln |u - v|) e^{i(u-v)t} - e^{i(u-v)\tau} \frac{dv}{u - v} = \int_{u - \sqrt{M}}^u (\ln |\xi|) e^{i\xi t} - e^{i\xi \tau} \frac{d\xi}{\xi}. \tag{9.12}
\]

The assertion of the proposition follows from (9.11), (9.12) and the equality:
\[
\int_0^\infty \frac{\sin \xi}{\xi} d\xi = \pi/2. \tag{9.13}
\]

\section{10 Series expansions of the entries of scattering matrices}

In cases of various diagrams, there is a series expansion
\[
s_{k\ell}(q, L, \varepsilon) = 1 + \varepsilon a_1(q) + \varepsilon^2 a_2(q, L) + \ldots, \tag{10.1}
\]
where
\[
a_2(q, L) = \int_\Omega F(p, q) d^4 p, \tag{10.2}
\]
\(\Omega\) is a four dimensional sphere (invariant region of integration) with radius \(L\), \(p = [-ip_0, p_1, p_2, p_3]\), and \(F\) is a rational function with respect to \(p\) and \(q\) (see, e.g., [11] p. 631] and [14]). We shall consider the cases where the limit on the right-hand side of (10.2) does not exist when \(L\) tends to infinity.

\textbf{Example 10.1} Let the relation
\[
a_2(q, L) = \int_\Omega F(p, q) d^4 p = i(\phi(q) \ln L + \psi(q) + o(1)) \tag{10.3}
\]
be valid for some real valued functions \(\phi(q)\) and \(\psi(q)\). Then, the integral in (10.3) diverges logarithmically, and the second term
\[
a_2(q) := \lim_{L \to \infty} \int_\Omega F(p, q) d^4 p \tag{10.4}
\]
in the power series

$$s_{k\ell}(q, \varepsilon) = 1 + \varepsilon a_1(q) + \varepsilon^2 a_2(q) + \ldots,$$

(10.5)

which corresponds to the entry $s_{k\ell}$ of the non-generalized scattering function, is equal to infinity.

In order to remove this divergence, we introduce

$$\tilde{s}_{k\ell}(q, L, \varepsilon) = L^{-\frac{i\varepsilon^2}{2}\phi(q)} s_{k\ell}(q, L, \varepsilon).$$

(10.6)

Using (10.1) and (10.6) we have

$$\tilde{s}_{k\ell}(q, L, \varepsilon) = 1 + \varepsilon a_1(q) + \varepsilon^2 \left( a_2(q, L) - i\phi(q) \ln L \right) + \ldots$$

(10.7)

It follows from (10.3) that the second term

$$\tilde{a}_2(q, L) := a_2(q, L) - i\phi(q) \ln L$$

(10.8)

of the power series (10.7) converges when $L \to \infty$.

We emphasize that

$$|s_{k\ell}(q, L, \varepsilon)| = |\tilde{s}_{k\ell}(q, L, \varepsilon)|.$$  

(10.9)

Remark 10.2 The factor $U_0(q, L, \varepsilon) = L^{\frac{i\varepsilon^2}{2}\phi(q)}$ is an analogue of the deviation factor $\tilde{W}_0(t, q)$ which we considered in the previous sections. Namely, the factor $U_0(q, L, \varepsilon)$ describes the deviation of the initial and final states of a system from the free state.

Remark 10.3 It is well known that condition (10.3) is fulfilled for many problems arising in the theory of collisions of particles.

Example 10.4 Let the relation

$$a_2(q, L) = \int_{\Omega} F(p, q) d^4 p = i(\phi(q)L^2 + \psi(q)L + \nu(q) \ln L + \mu(q) + o(1)),$$

(10.10)
where $\phi, \psi, \nu$ and $\mu$ are real valued functions, be valid. In this (more general) case, the factor $U_0(q, L, \varepsilon)$ has the form

$$U_0(q, L, \varepsilon) = e^{i\varepsilon^2(\phi(q)L^2 + \psi(q)L)}L^{i\varepsilon^2\nu(q)}. \quad (10.11)$$

Using $U_0$, we again introduce $\tilde{s}_{k\ell}$:

$$\tilde{s}_{k\ell}(q, L, \varepsilon) = U_0(q, L, \varepsilon)^{-1}s_{k\ell}(q, L, \varepsilon). \quad (10.12)$$

Now, in view of $(10.10)$, relations $(10.7)$ and $(10.8)$ take the forms:

$$\tilde{s}_{k\ell}(q, L, \varepsilon) = 1 + \varepsilon a_1(q) + \varepsilon^2 \tilde{a}_2(q, L) + ... \quad (10.13)$$

$$\tilde{a}_2(q, L) = a_2(q, L) - i(\phi(q)L^2 + \psi(q)L + \nu(q)\ln L) \quad (10.14)$$

It follows from $(10.10)$ and $(10.14)$ that the second term in the power series $(10.13)$ converges when $L \to \infty$.

Relation $(10.9)$ also holds for Example 10.4.

**Remark 10.5** In the cases of all irreducible diagrams, the relation $(10.10)$ is valid (see [1], Sect. 46, 47).

The simplest subcase of Example 10.4 is obtained by setting

$$\phi(q) = 0, \quad \nu(q) = 0, \quad \psi(q) = 1. \quad (10.15)$$

In this subcase we have

$$U_0(q, L, \varepsilon) = U_0(L, \varepsilon) = e^{i\varepsilon^2 L}. \quad (10.16)$$

### 11 Conclusion

We have shown that generalized scattering operators have an interesting inner structure (see Theorems 7.1 and 8.1). We believe that this structure (e.g., relation $(8.12)$) may be checked experimentally.

In Remark 3.1 and in Sections 9 and 10 we discuss the absence of divergences when one uses generalized scattering operators instead of the classical
scattering operators. Thus, generalized scattering operators present an important tool to avoid divergences. Moreover, generalized scattering operators exist for the cases where the initial and final states of the system are not free, and the classical scattering operators do not exist.

The dynamical and stationary deviation factors $W_0(t)$ and $V_0(r)$ describe the deviation of the initial and final states of the corresponding system from the free state. This fact can be checked, I think, by experimental way. Deviation factors are not uniquely defined. The non-uniqueness is contained in constant operator multipliers (multipliers $C_\pm$ for the case of $W_0(t)$ and multiplier $C$ for $V_0(r)$). The choice of multipliers $C_\pm$ and $C$ depends on the particular physical problem under consideration.

The following principles of choosing the constant multipliers $C_\pm$ and $C$ can be formulated for radial Schrödinger and Dirac equations. In the case of radial Schrödinger equations (considered in Sections 3 and 4), the operators $C_\pm$ and $C$ should not depend on parameter $\ell$. In the case of radial Dirac equation (considered in Section 5), the operators $C_\pm$ and $C$ should not depend on parameter $k$. Then, the effective cross section of the scattering does not depend on these multipliers.

Important ergodic interrelations between dynamical and stationary deviation factors are given in formulas (4.19), (5.16) and (6.43). We suppose that these interrelations may be confirmed experimentally and that similar interrelations hold in other important cases.

Recall that $S_{st}$ and $S_{dyn}$ stand for the generalized stationary and dynamical, respectively, scattering operators.

**Open problem.** Prove the equality

$$S_{st} = S_{dyn}.$$  \hfill (11.1)

We proved \cite{27, 28} the equality (11.1) for the cases of Schrödinger and Dirac equations with Coulomb potentials.

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