Weak perturbations of shock waves.

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Introduction.

The Cauchy problem is considered for the perturbed strictly hyperbolic 2x2 system of quasilinear equations

\[
\begin{cases}
  u_t + \lambda(u)u_x = \varepsilon f(u, v), \quad u|_{t=0} = \tilde{u}(x), \\
  v_t + \mu(u, v)v_x = \varepsilon g(u, v), \quad v|_{t=0} = \tilde{v}(x), \quad x \in \mathbb{R}.
\end{cases}
\]

Here \( \varepsilon \) is a small parameter. We assume, that the initial data \( \tilde{u}(x) \), \( \tilde{v}(x) \) have jumps at \( x = 0 \). The unperturbed problem (with \( \varepsilon = 0 \)) has a persistent solution with two discontinuity lines (shock waves \( \tilde{1} \)). Well-known Hugoniot conditions are necessary for a uniqueness of the solution of the problem (1). Both an asymptotics of shock waves position in the plane \((x,t)\) and an asymptotics of the perturbed problem solution are discussed, when \( \varepsilon \to 0 \).

In this case the asymptotic formula

\[
s^\pm(t, \varepsilon) \sim \sum_{i=0}^{\infty} \varepsilon^i s^\pm_i(t), \quad \varepsilon \to 0.
\]

holds for the discontinuity lines, and

\[
u(x, t, \varepsilon) \sim \sum_{i=0}^{\infty} \varepsilon^i u_i(x, t), \quad v(x, t, \varepsilon) \sim \sum_{i=0}^{\infty} \varepsilon^i v_i(x, t), \quad \varepsilon \to 0
\]

hold for the solution in the continuity domain. Here the leading terms \( s^\pm_0(t) \), \( u_0(x, t) \) and \( v_0(x, t) \) are the shock waves and the solution of the unperturbed problem respectively. The corrections in the asymptotic solution \( s^\pm_i(t) \), \( u_i(x, t) \), \( v_i(x, t) \) are found from linear equations.

We will use the following domains in the plain \((x, t)\)

\[
D^+ = \{(x, t) \mid t > 0, x > s^+(t, \varepsilon)\}, \quad D^- = \{(x, t) \mid t > 0, x < s^-(t, \varepsilon)\},
\]

\[
\tilde{D} = \{(x, t) \mid t > 0, s^-(t, \varepsilon) < x < s^+(t, \varepsilon)\}.
\]

1 We will name the lines of discontinuity of the solution in the plain \((x, t)\) as shock lines.
Asymptotic solution in domains $D^\pm$.

To obtain an asymptotics in the domains $D^\pm$ we consider two Cauchy problems for the system (1). The initial data are taken in the form

(A) $u|_{t=0} = \tilde{u} (x), |_{t=0} = \tilde{v} (x), \quad x < 0$

(B) $u|_{t=0} = \tilde{u} (x), |_{t=0} = \tilde{v} (x), \quad x > 0$

Each of these problems can be solved (see [1]) in some area. The Cauchy problem (1),(A) has a continuous solution in a domain $\tilde{D}^-$ in the plain $(x,t)$. Similarly, the Cauchy problem (1),(B) has a continuous solution in a domain $\tilde{D}^+$. 

In order to find the unknown coefficients in series (3), we substitute expansions (3) into (1). After that, as usually, we obtain a system of recurrence relations. We have the following Cauchy problems for leading terms and terms in order $\varepsilon$

\[
\begin{align*}
\begin{cases}
u_{0t} + \lambda(u_0)u_{0x} = 0, & u_0|_{t=0} = \tilde{u} (x), \ x > 0(0), \\
v_{0t} + \mu(u_0, v_0)v_{0x} = 0, & v_0|_{t=0} = \tilde{v} (x), \ x > 0(0), \\
u_{1t} + \lambda(u_0)u_{1x} + \lambda_u(u_0)u_{1} u_{0x} = f(u_0, v_0), & u_1|_{t=0} = 0, \ x > 0(0), \\
v_{1t} + \mu(u_0, v_0)v_{1x} + \mu_u(u_0, v_0)u_{1} v_{0x} + \mu_v(u_0, v_0)v_{1} v_{0x} = g(u_0, v_0), & v_0|_{t=0} = 0, \ x > 0(0).
\end{cases}
\end{align*}
\]

Two various Cauchy problems for each system here are written down. In the first case the initial data are taken on the semiaxis $x > 0$. In the second case the initial data are taken on the semiaxis $x < 0$. Thus, the asymptotic solution for the Cauchy problem (1),(2) in the domains $\tilde{D}^\pm$ have been constructed. Moreover, in the same way, we can find all functions $u_i, v_i$ in the domains $\tilde{D}^\pm$.

The following properties take place (see [1]): $D^- < \tilde{D}^-$ and $D^+ < \tilde{D}^+$. Thus the asymptotic solution for the Cauchy problem (1),(2) is found in the domains $D^\pm$.

An asymptotics of shock lines and an asymptotics of the solution in domain $\tilde{D}$.

Let’s remind, that the Hugoniot conditions follow from the conservation laws of system (1). In this case, the Hugoniot conditions looks as follows

\[
\begin{align*}
D^\pm \{u(t, s^\pm(t, \varepsilon), \varepsilon) - \bar{u}(t, s^\pm(t, \varepsilon), \varepsilon)\} &= \Lambda(u(t, s^\pm(t, \varepsilon), \varepsilon)) - \Lambda(\bar{u}(t, s^\pm(t, \varepsilon), \varepsilon)), \\
D^\pm \{\Phi(u(t, s^\pm(t, \varepsilon), \varepsilon), \varphi(t, s^\pm(t, \varepsilon), \varepsilon)) - \Phi(\bar{u}(t, s^\pm(t, \varepsilon), \varepsilon), \bar{\varphi}(t, s^\pm(t, \varepsilon), \varepsilon))\} &= \\
&= \Psi(u(t, s^\pm(t, \varepsilon), \varepsilon), \varphi(t, s^\pm(t, \varepsilon), \varepsilon)) - \Psi(\bar{u}(t, s^\pm(t, \varepsilon), \varepsilon), \bar{\varphi}(t, s^\pm(t, \varepsilon), \varepsilon)).
\end{align*}
\]

Here $\Lambda_u(u) = \lambda(u)$, $D^\pm(t, \varepsilon) = s^\pm(t, \varepsilon)$\cite{footnote}. From the first conservation law we have the equations (5). From the second we have the equations (6). The tilde notes that the value of the discontinuous function is taken from the $\tilde{D}$ domain. As usually, we substitute the

\[\text{---Footnotes---}
\]

\[\text{---Footnotes---}
\]

2The functions $\Phi$ and $\Psi$ are determined by a choice of the conservation laws of the system (1).
expansions (2),(3) into the (6),(7) and use the asymptotic expansion for all functions as \( \varepsilon \to 0 \). After that, collecting terms in order \( \varepsilon^0 \) and \( \varepsilon^1 \) we obtain

\[
D_0^\pm(u_0^\pm - \tilde{u}_0^\pm) = \Lambda(u_0^\pm) - \Lambda(\tilde{u}_0^\pm),
\]

\[
D_0^\pm\{\Phi(u_0^\pm, v_0^\pm) - \Phi(\tilde{u}_0^\pm, \tilde{v}_0^\pm)\} = \Psi(u_0^\pm, v_0^\pm) - \Psi(\tilde{u}_0^\pm, \tilde{v}_0^\pm), \quad (7.1\pm)
\]

\[
D_1^\pm(u_0^\pm - \tilde{u}_0^\pm) + D_1^\pm(u_1^\pm + u_{0x}^\pm s_1^\pm - \tilde{u}_1^\mp - \tilde{u}_{0x}^\pm s_1^\pm) = \Lambda_u(u_0^\pm)(u_1^\pm + u_{0x}^\pm s_1^\pm) - \Lambda_u(\tilde{u}_0^\pm)(\tilde{u}_1^\mp + \tilde{u}_{0x}^\pm s_1^\pm),
\]

\[
D_1^\pm\{\Phi(u_0^\pm, v_0^\pm) - \Phi(\tilde{u}_0^\pm, \tilde{v}_0^\pm)\} + D_0^\pm\{\Phi_u(u_0^\pm, v_0^\pm)(u_1^\pm + u_{0x}^\pm s_1^\pm) + \Phi_v(u_0^\pm, v_0^\pm)(v_1^\pm + v_{0x}^\pm s_1^\pm)\} - \Phi_u(\tilde{u}_0^\pm, \tilde{v}_0^\pm)(\tilde{u}_1^\mp + \tilde{u}_{0x}^\pm s_1^\pm) - \Phi_v(\tilde{u}_0^\pm, \tilde{v}_0^\pm)(\tilde{v}_1^\mp + \tilde{v}_{0x}^\pm s_1^\pm) = 
\]

\[
= \Psi_u(u_0^\pm, v_0^\pm)(u_1^\pm + u_{0x}^\pm s_1^\pm) + \Psi_v(u_0^\pm, v_0^\pm)(v_1^\pm + v_{0x}^\pm s_1^\pm) - \Psi_u(\tilde{u}_0^\pm, \tilde{v}_0^\pm)(\tilde{u}_1^\mp + \tilde{u}_{0x}^\pm s_1^\pm) - \Psi_v(\tilde{u}_0^\pm, \tilde{v}_0^\pm)(\tilde{v}_1^\mp + \tilde{v}_{0x}^\pm s_1^\pm), \quad (7.2\pm)
\]

All function with tilde and functions \( s_1^\pm, D_1^\pm = s_1^\pm \) are unknown here.

Thus, from the Hugoniot conditions, we have four equations for six unknown functions in every order of \( \varepsilon \). Two additional equations we can obtain from differential equations (1). These equations are the same as the equations (4), when we replace both \( u \) and \( v \) on \( \tilde{v} \) and \( \tilde{v} \) respectively. Moreover, it is necessary to add the initial data to the differential equations:

\[
\tilde{u}_1|_{x=s_0^\pm(t)} = \tilde{u}_1(t) \quad \tilde{v}_1|_{x=s_0^\pm(t)} = \tilde{v}_1(t). \quad (8.1)
\]

The leading terms \( \tilde{u}_0, \tilde{v}_0, s_0 \) are the unperturbed solution in domain \( \tilde{D} \).

Without loss of generality we can consider, that \( v_0 \) has a jump on the curve \( x = s_0^+(t) \), and \( u_0 \) is continuous function. Then on the curve \( x = s_0^-(t) \) the function \( u_0 \) has a jump and \( v_0 \) is continuous. Using this fact, we can determine all six unknown functions. For example the procedure of a construction of \( u_1, v_1, s_1^\pm \) in domain \( \tilde{D} \) looks as follows

1) The function \( u_0 \) is continuous on the curve \( x = s_0^+(t) \). Hence we can solve the first equation in (7.2+) to determine \( \tilde{u}_1^+ \), because \( D_0^+, u_1^+ \) are known.

2) To find the function \( u_1 \) inside of the domain \( \tilde{D} \) we must solve the Cauchy problem (4.2) for the first equation with the initial data (8.1). In the same time, we determine the function \( \tilde{u}_1^- \).

3) After that we can obtain the solution of the first equation in (7.2-) to find \( D_1^- \).

4) From the second equation in (7.2-) we can determine \( \tilde{v}_1^- \) in the same way as an item 1.

5) To find the function \( \tilde{v}_1 \) in the area \( \tilde{D} \) and \( \tilde{v}_1^+ \) respectively we must solve the Cauchy problem (4.2) with the initial data (8.1).

6) To obtain \( D_1^+ \) we solve the second equation in (7.2+).

This algorithm is convenient for determination of all coefficients of the asymptotic expansion.

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References

[1] Rogdestvensky B.L. Yanenko N.N. Systems of quasilinear equations. Moskow, ”Nauka”, 1978.