TRANSFERENCE FOR BANACH SPACE REPRESENTATIONS
OF NILPOTENT LIE GROUPS

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Abstract. We establish a general CCR (liminarity) property for uniformly bounded irreducible representations of connected simply connected nilpotent Lie groups on reflexive Banach spaces, extending the well known property of unitary irreducible representations. On this background, we study multipliers and \( L^p \)-boundedness properties of operators obtained by the Weyl-Pedersen transform for unitary irreducible representations of nilpotent Lie groups. The main results apply to all 1-dimensional central extensions of symplectic nilpotent Lie groups. We provide some infinite families of examples, in particular a continuous family of 3-step nilpotent Lie groups that are mutually non-isomorphic, for which our results are applicable. Our method relies upon an abstract approach blending the method of transference from abstract harmonic analysis and a systematic use of spaces of smooth vectors with respect to Lie group representations.

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1. Introduction

The problems addressed in this paper go back to the Weyl multipliers that were introduced in [Man80] and were more recently studied in [BT15]. These are multiplier operators defined with respect to the classical Weyl calculus of pseudo-differential operators, and the background for their construction is provided by the representation theory of the Heisenberg groups, which are precisely the 2-step

2000 Mathematics Subject Classification. Primary 43A15; Secondary 22E25, 22E27.
Key words and phrases. Banach space representation; Lie group; Weyl transform; Pedersen transform; multiplier; twisted convolution; Calderón-Zygmund theory; transference.

This research was partly supported by Project MTM2013-42105-P, fondos FEDER, Spain. The two first-named authors have also been supported by a Grant of the Romanian National Authority for Scientific Research, CNCS-UEFISCDI, project number PN-II-ID-PCE-2011-3-0131. The third-named author has also been supported by Project E-64, D.G. Aragón, Spain.
nilpotent Lie groups with 1-dimensional center. On the other hand, a quite remarkable Weyl calculus for unitary irreducible representations of arbitrary nilpotent Lie groups was constructed in [Ped94], and therefore it is natural to consider the corresponding multiplier operators and their applications.

More specifically, let \( \pi: G \to B(H) \) be a unitary irreducible representation of a nilpotent Lie group, where \( H := L^2(\mathbb{R}^{d/2}) \) and the even integer \( d \) is the dimension of the coadjoint orbit \( \mathcal{O} \) associated to \( \pi \) by the Kirillov correspondence. The image of the corresponding integrated \( \ast \)-representation \( \pi: L^1(G) \to B(H) \) is a dense subalgebra of the \( C^\ast \)-algebra of compact operators, whose multiplier algebra is the whole \( C^\ast \)-algebra \( B(H) \) of bounded linear operators on \( H \). That property of \( \pi \) is equivalent to the fact that the coadjoint orbit \( \mathcal{O} \) is a closed subset of the dual of the Lie algebra of \( G \), since the Kirillov correspondence is a homeomorphism between the unitary dual of \( G \) and the space of coadjoint orbits. Further enhancing the Kirillov correspondence, N.V. Pedersen [Ped94] later constructed a bijective operator calculus \( T^\pi: S'(\mathbb{R}^d) \to L(S(\mathbb{R}^{d/2}), S'(\mathbb{R}^{d/2})) \), where \( S(\bullet) \) is the space of Schwartz functions. The \( L^2 \)-continuity properties of the operators \( T^\pi(a) \) for \( a \in S(\mathbb{R}^d) \) were more recently investigated in [BB11], [BB12], and [BB15a].

In this paper we study how the above facts can be transferred from operators on \( L^2(\mathbb{R}^{d/2}) \) to operators on \( L^p(\mathbb{R}^{d/2}) \) for \( 1 < p < \infty \). To this end, we first prove that for any irreducible uniformly bounded representation of \( G \) on a reflexive Banach space \( X \) the image of its corresponding integrated representation \( L^1(G) \to B(X) \) is dense in the closed two-sided ideal of \( B(X) \) consisting of operators that can be approximated by finite-rank operators (Theorem 2.10). This two-sided ideal property motivates the notion of a Pedersen multiplier, meaning an operator \( M \in B(H) \) for which there exists a bounded linear operator \( C_M: L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d) \) such that \( T^\pi(C_M \phi) = MT^\pi(\phi) \) for every \( \phi \in S(\mathbb{R}^d) \). One of our main results is that every Pedersen multiplier is \( L^p \)-bounded if \( 1 < p < \infty \) and \( \pi \) is associated to a generic flat coadjoint orbit (Theorem 5.8). Specific examples of Pedersen multipliers are then produced (Corollary 6.12), using a version of Calderón-Zygmund theory with a twist (see Theorem 6.9) and continuity properties of the twisted convolution (see Proposition 6.11).

The nilpotent Lie groups \( G \) involved in our investigation can be alternatively described as 1-dimensional central extensions of nilpotent Lie groups \( G_0 \) endowed with a left invariant symplectic structure. The class of these symplectic nilpotent Lie groups \( G_0 \) is really huge. The simplest example is the abelian group \( G_0 = (\mathbb{R}^{2n}, +) \) with its canonical symplectic structure and in this case \( G \) is the \((2n + 1)\)-dimensional Heisenberg group involved in [Mau80] and [BT15]. But as we will show below in Section 7 many other examples exist, to which our results apply. For instance, there are uncountable families of symplectic 2-step nilpotent Lie groups \( G_0 \) that are mutually non-isomorphic. Their corresponding Lie groups \( G \) are 3-step nilpotent with 1-dimensional center, a class of Lie groups whose classification is still beyond reach. We have mentioned this point because one of the problems faced in the investigation on this level of generality is the lack of information on the structure constants, stratifications, or gradings of nilpotent Lie groups that are needed for the estimation techniques based on special functions and classical polynomials. This lack of information renders these techniques unusable, and therefore we will have to replace them by methods based on abstract harmonic analysis which are still available.
The techniques we use rely on intertwining operators between spaces of smooth vectors for representations of $G$ on various Banach spaces, and thereby transferring information between these representations. To sketch the idea, let $\pi: G \to B(\mathcal{X})$ and $\tau: G \to B(\mathcal{Y})$ be uniformly bounded continuous representations on the Banach spaces $\mathcal{X}$ and $\mathcal{Y}$, with $\mathcal{X}_\infty$ and $\mathcal{Y}_\infty$ the subspaces of smooth vectors of $\mathcal{X}$ and $\mathcal{Y}$, respectively. Assume that there is topological linear isomorphism $A: \mathcal{X}_\infty \to \mathcal{Y}_\infty$ such that for all $g \in G$ one has $A\pi(g)|_{\mathcal{X}_\infty} = \tau(g)A$. Under suitable conditions on $\pi$, if the norm-closure of $\pi(L^1(G))$ is an ideal in $B(\mathcal{X})$, then the existence of $A$ implies that the same is true for the norm-closure of $\tau(L^1(G))$ in $B(\mathcal{Y})$. If $F$ is a space of functions on $G$, then $\pi(L^1(G) \cap F)$ might not lead to an ideal. However, if $M \in B(\mathcal{X})$ is a multiplier of $\pi(L^1(G) \cap F)$, that is, $M\pi(L^1(G) \cap F) \subseteq \pi(L^1(G) \cap F)$, then we can find $T \in B(\mathcal{Y})$ such that $TA = \overline{AM}$ on $\mathcal{X}_\infty$, and $T\tau(L^1(G) \cap F) \subseteq \tau(L^1(G) \cap F)$. The main tool we use is the transference, based on the dual maps of the coefficient maps associated to $\pi$ and $\tau$, both defined on the space of convolution operators on $G$ (see Definitions 5.1 and 5.5 for details). Typically $\pi$ will be a unitary irreducible representation, and we will realize the above strategy with the convolution algebra $(L^1(G), \ast)$ replaced by a twisted convolution algebra.

**Structure of this paper.** The article is organized as follows. In Section 2 we prove our main results on irreducible uniformly bounded representations of nilpotent Lie groups on reflexive Banach spaces (see Theorem 2.9 and Theorem 2.10). $L^p$-properties of operators defined by the Pedersen transform is the subject of Sections 3–5. Section 4 gives the characterization of twisted convolution operators as representations of $\mathcal{G}(\mathcal{V})$ replaced by a twisted convolution algebra. Finally, Section 6 contains examples of Pedersen multipliers which are Pedersen transforms of certain functions that satisfy appropriate Calderón-Zygmund conditions. Section 7 contains some families of examples of groups to which earlier approaches are not applicable. In particular, we give a continuous family of 3-step nilpotent Lie groups that are mutually non-isomorphic and have generic flat orbits, and an example of a symplectic nilpotent Lie group that does not admit any homogeneous structure.

**Notation.** Throughout this paper we denote by $S(\mathcal{V})$ the Schwartz space on a finite-dimensional real vector space $\mathcal{V}$. That is, $S(\mathcal{V})$ is the set of all smooth functions that decay faster than any polynomial together with their partial derivatives of arbitrary order. Its topological dual — the space of tempered distributions on $\mathcal{V}$ — is denoted by $S'(\mathcal{V})$. We use $\langle \cdot, \cdot \rangle$ to denote any duality pairing between finite-dimensional real vector spaces whose meaning is clear from the context. For any topological space $T$ we denote by $C(T)$ the space of continuous complex-valued functions on $T$, and by $C_b(T)$ the Banach space of bounded functions in $C(T)$ with the sup norm.

We also use the convention that the Lie groups are denoted by upper case Latin letters and the Lie algebras are denoted by the corresponding lower case Gothic letters. For any finite-dimensional real Lie algebra $\mathfrak{g}$ we denote by $\mathfrak{g}_C := \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g}$ its complexification, which is a complex Lie algebra, and by $U(\mathfrak{g}_C)$ the universal enveloping algebra of $\mathfrak{g}_C$, which is a complex unital associative $\ast$-algebra and is isomorphic to the algebra of distributions with the support at the unit element $1 \in G$ for any Lie group $G$ whose Lie algebra is isomorphic to $\mathfrak{g}$.
2. Banach space representations of nilpotent Lie groups

The main result of this section is Theorem 2.10 which shows that any uniformly bounded irreducible representation of a nilpotent Lie group \( G \) on a reflexive Banach space is CCR, that is, the closure of the image of \( L^1(G) \) under the representation is the closure of finite-rank operators on that reflexive Banach space. The result is proved by transferring the well-known results for irreducible unitary representations on Hilbert spaces, and illustrates the role that unitary representation theory on Hilbert spaces plays in the general Banach space representation theory of nilpotent Lie groups, via transference.

As mentioned above, the main results of this section are about nilpotent Lie groups and we actually point out in Remark 2.11 that they cannot be directly extended further. Nevertheless, some auxiliary results and lemmas hold true for more general classes of groups hence we have stated and proved them on their natural level of generality as they may hold an independent interest.

Let \( G \) be any connected, simply connected, nilpotent Lie group with its Lie algebra \( \mathfrak{g} \). Then the exponential map \( \exp_G : \mathfrak{g} \to G \) is a diffeomorphism with the inverse denoted by \( \log_G : G \to \mathfrak{g} \).

Let \( \mathcal{X} \) be any complex Banach space with a fixed norm that defines its topology. Assume that \( \pi : G \to B(\mathcal{X}) \) is a continuous representation which is uniformly bounded, in the sense that \( \sup \{ \| \pi(g) \| : g \in G \} < \infty \). Continuity of \( \pi \) means that its corresponding map \( G \times \mathcal{X} \to \mathcal{X}, (g, x) \mapsto \pi(g)x \), is continuous. Then one can define the continuous homomorphism of Banach algebras

\[
\pi : L^1(G) \to B(\mathcal{X}), \quad \pi(\varphi) = \int_G \varphi(g)\pi(g)dg
\]

where the above integral is strongly convergent.

In the next definition we introduce the smooth vectors for Banach space representations, and the smooth operators for Hilbert space representations. For our present purposes, a convenient reference for these notions is [BB10].

**Definition 2.1.**

1. The space of smooth vectors for the representation \( \pi \) is

\[
\mathcal{X}_\infty := \{ x \in \mathcal{X} \mid \pi(\cdot)x \in C^\infty(G, \mathcal{X}) \}.
\]

The linear space \( \mathcal{X}_\infty \) is endowed with the linear topology which makes the linear injective map \( \mathcal{X}_\infty \to C^\infty(G, \mathcal{X}), \, x \mapsto \pi(\cdot)x \) into a linear topological isomorphism onto its image. Then \( \mathcal{X}_\infty \) is a Fréchet space which is continuously and densely embedded in \( \mathcal{X} \) (see for instance [BB10] Prop. 2.2 and the references therein). The derived representation

\[
d\pi : \mathfrak{g} \to \text{End}(\mathcal{X}_\infty), \quad d\pi(X)x = \left. \frac{d}{dt} \right|_{t=0} \pi(\exp_G(tX))x
\]

is a Lie algebra representation, so it extends to a unital homomorphism of complex associative algebras \( d\pi : U(\mathfrak{g}_C) \to \text{End}(\mathcal{X}_\infty) \).

2. Assume that \( \mathcal{X} \) is a Hilbert space. Then the set \( \mathcal{B}(\mathcal{X})_\infty \) of smooth operators for the representation \( \pi \) is

\[
\mathcal{B}(\mathcal{X})_\infty := \{ T \in \mathcal{B}(\mathcal{X}) \mid T(\mathcal{X}) \subseteq \mathcal{X}_\infty \text{ and } T^*(\mathcal{X}) \subseteq \mathcal{X}_\infty \}.
\]
(See [BB10, Sect. 3] for more details and [BB10, Cor. 3.1] for the natural
topology on $B(X)_\infty$ when $\pi$ is a unitary irreducible representation.)

It is easily checked that $\pi(S(G))X \subseteq X_\infty$. In particular, one has a representation
of the convolution algebra $S(G)$,
$$\pi_S: S(G) \to \text{End}(X_\infty), \quad \pi_S(\varphi) := \pi(\varphi)|_{X_\infty}.$$  

Now we can make the following definition.

**Definition 2.2.** With the above notation, the uniformly bounded continuous repre-
sentation $\pi: G \to B(X)$ is called strongly irreducible if its associated representation
$\pi_S: S(G) \to \text{End}(X_\infty)$ is algebraically irreducible, that is, the only linear subspaces
of $X_\infty$ that are invariant to all the operators in $\pi_S(S(G))$ are $\{0\}$ and $X_\infty$.

On the other hand, the representation $\pi$ is called topologically irreducible (for
short irreducible) if the only closed linear subspaces of $X$ that are invariant to all
the operators in $\pi(G)$ are $\{0\}$ and $X$.

**Example 2.3.** If $X$ is a Hilbert space and the bounded continuous representation
$\pi: G \to B(X)$ is topologically irreducible, then $\pi$ is also strongly irreducible.

In fact, since the group $G$ is amenable and $\pi$ is bounded, it follows that $\pi$ is
similar to a unitary representation. Then, it is known that unitary irreducible
representations of nilpotent Lie groups are strongly irreducible; see [How77, Cor.
3.4.1].

**Lemma 2.4.** Let $G$ be any connected Lie group with its Lie algebra $\mathfrak{g}$ and assume
that $\pi: G \to B(X)$ and $\tau: G \to B(Y)$ are uniformly bounded continuous representa-
tions on Banach spaces. Then for any topological linear isomorphism $A: X_\infty \to Y_\infty$
the following assertions are equivalent:

(i) For all $g \in G$ one has $A\pi(g)|_{X_\infty} = \tau(g)A$.

(ii) For all $X \in \mathfrak{g}$ one has $\text{Ad}(X) = d\tau(X)A$.

**Proof.** This is a special case of [Po72, Cor. 3.3]. $\square$

The following notion of almost equivalence of group representations is suggested
by [Po72, Def. 3.3].

**Definition 2.5.** In the setting of Lemma 2.4, we say the representations $\pi$ and $\tau$
are almost equivalent, and the operator $A$ is called an almost equivalence.

**Remark 2.6.** The almost equivalence relation is reflexive, symmetric, and transi-
tive, and for any pair of unitary representations this relation coincides with unitary
equivalence by [Po72, Th. 3.4].

It is interesting to study how various properties of Lie group representations are
transferred by almost equivalence. Indeed, we prove in Theorem 2.10 below that
in the case of nilpotent Lie groups the well-known CCR property of their unitary
irreducible representations propagates under a suitable form to all their uniformly
bounded irreducible representations on reflexive Banach spaces.

**Lemma 2.7.** In the setting of Lemma 2.4 the following assertions hold:

(i) If $A$ is regarded as a densely-defined linear operator from $X$ into $Y$, then its
closure $\overline{A}: D(\overline{A}) \to Y$ is an injective operator.

(ii) For every $\varphi \in C_0^\infty(G)$ and $x \in D(\overline{A})$ one has $\pi(\varphi)x \in X_\infty \subseteq D(\overline{A})$ and
$A\pi(\varphi)x = \overline{A}\pi(\varphi)x = \tau(\varphi)\overline{Ax}$. 
Theorem 2.9. Let \( \pi \) be a nilpotent Lie group, then the above assertion holds for \( \varphi \in \mathcal{S}(G) \).

Proof. That \( A \) is a closable operator follows by \cite[Th. 3.2]{Po72}, while injectivity of \( \hat{A} \) was noted in the proof of \cite[Cor. 3.3]{Po72}.

The second and third assertions follows by Hille’s theorem which roughly says that closed linear operators commute with the Bochner integrals (see \cite[Th. 3.7.12]{HiPl74}.

Lemma 2.8. Let \( G \) be any locally compact group with a set \( \mathcal{A} \) of absolutely integrable functions on \( G \) that is invariant under right translations by some dense subset of \( G \) and such that for every neighborhood \( V \) of \( 1 \in G \) there exists \( \varphi \in \mathcal{A} \) with \( \text{supp} \varphi \subseteq V \), \( 0 \leq \varphi \) almost everywhere on \( G \), and \( \int_G \varphi(g)dg = 1 \).

If \( \pi : G \to \mathcal{B}(X) \) is any bounded continuous representation with its corresponding Banach algebra representation \( \pi : L^1(G) \to \mathcal{B}(X) \), then for any closed linear subspace \( \mathcal{Y} \subseteq X \) one has \( \pi(G)\mathcal{Y} \subseteq \mathcal{Y} \iff \pi(\mathcal{A})\mathcal{Y} \subseteq \mathcal{Y} \).

Proof. The implication \( \Rightarrow \) is clear. For the converse implication use that fact that for any absolutely integrable function \( \varphi \) on \( G \) with \( 0 \leq \varphi \) almost everywhere on \( G \) and \( \int_G \varphi(g)dg = 1 \), one has

\[
\|\pi(\varphi)x - x\| \leq \int_G \varphi(g)\|\pi(g)x - x\|dg \leq \sup_{g \in V} \|\pi(g)x - x\|
\]

for all \( x \in X \) and \( V \subseteq G \) with \( \text{supp} \varphi \subseteq V \) and \( \phi \geq 0 \) almost everywhere on \( G \). Now, by using the hypothesis, there is such a function \( \varphi \in \mathcal{A} \). Thus, by using in addition the continuity of \( \pi : G \to \mathcal{B}(X) \), it follows that

\[
(\forall x \in X) \quad \inf_{\varphi \in \mathcal{A}} \|\pi(\varphi)x - x\| = 0.
\]

Then, by the hypothesis that \( \mathcal{A} \) is invariant under right translations by some dense subset \( \Gamma \subseteq G \), we obtain

\[
(\forall x \in X)(\forall g \in \Gamma) \quad \inf_{\varphi \in \mathcal{A}} \|\pi(\varphi)x - \pi(g)x\| = \inf_{\varphi \in \mathcal{A}} \|\pi(\varphi)\pi(g)x - \pi(g)x\| = 0.
\]

This implies that if \( \mathcal{Y} \subseteq X \) is any closed linear subspace with \( \pi(\mathcal{A})\mathcal{Y} \subseteq \mathcal{Y} \), then \( \pi(\Gamma)\mathcal{Y} \subseteq \mathcal{Y} \), hence also \( \pi(G)\mathcal{Y} \subseteq \mathcal{Y} \).

Theorem 2.9. Let \( \pi : G \to \mathcal{B}(Z) \) be any uniformly bounded continuous representation of a nilpotent Lie group \( G \). Then \( \pi \) is topologically irreducible if and only if it is strongly irreducible. In this case there exists a unique point in the unitary dual of \( G \) consisting of unitary irreducible representations of \( G \) which are almost equivalent to \( \pi \) and whose spaces of smooth vectors are isomorphic as \( \mathcal{S}(G) \)-modules with the space of smooth vectors of \( \pi \).

Proof. First let \( \pi : G \to \mathcal{B}(Z) \) be any strongly irreducible representation and \( \{0\} \neq \mathcal{Y} \subseteq Z \) be any closed linear subspace with \( \pi(G)\mathcal{Y} \subseteq \mathcal{Y} \). Then one has also \( \pi(S(G))\mathcal{Y} \subseteq \mathcal{Y} \) by Lemma 2.8. On the other hand \( \pi(S(G))\mathcal{Y} \subseteq \mathcal{Z}_\infty \), and \( \pi(S(G))\mathcal{Y} \) is a dense linear subspace of \( \mathcal{Y} \). Moreover, since \( \{0\} \neq \mathcal{Y} \), it easily checked that \( \{0\} \neq \pi(S(G))\mathcal{Y} \) hence \( \pi(S(G))\mathcal{Y} = \mathcal{Z}_\infty \) by the assumption that \( \pi \) is strongly irreducible along with the fact that the linear space \( \pi(S(G))\mathcal{Y} \) is invariant under the algebra representation \( \pi_S : S(G) \to \text{End}(\mathcal{Z}_\infty) \). Thus \( \mathcal{Z}_\infty = \pi(S(G))\mathcal{Y} \subseteq \mathcal{Y} \), which implies \( \mathcal{Y} = \mathcal{Z} \) and we are done with one of the asserted implications.
For the converse implication assume that \( \pi \) is topologically irreducible. Using Lemma 2.8 for \( \mathcal{A} = \mathcal{S}(G) \), it then follows that \( \pi : \mathcal{S}(G) \to \mathcal{B}(Z) \) is a topologically irreducible representation, and now the conclusion follows by \([Lu90\text{ Th.}]\). The uniqueness of the point in the unitary dual of \( G \) as in the statement follows by \([Po72\text{ Th. 3.4}]\).

Note that the second part of the above theorem says that one can find a unitary irreducible representation \( \pi_0 : G \to \mathcal{B}(\mathcal{H}) \) and a closed injective linear operator \( A : \mathcal{D}(A) \to Z \) whose domain \( \mathcal{D}(A) \) is dense in \( \mathcal{H} \) and contains \( \mathcal{H}_\infty \), with range \( \text{Ran} \ A \) dense in \( Z \) and contains \( Z_\infty \), and which defines by restriction a topological isomorphism of \( \mathcal{S}(G) \)-modules \( A|_{\mathcal{H}_\infty} : \mathcal{H}_\infty \to Z_\infty \). Next, we take advantage of the properties of \( \pi_0 \) to establish Theorem 2.10. In order to state the result we denote by \( K(\mathcal{X}) \) the norm-closure of the finite-rank operators in \( \mathcal{B}(\mathcal{X}) \), which is equal to the set of all compact operators on \( \mathcal{X} \) if the Banach space \( \mathcal{X} \) has the approximation property.

**Theorem 2.10.** Let \( \pi : G \to \mathcal{B}(Z) \) be any uniformly bounded continuous representation of a nilpotent Lie group \( G \). If \( \pi \) is topologically irreducible and the Banach space \( Z \) is reflexive, then \( \pi \) has the CCR property; that is, the norm closure of \( \pi(L^1(G)) \) is equal to \( K(Z) \).

**Proof.** We first prove that \( \pi(L^1(G)) \subseteq K(Z) \). It suffices to check that for arbitrary \( \varphi \in \mathcal{S}(G) \) one has \( \pi(\varphi) \in K(Z) \). We will actually prove something more, namely that the operator \( \pi(\varphi) : Z \to Z \) is nuclear. To this end note that \( \pi(\varphi)Z \subseteq Z_\infty \), and then \( \pi(\varphi) : Z \to Z_\infty \) is a continuous operator by the closed graph theorem, using the fact that the inclusion map \( Z_\infty \hookrightarrow Z \) is continuous and \( Z_\infty \) is a Fréchet space.

Then the operator \( \pi(\varphi) \) factorizes as \( Z \xrightarrow{\pi(\varphi)} Z_\infty \hookrightarrow Z \). Therefore the assertion will follow by \([Sch66\text{ Ch. III, Prop. 7.2 and its Cor. 2}]\) as soon as we will have proved that \( Z_\infty \) is a nuclear space.

For that, we use the fact that Theorem 2.8 gives a topological isomorphism of \( \mathcal{S}(G) \)-modules \( \mathcal{H}_\infty \cong Z_\infty \). The space of smooth vectors \( \mathcal{H}_\infty \) is well-known to be a nuclear Fréchet space (see for instance \([BB10\text{ Cor. 3.1(1.)}]\)), hence also \( Z_\infty \) is a nuclear space, and this completes the proof of the inclusion \( \pi(L^1(G)) \subseteq K(Z) \).

Now, to prove that the above inclusion is actually an equality, we will check that \( \pi(\mathcal{S}(G)) \) contains a dense subspace of \( K(Z) \). More precisely, for a suitable dense linear subspace \( \mathcal{Y} \subseteq Z^* \) we will prove that

\[
\mathcal{Y} \otimes Z_\infty \subseteq \pi(\mathcal{S}(G)). \tag{2.1}
\]

The set of finite-rank operators

\[
\overline{\mathcal{H}_\infty} \otimes \mathcal{H}_\infty := \text{span} \{ (\cdot | h)k \mid h, k \in \mathcal{H}_\infty \}
\]

is contained in \( \mathcal{B}(\mathcal{H})_\infty \), where \((\cdot | \cdot)\) is the scalar product of the Hilbert space \( \mathcal{H} \). Since the map \( \pi_0 : \mathcal{S}(G) \to \mathcal{B}(\mathcal{H})_\infty \) is surjective (see \([Low77\text{ Th. 3.4}]\)), for any \( h, k \in \mathcal{H}_\infty \) there exists \( \psi \in \mathcal{S}(G) \) such that one has \( \pi_0(\psi)v = (v | h)k \) for all \( v \in \mathcal{H}_\infty \). Using the topological isomorphism of \( \mathcal{S}(G) \)-modules \( A : \mathcal{H}_\infty \to Z_\infty \) we now obtain

\[
\pi(\psi)Av = A\pi_0(\psi)v = (v | h)Ak = (A^{-1}Av | h)Ak = \langle (A^{-1})^*\mathcal{H}, Av \rangle Ak
\]
where $\overline{H} := \langle \cdot \mid h \rangle \in H^* \subseteq (H_\infty)^*$ and $(A^{-1})^*: (H_\infty)^* \to (Z_\infty)^*$ is the transpose mapping of $A^{-1}$. It follows that

\[ (\forall w \in Z_\infty) \quad \pi(\psi)w = \langle (A^{-1})^*\overline{H}, w \rangle Ak. \quad (2.2) \]

Here we know that $\pi(\psi) \in B(Z)$ since $\psi \in S(G)$ and $\pi$ is uniformly bounded, hence using also the fact that $Z_\infty$ is dense in $Z$, it follows that for arbitrary $h \in H_\infty$ the functional $(A^{-1})^*\overline{H} \in (Z_\infty)^*$ is continuous with respect to the norm of $Z$ and then it uniquely extends to a functional $(A^{-1})^*\overline{H} \in Z^*$. Thus, denoting

\[ \mathcal{Y} := \text{span}\{ (A^{-1})^*\overline{H} \mid h \in H_\infty \} \subseteq Z^* \]

one obtains (2.1), and it remains to check that $\mathcal{Y}$ is a dense subspace of $Z^*$.

To this end we argue by contradiction. If $\mathcal{Y}$ were not dense in $Z^*$, then by the Hahn-Banach theorem there exists $\gamma \in (Z^*)^* \setminus \{0\}$ with $\gamma(y) = 0$ for all $y \in \mathcal{Y}$. Using the fact that $Z$ is a reflexive Banach space it then follows that there exists $x \in Z \setminus \{0\}$ with $\langle (A^{-1})^*\overline{H}, x \rangle = 0$ for all $h \in H_\infty$. By (2.2), this implies that $\pi(\psi)x = 0$ for every $\psi \in S(G)$ with $\pi_0(\psi) \in (H^*)_\infty \otimes H_\infty$.

But the surjective map $\pi_0: S(G) \to B(H)_\infty$ is open by the open mapping theorem, and on the other hand the set of finite-rank operators $\overline{H}_\infty \otimes H_\infty$ is dense in the Fréchet space $B(H)_\infty$ by [BB10] Cor. 3.3, and then it is straightforward to check that $\{ \psi \in S(G) \mid \pi_0(\psi) \in \overline{H}_\infty \otimes H_\infty \}$ is a dense subset of $S(G)$. This implies $\pi(\psi)x = 0$ for every $\psi \in S(G)$, hence $x = 0$, which is a contradiction with $x \in Z \setminus \{0\}$. This completes the proof of the theorem. \hfill \QED

In order to point out the significance of Theorem 2.10, we recall that many important Banach spaces, as for instance $L^p$-spaces or more general mixed-norm Lebesgue spaces $L^{p_1,\ldots,p_k}$, carry irreducible representations of nilpotent Lie groups or even exponential solvable Lie groups (that is, Lie groups $G$ whose exponential map $\exp_G: \mathfrak{g} \to G$ is bijective); see [LuMo01] and [LuMiMo03]. In this connection, let us also make the following remark that shows in particular that Theorem 2.10 cannot be extended beyond nilpotent Lie groups.

**Remark 2.11.** Let $G$ be any exponential solvable Lie group. Then the following assertions are equivalent:

(i) $G$ is a CCR (liminary) group, that is, every unitary irreducible representation of $G$ on a Hilbert space has the CCR property;

(ii) $G$ is type R, that is, for every $g \in G$ all the eigenvalues of the operator $\text{Ad}_G g : \mathfrak{g}_C \to \mathfrak{g}_C$ belong to the unit circle $T$ (equivalently, for every $x \in \mathfrak{g}$, all the eigenvalues of the operator $\text{ad}_g x : \mathfrak{g}_C \to \mathfrak{g}_C$ are purely imaginary);

(iii) every coadjoint orbit of $G$ is a closed subset of $\mathfrak{g}^*$;

(iv) $G$ is a nilpotent Lie group.

In fact, since $G$ is an exponential Lie group, it is a connected, simply connected, solvable Lie group of type I (see [AuKo71] Sect. 0, Rem. 3). Then, using also Glimm’s characterization of separable $C^*$-algebras of type I, we obtain that $G$ has the property GCR (is postliminary). Hence the first two assertions are equivalent by [AuMo66] Ch. V, Th. 1–2].

By [FuLu15] Prop. 5.2.13, Th. 5.2.16], since $G$ is an exponential Lie group, it follows that for every $x \in \mathfrak{g}$, the operator $\text{ad}_x : \mathfrak{g}_C \to \mathfrak{g}_C$ has no nonzero purely imaginary eigenvalues, hence the second and fourth assertion are equivalent. Finally, the equivalence between the first and third assertion is discussed in [FuLu15] Th. 5.3.31 infra [see also [Pu68] Th. 1].
3. The Pedersen Transform and Its Associated Multipliers

In order to define the Pedersen transform below, we first need to introduce the following notation (see [CG90] and [Pe94] for more details):

1. \( G \) is a again a connected simply connected nilpotent Lie group.
2. We denote by \( \mathfrak{g}^* \) the linear dual space to \( \mathfrak{g} \) and by \( \langle \cdot, \cdot \rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R} \) the natural duality pairing.
3. Let \( \xi_0 \in \mathfrak{g}^* \) with the corresponding coadjoint orbit \( O := \text{Ad}^*_G(\xi_0) \subseteq \mathfrak{g}^* \).
4. Let \( \pi : G \to B(\mathcal{H}) \) be any unitary irreducible representation associated with the coadjoint orbit \( O \).
5. The isotropy group at \( \xi_0 \) is \( G_{\xi_0} := \{ g \in G \mid \text{Ad}^*_G(g)\xi_0 = \xi_0 \} \) with the corresponding isotropy Lie algebra \( \mathfrak{g}_{\xi_0} = \{ X \in \mathfrak{g} \mid \xi_0 \circ \text{ad}_g X = 0 \} \). If we denote the center of \( \mathfrak{g} \) by \( \mathfrak{z} := \{ X \in \mathfrak{g} \mid [X, \mathfrak{g}] = \{ 0 \} \} \), then it is clear that \( \mathfrak{z} \subseteq \mathfrak{g}_{\xi_0} \).
6. Let \( n := \dim \mathfrak{g} \) and fix a sequence of ideals in \( \mathfrak{g} \),
   \[
   \{0\} = \mathfrak{g}_0 \subset \mathfrak{g}_1 \subset \cdots \subset \mathfrak{g}_n = \mathfrak{g}
   \]
   such that \( \dim(\mathfrak{g}_j/\mathfrak{g}_{j-1}) = 1 \) and \( [\mathfrak{g}, \mathfrak{g}_j] \subseteq \mathfrak{g}_{j-1} \) for \( j = 1, \ldots, n \).
7. Pick any \( X_j \in \mathfrak{g}_j \setminus \mathfrak{g}_{j-1} \) for \( j = 1, \ldots, n \), such that \( \langle \xi_0, X_1 \rangle = 1 \). Note that the set \( \{X_1, \ldots, X_n\} \) is a Jordan-Hölder basis in \( \mathfrak{g} \).

**Definition 3.1.** Consider the set of jump indices of the coadjoint orbit \( O \) with respect to the aforementioned Jordan-Hölder basis \( \{X_1, \ldots, X_n\} \subseteq \mathfrak{g} \),
\[
\epsilon := \{ j \in \{1, \ldots, n\} \mid \mathfrak{g}_j \not\subseteq \mathfrak{g}_{j-1} + \mathfrak{g}_{\xi_0} \} = \{ j \in \{1, \ldots, n\} \mid X_j \not\in \mathfrak{g}_{j-1} + \mathfrak{g}_{\xi_0} \}
\]
and then define the corresponding predual of the coadjoint orbit \( O \),
\[
\mathfrak{g}_\epsilon := \text{span} \{ X_j \mid j \in \epsilon \} \subseteq \mathfrak{g}.
\]
We note the direct sum decomposition \( \mathfrak{g} = \mathfrak{g}_\epsilon + \mathfrak{g}_\epsilon \).

We can now give the general construction of an operator calculus, which is due to N.V. Pedersen; see [Pe94, page 10].

**Definition 3.2.** The Pedersen transform \( T^\pi(\cdot) \) for the unitary representation \( \pi \) is defined for every \( b \in S(\mathfrak{g}_\epsilon) \) by
\[
T^\pi(b) = \int_{\mathfrak{g}_\epsilon} b(X)\pi(\exp_G X)dX \in B(\mathcal{H}).
\]
We call \( T^\pi(b) \) the operator with the Pedersen symbol \( b \in S(\mathfrak{g}_\epsilon) \).

**Theorem 3.3.** The Pedersen transform has the following properties:

1. For every symbol \( b \in S(\mathfrak{g}_\epsilon) \) we have \( T^\pi(b) \in B(\mathcal{H})_\infty \) and the mapping \( S(\mathfrak{g}_\epsilon) \to B(\mathcal{H})_\infty \), \( b \mapsto T^\pi(b) \) is a linear topological isomorphism.
2. For every \( B \in B(\mathcal{H})_\infty \) we have \( B = T^\pi(b) \), where \( b \in S(\mathfrak{g}_\epsilon) \) is given by \( b(X) = \text{Tr}(\pi(\exp_G X)^{-1}B) \) for every \( X \in \mathfrak{g}_\epsilon \).
3. For every \( a, b \in S(\mathfrak{g}_\epsilon) \) we have
   \begin{enumerate}
   \item \( T^\pi(b) = T^\pi(b)^* \), where \( \overline{b(X)} = \overline{b(-X)} \) for every \( X \in \mathfrak{g}_\epsilon \);
   \item \( \text{Tr}(T^\pi(b)) = b(0) \);
   \item \( \text{Tr}(T^\pi(a)T^\pi(b)^*) = \int_{\mathfrak{g}_\epsilon} a(X)b(X)dX \).
   \end{enumerate}

**Proof.** See [Pe94, subsects. 2.2–2.3].
Notation 3.4. Recall that $\mathcal{H}_{-\infty}$ is the space of continuous antilinear functionals on $\mathcal{H}_\infty$ and the corresponding pairing will be denoted by $(\cdot \mid \cdot) : \mathcal{H}_{-\infty} \times \mathcal{H}_\infty \to \mathbb{C}$ just as the scalar product in $\mathcal{H}$, since they agree on $\mathcal{H}_\infty \times \mathcal{H}_\infty$ if we think of the natural inclusions $\mathcal{H}_\infty \hookrightarrow \mathcal{H} \hookleftarrow \mathcal{H}_{-\infty}$. \hfill $\square$

Definition 3.5. Recall that $B(\mathcal{H})_\infty$ is an involutive associative subalgebra of $B(\mathcal{H})$. It then follows by Theorem 3.3 \cite{33} that there exists a uniquely defined bilinear associative twisted convolution product

$$S(\mathfrak{g}_c) \times S(\mathfrak{g}_c) \to S(\mathfrak{g}_c), \quad (a, b) \mapsto a \ast_c b$$

such that

$$(\forall a, b \in S(\mathfrak{g}_c)) \quad T^\pi(a \ast_c b) = T^\pi(a)T^\pi(b).$$

Thus $(S(\mathfrak{g}_c), \ast_c)$ is made into an involutive associative algebra such that the mapping $S(\mathfrak{g}_c) \to B(\mathcal{H})_\infty$, $a \mapsto T^\pi(a)$ is an algebra isomorphism.

Remark 3.6. Here are a few versions of twisted convolution products on Lie groups that occur in the earlier literature:

1. In the special case when the center of $G$ is 1-dimensional and the representation $\pi$ is square-integrable modulo the center of $G$, the above twisted convolution product agrees with the one introduced in \cite{33} Def. 1.
2. Twisted convolution products were studied in \cite{Mu84} in the special setting where $G$ is a homogeneous group with 1-dimensional center and its generic coadjoint orbits are flat.
3. In the case of the Heisenberg group, the twisted convolution product plays in important role in the description of various function spaces; see for instance \cite{Ga01} for Besov spaces.

Remark 3.7. Both $\mathcal{H}_\infty$ and $B(\mathcal{H})_\infty$ are nuclear Fréchet spaces and we have the canonical linear topological isomorphism $\mathcal{H}_\infty \hat{\otimes} \mathcal{H}_\infty \simeq B(\mathcal{H})_\infty$, $x \otimes y \mapsto (\cdot \mid y)x$ (see also \cite{BB10}). Let $B(\mathcal{H})^*_\infty \simeq \mathcal{L}(\mathcal{H}_\infty, \mathcal{H}_{-\infty})$ be the topological dual of $B(\mathcal{H})_\infty$ and denote by $\langle \cdot, \cdot \rangle$ either of the duality pairings

$$B(\mathcal{H})^*_\infty \times B(\mathcal{H})_\infty \to \mathbb{C} \text{ and } S'(\mathfrak{g}_c) \times S(\mathfrak{g}_c) \to \mathbb{C}.$$ 

Then for every tempered distribution $a \in S'(\mathfrak{g}_c)$ we can use Theorem 3.3 \cite{33} to define $T^\pi(a) \in B(\mathcal{H})^*_\infty$ such that

$$(\forall b \in S(\mathfrak{g}_c)) \quad \langle T^\pi(a), T^\pi(b) \rangle = (a, b)$$

Just as in Definition 3.2 we call $T^\pi(a) \in \mathcal{L}(\mathcal{H}_\infty, \mathcal{H}_{-\infty})$ the operator with the Pedersen symbol $a \in S'(\mathfrak{g}_c)$. Note that if actually $a \in S(\mathfrak{g}_c)$, then the present notation agrees with Definition 3.2 because of Theorem 3.3 \cite{33}.

We also note that the Pedersen transform extended as above is a linear topological isomorphism

$$T^\pi : S'(\mathfrak{g}_c) \to B(\mathcal{H})^*_\infty \simeq \mathcal{L}(\mathcal{H}_\infty, \mathcal{H}_{-\infty})$$

which restricts to a unitary operator

$$T^\pi : L^2(\mathfrak{g}_c) \to \mathcal{S}_2(\mathcal{H})$$

where $\mathcal{S}_2(\mathcal{H})$ stands for the Hilbert-Schmidt ideal on $\mathcal{H}$.

If we introduce the distribution space

$$L^\pi(\mathfrak{g}_c) := \{ b \in S'(\mathfrak{g}_c) \mid T^\pi(b) \in B(\mathcal{H}) \} \quad (3.1)$$
then it has the natural structure of a von Neumann algebra with respect to the twisted convolution product, defined by duality from Definition 3.9 such that $T^\pi : L^\pi(\mathfrak{g}_e) \to \mathcal{B}(\mathcal{H})$ is an isometric $\ast$-isomorphism. It is easily checked that $L^1(\mathfrak{g}_e) \subseteq L^\pi(\mathfrak{g}_e)$.

**Some basic problems on Pedersen multipliers.** Before we introduce the Pedersen multipliers, we prove a proposition that will be useful later on and serves as motivation for these multipliers.

**Proposition 3.8.** In the notation above, the following assertions hold.

(i) If $a, b \in L^2(\mathfrak{g}_e)$, then $a \ast_b b \in L^2(\mathfrak{g}_e)$ and

$$\| a \ast_b b \|_{L^2(\mathfrak{g}_e)} \leq \| a \|_{L^2(\mathfrak{g}_e)} \| b \|_{L^2(\mathfrak{g}_e)}.$$

(ii) The space $L^\pi$ consists precisely of those $u \in \mathcal{S}'(\mathfrak{g}_e)$ such that

$$u \ast f \in L^2(\mathfrak{g}_e) \quad \text{for all } f \in \mathcal{S}(\mathfrak{g}_e),$$

$$\mathcal{S}(\mathfrak{g}_e) \to L^2(\mathfrak{g}_e), \quad f \mapsto u \ast f \quad \text{extends continuously to } L^2(\mathfrak{g}_e). \quad (3.2)$$

**Proof.** Assertion (i) follows immediately from the fact that $T^\pi : L^2(\mathfrak{g}_e) \to \mathcal{S}_2(\mathcal{H})$ is a unitary operator (see Remark 3.7).

If $u \in \mathcal{S}'(\mathfrak{g}_e)$ satisfies condition (3.2), then for every $f \in \mathcal{S}(\mathfrak{g}_e)$ we have

$$\| T^\pi(u)T^\pi(f) \|_{\mathcal{S}_2(\mathcal{H})} = \| T^\pi(u \ast f) \|_{\mathcal{S}_2(\mathcal{H})} = \| u \ast f \|_{L^2(\mathfrak{g}_e)} \leq C \| f \|_{L^2(\mathfrak{g}_e)} = C \| T^\pi(f) \|_{\mathcal{S}_2(\mathcal{H})}.$$

Therefore, for every $f, h \in \mathcal{S}(\mathfrak{g}_e)$ we have

$$| \text{Tr}(T^\pi(u)T^\pi(f)^*T^\pi(h)^*) | \leq C \| T^\pi(f) \|_{\mathcal{S}_2(\mathcal{H})} \| T^\pi(h) \|_{\mathcal{S}_2(\mathcal{H})}. \quad (3.3)$$

Now, by Theorem 3.3 (1), for every $\xi, \eta, \eta' \in \mathcal{H}^\infty$ there exists $f, h \in \mathcal{S}(\mathfrak{g}_e)$ such that

$$T^\pi(f) = \overline{\eta} \otimes \xi, \quad T^\pi(f) = \overline{\eta'} \otimes \xi,$$

where $\overline{\eta} \otimes \xi$ denotes the operator $v \mapsto (v | \eta)\xi$. Choosing $\| \eta' \| = 1$, we get from (3.3) that

$$|(T^\pi(u)(\overline{\eta} \otimes \xi)^* \eta' | \xi) | \leq C \| \xi \| \| \eta \|,$$

that is,

$$|(T^\pi(u)\eta | \xi) | \leq C \| \xi \| \| \eta \|$$

for every $\xi, \eta \in \mathcal{H}^\infty$. Since $\mathcal{H}^\infty$ is dense in $\mathcal{H}$, we get that $T^\pi(u) \in \mathcal{B}(\mathcal{H})$, thus $u \in L^\pi$.

On the other hand, if $u \in L^\pi$ we have that for every $f \in L^2(\mathfrak{g}_e)$, $T^\pi(u \ast f)$ is a Hilbert-Schmidt operator. Thus $u \ast f \in L^2(\mathfrak{g}_e)$, and we can write

$$\| u \ast f \|_{L^2(\mathfrak{g}_e)} = \| T^\pi(u)T^\pi(f) \|_{\mathcal{S}_2(\mathcal{H})} \leq C \| T^\pi(u) \|_{\mathcal{B}(\mathcal{H})} \| f \|_{L^2(\mathfrak{g}_e)}.$$

Hence $u$ satisfies (3.2). \qed

The following definition is inspired by the notion of Weyl multiplier introduced at the beginning of [Mau80] Sect. 3.

**Definition 3.9.** Let $p \in [1, \infty)$. We say that an operator $M \in \mathcal{B}(\mathcal{H})$ is a Pedersen multiplier of $L^p(\mathfrak{g}_e)$ (with respect to the representation $\pi$) if there exists a bounded linear operator $C_M : L^p(\mathfrak{g}_e) \to L^p(\mathfrak{g}_e)$ such that for every $\phi \in \mathcal{S}(\mathfrak{g}_e)$ we have $T^\pi(C_M \phi) = MT^\pi(\phi)$.\[\]
Remark 3.10. It easy to see that if $M \in \mathcal{B}(\mathcal{H})$ is a Pedersen multiplier, the equality $T^\pi(C_M\phi) = MT^\pi(\phi)$ holds for every $\phi \in L^1(g_e) \cap L^p(g_e)$. This further implies that $C_M(L^1(g_e) \cap L^p(g_e)) \subseteq L^p(g_e)$. If $M$ is a Pedersen multiplier of $L^p(g_e)$, then there can exist at most one operator $C_M$ as in Definition 3.9 since $L^1(g_e) \cap L^p(g_e)$ is dense in $L^p(g_e)$.

Remark 3.11. The multiplier terminology in Definition 3.9 is motivated by the fact that the operator $C_M$ is a multiplier with respect to the twisted convolution, that is, we have

$$ (\forall \phi, \psi \in L^1(g_e) \cap L^p(g_e)) \quad C_M(\phi \ast_e \psi) = (C_M\phi) \ast_e \psi. \quad (3.4) $$

In fact we have

$$ MT^\pi(\phi)T^\pi(\psi) = MT^\pi(\phi \ast_e \psi) = T^\pi(C_M(\phi \ast_e \psi)) $$

and on the other hand

$$ MT^\pi(\phi)T^\pi(\psi) = T^\pi(C_M\phi)T^\pi(\psi) = T^\pi((C_M\phi) \ast_e \psi). $$

Since the mapping $T^\pi: L^\infty(g_e) \rightarrow \mathcal{B}(\mathcal{H})$ is a $*$-isomorphism (see Remark 3.7), we see that $(3.4)$ holds true.

Proposition 3.12. Let $M \in \mathcal{B}(\mathcal{H})_\infty^*$ such that there is a bounded linear operator $C_M: L^p(g_e) \rightarrow L^p(g_e)$ with $T^\pi(C_M\phi) = MT^\pi(\phi)$ for every $\phi \in S(g_e)$. Then there exists $u \in S'(g_e)$ such that for every $\phi \in S(g_e)$ we have $C_M\phi = u \ast_e \phi$. Moreover $M$ is a Pedersen multiplier of $L^p(g_e)$ if and only if $u \in L^p(g_e)$.

Proof. By Remark 3.7 there is $u \in S'(g_e)$ such that $M = T^\pi(u)$ since $M \in \mathcal{B}(\mathcal{H})_\infty^* \simeq L(H_\infty, H_{-\infty})$. We then have

$$ T^\pi(C_M\phi) = T^\pi(uT^\pi(\phi)) = T^\pi(u \ast_e \phi) $$

for all $\phi \in S(g_e)$, thus $C_M\phi = u \ast_e \phi$, by the injectivity of $T^\pi: S'(g_e) \rightarrow \mathcal{B}(\mathcal{H})_\infty^*$.

In addition $M = T^\pi(u) \in \mathcal{B}(\mathcal{H})$ if and only if $u \in L^p(g_e)$, which completes the proof. $\square$

We list here some basic and important problems for Pedersen multipliers. In the next section we will answer them in the case of generic flat coadjoint orbits.

Problem 3.13. Is it true that every bounded linear operator $C: L^p(g_e) \rightarrow L^q(g_e)$, $1 \leq p \leq q \leq \infty$, that satisfies the multiplier condition $(3.4)$, is of the form $C\phi = u \ast_e \phi$, $\phi \in S(g_e)$, for a certain $u \in S'(g_e)$?

Proposition 3.12 gives an affirmative answer in the case when $C = C_M$, with $M \in \mathcal{B}(\mathcal{H})_\infty^*$ such that $T^\pi(C_M\phi) = MT^\pi(\phi)$ for every $\phi \in S(g_e)$. $\square$

Problem 3.14. We recall from [ER95] Lemma 2.1 that by using suitable polarizations it is possible to perform a realization $\mathcal{H} \simeq L^2(\mathbb{R}^d)$ such that the representation $\pi$ gives rise to a continuous representation of $G$ by isometries of $L^p(\mathbb{R}^d)$ for arbitrary $p \in [1, \infty)$, where $d = \dim \mathcal{O}$. In this framework, it is possible to prove a version of [Mau80] Th. 5.2, to the effect that if $M \in \mathcal{B}(\mathcal{H})$ is a Pedersen multiplier of $L^p(g_e)$ then $M$ is also a bounded operator on $L^p(\mathbb{R}^d)$?

Proposition 3.12 immediately implies [Mau80] Prop. 3.1. Also, it allows us to study the Pedersen multipliers by using the operators defined by twisted convolution with a fixed function. To this end, one may need twisted versions of the results on convolution operators on amenable groups, for instance [He71] Th. C]; see also [De11].
4. Characterization of twisted convolution operators

The main result of this section is Proposition 4.5 below, which gives the characterization of twisted convolution operators for square-integrable representations.

For the sake of simplicity we identify $g$ to $G$ by means of the exponential map, so that the group multiplication $x \cdot y$ will be given by the Baker-Campbell-Hausdorff series in $g$; see for example [CG90] pages 11–12. We work in the following setting:

**Setting 4.1.** The center $Z$ of $G$ is 1-dimensional and the representation $\pi$ is square-integrable modulo $Z$.

In this setting we have $\mathfrak{g}_e = \mathfrak{z}$ (see [CG90]), which implies $e = \{2, \ldots, n\}$, hence $\mathfrak{g}_e = \mathbb{R}X_2 + \cdots + \mathbb{R}X_n \cong \mathfrak{g}/\mathfrak{z}$.

Then $\mathfrak{g}_e$ has the natural structure of a nilpotent Lie algebra (but not a Lie subalgebra of $\mathfrak{g}$) and we will denote its Baker-Campbell-Hausdorff multiplication by $\mathfrak{g}_e \times \mathfrak{g}_e \to \mathfrak{g}_e$, $(x, y) \mapsto x \cdot e y$.

Recall that $x \cdot e y$ is just the projection of $x \cdot y$ onto $\mathfrak{g}_e$ according to the direct sum decomposition $\mathfrak{g} = \mathfrak{g}_e + \mathfrak{z}$. In view of the relationship between $\xi_0$ and $X_1, \ldots, X_n$, it then follows that

$$\forall x, y \in \mathfrak{g}_e \quad x \cdot y = \underbrace{x \cdot e y + \langle \xi_0, x \cdot y \rangle X_1}_{\in \mathfrak{g}_e}.$$  \hfill (4.1)

Under the above assumption, the coadjoint orbit of $\xi_0$ is flat, that is,

$$O = \{ \xi \in \mathfrak{g}^* \mid \langle \xi, X_1 \rangle = 1 \}$$

and the fact that the representation $\pi$ is associated with $O$ means that

$$\forall t \in \mathbb{R} \quad \pi(tX_1) = e^{it}.$$  \hfill (4.2)

**Relationship to projective representations.** In the above setting, by using (4.1), we get

$$\forall x_1, x_2 \in \mathfrak{g}_e \quad \pi(x_1)\pi(x_2) = \pi(x_1 \cdot x_2) = e^{i\alpha(x_1, x_2)}\pi(x_1 \cdot e x_2)$$

where

$$\alpha : \mathfrak{g}_e \times \mathfrak{g}_e \to \mathbb{R}, \quad \alpha(x_1, x_2) = \langle \xi_0, x_1 \cdot x_2 \rangle.$$  \hfill (4.3)

Hence $\pi|_{\mathfrak{g}_e} : \mathfrak{g}_e \to B(\mathcal{H})$ is a projective representation of the nilpotent Lie group $G_e := (\mathfrak{g}_e, \cdot e)$ with the corresponding 2-cocycle $e^{i\alpha} : G_e \times G_e \to T$ defined as above. For later use we recall that the cocycle property is

$$\forall x, y, z \in \mathfrak{g}_e \quad \alpha(x, y) + \alpha(x \cdot e y, z) = \alpha(x, y \cdot e z) + \alpha(y, z).$$

We also have $\alpha(\lambda x, \mu x) = 0$ for all $x \in \mathfrak{g}_e$ and $\lambda, \mu \in \mathbb{R}$ (see [Mai07] Lemma 1 and Lemma 6.6 below).
Formulas for the twisted convolution product. By using (1.4), it is easily checked that the twisted convolution product introduced in Definition 3.5 agrees with the $e^{i\alpha}$-twisted convolution product from [Mü84 Eq. (5)], for the 2-cocycle $e^{i\alpha} : \mathfrak{g}_e \times \mathfrak{g}_e \to \mathbb{T}$ on the group $G_e$. Specifically, for $b_1, b_2 \in L^1(\mathfrak{g}_e)$ we have

$$ (b_1 \ast_e b_2)(x) = \int_{\mathfrak{g}_e} b_1(x \cdot_e (-y)) e^{i\alpha(x \cdot_e (-y), y)} b_2(y) dy = \int_{\mathfrak{g}_e} e^{-i\alpha(x, y)} b_1(x \cdot_e (-y)) b_2(y) dy = \int_{\mathfrak{g}_e} e^{i\alpha(-x, y)} b_1(y) b_2((-y) \cdot_e x) dy. \quad (4.3)$$

See also [Mai07, Eq. (4)] for $\hbar = 1$.

Lemma 4.2. In the Setting 4.1, the following assertions hold true:

1. The twisted convolution product turns the Schwartz space $S(\mathfrak{g}_e)$ into an associative Fréchet algebra.
2. The twisted convolution product extends to separately continuous bilinear maps $S^1(\mathfrak{g}_e) \times S^1(\mathfrak{g}_e) \to C^\infty(\mathfrak{g}_e)$ and $S(\mathfrak{g}_e) \times S(\mathfrak{g}_e) \to C^\infty(\mathfrak{g}_e)$, respectively.
3. If we denote by $\delta_0 \in S^1(\mathfrak{g}_e)$ the Dirac distribution at any $v \in \mathfrak{g}_e$, then for every $\phi \in S(\mathfrak{g}_e)$ and $v \in \mathfrak{g}_e$ we have $(\phi \ast_0 \delta)(x) = e^{-i\alpha(x, -v)} \phi(x - e \cdot (-v))$.
4. There exists a unique unitary representation $\rho : G \to \mathcal{B}(L^2(\mathfrak{g}_e))$ such that for every $v \in \mathfrak{g}_e$, $t \in \mathbb{R}$, and $\phi \in S(\mathfrak{g}_e)$ we have $\rho(tX_1 + v) \phi = e^{i\delta}(\phi \ast_0 \delta)$.

Moreover, the representation $\rho$ is the orthogonal direct sum of countably many copies of the irreducible representation $\pi$ and is unitarily equivalent to the representation of $G$ induced from the character $tX_1 \mapsto e^{it}$ of the center of $G$.

Proof. For Assertion (1), note that $S(\mathfrak{g}_e)$ is closed under the twisted convolution product since in (1.3) we have $\alpha(x, y) = (\xi_0, x \cdot y)$ for all $x, y \in \mathfrak{g}_e$, and the function $(x, y) \mapsto (\xi_0, x \cdot y)$ is a polynomial. The separate continuity of the twisted convolution product can be proved by using [CC90 Th. A.2.6], and then the joint continuity follows since $S(\mathfrak{g}_e)$ is a Fréchet space. See also [Mai07, Prop. 4].

Assertion (2) can be proved by using standard arguments.

For Assertion (3) we first use Assertion (2) to extend formula (4.3) to $b_2 \in S^1(\mathfrak{g}_e)$, and then specialize that formula for $b_1 = \phi \in S(\mathfrak{g}_e)$ and $b_2 = \delta_0 \in S^1(\mathfrak{g}_e)$.

For Assertion (4) it is straightforward to check that $\rho$ is a group homomorphism, and then by using Assertion (3) we can prove that $\rho$ is also continuous and unitary. The second part of the assertion can be obtained by using the arguments from the proof of [Mai07, Prop. 2].

The infinitesimal generators of the representation $\rho$ of Lemma 4.2(4) provide useful generalizations of the so-called symplectic differential operators from [Mai86], and their role was actually foreseen in the deep comment prior to [Mai86 Lemma 2]. More specifically, for $j \in \{2, \ldots, n\}$ let us denote by $X_j$ the first-order differential operator on $\mathfrak{g}_e$ defined by the left-invariant vector field on the group $G_e = (\mathfrak{g}_e, \cdot_e)$ whose value at $0 \in \mathfrak{g}_e$ is $X_j$ (see for instance [CC90, page 236]). Then for every $J = (j_2, \ldots, j_n) \in \mathbb{N}^{n-1}$ denote $d\rho(X)_J := d\rho(X_2)^{j_2} \cdots d\rho(X_n)^{j_n}$ and $\bar{X}^J := \bar{X}_2^{j_2} \cdots \bar{X}_n^{j_n}$, which are linear differential operators with polynomial...
coefficients on \( g_e \). Also pick a basis \( Q_0 \) in the complex linear space \( \mathcal{P}(g_e) \) of all complex-valued polynomial functions on \( g_e \).

**Lemma 4.3.** If \( 1 \leq p \leq \infty \), then the topology of the Schwartz space \( S(g_e) \) is determined by the family of seminorms \( \{ | \cdot |_{Q,J,p} \mid Q \in Q_0, J \in \mathbb{N}^{n-1} \} \), where we define

\[
|\phi|_{Q,J,p} := \| Q\tilde{X}^J \phi \|_{L^p(g_e)}
\]

for all \( \phi \in S(g_e) \), \( Q \in Q_0 \), and \( J \in \mathbb{N}^{n-1} \).

**Proof.** For every \( j \in \{ 2, \ldots, n \} \) it follows by the formula provided by Lemma 4.2(3) that for the polynomial function \( p_j(\cdot) := i \left[ \frac{d}{dt} \right]_{t=0} \alpha(\cdot, tX_j) \) we have

\[
(\forall \phi \in S(g_e)) \quad d\rho(X_j)\phi = \frac{d}{dt} |_{t=0} (\phi \ast e \delta_{tX_j}) = -\tilde{X}_j \phi + p_j \phi.
\]

Then, just as in [Mai86, Lemma 2] and the comment after it, we obtain formulas of the form

\[
d\rho(X)^J = (-1)^{|J|} \tilde{X}^J + \sum_{|S|<|J|} a_S(\cdot) \tilde{X}^S,
\]

\[
\tilde{X}^J = (-1)^{|J|} d\rho(X)^J + \sum_{|S|<|J|} b_S(\cdot) d\rho(X)^S
\]

with polynomial coefficients \( a_S(\cdot) \) and \( b_S(\cdot) \). Now the conclusion for \( p = \infty \) follows by using [CG90, Cor. A.2.3(b)] and then the general case follows by standard estimates; see for instance the ones prior to [CG90, Cor. A.2.4]). \( \square \)

For the statement of the following lemma we note that for \( 1 \leq p \leq \infty \) there exists a continuous isometric representation \( \rho : G \to B(L^p(g_e)) \) given by the same formula as the representation \( \rho \) provided by Lemma 4.2(3) for \( p = 2 \).

**Lemma 4.4.** If \( 1 \leq p \leq q < \infty \) and \( C : L^p(g_e) \to L^q(g_e) \) is a bounded linear operator, then the following properties are equivalent:

1. For all \( \phi, \psi \in L^1(g_e) \cap L^p(g_e) \) we have \( C(\phi \ast e \psi) = (C\phi) \ast e \psi \).
2. For all \( v \in g_e \) we have \( \rho(v)C = C\rho(v) \).

**Proof.** “(2) \Rightarrow (1)” It follows by formula 4.4 that for \( \phi, \psi \in L^1(g_e) \cap L^p(g_e) \) we have

\[
\phi \ast e \psi = \int_{g_e} \psi(y)\rho(y)\phi dy
\]

where the integral is convergent in \( L^p(g_e) \). Therefore

\[
C(\phi \ast e \psi) = \int_{g_e} \psi(y)C(\rho(y)\phi) dy = \int_{g_e} \psi(y)\rho(y)C\phi dy = (C\phi) \ast e \psi
\]

where the integrals are convergent in \( L^q(g_e) \) and the latter equality follows by the straightforward extension of formula 4.4 to \( b_1 \in L^q(g_e) \) and \( b_2 \in L^1(g_e) \).

“(1) \Rightarrow (2)” Let \( v \in g_e \) arbitrary and pick a neighbourhood base \( \{ U_r \}_{r \geq 1} \) of \( 0 \in g_e \) consisting of bounded sets. For every \( r \geq 1 \) let \( 0 \leq \psi_r \in L^1(g_e) \) with \( \text{supp} \psi_r \subseteq U_r \) and \( \int_{g_e} \psi_r dx = 1 \). Then by the definition of \( \rho(\cdot)\phi \) for \( \phi \in L^1(g_e) \cap L^p(g_e) \) we obtain

\[
(\rho(v)\phi) \ast e \psi_r = (\phi \ast e \delta_v) \ast e \psi_r = \phi \ast e (\delta_v \ast e \psi_r)
\]

hence by applying \( C \) to the extreme terms of these equalities and by using the assumption (1) we obtain

\[
C(\rho(v)\phi) \ast e \psi_r = (C\phi) \ast e (\delta_v \ast e \psi_r) = (\rho(v)C\phi) \ast e \psi_r.
\]
For $r \to \infty$ it then follows just as in the case of the non-twisted convolution (see [HR79, Th. 20.15] and also [EL69a, Th. 4]) that $C(\rho(v)\phi) = \rho(v)C\phi$ for arbitrary $\phi \in L^1(\mathfrak{g}_c) \cap L^p(\mathfrak{g}_c)$. Since $L^1(\mathfrak{g}_c) \cap L^p(\mathfrak{g}_c)$ is dense in $L^p(\mathfrak{g}_c)$, we see that Property $\mathcal{P}$ is satisfied, and this completes the proof.

We give now an affirmative answer to the above Problem 3.13 for square integrable representations. The method of proof of the following proposition goes back to [Hör60, Th. 1.2].

**Proposition 4.5.** Let $1 \leq p \leq q \leq \infty$ and assume that $C: L^p(\mathfrak{g}_c) \to L^q(\mathfrak{g}_c)$ is a bounded linear operator such that for all $\phi, \psi \in L^1(\mathfrak{g}_c) \cap L^p(\mathfrak{g}_c)$ we have $C(\phi * v \psi) = (C\phi) * v \psi$. Then there exists a unique $u \in \mathcal{S}'(\mathfrak{g}_c)$ such that for every $\phi \in \mathcal{S}(\mathfrak{g}_c)$ we have $C\phi = u * \phi$.

**Proof.** Assume that we have proved that the functional

$$u: \mathcal{S}(\mathfrak{g}_c) \to \mathbb{C}, \quad (u, \phi) = (C\phi)(0),$$

(4.6)
is well defined and it is a distribution, where we have denoted $\hat{\phi}(x) := \phi(-x)$. Then since for all $\phi, \psi \in \mathcal{S}(\mathfrak{g}_c)$ we have

$$(u * \phi, \psi) = (\hat{u}, \psi * \hat{\phi}) = (C(\phi * v \hat{\psi}))(0) = (C(\phi) * v \hat{\psi})(0) = (C(\phi), \psi)$$

it follows that $u * \phi = C(\phi)$ for all $\phi \in \mathcal{S}(\mathfrak{g}_c)$.

Therefore the only problem is to check that the linear functional $u$ in (4.6) is well defined and continuous.

It follows by the above Lemma 4.4 that for all $v \in \mathfrak{g}_c$ we have $\rho(v)C = C\rho(v)$. Moreover, it is easily seen that $\mathcal{S}(\mathfrak{g}_c)$ is contained in the space of smooth vectors for the representation $\rho$ on $L^p(\mathfrak{g}_c)$, and then by [Po72, Th. 3.1, Th. 5.1] we obtain that $C\mathcal{S}(\mathfrak{g}_c)$ is also contained in the space of smooth vectors for $\rho$ in $L^q(\mathfrak{g}_c)$. Then for all $X \in U((\mathfrak{g}_c)_C)$ and $\phi \in \mathcal{S}(\mathfrak{g}_c)$ the function $C\phi$ is smooth on $\mathfrak{g}_c$ and

$$d\rho(X)C\phi = C(d\rho(X)\phi) \quad \text{and} \quad d\rho(X)C\phi \in L^q(\mathfrak{g}_c).$$

(4.7)

It then follows that for every integer $k > n/q$ there exists a constant $c > 0$ such that for every $\phi \in \mathcal{S}(\mathfrak{g}_c)$ we have

$$|(C\phi)(0)| \leq c \sum_{|J| \leq k} \|\tilde{X}^J(C\phi)\|_{L^q(\mathfrak{g}_c)}$$

(see [Po72, Lemma 5.1] or the proof of [CG90, Lemma A.1.5]). By using Lemma 4.3 we see that, maybe by changing the constant $c > 0$, we also have

$$|(C\phi)(0)| \leq c \sum_{|J| \leq k} \|d\rho(X)^J(C\phi)\|_{L^q(\mathfrak{g}_c)}$$

$$= c \sum_{|J| \leq k} \|C d\rho(X)^J \phi\|_{L^q(\mathfrak{g}_c)}$$

$$\leq c \|C\| \sum_{|J| \leq k} \|d\rho(X)^J \phi\|_{L^p(\mathfrak{g}_c)}$$

and another application of Lemma 4.3 shows that the functional $\phi \mapsto (C\phi)(0)$ is continuous on the Schwartz space $\mathcal{S}(\mathfrak{g}_c)$, which concludes the proof.

**Remark 4.6.** In the case $p = 1$ the distribution $u \in \mathcal{S}'(\mathfrak{g}_c)$ in Proposition 4.5 is actually a measure; see [EL69a, Th. 1].
5. Pedersen multipliers are $L^p$-bounded

In this section we prove (see Theorem 5.8) that all Pedersen multipliers in the setting of Section 4 are bounded on the corresponding $L^p$-spaces for all $p \in (1, \infty)$. To this end we will adapt the transference method to projective representations, and this will allow us to obtain a generalization of [Mau80 Th. 5.2] to any nilpotent Lie group with square-integrable representations.

In this section, unless otherwise specified, $H$ is an arbitrary amenable locally compact group, with a fixed left-invariant Haar measure.

**Definition 5.1.** Let $\sigma : H \to B(\mathbb{Z})$ be any strongly continuous representation by isometries of a Banach space, and consider the duality pairing $\langle \cdot, \cdot \rangle : \mathbb{Z} \times \mathbb{Z}^* \to \mathbb{C}$. This gives rise to the “coefficient map”, that is, the bounded bilinear continuous map $\mathbb{Z} \times \mathbb{Z}^* \to C_b(H)$, $(z, u) \mapsto \langle \sigma(z) z, u \rangle$, which further leads to the bounded linear operator defined on the projective tensor product of Banach spaces,

$$\Psi_\sigma : \mathbb{Z} \hat{\otimes} \mathbb{Z}^* \to C_b(H).$$

We define $A_\sigma(H) := \text{Ran } \Psi_\sigma \subseteq C_b(H)$, regarded as a Banach space with the norm for which the canonical isomorphism

$$(\mathbb{Z} \hat{\otimes} \mathbb{Z}^*)/\text{Ker } \Psi_\sigma \to A_\sigma(H), \quad v + \text{Ker } \Psi_\sigma \mapsto \Psi_\sigma(v)$$

is an isometry.

If $p \in (1, \infty)$, $\mathbb{Z} = L^p(H)$, and $\sigma = \lambda : H \to B(L^p(H))$ is the left regular representation, then one denotes $A_p(H) := A_\lambda(H)$ and this is called the Figà-Talamanca-Herz algebra of the group $H$, also called the Fourier algebra in the case $p = 2$. Note that $A_p(H)$ can alternatively be described as the space of functions $F$ in $C(H)$ of the form $F = \sum_{n=1}^{\infty} f_n \ast g_n$, where $(f_n) \subseteq L^p(H)$, $(g_n) \subseteq L^q(H)$ with $(1/p) + (1/q) = 1$, such that $\sum_{n=1}^{\infty} \|f_n\|_p \|g_n\|_q < \infty$; see [De11 Def. 2 in Sect. 3.1; pp. 33, 34], though $A_p(\cdot)$ is defined there as the coefficient space of the right regular representation.

One also considers the multiplier algebra

$$B_p(H) := \{ f \in C(H) \mid \langle \psi f \in A_p(H) \rangle \} \quad f \varphi \in A_p(H)$$

with the norm $\|f\|_{B_p(H)}$ defined as the norm of the bounded linear operator $\varphi \mapsto f \varphi$ on $A_p(H)$. Denoting by $\rho : H \to B(L^p(H))$ the right regular representation, we also define the norm closed subalgebra $\text{Conv}_p(H)$ of the Banach space $B(L^p(H))$ by

$$\text{Conv}_p(H) := \{ S \in B(L^p(H)) \mid \langle \forall g \in H \rangle S \rho(g) = \rho(g) S \}.$$ 

**Remark 5.2.** For later use, we record the following properties that hold for every $p \in (1, \infty)$ and are based on the fact that the locally compact group $H$ is amenable:

1. One has $A_2(H) \subseteq A_p(H)$ (see [De11 Th. 9 in Sect. 8.3]).
2. The space $A_p(H)$ is a closed ideal of $B_p(H)$ (see [PI84 Prop. 19.8]).
3. There exists a natural bilinear map $\langle \cdot, \cdot \rangle : \text{Conv}_p(H) \times A_p(H) \to \mathbb{C}$ which gives rise to an isometric isomorphism of Banach spaces $\text{Conv}_p(H) \cong A_p(H)^*$ (see [PI84 Prop. 10.3, Prop. 19.7], or alternatively [De11 §4.1, Th. 6; §5.4, Cor. 3])

With the above notation, we now prove the following generalization of [Mau80 Lemma 5.1], which gives us the transference inclusion map $A_\sigma(H) \subseteq A_p(H)$. 


Proposition 5.3. Let $V$ be any finite-dimensional vector space, $p \in (1, \infty)$, and $\sigma: H \to B(L^p(V))$ be any representation for which there exist some continuous functions $P: H \times V \to \mathbb{R}$ and $Q: H \times V \to V$ satisfying the following conditions:

(i) For every $g \in H$ the function $P(g, \cdot)$ is a polynomial and the map $Q(g, \cdot)$ is a measure-preserving polynomial diffeomorphism of $V$.

(ii) One has $\langle \sigma(g) \varphi, \psi \rangle = e^{P(g, v)} \varphi(Q(g, v))$ for all $g \in H$, $v \in V$, and $\varphi \in L^p(V)$.

(iii) One has $\langle \sigma(g) \varphi, \psi \rangle \in L^2(H)$ for all $\varphi, \psi \in C_0^\infty(V)$.

Then $\sigma$ is a strongly continuous representation by isometries, and $A_p(H) \subseteq A_p(H)$.

Proof. It is easy to see that assumptions (i) and (ii) ensure that $\sigma: H \to B(L^p(V))$ is a strongly continuous representation by isometries. Hypothesis (i) ensures existence of the continuous unitary representation

$$\sigma_p: H \to B(L^2(\mathbb{V})), \quad \sigma_p(g) = e^{P(g, v)} \varphi(Q(g, v))$$

The representation $\sigma_p$ is square integrable by hypothesis (iii) (see for instance [CG90, page 169]), hence it is unitary equivalent to the restriction of the left regular representation of $H$ to an invariant subspace, and this implies $A_{\sigma_p}(H) \subseteq A_p(H)$. Using also Remark 5.2.1, we obtain $A_{\sigma_p}(H) \subseteq A_p(H)$.

Then for all $\varphi, \psi \in C_0^\infty(V) \subseteq L^2(V) \cap L^p(V)$ we have $\langle \sigma(g) \varphi, \psi \rangle = \langle \sigma_p(g) \varphi, \psi \rangle$ for arbitrary $g \in H$. If $q < 1$, then $\Psi_\sigma: L^p(V) \otimes L^q(V) \to B_p(H)$ is a bounded linear operator by [He71, Th. A, page 72]. On the other hand, $\Psi_\sigma(C_0^\infty(V) \otimes C_0^\infty(V)) \subseteq A_{\sigma_p}(H) \subseteq A_p(H)$, $C_0^\infty(V)$ is dense both in $L^p(V)$ and in $L^q(V)$, while $A_p(H)$ is a closed linear subspace of $B_p(H)$ by Remark 5.2.2. It then follows that $\text{Ran} \Psi_\sigma \subseteq A_p(H)$, that is, $A_\sigma(H) \subseteq A_p(H)$, and this completes the proof. \qed

We make the following definition.

Definition 5.4. In the framework of Proposition 5.3 and its proof the unitary representation $\sigma_p: H \to B(L^2(V))$ is an admissible realization of its unitary equivalence class.

For every $p \in (1, \infty)$ we will denote by $\sigma_p: H \to B(L^p(V))$ the representation denoted by $\sigma$ in the statement of Proposition 5.3.

Definition 5.5. Let $p \in (1, \infty)$ and $\sigma: H \to B(\mathcal{X})$ be any strongly continuous representation by isometries of a (reflexive) Banach space $\mathcal{X}$, with $A_\sigma(H) \subseteq A_p(H)$. Recalling Remark 5.2.4, for arbitrary $S \in \text{Conv}_p(H)$ we define its corresponding transferred operator $\Psi_\sigma^*(S): \mathcal{X} \to \mathcal{X}$ by

$$(\forall x \in \mathcal{X})(\forall \eta \in \mathcal{X}^*) \quad \langle \Psi_\sigma^*(S)x, \eta \rangle := \langle S, u_{x, \eta}^\sigma \rangle,$$

The left-hand side of the above equality uses the duality pairing $\langle \cdot, \cdot \rangle: \mathcal{X} \times \mathcal{X}^* \to \mathbb{C}$, the right-hand side involves the duality pairing $\langle \cdot, \cdot \rangle: \text{Conv}_p(H) \times A_p(H) \to \mathbb{C}$, and in the statement of Proposition 5.3. We also denoted $u_{x, \eta}^\sigma := (\sigma(\cdot)x, \eta) \in A_\sigma(H) \subseteq A_p(H)$ (by assumption), where we use the duality pairing $\langle \cdot, \cdot \rangle: \mathcal{X} \times \mathcal{X}^* \to \mathbb{C}$.

Example 5.6. In Definition 5.5 using the specific form of the duality provided by Remark 5.2.4, one can check that if $\lambda_p: H \to B(L^p(H))$ is the left regular representation and there exists $\varphi \in L^1(H)$ with $S = \lambda_p(\varphi)$, that is, $S\psi = \varphi * \psi$ for all $\psi \in L^p(H)$, then $\Psi_\sigma^*(S) = \int H \varphi(g) \sigma(g) dg = \sigma(\varphi)$. In particular this implies that $\Psi_\sigma^* \circ \lambda_p$ is continuous on $L^1(H)$. Note that for $p = 2$ the identity between
Remark 5.7. Let us explain the notation and some other facts on Definition 5.5.

(i) Since $A_p(H) \subseteq A_p(H)$ and we have continuous inclusions $A_p(H) \hookrightarrow C_0(H)$ and $A_p(H) \hookrightarrow C_0(H)$, it easily follows by the closed graph theorem that the inclusion map $A_p(H) \hookrightarrow A_p(H)$ is also continuous, hence there exists a constant $\kappa > 0$ such that $\|f\|_{A_p(H)} \leq \kappa \|f\|_{A_p(H)}$ for all $f \in A_p(H)$. Moreover,

$$u^\sigma_{x,\eta} = \Psi_\sigma(x \otimes \eta)$$

and this further implies $|\langle \sigma(x), \eta \rangle| \leq \kappa \|\eta\| \cdot \|x\|$, hence $\Psi^\sigma(S) \in \mathcal{B}(\mathcal{X})$ and $\|\Psi^\sigma(S)\| \leq \kappa \|S\|$.

(ii) The map $\Psi^\sigma: \text{Conv}_p(H) \to \mathcal{B}(\mathcal{X})$ is an algebra homomorphism and is equal to the dual map of $\Psi_\sigma: \mathcal{X} \otimes \mathcal{X}^* \to A_\sigma(H) \hookrightarrow A_p(H)$.

In fact, that duality property follows directly by Definition 5.5, and it implies that $\Psi^\sigma_\sigma$ is continuous with respect to the weak*-topologies on the dual spaces $\mathcal{B}(\mathcal{X}) \simeq (\mathcal{X} \otimes \mathcal{X}^*)^*$ and $\text{Conv}_p(H) \simeq A_p(H)^*$. Then, using also the easy checked fact that the multiplication maps in the algebras $\text{Conv}_p(H)$ and $\mathcal{B}(\mathcal{X})$ are separately weak*-continuous, it follows that it suffices to prove $\Psi^\sigma_\sigma(S_1S_2) = \Psi^\sigma_\sigma(S_1)\Psi^\sigma_\sigma(S_2)$ for all $S_1$ and $S_2$ in some weak*-dense subset of $\text{Conv}_p(H)$. But that property is a direct consequence of Example 5.6 if $S_j = \lambda_p(\varphi_j)$ with $\varphi_j \in L^1(H)$ for $j = 1, 2$, and on the other hand $\{\lambda_p(\varphi) \mid \varphi \in L^1(H)\}$ is weak*-dense in $\text{Conv}_p(H)$ by [De11, §4.1, Cor. 7(1); §5.4, Cor. 3], also using that $H$ is amenable.

In particular, for every $S \in \text{Conv}_p(H)$, we have the commutative diagram

$$\begin{array}{ccc}
A_p(H) = (L^p(H) \otimes L^p(H)^*) / \ker \Psi_\lambda & \xrightarrow{S} & \mathbb{C} \\
\uparrow & \quad & \uparrow 1_{\mathbb{C}} \\
A_\sigma(H) = (\mathcal{X} \otimes \mathcal{X}^*) / \ker \Psi_\sigma & \xrightarrow{\Psi^\sigma_\sigma(S)} & \mathbb{C}
\end{array}$$

with $\iota$ the inclusion mapping.

We can now state the following generalization of [Mau80, Th. 5.2]. In this statement we recall the setting we are working in, for the sake of clarity.

Theorem 5.8. Let $G$ be a nilpotent Lie group with 1-dimensional center and with generic flat coadjoint orbits, and $p \in (1, \infty)$. If $\pi: G \to \mathcal{B}(L^2(V))$ is an arbitrary admissible realization of a point in the unitary dual of $G$ corresponding to a generic coadjoint orbit, then for each of its corresponding Pedersen multipliers $M \in \mathcal{B}(L^2(V))$ its restriction $M|_{L^2(V) \cap L^p(V)}$ extends to an operator in $\mathcal{B}(L^p(V))$.

By Kirillov’s theory every irreducible unitary representation of $G$ is (up to unitary equivalence) an induced representation through some polarization of the Lie algebra of $G$ (see [CG90, page 46]). Then, for every point in the unitary dual of a nilpotent Lie group there is an admissible realization (see for instance [ER93, Lemma 2.1]). Moreover, in the conditions of Theorem 5.8 $\pi$ is square-integrable modulo the center (see [CG90, Th. 4.5.2]).

The proof of Theorem 5.8 requires some preparations.
First note that we work in the Setting 5.3 of Section 4. Moreover we denote
\[ \gamma(x_1, x_2) := e^{i\alpha(x_1, x_2)}, \quad \text{for all } x_1, x_2 \in g_e, \]
so that (5.1) takes on the form
\[ (\forall x_1, x_2 \in g_e) \quad \pi(x_1)\pi(x_2) = \gamma(x_1, x_2)\pi(x_1 \cdot e, x_2). \quad (5.1) \]

**Notation 5.9.** Recall that we have identified \( G_e \) and \( g_e \) via the exponential mapping. We denote \( \Gamma := G / \text{Ker} \pi \). This is a nilpotent Lie group which is not simply connected, and in fact its center \( Z / \text{Ker} \pi \) is isomorphic to the unit circle \( T \). One has a short exact sequence of Lie groups
\[ 1 \rightarrow T \rightarrow \Gamma \rightarrow G_e \rightarrow 1 \]
and the group \( \Gamma \) can be alternatively described as \( \Gamma = T \times G_e \) with the product given by
\[ (t_1, x_1) \cdot (t_2, x_2) = (t_1 t_2 \gamma(x_1, x_2), x_1 \cdot e x_2) \text{ for all } (t_1, x_1), (t_2, x_2) \in T \times G_e. \]

**Lemma 5.10.** For arbitrary \( p \in [1, \infty) \) define the maps
\[ \sharp: L^p(G_e) \rightarrow L^p(\Gamma), \quad \psi \mapsto \psi^\sharp, \quad \psi^\sharp(t, x) := \frac{1}{t} \psi(x) \]
and
\[ \flat: L^p(\Gamma) \rightarrow L^p(G_e), \quad \varphi \mapsto \varphi^\flat, \quad \varphi^\flat(x) := \int_T \varphi(s, x) ds. \]

Also let
\[ \mathcal{P}: L^p(\Gamma) \rightarrow L^p(\Gamma), \quad (\mathcal{P}\varphi)(t, x) = \frac{1}{t} \int_T \varphi(s, x) ds. \]

Then the following assertions hold:

(i) All the maps \( \sharp: (L^p(G_e), *) \rightarrow (L^p(\Gamma), *), \) \( (L^p(\Gamma), *) \rightarrow (L^p(G_e), *) \)
\( \mathcal{P}, 1 - \mathcal{P}: (L^p(\Gamma), *) \rightarrow (L^p(G_e), *) \) are norm non-increasing bounded operators, \( \sharp \) and \( \mathcal{P} \) are actually isometries, \( \mathcal{P} \) is a projection and the images of \( \mathcal{P} \) and \( \sharp \) are the same. For \( p = 1 \), the above operators are homomorphisms of Banach \(*\)-algebras, the image of \( \sharp \) is a closed two-sided ideal of \( (L^1(\Gamma), *) \), and also the image of \( 1 - \mathcal{P} \) is a closed two-sided ideal of \( (L^1(\Gamma), *) \).

(ii) For all \( \psi \in L^p(G_e) \) and \( \varphi \in L^q(\Gamma) \) one has \( (\psi^\sharp)^\flat = \psi \) and \( (\varphi^\flat)^\sharp = \mathcal{P}\varphi \).

(iii) If \( \sigma: \Gamma = G / \text{Ker} \pi \rightarrow B(L^2(V)) \) is the representation obtained by the factorization of \( \pi: G \rightarrow B(L^2(V)) \), then for every \( \psi \in L^1(G_e) \cup L^2(G_e) \) one has \( T^\pi(\psi) = \sigma(\psi^\flat) \).

**Proof.** Assertion (ii) follows by [5.69a, Lemmas 3.1–3.2, Th. 2]. Assertion (iii) is straightforward. For Assertion (iii), note that since \( \Gamma = T \times G_e \), we have \( \sigma(t, x) = t\pi(x) \) and \( \psi^\flat(t, x) = t^{-1}\psi(x) \), and therefore \( \psi \in L^q(G_e) \) if and only if \( \psi^\flat \in L^q(\Gamma) \) for \( q = 1, 2 \). Then the equality \( T^\pi(\psi) = \sigma(\psi^\flat) \) follows using the boundedness of \( \sigma \) when \( \psi \in L^1(G_e) \), or the fact that \( \sigma \) is square-integrable, when \( \psi \in L^2(G_e) \). \( \square \)

**Proof of Theorem 5.8.** Using Definition 5.3 for every \( \psi \in \mathcal{S}(G_e) \) we have
\[ MT^\pi(\psi) = T^\pi(C_M \psi). \]
Now use Lemma 5.10 to define \( T_M : L^p(\Gamma) \to L^p(\Gamma) \) via the commutative diagram

\[
\begin{array}{ccc}
L^p(G_e) & \xrightarrow{C_M} & L^p(G_e) \\
\downarrow & & \downarrow \mathbb{i} \\
L^p(\Gamma) & \xrightarrow{T_M} & L^p(\Gamma)
\end{array}
\]

hence for every \( \varphi_1, \varphi_2 \in L^1(\Gamma) \cap L^p(\Gamma) \) we have

\[
T_M(\varphi_1 \ast \varphi_2) = (C_M((\varphi_1 \ast \varphi_2)^\flat))^\sharp = (C_M(\varphi_1^\flat \ast_e \varphi_2^\flat))^\sharp = (C_M(\varphi_1^\flat) \ast_e \varphi_2^\flat)^\sharp
\]

The last of the above equalities is based on the fact that by Lemma 5.10, the maps \( \sharp \) and \( \mathcal{P} \) have the same image in \( L^1(\Gamma) \), while \( \mathcal{P}(L^1(\Gamma)) \) and \( (1 - \mathcal{P})(L^1(\Gamma)) \) are closed ideals of \( (L^1(\Gamma), \ast) \) whose intersection is \( \{0\} \), hence \( \mathcal{P}(\varphi_1) \ast (1 - \mathcal{P})(\varphi_2) = 0 \) for all \( \varphi_1 \in L^1(\Gamma) \cap L^p(\Gamma), \varphi_2 \in L^1(\Gamma) \). Now, the required equality follows because \( L^p(\Gamma) \) is a right Banach module of \( L^1(\Gamma), \mathcal{P}(L^1(\Gamma) \cap L^p(\Gamma)) \) is dense in \( \mathcal{P}(L^p(\Gamma)) \) and \( T_M(\varphi_1) \in \mathcal{P}(L^p(\Gamma)) \).

Since \( T_M \in \mathcal{B}(L^p(\Gamma)) \), it follows by the above equalities that \( T_M \in \text{Conv}_p(\Gamma) \), and then by Proposition 5.3 we may define the transferred operator \( \Psi^*_{\sigma_p}(T_M) \) in \( \mathcal{B}(L^p(\Gamma')) \). Recall that, for \( k \in \{p, 2\} \) and \( \sigma_2 = \sigma \), the mapping \( \Phi^*_{\sigma_k} \) is the dual operator

\[
\Psi^*_{\sigma_k} : \text{Conv}_k(\Gamma) \simeq A_k(\Gamma)^* \to (L^k(\mathcal{V}) \hat{\otimes} L^k(\mathcal{V}))^* \simeq \mathcal{B}(L^k(\mathcal{V}))
\]

of the coefficient map

\[
\Psi_{\sigma_k} : L^k(\mathcal{V}) \hat{\otimes} L^k(\mathcal{V})^* \to A_{\sigma_k}(\Gamma) \hookrightarrow A_k(\Gamma).
\]

Also, \( \text{Conv}_p(\Gamma) \hookrightarrow \text{Conv}_2(\Gamma) \) by \cite{De11} §8.3, Th. 6. Thus since \( \sigma_p(g) = \sigma_2(g) \) on \( \mathcal{S}(\mathcal{V}) \subseteq L^p(\mathcal{V}) \cap L^2(\mathcal{V}) \), it follows that \( \Psi^*_{\sigma_p} \) is equal to the composition

\[
\text{Conv}_p(\Gamma) \hookrightarrow \text{Conv}_2(\Gamma) \xrightarrow{\Psi^*_{\sigma_2}} (\mathcal{S}(\mathcal{V}) \hat{\otimes} \mathcal{S}(\mathcal{V}))^* \simeq \mathcal{L}(\mathcal{S}(\mathcal{V}), \mathcal{S}'(\mathcal{V})).
\]

In particular,

\[
(\Psi^*_{\sigma_p}(T_M) = \Psi^*_{\sigma_2}(T_M) \quad \text{on} \quad \mathcal{S}(\mathcal{V}) \subseteq L^p(\mathcal{V}) \cap L^2(\mathcal{V}).
\]

Moreover, we will show that

\[
\Psi^*_{\sigma_2}(T_M) = M.
\]

To this end, first recall as well that \( \Psi^*_{\sigma_2} \) satisfies

\[
\Psi^*_{\sigma_2}(\lambda(\varphi)) = \sigma(\varphi) \quad \text{for all} \quad \varphi \in L^1(\Gamma) \cup L^2(\Gamma),
\]

by Example 5.6. Furthermore, one has \( T^\pi(C_M \phi) = M T^\pi(\phi) \in \mathfrak{S}_2(L^2(\mathcal{V})) \) for every \( \phi \in L^2(G_e) \cap L^p(G_e) \), hence \( C_M \phi \in L^p(G_e) \cap L^2(G_e) \). This implies

\[
T_M(L^p(\Gamma) \cap L^2(\Gamma)) \subseteq L^p(\Gamma) \cap L^2(\Gamma).
\]

Then, for every \( \psi \in \mathcal{S}(G_e) \) one has \( M T^\pi(\psi) = T^\pi(C_M \psi) = \sigma((C_M \psi)^\sharp) \) by Lemma 6.10ii. Using the equality \( \varphi^\sharp = \psi \) for \( \varphi := \psi^\sharp \) (see Lemma 5.10i), one further obtains

\[
M T^\pi(\psi) = \sigma((C_M \psi)^\sharp) = \sigma(T_M \varphi) = \Psi^*_{\sigma_2}(\lambda(T_M \varphi)) = \Psi^*_{\sigma_2}(T_M \lambda(\varphi))
\]
where the last two equalities are based on (5.3) and on the fact that $T_M \in \text{Conv}_\rho(\Gamma)$, respectively. Now, since $\Psi^*_\sigma : \text{Conv}_2(\Gamma) \to \mathcal{B}(L^2(\mathcal{V}))$ is a morphism of Banach algebras (Remark 5.7), it follows that

$$MT^\pi(\psi) = \Psi^*_\sigma(T_M)\Psi^*_\sigma(\lambda(\varphi)) = \Psi^*_\sigma(T_M)\sigma(\varphi) = \Psi^*_\sigma(T_M)\sigma(\psi^2) = \Psi^*_\sigma(T_M)T^\pi(\psi).$$

Here we have used successively (5.3), $\varphi = \psi^2$, and Lemma 5.10 (iii). Since $\psi \in \mathcal{S}(G_c)$ is arbitrary and $T^\pi(\mathcal{S}(G_c))L^2(\mathcal{V})$ is dense in $L^2(\mathcal{V})$ the above equalities imply (5.3).

Now recall that $T_M \in \text{Conv}_\rho(\Gamma)$. Then Proposition [5.3] shows that it is transferred to the operator $\Psi^*_\sigma(T_M) \in \mathcal{B}(L^p(\mathcal{V}))$. On the other hand, (5.2) and (5.3) show that $M = \Psi^*_\sigma(T_M) = \Psi^*_\sigma(T_M) \in \mathcal{B}(L^p(\mathcal{V}))$, and this concludes the proof. \[\square\]

6. Calderón-Zygmund theory with a twist

The main result of this subsection is a simultaneous generalization of [Mau80, Th. 3.2] and [CW71, Th. III. 2.4], which is established in Corollary 6.10 below. As a consequence, we give a collection of examples of Pedersen multipliers in Corollary 6.12. Some methods we use in this section have common ideas with [CW71, Ch. III, §3], but we do not work here with compact groups. In fact, we will use these results for simply connected nilpotent Lie groups, without any assumption of homogeneity as in the earlier literature.

6.1. Pseudo-distances, cocycles and a Calderón-Zygmund decomposition for locally compact groups.

**Setting 6.1.** Unless otherwise mentioned, in this subsection we work in the setting provided by a unimodular locally compact group $G$ with a Haar measure denoted by $| \cdot |$. We also assume that there is a function $m : G \to [0, \infty]$ with the following properties:

(1) $m(x) = 0$ if and only if $x = 1$.

(2) $(\forall x \in G) \ m(x) = m(x^{-1})$;

(3) There is a constant $C$ such that

$$m(xy) \leq C \max\{m(x), m(y)\} \leq C(m(x) + m(y))$$

for every $x, y \in G$.

(4) For $r > 0$, $x \in G$, we denote

$$B(x, r) = \{y \in G \mid m(xy^{-1}) < r\}. $$

Then there is $C > 0$ such that for every $r > 0$

$$|B(1, 2r)| \leq C|B(1, r)|. $$

Note that in the Setting 6.1 the function $m : G \times G \to [0, \infty]$ defined by $m(x, y) = m(xy^{-1})$ is a right-invariant pseudo-distance on $G$ and the space $(G, m, | \cdot |)$ is a space of homogeneous type, in the sense of [CW71, Ch. III, §1]. The right invariance of $m$ also shows that for every $x \in G$ and $r > 0$ we have

$$|B(x, 2r)| \leq C|B(x, r)|. \quad (6.1)$$

**Example 6.2.** The first example records some facts from [CW71, Ch. III, §3], but we do not assume here that the group $G$ is compact.
Let $G$ be a unimodular locally compact group $G$ with a Haar measure denoted by $|\cdot|$, for which there exist a constant $C > 1$ and a neighbourhood basis consisting of relatively compact open sets $1 \in \cdots \subseteq U_2 \subseteq U_1 \subseteq G$ satisfying for every $j \geq 1$

(i) $U_j = U_j^{-1}$;
(ii) $U_j U_{j+1} \subseteq U_j$;
(iii) $|U_j| < C|U_{j+1}|$.

We then define $U_0 := G$, 

\[ m : G \to [0, \infty], \quad m(x) = \inf\{|U_j| \mid x \in U_j\} \]

and also 

\[ m : G \times G \to [0, \infty], \quad m(x, y) := m(xy^{-1}) \]

and 

\[(\forall x \in G)(\forall r \in (0, \infty]) \quad B(x, r) := \{ y \in G \mid m(x, y) < r \}.\]

We may have $m(x) = \infty$ at most for $x \in G \setminus U_1$, and this does happen if and only if $G$ is noncompact.

The function $m$ defined above satisfies the conditions in the Setting 6.1 above. It is easy to check that (1)–(3) hold true. To check (1), we show that $\{B(1, r) \mid 0 < r \leq |G|\} = \{U_j \mid j \geq 1\}$ and 

\[(\forall r > 0) \quad \frac{r}{C} < |B(1, r)| < r. \tag{6.2} \]

Indeed, note that $|U_j| < \infty$ as $j \to \infty$, and for all $0 < r \leq |G|$ and $j \geq 0$ the implication 

\[ |U_{j+1}| < r \leq |U_j| \implies B(1, r) = U_{j+1} \tag{6.3} \]

holds true. In fact, if $x \in B(1, r)$, then $m(x) < r \leq |G|$, hence there exists the least integer $k \geq 0$ with $x \in U_{k+1}$ and $|U_{k+1}| < r$. Since $k$ is the least with that property, we have $|U_{k+1}| < r \leq |U_k|$, hence actually $k = j$ and then $x \in U_{j+1}$. Conversely, if $x \in U_{j+1}$, then $m(x) \leq |U_{j+1}| < r$, hence $x \in B(1, r)$. This completes the proof of (6.3), which readily implies that (6.2) holds true.

**Example 6.3.** The present example provides a generalization of the examples in [CW71, Ch. III, §3, page 79], and shows that if $G$ is any connected Lie group, then there exists a fundamental system of neighborhoods $\{U_1\}_{1 \geq 0}$ of $1 \in G$ satisfying the conditions (i)–(iii) in Example 6.2.

In fact, let $\mathfrak{g}$ be the Lie algebra of $G$. Fix any Euclidean norm $\|\cdot\|$ on $\mathfrak{g}$ and endow $G$ with the corresponding left-invariant Riemannian structure and the corresponding distance function $d : G \times G \to [0, \infty)$, which is left invariant by [KN63, Prop. IV.2.5] and complete by [KN63, Th. IV.4.5]. Then select some $r_0 > 0$ for which the Riemannian exponential map at $1 \in G$, denoted by $\exp_{1, r} : V(0, r) \to U(1, r)$, is a diffeomorphism if $0 < r \leq r_0$, where $V(0, r) := \{ X \in \mathfrak{g} \mid \|X\| < r \}$ and $U(1, r) := \{ x \in G \mid d(1, x) < r \}$ for every $r > 0$ (see [KN63, Prop. IV.3.4]).

After replacing $r_0$ by a lesser value, we may assume that there exist some constants $0 < C_1 < C_2$ with 

\[ C_1 r^m \leq |U(1, r)| \leq C_2 r^m \text{ if } 0 < r \leq r_0 \tag{6.4} \]

where $m := \dim G$ and $|\cdot|$ denotes the left-invariant Haar measure on $G$ defined by the volume associated with the Riemannian metric. To see this, one may either use the fact that the differential of the Riemannian exponential map at $0 \in \mathfrak{g}$ is the identity map on $\mathfrak{g}$, or one can use the second formula in [Gr73, Th. 3.1].
On the other hand, using the fact that the distance function $d$ is left invariant on $G$, we obtain for all $x, y \in G$,
\[ d(1, xy) \leq d(1, x) + d(x, xy) = d(1, x) + d(1, y) \leq 2 \max\{d(1, x), d(1, y)\} \]
and this implies
\[ (\forall r > 0) \quad U(1, r) \cdot U(1, r) \subseteq U(1, 2r). \tag{6.5} \]
If we now fix any $r$ with $0 < r \leq r_0$ and define $U_j := U(1, r/2^j)$ for all $j \geq 0$, then the conditions (i)–(iii) in Setting 6.1 are satisfied. In fact, it follows by (6.4) that
\[ (\forall j \geq 0) \quad C_2 r^{m/2} \leq |U_j| \leq C_2 r^{m/2} \]
hence $|U_j| \leq C|U_{j+1}|$ for $C := 2^n C_2/C_1$. Using (6.5), we also obtain $U_{j+1} U_{j+1} \subseteq U_j$ for all $j \geq 0$, and we are done.

Let $L^1_{\text{comp}}(G)$ denote the vector subspace of $L^1(G)$ consisting of (equivalence classes of) integrable functions on $G$ whose essential support is compact.

**Proposition 6.4.** There exist constants $C', M > 0$ for which the following assertion holds true: If $0 \leq f \in L^1_{\text{comp}}(G)$ and $\alpha > 0$, then there exist a sequence $\{z_i\}_{i \geq 1}$ in $G$ and a sequence of positive real numbers $\{r_i\}_{i \geq 1}$ satisfying the following conditions:

1. $f \leq \alpha$ almost everywhere on $G \setminus \left( \bigcup_{i \geq 1} B(z_i, r_i) \right)$;
2. for every $i \geq 1$ we have $\frac{1}{|B(z_i, r_i)|} \int_{B(z_i, r_i)} f(x)dx \leq C' \alpha$;
3. $\sum_{i \geq 1} |B(z_i, r_i)| \leq \frac{C'}{\alpha} \int_G f(x)dx$;
4. for every $x \in G$ the set $\{i \geq 1 \mid x \in B(z_i, r_i)\}$ contains at most $M$ elements.

**Proof.** It follows by (6.1), along with the invariance of the Haar measure that there exists a constant $A > 0$ with the property
\[ (\forall x \in G)(\forall r > 0) \quad 0 < |B(x, r)| \leq A|B(x, r/2)|. \tag{6.6} \]
Therefore one can use [CW71, Th. III.(2.3)] to obtain the conclusion. $\square$

**Corollary 6.5.** There exists a constant $C'' > 0$ for which the following assertion holds true for any continuous function $\gamma: G \times G \to T$: If $0 \leq f \in L^1_{\text{comp}}(G)$, $\alpha > 0$, and $\{z_i\}_{i \geq 1}$ and $\{r_i\}_{i \geq 1}$ are the sequences provided by Proposition 6.4, then there exist $g, b_1, b_2, \ldots \in L^1(G)$ satisfying the following conditions:

1. $f = g + \sum_{i \geq 0} b_i$ almost everywhere on $G$;
2. $|g| \leq C'' \alpha$ almost everywhere on $G$;
3. $\|g\|_1 \leq C'' \|f\|_1$;
4. for every $i \geq 1$ we have $\text{supp} b_i \subseteq B(z_i, r_i)$;
5. for every $i \geq 1$ we have $\int_G b_i(z) \gamma(z, z^{-1})dz = 0$;
6. $\sum_{i \geq 1} \|b_i\|_1 \leq C'' \|f\|_1$.

**Proof.** Let $B := \bigcup_{i \geq 1} B_i$, where $B_i := B(z_i, r_i)$ for every $i \geq 1$. Also denote $\chi_i := \chi_{B_i}$ (the characteristic function of $B_i$) and define
\[ \eta_i(z) = \begin{cases} \frac{\chi_i(z)}{\sum_{i \geq 1} \chi_i(z)} & \text{if } z \in B, \\ 0 & \text{if } z \in G \setminus B. \end{cases} \]
Note that \( \sum_{i \geq 1} \eta_i = \chi_B \).

Then for arbitrary \( i \geq 1 \) and \( z \in G \) define

\[
b_i(z) := f(z)\eta_i(z) - \frac{1}{|B_1|} \left( \int_G f(u)\eta_i(u)\gamma(z,u^{-1})du \right) \chi_i(z)\gamma(z,z^{-1})^{-1}
\]

and

\[
g(z) = \begin{cases} 
\sum_{i \geq 1} \frac{1}{|B_1|} \left( \int_G f(u)\eta_i(u)\gamma(z,u^{-1})du \right) \chi_i(z)\gamma(z,z^{-1})^{-1} & \text{if } z \in B, \\
\frac{1}{f(z)} & \text{if } z \in G \setminus B.
\end{cases}
\]

We now check that the conditions of the statement are satisfied.

1. We have \( g + \sum_{i \geq 1} b_i = f \) on \( G \setminus B \), and on the other hand \( g + \sum_{i \geq 1} b_i = f + \sum_{i \geq 1} \eta_i = f\chi_B = f \) on \( B \).

2. If \( z \notin B \), then \( |g(z)| = |f(z)| \leq C\alpha \). On the other hand, if \( z \in B \), then also \( |g(z)| \leq \sum_{i \geq 1} \frac{1}{|B_1|} \int_B f(u)du \leq C\alpha \).

3. We have \( \int_{G \setminus B} |g(x)|dx = \int_{G \setminus B} |f(x)|dx \leq \|f\|_1 \) and

\[
\int_B |g(x)|dx \leq \sum_{i \geq 1} \int_{B_1} |f(x)|dx \leq M\|f\|_1
\]

where \( M > 0 \) is the constant provided by Proposition 6.4.

4. For this condition we have \( \text{supp } b_i \subseteq (\text{supp } \eta_i) \cup (\text{supp } \chi_i) \subseteq B(z_i, r_i) \).

5. For every \( z \in G \) we have

\[
b_i(z)\gamma(z,z^{-1}) = f(z)\eta_i(z)\gamma(z,z^{-1}) - \frac{1}{|B_1|} \left( \int_G f(u)\eta_i(u)\gamma(z,u^{-1})du \right) \chi_i(z)
\]

hence \( \int_G b_i(z)\gamma(z,z^{-1})dz = 0 \).

6. We have \( \|b_i\|_1 \leq \int_G f(z)\eta_i(z)dz + \int_{B_1} f(z)dz \) hence

\[
\sum_{i \geq 1} \|b_i\|_1 \leq \int_G f(z)dz + \sum_{i \geq 1} \int_{B_1} f(z)dz \leq (M + 1)\|f\|_1
\]

and this completes the proof. \( \square \)

The next lemma records some properties of a cocycle arising from a projective representation of a topological group.

**Lemma 6.6.** Let \( H \) be any topological group. Assume that \( H \) is any complex Hilbert space and \( \pi: H \to U(\mathcal{H}) \) and \( \gamma: H \times H \to \mathbb{T} \) are any maps satisfying

\[
(\forall x, y \in H) \quad \pi(x)\pi(y) = \gamma(x,y)\pi(xy), \quad \pi(x^{-1}) = \pi(x)^{-1}, \quad \text{and } \pi(1) = 1.
\]

Then the following assertions hold true for all \( x, y, z \in H \):

(i) \( \gamma(x,y)\gamma(xy,z) = \gamma(x,yz)\gamma(y,z) \);
(ii) \( \gamma(y^{-1},x^{-1}) = \gamma(x,y)^{-1} \);
(iii) \( \gamma(xy^{-1},y) = \gamma(x,y^{-1})^{-1} \);
(iv) \( \gamma(x,x^{-1}y) = \gamma(x^{-1},y)^{-1} \);
(v) \( \gamma(x,z^{-1})\gamma(xz^{-1},zy^{-1}) = \gamma(x,y^{-1})\gamma(y,z^{-1}) \).
We now still need to prove that 
\[ \gamma \] hence \[ \gamma \] Here we replace the assertion.
The relations \[ \gamma(x, z^{-1}) \gamma(xz^{-1}, z) = \gamma(x, 1) \gamma(z^{-1}, z). \]
On the other hand, applying \( \pi \) to \( z^{-1}z = 1 \), one has \( \pi(z^{-1})\pi(z) = \gamma(z^{-1}, z)\pi(1) \), hence \( \gamma(z^{-1}, z) = 1 \). Then, applying \( \pi \) to \( x1 = x \), one has \( \pi(x)\pi(1) = \gamma(x, 1)\pi(x) \), hence \( \gamma(x, 1) = 1 \). Thus the above displayed equality directly implies \( \text{iii} \) with \( z \) replaced by \( y \).

For \( \text{iv} \), one proceeds as above, first plugging in \( y := x^{-1} \) in \( \text{iii} \), and this implies
\[ \gamma(x, x^{-1})\gamma(1, z) = \gamma(x, x^{-1}z)\gamma(x^{-1}, z). \]
The relations \( \gamma(1, z) = \gamma(x, x^{-1}) = 1 \) can be established as above, and we obtain the assertion.

For \( \text{v} \), we plug in \( z := v \) and \( y := z^{-1} \) in \( \text{iii} \). This gives
\[ \gamma(x, z^{-1})\gamma(xz^{-1}, v) = \gamma(x, z^{-1}v)\gamma(z^{-1}, v). \]
Here we replace \( v := zy^{-1} \), and we thus obtain
\[ \gamma(x, z^{-1})\gamma(xz^{-1}, zy^{-1}) = \gamma(x, y^{-1})\gamma(z^{-1}, zy^{-1}). \]
We now still need to prove that \( \gamma(z^{-1}, zy^{-1}) = \gamma(y, z^{-1}) \). Using \( \text{iii} \), that equality is equivalent to \( \gamma(z^{-1}, zy^{-1}) = \gamma(z, y^{-1})^{-1} \). But this can be obtained directly from \( \text{vi} \) for \( z := x^{-1} \) and then replacing \( y \) by \( y^{-1} \). This completes the proof. \( \square \)

6.2. A continuity result for twisted convolutions on nilpotent Lie groups.
We now resume the study of twisted convolutions in the context of nilpotent Lie groups. In this subsection we assume the following setting.

Setting 6.7.
- \( G \) is a connected and simply connected nilpotent Lie group.
- \( \pi: G \to B(\mathcal{H}) \) is a continuous unitary projective representation of \( G \) with cocycle \( \gamma: G \times G \to \mathbb{T} \).
- \( \gamma \) is a smooth polynomial cocycle, that is, \( \gamma(x, y) = e^{i\alpha(x,y)} \), where \( \alpha \) is a real polynomial function on \( G \times G \).
- On \( G \) we consider a pseudometric \( m \) as in Setting 6.7 and we denote by \( C_m \) a constant for which
\[ m(xy) \leq C_m \max\{m(x), m(y)\}. \]

In the above Setting 6.7 for \( k \in S'(G) \) and \( f \in S(G) \) we can define \( k \ast_{\gamma} f \in S'(G) \cap C(G) \) by
\[ k \ast_{\gamma} f(x) = \langle k(y), \gamma(y^{-1}, x) f(y^{-1}x) \rangle. \]
Note that this twisted convolution extends for \( k \in L^1_{\text{loc}}(G) \) and \( f \in C_0(G) \) by
\[ (k \ast_{\gamma} f)(x) = \int_G \gamma(x, y^{-1}) k(xy^{-1}) f(y) dy, \]
where we have used Lemma 6.6 (\text{iii}, \text{iv}).

Definition 6.8. In the Setting 6.7 a distribution \( k \in S'(G) \) is called a Calderón-Zygmund kernel on \( G \) (with respect to \( \gamma \) and \( m \)) if it is locally integrable outside 1 on \( G \) and satisfies the following conditions:
(i) There exists a constant \( C_2 > 2C_m \) for which the function
\[
    u \mapsto \int_{m(z) > C_2 m(u)} |\gamma(z, u^{-1})k(zu^{-1}) - k(z)| \, dz
\]
is in \( L^\infty(G) \).

(ii) For \( f \in \mathcal{S}(G), k \ast f \in L^2(G) \), and the operator \( K: \mathcal{S}(G) \to L^2(G) \) defined by
\[
    (Kf)(x) = k \ast f, \quad f \in \mathcal{S}(G),
\]
extends to a bounded operator on \( L^2(G) \).

**Theorem 6.9.** In the Setting \( 6.7 \) let \( k \in \mathcal{S}'(G) \) be a Calderón-Zygmund kernel. Then there exists a constant \( A_1 > 0 \) with the property that for every \( f \in L^1_{\text{comp}}(G) \) we have
\[
    (\forall \alpha > 0) \quad \|\{(Kf)(\cdot)\} > \alpha\| \leq \frac{A_1}{\alpha} \|f\|_1.
\]

**Proof.** First note that for arbitrary \( f \in L^1_{\text{comp}}(G) \subseteq L^2(G) \) we have by hypothesis that \( Kf \in L^2(G) \).

Next, write \( f = g + \sum_{i \geq 1} b_i \) as in Corollary \( 6.5 \). Then for arbitrary \( \alpha > 0 \) we have
\[
    \|\{(Kf)(\cdot)\} > \alpha\| \leq \|\{(Kg)(\cdot)\} > \alpha/2\| + \left\| \sum_{i \geq 1} \{(Kb_i)(\cdot)\} > \alpha/2\right\|.
\]

Chebyshev’s inequality and Corollary \( 6.5 \) show that
\[
    \|\{(Kg)(\cdot)\} > \alpha/2\| \leq \frac{4(C_m' \|K\|)^2}{\alpha} \|f\|_1.
\]

On the other hand, by using Proposition \( 6.4 \) (3.) and \( 6.6 \), we obtain a constant \( A_2 > 0 \) for which
\[
    \|\sum_{i \geq 1} B(\bar{z}_i, C_2 r_i)\| \leq \sum_{i \geq 1} |B(\bar{z}_i, C_2 r_i)| \leq A_2 \sum_{i \geq 1} |B(\bar{z}_i, r_i)| \leq \frac{A_2 C_2'}{\alpha} \|f\|_1.
\]

Therefore, in order to estimate the second term in the right-hand side of \( 6.7 \), it suffices to estimate \( \|\{x \in G \setminus \bigcup_{i \geq 1} B_2^i \mid \sum_{i \geq 1} |(Kb_i)(x)| > \frac{\alpha}{2}\\| \), where we denoted \( B_2^i := B(\bar{z}_i, C_2 r_i) \) for every \( i \geq 1 \). We have
\[
    \|\{x \in G \setminus \bigcup_{i \geq 1} B_2^i \mid \sum_{i \geq 1} |(Kb_i)(x)| > \frac{\alpha}{2}\\|
    \leq \|\{x \in (G \setminus B_2^i) \mid \sum_{i \geq 1} |(Kb_i)(x)| > \frac{\alpha}{2}\}\|
    \leq \frac{2}{\alpha} \sum_{i \geq 1} \int_{G \setminus B_2^i} |(Kb_i)(x)| \, dx.
\]

Note that for \( x \in G \setminus B_2^i \) and \( y \in B_i \) we have that
\[
    2C_{r_i} \leq C_{2r_i} \leq m(z_i x_{-1}) \leq C(m(y z_{-1} z_i x_{-1}) + m(z_{i} y_{-1})) \leq C(m(y x_{-1}) + r_i)
\]
hence \( m(xy_{-1}) \geq r_i \). Then we can write
\[
    (Kb_i)(x) = \int_{G} \gamma(x, y_{-1})k(xy_{-1}) b_i(y) \, dy.
\]
Thus, for every $i \geq 1$ we have

$$\int_{G \setminus B_i^2} |(Kb_i)(x)| dx \leq \int_{m(xz_i^{-1}) > C_2 r_i} |(Kb_i)(x)| dx$$

$$= \int_{m(xz_i^{-1}) > C_2 r_i} \left| \int_{G} (\gamma(x, y^{-1})k(xy^{-1}) - \gamma(z_i, y^{-1})\gamma(x, z_i^{-1}k(xz_i^{-1})) b_i(y) dy \right| dx$$

$$\leq \int_{G} \int_{m(xz_i^{-1}) > C_2 r_i} |k(xy^{-1})\gamma(x, y^{-1})\gamma(x, z_i^{-1})^{-1}\gamma(z_i, y^{-1})^{-1} - k(xz_i^{-1})||b_i(y)|| dx dy$$

$$= \int_{G} \int_{m(z) > C_2 r_i} |k(zu^{-1})\gamma(zz_i, z_i^{-1}u^{-1})\gamma(zz_i, z_i^{-1})^{-1}\gamma(z_i, z_i^{-1}u^{-1})^{-1} - k(z)|| dx du$$

where we have used first Corollary 6.5 and then the change of variables $(x, y) = (zz_i, uz_i)$. Now note that by using Lemma 6.6(iv) and then Lemma 6.6(i) we have

$$\gamma(zz_i, z_i^{-1}u^{-1})\gamma(zz_i, z_i^{-1})^{-1}\gamma(z_i, z_i^{-1}u^{-1})^{-1} = \gamma(zz_i, z_i^{-1})^{-1}\gamma(zz_i, z_i^{-1}u^{-1})\gamma(z_i^{-1}, u^{-1})$$

$$= \gamma(z, u^{-1}).$$

Hence by using also the fact that supp $b_i \subseteq B(z_i, r_i)$, we obtain from the above estimates

$$\int_{G \setminus B_i^2} |(Kb_i)(x)| dx \leq \int_{G} \int_{m(z) > C_2 m(u)} |k(zu^{-1})\gamma(z, u^{-1}) - k(z)||b_i(uz_i)|| dz du$$

$$\leq C_3 ||b_i||_1$$

where $C_3 > 0$ is the supremum from the first hypothesis in the statement.

Now the conclusion follows by 6.7, 6.11 and Corollary 6.5(6). \qedbad\smallskip

The following corollary is a generalization of [Mau80, Th. III.(2.4)] which is not covered by [CW71, Th. III.(2.4)].

**Corollary 6.10.** In the setting of Theorem 6.9, the operator $K$ is bounded on $L^p(G)$ if $1 < p \leq 2$ and is of weak type $(1, 1)$.

**Proof.** This follows by Theorem 6.9 and Marcinkiewicz’ interpolation theorem. \qedbad

### 6.3. Examples of Pedersen multipliers

We now apply the results of Subsection 6.2 to the study of Pedersen multipliers for flat coadjoint orbits.

We thus resume the Setting 4.1 that is, $G$ is a connected and simply connected nilpotent Lie group with one-dimensional center $Z$, and $\pi: G \rightarrow \mathcal{B}(\mathcal{H})$ is a irreducible unitary representation that is square integrable modulo $Z$. Thus the group $G_e \cong g_e$ is a nilpotent Lie group. The restriction of $\pi$ to $G_e$ is a projective representation of $G_e$ with a smooth, polynomial cocycle $\gamma(x_1 x_2) = e^{i\alpha(x_1, x_2)}$ as in (4.2).

We then have that the twisted convolution $\ast_\gamma$ on $g_e \cong G_e$ is nothing else than the twisted convolution $\ast_{e}$.

The first result is a generalization of [Mau80, Prop. 3.4]. The proof is similar to the proof of the cited proposition, except that the needed $L^2$ continuity follows from Proposition 5.8 as the Fourier transform is not useful unless $G_e$ is abelian.
Proposition 6.11. Assume \( k \in L^{1+\epsilon}(g_e) \) for every \( 0 < \epsilon \leq \epsilon_0 \). Then the operator 
\[
 f \mapsto k \ast_e f
\]
is bounded on \( L^p(g_e) \) for every \( 1 < p < \infty \).

Proof. Proposition \([5,8]\) shows that the bilinear mapping 
\[
 L^2(g_e) \times L^2(g_e) \to L^2(g_e), \quad (k, f) \mapsto k \ast_e f,
\]
is continuous. The formula for twisted convolutions also shows easily that the mappings 
\[
 L^1(g_e) \times L^\infty(g_e) \to L^\infty(g_e), \quad (k, f) \mapsto k \ast_e f,
\]
\[
 L^1(g_e) \times L^1(g_e) \to L^1(g_e), \quad (k, f) \mapsto k \ast_e f,
\]
are continuous. Now the proposition follows by Thorin’s bilinear interpolation theorem. \( \square \)

Corollary 6.12. Assume the above setting and that \( g_e \) is endowed with a pseudo-metric \( m_\gamma \) as in Setting \([6,7]\) with its corresponding constant \( C_m \). Let 
\[
 k \in S'(g_e) \cap L^1_{\text{loc}}(g_e \setminus \{0\}), \quad k = k_1 + k_2, \quad \text{where}
\]
- \( k_1 \in L^{1+\epsilon}(g_e) \) for all \( 0 < \epsilon \leq \epsilon_0 \);
- \( k_2 \) is a Calderón-Zygmund kernel on \( g_e \) with respect to \( m_\gamma \).

Then the operator \( M = T^\pi(k) \) is bounded on \( H \) and is a Pedersen \( L^p \)-multiplier for every \( 1 < p \leq 2 \).

Proof. By Proposition \([6,11]\) and Corollary \([6,10]\) we get that \( Kf = k \ast_e f \) is bounded on \( L^p(g_e) \) for each \( 1 < p \leq 2 \). Then by Proposition \([5,8,11]\) we get that \( M \) is bounded on \( H \), and since we have that 
\[
 T^\pi(K\phi) = T^\pi(k \ast_e \phi) = MT^\pi(\phi)
\]
for every \( \phi \in L^1(g_e) \cap L^p(g_e) \), \( 1 \leq p \leq 2 \), we get that \( M \) is an \( L^p \)-Pedersen multiplier. \( \square \)

Remark 6.13. In the notation of Corollary \([6,12]\) above, the required \( L^2(g_e) \) continuity of the operator \( f \mapsto k_2 \ast_e f \) (see Definition \([6,8]\)) is equivalent with the fact that \( \pi(k_2) \in \mathcal{B}(H) \), by Proposition \([5,8]\) Calderón-Vaillancourt-type conditions on \( k_2 \) such that the later holds can be found in \([BB15a]\).

7. Some specific examples

In this section we give some families of examples of groups with square-integrable representations that satisfy the conditions in our main result. In particular, we indicate such an example of nilpotent Lie group (see Example \([7,5]\) of which we additionally prove that it does not admit any 1-parameter group of dilations. These groups are constructed as 1-dimensional central extensions of symplectic nilpotent Lie groups, therefore we actually focus on providing examples of the later.

Definition 7.1. Let \((g_0, \omega)\) be any symplectic nilpotent Lie algebra. This means that \( g_0 \) is a nilpotent Lie algebra and \( \omega : g_0 \times g_0 \to \mathbb{R} \) is a skew-symmetric nondegenerate bilinear functional satisfying the 2-cocycle condition 
\[
 (\forall x, y, z \in g_0) \quad \omega([x, [y, z]_{g_0}]) + \omega([y, [z, x]_{g_0}]) + \omega([z, [x, y]_{g_0}]) = 0.
\]
Denote by \( g := \mathbb{R} + \omega g_0 \) the corresponding 1-dimensional central extension, that is, \( g = \mathbb{R} \times g_0 \) as a vector space and the Lie bracket of \( g \) is defined by

\[
(\forall t, s \in \mathbb{R})(\forall x, y \in g_0) \quad [(t, x), (s, y)]_g := (\omega(x, y), [x, y]_{g_0}).
\]

We denote by \( G_0 \) and \( G \) the Lie groups obtained from \( g_0 \) and \( g \) by using the multiplication given by the Baker-Campbell-Hausdorff formula.

**Remark 7.2.** We also recall from [BB15b, Ex. 6.8] that if \( g_0 \) is an \( n \)-step nilpotent Lie algebra, then \( g \) is \((n + 1)\)-step nilpotent Lie algebra. Moreover, the center \( Z(g) \) of \( g \) is 1-dimensional and there exists an isomorphism of Lie algebras \( g_0 \simeq g/Z(g) \), hence the isomorphism class of \( g \) is uniquely determined by the isomorphism class of \( g_0 \). More precisely, for any Lie algebras \( g_1 \) and \( g_2 \), if \( g_1/Z(g_1) \not\simeq g_2/Z(g_2) \), then \( g_1 \not\simeq g_2 \).

Moreover, since \( \omega \) is non-degenerate, the coadjoint orbits of elements in \( Z(g)^* \) are flat, hence the corresponding unitary irreducible representations are square-integrable modulo \( Z(g) \) and non-isomorphic.

### 7.1. Two infinite classes of 3-step nilpotent Lie algebras.

In this subsection we provide two classes of mutually non-isomorphic 3-step nilpotent Lie algebras with 1-dimensional center and square-integrable representation modulo the center. To this end, using Definition 7.1 and Remark 7.2, it suffices to give examples of symplectic, 2-step nilpotent Lie algebras. We recall that the classification of 3-step nilpotent Lie algebras with 1-dimensional center is still an open problem, but some structure theory for them was developed in [BB15b].

In Example 7.3 below, we describe an uncountable family of symplectic, 2-step nilpotent Lie algebras of dimension 6. It is worth pointing out that \( n = 6 \) is the least possible dimension for which it is possible to find an infinite family of symplectic nilpotent Lie algebras of dimension \( n \) that are mutually nonisomorphic as symplectic Lie algebras. For, such Lie algebras lead via the construction in Definition 7.1 to mutually nonisomorphic Lie algebras of dimension \( n + 1 \), and it is well known that there only exist finitely many isomorphism classes of nilpotent Lie algebras of dimension \( n + 1 \) if \( n \leq 5 \).

**Example 7.3** ([La05, Ex. 3.5], [La06, Ex. 5.2]). For all \( s, t \in \mathbb{R} \setminus \{0\} \) let \( g_0(s, t) \) be the 6-dimensional 2-step nilpotent Lie algebra defined by the commutation relations

\[
[X_6, X_5] = sX_3, \quad [X_6, X_4] = (s + t)X_2, \quad [X_5, X_4] = tX_1.
\]

It follows by [La05] Ex. 3.5 that

\[
\{g_0(s, t) \mid s^2 + st + t^2 = 1, \ 0 < t \leq 1/\sqrt{3}\}
\]

is a family of isomorphic Lie algebras that are however pairwise non-isomorphic as symplectic Lie algebras with the common symplectic structure \( \omega \) given by the matrix

\[
J_\omega = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}
\]
that is, the skew-symmetric bilinear functional $\omega: g_0(s, t) \times g_0(s, t) \to \mathbb{R}$ is defined by $\omega(X_j, X_k) = \omega(X_2, X_3) = \omega(X_3, X_4) = 1$ and $\omega(X_j, X_k) = 0$ for all the other pairs $(j, k)$ with $1 \leq j < k \leq 6$.

In Example 7.4 below, we describe an infinite family of symplectic 2-step nilpotent Lie algebras whose dimensions can be arbitrarily high, in contrast to the Lie algebras from Example 7.3 above.

**Example 7.4.** Let $(V, E)$ be any simple finite graph, that is, $V$ is a finite set whose points are called vertices, and $E$ is a set of 2-element subsets of $V$ called edges of the graph. Let $V_0$ be any real vector space with a fixed basis labeled by the vertices of our graph, and define $V_1$ as the linear subspace of $\wedge^2 V_0$ generated by the edges, that is,

$$V_1 := \text{span} \{v \wedge w \mid \{v, w\} \in E\} \subseteq \wedge^2 V_0.$$  

The Lie algebra associated with the above graph is then defined as $g_0(V, E) := V_0 + V_1$ with its Lie bracket defined by $[V_0, V_1] = [V_1, V_1] = \{0\}$ and $[v, w] := v \wedge w \in V_1$ for all $v, w \in V_0$. Thus $g_0(V, E)$ is a 2-step nilpotent Lie algebra, and $\text{dim}(g_0(V, E)) = |V| + |E|$, where we denote by $|V|$ and $|E|$ the number of elements in the sets $V$ and $E$, respectively. It was established in [Man15, Th. 1] that the isomorphism class of the Lie algebra $g_0(V, E)$ uniquely determines the isomorphism class of the graph $(V, E)$, that is, non-isomorphic graphs give rise to non-isomorphic Lie algebras.

On the other hand, it follows by [PT09, Th. 3] that $g_0(V, E)$ admits a symplectic form as in Definition 7.1 above if and only if $\text{dim}(g_0(V, E))$ is even and for every connected component of the graph $(V, E)$ the number of its edges does not exceed the number of its vertices. We note however that if the graph $(V, E)$ satisfies that condition, then $g_0(V, E)$ may have many different symplectic structures. For instance, if $(V, E)$ is the triangle graph, that is, it has three vertices and three edges, then it is easily seen that $g_0(V, E)$ is isomorphic to any of the Lie algebras $g_0(s, t)$ from Example 7.4, and we have seen in that example that the corresponding Lie algebras are mutually nonisomorphic as symplectic Lie algebras.

### 7.2. Example of non-homogeneous nilpotent symplectic Lie algebra.

Let $g$ be any nilpotent $m$-dimensional real Lie algebra. We define as usual $g^0 := g$ and $g^k := [g, g^{k-1}]$ for every integer $k \geq 1$. We also recall that a finite-dimensional Lie algebra is **characteristically nilpotent** if all its derivations are nilpotent maps. In particular, a characteristically nilpotent Lie algebra does not admit any nonzero diagonalizable derivation, and it does not admit any 1-parameter group of dilations with diagonalizable infinitesimal generator.

The version of the following example for the ground field $\mathbb{C}$ is contained in [Bu00, Cor. 3.9] for $\mu_5^5$ without an explicit proof, so give here a method of proof that actually works over any ground field of characteristic zero.

**Example 7.5.** Let $g$ be the real Lie algebra with a basis $\{X_1, \ldots, X_8\}$ with

$$[X_1, X_k] = X_{k+1} \text{ for } k = 2, \ldots, 7,$$

$$[X_2, X_3] = X_6 + X_7, \quad [X_2, X_4] = X_7 + X_8, \quad [X_2, X_5] = X_8.$$

Then $g$ is a characteristically nilpotent symplectic Lie algebra with 1-dimensional center.
The center of $\mathfrak{g}$ is spanned by $X_8$. We first prove that every derivation $D$ of $\mathfrak{g}$ is nilpotent. There exist $a_{kj} \in \mathbb{R}$ with $D(X_k) = \sum_{j=1}^{8} a_{kj} X_j$ for $k = 1, \ldots, 8$. Since $\mathfrak{g}^k$ are characteristic ideals of $\mathfrak{g}$, hence $D(\mathfrak{g}^k) \subseteq \mathfrak{g}^k$, and $\dim(\mathfrak{g}^k/\mathfrak{g}^{k+1}) = 1$ for $k = 1, \ldots, 6$, it follows that $a_{kj} = 0$ if $3 \leq k \leq 8$ and $j < k$. This condition is trivially satisfied for $k = 1$, and to see that it is satisfied also for $k = 2$, we first note that $D(X_8) = D([X_2, X_5]) = [D(X_2), X_5] + [X_2, D(X_5)]$. By this equality, plugging in the above formula of $D(X_k)$ and using the above commutation relations, we obtain $a_{88} X_8 = a_{21} X_6 + a_{22} X_8 + a_{55} X_8$, hence

$$a_{88} = a_{22} + a_{55}$$  \hspace{1cm} (7.1)

and also $a_{21} = 0$, as we wished for. Thus

$$D(X_k) = \sum_{j=1}^{8} a_{kj} X_j \text{ for } k = 1, \ldots, 8. \hspace{1cm} (7.2)$$

As above, using $D(X_7) + D(X_8) = D([X_2, X_3]) = [D(X_2), X_3] + [X_2, D(X_3)]$, we obtain $a_{77} X_7 + (a_{87} + a_{88}) X_8 = a_{22} X_7 + X_8 + a_{44} (X_7 + X_8) + a_{45} X_8$ and then

$$a_{77} = a_{22} + a_{44} \text{ and } a_{78} + a_{88} = a_{22} + a_{44} + a_{45}. \hspace{1cm} (7.3)$$

For $k = 2, \ldots, 7$, using $D(X_{k+1}) = [D(X_1), X_k] + [X_1, D(X_k)]$, we obtain

$$\sum_{j=1}^{8} a_{k+1,j} X_j = \left[ \sum_{j=1}^{8} a_{1j} X_j, X_k \right] + [X_1, \sum_{j=1}^{8} a_{kj} X_j]$$

hence, computing the coefficient of $X_{k+1}$ in both sides of the above equation, we obtain

$$a_{k+1,k+1} = a_{11} + a_{kk} \text{ hence } a_{kk} = (k-2)a_{11} + a_{22}, \text{ for } k = 3, \ldots, 8. \hspace{1cm} (7.4)$$

Then by (7.1) and (7.4) we obtain $a_{22} = a_{88} - a_{55} = 3a_{11}$, hence

$$a_{kk} = (k+1)a_{11} \text{ for } k = 2, \ldots, 8. \hspace{1cm} (7.5)$$

Using $D(X_5) = [D(X_1), X_4] + [X_1, D(X_4)]$, we obtain $\sum_{j=5}^{8} a_{5j} X_j = \left[ \sum_{j=1}^{8} a_{1j} X_j, X_4 \right] + [X_1, \sum_{j=4}^{8} a_{4j} X_j]$ hence

$$a_{55} = a_{44} + a_{11}, a_{56} = a_{45}, a_{57} = a_{46} + a_{12}, a_{58} = a_{47} + a_{12}. \hspace{1cm} (7.6)$$

Using $D(X_6) = [D(X_1), X_5] + [X_1, D(X_5)]$, we obtain $\sum_{j=6}^{8} a_{6j} X_j = \left[ \sum_{j=1}^{8} a_{1j} X_j, X_5 \right] + [X_1, \sum_{j=5}^{8} a_{5j} X_j]$ hence

$$a_{66} = a_{55} + a_{11}, a_{67} = a_{56}, a_{68} = a_{57} + a_{12}. \hspace{1cm} (7.7)$$

Using $D(X_7) = [D(X_1), X_6] + [X_1, D(X_6)]$, we obtain $\sum_{j=7}^{8} a_{7j} X_j = \left[ \sum_{j=1}^{8} a_{1j} X_j, X_6 \right] + [X_1, \sum_{j=6}^{8} a_{6j} X_j]$ hence $a_{77} = a_{66} + a_{11}$ and $a_{78} = a_{67}$. Thus, using also (7.7), we obtain $a_{78} = a_{56} = a_{57} = a_{12}$. Then (7.3) implies
while if $k \leq j < k < \ell$ then $\omega(X_j, X_k) = 0$, and moreover $[g, g^3] \leq g^3$ and $[g^3, g^3] = \{0\}$. To prove that $\omega$ is a 2-cocycle of $g$, it suffices to check that if $1 \leq j < k < \ell \leq 8$, then the sum

$S := \omega([X_j, X_k], X_\ell) + \omega([X_k, X_\ell], X_j) + \omega([X_\ell, X_j], X_k)$

is equal to zero. To this end we discuss the cases that can occur.

**Case 1:** $1 \leq j < k < \ell \leq 4$. There are four possible situations of this type:

or

$S = \omega([X_2, X_3], X_4) + \omega([X_1, X_4], X_2) + \omega([X_4, X_2], X_3)$

or

$S = \omega([X_1, X_3], X_4) + \omega([X_3, X_4], X_1) + \omega([X_4, X_1], X_3)$

or

$S = \omega([X_1, X_3], X_4) + \omega([X_2, X_4], X_1) + \omega([X_4, X_1], X_2)$

$\omega(X_3, X_4) + \omega(X_7 + X_8, X_1) + \omega(-X_5, X_2) = 0$

$S = \omega(X_3, X_4) + \omega(X_7 + X_8, X_1) + \omega(-X_5, X_2)$

$\omega(X_8, X_1) - \omega(X_5, X_2) = 0.$

**Case 2:** $1 \leq j < k < 5 \leq \ell \leq 8$

Subcase 2a: $j = 1$. Then $S = \omega(X_{k+1}, X_\ell) + \omega([X_k, X_\ell], X_1) + \omega(-X_{\ell+1}, X_k)$, where $X_{\ell+1} := 0$ if $\ell = 8$.

If $k = 2$, then $S = \omega(X_3, X_\ell) + \omega([X_2, X_\ell], X_1) + \omega(-X_{\ell+1}, X_2)$ and therefore

$S = \begin{cases} 0 + \omega(X_8, X_1) - \omega(X_6, X_2) = 0 & \text{if } \ell = 5, \\ \omega(X_3, X_6) + \omega(0, X_1) - \omega(X_7, X_2) = 0 & \text{if } \ell = 6, \\ \omega(X_3, X_7) + \omega(0, X_1) - \omega(X_{\ell+1}, X_2) = 0 & \text{if } \ell = 7, 8. \end{cases}$

If $k = 3, 4$, then $[X_k, X_\ell] = 0$, hence $S = \omega(X_{k+1}, X_\ell) + \omega(-X_{\ell+1}, X_k)$, and then if $k = 3$, then

$S = \begin{cases} \omega(X_4, X_5) - \omega(X_6, X_3) = 0 & \text{if } \ell = 5, \\ 0 & \text{if } \ell = 6, 7, 8, \end{cases}$

while if $k = 4$, then $X_{k+1} = X_5 \in g^3$, hence $S = 0$ for $\ell = 5, 6, 7, 8$.

Subcase 2b: $j = 2 < 3 \leq k$. Then $[X_k, X_\ell] = 0$ and $[X_2, X_k] \in g^3$, hence $S = \omega([X_2, X_k], X_\ell) + \omega([X_\ell, X_2], X_k) = 0$. 

Subcase 2c: \( j = 3 < 4 = k \). Then \([X_j, X_k] = [X_k, X_\ell] = [X_\ell, X_j] = 0\) for \( \ell = 5, 6, 7, 8 \), hence \( S = 0 \).

Case 3: \( 1 \leq j < 5 \leq k < \ell \leq 8 \). Then \([X_k, X_\ell] = 0\) and \([X_j, X_k], [X_\ell, X_j] \in g^3\), hence \( S = 0 \).

Case 4: \( 5 \leq j < k < \ell \leq 8 \). Then \([X_j, X_k, X_\ell] \in g_3\), hence \( S = 0 \).

Acknowledgment. We wish to thank Jean Ludwig and Yuri Turovskii for their kind assistance with some pertinent references and remarks.

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