Acoustic wave propagation in an one-dimensional layered system

Pi-Gang Luan and Zhen Ye

Wave Phenomena Laboratory, Department of Physics, National Central University, Chung-li, Taiwan 320

(October 22, 2018)

Propagation of acoustic waves in an one-dimensional water duct containing many air filled blocks is studied by the transfer matrix formalism. Energy distribution and interface vibration of the air blocks are computed. For periodic arrangement band structure and transmission rate are calculated analytically, whereas the Lyapunov exponent and its variance are computed numerically for random situations. A distinct collective behavior for localized waves is found. The results are also compared with optical situations.

PACS numbers: 43.20., 71.55J, 03.40K

I. INTRODUCTION

Propagation of waves in periodic and disordered media has been and continues to be an interesting subject for physicists. When propagating in media with inhomogeneities, waves are subject to multiple scattering, which leads to many peculiar phenomena such as band structures in periodic media and wave localization in random media.

The propagation of waves in one dimensional (1D) systems has attracted particular interest from scientists because in higher dimensions the interaction between waves and scatterers is so complicated that the theoretical computation is rather involved and most solutions require a series of approximations which are not always justified, making it difficult to relate theoretical predictions to experimental observations. Yet wave localization in one dimension (1D) poses a more manageable problem which can be tackled in an exact manner by the transfer matrix method. Moreover, results from 1D can provide insight to the problem of wave localization in general and are suitable for testing various ideas. Indeed, over the past decades considerable progress has been made in understanding the localization behavior in 1D disordered systems. However, a number of important issues remained untouched. These issues include, for example, how waves are localized inside the media and whether there is a distinct feature for wave localization which would allow to differentiate the localization from residual absorption effect without ambiguity. Results from the statistical analysis of the scaling behavior in 1D random media is not conclusive. Another question could be whether the localized state is a phase state which would accommodate a more systematic interpretation. Furthermore, the study of acoustic propagation in 1D random media is relatively scarce. All these motivate us to consider wave propagation in 1D media further, with emphasis on the acoustic wave propagation.

In this paper we study the problem of acoustic wave propagation in an one dimensional water duct containing many air blocks either regularly or randomly but on average regularly distributed inside the duct. The frequency band structures and wave transmission are computed numerically. We show that while our results affirm the previous claim that all waves are localized inside an 1D medium with any amount of disorder, there are, however, a few distinctive features in our results. Among them, in contrast to optical case there is no universal scaling behavior in the present system. In addition, when waves are localized, a collective behaviour of the system emerges. We will also show the energy distribution in the water duct.

This paper is organized as follows. The next section we explain the the model employed, discuss the transfer matrix method and derive relevant formulas. In section III numerical results and discussions are given. We then summarize the paper in section IV.

II. MODEL AND METHOD

A. System setup

We study the system consisting air blocks inside a water duct. The system is chosen because air filled blocks are strong acoustic scatterers. This can be seen as follows. The scattering is largely controlled by the acoustic impedance which is defined as $\rho c$, with $\rho$ and $c$ being the mass density of the medium and the acoustic phase speed respectively. The acoustic impedance ratio between water and air is about 3500. This large contrast leads to strong scattering, making the system of air blocks in water an ideal candidate for the study of acoustic scattering.

The 1D acoustic system we consider is illustrated by Fig. 1. Assume that $N$ air blocks of thickness $a_j$ ($j = 1, \ldots, N$) are placed regularly or randomly in a water duct with length $L$ measured from the left boundary of the duct (LB). The distance from LB to the left interface of the first air block is $D$. For simplicity while without compromising generality, in later numerical computation we set the thickness of each air block to the same value $a$. The air fraction is clearly $\beta = Na/L$, the average distance between two adjacent air or water blocks is $\langle d \rangle = L/N = a/\beta$, and the average thickness of water blocks is $\langle b \rangle = (D + \sum_{j=1}^{N-1} a_j)/N$. The degree of randomness for the system is controlled by a parameter $\Delta$ in such a way that the thickness of the $j$-th water block is $b_j = \langle b \rangle (1 + \delta_j)$ with $\delta_j$ being a random number within the interval $[-\Delta, \Delta]$; the regular case corresponds to $\Delta = 0$. An acoustic source placed at LB generates
monochromatic waves with an oscillation $v(t) = ve^{-i\omega t}$. Transmitted waves propagate through the $N$ air blocks and travel to the right infinity. In order to avoid unnecessary confusion, possible effects from surface tension, viscosity or any absorption are neglected.

For convenience, we use the dimensionless quantity $k_{bg}$ to measure the frequency, where $k = \omega/c$ is the wave number in water blocks. Similarly, $k_g$ represents the wave number in air blocks. We also define the following parameters for later use

$$g = \frac{p_g}{\rho}, \quad h = \frac{c_g}{c} = \frac{k}{k_g}, \quad q^2 = gh, \quad \eta = \ln q.$$ (1)

### B. Wave propagation and state vector

We use the transfer matrix method[1] for solving the wave propagation in the system. Dropping out the time factor $e^{-i\omega t}$, the wave propagation obeys Helmholtz equation

$$p''_m(x) + k^2_p p_m(x) = 0,$$ (2)

in which $p_m(x)$ is the pressure field, and the subscript $m$ refers to the medium that can be either water or air, depending on where $x$ is located. Within any layer (air blocks or water blocks), the wave is

$$p_m(x) = A_m e^{ik_m x} + B_m e^{-ik_m x}.$$ (3)

where $A_m e^{ik_m x}$ represents the wave transmitted away from the source to the right and $B_m e^{-ik_m x}$ the wave reflected towards the source. In terms of $A_m$ and $B_m$, the corresponding velocity field $u_m$, which is another dynamical variable describing the oscillation of the medium, can be calculated as

$$u_m(x) = \frac{1}{i\omega \rho_m} p'_m(x) = \frac{1}{\rho_m c_m} [A_m e^{ik_m x} - B_m e^{-ik_m x}],$$ (4)

where $\rho_m$ refers to the mass density of medium $m$.

Define a state vector

$$S_m(x) = \begin{pmatrix} S^1_m(x) \\ S^2_m(x) \end{pmatrix} = \begin{pmatrix} A_m(x) \\ B_m(x) \end{pmatrix} = \begin{pmatrix} A_m e^{ik_m x} \\ B_m e^{-ik_m x} \end{pmatrix},$$ (5)

then $p_m(x)$ and $u_m(x)$ can be determined by $S_m(x)$:

$$p_m = S^1_m + S^2_m, \quad u_m = \frac{1}{\rho_m c_m} (S^1_m - S^2_m).$$ (6)

We denote the state vector in the $j$-th air block as $G_j(x)$ with two components $G^1_j(x) = G^1_j e^{ik_s x}$ and $G^2_j(x) = G^2_j e^{-ik_s x}$ and in the $j$-th water block as $W_j(x)$ with $W^1_j(x) = W^1_j e^{ik_s x}$ and $W^2_j(x) = W^2_j e^{-ik_s x}$.

By invoking the condition that the pressure and velocity fields are continuous across the interfaces separating water and air, one finds

$$W_{j-1}(x_j) = J G_j(x_j), \quad G_j(y_j) = J^{-1} W_j(y_j),$$

with

$$J = q^{-1} \begin{pmatrix} \cosh \eta & \sinh \eta \\ \sinh \eta & \cosh \eta \end{pmatrix}.$$ (7)

For wave propagation in the $j$-th air or water layer, we have

$$G_j(x_j) = U_g(a_j) G_j(y_j),$$ (8)

with

$$U_g(a_j) = \begin{pmatrix} e^{-ik_s a_j} & 0 \\ 0 & e^{ik_s a_j} \end{pmatrix},$$

and

$$W_j(y_j) = U(b_j) W_j(x_j+1),$$ (9)

with

$$U(b_j) = \begin{pmatrix} e^{-ik_b j} & 0 \\ 0 & e^{ik_b j} \end{pmatrix}.$$ (10)

From these results the transfer matrix $M_j$ for $j$-th unit is

$$M_j = J U_g(a_j) J^{-1} U(b_j),$$ (11)

and the state vectors in water blocks satisfies

$$W_{j-1}(x_j) = M_j W_j(x_j+1).$$ (12)

Therefore any two state vectors of the water blocks are connected as

$$W_{j1-1}(x_{j1}) = M_{j1,j2} W_{j2}(x_{j2}+1),$$ (13)

where

$$M_{j1,j2} = M_{j1} M_{j1+1} \cdots M_{j2}, \quad 1 \leq j_1 \leq j_2 \leq N.$$ (14)

From Eq.(10) and (13) one can easily prove that all $M_{j1,j2}$ are uni-modular matrices, a result of energy conservation. We denote

$$M_{1,N} \equiv M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}.$$ (15)

A simple reduction leads to

$$M_{21} = M_{12}^*, \quad M_{22} = M_{11}^*.$$ (16)

It is clear that the transfer matrix $M_{1,N}$ connects the first water block and the last water block.

Imposing the boundary conditions at LB

$$v = \frac{1}{\rho c} [W^1_N - W^2_N],$$ (17)

we obtain

$$W^2_N = 0,$$ (18)

The condition in (18) relates $W_0$ to the LB oscillation $v$, and the second relation (17) results from the fact that there is no reflection at the rightmost boundary in Fig. 1.
C. Parameterization for regular system

In the situation of regular arrangement of air blocks, \(a_j = a, \ b_j = b, \ d_j = d, \ M_j = M\) and \(M = M_{1,N} = M^N\). According to Eq. (11), \(M\) acquires the form

\[
M = \begin{pmatrix}
\alpha & \beta \\
\beta^* & \alpha^*
\end{pmatrix}
\]  
(19)

with

\[
\det(M) = 1,
\]
(20)

where

\[
\alpha = e^{-ikb}[\cos k_g a - i \sin k_g a \cosh 2\eta],
\]
(21)

\[
\beta = ie^{ikb} \sin k_g a \sinh 2\eta,
\]

are functions of three parameters: \(k_g a, kb\) and \(\eta\). In order to express physical quantities in a simple closed form, we use the parametrization given in Eq. (11). We define

\[
\text{tr}(M) = 2 \cos \theta,
\]

then the eigenvalue equation \(\lambda^2 - \text{tr}(M) \lambda + 1 = 0\) yields two eigenvalues \(\lambda = e^{\pm i\theta}\). Since \(\text{tr}(M) = \alpha + \alpha^* = 2\alpha_R\), and thus \(\alpha_R = \cos \theta\), Eq. (11) becomes

\[
\alpha_R^2 - |\beta|^2 = \sin^2 \theta,
\]

where \(\alpha_R = \text{Re}(\alpha)\) and \(\alpha_I = \text{Im}(\alpha)\) represent the real and imaginary parts of \(\alpha\), respectively.

Following Eq. (11), we adapt the parametrization,

\[
\alpha(\theta, \chi) \equiv \cos \theta - i \sin \theta \cos 2\chi,
\]

\[
\beta(\theta, \chi, \delta) \equiv i e^{i\delta} \sin \theta \sin 2\chi,
\]

where \(\theta\) and \(\chi\) in general are complex variables and \(\delta\) is chosen as \(kb\). Comparing Eq. (22) and (23), the following relations are established:

\[
\cos \theta = \cos k_g a \cos kb - \sin k_g a \sin b \cosh 2\eta,
\]

\[
\sin \theta \cos 2\chi = \cos k_g a \sin kb + \sin k_g a \cos kb \cosh 2\eta,
\]

\[
\sin \theta \sin 2\chi = \sin k_g a \sinh 2\eta.
\]

(23)

From these relations one finds that \(\cos \theta, \sin \theta \cos 2\chi\) and \(\sin \theta \sin 2\chi\) are all real, providing restrictions on the possible values of \(\theta\) and \(\chi\). Furthermore, using the definition (22), it is straightforward to verify that

\[
[M(\theta, \chi, \delta)]^n = M(n\theta, \chi, \delta)
\]

\[
= \begin{pmatrix}
\alpha(n\theta, \chi) & \beta(n\theta, \chi, \delta) \\
\beta^*(n\theta, \chi, \delta) & \alpha^*(n\theta, \chi)
\end{pmatrix},
\]

where \(n\) is an integer.

In this parametrization, matrix \(M\) becomes a function of \(\theta, \chi\), and \(\delta\). A direct computation shows that the two eigen-vectors of matrix \(M\) can be written as

\[
X_1 = \begin{pmatrix}
\cosh \chi \\
e^{-i\delta} \sinh \chi
\end{pmatrix}, \quad X_2 = \begin{pmatrix}
e^{i\delta} \sinh \chi \\
\cosh \chi
\end{pmatrix},
\]

(25)

for the eigen-values \(e^{\pm i\theta}\) respectively.

D. Determination of \(p\) and \(u\)

Denoting the pressure and velocity fields on left and right boundaries of \(j\)-th air block as \(p_j^L, p_j^R\) and \(u_j^L, u_j^R\), we have

\[
p_j^L = W_j^L(x_j) + W_j^{L-1}(x_j),
\]

\[
p_j^R = G_j^R(y_j) + G_{j-1}^R(y_j),
\]

(26)

\[
u_j^L = \frac{1}{\rho c} [W_j^L(x_j) - W_j^{L-1}(x_j)],
\]

\[
u_j^R = \frac{1}{\rho c} [G_j^R(y_j) - G_j^{R-1}(y_j)].
\]

The wave vectors \(W_j^L(x_j)\) and \(G_j(y_j)\) are determined by the following equations:

\[
W_0(D) = M_{1-j} W_{j-1}(x_j)
\]

\[
= M_{1-j} \frac{JU_g(a_j)}{G_j(y_j)}.
\]

(27)

For the case of regularly arranged air blocks, by Eqs. (24), (26), and (27), \(p_j^L, p_j^R\) and \(u_j^L, u_j^R\) are explicitly given by

\[
p_j^L = \rho c \left[ \frac{\alpha[(N - j + 1)\theta] + \beta^*[(N - j + 1)\theta]}{\alpha(N\theta) e^{-ikD} - \beta^*(N\theta) e^{ikD}} \right],
\]

\[
p_j^R = \rho c \left[ \frac{\alpha[(N - j)\theta] e^{-ikb} + \beta^*[(N - j)\theta] e^{ikb}}{\alpha(N\theta) e^{-ikD} - \beta^*(N\theta) e^{ikD}} \right],
\]

(28)

\[
u_j^L = \frac{1}{\rho c} \left[ \frac{\alpha[(N - j + 1)\theta] - \beta^*[(N - j + 1)\theta]}{\alpha(N\theta) e^{-ikD} - \beta^*(N\theta) e^{ikD}} \right],
\]

\[
u_j^R = \frac{1}{\rho c} \left[ \frac{\alpha[(N - j)\theta] e^{-ikb} - \beta^*[(N - j)\theta] e^{ikb}}{\alpha(N\theta) e^{-ikD} - \beta^*(N\theta) e^{ikD}} \right].
\]

E. Band structures

According to the Bloch theorem, the eigen-modes of wave field \(p\) and velocity field \(u\) in an infinite periodic medium can be written as

\[
p(x) = \xi(x)e^{ikx}, \quad u(x) = \zeta(x)e^{ikx}
\]

(29)

where \(\xi(x)\) and \(\zeta(x)\) are periodic functions satisfying \(\xi(x + d) = \xi(x)\), \(\zeta(x + d) = \zeta(x)\) and \(K\) is the usual Bloch wave number.

Eq. (29) implies

\[
e^{-iKd}W_j(x + d) = W_{j-1}(x_j)
\]

\[
= M(\theta, \chi, \delta) W_j(x_j + d).
\]

(30)

Therefore \(e^{-iKd}\) is the eigenvalue of matrix \(M\) and equals \(e^{i\theta}\). Substituting \(Kd = \theta\) in Eq. (23), we obtain the dispersion relation

\[
\cos Kd = \cos k_g a \cos kb - \cosh 2\eta \sin k_g a \sin kb,
\]

(31)
which describe the band structure. When

\[ |\cos k_A \cos k_B - \sin k_A \sin k_B \cos 2\eta| \leq 1, \]

the solution for \( K \) is a real number. When \n
\[ |\cos k_A \cos k_B - \sin k_A \sin k_B \cos 2\eta| > 1, \]

the solution for \( K \) is complex and is of the form

\[ K = \frac{n \pi}{\eta} + iK_t, \tag{32} \]

where \( n \) is an integer in order to satisfy Eq. (11). The frequency ranges within which real solutions for \( K \) can be deduced define the pass bands, while the ranges rendering the complex solutions for \( K \) determine the stop bands (band gaps).

**F. Transmission, reflection, energy flow and Lyapunov exponent**

Waves propagating through \( N \) air blocks will be scattered many times before they go out. The total transmission and reflection rates can be obtained from scattering matrix \( M^* \),

\[ M^* = \begin{pmatrix} 1/t & r^*/t^* \\ r/t & 1/t^* \end{pmatrix}, \]

where \( t \) and \( r \) are transmission and reflection coefficients, and \( T = |t|^2 \) and \( R = |r|^2 \) define the transmission and reflection rates. Based on the previous discussion

\[ M^* = M_{1,N} = \mathcal{M} \tag{33} \]

and

\[ T_N = \frac{1}{|M_{11}|^2}, \quad R_N = \frac{|M_{12}|^2}{|M_{11}|^2}. \tag{34} \]

Note that in Eq. (34) the relation \( T_N + R_N = 1 \) is satisfied, which is a consequence of energy conservation since the system we considered does not absorb any energy.

Another consequence of energy conservation is that energy flow along the whole duct is a constant. Recall that in the previous discussion

\[ |\alpha(N\theta)|^2 = 1 + \sin^2 N\theta \sin^2 2\chi, \tag{38} \]

and

\[ J_N = \frac{\rho v^2}{2} \begin{bmatrix} c \\ C + A \cos 2N\theta + B \sin 2N\theta \end{bmatrix} \tag{39} \]

where \( A, B, C, F, \phi \) are defined by

\[ A = -\sin^2 2\chi + \frac{1}{2} \cos(2kD - \delta) \sin 4\chi, \]

\[ B = -\sin(2kD - \delta) \sin 2\chi, \quad C = 1 - A, \]

\[ F = \sqrt{A^2 + B^2}, \quad \phi = \tan^{-1}\left(\frac{B}{A}\right). \tag{40} \]

It is well-known that in a random medium waves are always localized in space and the localization is characterized by the Lyapunov exponent (LE) \( \gamma \). The Lyapunov exponent is defined as

\[ \gamma = \lim_{N \to \infty} \langle \gamma_N \rangle, \tag{41} \]

with

\[ \gamma_N = \frac{1}{2N} \ln \left( \frac{1}{T_N} \right). \tag{42} \]

The fluctuation or variance of \( \gamma \) defined by

\[ \text{var}(\gamma) = \lim_{N \to \infty} \left( \langle \gamma_N^2 \rangle - \langle \gamma_N \rangle^2 \right) \tag{43} \]

is a quantity which as will be shown below gives important information about the system.

**G. Energy distribution in 1D systems**

The time averaged energy density \( \mathcal{E}(x) \) at \( x \) are defined by

\[ \mathcal{E}_m(x) = \frac{\rho_m}{4} \left[ |u_m|^2 + \frac{|p_m|^2}{\rho_m c_m^2} \right]. \tag{44} \]

By direct calculations we find that energy density in our 1D system is piece-wise constant and hence we can suppress the redundant variable \( x \). Energy density in \( j \)-th water block \( \mathcal{E}_w^j \) and in \( j \)-th air block \( \mathcal{E}_a^j \) are given by

\[ \mathcal{E}_w^j = \frac{1}{2\rho c^2} \left[ |W_j|^2 + |W_j^2|^2 \right] \tag{45} \]

and

\[ \mathcal{E}_a^j = \frac{1}{2\rho_a c_a^2} \left[ |G_j|^2 + |G_j^2|^2 \right]. \tag{46} \]

In the regular case Eq. (45) and (46) become
\[ \mathcal{E}_j^w = \frac{|f|^2}{2 \rho c^2} \left[ M^1[(N - j)\theta]M[(N - j)\theta]\right]_{11} \]  
(47)

and

\[ \mathcal{E}_j^g = \frac{|f|^2}{2 \rho c^2} \left[ M^1[(N - j + 1)\theta](J^{-1})^\dagger \right. 
\times J^{-1} M[(N - j + 1)\theta] \right]_{11}, \]  
(48)

where

\[ f = \frac{\rho cv}{M_{11} e^{-i k D} - M_{12}^* e^{i k D}} \]  
\[ = \frac{\alpha (N\theta) e^{-i k D} - \beta^* (N\theta) e^{i k D}}{\rho cv}. \]  
(49)

After lengthy but straightforward calculations Eq. (47) and (48) can be further simplified to

\[ \mathcal{E}_j^w = A_w + B_w \cos[2(N - j)\theta] \]  
(50)

and

\[ \mathcal{E}_j^g = C_g + A_g \cos[2(N - j + 1)\theta] \]  
\[ + B_g \sin[2(N - j + 1)\theta] \]  
\[ = C_g + D_g \cos[2(N - j + 1)\theta - \phi_g], \]  
(51)

where \( A_w, B_w, A_g, B_g, C_g, D_g \) and \( \phi_g \) are defined by

\[ A_w = \frac{\rho v^2}{2[C + F \cos(2N\theta - \phi)]] \cosh^2 2\chi, \]  
(52)

\[ B_w = -\frac{\rho v^2}{2[C + F \cos(2N\theta - \phi)]] \sinh^2 2\chi} \]  
(53)

\[ A_g = \frac{\rho v^2}{4h[C + F \cos(2N\theta - \phi)] \cosh 2\eta(1 - \cosh 4\chi) + \cos \delta \sinh 2\eta \sinh 4\chi], \]  
(54)

\[ B_g = \frac{\rho v^2 \sin \delta \sinh 2\eta \sinh 2\chi}{2h[C + F \cos(2N\theta - \phi)]}, \]  
(55)

\[ C_g = \frac{\rho v^2}{4h[C + F \cos(2N\theta - \phi)] \cosh 2\eta(1 + \cosh 4\chi) - \cos \delta \sinh 2\eta \sinh 4\chi], \]  
(56)

\[ \phi_g = \tan^{-1}\left(\frac{B_g}{A_g}\right), \quad D_g = \sqrt{A_g^2 + B_g^2}. \]  
(57)

The variables \( C, F, \phi \) in these equations are the same as in Eq. (40).

### H. Energy localization and collective behavior

When waves propagate through media alternated with different material compositions, multiple scattering of waves is established by an infinite recursive pattern of rescattering. Writing \( p(x) = A(x)e^{i\theta(x)} \) with \( A(x) \) and \( \theta \) being the amplitude and phase respectively, the energy flow \( J \sim \text{Re}\{p^*(x)\partial_x p(x)\} \) becomes \( J \sim A^2 \partial_x \theta \). Obviously, the energy flow will come to a complete halt and the waves are localized in space when \( A \) does not equal to zero but phase \( \theta \) is constant at least by domains.

From these observations, we propose to use the phase behavior of waves to characterize the wave localization. Expressing \( p(x) \) and \( u(x) \) as

\[ p(x) \equiv A_p(x)e^{i\theta_p(x)} \]  
(58)

and

\[ u(x) \equiv A_u(x)e^{i\theta_u(x)}. \]  
(59)

we construct two unit phase vectors as

\[ \tilde{v}_p \equiv \cos \theta_p \hat{e}_x + \sin \theta_p \hat{e}_y \]  
(60)

and

\[ \tilde{v}_u \equiv \cos \theta_u \hat{e}_x + \sin \theta_u \hat{e}_y. \]  
(61)

Physically, these phase vectors represent the oscillation behavior of the system. We can plot the phase vectors in a two-dimensional plane.

### III. NUMERICAL RESULTS AND DISCUSSION

#### A. Ordered cases

In the ordered case, the interference of multiply scattered waves leads to frequency band structures. For frequencies located in pass bands, waves propagate through the whole system, while for frequencies within a band gap, waves are evanescent.

In Fig. 2(a), \( \cos Kd \) versus \( kb/\pi \) from Eq. (37) are displayed for \( \beta = 10^{-4} \) (solid curve) and \( \beta = 3 \times 10^{-5} \) (broken curve). The segments of curves between \( \cos Kd = \pm 1 \) (the gray region) give real solutions of the Kd curve. The segments of curves between \( \cos Kd > 3 \) for \( \beta = 10^{-3} \) and \( kb > 3\pi \), the pass bands almost vanish. The band structures for various bands are shown in Fig. 2(c) and (d). We see that as the volume fraction \( \beta \) increases, the width of the pass bands decrease and the width of band gaps become larger. The air blocks are very strong acoustic scatterer in water leads to very wide band gaps and narrow pass bands shown in Fig. 2(c) and (d).

In Fig. 2(b) transmission and reflection versus frequency for different \( N \) are plotted according to Eqs. (57) and
Inside a pass band, transmission is significant. The transmission rate $T_N$ oscillates as the frequency varies, and it has $N - 1$ peaks as expected from Eq. (27). At these peaks $\sin N\theta = 0$, $T_N = 1$ and $R_N = 0$. That is, at the frequencies waves tunnel through the whole system completely, a phenomenon of resonant tunneling. Within a band gap, on contrast, although it is not exactly zero due to the finiteness of the system considered, the transmission is significantly prohibited. We have approximately $T_N \approx 0$ and $R_N \approx 1$. Using Eq. (53), energy flow $\mathcal{J}_N$ as function of the air-block number $N$ for different frequencies are plotted in Fig. 4. The frequencies appearing in diagrams (a), (b), (c), (d) are chosen from the first pass band. These diagrams show that the flow varies periodically as $N$ increases. When the frequency is low, for instance at $kb/\pi = 0.02$, a simple periodic behavior of $\mathcal{J}_N$ with long space period is observed. As we increase the frequency, the period becomes shorter, as illustrated in diagram (b). If we further increase the frequency, the oscillation of $\mathcal{J}_N$ becomes more rapid and the period is smaller than $d$. However, since $N$ is an integer, the behavior of $\mathcal{J}_N$ tends to be more complicated as $N$ is enlarged shown in diagrams (c) and (d).

When frequency enters into the band gap, waves become evanescent, as indicated in diagrams (e), (f), (g), and (h). Diagram (e) and (h) display the decay behavior of $\mathcal{J}_N$ at frequencies very close to the gap edges. The behavior of $\mathcal{J}_N$ at frequencies near the center of first band gap are illustrated in diagrams (f) and (g), showing an exponential decay behavior. We see that the decay rates in (f) and (g) are much more stronger than in (e) and (h), as expected.

Using Eq. (54) and (51), energy density can be calculated. Fig. 8 shows the energy density distribution along the duct for $N = 100$ air blocks. On the left panel of Fig. 8, i.e. diagrams (a1), (b1), (c1), and (d1), the energy density in water blocks are shown, whereas on the right panel, i.e. diagrams (a2), (b2), (c2), and (d2), the energy density in the air blocks are displayed. In the computation, the air fraction is taken as $\beta = 10^{-4}$. We find that for frequencies located in the pass bands, the energy density varies periodically along the traveling path. For frequencies within the frequency gaps, the energy is trapped near the acoustic source, and the energy density decays about exponentially along the path. Meanwhile, the energy flow is calculated to be close to zero.

We now look at the oscillation behavior of the blocks. As discussed above, zero energy flow leads to a phase coherence behavior of the medium. For the purpose we consider the behavior of the phase vectors $\vec{v}_p$ and $\vec{v}_v$ defined in Eqs. (4) and (5). Typical results of phase behavior are shown in Fig. 8. For further convenience, only the phase vectors at the interfaces of air and water are shown. Symbols $p^L$, $p^R$, $u^L$, $u^R$ appearing in Fig. 8 denote respectively the phase vectors for the pressure and the velocity fields on the left and right side of the air blocks. We set the vibration phase of LB to 1. First we discuss the cases shown by diagrams (a) and (b). The frequencies in (a) and (b) are chosen from the first pass band. The phase vectors in these two cases point to various directions along the duct and $\vec{v}_p \cdot \vec{v}_v \neq 0$, resulting in a nonzero energy flow. The waves are extended in the system. In diagrams (c) and (d), the frequencies are chosen from the first band gap in which waves are evanescent as shown in Fig. 8. In this case, all the phase vectors of the pressure field are pointing to either $\pi/2$ or $-\pi/2$, and are perpendicular to the phase vectors of the velocity field. The pressure at the two sides of any air block is almost in phase. Different from the higher dimensional cases in which all phase vectors of localized fields point to the same direction, the present phase vectors are constant only by domains. The velocity field in neighboring domains oscillate with a phase difference $\pi$. At the far end of the sample, however, the phase vectors become gradually disoriented, implying that the energy can leak out only at the boundary due to the finite sample size. We note here that such a phase ordering not only exists for the boundaries of the air blocks, but also appears inside the whole medium.

**B. Disordered situations**

Unlike in the ordered case for which waves can propagate through all air blocks if the frequency is located in pass band, in a disordered 1D system waves are always localized. In this section, we study numerically acoustic propagation in the disordered case.

Fig. 7(a) presents the typical results of the transmission rate as a function of $kb/\pi$ for various $\beta$ at a given randomness. At frequencies for which the wavelength is smaller than the averaged distance between air blocks, the transmission is significantly reduced by increasing air fraction.

Fig. 7(b) illustrates the effect of the randomness $\Delta$ on transmission for a given air-fraction. For comparison, the transmission in the corresponding regular array ($\Delta = 0$) is also plotted. The gaps are located between $k(b)/\pi = 0.4638$ and 1, 1.21 and 2, 2.128 and 3, and so on. We find that for frequencies located inside the band gaps of the corresponding periodic array, the disorder-induced localization effect competes yet reduces the band gap effect. To characterize wave localization in this case, both the band gap and the disorder effects should be considered, supporting the two parameter scaling theory. However, increasing disorder tends to smear out the band structures. When exceeding a certain amount, the effect from the disorder suppresses the band gap effect completely, and there is no distinction between the localization at frequencies within and outside the band gaps.

Fig. 8 shows that the localization behaviour depends crucially on whether the wavelength ($\lambda$) is greater than the average distance between air blocks. When $b/\lambda$ is less than $1/4$, the localization effect is weak. We also observe that with the added disorder, the transmission is enhanced in the middle of the gaps. Similar enhancement due to disorder has also been reported recently. Differing from, however, the transmission at frequencies within the gaps of the corresponding periodic arrays in the present system is not always enhanced by disorder. Instead the transmission is reduced further by the disorder near the band edges.
To further explore the transmission property, we calculate Lyapunov exponent $\gamma$ and its variance $\text{var}(\gamma)$ according to Eq. (41) and (43). The sample size is chosen in such a way that it is much larger than the localization length and the ensemble average is carried out over 2000 random configurations. Fig. 8 presents the results for Lyapunov exponent (LE) and its variance as a function of $k(b)/\pi$ for various randomness. As expected, when the randomness is small LE mimics the band structures and the variance of LE inside the gaps is small. Contrast to the optical case, there are no double maxima for the variance inside the gap. Rather, the double peaks appear in the allowed bands when the system is disordered. The double peak feature is more prominent in the low frequency bands. When the randomness exceeds a certain value, however, the double peaks emerge. The higher frequency, the lower is the critical value. For example, the double peaks are still visible in the first allowed band (c.f. Fig. 9(b)), while there is only one peak inside the higher pass bands. Meanwhile, the increasing disorder reduces the band gap effect and smears out the oscillation in LE, in accordance with Fig. 8. We also plot LE versus its variance in Fig. 8. When randomness is weak, several branches appear in the LE-variance relation. The frequency range of a branch covers a pass band. In the optical case, the frequency range of a branch covers a gap instead. A prominent feature in the present system is that when the double peaks in the variance are destroyed, the minima of LE correspond to the maxima of its variance. This is different from the optical case.

With increasing disorder, we do not observe the genuine linear dependence between LE and its variance, as expected from the single parameter scaling theory, indicating that the single parameter scaling law may not be applicable in the present system.

Eqs. (15) and (16) provide the formulas for studying energy distribution in any situation. As discussed in the regular case, when wave is localized, a kind of ordering of the phase vectors appears. From the alignment pattern of phase vectors we can know whether waves inside the system are localized. For air fraction $\beta = 10^{-4}$, three different cases with different $k(b)/\pi$, $\Delta$ and $N$ are shown in Fig. 9. To isolate the localization effect from the band gap effects, we choose the two frequencies located in the first allowed band: one is in the middle of the band and the other is near the lower band edge.

First, we note that the energy density is constant in each individual block. This is a special feature of 1D classical systems, and can be verified by a deduction from Eqs. (15) and (16). From these figures, we observe that when the sample size is sufficiently large, waves are always localized for any given amount of randomness. When localized, the waves are trapped inside the medium, but not necessarily confined at the site of the source, unless the band gap effect is dominant. The energy distribution does not follow an exponential decay along the path. This differs from situations in higher dimensions. It is also shown that the energy stored in the medium can be tremendous.

These figures also show that for low frequencies and when the randomness is small, to trap the waves a large number of air blocks is needed. Like in the regular cases, when waves are localized, the coherent behavior of the medium appears, and the phases are constant by domains. The phase vector domains are sensitive to the arrangement of the air blocks. Moreover, when localization is evident, increasing the sample size by adding more air blocks to the far end of the system will not change the patterns of the energy distribution and phase vectors. Therefore the energy localization and the phase coherence behavior are not caused by the boundary effect.

Fig. 8 shows the energy distribution inside the medium. Here we choose $k(b)/\pi = 0.25$, which is inside first pass band. We see that the energies are localized in both air and water blocks. However, unlike in the regular cases, the energy does not always decay exponentially along the path. In Fig. 11 we use the same parameters as in Fig. 8 except that we choose $k(b)/\pi = 0.46$, which is very close to the gap edge 0.4638. We see now that waves can be very easily trapped by using only $N = 50$ air blocks. In Fig. 11 we choose $k(b)/\pi = 0.25$ as used in Fig. 8 but increase the randomness to $\Delta = 1$, waves are trapped by $N = 150$ air blocks, fewer than in the case of Fig. 8.

IV. CONCLUDING REMARKS

In this paper we studied the propagation of acoustic waves in 1D layered system consisting of air and water blocks. Both regular and random arrangements of air blocks have been studied. We first derived the basic formulas for the general situations, and then applied these formulas to regular and random cases separately. For periodically placed air blocks, the band structures were studied and were shown to have large band gaps. The transmission, reflection, Lyapunov exponent and its variance, energy distribution, energy flow and medium vibration were also studied. For the case of randomly placed air blocks, the results pointed out that waves are always confined in a finite spatial region. The disorder leads a significant energy storage in the system. It is also indicated that the wave localization is related to a collective behavior of the system in the presence of multiple scattering, also observed for higher dimensions. The appearance of such a collective phenomenon may be regarded as an indication of a kind of classical Goldstone modes in the context of the field theory.

ACKNOWLEDGMENTS

The work received support from National Science Council (No. NSC89-2611-M008-002 and NSC89-2112-M008-008).
FIGURES

Graph 1

Graph 2

Graph 3

Graph 4

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FIG. 3. (a1)-(a3) Transmission rate versus frequency for $N = 3, 10, 100$. (b1)-(b3) Reflection rate versus frequency for $N = 3, 10, 100$. The volume fraction of air is $\beta = 10^{-4}$.

Figure 4

FIG. 4. (a)-(d) Energy flow vs number of air blocks $N$, four frequencies are chosen from first allowed band. (e)-(h) Log-Energy flow versus $N$ in first band gap. The volume fraction of air is $\beta = 10^{-4}$.

FIG. 5. Energy density distributions along the duct at four different frequencies: $kb/\pi = 0.02, 0.35, 0.4638, 0.9999$. Left (a1)-(d1): Energy density in water blocks. Right (a2)-(d2): Energy density in air blocks. Number of air blocks $N = 100$ and air fraction $\beta = 10^{-4}$.

Figure 6

FIG. 6. Phase vectors at air block boundaries. Total number of air blocks is $N = 15$, and the air fraction is $\beta = 10^{-4}$. Four frequencies are chosen according to Fig. 5 that is, $kb/\pi = 0.02$, and 0.35 in the first allow band and 0.4638, 0.9999 in the first gap. In the diagrams, the small arrows are used to represent the phase vectors.
Figure 7

**FIG. 7.** Transmission versus $k(b)/\pi$ for various air fractions at $\Delta = 0.3$ (a) and different disorders (b). The number of the air-blocks is 100.

Figure 8

**FIG. 8.** Diagrams (a), (b), and (c) show the Lyapunov exponent (LE) in broken lines and its variance in solid lines as a function of $k(b)/\pi$ for three random situations. Diagrams (d), (e), and (f) present the plots of the exponent versus its variance at the three random cases. Here $\beta = 10^{-4}$.

Figure 9

**FIG. 9.** Energy density distributions along the duct in air blocks (a) and in water blocks (b). Phase vectors at the interfaces for three spatial ranges of the medium are illustrated in (c). The unit of energy density is $J/m^3$, air fraction $\beta = 10^{-4}$, $k(b)/\pi = 0.25$, $\Delta = 0.3$, $N = 1500$.

Figure 10

**FIG. 10.** Energy density distributions along the duct in air blocks (a) and in water blocks (b). Phase vectors at the interfaces for three spatial ranges of the medium are illustrated in (c). The unit of energy density is $J/m^3$, air fraction $\beta = 10^{-4}$, $k(b)/\pi = 0.46$, $\Delta = 0.3$, $N = 50$. 

FIG. 11. Energy density distributions along the duct in air blocks (a) and in water blocks (b). Phase vectors at the interfaces for three spatial ranges of the medium are illustrated in (c). The unit of energy density is $J/m^3$, air fraction $\beta = 10^{-4}$, $k(b)/\pi = 0.25$, $\Delta = 1$, $N = 150$. 