A NEW TWO WEIGHT ESTIMATES FOR A VECTOR-VALUED POSITIVE OPERATOR

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Abstract. We give a new characterization of the two weight inequality for a vector-valued positive operator. Our characterization has a different flavor than the one of Scurry’s [5] and Hänninen’s [6]. The proof can be essentially derived from the scalar-valued case.

1. Introduction

1.1. Preliminaries. We start with the scalar-valued positive dyadic operators. Let $\alpha = \{\alpha_I : I \in \mathcal{D}\}$ be non-negative constants associated to dyadic cubes in $\mathbb{R}^d$. Let $\mu$ and $\nu$ be weights. For a cube $I \in \mathcal{D}$, set

$$E^\mu_I f := \left(\mu(I)^{-1} \int_I f \, d\mu\right) \cdot 1_I. \quad (1.1)$$

Consider the linear operator defined by

$$T_\alpha f := \sum_{I \in \mathcal{D}} \alpha_I \cdot E^\mu_I f. \quad (1.2)$$

Theorem 1.1. Let $1 < p < \infty$ and let $1/p + 1/p' = 1$. $T_\alpha : L^p(\mu) \to L^p(\nu)$ if and only if

$$\int_J \left| \sum_{I \in \mathcal{D} : I \subseteq J} \alpha_I \cdot 1_I \right|^p \, d\nu \leq C_1^p \cdot \mu(J), \quad J \in \mathcal{D} \quad (1.3)$$

$$\int_J \left| \sum_{I \in \mathcal{D} : I \subseteq J} \frac{\alpha_I \cdot \nu(I)}{\mu(I)} \cdot 1_I \right|^{p'} \, d\mu \leq C_2^{p'} \cdot \nu(J), \quad J \in \mathcal{D}. \quad (1.4)$$

In particular, $\|T_\alpha\|_{L^p(\mu) \to L^p(\nu)} \leq C_1 + C_2$.

In [1], Theorem 1.1, named as bilinear embedding theorem, is proved using Bellman function technique for $p = 2$. In [2], the case for all $1 < p < \infty$ is obtained, using a technique developed in [7]. In [3] and [4], the proof is significantly simplified. In fact, the proof given in [3] also works for more general measurable spaces, see Definition 1.2. In [5], a vector-valued extension is established, and in [6], a simplified proof is obtained using the same idea as [4].

1.2. The main problem. We are interested in the two weight estimates for the vector-valued case. Our basic setup is

Definition 1.2. For a measurable space $(\mathcal{X}, \mathcal{T})$, a lattice $\mathcal{L} \subseteq \mathcal{T}$ is a collection of measurable subsets of $\mathcal{X}$ with the following properties

(i) $\mathcal{L}$ is a union of generations $\mathcal{L}_n, n \in \mathbb{Z}$, where each generation is a collection of disjoint measurable sets, covering $\mathcal{X}$.

Key words and phrases. two weight, vector-valued positive operator, measurable space setting.
(ii) For each \( n \in \mathbb{Z} \), the covering \( \mathcal{L}_{n+1} \) is a countable refinement of the covering \( \mathcal{L}_n \), i.e. each set \( I \in \mathcal{L}_n \) is a countable union of disjoint sets \( J \in \mathcal{L}_{n+1} \). We allow the situation where there is only one such set \( J \), i.e. \( J = I \); this means that \( I \in \mathcal{L}_n \) also belongs to the generation \( \mathcal{L}_{n+1} \).

**Definition 1.3.** For a positive measure \( \mu \) on \((\mathcal{X}, \mathcal{T})\), define the averaging operator as

\[
(1.5) \quad E^\mu_I f := \left( \mu(I)^{-1} \int_I f \, d\mu \right) 1_I.
\]

From now on, we assume \((\mathcal{X}, \mathcal{T})\) is a measurable space, \( \mathcal{L} \subseteq \mathcal{T} \) is a lattice on \( \mathcal{X} \), and \( \mu, \nu \) are two positive measures.

**Definition 1.4.** Let \( \alpha = \{ \alpha_I : I \in \mathcal{L} \} \) be non-negative constants associated to a lattice \( \mathcal{L} \) on \((\mathcal{X}, \mathcal{T})\). Define a vector-valued operator

\[
(1.6) \quad T_\alpha f := \{ \alpha_I \cdot E^\mu_I f \}_{I \in \mathcal{L}}.
\]

**Theorem 1.5** (Two weight estimates for a vector-valued positive operator). Let \( 1 < p < \infty \) and \( 1 \leq q < \infty \).

\[
(1.7) \quad \int_\mathcal{X} \left[ \sum_{I \in \mathcal{L}} \left| \alpha_I \cdot E_I^\mu f \right|^q \right]^{\frac{p}{q}} \, d\nu \leq C_p \int_\mathcal{X} |f|^p \, d\mu,
\]

holds if and only if

(i) for the case \( 1 < p \leq q \), we have

\[
(1.8) \quad \int_J \left[ \sum_{I \in \mathcal{L} : I \subseteq J} \alpha^q_I \cdot 1_I \right]^{\frac{p}{q}} \, d\nu \leq C_p^p \cdot \mu(J), \quad J \in \mathcal{L}
\]

(ii) for the case \( q < p < \infty \), we have both \((1.8)\) and

\[
(1.9) \quad \int_J \left[ \sum_{I \in \mathcal{L} : I \subseteq J} \alpha^q_I \cdot \frac{\nu(I)}{\mu(I)} \cdot 1_I \right] \left( \frac{q}{p} \right)^p \, d\mu \leq C_p^q \cdot \nu(J), \quad J \in \mathcal{L}.
\]

In particular, \( C \asymp C_1 + C_2 \).

**Remark 1.6.** The case \( q = 1 \) is a generalization of Theorem 1.1 to the measurable space setting. The proof given in [3] adapts to this general situation.

**Remark 1.7.** In [5] and [6], to obtain the two testing conditions, they first rewrite \((1.7)\) into

\[
(1.10) \quad \sum_{I \in \mathcal{L}} \alpha_I \cdot E_I^\mu f \cdot E_I^\nu g_I \cdot \nu(I) \lesssim ||f||_{L^p(\mu)} \cdot ||\{g_I\}_{I \in \mathcal{L}}||_{L^{p'}(\nu, \mu)}.
\]

Setting \( f = 1_J \), one deduces \((1.8)\). For the second testing condition, one turns to consider the family of functions \( \{g_I\}_{I \in \mathcal{L}} \) supported on \( J \in \mathcal{L} \) with \( l^{p'} \)-norm equal to 1. This gives

\[
(1.11) \quad \int_J \left[ \sum_{I \in \mathcal{L} : I \subseteq J} \alpha_I \cdot \frac{\nu(I)}{\mu(I)} \cdot E_I^\nu g_I \right]^{p'} \, d\mu \lesssim \nu(J), \quad J \in \mathcal{L}.
\]

Compare Theorem 1.5 with the main results in [5] and [6]. We have a very different condition \((1.9)\) than \((1.11)\) with seemingly ‘wrong’ exponents. However, We will see that both \((1.8)\) and \((1.9)\) are testing conditions on some families of special functions.
2. The Case: $1 < p \leq q$

We will see in this section that when $1 < p \leq q$, (1.8) is equivalent to (1.7). On one hand, (1.8) can be deduced from (1.7) by setting $f = 1_{J^*}$. On the other hand, consider the maximal function

$$M_\mu f(x) := \sup_{x \in I, I \in \mathcal{L}} \|\mathbb{E}_I^\mu f(x)\|.$$  

The celebrated Doob’s martingale inequality asserts

$$\|M_\mu f\|_{L^p(\mu)} \leq p' \cdot \|f\|_{L^p(\mu)}.$$  

Let $E_k := \{x \in \mathcal{X} : M_\mu f(x) > 2^k\}$ and let $\mathcal{E}_k := \{I \in \mathcal{L} : I \in E_k\}$. Note that $E_k$ is a disjoint union of maximal sets in $\mathcal{E}_k$, maximal in the sense of inclusion. Denote these disjoint maximal sets by $\mathcal{E}_k^*$. Hence, $E_k = \bigcup_{J \in \mathcal{E}_k^*} J$.

$$\int_{\mathcal{X}} \left[ \sum_{I \in \mathcal{L}} |\alpha_I \cdot \mathbb{E}_I^\mu f|^q \right]^{\frac{p}{q}} d\nu \leq \sum_k \int_{E_k} \left[ \sum_{J \in \mathcal{E}_k \setminus \mathcal{E}_{k+1}} |\alpha_J \cdot \mathbb{E}_J^\mu f|^q \right]^{\frac{p}{q}} d\nu, \quad 1 < p \leq q$$

$$\leq \sum_k 2^{(k+1)p} \int_{E_k} \left[ \sum_{J \in \mathcal{E}_k \setminus \mathcal{E}_{k+1}} \alpha_J^q \cdot 1_J \right]^{\frac{p}{q}} d\nu$$

$$\leq \sum_k 2^{(k+1)p} \sum_{J \in \mathcal{E}_k} \int_J \left[ \sum_{I \in \mathcal{L} \setminus J} \alpha_I^q \cdot 1_I \right]^{\frac{p}{q}} d\nu$$

$$\leq C_p^q \cdot \sum_k 2^{(k+1)p} \cdot \mu(E_k), \quad (1.8)$$

$$\leq C_p^q \cdot \|M_\mu f\|_{L^p(\mu)}^p \|f\|_{L^p(\mu)}^p, \quad (2.2).$$

3. The Case: $q < p < \infty$, A Counterexample

In this section, we see that (1.8) itself is not sufficient for (1.7) for the case $q < p < \infty$.

Consider the real line $\mathbb{R}$ with the Borel $\sigma$-algebra $\mathcal{B}(\mathbb{R})$. Let the lattice be all the triadic intervals. We specify the positive measures $\mu, \nu$, the non-negative constants $\alpha = \{\alpha_I : I \in \mathcal{L}\}$, and the functions $f$ in the following way.

Let $C = \cap_{n \geq 0} C_n$ be the $1/3$-Cantor set, where $C_0 = [0, 1), C_1 = [0, 1/3) \cup [2/3, 1)$ and, in general, $C_n = \cup \{[x, x + 3^{-n}) : x = \sum_{j=1}^n \varepsilon_j 3^{-j}, \varepsilon_j \in \{0, 2\}\}$.

(i) The measure $\mu$ is the Lebesgue measure restricted on $[0, 1)$ and the measure $\nu$ is the Cantor measure, i.e. $\nu(I) = 2^{-n}$ for each $I$ belongs to a connect component of $C_n$.  

(ii) Define $\alpha_I = (2/3)^{n/p}$ for each $I$ belongs to a connect component of $C_n$.  

(iii) For the function $f$, consider the gap of $C$, i.e. $[0, 1) \setminus C$. This is a disjoint union of triadic intervals. Let $f = (3/2)^{n/p} \cdot n^{-r}$ for each $I \in [0, 1) \setminus C$ with length of $I$ equals $3^{-n}$, where $r$ is to be chosen later in the proof.

Claim 3.1. The above construction gives a counterexample with properly chosen $r$.  

Proof. We begin with checking (1.8). It suffices to check for every \( J \) belongs to a connected component of \( C_n \), and thus \( \mu(J) = 3^{-n} \). Note that

\[
\left| \sum_{I \in \mathcal{L} : I \subseteq J} \alpha_I \cdot 1_I \right| \overset{\frac{p}{q}}{\lesssim} \left| \sum_{k \geq n} \left( \frac{2}{3} \right)^{\frac{p}{r}} \right| \propto \left( \frac{2}{3} \right)^n.
\]

Hence,

\[
\int_J \left| \sum_{I \in \mathcal{L} : I \subseteq J} \alpha_I \cdot 1_I \right| \overset{\frac{p}{q}}{\lesssim} \left( \frac{2}{3} \right)^n \cdot \nu(J) = \mu(J).
\]

Next, we show that (1.7) fails. This requires a smart choice of \( r \) in the definition of \( f \). Picking \( r > \frac{1}{p} \), we have

\[
||f||_{L^p(\mu)}^p = \int_0^1 |f|^p dx = \sum_{n \geq 1} \left( \frac{3}{2} \right)^n \cdot \frac{1}{3^n} \cdot 2^n = \sum_{n \geq 1} n^{-pr} < \infty.
\]

Since \( q < p < \infty \), we can pick \( r \) such that \( \frac{1}{p} < r < \frac{1}{q} \). Note that for every \( I \) belongs to a connected component of \( C_n \), we have

\[
E_{\mu} f \geq \frac{1}{3} \left( \frac{3}{2} \right)^{\frac{n+1}{p}} (n+1)^{-r}.
\]

Hence, consider \( \mathcal{I}_n = \{ I : I \text{ is tri-adic with length less than or equal to } 3^{-n} \} \),

\[
\sum_{I \in \mathcal{I}_n} \left| \alpha_I \cdot E_{\mu} f \cdot 1_{C_n} \right| \overset{\frac{p}{q}}{\gtrsim} \sum_{k \leq n} \left| \frac{1}{3} \left( \frac{3}{2} \right)^{\frac{1}{p}} (k+1)^{-r} \right| \gtrsim \sum_{k \leq n} (k+1)^{-qr}.
\]

And so,

\[
\int \left[ \sum_{I \in \mathcal{I}_n} \left| \alpha_I \cdot E_{\mu} f \right|^q \right] \overset{\frac{p}{q}}{\gtrsim} \left[ \sum_{k \leq n} (k+1)^{-qr} \right] ^{\frac{p}{q}} \nu(C_n) \to \infty \text{ as } n \to \infty.
\]

We can see that the condition \( q < p < \infty \) is crucial in our construction. \( \square \)

4. The case: \( q < p < \infty \)

We discuss the case \( q < p < \infty \) of Theorem 1.5 in this section. In particular, we see that both (1.8) and (1.9) are testing conditions on some families of special functions.

To begin, since

\[
||T_{\alpha} f||_{L^p(\mu)}^q = \sup_{||g||_{L^q(\mu)}} \int_X \left[ \sum_{I \in \mathcal{L}} \left| \alpha_I \cdot E_{\mu} f \right|^q \right] g d\nu,
\]

we can write
Lemma 4.1.

\[
\|T_{\alpha}\|_{L^p(\mu)\rightarrow L^p(\nu)}^q = \sup_{\|g\|_{L^p(\mu)} = 1} \sup_{\|f\|_{L^p(\mu)}} = \int_X \left[ \sum_{I \in \mathcal{L}} |\alpha_I - \mathbb{E}_I(f)|^q \right] g d\nu.
\]

Without loss of generality, we assume that both \(f\) and \(g\) are non-negative. The following lemma reduces us to the scalar-valued case.

Lemma 4.2

An easy application of Hölder’s inequality shows that the LHS of (4.3) is no more than its RHS. The other half of this lemma depends on the following famous Rubio de Francia Algorithm.

Lemma 4.2 (Rubio de Francia Algorithm). For every \(q < p < \infty\) and \(f \in L^p(\mu)\), there exists a function \(F \in L^p(\mu)\), such that \(f \leq F\), \(\|F\|_{L^p(\mu)} \approx \|f\|_{L^p(\mu)}\) and

\[
\mu(I)^{-1} \int_I F^q d\mu \lesssim \inf_{x \in I} F(x), \quad I \in \mathcal{L}.
\]

Proof. Consider the maximal operator \(M_\mu\) defined in (2.1). Doob’s martingale inequality (2.2) implies

\[
\|M_\mu\|_{L^{p/q}(\mu)\rightarrow L^{p/q}(\mu)} \leq \left( \frac{p}{q} \right)^{\frac{1}{q}}.
\]

Denote \(M_\mu^{(0)} = Id\), \(M_\mu^{(1)} = M_\mu\) and \(M_\mu^{(k)} = M_\mu \circ M_\mu^{(k-1)}\). Define the function \(F\) by

\[
F = \left[ \sum_{k \geq 0} \left( 2 \|M_\mu\|_{L^{p/q}(\mu)\rightarrow L^{p/q}(\mu)} \right)^{-k} M_\mu^{(k)}(f^q) \right]^{\frac{1}{q}}.
\]

First we check the validity of the definition for \(F\). Note that

\[
\|F\|_{L^p(\mu)}^q = \left\{ \int_X \left[ \sum_{k \geq 0} \left( 2 \|M_\mu\|_{L^{p/q}(\mu)\rightarrow L^{p/q}(\mu)} \right)^{-k} M_\mu^{(k)}(f^q) \right]^{\frac{q}{p}} d\mu \right\}^{\frac{p}{q}}
\]

\[
\leq \sum_{k \geq 0} \left( 2 \|M_\mu\|_{L^{p/q}(\mu)\rightarrow L^{p/q}(\mu)} \right)^{-k} \left( \int_X \left| M_\mu^{(k)}(f)^{\frac{q}{p}} \right| d\mu \right)^{\frac{p}{q}}, \text{ Minkowski inequality}
\]

\[
\leq \sum_{k \geq 0} \left( 2 \|M_\mu\|_{L^{p/q}(\mu)\rightarrow L^{p/q}(\mu)} \right)^{-k} \left( \|M_\mu\|_{L^{p/q}(\mu)\rightarrow L^{p/q}(\mu)} \right)^k \|f\|_{L^p(\mu)}^q = 2\|f\|_{L^p(\mu)}^q.
\]

Hence, \(F\) is the \(L^{p/q}(\mu)\)-limit of the partial sums and thus well-defined. Moreover, we have also proved that \(\|F\|_{L^p(\mu)} \lesssim \|f\|_{L^p(\mu)}\).
Considering only $k = 0$ in the definition for $F$, we have $F \geq f$. And so $||F||_{L^p(\mu)} \asymp ||f||_{L^p(\mu)}$.

Finally, note that

$$\mu(I)^{-1} \int_I F^q d\mu \leq \inf_{x \in I} M_{\mu}(F^q)(x)$$

and

$$M_{\mu}(F^q) = \sum_{k \geq 0} \left( 2||M_{\mu}||_{L^{p/q}(\mu) \to L^{p/q}(\mu)} \right)^{-k} M_{\mu}^{(k+1)}(f^q) = 2||M_{\mu}||_{L^{p/q}(\mu) \to L^{p/q}(\mu)} (F^q - f^q) \lesssim F^q.$$  

Therefore, we deduce

$$\mu(I)^{-1} \int_I F^q d\mu \lesssim \inf_{x \in I} F^q(x), \ I \in \mathcal{L}.$$  

Now that our problem is reduced to determine a necessary and sufficient condition of

$$\int_X \left| \sum_{I \in \mathcal{L}} \alpha_I \cdot \mathbb{E}_{I}^\mu(f^q) \right| d\nu \lesssim \int_X |f|^p d\mu,$$

we may consult to the scalar-valued Theorem 1.1. Note that Theorem 1.1 still holds in the measurable space setting as is pointed out in [3]. Therefore, Theorem 1.5 follows from Theorem 1.1 for free, and both (1.8) and (1.9) are testing conditions with respect to the derived scalar-valued problem.

ACKNOWLEDGEMENT

The author would like to thank his PhD thesis advisor, Serguei Treil, for many enlightening and insightful discussions on this problem.

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