A note on the $\top$-Stein matrix equation

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Abstract

This note is concerned with the linear matrix equation $X = AX\top B + C$, where the operator $(\cdot)^\top$ denotes the transpose (\top) of a matrix. The first part of this paper set forth the necessary and sufficient conditions for the unique solvability of the solution $X$. The second part of this paper aims to provide a comprehensive treatment of the relationship between the theory of the generalized eigenvalue problem and the theory of the linear matrix equation. In the finally part of this paper starts with a briefly review of numerical methods for solving the linear matrix equation. Related to the computed methods, knowledge of the residual is discussed. An expression related to the backward error of an approximate solution is obtained; it shows that a small backward error implies a small residual. Just like for the discussion of linear matrix equations, perturbation bounds for solving the linear matrix equation are also proposed in this work.

Keywords: Sylvester equation; Stein equation; PQZ decomposition; deflating subspace; Smith method; perturbation bound; backward error

1 Introduction

Our purpose of this work is to study the so-called $\top$-Stein matrix equation

$$X = AX\top B + C,$$  \hfill (1.1)

where $A, B, C \in \mathbb{R}^{n \times n}$ are known matrices, and $X \in \mathbb{R}^{n \times n}$ is an unknown matrix to be determined. Our interest in the $\top$-Stein equation originates from the study of completely integrable mechanical systems, that is, the analysis of the $\top$-Sylvester equation

$$AX + X\top B = C,$$  \hfill (1.2)

where $A, B, C$ are matrices in $\mathbb{R}^{n \times n}$ \cite{[5, 14]}. By means of the generalized inverses or QZ decomposition \cite{[4]}, the solvability conditions of (1.2) are studies in \cite{[5, 14, 6]}. Suppose that the matrix pencil $A - \lambda B\top$ is regular, that is, $aA + bB\top$ is invertible for some scalars $a$ and $b$. The $\top$-Sylvester equation (1.2) can be written as

$$(aA + bB\top)X + X\top (aB + bA\top) = aC + bC\top.$$  \hfill (1.3)

Pre-multiplying both sides of (1.3) by $(aA + bB\top)^{-1}$, we have

$$X + UX\top V = D,$$  \hfill (1.4)

where $U = (aA + bB\top)^{-1}$, $V = aB + bA\top$ and $D = (aA + bB\top)^{-1}(aC + bC\top)$. This is of the form (1.1). In other words, numerical approaches for solving (1.2) can be obtained by transforming (1.2) into the form of (1.1), and then applying numerical methods to (1.1) for the solution \cite{[6, 17, 18]}. With this in mind, in this note we are interested in the study of $\top$-Stein matrix equation (1.1).

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Our major purpose in this work can be divided into three parts. First, we determine necessary and sufficient conditions for the unique solvability of the solution to (1.1). In doing so, Zhou et al. [21] transform (1.1) to the standard Stein equation
\[ W = AB^T W A B + AC^T B + C \] (1.5)
with respect to the unknown matrix \( W \in \mathbb{R}^{m \times n} \) and give the following necessary condition
\[ \mu \nu \neq 1, \quad \text{for all } \mu, \nu \in \sigma(A^T B). \] (1.6)
Here, \( \sigma(A^T B) \) be the set of all eigenvalues of \( A^T B \). Zhou shows that if (1.5) has a unique solution, then (1.1) has a unique solution. However, a counterexample is provided in [21] to show that the relation (1.6) is only a necessary condition for the unique solvability of (1.1).

In [6, 13], the periodic QZ (PQZ) decomposition [4] is applied to consider the necessary and sufficient conditions of the unique solvability of (1.1), conditions are given in [13] ignore the possibility of the existence of the unique solution, while 1 is a simple root of \( \sigma(A^T B) \). This condition is included in our subsequent discussion and the following remark is provided to support our observation.

**Remark 1.1** Let \( A = -1 \) and \( B = 1 \), that is, \( \sigma(AB^T) = \{-1\} \). It is clear that, the scalar equation \( X = -X^T + C \) has a unique solution \( X = C/2 \). But, condition (1.6) is not satisfied by choosing \( \mu = \nu = -1 \).

It can also be observed from Remark 1.1 that even if (1.1) is uniquely solvable, it does not imply (1.5) (namely, \( X = X + C - C \)) is uniquely solvable. Conditions in [6 (4.6)] provided that conditions for the unique solvability of the solution to (1.1) via a structured algorithm. In our work, we through a complete analysis for square coefficient matrices in terms of the analysis of the spectra of the matrix \( A^T B \), the new approach to the condition of unique solvability of the \( T \)-Stein equation (1.1) can be obtained.

Second, we present the invariant subspace method and, more generally, the deflating subspace method to solve the \( T \)-Stein equation. Our methods are based on the analysis of the eigeninformation for a matrix pencil. We carry out a thorough discussion to address the various eigeninformation encountered in the subspace methods. These ideas can be implemented into algorithms easily.

Finally, we take full account of the error analysis of Eq. (1.1). Expressions and implications such as the residual, the backward error, and perturbation bounds are derived in this work. Note that for an approximate solution \( Y \) of (1.1), the backward error tells us how much the matrices \( A, B \) and \( C \) must be perturbed. An important point found in Section 5 is that a small backward error indicates a small value for the residual \( R = Y - AY^T B - C \), but reverse is not usually true.

Beginning in Section 2, we formulate the necessary and sufficient conditions for the existence of the solution of (1.1) directly by means of the spectrum analysis. In Section 3 we provide an deflating subspace method for computing the solution of Eq. (1.1). Numerical methods for solving Eq. (1.1) and the related residual analysis are discussed in Section 4. The associated error analysis of Eq. (1.1) is given in Section 5 and concluding remarks are given in Section 6.

## 2 Solvability conditions of the Matrix Equation (1.1)

In order to formalize our discussion, let the notations \( A \otimes B \) be the Kronecker product of matrices \( A \) and \( B \), \( I_n \) be the \( n \times n \) identity matrix, and \( \| \cdot \|_F \) denotes the Frobenius norm.

With the Kronecker product, Eq. (1.1) can be written as the enlarged linear system
\[ (I_{n^2} - (B^T \otimes A)P) \text{vec}(X) = \text{vec}(C), \] (2.1)
where \( \text{vec}(X) \) stacks the columns of \( X \) into a column vector and \( P \) is the Kronecker permutation matrix [2] which maps \( \text{vec}(X) \) into \( \text{vec}(X^T) \), i.e.,
\[ P = \sum_{1 \leq i,j \leq n^2} e_j e_i^T \otimes e_i e_j^T, \]
Lemma 2.2 Let $C$ then apply it to discuss the eigenvalues of $(B \otimes A)$. In the following two lemmas, we first review the periodic QZ decomposition for two matrices and $B$ unitary matrices $P$, $Q \in \mathbb{C}^{n \times n}$ such that $A := P A Q$ and $B := Q H B^T P^H$ are two upper triangular matrices.

Lemma 2.2 Let $A$ and $B$ be two matrices in $\mathbb{R}^{n \times n}$. Then, there exist unitary matrices $P, Q \in \mathbb{C}^{n \times n}$ such that $A := P A Q$ and $B := Q H B^T P^H$ are two upper triangular matrices.

Proof. Part 1 follows immediately from Lemma since $A := P A Q$ and $B := Q H B^T P^H$ for some unitary matrices $P$ and $Q$, that is,

\[
(B^T \otimes A)P = (Q \otimes P^H)(U_A \otimes U_B)P(Q^H \otimes P)
\]

\[
= (Q \otimes P^H)(U_A \otimes U_B)(P \otimes Q^H)P
\]

Let the diagonal entries of $U_A$ and $U_B$ be denoted by $\{a_{ii}\}$ and $\{b_{jj}\}$, respectively. Then, $(U_A \otimes U_B)$ is an upper triangular matrix with given diagonal entries, specified by $a_{ii}$ and $b_{jj}$. After multiplying $(U_A \otimes U_B)$ with $P$ from the right, the position of the entry $a_{ii}b_{jj}$ is changed to be in the $j + n(i - 1)$-th row and the $i + n(j - 1)$-th column of the matrix $(U_A \otimes U_B)P$. They are then reshuffled by a sequence of permutation matrices to form a block upper triangular matrix with diagonal entries arranged in the following order

\[
\begin{bmatrix}
0 & a_{11}b_{11} \\
0 & a_{22}b_{11}
\end{bmatrix}, \ldots, \begin{bmatrix}
0 & a_{nn}b_{11} \\
0 & a_{nn}b_{22}
\end{bmatrix},
\begin{bmatrix}
0 & a_{11}b_{11} \\
0 & a_{22}b_{22}
\end{bmatrix}, \ldots, \begin{bmatrix}
0 & a_{nn}b_{nn} \\
0 & a_{nn}b_{nn}
\end{bmatrix}
\]

(2.3)

Note that the reshuffling process is not hard to see by following the ordering as used in matrix of size 2, that is, when $n = 2$, $U_A = \begin{bmatrix}
a_{11} & a_{12} \\
0 & a_{22}
\end{bmatrix}$ and $U_B = \begin{bmatrix}
b_{11} & b_{12} \\
0 & b_{22}
\end{bmatrix}$, we have

\[
(U_A \otimes U_B)P = \begin{bmatrix}
a_{11}b_{11} & a_{12}b_{11} & a_{11}b_{12} & a_{12}b_{12} \\
0 & 0 & a_{11}b_{22} & a_{12}b_{22} \\
0 & a_{22}b_{11} & 0 & a_{22}b_{12} \\
0 & 0 & 0 & a_{22}b_{22}
\end{bmatrix}.
\]
However, it is conceptually simple but operationally tedious to reoder \((U_A \otimes U_B)\mathcal{P}\) to show this result even for \(n = 3\) and that will be left as an exercise.

By (2.3), it can be seen that

\[
\sigma((B^T \otimes A)\mathcal{P}) = \left\{ a_{ii}b_{ii}, \pm \sqrt{a_{ii}a_{jj}b_{ii}b_{jj}}, 1 \leq i, j \leq n \right\}
\]

\[
= \left\{ \lambda_i, \pm \sqrt{\lambda_i \lambda_j}, 1 \leq i, j \leq n \right\}
\]

where \(\lambda_i = a_{ii}b_{ii} \in \sigma(A^\top B)\) for \(1 \leq i \leq n\).

Before demonstrating the unique solvability conditions, we need to define that a subset \(\Lambda = \{\lambda_1, \ldots, \lambda_n\}\) of complex numbers is said to be \(\top\)-reciprocal free if and only if whenever \(i \neq j\), \(\lambda_i \neq 1/\lambda_j\). This definition also regards \(0\) and \(\infty\) as reciprocals of each other. Then, we have the following solvability conditions of Eq. (1.1).

**Theorem 2.1** The \(\top\)-Stein matrix equation (1.1) is uniquely solvable if and only if the following conditions are satisfied:

a. The set of \(\sigma(A^\top B) \setminus \{-1\}\) is \(\top\)-reciprocal free.

b. \(-1\) can be an eigenvalue of the matrix \(A^\top B\), but must be simple.

**Proof.** From (2.1), we know that the \(\top\)-Stein matrix equation (1.1) is uniquely solvable if and only if

\[1 \notin \sigma((B^T \otimes A)\mathcal{P}). \quad (2.4)\]

By Lemma 2.2 if \(\lambda \in \sigma(A^\top B)\), then \(1/\lambda \notin \sigma(A^\top B)\). Otherwise, \(1 = \sqrt{\lambda \cdot 1/\lambda} \in ((B^T \otimes A)\mathcal{P})\). On the other hand, if \(-1 \in \sigma(A^\top B)\) and \(-1\) is not a simple eigenvalue, then \(1 \in \sigma((B^T \otimes A)\mathcal{P})\). This verifies (2.4) and the proof of the theorem is complete.

3 The connection between deflating subspace and Eq. (1.1)

The relationship between solution of matrix equations and the matrix eigenvalue problems has been widely studied in many applications. It is famous that solution of Riccati and polynomial matrix equations can be found by computing invariant subspaces of matrices and deflating subspaces of matrix pencils [3]. This reality leads us to finding some algorithms for computing solution of Eq. (1.1) based on the numerical computation of invariant or deflating subspaces.

Given a pair of \(n \times n\) matrices \(A\) and \(B\), recall that the function \(A - \lambda B\) in the variable \(\lambda\) is said to be the matrix pencil related to the pair \((A, B)\). For a \(k\)-dimensional subspace \(\mathcal{X} \subseteq \mathbb{C}^n\) is called a deflating subspace for the pencil \(A - \lambda B\) if there exists a \(k\)-dimensional subspace \(\mathcal{Y} \subseteq \mathbb{C}^n\) such that

\[AX \subseteq \mathcal{Y} \text{ and } BX \subseteq \mathcal{Y},\]

that is,

\[AX = YT_1 \text{ and } BX = YT_2, \quad (3.1)\]

where \(X, Y \in \mathbb{C}^{n \times k}\) are two full rank matrices whose columns span the spaces \(\mathcal{X}\) and \(\mathcal{Y}\), respectively, and matrices \(T_1, T_2 \in \mathbb{C}^{k \times k}\). In particular, if in (3.1), \(X = Y\) and \(B = T_2 = I\) for an \(n \times n\) identity matrix \(I\), then we have the simplified formula

\[AX = XT_1. \quad (3.2)\]

Here, the space \(\mathcal{X}\) spanned by the columns of the matrix \(X\) is called an invariant subspace for \(A\), and satisfies

\[A\mathcal{X} \subseteq \mathcal{X}.\]
One strategy to analyze the eigeninformation is to transform one matrix pencil to its simplified and equivalent form. That is, two matrix pencils $A - \lambda B$ and $\tilde{A} - \lambda \tilde{B}$ are said to be equivalent if and only if there exist two nonsingular matrices $P$ and $Q$ such that

$$P(A - \lambda B)Q = \tilde{A} - \lambda \tilde{B}.$$ 

In the subsequent discuss, we will use the notion $\sim$ to describe this equivalence relation, i.e., $A - \lambda B \sim \tilde{A} - \lambda \tilde{B}$.

Our task in this section is to identify eigenvectors of problem (3.1) and then associate these eigenvectors (left and right) with the solution of Eq. (1.1). We begin this analysis by studying the eigeninformation of two matrices $A$ and $B$, where $A - \lambda B$ is a regular matrix pencil.

Note that for the ordinary eigenvalue problem, if the eigenvalues are different then the eigen-

\begin{proof}

\textbf{Theorem 3.1} Given a pair of $n \times n$ matrix $A$ and $B$, if the matrix pencil $A - \lambda B$ is regular, then its Jordan chains corresponding to all finite and infinite eigenvalues carry the full spectral information about the matrix pencil and consists of $n$ linearly independent vectors.

\textbf{Lemma 3.1} Let $A - \lambda B \in \mathbb{C}^{n \times n}$ be a regular matrix pencil. Assume that matrices $X_i, Y_i \in \mathbb{C}^{n_i \times n_i}$, $i = 1, 2$, are full rank and satisfies the following equations

$$AX_i = Y_iR_i, \quad \quad (3.3a)$$
$$BX_i = Y_iS_i, \quad \quad (3.3b)$$

where $R_i$ and $S_i$, $i = 1, 2$, are square matrices of size $n_i \times n_i$. Then

i) $R_i - \lambda S_i \in \mathbb{C}^{n_i \times n_i}$ are regular matrix pencils for $i = 1, 2$.

ii) if $\sigma(R_1 - \lambda S_1) \cap \sigma(R_2 - \lambda S_2) = \phi$, then the matrix $[X_1 \quad X_2] \in \mathbb{R}^{n \times (n_1 + n_2)}$ is full rank.

We also need the following useful lemma.

\textbf{Lemma 3.2} Given two regular matrix pencils $A_i - \lambda B_i \in \mathbb{C}^{n_i \times n_i}$, $1 \leq i \leq 2$. Consider the following equations with respect to $U, V \in \mathbb{C}^{n_1 \times n_2}$

$$A_1U = VA_2, \quad \quad (3.4a)$$
$$B_1U = VB_2. \quad \quad (3.4b)$$

Then, if $\sigma(A_1 - \lambda B_1) \cap \sigma(A_2 - \lambda B_2) = \phi$, the equation (3.4a) has the unique solution $U = V = 0$.

\textbf{Proof}. For $n_2 = 1$, we get

$$A_1u = a_2v,$$
$$B_1u = b_2v,$$

where $a_2, b_2 \in \mathbb{C}, u, v \in \mathbb{C}^{n_1 \times 1}$. We may without loss of generality assume that $b_2 \neq 0$, then $A_1u = \frac{b_2}{a_2}B_1u$ and thus $u = v = 0$. Now, for any $n_2 > 1$, consider the generalized Schur decomposition of $A_2 - \lambda B_2$. We can assume that $A_2 = [a_{ij}]$ and $B_2 = [b_{ij}]$ are upper triangular matrices (i.e., $a_{ij} = b_{ij} = 0, 1 \leq j < i \leq n_2$). Denote that the $i$-th columns of $U$ and $V$ are $u_i$ and $v_i$, respectively. Thus,

$$A_1u_i = \sum_{k=1}^{i} a_{ki}v_k, \quad \quad (3.5a)$$
Theorem 3.1 Let $A, B$ and $C \in \mathbb{R}^{n \times n}$ are given in Eq. (1.1), let us write

$$
\mathcal{M} \begin{bmatrix} I_n \\ XA^T \end{bmatrix} = \begin{bmatrix} I_n \\ AX^T \end{bmatrix} BA^T,
$$

$$
\mathcal{L} \begin{bmatrix} I_n \\ XA^T \end{bmatrix} = \begin{bmatrix} I_n \\ AX^T \end{bmatrix}
$$

or if and only if its dual form

$$
[-AX^T \ I_n] \mathcal{M} = [-XA^T \ I_n],
$$

$$
[-AX^T \ I_n] \mathcal{L} = AB^T [-XA^T \ I_n].
$$

Armed with the property given in Theorem 3.1 and Lemma 3.2, we can now attack the problem of determine how the deflating subspace is related to the solution of Eq. (1.1).

**Theorem 3.2** Let $A, B$ and $C \in \mathbb{R}^{n \times n}$ are given in Eq. (1.1), let us write

$$
\mathcal{M} \begin{bmatrix} U_1 \\ V_1 \end{bmatrix} = \begin{bmatrix} U_2 \\ V_2 \end{bmatrix} T_1,
$$

$$
\mathcal{L} \begin{bmatrix} U_1 \\ V_1 \end{bmatrix} = \begin{bmatrix} U_2 \\ V_2 \end{bmatrix} T_2
$$

where $\begin{bmatrix} U_i \\ V_i \end{bmatrix}$ is full rank, $i = 1, 2$. Assume that the set of $\sigma(BA^T)$ is $\top$-reciprocal free. Then, we have
1. \( U_1 = U_2 = 0 \) if \( \sigma(T_1 - \lambda T_2) = \sigma(I_n - \lambda AB^\top) \).

2. \( U_1 \) and \( U_2 \) are nonsingular if \( T_1 - \lambda T_2 \sim BA^\top - \lambda I_n \). Moreover, if \( A \) is nonsingular, then
\[
X = V_1 U_1^{-1} A^{-\top} = U_2^{-\top} V_2 A^{-\top}
\]
is the unique solution of Eq. (1.1).

**Proof.** From (3.7) we get
\[
BA^\top U_1 = U_2 T_1, \tag{3.8a}
\]
\[
-CA^\top U_1 + V_1 = V_2 T_1, \tag{3.8b}
\]
\[
U_1 = U_2 T_2, \tag{3.8c}
\]
\[
AC^\top U_1 + AB^\top V_1 = V_2 T_2, \tag{3.8d}
\]
i) It follows from (3.8a) and (3.8c) that since \( \sigma(BA^\top - \lambda I_n) \cap \sigma(T_1 - \lambda T_2) = \emptyset \), we have \( U_1 = U_2 = 0 \) by Lemma 3.2.

ii) It can be seen that there exist two nonsingular matrices \( U \) and \( V \) such that
\[
M \begin{bmatrix} 0 \\ U \\ V \end{bmatrix} = \begin{bmatrix} 0 \\ U_2 \\ V_2 \end{bmatrix} T_2, \tag{3.7}
\]
\[
L \begin{bmatrix} 0 \\ U \\ V \end{bmatrix} = \begin{bmatrix} 0 \\ U_2 \\ V_2 \end{bmatrix} T_1. \tag{3.7}
\]

Hence, together with (3.7) we have
\[
M \begin{bmatrix} 0 \\ U \\ V \end{bmatrix} = \begin{bmatrix} 0 \\ U_2 \\ V_2 \end{bmatrix} T_2, \tag{3.7}
\]
\[
L \begin{bmatrix} 0 \\ U \\ V \end{bmatrix} = \begin{bmatrix} 0 \\ U_2 \\ V_2 \end{bmatrix} T_1. \tag{3.7}
\]

Since \( \sigma(M - \lambda L) = \sigma(BA^\top - \lambda I_n) \cup \sigma(I_n - \lambda AB^\top) \), by Theorem 3.1 and Lemma 3.1, the matrix \( \begin{bmatrix} 0 \\ U \\ V \end{bmatrix} \) is nonsingular. Together with (3.8c), \( U_1 \) and \( U_2 \) are nonsingular.

Let \( X_i = V_i U_i^{-1}, \ i = 1, 2 \), then form (3.8a) and (3.8d)
\[
AC^\top + AB^\top X_1 = V_2 T_2 U_1^{-1} = X_2,
\]
\[
-CA^\top + X_1 = V_2 T_1 U_1^{-1} = X_2 BA^\top,
\]
or
\[
AC^\top + AB^\top X_1 = X_2,
\]
\[
AC^\top + AB^\top X_2 = X_1^\top.
\]

Since the set of \( \sigma(AB^\top) = \sigma(BA^\top) \) is \( T \)-reciprocal free, together with
\[
X_1^\top - X_2 - AB^\top (X_1^\top - X_2)^\top = 0,
\]
we get \( X_1 = X_2^\top \). If \( A \) is nonsingular, it is easy verify that two matrices \( X_1 A^{-\top} \) and \( X_2^\top A^{-\top} \) are both satisfying \( T \)-Stein equation Eq. (1.1). The proof of part (ii) is complete.

**Remark 3.1** 1. It is easily seen that \( \begin{bmatrix} I_n \\ X A^\top \end{bmatrix} \) and \( \begin{bmatrix} U_1 \\ V_2 \end{bmatrix} \) both span the unique deflating subspace of \( M - \lambda L \) corresponding to the set of \( \sigma(BA^\top) \). Otherwise, in part (ii) we know that \( T_2 \) is
nonsingular. We then be able to transform the formulae defined in (3.7) into the generalized eigenvalue problem as follows.

\[
\mathcal{M} \begin{bmatrix} U_1 \\ V_1 \end{bmatrix} = \mathcal{L} \begin{bmatrix} U_1 \\ V_1 \end{bmatrix} BA^\top.
\]

That is, some numerical methods for the computation of the eigenspace of \( \mathcal{M} - \lambda \mathcal{L} \) corresponding to the set of \( \sigma(BA^\top) \) can be designed and solved Eq. (3.1).

2. Since the transport of the unique solution \( X \) of Eq. (1.1) is equal to the unique solution \( Y \) of the following matrix equation

\[
Y = B^\top YA^\top + C^\top.
\]

Analogous to the consequences of Theorem 3.2, the similar results can be obtained with respect to Eq. (3.11) if \( B \) is nonsingular. However, we point out that Eq. (1.1) can be solved by computing deflating subspaces of another matrix pencils. For instance we let

\[
\mathcal{M}_1 - \lambda \mathcal{L}_1 := \begin{bmatrix} A^\top B & 0 \\ -C - AC^\top B & I_n \end{bmatrix} - \lambda \begin{bmatrix} I_n & 0 \\ 0 & AB^\top \end{bmatrix}.
\]

Assume that the set of \( \sigma(BA^\top) \) is \( \top \)-reciprocal free, it can be shown that \( \mathcal{M}_1 \begin{bmatrix} I_n \\ X \end{bmatrix} = \mathcal{L}_1 \begin{bmatrix} I_n \\ X \end{bmatrix} A^\top B \) and it has similar results as the conclusion of Theorem 3.2. The unique solution \( X \) of (1.1) can be found by computing deflating subspaces of the matrix pencil \( \mathcal{M}_1 - \lambda \mathcal{L}_1 \) without the assumption of the singularity of \( A \) and \( B \).

4 Computational methods for solving Eq. (1.1)

Numerical methods for solving Eq. (1.1) has received great attention in theory and in practice and can be found in [18, 17] for Krylov subspace methods and in [16, 15, 20] for Smith-type iterative methods. In particular, Smith-type iterative methods are only workable in the case \( \rho(AB^\top) < 1 \), where \( \rho(AB^\top) \) denotes the spectral radius of \( AB^\top \). In the recent years, a structure algorithm has been studied for Eq. (1.1) [6] via PQZ decomposition, which consists of transforming and into Schur form by a PQZ decomposition, and then solving the resulting triangular system by way of back-substitution. In this section, we revisit these numerical methods and point out the advantages and drawbacks of all algorithms.

4.1 Krylov subspace methods

Since the \( \top \)-Stein equation is essentially a linear system (2.1), we certainly can use Krylov subspace methods to solve (2.1). See, e.g., [18, 17], and the reference cited therein. The general idea for applying Krylov subspace methods is by defining the \( \star \)-Stein operator \( \mathcal{T} : X \to X - AX^\top B \) and its adjoint liner operator \( \mathcal{T}^* : Y \to Y - BY^\top A \) such that \( \langle \mathcal{T}(X), Y \rangle = \langle X, \mathcal{T}^*(Y) \rangle \). Here, \( X, Y \in \mathbb{R}^{m \times n} \) and the notion \( \langle \cdot, \cdot \rangle \) is denoted as the Frobenius inner product. Then, the iterative method based on Krylov subspaces for Eq. (1.1) is as follows.

- The conjugate gradient (CG) method [17]:

\[
X_{k+1} = X_k + \frac{\|R_k\|^2}{\|P_k\|^2} P_k,
\]

\[
R_{k+1} = C - \mathcal{T}(X_{k+1}) = R_k - \frac{\|R_k\|^2}{\|P_k\|^2} T(P_k),
\]

\[
D_{k+1} = \mathcal{T}^*(R_{k+1}) + \frac{\|R_{k+1}\|^2}{\|R_k\|^2} D_k,
\]
with an initial matrix $X_0$ and the corresponding initial conditions

$$R_0 = C - T(X_0), \quad D_0 = T^*(R_0).$$

Note that when the solvability conditions of Theorem 2.1 are met, the CG method is guaranteed to converge in a finite number of iterations for any initial matrix $X_0$.

### 4.2 The Bartels-Stewart-like Algorithm \[1\]

In this subsection we focus on the discussion of the Bartels-Stewart algorithm, which is known to be a numerical stable algorithm, to solve $T$-Stein equations. This method is to solve Eq. (1.1) by means of the PQZ decomposition \[1\]. Its approach has been discussed in \[6\] and can be summarized as follows. From Lemma 2.2 we know that there exist two unitary matrices $P$ and $Q$ (see \[4\] for the computation procedure) such that

$$PXQ - PAQ \cdot Q^H X^\top P^\top \cdot P^\top Q = PCQ$$  \hspace{1cm} (4.1)

With $A = PAQ$ and $B^\top = Q^H B^\top P^H$ being upper-triangular, the transformed equation looks like

\[
\begin{bmatrix}
\hat{X}_{11} & \hat{x}_{12} \\
\hat{x}_{21} & \hat{X}_{22}
\end{bmatrix} -
\begin{bmatrix}
A_{11} & \hat{a}_{12} \\
0 & \hat{a}_{22}
\end{bmatrix}
\begin{bmatrix}
\hat{X}_{11}^\top & \hat{x}_{21}^\top \\
\hat{x}_{12}^\top & \hat{X}_{22}
\end{bmatrix}
\begin{bmatrix}
\hat{B}_{11} & 0 \\
\hat{b}_{21} & \hat{b}_{22}
\end{bmatrix} =
\begin{bmatrix}
\hat{C}_{11} & \hat{c}_{12} \\
\hat{c}_{21} & \hat{c}_{22}
\end{bmatrix}
\]

with $\hat{X} = \begin{bmatrix}
\hat{X}_{11} & \hat{x}_{12} \\
\hat{x}_{21} & \hat{X}_{22}
\end{bmatrix}$. We then have

\[
\hat{x}_{22} - \hat{a}_{22} \hat{x}_{22}^\top \hat{b}_{22} = \hat{c}_{22}, \quad \text{(4.2)}
\]

\[
\hat{x}_{21} - \hat{a}_{22} \hat{x}_{12} \hat{B}_{11} = \hat{c}_{21} + \hat{a}_{22} \hat{x}_{22}^\top \hat{b}_{21}, \quad \text{(4.3)}
\]

\[
\hat{x}_{12} - \hat{A}_{11} \hat{x}_{21} \hat{b}_{22} = \hat{c}_{12} + \hat{a}_{12} \hat{x}_{22}^\top \hat{b}_{21}, \quad \text{(4.4)}
\]

\[
\hat{X}_{11} - \hat{A}_{11} \hat{X}_{11} \hat{B}_{11} = \hat{C}_{11} + \hat{a}_{12} \hat{x}_{12} \hat{B}_{11} + \hat{A}_{11} \hat{x}_{21} \hat{b}_{21} + \hat{a}_{12} \hat{x}_{22} \hat{b}_{21}. \quad \text{(4.5)}
\]

Thus, the Bartels-Stewart algorithm can easily be constructed by first solving $\hat{x}_{22}$ from (4.2), using $\hat{x}_{22}$ to obtain $\hat{x}_{12}$ and $\hat{x}_{21}$ from (4.3) and (4.4), and then repeating the same discussion as (4.2)-(4.4) by taking advantage of the property of $A_{11}$ and $B_{11}$ being lower triangular matrices from (4.5).

### 4.3 Smith-type iterative methods

Originally, Smith-type iterative methods are developed to solve the standard Stein equation

$$X = AXB + C, \quad A, B, C \in \mathbb{R}^{n \times n}.$$ 

As mention before, the unknown $X$ is highly related to the generalized eigenspace problems

$$\begin{bmatrix} B & 0 \\ -C & I \end{bmatrix} \begin{bmatrix} I \\ 0 \end{bmatrix} X = \begin{bmatrix} I & 0 \\ 0 & A \end{bmatrix} X \begin{bmatrix} I \\ 0 \end{bmatrix} B.$$  \hspace{1cm} (4.6a)

or

$$A X \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} = X \begin{bmatrix} I & 0 \\ 0 & -C \end{bmatrix} A.$$  \hspace{1cm} (4.6b)

Pre-multiplying (4.6a) by the matrix $\begin{bmatrix} B & 0 \\ -AC & I \end{bmatrix}$ and post-multiplying (4.6b) by the matrix $\begin{bmatrix} I & 0 \\ -CB & A \end{bmatrix}$, we get

$$\begin{bmatrix} B^2 & 0 \\ -C - ACB & I_n \end{bmatrix} X = \begin{bmatrix} I_n & 0 \\ 0 & A^2 \end{bmatrix} X B^2.$$
\[ A^2 \begin{bmatrix} X & I_n \end{bmatrix} \begin{bmatrix} B^2 & 0 & 0 \\ 0 & I_n \end{bmatrix} = \begin{bmatrix} X & I_n \end{bmatrix} \begin{bmatrix} I_n & 0 \\ -C - ACB & A^2 \end{bmatrix}. \]

Then, for any positive integer \( k > 0 \), we obtain
\[
\begin{bmatrix} B^{2k-1} \\ -C_k & I_n \end{bmatrix} X = \begin{bmatrix} I_n & 0 \\ 0 & A^{2k-1} \end{bmatrix} X B^{2k-1},
\]
\[
A^{2k-1} \begin{bmatrix} X & I_n \end{bmatrix} \begin{bmatrix} B^{2k-1} & 0 \\ 0 & I_n \end{bmatrix} = \begin{bmatrix} X & I_n \end{bmatrix} \begin{bmatrix} I_n & 0 \\ -C_k & A^{2k-1} \end{bmatrix},
\]
where the sequence \( \{C_k\} \) is defined by
\[
C_k = C_{k-1} + A^{2k-1} C_{k-1} B^{2k-1}, \quad k \geq 1, \tag{4.7a}
\]
\[
C_0 = C. \tag{4.7b}
\]

The explicit expression of \( C_k \) is given as following
\[
C_k = \sum_{i=1}^{2^k-1} A^i CB^i.
\]

Under the condition \( \rho(A)\rho(B) < 1 \), it is easy to see that \( \{C_k\} \) is convergence, and
\[
\limsup_{k \to \infty} \sqrt[2^k]{\|X - C_k\|} \leq \rho(A)\rho(B),
\]
that is, \( C_k \) converges quadratically to \( X \) as \( k \to \infty \). This iterative method \[4.7\] is called Smith iteration \[10\]. In recent years, some modified iterative methods are so-called Smith-type iteration, which are based on Smith iteration and improve its speed of convergence. See, e.g., \[20\] and the references cited therein.

Since the condition \( \rho(A)\rho(B) < 1 \) implies that the assumptions of Theorem \[2.1\] hold, Eq. \[1.1\] is equivalent to Eq. \[1.5\]. We can apply Smith iteration to the Eq. \[1.1\] with the substitution \( (A, B, C) = (AB^\top, A^\top B, C + AC^\top B) \). One possible drawback of the Smith-type iterative methods is that it cannot always handle the case when there exist eigenvalues \( \lambda, \mu \in \sigma(A^\top B) \) such that \( \lambda \mu = -1 \) even the unique solution \( X \) exist. Based on the solvable conditions given in this work, it is possible to develop a specific technique working on the particular case and it is a subject currently under investigation.

### 5 Error analysis

Error analysis is a way for testing the stability of an numerical algorithm and evaluating the accuracy of an approximated solution. In the subsequent discussion, we want to consider the backward error and perturbation bounds for solving Eq. \[1.1\].

As indicated in \[1.1\], matrices \( \hat{A} \) and \( \hat{B}^\top \) are both upper-triangular. We can then apply the error analysis for triangular linear systems in \[10\] Section 3.1 \[11\] to obtain
\[
\|\hat{C} - (\hat{X} - \hat{A}\hat{X}^\top \hat{B})\|_F \leq c_{m,n} u (1 + \|\hat{A}\|_F \|\hat{B}\|_F) \|\hat{X}\|_F,
\]
where \( c_{m,n} \) is a content depending on the dimensions \( m \) and \( n \), \( u \) is the unit roundoff. Since the PQZ decomposition is a stable process, it is true that
\[
\|C - (X - AX^\top B)\|_F \leq c'_{m,n} u (1 + \|A\|_F \|B\|_F) \|X\|_F, \tag{5.1}
\]
with a modest multiple \( c'_{m,n} \).

Note that the inequality of the form \[5.1\] can be served as a stopping criterion for terminating iterations generated from Krylov subspace methods \[13\] \[17\] and Smith-type iterative methods \[10\] \[15\] \[20\]. In what follows, we shall derive the error associated with numerical algorithms, following the development in \[8\] \[11\].
5.1 Backward error

Like the discussion of ordinary Sylvester equations \[\text{II}\], the normwise backward error of an approximate solution \(Y\) of Eq. \[\text{II}\] is defined by

\[
\eta(Y) \equiv \min \{ \epsilon : Y = (C + \delta C) + (A + \delta A)Y^\top (B + \delta B) \},
\]

(5.2)

where \(\alpha \equiv \|A\|_F\), \(\beta \equiv \|B\|_F\) and \(\gamma \equiv \|C\|_F\). Let \(\mathcal{R} \equiv \delta C + \delta AY^\top B + AY^\top \delta B + \delta AY^\top \delta B\), which implies that \(\mathcal{R} = Y - AY^\top B - C\). It can be seen that the residual \(\mathcal{R}\) satisfies

\[
\|\mathcal{R}\|_F \leq \eta(Y)(\gamma + \|Y\|_F \alpha \beta(2 + \eta(Y))).
\]

(5.3)

From (5.3), we know that a small backward error indeed implies a small relative residual \(\mathcal{R}\). Since the coefficient matrices in Eq. \(\text{II}\) include nonlinearity, it appears to be an open problem to obtain the theoretical backward error with respect to the residual. Again, similar to the Sylvester equation discussed in \[\text{II}\], Section 16.2, the conditions under which a \(\top\)-Stein equation has a well-conditioned solution remain unknown.

5.2 Perturbation bounds

Consider the perturbed equation

\[
X + \delta X = (A + \delta A)(X + \delta X)^\top (B + \delta B) + (C + \delta C).
\]

(5.4)

Let \(S(X) = X - AX^\top B\) be the corresponding \(\top\)-Stein operator. We then have \(S(\delta X) = \delta C + A(X + \delta X)^\top \delta B + \delta A(X + \delta X)^\top (B + \delta B)\). With the application of norm, it follows that

\[
\|\delta X\|_F \leq \|S^{-1}\|_F \{\|\delta C\|_F + \|\delta S\|_F(\|X\|_F + \|\delta X\|_F)\},
\]

where \(\|\delta S\|_F \equiv \|A\|_F\|\delta B\|_F + \|\delta A\|_F(\|B\|_F + \|\delta B\|_F)\). When \(\|\delta S\|_F\) is small enough so that \(1 \geq \|S^{-1}\|_F \cdot \|\delta S\|_F\), we can rearrange the above result to

\[
\frac{\|\delta X\|_F}{\|X\|_F} \leq \frac{\|S^{-1}\|_F}{1 - \|S^{-1}\|_F \cdot \|\delta S\|_F} \left(\frac{\|\delta C\|_F}{\|C\|_F} + \frac{\|\delta S\|_F}{\|S\|_F}\right).
\]

With \(\|C\|_F = \|S(X)\|_F \leq \|S\|_F \cdot \|X\|_F\) and the condition number \(\kappa(S) \equiv \|S\|_F \cdot \|S^{-1}\|_F\), we arrive at the standard perturbation result

\[
\frac{\|\delta X\|_F}{\|X\|_F} \leq \frac{\kappa(S)}{1 - \kappa(S) \cdot \|\delta S\|_F / \|S\|_F} \left(\frac{\|\delta C\|_F}{\|C\|_F} + \frac{\|\delta S\|_F}{\|S\|_F}\right).
\]

(5.5)

Thus the relative error in \(X\) is controlled by those in \(A\), \(B\) and \(C\), magnified by the condition number \(\kappa(S)\).

On the other hand, we can also drop the high order terms in the perturbation to obtain

\[
\delta X - A\delta X^\top B = AX^\top \delta B + \delta AX^\top B + \delta C.
\]

We then rewrite the system in terms of

\[
Q \text{vec}(\delta X) = \left[ \begin{array}{ccc} (X^\top B)^\top & I_m \otimes I_n & I_m \otimes (AX^\top) \end{array} \right] \left[ \begin{array}{c} \text{vec}(\delta A) \\ \text{vec}(\delta B) \\ \text{vec}(\delta C) \end{array} \right],
\]

where \(Q = I_{mn} - (B^\top \otimes A)P\). Let \(\zeta = \max \left\{ \frac{\|\delta A\|_F}{\|A\|_F}, \frac{\|\delta B\|_F}{\|B\|_F}, \frac{\|\delta C\|_F}{\|C\|_F} \right\}\). It can be shown that

\[
\frac{\|\delta X\|_F}{\|X\|_F} \leq \sqrt{3} \zeta,
\]

(5.5)
where $\Psi = \|Q^{-1} \left[ \begin{array}{cc} \alpha (X^\top B)^\top \otimes I_m & \beta I_n \otimes (AX)^\top \\ \gamma I_m \otimes I_n \otimes (AX) \end{array} \right] \|_2 / \|X\|_F$.

A possible disadvantage of the perturbation bound (5.5), which ignores the consideration of the underlying structure of the problem, is to overestimate the effect of the perturbation on the data. But this “universal” perturbation bound is accessible to any given matrices $A$, $B$ and $C$ of Eq. (1.1).

Unlike the perturbation bound (5.5), it is desirable to obtain a posteriori error bound by assuming $\delta A = \delta B = 0$ and $\delta C = \hat{X} - A\hat{X}^\top B - C$ in (5.4). This assumption gives rise to

$$\frac{\|\delta X\|_F}{\|X\|_F} \leq \frac{\|P^{-1}\|_2 \|R\|_F}{\|X\|_F}. \quad (5.6)$$

It is true that while doing numerical computation, this bound given in (5.6) provides a simpler way for estimating the error of the solution of Eq. (1.1).

6 Conclusion

In this note, we propose a novel approach to the necessary and sufficient conditions for the unique solvability of the solution $X$ of the $\top$-Stein equation for square coefficient matrices in terms of the analysis of the spectra $\sigma(A^\top B)$. Solvability conditions have been derived and algorithms have been proposed in [6, 13] by using PQZ decomposition. On the other hand, one common procedure to solve the Stein-type equations is by means of the invariant subspace method. We believe that our discussion is the first which implements the techniques of the deflating subspace for solving $\top$-Stein matrix equation and might also gives rise to the possibility of developing an advanced and effective solver in the future. Also, we obtain the theoretical residual analysis, backward error analysis, and perturbation bounds for measuring accurately the error in the computed solution of Eq. (1.1).

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