Generation of even and odd nonlinear coherent states

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Abstract

We show that a class of even and odd nonlinear coherent states, defined as the eigenstates of product of a nonlinear function of the number operator and the square of the boson annihilation operator, can be generated in the center-of-mass motion of a trapped and bichromatically laser-driven ion. The nonclasscial properties of the states are studied.

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Coherent states are important in many fields of physics \[1,2\]. Coherent states \(|\alpha\rangle\), defined as the eigenstates of the harmonic oscillator annihilation operator \(\hat{a}\), \(\hat{a}|\alpha\rangle = \alpha |\alpha\rangle \[3\], have statistical properties like the classical radiation field. In a harmonic oscillator potential the centre of the coherent state wavepacket follows the classical trajectory. There exist states of the electromagnetic field whose properties, like squeezing, higher order squeezing, antibunching and sub-Poissonian statistics \[4,5\], are strictly quantum mechanical in nature. These states are called nonclassical states. The coherent states define the limit between the classical and nonclassical behaviour of the radiation field as far as the nonclassical effects are considered.

A generalization of the coherent states was done by \(q\)-deforming the basic commutation relation \([\hat{a}, \hat{a}^\dagger] = 1 \[6,7\]. A further generalization is to define states that are eigenstates of the operator \(f(\hat{n})\hat{a}\),

\[
f(\hat{n})\hat{a}|f, \alpha\rangle = \alpha |f, \alpha\rangle,
\]

where \(f(\hat{n})\) is a operator valued function of the number operator \(\hat{n} = \hat{a}^\dagger\hat{a}\). These eigenstates are called nonlinear coherent states and are nonclassical. In the linear limit, \(f(\hat{n}) = 1\), the nonlinear coherent states become the usual coherent states \(|\alpha\rangle\). The nonlinear coherent states were introduced, as f-coherent states, in connection with the study of the oscillator whose frequency depends on its energy \[8,9\]. A class of nonlinear coherent states can be realized physically as the stationary states of the center-of-mass motion of a trapped ion \[10\]. These nonlinear coherent states exhibit nonclassical features like squeezing and self-splitting.

Superposition of coherent states gives rise to states with nonclassical properties. An important case is the superposition of the coherent states \(|\alpha\rangle\) and \(|-\alpha\rangle\) and the resultant states are eigenstates of the operator \(\hat{a}^2 \[11\). The symmetric combination is the even coherent state (ECS), \(|\alpha, +\rangle\), and its number state expansion is

\[
|\alpha, +\rangle = [\cosh |\alpha|^2]^{-1/2} \sum_{n=0}^{\infty} \frac{\alpha^{2n}}{\sqrt{(2n)!}} |2n\rangle.
\]

The antisymmetric combination is the odd coherent state (OCS), \(|\alpha, -\rangle\), given by
\[ |\alpha, -\rangle = [\sinh |\alpha|^2]^{-1/2} \sum_{n=0}^{\infty} \frac{\alpha^{2n+1}}{\sqrt{(2n+1)!}} |2n + 1\rangle. \] (3)

The ECS has a squeezing but no antibunching effect. The OCS has an antibunching effect but no squeezing effect \[12,13]. Both ECS and OCS have oscillatory photon number distribution. These states can be generated by various schemes: propagation in Kerr medium \[14,15\], micromaser cavity experiments \[16\], quantum nondemolition experiments \[17\] and motion of a trapped ion \[18\]. The even and odd coherent states can be interpreted as Schrödinger cat states for appropriately large values of \(\alpha\) \[14\].

The notion of even and odd coherent states has been generalised to the case of nonlinear coherent states \[19,20\]. The even and odd nonlinear coherent states are defined as the eigenstates of the operator \(F(\hat{n})\hat{a}^2\), where \(F(\hat{n})\) is a operator valued function of the number operator \(\hat{n}\). We denote the eigenstates as \(|\alpha, F\rangle\) and they satisfy

\[ F(\hat{n})\hat{a}^2|\alpha, F\rangle = \alpha|\alpha, F\rangle, \] (4)

where \(\alpha\) is complex. The above equation gives rise to the recurrence relation

\[ \langle n + 2|\alpha, F\rangle = \alpha \frac{\langle n|\alpha, F\rangle}{F(n)\sqrt{(n+1)(n+2)}}, \] (5)

for \(n = 0, 1, 2, 3\ldots\) and the function \(F(n)\) is obtained by replacing the number operator \(\hat{n}\) in \(F(\hat{n})\) by the integer \(n\). The complex numbers \(\langle n + 2|\alpha, F\rangle\) \((n = 0, 1, 2, \ldots)\) are the expansion coefficients of the state \(|\alpha, F\rangle\) in the harmonic oscillator basis. The above recurrence relation between the expansion coefficients yields

\[ \langle 2n|\alpha, F\rangle = \alpha^n \frac{\langle 0|\alpha, F\rangle}{F(2(n-1))!!\sqrt{(2n)!}}, \] (6)

\[ \langle 2n + 1|\alpha, F\rangle = \alpha^n \frac{\langle 1|\alpha, F\rangle}{F(2n-1)!!\sqrt{(2n+1)!}}. \] (7)

where \(F(2(n-1))!! = F(0)F(2)F(4)\ldots F(2(n-1))\) and \(F(2n-1)!! = F(1)F(3)F(5)\ldots F(2n-1)\). The function \(F(k)!!\) is set equal to unity if the argument \(k\) is less than or equal to zero. The above relations yields all the coefficients, \(n = 1, 2, 3\ldots\), in terms of \(\langle 0|\alpha, F\rangle\) and...
\[\langle 1|\alpha, F\rangle.\] The coefficients \(\langle 0|\alpha, F\rangle\) and \(\langle 1|\alpha, F\rangle\) are fixed by normalizing the state \(|\alpha, F\rangle\). If we choose \(\langle 1|\alpha, F\rangle = 0\), the state \(|\alpha, F\rangle\) involves the superposition of even number (Fock) states and represents the even nonlinear coherent state. If \(\langle 0|\alpha, F\rangle = 0\), the state \(|\alpha, F\rangle\), the superposition of odd number states, is the odd nonlinear coherent state. We denote the even nonlinear coherent state as \(|\alpha, F, +\rangle\) and the odd nonlinear coherent state as \(|\alpha, F, -\rangle\).

In the linear limit, \(F(\hat{n}) = 1\), the even nonlinear coherent state becomes the even coherent state and the odd nonlinear coherent state becomes the odd coherent state. Depending on the form of \(F(\hat{n})\) the even and odd nonlinear coherent states may exhibit many of the nonclassical features. It is interesting to note that the squeezed vacuum and the squeezed first excited state of the harmonic oscillator can be interpreted as even and odd nonlinear coherent states respectively. The squeezed vacuum is the even nonlinear coherent state when \(F(\hat{n}) = 1/(1 + \hat{a}^\dagger \hat{a})\) and the squeezed first excited state is the odd nonlinear coherent state with \(F(\hat{n}) = 1/(2 + \hat{a}^\dagger \hat{a})\).

One of the interesting systems in quantum optics is the harmonically trapped and laser-driven ion wherein the interaction between the ion and the laser has nonlinear \(\hat{n}\) dependence. This system has been studied in very many contexts: nonlinear coherent states \([10]\), vibronic Jaynes-Cummings interaction \([21]\), nonlinear Jaynes-Cummings interaction \([22]\), generation of even and odd coherent states \([18]\), quantum signatures of chaos \([23]\), quantum nondemolition measurements \([24]\), quantum logic operations \([25]\), engineering of Hamiltonian \([26]\) and generation of amplitude-squared squeezed states \([27]\). In the present contribution we show that a class of even and odd nonlinear coherent states can be generated in the center-of-mass motion of a harmonically trapped ion via bichromatic laser excitation. We also study the nonclassical properties of the states produced.

We consider a single ion, having an electronic transition frequency \(\omega\) and a lower (second) vibrational sideband with respect to that frequency, trapped in harmonic potential of frequency \(\nu\). Two laser fields, tuned, respectively, to \(\omega\) and the vibrational sideband transition frequency interact with the ion. The Hamiltonian of this system in the optical rotating-wave approximation can be written as \([18]\).
\[ \hat{H} = \hat{H}_0 + \hat{H}_{int}(t), \]  

(8)

with

\[ \hat{H}_0 = \hbar \nu \hat{a}^\dagger \hat{a} + \hbar \omega \hat{\sigma}_{22}. \]  

(9)

The free Hamiltonian \( \hat{H}_0 \) describes the free motion of the internal and external degrees of freedom and the interaction Hamiltonian \( \hat{H}_{int} \),

\[ \hat{H}_{int}(t) = \lambda [E_0 e^{\frac{-i}{\hbar} (k_0 x - \omega t)} + E_1 e^{\frac{-i}{\hbar} (k_1 x - (\omega - 2\nu) t)}] \hat{\sigma}_{12} + H.c., \]  

(10)

describes the interaction of the ion with the two laser fields. The operators \( \hat{\sigma}_{ij} \) \((i, j = 1, 2)\) are the electronic flip operators corresponding to the transition \( |j\rangle \rightarrow |i\rangle \) and \( \hat{a} \) is the annihilation operator for the vibrational motion of the ion in the harmonic potential. The constant \( \lambda \) is the electronic coupling matrix element and \( k_0, k_1 \) are the wave vectors of the laser fields. The operator of the center-of-mass position of the ion is

\[ \hat{x} = \frac{\eta}{k_L} (\hat{a} + \hat{a}^\dagger), \]  

(11)

where \( \eta \) is the Lamb-Dicke parameter and \( k_L \approx k_0 \approx k_1 \) is the wave vector of the driving laser field.

In the interaction picture, defined by the unitary transformation \( \exp(-i \frac{\hat{H}_0 t}{\hbar}) \), the interaction Hamiltonian becomes

\[ \hat{H}'_{int} = \hbar \Omega_i \exp(-\frac{\eta^2}{2}) \hat{\sigma}_{12} \left[ \sum_{k,l=0}^{\infty} \frac{(i\eta)^{k+l}}{k!!l!!} e^{i(k-2\nu)t} \hat{a}^\dagger k \hat{a}^l + \frac{\Omega_i}{\Omega_1} \sum_{k,l=0}^{\infty} \frac{(i\eta)^{k+l}}{k!!l!!} e^{i(k-l)t} \hat{a}^\dagger k \hat{a}^l \right] + H.c. \]  

(13)

where \( \Omega_i = \frac{\lambda E_i}{\hbar} \) \((i = 1, 2)\) are the Rabi frequency of the two laser fields tuned to the electronic transition and the second sideband respectively. In the rotating wave approximation, neglecting terms rotating with frequencies \( \nu \) or more, the interaction picture Hamiltonian becomes

\[ \hat{H}'_{int} = \hbar \Omega_1 \exp(-\frac{\eta^2}{2}) \hat{\sigma}_{21} \hat{F} + H.c., \]  

(14)
\[
\hat{F} = \sum_{k=0}^{\infty} \frac{(i\eta)^{2k+2}}{k!(k+2)!} \hat{a}^\dagger k \hat{a}^k + \frac{\Omega_0}{\Omega_1} \sum_{k=0}^{\infty} \frac{(i\eta)^{2k}}{k!^2} \hat{a}^\dagger k \hat{a}^k.
\] (15)

The time evolution of the system is governed by the master equation for the vibronic density operator \(\hat{\rho}\),

\[
\frac{d}{dt} \hat{\rho} = -\frac{i}{\hbar} [\hat{H}_\text{int}, \hat{\rho}] + \frac{\Gamma}{2} (2\hat{\sigma}_{12} \hat{\rho} \hat{\sigma}_{21} - \hat{\sigma}_{22} \hat{\rho} - \hat{\rho} \hat{\sigma}_{22}),
\] (16)

where the second term is to include the effect of spontaneous emission with energy relaxation rate \(\Gamma\) and

\[
\hat{\rho}' = \frac{1}{2} \int_{-1}^{1} dy W(y) e^{i\eta(\hat{a}^\dagger + \hat{a}) y} \hat{\rho} e^{-i\eta(\hat{a}^\dagger + \hat{a}) y},
\] (17)

accounts for changes in the vibrational energy due to spontaneous emission. \(W(y)\) gives the angular distribution of spontaneous emission. The steady state solution \(\hat{\rho}_s\) of Eq. (16) is obtained by setting \(\frac{d}{dt} \hat{\rho} = 0\). If we make the ansatz that \(\hat{\rho}_s\) is given by

\[
\hat{\rho}_s = |1\rangle \langle \zeta| |\zeta\rangle |1\rangle,
\] (18)

where \(|1\rangle\) is the electronic ground state and \(|\zeta\rangle\) is the vibrational state of the ion, then the state \(|\zeta\rangle\) obeys

\[
\hat{F}|\zeta\rangle = 0.
\] (19)

Using \(\hat{F}\) given by Eq. (15) we get

\[
\langle n + 2 |\zeta\rangle = \frac{\Omega_0}{\Omega_1 \eta^2 \sqrt{(n+1)(n+2)L_n^0(\eta^2)}} L_n^0(\eta^2) \langle n |\zeta\rangle
\] (20)

where \(L_n^m\) is an associated Laguerre polynomial defined by

\[
L_n^m(x) = \sum_{l=0}^{\infty} \binom{n+m}{n-l} \frac{(-x)^l}{l!}.
\] (21)

The numbers \(\langle n + 2 |\zeta\rangle\) are the expansion coefficients for the state \(|\zeta\rangle\) in the Fock states basis. Comparing with Eq. (3) indicates that the state \(|\zeta\rangle\) is an even or odd nonlinear coherent state with
\[ \alpha = \frac{\Omega_0}{\Omega_1 \eta^2} \]  

and

\[ F(n) = L_n^2(\eta^2)[(n + 1)(n + 2)L_n^0(\eta^2)]^{-1}. \]  

In the limit \( \eta \to 0 \) the function \( F(n) \) becomes \( \frac{1}{2} \) for all \( n \). Hence in the small \( \eta \) limit the even and odd nonlinear coherent states become the ECS and OCS respectively.

If the initial state of the ion is a combination of even(odd) number states then the state of the system at later times will involve a superposition of even(odd) number states only as the master equation Eq. (16) contains only even powers of \( \hat{a} \) and \( \hat{a}^\dagger \). If the initial state of the ion is the vacuum state then the stationary state of the system is given by

\[ |\alpha, F, +\rangle = N \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{(2n)!F(2n-2)!!}} |2n\rangle; \quad N^{-1} = \sqrt{\sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{(2n)!(F(2n-2)!!)^2}} \]  

where \( \alpha \) and \( F(n) \) are defined by Eq.(22) and Eq.(23) respectively. This state is the even nonlinear coherent state for the vibrational motion of the center-of-mass of the ion in the harmonic potential. The behaviour of the expansion coefficients \( \langle n | \alpha, F, + \rangle \) is highly oscillatory becoming zero for odd \( n \). This oscillatory behaviour is one of the nonclassical features.

The ECS exhibits squeezing in the \( p \)--quadrature which is defined as \( i(\hat{a}^\dagger - \hat{a})/\sqrt{2} \). For even nonlinear coherent states the expectation values of \( \hat{a} \) and \( \hat{a}^\dagger \) become zero and the uncertainty in \( p \) is given by

\[ \langle (\Delta \hat{p})^2 \rangle = \langle \hat{p}^2 \rangle - \langle \hat{p} \rangle^2, \]

\[ = \frac{1}{2} \left[ 1 + 2\langle \hat{a}^\dagger \hat{a} \rangle - 2\langle \hat{a}^2 \rangle \right], \]

where the expansion coefficients of the even nonlinear coherent state in the harmonic oscillator basis are taken to be real. In Fig. (1) we have shown the uncertainty in \( p \) as a function of \( \eta \) for the states defined by Eq. (24). From Fig. (1) it is clear that the uncertainty in \( p \) is less than that of the vacuum state value of 0.5 indicating that the states exhibit
squeezing. As $\eta$ increases the uncertainty in $p$ approaches that of the vacuum state. The reason for this behaviour is the following. As $\eta$ increases the occupation number distribution $p(n) = |\langle n|\alpha, F, + \rangle|^2$ starts peaking near $n = 0$. To make this explicit we have shown in Fig. (2) the occupation number distribution $p(n)$ as a function of $n$ for various values of $\eta$.

If the initial state of the ion is the first excited state of the harmonic trap then the state of the system at later times will involve only odd number states. The resultant stationary state of the system is an odd nonlinear coherent state given by

$$\lvert \alpha, F, - \rvert = N \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{(2n+1)!F(2n-1)!!}} \rvert 2n+1 \rangle; \ N^{-1} = \sqrt{\sum_{n=0}^{\infty} \frac{\lvert \alpha \rvert^{2n}}{(2n+1)!(F(2n-1)!!)^2}},$$

(27)

where $F(n)$ and $\alpha$ are again defined by Eq.(23) and Eq.(22) respectively. As in the case of even nonlinear coherent states the behaviour of occupation number distribution of the odd nonlinear coherent state, Eq.(27), is oscillatory becoming zero for even $n$. The occupation number distribution $p(n)$ of the odd nonlinear coherent states is sub-Poissonian. A state is said to exhibit sub-Poissonian statistics if the $q$ parameter defined as

$$q = \frac{\langle \hat{n}^2 \rangle - \langle \hat{n} \rangle^2}{\langle \hat{n} \rangle} - 1,$$

(28)

is negative. Negative $q$ indicates that the state is nonclassical. For the coherent states $q$ is zero. Fig. (3) shows the $q$ parameter as a function of $\alpha$ for the odd nonlinear coherent states of Eq. (27). It is clear that the states are sub-Poissonian. It is interesting to note that the $q$ value for large values $\eta$ approaches the value of that of the first excited of the harmonic oscillator. The reason being that the occupation number distribution gets concentrated at $n = 1$ as $\eta$ increases.

In conclusion, we have shown that a class of even and odd nonlinear coherent states can be generated in the center-of-mass motion of a trapped and bichromatically laser driven ion. These even and odd nonlinear coherent states are nonclassical. The even nonlinear coherent state exhibits squeezing while the odd nonlinear coherent states exhibits sub-Poissonian
statistics. Both the even and odd nonlinear coherent states have oscillatory occupation number distribution.

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FIGURES

FIG. 1. Uncertainty S, $\langle p^2 \rangle - \langle p \rangle^2$, in $p$ as a function of $r$ for $\frac{\Omega_0}{\eta_1} = .001$ (solid) and $\frac{\Omega_0}{\eta_1} = .0001$ (dashed) for the state $|\alpha, F, +\rangle$. $r$ represents real $\eta$.

FIG. 2. Occupation number distribution $p(n)$ as a function of $n$ for the state $|\alpha, F, +\rangle$ for various $\eta$ values and $\frac{\Omega_0}{\eta_1} = .0001$. (a) $\eta = .008$, (b) $\eta = .012$, (c) $\eta = .02$, and (d) $\eta = .1$.

FIG. 3. Mandel’s $q$ parameter as a function of $r$ for $\frac{\Omega_0}{\eta_1} = .001$ for the state $|\alpha, F, -\rangle$. $r$ represents real $\eta$. 

