A note on a walk-based inequality for the index of a signed graph

https://doi.org/10.1515/spma-2020-0120
Received October 6, 2020; accepted December 6, 2020

Abstract: We derive an inequality that includes the largest eigenvalue of the adjacency matrix and walks of an arbitrary length of a signed graph. We also consider certain particular cases.

Keywords: Signed graph; walk; adjacency matrix; index; upper bound

MSC: 05C22; 05C50

1 Introduction

A signed graph $\hat{G}$ is a pair $(G, \sigma)$, where $G = (V, E)$ is an unsigned graph, called the underlying graph, and $\sigma : E \rightarrow \{-1, +1\}$ is the sign function. We denote the number of vertices of a signed graph by $n$. The edge set of a signed graph is composed of subsets of positive and negative edges. Throughout the paper we interpret an unsigned graph as a signed graph with all the edges being positive.

The $n \times n$ adjacency matrix $A_{\hat{G}}$ of $\hat{G}$ is obtained from the standard $(0, 1)$-adjacency matrix of $G$ by reversing the sign of all 1s which correspond to negative edges. The largest eigenvalue of $A_{\hat{G}}$ is called the index of $\hat{G}$ and denoted by $\lambda_1$. A detailed introduction to spectra of signed graphs can be found in [3].

Spectra of signed graph have received a great deal of attention in the recent years. In particular, some upper bounds for $\lambda_1$ appeared in our previous works [1, 2]. In this note we generalize the result of [2] concerning an upper bound for $\lambda_1$ in terms of certain standard invariants. Additional terminology and notation are given in Section 2. Our contribution and some consequences are given in Section 3.

2 Terminology and notation

If the vertices $i$ and $j$ are adjacent, then we write $i \sim j$. In particular, the existence of a positive (resp. negative) edge between these vertices is designated by $i \sim^+ j$ (resp. $i \sim^- j$). We use $d_i$ to denote the degree of a vertex $i \in V(\hat{G})$; in particular, we write $d_i^+$ and $d_i^-$ for the positive and negative vertex degree (i.e., the number of positive and negative edges incident with $i$), respectively. For (not necessary distinct) vertices $i$ and $j$, we use $c_{ij}^{++}$ to denote the number of their common neighbours joined to both of them by a positive edge, $c_{ij}^{--}$ to denote the the number of their common neighbours joined to $i$ by a positive edge and to $j$ by any edge. We also use the similar notation for all the remaining possibilities.

The definition of a walk in a signed graph does not deviate from the same definition in the case of graphs. So, a walk is a sequence of alternate vertices and edges such that consecutive vertices are incident with the corresponding edge. A walk in a signed graph is positive if the number of its negative edges (counted with

*Corresponding Author: Zoran Stanić: Faculty of Mathematics, University of Belgrade Studentski trg 16, 11 000 Belgrade, Serbia, E-mail: zstanic@math.rs

© 2021 Zoran Stanić, published by De Gruyter. This work is licensed under the Creative Commons Attribution alone 4.0 License.
their multiplicity if there are repeated edges) is not odd; otherwise, it is negative. In the same way we decide whether a cycle in a signed graph is positive or negative. We use \( w^+_i(i, j) \) and \( w^-_i(i) \) to denote the number of positive walks of length \( k \) starting at \( i \) and terminating at \( j \) and the number of positive walks of length \( k \) starting at \( i \), respectively, and similarly for the numbers of negative ones.

### 3 Results

Our main result reads as follows.

**Theorem 3.1.** For the index \( \lambda_1 \) of signed graph \( \hat{G} \),

\[
\lambda_1 (n_i^- + n_i^+ + \lambda_1^{-1}) \leq (n_i^- + n_i^+)d_i + \sum_{j=1}^{n}(w^+ + w^-)d_j - 2\left( \sum_{j: w^+_j \neq 0} (c_{ij}^- + c_{ij}^+) + \sum_{j: w^-_j \neq 0} (c_{ij}^- + c_{ij}^-) \right),
\]

where \( i \) is a vertex that corresponds to the largest absolute value of the coordinates of an eigenvector afforded by \( \lambda_1 \), \( r \) (\( r \geq 2 \)) is an integer, \( w^+ = w_{r-1}^+(i, j) \), \( w^- = w_{r-1}^-(i, j) \) and \( n_j^- \) (resp. \( n_j^+ \)) is the number of vertices \( j \) such that \( w^+ \neq 0 \) (resp. \( w^- \neq 0 \)).

**Proof.** Let \( x = (x_1, x_2, \ldots, x_n)^T \) be an eigenvector associated with \( \lambda_1 \) and let \( x_i \) be the coordinate that is largest in absolute value. Without loss of generality, we may assume that \( x_i = 1 \). Considering the \( i \)th and the \( j \)th equality of \( \lambda_1 x = A \hat{G} x \), we get

\[
\lambda_1 = \sum_{k \sim i} x_k - \sum_{k \sim j} x_k,
\]

and

\[
\lambda_1 x_j = \sum_{k \sim j} x_k - \sum_{k \sim i} x_k.
\]

By multiplying the equality (2) by \( w^+ = w_{r-1}^+(i, j) \) and adding to (1), we get

\[
\lambda_1 (1 + w^+ x_j) = \sum_{k \sim j} x_k - \sum_{k \sim i} x_k + w^+ \left( \sum_{k \sim j} x_k - \sum_{k \sim j} x_k \right)
\]

\[
= (1 + w^+ ) \left( \sum_{k \sim j} x_k - \sum_{k \sim i} x_k \right) + (1 - w^+ ) \left( \sum_{k \sim j} x_k - \sum_{k \sim j} x_k \right)
\]

\[
+ \sum_{k \sim j} x_k - \sum_{k \sim i} x_k + w^+ \left( \sum_{k \sim j} x_k - \sum_{k \sim j} x_k \right)
\]

\[
\leq (1 + w^+) (c_{ij}^+ + c_{ij}^-) + |1 - w^+| (c_{ij}^- + c_{ij}^-) + d_j - c_{ij}^+ + d_i - c_{ij}^- + w^+(d_j^- - c_{ij}^- + d_j^- - c_{ij}^-)
\]

\[
= d_j + w^+ d_j - (1 + w^- - |1 - w^-|) (c_{ij}^- + c_{ij}^-).
\]

Observe that, for \( w^+ \neq 0 \), the previous inequality reduces to

\[
\lambda_1 (1 + w^+ x_j) \leq d_j + w^+ d_j - 2(c_{ij}^- + c_{ij}^-).
\]

Taking the summation over all \( j \) such that \( w^+ \neq 0 \), we get

\[
\lambda_1 \left( n_i^- + \sum_{j: w^+_j \neq 0} w^+ x_j \right) \leq n_i d_i + \sum_{j: w^-_j \neq 0} (w^+ d_j - 2(c_{ij}^- + c_{ij}^-)). \tag{3}
\]

Similarly, by multiplying the equality (2) by \( w^- = w_{r-1}^-(i, j) \) and subtracting it from (1), we get

\[
\lambda_1 (1 - w^- x_j) \leq d_i + w^- d_j - (1 + w^- - |1 - w^-|) (c_{ij}^+ + c_{ij}^-),
\]
which, after taking the summation over all $j$ such that $w^- \neq 0$, leads to
\[
\lambda_1 \left(n_i^- - \sum_{j : w^- \neq 0} n_i^- w^- x_j \right) \leq \sum_{j : w^- \neq 0} n_i^- d_j + \sum_{j : w^- \neq 0} \left( w^- d_j - 2(c_{ij}^+ + c_{ij}^-) \right).
\tag{4}
\]

Since
\[
\lambda_1^{-1} A_1^{-1} x_i = \sum_{j=1}^{n}(w^+ - w^-) x_j = \sum_{j : w^+ \neq 0} x_j - \sum_{j : w^- \neq 0} x_j,
\]
by summing (3) and (4), we obtain
\[
\lambda_1 \left(n_i^+ + n_i^- + \lambda_1^{-1} \right) \leq \left(n_i^+ + n_i^- \right) d_i + \sum_{j=1}^{n}(w^+ + w^-) d_j - 2 \left( \sum_{j : w^+ \neq 0} (c_{ij}^+ + c_{ij}^+) + \sum_{j : w^- \neq 0} (c_{ij}^- + c_{ij}^-) \right),
\]
which completes the proof. \qed

The Laplacian matrix $L_G$ is defined as $L_G = D_G - A_G$, where $D_G$ is the diagonal matrix of vertex degrees. Observe that the counterparts to (1) and (2) in the case of the Laplacian matrix $L_G$ are given by $\mu_1 = d_i + \sum_{k \prec i} x_k - \sum_{k \succ i} x_k$ and $\mu_1 x_i = d_i x_i + \sum_{k \prec i} x_k - \sum_{k \succ i} x_k$ ($\mu_1$ being the largest eigenvalue of $L_G$). Now, with slight modifications in the previous proof, we get the following.

**Theorem 3.2.** For the Laplacian index $\mu_1$ of signed graph $\hat{G}$,
\[
\mu_1 \left(n_i^+ + n_i^- + \mu_1^{-1} \right) \leq \left(n_i^+ + n_i^- \right) d_i + \sum_{j=1}^{n}(w^+ + w^-) d_j - \sum_{j : w^+ \neq 0} (c_{ij}^+ + c_{ij}^+) - \sum_{j : w^- \neq 0} (c_{ij}^- + c_{ij}^-),
\]
with the notations of Theorem 3.1.

For $r = 2$, we have $n_i^+ = w^+ = d_i^+$, $n_i^- = w^- = d_i^-$, while $\sum_{j : w^+ \neq 0}(c_{ij}^+ + c_{ij}^+) + \sum_{j : w^- \neq 0}(c_{ij}^- + c_{ij}^-) = 2T_i$, i.e., this is twice the sum of negative triangles passing through $i$. Thus Theorem 3.1 gives $\lambda_1 (d_i^+ + \lambda_1) \leq d_i^2 + d_i m_i - 4T_i$, where $m_i$ is the average degree of the neighbours of $i$. This quadratic equation leads to
\[
\lambda_i^2 \leq \max_{1 \leq i \leq n} \left\{ \frac{1}{2} \left( \sqrt{5d_i^2 + 4(d_i m_i - 4T_i)} - d_i \right) \right\},
\]
the upper bound obtained in [2].

For $r = 3$, we get
\[
\lambda_1 \left(n_i^+ + n_i^- + \lambda_1^2 \right) \leq \left(n_i^+ + n_i^- \right) d_i + \sum_{j : w^+ \neq 0} (w^+ d_j - 2w^-) + \sum_{j : w^- \neq 0} (w^- d_j - 2w^+),
\]
as $c_{ij}^+ + c_{ij}^- = w^+$, $c_{ij}^+ + c_{ij}^- = w^-$. In particular case of graphs, the latter inequality reduces to
\[
\lambda_1 \left(d_2(i) + 1 + \lambda_1^2 \right) \leq (d_2(i) + 1)d_i + w_3(i),
\]
where $d_2(i)$ denotes the number of vertices at distance 2 from $i$ (and then $n_i^+ = d_2(i) + 1$) and $w_3(i)$ denotes the number of walks of length 3 starting at $i$.

**Acknowledgements:** Research is partially supported by Serbian Ministry of Education via Faculty of Mathematics, University of Belgrade.

**Data Availability Statement:** Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

**References**

[1] Z. Stanić, Bounding the largest eigenvalue of signed graphs, Linear Algebra Appl., 573 (2019), 80–89.
[2] Z. Stanić, Some bounds for the largest eigenvalue of a signed graph, Bull. Math. Soc. Sci. Math. Roumanie, 62(110) (2019), 183–189.
[3] T. Zaslavsky, Matrices in the theory of signed simple graphs, in B.D. Acharya, G.O.H. Katona, J. Nešetřil (Eds.), Advances in Discrete Mathematics and Applications: Mysore 2008, Ramanujan Math. Soc., Mysore, 2010, pp. 207–229.