Quantum oscillations in the black hole horizon

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Abstract

By applying Rosen’s quantization approach to the historical Oppenheimer and Snyder gravitational collapse and by setting the constraints for the formation of the Schwarzschild black hole (SBH), in a previous paper [1] two of the Authors (CC and FF) found the gravitational potential, the Schrödinger equation, the solution for the energy levels, the area quantum and the quantum representation of the ground state at the Planck scale of the SBH. Such results are consistent with previous ones in the literature. It was also shown that the traditional classical singularity in the core of the SBH is replaced by a quantum oscillator describing a non-singular two-particle system where the two components, named the “nucleus” and the “electron”, strongly interact with each other through a quantum gravitational interaction. In agreement with the de Broglie hypothesis, the “electron” is interpreted in terms of the quantum oscillations of the BH horizon. In other words, the SBH should be the gravitational analogous of the hydrogen atom. In this paper, it is shown that these results allow us to compute the SBH entropy as a function of the BH principal quantum number in terms of Bekenstein-Hawking entropy and three sub-leading corrections. In addition, the coefficient of the formula of Bekenstein-Hawking entropy is reduced to a quarter of the traditional value. Then, it is shown that, by performing a correct rescaling of the energy levels, the semi-classical Bohr-like approach to BH quantum physics, previously developed by one of the Authors (CC), is consistent with the obtained results for large values of the BH principal quantum number. After this, Hawking radiation will be analysed by discussing its connection with the BH quantum structure. Finally, it is shown that the time evolution of the above mentioned system solves the BH information paradox.

1 Introduction

Black holes are at all effects theoretical and conceptual laboratories where one discusses, tests and tries to understand the fundamental problems and potential contradictions that arise in the various attempts made to unify Einstein’s general theory of relativity with quantum mechanics. In a previous paper [1] two of the Authors (CC and FF) suggested a new model of quantum BH based on a mathematical analogy to that of the hydrogen atom obtained by using the same quantization approach proposed by the historical collaborator of Einstein, N. Rosen [2]. It is well-known that the canonical quantization of general relativity leads to the Wheeler-DeWitt equation introducing the so-called Superspace, an infinite-dimensional space of all possible 3-metrics; Rosen, instead, preferred to start his work from the classical cosmological equations using a simplified quantization scheme. He wanted to reduce, at least formally, the cosmological Einstein-Friedman equations of general relativity to a quantum mechanical system; if this issue holds, the Friedman equations can be recast as a Schrödinger equation and the cosmological solutions can be read as eigensolutions of such a “cosmological” Schrödinger equation. In this way Rosen found that, in the case of a
Universe filled with pressureless matter, the equation is like that for the $s$ states of a hydrogen-like atom [2]. It is important to recall that that attempts at quantising the FLRW universe date back at least to DeWitt’s first of his famous 1967 papers [3], where the interesting observation is made that one needs a lot of particles to ensure the semiclassical behaviour.

Actually, Rosen’s approach can be also applied to the historical Oppenheimer and Snyder gravitational collapse [4], that is the simple case of a pressureless “star of dust”. By applying the constraints for the formation of the SBH, in [1] it has been found the gravitational potential, the Schrödinger equation and the solution for the energy levels of the black hole. The energy spectrum that has been found is consistent with the one which was found by Bekenstein in 1974 [5] and with other ones in the literature [6, 7, 8, 9, 10]. The black hole area quantization has been also achieved by finding a result similar to the one obtained by Bekenstein but with a different coefficient. A quantum representation of the SBH ground state at the Planck scale has been obtained too. Finally, it has been shown that the traditional classical singularity in the core of the SBH is replaced, in a full quantum treatment, by a two-particle system where the two components strongly interact with each other through a quantum gravitational interaction. This system seems to be non-singular from the quantum point of view and it is analogous to the hydrogen atom because it consists of a “nucleus” and an “electron”. In agreement with the de Broglie hypothesis [11], the “electron” is interpreted in terms of the quantum oscillations of the BH horizon. Let us remark that the study of the collapse of a simple set of dust particles can be a first and valid approach to face the problem as it presents the advantage of finding and putting in evidence some fundamental properties that can be found also in other models characterized by, e.g., more articulated descriptions in terms of quantum fields that require a more complicated interpretation. This method, even if simplified, has a good validity as, up to now, there is not a unique and valid theory of quantum gravity that suggests a specific model to use. On the other hand, one must also stress that a very complex phenomenon such as the gravitational collapse of a compact body is here treated in the highly idealised Oppenheimer and Snyder model, where only one degree of freedom survives and is quantised. This makes the emerging picture of quantum BHs oversimplified as well and really consistent only for Schwarzschild BHs, which are the final result of the Oppenheimer and Snyder gravitational collapse. Moreover, matter in the Oppenheimer and Snyder model is dust, and this is consistent with the hydrogen-like potential energy which will be found in the next Section, if one takes the Newtonian form. Then, that BHs then look like hydrogen atoms should be no surprise given the premises. For the sake of completeness, one recalls that, concerning the Oppenheimer and Snyder model, some progress with respect to [1] has been realized in the recent attempt [12]. This paper is organized as it follows. In Section II, Rosen’s approach, which has been discussed in [1], is reviewed by also adding some interesting new insights; in Section III, it is shown that the same results can be obtained also through a path integral approach; in Section IV, the SBH entropy is calculated in terms of Bekenstein-Hawking entropy and three sub-leading corrections. It is also shown that the coefficient of the formula of Bekenstein-Hawking entropy is reduced to one half of its traditional value; in Section 5, it is shown that, by performing a rescaling of the energy levels, the semi-classical Bohr-like approach to BH quantum physics, previously developed by one of the Authors (CC) in [13, 14] and reviewed in [15], is consistent with the obtained results for large values of the BH principal quantum number; in Section VI, Hawking radiation will be analysed by discussing its connection with the BH quantum structure; in Section VII, we analyze the time evolution of the “gravitational hydrogen atom” (GHA), and discuss how the resolution of the BH information paradox can be achieved; finally, Section VIII is devoted to the conclusion remarks.

2 A review of Rosen’s approach

Classically, the gravitational collapse in the simple case of a pressureless “star of dust” with uniform density is well known [16]. From the historical point of view, it was originally analysed in the famous paper of Oppenheimer and Snyder [4]. For the interior of the collapsing star, one indeed uses the well-known Friedmann-Lemaître-Robertson-Walker (FLRW) line-element with comoving hyper-spherical coordinates $\chi, \theta, \varphi$ [16]. Thus, one writes down [16] (hereafter Planck units will be used, i.e. $G = c = k_B = \hbar = \frac{\chi}{4\pi G_0} = 1$)

$$ds^2 = d\tau^2 + a(\tau)(-d\chi^2 - \sin^2 \chi(d\theta^2 + \sin^2 \theta d\varphi^2),$$  

(1)
where the origin of coordinates is set at the centre of the star, and \( a(\tau) \) is the scale factor given by the familiar cycloidal relation \[16\]

\[
a = \frac{4}{3} a_m (1 + \cos \eta),
\]

\[
\tau = \frac{1}{2} a_m (\eta + \sin \eta),
\] (2)

while the density is given by \[16\]

\[
\rho = \left( \frac{3 a_m}{8 \pi} \right) a^{-3} = \left( \frac{3}{8 \pi a_m^2} \right) \left[ \frac{1}{2} (1 + \cos \eta) \right]^{-3}.
\] (3)

Setting \( \sin^2 \chi \) one chooses the case of positive curvature, which corresponds to a gas sphere whose dynamics begins at rest with a finite radius, and, in turn, it is the only one of interest \[16\]. Thus, the choice \( k = 1 \) is made for dynamical reasons (the initial rate of change of density is null, that means “momentum of maximum expansion” \[16\]), but the dynamics also depends on the field equations.

In order to discuss the simplest model of a “star of dust”, that is, the case of zero pressure, one sets the stress-energy tensor as \[16\]

\[
T = \rho u \otimes u,
\] (4)

where \( \rho \) is the density of the collapsing star and \( u \) the four-vector velocity of the matter. On the other hand, the external geometry is given by the Schwartzschild line-element \[16\]

\[
ds^2 = \left( 1 - \frac{2M}{r} \right) dt^2 - r^2 \left( \sin^2 \theta d\phi^2 + d\theta^2 \right) - \frac{dr^2}{1 - \frac{2M}{r}},
\] (5)

where \( M \) is the total mass of the collapsing star which is constant during the collapse \[16\]. As there are no pressure gradients, which can deflect the particles motion, the particles on the surface of any ball of dust move along radial geodesics in the exterior Schwarzschild space-time \[16\]. Considering a ball which begins at rest with finite radius (in terms of the Schwarzschild radial coordinate) \( r = r_i \) at the (Schwarzschild) time \( t = 0 \), the geodesics motion of its surface is given by the following equations \[16\]:

\[
r = \frac{1}{2} r_i (1 + \cos \eta),
\] (6)

\[
t = 2M \ln \left[ \frac{\sqrt{r_i^2 - 1 + \tan (\eta/2)}}{\sqrt{2M - 1} - \tan (\eta/2)} \right] + 2M \sqrt{\frac{r_i}{2M}} - 1 \left[ \eta + \left( \frac{r_i}{4M} \right) (\eta + \sin \eta) \right].
\] (7)

The proper time measured by a clock put on the surface of the collapsing star is \[16\]

\[
\tau = \sqrt{\frac{r_i^3}{8M}} (\eta + \sin \eta).
\] (9)

The collapse begins for \( r = r_i \), \( \eta = \tau = t = 0 \), and terminates at the singularity \( r = 0 \), \( \eta = \pi \) after a duration of proper time measured by the falling particles \[16\]

\[
\Delta \tau = \pi \sqrt{\frac{r_i^3}{8M}}.
\] (10)

which coincidentally corresponds, as it is well known, to the interval of Newtonian time for free-fall collapse in Newtonian theory. Differently from the cosmological case, where the solution is homogeneous and isotropic everywhere, here the internal homogeneity and isotropy of the FLRW line-element are broken at the star’s surface, that is, a some radius \( \chi = \chi_0 \) \[16\]. At that surface, which is a 3-dimensional world tube enclosing the star’s fluid \[16\], the interior FLRW geometry must match smoothly the exterior Schwarzschild geometry \[16\]. One considers a range of \( \chi \) given by \( 0 \leq \chi \leq \chi_0 \), with \( \chi_0 < \frac{\pi}{2} \) during the collapse \[16\]. For the pressureless case the match is
possible [16]. The external Schwarzschild solution predicts indeed a cycloidal relation for the star’s circumference [16]

\[ C = 2\pi r = 2\pi \left( \frac{4}{3} r_i (1 + \cos \eta) \right) , \]

\[ \tau = \sqrt{\frac{3}{4M}} (\eta + \sin \eta) . \]

The interior FLRW predicts a similar cycloidal relation [16]

\[ C = 2\pi r = 2\pi a \sin \chi_0 = \pi \sin \chi_0 a_m (1 + \cos \eta) , \]

\[ \tau = \frac{1}{2} a_m (\eta + \sin \eta) . \]

Therefore, the two predictions agree perfectly for all time if and only if [16]

\[ r_i = a_m \sin \chi_0 , \]

\[ M = \frac{1}{2} a_m \sin^3 \chi_0 , \]

where \( r_i \) and \( a_m \) are the values of the Schwarzschild radial coordinate in Eq. (5) and of the scale factor in Eq. (1) at the beginning of the collapse, respectively. Thus, Eqs. (13) represent the requested match, while the Schwarzschild radial coordinate, in the case of the matching between the internal and external geometries, is [16]

\[ r = a \sin \chi_0 . \]

One underlines that the Oppenheimer and Snyder gravitational collapse is not physical and can be considered as a toy model. But here the key point is that, as it is well known from the historical paper of Oppenheimer and Snyder [4] and by various subsequent works, the final state of this simplified gravitational collapse is the SBH, which, instead, has a fundamental role in quantum gravity. It is indeed a general conviction, arising from an idea of Bekenstein [17], that, in the search of a quantum gravity theory, the SBH should play a role similar to the hydrogen atom in quantum mechanics. It should be a “theoretical laboratory” where one discusses and tries to understand conceptual problems and potential contradictions in the attempts to unify Einstein’s general relativity with quantum mechanics. Thus, despite non-physical, the Oppenheimer and Snyder gravitational collapse must be here considered as a tool which permits to understand a fundamental physical system, that is the SBH. In fact, it will be shown that, by setting the constraints for the formation of the SBH in the quantized Oppenheimer and Snyder gravitational collapse, one arrives to quantize the SBH, and this will be a remarkable, important result in the quantum gravity’s search. In fact, the Oppenheimer and Snyder gravitational collapse and/or some of its modifications have currently a renovated interest among researchers in gravitation, see for example [18, 19, 20, 21, 22, 23, 24] and references within.

By rewriting the FLRW line-element (1) in spherical coordinates and comoving time as

\[ ds^2 = d\tau^2 - a^2(\tau) \left( \frac{dr^2}{1 - \frac{r^2}{a^2}} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right) , \]

and by using the Einstein field equation

\[ G_{\mu\nu} = -8\pi T_{\mu\nu} , \]

one gets the dynamical equations for the Oppenheimer and Snyder gravitational collapse as [1, 2]

\[ \ddot{a}^2 = \frac{8}{3} \pi a^2 \rho - 1 , \quad \dot{a} = -\frac{4}{3} \pi a \rho , \]

with \( \ddot{a} = \frac{da}{d\tau} \). Following [1, 2], for consistency, one gets

\[ \frac{d\rho}{da} = \frac{3\rho}{a} , \]

which, when integrated, gives

\[ \rho = \frac{C}{a^4} . \]
The constant \( C \) is determined by the initial conditions and it turns out to be
\[
C = \frac{3\pi a_0}{8\pi}.
\]
(20)

Thus, one rewrites Eq. (19) as
\[
\rho = \frac{3\pi a_0}{8\pi a^3}.
\]
(21)

The first of (17) can be written as
\[
\frac{1}{2}M\dot{a}^2 - \frac{4}{3}\pi Ma^2 \rho = -\frac{M}{2},
\]
(22)

which seems like the energy equation for for a particle in one-dimensional motion having coordinate \( a \):
\[
E = T + V,
\]
(23)

where
\[
T = \frac{M\dot{a}^2}{2}
\]
(24)

and
\[
V(a) = -\frac{4}{3}\pi Ma^2\rho,
\]
(25)

are the kinetic and potential energy, respectively. Then, the total energy is
\[
E = -\frac{M}{2}
\]
(26)

From the second of Eqs. (17) one gets the equation of motion of this particle, that is
\[
M\ddot{a} = -\frac{4}{3}M\pi a\rho.
\]
(27)

The momentum of the particle is
\[
P = M\dot{a},
\]
(28)

with an associated Hamiltonian
\[
H = \frac{p^2}{2M} + V.
\]
(29)

Till now the discussion has been classical. In order to start the quantum analysis, one needs to define a wave-function as [1, 2]
\[
\Psi \equiv \Psi (a, \tau).
\]
(30)

Hence, in correspondence of the classical equation (29), one finds the traditional Schrödinger equation:
\[
\frac{i}{\hbar}\frac{\partial \Psi}{\partial \tau} = -\frac{1}{2M} \frac{\partial^2 \Psi}{\partial a^2} + V \Psi.
\]
(31)

For a stationary state with energy \( E \) one gets
\[
\Psi = \Psi (a) \exp \left( -iE\tau \right),
\]
(32)

and Eq. (30) becomes
\[
-\frac{1}{2M} \frac{\partial^2 \Psi}{\partial a^2} + V \Psi = E \Psi.
\]
(33)

Inserting Eq. (21) into Eq. (25) one obtains
\[
V(a) = -\frac{Ma_0}{2a}.
\]
(34)

Setting
\[
\Psi = aX,
\]
(35)

Eq. (33) becomes
\[
-\frac{1}{2M} \left( \frac{\partial^2 X}{\partial a^2} + \frac{2}{a} \frac{\partial X}{\partial a} \right) + VX = EX.
\]
(36)
With $V$ given by Eq. (34), Eq. (36), is analogous to the Schrödinger equation in polar coordinates for the $s$ states ($l = 0$) of a hydrogen-like atom \[1, 2, 25\] in which the squared electron charge $e^2$ is replaced by $\frac{M a_0}{2}$. Thus, for the bound states ($E < 0$) the energy spectrum is

$$E_n = -\frac{a_0^2 M^3}{8n^2}, \tag{37}$$

where $n$ is the principal quantum number. Following again \[1, 2\], one inserts Eq. (26) into Eq. (37), obtaining the mass spectrum of the gravitational collapse:

$$M_n = \frac{a_0^2 M^3}{4n^2} \Rightarrow M_n = \frac{2n}{a_0}, \tag{38}$$

On the other hand, by using Eq. (26), one finds the energy levels of the collapsing star as

$$E_n = -\frac{n}{a_0}. \tag{39}$$

Eq. (38) represents the spectrum of the total mass of the collapsing star, while Eq. (39) represents the energy spectrum of the the collapsing star where the gravitational energy, which is given by Eq. (34), is included. The total energy of a quantum system with bound states is indeed negative. Concerning the gravitational energy, one recalls that, in general, the equivalence principle forbids its localization in general relativity, with the sole exception of a spherical star \[16\], which is exactly the case analysed in this paper. In this case the gravitational energy is indeed localized not by mathematical conventions, but by the circumstance that transfer of energy is detectable by local measures \[16\]. Thus, one can surely consider Eq. (34) as the gravitational potential energy of the collapsing star.

Now, let us consider the case of a completely collapsed star, i.e., a SBH; this means $\chi_0 = \frac{\pi}{2}$, $r = a$ and $r_i = a_0 = 2M = r_g$, in Eqs. (13) and (14). Therefore, Eqs. (34) and (35) become

$$V(r) = -\frac{M^2}{r}, \tag{40}$$

$$\Psi = rX, \tag{41}$$

Eq. (36) become

$$-\frac{1}{2M} \left( \frac{\partial^2 X}{\partial r^2} + \frac{2}{r} \frac{\partial X}{\partial r} \right) + VX = EX, \tag{42}$$

with

$$E_n = -\frac{r_i^2 M^3}{8n^2}, \tag{43}$$

$$M_n = \sqrt{n}, \tag{44}$$

$$E_n = -\sqrt{n}. \tag{45}$$

Eqs. (40), (42), (44) and (45) should be the exact gravitational potential energy, Schrödinger equation, mass spectrum and energy spectrum for the SBH interpreted as GHA, respectively. Actually, a further final correction is needed. To better clarify this point, one compares our Eq. (40) with the analogous potential energy of an hydrogen atom, which is \[25\]

$$V(r) = -\frac{e^2}{r}. \tag{46}$$

Eqs. (40) and (46) are formally identical, but there is an important difference. In (46), the electron’s charge is constant for all the energy levels of the hydrogen atom. Instead, in the case of Eq. (40), based on the emissions of Hawking quanta or on the absorptions of external particles, the BH mass changes during the jumps from an energy level to another. In fact, such a BH mass decreases for emissions and increases for absorptions. Thus, one must also consider this BH dynamical behavior. A good way to take into account the BH dynamical behavior is by introducing the BH effective state. This consists in introducing some effective quantities. Considering the initial BH mass before a BH transition (an emission of a Hawking quantum or an absorption of an external
particle), \( M \), and the final BH mass after the transition, \( M \pm \omega \), where \( \omega \) is the mass-energy of the particle involved in the transition (the sign + or − corresponds to an absorption or to an emission, respectively), one introduces the BH effective mass and effective horizon as \[ M_E \equiv M \pm \frac{\omega}{2}, \quad r_E \equiv 2M_E, \] (47)
respectively. The effective quantities (47) represent average quantities. The variable \( r_E \) is indeed the average of the initial and final horizons, while \( M_E \) is the average of the initial and final masses (see [1, 13, 14] for further details). They represent the BH mass and horizon during the BH contraction (expansion), i.e., during the emission (absorption) of a particle. Thus, on one hand, the introduction of the effective quantities (47) in the BH dynamical equations is very intuitive. On the other hand, it is rigorously justified through Hawking periodicity argument [26]. In order to take the BH dynamical behavior into due account, one must replace the BH mass \( M \) with the BH effective mass \( M_E \) in Eqs. (40), (42), (43), and (26), obtaining [1]
\[ V(r) = -\frac{M_E^2}{r}, \] (48)
and
\[ -\frac{1}{2M_E} \left( \frac{\partial^2 X}{\partial r^2} + \frac{2}{r} \frac{\partial X}{\partial r} \right) + VX = EX, \] (49)
with
\[ E_n = -\frac{E^2 M^3}{8n^2}, \] (50)
\[ E = -\frac{M_E}{2}. \] (51)
Now, from the quantum point of view, one wants to obtain the energy eigenvalues as being absorptions starting from the BH formation, that is from the BH having null mass, where with “the BH having null mass” one means the situation of the gravitational collapse before the formation of the first event horizon. This implies that one must replace \( M \to 0 \) and \( \omega \to M \) in Eqs. (47) and must take the plus sign (absorptions) in the same equations. Thus, one gets
\[ M_E \equiv \frac{M}{2}, \quad r_E \equiv 2M_E = M. \] (52)
Then, one inserts Eqs. (51) and (52) into Eq. (50), obtaining the BH mass spectrum as
\[ M_n = 2\sqrt{n}, \] (53)
and, by using Eq. (51), one finds the BH energy levels as
\[ E_n = -\frac{\sqrt{n}}{4}. \] (54)
In its absolute value, Eq. (54) is consistent with the BH energy spectrum found by Bekenstein in 1974 [5]. Bekenstein indeed obtained \( E_n \sim \sqrt{n} \) by using the Bohr-Sommerfeld quantization condition because he argued that the SBH behaves as an adiabatic invariant. In our opinion, the Authors of previous literature did not consider that the BH energy spectrum must have negative eigenvalues because the GHA is a quantum system composed by bound states. The total BH energy \( E = -\frac{M}{2} \) is negative and different from the BH inert mass \( M \). In fact, the quantization procedure in [1] splits the classical BH in a two-particle quantum system having a gravitational negative energy given by Eq. (48). It is indeed well known that the bound states of physical systems must have negative energies.
Now, one recalls that in the SBH the horizon area \( A \) is related to the mass by the relation \( A = 16\pi M^2 \). If one assumes that a neighboring particle is captured by the BH causing a transition from the state with \( n \) to the state with \( n + 1 \) (two neighboring levels) by using (53), one immediately gets the area quantum as
\[ \Delta A_{n \to n+1} = 64\pi (n + 1 - n) = 64\pi, \] (55)
which is similar to the original result of Bekenstein [5], but with a different coefficient. This is not surprising since there is no general consensus on the area quantum. For the BH ground state \((n = 1)\), from Eq. (53) one gets

\[ M_1 = 2 \tag{56} \]

in Planck units. Therefore, in standard units, one gets \(M_1 = 2m_P\), where \(m_P\) is the Planck mass. A total negative energy arising from Eq. (54) and a Schwarzschild radius are associated to the mass (56):

\[ E_1 = -\frac{1}{2}, \quad r_{s1} = 4. \tag{57} \]

This is the state having minimum mass and minimum energy (the energy of this state is minimum in absolute value; in its real value, being negative, it is maximum). It represents the smallest possible BH. In the case of Bohr’s semi-classical model of hydrogen atom, the Bohr radius, which represents the classical radius of the electron at the ground state, is [25]

\[ \text{Bohr radius} = \frac{1}{m_e e^2}, \tag{59} \]

where \(m_e\) is the electron mass. In order to obtain the correspondent “Bohr radius” for what we call the GHA, one needs to replace both \(m_e\) and \(e\) in Eq. (59) with the effective mass of the BH ground state, which is \(\frac{M_1}{2} = 1\). Thus, now the “Bohr radius” becomes

\[ b_1 = 1, \tag{60} \]

which in standard units reads \(b_1 = l_P\), where \(l_P\) is the Planck length. Thus, it has been found that the “Bohr radius” associated to the smallest possible BH is equal to the Planck length. Following again [2, 1], the wave-function associated to the BH ground state is

\[ \Psi_1 = 2b_1^{\frac{3}{2}} r \exp \left( -\frac{r}{b_1} \right) = 2r \exp (-r), \tag{61} \]

where \(\Psi_1\) is normalized as

\[ \int_0^{\infty} \Psi_1^2 dr = 1. \tag{62} \]

The size of this BH is of the order of

\[ \bar{r}_1 = \int_0^{\infty} \Psi_1^2 r dr = \frac{3}{2} b_1 = \frac{3}{2}. \tag{63} \]

It is also important to recall that in quantum mechanics the Bohr radius (60) gives the radius with the maximum radial probability density instead of its expected radial distance [27]. The latter is indeed 1.5 times the Bohr radius; this depends on the long tail of the radial wave function and it is so given by Eq. (63). Thus, remarkably, Eqs. (60) and (63) are in complete agreement with the standard quantum approach to the hydrogen atom.

Hence, an interesting quantum representation of the Schwarzschild BH ground state at the Planck scale has been obtained. This Schwarzschild BH ground state represents the BH minimum energy level which is compatible with the generalized uncertainty principle (GUP) [28]. The GUP indeed prevents a BH from its total evaporation by stopping Hawking’s evaporation process in exactly the same way that the usual Heisenberg uncertainty principle (HUP) prevents the hydrogen atom from total collapse [29].

Following again Rosen [2], one can also easily find the size, that is the "expected radial distance", of the quantum SBH excited at the level \(n\) as

\[ \bar{r}_n = \frac{3}{2} \sqrt{n}. \tag{64} \]

Let us clarify an important point. It is well known that the Schwarzschild coordinates break down when the radius of the star becomes the gravitational radius, which means that it is no longer possible to match the \(\chi_0\) surface with a space-like normal to a constant \(r\) surface which is null at the
gravitational radius and has a time-like normal for values of \( r \) which are less than the gravitational radius. Thus, we want to stress here that we never went beyond the gravitational radius. This issue is clarified as follows. The passage from a classical to a quantum analysis splits the classical BH in a two-particle system, the “nucleus” and the “electron”. Here the key point is the physical meaning of the second particle, the “electron” during the BH formation and in the following BH dynamical evolution. From a quantum point of view, the BH formation is interpreted as being the formation of the initial pair composed by the “nucleus” and the “electron”. Thus, it represents an absorption from the BH having null mass. This is not an instantaneous process, but, instead, it happens in a finite lapse of time. Let us assume that, during a little, but finite, lapse of comoving time (which is the time that we used in the above analysis), say \( \Delta \tau \), the gravitational collapse generates a BH having an initial mass, say \( M_0 = 2\sqrt{n} \). During \( \Delta \tau \) the BH is forming, which means that the two particles are evolving. When the BH is completely formed the two particles are in a stationary state, having an energy \( E_n = -\frac{n}{4} \). One notes that in the two-particle quantum system, the two particles are equal and can be mutually exchanged without varying the physical properties of the system. This means that, when the system is in the stationary state, one can assign to the “electron” the half of the total energy of the two-particle quantum system, obtaining

\[
\frac{E_n}{2} = -\frac{\sqrt{n}}{4} = -\frac{n}{2M_n},
\]

which, remarkably, corresponds exactly to a particle quantized with antiperiodic boundary conditions on a circle of length

\[
L = 4\pi M_n = 8\pi \sqrt{n}.
\]

This means that the radius of the circle is \( r_0 = 2M_n \), i.e., exactly the gravitational radius. Hence, when the system is in the stationary state, that is, when the BH is completely formed, the distance between the “nucleus” and the “electron” in terms of the Schwarzschild radial coordinate is exactly the gravitational radius.

Following the analogy with the hydrogen atom, one can evoke the de Broglie hypothesis [11] and consider the wave nature of the BH “electron”. This means that such particle does not orbit the nucleus in the same way as a planet orbits the Sun, but instead exists as standing wave. The correct analogy is that of a large and often oddly shaped “atmosphere” (the BH “electron”), distributed around a relatively tiny planet (the BH “nucleus”). The correct physical interpretation of such an “atmosphere” is nothing else than the BH horizon modes. In fact, the idea that the radius of the event horizon undergoes quantum oscillations has a longstanding history. Such horizon modes were introduced in a semi-classical framework about 50 years ago [30] in terms of BH quasi-normal modes (QNMs) which represent the BH back reaction to perturbations. Both of the absorptions of external particles and the emissions of Hawking quanta are BH perturbations and this allowed one of us, CC, to develop the Bohr-like approach to BH quantum physics which will be discussed and refined in Section 5 of this paper, starting from an original idea of Hod [31] which has been improved by Maggiore [32]. On one hand, the QNMs approach is a semi-classical approach similar to the approach that Bohr developed in 1913 [33, 34] concerning the structure of the hydrogen atom. On the other hand, the importance of horizon modes in a quantum gravity framework has been recently emphasized by considering them as being described by the periodic motion of their particle-like analogue [35], in full accordance with the de Broglie hypothesis. The Authors found an energy spectrum which scales as which scales as \( \sim \sqrt{n} \), which is consistent with the results obtained in this paper. The key point is that during both the processes of absorptions of external particles, included the original BH formation, and of emissions of Hawking quanta, the BH horizon is not fixed at a constant distance from the BH nucleus [6]. In fact, due to energy conservation, the BH contracts during the emission of a particle and expands during an absorption [6]. Such quantum contractions/expansions are not “one shot processes” [6]. They generates oscillations of the horizon instead [6]. Thus, the “Bohr radius”, Eq. (60), and the “expected radial distance”, Eq. (64), must be interpreted as dynamical quantities characterizing the quantum state of the horizon modes during a transition from a stationary state to another, while the “Bohr orbits” of the stationary states are characterized by Eq. (66). Clearly, if the second particle, that is the “electron”, is interpreted in terms of horizon oscillations, then the distance between the two particle is never less than the gravitational radius which oscillates.

Another fundamental issue is the following. The quantum BH expressed by the system of Eqs. (48) - (51) seems to be non-singular. It is indeed well known that, in the classical general relativistic framework, in the internal geometry all time-like radial geodesics of the collapsing star
terminate after a lapse of finite proper time in the termination point \( r = 0 \) and it is impossible to extend the internal space-time manifold beyond that termination point [16]. Thus, the point \( r = 0 \) represents a singularity based on the definition by Schmidt [36]. But what happens in the quantum framework in [1] is completely different. The above analysis has shown that the classical SBH is described in terms of a quantum harmonic oscillator that reflects several analogies to that describing a two-particle system where the two components strongly interact with each other through a quantum gravitational interaction. The system that has been analysed so far is indeed formally equal to the well known system of two quantum particles having finite distance with the mutual attraction of the form \( 1/r \) [25]. Similarly to the an hydrogen atom, this oscillator describes the behaviour of a two particle system where one particle behaves as the “nucleus” and the other as an “electron” of a GHA. Of course, following this mathematical analogy, one has to figure out what is the meaning of the word “particle” in a quantum framework. What this means is that within the tiny confines of the “gravitational atom”, the so-called “electron” cannot really be regarded as a “point-like particle” having a definite energy and location or an effective pair of particles as occurs in the real hydrogen atom [1]. Thus, it is somewhat misleading to talk about the BH “electron” “falling into” the BH “nucleus”. In other words, the Schwarzschild radial coordinate cannot become equal to zero. The GUP makes this last statement even stronger. In fact, it can be written down in the general form as (see [28] and the review [37])

\[
\Delta x \Delta p \geq \frac{1}{2} \left[ 1 + \eta (\Delta p)^2 + \ldots \right].
\]  

(67)

Eq. (67) implies a non-zero lower bound on the minimum value of the uncertainty on the particle’s position which is of order of the Planck length [28, 37]. In other words, the GUP implies the existence of a minimal length \( L \) in quantum gravity also when Einstein’s equations hold down to the Planck scales [38]. Following this hypothesis, instead of focusing on the energy–tensor quantity such as the momentum vector, we obtain an equivalent indetermination relationship from an invariant scalar quantity, the proper energy \( E \). To obtain \( E \) one integrates and averages Einstein’s equations over a 3D space-like hypersurface, \( \sigma \), with unit normal vector \( n \sim g^{-1/2} \).

The proper energy is averaged over a proper volume \( L^3 \), and is obtained from the integral of the energy momentum tensor taken over a chosen proper volume element of a space-like 3D hypersurface, giving

\[
\bar{E} \sim \frac{g^2}{L} R(4) = L \left( \Delta (\Delta g(g)^{-1}) + (\Delta g(g)^{-1})^2 \right).
\]  

(68)

Rescaling Eq. (68) down to the Planck scale \( L_p \) and defining, in natural units, the light crossing time \( \tau_p = L \) and the Planck Time \( \tau_p \), one obtains the indetermination relationship valid down to the Planck scale:

\[
\left( \frac{\tau_p}{\tau} \right)^2 \left( \frac{E \times \tau}{\hbar} \right) = \left( \frac{L_p}{L} \right)^2 \left( \frac{E \times \tau}{\hbar} \right) = \frac{L^2}{g^2} R(4)(g,L).
\]  

(69)

Here \( g \) is the covariant metric tensor, \( g^{-1} \) the contravariant metric tensor and the Riemann tensor, written in terms of the metric variations \( \Delta g \), is

\[
R(4)(g,L) \sim \frac{g^2}{L^2} \left( \Delta (\Delta g(g)^{-1}) + (\Delta g(g)^{-1})^2 \right),
\]  

(70)

where

\[
(\Delta g(g)^{-1})^2_{\{ki,ln-klm,il\}} = \Gamma^m_{\ kl} \Gamma^n_{\ lm} - \Gamma^m_{\ km} \Gamma^n_{\ il}
\]  

(71)

are the Christoffel symbols and \( \Delta (g^{-1} \Delta g) \) the second derivatives of the metric tensor with respect to the coordinates,

\[
\partial^2_{x\mu} g = \frac{g}{L^2} \Delta_k \Delta_{\mu} g = \frac{g^2}{L^2} \Delta_k (\Delta_l g(g)^{-1}).
\]  

(72)
The indetermination relationship of Eq. (69) corresponds to the presence of fluctuations of the averaged quantity of the proper energy $\bar{E}$ averaged over the volume $L^3$. If $\bar{E} = \Delta E^*$ and $\tau = \Delta t$, Eq. (69) becomes

$$\Delta E^* \times \Delta t = \left(\frac{\tau}{\tau_p}\right)^2 \frac{L^2}{g^2} R_{(4)}(g, L) = \frac{1}{g^2} \left(\frac{L^2}{L_p}\right)^2 R_{(4)}(g, L) \geq \frac{1}{2} \left[1 + \eta (\Delta E^*)^2 + \ldots\right],$$  

(73)

and at Planck scales becomes

$$\Delta E^* \times \Delta t = \hbar \frac{L^2}{g^2} R_{(4)}(g, L) = \Delta \left((\Delta g(g)^{-1}) + (\Delta g(g)^{-1})^2 \right) \geq \frac{1}{2} \left[1 + \eta (\Delta E^*)^2 + \ldots\right],$$  

(74)

where $\Delta E^* = \Delta E + \Delta g \Lambda + g \Delta \Lambda$ (with $\Lambda$ the cosmological constant) is averaged on the volume $L^3$ of the 3D space-like hypersurface $\sigma$. This would suggest that, starting from Einstein’s equations, already at the first order, one has a background curvature and fluctuations with wavelength $\lambda = L$ that are connections between events in the spacetime due to an exchange of virtual gravitons with wavelength $\lambda$, in the classical interpretation.

A more exotic interpretation can be given with the ER = EPR scenario, where the two events forming the spacetime texture (or the relationship between the two “particles” of our “gravitational hydrogen atom”) represent the connection of events in spacetime through an Einstein-Rosen (ER) wormhole supposed to be equivalent to a connection between events through Einstein-Podolski-Rosen quantum entangled states [38].

One notes also another important difference between the classical hydrogen atom of quantum mechanics [25] and the quantum SBH [1]. In the standard hydrogen atom the nucleus and the electron are real and different particles. In the quantum SBH here analysed the two actors described by our oscillator behave instead as equal particles, as one can easily find described in the system of equations (48) - (51). Thus, what we call the “nucleus” and the “electron”, in this mathematical description can be mutually exchanged without varying the actual physical properties of the system, a quantum oscillator. Therefore, the quantum state described by the quantum oscillator results even more uncertain, more non-deterministic, more smeared and more probabilistic than the corresponding quantum states of the particles of the hydrogen atom. The results in [1] are also in agreement with the general conviction that quantum gravity effects become fundamental in the presence of strong gravitational fields. In a certain sense, the results in [1] that have been reanalysed in this Section allow us to “see into” the SBH.

Recently, the analysis in [1] has been extended to the Reissner-Nordstrom BH [39]. Moreover, the stability of the solutions has been discussed, showing the existence of oscillatory regimes or exponential damping for the evolution of a small perturbation from a stable state.

The close relationship between GUP and the emergence of a minimal length suggests a scenario that already seems useful for a future quantum gravity. The analysis carried out so far can be summed up by saying that there is a non-local core in a BH, a “critical point” of the metric description in which classical concepts collapse and a quantum regime comes into play. In dynamic terms one can see it as the formation of a zone of high coherence [40]. A non-local state can be associated with a physical regime described by a non-switching geometry that results in typical oscillating processes [41, 42, 43, 44].

3 Path integral formulation

Alternatively, one can obtain the same results through a path integral approach, as expected. Let us consider the standard Einstein - Hilbert Lagrangian [45]:

$$\mathcal{L}_{EH} = \sqrt{-g} R \frac{16\pi}{16\pi}$$  

(75)

By using the FLRW line-element (15), one gets

$$\mathcal{L}_{FLRW} = \dot{a}^2 + \frac{8}{3} \pi a^2 \rho.$$  

(76)

Let us rescale Eq. (76) as

$$\mathcal{L} = \frac{1}{2} M \dot{a}^2 + \frac{4}{3} M \pi a^2 \rho,$$  

(77)
which can be written as
\[ \mathcal{L} = \frac{M \dot{a}^2}{2} - V(a), \]  
where
\[ V(a) \equiv -\frac{4}{3}M \pi a^2 \rho. \]  
The energy function associated to the Lagrangian is
\[ E = \frac{\partial \mathcal{L}}{\partial \dot{a}} \dot{a} - \mathcal{L}. \]  
Then, by inserting Eq. (77) in Eq. (80), and by using the first of Eqs. (17), one gets
\[ E = -\frac{M}{2}, \]  
which is exactly Eq. (26).

Now, by taking an infinitesimal time interval \([\tau, \tau + \delta \tau]\), the wave function \(\Psi\) for the system at the time \(\tau + \delta \tau\) in terms of its value at time \(\tau\) is determined by the integral equation
\[ \Psi (Q, \tau + \delta \tau) = \int \exp \left\{ i \delta \tau \mathcal{L} \left( \frac{Q - q}{\delta \tau}, Q \right) \right\} \Psi (q, \tau) \frac{\sqrt{g(q)} dq}{A(\delta \tau)}, \]  
where the transformation function \(q'_\tau + \delta \tau \equiv Q|q'_\tau \equiv q\), which connects the representations referring to the two different times \(\tau\) and \(\tau + \delta \tau\), corresponds in the classical theory to \(\exp (i \delta \tau \mathcal{L})\), \(\sqrt{g(q)} dq\) is the volume element in q-space and \(A(\delta \tau)\) is a suitable normalization constant, see [46] for further details. In accordance with Eq. (82), the wave function for the system must satisfy, for infinitesimal \(\varepsilon\), the equation (where one writes \(\varepsilon\) for \(\delta \tau\))
\[ \Psi (a, \tau + \varepsilon) = \int \exp \left\{ i \varepsilon \left[ \frac{M}{2} \left( \frac{a - y}{\varepsilon} \right)^2 - V(a) \right] \right\} \Psi (y, \tau) \frac{dy}{A}. \]  
Now, one replaces \(y = \eta + a\) in the integral, obtaining
\[ \Psi (a, \tau + \varepsilon) = \int \exp \left\{ i \left[ \frac{M \eta^2}{2 \varepsilon} - \varepsilon V(a) \right] \right\} \Psi (\eta + a, \tau) \frac{d\eta}{A}. \]  
Only values of \(\eta\) close to zero will contribute to the integral. Then, one expands \(\Psi (\eta + a, \tau)\) in a Taylor series around \(\eta = 0\). After rearranging, one gets
\[ \Psi (a, \tau + \varepsilon) = \frac{\exp (-i \varepsilon V(a))}{A} \int \exp \left( i \frac{M \eta^2}{2 \varepsilon} \right) \left[ \Psi + \eta \Psi' + \frac{\eta^2}{2} \Psi'' + \ldots \right] d\eta, \]  
where, for the sake of simplicity, we defined \(\Psi \equiv \Psi (a, \tau), \Psi' (a, \tau) \equiv \frac{\partial \Psi (a, \tau)}{\partial a}\) and \(\Psi'' (a, \tau) \equiv \frac{\partial^2 \Psi (a, \tau)}{\partial a^2}\). From Pierce's integral tables in Feynman’s PhD thesis [46] one gets
\[ \int_{-\infty}^{\infty} \exp \left( i \frac{M \eta^2}{2 \varepsilon} \right) d\eta = \sqrt{\frac{2 \pi i \varepsilon}{M}}, \]  
and, by differentiating both sides with respect to \(M\), one finds
\[ \int_{-\infty}^{\infty} \exp \left( i \frac{M \eta^2}{2 \varepsilon} \right) \eta^2 d\eta = \sqrt{\frac{2 \pi i \varepsilon}{M}} \frac{\varepsilon i}{M}. \]  
The integral with \(\eta\) in the integrand is the integral of an odd function, and so it is zero. Hence one obtains
\[ \Psi (a, \tau + \varepsilon) = \frac{1}{A} \sqrt{\frac{2 \pi i \varepsilon}{M}} \int_{-\infty}^{\infty} \exp [-i \varepsilon V(a)] \left[ \Psi + \frac{\varepsilon i}{2M} \Psi'' + \ldots \right]. \]  
The left hand side of Eq. (88) approaches \(\Psi (a, \tau)\) for small \(\varepsilon\). The equality holds if
\[ A (\varepsilon) = \sqrt{\frac{2 \pi i \varepsilon}{M}}, \]  
which can be written as
\[ \mathcal{L} = \frac{M \dot{a}^2}{2} - V(a), \]  
where
\[ V(a) \equiv -\frac{4}{3}M \pi a^2 \rho. \]  
The energy function associated to the Lagrangian is
\[ E = \frac{\partial \mathcal{L}}{\partial \dot{a}} \dot{a} - \mathcal{L}. \]  
Then, by inserting Eq. (77) in Eq. (80), and by using the first of Eqs. (17), one gets
\[ E = -\frac{M}{2}, \]  
which is exactly Eq. (26).
If one expands both sides of Eq. (88) in powers of $\varepsilon$ up to the first order, one finds

$$i \frac{\partial \Psi}{\partial \tau} = -\frac{1}{2M} \frac{\partial^2 \Psi}{\partial a^2} + V(a)\Psi,$$

(90)

which is exactly the Schrödinger equation (31). Hence, now one can remake the analysis below Eq. (31) in Section I obtaining the same results.

4 Black hole entropy

Now, one can improve the analysis made in Section II by discussing the SBH entropy. So, let us consider a SBH excited at the quantum level $n$; then, by putting $A_n \equiv 16\pi M^2 n$, the number of quanta of area is

$$N_n \equiv \frac{A_n}{\Delta A_n} = \frac{16\pi M^2 n}{64\pi} = \frac{1}{4} M^2 n = n,$$

(91)

where Eq. (53) has been used. Thus, one finds the intriguing (but also intuitive) result that the number of quanta of area is equal to the SBH principal quantum number. Then, the formula of the famous Bekenstein-Hawking entropy [48, 47] reads

$$(S_{BH})_n \equiv \frac{A_n}{4} = \frac{N_n \Delta A_n}{4} = 16\pi n,$$

(92)

that means that Bekenstein-Hawking is linear function of the SBH principal quantum number, i.e. of the BH excited state. By using the quantum tunnelling approach one can arrive to the sub-leading corrections at third order approximation [49]

$$S_{total} = S_{BH} - \ln S_{BH} + \frac{3}{2A} + \frac{2}{A^2}.$$  

(93)

Thus, by using Eq. (92) one gets

$$(S_{total})_n = 16\pi n - \ln (16\pi n) + \frac{3}{128\pi n} + \frac{2}{(64\pi n)^2}.$$  

(94)

One stresses that in the above analysis it has been implicitly assumed that the coefficient of Bekenstein-Hawking entropy has its traditional value $\frac{1}{4}$. On the other hand, following the analysis in [50], one can show that the mass and energy spectra given by Eqs. (53) and (54) imply that the value of such a coefficient is different. For the sake of completeness, here one must recall that the use of the microcanonical ensemble for describing Hawking radiation and BH evaporation was advocated by Hawking himself in [51] and later on revived in [52], way before [50].

One starts to recall that the microcanonical ensemble is the proper ensemble to describe BHs which are not in thermodynamic equilibrium, such as radiating BHs [50]. This choice of ensemble eliminates the problems, i.e., negative specific heat and loss of unitarity, encountered when the canonical ensemble is used [50].

Now, let energy eigenvalues $E_n$ with multiplicities $\nu(n)$ be [50]

$$E_n = \gamma \sqrt{n}, \quad \nu(n) = g^n, \quad n = 1, 2, \ldots$$

(95)

where $\gamma = -\frac{1}{4}$ from Eq. (54) and $g > 1$. The microcanonical partition function $\Omega(E)$ is equal to the number of eigenvalues below the energy $E$. Hence

$$\sum_{n=1}^{n(E)} g^n = \frac{g^{n(E)} - 1}{g - 1},$$

(96)

where [50]

$$n(E) = \frac{E^2}{\gamma^2} = 4E^2.$$  

(97)

One considers Bekenstein’s original intuition that the BH entropy can be written as [48]

$$S_{BH} = f(A) = cA,$$

(98)
where \( A \) is the area of the event horizon. The aim is to determine the constant \( c \) starting from the analysis developed in previous Sections. Starting from Eq. (98), it is natural to assume an \( \exp(cA) \)-fold degeneracy in the possible BH mass eigenvalues \( M_n \) [50]. This assumption of degeneracy is justified because entropy, in general, can be understood as a logarithm of the number of microstates corresponding to the same macrostate [50]. Since for a SBH with mass \( M \), \( A = 16\pi M^2 \), one is prompted to define \( \nu(M_n) \) as the number of degenerate states corresponding to the same mass eigenvalue \( M_n \) such that [50]

\[
\nu(M_n) = \nu(n) = \exp(16c\pi M_n^2). \tag{99}
\]

Considering Eq. (53) one can write [50]

\[
g = \exp(64c\pi). \tag{100}
\]

Now, one comes back to the microcanonical system. As in [50], to find the thermodynamics, one has to compute

\[
\omega = \frac{\delta \Omega}{\delta E} = \frac{\log(g)}{g-1}g^{4E^2}. \tag{101}
\]

Then, the microcanonical temperature is given by

\[
k_B T = \frac{\Omega}{\omega} = \frac{g^{4E^2} - 1}{8E\log(g)} \frac{1}{g^{4E^2}}. \tag{102}
\]

Its inverse is equal to

\[
\beta = 8E\log(g) \left[ 1 + \mathcal{O} \left(g^{-4E^2}\right) \right]. \tag{103}
\]

From Section II one has \( M = |4E| \). Then

\[
\beta = 2M \log(g) = 128Mc\pi, \tag{104}
\]

up to the exponentially small corrections in Eq. (103). By imposing \( \beta = \beta_H = 8\pi M \) (in Planck units), one finds

\[
c = \frac{1}{16}. \tag{105}
\]

Thus, from Eq. (98), one gets

\[
S_{BH} = \frac{A}{16}. \tag{106}
\]

Hence, it has been shown that the approach to BH quantum physics that has been developed in previous Sections implies that the coefficient of the Bekenstein-Hawking entropy is a quarter of its traditional value. Then, Eqs. (92) and (94) becomes

\[
(S_{BH})_n \equiv \frac{A_n}{16} = \frac{N_n \Delta A_n}{16} = 4\pi n, \tag{107}
\]

and

\[
(S_{total})_n = 4\pi n - \ln (4\pi n) + \frac{3}{32\pi n} + \frac{2}{(16\pi n)^2}, \tag{108}
\]

respectively.

### 5 Consistence with the semi-classical Bohr-like approach to black hole quantum physics

The Bohr-like approach to BH quantum physics has been previously developed by one of the Authors (CC) (see [13, 14] for further details) and reviewed in [15], starting from the pioneering works [53, 54]. This approach is founded on the concept of BH effective state, which has been partially introduced in Section II of this paper through the definitions of the BH effective mass and effective horizon, and the natural correspondence between Hawking radiation and BH quasi-normal modes (QNMs). Such a correspondence allows indeed to naturally interpret BH QNMs as quantum levels in a semi-classical approach. This is an approach to BH quantum physics somewhat
similar to the historical semi-classical approach to the structure of a hydrogen atom introduced by Bohr in 1913 [33, 34]; in a certain sense, QNMs represent the “electron” which jumps from a level to another one and the absolute values of the QNMs frequencies, “triggered” by emissions (Hawking radiation) and absorption of external particles, represent the energy “shells” of the GHA. After shortly reviewing the results in [13, 14, 15, 53, 54], the approach will be refined through an important rescaling of the quantum levels and some further modify which will take into account the real physical behaviors of the SBH. In that way, it will be shown that this approach to BH physics will be completely consistent with the full quantum treatment of previous Sections. This will give a remarkable physical insight; to clarify this point, let us again consider the analogy between the potential energy of a hydrogen atom, given by Eq. (46), and the potential energy of the GHA given by Eq. (48). Eq. (46) represents the interaction between the nucleus of the hydrogen atom, having a charge $e$ and the electron, having a charge $-e$ while Eq. (48) represents the interaction between the nucleus of the GHA, i.e., the BH, having an effective, dynamical mass $M_E$, and another, mysterious, particle, i.e., the “electron” of the GHA having again an effective, dynamical mass $M_E$. The Bohr-like approach to BH quantum physics shows that the “electron states” of the BH are exactly the BH QNMs “triggered” by emissions of Hawking quanta and absorption of external particles. Remarkably, the QNM jumping from a level to another one has been indeed interpreted in terms of a particle quantized on a circle, which is analogous to the electron traveling in circular orbits around the hydrogen nucleus, similar in structure to the solar system, of Bohr’s semi-classical model of the hydrogen atom [33, 34].

Let us recall that Hawking radiation is today largely analysed through the tunneling mechanism (see [55, 56, 57, 58, 59, 60, 61] and references within); by considering a classically stable object, if one sees that it becomes quantum-mechanically unstable, suspecting tunneling is natural. Then, particles creation by BH can be described as tunneling generated by vacuum fluctuations near the BH horizon [55, 56, 57, 58, 59, 60, 61]. If a virtual particle pair originates just inside the BH horizon, the virtual particle with positive energy can tunnel outside the BH becoming a real particle. The same happens when a virtual particle pair originates just outside the horizon. Then, the particle with negative energy can tunnel inwards. It results that the particle with negative energy is absorbed by the BH in both cases. Consequently, the BH mass decreases and the particle with positive energy moves towards infinity. This is exactly the mechanism of emission of Hawking radiation. Working in strictly thermal approximation, the probability of emission of Hawking quanta is [47, 55]

$$\Gamma \sim \exp \left(-\frac{\omega}{T_H}\right),$$

(109)

where $\omega$ is the energy-frequency of the emitted particle and $T_H \equiv \frac{1}{8\pi M}$ is the Hawking temperature. But, if one considers the energy conservation, that means the BH contraction which enables a varying BH geometry, one obtains the following correction [55]:

$$\Gamma \sim \exp \left[-\frac{\omega}{T_H} \left(1 - \frac{\omega}{2M}\right)\right],$$

(110)

and so

$$\Gamma = \alpha \exp \left[-\frac{\omega}{T_H} \left(1 - \frac{\omega}{2M}\right)\right],$$

(111)

where $\alpha \sim 1$ and one gets the additional term $\frac{\omega}{2M}$. The tunnelling picture can be improved by showing that the probability of emission (94) is really associated to the two distributions [13, 60]

$$\langle N \rangle_{boson} = \frac{1}{\exp \left[4\pi (2M - \omega) \omega\right] - 1},$$

(112)

$$\langle N \rangle_{fermion} = \frac{1}{\exp \left[4\pi (2M - \omega) \omega\right] + 1},$$

(113)

for bosons and fermions, respectively, which are now strictly thermal.

Now, one considers Dirac delta perturbations representing subsequent absorptions of particles with negative energies [13, 14]. Those perturbations are associated to the emission of Hawking radiation [13, 14]; BH response to perturbations are QNMs [13, 14, 15, 53, 54]. These are frequencies of radial spin-$j$ perturbations which obeys a time independent Schrödinger-like equation, see references [13, 14, 15, 53, 54] for further details. QNMs represent indeed BH modes of energy dissipation having complex frequency [13, 14, 15, 53, 54]. The first attempt to model the
semi-classical BH in terms of BH QNMs can be found in reference [62]. For large values of the principal quantum number \( n \), with \( n = 1, 2, \ldots \), QNMs result independent of both the spin and the angular momentum quantum numbers [13, 14, 15, 53, 54]. This is in agreement with Bohr’s Correspondence Principle [63], which is one of the most important principle concerning the approach of quantum theory through semi-classical approximation. It states that “transition frequencies at large quantum numbers should equal classical oscillation frequencies” [63]. In other words, Bohr’s Correspondence Principle permits a precise semi-classical analysis of quantum phenomena when the values of the principal quantum number \( n \) are large. In our case, it enables to realize a precise analysis of excited BHs [13, 14, 15, 53, 54]. By considering this important principle, it has been indeed shown that QNMs can supply information on area quantization because QNMs perturbations can be associated to absorption of external particles [31]. The analysis in [31] was then improved in [32] by solving some important issues. In any case, an important problem was that, being QNMs countable frequencies, the continuous character of Hawking radiation did not originally allow to interpret QNMs in terms of emitted quanta [65]. This avoided to associate QNMs to Hawking radiation [65]. More recently, it has been observed (see [13, 14, 15, 53, 54]) that an important consequence of the non-thermal spectrum in [55] is the countable character of subsequent emissions of Hawking particles. This generates a natural correspondence between Hawking radiation and BH QNMs which allows to interpret QNMs also in terms of emitted quanta. In fact, the key point is that Dirac delta perturbations arising from discrete subsequent absorptions of particles with negative energies, which are associated to Hawking emissions, generates indeed a BH back reaction in terms of QNMs. Thus, the BH contraction arising from energy conservation has not to be considered as being a “one shot process”. It consists instead in oscillations of the BH horizon which are the BH QNMs. It is indeed well known that the correspondence between emitted radiation and proper oscillation of the emitting body is considered a fundamental behavior of every radiation process in Nature. In other words, one can extend the analysis in [31, 32] by considering QNMs in terms of quantum levels also for emitted energies [13, 14, 15, 53, 54]. This is the fundamental idea on which the Bohr-like approach to BH quantum physics is founded [13, 14, 15, 53, 54].

Let us shortly review how the Bohr-like approach works. One introduces the effective temperature

\[
T_E(\omega) \equiv \frac{2M}{2M - \omega} T_H = \frac{1}{4\pi(2M - \omega)},
\]

and rewrites Eq. (114) in a Boltzmann-like form similar to Eq. (109):

\[
\Gamma = \alpha \exp[-\beta_E(\omega)\omega] = \alpha \exp \left( -\frac{\omega}{T_E(\omega)} \right),
\]

where \( \exp[-\beta_E(\omega)\omega] \) is the effective Boltzmann factor, with \( \beta_E(\omega) \equiv \frac{1}{T_E(\omega)} \). Hence, \( T_E(\omega) \) replaces \( T_H \) in the equation of the probability of emission. It is important to recall that in various fields of science the deviation from the thermal spectrum of an emitting body can be taken into account through the introduction of an effective temperature representing the temperature of a black body that emits the same total amount of radiation. The concept of effective temperature in BH physics has been introduced in the pioneering works [53, 54] and then used in [13, 14, 39, 15] and in other works (references within [15]). The effective temperature \( T_E(\omega) \) depends on the energy-frequency of the emitted radiation and the deviation of the BH radiation spectrum from the strictly thermal feature can be quantified through the ratio

\[
\frac{T_E(\omega)}{T_H} = \frac{2M}{2M - \omega}.
\]

It was exactly the introduction of the effective temperature which permitted the introduction of the other effective quantities (see Eq. (47). Thus, in order to finalize the discussion on the BH effective state of Section II, one recalls that the effective temperature \( T_E \) is the inverse of the average value of the inverses of the initial and final Hawking temperatures. (Before the emission we have

\[
T_H^I = \frac{1}{8\pi M},
\]

and after the emission

\[
T_H^F = \frac{1}{8\pi (M - \omega)}.
\]
As stated before, by considering an excited SBH, which means a SBH having large values of the principal quantum number $n$, then, the QNMs expression results independent on the angular momentum quantum number, in full agreement with the above cited Bohr’s Correspondence Principle. Then, if one wants to take into account the non-strictly thermal behavior of the radiation spectrum, one replaces $T_H$ with $T_E$ in the standard, strictly thermal, equation for the QNMs:

$$\omega_n = a + ib + 2\pi i n T_E(|\omega_n|) \approx 2\pi i n T_E(|\omega_n|) = \frac{in}{4M - 2|\omega_n|} = \frac{in}{4M_E},$$  \hspace{1cm} (119)$$

where $a = \ln(3)T_E(|\omega_n|), b = \pi T_E(|\omega_n|)$ for $j = 0, 2$ (scalar and gravitational perturbations), $a = 0, b = 0$ for $j = 1$ (vector perturbations) and $a = 0, b = \pi T_E(|\omega_n|)$ for half-integer values of $j$. In Eq. (119), $M_E$ is the BH effective mass (see also Eq. (47)), where the minus sign has been taken into account since an emission is considered. On the other hand, as $a, b \ll |2\pi i n T_E(|\omega_n|)|$ for large $n$, the leading term in the imaginary part of the complex frequencies well approximates the QNMs (119). Therefore, the spin content of the perturbation does not influence the leading asymptotic behavior of $|\omega_n|$. This is again consistent with the Bohr’s Correspondence Principle.

At order of leading asymptotic behavior the solution of Eq. (119) is [13, 14, 15, 53, 54]

$$|\omega_n| = M - \sqrt{M^2 - \frac{n}{2}},$$  \hspace{1cm} (120)$$

In [13, 14, 15, 53, 54], $|\omega_n|$ has been interpreted like the total energy emitted for a BH excited at a level $n$. Now, considering an emission from a BH at rest to a state with large $n = n_1$, and using Eq. (120), one sees that the BH mass changes from $M$ to

$$M_{n_1} \equiv M - |\omega_{n_1}| = \sqrt{M^2 - \frac{n_1}{2}}.$$  \hspace{1cm} (121)$$

If we consider a subsequent emission from the quantum level $n = n_1$ to the quantum level $n = n_2$, with $n_2 > n_1$, then the BH mass changes again from $M_{n_1}$ to

$$M_{n_2} \equiv M_{n_1} - \Delta E_{n_1 \rightarrow n_2} = M - |\omega_{n_2}| = \sqrt{M^2 - \frac{n_2}{2}},$$  \hspace{1cm} (122)$$

with

$$\Delta E_{n_1 \rightarrow n_2} \equiv |\omega_{n_2}| - |\omega_{n_1}| = M_{n_1} - M_{n_2} = \sqrt{M^2 - \frac{n_1}{2}} - \sqrt{M^2 - \frac{n_2}{2}},$$  \hspace{1cm} (123)$$

which is the quantum jump between the two BH energy levels due to the emission of a particle having frequency $\Delta E_{n_1 \rightarrow n_2}$. Hence, the quantum jump implies the emission of a discrete amount of energy. In the case of Bohr’s hydrogen atom [33, 34], the electron only gains and loses energy when it jumps from one allowed energy shell to another. It can absorb or emit radiation having an energy difference of the levels governed by the Planck relation (in standard units) $E = hf$, being $h$ the Planck constant and $f$ the transition frequency. In the Bohr-like approach to BH physics, QNMs can only gain and lose energy by jumping from one allowed energy shell to another. Thus, they absorb or emit radiation (emitted radiation is given by Hawking quanta) with an energy difference of the levels given by to Eq. (123). Remarkably, one can interpret Eq. (120) in terms of a particle, “the electron”, quantized on a circle of length [13, 14, 15, 53, 54]

$$L = \frac{1}{T_E(|\omega_n|)} = 4\pi \left( M + \sqrt{M^2 - \frac{n}{2}} \right).$$  \hspace{1cm} (124)$$

This is analogous to the electron traveling in circular orbits around the hydrogen nucleus.

Now, the semi-classical Bohr-like approach to BH physics will be refined through some simple, but important physical observation, and this will allow us to find a perfect consistence the approaches which have been analysed in Sections II and III. Let us observe that the Bohr-like approach has been developed starting from the assumption of a SBH at rest which starts to emit Hawking radiation. But the key point is that a BH can never be at rest. Let us clarify this point. After the BH formation, there is a “settlement” phase due to “gravitational recoil”. As one is considering SBHs, which arise from a radial gravitational collapse having spherical symmetry, there is no gravitational wave (GW) emission in such a “settlement” phase. In fact, in general relativity, GWs are quadrupole waves and a perfect spherical symmetric source cannot emit GWs [16]. On
the other hand, from the quantum point of view, one can consider the BH formation as subsequent absorptions of separated particles starting from a BH having null mass. In that case, the approach in [31, 32] implies that particles absorptions by the BH generates perturbations and that the BH back reaction to such perturbations is given by QNMs. In other words, the “settlement” phase after the BH formation is completely described by the BH QNMs “triggered” by the absorption of particles. When will this phase end? In order to answer this question one has to recall that a BH is not immerged in vacuum. Instead, it is immerged in the thermal bath of the Cosmic Background Radiation (CBR). Thus, it will continue to absorb CBR photons (and other potential external particles) until the CBR temperature will be higher than the BH Hawking temperature. In other words, the “settlement” phase will end when the BH Hawking temperature will become higher than the CBR temperature. But, at that point, the BH will start to emit Hawking radiation and the BH QNMs will be “triggered” by the absorption of Hawking quanta having negative energies. The process will continue again and again, until the Planck distance and the Planck mass are approached. At that point, as it has been stressed in Section II, the GUP prevents the BH from its total evaporation by stopping Hawking’s evaporation process in exactly the same way that the HUP prevents the hydrogen atom from total collapse [29]. Therefore, the BH arrives at the minimum energy level which is compatible with the GUP; such an energy level corresponds to our Eq. (57).

From the above discussion one argues that the only BH at rest is the BH having null mass and one must consider absorptions instead of emissions. This means that one must replace \( M \rightarrow 0 \) and \(-\omega_n \rightarrow M\) in the right hand side of Eq. (119). One also recalls that the principal quantum number increases for absorptions instead of emissions. Thus, one obtains

\[
\omega_n = \frac{in}{2M} = \frac{in}{4M_E}, \tag{125}
\]

and, again, one gets

\[
M_E = \frac{M}{2} \tag{126}
\]

One gets complete consistence with the two-particle quantum system of Eqs. (48) and (49) by interpreting the quasi-normal frequency (125) as being a particle, the “electron”, quantized on a circle. But it has been previously observed that, in such a two-particle quantum system, the two particles are equal and can be mutually exchanged without varying the physical properties of the system. This means that one can assign to the QNMs in Eq. (125) the half of the total energy of the two-particle quantum system, obtaining

\[
\omega_n = \frac{E_n}{2} = \frac{in}{2M} = \frac{in}{4M_E} \tag{127}
\]

Let us also remark that the total BH energy \( E = -\frac{M}{2} \) is negative and different from the BH inert mass \( M \). Thus, by using Eq. (51), one gets from Eq. (127)

\[
\frac{M_E}{4} = \left| \frac{in}{4M_E} \right|, \tag{128}
\]

that is

\[
M_n = 2\sqrt{n} \tag{129}
\]

Remarkably, Eq. (129) is exactly the same Eq. (53). Thus, one can use again Eq. (51) in order to re-obtain Eq. (54).

In complete consistence with Section 2, Eq. (127) is interpreted in terms of a particle, the “electron”, quantized on a circle of length

\[
L = \frac{1}{T_E} = 8\pi M_E = 8\pi \sqrt{n}, \tag{130}
\]

which is equal to Eq. (66) which was derived in Section 2 through a full quantum analysis. The semi-classical physical interpretation of the QNMs in Eq. (125) is of a collection of damped harmonic degrees of freedom, which releases an intuitive physical picture of a BH as a whole [32]. The larger is \(|\omega_n|\), the shorter is the lifetime, as one expects from physical intuition [32]. Thus, the increasing of absorptions “triggers” shorter and shorter lived modes, while subsequent emissions of Hawking quanta “trigger” longer and longer lived modes.
It has been shown that different approaches give the same results concerning the BH mass and energy spectra. It also means that one gets different pictures concerning the SBH. In the classical framework of Einstein’s general relativity, a BH is a “dead object” with inert mass. It is a definitive prison where anything that enters cannot escape. This means that it can only become more massive and bigger with time. In the semi-classical approximation, Hawking has shown, in one of his most famous papers [47], that BHs may actually emit quanta. The emission of Hawking radiation implies that BHs lose energy and decrease their mass until eventually evaporating. Hawking’s semi-classical picture can be refined through the Bohr-like approach that has been developed in this Section. The “electron”, which is given by the BH QNMs “triggered” by the absorptions of external particles and/or by the emissions of Hawking quanta, travels in circular orbits around the “nucleus”. The circular orbits become more external in the case of absorptions and more internal in the case of emissions, see Eq. (130). In other words, the “electron” jumps from an orbit to another one due to absorptions and/or emissions. The full quantum picture discussed in Section 2 is completely consistent with the Bohr-like approach analysed in this Section. In particular it permitted to obtain an interesting quantum representation of the Schwarzschild BH ground state at the Planck scale which also concerns the interpretation of the BH origin in terms of evolving particles pair.

6 Hawking radiation and quantum structure of a black hole

It is interesting to make some considerations on the spacing of the mass levels. In the case of an absorption, from Eq. (53), one gets for two neighboring levels

$$\Delta M_{n \rightarrow n+1} = 2 (\sqrt{n+1} - \sqrt{n}) = \frac{2 (n + 1 - n)}{(\sqrt{n+1} + \sqrt{n})} = \frac{2}{(\sqrt{n+1} + \sqrt{n})},$$

which means that the spacing of the mass levels decreases with increasing $n$, and that

$$\lim_{n \to \infty} \Delta M_{n \rightarrow n+1} = 0.$$  

For large $n$ (highly excited BHs), one gets $\sqrt{n} \simeq \sqrt{n + 1}$. Thus,

$$\Delta M_{n \rightarrow n+1} \simeq \frac{1}{\sqrt{n}}.$$  

(133)

If one considers an astrophysical BH having mass of the order of ten solar masses, from Eq. (53), one gets $\sqrt{n} \sim 10^{38}$, which means that $\Delta M_{n \rightarrow n+1} \simeq 10^{-38}m_P$. This is an infinitesimal quantity, and this implies that, considering astrophysical BHs, on one hand their mass spectrum becomes practically continuous and, on the other hand, that the emission of Hawking radiation will be practically thermal. This is consistent with the original semi-classical computation of Hawking that can be found in [47] and it is interesting for the following reason. In [8] Bekenstein and Mukhanov claimed that a modification of the Hawking radiance spectrum should be necessary in order to take into account that BHs should have a discrete mass spectrum with concomitant line emission. Instead, here it has been shown that such a modification is not strictly necessary because the spacing of the levels become infinitesimal for large $n$. Thus, one can also set $M_n \simeq M_{n+1} \equiv M$. Thus, from Eq. (133), one finds a minimum energy for the emissions of Hawking quanta given by

$$E_{\text{min}} \simeq \frac{2}{M} = 16\pi T_H.$$  

(134)

One of the most interesting results we find is that the quantization of the area is compatible with the thermality of the spectrum since, for large values of the main quantum number, the variation of the radius of the black hole corresponding to a Hawking emission is infinitesimal. Another important issue is that Eq. (133) is the same as the one at the foundation of the interesting approach of Dvali and Gomez’s to the quantum description of BHs [64]. The approach in [64] seems indeed consistent with the results of this paper (for instance, information is also not lost during the evaporation by construction).

Starting from these considerations, let us consider an interesting analogy with loop quantum gravity (LQG). One observes that the results derived in previous Sections seem consistent with the idea that the Hawking radiation spectrum should be discrete if one quantizes the area spectrum in a
way that the allowed area is the integer multiples of a single unit area, as it has been originally suggested in [8]. On the other hand, the Hawking radiation spectrum seems to be continuous in LQG if the area spectrum is quantized in such a way that there is not only a single unit area [70]. But, in [71], by assuming the locality of photon emission in a BH, the Author suggested that the Hawking radiation spectrum is generally countable in the LQG framework even if the allowed area is not simply the integer multiples of a single unit area. This result arises from the selection rule for quantum BHs. Such an analysis shows that the Hawking radiation spectrum is truncated below a certain frequency and hence there is a minimum energy of an emitted particle [71]:

$$E_{\text{min}} \approx \beta T_H = \frac{\beta}{8\pi M}. \quad (135)$$

Eq. (135) is consistent with the result of Eq. (134) for $\beta = 16\pi$.

Now, define as $n_{\text{max}}$ the maximum value of the BH principal quantum number that corresponds to the instant when the BH temperature equals the temperature of the surrounding CBR. After such an instant, the BH stops to absorb CBR photons and starts to emit Hawking radiation. Thus, from Eq. (53), one gets

$$M_{n_{\text{max}}} \equiv 2\sqrt{n_{\text{max}}} \Rightarrow n_{\text{max}} = \frac{M_{n_{\text{max}}^2}}{4}, \quad (136)$$

and sees that, for example, in the case of an astrophysical BH having a mass of the order of ten solar masses, it is $n_{\text{max}} \sim 10^{76}$.

Now, one can compute the pre-factor $\alpha$ in Eq. (111) by refining the analysis in [13] in the light of the results of previous Sections. Let us focus on Eqs. (93) and (94). In this kind of leading order tunneling calculations, the exponent is indeed due to the classical action and the pre-factor is a correction of the order of the Planck constant. Then, in the case of emission of Hawking radiation, the variation of the Bekenstein-Hawking entropy [13, 55]

$$\Gamma = \alpha \exp \Delta S_{BH} = \alpha \exp \left[ -\frac{\omega}{T_H} \left( 1 - \frac{\omega}{2M} \right) \right], \quad (137)$$

is the order of unity for an emitted particle having energy of the order of Hawking temperature. Consequently, the exponent appearing in the right hand side of Eqs. (111) and (137) is the order of unity. Thus, one can ask what is the real meaning of such a scaling if the pre-factor is unknown. By refining the analyses in [13, 14], one shows that, fixed two quantum levels $m$ and $n$, the mass-energy $-\Delta M_{m \rightarrow n} = \omega_{m \rightarrow n}$ emitted in an arbitrary transition $m \rightarrow n$, with $n < m$, is proportional to the effective temperature associated to the transition, and that the constant of proportionality depends only on the difference $m - n$. Setting [13, 14]

$$\omega_{m \rightarrow n} = M_m - M_n = K \left[ T_E \right]_{m \rightarrow n}, \quad (138)$$

and considering that, from Eq. (53), it is

$$M_m = 2\sqrt{m}, \quad M_n = 2\sqrt{n}, \quad (139)$$

respectively, then one can see if there are values of the constant $K$ for which Eq. (138) is satisfied. As the effective temperature is the inverse of the average value of the inverses of the initial and final Hawking temperatures, it is [13, 14]

$$[T_E]_{m \rightarrow n} = \frac{1}{4\pi (M_m + M_n)}, \quad (140)$$

which implies

$$M_m^2 - M_n^2 = \frac{K}{4\pi}. \quad (141)$$

By using Eqs. (139), Eq. (141) becomes

$$4(m - n) = \frac{K}{4\pi}. \quad (142)$$

From last expression and from Eq. (138) one deduces that $K = 16\pi (m - n)$. Hence, one finds

$$\omega_{m \rightarrow n} = M_m - M_n = 16\pi (m - n) [T_E]_{m \rightarrow n}. \quad (143)$$
Using Eq. (115), the probability of emission between the two levels $m$ and $n$ can be written as

$$
\Gamma_{m \rightarrow n} = \alpha \exp \left\{ - \frac{\Delta E_{m \rightarrow n}}{|\Delta E_{m \rightarrow n}|} \right\} = \alpha \exp [16\pi (n - m)].
$$

(144)

Thus, the probability of emission between two arbitrary SBH quantum levels characterized by the two principal quantum numbers $m$ and $n$ scales like $\exp [16\pi (n - m)]$. In particular, for $n = m - 1$, the probability of emission has its maximum value $\sim \exp (-16\pi)$. This means that the probability is maximum for two adjacent levels, as one intuitively expects.

Now, in order to compute the pre-factor $\alpha$, one assumes the unitarity of BH evaporation [13]. Let us remark that in the next Section the BH information paradox will be solved without using the results that it will be obtained hereafter in this Section. This will imply the unitarity of BH evaporation by confirming the correctness of the assumption that one makes here. One also recalls that the majority of researchers today think that BH evaporation is unitary. On the other hand, results in previous Sections showed the SBH in terms of a well defined quantum mechanical system, having an ordered, discrete quantum spectrum. This seems consistent with the unitarity of BH evaporation. Such a unitarity implies that, for a generic BH principal quantum number $m$, one must have

$$
\sum_{n=1}^{m} \Gamma_{m \rightarrow n} = 1,
$$

(145)

where the sum in the last expression has been stopped at $n = 1$ because, as it has been previously stressed, the GUP prevents a BH from its total evaporation by stopping the evaporation process at the Planck scale, while in [1] and in Section I of this paper it has been shown that the Planck scale is reached at $n = 1$ (BH ground state). One also notes that $n = m$ corresponds to the probability that the BH does not emit [13].

Now, using Eq. (144), one gets

$$
\sum_{n=1}^{m} \Gamma_{m \rightarrow n} = \alpha \exp [16\pi (m - n)] = 1.
$$

(146)

Setting $k = m - n$ and $\exp (-16\pi) = X$, Eq. (146) becomes

$$
\alpha \sum_{k=0}^{m-1} X^k = 1.
$$

(147)

This sum is the $k$-th partial sum of the geometric series and can be solved as

$$
\sum_{k=0}^{m-1} X^k = \frac{1 - X^m}{1 - X}.
$$

(148)

Thus, one gets

$$
\frac{1 - X^m}{1 - X} = 1,
$$

(149)

that is

$$
\alpha \equiv \alpha_m = \frac{1 - X}{1 - X^m} = \frac{1 - \exp (-16\pi)}{1 - \exp (-16\pi m)}.
$$

(150)

Hence, one finds that the pre-factor $\alpha$ depends on the BH quantum level $m$. Now, if one inserts this result in Eq. (144), one fixes the probability of emission between the two levels $m$ and $n$ as

$$
\Gamma_{m \rightarrow n} = \alpha_m \exp [16\pi (n - m)] = \left[ \frac{1 - \exp (-16\pi)}{1 - \exp (-16\pi m)} \right] \exp [16\pi (n - m)].
$$

(151)

From the quantum mechanical point of view, one physically interprets Hawking radiation like energies of quantum jumps among the unperturbed levels. One also notice that, for large $m$, it is

$$
\alpha_m \simeq \text{constant} = 1 - \exp (-16\pi).
$$

(152)

Thus, Eq. (151) is well approximated by

$$
\Gamma_{m \rightarrow n} = \alpha_m \exp [16\pi (n - m)] \simeq [1 - \exp (-16\pi)] \exp [16\pi (n - m)].
$$

(153)
One can also rewrite Eqs. (112) and (113) in terms of the effective mass

\[ [M_E]_{m \to n} = \frac{M_m + M_n}{2} \]  

(154)

which is associated to the transition \( m \to n \), with \( n < m \), as

\[
\langle n \rangle_{\text{boson}} = \frac{1}{\exp\left([8\pi [M_E]_{m \to n}] \omega - 1\right)}.
\]  

(155)

\[
\langle n \rangle_{\text{fermion}} = \frac{1}{\exp\left([8\pi [M_E]_{m \to n}] \omega + 1\right)}.
\]  

(156)

In order to finalize the discussion of this Section, it is important to raise a very important point. In previous discussion, fixed two quantum levels \( m \) and \( n \), it has been considered the emission of Hawking quanta through the variation of the BH mass-energy

\[
\Delta M_{m \to n} = M_n - M_m = 2(\sqrt{n} - \sqrt{m}) = -\omega_{m \to n}
\]  

(157)

in an arbitrary transition \( m \to n \), with \( n < m \). But in this paper (and in the Authors’ knowledge for the first time in the literature) it has been shown that the total BH energy \( E = -\frac{M_\infty}{2} \) is negative and different from the BH inert mass \( M \), see Section II for details. Thus, in correspondence of the variation of the BH inert mass \( \Delta M_{m \to n} \), one must also consider the variation of the BH total energy, which can be calculated through Eq. (54) as

\[
\Delta E_{m \to n} \equiv -\frac{1}{2}(\sqrt{n} - \sqrt{m})
\]  

(158)

Thus, the total negative energy which is carried out by Hawking radiation when the BH is subjected to an arbitrary transition \( m \to n \) will be \(-\Delta E_{m \to n}\). In other words, the total energy associated to the emissions of Hawking quanta is negative. Let us clarify this issue. One starts to consider the absorption of an external particle which generates a quantum transition; in a classical framework, a particle having frequency \( \omega \) which falls into a SBH has also a total energy \( \omega + E_g \), where [16]

\[
E_g = \frac{\omega}{2} \ln \left(1 - \frac{2M}{r}\right)
\]  

(159)

is the gravitational energy and \( M \) is the BH mass. One sees that \( E_g \) diverges to \(-\infty\) when the particle approaches the gravitational radius. From the quantum point of view (see previous Sections), one considers the variation of the BH mass

\[
\omega_{m \to n} = \Delta M_{n \to m} = -\Delta M_{m \to n} = 2(\sqrt{n} - \sqrt{m})
\]  

(160)

in an arbitrary transition \( n \to m \), with \( n < m \). This implies a variation of the BH total energy

\[
\Delta E_{n \to m} \equiv -\Delta E_{m \to n} = \frac{1}{2}(\sqrt{n} - \sqrt{m}).
\]  

(161)

In Eq. (160) \( \omega_{m \to n} = \Delta M_{n \to m} \) is the kinetic energy of the particle which falls into the BH. On the other hand, such a particle has a total energy given by \( \omega_{m \to n} + E_g(n \to m) \). Hence, the quantum version of (159) is

\[
E_g(n \to m) = \Delta E_{n \to m} - \omega_{m \to n} = \frac{5}{2}(\sqrt{n} - \sqrt{m}).
\]  

(162)

This is the gravitational energy of the interaction between the particle and the BH which, being at the excited state \( n \), will have a mass \( M_n \) and a corresponding gravitational radius \( 2M_n \). In fact, our quantum analysis shows that an absorption of an external particle having mass \( \Delta M_{n \to m} \) increases the total BH energy of a negative quantity \( \Delta E_{n \to m} \) given by Eq. (161). Thus, the maximum quantity of the gravitational energy that can be absorbed by the BH when it absorbs the external particle is given by Eq. (162). A completely symmetric analysis works in the case of an emission; in that case, the BH mass decreases of the quantity given by Eq. (157), its total energy decreases of a quantity given by Eq. (158) and its gravitational energy decreases of the quantity

\[
E_g(n \to m) = \Delta E_{n \to m} + \omega_{m \to n} = \frac{5}{2}(\sqrt{n} - \sqrt{m}),
\]  

(163)
which means that a negative gravitational energy is associated to the emitted Hawking quantum having positive kinetic energy
\[ \omega_{m \to n} = 2 \left( \sqrt{m} - \sqrt{n} \right). \] (164)
In the standard calculations concerning Hawking radiation the negative gravitational energy is implicitly taken into account by considering that the emitted Hawking quanta are subjected to the BH strong gravitational redshift. Thus, when one refers to “Hawking quanta having positive energies”, one refers to kinetic energies.

In summary, the analysis in this paper allows to obtain some very interesting insights also on the gravitational energy, which represents one of the more mysterious and controversial issues of gravitational physics [16].

7 Solution to the black hole information paradox

In 1976 Stephen Hawking claimed that “Because part of the information about the state of the system is lost down the hole, the final situation is represented by a density matrix rather than a pure quantum state” (verbatim from ref. [72]). This was the starting point of the popular “BH information paradox”. After Hawking’s original claim, enormous time and effort was and is currently devoted to solve the paradox. Consequences of the BH information paradox are indeed not trivial. By assuming that information is loss in BH evaporation, pure quantum states arising from collapsed matter would decay into mixed states arising from BH evaporation [73]. The devastating consequence is that quantum gravity should not be unitary [73]. Various physicists worked and currently work on this issue. Some of them remain convinced that quantum information is destroyed in BH evaporation. Other ones claim that Hawking’s original statement was wrong and information must be, instead, preserved.

In this Section the solution which has been derived in [14] will be refined and adapted to the SBH quantum equations that have been obtained in previous Sections. Let us consider the instant when the BH Hawking temperature will become higher than the CBR temperature. At that point, the BH will start to emit Hawking radiation. Thus, one assumes a first emission from the BH excited state corresponding to the BH principal quantum number, say \( n \), which has a total energy \( E_n = -\frac{1}{2} \sqrt{m} \) given by Eq. (54), and a corresponding total mass \( M_n = 2\sqrt{m} \), to a state with \( n < m \), which corresponds to a total energy \( E_m = -\frac{1}{2} \sqrt{m} \) and to a total mass \( M_m = 2\sqrt{m} < M_m \). From the quantum mechanical point of view, one physically interprets Hawking radiation like energies of quantum jumps among the unperturbed levels of Eq. (54) [25, 14, 27]. In quantum mechanics, time evolution of perturbations can be described by an operator [14, 74]
\[ U(t) = \begin{cases} W(t) & \text{for } t \leq \tau \\ 0 & \text{for } t < 0 \text{ and } t > \tau \end{cases} \] (165)
Then, the complete (time dependent) Hamiltonian is described by the operator [14, 74]
\[ H(r, t) \equiv V(r) + U(t), \] (166)
where \( V(r) \) is given by Eq. (48). Thus, considering a wave function \( \psi(r, t) \), one can write the correspondent time dependent Schrödinger equation for the system:
\[ i \frac{d}{dt} |\psi(r, t)\rangle = [V(r) + U(t)] |\psi(r, t)\rangle = H(r, t) |\psi(r, t)\rangle. \] (167)

The state which satisfies Eq. (167) is
\[ |\psi(r, t)\rangle = \sum_n a_n(t) \exp(-iE_n t) |\varphi_n(r)\rangle, \] (168)
where the \( \varphi_n(r) \)'s are the eigenfunctions of the time independent Schrödinger equation (49) and the \( E_n \) are the correspondent eigenvalues. In the basis \( |\varphi_n(r)\rangle \), the matrix elements of \( W(t) \) can be written as
\[ W_{ij}(t) \equiv A_{ij}\delta(t), \] (169)
where \( W_{ij}(t) = \langle \varphi_i(r)|W(t)|\varphi_j(r)\rangle \) and the \( A_{ij} \)'s are real. In order to solve the complete quantum mechanical problem described by the operator (166), one needs to know the probability amplitudes
Therefore, the probability to find the system in an eigenstate having energy \(\psi_m(r,t)\) is given by

$$|\psi_m(r,t)\rangle = \exp(-iE_m t) |\varphi_m(r)\rangle,$$

is non-zero for \( t < 0 \). This implies \( a_n(t) = \delta_{nm} \) for \( t < 0 \). When the perturbation operator (165) stops to work, i.e., after the emission, for \( t > \tau \) the probability amplitudes \( a_n(t) \) return to be time independent, having the value \( a_{m\rightarrow n}(\tau) \). In other words, for \( t > \tau \) the system is described by the wave function \( \psi_f(r,t) \), which corresponds to the state

$$|\psi_f(r,t)\rangle = \sum_{n=1}^{m} a_{m\rightarrow n}(\tau) \exp(-iE_n t) |\varphi_n(r)\rangle.$$

Therefore, the probability to find the system in an eigenstate having energy \( E_n = -\frac{1}{2} \sqrt{n} \), with \( n < m \) for emissions, is given by

$$\Gamma_{m\rightarrow n}(\tau) = |a_{m\rightarrow n}(\tau)|^2.$$

By using a standard analysis, one obtains the following differential equation from Eq. (171):

$$i\dot{a}_{m\rightarrow n}(t) = \sum_{l=1}^{n} W_{nl} a_{m\rightarrow l}(t) \exp \left[ i (\Delta E_{l\rightarrow n}) t \right],$$

where the dot over \( a_{m\rightarrow n}(t) \) denotes the derivative with respect to time. To first order in \( U(t) \), by using the Dyson series [74], one gets the following solution:

$$a_{m\rightarrow n} = -i \int_0^t \{ W_{nm}(t') \exp \left[ i (\Delta E_{m\rightarrow n}) t' \right] \} dt'.$$

By inserting Eq. (169) in Eq. (174) one obtains

$$a_{m\rightarrow n} = i A_{nm} \int_0^t \{ \delta(t') \exp \left[ i (\Delta E_{m\rightarrow n}) t' \right] \} dt' = \frac{i}{2} A_{nm}.$$  

Combining this equation with Eqs. (144) and (172) one obtains

$$A_{nm} = 2\sqrt{\alpha} \exp \left[ 8\pi (n - m) \right],$$

and so

$$a_{m\rightarrow n} = -i \sqrt{\alpha} \exp \left[ 8\pi (n - m) \right].$$

As \( \sqrt{\alpha} \approx 1 \), one gets \( A_{nm} \approx 10^{-11} \) for \( n = m - 1 \), i.e., when the probability of emission has its maximum value. This implies that second order terms in \( U(t) \) can be neglected. Clearly, for \( n < m - 1 \), the approximation is better because the \( A_{nm} \)'s are even smaller than \( 10^{-11} \). Thus, one can write down the final form of the ket representing the state as

$$|\psi_f(r,t)\rangle = \sum_{n=1}^{m} a_{m\rightarrow n} \exp \left[ -iE_n t \right] |\varphi_n(r)\rangle.$$

Eq. (178) represents a pure final state instead of a mixed final state. Then, the states are written in terms of a unitary evolution matrix instead of a density matrix and this implies the fundamental conclusion that information is not lost in BH evaporation. This result is consistent with 't Hooft’s idea that Schrödinger equations can be used universally for all dynamics in the universe [75] and dismisses the claim of Hawking [72] that has been cited at the starting of this Section. The above final state is due to potential arbitrary transitions \( m \rightarrow n \), with \( m > n \). Then, the subsequent collapse of the wave function to a new stationary state, at the quantum level \( n \),

$$|\psi_n(r,t)\rangle = \exp(-iE_n t) |\varphi_n(r)\rangle,$$

implies that the wave function of the infalling particle in Hawking’s mechanism of particles creation by BH has been transferred to the two-particle system described in the previous Sections, and it is given by

$$|\psi_{(m\rightarrow n)}(r,t)\rangle \equiv \exp(-iE_n t) |\varphi_n(r)\rangle - \exp(-iE_m t) |\varphi_m(r)\rangle.$$
This wave function results entangled with the wave function of the particle which propagates towards infinity. Now, it will be shown that this key point solves the entanglement problem connected with the information paradox. This problem concerns the entanglement structure of the wave function associated to the particle pair creation [76]. In other terms, in order to solve the paradox, one needs to know the part of the wave function in the interior of the horizon, i.e., the part of the wave function associated to the interior, infalling mode. This is exactly the part of the wave function which in the Hawking computation gets entangled with the part of the wave function outside, i.e., the part of the wave function associated to the particle which escapes from the BH [76]. Here the key point is that the particle which falls into the BH transfers its part of the wave function and, in turn, the information encoded in it, to the two-particle system governed by Eqs. (48) - (51). Hence, the emitted radiation results to be entangled with such a quantum system which concerns the “excited states” of the “gravitational atom”. Thus, one argues that the BH response to the absorption of an interior, infalling mode is to add energy to the “atom’s excited state” corresponding to the energy level $E_m$, in order to allow it to jump to the “atom’s excited state” corresponding to the energy level $E_n$. In that way, the interior part of the wave function is now “within” the two-particle system, which is now at the quantum level $E_n$. Let us clarify this point. Again, let us consider the instant when the BH Hawking temperature is higher than the CBR temperature. The BH will be in a stationary state given by Eq. (170). Then, the BH will start to emit Hawking radiation and one assumes a first emission from the BH excited state corresponding to the BH principal quantum number $n = m$, which has a total energy $E_m = -\frac{1}{\sqrt{m}}$, and a corresponding total mass $M_m = 2\sqrt{m}$, given by Eq. (53), to a state with $n = m_1 < m$, which corresponds to a total energy $E_{m_1} = -\frac{1}{\sqrt{m_1}}$ and to a total mass $M_{m_1} = 2\sqrt{m_1}$. It will be $M_{m_1} < M_m$ because the infalling particle in Hawking’s mechanism of particles creation by BH has negative mass. The energy jump between the two levels is

$$\Delta E_{m \rightarrow m_1} \equiv E_{m_1} - E_m = \frac{1}{2} (\sqrt{m} - \sqrt{m_1})$$  \hspace{1cm} (181)

In other words, the energy of the first absorbed particle having negative mass is transferred, together with its part of the wave function, to the two-particle system, which is now entangled with the emitted particle. Now, by using Eq. (180), and by setting $n = m_1$, one finds that the part of the wave function in the interior of the horizon is

$$|\psi_{(m \rightarrow m_1)}(r, t)\rangle \equiv \exp (-iE_{m_1}t)|\varphi_{m_1}(r)\rangle - \exp (-iE_m t)|\varphi_m(r)\rangle.$$  \hspace{1cm} (182)

Let us consider a second emission, which corresponds to the transition from the state with $n = m_1$ to a state with, say, $n = m_2 < m_1$. The BH total energy changes from $E_{m_1} = -\frac{1}{\sqrt{m_1}}$ to $E_{m_2} = -\frac{1}{\sqrt{m_2}}$, while the BH mass changes from $M_{m_1} = 2\sqrt{m_1}$ to $M_{m_2} = 2\sqrt{m_2} < M_{m_1}$. The energy jump between the two levels is

$$\Delta E_{m_1 \rightarrow m_2} \equiv E_{m_2} - E_{m_1} = \frac{1}{2} (\sqrt{m_1} - \sqrt{m_2}).$$  \hspace{1cm} (183)

As before, the energy of the second absorbed particle having negative mass is transferred, together with its part of the wave function, to the two-particle system, which is now entangled with both of the emitted particles. By using again Eq. (180) and setting $n = m_2$ and $m = m_1$, one finds that the part of the wave function of the second infalling mode is

$$|\psi_{(m_1 \rightarrow m_2)}(r, t)\rangle \equiv \exp (-iE_{m_2}t)|\varphi_{m_2}(r)\rangle - \exp (-iE_{m_1}t)|\varphi_{m_1}(r)\rangle.$$  \hspace{1cm} (184)

Let us consider a third emission, which corresponds to the transition from the state with $n = m_2$ to a state with, say, $n = m_3 < m_2$. The BH total energy changes from $E_{m_2} = -\frac{1}{\sqrt{m_2}}$ to $E_{m_3} = -\frac{1}{\sqrt{m_3}}$, while the BH mass changes from $M_{m_2} = 2\sqrt{m_2}$ to $M_{m_3} = 2\sqrt{m_3} < M_{m_2}$. The energy jump between the two levels is

$$\Delta E_{m_2 \rightarrow m_3} \equiv E_{m_3} - E_{m_2} = \frac{1}{2} (\sqrt{m_2} - \sqrt{m_3}).$$  \hspace{1cm} (185)

The energy of the third absorbed particle having negative mass is transferred to the two-particle, which is now entangled with the third emitted particles. By using again Eq. (180), and setting $n = m_3$ and $m = m_2$, one finds that the part of the wave function of the third infalling mode is

$$|\psi_{(m_2 \rightarrow m_3)}(r, t)\rangle \equiv \exp (-iE_{m_3}t)|\varphi_{m_3}(r)\rangle - \exp (-iE_{m_2}t)|\varphi_{m_2}(r)\rangle.$$  \hspace{1cm} (186)
The process will continue again and again till the Planck distance and the Planck mass are approached by the evaporating BH. At that point, as it has been stressed in previous Sections, the GUP prevents the total BH evaporation in exactly the same way that the HUP prevents the hydrogen atom from total collapse, and the two-particle system is entangled with all the Hawking quanta emitted at that time. Hence, the BH arrives at the minimum energy level which is compatible with the GUP. Such an energy level corresponds to Eq. (57) and consists of a negative energy having the absolute value of the Planck energy. In other words, subsequent “absorptions” by the BH of the wave functions of absorbed particles having negative mass-energy in Hawking’s mechanism lead the total BH wave function, which at the starting of the emission process was given by Eq. (170), to evolve till the BH wave-function of the minimum energy level is

$$|\psi_1(r,t)\rangle = \exp (-iE_1t) |\varphi_1(r)\rangle .$$  \hspace{1cm} (187)

Clearly, the evolution of BH evaporation that it has been discussed above is unitary.

Therefore, it has been shown that SBHs, which are considered the fundamental bricks of quantum gravity, are well defined quantum mechanical systems, having ordered, discrete quantum spectra, which preserve physical information by restoring predictability in gravitational collapse.

8 Conclusion remarks

Rosen’s approach has been originally applied to the historical Oppenheimer and Snyder gravitational collapse in [1]. By setting the constraints for the formation of the SBH, the gravitational potential, the Schrödinger equation, the solution for the energy levels, and the area quantum have been found, by also discussing the quantum representation of the BH’s ground state at the Planck scale. Such results are consistent with previous ones in the literature. It was also shown that the traditional classical singularity in the core of the SBH is replaced by a non-singular two-particle system where the two components, the “nucleus” and the “electron”, strongly interact with each other. In agreement with the de Broglie hypothesis [11], the “electron” has been interpreted in terms of the quantum oscillations of the BH horizon. In Section II, the results obtained in [1] have been reviewed by also adding some new insights; in Section III it has been indeed shown that the same results can be obtained through a path integral approach [46]; in Section IV it has been shown that such results allow to compute the SBH entropy as a function of the BH principal quantum number in terms of Bekenstein-Hawking entropy and three sub-leading corrections. In addition, the coefficient of the formula of Bekenstein-Hawking entropy is reduced to a quarter of its traditional value; in Section V it has been shown that, by performing a correct rescaling of the energy levels, the semi-classical Bohr-like approach to BH quantum physics, previously developed by one of the Authors (CC) in [13, 14, 15, 53, 54], is consistent with the obtained results for large values of the BH principal quantum number. After this, Hawking radiation has been analysed by discussing its connection with the BH quantum structure in Section VI; finally, in Section VII, by analyzing the time evolution of the GHA, the solution of the BH information paradox is discussed. In fact, on one hand, it has been shown that, contrary to the famous claim of Hawking in [72], the final situation of BH evaporation is represented by a pure quantum state given by Eq. (178). On the other hand, also the entanglement problem connected with the information paradox has been solved because emitted Hawking radiation results entangled with the two-particle system governed by Eqs. (48) - (51) in a unitary process which allows the BH total wave function at the starting of the emission process to evolve till the BH wave-function of the minimum energy level. Therefore, the results obtained here seem consistent with the interesting approach of Hajicek and Kiefer (see [19, 20] for details). In such works, the Authors indeed discussed the quantization of a spherical dust shell by constructing a well-defined self-adjoint extension for the Hamilton operator. As a result, the evolution is unitary and the singularity is avoided.

Finally one takes the chance to recall that, in a series of interesting papers [77, 78, 79], the Authors wrote down the Schrödinger equation for a collapsing object and showed by explicit calculations that quantum mechanics is perhaps able to remove the singularity at the BH center (in various space-time slicings); this is consistent with our analysis. Moreover, they also proved (among the other things) that the wave function of the collapsing object is non-singular at the center even when the radius of the collapsing object (classically) reaches zero. In [79], they considered charged BHs.

In regard to the area quantization, another interesting approach, based on graph theory, can be
found in [80]. Here, the Bekenstein-Hawking area entropy accompanied with a proper logarithmic term (subleading correction) is obtained, and the size of the unit horizon area is fixed.

Curiously, Davidson also found a hydrogen-like spectrum in a totally different contest [81]. It seems that the results found in this paper are in agreement with the previous literature.

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References

[1] C. Corda, F. Feleppa, arXiv:1912.06478 (2020).
[2] N. Rosen, Int. Journ. Theor. Phys. 32, 8, (1993).
[3] B. S. DeWitt, Phys. Rev. 160, 1113 (1967).
[4] J. R. Oppenheimer, H. Snyder, Phys. Rev. 56, 455 (1939).
[5] J. Bekenstein, Lett. Nuovo Cimento, 11, 467 (1974).
[6] M. Maggiore, Nucl. Phys. B 429, 205 (1994).
[7] V. Mukhanov, JETP Letters 44, 63 (1986).
[8] J. D. Bekenstein, V. Mukhanov, Phys. Lett. B 360, 7 (1995).
[9] S. Das, P. Ramadevi, U. A. Yajnik, Mod. Phys. Lett. A 17, 993 (2002).
[10] A. Barvinski, S. Das, G. Kunstatter, Phys.Lett. B 517, 415 (2001).
[11] L. de Broglie, Ann. de Physique (10) 3, 22 (1925).
[12] W. Piechocki and T. Schmitz, Phys. Rev. 102, 046004 (2020).
[13] C. Corda, Class. Quantum Grav. 32, 195007 (2015).
[14] C. Corda, Ann. Phys. 353, 71 (2015).
[15] C. Corda, Adv. High En. Phys. 867601 (2015).
[16] C. W. Misner, K. S. Thorne, J. A. Wheeler, Gravitation (W. H. Feeman and Co., 1973).
[17] J. D. Bekenstein, in Proceedeings of the Eight Marcel Grossmann Meeting, T. Piran and R. Ruffini, eds., pp. 92-111 (World Scientific Singapore 1999).
[18] C. Kiefer and T. Schmitz, Phys. Rev. D 99, 126010 (2019).
[19] P. Hajicek and C. Kiefer, Nucl. Phys. B 603, 531 (2001).
[20] P. Hajicek and C. Kiefer, Int. J. Mod. Phys. D 10, 775 (2001).
[21] T. Schmitz, Phys. Rev. D 101, 026016 (2020).
[22] F. Alford, Ann. Henri Poincaré 21, 2031 (2020).
[23] I. Bengtsson, E. Jakobsson, J. M. M. Senovilla, Phys. Rev. D 88, 064012 (2013).
[24] L. Brewin, J. Kajtar, Phys. Rev. D 80, 104004 (2009).
[25] A. Messiah, Quantum Mechanics, Vol. 1, North-Holland, Amsterdam (1961).
[26] S. W. Hawking. “The Path Integral Approach to Quantum Gravity”, in General Relativity: An Einstein Centenary Survey, eds. S. W. Hawking and W. Israel, (Cambridge University Press, 1979).
[27] N. Zettili, “Quantum Mechanics: Concepts and Applications” (2nd ed.), Chichester: Wiley (2009).
[28] A. Kempf, G. Mangano, R. B. Mann, Phys. Rev. D 52, 1108 (1995).
[29] R. J. Adler, P. Chen, D. I. Santiago, Gen. Rel. Grav. 33, 2101 (2001).
[30] W. H. Press, Astrophys. J. 170 L105 (1971).
[31] S. Hod, Phys. Rev. Lett. 81, 4293 (1998).
[32] M. Maggiore, Phys. Rev. Lett. 100, 141301 (2008).
[33] N. Bohr, Philos. Mag. 26, 1 (1913).
[34] N. Bohr, Philos. Mag. 26, 476 (1913).
[35] E. Spallucci, A. Smailagic, Phys. Lett B 816, 136180 (2021).
[36] B. G. Schmidt, Gen. Rel. Grav. 1, 269 (1971).
[37] A. N. Tawfi kand and A. M. Diab, Rep. Prog. Phys. 78, 126001 (2015).
[38] F. Tamburini and I. Licata, Entropy 22, 3 (2020).
[39] C. Corda, F. Feleppa, F. Tamburini, EPL 132, 30001 (2020).
[40] J. Linde say, “Foundations of Quantum Gravity”, Cambridge University Press (2013).
[41] B. Muthukumar, P. Mitra, Phys. Rev. D 66 (2002).
[42] S. Haldar, C. Corda, S. Chakraborty, Adv. High Energy Phys., 9851598 (2018).
[43] Lin Bing-Sheng, Heng Tai-Hua, Chinese Phys. Lett. 28, 070303 (2011).
[44] I. Dadic, L. Jonke, S. Meljanac, Acta Phys. Slov. 55 (2005).
[45] L. D. Landau, E. M. Lifshitz, “The Classical Theory of Fields”, 2nd edition, Pergamon Press (1962).
[46] R. Feyn man, “Feynm an’s Thesis – A New Approach to Quantum Theory”, Edited by Laurie M Brown, World Scientific (2005).
[47] S. W. Hawking, Commun. Math. Phys. 43, 199 (1975).
[48] J. D. Bekenstein, Phys. Rev. D 7, 2333 (1973).
[49] Hao-Peng Yan, Wen-Biao Liu, Phys. Lett. B 759, 293 (2016).
[50] G. Scharf, Nuovo Cim. B 113, 821 (1998).
[51] S. W. Haw king, Phys. Rev. D 13, 191 (1976).
[52] B. Harms and Y. Leblanc, Phys. Rev. D 46, 2334 (1992).
[53] C. Corda, Int. Journ. Mod. Phys. D 21, 124023 (2012).
[54] C. Corda, JHEP 8, 101 (2011).
[55] M. K. Parikh, F. Wilczek, Phys. Rev. Lett. 85, 5042 (2000).
[56] R. Banerjee, B.R. Majhi, JHEP 0806, 095 (2008).
[57] M. Angheben, M. Nadalini, L. Vanzo and S. Zerbini, JHEP 0505, 014 (2005).
[58] M. Arzano, A. J. M. Medved, E. C. Vagenas, JHEP 0509, 037 (2005).
[59] R. Banerjee, B.R. Majhi, Phys. Lett. B 675, 243 (2009).
[60] C. Corda, Ann. Phys. 337, 49 (2013).
[61] S. Z. Yang, H. L. Li, Q. Q. Jiang and M. Q. Liu, Sci China-Phys. Mech. Astron. 50, 2, 249 (2007).
[62] J. York Jr., Phys. Rev. D 28, 2929 (1983).
[63] N. Bohr, Zeits. Phys. 2, 423 (1920).
[64] G. Dvali and C. Gomez, Fortsch. Phys. 61, 742 (2013).
[65] L. Motl, Adv. Theor. Math. Phys. 6, 1135 (2003).
[66] B. Zhang, Q.-Y. Cai, L. You, M. S. Zhan, Phys. Lett. B 675, 98 (2009).
[67] B. Zhang, Q.-Y. Cai, M. S. Zhan, L. You, Ann. Phys. 326, 350 (2011).
[68] X.-K. Guo, Q.-Y., Cai, Int. Journ. Theor. Phys. 53, 2980 (2014).
[69] B. Zhang, Q. Y. Cai, M. S. Zhan, L. You, Int. Journ. Mod. Phys. D 22, 1341014 (2013).
[70] K. V. Krasnov, Class. Quant. Grav. 16, 563 (1999).
[71] Y. Yoon, Jour. Kor. Phys. Soc. 68, 730 (2016).
[72] S. W. Hawking, Phys. Rev. D 14, 2460 (1976).
[73] S. W. Hawking, Phys. Rev. D 72, 084013 (2005).
[74] J. J. Sakurai, Modern Quantum Mechanics, Pearson Education (2006).
[75] G. ’t Hooft, Int. J. Mod. Phys. A 11, 4623 (1996).
[76] Samir D. Mathur, Pramana 79, 1059 (2012).
[77] A. Saini and D. Stojkovic, Phys. Rev. D 89, 044003 (2014).
[78] E. Greenwood and D. Stojkovic, JHEP 0806, 042 (2008).
[79] J. E. Wang, E. Greenwood and D. Stojkovic, Phys. Rev. D 80, 124027 (2009).
[80] A. Davidson, Phys. Rev. D 100, 081502 (2019).
[81] A. Davidson, Phys. Lett. B 780, 29 (2018).