Depolarization for quantum channels with higher symmetries

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Abstract

The depolarization channel is usually modelled as a quantum operation that destroys all input information, replacing it by a completely chaotic state. For qubits this has a quite intuitive interpretation as a shrinking of the Bloch sphere. We propose a way of dealing with depolarizing dynamics (in the Markov approximation) for systems with arbitrary symmetries.

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Quantum systems evolve via unitary operations determined by the Schrödinger equation. This is also true for composite systems, e.g. a small quantum system $S$ that is surrounded by some environment $E$, with which it interacts. However, while the evolution of the total system is described by a unitary operation $\hat{U}_{SE}(t)$, the dynamics of the system $S$ alone (obtained by tracing out the uncontrollable degrees of freedom of $E$) is, in general, no longer unitary (Breuer and Petruccione 2007, Alicki and Lendi 2007, Weiss 2008). The system–environment interaction leads to entanglement between them, and this is reflected in the fact that $\hat{U}_{SE}(t) \neq \hat{U}_S(t) \otimes \hat{U}_E(t)$. From the perspective of quantum information processing, such an interaction is undesirable and causes errors and noise in the system (Nielsen and Chuang 2000).

The dynamics of $S$ can be described by a finite-time trace-preserving completely positive map (CPM) (Stinespring 1955, Choi 1972) that transforms input states $\hat{\rho}_m \mapsto \hat{\rho}_m = \hat{\rho}(0)$ into output states $\hat{\rho}_o = \hat{\rho}(t)$, i.e.

$$\hat{\rho}_m \mapsto \hat{\rho}_o = \mathcal{E}_t(\hat{\rho}_m),$$

which is also known as a quantum channel. One may think of the environment as extracting information from the system, as it typically maps pure states into mixed states.

This noise process can also be described by a quantum operation involving only operators of the system of interest. This is called a Kraus decomposition (Kraus 1983) and has the form (omitting all the unnecessary subscripts)

$$\mathcal{E}(\hat{\rho}) = \sum_r \hat{K}_r \hat{\rho} \hat{K}_r^\dagger,$$

where the Kraus operators satisfy the condition

$$\sum_r \hat{K}_r^\dagger \hat{K}_r = \hat{1},$$

which ensures that unit trace is preserved for all times.

A channel is Markovian when the coupling of the system $S$ with the environment $E$ can be treated under the Markov and Born approximations. The evolution of the state at a given instant is then fully determined by the state at that instant; hence the process has no ‘memory’ of its past. This is a commonly used approximation in quantum optics and leads to the well-known Lindblad form of a master equation (Lindblad 1976, Gorini et al 1976). For these Markovian channels, one can always write

$$\mathcal{E}_t(\hat{\rho}) = \exp(\mathcal{L}_t) \hat{\rho}(0),$$

where the Lindblad superoperator $\mathcal{L}$ is

$$\mathcal{L}(\hat{\rho}) = -i[H, \hat{\rho}] + \frac{1}{2} \sum_r a_r (\hat{L}_r \hat{\rho} \hat{L}_r^\dagger + \hat{L}_r^\dagger \hat{L}_r \hat{\rho}) + [\hat{L}_r \hat{\rho} \hat{L}_r^\dagger].$$

Here $\hat{H}$ is the Hamiltonian of the undamped system $S$, $\hat{L}_r$ is the system operator defined to model the effective dissipative interaction with the environment, and $a_r \geq 0$ is a constant that accounts for decoherence rates. In this form, $\mathcal{L}$ appears as the generator of a CPM. In fact, expanding equation (5) to first order in the short-time interval $\tau$, one can immediately find the corresponding Kraus operators (Shabani and Lidar 2005)

$$\hat{K}_0 = 1 - \tau \left( \frac{i}{\hbar} + \frac{1}{2} \sum_r \hat{L}_r^\dagger \hat{L}_r \right),$$

$$\hat{K}_r = \sqrt{\tau} \hat{L}_r.$$
For obvious reasons, the characterization and classification of these maps have attracted a great deal of interest in recent years (Keyl 2002). The majority of the results obtained so far relate to two specific classes: the qubit channels and the bosonic Gaussian channels (Caruso and Giovannetti 2007). In this paper, we shall be mainly concerned with the former.

In classical computation, the only error that can occur is the bit flip 0 ↔ 1. In quantum computation, however, the existence of superposition states also brings the possibility of other basic errors for a single qubit. They are the phase flip and the bit–phase flip. The first changes the phase of the state, and the latter combines phase and bit flips. The set of Kraus operators for each one of these channels is given by

\[ \hat{K}_0 = \sqrt{1 - p/2} \hat{1}, \quad \hat{K}_1 = \sqrt{p/2} \hat{\sigma}_j, \]

where \( j = x \) gives the bit flip, \( j = z \) the phase flip and \( j = y \) the phase–bit flip. They can also be interpreted as corresponding to a probability \( 1 - p/2 \) of remaining in the same state, and a probability \( p/2 \) of having an error. It is not difficult to represent these channels in terms of a superoperator

\[ \dot{\rho} = \Gamma (\dot{\rho} - \frac{1}{2} \hat{1} \otimes \hat{1}), \]

while the associated Lindblad equation is

\[ \dot{\rho} = -\Gamma \left( \hat{s} - \frac{1}{2} \hat{1} \right), \]

where we have omitted the free evolution of the system, since it is irrelevant for our purposes here, and \( \Gamma \) is a constant. If we use the standard Bloch parametrization for the density matrix \( \dot{\rho} = (\hat{1} + \hat{s} \cdot \hat{\sigma})/2 \), we immediately get

\[ \dot{s} = -\Gamma s, \]

which clearly shows that the action of the channel is to contract the sphere with a lifetime \( \Gamma^{-1} \). This is precisely the idea behind depolarization: for long times the system will end in a fully depolarized or chaotic state, whose density matrix is diagonal \( \dot{\rho}_{\text{unpol}} = \frac{1}{2} \hat{1} \).

For \( n \) qubits, a possible extension is to assume that the error operators can be represented by the \( n \)-qubit Pauli group (Nielsen and Chuang 2000) \( \mathcal{P}_n = \{ \hat{1}, \hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z \} \otimes^n \), where \( \otimes^n \) denotes the \( n \)-fold tensor product. This means that each qubit is acted by the identical independent depolarizing channels. However, alternative models, such as collective or correlated depolarization, have been proposed (Reina et al 2002, Banaszek et al 2004, Ball et al 2004, Ball and Banaszek 2005, Ban and Shibata 2006), which do not fit in this simple extension. The point we want to stress is that the final result depends obviously on the symmetry of the channel and this, in turn, depends on physical considerations.

The purpose of this work is to address this problem and identify the proper depolarizing channel for more general situations. To this end, consider a system whose Hamiltonian is a function of a symmetry Lie algebra \( \mathfrak{A} \). In order to keep the discussion as simple as possible, we assume that this algebra is semisimple and denote by \( \Delta \) the set of nonzero roots (Erdmann and Wildon 2006). We use the standard Cartesian–Weyl basis \( \{ \hat{h}_i, \hat{\epsilon}_a \} \) in terms of which we have the commutation relations

\[ \{ \hat{h}_i, \hat{h}_j \} = 0, \quad [\hat{h}_i, \hat{\epsilon}_a] = \alpha(\hat{h}_i)\hat{\epsilon}_a, \quad [\hat{\epsilon}_a, \hat{\epsilon}_b] = N_{ab}\hat{\epsilon}_{a+b}. \]

where the last one is valid only when \( a + b \in \Delta \). The operators \( \{ \hat{h}_i \} \) (i runs from 1 to \( \ell \), where \( \ell \) is the rank of the group) constitute the Cartan subalgebra and may be taken diagonal in any irreducible representation. On the other hand, \( \{ \hat{\epsilon}_a, \hat{\epsilon}_{-a} \} \) are raising and lowering operators and we can always choose \( \hat{\epsilon}_a = \hat{\epsilon}_{-a} \).

The Hilbert space decomposes into finite-dimensional subspaces \( \mathcal{H} = \bigoplus \mathcal{H}_i \) so that in each \( \mathcal{H}_i \), the operators of \( \mathfrak{A} \) act irreducibly. Let \( \{ \hat{h}; \lambda \} \) be an orthonormal basis in \( \mathcal{H}_i \), where \( \hat{h} = (h_1, \ldots, h_j, \ldots, h_{\ell}) \). Then, we have that

\[ \hat{h}_j |\hat{h}; \lambda \rangle = h_j |\hat{h}; \lambda \rangle. \]

Notice also that the density operator of any quantum unpolarized state can be written in terms of these invariant subspaces

\[ \hat{\rho}_{\text{unpol}} = \bigoplus \rho_{\lambda} \hat{1}_\lambda, \]

where \( \hat{1}_\lambda \) denotes the unity in the corresponding subspace and all the coefficients \( r_{\lambda} \) are real and nonnegative and fulfill the unit-trace condition.

Since all the error operators are in the algebra \( \mathfrak{A} \), the only possible Kraus operators are elements of \( \mathfrak{A} \), and they induce a local Lindblad equation fully analogous to (5). Very different processes can be described in terms of this master equation. The first one can be called a pure dephasing channel, represented by

\[ \dot{L}_r = \hat{h}_r. \]

Such a map obviously preserves the diagonal operators, \( L(\hat{h}_r) = 0 \), and the asymptotic limit of the channel is given by

\[ \lim_{n \to \infty} \rho^n(\hat{\rho}) = \sum_{\hat{h}, \lambda} \phi(h; |\hat{h}; \lambda \rangle \langle \hat{h}; \lambda \rangle, \]

where \( \phi(h; \lambda) \) are just precisely the occupation probabilities in each invariant subspace. For the case of a symmetry algebra
SU(2), generated by \{\hat{\mathbb{S}}_x, \hat{\mathbb{S}}_y, \hat{\mathbb{S}}_z\}, the archetypal example of this situation is

\[ \mathcal{L}(\hat{\rho}) = \hat{\mathbb{S}}_z \hat{\rho} \hat{\mathbb{S}}_z - \frac{1}{2} (\hat{\mathbb{S}}_x^2 \hat{\rho} + \hat{\rho} \hat{\mathbb{S}}_x^2) = \hat{\mathbb{S}}_x \hat{\rho} \hat{\mathbb{S}}_x - \hat{\rho} , \]

which is a standard way of accounting for processes that lead to a loss of coherence without changing the level populations (Briegel and Englert 1993).

Next we consider a generalized amplitude-damping channel, for which

\[
\hat{L}_x = \hat{e}_{-\alpha} ,
\]

In the asymptotic limit, this leads to the ground state in each invariant subspace

\[
\lim_{\lambda \to \infty} E^\mu(\hat{\rho}) = \sum_{\lambda} \text{Tr}[\hat{\rho}(\lambda)] \hat{\rho}_{\lambda} = \hat{\rho}_{\text{unpol}},
\]

where \{\hat{\rho}_{\text{unpol}}\} is the lowest-weight state that is annihilated by all the lowering operators: \hat{e}_{-\alpha} |\text{unpol}\rangle = 0. It is clear that now no Lindblad (or Kraus) map preserves the diagonal operators. In particular, for SU(2) symmetry, the textbook example is

\[
\mathcal{L}(\hat{\rho}) = \hat{\mathbb{S}}_z \hat{\rho} \hat{\mathbb{S}}_z - \frac{1}{2} (\hat{\mathbb{S}}_x \hat{\mathbb{S}}_x + \hat{\mathbb{S}}_y \hat{\mathbb{S}}_y + \hat{\mathbb{S}}_z \hat{\mathbb{S}}_z) ,
\]

which is a standard model for a depolarizing channel (Agarwal 1974).

While these generalizations are more or less obvious, the corresponding one for a depolarization channel is far from trivial. Such a situation may be described by the action of a Lindblad operator proportional to \hat{e}_{-\alpha} , followed by others proportional to \hat{e}_{\alpha} . One can check that only in this way we asymptotically obtain an unpolarized state

\[
\lim_{\lambda \to \infty} E^\mu(\hat{\rho}) = \sum_{\lambda} \text{Tr}[\hat{\rho}(\lambda)] \hat{\rho}_{\lambda} = \hat{\rho}_{\text{unpol}},
\]

where the trace operation is taken in each invariant subspace. The associated Lindblad operator is

\[
\mathcal{L}(\hat{\rho}) = \frac{1}{2} \sum_{\lambda} \rho_x (2 \hat{e}_{-\alpha} \hat{\rho} \hat{e}_{-\alpha} + 2 \hat{\rho} \hat{e}_{-\alpha} \hat{e}_{-\alpha} - \{\hat{e}_{\alpha} , \hat{e}_{\alpha} \}) - \{\hat{\rho} \rho_{\text{unpol}} \}
\]

and one can check that \mathcal{L}(\hat{\rho}_{\text{unpol}}) = 0. An example of this situation for qubit systems parallels completely (22), but with the corresponding spin-like operators is

\[
\mathcal{L}(\hat{\rho}) = \hat{\mathbb{S}}_z \hat{\rho} \hat{\mathbb{S}}_z + \hat{\mathbb{S}}_x \hat{\rho} \hat{\mathbb{S}}_x - \frac{1}{2} ((\hat{\mathbb{S}}_x + \hat{\mathbb{S}}_x) \hat{\rho} \hat{\mathbb{S}}_x + \hat{\rho} \hat{\mathbb{S}}_x , \hat{\mathbb{S}}_x) .
\]

It is worth noting that such a Lindblad operator appears as a limit case of decaying into a bath at infinite temperature.

An effective depolarization channel is actually a common situation in driven dissipative systems. Consider a typical evolution equation

\[
\dot{\hat{\rho}} = -\gamma [\hat{\mathbb{S}}_x , \hat{\rho}] + \gamma [\hat{\mathbb{S}}_x \hat{\rho} \hat{\mathbb{S}}_x - \{\hat{\mathbb{S}}_x + \hat{\mathbb{S}}_x \} \hat{\rho} \hat{\mathbb{S}}_x + \hat{\rho} \hat{\mathbb{S}}_x , \hat{\mathbb{S}}_x ] ,
\]

which describes the decay (with rate \gamma ) of an externally driven (with coupling constant \kappa ) collective spin into a zero-temperature bath (Agarwal 1974). In the strong-pumping limit (\kappa \gg \gamma ), the above equation can be easily diagonalized

(Klimov and Sánchez-Soto 2000): it is enough to apply the rotation \( \hat{U} = \exp(\Im \hat{\mathbb{S}}_x / 2) \) and to make the rotating wave approximation. The final result is

\[
\dot{\hat{\rho}} = -\gamma [\hat{\mathbb{S}}_x , \hat{\rho}] + \frac{\gamma}{2} (2 \hat{\mathbb{S}}_x \hat{\rho} \hat{\mathbb{S}}_x - \hat{\rho} \hat{\mathbb{S}}_x + \hat{\mathbb{S}}_x \hat{\rho} \hat{\rho} \hat{\mathbb{S}}_x + \hat{\rho} \hat{\mathbb{S}}_x , \hat{\mathbb{S}}_x ) ,
\]

where \( \hat{\rho} = \hat{U} \hat{\rho} \hat{U}^\dagger \) is the density matrix in the rotated frame. We can clearly observe the emergence of a pure dephasing (first line) and a depolarizing channel (second line).

In summary, what we expect to have accomplished in this paper is to provide a construction of the depolarizing channel for systems with arbitrary symmetries. This may be more than an academic curiosity for more involved systems currently under investigation as candidates for quantum information processing.

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