On hybrid states of two and three level atoms

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Abstract We calculate atom-photon resonances in the Wigner-Weisskopf model, admitting two photons and choosing a particular coupling function. We also present a rough description of the set of resonances in a model for a three-level atom coupled to a scalar-photon field. We give a general picture of matter-field resonances these results fit into.

Subject area: Quantum Field Theory
I. Introduction

Last years there has been a renewed interest for atom-radiation states in non-relativistic quantum electrodynamics. After the rigorous proof of the existence, for small coupling constants, of resonances (singularities of an analytic continuation of the Hamiltonian resolvent), resonances coming from the naked-atom-Hamiltonian eigenvalues, many studies have been concentrated on the study of the Hamiltonian spectrum and more particularly on the existence of the fundamental state in various models, for arbitrary coupling constants. This latter question is a non trivial problem since it has long been known that negative Hamiltonian eigenvalues may appear when the coupling constant increases. In preceding works (see Ref. 10 and references therein), for our part, we were interested in a method for calculating resonances for arbitrary values of the coupling constant. It is a way of catching the above mentioned particular eigenvalues. The result we obtained can be generally stated as follows. The coupling of a discrete-level system $S$ to a zero-mass field does not only shift the level energies into the complex plane. Certainly these energies become resonances. But the coupling also creates other poles of the resolvent (or of its continuations), which have to be placed on the same footing as the preceding ones. For certain values of the parameters of $S$, or certain values of the ($S$+field) coupling constant, these latter poles may be eigenvalues of the coupled-system Hamiltonian. (In the paper, the word resonance will refer to such a pole or, by extension, to an eigenvalue, when there is an obvious possible continuous transition from the one to the other.)

The presence of these poles is well-known for an atom or a molecule in an environment in which the only emitted or absorbed photons are monochromatic (lasers or cavities). Indeed, for a two-level atom with energies $e_0$ and $e_1$, and photons with energy $e_1 - e_0$, the upper level is split into two levels by the coupling, when at most one photon is considered (vacuum-field Rabi splitting). The coupling increases the number of Hamiltonian eigenvalues because it splits the degeneracy of each of the eigenvalues $e_1, 2e_1 - e_0, \cdots, e_1 + n(e_1 - e_0), \cdots$ of the uncoupled atom-photon system. This phenomenon occurs in different but analogous situations: for instance the coupling of an exciton to the mode of a cavity, and it can be detected by spectroscopic means. It is also present in electron-phonon interactions. Let us note that the final number of atom-photon states is simply a consequence of the combination of discrete atom states with discrete photon-states.

Many papers have studied the coupling of a two-level system to another system which has either discrete levels or a continuum of levels, often focusing on the continuous transition from the vacuum Rabi splitting to the Fermi’s golden rule. In the present work, a splitting is exhibited also in the continuum case. The eigenstates of the naked atom we are used to are thus but a small part of all the possible states.

In fact, the appearance of new resonances when the system $S$ is coupled to a field is a general phenomenon whose explanation is given in Section II. It is, of course, not due to the smallness of the photon-state width. A numerical method, explained and illustrated in previous publications, enabled us to study and calculate these resonances in various simple models: $S$ was either a harmonic oscillator or a two-level atom. But in this latter case, we almost always limited ourselves to considering physical states with at most one photon.

A reason to carry on with the study is that the new resonances appearing with the
coupling might play a role for systems having a large spatial extension: for instance Rydberg atoms or, a case maybe more important, molecular orbitals of big molecules. Indeed, in a very simplified model for an interaction of $S$ with the field, a model in which the spatial extension $\delta$ of $S$ may be varied ($S$ is a charged harmonic oscillator whose mass and spring constant may be varied), we noted that some among the poles we speak about come to the negative real axis when $\delta$ gets large enough\textsuperscript{18}. These poles then correspond to stable states and are therefore important states. Their wave function can be written: it is a mixing of electron and photon states. Analogous phenomena may be expected for more complex extended systems.

In this perspective, it is important to be able to start the study of the resonances in the two following situations: a system with more than two levels and, in the case of two levels, a system with several (non monochromatic) photons. Section III deals with this second question. To the best of our knowledge, it is tackled in the literature only when the fundamental state question is discussed\textsuperscript{4}. Here we calculate some mixed (or hybrid) states with several photons, or the resonances which correspond to them. Section IV tackles the same problem for three-level atoms, but since the situation is more complicated, we limit ourselves to a qualitative description of the numerous resonances. Both studies lead to the reasonable conjecture that the number of resonances should be the product of the number of atom states by the number of independent radiation states actually coupled to the atom.

II. Notation of the resonances

We need a precise notation for the different poles.

It will follow from a very general argument, which also explains why every atomic level should give rise to a double infinity of resonances. Let us consider a material quantum system $S$ whose Hamiltonian, $H_S$, has eigenstates $|0\rangle$, $|1\rangle$, $|2\rangle$, $\cdots$, with energies $e_0, e_1, e_2, \cdots$. Let us suppose that this system is coupled to the field of a massless boson, here the photon. We denote the state-space of the $(S + \text{field})$-system by $\mathcal{E}$.

Let us consider a very general form for the Hamiltonian of the coupled system:

$$H(\lambda) = H_S \otimes 1 + 1 \otimes H_{\text{rad}} + \lambda V$$ \hspace{1cm} (2.1)

where $H_{\text{rad}}$ is the energy operator for the photon field and $\lambda V$ represents the coupling of $S$ to the field. Let us introduce the auxiliary Hamiltonian.

$$H(\lambda, \mu) = H_S \otimes 1 + \mu 1 \otimes H_{\text{rad}} + \lambda V$$ \hspace{1cm} (2.2)

where $\mu$ is a parameter which may be zero or positive.

If $\mu = 0$ and $\lambda = 0$, the energy levels $e_0, e_1, \cdots$ are infinitely degenerated (in $\mathcal{E}$), as the number of photons accompanying state $|m\rangle$ may be any integer and, moreover, the dimension of the space of possible photon states is infinite. Thus the dimension of the eigenspace $\mathcal{H}_i$ associated with the eigenvalue $e_i$ of $H(0,0)$ is infinite. Let us underline the fact that the degeneracy we speak about here is different from the one we mentioned in the second paragraph of the introduction.

The idea underlying our present study, as well as the preceding ones, is to perturb $H(0,0)$ with respect to $\lambda$ and $\mu$ successively. A priori, the perturbation with respect to $\lambda$ removes
the degeneracy, although it may be only partially, in some particular cases. This leads to
the following notation.

**Notation 1** \( \zeta_{i,1}(\lambda), \zeta_{i,2}(\lambda), \cdots \) denote the \( \mathbb{N}^2 \) \((= \mathbb{N})\) eigenvalues of \( H(\lambda, 0) \) which
tend to \( e_i \) when \( \lambda \) tends to 0.

The choice of the second index, or an equivalent choice, will be described in the
different contexts.

The perturbation with respect to \( \mu \) then leads us to the following notation for the
resonances we are interested in:

**Notation 2** When we have selected, in a way which remains to be defined, one pole
among those of the resolvent of \( H(\lambda, \mu) \) which tend to \( \zeta_{i,j}(\lambda) \) when \( \mu \) tends to 0, we denote
it by \( z_{i,j}(\lambda, \mu) \).

There are two reasons for having to make a selection. The first one is that if the \( \zeta_{i,j} \)'s
remain degenerated, and this will be the case in the model of Section III (see Proposition
3.2), then the degeneracy may be removed for \( \mu \neq 0 \); in other words, for \( \mu \neq 0 \), there
may be several resonances (or eigenvalues) going to the same \( \zeta_{i,j} \) when \( \mu \) goes to 0. (See
a more precise description at the beginning of Section III.4.2.) The one we call \( z_{i,j}(\lambda, \mu) \)
will be a particular one, selected in a way which will be made precise in the context. The
second reason is that the matrix elements of \( z - H(\lambda, \mu) \) are multivalued functions of
\( z \) for \( \mu \neq 0 \). Therefore, two poles of two different determinations may have the same limit
when \( \mu \) goes to 0. However, we assume that there is only one pole tending to \( \zeta_{i,j} \) for a
given determination, once the first choice has been made. This will enable us to get the
\( z_{i,j}(\lambda, \mu) \)'s by numerical calculus, starting from their germs \( \zeta_{i,j}(\lambda) \)'s. The notations of the
poles of the different branches of the resolvent matrix-elements will be made precise later
on, by adding an upper index which refers to the determination (see Section III.4.2.3).

**III Hybrid states for a two-level atom in the Wigner-Weisskopf model**

Two resonances (distinct or not, see below) are well-known in the Wigner-Weisskopf
model, or in the Friedrichs model, which describe a two-level atom coupled to the field of a
massless scalar boson\(^4,8,9\). These resonances can be seen without considering two-photon
states. One can be associated, for small coupling constants, to the excited state of the
atom; the other one is an eigenvalue corresponding to a stable state which differs from
the unperturbed fundamental state. This eigenvalue appears when the coupling constant
gets large enough. (Up to recently, it was not clear whether this two resonances were
two occurrences of the same resonance or not. In fact this depends on the parameters of the
physical system, but in any case it can be shown that there are actually two different
resonances\(^10\), for a given value of the coupling constant). In this section we want to study
other resonances by taking into account several photons. The existence of such states
is already alluded to in Ref. 4, where Theorem 2.2 shows that the ground state of the
coupled system must take several-photon states into account. In a particular coupling, we
will construct two of these states or resonances. It is the subject of Section III.4.2. In
accordance with the general notations introduced in Section II, they will be denoted by
\( z_{0,2}(\lambda, 1) \) and \( z_{1,2}(\lambda, 1) \), tending respectively to the energies 0 and 1 of the naked atom.
The index 2 will be explained later on.
III.1 The model and some notations

The Hamiltonian of the model is the energy operator in $F$. The annihilation operator is $H$. Note that in (3.1) by the dilated function $g$. In order to have the mixed states appear readily, we introduce $H$ by $F$. For each $n$, we set $\lambda, \mu$ thus not so arbitrary as it may seem since $H$ is in some way related to the width of the coupling function. But note that the position of the peak of $g$ is also moved. This is a departure from a study which would start with monochromatic photons and enlarge the width of their spectrum little by little, without changing the position of the peak. This Hamiltonian has invariant subspaces. To describe them and, in the same way, prepare the notations for Section IV, let us set $H^{(1)}$, the space spanned by $g$.

$F^{(1)}_{n} := (H_{rad}^{(1)})^{\oplus n}$, the space of $n$ photon states, but each photon being in state $g$. $E_0 := |0\rangle \otimes F_0$, and for $n \geq 1$, $E_n := |0\rangle \otimes F_n \oplus |1\rangle \otimes F_{n-1}$, a space we call the "$n$-excitation space".

$E_n^{(1)} := |0\rangle \otimes F_n^{(1)} \oplus |1\rangle \otimes F_{n-1}$, $n \geq 1$, $E_0^{(1)} = E_0$

$E^{(1)} := \bigoplus E_n^{(1)}$, $E := \bigoplus E_n$

Lemma $E_n$ is invariant by $H(\lambda,\mu)$, for all $n \geq 0$ and all $\mu \geq 0$; $E_n^{(1)} \subset E_n$ is invariant by $H(\lambda,0)$, for all $n \geq 0$.

As we said when we introduced Notations 1 and 2 in Section II, the method for studying the resonances of $H(\lambda,\mu)$ is a numerical method in which $\mu$ takes greater and greater values, starting from 0. So we begin with giving some properties of $H(\lambda,0)$.

III.2 Construction of resonances of $H(\lambda,\mu)$ from eigenvalues of $H(\lambda,0)$. Setting up

Proposition 3.1 $H(\lambda,0) \mid_{E^{(1)}}$, the restriction of $H(\lambda,0)$ to $E^{(1)}$, has a double infinity of eigenvalues

$$
\zeta_{0,n} (\lambda) = 2^{-1}(1 - \sqrt{1 + 4n\lambda^2}), \quad \zeta_{1,n} (\lambda) = 2^{-1}(1 + \sqrt{1 + 4n\lambda^2})
$$

(3.3)

For each $n$, an eigenvector of $H(\lambda,0) \mid_{E^{(1)}}$ associated with $\zeta_{i,n} (\lambda)$ is

$$
\phi_{i,n}^{(0)} := \left(1 + n\lambda^2\zeta_{2,n}^{-2}(\lambda)\right)^{-1} \left( |1, g^{\otimes (n-1)} + \sqrt{n} \lambda \zeta_{0,n}^{-1}(\lambda) |0, g^{\otimes n}\right) \in E_n^{(1)}
$$

(3.4)
Proof. $E_n^{(1)}$, two dimensional, is invariant by $H(\lambda, 0)$. (3.3) is thus obtained through diagonalizing a two-by-two matrix. 

The choice of the first index in the notation of the eigenvalues is in accordance with the principle stated at the end of Section II. By the choice of the second index, we indicate that the eigenvector belongs to $E_n^{(1)}$.

We note that only some part of the double degeneracy mentioned in Section II is removed here. Indeed, each $\zeta_{i,n}(\lambda)$ is still degenerated, since adding an arbitrary number of photons orthogonal to $g$ to some state does not change the energy of this state. This follows from $[H(\lambda, 0), 1 \otimes c^*(h)] = 0$, when $(g, h) = 0$. Let us develop this in order to introduce some notations.

Let $g_1, g_2, \ldots$ be a basis of functions orthogonal to $g$. For $p \geq 1$, let $G_{0,p}$ be the subspace of $E$ spanned by

$$\phi_{0,n}^{(n_1, n_2, \ldots)} := \prod_{i=1}^{\infty} (1 \otimes c^*(g_i))^{n_i} \phi_{0,0}^{(0)} \in E_n$$

when $n$ varies from $p$ to infinity; $n_i$ are $k$ non-negative integers, $k$ being arbitrary and

$$\sum_{i=1}^{k} n_i = n - p.$$

In the same way, let $G_{1,p}$ be the subspace spanned by

$$\phi_{1,n}^{(n_1, n_2, \ldots)} := \prod_{i=1}^{\infty} (1 \otimes c^*(g_i))^{n_i} \phi_{1,0}^{(0)} \in E_n$$

Proposition 3.2 For generic $\lambda$-values, $G_{i,p}$ is the eigenspace of $H(\lambda, 0)$ associated with the eigenvalue $\zeta_{i,p}$, for $i = 0, 1$ and $p \geq 1$.

See hints for the proof in Appendix B1, which is devoted to the proof of an analogous property.

Let us thus note that eigenstates of $H(\lambda, 0)$ associated with $\zeta_{i,n}$ not only are not $n$-photon states but are neither necessarily $n$-excitation states. This complicates the picture of these eigenvalues and of the resonances $z_{i,n}(\lambda, \mu)$ they give rise to.

To be complete, let us still mention that 0 is an eigenvalue of $H(\lambda, 0)$, with, if $\sum n_i = n$, $n$ arbitrary, the associated eigenvectors

$$\phi_{0,0}^{(n_1, n_2, \ldots)} := \prod_{i=1}^{\infty} (1 \otimes c^*(g_i))^{n_i} |0\rangle \otimes \Omega \in E_n$$

where $\Omega$ denotes the vacuum state of the field.

Let us now turn to the $\mu$-variation, in order to define and construct resonances $z_{0,1}(\lambda, \mu)$, $z_{0,2}(\lambda, \mu), \ldots$ and $z_{1,1}(\lambda, \mu)$, $z_{1,2}(\lambda, \mu), \ldots$, from their germs $\zeta_{i,j}(\lambda)$ (for the notation, see Notation 2 in Section II).

This construction, as we said, is purely numerical for the moment. It goes step-by-step, starting from $\mu = 0$. We do not look for any existence theorem nor for a complete description, difficult to get because of the complicated structure of the set of resonances. We just
want to obtain numerical values for resonances which have not been considered up to now, and might be important.

This requires a particular choice for \( g \).

Before that, let us nevertheless give some general indications.

### III.3 General remarks about the poles of the resolvent of \( H(\lambda, \mu) \)

In front of so many resonances, it is natural to ask oneself the question: which one is the one we are used to, that is to say is there any which could be “associated” with the excited state \( |1\rangle \), for the Hamiltonian \( H(\lambda, 1) \)? Without going into details here, let us explain shortly why the issue is not simple. In particular, given a value \( \lambda_0 \) of the coupling constant, even a small one, why \( z_{1,1}(\lambda_0, \mu) \) is not necessarily the resonance we are used to. The resonance we are used to is obtained through restricting \( H(\lambda, \mu) \) to \( \mathcal{E}_1 \) and following the resonance which sits at 1 for \( \mu = 1 \) and \( \lambda = 0 \). We follow it as \( \lambda \) increases from 0 to \( \lambda_0 \). Since its position at \( \lambda = 0 \) does not depend on \( \mu \), it amounts to following the resonance along the path of Figure 1a, from its value 1 at the origin of the path.

It is the usual perturbative approach, in which there is no need to introduce \( \mu \). On the contrary, \( z_{1,1}(\lambda_0, 1) \), also defined with \( \mathcal{E}_1 \) (see III.4.1.2), is the limit of \( z_{1,1}(\lambda_0, \mu) \), when \( \mu \) increases from 0 (then \( z_{1,1}(\lambda_0, 0) = \zeta_{1,1}(\lambda_0) \)) to 1. Since \( \zeta_{1,1}(\lambda_0) \), by definition, tends to 1 when \( \lambda_0 \) tends to 0, \( z_{1,1}(\lambda_0, 1) \) is the resonance obtained through following the resonance along the path of Figure 1b, from the value 1 at the origin of this path. Now, in some examples, functions \( z_{0,1}(\lambda, \mu) \) and \( z_{1,1}(\lambda, \mu) \) are two branches of a unique analytic 2-variable function, which, even when restricted to \( \mathbb{R}^2 \), has branch points. These real branch points may lie inside the rectangle \([0, \lambda_0] \times [0, 1]\) of the \((\lambda, \mu)\) plane, so that the two previous values of the resonances at the common end \((\lambda_0, 1)\) of the two previous paths may be different. The \( \lambda \) and \( \mu \) variations do not necessarily commute\(^{10}\). (Let us note here that it is thus dangerous to move \( \lambda \) and \( \mu \) simultaneously without caution, as we did it for instance in the second paragraph of Section III of Ref. 17). This phenomenon is at the origin of the remark in parentheses in the first paragraph of Section III.

This seems to be a drawback of the introduction of parameter \( \mu \). But this introduction enables us to see resonances for the physical Hamiltonian \((\mu = 1)\) which the perturbative approach does not give so easily. Indeed, when \( \lambda \) is the only parameter and when it is varied from \( \lambda_0 \) to 0, the continuous variation of the resonance sitting at \( z_{i,1}(\lambda_0, \mu) \) for \( \lambda = \lambda_0 \) may lead to 1, but also to infinity, or to a pole of \( g \) (see Figure 2, where the pole is \(-i\)). The result depends on \( \mu \) and \( g \). But these two latter limits are too singular points to start a calculation in their neighbourhood.
Since the difficulties of a general study are due to branch points whose positions depend on \( g \), we choose a particular \( g \). We now come back to the study of the announced particular case.

**III.4 Construction of resonances of \( H(\lambda, \mu) \) from eigenvalues of \( H(\lambda, 0) \). Calculations for a particular \( g \).**

We fix the coupling constant at 0.1 and take the particular function we already used\(^{10}\):

\[
g(p) = \sqrt{\frac{2}{\pi}} \frac{p}{1 + p^2}
\]

This is a simple rational function which exhibits the type of singularity that actual coupling functions may have. (See for instance matrix elements of the interaction Hamiltonian for hydrogenic atoms in the electromagnetic field\(^{19}\))

In Section III.4.1, we recall the definition of \( z_{0,1}(\lambda, \mu) \) and \( z_{1,1}(\lambda, \mu) \) and the known formulas through which they are obtained. We give their values for various \( \mu \) and \( \lambda \). We use them later on. Then, in Section III.4.2, we define resonances \( z_{0,2}(\lambda, \mu) \) and \( z_{1,2}(\lambda, \mu) \) and give some approximate values. An evaluation of the errors is given in Appendix A.

**III.4.1 Brief review about two resonances obtained with only one photon:**

\( z_{0,1}(0.1, \mu) \) and \( z_{1,1}(0.1, \mu) \)

They are poles of \( \langle 1 \otimes \Omega \mid [z - H \mid \varepsilon_i]^{-1} \mid 1 \otimes \Omega \rangle \) or of its analytic continuation.

**III.4.1.1 Resonance \( z_{0,1}(0.1, \mu) \)**

\( z_{i,1}(0.1, \mu) \) are zeros of \( f(\lambda, \mu, \cdot) \), defined by

\[
f(\lambda, \mu, z) := z - 1 - 2\lambda^2 \int_0^\infty \frac{g(p)^2}{z - \mu p} dp \tag{3.9}
\]

in the cut plane \( \mathbb{C}/\mathbb{R}^+ \), or of its analytic continuation through the cut, clockwise\(^{9,20,21}\). For \( \mu \leq \mu_c(\lambda) := 2\lambda^2 \int_0^\infty \frac{g(p)^2}{p} dp \), \( f(\lambda, \mu, \cdot) \) has only one zero; it is on the negative real axis and is 0 if and only if \( \mu = \mu_c(\lambda) \). When \( \mu \) tends to 0, it tends to \( \zeta_{0,1}(0.1) \) and therefore we denote it by \( z_{0,1}(0.1, \mu) \). We recall that the corresponding normed eigenvector, which we denote by \( \psi_{0,1}^\lambda(\lambda, \mu) \), is proportional to

\[
\psi_{0,1}^\lambda(\lambda, \mu) := \langle 1, \Omega \rangle + \lambda \langle 0, \frac{g}{z_{0,1}(\lambda, \mu) - \mu} \rangle \tag{3.10}
\]

For \( \mu > \mu_c(\lambda) \), \( z_{0,1}(\lambda, \cdot) \) is defined as a zero of the analytic continuation

\[
f_+(\lambda, \mu, z) := z - 1 - 2\lambda^2 \int_0^\infty \frac{g(p)^2}{z - \mu p} dp + 4i\pi \frac{\lambda^2 g(z)^2}{\mu} \tag{3.11}
\]

of \( f(\lambda, \mu, \cdot) \), clockwise through the cut. When \( \mu \to \mu_c(\lambda) \), it connects to the values for \( \mu < \mu_c(\lambda) \). For \( \mu \) varying from 0 to 2, some values of \( z_{0,1}(0.1, \mu) \) are given in Table 1 and Figure 1.

| \( 10^4 \mu \) | 0   | 0.1 | 1   | 3   | 6   | 6.2  | 6.36 | 6.366 |
|-----------------|-----|-----|-----|-----|-----|------|------|-------|
| \( 10^4 z_{0,1}(0.1, \mu) \) | -99 | -94 | -68 | -34 | -2.5| -1.1 | 0.04 | 0.001 |

Table 1. Values of \( z_{0,1}(0.1, \mu) \), for some values of \( \mu \) below \( \mu_c(0.1) \).
Figure 2. The resonance \( z_{0,1}(0.1, \mu) \), for \( \mu \in [0, 2] \), first in \( \mathbb{R}^- \), then in the second sheet of the complex plane.

The physical value for \( \mu = 1 \) is \( 0.11 - 0.95 \, i \). For \( \mu = 0 \), \( z_{0,1}(0.1, 0) = \zeta_{0,1}(0.1) \simeq -0.0099 \); the graph goes through 0 for \( \mu \simeq 6.36 \times 10^{-3} \) and \( z_{0,1}(0.1, 2) = 0.13 - 1.97 \, i \).

In order to connect these results to the usual perturbative treatment, we drew a dashed line in Figure 2. It describes the continuous move of the resonance which sits at the point \( z_{0,1}(0.1, 1) \) for \( \lambda = 0.1 \), when \( \lambda \) decreases from this value to 0. One can see that the limit is not 1, the energy of the naked excited state, but \(-i\), a pole of \( g \). This makes a difference with what occurs for \( z_{1,1}(\lambda, 1) \), as we will see just below. To distinguish the two behaviours, we called a resonance which, as a function of the only \( \lambda \) variable, does not tend to the excited state energy when \( \lambda \) tends to 0 a “non standard” resonance\(^{10}\). Be careful that a resonance may be standard for some \( \mu \)'s and non standard for others.

**III.4.1.2 Resonance** \( z_{1,1}(0.1, \mu) \)

It is another zero of \( f_+(\lambda, \mu, .) \), which tends to \( \zeta_{1,1}(0.1) \), when \( \mu \to 0 \). Its displacement when \( \mu \) varies is given by the full line of Figure 3.

Figure 3. \( z_{1,1}(0.1, \mu) \), in the second sheet, for \( \mu \in [0, 2] \) (full line)

We have \( z_{1,1}(0.1, 0) = 1.0099 \), \( z_{1,1}(0.1, 1) = 0.997 - 0.010 \, i \), \( z_{1,1}(0.1, 2) = 0.995 - 0.0032 \, i \). It can be shown that the resonance which sits at point \( z_{1,1}(\lambda, 1) \) for \( \lambda = 0.1 \) moves continuously to 1 along the dashed line of Figure 3 when \( \lambda \) goes to 0. Therefore, referring to the discussion of Section III.3, we can say that \( z_{1,1}(0.1, 1) \) is the resonance usually associated with the excited state.
We now come to the original part of this Section III.

III.4.2 Two other resonances, obtained with two photons

Among the various resonances tending to $\zeta_{i,2}(\lambda)$ when $\mu$ goes to 0, we select poles of matrix elements of $[z - H(\lambda, \mu) |_{E_2}]^{-1}$, or of one of their analytic continuation. Note that another choice among those mentioned after Notation 2 would have been to look for poles of matrix elements of $[z - H(\lambda, \mu) |_{E_3}]^{-1}$, since we have eigenvectors of $H(\lambda, 0)$ in $E_3$ associated with the same eigenvalue $\zeta_{i,2}(\lambda)$: for instance $(1 \otimes c^\ast(g_1))\phi_{i,2}$. But we make no claim to being complete. We just want to give an example of two resonances which are not considered usually.

Two new difficulties now appear. The first one is that, contrary to the $E_1$-sector case, resonances in the $E_2$-sector are given by zeros of a function $D(\lambda, \mu, \cdot)$ which is no longer explicit, since we will see that it is the sum of a series. Up to now, we can only calculate these zeros approximately, through cutting the series after the first non trivial term. The zeros of this truncated function are the approximate values we consider, for the resonances in this sector.

The second difficulty is that this truncated function has several branch points; as a consequence, it has several analytic continuations. There is no reason why it should not be the same for $D(\lambda, \mu, \cdot)$ itself, although it is possible. Therefore, when considering a zero, one must tell which branch is in question; we give a notation for the zeros of the various branches later on.

III.4.2.1 These resonances as zeros of a function $D(\lambda, \mu, \cdot)$

In $E_2$, looking for the eigenvalues leads to the following proposition.

**Proposition 3.3** (Eigenvalues of $H(\lambda, \mu) |_{E_2}$ as zeros of a multivalued function) $\lambda$ being fixed and $\mu$ and $z$ being two parameters satisfying $\mu > 0$ and $z \in \mathbb{R}^-$, let $D_{\mu, z}$ be the Fredholm function (see Ref. 22, p. 68) associated with the integral equation

$$
\varphi_{\lambda, \mu}(p) - \lambda^2 \int K_{\mu, z}(p, q) \varphi_{\lambda, \mu}(q) \, dq = 0
$$

(3.12)

with

$$
K_{\mu, z}(p, q) = \frac{g(p) \overline{\varphi}(q)}{(z - 1 - \mu |p| - \lambda^2 (T_{\mu, z}g)(p))(z - \mu(|p| + |q|))}
$$

(3.13)

where

$$
(T_{\mu, z}f)(p) = \int \frac{\overline{\varphi}(q')f(q')}{z - \mu(|p| + |q'|)} \, dq'
$$

(3.14)

Let $z \to D(\lambda, \mu, z)$ be the multivalued function which, for $z \in \mathbb{R}^-$, equals $D_{\mu, z}(\lambda^2)$.

In the case where one of the zeros of $D(\lambda, \mu, \cdot)$ is real negative, this zero is an eigenvalue of $H(\lambda, \mu)$. Let us denote it by $\xi_{0,2}(\lambda, \mu)$. The associated eigenvector is in $E_2$, proportional to

$$
\psi_{0,2}^{(0)}(\lambda, \mu) := |1, \varphi_{\lambda, \mu}| + \sqrt{2} \lambda |0, \frac{g \vee \varphi_{\lambda, \mu}}{\xi_{0,2}(\lambda, \mu) - \mu(|p_1| + |p_2|)}
$$

(3.15)

where $\varphi_{\lambda, \mu}$ is a solution of

$$
\varphi_{\lambda, \mu}(p) - \lambda^2 \int K_{\mu, \xi_{0,2}(\lambda, \mu)}(p, q) \varphi_{\lambda, \mu}(q) \, dq = 0
$$

(3.16)
Proof: \( |1, \varphi_{\lambda, \mu} \rangle + |0, \chi_{\lambda, \mu} \rangle \) is an eigenvector of \( H(\lambda, \mu) \rvert_{\mathcal{E}_2} \) associated with the eigenvalue \( z \) if and only if (3.12) holds and \( \chi_{\lambda, \mu} = \sqrt{2} \lambda (z - \mu (|p_1| + |p_2|))^{-1} g \vee \varphi_{\lambda, \mu} \). According to Fredholm’s theory, (3.12) has a non trivial solution only if \( D_{\mu, z}(\lambda^2) = 0 \). The Proposition follows from the fact that (3.14) and (3.13) are defined if \( z \) is a negative number.

Remark: If \( \mu \) is set to 0 in (3.12), (3.13) et (3.14), one finds that (3.12) implies \( z = \zeta_{0,2}(\lambda) \). If the continuity with respect to \( \mu \) could be proved, we would get that the limit of \( \xi_{0,2}(\lambda, \mu) \) when \( \mu \) goes to 0 is \( \zeta_{0,2}(\lambda) \). We do not know how to prove this at the moment, since we do not know all the zeros of \( D(\lambda, \mu, .) \). Nevertheless, the result of the approximate calculation of Section III.4.2.4 is along this line. That is why we eventually change notation \( \xi_{0,2}(\lambda, \mu) \) for \( z_{0,2}(\lambda, \mu) \), according to Notation 2. (The upper index is explained later on.)

To switch from eigenvalues to resonances, let us take \( h \in L^2(\mathbb{R}^2) \), \( z \) s.t. \( \Im z > 0 \) and \( \psi := |1, h \rangle \). Let us introduce \( H_{\mu, z}(p, q, \lambda^2) \), the resolvent kernel of equation (3.12) (see Ref. 22, p. 63). It can be shown that

\[
(\psi, [z - H(\lambda, \mu)]^{-1} \psi) = 1 - \lambda^2 \int H_{\mu, z}(p, q, \lambda^2) h(p) h(q) \, dp dq
\]

(3.17)

We know (Ref. 22, p. 58 and 63) that the only singular points of \( H_{\mu, z}(p, q, .) \) are the solutions of \( D_{\mu, z}(\lambda) = 0 \). The zeros of analytic continuations of function \( D(\lambda, \mu, .) \) of Proposition 3.3 will thus give us poles of the left-hand side of (3.17), that is to say resonances. The calculation will be an approximate one. The result for \( \mu = 1 \) is given by lines two and four of Table 3, in Section III.5.

### III.4.2.2 Zeros of \( D(\lambda, \mu, .) \) approached by zeros of a function \( D^{(1)}(\lambda, \mu, .) \)

#### Proposition 3.4
For \( z < 0 \), \( D(\lambda, \mu, z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} C_n(\lambda, \mu, z) \) with \( C_0 = 1 \),

\[
C_n(\lambda, \mu, z) = \lambda^{2n} \int D_n(\mu, z, p_1, \cdots, p_n) \, dp_1 \cdots dp_n
\]

(3.18)

and, for \( 1 \leq n \leq 3 \),

\[
D_n(\mu, z, p_1, \cdots, p_n) = \mu^{-n} \prod_{i=1}^{i=n} \frac{|g(p_i)|^2}{f(\lambda, \mu, z - \mu |p_i|)(z - 2|p_i|) \prod_{i<j} (\frac{z}{\mu} - (|p_i| + |p_j|))^2}
\]

(3.19)

Proof: The \( D_n \)'s are determinants given by Fredholm’s theory. We calculated them for \( n \leq 3 \); the result is given in (3.19).
search for zeros of \( D(\lambda, \mu, z) \) by the search for the zeros of the sum of the first two terms of the series:

\[
D^{(1)}(\lambda, \mu, z) := 1 - C_1(\lambda, \mu, z) \tag{3.20}
\]

The first corrections to this approximation are discussed in Appendix A. From Proposition 3.3, we get, if \( \mu \neq 0 \),

\[
D^{(1)}(\lambda, \mu, z) = 1 - \frac{2\lambda^2}{\mu} \int_0^\infty \psi(\lambda, \mu, z, q) \, dq \tag{3.21}
\]

where

\[
\psi(\lambda, \mu, z, q) := \frac{g(\mu^{-1}q)^2}{(z - 2|q|) \, f(\lambda, \mu, z - |q|)} \tag{3.21'}
\]

and \( D^{(1)}(\lambda, 0, z) = \frac{z(z-1) - 2\lambda^2}{z(z-1) - \lambda^2} \).

**III.4.2.3 Analytic properties of \( D^{(1)} \)**

**Proposition 3.5** For \( \mu \neq 0 \), \( D^{(1)}(\lambda, \mu, .) \) has at least three branch points: 0, \( z_{0,1}(\lambda, \mu) \) and \( z_{1,1}(\lambda, \mu) \).

**Proof** Let us recall that \( f(\lambda, \mu, .) \) is analytic in the complex plane cut along \( \mathbb{R}^+ \), with a branch point at 0. The analytic continuation from the upper half-plane has a pole at \( -i\mu \). An explicit expression of \( f \) in the upper half-plane is, for \( z \neq i\mu \),

\[
f(\lambda, \mu, z) = z - 1 + \lambda^2 \frac{2\mu^3 + \mu^2\pi z + 2\mu z^2 - \pi z^3 + 4\mu z^2(\log \mu - \log(-z))}{\pi(\mu^2 + z^2)^2} \tag{3.22}
\]

The continuation clockwise across the cut, which we denote by \( f_+(\lambda, \mu, z) \), is obtained through adding \( 4i\pi \frac{\lambda^2}{\mu} g(\mu^{-1}z)^2 \) to the above expression. It is convenient to rather introduce the function \( \tilde{f}(\lambda, \mu, z) \) which coincides with \( f(\lambda, \mu, z) \) in the upper half-plane and has a cut along \( i\mathbb{R}^- \).

For \( \Im z > 0 \), the poles of \( \psi(\lambda, \mu, z, .) \) (see (3.21')) in the complex plane cut along \( z + i\mathbb{R}^+ \) are \( \pm i\mu, \, z/2 \) and \( q_i(z, \mu, \lambda) := z - z_{i,1}(\lambda, \mu) \). Depending on whether \( \mu \) is smaller or greater than \( \mu_c(\lambda) \), the \( z \)-dependent poles sit at places schematically shown in Figure 4.

![Figure 4. Branch point and poles of \( \psi(\lambda, \mu, z, .) \) defined by (3.21')]
This follows from the position of \( z_{i,1}(\lambda, \mu) \), given by the curves in Figures 2 and 3. If \( \mu = \mu_c(\lambda) \), the pole \( q_0(z, \mu) \) coincides with \( z \). (In Figure 4, \( z \) has a positive real part, but it could be negative, as well.)

When \( z \) enters the lower half-plane along a path \( \gamma \) (for instance the dotted lines in Figures 7 and 8), the integration path in (3.21) may have to be deformed in order to be kept away from some of the three poles or from the branch point \( z \) of \( \psi(\lambda, \mu, z, \cdot) \). Two different paths \( \gamma \) and \( \gamma' \) will not necessarily yield the same result. For instance, if \( \gamma \) crosses \( z_{i,1}(\lambda, \mu) + \mathbb{R}^+ \), \( q_i(z, \mu) \) crosses \( \mathbb{R}^+ \); the integration path has thus to be deformed, whereas it is not the case if \( \gamma \) crosses \( z_{i,1}(\lambda, \mu) - \mathbb{R}^+ \). As a consequence, \( z_{0,1}(\lambda, \mu) \) and \( z_{1,1}(\lambda, \mu) \) are branch points.

If \( z \) comes to \( z_{0,1}(\lambda, \mu) \) (resp. \( z_{1,1}(\lambda, \mu) \)), then the pole \( q_0(z, \mu) \) (resp. \( q_1(z, \mu) \)) comes to 0 and the integral in (3.21) is singular.

To describe the various branches of \( D^{(1)} \) readily, we need a notation for the homotopy classes of paths in \( X_{\lambda, \mu} := \mathbb{C} \setminus \{0, z_{0,1}(\lambda, \mu), z_{1,1}(\lambda, \mu)\} \). (Two paths are homotopic if they can be continuously deformed into one another, in \( X_{\lambda, \mu} \).) Depending on whether \( \mu \) is smaller or greater than \( \mu_c(\lambda) \), the relative position of the three branch points is different.

We refer to Figure 5 for the definition of the fundamental class of paths \( a, a_0, a_1 \). The base point \( B \) is chosen real and smaller than \( \zeta_{0,2}(\lambda) \).

The path \( \gamma_0(z) \) is defined as the polyline going through points \( B, B + i\epsilon \), \( \mathbb{R}(z) + i\epsilon \) and ending at \( z \), for an arbitrary \( \epsilon > 0 \). Every path \( \gamma \) from \( B \) to \( z \) can be expressed by means of \( a, a_0, a_1 \) and \( \gamma_0(z) \). It defines a homotopy class \( [\gamma_0(z)]^{-1}\gamma \in \pi_1(X_{\lambda, \mu}) \). Conversely, with each class \( [l] \) of \( \pi_1(X_{\lambda, \mu}) \), we can associate the homotopy class of paths \( \gamma_0(z)l \) going from \( B \) to \( z \). With each pair consisting of \( [l] \) in \( \pi_1(X_{\lambda, \mu}) \) and \( z \), we can associate the analytic continuation in \( z \) of \( D^{(1)}(\mu, \cdot, \cdot) \) along the path \( \gamma_0(z)l \). We denote the value at the end of the path by \( D^{(1)}_{[l]}(\mu, \cdot, \cdot) \). From now on, we will omit to mention the \( \lambda \) variable (which has been fixed to 0.1.)

We denote by \( z[l](\mu) \) a point such that \( D^{(1)}_{[l]}(\mu, z[l](\mu)) = 0 \). When \( \mu \) varies, we assume that this point varies continuously. It is denoted by \( z_{i,2}[l](\mu) \) if its limit when \( \mu \) goes to 0 is \( \zeta_i \).

These \( z_{i,2}[l](\mu) \) are approximate values for the resonances we are considering. As regards their physical meaning, we refer to the short comment in Section 3.5.

We now give some values of these functions for various \( \mu \) values.
III.4.2.4 Values of resonance $z_{0.2}^{(1)}(\mu)$

(i) $\mu < 6.3662 \times 10^{-3}$

(a) $[\ell] = 1$. For $\mu$ close to 0, we look for a zero of $D^{(1)}(\mu, \cdot)$ in the neighbourhood of $\zeta_{0.2} = -1.96 \times 10^{-2}$. $D^{(1)}(\mu, \cdot)$ is well defined by (3.21) in the neighbourhood of every negative real number. A calculation on a computer yields negative real zeros of this expression, for $\mu$ small (see Table 2, first column). When $\mu$ increases up to a certain value $\mu_{c,2}^a$, close to $\mu_c := \mu_c(10^{-1})$, the same formula still gives a negative real zero. The first column of Table 2 gives its values for $0 \leq \mu \leq 6.3662 \times 10^{-3} \simeq \mu_c$.

| $10^3 \mu$ | $10^4 z_{0.2}^{(1)}(\mu)$ | $\frac{C_2}{2}(\mu, z_{0.2}^{(1)}(\mu))$ | $\frac{M_2}{6}(\mu, z_{0.2}^{(1)}(\mu))$ | $\frac{M_4}{24}(\mu, z_{0.2}^{(1)}(\mu))$ | $\partial_z C_1(\mu, z_{0.2}^{(1)}(\mu))$ |
|-----------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 0         | $-196$          | 0               | 0               | 0               | 103.9           |
| 0.1       | $-185.5$        | $1.65 \times 10^{-3}$ | $3.95 \times 10^{-6}$ | $1.04 \times 10^{-8}$ | 105.3           |
| 1         | $-135.7$        | $13.14 \times 10^{-3}$ | $10^{-4}$       | $6.26 \times 10^{-7}$ | 116             |
| 6         | $-9.3$          | $0.099$         | $7.93 \times 10^{-4}$ | $10^{-4}$       | 143.8           |
| 6.2       | $-6.6$          | $0.106$         | $9.3 \times 10^{-3}$ | $5.2 \times 10^{-4}$ | 247.1           |
| 6.36      | $-4.4$          | $0.111$         | $1.09 \times 10^{-2}$ | $7.1 \times 10^{-4}$ | 284.2           |
| 6.3662    | $-4.328$        | $0.112$         | $1.10 \times 10^{-2}$ | $7.17 \times 10^{-4}$ | 285             |

Table 2. Values of $z_{0.2}^{(1)}(\mu)$, for some values of $\mu$ below $\mu_c$.

Through comparing Table 2 with Table 1, one sees that this zero is smaller than $z_{0.1}(\mu)$. In accordance with the notations of the end of Section III.4.2.3, and with the upper index 1 denoting the unity element in $\pi_1(X_\mu)$, this zero may then be denoted by $z_{0.2}^{(1)}(\mu)$, since the branch point $z_{0.1}(\mu)$ does not belong to the interval $Bz_{0.2}^{(1)}(\mu)$. Figure 6a shows the three cuts of $D^{(1)}(\mu, \cdot)$ for $\mu < 6.3662 \times 10^{-3}$ and this zero $z_{0.2}^{(1)}(\mu)$.

It has not been possible to determine the place of $\mu_{c,2}^a$ with respect to $\mu_c(10^{-1})$. We will come back to this point in Appendix A when we estimate the errors made in the approximation.

(b) $[\ell] = a_0$: a zero associated with another branch. Figure 6a also schematically shows another zero: $z_{0.2}^{(a_0)}(\mu)$. The notation indicates that this value is now the zero of the continuation of $D^{(1)}(\mu, \cdot)$ along the path $\gamma' := \gamma_0(z_{0.2}^{(a_0)}(\mu))$ $a_0$, a path which goes around the branch point $z_{0.1}(\mu)$. Figure 6b shows the two paths followed by $q_0(z, \mu)$ when $z$ goes along $\gamma_0(z_{0.2}^{(1)}(\mu))$ or $\gamma'$.

Figure 6. Paths defining resonances $z_{0.2}^{(1)}(\mu)$ for $\mu < 6 \times 10^{-3}$.
This \( z_{0,2}^{a_0}(\mu) \), also close to \( \zeta_{0,2} \) for \( \mu \approx 0 \), is a zero of \( D_{a_0}^{(1)}(\mu,.) \) which, for \( \Re(z) < 0 \) (and also for \( \Re(z) > 0, \Im(z) > 0 \)), reads

\[
D_{a_0}^{(1)}(\mu, z) = 1 - \frac{2\lambda^2}{\mu} \int_0^\infty \frac{g(\mu^{-1}q)^2}{(z-2|q|) f(\mu, z-|q|)} \, dq + 4i\pi \frac{\lambda^2}{\mu} \frac{g(q_0(z,\mu)/\mu)^2}{(z-2q_0(z,\mu)) \partial_z f(z_{0,1}(\mu,\lambda), \mu, \lambda)}
\]

with \( \lambda = 0.1 \). This zero is no longer real.

(ii) For \( \mu \approx \mu_c \), things are not clear.

(iii) For \( \mu > 7 \cdot 10^{-3} \), the branch point \( z_{0,1} \) of \( D^{(1)} \) is in \( \{ z; \Re(z) > 0, \Im(z) < 0 \} \) (see Figure 2); it is shown in Figure 7. Among the various analytic continuations of \( D^{(1)} \) across \( \Re^+ \), we consider \( D_1^{(1)} \). Expressions are given below (formula (3.24) and the two last lines of the section). The zero of \( D_1^{(1)} \), denoted by \( z_{1,2}^{1}(\mu) \), follows the curve of Figure 7 when \( \mu \) varies from \( 7 \cdot 10^{-3} \) to 1. For \( \mu = 1 \), the way \( D^{(1)} \) is analytically continued to the zero of \( D_1^{(1)} \) is shown by the dotted line.

![Figure 7. \( z_{1,2}^{1}(\mu) \), in the complex plane, for \( \mu \in [0.007, 1] \)](image)

For \( \mu = 7 \cdot 10^{-3} \), \( z_{1,2}^{1}(\mu) = 2.8 \cdot 10^{-4} - 2.4 \cdot 10^{-5} i \), close to 0. Values of \( z_{1,2}^{1}(\mu) \) for \( \mu \) between \( 0.37 \cdot 10^{-3} \) and \( 7 \cdot 10^{-3} \) are difficult to get. Actually, their determination is not useful since \( z_{1,2}^{1}(\mu) \) is only an approximate value of the exact resonance. We have \( z_{1,2}^{1}(0.1,1) = 0.216 - 1.9 i \). For \( \Im z < 0, \Re(z) > 0 \) and \( \mu > 7.35 \cdot 10^{-3} \), still with \( \lambda = 0.1 \),

\[
D_1^{(1)}(\mu, z) = 1 - \frac{2\lambda^2}{\mu} \left( \int_{[0,\Re(z)] \cup [\Re(z),z]} \frac{g^2(\mu^{-1}q)}{(z-2q) f(\mu, z-q)} + \int_{[0,\Re(z)] \cup [\Re(z),z]} \frac{g^2(\mu^{-1}q)}{(z-2q) f(\mu, z-q)} + i\pi g^2(\frac{z}{2\mu}) f(\mu, z/2)^{-1} + 2i\pi g^2(\frac{z-z_{0,1}(\mu)}{\mu}) (2z_{0,1}(\mu) - z)^{-1} \partial_z f(\mu, z_{0,1}(\mu))^{-1} \right)
\]

(3.24)

This formula is derived from (3.21) in the following way: when \( z \) enters the quadrant \( \{ z; \Re(z) > 0, \Im(z) < 0 \} \), the cut in Figure 4b drags the integration contour along, which
yields the first two terms. The last two terms come from the residues of the poles \( z/2 \) and \( q_0(z, \mu) \) which cross \( \mathbb{R}^+ \) (\( q_1(z, \mu) \) does not cross \( \mathbb{R}^+ \)). For \( 10^3 \mu \in [7, 7.3] \), the expression of \( D_1^{(1)}(\mu, z) \) does not contain the residue term at \( q_0 \) since \( \gamma_0(z_{0,1}^{(1)}(\mu)) \) does not cross \( z_{0,1}(\mu) + \mathbb{R}^+ \).

**III.4.2.5 Values of resonance \( z_{1,2}^{(1)}(\mu) \)**

It can be shown numerically that the zero of \( D_1^{(1)}(\mu, \cdot) \) which tends to \( \zeta_{1,2} \) when \( \mu \) tends to 0, denoted by \( z_{1,2}^{(1)}(\mu) \), is a zero in \( \{ z; \Re(z) > 0, \Im(z) < 0 \} \) of \( \Delta(\mu, \cdot) \), where

\[
\Delta(\mu, z) := 1 - \frac{2\lambda^2}{\mu} \left( \int_{[0, \Re(z)] \cup [\Re(z), z]} \frac{g^2(\mu - q)}{(z - 2q) f_+(\mu, z - q)} + \int_{[0, \Re(z)] \cup [\Re(z), z]} \frac{g^2(\mu - q)}{(z - 2q) f_+(\mu, z - q)} + i\pi g^2(\frac{z}{2\mu}) f_+(\mu, z/2) - 1 + 2i\pi g^2(\frac{z - z_{0,1}(\mu)}{\mu}) (2z_{0,1}(\mu) - z)^{-1} \partial_z f_+(\mu, z_{0,1}(\mu))^{-1} + 2i\pi g^2(\frac{z - z_{1,1}(\mu)}{\mu}) (2z_{1,1}(\mu) - z)^{-1} \partial_z f_+(\mu, z_{1,1}(\mu))^{-1} \right)
\]

\( \Delta(\mu, z) \) is the expression of \( D_1^{(1)}(\mu, z) \) in the neighbourhood of the considered zero, but not everywhere in the lower half-plane; for example, these two functions differ at a point \( z \) such that \( \Im(z) < \Im(z_{0,1}(\mu)) \). The variation of \( z_{1,2}^{(1)}(\mu) \) with \( \mu \) is given by Figure 8.

![Figure 8. \( z_{1,2}^{(1)}(\mu) \), in the complex plane, for \( \mu \in [3 \cdot 10^{-4}, 2] \)](image)

This resonance starts from \( 2^{-1}(1 + \sqrt{1 + 8\lambda^2}) = 1.01962 \) for \( \mu = 0 \) and goes through \( 1.043 - 1.127i \) for \( \mu = 1 \). We note that this value is much farther from the real axis than the resonance \( z_{1,1}(0.1, 1) \) of sector \( \mathcal{E}_1 \). As in Figure 7, the dotted line in Figure 8 indicates the path of the analytic continuation of \( D^{(1)} \) to the zero (of \( D_1^{(1)} \)), for \( \mu = 1 \). The origin is a branch point. The other two branch points, \( z_{0,1}(0.1, 1) \) and \( z_{1,1}(0.1, 1) \), whose values are recalled in Table 3, cannot be drawn at the scale on the figure.
### III.5 Conclusion of Section III

Table 3 gathers the four results we obtained for $\lambda = 0.1$ and $\mu = 1$. It gives four resonances among all those of the physical (i.e $\mu = 1$) Hamiltonian, when the function $g$ is given by (3.8).

| $z_{0,1}$ | $0.13 - 1.97i$ |
| $z_{0,2}$ | $0.216 - 1.9i$ |
| $z_{1,1}$ | $0.997 - 0.010i$ |
| $z_{1,2}$ | $1.043 - 1.127i$ |

Table 3. Four resonances (approximate values for two of them), for $\lambda = 0.1$ and $\mu = 1$

We see that the usual resonance $z_{1,1}$ is the only one near the real axis. However, for small values of $\mu$, this is no longer true, as it can be seen from Figures 2, 7 and 8, or Table 2. Therefore, resonances for small $\mu$ may play an important role. Now, in the harmonic-oscillator example mentioned in the introduction, what plays the role of parameter $\mu$ is $\delta^{-1}$, the inverse of the spatial extension of the states. Thus, for extended states, $\mu$ is small, and we recover our motivation for the study of all the resonances of atom-field Hamiltonians.

One should of course discuss which of these resonances have a physical meaning and make this meaning precise. One should also find actual situations in which these resonances or eigenvalues can be seen readily. Unfortunately, some more work has to be done. We think that the physical meaning should appear quite easily in the case of eigenvectors of the Hamiltonian. In $\mathcal{E}_1$ (see (3.10), Table 1 and Figure 2), we have an example of such a state. As regards the restriction of $H(\lambda, \mu)$ to $\mathcal{E}_2$, we did find approximate real values for the resonances. But we have not shown that (real) eigenvalues do exist. The study has to be carried on. Now, as pure resonances are concerned, we think that their study cannot simply be an academic question, since eigenvectors change continuously into resonances when parameters of the physical system are varied.

In any case, the 2-level model is perhaps too simple to find a concrete application. The next section is a step towards more realistic models.

### IV Hybrid states for a three-level atom coupled to photons

To be in a position to describe mixed states in more realistic models, we not only must be able to consider several photons, as in Section III, but we must also be able to consider several atomic or molecular levels. The present section is a preliminary study devoted to mixed states when $S$ is a three-level atom coupled to the radiation by a Hamiltonian of type (2.2).

The reason why the study is only a preliminary one is that there come some additional difficulty, together with those already mentioned in Section III: for $\mu \geq 0$, none of the spaces with a bounded excitation number is stable by the evolution operator, contrary to the two-level case. This is due to the possibility of the transition $|0\rangle + photon \rightarrow |2\rangle$ ($|2\rangle$ is the second excited state), through which the excitation number may increase. For this reason, the determination of the Hamiltonian eigenvalues is already not simple for $\mu = 0$. Since we assume that the structure of the resonance set will roughly be conserved when $\mu$ takes non-zero values, we must study the ($\mu = 0$)-problem first. This is the subject of this Section IV. The displacement of these resonances when $\mu$ becomes non-zero will not be examined in the paper.
A typical result is illustrated by Figure 12.

**IV.1 Notations and Hamiltonian**

The atom has three levels with energies \( e_0 = 0, e_1 \) and \( e_2 \), corresponding to states \(|0\rangle, |1\rangle \) and \(|2\rangle\). \( f_{01}, f_{12}, f_{02} \) being three normed functions in \( L^2(\mathbb{R}) \), the Hamiltonian is

\[
H(\lambda, \mu) := H_0(\lambda, \mu) + \lambda_{02} V
\]

with

\[
H_0(\lambda, \mu) := (e_1 |1\rangle\langle 1| + e_2 |2\rangle\langle 2|) \otimes 1 + \mu 1 \otimes H_{\text{rad}} + \\
\lambda_{01} \left( |1\rangle\langle 0| \otimes c(f_{01}) + |0\rangle\langle 1| \otimes c^*(f_{01}) \right) + \lambda_{12} \left( |2\rangle\langle 1| \otimes c(f_{12}) + |1\rangle\langle 2| \otimes c^*(f_{12}) \right)
\]

\[
V := |2\rangle\langle 0| \otimes c(f_{02}) + |0\rangle\langle 2| \otimes c^*(f_{02})
\]

\( H_{\text{rad}} \) is as in Section III.1. Transition \(|0\rangle \rightarrow |2\rangle\) is distinguished from \(|0\rangle \rightarrow |1\rangle\) and \(|1\rangle \rightarrow |2\rangle\) for a physical reason and also to prepare a perturbative calculus.

We use the following notations, modeled on those of Section II.

\( H_{\text{rad}}^{(2)} \): the space spanned by \( f_{01}, f_{12}, F_2^{(2)} := (H_{\text{rad}}^{(2)})^\vee \): the \( n \) photon-state space, but each photon being restricted to be in \( H_{\text{rad}}^{(2)} \) and \( F^{(2)} := \bigoplus F_n^{(2)} \).

\( H_{\text{rad}}^{(3)} \): the space spanned by \( f_{01}, f_{12}, f_{02} \), \( F_3^{(3)} := (H_{\text{rad}}^{(3)})^\vee \): each photon is restricted to be in \( H_{\text{rad}}^{(3)} \) and \( F^{(3)} := \bigoplus F_n^{(3)} \).

As regards the atom-photon system, \( \mathcal{E}_n \) and \( \mathcal{E} \) have been defined in Section III.1. We set

\[
\mathcal{E}_1^{(k)} := |0\rangle \otimes F_1^{(k)} \oplus |1\rangle \otimes F_0^{(k)}, \quad k = 2, 3
\]

\[
\mathcal{E}_n^{(k)} := |0\rangle \otimes F_n^{(k)} \oplus |1\rangle \otimes F_{n-1}^{(k)} \oplus |2\rangle \otimes F_{n-2}^{(k)}, \quad k = 2, 3, \quad n \geq 2
\]

\[
\mathcal{E}^{(k)} := \bigoplus_{n=0}^\infty \mathcal{E}_n^{(k)}
\]

\[
\lambda := (\lambda_{01}, \lambda_{12}, \lambda_{02})
\]

We also introduce \( s_0 := (f_{01}, f_{02}), s_1 := (f_{01}, f_{12}), s_2 := (f_{02}, f_{12}) \).

We will use the letter \( \phi \) to indicate eigenvectors of \( H_0(\lambda, 0) \); a priori, they depend on \( \lambda_{01} \) and \( \lambda_{12} \). We will use \( \chi \) to indicate eigenvectors of \( H(\lambda, 0) \); they also depend on \( \lambda_{02} \).

We aim at getting the eigenvalues of \( H(\lambda, 0) \), an operator which we simply write \( H(\lambda) \), from now on. When the variable \( \mu \) is not mentioned, it will be assumed to be 0.

To this end, we take up the idea mentioned in Section II consisting in perturbing \( H(0) \) through introducing the interaction step by step. The doubly infinite degeneracy of the eigenvalues \( e_0, e_1 \) and \( e_2 \) of \( H(0) \), due to the arbitrariness of the number of photons and the arbitrariness of the state of each photon in the corresponding eigenvectors, is partially removed at each step. The first perturbation will be the addition of the \( \lambda_{01} \) and \( \lambda_{12} \) terms of (4.2) to \( H(0) \). It is described in Section IV.2. The second perturbation will be the supplementary addition of \( \lambda_{02} V \). It is described in Section IV.3.
IV.2 Perturbation with respect to $\lambda_0$ and $\lambda_1$. First splitting of $e_0$, $e_1$ and $e_2$

Here we are interested in $H_0(\lambda) := H_0(\lambda,0)$; the interaction $\lambda_{02}V$ is switched off. First, in $\mathcal{E}_0$, $|0, \Omega\rangle$, is an eigenvector associated with the eigenvalue 0.

IV.2.1 Three eigenvectors of $H_0(\lambda)$ in the 1-excitation space $\mathcal{E}_1^{(2)}$ and the three associated eigenvalues

The space $\mathcal{E}_1^{(2)}$, three-dimensional, is invariant by $H_0(\lambda)$. It is the direct sum of the eigen-subspaces $|1\rangle \otimes \mathcal{F}_0^{(2)}$ and $|0\rangle \otimes \mathcal{F}_1^{(2)}$ of $H_0(0)$, associated with the eigenvalues $e_1$ and $e_0$ respectively, and dim. $\mathcal{F}_0^{(2)} = 1$ and dim. $\mathcal{F}_1^{(2)} = 2$. The first perturbation will shift $e_1$ and split $e_0$ into two eigenvalues, as it is represented in the first two columns of Figure 9.

![Diagram](image)

Figure 9. Levels associated with eigenvectors of $H_0(\lambda)$ in $\mathcal{E}_1^{(2)}$, therefore without any spectator-photon.

The third column gives the notation of the perturbed eigenvalues. In the fourth column, we recalled the above mentioned number 1 and 2, which are the dimensions of the projections of $\mathcal{E}_1^{(2)}$ on $|i\rangle \otimes \mathcal{F}^{(2)}$. They also are the sum of the dimensions of the eigenspaces of $H(\lambda) \upharpoonright_{\mathcal{E}_1^{(2)}}$ associated with eigenvalues which tend to $e_i$ when $\lambda$ tends to 0.

"Apart from degeneracy" indicates that in considering $\mathcal{E}_1^{(2)}$ we do not take account of the additional degeneracy of the eigenvalues due to photons in states orthogonal to $f_{01}$ and $f_{12}$; we call these photons spectator-photons. This degeneracy is explained in Proposition 4.2 and is removed by the second perturbation studied in Section IV.3.

The eigenvalues obtained through the first perturbation and the associated eigenvectors are given by the following proposition.

**Proposition 4.1** In $\mathcal{E}_1^{(2)}$, if $f_{01} \neq f_{12}$, $H_0(\lambda)$ has three eigenvalues:

$$\begin{align*}
(\zeta_{0,1})_1(\lambda_{01}, \lambda_{12}) &= 2^{-1}(e_1 - \sqrt{e_1^2 + 4\lambda_{01}^2}) \quad (4.4) \\
(\zeta_{0,1})_2(\lambda_{01}, \lambda_{12}) &= 0 \quad (4.5)
\end{align*}$$

(1) $|i, n\rangle$ schematically indicates a $n$-photon state with the atom in state $i$. 

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\[ \zeta_{1,1}(\lambda_{01}, \lambda_{12}) = 2^{-1}(e_1 + \sqrt{e_1^2 + 4\lambda_{01}^2}) \]

The associated normed eigenvectors are
\[
(\phi_{0,1}^{(0)})_1(\lambda_{01}, \lambda_{12}) = (1 + (\zeta_{0,1})_1^{-2}\lambda_{01}^2)^{-1/2}( | 1, \Omega \rangle + (\zeta_{0,1})_1^{-1}\lambda_{01} | 0, f_{01} \rangle ) \tag{4.7}
\]
\[
(\phi_{0,1}^{(0)})_2(\lambda_{01}, \lambda_{12}) = (1 - |s_1|^2)^{-1/2} | 0, f_{12} - s_1f_{01} \rangle. \tag{4.8}
\]
\[
\phi_{1,1}(\lambda_{01}, \lambda_{12}) = (1 + (\zeta_{1,1})^{-2}\lambda_{01}^2)^{-1/2}( | 1, \Omega \rangle + (\zeta_{1,1})^{-1}\lambda_{01} | 0, f_{01} \rangle ) \tag{4.9}
\]

The calculus is straightforward since the dimension of \( E_1^{(2)} \) is three.

To the second order with respect to \( \lambda \), we get \((\zeta_{0,1})_1 = -\frac{1}{e_1}\lambda_{01}^2 \) and \( \zeta_{1,1} = e_1 + \frac{1}{e_1}\lambda_{01}^2 \). Note that the perturbed eigenvalues (4.4) to (4.6) do not depend on \( \lambda_{12} \). They tend to \( 0, 0 \) and \( e_1 \), respectively, if \( \lambda_{01} \) tends to \( 0 \). The notation \( \zeta_i \) is thus in accordance with the rules stated in Section II, as far as the first index is concerned; index \( i \) refers to the unperturbed level in the following way: \( \lim_{\lambda_{01} \to 0}(\zeta_{0,1})_j = 0 \), \lim_{\lambda_{01} \to 0} \zeta_{1,1} = e_1 \). To distinguish perturbed eigenvalues which tend to the same \( e_i \) if \( (\lambda_{01}, \lambda_{12}) \to (0, 0) \), we chose to put the index \( p \), indicating that the eigenvectors are in \( E_p \) (as in Section III.2). Since several eigenvalues with the same indices \( i \) and \( p \) may still have the same limit when \( (\lambda_{01}, \lambda_{12}) \) tends to \( (0, 0) \), an additional index \( j \) is used to number them.

We have \((\phi_{0,1}^{(0)})_1(\lambda_{01}, \lambda_{12}) \to | 0, f_{01} \rangle \) and \( \phi_{1,1}(\lambda_{01}, \lambda_{12}) \to | 1, \Omega \rangle \), if \( \lambda_{01} \) tends to \( 0 \).

If \( f_{01} = f_{12} \), the dimension of \( E_1^{(2)} \) is two and the eigenvector \((\phi_{0,1}^{(0)})_2 \) disappears.

As we said before, these eigenvalues are actually infinitely degenerated; indeed, adding photons orthogonal to \( \{f_{01}, f_{12}\} \) gives an eigenstate with the same energy, since the energy of the photons is not taken into account in \( H_0(\lambda) \). Let us state this fact precisely, with some notations which will be useful later on.

IV.2.2 Other eigenvectors of \( H_0(\lambda) \) in the \((n+1)\)-excitation space \( E_{n+1} \), \( n > 0 \), associated with the same eigenvalues.

**Proposition 4.2** Let \( g_1, g_2, \cdots \) be an orthonormal basis of functions orthogonal to \( f_{01} \) and \( f_{12} \).

(i) Let \( \mathcal{G}_{0,1,1} \) be the subspace of \( \mathcal{E} \) spanned by the normed vectors
\[
(\phi_{0,n+1,\cdots}^{(n_1,n_2,\cdots)})_1 := \prod_{i=1}^{\infty} (n_i!)^{-\frac{1}{2}} (1 \otimes c^*(g_i))^{n_i} (\phi_{0,1}^{(0)})_1 \tag{4.10}
\]
where the \( n_i \)'s are \( k \) non-negative integers and \( k \) is arbitrary. These vectors are in \( E_{n+1} \) if \( n = \sum_{i=1}^{k} n_i \). For generic values of \( \lambda_{01} \) and \( \lambda_{12} \), \( \mathcal{G}_{0,1,1} \) is the eigenspace of \( H_0(\lambda) \) associated with eigenvalue \((\zeta_{0,1})_1 \).

(ii) If \( f_{01} \neq f_{12} \), let us set \( g_0 := (1 - |s_1|^2)^{-\frac{1}{2}} (f_{12} - s_1f_{01}) \) and let \( f_{01}^\perp \) be the subspace of \( \mathcal{H}_{rad} \) orthogonal to \( f_{01} \), spanned by the \( g_i \)'s, \( i = 0, 1, \cdots \). The eigenspace of \( H_0(\lambda) \) associated with eigenvalue \((\zeta_{0,1})_2 \) is \( | 0 \rangle \otimes \mathcal{F}(f_{01}^\perp) \), where \( \mathcal{F}(f_{01}^\perp) \) is the Fock space built with \( f_{01}^\perp \). We set
\[
(\phi_{0,n+1,\cdots}^{(n_0,n_1,\cdots)})_2 := \prod_{i=0}^{\infty} (n_i!)^{-\frac{1}{2}} (1 \otimes c^*(g_i))^{n_i} (\phi_{0,1}^{(0)})_2 \tag{4.11}
\]
(iii) Lastly, let $G_{1,1}$ be the subspace of $E$ spanned by the normed vectors

$$\phi_{1,n+1}^{(n_1,n_2,\ldots)} := \prod_{i=1}^{\infty} (n_i)!^{-\frac{1}{2}} (1 \otimes c^*(g_i))^{n_i} \phi_{1,1}^{(0)}$$

(4.12)

For generic values of $\lambda_{01}$ and $\lambda_{12}$, $G_{1,1}$ is the eigenspace of $H_0(\lambda)$ associated with eigenvalue $\zeta_{1,1}$.

That $\prod_{i=1}^{\infty} (1 \otimes c^*(g_i))^{n_i} (\phi_{0,1})_{1,1}, \prod_{i=0}^{\infty} (1 \otimes c^*(g_i))^{n_i} (\phi_{0,1})_{2,1}$ and $\prod_{i=1}^{\infty} (1 \otimes c^*(g_i))^{n_i} \phi_{1,1}^{(0)}$ are eigenvectors follows from the fact that $[1 \otimes c^*(g), H_0(\lambda)] = 0$ if $g$ is orthogonal to $H_{\text{rad}}^{(2)}$. In Appendix B1, we explain how the whole eigenspace can be determined, for each of the three eigenvalues.

Let $g$ (which is no longer function (3.8)) be a linear combination of the $g_i$'s. Proposition 4.2 may be stated in the following terms: the three eigenvalues are twice infinitely degenerated; firstly through the number of spectator-photons (the $n$ variable), and secondly through the infinity of possible states for each spectator-photon (the $g$ variable). In the level diagrams, if we symbolise the degeneracy of an eigenvalue (due to the possibility of one spectator-photon in the eigenvector) by a dotted line, then in the case where at most one spectator-photon is present, the degeneracy of the levels may be represented in the following way:

Figure 10. Levels of Figure 9, with eigenvectors of $H_0(\lambda)$ in $E_2$; at most one spectator-photon

The degeneracy of the eigenvalue is greater if we admit eigenvectors with a greater number of spectator-photons, with all their possible states.

But considering a total number of photons greater than one yields other eigenvalues. For instance, we claim that eigenvectors in $E_2$ are not necessarily of the form given in the last column of Figure 10. We are going to see that the set of perturbed eigenvalues coming from one given unperturbed energy $e_i$ changes when the number of photons coupled to the atom changes. This was already the case for the two-level atom of Section III (see for

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(1) dotted lines are slightly shifted from the level for reading purpose
instance Proposition 3.1 of Section III.2, for \( \mu = 0 \), where the eigenvalue depended on the space in which the eigenvector was looked for).

**IV.2.3 Other eigenvectors of** \( H_0(\lambda) \) **in** \( \mathcal{E}_{n+1} \) **and other eigenvalues**

In this section, we assume \( f_{01} \neq f_{12} \).

**Proposition 4.3** \( H_0(\lambda) \) **has an infinity of eigenvalues different from** \((\zeta_{0,1}), (\zeta_{0,1})_2 \text{ et } \zeta_{1,1}\).

**Proof** \( \mathcal{E}_n \) is invariant by \( H_0(\lambda) \). The subspace \( \mathcal{E}_n^{(2)} \) of \( \mathcal{E}_n \) is also invariant.

Let us first consider \( \mathcal{E}_2^{(2)} \), six-dimensional. If the coupling constants are small, there are six eigenvalues: when \( \lambda_{01} \) and \( \lambda_{12} \) tend to 0, one of them tends to \( e_2 \) (it is denoted by \( \zeta_{2,2} \)), two tend to \( e_1 \) (denoted by \((\zeta_{1,2})_1 \) and \((\zeta_{1,2})_2 \)) and three, one of which is zero, tend to 0 (denoted by \((\zeta_{0,2})_1 = 0\), \((\zeta_{0,2})_2 \) and \((\zeta_{0,2})_3 \)). They are obtained through diagonalizing a six by six matrix (see Appendix B2). The calculation is straightforward although the result has not a simple expression. In this 2-excitation space, that two eigenvalues tend to \( e_1 \) is due to the fact that there are two possible photon states, and that three eigenvalues tend to 0 is due to the fact that there are three possible independent states for the two photons. The fourth column of Figure 11 recall these numbers. The eigenvector associated with \((\zeta_{0,2})_1 = 0\) is due to the fact that there are two possible independent states for the two photons.

We already found other eigenvectors in the same space \( \mathcal{E}_2 \). They were built from eigenvectors in \( \mathcal{E}_1^{(2)} \). They are for instance \((\phi_{0,1}^{(1,0,...)})_1 \) and \((\phi_{1,1}^{(1,0,...)}) \), with notations of Proposition 4.2; they are in \( \mathcal{E}_2^{(3)} \) if \( g_1 \in \mathcal{H}^{(3)}_{\text{rad}} \). The corresponding eigenvalues, \((\zeta_{0,1})_1 \) and \( \zeta_{1,1} \), are different from the six we just saw (see Appendix B2), except possibly for particular values of \( \lambda \).

Since the notations are a bit heavy, we again represent the levels in Figure 11.

| \( e_2 \) ----- | \( \zeta_{2,2} \) | 1 | \( \phi_{2,2}^{(0)}, |2,0\rangle, |1,1\rangle, |0,2\rangle \) |
| \( e_1 \) ----- | \( (\zeta_{1,2})_2 \) | 2 | \( (\phi_{1,2}^{(0)})_2 \) " |
| \( (\zeta_{1,2})_1 \) | \( (\phi_{1,2}^{(0)})_1 \) " |
| 0 ----- | \( (\zeta_{0,2})_3 \) | 3 | \( (\phi_{0,2}^{(0)})_3 \) " |
| \( (\zeta_{0,2})_1 \) | \( (\phi_{0,2}^{(0)})_1 \) " |
| \( (\zeta_{0,2})_2 \) | \( (\phi_{0,2}^{(0)})_2 \) " |

levels of the decoupled 2-excitation eigen-field state-space eigenvectors
atom mixed levels values (apart from deg.) and their contents

Figure 11. Levels associated with eigenvectors of \( H_0(\lambda) \) in \( \mathcal{E}_2^{(2)} \), without any spectator-photon

Figures in the fourth column are the dimensions of the projections of \( \mathcal{E}_2^{(2)} \) on \( |i\rangle \otimes \mathcal{F}^{(2)} \).

The changes in the levels \( e_0, e_1 \) and \( e_2 \) of Figures 10 and 11 cannot be superimposed on one another, in general. Both are to be considered in describing the levels of \( H_0(\lambda) \).

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To complete the study, we have to take an arbitrary number of photons into account. The study of the spectrum of $H_0(\lambda)$ is completed when one has also considered eigenvectors in $\mathcal{E}_2^{(2)}$, $\mathcal{E}_4^{(2)}$, etc.; They give new levels. Eventually, there is a very great number of levels. Let $S(n, p)$ be the number of independent symmetric states that can be formed with $n$ photons, each photon being in $p$ possible states. Let us consider $1, 2, \cdots, n$ photons successively. The fundamental state is split into a doublet, a triplet, $\cdots$ a $S(n, 2)$-multiplet. If we look for all possible states, all these levels must be considered. As regards level 1, a shift, then a doublet, a triplet, $\cdots$, a $S(n - 1, 2)$-multiplet. Lastly, for level 2, we get no change, then a shift, a doublet, $\cdots$, a $S(n - 1, 2)$-multiplet. To be complete, let us recall level $e_0 = 0$, with eigenvector $|0, \Omega\rangle$.

The doubly infinite degeneracy due to spectator-photons still remains. With the six eigenvalues $\zeta_{2,2}$, $(\zeta_{1,2})_1$ and $(\zeta_{1,2})_2$, and lastly $(\zeta_{0,2})_1$, $(\zeta_{0,2})_2$ et $(\zeta_{0,2})_3$, are associated eigenvectors $\phi_{2,2}^{(0)}$, $(\phi_{1,2}^{(0)})_1$ and $(\phi_{1,2}^{(0)})_2$, and lastly $(\phi_{0,2}^{(0)})_1$, $(\phi_{0,2}^{(0)})_2$ et $(\phi_{0,2}^{(0)})_3$. As previously, through application of $(1 \otimes c^*(g))^{n_1}$ with $g$ orthogonal to $\{f_01, f_{12}\}$, or more generally application of $\prod (1 \otimes c^*(g_i))^{n_i}$, one gets other eigenvectors associated with the same eigenvalues. In the same way that we built Figure 10 from Figure 9, we could illustrate this degeneracy graphically through adding dotted lines in Figure 11.

Let us come back to our initial problem, which is to determine the spectrum of $H(\lambda)$, at least roughly. It may be expected that the perturbation $\lambda_{02}$ $V$ partially removes the degeneracy of each of the six above mentioned eigenvalues, as well as degeneracis of the same type, for instance those of the three eigenvalues $(\zeta_{0,1})_1$, $(\zeta_{0,1})_2$ and $\zeta_{1,1}$ that we obtained previously. It is this simpler question that we now examine. We are going to show that the coupling $\lambda_{02}$ $V$ splits the first three levels of Figure 10, eigenvalues $(\zeta_{0,1})_1$, $(\zeta_{0,1})_2$ and $\zeta_{1,1}$ of Proposition 4.2, into an infinity of levels and calculate the splittings of $(\zeta_{0,1})_1$ and $\zeta_{1,1}$, at the lowest order in $\lambda_{02}$.

**IV.3 Perturbation with respect to $\lambda_{02}$. Second removal of degeneracy**

We are now interested in $H(\lambda)$. The function $f_{02}$ comes into play. Hence we assume that the first vector of the basis $g_1, g_2, \cdots$ of Proposition 4.2 is in $\mathcal{H}_{\text{rad}}^{(3)}$. The $g_i$'s, $i > 1$ are thus orthogonal to $\mathcal{H}_{\text{rad}}^{(3)}$. We saw that the $(\phi_{i,p}^{(q,0,\cdots)})_k, q = 0, 1, \cdots$, which are in $\mathcal{E}^{(3)}$, are associated with a unique eigenvalue $(\zeta_{i,p})_k$. In the simple cases $(i, p) = (1, 1)$ and $(i, p, k) = (0, 1, 1)$, we are going to show that the degeneracy is removed. We calculate the approximations of order two in $\lambda_{02}$ of those eigenvalues of $H(\lambda)$ which tend to $(\zeta_{i,p})_k$ when $\lambda_{02}$ tends to 0. These eigenvalues depend on $q$. The approximations are denoted by $(z_{i,p}^{(q)})^{(\leq 2)}$. The corresponding eigenvectors are denoted by $(\chi_{i,p}^{(q)})_k$ and their 2-order approximations by $(\chi_{i,p}^{(q)})^{(\leq 2)}$. We will have

$$\lim_{\lambda_{02} \to 0} (\chi_{i,p}^{(q)})^{(\leq 2)}_k = (\phi_{i,p}^{(q,0,\cdots)})_k = (1 \otimes c^*(g_1))^{q} (\phi_{i,p}^{(0)})_k$$

(4.13)

(Here, $z$ has not the same meaning as in Section III; $\mu$ remains zero.)

Note that, if $\chi \in \mathcal{H}^{(3)}$ is an eigenvector of $H(\lambda)$, then for every $g$ orthogonal to $\mathcal{H}_{\text{rad}}^{(3)}$, $(1 \otimes (c^*(g))^n) \chi$ is still an eigenvector, associated with the same eigenvalue. This second degeneracy removal is thus only very partial.
We assume that \( p = 1 \) and we limit ourselves to perturbing \((\zeta_{0,1})_1, (\zeta_{0,1})_2\) if \( f_{01} \neq f_{12} \), and \( \zeta_{1,1} \).

If \( f_{01} \neq f_{12} \), the diagram in Figure 10 transforms into the one of Figure 12.

![Diagram](image)

Figure 12. Qualitative description of the perturbation of Figure 10 first levels, due to the \( \lambda_0 V \) term; \( (g_i = g_1) \)

The case of \((\zeta_{0,1})_2\) is particular as we will see later on.

Note that the exact eigenvector \((\chi^{(q)}_{i,\mu})_{k}\) is no longer in \( E^{(3)}_{q+1} \) but in \( \oplus_{r=0}^{\infty} E^{(3)}_{r+1} \).

More precisely, we have

**Proposition 4.4** Through the perturbation \( \lambda_0 V \), each of the eigenvalues \((\zeta_{0,1})_1(\lambda_0, \lambda_{12})\) and \( \zeta_{1,1}(\lambda_0, \lambda_{12}) \) is at least split into an infinity of eigenvalues \((\zeta^{(n)}_{0,1})_1(\lambda)\) and \( \zeta^{(n)}_{1,1}(\lambda) \), given at second order in \( \lambda_0 \) by the following formulas

\[
(i) \quad (\zeta^{(n)}_{0,1})_1^2(\lambda) = (\zeta_{0,1})_1(\lambda_0, \lambda_{12}) + \lambda_0^2 \left( 1 + n! \frac{\lambda_0^2}{(\zeta_{0,1})_1^2} \right)^{-1} A^{(n)}_{0,1,1}(\lambda_0, \lambda_{12})
\]

with, if \( f_{01} \neq f_{12},

\[
A^{(n)}_{0,1,1}(\lambda_0, \lambda_{12}) := n \frac{\lambda_0^4 |(f_{02}, g_1)|^2}{((\zeta_{0,1})_1^2) \left( \lambda_{12}^2 (\zeta_{0,1})_1 + \lambda_0^2 ((\zeta_{0,1})_1 - e_2) \right)}
\]

where \( |(f_{02}, g_1)| \) can be expressed with the \( s_i \)'s,

and, if \( f_{01} = f_{12}, \) and thus \( g_1 = (1 + |s_0|^2)^{-\frac{1}{2}}(f_{02} - s_0 f_{01}),

\[
A^{(n)}_{0,1,1}(\lambda_0, \lambda_{12}) := \frac{\lambda_0^4}{((\zeta_{0,1})_1^2) \left( \lambda_{12}^2 (\zeta_{0,1})_1 + \lambda_0^2 ((\zeta_{0,1})_1 - e_2) \right)} (n |(f_{02}, g_1)|^2 + |s_0|^2)
\]

(ii) \( (\zeta^{(n)}_{1,1})_1^2 \) is obtained through replacing \((\zeta_{0,1})_1\) by \( \zeta_{1,1} \) in expressions giving \((\zeta^{(n)}_{0,1})_1^2\).

---

(1) The residual degeneracy is not mentioned.
(iii) \((\zeta_{0,1})_2\), which is more degenerated than the two previous eigenvalues, is also split. The 2-order approximations of the perturbed eigenvalues, \((\zeta_{0,1}^{(n,m)})_2\), now depending on two indices, are obtained through the vanishing of an infinite order determinant.

The broad lines of the proof are given in Appendix B3, together with the method for calculating the corresponding eigenvectors \((\chi_{0,1}^{(n)})_1 \leq 2(\lambda)\), \((\chi_{0,1}^{(n,m)})_2 \leq 2(\lambda)\), and \((\chi_{1,1}^{(n)}) \leq 2(\lambda)\)

IV.4 Conclusion of Section IV

Section IV.3 described the splitting of the eigenvalues \((\zeta_{0,1})_1\), \((\zeta_{0,1})_2\) and \(\zeta_{1,1}\) of Proposition 4.2. It gave a small part of the spectrum of \(H(\lambda,0)\), described in the second column of Figure 12. In view of these results one may reasonably surmise the following points.

This splitting of the levels of the second column of Figure 9 (or 10) will reproduce for those of the second column of Figure 11, which are different. In other terms, the degeneracy of the latter levels, due to spectator-photons, will also be removed by the coupling \(\lambda_{02}V\). More generally, each level multiplet which was mentioned at the end of the proof of Proposition 4.3 is also split when the interaction is totally switched on. This rough description of the two level splittings we get by successively taking the two parts of the interaction Hamiltonian into account eventually yields quite a complicated spectrum for \(H(\lambda,0)\). But the reason of the great number of levels is simple; it is recalled in Section V below.

When \(\mu\) increases from 0, we expect that the eigenvalues we found move into eigenvalues or resonances of \(H(\lambda,\mu)\). The set of these eigenvalues or resonances thus likely has the same rich structure.

V General conclusion

Results of Sections III and IV fit into the same frame. They lead us to expect that the number of eigenstates or resonances of the atom-photon system is formally, apart from accidental degeneracy, the product of the dimensions of the atom state-space and the field state-space. The description is complicated because of the multivaluedness of the resolvent matrix elements, as functions of the \(\lambda\) and \(\mu\) parameters.

This picture may be illustrated through the following argument: if the energy of each level of the isolated atom is considered as an eigenvalue of \(H_{\text{atom}} \otimes 1_{\text{field}}\), this level is twice infinitely degenerated (number of photons and state of each photon). This degeneracy is removed with the actual Hamiltonian which describes the coupling of the atom to the field. The shift of the naked-atom levels by the coupling of the atom to the photon field is thus not the main feature in the change in the Hamiltonian “spectrum”. The main feature is more the emergence of numerous resonances, as in the monochromatic-photon case.

For a two-level atom and Hamiltonian (3.1) (Section III), only one photon state comes into play. The subspace of \(H_{\text{rad}}\) to be considered is \(H^{(1)}_{\text{rad}}\) and the two degeneracies which are removed, same energy for states \(|0,0\rangle\), \(|0,g\rangle,\ldots,|0,g^{\nu n}\rangle,\ldots\) on the one hand and \(|1,0\rangle\), \(|1,g\rangle,\ldots,|1,g^{\nu n}\rangle,\ldots\) on the other hand, only concern the number of photons. A great degeneracy remains since adding photons in states orthogonal to the distinguished \(g\)-state do not change the energies.

For a three-level atom, the number of coupling functions in the Hamiltonian is greater and this forced us to start to pay attention to different photon states. As a consequence, the just mentioned degeneracy now starts being removed.
For a real atom, with its infinity of levels, the splitting will still be greater. Our perturbative treatment illustrates how the different photon states may be taken into account successively. Calculations will of course be impossible if some physically justified simplifications are not made.

The present limits of the study are the following.

Section III described resonances for a realistic Hamiltonian ($\mu \neq 0$), but for a system with only two levels. However, even in that simplified case, we are far from having found all the resonances since we considered only one-or-two-excitation spaces, $\mathcal{E}_1$ or $\mathcal{E}_2$. It would be necessary to take more than two photons into account. But the resonances are then given by more and more complicated equations.

In Section IV, to be able to present a qualitative description of resonances in a three-level system, we had to work in the limit $\mu = 0$. This first stage seems unavoidable to us if one wants to solve the question completely. Doing this, we were able to take an infinity of photons into account. But the calculations are only carried to the second order in $\lambda_{02}$ and also the displacement of the resonances when $\mu$ becomes non-zero is just qualitatively mentioned.

However, we have seen that these partial results give new information about hybrid states which are present in matter-field interactions such as the interaction to which we borrowed our terminology: the interaction of atoms (or molecules) with the electromagnetic field. We hope that concrete problems will justify approximations making calculations possible.

**Appendix A. Estimation of corrective terms in the Fredholm expansion of Proposition 3.4**

**Proposition** Set $\varphi_2(\mu, z, p, q) := \mu^2 \ D_2(\mu, z, p, q)$ and

\[
\varphi_3(\mu, z, p, q) := \frac{|g(p)| \ |g(q)| \ (1 + |p|)^{1/4} \ (1 + |q|)^{1/4} \ (|p| - |q|)^2}{|f(z - \mu|p|)|^{1/2} \ |f(z - \mu|q|)|^{1/2} \ |\bar{z} - 2|p||^{1/2} \ |\bar{z} - 2|q||^{1/2} \ |\bar{z} - |p| - |q||^2}
\]

\[
\varphi_4(\mu, z, p, q) := \frac{|g(p)|^{2/3} \ |g(q)|^{2/3} \ (1 + |p|)^{1/4} \ (1 + |q|)^{1/4} \ (|p| - |q|)^2}{|f(z - \mu|p|)|^{1/3} \ |f(z - \mu|q|)|^{1/3} \ |\bar{z} - 2|p||^{1/3} \ |\bar{z} - 2|q||^{1/3} \ |\bar{z} - |p| - |q||^2}
\]

defined for $z < 0$. we have

\[
C_2(\lambda, \mu, z) = (\lambda^2 / \mu)^2 \ ||\varphi_2(\mu, z, \ldots)||_1 \tag{A1}
\]

\[
|C_3(\lambda, \mu, z)| < M_3(\lambda, \mu, z) := 4 \ (\lambda^2 / \mu)^3 \ ||\varphi_3(\mu, z, \ldots)||_3^3 \tag{A2}
\]

Assuming that (3.19) also holds for $n = 4$, we also have

\[
|C_4(\lambda, \mu, z)| < M_4(\lambda, \mu, z) := 4^2 \ (\lambda^2 / \mu)^4 \ ||\varphi_4(\mu, z, \ldots)||_6^6 \tag{A3}
\]

**Proof** Use (3.19) and Hölder’s inequalities.
We omit \( \lambda \), which is fixed to 0.1. Table 2 gives values for \( C_2(\mu, z_{0,2}^1(\mu)) \), \( M_3(\mu, z_{0,2}^1(\mu)) \), \( M_4(\mu, z_{0,2}^1(\mu)) \) and \( \partial_z C_1(\mu, z_{0,2}^2(\mu)) \), from

\[
\partial_z C_1(\mu, z) = -\lambda^2 \int \frac{|g(p)|^2}{f(\mu, z - \mu |p|)(z - 2\mu |p|)} \left( \frac{1}{f(\mu, z - \mu |p|)} - \frac{1}{z - 2\mu |p|} \right) dp - \lambda^2 \int \int \frac{|g(p)|^2 |g(q)|^2 dp dq}{f^2(\mu, z - \mu |p|)(z - 2\mu |p|)(z - \mu (p + q))^2}
\]

By definition, \( C_1(\mu, z_{0,1}^1(\mu)) = 1 \). We see how the terms of the Fredholm expansion decrease with the order.

For \( \mu = \mu_c \), \( D(\mu_c, 0) \) is close to 0. Let us recall that we have \( f(\lambda, \mu_c(\lambda), 0) = 0 \). It would be interesting to see whether \( D(\mu_c, 0) \) vanishes or not. To try and answer this question, let us estimate the error we made in calculating the zero of \( C_1(z) \) by the truncated series. Let us consider the following expansion of \( D(\mu, z) \) near \( z_{0,2}^1(\mu) \)

\[
1 - C_1(\mu, z_{0,2}^1(\mu)) - (z - z_{0,2}^1(\mu)) \partial_z C_1(\mu, z_{0,2}^1(\mu)) + \frac{1}{2} C_2(\mu, z_{0,2}^1(\mu)) + \\
\frac{1}{2} (z - z_{0,2}^1(\mu)) \partial_z C_2(\mu, z_{0,2}^1(\mu)) - \frac{1}{6} \left( C_3(\mu, z_{0,2}^1(\mu)) + (z - z_{0,2}^1(\mu)) \partial_z C_3(\mu, z_{0,2}^1(\mu)) \right)
\]

Let us assume that terms \((z - z_{0,2}^1(\mu)) \partial_z C_2(\mu, z_{0,2}^1(\mu))\) and \((z - z_{0,2}^1(\mu)) \partial_z C_3(\mu, z_{0,2}^1(\mu))\) can be neglected. Then the correction to the zero is

\[
\frac{1}{\partial_z C_1(\mu, z_{0,2}^1(\mu))} \left( -\frac{1}{2} C_2(\mu, z_{0,2}^1(\mu)) + \frac{1}{6} C_3(\mu, z_{0,2}^1(\mu)) \right)
\]

whose principal term is of the order of \( 4 \times 10^{-4} \). Since it is precisely the order of \( z_{0,2}^1(\mu_c) \), it is not possible to answer the question.

**Appendix B Sketches of proofs of results in the 3-level case**

**B1 Sketch of the proof of Proposition 4.2**

Let \( \Phi := |2, F_n\rangle + |1, F_{n+1}\rangle + |0, F_{n+2}\rangle \) be an eigenvector of \( H_0(\lambda) \) in \( \mathcal{E}_{n+2} \) associated with \( z \), one of the two eigenvalues \( (\zeta_{0,1})_1 \) or \( \zeta_{1,1} \). It can be shown that

\[
(z - e_2)^{-1} \lambda^2_{12}(f_{12}, F_{n+1}) \vee f_{12} = z^{-1} \lambda^2_{01}(f_{01}, F_{n+1}) \vee f_{01}
\]

where

\[
(f_{12}, F_{n+1})(p_1, p_2, \ldots, p_n) = \int \mathcal{F}_{12}(p) F_{n+1}(p, p_1, p_2, \ldots, p_n) dp
\]

\[
(f_{01}^{\vee n} : h)(p_1, p_2, \ldots, p_{n+1}) = \frac{1}{n + 1} \sum_{i=1}^{n+1} \left( \prod_{j \neq i} f(p_j) \right) h(p_i)
\]

From (B1.1), through decomposing \( F_{n+1} \) on a basis of the \((n + 1)\)-photon space built with \( f_{01}, f_{12} \) and the \( g_i \)'s, we get that \( F_{n+1} \) is a sum of states \( \prod_{i=1}^{\infty} (c^*(g_i))^{n_i} \Omega \) with \( \sum_i n_i = n + 1 \). From relations expressing that \( \Phi \) is an eigenvector, we derive \( F_n = 0 \) and, if \( z \neq 0 \), \( F_{n+2} = z^{-1} \sqrt{n + 2} \lambda_0 F_{n+1} \vee f_{01} \); this implies that \( \Phi \) is in \( \mathcal{G}_{0,1,1} \) if \( z = (\zeta_{0,1})_1 \) or in \( \mathcal{G}_{1,1} \) if \( z = \zeta_{1,1} \). Hence (i) and (iii).

If \( z = (\zeta_{0,1})_2 = 0 \), then \( \Phi \) is an eigenvector if and only if \( F_n = 0 \), \( F_{n+1} = 0 \) and \( (f_{01}, F_{n+2}) = 0 \). Hence (ii).
B2 The six eigenvalues $\zeta_{2,2}$, $(\zeta_{1,2})_1$, $(\zeta_{1,2})_2$, $(\zeta_{0,2})_1$, $(\zeta_{0,2})_2$ and $(\zeta_{0,2})_3$, with eigenvectors in $\mathcal{E}_2^{(2)}$.

The space spanned by vectors

$$\begin{align*}
|2, \Omega\rangle, & \quad |1, f_{12}\rangle, \quad |1, f_{01}\rangle, \quad |0, f_{12} \vee f_{01}\rangle, \quad |0, f_{01} \vee f_{01}\rangle
\end{align*}$$

is invariant. Through adding $|0, f_{12} \vee f_{12}\rangle$, we get a basis of $\mathcal{E}_2^{(2)}$, in which the matrix of $H_0$ is

$$
\begin{pmatrix}
\varepsilon_2 & \lambda_{12} & s_1 \lambda_{12} & 0 & 0 & 0 \\
\lambda_{12} & e_1 & 0 & \frac{\lambda_0}{\sqrt{2}} & 0 & \sqrt{2} s_1 \lambda_{01} \\
0 & 0 & e_1 & s_1 \lambda_0 & 0 & 0 \\
0 & \sqrt{2} \lambda_0 & 0 & 0 & 0 & 0 \\
0 & 0 & \sqrt{2} \lambda_{01} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

One of the eigenvalues is $0$ and the eigenvector is the one mentioned in the text. The other eigenvalues are the $\zeta$'s for which

$$
\zeta(\zeta - e_1) \left( \zeta(\zeta - e_1)(\zeta - e_2) - 3 \lambda^2_{01}(\zeta - e_2) - \lambda^2_{12} \zeta \right) + \zeta \lambda^2_{01} \left( 2 \lambda^2_{01} + (2 - |s_1|^2) \lambda^2_{12} \right) - 2 e_2 \lambda^4_{01}
$$

vanishes. Neglecting fourth order terms, we get the following solutions at second order in $\lambda$

$$
(\zeta^0_{0,2})_2 = 0, \quad (\zeta^0_{0,2})_3 = -\frac{3}{e_1} \lambda^2_{01}, \quad (\zeta^0_{1,2})_1 = e_1, \quad (\zeta^0_{1,2})_2 = e_1 + \frac{3}{e_1} \lambda^2_{01} + \frac{1}{e_1 - e_2} \lambda^2_{12},
$$

$$
\zeta_{2,2} = e_2 + \frac{1}{e_2 - e_1} \lambda^2_{12}
$$

This proves that these eigenvalues are different from those found in $\mathcal{E}_1^{(2)}$.

B3 Sketch of the proof of Proposition 4.4.

Let us use Kato's method to determine the three (infinite) sets of perturbed eigenvalues which tend to each of the unperturbed eigenvalues. (See a short account in Ref. 24.)

(i) Let us first consider $(\zeta_{0,1})_1$. The unperturbed eigenspace is $\mathcal{G}_{0,1,1}$ (Proposition 4.2). Let $P_{0,1,1}^0$ be the projector on this space and $Q_{0,1,1}^0 = 1 - P_{0,1,1}^0$. We need operator $Q_{0,1,1}^0$ which is sometimes written as $Q_{0,1,1}^0[(\zeta_{0,1})_1 - H_0]^{-1}Q_{0,1,1}^0$; for $x \in \mathcal{E}$, it is defined by $Q_{0,1,1}^0 x = Q_{0,1,1}^0 z$, where $z$ is any vector in $\mathcal{E}$ satisfying $[(\zeta_{0,1})_1 - H_0] z = Q_{0,1,1}^0 x$. Let $\mathcal{H}_{0,1,1}$ be the direct sum of the eigenspaces associated with eigenvalues of $H(\lambda)$ which tend to $(\zeta_{0,1})_1$ when $\lambda_{02}$ tends to $0$. Let $P_{0,1,1}^0(\lambda_{02})$ be the projector on $\mathcal{H}_{0,1,1}$. Therefore $P_{0,1,1}^0(\lambda_{02}) \rightarrow P_{0,1,1}^0$. We assume that $P_{0,1,1}^0$ and $P_{0,1,1}^0(\lambda_{02})$ establish one-to-one correspondences $\mathcal{H}_{0,1,1} \rightarrow \mathcal{G}_{0,1,1}$ and $\mathcal{G}_{0,1,1} \rightarrow \mathcal{H}_{0,1,1}$. Set

$$
L_{0,1,1} = P_{0,1,1}^0 H P_{0,1,1}^0 P_{0,1,1}^0 \quad \text{and} \quad K_{0,1,1} = P_{0,1,1}^0 P_{0,1,1}^0 P_{0,1,1}^0
$$

(B3.1)
We recall the expansion of $P^{0,1,1}$: with $S^{(0)} := -P^{0,1,1}_0$ and $S^{(k)} := (\tilde{Q}^{0,1,1}_0)^k$,

$$P^{0,1,1} = P^{0,1,1}_0 - \sum_n \lambda^{(n)}_{02} \sum_{k_0 \geq 0, k_0 + \cdots + k_n = n} S^{(k_0)} V S^{(k_1)} \cdots V S^{(k_n)} \tag{B3.2}$$

A necessary and sufficient condition for $\chi \in \mathcal{H}_{0,1,1}$ to be an eigenvector of $H(\lambda)$ associated with the eigenvalue $z$ is that there exists $\phi \in \mathcal{G}_{0,1,1}$ satisfying $\chi = P^{0,1,1}_0 \phi$ and

$$L_{0,1,1} \phi = z K_{0,1,1} \phi \tag{B3.3}$$

$L_{0,1,1}$ and $K_{0,1,1}$ are operators in $\mathcal{G}_{0,1,1}$ and the problem of the perturbation of $(\zeta_{0,1})_1$ is turned into finding such $z$. We still have a degeneracy due to photon-states in $(\mathcal{H}_{\text{rad}}^{(3)})^\perp$; indeed, if $\phi$ is a solution for (B3.3), then $\Pi_{i \geq 2} \left(1 \otimes c^*(g_i)\right)^n \phi$ is still a solution. Thus we are not going to look for all $\phi$’s, but only for those in an invariant subspace of $\mathcal{G}_{0,1,1}$. Lemmas B3.1 to B3.6 prepare the calculation of eigenvalues into which $(\zeta_{0,1})_1$ splits. The result for $\zeta_{1,1}$ will be obtained through a simple change in the notations. The splitting of $(\zeta_{0,1})_2$ is just outlined.

Let $g_1$ be the function of $\mathcal{H}_{\text{rad}}^{(3)}$ orthogonal to $\mathcal{H}_{\text{rad}}^{(2)}$ and let $\mathcal{G}^{(1)}_{0,1,1}$ be the subspace of $\mathcal{G}_{0,1,1}$ spanned by $(1 \otimes c^*(g_1))^n (\phi^{(0)}_{0,1,1})$, $n \geq 0$. Let us denote the approximations of $K_{0,1,1}$ and $L_{0,1,1}$ to order $q$ in $\lambda_{02}$ by $K_{0,1,1}^{\leq_q}$ and $L_{0,1,1}^{\leq_q}$.

**Lemma B3.1** For all $q$, $\mathcal{G}^{(1)}_{0,1,1}$ is invariant by $K_{0,1,1}^{\leq_q}$ and $L_{0,1,1}^{\leq_q}$.

**Proof** Operators $1 \otimes c^*(g_1)$ and $V$ sent $\mathcal{E}^{(3)}$ into $\mathcal{E}^{(3)}$, since functions orthogonal to $\mathcal{H}_{\text{rad}}^{(3)}$ do not play any part. Now, $P^{0,1,1}_0$ sends $\mathcal{E}^{(3)}$ into $\mathcal{G}^{(1)}_{0,1,1} \subset \mathcal{E}^{(3)}$. Thus, if $X$ is any endomorphism of $\mathcal{E}^{(3)}$, $P^{0,1,1}_0 X P^{0,1,1}_0$ is an endomorphism of $\mathcal{G}^{(1)}_{0,1,1}$. $Q^{0,1,1}$ also leaves $\mathcal{E}^{(3)}$ invariant. The same is true for $\mathcal{G}^{0,1,1}$ and therefore for $P^{0,1,1}_0$. Hence the Lemma.

Expressions to the second order in $\lambda_{02}$ of $L_{0,1,1}$ and $K_{0,1,1}$ are

$$K_{0,1,1}^{\leq 2} = P^{0,1,1}_0 - \lambda_{02}^2 P^{0,1,1}_0 V (\tilde{Q}^{0,1,1}_0)^2 V P^{0,1,1}_0 \tag{B3.4}$$

$$L_{0,1,1}^{\leq 2} = (\zeta_{0,1})_1 K_{0,1,1}^{\leq 2} + \lambda_{02}^2 P^{0,1,1}_0 V \tilde{Q}^{0,1,1}_0 V P^{0,1,1}_0 \tag{B3.5}$$

To calculate $K_{0,1,1}^{\leq 2}(\phi^{(n)}_{0,n+1})_1$ and $L_{0,1,1}^{\leq 2}(\phi^{(n)}_{0,n+1})_1$, we need the following lemma:

**Lemma B3.2**

$$V (\phi^{(n)}_{0,n+1})_1 = 2 (\phi^{(n)}_{0,1})_1 \tag{B3.6}$$

with $(\phi^{(0)}_{0,1})_1 = N_1 \frac{\lambda_{01}}{(\zeta_{0,1})_1} \bar{\sigma}_0$ and, for $n \geq 1$,

$$(\phi^{(n)}_{0,1})_1 = N_1 \frac{\lambda_{01}}{(\zeta_{0,1})_1} \left( \bar{\sigma}_0 g_1^{\nu^n} + n(f_{02}, g_1) f_{01} \lor g_1^{\nu^{n-1}} \right) \tag{B3.7}$$

where $N_1 = (1 + (\zeta_{0,1})_1^{-2} \lambda_{01}^2)^{-1/2}$
Lemma B3.3 \( K_{0,1,1}^{\leq 2} \) and \( L_{0,1,1}^{\leq 2} \) are diagonal in the basis \((\phi_{0,n+1}^{(n)})_{1}\) of \(G_{0,1,1}^{(1)}\).

**Proof** \( V(\phi_{0,n+1}^{(n)})_{1} \in \mathcal{E}_{n+2} \), since \((\phi_{0,n+1}^{(0)})_{1}\) has no component on \( |2\rangle \otimes \mathcal{F} \). Through using
\( Q_{0}^{0,1,1} \mathcal{E}_{n+3} \subset \mathcal{E}_{n+2}^{(3)} \), we then get
\[
(Q_{0}^{0,1,1} V P_{0}^{0,1,1}(\phi_{0,n+1}^{(n)}), Q_{0}^{0,1,1} V P_{0}^{0,1,1}(\phi_{0,n+1}^{(m)}) = 0 , \text{ if } m \neq n
\]
and \( K_{0,1,1}^{\leq 2} \) is diagonal. The same is true for \( L_{0,1,1}^{\leq 2} \).

**Lemma B3.4** Let us set \( u_{1,n} := Q_{0}^{0,1,1} V(\phi_{0,n+1}^{(n)})_{1} \) and define \( \kappa_{1,n}, \theta_{1,n+1} \) and \( h_{1,n+2} \) by \( u_{1,n} = | \ 2, \kappa_{1,n} \rangle + | \ 1, \theta_{1,n+1} \rangle + | \ 0, h_{1,n+2} \rangle \). We denote the expression of \( z \) to the second order in \( \lambda_{02} \) by \( (z_{0,1}^{(n)})_{1}^{\leq 2} \). We have
\[
(z_{0,1}^{(n)})_{1}^{\leq 2} = (\zeta_{0,1})_{1} + \lambda_{02}^{2}(\phi_{0,1}^{(n)}, \kappa_{1,n}) \tag{B3.8}
\]

**Proof** From Lemma B3.3, it follows that \((\phi_{0,n+1}^{(n)})_{1}\) are eigenvectors of \( L_{0,1,1}^{\leq 2} \) and \( K_{0,1,1}^{\leq 2} \), associated with eigenvalues which we denote by \((l_{0,1}^{(n)})_{1}^{\leq 2} \) and \((k_{0,1}^{(n)})_{1}^{\leq 2} \) respectively. Vectors \( \phi \) in \( G_{0,1,1}^{(1)} \) satisfying
\( (L_{0,1,1}^{\leq 2} - zK_{0,1,1}^{\leq 2}) \phi = 0 \) are necessarily these \((\phi_{0,n+1}^{(n)})_{1}\); the corresponding \( z \)-value, for each \( n \), is \(( (k_{0,1}^{(n)})_{1}^{\leq 2} )^{-1} (l_{0,1}^{(n)})_{1}^{\leq 2} \). Through using (B3.4) and (B3.6), we get
\[
(k_{0,1}^{(n)})_{1}^{\leq 2} = ( (\phi_{0,n+1}^{(n)}), K_{0,1,1}^{\leq 2} (\phi_{0,n+1}^{(n)})) = 1 - \lambda_{02}^{2} ||u_{1,n}||^2 \tag{B3.9}
\]
\[
(l_{0,1}^{(n)})_{1}^{\leq 2} = (\zeta_{0,1})_{1} ( 1 - \lambda_{02}^{2} ||u_{1,n}||^2 ) + \lambda_{02}^{2} (V(\phi_{0,n+1}^{(n)}), u_{1,n})
\]
As a consequence,
\[
(l_{0,1}^{(n)})_{1}^{\leq 2} = (\zeta_{0,1})_{1} ( 1 - \lambda_{02}^{2} ||u_{1,n}||^2 ) + \lambda_{02}^{2} ((\phi_{0,1}^{(n)}), \kappa_{1,n}) \tag{B3.10}
\]
Hence (B3.8) holds since \( 1 - ||u_{1,n}||^2 \) is to be replaced by \( 1 \), to the considered approximation.

**Lemma B3.5**

(a) Let us set \( M_{1} := \left( \frac{\lambda_{12}^{2}}{(\zeta_{0,1} - e_{2})} + \frac{\lambda_{01}^{2}}{(\zeta_{0,1})_{1}} \right)^{-1} \). For \( n \geq 0 \), a vector \( v \) satisfying
\( [ (\zeta_{0,1} - H_{0}) v = | 2, (\phi_{0,1}^{(n)})_{1} \rangle \) is \( v_{1,n} = | 2, \kappa_{1,n}^{'} \rangle + | 1, \theta_{1,n+1}^{'} \rangle + | 0, h_{1,n+2}^{'} \rangle \), with
\[
\kappa_{1,n}^{'} = \frac{(\phi_{0,1}^{(n)})_{1}}{(\zeta_{0,1} - e_{2})} + \sqrt{n + 1} \lambda_{12} \left( f_{12, \theta_{1,n+1}^{'}}, (\zeta_{0,1})_{1} - e_{2} \right) \tag{B3.11}
\]
\[
h_{1,n+2}^{'} = \sqrt{n + 2} \lambda_{01} \frac{\theta_{1,n+1}^{'} \lor f_{01}}{(\zeta_{0,1})_{1}} \tag{B3.12}
\]
\[
\theta_{1,n+1}^{'} = N_{1} \sqrt{n + 1} \frac{\lambda_{01} \lambda_{12}}{((\zeta_{0,1})_{1} - e_{2})(\zeta_{0,1})_{1}} \left( s_{0}(\theta_{1,n+1})_{1}^{(1)} + n(f_{02}, g_{1})(\theta_{1,n+1}^{2}) \right) \tag{B3.13}
\]

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Relations (B3.11) and (B3.12) are obtained through projecting the equality defining $\theta$ on $v$ where, if

$$
(\theta'_{1,n+1})^{(1)} = \frac{(\zeta_{0,1} - e_2)}{\lambda_{12}^2} \frac{1}{1 - |s_1|^2} (s_1 f_{01} - f_{12}) \vee g_1^{\gamma^n}
$$

\hspace{1cm} (B3.14)

$$(\theta'_1)^{(2)} = 0, \quad (\theta'_{1,n+1})^{(2)} = \frac{M_1}{2(1 - |s_1|^2)} \left( s_1 f_{01} \vee f_{01} - 2 f_{01} \vee f_{12} + \bar{s}_1 f_{12} \vee f_{12} \right) \vee g_1^{\gamma^{n-1}}
$$

(B3.15)

If $f_{01} = f_{12}$,

$$(\theta'_{1,n+1})^{(1)} = -M_1 f_{01} \vee g_1^{\gamma^n}
$$

(B3.16)

$$(\theta'_1)^{(2)} = 0, \quad (\theta'_{1,n+1})^{(2)} = -\frac{1}{2} M_1 f_{01} \vee f_{01} \vee g_1^{\gamma^{(n-1)}}
$$

(B3.17)

(b) $u_{1,n}$ of Lemma B3.4 is equal to $v_{1,n}$

Proof

Relations (B3.11) and (B3.12) are obtained through projecting the equality defining $\theta$ on $|2\rangle \otimes \mathcal{H}_{\text{rad}}$ and $|0\rangle \otimes \mathcal{H}_{\text{rad}}$, respectively. The projection on $|1\rangle \otimes \mathcal{H}_{\text{rad}}$ implies that $\theta'_{1,n+1}$ satisfies

$$
\mathcal{L}_{1,n+1} \theta'_{1,n+1} = \sqrt{n + 1} \lambda_{12} (\zeta_{0,1} - e_2) \vee f_{12}
$$

(B3.18)

where

$$
\mathcal{L}_{1,n+1} \theta'_{1,n+1} := -\frac{(n + 1) \lambda_{12}^2}{(\zeta_{0,1} - e_2)} f_{12} \vee (f_{12}, \theta'_{1,n+1}) - \frac{(n + 1) \lambda_{01}^2}{(\zeta_{0,1} - e_2)} f_{01} \vee (f_{01}, \theta'_{1,n+1})
$$

(B3.19)

For $n \geq 1$, $(\varphi_{0,1}^{(n)},)$, given by (B3.7) is decomposed into two parts. Hence we introduce two functions $(\theta'_{1,n+1})^{(1)}$ and $(\theta'_{1,n+1})^{(2)}$ satisfying

$$
\mathcal{L}_{1,n+1} (\theta'_{1,n+1})^{(1)} = g_1^{\gamma^n} \vee f_{12}
$$

(B3.20)

$$
\mathcal{L}_{1,n+1} (\theta'_{1,n+1})^{(2)} = g_1^{\gamma^{(n-1)}} \vee f_{01} \vee f_{12}
$$

(B3.21)

so that a solution of (B3.18) will be given by (B3.13). To prove (a) if $f_{01} \neq f_{12}$, we check that (B3.14) satisfies (B3.20) and that (B3.15) satisfies (B3.21). We proceed in the same way with (B3.16) and (B3.17), if $f_{01} = f_{12}$, (B3.18) and (B3.19) being still true. Only $(\theta'_{1,n+1})^{(1)}$ plays a part if $n = 0$.

To prove (b), we note that vectors in $G_{0,1,1}$ are linear combinations of $|1, g_{i_1} \vee \cdots \vee g_{i_n}\rangle$ and $|0, f_{01} \vee g_{i_1} \vee \cdots \vee g_{i_n}\rangle$. Since (a) implies that $\theta'_{1,n+1}$ is a sum of symmetric products of terms one of which at least is in $H_{\text{rad}}(2)$, we have $(\theta'_{1,n+1}, g_{i_1} \vee \cdots \vee g_{i_{n+1}}) = 0$ and $(\theta'_{1,n+2}, f_{01} \vee g_{i_1} \vee \cdots \vee g_{i_{n+1}}) = 0$. As a consequence, $|2, \kappa'_{\nu,n}\rangle$, $|1, \theta'_{1,n+1}\rangle$ and $|0, h'_{1,n+2}\rangle$ are orthogonal to $G_{0,1,1}$ and $u_{1,n} = G_{0,1,1}^{0,1} v_{1,n} = v_{1,n}$.

Lemma B3.6 Take $n \geq 0$. For $f_{01} \neq f_{12}$,

$$
((\varphi_{0,1}^{(n)}, \kappa_{1,n}) = n N^2 \frac{\lambda_{01}^4 \lambda_{12}^2 (f_{02}, g_1)^2}{(\zeta_{0,1} - e_2) (\lambda_{12}^2 (\zeta_{0,1} - e_2) + \lambda_{01}^2 (\zeta_{0,1} - e_2))}
$$

(B3.22)
For $f_{01} = f_{12}$,

\[
\left(\varphi_{0,1}^{(n)}, \kappa_{1,n}\right) = N_1^2 \frac{\lambda_{01}^4}{((\zeta_{0,1})_1)^2 (\lambda_{12}^2 (\zeta_{0,1})_1 + \lambda_{01}^2 ((\zeta_{0,1})_1 - e_2))} (n|f_{02}, g_1|^2 + |s_0|^2)
\]

(B3.22')

Proof Use (B3.7) and Lemma B3.5. Formulas (4.14) and (4.15) then follow from (B3.10) and (B3.22). In the same way, (4.16) follows from (B3.22').

(ii) All that has been written up to now can be transposed from $(\zeta_{0,1})_1$ to $\zeta_{1,1}$, with the following changes

\[
P_0^{0,1,1} \rightarrow P_0^{1,1}, \quad P_0^{0,1,1} \rightarrow P_1^{1,1}, \quad Q_0^{0,1,1} \rightarrow Q_0^{1,1}, \quad \tilde{Q}_0^{0,1,1} \rightarrow \tilde{Q}_0^{1,1}, \quad \mathcal{H}_{0,1,1} \rightarrow \mathcal{H}_{1,1,1},
\]

\[
(\phi_{0,n+1})_1 \rightarrow (\phi_{1,n+1})_1, \quad (\xi_{0,n+1})_1 \rightarrow (\xi_{1,n+1})_1, \quad (\lambda_{n+1}^{(n)}) \rightarrow (\lambda_{1,n}^{(n)}) \leq 2 \rightarrow (\lambda_{1,1}^{(n)}) \leq 2
\]

$N_1, M_1, (\varphi_{0,1}^{(n)})_1$ are thus also changed, as well as $L_{1,n+1}$ and the solutions $\theta_{1,n+1}$ $h_{1,n+2}$ and $\kappa_{1,n}$.

One then get (ii) of Proposition 4.4.

(iii) Let us come on now to the splitting of $(\zeta_{0,1})_2$, an eigenvalue which is zero and exists only if $f_{01} \neq f_{12}$. The issue is more complicated due to the fact that the unperturbed eigenspace of interest is no longer just spanned by the $\Pi(1 \otimes c^*(g_1))^{n}(\phi_{0,1}^{(n)})_2$. It is spanned by the $|0, \Pi(c^*(g_i))^{n_i}\Omega_i|$, with $i = 0$ or 1. Let us denote this space by $\mathcal{G}$. Let $P_0^{0,1,2}$ be the projector on $\mathcal{G}$. With $K_{0,1,2}^{\leq 2}$ and $L_{0,1,2}^{\leq 2}$ defined as in (B3.4) and (B3.5), we can see that $L_{0,1,2}^{\leq 2}$ has non-vanishing matrix elements between say $g_0^{\nu} \vee g_1^{\nu(p+2)}$ and $g_0^{\nu(p+2)} \vee g_1^{\nu(p+1)}$, or between $g_0^{\nu(p+1)} \vee g_1^{\nu(p+1)}$ and $g_0^{\nu(p+2)} \vee g_1^{\nu(p)}$ or also between $g_0^{\nu(p+2)} \vee g_1^{\nu(p)}$ and $g_0^{\nu(p+3)} \vee g_1^{\nu(p+1)}$. This makes the computation of the z’s satisfying det($L_{0,1,2}^{\leq 2} - z K_{0,1,2}^{\leq 2}$) = 0 more intricate and we don’t calculate them here.

This completes the proof of Proposition 4.4.

Eigenvectors $(\chi_{0,1}^{(n)})_1^{(2)}, (\chi_{1,1}^{(n)})_1^{(2)}$ are obtained from the correspondences $\mathcal{G}_{0,1,1} \rightarrow \mathcal{H}_{0,1,1}$, and $\mathcal{G}_{1,1} \rightarrow \mathcal{H}_{1,1,1}$, through the second order expansion of operators $P_0^{0,1,1}(\lambda_{02})$ and $P_0^{1,1}(\lambda_{02})$ which perform these correspondences. For example, we get (see Ref 24 p. 614)

\[
(\chi_{0,1}^{(n)})_1^{(2)} = (\phi_{0,n+1})_1^{(n)} + \chi_{02}^2 \left(-P_0^{0,1,1}V(\tilde{Q}_0^{0,1,1})^2V + \tilde{Q}_0^{0,1,1}V \tilde{Q}_0^{0,1,1}V\right)(\phi_{0,n+1})_1^{(n)}
\]
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