Spin-rotation coupling and Thomas precession of ions in a storage ring

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Abstract. We discuss the role of spin-rotation coupling and Thomas precession in the decay of hydrogen-like heavy ions that are injected in a storage ring. We find that the decay curve has a time modulation that vanishes if the magnetic field also disappears. The results can be applied to the recently reported modulation in the decay of the hydrogen-like ions \textsuperscript{140}Pr\textsuperscript{58+}, \textsuperscript{142}Pm\textsuperscript{60+} and \textsuperscript{122}I\textsuperscript{52+}, where rotation couples to the spins of electron and nucleus. Spin-rotation coupling and Thomas precession also generate oscillations in the decay of a muon bound to a nucleus rotating in a storage ring.

1. Introduction

In this work we study the effect of spin-rotation coupling (SRC) and Thomas precession on the behaviour of bound particles accelerating in a storage ring (SR) in which there is a magnetic field perpendicular to the circular orbits of the particles. We consider, in particular, the case of an electron bound to a heavy nucleus and dragged by it along its path in the SR.

This problem also involves some basic issues of quantum mechanics since the relevant states are a superposition of hyperfine states (here denoted with $F = 3/2$ and $F = 1/2$) that evolve in time once the nucleus-electron system is injected in the SR. The decay curves obtained in this way show a modulation that has its origin in the spin-dependent part of the Thomas precession, the interaction of the rotation field with the spins of electron and nuclei and their respective magnetic moments. We consider the case in which the decay occurs via electron capture (EC) \cite{1,2,3,4,5}. This is possible only for states with spin $F = 1/2$ because decay from spin $3/2$-states is forbidden by the conservation of the $F$ quantum number \cite{2}. We specifically refer to the hydrogen-like ions \textsuperscript{140}Pr\textsuperscript{58+}, \textsuperscript{142}Pm\textsuperscript{60+} and \textsuperscript{122}I\textsuperscript{52+}.

In what follows we take into account bound states kinematics, dragging effects, and QED effects in the calculation of the magnetic moment of the bound electron that also contribute to the probability modulation mentioned above.

2. The model

The full Hamiltonian that describes the behaviour of nucleus and bound electron in the external field $\mathbf{B}$ of the ring is $H = H_0 + H_1$, where $H_0$ contains all the usual standard terms (Coulomb...
potential, spin–orbit coupling, etc.), and $H_1$ is (in units $\hbar = c = 1$)

$$H_1 = -\mathcal{A} \mathbf{s} \cdot \mathbf{I} - \mathbf{s} \cdot \mathbf{\Omega}_e - \mathbf{I} \cdot \mathbf{\Omega}_n,$$  

(1)

where $\mathcal{A} \approx Z^4 A^4 g_n^3 \sim N \times 10^{14}$ Hz, $N \sim O(1)$ is the strength of spin–spin coupling, while

$$\mathbf{\Omega}_e \equiv \omega_{ge} + \omega_{Th}^{(e)} - \omega_{e}^{(c)},$$

$$\mathbf{\Omega}_n \equiv \omega_{gn} + \omega_{Th}^{(n)} - \omega_{e}^{(c)},$$

(2)

(3)

represent the precession of the electron spin and the usual spin precession of the nucleus in its motion in a SR assumed circular for simplicity. In (2) and (3), $\omega_{ge,n}$ are the electron and nucleus spin precession frequencies due to the respective magnetic moments $g_e,n$ and $\omega_{e}^{(c,n)}$ are the angular cyclotron frequencies. The explicit expressions of all these quantities are given below. We refer (1) to a frame rotating about the $x_3$-axis in the clockwise direction of the ions, with the $x_2$-axis tangent to the ion orbit in the direction of its momentum and write $\mathbf{B} = B \mathbf{\hat{u}}_3$.

The Thomas precession $\omega_{Th}^{(e,n)}$ is a kinematic effect. It occurs because two successive Lorentz transformations in different directions are equivalent to a Lorentz transformation plus a three-dimensional rotation $[7]$. The SRC can be derived for any particle with spin $[8, 9, 10]$. It follows, for the electron, from the spin connection coefficients of the Dirac equation in a rotating frame $[11, 10, 12]$ (for other applications see $[13]$). In our calculation, we neglect any stray electric fields and electric fields needed to stabilize the nucleus orbits, as well as all other effects which could affect the Thomas precession $[14]$.

We indicate by $\beta$ and $\beta_n$ the velocities of electron and nucleus relative to the lab frame. Using the composition of velocities, the Lorentz factor $\gamma = 1/\sqrt{1 - \beta^2}$ of the electron can be written in the form $\gamma = \gamma_n \gamma_e (1 + \Pi)$, where $\Pi = \beta_n \cdot \beta_{e,n} = \beta_n/\beta_{e,n} \cos \theta$, $\beta_{e,n}$ is the velocity of the electron relative to the nucleus, $\gamma_n = 1/\sqrt{1 - \beta_n^2}$, and $\gamma_e = 1/\sqrt{1 - \beta_e^2}$. The explicit expression of $\gamma$ is also useful

$$\beta = \frac{1}{\gamma_n (1 + \Pi)} \left[ \frac{\gamma_n^2 \Pi}{\gamma_n + 1} \beta_n + \gamma_n \beta_n \right].$$

(4)

The Thomas precession of the electron in the lab frame is given by $\omega_{Th} = -\frac{\gamma^2}{\gamma + 1} \frac{d\beta}{dt} \wedge \beta$. The field $\mathbf{B}$ in the lab frame (where $\mathbf{E} = 0$) is transformed to the nucleus rest frame and gives $\mathbf{E}' = \gamma_n \beta_n \wedge \mathbf{B}$ and $\mathbf{B}' = \gamma_n \mathbf{B}$ on account of $\beta_n \cdot \mathbf{B} = 0$. The equations of motion are $d\beta_{e,n} = \frac{\mathbf{f}_{e,n}}{m \gamma_e (1 + \Pi)}$ for the electron with respect to the nucleus and $d\beta_n = \frac{Q}{M \gamma_n} \beta_n \wedge \mathbf{B}$ for the nucleus with respect to the lab frame. Here $\mathbf{f}_{e,n} = -e (\mathbf{E}' + \beta_{e,n} \wedge \mathbf{B}') = -e \gamma_n (\beta_{e,n} + \beta_n) \wedge \mathbf{B}$. Using $dt = \gamma_n (1 + \Pi) d\tau_n$, taking $\beta_{e,n} \cdot \mathbf{B} = 0$, $\mathbf{E}_{e,n} \wedge \beta_{e,n} = 0$ and $\mathbf{E}_{e,n} \wedge \beta_n = 0$ (averaged over the decay time of the ion in the SR), we find

$$\frac{d\beta_{e,n}}{dt} = -\frac{e}{m \gamma_e (1 + \Pi)} [\beta_{e,n} + \beta_n] \wedge \mathbf{B}. $$

Neglecting spin–orbit coupling, $\omega_{Th}$ can be written as

$$\omega_{Th} = \frac{e \mathbf{B}}{m \gamma_e (1 + \Pi)} I_e - \frac{Q \mathbf{B}}{M \gamma_n} I_Q,$$

(5)

where

$$I_e = \frac{\gamma_e (\gamma_e + \gamma_n)^2 \left( \beta_n^2 + \frac{\beta_{e,n}^2}{\gamma_e} + 2 \Pi + \frac{\gamma_n \Pi^2}{\gamma_n + 1} - \gamma_e \right)}{(1 + \Pi)[\gamma_e (\gamma_e + \gamma_n)^2 (1 + \Pi) + 1]}.$$
From (2), (5) and (8) we obtain
\[ I_Q = \frac{\beta_n^2 (1 + \frac{\gamma_n \Pi}{\gamma_n + 1})^2 - X}{\gamma_{e|n} \gamma_n (1 + \Pi) + 1} , \]
\[ Y = \frac{\beta_{e|n} (2 - \cos^2 \theta - \sin^2 \theta)}{3 \gamma_n^2} , \quad X = \frac{\beta_{e|n}^2 \beta_n^2 \sin^2 \theta}{3 (\gamma_{n} + 1)} . \]

The Coulomb interaction also contributes to the Thomas precession. It generates the spin-orbit coupling term in the Hamiltonian of the electron \( H_C \sim \frac{g_e}{2m_e} (dV/dr) \mathbf{s} \cdot \mathbf{L} \). However \( \mathbf{s} \cdot \mathbf{L} = \frac{1}{2}(j(j+1) - l(l+1) - s(s+1)) \) vanishes when \( j = s \) for \( l = 0 \) and \( j = l \pm 1/2 \) for \( l \neq 0 \), therefore \( H_C = 0 \) in the ground state. Moreover, the effect of \( \mathbf{A} \) in \( \mathbf{p} = \frac{1}{m}(\mathbf{p} - e\mathbf{A}) \) is negligible in the present context.

The coupling of the electron magnetic moment \( \mathbf{\mu}_e = -\frac{g_e e}{2 m_e} \mathbf{s} \) with the magnetic field is described by
\[ H_{g_e} = \frac{1}{\gamma} \mathbf{\mu}_e \cdot \mathbf{B} = \mathbf{\mu}_e \cdot \left[ \mathbf{B} - \frac{\gamma}{\gamma + 1} \beta (\beta \cdot \mathbf{B}) \right] . \]

Keeping only the quadratic term in \( \beta_{e|n} \) and using \( \beta (\beta \cdot \mathbf{B}) = \frac{(\beta_{e|n} \cdot \hat{\mathbf{u}}_3)^2}{\gamma_n^2 (1 + \Pi)^2} \mathbf{B} \), we obtain
\[ H_{g_e} = \Upsilon \mathbf{\mu}_e \cdot \mathbf{B} , \quad \Upsilon = 1 - \frac{\gamma_n^2 (\beta_{e|n} \cdot \hat{\mathbf{u}}_3)^2}{\gamma (\gamma + 1)} , \quad (6) \]
and from it \( ds/dt = i[H_{g_e}, \mathbf{s}] = \omega_{g_e} \wedge \mathbf{s} = \Upsilon \mathbf{\mu}_e \wedge \mathbf{B} \) which yields
\[ \omega_{g_e} = -\frac{g_e e}{2 m_e} \Upsilon \mathbf{B} . \quad (7) \]

In order to refer the spin precession to the particle orbit, the effective cyclotron frequency \( \omega_{e}^{(c)} \) must now be subtracted. Its value is obtained by computing the instantaneous acceleration \( d\beta/dt = \omega_{e}^{(c)} \beta \hat{\mathbf{u}}_1 \). Omitting terms like \( \mathbf{a}_{e|n} = \frac{g_e}{m_e} \mathbf{E}_{e|n} \), \( \beta_{e|n} \wedge \mathbf{B} \) and \( [\mathbf{B} \cdot (\beta_{e|n} \wedge \beta_n)]\beta_n \) that vanish when averaged, we find
\[ \omega_{e}^{(c)} = \left[ \frac{eB \beta_n}{m_e \beta} \frac{1 - (\beta_{e|n} \cdot \hat{\mathbf{u}}_3)^2}{\gamma_{e|n} \gamma_n (1 + \Pi)^2} + \frac{QB \beta_n \Xi_n}{M \gamma_n} \right] \hat{\mathbf{u}}_1 , \quad (8) \]
where \( \Xi_n = \frac{1}{\gamma_{n} + 1} \left( 1 + \frac{\gamma_n \Pi}{\gamma_n + 1} \right) \) and
\[ \beta = \sqrt{\frac{\beta_n^2 + \frac{\beta_{e|n}^2}{\gamma_n} + 2 \Pi + \Pi^2}{(1 + \Pi)^2}} . \]

From (2), (5) and (8) we obtain
\[ \Omega_e = -\frac{eB}{m_e} \left( \frac{g_e}{2} \Upsilon - \frac{I_e}{\gamma_{e|n} \gamma_n} - U \right) - \frac{QB I_Q + V}{M \gamma_n} , \quad (9) \]
where \( \Upsilon \) is defined in (6) and
\[ U = \frac{1 - (\beta_{e|n} \cdot \hat{\mathbf{u}}_3)^2 \beta_n}{\gamma_{e|n} \gamma_n (1 + \Pi)^2 \beta} , \quad V = \frac{\beta_n}{\beta (1 + \Pi)} \left( 1 + \frac{\gamma_n \Pi}{\gamma_n + 1} \right) . \quad (10) \]
The standard result $\Omega_e = -eB\alpha_e/m_e$, where $\alpha_e = (|g_e| - 2)/2$ is the electron magnetic moment anomaly, is recovered in the limit $Q = 0$.

The calculation of $g_e$-factors, based on bound state (BS) QED, can be carried out with accuracy even though, in our case, the expansion parameter is $Z\alpha \simeq 0.4$. The BS-QED calculation gives \cite{15, 16}

$$ g_b^\prime = 2 \left[ 1 + 2\sqrt{1 - (\alpha Z)^2} + \frac{\alpha Z}{\pi} C^{(2)}(\alpha Z) \right], \quad (11) $$

where $C^{(2)}(\alpha Z) \simeq \frac{1}{2} + \frac{1}{12}(\alpha Z)^2 + \frac{7}{16}(\alpha Z)^4$. From (11) we obtain the values $\alpha_e \simeq -0.065122$, $a_e \simeq -0.0682112$ and $d_e = 0.0505352$ for $^{140}\text{Pr}^{58+}$, $^{142}\text{Pm}^{60+}$ and $^{122}\text{I}^{52+}$ respectively. The addition of more expansion terms \cite{17} does not change these results appreciably.

Consider now the nucleus with spin $I$. The terms of (3) are $\omega_{g_n} = g_n\mu N_B$, where $\mu_N = \frac{|e|}{2m_p}$,

$$ \omega_T^{(n)} = -\frac{\gamma_n - 1}{\gamma_n} \frac{Q_B}{M}, \quad \omega_C^{(n)} = \frac{Q_B}{M\gamma_n} \text{ and give} $$

$$ \Omega_n = \frac{Q_B}{M} \left( \frac{g_n A}{2Z} - 1 \right). \quad (12) $$

### 3. Probability and modulation

Let $|I, m_I\rangle$ and $|s, m_s\rangle$ be the eigenstates of the operators $\hat{I}$ and $\hat{s}$. The total angular momentum operator is $\hat{F} = \hat{s} + \hat{I}$. The angular momentum $F$ assumes the values $F = 3/2, 1/2$, $m_F = 3/2, \pm 1/2$, and $m_F = 1/2, \pm 1/2$ because $I = 1, m_I = \pm 1, 0$ and $s = 1/2, m_s = \pm 1/2$.

By making use of the raising and lowering operators $\hat{F}_\pm = \hat{I}_\pm + \hat{s}_\pm$, we construct the normalized and orthogonal states

$$ \phi_1 \equiv \left| \frac{3}{2}, \frac{3}{2}\right>_F = |1, 1\rangle_I \left| \frac{1}{2}, \frac{1}{2}\right>_s $$

$$ \phi_2 \equiv \left| \frac{3}{2}, \frac{1}{2}\right>_F = \sqrt{\frac{7}{3}} |1, 0\rangle_I \left| \frac{1}{2}, \frac{1}{2}\right>_s + \sqrt{\frac{1}{3}} |1, 1\rangle_I \left| \frac{1}{2}, -\frac{1}{2}\right>_s $$

$$ \phi_3 \equiv \left| \frac{3}{2}, -\frac{1}{2}\right>_F = \sqrt{\frac{7}{3}} |1, 0\rangle_I \left| \frac{1}{2}, -\frac{1}{2}\right>_s + \sqrt{\frac{1}{3}} |1, 1\rangle_I \left| \frac{1}{2}, \frac{1}{2}\right>_s $$

$$ \phi_4 \equiv \left| \frac{3}{2}, -\frac{3}{2}\right>_F = |1, -1\rangle_I \left| \frac{1}{2}, -\frac{1}{2}\right>_s $$

$$ \phi_5 \equiv \left| \frac{1}{2}, \frac{1}{2}\right>_F = \sqrt{\frac{7}{3}} |1, 0\rangle_I \left| \frac{1}{2}, \frac{1}{2}\right>_s - \sqrt{\frac{1}{3}} |1, 1\rangle_I \left| \frac{1}{2}, -\frac{1}{2}\right>_s $$

$$ \phi_6 \equiv \left| \frac{1}{2}, -\frac{1}{2}\right>_F = -\sqrt{\frac{2}{3}} |1, -1\rangle_I \left| \frac{1}{2}, \frac{1}{2}\right>_s + \sqrt{\frac{1}{3}} |1, 0\rangle_I \left| \frac{1}{2}, -\frac{1}{2}\right>_s. $$

The $(6 \times 6)$ matrix with elements $\langle \phi_i | \hat{H}_I | \phi_j \rangle$ has the eigenvalues

$$ \lambda_{1, 4} = -\frac{A}{2} \pm \left( \frac{\Omega_n}{2} + \Omega_e \right), \quad \lambda_{2, 3} = \frac{A}{4} + \frac{\Omega_n}{2} - \frac{\sqrt{\Delta_\pm}}{4}, \quad \lambda_{5, 6} = \frac{A}{4} + \frac{\Omega_n}{2} + \frac{\sqrt{\Delta_\pm}}{4}, \quad (13) $$


\[ \Delta_{\pm} = 9A^2 \pm 4A\Omega_e + 4(\Omega_e - \Omega_n)^2, \]

and the corresponding eigenstates \( |i \rangle \) \((i = 1, \ldots, 6)\)

\[
|1, 4\rangle = \phi_{1,4}, \quad |2, 5\rangle = \frac{B_{\pm}}{\sqrt{1 + B_{\pm}^2}} \phi_2 + \frac{1}{\sqrt{1 + B_{\pm}^2}} \phi_5, \\
|3, 6\rangle = -\frac{A_{\pm}}{\sqrt{1 + A_{\pm}^2}} \phi_3 + \frac{1}{\sqrt{1 + A_{\pm}^2}} \phi_6,
\]

where

\[
A_{\pm} = \frac{9A - 2\Omega_e + 2\Omega_n \pm 3\sqrt{\Delta_{\pm}}}{4\sqrt{2}(\Omega_e - \Omega_n)}, \quad B_{\pm} = \frac{9A + 2\Omega_e - 2\Omega_n \pm 3\sqrt{\Delta_{\pm}}}{4\sqrt{2}(\Omega_e - \Omega_n)}.
\]

In the limit \( A \gg \Omega_{e,n} \) we obtain

\[
|i\rangle \simeq |\phi_i\rangle,
\]

and

\[
\lambda_1 - \lambda_2 = \lambda_2 - \lambda_3 = \frac{\lambda_1 - \lambda_3}{2} = -\frac{\Omega_e + 2\Omega_n}{3}, \quad \lambda_1 - \lambda_4 = -\Omega_e + 2\Omega_n, \quad \lambda_5 - \lambda_6 = \frac{\Omega_e - 4\Omega_n}{3}.
\]

Notice that in these expressions the \( A \)-terms coming from the spin-spin coupling cancel out.

### 3.1. Modulation induced by quantum beats

These results must be now applied to the GSI experiment. Since the heavy nucleus decays via EC, only the states with \( F = 1/2 \) are relevant. For simplicity we confine ourselves to the Hilbert subspace spanned by the states \( \{|5\rangle, |6\rangle\} \). Here we follow [6]. The decay processes involved in the GSI experiment

\[ ^{140}\text{Pr}^{58+} \rightarrow ^{140}\text{Ce}^{58+} + \nu_e, \quad ^{142}\text{Pm}^{60+} \rightarrow ^{140}\text{Nd}^{58+} + \nu_e, \quad ^{122}\text{I}^{52+} \rightarrow ^{122}\text{Te}^{52+} + \nu_e, \]

can be schematically represented as

\[
\mathcal{I} \rightarrow \mathcal{F} + \nu_e,
\]

where \( \mathcal{I} \) and \( \mathcal{F} \) stand for initial and final states. At the initial instant \( t = 0 \) (before injection into the SR) the system nucleus-electron is produced in a superposition of the states \( \{|5\rangle, |6\rangle\} \),

\[
|\mathcal{I}(0)\rangle = \sum_{a=5}^{6} c_a |a\rangle = c_5 |5\rangle + c_6 |6\rangle.
\]

with \( |c_5|^2 + |c_6|^2 = 1 \). Assuming for simplicity that the two states with energies \( \lambda_5 \) and \( \lambda_6 \) decay with the same rate \( \Gamma \), at the time \( t \) the system evolves to the state

\[
|\mathcal{I}(t)\rangle = e^{-\Gamma t/2} \left( c_5 e^{-i\lambda_5 t} |5\rangle + c_6 e^{-i\lambda_6 t} |6\rangle \right).
\]

The probability of EC at time \( t \) becomes

\[
P_{EC}(t) = e^{-\Gamma t} \left| \langle \nu_e, \mathcal{F}|S|\mathcal{I}(t)\rangle \right|^2 = e^{-\Gamma t} \tilde{P}_{EC} \left[ 1 + a_{56} \cos (\omega_{56} t + \varsigma) \right],
\]

where (see Eq. (16))

\[
\omega_{56} = |\lambda_5 - \lambda_6| = \frac{\Omega_e - 4\Omega_n}{3},
\]
\[ \tilde{P}_{EC} = |\langle \nu_e, F | S | 5 \rangle|^2 = |\langle \nu_e, F | S | 6 \rangle|^2, \quad a_{56} = 2|c_5||c_6|, \]
and finally \( S \) is the interaction operator\(^1\).

The phase \( \zeta \) comes from possible phase differences of the amplitude \( c_1 \) and \( c_2 \) and of \(|\langle \nu_e, F | S | 5 \rangle|\) and \(|\langle \nu_e, F | S | 6 \rangle|\). As (18) and (19) show, the modulation of the decay probability does not depend on \( A \).

### 3.2. Estimate of \( \gamma_{e|n} \)

We now compare the frequencies \( \omega_{56}/2\pi \), given by (19), with the experimental values \( \sim 0.14 \) Hz found for \(^{140}\text{Pb}\)\(^{58+}\) and \(^{142}\text{Pm}\)\(^{60+}\) and \( \sim 0.16 \) Hz for \(^{122}\text{I}\)\(^{52+}\) and consider first the case \( \omega_{56} = \lambda_5 - \lambda_6 \). We find

\[
\frac{|e|B}{3m_p} \left[ \frac{m_p}{m_e} \left( \tilde{Y}(a_e + 1) - \frac{i}{\gamma_{e|n}} - \tilde{U} \right) + \frac{I_Q + \tilde{V} Z}{\gamma_n} + 4 Z \left( \frac{g_n A}{2} - 1 \right) \right] = 2\pi 0.14 \text{ Hz}, \tag{20}
\]

where a bar on top means average values. These are computed by first expanding the quantities \( I_{eQ}, \tilde{Y}, U \) and \( V \) in terms of \( P_i < 1 \) and then averaging over the angle by means of

\[
\langle \cos^n \theta \rangle = \frac{1+i-(1)^n}{2(n+1)}. \]

Using \( (\beta_{e|n} \cdot \hat{u}_i)^2 = \frac{1}{3} \beta_{e|n}^2, \) \( i = 1, 2, 3, \) \( \gamma_{e|n}^{(Pm)} = 1.43, \) \( g_n^{(Pr,Pm)} = 2.5, \)

\( g_n^{(I)} = 0.94, \) and \( B = 1.197T \) (we are assuming that this is its average value over the SR circumference), up to \( \mathcal{O}(\Pi^6) \) we obtain from (20) the numerical solutions

\[
\gamma_{e|n}^{(Pr)} \sim 1.07904, \quad \gamma_{e|n}^{(Pm)} \sim 1.08435, \quad \gamma_{e|n}^{(I)} \sim 1.05902, \tag{21}
\]

which must be compared with the Lorentz factors of the bound electron in the Bohr model \( \gamma_{e|n}^{(Pr)} \sim 1.0970, \) \( \gamma_{e|n}^{(Pm)} \sim 1.1040, \) and \( \gamma_{e|n}^{(I)} \sim 1.0776. \) The values (21) imply that the binding energies \( E = T + E_p = -m[1 - \gamma_{e|n} + (\alpha Z)^2], \) where \( T \) and \( E_p \) are kinetic and potential energies of the bound electron, are given by

\[
E^{(Pr)} \sim -54.4\text{keV}, \quad E^{(Pm)} \sim -58.2\text{keV}, \quad E^{(I)} \sim -46.3\text{keV},
\]

in agreement with the values

\[
E^{(Pr)}_R = -49.5\text{keV}, \quad E^{(Pm)}_R = -53.1\text{keV}, \quad E^{(I)}_R = -39.6\text{keV}, \tag{22}
\]

derived from the relativistic equation \([21]\)

\[
E_R = -\frac{RZ^2}{n^2} \left[ 1 + \frac{(\alpha Z)^2}{n} \left( 1 - \frac{3}{4n} \right) \right], \tag{23}
\]

where \( R = 13.6057\text{eV} \) and \( n = 1. \) Results are summarized in Table 1.

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\(^1\) If we consider the Hilbert space spanned by the states \((14), \{|1\}, \ldots, \{6\}\), then the probability (18) assumes the form \( P_{EC}(t) \sim [1 + \sum_{i<j} a_{ij} \cos \omega_{ij} t], \) with \( \omega_{ij} = |\lambda_i - \lambda_j| \) and \( i, j = 1, \ldots, 5 \). It contains 5 terms of which only one (that due to (15) and (16)) contributes to the probability, while the others vanish because of EW selection rules. Assuming, therefore, that the states are equiprobable, the magnitudes take the value \( a_{ij} \approx 0.2, \) with \( \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{5, 6\} \). The values obtained in the GSI experiments are: \( a(Pr) = 0.18(3), a(Pm) = 0.23(4), a(I) = 0.22(2) \) \([19]\).
Table 1. Estimates of $\gamma_{e|n}$ for Pr, Pm, and I. We determine the binding energy by using $E = T + E_p = -m[1 - \gamma_{e|n} + (\alpha Z)^2]$, and compare it with $E_{\text{binding}} = -\frac{R Z^2}{\gamma^2}[1 + \frac{(\alpha Z)^2}{n^2}(1 + \frac{3}{4} n)]$.

| ion       | $\gamma_{e|n}$ | $E$(keV) | $E_{\text{binding}}$(keV) |
|-----------|----------------|----------|---------------------------|
| $^{140}$Pr$^{58+}$ | 1.07904 | -54.3834 | -49.5574 |
| $^{142}$Pm$^{60+}$ | 1.08435 | -58.2041 | -53.1360 |
| $^{122}$I$^{52+}$ | 1.05902 | -46.3179 | -39.6484 |

4. Spin-rotation coupling in compound spin objects

SRC can be generalized to other compound spin systems. Consider, for instance muons bound to nuclei again moving in storage rings. The behaviour of muons can be monitored in this case through their decay products. For negative muon, it is convenient to detect the electron emitted along the path of the nuclei. The quantity of interest is now

$$\Omega_\mu \equiv \omega_{Th}^{(\mu)} + \omega_{g_\mu} - \omega_{c}^{(\mu)}.$$  \hspace{1cm} (24)

where $\omega_{Th}^{(\mu)}$, $\omega_{g_\mu}$ and $\omega_{c}^{(\mu)}$ follow from (5), (7) and (8) now referred to the muon rather than the electron. The muon magnetic moment can also be calculated using (11). The physical situation is analogous to a $g$-2 experiment where oscillations in the decay of the muon along its orbit in a SR are used to measure the anomaly $a_\mu = (g_\mu - 2)/2$. In fact, in the absence of the nucleus ($Q = 0$), we obtain

$$\Omega_{\mu,l} = \omega_{Th,l}^{(\mu)} + \omega_{g_\mu,l} - \omega_{c,l}^{(\mu)} = -\frac{e B}{\tilde{m}} \left( -\frac{\gamma - 1}{\gamma} + \frac{g_\mu}{2} - \frac{1}{\gamma} \right) = -\frac{e B}{\tilde{m}} a_\mu,$$  \hspace{1cm} (25)

where $\tilde{m}$ is the mass of the muon. Neglecting the magnetic field components along the $x_2$ and $x_3$-axes, we can write (24) in the form $\Omega_\mu = \Omega_{\mu} \hat{u}_3$, where

$$\Omega_{\mu} = -\frac{e B}{\tilde{m}} \left[ \left( \frac{g_\mu}{2} \gamma_{1} - \varepsilon_2 \frac{I_{\mu}}{\gamma_{1}} - U \varepsilon_3 \right) + Z \frac{\tilde{m}}{2} \frac{\varepsilon_1 I_{Q} + \varepsilon_5 V}{m_p} \right],$$  \hspace{1cm} (26)

and the $\varepsilon$’s are tags introduced to distinguish the various contributions. In particular $\varepsilon_1$ tags the $g_\mu$ contribution, $\varepsilon_2$ and $\varepsilon_4$ those due to $\omega_{Th}^{(\mu)}$, while $\varepsilon_3$ and $\varepsilon_5$ refer to $\omega_{c}^{(\mu)}$. On carrying out averages over the angles and taking $\gamma_n \sim 1.6$, we obtain Table 2. The $\varepsilon_1, \varepsilon_2, \varepsilon_3$ terms tend to balance one another. The larger contributions come from $\varepsilon_4$ and $\varepsilon_5$ and remain dominant so far as $Z/A \sim 0.5$. The last two terms are not present in the original derivation of SRC in which the fermion is not bound. They are entirely due to the presence of the nucleus to which the muon is attached and their contributions, as mentioned above, can be traced back to $\omega_{Th}^{(\mu)}$ and $\omega_{c}^{(\mu)}$.

Generalizations of SRC can also be obtained for bound bosons following the procedure outlined above. The Lorentz factor $\gamma_{\mu|n}$ that appears in (26) is the only free parameter of the entire calculation. It cannot, in fact be determined, as in [4], by comparing (26) with experimental data that, as yet, do not exist. We can however give some estimates. Choosing $\gamma_{\mu|n}$ equal to the Bohr atom value we obtain the value $\gamma_{\mu|n} = 1.0011$ that substituted in (26) gives the results reported in Table 2. The same value of $\gamma_{\mu|n}$ leads to the Bohr atom energy $E = -10.8952$KeV for He in good agreement with the relativistic value $E_R = -11.2755$KeV.
Table 2. Partial and total contributions to $\Omega_{\mu}$ and comparison of $^4\text{He}^{1+}$ with a few ions for which $Z/A \sim 1/2$.

| $\epsilon$ (Hz) | $^4\text{He}^{1+}$ | $^{142}\text{Pr}^{60+}$ | $^{140}\text{Pm}^{58+}$ | $^{122}\text{I}^{52+}$ |
|-----------------|---------------------|----------------------|----------------------|----------------------|
| $\epsilon_1$    | 5.6374 $10^7$       | 5.1851 $10^7$        | 5.1534 $10^7$        | 5.2735 $10^7$        |
| $\epsilon_2$    | $-1.9405 10^7$      | $-1.9929 10^7$       | $-2.0122 10^7$       | $-1.9383 10^7$       |
| $\epsilon_3$    | $-3.6903 10^7$      | $-3.1395 10^7$       | $-3.0947 10^7$       | $-3.2708 10^7$       |
| $\epsilon_4$    | $1.0940 10^6$       | $0.8968 10^6$        | $0.9220 10^6$        | $0.9066 10^6$        |
| $\epsilon_5$    | $2.0811 10^6$       | $1.7345 10^6$        | $1.7605 10^6$        | $1.8105 10^6$        |
| $\Omega_{\mu}$ (Hz) | $3.2417 10^6$       | $3.1591 10^6$        | $3.1475 10^6$        | $3.3613 10^6$        |

appears from Table 2 that a typical modulation frequency $\sim 3.5 \times 10^6\text{Hz}$ is superimposed on the usual exponential decay of the muon dragged along the ion orbit, while the muon polarization is approximately $\beta \sim \beta_\mu \sim 0.75$. As a comparison, the typical modulation in the $g-2$ experiment is $\Omega_{\mu}^{(g-2)} \sim 2.3 \times 10^3\text{Hz}$ with a polarization $\beta^{(g-2)} \sim 0.9$.

5. Conclusions

In this paper we have discussed the decay of hydrogen-like ions injected in a SR. We consider first the case of an electron bound to a nucleus, which after EC decays. The decay is modulated, and in the case of the ions considered, the period is of the order of 7 sec. These calculations reproduce well the results reported in [2].

In the second case, we consider a muon that is also attached to a nucleus rotating in a SR. As the muon decays along the nucleus path, oscillations also appear due to the combined action of $\omega_{Th}^{(\mu)}$, $\omega_{g\mu}$, and $\omega_c^{(\mu)}$.

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