THE JACOBI OPERATOR AND ITS DONOGHUE m-FUNCTIONS

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Abstract. In this paper we construct Donoghue $m$-functions for the Jacobi differential operator in $L^2((-1,1);(1-x)^\alpha(1+x)^\beta dx)$, associated to the differential expression

$$\tau_{\alpha,\beta} = -(1-x)^{-\alpha}(1+x)^{-\beta}(d/dx)((1-x)^{\alpha+1}(1+x)^{\beta+1})(d/dx),$$

whenever at least one endpoint, $x = \pm 1$, is in the limit circle case. In doing so, we provide a full treatment of the Jacobi operator’s $m$-functions corresponding to coupled boundary conditions whenever both endpoints are in the limit circle case, a topic not covered in the literature.

1. Introduction

This paper should be regarded as a sequel to the recent [27] in which the Donoghue $m$-function was derived for singular Sturm–Liouville operators. To illustrate the theory, we now apply it to a representative example, the Jacobi differential operator associated with $L^2((-1,1);(1-x)^\alpha(1+x)^\beta dx)$-realizations of the differential expression,

$$\tau_{\alpha,\beta} = -(1-x)^{-\alpha}(1+x)^{-\beta}(d/dx)((1-x)^{\alpha+1}(1+x)^{\beta+1})(d/dx),$$

whenever at least one endpoint, $x = \pm 1$, is in the limit circle case (see, e.g. [1, Ch. 22], [4], [9], [20, Sect. 23], [38, Ch. 4], [44, Sects. VII.6.1, XIV.2], [62, Ch. 18], [66, Ch. 7], [69, Ch. 9]).

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[71, Ch. IV]). In particular, this provides a full treatment of \( m \)-functions corresponding to coupled boundary conditions whenever both endpoints are in the limit circle case, a new result.

To set the stage we briefly discuss abstract Donoghue \( m \)-functions following [29, 25, 27, and 28]. Given a self-adjoint extension \( A \) of a densely defined, closed, symmetric operator \( \dot{A} \) in \( \mathcal{H} \) (a complex, separable Hilbert space) with equal deficiency indices and the deficiency subspace \( N'_i \) of \( \dot{A} \) in \( \mathcal{H} \), with

\[
N'_i = \ker ((A)^* - iI_\mathcal{H}), \quad \dim (N'_i) = k \in N \cup \{\infty\},
\]

the Donoghue \( m \)-operator \( M^{Do}_{A,N'_i} (\cdot) \in \mathcal{B}(N'_i) \) associated with the pair \((A,N'_i)\) is given by

\[
M^{Do}_{A,N'_i} (z) = P_{N'_i}(zA + I_\mathcal{H})(A - zI_\mathcal{H})^{-1}P_{N'_i} |_{N'_i},
\]

\[
= zI_{N'_i} + (z^2 + 1)P_{N'_i}(A - zI_\mathcal{H})^{-1}P_{N'_i} |_{N'_i}, \quad z \in \mathbb{C} \setminus \mathbb{R},
\]

with \( I_{N'_i} \) the identity operator in \( N'_i \), and \( P_{N'_i} \) the orthogonal projection in \( \mathcal{H} \) onto \( N'_i \). The special case \( k = 1 \), was discussed in detail by Donoghue [17]; for the case \( k \in \mathbb{N} \) we refer to [31].

More generally, given a self-adjoint extension \( A \) of \( \dot{A} \) in \( \mathcal{H} \) and a closed, linear subspace \( \mathcal{N} \) of \( N'_i \), the Donoghue \( m \)-operator \( M^{Do}_{A,N} (\cdot) \in \mathcal{B}(\mathcal{N}) \) associated with the pair \((A,\mathcal{N})\) is defined by

\[
M^{Do}_{A,N} (z) = P_{\mathcal{N}}(zA + I_\mathcal{H})(A - zI_\mathcal{H})^{-1}P_{\mathcal{N}} |_{\mathcal{N}} = zI_{\mathcal{N}} + (z^2 + 1)P_{\mathcal{N}}(A - zI_\mathcal{H})^{-1}P_{\mathcal{N}} |_{\mathcal{N}}, \quad z \in \mathbb{C} \setminus \mathbb{R},
\]

with \( I_{\mathcal{N}} \) the identity operator in \( \mathcal{N} \) and \( P_{\mathcal{N}} \) the orthogonal projection in \( \mathcal{H} \) onto \( \mathcal{N} \).

Since \( M^{Do}_{A,N} (z) \) is analytic for \( z \in \mathbb{C} \setminus \mathbb{R} \) and satisfies (see [29, Theorem 5.3])

\[
[\text{Im}(z)]^{-1}\text{Im}(M^{Do}_{A,N}(z)) \geq 2 \left[ (|z|^2 + 1) + \left( (|z|^2 - 1)^2 + 4(\text{Re}(z))^2 \right)^{1/2} \right]^{-1} I_{\mathcal{N}},
\]

\[
z \in \mathbb{C} \setminus \mathbb{R},
\]

\( M^{Do}_{A,N}(\cdot) \) is a \( \mathcal{B}(\mathcal{N}) \)-valued Nevanlinna–Herglotz function. Thus, \( M^{Do}_{A,N}(\cdot) \) admits the representation

\[
M^{Do}_{A,N}(z) = \int_{\mathbb{R}} d\Omega^{Do}_{A,N}(\lambda) \left[ \frac{1}{\lambda - z} - \frac{\lambda}{\lambda^2 + 1} \right], \quad z \in \mathbb{C} \setminus \mathbb{R},
\]

where the \( \mathcal{B}(\mathcal{N}) \)-valued measure \( \Omega^{Do}_{A,N}(\cdot) \) satisfies

\[
\Omega^{Do}_{A,N}(\lambda) = (\lambda^2 + 1)(P_{\mathcal{N}} E_A(\lambda) P_{\mathcal{N}} |_{\mathcal{N}}),
\]

\[
\int_{\mathbb{R}} d\Omega^{Do}_{A,N}(\lambda)(1 + \lambda^2)^{-1} = I_{\mathcal{N}},
\]

\[
\int_{\mathbb{R}} d(\xi, \Omega^{Do}_{A,N}(\lambda)\xi)_{\mathcal{N}} = \infty \quad \text{for all} \; \xi \in \mathcal{N} \setminus \{0\},
\]

with \( E_A(\cdot) \) the family of strongly right-continuous spectral projections of \( A \) in \( \mathcal{H} \) (see [25] for details). Operators of the type \( M^{Do}_{A,N}(\cdot) \) and some of its variants have attracted considerable attention in the literature. They appear to go back to Krein [45] (see also [46]), Saakjan [70], and independently, Donoghue [17]. The interested reader can find a wealth of additional information in the context of (1.3)–(1.9) in
without going into further details (see [29, Corollary 5.8] for details) we note that the prime reason for the interest in $M \cdot \Omega_{\mathcal{A}, N_i}(\cdot)$ lies in the fundamental fact that the entire spectral information of $\mathcal{A}$ contained in its family of spectral projections $E_{\mathcal{A}}(\cdot)$, is already encoded in the $B(N_i)$-valued measure $\Omega_{\mathcal{A}, N_i}(\cdot)$ (including multiplicity properties of the spectrum of $\mathcal{A}$) if and only if $\mathcal{A}$ is completely non-self-adjoint in $\mathcal{H}$ (that is, if and only if $\mathcal{A}$ has no invariant subspace on which it is self-adjoint, see [29, Lemma 5.4]).

We also note that a particularly attractive feature of the Donoghue $m$-operator, that distinguishes it from the Weyl–Titchmarsh–Kodaira $m$-operator, consists of the explicit appearance of the resolvent $(\mathcal{A} - zI)_{\mathcal{H}}^{-1}$, $z \in \mathbb{C} \setminus \mathbb{R}$, in it’s definition (1.3) (resp., (1.4)).

In the remainder of this paper, we will exclusively focus on the particular case $\mathcal{N} = \mathcal{N}_i = \ker((\mathcal{A})^* - iI_{\mathcal{H}})$, with $\mathcal{A}$ being a singular Sturm–Liouville operator.

Turning to the content of each section, we discuss the necessary background in connection to singular Sturm–Liouville operators in Section 2. In Sections 3 and 4 we recall the Donoghue $m$-functions in the two limit circle and one limit circle endpoint cases, respectively, following [27]. The Jacobi operator and its Donoghue $m$-functions are the topic of Section 5, with Appendices A–C providing a detailed treatment of solutions of the Jacobi differential equation and the associated hypergeometric differential equations.

Finally, some comments on some of the basic notation used throughout this paper. If $T$ is a linear operator mapping (a subspace of) a Hilbert space into another, then $\text{dom}(T)$ and $\ker(T)$ denote the domain and kernel (i.e., null space) of $T$. The spectrum and resolvent set of a closed linear operator in a Hilbert space will be denoted by $\sigma(\cdot)$ and $\rho(\cdot)$, respectively. Moreover, we denote the scalar product and norm in $L^2((a,b);dx)$ by $(\cdot, \cdot)_{L^2((a,b);dx)}$ (linear in the second argument) and $\| \cdot \|_{L^2((a,b);dx)}$.

2. Some Background

In this section we briefly recall the basics of singular Sturm–Liouville operators. The material is standard and can be found, for instance, in [5, Ch. 6], [14, Chs. 8, 9], [18, Sects. 13.6, 13.9, 13.10], [19], [33, Ch. 4], [39, Ch. III], [59, Ch. V], [60], [64, Ch. 6], [72, Ch. 9], [73, Sect. 8.3], [74, Ch. 13], [76, Chs. 4, 6–8].

Throughout this section we make the following assumptions:

Hypothesis 2.1. Let $(a, b) \subseteq \mathbb{R}$ and suppose that $p, q, r$ are (Lebesgue) measurable functions on $(a, b)$ such that the following items (i)–(iii) hold:

(i) $r > 0$ a.e. on $(a, b)$, $r \in L^1_{\text{loc}}((a,b);dx)$.
(ii) $p > 0$ a.e. on $(a, b)$, $1/p \in L^1_{\text{loc}}((a,b);dx)$.
(iii) $q$ is real-valued a.e. on $(a, b)$, $q \in L^1_{\text{loc}}((a,b);dx)$.

Given Hypothesis 2.1, we study Sturm–Liouville operators associated with the general, three-coefficient differential expression $\tau$ of the form

$$\tau = \frac{1}{r(x)} \left[ \frac{d}{dx} p(x) \frac{d}{dx} + q(x) \right] \text{ for a.e. } x \in (a,b) \subseteq \mathbb{R}. \quad (2.1)$$
If \( f \in AC_{\text{loc}}((a,b)) \), then the quasi-derivative of \( f \) is defined to be \( f^{[1]} := pf' \). Moreover, the Wronskian of two functions \( f, g \in AC_{\text{loc}}((a,b)) \) is defined by

\[
W(f, g)(x) = f(x)g^{[1]}(x) - f^{[1]}(x)g(x) \quad \text{for a.e. } x \in (a,b). \tag{2.2}
\]

Assuming Hypothesis 2.1, the maximal operator \( T_{\text{max}} \) in \( L^2((a,b); rdx) \) associated with \( \tau \) is defined by

\[
T_{\text{max}}f = \tau f, \quad f \in \text{dom}(T_{\text{max}}) = \{ g \in L^2((a,b); rdx) \mid g, g^{[1]} \in AC_{\text{loc}}((a,b)); \tau g \in L^2((a,b); rdx) \}. \tag{2.3}
\]

The preminimal operator \( \hat{T}_{\text{min}} \) in \( L^2((a,b); rdx) \) associated with \( \tau \) is defined by

\[
\hat{T}_{\text{min}}f = \tau f, \quad f \in \text{dom}(\hat{T}_{\text{min}}) = \{ g \in L^2((a,b); rdx) \mid g, g^{[1]} \in AC_{\text{loc}}((a,b)); \text{supp}(g) \subset (a,b) \text{ is compact}; \tau g \in L^2((a,b); rdx) \}. \tag{2.4}
\]

One can prove that \( \hat{T}_{\text{min}} \) is closable, and one then defines the minimal operator \( T_{\text{min}} \) by \( T_{\text{min}} = \hat{T}_{\text{min}} \).

Still assuming Hypothesis 2.1, one can prove the following basic fact,

\[
(\hat{T}_{\text{min}})^* = T_{\text{max}}, \tag{2.5}
\]

and hence \( T_{\text{max}} \) is closed. Moreover, \( \hat{T}_{\text{min}} \) is essentially self-adjoint if and only if \( T_{\text{max}} \) is symmetric, and then \( \hat{T}_{\text{min}} = T_{\text{min}} = T_{\text{max}} \).

The celebrated Weyl alternative can be stated as follows:

**Theorem 2.2** (Weyl’s Alternative).
Assume Hypothesis 2.1. Then the following alternative holds: Either

(i) for every \( z \in \mathbb{C} \), all solutions \( u \) of \((\tau - z)u = 0\) are in \( L^2((a,b); rdx) \) near \( b \) (resp., near \( a \)),

or,

(ii) for every \( z \in \mathbb{C} \), there exists at least one solution \( u \) of \((\tau - z)u = 0\) which is not in \( L^2((a,b); rdx) \) near \( b \) (resp., near \( a \)). In this case, for each \( z \in \mathbb{C} \setminus \mathbb{R} \), there exists precisely one solution \( u_b \) (resp., \( u_a \)) of \((\tau - z)u = 0 \) (up to constant multiples) which lies in \( L^2((a,b); rdx) \) near \( b \) (resp., near \( a \)).

This yields the limit circle/limit point classification of \( \tau \) at an interval endpoint as follows.

**Definition 2.3.** Assume Hypothesis 2.1.
In case (i) in Theorem 2.2, \( \tau \) is said to be in the limit circle case at \( b \) (resp., \( a \)). (Frequently, \( \tau \) is then called quasi-regular at \( b \) (resp., \( a \)).)

In case (ii) in Theorem 2.2, \( \tau \) is said to be in the limit point case at \( b \) (resp., \( a \)).

If \( \tau \) is in the limit circle case at \( a \) and \( b \) then \( \tau \) is also called quasi-regular on \((a,b)\).

The next result links self-adjointness of \( T_{\text{min}} \) (resp., \( T_{\text{max}} \)) and the limit point property of \( \tau \) at both endpoints. Here, and throughout, we shall employ the notation

\[
N_z = \ker(T_{\text{max}} - zI_{L^2((a,b); rdx)}), \quad z \in \mathbb{C}. \tag{2.6}
\]
Theorem 2.4. Assume Hypothesis 2.1, then the following items (i) and (ii) hold:

(i) If $\tau$ is in the limit point case at a (resp., b), then
\[ W(f,g)(a) = 0 \quad \text{(resp., } W(f,g)(b) = 0) \] for all $f, g \in \text{dom}(T_{\text{max}}). \] (2.7)

(ii) Let $T_{\text{min}} = \overline{T_{\text{min}}}$. Then
\[ n_{\pm}(T_{\text{min}}) = \dim (N_{\pm}) \]
\[ = \begin{cases} 2 & \text{if } \tau \text{ is in the limit circle case at } a \text{ and } b, \\ 1 & \text{if } \tau \text{ is in the limit circle case at } a \\ & \text{and in the limit point case at } b, \text{ or vice versa,} \\ 0 & \text{if } \tau \text{ is in the limit point case at } a \text{ and } b. \end{cases} \] (2.8)

In particular, $T_{\text{min}} = T_{\text{max}}$ is self-adjoint if and only if $\tau$ is in the limit point case at $a$ and $b$.

All self-adjoint extensions of $T_{\text{min}}$ are then described as follows:

Theorem 2.5. Assume Hypothesis 2.1 and that $\tau$ is in the limit circle case at $a$ and $b$ (i.e., $\tau$ is quasi-regular on $(a,b)$). In addition, assume that $v_j \in \text{dom}(T_{\text{max}})$, $j = 1, 2$, satisfy
\[ W(\overline{v_1}, v_2)(a) = W(\overline{v_1}, v_2)(b) = 1, \quad W(\overline{v_1}, v_2)(a) = W(\overline{v_1}, v_2)(b) = 0, \quad j = 1, 2. \] (2.9)

(E.g., real-valued solutions $v_j$, $j = 1, 2$, of $(\tau - \lambda)u = 0$ with $\lambda \in \mathbb{R}$, such that $W(v_1, v_2) = 1$.) For $g \in \text{dom}(T_{\text{max}})$ we introduce the generalized boundary values
\[ \tilde{g}_1(a) = -W(v_2, g)(a), \quad \tilde{g}_1(b) = -W(v_2, g)(b), \quad \tilde{g}_2(a) = W(v_1, g)(a), \quad \tilde{g}_2(b) = W(v_1, g)(b). \] (2.10)

Then the following items (i)–(iii) hold:

(i) All self-adjoint extensions $T_{\gamma, \delta}$ of $T_{\text{min}}$ with separated boundary conditions are of the form
\[ T_{\gamma, \delta} f = \tau f, \quad \gamma, \delta \in [0, \pi), \] (2.11)
\[ f \in \text{dom}(T_{\gamma, \delta}) = \left\{ g \in \text{dom}(T_{\text{max}}) \left| \begin{array}{l} \cos(\gamma)\tilde{g}_1(a) + \sin(\gamma)\tilde{g}_2(a) = 0, \\
\cos(\delta)\tilde{g}_1(b) + \sin(\delta)\tilde{g}_2(b) = 0 \end{array} \right. \right\}. \]

(ii) All self-adjoint extensions $T_{\varphi, R}$ of $T_{\text{min}}$ with coupled boundary conditions are of the type
\[ T_{\varphi, R} f = \tau f, \] (2.12)
\[ f \in \text{dom}(T_{\varphi, R}) = \left\{ g \in \text{dom}(T_{\text{max}}) \left| \begin{array}{l} \left( \begin{array}{c} \tilde{g}_1(b) \\ \tilde{g}_2(b) \end{array} \right) = e^{i\varphi} R \left( \begin{array}{c} \tilde{g}_1(a) \\ \tilde{g}_2(a) \end{array} \right) \end{array} \right. \right\}, \]
where $\varphi \in [0, \pi)$, and $R$ is a real $2 \times 2$ matrix with $\det(R) = 1$ (i.e., $R \in SL(2, \mathbb{R})$).

(iii) Every self-adjoint extension of $T_{\text{min}}$ is either of type (i) (i.e., separated) or of type (ii) (i.e., coupled).

One can now detail the characterization of $\text{dom}(T_{\text{min}})$ by
\[ T_{\text{min}} f = \tau f, \] (2.13)
\[ f \in \text{dom}(T_{\text{min}}) = \left\{ g \in \text{dom}(T_{\text{max}}) \left| \tilde{g}_1(a) = \tilde{g}_2(a) = \tilde{g}_1(b) = \tilde{g}_2(b) = 0 \right. \right\}. \]
Finally, we turn to the characterization of generalized boundary values in the case where $T_{\min}$ is bounded from below following [26] and [60].

We briefly recall the basics of oscillation theory with particular emphasis on principal and nonprincipal solutions, a notion originally due to Leighton and Morse [50] (see also Rellich [66], [67] and Hartman and Wintner [36, Appendix]). Our outline below follows [13], [18, Sects. 13.6, 13.9, 13.10], [33, Ch. 7, [35, Ch. XI], [60], [76, Chs. 4, 6–8].

**Definition 2.6.** Assume Hypothesis 2.1.

(i) Fix $c \in (a, b)$ and $\lambda \in \mathbb{R}$. Then $\tau - \lambda$ is called nonoscillatory at $a$ (resp., $b$), if every real-valued solution $u(\lambda, \cdot)$ of $\tau u = \lambda u$ has finitely many zeros in $(a, c)$ (resp., $(c, b)$). Otherwise, $\tau - \lambda$ is called oscillatory at $a$ (resp., $b$).

(ii) Let $\lambda_0 \in \mathbb{R}$. Then $T_{\min}$ is called bounded from below by $\lambda_0$, and one writes $T_{\min} \geq \lambda_0 I_{L^2((a,b);rdx)}$, if

$$
(u, [T_{\min} - \lambda_0 I_{L^2((a,b);rdx)}]u)_{L^2((a,b);rdx)} \geq 0, \quad u \in \text{dom}(T_{\min}).
$$

The following is a key result.

**Theorem 2.7.** Assume Hypothesis 2.1. Then the following items (i)–(iii) are equivalent:

(i) $T_{\min}$ (and hence any symmetric extension of $T_{\min}$) is bounded from below.

(ii) There exists a $\nu_0 \in \mathbb{R}$ such that for all $\lambda < \nu_0$, $\tau - \lambda$ is nonoscillatory at $a$ and $b$.

(iii) For fixed $c, d \in (a, b)$, $c \leq d$, there exists a $\nu_0 \in \mathbb{R}$ such that for all $\lambda < \nu_0$, $\tau u = \lambda u$ has (real-valued) nonvanishing solutions $u_a(\lambda, \cdot) \neq 0, \tilde{u}_a(\lambda, \cdot) \neq 0$ in the neighborhood $(a, c]$ of $a$, and (real-valued) nonvanishing solutions $u_b(\lambda, \cdot) \neq 0, \tilde{u}_b(\lambda, \cdot) \neq 0$ in the neighborhood $[d, b)$ of $b$, such that

$$
W(\tilde{u}_a(\lambda, \cdot), u_a(\lambda, \cdot)) = 1, \quad u_a(\lambda, x) = o(\tilde{u}_a(\lambda, x)) \quad x \downarrow a, 
$$

$$
W(\tilde{u}_b(\lambda, \cdot), u_b(\lambda, \cdot)) = 1, \quad u_b(\lambda, x) = o(\tilde{u}_b(\lambda, x)) \quad x \uparrow b, 
$$

$$
\int_a^c dx \frac{1}{p(x)^{-1}u_a(\lambda,x)^{-2}} = \int_d^b dx \frac{1}{p(x)^{-1}u_b(\lambda,x)^{-2}} = \infty, 
$$

$$
\int_a^c dx \frac{1}{p(x)^{-1}\tilde{u}_a(\lambda,x)^{-2}} < \infty, \quad \int_d^b dx \frac{1}{p(x)^{-1}\tilde{u}_b(\lambda,x)^{-2}} < \infty. 
$$

**Definition 2.8.** Assume Hypothesis 2.1, suppose that $T_{\min}$ is bounded from below, and let $\lambda \in \mathbb{R}$. Then $u_a(\lambda, \cdot)$ (resp., $u_b(\lambda, \cdot)$) in Theorem 2.7 (iii) is called a principal (or minimal) solution of $\tau u = \lambda u$ at $a$ (resp., $b$). A real-valued solution $\tilde{u}_a(\lambda, \cdot)$ (resp., $\tilde{u}_b(\lambda, \cdot)$) of $\tau = \lambda u$ linearly independent of $u_a(\lambda, \cdot)$ (resp., $u_b(\lambda, \cdot)$) is called nonprincipal at $a$ (resp., $b$). In particular, $\tilde{u}_a(\lambda, \cdot)$ (resp., $\tilde{u}_b(\lambda, \cdot)$) in (2.15)–(2.18) are nonprincipal solutions at $a$ (resp., $b$).

Next, we revisit in Theorem 2.5 how the generalized boundary values are utilized in the description of all self-adjoint extensions of $T_{\min}$ in the case where $T_{\min}$ is bounded from below.

**Theorem 2.9** ([26, Theorem 4.5]). Assume Hypothesis 2.1 and that $\tau$ is in the limit circle case at $a$ and $b$ (i.e., $\tau$ is quasi-regular on $(a,b)$). In addition, assume that $T_{\min} \geq \lambda_0 I$ for some $\lambda_0 \in \mathbb{R}$, and denote by $u_a(\lambda_0, \cdot)$ and $\tilde{u}_a(\lambda_0, \cdot)$ (resp.,
one obtains for all $g \in (2.21)$

\[ W(\tilde{u}_a(\lambda_0, \cdot), u_a(\lambda_0, \cdot)) = W(\tilde{u}_b(\lambda_0, \cdot), u_b(\lambda_0, \cdot)) = 1. \]  

(2.19)

Introducing $v_j \in \text{dom}(T_{\max})$, $j = 1, 2$, via

\[ v_1(x) = \begin{cases} \tilde{u}_a(\lambda_0, x), & \text{for } x \text{ near } a, \\ \tilde{u}_b(\lambda_0, x), & \text{for } x \text{ near } b, \end{cases} \]

\[ v_2(x) = \begin{cases} u_a(\lambda_0, x), & \text{for } x \text{ near } a, \\ u_b(\lambda_0, x), & \text{for } x \text{ near } b, \end{cases} \]  

(2.20)

one obtains for all $g \in \text{dom}(T_{\max})$,

\[ \tilde{g}(a) = -W(v_2, g)(a) = \tilde{g}_1(a) = -W(u_a(\lambda_0, \cdot), g)(a) = \lim_{x \to a} \frac{g(x)}{u_a(\lambda_0, x)}, \]

\[ \tilde{g}(b) = -W(v_2, g)(b) = \tilde{g}_b(b) = -W(u_b(\lambda_0, \cdot), g)(b) = \lim_{x \to b} \frac{g(x)}{u_b(\lambda_0, x)}, \]  

(2.21)

\[ \tilde{g}'(a) = W(v_1, g)(a) = \tilde{g}_2(a) = W(\tilde{u}_a(\lambda_0, \cdot), g)(a) = \lim_{x \to a} \frac{g(x) - \tilde{g}(a)\tilde{u}_a(\lambda_0, x)}{u_a(\lambda_0, x)}, \]

\[ \tilde{g}'(b) = W(v_1, g)(b) = \tilde{g}_b(b) = W(\tilde{u}_b(\lambda_0, \cdot), g)(b) = \lim_{x \to b} \frac{g(x) - \tilde{g}(b)\tilde{u}_b(\lambda_0, x)}{u_b(\lambda_0, x)}. \]  

(2.22)

In particular, the limits on the right-hand sides in (2.21), (2.22) exist.

The Friedrichs extension $T_F$ of $T_{\min}$ now permits a particularly simple characterization in terms of the generalized boundary values $\tilde{g}(a), \tilde{g}(b)$ as derived by Niessen and Zettl [60] (see also [30], [40], [41], [42], [55], [67], [68], [75]):

**Theorem 2.10.** Assume Hypothesis 2.1 and that $\tau$ is in the limit circle case at $a$ and $b$ (i.e., $\tau$ is quasi-regular on $(a, b)$). In addition, assume that $T_{\min} \geq \lambda_0 I$ for some $\lambda_0 \in \mathbb{R}$. Then the Friedrichs extension $T_F = T_{0, 0}$ of $T_{\min}$ is characterized by

\[ T_F f = \tau f, \quad f \in \text{dom}(T_F) = \{ g \in \text{dom}(T_{\max}) \mid \tilde{g}(a) = \tilde{g}(b) = 0 \}. \]  

(2.23)

3. Donoghue $m$-functions: Two Limit Circle Endpoints

The Donoghue $m$-functions in the case where $\tau$ is in the limit circle case at $a$ and $b$ is the primary topic of this section following [27, Sect. 6].

**Hypothesis 3.1.** In addition to Hypothesis 2.1 assume that $\tau$ is in the limit circle case at $a$ and $b$. Moreover, for $z \in \rho(T_{0, 0})$, let $\{u_j(z, \cdot)\}_{j=1, 2}$ denote solutions to $\tau u = zu$ which satisfy the boundary conditions

\[ \tilde{u}_1(z, a) = 0, \quad \tilde{u}_1(z, b) = 1, \]

\[ \tilde{u}_2(z, a) = 1, \quad \tilde{u}_2(z, b) = 0. \]  

(3.1)

Assume Hypotheses 3.1. By Theorem 2.5 or Theorem 2.9, the following statements (i)-(iii) hold.

(i) If $\gamma, \delta \in [0, \pi)$, then the operator $T_{\gamma, \delta}$ defined by

\[ T_{\gamma, \delta} f = T_{\max} f, \]

\[ f \in \text{dom}(T_{\gamma, \delta}) = \left\{ g \in \text{dom}(T_{\max}) \mid \begin{align*} \cos(\gamma)\tilde{g}(a) + \sin(\gamma)\tilde{g}'(a) = 0, \\ \cos(\delta)\tilde{g}(b) + \sin(\delta)\tilde{g}'(b) = 0 \end{align*} \right\}, \]

is a self-adjoint extension of $T_{\min}$. 

(ii) If $\varphi \in [0, \pi)$ and $R \in \text{SL}(2, \mathbb{R})$, then the operator $T_{\varphi, R}$ defined by

$$T_{\varphi, R} f = T_{\text{max}} f,$$

$$f \in \text{dom}(T_{\varphi, R}) = \left\{ g \in \text{dom}(T_{\text{max}}) \mid \left( \bar{g}(b) \overline{g'(b)} \right) = e^{i\varphi} R \left( \bar{g}(a) \overline{g'(a)} \right) \right\},$$

is a self-adjoint extension of $T_{\text{min}}$.

(iii) If $T$ is a self-adjoint extension of $T_{\text{min}}$, then either $T = T_{\gamma, \delta}$ for some $\gamma, \delta \in [0, \pi)$, or $T = T_{\varphi, R}$ for some $\varphi \in [0, \pi)$ and some $R \in \text{SL}(2, \mathbb{R})$.

**Notational Convention.** To describe all possible self-adjoint boundary conditions associated with self-adjoint extensions of $T_{\text{min}}$ effectively, we will frequently employ the notation $T_{A, B, M^P_{A, B} (\cdot)}$, etc., where $A, B$ represents $\gamma, \delta$ in the case of separated boundary conditions and $\varphi, R$ in the context of coupled boundary conditions.

Choosing $\gamma = \delta = 0$ in (3.2) yields the self-adjoint extension with Dirichlet-type boundary conditions at $a$ and $b$, equivalently, the Friedrichs extension $T_{F}$ of $T_{\text{min}}$:

$$\text{dom}(T_{0, 0}) = \text{dom}(T_{F}) = \left\{ g \in \text{dom}(T_{\text{max}}) \mid \bar{g}(a) = \bar{g}(b) = 0 \right\}.$$  

Since the coefficients of the Sturm–Liouville differential expression are real, the following conjugation property holds:

$$\overline{u_j(z, \cdot)} = u_j(\overline{z}, \cdot), \quad z \in \rho(T_{0, 0}), \quad j \in \{1, 2\}.$$  

Applying (3.1), one computes

$$W(u_1(z, \cdot), u_2(z, \cdot))(a) = -\overline{u_1'(z, a)},$$

$$W(u_1(z, \cdot), u_2(z, \cdot))(b) = \overline{u_2'(z, b)}, \quad z \in \rho(T_{0, 0}).$$  

In particular, since the Wronskian of two solutions is constant,

$$\overline{u_2'(z, b)} = -\overline{u_1'(z, a)}, \quad z \in \rho(T_{0, 0}).$$  

We begin by recalling the orthonormal basis for $\mathcal{N}_{\pm i}$ given by $\{v_j(\pm i, \cdot)\}_{j=1,2}$,

$$v_1(\pm i, \cdot) = c_1(\pm i) u_1(\pm i, \cdot),$$

$$v_2(\pm i, \cdot) = c_2(\pm i) \left[ u_2(\pm i, \cdot) - \frac{(u_1(\pm i, \cdot), u_2(\pm i, \cdot))_{L^2((a, b); rdrx)}}{||u_1(\pm i, \cdot)||_{L^2((a, b); rdrx)}^2} u_1(\pm i, \cdot) \right].$$

with

$$c_1(\pm i) = \left[ ||u_1(\pm i, \cdot)||_{L^2((a, b); rdrx)}^2 \right]^{-1/2},$$

$$c_2(\pm i) = \left[ \frac{\text{Im}(\overline{u_2'(i, b)})}{\text{Im}(\overline{u_1'(i, b))}} \right] u_1(\pm i, \cdot).$$

$$= \left[ \pm \text{Im}(\overline{u_2'(i, a)}) \pm \frac{\text{Im}(\overline{u_2'(i, b)})^2}{\text{Im}(\overline{u_1'(i, b))}} \right].$$

The Donoghue $m$-function $M^P_{T_{A, B, \mathcal{N}_{{\gamma}}(\cdot)}}$ with $T_{A, B}$ any self-adjoint extension of $T_{\text{min}}$ is provided next (cf. Theorems 6.1–6.3 in [27]).
Theorem 3.2. Assume Hypothesis 3.1 and let \( \{v_j(i, \cdot)\}_{j=1,2} \) be the orthonormal basis for \( N_i \) defined in (3.8)–(3.11). The Donoghue m-function \( M^D_{T_{\theta,0},N_i}(\cdot) : \mathbb{C}\setminus \mathbb{R} \to \mathcal{B}(N_i) \) for \( T_{0,0} \) satisfies

\[
M^D_{T_{0,0},N_i}(\pm i) = \pm iI_{N_i},
\]

\[
M^D_{T_{0,0},N_i}(z) = -\sum_{j,k=1}^2 \left[ i\tilde{\delta}_{j,k} + W_{j,k}(z)\right](v_k(i, \cdot), \cdot)_{L^2((a,b);rdx)}v_j(i, \cdot)_{N'_i}, \tag{3.12}
\]

\[
= -iI_{N'_i} - \sum_{j,k=1}^2 W_{j,k}(z)(v_k(i, \cdot), \cdot)_{L^2((a,b);rdx)}v_j(i, \cdot)_{N'_i},
\]

\[
z \in \mathbb{C}\setminus \mathbb{R}, \ z \neq \pm i,
\]

where the matrix \( (W_{j,k}(\cdot))_{j,k=1}^2 \), \( z \in \mathbb{C}\setminus \mathbb{R}, \ z \neq \pm i \), is given by

\[
W_{1,1}(z) = [c_1(i)]^2\left[ \tilde{u}'_1(z, b) - \tilde{u}'_1(-i, b) \right], \tag{3.13}
\]

\[
W_{1,2}(z) = c_1(i)c_2(i)\left\{ \frac{\text{Im}(\tilde{u}'_2(i, b))}{\text{Im}(\tilde{u}'_1(i, b))}\left[ \tilde{u}'_1(-i, b) - \tilde{u}'_1(z, b) \right] + \tilde{u}'_2(z, b) + \tilde{u}'_1(-i, a) \right\}, \tag{3.14}
\]

\[
W_{2,1}(z) = -c_1(i)c_2(i)\left\{ \frac{\text{Im}(\tilde{u}'_2(i, b))}{\text{Im}(\tilde{u}'_1(i, b))}\left[ \tilde{u}'_1(z, b) - \tilde{u}'_1(-i, b) \right] + \tilde{u}'_2(-i, b) + \tilde{u}'_1(z, a) \right\}, \tag{3.15}
\]

\[
W_{2,2}(z) = [c_2(i)]^2\left[ \tilde{u}'_2(-i, b) - \tilde{u}'_2(z, b) + \frac{\text{Im}(\tilde{u}'_2(i, b))}{\text{Im}(\tilde{u}'_1(i, b))}\left[ \tilde{u}'_1(z, b) - \tilde{u}'_1(-i, b) \right] \frac{\text{Im}(\tilde{u}'_2(i, b))}{\text{Im}(\tilde{u}'_1(i, b))}\left[ \tilde{u}'_1(z, a) - \tilde{u}'_1(-i, a) \right] + \tilde{u}'_2(-i, a) - \tilde{u}'_2(z, a) + \frac{\text{Im}(\tilde{u}'_2(i, b))}{\text{Im}(\tilde{u}'_1(i, b))}\left[ \tilde{u}'_1(z, a) - \tilde{u}'_1(-i, a) \right] \right]. \tag{3.16}
\]

Furthermore, the following items \((i)–(v)\) hold.

(i) If \( \gamma, \delta \in (0, \pi) \), then the Donoghue m-function \( M^D_{T_{\gamma,\delta},N_i}(\cdot) : \mathbb{C}\setminus \mathbb{R} \to \mathcal{B}(N_i) \) for \( T_{\gamma,\delta} \) satisfies

\[
M^D_{T_{\gamma,\delta},N_i}(\pm i) = \pm iI_{N_i},
\]

\[
M^D_{T_{\gamma,\delta},N_i}(z) = M^D_{T_{0,0},N_i}(z) + (i - z) \sum_{j,k,\ell=1}^2 \left[ K_{\gamma,\delta}(z)^{-1} \right]_{j,k} W^K_{\ell,\ell}(z)(u_\ell(z, \cdot), \cdot)_{L^2((a,b);rdx)}v_j(i, \cdot)_{N'_i},
\]

\[
z \in \mathbb{C}\setminus \mathbb{R}, \ z \neq \pm i,
\]

\[
(3.17)
\]
where the invertible matrix \(K_{\gamma, \delta} (\cdot)\) and \((W_{\ell, k}^{K_r} (\cdot))_{\ell, k=1}^2\) are given by

\[
K_{\gamma, \delta}(z) = \begin{pmatrix}
\cot(\delta) + \overline{u}_1'(z, b) & -\overline{u}_1'(z, a) \\
-\overline{u}_2'(z, b) & -\cot(\gamma) - \overline{u}_2'(z, a)
\end{pmatrix},
\]

where \(0 < \delta < \pi\) and \(\gamma, \delta\) are given by

\[
\begin{align*}
W_{1,1}^{K_r}(z) &= c_1(i)[\overline{u}_1'(z, b) - \overline{u}_1'(-i, b)], \\
W_{1,2}^{K_r}(z) &= c_1(i)[\overline{u}_1'(z, b) + \overline{u}_1'(-i, a)], \\
W_{2,1}^{K_r}(z) &= \overline{v}_2(-i, b)\overline{u}_1'(z, b) - \overline{v}_2'(-i, b) - \overline{v}_2(-i, a)\overline{u}_1'(z, a) \\
&= -c_2(i)\left\{\frac{\text{Im}(\overline{u}_2'(i, b))}{\text{Im}(\overline{u}_1'(i, b))}\right\} + \overline{u}_1'(-i, b) + \overline{u}_1'(z, a)
\end{align*}
\]

\[
W_{2,2}^{K_r}(z) = \overline{v}_2(-i, b)\overline{u}_2'(z, b) - \overline{v}_2(-i, a)\overline{u}_2'(z, a) + \overline{v}_2'(i, a)
\]

(ii) If \(\varphi \in [0, \pi]\) and \(R \in \mathcal{S}(2, \mathbb{R})\) with \(R_{1,2} \neq 0\), then the Donoghue \(m\)-function \(M_{T_{\varphi, R}, N_i}(\cdot) : \mathbb{C} \setminus \mathbb{R} \rightarrow \mathcal{B}(N_i)\) for \(T_{\varphi, R}\) satisfies

\[
M_{T_{\varphi, R}, N_i}(\pm i) = \pm i N_i,
\]

\[
M_{T_{\varphi, R}, N_i}(z) = M_{T_{0,0}, N_i}(z) + (i - z) \sum_{j,k=1}^2 [K_{\varphi, R}(z)^{-1}]_{j,k} W_{L^2((a,b),rdx)}^{K_r}(u_j(\overline{\varphi}, \cdot); u_k(i, \cdot))|_{N_i},
\]

where \((W_{L^2}^{K_r}(\cdot))_{\ell, k=1}^2\) is once again given in (3.19)–(3.22) and the invertible matrix \(K_{\varphi, R}(\cdot)\) is given by

\[
K_{\varphi, R}(z) = \begin{pmatrix}
\frac{-R_{2,2}}{R_{1,2}} + \overline{u}_1'(z, b) & e^{-i\varphi} \frac{R_{1,2}}{R_{1,2}} - \overline{u}_1'(z, a) \\
e^{i\varphi} \frac{R_{1,2}}{R_{1,2}} + \overline{u}_2'(z, b) & -\frac{R_{1,1}}{R_{1,2}} - \overline{u}_2'(z, a)
\end{pmatrix}.
\]

(iii) If \(\gamma \in (0, \pi)\), then the Donoghue \(m\)-function \(M_{T_{0,0}, N_i}(\cdot) : \mathbb{C} \setminus \mathbb{R} \rightarrow \mathcal{B}(N_i)\) for \(T_{0,0}\) satisfies

\[
M_{T_{0,0}, N_i}(\pm i) = \pm i N_i,
\]

\[
M_{T_{0,0}, N_i}(z) = M_{T_{0,0}, N_i}(z) + \frac{z - i}{\cot(\gamma) + \overline{u}_2'(z, a)} \sum_{\ell=1}^2 [u_2(\overline{\varphi}, \cdot); u_k(i, \cdot)]L^2((a,b);rdx) |_{N_i},
\]

where \(\cot(\gamma) + \overline{u}_2'(z, a) \neq 0\) and the scalars \(\{W_{L^2}^{K_r}(\cdot)\}_{\ell=1,2}\) are given by (3.20) and (3.22).

(iv) If \(\delta \in (0, \pi)\), then the Donoghue \(m\)-function \(M_{T_{0,\delta}, N_i}(\cdot) : \mathbb{C} \setminus \mathbb{R} \rightarrow \mathcal{B}(N_i)\) for \(T_{0,\delta}\) satisfies

\[
M_{T_{0,\delta}, N_i}(\pm i) = i N_i,
\]
where \( \cot(\delta) + \tilde{u}_1(z, b) \neq 0 \) and the scalars \( \{W_{\ell,1}(\cdot)\}_{\ell=1}^2 \) are given by (3.19) and (3.21).

(v) If \( \varphi \in [0, \pi) \) and \( R \in SL(2, \mathbb{R}) \) with \( R_{1,2} = 0 \), then the Donoghue \( m \)-function \( M_{T_0,R,N_\ell}^D(\pm i) = \pm i I_{N_\ell} \),

\[
M_{T_0,R,N_\ell}^D(z) = M_{T_0,R,N_\ell}^D(z)
\]

\[
- \frac{z - i}{k_{\varphi,R}(z)}(u_{\varphi,R}(\zeta, \cdot) + W_{\ell,1}(z) u_{\varphi,R}(\zeta, \cdot)) \left\{ e^{i\varphi} R_{2,1} K_{0,k}^R(z) + R_{2,2} K_{0,k}^R(z) \right\} u_{\varphi,R}(z, b),
\]

where the matrix \( \{W_{\ell,1}(\cdot)\}_{\ell=1}^2 \) is once again given in (3.19)–(3.22) and the nonzero scalar \( k_{\varphi,R}(\cdot) \) is given by

\[
k_{\varphi,R}(z) = -R_{2,1} R_{2,2} - e^{i\varphi} R_{2,2} \tilde{u}_{\varphi,R}(z, a) + \tilde{u}_{\varphi,R}(z, b),
\]

where

\[
u_{\varphi,R}(\zeta, \cdot) = e^{-i\varphi} R_{2,2} u_2(\zeta, \cdot) + u_1(\zeta, \cdot), \quad \zeta \in \rho(T_0,0).
\]

Remark 3.3. For the Krein extension, \( T_{0,R_K} \), under the additional assumption that \( T_{min} \geq \varepsilon I \left| L^2((a,b);rdx) \right| \) for some \( \varepsilon > 0 \), applying [24, Theorem 3.5(ii)], one computes for the matrix \( K_{0,R_K} \),

\[
K_{0,R_K}(z) = \begin{pmatrix} \tilde{u}_1^j(z,b) - \tilde{u}_1^j(0,b) & \tilde{u}_1^j(0,a) - \tilde{u}_1^j(z,a) \\ \tilde{u}_2^j(z,b) - \tilde{u}_2^j(0,b) & \tilde{u}_2^j(0,a) - \tilde{u}_2^j(z,a) \end{pmatrix}, \quad z \in \rho(T_0,0) \cap \rho(T_{0,R_K}),
\]

where we note that \( 0 \in \sigma(T_{0,R_K}) \).

4. DONOGHUE \( m \)-FUNCTIONS: ONE LIMIT CIRCLE ENDPOINT

In this section we recall the Donoghue \( m \)-functions in the case where \( \tau \) is in the limit circle case at precisely one endpoint (which we choose to be \( a \) without loss of generality) following [27, Sect. 5].

Hypothesis 4.1. In addition to Hypothesis 2.1 assume that \( \tau \) is in the limit circle case at \( a \) and in the limit point case at \( b \). Moreover, for \( z \in \rho(T_0) \), let \( \psi(z, \cdot) \) denote the unique solution to \((\tau - z) y = 0 \) that satisfies \( \psi(z, \cdot) \in L^2((a,b);rdx) \) and \( \tilde{\psi}(z, a) = 1 \).

Assume Hypothesis 4.1. By Theorem 2.5 or Theorem 2.9, the following statements (i) and (ii) hold.
Theorem 4.2. Assume Hypothesis

\[ T_\gamma f = T_{\max} f, \]

\[ f \in \text{dom}(T_\gamma) = \{ g \in \text{dom}(T_{\max}) \mid \cos(\gamma)g(a) + \sin(\gamma)g'(a) = 0 \}, \] (4.1)

is a self-adjoint extension of \( T_{\min} \).

(ii) If \( T \) is a self-adjoint extension of \( T_{\min} \), then \( T = T_\gamma \) for some \( \gamma \in [0, \pi) \).

Statements analogous to (i) and (ii) hold if \( \tau \) is in the limit point case at \( a \) and in the limit circle case at \( b \); for brevity we omit the details.

Choosing \( \gamma = 0 \) in (4.1) yields the self-adjoint extension \( T_0 \) with a Dirichlet-type boundary condition at \( a \):

\[ \text{dom}(T_0) = \{ g \in \text{dom}(T_{\max}) \mid g(a) = 0 \}. \] (4.2)

Since the coefficients \( p, q, \) and \( r \) are real-valued, the solution \( \psi(z, \cdot) \) has the following conjugation property:

\[ \overline{\psi(z, \cdot)} = \psi(\overline{z}, \cdot), \quad z \in \rho(T_0). \] (4.3)

We now turn to the Donoghue \( m \)-function \( M^D_{\gamma, N_0}(\cdot) \) with \( T_\gamma \) any self-adjoint extension of \( T_{\min} \) (cf. Theorems 5.1 and 5.2 in [27]).

**Theorem 4.2.** Assume Hypothesis 4.1 and let \( \gamma \in [0, \pi) \). The Donoghue \( m \)-function \( M^D_{\gamma, N_0}(\cdot) : \mathbb{C} \setminus \mathbb{R} \to \mathcal{B}(N_0) \) for \( T_\gamma \) satisfies

\[ M^D_{\gamma, N_0}(\pm i) = \pm iN_0, \quad \gamma \in [0, \pi), \]

\[ M^D_{\gamma, N_0}(z) = \left[ -i + \frac{\overline{\psi}'(z, a) - \overline{\psi}'(-i, a)}{\text{Im}(\psi(i, a))} \right] N_0, \quad z \in \mathbb{C} \setminus \mathbb{R}, \quad z \neq \pm i, \]

\[ M^D_{\gamma, N_0}(z) = M^D_{\gamma, N_0}(\overline{z}) \quad \text{if} \quad z \in \mathbb{R}, \quad z \neq \pm i. \] (4.4)

5. The Jacobi Operator and its Donoghue \( m \)-functions

We now turn to the principal topic of this paper, the Jacobi differential expression

\[ \tau_{\alpha, \beta} = -(1 - x)^{-\alpha}(1 + x)^{-\beta}(d/dx)((1 - x)^{\alpha+1}(1 + x)^{\beta+1})(d/dx), \]

\[ x \in (-1, 1), \quad \alpha, \beta \in \mathbb{R}, \] (5.1)

that is, in connection with Sections 2 one now has

\[ a = -1, \quad b = 1, \]

\[ p(x) = p_{\alpha, \beta}(x) = (1 - x)^{\alpha+1}(1 + x)^{\beta+1}, \quad q(x) = q_{\alpha, \beta}(x) = 0, \] (5.2)

\[ r(x) = r_{\alpha, \beta}(x) = (1 - x)^{\alpha}(1 + x)^{\beta}, \quad x \in (-1, 1), \quad \alpha, \beta \in \mathbb{R} \]

(see, e.g. [1, Ch. 22], [4], [9], [20, Sect. 23], [38, Ch. 4], [44, Sects. VII.6.1, XIV.2], [62, Ch. 18], [71, Ch. IV]).

\( L^2 \)-realizations of \( \tau_{\alpha, \beta} \) are thus most naturally associated with the Hilbert space \( L^2((-1, 1); r_{\alpha, \beta}dx) \). However, occasionally the weight function is absorbed into the Hilbert space leading to an equivalent differential expression in the Hilbert
Returning to the concrete Jacobi case at hand, one can choose

\[ y_1(x) = 1, \quad x \in (-1, 1), \]

\[ y_2(x) = \int_0^x dx' (1 - x')^{-1-\alpha}(1 + x')^{-1-\beta} \]

(5.4)

Thus, one has the classification,

\[
\tau_{\alpha,\beta} \begin{cases} 
\text{regular at } -1 \text{ if and only if } \alpha \in \mathbb{R}, \beta \in (-1, 0), \\
\text{in the limit circle case at } -1 \text{ if and only if } \alpha \in \mathbb{R}, \beta \in [0, 1), \\
\text{in the limit point case at } -1 \text{ if and only if } \alpha \in \mathbb{R}, \beta \in (-1, 1), \\
\text{regular at } +1 \text{ if and only if } \alpha \in (-1, 0), \beta \in \mathbb{R}, \\
\text{in the limit circle case at } +1 \text{ if and only if } \alpha \in [0, 1), \beta \in \mathbb{R}, \\
\text{in the limit point case at } +1 \text{ if and only if } \alpha \in \mathbb{R} \setminus (-1, 1), \beta \in \mathbb{R}.
\end{cases}
\]

(5.5)

The maximal and preminimal operators, \( T_{\max,\alpha,\beta} \) and \( T_{\min,0,\alpha,\beta} \), associated to \( \tau_{\alpha,\beta} \) in \( L^2((-1, 1); r_{\alpha,\beta} dx) \) are then given by

\[
T_{\max,\alpha,\beta} f = \tau_{\alpha,\beta} f, \\
f \in \text{dom}(T_{\max,\alpha,\beta}) = \{ g \in L^2((-1, 1); r_{\alpha,\beta} dx) \mid g, g^{[1]} \in AC_{loc}((-1, 1)); \tau_{\alpha,\beta} g \in L^2((-1, 1); r_{\alpha,\beta} dx) \},
\]

(5.6)

and

\[
T_{\min,0,\alpha,\beta} f = \tau_{\alpha,\beta} f, \\
f \in \text{dom}(T_{\min,0,\alpha,\beta}) = \{ g \in L^2((-1, 1); r_{\alpha,\beta} dx) \mid g, g^{[1]} \in AC_{loc}((-1, 1)); \tau_{\alpha,\beta} g \in L^2((-1, 1); r_{\alpha,\beta} dx) \}.
\]

(5.7)
The fact (5.4) naturally leads to principal and nonprincipal solutions $u_{\pm 1, \alpha, \beta}(0, x)$ and $\tilde{u}_{\pm 1, \alpha, \beta}(0, x)$ of $\tau_{\alpha, \beta} y = 0$ near $\pm 1$ as follows:

$$
u_{-1, \alpha, \beta}(0, x) = \begin{cases} -2^{-\alpha-1} \beta^{-1}(1 + x)^{-\beta}[1 + O(1 + x)], & \beta \in (-\infty, 0), \\ 1, & \beta \in [0, \infty), \end{cases} \quad \alpha \in \mathbb{R}$$

$$\tilde{u}_{-1, \alpha, \beta}(0, x) = \begin{cases} 1, & \beta \in (-\infty, 0), \alpha \in \mathbb{R}, \\ -2^{-\alpha-1} \ln((1 + x)/2), & \beta = 0, \\ 2^{-\alpha-1} \beta^{-1}(1 + x)^{-\beta}[1 + O(1 + x)], & \beta \in (0, \infty), \end{cases}$$

(5.8)

and

$$u_{+1, \alpha, \beta}(0, x) = \begin{cases} 2^{-\beta-1} \alpha^{-1}(1 - x)^{-\alpha}[1 + O(1 - x)], & \alpha \in (-\infty, 0), \\ 1, & \alpha \in [0, \infty), \end{cases} \quad \beta \in \mathbb{R}$$

$$\tilde{u}_{+1, \alpha, \beta}(0, x) = \begin{cases} 1, & \alpha \in (-\infty, 0), \beta \in \mathbb{R}, \\ 2^{-\beta-1} \ln((1 - x)/2), & \alpha = 0, \\ -2^{-\beta-1} \alpha^{-1}(1 - x)^{-\alpha}[1 + O(1 - x)], & \alpha \in (0, \infty), \end{cases}$$

(5.9)

Combining the fact (5.5) with Theorem 2.2, $T_{\min, 0, \alpha, \beta}$ is essentially self-adjoint in $L^2((-1, 1); r_{\alpha, \beta} dx)$ if and only if $\alpha, \beta \in \mathbb{R} \setminus (-1, 1)$. Thus, boundary values for $T_{\max, \alpha, \beta}$ at $-1$ exist if and only if $\alpha \in \mathbb{R}$, $\beta \in (-1, 1)$, and similarly, boundary values for $T_{\max, \alpha, \beta}$ at $+1$ exist if and only if $\alpha \in (-1, 1)$, $\beta \in \mathbb{R}$.

Employing the principal and nonprincipal solutions (5.8), (5.9) at $\pm 1$, according to (2.22), (2.23), generalized boundary values for $g \in \text{dom}(T_{\max, \alpha, \beta})$ are of the form

$$\overline{g}(-1) = \begin{cases} g(-1), & \beta \in (-1, 0), \\ -2^{\alpha+1} \lim_{x \downarrow -1} g(x)/\ln((1 + x)/2), & \beta = 0, \\ 2^{\beta+1} \lim_{x \downarrow -1} (1 + x)^{\beta} g(x), & \beta \in (0, 1), \end{cases} \quad \alpha \in \mathbb{R},$$

$$\overline{g}'(-1) = \begin{cases} g^{[1]}(-1), & \beta \in (-1, 0), \\ \lim_{x \downarrow -1} \left[ g(x) + \overline{g}(-1) 2^{\alpha-1} \ln((1 + x)/2) \right], & \beta = 0, \\ \lim_{x \downarrow -1} \left[ g(x) - \overline{g}(-1) 2^{\alpha-1} \beta^{-1}(1 + x)^{-\beta} \right], & \beta \in (0, 1), \end{cases}$$

(5.10)

$$\overline{g}(1) = \begin{cases} g(1), & \alpha \in (-1, 0), \\ 2^{\beta+1} \lim_{x \uparrow 1} g(x)/\ln((1 - x)/2), & \alpha = 0, \\ -\alpha 2^{\beta+1} \lim_{x \uparrow 1} (1 - x)^{\alpha} g(x), & \alpha \in (0, 1), \end{cases} \quad \beta \in \mathbb{R},$$

$$\overline{g}'(1) = \begin{cases} g^{[1]}(1), & \alpha \in (-1, 0), \\ \lim_{x \uparrow 1} \left[ g(x) - \overline{g}(1) 2^{-\beta-1} \ln((1 - x)/2) \right], & \alpha = 0, \\ \lim_{x \uparrow 1} \left[ g(x) + \overline{g}(1) 2^{-\beta-1} \alpha^{-1}(1 - x)^{-\alpha} \right], & \alpha \in (0, 1), \end{cases}$$

(5.11)

As a result, the minimal operator $T_{\min}$ associated to $\tau_{\alpha, \beta}$, that is, $T_{\min} = T_{\min, 0}$, is thus given by

$$T_{\min, \alpha, \beta} f = \tau_{\alpha, \beta} f, \quad f \in \text{dom}(T_{\min, \alpha, \beta}) = \{ g \in L^2((-1, 1); r_{\alpha, \beta} dx) \mid g, g^{[1]} \in AC_{\text{loc}}((-1, 1)) \},$$

(5.12)
The Jacobi Donoghue $m$-function is given by

\[
\bar{g}(-1) = \bar{g}'(-1) = \bar{g}(1) = \bar{g}'(1) = 0; \quad \tau_{\alpha,\beta} g \in L^2((-1, 1); r_{\alpha,\beta} dx). \]

For a detailed treatment of solutions of the Jacobi differential equation and the associated hypergeometric differential equations we refer to Appendices A–C.

Remark 5.1. We now mention a few special cases of interest. The Legendre equation ($\alpha = \beta = 0$) has frequently been discussed in the literature, see, for instance, [26] and the extensive list of references cited therein. The Gegenbauer, or ultraspherical, equation (see, e.g., [1, Ch. 22], [62, Ch. 18], [71, Ch. IV]) can be realized by choosing $\mu = 1$ in the Gegenbauer equation, or $\mu = 0$ in the Jacobi equation, or $\alpha = \beta = -1/2$ in the Jacobi equation (see, e.g., [1, Ch. 22], [62, Ch. 18], [71, Ch. IV]), whereas the second kind is realized by choosing $\alpha = 1$ in the Gegenbauer equation, or $\alpha = \beta = 1/2$ in the Jacobi equation (see, e.g., [1, Ch. 22], [62, Ch. 18], [71, Ch. IV]).

We now determine the solutions $\phi_{0,\alpha,\beta}(z, \cdot)$ and $\theta_{0,\alpha,\beta}(z, \cdot)$ of $\tau_{\alpha,\beta} u = zu, z \in \mathbb{C}$, that are subject to the conditions

\[
\begin{align*}
\bar{\phi}_{0,\alpha,\beta}(z, -1) &= 0, & \bar{\phi}'_{0,\alpha,\beta}(z, -1) &= 1, \\
\bar{\theta}_{0,\alpha,\beta}(z, -1) &= 1, & \bar{\theta}'_{0,\alpha,\beta}(z, -1) &= 0.
\end{align*}
\]

In particular, one obtains from (C.1),

\[
\begin{align*}
\phi_{0,\alpha,\beta}(z, x) &= \begin{cases} 2^{-\alpha-1}\beta^{-1}y_{2,\alpha,\beta,-1}(z, x), & \beta \in (-1, 0), \\
y_{1,\alpha,\beta,-1}(z, x), & \beta \in [0, 1), \end{cases} \\
\theta_{0,\alpha,\beta}(z, x) &= \begin{cases} y_{1,\alpha,\beta,-1}(z, x), & \beta \in (-1, 0), \\
2^{-\alpha-1}y_{2,\alpha,0,-1}(z, x), & \beta = 0, \\
2^{-\alpha-1}\beta^{-1}y_{2,\alpha,\beta,-1}(z, x), & \beta \in (0, 1), \end{cases}
\end{align*}
\]

5.1. The Regular and Limit Circle Case $\alpha, \beta \in (-1, 1)$. In this section we compute the Donoghue $m$-function when the Jacobi problem considered is either in the regular or limit circle case at $\pm 1$.

Using (5.13), the solutions in (3.1) for this example are given by

\[
\begin{align*}
u_{1,\alpha,\beta}(z, x) &= \frac{\phi_{0,\alpha,\beta}(z, x)}{\bar{\phi}_{0,\alpha,\beta}(z, 1)} \\
&= \begin{cases} y_{2,\alpha,\beta,-1}(z, x)/\bar{y}_{2,\alpha,\beta,-1}(z, 1), & \beta \in (-1, 0), \\
y_{1,\alpha,\beta,-1}(z, x)/\bar{y}_{1,\alpha,\beta,-1}(z, 1), & \beta \in [0, 1), \end{cases} \\
u_{2,\alpha,\beta}(z, x) &= \theta_{0,\alpha,\beta}(z, x) - \bar{\theta}_{0,\alpha,\beta}(z, 1)/\bar{\phi}_{0,\alpha,\beta}(z, 1)\phi_{0,\alpha,\beta}(z, x) \\
&= \begin{cases} y_{1,\alpha,\beta,-1}(z, x) - [\bar{y}_{1,\alpha,\beta,-1}(z, 1)/\bar{y}_{2,\alpha,\beta,-1}(z, 1)]y_{2,\alpha,\beta,-1}(z, x), & \beta \in (-1, 0), \\
2^{-\alpha-1}y_{2,\alpha,0,-1}(z, x) - [\bar{y}_{2,\alpha,0,-1}(z, 1)/\bar{y}_{1,\alpha,0,-1}(z, 1)]y_{1,\alpha,0,-1}(z, x), & \beta = 0, \\
2^{-\alpha-1}\beta^{-1}y_{2,\alpha,\beta,-1}(z, x) - [\bar{y}_{2,\alpha,\beta,-1}(z, 1)/\bar{y}_{1,\alpha,\beta,-1}(z, 1)]y_{1,\alpha,\beta,-1}(z, x), & \beta \in (0, 1), \end{cases}
\end{align*}
\]
where the generalized boundary values are given in (C.2)–(C.4). Hence substituting (5.15) into (3.8)–(3.11) and applying Theorem 3.2 yields the (Nevanlinna–Herglotz) Donoghue $m$-function $M_{T_{0,RK,\alpha,\beta}}^{\sigma}(\cdot)$ for any self-adjoint extension $T_{\alpha,B,\alpha,\beta}$ of $T_{\text{min}}$ with $\alpha, \beta \in (-1, 1)$.

As an example of coupled boundary conditions, we consider the Krein–von Neumann extension following Example 4.3 found in [24]. For $\alpha, \beta \in (-1, 1)$, the following five cases are associated with a strictly positive minimal operator $T_{\text{min},\alpha,\beta}$ and we now provide the corresponding choices of $R_{K,\alpha,\beta}$ for the Krein–von Neumann extension $T_{0,RK,\alpha,\beta}$ of $T_{\text{min},\alpha,\beta}$:

\[ T_{0,RK,\alpha,\beta} f = \tau_{\alpha,\beta} f, \quad (5.16) \]

\[ f \in \text{dom}(T_{0,RK,\alpha,\beta}) = \left\{ g \in \text{dom}(T_{\text{max},\alpha,\beta}) \mid \begin{array}{l}
\left(1 - 2^{1-\alpha} - 1, \frac{\Gamma(-\alpha)\Gamma(-\beta)}{\Gamma(-\alpha - \beta)} \right)
\end{array} \right\}, \quad \alpha, \beta \in (-1, 0), \]

\[ R_{K,\alpha,\beta} = \begin{cases} 
0 & \alpha \in (0, 1), \beta \in (-1, 0), \\
1 & \alpha \in (-1, 0), \beta = 0,
\end{cases} \quad (5.17) \]

where we interpret $1/\Gamma(0) = 0, \psi(\cdot) = \Gamma'/(\cdot) \Gamma(\cdot)$ denotes the Digamma function, and $\gamma_E = -\psi(1) = 0.57721\ldots$ represents Euler's constant. Obviously, $\det(R_{K,\alpha,\beta}) = 1$ in all five cases. Furthermore, as $R_{1,2} \neq 0$ for each case, Theorem 3.2 (ii) applies and one obtains the Donoghue $m$-function $M_{T_{0,RK,\alpha,\beta}}^{\sigma}(\cdot)$ for the Krein–von Neumann extension $T_{0,RK,\alpha,\beta}$ by utilizing (5.15) and (5.17) as well as the explicit form of $K_{0,RK} (\cdot)$ in (3.30). We note once again that $M_{T_{0,RK,\alpha,\beta}}^{\sigma}(\cdot)$ is a Nevanlinna–Herglotz function.

In the remaining four cases not covered by (5.17), given by all combinations of $\alpha = 0, \beta = 0, \alpha \in (0, 1)$, and $\beta \in (0, 1)$, one observes that [24, Theorem 3.5] is not applicable as the underlying minimal operator, $T_{\text{min},\alpha,\beta}$, is nonnegative but not strictly positive. In particular, the Jacobi polynomials satisfy Friedrichs boundary conditions for $\alpha, \beta \in [0, 1)$, hence $0 \in \sigma(T_{F,\alpha,\beta}), \alpha, \beta \in [0, 1)$ and $T_{\text{min},\alpha,\beta} \geq 0$ is nonnegative, but not strictly positive when $\alpha, \beta \in [0, 1)$.

5.2. Precisely One Interval Endpoint in the Limit Point Case. In this section we determine the Donoghue $m$-function in all situations where precisely
one interval endpoint is in the limit point case. We will focus on the case when
$\alpha \in (-\infty, -1)$ or $\alpha \in [1, \infty)$, so that the right endpoint $x = 1$ represents
the limit point case. The converse situation can be obtained by reflection with respect to
the origin (i.e., considering the transform $(-1, 1) \ni x \mapsto -x \in (-1, 1)$).

We recall from [26, Sect. 6] that the Weyl–Titchmarsh–Kodaira solution and
$m$-function corresponding to the Friedrichs (resp., Dirichlet) boundary condition at
$x = -1$ is determined via the requirement

$$\psi_{0, \alpha, \beta}(z, \cdot) = \theta_{0, \alpha, \beta}(z, \cdot) + m_{0, \alpha, \beta}(z)\phi_{0, \alpha, \beta}(z, \cdot) \in L^2((c, 1); r_{\alpha, \beta}dx),$$

$z \in \mathbb{C} \backslash \sigma(T_{F, \alpha, \beta}), \ \alpha \in (-\infty, -1) \cup [1, \infty), \ \beta \in (-1, 1), \ c \in (-1, 1)$.  

(5.18)

In particular, since $\widetilde{\psi}_{0, \alpha, \beta}(z, -1) = m_{0, \alpha, \beta}(z)$ one finds from Theorem 4.2,

$$M_{T_{0, \alpha, \beta}, N_i}^D(z) = \left[ -i + \frac{m_{0, \alpha, \beta}(z) - m_{0, \alpha, \beta}(-i)}{\text{Im}(m_{0, \alpha, \beta}(i))} \right] I_{N_i},$$

$$M_{T_{0, \alpha, \beta}, N_i}^D(z) = M_{T_{0, \alpha, \beta}, N_i}^D(z) + (i - z) \frac{m_{0, \alpha, \beta}(z) - m_{0, \alpha, \beta}(-i)}{\text{cot}(\gamma) + m_{0, \alpha, \beta}(z)} \times \langle \psi_{0, \alpha, \beta}(z, \cdot) \rangle |_{N_i}, \ \gamma \in (0, \pi),$$

$\alpha \in (-\infty, -1) \cup [1, \infty), \ \beta \in (-1, 1), \ z \in \mathbb{C} \backslash \mathbb{R}$,  

(5.19)

where $\psi_{0, \alpha, \beta}(z, \cdot)$ and $m_{0, \alpha, \beta}(z, \cdot)$ are given by the following:

**(I) The Case $\alpha \in [1, \infty)$ and $\beta \in (-1, 0)$:**

$$\psi_{0, \alpha, \beta}(z, x) = y_{1, \alpha, \beta, -1}(z, x) - 2^{-\alpha - 1}y_{2, \alpha, \beta, -1}(z, x)m_{0, \alpha, \beta}(z),$$

$$m_{0, \alpha, \beta}(z) = 2^{1 + \alpha + \beta} \Gamma(1 + \beta) \frac{1}{\Gamma(1 - \beta)} \times \frac{\Gamma(1 + \alpha - \beta + \sigma_{\alpha, \beta}(z))/2}{\Gamma(1 + \alpha + \beta + \sigma_{\alpha, \beta}(z))/2} \Gamma(1 + \alpha - \beta - \sigma_{\alpha, \beta}(z))/2 \Gamma(1 + \alpha + \beta - \sigma_{\alpha, \beta}(z))/2,$$

$z \in \rho(T_{F, \alpha, \beta}), \ \alpha \in [1, \infty), \ \beta \in (-1, 0)$,

$$\sigma(T_{F, \alpha, \beta}) = \{ (n - \beta)(n + 1 + \alpha) \}_{n \in \mathbb{N}_0}, \ \alpha \in [1, \infty), \ \beta \in (-1, 0),$$

(5.20)

with

$$\sigma_{\alpha, \beta}(z) = [(1 + \alpha + \beta)^2 + 4z]^{1/2}.$$  

(5.21)

**(II) The Case $\alpha \in [1, \infty)$ and $\beta = 0$:**

$$\psi_{0, \alpha, 0}(z, x) = -2^{-\alpha - 1}y_{2, \alpha, 0, -1}(z, x) + y_{1, \alpha, 0, -1}(z, x)m_{0, \alpha, 0}(z),$$

$$m_{0, \alpha, 0}(z) = -2^{-\alpha - 1} \{ 2\gamma_E + \psi([1 + \alpha + \sigma_{\alpha, 0}(z])/2) + \psi([1 + \alpha - \sigma_{\alpha, 0}(z])/2) \},$$

$z \in \rho(T_{F, \alpha, 0}), \ \alpha \in [1, \infty), \ \beta = 0$,

$$\sigma(T_{F, \alpha, 0}) = \{ n(n + 1 + \alpha) \}_{n \in \mathbb{N}_0}, \ \alpha \in [1, \infty), \ \beta = 0.$$  

(5.22)
(III) The Case $\alpha \in [1, \infty) \text{ and } \beta \in (0, 1)$:

$$
\psi_{0,\alpha,\beta}(z, x) = 2^{-\alpha -1}\beta^{-1}y_{2,\alpha,\beta,-1}(z, x) + y_{1,\alpha,\beta,-1}(z, x)m_{0,\alpha,\beta}(z),
$$

$$
m_{0,\alpha,\beta}(z) = \beta^{-1}2^{-1-\alpha-\beta}\Gamma(1-\beta)
\frac{\Gamma(1+\alpha+\beta)}{\Gamma(1+\beta)} \times \frac{\Gamma(1+\alpha+\beta-\sigma_{\alpha,\beta}(z))/2}{\Gamma(1+\alpha-\beta-\sigma_{\alpha,\beta}(z))/2} \frac{\Gamma((1+\alpha-\beta-\sigma_{\alpha,\beta}(z))/2)}{} \left(5.23\right)
$$

$$
z \in \rho(T_{F,\alpha,\beta}), \ \alpha \in [1, \infty), \ \beta \in (0, 1),
$$

$$
\sigma(T_{F,\alpha,\beta}) = \{n(n+1+\alpha+\beta)\}_{n \in \mathbb{N}_0}, \ \alpha \in [1, \infty), \ \beta \in (0, 1).
$$

(IV) The Case $\alpha \in (-\infty, -1) \text{ and } \beta \in (-1, 0)$:

$$
\psi_{0,\alpha,\beta}(z, x) = y_{1,\alpha,\beta,-1}(z, x) - 2^{-\alpha -1}\beta^{-1}y_{2,\alpha,\beta,-1}(z, x)m_{0,\alpha,\beta}(z),
$$

$$
m_{0,\alpha,\beta}(z) = 2^{1+\alpha+\beta}\beta\Gamma(1+\beta)
\frac{\Gamma(1-\beta)}{\Gamma(1+\beta)} \times \frac{\Gamma(1-\alpha-\beta-\sigma_{\alpha,\beta}(z))/2}{\Gamma(1-\alpha-\beta+\sigma_{\alpha,\beta}(z))/2} \frac{\Gamma((1-\alpha-\beta+\sigma_{\alpha,\beta}(z))/2)}{\Gamma((1-\alpha+\beta+\sigma_{\alpha,\beta}(z))/2)} \left(5.24\right)
$$

$$
z \in \rho(T_{F,\alpha,\beta}), \ \alpha \in (-\infty, -1), \ \beta \in (-1, 0),
$$

$$
\sigma(T_{F,\alpha,\beta}) = \{(n+\alpha+\beta)(n+1)\}_{n \in \mathbb{N}_0}, \ \alpha \in (-\infty, -1), \ \beta \in (-1, 0).
$$

(V) The Case $\alpha \in (-\infty, -1) \text{ and } \beta = 0$:

$$
\psi_{0,\alpha,0}(z, x) = -2^{-\alpha -1}y_{2,\alpha,0,-1}(z, x) + y_{1,\alpha,0,-1}(z, x)m_{0,\alpha,0}(z),
$$

$$
m_{0,\alpha,0}(z) = -2^{-\alpha -1}\{2\gamma_E + \psi(1-\alpha+\sigma_{\alpha,0}(z))/2 + \psi(1-\alpha-\sigma_{\alpha,0}(z))/2\},
$$

$$
z \in \rho(T_{F,\alpha,0}), \ \alpha \in (-\infty, -1), \ \beta = 0,
$$

$$
\sigma(T_{F,\alpha,0}) = \{(n+\alpha)(n+1)\}_{n \in \mathbb{N}_0}, \ \alpha \in (-\infty, -1), \ \beta = 0. \left(5.25\right)
$$

(VI) The Case $\alpha \in (-\infty, -1) \text{ and } \beta \in (0, 1)$:

$$
\psi_{0,\alpha,\beta}(z, x) = 2^{-\alpha -1}\beta^{-1}y_{2,\alpha,\beta,-1}(z, x) + y_{1,\alpha,\beta,-1}(z, x)m_{0,\alpha,\beta}(z),
$$

$$
m_{0,\alpha,\beta}(z) = -\beta^{-1}2^{-1-\alpha-\beta}\Gamma(1-\beta)
\frac{\Gamma(1+\beta)}{\Gamma(1+\beta)} \times \frac{\Gamma(1+\beta+\alpha+\sigma_{\alpha,\beta}(z))/2}{\Gamma(1+\beta-\alpha-\sigma_{\alpha,\beta}(z))/2} \frac{\Gamma((1+\beta-\alpha-\sigma_{\alpha,\beta}(z))/2)}{\Gamma((1+\beta+\alpha+\sigma_{\alpha,\beta}(z))/2)} \left(5.26\right)
$$

$$
z \in \rho(T_{F,\alpha,\beta}), \ \alpha \in (-\infty, -1), \ \beta \in (0, 1),
$$

$$
\sigma(T_{F,\alpha,\beta}) = \{(n+\alpha)(n+1+\beta)\}_{n \in \mathbb{N}_0}, \ \alpha \in (-\infty, -1), \ \beta \in (0, 1).
$$

APPENDIX A. THE HYPERGEOMETRIC AND JACOBI DIFFERENTIAL EQUATIONS

In this appendix we provide the connection between the hypergeometric differential equation (cf. [1, Sect. 15.5])

$$
\xi(1-\xi)\ddot{\psi}(\xi) + [c(a+b+1)]\dot{\psi}(\xi) - abw(\xi) = 0, \ \xi \in (0, 1), \tag{A.1}
$$

(21) and the Jacobi differential equation

$$
\tau_{\alpha,\beta}y(z, x) = -(1-x^2)y''(z, x) + [\alpha - \beta + (\alpha + \beta + 2)x]y'(z, x) = zy(z, x), \ \alpha, \beta \in \mathbb{R}, \ x \in (-1, 1), \tag{A.2}
$$

\( A.1 \) and \( A.2 \)
(where \( t = d/dz \)). Making the substitution \( \xi = (1 + x)/2 \) in (A.2) yields
\[
\xi(1 - \xi) \ddot{y}(z, \xi) + [\beta + 1 - (\alpha + \beta + 2)\xi]\dot{y}(z, \xi) + zy(z, \xi) = 0,
\]
\( \alpha, \beta \in \mathbb{R}, \, \xi \in (0,1). \) \hspace{1cm} (A.3)

which is equal to (A.1) once one identifies,
\[
a = [1 + \alpha + \beta + \sigma_{\alpha, \beta}(z)]/2, \quad b = [1 + \alpha + \beta - \sigma_{\alpha, \beta}(z)]/2, \quad c = 1 + \beta,
\]
\( \sigma_{\alpha, \beta}(z) = [(1 + \alpha + \beta)^2 + 4z]^{1/2}. \) \hspace{1cm} (A.4)

At the endpoint \( x = -1 \) of the Jacobi equation the substitution used to arrive at (A.3) yields \( \xi = 0 \), hence we next consider solutions of (A.1) near \( \xi = 0 \) (cf. [1, Eqs. 15.5.3, 15.5.4]) (analogous solutions near \( \xi = 1 \) are found in (A.13))
\[
w_{1,0}(\xi) = F(a, b; c; \xi) = \sum_{n \in \mathbb{N}_0} \frac{(a)_n(b)_n \xi^n}{(c)_n n!}, \quad a, b \in \mathbb{C}, \, c \in \mathbb{C}(\mathbb{N}_0),
\]
\[
w_{2,0}(\xi) = \xi^{1-c} F(a-c+1, b-c+1; 2-c; \xi), \quad a, b \in \mathbb{C}, \, (c-1) \in \mathbb{C}\backslash \mathbb{N}, \quad \xi \in (0,1). \) \hspace{1cm} (A.5)

Here \( F(\cdot, \cdot; \cdot; \cdot) \) (frequently written as \( {}_2F_1(\cdot, \cdot; \cdot; \cdot) \)) denotes the hypergeometric function (see, e.g., [1, Ch. 15]), \( \psi(\cdot) = \Gamma'(\cdot)/\Gamma(\cdot) \) the Digamma function, \( \gamma_E = -\psi(1) = 0.57721 \ldots \) represents Euler’s constant, and
\[
(\zeta)_0 = 1, \quad (\zeta)_n = \Gamma(\zeta + n)/\Gamma(\zeta), \quad n \in \mathbb{N}, \quad \zeta \in \mathbb{C}(\mathbb{N}_0), \) \( \zeta \in (0,1). \) \hspace{1cm} (A.6)

abbreviates Pochhammer’s symbol (see, e.g., [1, Ch. 6]).

In addition,
\[
w_{1,0} \text{ and } w_{2,0} \text{ are linearly independent if } c \in \mathbb{C}\backslash \mathbb{Z}, \) \hspace{1cm} (A.7)

which can be seen by noticing the different behaviors of \( w_{1,0}(\xi), \, w_{2,0}(\xi) \) around \( \xi = 0 \). One notes that only the case \( c = 1 + \beta \in (0,2) \) is needed. Thus, for \( c = 1 \) we will use instead
\[
w_{1,0}(\xi) = F(a, b; 1; \xi), \quad a, b \in \mathbb{C},
\]
\[
w_{2,0}^{\ln}(\xi) = F(a, b; 1; \xi) \ln(\xi) + \sum_{n \in \mathbb{N}} \frac{(a)_n(b)_n \xi^n}{(n)_n (n+1)^2} \xi^n \times [\psi(a + n) - \psi(a) + \psi(b + n) - \psi(b) - 2\psi(n+1) - 2\gamma_E], \quad a, b \in \mathbb{C}\backslash \mathbb{N}_0, \quad \xi \in (0,1), \) \hspace{1cm} (A.8)

where the superscript \( \text{“ln”} \) indicates the presence of a logarithmic term (familiar from Frobenius theory).

Using (A.4) in formulas (A.5) and (A.8), one obtains for the solutions of the Jacobi differential equation \( \gamma_{\alpha, \beta} y(z, \cdot) = zy(z, \cdot) \) (cf. (A.2)) near \( x = -1, \)
\[
y_{1,0,\beta,-1}(z, x) = F(a_{\alpha, \beta, \sigma_{\alpha, \beta}(z)}; a_{\alpha, \beta, -\sigma_{\alpha, \beta}(z)}; 1 + \beta; (1 + x)/2), \quad \beta \in \mathbb{R}\backslash \mathbb{N}, \) \hspace{1cm} (A.9)
\]
\[
y_{2,0,\beta,-1}(z, x) = (1 + x)^{-\beta} F(a_{\alpha, \beta, -\sigma_{\alpha, \beta}(z)}; a_{\alpha, -\beta, -\sigma_{\alpha, \beta}(z)}; 1 - \beta; (1 + x)/2), \quad \beta \in \mathbb{R}\backslash \mathbb{N}_0, \) \hspace{1cm} (A.10)
\]
\[
y_{2,0,0,-1}(z, x) = F(a_{\alpha, 0, \sigma_{\alpha, 0}(z)}; a_{\alpha, 0, -\sigma_{\alpha, 0}(z)}; 1; (1 + x)/2) \ln((1 + x)/2)
\]
Again, for independent 
\( \beta \in \mathbb{R} \) and for
\[
\begin{align*}
  &\alpha \in \mathbb{R}, \ z \in \mathbb{C}, \ x \in (-1, 1),
\end{align*}
\]
where we abbreviated
\[
\alpha_{\mu,\nu,\pm\sigma} = \lfloor 1 + \mu + \nu \pm \sigma \rfloor/2, \quad \mu, \nu, \sigma \in \mathbb{C}
\]
(A.12)
Again one observes that for 
\( z \in \mathbb{C}, \ y_{1,\alpha,\beta,-1}(z, \cdot) \) and \( y_{2,\alpha,\beta,-1}(z, \cdot) \) are linearly independent for \( \alpha \in \mathbb{R}, \ \beta \in \mathbb{R}\setminus\mathbb{Z} \). Similarly, for \( z \in \mathbb{C}, \ y_{1,\alpha,0,-1}(z, \cdot) \) and \( y_{2,\alpha,0,-1}(z, \cdot) \) are linearly independent for \( \alpha \in \mathbb{R} \).
In precisely the same manner solutions of (A.1) are given by
\[
\begin{align*}
  w_{1,1}(\xi) &= F(a, b; a + b - c + 1; 1 - \xi), \quad a, b \in \mathbb{C}, \ c - a - b \in \mathbb{C}\setminus\mathbb{N}, \\
  w_{2,1}(\xi) &= (1 - \xi)^{c-a-b} F(c - a, c - b; c - a - b + 1; 1 - \xi),
  \quad a, b \in \mathbb{C}, \ a + b - c \in \mathbb{C}\setminus\mathbb{N},
\end{align*}
\]
and for \( a + b = c \),
\[
\begin{align*}
  w_{1,1}(\xi) &= F(a, b; 1; 1 - \xi), \quad a, b \in \mathbb{C}, \\
  w_{2,1}^{(n)}(\xi) &= F(a, b; 1; 1 - \xi) \ln(1 - \xi) + \sum_{n \in \mathbb{N}} \frac{(a)_n (b)_n}{(n!)^2} (1 - \xi)^n \\
  &\times [\psi(a + n) - \psi(a) + \psi(b + n) - \psi(b) - 2\psi(n + 1) - 2\gamma_E],
  \quad a, b \in \mathbb{C}, \ \xi \in (0, 1).
\end{align*}
\]
which are obtained from (A.5) and (A.8) by the change of variables
\[
(a, b, c, \xi) \rightarrow (a, b, a + b - c + 1, 1 - \xi).
\]
Together with the identification \( x = (1 + \xi)/2 \) and (A.4) one obtains the following solutions of \( \tau_{a,\beta} y(z, \cdot) = zy(z, \cdot) \) near \( x = +1 \),
\[
\begin{align*}
  y_{1,\alpha,\beta,+1}(z, x) &= F(a_{\alpha,\beta,\sigma_{a,\beta}}(z), a_{\alpha,\beta,-\sigma_{a,\beta}}(z); 1 + \alpha; (1 - x)/2), \quad \alpha \in \mathbb{R}\setminus(-\mathbb{N}), \\
  y_{2,\alpha,\beta,+1}(z, x) &= (1 - x)^{-\alpha} F(a_{-\alpha,\beta,\sigma_{a,\beta}}(z), a_{-\alpha,\beta,-\sigma_{a,\beta}}(z); 1 - \alpha; (1 - x)/2), \quad \alpha \in \mathbb{R}\setminus\mathbb{N},
\end{align*}
\]
\[
\begin{align*}
  y_{2,0,\beta,+1}(z, x) &= F(a_{0,\beta,\sigma_{0,\beta}}(z), a_{0,\beta,-\sigma_{0,\beta}}(z); 1; (1 - x)/2) \ln((1 - x)/2) \\
  + \sum_{n \in \mathbb{N}} \frac{(a_{0,\beta,\sigma_{0,\beta}}(z)_n (a_{0,\beta,-\sigma_{0,\beta}}(z)_n)}{(2n(n!)^2} (1 - x)^n \\
  &\times [\psi(a_{0,\beta,\sigma_{0,\beta}}(z) + n) - \psi(a_{0,\beta,\sigma_{0,\beta}}(z)) + \psi(a_{0,\beta,-\sigma_{0,\beta}}(z) + n) \\
  - \psi(a_{0,\beta,-\sigma_{0,\beta}}(z)) - 2\psi(n + 1) - 2\gamma_E], \quad \alpha = 0, \\
  \beta \in \mathbb{R}, \ z \in \mathbb{C}, \ x \in (-1, 1).
\end{align*}
\]
Again, for \( z \in \mathbb{C}, \ y_{1,\alpha,\beta,+1}(z, \cdot) \) and \( y_{2,\alpha,\beta,+1}(z, \cdot) \) are linearly independent for \( \alpha \in \mathbb{R}\setminus\mathbb{Z}, \ \beta \in \mathbb{R} \). Similarly, for \( z \in \mathbb{C}, \ y_{1,0,\beta,+1}(z, \cdot) \) and \( y_{2,0,\beta,+1}(z, \cdot) \) are linearly independent for \( \beta \in \mathbb{R} \).
where we used the fact

\[ \text{Eq. 18.5.7} \]

and quasi-rational eigenfunctions. The A.1 Remark

\[ \alpha \geq x, \]

\[ x \]

differential equation (A.2) with Neumann boundary conditions at

\[ \text{In particular, one can verify that the Jacobi polynomials are solutions of the Jacobi} \]

with

\[ n \leq 1 \] is a polynomial of degree at most \( \alpha, \beta \)

and can be defined by continuity for all parameters \( \alpha, \beta \in \mathbb{R} \). Note that \( P_n^{\alpha, \beta}(x) \)

one infers that

\[ \text{Moreover } y_{j, \alpha, \beta, \pm 1}(z, x), \] for \( j = 1, 2 \), are entire with respect to \( z \in \mathbb{C} \).

\[ \text{(A.19)} \]

Moreover \( y_{j, \alpha, \beta, \pm 1}(z, x) \) satisfy the relations (cf. (A.28))

\[ y_{1, \alpha, \beta, -1}(z, x) = (1 + x)^{-\beta}y_{2, \alpha, -\beta, -1}(z + (1 + \alpha)\beta, x), \] \( \alpha \in \mathbb{R}, \beta \in \mathbb{R} \setminus \{0\} \),

\[ y_{2, \alpha, \beta, -1}(z, x) = (1 + x)^{-\beta}y_{1, \alpha, -\beta, -1}(z + (1 + \alpha)\beta, x), \] \( \alpha \in \mathbb{R}, \beta \in \mathbb{R} \setminus \{0\} \),

\[ y_{1, \alpha, \beta, +1}(z, x) = (1 - x)^{-\alpha}y_{2, -\alpha, \beta, +1}(z + (1 + \beta)\alpha, x), \] \( \alpha \in \mathbb{R} \setminus \{0\}, \beta \in \mathbb{R} \),

\[ y_{2, \alpha, \beta, +1}(z, x) = (1 - x)^{-\alpha}y_{1, -\alpha, \beta, +1}(z + (1 + \beta)\alpha, x), \] \( \alpha \in \mathbb{R} \setminus \{0\}, \beta \in \mathbb{R} \),

where we used the fact

\[ \sigma_{\alpha, \beta}(z) = \begin{cases} 
\sigma_{\alpha, \beta}(z + (1 + \alpha)\beta), \\
\sigma_{\alpha, \beta}(z + (1 + \beta)\alpha), \\
\sigma_{\alpha, \beta}(z + \alpha + \beta).
\end{cases} \] \( \text{(A.24)} \)

Remark A.1. We conclude this appendix by briefly discussing Jacobi polynomials and quasi-rational eigenfunctions. The \( n \)th Jacobi polynomial is defined as (see [61, Eq. 18.5.7])

\[ P_n^{\alpha, \beta}(x) := \frac{(\alpha + 1)n}{n!}F(-n, n + \alpha + \beta + 1; \alpha + 1; (1 - x)/2), \] \( n \in \mathbb{N}_0, -\alpha \notin \mathbb{N}, -n - \alpha - \beta - 1 \notin \mathbb{N}, \)

and can be defined by continuity for all parameters \( \alpha, \beta \in \mathbb{R} \). Note that \( P_n^{\alpha, \beta}(x) \)

it satisfies the equation

\[ \tau_{\alpha, \beta}P_n^{\alpha, \beta}(x) = \lambda_n^{\alpha, \beta}P_n^{\alpha, \beta}(x), \]

with

\[ \lambda_n^{\alpha, \beta} := n(n + 1 + \alpha + \beta). \] \( \text{(A.27)} \)

In particular, one can verify that the Jacobi polynomials are solutions of the Jacobi differential equation (A.2) with Neumann boundary conditions at \( x = +1 \) (resp. \( x = -1 \)) if \( \alpha \in (-1, 0) \) (resp. \( \beta \in (-1, 0) \)) and Friedrichs boundary conditions if \( \alpha \geq 0 \) (resp. \( \beta \geq 0 \)).
One notes that poles occur on the right-hand side of (B.1), (B.2) whenever $(1 + x)^{± \beta}$ and $(1 - x)^{± \alpha}$ are regarded as formal multiplication operators. This is summarized in Table 1, which is taken from [8]. Here $(1 - x)^{-\alpha} P_n^{\alpha,\beta}(x)$ satisfy at $x = +1$ the Friedrichs boundary condition for $\alpha \leq 0$ and Neumann for $\alpha \in (0,1)$, while at $x = -1$ they satisfy the Friedrichs for $\beta \geq 0$ and Neumann for $\beta \in (-1,0)$. For $(1 + x)^{-\beta} P_n^{\alpha,\beta}(x)$ the roles of $\alpha$ and $\beta$ interchange compared to the last case, meaning Friedrichs at $x = +1$ for $\alpha \geq 0$, Neumann for $\alpha \in (-1,0)$, and at $x = -1$, Friedrichs for $\beta \leq 0$, Neumann for $\beta \in (0,1)$. Finally $(1 - x)^{-\alpha}(1 + x)^{-\beta} P_n^{\alpha,\beta}(x)$ satisfy at $x = +1$ (resp. $x = -1$) the Friedrichs boundary condition for $\alpha \leq 0$ (resp. $\beta \leq 0$) and Neumann for $\alpha \in (0,1)$ (resp. $\beta \in (0,1)$).

Table 1. Formal quasi-rational eigensolutions of $\tau_{\alpha,\beta}$

| Eigenfunctions | Eigenvalues |
|----------------|-------------|
| $P_n^{\alpha,\beta}(x)$ | $n(n + 1 + \alpha + \beta)$ |
| $(1 - x)^{-\alpha} P_n^{\alpha,\beta}(x)$ | $n(n + 1 - \alpha + \beta) - \alpha(1 + \beta)$ |
| $(1 + x)^{-\beta} P_n^{\alpha,\beta}(x)$ | $n(n + 1 + \alpha - \beta) - \beta(1 + \alpha)$ |
| $(1 - x)^{-\alpha}(1 + x)^{-\beta} P_n^{\alpha,\beta}(x)$ | $n(n + 1 - \alpha - \beta) - (\alpha + \beta)$ |

More generally, all quasi-rational solutions, meaning the logarithmic derivative being rational, can be derived from the the Jacobi polynomials together with

\[(1 + x)^{-\beta} \circ \tau_{\alpha,\beta} \circ (1 + x)^{\beta} = \tau_{\alpha,\beta} + (1 + \alpha)\beta,\]
\[(1 - x)^{-\alpha} \circ \tau_{\alpha,\beta} \circ (1 - x)^{\alpha} = \tau_{\alpha,\beta} + (1 + \beta)\alpha,\]
\[(1 - x)^{-\alpha}(1 + x)^{-\beta} \circ \tau_{\alpha,\beta} \circ (1 - x)^{\alpha}(1 + x)^{\beta} = \tau_{\alpha,\beta} + \alpha + \beta,\]

where $(1 + x)^{\pm \beta}$ and $(1 - x)^{\pm \alpha}$ are regarded as formal multiplication operators. This is summarized in Table 1, which is taken from [8]. Here $(1 - x)^{-\alpha} P_n^{\alpha,\beta}(x)$ satisfy at $x = +1$ the Friedrichs boundary condition for $\alpha \leq 0$ and Neumann for $\alpha \in (0,1)$, while at $x = -1$ they satisfy the Friedrichs for $\beta \geq 0$ and Neumann for $\beta \in (-1,0)$. For $(1 + x)^{-\beta} P_n^{\alpha,\beta}(x)$ the roles of $\alpha$ and $\beta$ interchange compared to the last case, meaning Friedrichs at $x = +1$ for $\alpha \geq 0$, Neumann for $\alpha \in (-1,0)$, and at $x = -1$, Friedrichs for $\beta \leq 0$, Neumann for $\beta \in (0,1)$. Finally $(1 - x)^{-\alpha}(1 + x)^{-\beta} P_n^{\alpha,\beta}(x)$ satisfy at $x = +1$ (resp. $x = -1$) the Friedrichs boundary condition for $\alpha \leq 0$ (resp. $\beta \leq 0$) and Neumann for $\alpha \in (0,1)$ (resp. $\beta \in (0,1)$).

Appendix B. Connection Formulas

In this appendix we provide the connection formulas utilized to find the solution behaviors in Appendix C. We express them using $w_{1,0}(\xi)$ and $w_{2,0}(\xi)$ ($w_{2,0}^n(\xi)$) and their analogs $w_{1,1}(\xi)$ and $w_{2,1}(\xi)$ ($w_{2,1}^n(\xi)$) at the endpoint $\xi = 1$.

We recall the relations (A.4) connecting the parameters $a, b, c$ and $\alpha, \beta$.

(I) The case $\alpha \in \mathbb{R}\setminus\mathbb{Z}$, $\beta \in (-1,1)\setminus\{0\}$, that is, $c \in (0,2)\setminus\{1\}$, $a + b - c \in \mathbb{R}\setminus\mathbb{Z}$:

The two connection formulas are given by (cf. [62, Eq. 15.10.21–22])

\[w_{1,0}(\xi) = \frac{\Gamma(c)(c - a - b)}{\Gamma(c - a)\Gamma(c - b)} w_{1,1}(\xi) + \frac{\Gamma(c)(c - a - b)}{\Gamma(a)\Gamma(b)} w_{2,1}(\xi),\]  
\[w_{2,0}(\xi) = \frac{\Gamma(2 - c)(c - a - b)}{\Gamma(1 - a)\Gamma(1 - b)} w_{1,1}(\xi) + \frac{\Gamma(2 - c)(c - a - b)}{\Gamma(a - c + 1)\Gamma(b - c + 1)} w_{2,1}(\xi).\]  

One notes that poles occur on the right-hand side of (B.1), (B.2) whenever $(a + b - c) \in \mathbb{Z}$. Using (A.15) one can also express $w_{1,1}(\xi)$ or $w_{2,1}(\xi)$ as a linear combination of $w_{1,0}(\xi)$ and $w_{2,0}(\xi)$:

\[w_{1,1}(\xi) = \frac{\Gamma(a + b - c + 1)(1 - c)}{\Gamma(a - c + 1)(b - c + 1)} w_{1,0}(\xi) + \frac{\Gamma(a + b - c + 1)(c - 1)}{\Gamma(a)\Gamma(b)} w_{2,0}(\xi),\]
\[w_{2,1}(\xi) = \frac{\Gamma(1 + c - a - b)(1 - c)}{\Gamma(1 - a)(1 - b)} w_{1,0}(\xi) + \frac{\Gamma(1 + c - a - b)(c - 1)}{\Gamma(c - a)(c - b)} w_{2,0}(\xi),\]  

(B.4)
The two relations (B.9) immediately imply differential expression. Since this case was treated in detail in [26], we shall only present the connection formulas for completeness.

$$w_{1,0}(\xi) = F(a, b; a + b; \xi)$$ can be expanded at $\xi = 1$ (cf. [1, Eq. 15.3.10]):

$$F(a, b; a + b; \xi) = \frac{\Gamma(a + b)}{\Gamma(a)\Gamma(b)} \sum_{n \in \mathbb{N}_0} \frac{(a)(a)_{n}(b)_{n}}{(n!)^{2}} [2\psi(n + 1) - \psi(a + n) - \psi(b + n) - \ln(1 - \xi)](1 - \xi)^{n}. \quad (B.5)$$

Meanwhile, two linearly independent solutions at $\xi = 1$ are taken from (A.14). The connection formula for $w_{1,1}(\xi)$ is given by (B.3) with $a + b = c$. To obtain a second connection formula one compares the expansion of $w_{1,1}^{ln}(\xi)$ at $\xi = 1$ with the expansion of $F(a, b; a + b; \xi)$ at $\xi = 1$, using (B.5), and then obtains

$$w_{2,1}^{ln}(\xi) = -[\psi(1 - a) + \psi(1 - b) + 2\gamma_{E}] \frac{\Gamma(1 - a - b)}{\Gamma(1 - a)\Gamma(1 - b)} w_{1,0}(\xi)$$

$$- [\psi(a) + \psi(b) + 2\gamma_{E}] \frac{\Gamma(a + b - 1)}{\Gamma(a)\Gamma(b)} w_{2,0}(\xi). \quad (B.6)$$

The case $\alpha = 0$, $\beta = 0$, that is, $c = 1$, $a + b \in \mathbb{R}\setminus\mathbb{Z}$:

This case is analogous to the previous case, with the roles of $\alpha$ and $\beta$ interchanged. Concretely, this means that the connection formulas (B.5) and (B.6) must be changed through the renaming (A.15) with $c \rightarrow a + b - c + 1 = a + b$, as $c = 1$. As $c$ does not appear in (B.5) and (B.6) (it was eliminated via $c = a + b$), one can adopt the aforementioned formulas directly, only changing the second index in the $w^{s}$

$$w_{1,0}(\xi) = \frac{\Gamma(1 - a - b)}{\Gamma(1 - a)\Gamma(1 - b)} w_{1,1}(\xi) + \frac{\Gamma(a + b - 1)}{\Gamma(a)\Gamma(b)} w_{2,1}(\xi), \quad (B.7)$$

$$w_{2,0}(\xi) = -[\psi(1 - a) + \psi(1 - b) + 2\gamma_{E}] \frac{\Gamma(1 - a - b)}{\Gamma(1 - a)\Gamma(1 - b)} w_{1,1}(\xi)$$

$$- [\psi(a) + \psi(b) + 2\gamma_{E}] \frac{\Gamma(a + b - 1)}{\Gamma(a)\Gamma(b)} w_{2,1}(\xi). \quad (B.8)$$

The case $\alpha = \beta = 0$, that is, $a + b = c = 1$:

For $\alpha = 0$ and $\beta = 0$ the Jacobi differential expression (5.1) becomes the Legendre differential expression. Since this case was treated in detail in [26], we shall only present the connection formulas for completeness.

The special solutions $w_{1,i}(\xi)$ and $w_{2,i}^{ln}(\xi)$ for $i = 1, 2$ are given by (A.8) and (A.14), respectively. Note that the following relations hold

$$w_{1,1}(\xi) = w_{1,0}(1 - \xi), \quad w_{2,1}^{ln}(\xi) = w_{2,0}(1 - \xi). \quad (B.9)$$

Using [1, Eq. 15.3.10] together with $w_{1,0} = F(a, b; a + b, \xi)$ and Euler’s famous reflection formula, $\Gamma(\pi)\Gamma(1 - \pi) = \pi \csc(\pi\pi)$ (cf. [1, Eq. 6.1.17]), one obtains

$$w_{1,0}(\xi) = -\pi^{-1} \sin(\pi a) (\psi(a) + \psi(b) + 2\gamma_{E}) w_{1,1}(\xi) + w_{2,1}(\xi). \quad (B.10)$$

The two relations (B.9) immediately imply

$$w_{1,1}(\xi) = -\pi^{-1} \sin(\pi a) (\psi(a) + \psi(b) + 2\gamma_{E}) w_{1,0}(\xi) + w_{2,0}(\xi), \quad (B.11)$$

1Formula (B.7) could have been obtained directly from (B.1) by setting $c = 1$. 

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**References**

1. JACOBI DONOGHUE m-FUNCTIONS, m-Functions, 23.
\[ w_{2,1}^{1n}(\xi) = \pi^{-1} \sin(\pi a) \left[ (\psi(a) + \psi(b) + 2\gamma_E)^2 - \pi^2 [\sin(\pi a)]^{-2} \right] w_{1,0}(\xi) + \psi(a) + \psi(b) + 2\gamma_E w_{2,1}^{1n}(\xi). \]

### Appendix C. Behavior of \( y_{j,a,b,\pm 1}(z,x) \), \( j = 1, 2 \), near \( x = \pm 1 \)

In this appendix we focus on the generalized boundary values for the solutions \( y_{j,a,b,-1}(z,x) \), \( j = 1, 2 \) at \( x = \mp 1 \). One obtains for \( z \in \mathbb{C} \),

\[
\bar{y}_{1,a,b,-1}(z,-1) = \begin{cases} 
1, & \beta \in (-1,0), \\
0, & \beta = 0, \\
0, & \beta \in (0,1),
\end{cases} \\
\bar{y}'_{1,a,b,-1}(z,-1) = \begin{cases} 
0, & \beta \in (-1,0), \\
1, & \beta = 0, \\
1, & \beta \in (0,1),
\end{cases} \\
\bar{y}_{2,a,b,-1}(z,-1) = \begin{cases} 
0, & \beta \in (-1,0), \\
-2^{\alpha+1}, & \beta = 0, \\
-2^{\alpha+1}, & \beta \in (0,1),
\end{cases} \\
\bar{y}'_{2,a,b,-1}(z,-1) = \begin{cases} 
0, & \beta \in (-1,0), \\
\beta = 0, \\
0, & \beta \in (0,1),
\end{cases}
\]

and employing connection formulas for the endpoint \( x = +1 \),

\[
\bar{y}_{1,a,b,-1}(z,1) = \begin{cases} 
\frac{\Gamma(1+\beta)\Gamma(-\alpha)}{\Gamma(a_{-a,b,\sigma_{a,b}}(z))\Gamma(a_{-a,b,-\sigma_{a,b}}(z))}, & \alpha \in (-1,0), \\
-2^{1+\alpha+\beta} \Gamma(1+\alpha)\Gamma(1+\beta), & \alpha \in [0,1),
\end{cases} \\
\bar{y}'_{1,a,b,-1}(z,1) = \begin{cases} 
\frac{2^{1+\alpha+\beta} \Gamma(1+\alpha)\Gamma(1+\beta)}{\Gamma(a_{-a,b,\sigma_{a,b}}(z))\Gamma(a_{-a,b,-\sigma_{a,b}}(z))}, & \alpha \in (-1,0), \\
\frac{\Gamma(a_{0,b,\sigma_{a,b}}(z))\Gamma(a_{0,b,-\sigma_{a,b}}(z))}{\Gamma(a_{0,b,\sigma_{a,b}}(z))\Gamma(a_{0,b,-\sigma_{a,b}}(z))} \left[2\gamma_E + \psi(a_{0,b,\sigma_{a,b}}(z)) + \psi(a_{0,b,-\sigma_{a,b}}(z))\right], & \alpha = 0, \\
\frac{\Gamma(1+\beta)\Gamma(-\alpha)}{\Gamma(a_{-a,b,\sigma_{a,b}}(z))\Gamma(a_{-a,b,-\sigma_{a,b}}(z))}, & \alpha \in (0,1), \\
\end{cases}
\]

\[
\bar{y}_{2,a,b,-1}(z,1) = \begin{cases} 
\frac{-2^{\beta} \Gamma(1-\beta)\Gamma(-\alpha)}{\Gamma(a_{-a,b,\sigma_{a,b}}(z))\Gamma(a_{-a,b,-\sigma_{a,b}}(z))}, & \alpha \in (-1,0), \\
\frac{-2^{\alpha+1} \Gamma(1+\alpha)\Gamma(1-\beta)}{\Gamma(a_{-a,b,\sigma_{a,b}}(z))\Gamma(a_{-a,b,-\sigma_{a,b}}(z))}, & \alpha \in [0,1),
\end{cases}
\]
\[
\bar{y}_{2,a,\beta,-1}(z,1) = \begin{cases}
\frac{2^{\alpha+1}\Gamma(1+\alpha)\Gamma(1-\beta) - 2^{\beta}\Gamma(1-\beta)}{\Gamma(a_{\alpha,-\beta,\sigma_{\alpha,\beta}(z)})\Gamma(a_{\alpha,-\beta,-\sigma_{\alpha,\beta}(z)})}, & \alpha \in (-1, 0), \\
\frac{2^{\gamma_E} + \psi(a_{\alpha,0,\sigma_{\alpha,0}(z)}) + \psi(a_{\alpha,-\sigma_{\alpha,0}(z)})\Gamma(-\alpha)}{\Gamma(a_{\alpha,0,\sigma_{\alpha,0}(z)})\Gamma(a_{\alpha,0,-\sigma_{\alpha,0}(z)})}, & \alpha \in (0, 1), \\
\beta \in (-1, 0) \cup \{0\}, & \\end{cases}
\]

\[\bar{y}_{2,a,0,-1}(z,1) = \begin{cases}
\frac{-2^{\gamma_E} + \psi(a_{\alpha,0,\sigma_{\alpha,0}(z)}) + \psi(a_{\alpha,-\sigma_{\alpha,0}(z)})\Gamma(-\alpha)}{\Gamma(a_{\alpha,0,\sigma_{\alpha,0}(z)})\Gamma(a_{\alpha,0,-\sigma_{\alpha,0}(z)})}, & \alpha \in (-1, 0), \\
\frac{2^{\gamma_E} + \psi((1 + \sigma_{0,0}(z))/2)\Gamma((1 - \sigma_{0,0}(z))/2) + \psi((1 - \sigma_{0,0}(z))/2)\Gamma((1 + \sigma_{0,0}(z))/2)}{\Gamma(a_{\alpha,0,\sigma_{\alpha,0}(z)})\Gamma(a_{\alpha,0,-\sigma_{\alpha,0}(z)})}, & \alpha \in (0, 1), \\
\beta = 0. & \\end{cases}\]

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