Inclusive semileptonic $B$ decays from QCD with NLO accuracy for power suppressed terms

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We present the results of a calculation of the perturbative QCD corrections for the semileptonic inclusive width of a heavy flavored meson. Within the Heavy Quark Expansion we analytically compute the QCD correction to the coefficient of power suppressed contribution of the chromo-magnetic operator in the limit of vanishing mass of the final state quark. The important phenomenological applications are decays of bottom mesons, and to the less extend, charmed mesons.

INTRODUCTION

With the success of the LHC mission and the Higgs boson discovery the validity of Standard Model (SM) as the theory of particle interactions at energies below 1 TeV has been convincingly proven. However, it is hard to expect that we shall be able to explore still higher energy regions in the same manner, namely by a direct observation of new physics phenomena. It is conceivable that new phenomena beyond the SM can only be identified through detecting slight discrepancies between theoretical predictions within the SM and precision measurements at low energy with available machines. For this program to succeed, accurate theoretical predictions withing the SM are of crucial importance, especially precise numerical values of key parameters of the SM are necessary.

In this respect, there are a few places that provide valuable information. The muon decay is important for the determination of the Fermi constant $G_F$ with high accuracy [1,2]. To match the precision of the experimental data in this case, the theoretical calculations have to be performed with very high accuracy. In this case this is feasible, since the purely leptonic decays are well described with perturbation theory and the expansion parameter $\alpha \approx 1/137$ is small. The latest theoretical result includes the second order (NNLO) radiative correction in the fine structure constant expansion [3]

$$\Gamma(\mu \to \nu_{\mu} e\bar{\nu}_e)/\Gamma^0 = 1 + \left(\frac{25}{8} - \frac{\pi^2}{2}\right) \alpha \pi + 6.74 \left(\frac{\alpha}{\pi}\right)^2$$

with $\Gamma^0 = G_F^2 m_{\mu}^5/(192\pi^3)$ and $m_{\mu}$ is the muon mass. It results in an $O(1\text{ppm})$ accuracy of theoretical expression that is competitive for comparison with experimental data.

There is a common belief that the flavor physics of quarks is one of the most promising places for search of new physics [4]. The relevant SM parameters in this sector are the Fermi constant and quark mixing parameters gathered in the CKM matrix. While the quark weak decays are mediated through charged currents at tree level (which are believed not to have sizable contributions of possible new physics), their study is of paramount importance for precise determination of the numerical values of the CKM matrix elements. In contrast to leptons, obtaining a theoretical prediction for processes with quarks requires the use of genuinely nonperturbative computational methods (like QCD lattice calculations) due to confinement. Nevertheless, for heavy mesons the theoretical treatment is somewhat easier because the large mass of the heavy quark opens the possibility for an expansion in powers of $\Lambda/m_Q$ where $m_Q$ is the quark mass and $\Lambda \sim 500\text{ MeV}$ is a hadronic scale [5]. Top quarks do not form mesons due to the short top quark lifetime, charmed mesons are probably not heavy enough, rendering the application of the Heavy-Quark Expansion (HQE) marginal, but the case of bottom-meson decays is certainly tractable in this way and thus has been intensively studied. The technique is applicable to $b \to u$ and $b \to c$ transition and both to semileptonic and purely hadronic decays. For definiteness, we will stick to semileptonic $b \to c$ decays.

Over the last ten years the HQE in inclusive semileptonic $b \to c$ decays has been refined to such an extend that the remaining theoretical uncertainty in the prediction of the total inclusive rate for $B \to X_c \ell \nu$ has reached a level of less than two percent. The structure of the HQE in the case at hand is given by [6]

$$\Gamma(B \to X_c \ell \nu) / \Gamma^0 = |V_{cb}|^2 \left[ a_0(1 + \frac{\mu_G^2}{2m_b^2}) + a_2 \frac{\mu_G^2}{m_b^2} + a_3 \frac{\rho}{m_b} + O\left(\frac{\Lambda^4}{m_b^4}\right)\right]$$

where $\Gamma^0 = G_F^2 m_b^5/(192\pi^3)$, $m_b$ is the $b$-quark mass, $\mu_G$ (the kinetic energy parameter), $\mu_G$ (the chromo-magnetic parameter), and $\rho$ are nonperturbative contributions with numerical values of the order of $\Lambda$. The coefficients $a_i$ are functions of the quark (and, in general, lepton) masses and have a perturbative expansion in the strong coupling constant $\alpha_s(m_b)$. The leading term $a_0$ is known analytically to $O(\alpha_s^2)$ precision in the massless limit of the final state quark [7]. At NNLO the mass corrections have been analytically accounted for as an expansion in ref. [8] and numerically in [9]. The coefficient of the kinetic energy parameter is linked to $a_0$ by Lorentz invariance, see the explicit analysis in [10].
The parametrically largest contribution to the width currently unknown is the $\alpha_s$ correction to the coefficient of the chromo-magnetic parameter $a_2$, which has been investigated recently in [11], where a numerical result for this contribution has been obtained. From the numerical study performed in [11] one can infer that the $\alpha_s$ corrections to $a_2$ are of the expected size.

In this letter we report on an analytical calculation of corrections to $a_2$ in the limit of vanishing charmed quark mass. As it turns out, the precision gained in this approximation is sufficient for phenomenological applications.

**OUTLINE OF THE CALCULATION**

The rate (1) is obtained from taking the absorptive part of the forward matrix element of the transition operator $T$ [12],

$$T = i \int d^4x T[H_{\text{eff}}(x)H_{\text{eff}}(0)] \quad (2)$$

where $H_{\text{eff}}$ is the effective Hamiltonian for the semileptonic transition

$$H_{\text{eff}} = 2\sqrt{2}G_F V_{cb}(\bar{b}_L\gamma_\mu c_L)(\bar{v}_L\gamma^\mu \ell_L). \quad (3)$$

In order to make the dependence of the width on the heavy quark mass $m_b$ explicit and to build up an expansion in $\lambda/m_b$, one matches a time-ordered product of full QCD operators $H_{\text{eff}}$ in (3) on an expansion in terms of Heavy Quark Effective Theory (HQET) [13 [14]

$$(\text{Im}T)/R_0 = C_0\mathcal{O}_0 + C_v\mathcal{O}_v + C_\pi\mathcal{O}_\pi + C_G\mathcal{O}_G + 1/2m_b^2 \quad (4)$$

where $R_0 = \pi \Gamma_0 |V_{cb}|^2$. The local operators $\mathcal{O}_i$ in the expansion (1) are ordered by their dimensionality $\mathcal{O}_0 = h_0 h_v, \mathcal{O}_v = h_v\pi h_v, \mathcal{O}_\pi = h_\pi h_v, \mathcal{O}_G = h_\pi 2/|\not\!\not\!\not h_v$. Here $v$ is the velocity of the heavy hadron appearing in the HQET construction, $\pi_\mu = i\partial_\mu + g_\sigma A_\mu$ is the covariant derivative of QCD, $\pi^{\mu} = \pi^{\mu} + \pi^{\mu}_L$, and $h_v$ is the heavy-quark field entering the HQET Lagrangian [13 [14]. The expansion (1) is a matching relation from QCD to HQET with proper operators up to dimension five with the corresponding coefficient functions. Note that the operator $\mathcal{O}_\pi$ will be eliminated by using the equation of motion for $h_v$ once the forward matrix elements with meson states are taken. The Lagrangian for the modes $h_v$ is given by

$$\mathcal{L} = \mathcal{O}_v + \frac{1}{2m_b}(\mathcal{O}_\pi + C_m(\mu)\mathcal{O}_G) + O \left( \frac{\Lambda^2}{m_b^2} \right) \quad (5)$$

with

$$C_m(\mu) = 1 + \frac{\alpha_s(\mu)}{2\pi} \left\{ C_F + C_A \left( 1 + \ln \frac{\mu}{m_b} \right) \right\} \quad (6)$$

being the coefficient of the chromo-magnetic operator $\mathcal{O}_G$ including the $O(\alpha_s)$ QCD correction [14]. Note that we define the modes $h_v$ such that terms of the order $O(1/m_b^2)$ in the Lagrangian contain no time derivative [14 [16].

It is convenient to choose the local operator $b\gamma \ell$ (defined in full QCD) as a leading term of heavy quark expansion [17]. Indeed, the current $b\gamma \ell$ is conserved and thus its forward matrix element with hadronic states is absolutely normalized. For implementing this one needs an expansion (matching) of a full QCD local operator $b\gamma \ell$ in HQE through HQET operators. The expansion reads

$$b\gamma \ell = \mathcal{O}_0 + \mathcal{O}_\pi + \mathcal{O}_G \mathcal{O}_G + O(1/m_b^2) \quad (7)$$

and is valid including the radiative corrections of order $\alpha_s$. Thus, the leading power operator has no corrections and the kinetic operator has the same coefficient as the leading one due to Lorentz invariance.

Substituting expansion (7) into (4) one obtains after using the equation of motion for the operator $\mathcal{O}_v$ in the forward matrix elements

$$(\text{Im}T)/R_0 = C_0 \left\{ b\gamma \ell - \frac{\mathcal{O}_\pi}{2m_b^2} \right\} + \left\{ -C_v C_m + C_G - \mathcal{O}_G \mathcal{O}_G \right\} \frac{1}{2m_b^2}. \quad (8)$$

The numerical value for the chromo-magnetic moment parameter $m_B^2$ related to the forward matrix element of the operator $\mathcal{O}_G$ is usually taken from the mass splitting between the pseudoscalar and vector ground-state mesons. The mass difference of bottom mesons $m_B^2 - m_B^2 = \Delta m_B^2 = 0.49$ GeV$^2$ is given by

$$\frac{1}{2M_B} C_m(\mu) (B(p)|\mathcal{O}_G|B(p)) \frac{\mu^2}{2m_b^2} \quad (9)$$

where we use the usual relativistic normalization of the states.

Taking the forward matrix element of (8) one gets

$$\Gamma(B \rightarrow X_v\nu\ell) = \Gamma_0 |V_{cb}|^2 \left\{ C_0 \left( 1 + \frac{\mu^2}{2m_b^2} \right) + \left( -C_v + C_G - \frac{C_G C_0}{C_m} \right) \frac{3\Delta m_B^2}{8m_b^2} \right\}. \quad (10)$$

The matching procedure is straightforward and consists in computing matrix elements with partonic states (quarks and gluons on shell) at both sides of the expansion (4). In this way the coefficient function $C_0$ of the dimension three operator $h_\pi h_v$ determines the total width of the heavy quark and at the same time the leading contribution to the width of a bottom hadron. Going to order $\alpha_s$, the calculation of the transition operator $T$ in (2) requires to consider three-loop diagrams with external heavy quark lines on shell. The leading order
result is well known and requires the calculation of the two-loop Feynman integrals of the simplest topology – the sunset type ones [13]. At the NLO level one needs the on-shell tree-loop integrals with massive lines. The computation has been performed in dimensional regularization used for both ultraviolet and infrared singularities. We used the systems of symbolic manipulations REDUCE [19] and Mathematica [20] with original codes written for the calculation. The reduction to master integrals has been used for checking and further application to complicated vertex diagrams. The master integrals have been computed directly and then checked with the program HypExp [22]. The renormalization is performed on-shell by the multiplication of the bare (direct from diagrams) results by the renormalization constant $Z^{OS}_{2}$

$$Z^{OS}_{2} = 1 - C_F \frac{\alpha_s}{4\pi} \left( \frac{3}{\epsilon} + 3\ln\left( \frac{\mu^2}{m_F^2} \right) + 4 \right).$$

(11)

In fig. 1 we show some typical three loop diagrams. By using the described methods one reproduces the known result [10]

$$C_0 = 1 + \Delta^{(0)}_{\rho} + C_F \frac{\alpha_s}{\pi} \left\{ \left( \frac{25}{8} - \frac{\pi^2}{2} \right) + \Delta^{(1)}_{\rho} \right\}$$

(12)

with $C_F = 4/3$ and $\rho = m_{C}^2/m_{b}^2$. Here $\Delta^{(0)}_{\rho}$ and $\Delta^{(1)}_{\rho}$ are corrections due to charmed quark mass at LO and NLO respectively. They are known analytically and normalized such that $\Delta^{(0)}_{\rho}(0) = \Delta^{(1)}_{\rho}(0) = 0$.

The coefficient $C_v$ is singled out by taking the matrix element between quarks on shell and one gluon with vanishing momentum and longitudinal polarization. The coefficient $C_v$ reads

$$C_v = 5 + C_F \frac{\alpha_s}{\pi} \left\{ \frac{25}{24} - \frac{\pi^2}{2} \right\}.$$  

(13)

It has no $\mu$ dependence and no $C_A$ color contribution. This matches also the possibility to compute this coefficient using small momentum expansion near the quark mass shell, $p = m v + k$. A powerful check of the result is an explicit cancellation of the contribution proportional to the color structure $C_A$ and the renormalization (cancellation of $c$-poles) with the same renormalization constant $Z^{OS}_{2}$ shown in [11].

The final coefficient of the chromo-magnetic operator multiplied by $C_m$ (see eq. (10)) reads

$$C_{\text{fin}} = -C_v + \left( C_G - \bar{C}_G C_0 \right)/C_m$$

(14)

and

$$C_{\text{fin}} = -3 + \alpha_s\Delta^{(0)}_{G}(m_c) + \frac{\alpha_s}{\pi} \Delta^{(1)}_{G}(m_c)$$

(15)

$$+ \frac{\alpha_s}{\pi} \left\{ C_A \left( \frac{31}{18} - \frac{\pi^2}{9} \right) + C_F \left( \frac{43}{144} - \frac{19\pi^2}{36} \right) \right\}.$$  

The function $\Delta^{(0)}_{G}(\rho)$ is known analytically. The function $\Delta^{(1)}_{G}(\rho)$ emerges in the analysis of ref. [11] where the analytical result for the coefficient of the chromo-magnetic operator at the level of hadronic structure functions has been obtained. Both functions are chosen such that they vanish at $m_c = 0$. The final integration over the phase space in ref. [11] has been done numerically that prevents us from a direct comparison between the two results. Numerically we obtain at $m_c = 0$

$$C_{\text{fin}} = -3 + \frac{\alpha_s}{\pi} \left( 0.63 C_A - 4.91 C_F \right)$$

(16)

$$= -3 + \frac{\alpha_s}{\pi} (-4.67) = -3(1 + 1.56 \frac{\alpha_s}{\pi}).$$

The $\mu$ dependence of the prefactor of $\mathcal{O}_G$ in (8) matches the leading order anomalous dimension of the chromo-magnetic operator [13], such that $C_{\text{fin}}$ is $\mu$ independent. Furthermore, the mass parameter of the heavy quark $m_c$ is chosen to be the pole mass which is a proper formal parameter for perturbative computations in HQET (see, discussion in [6]). After having obtained the results of perturbation theory computation for the coefficients of QE, one is free to change this parameter to any other [24].

Our results [15,16] still depend on $\mu$ through the strong coupling $\alpha_s$ defined in the MS-scheme; however, this remaining scale dependence can only be resolved at the next order in $\alpha_s$.

**DISCUSSION OF THE RESULTS**

The radiative corrections are of reasonable magnitude and are well under control for the numerical values of the coupling constant for $\mu \sim 2 - 4$ GeV. This provides a clean application of the results to decay into light quarks $u$ for bottom mesons and $d$ for charmed mesons. For application to $b \rightarrow c$ transition the important question is the magnitude of corrections due to nonvanishing charmed quark mass. It seems that mass corrections are important but still under control. The small $\rho$ expansion reads $\Delta^{(0)}_{G}(\rho) = 8\rho + \ldots$, and $\Delta^{(1)}_{G}(\rho) = A_0 + \ldots$ where the factor $A$ is not known analytically. Assuming $|A| \leq 50$ one sees that the massless approximation dominates the radiative correction for typical values of $\rho$ in
the range $\rho = 0.06 \pm 0.02$ \cite{26},

$$C_{fin} = -3 + \frac{\alpha_s}{\pi} (-4.67 + \rho A) \ .$$  \hspace{1cm} (17)

The numerical value of the coefficient can change significantly only in the case of negative (and rather large) contribution due to $c$-quark mass.

At present the value of $|V_{cb}|$ from inclusive decays is $|V_{cb}| = (4.41 \pm 0.15 \pm 0.16) \times 10^{-3}$ \cite{27} while the extraction from exclusive $B \rightarrow \pi\ell\nu$ yields $|V_{cb}| = (3.23 \pm 0.31) \times 10^{-3}$. However, the exclusive determination does not rely on the local OPE considered here, so from our results we cannot really draw a definite conclusion. Nevertheless, if our result indicates the size of the expected corrections, it cannot resolve the tension between the inclusive and the exclusive value.

More important are the implications for inclusive semileptonic $B$ meson decays to charm, since here the precision is high enough to worry about the correction computed above. Indeed, the inclusive determination has a precision at the level of roughly 2\%, the value being $|V_{cb}| = (42.4 \pm 0.9) \times 10^{-3}$ \cite{27,28}. Since we only have the analytical result in the limit $m_c \rightarrow 0$ at hand, we estimate the impact of our correction in a simplified manner. Because it is a small correction, we only account for charmed quark mass at tree approximation, taking into account the kinematic function $\Delta_0^{(0)}(\rho) = -8\rho - 12\rho^2 \ln \rho + 8\rho^3 - \rho^4$. The determination of $|V_{cb}|$ uses the total rate only, so we get for the shift in $|V_{cb}|$ through the $\alpha_s$ correction in the coefficient of the chromo-magnetic operator

$$\frac{\Delta |V_{cb}|}{|V_{cb}|} = 4.67 \frac{\alpha_s}{\pi} \frac{3\Delta m_B^2}{8\alpha_s^2} \frac{1}{2(1 + \Delta_0^{(0)}(\rho))}$$  \hspace{1cm} (18)

which yields for $\rho = 0.07$ and $\alpha_s/\pi = 0.1$ a relative shift of $+0.3\%$ in the value of $|V_{cb}|$. This result is compatible with the study in \cite{11}, which includes the charmed quark mass. A preliminary comparison of extrapolation of the results of ref. \cite{11} to small mass limit shows a reasonable agreement.

The shift in $|V_{cb}|$ has to be compared to the corrections of order $(\Lambda/m_b)^n$, $n = 3, 4$ at tree level. The $(\Lambda/m_b)^3$ contributions induce a relative shift in $|V_{cb}|$ of about $-1.5\%$ which is included in the current analysis. The terms of order $(\Lambda/m_b)^4$ are not yet included and shift the value of $|V_{cb}|$ by about $0.3\%$ \cite{29}, which is roughly of the same order as the corrections considered here.

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