Quantization over boson operator spaces

Tomaž Prosen\textsuperscript{1,2} and Thomas H Seligman\textsuperscript{3,4}

\textsuperscript{1} Department of Physics, FMF, University of Ljubljana, Ljubljana, Slovenia
\textsuperscript{2} Department of Physics and Astronomy, University of Potsdam, Potsdam, Germany
\textsuperscript{3} Instituto de Ciencias Físicas, Universidad Nacional Autónoma de México, Cuernavaca, Morelos, Mexico
\textsuperscript{4} Centro Internacional de Ciencias, Cuernavaca, Morelos, Mexico

Received 9 June 2010, in final form 17 August 2010
Published 1 September 2010
Online at stacks.iop.org/JPhysA/43/392004

Abstract

The framework of third quantization—canonical quantization in the Liouville space—is developed for open many-body bosonic systems. We show how to diagonalize the quantum Liouvillean for an arbitrary quadratic $n$-boson Hamiltonian with arbitrary linear Lindblad couplings to the baths and, as an example, explicitly work out a general case of a single boson.

PACS numbers: 03.65.Fd, 05.30.Jp, 03.65.Yz

1. Introduction

Liouville spaces of operators have proven very useful in the analysis of many-body problems [1], and in particular in optical spectroscopy [2]. In such approaches one equips the Liouville space of quantum operators (representing observables or density matrices) with an inner product such that the Liouville space in fact becomes a Hilbert space. Therefore, for many body problems, a second quantization over such spaces is tempting and indeed was presented by one of the authors [3], and coined as third quantization, for the finite-dimensional case of Fermi operators. It was successfully used to explicitly and efficiently solve situations with quadratic (or quasi-free) Hamiltonians and linear coupling to an environment via the Lindblad operators [3, 4] or via the Redfield model [5]. In particular, the method already provided exciting new physics, such as a discovery of a quantum-phase transition far from equilibrium in XY spin chain [4].

The present communication deals with developing a similar framework for bosons. This is of considerable importance for several reasons. First, there exists a somewhat dated but vast literature on harmonic oscillators and more general quadratic Hamiltonians reported in two books [6, 7] essential to the development of quantum optics and many other fields. Second, oscillators and linear couplings to the environment are widely used in the theory of decoherence [8]. Finally, the development of the technique of third quantization over infinite-dimensional boson spaces seems essential for a future development of a supersymmetric theory...
of open many-body systems involving fermions, bosons, their interaction and coupling to the environment. For quadratic interactions and linear coupling to the environment closed form solutions will likely emerge, whereas for nonlinearly interacting systems perturbative and non-perturbative methods of many-body physics can be used in conjunction with our approach. Though many aspects of the open bosonic problems with linear environment coupling operators have been treated before, the elegance and versatility of the third quantization treatment distinguish this treatment from others.

2. Preliminaries

Let us consider a Hilbert–Fock space $\mathcal{H}$ of $n$ bosons. The representation of $\mathcal{H}$ can be generated by a special element $\psi_0 \in \mathcal{H}$, called a vacuum pure state, and a set of $n$ unbounded operators over $\mathcal{H}$, $a_1, \ldots, a_n$, and their Hermitian adjoints $a_1^\dagger, \ldots, a_n^\dagger$, satisfying canonical commutation relations (CCR)

$$[a_j, a_k^\dagger] = \delta_{j,k}, \quad [a_j, a_k] = [a_j^\dagger, a_k^\dagger] = 0.$$  \(1\)

Let us define a pair of vector spaces $\mathcal{K}$ and $\mathcal{K}'$, such that $\mathcal{K}$ contains trace class operators, for example density matrices, and $\mathcal{K}'$ contains unbounded operators which we need as physical observables. After choosing a specific space of observables $\mathcal{K}'$, we define the space $\mathcal{K}$ as a subspace of trace class operators over $\mathcal{H}$, such that $\rho \in \mathcal{K}$ if and only if $A\rho$ is a trace class for any $A \in \mathcal{K}'$. In this sense $\mathcal{K}'$ is a dual space to $\mathcal{K}$.

For instance, we may chose $\mathcal{K}'$ as a linear space of all (unbounded) operators whose Weyl symbol, i.e. the phase-space representation of the operator, is an entire function on $\mathbb{C}^{2n}$. Then $\mathcal{K}$ must be limited to operators with finite support in the number operator basis, i.e. to operators which have a finite number of non-vanishing matrix elements in this basis. Such a constraint on density matrices may be too restrictive for certain applications. We shall show later (using equation (9)) that it can be relaxed by restricting $\mathcal{K}'$.

Since the development in the paper will be purely algebraic, we shall adopt Dirac notation and write an element of $\mathcal{K}$ as $\ket{\rho}$ and an element of $\mathcal{K}'$ as $\bra{A}$, such that their contraction gives the expectation value of $A$ for a state $\rho$,

$$\bra{A}\rho\rangle = \text{tr} A\rho.$$  \(2\)

We keep distinct styles of brackets to emphasize the manifest difference between the spaces from which the ket and the bra have to be chosen.

If $b$ is any of $a_j, a_j^\dagger$, then for each $\rho \in \mathcal{K}$ and $A \in \mathcal{K}'$, $b\rho$, $\rho b$ and $Ab$, $bA$ are also the elements of $\mathcal{K}$ and $\mathcal{K}'$ respectively, so one can define the left- and right-multiplication maps $\hat{b}^L$ and $\hat{b}^R$ over $\mathcal{K}$:

$$\hat{b}^L\ket{\rho} = \ket{b\rho}, \quad \hat{b}^R\ket{\rho} = \ket{\rho b}.$$  \(3\)

The action of their adjoint maps on $\mathcal{K}'$ follows from definition (2) and cyclicity of trace,

$$\bra{A}\hat{b}^L = \bra{Ab}, \quad \bra{A}\hat{b}^R = \bra{bA}.$$  \(4\)

With a slight abuse of notation, we can also write $(\hat{b}^L)^* = \hat{b}^R$ and $(\hat{b}^R)^* = \hat{b}^L$.

Let us now define the set of $4n$ maps $\hat{a}_{\nu,j}$, $\hat{a}'_{\nu,j}$, $j = 1, \ldots, n$, $\nu = 0, 1$,

$$\hat{a}_{0,j} = a_j^L, \quad \hat{a}'_{0,j} = a_j^{\dagger L} - a_j^R, \quad \hat{a}_{1,j} = a_j^R, \quad \hat{a}'_{1,j} = a_j^{\dagger R} - a_j^L,$$  \(5\)

satisfying the unique properties as follows:
\[ [\hat{a}_{\nu,j}, \hat{a}_{\mu,k}^\dagger] = \delta_{\nu,\mu}\delta_{j,k}, \quad [\hat{a}_{\nu,j}, \hat{a}_{\mu,k}] = [\hat{a}_{\nu,j}^\dagger, \hat{a}_{\mu,k}^\dagger] = 0, \]  

\[ (1) \hat{a}_{\nu,j}^\dagger = 0, \]  

\[ (iii) \hat{a}_{\nu,j} \text{ right-annihilate the vacuum pure state } |\rho_0\rangle \equiv |\psi_0\rangle \langle \psi_0| \]  

Writing a 2n component multi-index \( m = (m_{\nu,j} \in \mathbb{Z}_+; \nu \in \{0, 1\}, j \in \{1, \ldots, n\})^T \) we can define convenient dual-Fock bases of the spaces \( K, K' \) as

\[ |m\rangle = \prod_{v,j} \frac{\hat{a}_{\nu,j}^{m_{\nu,j}}}{\sqrt{m_{\nu,j}}} |\rho_0\rangle, \quad |m\rangle = (1) \prod_{v,j} \frac{(\hat{a}_{\nu,j})^{m_{\nu,j}}}{\sqrt{m_{\nu,j}}}, \]

whose bi-orthonormality \( \langle m' | m \rangle = \delta_{m', m} \) is simply guaranteed by almost-CCR (6). Here and below, \( \chi = (x_1, x_2, \ldots)^T \) designates a vector (column) of any, scalar-, operator- or map-valued symbols, \( \chi \cdot y = x_1y_1 + x_2y_2 + \cdots \) designates a dot product and bold upright letters \( \mathbf{A} \) shall be used for complex-valued matrices.

The explicit construction of the bases (9) allows us to enlarge the space \( K \) and restrict the space \( K' \) as compared to the above example. Namely, we now identify the space \( K \) with the \( \ell^2 \) Hilbert space of vectors of coefficients \( \{\sigma_m\}, K \ni |\sigma\rangle = \sum_m \sigma_m |m\rangle \), and the space \( K' \) with the \( \ell^2 \) Hilbert space of vectors of coefficients \( \{|S_m\}, \ K' \ni |S\rangle = \sum_m S_m |m\rangle \). Then, clearly by Cauchy–Schwartz inequality, \( \text{tr} S\sigma \rangle = \big| \sum_m S_m \sigma_m \big| < \infty \) and hence \( K \) and \( K' \) are dual in the required sense.

3. Explicit solution of the Lindblad equation for quadratic bosonic systems

Our goal is to present an exact solution of the master equation of an open \( n \)-particle system, say of the Lindblad form [9–11]

\[ \frac{d\rho}{dt} = \hat{\mathcal{L}}\rho := -i[H, \rho] + \sum_{\mu} \left(2L_{\mu}^\dagger \rho L_{\mu} - \{L_{\mu}^\dagger L_{\mu}, \rho\} \right), \]  

where \( H \) is a Hermitian operator (Hamiltonian), \( \{x, y\} := xy + yx \), and \( L_{\mu} \) are arbitrary (Lindblad) operators representing couplings to different baths. We are going to describe a general method for the explicit solution of (10) for an arbitrary quadratic system of \( n \) bosons with linear bath operators

\[ H = a^\dagger \cdot \mathbf{H} + a \cdot \mathbf{K} a + a^\dagger \cdot \mathbf{K} a^\dagger \]  

\[ L_{\mu} = L_{\mu}^\dagger \cdot a + k_{\mu} \cdot a^\dagger. \]

Note that the hermiticity of the Hamiltonian and CCR (1) implies that the matrix \( \mathbf{H} \) is Hermitian \( H^\dagger = \mathbf{H}^T = \mathbf{H} \) and the matrix \( \mathbf{K} \) is symmetric \( \mathbf{K} = \mathbf{K}^T \). For bound oscillator systems, one can always choose \( \mathbf{K} = 0 \); however, in order to be able to describe freely moving or inverted oscillator modes, we shall keep the general form.

Following the rules of the algebra (3)–(5) we straightforwardly express the Liouvillean (10) in terms of the canonical operators

\[ \hat{\mathcal{L}} = -i\hat{H}^L + i\hat{H}^R + \sum_{\mu} 2\hat{L}_{\mu}^L \hat{L}_{\mu}^R - \hat{L}_{\mu}^L \hat{L}_{\mu}^L - \hat{L}_{\mu}^R \hat{L}_{\mu}^R \]
where we define $n \times n$ matrices
\[
M := \sum_{\mu} L_{\mu} \otimes \bar{L}_{\mu} = M^1, \quad N := \sum_{\mu} k_{\mu} \otimes \bar{k}_{\mu} = N^1, \quad L := \sum_{\mu} L_{\mu} \otimes \bar{L}_{\mu}.
\]

The fact that the primed maps $\hat{a}'$ always appear on the left-hand side in each term manifestly expresses the trace preservation of the flow (10), i.e. $(1|\hat{L} = 0$.

Note that adding inhomogeneous terms (linear forces) to the Hamiltonian $H \rightarrow H + \hat{f} \cdot \hat{a} + \hat{f}' \cdot \hat{a}'$ and the Lindblad operators $L_{\mu} \rightarrow L_{\mu} + \lambda_{\mu} \hat{1}$, where $\hat{f}$ is a $c$-vector and $\lambda_{\mu}$ are the $c$-numbers, simply results in an extra linear term in the Liouvillean $\hat{L} \rightarrow \hat{L} + \hat{g} \cdot \hat{a}'$, which must be removed by a canonical transformation affecting the annihilation maps $\hat{a}_{\nu,j}$ only, namely $\hat{a} \rightarrow \hat{a} + \hat{g} \cdot \hat{1}$, $\hat{a}' \rightarrow \hat{a}'$ and thus preserving the important relations (6), (7), while the right vacuum state $|\beta_0\rangle$ (which is unimportant for the following discussion) is trivially shifted.

Writing $4n$ vector of the canonical maps as $\hat{b} = (\hat{a}, \hat{a}')^T = (\hat{a}_{0j}, \hat{a}_{j0}, \hat{a}_{j1}, \hat{a}_{1j})^T$ the Liouvillean (14) can be compactly rewritten in terms of a symmetric form
\[
\hat{L} = \hat{b} \cdot \hat{S} \hat{b} - S_0 \hat{1},
\]
where $\hat{S}$ is a complex symmetric $4n \times 4n$ matrix which can be conveniently written in terms of two $2n \times 2n$ matrices
\[
\hat{S} = \begin{pmatrix} 0 & -\hat{X} \\ \hat{X}^T & Y \end{pmatrix}.
\]

namely
\[
\hat{X} := \frac{1}{2} \begin{pmatrix} i\hat{H} - \hat{N} + M & -2iK - L + L^T \\ 2iK - L + L^T & -i\hat{H} - N + \bar{M} \end{pmatrix}
\]

and
\[
\hat{Y} := \frac{1}{2} \begin{pmatrix} -2i\bar{K} - \bar{L} - \bar{L}^T & 2N \\ 2iK - L - L^T & 2i\hat{K} - \hat{L} + \hat{L}^T \end{pmatrix} = \hat{Y}^T.
\]

The scalar $S_0$ (16) stemming from reordering of maps equals $S_0 = \text{tr} \hat{X} = \text{tr} \hat{M} = \text{tr} \hat{N}$.

For the rest of our discussion we shall assume that the matrix $\hat{X}$ is diagonalizable 5, i.e. it is similar to a diagonal matrix
\[
\hat{X} = \hat{P} \Delta \hat{P}^{-1}, \quad \Delta = \text{diag}\{\beta_1, \ldots, \beta_{2n}\}
\]

where the $2n$ complex eigenvalues $\beta_j$ shall be named as rapidities in analogy to the fermionic case [3]. Let $\hat{J} = i\sigma_y \otimes I_{2n}$ denote the symplectic unit, satisfying $\hat{J}^2 = -I_{4n}$. It is straightforward to check that the matrix $\hat{J} \hat{S}$ belongs to the symplectic algebra, namely $\hat{J} (\hat{J} \hat{S}) = - (\hat{J} \hat{S})^T \hat{J}$, and can be diagonalized as
\[
\hat{J} \hat{S} = \hat{V}^{-1} [(-\Delta) \otimes \Delta] \hat{V}, \quad \hat{V} = [\hat{P}^T \otimes \hat{P}^{-1}] \begin{pmatrix} I_{2n} & -\hat{Z} \\ 0 & I_{2n} \end{pmatrix},
\]

with the eigenvector matrix $\hat{V}$ belonging to the complex symplectic group
\[
\hat{V}^T \hat{J} \hat{V} = \hat{J}.
\]

5 The more general case allowing for a possible non-generic non-diagonalizability of $\hat{X}$ shall be treated in future work.
The $2n \times 2n$ complex symmetric matrix $Z = Z^T$ is a solution of the continuous Lyapunov equation [12]

$$X^T Z + ZX = Y.$$  \hspace{1cm} (23)

It is known that the solution of (23) exists and is unique if no pairs of eigenvalues, rapidities $\beta_j, \beta_j'$, exist, such that $\beta_j + \beta_j' = 0$. We note that $X$, and also $Y$, are (unitarily) similar to real matrices, namely if $U := \frac{1}{\sqrt{2}}(1 + i\sigma_x) \otimes I_{2n}$ then $UXU^{-1}$, and $UYU^{-1}$, are real matrices. Thus, the rapidities should come in complex conjugate pairs $\beta_j, \bar{\beta}_j$. Existence and uniqueness of the solution of (23) and further, the decomposition (21), are thus guaranteed if all the rapidities lie away from the imaginary axis, $Re \beta_j \neq 0$. Diagonalizing $X$ (20) is then also the essential part of efficiently computing the solution $Z$ of (23) [13].

Defining the $4n$ normal master-mode maps (NMM) as $(\hat{\zeta}, \hat{\zeta}')^T := V\hat{\xi}$, or

$$\hat{\zeta} = P^{-1} \hat{a}'$$

which, due to symplecticity (22), satisfy almost-CCR

$$[\hat{\zeta}_r, \hat{\zeta}'_s] = \delta_{rs}, \quad [\hat{\zeta}_r, \hat{\zeta}'_s] = [\hat{\zeta}'_r, \hat{\zeta}'_s] = 0.$$  \hspace{1cm} (25)

brings the Liouvillean to the normal form $\hat{L} = J\hat{\xi} \cdot JS\hat{\xi} - S_0 \mathbb{1} = J\hat{\xi} \cdot [(-\Delta) \oplus \Delta |\hat{\xi}\rangle - S_0 \mathbb{1}$ and finally

$$\hat{L} = -2 \sum_{r=1}^{2n} \beta_r \hat{\zeta}_r \hat{\zeta}_r.$$  \hspace{1cm} (26)

In contrast to the (finite-dimensional) fermionic case [3], the existence of a stable fixed point of the Liouvillean dynamics is not guaranteed here. Since our (central) system is infinite dimensional, it can absorb excitations from the environment and amplify indefinitely. Indeed, this is signaled by at least one of the rapidities being to the left of the imaginary axis, $\exists j, Re \beta_j < 0$.

However, if we assume that all the rapidities lie to the right of the imaginary axis, $\forall j, Re \beta_j > 0$, then we can make the following statements:

1. A unique non-equilibrium steady state (NESS) exists $|\text{NESS}\rangle \in \mathcal{K}$ namely the ‘right vacuum state’ of the Liouvillean

$$\hat{L}|\text{NESS}\rangle = 0.$$  \hspace{1cm} (27)

All the physical properties of NESS are essentially determined by the NMM annihilation relations

$$|1\rangle\hat{\zeta}_r = 0, \quad \hat{\zeta}_r|\text{NESS}\rangle = 0.$$  \hspace{1cm} (28)

2. The complete (point) spectrum of the decay modes of the Liouvillean is given in terms of a $2n$ component multi-index of super-quantum numbers $m \in \mathbb{Z}_{2n}$:

$$\lambda_m = -2 \sum_r m_r \beta_r, \quad \hat{L}|\Theta_m\rangle = \lambda_m|\Theta_m\rangle, \quad (|\Theta_m\rangle|\hat{L} = \lambda_m|\Theta_m\rangle),$$  \hspace{1cm} (29)

where

$$|\Theta_m\rangle = \prod_r \frac{(|\hat{\zeta}_r\rangle)^{m_r}}{\sqrt{m_r!}}|\text{NESS}\rangle, \quad (|\Theta_m\rangle| = (1) \prod_r \frac{(|\hat{\zeta}_r\rangle)^{m_r}}{\sqrt{m_r!}}).$$  \hspace{1cm} (30)

6 In fact, if we were working in terms of the Hermitian ‘coordinate’ and ‘momentum’ operators, $a_j + a_j^\dagger$, $(a_j - a_j^\dagger)$, from the outset, then a very similar formalism could have been developed with matrices $X$ and $Y$ being automatically real.
(3) The two-point correlator of the NESS is given by the solution of the Lyapunov equation (23). If \( b = (a, a^\dagger)^T \) designates a \( 2n \) column of the canonical operators, then

\[
\text{tr} : b^T b : \rho_{\text{NESS}} = (\langle \hat a \hat a^\dagger | \text{NESS} \rangle = (1|\hat a \hat a^\dagger |\text{NESS} \rangle) = Z_{r,s}. \tag{31}
\]

**Proof.** Using purely algebraic manipulations (i) and (ii) can be proved in analogy to the fermionic case [3] and are in fact equivalent to the standard construction of the Fock ground state and quasi-particle excitations with a restriction of non-normality. (iii) The first two equality signs are just definitions (2) and (5), whereas the last equality follows after expressing \( \hat a^\dagger \) in terms of the NMM maps (24), and using (28). \( \square \)

We note that the NESS is essentially a Gaussian state—namely it is a ‘ground state’ of a quadratic Liouvillean—and hence the Wick theorem can be used to express any higher order correlator in terms of two-point contractions (31).

The correlation matrix \( Z \) is in fact equivalent to the so-called covariance matrix which encodes all the second moments of the canonical coordinate and momentum operators (assuming, as we have done here for simplicity, that the terms linear in field operators and thus the first moments in the NESS vanish). In fact, there is a simple non-homogeneous (due to CCR) linear relationship between \( Z \) and the covariance matrix. We note that the equation of motion for the covariance matrix—which essentially again amounts to the Lyapunov-like equation (equivalent to equation (23))—can be obtained independently from our approach in a straightforward but tedious calculation [14]. Nevertheless, our approach here is much richer. It allows one to obtain with the same calculation the complete information about the Liouvillean excitation (decay) spectrum and accommodates for a general setup in which the interaction (Liouvilleans involving terms higher than second order in \( a_j, a_j^\dagger \)) can be incorporated systematically.

4. Example: a general Lindblad equation for a single-quantum oscillator

Take the simplest case, namely a single-quantum harmonic oscillator \( n = 1 \). We choose \( H \equiv \omega, K \equiv 0, M \equiv u = \sum |\mu|^2 > 0, N \equiv v = \sum |k|^2 > 0, L \equiv w = \sum l_k k \mu \in \mathbb{C} \).

Matrices (18) and (19) now read

\[
X = \frac{1}{2} \begin{pmatrix} i\omega + u - v & 0 \\ 0 & -i\omega + u - v \end{pmatrix}, \quad Y = \begin{pmatrix} -\bar w & v \\ v & -w \end{pmatrix}. \tag{32}
\]

The rapidity spectrum \( \beta_k = \frac{1}{2}(u - v \pm i\omega) \) indicates that the problem has a stable fixed point if and only if \( u > v \). Then, the NESS is approached at an exponential rate \( \sim \exp[-(u - v)t] \) and the complete correlator (31), as given by the straightforward solution of the Lyapunov equation (23) (now a linear system of three equations), read

\[
\text{tr} \ a^2 \rho_{\text{NESS}} = Z_{11} = \frac{\bar w}{u - v + i\omega} = \text{tr} (a^2)^2 \rho_{\text{NESS}}, \quad \text{tr} \ a^\dagger a \rho_{\text{NESS}} = Z_{12} = \frac{v}{u - v}. \tag{33}
\]

It is perhaps worth noting that this is not a Gibbs thermal state for any combination of Lindblad parameters, the fact which can be viewed as a manifestation of non-ergodicity of the harmonic oscillator. Furthermore, since the NESS is a generalized Gaussian state, expectation values of all higher moments of the field are calculated trivially.
5. Discussion

In the present communication, we outlined a simple and mathematically consistent framework for diagonalizing quantum-Liouvilleans for arbitrary bi-linear systems of bosons. Our method in some sense resembles the technique known as non-equilibrium thermo-field dynamics [15, 16], which has been recently applied to a similar problem of diagonalizing the Lindblad equation [17]. However, there is an important difference, namely in thermo-field dynamics, one treats a density matrix as an element of a tensor product Hilbert space, whereas the operator (observable) of choice is fixed in a given calculation. In our approach, we consider (dual) spaces of both simultaneously, so the canonical structure of our formalism is simpler and more transparent. This, in turn, is reflected in the ease of getting simple general results as outlined in the example in section 4).

Note that our formalism is flexible with respect to taking different realizations of the dual spaces \( \mathcal{K} \) and \( \mathcal{K}' \), namely for some physics applications it may be advantageous to take a larger space of observables \( \mathcal{K}' \) at the expense of a smaller dual space \( \mathcal{K} \), or vice versa. In analogy to the fermionic case [5] this formalism can be extended to explicitly time-dependent problems. For example, for periodic time-dependent problems [18], the Floquet representation of the correlation matrix leads to a discrete Lyapunov equation rather than a continuous one that appears here.

The approach presented here differs essentially from the framework proposed earlier for fermionic systems [3], where the density operators and the observables belong to the same Hilbert space. For bosonic systems, due to infinite dimensionality, this is not possible, and the symmetry between density operator space and observable operator space is broken. However, it may be desirable for certain purposes to have formally similar construction for bosonic and fermionic systems. Thus, we point out an equivalent version of fermionic third quantization, which follows exactly the steps of the present communication, but starting instead from a set of fermionic operators \( c_j, c_j^\dagger \), obeying canonical anti-commutation relations (CAR),

\[
\{ c_j, c_k^\dagger \} = \delta_{j,k}, \quad \{ c_j, c_k \} = \{ c_j^\dagger, c_k^\dagger \} = 0, \quad (34)
\]

Introducing the dual spaces of density operators and observables, stating (2)–(4) and defining the canonical adjoint fermionic maps

\[
\hat{c}_{0,j} = \hat{c}_j^\dagger, \quad \hat{c}'_{0,j} = \hat{c}_j^\dagger - \hat{c}_j^R \hat{\rho}, \\
\hat{c}_{1,j} = \hat{c}_j^R \hat{\rho}, \quad \hat{c}'_{1,j} = \hat{c}_j^R \hat{\rho} - \hat{c}_j^L, \quad (35)
\]

satisfying almost-CAR

\[
\{ \hat{c}_{v,j}, \hat{c}_{\mu,k} \} = \delta_{v,\mu} \delta_{j,k}, \quad \{ \hat{c}_{v,j}, \hat{c}_{\mu,k} \} = \{ \hat{c}'_{v,j}, \hat{c}'_{\mu,k} \} = 0, \quad (36)
\]

and the properties (7) and (8). The parity superoperator \( \hat{P} \) is uniquely defined by the dual vacuum states \( |\hat{P} \rangle = (1| \hat{P} | \rho_0 \rangle = | \rho_0 \rangle \), and anticommutes with all the elements of the adjoint algebra \( \{ \hat{P}, \hat{c} \} = \{ \hat{P}, \hat{c}' \} = 0 \). The difference to the more symmetric approach [3] is that now the canonical conjugate adjoint maps are not the Hermitian adjoint maps \( \hat{c}'_{v,j} \neq \hat{c}_v^\dagger \), which is however of no consequence as we are anyway dealing with the problems in which non-normal operators enter in an essential way.

Acknowledgments

We acknowledge discussions with F Leyvraz, J Eisert and V I Man’ko. This work was supported by the programme P1-0044, and the grant J1-2208, of Slovenian Research Agency, and by CONACyT, Mexico, project 57334 as well as the University of Mexico, PAPIIT project IN114310.
References

[1] Fano U 1964 *Lectures on the Many-Body Problem* ed E R Caianiello (New York: Academic)
[2] Mukamel S 1995 *Principles of Nonlinear Optical Spectroscopy* (Oxford: Oxford University Press)
[3] Prosen T 2008 *New J. Phys.* 10 043026
[4] Prosen T and Pižorn I 2008 *Phys. Rev. Lett.* 101 105701
[5] Prosen T and Žunkovič B 2010 *New J. Phys.* 12 025016
[6] Malkin I A and Manko V I 1979 *Dynamic Symmetry and Coherent States of Quantum Systems* (Moscow: Nauka)
[7] Moshinsky M and Smirnov Y F 1996 *The Harmonic Oscillator in Modern Physics* (New York: Harwood Academic)
[8] Davidovich L, Brune M, Raimond J M and Haroche S 1996 *Phys. Rev. A* 53 1295
[9] Kossakowski A 1972 *Rep. Math. Phys.* 3 247
[10] Gorini V, Kossakowski A and Sudarshan E C G 1976 *J. Math. Phys.* 17 821
[11] Lindblad G 1976 *Commun. Math. Phys.* 48 119
[12] http://en.wikipedia.org/wiki/Lyapunov_equation
[13] http://en.wikipedia.org/wiki/Sylvester_equation
[14] Eisert J and Prosen T 2010 Noise induced criticality (preprint)
[15] Umezawa H 1992 *Advanced Field Theory—Micro, Macro and Thermal Physics* (New York: AIP)
[16] Arita T 1994 *Condens. Matter Phys.* 4 26
[17] Ban M 2009 *J. Mod. Opt.* 56 577
[18] Ilievski E 2010 Phase transition in a time-dependent quantum spin chain far from equilibrium *Diploma Thesis* University of Ljubljana (in Slovenian)