Time and temperature-dependent correlation function of an impurity in one-dimensional Fermi and Tonks–Girardeau gases as a Fredholm determinant

Oleksandr Gamayun 1, 5, Andrei G Pronko 1 and Mikhail B Zvonarev 2, 4

1 Institut-Lorentz, Universiteit Leiden, PO Box 9506, 2300 RA Leiden, The Netherlands
2 Steklov Mathematical Institute, Fontanka 27, St. Petersburg, 191023, Russia
3 LPTMS, CNRS, University of Paris-Sud, Université Paris-Saclay, F-91405 Orsay, France
4 ITMO University, 197101, Saint-Petersburg, Russia
5 Author to whom any correspondence should be addressed.

E-mail: Gamayun@Lorentz.LeidenUniv.nl, agp@pdmi.ras.ru and mikhail.zvonarev@gmail.com

Keywords: impurity dynamics, Fredholm determinants, one-dimensional systems, quantum gases, Bethe ansatz

Abstract

We investigate a free one-dimensional spinless Fermi gas, and the Tonks–Girardeau gas interacting with a single impurity particle of equal mass. We obtain a Fredholm determinant representation for the time-dependent correlation function of the impurity particle. This representation is valid for an arbitrary temperature and an arbitrary repulsive or attractive impurity–gas δ-function interaction potential. It includes, as particular cases, the representations obtained for zero temperature and arbitrary repulsion in (Gamayun et al 2015 Nucl. Phys. B 892 83–104), and for arbitrary temperature and infinite repulsion in (Izergin and Pronko 1998 Nucl. Phys. B 520 594–632).

1. Introduction

Quantum many-body interacting systems in one spatial dimension solved by the Bethe ansatz are unique in that their complete set of eigenfunctions and eigenstates are known explicitly [1]. The use of this property constitutes a whole research field by now. Describing the time evolution of the observables is one of the most challenging problems in the field, remaining largely open.

The present paper is devoted to a particular Bethe ansatz-solvable model: a single mobile impurity interacting with a free Fermi gas in one spatial dimension through a δ-function potential of an arbitrary (positive or negative) strength g. Its eigenfunctions and spectrum have been found in [2, 3]. The eigenstates were represented as a sum of a product of plane waves with special coefficients. Such a representation is common for the Bethe ansatz-solvable models. The solution [2, 3] may be obtained from the one for the Gaudin–Yang model [4–6] as a particular case. However, the model we consider here has its own specificity: there exists a representation for its eigenstates through a single determinant resembling the Slater determinant for the free Fermi gas [7–9]. We use this representation as a starting point to investigate the time-dependent two-point impurity correlation function. We also argue that this function is the same as the one for a single mobile impurity interacting with the Tonks–Gurardeau gas [10, 11].

Our main result is an exact expression for the aforementioned correlation function in the limit of infinite system size, $L \to \infty$, and for arbitrary chemical potential (or density). This representation is obtained for the repulsive as well as attractive impurity–gas δ-function interaction potential of arbitrary strength $g$, and for arbitrary temperature. The key ingredient is the Fredholm determinant of a linear integral operator of integrable type (see, e.g, section XIV.1 of [1]). The present solution encompasses two particular cases known so far (i) infinite repulsion $g \to \infty$ [12, 13] (ii) arbitrary repulsion $g \geq 0$ at zero temperature [14].

The paper is organized as follows: in section 2 we define the model under consideration. In section 3 we summarize our main results. In section 4 we explain the transformation from the laboratory to the mobile impurity reference frame. In section 5 we write the eigenfunctions in the mobile impurity reference frame. They
are represented as a determinant differing from the Slater determinant representation of the free Fermi gas by a single row only. Using this representation we obtain the form-factors (the overlaps of the eigenfunctions of the model under consideration with the eigenfunctions of the free Fermi gas) in the form of the determinants of some finite-dimensional matrices. In section 6 we perform a form-factors summation in the case \( g \geq 0 \) which leads us to the Fredholm determinant representation of the impurity correlation function. In section 7 we extend the results obtained in section 6 to the case \( g < 0 \). Some properties of form-factors and Fredholm determinants, which we use to derive our results, are given in the appendices.

2. Model

In this section we define our model. We consider a free one-dimensional spinless Fermi gas at temperature \( T \) in the presence of a single distinct quantum particle (impurity) propagating through it. We investigate the correlation function

\[
G(x, t) = \frac{1}{Z} \langle \psi_\uparrow(x, t) \psi_\downarrow^\dagger(0, 0) \rangle_T.
\]

(2.1)

The canonical creation (annihilation) operators \( \psi^\dagger_\sigma \) (\( \psi_\sigma \)) carry the subscript \( \sigma = \uparrow \) for the Fermi gas, and \( \sigma = \downarrow \) for the impurity. The canonical anticommutation relations read

\[
[\psi_\sigma(x), \psi^\dagger_\sigma(x')] = \delta_{\sigma\sigma'} \delta(x - x').
\]

(2.2)

The temperature-weighted average \( \langle \cdots \rangle_T \) in equation (2.1) is defined as

\[
\langle \cdots \rangle_T = \sum_{|N|} \langle 0 \rangle \otimes \langle N| e^{-\beta(E_N - \mu N)} \cdots \langle N| \otimes |0\rangle.
\]

(2.3)

Here

\[
|N\rangle = c^\dagger_{p_1} \cdots c^\dagger_{p_N} |0\rangle
\]

(2.4)

is the free Fermi gas state containing \( N \) fermions with the momenta \( p_1, \ldots, p_N \). The gas is on a ring of circumference \( L \), and periodic boundary conditions are imposed. We have

\[
\psi^\dagger_\sigma(x) = \frac{1}{\sqrt{L}} \sum_p e^{-i p x/L} c^\dagger_{p\sigma}, \quad p = \frac{2\pi n}{L}, \quad n = 0, \pm 1, \pm 2, \ldots
\]

(2.5)

The vacuum \( |0\rangle \) is the state with no particles, \( c_0|0\rangle = 0 \). The sum in equation (2.3) runs through all possible values of \( p_1, \ldots, p_N \), and \( N \):

\[
\sum_{|N|} = \sum_{N=0}^{\infty} \cdots \sum_{p_N}.
\]

(2.6)

The parameter \( \beta \) is the inverse temperature, \( \beta = 1/T \). The Boltzmann constant, \( k_B \), and the Planck constant, \( \hbar \), are equal to one in our units. The Boltzmann–Gibbs weight \( e^{-\beta(E_N - \mu N)} \) is defined by the value of the free Fermi gas energy \( E_N \) in the state (2.4):

\[
H|N\rangle = E_N |N\rangle, \quad E_N \equiv E_N(p_1, \ldots, p_N)
\]

(2.7)

and by the chemical potential \( \mu \). The Hamiltonian of the free Fermi gas reads

\[
H = \int_0^L dx \psi^\dagger_{\uparrow}(x) \left( -\frac{1}{2m} \frac{\partial^2}{\partial x^2} \right) \psi_{\uparrow}(x),
\]

(2.8)

where \( m \) is the particle mass. Therefore

\[
E_N = \sum_{j=1}^{N} \epsilon(p_j),
\]

(2.9)

where

\[
\epsilon(q) = \frac{q^2}{2m}.
\]

(2.10)

The grand partition function \( Z \) in equation (2.1) is

\[
Z = \sum_{|N|} e^{-\beta(E_N - \mu N)} = \prod_p (1 + e^{-\beta(\epsilon(p) - \mu)}).
\]

(2.11)

Here, the product is taken over all possible values of the single-particle momentum, \( p = 2\pi n/L, \quad n = 0, \pm 1, \pm 2, \ldots \). The operator \( \psi_\uparrow \) in equation (2.1) evolves with time \( t \) as

\[
\psi_\uparrow(x, t) = e^{iHt} \psi_\uparrow(x, 0) e^{-iHt},
\]

where
\[ H = H_I + H_{\text{imp}} \]  

(2.12)

is the Hamiltonian of the entire system, and

\[ H_{\text{imp}} = \int_0^L dx \left[ \psi_I^\dagger(x) \left( -\frac{1}{2m} \frac{\partial^2}{\partial x^2} \right) \psi_I(x) + g \psi_I^\dagger(x) \psi_I(x) \psi^\dagger(x) \psi(x) \right]. \]  

(2.13)

The Hamiltonian (2.12) defines the fermionic Gaudin–Yang model \([4–6]\), in which the number of impurity particles

\[ N_I = \int_0^L dx \psi_I^\dagger(x) \psi_I(x), \]  

(2.14)

is arbitrary. However, only states of the Hamiltonian (2.12) with \( N_I = 0 \) and \( N_I = 1 \) are relevant for the dynamics described by the correlation function (2.1). The parameter \( g \) in equation (2.13) gives the strength of the impurity-gas interaction. We consider the case of both repulsive, \( g \geq 0 \), and attractive, \( g < 0 \), interactions. The first-quantized form of the Hamiltonian (2.12) with \( N_I = 1 \) is

\[ H = -\frac{1}{2m} \sum_{j=1}^N \frac{\partial^2}{\partial x_j^2} + \frac{1}{2m} \frac{\partial^2}{\partial x_1^2} + g \sum_{j=1}^N \delta(x_j - x_I). \]  

(2.15)

Here, \( x_1, \ldots, x_N \) are coordinates of the gas particles, and \( x_I \) is the coordinate of the impurity.

Let us now consider the Tonks–Girardeau gas \([10, 11]\). It consists of bosons interacting with each other through a \( \delta \)-function repulsive potential of infinite strength. The eigenfunctions of the Tonks–Girardeau gas, \( \Psi_B(x_1, \ldots, x_N) \), and those of the free Fermi gas, \( \Psi_F(x_1, \ldots, x_N) \), are in one-to-one correspondence:

\[ \Psi_F(x_1, x_2, \ldots, x_N) = \Psi_B(x_1, x_2, \ldots, x_N) \prod_{1 \leq i < j \leq N} \text{sign}(x_j - x_i). \]  

(2.16)

Here, \( \text{sign}(x) \) stands for the sign function, equal to one (minus one) for \( x \) positive (negative). Adding a single mobile impurity to the Tonks–Girardeau gas leads to the model whose eigenstates \( \Psi_B(x_1, x_2, \ldots, x_N) \) satisfy

\[ \Psi_F(x_1, x_2, \ldots, x_N) = \Psi_B(x_1, x_2, \ldots, x_N) \prod_{1 \leq i < j \leq N} \text{sign}(x_j - x_i), \]  

(2.17)

where \( \Psi_F(x_1, x_2, \ldots, x_N) \) is an eigenstate of the Hamiltonian (2.15). We stress that \( \Psi_B(x_1, x_2, \ldots, x_N) \) (\( \Psi_F(x_1, x_2, \ldots, x_N) \)) is symmetric (antisymmetric) with respect to any permutation of \( x_1, \ldots, x_N \). The impurity is distinguishable from the gas particles, therefore no symmetry restriction needs to be enforced when exchanging \( x_I \) with \( x_1, \ldots, x_N \). Equations (2.16) and (2.17) imply that the correlation function (2.1) is the same whether the impurity is injected into the free Fermi gas or the Tonks–Girardeau gas. Note that this statement holds true for certain other observables (for example, the average impurity momentum [15]). We present our calculations for an impurity injected into a free Fermi gas, bearing in mind that our results, which are summarized in section 3, are applicable to the case of the impurity injected into the Tonks–Girardeau gas as well.

3. Results

In this section we show the main result of our paper: the Fredholm determinant representation of the correlation function (2.1) at a given temperature \( T \) and chemical potential \( \mu \). The structure of the representation crucially depends on the sign of the impurity-gas interaction \( g \).

In section 3.1 we give the representation for the repulsive impurity-gas interaction, \( g \geq 0 \). The result for the attractive interaction, \( g < 0 \), is given in section 3.2. The particular cases of the infinitely strong repulsion, \( g \rightarrow \infty \), and attraction, \( g \rightarrow -\infty \), are discussed in a separate section 3.3.

3.1. Impurity-gas repulsion, \( g \geq 0 \)

The Fredholm determinant representation for the correlation function (2.1) in the case of the repulsive impurity-gas interaction, \( g \geq 0 \), reads

\[ G_{\text{rep}}(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{d} \Lambda \left[ (h - 1) \det \left( \hat{I} + \hat{V} + \hat{\mu} \right) \right]. \]  

(3.1)

Here

\[ h = \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{d} q \ e^{-ir(q)} \left[ \nu_{-}(q) - \nu_{+}(q) \right] = \frac{g^2}{4} \int_{-\infty}^{\infty} \text{d} q \ \frac{e^{-ir(q)}}{|k_+ - q|^{1}}, \]  

(3.2)

where

\[ \tau(q) = te(q) - xq, \]  

(3.3)
the function $\epsilon(q)$ is defined by equation (2.10),

$$\nu_{\pm}(q) = \frac{g}{2} \frac{1}{q - k_{\pm}},$$

(3.4)

and

$$k_{\pm} = \frac{g}{2}(\Lambda \pm i).$$

(3.5)

The identity operator is denoted by $\hat{1}$. The kernels of the linear integral operators $\hat{V}$ and $\hat{W}$, on the domain $[-\infty, \infty] \times [-\infty, \infty]$, are defined by

$$V(q, q') = \frac{\epsilon_{+}(q)\epsilon_{-}(q') - \epsilon_{-}(q)\epsilon_{+}(q')}{q - q'},$$

(3.6)

and

$$W(q, q') = \epsilon_{+}(q)\epsilon_{+}(q'),$$

(3.7)

respectively. The functions $\epsilon_{\pm}$ are defined as

$$\epsilon_{+}(q) = \epsilon(q)\epsilon_{-}(q), \quad \epsilon_{-}(q) = \frac{1}{\sqrt{\pi}} \sqrt{\theta(q)} e^{i\epsilon(q)/2},$$

(3.8)

where

$$\epsilon(q) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} dq' e^{-i\epsilon(q')\left(\frac{\nu_{+}(q')}{q' - q - i0} - \frac{\nu_{-}(q')}{q' - q + i0}\right)},$$

(3.9)

and

$$\theta(q) = \frac{1}{e^{\beta(\epsilon(q) - \mu)} + 1}$$

(3.10)

is the Fermi weight. The determinant—‘det’ symbol in equation (3.1)—stands for the Fredholm determinant of the corresponding linear integral operator. For completeness, we discuss the properties of Fredholm determinants, which we use to derive our results, in appendix A.

The kernel (3.6) belongs to a class of integrable kernels [1, 16]. Due to this property the resolvent operator $\hat{R}$, defined as

$$\hat{I} - \hat{R} = (\hat{I} + \hat{V})^{-1},$$

(3.11)

is also integrable:

$$R(q, q') = \frac{f_{+}(q)f_{-}(q') - f_{-}(q)f_{+}(q')}{q - q'}.$$  

(3.12)

The functions $f_{\pm}$ are the solutions to the integral equations

$$f_{\pm}(q) + \int_{-\infty}^{\infty} dq' V(q, q')f_{\pm}(q') = \epsilon_{\pm}(q).$$

(3.13)

Using the fact that the operator $\hat{W}$, with the kernel (3.7), has rank one, we recast the representation (3.1) into the following form:

$$G_{\text{rep}}(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\Lambda \left( h - B_{++}\right) \det(\hat{I} + \hat{V}),$$

(3.14)

where

$$B_{++} = \int_{-\infty}^{\infty} dq \epsilon_{+}(q)f_{+}(q).$$

(3.15)

3.2. Impurity-gas attraction, $g < 0$

The Fredholm determinant representation for the correlation function (2.1) in the case of the attractive impurity-gas interaction, $g < 0$, reads

$$G_{\text{attr}}(x, t) = G_{\text{rep}}(x, t) + G^{i}(x, t).$$

(3.16)

Here, $G_{\text{rep}}$ is given by equation (3.1) in which we let $g$ be negative. The function $G^{i}$ in equation (3.16) is given by the following expression:
\[
G^i(x, t) = \frac{g^2}{4} \frac{1}{2\pi} \int_{-\infty}^{\infty} d\lambda \, e^{-i[r(k) + r(k^{-1})]} \left[ \det(\hat{I} + \hat{V} + \hat{V}_1) - \det(\hat{I} + \hat{V} + \hat{V}_1 - \hat{V}_2) \right]. \tag{3.17}
\]

The kernels of the operators entering equation (3.17) are defined on \([-\infty, \infty] \times [-\infty, \infty]\). The kernel of \(\hat{V}\) is given by equation (3.6), and those of \(\hat{V}_1\) and \(\hat{V}_2\) are given by

\[
V_1(q, q') = \frac{h e_{-}(q) e_{-}(q')}{{(k_+ - q)(k_- - q')}} - \frac{e_{-}(q) e_{-}(q')}{k_- - q} - \frac{e_{+}(q) e_{-}(q')}{k_+ - q'}, \tag{3.18}
\]

and

\[
V_2(q, q') = g^2 \frac{e_{-}(q) e_{-}(q')}{{|k_+ - q|^2|k_+ - q'|^2}}, \tag{3.19}
\]

respectively. The functions entering equations (3.18) and (3.19) are defined in section 3.1. Note that \(G^i\) accounts for the bound states formed by the impurity and the gas particles, which exist for the Hamiltonian (2.15) with \(g < 0\). This is explicitly shown in section 7.

### 3.3. The limiting cases of infinitely strong repulsion and attraction, \(g \rightarrow \pm \infty\)

In this section we consider the Fredholm determinant representation for the correlation function (2.1) in the limiting cases of infinitely strong impurity-gas repulsion, \(g \rightarrow \infty\), and attraction, \(g \rightarrow -\infty\). In the case \(g \rightarrow \infty\) our representation coincide with the one given in [12, 13]. In the previously unexplored case \(g \rightarrow -\infty\) the representation is shown to be identical with the one obtained for \(g \rightarrow \infty\).

In the \(g \rightarrow \pm \infty\) limits the function (3.2) reads as follows:

\[
\lim_{g \rightarrow \pm \infty} h = 2 \left( \sin \frac{\eta}{2} \right)^2 G_0(x, t). \tag{3.20}
\]

Here, the variable \(\eta\) is related to \(\Lambda\) by the formula

\[
\Lambda = -\cot \left( \frac{\eta}{2} \right). \tag{3.21}
\]

and \(G_0\) is the Green’s function of a free particle

\[
G_0(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \, e^{-i r(k)} = \begin{cases} e^{-i \frac{m}{2\pi} \sqrt{\frac{\eta}{2}} t} & \text{if } t \neq 0, \\ \delta(x) & \text{if } t = 0. \end{cases} \tag{3.22}
\]

Let us consider the function \(G_{\text{rep}}(x, t)\), which is defined by equation (3.1) for both repulsive, \(g \geq 0\), and attractive, \(g < 0\), interactions. The \(g \rightarrow \pm \infty\) limits of this equation read

\[
G_{\infty}(x, t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\eta \left\{ [G_0(x, t) - 1] \det(\hat{I} + \hat{V}_{\infty}) + \det(\hat{I} + \hat{V}_{\infty} - \hat{W}_{\infty}) \right\}, \quad g \rightarrow \pm \infty. \tag{3.23}
\]

The kernels of the operators \(\hat{V}_{\infty}\) and \(\hat{W}_{\infty}\) are

\[
V_{\infty}(q, q') \equiv \lim_{g \rightarrow \pm \infty} V(q, q'), \tag{3.24}
\]

and

\[
W_{\infty}(q, q') \equiv \lim_{g \rightarrow \pm \infty} W(q, q'), \tag{3.25}
\]

where \(V(q, q')\) and \(W(q, q')\) are defined by equations (3.6) and (3.7), respectively. In the \(g \rightarrow \pm \infty\) limits the function (3.9) reads as follows:

\[
e_{\infty}(q) \equiv \lim_{g \rightarrow \pm \infty} e(q) = \left( \sin \frac{\eta}{2} \right)^2 \frac{1}{\pi} \text{p.v.} \int_{-\infty}^{\infty} dq' e^{-i r(q')} \left( \frac{\sin \frac{\eta}{2} \cos \frac{\eta}{2} e^{-i r(q)}}{q' - q} \right). \tag{3.26}
\]

where the symbol ‘p.v.’ indicates that the integral has to be interpreted as the Cauchy principal value. Note that the representation (3.23) coincides with equation (5.63) of [13] in the limit \(B \rightarrow \infty\) and \(h \rightarrow -\infty\) while keeping \(h + B = \text{const} = \mu\), where \(\mu\) is the chemical potential in our model.

The function \(G^i(x, t)\) entering equation (3.16), and defined by equation (3.17), vanishes in the \(g \rightarrow -\infty\) limit

\[
G^i(x, t) \sim O \left( \frac{1}{g} \right), \quad g \rightarrow -\infty. \tag{3.27}
\]

Let us prove this statement. Letting \(g \rightarrow -\infty\) in equations (3.6), (3.18), and (3.19), after some elementary algebra, we get from equation (3.17)
\[ G^i(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\Lambda \ e^{-i[\tau(k_\perp) + \tau(k_\parallel)]} u(\Lambda), \quad g \to -\infty, \]  \hfill (3.28)

where
\[ u(\Lambda) = \frac{4}{(\Lambda^2 + 1)^2} \det (\hat{I} + \hat{V}_\infty) B_{\ldots}. \]  \hfill (3.29)

Here, the kernel of the operator \( \hat{V}_\infty \) is defined by equation (3.24), and
\[ B_{\ldots} = \int_{-\infty}^{\infty} dq \ e^{-\langle q \rangle f_\perp (q)}, \]  \hfill (3.30)

where \( f_\perp \) is defined by equation (3.13). We have
\[ \tau (k_\parallel) + \tau (k_\perp) = \frac{t}{m} \left( \frac{8}{4} (\Lambda^2 - 1) - 2x\Lambda, \right. \]  \hfill (3.31)

where \( \tau \) is defined by equation (3.3), and \( k_\pm \) by equation (3.5). Therefore, the right-hand side of equation (3.28) decays at least as fast as \( 1/g \) if
\[ |u(0)| < \infty. \]  \hfill (3.32)

The analytic properties of \( u(\Lambda) \) discussed in section 2.3 of [17] imply the validity of equation (3.32), thus completing the proof of equation (3.27).

We found that the correlation function (2.1) is given by the same expression (3.23) for both \( g \to \infty \) and \( g \to -\infty \) limits. This implies, in particular, that the results for \( G_\infty(x, t) \) obtained by the technique developed in [18] (which is specific to the \( g \to \infty \) limit) are valid in the \( g \to -\infty \) limit.

4. Mobile impurity reference frame

In this section we take our model defined in the laboratory reference frame, and write it in the mobile impurity reference frame, following [8]. We obtain the Hamiltonian which does not contain the impurity coordinate and contains the total momentum (a good quantum number) explicitly. The calculations performed in the rest of our paper are based on this representation of the Hamiltonian.

Let us introduce the operator
\[ \mathcal{Q} = e^{iP_\perp X_\perp}, \]  \hfill (4.1)

where
\[ X_\sigma = \int_0^L dx \ x \psi^\dagger_\sigma (x) \psi_\sigma (x), \quad P_\sigma = \int_0^L dx \ \psi^\dagger_\sigma (x) \left( -i \frac{\partial}{\partial x} \right) \psi_\sigma (x), \quad \sigma = \uparrow, \downarrow . \]  \hfill (4.2)

The transformation of an arbitrary operator \( \mathcal{O} \) from the laboratory to the mobile impurity reference frame reads
\[ \mathcal{O} \to \mathcal{O}_\perp = \mathcal{Q} \mathcal{O} \mathcal{Q}^{-1}. \]  \hfill (4.3)

It is often referred to as a polaron, or the Lee–Low–Pines transformation [19]. For single-particle operators, we get
\[ \psi_{\uparrow} \mathcal{Q} (x) = e^{-i\hbar x} \psi_{\uparrow} (x) \]  \hfill (4.4)

\[ \psi_{\downarrow} \mathcal{Q} (x) = \psi_{\downarrow} (x - X_\uparrow), \]  \hfill (4.5)

and for the Hamiltonian (2.12) subjected to the condition \( N_\uparrow = 1 \), we get
\[ H_\mathcal{Q} = H_{\uparrow} \mathcal{Q} + H_{\text{imp}} \mathcal{Q}, \quad N_\uparrow = 1. \]  \hfill (4.6)

Here
\[ H_{\uparrow} \mathcal{Q} = H_{\uparrow} \]  \hfill (4.7)

and
\[ H_{\text{imp}} \mathcal{Q} = \left( \frac{P_\uparrow - P_\downarrow}{2m} \right)^2 + g \rho_{\downarrow} (0), \quad N_\downarrow = 1. \]  \hfill (4.8)

The total momentum \( P \) is conserved in the model (2.12):
\[ [P, H] = 0, \quad P = P_\uparrow + P_\downarrow. \]  \hfill (4.9)

Equation (4.9), written in the mobile impurity reference frame reads
\[ [P_\uparrow, H_{\uparrow} \mathcal{Q}] = 0, \quad P_\downarrow = P_\downarrow, \quad N_\downarrow = 1. \]  \hfill (4.10)

This means that \( P_\uparrow \) is the total momentum of the model written in the mobile impurity reference frame.
The state (2.4) is an eigenfunction of the momentum operator for the free Fermi gas
\[ P_i |N \rangle = P_i |N \rangle, \quad P_i = \sum_{j=1}^{N} p_j. \] (4.11)

Using that \( X_i |0 \rangle = 0 \) we get for the operator (4.1)
\[ \mathcal{Q} |N \rangle \otimes |0 \rangle = |N \rangle \otimes |0 \rangle. \] (4.12)

We note that
\[ H |N \rangle \otimes |0 \rangle = E_0 |N \rangle \otimes |0 \rangle, \] (4.13)
where the energy \( E_N \) of the free Fermi gas in the state \( |N \rangle \) is defined by equations (2.7) and (2.9). Applying the transformation (4.3) to the operators entering equation (2.1) and using equations (4.4) and (4.11)–(4.13) we get
\[ G(x, t) = \frac{1}{Z} \sum_{|N\rangle} e^{iE_N t} e^{-iP_N x} e^{-\beta E_N - \mu N} \langle 0 \rangle \otimes \langle N | \psi_i(x) e^{-iH_0 t} \psi^\dagger_i(0) |N \rangle \otimes |0 \rangle. \] (4.14)

Using the momentum space decomposition (2.5) for the operators \( \psi_i \) and \( \psi^\dagger_i \) entering equation (4.14), we get
\[ G(x, t) = \frac{1}{Z} \sum_{|N\rangle} e^{-\beta E_N - \mu N} e^{iEt} \frac{1}{L} \sum_{p} e^{-i(P_N - p)x} \langle N | e^{-iH_0 t} p |N \rangle. \] (4.15)

Here, \( H_Q(p) \) is obtained from \( H_Q \), equation (4.6), by projecting the operator \( P_i \) entering equation (4.8) onto the state \( r^{\dagger}_{p_i} |0 \rangle \) having momentum \( p \):
\[ H_Q(p) = H_1 + H_{\text{imp}} Q(p), \quad H_{\text{imp}} Q(p) = \frac{(p - P_i)^2}{2m} + g_{\rho_i}(0). \] (4.16)

By using the mobile impurity reference frame we reduce the problem of calculating the correlation function (2.1) to the analysis of the matrix elements \( \langle N | e^{-iH_0 t} p |N \rangle \) entering equation (4.15). The impurity-gas interaction term, \( g_{\rho_i}(0) \), in equation (4.16) has the form of a static potential scattering off the gas particles. Using the completeness condition
\[ \sum_{f_p} |f_p \rangle \langle f_p| = 1 \] (4.17)
for the eigenfunctions \( |f_p \rangle \) of the Hamiltonian \( H_Q(p) \) at a given \( p \),
\[ H_Q(p) |f_p \rangle = E_f |f_p \rangle, \] (4.18)
we write
\[ \langle N | e^{-iH_0 t} p |N \rangle = \sum_{f_p} |N |f_p \rangle \langle f_p| e^{-iE_f}. \] (4.19)

The overlaps \( \langle N |f_p \rangle \), often called form-factors, vanish in the large \( N \) limit for any \( g \neq 0 \) (the decay rate for some \( \langle N |f_p \rangle \) is found in [8]). This vanishing is an example of the Anderson orthogonality catastrophe [20]. We give an explicit expression for \( \langle N |f_p \rangle \) in section 5. The sum over \( f_p \) in equation (4.19) contains infinitely many terms, and does not vanish as \( N \to \infty \). We calculate this sum in sections 6 and 7.

5. Bethe ansatz

In section 5.1 we present the eigenfunctions and spectrum of the Hamiltonian (4.16). In section 5.2 we discuss the solutions of the Bethe ansatz equations. In section 5.3 we calculate the overlaps between the eigenfunctions of the Hamiltonian (4.16) and the eigenfunctions (2.4) of the free Fermi gas.

5.1. Eigenfunctions and spectrum

The eigenfunctions and spectrum of the mobile impurity model (2.15) were found exactly in [2, 3]. The eigenfunctions in the coordinate representation could be written as a sum running over all permutations of \( N \) particles, and containing \( N! \) terms. Each of the terms is a product of plane waves multiplied by a factor which does not depend on the coordinates of the particles. Such a structure of the eigenfunctions is common for the Bethe ansatz solvable models [1]. For example, this is the case in the Gaudin–Yang model [4–6, 21], which contains the mobile impurity model (2.15) as a particular instance. However, specific to the model (2.15), the eigenfunctions can be written in the mobile impurity reference frame as a single determinant, much resembling the Slater determinant for the eigenfunctions of a free Fermi gas. We demonstrate this explicitly in the present section.
We let the particle mass

\[ m = 1 \]  

through the rest of the paper in order to lighten notations. We write the eigenfunctions \( f_p \) defined by equation (4.18), in the coordinate representation as the determinant of the \((N + 1) \times (N + 1)\) matrix

\[
f_p(x_1, \ldots, x_N) = \frac{Y_f}{\sqrt{N!}} \begin{vmatrix} e^{ik_1} & \cdots & e^{ik_N} & e^{ik_{N+1}} \\ \vdots & \ddots & \vdots & \vdots \\ e^{ik_N} & \cdots & e^{ik_{N+1}} \\ \nu_-(k_1) & \cdots & \nu_-(k_N) & \nu_-(k_{N+1}) \end{vmatrix} \tag{5.2}
\]

Here, \( \nu_\cdot \) is defined by equation (3.4). The factor \( Y_f \) ensures the normalization condition

\[
(f_p | f'_{p'}) = \delta_{pp'}.
\tag{5.3}
\]

Note that \( (f_p | f'_{p'}) = \delta_{pp'} \) for \( p = p' \). We calculate \( Y_f \) in section 5.3. The set of quasi-momenta \( k_1, \ldots, k_{N+1} \) satisfies a system of nonlinear equations (Bethe equations)

\[
\cot \frac{k_j L}{2} = \frac{2k_j}{g} - \Lambda, \quad j = 1, 2, \ldots, N + 1,
\tag{5.4}
\]

and

\[
p = \sum_{j=1}^{N+1} k_j.
\tag{5.5}
\]

Recall that \( p \) is the total momentum of the system in the laboratory reference frame, and it is quantized as follows

\[
p = \frac{2\pi}{L} n, \quad n = 0, \pm 1, \pm 2, \ldots. \tag{5.6}
\]

The energy \( E_f \) of the state \( | f_p \rangle \) is

\[
E_f = \frac{1}{2} \sum_{j=1}^{N+1} k_j^2.
\tag{5.7}
\]

The determinant in equation (5.2) differs from the Slater determinant for the eigenfunctions of a free Fermi gas by the last row only. Note that the explicit form of the matrix under the determinant can be changed by a unitary transformation; its size can also be reduced by decomposing the determinant. This explains the variety of the representations equivalent to the one given by equation (5.2), found in the literature [7–9, 14, 15].

### 5.2. Solutions of the Bethe equations

In this section we discuss the properties of the solutions of the Bethe equations (5.4) and (5.5) that we use to derive our results.

Any solution \( k_1, \ldots, k_{N+1}, \Lambda \) of the Bethe equations (5.4) and (5.5) has the following properties [3]: (i) \( \Lambda \) is real. (ii) If \( g \geq 0 \) all \( k_j \)'s are real. (iii) If \( g < 0 \) either all \( k_j \)'s are real, or \( k_1, \ldots, k_{N-1} \) are real, while \( k_N \) and \( k_{N+1} \) have a non-zero imaginary part, and \( k_N = k_{N+1}^* \). Furthermore, if \( k_N \) and \( k_{N+1} \) are complex, they read as follows in the large \( L \) limit:

\[
k_N = k_+ + \mathcal{O}(e^{-|g|L}), \quad k_{N+1} = k_- + \mathcal{O}(e^{-|g|L}), \tag{5.8}
\]

where \( k_\pm \) are defined by equation (3.5).

We will often use the following representation of the Bethe equations (5.4):

\[
\nu_j(k_j) = e^{ik_j L} = \frac{\nu_j'(k_j)}{\nu_j(k_j)}, \quad j = 1, \ldots, N + 1, \tag{5.9}
\]

where \( \nu_j \) are given by equation (3.4). Taking the derivative of equation (5.9) with respect to \( \Lambda \) we get

\[
\frac{\partial k_j}{\partial \Lambda} = \frac{\nu_j(k_j) - \nu_k(k_j) \nu_j'(k_j)}{\nu_j(k_j) + \frac{4}{L} \nu_j'(k_j)}, \quad j = 1, \ldots, N + 1. \tag{5.10}
\]

Substituting equation (5.8) into (5.10) we obtain

\[
\frac{\partial k_N}{\partial \Lambda} = \frac{\partial k_{N+1}}{\partial \Lambda} = \frac{g}{2} + \mathcal{O}(e^{-|g|L}), \quad k_N^2 = k_{N+1}. \tag{5.11}
\]
Let us introduce the real-valued function $R(x)$ as a solution of the equation
\[ \cot R(x) = x + \frac{4}{gL} R(x). \] (5.12)

This function is single-valued for $-\infty < x < \infty$ if
\[ \frac{4}{gL} \geq -1. \] (5.13)

For the rest of the paper we assume that equation (5.13) holds true. We note that the results obtained using this condition are valid for arbitrary $g$ in the $L \to \infty$ limit.

The function $R(x)$ decays monotonously
\[ \frac{\partial R(x)}{\partial x} < 0, \quad -\infty < x < \infty \] (5.14)
and takes the values between 0 and $\pi$:
\[ R(-\infty) = \pi, \quad R(\infty) = 0. \] (5.15)

Using this function, any real-valued quasi-momenta $k_j$ entering equation (5.4) can be parametrized as follows
\[ k_j = \frac{2\pi}{L} \left( n_j - \frac{\delta_j}{\pi} \right), \quad n_j = 0, \pm 1, \pm 2, \ldots, \] (5.16)
where
\[ \delta_j = R \left( \Lambda - \frac{4\pi}{gL} n_j \right), \quad 0 \leq \delta_j < \pi. \] (5.17)

Combining equations (5.16), (5.17), and (5.14) one can immediately see that
\[ \frac{\partial k_j}{\partial \Lambda} > 0, \quad -\infty < \Lambda < \infty, \quad j = 1, \ldots, N + 1 \] (5.18)
for any real-valued quasi-momentum $k_j$.

Evidently, real quasi-momenta $k_j$’s, are in one-to-one correspondence with $n_j$’s. The $n_j$’s should be different from each other, otherwise the function (5.2) vanishes. This implies that the $n_j$’s should be different as well. Therefore, each eigenstate (5.2) is uniquely characterized by an ordered set $n_1 < \cdots < n_{N+1}$ (or by $n_1 < \cdots < n_{N-1}$ if $k_N = k_{N+1}$) for a given $p$. We also stress that taking the limits $\Lambda \to \infty$ and $\Lambda \to -\infty$ one gets the same subset of eigenstates (5.2). The condition $\delta_j < \pi$ imposed in equation (5.17) ensures that this subset is taken into account only once.

### 5.3. Determinant representation for $\langle N|f_p^\dagger f_p^\dagger \rangle$

In this section we present an explicit formula for the overlaps $\langle N|f_p^\dagger f_p^\dagger \rangle$ between the eigenfunctions (2.4) of the free Fermi gas and the eigenfunctions (5.2) of the mobile impurity model. We also present an explicit formula for the normalization constant $Y_f$ in equation (5.2).

Straightforward calculations (see appendix B.1 for details) lead us to the following result:
\[ \langle N|f_p^\dagger f_p^\dagger \rangle = Y_f \left( \frac{2}{L} \right)^N \det D_f \prod_{j=1}^{N+1} \nu_-(k_j), \] (5.19)
where
\[ \det D_f = \begin{vmatrix} \frac{1}{k_1 - p_1} & \cdots & \frac{1}{k_{N+1} - p_1} \\ \vdots & \ddots & \vdots \\ \frac{1}{k_1 - p_N} & \cdots & \frac{1}{k_{N+1} - p_N} \end{vmatrix} \] (5.20)
is the determinant of the $(N + 1) \times (N + 1)$ matrix, and $\nu_-$ is defined by equation (3.4). We get from equation (5.19)
\[ |\langle N|f_p^\dagger f_p^\dagger \rangle|^2 = |Y_f|^2 \left( \frac{2}{L} \right)^{2N} |\det D_f|^2 \prod_{j=1}^{N+1} |\nu_-(k_j)|^2 |\nu_+(k_j)|. \] (5.21)
Now, using equation (B16) from appendix B.2 we write an explicit expression for $|Y_j|^2$:

$$
|Y_j|^2 = \prod_{j=1}^{N+1} \left[ 1 + \frac{4}{gL} \nu_-(k_j) \nu_+(k_j) \right] \frac{\nu_-(k_j) \nu_+(k_j)}{\nu_+(k_j) \nu_+(k_j)}.
$$

(5.22)

Substituting this expression into equation (5.21) and using equation (3.10) we get

$$
|\langle N | f_p \rangle|^2 = \left( \frac{2}{L} \right)^N \left| \det D_f \right|^2 \left| \sum_{j=1}^{N+1} \frac{\partial k_j}{\partial \Lambda} \right|^{-1} \left| \prod_{j=1}^{N+1} \frac{\partial k_j}{\partial \Lambda} \right|.
$$

(5.23)

Recall that $k_0, \ldots, k_{N+1}$, $\Lambda$ solve the Bethe equations (5.4) and (5.5).

In sections 6 and 7 we employ the representation (5.23) to perform the summations in equations (4.19) and (4.15) and obtain the result expressed in terms of the Fredholm determinants.

6. Summation in case of impurity-gas repulsion, $g \geq 0$

In this section we perform the summations in equations (4.19) and (4.15). This results in the Fredholm determinant representation for the correlation function (2.1). We assume that the impurity-gas interaction is repulsive, $g \geq 0$. The case of the attractive interaction, $g < 0$, is considered in section 7.

In section 6.1 we perform the summation in equation (4.19) using the explicit formula (5.23). In section 6.2 we transform the representations obtained in section 6.1, using the fact that the system size $L \to \infty$. In section 6.3 we perform the summations in equation (4.15).

6.1. Summation over $f_p$

In this section we perform the summation over $f_p$ in equation (4.19).

The quasi-momenta $k_0, \ldots, k_{N+1}$ characterizing the eigenfunctions $f_p$ for a given $p$ are coupled to each other by the condition (5.5). There, the total momentum $p$ is quantized, as given by equation (5.6). The quantization is due to a finite system size, $L$, and should play no role for the observables as $L \to \infty$. We can therefore write the following identity

$$
\delta \left( \sum_{j=1}^{N+1} k_j - p \right) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dz \, e^{i \left( \sum_{j=1}^{N+1} k_j - p \right) z}, \quad L \to \infty.
$$

(6.1)

The argument $\sum_{j=1}^{N+1} k_j - p$ of the Dirac $\delta$-function on the left-hand side of this expression is equal to zero when equation (5.5) is satisfied. Using equation (6.1) to calculate the correlation function (2.1) produces results which are exact in the $L \to \infty$ limit.

Let us now introduce the function

$$
\Delta_p(n_0, \ldots, n_{N+1}) = \int_{-\infty}^{\infty} d\Lambda \left| \sum_{j=1}^{N+1} \frac{\partial k_j}{\partial \Lambda} \right| \delta \left( \sum_{j=1}^{N+1} k_j - p \right).
$$

(6.2)

Here, $n_0, \ldots, n_{N+1}$ are integers which define $k_0, \ldots, k_{N+1}$ as given by equations (5.16) and (5.17). We stress that $k_j$'s depend on $\Lambda$ through $\delta$'s as given by equation (5.17). Therefore, $\Delta_p(n_0, \ldots, n_{N+1}) = 1$ if there exists a solution to the Bethe equations (5.4) and (5.5) for a given set $n_0, \ldots, n_{N+1}$, and $p$, and is equal to zero otherwise:

$$
\Delta_p(n_0, \ldots, n_{N+1}) = \begin{cases} 
1 & \text{if $n_0, \ldots, n_{N+1}$, and $p$ solve equations (5.4) and (5.5),} \\
0 & \text{otherwise.}
\end{cases}
$$

(6.3)

Defining

$$
T_p(n_0, \ldots, n_{N+1}) = \begin{cases} 
|\langle N | f_p \rangle|^2 & \text{if $n_0, \ldots, n_{N+1}$, and $p$ solve equations (5.4) and (5.5),} \\
0 & \text{otherwise,}
\end{cases}
$$

(6.4)

and using equations (5.23) and (6.2) we get

$$
T_p(n_0, \ldots, n_{N+1}) = \int_{-\infty}^{\infty} d\Lambda \left( \frac{2}{L} \right)^N \left| \det D_f \right|^2 \delta \left( \sum_{j=1}^{N+1} k_j - p \right).
$$

(6.5)

We stress that $T_p$ is symmetric with respect to the permutations of $n_0, \ldots, n_{N+1}$, and vanishes if any two integers from this set coincide. Using equation (6.5) we write the right-hand side of equation (4.19) as follows:
\[
\langle N | e^{-iH_0(p)} | N \rangle = \frac{1}{(N+1)!} \sum_{n_0, \ldots, n_{N+1}} F_p (n_1, \ldots, n_{N+1}) e^{-iE_0}. \tag{6.6}
\]

On the right-hand side of this expression the summation over each \( n_j \) goes independently over all integers. The factor \( 1/(N+1)! \) compensates the multiple counting of the Bethe ansatz solutions stemming from the removal of the ordering condition \( n_1 < \cdots < n_{N+1} \). Substituting the identity (6.1) into equation (6.6) we get

\[
\langle N | e^{-iH_0(p)} | N \rangle = \frac{1}{(N+1)!} \int_{-\infty}^{\infty} d\Lambda \
\times \sum_{n_0, \ldots, n_{N+1}} \int_{-\infty}^{\infty} \frac{dz}{2\pi} e^{-ipz} \left( \frac{2}{L} \right)^N | \det D_j |^2 \prod_{j=1}^{N+1} \left[ e^{-i\tau(k_j)} \frac{\partial k_j}{\partial \Lambda} \right]. \tag{6.7}
\]

where

\[
\tau(k) = \frac{k^2}{2} - zk. \tag{6.8}
\]

We stress that each \( k_j \) depends on \( n_j \) as given by equations (5.16) and (5.17). Since \( k_0, \ldots, k_{N+1} \) are real we have

\[
| \det D_j |^2 = (\det D_j)^2 \tag{6.9}
\]

and equation (5.18) is applicable. We can, therefore, rewrite equation (6.7) as follows

\[
\langle N | e^{-iH_0(p)} | N \rangle = \frac{1}{(N+1)!} \int_{-\infty}^{\infty} d\Lambda \
\times \sum_{n_0, \ldots, n_{N+1}} \int_{-\infty}^{\infty} \frac{dz}{2\pi} e^{-ipz} \left( \frac{2}{L} \right)^N (\det D_j)^2 \prod_{j=1}^{N+1} \left[ e^{-i\tau(k_j)} \frac{\partial k_j}{\partial \Lambda} \right]. \tag{6.10}
\]

We now transform the summation over \( n_0, \ldots, n_{N+1} \) in equation (6.10) using a technique explained in appendix B.3. Employing equation (B23) we get

\[
\left( \frac{2}{L} \right)^N \sum_{n_0, \ldots, n_{N+1}} (\det D_j)^2 \prod_{j=1}^{N+1} \left[ e^{-i\tau(k_j)} \frac{\partial k_j}{\partial \Lambda} \right] = (h - 1) \det A + \det (A - B). \tag{6.11}
\]

Here

\[
h = \sum_n \frac{\partial k}{\partial \Lambda} e^{-i\tau(k)}, \tag{6.12}
\]

\[
A_{jl} = \frac{2}{L} \sum_n \frac{\partial k}{\partial \Lambda} \frac{e^{-i\tau(k)}}{(k - p_j)(k - p_l)}, \quad j, l = 1, \ldots, N, \tag{6.13}
\]

and

\[
B_{jl} = \frac{2}{L} e(p_j) e(p_l), \quad j, l = 1, \ldots, N, \tag{6.14}
\]

where

\[
e(q) = \sum_n \frac{\partial k}{\partial \Lambda} e^{-i\tau(k)} \quad q = \frac{2\pi}{L} n, \quad n = 0, \pm 1, \pm 2, \ldots. \tag{6.15}
\]

We stress that the summation index \( n \) in equations (6.12), (6.13), and (6.15) is related to \( k \) by equation (5.16). We rewrite equation (6.13) in a form used in our subsequent calculations

\[
A_{jl} = \frac{2}{L} \left[ \delta_{jl} \sum_n \frac{\partial k}{\partial \Lambda} \frac{e^{-i\tau(k)}}{(k - p_j)^2} + (1 - \delta_{jl}) \frac{e(p_j) - e(p_l)}{p_j - p_l} \right], \quad j, l = 1, \ldots, N. \tag{6.16}
\]

Substituting equation (6.11) into (6.10) we get

\[
\langle N | e^{-iH_0(p)} | N \rangle = \int_{-\infty}^{\infty} d\Lambda \int_{-\infty}^{\infty} \frac{dz}{2\pi} e^{-ipz} [ (h - 1) \det A + \det (A - B) ]. \tag{6.17}
\]

### 6.2. Large L limit

In this section we use the fact that the system size \( L \to \infty \) to transform the representations obtained in section 6.1.
We introduce the function

\[ b(k) = \frac{2k}{g} - \Lambda - \cot \frac{kL}{2} \]  

(6.18)

whose zeroes \( b(k) = 0 \) are the solutions to the Bethe equations (5.4). It satisfies the following identity

\[ \frac{\partial b(k)}{\partial k} = \left( \frac{\partial k}{\partial \Lambda} \right)^{-1} \quad \text{at} \quad b(k) = 0. \]  

(6.19)

Using this identity we transform the summation in equations (6.12), (6.16), and (6.15) as follows

\[ \sum_{\pi} \frac{\partial k}{\partial \Lambda} f(k) = \frac{1}{2\pi i} \oint_c \frac{dk}{b(k)} f(k). \]  

(6.20)

The contour \( \gamma \) is oriented counter-clockwise. It is chosen in such a way that the zeroes of \( b(k) \) are outside the domain bounded by \( \gamma \), and the poles of the function \( f(k) \) are outside.

We now take the large \( L \) limit. This will remove the effects of the boundary conditions, and further simplify the formulas. The function \( \cot (kL/2) \) entering \( b(k) \) is periodic with the period \( 2\pi/L \). The variation over the period of the other terms entering \( b(k) \) can be neglected in the integrals containing \( b(k) \) to the leading order of the 1\( /L \) expansion. Taking into account that

\[ \int_0^{\pi} \frac{dk}{k - \cot k} = \frac{1}{z + \text{isign(Im} z)} \quad \text{for} \quad \text{Im} \ z = 0 \]  

(6.21)

we can replace \( \cot (kL/2) \) with \( i \) if \( \text{Im} \ \gamma > 0 \) and with \(-i\) if \( \text{Im} \ \gamma < 0 \) in the function \( b(k) \) on the right-hand side of equation (6.20).

We begin with applying the arguments from the previous paragraph to the representation (6.20) of the function (6.12). We take the contour \( \gamma \) consisting of two straight lines, \( \gamma^+ \) and \( \gamma^- \). Here, \( \gamma^+ \) runs from \( \infty + i0 \) to \(-\infty + i0 \), and \( \gamma^- \) runs from \(-\infty - i0 \) to \( \infty - i0 \). Therefore

\[ h = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{dk}{2k/g - \Lambda - i} - \frac{1}{2k/g - \Lambda + i}. \]  

(6.22)

We now perform the same procedure with the function \( e(q) \), defined by equation (6.15). We take the contour \( \gamma \) consisting of \( \gamma^+ \), \( \gamma^- \), and \( c \). Here, \( c \) is a clockwise-oriented closed contour around the point \( q \). We have

\[ \oint_c \frac{dk}{b(k) - q} = 0 \quad \text{at} \quad q = \frac{2\pi}{L} n, \quad n = 0, \pm 1, \pm 2, \ldots. \]  

(6.23)

Therefore

\[ e(q) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{dk}{2k/g - \Lambda - i} - \frac{1}{2k/g - \Lambda + i}. \]  

(6.24)

where p.v. stands for the Cauchy principal value of the integral.

Now we transform the diagonal entries of the matrix (6.16):

\[ A_{jj} \equiv \frac{2}{L} \sum_{n} \frac{\partial k}{\partial \Lambda} \frac{e^{i\tau(k)}}{(k - p_j)^2} = \frac{1}{2\pi i} \oint_{\gamma} \frac{dk}{2k/g - \Lambda - \cot \frac{kL}{2}} \frac{e^{i\tau(k)}}{(k - p_j)^2}, \]  

(6.25)

We take the same contour \( \gamma \) as for \( e(q) \). The contour \( c \) gives now a non-vanishing contribution. We have

\[ A_{jj} = e^{-i\tau(p_j)} + \frac{2}{L} \frac{\partial q}{q} e(q) \bigg|_{q=p_j}. \]  

(6.26)

Using

\[ \prod_{j=1}^{N} e^{-i\tau(p_j)} = e^{-i\tau E_{0} + i\tau N z}, \]  

(6.27)

we arrive at the following expressions for the determinants entering equation (6.17):

\[ \det A = e^{-i\tau E_{0} + i\tau N z} \det (I + \tilde{V}), \]  

(6.28)
and

$$\det(A - B) = e^{-i\theta x^2} \det(I + \hat{V} - \hat{W}).$$  \hfill (6.29)

Here

$$\hat{V}_j = \frac{2\pi}{L} E_+(p_j) E_-(p_j) - E_+(p_j) E_-(p_j), \quad j, l = 1, \ldots, N$$  \hfill (6.30)

and

$$\hat{W}_j = \frac{2\pi}{L} E_+(p_j) E_+(p_j), \quad j, l = 1, \ldots, N.$$  \hfill (6.31)

The functions $E_\pm$ are defined as

$$E_+(q) = e(q) E_-(q), \quad E_-(q) = \frac{1}{\sqrt{\pi}} e^{q^2/2},$$  \hfill (6.32)

and the functions $h$, $e(q)$, and $\tau(q)$ by equations (6.22), (6.15), and (6.8), respectively. Substituting equations (6.28) and (6.29) into equation (6.17) we get

$$\langle N | e^{-iH_0(p)} | N \rangle = \int_{-\infty}^{\infty} d\Lambda \times \int_{-\infty}^{\infty} \frac{dz}{2\pi} e^{i(p^\prime - p)z} e^{-i\theta z^2} \{ (h - 1) \det(I + \hat{V}) + \det(I + \hat{V} - \hat{W}) \}. \hfill (6.33)$$

6.3. Summation over $p$ and $\{N \}$

In this section we perform the summation over $p$ and $\{N \}$ in equation (4.15) and obtain the Fredholm determinant representation for the correlation function (2.1).

We replace the sum over $p$ with an integral

$$\frac{1}{L} \sum p \to \frac{1}{2\pi} \int_{-\infty}^{\infty} dp,$$  \hfill (6.34)

while taking the $L \to \infty$ limit of equation (4.15), and get

$$G(x, t) = \frac{1}{Z} \sum_{\{N \}} e^{-\beta(E_N - \mu N)} e^{ixt} \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{-i(p - p_0)x} \langle N | e^{-itH_0(p)} | N \rangle.$$  \hfill (6.35)

The right-hand side of equation (6.33) depends on $p$ through the function $e^{ipx}$ only. Taking this into account and substituting equation (6.33) into (6.35) we perform the integration over $p$,

$$\int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{ip(x - z)} = \delta(x - z),$$  \hfill (6.36)

and then the integration over $z$, to obtain the following expression

$$G(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\Lambda \frac{1}{Z} \sum_{\{N \}} e^{-\beta(E_N - \mu N)} \{ (h - 1) \det(I + \hat{V}) + \det(I + \hat{V} - \hat{W}) \}. \hfill (6.37)$$

Here, $h$, $\hat{V}$, and $\hat{W}$ are given by equations (6.22), (6.30), and (6.31), respectively.

Equation (6.37) is the desired Fredholm determinant representation for the correlation function (2.1). This representation may be written in the following equivalent form (see appendix A for a proof)

$$G(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\Lambda \{ (h - 1) \det(I + \hat{V}) + \det(I + \hat{V} - \hat{W}) \}. \hfill (6.38)$$

Here, $I$ is the identity operator, and $\hat{V}$ and $\hat{W}$ are linear integral operators with the kernels

$$V(q, q') = \sqrt{\vartheta(q)} \hat{V}(q, q') \sqrt{\vartheta(q')},$$  \hfill (6.39)

$$W(q, q') = \sqrt{\vartheta(q)} \hat{W}(q, q') \sqrt{\vartheta(q')},$$

where

$$\vartheta(q) = \frac{1}{e^{\beta|q - \mu|} + 1}.$$  \hfill (6.40)
is the Fermi weight, and $\epsilon(q) = q^2/2$ is the single-particle energy. The kernels

$$V(q, q') = \frac{E_+(q)E_-(q') - E_-(q)E_+(q')}{q - q'}$$

and

$$W(q, q') = E_+(q)E_+(q')$$

are obtained from equations (6.30) and (6.31), respectively.

### 7. Summation in case of impurity-gas attraction, $g < 0$

In the present section we adapt the approach of section 6 to the case of attractive impurity-gas interaction, $g < 0$. We make our analysis for $L$ and $g$ satisfying equation (5.13). Our results are applicable for all values of $g$ in the $L \to \infty$ limit.

The quasi-momenta $k_N$ and $k_{N+1}$ could be complex for $g < 0$ (this is discussed in section 5.1). We split the sum on the right-hand side of equation (4.11) into two parts:

$$\langle N|e^{-iH_0(p)}|N \rangle = \sum_{f_p} |\langle N|f_p\rangle|^2 e^{-i\epsilon_f} + \sum_{i_p} |\langle N|i_p\rangle|^2 e^{-i\epsilon_i}.$$  \hfill (7.1)

The sum over $f_p$ runs over the real sets $k_0, \ldots, k_{N-1}$, while the sets with $k_N = k_{N+1}$ constitute the sum over $i_p$. The approach of section 6 can be applied to the sum over $f_p$ directly. On substituting this sum into equation (4.15) it leads to the same Fredholm determinant representation for $G(x, t)$, as given by equation (6.38) in the case $g \geq 0$.

Let us now consider the sum over $f_p$. Proceeding as in section 6 we get the analogue of equation (6.10):

$$\sum_{f_p} |\langle N|f_p\rangle|^2 e^{-i\epsilon_f} = \frac{1}{(N - 1)!} \int_{-\infty}^{\infty} d\Lambda \int_{-\infty}^{\infty} \frac{dz}{2\pi} e^{-ipz} e^{-i\epsilon_f} \left( \frac{2}{L} \right)^N \sum_{n_0, \ldots, n_{N-1}} |det D_f|^2 \prod_{j=1}^{N-1} \left[ e^{-ir(k_j)} \frac{\partial k_j}{\partial \Lambda} \right].$$

Since $k_0, \ldots, k_{N-1}$ are real, and $k_N = k_{N+1}$, we have

$$|det D_f|^2 = -(det D_f)^2.$$  \hfill (7.3)

Using equation (5.18) for $k_0, \ldots, k_{N-1}$, equation (5.11) for $k_N$ and $k_{N+1}$, and taking into account equation (7.3), we transform equation (7.2) into

$$\sum_{f_p} |\langle N|f_p\rangle|^2 e^{-i\epsilon_f} = -\frac{1}{(N - 1)!} \frac{2^2}{4} \int_{-\infty}^{\infty} d\Lambda \int_{-\infty}^{\infty} \frac{dz}{2\pi} e^{-ipz} e^{-i[\epsilon_f(k_{N+1}) + \epsilon_f(k_0)]} \left( \frac{2}{L} \right)^N \sum_{n_0, \ldots, n_{N-1}} \left( det D_f \right)^2 \prod_{j=1}^{N-1} \left[ e^{-ir(k_j)} \frac{\partial k_j}{\partial \Lambda} \right].$$  \hfill (7.4)

We now transform the summation over $n_0, \ldots, n_{N-1}$ in equation (7.4) using the technique explained in appendix B.3. Employing equation (B30) and the asymptotic formula (5.8) we get

$$\left( \frac{2}{L} \right)^N \frac{1}{(N - 1)!} \sum_{n_0, \ldots, n_{N-1}} (det D_f)^2 \prod_{j=1}^{N-1} \left[ e^{-ir(k_j)} \frac{\partial k_j}{\partial \Lambda} \right] = -det C + det (C + D).$$  \hfill (7.5)

Here

$$C_{jl} = \frac{2}{L} \sum_k \frac{\partial k}{\partial \Lambda} \frac{e^{-ir(k-j)(k+l)}}{(k-j)(k+l)}, \quad j, l = 1, \ldots, N,$$

and

$$D_{jl} = \frac{2}{L} \nu(p_j) \nu(p_l), \quad j, l = 1, \ldots, N,$$

where

$$\nu(q) = \frac{k_- - k_+}{(k_- - q)(k_+ - q)}, \quad q = \frac{2\pi}{L} n, \quad n = 0, \pm 1, \pm 2, \ldots.$$  \hfill (7.8)

We stress that the summation index $n$ in equation (7.6) is related to $k$ by equation (5.16). We rewrite equation (7.6) in a form used in our subsequent calculations:
\[ C_{jl} = A_{jl} - \frac{2}{L} \frac{e(p_j)}{k_+ - p_j} - \frac{2}{L} \frac{e(p_j)}{k_- - p_j} + \frac{2}{L} \frac{h}{(k_+ - p_j)(k_- - p_j)}, \quad j, l = 1, \ldots, N. \]  

Here, \( h \), \( A \), and \( e \) are defined by equations (6.12), (6.13), and (6.15), respectively. Substituting equation (7.5) into (7.4) we get

\[
\sum_{j, l} |N(f_p)|^2 e^{-i\varepsilon_{jl}} = \frac{\sqrt{2}}{4} \int_{-\infty}^{\infty} d\Lambda \int_{-\infty}^{\infty} \frac{dz}{2\pi} e^{-i\varphi} e^{-i[\tau(k_+) + \tau(k_-)]} \times [\det C - \det (C + D)].
\]

Further transformations of this sum into the Fredholm determinant representation of the function (2.1) can be performed using steps similar to those performed in sections 6.2 and 6.3, and we arrive at the results summarized in section 3.2.

8. Discussion and outlook

The main result of the present paper is the Fredholm determinant representation for the time-dependent two-point impurity correlation function (2.1). We consider a particular Bethe ansatz solvable model: A mobile impurity interacting with a free Fermi gas (or the Tonks–Girardeau gas [10, 11]) through a \( \delta \)-function potential of arbitrary strength \( g \). We extend the approach of [14] to the case where the impurity-gas interaction can be attractive, \( g < 0 \), and the temperature can be finite.

Let us discuss how our results can be used to investigate the mobile impurity dynamics. Various asymptotic formulas for a Fredholm determinant can be obtained by formulating the matrix Riemann–Hilbert problem and solving it asymptotically [1, 22–25]. In model (2.1) Fredholm determinant representations are known in the \( g \to \infty \) limit [12, 13], and corresponding asymptotic solutions of the matrix Riemann–Hilbert problem have been discussed in [17, 26–28]. However, these solutions are specific to certain regimes in which the temperature-weighted average is different from the one defined by equation (2.3) (see [29, 30] for further description of these regimes). The results obtained in [14] and in the present paper make it possible to formulate the matrix Riemann–Hilbert problem at arbitrary \( g \), in the regime where the temperature-weighted average is defined by equation (2.3). The large time and distance asymptotic solution of the matrix Riemann–Hilbert problem should be compared with the predictions of [18] (see equation (6) therein) valid when \( g \to \infty \) and the temperature is zero. The Fredholm determinant representation is also promising for investigating the spectral function (defined by the Fourier transform of the correlation function (2.1)) from space and time to momentum and frequency variables. All existing approaches based on a mapping between microscopic interacting theories and effective free field theories describe a shape of the spectral function only in some vicinity of a singularity (see [31, 32] and references therein). In contrast, a numerical evaluation of the Fredholm determinant, combined with asymptotic expansions, has been shown for several models to give precise data for correlation and spectral functions everywhere in space–time and momentum–frequency domains [28, 33, 34]. We expect that this approach will give precise data for the impurity spectral function in our model as well. Having such data may help verify the phenomena predicted in [35].

The approach developed in the present paper could be applied to other observables in the model. Of particular interest is the time-dependent average momentum of an impurity particle injected with some initial momentum into a free Fermi gas. This problem has been investigated using the Bethe ansatz [15]. However, the form-factors summation (which we do analytically in section 6 for the impurity correlation function) was done numerically in [15]. Later, the same problem was addressed with other techniques [36–40]. The existing results were obtained for particular time scales (for example, short time in the case of simulations using time-dependent density matrix renormalization group) and values of the external parameters (for example, weak impurity–gas coupling in the case of calculations done by diagrammatic methods). We expect that, using the Fredholm determinant representation, the average momentum of the impurity particle could be precisely calculated at all times and for all values of the external parameters.

Let us now discuss how our results could help further develop the Bethe ansatz approach. Although the Fredholm determinant representation for correlation functions of the Tonks–Girardeau gas was found [41–43], the search still continues for other Bethe ansatz solvable models. This search has been particularly successful for those models whose many-body wave functions vanish as any two particles approach each other, either on the lattice or in the continuum. In that case, Fredholm determinant representations can be found using the same techniques as those for a free Fermi gas (for example, Wick’s theorem can be employed [44]). In the model we consider here the wave functions do not vanish as the impurity approaches the gas particles, unless \( g \to \infty \). Nonetheless, we find that the Fredholm determinant representation for the correlation function, given in
equation (2.1), exists for any \( g \). Whether its existence is due to some mapping between our model and a non-interacting quantum field theory remains an open problem.

The linear integral operators entering the Fredholm determinant representation given in the present paper are of a special, integrable type (see, e.g, section XIV.1 of [1]). Exploiting the properties of operators of this type, helped represent correlation functions of several models as solutions to nonlinear differential equations [16, 45, 46]. Our results make it possible to try applying this approach to the mobile impurity model considered here.

In the case \( g \to \infty \), Fredholm determinant representations for the correlation function (2.1) of the free Fermi gas and the Tonks–Girardeau gas on the lattice are found in [47, 48], respectively. A straightforward modification of the approach given in the present paper would lead to a Fredholm determinant representation for the same models with arbitrary \( g \).

Acknowledgments

We thank A Maitra, S Majumdar, G Schehr, and A Sykes for careful reading of the manuscript. The work of OG and MBZ is part of the Delta ITP consortium, a program of the Netherlands Organisation for Scientific Research (NWO) that is funded by the Dutch Ministry of Education, Culture and Science (OCW). The work of AGP is partially supported by the Russian Science Foundation, Grant No. 14-11-00598.

Appendix A. Fredholm determinants for the free Fermi gas

In this appendix we recall how a Fredholm determinant is defined and prove an identity used in sections 6 and 7.

Take an arbitrary \( M \times M \) matrix \( V \). We have

\[
\det (\delta_{ij} + V_{ij}) = \sum_{N=0}^{M} \frac{1}{N!} \sum_{a_1=1}^{M} \cdots \sum_{a_N=1}^{M} \det V_{a_1 \cdots a_N}. \tag{A1}
\]

The limit \( M \to \infty \) in this expression defines the Fredholm determinant of the operator \( \hat{I} + \hat{V} \):

\[
\det (\hat{I} + \hat{V}) = \sum_{N=0}^{\infty} \frac{1}{N!} \sum_{a_1=1}^{\infty} \cdots \sum_{a_N=1}^{\infty} \det V_{a_1 \cdots a_N}. \tag{A2}
\]

The limit \( M \to \infty \) may not exist, and even if it does, the expression on the right-hand side of equation (A2) may diverge for some \( V \). We assume, however, that the necessary existence and convergence conditions are fulfilled for the operators encountered in our paper. These operators are linear integral operators, and we therefore use the following rigorous formula to define the Fredholm determinant (see, e.g., [49], vol IV, p 24):

\[
\det (\hat{I} + \hat{V}) = \lim_{N \to \infty} \frac{1}{N!} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \det \begin{vmatrix} V(k_1, k_1) & \cdots & V(k_1, k_N) \\ \vdots & \ddots & \vdots \\ V(k_N, k_1) & \cdots & V(k_N, k_N) \end{vmatrix}. \tag{A3}
\]

Let us now prove an identity which we use in sections 6 and 7. We take

\[
Z_K = \frac{1}{Z} \sum_{\{N\}} e^{-\beta (E_N - \mu N)} \det (I + \hat{K}). \tag{A4}
\]

Here

\[
Z = \sum_{\{N\}} e^{-\beta (E_N - \mu N)} \tag{A5}
\]

and \( \hat{K} \) is an arbitrary \( N \times N \) matrix whose entries \( \hat{K}_{ij} \equiv \hat{K}_{p_i p_j} \) are functions of \( p_1, \ldots, p_N \). The energy \( E_N \) is the sum of single-particle energies

\[
E_N = \sum_{j=1}^{N} \epsilon (p_j). \tag{A6}
\]

The momenta \( p_1, \ldots, p_N \) are quantized

\[
p_j = \frac{2\pi}{L} n_j, \quad n_j = 0, \pm 1, \pm 2, \ldots \tag{A7}
\]
and the summation over \( \{ N \} \) is defined as follows
\[
\sum_{\{ N \}} = \sum_{N=0}^{\infty} \sum_{p_1} \cdots \sum_{p_N} \sum_{p_N} \ldots (A8)
\]

The parameter \( \mu \) is the chemical potential, and \( \beta \) is the inverse temperature. We are going to demonstrate that equation \( (A4) \) can be represented as a Fredholm determinant
\[
Z_K = \det(\tilde{I} + \tilde{K}) \tag{A9}
\]

for which the operator \( \tilde{K} \) has the kernel
\[
K(q, q') = \sqrt{\vartheta(q)\vartheta(q')}\sqrt{\vartheta(q')}. \tag{A10}
\]

Here
\[
\vartheta(q) = \frac{1}{e^{\beta |q|} + 1} \tag{A11}
\]
is the Fermi weight, and the kernel \( \tilde{K}(q, q') \) of the operator \( \tilde{K} \) is obtained from the entries of the matrix \( \tilde{K}_{p_1p_1'} \) as follows:
\[
\tilde{K}_{p_1p_1'} = \frac{2\pi}{L} \tilde{K}(q, q'), \quad q = p_1, \quad q' = p_1'. \tag{A12}
\]

We stress that the equivalence of the representations \( (A4) \) and \( (A9) \) is known, see, for example, [43]. We prove it here to make our paper self-contained.

We begin showing the equivalence of equations \( (A4) \) and \( (A9) \) with taking a determinant from equation \( (A4) \) and writing it as
\[
\det (\delta_{ll'} + \tilde{K}_{p_1p_1'}) = \det \langle 0| c_{l_1}^\dagger \ldots c_{l_N}^\dagger | 0 \rangle \cdot e^{\sum_{l,l',p} \tilde{c}_{l_1}^\dagger \tilde{c}_{l_1'}^\dagger | 0 \rangle} = \langle N| e^{\sum_{l,l',p} \tilde{c}_{l_1}^\dagger \tilde{c}_{l_1'}^\dagger | 0 \rangle} = | N \rangle. \tag{A13}
\]

The symbol \( : \cdots : \) stands for the normal ordering. The state
\[
| N \rangle = e_{p_1} \cdots e_{p_N} | 0 \rangle, \tag{A14}
\]
of a free spinless Fermi gas contains \( N \) fermions with the momenta \( p_1, \ldots, p_N \), and \( | 0 \rangle \) is the vacuum state containing no fermions. We therefore get for equation \( (A4) \)
\[
Z_K = \frac{1}{Z} \sum_{\{ N \}} \langle N| e^{-\beta (H - \mu N)} e^{\sum_{l,l',p} \tilde{c}_{l_1}^\dagger \tilde{c}_{l_1'}^\dagger | 0 \rangle} = \langle N| e^{\sum_{l,l',p} \tilde{c}_{l_1}^\dagger \tilde{c}_{l_1'}^\dagger | 0 \rangle} \cdot | N \rangle. \tag{A15}
\]

Here, \( H \) is the Hamiltonian of a free spinless Fermi gas
\[
H = \mu N = \sum_p [e^\beta (\varphi(p) - \mu)] c_p^\dagger c_p = \sum_{p,p'} A_{pp'} c_{p'}^\dagger c_{p'}^\dagger c_{p'}^\dagger c_{p'} \tag{A16}
\]

We now take the identity (see, for example, [50] for a proof)
\[
e^{\sum_{l,l',p} \tilde{c}_{l_1}^\dagger \tilde{c}_{l_1'}^\dagger | 0 \rangle} = e^{\sum_{l,l',p} \tilde{c}_{l_1}^\dagger (e^{-\beta I})_{l_1l_1'}}, \tag{A17}
\]

where \( I \) is the identity operator, \( I_{p_1p_1'} = \delta_{p_1p_1} \), and \( A_{pp'} \) is an arbitrary function of \( p \) and \( p' \). Using this identity we bring the exponent of equation \( (A16) \) to the normal-ordered form
\[
e^{-\beta (H - \mu N)} = e^{\sum_{l,l',p} \tilde{c}_{l_1}^\dagger (e^{-\beta I})_{l_1l_1'}}, \tag{A18}
\]

We now substitute equation \( (A18) \) into \( (A15) \) and use the following identity (see, for example, [50] for a proof)
\[
e^{\sum_{l,l',p} \tilde{c}_{l_1}^\dagger (e^{-\beta I})_{l_1l_1'} c_{p'}^\dagger c_{p'}^\dagger c_{p'}^\dagger c_{p'} \cdot c_{p'}^\dagger \cdot} = e^{\sum_{l,l',p} \tilde{c}_{l_1}^\dagger (A_{pp'} + B_{pp'} + \sum_{p''} A_{pp''} B_{pp''}) c_{p'}^\dagger c_{p'}^\dagger \cdot c_{p'}^\dagger \cdot} \tag{A19}
\]

Here, \( A_{pp'} \) and \( B_{pp'} \) are arbitrary functions of \( p \) and \( p' \). We get
\[
Z_K = \frac{1}{Z} \sum_{\{ N \}} \langle N| e^{\sum_{l,l',p} \tilde{c}_{l_1}^\dagger (e^{-\beta I + \tilde{K}})_{l_1l_1'} c_{l_1'}^\dagger c_{l_1'}^\dagger | 0 \rangle} \cdot | N \rangle. \tag{A20}
\]

The sum over \( \{ N \} \) is the trace over all fermionic states. Therefore equation \( (A20) \) can be written as
\[
Z_K = \frac{\text{Tr}[e^{\sum_{l,l',p} \tilde{c}_{l_1}^\dagger (e^{-\beta I + \tilde{K}})_{l_1l_1'} c_{l_1'}^\dagger c_{l_1'}^\dagger}]}{\text{Tr}[e^{\sum_{l,l',p} \tilde{c}_{l_1}^\dagger (e^{-\beta I} - \tilde{K})_{l_1l_1'} c_{l_1'}^\dagger c_{l_1'}^\dagger}]} = \frac{e^{-\beta \tilde{h}}}{e^{-\beta (I + \tilde{K})}}, \tag{A21}
\]

where \( \tilde{h} \) is defined by the formula
\[
e^{-\beta \tilde{h}} = e^{-\beta (I + \tilde{K})}. \tag{A22}
\]
We recognize the numerator and denominator on the right-hand side of equation (A21) as the partition functions of the corresponding fermionic models, rewrite them using a standard formula found in statistical mechanics textbooks, and get

\[
Z_K = \frac{\det (\hat{I} + e^{-\beta \hat{H}})}{\det (\hat{I} + e^{-\beta \hat{K}})} = \frac{\det (\hat{I} + e^{-\beta (\hat{I} + \hat{K})})}{\det (\hat{I} + e^{-\beta \hat{K}})} = \det (\hat{I} + \hat{K}),
\]

(A23)

where \( \hat{K} \) is related to \( \hat{K} \) by equation (A10). We thus proved that equations (A9) and (A4) are equivalent.

**Appendix B. Summation formulas at finite N**

In this appendix we explain how we insert the summations (or integrations) into the product of two determinants. We use the following definition of the determinant of an arbitrary \( N \times N \) matrix \( A \):

\[
\det A = \sum_{a_1, \ldots, a_N=1}^N \epsilon_{a_1 \ldots a_N} A_{a_1} \cdots A_{a_N},
\]

(B1)

Here, \( \epsilon_{a_1 \ldots a_N} \) is the Levi-Civita symbol

\[
\epsilon_{a_1 \ldots a_N} = \begin{cases} +1 & \text{if } a_1, \ldots, a_N \text{ is an even permutation of } 1, \ldots, N, \\ -1 & \text{if } a_1, \ldots, a_N \text{ is an odd permutation of } 1, \ldots, N, \\ 0 & \text{otherwise}. \end{cases}
\]

(B2)

The product of two Levi-Civita symbols with \( a_1, \ldots, a_N \) and \( b_1, \ldots, b_N \) taking values from the set \( 1, \ldots, N \) can be written as the determinant of the \( N \times N \) matrix whose entries are the Kronecker delta symbols:

\[
\epsilon_{a_1 \ldots a_N} \epsilon_{b_1 \ldots b_N} = \begin{vmatrix} \delta_{a_1 b_1} & \cdots & \delta_{a_1 b_N} \\ \vdots & \ddots & \vdots \\ \delta_{a_N b_1} & \cdots & \delta_{a_N b_N} \end{vmatrix}.
\]

(B3)

**B.1. Summation in the overlap \( \langle N | f_p \rangle \)**

We write equation (5.2) as

\[
| f_p \rangle = \frac{Y_f}{\sqrt{N! L^N}} \prod_{j=1}^{N+1} \nu_+ (k_j) \sum_{a_1, \ldots, a_N=1}^N \epsilon_{a_1 \ldots a_N} \prod_{j=1}^N \left[ \frac{e^{i k_j x_j}}{\nu_+ (k_j)} - \frac{e^{i k_{N+1} x_j}}{\nu_- (k_{N+1})} \right]
\]

and the coordinate representation of state (2.4) as

\[
| N \rangle = \frac{1}{\sqrt{N! L^N}} \sum_{a_1, \ldots, a_N=1}^N \epsilon_{a_1 \ldots a_N} \prod_{j=1}^N e^{i p_j x_j},
\]

(B5)

Recall that the function \( \nu_- \) is defined by equation (3.4). Taking the product of equation (B4) with the complex conjugate of equation (B5), integrating the result over \( x_1, \ldots, x_N \), and using identity (B3) we get

\[
\langle N | f_p \rangle = \frac{Y_f}{N!} \prod_{j=1}^{N+1} \nu_+ (k_j) \sum_{a_1, \ldots, a_N=1}^N \sum_{b_1, \ldots, b_N=1}^N \epsilon_{a_1 \ldots a_N} \epsilon_{b_1 \ldots b_N} \prod_{j=1}^N \left[ \phi (k_{a_j}, p_{b_j}) \nu_- (k_{a_j}) - \phi (k_{b_j}, p_{a_j}) \nu_+ (k_{b_j}) \right]
\]

\[
\times \prod_{j=1}^N \left[ \phi (k_{N+1}, p_{b_j}) \nu_- (k_{N+1}) - \phi (k_{b_j}, p_{N+1}) \nu_+ (k_{b_j}) \right] = Y_f \begin{vmatrix} \phi (k_1, p_1) & \cdots & \phi (k_{N+1}, p_1) \\ \vdots & \ddots & \vdots \\ \phi (k_1, p_{N+1}) & \cdots & \phi (k_{N+1}, p_{N+1}) \end{vmatrix}.
\]

(B6)

Here, the function \( \phi \) is defined as

\[
\phi (z, z') = \frac{1}{L} \int_0^L \! \! dx \: e^{i x (z - z')}.
\]

(B7)

Using the quantization condition for the free Fermi gas momenta, \( \exp \{ iLp_j \} = 1, j = 1, \ldots, N \), and equations (5.9) and (3.4) we arrive at

\[
\phi (k_a, p_b) = \frac{2i}{L} \frac{\nu_- (k_a)}{k_a - p_b}, \quad a = 1, \ldots, N+1, \quad b = 1, \ldots, N.
\]

(B8)
Substituting equation (B8) into (B6) we have
\[
\langle N | f_p \rangle = Y_f \left( \frac{2 \pi}{L} \right)^N \det D_f \prod_{j=1}^{N+1} \nu_-(k_j),
\]
where
\[
\det D_f = \begin{vmatrix}
\frac{1}{k_1 - p_1} & \cdots & \frac{1}{k_{N+1} - p_1} \\
\vdots & \ddots & \vdots \\
\frac{1}{k_1 - p_N} & \cdots & \frac{1}{k_{N+1} - p_N}
\end{vmatrix}
\]
is the determinant of the \((N + 1) \times (N + 1)\) matrix.

B.2. Summation in the normalization factor \(|Y_f|^2\)

We write equation (5.2) as
\[
|f_p| = \frac{Y_f}{\sqrt{N!L^N}} \sum_{a_1, \ldots, a_{N+1}=1}^{N+1} \epsilon_{a_1 \ldots a_{N+1}} \nu_-(k_{a_{N+1}}) \prod_{j=1}^{N} e^{i \alpha_j x_j}.
\]
Recall that the function \(\nu_-\) is defined by equation (3.4). Taking the product of equation (B11) with its complex conjugate, integrating the result over \(x_1, \ldots, x_N\), and using identity (B3) we get
\[
|Y_f|^2 = \frac{1}{N!} \sum_{a_1, \ldots, a_{N+1}=1}^{N+1} \sum_{b_1, \ldots, b_{N+1}=1}^{N+1} \epsilon_{a_1 \ldots a_{N+1}} \epsilon_{b_1 \ldots b_{N+1}} \nu_-(k_{a_{N+1}}) \nu_+(k_{b_{N+1}}) \prod_{j=1}^{N} \phi(k_{a_j}, k_{b_j}).
\]

Here, the function \(\phi\) is given by equation (B7).

Let us now use the fact that \(k_1, \ldots, k_{N+1}\) are solutions to the Bethe equations (5.4) and (5.5). We established in section 5.3 that \(k_1, \ldots, k_{N-1}\) are always real, and \(k_N \) and \(k_{N+1}\) are either both real or complex-conjugated to each other, \(k_N = k_{N+1}^{*}\). We thus eliminate the complex conjugation from equation (B12):
\[
|Y_f|^2 = -\alpha \begin{vmatrix}
\phi(k_1, k_1) & \cdots & \phi(k_{N+1}, k_1) & \nu_-(k_1) \\
\vdots & \ddots & \vdots & \vdots \\
\phi(k_1, k_{N+1}) & \cdots & \phi(k_{N+1}, k_{N+1}) & \nu_+(k_{N+1}) \\
\nu_-(k_1) & \cdots & \nu_-(k_{N+1}) & 0
\end{vmatrix}.
\]

Here
\[
\alpha = \begin{cases} 
1 & \text{Im } k_N = 0 \\
-1 & \text{Im } k_N = 0
\end{cases}.
\]

Using equations (5.9) and (3.4) we get from equation (B7) the expression
\[
\phi(k_a, k_b) = \delta_{ab} - (1 - \delta_{ab}) \frac{4}{Lg} \nu_-(k_a) \nu_+(k_b), \quad a, b = 1, \ldots, N + 1
\]
valid for any quasi-momenta \(k_a\) and \(k_b\) from a given solution \(k_1, \ldots, k_{N+1}\) of the Bethe equations (5.4) and (5.5). Substituting equation (B15) into (B13) we get after some elementary algebra
\[
|Y_f|^2 = \alpha \prod_{j=1}^{N+1} \left[ 1 + \frac{4}{Lg} \nu_-(k_j) \nu_+(k_j) \right] \sum_{j=1}^{N+1} \nu_-(k_j) \nu_+(k_j) \frac{1}{1 + \frac{4}{Lg} \nu_-(k_j) \nu_+(k_j)}.
\]

B.3. Two summation formulas with \((\det D_f)^2\)

We consider
\[
S = \frac{1}{(N + 1)!} \sum_{a_1, \ldots, a_{N+1}} \det D_f \prod_{j=1}^{N+1} f(k_j),
\]
where \(\det D_f\) is defined by equation (5.20), and \(k_j^p\)’s and \(\eta_j^p\)’s by equation (5.16). The function \(f(k_j)\) is arbitrary. We write equation (5.20) as
\[ \det D_f = \prod_{j=1}^{N} \frac{1}{k_{a_j} - p_j}. \]  

Substituting this representation into equation (B17) and using identity (B3) we get after some elementary algebra

\[
S = \begin{vmatrix}
\tilde{A}_{11} & \ldots & \tilde{A}_{1N} & e_1 \\
\vdots & \ddots & \vdots & \vdots \\
\tilde{A}_{N1} & \ldots & \tilde{A}_{NN} & e_N \\
e_1 & \ldots & e_N & h 
\end{vmatrix}.
\]  

Here

\[
\tilde{A}_{jl} = \sum_{n} \frac{f(k)}{(k - p_j)(k - p_l)}, \quad j, l = 1, \ldots, N,
\]

\[
e_j = \sum_{n} \frac{f(k)}{k - p_j}, \quad j = 1, \ldots, N,
\]

and

\[
h = \sum_{n} f(k).
\]

We finally write equation (B19) as

\[
S = (h - 1) \det_{1 \leq j, l \leq N} \tilde{A}_{jl} + \det_{1 \leq j, l \leq N} (\tilde{A}_{jl} - e_j e_l).
\]  

We now consider

\[
s = \frac{1}{(N - 1)!} \sum_{n_{a_1}, \ldots, n_{a_{N-1}}} (\det D_f)^2 \prod_{j=1}^{N-1} f(k_j),
\]

where \( \det D_f \) is defined by equation (5.20). We stress that equation (B24) contains the parameters \( k_N \) and \( k_{N+1} \) without a summation over them, in contrast to equation (B17). We use the following two representations for equation (5.20):

\[
\det D_f = \sum_{a_1, \ldots, a_N} \epsilon_{a_1 \ldots a_N} \frac{k_{N+1} - k_N}{(k_N - p_{a_N})(k_{N+1} - p_{a_N})} \prod_{j=1}^{N-1} \frac{k_{N+1} - k_j}{(k_j - p_j)(k_{N+1} - p_j)},
\]

and

\[
\det D_f = \sum_{a_1, \ldots, a_N} \epsilon_{a_1 \ldots a_N} \frac{k_{N+1} - k_N}{(k_N - p_{a_N})(k_{N+1} - p_{a_N})} \prod_{j=1}^{N-1} \frac{k_N - k_j}{(k_j - p_j)(k_N - p_j)}.
\]

Substituting the product of these representations into equation (B24) and using the identity (B3) we get

\[
s = - \begin{vmatrix}
\tilde{C}_{a_1} & \ldots & \tilde{C}_{a_N} & v_1 \\
\vdots & \ddots & \vdots & \vdots \\
\tilde{C}_{a_N} & \ldots & \tilde{C}_{NN} & v_N \\
v_1 & \ldots & v_N & 0 
\end{vmatrix}.
\]  

Here

\[
\tilde{C}_{jl} = \sum_{n} \frac{f(k)(k_{N+1} - k)(k_N - k)}{(k - p_j)(k - p_l)(k_{N+1} - p_j)(k_N - p_l)}, \quad j, l = 1, \ldots, N,
\]

and

\[
v_j = \frac{k_{N+1} - k_N}{(k_{N+1} - p_j)(k_N - p_j)}, \quad j = 1, \ldots, N.
\]

We finally write equation (B27) as

\[
s = - \det_{1 \leq j, l \leq N} \tilde{C}_{jl} + \det_{1 \leq j, l \leq N} (\tilde{C}_{jl} + v_j v_l).
\]
References

[1] Korepin V E, Bogoliubov N M and Izergin A G 1993 Quantum Inverse Scattering Method and Correlation Functions (Cambridge: Cambridge University Press)
[2] McGuire J B 1965 Interacting fermions in one dimension: I. Repulsive potential J. Math. Phys. 6 432–9
[3] McGuire J B 1966 Interacting fermions in one dimension: II. Attractive potential J. Math. Phys. 7 123–32
[4] Gaudin M 1967 Un système a une dimension de fermions en interaction Phys. Lett. A 24 53–6
[5] Yang C N 1967 Some exact results for the many-body problem in one dimension with repulsive delta-function interaction Phys. Rev. Lett. 19 1312–4
[6] Gaudin M 1983 La fonction D’onde de Bethe (Paris: Masson)
[7] Edwards D M 1990 Magnetism in single-band models Prog. Theor. Phys. Suppl. 101 453–61
[8] Castella H and Zotos X 1993 Exact calculation of spectral properties of a particle interacting with a one-dimensional fermionic system Phys. Rev. B 47 16186
[9] Roeder C and Kohler H 2012 From hardcore bosons to free fermions with Painlevé V J. Stat. Mech. 542–64
[10] Tonks L 1936 The complete equation of state of one, two and three-dimensional gases of hard elastic spheres Phys. Rev. 50 955–63
[11] Girardeau M D 1960 Relationship between systems of impenetrable bosons and fermions in one dimension J. Math. Phys. 1 516–23
[12] Izergin A G and Pronko A G 1997 Correlators in the one-dimensional two-component Bose and Fermi gases Phys. Lett. A 236 445–54
[13] Izergin A G and Pronko A G 1998 Temperature correlators in the two-component one-dimensional gas Nucl. Phys. B 520 594–632
[14] Gamayun O, Pronko A G and Zvonarev M B 2013 Impurity Green’s function of a one-dimensional Fermi gas Nucl. Phys. B 892 83–104
[15] Mathy C M, Zvonarev M B and Demler E 2012 Quantum flutter of supersonic particles in one-dimensional quantum liquids Nat. Phys. 8 881–6
[16] Its A R, Izergin A G, Korepin V E and Slavnov N A 1990 Differential equations for quantum correlation functions Int. J. Mod. Phys. B 4 1033–37
[17] Cheianov V V and Zvonarev M B 2004 Zero temperature correlation functions for the impenetrable fermion gas J. Phys. A: Math. Gen. 37 2261–97
[18] Zvonarev M B, Cheianov V V and Giamarchi T 2007 Spin dynamics in a one-dimensional ferromagnetic Bose gas Phys. Rev. Lett. 99 240404
[19] Devreese J T and Alexandrov A S 2009 Fröhlich polaron and bipolaron: recent developments Rep. Prog. Phys. 72 066501
[20] Anderson P W 1967 Infrared catastrophe in Fermi gases with local scattering potentials Phys. Rev. Lett. 18 1049–51
[21] Takahashi M 1999 Thermodynamics of One-Dimensional Solvable Models (Cambridge: Cambridge University Press)
[22] Deißl P, Its A R and Zhou X 1997 A Riemann–Hilbert approach to asymptotic problems arising in the theory of random matrix models, and also in the theory of integrable statistical mechanics Annt. Math. 146 149–235
[23] Kitanine N, Kozłowski K K, Maillet J M, Slavnov N A and Terras V 2009 Riemann–Hilbert approach to a generalised sine kernel and applications Commun. Math. Phys. 291 691–761
[24] Slavnov N A 2010 Integral operators with the generalized sine kernel on the real axis Theor. Math. Phys. 165 1262–74
[25] Kozlowski K K 2011 Riemann–Hilbert approach to the time–dependent generalized sine kernel Adv. Theor. Math. Phys. 15 1655–743
[26] Gohmann F, Izergin A G, Korepin V E and Pronko A G 1998 Time and temperature dependent correlation functions of the one-dimensional impenetrable electron gas Int. J. Mod. Phys. B 12 2409–33
[27] Gohmann F, Its A R and Korepin V E 1998 Correlations in the impenetrable electron gas Phys. Lett. A 249 117–25
[28] Cheianov V V and Zvonarev M B 2008 One-particle equal time correlation function for the spin-incoherent infinite U Hubbard chain J. Phys. A: Math. Theor. 41 045002
[29] Gohmann F and Korepin V E 1999 universal correlations of one-dimensional interacting electrons in the gas phase Phys. Lett. A 260 516–21
[30] Cheianov V V and Zvonarev M B 2004 Nonunitary spin–charge separation in a one-dimensional fermion gas Phys. Rev. Lett. 92 176401
[31] Lamacraft A 2009 Dispersion relation and spectral function of an impurity in a one-dimensional quantum liquid Phys. Rev. B 79 241105
[32] Zvonarev M B, Cheianov V V and Giamarchi T 2009 Edge exponent in the dynamic spin structure factor of the Yang–Gaudin model Phys. Rev. B 80 201102
[33] Cheianov V V, Smith H and Zvonarev M B 2005 Low-temperature crossover in the momentum distribution of cold atomic gases in one dimension Phys. Rev. A 71 033610
[34] Zvonarev M B, Cheianov V V and Giamarchi T 2009 Dynamical properties of the one-dimensional spin-1/2 Bose–Hubbard model near a Mott–insulator to fermionic-liquid transition Phys. Rev. Lett. 103 110401
[35] Kantian A, Schollwöck U and Giamarchi T 2014 Competing regimes of motion of 1D mobile impurities Phys. Rev. Lett. 113 070601
[36] Knapp M, Mathy C J M, Ganahl M, Zvonarev M B and Demler E 2014 Quantum flutter: signatures and robustness Phys. Rev. Lett. 112 015302
[37] Burovski E, Cheianov V, Gamayun O and Lychkovski O 2014 Momentum relaxation of a mobile impurity in a one-dimensional quantum gas Phys. Rev. A 89 041601
[38] Lychkovski O 2014 Perpetual motion of a mobile impurity in a one-dimensional quantum gas Phys. Rev. A 89 033619
[39] Gamayun O 2014 Quantum Boltzmann equation for a mobile impurity in a degenerate Tonks–Girardeau gas Phys. Rev. A 89 096327
[40] Gamayun O, Lychkovski O and Cheianov V 2014 Kinetic theory for a mobile impurity in a degenerate Tonks–Girardeau gas Phys. Rev. E 90 032132
[41] Schlüter T D 1963 Note on the one-dimensional gas of impenetrable point–particle bosons J. Math. Phys. 4 666–71
[42] Lenard A 1964 Momentum distribution in the ground state of the one-dimensional system of impenetrable bosons J. Math. Phys. 5 930–43
[43] Lenard A 1966 One-dimensional impenetrable bosons in thermal equilibrium J. Math. Phys. 7 1268–72
[44] Zvonarev M B, Cheianov V V and Giamarchi T 2009 The time–dependent correlation function of the Jordan–Wigner operator as a fredholm determinant J. Stat. Mech. P07035
[45] Sato M, Miwa T and Jimbo M 1979 Holonomic quantum fields: II. The Riemann–Hilbert problem Publ. RIMS Kyoto Univ. 15 201–78
[46] Jimbo M, Miwa T, Möri Y and Sato M 1980 Density matrix of an impenetrable Bose gas and the fifth Painlevé transcendent Physica D 1 80–158
[47] Izergin A G, Pronko A G and Abarenkov N I 1998 Temperature correlators in the one-dimensional Hubbard model in the strong coupling limit Phys. Lett. A 245 537–47
[48] Abarenkova N I, Izergin A G and Pronko A G 2001 Correlators of the ladder spin model in the strong coupling limit J. Math. Sci. 104 1087–96

[49] Smirnov V I 1964 A Course of Higher Mathematics (Oxford: Pergamon)

[50] Alexandrov A and Zabrodin A 2013 Free fermions and tau-functions J. Geom. Phys. 67 37–80