Rectilinear Link Diameter and Radius in a Rectilinear Polygonal Domain*

Man-Kwun Chiu1,2, Elena Khramtcova3, Matias Korman4, Aleksandar Markovic5, Yoshio Okamoto6, Aurélien Ooms3, André van Renssen1,2, and Marcel Roeloffzen1,2

1National Institute of Informatics, Tokyo, Japan, {chiumk, andre, marcel}@nii.ac.jp
2JST, ERATO, Kawarabayashi Large Graph Project
3Université libre de Bruxelles (ULB), Brussels, Belgium, elena.khramtsova@gmail.com, aureooms@ulb.ac.be
4Tohoku University, Sendai, Japan, mati@dais.is.tohoku.ac.jp
5TU Eindhoven, Eindhoven, the Netherlands, a.markovic@tue.nl
6University of Electro-Communications, Tokyo, Japan, okamotoy@uec.ac.jp

Abstract

We study the computation of the diameter and radius under the rectilinear link distance within a rectilinear polygonal domain of \( n \) vertices and \( h \) holes. We introduce a graph of oriented distances to encode the distance between pairs of points of the domain. This helps us transform the problem so that we can search through the candidates more efficiently. Our algorithm computes both the diameter and the radius in \( O(n^{\omega}) \) time, where \( \omega < 2.373 \) denotes the matrix multiplication exponent and \( \chi \in \Omega(n) \cap O(n^{2}) \) is the number of edges of the graph of oriented distances. We also provide a faster algorithm for computing the diameter that runs in \( O(n^2 \log n) \) time.

1 Introduction

Diameters and radii are popular characteristics of metric spaces. For a compact set \( S \) with a metric \( d: S \times S \to \mathbb{R}^+ \), its diameter is defined as \( \text{diam}(S) := \max_{p \in S} \max_{q \in S} d(p, q) \), and its radius is defined as \( \text{rad}(S) := \min_{p \in S} \max_{q \in S} d(p, q) \). The points that realize these distances are called the diametral pair and center, respectively. All of these terms are the natural extension of the same concepts in a disk and give some interesting properties of the environment, such as the worst-case response time or ideal location of a serving facility.

Much research has been devoted towards finding efficient algorithms to compute the diameter and radius for various types of sets and metrics. In computational geometry, one of the most well-studied and natural metric spaces is a polygon in the plane. This paper focuses on the computation of the diameter and the radius of a rectilinear polygon, possibly with holes (i.e., a rectilinear polygonal domain) under the rectilinear link distance. Intuitively, this metric measures the minimum number of links (segments) required in any rectilinear path connecting two points in the domain, where rectilinear indicates that we are restricted to horizontal and vertical segments only.

* MC, AvR and MR were supported by JST ERATO Grant Number JPMJER1201, Japan. EK was partially supported by the SNF Early Postdoc Mobility grant P2TIP2-168563, Switzerland, and F.R.S.-FNRS, Belgium. MK was supported in part by KAKENHI Nos. 15H02665 and 17K12635, Japan. AM was supported by the Netherlands’ Organisation for Scientific Research (NWO) under project no. 024.002.003. YO was partially supported by JSPS KAKENHI Grant Number 15K00009 and JST CREST Grant Number JPMJCR1402, and Kayamori Foundation of Informational Science Advancement. AO was supported by the Fund for Research Training in Industry and Agriculture (FRIA).
Table 1: Summary of the best known results for computing the diameter and radius of a polygonal domain of \( n \) vertices and \( h \) holes under different metrics. In the table, \( \omega < 2.373 \) is the matrix multiplication exponent.

| Metric   | Simple polygon | Polygonal domain |
|----------|---------------|------------------|
| Euclidean | \( O(n) \) [9] | \( O(n^{1.78}) \) [5] |
| \( L_1 \) | \( O(n) \) [6] | \( O(n^2 + h^2) \) [3] |
| Link     | \( O(n \log n) \) [21] | open |
| Rectilinear link | \( O(n) \) [18] | \( O(n^2 \log n) \) (Thm. 3) | \( O(n^\omega) \) (Thm. 4) |

1.1 Previous Work

The link distance is a very natural metric and simple to describe. Initially, the interest was motivated by the potential robotics applications (i.e., having some kind of robot with wheels for which moving in a straight line is easy, but making turns is costly in time or energy). Since then, it has attracted a lot of attention from a theoretical perspective.

Indeed, many problems that are easy under the \( L_1 \) or Euclidean metric turn out to be more challenging under the link distance. For example, computing the shortest path between two points in a polygonal domain can be done in \( O(n \log n) \) time for both Euclidean [10] and \( L_1 \) metrics [14, 15]. However, even approximating the same under the link distance is 3-SUM hard [16], and thus it is unlikely that a significantly subquadratic-time algorithm is possible.

Computing the diameter and radius is no exception: when considering simple polygons (i.e., polygons without holes) of \( n \) vertices, the diameter and center can be found in linear time for both Euclidean [2, 9] and \( L_1 \) metrics [6]. However, the best known algorithms for the link distance run in \( O(n \log n) \) time [7, 21]. Lowering the running times or proving the impossibility of this is a longstanding open problem in the field. The only partial answer to this question was given by Nilsson and Schuierer [18, 19]; they showed that the diameter and center can be found in linear time when we are only allowed to use rectilinear paths.

If we consider polygons with holes, the difference becomes even bigger: no algorithm for computing the diameter and radius under the link distance is known, not even one that runs in exponential time. In comparison, polynomial-time algorithms are known both for diameter and radius under \( L_1 \) and Euclidean metrics. A summary of the best running time for each case can be found in Table 1.

Although the change in metric seems minor, and one would venture that somehow the known algorithms for other metrics extend to this one, there are two major difficulties to overcome. First, the presence of holes in the polygonal domain removes many properties that hold for simple polygons. For example, a cornerstone for the algorithm that computes the radius under rectilinear link distance [19] is that any simple polygon whose diameter is \( D \) must have radius equal to either \( \lceil D/2 \rceil \) or \( \lceil D/2 \rceil + 1 \). Unfortunately, this does not hold when we consider domains with holes as illustrated in Figure 1. Thus, it is hard to extend the algorithms for simple polygons to general domains.

More importantly, when considering regions with holes, it is difficult to discretize the search space. Indeed, it is known that the diameter and radius can be uniquely realized by points in the interior of a polygonal domain [23]. The example in Figure 2 can be extended to the link distance by making the regions much thinner and thus preventing the possibility of using non-rectilinear “shortcuts”. As is the case for link distance, there is no clear discretization of the diameter or radius under both Euclidean and \( L_1 \) metrics and the different algorithms known for computing the diameter and center somehow discretize the topology of the shortest paths involved [3, 4, 5, 22]. For a single topology we get a small number of candidates and thus the problem shifts to efficiently searching through them.

This brings a second difficulty: although in most metrics the topology of the path is determined by the first and last corners of the domain traversed in the shortest path, this is not true for the link distance. Indeed, shortest paths under this metric need not pass through corners of the polygon; if we restrict ourselves to such paths, we may get a path that uses \( \Omega(n) \) more links [12]. What is even worse, depending on the model of computation, the shortest path has to turn at some interior points for which it is nontrivial to have an explicit representation [11].

Because of these difficulties no algorithm that can compute the exact link diameter or radius of a polygonal domain has been found. We emphasize that not even an exponential-time algorithm is known.
Figure 1: An example with diameter 8 (crossed points) and radius 7 (dotted point). By increasing the number of bends in the holes we can show that, unlike in simple polygons, the diameter and radius can be arbitrarily close to each other. Note that any point in the domain is either a part of a diametral pair or a center.

Figure 2: An example showing no diametral pair lies on the boundary of the polygonal domain. Any pair of points in the dashed blue regions will have distance 6 from each other (out of the 4 shortest paths connecting them two are shown) whereas other pairs will have distance 5 or less.

1.2 Results

As a stepping stone in further understanding the link distance, we study how to compute both the diameter and radius under the rectilinear link distance. In our work we introduce the graph of oriented distances, a graph that implicitly encodes the distance between regions of the domain. In Section 3 we use this graph to transform the problem: rather than searching pairwise distances in a list of potential candidates for diameter or center, we transform the problem into a rectangle intersection problem. Intuitively speaking, we cover the domain with several rectangles, and we find two pairs of rectangles that pairwise intersect (and satisfy other properties). In particular, once we have found the diametral pair, the four rectangles that satisfy the property can be used as a witness. To the best of our knowledge, this is the first time that such a transformation is used: previous methods generate a list of candidates for the diameter/radius and select the largest. The only way of verifying that the result is indeed correct is by checking that no other candidate has a larger/smaller distance.

This transformation leads to an algorithm for computing both the rectilinear link diameter and radius of a rectilinear polygonal domain with \( n \) vertices and \( h \) holes. The algorithm is described in Section 4 and runs in \( O(n^\omega) \) time, where \( \omega < 2.373 \) is the matrix multiplication exponent (Le Gall [13] provided the best known bound on \( \omega \)). Alternatively, we can also bound the running time in terms of the number \( \chi \) of edges of the graph of oriented distances (\( \chi \) will range from \( \Omega(n) \) to \( O(n^2) \) depending on \( P \)). With this parameter the running time becomes \( O(n^2 + nh \log h + \chi^2) \). Naturally, this second version is desirable when \( \chi \in o(n^{\omega/2}) \approx o(n^{1.18}) \). In Section 5 we use a different approach to obtain a faster algorithm. This algorithm, which runs in \( O(n^2 \log n) \) time, exploits properties of the diameter. Specifically, we heavily use that this value is a maximum over a maximum of distances, hence it can only be used for the diameter (recall that we have a minimum-maximum alternation in the definition of the radius). All of the algorithms presented in the paper can be modified to return not only diameter or radius, but also the points that realize it (i.e., diametral pair and center).

Note that in other metrics, the running time for simple and polygonal domains differs by at least a cubic factor. However, the running time of our three algorithms increases by a slightly superlinear factor when compared to the case of simple polygons [18, 19]. This is partially due to the fact that rectilinear link distance is much easier than the link distance, but also because we can transform the problem using the
graph of oriented distances. We believe this to be our main contribution and hope that it motivates similar transformations in other metrics.

1.3 Preliminaries

A rectilinear simple polygon (also called an orthogonal polygon) is a simple polygon that has horizontal and vertical edges only. A rectilinear polygonal domain \( P \) with \( h \) pairwise disjoint holes and \( n \) vertices is a connected and compact subset of \( \mathbb{R}^2 \) with \( h \) pairwise disjoint holes, in which the boundary of each hole is a simple closed rectilinear curve. Thus, the boundary \( \partial P \) of \( P \) consists of \( n \) line segments.

Each of the holes as well as the outer boundary of \( P \) is regarded as an obstacle that paths in \( P \) are not allowed to cross. A rectilinear path \( \pi \) from \( p \in P \) to \( q \in P \) is a path from \( p \) to \( q \) that consists of vertical and horizontal segments, each contained in \( P \), and such that along \( \pi \) each vertical segment is followed by a horizontal one and vice versa. Recall that \( P \) is a closed set, so \( \pi \) can traverse the boundary of \( P \) (along the outer face and any of the \( h \) obstacles).

We define the link length of such a path to be the number of segments composing it. The rectilinear link distance between points \( p, q \in P \) is defined as the minimum link length of a rectilinear path from \( p \) to \( q \) in \( P \), and denoted by \( \ell_P(p, q) \). It is well known that in rectilinear polygonal domains there always exists a rectilinear polygonal path between any two points \( p, q \in P \), and thus the distance is well defined. Once the distance is defined, the definitions of rectilinear link diameter \( \text{diam}(P) \) and rectilinear link radius \( \text{rad}(P) \) directly follow.

For simplicity in the description, we assume that a pair of vertices do not share the same \( x \)- or \( y \)-coordinate unless they are connected by an edge. This general position assumption can be removed with classic symbolic perturbation techniques. Also notice that, since we are considering rectilinear polygons, no edge has length 0. However, for simplicity in the analysis we will allow edges in a rectilinear path to have length 0. These edges of length 0 are considered as edges and thus potentially contribute to the link distance (naturally, no shortest path will ever have such an edge). From now on, for ease of reading, we will refer to rectilinear simple polygons and rectilinear polygonal domains as “simple polygon” and “domain.” Similarly, we will use the term “distance” to refer to the rectilinear link distance.

2 Graph of Oriented distances

For any domain \( P \), we virtually shoot a ray left and right from any horizontal segment of the domain until it hits another segment of \( P \), partitioning it into rectangles. We call this partition the horizontal decomposition. Let \( \mathcal{H}(P) \) be the set containing those rectangles. Similarly, if we shoot rays up and down from vertical segments, we get the vertical decomposition. Let \( \mathcal{V}(P) \) be the set of rectangles in this second decomposition (see Figure 3). Observe that both decompositions have linear size and can be computed in \( O(n \log n) \) time with a plane sweep.

Given two rectangles \( i, j \in \mathcal{H}(P) \cup \mathcal{V}(P) \), we use \( i \cap j \) to denote the boolean operation which returns \( true \) if and only if (1) the rectangles \( i \) and \( j \) properly intersect (i.e. their intersection has non-zero area), and (2) one of \( i, j \) belongs to \( \mathcal{H}(P) \), and the other to \( \mathcal{V}(P) \).

**Definition 1** (Graph of Oriented Distances). Given a rectilinear polygonal domain \( P \), let \( \mathcal{G}(P) \) be the unweighted undirected graph defined as \( \mathcal{G}(P) = (\mathcal{H}(P) \cup \mathcal{V}(P), \{ (h, v) \in \mathcal{H}(P) \times \mathcal{V}(P) : h \cap v \} \).  

In other words, vertices of \( \mathcal{G}(P) \) correspond to rectangles of the horizontal and the vertical decompositions of \( P \). We add an edge between two vertices if and only if the corresponding rectangles properly intersect. Note that this graph is bipartite, and has \( O(n) \) vertices. From now on, we make a slight abuse of notation and identify a rectangle with its corresponding vertex (thus, we talk about the neighbors of a rectangle \( i \in \mathcal{H}(P) \) in \( \mathcal{G}(P) \), for example).

The name **Graph of Oriented Distances** is easily explained: consider a rectilinear path \( \pi \) between two points in \( P \). Each horizontal edge of \( \pi \) is contained in a rectangle of \( \mathcal{H}(P) \) and each vertical edge is contained in a rectangle of \( \mathcal{V}(P) \). A bend in the path takes place in the intersection of the rectangles containing the two adjacent edges and corresponds to an edge of \( \mathcal{G}(P) \). So every rectilinear path \( \pi \) has a corresponding path \( \pi' \) in \( \mathcal{G}(P) \) (and vice versa). Moreover, each bend of \( \pi \) is associated with an edge of \( \pi' \).  

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Figure 3: The horizontal and vertical decomposition of the domain. Note that, because of our general position assumption, both subdivisions have the same number of rectangles.

**Definition 2** (Oriented distance). Given a rectilinear polygonal domain $P$, let $i$ and $j$ be two vertices of $\mathcal{G}(P)$, let $\Delta(i, j)$ to be the length of the shortest path between $i$ and $j$ in graph $\mathcal{G}(P)$ plus one. We also define $\Delta(i, i) = 1$.

The reason why we add the extra unit is to make sure that the link distance and the oriented distance match (see Lemma 2 below). We first list some useful properties of the oriented distance, which directly follow from the definition. Then we show the relationship between the oriented distance $\Delta(\cdot, \cdot)$ in $\mathcal{G}(P)$ and the link distance $\ell_P(\cdot, \cdot)$ in $P$.

**Lemma 1.** Let $i, j, i', j'$ be any (not necessarily distinct) rectangles in $\mathcal{H}(P) \cup \mathcal{V}(P)$ such that $i \cap i'$, and $j \cap j'$. Then, the following hold.

(a) $\Delta(i, j) = \Delta(j, i)$.

(b) $\Delta(i', j) \in \{ \Delta(i, j) - 1, \Delta(i, j) + 1 \}$.

(c) $\Delta(i', j') \in \{ \Delta(i, j) - 2, \Delta(i, j), \Delta(i, j) + 2 \}$.

**Lemma 2.** Let $p$ and $q$ be two points of the rectilinear polygonal domain $P$. The rectilinear link distance $\ell_P(p, q)$ between $p$ and $q$ can be characterized as follows. If $p$ and $q$ lie in the same vertical or horizontal rectangle of $\mathcal{V}(P)$ or $\mathcal{H}(P)$ then $\ell_P(p, q) = 1$ (if $p$ and $q$ share a coordinate) or $\ell_P(p, q) = 2$ (if both $x$- and $y$-coordinates of $p$ and $q$ are distinct). Otherwise, let $i \in \mathcal{H}(P)$, $i' \in \mathcal{V}(P)$, $j \in \mathcal{H}(P)$ and $j' \in \mathcal{V}(P)$ be vertices of the graph of oriented distances such that $p \in i \cap i'$ and $q \in j \cap j'$. Then

$$\ell_P(p, q) = \min\{ \Delta(i, j), \Delta(i', j'), \Delta(i', j), \Delta(i', j') \}.$$ 

**Proof.** The case in which $p$ and $q$ lie in the same rectangle $R$ is easy: in either case we can connect them with either a single segment or a path with exactly one bend and stay within $R$. Since we are within $R$ the path is feasible and has the minimum number of links possible.

Now suppose that there is no rectangle that contains both $p$ and $q$. Let $\pi$ be a shortest rectilinear path from $p$ to $q$ in $P$. It cannot lie entirely in one rectangle. Assume that the first and the last link of $\pi$ are horizontal. Then the length of $\pi$ is equal to $\Delta(i, j)$. Moreover, if one of $\Delta(i, j'), \Delta(i', j), \Delta(i', j')$ was strictly smaller than $\Delta(i, j)$, then it would correspond to a path $\pi'$ shorter than $\pi$. Analogous arguments for other orientations of the initial and final links of $\pi$ complete the proof. \qed

Intuitively speaking, if we are given two disjoint rectangles $i, j \in \mathcal{H}(P)$, then $\Delta(i, j)$ denotes the minimum number of links needed to connect any two points $p \in i$ and $q \in j$ under the constraint that the first and the last segments of the path are horizontal. If we looked for rectangles in $\mathcal{V}(P)$ we would instead require that
the path starts (or ends) with vertical segments. It follows that the link distance is the minimum of the four possible options.

In our algorithms we will often look for oriented distances between rectangles, so we compute it and store them in a preprocessing phase. Fortunately, a similar decomposition was used by Mitchell et al. \cite{17}. Specifically, they show how to compute the distance from a single rectangle to all other rectangles in \(O(n + h \log h)\) time with an \(O(n)\) size data structure\footnote{In fact, Mitchell et al. \cite{17} show how to compute the distance from a single point to all other rectangles, but their method can be easily adapted to compute the distance from a rectangle as well.}

**Lemma 3.** \cite{17} Given the horizontal and vertical decompositions \(\mathcal{H}(P)\) and \(\mathcal{V}(P)\) we can compute for a single rectangle \(i\) in either decomposition the oriented distance \(\Delta(i, j)\) to every other rectangle \(j\) in \(O(n + h \log h)\) time.

We construct this data structure for each of the \(O(n)\) rectangles, thus leading to a data structure that takes \(O(n^2)\) space and \(O(n^2 + nh \log h)\) time to compute.

### 3 Characterization via Boolean Formulas

Let \(\hat{d} = \max_{i,j \in \mathcal{H}(P) \cup \mathcal{V}(P)} \Delta(i, j)\) be the largest distance between vertices of \(\mathcal{G}(P)\). Similarly, we define \(\hat{r} = \min_{i,j \in \mathcal{H}(P) \cup \mathcal{V}(P)} \max_{j \in \mathcal{H}(P) \cup \mathcal{V}(P)} \Delta(i, j)\). Note that these two values are the diameter and the radius of \(\mathcal{G}(P)\) plus one (recall that we add one unit to the graph distance when defining \(\Delta\)). We use \(\hat{d}\) and \(\hat{r}\) to approximate the diameter \(\text{diam}(P)\) and radius \(\text{rad}(P)\) of a domain \(P\) under the rectilinear link distance.

First, we relate the distance between two points \(p, q \in P\) to the oriented distances between the rectangles that contain \(p\) and \(q\). Specifically, from Lemma 2 we know that \(\ell_P(p, q) = \min\{\Delta(i, j), \Delta(i, j'), \Delta(i', j), \Delta(i', j')\}\), where \(i, j \in \mathcal{H}(P)\) are the horizontal rectangles containing \(p\) and \(q\), respectively, and \(i', j', j' \in \mathcal{V}(P)\) are the vertical rectangles containing \(p\) and \(q\). Similarly, we define \(\hat{\ell}(p, q) = \max\{\Delta(i, j), \Delta(i, j'), \Delta(i', j), \Delta(i', j')\}\). It then follows from Lemma 3 that these two values differ by at most 2.

**Lemma 4.** For any two points \(p, q \in P\), let \(i, j \in \mathcal{H}(P)\) and \(i', j' \in \mathcal{V}(P)\) be the rectangles containing \(p\) and \(q\), i.e., \(p \in i \cap i'\) and \(q \in j \cap j'\). Then, it holds that \(\hat{\ell}(p, q) - 2 \leq \ell_P(p, q) \leq \hat{\ell}(p, q) - 1\).

This relation allows us to express the rectilinear link diameter of a domain in terms of \(\hat{d}\).

**Theorem 1.** The rectilinear link diameter \(\text{diam}(P)\) of a rectilinear polygonal domain \(P\) satisfies \(\text{diam}(P) = \hat{d} - 1\) if and only if there exist \(i, i', j, j' \in \mathcal{H}(P) \cup \mathcal{V}(P)\) with \(i \cap i'\) and \(j \cap j'\) such that \(\Delta(i, j) = \hat{d}\) and \(\Delta(i', j') = \hat{d}\). Otherwise, \(\text{diam}(P) = \hat{d} - 2\).

**Proof.** First observe that for any pair of points \(p, q \in P\) we have \(\ell_P(p, q) \leq \hat{\ell}(p, q) - 1 \leq \hat{d} - 1\) by Lemma 4. Hence, the diameter of \(P\) is at most \(\hat{d} - 1\). Similarly, by the definitions of \(\hat{d}\) and \(\hat{\ell}(\cdot, \cdot)\), there must be a pair of points \(p, q \in P\) so that \(\hat{\ell}(p, q) = \hat{d}\). Again by Lemma 4 it follows that \(\text{diam}(P) \geq \ell_P(p, q) \geq \hat{\ell}(p, q) - 2 = \hat{d} - 2\).

Next we show that the diameter is \(\hat{d} - 1\) if and only if the above condition holds. If \(\Delta(i, j) = \hat{d}\) and \(\Delta(i', j') = \hat{d}\), then by Lemma 4 and the fact that neither \(\Delta(i, j')\) nor \(\Delta(i', j)\) can be larger than \(\hat{d}\), we know that \(\Delta(i, j') = \Delta(i', j) = \hat{d} - 1\). This implies that a pair of points \(p \in i \cap i'\) and \(q \in j \cap j'\) have \(\ell_P(p, q) = \hat{d} - 1\). Thus, the diameter is \(\hat{d} - 1\).

Now consider any pair \(p, q\) and the set of rectangles \(i, j \in \mathcal{H}(P)\) and \(i', j' \in \mathcal{V}(P)\) with \(p \in i \cap i'\) and \(q \in j \cap j'\). Recall that \(\ell_P(p, q) = \min\{\Delta(i, j), \Delta(i, j'), \Delta(i', j), \Delta(i', j')\}\). By Lemma 1 \(\Delta(i, j)\) and \(\Delta(i', j')\) must differ by exactly one from \(\Delta(i', j)\) and \(\Delta(i, j')\). That implies that two distances may be \(\hat{d} - 1\), but if the condition in the lemma is not satisfied, at most one can be \(\hat{d}\) and the fourth must be \(\hat{d} - 2\) or less. Therefore, if the condition is not satisfied for \(i, i', j, j'\), then the diameter is indeed \(\hat{d} - 2\).

For the radius we can make a similar argument.

**Theorem 2.** The rectilinear link radius \(\text{rad}(P)\) of a rectilinear polygonal domain \(P\) satisfies \(\text{rad}(P) = \hat{r} - 1\) if and only if for all \(i, i' \in \mathcal{H}(P) \cup \mathcal{V}(P)\) with \(i \cap i'\) there exist \(j, j' \in \mathcal{H}(P) \cup \mathcal{V}(P)\) with \(j \cap j'\) such that \(\Delta(i, j) \geq \hat{r}\) and \(\Delta(i', j') \geq \hat{r}\). Otherwise, \(\text{rad}(P) = \hat{r} - 2\).
Proof. We first show by contradiction that the real radius satisfies $\text{rad}(P) \leq \hat{r} - 1$. Suppose the radius is greater than or equal to $\hat{r}$. Then, for all $p \in P$ there exists a point $q \in P$ such that $\ell_P(p, q) \geq \hat{r}$. Now consider a rectangle $i \in \mathcal{H}(P) \cup \mathcal{V}(P)$, a point $p \in i$ and a point $q$ at distance $\hat{r}$ from $p$. Consider the two rectangles $j \in \mathcal{H}(P)$ and $j' \in \mathcal{V}(P)$ so that $q \in j \cap j'$. By Lemma 4 we know that $\Delta(i, j) \geq \ell_P(p, q) \geq \hat{r}$ and $\Delta(i, j') \geq \ell_P(p, q) \geq \hat{r}$. By Lemma 1, $\Delta(i, j)$ and $\Delta(i, j')$ differ by one, and thus one of them must be at least $\hat{r} + 1$. That is, for any rectangle $i$ we can find a second rectangle at oriented distance $\hat{r} + 1$. This implies that $\hat{r} = \min_{i \in \mathcal{H}(P) \cup \mathcal{V}(P)} \max_{j \in \mathcal{H}(P) \cup \mathcal{V}(P)} \Delta(i, j) \geq \hat{r} + 1$, which is a contradiction. Therefore, our initial assumption that $\text{rad}(P) \geq \hat{r}$ is false and we conclude that $\text{rad}(P) \leq \hat{r} - 1$.

Next we show that $\text{rad}(P) \geq \hat{r} - 2$. Consider any point $p$ and a rectangle $i \in \mathcal{H}(P)$ that contains it. By definition of $\hat{r}$ there is a rectangle $j \in \mathcal{H}(P) \cup \mathcal{V}(P)$ so that $\Delta(i, j) \geq \hat{r}$. Let $q$ be any point in $j$. From Lemma 4 we get that $\ell_P(p, q) \geq \ell(p, q) - 2 \geq \Delta(i, j) - 2 \geq \hat{r} - 2$. Hence for any point $p$, there is a point $q$ that is at distance at least $\hat{r} - 2$, which implies $\text{rad}(P) \geq \hat{r} - 2$.

Now we show that if the above condition is satisfied, then it must hold that $\text{rad}(P) = \hat{r} - 1$. Assume the condition holds and consider any point $p$ and two rectangles $i, i' \in \mathcal{H}(P) \cup \mathcal{V}(P)$ so that $i \cap i'$ and $p \in i \cap i'$. There exist $j, j' \in \mathcal{H}(P) \cup \mathcal{V}(P)$ so that $j \cap j'$, $\Delta(i, j) \geq \hat{r}$, and $\Delta(i, j') \geq \hat{r}$. By Lemma 1 we know that $\Delta(i, j')$ and $\Delta(i', j)$ must be at least $\hat{r} - 1$. Therefore $\ell_P(p, q) \geq \hat{r} - 1$ for any point $q \in j \cap j'$. This shows that for any point $p$ there is a point $q$ whose link distance to $p$ is at least $\hat{r} - 1$, giving a lower bound on the radius. Combining this with the upper bound shown above, we obtain that $\text{rad}(P) = \hat{r} - 1$ as claimed.

If the condition is not true, then we know there exist rectangles $i, i' \in \mathcal{H}(P) \cup \mathcal{V}(P)$ so that $i \cap i'$, and for every $j, j' \in \mathcal{H}(P) \cup \mathcal{V}(P)$ with $j \cap j'$ the above statement is not true. Now consider a point $p \in i \cap i'$. We argue that $p$ has distance at most $\hat{r} - 2$ to any other point $q \in P$. Consider any point $q$ and let $j, j' \in \mathcal{H}(P) \cup \mathcal{V}(P)$ be the rectangles containing $q$. We perform a case analysis on the value of $\Delta(i, j)$. First consider the case $\Delta(i, j) \geq \hat{r} + 1$. In this case $\Delta(i', j) \geq \hat{r}$ and $\Delta(i', j') \geq \hat{r}$ which contradicts our assumption that the above statement is not true for every $(j, j')$. If $\Delta(i, j) = \hat{r}$, then by Lemma 1 and the assumption that not both $\Delta(i, j) \geq \hat{r}$ and $\Delta(i', j') \geq \hat{r}$ we find that $\Delta(i', j') = \hat{r} - 2$ which implies that $\ell_P(p, q) \leq \hat{r} - 2$. If $\Delta(i, j) = \hat{r} - 1$, then by Lemma 1 both $\Delta(i, j')$ and $\Delta(i', j')$ differ from $\Delta(i, j)$ by 1, but by our assumption that not both $\Delta(i, j') \geq \hat{r}$ and $\Delta(i', j) \geq \hat{r}$, one of them must be $\hat{r} - 2$. Lastly, if $\Delta(i, j) \leq \hat{r} - 2$, we can already conclude that $\ell_P(p, q) \leq \hat{r} - 2$. This shows that from $p$ any other point $q$ is at most distance $\hat{r} - 2$ away, hence the radius is at most $\hat{r} - 2$. Combining this with the lower bound of $\hat{r} - 2$ (shown above), we conclude that the radius must be $\hat{r} - 2$.

With the above characterization, we can naively compute the diameter and the radius by checking all $O(n^4)$ quadruples $(i, i', j, j') \in \mathcal{H}(P) \times \mathcal{V}(P) \times \mathcal{H}(P) \times \mathcal{V}(P)$. However, the approach can be improved by using $G(P)$.

**Theorem 3.** The rectilinear link distance $\text{diam}(P)$ and radius $\text{rad}(P)$ of a rectilinear polygonal domain $P$ consisting of $n$ vertices and $h$ holes can be computed in $O(n^2 + nh \log h + \chi^2)$ time, where $\chi$ is the number of edges of $G(P)$ (i.e., the number of pairs of intersecting rectangles of $\mathcal{H}(P)$ and $\mathcal{V}(P)$).

**Proof.** First, we use Lemma 5 to compute the oriented distance $\Delta(\cdot, \cdot)$ between each pair of oriented rectangles from the horizontal and the vertical decompositions in $O(n^2 + nh \log h)$ time.

Recall that $G(P)$ adds an edge between two rectangles $i, i'$ if and only if $i \cap i'$. Thus, rather than looking at all quadruples $(i, i', j, j') \in \mathcal{H}(P) \times \mathcal{V}(P) \times \mathcal{H}(P) \times \mathcal{V}(P)$, we can look at pairs of edges of $G(P)$. For each of the $O(\chi^2)$ pairs we can test the conditions from Theorems 1 and 2 in a brute-force manner in constant time which gives us the claimed running time.

As we discuss later, this method is only useful when $\chi$ is very small, i.e. almost linear size or smaller.

**Remark on the interior realization of the diameter/radius.** Theorems 1 and 2 together with Lemma 1 imply that a necessary condition for the diameter to be uniquely realized by pairs of interior points is that $\text{diam}(P) = \hat{d} - 1$. Similarly, for all centerpoints to be determined by points in the interior we must have $\text{rad}(P) = \hat{r} - 1$. However, neither condition is sufficient. This transformation of the problem into a search of quadruples of rectangles allows us to handle the interior cases in the same way as the boundary cases.
4 Computation via Matrix Multiplication

In this section we provide an alternative method to compute the diameter and radius. These methods also use the conditions in Theorems 1 and 2, but instead exploit the behavior of matrix multiplication on (0,1)-matrices. Recall that, given two (0,1)-matrices $A$ and $B$, their product is $(AB)_{i,j} = \sum_k (A_{i,k} \cdot B_{k,j}) = |\{ k : A_{i,k} = 1 \land B_{k,j} = 1 \}|$.

We define a (0,1)-matrix $I$, which is used to compute both the diameter and radius:

$$I_{i,j} = \begin{cases} 1 & \text{if } i \cap j, \\ 0 & \text{otherwise.} \end{cases}$$

In other words, for each pair $i, j$ of rectangles in $\mathcal{H}(P) \cup \mathcal{V}(P)$, the matrix $I$ indicates whether $i$ and $j$ intersect and have different orientations (one horizontal, one vertical). Note that, for ease of explanation, we have slightly abused the notation and identified rectangles of $\mathcal{H}(P) \cup \mathcal{V}(P)$ with indices in the matrix.

4.1 Computing the Diameter

We use Theorem 1 to compute the diameter. Thus, we need to determine if there exist four rectangles in $\mathcal{H}(P) \cup \mathcal{V}(P)$ that satisfy the condition of Theorem 1. If so, the diameter will be $\hat{d} - 1$; otherwise, $\hat{d} - 2$.

In order to do so, we define the (0,1)-matrix $D$ that indicates, for a pair $i, j$ of rectangles in $\mathcal{H}(P) \cup \mathcal{V}(P)$, whether the oriented distance between them is $\hat{d}$:

$$D_{i,j} = \begin{cases} 1 & \text{if } \Delta(i, j) = \hat{d}, \\ 0 & \text{otherwise.} \end{cases}$$

By multiplying $I$ and $D$, we obtain $(ID)_{i,j} = |\{ i' : (i \cap i') \land (\Delta(i', j) = \hat{d}) \}|$.

In other words, the entry at $(i,j)'$ of the product $ID$ counts the number of rectangles in $\mathcal{H}(P) \cup \mathcal{V}(P)$ that intersect rectangle $i$ and are oriented differently from it, and at the same time are at oriented distance $\hat{d}$ from rectangle $j'$. We construct the (0,1)-matrix $M$ that records when an entry in the product $ID$ is non-zero:

$$M_{i,j} = \begin{cases} 1 & \text{if } (ID)_{i,j} > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Finally, we look at the product $DM$. Note that $(DM)_{i,j} > 0$ if and only if there are two rectangles $j$ and $j'$ with $j \cap j'$ such that $\Delta(i, j) = \hat{d}$ and $\Delta(i', j') = \hat{d}$.

The quantifier on $j'$ and the condition on its intersection with $j$ can be moved just to the right of the quantifier on $j$ without altering the meaning of the formula, since both of them are existential quantifiers. Therefore, the condition in Theorem 1 is satisfied if and only if there exists a 1-entry in $I$ whose corresponding entry in $DM$ is non-zero. This condition can be checked in quadratic time (once matrix $DM$ has been computed) by iterating over the entries of the matrices in quadratic time since the matrices have linearly many rows and columns.

4.2 Computing the Radius

A similar construction can be used to verify the condition in Theorem 2 and compute the radius. Similar to the matrix $D$ given above, we define the (0,1)-matrix $R$ which indicates whether a pair of rectangles is at oriented distance at least $\hat{r}$ from each other:

$$R_{i,j} = \begin{cases} 1 & \text{if } \Delta(i, j) \geq \hat{r}, \\ 0 & \text{otherwise.} \end{cases}$$

By multiplying $I$ and $R$, we obtain $(IR)_{i,j} = |\{ i' : (i \cap i') \land (\Delta(i', j') \geq \hat{r}) \}|$. 

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Analogous to matrix $M$, we construct the $(0,1)$-matrix $N$ that indicates whether the corresponding entry of $IR$ is non-zero, as follows:

$$N_{i,j} = \begin{cases} 1 & \text{if } (IR)_{i,j} > 0, \\ 0 & \text{otherwise.} \end{cases}$$

We now look at the product $RN$. Note that $(RN)_{i',i} > 0$ if and only if there are two rectangles $j$ and $j'$ with $j \cap j'$ such that $\Delta(i,j) \geq \hat{r}$ and $\Delta(i',j') \geq \hat{r}$.

By a similar argument as in the diameter case, the condition on Theorem 2 is satisfied if and only if for each 1-entry in $I$ the corresponding entry in $RN$ is non-zero. As before, this condition can be checked by iterating over the entries of the matrices in quadratic time once the matrix $RN$ has been computed.

Note that the time taken by the computation of the various matrix products dominates the time taken by the other loops and operations. All matrices are linear in dimensions and the product of two $O(n) \times O(n)$ matrices can be computed in $O(n^2)$ time. We summarize the results of this section in the following theorem.

**Theorem 4.** The rectilinear link diameter $\text{diam}(P)$ and radius $\text{rad}(P)$ of a rectilinear polygonal domain $P$ consisting of $n$ vertices can be computed in $O(n^2 \log n)$ time.

## 5 Computing the Diameter Faster

We present a faster method for computing the diameter. This method uses the fact that the diameter is defined as a maximum over maximums which allows us to reduce the running time to $O(n^2 \log n)$. Recall that the radius is a minimum over maximums, thus the algorithm of this section does not trivially extend to the computation of the radius. The rest of this section is the proof of the following statement.

**Theorem 5.** The rectilinear link diameter $\text{diam}(P)$ of a rectilinear polygonal domain $P$ of $n$ vertices can be computed in $O(n^2 \log n)$ time.

By Theorem 4 after we compute the oriented diameter $\hat{d}$, we only need to consider $\hat{d} - 1$ or $\hat{d} - 2$ as candidates to be $\text{diam}(P)$. The following corollary of Theorem 4 can be obtained by applying Lemma 1:

**Corollary 1.** The diameter $\text{diam}(P)$ equals $\hat{d} - 2$ if and only if for all rectangles $i$ and $j$ with $\Delta(i,j) = \hat{d}$, and for all rectangles $i'$ and $j'$ with $i \cap i'$ and $j \cap j'$, we have $\Delta(i',j') = \hat{d} - 2$. Otherwise, $\text{diam}(P) = \hat{d} - 1$.

This condition can be checked in $O(n^4)$ time in a brute-force manner as follows. We iterate over every pair $(i,j)$ with $\Delta(i,j) = \hat{d}$. For each such pair we find the sets $\text{cover}(i) = \{i' : i \cap i'\}$ and $\text{cover}(j) = \{j' : j \cap j'\}$. Then for each pair $(i',j') \in \text{cover}(i) \times \text{cover}(j)$ we check if $\Delta(i',j') = \hat{d} - 2$. If there is a pair for which this is not the case, then by the above corollary the diameter is $\hat{d} - 1$. Since each of the covers may have linear size the running time is $\Omega(n^4)$.

The key observation that allows us to reduce this to $O(n^2 \log n)$ time is that in the end there are only $O(n^2)$ unique pairs to test. Indeed, what we are checking is the distance of every pair $(i',j')$ in the set

$$\mathcal{T} = \{(i',j') : \exists i,j \text{ such that } (i \cap i', j \cap j', \Delta(i,j) = \hat{d})\}$$

which clearly has only quadratic size. Next we show that this set has more structure than just being an arbitrary set of rectangles, which allows us to compute it more quickly.

First, instead of iterating over every pair $(i,j)$ with $\Delta(i,j) = \hat{d}$ and computing all pairs in the $\text{cover}(i) \times \text{cover}(j)$, we iterate over $i$ and compute all pairs in $\text{cover}(i) \times \bigcup_{j : \Delta(i,j) = \hat{d}} \text{cover}(j)$. For a rectangle $i \in \mathcal{H}(P) \cup \mathcal{V}(P)$, let $\mathcal{S}_i$ denote the set of rectangles at oriented distance $\hat{d}$ from $i$. Now let

$$\mathcal{T} = \bigcup_i \mathcal{T}_i = \bigcup_i \{(i',j') : \exists j \text{ such that } (i' \cap i, j' \cap j, j \in \mathcal{S}_i)\}.$$ 

Note that the rectangles fulfilling the role of $i'$ are easily found (i.e., they must intersect $i$ and must have different orientation), but naively computing the ones that fulfill the role of $j'$ leads to a quadratic runtime. That is, if we were to compute for each $j \in \mathcal{S}_i$ its cover, then this may take $\Omega(n^2)$ time. However, there are only $O(n)$ rectangles that can fulfill the role of $j'$ and we show how to find them in $O(n \log n)$ time.
For this purpose we use an orthogonal segment intersection reporting data structure, derived from a known dynamic ray shooting data structure [8]. The data structure we use stores horizontal line segments. It allows to add or remove horizontal line segments in \( O(\log n) \) time per segment. The structure reports the first segment hit by a query ray in \( O(\log n) \) time. By repeatedly using the structure, we can find all \( z \) horizontal line segments intersected by a vertical line segment in \( O((z+1)\log n) \) time. While performing the query, we also remove all the reported segments from the data structure in the same time complexity.

For a rectangle \( k \), we define the middle segment \( \ell_k \) of \( k \). If \( k \) is a horizontal rectangle, \( \ell_k \) is the line segment connecting the midpoints of its left and right boundary; if \( k \) is a vertical rectangle, \( \ell_k \) is the segment connecting the midpoints of its top and bottom boundary.

We fix a rectangle \( i \) and assume without loss of generality that the rectangles in \( S_i \) are vertical. Insert the middle segments of all horizontal rectangles in \( H(P) \) into the intersection reporting data structure. Then, for each rectangle \( j \in S_i \), we query its corresponding middle segment. By the definition of middle segments, each reported horizontal segment corresponds to a rectangle \( j' \) intersecting \( j \). Since we remove each segment as we find it, no rectangle is reported twice. Repeating this for all \( j \in S_i \) finds the set \( C_i = \{ j' : j' \cap j, j \in S_i \} \) of all horizontal rectangles that intersect at least one rectangle in \( S_i \). Each query can be charged either to the horizontal segment that is deleted from the data structure or, in case \( z = 0 \), to the rectangle \( j \in S_i \) that we are querying. Hence, the total query time sums to \( O(n \log n) \).

For each rectangle in the set \( C_i \), we should check the distance to every rectangle \( i' \) such that \( i' \cap i \). Doing this explicitly takes \( O(n^2) \) time. Thus, summing over all rectangles \( i \), we get the total running time of \( O(n^3) \).

To bring the running time down to \( O(n^2 \log n) \), we create a reverse map of the map \( i \mapsto C_i \). For each rectangle \( k \), we build a collection \( L_k \) that contains \( i \) if and only if \( k \) belongs to \( C_i \). Given a rectangle \( j' \), we need to check the distance between \( j' \) and \( i' \) for any \( (i,i') \) with \( i \in L_{j'} \) and \( i \cap i' \). Using the intersection reporting data structure, we compute for each rectangle \( j' \) the set \( D_{j'} \), which is the set of all rectangles intersecting those in \( L_{j'} \). For each rectangle \( i' \in D_{j'} \), we test if \( \Delta(i', j') = d - 2 \). Again recall that if we find a pair whose is \( d \), then the diameter must be \( d - 1 \) (otherwise, the diameter is \( d - 2 \)). This proves Theorem 5.

6 Conclusions

Our algorithms heavily rely on Theorem 1 and 2. They implicitly do a search among a list of candidates for the diametral pair or radius, but it is not just an exhaustive search. For example, if we are looking for the diameter and have found the four rectangles that satisfy the condition of Theorem 1, we can stop the search and return that the diameter is \( \hat{d} \).

We note that \( \hat{d} \) and \( \hat{r} \) can be used to give an approximation of the diameter and radius, respectively, with an additive error of only one unit. However, the running time of computing these two values is almost as large as computing the exact values. It would be interesting to see if there is another way to approximate the diameter or radius.

This consideration, together with our results, reminds us of recent lower bound results in fine-grained complexity. For \( n \)-vertex sparse unweighted undirected graphs, under the orthogonal vectors conjecture, computing the diameter and even approximating it within a factor of \( 3/2 - \varepsilon \) cannot be done in \( O(n^{2-o(1)}) \) time for any \( \varepsilon > 0 \) [20], and under the hitting set conjecture, computing the radius and even approximating it within a factor of \( 3/2 - \varepsilon \) cannot be done in \( O(n^{2-o(1)}) \) time for any \( \varepsilon > 0 \) [11]. It is known that both the strong exponential-time hypothesis and the hitting set conjecture individually imply the orthogonal vectors conjecture (see Vassilevska Williams’ survey [22]). As we already pointed out, the (not rectilinear) link distance computation is 3-SUM hard [10], and it is straightforward to adapt the proof to show that the (not rectilinear) link diameter computation is 3-SUM hard, too. However, we have been unable to show the 3-SUM hardness of computing the rectilinear link diameter or radius, nor any hardness of having an \( O(n^{2-o(1)}) \)-time algorithm based on the strong exponential-time hypothesis, the orthogonal vectors conjecture, or the hitting set conjecture so far. Such a result would show that our algorithms are close to optimal.
A natural way to extend our results would be to consider the $c$-oriented link distance. In this distance we are allowed to use $c$ slopes in our path, normally in steps of $2\pi/c$ radians. In general allowing more directions would create more natural paths and potentially much shorter paths. Unfortunately non-orthogonal directions create some problems. The duality between the graph $G(P)$ and oriented distance relies on the observation that within a horizontal rectangle we can reach every point with just one bend, regardless of where we enter the rectangle with a vertical ray. When the directions are no longer orthogonal we cannot give this guarantee anymore (see Figure 4). This may be solved by counting bends that “skip” over orientations to count heavier. That is, within the path, we force adjacent edges to have adjacent orientations, even if those edges have length 0. For example, in the 8-oriented distance, an L-shaped path would have a cost of 3. However, it is not clear how to extend our results to this model. More importantly, this distance would differ a lot from the classical link distance.

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