The Relation Between the Associate Almost Complex Structure to $HM'$ and $(HM', S, T)$-Cartan Connections

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Abstract. In the present paper, the $(HM', S, T)$-Cartan connections on pseudo-Finsler manifolds, introduced by A. Bejancu and H.R. Farran, are obtained by the natural almost complex structure arising from the nonlinear connection $HM'$. We prove that the natural almost complex linear connection associated to a $(HM', S, T)$-Cartan connection is a metric linear connection with respect to the Sasaki metric $G$. Finally we give some conditions for $(M', J, G)$ to be a Kähler manifold.

Key words: almost complex structure; Kähler and pseudo-Finsler manifolds; $(HM', S, T)$-Cartan connection

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1 Introduction

Almost complex structures are important structures in differential geometry [8, 9]. These structures have found many applications in physics. H.E. Brandt has shown that the spacetime tangent bundle, in the case of Finsler spacetime manifold, is almost complex [4, 5, 6]. Also he demonstrated that in this case the spacetime tangent bundle is complex provided that the gauge curvature field vanishes [3]. In [1], for a pseudo-Finsler manifold $F^m = (M, M', F^*)$ with a nonlinear connection $HM'$ and any two skew-symmetric Finsler tensor fields of type $(1, 2)$ on $F^m$, A. Bejancu and H.R. Farran introduced a notion of Finsler connections which named “$(HM', S, T)$-Cartan connections”. If, in particular, $HM'$ is the canonical nonlinear connection $GM'$ of $F^m$ and $S = T = 0$, the Finsler connection is called the Cartan connection and it is denoted by $FC^* = (GM', \nabla^*)$ (see [1]). They showed that $\nabla^*$ is the projection of the Levi-Civita connection of the Sasaki metric $G$ on the vertical vector bundle. Also they proved that the associate linear connection $D^*$ to the Cartan connection $FC^*$ is a metric linear connection with respect to $G$ [1]. In this paper we obtain the $(HM', S, T)$-Cartan connections by using the natural almost complex structure arising from the nonlinear connection $HM'$, then the natural almost complex linear connection associated to a $(HM', S, T)$-Cartan connection is defined. We prove that the natural almost complex linear connection associated to a $(HM', S, T)$-Cartan connection is a metric linear connection with respect to the Sasaki metric $G$. Kähler and para-Kähler structures associated with Finsler spaces and their relations with flag curvature were studied by M. Crampin and B.Y. Wu (see [7, 12]). They have found some interesting results on this matter. In [12], B.Y. Wu gives some equivalent statements to the Kählerity of $(M', G, J)$. In the present paper we give other conditions for the Kählerity of $(M', G, J)$, which extend the previous results.
2 The associate almost complex structure to $HM'$

Let $M$ be a real $m$-dimensional smooth manifold and $TM$ be the tangent bundle of $M$. Let $M'$ be a nonempty open submanifold of $TM$ such that $\pi(M') = M$ and $\theta(M) \cap M' = \emptyset$, where $\theta$ is the zero section of $TM$. Suppose that $F^m = (M, M', F^*)$ is a pseudo-Finsler manifold where $F^* : M' \to \mathbb{R}$ is a smooth function which in any coordinate system $\{(U', \Phi') : x', y'\}$ in $M'$, the following conditions are fulfilled:

- $F^*$ is positively homogeneous of degree two with respect to $(y^1, \ldots, y^m)$, i.e., we have
  
  $$F^*(x^1, \ldots, x^m, ky^1, \ldots, ky^m) = k^2 F^*(x^1, \ldots, x^m, y^1, \ldots, y^m)$$

  for any point $(x, y) \in (\Phi', U')$ and $k > 0$.

- At any point $(x, y) \in (\Phi', U')$, $g_{ij}$ are the components of a quadratic form on $\mathbb{R}^m$ with $q$ negative eigenvalues and $m - q$ positive eigenvalues, $0 < q < m$ (see [1]).

Consider the tangent mapping $\pi_* : TM' \to TM$ of the submersion $\pi : M' \to M$ and define the vector bundle $VM' = \ker \pi_*$. A complementary distribution $HM'$ to $VM'$ in $TM'$ is called a nonlinear connection or a horizontal distribution on $M'$

$$TM' = HM' \oplus VM'.$$

A nonlinear connection $HM'$ enables us to define an almost complex structure on $M'$ as follows:

$$J : \Gamma(TM') \to \Gamma(TM'),$$

$$J \left( \frac{\delta}{\delta x^i} \right) = - \frac{\partial}{\partial y^i}, \quad J \left( \frac{\partial}{\partial y^i} \right) = \frac{\delta}{\delta x^i},$$

where $\{ \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^j_i(x, y) \frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^i} \}$ is assumed as a local frame field of $TM'$ and $\Gamma(TM')$ is the space of smooth sections of the vector bundle $TM'$. We call $J$ the associate almost complex structure to $HM'$. Obviously we have $J^2 = -1 \id_{TM'}$, also we can assume the conjugate of $J$, $J' = -J$, as an almost complex structure. Now we give the following proposition which was proved by B.Y. Wu [12].

**Proposition 1.** Let $F^m = (M, M', F)$ be a Finsler manifold. Then the following statements are mutually equivalent:

1) $F^m = (M, M', F)$ has zero flag curvature;
2) $J$ is integrable;
3) $\nabla J = 0$, where $\nabla$ is the Levi-Civita connection of the Sasaki metric $G$;
4) $(M', J, G)$ is Kählerian.

**Corollary 1.** Let the associate almost complex structure to $J$ (or $J'$) be a complex structure; then we have

$$\frac{\delta N^k_j}{\delta x^i} = N^k_j \frac{\delta}{\delta x^i}, \quad \frac{\partial N^k_j}{\partial y^i} = \partial N^k_j \frac{\partial}{\partial y^i}.$$ 

So in this case the horizontal distribution is integrable.
3 \( (HM', S, T)\)-Cartan connection by using the associate almost complex structure \( J \)

In this section we give another way to define \((HM', S, T)\)-Cartan connection by using the associate almost complex structure \( J \) on \( M' \). Then we study the Kählerity of \((M', J, G)\), where \( G \) is the Sasaki metric and \( F^m = (M, M', F^*) \) is a Finsler manifold.

Let \( F^m = (M, M', F^*) \) be a pseudo-Finsler manifold. Then a Finsler connection on \( F^m \) is a pair \( FC = (HM', \nabla) \) where \( HM' \) is a nonlinear connection on \( M' \) and \( \nabla \) is a linear connection on the vertical vector bundle \( VM' \) (see [1]).

**Theorem 1.** Let \( \nabla \) be a FC on \( M' \). The differential operator \( \mathcal{D} \) defined by

\[
\mathcal{D} X Y = \nabla_X v Y - J \nabla_X J h Y \quad \forall X, Y \in \Gamma(TM')
\]

is a linear connection on \( M' \). Also \( J \) is parallel with respect to \( \mathcal{D} \), that is

\[
(\mathcal{D} X J) Y = 0 \quad \forall X, Y \in \Gamma(TM').
\]

We call \( \mathcal{D} \) the natural almost complex linear connection associated to FC \( \nabla \) on \( M' \).

**Proof.** For any \( X, Y, Z \in \Gamma(TM') \) and \( f \in C^\infty(M') \) we have

\[
\mathcal{D}_f X Y = f \nabla_X v Y + \nabla_Y v Z - J (f \nabla_X J h Z + \nabla_Y J h Z)
\]

\[
= f (\nabla_X v Z - J \nabla_X J h Z) + \nabla_Y v Z - J \nabla_Y J h Z = f \mathcal{D}_X Z + \mathcal{D}_Y Z,
\]

\[
\mathcal{D}_X (f Y + Z) = X f (J h Y) + f (\nabla_X v Y - J \nabla_X J h Y) + \nabla_X v Z - J \nabla_X J h Z
\]

\[
= (X f) Y + f \mathcal{D}_X Y + \mathcal{D}_X Z.
\]

Therefore \( \mathcal{D} \) is a linear connection on \( M' \).

Also we have

\[
(\mathcal{D} X J)(Z) = \mathcal{D}_X (J(Z)) - J(\mathcal{D} X Z)
\]

\[
= \nabla_X v J(Z) - J \nabla_X J(h(J(Z))) - J \nabla_X v Z - \nabla_X J h Z
\]

\[
= \nabla_X \left( -Z^i \frac{\partial}{\partial y^i} \right) - J \nabla_X \left( -\tilde{Z}^i \frac{\partial}{\partial y^i} \right) - J \nabla_X \left( \tilde{Z}^i \frac{\partial}{\partial y^i} \right) = 0,
\]

where in local coordinates \( Z = Z^i \frac{\partial}{\partial x^i} + \tilde{Z}^i \frac{\partial}{\partial y^i} \).

Note that the torsion of \( \mathcal{D} \) is given by the following expression:

\[
T^\mathcal{D}(X, Y) = \nabla_X v Y - \nabla_Y v X - v[X, Y] - J(\nabla_X J h Y - \nabla_Y J h X - J h[X, Y]).
\]

**Theorem 2.** Let \( HM' \) be a nonlinear connection on \( M' \) and \( S \) and \( T \) be any two skew-symmetric Finsler tensor fields of type \((1, 2)\) on \( F^m \). Then there exists a unique linear connection \( \nabla \) on \( VM' \) satisfying the conditions:

(i) \( \nabla \) is a metric connection;

(ii) \( T^\mathcal{D} \), \( S \) and \( T \) satisfy

\[
(a) \quad T^\mathcal{D}(v X, v Y) = S(v X, v Y), \quad (b) \quad h T^\mathcal{D}(h X, h Y) = J T(J h X, J h Y)
\]

for any \( X, Y \in \Gamma(TM') \), where \( J \) is the associate almost complex structure to \( HM' \).
The above computation shows that the connection $\nabla$ defined by (2) and (3) is a metric connection.

Locally we set $\nabla_{\delta_{xy}} \frac{\partial}{\partial y} = F_{ij}^k(x, y) \frac{\partial}{\partial y^i}$. Then we can obtain the following expression for the coefficients $C_{ijm}^n$:

$$C_{ijm}^n = \frac{1}{2} \left\{ \frac{\partial g_{ji}}{\partial y^j} + \frac{\partial g_{ij}}{\partial y^j} - \frac{\partial g_{ji}}{\partial x^j} + S_{ij}^h g_{ih} + S_{ij}^h g_{ih} - S_{ih}^h g_{ij} \right\} g^{lm}.$$ 

Also in (3) let $X = \delta_{xy}$, $Y = \delta_{xz}$ and $Z = \delta_{zx}$. Then we can obtain the following expression for the coefficients $F_{ijm}^n$:

$$F_{ijm}^n = \frac{1}{2} \left\{ \frac{\delta g_{ji}}{\delta x^j} + \frac{\delta g_{ij}}{\delta x^j} - \frac{\delta g_{ji}}{\delta x^j} - T_{ij}^h g_{ih} - T_{ij}^h g_{ih} + T_{ih}^h g_{ij} \right\} g^{lm}.$$ 

By using the relations $J \circ v = h \circ J$, $v \circ J = J \circ h$ and (1) we have

$$T^D(vX, vY) = \nabla_{vX} vY - \nabla_{vY} vX - [vX, vY],$$
$$hT^D(hX, hY) = J(\nabla_{hY} hX - \nabla_{hX} hY + h[hX, hY]).$$

Suppose that $X, Y \in \Gamma(TM')$ are two arbitrary vector fields on $M'$. In local coordinates, let $X = X^i \delta_{xi} + \bar{X}^i \delta_{yi}$ and $Y = Y^i \delta_{xi} + \bar{Y}^i \delta_{yi}$, after performing some computations we have:

$$T^D \left( \bar{X}^i \delta_{yi}, \bar{Y}^i \delta_{yi} \right) = S \left( \bar{X}^i \delta_{yi}, \bar{Y}^i \delta_{yi} \right),$$
$$hT^D \left( X^i \delta_{xi}, Y^i \delta_{xi} \right) = JT \left( J \left( X^i \delta_{xi} \right), J \left( Y^i \delta_{xi} \right) \right).$$

The relations (6) and (7) show that $\nabla$ satisfies (ii) of Theorem 1.
Now let $\tilde{\nabla}$ be another linear connection on $VM'$ which satisfies (i) and (ii). By using the relations (i), (ii), (10) and this fact that

\[
\tilde{\nabla}X(g(vY,vZ)) + \tilde{\nabla}Y(g(vZ,vX)) - \tilde{\nabla}(g(vX,vY)) \]

\[
= g(\tilde{\nabla}_{vX}vY + \tilde{\nabla}_{vY}vX - T^G(vX,vY) - [vX,vY], vZ) \]

\[
+ g(T^G(vX,vZ) + [vX,vZ], vY) + g(T^G(vY,vZ) + [vY,vZ], vX), \]

\[
hX(g(vY,vJZ)) + hY(g(vZ,vJX)) - hZ(g(vJX,vJY)) \]

\[
= g(\tilde{\nabla}_{hX}hY + \tilde{\nabla}_{hY}hX - JT(hX,hY) - h[\nabla hX,hY], hZ) \]

\[
+ g(JT(hX,hZ) + h[\nabla hX,hZ], hY) + g(JT(hY,hZ) + h[\nabla hY,hZ], hX). \]

The relations (8) and (9) show that $\tilde{\nabla}$ satisfies (2) and (3), respectively. Therefore $\nabla = \tilde{\nabla}$.

The Finsler connection $FC = (HM', \nabla)$ where $\nabla$ is given by Theorem 1 is called the $(HM', S, T)$-Cartan connection (see [12]) which in this case is obtained by the associate almost complex structure to $HM'$. If, in particular, $HM'$ is just the canonical nonlinear connection $GM'$ of $\mathbb{R}^m$ (for more details about $GM'$ see [1]) and $S = T = 0$, the $FC$ is called the Cartan connection and it is denoted by $FC^* = (GM', \nabla^*)$.

By means of the pseudo-Riemannian metric $g$ on $VM'$ we consider a pseudo-Riemannian metric on the vector bundle $TM'$ similar to the Sasaki one and denote it by $G$, that is

\[
G = g_{ij}(x,y)dx^i dx^j + g_{ij}(x,y)\delta y^j \delta y^i,
\]

where $\delta y^i = dy^i + N^i_j(x,y)dx^j$. Denote by $\nabla'$ the Levi-Civita connection on $M'$ with respect to $G$. A. Bejancu and H.R. Farran showed $\nabla^*$ is the projection of the Levi-Civita connection $\nabla'$ on the vertical vector bundle also they proved the following theorem (see [1]).

**Theorem 3.** The associate linear connection $D^*$ to the Cartan connection $FC^* = (GM', \nabla^*)$ is a metric linear connection with respect to $G$.

Now we give the following theorem which shows the natural almost complex linear connections associated to $(HM', S, T)$-Cartan connections are metric linear connections with respect to $G$.

**Theorem 4.** The natural almost complex linear connection $D$ associated to a $(HM', S, T)$-Cartan connection $FC = (HM', \nabla)$ is a metric linear connection with respect to $G$.

**Proof.** For any $X,X_1, X_2 \in \Gamma(TM')$ we have

\[
D_XG(X_1, X_2) = XG(X_1, X_2) - G(D_X X_1, X_2) - G(X_1, D_X X_2) \]

\[
= X(G(X_1, X_2)) - G(\nabla_X vX_1, X_2) + G(J\nabla_X hX_1, X_2) \]

\[
- G(X_1, \nabla_X vX_2) + G(X_1, J\nabla_X hX_2). \quad (10)
\]

By (10) and this fact that $S$ and $T$ are skew-symmetric we have:

\[
D \frac{\partial}{\partial y^j} G \left( \frac{\partial}{\partial y^j} \frac{\delta}{\delta x^k} \right) = D \frac{\delta}{\delta x^j} G \left( \frac{\partial}{\partial y^j} \frac{\delta}{\delta x^k} \right) = 0,
\]

\[
D \frac{\partial}{\partial y^j} G \left( \frac{\partial}{\partial y^j} \frac{\delta}{\delta x^k} \right) = D \frac{\partial}{\partial y^j} G \left( \frac{\delta}{\delta x^j} \frac{\delta}{\delta x^k} \right) = \frac{\partial g_{jk}}{\partial y^j} C^j_{hk} g_{kh} - C^j_{kh} g_{jh} = 0,
\]

\[
D \frac{\delta}{\delta x^j} G \left( \frac{\partial}{\partial y^j} \frac{\delta}{\delta y^k} \right) = D \frac{\delta}{\delta x^j} G \left( \frac{\delta}{\delta x^j} \frac{\delta}{\delta y^k} \right) = \frac{\delta g_{jk}}{\delta x^j} F^h_{jk} g_{hk} - F^h_{jk} g_{jh} = 0.
\]

Therefore $D_X G = 0$ for any $X \in \Gamma(TM')$. ■
Let $F^m = (M, M', F)$ be a Finsler manifold. We can easily check that the pair $(J, G)$ defines an almost Hermitian metric on $M'$. In the following theorem we give a sufficient condition for Finsler tensor fields $S$ and $T$ such that $\mathcal{D}$ be the Levi-Civita connection arising from $G$.

**Theorem 5.** The natural almost complex linear connection $\mathcal{D}$ associated to a $(HM', S, T)$-Cartan connection $FC = (HM', \nabla)$ is the Levi-Civita connection arising from $G$ if $T^D(X, Y) = 0$ for any $X, Y \in \Gamma(TM')$ or equivalently if

$$S = T = 0, \quad C^{k}_{ij} = R^{k}_{ij} = 0, \quad F^{k}_{ij} = \frac{\partial N^{k}_{ij}}{\partial y^i},$$

where $R^{k}_{ij} = \frac{\delta N^{k}_{ij}}{\delta x^j} - \frac{\delta N^{k}_{ik}}{\delta x^j}$.

**Proof.** By Theorem 4, $\mathcal{D}$ is a metric linear connection with respect to $G$. Therefore if $T^D = 0$ then $\mathcal{D}$ is the Levi-Civita connection. In local coordinates we have

$$T^D \left( \frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^i} \right) = S^{k}_{ij} \frac{\partial}{\partial y^k},$$

$$T^D \left( \frac{\partial}{\partial y^j}, \frac{\delta}{\delta x^i} \right) = C^{k}_{ji} \frac{\delta}{\delta x^k} + \left( \frac{\partial N^{k}_{ji}}{\partial y^i} - F^{k}_{ij} \right) \frac{\partial}{\partial y^k},$$

$$T^D \left( \frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j} \right) = T^{k}_{ij} \frac{\delta}{\delta x^k} + \left( \frac{\delta N^{k}_{ij}}{\delta x^i} - \frac{\delta N^{k}_{ij}}{\delta x^j} \right) \frac{\partial}{\partial y^k}.$$ 

Therefore the proof is completed.

**Corollary 2.** If $T^D = 0$ then $(M', J, G)$ is a Kähler manifold.

**Proof.** If $T^D = 0$ then $\mathcal{D}$ is the Levi-Civita connection of $G$. Also $J$ is parallel with respect to $\mathcal{D}$. Therefore $\mathcal{D}$ (the Levi-Civita connection of $G$) is almost complex. Consequently by using Theorem 4.3 of [10], $(M', J, G)$ is a Kähler manifold.

We know that the almost Hermitian manifold $(M', J, G)$ is an almost Kähler manifold if and only if the fundamental 2-form $\Phi$ is closed ($\Phi$ is defined by $\Phi(X, Y) = G(X, JY)$ for all $X, Y \in \Gamma(TM')$). Therefore we can give the following theorem.

**Theorem 6.** The almost Hermitian manifold $(M', J, G)$ is an almost Kähler manifold if and only if

$$\frac{\delta g_{ik}}{\delta x^j} + \frac{\partial N^h_{ik}}{\partial y^i} g_{hk} - \left( \frac{\delta g_{ij}}{\delta x^k} + \frac{\partial N^h_{ij}}{\partial y^i} g_{hk} \right) = 0 \quad (11)$$

and

$$R^h_{ijk} g_{kk} - R^h_{ik} g_{kj} + R^h_{ij} g_{hi} = 0. \quad (12)$$

**Proof.** Let $X_0, X_1, X_2 \in \Gamma(TM')$. Then we have

$$d\Phi(X_0, X_1, X_2) = X_0 G(X_1, JX_2) - X_1 G(X_0, JX_2) + X_2 G(X_0, JX_1) - G([X_0, X_1], JX_2) + G([X_0, X_2], JX_1) - G([X_1, X_2], JX_0).$$

By using the above relation in local coordinates we have:

$$d\Phi \left( \frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^k} \right) = d\Phi \left( \frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}, \frac{\delta}{\delta x^k} \right) = 0.$$
The Relation Between the Associate Almost Complex Structure

\[
d\Phi \left( \frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^k} \right) = \delta g_{ik} \delta x^j + \frac{\partial N^h_k}{\delta y^i} g_{hk} - \left( \frac{\delta g_{ij}}{\delta x^k} + \frac{\partial N^h_j}{\delta y^i} g_{hk} \right),
\]

\[
d\Phi \left( \frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^k} \right) = \left( \frac{\delta N^h_i}{\delta x^j} - \frac{\delta N^h_j}{\delta x^i} \right) g_{hk} - \left( \frac{\delta N^h_j}{\delta x^k} - \frac{\delta N^h_k}{\delta x^j} \right) g_{hi}.
\]

Therefore the fundamental 2-form \( \Phi \) is closed if and only if the equations (11) and (12) are confirmed.

Now, by using Proposition 1 and Corollary 2, we have the following corollary.

**Corollary 3.** Let \( F^m = (M, M', F) \) be a Finsler manifold. If \( T^D = 0 \) then,

1) \( F^m = (M, M', F) \) has zero flag curvature;
2) \( J \) is integrable;
3) \( \nabla J = 0 \), where \( \nabla \) is the Levi-Civita connection of the Sasaki metric \( G \);
4) \( (M', J, G) \) is Kählerian.

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