Stable commutator length in right-angled Artin and Coxeter groups

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Abstract
We establish a spectral gap for stable commutator length (scl) of integral chains in right-angled Artin groups (RAAGs). We show that this gap is \textit{not} uniform, that is, there are RAAGs and integral chains with scl arbitrarily close to zero. We determine the size of this gap up to a multiplicative constant in terms of the \textit{opposite path length} of the defining graph. This result is in stark contrast with the known uniform gap 1/2 for elements in RAAGs. We prove an analogous result for right-angled Coxeter groups. In a second part of this paper, we relate certain integral chains in RAAGs to the \textit{fractional stability number} of graphs. This has several consequences: First, we show that every rational number $q \geq 1$ arises as the scl of an integral chain in some RAAG. Second, we show that computing scl of elements and chains in RAAGs is NP hard. Finally, we heuristically relate the distribution of scl for random elements in the free group to the distribution of fractional stability number in random graphs. We prove all of our results in the general setting of graph products. In particular, all above results hold verbatim for right-angled Coxeter groups.

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INTRODUCTION

The stable commutator length (scl) is a relative version of the Gromov–Thurston norm. For a finite collection of loops \( \gamma_1, \ldots, \gamma_k \) in a topological space \( X \), its scl is the least complexity of surfaces bounding it, measured in terms of Euler characteristics (see Definition 2.1). This only depends on the fundamental group \( G = \pi_1(X) \) and the conjugacy classes \( g_1, \ldots, g_k \) representing the free homotopy classes of \( \gamma_1, \ldots, \gamma_k \), and it is denoted as \( \text{scl}_G(g_1 + \cdots + g_k) \). We call this the scl of the (integral) chain \( g_1 + \cdots + g_k \).

The scl arises naturally in geometry, topology and dynamics and has seen a vast development in recent years by Calegari and others [4, 9, 10, 13, 28].

A group \( G \) has a spectral gap \( C > 0 \) for elements (respectively, chains) if \( \text{scl}_G(g) \notin (0, C) \) for any element (respectively, any chain) \( g \) in \( G \). The largest such \( C \) is called the optimal spectral gap of \( G \) for elements (respectively, chains). Various kinds of groups are known to have a gap for elements: word-hyperbolic groups [10], finite index subgroups of mapping class groups [4], subgroups of right-angled Artin groups (RAAGs; defined below) [26] and 3-manifold groups [14]; see Theorem 2.17. The spectral gap property can be used to obstruct group homomorphisms since the scl is non-increasing under homomorphisms.

In contrast, much less is known about spectral gaps for chains. Calegari–Fujiwara [10] showed that hyperbolic groups have a spectral gap for chains. Their estimates have been made uniform and explicit in certain families of hyperbolic groups (Theorem 2.18). To our best knowledge, all previously known nontrivial examples with a spectral gap for chains are direct products of hyperbolic groups.

In this article, we establish a spectral gap for chains in RAAGs. The RAAG \( A(\Gamma) \) associated to a simplicial graph \( \Gamma \) is the group with presentation

\[
A(\Gamma) = \langle V(\Gamma) \mid [v, w]; (v, w) \in E(\Gamma) \rangle,
\]

which is not hyperbolic unless the graph contains no edge. Such groups are of importance due to their rich subgroup structure [1, 5, 6, 24, 37].

The gap is controlled by an invariant \( \Delta(\Gamma) \geq 0 \) of the defining simplicial graph \( \Gamma \) that we introduce, called the opposite path length; see Subsection 1.1.

**Theorem A.** Let \( G \) be the RAAGs associated to a simplicial graph \( \Gamma \). Then the optimal spectral gap for integral chains in \( G \) is at least \( \frac{1}{24 + 12\Delta(\Gamma)} \) and at most \( \frac{1}{2\Delta(\Gamma)} \).
The gap cannot be uniform among all RAAGs as there is an explicit finite graph \( \Delta_m \) with \( \Delta(\Delta_m) = m \) for any \( m \in \mathbb{Z}_+ \). The nonuniformness of the gap for chains is striking since RAAGs are known to have a uniform spectral gap 1/2 for elements [19, 26]. This is the first class of groups where the optimal gap for chains is known to be different from the optimal gap for elements. Using the nonuniformness, we construct countable groups where this difference becomes more apparent (Subsection 1.3).

We prove these results in the much more general setting of graph products (Theorem D). In particular, Theorem A holds verbatim for right-angled Coxeter groups, which are defined in the same way as RAAGs except that generators have order 2. For right-angled Coxeter groups, no gap was previously known in general, even for elements.

For a simplicial graph \( \Gamma \) we will construct a graph \( D_\Gamma \) and a chain \( c_\Gamma \) in \( A(D_\Gamma) \), called the double chain of \( \Gamma \); see Definition 1.1. We will relate the scl of this chain linearly to the fractional stability number (Definition 1.2) of \( \Gamma \) (Theorem H). The latter invariant is well-studied [32]. It is known that computing the fractional stability number is NP hard [22] and that every rational number \( q \geq 2 \) is the fractional stability number of some graph [32, Proposition 3.2.2]. As consequences of this connection, we obtain the following two theorems.

**Theorem B** (NP-hardness, Theorem 7.14). Unless \( P=NP \), there is no algorithm that, given a simplicial graph \( \Gamma \), an element \( w \in A(\Gamma) \) and a rational number \( q \in \mathbb{Q}^+ \), decides if \( scl_{A(\Gamma)}(w) \leq q \) with polynomial run time in \( |V(\Gamma)| + |w| \). The same holds for chains.

This is in stark contrast to the case of free groups, as there is an algorithm by Calegari computing \( scl \) with polynomial run time in the word length of the input [9, 11]; also compare to [27].

**Theorem C** (Rational realization, Theorem 7.13). For every rational \( q \in \mathbb{Q}_{\geq 1} \) there is an integral chain \( c \) in a RAAGs \( A(\Gamma) \) such that \( scl_{A(\Gamma)}(c) = q \).

In the case of free groups, it is an unsolved conjecture of Calegari–Walker that the set of values of \( scl \) is dense in some intervals.

In the following subsections, we will describe the generalization of our results to graph products, and collect some further results.

### 1.1 Spectral gaps for integral chains: Overview of the proof

We now state the generalization of Theorem A to graph products and describe the main steps in its proof.

For a simplicial graph \( \Gamma \), let \( \{G_v\}_{v \in V(\Gamma)} \) be a family of groups indexed by the vertex set \( V(\Gamma) \) of \( \Gamma \). The graph product for these data is the free product \( \star_{v \in V(\Gamma)} G_v \) subject to the relations \([g_v, h_w]\) for every \( g_v \in G_v \) and \( h_w \in G_w \) whenever \((v, w) \in E(\Gamma)\) is an edge of \( \Gamma \). Graph products are generalizations of both RAAGs (which have vertex groups \( \mathbb{Z} \)) and right-angled Coxeter groups (which have vertex groups \( \mathbb{Z}/2 \)).

For an integer \( m \geq 1 \), the opposite path of length \( m \) is the simplicial graph \( \Delta_m \) with vertex set \( V(\Delta_m) = \{v_0, \ldots, v_m\} \) and edge set \( E(\Delta_m) = \{(v_i, v_j) \mid |i - j| \geq 2\} \). We define the opposite path length of a simplicial graph \( \Gamma \) to be

\[
\Delta(\Gamma) := \max\{m \mid \Delta_m \text{ is an induced subgraph of } \Gamma\}.
\]

Here a subgraph \( \Lambda \) of \( \Gamma \) is induced if any edge in \( \Gamma \) connecting \( u, v \in \Lambda \) belongs to \( \Lambda \).
**Theorem D** (Theorem 6.2). Let $\Gamma$ be a simplicial graph, let $\{G_v\}_{v \in V(\Gamma)}$ be a family of groups and let $G(\Gamma)$ be the associated graph product. If $c$ is an integral chain in $G(\Gamma)$, then either $\text{scl}_{G(\Gamma)}(c) \geq \frac{1}{12\Delta(\Gamma)+24}$ or $c$ is equivalent (see below) to a chain supported on the vertex groups, called a vertex chain.

For vertex chains, there is an algorithm to compute $\text{scl}_{G(\Gamma)}(c)$ in terms of the scls in the vertex groups.

Moreover, there is an integral chain $\delta$ on $G(\Gamma)$ such that

$$\frac{1}{12(\Delta(\Gamma) + 2)} \leq \text{scl}_{G(\Gamma)}(\delta) \leq \frac{1}{\Delta(\Gamma)}.$$ 


The equivalence relation of chains, roughly speaking, is based on the following moves that does not change the scl. In an arbitrary group $G$ with a chain $c$ and elements $g, h \in G$ we have $\text{scl}_G(c + g^n) = \text{scl}_G(c + n \cdot g)$ for every $n \in \mathbb{Z}$ and $\text{scl}_G(c + g) = \text{scl}_G(c + gh^{-1})$. If in addition $g$ and $h$ commute, we have $\text{scl}_G(c + g \cdot h) = \text{scl}_G(c + g + h)$. We say that two chains $c, c'$ in $G$ are **equivalent**, if $c$ can be transformed into $c'$ by a finite sequence of these identities. See Definition 2.4 for the precise definition.

Formally, a **vertex chain** is of the form $c = \sum_v c_v$, where each $c_v$ is a chain in the vertex group $G_v$. For RAAGs and right-angled Coxeter groups, any null-homologous vertex chain is equivalent to the zero chain and has zero scl. Thus, Theorem A immediately follows from Theorem D. Moreover, we have a uniform gap $1/60$ for all hyperbolic right-angled Coxeter groups; see Corollary 6.19.

In particular, Theorem D implies that groups with a gap for integral chains are preserved under taking graph products over finite graphs; see Corollary 6.4.

### 1.1.1 Gaps for chains in graphs of groups

The spectral gap result in Theorem D is based on a simple criterion for spectral gaps in graphs of groups that we prove. For simplicity, we state it for amalgamations.

**Theorem E** (Theorem 4.1, Long pairings). Let $G = A \star_C B$ be an amalgamation and let $\sum_{i \in I} g_i$ be an integral chain. Then either

$$\text{scl}_G(c) \geq \frac{1}{12N}$$

or $c = \sum_{i \in I} g_i$ has a term $g = g_i$ such that $g^N = h^k h' d$ as reduced elements, where $h$ is cyclically conjugate to the inverse of some term $g_j$ in $c$, $h'$ is a prefix (Definition 2.20) of $h$ and $d \in C$.

We give two proofs of this criterion in Section 4, one using surfaces and the other using quasimorphisms.

To make use of this criterion, we reduce chains so that the exceptional algebraic relation $g^N = h^k h' d$ does not occur for a suitable $N$. In Section 5, we develop tools to achieve this goal for $N = D + 2$, provided that all edge groups are BCMS-D subgroups (Definition 5.8). BCMS-D subgroups are generalizations of malnormal and central subgroups. In particular, malnormal subgroups are BCMS-1 and central subgroups are BCMS-0.

As key examples, for a graph product over a graph $\Gamma$, the subgroup corresponding to any induced subgraph is BCMS-D if the opposite path length $\Delta(\Gamma) = D$. Theorem D is obtained from the following estimate.
**Theorem F** (Theorem 5.1, BCMS gap). Let $G$ be a graph of groups such that the embedding of every edge group $C \leq G$ is a BCMS-D subgroup. Then for any integral chain $c$ in $G$, either $c$ is equivalent to a chain supported on vertex groups, or

$$\text{scl}_G(c) \geq \frac{1}{12(D + 2)}.$$  

The special case where every edge group is malnormal in $G$ is equivalent to that the fixed point set of each $g \neq id \in G$ has diameter at most 1 for the action on the Bass–Serre tree. In this case, we have the following corollary

**Corollary G.** Let $G$ be a graph of groups such that the embedding of each edge group $C \leq G$ is malnormal. Then for any integral chain $c$ in $G$, either $c$ is equivalent to a chain supported on vertex groups, or

$$\text{scl}_G(c) \geq \frac{1}{36}.$$  

A similar result was obtained by Clay–Forester–Louwsma for hyperbolic elements in a group acting $K$-acylindrically on a simplicial tree; see [15, Theorem 6.11].

### 1.2 Fractional stability number and scl

The algorithm mentioned in Theorem D computes scls of vertex chains as certain graph-theoretic quantities; see Section 7. As a special case, we discover a connection between scls of certain chains in RAAGs and the fractional stability numbers of graphs.

**Definition 1.1** (Double graphs and double chains). For a simplicial graph $\Gamma$ with vertices $V(\Gamma)$ and edges $E(\Gamma)$ we define the double graph $D_\Gamma$ as the graph with vertex and edge set

$$V(D_\Gamma) = \{ a_v, b_v \mid v \in V(\Gamma) \} \text{ and } E(D_\Gamma) = \{ (a_v, a_w), (a_v, b_w), (b_v, a_w), (b_v, b_w) \mid (v, w) \in E(\Gamma) \}.$$  

Let $d_\Gamma$ be the integral chain $\sum_{v \in V(\Gamma)} [a_v, b_v]$. We call $D_\Gamma$ the double graph and $d_\Gamma$ the double chain in $A(D_\Gamma)$.

A key feature of this construction is that $A(D_\Gamma)$ is the graph product over the graph $\Gamma$ with vertex groups $F(a_v, b_v)$.

**Definition 1.2** (Fractional stability number). A stable measure is a collection of non-negative weights $x = \{ x_v \}_{v \in V}$ assigned to vertices of $\Gamma$ such that for any clique $C$ (that is, a complete subgraph) in $\Gamma$ we have that $\sum_{v \in C} x_v \leq 1$. The fractional stability number of $\Gamma$ is the supremum of $\sum_{v \in V} x_v$ over all stable measures and denoted by $\text{fsn}(\Gamma)$.

**Theorem H** (Theorem 7.11). Let $\Gamma$ be a graph and let $D_\Gamma$ and $d_\Gamma$ be the associated double graph and double chain, respectively. Then
FIGURE 1  scl for random words in the free group versus scl of random chains \(d_T\) in RAAGs \(A(D_T)\). In both cases, scl is rational and values with small denominator appear more frequent and the histogram exhibits a fractal behavior. In Subsection 7.3, we explain this distribution as the interference of (rounded) Gaussian distributions. (a) Histogram of \(\text{scl}_{F_2}(w)\) for 50,000 random words \(w\) uniformly chosen of length 24 in \(F_2\). (b) Histogram of \(\text{scl}_{A(D_T)}(d_T) = \frac{1}{2} \cdot \text{fsn}(\Gamma)\) for 50,000 random graphs \(\Gamma\) uniformly chosen on 24 vertices. Here, \(d_T\) is the double chain in \(A(D_T)\) (Definition 1.1).

\[
\text{scl}_{A(D_T)}(d_T) = \frac{1}{2} \cdot \text{fsn}(\Gamma).
\]

Combining with known results about fractional stability numbers, we deduce Theorems B and C. See Section 7 for the more general results about computations of scls of vertex chains in graph products.

The distributions of scl of random elements in free groups and fsn of random graphs are depicted in Figure 1. They exhibit a strikingly similar behavior: For both distributions values with small denominators appear more frequently, and the histograms exhibit some self-similarity. In Subsection 7.3, we analyze the distribution of scl and fsn further. This analysis allows us to describe a 5-parameter random variable \(X\) (Definition 7.15) which exhibits qualitatively the same distribution as scl and fsn. We use \(X\) to model both scl and fsn in Figure 7. While this is purely heuristic, it suggests that the distribution of scl and fsn converge to a similar distribution for large words or graph sizes; see Question 7.16.

1.3  Groups with interesting scl spectrum

The scl spectrum of a group is the range of the map \(\text{scl}_G : [G, G] \to \mathbb{R}_{\geq 0}\). The nonuniformness of spectral gap in Theorem A allows us to construct groups with interesting spectrum. There are few (classes of) groups where the spectrum of scl is fully known; see [8, Remark 5.20; 25; 38].
Theorem I. There is a countable (right-angled Artin) group $G$ such that $\text{scl}_G(g) \geq 1/2$ for all $g \neq \text{id} \in [G, G]$ but there is no spectral gap for chains in $G$.

Theorem J. There is a countable group $G$ such that $\text{scl}_G(g) \geq 1/2$ for all $g \neq \text{id} \in [G, G]$, and its scl spectrum is dense in $[3/2, \infty)$.

To the authors’ best knowledge there was no group known that has a spectral gap for elements and the spectrum of elements becomes eventually dense, though free groups are conjectured to have this property.

These results are proved in Subsection 6.6.

1.4 Organization

This article is organized as follows. In Section 2, we recall basic results of scl, graph of groups and graph products, respectively. In Section 4, we prove Theorem E estimating scl in graphs of groups. In Section 5, we will develop the theory of BCMS-D subgroups and prove Theorem F. Then we apply this to graph products of groups and prove Theorem D in Section 6. Finally, in Section 7 we compute scls of vertex chains in graph products and relate them to fractional stability numbers.

2 BACKGROUND

We briefly introduce several concepts and set up some notations related to scl and graphs of groups. All results in this section are standard. Readers familiar with these topics may skip this section and refer to it when necessary.

2.1 Stable commutator length

We give the precise definition of the scl and recall some basic results. The reader may refer to [8, chapter 2] for details.

Given a group $G$, let $X$ be a topological space with fundamental group $G$. An integral chain is a finite formal sum of elements in $G$. Given an integral chain $\sum g_j$, consider loops $\gamma_j$ in $X$ so that the free homotopy class of $\gamma_j$ represents the conjugacy class of $g_j$ for each $j$.

An admissible surface is a pair $(S, f)$, where $S$ is a compact oriented surface and $f : S \to X$ is a continuous map such that the following diagram commutes and $\delta f_+ [\delta S] = n(S, f)[\sqcup S_j^1]$ for some integer $n(S, f) > 0$, called the degree of the admissible surface.

\[
\begin{array}{ccc}
\delta S & \xrightarrow{i} & S \\
\delta f \downarrow & & f \downarrow \\
\sqcup S_j^1 & \xrightarrow{\cup \gamma_j} & X.
\end{array}
\]

Admissible surfaces exist if the chain is null-homologous, that is, $\sum [g_j] = 0 \in H_1(G; \mathbb{Q})$. Let $\chi^-(S)$ be the Euler characteristic of $S$ after removing disk and sphere components.
**Definition 2.1.** For any null-homologous integral chain $\sum g_j$ in $G$, we define

$$scl_G \left( \sum g_j \right) := \inf_{(S,f)} \frac{-\chi(S)}{2 \cdot n(S,f)}.$$ 

When the chain represents a nontrivial rational homology class, we make the convention that $scl_G(\sum g_j) = +\infty$.

We often omit the map $f$ and refer to an admissible surface $(S, f)$ simply as $S$.

In the special case where the chain is an element $g \in [G, G]$, this agrees with the algebraic definition using commutator lengths. See [8, chapter 2] for more details as well as an algebraic definition for $scl$ of integral chains.

**Lemma 2.2.** For an integral chain $c = gh - g - h$ with $g, h \in G$, we have $scl_G(c) \leq 1/2$.

**Proof.** The fundamental group of a pair of pants $S$ is the free group of rank 2, where the generators $a, b$ can be chosen so that the boundary loops with the induced orientation are represented by $ab, a^{-1}$ and $b^{-1}$, respectively. The homomorphism $F_2 \rightarrow G$ determined by $a \mapsto g$ and $b \mapsto h$ corresponds to a map $f : S \rightarrow X$, which provides an admissible surface for $c$ of degree one. Hence,

$$scl_G(c) \leq \frac{-\chi(S)}{2} = \frac{1}{2}. \quad \square$$

Let $C_1(G)$ be the space of real 1-chains. By identifying $g^{-1}$ with $-g$ in $C_1(G)$, $scl$ is defined for any finite sum $\sum t_ig_i \in C_1(G)$, where $t_i \in \mathbb{Z}$. It is known that $scl$ is linear on rays and satisfies the triangle inequality, and thus extends to a (semi-)norm on $C_1(G)$.

**Proposition 2.3** (Scl as a norm). Scl is a semi-norm on $C_1(G)$. In particular, $scl(c_1 + c_2) \leq scl(c_1) + scl(c_2)$ for any $c_1, c_2 \in C_1(G)$.

**Definition 2.4** (Equivalent chains). Let $E(G)$ be the subspace of $C_1(G)$ spanned by elements of the following forms:

1. $g^n - n \cdot g$, where $n \in \mathbb{Z}$ and $g \in G$,
2. $hgh^{-1} - g$, where $g, h \in G$,
3. $gh - g - h$, where $g$ and $h$ are commuting elements in $G$.

We say two chains $c$ and $c'$ are equivalent if they differ by an element in $E(G)$.

Note that this is slightly different from the usual definition (for example, [8, Definition 2.78]) by adding (3).

**Proposition 2.5** (scl of equivalent chains). If $c$ and $c'$ are equivalent chains, then

$$scl_G(c) = scl_G(c').$$
Proof. Since scl is a semi-norm, this is to show that scl vanishes on each basis element of $E(G)$. For chains of the first two kinds, see [8, section 2.6]. For a chain $gh - g - h$, where $g$ and $h$ commute, since $(gh)^n = g^n \cdot h^n$ for any $n \in \mathbb{Z}^+$, there is a thrice-punctured sphere with boundary components representing $(gh)^n$, $g^{-n}$ and $h^{-n}$, respectively. This gives rise to an admissible surface $S$ for the chain $gh - g - h$ of degree $n$, which has $-\chi(S) = 1$. Letting $n$ go to infinity, we have $\text{scl}_G(gh - g - h) = 0$.

We collect a few properties of scl. The main reference is [8].

**Proposition 2.6** (Monotonicity and retract). Let $H, G$ be groups and let $f : H \to G$ be a homomorphism. Then for any chain $c$ in $C_1(H)$ we have $\text{scl}_H(c) \geq \text{scl}_G(f(c))$. If in addition $H$ is a retract of $G$, that is, there is a homomorphism $r : G \to H$ such that $r \circ f = \text{id}_H$, then for any chain $c$ in $H$ we have that $\text{scl}_H(c) = \text{scl}_G(c)$.

**Proposition 2.7.** If $c = c_1 + c_2$ is a chain in $G = G_1 \rtimes G_2$, where $c_1$ is supported on $G_1$ and $c_2$ is supported on $G_2$, then $\text{scl}_G(c) = \text{scl}_{G_1}(c_1) + \text{scl}_{G_2}(c_2)$.

**Proof.** This is a special case of [14, Theorem 6.2] since $G$ is a graph of groups with vertex groups $G_1, G_2$ and a trivial edge group.

**Proposition 2.8** [8, Theorem 2.101]. Let $G$ be a group and let $c = \sum_{i=1}^n g_i$ be a chain. Let $G = G \rtimes \langle t_1 \rangle \rtimes \cdots \rtimes \langle t_{n-1} \rangle$ be the free product of $G$ with $n - 1$ infinite cyclic groups. Then

$$\text{scl}_G(c) = \text{scl}_G\left( g_1 \cdot \prod_{i=1}^{n-1} t_i g_{i+1} t_i^{-1} \right) - \frac{n-1}{2}.$$

**Proposition 2.9** (Index formula [8, Corollary 2.81]). Let $H \trianglelefteq G$ be a finite index normal subgroup. The quotient $F = G / H$ acts on $H$ by outer-automorphisms $h \mapsto f \cdot h$, where $f \cdot h$ is a well-defined conjugacy class in $H$. Then for any $h \in H$, we have

$$\text{scl}_G(h) = \frac{1}{|F|} \text{scl}_H\left( \sum_{f \in F} f \cdot h \right).$$

2.2 Quasimorphisms

Let $G$ be a group. A map $\phi : G \to \mathbb{R}$ is called a quasimorphism if there is a constant $D > 0$ such that $|\phi(g) + \phi(h) - \phi(gh)| \leq D$ for all $g, h \in G$. The infimum of all such $D$ is called the defect of $\phi$ and denoted by $D(\phi)$. Every bounded map and every homomorphism to $\mathbb{R}$ are trivially quasimorphisms but there are many nontrivial examples; see Example 2.15. A quasimorphism is called homogeneous if $\phi(g^n) = n \cdot \phi(g)$ for every $g \in G$ and $n \in \mathbb{Z}$. Every quasimorphism $\phi : G \to \mathbb{R}$ has a unique associated homogeneous quasimorphism $\tilde{\phi}$ defined via

$$\tilde{\phi}(g) := \lim_{n \to \infty} \frac{\phi(g^n)}{n}$$

which we call the homogeneous representative of $\phi$.
**Proposition 2.10** (Homogeneous representative [8, Lemma 2.58]). Let \( \phi : G \to \mathbb{R} \) be a quasimorphism with defect \( D(\phi) \). Then the homogeneous representative \( \tilde{\phi} \) is in bounded distance to \( \phi \) and satisfies \( D(\tilde{\phi}) \leq 2D(\phi) \).

Here two quasimorphisms \( \phi, \psi : G \to \mathbb{R} \) are in bounded distance if \( \phi - \psi \) is bounded in the supremum norm.

Quasimorphisms are intimately connected to \( \text{scl} \) through Bavard’s duality:

**Theorem 2.11** (Bavard’s Duality Theorem [3; 8, Theorem 2.79]). For any chain \( c = \sum_{i \in I} n_i g_i \) with real coefficients \( n_i \in \mathbb{R} \) we have

\[
\text{scl}_G(c) = \sup_{\phi} \frac{\sum_{i \in I} n_i \phi(g_i)}{2D(\phi)},
\]

where the supremum is taken over all homogeneous quasimorphisms \( \phi : G \to \mathbb{R} \). Moreover, this supremum is achieved.

One can actually choose the homogeneous quasimorphism achieving the supremum in Bavard’s duality to be the homogenization of a quasimorphism with nice properties. A quasimorphism \( \phi \) is called \( \text{antisymmetric} \) if \( \phi(g) = -\phi(g^{-1}) \) for all \( g \in G \).

**Proposition 2.12** (Extremal quasimorphisms). Let \( G \) be a group. For any chain \( c \) in \( G \) there is a quasimorphism \( \phi : G \to \mathbb{R} \) with \( D(\phi) = 1/4 \) that achieves the supremum of Bavard’s duality, that is, such that

\[
\text{scl}_G(c) = \tilde{\phi}(c)
\]

where \( \tilde{\phi} \) is the homogenization of \( \phi \). Moreover, we may choose \( \phi \) to be antisymmetric.

**Proof.** The statement without the moreover part is well-known, and follows from the proof of [8, Theorem 2.70]. Now suppose \( \psi \) is such a quasimorphism with \( D(\psi) = 1/4 \) and \( \tilde{\psi}(c) = \text{scl}_G(c) \).

Let \( \phi(g) := (\psi(g) - \psi(g^{-1}))/2 \). Then \( \phi \) is an antisymmetric quasimorphism with \( D(\phi) \leq D(\tilde{\psi}) = 1/4 \). It also follows by definition that \( \tilde{\phi} = \tilde{\psi} \), and in particular \( \tilde{\phi}(c) = \tilde{\psi}(c) = \text{scl}_G(c) \). Thus, by Barvard’s duality, we must also have \( D(\phi) \geq 1/4 \) and hence \( D(\phi) = 1/4 \). This gives us the desired quasimorphism \( \phi \).

\( \square \)

**Lemma 2.13.** For any homogeneous quasimorphism \( \phi \) on \( G \), we have \( \phi(gh) = \phi(g) + \phi(h) \) if \( g \) and \( h \) commute.

**Proof.** Note that for any \( n \in \mathbb{Z}_+ \) we have \( (gh)^n = g^n h^n \) and

\[
|\phi(gh) - \phi(g) - \phi(h)| = \frac{1}{n} |\phi(g^n h^n) - \phi(g^n) - \phi(h^n)| \leq D(\phi)/n.
\]

Taking \( n \to \infty \) we have \( \phi(gh) = \phi(g) + \phi(h) \). \( \square \)
Proposition 2.14. Let $c$ be a chain in $G \cong G_1 \times G_2$. Then $c$ is equivalent to a chain $c_1 + c_2$ where $c_1$ is supported on $G_1$ and $c_2$ is supported on $G_2$, and $c_1, c_2$ are integral chains if $c$ is. Moreover,

$$\text{scl}_G(c) = \max\{\text{scl}_{G_1}(c_1), \text{scl}_{G_2}(c_2)\}.$$ 

Proof. Each element $g \in G$ can be written as $g_1 g_2$ for some $g_1 \in G_1$ and $g_2 \in G_2$, and thus $g$ is equivalent to $g_1 + g_2$ as chains. The first claim easily follows from this.

Every homogeneous quasimorphism $\phi$ on $G$ restricts to quasimorphisms $\phi_1$ and $\phi_2$ on $G_1$ and $G_2$, respectively. Then for the decomposition $g = g_1 g_2$ above for any $g \in G$, we have $\phi(g) = \phi(g_1) + \phi(g_2) = \phi_1(g_1) + \phi_2(g_2)$ by Lemma 2.13. It follows that $D(\phi) = D(\phi_1) + D(\phi_2)$ and $\phi(c) = \phi_1(c_1) + \phi_2(c_2)$ for the decomposition above.

Let $\phi$ be an extremal homogeneous quasimorphism for a chain $c$. For the decomposition $c = c_1 + c_2$ and $\phi = \phi_1 + \phi_2$, we have

$$\text{scl}_G(c) = \frac{\phi(c_1 + c_2)}{2D(\phi)} \leq \frac{|\phi_1(c_1)| + |\phi_2(c_2)|}{D(\phi_1) + D(\phi_2)} \leq \max\left\{\frac{|\phi_1(c_1)|}{D(\phi_1)}, \frac{|\phi_2(c_2)|}{D(\phi_2)}\right\}$$

$$\leq \max\{\text{scl}_{G_1}(c_1), \text{scl}_{G_2}(c_2)\}$$

by Bavard’s duality. This proves the second claim since the other direction $\text{scl}_G(c_1 + c_2) \geq \text{scl}_{G_i}(c_i)$ follows by the monotonicity of $\text{scl}$ under the projection $G \to G_i$, where $i = 1, 2$. □

Example 2.15 (Brooks Quasimorphisms). We describe a family of quasimorphisms on nonabelian free groups that certify a spectral gap in free groups. Let $F(S)$ be the free group on a generating set $S$ and let $w \in F(S)$ be a reduced word. For an element $x \in F(S)$, let $\nu_w(x)$ be the maximal number of times that $w$ is a subword of $x$ that is, the maximal $n$ such that $x = x_0 w x_1 \cdots w x_n$, where $x_0, \ldots, x_n \in F(S)$ and this expression is reduced. We define $\phi_w : F(S) \to \mathbb{Z}$ via $\phi_w : x \mapsto \nu_w(x) - \nu_{w^{-1}}(x)$. This map is called the Brooks quasimorphism for $w$. The family of these maps were introduced by Brooks in [7] to show that the vector space of quasimorphisms is infinite dimensional. We will generalize Brooks quasimorphisms from free groups to amalgamated free products and HNN extensions in Subsection 4.1.

2.3 Spectral gaps in scl

We summarize some known methods and results on $\text{scl}$ spectral gaps.

Definition 2.16. We say a group $G$ has a spectral gap $C > 0$ for elements (respectively, integral chains) if $\text{scl}_G(c) \notin (0, C)$ for all elements (respectively, integral chains) $c$ in $G$.

The spectral gap property can be used to obstruct certain homomorphisms using monotonicity of $\text{scl}$ (Proposition 2.6). A gap result for integral chains can also be used to estimate index of certain kinds of subgroups using the index formula (Proposition 2.9).

There are two main approaches to prove spectral gap results in a group $G$.

In light of Theorem 2.11 one approach is to construct for a given element $g$ (respectively, chain $c$) a homogeneous quasimorphism $\phi_g$ (respectively, $\phi_c$) of unit defect such that $\phi_g(g) \geq C$ (respectively, $\phi_c(c) \geq C$) for a uniform $C > 0$. However, it is notoriously difficult to construct these
maps which witness the optimal gap. For the free group, only two such constructions are available [14, 26].

The other approach is to give a uniform lower bound of the complexity of all admissible surfaces. This is usually done by first simplifying admissible surfaces (sometimes in the language of disk diagrams) into certain normal form and then making use of a particular structure of the normal form; See, for instance, [12, 14, 17, 18, 20, 29].

Here we list some known spectral gap results for elements in Theorem 2.17 and for chains in Theorem 2.18. The list is by no means extensive.

**Theorem 2.17.**

1. (Calegari–Fujiwara [10, Theorem A]) Any \(\delta\)-hyperbolic group with a generating set \(S\) has a spectral gap \(C = C(|S|, \delta)\) for elements. Moreover, an element \(g\) has \(\text{scl}_C(g) = 0\) if and only if \(g^n\) is conjugate to \(g^{-n}\) for some \(n \in \mathbb{Z}_+\).

2. (Bestvina–Bromberg–Fujiwara [4, Theorem B]) Let \(G\) be a finite index subgroup of the mapping class group \(\text{Mod}(\Sigma)\) of a possibly punctured closed orientable surface \(\Sigma\). Then \(G\) has a spectral gap \(C(G)\) for elements.

3. (Chen–Heuer [14, Theorem C]) For any orientable 3-manifold \(M\), its fundamental group has a spectral gap \(C(M)\) for elements.

4. (Heuer [26, Theorem 7.3]) Any (subgroup of a) RAAG has a spectral gap \(1/2\) for elements. Moreover, any nontrivial element has positive \(\text{scl}\). A new topological proof is given in [14]. Weaker results are obtained in [19, 20].

5. (Clay–Forester–Louwsma [15, Theorem 6.9]) Let \(\{G_v\}\) be a family of groups with a uniform gap for elements. Then their free product also has a spectral gap for elements.

6. (Chen–Heuer [14, Theorem F]) Let \(\{G_v\}\) be a family of groups without 2-torsion such that they have a uniform gap for elements. Then their graph product also has a spectral gap for elements. The assumption on 2-torsion is unnecessary by our Theorem 6.2.

**Theorem 2.18.**

1. (Calegari–Fujiwara [10, Theorem A ‘]) Any \(\delta\)-hyperbolic group with generating set \(S\) has a spectral gap \(C = C(|S|, \delta)\) for integral chains. Moreover, an integral chain has zero \(\text{scl}\) if and only if it is equivalent to the zero chain. The following families of hyperbolic groups have uniform gaps even though the numbers of generators are unbounded.

2. (Tao [36, Theorem 1.1]) Any free group has a spectral gap \(C = 1/8\) for integral chains.

3. (Chen–Heuer [14, Proposition 9.1]) Free products of cyclic groups have a spectral gap \(C = 1/12\) for integral chains. This is sharp for \(\mathbb{Z}/2 \ast \mathbb{Z}/3\).

4. (Chen–Heuer [14, Theorem 9.5]) There is a uniform constant \(C > 0\) such that the orbifold fundamental group of any closed hyperbolic 2-dimensional orbifold has a spectral gap \(C\) for integral chains.

Note by Proposition 2.14 that groups with spectral gaps for chains is closed under direct products. Corollary 6.4 generalizes this to graph products. The authors are unaware of any groups that were previously known to have a spectral gap for chains other than direct products of hyperbolic groups.
2.4 | Amalgamated free products

Let $G = A \star_C B$ be the amalgamated free product of groups $A$ and $B$ over a subgroup $C$. For any $g \in G \setminus C$, we may write

$$g = w_1 \cdots w_n,$$

where $w_i \in A \setminus C$ or $w_i \in B \setminus C$ for all $i \in \{1, \ldots, n\}$ such that the $w_i$’s alternate between $A \setminus C$ and $B \setminus C$.

**Remark 2.19.** In an amalgamated free product $G = A \star_C B$ we use text font (for example, $a, b$) to denote elements of $A \setminus C$ or $B \setminus C$. We refer to those elements as vertex elements. Ordinary roman letters (for example, $a, b$) denote generic elements in $G$.

**Definition 2.20** (Cyclically reduced form for amalgamated free products). We say that for an element $g \in G \setminus C$ the expression (2.1) is the reduced form of $g$. We define the length of $g$ as $n$ and denote it by $|g|$. Given the normal form (2.1) a prefix of $g$ is an element $h \in G \setminus C$ with normal form $h = w_1 \cdots w_m$ where $0 \leq m < n$.

If $w_i$ and $w_n$ as in the reduced form (2.1) lie in different sets $A \setminus C$ and $B \setminus C$, then we say that $g$ is cyclically reduced.

For $x_1, \ldots, x_m \in G \setminus C$ we say that the expression $g = x_1 \cdots x_m$ is a reduced decomposition of $g$ if there are reduced forms of each $x_i$ such that their concatenation is a reduced form of $g$. Observe that $g$ is cyclically reduced if and only if the expression $g \cdot g$ is reduced.

The reduced forms of an element are unique up to multiplication by $C$:

**Proposition 2.21** (Reduced form for amalgamated free products [34]). Let $G = A \star_C B$ be an amalgamated free product and suppose that

$$w_1 \cdots w_n = w'_1 \cdots w'_n,$$

where all $w$ terms alternate between $A \setminus C$ and $B \setminus C$. Then $n = n'$ and there are elements $d_0, \ldots, d_n \in C$ with $d_0 = e = d_n$ such that $w_i = d_{i-1} w_i' d_i^{-1}$ for all $i \in \{1, \ldots, n\}$.

**Corollary 2.22.** Let $G = A \star_C B$ be an amalgamated free product. Suppose that

$$x_1 \cdots x_n = x'_1 \cdots x'_n$$

are two reduced decompositions such that $|x_i| = |x'_i|$ for all $i \in \{1, \ldots, n\}$. Then there are elements $d_0, \ldots, d_n \in C$ with $d_0 = e = d_n$ such that $x_i = d_{i-1} x_i' d_i^{-1}$ for all $i \in \{1, \ldots, n\}$.

We will also need the following result later.

**Proposition 2.23.** Let $g, h \in G$ be two elements. Then there are elements $y_1, y_2, y_3 \in G$ in reduced form and vertex elements (see Remark 2.19) $x_1, x_2, x_3 \in G$ such that

$$g = y_1^{-1} x_1 y_2.$$
We denote such types vertex elements and denote them with text font, for example, \(a, b\);

(2) the possible types of any pair \((w_i, w_{i+1})\) are indicated by the oriented edges in Figure 2.

Note that for any \(c_1, c_2 \in C\), the word \(c_1w_ic_2\) can be rewritten into one of the same type as \(w_i\), for instance, \(c_1 \cdot t^{-1}a't \cdot c_2 = t^{-1}a''t\) with \(a'' = \phi(c_1)a'\phi(c_2)\).
**Definition 2.24** (cyclically) reduced form for HNN extensions. We say an expression as in (2.2) is a reduced form of \( g \). Define the length of \( g \) to be \( n \) in (2.2), denoted as \(|g|\). Given the reduced form (2.2) of \( g \), a prefix of \( g \) is some \( h = w_1 \cdots w_m \) with \( 0 \leq m < n \). We say that \( g \) is cyclically reduced if the reduced expression \( g = w_1 \cdots w_n \) satisfies in addition that \((w_n, w_1)\) is as in Figure 2 and we call such an expression a cyclically reduced form. We say that \( h \) is a cyclic conjugate of \( g \) if the reduced form of \( h \) is a cyclic permutation of the reduced form of \( g \).

For a reduced element \( g \), we say an expression \( g = x_1 \cdots x_m \) is a reduced decomposition if there are reduced forms of each \( x_i \) so that the concatenation is a reduced form of \( g \). Observe that \( g \) is cyclically reduced if and only if \( g \cdot g \) is a reduced decomposition.

Reduced forms of a given element \( g \) is essentially unique:

**Lemma 2.25** (Britton’s lemma [30]). Let \( G = A \ast_C \) be an HNN extension and suppose that
\[
w_1 \cdots w_n = w'_1 \cdots w'_{n'}\]
are two reduced forms of \( g \in G \setminus C \). Then \( n = n' \) and there are elements \( d_0, \ldots, d_n \in C \) with \( d_0 = e = d_n \) such that \( w_i = d_{i-1} w'_i d_{i-1}^{-1} \) for all \( i \in \{1, \ldots, n\} \).

From this we see that \(|g|\) does not depend on the choice of reduced forms, and a reduced decomposition \( g = x_1 \cdots x_m \) does not depend on the choice of reduced forms of \( x_i \)'s.

**Corollary 2.26.** Let \( G = A \ast_C \) be an HNN extension. Suppose that
\[
x_1 \cdots x_n = x'_1 \cdots x'_n\]
are two reduced decompositions such that \(|x_i| = |x'_i|\) for all \( i \in \{1, \ldots, n\} \). Then there are elements \( d_0, \ldots, d_n \in C \) with \( d_0 = e = d_n \) such that \( x_i = d_{i-1} x'_i d_{i-1}^{-1} \) for all \( i \in \{1, \ldots, n\} \).

**Proposition 2.27.** Let \( g, h \in G \) be two elements. Then there are elements \( y_1, y_2, y_3 \in G \) and vertex elements \( x_1, x_2, x_3 \in G \) such that
\[
g = y_1^{-1} x_1 y_2
\]
\[
h = y_2^{-1} x_2 y_3
\]
\[
(gh)^{-1} = y_3^{-1} x_3 y_1
\]
as reduced expressions, where \( y_i \) and \( x_i \) might be the identity.

**Proof.** The proof of Proposition 2.23 works verbatim, interpreting vertex elements and reduced forms in the HNN extension context. \( \square \)

### 2.6 Graphs of groups

We briefly introduce graphs of groups to state the results of Sections 4 and 5 more compactly. Graph of groups is a generalization of both amalgamated free products and HNN extensions discussed in the previous sections. We refer to [34] for details.
Let \( \Gamma \) be an oriented connected graph with vertex set \( V \) and edge set \( E \). Each edge \( e \in E \) is oriented with origin \( o(e) \) and terminus \( t(e) \). Denote the same edge with opposite orientation by \( \bar{e} \), which provides an involution on \( E \) satisfying \( t(\bar{e}) = o(e) \) and \( o(\bar{e}) = t(e) \).

A graph of groups with underlying graph \( \Gamma \) is a collection of vertex groups \( \{ G_v \}_{v \in V} \), edge groups \( \{ G_e \}_{e \in E} \), and injections \( t_e : G_e \to G_{t(e)} \), such that \( G_e = G_{\bar{e}} \). Fix a pointed \( K(\pi_1 G_v, 1) \) space \( X_v \) for each \( v \) and a pointed \( K(\pi_1 G_e, 1) \) space \( X_e \) for each \( e \). Each injection \( t_e \) determines a map \( X_e \to X_{t(e)} \), based on which we can form a mapping cylinder \( M_{e,t(e)} \), where we think of \( X_e \) and \( X_{t(e)} \) as the subspaces on its boundary. Glue all such mapping cylinders along their boundary by identifying \( X_v \) in all \( M_{e,v} \) (with \( t(e) = v \)) and identifying \( X_e \) with \( X_{\bar{e}} \) in \( M_{e,t(e)} \) and \( M_{\bar{e},t(\bar{e})} \).

We refer to the resulting space \( X \) as the graph of spaces associated to the graph of groups, where the image of each \( X_e \) is called an edge space. The fundamental group \( \pi_1(X) \) is called the fundamental group of the graph of groups. When there is no danger of ambiguity, we will simply refer to \( G \) as the graph of groups.

**Theorem 2.28** [34]. Every fundamental group of a graph of groups can be written as a sequence of amalgamated free products and HNN extensions over the edge groups.

### 3 | GRAPH PRODUCTS OF GROUPS

Graph products of groups generalize both right-angled Artin and right-angled Coxeter groups. They were introduced by Green in her thesis [21]. We go through some basic concepts and then establish the pure factor decomposition and the centralizer theorem (Theorem 3.7). We will need these results in Sections 6 and 7.

Let \( \Gamma \) be a finite simplicial graph with vertex set \( V(\Gamma) \) and edge set \( E(\Gamma) \) and let \( \{ G_v \}_{v \in V(\Gamma)} \) be a family of groups. Then, the graph product \( \mathcal{G}(\Gamma, \{ G_v \}_{v \in V(\Gamma)}) \) associated to these data is defined as the free product \( \star_v G_v \) of the vertex groups subject to the relations \( [g_v, g_w] \) for every \( g_v \in G_v, g_w \in G_w \) with \( (v, w) \in E(\Gamma) \). If the family of groups \( G_v \) is understood we will simply denote the group as \( \mathcal{G}(\Gamma) \).

When all \( G_v = \mathbb{Z} \) (respectively, \( \mathbb{Z}/2 \)), we refer to \( \mathcal{G}(\Gamma) \) as the right-angled Artin (respectively, Coxeter) group, denoted as \( A(\Gamma) \) (respectively, \( C(\Gamma) \)). We use RAAG(s) and RACG(s) as abbreviations for these two types of groups.

A normal form of elements is developed in [21]. Every element \( g \in \mathcal{G}(\Gamma) \) can be written as a product \( g_1 \cdots g_n \) where each \( g_i \) is in some vertex group. Following [21, Definition 3.5], we say that \( n \) is the syllable length in such an expression. There are three types of moves on the set of words representing the same element.

- **(Syllable shuffling)** if there is a subsequence \( g_i \cdots g_j \) with \( 1 \leq i < j \leq n \) and \( g_j \in G_{v_j} \) such that every \( g_k \) lies in a vertex group \( G_{v_k} \) and \( v_k \) is adjacent to \( v_j \) for all \( i < k < j \), then we can replace it by \( g_i g_j g_{i+1} \cdots g_{j-1} \), and similarly if every \( v_k \) is adjacent to \( v_i \).
- **(Merging)** if two consecutive letters \( g_i, g_{i+1} \) lie in the same vertex group \( G_v \), we can merge them into a single letter \( g_i g_{i+1} \in G_v \).
- **(Deleting)** if some \( g_i = 1 \), then we can delete it.

Note that syllable shuffling preserves the syllable length while the other two moves reduce it.

We say that an expression \( g_1 \cdots g_n \) is reduced if

- each \( g_i \) is nontrivial, and
there is no subsequence \( g_i \cdots g_j \) with \( 1 \leq i < j \leq n \) such that \( g_i, g_j \) lie in the same vertex group \( G_v \), and every \( g_k \) lies in a vertex group \( G_{v_k} \) with \( v_k \) adjacent to \( v \) for all \( i < k < j \).

**Lemma 3.1** [21, Theorem 3.9]. Every element \( g \in G(\Gamma) \) can be written as a reduced expression. This expression has minimal syllable length, and is unique up to syllable shufflings.

The minimal syllable length of words representing \( g \) is denoted \(|g|\), which is achieved by a word representing \( g \) if and only if the word is reduced.

Similarly, a word is (proper) cyclically reduced if every cyclic permutation of its letters is reduced.

**Lemma 3.2** (Proof of [21, Theorem 3.24]). Every conjugacy class in \( G(\Gamma) \) contains an element represented by a cyclically reduced word. Any two cyclically reduced words in the same conjugacy class differ by a cyclic permutation of the letters and syllable shuffling.

Given a reduced expression \( g = g_1 \cdots g_n \), its support is the induced subgraph consisting of vertices \( v \) such that some \( g_i \) lies in \( G_v \). Since syllable shuffling does not change the support, by Lemma 3.1, the support does not depend on the choice of reduced expressions. We denote it by \( \text{supp}(g) \).

For an element \( g \in G(\Gamma) \) some conjugate \( \tilde{g} = p^{-1} gp \) is represented by a cyclically reduced word. By Lemma 3.2, the support \( \text{supp}(\tilde{g}) \) does not depend on the choice of \( p \) and we set \( \Theta(g) := \text{supp}(\tilde{g}) \).

The process of putting a word into a cyclically reduced word in the same conjugacy class does not enlarge the support (see the proof of [21, Theorem 3.24]), thus \( \Theta(g) \) is the smallest support of elements in the conjugacy class of \( g \).

**Lemma 3.3.** For any \( g \in G(\Gamma) \), we have \( \Theta(g) \subset \text{supp}(g) \).

Given two elements \( g, h \in G(\Gamma) \), we can relate the normal form of \( g \cdot h \) to the reduced expressions of \( g, h \) as follows.

**Proposition 3.4.** For any elements \( g, h \in G(\Gamma) \), there is a (possibly empty) clique \( q = \{v_1, \ldots, v_k\} \) for some \( k \geq 0 \) such that we may write \( g = g_0 q g_x \) and \( h = x^{-1} q_h h_0 \) as reduced expressions with \( q_g = g_1 \cdots g_k \) and \( q_h = h_1 \cdots h_k \) with \( g_i, h_i \in G_{v_i} \) and none of \( g_i, h_i, g_i \cdot h_i \) is the identity for all \( i \in \{1, \ldots, k\} \) such that a reduced expression for \( gh \) is given by

\[
g \cdot h = g_0 \cdot q_{gh} \cdot h_0,
\]

where \( q_{gh} \) is given by \( q_{gh} = (g_1 h_1) \cdots (g_k h_k) \).

**Proof.** Given \( g, h \in G(\Gamma) \) as in the proposition, choose \( x \) to be a word with the maximal syllable length such that \( g = g' x \) and \( h = x^{-1} h' \) are reduced expressions for some words \( g', h' \). Given \( g' \) and \( h' \), choose \( q_g \) and \( q_h \) to be words with maximal syllable length such that the support of \( q_g \) and \( q_h \) is a clique \( q = \{v_1, \ldots, v_k\} \) and we can write \( g' = g_0 q_g, h' = q_h h_0 \) as reduced expressions for some words \( g_0, h_0 \). Define \( q_{gh} \) as in the proposition. By the maximality of \( x \), none of the terms in \( q_g \) and \( q_h \) cancel and thus the support of \( q_{gh} \) is also equal to \( q \). Note that for the expression

\[
g_0 \cdot q_{gh} \cdot h_0,
\]
$g_0 \cdot q_{gh}$ is reduced since $g' = g_0 \cdot q_g$ is reduced and has the same support. Similarly, $q_{gh} \cdot h_0$ is reduced. Finally, one cannot shuffle a letter in $g_0$ to merge with another in $h_0$ since this would contradict the maximality of $g_q$ and $h_q$ by Lemma 3.1. Thus, $g_0 \cdot q_{gh} \cdot h_0$ is a reduced expression for $g \cdot h$.

Next we introduce the pure factor decomposition. For a graph $\Gamma$, the opposite graph $\Gamma^{opp}$ is the graph with the same vertices as $\Gamma$ and where two vertices are adjacent if and only if they are not adjacent in $\Gamma$.

Let $C^*_1, \ldots, C^*_\ell$ be the connected components of $\Theta(g)^{opp}$ each consisting of a single vertex and let $C_1, \ldots, C_\ell$ be the connected components of $\Theta(g)^{opp}$ with more than one vertices. Letters of $\tilde{g}$ in different components can be shuffled across. By combining letters in the same component via shuffling, we can write $\tilde{g}$ as $\gamma^*_1 \cdots \gamma^*_\ell \cdot g_1 \cdots g_\ell$ with supp($\gamma^*_i$) = $C^*_i$ and supp($g_i$) = $C_i$. Then it is easy to see that every $g_i$ is cyclically reduced.

Now write $g_i = \gamma^{e_i}_i$ such that $e_i \in \mathbb{Z}_+$ and $\langle \gamma_i \rangle$ is maximal cyclic. Such an expression exists by [2, Corollary 47]. We get

$$g = p \cdot \gamma^*_1 \cdots \gamma^*_\ell \cdot \gamma^{e_1}_1 \cdots \gamma^{e_\ell}_\ell \cdot p^{-1}. \quad (3.1)$$

**Definition 3.5** (Pure factor decomposition, pure factors, and primitive pure factors). For any element $g$, an expression (3.1) is called a pure factor decomposition of $g$, where each $\gamma^*_i$ and $g_i = \gamma^{e_i}_i$ is called a pure factor of $g$.

If $g = g^{e_1}$ with $e_1 = 1$ is its own pure factor decomposition and $|g| \geq 2$, then $g$ is called a primitive pure factor.

**Lemma 3.6.** Each pure factor of $g$ is unique up to cyclic conjugation. The set of pure factors of $g$ up to cyclic conjugation is uniquely determined by $g$.

**Proof.** This directly follows from Lemma 3.2 and the fact that letters in different pure factors commute with each other. \qed

### 3.1 Centralizers in graph products

The goal of this subsection is to describe the centralizer of any element $g$ in a graph product $G(\Gamma)$.

Recall that $\Theta(g)$ is the support of any cyclically reduced representative of $g$, that $C^*_1, \ldots, C^*_\ell$ denote the connected components of $\Theta(g)^{opp}$ that each consists of a single vertex and that $C_1, \ldots, C_\ell$ denote the connected components of $\Theta(g)^{opp}$ containing more than one vertex. Finally, let $D(g)$ be the subset of $V(\Gamma) \setminus V(\Theta(g))$ consisting of vertices which are adjacent to every vertex of $\Theta(g)$.

The following result fully characterizes the centralizer of an element in terms of the pure factors.

**Theorem 3.7** (Centralizer theorem). Let $g \in G(\Gamma)$ be an element with pure factor decomposition

$$g = p \cdot \gamma^*_1 \cdots \gamma^*_\ell \cdot \gamma^{e_1}_1 \cdots \gamma^{e_\ell}_\ell \cdot p^{-1},$$

where $p = p_1 \cdots p_\ell$ and $\gamma_1, \ldots, \gamma_\ell$ is a pure factor decomposition of $g$.
where \( \text{supp}(\gamma_i^*) = C_i^* \), \( \text{supp}(\gamma_i) = C_i \) and let \( D(g) \) be defined as above. Then an element \( h \in G(\Gamma) \) commutes with \( g \) if and only if

\[
h = p \cdot \xi_1^* \cdots \xi_n^* \cdot \gamma_1^{f_1} \cdots \gamma_\ell^{f_\ell} \cdot z \cdot p^{-1},
\]

where \( \xi_i \) lies in the centralizer \( Z_{C_i^*}(\gamma_i^*) \), where \( G_i^* \) is the vertex group of \( C_i^* \), \( f_i \in \mathbb{Z} \), and \( \text{supp}(z) \subset D(g) \).

This generalizes several similar results: In the case where \( g \) is itself a single pure factor, this is proved by Barkauskas [2, Theorem 53]. In the case of RAAGs this has been done by Droms–Servatius–Servatius [35] and in the case of graph products of abelian groups this has been done by Corredor–Gutierrez [16, Centralizer Theorem].

**Lemma 3.8.** If \( g \) is cyclically reduced with pure factor decomposition

\[
g = \gamma_1^* \cdots \gamma_n^* \cdot \gamma_1^{e_1} \cdots \gamma_\ell^{e_\ell},
\]

where \( \text{supp}(\gamma_i^*) = C_i^* \), \( \text{supp}(\gamma_i) = C_i \) and the set \( D(g) \) is defined as above. If \( h \in G(\Gamma) \) commutes with \( g \) and is supported on \( \Theta(g) \cup D(g) \), then

\[
h = \xi_1^* \cdots \xi_n^* \cdot \gamma_1^{f_1} \cdots \gamma_\ell^{f_\ell} \cdot z,
\]

where \( \xi_i \in Z_{C_i^*}(\gamma_i^*) \), where \( G_i^* \) is the vertex group of \( C_i^* \), \( f_i \in \mathbb{Z} \), and \( \text{supp}(z) \subset D(g) \).

**Proof.** Recall that every vertex in \( D(g) \) is adjacent to all vertices in \( \Theta(g) \) and that letters supported on different components of \( \Theta(g) \) commute with each other. So, we can express \( h \) as a reduced expression \( h = h_1^* \cdots h_n^* \cdot h_1 \cdots h_\ell \cdot z \), where each \( h_i^* \) (respectively, \( h_i \)) is a reduced word with \( \text{supp}(h_i^*) \subset C_i^* \) (respectively, \( \text{supp}(h_i) \subset C_i \)), and \( z \) is a reduced word with \( \text{supp}(z) \subset D(g) \). Then

\[
hgh^{-1} = \prod h_i^* \gamma_i^*(h_i^*)^{-1} \cdot \prod h_i \gamma_i^{e_i} h_i^{-1}.
\]

Since different factors have disjoint support, we observe that \( hgh^{-1} = g \) if and only if \( h_i^* \gamma_i^*(h_i^*)^{-1} = \gamma_i^* \) and \( h_i \gamma_i^{e_i} h_i^{-1} = \gamma_i^{f_i} \) for each \( i \).

This reduces the problem to the case of a single pure factor. Hence, by [2, Theorem 53], we must have \( h_i^* \in Z_{C_i^*}(\gamma_i^*) \) where \( G_i^* \) is the group associated to the vertex \( C_i^* \) and \( h_i = \gamma_i^{f_i} \) for some \( f_i \in \mathbb{Z} \).

**Lemma 3.9.** Suppose \( g \) and \( h \) are reduced words where the last letter \( h_v \) of \( h \) lies in \( G_v \) for some vertex \( v \not\in \text{supp}(g) \) that is not adjacent to some \( u \in \text{supp}(g) \). Then \( g \) and \( h \) do not commute.

**Proof.** Express \( G(\Gamma) \) as an amalgam \( A \star_C B \), where \( A = G(\text{St}(u)) \), \( B = G(\Gamma \setminus \{u\}) \), and \( C = G(\text{Lk}(u)) \). Here \( \text{St}(u) \) and \( \text{Lk}(u) \) denote the star and the link of \( u \) in \( \Gamma \), respectively. For a reduced expression \( g = g_1 \cdots g_n \), we can pick out letters in \( G_u \) to obtain \( g_1 \cdots g_n = b_0 g_k b_1 \cdots b_{s-1} g_k b_s \), where each \( g_k \in G_u \) and each \( b_k \) is the product of letters outside \( G_u \) sitting in between \( g_k \) and \( g_{k+1} \). Note that \( b_k \in B \setminus C \) for all \( k \neq 0, s \) since we start with a reduced expression of \( g \). If \( b_0 \in C \),
then we can shuffle it across $a_1$. Thus, we assume either $b_0 \in B \setminus C$ or $b_0 = id$. The same can be done for $b_s$, except for the case where $s = 1$ and both $b_0, b_s \in C$, in which we may assume one of them to be the identity.

In summary, this naturally expresses $g$ as a reduced word $g = b_0a_1b_1 \cdots b_{s-1}a_sb_s$ in the amalgam $A \ast_C B$, where $s \geq 1$ and $a_k = g_{k} \in G_v \subset A \setminus C$ and $b_k \in B \setminus C$ for each $k$, except that possibly $b_0, b_s = id$, or one of them is the identity and the other lies in $C$ when $s = 1$.

Similarly we have $h = \beta_0 \alpha_1 \beta_1 \cdots \alpha_t \beta_t$ for some $t \geq 0$, where each $\alpha_i \in A \setminus C$ and $\beta_i \in B \setminus C$ except possibly $\beta_0 = id$. Note that we must have $\beta_i \in B \setminus C$ since $h_v$ is the last letter of $h$ as a reduced word in the graph product and $v \notin St(u)$.

As words in the amalgam, we have

$$gh = b_0 \cdots a_1 \cdots (b_s \beta_0) \alpha_1 \cdots \alpha_t \beta_t,$$

$$hg = \beta_0 \alpha_1 \cdots \alpha_t (\beta_s \beta_0) \cdots a_1 \cdots a_s b_s.$$ 

Since $\beta_t$ as a reduced word in the graph product contains $h_v$ and $v \notin supp(g)$, while $supp(b_0) \subset supp(g)$, we know $\beta_t \cdot b_0 \in B \setminus C$. Thus, $hg$ is a reduced word in the amalgam except that possibly $\beta_0, b_s = id$.

If $gh = hg$, when written as reduced words in the amalgam they must have the same length and start and end on elements in the same factor groups (that is, $A$ and $B$). There are eight cases depending on whether $b_0, b_s, \beta_0 \in C$, but there are only two cases where $gh$ and $hg$ can be written as reduced words of the same type and length:

1. $b_0 = \beta_0 = id$ and $b_s \notin C$, or
2. $b_0, b_s, \beta_0 \notin C$, where $b_s \beta_0 \notin C$.

In both cases, $hg$ ends with $b_s$ and $gh$ ends with $\beta_t$ (or $b_s \beta_0$ when $t = 0$). If $gh = hg$, then we must have $\beta_t \in Cb_sC$ (or $\beta_0 \in b_s^{-1}Cb_sC$ when $t = 0$). Any element in $Cb_sC$ (or $b_s^{-1}Cb_sC$) as a word in the graph product is supported on $supp(g) \cup St(u)$, however, $\beta_t$ contains $h_v$ and $v \notin supp(g) \cup St(u)$. This is a contradiction. Hence, $gh \neq hg$. 

\begin{lemma}
Suppose $g$ is cyclically reduced. Let $D(g)$ be the set of vertices outside $supp(g)$ and adjacent to all those in $supp(g) = \Theta(g)$ as above. If $h \in \mathcal{G}(\Gamma)$ commutes with $g$, then $h$ is supported on $\Theta(g) \cup D(g)$.
\end{lemma}

\begin{proof}
Write $h$ in a reduced expression. Denote $supp(g) \cup D(g)$ by $\Delta$ and suppose $supp(h) \notin \Delta$. Let $h_v$ be the last letter in $h$ with the property that $h_v \in G_v$ for some $v \notin \Delta$. Then $h_v$ cuts $h$ into a reduced expression $h_p h_v h_s$, where $supp(h_s) \subset \Delta$. As vertices in $D(g)$ are adjacent to all vertices in $\Theta(g)$, by shuffling letters of $h_s$ in $D(g)$ to the end, we may represent $h_s = h'_s h_z$ as a reduced word so that $supp(h'_s) \subset \Theta(g)$ and $supp(h_z) \subset D(g)$.

As $h_z$ commutes with $g$, we know $h' = h'_p h_v h'_s$ also commutes with $g$, and $h_v$ is also the last letter in $h'$ supported outside $\Delta$. Then the conjugate $h'_s h_p h_v$ must commute with $g' = h'_s g(h'_s)^{-1}$. Note that $supp(g') = \Theta(g)$ by Lemma 3.3 since we know $supp(g') \subset supp(g) = \Theta(g)$ as $supp(h'_s) \subset \Theta(g)$. Applying Lemma 3.9 to $h'_s h_p h_v$ and $g'$ we get a contradiction. Thus, we must have $supp(h) \subset \Delta$.
\end{proof}

Now we prove Theorem 3.7.
Proof of Theorem 3.7. Since the centralizer of $p^{-1}g p$ is $p^{-1}Z_{\mathcal{G}(\Gamma)}(g)p$, it suffices to prove the theorem assuming $g = \tilde{g}$ is cyclically reduced, that is, $p = id$. Then by Lemma 3.10, any $h$ commuting with $g$ must be supported in $\Theta(g) \cup D(g)$. Thus, the result follows from Lemma 3.8.

Definition 3.11 (Pure factor chain). Suppose that $g \in \mathcal{G}(\Gamma)$ has an associated pure factor decomposition

$$g = p \cdot \gamma_1^s \cdots \gamma_{e_1}^s \cdot \gamma_1^{e_1} \cdots \gamma_{e_{\ell}}^{e_{\ell}} \cdot p^{-1},$$

where $\gamma_i^s$ and $\gamma_i$ and $e_i$ are as in Equation (3.1). Then we define the associated pure factor chain $g^{\text{pf}}$ of $g$ initially as

$$g^{\text{pf}} = \gamma_1^s + \cdots + \gamma_{e_{\ell}}^s + e_1 \gamma_1 + \cdots e_{\ell} \gamma_{e_{\ell}},$$

and then remove $e_i \gamma_i$ (respectively, $\gamma_i^s$) if $\gamma_i$ (respectively, $\gamma_i^s$) is conjugate to its inverse.

For an integral chain $c = \sum_{i=1}^n c_i$ we define the associated pure factor chain $c^{\text{pf}}$ as follows: Set $c^1 = \sum_{i=1}^n c_i^{\text{pf}}$. If there is a term $g_i^{-1}$ and $h_i g_i h_i^{-1}$ for some $g_i, h_i \in \mathcal{G}(\Gamma)$ in $c^1$, define $c^2$ as the chain $c^1$ without $g_i^{-1}$ and $g_i g_i h_i^{-1}$. If $c^1$ is defined but still has terms $g_i^{-1}$ and $h_i g_i h_i^{-1}$ for some $g_i, h_i \in \mathcal{G}(\Gamma)$, define $c^{i+1}$ as $c^i$ without $g_i^{-1}$ and $h_i g_i h_i^{-1}$. Every such step reduces the number of terms by two, and thus, this process will eventually stop. We call the resulting chain the pure factor chain $c^{\text{pf}}$ associated to $c$. Note that $c^{\text{pf}}$ is equivalent to $c$.

By Lemma 3.6, the pure factor chains for different pure factor decompositions are equivalent in the sense of Definition 2.4.

Proposition 3.12. Let $c$ be an integral chain in $\mathcal{G}(\Gamma)$ equivalent (Definition 2.4) to a chain that consists of terms just supported on the vertex groups. Let $c^{\text{pf}}$ be a pure factor chain. Then $c^{\text{pf}}$ consists of terms which are just supported on vertex groups.

Proof. Let $\gamma \in \mathcal{G}(\Gamma)$ be a primitive pure factor (Definition 3.5) so that $\gamma$ is not conjugate to $\gamma^{-1}$. For any element $g \in \mathcal{G}(\Gamma)$ we define $\sigma_\gamma(g) = n$ if $\gamma^n$ up to cyclic conjugation is a pure factor of $g$ for some $n \in \mathbb{Z} \setminus \{0\}$. This is well-defined by Lemma 3.6. The number $n$ is uniquely determined since $\gamma^n$ is cyclically reduced and has length $|n||\gamma|$, and $\gamma$ is not conjugate to $\gamma^{-1}$. Set $\sigma_\gamma(g) = 0$ if no conjugate of $\gamma^n$ for any $n$ is a pure factor of $g$.

For a chain $c = \sum_{i \in I} \lambda_i c_i$ set $\sigma_\gamma(c) := \sum_{i \in I} \lambda_i \sigma_\gamma(c_i)$.

Claim 3.13. If $c$ and $c'$ are equivalent chains (Definition 2.4). Then $\sigma_\gamma(c) = \sigma_\gamma(c')$.

Proof. It suffices to show that $\sigma_\gamma(c) = 0$ for each basis element $c$ in $E(G)$ as in Definition 2.4. Apparently $\sigma_\gamma(g) = \sigma_\gamma(pg^{-1}p^{-1})$ since the pure factors of $g$ up to cyclic conjugation only depends on the conjugacy class of $g$. The fact that $\sigma_\gamma(g^n) = n \sigma_\gamma(g)$ for all $n \in \mathbb{Z}$ follows from the definition.

It remains to show that $\sigma_\gamma(x_1) + \sigma_\gamma(x_2) = \sigma_\gamma(x_1 x_2)$ for two commuting elements $x, x_2 \in G$. If $\sigma_\gamma(x_1) = \sigma_\gamma(x_2) = \sigma_\gamma(x_1 \cdot x_2) = 0$, then the result trivially holds.

Without loss of generality, assume that $\sigma_\gamma(x_1) \neq 0$. Let

$$x_1 = p \cdot \gamma_1^s \cdots \gamma_{e_1}^s \cdot \gamma_1^{e_1} \cdots \gamma_{e_{\ell}}^{e_{\ell}} \cdot p^{-1}$$
be the pure factor decomposition of $x_1$ with $\gamma_1 = \gamma$. Then $x_2$ has to be of the form

$$x_2 = p \cdot \xi_1^* \cdots \xi_{\ell}^* \cdot \gamma_1^{f_1} \cdots \gamma_{\ell}^{f_\ell} \cdot z \cdot p^{-1}$$

by Theorem 3.7. Note by the definition of $D(g)$ that $\text{supp}(z)$ is disjoint from the support of any $\gamma_i^*$ and $\gamma_i$. Thus, $z$ does not contribute to $\sigma_\gamma(x_2)$ and hence $\sigma_\gamma(x_2) = f_1$. For the same reason, we have $\sigma_\gamma(x_1x_2) = e_1 + f_1$ from the expression

$$x_1x_2 = p \cdot (\gamma_1^{e_1} \xi_1) \cdots (\gamma_{\ell}^{e_\ell} \xi_{\ell}^*) \cdot \gamma_1^{f_1} \cdots \gamma_{\ell}^{f_\ell} \cdot z \cdot p^{-1}.$$ 

Thus, $\sigma_\gamma(x_1x_2) = e_1 + f_1 = \sigma_\gamma(x_1) + \sigma_\gamma(x_2)$. This shows the claim. □

To conclude the proof of Proposition 3.12, let $c$ be an integral chain which is equivalent to a chain $c'$ where every term is supported on a vertex. Let $c_{pf}$ be a pure factor chain associated to $c$. If $c_{pf}$ has a term $\gamma$ which is not supported on vertices, then it is not conjugate to its inverse as such terms are removed in the beginning of the construction of $c_{pf}$. This term gives us a primitive pure factor $\gamma$ such that $\sigma_\gamma(c_{pf}) \neq 0$ since the number of terms in $c_{pf}$ cannot be further reduced. On the other hand, we have $|\gamma| \geq 2$, since $\gamma$ is not supported in a vertex. Thus, $\sigma_\gamma(c') = 0$. This contradicts the above claim since $c$ and $c'$ are equivalent. □

We show in Corollary 6.18 that an element $g$ in a RACG has $\text{scl}(g) = 0$ if and only if $g$ is equivalent to the zero chain. Jing Tao asked us if this can be characterized more explicitly in the following form. We confirm this explicit characterization.

**Proposition 3.14.** For a RACG $C(\Gamma)$, an element $g$ is equivalent to the zero chain if and only if $g$ is conjugate to $g^{-1}$. Moreover, this is equivalent to $g = ab$ with $a^2 = b^2 = id$.

**Proof.** In any group, if $g = ab$ with $a^2 = b^2 = id$, then $g^{-1} = ba$ is conjugate to $g = ab$. It is also clear that if $g$ is conjugate to $g^{-1}$, then $g$ is equivalent to the zero chain.

Let $g$ be any element in a RACG $C(\Gamma)$ with pure factorization

$$g = p \cdot \gamma_1^* \cdots \gamma_{\ell}^* \cdot \gamma_1^{e_1} \cdots \gamma_{\ell}^{e_\ell} \cdot p^{-1}.$$ 

As each $\gamma_i^*$ necessarily has order 2 as it lies in a vertex group $\mathbb{Z}/2$, the pure factor chain $g_{pf} = \sum e_i \gamma_i$, where the summation runs over $i$ such that $\gamma_i$ that is not conjugate to $\gamma_i^{-1}$. By Proposition 3.12, if $g$ is equivalent to the zero chain, then $g_{pf}$ is literally the zero chain, so $\gamma_i$ is conjugate to $\gamma_i^{-1}$ for all $1 \leq i \leq \ell$. As $\gamma_i$'s and $\gamma_i^*$'s all commute, it follows that $g$ and $g^{-1}$ are conjugate.

Moreover, to see that $g = ab$ for some $a^2 = b^2 = id$, it suffices to show this for each $\gamma_i$ due to the commutativity. Each $\gamma_i$ is written as a cyclically reduced word $w$, and reversing the order of the letters gives a word $\bar{w}$ representing $\gamma_i^{-1}$, which must also be cyclically reduced. As these two cyclically reduced words represent the same conjugacy class since $\gamma_i$ is conjugate to $\gamma_i^{-1}$, by Lemma 3.2, we know up to syllable shuffling $w$ and $\bar{w}$, they differ by a cyclic permutation. That is, there is a reduced expression $uv$ equivalent to $w$ so that $vu$ is equivalent to $\bar{w}$, where $u$, $v$ are reduced words. It has the property that $uvvu = \gamma_i \cdot \gamma_i^{-1} = id$. By the following claim (with $n = 2$), we conclude that $u^2 = v^2 = id$ as desired.

**Claim 3.15.** For any $n \geq 1$, suppose both $u_1 u_2 \cdots u_n$ and $u_n \cdots u_2 u_1$ are reduced expressions in $C(\Gamma)$, where each $u_i$ is a reduced word. If $u_1 u_2 \cdots u_n \cdot u_n \cdots u_2 u_1 = id$, then $u_i^2 = id$ for all $i$. 
Proof. We proceed by induction on the total length \( \sum |u_i| \) of the word \( u_1 \cdots u_n \). The result is immediate if the total length is 1. Suppose the result holds when the total length is at most \( L - 1 \) for \( L \geq 2 \), and consider such an expression with total length \( L \). The expression \( u_1 u_2 \cdots u_n \cdot u_n \cdots u_2 u_1 \) must be reducible by assumption. Each \( u_i \) appears twice in the expression, we distinguish them by denoting the copy on the right as \( u'_i \) to avoid confusion.

As \( u_1 u_2 \cdots u_n \cdot u'_n \cdots u'_2 u'_1 \) is the product of two reduced expressions, it must be the case that some letter \( g_i \) in some \( u_i \) can be shuffled all the way to merge with a letter \( g_j \) in some \( u'_j \), where \( v \) is a vertex in \( \Gamma' \), and \( 1 \leq i, j \leq n \). We necessarily have \( g_i = g'_j \) as the vertex group \( G_v = \mathbb{Z}/2 \).

We first show \( i = j \). If \( i < j \), then \( u_j \) sits in between \( u_i \) and \( u'_j \), so the \( g_i \) in \( u_i \) can be shuffled across \( u_j \) which also contains a copy of \( g_i \), contradicting that \( u_1 \cdots u_n \) is reduced. If \( i > j \), then the \( g_i \) in \( u'_j \) can be shuffled across \( u'_i \) which contains a copy of \( g_i \), contradicting that \( u'_n \cdots u'_1 \) is reduced.

Now given \( i = j \), suppose \( u_i = g_1 \cdots g_k \) as a reduced word, where \( g_i \) is the generator of the vertex group \( G_{v_i} \) of some vertex \( v_i \). Then for some \( 1 \leq s, t \leq k \), the letter \( g_s = g_t \) in \( u_i \) can be shuffled across \( (g_{s+1} \cdots g_k)u_{i+1} \cdots u_n \cdot u'_n \cdots u'_{i+1}(g_1 \cdots g_{t-1}) \) to cancel with \( g_t = g'_s \) inside \( u'_i \), where \( v = v_s = v_t \). We must have \( s \geq t \) as otherwise \( g_s \) can be shuffled across \( g_{s+1} \cdots g_{t-1} \) inside \( u_i \) to cancel \( g_t \), contradicting that \( u_i \) is reduced.

First consider the case \( s > t \). Then \( g_s \) can be shuffled to the end of \( u_i \) and \( g_t \) can be shuffled to the head of \( u'_j = u_j \), so \( u_i \) is equivalent to \( v_i x \) as a reduced expression for some reduced word \( v_i \), where \( x = g_s = g_t \). It follows that

\[
 u_1 \cdots u_n = u_1 \cdots u_{i-1} x v_i x u_{i+1} \cdots u_n = u_1 \cdots u_{i-1} x v_i u_{i+1} \cdots u_n x
\]

as words equivalent up to syllable shuffling. It follows that the subword \( u_1 \cdots u_{i-1} x v_i u_{i+1} \cdots u_n \) of total length \( L - 1 \) is reduced, as part of the last reduced expression. Similarly,

\[
 u'_n \cdots u'_i = u'_n \cdots u'_{i+1} x v'_i x u'_{i-1} \cdots u'_1 = x u'_{i+1} v'_i u'_{i-1} \cdots u'_1
\]

as equivalent reduced words, and \( u'_n \cdots u'_{i+1} v'_i u'_{i-1} \cdots u'_1 \) is reduced, where \( v'_i = v_i \). Hence, by the induction hypothesis, as

\[
 (u_1 \cdots u_{i-1} x v_i u_{i+1} \cdots u_n)(u'_n \cdots u'_{i+1} v'_i u'_{i-1} \cdots u'_1)
 = (u_1 \cdots u_{i-1} x v_i u_{i+1} \cdots u_n x)(x u'_{i+1} v'_i u'_{i-1} \cdots u'_1)
 = u_1 u_2 \cdots u_{i-1} x (u_n \cdots u_{i+1}) v'_i = id,
\]

where we think of \( x \) and \( v_i \) both as reduced words in the reduced expression, we must have \( u_j^2 = id \) for all \( j \neq i \) and \( v_i^2 = id \), which implies \( u_i^2 = x v_i^2 x = x^2 = id \).

Now consider the remaining case \( s = t \). In this case, \( g_s = g_t \) commutes with all remaining letters in \( u_i \) as well as those in \( u_j \) for \( j > i \) by the same analysis as above. So, we can write \( u_i \) equivalently as \( v_i x \) and \( x v_i \), which are reduced expressions where \( x = g_s \) commutes with \( v_i \). By the same argument, \( u_1 \cdots u_{i-1} v_i u_{i+1} \cdots u_n \) is a reduced word of total length \( L - 1 \) as a subword of \( u_1 \cdots u_{i-1} v_i u_{i+1} \cdots u_n x \) and similarly \( u'_n \cdots u'_{i+1} v'_i u'_{i-1} \cdots u'_1 \) is reduced. As the product of them is the identity, the induction hypothesis implies \( u_j^2 = id \) for all \( j \neq i \) and \( v_i^2 = id \), which implies \( u_i^2 = v_i^2 x^2 = x^2 = id \).

\[\square\]
FIGURE 3  Pieces with or without a hole in the interior

4  GAPS FROM SHORT OVERLAPS

Let $G$ be a group splitting over a subgroup $C$, that is, $G$ is either an amalgam $A \star_C B$ or an HNN extension $A \ast_C$. In either case, $G$ is a graph of groups with a unique edge group $C$, realized as a graph of spaces $X$ with a single edge space.

Consider an integral chain $d = \sum g(i)$, where each $g(i) = w_1(i) \cdots w_L(i)$ is a cyclically reduced word and does not lie in the vertex groups. For any integral chain $d + d'$, where $d'$ is a sum of elements in vertex groups, any admissable surface $S$ of degree $n$ for $d + d'$ can be considered as an admissable surface for $d$ of the same degree with extra boundary components representing curves in vertex groups. This is called an admissable surface for $d$ relative to the vertex groups.

Then $S$ can be simplified into the simple normal form in the sense of [13, section 3.2], which does not increase $-\chi(S)$ and does not change the degree. This means that $S$ is obtained by gluing pieces together, where each piece is a polygon possibly containing a hole in the interior, with $2k$ sides alternating between arcs and turns for some $k \in \mathbb{Z}_+$; see Figure 3. Topologically, each piece is either a disk or an annulus. Turns are places that these pieces glue along, and arcs are part of $\partial S$. They carry labels that we describe as follows.

In the case of an amalgam, each piece is either supported in $A$ or $B$. If a piece is supported in $A$, then each arc is labeled by some $w_i(k) \in A \setminus C$, and each turn is labeled by some element $c \in C$, which we refer to as the winding number of the turn. The product of labels on the polygonal boundary of each piece supported in $A$ (respectively, $B$) defines a conjugacy class in $A$ (respectively, $B$), which is $id$ if and only if the piece is a disk (that is, has no hole inside).

In the case of an HNN extension, each piece is supported in the vertex group $A$. Each arc is labeled by some $w_i(k) \in A \setminus C$, and each turn is labeled by some element $c \in C$, the winding number of the turn. Recall from Subsection 2.5 that each $w_i(k)$ falls into one of four types. If a turn travels from some $w_i(k)$ to $w_j(\ell)$, then the possible types of $(w_i(k), w_j(\ell))$ are

\[(at \text{ or } a, t^{-1}a \text{ or } a) \text{ and } (t^{-1}a \text{ or } t^{-1}at, at \text{ or } t^{-1}at).\]

It follows that the product of labels on the polygonal boundary defines a conjugacy class in $A$. The conjugacy class is $id$ if and only if the piece is a disk.

In both cases, each disk piece has at least two turns since each $w_i(k) \notin C$.

Pieces are glued together along paired turns to form $S$. Here a turn from $w_i(k)$ to $w_j(\ell)$ with winding number $c \in C$ is uniquely paired with a turn from $w_{j-1}(\ell)$ to $w_{i+1}(k)$ with winding number $c^{-1}$ (Figure 4). The gluing guarantees that each boundary component of $S$ is labeled by a conjugate of $g(i)^k$ for some $k \in \mathbb{Z}_+$. The way we glue pieces together is encoded by the gluing graph $\Gamma_S$, where each vertex corresponds to a piece and each edge corresponds to a gluing of two paired turns. For each vertex $v$, let $d(v)$ be its valence in $\Gamma_S$, and let $\delta(v) = 1$ if the corresponding piece is a disk and $\delta(v) = 0$ otherwise (that is, for an annulus piece).
If we cap off the hole in each annulus piece in \( S \), then the surface deformation retracts to the graph \( \Gamma_S \). Recall that the Euler characteristic \( \chi(\Gamma_S) \) can be computed as \( \sum_v [1 - d(v)/2] \), so we have

\[
-\chi(S) = -\chi(\Gamma_S) + |V_A| = \sum_v [d(v)/2 - \delta(v)], \tag{4.1}
\]

where \( |V_A| \) is the number of annuli pieces. Note that \( d(v)/2 - \delta(v) \geq 0 \), and the equality holds if and only if \( v \) is a disk piece with two turns (that is, \( v \) has valence 2 in \( \Gamma_S \)).

**Theorem 4.1.** Suppose \( G \) is a group that splits over a subgroup \( C \). Let \( c = \sum_{i=1}^n g(i) \) be an integral chain in \( G \) where each term either lies in a vertex group or is cyclically reduced.

Fix an integer \( N \in \mathbb{Z}^+ \). Then either

\[
scl_G(c) \geq \frac{1}{12N}
\]

or for any cyclically reduced \( g = g(i), i \in \{1, \ldots, n\} \), we have

\[
g^N = h^k h' d,
\]

where

- \( h \) is a cyclically reduced word conjugate to \( g(j)^{-1} \) for some \( j \in \{1, \ldots, n\} \), and \( k \in \mathbb{Z}_{\geq 0} \),
- \( h' \) is a prefix of \( h \) and
- \( d \in C \).

**Proof.** Without loss of generality, assume \( g(1) \) is cyclically reduced and no such equations hold for \( g = g(1) \). We show \( scl_G(c) \geq \frac{1}{12N} \).

Start with any admissible surface \( S \) for \( c \) without sphere or disk components. For any large integer \( M \), there is a finite normal cover \( \tilde{S} \) of \( S \) where each component of \( \partial S \) covers some component of \( \partial S \) with degree greater than \( M \). In particular, this shows that, up to taking finite covers, any boundary component of \( S \) winding around \( g(1) \) represents \( g(1)^q r \) for some \( q, r \in \mathbb{Z}^+ \) where the remainder \( r \) is negligible compared to \( q \). Thus, in the following estimate, we will assume for simplicity that whenever a boundary component of \( S \) winds around \( g(1) \), it actually winds around \( g(1) \) some \( N \)-multiple of times.

Remove elements in \( c \) supported on vertex groups to obtain an integral chain \( c_0 \). Then as explained above, we can think of \( S \) as an admissible surface for \( c_0 \) relative to the vertex groups. Up to homotopy and compression, we can put \( S \) into the simple normal form, which does not affect the boundary; see [13, Lemma 3.7]. For each boundary component representing \( g(1)^q \), cut it into \( q \) segments, so that each segment is labeled by the cyclically reduced word representing \( g(1)^N \).
Each segment consists of $L_1 \cdot N$ distinct arcs (some with same labels) and thus witnesses $L_1 \cdot N$ pieces, some of which might be counted multiple times (since some arcs might lie on the same piece).

We claim that at least one of these pieces witnessed along a segment is represented by a vertex $v$ in the gluing graph $\Gamma_S$ such that $d(v)/2 - \delta(v) > 0$. If not, then each such a piece is a disk with two turns. Such rectangles glue to a long strip (see Figure 5), whose boundary shows that $g(1)^Nc_1wc_2 = id$ for some $c_1, c_2 \in C$, where $w$ is the word on the opposite side of $g(1)^N$ and must be a reduced subword of some $g(j)^m$ ($g(j)$ represents the loop that the boundary component on the opposite side of the strip maps onto). In algebraic terms, this implies an equation that should not exist by our assumption.

Therefore, for each segment $\sigma$ as above, we can choose a piece $v(\sigma)$ witnessed by $\sigma$ so that $d(v)/2 - \delta(v) > 0$. It is possible that $v(\sigma) = v(\sigma')$ for distinct segments $\sigma, \sigma'$. Thinking of such pieces as vertices on $\Gamma_S$, each $v = v(\sigma)$ either has $d(v) \geq 3$ or has $d(v) \leq 2$ and $\delta(v) = 0$. In the former case, such a vertex is witnessed by at most $d(v)$ segments, and hence each segment witnessing $v$ contributes at least $\frac{1}{d(v)} [d(v)/2 - \delta(v)] \geq \frac{d(v) - 2}{2d(v)} \geq 1/6$ to the right-hand side of Equation (4.1). In the latter case, such a vertex has $\delta(v) = 0$, and hence each segment witnessing $v$ contributes at least $\frac{1}{d(v)} [d(v)/2 - \delta(v)] = 1/2$ to the right-hand side of (4.1). Thus, in any case, each segment contributes at least $1/6$ to $-\delta(S)$, and the total number of such segments is $n/N$, where $n$ is the degree of $S$.

Hence, we obtain

$$\frac{-\chi(S)}{2n} \geq \frac{1}{6} \cdot \frac{n}{N} \cdot \frac{1}{2n} = \frac{1}{12N}.$$ 

Since $S$ is arbitrary, this gives the desired estimate. \hfill \Box

### 4.1 Proof of Theorem 4.1 using quasimorphisms

In this section, we will give an alternative proof to Theorem 4.1 using explicit quasimorphisms. The quasimorphisms will be similar to the counting quasimorphisms discovered by Brooks [7]; see also Example 2.15. For amalgamated free products, this is also similar to [15].

Let $G$ be an amalgamated free product or HNN extension which splits over a group $C$. Let $w \in G$ be a cyclically reduced element. Then we define $v_w : G \to \mathbb{N}$ as follows. For any $g \in G$ let $v_w(g)$ be the largest integer $n$ such that $g$ has reduced decomposition

$$g = g_0w_1g_1 \cdots w_ng_n,$$
where \( g \in G \) is possibly the empty word and \( w_i \in CwC \). We define \( \phi_w = \nu_w - \nu_{w^{-1}} \).

**Proposition 4.2.** The map \( \phi_w : G \to \mathbb{R} \) is a quasimorphism with defect \( D(\phi_w) \leq 3 \).

**Proof.** We need the following claim for the proof.

**Claim 4.3.** Let \( yxy' \) be a reduced expression where \( x \) is a vertex element. Then
\[
\nu_w(yxy') - \nu_w(y) - \nu_w(y') \in \{0, 1\}.
\]

**Proof.** Suppose \( \nu_w(y) = n \) with \( y = y_0w_1y_1 \cdots w_ny_n \) and \( \nu_w(y') = n' \) with \( y' = y'_0w'_1y'_1 \cdots w'_n'y'_n \), where \( w_i, w'_i \in CwC \). Then
\[
yxy' = y_0 w_1 y_1 \cdots w_n (y_n x y_0 w'_1 y'_1 \cdots w'_n y'_n)
\]
is a reduced decomposition and thus \( \nu_w(yxy') \geq \nu_w(y) + \nu_w(y') \).

On the other hand, suppose that \( \nu_w(yxy') = m \) and we have a reduced decomposition of \( yxy' \) that contains \( m \) disjoint copies of words in \( CwC \). We also have a reduced expression of \( yxy' \) induced from arbitrary reduced words representing \( y \) and \( y' \). By chopping up the second reduced expression so that subwords have lengths matching the first reduced decomposition, it follows from Corollaries 2.22 and 2.26 that all the subwords in the first expression representing elements in \( CwC \) give disjoint subwords of \( y \) or \( y' \), except when the subword intersects \( x \), which can occur for at most one subword. Thus,
\[
\nu_w(yxy') \leq \nu_w(y) + \nu_w(y') + 1,
\]
which shows the claim. \( \square \)

Let \( g, h \in G \). Using Propositions 2.23 and 2.27, we see that there are elements \( y_1, y_2, y_3 \in G \) and vertex elements \( x_1, x_2, x_3 \) such that
\[
g = y_1^{-1} x_1 y_2
\]
\[
h = y_2^{-1} x_2 y_3
\]
\[
(gh)^{-1} = y_3^{-1} x_3 y_1
\]
as reduced expressions.

Using Claim 4.3, we see that
\[
|\phi_w(g) - \nu_w(y_1^{-1}) - \nu_w(y_2) + \nu_{w^{-1}}(y_1^{-1}) + \nu_{w^{-1}}(y_2)| \leq 1
\]
\[
|\phi_w(h) - \nu_w(y_2^{-1}) - \nu_w(y_3) + \nu_{w^{-1}}(y_2^{-1}) + \nu_{w^{-1}}(y_3)| \leq 1
\]
\[
|\phi_w(gh) - \nu_w(y_3^{-1}) - \nu_w(y_3) + \nu_{w^{-1}}(y_3^{-1}) + \nu_{w^{-1}}(y_3)| \leq 1.
\]
Using that \( \nu_{w^{-1}}(g) = \nu_w(g^{-1}) \) for any \( g \), we obtain
\[
|\phi_w(g) + \phi_w(h) - \phi_w(gh)| \leq 3,
\]
which shows the proposition. \( \square \)
We can now prove Theorem 4.1 using quasimorphisms:

**Proof of Theorem 4.1.** Let $G$ be a group which splits over $C$ and let $\sum_{i=1}^{n} g(1)$ be some integral chain where every term either lies in a vertex group or is cyclically reduced and let $N \in \mathbb{Z}^+$ be some integer. Suppose for some $g = g(i)$ cyclically reduced, the equation $g^N = h^k h' d$ as in the theorem does not hold. Then for $w = g^N$, we know $v_{w^{-1}}(g(j)m) = 0$ for all $j$ and $m \in \mathbb{Z}^+$. It follows that

$$\phi_w(g(j)m) \geq 0$$

for any $j \in \{1, \ldots, n\}$. Moreover,

$$\phi_w(g(i)m) \geq \left\lfloor \frac{m}{N} \right\rfloor$$

and thus $\bar{\phi}_w(g(i)) \geq \frac{1}{N}$ for the homogenization.

We conclude that

$$\sum_{j=1}^{n} \bar{\phi}_w(g(j)) \geq \frac{1}{N}.$$ 

On the other hand, we have that $D(\phi_w) \leq 3$ and thus $D(\bar{\phi}_w) \leq 6$ by Proposition 2.10. By Bavard’s Duality Theorem (Theorem 2.11), we obtain

$$\text{scl}_G \left( \sum_{i=1}^{n} g(i) \right) \geq \frac{1}{12N},$$

which completes the proof of Theorem 4.1. \hfill $\square$

## 5 CENTRAL/MALNORMAL SUBGROUPS

In this section, we will use Theorem 4.1 to give a criterion for chains in certain amalgamated free products and HNN extensions to have a gap in scl.

To apply Theorem 4.1, we need to solve the following equation for some fixed integer $N \in \mathbb{N}$

$$g^N = h^k h' c,$$  \hspace{1cm} (5.1)

where both sides are reduced decompositions (see Definitions 2.20 and 2.24), where $|g| \geq |h|$, $h'$ is a prefix of $h$, $c \in C$ and $k \geq N$.

To solve Equation (5.1), we define and study BCMS-$D$ subgroups $H$ of a group $G$ for an integer $D$ (Definition 5.8). Central subgroups are BCMS-0 and malnormal subgroups are BCMS-1. As a key example, if $\Lambda \subset \Gamma$ is an induced subgraph of a graph $\Gamma$, then the associated subgroup $A(\Lambda)$ of the RAAG $A(\Gamma)$ is a BCMS-$D$ subgroup for some $D$; see Lemma 6.6.

If the subgroup $C$ that $G$ splits over is BCMS-$D$, then for $N = D + 2$ we solve Equation (5.1) as follows:

- if $|g| = |h|$, then Equation (5.1) reduces to $g^N = h^N c$ for some $c \in C$. We show that there is some element $z \in C$ which commutes with $g$ such that $g = h z$, so that $c = z^N$; see Proposition 5.19;
- if $|g| > |h|$, then Equation (5.1) implies that there is some element $x \in G$ and an element $c \in C$ which commutes with $x$ such that $g = x^m c$ for some $m \geq 2$; see Proposition 5.21.
In both cases, Equation (5.1) only holds when \( g \) can be replaced by a simpler equivalent integer chain. This way we show the following.

**Theorem 5.1.** Let \( G \) be the fundamental group of a graph of groups such that the embedding of every edge group \( C \leq G \) has property BCMS-\( D \). Let \( c \) be an integral chain in \( G \). Then either \( c \) is equivalent (Definition 2.4) to an integral chain \( \tilde{c} \) such that every term lies in a vertex group or

\[
\text{scl}_G(c) \geq 1 \quad \frac{1}{12(D + 2)}.
\]

This section is organized as follows. In Subsections 5.1 and 5.2, we define CM-subgroups and BCMS-\( D \) subgroups, respectively. In Subsections 5.3 and 5.4, we prove properties of BCMS-\( D \) subgroups related to Equation (5.1). Then we solve Equation (5.1) in Subsection 5.5 and prove Theorem 5.1 in Subsection 5.6.

### 5.1 CM-subgroups

In this section, we introduce central/malnormal subgroups (CM-subgroups). CM-subgroups are generalizations of two very different types of subgroups: central subgroups and malnormal subgroups. Recall that a subgroup \( H \leq G \) is **central**, if for every element \( g \in G \) and every element \( h \in H \) we have that \( ghg^{-1} = h \). On the other hand, a subgroup \( H \leq G \) is **malnormal**, if for every element \( g \in G \setminus H \) and every element \( h \in H \) we have that \( ghg^{-1} \not\in H \).

We say that an element \( g \in G \) is a **CM-representative for** \( H \leq G \), if for every \( h \in H \) either

(i) \( ghg^{-1} = h \), or

(ii) \( ghg^{-1} \not\in H \).

For a subset \( S \) of \( G \), let \( Z_H(S) \) be the subgroup of elements in \( H \) commuting with all elements of \( S \). When \( S = \{g\} \), we simply denote it as \( Z_H(g) \). Then \( g \) is a CM-representative for \( H \) if and only if \( gHg^{-1} \cap H = Z_H(g) \).

**Proposition 5.2 (Uniqueness of CM-representatives).** Let \( g \) be a CM-representative for \( H \leq G \). Then \( g' \in H \) is a CM-representative if and only if there are elements \( h \in H, z \in Z_H(Z_H(g)) \) such that \( g' = hzgh^{-1} \). In this case, we have that \( Z_H(g') = hZ_H(g)h^{-1} \).

**Proof.** First assume that \( g \) is a CM-representative and let \( g' = hzgh^{-1} \) for some \( h \in H \) and \( z \in Z_H(Z_H(g)) \). We show that \( g' \) is a CM-representative. For any \( x \in H \), we have \( g'xg'^{-1} = hz(g^{-1}hx)g^{-1}z^{-1}h^{-1} \). Since \( h^{-1}xh \in H \) and \( g \) is a CM-representative, either \( g(h^{-1}xh)g^{-1} \not\in H \) or \( g(h^{-1}xh)g^{-1} = h^{-1}xh \). In the former case we have \( g'xg'^{-1} \not\in H \) since \( hz \in H \), while in the latter case we have \( h^{-1}xh \in Z_H(g) \) and \( g'xg'^{-1} = hz(h^{-1}xh)z^{-1}h^{-1} = h(h^{-1}xh)h^{-1} = x \). Thus, \( g' \) is a CM-representative, and the calculation shows that \( x \in Z_H(g') \) if and only if \( h^{-1}xh \in Z_H(g) \), that is, \( x \in hZ_H(g)h^{-1} \).

Conversely, if \( g' = h_1gh_2 \) is a CM-representative for some \( h_1, h_2 \in H \), then by what we proved above, so is \( g'' = hg \), where \( h = h_2h_1 \). Then for any \( x \in Z_H(g) \), we have \( g''xg'^{-1} = hgxg^{-1}h^{-1} = hxh^{-1} \in H \). Since \( g'' \) is a CM-representative, we must have \( hxh^{-1} = g''xg'^{-1} = x \) for all \( x \in Z_H(g) \). Hence, \( h \in Z_H(Z_H(g)) \). \( \square \)
Definition 5.3 (CM-subgroups and CM-choice). We say that \( H \leq G \) is a CM-subgroup of \( G \), if for every \( g \in G \) there is an element \( \bar{g} \in HgH \) such that \( \bar{g} \) is a CM-representative for \( H \).

A CM-choice for a CM-subgroup \( H \leq G \) is a choice of one CM-representative for each double coset \( HgH \) with \( g \in G \).

Every central or malnormal subgroup \( H \leq G \) is a CM-subgroup. The motivating example for CM-subgroups comes from RAAGs: We will see that for any induced subgraph \( \Lambda \subset \Gamma \) the associated RAAG \( A(\Lambda) \) is a CM-subgroup of \( A(\Gamma) \) (Lemma 6.6). We will have this application in mind throughout this section.

Example 5.4. Consider the graph \( \Delta_1 \) with vertex set \( \{v_0, v_1\} \) and empty edge set and the graph \( \Delta_2 \) with vertex set \( \{v_0, v_1, v_2\} \) and a single edge \((v_0, v_2)\). The associated RAAGs are \( A(\Delta_1) \cong \mathbb{Z} \rtimes \mathbb{Z} \) and \( A(\Delta_2) \cong \mathbb{Z} \rtimes \mathbb{Z}^2 \).

The subgroup \( A(\Delta_1) \) arises naturally as a subgroup of \( A(\Delta_2) \) and is neither central nor malnormal, but it is a CM-subgroup by Lemma 6.6. Not every element of \( A(\Delta_2) \setminus A(\Delta_1) \) is a CM-representative, such as \( v_1v_2 \in A(\Delta_2) \setminus A(\Delta_1) \): For \( v_0 \in A(\Delta_1) \) we have that \( (v_1v_2)v_0(v_1v_2)^{-1} = v_1v_0v_1^{-1} \in A(\Delta_1) \), but \( (v_1v_2)v_0(v_1v_2)^{-1} \neq v_0 \). However, \( v_2 \in A(\Delta_1)(v_1v_2)A(\Delta_1) \) is a CM-representative.

We will see that for every double coset \( A(\Delta_1)gA(\Delta_1) \), an element with the shortest word length in the double coset is a CM-representative (Lemma 6.6). This yields a natural CM-choice.

Proposition 5.5 (Inheritance properties of CM-subgroups). Let \( K \leq H \leq G \) be nested subgroups.

- If \( K \leq G \) is a CM-subgroup, then \( K \leq H \) is a CM-subgroup.
- If \( K \leq H \) is a CM-subgroup and \( H \leq G \) is a CM-subgroup, then \( K \leq G \) is a CM-subgroup.

Proof. The first item is immediate. For the second item, for any \( g \in G \setminus K \) we need to find a CM-representative in \( KgK \). As \( H \leq G \) is a CM-subgroup, there is a CM-representative in \( HgH \) for \( H \leq G \). By Proposition 5.2, there is a CM-representative of the form \( \bar{g} = gh \) for some \( h \in H \). Similarly, since \( K \leq H \) is a CM-subgroup, we have a CM-representative \( \bar{h} = kh \) for \( h \) with \( k \in K \).

Then \( \bar{g}' = gk^{-1} = \bar{g}\bar{h}^{-1} \) is a CM-representative in \( KgK \). Indeed, for any \( k_0 \in K \), we have

\[
g'k_0g'^{-1} = \bar{g}\bar{h}^{-1}k_0\bar{h}\bar{g}^{-1}.
\]

As \( g'k_0g'^{-1} \) is the conjugate of \( \bar{h}^{-1}k_0\bar{h} \in H \) by \( \bar{g} \), it is either outside \( H \) and hence outside \( K \) or equal to \( \bar{h}^{-1}k_0\bar{h} \). In the latter case, either \( \bar{h}^{-1}k_0\bar{h} \not\in K \) or \( \bar{h}^{-1}k_0\bar{h} = k_0 \) since \( \bar{h} \) is a CM-representative.

5.2 | BCMS-D subgroups

Given a proper CM-subgroup \( H \) of a group \( G \) and a CM-representative \( g \in G \setminus H \) the centralizer \( Z_H(g) \) measures how much the subgroup \( H < G \) fails to be malnormal for the element \( g \). It has an interesting structure in the motivating example of RAAGs.

Example 5.6. Let \( \Delta_1 \) and \( \Delta_2 \) be the graphs defined in Example 5.4. We have seen that \( v_2 \in A(\Delta_2) \setminus A(\Delta_1) \) is a CM-representative. Here \( Z_{A(\Delta_1)}(v_2) = A(\Delta_0) \) where \( \Delta_0 \) is the graph with single vertex \( v_0 \).
More generally, we will see that if we choose CM-representatives to be elements in each double coset of minimal length then every such centralizer is the RAAG on an induced subgraph of the defining graph (Lemma 6.6) and thus it is again a CM-subgroup.

On the other hand, if we choose the CM-representatives in a different way, the centralizers may not have this structure, but they only differ by conjugations according to Proposition 5.2

Let \( H_0 \) be a group and let \( H_1 \) be a proper CM-subgroup of \( H_0 \). Let \( h_0 \in H_0 \setminus H_1 \) be a CM-representative. Then \( H_2 := Z_{H_1}(h_0) \) is a subgroup of \( H_1 \). There are three cases:

(i) if \( H_2 = H_1 \), then \( H_1 \) lies in the centralizer of the element \( h_0 \),
(ii) if \( H_2 = \{ e \} \), then \( H_1 \) behaves like a malnormal subgroup with respect to the element \( h_0 \) or
(iii) \( \{ e \} \neq H_2 < H_1 \) is a proper nontrivial subgroup.

If \( h_0 \) is as in case (iii) and \( H_2 \) is a CM-subgroup of \( H_1 \), then we may continue this process: Given a CM-representative \( h_1 \in H_1 \setminus H_2 \), define \( H_3 = Z_{H_2}(h_1) \).

Informally, if this process always yields CM-subgroups and eventually stops (in about \( D \) steps), then we say the subgroup \( H_1 \) has bounded CM-subgroup sequence of depth \( D \), which we abbreviate as BCMS-\( D \). We make this precise in the following definitions.

**Definition 5.7** (CM-subgroup sequence). In a group \( H \), a CM-subgroup sequence of length \( m + 1 \) is a sequence of nested subgroups \( H = H_0 > H_1 > \cdots > H_{m+1} > H_{m+2} \) such that \( H_{i+1} \) is a proper CM-subgroup of \( H_i \) for all \( 0 \leq i \leq m \) (not including \( i = m + 1 \)) and \( H_{i+2} = Z_{H_{i+1}}(g_i) \) for some \( g_i \in H_i \setminus H_{i+1} \).

For any CM-subgroup sequence \( H_{m+2} \leq \cdots \leq H_0 \), if \( H_1 \) is central we must have \( H_2 = H_1 \), which forces \( m = 0 \). If \( H_1 \) is malnormal, then we have \( H_2 = \{ e \} = H_3 \), forcing \( m \leq 1 \). Note that not every nested sequence of proper CM-subgroups is a CM-subgroup sequence due to the requirement \( H_{i+2} = Z_{H_{i+1}}(g_i) \). For instance, \( \{ e \} \leq Z \leq Z^2 \) is a nested sequence of proper CM-subgroups, but \( Z_Z(g) \neq \{ e \} \) for all \( g \in Z^2 \).

It is important to note that, in the definition of CM-subgroup sequences, the only requirement on \( H_{m+2} \) is that \( H_{m+2} = Z_{H_{m+1}}(g_m) \) for some \( g_m \in H_m \setminus H_{m+1} \), and in general it may not be a CM-subgroup of \( H_{m+1} \). It is part of the definition of BCMS subgroups below that \( H_{m+2} \) is required to be a proper CM-subgroup of \( H_{m+1} \) except when \( H_{m+2} = H_{m+1} \).

**Definition 5.8** (BCMS-D). Let \( D \in \mathbb{Z}_{\geq 0} \) be an integer, and let \( H_0 \) be a group. We say that a subgroup \( H_1 \leq H_0 \) (or really the pair \( (H_0, H_1) \)) has bounded CM-subgroup sequences of depth \( D \) (BCMS-D) if \( H_1 \) is a CM-subgroup and for every CM-subgroup sequence \( H_{m+2} \leq \cdots \leq H_0 \) we have that either \( H_{m+2} = H_{m+1} \) or that \( H_{m+2} < H_{m+1} \) is a proper CM-subgroup. Moreover, we require that every CM-subgroup sequence has length at most \( D + 1 \), that is, if \( H_{m+2} \leq \cdots \leq H_0 \) is a CM-subgroup sequence, then \( m \leq D \). We also say \( H_1 \) is a BCMS-D subgroup.

We see that central subgroups have BCMS-0 and malnormal subgroups have BCMS-1. Note that by definition a subgroup \( H \) has BCMS-D then it also has BCMS-D' if \( D' \geq D \), that is, we do not require \( D \) to be the optimal upper bound.

In general, verifying whether a CM-subgroup \( H \leq G \) has BCMS-D requires one to check all CM-subgroup sequences. As we saw in Example 5.6, certain choices of CM-representatives have centralizers that are easier to study in some cases. We will show that one can restrict
attention to some special families of CM-subgroup sequences corresponding to nice choices of CM-representatives to verify whether a CM-subgroup has BCMS-$D$.

We incorporate the choice of CM-representatives into the following notion.

**Definition 5.9** (CM-subgroup-choice). A CM-subgroup-choice $I(G)$ is a CM-choice (Definition 5.3) for every proper CM-subgroup $H < G$.

Given a CM-subgroup-choice $I(G)$, whenever we have a chain of proper subgroups $K < H < G$ such that $K < H$ and $H < G$ are CM-subgroups. Then $K < G$ is also a CM-subgroup by Proposition 5.5. For any element $h \in H \setminus K$ the CM-subgroup-choice $I(G)$ gives us a CM-representative of $h$ for $K < G$ which is also a CM-representative for $K < H$.

**Definition 5.10** (CM-sequence). Given a proper CM-subgroup $H_1 < H_0$ and a CM-subgroup-choice $I(H_0)$, a CM-sequence of length $m + 1$ is a sequence of elements $(h_0, \ldots, h_m)$ in $G$ such that there is a CM-subgroup sequence $H_{m+2} \leq H_{m+1} < \cdots < H_0$ satisfying

- $h_i \in H_1 \setminus H_{i+1}$ is the CM-representative for $H_{i+1}$ provided by $I(G)$ for all $0 \leq i \leq m$, and
- $H_{i+2} = Z_{H_{i+1}}(h_i)$ for all $0 \leq i \leq m$.

Given a CM-sequence $(h_0, \ldots, h_m)$, it uniquely determines the CM-subgroup sequence $H_{m+2} \leq H_{m+1} < \cdots < H_0$ by the relation $H_{i+2} = Z_{H_{i+1}}(h_i)$, which we refer to as the associated CM-subgroup sequence.

Apparently, if $H_1$ is a BCMS-$D$ subgroup, then any CM-sequence $(h_0, \ldots, h_m)$ has length at most $D + 1$, that is, $m \leq D$. Conversely, given a CM-subgroup-choice $I(G)$, not every CM-subgroup sequence appears as one associated to some CM-sequence $(h_0, \ldots, h_m)$. However, it suffices to consider CM-subgroup sequences associated to CM-sequences to show that $H_1$ is a BCMS-$D$ subgroup.

**Proposition 5.11.** Fix a CM-subgroup-choice $I(H_0)$. Suppose $H_1$ is a proper CM-subgroup of $H_0$, and for the CM-subgroup sequence $H_{m+2} \leq \cdots \leq H_0$ associated to any CM-sequence $(h_0, \ldots, h_m)$, we have that either $H_{m+2} = H_{m+1}$ or that $H_{m+2} < H_{m+1}$ is a proper CM-subgroup. Then $H_1$ has BCMS-$D$ if and only if every CM-sequence $(h_0, \ldots, h_m)$ has $m \leq D$.

**Proof.** Given any CM-subgroup sequence $H_{m+2} \leq \cdots \leq H_0$, we claim that for any $0 \leq k \leq m$, there is a CM-subgroup sequence $H'_{m+2} \leq \cdots \leq H'_0$ with $H'_1 = H_1$ and $H'_0 = H_0$ such that

- $H_{m+2} = H_{m+1}$ if and only if $H'_{m+2} = H'_{m+1}$, and $H_{m+2} \leq H_{m+1}$ is a proper CM-subgroup if and only if $H'_{m+2} \leq H'_{m+1}$ is a proper CM-subgroup;
- there is a CM-sequence $(\tilde{h}_0, \ldots, \tilde{h}_k)$ whose associated CM-subgroup is $H'_{k+2} \leq \cdots \leq H'_0$.

The claim with $k = m$ together with our assumption shows that, whenever we have a CM-subgroup sequence $H_{m+2} \leq \cdots \leq H_0$, we have that either $H_{m+2} = H_{m+1}$ or that $H_{m+2} \leq H_{m+1}$ is a proper CM-subgroup. Moreover, there is a CM-sequence $(\tilde{h}_0, \ldots, \tilde{h}_m)$ of the same length, which proves the proposition.

Thus, it suffices to prove this claim, which we show by induction on $k$. For the base case $k = 0$, by definition there is a CM-representative $h_0$ for $H_1 < H_0$ (not necessarily from $I(H_0)$) such that $Z_{H_1}(h_0) = H_2$. Let $\tilde{h}_0$ be the CM-representative in $H_1 h_0 H_1$ chosen by $I(H_0)$. By Proposition 5.2, there is some $h \in H_1$ and $z \in Z_{H_1}(H_2)$ such that $\tilde{h}_0 = h z h_0 h^{-1}$. In this case, $h H_{m+2} h^{-1} \leq \cdots \leq H_{m+1}$...
$hH_2h^{-1} \leq H_1 \leq H_0$ is a CM-subgroup sequence where $hH_2h^{-1} \leq H_1 \leq H_0$ is the CM-subgroup sequence associated to the CM-sequence $(\bar{h}_0)$ since $Z_{H_1}(\bar{h}_0) = hH_2h^{-1}$ by Proposition 5.2.

Suppose the claim holds for some $0 \leq k < m$, that is, there is a CM-subgroup sequence $H_{m+2} \leq \cdots \leq H_0$ with $H_1 = H_1$ and $H_0 = H_0$, such that the relation between $H_{m+2}$ and $H_{m+1}$ corresponds to the relation between $H_{m+2}$ and $H_{m+1}$, and there is a CM-sequence $(\bar{h}_0, \ldots, \bar{h}_k)$ whose associated CM-subgroup is $H_{k+2} \leq \cdots \leq H_0$. Since $k < m$, there is a CM-representative $\bar{h}_{k+1} \in H_{k+1} \setminus H_{k+2}$ such that $Z_{H_{k+2}}(\bar{h}_{k+1}) = H_{k+3}$. Let $\bar{h}_{k+1} = hzh_{k+1}h^{-1}$ be the CM-representative in $H_{k+2}h_{k+1}H_{k+2}$, where $h \in H_{k+2}$ and $z \in Z_{H_{k+2}}(H_{k+3})$. Then $H_{m+2}h^{-1} \leq \cdots \leq H_{k+3}h^{-1} \leq H_{k+2} \leq \cdots \leq H_0$ is a CM-subgroup sequence associated to the CM-sequence $(\bar{h}_0, \ldots, \bar{h}_k, \bar{h}_{k+1})$ since $Z_{H_{k+2}}(\bar{h}_{k+1}) = hH_{k+3}h^{-1}$ by Proposition 5.2. This completes the induction and proves the proposition. □

In what follows, we will use the proposition above as an alternative definition of BCMS-D subgroups since it is easier to check. In practice, only certain subgroups arise as $H_i$ in some CM-subgroup sequence associated to a CM-subgroup, and thus one only needs to fix the CM-subgroup-choice for these CM-subgroups of $H_0$. See the example below.

For every $D \in \mathbb{Z}_+$ there is a BCMS-D subgroup of a group which is not a BCMS-$(D-1)$-subgroup.

Example 5.12. For $n \in \mathbb{N}$, let $\Delta_n$ be the graph with vertex and edge set

$$V(\Delta_n) = \{v_0, \ldots, v_n\}$$

and

$$E(\Delta_n) = \{(v_i, v_j) \mid |i - j| \geq 2\}.$$ 

For $n \in \mathbb{N}$ and $i \in \{1, \ldots, n\}$ let $\Delta_i^i$ be the induced subgraph of $\Delta_n$ with vertex set

$$V(\Delta_i^i) = \{v_i, \ldots, v_n\}.$$ 

By Lemma 6.6, we have that $A(\Delta_i^i) < A(\Delta_n)$ is a CM-subgroup and that $v_0$ is a CM-representative. We compute that $Z_{A(\Delta_i^i)}(v_0) = A(\Delta_i^i)$. More generally, we will see that $A(\Delta_i^i)$ is a CM-subgroup of $A(\Delta_n)$, that $v_{i-1}$ is a CM-representative and that $Z_{A(\Delta_i^i)}(v_{i-1}) = A(\Delta_n^i)$ for $1 \leq i \leq n-1$.

Thus, $(v_0, \ldots, v_n)$ is a CM-sequence of length $n + 1$ and the associated CM-subgroup sequence is

$$\{e\} \leq \cdots \leq A(\Delta_n^i) \leq A(\Delta_n).$$

We will see that those are, in some sense, the longest CM-sequence for subgroups associated to induced subgraphs on RAAGs (Lemma 6.8).

5.3 Normal forms for elements in BCMS-D subgroups

If $H_1$ is a BCMS-D subgroup of $H_0$, given a CM-subgroup-choice $I(H_0)$, then we may write every element as a product of CM-representatives up to conjugation as follows:

Proposition 5.13 (Normal form for elements). Let $H_1$ be a BCMS-D subgroup of $H_0$ with a CM-subgroup-choice $I(H_0)$, and let $g \in H_0 \setminus H_1$ be an element.
Then there is \( n \leq D \), a CM-sequence \((h_0, \ldots, h_n)\) with associated CM-subgroup sequence \( H_{n+2} \leq \cdots \leq H_0 \), and a conjugate \( g' \) of \( g \) by an element of \( H_1 \) such that

\[
g' = h_0 \cdots h_n e_n
\]

with \( e_n \in H_{n+2} \). Moreover, the integer \( n \) and the CM-sequence \((h_0, \ldots, h_n)\) are uniquely determined by \( g \), and \( e_n \) is unique up to conjugation in \( H_{n+2} \).

**Proof.** We inductively prove the following statement:

**Claim 5.14.** For every \( m \geq 0 \), there is a conjugate \( g' \) of \( g \) by an element of \( H_1 \) such that either

(i) \( g' = h_0 \cdots h_m e_m \) for some \( e_m \in H_{m+1} \), where \((h_0, \ldots, h_m)\) is a CM-sequence with \( H_{m+2} \leq \cdots \leq H_0 \) as the associated CM-subgroup sequence, or

(ii) \( g' = h_0 \cdots h_j e_j \) for some \( e_j \in H_{j+2} \) and \( j \leq m \), where \((h_0, \ldots, h_j)\) is a CM-sequence with \( H_{j+2} \leq \cdots \leq H_0 \) as the associated CM-subgroup sequence.

**Proof.** Let \( h_0 = h_0 h' \) be the CM-representative in \( H_1 g H_1 \) provided by \( I(H_0) \), where \( h, h' \in H_1 \). Then \( h_0 = g' h h' \) for \( g' := h_0 h^{-1} \), and thus \( g' = h_0 e_0 \) with \( e_0 := (h h')^{-1} \in H_1 \). This shows the claim for \( m = 0 \).

Suppose the claim is true for some \( m \geq 0 \). If statement (ii) holds for \( m \), then it also holds for \( m + 1 \) and we are done. Thus, assume that \( g' = h_0 \cdots h_m e_m \) where \( g' \) is a conjugate of \( g \) by some element in \( H_1 \), \((h_0, \ldots, h_m)\) is a CM-sequence with associated CM-subgroup sequence \( H_{m+2} \leq \cdots \leq H_0 \), and \( e_m \in H_{m+1} \).

If \( e_m \in H_{m+2} \) we are done as in case (ii) as well. Otherwise, let \( h_{m+1} \in H_{m+1} \setminus H_{m+2} \) be the CM-representative in \( H_{m+2} e_m H_{m+2} \) given by \( I(H_0) \). Then \( h^l h_{m+1} h^r = e_m \) for some \( h^l, h^r \in H_{m+2} \). Thus,

\[
g' = h_0 \cdots h_m h^l h_{m+1} h^r = h^l h_0 \cdots h_m h_{m+1} h^r
\]

as \( h^l \) commutes with all \( h_0, \ldots, h_m \) by the definition of \( H_{m+2} \). Conjugating both sides of the equation above by \( h^l \) and setting \( e_{m+1} = h^r h^l \) proves the claim.

Now as \( H_1 < H_0 \) has property BCMS-D, we will arrive at item (ii) of the claim eventually (if \( m \geq D \)).

The uniqueness can be observed in the inductive construction above as follows. Note that \( h_0 \) is uniquely determined as the CM-representative in \( H_1 g H_1 \) since \( I(H_0) \) is fixed. Next we show that \( e_0 \) is uniquely determined up to conjugation by an element in \( H_2 \). Suppose there is a different choice \( e'_0 \in H_1 \) such that \( h_0 e'_0 = h h_0 e_0 h^{-1} \) for some \( h \in H_1 \), then \( h^{-1} = h_0 [e_0 h^{-1} (e'_0)^{-1}] h_0^{-1} \), which forces \( h^{-1} = e_0 h^{-1} (e'_0)^{-1} \) as \( h_0 \) is a CM-representative for \( H_1 \). Thus, \( h^{-1} e'_0 h = e_0 \), so \( h_0 e'_0 = h h_0 e_0 h^{-1} = h h_0 h^{-1} e'_0 \), which implies \( h_0 = h h_0 h^{-1} \), that is, \( h \in Z_{H_1} (h_0) = H_2 \). This proves that \( e'_0 \) differs from \( e_0 \) via conjugation by some \( h \in H_2 \). In particular, the double coset \( H_2 e_0 H_2 \) is uniquely determined and so is \( h_1 \). Continuing this process, one can observe that each \( h_i \) in the expression is uniquely determined, each element \( e_i \) is unique up to conjugation by an element of \( H_{i+2} \), and the integer \( n \) is characterized as the first \( n \) such that \( e_n \in H_{n+2} \) (which is not ambiguous by the uniqueness up to conjugation).

**Definition 5.15** (CM-reduced element). Suppose that \( H_1 \) is a BCMS-D subgroup of \( H_0 \) with CM-subgroup-choice \( I(H_0) \). For any \( g \in H_0 \setminus H_1 \), we say that \( g \) is **CM-reduced** if we have \( e_n = 1 \) when
Proposition 5.16. Let $H_1$ be a BCMS-$D$ subgroup of $H_0$ with CM-subgroup-choice $I(H_0)$. Let $g \in H_0$ be an element and let $g'$, $e_n$ and $(h_0, \ldots, h_n)$ be as in the normal form from Proposition 5.13. Then $g$ is conjugate to $h e_n$ by an element of $H_1$ and equivalent to $h + e_n$ as a chain, where $h = h_0 \cdots h_n$ is CM-reduced.

Proof. This follows immediately from Proposition 5.13 and Definition 2.4 noting that $e_n$ commutes with $h_0, \ldots, h_n$. □

5.4 Equations in amalgamated free products or HNN extensions

In the rest of Section 5, we will consider a group $G$ that splits over a BCMS-$D$ subgroup $C$. Note that if $g \in G \setminus C$ can be written as a cyclically reduced word in the sense of Definitions 2.20 or 2.24, then naturally any element $c g c' \in C g C$ also has this property. In particular, in this case, any CM-representative in $C g C$ with respect to the CM-subgroup $C$ can be written as a cyclically reduced word.

Similarly, if $g$ is CM-reduced with $g = c_0 \cdots c_m$ for a CM-sequence $(c_0, \ldots, c_m)$, then $g$ is cyclically reduced if and only if $c_0$ is. So, we say $g$ is cyclically reduced and CM-reduced (for example, in Proposition 5.19) if $g$ can be written this way with $c_0$ cyclically reduced.

We will need the following proposition to compare terms in certain expressions in a group $G$ that splits over a BCMS-$D$ subgroup $C$. This is similar to Corollaries 2.22 and 2.26.

Proposition 5.17. Let $G$ be a group that splits over a BCMS-$D$ subgroup $C$. Let $I(G)$ be a CM-subgroup-choice. For some $m \leq D$ let $(c_0, \ldots, c_m)$ is a CM-sequence and let $C_{m+2} \leq \cdots \leq C_1 := C \leq C_0 := G$ be the associated CM-subgroup-sequence.

Suppose $c_0 \in G \setminus C$ (or equivalently $g$) can be written as a cyclically reduced word. Let $n \geq m + 2$ and suppose there are elements $x_1, \ldots, x_{n-1}, x'_1, \ldots, x'_{n-1} \in C_{m+1}$ and $x_0, x'_0, x_n, x'_n \in C_1$ such that

$$x_0 c^{(m)} x_1 \cdots c^{(m)} x_n = x'_0 c^{(m)} x'_1 \cdots c^{(m)} x'_n,$$

where $c^{(m)} = c_0 \cdots c_m$. Then there are $d_1, \ldots, d_{n-m} \in C_{m+2}$ such that

$$d_{i-1} x'_i d_i^{-1} = x_i$$

for all $2 \leq i \leq n - m$.

Proof. We observe that by Corollaries 2.22 and 2.26, there are elements $d_0, \ldots, d_n \in C_1$ with $d_0 = e = d_n$ such that $x_0 c^{(m)} x_1 = d_0 x'_0 c^{(m)} x'_1 d_1^{-1}$ and $c^{(m)} x_i = d_{i-1} c^{(m)} x'_i d_i^{-1}$ for all $i \in \{2, \ldots, n\}$.

Claim 5.18. For every $j \in \{0, \ldots, m\}$ we have that $d_i \in C_{j+2}$ for all $i \in \{1, \ldots, n - j\}$.

Proof. We proceed by induction. For $j = 0$ we write $c^{(m)} = c_0 c''$ with $c'' = c_1 \cdots c_m$. Then we obtain

$$c_0^{-1} d_{i-1} c_0 = c'' x_i d_i x_i'^{-1} c''^{-1}$$
for all \( i \in \{2, \ldots, n\} \) from \( c^{(m)}x_i = d_{i-1}c^{(m)}x_i'd_i^{-1} \). Observe that both \( d_{i-1} \in C_1 \) and \( c''x_id_{i-1}'c''^{-1} \in C_1 \). Since \( c_0 \) is a CM-representative for \( C_1 < C_0 \) we have \( d_{i-1} \in C_2 = Z_{C_1}(c_0) \). Since \( d_n = e \) we conclude that \( d_i \in C_2 \) for all \( i \in \{1, \ldots, n\} \).

Suppose the claim is true for some \( j-1 \in \{0, \ldots, m-1\} \). We wish to show that it is true for \( j \) as well. We may write \( c^{(m)} = c'c_jc'' \) for \( c' = c_0 \cdots c_{j-1} \) and \( c'' = c_{j+1} \cdots c_m \). By the induction hypothesis we have that \( d_i \in C_{j+1} \) for all \( i \in \{1, \ldots, n-j+1\} \), so all such \( d_i \) commute with \( c' \). As above, we obtain

\[
   d^{-1}_i c_j = c'' x_i d_i' c''^{-1}
\]

for all \( i \in \{2, \ldots, n-j+1\} \) from \( c^{(m)}x_i = d_{i-1}c^{(m)}x_i'd_i^{-1} \). Note that \( d_{i-1}, d_i, x_i', c'' \in C_{j+1} \) for all \( i \in \{2, \ldots, n-j+1\} \). Thus, as \( c_j \) is a CM-representative, we see that \( d_i \in C_{j+2} \) for all \( i \in \{1, \ldots, n-j\} \). This completes the induction. \( \Box \)

For \( j = m \) the claim implies that \( d_i \in C_{m+2} \) for all \( i \in \{1, \ldots, n-m\} \) and thus all such \( d_i \) commute with \( c^{(m)} \). Hence, from \( c^{(m)}x_i = d_{i-1}c^{(m)}x_i'd_i^{-1} \) we have

\[
   x_i = d_{i-1}x_i' d_i^{-1}
\]

for all \( i \in \{2, \ldots, n-m\} \). This finishes the proof. \( \Box \)

### 5.5 Solutions to Equation (5.1)

**Proposition 5.19.** Let \( G \) be a group that splits over a BCMS-D subgroup \( C \), and let \( I(G) \) be a CM-subgroup choice. Suppose \( g \in G \) is cyclically reduced and CM-reduced. Suppose there is a cyclically reduced word \( h \in G \) with \( g^N = h^N c \) for some \( c \in C \) and \( N > D + 2 \). Then there is an element \( z \) which commutes with \( g \) such that \( g = hz \).

**Proof.** By our assumption, we have that \( g = c_0 \cdots c_m \) where \( c_0 \) is cyclically reduced and \( (c_0, \ldots, c_m) \) is a CM-sequence with associated CM-subgroup sequence \( C_{m+2} \leq \cdots \leq C_0 \), where \( C_0 = G \) and \( C_1 = C \). Note that \( m \leq D \) since \( C \) is a BCMS-D subgroup.

By Corollaries 2.22 and 2.26, there are \( d_i \in C \) for \( 0 \leq i \leq N \) with \( d_0 = e = d_N \) such that \( g = d_{i-1}hd_i^{-1} \) for \( 1 \leq i \leq N-1 \) and \( g = d_{N-1}hc d_N^{-1} \). Redefining \( d_N^{-1} \) to be \( cd_N^{-1} \) we get that \( g = d_{i-1}hd_i^{-1} \) for \( 1 \leq i \leq N \) and thus

\[
   d_{i-1}^{-1}gd_i = d_i^{-1}gd_{i+1}
\]

for all \( 1 \leq i \leq N-1 \).

**Claim 5.20.** For every \( 0 \leq j \leq m + 1 \) we have that \( d_i \in C_{j+1} \) for all \( 0 \leq i \leq N - j \).

**Proof.** We proceed by induction. For \( j = 0 \) the claim is immediate as all terms are in \( C_1 = C \).

Suppose the claim is true for some \( 0 \leq j \leq m \). Write \( g = c_0 \cdots c_m = c'c_jc'' \) for \( c' = c_0 \cdots c_{j-1} \) and \( c'' = c_{j+1} \cdots c_m \). Observe that by the induction hypothesis, \( c' \) commutes with \( d_i \) for all \( 0 \leq i \leq N-j \). Thus, for all \( 1 \leq i \leq N-j-1 \), we deduce from Equation (5.2) that

\[
   c_j^{-1}(d_i d_{i-1}^{-1})c_j = c'' d_{i+1}d_i^{-1} c''^{-1}.
\]
By the induction hypothesis, \( c''d_i d_i^{-1}c''^{-1} \in C_{j+1} \) for all such \( i \). Thus \( d_i d_i^{-1} \in C_{j+1} \) for all \( 1 \leq i \leq N - j - 1 \) since \( c_j \) is a CM-representative. Recall that \( d_0 = e \). Thus, for every \( i \in \{1, \ldots, N - j - 1\} \) we have that
\[
d_i = d_i d_0^{-1} = (d_i d_{i-1}^{-1})(d_{i-1} d_{i-2}^{-1}) \cdots (d_1 d_0^{-1}) \in C_{j+2}.
\]
This shows the claim. \( \square \)

In particular, for \( j = m + 1 \) the claim implies that \( d_i \in C_{m+2} \) for all \( 0 \leq i \leq N - m - 1 \). Since \( m \leq D \) and \( D + 2 \leq N \), we have that \( d_1 \in C_{m+2} \). Thus, \( d_1 \) commutes with \( g \). This concludes the proof of Proposition 5.19 as \( g = d_0^{-1} hd_1 = hd_1 \). \( \square \)

**Proposition 5.21.** Let \( G \) be a group that splits over a BCMS-\( D \) subgroup \( C \), and let \( T(G) \) be a CM-subgroup-choice. Let \( g, h \in G \) be cyclically reduced words with \( |g| > |h| \) and let \( h' \) be a prefix of \( h \). Suppose
\[
g^N = h^k h' c
\]
for some \( c \in C \) and \( N \geq D + 2 \).

Then there is a cyclically reduced element \( x \in G \) such that \( g = x^{n_g} c \) for some \( n_g \geq 2 \) and \( c \in C \) that commutes with \( x \).

**Proof.** Let \( C_0 = G \) and \( C_1 = C \). We inductively prove the following claim:

**Claim 5.22.** There are two coprime integers \( n_g, n_h \in \mathbb{Z}_+ \) and \( 0 \leq n'_h < n_h \) such that for every \( m \geq 0 \) either

(i) there is a CM-sequence \( (c_0, \ldots, c_m) \) with the associated CM-subgroup sequence \( C_{m+2} \leq \cdots \leq C_0 \) and elements \( d_g, d_h \in C \) such that
\[
d_g g d_g^{-1} = c^{(m)} z_1 \cdots c^{(m)} z_{n_g} \text{, and} \\
d_h h d_h^{-1} = c^{(m)} z'_1 \cdots c^{(m)} z'_{n_h},
\]
for \( c^{(m)} = c_0 \cdots c_m \) and \( z_i, z'_i \in C_{m+1} \), or

(ii) there is an \( n \leq m \) and a CM-sequence \( (c_0, \ldots, c_n) \) with the associated CM-subgroup-sequence \( C_{n+2} \leq \cdots \leq C_0 \) and elements \( d_g, d_h \in C \) such that
\[
d_g g d_g^{-1} = c^{(n)} z_1 \cdots c^{(n)} z_{n_g} \text{, and} \\
d_h h d_h^{-1} = c^{(n)} z'_1 \cdots c^{(n)} z'_{n_h},
\]
for \( c^{(n)} = c_0 \cdots c_n \) and \( z_i, z'_i \in C_{n+2} \).

**Proof.** We first show that the claim is true for \( m = 0 \). Let \( d \) be the greatest common divisor of \( |g| \) and \( |h| \). Note that \( c \) lies in \( C \), which is the subgroup that \( G \) splits over, so it can be ignored whenever we measure the length of a reduced word. Since both \( g \) and \( h \) are cyclically reduced and \( h' \) is a prefix of \( h \), we have \( N|g| = |g^N| = |h^k h'| = k|h| + |h'| \). Hence, \( d \) also divides \( |h'| \). Thus, we can
write \( g = g_1 \cdots g_{n_g}, h = h_1 \cdots h_{n_h} \) and \( h' = h_1 \cdots h'_{n_h}, \) where \( n_g = |g|/d, \) \( n_h = |h|/d, \) \( n'_h = |h'|/d \) and all the \( g_i \) and \( h_i \) are reduced words of length \( d. \) Note that \( n_g > n_h \geq 1 \) since \( |h| < |g| \).

Then we have reduced decompositions

\[
(g_1 \cdots g_{n_g})^N = (h_1 \cdots h_{n_h})^k h_1 \cdots h'_{n_h-1}(h_{n_h}^c).
\]

By Corollaries 2.22 and 2.26, there are elements \( d_0, \ldots, d_{Nn_g} \in C_1 = C \) with \( d_0 = e \) and \( d_{Nn_g} = c^{-1} \) such that \( g_i = d_{i-1} h_i d_i^{-1} \) for all \( 1 \leq i \leq Nn_g, \) where the index \( i \) in \( g_i \) and \( h_i \) is taken mod \( n_g \) and \( n_h, \) respectively. Thus, for all \( 1 \leq i \leq n_g, \) we have

\[
g_i = d_{i-1} h_i d_i^{-1} = d_{i-1} h_i + n_h d_i^{-1} = d_{i-1} d_{i-1 + n_h}^{-1} g_i + n_h d_i + n_h d_i^{-1},
\]

and hence \( g_i \in C_{g_i + n_h} C. \) As \( n_h \) and \( n_g \) are coprime we see that \( g_i \in C g_i C \) for all \( 1 \leq i \leq n_g, \) and by \( g_i = d_{i-1} h_i d_i^{-1} \) we have \( h_i \in C g_i C \) for all \( 1 \leq i \leq n_h. \) Let \( c_0 \in C \) be the CM-representative of \( C g_1 C \) provided by \( I(G). \) Then the above calculations show that

\[
g = z_0 c_0 z_1 \cdots c_0 z_{n_g}, \text{ and}
\]

\[
h = z'_0 c_0 z'_1 \cdots c_0 z'_{n_h},
\]

for some \( z_i, z'_i \in C = C_1. \) Conjugating \( g \) and \( h \) by \( z_0 \) and \( z'_0, \) respectively, and possibly changing \( z_{n_g} \) and \( z'_{n_h} \) we achieve case (i) of the claim with \( m = 0. \) Note that \( c_0 \) is cyclically reduced by the expression above since \( g \) is cyclically reduced and \( |g| = n_g |g_1| = n_g |c_0|. \)

Now suppose that the claim is true for some \( m > 0. \) We prove it for \( m + 1. \) If item (ii) of the claim holds for \( m, \) then clearly it holds for \( m + 1 \) and we are done. Thus, suppose that item (i) holds for \( m. \) We will argue similarly as in the case of \( m = 0. \) By the induction hypothesis, we have

\[
g = d_g^{-1} c^{(m)} z_1 \cdots c^{(m)} z_{n_g} d_g, \text{ and}
\]

\[
h = d_h^{-1} c^{(m)} z'_1 \cdots c^{(m)} z_{n_h} d_h,
\]

for some \( d_g, d_h \in C_1, \) \( c^{(m)} = c_0 \cdots c_m, \) and \( z_i, z'_i \in C_{m+1}. \) Since \( h' \) is a prefix of \( h, \) we have a reduced decomposition \( h = h' \cdot h'' \) for some reduced word \( h''. \) Comparing it to the reduced decomposition

\[
h = \left( d_h^{-1} c^{(m)} z'_1 \cdots c^{(m)} z_{n_h} d_h \right) \left( c^{(m)} z_{n_h+1} c^{(m)} z_{n_h} d_h \right),
\]

by Corollaries 2.22 and 2.26, we observe that \( h' = d_h^{-1} c^{(m)} z'_1 \cdots c^{(m)} z_{n_h} d_h' \) for some \( d_h' \in C_1. \) Thus,

\[
d_g^{-1} \left( c^{(m)} z_1 \cdots c^{(m)} z_{n_g} \right)^N d_g = d_h^{-1} \left( c^{(m)} z'_1 \cdots c^{(m)} z'_{n_h} \right)^k \left( c^{(m)} z'_1 \cdots c^{(m)} z'_{n_h} \right) d_h' c.
\]

Applying Proposition 5.17 to this equation with \( n = N \cdot n_g, \) we obtain elements \( d_1, \ldots, d_{Nn_g - m} \in C_{m+1} = Z_{c^{(m+1)}}(c_m) \) such that \( z_i = d_{i-1} z'_i d_i^{-1} \) for all \( i \in \{2, \ldots, Nn_g - m\}, \) where the index \( i \) in \( z_i \) and \( z'_i \) is taken mod \( n_g \) and \( n_h, \) respectively.

Note that \( m \leq D \) since \( (c_0, \ldots, c_m) \) is a CM-sequence, and thus \( m + 2 \leq D + 2 \leq N. \) It follows that \( (N-2)n_g > m \cdot n_g > m \) since \( n_g \geq 2. \) That is, we have \( 2n_g < Nn_g - m \) and thus \( n_g + 1 + n_h \leq Nn_g - m \) as \( |h| < |g|. \)
Hence,

\[ z_i = d_{i-1} z'_i d_{i-1}^{-1} = d_{i-1} z_{i+n_h} d_{i-1}^{-1} = d_{i-1} d_{i+n_h-1}^{-1} z_{i+n_h} d_{i+n_h} d_{i-1}^{-1} \]

for all \( 2 \leq i \leq n_g + 1 \) where indices in \( z_i \) are taken mod \( n_g \). As \( n_h \) and \( n_g \) are coprime we see that \( z_i \in C_{m+2} z_1 C_{m+2} \) for all \( 1 \leq i \leq n_g \). Combining with \( z_i = d_{i-1} z'_i d_{i-1}^{-1} \) we have \( z'_i \in C_{m+2} z_1 C_{m+2} \) for all \( 1 \leq i \leq n_h \).

If \( z_1 \in C_{m+2} \), then all \( z_i, z'_i \in C_{m+2} \) and we achieve item (ii) of the claim with \( n = m \) and thus we are done.

Otherwise, let \( c_{m+1} \in C_{m+1} \setminus C_{m+2} \) be the CM-representative of \( C_{m+2} z_1 C_{m+2} \) provided by \( I(G) \). Using the fact that elements in \( C_{m+2} \) commute with \( c^{(m)} \), it follows that there are \( y_i, y'_i \in C_{m+2} \) such that

\[ d_{g} g d_{g}^{-1} = y_0 c^{(m+1)} y_1 \cdots c^{(m+1)} y_{n_g} \text{ and } d_{h} h d_{h}^{-1} = y'_0 c^{(m+1)} y'_1 \cdots c^{(m+1)} y'_{n_h} \]

for \( c^{(m+1)} = c^{(m)} c_{m+1} \).

Conjugating \( d_{g} g d_{g}^{-1} \) and \( d_{h} h d_{h}^{-1} \) by \( y_0 \) and \( y'_0 \), respectively, we achieve item (i) of the claim for \( m + 1 \) and thus the result follows.

As \( C < G \) is a BCMS-D subgroup, by the claim above, there is some \( n \leq D, d_g \in C \), and a CM-sequence \((c_0, \ldots, c_n)\) such that

\[ d_{g} g d_{g}^{-1} = c^{(n)} z_1 \cdots c^{(n)} z_{n_g} \]

with \( z_i \in C_{n+2} \) and \( c^{(n)} = c_0 \cdots c_n \). Thus, all \( z_i \) commute with \( c^{(n)} \) and we have

\[ d_{g} g d_{g}^{-1} = \left( c^{(n)} \right)^{n_g} z \]

with \( z = z_1 \cdots z_{n_g} \). Let \( x = d_{g}^{-1} c^{(n)} d_{g} \) and \( c = d_{g}^{-1} z d_{g} \). Then \( g = x^{n_g} c \) and \( c \) commutes with \( x \). By construction, we have \( |x| = |c_0| = |g| = |g|/n_g \) and \( |x^{n_g}| = |g| \), thus \( x \) is cyclically reduced. This finishes the proof of Proposition 5.21.

\[ \square \]

### 5.6 Proof of Theorem 5.1

We use the following reduced form of integral chains to prove Theorem 5.1.

**Lemma 5.23.** Let \( G \) be a group that splits over a BCMS-D subgroup \( C \). Any integral chain \( d \) is equivalent to a chain \( d' = d_1 + d_2 \), where

1. \( d_1 = \sum_{i=1}^{n} g_i \) for some \( n \geq 0 \), where every \( g_i \) is cyclically reduced (see Definitions 2.20 and 2.24) and does not conjugate into any vertex group,
2. every term of \( d_2 \) lies in some vertex group,
3. there is no \( 1 \leq i \leq j \leq n \) such that \( g_i = g' c \) where \( g' \) is a conjugate of \( g_j^{-1} \) and \( c \in C \) commutes with \( g' \).
(4) there is no $1 \leq i \leq n$ such that $g_i = x^m c$ for some $m > 1$, $x \in G$, and $c \in C$ so that $x$ and $c$ commute,
(5) for every $1 \leq i \leq n$ we have that $g_i$ is CM-reduced (Definition 5.15).

Proof. Given an expression $d' = d_1 + d_2$ of integral chains, where $d_1 = \sum_{j=1}^{m} k_j h_j$ with cyclically reduced words $h_j \in G$ and $k_j \in \mathbb{Z}_+$, and every term of $d_2$ lies in some vertex group, associate a complexity $n(d') = \sum_{j=1}^{m} |h_j|$.

There exists a chain equivalent to $d$ that admits such an expression by replacing elements in $d$ by suitable conjugates so that they are either cyclically reduced or in a vertex group.

Let $d' = d_1 + d_2$ with $d_1 = \sum_{i=1}^{n} g_i$ be an expression of this form for a chain equivalent to $d$ where $n(d')$ is minimal among such equivalent chains. We claim that $d'$ satisfies the conditions (1)–(4). Each $g_i$ is cyclically reduced by our requirement, and the first two conditions are easy to verify. If there are $1 \leq i \leq j \leq n$ such that $g_i = g' c$ where $c$ commutes with $g'$ and $g'$ is conjugate to $g_j^{-1}$, then $g_i$ is equivalent to the chain $g' + c$ by (3) of Definition 2.4 and equivalent to $-g_j + c$ by equivalence (1) and (2) of Definition 2.4. Thus, we may cancel $g_i$ and $g_j$ at the cost of changing $d_2$ until one term has coefficient zero to reduce $n(d')$. Similarly we see that if $g_i = x^m c$ where $m > 1$ and $c$ commutes with $x$, then we may replace $k_i g_i$ by $mk_i x + c$, which has smaller complexity since $|x| < m |x| = |x^m| = |g_i|$.

Finally, we can always make the chain $d'$ above further satisfy (5): by Proposition 5.16 we may replace every (cyclically reduced) $g_i$ by $h_i + c_i$ where $h_i$ is CM-reduced, $c_i \in C$ lies in the edge group (and thus in a vertex group). Moreover, Proposition 5.16 shows that $g_i$ is conjugate to $h_i c_i$ by an element of $C$, so $h_i \in C g_i C$ must be represented by a cyclically reduced word as $g_i$ is, and we have $|h_i| = |g_i|$. This operation does not affect the complexity of the expression and thus the chain $d'$ admits a desired expression. □

We can now prove Theorem 5.1:

**Theorem 5.1.** Let $G$ be a graph of groups where each edge group is a BCMS-D subgroup of $G$. Let $c$ be an integral chain in $G$. Then either $c$ is equivalent (Definition 2.4) to a chain $\tilde{c}$ such that every term lies in a vertex group or

$$\text{scl}_G(c) \geq \frac{1}{12(D + 2)}.$$ 

Proof. Fix a CM-subgroup choice $I(G)$. Assume first that the graph of groups is either an amalgamated free product or an HNN extension over a BCMS-D subgroup $C$.

Let $c' = c_1 + c_2$ be a chain equivalent to $c$ as in Lemma 5.23 with $c_1 = \sum_{i=1}^{n} g_i$.

Suppose $n > 0$ and without loss of generality assume that $g_1$ has the longest length. Set $N = D + 2$ and suppose that

$$\text{scl}_G(c) < \frac{1}{12N}.$$ 

By Theorem 4.1, there is some $1 \leq j \leq n$ and a cyclic conjugate $h$ of $g_j^{-1}$ such that

$$g_j^N = h k h' c,$$

where $h'$ is a prefix of $h$ and $c \in C$. Since $|g_1|$ is maximal among all $g_i$ we conclude that $|g| \geq |h|$. Now consider the following two cases.
• $|g| = |h|$. Since all of $g$, $h$ and $h'$ are cyclically reduced, we must have $g^N = h^N c$ in this case. Since $g$ is CM-reduced, by Proposition 5.19 there is some $z \in C$ which commutes with $g$ such that $g = h z$. This contradicts (3) of Lemma 5.23.

• $|g| > |h|$. In this case, Proposition 5.21 implies that there is some $x \in G$, $m \geq 2$ and $c \in C$ such that $g = x^m c$. This contradicts (4) of Lemma 5.23.

Therefore, we must have

$$\text{scl}_G(c) \geq \frac{1}{12N} = \frac{1}{12(D + 2)},$$

unless $c$ is equivalent to a chain where all terms lie in vertex groups.

When $G$ is a general graph of groups, the chain is supported on a finite subgraph, so we can proceed by induction on the number of edges in the support. At each step, any chosen edge group $C$ splits the group as an amalgamated free product or an HNN extension over $C$, depending on whether the edge separates the graph. Note that any BCMS-$D$ edge subgroup of $G$ lying in a subgroup $H$ is also a BCMS-$D$ subgroup of $H$. Thus, either at some stage what we have shown above implies the desired gap, or we can keep replacing the chain by equivalent ones supported in subgraphs with strictly smaller number of edges until every term lies in vertex groups. □

6 | GAPS FOR GRAPH PRODUCTS OF GROUPS

In this section, we apply Theorem 5.1 from the previous section to obtain gap results for graph products. We will use basic notions and properties of graph products in Section 3.

The lower bounds of scl for integral chains depends on the existence of certain induced subgraphs. Let $\Delta_n$ be the simplicial graph with vertex set $V(\Delta_n) = \{v_0, \ldots, v_n\}$ and edge set $E(\Delta_n) = \{(v_i, v_j) : |i - j| > 2\}$. We call this graph the opposite path of length $n$. For any simplicial graph $\Gamma$ we define

$$\Delta(\Gamma) := \max\{n \mid \Delta_n \text{ is an induced subgraph of } \Gamma\}.$$ 

The lower bound we establish has size determined by $\Delta(\Gamma)$. The bound applies to all integral chains except for those equivalent (Definition 2.4) to vertex chains.

**Definition 6.1.** A vertex chain is a chain of the form $c = \sum_{v \in V} c_v$, where each $c_v$ is a chain in the vertex group $G_v$.

**Theorem 6.2 (Gaps for graph products of groups).** Let $\mathcal{G}(\Gamma)$ be a graph product and let $c$ be an integral chain of $\mathcal{G}(\Gamma)$. Then either

$$\text{scl}_{\mathcal{G}(\Gamma)}(c) \geq \frac{1}{12(\Delta(\Gamma) + 2)},$$

or one of the following equivalent statements holds:

(i) $c$ is equivalent (Definition 2.4) to a vertex chain,
(ii) the pure factor chain $c^{pf}$ (Definition 3.11) is a vertex chain.
We will study vertex chains in detail in Section 7. In particular, we prove the following theorem that computes the scl of a vertex chain \( c = \sum_{v \in V} c_v \) in terms of \( \text{scl}_{G_v}(c_v) \) and the structure of the defining graph.

**Theorem 6.3** (Vertex chains). Let \( G(\Gamma) \) be a graph product of groups and let \( c = \sum_{v \in V(\Gamma)} c_v \) be a vertex chain, where each \( c_v \) is a chain in the vertex group \( G_v \). Then \( \text{scl}_{G(\Gamma)}(c) \) can be computed as a linear programming problem if each \( \text{scl}_{G_v}(c_v) \) is known, and it is rational if each \( \text{scl}_{G_v}(c_v) \) is. Moreover,

\[
\text{scl}_{G(\Gamma)}(c) \geq \text{scl}_{G_v}(c_v)
\]

for any vertex \( v \).

See the end of Subsection 7.1 for a proof.

Combining with Theorem 6.2, we have:

**Corollary 6.4.** Let \( G = G(\Gamma) \) be a graph product of groups over a finite graph \( \Gamma \), where each vertex group \( G_v \) has a spectral gap \( C_v > 0 \) for integral chains. Then \( G \) also has a gap \( C = \min \left\{ \frac{1}{12(\Delta(\Gamma)+2)}, C_v \right\} \) for integral chains.

In particular, we have a gap theorem for RAAGs and RACGs; see Theorem 6.16.

We can also construct integral chains with small scl.

**Theorem 6.5** (Chains with small scl). Let \( G(\Gamma) \) be a graph product of groups and let \( \Delta(\Gamma) \) be as above. Then there is an explicit integral chain \( \delta \) in \( G(\Gamma) \) such that

\[
\frac{1}{12(\Delta(\Gamma)+2)} \leq \text{scl}_G(\delta) \leq \frac{1}{\Delta(\Gamma)}.
\]

This shows that the estimate in Theorem 6.2 is accurate up to a scale of 12.

This section is organized as follows. In Subsection 6.1, we define the canonical CM-subgroup choice in a graph product \( G(\Gamma) \) and show the nice behavior of CM-subgroup sequences with respect to this choice. In Subsection 6.2, we show that the subgroup \( G(\Lambda) \) associated to any induced subgraph \( \Lambda \subset \Gamma \) has BCMS-\( \Delta(\Gamma) \). In Subsection 6.3, we will see that opposite paths are sources of integral chains with small scl. Then we prove Theorems 6.2 and 6.5 in Subsection 6.4. In Subsection 6.5, we deduce the gap results in the special case of RAAGs and RACGs. Finally as applications, we construct groups with interesting scl spectra in Subsection 6.6.

### 6.1 Canonical CM-choice

Let \( G(\Gamma) \) be a graph product of groups. Every induced subgraph \( \Lambda \subset \Gamma \) induces a subgroup \( G(\Lambda) \subset G(\Gamma) \). We find nice CM-representatives with respect to such subgroups.

**Lemma 6.6.** Let \( \Lambda \subset \Gamma \) be an induced subgraph of \( \Gamma \), let \( g \in G(\Gamma) \setminus G(\Lambda) \) and let \( \bar{g} \) be the element with the shortest length among all elements in \( G(\Lambda)gG(\Lambda) \). Then

1. \( \bar{g} \) is a CM-representative, and
the centralizer $Z_{G(\Lambda)}(\tilde{g}) = G(\Theta)$ where $\Theta$ is the induced subgraph of $\Lambda$ that consists of all vertices of $\Lambda$ adjacent to all vertices in the support of $\tilde{g}$.

Proof. Let $\tilde{g}$ be a word of minimal syllable length in $G(\Lambda)gG(\Lambda)$. Then $\tilde{g}$ is in particular reduced by Lemma 3.1.

Suppose that there are some $h_1, h_2 \in G(\Lambda)$ such that $\tilde{g}h_1\tilde{g}^{-1} = h_2^{-1}$. Then $h_2\tilde{g}h_1 = \tilde{g}$. We may assume that $h_1, h_2$ are written as reduced words. By Lemma 3.1, there are three cases:

- some letter in $h_2$ merges with another in $\tilde{g}$ and commutes with all the letters in between;
- some letter in $h_1$ merges with another in $\tilde{g}$ and commutes with all the letters in between; or
- some letter in $h_2$ merges with another in $h_1$ and commutes with all the letters in between.

The first two cases can not occur by our choice of $\tilde{g}$ as we can remove the letter that merges with $h_1$ or $h_2$ in $\tilde{g}$. Thus, we should keep having the last case until the word $h_1\tilde{g}h_2$ reduces to $\tilde{g}$. The process implies that $h_1 = h_2^{-1}$ and both commute with every letter of $\tilde{g}$. This shows that $\tilde{g}$ is a CM-representative.

The observation above also implies that a reduced word $h \in G(\Lambda)$ commutes with $\tilde{g}$ if and only if every letter in it commutes with all those in $\tilde{g}$. This shows $Z_{G(\Lambda)}(\tilde{g}) = G(\Theta)$ as in (2). □

As the minimal representatives in the double cosets yields nice and controlled centralizers, we always use them as our CM-choice in what follows. It is not important for our purposes but the method above shows that $\tilde{g}$ is the unique element of minimal length in $G(\Lambda)gG(\Lambda)$.

Definition 6.7 (Canonical CM-choice). Let $\Gamma$ be a simplicial graph and let $G(\Gamma)$ be a graph product of groups. We define the canonical CM-subgroup choice $I(G(\Gamma))$ as follows: For any induced subgraph $\Lambda \subset \Gamma$ and any $g \in G(\Gamma)$ we choose $\tilde{g}$ a CM-representative of $g$ for $G(\Lambda) \leq G(\Gamma)$ as an element with the smallest syllable length in $G(\Lambda)gG(\Lambda)$. For any other CM-subgroups, we choose the CM-representatives arbitrarily.

Note that for any induced subgraph $\Lambda$ of $\Gamma$, item (1) of Lemma 6.6 shows that $G(\Lambda) \leq G(\Gamma)$ is a CM-subgroup. Moreover, under the canonical choice all CM-subgroup sequences have the form

$$G(\Lambda_{n+2}) \leq G(\Lambda_{n+1}) \leq \ldots \leq G(\Lambda_1) \leq G(\Gamma),$$

where $n \geq 0$ and $\Lambda_{n+2} \subset \ldots \subset \Lambda_1 = \Lambda$ is a proper nested sequence of induced subgraphs except that possibly $\Lambda_{n+2} = \Lambda_{n+1}$. Thus, either $G(\Lambda_{n+2}) = G(\Lambda_{n+1})$ or $G(\Lambda_{n+2})$ is a proper CM-subgroup by Lemma 6.6.

Thus, to show that $G(\Lambda) \leq G(\Gamma)$ is a BCMS-$D$ subgroup, we need to control the length of CM-sequences with respect to the canonical choice. This is what we do in the next subsection.

### 6.2 The opposite paths $\Delta_m$ and lengths of CM-sequences

Now we find the maximal length of CM-sequences in a given graph product on a graph $\Gamma$ with respect to the canonical CM-choice. Then we show that the subgroup associated to any induced subgraph of $\Gamma$ is BCMS-$D$ for $D = \Delta(\Gamma)$.

Recall that for a graph $\Gamma$, we define $\Delta(\Gamma)$ to be the largest number $m \in \mathbb{Z}_+$ such that $\Delta_m$ is an induced subgraph of $\Gamma$. The only graphs where $\Delta_1$ does not embed as an induced subgraph are
complete graphs (including the graph with a single vertex). We set $\Delta(\Gamma) = 0$ if $\Gamma$ is a complete graph. If all $\Delta_m$ are induced subgraphs of $\Gamma$, then $\Gamma$ is necessarily infinite, and we set $\Delta(\Gamma) = \infty$.

For example we see that $\Delta(\Delta_m) = m$. Observe also that $\Delta(\Gamma) \leq |\Gamma| - 1$. We will see that $\Delta(\Gamma)$ controls the length of the longest CM-sequence in subgroups of $\mathcal{G}(\Gamma)$ associated to induced subgraphs.

On the one hand, for arbitrary nontrivial vertex groups, a graph product on the graph $\Delta_n$ has a CM-subgroup sequence of length $n + 1$. For $n \in \mathbb{Z}^+$ and $i \in \{1, \ldots, n\}$ let $\Delta_n^i$ be the induced subgraph of $\Delta_n$ with vertex set

$$V(\Delta_n^i) = \{v_i, \ldots, v_n\}.$$ 

For arbitrary nontrivial elements $g_i \in G_{v_i}$, we have a CM-sequence $(g_0, \ldots, g_n)$ of length $n + 1$, and the associated CM-subgroup sequence is

$$\{e\} \leq \{e\} \leq \mathcal{G}(\Delta_n^m) \leq \cdots \leq \mathcal{G}(\Delta_n^1) \leq \mathcal{G}(\Delta_n).$$

On the other hand, we can find an induced subgraph isomorphic to some $\Delta_m$ from a CM-sequence.

**Lemma 6.8.** Let $\Gamma_0$ be a graph and let $\Gamma_1 \subset \Gamma_0$ be an induced proper subgraph. Fix arbitrary nontrivial vertex groups to form a graph product $\mathcal{G}(\Gamma_0)$. For the canonical CM-choice, let $(c_0, \ldots, c_m)$ be a CM-sequence with respect to $\mathcal{G}(\Gamma_1) < \mathcal{G}(\Gamma_0)$ of length $m + 1$, and let $C_{m+2} \leq \cdots \leq C_0$ be the associated CM-subgroup sequence. Then there is an induced subgraph $\Delta_m$ of $\Gamma$.

To prove Lemma 6.8, we first observe some basic relationship between the graphs defining the subgroups $C_i$ and those supporting $c_i$. Recall that $C_{i+2} = Z_{C_{i+1}}(c_i)$ for all $0 \leq i \leq m$.

**Lemma 6.9.** In the setting of Lemma 6.8, there are induced subgraphs $\Gamma_{m+2} \subset \cdots \subset \Gamma_1 \subset \Gamma_0$ such that for each $0 \leq i \leq m$ there is an induced subgraph $\Lambda_i \subset \Gamma_i$ with the following properties.

1. $\Lambda_i$ is the induced subgraph on the support of $c_i$ for all $0 \leq i \leq m$.
2. For each $0 \leq i \leq m$, $\Gamma_{i+2}$ is the induced subgraph consisting of vertices in $\Gamma_{i+1}$ adjacent to all those in $\Lambda_i$.
3. $C_i = \mathcal{G}(\Gamma_i)$ for all $0 \leq i \leq m + 2$.
4. $\Lambda_i \setminus \Gamma_{i+1} \neq \emptyset$ for any $0 \leq i \leq m$.
5. $\Lambda_i \subset \Gamma_i \setminus \Gamma_{i+2}$ for every $0 \leq i \leq m$.

**Proof.** Bullet (3) holds for $i \in \{0, 1\}$ by definition. Now we consider $i \geq 2$. Inductively from $i = 2$ to $i = m + 2$, we take bullet (1) as the definition of $\Lambda_i$, based on which we define $\Gamma_{i+2}$ as in bullet (2). Then $\Lambda_i \subset \Gamma_i$ and $\Gamma_{i+2} \subset \Gamma_{i+1}$ by definition, and bullet (3) follows from Lemma 6.6. Then bullet (4) holds since $c_i \not\in C_{i+1}$ (as a CM-representative).

To see bullet (5), recall that every vertex of $\Gamma_{i+2}$ is adjacent to all vertices in $\Lambda_i$. If a reduced expression of $c_i$ contains a letter in $G_v$ for some $v \in \Gamma_{i+2}$, then we can shuffle it to the end of $c_i$, contradicting to the choice of $c_i$. \hfill $\square$

Lemma 6.8 follows from the case $i = m$ in Lemma 6.10, which is stated in a way to suit its proof by induction. In below, we say the induced subgraph of $\Gamma_0$ on a sequence of (distinct) vertices $(v_0, \ldots, v_i)$ is isomorphic to $\Delta_i$ as labeled graphs if $v_j$ and $v_k$ are adjacent in $\Gamma_0$ if and only if $|j - k| \geq 2$. 
Lemma 6.10. In the setup of Lemmas 6.8 and 6.9, for each $1 \leq i \leq m$, there is a sequence of distinct vertices $V_i = (v_0, \ldots, v_i)$ of $\Gamma_0$ such that

- $v_1, \ldots, v_i \in \Gamma_{m-i+1}$,
- $v_0 \in \Lambda_{m-i} \setminus \Gamma_{m-i+1}$,

and the induced subgraph of $\Gamma_0$ on $V_i$ is isomorphic to $\Delta_i$ as labeled graphs.

Proof. We show this lemma by induction on $i$. First consider the base case $i = 1$. Let $u$ be an arbitrary vertex in $\Lambda_m \setminus \Gamma_{m+1}$, which exists by bullet (4) of Lemma 6.9. There are following two possibilities.

- If $\Lambda_{m-1} \cap \Gamma_m = \emptyset$, there is some $v_0 \in \Lambda_{m-1}$ not adjacent to $u$ since $u \not\in \Gamma_{m+1}$; See bullet (2) of Lemma 6.9. Then $v_0 \in \Lambda_{m-1} \setminus \Gamma_m = \Lambda_{m-1}$ and $V_1 = (v_0, u)$ satisfies the desired properties.
- If $\Lambda_{m-1} \cap \Gamma_m \neq \emptyset$, write $c_{m-1}$ as a reduced word and let $g_{v_1}$ be the last letter in $c_{m-1}$ that is supported on some $v_1 \in \Gamma_m$. Then there must be some letter $g_{v_0}$ in $c_{m-1}$ supported on $v_0 \in \Lambda_{m-1}$ appearing after $g_{v_1}$ such that $v_0$ and $v_1$ are not adjacent, since otherwise we can shuffle $g_{v_1}$ all the way to the end of $c_{m-1}$ contradicting the fact that $c_{m-1}$ has the shortest syllable length in $C_m c_{m-1} C_m$ and $C_m = \mathcal{G}(\Gamma_m)$. Note that $v_0 \not\in \Gamma_m$ since $g_{v_1}$ is the last letter on a vertex in $\Gamma_m$. Thus, $V_1 = (v_0, v_1)$ satisfies the desired properties.

Suppose the lemma holds for some $1 \leq i < m$ with a sequence of vertices $V_i = (v_0, \ldots, v_i)$. The simplest attempt to obtain $V_{i+1}$ is to add a suitable vertex $w_0$ at the beginning of $V_i$. Since $v_0 \not\in \Gamma_{m-i+1}$, there is some vertex $w_0$ in $\Lambda_{m-i} \cap \Gamma_{m-i}$ that is not adjacent to $v_0$. Note that $v_0 \not\in \Lambda_{m-i-1}$ since otherwise it must be adjacent to all vertices in $\Gamma_{m-i+1}$ and in particular to $v_1$, contradicting the induction hypothesis. Combining with $v_s \in \Gamma_{m-i+1}$ for $s \geq 1$ and $\Gamma_{m-i+1} \cap \Lambda_{m-i-1} = \emptyset$ by bullet (5) of Lemma 6.9, all vertices $w_\ell \in \Lambda_{m-i-1}$ we construct below are distinct from those in $V_i$.

Ideally we would like to choose $w_0$ above so that it lies in $\Lambda_{m-i-1} \setminus \Gamma_{m-i}$, in which case $V_{i+1} := (w_0, v_0, v_1, \ldots, v_i)$ is a desired sequence: Observe that $w_0 \in \Lambda_{m-i-1}$ is adjacent to all $v_1, \ldots, v_i \in \Gamma_{m-i+1}$ but not to $v_0$.

The remaining (harder) case is when every vertex in $\Lambda_{m-i-1} \setminus \Gamma_{m-i}$ is adjacent to $v_0$. In this case, we show the following claim to construct another sequence $W$ of vertices so that the concatenated sequence $(W, V)$ has the desired properties once we cut it down to have exactly $i + 1$ vertices by removing some vertices in the tail. The vertices in $W$ are listed in reverse order to reflect the order they appear in the inductive process below.

Claim 6.11. There is a sequence $W = (w_k, \ldots, w_0)$ of vertices for some $k \geq 1$ such that

- $w_0, \ldots, w_{k-1} \in \Lambda_{m-i-1} \cap \Gamma_{m-i}$,
- $w_k \in \Lambda_{m-i-1} \setminus \Gamma_{m-i}$,
- the induced subgraph on $W$ is isomorphic to $\Delta_k$ as labeled graphs, that is, for $0 \leq s < t \leq k$ the vertices $w_s$ and $w_t$ are adjacent in $\Gamma_0$ if and only if $|s - t| \geq 2$,
- $v_0$ is adjacent to $w_\ell$ if and only if $\ell > 0$.

Proof of Claim 6.11. By our assumption, there is some $w_0 \in \Lambda_{m-i-1} \cap \Gamma_{m-i}$ not adjacent to $v_0$. Choose $g_{w_0}$ to be the last letter on $c_{m-i-1}$ supported on a vertex $w_0$ with this property. Now inductively we can find letters $g_{w_1}, \ldots, g_{w_k}$ of $c_{m-i-1}$ supported on vertices $w_1, \ldots, w_k \in \Lambda_{m-i-1}$ such that
• for each \( 1 \leq \ell \leq k \), \( g_{w_{\ell}} \) is the last letter on \( c_{m-i-1} \) after \( g_{w_{\ell-1}} \) such that \( w_{\ell} \) is not adjacent to \( w_{\ell-1} \),
• \( w_{\ell} \in \Lambda_{m-i-1} \cap \Gamma_{m-i} \) for all \( \ell < k \),
• \( w_k \in \Lambda_{m-i-1} \setminus \Gamma_{m-i} \).

We are guaranteed to end up with some \( w_k \notin \Gamma_{m-i} \): if \( w_k \in \Gamma_{m-i} \), \( g_{w_k} \) cannot commute with all letters after it on \( c_{m-i-1} \) by the minimality of \( c_{m-i-1} \), so we can continue the sequence by adding the last letter \( g_{w_{k+1}} \) on \( c_{m-i-1} \) after \( g_{w_k} \) with the property that \( w_{k+1} \) is not adjacent to \( w_k \).

Then by construction \( W = (w_k, \ldots, w_0) \) consists of distinct vertices and the corresponding induced subgraph in \( \Gamma_0 \) is isomorphic to \( \Delta_k \) as labeled graphs. By our choice of \( w_0 \), we see \( w_{\ell} \) is adjacent to \( v_0 \) if and only if \( \ell > 0 \). This constructs the desired sequence \( W \) in Claim 6.11.

Now we finish the proof of Lemma 6.10. By Claim 6.11, for all \( 0 \leq \ell \leq k \), \( w_{\ell} \) is adjacent to \( v_1, \ldots, v_i \) as \( w_{\ell} \in \Lambda_{m-i-1} \) and \( v_1, \ldots, v_i \in \Gamma_{m-i+1} \). Then for the concatenated sequence \( \bar{V}_{i+1} : = (W, V) \), its corresponding induced subgraph of \( \Gamma_0 \) is isomorphic to \( \Delta_{i+k+1} \) as labeled graphs, and all vertices lie in \( \Gamma_{m-i} \) except that the first vertex \( w_k \) lies in \( \Lambda_{m-i-1} \setminus \Gamma_{m-i} \). Thus, by taking the first \( i + 2 \) vertices in the sequence \( \bar{V}_{i+1} \) as our \( V_{i+1} \), this finishes the inductive proof of Lemma 6.10.

Now we deduce Lemma 6.8 from Lemma 6.10.

**Proof of Lemma 6.8.** The case of \( i = m \) in Lemma 6.10 implies that the induced subgraph of \( \Gamma_0 \) with vertex set \( V_m = (v_0, \ldots, v_m) \) is \( \Delta_m \).

**Proposition 6.12.** Let \( \Gamma \) be a simplicial graph where \( D : = \Delta(\Gamma) < \infty \). Let \( G(\Gamma) \) be a graph product on \( \Gamma \) with arbitrary fixed nontrivial vertex groups. Then for any induced subgraph \( \Lambda \) of \( \Gamma \), the subgroup \( G(\Lambda) \) has property BCMS-\( D \).

**Proof.** It is enough to check the BCMS-\( D \) property using the canonical CM-subgroup choice by Proposition 5.11. As we explained at the end of Subsection 6.1, it suffices to control the length of CM-sequences. By Lemma 6.8, for any CM-sequence \( (c_0, c_1, \ldots, c_m) \), there is an induced subgraph of \( \Gamma \) isomorphic to \( \Delta_m \). Thus, by definition \( D = \Delta(\Gamma) \geq m \). Hence, \( G(\Lambda) \) is a BCMS-\( D \) subgroup.

**6.3 Scl in opposite paths**

Let \( \Delta_m \) be the opposite path on the vertices \( \{v_0, \ldots, v_m\} \) as described above. In this section, we will see that for any (nontrivial) vertex groups \( (G_v)_{v \in V(\Delta_m)} \), the associated graph product \( G(\Delta_m) \) has an integral chain with small scl. Choose a nontrivial element \( g_i \in G_{v_i} \) for every vertex \( v_i \) of \( \Delta_m \). For any \( m \geq 2 \), define a chain \( \delta_m \) in \( G(\Delta_m) \) as

\[
\delta_m := g_{0,m} - g_{0,m-1} - g_{1,m} + g_{1,m-1},
\]

where \( g_{i,j} := g_i \cdots g_j \).

The following computation leads to an upper bound of \( \text{scl}(\delta_m) \).

**Lemma 6.13.** Given \( m \geq 2 \) and \( 0 \leq i \leq m \), for every \( 1 \leq j \leq m - i + 1 \) we have

\[
g_{i,m}^j = g_{i,m-1}^j e_j,
\]
where \( c_j \) is recursively defined as follows: \( c_1 = g_m \) and for \( 1 \leq j \leq m - i \)

\[
c_{j+1} := g_{m-j,m-1}^{-1}c_j g_{m-j,m}.
\]

**Proof.** We proceed by induction. For \( j = 1 \) the result is obvious. Suppose the conclusion holds for some \( j \in \{1, \ldots, m-i\} \). Then

\[
g_{i,m}^{j+1} = g_{i,m}^j \cdot g_{i,m} = g_{i,m-1}^j c_j g_{i,m}.
\]

Since \( c_j \) commutes with all \( g_i, \ldots, g_{m-j-1} \) as it is a product of terms \( g_k \) for \( k \geq m-j+1 \), we see that

\[
g_{i,m}^{j+1} = g_{i,m-1}^j c_j \cdot g_{i,m} = g_{i,m-1}^j g_{i,m-j-1} c_j g_{m-j,m} = g_{i,m-1}^j c_{j+1}.
\]

\( \square \)

**Proposition 6.14.** Let \( m \geq 2 \) and \( \delta_m \) be the chain in \( G(\Delta_m) \) defined as above. Then

\[
\frac{1}{12(m+2)} \leq \text{scl}_{G(\Delta_m)}(\delta_m) \leq \frac{1}{m}.
\]

**Proof.** By Lemma 6.13, we have \( g_{i,m}^m = g_{i,m-1}^m c_m \) for \( i \in \{0, 1\} \), and \( c_m \) does not depend on \( i \). Thus, by Lemma 2.2 we have

\[
\text{scl}(g_{i,m}^m - g_{i,m-1}^m - c_m) \leq \frac{1}{2}
\]

for \( i \in \{0, 1\} \). Therefore,

\[
\text{scl}(m \cdot \delta_m) = \text{scl} \left( g_{0,m}^m - g_{0,m-1}^m - g_{1,m}^m + g_{1,m-1}^m \right)
\]

\[
\leq \text{scl} \left( (g_{0,m}^m - g_{0,m-1}^m - c_m) - (g_{1,m}^m - g_{1,m-1}^m - c_m) \right) \leq 1
\]

by the triangle inequality. Hence, we conclude that

\[
\text{scl}(\delta_m) \leq \frac{1}{m}.
\]

On the other hand, we see that \( \Delta(\Delta_m) = m \), and thus by Theorem 6.2 (proved below) we have \( \text{scl}(\delta_m) \geq \frac{1}{12(m+2)} \), since \( \delta_m \) is already a pure factor chain that is not a vertex chain. This finishes the proof. \( \square \)

### 6.4 Proofs of Theorems 6.2 and Theorem 6.5

We now prove Theorems 6.2 and 6.5.

We first prove the following lemma dealing with an essential part of Theorem 6.2.
Lemma 6.15. Fix an integer $D \geq 1$. If a graph $\Gamma$ satisfies $\Delta(\Gamma) \leq D$, then every integral chain $c$ in $\mathcal{G}(\Gamma)$ either has $\text{scl}_{\mathcal{G}(\Gamma)}(c) \geq \frac{1}{12(D+2)}$ or is equivalent to an integral vertex chain.

Proof. For any integral chain $c = \sum_i g_i$, define its support $\text{supp}(c)$ to be the union of $\text{supp}(g_i)$. Let $\Lambda$ be the induced subgraph of $\Gamma$ on $\text{supp}(c)$, which is finite and $\Delta(\Lambda) \leq \Delta(\Gamma)$ by definition. We may reduce the assertion to the case $\Gamma = \Lambda$ as follows. Note that $\text{scl}_{\mathcal{G}(\Lambda)}(c) = \text{scl}_{\mathcal{G}(\Gamma)}(c)$ since $\mathcal{G}(\Lambda)$ is a retract of $\mathcal{G}(\Gamma)$. If $c$ is not equivalent to a vertex chain in $\mathcal{G}(\Gamma)$, neither is it as a chain in $\mathcal{G}(\Lambda)$. Hence, it suffices to prove the lemma assuming $\Gamma$ to be a finite graph.

We proceed by induction on the size $|\Gamma|$. The assertion trivially holds when $|\Gamma| = 1$ since $c$ must be a vertex chain in this case.

Suppose for some $n \geq 1$ the assertion holds for all integral chains $c$ in any graph product $\mathcal{G}(\Gamma)$ with $|\Gamma| \leq n$. Consider an integral chain $c$ in some graph product $\mathcal{G}(\Gamma)$ with $|\Gamma| = n + 1$ and $\Delta(\Gamma) \leq D$ such that $c$ is not equivalent to a vertex chain. We need to show

$$\text{scl}_{\mathcal{G}(\Gamma)}(c) \geq \frac{1}{12(D+2)}.$$  

Pick any vertex $v$ in $\Gamma$. If $\Gamma = \text{St}(v)$, where $\text{St}(v)$ denotes the star of $v$, then $\mathcal{G}(\Gamma) = G_v \times \mathcal{G}(\text{Lk}(v))$, where $\text{Lk}(v)$ denotes the link of $v$. Then by Proposition 2.14, $c$ is equivalent to a sum of integral chains $c_v + c'$, where $c_v$ is supported on $G_v$ and $c'$ is supported on $\text{Lk}(v)$. Here $c'$ cannot be equivalent to a vertex chain since $c$ is not. Note that $\Delta(\text{Lk}(v)) \leq \Delta(\Gamma) \leq D$ since $\text{Lk}(v)$ is an induced subgraph of $\Gamma$. Thus, by the induction hypothesis and Proposition 2.14, we have $\text{scl}_{\mathcal{G}(\Gamma)}(c) = \text{scl}_{\mathcal{G}(\text{Lk}(v))}(c') \geq \frac{1}{12(D+2)}$.

Now assume $\Gamma \neq \text{St}(v)$. Then $\mathcal{G}(\Gamma)$ splits non-trivially as an amalgam $\mathcal{G}(\Gamma) = \mathcal{G}(\text{St}(v)) \star \mathcal{G}(\text{Lk}(v)) \mathcal{G}(\Gamma \setminus v)$. We know that $\mathcal{G}(\text{Lk}(v)) < \mathcal{G}(\Gamma)$ is a BCMS-$\Delta(\Gamma)$ subgroup by Proposition 6.12. Thus, by Theorem 5.1, it suffices to consider the case where $c$ is equivalent to an integral chain $\bar{c}$ such that every term of $\bar{c}$ lies in $\mathcal{G}(\text{St}(v))$ or $\mathcal{G}(\Gamma \setminus v)$. We may again split every term supported on $\mathcal{G}(\text{St}(v))$ into terms in $G_v$ and in $\mathcal{G}(\text{Lk}(v)) < \mathcal{G}(\Gamma \setminus v)$. Thus, $\bar{c}$ and $c$ are equivalent to a chain $c' + c_v$, where $c'$ is supported on $\Gamma \setminus v$ and $c_v$ is supported on $G_v$. By the monotonicity of $\text{scl}$ for the retraction $\mathcal{G}(\Gamma) \to \mathcal{G}(\Gamma \setminus v)$, we deduce that

$$\text{scl}_{\mathcal{G}(\Gamma)}(c) = \text{scl}_{\mathcal{G}(\Gamma)}(c' + c_v) \geq \text{scl}_{\mathcal{G}(\Gamma \setminus v)}(c').$$

As $c'$ is not equivalent to a vertex chain since $c$ is not, we have $\text{scl}_{\mathcal{G}(\Gamma \setminus v)}(c') \geq \frac{1}{12(D+2)}$ by the induction hypothesis. □

Proof of Theorem 6.2. Let $c$ be an integral chain in a graph product $\mathcal{G}(\Gamma)$. By Lemma 6.15, either $\text{scl}_{\mathcal{G}(\Gamma)}(c) \geq \frac{1}{12(\Delta(\Gamma)+2)}$ or $c$ is equivalent to a vertex chain. By Proposition 3.12, $c$ is equivalent to such a vertex chain if and only if $c^{\text{pf}}$ is a vertex chain. □

Proof of Theorem 6.5. For any graph $\Gamma$ and a graph product $\mathcal{G}(\Gamma)$ on $\Gamma$, the inclusion $i_m : \mathcal{G}(\Delta_m) \to \mathcal{G}(\Gamma)$ is a retract, where $m = \Delta(\Gamma)$. By Proposition 6.14, there is an integral chain $\delta_m$ in $\mathcal{G}(\Delta_m)$ such that

$$\frac{1}{12(m+2)} \leq \text{scl}_{\mathcal{G}(\Delta_m)}(\delta_m) \leq \frac{1}{m}.$$  

Since a chain in the retract has the same $\text{scl}$ as in the whole group (Proposition 2.6), we conclude that $\delta = i_m(\delta_m)$ has the same property. This concludes the proof. □
6.5 Applications to RAAGs and right-angled Coxeter groups

Our gap theorems can be simplified in the case of RAAGs and right-angled Coxeter groups.

**Theorem 6.16** (RAAGs and RACGs). Let $G$ be the right-angled Artin (or Coxeter) group with defining graph $\Gamma$. Then for any integral chain $c$ not equivalent to the zero chain, we have $\text{scl}_G(c) \geq \frac{1}{12(\Delta(\Gamma)+2)}$.

**Proof.** Note that any null-homologous chain of the form $\sum_v c_v$ in $G$ is equivalent to the zero chain since each vertex group is abelian, where each $c_v$ is a chain in the vertex group $G_v$. Thus, the result follows from Theorem 6.2.

By Theorem 6.5, the gap above cannot be uniform in the class of RAAGs, although there is uniform gap $1/2$ for elements in RAAGs [26]. It is natural to ask whether this holds analogously for RACGs.

**Question 6.17.** Is there a uniform spectral gap for elements in RACGs?

Note that there is a uniform gap theorem [14, Theorem F] for elements in many graph products, but it does not apply to RACGs because of the existence of 2-torsion. However, we are able to characterize elements in RACGs with zero scl.

**Corollary 6.18.** Let $G$ be the right-angled Coxeter group with defining graph $\Gamma$. Then for any element $g \in G$, we have either $\text{scl}_G(g) \geq \frac{1}{12(\Delta(\Gamma)+2)}$ or $\text{scl}_G(g) = 0$. Moreover, the latter case occurs if and only if $g$ is conjugate to $g^{-1}$, or more precisely, $g = ab$ with $a^2 = \text{id}$ and $b^2 = \text{id}$.

**Proof.** The first assertion directly follows from Theorem 6.16. It also implies that $\text{scl}_G(g) = 0$ if and only if $g$ is equivalent to the zero chain. Hence, by Proposition 3.14, we obtain the more explicit characterization of such $g$. One can similarly characterize elements with zero scl in other graph products if elements with zero scl are understood in vertex groups.

We also get a uniform gap for integral chains if we add a hyperbolicity assumption. Since the only hyperbolic RAAGs are free groups, we focus on hyperbolic RACGs below.

**Corollary 6.19.** Let $G = C(\Gamma)$ be a hyperbolic right-angled Coxeter group. Then $\text{scl}_G(c) \geq \frac{1}{60}$ for any integral chain not equivalent to the zero chain.

**Proof.** It is known by [31] that $C(\Gamma)$ is hyperbolic if and only if the graph $\Gamma$ has no induced subgraph isomorphic to the cyclic graph of length 4. Note that the graph $\Delta_4$ contains such an induced subgraph with vertices $v_0, v_1, v_3, v_4$. Thus, $\Delta(\Gamma) \leq 3$ if $C(\Gamma)$ is hyperbolic. Hence, the result follows from Theorem 6.16.

Based on this, we make the following conjecture.

**Conjecture 6.20.** There is a uniform constant $B > 0$ such that any hyperbolic $C$-special (or $A$-special, see [24] for definitions) group has a spectral gap $B$ for integral chains.
If the conjecture holds true, one can use it and the index formula (Proposition 2.9) to establish effective lower bounds for the index of special subgroups in hyperbolic groups. For instance, it is a well-known theorem that every hyperbolic 3-manifold group contains a finite index subgroup that is special (and hyperbolic) [1], but it is unknown whether the index has a uniform upper bound independent of the manifold. This connection was suggested to us by Danny Calegari and motivated this work on scl of integral chains in RAAGs, but we did not anticipate the spectral gap to be non-uniform.

One can also bound \( \Delta(\Gamma) \) in terms of other invariants of the graph \( \Gamma \).

**Corollary 6.21.** If \( \Gamma \) is a simplicial graph where each vertex has valence at most \( m \geq 0 \), then integral chains in \( A(\Gamma) \) and \( C(\Gamma) \) have a gap \( \frac{1}{12(m+3)} \).

**Proof.** Note that in \( \Delta_{m+2} \) the vertex \( v_0 \) is adjacent to \( m+1 \) vertices \( v_2, v_3, \ldots, v_{m+2} \). Thus, we must have \( \Delta(\Gamma) \leq m+1 \). We conclude by Theorem 6.16. \( \square \)

The dimension of a right-angled Artin (respectively, Coxeter) group \( A(\Gamma) \) (respectively, \( C(\Gamma) \)) associated to some simplicial graph \( \Gamma \) is the largest size of cliques in \( \Gamma \).

**Corollary 6.22.** Any right-angled Artin (respectively, Coxeter) group \( G = A(\Gamma) \) (respectively, \( G = C(\Gamma) \)) of dimension at most \( d \) has a gap \( \frac{1}{12(2d+1)} \) for integral chains.

**Proof.** Note from the definition that \( \Delta_{2d} \) contains a clique of size \( d+1 \) with vertices \( v_0, v_2, \ldots, v_{2d} \). Thus, \( \Delta(\Gamma) \leq 2d - 1 \), and the result follows from Theorem 6.16. \( \square \)

### 6.6 Groups with interesting scl spectra

Theorem 6.16 implies interesting properties of the spectrum of the infinitely generated RAAG \( A(\Delta_{\infty}) \).

**Proposition 6.23.** The set of values obtained as scl of integral chains in \( A(\Delta_{\infty}) \) is dense in \( \mathbb{R}_{\geq 0} \), and in particular there is no spectral gap. However, there is a gap \( 1/2 \) for elements in \( A(\Delta_{\infty}) \).

**Proof.** Note that \( A(\Delta_{\infty}) \) retracts to \( A(\Delta_m) \) for any \( m \in \mathbb{Z}_+ \). Thus,

\[
\text{scl}_{A(\Delta_{\infty})}(\delta_m) = \text{scl}_{A(\Delta_m)}(\delta_m) \in \left[ \frac{1}{12(m+2)}, \frac{1}{m} \right]
\]

by Proposition 6.14, where \( \delta_m \) is defined in Subsection 6.3. Thus, we obtain a sequence of integral chains whose scl is positive and converges to 0. Taking integer multiples of such integral chains proves the density. The gap \( 1/2 \) for elements in \( A(\Delta_{\infty}) \) is shown in [26]. \( \square \)

No groups were previously known to have a gap for elements but no gap for integral chains.

With a small modification to the group, we can make scl values of elements eventually dense in \( \mathbb{R}_{\geq 0} \).
Proposition 6.24. Let $G = A(\Delta_\infty) \ast F_3$, where $F_3$ is the free group generated by $a, b, c$. Then $\text{scl}_G(g) \geq 1/2$ for all $g \neq id \in G$, and the set $\{\text{scl}_G(g) \mid g \in [G, G]\}$ is dense in $[3/2, \infty)\).

Proof. If $g \neq id$ conjugates into $A(\Delta_\infty)$ or $F_3$, the lower bound $1/2$ is known by [18, 26]. Otherwise, the lower bound $1/2$ follows from [12] since both factor groups are torsion-free.

As for the density, recall that the integral chain $\delta_m = g_0, m - g_1, m - g_0, m - 1 + g_1, m - 1$ has $\text{scl}$ between $1/(12(m + 2))$ and $1/m$. Applying Proposition 2.8 to $g = c b a g_0, m^{-1} a^{-1} g_1, m^{-1} b^{-1} g_0, m - 1 c^{-1} g_1, m^{-1}$ for any $n \in \mathbb{Z}^+$, we have

$$\text{scl}_G(g) = \text{scl}_{A(\Delta_\infty)}(n \delta_m) + \frac{3}{2} \in \left[ \frac{3}{2} + \frac{n}{12(m + 2)}, \frac{3}{2} + \frac{n}{m} \right].$$

The density follows since $m$ and $n$ are arbitrary positive integers. \hfill \Box

7  |  SCL OF VERTEX CHAINS

We describe an algorithm to compute $\text{scl}$ of vertex chains in Subsection 7.1. This allows us to relate $\text{scl}$ to the fractional stability number (fsn) of graphs in Subsection 7.2. In Subsection 7.3, we observe and explain the similarity in histograms of $\text{scl}$ and fsn on random words and graphs, respectively (Figure 1).

7.1  |  Computation by linear programming

Given any vertex chain $c$, we will give two linear programming problems ($P_c$) and $(P^*_c)$ that both compute $\text{scl}(c)$. They are dual to each other and thus yield dual solutions. Moreover, feasible solutions of $(P_c)$ yield quasimorphisms with controlled defects and feasible solutions to $(P^*_c)$ yield admissible surfaces.

To describe the linear programming problems, we introduce the following notion.

Definition 7.1. A stable measure on a graph $\Gamma$ is a list of numbers $\mu = (\mu_v)_{v \in V}$, one for each vertex, such that

- the sum of $\mu_v$ over all vertices in any given clique $q$ of $\Gamma$ is at most 1;
- $\mu_v \geq 0$ for each $v$.

A set $S$ of vertices is called a stable set if they are pairwise non-adjacent in $\Gamma$. Equivalently, each clique contains at most one vertex in $S$. Thus, the indicator function of any stable set is a stable measure.

Given a stable measure $\mu$ and a vertex chain $c$, let

$$|\mu|_c := \sum_{v \in V} \mu_v \cdot \text{scl}_G(c_v),$$

which is linear in $\mu_v$. Then maximizing $|\mu|_c$ among stable measures is a linear programming problem $(P_c)$ since the defining properties of a stable measure are linear inequalities in $\mu_v$. Note
that the set of stable measures is a compact convex rational polyhedron in $\mathbb{R}^{|V|}$, and thus the problem $(P_\nu)$ has an optimal solution at a rational point.

In general, one can replace \( \text{scl}_G(c_v) \) by other weights on vertices, and the corresponding problem is called the \textit{fractional weighted stability number} in graph theory; see [23, p. 333]. Note that the result is rational if the weights are, since the feasible set is a rational polyhedron.

To describe its dual problem $(P_\nu^*)$, we introduce weighted clique cover.

**Definition 7.2.** Given a vertex chain $c$, a \textit{weighted clique cover} with respect to $c$ is a list of real numbers (considered as weights) $y = \{y_q\}$, indexed by the cliques $q$ of $\Gamma$ such that

- the sum of $y_q$ over all cliques containing any given vertex $v$ is at least $\text{scl}_G(c_v)$,
- $y_q \geq 0$ for every clique $q$.

For any weighted clique cover $y$, let $|y| := \sum y_q$, where the sum is taken over all cliques $q$ of $\Gamma$.

Then minimizing $|y|$ over all weighted clique covers with respect to $c$ is the linear programming problem $(P_\nu^*)$ dual to the problem $(P_\nu)$. Thus, they have the same optimal value by the strong duality theorem of linear programming, explained as follows.

**Lemma 7.3.** For any vertex chain $c$ in a graph product $G(\Gamma)$, we have

$$\max_\mu |\mu|_c = \min_y |y|,$$

where the maximization is taken over stable measures $\mu$ and the minimization is taken over weighted clique covers $y$.

**Proof.** Let $CL(\Gamma)$ be the set of cliques of $\Gamma$. Let $M_\Gamma$ be the 0–1 matrix where the columns are indexed by the vertices $v \in V(\Gamma)$ and the rows are indexed by all cliques $q \in CL(\Gamma)$ such that the $(q, v)$-entry is 1 if and only if $v \in q$.

Then in matrix form, the problem $(P_\nu)$ is to maximize $s^T \cdot \mu$ subject to $M_\Gamma \cdot \mu \leq 1_{CL(\Gamma)}$ and $\mu \geq 0$. Here $1_{CL(\Gamma)}$ is the vector of 1’s of length $|CL(\Gamma)|$ and $s$ is the vector indexed by $V(\Gamma)$ with entry $\text{scl}_G(c_v)$ at vertex $v$. By the strong duality theorem of linear programming [33, p. 91 (19)] the optimal value agrees with the minimal value of $1^T_{CL(\Gamma)} \cdot y$ subject to $M_\Gamma^T \cdot y \geq s$ and $y \geq 0$, which is the matrix form of $(P_\nu^*)$. \qed

The main result of this subsection is that both $(P_\nu)$ and $(P_\nu^*)$ compute $\text{scl}_{G(\Gamma)}(c)$.

**Theorem 7.4.** For any vertex chain $c$ in a graph product $G = G(\Gamma)$, we have

$$\text{scl}_G(c) = \max_\mu |\mu|_c = \min_y |y|,$$

where the maximization is taken over stable measures $\mu$ and the minimization is taken over weighted clique covers $y$.

By Lemma 7.3, to prove Theorem 7.4, it suffices to establish the following two lemmas.

**Lemma 7.5.** For any vertex chain $c$ in a graph product $G = G(\Gamma)$, we have $\text{scl}_G(c) \leq |y|$ for any weighted clique cover $y$ with respect to $c$. 


**Lemma 7.6.** For any vertex chain $c$ in a graph product $G = G(\Gamma)$, we have $\text{scl}_G(c) \geq |\mu|_c$ for any stable measure $\mu$.

To prove Lemma 7.5, we first show that $\text{scl}$ of a vertex chain is increasing in the coefficients.

**Lemma 7.7.** Fix a chain $c_v$ in each vertex group $G_v$. Given numbers $\lambda_v \geq \lambda'_v \geq 0$ for each vertex $v$, we have $\text{scl}_G(\sum_v \lambda_v c_v) \geq \text{scl}_G(\sum_v \lambda'_v c_v)$.

**Proof.** It suffices to show that $\text{scl}$ is non-decreasing in every single $\lambda_u$, fixing $\lambda_v$ for all $v \neq u$. Split $G$ as an amalgam $G(\text{St}(u)) \star G(Lk(u)) \star G(\Gamma \setminus \{u\})$. Then we think of the vertex chain $c = \sum_u \lambda_v c_v = \lambda_u c_u + \sum_{v \neq u} \lambda'_v c_v$ as a sum of two chains supported on the two factor groups.

By [14, Theorem 6.2], we have

$$\text{scl}_G(c) = \inf_d \left[ \text{scl}_{G(\text{St}(u))}(\lambda_u c_u + d) + \text{scl}_{G(\Gamma \setminus \{u\})}(-d + \sum_{v \neq u} \lambda'_v c_v) \right],$$

where the infimum is taken over all chains $d$ in $G(Lk(v))$.

Since $G(\text{St}(u)) = G_u \times G(Lk(u))$ is a direct product, by Proposition 2.14 we have $\text{scl}_{G(\text{St}(u))}(\lambda_u c_u + d) = \max(\lambda_u \text{scl}_{G_u}(c_u), \text{scl}_{G(\text{St}(u))}(d))$, which is clearly non-decreasing in $\lambda_u$. Thus, $\text{scl}_G(c)$ is non-decreasing in $\lambda_u$ by the formula above. $\square$

**Proof of Lemma 7.5.** Recall that each induced subgroup is a retract in a graph product, so $\text{scl}$ in the two groups agree for any chain in the subgroup (Proposition 2.6). Without loss of generality, assume $\text{scl}_{G_v}(c_v) > 0$ for each vertex $v$, as otherwise we may consider the problem on the induced subgroup supported on those vertices with this property. Given a weighted clique cover $y$, for each clique $q$, define a vertex chain $d_q = \sum_{v \in q} \frac{y_q}{\text{scl}_{G_v}(c_v)} c_v$. Since $G(q)$ is the direct product of vertex groups $G_v$ for $v \in q$, by Proposition 2.14, we have

$$\text{scl}_{G(q)}(d_q) = \text{scl}_{G(q)}(d_q) = \max_{v \in q} \text{scl}_{G_v}\left( \frac{y_q}{\text{scl}_{G_v}(c_v)} c_v \right) = y_q.$$

Consider the vertex chain $\sum_q d_q = \sum_v \frac{\sum_{q \ni v} y_q}{\text{scl}_{G_v}(c_v)} c_v$. Note that the coefficient of $c_v$ is $\frac{\sum_{q \ni v} y_q}{\text{scl}_{G_v}(c_v)}$, which is no less than 1, the coefficient of $c_v$ in $c$, by the definition of weighted clique cover. Thus, by Lemma 7.7, we have

$$\text{scl}(c) = \text{scl}\left( \sum_v c_v \right) \leq \text{scl}\left( \sum_q d_q \right) \leq \sum_q \text{scl}(d_q) = \sum_q y_q = |y|.$$

$\square$

To prove Lemma 7.6, we construct quasimorphisms and use Bavard’s duality.

Given a quasimorphism $f_v$ on each vertex group $G_v$, we can combine them to obtain a quasimorphism $f$ on the graph product $G = G(\Gamma)$ as follows.

For each element $g \in G$ with reduced expression $g = g_1 \cdots g_n$, we naturally have a vertex chain $s(g) := \sum_i g_i$. This only depends on $g$ since reduced expressions are unique up to syllable shuffling.
Define \( f(c) := \sum_v f_v(c_v) \) for all vertex chains and extend \( f \) to \( G(\Gamma) \) by setting
\[
f(g) := f(s(g))
\]
using the splitting above.

**Lemma 7.8.** If each \( f_v \) is antisymmetric, then the function \( f \) defined above is a quasimorphism on \( G \) with defect \( D(f) = \sup_q \sum_{v \in q} D(f_v) \), where the supremum is taken over all cliques \( q \) of \( \Gamma \).

**Proof.** For each clique \( q \) and each vertex \( v \in q \), we can find elements \( g_q, h_q \in G_v \) with \( f_v(g_q) + f_v(h_q) - f_v(g_qh_q) \) arbitrarily close to \( D(f_v) \). Then for \( g_q := \prod_{v \in q} g_v \) and \( h_q := \prod_{v \in q} h_v \), we have
\[
f(g_q) + f(h_q) - f(g_qh_q) = \sum_{v \in q} [f_v(g_v) + f_v(h_v) - f_v(g_vh_v)],
\]
which can be made arbitrarily close to \( \sum_{v \in q} D(f_v) \). This proves the ‘\( \geq \)’ direction.

For the reversed direction, for any \( g, h \in G \), we have reduced expressions \( g = g_0q_gg_\chi \) and \( h = h_\chi^{-1}q_\chi h_0 \) as in Proposition 3.4, where \( supp(q_g) = supp(q_\chi) = q = \{v_1, ..., v_k\} \) is a clique, \( q_g = g_1 \cdots g_k, q_\chi = h_1 \cdots h_k \) with \( g_i, h_i \in G_{v_i} \), and \( gh \) admits a reduced expression \( gh = g_0q_\chi h_0 \). Since each \( f_i \) is antisymmetric, we have \( f(x) + f(x^{-1}) = 0 \). The definition of \( f \) implies that for the reduced expression \( g = g_0q_g \chi \) we have \( f(g) = f(g_0) + f(q_g) + f(\chi) \) and similarly for \( h \) and \( gh \). Hence,
\[
|f(g) + f(h) - f(gh)| = |f(q_g) + f(q_\chi) - f(q_\chi g)| = \sum_{i=1}^k |f_i(g_i + h_i - g_ih_i)| \leq \sum_{i=1}^k D(f_v),
\]
where the second inequality uses the formula derived in the first part of the proof. This proves the equality. \( \square \)

Now we are in a place to prove Lemma 7.6.

**Proof of Lemma 7.6.** For each vertex \( v \), let \( \phi_v \) be an extremal antisymmetric quasimorphism as in Proposition 2.12 for the chain \( c_v \), that is, we have \( \tilde{\phi}_v(c_v) = scl_{G_v}(c_v) \) and \( D(\phi_v) = 1/4 \).

Given any stable measure \( \mu = (\mu_v) \), let \( f_v = \mu_v \cdot \phi_v \) and let \( f \) be the quasimorphism obtained as above by combining \( f_v \)’s. Then for each clique \( q \), we have \( \sum_{v \in q} D(f_v) = \frac{1}{4} \sum_{v \in q} \mu_v \leq 1/4 \) by the definition of stable measures. Thus, by Lemma 7.8, we have \( D(f) \leq 1/4 \), and thus \( D(\tilde{f}) \leq 1/2 \) by Proposition 2.10. Note that \( \tilde{f}(g_v) = \mu_v \cdot \tilde{\phi}_v(g_v) \) for each \( g_v \in G_v \) and similarly for any chain in \( G_v \). By Bavard’s duality, we have
\[
scl_{G}(c) \geq \frac{\tilde{f}(c)}{2D(\tilde{f})} \geq \tilde{f}(c) = \sum_v \mu_v \cdot \tilde{\phi}_v(c_v) = \sum_v \mu_v \cdot scl_{G_v}(c_v) = |\mu|_c.
\]

**Proof of Theorem 7.4.** We have \( \max_x |\mu|_c \leq scl_{G(\Gamma)}(c) \leq \min_y |y| \) by Lemmas 7.5 and 7.6. By Lemma 7.3, we know \( \max_x |x|_c = \min_y |y| \), which proves the equality. \( \square \)

Summarizing the results in this subsection, we give a proof of Theorem 6.3.

**Proof of Theorem 6.3.** The linear programming problems \((P_c)\) and \((P^*_c)\) both compute \( scl_{G(\Gamma)}(c) \) by Theorem 7.4. Since the optimal solution of \((P_c)\) is achieved at some rational point, we see that
Theorem 7.4 yields an algorithm to compute scl of vertex chains. This algorithm has been implemented in Python. The code may be found on the second author's website.

7.2 Scl and fractional stability number

In this section, we consider the case where all the vertex terms in the vertex chain have the same scl. This relates scl to well-studied invariants in graph theory.

To be explicit, we construct for a given graph \( \Gamma \) a graph \( D_\Gamma \) and a chain \( d_\Gamma \) in the RAAG \( A(D_\Gamma) \), such that \( A(D_\Gamma) \) can be also viewed as a graph product over \( \Gamma \) and \( d_\Gamma \) is a vertex chain where each term has scl 1/2.

**Definition 7.9 (Double graph).** For a graph \( \Gamma \) with vertex and edge set \( V(\Gamma) \) and \( E(\Gamma) \), let \( D_\Gamma \) be the graph with vertex and edge set

\[
V(D_\Gamma) = \{a_u, b_u \mid u \in V(\Gamma)\} \quad \text{and} \quad E(D_\Gamma) = \{(a_u, a_w), (a_u, b_w), (b_u, a_w), (b_v, b_w) \mid (u, w) \in E(\Gamma)\}.
\]

Moreover, let \( d_\Gamma = \sum_{v \in V(\Gamma)} [a_v, b_v] \) in \( A(D_\Gamma) \). Then \( D_\Gamma \) is called the double graph and \( d_\Gamma \) the double chain associated to \( \Gamma \).

**Definition 7.10 (Fractional stability number).** Let \( \Gamma \) be a graph. Then the fractional stability number of \( \Gamma \) is defined as

\[
\text{fsn}(\Gamma) := \max_{\mu} \sum_{u} \mu_{u},
\]

where the maximum is taken over all stable measures \( \mu \).

The fractional stability number of a graph is the fractional chromatic number of its opposite graph. This invariant appears more frequently in the literature. For a reference to fractional stability number see [32]. The results of the previous section implies:

**Theorem 7.11 (scl and fsn).** Let \( \Gamma \) be a graph and let \( D_\Gamma \) and \( d_\Gamma \) be the associated double graph and double chain, respectively. Then

\[
scl_{A(D_\Gamma)}(d_\Gamma) = \frac{1}{2} \text{fsn}(\Gamma),
\]

where \( \text{fsn}(\Gamma) \) is the fractional stability number of \( \Gamma \).

\( \dagger \) https://www.nicolausheuer.com/code.html
Proof. Note that for each \( v \in \Gamma \), the vertices \( a_v \) and \( b_v \) are not adjacent and hence \( A(\{a_v, b_v\}) \) is a free group \( F_v = F(a_v, b_v) \) of rank two. Also note that \( a_v \) and \( b_v \) are both (respectively, not) adjacent to \( a_u \) and \( b_u \), if \( v \) is (respectively, not) adjacent to \( u \). Thus, we observe that \( A(D_{\Gamma}) \) is a graph product over \( \Gamma \), where the vertex groups are the free groups \( F_v \). In this view, \( d_{\Gamma} \) is a vertex chain where each vertex term is \( [a_v, b_v] \), which satisfies \( \text{scl}_{F_v}(\{a_v, b_v\}) = 1/2 \). Thus, the result follows from Theorem 7.4.

For the rest of this subsection, we apply known results of \( \text{fsn} \) on graphs to deduce properties of \( \text{scl} \) in such groups. We first describe the full spectrum of \( \text{fsn} \) on graphs. Note that the full spectrum of \( \text{scl} \) is not known even in the best understood case of free groups.

**Proposition 7.12** see also [32, Proposition 3.2.2]. The set of numbers that appear as \( \text{fsn}(\Gamma) \) for some nonempty graph \( \Gamma \) is

\[
\{1\} \cup [2, \infty) \cap \mathbb{Q}.
\]

Proof. We already know that \( \text{fsn}(\Gamma) \) is always rational since the feasible set is a rational polyhedron. It is also easy to note that \( \text{fsn}(\Gamma) \geq 1 \) since each vertex is a stable set, and that \( \text{fsn}(\Gamma) \geq 2 \) whenever there are two non-adjacent vertices.

So, it suffices to construct graphs to achieve all rational numbers \( r \geq 2 \). For any \( m \geq 2 \) and \( n \geq 2m \), let \( \Gamma_{m,n} \) be the graph with \( n \) vertices \( v_1, \ldots, v_n \) such that it is the union of cliques on \( v_{i+1}, \ldots, v_{i+m} \) for all \( 1 \leq i \leq n \), where indices are taken mod \( n \). We claim that \( \text{fsn}(\Gamma_{m,n}) = n/m \), from which the result would follow.

Using \( n \geq 2m \), it is straightforward to check that the cliques used to described \( \Gamma_{m,n} \) are all the maximal cliques. Thus, having weight \( 1/m \) on all vertices is a stable measure, which shows \( \text{fsn}(\Gamma_{m,n}) \geq n/m \).

On the other hand, assigning weight \( 1/m \) to each maximal clique (and 0 to all smaller cliques) is a weighted clique cover, and hence \( \text{fsn}(\Gamma_{m,n}) \leq n/m \) by the dual problem. Thus, \( \text{fsn}(\Gamma_{m,n}) = n/m \).

For comparison, it is known that \( \text{scl} \) in free groups has a sharp lower bound \( 1/2 \), and based on experiments, the spectrum seems to be proper in \( [1/2, 3/4) \) and dense in \( [3/4, \infty) \). However, it appears to be much harder if possible at all to construct families of elements or integral chains in free groups with \( \text{scl} \) achieving arbitrary rational numbers greater than 1.

Combining Theorem 7.11 and Proposition 7.12, we deduce:

**Theorem 7.13** (Rational realization). For every rational number \( q \geq 1 \) there is an integral chain \( c \) in a RAAG \( A(\Gamma) \) such that \( \text{scl}_{A(\Gamma)}(c) = q \).

Computing the fractional stability number is NP-hard [22]. This implies that computing \( \text{scl} \) in RAAGs is also NP-hard.

**Theorem 14** (NP-hardness). Unless \( P = NP \), there is no algorithm which, given a simplicial graph \( \Gamma \), an element \( g \in A(\Gamma) \) and a rational number \( q \in \mathbb{Q}^+ \) decides if \( \text{scl}_{A(\Gamma)}(g) \leq q \) in polynomial time in \( |V(\Gamma)| + |g| \). The same holds for chains.

Proof. It is known [22] that computing \( \text{fsn} \) for a graph \( \Gamma \) is NP-hard. Given a graph \( \Gamma \), we may in polynomial time construct the double graph and the double chain \( d_{\Gamma} \in A(D_{\Gamma}) \). By Theorem 7.11, computing \( \text{scl}(d_{\Gamma}) = 1/2 \text{fsn}(\Gamma) \) is NP-hard as well.
Statistical analysis of SCL and FSN: Let $X$ be either a random scl in $F_2$ on words of length 24 or a fsn of a random graph on 25 vertices. Let $X_n$ denote the set of elements with denominator exactly $n$. (a) This figure plots $e_n = \# X_n$, the number of elements having denominator $n$ for 50000 random samples. (b) This figure shows the distribution of SCL having denominator 13, 23 and 29 for 50000 samples. (c) This figure shows the different standard sample deviations of $X_n$.

Let $\bar{D}_\Gamma$ be the graph obtained from $D_\Gamma$ by adding $|V(\Gamma)|$ isolated vertices. Then $A(\bar{D}_\Gamma)$ is a free product $A(D_\Gamma) \star F_{|V(\Gamma)|}$. Using Proposition 2.8, we may in polynomial time construct an element $\bar{d}$ in $A(\bar{D}_\Gamma)$ such that $\text{scl}(\bar{d}) = \text{scl}(d_\Gamma) + \frac{|V(\Gamma)|-1}{2}$. Thus, computing scl of elements in RAAGs is also NP-hard.

7.3 | Histograms of scl and fsn

Although it is NP-hard, we may compute fsn relatively quickly for graphs with up to 30 vertices. This allows us to perform computer experiments on the distribution of fsn for random graphs. The result of these experiments is recorded (rescaled by $1/2$) in Figure 1b in the introduction. Here we considered 50000 random graphs with 25 vertices, where between every two vertices there is an edge with probability $1/2$. This reveals an interesting distribution of fsn on random graphs: Values with low denominator appear much more frequently and the histogram exhibits a self-similar behavior.

The same type of histogram has been observed for scl of random elements in the free group (Figure 1a). Here we consider 50 000 uniformly chosen random words of length 24 in the commutator subgroup of the free group $F_2$. See [8, section 4.1.9] for a discussion of this phenomenon and comparison to Arnold’s tongue. Explanations of these patterns in the frequency for either scl or fsn are not known.

In this section, we will give a brief statistical analysis of both scl and fsn. We show that both scl and fsn can be modeled using the same type of distributions which we describe in Definition 7.15. While this is purely heuristic, it indicates that fsn and scl converge for large graph sizes / word lengths to a similar distribution; see Question 7.16.

Let $SCL$ denote the random variable $2 \cdot \text{scl}(W)$ where $W$ is the random variable with a uniform distribution on $\{w \in F_2 \mid |w| \leq 24\}$ and let $FSN$ be the random variable $\text{fsn}(\Gamma)$ where $\Gamma$ is a random variable with uniform distribution $\{\Gamma \mid V(\Gamma) = 25\}$. We note that the factor of 2 for scl is intended and indeed necessary. In light of the relationship to Euler characteristic (Definition 2.1) and Bavard’s Duality Theorem (Theorem 2.11), $2 \cdot \text{scl}$ seems to be the more natural invariant. The histograms of 50 000 independent instances of $SCL$ and $FSN$ may be found in Figure 7.
We make the following two crucial heuristic observations.

1. For large integers \( n \), we observe that \( P(X \text{ has denominator } n) \sim \frac{\phi(n)}{n^d} \), where \( \phi \) is Euler’s Totient function and \( X \) is SCL or FSN. This is depicted in Figure 6a. Experimentally we may estimate that \( d \sim 1.7 \) for SCL and \( d \sim 2.5 \) for FSN. It is also apparent that for smaller \( n \) this heuristic does not hold, and that instead this coefficient is much smaller. This suggests that the exponent may be approximated by \( d \cdot (1 - n^{\beta}) \) for some negative \( \beta \).

2. For a fixed denominator \( n \), let \( X_n \) be the random variable of \( X \) conditioned on that \( X \) has denominator \( n \). Then \( X_n \) follows roughly a normal distribution \( B_n \) (rounded to the closest rational in \( 1/n \)) with fixed mean \( \mu \) and standard deviation \( \sigma_n \); see Figure 6b. The standard deviation appears to be roughly of the form \( \sigma_n = c_1 \cdot n^{c_2} \); see Figure 6c.

This suggests that both the histogram of scl and fsn are the result of an interference of several (rounded) ‘normal’ distributions \( B_n \).

These observations lead us to the following construction of a random variable \( X \) depending on real parameters \( d, \beta, \mu, c_1, c_2 \).
Definition 7.15 (The distribution \( X \)). Let \( d < -1, \beta < 0, c_1 > 0, c_2 < 0, \) and \( \mu \) be real parameters. Define the random variable \( X = X(d, \beta, \mu, c_1, c_2) \) as follows:

Set \( p(n, \beta, d) = n^{(1-\beta) d} \). Choose with probability \( p(n, \beta, d) / \sum_{n=1}^{\infty} p(n, \beta, d) \) an integer \( n \in \mathbb{N} \). Choose the rational \( X \) in \( \frac{1}{n} \mathbb{Z} \) as follows: Let \( N_n \) be the random variable with distribution \( \mathcal{N}(\mu, (c_1 \cdot n^2)^2) \), the normal distribution with mean \( \mu \) and standard deviation \( c_1 \cdot n^2 \). Set \( X \) to be the number in \( \frac{1}{n} \mathbb{Z} \) closest to \( N_n \).

The distribution of \( X \) may be found on the second authors website\(^1\). We may use this to fit \( X \) to SCL and FSN. The result of this experiment is shown in Figure 7. At least qualitatively, \( X \) is a good approximation of the distribution of SCL and FSN.

Based on this, we ask:

Question 7.16. Is there a natural distribution \( Y \) indexed by some parameter set \( P \) such that there are sequences of parameters \( s_n, f_n \) for \( n \in \mathbb{N} \) such that as \( n \to \infty \), both the random variable \( \text{scl}(w) \), for \( w \) uniformly chosen from \( \{w \in [F_2, F_2] \mid |w| = 2 \cdot n\} \), and \( \text{fsn}(\Gamma) \) where \( \Gamma \) is uniformly chosen among all graphs with \( n \) vertices converge almost surely to \( Y(s_n) \) and \( Y(f_n) \), respectively?

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\(^1\) https://www.nicolausheuer.com/code.html
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