On convex hull and winding number of self similar processes

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Abstract:
It is well known that for a standard Brownian motion (BM) \( \{B(t), t \geq 0\} \) with values in \( \mathbb{R}^d \), its convex hull \( V(t) = \text{conv}\{B(s), s \leq t\} \) with probability 1 for each \( t > 0 \) contains 0 as an interior point (see Evans (1985)). We also know that the winding number of a typical path of a 2-dimensional BM is equal to \( +\infty \).

The aim of this article is to show that these properties aren’t specifically "Brownian", but hold for a much larger class of \( d \)-dimensional self similar processes. This class contains in particular \( d \)-dimensional fractional Brownian motions and (concerning convex hulls) strictly stable Levy processes.

Key-words: Brownian motion, multi-dimensional fractional Brownian motion, stable Levy processes, convex hull, winding number.

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1 Introduction

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a basic probability space. Consider a \( d \)-dimensional process \( X = \{X(t), t \geq 0\} \) defined on \( \Omega \) which is self-similar of index \( H > 0 \). It means that for each constant \( c > 0 \) the process \( \{X(ct), t \geq 0\} \) has the same distribution as \( \{c^H X(t), t \geq 0\} \).

Let \( L = \{L(u), u \in \mathbb{R}^1\} \) be the strictly stationary process obtained from \( X \) by Lamperti transformation:

\[
L(u) = e^{-Hu}X(e^u), \quad u \in \mathbb{R}^1.
\]

Equivalently,

\[
X(t) = t^H L(\log t), \quad t \in \mathbb{R}^+.
\]

Let \( \Theta = \{0, 1\}^d \) be the set of all dyadic sequences of length \( d \). Denote by \( D_\theta \).

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\( \theta \in \Theta \), the quadrant
\[
D_{\theta} = \prod_{i=1}^{d} \mathbb{R}_{\theta_i},
\]
where \( \mathbb{R}_{\theta_i} = [0, \infty) \) if \( \theta_i = 1 \), and \( \mathbb{R}_{\theta_i} = (-\infty, 0] \) if \( \theta_i = 0 \).

The positive quadrant \( D_{(1,1,...,1)} \) for simplicity is denoted by \( D \).

We say that the process \( X \) is non degenerate if for all \( \theta \in \Theta \)
\[
\mathbb{P}\{X(1) \in D_{\theta}\} > 0.
\]

Two important examples of self similar processes are fractional Brownian motion and stable Levy process.

**Definition 1** We call a self-similar (of index \( H > 0 \)) process \( B^H \) fractional Brownian motion (FBM) if for each \( e \in \mathbb{R}^d \) the scalar process \( t \rightarrow \langle B^H(t), e \rangle \) is a standard one-dimensional FBM of index \( H \) up to a constant \( c(e) \).

It is easy to see that in this case \( c^2(e) = \langle Qe, e \rangle \), where \( Q \) is the covariance matrix of \( B^H(1) \), and hence
\[
\mathbb{E}\langle B^H(t), e \rangle \langle B^H(s), e \rangle = \langle Qe, e \rangle \frac{1}{2} (t^{2H} + s^{2H} - |t - s|^{2H}), \quad t, s \geq 0; \quad e \in \mathbb{R}^d.
\]

The process \( B^H \) is non degenerate iff the rank of the matrix \( Q \) is equal to \( d \). If \( H = \frac{1}{2}, Q = I_d \), then \( B^H \) is a standard Brownian motion.

(See A. Xiao (2013), Račkhauskas and Ch. Suquet (2011), F. Lavancier et al. (2009) and references therein for more general definitions of operator self-similar FBM).

**Definition 2** We call \( S(t), t \in \mathbb{R}_+ \) \( \alpha \)-strictly stable Levy process (StS) if
1) \( S(1) \) has a \( \alpha \)-strictly stable distribution in \( \mathbb{R}^d \);
2) it has independent and stationary increments;
3) it is continuous in probability.

Then for each \( t \in \mathbb{R}_+ \) the random variable \( S(t) \) has the same distribution as \( t^\frac{1}{\alpha} S(1) \).
The cadlag version of $S$ on $[0, 1]$ can be obtained with the help of LePage series representation (see [7] for more details). If $\alpha \in (0, 1)$ or if $\alpha \in (1, 2)$ and $EX(1) = 0$, then we have:

$$
\{S(t), t \in [0, 1]\} \overset{\mathcal{D}}{=} \{c \sum_{k=1}^{\infty} \Delta_{k}^{-1/\alpha} \varepsilon_{k} 1_{[0, t]}(\eta_{k}), \; t \in [0, 1]\},
$$

where $c$ is a constant, $\Delta_{k} = \sum_{j=1}^{k} \gamma_{j}$, $\{\gamma_{j}\}$ is a sequence of i.i.d. random variables with common standard exponential distribution, $\{\varepsilon_{k}\}$ is a sequence of i.i.d. random variables with common distribution $\sigma$ concentrated on unit sphere $S^{d-1}$, $\{\eta_{k}\}$ is a sequence of $[0, 1]$-uniformly distributed i.i.d. random variables, and the three sequences $\{\gamma_{j}\}$, $\{\varepsilon_{k}\}$, $\{\eta_{k}\}$ are supposed to be independent.

The measure $\sigma$ is called spectral measure of $S$. It is easy to see that if \[\overset{\mathcal{D}}{=}\] takes place, the process $X$ is non degenerate iff $\text{vect}\{\text{supp} \sigma\} = \mathbb{R}^{1}$.

In Section 2 the object of our interest is the convex hull process $V = \{V(t)\}$ associated with $X$. We show that under very sharp conditions with probability 1 for all $t > 0$ the convex set $V(t)$ contains 0 as its interior point. From this result some interesting corollaries are deduced.

Section 3 is devoted to studying the winding numbers of two-dimensional self similar processes. As a corollary of our main result we show that for the typical path of a standard two-dimensional FBM the number of its clockwise and anti-clockwise winds around 0 in the neighborhood of zero or at infinity is equal to $+\infty$.

## 2 Convex hulls

For a Borel set $A \subset \mathbb{R}^{d}$ we denote by $\text{conv}(A)$ the closed convex hull of $A$ and define the convex hull process related to $X$:

$$
V(t) = \text{conv}\{X(s), \; s \leq t\}.
$$

**Theorem 1** Let $X$ be a non degenerate self similar process such that the strictly stationary process $L$ generating $X$ is ergodic. Then with probability 1 for all $t > 0$ the point 0 is an interior point of $V(t)$.

**Application to FBM.** Let $B^{H}$ be a FBM with index $H$. The next properties follow from the definition without difficulties.

1) **Continuity.** The process $X$ has a continuous version.

Below we always suppose $B^{H}$ to be continuous.
2) **Reversibility.** If the process $Y$ is defined by

$$Y(t) = B^H(1) - B^H(1-t), \quad t \in [0,1],$$

then \(\{Y(t), \ t \in [0,1]\} \overset{L}{=} \{B^H(t), \ t \in [0,1]\}\), where \(\overset{L}{=}\) means equality in law.

3) **Ergodicity.** Let \(L = \{L(u), \ u \in \mathbb{R}^1\}\) be the strictly stationary Gaussian process obtained from \(B^H\) by Lamperti transformation \(\Box\).

Then \(L\) is ergodic (see Cornfeld et al. (1982), Ch. 14, §2, Th.1, Th.2).

It is supposed below that the process \(B^H\) is non degenerate.

**Corollary 1** Let \(V\) be the convex hull process related to \(B^H\). Then with probability 1 for all \(t > 0\) the point 0 is an interior point of \(V(t)\).

This follows immediately from Th.1.

**Corollary 2** Let \(V\) be the convex hull process related to \(B^H\). Then for each \(t > 0\) with probability 1 the point \(B^H(t)\) is an interior point of \(V(t)\).

**Proof of Corollary 2.** Denote by \(A^\circ\) the interior of \(A\). By self-similarity of the process \(B^H\) it is sufficient to state this property for \(t = 1\). Then, due to the reversibility of \(B^H\) by Th. 1., a.s.

$$0 \in \text{conv}\{ B^H(1) - B^H(1-t), \ t \in [0,1]\}^\circ.$$  \hspace{1cm} (3)

As

\[
\text{conv}\{ B^H(1) - B^H(1-t), \ t \in [0,1]\} = B^H(1) - \text{conv}\{ B^H(1-s), \ s \in [0,1]\},
\]

the relation (3) is equivalent to

$$B^H(1) \in \text{conv}\{ B^H(s), \ s \in [0,1]\}^\circ,$$

which concludes the proof.  \(\Box\)

Let \(\mathcal{K}_d\) be the family of all compact convex subsets of \(\mathbb{R}^d\). It is well known that \(\mathcal{K}_d\) equipped with Hausdorff metric is a Polish space.

We say that a function \(f : [0,1] \to \mathcal{K}_d\) is increasing, if \(f(t) \subset f(s)\) for \(0 \leq t < s \leq 1\).
We say that a function \( f : [0, 1] \to K_d \) is almost everywhere constant, if \( f \) is such that for almost every \( t \in [0, 1] \) there exists an interval \((t - \varepsilon, t + \varepsilon)\) where \( f \) is constant.

We say that a function \( f : [0, 1] \to K_d \) is a Cantor - staircase (C-S), if \( f \) is continuous, increasing and almost everywhere constant.

The next statement follows easily from Corollary 2.

**Corollary 3** Let \( V \) be the convex hull process related to \( B^H \). Then with probability 1 the paths of the process \( t \to V(t) \) are C-S functions.

**Remark 1** Let \( h : K \to \mathbb{R}^1 \) be an increasing continuous function. Then almost all paths of the process \( t \to h(V(t)) \) are C-S real functions. This obvious fact may be applied to all reasonable geometrical characteristics of \( V(t) \), such as volume, surface area, diameter, ...

**Application to StS.** Let now \( S \) be a StS process with exponent \( \alpha < 2 \). The following properties are more or less known.

1) **Right continuity.** The process \( S \) has a cadlag version (see remark above just after the definition).

2) **Reversibility.** Let \[ Y(t) = S(1) - S(1 - t), \quad t \in [0, 1]. \]

Then \( \{Y(t), \; t \in [0, 1]\} \overset{\mathcal{L}}{=} \{S(t), \; t \in [0, 1]\} \).

3) **Self-similarity.** The process \( S \) is self-similar of index \( H = \frac{1}{\alpha} \).

4) **Ergodicity.** Let \( L = \{L(u), \; u \in \mathbb{R}^1\} \) be the strictly stationary process obtained from \( S \) by Lamperti transformation \[1\]. Then \( L \) is ergodic.

We suppose that the law of \( S(1) \) is non degenerate.

**Corollary 4** Let \( V \) be the convex hull process related to \( S \). Then with probability 1 for all \( t > 0 \) the point 0 is an interior point of \( V(t) \).

**Corollary 5** Let \( V \) be the convex hull process related to \( S \). Then for each \( t > 0 \) with probability 1 the point \( X(t) \) is an interior point of \( V(t) \).
Corollary 6 Let $V$ be the convex hull process related to $S$. Then with probability 1 the paths of the process $t \to V(t)$ are right continuous almost everywhere constant functions.

We omit proofs of these statements as they are similar to proofs of Corollaries 1 - 3.

Proof of Theorem 1. We first show that

$$p \overset{\text{def}}{=} \mathbb{P}\{ \exists t \in (0,1] | X(t) \in D^\circ \} = 1. \quad (4)$$

Remark that $p$ is strictly positive:

$$p \geq \mathbb{P}\{X(1) \in D^\circ \} > 0 \quad (5)$$
due to the hypothesis that the law of $X(1)$ is non degenerate.

By self similarity

$$\mathbb{P}\{D^\circ \cap \{X(t), t \in [0,T]\} = \emptyset\} = 1 - p$$

for every $T > 0$. The sequence of events $(A_n)_{n \in \mathbb{N}}$, 

$$A_n = \{D^\circ \cap \{X(t), t \in [0,n]\} = \emptyset\},$$

being decreasing, it follows that

$$1 - p = \lim \mathbb{P}(A_n) = \mathbb{P}(\cap_n A_n) = \mathbb{P}\{X(t) \notin D^\circ, \forall t \geq 0\}.$$

In terms of the stationary process $L$ from Lamperti representation it means that

$$\mathbb{P}\{L(s) \notin D^\circ, \forall s \in \mathbb{R}^1\} = 1 - p.$$ 

As this event is invariant, by ergodicity of $L$ and due to (5) we see that the value $p = 1$ is the only one possible.

Applying the similar arguments to another quadrants $D_{\theta}, \theta \in \Theta$, we get that with probability 1 there exists points $t_\theta \in [0,1]$, such that $X(t_\theta) \in D_{\theta}^\circ$, $\theta \in \Theta$. Now, to end the proof it is sufficient to remark that

$$V(1) = \text{conv}\{X(t), t \in [0,1]\}^\circ \supset \text{conv}\{X(t_\theta), \theta \in \Theta\}^\circ$$

and that the last set evidently contains 0.
3 Winding numbers

Now we consider a 2-dimensional self similar process \( X = \{X(t), \ t \geq 0\} \). It is supposed that the following properties are fulfilled:

1) Process \( X \) is continuous.
2) Process \( X \) is non-degenerate.
3) Process \( X \) is symmetric: \( X \) and \( -X \) have the same law.
4) The stationary process \( L \) associated with \( X \) is ergodic.
5) Starting from \( 0 \) the process \( X \) with probability 1 never come back:
   \[
   \mathbb{P}\{X(t) \neq 0, \ \forall \ t > 0\} = 1.
   \] (6)

Due to the last hypothesis, considering \( \mathbb{R}^2 \) as complex plane, we can define the winding numbers (around \( 0 \)) \( \nu[\ s, \ t\] , \( 0 < s < t \), by the usual way (see [5], Ch.5):

\[
\nu[\ s, \ t\] = \arg (X(t)) - \arg (X(s)).
\]

We set

\[
\nu_+(0, t] = \limsup_{s \downarrow 0} \nu[\ s, \ t\] , \ \nu_-(0, t] = \liminf_{s \downarrow 0} \nu[\ s, \ t\]
\]

\[
\nu_+[s, \infty) = \limsup_{t \to \infty} \nu[s, t], \ \nu_-[s, \infty) = \liminf_{t \to \infty} \nu[s, t].
\]

The values \( \nu_+(0, t], \ \nu_-(0, t] \) represent respectively the number of clockwise and anti-clockwise winds around \( 0 \) in the neighborhood of the starting point, while \( \nu_+[s, \infty), \ \nu_-[s, \infty) \) are the similar winding numbers at infinity.

**Theorem 2** Let \( X \) be a 2-dimensional self similar process with the properties 1)–5) mentioned above. Then with probability one for all \( t > 0 \)

\[
\nu_+(0, t] = \nu_+[t, \infty) = -\nu_-(0, t] = -\nu_-[t, \infty) = +\infty.
\] (7)

**Corollary 7** Let \( B^H \) be a 2-dimensional standard FBM and assume that \( H \in [1/2, 1) \). Then with probability one for all \( t > 0 \) the equalities (7) take place.

**Proof.** Case \( H = 1/2 \) is well known, see [5], Ch. 5, which contains exhaustive information on Brownian winding numbers.

If \( H \in (1/2, 1) \), we apply Theorem 2 as all hypothesis 1)–5) are fulfilled: indeed, the properties 1)–3) are obvious; the ergodicity of \( L \), \( L(t) = (L_1(t), L_2(t)) \), follows from the fact that \( EL_1(t)L_1(0) \to 0 \) when \( t \to \infty \) (see, [11], Ch. 14, Sec. 2, Th.2); The property 5) can be deduced from Th. 11 of [8] (see also Th. 4.2 of [9] and Th. 2.6 of [10]).
Remark 2 If $H \in (0, \frac{1}{2})$, the process $t \to \arg B^H(t) - \arg B^H(0)$ is not continuous with positive probability as the set $\{ t \in (0,1) \mid B^H(t) = 0 \}$ is not empty (see [8], Th. 11). It means that in this case the winding numbers could be defined only for the excursions of $B^H$, and we need for its study more sophisticated methods.

Proof of Theorem 2. By 5) we have
\[ \mathbb{P}\{L(t) \neq 0, \; \forall \; t \in \mathbb{R}^1\} = 1. \]
Hence as above we can define for $L$ the winding numbers $\nu^L_+(-\infty, t], \nu^L_+t, \infty)$, and besides we have
\[ \nu^L_+(-\infty, t] = \nu^L_+(0, e^t], \quad \nu^L_+[t, \infty) = \nu^L_+[e^t, \infty). \]
Therefore from now on we can work with the process $L$ and will omit the index $L$ in the notation of winding numbers.

Let us show that
\[ \mathbb{P}\{|\nu^L_+[t, \infty)| = \infty, \; \forall \; t \in \mathbb{R}^1\} = 1. \] (8)

By symmetry (property 3)) it is sufficient to state that
\[ \mathbb{P}\{\nu^L_+[t, \infty) = \infty, \; \forall \; t \in \mathbb{R}^1\} = 1. \] (9)

Using the arguments from the proof of Th. 1 we remark that the process $L$ visits infinitely often each of four basic quadrants. It follows by continuity that at least one of two events $A, B$,
\[ A = \{ \exists t > 0, \; \text{such that} \; \arg X(t) - \arg X(0) > \frac{\pi}{2} \}, \]
\[ B = \{ \exists t > 0, \; \text{such that} \; \arg X(t) - \arg X(0) < \frac{\pi}{2} \}, \]
has probability 1. By symmetry (property 3)) $\mathbb{P}(A) = \mathbb{P}(B)$. Thus,
\[ \mathbb{P}\{\exists t > 0, \; \text{such that} \; \arg X(t) - \arg X(0) > \frac{\pi}{2} \} = 1. \]

From this follows by stationarity that for all $s \in \mathbb{R}^1$,
\[ \mathbb{P}\{\exists t > s, \; \text{such that} \; \arg X(t) - \arg X(s) > \frac{\pi}{2} \} = 1. \]
The set
\[ E = \{ (s, \omega) \in \mathbb{R}^1 \times \Omega \mid \exists t > s, \ \text{such that} \ \arg X(t) - \arg X(s) > \frac{\pi}{2} \} \]
is measurable as the process \( s \to \sup_{t > s} (\arg X(t) - \arg X(s)) \) is continuous.

Based on the aforementioned and due to Fubini theorem, the set \( E \) is such that \( \lambda \times \mathbb{P}(E^c) = 0 \), \( \lambda \) being the Lebesgue measure. Therefore there exists \( \Omega' \subset \Omega, \ \mathbb{P}(\Omega') = 1 \) such that for each \( \omega \in \Omega' \), for almost all \( s \in \mathbb{R}^1 \), there exists \( t > s \) for which \( \arg X(t) - \arg X(s) > \frac{\pi}{2} \). Take \( \omega \in \Omega' \). Let us denote \( E_\omega \) the corresponding \( \omega \)-section of \( E \). Without loss of generality, we can suppose that for each \( \omega \in \Omega' \), the point 0 belongs to \( E_\omega \). As \( \lambda \times \mathbb{P}(E_\omega^c) = 0 \), \( E_\omega \) is dense in \( \mathbb{R}^1 \).

Let \( u > 0 \) be such that \( \arg X(u) - \arg X(0) > \frac{\pi}{2} \). By continuity, \( \arg X(t) - \arg X(0) > \frac{\pi}{2} \) for all \( t \) in a sufficiently small neighborhood of \( u \) and therefore, there exists \( t_1 \in E_\omega \) for which \( \arg X(t_1) - \arg X(0) > \frac{\pi}{2} \).

Repeating this reasoning, we can build an increasing sequence \( (t_n) \) such that \( t_1 = 0 \) and \( t_n \in E_\omega \). Since for each \( n \), \( \arg X(t_n) - \arg X(t_{n-1}) > \frac{\pi}{2} \), we get \( \sup_{t > 0} (\arg X(t) - \arg X(0)) = +\infty \).

Thus, it is proved that for each \( t \)
\[ \mathbb{P}\{\nu_+[t, \infty) = +\infty\} = 1. \tag{10} \]

Now to show that
\[ \mathbb{P}\{\nu_+[t, \infty) = +\infty, \ \forall t \in \mathbb{R}^1\} = 1 \]
it is sufficient to remark that for each \( \omega \) from \( \Omega' \) the \( \omega \)-section \( E_\omega = \mathbb{R}^1 \).
Indeed, supposing that there exists \( u \in E_\omega^c \) we should have
\[ \arg X(s) - \arg X(u) \leq \frac{\pi}{2} \]
for each \( s > t \), but that is in contradiction with the existence of \( t \in E_\omega, \ t > u \), for which (10) holds. Thus (9) is proved. Applying the previous reasonings to the process \( \{L(-t), \ t \in \mathbb{R}^1\} \), we prove the remaining equalities of (7).

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