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Pose control of multi-rigid-body systems with distance-dependent topologies

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Abstract
This paper investigates the pose control problem of multi-rigid-body systems, where the distance-dependent graphs are adopted to describe the interaction relations between rigid bodies. The authors design distributed control laws for torque and force of rigid bodies, with back-stepping technique and potential function method being utilized. An admissible set for initial states of rigid bodies is presented, on which the theoretical results about attitude synchronization, collision avoidance and connectivity maintenance are established without relying on the prior connectivity assumption of the dynamical neighbour graphs. The authors introduce the initial neighbourhoods or the neighbourhoods corresponding to a minimum spanning tree extracted from the initial neighbour graph into the controller, to reduce the communication pressure on system and the constraints on motion of rigid bodies. Furthermore, a leader is introduced into the system to guide all rigid bodies to a desired attitude. Simulation examples are given to verify the theoretical results.

1 INTRODUCTION

In recent decades, the cooperative control problem of multi-agent systems has generated great interest in natural systems like birds and fishes and practical systems like robotics and unmanned aerial vehicle (UAVs). Most existing works view the agents as mass points, which is no longer appropriate for some systems such as robotics, UAVs and satellites; under such a case, modeling the agents as rigid bodies will be more suitable. One of the important topics in the study of multiple rigid bodies is designing a distributed control law for each rigid body such that the whole system cooperates to reach a certain pose coordination state.

The motion of a rigid body in three-dimensional space includes rotation and translation, which are described, respectively, by attitude and position. The pair of attitude and position is the pose of rigid body. Note that the attitude of rigid bodies can only be represented by rotation matrix globally and uniquely. Generally speaking, based on the control inputs of the system, the pose control problem can be classified into two categories, that are controller on the kinematic level and on the dynamical level. For the kinematic level category, control inputs are torque and force. There are some works that investigated the pose control problem of rigid bodies on the kinematic level. For example, Igarashi et al. [1] and Hatanaka et al. [2] studied the pose synchronization problem under fixed and strong connected neighbour graphs, and then extended to the switching topology case where the time length that the graphs are not strong connected is assumed to satisfy a certain condition. Thunberg et al. [3] investigated the pose formation control problem under both fixed and switching topologies. Our previous works (cf. [4]) also considered the pose control problem of multiple rigid bodies under switching topology. Note that all of the above works directly assume the dynamical connectivity of neighbour graphs, which is hard to verify and guarantee.

From a physical perspective, control on the dynamical level for the rigid bodies is more meaningful and practical. Some results about the pose control problem have been obtained. Lee et al. [5] considered the formation tracking problem for spacecrafts formation flying; Nazari et al. [6] considered the tracking control of spacecraft formation with time delays. However, the control laws in both of the works are decentralized. We note that most of the relevant works only considered the attitude control problem; for example, Sarlette et al. [7] and [8]...
considered the attitude synchronization problem under fixed and switching topologies; Guo et al. [9] achieved finite-time attitude synchronization with undirected topology, and then extended to a more complex case where the angular velocity of rigid bodies cannot be measured; Weng et al. in [10] designed an event-triggered control law to achieve attitude synchronization. There are also some works that used the parameterizations of rotation matrix such as rotation vector (cf. [11]), Rodrigues parameters (cf. [12]), modified Rodrigues parameters (MRPs) (cf. [13–15]) and unit quaternion (cf. [16–18]) to investigate the attitude control problem. Also, these results rely on the dynamical connectivity assumptions of neighbour graphs.

Connectivity of neighbour graphs plays a key role in guaranteeing the coordination of multi-agent systems. In order to remove the prior assumptions on dynamical connectivity of neighbour graphs, researchers turn to looking for methods to preserve the connectivity. As far as we know, the most commonly used approach to maintain connectivity of the dynamical neighbour graphs is the potential function method, and the most commonly used approach to maintain connectivity is the event-triggered control law to achieve attitude synchronization. Also, these results rely on the dynamical connectivity assumptions of neighbour graphs.

There are also some works that used the parameterizations of rotation matrix such as rotation vector, Rodrigues parameters, MRPs and unit quaternion to represent the attitude of rigid bodies. We apply the back-stepping technique and high gain control method to control the attitude of rigid bodies to reach synchronization; meanwhile, we utilize the potential function method to avoid collision between rigid bodies and preserve connectivity of neighbour graphs. In order to reduce communication pressure, in the design of attractive control term for connectivity preservation, we introduce the initial neighbourhoods to release the constraint on those newly formed edges so that they are not allowed to break. Besides, we extract a minimum spanning tree from the initial graph and then use its corresponding neighbourhoods to replace the initial neighbourhoods, to further reduce the communication of system and limitation on motion of rigid bodies. Admissible set for initial states of rigid bodies is found to guarantee attitude synchronization, collision avoidance and connectivity maintenance. Furthermore, we introduce a leader into the system to guide all rigid bodies to the desired attitude. The main contributions of this work are summarized as follows:

1) Most existing works about rigid bodies used parameterizations of rotation matrix such as rotation vector, Rodrigues parameters, MRPs and unit quaternion to represent the attitude of rigid bodies (cf. [11–18]); however, these parameterizations are often local and are not one-to-one. Moreover, most relevant papers only considered the attitude control subproblem to simplify their analysis. In this work, we use rotation matrix to globally and uniquely represent the attitude of rigid bodies and simultaneously control the attitude and position of rigid bodies. Note that there is a coupled relationship between the rotation matrices and positions of rigid bodies, which makes the theoretical analysis very difficult.

2) Compared to fixed or switching topology that almost all existing works about rigid bodies adopted, this work used the distance-dependent topologies to describe the neighbour relations, which is more in line with the actual communication between rigid bodies. For both the leaderless and leader–follower case, sufficient conditions for attitude synchronization and collision avoidance were established without assuming a prior connectivity of the dynamical neighbour graphs.

3) We introduced the initial neighbourhoods, or even the neighbourhoods corresponding to a minimum spanning tree extracted from the initial neighbour graph, into the control term for connectivity maintenance. Compared with the existing results, the newly formed edges or even the initial edges that not in the extracted minimum spanning tree can be allowed to be lost, which may reduce the communication pressure on system and motion constraints on rigid bodies.

This paper proceeds as follows. In Section 2, we provide preliminaries including rigid bodies and graphs, and formulate the pose control problem. In Section 3, the control laws for attitude and position of rigid bodies are designed. Section 4 presents the main results and the analysis. Two extension cases are considered in Section 5. Section 6 gives simulation examples to illustrate the theoretical results. Conclusions are summarized in Section 7.

## 2 PRELIMINARIES AND PROBLEM FORMULATION

Let us consider a system composed of $n$ rigid bodies moving in the three-dimensional space. We want to design distributed control laws for rigid bodies such that the attitudes of rigid bodies reach synchronization asymptotically while the connectivity of the dynamical neighbour graphs being preserved and the collision among rigid bodies being avoided.

### 2.1 Models of rigid bodies

Denote the inertial coordination frame as $\Sigma_w$, and fix a frame $\Sigma_i$ on the rigid body $i$ ($i = 1, 2, \ldots, n$). This paper assumes $\Sigma_w$ and $\Sigma_i$ are right-handed and Cartesian. Then, from [24], the kinematic model of rigid bodies are given as follows:

$$
\begin{align}
\dot{R}_i(t) &= R_i(t)\hat{\omega}_i(t) \\
\dot{p}_i(t) &= R_i(t)\dot{u}_i(t)
\end{align}
$$

(1)
where \( R_i(t) \in \mathbb{R}^{3 \times 3} \) is the attitude of rigid body \( i \), representing the rotation of \( \Sigma_i \) relative to \( \Sigma_n \) at time \( t \). Murray [24] shows that rotation matrix has the following properties: 1) \( R_i(t)R_j^T(t) = I_3 \); 2) \( \det(R_i(t)) = 1 \) with \( \det(\cdot) \) being the determinant of the corresponding matrix. The set formed by all rotation matrices is the Lie group \( SO(3) = \{ R \in \mathbb{R}^{3 \times 3} : RR^T = I_3, \det(R) = 1 \} \).

\( p_i(t) \in \mathbb{R}^3 \) is the position of rigid body \( i \)'s center at time \( t \) in the inertial frame \( \Sigma_n \). The pair of attitude and position \( (R_i, p_i(t)) \) is the pose of rigid body \( i \), and all the poses form the set \( SE(3) = \{(R, p) : R \in SO(3), p \in \mathbb{R}^3 \} \), which is the Cartesian product of \( SO(3) \) and \( \mathbb{R}^3 \). \( \omega_i(t) \in \mathbb{R}^3 \) is the angular velocity of \( \Sigma_i \) at time \( t \) relative to \( \Sigma_n \), also viewed in the body frame \( \Sigma_i \). In model (1), \( R_i(t) \) and \( p_i(t) \) are the state systems, and \( \omega_i(t), \dot{u}_i(t) \) are the inputs. The notation \( \hat{\omega} \) denotes the skew symmetric matrix generated by the vector \( \omega = [\omega_1, \omega_2, \omega_3]^T \in \mathbb{R}^3 \) as follows:

\[
\hat{\omega} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\
\omega_3 & 0 & -\omega_1 \\
-\omega_2 & \omega_1 & 0 \end{bmatrix}
\]

(2)

i.e. \( \wedge : \mathbb{R}^3 \rightarrow so(3) \) with \( so(3) = \{ A \in \mathbb{R}^{3 \times 3} : A^T = -A \} \) being the set of skew symmetric matrices. Denote the inverse operator to \( \wedge \) as \( \vee : so(3) \rightarrow \mathbb{R}^3 \), then for any \( A \in so(3) \), \( A^\vee \) is expressed as follows:

\[
A^\vee = \begin{bmatrix} -a_{23} & a_{12} & a_{13} \\
a_{13} & -a_{12} & a_{23} \\
a_{12} & a_{23} & 0 \end{bmatrix}
\]

(3)

The dynamical model of rigid bodies are given by

\[
\begin{align*}
\dot{\omega}_i(t) &= f_i^{-1}\left[ -\hat{\omega}_i(t)\omega_i(t) + \tau_i(t) \right] \\
\dot{u}_i(t) &= f_i(t)/m_i - \hat{\omega}_i(t)u_i(t)
\end{align*}
\]

(4)

where \( f_i \in \mathbb{R}^{3 \times 3} \) is the constant inertia matrix of rigid body \( i \), \( m_i \) is the mass of \( i \), \( \tau_i(t) \in \mathbb{R}^3 \) and \( f_i(t) \in \mathbb{R}^3 \) are the control torque and control force at time \( t \) expressed in \( \Sigma_i \), respectively, and they are the control inputs.

### 2.2 Distance-dependent graphs

The distance-dependent graphs are adopted to describe the interaction of rigid bodies. Denote the interaction radius as \( r \), then two rigid bodies \( i \) and \( j \) are said to be neighbours if and only if their distance is less than \( r \), i.e. \( \| p_{ij}(t) \| < r \), where \( p_{ij} \) represents the relative position between rigid bodies \( i \) and \( j \) viewed in \( \Sigma_i \) and \( p_{ij}(t) = R_i^{-1}(t)(p_i(t) - p_j(t)) \). Then, the neighbour set of rigid body \( i \) can be denoted as \( N_i = \{ j : \| p_{ij}(t) \| < r, j \neq i \} \). Note that \( N_i \) can be based on either communication or estimation and is to be clarified by the context. If communication is used, the rigid bodies can receive information from their neighbours, whereas if estimation is used, they observe their neighbours by, for example, vision.

Denote \( p(t) = (p_1^T(t), \ldots, p_n^T(t))^T \), then the interaction between rigid bodies can be described by a distance-dependent graph sequence \( G(p(t), r) = (V, E(p(t), A(p(t))) \), where the vertex set \( V = \{ 1, \ldots, n \} \), the edge set containing pairs of neighbouring rigid bodies is defined as \( E(p(t)) = \{ (i, j) : \| p_{ij}(t) \| < r \} \) and \( A(p(t)) \) is the weight set. The interaction weight \( a_{ij}(p(t)) \) is a continuous and differential function varying from 0 to 1, and \( a_{ij}(p(t)) > 0 \) if and only if \( \| p_{ij}(t) \| < r \). This paper uses the function introduced in [21] to define the weights \( a_{ij}(p(t)) \)

\[
a_{ij}(p(t)) = \rho \left( \frac{\| p_{ij}(t) \|}{\| r \|} \right) \in [0, 1], \quad i \neq j
\]

where

\[
\rho(z) = \begin{cases} 1, & z \in [0, b] \\
\frac{1}{2} \left( 1 + \cos \left( \pi \frac{z-h}{1-h} \right) \right), & z \in [b, 1] \\
0, & \text{otherwise}
\end{cases}
\]

\[
\| r \| = \frac{1}{\epsilon} \left( \sqrt{1 + \epsilon \| z \|^2} - 1 \right)
\]

with \( b \) being a positive constant satisfying \( b \in (0, 1) \) and \( \epsilon > 0 \) being a fixed parameter. The \( \sigma \)-norm is not a strict norm, but it is differentiable everywhere, overcoming the drawback that the ordinary Euclidean norm is not differentiable at zero. We have its gradient \( \sigma_z(\| z \|) = \| z \| (1 + \epsilon \| z \|^2) \). Obviously, \( \| p_{ij}(t) \|_\sigma = \| p_{ij}(t) - p_i(t) \|_\sigma \) and \( \sigma_z(\| p_{ij}(t) \|) = R_i^{-1}(t) \cdot \sigma_z(\| p_{ij}(t) - p_i(t) \|) \).

Define \( D(p(t)) \) as the degree matrix with diagonal elements \( d_i(p(t)) = \sum_{j=1}^{n} a_{ij}(p(t)) \), then the Laplacian matrix corresponding to graph \( G(p(t), r) \) is defined as \( L(p(t)) = D(p(t)) - A(p(t)) \). It is clear that \( L(p(t)) \) is symmetric for all \( t \geq 0 \) and \( 1_z^T L(p(t)) 1_z = 0, L(p(t))^1_z = 0 \), where \( 1_z \) represents the \( n \)-dimensional vector with all elements being 1.

### 2.3 Problem formulation

The pose control problem of rigid bodies is concerned with designing distributed control laws for torque and force of rigid bodies such that their poses reach the desired shape. For this purpose, we assume the neighbour set \( N_i \) in this paper be based on both communication and estimation. Assume that the rigid body \( i \) can measure its own velocities \( \{ \omega_i(t), v_i(t) \} \) by using devices such as gyroscopes and accelerators, and then transmit this information to its neighbours \( j \in N_i \). In addition, we assume that the rigid body \( i \) can observe the relative information \( \{ R_{ij}(t), p_{ij}(t), j \in N_i \} \) of its neighbours, where \( R_{ij}(t) = R_i^{-1}(t) R_j(t) \) represents the rotation matrix of \( \Sigma_i \) relative to \( \Sigma_j \) and \( p_{ij}(t) \) represents the relative position viewed in
Then, the purpose of this paper is to design control laws for torque $\tau_i(t)$ and force $f_i(t)$ based on the information received and observed by $i$, such that the pose of rigid bodies fulfills the following tasks:

1) The system reaches attitude synchronization, meaning the attitudes of all rigid bodies converge to a common attitude asymptotically, i.e. $R_{ij}(t) \to I_3$, $\forall i,j$ as $t \to \infty$.
2) The angular velocity and linear velocity of rigid bodies reach the same, respectively.
3) There is no collision between rigid bodies when they are in motion, i.e. $\|p_{ij}(t)\| > s$, $\forall i \neq j, \forall t \geq 0$, where the positive constant $s$ satisfying $s < r$ denotes the safe distance between rigid bodies.
4) The connectivity of the dynamical neighbour graphs is preserved, i.e. $G(p(t), r)$ is connected for all $t \geq 0$.

Remark 1. Reynolds in 1980s (cf. [25]) proposed the classical flocking model for multi-agent systems, which consists of three rules, that are separation, alignment and cohesion. We believe that the collision avoidance; attitude synchronization and velocity consensus; and connectivity preservation tasks are corresponding to these three rules. Flocking of multi-rigid-body systems has wide practical applications such as rescue and reconnaissance. To fulfill such tasks, it is essential that rigid bodies maintain relative or the same attitudes; at the same time, rigid bodies need to avoid collision with each other and aggregate together to maintain communication.

Remark 2. In this work, we use rotation matrix to represent the attitude of rigid bodies, which is different from most existing results where the parameterized representations of rotation matrices, such as rotation vectors, Rodrigues parameters, MRPs and unit quaternions (cf. [11–18]), are adopted. Compared with the parameterized representations, rotation matrix can globally and uniquely represent the attitude of rigid bodies. However, it is worthy to notice that there is complex coupling relationship between the attitude and position of rigid bodies whose interaction is described by dependent-graphs, and how to deal with the coupled relationship is the key point of our analysis.

3 | DESIGN OF DISTRIBUTED CONTROL

In this section, we will design control laws for torque $\tau_i(t)$ and force $f_i(t)$, which are used for attitude synchronization and position coordination, respectively. We will continue by first presenting the control for attitudes.

3.1 | Control for attitudes

The control for attitudes is mainly concerned about the attitude synchronization task. In [26], the distributed control law for $\omega_i(t)$ could synchronize the attitude of rigid bodies on the kinematic level. In order to simplify the design of control law for $\tau_i(t)$, we introduce the back-stepping technique. The main idea of back-stepping is to decompose the complex nonlinear system into a series of subsystems, and then design part of Lyapunov function and intermediate virtual control law for each subsystems and, finally, integrate them to complete the design of control law for the whole system. Following this idea, we take the control law $\omega_i(t)$ in [26] as the virtual control for the kinematic model subsystem and design $\tau_i(t)$ to track it. To the end, we first define the following kinematic control law introduced in [26] as a desired angular velocity for rigid body $i$

$$\dot{\omega}_i^d(t) = \sum_{j \in \mathcal{N}_i(t)} a_{ij}(p(t)) sk(R_{ij}(t))^\nu$$

where $sk(R) = \frac{1}{2}(R - R^T)$ and the notation $\nu'$ is defined by (3). Then, the error between the actual angular velocity and the desired one can be obtained as follows:

$$\ddot{\omega}_i(t) = \omega_i(t) - \omega_i^d(t)$$

The aim for the control law of torque is to decrease the error $\ddot{\omega}_i(t)$ to zero. In the following, we will utilize high gain control method to finish this task. The idea of high gain control is to make $\ddot{\omega}_i(t)$ quickly converge to zero by choosing a proper gain parameter $h_i$.

Remark 3. The control law (5) used the information of $\dot{\omega}_i^d(t)$, which can be obtained as follows:

$$\dot{\omega}_i^d(t) = \sum_{j \in \mathcal{N}_i(t)} \dot{a}_{ij}(p(t)) sk(R_{ij}(t))^\nu$$

$$+ \sum_{j \in \mathcal{N}_i(t)} a_{ij}(p(t)) \frac{d}{dt} sk(R_{ij}(t))^\nu$$

$$= \sum_{j \in \mathcal{N}_i(t)} \left[ \dot{p}(\frac{\|p_{ij}(t)\|}{\|r\|}) \cdot \sigma_i(p_{ij}(t)) \cdot \left( R_{ij}(t) \omega_j(t) - \omega_i(t) \right) \right]$$
we know that the positive definiteness of \( R \) can be found in [26]. By the definition of \( \omega_D \), it is used in the control law of \( \tau_i(t) \).

In the following analysis, the positive definiteness of the rotation matrices \( \{R_i(t), i \in V\} \) plays a key role to ensure attitude synchronization; here the positive definiteness of \( R_i \) (not necessarily symmetric) means that \( x^T R_i x > 0 \) for all nonzero vector \( x \) (cf. [1]). Define \( \varphi(R_i(t)) = \frac{1}{2} \text{tr}(I_i - R_i(t)) \), where \( \text{tr}(\cdot) \) represents the trace of the corresponding matrix. From [26], we know that the positive definiteness of \( R_i(t) \) is equivalent to \( \varphi(R_i(t)) < 1 \). Now we present the following lemma to show that the positive definiteness of the rotation matrices can be preserved under some mild conditions on the gain parameter \( \beta_i \) in (5).

**Lemma 1.** For systems (1) and (4) with the control torque (5), suppose that the gain parameter satisfies \( \beta_i \geq \frac{2 \|\omega_A(0)\|}{(1 - \varphi(R_i(0))^{2})} \). If the rotation matrices at the initial time are positive definite, then they remain to be positive definite for all \( t > 0 \).

**Proof.** Let \( \alpha(t) = \arg \max_{\alpha \in V} \varphi(R_i(t)) \), then the function \( \varphi(R_{\text{max}}(t)) \) is continuous and it is differential everywhere except for the changing points for \( \alpha(t) \) which we denote as \( t_1, t_2, \ldots, t_f \). For \( t \in [0, t_f] \), we have

\[
\dot{\varphi}(R_{\text{max}}(t)) = (sk(R_{\text{max}}(t))^T \omega_{\text{max}}(t) = (sk(R_{\text{max}}(t))^T \sum_{j \in N_{\text{max}}(t)} a_{ij}(p(t))sk(R_{\text{max}}(t))^T + \dot{\omega}_{\text{max}}(t)) \\
= (sk(R_{\text{max}}(t))^T \omega_{\text{max}}(t) + \sum_{j \in N_{\text{max}}(t)} a_{ij}(p(t)) \left[-\varphi(R_{\text{max}}(t)) - \frac{1}{2} \lambda_{\text{min}}(R_{\text{max}}(t) + R_{\text{max}}(t)^T)\varphi(R_{\text{max}}(t)) \right],
\]

where \( \lambda_{\text{min}}(\cdot) \) represents the minimum eigenvalue of the corresponding matrix. The detailed derivation of the above inequality can be found in [26]. By the definition of \( \alpha(t) \), we know \( \dot{\varphi}(R_{\text{max}}(t)) + \varphi(R_{\text{max}}(t)) \leq 0 \). From \( \varphi(R_{\text{max}}(0)) < 1 \) and the continuity of \( \varphi(R_{\text{max}}(t)) \), we see that there exists \( 0 < \delta_1 \leq t_f \) such that \( \varphi(R_{\text{max}}(t)) \leq 1 \) for \( t \in [0, \delta_1] \), which implies that \( \lambda_{\min}(R_{\text{max}}(t) + R_{\text{max}}(t)^T) \geq 0 \). Hence,

\[
\dot{\varphi}(R_{\text{max}}(t)) \leq (sk(R_{\text{max}}(t))^T \omega_{\text{max}}(t) \leq \|sk(R_{\text{max}}(t))^T\| \cdot \|\omega_{\text{max}}(t)\| \leq \|\omega_{\text{max}}(0)\| \leq \frac{2 \|\omega_A(0)\|}{(1 - \varphi(R_{\text{max}}(0))^{2})}, \quad t \in [0, \delta_1].
\]

Then, by comparison theorem [27], we have

\[
\varphi(R_{\text{max}}(t)) \leq \varphi(R_{\text{max}}(0)) \leq \sqrt{\varphi(R_{\text{max}}(0))^2 + \frac{2 \|\omega_A(0)\|}{\beta_i}} \leq 1, \quad t \in [0, \delta_1]
\]

which means that \( R_i(t) \) is positive definite in \([0, \delta_1]\).

Repeating the above process, we see that there exists \( \delta_1 < \delta_2 < t_2 \), such that \( \varphi(R_{\text{max}}(t)) \leq 1 \) for \( t \in [\delta_1, \delta_2] \). Hence,

\[
\dot{\varphi}(R_{\text{max}}(t)) \leq \frac{\|\omega_A(0)\| \varphi(R_{\text{max}}(t)) \cdot e^{-\beta_i t}}{\beta_i} \leq 1 \quad t \in [0, \delta_2]
\]

which we have \( \varphi(R_{\text{max}}(t)) < 1 \) for \( t \in [0, \delta_2] \).

We assert that there exists a time sequence \( \delta_2, \delta_3, \ldots, \delta_n \) with \( \delta_n \) being a positive constant, such that \( \delta_n > t_f \) and \( \varphi(R_{\text{max}}(t)) < 1 \) for \( t \in [0, \delta_n] \). Then, we can prove the lemma by repeating the above process in intervals \([t_f, t_2], [t_2, t_3], \ldots, [t_{n-1}, t_f]\). \( \square \)

### 3.2 Control for positions

Let us proceed with the control for position of rigid bodies, whose main purpose is to achieve collision avoidance among rigid bodies and connectivity maintenance of the dynamical neighbour graphs. In this paper, we adopt the potential function method to address the problem. The basic idea of maintaining connectivity with potential function method is forcing agents that constitute a distance-based interaction link to remain within a certain distance for all time by producing an unbounded or big enough attractive potential between the pair of agents when they tend to move away from the interaction threshold distance. Similarly, the idea of avoiding collision is forcing agents whose distance is close to the safe threshold distance to stay away with each other by producing an unbounded or big enough repulsive potential. Following those thoughts, we first introduce the following two continuous action functions, which denote the repulsive force and the attractive force, respectively.

\[
\phi_A(z) = \begin{cases} 
0, & z \in [0, d_1] \\
<0, & z \in (d_1, d_1 + d_2) \\
0, & z \in [d_1 + d_2, \infty)
\end{cases} \quad \phi_B(z) = \begin{cases} 
0, & z \in [0, d_2] \\
>0, & z \in (d_2, \infty)
\end{cases}
\]
where the constants $b_1$ and $b_2$ are taken to satisfy $0 < s < b_1 < b_2 < r$. The relationship of the parameters $s, r, b_1, b_2$ are depicted in Figure 1. Based on $\phi_\alpha$ and $\phi_\beta$, the control law for force $f_i(t)$ can be designed as follows:

\[
   f_i(t) = \sum_{j \in \mathcal{N}_i(t)} \phi_\alpha (\| p_j(t) - p_i(t) \|_2) \sigma_\alpha (p_j(t)) \\
   + \sum_{j \in \mathcal{N}_i(t)} \phi_\beta (\| p_j(t) - p_i(t) \|_2) \sigma_\beta (p_j(t)) \\
   + \sum_{j \in \mathcal{N}_i(t)} a_{ij}(p(t)) (R_{ij}(t) v_j(t) - u_j(t)),
\]

where $\mathcal{N}_i(0)$ and $\mathcal{N}_i(t)$, respectively, denote the neighbour set of rigid body $i$ at the initial time and at time $t$. On the basis of the above analysis, we know the first and second terms of (7) are used for collision avoidance and connectivity maintenance, respectively. The third is designed to synchronize the linear velocity of rigid bodies. The set $\mathcal{N}_i(0) \bigcap \mathcal{N}_i(t)$ instead of $\mathcal{N}_i(t)$ is used in the attractive term of (7) to allow new edges to be lost after they are formed while preserving connectivity of the dynamical neighbour graphs.

4 \ MAIN RESULTS

In this section, we present the main results about the pose coordination of rigid bodies. First, denote $u(t) = (u_1^T(t), ..., u_d^T(t))^T$ and define the potential function of position as

\[
   V(p(0), p(t), u(t)) = \frac{1}{2} \sum_{i} \sum_{j \neq i} \psi_\alpha (\| p_j(t) - p_i(t) \|_2) \\
   + \frac{1}{2} \sum_{i} \sum_{j \in \mathcal{N}_i(t)} \psi_\beta (\| p_j(t) - p_i(t) \|_2) \\
   + \frac{1}{2} \sum_{j} u_j^T(t) u_j(t)
\]

where

\[
   \psi_\alpha (z) = \int_{||z||_2}^{\xi} \phi_\alpha (\mu) d\mu, \quad \psi_\beta (z) = \int_{0}^{\xi} \phi_\beta (\mu) d\mu
\]

which are regarded as potential functions that correspond to the action functions $\phi_\alpha$ and $\phi_\beta$, respectively. It is not hard to see that $\psi_\alpha (z)$ and $\psi_\beta (z)$ are bounded above by $\max \{ \psi_\alpha (||z||_2), \psi_\beta (||z||_2) \}$.

Denote $R(t) = (R_1^T(t), ..., R_d^T(t))^T, \omega(t) = (\omega_1^T(t), ..., \omega_d^T(t))^T$. Now we present the main result concerning attitude synchronization, collision avoidance and connectivity maintenance.

**Theorem 1.** For system (1) and (4) with the control laws (5) and (7), suppose that the gain parameter satisfies $b_1 \geq \max \{ 2/||\omega(0)||, 2||\omega(0)||/ (1 - \varphi(0)) \}$ and $s + b_2 < r$ holds true. If the initial state is taken from an admissible set defined as follows,

\[
   \Omega = \{ (R, p, \omega, u) : R_i \text{ is positive definite for } i \in V \\
   \| p_i - p_j \| > s \text{ for } i \neq j, \quad G(p, r) \text{ is connected} \\
   V(p, u, p) < c^* \text{ with } c^* = \min \{ \psi_\alpha (||z||_2), \psi_\beta (||p||_2) \} \}
\]

Then,

(i) The system reaches attitude synchronization.
(ii) The angular velocities of rigid bodies converge to zero, and the linear velocities in the inertial frame $(R, \omega)$ are synchronized.
(iii) There is no collision among rigid bodies when they are in motion.
(iv) The connectivity of the dynamical neighbour graphs is preserved.

**Proof.** It is not hard to see that the potential function $V(p(0), p(t), u(t))$ will suffer an abrupt decrease when an existing edge $(i, j)$ with $j \in \mathcal{N}_i(0) \bigcap \mathcal{N}_j(t)$ is lost. Therefore, we cannot use it as part of Lyapunov candidate. To solve this problem, in the following, we will find a way to preserve all edges of the initial graph $G(p(0), r)$, and hence ensure the continuity of function $V(p(0), p(t), u(t))$.

First, take the initial state $(R(0), p(0), \omega(0), u(0))$ from the admissible set $\Omega$, denote $c^* \triangleq V(p(0), p(0), u(0))$, then $c < c^*$, which implies that there exists a constant $\delta \in (s, r, \min \{ 1/2, b_1/r, 1 - b_2/r \})$, such that each point $(p, u)$ in the set \{ $(p, u)$ : there exists $i \neq j$ such that $\| p_i - p_j \| < \delta r$ or there exists an edge $(k, l) \in E(p(0))$ such that $1 - \delta) r < \| p_k - p_l \| < r \} satisfies $V(p(0), p, u) > c$. Note that the...
non-emptiness of \( \{s/r, \min\{1/2, b_1/r, 1 - b_2/r\}\} \) can be guaranteed by the condition \( s + b_2 < r \). We then define

\[
\Gamma = \left\{ (p_i, u) : \|p_i - p_j\| \geq \delta r \text{ for } i \neq j, \|p_k - p_i\| \leq (1 - \delta) r \right\}
\]

for \( (k, l) \in E(p(0)) \) and \( k \neq l, V(p(0), p, u) \leq c \}. \)

Obviously, \( \Gamma \) is a closed set.

Now we apply the contradiction method to show that \( (p(t), u(t)) \in \Gamma \) for all \( t \geq 0 \). First, assume that there exists a time \( t^0 \) satisfies \( (p(t^0), u(t^0)) \notin \Gamma \), then there exists \( 0 < t^0 \leq t^0 \), such that (i) \( V(p(0), p(t^0), u(t^0)) > c \), (ii) \( G(p(t^0), t) \) is a subgraph of \( G(p(t), t) \) for \( t \in [0, t^0] \). Hence, \( V(p(0), p(t), u(t)) \) is continuous differentiable on the set \( \Gamma \) for \( t \in [0, t^0] \) and its derivative is given by

\[
\dot{V}(p(0), p(t), u(t)) = \sum_i (R_i(t)u_i(t))^T \nabla_{p_i} V(p(0), p(t), u(t)) + \sum_i u_i^T(t)\dot{u}_i(t)
\]

\[
= \sum_i \left( R_i(t)u_i(t) \right)^T \left[ -\sum_{j \in N_i(t)} \phi_\delta (\|p_j(t) - p_i(t)\|_\sigma) \right. \\
\quad \times \sigma_\delta (p_j(t) - p_i(t)) - \sum_{j \in N_i(t) \cap N_j(t)} \phi_\delta (\|p_j(t) - p_i(t)\|_\sigma) \\
\left. \times \sigma_\delta (p_j(t) - p_i(t)) \right]
\]

\[
+ \sum_i \left[ \sum_{j \in N_i(t)} \phi_\delta (\|p_j(t)\|_\sigma) \sigma_\delta (p_j(t)) \\
+ \sum_{j \in N_i(t) \cap N_j(t)} \phi_\delta (\|p_j(t)\|_\sigma) \sigma_\delta (p_j(t)) \\
+ \sum_{j \in N_i(t)} a_{ij}(p(t))(R_j(t)u_j(t) - u_i(t)) - \nabla_i (t)u_i(t) \right]
\]

\[
= \sum_i \sum_{j \in N_i(t)} a_{ij}(p(t))(R_j(t)u_j(t) - R_i(t)u_i(t))
\]

\[
= -\frac{1}{2} \sum_i \sum_{j \in N_i(t)} a_{ij}(p(t))\|R_j(t)u_j(t) - R_i(t)u_i(t)\|^2
\]

\[
\leq 0
\]

(8)

Equation (8) means that the function \( V(p(0), p(t), u(t)) \) is non-increasing in \([0, t^0] \) and hence \( V(p(0), p(t^0), u(t^0)) \leq V(p(0), p(0), u(0)) = c \), which, in conjunction with \( V(p(0), p(t^0), u(t^0)) > c \), leads to the contradiction.

The above analysis shows that \( (p(t), u(t)) \in \Gamma \) for \( t \geq 0 \), implying that \( \|p_i(t) - p_j(t)\| \geq \delta r > s \) for all \( i \neq j \) and \( \|p_k(t) - p_i(t)\| \leq (1 - \delta) r \) for all \( (k, l) \in E(p(0)) \). Hence, there is no collision among rigid bodies when they are in motion and the connectivity of the dynamical neighbour graph \( G(p(t), t) \) can be guaranteed; furthermore, the potential function \( V(p(0), p(t), u(t)) \) is continuous and differentiable on the set \( \Gamma \) for \( t \geq 0 \).

Let us proceed by introducing a Hamilton function to play the role of Lyapunov function for the whole system. For this purpose, define

\[
H(R(t), \omega(t), p(0), p(t), u(t)) = \sum_i |\phi(R_i(t))|
\]

\[
+ \frac{1}{2} \sum_i \omega_i^T(t)\omega_i(t)(1 + e^{\beta t}) + V(p(0), p(t), u(t))
\]

where \( \omega_i(t) \) is given by (6). Obviously, the Hamilton function \( H(R(t), \omega(t), p(0), p(t), u(t)) \) is continuous differentiable for \( t \geq 0 \) and its derivative is given by

\[
\dot{H}(R(t), \omega(t), p(0), p(t), u(t))
\]

\[
= \sum_i \phi(R_i(t)) + \sum_i \omega_i^T(t)\omega_i(t)(1 + e^{\beta t})
\]

\[
+ \frac{1}{2} \sum_i b_i e^{\beta t} \omega_i^T(t)\omega_i(t) + \dot{V}(p(0), p(t), u(t))
\]

\[
\leq \sum_i \sum_{j \in N_i(t)} a_{ij}(p(t)) \left[ -\phi(R_i(t)) + \phi(R_j(t)) \right]
\]

\[
- \frac{1}{2} \lambda_{\min}(R_i(t) + R_j^T(t))\phi(R_j(t))
\]

\[
+ \sum_i \omega_i^T(t)\omega_i(t) - \sum_i b_i e^{\beta t} \omega_i^T(t)\omega_i(t)
\]

\[
- \frac{1}{2} \sum_i b_i e^{\beta t} \omega_i^T(t)\omega_i(t) + \dot{V}(p(0), p(t), u(t))
\]

\[
\leq -\frac{1}{2} \sum_i \sum_{j \in N_i(t)} a_{ij}(p(t))\lambda_{\min}(R_i(t) + R_j^T(t))\phi(R_j(t))
\]

\[
- \frac{1}{2} \sum_i \sum_{j \in N_i(t)} a_{ij}(p(t))\|R_i(t)u_i(t) - R_j(t)u_j(t)\|^2
\]

\[
- \sum_i b_i \omega_i^T(t)\omega_i(t)
\]

(9)

In the above deduction, we use \( \phi(R_i(t)) \), which is given in the proof of Lemma 1, to obtain the first inequality. Furthermore,

\[
\sum_i \sum_{j \in N_i(t)} a_{ij}(p(t)) \left[ -\phi(R_i(t)) + \phi(R_j(t)) \right]
\]

\[
- \frac{1}{2} \lambda_{\min}(R_i(t) + R_j^T(t))\phi(R_j(t))
\]

\[
+ \sum_i \omega_i^T(t)\omega_i(t) - \sum_i b_i e^{\beta t} \omega_i^T(t)\omega_i(t)
\]

\[
- \frac{1}{2} \sum_i b_i e^{\beta t} \omega_i^T(t)\omega_i(t) + \dot{V}(p(0), p(t), u(t))
\]

\[
\leq -\frac{1}{2} \sum_i \sum_{j \in N_i(t)} a_{ij}(p(t))\lambda_{\min}(R_i(t) + R_j^T(t))\phi(R_j(t))
\]

\[
- \frac{1}{2} \sum_i \sum_{j \in N_i(t)} a_{ij}(p(t))\|R_i(t)u_i(t) - R_j(t)u_j(t)\|^2
\]

\[
- \sum_i b_i \omega_i^T(t)\omega_i(t)
\]
Note that (6) and the condition $b_i \geq 2/\|\vec{\omega}_i(0)\|$ is used in the above calculation. Hence, (9) is satisfied.

The initial state $(R(0), p(0), \omega(0), v(0)) \in \Omega$, namely, $R_i(0)$ is positive definite, then from Lemma 1, $R_i(t)$ is positive definite for all $t \geq 0$, which implies that $\dot{H}(R(t), \omega(t), p(t), v(t), u(t)) \leq 0$ for $t \geq 0$. Thus, $\dot{H}(R(t), \omega(t), p(t), v(t), u(t))$ has a finite limit as $t \to \infty$. Furthermore, from our analysis, we know that $\dot{H}(R(t), \omega(t), p(t), v(t), u(t))$ is uniformly continuous. Then, by Barbalat’s lemma [28], $\lim_{t \to \infty} \dot{H}(R(t), \omega(t), p(t), v(t), u(t)) = 0$, from which we have

$$\lim_{t \to \infty} \left\{ -\frac{1}{2} \sum_{j \in \hat{N}_i(t)} \sum_{j \in \hat{N}_i(t)} a_{ij}(p(t)) \lambda_{\min}(R_i(t) + R_j^T(t)) \varphi(R_j(t)) - \frac{1}{2} \sum_{j \in \hat{N}_i(t)} \sum_{j \in \hat{N}_i(t)} a_{ij}(p(t)) \|R_j(t) \omega_j(t) - R_i(t) \omega_i(t)\|^2 - \sum_{j \in \hat{N}_i(t)} b_i \vec{\omega}_j^T(t) \vec{\omega}_j(t) \right\} = 0$$

Hence,

$$R_j(t) \to I, \quad \omega_j(t) \to \omega_i(t), \quad v_j(t) \to v_i(t) \to 0, \quad \text{as } t \to \infty$$

which implies that the attitude synchronization is achieved, the angular velocities of rigid bodies converge to zero and the linear velocities in the inertial frame are synchronized.

**Remark 4.** From the definition of $\psi_2, \psi_3$ and $V(p, \tilde{p}, v)$, we see $V(p, \tilde{p}, v)$ equals to $\frac{1}{2} \|\vec{u}\|^2$ if the distance between each pair of neighbouring rigid bodies is within the range $[b_1, b_2]$. In such a situation, the last item in $\Omega$, namely, $V(p, \tilde{p}, v) < \varepsilon^s$, can be ensured by the condition that $\frac{1}{2} \|\vec{u}\|^2 < \varepsilon^s$, which is satisfied if the initial linear velocity of rigid bodies are small or the value $\varepsilon^s$ is big enough. In general, we can ensure that $V(p, \tilde{p}, v) < \varepsilon^s$ holds true for most initial position $p$ by designing appropriate action functions $\varphi_2, \varphi_3$ and choosing appropriate initial linear velocities. Hence, the admissible set $\Omega$ contains many elements that the initial state of rigid bodies could take.

### 5 EXTRCTIONS

#### 5.1 Minimum spanning tree-based control

The potential function method is the most commonly used method to maintain connectivity of the dynamical neighbour graphs for continuous-time systems. However, in most of the related works, the edges cannot be allowed to break once they are formed, leading to a growing number of edges in the neighbour graphs, and hence bringing burden to the communication capacity and limitation to the motion of rigid bodies. To overcome these drawbacks, we introduced the initial neighbour set $\hat{\mathcal{N}}(0)$ in the control design for $\psi(t)$ and used $\hat{\mathcal{N}}(0) \bigcap \mathcal{N}(t)$ instead of $\mathcal{N}(t)$ to preserve the connectivity of neighbour graphs. From the above analysis, we see only the edges in $\mathcal{N}(0)$ cannot be allowed to break.

However, if the interaction radius $r$ is large or the initial distance between rigid bodies is small, the number of edges in the initial neighbour graph is large. In such a situation, there is still a large requirement for the communication capacity and the motion of rigid bodies is still limited, even though $\hat{\mathcal{N}}(0) \bigcap \mathcal{N}(t)$ is used to maintain connectivity. To solve this problem, in this section, we first extract a spanning tree, denoted as $\hat{G}(0)$, from the initial neighbour graph $G(p(0), r)$ and then preserve all the edges of $\hat{G}(0)$ by using $\hat{\mathcal{N}}(0) \bigcap \mathcal{N}(t)$ in the control law for $\psi(t)$, where $\hat{\mathcal{N}}(0)$ is the initial neighbour set of rigid body $i$ corresponding to $\hat{G}(0)$. Note that spanning tree is the sparsest connected subgraph for any connected graphs. Hence, our method can preserve connectivity of the dynamical neighbour graphs, and at the same time reduce the communication burden of system and the restriction on the motion of rigid bodies. The classical algorithms to construct minimum spanning tree are Prim (cf. [29]) and Kruskal (cf. [30]), here minimum means that the total edge cost which we define as $\sum_{(i,j) \in E(0)} (1 - a_{ij}(p(0)))$ is minimum, where $E(0)$ represents the edge set of $\hat{G}(0)$. This paper uses the Prim algorithm to extract the minimum spanning tree $\hat{G}(0)$ from $G(p(0), r)$, and we conclude it in Algorithm 1.

The Prim algorithm is not distributed, and one can find distributed algorithms for minimum weight spanning tree in papers

---

**Algorithm 1 Prim Algorithm**

**Input:** the initial neighbour graph $G(p(0), r) = (V, E(p(0)), \mathcal{A}(p(0)))$;

**Initialization:** $\hat{G}(0) = \{\emptyset\}$, where $k$ is a node randomly chosen from $V$, $\hat{E}(0) = \emptyset$;

**while** $\hat{G}(0) \neq V$ **do**

1. Select the minimum weight edge $(i, j)$ with weight defined as $1 - a_{ij}(p(0))$ from $E(p(0))$, where $i \in \hat{V}(0), j \notin \hat{V}(0)$ and $j \in V$. If there is more than one edge meets the condition and the weight is the same, then choose one randomly;

2. Add $(i, j)$ into $\hat{G}(0)$, and add $(i, j)$ into $\hat{E}(0)$;

**end while**

**Output:** the minimum spanning tree $\hat{G}(0) = (\hat{V}(0), \hat{E}(0))$. 
such as [31]. Based on $\mathcal{G}(0)$ extracted by the Prim algorithm, we then modify the control law for force (7) by replacing $\mathcal{N}_j(0)$ with $\widetilde{\mathcal{N}}_j(0)$, and correspondingly modify $V' (p(0), p(t), \mathbf{u}(t))$, then Theorem 1 still holds, and only the edges in $\mathcal{G}(0)$ cannot be allowed to break while preserving connectivity of the dynamical neighbour graphs.

5.2 | Leader-following framework

In the above work, we have studied the pose control problem of multi-rigid-body systems. It is worthy to notice that the final attitude that all rigid bodies converge to is determined by the initial states and model parameters of the system, once the control laws are given. However, the final attitude may not be what we expect. To solve this issue, we consider the leader-following framework in this section and introduce a leader rigid body labeled 0 into the system to guide all the rigid bodies converge to the expected value.

Denote the expected attitude as $R^e$, and assume that the leader rigid body evolves according to the following dynamics:

$$
\begin{align*}
R_0(t) &= 0, R_0(0) = R^e, \omega_0(t) = 0 \\
p_0(t) &= R_0(t) u_0(t) \\
\dot{u}_0(t) &= f_0(t)/m_0 - \hat{\omega}_0(t) u_0(t)
\end{align*}
$$

(10)

Suppose that the leader interacts with the other $n$ rigid bodies (called followers) if their distance is less than $r$, that is to say, the neighbour set of the leader is $\mathcal{N}_0(t) = \{i \in V : \|p_0(t)\| < r\}$. In such a situation, the neighbour set of the follower $i(i \in V)$ is updated to $\mathcal{N}_i(t) = \{j \in V \cup \{0\} : \|p_j(t)\| < r, j \neq i\}$.

In this section, we still use $R(t), \omega(t), p(t)$ and $\mathbf{u}(t)$ to, respectively, represent the collection of the rotation matrices, angular velocities, positions and linear velocities of all the $n + 1$ rigid bodies, then the force of the leader is designed as

$$
f_0(t) = m_0 \left\{ \sum_{j \in \mathcal{N}_0(t)} \phi_\alpha (\|p_j(t)\|_2) \sigma_\epsilon (p_{0j}(t)) \\
+ \sum_{j \in \mathcal{N}_0(0) \setminus \mathcal{N}_0(t)} \phi_\beta (\|p_j(t)\|_2) \sigma_\epsilon (p_{0j}(t)) \\
+ \sum_{j \in \mathcal{N}_0(t)} \phi_\alpha (p(t))(R_{0j}(t)u_{0j}(t) - u_0(t)) \right\}
$$

(11)

and the potential function of position is defined as

$$
V_{LF}(p(0), p(t), \mathbf{u}(t)) = \frac{1}{2} \sum_{i=0}^{n} \sum_{j \neq \ell} \psi_\alpha (\|p_j(t) - p_\ell(t)\|_2) \\
+ \frac{1}{2} \sum_{i=0}^{n} \sum_{j \in \mathcal{N}_0(t) \setminus \mathcal{N}_i(t)} \psi_\beta (\|p_j(t) - p_\ell(t)\|_2) \\
+ \frac{1}{2} \sum_{i=0}^{n} \mathbf{u}_j^T(t)\mathbf{u}_\ell(t).
$$

The control laws for torque and force of the $n$ followers are described by (5) and (7) with the updated neighbour set. Then, we have the following results for the leader-following framework we considered.

**Lemma 2.** For the leader-following framework under consideration, assume that the gain parameter in (5) satisfies $b_i \geq 2\|\hat{\omega}_{i}(0)\|/ (1 - \varphi(R_{i0}(0))^2)$. If for $i = 1, 2, \ldots, n$, the relative attitude $R_{i0}$ is positive definite at the initial time, then they remain positive definite for all $i > 0$.

The proof of Lemma 2 can be obtained by a similar analysis to that of Lemma 1.

**Theorem 2.** For system (1), (4) and (10) with the control laws (5), (7) and (11), suppose that the gain parameter satisfies $b_i \geq \max[2/\|\hat{\omega}_{i}(0)\|, 2\|\hat{\omega}_{i}(0)\|/(1 - \varphi(R_{i0}(0))^2)] \quad$ and $s + b_2 < r$ holds true. If the initial state is taken from an admissible set defined as follows:

$$
\Omega_{LF} = \left\{ (R, p, \omega, \mathbf{u}) : R_{i0} \text{ is positive definite for } i \in V; \\
\|p_i - p_j\| > s \text{ for } i \neq j; \mathcal{G}(p, r) \text{ is connected}, \\
V'_{LF}(p_i, p_j, \mathbf{u}) < \varepsilon^k \right\}.
$$

Then,

(i) The followers can track the leader's attitude asymptotically and all the rigid bodies converge to the expected attitude, i.e. $R_{i0}(t) \rightarrow I_3, \forall i \in V$ as $t \rightarrow \infty$.

(ii) The angular velocities of rigid bodies converge to zero, and the linear velocities of the $n + 1$ rigid bodies in the inertial frame are synchronized.

(iii) There is no collision among the $n + 1$ rigid bodies when they are in motion.

(iv) The connectivity of the dynamical neighbour graphs is preserved.

**Proof.** Following the analysis similar to that of Theorem 1, we obtain that $\|p_j(t) - p_\ell(t)\| \geq \delta_r > s$ for all $i \neq j$ and $\|p_j(t) - p_\ell(t)\| \leq (1 - \delta_r)r$ for all $i, \ell \in E(p(0))$; moreover, the potential function $V'_{LF}(p(0), p(t), \mathbf{u}(t))$ is continuous and differentiable for $t \geq 0$.

We then introduce the following Hamilton function to play the role of Lyapunov function for our analysis:

$$
H_{LF}(R(t), \omega(t), p(0), p(t), \mathbf{u}(t)) = \sum_{i=1}^{n} \varphi(R_{i0}(t)) \\
+ \frac{1}{2} \sum_{i=1}^{n} \mathbf{\hat{\omega}}_i^T(t)\mathbf{\hat{\omega}}_i(t)(1 + \rho_i t) + V'_{LF}(p(0), p(t), \mathbf{u}(t))
$$

Obviously, $H_{LF}(R(t), \omega(t), p(0), p(t), \mathbf{u}(t))$ is continuous differentiable for $t \geq 0$. In the following, we focus on analyzing its derivative.
First, define

\[ U(R(t)) = \sum_{i=1}^{n} \varphi(R_{0i}(t)) \]

then,

\[ \dot{U}(R(t)) = \sum_{i=1}^{n} \left( (sk(R_{0i}(t))^\top R_{0i}(t) \left( \sum_{j \in \mathcal{I}_i(t)} a_{ij}(p(t))sk(R_{ij}(t))^\top \right) \right) \omega_{ij}(t) \]

\[ + \sum_{i=1}^{n} \left( (sk(R_{0i}(t))^\top R_{0i}(t) \omega_i(t) \right) \]

\[ = \sum_{i=1}^{n} \sum_{j \in \mathcal{I}_i(t)} a_{ij}(p(t)) \left[ -\varphi(R_{0i}(t)) + \varphi(R_{ij}(t)) \right. \]

\[ - \frac{1}{4} \text{tr}\left( (R_{0i}(t) + R_{ij}(t))(I_3 - R_i(t)) \right) \]

The rest of the proof is similar to Theorem 1, we then finish the proof. \(\square\)

6 | SIMULATION RESULTS

In this section, we present simulation examples to illustrate our results. Assume the system consists of four rigid bodies, each of which is described by (1) and (4). The control law for torque is designed according to (5) and the force controller is (7), where the two action functions \(\phi_\alpha\) and \(\phi_\beta\) are taken in (13) and (14) with \(i = 1, b_1 = 3, b_2 = 5, r = 8\). The parameters in \(\rho(\cdot)\) and \(\| \cdot \|_\sigma\) are taken as \(b = 0.5, \varepsilon = 0.5\). The initial states are taken from the admissible set \(\Omega\), and the exact values are shown in Table 1. Figure 2(a) illustrates the initial poses and neighbour relations of the four rigid bodies.

\[ \phi_\alpha(\zeta) = \begin{cases} 0, & \zeta \in [0, \| \cdot \|_\sigma] \\ 20(\zeta^2 - (\| \cdot \|_\sigma + 2b_1 |\cdot|_\sigma)^2 - b_2 |\cdot|_\sigma) & \zeta \in (\| \cdot \|_\sigma, b_1 |\cdot|_\sigma) \\ 0, & \zeta \in b_1 |\cdot|_\sigma, \infty \end{cases} \] (13)

\[ \phi_\beta(\zeta) = \begin{cases} 0, & \zeta \in [0, b_2 |\cdot|_\sigma] \\ 20(\zeta^2 + 2b_1 |\cdot|_\sigma + 2r |\cdot|_\sigma - b_2 |\cdot|_\sigma) & \zeta \in (b_2 |\cdot|_\sigma, r |\cdot|_\sigma) \\ 0, & \zeta \in r |\cdot|_\sigma, \infty \end{cases} \] (14)
### TABLE 1 Initial value of the states of rigid bodies

| Rigid body | Initial rotation matrix | Initial position | Initial angular velocity | Initial linear velocity |
|------------|-------------------------|------------------|-------------------------|------------------------|
| 1          | \[
    \begin{bmatrix}
    0.9336 & 0.3040 & 0.1895 \\
    -0.3118 & 0.9501 & 0.0116 \\
    -0.1765 & 0.0699 & 0.9818 \\
    \end{bmatrix},
    \begin{bmatrix}
    0.9997 & 0.0177 & 0.0141 \\
    0.0167 & 0.9786 & 0.0638 \\
    -0.0152 & 0.0635 & 0.9979 \\
    \end{bmatrix},
    \begin{bmatrix}
    0.5545 & 0.7282 & 0.4028 \\
    0.5832 & 0.5327 & 0.5045 \\
    0.5327 & 0.5327 & 0.5045 \\
    \end{bmatrix}
\] | \[3 0 0]^T\] | \[2 3 4]^T\] | \[-4 1 1]^T\] |
| 2          | \[0.0167 & 0.9786 & 0.0638 \] | \[0 0 3]^T\] | \[0 5 4]^T\] | \[-1 3 1]^T\] |
| 3          | \[-0.8100 & 0.5832 & 0.0609 \] | \[0 0 3]^T\] | \[2 2 2]^T\] | \[7 -1 -1]^T\] |
| 4          | \[-0.6043 & 0.4357 & 0.6671 \] | \[7 -1 -1]^T\] | \[2 3 3]^T\] | \[2 2 2]^T\] |

**FIGURE 2** Poses and neighbour relations of rigid bodies in leaderless case. (a) Initial poses and neighbour relations. (b) Poses and neighbour relations at \(t = 1\). (c) Poses and neighbour relations at \(t = 5\). The green dotted lines represent the neighbour relations, and the red coordinates represent the attitudes of rigid bodies.

Set \(b_i = \max\{2/\|\bar{ω}_i(0)\|, 2/\|\bar{ω}_i(0)\|/(1 - φ(R_i(0))^2)\}\), then the simulation results of the aforementioned system are obtained as shown in Figures 2–4. Figure 2(b) and (c) depicts the poses and neighbour relations of rigid bodies, from which we see the initial edges are preserved and attitude synchronization is reached. Figures 3 and 4 show the detailed results. From Figure 3(a), we see that attitude synchronization is reached, where the rotation vector is used to represent the trajectory of the rotation matrix; Figure 3(b) shows that the minimum distance between rigid bodies is greater than \(s\), implying that no collision occurs; Figure 3(c) shows that the second smallest eigenvalue of the Laplacian matrix of the corresponding neighbour graphs is larger than 0 for \(t \geq 0\), meaning that the neighbour graphs are connected; moreover, the characteristics of the curve increasing first and then decreasing indicates that some newly formed edges break again in the process of preserving connectivity, which is also illustrated in Figure 2(b) and (c) and proves that the introduction of \(\mathcal{N}_i(0) \cap \mathcal{N}_i(t)\) into the control...
design is effective. Figure 4 depicts the velocities of rigid bodies, from which we see the angular velocities converge to zero, the linear velocities are synchronized and the angular velocity errors exponentially converge to zero.

Now we present an example to illustrate the results of minimum spanning tree-based control. In order to make the simulation results more intuitive, we set \( r = 7 \) in this example. Other parameters and initial states of rigid bodies keep unchanged. Figure 5(a) depicts the initial poses and neighbour relations of rigid bodies. Apply the Prim algorithm to extract minimum spanning tree \( \hat{\mathcal{G}}(0) \) from \( \mathcal{G}(p(0), r) \), and depict it in Figure 5(b), then the simulation results of system (1) and (4) under control law (5) and (7) with \( \hat{\mathcal{N}}(0) \) replacing \( \mathcal{N}(0) \) are obtained and are shown in Figure 5(c). The detailed simulation results are omitted to save space. From Figure 5, we see only the initial edges belonging to the minimum spanning tree \( \hat{\mathcal{G}}(0) \) cannot
be lost while preserving connectivity of the dynamical neighbour graphs.

In the end, we illustrate the results of leader-following framework. All the parameters and initial states of rigid bodies remain the same as the leaderless case. The leader’s initial states are taken as \( R_0 = I, \omega_0 = [0 \ 0 \ 0]^T, p_0 = [4 \ 2 \ 2]^T, \omega_0 = [-1 \ -1 \ -1]^T \), set \( b_i = \max\{2/\|\ddot{\omega}_i(0)\|^2, 2/\|\ddot{\omega}_i(0)\|/ (1 - \varphi(R_0(0)))^5\} \), then the conditions of Theorem 2 are satisfied. Figures 6 and 7 show the detailed simulation results, verifying Theorem 2.

7 CONCLUDING REMARKS

This paper addressed the pose control problem of multi-rigid-body systems with distance-dependent graphs. The authors presented control laws for torque and force of rigid bodies to achieve attitude synchronization and collision avoidance while preserving connectivity of the dynamical neighbour graphs. Theoretical results were established with only relying on system parameters and initial states of rigid bodies. Besides, minimum spanning tree-based control was considered to further reduce the communication burden of system, and the leader-following framework was investigated to guide the rigid bodies converge to a desired attitude. There are lots of problems worthy of further study, for example, designing distributed adaptive controller when the inertial matrix \( J \) is unknown.

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