Investigation of a class of two-dimensional conjugate integral equation with fixed super-singular kernels

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INVESTIGATION OF A CLASS OF TWO-DIMENSIONAL CONJUGATE INTEGRAL EQUATION WITH FIXED SUPER-SINGULAR KERNELS

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Abstract. In this paper, two-dimensional linear conjugate Volterra integral equations containing super-singularities in the kernels are considered. The existence of a unique solution in a certain function class is established. Formulas representing the solution are given.

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1. INTRODUCTION

Let $D$ denote the rectangle $D = \{a < x < a_0, b_0 < y < b\}$ and introduce the sets $\Gamma_1 = \{a < x < a_0, y = b\}$ and $\Gamma_2 = \{x = a, b_0 < y < b\}$. In $D$, we consider the two-dimensional integral equation

$$u(x,y) + \lambda \int_a^x \frac{u(t,y)}{(t-a)^\alpha} \, dt - \mu \int_y^b \frac{u(x,s)}{(b-s)^\beta} \, ds$$

$$+ \delta \int_a^x \frac{dt}{(t-a)^\alpha} \int_y^b \frac{u(t,s)}{(b-s)^\beta} \, ds = f(x,y), \quad (1.1)$$

and the integral equation conjugate to equation (1.1)

$$T^{\alpha,\beta}_{\lambda,\mu}(v) = v(x,y) + \lambda \int_x^{a_0} v(t,y) \, dt - \frac{\mu}{(b-y)^\beta} \int_{b_0}^y v(x,s) \, ds$$

$$+ \frac{\delta}{(x-a)^\alpha(b-y)^\beta} \int_x^{a_0} dt \int_{b_0}^y v(t,s) \, ds = g(x,y), \quad (1.2)$$

where $\alpha > 1, \beta > 1, \{\lambda, \mu, \delta\} \subset \mathbb{R}$, $f(x,y), g(x,y)$ are the given functions and $u(x,y), v(x,y)$ are the unknown functions.
By the study of the solutions of the integral equations (1.1) and (1.2), the problem can be reduced to the determination of the continuous solution of a hyperbolic equation with two super-singular lines and its conjugate equation in D.

We seek a solution of equation (1.1) in the class of functions \( u(x, y) \in C(D) \) that vanish on the singular lines \( I_1 \) and \( I_2 \). Moreover, we will assume that the unknown function \( u(x, y) \) vanishes as \( x \to a \) by an order higher than \( \alpha - 1 \), and it vanishes as \( y \to b \) by an order higher than \( \beta - 1 \).

We note that the one-dimensional integral equations of types (1.1) and (1.2) are studied in [2, 4–6]. Two-dimensional, three-dimensional and some many-dimensional Volterra type integral equations of type (1.1) are studied in [2, 3, 7–9]. One-dimensional singular integral equations with Cauchy kernels are considered in [1].

2. THE CASE WHERE \( \delta = -\lambda \mu \)

In this case, the integral equation (1.2) can be represented in the following form:

\[
v(x, y) - \frac{\mu}{(b - y)^\beta} \int_{b_0}^{y} v(x, s) ds + \frac{\lambda}{(x - a)^\alpha} \int_{x}^{a_0} [v(t, y) - \frac{\mu}{(b - y)^\beta} \int_{b_0}^{y} v(t, s) ds] dt = g(x, y).
\]

If we introduce a new unknown function

\[
W(x, y) = v(x, y) - \frac{\mu}{(b - y)^\beta} \int_{b_0}^{y} v(x, s) ds,
\]

we arrive to a one-dimensional conjugate Volterra type integral equation

\[
W(x, y) + \frac{\lambda}{(x - a)^\alpha} \int_{x}^{a_0} W(t, y) dt = g(x, y).
\]

In the case where \( a < x < a_0 \), according to [3], the integral equation (2.2) has a unique solution which is given by the formula

\[
W(x, y) = g(x, y) - \frac{\lambda}{(x - a)^\alpha} \int_{x}^{a_0} \exp[\lambda(\omega_a^\alpha(t) - \omega_a^\alpha(x))] g(t, y) dt,
\]

where \( \omega_a^\alpha(x) = [(\alpha - 1)(x - a)^{\alpha - 1}]^{-1} \).

Analogously, the solution of the integral equation (2.1), for \( b_0 < y < b \), is given by the formula

\[
v(x, y) = W(x, y) + \frac{\mu}{(b - y)^\beta} \int_{b_0}^{y} \exp[\mu(\omega_b^\beta(y) - \omega_b^\beta(s))] W(t, y) ds,
\]

where \( \omega_b^\beta(y) = [(\beta - 1)(b - y)^{\beta - 1}]^{-1} \).

By substituting the value of \( W(x, y) \) into (2.4), we obtain a general solution of the integral equation (1.2) in the form:
\[ v(x, y) = g(x, y) - \frac{\lambda}{(x-a)^\alpha} \int_x^{a_0} \exp[\lambda(\omega_a^\alpha(t) - \omega_a^\alpha(x))]g(t, y)dt \]
\[ + \frac{\mu}{(b-y)^\beta} \int_{b_0}^{y} \exp[\mu(\omega_b^\beta(y) - \omega_b^\beta(s))]g(x, s)ds \]
\[ - \frac{\lambda \mu}{(x-a)^\alpha(b-y)^\beta} \int_x^{a_0} \exp[\lambda(\omega_a^\alpha(t) - \omega_a^\alpha(x))]dt \]
\[ \times \int_{b_0}^{y} \exp[\mu(\omega_b^\beta(y) - \omega_b^\beta(s))]g(t, s)ds \equiv (T_{\lambda, \mu}^{\alpha, \beta})^{-1} g. \]  

We thus obtain the following

**Theorem 1.** Assume that in equation (1.2) the parameters are related by the equality \( \delta = -\lambda \mu, \) and \( g(x, y) \in C(D). \) Then the non-homogeneous integral equation (1.2) has a unique solution in class \( C(D), \) which is given by formula (2.5).

3. THE CASE WHERE \( \delta \neq -\lambda \mu \)

In this case, the integral equation (1.2) can be represented in the following form:

\[ T_{\lambda, \mu}^{\alpha, \beta}(v) \equiv v(x, y) + \frac{\lambda}{(x-a)^\alpha} \int_x^{a_0} v(t, y)dt \]
\[ - \frac{\mu}{(b-y)^\beta} \int_{b_0}^{y} v(x, s)ds - \frac{\lambda \mu}{(x-a)^\alpha(b-y)^\beta} \int_x^{a_0} dt \int_{b_0}^{y} v(t, s)ds \]
\[ = g(x, y) + \frac{\delta_1}{(x-a)^\alpha(b-y)^\beta} \int_x^{a_0} dt \int_{b_0}^{y} v(t, s)ds. \]

Let us introduce the function

\[ g_1(x, y) = g(x, y) - \frac{\delta_1}{(x-a)^\alpha(b-y)^\beta} \int_x^{a_0} dt \int_{b_0}^{y} v(t, s)ds, \]

where \( \delta_1 = \delta + \lambda \mu. \) Clearly, \( g_1(x, y) \in C(D). \) Then the solution of the integral equation \( T_{\lambda, \mu}^{\alpha, \beta}(v) = g_1(x, y) \) is as follows:

\[ v(x, y) = g_1(x, y) - \frac{\lambda}{(x-a)^\alpha} \int_x^{a_0} \exp[\lambda(\omega_a^\alpha(t) - \omega_a^\alpha(x))]g_1(t, y)dt \]
\[ + \frac{\mu}{(b-y)^\beta} \int_{b_0}^{y} \exp[\mu(\omega_b^\beta(y) - \omega_b^\beta(s))]g_1(x, s)ds \]
\[ - \frac{\lambda \mu}{(x-a)^\alpha(b-y)^\beta} \int_x^{a_0} \exp[\lambda(\omega_a^\alpha(t) - \omega_a^\alpha(x))]dt \]
\[ \times \int_{b_0}^{y} \exp[\mu(\omega_b^\beta(y) - \omega_b^\beta(s))]g_1(t, s)ds \equiv (T_{\lambda, \mu}^{\alpha, \beta})^{-1} g_1(x, y). \]  

(3.1)
In formula (3.1), by substituting the value of \( g_1(x,y) \) and rearranging the appropriate terms, we arrive to the solution of the integral equation

\[
\varphi(x,y) + \frac{\delta_1}{(x-a)^\alpha(b-y)^\beta} \int_x^{a_0} dt \int_{b_0}^y \Phi(t,s) ds = E_{\lambda,\mu}^{\alpha,\beta}[g(x,y)],
\]

where

\[
E_{\lambda,\mu}^{\alpha,\beta}[g(x,y)] = \exp[\lambda \omega_a^\alpha(x) - \mu \omega_b^\beta(y)](T_{\lambda,\mu}^{\alpha,\beta})^{-1} g(x,y)
\]

\[
\begin{align*}
&= \exp[\lambda \omega_a^\alpha(x) - \mu \omega_b^\beta(y)]g(x,y) \\
&\quad - \frac{\lambda}{(x-a)^\alpha} \exp[-\mu \omega_b^\beta(y)] \int_x^{a_0} \exp[\lambda \omega_a^\alpha(t)] g(t,y) dt \\
&\quad + \frac{\mu}{(b-y)^\beta} \exp[\lambda \omega_a^\alpha(x)] \int_{b_0}^y \exp[-\mu \omega_b^\beta(s)] g(x,s) ds \\
&\quad - \frac{\lambda \mu}{(x-a)^\alpha(b-y)^\beta} \int_x^{a_0} \exp[\lambda \omega_a^\alpha(t)] dt \int_{b_0}^y \exp[-\mu \omega_b^\beta(s)] g(t,s) ds
\end{align*}
\]

and

\[
\Phi(x,y) = \exp[\lambda \omega_a^\alpha(x) - \mu \omega_b^\beta(y)] v(x,y).
\]

4. Representation of a solution by functional series of \( \exp(-\omega_a^\alpha(x)) \)

We seek a solution for integral equation (3.2) in the class of functions that can be represented in the form

\[
\Phi(x,y) = \sum_{n=1}^{\infty} (\exp(-\omega_a^\alpha(x)))^n \Phi_n(y)(x-a)^{-\alpha},
\]

where \( \Phi_n(y) \) are unknown functions.

We assume that function \( g(x,y) \) admits representation in the form

\[
g(x,y) = \exp[-\lambda \omega_a^\alpha(x) + \mu \omega_b^\beta(y)] \sum_{n=1}^{\infty} \frac{[\exp(-\omega_a^\alpha(x))]^n}{n!} (x-a)^{-\alpha} g_n(y),
\]

where \( g_n(y) \) are known functions. Moreover, assume that the series (4.2) converges absolutely and uniformly. By substituting this value \( g(x,y) \) into (3.3), we have

\[
E_{\lambda,\mu}^{\alpha,\beta}[g(x,y)] = \sum_{n=1}^{\infty} [\exp(-\omega_a^\alpha(x))]^n (x-a)^{-\alpha} \\
\times \frac{n + \lambda}{n} \left[ g_n(y) + \mu (b-y)^{-\mu} \int_{b_0}^y g_n(s) ds \right] \\
- (x-a)^{-\alpha} \frac{\lambda}{n-1} \sum_{n=1}^{\infty} [n-1]! (a_0)^{n-1} \left[ g_n(y) + \mu (b-y)^{-\mu} \int_{b_0}^y g_n(s) ds \right].
\]
By substituting the values of $\Phi(x, y)$ and $E^\alpha_{\omega, \mu}[g(x, y)]$ into the integral equation (3.2) and equating the coefficients at $[\exp(-\omega_a^\alpha(x))]^k$, $k = 0, 1, 2, \ldots$, we obtain the following relations between the functions $\Phi_n(y)$ and $g_n(y)$, $n = 0, 1, 2, \ldots$:

\[
\delta_1(b - y)^{-\beta} \sum_{n=1}^{\infty} n^{-1}[\exp(-\omega_a^\alpha(a_0))]^n \int_{b_0}^{y} \Phi_n(s)ds
\]
\[
= -\sum_{n=1}^{\infty} n^{-1}[\exp(-\omega_a^\alpha(a_0))]^n \lambda [g_n(y) + (b - y)^{-\beta} \mu \int_{b_0}^{y} g_n(s)ds] \quad (4.3)
\]
and

\[
\Phi_n(y) - \frac{\delta_1}{n(b - y)^{\beta}} \int_{b_0}^{y} \Phi_n(s)ds = \left(\frac{n + \lambda}{n}\right) g_n(y) + \frac{\mu(n + \lambda)}{n(b - y)^{\mu}} \int_{b_0}^{y} g_n(s)ds.
\quad (4.4)
\]

According to [3], if the system of integral equation (4.4) has a solution, then it can be represented in the form

\[
\Phi_n(y) = \frac{n + \lambda}{n} g_n(y) + \left(\frac{\mu n - \delta_1}{n(b - y)^{\beta}}\right) \int_{b_0}^{y} \exp\left[\frac{\delta_1}{n}(\omega_b^\beta(s) - \omega_b^\beta(y))\right] g_n(s)ds, \quad (4.5)
\]
where $n = 0, 1, 2, \ldots$, $\omega_b^\beta(y) = [(\beta - 1)(b - y)^{\beta - 1}]^{-1}$.

Furthermore, it follows from equality (4.3) that

\[
\delta_1(b - y)^{-\beta} \int_{b_0}^{y} \Phi_n(s)ds = -\lambda [g_n(y) + (b - y)^{-\beta} \int_{b_0}^{y} g_n(s)ds] \quad (4.6)
\]
for $n = 0, 1, 2, \ldots$. From expression (4.6), by substituting the values of $\Phi_n(s)$ according to formula (4.5), we obtain the equality

\[
\left(\frac{n + \lambda}{n}\right) \left[\frac{\mu n}{(b - y)^{\beta}} \int_{b_0}^{y} g_n(s)ds - \frac{\delta_1}{(b - y)^{\beta}} \int_{b_0}^{y} \exp\left(\frac{\delta_1}{n}(\omega_b^\beta(s) - \omega_b^\beta(y))\right) g_n(s)ds\right]
\]
\[
= -\lambda \left[g_n(y) + \frac{\mu}{(b - y)^{\beta}} \int_{b_0}^{y} g_n(s)ds\right]. \quad n = 0, 1, 2, 3, \ldots \quad (4.7)
\]

From formula (4.1), by substituting the value $\Phi_n(y)$ from equality (4.5), where $\Phi(x, y) = v(x, y) \exp[\lambda \omega_a^\alpha(x) - \mu \omega_b^\beta(y)]$, we find
Thus, we arrive at the following conclusion.

**Theorem 2.** Assume that in the integral equation (1.2) \( \delta \neq -\lambda \mu \), and that the function \( g(x, y) \) is represented by series (4.2), which converges absolutely and uniformly. Then the integral equation (1.2) has a solution in the class of functions \( v(x, y) \) that are representable in the form

\[
v(x, y) = \exp[-\lambda \omega_a^\alpha(x) + \mu \omega_b^\beta(y)] \sum_{n=1}^{\infty} \frac{\exp(-\omega_a^\alpha(x))n}{(x-a)^\alpha} \left( \frac{n + \lambda}{n} \right) \times \left[ g_n(y) + \frac{\mu n - \delta}{n(b-y)^\beta} \int_{b_0}^{b} (\exp(\omega_b^\beta(s) - \omega_b^\beta(y))) \frac{\delta}{\pi} g_n(s) ds \right].
\]

(4.8)

Moreover, if the functions \( g_k(y), k = 1, 2, \ldots, \) in (4.2) satisfy the infinite system of solvability conditions (4.7), then that solution is unique and can be represented by formula (4.8).

**Remark 1.** In the case where \( \delta \neq -\lambda \mu \), the solution of the integral equation (3.2) could be sought in the class of functions that are representable by a functional series of \( \exp(-\omega_b^\beta(y)) \), i.e.,

\[
\Phi(x, y) = \sum_{n=1}^{\infty} \exp(-n \omega_b^\beta(y))(b - y)^{-\beta} W_n(x)
\]

where \( W_n(x) \) are unknown functions. Then one assumes that the function \( g(x, y) \) is represented in the form

\[
g(x, y) = \exp[-\lambda \omega_a^\alpha(x) + \mu \omega_b^\beta(y)] \sum_{n=1}^{\infty} \frac{\exp(-n \omega_b^\beta(y))}{(b - y)^\beta} g_n(x).
\]

By modifying suitably the argument above, in that case, one can also obtain a statement similar to Theorem 2.

5. **Remarks on a Non-model Integral Equation**

In the domain \( D \), we consider the two-dimensional integral equation

\[
u(x, y) + \int_a^x \frac{A(t)u(t, y)}{(t-a)^\alpha} dt - \int_y^b \frac{B(s)u(x, s)}{(b-s)^\beta} ds \]

\[
+ \int_a^x \frac{dt}{(t-a)^\alpha} \int_y^b \frac{c(t, s)u(t, s)}{(b-s)^\beta} ds = f(x, y),
\]

(5.1)

and its conjugate equation
Integral equations of form (5.1) are studied in [8].

Remark 2. One can find a solution of the integral equation (5.2) if \( c(x, y) \equiv A(x)B(y) \). In that case, as is shown in [6], the question is reduced to finding a solution of two split systems of one-dimensional conjugate integral equations of type (5.2).

Remark 3. In the case where \( c(x, y) \neq A(x)B(y) \), the problem of finding solution for integral equation (5.2) is reduced to the problem of the determination of a solution of the integral equation

\[
v(x, y) + \frac{c(x, y)}{(x-a)^\alpha (b-y)^\beta} \int_x^{a_0} \exp\left[A(a)\left(\omega_\alpha(t) - \omega_\alpha(x)\right) - W_{A,\alpha}^{-}(t) - W_{A,\alpha}^{-}(x)\right] dt \\
\times \int_{b_0}^{b} \exp\left[B(b)\left(\omega_\beta(s) - \omega_\beta(y)\right) + W_{B,\beta}^{-}(s) - W_{B,\beta}^{-}(y)\right] v(t, s) ds \\
\equiv \left(T_{A(x),B(y)}^{\alpha,\beta}\right)^{-1}(g(x, y)),
\]

for any \((x, y) \in D\), where \((T_{A(x),B(y)}^{\alpha,\beta})^{-1}\) is a known integral operator.

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