Opinion Dynamics in Networks: Convergence, Stability and Lack of Explosion

Tung Mai
Georgia Institute of Technology
maitung89@gatech.edu

Ioannis Panageas
Georgia Institute of Technology
ioannis@gatech.edu

Vijay V. Vazirani
Georgia Institute of Technology
vazirani@cc.gatech.edu

Abstract

Inspired by the work of Kempe et. al. [15], we introduce and analyze a model on opinion formation; the update rule of our dynamics is a simplified version of that of [15]. We assume that the population is partitioned into types whose interaction pattern is specified by a graph. Interaction leads to population mass moving from types of smaller mass to those of bigger. We show that starting uniformly at random over all population vectors on the simplex, our dynamics converges point-wise with probability one to an independent set. This settles an open problem of [15], as applicable to our dynamics. We believe that our techniques can be used to settle the open problem for the Kempe et. al. dynamics as well.

Next, we extend the model of Kempe et. al. by introducing the notion of birth and death of types, with the interaction graph evolving appropriately. Birth of types is determined by a Bernoulli process and types die when their population mass is less than a parameter $\epsilon$. We show that if the births are infrequent, then there are long periods of “stability” in which there is no population mass that moves. Finally we show that even if births are frequent and “stability” is not attained, the total number of types does not explode: it remains logarithmic in $1/\epsilon$. 

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1 Introduction

The birth, growth and death of political parties, organizations, social communities and product adoption groups (e.g., whether to use Windows, Mac OS or Linux) often follows common patterns, leading to the belief that the dynamics underlying these processes has much in common. Understanding this commonality is important for the purposes of predictability and hence has been the subject of study in mathematical social science for many years \[4,7,8,13,26\]. In recent years, the growth of social communities on the Internet, and their increasing economic and social value, has provided fresh impetus to this study \[1,2,5,17\].

In this paper, we continue along these lines by building on a natural model proposed by Kempe et. al. \[15\]. Their model consists of an influence graph \(G\) on \(n\) vertices (types, parties) into which the entire population mass is partitioned. Their main tenet is that individuals in smaller parties tend to get influenced by those in bigger parties. Individuals in the two vertices connected by an edge can interact with each other. These interactions result in individuals moving from smaller to bigger in population vertices. Kempe et. al. characterize stable equilibria of this dynamics via the notion of Lyapunov stability, and they show that under any stable equilibrium, the entire mass lies in an independent set, i.e., the population breaks into non-interacting islands. The message of this result is clear: a population is (Lyapunov) stable, in the sense that the system does not change by much under small perturbations, only if people of different opinions do not interact. They also showed convergence to a fixed point, not necessarily an independent set, starting from any initial population vector and influence graph. One of their main open problems was to determine whether starting uniformly at random over all population vectors on the unit simplex, their dynamics converge with probability one to an independent set.

We first settle this open problem in the affirmative for a modification of the dynamics, which however is similar as that of Kempe et. al. in spirit in that it moves mass from smaller to bigger parties (the dynamics is defined below along with a justification). We believe that the ideas behind our analysis can be used to settle the open problem for the dynamics of Kempe et. al. as well, via a more complicated spectral analysis of the Jacobian of the update rule of the dynamics (see Section 3.2).

Whereas the model of Kempe et. al. captures and studies the effects of migration of individuals across types in a very satisfactory manner, it is quite limited in that it does not include the birth and death of types. In this paper, we model birth and death of types. In order to arrive at realistic definitions of these notions, we first conducted case studies of political parties in several countries. We present below a case study on Greek politics, but similar phenomena arise in India, Spain, Italy and Holland (see Wikipedia pages).

The Siriza party in Greece provides an excellent example of birth of a party (this information is readily available in Wikipedia pages). This party was essentially in a dormant state until the first 2012 elections in which it got 16.8% of the vote, mostly taken away from the Pasok party, which dropped from 43.9% to 13.2% in the process (Wikipedia). In the second election in 2012, Siriza increased its vote to 26.9% and Pasok dropped to 12.3%. Finally, in 2015, Siriza increased to 36.3% and Pasok dropped further to 4.7%. Another party, Potami, was formed in 2015 and got 6.1% of the vote, again mainly from Pasok. However, in a major 2016 poll, it seems to have collapsed and is likely be absorbed by other parties. In contrast, the KKE party in Greece, which had almost no interactions with the rest of the parties (and was like a disconnected component), has remained between 4.5-8.5% of the vote over the last 26 years.

Motivated by these examples, we have modeled birth and death of types in the following manner. We model population as a continuum, as is standard in population dynamics, and time is discrete. This is the same as arXiv Version 1 of \[15\], which is what we will refer to throughout this paper; the
later versions study the continuous time analog. The birth of a new type in our model is determined by a Bernoulli process, with parameter \( p \). The newly born type absorbs mass from all other types via a randomized process given by an arbitrary distribution with finite support (see Section 2.2). After birth, the new type is connected to an arbitrary, though non-empty, set of other types. Our model has a parameter \( \epsilon \), and when the size of a type drops below \( \epsilon \), it simply dies, moving its mass equally among its neighbors.

Our rule for migration of mass, which is somewhat different from that of Kempe et. al. is motivated by the following considerations. For a type \( u \), \( x_u \) will denote the fraction of population that is of type \( u \). Assume that types \( u \) and \( v \) have an edge, i.e., their populations interact. If so, we will assume that some individuals of the smaller type get influenced by the larger one and move to the larger one. The question is what is a reasonable assumption on the population mass that moves.

For arriving at the rule proposed in this paper, consider three situations. If \( x_u = .02 \) and \( x_v = .25 \), i.e., the smaller type is very small, then clearly not many people will move. If \( x_u = .22 \) and \( x_v = .25 \), i.e., the types are approximately of the same population size, then again we expect not many people to move. Finally, if \( x_u = .15 \) and \( x_v = .25 \), i.e., both types are reasonably big and their difference is also reasonably big, then we expect several people to move from the smaller to the bigger type. From these considerations, we get the rule that the amount of population mass moving from \( v \) to \( u \), assuming \( x_v < x_u \), is given by

\[
f_{v \rightarrow u}^{(t)} = \alpha_{uv} x_u^{(t)} (x_u^{(t)} - x_v^{(t)}),
\]

where \( 0 \leq \alpha_{uv} = \alpha_{vu} \leq 1 \) captures the level of influence between \( u, v \). We assume that the system is closed, i.e., that it does not get influence from outside factors. This is often, though not always, the case.

### 1.1 Our results and techniques

We first study our migration dynamics without birth and death and settle the open problem of Kempe et. al., as it applies to our dynamics. We show that the dynamics converges set-wise to a fixed point, i.e., there is a set \( S \) containing only fixed points such that the distance between the trajectory of the dynamics and \( S \) goes to zero for all starting population vectors. To show this convergence result, we use a simple potential function of the population mass namely, the \( \ell_2^2 \) norm of the population vector, and we show that this potential is strictly increasing at each time step (unless the dynamics is at a fixed point). Moreover, the potential is bounded, hence the result follows. Next, we strengthen this result by showing point-wise convergence as well. The latter result is technically deeper and more difficult, since it means that every trajectory converges to a specific fixed point \( p \). We show point-wise convergence by constructing a local potential function that is decreasing in a small neighborhood of the limit point \( p \). The potential function is always non-zero in that small neighborhood and is zero only at \( p \).

Using the latter result and one of the most important theorems in dynamical systems, the Center Stable Manifold Theorem, we prove that with probability one, under an initial population vector picked uniformly at random from the unit simplex, our dynamics converges point-wise to a fixed point \( p \), where the active types \( w \) in \( p \), i.e., \( w \in V(G) \) so that \( p_w > 0 \), form an independent set of \( G \). This involves characterization of the linearly stable (see Section 2.3 for definition) fixed points and proving that the update rule of the dynamics is a local diffeomorphism\(^1\). This settles the open problem of Kempe et. al., mentioned in the Introduction, for our dynamics. This result is important because it allows us to perform a long-term average case analysis of the behavior of our dynamics and make predictions.

\(^1\)Continuously differentiable, the inverse exists and is also continuously differentiable.
Next, we introduce birth and death in our model. Clearly there will be no convergence in this case since new parties are created all the time. Instead we define and study a notion of “stability” which is different from the classical notions that appear in dynamical systems (see Section 2.3 for the definition of the classical notion and Definition 4.2 for our notion). A dynamics is \((T, d)\)-stable if and only if \(\forall t: T \leq t \leq T + d\), no population mass moves at step \(t\). We show that despite birth and death, there are arbitrarily long periods of “stability” with high probability, for a sufficiently small \(p\). Finally, we show that in the long run, with high probability, for a sufficiently large \(p\), the number of types in the population will be \(O(\log(1/\epsilon))\). This may seem counter-intuitive, since with a large \(p\) new types will be created often; however, since new types absorb mass from old types, the old types die frequently. In contrast, in the short term (from the definition of \(\epsilon\)) we can have up to \(\Theta(1/\epsilon)\) types.

Let us give an interpretation of the results of the previous paragraph in terms of political parties of certain countries (information obtained from Wikipedia). Countries do have periods of political stability, e.g., during 1981-85, 2004-07, no new major (with more than 1% of the vote) parties were formed in Greece, moreover there was no substantial change in the percentage of votes won by parties in successive elections. The parameter \(\epsilon\) can be interpreted as the fraction of people that can form a party that participates in elections. The minimum size of a party arises for organizational and legal reasons, and is \(\Theta(1/Q)\), where \(Q\) is the population of the country and therefore \(\epsilon\) is inversely related to population. The message of the latter theorem is that the number of political parties grows as the logarithm of the population of the country, i.e. \(O(\log Q)\). The following data supports this fact. The population of Greece, Spain and India in 2015 was 1.1e7, 4.6e7 and 1.2e9, respectively, and the number of parties that participated in the general elections was 20, 32 and 50, respectively.

### 1.2 Related work

As stated above, we build on the work of Kempe et. al. \[15\]. They model their dynamics in a similar way, i.e., there is a flow of population for every interacting pair of types \(u, v\). The flow goes from smaller to bigger types; in our case the mass is just the population of a type. One very interesting common trait between the two dynamics is that the fixed points have similar description: all types with positive mass belonging to the same connected component \(C\) have the same mass. Stable fixed points also have the same properties in both dynamics, namely they are independent sets. The update rules of the two dynamics are somewhat different; our simpler dynamics helps us in proving stronger results.

One of the most studied models is the following: there is a graph \(G\) in which each vertex denotes an individual having two possible opinions. At each time step, an individual is chosen at random who next chooses his opinion according to the majority (best response) opinion among his neighbors. This has appeared in \[9, 22\], where they address the question: in which classes of graphs do individuals reach consensus. The same dynamics, but with each agent choosing his opinion according to noisy best response (the dynamics is a Markov chain) has been studied in \[16, 21\] and many other papers referenced therein. They give bounds for the hitting time and expected time of the consensus state (risk dominant) respectively.

Another well-known model for the dynamics of opinion formation in multiagent systems is Hegselmann-Krause \[10\]. Individuals are discrete entities and are modeled as points in some opinion space (e.g., real line). At every time step, each individual moves to the mass center of all the individuals within unit distance. Typical questions are related to the rate of convergence (see \[6\] and references therein). Finally, another classic model is the voter model, where there is a fixed graph \(G\) among the individuals, and at every time step, a random individual selects a random neighbor and
adopts his opinion [11]. For more information on opinion formation dynamics of an individual using information learned from his neighbors, see [12] for a survey.

Other works, including dynamical systems that show convergence to fixed points, are [18, 20, 23, 24, 27]. [27] focuses on quadratic dynamics and they show convergence in the limit. On the other hand [3] shows that sampling from the distribution this dynamics induces at a given time step is PSPACE-complete. In [19, 23], it is shown that replicator dynamics in linear congestion and 2-player coordination games converges to pure Nash equilibria, and in [18, 24] it is shown that gradient descent converges to local minima, avoiding saddle points even in the case where the fixed points are uncountably many.

Organization: In Section 2 we describe our dynamics formally and give the necessary definitions about dynamical systems. In Section 3 we show that our dynamics without births/deaths converges with probability one to fixed points \( p \) so that the set of types with positive population, i.e., active types, form an independent set of \( G \). Finally, in Section 4 we first show that there is no explosion in the number of types (i.e., the order never becomes \( \Theta(1/\epsilon) \)) and also we perform stability analysis using our notion.

2 Preliminaries

Notation: We denote the probability simplex on a set of size \( n \) as \( \Delta_n \). Vectors in \( \mathbb{R}^n \) are denoted in boldface and \( x_j \) denotes the \( j \)-th coordinate of a given vector \( x \). Time indices are denoted by superscripts. Thus, a time indexed vector \( x \) at time \( t \) is denoted as \( x^{(t)} \). We use the letters \( J, \bar{J} \) to denote the Jacobian of a function and finally we use \( f^t \) to denote the composition of \( f \) by itself \( t \) times.

2.1 Migration dynamics

Let \( G = (V, E) \) be an undirected graph on \( n \) vertices (which we also call types), and let \( N_v \) denote the set of neighbors of \( v \) in \( G \). During the whole dynamical process, each vertex \( v \) has a non-negative population mass representing the fraction of the population of type \( v \). We consider a discrete-time process and let \( x_v^{(t)} \) denote the mass of \( v \) time step \( t \). It follows that the condition

\[
\sum_{v \in V} x_v^{(t)} = 1,
\]

must be maintained for all \( t \), i.e., \( x^{(t)} \in \Delta_n \) for all \( t \in \mathbb{N} \).

Additionally, we consider a dynamical migration rule where the populations can move along edges of \( G \) at each step. The movement at step \( t \) is determined by \( x^{(t)} \). Specifically, for \( uv \in E(G) \), the amount of mass moving from \( v \) to \( u \) at step \( t \) is given by

\[
f_{v \rightarrow u}^{(t)} = \alpha_{uv} x_u^{(t)} x_v^{(t)} (x_u^{(t)} - x_v^{(t)}).
\]

Note that \( f_{v \rightarrow u}^{(t)} > 0 \) implies that population is moving from \( v \) to \( u \), and \( f_{v \rightarrow u}^{(t)} < 0 \) implies that populations is moving in the other direction. The update rule for the population of type \( u \) can be

\(\Delta_n \) denotes the simplex of size \( |V(G)| = n \).
We denote the update rule of the dynamics as $g$.

To model birth of new types, at each time step, with probability $x$, we consider it to be dead and move its mass equally to all its neighbors. Formally, if $x$ is easy to see $x \sim D$ where $D$ is a distribution with support $[\beta_{min}, \beta_{max}]$. Specifically, let $Z_u \sim D$ where $D$ is a distribution with support $[\beta_{min}, \beta_{max}]$, the amount of mass going from $u$ to $v$ is $Z_u x_v$.

Formally, if $x_t \in \Delta_n$ then $x_t \in \Delta_n$.

Therefore it holds that $x_t = x'(x_0)$, where $x'$ denotes the composition of $g$ by itself $t$ times. It is easy to see $g$ is well-defined for $\alpha_{uv} \leq 1$ for all $uv \in E(G)$, in the sense that if $x_t \in \Delta_n$ then $x_{t+1} \in \Delta_n$. This is true since for all $u$ we get (using induction, i.e., $x_t \in \Delta_n$)

$$x_{t+1} = x_t + \sum_{v \in N_u} \alpha_{uv} x_u x_v (x_u - x_v) \geq x_t - \sum_{v \in N_u} x_t x_v \geq x_t - x_t (1 - x_u) \geq 0,$$

moreover it holds

$$x_{t+1} = x_t + \sum_{v \in N_u} \alpha_{uv} x_u x_v (x_u - x_v) \leq x_t + \sum_{v \in N_u} x_t x_v \leq x_t + x_t (1 - x_u) \leq x_t + 1 - x_u = 1,$$

and also $\sum_u x_{t+1} = \sum_u x_t = 1$ (the other terms cancel out).

### 2.2 Birth and death of types

Political parties or social communities don’t tend to survive once their size becomes “small” and hence there is a need to incorporate death of parties in our model. We will define a global parameter $\epsilon$ in our model. When the population mass of a type $v$ becomes smaller than some fixed value $\epsilon$, we consider it to be dead and move its mass equally to all its neighbors. Formally, if $x_t \leq \epsilon$ then $x_t \leftarrow 0$ and $x_t = x_u + x_v / |N_v|$ for all $u \in N_v$. Also, vertex $v$ is removed and a clique is added on its neighbors to ensure connectivity of the resulting graph.

Every so often, new political opinions emerge and like-minded people move from the existing parties to create a new party, which then follows the normal dynamics to either survive or die out. To model birth of new types, at each time step, with probability $p$, we create a new type $v$ such that $v$ takes a portion of mass from each existing type. The amount of mass going to $v$ from each $u$ follows an arbitrary distribution in the range $[\beta_{min}, \beta_{max}]$. Specifically, let $Z_u \sim D$ where $D$ is a distribution with support $[\beta_{min}, \beta_{max}]$, the amount of mass going from $u$ to $v$ is $Z_u x_v$. We denote the update rule of the dynamics as $g : \Delta_n \rightarrow \Delta_n$, i.e., we have that

$$x_{t+1} = g(x_t).$$
We connect \( v \) to the existing graph arbitrarily such that it remains connected.

Additionally, we make a small change to the migration dynamics defined in Section 2.1 to make it more realistic. Our tenet is that population mass migrates from smaller to bigger types because of influence. However, if the two types are of approximately the same size, the difference is size is not discernible and hence migration should not happen. To incorporate this, we introduce a new parameter \( \delta > 0 \) and if \( |x_u - x_v| \leq \delta \), we assume that no population moves from \( u \) to \( v \). Finally, at each step, the order of actions on \( G \) is: migration \( \rightarrow \) birth \( \rightarrow \) death.

**Remark 2.1.** It is not hard to see that the maximum number of types is \( 1/\epsilon \) (by definition). We say that we have explosion in the number of types if they are of \( \Theta(1/\epsilon) \). In Theorem 4.7 we show that in the long run, the number of types is much smaller, it is \( O(\log(1/\epsilon)) \) with high probability.

### 2.3 Definitions and basics

A recurrence relation of the form \( x(t+1) = f(x(t)) \) is a discrete time dynamical system, with update rule \( f : S \rightarrow S \) (for our purposes, the set \( S \) is \( \Delta_n \)). The point \( z \) is called a fixed point or equilibrium of \( f \) if \( f(z) = z \). A fixed point \( z \) is called Lyapunov stable (or just stable) if for every \( \varepsilon > 0 \), there exists a \( \zeta = \zeta(\varepsilon) > 0 \) such that for all \( x \) with \( \|x - z\| < \zeta \) we have that \( \|f^k(x) - z\| < \varepsilon \) for every \( k \geq 0 \). We call a fixed point \( z \) linearly stable if, for the Jacobian \( J(z) \) of \( f \), it holds that its spectral radius is at most one. It is true that if a fixed point \( z \) is stable then it is linearly stable but the converse does not hold in general [25]. A sequence \((f^t(x(0)))_{t \in \mathbb{N}}\) is called a trajectory of the dynamics with \( x(0) \) as starting point. A common technique to show that a dynamical system converges to a fixed point is to construct a function \( P : \Delta_m \rightarrow \mathbb{R} \) such that \( P(f(x)) > P(x) \) unless \( x \) is a fixed point. We call \( P \) a potential or Lyapunov function.

### 3 Convergence to independent sets almost surely

In this section we prove that the deterministic dynamics (assuming no death/birth of types, namely the graph \( G \) remains fixed) converges point-wise to fixed points \( p \) where \( \{v : p_v > 0\} \) (set of active types) is an independent set of the graph \( G \), with probability one assuming that the starting point \( x(0) \) follows an atomless distribution with support in \( \Delta_n \). To do that, we show that for all starting points \( x(0) \), the dynamics converges point-wise to fixed points. Moreover we prove that the update rule of the dynamics is a diffeomorphism and that the linearly stable fixed points \( p \) of the dynamics satisfy the fact that the set of active types in \( p \) is an independent set of \( G \). Finally, our main claim of the section follows by using a well-known theorem in dynamical systems, called Center-Stable Manifold theorem.

**Structure of fixed points.** The fixed points of the dynamics \( [1] \) are vectors \( p \) such that for each \( uv \in E(G) \), at least one of the following conditions must hold:

1. \( p_v = p_u \),
2. \( p_v = 0 \),
3. \( p_u = 0 \).

Therefore, for each fixed point \( p \), the set of active types (types with non-zero population mass) with respect to \( p \) must form a set of connected components such that all types in each component have the same population mass. We first prove that the dynamics converges point-wise to fixed points.

#### 3.1 Point-wise convergence

First we consider the following function

\[
\Phi(x) = \sum_v x_v^2
\]
and state the following lemma on Φ.

**Lemma 3.1 (Lyapunov (potential) function).** Let \( x \) be a vector with \( x_u > x_v \). Let \( y \) be another vector such that \( y_v = x_v - d, y_u = x_u + d \) for some \( 0 < d \leq x_v \) and \( y_z = x_z \) for all \( z \neq u, v \). Then

\[
Φ(x) < Φ(y).
\]

**Proof.** By the definition of \( Φ \),

\[
Φ(y) = y_v^2 + y_u^2 + \sum_{z \neq u,v} y_z^2
= (x_v - d)^2 + (x_u + d)^2 + \sum_{z \neq u,v} y_z^2
= x_v^2 - 2dx_v + d^2 + x_u^2 - 2dx_u + d^2 + \sum_{z \neq u,v} x_z^2
= Φ(x) + 2d(x_u - x_v) + 2d^2
> Φ(x).
\]

The inequality follows because \( d > 0 \) and \( x_u > x_v \).

If we think of \( x \) as a population vector, Lemma 3.1 implies that \( Φ(x) \) increases if population is moving from a smaller type to a bigger type.

**Theorem 3.2 (Set-wise convergence).** \( Φ(x^{(t)}) \) is strictly increasing along every nontrivial trajectory, i.e., \( Φ(x^{(t+1)}) = Φ(g(x^{(t)})) ≥ Φ(x^{(t)}) \) with equality only when \( x^{(t)} \) is a fixed point. As a corollary, the dynamics converges to fixed points (set-wise convergence).

**Proof.** First we prove that the dynamical process converges to a set of fixed points by showing that \( Φ(x^{(t)}) \) is strictly increasing as \( t \) grows. The idea is breaking a migration step from \( x^{(t)} \) to \( x^{(t+1)} \) into multiple steps such that each small step only involves migration between two types. Moreover, in each small step, population is moving from a smaller type to a bigger type. Lemma 3.1 guarantees that \( Φ \) is strictly increasing in every small step, and thus strictly increasing in the combined step from \( x^{(t)} \) to \( x^{(t+1)} \).

Let \( D \) be the directed graph representing the migration movement from \( x^{(t)} \) to \( x^{(t+1)} \). Formally, for each edge \( uv \in E \) we direct \( uv \) in both directions and let \( f_{v\rightarrow u} = \max(0, f_{v\rightarrow u}^{(t)}) \). Define the following process on \( D \):

1. If there exists a directed path \( v \rightarrow u \rightarrow z \) of length 3 in \( D \) such that \( f_{v\rightarrow u} \) and \( f_{u\rightarrow z} \) are both positive, we make the following modification to the flow in \( D \).

   Let \( Δ = \min(f_{v\rightarrow u}, f_{u\rightarrow z}) \), and

   \[
   f_{v\rightarrow u} \leftarrow f_{v\rightarrow u} - Δ
   f_{u\rightarrow z} \leftarrow f_{u\rightarrow z} - Δ
   f_{v\rightarrow z} \leftarrow f_{v\rightarrow z} + Δ.
   \]

2. Keep repeating the previous step until there is no path of length 3 carrying positive flow.
The above process must terminate since the function
\[ \sum_{v \to u} f_{v \to u}(x_u^{(t)} - x_v^{(t)})^2 \]
strictly increases in each modification. At the end of the process, there is no path of length 3 carrying positive flow. In other words, each type can not have both flows coming into it and flows coming out of it. Furthermore, it is easy to see that the net flow \( \sum_{v \in V} f_{v \to u} \) at each type \( u \) is preserved and \( f_{v \to u} > 0 \) only if \( x_u^{(t)} > x_v^{(t)} \). We can break a migration step from \( x^{(t)} \) to \( x^{(t+1)} \) into multiple migrations such that each small migration corresponds to a flow \( f_{v \to u} > 0 \) at the end of the process. It follows that in each small migration, population is moving from one smaller type to one bigger type.

To finish the proof we proceed in a standard manner as follows: Let \( \Omega \subset \Delta_n \) be the set of limit points of a trajectory \( x^{(t)} \). \( \Phi(x^{(t)}) \) is increasing with respect to time \( t \) by above and so, because \( \Phi \) is bounded on \( \Delta_n \), \( \Phi(x^{(t)}) \) converges as \( t \to \infty \) to \( \Phi^* = \sup \{ \Phi(x^{(t)}) \} \). By continuity of \( \Phi \) we get that \( \Phi(p) = \lim_{t \to \infty} \Phi(x^{(t)}) = \Phi^* \) for all \( p \in \Omega \), namely \( \Phi \) is constant on \( \Omega \). Also \( p^{(t)} = \lim_{k \to \infty} x^{(t_k+t)} \) as \( k \to \infty \) for some sequence of times \( \{t_i\} \), with \( p^{(0)} = \lim_{k \to \infty} x^{(t_k)} \) and \( p^{(0)} \in \Omega \). Therefore \( p^{(t)} \) lies in \( \Omega \) for all \( t \in \mathbb{N} \), i.e., \( \Omega \) is invariant. Thus, since \( p^{(0)} \in \Omega \) and the orbit \( p^{(t)} \) lies in \( \Omega \), we get that \( \Phi(p^{(t)}) = \Phi^* \) on the orbit. But \( \Phi \) is strictly increasing except on fixed points and so \( \Omega \) consists entirely of fixed points.

\[ \text{Theorem 3.3 (Point-wise convergence). \textbf{The dynamics converges point-wise to fixed points.}} \]

\[ \text{Proof.} \] We first construct a local potential function \( \Psi \) such that \( \Psi(x,p) \) is strictly decreasing in some small neighborhood of a limit point (fixed point) \( p \). Formally we initially prove the following:

\[ \text{Claim 3.4 (Local Lyapunov function). Let } p \text{ be a limit point (which will be a fixed point of the dynamics by Theorem 3.2) of trajectory } y^{(t)}, \text{ and } \Psi(x,p) \text{ be the following function} \]
\[ \Psi(x,p) = \sum_{v:p_v>0} (p_v - x_v). \]

There exists a small \( \varepsilon > 0 \) such that if \( \|y^{(t)} - p\|_1 \leq \varepsilon \) then \( \Psi(y^{(t+1)},p) \leq \Psi(y^{(t)},p) \).

\[ \text{Proof of Claim 3.4.} \] We know that the set of active types in \( p \) must form a set of connected components such that all types in each component have the same population mass with respect to \( p \). Let \( C \) be one such component and let \( \delta(C) = \{v : v \notin C, u \in C, uv \in E\} \). We choose \( \varepsilon \) to be so small so that \( y_u^{(t)} > y_v^{(t)} \) for all \( u \in C \) and \( v \in \delta(C) \), because \( y_v^{(t)} < \varepsilon \) since \( p_v = 0 \) (thus is arbitrarily close to zero) for \( v \in \delta(C) \). Therefore, the net flow into \( C \) must be non-negative, thus \( \Psi(g(y^{(t)}),p) \leq \Psi(y^{(t)},p) \).

Additionally, we have that \( y_u^{(t)} \leq p_u \) for all \( u \in C \). Suppose otherwise, then there exists a type \( w \triangleq w(t) \in C \) with \( y_{w(t)}^{(t)} > p_{w(t)} \), which is a contradiction. To see why, consider \( w(t') \) to be \( \arg\max_{z \in C} y_z^{(t')} \), then it should hold that \( y_{w(t')}^{(t')} \geq p_{w(t')} + s \) for some constant \( s > 0 \) independent of \( t' \) and \( t' \geq t \), because \( y_{w(t')}^{(t')} \) is increasing and therefore \( p \) cannot be a limit point of the trajectory \( y^{(t)} \). Hence, \( \Psi(y^{(t)},p) \) must be non-negative and only equal to zero when \( y^{(t)} = p \) (i).

To finish the proof of the theorem, if \( p \) is a limit point of \( y^{(t)} \), there exists an increasing sequence of times \( \{t_n\} \), with \( t_n \to \infty \) and \( y^{(t_n)} \to p \). We consider \( \varepsilon' \) such that the set \( S = \{x : x \leq p \text{ and } \Psi(x,p) \leq \varepsilon'\} \) is inside \( B = \|x - p\|_1 < \varepsilon \) where \( \varepsilon \) is from Claim 3.4 about the local potential.
Since \( y^{(t_n)} \to p \), consider a time \( t_N \) where \( y^{(t_N)} \) is inside \( S \). From Claim 3.4 we get that if \( y^{(t)} \in B \) then \( \Psi(y^{(t+1)}, p) \leq \Psi(y^{(t)}, p) \) (and also \( y^{(t)} \in S \), thus \( \Psi(y^{(t)}, p) \leq \epsilon' \) and \( y^{(t)} \leq p \) for all \( t \geq t_N \) (namely the orbit remains in \( S \); we use Claim 3.4 inductively). Therefore \( \Psi(x^{(t)}, p) \) is decreasing in \( S \) and since \( \Psi(y^{(t_n)}, p) \to \Psi(p, p) = 0 \), it follows that \( \Psi(y^{(t)}, p) \to 0 \) as \( t \to \infty \).

Hence \( y^{(t)} \to p \) as \( t \to \infty \) using (i).

\[ \square \]

### 3.2 Diffeomorphism and stability analysis via Jacobian

In this section we compute the Jacobian \( J \) of \( g \) and then perform spectral analysis on \( J \). The Jacobian of \( g \) is the following:

\[
\frac{\partial g_u}{\partial x_u} = J_{u,u} = 1 + \sum_{v \in N_u} \alpha_{uv}(2x_vx_u - x_v^2),
\]

\[
\frac{\partial g_u}{\partial x_v} = J_{u,v} = \alpha_{uv}(x_u^2 - 2x_u x_v) \text{ if } uv \in E \text{ else } 0.
\]

**Lemma 3.5 (Local Diffeomorphism).** The Jacobian is invertible on the subspace \( \sum_v x_v = 1 \), for \( \alpha_{uv} < \frac{1}{2} \) for each \( uv \in E(G) \). Moreover, \( g \) is a local diffeomorphism in a neighborhood of \( \Delta_n \).

**Proof.** First we have that \( \sum_{v \in N_u} (2x_u x_v - x_v^2) \geq -\sum_{v \in N_u} x_v^2 \geq -1 \) and hence \( J_{u,u} > 0 \) for all \( u \) and \( x \in \Delta_n \). Additionally, we get that

\[
|J_{u,u}| - \sum_{v \neq u} |J_{v,u}| = 1 + \sum_{v \in N_u, x_v > 2x_u} \alpha_{uv}(4x_u x_v - 2x_v^2)
\]

\[
\geq 1 - 2\sum_{v \in N_u, x_v > 2x_u} \alpha_{uv}x_v^2 > 0,
\]

where we used the fact that \( a < \frac{1}{2} \) and that \( \sum_v x_v^2 \leq 1 \). Therefore we conclude that \( J^\top \) is diagonally dominant.

Finally, assume that \( J \) is not invertible, then there exists a nonzero vector \( y \) so that \( J^\top y = 0 \) (ii). We consider the index type with the maximum absolute value in \( y \), say \( w \). Hence we have \( |y_w| \geq |y_v| \) for all \( v \in V(G) \). Finally, using (ii) we have that \( J_{w,u} y_w = -\sum_{v \neq w} J_{v,w} y_v \), thus \( J_{w,w} \leq \sum_{v \neq w} |J_{v,w}| |y_v| |y_w| \leq \sum_{v \neq w} |J_{w,v}| \) (first inequality is triangle inequality and second comes from assumption on \( w \)). We reach a contradiction because we showed before that \( J^\top \) is diagonally dominant.

Therefore \( J(x) \) is invertible for all \( x \in \Delta_n \). Moreover, from inverse function theorem we get the claim that the update rule of the dynamics is a local diffeomorphism in a neighborhood of \( \Delta_n \). \( \square \)

**Lemma 3.6 (Linearly stable fixed point \( \Rightarrow \) independent set).** Let \( p \) be a fixed point so that there exists a connected component \( C \), \( |C| > 1 \) with \( p_v > 0 \) some population mass for all \( v \in C \). Then the Jacobian at \( p \) has an eigenvalue with absolute value greater than one.

**Proof.** The Jacobian at \( p \) has equations:

1. Assume \( p_u = 0 \) then

\[
J_{u,u} = 1 - \sum_{v \in N_u} \alpha_{uv}p_v^2,
\]

\[
J_{u,v} = 0 \text{ for all } v \neq u.
\]
2. Assume $p_u > 0$ then

$$J_{u,u} = 1 + \sum_{v \in N_u} \alpha_{uv} p_v^2.$$ 

For all $v \in V(G)$ so that $p_v = 0$, it follows that $J_{v,v'}$ is nonzero only when $v' = v$ (diagonal entry) and thus the corresponding eigenvalue of $J$ is $1 - \sum_{v' \in N_v} \alpha_{vv'} p_{v'}^2 \leq 1$ with left eigenvector $(0,...,0,1,0,...,0)$. Hence the characteristic polynomial of $J$ at $p$ is equal to

$$\prod_{v \cdot p_v = 0} \left( \lambda - (1 - \sum_{v' \in N_v} \alpha_{vv'} p_{v'}^2) \right) \times \det(\lambda I - J),$$

where $J$ corresponds to $J$ at $p$ by deleting rows corresponding to types $v$ such that $p_v = 0$ and $I$ the identity matrix. So it suffices to prove that $J$ has an eigenvalue with absolute value greater than one.

Assume $J$ has size $l \times l$, in other words the number of active types is $l$ in $p$. Every type $v$ that has no active neighbors satisfies $J_{v,v} = 1$ and every type $v$ that has at least one active neighbor satisfies $J_{v,v} = 1 + \sum_{v' \in N_v} \alpha_{vv'} p_{v'}^2 > 1$. Therefore $\text{trace}(J) > l$ by assumption on $p$. Hence the sum of the eigenvalues of $J$ is greater than $l$, thus there exists an eigenvalue with absolute size greater than 1.

### 3.3 Center-stable manifold and average case analysis

In this section we prove our first main result, Corollary 3.11 which is a consequence of the following theorem:

**Theorem 3.7.** The set of initial conditions so that dynamics $\mathbb{F}$ converges to a fixed point $p$ where the active types in $p$ do not form an independent set of $G$ has measure zero.

To prove Theorem 3.7, we are going to use arguably one of the most important theorems in dynamical systems, called Center Stable Manifold Theorem:

**Theorem 3.8 (Center-stable Manifold Theorem) [28].** Let $p$ be a fixed point for the $C^r$ local diffeomorphism $f : U \to \mathbb{R}^m$ where $U \subset \mathbb{R}^m$ is an open neighborhood of $p$ in $\mathbb{R}^m$ and $r \geq 1$. Let $E^s \oplus E^c \oplus E^u$ be the invariant splitting of $\mathbb{R}^m$ into generalized eigenspaces of the Jacobian of $g$, $J(p)$ corresponding to eigenvalues of absolute value less than one, equal to one, and greater than one. To the $J(p)$ invariant subspace $E^s \oplus E^c$ there is an associated local $f$ invariant $C^r$ embedded disk $W_{loc}^{sc}$ tangent to the linear subspace at $p$ and a ball $B$ around $p$ such that:

$$f(W_{loc}^{sc}) \cap B \subset W_{loc}^{sc}. \text{ If } f^m(x) \in B \text{ for all } m \geq 0, \text{ then } x \in W_{loc}^{sc}. \quad (3)$$

Since an $n$-dimensional simplex $\Delta_n$ in $\mathbb{R}^n$ has dimension $n-1$, we need to take a projection of the domain space ($\sum_x x_v = 1$) and accordingly redefine the map $g$. Let $x$ be a point mass in $\Delta_n$. Let $u$ be a fixed type and define $h : \mathbb{R}^n \to \mathbb{R}^{n-1}$ so that we exclude the variable $x_u$ from $x$, i.e., $h(x) = x_{-u}$. We substitute the variable $x_u$ with $1 - \sum_{v \neq u} x_v$ and let $g'$ be the resulting update rule of the dynamics $g'(x_{-u}) = g(x)$. The following lemma gives a relation between the eigenvalues of the Jacobians of functions $g$ and $g'$.

**Lemma 3.9.** Let $J, J'$ be the Jacobian of $g, g'$ respectively. Let $\lambda$ be an eigenvalue of $J$ so that $\lambda$ does not correspond to left eigenvector $(1,...,1)$ (with eigenvalue 1). Then $J'$ has also $\lambda$ as an eigenvalue.
Proof. By chain rule, the equations of \( J' \) are as follows:

\[
\frac{\partial g_j}{\partial x_w} = J'_{v,w}(x-u) = J_{v,w}(x-u, 1 - \sum_{v \neq u} x_v) - J_{v,u}(x-u, 1 - \sum_{v \neq u} x_v).
\]

Assume \( \lambda \) is associated with left eigenvector \( r \overset{\text{def}}{=} (r_1, \ldots, r_{n-1}, r_n) \) (we label the types with numbers \( 1, \ldots, n \) with \( u \) taking index \( n \)). We claim that \( \lambda \) is an eigenvalue of \( J \) with right eigenvector \( r' \overset{\text{def}}{=} (r_1 - r_n, \ldots, r_{n-1} - r_n) \). First it is easy to see that

\[
\sum_{j=1}^{n-1} J'_{ji}(r_j - r_n) = \sum_{j=1}^{n-1} (J_{ji} - J_{jn})(r_j - r_n)
\]

\[
= \sum_{j=1}^{n-1} J_{ji} r_j - \sum_{j=1}^{n-1} J_{jn} r_j - \sum_{j=1}^{n-1} J_{ji} r_n + \sum_{j=1}^{n-1} J_{jn} r_n.
\]

Since \( r \) is a left eigenvector, we get that \( \sum_{j=1}^{n} J_{ji} r_j = \lambda r_i \) for all \( i \in [n] \) and also it holds that \( \sum_{j=1}^{n} J_{ji} = 1 \) for all \( i \in [n] \). Therefore

\[
\sum_{j=1}^{n-1} J'_{ji}(r_j - r_n) = (\lambda r_i - J_{ni} r_n) - (\lambda r_n - J_{nn} r_n) - (1 - J_{ni}) r_n + (1 - J_{nn}) r_n
\]

\[
= \lambda (r_i - r_n),
\]

namely \( r'J' = \lambda r' \) and the lemma follows. \( \square \)

Before we proceed with the proof of Theorem 3.7, we state the following which is a corollary of Lemmas 3.5, 3.6 and 3.9, and also uses classic properties for determinants of matrices.

Corollary 3.10. Let \( p \) be a fixed so that the active types are not an independent set in \( G \), then \( J' \) at \( h(p) \) has an eigenvalue with absolute value greater than one. Additionally, The Jacobian \( J' \) of \( g' \) is invertible in \( h(\Delta_n) \) and as a result \( g' \) is a local diffeomorphism in a neighborhood of \( h(\Delta_n) \).

Proof. If \( p \) is a fixed point where the active types are not an independent set in \( G \), then by Lemma 3.6, we get that \( J \) at \( p \) has an eigenvalue with absolute value greater than one, hence using Lemma 3.9, it follows that \( J' \) at \( h(p) \) has an eigenvalue with absolute value greater than one.

Let \( B \) be the resulting matrix if we add all the first \( n-1 \) rows to the \( n \)-th row and then subtract the \( n \)-th column from all other columns in matrix \( J \). It is clear that \( \det(B) = \det(J) \neq 0 \) (determinant not zero since \( J \) is invertible from Lemma 3.5). Additionally, the last row of \( B \) is all 0’s and \( B_{nn} = 1 \), so \( \det(B) = \det(B') \) where \( B' \) is the resulting matrix if we delete from \( B \) last row.column. But \( B' = J' \), hence \( 0 \neq \det(J) = \det(B) = \det(J') \) and thus \( J' \) is invertible, therefore \( g' \) is a local diffeomorphism in a neighborhood of \( h(\Delta_n) \) (by Inverse function theorem). \( \square \)

Proof of Theorem 3.7. Let \( p \) be a fixed point of function \( g(x) \) so that the set of active types is not an independent set. We consider the projected fixed point \( p' \overset{\text{def}}{=} h(p) \) of function \( g' \). Then \( p' \) is a linearly unstable fixed point. Let \( B_{p'} \) be the (open) ball (in the set \( \mathbb{R}^{n-1} \)) that is derived from center-stable manifold theorem. We consider the union of these balls

\[
A = \cup_{p'} B_{p'}.
\]
Due to Lindelöf’s lemma stated in the appendix, we can find a countable subcover for \( A \), i.e., there exist fixed points \( p'_1, p'_2, \ldots \) such that \( A = \bigcup_{m=1}^{\infty} B_{p'_m} \). Starting from a point \( x' \in \Delta' \), there must exist a \( t_0 \) and \( m \) so that \( g'^{-t}(x) \in B_{p'_m} \) for all \( t \geq t_0 \) because of Theorem 3.3, i.e., the dynamics converges point-wise. From center-stable manifold theorem we get that \( g'^{-t}(p') \in W_{loc}^{sc}(p'_m) \cap \Delta' \) where we used the fact that \( g'((\Delta')) \subseteq \Delta' \) (the population vector is always in simplex, see Section 2.1), namely the trajectory remains in \( \Delta' \) for all times.

By setting \( D_1(p'_m) = g'^{-1}(W_{loc}^{sc}(p'_m) \cap \Delta') \) and \( D_{i+1}(p'_m) = g'^{-1}(D_i(p'_m) \cap \Delta') \) we get that \( x' \in D_i(p'_m) \) for all \( t \geq t_0 \). Hence the set of initial points in \( \Delta' \) so that dynamics converges to a fixed point \( p' \) so that the set of active types is not an independent set of \( G \) is a subset of

\[
P = \bigcup_{m=1}^{\infty} \bigcup_{i=0}^{\infty} D_i(p'_m). \tag{4}
\]

Since \( p'_m \) is linearly unstable fixed point (for all \( m \)), the Jacobian \( J' \) has an eigenvalue greater than 1, and therefore the dimension of \( W_{loc}^{sc}(p'_m) \) is at most \( n-2 \). Thus, the set \( W_{loc}^{sc}(p'_m) \cap \Delta' \) has Lebesgue measure zero in \( \mathbb{R}^{n-1} \). Finally since \( g' \) is a local diffeomorphism, \( g'^{-1} \) is locally Lipschitz (see [25] p.71). \( g'^{-1} \) preserves the null-sets (by Lemma A.2 that appears in the appendix) and hence (by induction) \( D_i(p'_m) \) has measure zero for all \( i \). Thereby we get that \( P \) is a countable union of measure zero sets, i.e., is measure zero as well.

**Corollary 3.11 (Convergence to independent sets).** Assume the initial mass vector \( x^{(0)} \in \Delta_n \) is chosen from an atomless distribution, then the dynamics converges point-wise with probability 1 to a point \( p \) so that the active types form an independent set in \( G \).

**Proof.** The proof comes from Theorem 3.3 and Theorem 3.7. From Theorem 3.3 we have that \( \lim_{t \to \infty} x^{(t)} \) exists for all \( x^{(0)} \in \Delta_n \) and from Theorem 3.7 we get the probability that the dynamics converges to fixed points where the active types are not an independent set is zero. Hence the dynamics converges to fixed points where the active types is an independent set, with probability one.

Corollary 3.11 is illustrated in Figure 1 for the case of a 3-path and a triangle. As shown in the figure, if the initial condition is chosen uniformly at random from a point in the simplex, the dynamics converges to an independent set with probability one.

### 4 Stability and bound on the number of types

In this section, we consider dynamical systems with migration, death and birth and prove two probabilistic statements on stability and number of types. The following direct application of Chernoff’s bound is used intensively to attain probabilistic guarantees.

**Lemma 4.1.** Consider a period of \( t \) steps.

1. There are at least \( tp/2 \) births with probability at least 1 \(- e^{-tp/8} \).
2. There are at most \( 3tp/2 \) births with probability at least 1 \(- e^{-tp/6} \).

**Proof.** Let \( X_i = 1 \) if there is a birth at step \( i \), and \( X_i = 0 \) otherwise. The number of births in \( k \) step is

\[
X = \sum_{i=1}^{k} X_i
\]

\( W_{loc}^{sc}(p'_m) \) denotes the center stable manifold of fixed point \( p'_m \).
It follows that $E[X] = tp$. From Chernoff’s bound,

$$\Pr(X \leq tp/2) \leq e^{-tp/8}.$$ 

In other words,

$$\Pr(X \geq tp/2) \geq 1 - e^{-tp/8}.$$ 

Applying Chernoff’s bound again, we have

$$\Pr(X \geq 3tp/2) \leq e^{-tp/6},$$

and

$$\Pr(X \leq 3tp/2) \geq 1 - e^{-tp/6}.$$ 

**4.1 Stability**

We define the notion of stability and give a stability result for a dynamical system involving migration, death and birth. For the rest of the paper, we denote by $\alpha_{\min} = \min_{uv \in E(G)} \alpha_{uv}$ and $\alpha_{\max} = \max_{uv \in E(G)} \alpha_{uv}$.

**Definition 4.2** $(T, d)$-Stable dynamics. A dynamics is $(T, d)$-stable if and only if $\forall T \leq t \leq T + d$, no population mass moves at step $t$.

**Remark 4.3.** In words, dynamics is $(T, d)$-stable if it has the same population vector $p$ for a period of $d + 1$ steps starting from $T$, where $p$ is an approximate fixed point of the initial dynamics (without births/deaths).

We state the following two lemmas whose proofs are straightforward from the definition of $\Phi$.

**Lemma 4.4.** If the dynamics is not $(t, 0)$-stable, the migration phase at time $t$ increases $\Phi$ by at least $2\alpha_{\min} \epsilon \delta^3$.

**Proof.** By Theorem 3.2, we know that $\Phi$ is strictly increasing in each migration when the dynamics is not at a fixed point. Moreover, it is easy to see that the increase is minimized when there is only one edge $uv$ carrying population flow. From the proof of Lemma 3.1, the additive increase in $\Phi$ is at least

$$2d |x_u - x_v|,$$

where $d$ is the amount of mass moving along $uv$. Without loss of generality, we may assume that $x_u > x_v$. Since $x_v \geq \epsilon$ and $x_u - x_v \geq \delta$,

$$d = \alpha_{uv} x_u x_v |x_u - x_v| \geq \alpha_{\min} \epsilon \delta^2.$$

It follows that

$$2d |x_u - x_v| \geq 2\alpha_{\min} \epsilon \delta^3.$$

**Lemma 4.5.** Each birth can decrease $\Phi$ by at most $2\beta_{\max}$.
Proof. Recall that
\[ \Phi(x) = \sum_v x_v^2. \]
The potential after a new type is created is
\[ \sum_v x_v^2 (1 - Z_v)^2 + \sum_v Z_v^2 x_v^2. \]
Therefore, the net decrease in potential is
\[ \Delta \Phi = \sum_v x_v^2 - \sum_v x_v^2 (1 - Z_v)^2 - \sum_v Z_v^2 x_v^2 \]
\[ = \sum_v Z_v x_v^2 - \sum_v Z_v^2 x_v^2 - \sum_v Z_v^2 x_v^2 \]
\[ \leq 2 \sum_v Z_v x_v^2 \leq 2 \beta_{\text{max}} \sum_v x_v^2 \]
\[ \leq 2 \beta_{\text{max}} \left( \sum_v x_v \right)^2 = 2 \beta_{\text{max}}. \]

With the two above lemmas, we can give a theorem on the stability of the dynamics:

**Theorem 4.6 ("Stable" for long enough).** Let \( p < \min \left( \frac{\epsilon \delta \alpha_{\min}}{3 \beta_{\max}}, \frac{2}{3} \right) \) and \( t > \frac{1}{\epsilon \delta \alpha_{\min} - 3p \beta_{\max}}. \)

With probability at least \( 1 - e^{-tp/6} \) the dynamics is \( (T, \frac{1}{3p}) \)-stable for some \( T \leq t \).

**Proof.** Consider a period of \( t \) steps. By Lemma 4.1, there are at most \( 3tp/2 \) births in the period with probability at least \( 1 - e^{-tp/6} \). Note that \( p < 2/3 \) guarantees that \( 3tp/2 < t \). In the migration phases of the period, \( \Phi \) can either increase if there is a migration or remain unchanged otherwise.

Assume that \( \Phi \) increases in more than \( t/2 \) migration phases. By Lemma 4.4, the amount of increase due to migrations is at least
\[ \frac{t}{2} 2 \alpha_{\min} \epsilon \delta^3 = t \alpha_{\min} \epsilon \delta^3. \]
Since there are at most \( 3tp/2 \) births, Lemma 4.5 guarantees that the amount of decrease due to births is at most
\[ \frac{3tp}{2} 2 \beta_{\max} = 3tp \beta_{\max}. \]
Therefore, the net increase of \( \Phi \) is least
\[ t \alpha_{\min} \epsilon \delta^3 - 3tp \beta_{\max} = t(\alpha_{\min} \epsilon \delta^3 - 3p \beta_{\max}). \]
Since \( t > \frac{1}{\epsilon \delta \alpha_{\min} - 3p \beta_{\max}} \), the net increase in \( \Phi \) is greater than 1, which is a contradiction.

It follows that \( \Phi \) cannot increase in more than \( t/2 \) migration phases, and must remain unchanged in at least \( t/2 \) migration phases. Note that if the dynamics is \((t', 0)\)-stable for some \( t' \), it will be \((t', d)\)-stable until the next birth at \( t'' = t' + d + 1 \). Since there are at most \( 3tp/2 \) births, there must be no migration in a period of
\[ \frac{t/2}{3tp/2} = \frac{1}{3p} \]
consecutive steps.
4.2 Bound on the number of types

In this section, we investigate a behavior of the dynamics following a long period of time. Specifically, we show that after a large number of steps, the number of types can not be too high. Our goal is to prove the following theorem:

**Theorem 4.7 (Lack of explosion).** Suppose that $\alpha_{\text{max}} \leq p/512$. For every $t \geq (16/p) \log^2(1/\epsilon)$, the dynamics at step $t$ can have at most $48 \log(1/\epsilon)$ types with probability at least $1 - 3\epsilon$.

First we give the following lemma:

**Lemma 4.8.** Suppose that $\alpha_{\text{max}} \leq p/512$, and let $k$ be the number of types at step $t_0$. If $k \geq 48 \log(1/\epsilon)$, with probability at least $1 - 2\epsilon^2$, the number of types at step $t_0 + (16/p) \log(1/\epsilon)$ is at most $k/2 + 24 \log(1/\epsilon)$.

**Proof.** Without loss of generality, we may assume $t_0 = 0$. Our goal is to show that with high probability, at least half of the original types (types at step 0) will die from step 0 to step $(16/p) \log(1/\epsilon)$. From Lemma 4.1, with probability at least $1 - 2\epsilon^2$, there are at least $8 \log(1/\epsilon)$ and at most $24 \log(1/\epsilon)$ births in that period. Therefore, we may assume that the number of births in the period is between $8 \log(1/\epsilon)$ and $24 \log(1/\epsilon)$. Since the number of new types created is at most $24 \log(1/\epsilon)$, the lemma immediately follows. Assume that the number of original types dying in the period is less than $k/2$ for the sake of contradiction. It follows that the number of remaining original types at the end of the period is at least $k/2$.

Since $24 \log(1/\epsilon) \leq k/2$, the total number of types dying out in the period is at most $k/2 + 24 \log(1/\epsilon) \leq k$. Therefore, the total mass that can be added to the remaining types from the dead types is at most $k\epsilon$. It follows that on average, a remaining original type receive at most $2\epsilon$ from the dead types. Markov’s inequality says that at most half of the remaining original types can receive more than $4\epsilon$ from the dead types. Therefore, at most $k/4$ original types receive more than $4\epsilon$ from the dead types. Since the average mass of an original type is $1/k$, Markov’s inequality again guarantees that the number of original types having mass greater than $4/k$ is at most $k/4$. Combining the above two reasons, we can conclude that at least $k/2$ original types have initial mass less than $4\epsilon$ and receive less than $4\epsilon$ from the dead types in the whole period.

We will prove that those types can not remain at the end of the period. The idea is to bound the total increase in the masses of those types, and argue that after at least $8 \log(1/\epsilon)$ births, their masses will all be less than $\epsilon$.

Consider an original type $v$ such that $x_v^{(0)} \leq 4/k$ and $v$ receives less than $4\epsilon$ from the dead types from step 0 to step $(16/p) \log(1/\epsilon)$. Recall that the change in mass of $v$ at step $t$ is

$$\Delta x_v^{(t)} = \sum_{u \in N_v} \alpha_{uv} x_v^{(t)} x_u^{(t)} (x_v^{(t)} - x_u^{(t)}) \leq \sum_{u \in N_v} \alpha_{\text{max}} (x_v^{(t)})^3 \leq (k + 24 \log(1/\epsilon)) \alpha_{\text{max}} (x_v^{(t)})^3 < 2k\alpha_{\text{max}} (x_v^{(t)})^3.$$  

It follows that

$$x_v^{(t+1)} \leq x_v^{(t)} + 2k\alpha_{\text{max}} (x_v^{(t)})^3.$$
Ignoring the effect of births from step 0 to step \( t < (16/p) \log(1/\epsilon) \), we claim that
\[
x_v^{(t)} \leq \frac{8}{k} + 16\alpha_{\text{max}}kt \left( \frac{8}{k} \right)^3.
\]

We will prove the above claim by induction. We may assume that \( v \) receives all the mass from dead types at the beginning, since this assumption can only increase \( x_v^{(t)} \). We have
\[
x_v^{(0)} \leq \frac{4}{k} + 4\epsilon \leq \frac{8}{k},
\]
where the last inequality follows since \( k \leq 1/\epsilon \). Hence, the base case is satisfied. Suppose that the claim is true for \( t \), then we have
\[
x_v^{(t+1)} \leq x_v^{(t)} + 2k\alpha_{\text{max}} \left( x_v^{(t)} \right)^3 \\
\leq \frac{8}{k} + 16\alpha_{\text{max}}kt \left( \frac{8}{k} \right)^3 + 2k\alpha_{\text{max}} \left( \frac{8}{k} + 16\alpha_{\text{max}}kt \left( \frac{8}{k} \right)^3 \right)^3 \\
\leq \frac{8}{k} + 16\alpha_{\text{max}}kt \left( \frac{8}{k} \right)^3 + 2k\alpha_{\text{max}} \left( 2 \frac{8}{k} \right)^3 \\
= \frac{8}{k} + 16\alpha_{\text{max}}k(t + 1) \left( \frac{8}{k} \right)^3.
\]
The last inequality follows because
\[
k \geq 48 \log(1/\epsilon) \geq 48 \log(1/\epsilon)(512\alpha_{\text{max}}/p) > 1024\alpha_{\text{max}}t
\]
and thus,
\[
\frac{8}{k} = k \frac{8}{k^2} > 1024\alpha_{\text{max}}t \frac{8}{k^2} = 16\alpha_{\text{max}}t k \left( \frac{8}{k} \right)^3
\]

Now the mass of \( v \) decreases by at least a multiplicative factor of \( 1 - \beta_{\text{min}} \) at each birth. We may assume that the decreases on \( x_v \) happen after the increases since this assumption can only increase the bound on \( x_v \). We have
\[
x_v^{(t)} \leq \left( \frac{8}{k} + 16\alpha_{\text{max}}kt \left( \frac{8}{k} \right)^3 \right) (1 - \beta_{\text{min}})^B,
\]
where \( B \) is the number of births in the period of \( t \) steps. Setting \( t = (16/p) \log(1/\epsilon) \) and \( B = 8 \log(1/\epsilon) \) gives
\[
x_v^{(t)} \leq \left( \frac{8}{k} + 16\alpha_{\text{max}}k(16/p) \log(1/\epsilon) \left( \frac{8}{k} \right)^3 \right) (1 - \beta_{\text{min}})^{8 \log(1/\epsilon)} \\
\leq \left( \frac{8}{k} + 256 \log(1/\epsilon) \frac{1}{k^2} \right) \epsilon \\
= \frac{16\epsilon}{k}
\]
Therefore, \( x_v^{(t)} < \epsilon \) as desired. \( \square \)
Now we can prove Theorem 4.7.

**Proof of Theorem 4.7.** We only consider the last \((16/p)\log^2(1/\epsilon)\) steps and assume that \(t = (16/p)\log^2(1/\epsilon)\).

We call a period of \((16/p)\log(1/\epsilon)\) steps a *decreasing period* if it satisfies the condition in Lemma 4.8, i.e., if the number of types \(k\) at the beginning of the period is at least \(48\log(1/\epsilon)\), and the number of types at the end of the period is at most \(k/2 + 24\log(1/\epsilon)\). Construct a set \(P\) of periods of length \((16/p)\log(1/\epsilon)\) as follows:

1. Start with \(t' = 0\).
2. Repeat the following step until \(t' = t\):
   
   (a) If \(t' + (16/p)\log(1/\epsilon) \leq t\) and the number of types at \(t'\) is at least \(48\log(1/\epsilon)\), let \(i\) be the period from \(t'\) to \(t' + (16/p)\log(1/\epsilon)\), and add \(i\) to \(P\). Update \(t' \leftarrow t' + (16/p)\log(1/\epsilon)\).

   (b) Else update \(t' \leftarrow t' + 1\).

Assume that all periods in \(P\) are decreasing periods. By Lemma 4.8 the probability of such an outcome occurring is at least

\[
(1 - 2\epsilon^2)^{\log(1/\epsilon)} \geq 1 - 2\epsilon^2 \log(1/\epsilon) \geq 1 - 2\epsilon.
\]

With that assumption, if the number of types ever becomes smaller than \(48\log(1/\epsilon)\) and reaches \(48\log(1/\epsilon)\) again, it will be at least \(48\log(1/\epsilon)\) after a period of \((16/p)\log(1/\epsilon)\) steps assuming there are at least \((16/p)\log(1/\epsilon)\) subsequent steps. If there are less than \((16/p)\log(1/\epsilon)\) subsequent steps after the number of types reaches \(48\log(1/\epsilon)\), by Lemma 4.1 the probability that in those remaining steps, there are at most 24 \(\log(1/\epsilon)\) births is at least

\[
1 - e^{-16\log(1/\epsilon)/6} \geq 1 - \epsilon.
\]

By union bound, the probability of both outcomes occurring is at least

\[
1 - (2\epsilon + \epsilon) = 1 - 3\epsilon.
\]

Moreover, the number of types must become smaller than \(48\log(1/\epsilon)\) at some step of the dynamic. This follows because the number of types at the beginning is at most \(1/\epsilon\) and all periods in \(P\) are decreasing periods.

Theorems 4.6 and 4.7 are summarized in Figure 2.

5 Conclusion

In this paper, we introduce and analyze a model on opinion formation. In the first part, in which the dynamics is deterministic and we don’t have either deaths or births of types, we show that the dynamics converges point-wise to fixed points \(p\), where the set of active types in \(p\) forms an independent set of \(G\). After introducing births and deaths of types, we show that with high probability in the long run we reach a state in which there is no movement of population mass for a long period of time (aka “stable”). We also show that the number of types is logarithmic in \(1/\epsilon\), where \(\epsilon\) is the size of a type at which it dies.

A host of novel questions arise from this model and there is much scope for future work:
• **Understanding the behavior of the dynamics:** From Figure 2 we see that when \( p \) lies in the regime \((\Theta(\epsilon), \Theta(1))\), we don’t understand the behavior of the system, e.g., we don’t know if we have explosion in the number of types (i.e., having \( \Theta(1/\epsilon) \) types) in the long run. Moreover, we don’t know if the system reaches “stability” (in our notion).

• **Rate of convergence (without births and deaths):** How fast does our migration dynamics converge point-wise to fixed points \( p \)? How does the structure of \( G \) influence the time needed for convergence? Do the values of \( \alpha_{uv} \)’s affect the speed of convergence?

• **Average case analysis:** Theorem 3.7 gives qualitative information for the behavior of the dynamics assuming no births and deaths of types. However, it is not clear which independent sets are more likely to occur if we start at random in the simplex. Do the values of \( \alpha_{uv} \)’s affect the likelihood of the linearly stable fixed points?

• **Relaxing the notion of stability:** If we relax the notion of “stability” so that \( \gamma \)-fraction of the population is allowed to move (\( 1 - \gamma \) does not move for some \( 0 < \gamma < 1 \) ), can we give better guarantees in Theorem 4.6?

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\[ *this \text{ likelihood of a fixed point is called region of attraction.} \]
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A Appendix

The following theorem holds for every separable metric space, i.e., every metric space that contains a countable, dense subset. In particular, we use this theorem for $\mathbb{R}^{n-1}$ where $n$ is the number of types in the proof of Theorem 3.7.

**Theorem A.1** (Lindelöf’s lemma [14]). For every open cover there is a countable subcover.

The following lemma is used in Theorem 3.7 when we argue that the set of initial population vectors so that the dynamics converges to fixed points with an unstable direction, has measure zero. It roughly states that if a function $h$ is locally Lipschitz, then it preserves the measure zero sets (measure zero sets are mapped to measure zero sets).

**Lemma A.2** (Null-set preserving, Appendix of [19]). Let $h: S \rightarrow \mathbb{R}^m$ be a locally Lipschitz function with $S \subseteq \mathbb{R}^m$, then $h$ is null-set preserving, i.e., for $E \subset S$ if $E$ has measure zero then $h(E)$ has also measure zero.

**Proof.** Let $B_\gamma$ be an open ball such that $||h(\bar{y}) - h(\bar{x})|| \leq K_\gamma ||\bar{y} - \bar{x}||$ for all $\bar{x}, \bar{y} \in B_\gamma$. We consider the union $\bigcup_\gamma B_\gamma$ which cover $\mathbb{R}^m$ by the assumption that $h$ is locally Lipschitz. By Lindelöf’s lemma we have a countable subcover, i.e., $\bigcup_{i=1}^{\infty} B_i$. Let $E_i = E \cap B_i$. We will prove that $h(E_i)$ has measure zero. Fix an $\epsilon > 0$. Since $E_i \subset E$, we have that $E_i$ has measure zero, hence we can find a countable cover of open balls $C_1, C_2, \ldots$ for $E_i$, namely $E_i \subset \bigcup_{j=1}^{\infty} C_j$ so that $C_j \subset B_i$ for all $j$ and also $\sum_{j=1}^{\infty} \mu(C_j) < \frac{\epsilon}{K_m^i}$. Since $E_i \subset \bigcup_{j=1}^{\infty} C_j$ we get that $h(E_i) \subset \bigcup_{j=1}^{\infty} h(C_j)$, namely $h(C_1), h(C_2), \ldots$ cover $h(E_i)$ and also $h(C_j) \subset h(B_i)$ for all $j$. Assuming that ball $C_j \equiv B(\bar{x}, r)$ (center $\bar{x}$ and radius $r$) then it is clear that $h(C_j) \subset B(h(\bar{x}), K_i r)$ ($h$ maps the center $\bar{x}$ to $h(\bar{x})$ and the radius $r$ to $K_i r$ because of Lipschitz assumption). But $\mu(B(h(\bar{x}), K_i r)) = K_i^m \mu(B(\bar{x}, r)) = K_i^m \mu(C_j)$, therefore $\mu(h(C_j)) \leq K_i^m \mu(C_j)$ and so we conclude that

$$\mu(h(E_i)) \leq \sum_{j=1}^{\infty} \mu(h(C_j)) \leq K_i^m \sum_{j=1}^{\infty} \mu(C_j) < \epsilon$$

Since $\epsilon$ was arbitrary, it follows that $\mu(h(E_i)) = 0$. To finish the proof, observe that $h(E) = \bigcup_{i=1}^{\infty} h(E_i)$ therefore $\mu(h(E)) \leq \sum_{i=1}^{\infty} \mu(h(E_i)) = 0$. \qed

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B Figures

(a) The region with “C” corresponds to the initial population vectors so that the dynamics converges to the fixed point where all the population is of type C. The region “A+B” corresponds to the initial population masses so that the dynamics converges to a fixed point where part of the population is of type A and the rest of type B.

(b) Each region “A”, “B”, “C” corresponds to the initial population vectors so that the dynamics converges to all the population being of type A, B, C respectively. It is easy to see that an initial vector \((x_A, x_B, x_C)\) converges to the fixed point where all population is of type \(\arg\max_{i \in \{A,B,C\}} x_i\). In case of ties, the limit population is split equally among the tied types (symmetry).

Figure 1: Migration dynamics phase portrait for path and triangle of 3 types A, B, C respectively and for \(\alpha_{uv} = 0.5\) for all \(uv \in E(G)\). The black points and the line correspond to the fixed points. \(x_A, x_B\) correspond to the frequencies of people that are of type A, B. We omit \(x_C\) since \(x_C = 1 - x_A - x_B\).

Figure 2: In the red interval where \(p\) is \(O(\epsilon)\) we have that in the long run the system reaches a state where there is no migration of population mass for a long period of time. In the green interval where \(p\) is \(\Theta(1)\) we have that there is no explosion in the number of types, namely the number is at most \(O(\log(1/\epsilon))\) in the long term. In the blue interval, we don’t have a qualitative or quantitative characterization of the system.