Lower Bounds and Hardness Magnification for Sublinear-Time Shrinking Cellular Automata

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Abstract

The minimum circuit size problem (MCSP) is a string compression problem with a parameter \( s \) in which, given the truth table of a Boolean function over inputs of length \( n \), one must answer whether it can be computed by a Boolean circuit of size at most \( s(n) \geq n \).

Recently, McKay, Murray, and Williams (STOC, 2019) proved a hardness magnification result for MCSP involving (one-pass) streaming algorithms: For any reasonable \( s \), if there is no \( \text{poly}(s(n)) \)-space streaming algorithm with \( \text{poly}(s(n)) \) update time for \( \text{MCSP}[s] \), then \( \mathbf{P} \neq \mathbf{NP} \).

We prove an equivalent result for the (provably) strictly less capable model of shrinking cellular automata (SCAs), which are cellular automata whose cells can spontaneously delete themselves. We show every language accepted by an SCA can also be accepted by a streaming algorithm of similar complexity, and we identify two different aspects in which SCAs are more restricted than streaming algorithms. We also show there is a language which cannot be accepted by any SCA in \( o(n/\log n) \) time, even though it admits an \( O(\log n) \)-space streaming algorithm with \( O(\log n) \) update time, where \( n \) is the input length.

1 Introduction

The quest for lower bounds in computational complexity theory has been an arduous but by no means unfruitful one. The most recent developments have revealed a phenomenon dubbed hardness magnification [5, 6, 7, 17, 22, 23], giving several examples of natural problems for which establishing even slightly non-trivial lower bounds is at least as hard as proving major complexity class separations such as \( \mathbf{P} \neq \mathbf{NP} \). Among these, the preeminent example appears to be the minimum circuit size problem:

**Definition 1** (MCSP). For a Boolean function \( f : \{0, 1\}^n \to \{0, 1\} \), let \( \text{tt}(f) \) denote the truth table representation of \( f \) (as a binary string in \( \{0, 1\}^+ \) of length \( |\text{tt}(f)| = 2^n \)). For \( s : \mathbb{N}_+ \to \mathbb{N}_+ \), the minimum circuit size problem \( \text{MCSP}[s] \) is the problem where, given such a truth table \( \text{tt}(f) \), one must answer whether there is a Boolean circuit \( C \) on inputs of length \( n \) and size at most \( s(n) \) that computes \( f \), that is, \( C(x) = f(x) \) for every input \( x \in \{0, 1\}^n \).

It is a well-known fact that there is a constant \( K > 0 \) such that, for any function \( f \) on \( n \) variables as above, there is a circuit of size at most \( K \cdot 2^n/n \) that computes \( f \); hence, \( \text{MCSP}[s] \) is only non-trivial for \( s(n) < K \cdot 2^n/n \). Furthermore, \( \text{MCSP}[s] \in \mathbf{NP} \) for any constructible \( s \) and, since every circuit of size at most \( s(n) \) can be described by a binary string of \( O(s(n) \log s(n)) \) length, if \( 2^{O(s(n) \log s(n))} \subseteq \text{poly}(2^n) \) (e.g., \( s(n) \in O(n/\log n) \)), then \( \text{MCSP}[s] \in \mathbf{P} \) (by enumerating all possibilities). For large enough \( s(n) < K \cdot 2^n/n \) (e.g., \( s(n) \geq n \)), it remains unclear whether
MCSP\([s]\) is NP-complete (in the standard sense, i.e., under polynomial-time many-one reductions); see also \([13, 21]\). Still, it deserves mention that there has been some recent progress \([12]\) towards establishing NP-completeness under randomized many-one reductions for certain variants of MCSP.

Oliveira and Santhanam \([23]\) and Oliveira, Pich, and Santhanam \([22]\) have recently analyzed hardness magnification in the average-case as well as in the worst-case approximation (i.e., gap) settings of MCSP for various (uniform and non-uniform) computational models. Meanwhile, McKay, Murray, and Williams \([17]\) showed similar results hold in the standard (i.e., exact or gapless) worst-case setting and, in addition, proved the following magnification result for (single-pass) streaming algorithms, which are a very restricted uniform model; indeed, as mentioned in \([17]\), even string equality (i.e., the problem of accepting the language \(\{ww \mid w \in \{0, 1\}^+\}\)) cannot be solved by streaming algorithms (with limited space).

**Theorem 2** \([17]\). Let \(s : \mathbb{N}_+ \to \mathbb{N}_+\) be time constructible and \(s(n) \geq n\). If there is no \(\text{poly}(s(n))\)-space streaming algorithm with \(\text{poly}(s(n))\) update time for (the search version of) MCSP\([s]\), then \(P \neq \text{NP}\).

In this paper, we present an equivalent hardness magnification result (Theorem 24 and Corollary 25) for a (uniform) computational model which is provably even more restricted than streaming algorithms: shrinking cellular automata (SCAs). We show every language accepted by a sublinear-time SCA can also be accepted by a streaming algorithm with comparable complexity (Theorem 23) and, furthermore, we identify and prove two distinct limitations of SCAs compared to streaming algorithms (under sublinear-time constraints):

1. They are insensitive to the length of long unary substrings in their input (Lemma 14), which means (standard versions of) fundamental problems such as parity, modulo, majority, and threshold cannot be solved in sublinear time (Proposition 16 and Corollary 18).

2. Only a limited amount of information can be transferred between cells which are far apart (Lemma 22).

Both limitations are inherited from the underlying model of cellular automata. The first of these is circumvented by presenting the input in a special format that is efficiently verifiable by the SCA, which we motivate and adopt as part of the model (see Section 1.1.2 below); the second one is more dramatic and yields lower bounds even for languages presented in this format (Theorem 20).

It follows that any proof of \(P \neq \text{NP}\) based on a lower bound for solving MCSP\([s]\) with streaming algorithms and Theorem 2 must implicitly contain a proof of a lower bound for solving MCSP\([s]\) with SCAs. From a more “optimistic” perspective (with an eventual proof of \(P \neq \text{NP}\) in mind), although not as widely studied as streaming algorithms, SCAs are therefore at least as good as a “target” for proving lower bounds against and, in fact, should be an easier one if we are able to exploit their aforementioned limitations. (Refer to Section 6 for a further discussion on this topic, where we take into account a recently proposed barrier \([6]\) to existing techniques and which also applies to the lower bounds we prove for SCAs.)

From the perspective of cellular automata theory, our work furthers knowledge in sublinear-time cellular automata models, a topic seemingly neglected by the CA community at large (as discussed in, e.g., \([19]\)). Although this is certainly not the first result in which complexity-theoretical results for cellular automata and their variants have consequences for classical models (see, e.g., \([15, 24]\) for results in this sense), to the best of our knowledge said results address only necessary conditions for separating classical complexity classes. Hence, our result is also novel in providing an implication in the other direction, that is, a sufficient condition for said separations based on lower bounds for cellular automata models.
1.1 The Model

(One-dimensional) cellular automata (CAs) are a parallel computational model composed of identical cells arranged in an array. Each cell operates as a deterministic finite automaton (DFA) that is connected with its left and right neighbors and operates according to the same local rule. In classical CAs, the cell structure is immutable; shrinking CAs relax the model in that regard by allowing cells to spontaneously vanish (with their contents being irrecoverably lost). The array structure is conserved by reconnecting every cell with deleted neighbors to the nearest non-deleted ones in either direction.

SCAs were introduced by Rosenfeld, Wu, and Dubitzki in 1983 [25], but it was not until recent years that the model received greater attention by the CA community [16, 20]. SCAs are a natural and robust model of parallel computation which, unlike classical CAs, admit (non-trivial) sublinear-time computations.

We give a brief intuition as to how shrinking augments the classical CA model in a significant way. Intuitively speaking, any two cells in a CA can only communicate by signals, which necessarily requires time proportional to the distance between them. Assuming the entirety of the input is relevant towards acceptance, this gives us a linear time lower bound on every word accepted by the CA. In SCAs, however, said distance can be shortened as the computation evolves, thus allowing acceptance in sublinear time. As a matter of fact, the more cells are deleted, the faster distant cells can communicate and the computation can evolve, resulting in a trade-off between space (i.e., cells containing information) and time (i.e., amount of cells deleted).

1.1.1 Comparison with Related Models

Unlike other parallel models such as random access machines, SCAs are incapable of random access to their input. In a similar sense, SCAs are constrained by the distance between cells, which is an aspect usually disregarded in circuits and related models except perhaps for VLSI complexity [4, 28], for instance. In contrast to VLSI circuits, however, in SCAs distance is a fluid aspect, changing dynamically (and potentially quite dramatically) as the computation evolves. Also of note is that SCAs are a local computational model in a quite literal sense of locality that is coupled with the above concept of distance (instead of more abstract notions such as that from [30], for example).

These limitations hold not only for SCAs but also for standard CAs. Nevertheless, SCAs are more powerful than other CA models capable of sublinear-time computation such as ACAs [11, 19], which are CAs with their acceptance behavior such that the CA accepts if and only if all cells simultaneously accept. This is because SCAs can efficiently aggregate results computed in parallel (by combining them using some efficiently computable function); in ACAs any such form of aggregation is fairly limited as the underlying cell structure is static.

1.1.2 Block Words

As mentioned above, there is an input format which allows us to circumvent the first of the limitations of SCAs compared to streaming algorithms and which is essential in order to obtain a more serious computational model. In this format, the input is subdivided into blocks of the same size (for the same input length) and which are separated by delimiters and numbered in ascending order from left to right. Words with this structure are dubbed block words accordingly, and a set of such words is a block language. There is a natural presentation of any (ordinary) word as a block word (by mapping every symbol to its own block), which means there is a block language version to any (ordinary) language. (See Section 3.1.)
The concept of block words seems to arise naturally in the context of sublinear-time (both shrinking and standard) cellular automata [11, 19]. The syntax of block words is very efficiently verifiable (more precisely, in time linear in the block length) by a CA (without need of shrinking). In addition, the translation of a language to its block version (and its inverse) is a very simple map; one may frame it, for instance, as an $AC^0$ reduction. Hence, the difference between a language and its block version is solely in presentation.

Coupling block words with cellular automata yields a computational paradigm that appears to be substantially diverse from linear- and real-time cellular automata computation (see [19] for examples). Often we shall describe operations on a block (rather than on a cell) level and, by making use of block numbering, two blocks with distinct numbers may operate differently even though their contents are the same; this would be impossible at a cell level due to the locality of CA rules. In combination with shrinking, certain block languages admit merging groups of blocks in parallel; this gives rise to a form of reduction we call blockwise reductions and which we employ in a manner akin to downward self-reducibility as in [1].

An additional technicality which arises (and may potentially be seen as an additional limitation) is that the number of cells in a block is fixed at the start of the computation; this means a block cannot “allocate extra space” (beyond a constant multiple of the block length). This is the same limitation as that of linear bounded automata (LBAs) compared to Turing machines with unbounded space, for example. We circumvent this by increasing the block length in the problem instances as needed, that is, by padding each block so that enough space is available from the outset. This is still in line with the considerations above; for instance, the resulting language is still $AC^0$ reducible to the original one (and vice-versa).

1.2 Organization

The rest of the paper is organized as follows: Section 2 presents the basic definitions. In Section 3 we introduce block words and related concepts and discuss the aforementioned limitations of sublinear-time SCAs. Following that, in Section 4 we show how to obtain streaming algorithms from sublinear-time SCAs with comparable complexities. Finally, in Section 5 we prove the hardness magnification result for SCAs, and Section 6 concludes.

2 Preliminaries

$\mathbb{Z}$ denotes the set of integers, $\mathbb{N}_+$ that of positive integers, and $\mathbb{N}_0 = \mathbb{N}_+ \cup \{0\}$. For $a, b \in \mathbb{N}_0$, $[a, b] = \{x \in \mathbb{N}_0 \mid a \leq x \leq b\}$ and $[a, b) = \{x \in \mathbb{N}_0 \mid a \leq x < b\}$. For sets $A$ and $B$, $B^A$ is the set of functions $A \to B$.

We assume the reader is familiar with cellular automata as well as with basic notions in computational complexity theory (see, e.g., standard reference works such as [2, 8, 10]). Words are indexed starting with index zero. For a finite, non-empty set $\Sigma$ and a word $w \in \Sigma^*$, we write $w(i)$ for the $i$-th symbol of $w$ (and, in general, $w_i$ refers to another word altogether, not the $i$-th symbol of $w$). For $a, b \in \mathbb{N}_0$, $w[a, b]$ denotes the subword $w(a)w(a + 1)\cdots w(b - 1)w(b)$ of $w$ (where $w[a, b] = \varepsilon$ for $a > b$). $|w|_x$ is the number of occurrences of $x \in \Sigma$ in $w$. $\text{bin}_n(x)$ denotes the binary representation of $x \in \mathbb{N}_0$, $x < 2^n$, of length $n \in \mathbb{N}_+$ (padded with leading zeros). $\text{poly}(n)$ is the class of functions polynomial in $n \in \mathbb{N}_0$. $\text{REG}$ denotes the class of regular languages and $\text{TISP}[t, s]$ (resp., $\text{TIME}[t]$) the class of problems decidable by a Turing machine (with one

1 An alternative way of coping with this limitation is allowing the CA to “expand” by dynamically creating new cells between existing ones; however, this may result in a computational model which (under standard complexity assumptions) is dramatically more powerful than standard CAs [18, 20].
tape and one read-write head) in $O(t)$ time and $O(s)$ space (resp., unbounded space). Without restriction, we assume the empty word is not a member of any of the languages considered.

An $\omega$-word is a function $\mathbb{N}_0 \to \Sigma$, and a $\omega\omega$-word is a function $\mathbb{Z} \to \Sigma$. We write $\Sigma^\omega = \Sigma^{\mathbb{N}_0}$ for the set of $\omega$-words over $\Sigma$. To each $\omega\omega$-word corresponds a unique pair $(w_-, w_+)$ of $\omega$-words $w_-, w_+ \in \Sigma^\omega$ with $w_+(i) = w(i)$ for $i \geq 0$ and $w_-(i) = w(-i - 1)$ for $i < 0$. (Partial) $\omega$-word homomorphisms are extendable to (partial) $\omega\omega$-word homomorphisms as follows: Let $f: \Sigma^\omega \to \Sigma^\omega$ be an $\omega$-word homomorphism; then there is a unique map $f_{\omega\omega}: \Sigma^\omega \to \Sigma^\omega$ such that, for every $w \in \Sigma^\omega$, $w' = f_{\omega\omega}(w)$ is the (unique) $\omega\omega$-word with $w'_+ = f(w_+)$ and $w'_- = f(w_-)$.

For a circuit $C$, $|C|$ denotes the size of $C$, that is, the total number of gates in $C$. Any Boolean circuit $C$ can be described by a binary string of $O(|C| \log |C|)$ length.

**Definition 3** (Streaming algorithm). Let $s, u, r: \mathbb{N}_+ \to \mathbb{N}_+$ be functions. An $s$-space streaming algorithm $A$ is a random access machine which, on input $w$, works in $O(s(|w|))$ space and, on every step, can either perform an operation on a constant number of bits in memory or read the next symbol of $w$. $A$ has a update time if, for every $w$, the number of operations it performs between reading $w(i)$ and $w(i + 1)$ is at most $u(|w|)$. $A$ has $r$ reporting time if it performs at most $r(|w|)$ operations after having read $w(|w| - 1)$ (until it terminates).

In our discussions, our interest will be in $s$-space streaming algorithms with $\text{poly}(s(|w|))$ update and reporting time for sublinear $s$ (i.e., $s(|w|) \in o(|w|)$).

### 2.1 Cellular automata

We consider only CAs with the standard neighborhood. The symbols of an input word $w$ are provided from left to right in the cells 0 to $|w| - 1$ and are surrounded by inactive cells, which conserve their state during the entire computation (i.e., the CA is bounded). Acceptance of the input is signaled by cell zero (i.e., the leftmost input cell).

**Definition 4** (Cellular automaton). A cellular automaton (CA) $C$ is a tuple $(Q, \delta, \Sigma, q, A)$ where:

- $Q$ is a non-empty and finite set of states;
- $\delta: Q^3 \to Q$ is the local transition function;
- $\Sigma \subseteq Q$ is the input alphabet of $C$;
- $q \in Q \setminus \Sigma$ is the inactive state, that is, $\delta(q_1, q, q_2) = q$ for every $q_1, q_2 \in Q$; and
- $A \subseteq Q \setminus \{q\}$ is the set of accepting states of $C$.

A cell which is not in the inactive state is said to be active. The elements of $Q^Z$ are the (global) configurations of $C$. $\delta$ induces the global transition function $\Delta: Q^Z \to Q^Z$ of $C$ by $\Delta(c)(i) = \delta(c(i - 1), c(i), c(i + 1))$ for every cell $i \in \mathbb{Z}$ and configuration $c \in Q^Z$.

$C$ is said to accept an input $w \in \Sigma^+$ if cell zero is eventually in an accepting state, that is, there is $t \in \mathbb{N}_0$ such that $(\Delta^t(c_0))(0) \in A$, where $c_0 = c_0(w)$ is the initial configuration (for $w$): $c_0(i) = w(i)$ for $i \in [0, |w| - 1]$, and $c_0(i) = q$ otherwise. For a minimal such $t$, we say $C$ accepts $w$ with time complexity $t$. $L(A) \subseteq \Sigma^+$ denotes the set of words accepted by $C$. For a function $t: \mathbb{N}_+ \to \mathbb{N}_0$, $CA[t]$ is the class of languages accepted by CAs with time complexity $O(t(n))$, where $n$ is the input length.

Some remarks concerning the classes $CA[t]$: $CA[\text{poly}] = \text{TISP}[\text{poly}, n]$ (i.e., the class of polynomial-time LBAs), and $CA[t] = CA[1] \subseteq \text{REG}$ for every sublinear $t$. Furthermore, $CA[t] \subseteq \text{TISP}[t^2, n]$ (where $t^2(n) = (t(n))^2$) and $\text{TISP}[t, n] \subseteq CA[t]$.
**Definition 5** (Shrinking CA). A shrinking CA (SCA) $S$ is a cellular automaton which has a special delete state $\otimes \in Q \setminus (\Sigma \cup \{q\})$. In addition, the global transition function $\Delta_S$ of $S$ is given by applying the standard CA global transition function $\Delta$ (as in Definition 3) followed by removing all cells in the state $\otimes$, that is, $\Delta_S = \Phi \circ \Delta$, where $\Phi : Q^2 \to Q^2$ is the (partial) $\omega\omega$-word homomorphism for which $\Phi(\otimes) = \varepsilon$ and $\Phi(x) = x$ for every $x \in Q \setminus \{\otimes\}$. For a function $t : \mathbb{N}_+ \to \mathbb{N}_0$, SCA[$t$] denotes the class of languages accepted by SCAs with time complexity $O(t(n))$, where $n$ is the input length.

Note that $\Phi$ is only partial since, for instance, any $\omega\omega$-word in $\otimes^* \cdot \Sigma^* \cdot \otimes^*$ has no proper image (as it is not mapped to a $\omega\omega$-word). Hence, $\Delta_S$ is also only a partial function (on $Q^2$); nevertheless, $\Phi$ is total on the set of $\omega\omega$-words in which $\otimes$ occurs only finitely often and, in particular, $\Delta_S$ is total on the set of configurations arising from initial configurations for finite input words (which is the setting we are interested in). For convenience, we extend $\Delta$ in the obvious manner (i.e., as a map induced by $\delta$) so it is also defined for every (finite) word $w \in Q^*$. For $|w| \leq 2$, we set $\Delta(w) = \varepsilon$; for longer words, $|\Delta(w)| = |w| - 2$ holds.

The acceptance condition of SCAs is the same as in Definition 3 (i.e., acceptance is dictated by cell zero). Note that, unlike standard CAs, the index of one same cell can differ from one configuration to the next, that is, a cell index does not uniquely determine a cell on its own (rather, only when coupled with a time step). This is a consequence of applying $\Phi$, which (as a $\omega\omega$-word homomorphism) contracts the global configuration towards cell zero. More precisely, for a configuration $c \in Q^2$, the cell with index $i \geq 0$ in $\Delta(c)$ is the same as that with index $i + d_i$ in $c$, where $d_i$ is the number of cells with index $\leq i$ in $c$ that were deleted in the transition to $\Delta(c)$ (and an analogous observation holds for $i < 0$). This also implies the cell with index zero in $\Delta(c)$ is the same as that in $c$ with minimal positive index that was not deleted in the transition to $\Delta(c)$; thus, in any time step, cell zero is the leftmost active cell (unless all cells are inactive; in fact, cell zero is inactive if and only if all other cells are inactive). Granted, what indices a cell has is of little importance when one is interested only in the SCA’s configurations and their evolution; nevertheless, they are relevant when simulating an SCA with another machine model (as we do in Sections 3.3 and 4).

Naturally, $CA[t] \subseteq SCA[t]$ for every function $t$, and $SCA[poly] = CA[poly]$. For sublinear $t$, SCA[$t$] contains non-regular languages if, for instance, $t \in O(\log n)$ (see below); hence, the inclusion of $CA[t]$ in SCA[$t$] in strict. In fact, this is the case even if we consider only regular languages. One simple example is $L = \{w \in \{0, 1\}^+ \mid w(0) = w(|w| - 1)\}$, which is in SCA[$1$] and regular but not in CA[$o(n)$]. One obtains an SCA for $L$ by having all cells whose both neighbors are active cells delete themselves in the first step, followed by the two remaining cells comparing their states, with cell zero accepting if and only if this comparison succeeds or if the input has length 1 (which it can notice immediately since it is only for such words that it has two inactive neighbors). More precisely, the local transition function $\delta$ is such that, for $z_1, z_3 \in \{0, 1, q\}$ and $z_2 \in \{0, 1\}$,

$$\delta(z_1, z_2, z_3) = \begin{cases} \otimes, & z_1, z_3 \in \{0, 1\} \\ z_2, & \text{otherwise} \end{cases}$$

and $\delta(q, z'_2, z'_3) = \delta(q, z'_2, q) = a$; otherwise, $\delta$ simply conserves the cell’s state. See Figure 1 for an example.

Using a textbook technique to simulate a (bounded) CA with an LBA (and simply ignoring deleted cells), we have:

**Proposition 6.** For every function $t : \mathbb{N}_+ \to \mathbb{N}_+$ computable by an LBA in $O(n \cdot t(n))$ time, SCA[$t$] $\subseteq$ TISP[$n \cdot t(n), n$].
We prove this is a proper inclusion in Section 3.2 (Corollary 17). Using the well-known result $\text{TIME}[o(n \log n)] = \text{REG}$ [14], it follows that at least a logarithmic time bound is needed for SCAs to recognize languages which are not regular:

**Corollary 7.** SCA[$o(\log n)$] $\subsetneq$ REG.

This bound is tight: It is relatively easy to show that any language accepted by ACAs (see Section 1.1.1) in at most $t(n)$ time can also be accepted by an SCA in at most $t(n) + O(1)$ steps. Since there is a non-regular language recognizable by ACAs [11] in $O(\log n)$ time, the same language is recognizable by an SCA in $O(\log n)$ time.

For any finite, non-empty set $\Sigma$, we say a function $f: \Sigma^+ \to \Sigma^+$ is **computable in place** by an (S)CA if there is an (S)CA $S$ which, given $x \in \Sigma^+$ as input (surrounded by inactive cells), produces $f(x)$. We say a function $f: \mathbb{N}_+ \to \mathbb{N}_+$ is **constructible in place** by an (S)CA if $f(n) \leq 2^n$ and there is an (S)CA $S$ which, given $n \in \mathbb{N}_0$ in unary, produces $\text{bin}_n(f(n) - 1)$ (i.e., $f(n) - 1$ in binary). Note the set of functions computable or constructible in place by an (S)CA in at most $t(n)$ time, where $n$ is the input length and $t: \mathbb{N}_+ \to \mathbb{N}_+$ is some function, includes (but is not limited to) all functions computable by an LBA in at most $t(n)$ time.

## 3 Capabilities and Limitations of Sublinear-Time SCAs

### 3.1 Block Languages

Let $\Sigma$ be a finite, non-empty set. For $\Sigma_\epsilon = \Sigma \cup \{\epsilon\}$ and $x, y \in \Sigma^+$, $(i)_\epsilon \in (\Sigma_\epsilon \times \Sigma_\epsilon)^+$ denotes the (unique) word of length $\max\{|x|, |y|\}$ for which $(i)_\epsilon(i) = (x(i), y(i))$, where $x(i) = y(j) = \epsilon$ for $i \geq |x|$ and $j \geq |y|$.

**Definition 8** (Block word). Let $n, m, b \in \mathbb{N}_+$ be such that $b \geq n$ and $m \leq 2^n$. A word $w$ is said to be an $(n, m, b)$-**block word** (or simply **block word**) (over $\Sigma$) if it is of the form $w = w_0 \# w_1 \# \cdots \# w_{m-1}$ and $w_i = (\text{bin}_y(x)i)$, where $x_0 \geq 0$, $x_{i+1} = x_i + 1$ for every $i$, $x_{m-1} < 2^n$, and $y_i \in \Sigma^b$. In this context, $w_i$ is the $i$-th block of $w$.

Hence, every $(n, m, b)$-block word $w$ has $m$ many blocks of length $b$, and its total length is $|w| = (b + 1) \cdot m - 1 \in \Theta(bm)$. $n$ is implicitly encoded by the entries in the upper track (i.e., the $x_i$) and we shall see $m$ and $b$ as parameters thereof (see Definition 9 below), so the structure and lengths of each block can be verified locally (i.e., by only inspecting a block and its immediate neighbors). Note the block numbering starts with an arbitrary $x_0$ (instead of simply zero); this is intended so that, for $m' < m$, an $(n, m, b)$-block word admits $(n, m', b)$-block words as infixes.
(which would not be the case if we required \( x_0 = 0 \)). For example,

\[
  w = \left( \begin{array}{c} 01 \\ 0100 \end{array} \right) \# \left( \begin{array}{c} 10 \\ 1100 \end{array} \right) \# \left( \begin{array}{c} 11 \\ 1000 \end{array} \right)
\]

is a \((2, 3, 4)\)-block word with \( x_0 = 1, y_0 = 0100, y_1 = 1100, \) and \( y_2 = 1000 \).

When referring to block words, we use \( N \) for the block word length \( |w| \) and reserve \( n \) for indexing block words of different block length, overall length, or total number of blocks (or any combinations thereof). With \( m \) and \( b \) as parameters of \( n \), we obtain sets of block words:

**Definition 9** (Block language). Let \( m, b : \mathbb{N}_+ \to \mathbb{N}_+ \) be non-decreasing and constructible in place by a CA in \( O(m(n) + b(n)) \) time. Furthermore, let \( b(n) \geq n \) and \( m(n) \leq 2^n \). Then, \( \mathcal{B}^m_b \) denotes the set of all \((n, m(n), b(n))\)-block words for \( n \in \mathbb{N}_+ \), and every subset \( L \subseteq \mathcal{B}^m_b \) is an \(((n, m, b)\)-block language (over \( \Sigma \)).

An SCA can verify its input is a valid block word in \( O(b(n)) \) time, where with “verify” we mean every block checks locally that its structure and contents are consistent (with Definition 8). This can be realized using standard CA techniques and, in particular, does not require shrinking; see, for instance, [11, 19] for detailed constructions. Recall that in Definition 9 we did not require an SCA \( S \) to explicitly reject inputs not in \( L(S) \); hence, the time complexity of \( S \) on an input \( w \) counts only towards the complexity of \( S \) if \( w \in L(S) \), that is, the time complexity of \( S \) on any \( w \not\in L(S) \) is ignored. As a result, when constructing an SCA \( S \) for a block language \( L \), the complexity of verifying an input \( w \) is a block word counts towards the time complexity of \( S \) only if \( w \) is a yes-instance of \( L \) (i.e., \( w \in L \)) and, in particular, if \( w \) is a (valid) block word. Provided the state of every cell in \( S \) is relevant towards it accepting (which is the case for all constructions we describe), it suffices to have cells that detect a violation to \( w \) being a block word mark themselves with a special error flag (even if other cells continue their operation as normal, that is, as if \( w \) was a valid block word); since every cell is relevant towards acceptance, this is guaranteed to eventually prevent \( S \) from accepting (and, since there is no need to explicitly reject \( w \), it is irrelevant how long it takes for this to occur). Thus, without restriction, we assume for the rest of this paper that, in all SCA constructions we describe where the target language is a block language, the SCA always checks its input word is a valid block word (despite not explicitly mentioning it).

As stated in the introduction, our interest in block words is as a special format to provide inputs. There is a natural bijection between any language and a block version of it, namely by mapping each word \( z \) to a block word \( w \) in which each block \( w_i \) contains a symbol \( z(i) \) of \( z \) (padded up to the block length \( b \)) and the blocks are numbered from 0 to \(|z| - 1\):

**Definition 10** (Block version of a language). Let \( L \subseteq \Sigma^+ \) be a language, and let \( b \) be as in Definition 9. The **block version** \( \text{Block}_b(L) \) of \( L \) (with blocks of length \( b \)) is the block language for which, for every \( z \in \Sigma^+ \), \( z \in L \) if and only if we have \( w \in \text{Block}_b(L) \) for the \((n, m, b(n))\)-block word \( w = w_z \) (as in Definition 8) for which \( m = |z|, n = \lfloor \log m \rfloor, x_0 = 0, \) and \( y_i = z(i)0^{b(n)-1} \) for every \( i \in [0, m - 1] \).

Note \( \text{Block}_b(L) \not\in \text{REG} \) for any \( b \); hence, \( \text{Block}_b(L) \in \text{SCA}[t] \) only for \( t \in \Omega(\log n) \) (and constructible). For \( b(n) = n \), \( \text{Block}_n(L) \) is the block version with minimal padding.

For any two finite, non-empty sets \( \Sigma_1 \) and \( \Sigma_2 \), say a function \( f : \Sigma_1^+ \to \Sigma_2^+ \) is **non-stretching** if \( |f(x)| \leq |x| \) for every \( x \in \Sigma_1^+ \). We now define \( k\)-blockwise maps, which are maps that operate on block words by grouping \( k(n) \) many blocks together and mapping each such group (in a non-stretching manner) to a single block of length at most \((b(n) + 1) \cdot k(n) - 1\).
Let \( k : \mathbb{N}_+ \to \mathbb{N}_0, k(n) \geq 2 \), be non-decreasing and constructible in place by a CA in \( O(k(n)) \) time. A \( k \)-blockwise map is a map \( g : \mathbb{B}_b^{km} \to \mathbb{B}_b^{m} \) for which there is a non-stretching \( g' : \mathbb{B}_b^k \to \Sigma^+ \) such that, for every \( w \in \mathbb{B}_b^{km} \) (as in Definition 8) and \( w' = w_{ik} \# \cdots \# w_{(i+1)k-1} \), we have:

\[
g(w) = \left( \bin_m(x_0), \ldots, \bin_m(x_{m-1}) \right).
\]

Using blockwise maps, we obtain a very natural form of reduction operating on block words and which is highly compatible with SCAs as a computational model. The reduction divides an \((n, km, b)\)-block word in \( m \) many groups of \( k \) many contiguous blocks and, as a \( k \)-blockwise map, maps each such group to a single block (of length \( b \)):

**Definition 12** (Blockwise reducible). For block languages \( L \) and \( L' \), \( L \) is said to be \((k-)\)blockwise reducible to \( L' \) if there is a computable \( k\)-blockwise map \( g : \mathbb{B}_b^{km} \to \mathbb{B}_b^{m} \) such that, for every \( w \in \mathbb{B}_b^{km} \), we have \( w \in L \) if and only if \( g(w) \in L' \).

Since every application of the reduction shortens an instance to another whose length is approximately a \( 1/k \) fraction of the original one, logarithmically many applications suffice to produce a trivial instance (i.e., an instance consisting of a single block). This gives us the following computational paradigm of chaining blockwise reductions together:

**Lemma 13.** Let \( k, r : \mathbb{N}_+ \to \mathbb{N}_0 \) be functions, and let \( L \subseteq \mathbb{B}_b^{kr} \) be such that there is a series \( L = L_0, L_1, \ldots, L_{r(n)} \) of languages with \( L_i \subseteq \mathbb{B}_b^{kr-i} \) and such that \( L_i \) is \( k(n) \)-blockwise reducible to \( L_{i+1} \) via the (same) blockwise reduction \( g_i \). Furthermore, let \( g' \) be as in Definition 11 and let \( t_{g'} : \mathbb{N}_+ \to \mathbb{N}_+ \) be non-decreasing and such that, for every \( w' \in \mathbb{B}_b^r \), \( g'(w') \) is computable in place by an SCA in \( O(t_{g'}(|w'|)) \) time. Finally, let \( L_{r(n)} \in \text{SCA}[t] \) for some function \( t : \mathbb{N}_+ \to \mathbb{N}_+ \). Then, \( L \in \text{SCA}[r(n) \cdot t_{g'}(O(k(n) \cdot b(n))) + O(b(n)) + t(b(n))] \).

**Proof.** We construct an SCA \( S \) for \( L \) with the purported time complexity. Given \( w \in \mathbb{B}_b^{kr} \), \( S \) computes one blockwise reduction after the next. Each application of \( g \) is computed by applying \( g' \) on each group of relevant blocks (i.e., the \( w_i' \) from Definition 11) in parallel.

One detail to note is that this results in the same procedure \( P \) being applied to different groups of blocks in parallel, but it may be so that \( P \) requires more time for one group of blocks than for the other. Thus, we allow the entire process to be carried out asynchronously while requiring that, for each group of blocks, the respective results be present before each execution of \( P \) is started. (One way of realizing this, for instance, is having the first block in the group send a signal across the whole group to ensure all inputs are available and, when it arrives at the last block in the group, another signal is sent to trigger the start of \( P \).)

Using that \( t_{g'} \) is non-decreasing and that \( g' \) is non-stretching, the time needed for each execution of \( P \) is \( t_{g'}(|w_i'|) \in t_{g'}(O(k(n) \cdot b(n))) \) (which is not impacted by the considerations above) and, since there are \( r(n) \) reductions in total, we have \( r(n) \cdot t_{g'}(O(k(n) \cdot b(n))) \) time in total. Once a single block is left, the cells of \( S \) are synchronized (in \( O(b(n)) \) time) and then behave as in the SCA for \( L_{r(n)} \) guaranteed by the assumption \((t(b(n)))\).

### 3.2 Block Languages and Parallel Computation

In this section, we prove the first limitation of SCAs discussed in the introduction (Lemma 14) and which makes it impossible to accept the languages PARITY, MOD, MAJ, and THR (defined next) in sublinear time. Nevertheless, as is shown in Proposition 19, the block versions of these languages can be accepted quite efficiently. This motivates the block word presentation for inputs,
thus effectively eliminating a limitation which concerns (in a sense) only the presentation of instances (and, hence, is not a computational limitation of SCAs).

Let \( q > 2 \) and let \( k : \mathbb{N}_+ \to \mathbb{N}_+ \) be constructible in place by a CA in at most \( t_k(n) \) time for some function \( t_k : \mathbb{N}_+ \to \mathbb{N}_+ \). Additionally, let PARITY (resp., MOD\(_q\); resp., MAJ; resp., THR\(_k\)) be the language over \( \{0, 1\}^+ \) that contains exactly every word \( w \) for which \( |w|_1 \) is even (resp., \( |w|_1 = 0 \) (mod \( q \)); resp., \( |w|_1 \geq |w|_0 \); resp., \( |w|_1 \geq k(|w|) \)).

The following is a simple limitation of sublinear-time CA models such as ACAs (see Section 1.1.1 and also [26]), which we show also to hold for shrinking CAs.

**Lemma 14.** Let \( S \) be an SCA with input alphabet \( \Sigma \), and let \( x \in \Sigma \) be such that there is a (minimal) \( t \in \mathbb{N}_+ \) for which \( \Delta^t_S(y) = \varepsilon \) holds, where \( y = x^{2t+1} \) (i.e., the unique word of length \( 2t+1 \) over the unary alphabet \( \{x\} \)). Then, for every \( z_1, z_2 \in \Sigma^+ \), \( w = z_1yz_2 \in L(S) \) holds if and only if for every \( i \in \mathbb{N}_0 \) we have \( w_i = z_1yxz_2 \in L(S) \).

**Proof.** Letting \( w \in L(S) \) and \( i \in \mathbb{N}_0 \), we show \( w_i \in L(S) \); the converse is trivial. Since \( w \) and \( w_i \) both have \( z_1y \) as prefix and \( \Delta^t_S(y) \neq \varepsilon \) for \( t' < t \), if \( S \) accepts \( w \) in \( t' \) steps, then it also accepts \( w_i \) (in \( t' \) steps). Thus, assume \( S \) accepts \( w \) in \( t' \geq t \) steps. In this case, it suffices to show \( \Delta^t_S(w) = \Delta^t_S(w_i) \). To this end, let \( \alpha_j \) for \( j \in [0, t] \) be such that \( \alpha_0 = x \) and \( \alpha_{j+1} = \delta(\alpha_j, \alpha_j, \alpha_j) \). Thus, \( \Delta^t_S(a_j^{k+2}) = \alpha_j^{k+1} \) for every \( k \in \mathbb{N}_+ \) (and \( j < t \)) and, by an inductive argument as well as by the assumption on \( y \) (i.e., \( \alpha_t = \varepsilon \)), \( \Delta^t_S(x^{2t+1}) = \Delta^t_S(a_0^{2t+1}) = \varepsilon \). Since \( |y| \geq t \) and \( y \in \{x\}^+ \), we have \( \Delta^t_S(q_1z_1yx) = \Delta^t_S(q_1z_1y) \) and \( \Delta^t_S(yxz_2q) = \Delta^t_S(xyz_2q) = \Delta^t_S(yzzq) \); hence, \( \Delta^t_S(w) = \Delta^t_S(w_i) \) follows.

A direct implication of Lemma 14 is the following:

**Proposition 15.** Every unary language \( U \in \text{SCA}[o(n)] \) is either finite or cofinite.

Since PARITY has a unary subset which is neither finite nor cofinite, we can prove:

**Proposition 16.** \( \text{PARITY} \notin \text{SCA}[o(n)] \) (where \( n \) is the input length).

**Proof.** Let \( S \) be an SCA with \( L(S) = \text{PARITY} \). We show the time complexity of \( S \) cannot be \( o(n) \) in the infinite set \( U = \{1^{2m} \mid m \in \mathbb{N}_+ \} \subseteq \text{PARITY} \). If \( \Delta^t(1^{2t+1}) = \varepsilon \) for some \( t \in \mathbb{N}_0 \), then, by Lemma 14, \( L(S) \cap U \) is either finite or cofinite, thus contradicting \( L(S) = \text{PARITY} \). Hence, \( \Delta^t(1^{2t+1}) \neq \varepsilon \) for every \( t \in \mathbb{N}_0 \). In this case, if \( S \) has time complexity \( o(n) \) for the words in \( U \), then \( S \) must have time complexity \( O(1) \) in \( U \) (by an argument analogous to showing every language decidable by a TM in \( o(n) \) time is in \( \text{TIME}[1] \)), which also contradicts \( L(S) = \text{PARITY} \). It follows that \( S \) does not have time complexity \( o(n) \) in \( U \), as desired.

**Corollary 17.** \( \text{REG} \subsetneq \text{SCA}[o(n)] \).

The argument in the proof above generalizes to MOD\(_q\), MAJ, and THR\(_k\) with \( k \in \omega(1) \). For MOD\(_q\), consider the set \( U = \{1^{2m} \mid m \in \mathbb{N}_+ \} \). For MAJ and THR\(_k\), set \( U = \{0^m1^m \mid m \in \mathbb{N}_+ \} \) and \( U = \{0^{n-k(n)}1^{k(n)} \mid n \in \mathbb{N}_+ \} \), respectively; in this case, \( U \) is not unary, but the argument easily extends to the unary suffixes of the words in \( U \). Hence:

**Corollary 18.** \( \text{MOD}_q, \text{MAJ} \notin \text{SCA}[o(n)] \). Also, \( \text{THR}_k \in \text{SCA}[o(n)] \) if and only if \( k \in O(1) \).

The block versions of these languages, however, are not subject to the limitation above:

**Proposition 19.** For \( L \in \{\text{PARITY}, \text{MOD}_q, \text{MAJ} \} \), \( \text{Block}_n(L) \in \text{SCA}[\log N]^2 \) (where \( N = N(n) \) is the instance length). Also, \( \text{Block}_n(\text{THR}_k) \in \text{SCA}[\log N]^2 + t_k(n) \).
Proof. Given $L \in \{\text{PARITY}, \text{MOD}_q, \text{MAJ}, \text{THR}_k\}$, we construct an SCA $S$ for $L' = \text{Block}_n(L)$ with the purported time complexity. Let $w \in \mathcal{B}^m_2$ be an input of $S$. For simplicity, we assume that, for every such $w$, $m = m(n) = 2^n$ is a power of two; the argument extends to the general case in a simple manner. Hence, we have $N = |w| = n \cdot m$ and $n = \log m \in \Theta(\log N)$.

Let $L_0 \subset \mathcal{B}^m_2$ be the language containing every such block word $w \in \mathcal{B}^m_2$ for which, for $y_i$ as in Definition 8 and $y = \sum_{i=0}^{m-1} y_i$, we have $f_L(y) = f_{L_0}(y) = 0$, where $f_{\text{PARITY}}(y) = y \mod 2$, $f_{\text{MOD}}(y) = y \mod q$, $f_{\text{MAJ}}(y) = 0$ if and only if $y \geq 2^{n-1}$, and $f_{\text{THR}}(y) = 0$ if and only if $y \geq k(n)$. Thus, (under the previous assumption) we have $L_0 = L'$ (and, in the general case, $L_0 = L' \cap \mathcal{B}^m_2$).

Then, $L_0$ is $2$-blockwise reducible to a language $L_1 \subseteq \mathcal{B}^{m/2}_2$ by mapping every $(n, 2, n)$-block word of the form $(\text{bin}(n, 2x))^\#(\text{bin}(n, 2x+1))$ with $x \in [0, 2^{n-1} - 1]$ to $(\text{bin}(n, x))$. Using basic CA arithmetic and cell communication techniques, this requires $O(n)$ time. Repeating this procedure, we obtain a chain of languages $L_0, \ldots, L_n$ such that $L_i$ is $2$-blockwise reducible to $L_{i+1}$ in $O(n)$ time. By Lemma 15, $L' \in \text{SCA}[n^2 + t(n)]$ follows, where $t: \mathbb{N} \rightarrow \mathbb{N}$ is such that $L_n \in \text{SCA}[t]$. For $L \in \{\text{PARITY}, \text{MOD}_q, \text{MAJ}\}$, checking the above condition on $f_L(y)$ can be done in $t(n) \in O(n)$ time; as for $L = \text{THR}_k$, we must also compute $k$, so we have $t(n) \in O(n + k(n))$.

The general case follows from adapting the above reductions so that words with an odd number of blocks are also accounted for (e.g., by ignoring the last block of $w$ and applying the reduction on the first $m - 1$ blocks).

\[ \square \]

3.3 An Optimal SCA Lower Bound for a Block Language

Corollary 17 immediately implies SCAs are strictly more limited than streaming algorithms. However, the argument bases exclusively on SCAs being unable to cope with long unary subwords in the input (i.e., Lemma 14) and, hence, does not apply to block languages. In this section, we investigate an additional limitation of SCAs compared to streaming algorithms even considering only block languages. In particular, we show there is a language $L$ that is efficiently recognizable by streaming algorithms but such that $\text{Block}_n(L) \notin \text{SCA}[t]$ for nearly every sublinear $t$ (Theorem 20).

Let $L_1$ be the language of words $w \in \{0, 1\}^+$ such that $|w| = 2^n$ is a power of two and, for $i = w(0)w(1) \cdots w(n - 1)$ (seen as an $n$-bit binary integer), $w(i) = 1$. Consider the block version $\text{Block}_n(L_1)$ of $L_1$. It is not hard to show $\text{Block}_n(L_1)$ can be accepted by an $O(\log m)$-space streaming algorithm for $\text{Block}_n(L_1)$ with $O(\log m)$ update time. Nevertheless:

**Theorem 20.** $\text{Block}_n(L_1) \notin \text{SCA}[o(\log N)]$.

This lower bound is optimal since there is an SCA for $\text{Block}_n(L_1)$ with time complexity $O(N/\log N)$: Shrink every block to its respective bit (i.e., the $y_i$ from Definition 8), reducing the input to a word $w'$ of $O(N/\log N)$ length; while doing so, mark the bit corresponding to the $n$-th block. Then shift the contents of the first $n$ bits as a counter that decrements itself every new cell it visits and, when it reaches zero, signals acceptance if the cell it is currently at contains a 1. Using counter techniques as in [27, 29], this process takes $O(|w'|)$ time.

The proof employs some ideas from communication complexity. The basic setting is a game with two players $A$ and $B$ (both with unlimited computational resources) which receive inputs $w_A$ and $w_B$, respectively, and must produce an answer to the problem at hand while exchanging a limited amount of bits. We are interested in the case where the concatenation $w = w_Aw_B$ of the inputs of $A$ and $B$ is an input to an SCA and $A$ must output whether the SCA accepts $w$. More importantly, we analyze the case where only $B$ is allowed to send messages, that is, the case of one-way communication\(^2\).

\(^2\)One-way communication complexity can also be defined as the maximum over both communication directions
Definition 21 (One-way communication complexity). Let \( m, f : \mathbb{N}^+ \to \mathbb{N}^+ \) be functions with \( 0 < m(N) \leq N \), and let \( L \subseteq \Sigma^+ \) be a language. \( L \) is said to have \((m, f)\)-one-way communication complexity \( f \) if there are two families of algorithms (with unlimited computational resources) \( (A_N)_{N \in \mathbb{N}^+} \) and \( (B_N)_{N \in \mathbb{N}^+} \) such that the following holds for every \( w \in \Sigma^* \) of length \( |w| = N \), where \( w_A = w[0, m(N) - 1] \) and \( w_B = w[m(N), N - 1] \):

1. \( |B_N(w_B)| \leq f(N) \); and
2. \( A_N(w_A, B(w_B)) = 1 \) (i.e., accept) if and only if \( w \in L \).

\( C_{\text{ow}}^m(L) \) indicates the (pointwise) minimum over all such functions \( f \).

Note that \( A_N \) and \( B_N \) are nonuniform, so the length \( N \) of the (complete) input \( w \) is known implicitly by both algorithms.

Lemma 22. For any computable \( t : \mathbb{N}^+ \to \mathbb{N}^+ \) and \( m \) as in Definition 21, if \( L \in \text{SCA}[t] \), then \( C_{\text{ow}}^m(L)(N) \in O(t(N)) \).

The proof idea is to have \( A \) and \( B \) simulate the SCA for \( L \) simultaneously, with \( A \) maintaining the first half \( c_A \) of the SCA configuration and \( B \) the second half \( c_B \). (Hence, \( A \) is aware of the leftmost active state in the SCA and can thus detect whether the SCA accepts or not.) The main difficulty lies in guaranteeing that \( A \) and \( B \) can determine the states of the cells on the right (resp., left) end of \( c_A \) (resp., \( c_B \)) despite the local configurations of these cells “overstepping the boundary” between \( c_A \) and \( c_B \). Hence, for each step simulated, we have \( B \) communicate to \( A \) the states of the two leftmost cells in \( c_B \); with this, \( A \) can compute the states of all cells in \( c_A \) in the next configuration and, in addition, that of the leftmost cell \( \alpha \) of \( c_B \), which is added to \( c_A \).

(See Figure 2 for an illustration.) This last technicality is needed due to one-way communication, which makes it impossible for \( B \) to determine the next state of \( \alpha \) (since its left neighbor is in \( c_A \) and \( B \) cannot receive messages from \( A \)). As the simulation requires at most \( t(N) \) steps and we exchange only \( O(1) \) information at each step, this yields the purported \( O(t(N)) \) upper bound. The attentive reader may have noticed this discussion apparently ignores the fact that the SCA may shrink; indeed, we also prove that shrinking does not interfere with this strategy.

Proof. Let \( S \) be an SCA for \( L \) with time complexity \( O(t) \). Furthermore, let \( Q \) be the state set of \( S \), and let \( q \in Q \) be its inactive state (as in Definition 4). We construct algorithms \( A_N \) and \( B_N \) as in Definition 21 and such that \( |B_N(w_B)| \leq 2 \log(|Q|) \cdot t(N) \).

Fix \( N \in \mathbb{N}^+ \) and an input \( w \in \Sigma^N \). For \( w_B^0 = w_B q^{2(N)+2} \) and \( w_B^{i+1} = \Delta_S(w_B^i) \) for \( i \in \mathbb{N}_0 \), \( B_N \) computes and outputs the concatenation

\[
B_N(w_B) = w_B^0(0) w_B^1(1) w_B^2(0) w_B^3(1) \cdots w_B^{t(N)}(0) w_B^{t(N)}(1),
\]

(i.e., \( B \) to \( A \) and \( A \) to \( B \); see 9 for an example in the setting of cellular automata). Since here we are interested in proving a lower bound based on communication complexity, it suffices to consider a single (arbitrary) direction (in this case \( B \) to \( A \)).
The computability follows from $t$ and $\Delta_S$ being computable. Similarly, let $w_i^d = q^{2t(N)+2}w_A$ and $w_i^{d+1} = \Delta_S(w_i^d w_B^d(0)w_B^d(1))$ for $i \in \mathbb{N}_0$. $A$ computes $t(N)$ and $w_i^d$ for $i \in [0, t(N)]$ and accepts if there is any $j$ such that $w_i^A(j)$ is an accept state of $S$ and $w_i^A(j') = q$ for all $j' < j$; otherwise, $A$ rejects.

To prove the correctness of $A$, we show $w_i^A w_B^i = \Delta_S(q^{2t(n)+2}wq^{2(n)+2})$. Hence, the $w_i^A(j)$ of above corresponds to the state of cell zero in step $i$ of $S$, and it follows that $A$ accepts if and only if $S$ does.

The claim is shown by induction on $i \in \mathbb{N}_0$. The induction basis is trivial. For the induction step, let $w' = \Delta_S(w_i^A w_B^i)$. Using the induction hypothesis, it suffices to prove $w_i^{d+1} = w'$.

Note first that, due to the definition of $w_i^{d+1}$ and $w_B^{d+1}$, we have $w' = \Delta_S(w_i^A)\alpha\beta\Delta_S(w_B^i)$, where $\alpha, \beta \in Q \cup \{\varepsilon\}$. Let $\alpha_1 = w_i^A(|w_i^A| - 2), \alpha_2 = w_i^A(|w_i^A| - 1)$, and $\alpha_3 = w_B^i(0)$ and notice $\alpha = \delta(\alpha_1, \alpha_2, \alpha_3)$; the same is true for $\beta$ and $\beta_1 = \alpha_2, \beta_2 = \alpha_3, \beta_3 = w_B^i(1)$. Hence, we have $w_i^{d+1} = \Delta_S(w_i^A)\alpha\beta$, and the claim follows. \hfill \square

We are now in position to prove Theorem 20.

**Proof of Theorem 20.** We prove that, for our language $L_1$ of before, $\mathcal{C}_{op}(\text{Block}_n(L_1))(N) \geq 2^n - n$, where $m(n) = n \cdot (n + 1)$ (i.e., $A_N$ receives the first $n$ input blocks). Since for the input length $N$ we have $N \in \Theta(n \cdot 2^n)$, the claim then follows from the contrapositive of Lemma 22.

The proof is by a counting argument. Let $A_N$ and $B_N$ be as in Definition 21. The basic idea is that, for the same input $w_A$, if $B_N$ is given different inputs $w_B$ and $w'_B$ but $B_N(w_B) = B_N(w'_B)$, then $w = w_A w_B$ is accepted if and only if $w = w_A w'_B$ is accepted. Hence, for any $y, y' \in \{0, 1\}^{2^n - n}$ with $y \neq y'$, we must have $B_N(w_B) \neq B_N(w'_B)$, where $w_B, w'_B \in \mathfrak{S}_n^{2^n - n}$ are the block word versions of $y$ and $y'$, respectively. This is because, letting $j \in [0, 2^n - n]$ be such that $y(j) \neq y'(j)$ and $z = \text{bin}_n(n + j)$, precisely one of the words $zy$ and $zy'$ is in $L_1$ (and the other not). Since there are $2^{2^n - n}$ many such words and there is a bijection between them and block words in $\mathfrak{S}_n^{2^n - n}$ whose block numbering starts with $n + 1$ (i.e., $x_0 = n + 1$, where $x_0$ is as in Definition 8), the claim follows. \hfill \square

## 4 Simulation of an SCA by a Streaming Algorithm

**Theorem 23.** Let $t : \mathbb{N}_+ \rightarrow \mathbb{N}_+$ be computable by an $O(t)$-space random access machine as in Definition 2 in $O(t \log t)$ time. Then, if $L \in \text{SCA}[t]$, there is an $O(t)$-space streaming algorithm for $L$ with $O(\log t)$ update and $O(t^2 \log t)$ reporting time.

Before we state the proof, we first introduce some notation. Having fixed an input $w$, let $c_i(t)$ denote the state of cell $i$ in step $t$ on input $w$ and, more generally, $c_i(X) = c_i(x_1) \cdots c_i(x_k)$ for every finite set $X = \{x_1, \ldots, x_k\}$ with $x_i < x_j$ for $i < j$. Note that using $c_i$ we keep deleted states in the configuration and disregard any changes in the cell index caused by cell deletion; that is, $c_i(t)$ refers to the same cell $i$ as in the initial configuration $c_0$ (of Definition 2). For a finite, non-empty $I = [a, b] \subseteq \mathbb{Z}$ and $t \in \mathbb{N}_0$, let $\text{mdcr}_t(I) = \max\{i \mid i < a, c_i(t) \neq \odot\}$ denote the nearest non-deleted cell to the left of $I$; similarly, $\text{ndcr}_t(I) = \min\{i \mid i > b, c_i(t) \neq \odot\}$ is the nearest such cell to the right of $I$.

**Proof.** Let $S$ be an $O(t)$-time SCA for $L$. Using $S$, we construct a streaming algorithm $A$ (Algorithm 1) for $L$ and prove it has the purported complexities.
Algorithm 1: Streaming algorithm A

Compute \( t(|w|) \);
Initialize lists \( A_1, A_2, B_1, \) and \( B_2 \);
\( A_1[0] \leftarrow -1; B_1[0] \leftarrow q; A_2[0] \leftarrow 0; B_2[0] \leftarrow w(0); i \leftarrow 1; j_0 \leftarrow 0; \)
for \( \tau \leftarrow 0, \ldots, t(|w|) - 1 \) do

A \( \quad j \leftarrow j_0; \)

B \( \quad \text{if } i < |w| \text{ then} \)

C \( \quad r \leftarrow i; s \leftarrow w(i); i \leftarrow i + 1; \) else

D \( \quad r \leftarrow |w|; s \leftarrow q; j_0 \leftarrow j_0 + 1; \)

end while \( j \leq \tau \) do

E \( \quad r' \leftarrow A_2[j]; s' \leftarrow \delta(B_1[j], B_2[j], s); \)
\( \quad A_1[j] \leftarrow A_2[j]; B_1[j] \leftarrow B_2[j]; A_2[j] \leftarrow r; B_2[j] \leftarrow s; \)
\( \quad r \leftarrow r'; s \leftarrow s'; \)
\( \quad \text{if } s = \otimes \text{ then goto A;} \)
\( \quad j \leftarrow j + 1; \)

end

F \( \quad A_1[\tau + 1] \leftarrow -1; B_1[\tau + 1] \leftarrow q; A_2[\tau + 1] \leftarrow r; B_2[\tau + 1] \leftarrow s; \)

end

\textbf{Construction.} Let \( w \) be an input to \( A \). To decide \( L \), \( A \) computes the states of the cells of \( S \) in the time steps up to \( t(|w|) \). In particular, \( A \) sequentially determines the state of the leftmost active cell in each of these time steps (starting from the initial configuration) and accepts if and only if at least one of these states is accepting. To compute these states efficiently, we use an approach based on dynamic programming, reusing space as the computation evolves.

\( A \) maintains lists \( A_k \) and \( B_k \) for \( k \in \{1, 2\} \) and which are indexed by every step \( j \) starting with step zero and up to the current step \( \tau \). \( A_k \) stores cell indices and \( B_k \) the respective states, that is, \( B_k[j] = c_j(A_k[j]) \). Recall the state \( c_{j+1}(y) \) of a cell \( y \) in step \( j + 1 \) is determined exclusively by the previous state \( c_j(y) \) of \( y \) as well as the states \( c_j(x) \) and \( c_j(z) \) of the left and right neighbors \( x \) and \( z \) (respectively) of \( y \) in the previous step \( j \) (i.e., \( x = ntlc_j(y) \) and \( z = ntrc_j(y) \)). In the variables maintained by \( A \), \( x \) and \( c_j(x) \) correspond to \( A_1[j] \) and \( B_1[j] \), respectively, and \( y \) and \( c_j(y) \) to \( A_2[j] \) and \( B_2[j] \), respectively; \( z \) and \( c_j(z) \) are not stored in lists but, rather, in the variables \( r \) and \( s \) (and are determined dynamically). The cell indices computed (i.e., the contents of the list \( A_k \) and the variables \( r \) and \( r' \)) are not actually used by \( A \) to compute states and are inessential to the algorithm itself; we use them only to simplify the proof of correctness below (and do not count them towards the space complexity of \( A \)).

In each iteration of the \textit{for} loop, \( A \) determines \( c_{r+1}(z^r_0) \), where \( z^r_0 \) is the leftmost active cell of \( S \) in step \( \tau \), and stores it \( B_2[\tau + 1] \). \( i \) is the index of the next symbol of \( w \) to be read (or \( |w| \) once every symbol has been read), and \( j_0 \) is the minimal time step containing a cell whose state must be known to determine \( c_{r+1}(z^r_0) \) and remains 0 as long as \( i < |w| \). Hence, the termination of \( A \) is guaranteed by the finiteness of \( w \), that is, \( i \) can only be increased a finite number of times and, once all symbols of \( w \) have been read (i.e., the condition in line 3 no longer holds), by the increment of \( j_0 \) in line 4.
Correctness. The following invariants hold for both loops in $A$:

1. $A_2[\tau] = \min\{z \in \mathbb{N}_0 \mid c_r(z) \neq \varnothing\}$, that is, $A_2[\tau]$ is the leftmost active cell of $S$ in step $j$.
2. If $j \leq \tau$, then $r = n\text{dcr}_j(A_2[j])$ and $s = c_j(r)$.
3. For every $j' \in [j_0, \tau]$, $A_1[j'] = n\text{dcr}_{j'}(A_2[j'])$, $B_1[j'] = c_{j'}(A_1[j'])$, and $B_2[j'] = c_{j'}(A_2[j'])$.

These can be shown together with the observation that, following the assignment of $r'$ and $s'$ in line 2, we have $s' = c_{j+1}(r')$ and, if $s' \neq \varnothing$ and $j < \tau$, then also $r' = n\text{dcr}_j(A_2[j + 1])$. Using the above, it follows that after the execution of the \textbf{while} loop we have $j = \tau + 1$, $s \neq \varnothing$, and $s = c_{\tau+1}(r)$. Since then $r = A_2[j - 1] = A_2[\tau]$, we obtain $r = \min\{z \in \mathbb{N}_0 \mid c_{\tau+1}(z) \neq \varnothing\}$. Hence, as $B_2[\tau + 1] = s = c_{\tau+1}(r)$ holds in line [F] if $A$ then accepts, so does $S$ accept $w$ in step $\tau$. Conversely, if $A$ rejects, then $S$ does not accept $w$ in any step $\tau' \leq t(|w|)$.

Complexity. The space complexity of $A$ is clear since it is dominated by the list $B_k$, which has $O(t(|w|))$ many entries of $O(1)$ size. As mentioned above, we ignore the space used by the list $A_0$ and the variables $r$ and $r'$ since they are inessential to the algorithm (i.e., if we remove them as well as all instructions in which they appear, the algorithm obtained is equivalent to $A$).

As for the update time, note each list access or arithmetic operation costs $O(\log t(|w|))$ time (since $t(|w|)$ upper bounds all numeric variables). Every execution of the \textbf{while} loop body requires then $O(\log t(|w|))$ time and, since, there are at most $O(t(|w|))$ executions between any two subsequent reads (i.e., line [C]), this gives us the purported $O(t(|w|)\log t(|w|))$ update time.

Finally, for the reporting time of $A$, as soon as $i = |w|$ holds after execution of line [C] (i.e., $A$ has completed reading its input) we have that the \textbf{while} loop body is executed at most $\tau - j + 1$ times before line [C] is reached again. Every time this occurs (depending on whether line [C] is reached by the \texttt{goto} instruction or not), either $j_0$ or both $j_0$ and $\tau$ are incremented. Hence, since $\tau \leq t(|w|)$, we have an upper bound of $O(t(|w|)^2)$ executions of the \textbf{while} loop body, resulting (as above) in an $O(t(|w|)^2\log t(|w|))$ reporting time in total.

5 Hardness Magnification for Sublinear-Time SCAs

Let $K > 0$ be constant such that, for any function $s: \mathbb{N}_+ \to \mathbb{N}_+$, every circuit of size at most $s(n)$ can be described by a binary string of length at most $\ell = t(n) = K s(n) \log s(n)$. In addition, let $\perp$ denote a string (of length at most $\ell$) such that no circuit of size at most $s(n)$ has $\perp$ as its description. Furthermore, let $\text{Merge}[s]$ denote the following decision problem (adapted from [17]):

\textbf{Given:} (the binary representation of) $n \in \mathbb{N}_+$, (the respective descriptions of) circuits $C_0$ and $C_1$ (padded to length $\ell$) such that $|C_i| \leq s(n)$, and $\alpha, \beta, \gamma \in \{0, 1\}^n$ with $\alpha \leq \beta \leq \gamma < 2^n$ (as seen by binary integers).

\textbf{Decide:} Is there a circuit $C$ with $|C| \leq s(n)$ and such that $\forall x \in [\alpha, \beta] : C(x) = C_0(x)$ and $\forall x \in [\beta, \gamma] : C(x) = C_1(x)$?

Note $\text{Merge}[s] \in \Sigma_2^p$. Moreover, the search version of $\text{Merge}[s]$ is Turing-reducible (in polynomial time) to a decision problem very similar to $\text{Merge}[s]$ and which is also in $\Sigma_2^p$.

We are now in position to formulate our main theorem concerning SCAs and MCSP:

\footnote{Namely the same setting as in $\text{Merge}[s]$ but with the additional requirement that the description of $C$ has a given string as its prefix. This is a fairly common construction in complexity theory for reducing search to decision problems; see, for instance, [2][10] for the same idea applied in other contexts.}
Theorem 24. Let \( s : \mathbb{N}_+ \rightarrow \mathbb{N}_+ \) be constructible in place by a CA in \( O(s(n)) \) time. Furthermore, let \( m = m(n) \) denote the (maximum) instance length of \( \text{Merge}[s] \), and let \( f, g : \mathbb{N}_+ \rightarrow \mathbb{N}_+ \) with \( f(m) \geq g(m) \geq m \) also be constructible in place by a CA in \( O(f(m)) \) and \( O(g(m)) \) space. Then, for \( b(n) = \lceil g(m)/2 \rceil \), if (the search version of) \( \text{Merge}[s] \) is computable in place by a CA in at most \( f(m) \) time and \( g(m) \) space, then (the search version of) \( \text{Block}_b(\text{MCSP}[s]) \in \text{SCA}[n \cdot f(m)] \) (where the instance size is \( N \in \mathbb{O}(2^n \cdot b(n)) \)).

We are particularly interested in the contrapositive of Theorem 24. Since \( P = \text{NP} \) implies the collapse of the polynomial hierarchy (and \( P \) is closed under polynomial-time Turing reductions), it directly implies the following hardness magnification result:

Corollary 25. If \( \text{Block}_b(\text{MCSP}[s]) \not\in \text{SCA}[n \cdot f(m)] \) for every \( f(m), g(m) \in \text{poly}(m) = \text{poly}(s(n)) \), then \( P \not= \text{NP} \).

We first address the proof of Theorem 24, which follows closely. Refer to the brief discussion further below for how the statements above and the proof that follows relate to 17.

Let \( r : \mathbb{N}_+ \rightarrow \mathbb{N}_+ \) be a function. There is a very simple 1-blockwise reduction (in the extended sense of search problems) from (the search version of) \( \text{MCSP}[s] \) to (the block version of) the following (search) problem \( \text{Merge}_r[s] \) (in particular using \( r(n) = 2^n \)):

Given: \((\text{binary representation of}) n \in \mathbb{N}_+ \) and \((\text{the respective descriptions of}) C_1, \ldots, C_r \) (all padded to length \( \ell \)), where \( |C_i| \leq s(n) \) for every \( i \) (and \( r = r(n) \)).

Find: \((\text{the description of}) a \text{ circuit } C \) with \( |C| \leq s(n) \) and such that, for every \( i \) and every \( x \in [(i-1) \cdot 2^n/r, i \cdot 2^n/r), \) \( C(x) = C_i(x) \); if no such \( C \) exists or \( C_i = \bot \) for any \( i, \) answer with \( \bot \).

Evidently, \( \text{Merge}_r[s] \) is a generalization of the problem \( \text{Merge}[s] \) defined above and, more importantly, every instance of \( \text{Merge}_r[s] \) is effectively a concatenation of \( r/2 \) many \( \text{Merge}[s] \) instances (where \( \alpha, \beta, \gamma \) and \( \gamma \) are implicit). Using the polynomial-time CA algorithm guaranteed by the assumption, we can solve each such instance in parallel, thus producing an instance of \( \text{Merge}_{r/2}[s] \) (i.e., halving \( r \)). This yields a 2-blockwise reduction from \( \text{Merge}_r[s] \) to \( \text{Merge}_{r/2}[s] \) (cf. the proof of Proposition 19). Using Lemma 13 (and that \( \text{Merge}_r[s] \) for \( r = 1 \) is trivial), we obtain the purported SCA for \( \text{MCSP}[s] \).

Proof. Let \( n \) be fixed. First, we describe the 1-blockwise reduction from \( \text{MCSP}[s] \) to \( \text{Merge}_r[s] \), where \( r = 2^n \). Let \( T_a \) denote the (description of the) trivial circuit that is constant \( a \). Then we map each pair \( (\binom{\text{bin}_a(x)}{y}, \gamma) \) (where \( x \in \{0, 1\}^n \) and \( y \in \{0, 1\} \)) to \( \binom{\text{bin}_a(x)}{C_\gamma} \), where \( \pi(1)^* \) is a padding string so that the block length \( b(n) \) is preserved. (This is needed to ensure enough space is available for the construction; see also the details further below.) It is evident this can be done in time \( O(b(n)) \) and that the reduction is correct (i.e., every solution to the original MCSP instance must be also a solution of the produced instance of \( \text{Merge}_r[s] \) and vice-versa).

Next, we construct the 2-blockwise reduction from \( \text{Merge}_r[s] \) to \( \text{Merge}_{r/2}[s] \), where \( r = 2^n \) for some \( k \in [1, n] \). Let \( A \) denote the algorithm for \( \text{Merge}[s] \) from the assumption. Then, for \( j \in [0, \rho/2] \), we map each pair \((\binom{\text{bin}_a(x)}{y}, \gamma) \) (where \( \pi(1)^* \) is a padding string (as above) and \( C \) is the circuit produced by \( A \) for \( \alpha = 2j \cdot 2^n/\rho, \beta = (2j + 1) \cdot 2^n/\rho, \gamma = (2j + 2) \cdot 2^n/\rho. \)

To actually execute \( A \), we need \( g(m) \) space (which is guaranteed by the block length \( b(n) \)) and, in addition, to prepare the input so it is in the format expected by \( A \) (i.e., eliminating the padding between the two circuit descriptions and writing the representations of \( \alpha, \beta, \gamma \)), which can
be performed in $O(b(n)) \subseteq O(g(m)) \subseteq O(f(m))$ time. For the correctness, suppose the above reduces an instance of $\text{Merge}_s[s]$ with circuits $C_1, \ldots, C_ρ$ to an instance of $\text{Merge}_ρ/2[s]$ with circuits $D_1, \ldots, D_ρ/2$ (and no left was produced). Then, a circuit $E$ is a solution to the latter if and only if $E(x) = D_i(x)$ for every $i$ and $x \in [(i - 1) \cdot 2^n/(\rho/2), i \cdot 2^n/(\rho/2))$. Using the definition of $\text{Merge}_s[s]$, every $D_i$ must satisfy $D_i(x) = C_{2i-1}(x)$ and $D_i(y) = C_{2i}(y)$ for $x \in [(2i - 2) \cdot 2^n/\rho, (2i - 1) \cdot 2^n/\rho)$ and $y \in [(2i - 1) \cdot 2^n/\rho, 2i \cdot 2^n/\rho)$. Hence, $E$ agrees with $C_1, \ldots, C_ρ$ if and only if it agrees with $D_1, \ldots, D_ρ/2$ (on the respective intervals).

Since $s(n) \geq n$ and the language $\text{Merge}_s[s]$ is trivial (i.e., it can be accepted in $O(b(n))$ time), applying Lemma 13 completes the proof.

**Comparison with [17].** We first compare the statements of Theorem 24 and Corollary 25 and the related result from [17] (i.e., Theorem 2). The premise of $\text{Merge}_s[s]$ being computable by a CA is slightly weaker than $\text{Merge}_s[s]$ being computable by a TM (for the same complexities) and, thus, yields a slightly stronger implication (from a fine-grained point of view). Furthermore, the statement of Theorem 24 makes the connection to the space complexity of solving $\text{Merge}_s[s]$ explicit (i.e., choice of the block length).

As for the proof, notice that we need only merge two circuits at a time, thus making for a smaller instance size $m$ (of $\text{Merge}_s[s]$); this not only yields a simpler proof but also minimizes the resulting time complexity of the SCA (as $f(m)$ is then smaller). Also, in our case, there is no need for additional assumptions regarding the first reduction from $\text{MCSP}_s$ to $\text{Merge}_s[s]$; in fact, this reduction can be performed unconditionally. An additional positive aspect of our proof is that it makes all blockwise reductions (here employed in a manner similar to the self-reductions of [1]) explicit. Despite these simplifications, Theorems 24 and Corollary 25 and the proof could be easily adapted for generalizations of MCSP with similar structure and instance size (e.g., MCSP in the setting of Boolean circuits with oracle gates as in [17] or MCSP for multi-output Boolean functions as in [12]).

6 Concluding Remarks

**Proving SCA lower bounds for MCSP[s].** Using the language $L_1$ from Section 3.3 we might consider the intersection $L_1[s] = L_1 \cap \text{MCSP}_s[s]$. Evidently, $L_1[s]$ is comparable in hardness to $\text{MCSP}_s[s]$; for instance, it is polynomial-time reducible to $\text{MCSP}_s[s]$ (by a reduction which yields solutions of the latter; e.g., by a Turing reduction). Adapting the proof of Theorem 24 by using the same construction with an additional step at the end (using the circuit produced) that checks the property required by $L_1$ (thus requiring polynomial time in the length of the circuit description, i.e., $\text{poly}(s(n))$ time), we can derive a hardness magnification result for $L_1[s]$ too: If $\text{Block}_k(L_1[s]) \notin \text{SCA}[\text{poly}(s(n))]$ (for every $b \in \text{poly}(s(n))$), then $P \neq \text{NP}$. Using the methods from Section 3.3 and that there are $2^{\Theta(s(n) \log s(n))}$ many (unique) circuits of size $s(n)$ or less\footnote{Let $K > 0$ be constant such that every Boolean function on $m$ variables admits a circuit of size at most $K \cdot 2^m/m$. Further, let $m = m(n)$ be maximal such that $s(n) \geq K \cdot 2^m/m$ and notice every Boolean function on $m$ variables admits a circuit of size at most $s(n)$. Since there are $2^{2^m}$ (unique) Boolean functions and $m \in \Theta(\log s(n))$, it follows there are $2^{\Omega(s(n) \log s(n))}$ (unique) circuits of size at most $s(n)$.}, we can show that if $\text{Block}_k(L_1[s]) \in \text{SCA}[t(n)]$ for some $b \in \text{poly}(n)$ and some function $t: \mathbb{N}_+ \to \mathbb{N}_+$, then $t \in \Omega(s(n) \log s(n))$. Hence, for an eventual proof of $P \neq \text{NP}$ based on Corollary 25, we would need to develop new techniques (see also the discussion below) that raise this bound beyond $\text{poly}(s(n))$.

Seen from another angle, this demonstrates that, although we can prove a tight SCA worst-case lower bound for $L_1$ (Theorem 20), establishing similar lower bounds on instances of $L_1$ with
low circuit complexity (i.e., instances which are also in \( \text{MCSP}[s] \)) is at least as hard as showing \( P \neq NP \). In other words, it is straightforward to establish a lower bound for \( L_1 \) using arbitrary instances, but it is absolutely non-trivial to establish similar lower bounds for \( L_1 \) where instance hardness is measured in terms of circuit complexity.

The proof of Theorem 24 and the locality barrier. In a recent paper [6], Chen et al. propose the concept of a locality barrier to explain why current lower bound proof techniques (for a variety of non-uniform computational models) do not suffice to show the lower bounds needed for separating complexity classes in conjunction with hardness magnification (i.e., in our case above a \( \text{poly}(s(n)) \) lower bound that proves \( P \neq NP \)). In a nutshell, the barrier arises from proof techniques relativizing with respect to local aspects of the computational model at hand (in [6], concretely speaking, oracle gates of small fan-in), whereas it is known that a proof of \( P \neq NP \) must not relativize [3].

The proof of Theorem 24 confirms the presence of such a barrier also in the uniform setting and concerning the separation of \( P \) from \( NP \). Indeed, the proof mostly concerns the construction of an SCA where the overall computational paradigm of blockwise reductions (using Lemma 13) is unconditionally compatible with the SCA model (as exemplified in Proposition 19): the \( P = NP \) assumption is needed exclusively so that the local algorithm for \( \text{Merge}[s] \) in the statement of the theorem exists. Hence, the result also holds unconditionally for SCAs that are, say, augmented with oracle access (in a plausible manner, e.g., by using an additional oracle query track and special oracle query states) to \( \text{Merge}[s] \). (Incidentally, the same argument also applies to the proof of the hardness magnification result for streaming algorithms (i.e., Theorem 2) in [17], which also builds on the existence of a similar locally computable function.) In particular, this means the lower bound techniques from the proof of Theorem 24 do not suffice since they extend to SCAs having oracle access to any computable function.

Open questions. We conclude with a few open questions:

- By weakening SCAs in some aspect, certainly we can establish an unconditional MCSP lower bound for the weakened model which, were it to hold for SCAs, would imply the separation \( P \neq NP \) (using Corollary 25). What forms of weakening (conceptually speaking) are needed for these lower bounds? How are these related to the locality barrier discussed above?

- Secondly, we saw SCAs are strictly more limited than streaming algorithms. Proceeding further in this direction, can we identify further (natural) models of computation that are more restricted than SCAs and for which we can prove results similar to Theorem 24?

- Finally, besides MCSP, what other (natural) problems admit similar SCA hardness magnification results? More importantly, can we identify some essential property of these problems that would explain these results? For instance, in the case of MCSP there appears to be some connection to the length of (minimal) witnesses being much smaller than the instance length. Indeed, one sufficient condition in this sense (disregarding SCAs) is sparsity [5]; nevertheless, it seems rather implausible that this would be the sole property responsible for all hardness magnification phenomena.

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