THE \textit{RO(\Pi B)}-GRADED $C_2$-EQUIVARIANT ORDINARY COHOMOLOGY OF $B_{C_2}U(1)$

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\textbf{Abstract.} We calculate the ordinary $C_2$-cohomology, with Burnside ring coefficients, of $\mathbb{CP}_{C_2}^\infty = B_{C_2}U(1)$, the complex projective space, a model for the classifying space for $C_2$-equivariant complex line bundles. The $RO(C_2)$-graded Bredon ordinary cohomology was calculated by Gaunce Lewis, but here we extend to a larger grading in order to capture a more natural set of generators. These generators include the Euler class of the tautological bundle, which lies outside of the $RO(C_2)$-graded theory.

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Introduction

There has been a recent resurgence of interest in calculations in Bredon’s equivariant ordinary cohomology. See, for example, Dugger [7] and [8], Kronholm [15], Hogle [13], Hazel [11], and particularly the use of such calculations by Hill, Hopkins, and Ravenel to solve the Kervaire invariant problem [12]. A large influence on the present paper was the earlier calculation of the $C_2$-equivariant cohomology of projective spaces by Lewis in [16]. All of these calculations used $RO(G)$ grading.

A general problem with $RO(G)$-graded ordinary cohomology is that it does not have a Thom isomorphism, hence does not have Euler classes, for arbitrary vector bundles. It does have such for vector bundles all of whose fibers are modeled on a single representation $V$ of $G$. In that case, the Thom isomorphism shifts grading by $V$ and the Euler class of the bundle lives in grading $V$. But many naturally occurring bundles, like the tautological bundle on a projective space, have varying representations over the space, and there is no Thom isomorphism or Euler class for such a bundle using $RO(G)$ grading. Recalling that the nonequivariant cohomology of a projective space is generated by the Euler class of its tautological bundle, it is clear that not having such a class equivariantly is a serious lacuna.

It was with this in mind that Stefan Waner and I wrote [5], defining and exploring an ordinary cohomology theory with a grading expanded beyond $RO(G)$, in which there are natural gradings to use as the dimensions of arbitrary vector bundles. This theory does possess a Thom isomorphism theorem for any vector bundle, which allows us to define Euler classes of arbitrary vector bundles.

The main purpose of this paper is to calculate, using the expanded grading from [5], the ordinary cohomology of $B_{C_2}U(1)$, the classifying space for complex $C_2$-line bundles, which we can model as an infinite complex projective space. The result is Theorem 11.3 in which, as expected, the Euler class of the tautological bundle plays a central role as one of the multiplicative generators.

Applications of these results have already been made in [3], where Thomas Hudson, Sean Tilson, and I used this calculation as input to the calculation of the cohomologies of finite projective spaces. One interesting point is that the cohomologies of the finite projective spaces are not just quotient rings of the infinite case, but involve some new elements. The phenomenon that leads to these new elements is discussed in §16. We used those calculations to give an equivariant version of Bézout’s theorem, an important part of which was the calculation of the Euler classes of line bundles associated with hypersurfaces. Once again, those Euler classes live in gradings outside the $RO(C_2)$ grading, so use of the expanded grading was crucial. We are working on generalizing those results further, including the possibility of developing an equivariant Schubert calculus.

This paper is also meant to be a first step toward a useful theory of characteristic classes for equivariant vector bundles. The next steps will require similar calculations for the classifying spaces $B_GU(n)$ for larger $n$.

This paper is organized as follows. Part 1 reviews some necessary background material and describes ordinary cohomology with the expanded grading. The equivariant cohomology of a $C_2$-space is a module over the $RO(C_2)$-graded equivariant cohomology of a point, which is distinctly nontrivial away from the integer-graded part. The $C_2$-cohomology of a point was calculated by Stong, in an unpublished manuscript, and first published by Lewis in [16]. We summarize the calculation...
in §4, where we also give the cohomologies of $EC_2$ and $\hat{EC}_2$; part of our calculation of the cohomology of $B_{C_2}U(1)$ is based on isotropy separation, that is, on the cofibration sequence $(EC_2)_+ \to S^0 \to \hat{EC}_2$.

Part 2 of the paper gives the calculation of the cohomology of $B_{C_2}U(1)$. The main result is Theorem 11.3, which gives a simple description of this cohomology as an algebra over the cohomology of a point, in terms of multiplicative generators and relations. We also prove directly that it is a free module over the cohomology of a point, which was expected, given general freeness results like [10] in the $RO(C_2)$-graded case. The expanded grading includes $RO(C_2)$ as a subgroup and we compare our results to Lewis’s $RO(C_2)$-graded calculation in §12, where we will be able to see his result as a restriction of ours, but a restriction that does not see the most natural generators. We also describe the algebra obtained when using other coefficient systems, including constant $\mathbb{Z}$ coefficients.

Part 3 calculates the cohomology of a related space that carries information about the component structure only. This calculation helps explain the presence of some of the generators of the cohomology of $B_{C_2}U(1)$. It also helps explain a divisibility phenomenon that shows up in the cohomology of finite projective spaces. We use the calculation as well to explain why, equivariantly, $B_{C_2}U(1)$ does not also represent an ordinary cohomology group, as opposed to the convenient fact that, nonequivariantly, $BU(1)$ represents $H^2(\cdot; \mathbb{Z})$.

This work owes a large debt to Gaunce Lewis, of course. I would like to think that he would have enjoyed this paper, if he hadn’t written it himself first. This paper is also founded on my long and continuing collaboration with Stefan Waner that produced [5], which in turn came out of joint work with Peter May on equivariant orientation theory, which culminated in [4]. Finally, a word of praise for serendipity and the MathOverflow website, where I stumbled across a discussion of Bergman’s diamond lemma [2], which greatly simplified and improved the line of argument used in Sections 10 and 11.

Part 1. Equivariant ordinary cohomology

1. The representation ring and the Burnside ring

Throughout this paper our ambient group will be $C_2 = \{1, t\}$ (written multiplicatively), unless we say explicitly that we are stating results for more general groups. We begin by introducing notations for some representations of $C_2$.

**Definition 1.1.**

1. Let $\mathbb{R}$ denote the one-dimensional (real) representation of $C_2$ with trivial action.
2. Let $\mathbb{C}$ denote the one-dimensional complex representation of $C_2$ with trivial action. We fix once and for all an identification $\mathbb{C} = \mathbb{R}^2$.
3. Let $\mathbb{R}^\sigma$ denote the nontrivial irreducible real representation of $C_2$, that is, $\mathbb{R}$ with $t$ acting by multiplication by $-1$. We fix once and for all a nonequivariant identification $\mathbb{R} = \mathbb{R}^\sigma$.
4. Let $\mathbb{C}^\sigma$ denote the nontrivial complex representation of $C_2$, that is, $\mathbb{C}$ with $t$ acting by multiplication by $-1$. With our fixed identification $\mathbb{C} = \mathbb{R}^2$, we also think of $\mathbb{C}^\sigma = (\mathbb{R}^\sigma)^2$ as a real representation.
Thus, the real representation ring $RO(C_2)$ is a free abelian group on two generators, which we call $1$ (the class of $\mathbb{R}$) and $\sigma$ (the class of $\mathbb{R}\sigma$).

**Definition 1.2.** If $\alpha \in RO(C_2)$, let $|\alpha| \in \mathbb{Z}$ denote the (nonequivariant) dimension of $\alpha$ and let $\alpha^{C_2} \in \mathbb{Z}$ denote the dimension of its fixed set.

The Burnside ring $A(C_2)$ is also free abelian on two generators, which we call $1$ (the class of the orbit $C_2/C_2$) and $g$ (the class of the orbit $C_2/e$), with $g^2 = 2g$. We let $\kappa = 2 - g$, so $\kappa^2 = 2\kappa$ and $\{1, \kappa\}$ is another basis of $A(C_2)$. We let $\epsilon : A(C_2) \to \mathbb{Z}$ denote the augmentation map, with $\epsilon(1) = 1$, $\epsilon(g) = 2$, and $\epsilon(\kappa) = 0$.

Segal [19] showed that, for finite $G$, $A(G)$ is isomorphic to the ring of the orbit category to the category of abelian groups.

2. **Some Mackey functors**

Equivariant ordinary cohomology uses Mackey functors as coefficients, and we will view it as Mackey functor–valued, so we review some basic facts about such functors and establish some notation.

**Definition 2.1.** Let $\mathcal{O}_{C_2}$ denote the orbit category of $C_2$ and let $\hat{\mathcal{O}}_{C_2}$ denote the stable orbit category, i.e., the category of orbits of $C_2$ and stable $C_2$-maps between them.

$\mathcal{O}_{C_2}$ and $\hat{\mathcal{O}}_{C_2}$ each have two objects, $C_2/C_2$ and $C_2/e$. We picture the maps as follows:

$$
\begin{array}{ccc}
C_2/C_2 & \xrightarrow{\rho} & C_2/C_2 \\
\downarrow{\rho} & & \downarrow{\tau} \\
C_2/e & \xrightarrow{\tau} & C_2/e \\
\end{array}
$$

That is, in the stable orbit category, the ring of self maps of $C_2/C_2$ is $A(C_2)$ while the ring of self maps of $C_2/e$ is isomorphic to the group ring $\mathbb{Z}[C_2] \cong \mathbb{Z}[t]/(t^2)$. The group of maps $C_2/e \to C_2/C_2$ is free abelian on the projection $\rho$ while the group of maps $C_2/C_2 \to C_2/e$ is free abelian on the transfer map $\tau$. We have $\rho \circ \tau = g$ and $\tau \circ \rho = 1 + t$. Finally, $\rho t = \rho$, $t \tau = \tau$, $g \rho = 2\rho$, and $\tau g = 2\tau$.

**Definition 2.2.** A **Mackey functor** is a contravariant additive functor from the stable orbit category to the category of abelian groups.
Following common practice, we will denote Mackey functors using an underline, for example, $\underline{T}$. In [5] we used overlines for contravariant functors and underlines for covariant functors. In the case of finite groups, as here, we need not make the distinction because the stable orbit category is self-dual.

We will generally picture a Mackey functor $\underline{T}$ using a diagram of the following form:

$$
\begin{array}{c}
\underline{T}(\mathbb{C}_2/\mathbb{C}_2) \\
\rho \downarrow \tau \\
\underline{T}(\mathbb{C}_2/e) \\
\end{array}
$$

Here, $\rho$ and $\tau$ are the maps induced by the maps of the same name in $\hat{\theta}_{\mathbb{C}_2}$. $\underline{T}(\mathbb{C}_2/\mathbb{C}_2)$ should be a module over the Burnside ring; the action is specified by this diagram because the action of $g$ is given by $\tau \circ \rho$.

We now review and give names to the Mackey functors that will appear in our calculations, beginning with the following two:

$$
\underline{A}_{\mathbb{C}_2/\mathbb{C}_2} = \hat{\theta}_{\mathbb{C}_2}(-, \mathbb{C}_2/\mathbb{C}_2): \mathbb{Z}[\mathbb{C}_2] \to \mathbb{Z} \\
\underline{A}_{\mathbb{C}_2/e} = \hat{\theta}_{\mathbb{C}_2}(-, \mathbb{C}_2/e): \mathbb{Z}[\mathbb{C}_2] \to \mathbb{Z}
$$

We call $\underline{A}_{\mathbb{C}_2/\mathbb{C}_2}$ the *Burnside ring Mackey functor* and often write $A = \underline{A}_{\mathbb{C}_2/\mathbb{C}_2}$ for brevity. In $\underline{A}_{\mathbb{C}_2/e}$, $\epsilon: \mathbb{Z}[\mathbb{C}_2] \to \mathbb{Z}$ is

$$
\epsilon(a_0 + a_1 t) = a_0 + a_1.
$$

$A$ and $\underline{A}_{\mathbb{C}_2/e}$, being represented functors, are both projective, with

$$
\text{Hom}_{\hat{\theta}_{\mathbb{C}_2}}(\underline{A}, \underline{T}) \cong \underline{T}(\mathbb{C}_2/\mathbb{C}_2) \quad \text{and} \quad \text{Hom}_{\hat{\theta}_{\mathbb{C}_2}}(\underline{A}_{\mathbb{C}_2/e}, \underline{T}) \cong \underline{T}(\mathbb{C}_2/e).
$$

The next two are examples of a general construction $\langle C \rangle$ for any abelian group $C$, but these are the two cases that will occur in our calculations:

$$
\langle \mathbb{Z} \rangle: \mathbb{Z} \quad \langle \mathbb{Z}/2 \rangle: \mathbb{Z}/2
$$
The last four Mackey functors that will occur are the following.

\[
\begin{array}{c|c|c|c}
\mathbb{Z}: & \mathbb{Z} & \mathbb{Z}_- : & 0 \\
1 & \downarrow & 1 & \downarrow \\
\mathbb{Z} & \mathbb{Z} & & \mathbb{Z} \\
\end{array}
\]

\[
\begin{array}{c|c|c|c}
\mathbb{Z}' : & \mathbb{Z} & \mathbb{Z}'_ : & \mathbb{Z}/2 \\
2 & \downarrow & 0 & \downarrow \\
\mathbb{Z} & \mathbb{Z} & \pi & \mathbb{Z} \\
\end{array}
\]

\(\mathbb{Z}\) is the functor usually referred to as constant \(\mathbb{Z}\) coefficients.

**Multiplicative structures.** Let \(S \Box T\) denote the box product of Mackey functors as described, for example, in [16]. For us it suffices to know that a map \(S \Box T \to U\) is equivalent to a pair of maps

\[
\begin{array}{c}
S(C_2/C_2) \otimes T(C_2/C_2) \to U(C_2/C_2) \\
S(C_2/e) \otimes T(C_2/e) \to U(C_2/e)
\end{array}
\]

satisfying the following conditions, where we write \(xy\) for the image of \(x \otimes y\) under the appropriate one of these maps:

\[
\begin{align*}
t(xy) &= (tx)(ty), \\
\rho(xy) &= \rho(x)\rho(y), \\
\tau(x\rho(y)) &= \tau(x)y, \quad \text{and} \\
\tau(\rho(x)y) &= x\tau(y).
\end{align*}
\]

The last two conditions are called the Frobenius relations. By convention, if \(x \in S(C_2/C_2)\) and \(y \in T(C_2/e)\), we will write \(xy\) for \(\rho(x)y \in U(C_2/e)\).

The functor \(A\) has a self-pairing \(A \Box A \to A\) using the usual ring structures on \(A(C_2)\) and \(\mathbb{Z}\). A unital ring is a Mackey functor \(T\) with an associative and unital pairing \(T \Box T \to T\), where the unit is given by a map \(A \to T\). (Here, we use that \(A\) is the unit for \(\Box\), meaning that \(A \Box T \cong T\) for any Mackey functor \(T\).) The conditions above say that this is equivalent to \(T(C_2/C_2)\) being a unital ring, \(T(C_2/e)\) being a unital ring (with the action of \(t\) being a ring map), \(\rho: T(C_2/C_2) \to T(C_2/e)\) being a ring map, and \(\tau: T(C_2/e) \to T(C_2/C_2)\) being a left and right \(T(C_2/C_2)\)-module map. Clearly, \(A\) is itself a unital ring. Every Mackey functor is a module over \(A\) in the obvious sense.

\(\mathbb{Z}\) is a unital ring with the usual ring structure on \(\mathbb{Z}\). A module over \(\mathbb{Z}\) is precisely a Mackey functor such that \(\tau \circ \rho\) is multiplication by 2. The functors \(A_{C_2/e}, \mathbb{Z}'_-, \mathbb{Z}_, \mathbb{Z}/2\) are all modules over \(\mathbb{Z}\). \(\mathbb{Z}\) and \(A_{C_2/e}\) are projective \(\mathbb{Z}\)-modules.

**Generators and relations.** We will want to describe the results of our calculations in terms of generators and relations. When doing so, we identify elements of \(T(C_2/C_2)\) with maps \(A \to T\) and elements of \(T(C_2/e)\) with maps \(A_{C_2/e} \to T\). For
example, we can say that \( \mathbb{Z} \) is generated by an element \( \xi \) at level \( C_2/C_2 \) subject to the relation \( \kappa \xi = 0 \). By this we mean that the following sequence is exact:

\[
\mathbb{A} \xrightarrow{\kappa} \mathbb{A} \xrightarrow{\xi} \mathbb{Z} \to 0.
\]

Here, \( \kappa \) is the map corresponding to \( \kappa \in \mathbb{A}(C_2/C_2) \) and \( \xi \) is the map corresponding to \( 1 \in \mathbb{Z}(C_2/C_2) \), which we are also calling \( \xi \) while thinking of it as an abstract generator.

Generators may occur at either level \( C_2/C_2 \) or level \( C_2/e \), and similarly for relations. Here are descriptions of the other examples in terms of generators and relations:

- (\( \mathbb{Z} \)): Generated by an element \( e \) at level \( C_2/C_2 \) subject to \( \rho(e) = 0 \). That is, there is an exact sequence
  \[
  \mathbb{A}_{G/e} \to \mathbb{A} \to \langle \mathbb{Z} \rangle \to 0
  \]
  where the first map is specified at level \( C_2/e \) by \( \epsilon: \mathbb{Z}[G] \to \mathbb{Z} \).
- (\( \mathbb{Z}/2 \)): Generated by an element \( e \) at level \( C_2/C_2 \) subject to \( \rho(e) = 0 \) and \( 2e = 0 \).
- \( \mathbb{Z}_- \): Generated by an element \( \iota \) at level \( C_2/e \) such that \( \tau \iota = 0 \).
- \( \mathbb{Z}'_- \): Generated by an element \( \iota \) at level \( C_2/e \) such that \( \tau \iota = \iota \).
- \( \mathbb{Z}_- \): Generated by an element \( \iota \) at level \( C_2/e \) such that \( \tau \iota = -\iota \).

We noted that several of these Mackey functors are modules over the ring \( \mathbb{Z} \). We can describe modules over \( \mathbb{Z} \) in terms of generators and relations as well, where a generator at level \( C_2/C_2 \) gives a copy of \( \mathbb{Z} \) while a generator at level \( C_2/e \) gives a copy of \( \mathbb{A}_{C_2/e} \). The modules \( \mathbb{Z}_- \), \( \mathbb{Z}'_- \), and \( \mathbb{Z}_- \) are described by the same generators and relations as above. However, we can simplify the description of \( \langle \mathbb{Z}/2 \rangle \): As an \( \mathbb{Z} \)-module, it is generated by an element \( e \) at level \( C_2/C_2 \) such that \( \rho(e) = 0 \).

3. Equivariant ordinary RO(IIB)-graded cohomology

In [5], Stefan Waner and the author gave a detailed exposition of equivariant ordinary cohomology graded on “representations of the fundamental groupoid.” In this section we review some of the basic definitions and properties. We assume that \( G \) is an arbitrary finite group throughout this section, though [5] is written in the more general context of compact Lie groups.

**The equivariant fundamental groupoid and its representations.** When \( X \) is a \( G \)-space, we have the following definition, given originally by tom Dieck [6] and used extensively in [4] and [5].

**Definition 3.1.** The equivariant fundamental groupoid of \( X \), denoted \( \Pi G X \) (or \( \Pi X \) when \( G \) is understood), is the category whose objects are the \( G \)-maps \( x: G/H \to X \) for all the orbits \( G/H \) of \( G \), and whose maps from \( x \) to \( y: G/K \to X \) are pairs \((\omega, \alpha)\), where \( \alpha: G/H \to G/K \) is a \( G \)-map and \( \omega \) is a \( G \)-homotopy class of paths, rel endpoints, from \( x \) to \( y \circ \alpha \). Composition is induced by composition of maps of orbits and the usual composition of path classes.

\( \Pi G \) is a 2-functor, taking \( G \)-maps to functors and homotopies to natural isomorphisms. There is a functor \( \pi: \Pi G X \to \mathcal{O}_G \), with \( \pi(x: G/H \to X) = G/H \) and \( \pi(\omega, \alpha) = \alpha \). This makes \( \Pi G X \) a bundle of groupoids over \( \mathcal{O}_G \) in the language of [4].
Definition 3.2. For \( n \) an integer, let \( \mathcal{V}_G(n) \) denote the category of virtual \( n \)-dimensional orthogonal bundles over orbits as defined in [5, §1.3]. Its objects are classes of formal differences \( G \times_H V - G \times_H W \) under an appropriate stabilization relation. Maps are given by maps of virtual bundles covering given maps of orbits.

We have a functor \( \pi: \mathcal{V}_G(n) \to \mathcal{O}_G \) that gives the underlying orbits and maps of orbits. This makes \( \mathcal{V}_G(n) \) into a bundle of groupoids over \( \mathcal{O}_G \).

Definition 3.3. A virtual \( n \)-dimensional orthogonal representation of \( \Pi_G X \) is a functor \( \Pi_G X \to \mathcal{V}_G(n) \) over \( \mathcal{O}_G \). These form the category of virtual \( n \)-dimensional representations of \( \Pi_G X \) when we take morphisms to be natural isomorphisms.

For each representation \( V \) of \( G \), there is a representation of \( \Pi_G X \) denoted \( \mathcal{V}_G(V) \) over \( \mathcal{O}_G \). This generalizes to virtual representations of \( G \) to give representations of \( \Pi_G X \) when we take morphisms to be natural isomorphisms.

Less trivially, if \( \xi: E \to X \) is a \( G \)-vector bundle over \( X \), there is an associated representation of \( \Pi_G X \), denoted \( \xi^* \), given by \( \xi^*(x: G/H \to X) = x^*(\xi) \).

Definition 3.4. \( \text{RO}(\Pi_G X) \), the orthogonal representation ring of \( \Pi_G X \), is the ring whose elements are the isomorphism classes of the virtual orthogonal representations of \( \Pi_G X \) of all dimensions. Addition is given by direct sum of bundles and multiplication by tensor product.

In particular, \( \text{RO}(\Pi_G(\ast)) \cong \text{RO}(G) \) when \( \ast \) denotes the one-point \( G \)-space. In general, the map \( X \to \ast \) induces a map \( \text{RO}(G) \to \text{RO}(\Pi_G X) \), taking \( V \oplus W \) to \( \mathcal{V} \oplus \mathcal{W} \) as above.

Ordinary cohomology. Let \( B \) be a \( G \)-space and let \( (X, q, \sigma) \) be an ex-space over \( B \), which is to say that \( q: X \to B \) is a \( G \)-map and \( \sigma: B \to X \) is a section of \( q \). Suppose also given a virtual representation \( \gamma \) of \( \Pi_G B \) and a Mackey functor \( T \). In [5] we defined the \( \gamma \)-th reduced ordinary cohomology group of \( X \) with coefficients in \( T \), \( H^\gamma_G(X; T) \), a contravariant functor of \( X \) and a covariant functor of \( \gamma \) and of \( T \).

(Because we will be using only reduced cohomology, we write \( H \) rather than \( \hat{H} \).

If we have a parametrized space \( q: X \to B \), rather than an ex-space, we can form the ex-space \((X, q)_+\), which we will write \( X_+ \) when \( q \) is understood, given by \((X \sqcup B, q \sqcup 1, \sigma)\), where \( \sigma: B \to X \sqcup B \) is the evident inclusion. In particular, we may consider \( H^\gamma_G(B_+; T) \).

The collection of groups \( H^\gamma_G(X; T) \) as \( \gamma \) varies gives the \( \text{RO}(\Pi_G B) \)-graded ordinary cohomology of the ex-space \( X \) over \( B \). (In [5] we allow more general coefficient systems than just Mackey functors, but in this paper we will stick to the simpler case.) In particular, when we restrict the virtual representation \( \gamma \) to be of the form \( \mathcal{V} \subset \mathcal{W} \), the resulting groups are exactly the \( \text{RO}(G) \)-graded ordinary cohomology of \( X \) discussed in [17], [18], and [5], which generalizes Bredon’s original integer-graded theory. In particular, this theory obeys a dimension axiom that takes the following form: For \( x: G/H \to B \) and integers \( n \), we have

\[
H^\gamma_G((G/H, x)_+; T) \cong \begin{cases} T(G/H) & \text{if } n = 0 \\ 0 & \text{if } n \neq 0, \end{cases}
\]

naturally in \( x \).

The \( \text{RO}(\Pi_G B) \)-graded theory has many nice properties, discussed in detail in [5]. One of the main reasons for introducing the enlarged grading is to get general Thom
isomorphism and Poincaré duality theorems. In particular, the Thom isomorphism takes the following form: If \( \xi: E \to B \) is a \( G \)-vector bundle, let \( T(\xi) \) denote the ex-space over \( B \) given by taking the one-point compactification of each fiber, with the section given by the compactification points. A Thom class for \( \xi \) is a class \( t \in H^*_G(T(\xi); A) \) such that, for every \( x: G/K \to B \), the element
\[
x^*(t) \in H^*_G(T(x^*\xi); A)
\]
\[
\cong H^*_G(x^*\xi) \otimes (G \times K S^V; A)
\]
\[
\cong H^V_G(S^V; A_{K/K})
\]
\[
\cong A(K)
\]
is a generator. Here, \( x^* \) is the pullback of \( \xi \) along \( x \), so \( x^*\xi = G \times K V \) for some representation \( V \) of \( K \); \( x^*\xi^* \) denotes the representation of \( RO(\Pi G/K) \cong RO(K) \) associated to \( x^*\xi \).

**Theorem 3.5** (Thom Isomorphism [5, 3.11.3]). If \( \xi: E \to B \) is a \( G \)-vector bundle, then there exists a Thom class \( t \in H^*_G(T(\xi); A) \). For every Thom class \( t \), the map
\[
t \cup - : H^*_G(B_+; T) \to H^*_{G} (T(\xi); T)
\]
is an isomorphism for every representation \( \gamma \) and Mackey functor \( T \).

As usual, given a Thom class \( t(\xi) \) for a \( G \)-bundle \( \xi \), we define the corresponding Euler class \( e(\xi) \in H^*_G(B_+; A) \) to be the restriction of \( t(\xi) \) along the zero section \( B_+ \to T(\xi) \).

For computational purposes, it is useful to view cohomology not as group valued, but as Mackey functor valued. We let \( H^*_G(X; T) \) be the Mackey functor defined by
\[
H^*_G(X; T)(G/K) = H^*_G(G/K \times_B X; T).
\]
Because \( H^*_G(\_ \_ \_; \_ \_ \_ ) \) is stable (it has suspension isomorphisms for all representations), this is actually a functor on stable maps between orbits, so does define a Mackey functor. On the other hand, the Wirthmüller isomorphism ([5, 3.9.5]) allows us to write this as
\[
H^*_G(X; T)(G/K) \cong H^*_K(K \times_B X; T|K),
\]
where \( \gamma|K \) and \( T|K \) are the restrictions from \( G \) to \( K \) defined in the most obvious ways. Thus, treating ordinary cohomology as a Mackey functor amounts to considering the cohomologies of \( X \) for all subgroups of \( G \) simultaneously, along with the associated restriction and transfer maps. As often happens, the more structure present, the more limited the possibilities, hence the easier the computations.

It will be useful to distinguish between, say, cohomology graded on \( RO(\Pi G B) \) and cohomology graded on \( RO(G) \). There are traditional notations like \( E^* \) and various ad hoc variations like \( E^\bullet \) or \( E^* \) to denote grading on different groups. We adopt instead the following.

**Notation 3.6.** If \( E \) is an object graded on a group \( R \), and \( Q \subset R \) is any subset, we write \( E^Q \) for the collection of those parts of \( E \) graded on elements in \( Q \). In particular, we write \( E^R \) for the whole graded object.

For example, we will write \( H^*_G(RO(\Pi B)) \) for cohomology graded on \( RO(\Pi G B) \) (for a specified base space \( B \)) and \( H^*_G(RO(G)) \) for the part graded only on \( RO(G) \). If
\( \alpha \in RO(\Pi_G B) \), it is often useful to look at \( H^{q+RO(G)}_G \), the part of the cohomology graded on the coset \( \alpha + RO(G) \subset RO(\Pi_G B) \).

We also need change-of-grading, defined as follows.

**Definition 3.7.** If \( E^R \) is an object graded on a group \( R \) and \( \varphi : Q \to R \) is a group homomorphism, we define the object \( E^Q \) graded on \( Q \) by \( E^q = E^{\varphi(q)} \) for \( q \in Q \).

In particular, we will see the following occur in several places in our calculations.

**Proposition 3.8.** If \( E^R \) is a graded ring with identity and \( \varphi : Q \to R \) is a group homomorphism, then \( E^Q \) is again a graded ring with identity, and, for every \( q \in \ker \varphi \), there is a distinguished multiplicative unit \( \zeta_q \in E^q \), with \( \zeta_0 = 1 \) and \( \zeta_q \zeta_{q'} = \zeta_{q+q'} \).

**Proof.** If \( x \in E^q \) and \( y \in E^{q'} \), with \( q, q' \in Q \), then \( x \in E^{\varphi(q)} \) and \( y \in E^{\varphi(q')} \), so
\[
xy \in E^{\varphi(q)+\varphi(q')} = E^{q+q'}.
\]
It is straightforward to check that, with this product, \( E^Q \) is a graded unital ring, with identity \( 1 \in E^0 = E^{\varphi(0)} \).

For each \( q \in \ker \varphi \), let \( \zeta_q \in E^q = E^{\varphi(q)} = E^0 \) be the element corresponding to the identity \( 1 \in E^0 \). The equality \( \zeta_q \zeta_{q'} = \zeta_{q+q'} \) then follows by definition. In particular, \( \zeta_q \zeta_{-q} = \zeta_0 = 1 \), so each \( \zeta_q \) is a unit. \( \Box \)

As an example, if \( B \) is a \( G \)-space, then the integer-graded nonequivariant cohomology \( H^P(B_+; \mathbb{Z}) \) can be regraded on \( RO(G) \) via the dimension homomorphism \( RO(G) \to \mathbb{Z} \). This is precisely \( H^P_G(B_+; A) \)(\( G/e \)), the Mackey-functor valued \( RO(G) \)-graded equivariant cohomology of \( B \) evaluated at the orbit \( G/e \). There will be a unit in \( H^{RO(G)}(B_+; \mathbb{Z}) \) for every virtual representation \( V - W \) of dimension zero, encoding the fact that nonequivariant cohomology cannot tell the difference between \( V \) and \( W \). Similar units appear in \( H^{RO(\Pi B)}_G(B_+; A) \)(\( G/e \)).

4. The cohomologies of a point, \( EC_2 \), and \( \tilde{EC}_2 \)

We now return to the specific case of \( G = C_2 \). From this point on, all cohomology will be assumed to have coefficients in \( A \) unless stated otherwise, and we write \( H^P_G(X) \) for \( H^P_G(X ; A) \).

In Part 2 we shall calculate the cohomology of \( BC_2 U(1) \) as an algebra over the cohomology of a point, the latter being the reduced cohomology of \( S^0 \). The calculation will use the cofibration sequence \( (EC_2)_+ \to S^0 \to \tilde{EC}_2 \), where \( EC_2 \) is a nonequivariantly contractible free \( C_2 \)-space. To that end, we need explicit descriptions of the \( RO(C_2) \)-graded cohomologies of \( (EC_2)_+ \), \( S^0 \), and \( \tilde{EC}_2 \), and the maps in the long exact sequence induced by the cofibration sequence. The calculation of the \( C_2 \)-cohomology of a point with Burnside ring coefficients is due to Stong but first appeared in the literature in [16]. The description we give here is essentially Lewis’s, but with some changes in notation.

We first introduce some elements.

**Definition 4.1.** Let
\[
\iota \in H^{r-1}_G(S^0)(C_2/e)
\]
be the unit given by Proposition 3.8 (and the comments following it) by the fact that \( \sigma - 1 \) is in the kernel of the dimension homomorphism \( RO(C_2) \to \mathbb{Z} \). Its
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inverse is an element \( \iota^{-1} \in HC2_{\sigma}(S^0)(C_2/e) \) such that \( \iota \cdot \iota^{-1} \) is the identity in \( HC2_{\sigma}(S^0)(C_2/e) \).

**Definition 4.2.** Let \( e \in HC2_{\sigma}(S^0)(C_2/C_2) \) be the Euler class of \( \mathbb{R}^\sigma \), that is, the image of the unit under the map

\[
HC2_{\sigma}(S^0) \cong HC2_{\sigma}(S) \to HC2_{\sigma}(S^0),
\]

where the last map is restriction along the inclusion.

**Theorem 4.3** ([16, 2.1 & 4.3, with proofs in the Appendix]). Additively, for \( \alpha \in RO(C_2) \),

\[
HC2_{\sigma}(S^0) \cong \begin{cases} 
A & \text{if } \alpha = 0 \\
\mathbb{Z} & \text{if } |\alpha| = 0 \text{ and } \alpha C_2 < 0 \text{ is even} \\
\mathbb{Z}_- & \text{if } |\alpha| = 0 \text{ and } \alpha C_2 \leq 1 \text{ is odd} \\
\mathbb{Z}' & \text{if } |\alpha| = 0 \text{ and } \alpha C_2 > 0 \text{ is even} \\
\mathbb{Z}'_- & \text{if } |\alpha| = 0 \text{ and } \alpha C_2 \geq 3 \text{ is odd} \\
\langle \mathbb{Z} \rangle & \text{if } |\alpha| \neq 0 \text{ and } \alpha C_2 = 0 \\
\langle \mathbb{Z}/2 \rangle & \text{if } |\alpha| > 0 \text{ and } \alpha C_2 < 0 \text{ is even} \\
\langle \mathbb{Z}/2 \rangle & \text{if } |\alpha| < 0 \text{ and } \alpha C_2 \geq 3 \text{ is odd} \\
0 & \text{otherwise.}
\end{cases}
\]

Multiplicatively, \( H^{RO(C_2)}_{C_2}(S^0) \) is a strictly commutative RO(C_2)-graded ring, generated by elements

\[
\iota \in HC2_{\sigma}(S^0)(C_2/e) \\
\iota^{-1} \in HC2_{\sigma}(S^0)(C_2/e) \\
\xi \in HC2_{\sigma}(S^0)(C_2/C_2) \\
e \in HC2_{\sigma}(S^0)(C_2/C_2) \\
e^{-m}\kappa \in HC2_{\sigma}(S^0)(C_2/C_2) \quad m \geq 1 \\
e^{-m}\delta\xi^{-n} \in HC2_{\sigma}(S^0)(C_2/C_2) \quad m, n \geq 1.
\]

These generators satisfy the following structural relations:

\[
\tau(\iota^{-1}) = 0 \\
\tau(\iota^{-2n-1}) = e^{-1}\delta\xi^{-n} \quad \text{for } n \geq 1 \\
\kappa\xi = 0 \\
\rho(\xi) = \iota^2 \\
\rho(e) = 0 \\
\rho(e^{-m}\kappa) = 0 \quad \text{for } m \geq 1 \\
\rho(e^{-m}\delta\xi^{-n}) = 0 \quad \text{for } m \geq 2 \text{ and } n \geq 1 \\
2e^{-m}\delta\xi^{-n} = 0 \quad \text{for } m \geq 2 \text{ and } n \geq 1
\]

and the following multiplicative relations:

\[
\iota \cdot \iota^{-1} = \rho(1)
\]
\[ e \cdot e^{-m} \kappa = e^{-m+1} \kappa \quad \text{for } m \geq 1 \]
\[ \xi \cdot e^{-m} \kappa = 0 \quad \text{for } m \geq 1 \]
\[ e^{-m} \kappa \cdot e^{-n} \kappa = 2 e^{-m-n} \kappa \quad \text{for } m \geq 0 \text{ and } n \geq 0 \]
\[ e \cdot e^{-m} \delta \xi^{-n} = e^{-m+1} \delta \xi^{-n} \quad \text{for } m \geq 2 \text{ and } n \geq 1 \]
\[ \xi \cdot e^{-m} \delta \xi^{-n} = e^{-m} \delta \xi^{-n+1} \quad \text{for } m \geq 1 \text{ and } n \geq 2 \]
\[ \xi \cdot e^{-m} \delta \xi^{-1} = 0 \quad \text{for } m \geq 2 \]
\[ e^{-m} \kappa \cdot e^{-n} \delta \xi^{-k} = 0 \quad \text{if } m \geq 0, n \geq 1, \text{ and } k \geq 1 \]
\[ e^{-m} \delta \xi^{-k} \cdot e^{-n} \delta \xi^{-e} = 0 \quad \text{if } m, n, k, e \geq 1 \]

The following relations are implied by the preceding ones:
\[
\kappa e = 2e
\]
\[
2e^m \xi^n = 0 \quad \text{if } m > 0 \text{ and } n > 0
\]
\[
\iota^k = (-1)^k \iota^k \quad \text{for all } k
\]
\[
\xi \cdot \tau(\iota^k) = \tau(\iota^{k+2}) \quad \text{for all } k
\]
\[
e \cdot \tau(\iota^k) = 0 \quad \text{for all } k
\]
\[
e^{-m} \kappa \cdot \tau(\iota^k) = 0 \quad \text{for all } m \geq 1 \text{ and } k
\]
\[
e^{-m} \delta \xi^{-n} \cdot \tau(\iota^k) = 0 \quad \text{for all } m, n \geq 1 \text{ and } k
\]
\[
\tau(\iota^k) \cdot \tau(\iota^l) = 0 \quad \text{if } k \text{ or } l \text{ is odd}
\]
\[
\tau(\iota^{2k}) \cdot \tau(\iota^{2l}) = 2 \tau(\iota^{2(k+l)}) \quad \text{for all } k \text{ and } l
\]
\[
\tau(\iota^{2k+1}) = 0 \quad \text{if } k \geq 0
\]
\[
e \cdot e^{-1} \delta \xi^{-n} = 0 \quad \text{if } n \geq 1
\]

The notation \( e^{-m} \kappa \) comes from the fact that \( e^m \cdot e^{-m} \kappa = \kappa \in H_0^{C_2} (S^0) = A(C_2) \).
The reason for the notation \( e^{-m} \delta \xi^{-n} \) should become clearer shortly.

It helps to have a way to visualize these calculations, and the common way of doing so is to plot the groups or Mackey functors on a grid. Different authors, however, have used different axes. Lewis, in [16], uses \( \alpha^{C_2} \) as the horizontal axis and \( |\alpha| \) as the vertical axis. Dugger [7] and Kronholm [15] use the so-called motivic grading, plotting \( |\alpha| \) vs \( |\alpha| - \alpha^{C_2} \), although Dugger uses \( |\alpha| \) as the vertical axis while Kronholm uses it as the horizontal axis. Thinking of \( \alpha = a + b \sigma \), we will use \( a = \alpha^{C_2} \) for the horizontal axis and \( b = |\alpha| - \alpha^{C_2} \) for the vertical axis, consistent with the diagrams given in [3].

Figure 1 shows the Mackey functors \( H_0^{C_2} (S^0) \), with dots representing zero functors. Figure 2 gives the elements that generate the corresponding Mackey functors. Generators shown in parentheses represent elements at level \( C_2/e \).

**Remark 4.4 (Relationship with other common notations).** The related and somewhat simpler ring \( H_{C_2}^{RO(C_2)} (S^0; \mathbb{Z}) \) has been used extensively in recent literature. The map \( \Delta \rightarrow \mathbb{Z} \) that is \( \epsilon \) at level \( C_2/C_2 \) induces a map
\[
\epsilon_*: H_{C_2}^{RO(C_2)} (S^0; \Delta) \rightarrow H_{C_2}^{RO(C_2)} (S^0; \mathbb{Z}).
\]
Elements of $H^*_{RO(C_2)}(S^0; \mathbb{Z})$ have been given a multitude of different names: Under $\epsilon_+$, $e$ maps to the element called $a$ in [14], $a_\sigma$ in [12], and $\rho$ in the motivic literature, for example in [8] (in which the coefficients are further reduced to the constant $\mathbb{Z}/2$ functor). The element $\xi$ maps to the element called $u_{2\sigma}$ in [12] and the element $\tau^2$ in [8], whereas $\tau(\iota^{-2})$ maps to the element $\theta$ in [8]. In [7], Dugger uses constant $\mathbb{Z}$ coefficients but a completely different set of names. The element he calls $\theta$ there is the image of our $\tau(\iota^{-3})$, while the image of $\tau(\iota^{-2})$ is called $\alpha$.

We will also need to know the $RO(C_2)$-graded cohomology of $EC_2$.

**Theorem 4.5.** Additively, for $\alpha \in RO(C_2)$,

$$H^\alpha_{C_2}((EC_2)_+) \cong \begin{cases} \mathbb{Z} & \text{if } |\alpha| = 0 \text{ and } \alpha^{C_2} \text{ is even} \\ \mathbb{Z} & \text{if } |\alpha| = 0 \text{ and } \alpha^{C_2} \text{ is odd} \\ \langle \mathbb{Z}/2 \rangle & \text{if } |\alpha| > 0 \text{ and } \alpha^{C_2} \text{ is even} \\ 0 & \text{otherwise.} \end{cases}$$

Multiplicatively, $H^*_{C_2}((EC_2)_+)$ is a strictly commutative $RO(C_2)$-graded $\mathbb{Z}$-algebra generated by

$$e \in H^2_{C_2}((EC_2)_+)(C_2/C_2),$$

$$\iota \in H^{\iota^{-1}}_{C_2}((EC_2)_+)(C_2/e).$$
\[
\begin{array}{ccccccccccc}
\varepsilon_1^2 & \cdots & \varepsilon_3 & \varepsilon_5 & \cdots & 1 \\
\varepsilon_4 & \varepsilon_6 & \cdots & 1 & 1 & \cdots & 1
\end{array}
\]

Figure 2: The generators of \( H_{\bar{C}_2}^{RO(C_2)}(S^0) \)

\[
\begin{align*}
\iota^{-1} & \in H_{\bar{C}_2}^{1-\sigma}(\text{EC}_2_+) + (C_2/e), \\
\xi & \in H_{\bar{C}_2}^{2(1-\sigma)}((\text{EC}_2_+ + (C_2/C_2)), \quad \text{and} \\
\xi^{-1} & \in H_{\bar{C}_2}^{2(1-\sigma)}((\text{EC}_2_+ + (C_2/C_2)),
\end{align*}
\]

subject to the relations

\[
\begin{align*}
\rho(e) &= 0, \\
\tau(i) &= 0, \\
\rho(\xi) &= \iota^2, \\
\iota \cdot \iota^{-1} &= \rho(1), \quad \text{and} \\
\xi \cdot \xi^{-1} &= 1.
\end{align*}
\]

Figures 3 and 4 show \( H_{\bar{C}_2}^{RO(C_2)}((\text{EC}_2_+)) \) and its generators. It follows from the calculation, and the action of \( H_{\bar{C}_2}^{RO(C_2)}(S^0) \) given by Proposition 4.8, that \( H_{\bar{C}_2}^{RO(C_2)}((\text{EC}_2_+)) \cong H_{\bar{C}_2}^{RO(C_2)}(S^0)[\xi^{-1}] \).

Proof of Theorem 4.5. Because \( EC_2 \) is free, we have in integer gradings that

\[
H_{\bar{C}_2}^n((\text{EC}_2_+)) \cong H^n((\text{EC}_2/C_2)_+; \mathbb{Z}) = H^n((BC_2)_+; \mathbb{Z})
\]
Because $EC_2$ is nonequivariantly contractible, we have

$$H^n((EC_2)_+;\mathbb{Z}) \cong H^n(S^0;\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } n = 0 \\ \mathbb{Z}/2 & \text{if } n > 0 \text{ is even} \\ 0 & \text{otherwise.} \end{cases}$$

The map $C_2 \times EC_2 \to EC_2$ induces the restriction map from $H^n_{C_2}((EC_2)_+)$ to $H^n((EC_2)_+;\mathbb{Z}) \cong H^n(S^0;\mathbb{Z})$; on taking orbits by $C_2$ this is the map induced by the nonequivariant map $S^0 \simeq (EC_2)_+ \to (BC_2)_+$, which leads to the calculation

$$H^n_{C_2}((EC_2)_+) \cong \begin{cases} \mathbb{Z} & \text{if } n = 0 \\ \langle \mathbb{Z}/2 \rangle & \text{if } n > 0 \text{ is even} \\ 0 & \text{otherwise.} \end{cases}$$

Now consider, again for $n$ an integer,

$$H^{n-\sigma+1}_{C_2}((EC_2)_+) \cong H^{n+1}(EC_2)_+ \wedge_{C_2} S^\sigma;\mathbb{Z}) \cong H^n((BC_2)_+;\mathbb{Z})$$

$$= \begin{cases} \mathbb{Z}/2 & \text{if } n > 0 \text{ is odd} \\ 0 & \text{otherwise,} \end{cases}$$
where $\tilde{Z}$ is the nontrivial twisted coefficient system on $BC_2$. On the other hand,

$$H^{n-\sigma+1}_C((C_2 \times EC_2)_+) \cong H_{C_2}^{n+1}((C_2)_{+} \wedge C_2 S^\sigma; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}$$

with the induced action of $\mathbb{Z}[C_2]$ on $H^1((C_2)_{+} \wedge C_2 S^\sigma; \mathbb{Z}) = \mathbb{Z}$ being the nontrivial one. This gives the calculation

$$H^{n-\sigma+1}_C \cong \begin{cases} \mathbb{Z} & \text{if } n = 0 \\ \langle \mathbb{Z}/2 \rangle & \text{if } n > 0 \text{ is odd} \end{cases}$$

To calculate the groups in the remaining gradings, consider the equivariant bundle $q: EC_2 \times \mathbb{R}^2 \to EC_2$. Because $EC_2$ is free and the action of $C_2$ on $\mathbb{R}^{2\sigma}$ preserves nonequivariant orientation, $q$ can be considered an $\mathbb{R}^{2\sigma}$-dimensional bundle in the sense of [4]. By [5], it has a Thom class $\xi \in H^{2\sigma}_C(\Sigma^2(EC_2)_+) \cong H^{2\sigma-2}_C((EC_2)_+)$, the choice of which is determined by our nonequivariant identification of $\mathbb{R}^{2\sigma}$ and $\mathbb{R}^2$, and multiplication by $\xi$ gives the Thom isomorphism

$$- \cup \xi: H^n_{C_2}((EC_2)_+) \xrightarrow{\cong} H^{n+2\sigma}_G(\Sigma^2(EC_2)_+) \cong H^{n+2\sigma-2}_G((EC_2)_+).$$

Because multiplication by $\xi$ is an isomorphism, $\xi$ is an invertible element of $H^{RO(C_2)}_{C_2}((EC_2)_+)$. This allows us to extend the calculations above to all gradings in $RO(C_2)$, giving the additive part of the theorem.
To complete the multiplicative structure, we have just constructed the element \( \xi \) and we let \( e \) and \( \iota \) be the images of the elements of the same name from the cohomology of a point, restricted along the map \( EC_2 \to \ast \). Because \( EC_2 \to \ast \) is a nonequivariant equivalence, the (positive and negative) powers of \( \iota \) generate \( H_{C_2}^{RO(C_2)}((EC_2)_+)(C_2/E) \) just as they do in the cohomology of a point. Because we constructed \( \xi \) using our chosen nonequivariant identification of \( \mathbb{R}^{2\sigma} \) with \( \mathbb{R}^2 \), we have that \( \rho(\xi) = \iota^2 \). Since the same is true in the cohomology of a point, we must have that the element named \( \xi \) in the cohomology of a point maps to the one of the same name in the cohomology of \( EC_2 \). Because \( e \) is the Euler class of \( \mathbb{R}^\sigma \), our calculation of \( H_{C_2}^{\sigma}((EC_2)_+) \) shows that \( e \) must be the nonzero element there. The various relations can now be seen as inherited from the cohomology of a point. \( \square \)

Turning to \( H_{C_2}^{RO(C_2)}(\tilde{EC}_2) \), its ring structure is not that useful to us, partly because it is a ring without an identity. We will, instead, describe it as a module over \( H_{C_2}^{RO(C_2)}(S^0) \). In fact, it is a module over the following localization.

**Proposition 4.6.** On inverting \( e \) in \( H_{C_2}^{RO(C_2)}(S^0) \) we get

\[
H_{C_2}^{RO(C_2)}(S^0)[e^{-1}] \cong (\mathbb{Z})[e, e^{-1}, \xi]/(2\xi).
\]

**Proof.** Because \( \rho(e) = 0 \), inverting \( e \) kills \( H_{C_2}^{RO(C_2)}(S^0)(C_2/e) \), hence the result is a module over \( (\mathbb{Z}) \). In \( H_{C_2}^{RO(C_2)}(S^0) \), every element is annihilated by a high enough power of \( e \) except the terms \( e^{-m}\kappa \) and \( e^m\xi^n \) for \( m \geq 0 \) and \( n \geq 0 \). We have \( e^{m+1} \cdot e^{-m}\kappa = 2e \), so \( e^{-m}\kappa = 2e^{-m} \) in \( H_{C_2}^{RO(C_2)}(S^0)[e^{-1}] \). We have \( 2e\xi = 0 \), so \( 2\xi = 0 \) in \( H_{C_2}^{RO(C_2)}(S^0)[e^{-1}] \). \( \square \)

**Theorem 4.7.** Additively, for \( \alpha \in RO(C_2) \),

\[
H_{C_2}^{\alpha}(\tilde{EC}_2) \cong \begin{cases} 
\langle \mathbb{Z} \rangle & \text{if } \alpha^{C_2} = 0 \\
\langle \mathbb{Z}/2 \rangle & \text{if } \alpha^{C_2} \geq 3 \text{ is odd} \\
0 & \text{otherwise}.
\end{cases}
\]

Multiplication by \( e \in H_{C_2}^{RO(C_2)}(S^0) \) is an isomorphism on \( H_{C_2}^{RO(C_2)}(\tilde{EC}_2) \), so the latter is a module over \( H_{C_2}^{RO(C_2)}(S^0)[e^{-1}] \). As such, it is generated by elements

\[
\kappa \in H_{C_2}^0(\tilde{EC}_2)(C_2/C_2) \quad \text{and} \quad \delta \xi^{-k} \in H_{C_2}^{1-2k(\sigma-1)}(\tilde{EC}_2)(C_2/C_2) \quad k \geq 1
\]

such that

\[
\xi \cdot \kappa = 0 \\
\xi \cdot \delta \xi^{-k} = \delta \xi^{-(k-1)} \quad k > 1, \text{ and} \\
\xi \cdot \delta \xi^{-1} = 0.
\]

\( \square \)

Figures 5 and 6 show \( H_{C_2}^{RO(C_2)}(\tilde{EC}_2) \) and its generators.

**Proof of Theorem 4.7.** Consider the long exact sequence

\[
\cdots \to H_{C_2}^{n-1}((EC_2)_+) \overset{\delta}{\to} H_{C_2}^n(\tilde{EC}_2) \overset{\psi}{\to} H_{C_2}^n(S^0) \overset{\xi}{\to} H_{C_2}^n((EC_2)_+) \to \cdots
\]
is the epimorphism $H^{C_0}_C(\tilde{E}_C)$ which is true for any space and $H^{C_0}_C(\tilde{E}_C) = 0$ or 1. On the other hand, we know that $H^{C_0}_C(\tilde{E}_C) = 0$ (which is true for any space) and $\varphi: H^{0}_C(S^0) \to H^{0}_C((EC_2)_+)$ is the epimorphism $\epsilon: A \to \mathbb{Z}$. Therefore, we have the short exact sequence

$$
0 \longrightarrow H^{0}_{C_2}(\tilde{E}_C) \longrightarrow H^{0}_{C_2}(S^0) \longrightarrow H^{0}_{C_2}((EC_2)_+) \longrightarrow 0
$$

\[ \xymatrix{ & A \ar[rr]^\epsilon & & \mathbb{Z} & } \]
The kernel of $\epsilon$ is the copy of $\langle \mathbb{Z} \rangle$ generated by $\kappa$. This gives us the additive calculation of $H^n_{C_2}(\tilde{E}C_2)$ for $n \in \mathbb{Z}$ and also identifies the generators as $\kappa$ when $n = 0$ and $\delta(e^{-n+1}\xi(n-1)/2) = e^{n-1}\delta\xi(n-1)/2$ when $n \geq 3$ is odd.

Now $S^0 \to S^3$ is an equivalence on taking fixed points by $C_2$, hence the map $\tilde{E}C_2 \to \tilde{E}C_2 \wedge S^3$ obtained by smashing with $\tilde{E}C_2$ is a $C_2$-equivalence. The induced map in cohomology is given by multiplication by $e$, so multiplication by $e$ is an isomorphism on $H^{RO(C_2)}_{C_2}(\tilde{E}C_2)$. The theorem follows.

To complete the picture, we need to describe the maps in the $RO(C_2)$-graded long exact sequence

$$\cdots \to \Sigma H^{RO(C_2)}_{C_2}((EC_2)_+) \xrightarrow{\delta} H^{RO(C_2)}_{C_2}(\tilde{E}C_2) \xrightarrow{\psi} H^{RO(C_2)}_{C_2}(S^0) \xrightarrow{\varphi} H^{RO(C_2)}_{C_2}((EC_2)_+) \to \cdots.$$ 

It helps to look at Figures 1–6 when reading the following result, which follows from the proofs of the computations done above.

**Proposition 4.8.** $\delta: \Sigma H^{RO(C_2)}_{C_2}((EC_2)_+) \to H^{RO(C_2)}_{C_2}(\tilde{E}C_2)$ is given by

$$\delta((\kappa)) = 0 \text{ for all } k \text{ and }$$

$$\delta(e^m\xi^n) = \begin{cases} e^m\delta\xi^n & \text{if } n \leq -1 \\ 0 & \text{otherwise.} \end{cases}$$

$\psi: H^{RO(C_2)}_{C_2}(\tilde{E}C_2) \to H^{RO(C_2)}_{C_2}(S^0)$ is given by

$$\psi(e^m\kappa) = e^m\kappa$$

$$\psi(e^m\delta\xi^n) = \begin{cases} e^m\delta\xi^{-n} & \text{if } m \leq -1 \\ 0 & \text{otherwise.} \end{cases}$$

$\varphi: H^{RO(C_2)}_{C_2}(S^0) \to H^{RO(C_2)}_{C_2}((EC_2)_+)$ is given by

$$\varphi((\kappa)) = \kappa$$

$$\varphi(e^m\xi^n) = e^m\xi^n \text{ for } m \geq 0 \text{ and } n \geq 0$$

$$\varphi(e^{-m}\kappa) = 0 \text{ for } m \geq 1$$

$$\varphi(e^{-m}\delta\xi^{-n}) = 0.$$ 

\[\square\]

5. The Cohomology of a Point with Other Coefficient Systems

As mentioned, the cohomology of a point with constant $\mathbb{Z}$ coefficients has been used frequently in the recent literature. We review its calculation and point out that several other calculations follow from it easily.

First, let $C$ be any abelian group and consider $\langle C \rangle$, the Mackey functor with $\langle C \rangle(C_2/C_2) = C$ and $\langle C \rangle(C_2/e) = 0$.

**Proposition 5.1.** There is a natural isomorphism

$$H^n_{C_2}(X; \langle C \rangle) \cong H^n_{C_2}(X^{C_2}; C),$$

for any $\alpha \in RO(C_2)$. 

\[\square\]
This implies that, for \( n \) but equal to 0 if \( \langle \alpha \rangle \in A \) obeys a dimension axiom in integer grading, with \( H^n((C_2/C_2)_0^0; C) = \mathbb{C} \) if \( n = 0 \) but equal to 0 if \( n \neq 0 \), and \( H^n((C_2/e)_0^0; C) = 0 \) for all \( n \). This is precisely the dimension axiom satisfied by \( H^{RO(C_2)}_C(X; \langle C \rangle) \), so the two theories must be naturally isomorphic by the uniqueness of equivariant ordinary cohomology.

Proof. This is a special case of [5, 1.13.22], or can be seen directly as follows: \( H^{RO(C_2)}(X_{C_2}; C) \) is an \( RO(C_2) \)-graded cohomology theory in based \( C_2 \)-spaces \( X \). It obeys a dimension axiom in integer grading, with \( H^n((C_2/C_2)_0^0; C) = \mathbb{C} \) if \( n = 0 \) but equal to 0 if \( n \neq 0 \), and \( H^n((C_2/e)_0^0; C) = 0 \) for all \( n \). This is precisely the dimension axiom satisfied by \( H^{RO(C_2)}_C(X; \langle C \rangle) \), so the two theories must be naturally isomorphic by the uniqueness of equivariant ordinary cohomology.

In particular, consider \( \langle \mathbb{Z} \rangle \) and the short exact sequence
\[
0 \to \langle \mathbb{Z} \rangle \xrightarrow{\kappa} A \xrightarrow{\zeta} \mathbb{Z} \to 0,
\]
where the map \( \kappa \) takes 1 to \( \kappa \in A(C_2) \). We use this to think of \( \langle \mathbb{Z} \rangle \) as the submodule of \( A \) consisting of the multiples of \( \kappa \).

Proposition 5.2. Additively,
\[
H^{RO(C_2)}_C(S^0; \langle \mathbb{Z} \rangle) \equiv \begin{cases} 
\langle \mathbb{Z} \rangle & \text{if } \alpha^C = 0 \\
0 & \text{otherwise.}
\end{cases}
\]
The map \( H^{RO(C_2)}_C(S^0; \langle \mathbb{Z} \rangle) \to H^{RO(C_2)}_C(S^0; A) \) is injective with image the ideal
\[
\langle e^n \kappa \mid n \in \mathbb{Z} \rangle \subset H^{RO(C_2)}_C(S^0; A),
\]
so we identify \( H^{RO(C_2)}_C(S^0; \langle \mathbb{Z} \rangle) \) with this ideal. (Recall that \( e^n \kappa = 2e^n \) if \( n \geq 1 \).) The inclusion factors through \( H^{RO(C_2)}_C(\check{E}C_2; A) \), with image there the submodule of elements in gradings \( \alpha \) with \( \alpha^C = 0 \).

Proof. The additive calculation is immediate from Proposition 5.1. The dimension axiom tells us that \( H^{RO(C_2)}_C(S^0; \langle \mathbb{Z} \rangle) \to H^{RO(C_2)}_C(S^0; A) \) is the inclusion \( \langle \mathbb{Z} \rangle \to A \). Write \( \kappa \in H^{RO(C_2)}_C(S^0; \langle \mathbb{Z} \rangle) \) for the generator that maps to \( \kappa \in H^{RO(C_2)}_C(S^0; A) \).

Consider the inclusion \( S^0 \to S^\sigma \). Again by Proposition 5.1, we have that the induced map \( H^{RO(C_2)}_C(S^\sigma; \langle \mathbb{Z} \rangle) \to H^{RO(C_2)}_C(S^0; \langle \mathbb{Z} \rangle) \) is an isomorphism. But this map is multiplication by \( e \), so multiplication by \( e \) is an isomorphism on \( H^{RO(C_2)}_C(S^0; \langle \mathbb{Z} \rangle) \). This implies that, for \( n \in \mathbb{Z} \), \( H^{RO(C_2)}_C(S^0; \langle \mathbb{Z} \rangle) \) is generated by \( e^n \kappa \), an element that maps to \( e^n \kappa \in H^{RO(C_2)}_C(S^0; A) \).

We can now see that the map \( H^{RO(C_2)}_C(S^0; \langle \mathbb{Z} \rangle) \to H^{RO(C_2)}_C(S^0; A) \) is injective with image the ideal \( \langle e^n \kappa \mid n \in \mathbb{Z} \rangle \).

For the last statement, Proposition 5.1 implies that
\[
H^{RO(C_2)}_C(S^0; \langle \mathbb{Z} \rangle) \equiv H^{RO(C_2)}_C(\check{E}C_2; \langle \mathbb{Z} \rangle)
\]
because \( \check{E}C_2 \simeq S^0 \). Thus, we get a factorization
\[
H^{RO(C_2)}_C(S^0; \langle \mathbb{Z} \rangle) \equiv H^{RO(C_2)}_C(\check{E}C_2; \langle \mathbb{Z} \rangle) \to H^{RO(C_2)}_C(\check{E}C_2; A) \to H^{RO(C_2)}_C(S^0; A).
\]
The image in \( H^{RO(C_2)}_C(\check{E}C_2; A) \) is clear from our computations.

It’s useful to notice that the image of \( H^{RO(C_2)}_C(S^0; \langle \mathbb{Z} \rangle) \to H^{RO(C_2)}_C(\check{E}C_2; A) \) is a direct summand. The other summand is given by the groups in gradings \( \alpha \) with \( \alpha^C \neq 0 \); it’s straightforward to check that this is a submodule.
Theorem 5.3. Additively,
\[
H^a_C(S^0; \mathbb{Z}) \cong \begin{cases} 
\mathbb{Z} & \text{if } |\alpha| = 0 \text{ and } \alpha C_2 \leq 0 \text{ is even} \\
\mathbb{Z}' & \text{if } |\alpha| = 0 \text{ and } \alpha C_2 \leq 1 \text{ is odd} \\
\mathbb{Z}' & \text{if } |\alpha| = 0 \text{ and } \alpha C_2 > 0 \text{ is even} \\
\langle \mathbb{Z}/2 \rangle & \text{if } |\alpha| > 0 \text{ and } \alpha C_2 \leq 0 \text{ is even} \\
\langle \mathbb{Z}/2 \rangle & \text{if } |\alpha| < 0 \text{ and } \alpha C_2 \geq 3 \text{ is odd} \\
0 & \text{otherwise.} 
\end{cases}
\]

\(H^C_{RO}(S^0; \mathbb{Z})\) is a strictly commutative RO\((C_2)\)-graded algebra over \(\mathbb{Z}\), generated multiplicatively by elements
\[
\iota \in H^C_{-1}(S^0; \mathbb{Z})(C_2/e) \\
\iota^{-1} \in H^C_{-1}(S^0; \mathbb{Z})(C_2/e) \\
\xi \in H^C_2(S^0; \mathbb{Z})(C_2/C_2) \\
e \in H^C_2(S^0; \mathbb{Z})(C_2/C_2) \\
e^{-m} \delta^{-n} \in H^C_{-m+2n}(S^0; \mathbb{Z})(C_2/C_2), \quad m, n \geq 1.
\]

These generators satisfy the following structural relations:
\[
\tau(\iota^{-1}) = 0 \\
\rho(\xi) = \iota^2 \\
\rho(e) = 0 \\
e^{-1} \delta^n = \tau(\iota^{-2n}) \quad \text{for } n \geq 1 \\
\rho(e^{-m} \delta^{-n}) = 0 \quad \text{for } m \geq 2 \text{ and } n \geq 1
\]
and the following multiplicative relations:
\[
\iota \cdot \iota^{-1} = \rho(1) \\
e \cdot e^{-m} \delta^{-n} = e^{-m+1} \delta^{-n} \quad \text{for } m \geq 2 \text{ and } n \geq 1 \\
\xi \cdot e^{-m} \delta^{-n} = e^{-m} \delta^{-n+1} \quad \text{for } m \geq 1 \text{ and } n \geq 2 \\
\xi \cdot e^{-m} \delta^{-1} = 0 \quad \text{for } m \geq 2 \\
e^{-m} \delta^{-k} \cdot e^{-n} \delta^{-\ell} = 0 \quad \text{if } m, n, k, \ell \geq 1
\]
The following relations are implied by the preceding ones:
\[
2e^m \xi^n = 0 \quad \text{if } m > 0 \text{ and } n \geq 0 \\
2e^{-m} \delta^{-n} = 0 \quad \text{if } m \geq 2 \text{ and } n \geq 1 \\
\iota \cdot (-1)^k \cdot \iota^k \quad \text{for all } k \\
\xi \cdot \tau(i^k) = \tau(i^{k+2}) \quad \text{for all } k \\
e \cdot \tau(i^k) = 0 \quad \text{for all } k \\
e^{-m} \delta^{-n} \cdot \tau(i^k) = 0 \quad \text{for all } m, n \geq 1 \text{ and } k \\
\tau(i^k) \cdot \tau(i^\ell) = 0 \quad \text{if } k \text{ or } \ell \text{ is odd} \\
\tau(i^{2k}) \cdot \tau(i^{2\ell}) = 2\tau(i^{2(k+\ell)}) \quad \text{for all } k \text{ and } \ell
\]
\[
\tau(2^{k+1}) = 0 \quad \text{if } k \geq 0 \\
e^\cdot e^{-1} \delta \xi^{-n} = 0 \quad \text{if } n \geq 1
\]

**Proof.** By Proposition 5.2, we have a short exact sequence

\[
0 \rightarrow H_{C_2}^{RO(C_2)}(S^0; \langle \mathbb{Z} \rangle) \rightarrow H_{C_2}^{RO(C_2)}(S^0; A) \rightarrow H_{C_2}^{RO(C_2)}(S^0; \mathbb{Z}) \rightarrow 0
\]

exhibiting \(H_{C_2}^{RO(C_2)}(S^0; \mathbb{Z})\) as a quotient ring of \(H_{C_2}^{RO(C_2)}(S^0; A)\). The additive calculation follows by noticing that the elements killed in the quotient are all of the \(e^{-m}\kappa\) for \(m \geq 0\) and the elements \(2e^m\) for \(m \geq 1\). The rest is just seeing what relations are still necessary from those in Theorem 4.3. We also simplify a bit by noticing that, for a \(\mathbb{Z}\)-module generated by an element \(x\) at level \(C_2/C_2\), \(\rho(x) = 0\) implies that \(2x = 0\). \(\square\)

![Figure 7: \(H_{C_2}^{RO(C_2)}(S^0; \mathbb{Z})\)](image)

Figures 7 and 8 show \(H_{C_2}^{RO(C_2)}(S^0; \mathbb{Z})\) and its generators. You can see something interesting in these figures. Recall that the integer-graded part lies along the horizontal axis. As it must, because of the dimension axiom, it contains only one nonzero group, which is \(\mathbb{Z}\). Now look at the horizontal line just below the axis. It also contains only one nonzero group, which is \(\mathbb{Z}_-\). So, if we were to shift the \(\mathbb{Z}\) cohomology up and to the left by one, uniqueness shows that we get cohomology...
THE RO(II)B-GRADED $C_2$-EQUIVARIANT ORDINARY COHOMOLOGY OF $B_{C_2}U(1)$

Figure 8: The generators of $H_{C_2}^{RO(C_2)}(S^0; \mathbb{Z})$

with coefficients in $\mathbb{Z}_\infty$. This continues for the two horizontal lines below that and gives us the following result.

**Corollary 5.4.** There are natural isomorphisms

$$H_{C_2}^\alpha(X; \mathbb{Z}) \cong H_{C_2}^{\alpha+\sigma(-1)}(X; \mathbb{Z}_\infty)$$

$$\cong H_{C_2}^{\alpha+2\sigma(-1)}(X; \mathbb{Z}')$$

$$\cong H_{C_2}^{\alpha+3\sigma(-1)}(X; \mathbb{Z}'').$$

Moreover, these are isomorphisms of modules over $H_{C_2}^{RO(C_2)}(S^0; \mathbb{Z})$.

**Proof.** The proof of the isomorphisms was given above. That all these theories are modules over $H_{C_2}^{RO(C_2)}(S^0; \mathbb{Z})$ follows from the fact that $\mathbb{Z}_\infty$, $\mathbb{Z}'$, and $\mathbb{Z}''$ are all modules over $\mathbb{Z}$, as noted in §2, or as exhibited in the fact that all appear as groups within $H_{C_2}^{RO(C_2)}(S^0; \mathbb{Z})$.

**Part 2. The cohomology of $B_{C_2}U(1)$**

6. **The topology of $B_{C_2}U(1)$**

We introduce an explicit model for the classifying space $B_{C_2}U(1)$ and discuss its topology and its fundamental groupoid.
Recall from Definition 1.1 that we write $C$ for the trivial complex representation of $G$ and $C^\sigma$ for the nontrivial irreducible complex representation. As a model for $B_{C_2}U(1)$ we take

$$B_{C_2}U(1) = CP_C^\infty = CP(C^\infty \oplus (C^\sigma)^\infty),$$

the projective space of complex lines in $C^\infty \oplus (C^\sigma)^\infty$. Nonequivariantly, this is $CP^\infty$. Its fixed sets are

$$B_{C_2}U(1)^{C_2} = CP(C^\infty) \sqcup CP((C^\sigma)^\infty),$$

the disjoint union of two copies of $CP^\infty$. For notational simplicity we shall write $B = B_{C_2}U(1)$, $B^0 = CP(C^\infty)$, and $B^1 = CP((C^\sigma)^\infty)$, so $B^{C_2} = B^0 \sqcup B^1$.

Let $\omega$ denote the tautological complex line bundle over $B$. Nonequivariantly, it is the usual tautological line bundle over the infinite complex projective space. Its restriction to $B^0$ is the nonequivariant tautological bundle with $C_2$ acting trivially on the fibers; its restriction to $B^1$ is the nonequivariant tautological bundle with $C_2$ acting on each fiber by negation, as it does on $C^\sigma$.

There is an equivariant involution of $B$ we will want to take into account in our calculations. Write elements of $B = CP(C^\infty \oplus (C^\sigma)^\infty)$ as equivalence classes of pairs $[z_0 : z_1]$, with $z_0 \in C^\infty$ and $z_1 \in (C^\sigma)^\infty$ not both 0. We define a $G$-map $\chi : B \to B$ by

$$\chi[z_0 : z_1] = [z_1 : z_0],$$

using our chosen nonequivariant identification of $C$ with $C^\sigma$. It is straightforward to check that $\chi$ is equivariant, and clearly $\chi^2$ is the identity. Further, $\chi^{C_2}$ swaps $B^0$ and $B^1$ via a homeomorphism between them. Notice also that $\chi^*\omega \cong \omega \otimes_C C^\sigma$. More generally, if $f : X \to B$ classifies the complex line bundle $\theta$ over $X$, then $\chi f$ classifies $\theta \otimes_C C^\sigma$.

The fundamental groupoid $\pi : \Pi B \to \mathcal{O}_{C_2}$ is relatively simple because $B$ and the components of its fixed set are all simply connected. It is equivalent to a category over $\mathcal{O}_{C_2}$ having one object $b$ over $C_2/e$ and two objects, $b_0$ and $b_1$, over $C_2/C_2$, corresponding to the two components of $B^{C_2}$. The self-maps of $b$ map isomorphically to the self-maps of $C_2/e$; there is one map $b \to b_k$ for each $k$, over the map $C_2/e \to C_2/C_2$; and there are no non-identity self-maps of $b_k$. We can picture the category as follows:

![Diagram](attachment://diagram.png)

There is also an action of $\chi$, fixing $b$ and exchanging the objects $b_0$ and $b_1$. Note that $\rho_k t = \rho_k$ for $k = 0$ or 1.

We now want to compute the representation ring $RO(\Pi B)$. Let $\alpha$ be a real virtual representation of $\Pi B$. We have $\alpha(b) = (C_2 \times \mathbb{R}) \oplus (C_2 \times \mathbb{R})$ and, by an abuse of notation, we shall write this as $\alpha(b) = C_2 \times \mathbb{R}^\alpha$ where $\alpha = n \in \mathbb{Z}$. (For this discussion, we care only about the isomorphism class of $\alpha$.) The map $t$ acts on this bundle by its nontrivial action on $C_2$ and the homotopy class
of a linear map on $\mathbb{R}^n$ whose square is homotopically trivial. There are, therefore, two possible actions of $t$ on $\alpha(b)$: the map $t_+$ in which $t$ acts (homotopically) trivially on $\mathbb{R}^n$ and the map $t_-$ in which $t$ acts on $\mathbb{R}^n$ by any orientation-reversing linear map. If $\alpha(b_0) = \mathbb{R}^{n_0} \oplus (\mathbb{R}^n)^{n_1}$ with $n_0 + n_1 = n$, when is there a $C_2$-map $\alpha(b_1): C_2 \times \mathbb{R}^n \to \mathbb{R}^{n_0} \oplus (\mathbb{R}^n)^{n_1}$ such that $\alpha(b_0)\alpha(t) = \alpha(b_0)$, so that $\alpha$ is a functor? If the action of $t$ on $C_2 \times \mathbb{R}^n$ is by $t_+$, then we must have $n_1$ even, whereas, if the action of $t$ is by $t_-$, we must have $n_1$ odd. The same applies to $\alpha(b_1)$. Thus, $\alpha$ is entirely determined by its values $\alpha(b_0)$ and $\alpha(b_1)$, with the restriction that the parity of their nontrivial parts must agree. This gives us the following.

**Proposition 6.1.**

$RO(\Pi B) \cong \{ (\alpha_0, \alpha_1) \mid \alpha_k \in RO(C_2), |\alpha_0| = |\alpha_1|, \text{ and } \alpha_0 \bar{C}_2 \equiv \alpha_1 \bar{C}_2 \pmod{2} \}$

is a free abelian group of rank 3. It has as a basis the elements

\[
1 = (1, 1) \\
\sigma = (\sigma, \sigma) \\
\text{and} \\
\Omega = (1 - \sigma, \sigma - 1).
\]

**Proof.** The argument that $RO(\Pi B)$ is the indicated subgroup of $RO(C_2)^2$ was given just before the statement of the proposition, where we are now writing $\alpha_0 = \alpha(b_0)$ and $\alpha_1 = \alpha(b_1)$. Note that, if $|\alpha_0| = |\alpha_1|$, then $\alpha_0 \bar{C}_2 \equiv \alpha_1 \bar{C}_2 \pmod{2}$ if and only if $|\alpha_0 - \alpha_0 \bar{C}_2| \equiv |\alpha_1 - \alpha_1 \bar{C}_2| \pmod{2}$.

Now, for any $(\alpha_0, \alpha_1) \in RO(\Pi B)$, let $n = (\alpha_0 \bar{C}_2 - \alpha_1 \bar{C}_2)/2$. Then

\[
(\alpha_0, \alpha_1) = (\alpha_0 \bar{C}_2 - n) \cdot 1 + (|\alpha_0| - \alpha_0 \bar{C}_2 + n) \cdot \sigma + n \cdot \Omega.
\]

On the other hand, if $a \cdot 1 + b \cdot \sigma + c \cdot \Omega = 0$, then, looking at components, it is easy to see that we must have $a = b = c = 0$. Hence, $\{1, \sigma, \Omega\}$ is a basis. \qed

It will be useful to introduce the following elements:

\[
\Omega_0 = (2\sigma - 2, 0) = -1 + \sigma - \Omega \\
\Omega_1 = (0, 2\sigma - 2) = -1 + \sigma + \Omega.
\]

We can then write

\[
RO(\Pi B) \cong \mathbb{Z}\{1, \sigma, \Omega_0, \Omega_1\}/\langle \Omega_0 + \Omega_1 = 2\sigma - 2 \rangle
\]

where $\mathbb{Z}\{1, \sigma, \Omega_0, \Omega_1\}$ indicates the free abelian group on the generators $1$, $\sigma$, $\Omega_0$ and $\Omega_1$. This is the presentation of $RO(\Pi B)$ we will use most often.

The involution $\chi$ acts on $RO(\Pi B)$ by $\chi(\alpha_0, \alpha_1) = (\alpha_1, \alpha_0)$, so we have

\[
\chi(1) = 1 \\
\chi(\sigma) = \sigma \\
\chi(\Omega_0) = \Omega_1 \\
\chi(\Omega_1) = \Omega_0 \\
\chi(\Omega) = -\Omega.
\]

The tautological line bundle $\omega$ induces a representation $\omega^* \in RO(\Pi B)$, with

\[
\omega^* = (2, 2\sigma) = 2 + \Omega_1.
\]

Similarly,

\[
(\chi \omega)^* = (2\sigma, 2) = 2 + \Omega_0.
\]
For simplicity of notation, we will often write $\omega$ for $\omega^*$ and $\chi_\omega$ for $(\chi_\omega)^*$ when the context makes clear that we are speaking of the associated representation of the bundle, not the bundle itself.

**Lemma 6.2.** The inclusion $B^0 \to B$ induces the map $RO(\Pi B) \to RO(\Pi B^0) \cong RO(C_2)$ given by $(\alpha_0, \alpha_1) \mapsto \alpha_0$. Its kernel is the free abelian subgroup generated by $\Omega_1$.

Similarly, the inclusion $B^1 \to B$ induces the map $RO(\Pi B) \to RO(C_2)$ given by $(\alpha_0, \alpha_1) \mapsto \alpha_1$, and the kernel of this map is generated by $\Omega_0$.

**Proof.** That the inclusions $B^k \to B$ induce the maps claimed follows from the way we identified elements of $RO(\Pi B)$ with pairs $(\alpha_0, \alpha_1)$.

To identify the kernels, let $k = 0$ or $1$. If $\alpha \in RO(\Pi B)$ and $\alpha_k = 0$, then $|\alpha_{1-k}| = 0$ because the dimensions are equal, and $\alpha_{1-k}$ is even because the fixed sets have the same parity. The set of elements $\beta \in RO(C_2)$ with $|\beta| = 0$ and $\beta^{C_2}$ even is the subgroup generated by $2\sigma - 2$. This proves the lemma. \hfill $\square$

Our calculation of the cohomology of $B$ will use a combination of separation of isotropy and restrictions to fixed points. These give us the following diagram, in which we write $R = RO(\Pi B)$ for brevity, whose rows and columns are parts of long exact sequences.

$$
\begin{array}{c}
\xymatrix{
\underline{H}_C^R(B_+ \wedge_B \hat{EC}_2) \ar[r]^\eta \ar[d]^{\psi} & \underline{H}_C^R(B_{-}^{C_2} \wedge_B \hat{EC}_2) \ar[d]^{\psi} & \\
H_C^R(B_+) \ar[r]^\eta & H_C^R(B_{-}^{C_2}) & \Sigma^{-1}H_C^R(B/B^{C_2}) \ar[d]^{\psi} & \\
\underline{H}_C^R(B_+ \wedge_B (EC_2)_+) \ar[r]^\eta \ar[d]^{\delta} & \underline{H}_C^R(B_{-}^{C_2} \wedge_B (EC_2)_+) \ar[d]^{\delta} & \\
\Sigma^{-1}H_C^R(B_+ \wedge_B \hat{EC}_2) & \Sigma^{-1}H_C^R(B_{-}^{C_2} \wedge_B \hat{EC}_2)
}\end{array}
$$

(The missing corners are 0, as we shall see.) With that in mind, in the next two sections we calculate the cohomologies that appear in this diagram other than the cohomology of $B$ itself.

**7. The Cohomology of $B_{C_2}U(1)^{C_2}$**

In this section we calculate the $RO(\Pi B)$-graded cohomologies of the spaces $B_{+}^{C_2}$, $B_+^{C_2} \wedge_B (EC_2)_+$, and $B_+^{C_2} \wedge_B \hat{EC}_2$, as ex-spaces over $B$. Recall that $B^{C_2} = B^0 \sqcup B^1$, where each $B^k$ is a copy of $\mathbb{C}P^\infty$. So, we begin by considering the equivariant cohomology of this nonequivariant space, for which we need some results that hold for all groups $G$.

**Proposition 7.1.** Let $X$ be a based space with trivial $G$-action and let $\underline{T}$ be a Mackey functor. Then, in integer grading,

$$
H^n_G(X; \underline{T})(G/K) \cong H^n(X; \underline{T}(G/K)),
$$

naturally in $X$ and $K$. 

Proof. Both sides may be considered as integer-graded nonequivariant cohomology theories in $X$. They obey the same dimension axiom, so, by the uniqueness of ordinary cohomology, they are naturally isomorphic. Naturality in $K$ follows similarly.

Proposition 7.2. Let $X$ be a based space with trivial $G$-action, let $A$ be a $G$-space, and let $Y 	o A$ be an ex-space over $A$. Let $R$ be a commutative ring and suppose that the nonequivariant cohomology groups $H^n(X; R)$ are flat $R$-modules for all $n$. Further, suppose that $\mathcal{T}$ is a Mackey R-module, that is, a functor from $\mathcal{O}_G$ to $R$-modules. Then

$$H^{\mathcal{O}}_G(X \wedge Y; \mathcal{T}) \cong H^Z(X; R) \otimes_R H^{\mathcal{O}}_G(Y; \mathcal{T}).$$

Proof. Extending the usual notation, by the tensor product on the right we mean the $RO(\Pi A)$-graded Mackey functor defined by

$$\left( H^Z(X; R) \otimes_R H^{\mathcal{O}}_G(Y; \mathcal{T}) \right)^\alpha = \bigoplus_{n \in \mathbb{Z}} H^n(X; R) \otimes_R H^{\mathcal{O}}_G^{-n}(Y; \mathcal{T}).$$

Now, both sides of the claimed isomorphism are cohomology theories in ex-spaces $Y$ over $A$, the right side because of the flatness assumption. There is a pairing carrying the right side to the left, which is an isomorphism in integer grading for $Y = G/K_+ \to A$, using Proposition 7.1. There are Atiyah-Hirzebruch spectral sequences from [5, 3.6.1] of the form

$$H^0_G(Y; H^q_G(X; T)) \Rightarrow H^{0+q}_G(X \wedge Y; T)$$

and

$$H^\alpha_G \left( \left( H^Z(X; R) \otimes_R H^{\mathcal{O}}_G(Y; \mathcal{T}) \right)^Y \right) \Rightarrow \left( H^Z(X; R) \otimes_R H^{\mathcal{O}}_G(Y; \mathcal{T}) \right)^{\alpha+q}$$

whose $E_2$ pages are isomorphic via the pairing, showing that the two cohomology theories are isomorphic for all $Y$. □

We now return to our specific case of $G = C_2$.

Proposition 7.3. Let $\mathbb{C}P^\infty$ be the infinite complex projective space considered as a $C_2$-space with trivial $G$-action. We have

$$H^{RO(C_2)}_{C_2}(\mathbb{C}P^\infty) \cong H^{RO(C_2)}_{C_2}(S^0)[c],$$

$$H^{RO(C_2)}_{C_2}(\mathbb{C}P^\infty \wedge (EC_2)_+) \cong H^{RO(C_2)}_{C_2}(S^0)[c] \otimes H^{RO(C_2)}_{C_2}(S^0) H^{RO(C_2)}_{C_2}((EC_2)_+) \cong H^{RO(C_2)}_{C_2}((EC_2)_+)[c],$$

and

$$H^{RO(C_2)}_{C_2}(\mathbb{C}P^\infty \wedge \tilde{EC}_2) \cong H^{RO(C_2)}_{C_2}(S^0)[c] \otimes H^{RO(C_2)}_{C_2}(S^0) H^{RO(C_2)}_{C_2}((\tilde{EC}_2)),$$

where $c$ is the Euler class of the tautological complex line bundle $\omega$ over $\mathbb{C}P^\infty$ with trivial $C_2$-action, so $|c| = 2$. Moreover, the long exact sequence coming from the cofibration sequence

$$\mathbb{C}P^\infty \wedge (EC_2)_+ \to \mathbb{C}P^\infty \to \mathbb{C}P^\infty \wedge \tilde{EC}_2$$
is given by tensoring $H^{RO(\mathbb{C})}_{C_2}(S^0)[c]$ with the long exact sequence coming from the cofibration sequence $(EC_2)_+ \to S^0 \to \tilde{E}C_2$.

Proof. This follows from the preceding proposition, taking $X = \mathbb{C}P^\infty_+$ and $Y = S^0$, $(EC_2)_+$, or $\tilde{E}C_2$, using the nonequivariant calculation $H^2(\mathbb{C}P^\infty_+; \mathbb{Z}) \cong \mathbb{Z}[c]$, where $c$ is the nonequivariant Euler class of the tautological line bundle.

Now consider $B^0$ and $B^1$ as spaces over $B$. The following result calculates their $RO(\Pi B)$-graded cohomologies.

**Proposition 7.4.**

\[
H^{RO(\Pi B)}_{C_2}(B^0_+) \cong H^{RO(C_2)}_{C_2}(S^0)[c, \zeta_1, \zeta_1^{-1}]
\]

\[
H^{RO(\Pi B)}_{C_2}(B^0_+ \wedge_B (EC_2)_+) \cong H^{RO(\Pi B)}_{C_2}(B^0_+) \otimes H^{RO(C_2)}_{C_2}(S^0) H^{RO(C_2)}_{C_2}((EC_2)_+)
\]

\[
H^{RO(\Pi B)}_{C_2}(B^0_+ \wedge_B \tilde{E}C_2) \cong H^{RO(\Pi B)}_{C_2}(B^0_+) \otimes H^{RO(C_2)}_{C_2}(S^0) H^{RO(C_2)}_{C_2}(\tilde{E}C_2),
\]

where $|c| = 2$ and $|\zeta_1| = \Omega_1$. Similarly,

\[
H^{RO(\Pi B)}_{C_2}(B^0_+) \cong H^{RO(C_2)}_{C_2}(S^0)[c, \zeta_0, \zeta_0^{-1}]
\]

\[
H^{RO(\Pi B)}_{C_2}(B^0_+ \wedge_B (EC_2)_+) \cong H^{RO(\Pi B)}_{C_2}(B^0_+) \otimes H^{RO(C_2)}_{C_2}(S^0) H^{RO(C_2)}_{C_2}((EC_2)_+)
\]

\[
H^{RO(\Pi B)}_{C_2}(B^0_+ \wedge_B \tilde{E}C_2) \cong H^{RO(\Pi B)}_{C_2}(B^0_+) \otimes H^{RO(C_2)}_{C_2}(S^0) H^{RO(C_2)}_{C_2}(\tilde{E}C_2),
\]

where $|c| = 2$ and $|\zeta_0| = \Omega_0$. This gives the following calculations of the cohomology of $B^{C_2} = B^0 \sqcup B^1$:

\[
H^{RO(\Pi B)}_{C_2}(B^{C_2}_+) \cong H^{RO(\Pi B)}_{C_2}(B^0_+) \otimes H^{RO(\Pi B)}_{C_2}(B^1_+)
\]

\[
H^{RO(\Pi B)}_{C_2}(B^{C_2}_+ \wedge_B (EC_2)_+) \cong H^{RO(\Pi B)}_{C_2}(B^{C_2}_+) \otimes H^{RO(C_2)}_{C_2}(S^0) H^{RO(C_2)}_{C_2}((EC_2)_+)
\]

\[
H^{RO(\Pi B)}_{C_2}(B^{C_2}_+ \wedge_B \tilde{E}C_2) \cong H^{RO(\Pi B)}_{C_2}(B^{C_2}_+) \otimes H^{RO(C_2)}_{C_2}(S^0) H^{RO(C_2)}_{C_2}(\tilde{E}C_2),
\]

Proof. Consider the case of $B^0$. Because $B^0$ is simply connected and has trivial $C_2$-action, $RO(\Pi B^0) \cong RO(C_2)$. By Lemma 6.2, the kernel of the induced map $RO(\Pi B) \to RO(\Pi B^0)$ is the free abelian subgroup generated by the element $\Omega_1$. Let $\zeta_1 \in H^{\omega}_{C_2}(B^0_+)$ be the unit given by Proposition 3.8. Because $RO(\Pi B)$ is generated by $RO(G)$ and $\Omega_1$, we see that the $RO(\Pi B)$-graded cohomology of $B^0$ is completely determined by its $RO(G)$-graded part together with the invertible element $\zeta_1$. Together with the preceding proposition, this gives the calculations stated. The case of $B^1$ is similar, and the case of $B^{C_2} = B^0 \sqcup B^1$ follows.

The next calculation is key to much of what follows. Note that $\omega$ is an $\omega^*$-dimensional bundle in the sense of [4] and [5], so we have the Euler class $c_\omega = e(\omega) \in H^\omega_{C_2}(B_+)$. 

**Proposition 7.5.** The restriction of $c_\omega$ to $B^0$ is

\[
c_\omega| B^0 = c_\zeta_1 \in H^\omega_{C_2}(B^0_+).
\]

The restriction of $c_\omega$ to $B^1$ is

\[
c_\omega| B^1 = (e^2 + \xi c_0^{-1}) \zeta_1^{-1} \in H^\omega_{C_2}(B^1_+).
\]
Put another way,
\[ c_\omega | B^{C_2} = (c \zeta_1, (e^2 + \xi c) \zeta_0^{-1}). \]

**Proof.** Consider first the case of \( c_\omega | B^0 \). This is the Euler class of the tautological line bundle over \( B^0 \) with trivial \( C_2 \)-action. Grading over \( RO(C_2) \), this is how \( c \in H^2_{C_2}(B^0) \) was defined. When we grade on \( RO(\Pi B) \) via the restriction \( RO(\Pi B^0) = RO(C_2) \), the corresponding element in grading \( \omega = 2 + \Omega_1 \) is \( c \zeta_1 \), so we have the first equality claimed.

For \( c_\omega | B^1 \), we use the preceding proposition and the calculation of the cohomology of a point to see that
\[ H^\omega_{C_2}(B^1_+) \cong (\mathbb{Z} \oplus \mathbb{Z}) \]
with the first summand generated by \( e^2 \zeta_0^{-1} \) and the second by \( \xi c \zeta_0^{-1} \). Thus, \( c_\omega | B^1 = ae^2 \zeta_0^{-1} + b \xi c \zeta_0^{-1} \) for some integers \( a \) and \( b \). Restricting to nonequivariant cohomology, \( c_\omega | B^1 \) restricts to the first nonequivariant Chern class of the tautological bundle, \( e^2 \) restricts to \( 0 \), and \( \xi c \zeta_0^{-1} \) also restricts to the first nonequivariant Chern class of the tautological bundle. Thus, \( b = 1 \). To determine \( a \), consider the (equivariant) restriction to a single point in \( B^1 \). This time, \( c_\omega \) must restrict to the Euler class of the fiber over that point, which is a copy of \( \mathbb{C}^* \), whose Euler class is \( e^2 \zeta_0^{-1} \). Because \( c \) restricts to \( 0 \) at any point, being the Euler class of the tautological bundle with trivial \( C_2 \)-action, \( ae^2 \zeta_0^{-1} + \xi c \zeta_0^{-1} \) restricts to \( ae^2 \zeta_0^{-1} \), so we must have \( a = 1 \), giving the second equality claimed. \( \square \)

Because \( \chi \) exchanges the fixed-point components, we get the following immediate corollary, where we write \( c_{\chi \omega} \) for the Euler class of \( \chi \omega = \omega \otimes \mathbb{C}^\sigma \). Note that \( c_{\chi \omega} = \chi^* c_\omega \) because \( \chi \) classifies the operation \( - \otimes \mathbb{C}^\sigma \).

**Corollary 7.6.** \( c_{\chi \omega} | B^{C_2} = ((e^2 + \xi c) \zeta_1^{-1}, c \zeta_0) \in H^{\chi \omega}_{C_2}(B^{C_2}_+) \). \( \square \)

8. The Cohomologies of \( B_{C_2}U(1)_+ \wedge (EC_2)_+ \) and \( B_{C_2}U(1)_+ \wedge \tilde{EC}_2 \)

We start with an easy result.

**Proposition 8.1.** The inclusion \( B^{C_2} \to B \) induces an isomorphism
\[ H^*_{C_2}(B_+ \wedge_B \tilde{EC}_2) \cong H^*_{C_2}(B_+ \wedge_B EC_2) \]
where the latter is calculated in Proposition 7.4.

**Proof.** For any ex-\( C_2 \)-space \( X \) over \( B \), the inclusion \( X^{C_2} \to X \) induces a weak equivalence \( X^{C_2} \wedge_B EC_2 \to X \wedge_B EC_2 \), from which the isomorphism follows. \( \square \)

In one sense, the calculation of the cohomology of \( B_+ \wedge_B (EC_2)_+ \) is just as easy, as we have the following result.

**Proposition 8.2.** Each inclusion \( B^k \to B, k = 0 \) or \( 1 \), induces an isomorphism
\[ H^*_{C_2}(B_+ \wedge_B (EC_2)_+) \cong H^*_{C_2}(B^k_+ \wedge_B (EC_2)_+) \]
Consequently, the inclusion \( B^{C_2} \to B \) induces a split short exact sequence
\[ 0 \to H^*_{C_2}(B_+ \wedge_B (EC_2)_+) \xrightarrow{\eta} H^*_{C_2}(B^{C_2}_+ \wedge_B (EC_2)_+) \xrightarrow{\theta} H^*_{C_2}(B_+ \wedge_B B^{C_2} \wedge_B (EC_2)_+) \to 0. \]
For the last group we have an isomorphism

\[ \Sigma^{-1} H^{RO(\Pi B)}_{C_2}(B/\mathbb{B} B^C \wedge_B (EC_2)_+) \cong H^{RO(\Pi B)}_{C_2}(B^C_+ \wedge_B (EC_2)_+) \]

**Proof.** For each \( k \), the inclusion \( B^k \to B \) is a nonequivariant equivalence, hence \( B^k \times EC_2 \to B \times EC_2 \) is an equivariant equivalence, giving the first isomorphism.

We can take as a splitting of \( \eta \) the map

\[ \eta': H^{RO(\Pi B)}_{C_2}(B^C_+ \wedge_B (EC_2)_+) \]

\[ \cong H^{RO(\Pi B)}_{C_2}(B^C_+ \wedge_B (EC_2)_+) \oplus H^{RO(\Pi B)}_{C_2}(B^C_+ \wedge_B (EC_2)_+)
\]

\[ \to H^{RO(\Pi B)}_{C_2}(B^C_+ \wedge_B (EC_2)_+) \cong H^{RO(\Pi B)}_{C_2}(B^C_+ \wedge_B (EC_2)_+) \]

given by projection to the \( B^0 \) summand. We then have

\[ H^{RO(\Pi B)}_{C_2}(B/\mathbb{B} B^C \wedge_B (EC_2)_+) \cong \Sigma \ker \eta' \]

giving the rest of the proposition. \( \square \)

We will need to know exactly what the inclusion

\[ \eta: H^{RO(\Pi B)}_{C_2}(B^C_+ \wedge_B (EC_2)_+) \to H^{RO(\Pi B)}_{C_2}(B^C_+ \wedge_B (EC_2)_+) \]

does. First, we write the source in the following way.

**Definition 8.3.** Let \( c_\omega \in H^{c_\omega}_{C_2}(B^C_+ \wedge (EC_2)_+) \) denote the image of \( c_\omega \in H^{c_\omega}_{C_2}(B^C_+) \) under pullback along the projection \( B \times EC_2 \to B \). Let \( \zeta_1 \in H^{\Omega_0}_{C_2}(B^C_+ \wedge (EC_2)_+) \) be the element that restricts to \( \zeta_1 \in H^{\Omega_0}_{C_2}(B^0_+ \wedge (EC_2)_+) \).

**Corollary 8.4.**

\[ H^{RO(\Pi B)}_{C_2}(B^C_+ \wedge_B (EC_2)_+) \cong H^{RO(C_2)}_{C_2}((EC_2)_+)[c_\omega, \zeta_1, \zeta_1^{-1}] \]

**Proof.** This is clear from the isomorphism

\[ H^{RO(\Pi B)}_{C_2}(B^C_+ \wedge_B (EC_2)_+) \cong H^{RO(\Pi B)}_{C_2}(B^C_+ \wedge_B (EC_2)_+) \]

given by the preceding proposition, together with the calculation of Proposition 7.4, except possibly for the use of \( c_\omega \) as one of the generators. This is justified by Proposition 7.5, which tells us that \( c_\omega \) maps to \( c_\omega \in H^{c_\omega}_{C_2}(B^C_+ \wedge_B (EC_2)_+) \).

We also define the following element.

**Definition 8.5.** Let

\[ \zeta_0 = \xi \zeta_1^{-1} \in H^{\Omega_0}_{C_2}(B^C_+ \wedge_B (EC_2)_+) \]

Recalling that \( \xi \) is invertible in \( H^{RO(C_2)}_{C_2}((EC_2)_+) \), we see that \( \zeta_0 \) is invertible in \( H^{RO(\Pi B)}_{C_2}(B^C_+ \wedge_B (EC_2)_+) \) and that

\[ \zeta_0 \cdot \zeta_1 = \xi \]

**Proposition 8.6.** Under the inclusion

\[ \eta: H^{RO(\Pi B)}_{C_2}(B^C_+ \wedge_B (EC_2)_+) \to H^{RO(\Pi B)}_{C_2}(B^C_+ \wedge_B (EC_2)_+) \]

we have

\[ \eta(c_\omega) = (c\zeta_1, (c^2 + \xi c)\zeta_0^{-1}) \]

\[ \eta(\zeta_0) = (\xi \zeta_1^{-1}, \zeta_0) \]

and

\[ \eta(\zeta_1) = (\zeta_1, \xi \zeta_0^{-1}) \].
Proof. The calculation of $\eta(c_\infty)$ was done in Proposition 7.5. We have that

$$\zeta_1|B^0 = \zeta_1$$

and

$$\zeta_0|B^0 = \xi \zeta_1^{-1}$$

by definition. Now $\zeta_1|B^1$ is a unit in $H^{O_2}_C(B^1_+ \wedge_B (EC_2)_+)$, but this is a copy of $\mathbb{Z}$ generated by $\xi \zeta_1^{-1}$. Both $\zeta_1|B^2$ and $\xi \zeta_1^{-1}$ restrict to $i^2 \xi \zeta_1^{-1}$ nonequivariantly, so we must have $\zeta_1|B^1 = \xi \zeta_1^{-1}$. The equation $\zeta_0|B^1 = \zeta_0$ then follows from the definition that $\zeta_0 = \xi \zeta_1^{-1}$.

9. Preliminary results on the cohomology of $BC_2 U(1)$

Many of our arguments in the remainder of the calculation will be based on diagram (6.3), which we repeat here, with $R = RO(\Pi B)$ again.

The rows and columns are all parts of long exact sequences. That the map $\eta$ in the top and bottom rows of the diagram is an isomorphism is Proposition 8.1. That implies that $H^{RO(\Pi B)}_C(B/B C^2 \wedge_B EC_2)$, the group that would appear in the top and bottom right corners, is zero; alternatively, this vanishing follows from the fact that $(B/B C^2)^{C_2}$ is the trivial ex-space over $B^{C_2}$. In turn, this vanishing implies that the map $\phi$ on the far right is an isomorphism. In the third row, Proposition 8.2 shows that $\eta$ is a monomorphism and $\delta$ is an epimorphism.

**Proposition 9.1.** $\eta: H^\alpha_{C_2}(B_+) \to H^\alpha_{C_2}(B^{C_2}_+)$ is a monomorphism for $|\alpha| \leq 0$. It is also a monomorphism at level $C_2/C_2$ if $|\alpha| > 0$ and $\alpha_0^C$, and $\alpha_1^C$ are even.

**Proof.** From the long exact sequence, we see that $\eta$ is a monomorphism if the group $H^\alpha_{C_2}(B/B C^2) = 0$. In the diagram above we noted that $H^{RO(\Pi B)}_{C_2}(B/B C^2) \cong H^{RO(\Pi B)}_{C_2}(B/B C^2 \wedge_B (EC_2)_+)$ and we calculated the latter in Proposition 8.2. As a module over the ring $H^{RO(C_2)}_{C_2}((EC_2)_+)$, it is the suspension of an algebra with multiplicative generators in gradings 2 and $\Omega_0$. Further, $H^\beta_{C_2}((EC_2)_+)$ is zero if $|\beta| > 0$; at level $C_2/C_2$ it is also zero for those $|\beta| \geq 0$ for which $\beta C_2$ is odd. It follows that

$$H^\alpha_{C_2}(B/B C^2) \cong H^\alpha_{C_2}(B/B C^2 \wedge_B (EC_2)_+) = 0$$

for the $\alpha$ specified in the statement of the proposition.
The following is then a diagram chase.

**Corollary 9.2.** The diagram

\[
\begin{array}{ccc}
H^\alpha_{C_2}(B_+) & \xrightarrow{\eta} & H^\alpha_{C_2}(B^C_+) \\
\varphi \downarrow & & \varphi \downarrow \\
H^\alpha_{C_2}(B_+ \wedge_B (EC_2)_+) & \xrightarrow{\eta} & H^\alpha_{C_2}(B^C_+ \wedge_B (EC_2)_+)
\end{array}
\]

is a pullback square for \(|\alpha| < 0\) and also for those \(|\alpha| > 0\) for which \(C^2\) and \(C^1\) are even.

**Corollary 9.3.** There are unique elements \(\zeta_0 \in H^\alpha_{C_2}(B_+)\) and \(\zeta_1 \in H^{\alpha 1}_{C_2}(B_+)\) such that

\[
\begin{align*}
\eta(\zeta_0) &= (\xi_1^{-1}, \zeta_0), \\
\eta(\zeta_1) &= (\zeta_1, \xi_0^{-1})
\end{align*}
\]

\[
\begin{align*}
\varphi(\zeta_0) &= \zeta_0, \\
\varphi(\zeta_1) &= \zeta_1.
\end{align*}
\]

**Proof.** By Proposition 8.6, the element \(\zeta_0 \in H^\alpha_{C_2}(B_+ \wedge_B (EC_2)_+)\) satisfies \(\eta(\zeta_0) = (\xi_1^{-1}, \zeta_0)\). On the other hand, we have the element \((\xi_1^{-1}, \zeta_0) \in H^\alpha_{C_2}(B^C_+),\) with \(\varphi(\xi_1^{-1}, \zeta_0) = \eta(\zeta_0)\). It follows from Corollary 9.2 that there is a unique element \(\zeta_0 \in H^\alpha_{C_2}(B_+\) with the claimed images under \(\eta\) and \(\varphi\). The existence and uniqueness of \(\zeta_1\) follow in the same way.

We emphasize that \(\zeta_0\) and \(\zeta_1\) are not invertible as elements of \(H^\alpha_{C_2}(B_+)\), unlike their images in \(H^\alpha_{C_2}(B_+ \wedge_B (EC_2)_+)\)—their images in \(H^{\alpha 1}_{C_2}(B_+)\) are not invertible.

The action of \(\chi\) on these elements is given as follows.

**Proposition 9.4.** In \(H^{RO(1B)}_{C_2}(B_+)\) we have

\[
\chi(\zeta_0) = \zeta_1, \quad \chi(\zeta_1) = \zeta_0, \quad \chi(c_\omega) = c_\chi \omega, \quad \chi(c_\chi \omega) = c_\omega.
\]

**Proof.** These elements all live in gradings in which \(\eta\) is a monomorphism, so it suffices to check that the equalities are true after applying \(\eta\). We have

\[
\eta(\chi(\zeta_0)) = \chi \eta(\zeta_0) = \chi(\xi_1^{-1}, \zeta_0) = (\zeta_1, \xi_0^{-1}) = \eta(\zeta_1),
\]

hence \(\chi \zeta_0 = \zeta_1\). That \(\chi \zeta_1 = \zeta_0\) follows because \(\chi^2 = 1\).

We have already noted that \(\chi c_\omega = c_\chi \omega\) because \(\chi\) classifies \(- \otimes C \, C^\sigma\).

10. The proposed ring structure

In §11 we shall show that \(H^{RO(1B)}_{C_2}(B_+)\) is generated multiplicatively by the elements \(\zeta_0, \zeta_1, c_\omega,\) and \(c_\chi \omega,\) subject to the two relations given in the following result.

**Proposition 10.1.** In \(H^{RO(1B)}_{C_2}(B_+)\) we have the relations

\[
\zeta_0 \zeta_1 = \xi
\]

and

\[
\zeta_1 c_\omega = (1 - \kappa) \zeta_0 c_\omega + e^2.
\]
Proof. To show that the relations hold, we note that both take place in gradings where Proposition 9.1 says that $\eta: B_+ \to B_+$ is a monomorphism, so it suffices to show that the relations hold after applying $\eta$. Recall that

$$\eta(\zeta_0) = (\zeta_1^{-1}, \zeta_0),$$
$$\eta(\zeta_1) = (\zeta_1, \zeta_0^{-1}),$$
$$\eta(c_\omega) = (c_\zeta, (e^2 + \zeta e)\zeta_0^{-1}),$$
$$\eta(c_{\chi\omega}) = ((e^2 + \zeta e)\zeta_1^{-1}, c_\zeta).$$

We see immediately that

$$\eta(\zeta_0 \zeta_1) = (\zeta, \zeta) = \eta(\zeta),$$

hence $\zeta_0 \zeta_1 = \zeta$. For the second relation, we have

$$\eta(\zeta_1 c_{\chi\omega} - (1 - \kappa)\zeta_0 c_\omega) = (e^2 + \zeta e, e\zeta) - (1 - \kappa)(\zeta e, e^2 + \zeta e)$$
$$= (e^2 + \zeta e, e\zeta) - (\zeta e, -e^2 + \zeta e)$$
$$= (e^2, e^2)$$
$$= \eta(e^2),$$

using that $(1 - \kappa)\zeta = \zeta$ and $(1 - \kappa)e = -e$, hence the second relation also holds. \(\Box\)

Multiplying the second relation by $(1 - \kappa)$ and rearranging gives the similar relation

$$\zeta_0 c_\omega = (1 - \kappa)\zeta_1 c_{\chi\omega} + e^2.$$

Definition 10.2. Let $L^{RO(\Pi B)}$ be the algebra defined by

$$L^{RO(\Pi B)} = H^{RO(C_2)}(S^0)[\zeta_0, \zeta_1, c_\omega, c_{\chi\omega}] / \langle \zeta_0 \zeta_1 - \zeta, \zeta_1 c_{\chi\omega} - (1 - \kappa)\zeta_0 c_\omega - e^2 \rangle.$$

In the following section we shall show that this is isomorphic to the cohomology of $B$. The main result of this section is the following.

Theorem 10.3. $L^{RO(\Pi B)}$ is a free $H^{RO(C_2)}(S^0)$-module on a basis consisting of the images of those monomials $\zeta_0^k \zeta_1^m c_\omega^{n_1} c_{\chi\omega}^{n_2}$ that are not multiples of

- $\zeta_0 \zeta_1$
- $\zeta_1 c_{\chi\omega}$, or
- $\zeta_0^2 c_\omega$.

Proof. Write $I = \langle \zeta_0 \zeta_1 - \zeta, \zeta_1 c_{\chi\omega} - (1 - \kappa)\zeta_0 c_\omega - e^2 \rangle$. Rather than “write out tortuous verifications” [2, Introduction], we use Bergman’s diamond lemma, [2, Theorem 1.2], as modified for the commutative case by the comments in his §10.3. The diamond lemma will show that, using the relations, every element of $L^{RO(\Pi B)}$ can be reduced in a unique way to a linear combination of the basis elements specified in the statement of the theorem. (Note: We should presumably be working with anti-commutative rings and algebras, but Theorem 4.3 tells us that the cohomology of a point is strictly commutative, and our polynomial generators $\zeta_0$, etc., are in “even” gradings where no signs will be introduced.) To use the diamond lemma we need several things: First, we need a partial ordering $\approx$ on the set of all monomials in $H^{RO(C_2)}(S^0)[\zeta_0, \zeta_1, c_\omega, c_{\chi\omega}]$ such that, if $A \approx B$ then $CA \approx CB$ for any monomial $C$. (By a monomial we shall always mean a product $\zeta_0^{k_0} \zeta_1^{k_1} c_\omega^{m_1} c_{\chi\omega}^{n_1}$ with no coefficient.) We define $\approx$ by ordering by total degree and, within the same degree, saying that $\zeta_0^{k_0} \zeta_1^{k_1} c_\omega^{m_1} c_{\chi\omega}^{n_1} \prec \zeta_0^{k_2} \zeta_1^{k_2} c_\omega^{m_2} c_{\chi\omega}^{n_2}$.
if \( n_1 < n_2 \). Note that, as required for the diamond lemma, this partial order satisfies the descending chain condition.

Next, we need a reduction system \( S \), consisting of pairs \((W, f)\), where \( W \) is a monomial, \( f \) is a polynomial, and \( I \) is generated by the collection of elements \( W - f, (W, f) \in S \). We take \( S \) to be the set containing the three pairs

\[
\begin{align*}
\sigma_1 &= (W_1, f_1) = (\zeta_0\zeta_1, \xi) \\
\sigma_2 &= (W_2, f_2) = (\zeta_1c_{\chi\omega}, (1 - \kappa)\zeta_0c_{\omega} + e^2) \quad \text{and} \\
\sigma_3 &= (W_3, f_3) = (\zeta_0^2c_{\omega}, \xi c_{\chi\omega} + e^2\zeta_0).
\end{align*}
\]

The first two satisfy \( W - f \in I \) by the definition of \( I \); for the third, note that

\[
\begin{align*}
\zeta_0^2c_{\omega} &= \zeta_0(\zeta_0c_{\omega}) \\
&\equiv \zeta_0((1 - \kappa)\zeta_1c_{\chi\omega} + e^2) \pmod{I} \\
&\equiv \xi c_{\chi\omega} + e^2\zeta_0 \pmod{I}.
\end{align*}
\]

Note that \( W_1 - f_1 \) and \( W_2 - f_2 \) already generate \( I \), so it does no harm to add \( \sigma_3 \). (We will see in a moment why we need it.) We also need that the partial ordering on the monomials is compatible with the reduction system, meaning that, for each pair \((W, f_1), f_1 \) is a linear combination of monomials \( \sim W_1 \), which the reader can quickly check.

As discussed in [2], \( S \) defines reduction maps on \( H^{RQ(C)}_{\mathcal{C}_2}(S^0)[\zeta_0, \zeta_1, c_{\omega}, c_{\chi\omega}] \):

Given \((W, f)\), we define the reduction \( r \) on monomials by defining \( r(AW) = Af \), \( r(A) = A \) if \( A \) is not a multiple of \( W \), and then extending linearly to polynomials. Write \( r_1, r_2, \) and \( r_3 \) for the reduction maps defined by \( \sigma_1, \sigma_2, \) and \( \sigma_3 \), respectively. The last thing we need to verify in order to use the diamond lemma is that all ambiguities of the reduction system are resolvable, meaning that, if \((W_i, f_i) \) and \((W_j, f_j) \) are pairs in our reduction system, with corresponding reductions \( r_i \) and \( r_j \), and \( W \) is the least common multiple of \( W_i \) and \( W_j \), then \( r_i(W) \) and \( r_j(W) \) can be further reduced to give the same polynomial. There are three cases we need to check:

1. \( \sigma_1 \) and \( \sigma_2 \): \( \zeta_0\zeta_1c_{\chi\omega} \) is the least common multiple of \( \zeta_0\zeta_1 \) and \( \zeta_1c_{\chi\omega} \). In this case, we have

\[
r_1(\zeta_0\zeta_1c_{\chi\omega}) = \xi c_{\chi\omega}
\]

and

\[
r_2(\zeta_0\zeta_1c_{\chi\omega}) = (1 - \kappa)\zeta_0^2c_{\omega} + e^2\zeta_0.
\]

Applying \( r_3 \) to the latter gives

\[
r_3r_2(\zeta_0\zeta_1c_{\chi\omega}) = (1 - \kappa)\xi c_{\chi\omega} + (1 - \kappa)e^2\zeta_0 + e^2\zeta_0 = \xi c_{\chi\omega},
\]

resolving the ambiguity. (This is why we need to include \( \sigma_3 \) in our reduction system.)

2. \( \sigma_1 \) and \( \sigma_3 \): \( \zeta_0^2\zeta_1c_{\omega} \) is the least common multiple in question, and

\[
r_1(\zeta_0^2\zeta_1c_{\omega}) = \xi c_{\omega}
\]

while

\[
r_3(\zeta_0^2\zeta_1c_{\omega}) = \xi \zeta_1c_{\chi\omega} + e^2\zeta_0\zeta_1.
\]

Applying \( r_1r_2 \) to the second gives

\[
r_1r_2r_3(\zeta_0^2\zeta_1c_{\omega}) = r_1(\xi\zeta_0c_{\omega} + e^2\xi + e^2\zeta_0\zeta_1)
\]
In each case, the monomials are listed in order so that the grading of each is either

\[ n < 2 \]

or

\[ n \]

resolving this ambiguity.

(3) \( \sigma_2 \) and \( \sigma_3 \): \( \zeta_0^2 \zeta_1 c_\omega c_{\chi_\omega} \) is the least common multiple,

\[ r_2(\zeta_0^2 \zeta_1 c_\omega c_{\chi_\omega}) = (1 - \kappa) \zeta_0^2 c_\omega + e^2 \zeta_0^2 c_{\chi_\omega} \]

and

\[ r_3(\zeta_0^2 \zeta_1 c_\omega c_{\chi_\omega}) = \xi \zeta_1 c_{\chi_\omega} + e^2 \zeta_0 \zeta_1 c_{\chi_\omega} \]

In this case, we find that

\[ r_3 p_2(\zeta_0^2 \zeta_1 c_\omega c_{\chi_\omega}) = \xi \zeta_0 c_\omega c_{\chi_\omega} - e^2 \zeta_0^3 c_\omega + e^2 \xi c_{\chi_\omega} + e^4 \zeta_0 \]

\[ = r_2 r_3(\zeta_0^2 \zeta_1 c_\omega c_{\chi_\omega}). \]

Thus, we can resolve all the ambiguities, which completes our verification of the hypotheses of the diamond lemma.

The conclusion of the diamond lemma is that, as a module over \( \text{H}_{C_2}(\Pi B) \), \( \text{H}_{C_2}(S_0)[\zeta_0, \zeta_1, c_\omega, c_{\chi_\omega}] \) is the direct sum of \( I \) and the submodule generated by the irreducible monomials, which are precisely those not divisible by \( W_1 \), \( W_2 \), or \( W_3 \). Thus, the quotient ring \( \text{F}^{RO(\Pi B)} \) is free on the images of the irreducible monomials, proving the theorem.

\[ \square \]

Definition 10.4. We call a monomial in \( \zeta_0, \zeta_1, c_\omega, \) and \( c_{\chi_\omega} \) basic if it is not a multiple of

- \( \zeta_0 \zeta_1 \),
- \( \zeta_1 c_{\chi_\omega} \), or
- \( \zeta_0^2 c_\omega \).

The preceding theorem says that the basic monomials form a basis for \( \text{F}^{RO(\Pi B)} \) over the cohomology of a point. In fact, the diamond lemma provides an algorithm for reducing any element to “normal form,” that is, to write any element in terms of basic monomials: Simply apply the reductions \( r_1 \), \( r_2 \), and \( r_3 \) in any order until no further reductions can be achieved. This process is guaranteed to stop after finitely many steps, with the end result not depending on the order in which the reductions are applied.

The following result lists the basic monomials more explicitly, organized by cosets of \( RO(C_2) \) in \( RO(\Pi B) \).

Corollary 10.5. If \( n > 0 \), then the basic monomials with gradings in \( n\omega + RO(C_2) \) are

\[ \{ \zeta_1^n, \zeta_1^{n-1} c_\omega, \ldots, \zeta_1 c_\omega^{n-1}, c_\omega^n, \zeta_0 c_\omega^{n+1}, c_\omega^{n+1} c_{\chi_\omega}, \zeta_0 c_\omega^{n+2} c_{\chi_\omega}, c_\omega^{n+2} c_{\chi_\omega}^2, \ldots \}. \]

The basic monomials with gradings in \( RO(C_2) \) are

\[ \{ 1, \zeta_0 c_\omega, c_\omega c_{\chi_\omega}, \zeta_0 c_\omega^2 c_{\chi_\omega}, c_\omega^2 c_{\chi_\omega}, \zeta_0 c_\omega^{3} c_{\chi_\omega}^2, \ldots \}. \]

If \( n < 0 \), the basic monomials with gradings in \( n\omega + RO(C_2) \) are

\[ \{ \zeta_0^n, \zeta_0^{n-1} c_{\chi_\omega}, \ldots, \zeta_0 c_{\chi_\omega}^{n-1}, c_{\chi_\omega}^n, \zeta_0 c_\omega c_{\chi_\omega}^{n+1}, c_\omega c_{\chi_\omega}^{n+1}, \zeta_0 c_\omega^2 c_{\chi_\omega}^{n+1}, c_\omega^2 c_{\chi_\omega}^{n+2}, \ldots \}. \]

In each case, the monomials are listed in order so that the grading of each is either 2 or \( 2\sigma \) larger than the grading of the one preceding it. \[ \square \]
The proof is an exercise in computing the gradings of the basic monomials and arranging them in order, and is omitted.

11. The cohomology of $B_{C_2}U(1)$

There is an algebra map

$$f: P^{RO(ΠB)} \to H^{RO(ΠB)}_{C_2}(B_+)$$

defined by taking each of the generators $ζ_0$, $ζ_1$, $c_ω$, and $c_{χω}$ to the element of the same name. Proposition 10.1 shows that this does define an algebra map. The goal of this section is to show that $f$ is an isomorphism.

There are several ways the proof could proceed. One would be to take the approach used in [3] and verify that the basis found in Theorem 10.3 maps to a basis for the nonequivariant cohomology of $B$ and to a basis for the nonequivariant cohomology of $B_{C_2}$. By the argument given in that paper, that suffices to show that these elements form a basis for the equivariant cohomology of $B$. This could be carried out using the explicit description of the basis given in Corollary 10.5.

We will give a different argument here, partly for variety but, more importantly, because this argument will generalize to allow a similar computation in Part 3, where the cohomology being computed is not free, hence the preceding argument would not work.

What we shall do is the following. Because $P^{RO(ΠB)}$ is a free $H^{RO(G)}_{C_2}(S^0)$-module, tensoring with the long exact sequence of the cofibration $(EC_2)_+ \to S^0 \to \tilde{EC}_2$ gives us a long exact sequence

$$\cdots \to P^{RO(ΠB)} \otimes H^{RO(C_2)}_{C_2}(S^0) H^{RO(C_2)}_{C_2} \tilde{EC}_2 \to P^{RO(ΠB)} \to \cdots$$

The map $f$ induces the following map of long exact sequences:

$$\cdots \to P^{RO(ΠB)} \otimes H^{RO(C_2)}_{C_2}(S^0) H^{RO(C_2)}_{C_2} \tilde{EC}_2 \to P^{RO(ΠB)} \to \cdots$$

We will show that both $f_{(EC_2)_+}$ and $f_{\tilde{EC}_2}$ are isomorphisms, so conclude that $f$ is an isomorphism. The first isomorphism is straightforward, the second is less so.
Proposition 11.1.

\[ f_{(EC_2)^+} : P^{\text{RO}(\text{IB})} \otimes H_{C_2}^{\text{RO}(C_2)}(S^0) H_{C_2}^{\text{RO}(C_2)}((EC_2)_+) \to H_{C_2}^{\text{RO}(\text{IB})}(B_+ \wedge_B (EC_2)_+) \]

is an isomorphism.

Proof. Write

\[ P^{\text{RO}(\text{IB})} \otimes H_{C_2}^{\text{RO}(C_2)}(S^0) H_{C_2}^{\text{RO}(C_2)}((EC_2)_+) \]

\[ \cong H_{C_2}^{\text{RO}(C_2)}((EC_2)_+)[c_0, c_1, c_\omega, c_\chi_\omega]/(c_0 c_1 - \xi, c_1 c_\chi_\omega - \xi c_\omega - c_\omega^2) \]

and remember that \( H_{C_2}^{\text{RO}(C_2)}((EC_2)_+) \cong H_{C_2}^{\text{RO}(C_2)}(S^0)[\xi^{-1}] \). (Note that \( \kappa = 0 \) and \( 2c_\omega^2 = 0 \) in this ring.) The relation \( c_0 c_1 = \xi \) tells us that both \( c_0 \) and \( c_1 \) are invertible in this ring and that

\[ c_0 = \xi c_1^{-1}. \]

Further, the second relation can be written as

\[ c_\chi_\omega = c_0 c_1^{-1} c_\omega + c_1^{-1} c_\omega^2 = \xi c_1^{-2} c_\omega + c_1^{-1} c_\omega^2. \]

Thus, we have that

\[ P^{\text{RO}(\text{IB})} \otimes H_{C_2}^{\text{RO}(C_2)}(S^0) H_{C_2}^{\text{RO}(C_2)}((EC_2)_+) \cong H_{C_2}^{\text{RO}(C_2)}((EC_2)_+)[c_\omega, \xi, c_1^{-1}]. \]

The isomorphism now follows from Corollary 8.4.

Proposition 11.2.

\[ f_{\tilde{EC}_2} : P^{\text{RO}(\text{IB})} \otimes H_{C_2}^{\text{RO}(C_2)}(S^0) H_{C_2}^{\text{RO}(\tilde{G})}((\tilde{EC}_2) \to H_{C_2}^{\text{RO}(\text{IB})}(B_+ \wedge_B \tilde{EC}_2) \]

is an isomorphism.

Proof. By Proposition 8.1, we can write \( f_{\tilde{EC}_2} \) as

\[ f_{\tilde{EC}_2} : P^{\text{RO}(\text{IB})} \otimes H_{C_2}^{\text{RO}(C_2)}(S^0) H_{C_2}^{\text{RO}(C_2)}((\tilde{EC}_2) \]

\[ \to H_{C_2}^{\text{RO}(C_2)}(S^0)[c, c_1, c_1^{-1}] \otimes H_{C_2}^{\text{RO}(C_2)}(S^0) H_{C_2}^{\text{RO}(C_2)}((\tilde{EC}_2) \]

\[ \oplus H_{C_2}^{\text{RO}(C_2)}(S^0)[c, \xi, c_1^{-1}] \otimes H_{C_2}^{\text{RO}(C_2)}(S^0) H_{C_2}^{\text{RO}(\tilde{G})}((\tilde{EC}_2) \]

and, by previous calculations, it is induced by the map

\[ P^{\text{RO}(\text{IB})} \to H_{C_2}^{\text{RO}(C_2)}(S^0)[c, c_1, c_1^{-1}] \oplus H_{C_2}^{\text{RO}(C_2)}(S^0)[c, \xi, c_1^{-1}] \]

given by

\[ c_0 \mapsto (\xi c_1^{-1}, c_0) \]

\[ c_1 \mapsto (c_1, c_0) \]

\[ c_\omega \mapsto (c_\xi_1, (c^2 + \xi c_\xi_0) c_\xi^{-1}) \]

\[ c_\chi_\omega \mapsto ((c^2 + \xi c_\xi_0) c_\xi^{-1}, c_\xi_0). \]

Because \( e \) acts by isomorphisms on \( H_{C_2}^{\text{RO}(C_2)}((\tilde{EC}_2) \), we may as well invert \( e \) on both sides of this map and consider the resulting map of modules over

\[ H_{C_2}^{\text{RO}(C_2)}(S^0)[c_\xi^{-1}] \cong (\mathbb{Z})[\xi, c_\xi^{-1}] / (2\xi). \]
where we find an explicit inverse. To show that we therefore consider the map as the continuous algebra map determined by 

\[
\xi = 2 \text{ after inverting } c,
\]

so that 2\,\xi = 0. It follows that, on applying \( \bar{f} \) \( \in \tilde{P} \), we have that \( \bar{f}^2 = 0 \). Hence, we can write \( \bar{f} \) as an isomorphism, it suffices to show that \( \bar{f} \) is an isomorphism. First calculate

\[
\bar{f} \circ \bar{f} = \bar{f}(c, c) + e^{-2\xi} \bar{f}(c, c),
\]

so

\[
\bar{f} \circ \bar{f} = (c, c) + e^{-2\xi} (c, c)^2,
\]

using that \( 2\xi = 0 \). It follows that, on applying \( \bar{f} \), the infinite sum in the definition of \( \bar{f} \) telescopes and converges to \((c, c)\). We then have

\[
\bar{f} \circ (c, -\xi_1 c) = (0, e^2 \xi_0^{-1})
\]

and

\[
\bar{f} \circ (c, -\xi_0 c) = (e^2 \xi_1^{-1}, 0).
\]

Define

\[
g : H_{C_2}^{RO(C_2)}(S^0)[e^{-1}][c, \xi_1, \xi_0] \oplus H_{C_2}^{RO(C_2)}(S^0)[e^{-1}][c, \xi_0, \xi_0] \rightarrow P^{RO(\Pi B)}[e^{-1}]_{\xi}.
\]

as the continuous algebra map determined by

\[
g(1, 0) = e^{-2\xi_1}(c, -\xi_0 c)
\]

\[
g(\xi_1, 0) = e^{-2\xi_1^2}(c, -\xi_0 c)
\]

\[
g(\xi_1^{-1}, 0) = e^{-2}(c, -\xi_0 c)
\]

\[
g(c, 0) = e^{-2\xi_1} c, (c, -\xi_0 c)
\]
\[ \tilde{g}(0, 1) = e^{-2} \zeta_0 (c_\omega - \zeta_1 c) \]
\[ \tilde{g}(0, \zeta_0) = e^{-2} \zeta_0^2 (c_\omega - \zeta_1 c) \]
\[ \tilde{g}(0, \zeta_0^{-1}) = e^{-2} (c_\omega - \zeta_1 c) \]
\[ \tilde{g}(0, c) = e^{-2} \zeta_0 c (c_\omega - \zeta_1 c) . \]

It is now a matter of calculation to check that \( \tilde{g} \) is the inverse of \( \tilde{f}_E \).

Finally, we get to our main result.

**Theorem 11.3.** \( H_{C_2}^{RO(\Pi B)}(B_+) \) is the commutative \( H_{C_2}^{RO(C)}(S^0) \)-algebra generated by the elements \( \zeta_0, \zeta_1, c_\omega \), and \( c_{\chi \omega} \) subject to the two relations
\[ \zeta_0 \zeta_1 = \xi \]
and
\[ \zeta_1 c_\omega = (1 - \kappa) \zeta_0 c_\omega + e^2 . \]

It is free as a module over \( H_{C_2}^{RO(C)}(S^0) \) on the basic monomials given in Definition 10.4 and enumerated in Corollary 10.5.

**Proof.** It follows from Propositions 11.1 and 11.2 and the comparison of long exact sequences outlined at the beginning of this section that \( f_+ : P^{RO(\Pi B)} \to H_{C_2}^{RO(\Pi B)}(B_+) \) is a ring isomorphism. The theorem then follows from Theorem 10.3.

\[ \square \]

**Corollary 11.4.** The pairings
\[ H_{C_2}^{RO(\Pi B)}(B_+) \otimes H_{C_2}^{RO(C)}(S^0) \to H_{C_2}^{RO(\Pi B)}(B_+ \wedge_B (EC_2)_+) \]
and
\[ H_{C_2}^{RO(\Pi B)}(B_+) \otimes H_{C_2}^{RO(C)}(\tilde{EC}_2) \to H_{C_2}^{RO(\Pi B)}(B_+ \wedge_B \tilde{EC}_2) \]
are isomorphisms. The long exact sequence for the cofibration \( B_+ \wedge_B (EC_2)_+ \to B_+ \to B_+ \wedge_B \tilde{EC}_2 \) is isomorphic to the long exact sequence for the cofibration \( (EC_2)_+ \to S^0 \to \tilde{EC}_2 \) tensored with \( H_{C_2}^{RO(\Pi B)}(B_+) \).

**Proof.** This follows from the isomorphism \( P^{RO(\Pi B)} \cong H_{C_2}^{RO(\Pi B)}(B_+) \) and the isomorphism of the long exact sequences displayed above Proposition 11.1.

\[ \square \]

The cohomology group \( H_{C_2}^0(B_+) \) has an interesting structure. Additively, it is \( A \oplus \langle \mathbb{Z} \rangle \), where the second summand is generated by
\[ \epsilon = e^{-2} \kappa \zeta_0 c_\omega . \]

The following result is needed in the computation to follow of the Euler classes of the duals of \( \omega \) and \( \chi \omega \).

**Proposition 11.5.** The units in \( H_{C_2}^0(B_+) \) are
\[ \pm 1, \pm (1 - \kappa), \pm (1 - \epsilon), \text{ and } \pm (1 - \kappa + \epsilon) = \pm (1 - \kappa)(1 - \epsilon) . \]

Each of these elements squares to 1.
Proof. We know that $\kappa^2 = 2\kappa$ and we also have
\[
e^2 = (e^{-2}\kappa_0 c_\omega)^2
= 2e^{-4}\kappa(\kappa_0 c_\omega)^2
= 2e^{-4}\kappa_0 c_\omega(e^2 + (1 - \kappa)\zeta_1 c_{\chi_\omega})
= 2e^{-2}\kappa_0 c_\omega + 2(1 - \kappa)e^{-4}\kappa_0 c_\omega c_{\chi_\omega}
= 2e^{-2}\kappa_0 c_\omega
= 2e,
\]
using the fact that $e^{-4}\kappa_0 \xi = 0$. From this it follows that $(1 - \kappa)^2 = 1$ and $(1 - \epsilon)^2 = 1$. We also then have that $[(1 - \kappa)(1 - \epsilon)]^2 = 1$, but $(1 - \kappa)(1 - \epsilon) = 1 - \kappa + \epsilon$ because $\kappa \epsilon = 2\epsilon$. Therefore, the elements listed all square to 1 and are all units.

On the other hand, an arbitrary element of $H^0 G(B +)$ can be written as a sum $a + b\kappa + c\epsilon$ for integers $a$, $b$, and $c$, and consideration of when two such elements can multiply to give 1 leads to the conclusion that only the eight elements above can be units.

Notice that
\[
\kappa - \epsilon = \kappa - e^{-2}\kappa_0 c_\omega
= \kappa - e^{-2}\kappa((1 - \kappa)\zeta_1 c_{\chi_\omega} + e^2)
= e^{-2}\kappa_1 c_{\chi_\omega}
= \chi^2,
\]
so, where $\epsilon$ appears above, we can also expect to see $\kappa - \epsilon$, and we do.

Although the tautological bundle $\omega$ is a natural thing to consider, in some applications, and in algebraic geometry, the dual of $\omega$ is important as well. An example is when considering finite complex projective spaces as in [3]. We write $\omega^\vee = \text{Hom}(\omega, \mathbb{C})$.

Note that
\[
(\chi \omega)^\vee = \chi(\omega^\vee) = \text{Hom}(\omega, \mathbb{C}^\sigma),
\]
so we can write $\chi \omega^\vee$ unambiguously. For applications we need to know the Euler classes of these bundles.

**Proposition 11.6.**
\[
e(\omega^\vee) = -(1 - \epsilon)c_\omega
\]
and
\[
e(\chi \omega^\vee) = -(1 - \kappa)(1 - \epsilon)c_{\chi_\omega}.
\]

**Proof.** The bundle $\omega^\vee$ gives the same representation of $\Pi B$ as does $\omega$, so $e(\omega^\vee)$ lies in grading $\omega$. In this grading, $\eta: H^{\omega}_{C_2}(B^+) \to H^{\omega}_{C_2}(B^C_+)$ is monomorphic, so it suffices to show the desired equality in $H^{\omega}_{C_2}(B^C_+)$. Consider $\omega^\vee|B^0 = (\omega|B^0)^\vee$. As is true nonequivariantly, $\omega^\vee$ is the same underlying real bundle as $\omega$ but with the complex structure conjugated. The bundle $\omega|B^0$ is the nonequivariant tautological bundle over $\mathbb{C}P^\infty$ with trivial $C_2$-action on its fiber, so conjugating the $C_2$-action on the fibers has the same effect on the Euler class as locally applying an orientation-reversing map $\mathbb{R}^2 \to \mathbb{R}^2$ on each fiber. Thinking of the effect on the Thom class,
because the resulting map \( S^2 \to S^2 \) represents \(-1\) in \( A(C_2) \), the Thom class of \( \omega^\vee|B^0 \) is the negative of the Thom class of \( \omega|B^0 \), hence the same is true for the Euler classes. That is,

\[
e(\omega^\vee)|B^0 = e(\omega^\vee|B^0) = -e(\omega|B^0).
\]

On the other hand, \( \omega|B^1 \) is the nonequivariant tautological bundle with each fiber isomorphic to \( C^\sigma = \mathbb{R}^{2\sigma} \) as a representation of \( C_2 \). Conjugating the complex structure amounts locally to applying an orientation-reversing map \( \mathbb{R}^{2\sigma} \to \mathbb{R}^{2\sigma} \) on each fiber. The resulting map \( S^{2\sigma} \to S^{2\sigma} \) represents the unit \( \kappa - 1 \) in \( A(C_2) \) rather than \(-1\) (see the discussion at the end of §1), so

\[
e(\omega^\vee)|B^1 = (\kappa - 1) e(\omega|B^1).
\]

Putting together \( e(\omega^\vee)|B^0 \) and \( e(\omega^\vee)|B^1 \), we get

\[
\eta(e(\omega^\vee)) = (-1, \kappa - 1) \eta(c_\omega).
\]

We also calculate

\[
\eta(1 - \epsilon) = \eta(1 - e^{-2} \kappa \zeta_0 c_\omega) = (1, 1) - (0, \kappa) = (1, 1 - \kappa),
\]

so

\[
\eta(-1 - \epsilon) c_\omega = (-1, \kappa - 1) \eta(c_\omega) = \eta(e(\omega^\vee)),
\]

showing that \( e(\omega^\vee) = -(1 - \epsilon) c_\omega \) as claimed.

The second equality follows on applying \( \chi \):

\[
e(\chi \omega^\vee) = \chi e(\omega^\vee) = \chi(-1 - \epsilon) c_\omega = -(1 - \kappa + \epsilon) c_{\chi \omega} = -(1 - \kappa)(1 - \epsilon) c_{\chi \omega}.
\]

We can look at the calculation of the preceding proposition in the following way as well. Recalling that

\[
\eta(c_\omega) = (c_\zeta_1, (e^2 + \xi c) \zeta_0^{-1})
\]

and

\[
\eta(c_{\chi \omega}) = ((e^2 + \xi c) \zeta_1^{-1}, c_\zeta_0),
\]

we have

\[
\eta(e(\omega^\vee)) = -(1 - \epsilon) \eta(c_\omega) = (-c_\zeta_1, (e^2 - \xi c) \zeta_0^{-1})
\]

and

\[
\eta(e(\chi \omega^\vee)) = -(1 - \kappa)(1 - \epsilon) \eta(c_{\chi \omega}) = ((e^2 - \xi c) \zeta_1^{-1}, -c_\zeta_0).
\]

Dualizing negates the nonequivariant Chern class \( c \), but multiplies the Euler class \( e^2 \) of \( C^\sigma \) by \( \kappa - 1 \), which leaves it unchanged.

12. Comparison to Lewis’s Calculation in \( RO(C_2) \)-Grading

In [16], Gaunce Lewis calculated \( H^{RO(C_2)}_2(B_+) \), the \( RO(C_2) \)-graded part of the cohomology of \( B \). We here recover his results from ours.

Part of Lewis’s Theorem 5.1 shows that \( H^{RO(C_2)}_2(B_+) \) is generated by two elements we shall call \( \gamma \in H^{2\sigma}_2(B_+) \) (Lewis’s \( c \)) and \( \Gamma \in H^{2\sigma+2}_2(B_+) \) (Lewis’s \( C(1) \)), with the single relation

\[
\gamma^2 = e^2 \gamma + \xi \Gamma.
\]

This results in an additive basis consisting of

\[
1, \gamma, \Gamma, \gamma \Gamma, \Gamma^2, \gamma \Gamma^2, \ldots
\]

Our additive calculation gave a basis for \( H^{RO(C_2)}_2(B_+) \) consisting of

\[
1, \zeta_0 c_\omega, c_\omega c_{\chi \omega}, \zeta_0 c^2_\omega c_{\chi \omega}, c^2_\omega c^2_{\chi \omega}, \zeta_0 c^3_\omega c_{\chi \omega}, \ldots
\]
(See Corollary 10.5.) If we let $\gamma = \zeta_0 c_\omega$ and $\Gamma = c_\omega c_{\chi_0}$, we recover Lewis’s generators. Moreover, his relation between these generators can be seen as follows:

$$\gamma^2 = (\zeta_0 c_\omega)^2$$
$$= \zeta_0 c_\omega (e^2 + (1 - \kappa) \zeta_1 c_{\chi_0})$$
$$= e^2 \zeta_0 c_\omega + (1 - \kappa) \xi c_\omega c_{\chi_0}$$
$$= e^2 \gamma + \xi \Gamma,$$

using the fact that $(1 - \kappa) \zeta_0 \zeta_1 = (1 - \kappa) \xi = \xi$.

In his Remark 5.3, Lewis introduces an element $\tilde{\gamma} = e^2 + (1 - \kappa) \gamma$ (his $\bar{c}$) and points out that $\tilde{\gamma}$ could be used as a generator in place of $\gamma$. In fact, $\tilde{\gamma} = \zeta_1 c_{\chi_0}$, and his equation relating $\gamma$ and $\tilde{\gamma}$ is our basic relation between $\zeta_0 c_\omega$ and $\zeta_1 c_{\chi_0}$.

In [3] we also generalize Lewis’s calculation for finite projective spaces to the $RO(\Pi B)$-grading. In that case, our description is considerably simpler than the one he obtained.

13. Cohomology with other coefficient systems

The fact that the cohomology of $B$ with $A$ coefficients is a free $H_{G}^{RO(G)}(S^0)$-module allows us to calculate its cohomology with any coefficient system. We begin with a general result along the lines of Adams’ splitting [1]. For the next several results, we let $G$ be any finite group, $\alpha \in RO(\Pi B)$, and let $X \to B$ be any ex-$G$-space. By [5, §3.7], both $H^{G+RO(G)}_{\alpha}(-; A)$ and $H^{G+RO(G)}_{\alpha}(-; A)$ are represented by a spectrum $H_{A}^{\alpha}$ parametrized by $B$. Let $\rho: B \to *$ denote the map to a $G$-fixed point.

**Proposition 13.1.** Suppose that $H^{G}_{\alpha+RO(G)}(X; A)$ is free as a module over the ring $H^{G}_{RO(G)}(S^0; A)$. Then the (nonparametrized) spectrum $\rho_{!}(X \wedge_{B} H_{A}^{\alpha})$ is equivalent to a wedge of suspensions of the spectrum $H_{A}^{\alpha}$ representing $RO(G)$-graded cohomology.

Similarly, if $H^{G+RO(G)}_{\alpha}(X; A)$ is free as a module over $H^{G+RO(G)}_{\alpha}(S^0; A)$, then the spectrum $F_{B}(X, H_{A}^{\alpha})$ is equivalent to a wedge of suspensions of $H_{A}^{\alpha}$.

**Proof.** Recall from [5] that

$$H^{G}_{\alpha+\beta}(X; A) \cong \{S^\beta, \rho_{!}(X \wedge_{B} H_{A}^{\alpha})\}_{G}$$

for $\beta \in RO(G)$. Take a basis for $H^{G}_{\alpha+RO(G)}(X; A)$ over $H^{G}_{RO(G)}(S^0; A)$ and represent each basis element as a map $S^\beta \to \rho_{!}(X \wedge_{B} H_{A}^{\alpha})$. This gives a map

$$\bigvee S^\beta \to \rho_{!}(X \wedge_{B} H_{A}^{\alpha}),$$

where the wedge is taken over the basis. Using the fact that $H_{A}^{\alpha}$ is a module over $H_{A}$ (or $\rho^* H_{A}$), we then have the composite

$$\bigvee S^\beta \wedge H_{A} \to \rho_{!}(X \wedge_{B} H_{A}^{\alpha}) \wedge H_{A} \to \rho_{!}(X \wedge_{B} H_{A}^{\alpha}),$$

which is a weak equivalence.

The proof for cohomology is similar, using

$$H^{G+\beta}_{\alpha}(X; A) \cong \{S^\alpha, F_{B}(X, H_{A}^{\alpha})\}_{G}. \quad \square$$
Proposition 13.2. Let $G$ be any finite group, let $X \rightarrow B$ be an ex-$G$-space, and suppose that $H^G_{\alpha+RO(G)}(X;A)$ is a free module over $H^G_{RO(G)}(S^0;A)$ for an $\alpha \in RO(\Pi B)$. If $T$ is any Mackey functor, then

$$H^G_{\alpha+RO(G)}(X;T) \cong H^G_{\alpha+RO(G)}(X;A) \otimes_{H^G_{RO(G)}(S^0;A)} H^G_{RO(G)}(S^0;T).$$

Similarly, if $H^{\alpha+RO(G)}_G(X;A)$ is a free module over $H^G_{RO(G)}(S^0;A)$, then

$$H^{\alpha+RO(G)}_G(X;T) \cong H^{\alpha+RO(G)}_G(X;A) \otimes_{H^G_{RO(G)}(S^0;A)} H^G_{RO(G)}(S^0;T).$$

Proof. Consider homology, where we have the representing spectrum $HT^\alpha$, meaning that

$$H^G_{\alpha+\beta}(X;T) \cong [S^\beta, \rho_!(X \wedge_B HT^\alpha)]_G$$

for $\beta \in RO(G)$. Using the preceding proposition, we have

$$\rho_!(X \wedge_B HT^\alpha) \cong \rho_!(X \wedge_B H A^\alpha \wedge H \rho^* HT^\alpha)$$

$$\cong \rho_!(X \wedge_B H A^\alpha) \wedge_H HT$$

$$\cong (\bigvee S^\beta \wedge H A) \wedge_H HT$$

with the wedge taken over a basis for $H^G_{\alpha+RO(G)}(X;A)$ over $H^G_{RO(G)}(S^0;A)$. This implies the algebraic claim of the proposition.

The proof for cohomology is similar, using $F_B(X, HT^\alpha)$.

Returning to the case $G = C_2$ and $B = CP^\infty_C$, we can now easily compute the cohomology of $B$ with any coefficient system. The one that is probably of most interest is the constant $\mathbb{Z}$ functor $\mathbb{Z}$. Refer to Theorem 5.3 for the structure of $H^{RO(C_2)}_{C_2}(S^0;\mathbb{Z})$.

Theorem 13.3. $H^{RO(\Pi B)}_{C_2}(B_+;\mathbb{Z})$ is the commutative $H^{RO(C_2)}_{C_2}(S^0;\mathbb{Z})$-algebra generated by the elements $\zeta_0$, $\zeta_1$, $e_\omega$, and $c_{\chi_\omega}$ subject to the two relations

$$\zeta_0\zeta_1 = \xi$$

and

$$\zeta_1c_{\chi_\omega} = \zeta_0c_\omega + e^2.$$

It is free as a module over $H^{RO(C_2)}_{C_2}(S^0;\mathbb{Z})$.

Proof. It follows from Proposition 13.2 that

$$H^{RO(\Pi B)}_{C_2}(B_+;\mathbb{Z}) \cong H^{RO(C_2)}_{C_2}(B_+;\mathbb{Z}) \otimes_{H^{RO(C_2)}_{C_2}(S^0;\mathbb{Z})} H^{RO(C_2)}_{C_2}(S^0;\mathbb{Z}).$$

By the proof of Theorem 5.3, $H^{RO(C_2)}_{C_2}(S^0;\mathbb{Z})$ is the quotient of $H^{RO(C_2)}_{C_2}(S^0)$ by the ideal consisting of the elements $e^k\kappa$, $k \in \mathbb{Z}$. (Recall that $e^k\kappa = 2e^k$ for $k > 0$.)

The result then follows readily from Theorem 11.3, noting that the second relation simplifies because $\kappa = 0$ in the quotient.

Corollary 5.4 implies calculations of $H^*_C(B_+;T)$ with $T$ equal to $\mathbb{Z}'$, $\mathbb{Z}'$, or $\mathbb{Z}'$. Each of these is probably best understood as a shifted version of $H^*_C(B_+;\mathbb{Z})$, that is, as a free module over $H^*_C(B_+;\mathbb{Z})$ on a single generator.
Part 3. Invariants of the component structure

14. The space $K(2)$

The elements $\zeta_0$ and $\zeta_1$ that appear in the cohomology of $BC_2U(1)$ are examples of a much more general phenomenon. They come from the cohomology of another space $K(2)$ whose cohomology we will compute. Its structure as a space is fairly simple.

Definition 14.1. Let $K = K(2)$ be a $C_2$-space of the homotopy type of a $C_2$-CW complex such that $K$ is nonequivariantly contractible and $K^{C_2}$ consists of two contractible components.

Such a $C_2$-space exists and is unique up to $C_2$-homotopy equivalence using Elmendorf’s construction [9]. We could similarly construct spaces $K(n)$ with $K(n)^{C_2}$ having $n$ components, but the case $n = 2$ will suffice for our purposes here. We could also construct $K$ as the classifying space $B\Pi C_2C_2U(1)$ using the construction given in [4, 24.1].

Remark 14.2. Another model for $K(2)$ is $\tilde{EC}_2$ considered as an unbased space. We computed the $RO(C_2)$-graded (reduced) cohomology of $\tilde{EC}_2$ in Theorem 4.7; using a retraction $S^0 \to (\tilde{EC}_2)_+ \to S^0$, we see that the cohomology of $(\tilde{EC}_2)_+$ is the direct sum of the cohomology of $\tilde{EC}_2$ and the cohomology of a point, in the $RO(C_2)$ grading. However, this argument does not extend to other gradings because the projection $(\tilde{EC}_2)_+ \to S^0$ does not induce a map in cohomology in those other gradings. Instead, we need an argument similar to the one we used in calculating the cohomology of $BC_2U(1)$. We use the notation $K(2)$ to distinguish this unbased space from the based space $\tilde{EC}_2$.

Write $K^{C_2} = K^0 \sqcup K^1$. Elmendorf’s construction also gives us a $C_2$-map $B = BC_2U(1) \to K$, unique up to $C_2$-homotopy, such that $B^0$ maps to $K^0$ and $B^1$ maps to $K^1$. This map induces an isomorphism $RO(\Pi K) \cong RO(\Pi B)$, so we will use the same notation for elements of $RO(\Pi K)$ as we did for $RO(\Pi B)$.

The calculation of $H^R_{C_2}(K^0_+)$ is similar to the calculation of the cohomology of $B$, with some parts much simpler but some parts, including the final result, actually more complicated. We begin with some of the simple calculations.

Proposition 14.3. For $k = 0$ or 1, we have the following calculation:

\[
H^R_{C_2}(K^k_+) \cong H^R_{C_2}(S^0)[\zeta_{1-k}, \zeta_{1-k}^{-1}]
\]

\[
H^R_{C_2}(K^k_+ \wedge_K (EC_2)_+) \cong H^R_{C_2}(K^k_+) \otimes H^R_{C_2}(S^0) H^R_{C_2}(EC_2)_+
\]

\[
H^R_{C_2}(K^k_+ \wedge_K \tilde{EC}_2) \cong H^R_{C_2}(K^k_+) \otimes H^R_{C_2}(S^0) H^R_{C_2}(\tilde{EC}_2),
\]

where $|\zeta_i| = \Omega_i$. Further,

\[
H^R_{C_2}(K^{C_2}_+) \cong H^R_{C_2}(K^0_+) \oplus H^R_{C_2}(K^1_+)
\]

\[
H^R_{C_2}(K^{C_2}_+ \wedge_K (EC_2)_+) \cong H^R_{C_2}(K^{C_2}_+) \otimes H^R_{C_2}(S^0) H^R_{C_2}(EC_2)_+
\]

\[
H^R_{C_2}(K^{C_2}_+ \wedge_K \tilde{EC}_2) \cong H^R_{C_2}(K^{C_2}_+) \otimes H^R_{C_2}(S^0) H^R_{C_2}(\tilde{EC}_2).
\]

Proof. The proof is essentially the same as that of Proposition 7.4, simplified by the fact that each $K^k$ is contractible. 

Also simple are the cohomologies of $K_+ \wedge_K (EC_2)_+$ and $K_+ \wedge_K \tilde{EC}_2$.

**Proposition 14.4.** We have

$$H_{C_2}^{RO(\Pi K)}(K_+ \wedge_K (EC_2)_+) \cong H_{C_2}^{RO(C_2)}((EC_2)_+)[\zeta_0, \zeta_1]/\langle \zeta_0 \zeta_1 = \xi \rangle.$$

The long exact sequence of the pair $(K, K^{C_2})$ reduces to a split short exact sequence

$$0 \to H_{C_2}^{RO(\Pi K)}(K_+ \wedge_K (EC_2)_+) \overset{\eta}{\to} H_{C_2}^{RO(\Pi K)}(K^{C_2}_+ \wedge_K (EC_2)_+) \to H_{C_2}^{RO(\Pi K)}(K/K^{C_2} \wedge_K (EC_2)_+) \to 0$$

with

$$\eta(\zeta_0) = (\xi_1^{-1}, \zeta_0) \quad \text{and} \quad \eta(\zeta_1) = (\zeta_1, \xi_0^{-1}).$$

The proof is the similar to the proofs of the corresponding statements about $B_+ \wedge_B (EC_2)_+$ in §8, but simpler.

Note that, because $\xi$ is invertible in $H_{C_2}^{RO(C_2)}((EC_2)_+)$, the relation $\zeta_0 \zeta_1 = \xi$ implies that both $\zeta_0$ and $\zeta_1$ are invertible.

**Proposition 14.5.** The inclusion $K^{C_2} \to K$ induces an isomorphism

$$H_{C_2}^{RO(\Pi K)}(K_+ \wedge_K \tilde{EC}_2) \cong H_{C_2}^{RO(\Pi K)}(K^{C_2}_+ \wedge_K \tilde{EC}_2) \cong (H_{C_2}^{RO(C_2)}(S^0)[\zeta_1, \zeta_1^{-1}] \oplus H_{C_2}^{RO(C_2)}(S^0)[\zeta_0, \zeta_0^{-1}]) \oplus H_{C_2}^{RO(C_2)}(\tilde{EC}_2).$$

**Proof.** As in Proposition 8.1, this is just the observation that the inclusion induces a weak equivalence $K^{C_2}_+ \wedge \tilde{EC}_2 \to K_+ \wedge \tilde{EC}_2$. \qed

15. THE COHOMOLOGY OF $K(2)$

Unlike $B$, the cohomology of $K$ is not a free module over $H_{C_2}^{RO(C_2)}(S^0)$. This is where the argument gets more complicated.

Diagram (6.3) works equally well with $K$ in place of $B$. By the same argument used for the cohomology of $B$, we can find elements $\zeta_i \in H_{C_2}(K_+), k = 0$ and $1$, characterized by

$$\eta(\zeta_0) = (\xi_1^{-1}, \zeta_0) \quad \text{and} \quad \eta(\zeta_1) = (\xi_1, \xi_0^{-1}).$$

These map to the elements of $H_{C_2}^{RO(\Pi K)}(K_+ \wedge_K (EC_2)_+)$ of the same names that we saw in the preceding section. We have again that $\zeta_0 \zeta_1 = \xi$.

We define

$$Q^{RO(\Pi K)} = H_{C_2}^{RO(C_2)}(S^0)[\zeta_0, \zeta_1]/\langle \zeta_0 \zeta_1 = \xi \rangle.$$

The following is easily proved.

**Proposition 15.1.** $Q^{RO(\Pi K)}$ is a free $H_{C_2}^{RO(C_2)}(S^0)$-module on a basis consisting of the elements $1$, $\zeta_0^n$ for $n \geq 1$, and $\zeta_1^n$ for $n \geq 1$. \qed
To have a simple way of referring to this basis, we make the following definition.

**Definition 15.2.** We say that a monomial $\zeta_0^m \zeta_1^n$ is basic if $m = 0$ or $n = 0$.

These basis elements are distributed nicely:

**Lemma 15.3.** Let $\alpha \in \text{RO}(\Pi K)$. Then there is exactly one basic monomial with grading in the coset $\alpha \in \text{RO}(\Pi K)$.

**Proof.** Write $\alpha = m\Omega_0 + n\Omega_1 + \gamma$ where $\gamma \in \text{RO}(C_2)$. If $m \geq n$, then

$$m\Omega_0 + n\Omega_1 + \gamma = (m-n)\Omega_0 + \gamma + n(2\sigma - 2),$$

and the only basic monomial having grading in $(m-n)\Omega_0$ is $\zeta_0^m \zeta_1^n$.

Similarly, if $m < n$, the only basic monomial with grading in $\alpha$ is $\zeta_0^n \zeta_1^m$.

□

Consider the algebra map $f : Q_{\text{RO}(\Pi K)} \to H_{C_2}^{\text{RO}(\Pi K)}(K_+)$ given by taking each $\zeta_i$ to the element of the same name. As in the case of $B$, we have a map of long exact sequences:

$$\cdots \to Q_{\text{RO}(\Pi K)} \otimes H_{C_2}^{\text{RO}(C_2)}(S^0) \to H_{C_2}^{\text{RO}(C_2)}(\tilde{E}C_2) \to f_{EC_2} H_{C_2}^{\text{RO}(\Pi K)}(K_+ \wedge K \tilde{E}C_2) \to H_{C_2}^{\text{RO}(\Pi K)}(K_+ \wedge K \tilde{EC}_2) \to \cdots$$

We shall show that $f_{(EC_2)_+}$ is again an isomorphism, but in this case $f_{\tilde{E}C_2}$ is not, and we shall see where that leads us.

**Proposition 15.4.**

$$f_{(EC_2)_+} : Q_{\text{RO}(\Pi K)} \otimes H_{C_2}^{\text{RO}(C_2)}(S^0) H_{C_2}^{\text{RO}(C_2)}((EC_2)_+) \to H_{C_2}^{\text{RO}(\Pi K)}(K_+ \wedge K (EC_2)_+)$$

is an isomorphism.

**Proof.** We have that

$$Q_{\text{RO}(\Pi K)} \otimes H_{C_2}^{\text{RO}(C_2)}(S^0) H_{C_2}^{\text{RO}(C_2)}((EC_2)_+) \cong H_{C_2}^{\text{RO}(C_2)}((EC_2)_+) / \langle \zeta_0 \zeta_1 = \xi \rangle,$$

and the result now follows from Proposition 14.4. □
Proposition 15.5.

$$f_{\tilde{E}C_2} : \mathcal{Q}^{RO(\Pi K)} \otimes_{\mathcal{H}_{C_2}^{RO}(S^0)} \mathcal{H}_{C_2}^{RO}(\tilde{E}C_2) \to \mathcal{H}_{C_2}^{RO(\Pi K)}(K_+ \wedge K \tilde{E}C_2)$$

is a monomorphism, but not an isomorphism. It is split as a map of modules over $\mathcal{H}_{C_2}^{RO}(C_2)(S^0)$.

Proof. Consider a grading $m\Omega_0 + RO(C_2)$, $m \geq 0$, and the corresponding basic monomial $c_0^m$ that lives in this grading. (The case of $n\Omega_1 + RO(C_2)$ is similar.)

From the calculation in the preceding section, in gradings $m\Omega_0 + RO(C_2)$ the target of $f_{\tilde{E}C_2}$ is

$$\left(\mathcal{H}_{C_2}^{RO(C_2)}(S^0)\{c_0^{-m}\} \oplus \mathcal{H}_{C_2}^{RO(C_2)}(S^0)\{c_0^m\}\right) \otimes \mathcal{H}_{C_2}^{RO(C_2)}(\tilde{E}C_2).$$

If $x \in \mathcal{H}_{C_2}^{RO(C_2)}(\tilde{E}C_2)$, we have

$$f_{\tilde{E}C_2}(c_0^m \otimes x) = (\xi^m c_0^{-m} \otimes x, c_0^m \otimes x).$$

From this we can see that projection to the second summand is a splitting of $f_{\tilde{E}C_2}$ as a map of $\mathcal{H}_{C_2}^{RO(C_2)}(S^0)$-modules in gradings $m\Omega_0 + RO(C_2)$.

The cokernel of $f_{\tilde{E}C_2}$ in these gradings is isomorphic to the first summand, so is nontrivial, hence $f_{\tilde{E}C_2}$ is not an isomorphism.

We can now calculate the cohomology of $K$.

Theorem 15.6. As a module over $\mathcal{H}_{C_2}^{RO(C_2)}(S^0)$, $\mathcal{H}_{C_2}^{RO(\Pi K)}(K_+)$ is the pushout in the diagram

$$\begin{array}{ccc}
\mathcal{Q}^{RO(\Pi K)} \otimes_{\mathcal{H}_{C_2}^{RO}(S^0)} \mathcal{H}_{C_2}^{RO}(\tilde{E}C_2) & \longrightarrow & \mathcal{Q}^{RO(\Pi K)} \\
f_{\tilde{E}C_2} \downarrow & & \downarrow f \\
\mathcal{H}_{C_2}^{RO(\Pi K)}(K_+ \wedge K \tilde{E}C_2) & \longrightarrow & \mathcal{H}_{C_2}^{RO(\Pi K)}(K_+) \\
\end{array}$$

in which the vertical arrows are both split monomorphisms. So $\mathcal{H}_{C_2}^{RO(\Pi K)}(K_+)$ splits as a direct sum with $\mathcal{Q}^{RO(\Pi K)}$ as one summand, but the other summand is isomorphic to the direct sum of countably infinitely many copies of $\mathcal{H}_{C_2}^{RO(C_2)}(\tilde{E}C_2)$, with one copy in each coset of gradings $\alpha + RO(C_2) \subset RO(\Pi K)$.

As an algebra over $\mathcal{H}_{C_2}^{RO(C_2)}(S^0)$, $\mathcal{H}_{C_2}^{RO(\Pi K)}(K_+)$ is generated by elements $\zeta_0, \zeta_1$, with $\deg \zeta_0 = \Omega_0$ and $\deg \zeta_1 = \Omega_1$, and

$$x\psi_0^m, x\psi_1^m \quad x \in \mathcal{H}_{C_2}^{RO(C_2)}(\tilde{E}C_2), \quad m \in \mathbb{Z}$$

with $\deg x\psi_0^m = \deg x + m\Omega_1$ and $\deg x\psi_1^m = \deg x + m\Omega_0$. All relations among these generators are consequences of the following relations:

$$\zeta_0 \zeta_1 = \xi$$
$$y(x\psi_k^m) = (yx)\psi_k^m \quad \text{if } y \in \mathcal{H}_{C_2}^{RO(C_2)}(S^0)$$
$$(x\psi_0^m)(y\psi_0^n) = xy\psi_0^{m+n}$$
$$(x\psi_1^m)(y\psi_1^n) = xy\psi_1^{m+n}$$
\[(x\psi_0^m)(y\psi_1^n) = 0\]
\[\zeta_0 \cdot x\psi_0^m = \xi x\psi_0^{m-1}\]
\[\zeta_1 \cdot x\psi_0^m = x\psi_0^{m+1}\]
\[\zeta_0 \cdot x\psi_1^m = x\psi_1^{m+1}\]
\[\zeta_1 \cdot x\psi_1^m = \xi x\psi_1^{m-1}\]
\[\psi(x)\zeta_0 = \xi x\psi_0^{-1} + x\psi_1^1 \quad \text{for } x \in H_{C_2}^{RO(C_2)}(\tilde{E}C_2)\]
\[\psi(x)\zeta_1 = x\psi_1^1 + \xi x\psi_1^{-1} \quad \text{for } x \in H_{C_2}^{RO(C_2)}(\tilde{E}C_2)\]

In the last two relations, \(\psi(x)\) is the image of \(x\) under the map

\[\psi: H_{C_2}^{RO(C_2)}(\tilde{E}C_2) \to H_{C_2}^{RO(C_2)}(S^0).\]

**Proof.** In the map of long exact sequences displayed after Definition 15.2, we now know that \(f_{EC_2+}\) is an isomorphism and \(f_{\tilde{E}C_2}\) is a monomorphism. A diagram chase shows that these imply that the square involving \(f_{\tilde{E}C_2}\) and \(f\) is a pushout square, as claimed in the theorem. Moreover, because \(f_{\tilde{E}C_2}\) is a split monomorphism, its pushout, \(f\), is as well.

Note that the cokernel of \(f\) is therefore isomorphic to the cokernel of \(f_{EC_2}\), which is a direct sum of countably infinitely many copies of \(H_{C_2}^{RO(C_2)}(\tilde{E}C_2)\), with one copy in each coset of gradings \(\alpha + RO(C_2) \subset RO(\Pi K)\).

The analogue of Proposition 9.1 holds for \(K\), by a similar proof as for \(B\). We have already defined the elements \(\zeta_i \in H_{C_2}^{RO(\Pi K)}(K_+)\). We define the element \(x\psi_0^m\) to be the image in \(H_{C_2}^{RO(\Pi K)}(K)\) of \(\psi_0^m \otimes x \in H_{C_2}^{RO(\Pi K)}(K_+ \wedge K \tilde{E}C_2)\), where

\[\psi_0^m = (\zeta_0^m, 0)\]
\[\psi_1^m = (0, \zeta_1^m).\]

(See Proposition 14.5.) It follows that

\[\eta(x\psi_0^m) = (\psi(x)\zeta_0^m, 0)\]
\[\eta(x\psi_1^m) = (0, \psi(x)\zeta_1^m).\]

From the definition of \(Q^{RO(\Pi K)}\) and our calculation of \(H_{C_2}^{RO(\Pi K)}(K_+ \wedge K \tilde{E}C_2)\), we see that the elements \(\zeta_i\) and \(x\psi_k^m\) generate \(H_{C_2}^{RO(\Pi K)}(K_+)\) multiplicatively.

The relation \(\zeta_0\zeta_1 = \xi\) holds because it is true on applying \(\eta\) and takes place in a grading in which \(\eta\) is a monomorphism. The next four relations listed in the theorem hold because they do in \(H_{C_2}^{RO(\Pi K)}(K_+ \wedge K \tilde{E}C_2)\).
To verify the formulas for $\zeta_i \cdot x\psi^n_k$, consider the following diagram:

$$
\begin{array}{ccc}
H^0_{C^2}(K_+) \otimes H^0_{C^2}(K_+ \land \hat{E}C_2) & \xrightarrow{\eta} & H^0_{C^2}(K_+^2 \land \hat{E}C_2) \\
\downarrow & & \downarrow \\
H^0_{C^2}(K_+ \land \hat{E}C_2) & \xrightarrow{=} & H^0_{C^2}(K_+^2 \land \hat{E}C_2) \\
\downarrow & & \downarrow \\
H^0_{C^2}(K_+ \land \hat{E}C_2) & \xrightarrow{=} & H^0_{C^2}(K_+^2 \land \hat{E}C_2) \\
\downarrow & & \downarrow \\
H^0_{C^2}(K_+) & & \\
\end{array}
$$

This diagram implies that it suffices to check that the formula holds in the cohomology group $H^0_{C^2}(K_+^2 \land \hat{E}C_2)$, where it is easy to check component-wise. The relation $\psi(x)\zeta_0 = x\psi_0^{-1} + x\psi_1$ follows from the commutativity of the diagram in the theorem, as the two sides are the images of $\zeta_0 \cdot x$ around the two sides of the diagram. Similarly for the last relation.

That the listed relations imply all relations follows from the fact that they easily allow us to write any product of generators as a linear combination of basic monomials from $Q^0_{C^2}$ and elements $x\psi^n_k$ from $H^0_{C^2}(K_+ \land \hat{E}C_2)$. \hfill \Box

16. $\zeta_0$ and $\zeta_1$ as Invariants of the Component Structure

Consider any $C_2$-space $X$ and suppose that we have a decomposition $X^{C_2} = X^0 \sqcup X^1$, so each of $X^0$ and $X^1$ is a union of components of $X^{C_2}$. There is a $C_2$-map $k: X \to K(2)$, unique up to $C_2$-homotopy, sending $X^0$ to $K^0$ and $X^1$ to $K^1$. Then we have the elements $k^*\zeta_0 \in H^0_{C^2}(X_+)$ and $k^*\zeta_1 \in H^0_{C^2}(X_+)$, where we also write $k^*: RO(\Pi K) \to RO(\Pi X)$ for the induced map. We think of $k^*\zeta_0$ and $k^*\zeta_1$ as invariants of the component structure of $X$ and write $\zeta_0 = k^*\zeta_0$ and $\zeta_1 = k^*\zeta_1$ for simplicity of notation. This is exactly how the elements $\zeta_0$ and $\zeta_1$ in the cohomology of $BC_2U(1)$ arise. Note that we will always have $\zeta_0\zeta_1 = \xi$ because that relation is true in the cohomology of $K$.

One phenomenon that showed up in the computation of the cohomology of finite projective spaces in [3] that does not occur in the cohomology of $BC_2U(1)$ was that some nonzero cohomology elements were divisible by $\zeta_0$ or $\zeta_1$. A general result is the following. For a $C_2$-homology element, let $x|X^0$ be the image of $x$ under the restriction map $H^0_{C^2}(X^0_+) \to H^0_{C^2}(X^0_+)$. \hfill \textbf{Proposition 16.1.} Let $x \in H^0_{C^2}(X^0_+)$ be an element such that $x|X^0 = 0$. Then $x$ is infinitely divisible by $\zeta_0$.

A similar statement is true for $\zeta_1$, by symmetry. We first need a lemma.

\textbf{Lemma 16.2.} Let $f: Y \to X$ be an ex-$C_2$-space over $X$ and suppose that $f^{-1}(X^0)$ is contractible to the base section. Then $f^*\zeta_0$ is an invertible element of $H^0_{C^2}(Y)$. \hfill \textbf{Proof.} Let $i: * \to K$ be the inclusion of a fixed point in $K^1$. By its construction, $i^*\zeta_0 \in H^0_{C^2}(S^0)$ is invertible.
Using Elmendorf’s construction, we can construct a $C_2$-map $h: \tilde{X} \to X$ such that $h$ is a nonequivariant equivalence and $hC_2$ is equivalent to the inclusion $X^{C_2} \setminus X^0 \hookrightarrow X^{C_2}$. The composite $k \circ h: \tilde{X} \to K$ factors, up to homotopy, through $i$. Therefore, $h^*\zeta_0$ is invertible in $H^{RO(\Pi X)}_{C_2}(\tilde{X}_+)$. Let $Y^0 = (f^{C_2})^{-1}(X^0)$, which we are assuming is contractible to the base section. Using Elmendorf’s construction again, we can construct an ex-$C_2$-space $h: \tilde{Y} \to \tilde{X}$ and a diagram

$$
\begin{array}{ccc}
\tilde{Y} & \xrightarrow{\ell} & Y \\
\downarrow f & & \downarrow f \\
\tilde{X} & \xrightarrow{h} & X
\end{array}
$$

such that the map $h_0\tilde{Y} \to Y$ over $X$ is a nonequivariant equivalence and, on fixed points, is equivalent to the inclusion $(Y^{C_2} \setminus Y^0)_+ \hookrightarrow Y^{C_2}$; hence is an equivalence by the assumption on $Y$. (Here, $(Y^{C_2} \setminus Y^0)_+$ is the pushout $(Y^{C_2} \setminus Y^0) \cup_{\sigma(X) \setminus Y^0} X$.) It follows that

$$\ell^*: H^{RO(\Pi X)}_{C_2}(Y) \cong H^{RO(\Pi X)}_{C_2}(h_0\tilde{Y})\,.$$

But $\ell^*f^*\zeta_0 = \tilde{f}^*h^*\zeta_0$ is invertible, hence $f^*\zeta_0$ is invertible. \qed

**Proof of Proposition 16.1.** Let $i: X^0 \to X$ be the inclusion, let $Ci$ be the cofiber over $X$ of $i$, and let $j: X_+ \to Ci$ be inclusion in the cofiber. If $i^*x = 0$, then $x = j^*y$ for some $y$. Now $\zeta_0$ is invertible in the cohomology of $Ci$ by the preceding lemma, because the part of $(Ci)^{C_2}$ lying over $X^0$ is contractible to the base section. So, for each $n > 0$, there is a $z$ such that $y = \zeta_0^n z$. It follows that $x = \zeta_0^n j^* z$, so $x$ is divisible by $\zeta_0^n$. \qed

Using the spaces $K(n)$ for $n > 2$ we can generalize this construction. If we have a decomposition $X^{C_2} = \prod_{k=0}^{n-1} X^k$, we can define elements $\zeta_k$ associated to each $X^k$ satisfying the appropriate version of Proposition 16.1 and with $\prod_{k=0}^{n-1} \zeta_k = \xi$.

Note that Proposition 16.1 is not an “if and only if” result because $\zeta_0|X^0 \neq 0$ in general (for example, in $K$). In the applications so far, the present proposition has been sufficient.

**17. $B_{C_2}U(1)$ does not represent cohomology**

We noted earlier that we have a map $B = B_{C_2}U(1) \to K = K(2)$ mapping $B^0$ to $K^0$ and $B^1$ to $K^1$ and inducing an isomorphism $RO(\Pi B) \cong RO(\Pi K)$. In this way we consider $B$ as a space parametrized by $K$, which allows us to consider $H^{RO(\Pi B)}_{C_2}(B_+)$ as an algebra over $H^{RO(\Pi K)}_{C_2}(K_+)$, via a ring map

$$H^{RO(\Pi K)}_{C_2}(K_+) \to H^{RO(\Pi B)}_{C_2}(B_+).$$

We know that $\zeta_0 \mapsto \zeta_0$ and $\zeta_1 \mapsto \zeta_1$, but what about the elements $x\psi^m_k \in H^{RO(\Pi K)}_{C_2}(K_+)$? Recall that

$$\eta(x\psi^m_k) = (\psi(x)\zeta_1^m, 0) \quad \text{and} \quad \eta(x\psi^m_k) = (0, \psi(x)\zeta_0^m).$$
From this and the formulas used in the proof of Theorem 11.3 it is possible to work out the image of $x\psi^n_k$ in $\mathcal{H}_{C_2}^{\text{RO}(IB)}(B_+)$. It suffices for our purposes here that $x\psi^n_k$ maps to 0 in $\mathcal{H}_{C_2}^{\text{RO}(IB)}(B_+)$ if $\psi(x) = 0$.

From our calculations we get the following result.

**Proposition 17.1.** There is no equivariant section of the map $B \to K$, even up to homotopy.

**Proof.** If there were a homotopy section, then the ring map $\mathcal{H}_{C_2}^{\text{RO}(IK)}(K_+) \to \mathcal{H}_{C_2}^{\text{RO}(IB)}(B_+)$ would be injective. However, the element $(\delta \xi^{-1})\psi^n_0$ is nonzero in $\mathcal{H}_{C_2}^{\text{RO}(IK)}(K_+)$, but maps to 0 in $\mathcal{H}_{C_2}^{\text{RO}(IB)}(B_+)$ because $\psi(\delta \xi^{-1}) = 0$. □

**Remark 17.2.** This result settles a point on which the author was confused for a while. The $C_2$-space $B$ is nonequivariantly equivalent to $K(\mathbb{Z}, 2)$ and $B^{C_2}$ is equivalent to the disjoint union of two copies of $K(\mathbb{Z}, 2)$. This suggests that $B$ should represent the functor $H^2_{C_2}(-; \mathbb{Z})$ on spaces over $K$, but the cohomology of $B$ just seems wildly wrong for that: What would be the universal cohomology element in $H^2_{C_2}(B; \mathbb{Z})$? But the space representing $H^2_{C_2}(-; \mathbb{Z})$ must first be an ex-space over $K$, and the proposition shows that $B$ cannot be made into one, hence it is not a candidate to represent any cohomology group.

In fact, the ex-space representing $H^2_{C_2}(-; \mathbb{Z})$ is simply $K \times K(\mathbb{Z}, 2)$ for a nonequivariant $K(\mathbb{Z}, 2)$, say $\mathbb{C}P^\infty$, taken with trivial $C_2$-action, and with a section given by any chosen basepoint in $K(\mathbb{Z}, 2)$. The proposition shows that $B$ is not equivariantly equivalent to $K \times K(\mathbb{Z}, 2)$.

Thus, unlike the nonequivariant case, $C_2$-equivariant complex line bundles are not classified by a cohomology class (the Euler class in the nonequivariant case), as $B_{C_2}U(1)$ is not the representing space for any cohomology group.

**Corollary 17.3.** There is no complex line bundle over $K$ who associated representation is $\omega \in \text{RO}(IK) = \text{RO}(IB)$.

**Proof.** If there were such a line bundle, its classifying map $K \to B_{C_2}U(1)$ would be a (homotopy) section of the projection $B \to K$. But the preceding proposition shows that no such map exists. □

**Remark 17.4.** It is often convenient to know that a given representation of a fundamental groupoid comes from a bundle over the space, but this result shows that, at least for complex bundles, this is not something that is guaranteed to happen.

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