CALIBRATED MANIFOLDS AND GAUGE THEORY

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Abstract. By a theorem of Mclean, the deformation space of an associative submanifold $Y$ of an integrable $G_2$-manifold $(M, \varphi)$ can be identified with the kernel of a Dirac operator $D : \Omega^0(\nu) \to \Omega^0(\nu)$ on the normal bundle $\nu$ of $Y$. Here, we generalize this to the non-integrable case, and also show that the deformation space becomes smooth after perturbing it by natural parameters, which corresponds to moving $Y$ through 'pseudo-associative' submanifolds. Infinitesimally, this corresponds to twisting the Dirac operator $D \mapsto D_A$ with connections $A$ of $\nu$. Furthermore, the normal bundles of the associative submanifolds with $Spin^c$ structure have natural complex structures, which helps us to relate their deformations to Seiberg-Witten type equations.

If we consider $G_2$ manifolds with 2-plane fields $(M, \varphi, \Lambda)$ (they always exist) we can split the tangent space $TM$ as a direct sum of an associative 3-plane bundle and a complex 4-plane bundle. This allows us to define (almost) $\Lambda$-associative submanifolds of $M$, whose deformation equations, when perturbed, reduce to Seiberg-Witten equations, hence we can assign local invariants to these submanifolds. Using this we can assign an invariant to $(M, \varphi, \Lambda)$. These Seiberg-Witten equations on the submanifolds are restrictions of global equations on $M$. We also discuss similar results for the Cayley submanifolds of a $Spin(7)$ manifold.

0. Introduction

We first study deformations of associative submanifolds $Y^3$ of a $G_2$ manifold $(M^7, \varphi)$, where $\varphi \in \Omega^3(M)$ is the $G_2$ structure. We prove a generalized version of the McLean’s theorem where integrability condition of the underlying $G_2$ structure is not necessary. This deformation space might be singular, but by perturbing it with some natural parameters it can be made smooth. This amounts to deforming $Y$ through the associatives in $(M, \varphi)$ with varying $\varphi$, or alternatively deforming $Y$ through the pseudo-associative submanifolds ($Y$’s whose tangent planes become associative after rotating by a generic element of the gauge group of $TM$). Infinitesimally, these perturbed deformations correspond to the kernel of the twisted Dirac operator $D_A : \Omega^0(\nu) \to \Omega^0(\nu)$, twisted by some connection $A$ in $\nu(Y)$.

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The associative submanifolds with $\text{Spin}^c$ structures in $(M, \varphi)$ are useful objects to study, because their normal bundles have natural complex structures. Also we can view $(M, \varphi)$ as an analog of a symplectic manifold, and view a non-vanishing 2-plane field $\Lambda$ on $M$ as an analog of a complex structure taming $\varphi$. Note that 2-plane fields are stronger versions of $\text{Spin}^c$ structures on $M^7$, and they always exist by [1]. The data $(M^7, \varphi, \Lambda)$ determines an interesting splitting of the tangent bundle $TM = E \oplus V$, where $E$ is the bundle of associative 3-planes, and $V$ is the complementary 4-plane bundle with a complex structure, which is a spinor bundle of $E$. Then the integral submanifolds $Y^3$ of $E$, which we call $\Lambda$-associative submanifolds, can be viewed as analogues of J-holomorphic curves; because their normal bundles come with an almost complex structure. Even if they may not always exist, their perturbed versions, i.e. almost $\Lambda$-associative submanifolds, always do. Almost $\Lambda$-associative submanifolds are the transverse sections of the bundle $V \to M$. We can deform such $Y$ by using the connections in the determinant line bundle of $\nu(Y)$ and get a smooth deformation space, which is described by the twisted Dirac equation. Then by constraining this new variable with another natural equation we arrive to Seiberg-Witten type equations for $Y$. So we can assign an integer to $Y$, which is invariant under small isotopies through almost $\Lambda$-associative submanifolds.

In fact it turns out that $(M^7, \varphi, \Lambda)$ gives a finer splitting $TM = \bar{E} \oplus \xi$, where $\bar{E}$ is a 6-plane bundle with a complex structure, and $\xi$ is a real line bundle. In a way this structure of $(M, \varphi)$ mimics the structure of $(\text{Calabi-Yau}) \times S^1$ manifolds, and by ‘rotating’ $\xi$ inside of $TM$ we get a new insight for so-called “Mirror manifolds” which is investigated in [AS1].

There is a similar process for the deformations of Cayley submanifolds $X^4 \subset N^8$ of a Spin(7) manifold $(N^8, \Psi)$, which we discuss at the end. So in a way $\Lambda$-associative (or Cayley) manifolds in a $G_2$ (or Spin(7)) manifold, behave much like higher dimensional analogue of holomorphic curves in a Calabi-Yau manifold.

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1. Preliminaries

Here we first review basic properties of the manifolds with special holonomy (most material can be found in [B2], [B3], [H], [HL]), and then proceed to prove some results. Recall that the set of octonions \( \mathbb{O} = \mathbb{H} \oplus i\mathbb{H} = \mathbb{R}^8 \) is an 8-dimensional division algebra generated by \(<1, i, j, k, l, li, lj, lk>\). On the set of the imaginary octonions \( im\mathbb{O} = \mathbb{R}^7 \) we have the cross product operation \( \times : \mathbb{R}^7 \times \mathbb{R}^7 \rightarrow \mathbb{R}^7 \), defined by \( u \times v = im(\bar{v} \cdot u) \). The exceptional Lie group \( G_2 \) can be defined as the linear automorphisms of \( im\mathbb{O} \) preserving this cross product operation, \( G_2 = Aut(\mathbb{R}^7, \times) \).

There is also another useful description in terms of the orthogonal 3-frames in \( \mathbb{R}^7 \):

\[
\text{(1)} \quad G_2 = \{(u_1, u_2, u_3) \in (im\mathbb{O})^3 \mid <u_i, u_j> = \delta_{ij}, \; <u_1 \times u_2, u_3> = 0 \}
\]

Alternatively, \( G_2 \) can be defined as the subgroup of the linear group \( GL(7, \mathbb{R}) \) which fixes a particular 3-form \( \varphi_0 \in \Omega^3(\mathbb{R}^7) \). Denote \( e^{ijk} = dx^i \wedge dx^j \wedge dx^k \in \Omega^3(\mathbb{R}^7) \), then

\[
G_2 = \{A \in GL(7, \mathbb{R}) \mid A^* \varphi_0 = \varphi_0 \}
\]

\[
\varphi_0 = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356}
\]

**Definition 1.** A smooth 7-manifold \( M^7 \) has a \( G_2 \) structure if its tangent frame bundle reduces to a \( G_2 \) bundle. Equivalently, \( M^7 \) has a \( G_2 \) structure if there is a 3-form \( \varphi \in \Omega^3(M) \) such that at each \( x \in M \) the pair \( (T_x(M), \varphi(x)) \) is isomorphic to \( (T_0(\mathbb{R}^7), \varphi_0) \).

Here are some useful properties, discussed more fully in [B2]: Any \( G_2 \) structure \( \varphi \) on \( M^7 \) gives an orientation \( \mu \in \Omega^7(M) \) on \( M \), and this \( \mu \) determines a metric \( g = \langle , \rangle \) on \( M \), and a cross product structure \( \times \) on its tangent bundle of \( M \) as follows: Let \( i_v \) denote the interior product with a vector \( v \) then

\[
\text{(3)} \quad \langle u, v \rangle = [i_u(\varphi) \wedge i_v(\varphi) \wedge \varphi] / 6 \mu
\]

\[
\text{(4)} \quad \varphi(u, v, w) = \langle u \times v, w \rangle
\]

To emphasize the dependency on \( \varphi \) sometimes \( g \) is denoted by \( g_\varphi \). In particular, the 14-dimensional Lie group \( G_2 \) imbeds into \( SO(7) \) subgroup of \( GL(7, \mathbb{R}) \). Note that because of the way we defined \( G_2 = G_2^{\varphi_0} \), this imbedding is determined by \( \varphi_0 \).

Since \( GL(7, \mathbb{R}) \) acts on \( \Lambda^3(\mathbb{R}^7) \) with stabilizer \( G_2 \), its orbit \( \Lambda^3_+(\mathbb{R}^7) \) is open for dimension reasons, so the choice of \( \varphi_0 \) in the above definition is generic (in fact it has two orbits containing \( \pm \varphi_0 \)). \( G_2 \) has many copies \( G_2^\varphi \) inside \( GL(7, \mathbb{R}) \), which are all conjugate to each other, since \( G_2 \) has only one 7 dimensional representation. Hence the space of \( G_2 \) structures on \( M^7 \) are identified with the sections of the bundle:

\[
\text{(5)} \quad \mathbb{R}P^7 \cong GL(7, \mathbb{R})/G_2 \rightarrow \Lambda^3_+(M) \rightarrow M
\]
which are called the positive 3-forms, these are the set of 3-forms $\Omega^3_+(M)$ that can be identified pointwise by $\varphi_0$. Each $G_2^\omega$ imbeds into a conjugate of one standard copy $SO(7) \subset GL(7, \mathbb{R})$. The space of $G_2$ structures $\varphi$ on $M$, which induce the same metric on $M$, that is all $\varphi$’s for which the corresponding $G_2^\omega$ lies in the standard $SO(7)$, are the sections of the bundle (whose fiber is the orbit of $\varphi_0$ under $SO(7)$):

$$\mathbb{R}P^7 = SO(7)/G_2 \to \hat{\Lambda}^3_+(M) \to M$$

which we will denote by $\hat{\Omega}^3_+(M)$. The set of smooth 7-manifolds with $G_2$-structures coincides with the set of 7-manifolds with spin structure, though this correspondence is not $1 - 1$. This is because $Spin(7)$ acts on $S^7$ with stabilizer $G_2$ inducing the fibrations

$$G_2 \to Spin(7) \to S^7 \to BG_2 \to BSpin(7)$$

and so there is no obstruction to lifting maps $M^7 \to BSpin(7)$ to $BG_2$, and there are many liftings. Cotangent frame bundle $P^*(M) \to M$ of a manifold with $G_2$ structure $(M, \varphi)$ can be expressed as $P^*(M) = \bigcup_{x \in M} P^*_x(M)$, where each fiber is:

$$P^*_x(M) = \{ u \in Hom(T_x(M), \mathbb{R}^7) \mid u^*(\varphi_0) = \varphi(x) \}$$

Throughout this paper we will denote the cotangent frame bundle by $P^*(M) \to M$ and its adapted frame bundle by $P(M)$. They can be $G_2$ or $SO(7)$ frame bundles; to emphasize it sometimes we will specify them by the notations $P_{SO(7)}(M)$ or $P_{G_2}(M)$. Also we will denote the sections of a bundle $\xi \to Y$ by $\Omega^p(Y, \xi)$ or simply by $\Omega^p(\xi)$, and the bundle valued $p$-forms by $\Omega^p(\xi) = \Omega^p(\Lambda^pT^*Y \otimes \xi)$, and the sphere bundle of $\xi$ by $S(\xi)$. There is a notion of a $G_2$ structure $\varphi$ on $M^7$ being integrable, which corresponds to $\varphi$ being an harmonic form:

**Definition 2.** A manifold with $G_2$ structure $(M, \varphi)$ is called a $G_2$ manifold if the holonomy group of the Levi-Civita connection (of the metric $g_\varphi$) lies inside of $G_2$. Equivalently $(M, \varphi)$ is a $G_2$ manifold if $\varphi$ is parallel with respect to the metric $g_\varphi$ i.e. $\nabla_{g_\varphi}(\varphi) = 0$; this condition is equivalent to $d\varphi = 0 = d(*_{g_\varphi}\varphi)$.

In short one can define a $G_2$ manifold to be any Riemannian manifold $(M^7, g)$ whose holonomy group is contained in $G_2$, then $\varphi$ and the cross product $\times$ come as a consequence. It turns out that the condition $\varphi$ being harmonic is equivalent to the condition that at each point $x_0 \in M$ there is a chart $(U, x_0) \to (\mathbb{R}^7, 0)$ on which $\varphi$ equals to $\varphi_0$ up to second order term, i.e. on the image of $U$

$$\varphi(x) = \varphi_0 + O(|x|^2)$$

**Remark 1.** For example if $(X^6, \omega, \Omega)$ is a complex 3-dimensional Calabi-Yau manifold with Kähler form $\omega$, and a nowhere vanishing holomorphic 3-form $\Omega$, then $X \times S^1$ has holonomy group $SU(3) \subset G_2$, hence is a $G_2$ manifold. In this case

$$\varphi = Re \Omega + \omega \wedge dt.$$
Definition 3. Let \((M, \varphi)\) be a manifold with a \(G_2\) structure. A 4-dimensional submanifold \(X \subset M\) is called an co-associative if \(\varphi|_X = 0\). A 3-dimensional submanifold \(Y \subset M\) is called an associative if \(\varphi|_Y \equiv \text{vol}(Y)\); this condition is equivalent to \(\chi|_Y \equiv 0\), where \(\chi \in \Omega^3(M, TM)\) is the tangent bundle valued 3-form defined by the identity:

\[
\langle \chi(u, v, w), z \rangle = *\varphi(u, v, w, z)
\]

The equivalence of these conditions follows from the ‘associator equality’ of [HL](10)

\[
\varphi(u, v, w)^2 + |\chi(u, v, w)|^2/4 = |u \wedge v \wedge w|^2
\]

In general, if \(\{e^1, e^2, ..., e^7\}\) is any orthonormal coframe on \((M, \varphi)\), then the expression (2) for \(\varphi\) hold on a chart. By calculation \(*\varphi\), and using (9) we can calculate the expression of \(\chi\) (note the error in the second term of 6th line of the corresponding formula (5.4) of [M]):

\[
* \varphi = e^{4567} + e^{2367} + e^{2345} + e^{1357} - e^{1346} - e^{1256} - e^{1247}
\]

Also \(\chi\) can be expressed in terms of cross product operation (c.f. [LI], [HL], [K]):

\[
\chi(u, v, w) = -u \times (v \times w) - \langle u, v \rangle w + \langle u, v \rangle v
\]

When \(d\varphi = 0\), the associative submanifolds are volume minimizing submanifolds of \(M\) (calibrated by \(\varphi\)). Even in the general case of a manifold with a \(G_2\) structure \((M, \varphi)\), the form \(\chi\) imposes an interesting structure near associative submanifolds:

Notice (9) implies that, \(\chi\) maps every oriented 3-plane in \(T_x(M)\) to the orthogonal subspace \(T_x(M)^\perp\), so if we choose local coordinates \((x_1, ..., x_7)\) for \(M\) we get

\[
\chi = \sum a^\alpha_J \, dx^J \otimes \frac{\partial}{\partial x_\alpha}
\]

where \(dx^J = dx^i \wedge dx^j \wedge dx^k\), and the summation is taken over the multi-index \(J = \{i, j, k\}\) and \(\alpha\) such that \(\alpha \notin J\). So if \(Y \subset M\) is given by \((x_1, x_2, x_3)\) coordinates, then locally the condition \(Y\) to be associative is given by the equations:

\[
a^\alpha_{123} = 0
\]
From (9) it is easy to calculate $a_{ijk}^\alpha = *\varphi_{ijks}g^{s\alpha}$, where $g^{-1} = (g^{ij})$ is the inverse of the metric $g = (g_{ij})$, and of course the metric $g$ can be expressed in terms of $\varphi$. By evaluating $\chi$ on the orientation form of $Y$ we get a normal vector field so:

**Lemma 1.** To any 3-dimensional submanifold $Y^3 \subset (M, \varphi)$, $\chi$ associates a normal vector field, which vanishes when $Y$ is associative.

Hence $\chi$ defines an interesting flow on 3 dimensional submanifolds of $(M, \varphi)$, fixing associative submanifolds. On the associative submanifolds with a Spin$^c$ structure, $\chi$ rotates their normal bundles and imposes a complex structure on them:

**Lemma 2.** To any associative manifold $Y^3 \subset (M, \varphi)$ with a non-vanishing oriented 2-plane field, $\chi$ defines an almost complex structure on its normal bundle $\nu(Y)$ (notice that in particular any coassociative submanifold $X \subset M$ has an almost complex structure if its normal bundle has a non-vanishing section).

**Proof.** Let $L \subset \mathbb{R}^7$ be an associative 3-plane, that is $\varphi|_L = vol(L)$. Then to every pair of orthonormal vectors $\{u, v\} \subset L$, the form $\chi$ defines a complex structure on the orthogonal 4-plane $L^\perp$, as follows: Define $j : L^\perp \to L^\perp$ by

$$j(X) = \chi(u, v, X)$$

This is well defined, i.e. $j(X) \in L^\perp$, because when $w \in L$ we have:

$$\langle \chi(u, v, X), w \rangle = *\varphi(u, v, X, w) = - *\varphi(u, v, w, X) = \langle \chi(u, v, w), X \rangle = 0$$

Also $j^2(X) = j(\chi(u, v, X)) = \chi(u, v, \chi(u, v, X)) = -X$. We can check the last equality by taking an orthonormal basis $\{X_j\} \subset L^\perp$ and calculating

$$\langle \chi(u, v, \chi(u, v, X_j)), X_j \rangle = *\varphi(u, v, \chi(u, v, X_j), X_j) =$$

$$- *\varphi(u, v, X_j, \chi(u, v, X_j)) = - \langle \chi(u, v, X_j), \chi(u, v, X_j) \rangle = - \delta_{ij}$$

The last equality holds since the map $j$ is orthogonal, and the orthogonality can be seen by polarizing the associator equality (10), and by noticing $\varphi(u, v, X_i) = 0$. Observe that the map $j$ only depends on the oriented 2-plane $l = \langle u, v \rangle$ generated by $\{u, v\}$. So the result follows.

In fact, for any unit vector field $\xi$ on an associative $Y$ (i.e. a Spin$^c$ structure) defines a complex structure $J_\xi : \nu(Y) \to \nu(Y)$ by $J_\xi(z) = z \times \xi$, and the complex structure defined in Lemma 2 corresponds to $J_{u \times v}$, because from (12):

$$\chi(u, v, z) = \chi(z, u, v) = -z \times (u \times v) = \langle z, u \rangle v + \langle z, v \rangle u = J_{u \times v}(z).$$

Also recall that the complex structures on any $SO(4)$ bundle such as $\nu \to Y$ are given by the unit sections of the associated $SO(3)$ bundle $\lambda^+(\nu) \to Y$, which is induced by the left reductions $SO(4) = (SU(2) \times SU(2))/\mathbb{Z}_2 \to SU(2)/\mathbb{Z}_2 = SO(3)$. 

Definition 4. A Riemannian 8-manifold \((N^8, g)\) is called a Spin(7) manifold if the holonomy group of its Levi-Civita connection lies in Spin(7) \(\subset GL(8, \mathbb{R})\).

Equivalently a Spin(7) manifold \((N, \Psi)\) is a Riemannian 8-manifold with a triple cross product \(\times\) on its tangent bundle, and a harmonic 4-form \(\Psi \in \Omega^4(N)\) with
\[
\Psi(u, v, w, z) = g(u \times v \times w, z)
\]

It is easily checked that if \((M, \varphi)\) is a G2 manifold, then \((M \times S^1, \Psi)\) is a Spin(7) manifold where \(\Psi = \varphi \wedge dt - *\varphi\).

Definition 5. A 4-dimensional submanifold \(X\) of a Spin(7) manifold \((N, \Psi)\) is called Cayley if \(\Psi|_X \equiv \text{vol}(X)\). This is equivalent to \(\tau|_X \equiv 0\) where \(\tau \in \Omega^4(N, E)\) is a certain vector-bundle valued 4-form defined by the “four-fold cross product” of the imaginary octonions \(\tau(v_1, v_2, v_3, v_4) = v_1 \times v_2 \times v_3 \times v_4\) (see [M], [HL]).

2. Grassmann Bundles

Let \(G(3, 7)\) be the Grassmann manifold of oriented 3-planes in \(\mathbb{R}^7\). Let \(M^7\) be an oriented smooth 7-manifold, and let \(\tilde{M} \to M\) be the bundle oriented 3-planes in \(TM\), which is defined by the identification 
\[
(p, L) = (pg, g^{-1}L) \in \tilde{M}: \quad (16)
\]
\[
\tilde{M} = \mathcal{P}_{SO(7)}(M) \times_{SO(7)} G(3, 7) \to M.
\]
This is just the bundle \(\tilde{M} = \mathcal{P}_{SO(7)}(M)/SO(3) \times SO(4) \to \mathcal{P}_{SO(7)}(M)/SO(7) = M\). Let \(\xi \to G(3, 7)\) be the universal \(\mathbb{R}^3\) bundle, and \(\nu = \xi^\perp \to G(3, 7)\) be the dual \(\mathbb{R}^4\) bundle. Therefore, \(\text{Hom}(\xi, \nu) = \xi^* \otimes \nu \to G(3, 7)\) is the tangent bundle \(TG(3, 7)\).

\(\xi, \nu\) extend fiberwise to give bundles \(\Xi \to \tilde{M}, \mathcal{V} \to \tilde{M}\) respectively, and let \(\Xi^*\) be the dual of \(\Xi\). Notice that \(\text{Hom}(\Xi, \mathcal{V}) = \Xi^* \otimes \mathcal{V} \to \tilde{M}\) is the bundle of vertical vectors \(T^v(\tilde{M})\) of \(T(\tilde{M}) \to M\), i.e. the tangents to the fibers of \(\pi: \tilde{M} \to M\), hence
\[
(17) \quad T\tilde{M} \cong T^v(\tilde{M}) \oplus \pi^*TM = (\Xi^* \otimes \mathcal{V}) \oplus \Xi \oplus \mathcal{V}.
\]

That is, \(T\tilde{M}\) is the vector bundle associated to principal \(SO(3) \times SO(4)\) bundle \(\mathcal{P}_{SO(7)} \to \tilde{M}\) by the obvious representation of \(SO(3) \times SO(4)\) to \((\mathbb{R}^3)^* \otimes \mathbb{R}^4 + \mathbb{R}^3 + \mathbb{R}^4\). The identification (17) is defined up to gauge automorphisms of bundles \(\Xi\) and \(\mathcal{V}\).

Note that the bundle \(\mathcal{V} = \Xi^\perp\) depends on the metric, and hence it depends on \(\varphi\) when metric is induced from a G2 structure \((M, \varphi)\). To emphasize this fact we can denote it by \(\mathcal{V}_\varphi \to \tilde{M}\). But when we are considering G2 structures coming from G2 subgroups of a fixed copy of \(SO(7) \subset GL(7, \mathbb{R})\), they induce the same metric and so this distinction is not necessary.
Let $\mathcal{P}(\mathbb{V}) \to \tilde{M}$ be the $SO(4)$ frame bundle of the vector bundle $\mathbb{V}$, identify $\mathbb{R}^4$ with the quaternions $\mathbb{H}$, and identify $SU(2)$ with the unit quaternions $Sp(1) = S^3$. Recall that $SO(4)$ is the equivalence classes of pairs $[q, \lambda]$ of unit quaternions $SO(4) = (SU(2) \times SU(2))/\mathbb{Z}_2$

Hence $\mathbb{V} \to \tilde{M}$ is the associated vector bundle to $\mathcal{P}(\mathbb{V})$ via the $SO(4)$ representation

$$ x \mapsto qx\lambda^{-1} $$

There is a pair of $\mathbb{R}^3 = im(\mathbb{H})$ bundles over $\tilde{M}$ corresponding to the left and right $SO(3)$ reductions of $SO(4)$, which are given by the $SO(3)$ representations

$$ \lambda_+(\mathbb{V}) : x \mapsto qx q^{-1} $$
$$ \lambda_-(\mathbb{V}) : y \mapsto \lambda y \lambda^{-1} $$

The map $x \otimes y \mapsto xy$ gives actions $\lambda_+(\mathbb{V}) \otimes \mathbb{V} \to \mathbb{V}$ and $\mathbb{V} \otimes \lambda_-(\mathbb{V}) \to \mathbb{V}$; by combining we can think of them as one conjugation action

$$ (\lambda_+(\mathbb{V}) \otimes \lambda_-(\mathbb{V})) \otimes \mathbb{V} \to \mathbb{V} $$

If the $SO(4)$ bundle $\mathcal{P}(\mathbb{V}) \to \tilde{M}$ lifts to a $Spin(4) = SU(2) \times SU(2)$ bundle (locally it does), we get two additional bundles over $\tilde{M}$

$$ S : y \mapsto qy $$
$$ E : y \mapsto y\lambda^{-1} $$

They identify $\mathbb{V}$ as a tensor product of two quaternionic line bundles $\mathbb{V} = S \otimes \mathbb{H} E$. In particular, $\lambda_+(\mathbb{V}) = ad(S)$ and $\lambda_-(\mathbb{V}) = ad(E)$, i.e. they are the $SO(3)$ reductions of the $SU(2)$ bundles $S$ and $E$. Also there is a multiplication map $S \otimes E \to \mathbb{V}$. Recall the identifications: $\Lambda^2(\mathbb{V}) = \Lambda_+^2(\mathbb{V}) \oplus \Lambda_-^2(\mathbb{V}) = \lambda_-(\mathbb{V}) \oplus \lambda_+(\mathbb{V}) = \lambda(\mathbb{V}) = gl(\mathbb{V}) = ad(\mathbb{V})$.

### 2.1. Associative Grassmann Bundles.

Now consider the Grassmannian of associative 3-planes $G^\rho(3, 7)$ in $\mathbb{R}^7$, consisting of elements $L \in G(3, 7)$ with the property $\varphi_0|_L = vol(L)$ (or equivalently $\chi_0|_L = 0$). $G_2$ acts on $G^\rho(3, 7)$ transitively with the stabilizer $SO(4)$, so it gives the identification $G^\rho(3, 7) = G_2/SO(4)$. If we identify the imaginary octonions by $\mathbb{R}^7 = im(\mathbb{O}) \cong im(\mathbb{H}) \oplus \mathbb{H}$, then the action of the subgroup $SO(4) \subset G_2$ on $\mathbb{R}^7$ is

$$ \left( \begin{array}{cc} \rho(A) & 0 \\ 0 & A \end{array} \right) $$

where $\rho : SO(4) = (SU(2) \times SU(2))/\mathbb{Z}_2 \to SO(3)$ is the projection of the first factor [H], that is for $[q, \lambda] \in SO(4)$ the action is given by $(x, y) \mapsto (qxq^{-1}, qy\lambda^{-1})$. So the action of $SO(4)$ on the 3-plane $L = im(\mathbb{H})$ is determined by its action on $L^\perp$. Now let $M^\perp$ be a $G_2$ manifold. Similar to the construction before, we can construct the bundle of associative Grassmannians over $\tilde{M}$ (which is a submanifold of $\tilde{M}$):

$$ \tilde{M}_\varphi = P_{G_2}(M) \times_{G_2} G^\rho(3, 7) \to M $$
which is just the quotient bundle $\tilde{M}_\varphi = \mathcal{P}_{G_2}(M)/SO(4) \rightarrow \mathcal{P}_{G_2}(M)/G_2 = M$. As in the previous section, the restriction of the universal bundles $\xi, \nu = \xi_\perp \rightarrow G^\varphi(3, 7)$ induce 3 and 4 plane bundles $\Xi \rightarrow \tilde{M}_\varphi$ and $\mathbb{V} \rightarrow \tilde{M}_\varphi$ (by restricting from $\tilde{M}$). Also

$$T\tilde{M}_\varphi \cong T^v(\tilde{M}_\varphi) \oplus \Xi \oplus \mathbb{V}$$

(24)

From (22) we see that in the associative case, we have an important identification: $\Xi = \lambda_+ (\mathbb{V})$ (as bundles over $\cdot \varphi$), and the dual of the action $\lambda_+ (\mathbb{V}) \otimes \mathbb{V} \rightarrow \mathbb{V}$ gives a Clifford multiplication:

$$\Xi^* \otimes \mathbb{V} \rightarrow \mathbb{V}$$

(25)

In fact this is just the map induced from the cross product operation [AS2]. Recall that $T^v(M) = \Xi^* \otimes \mathbb{V} \rightarrow \tilde{M}$ is the subbundle of vertical vectors of $T(M) \rightarrow M$. The total space $E(\nu_\varphi)$ of the normal bundle of the imbedding $\tilde{M}_\varphi \subset \tilde{M}$ should be thought of an open tubular neighborhood of $\tilde{M}_\varphi$ in $\tilde{M}$, and it has a nice description:

**Lemma 3.** ([$\text{M}$]) Normal bundle $\nu_\varphi$ of $\tilde{M}_\varphi \subset \tilde{M}$ is isomorphic to $\mathbb{V}$, and the bundle of vertical vectors $T^v(\tilde{M}_\varphi)$ is the kernel of the Clifford multiplication $c : \Xi^* \otimes \mathbb{V} \rightarrow \mathbb{V}$. We have $T^v(M)|_{\tilde{M}_\varphi} = T^v(\tilde{M}_\varphi) \oplus \nu_\varphi$, and the following exact sequence over $\tilde{M}_\varphi$

$$T^v(\tilde{M}_\varphi) \rightarrow \Xi^* \otimes \mathbb{V}|_{\tilde{M}_\varphi} \xrightarrow{c} \mathbb{V}|_{\tilde{M}_\varphi} \rightarrow 0$$

Hence the quotient bundle, $T^v(\tilde{M})/T^v(\tilde{M}_\varphi)$ is isomorphic to $\mathbb{V}$.

**Proof.** This is because the Lie algebra inclusion $g_2 \subset so(7)$ is given by

$$\begin{pmatrix} a & \beta \\ -\beta^t & \rho(a) \end{pmatrix}$$

where $a \in so(4)$ is $y \mapsto qy - y\lambda$, and $\rho(a) \in so(3)$ is $x \mapsto qx - xq$. So the tangent space inclusion of $G_2/\text{SO}(4) \subset SO(7)/SO(4) \times SO(3)$ is given by the matrix $\beta \in (im\mathbb{H})^* \otimes \mathbb{H}$. Therefore, if we write $\beta$ as column vectors of three quaternions $\beta = (\beta_1, \beta_2, \beta_3) = i^* \beta_1 + j^* \beta_2 + k^* \beta_3$, then $\beta_1 i + \beta_2 j + \beta_3 k = 0$ ([M], [Mc]). □

The reader can consult Lemma 5 of [AS2] for a more self contained proof of this fact, where the Clifford multiplication is identified with the cross product operation.

### 3. Associative Submanifolds

Any imbedding of a 3-manifold $f : Y^3 \hookrightarrow M^7$ induces an imbedding $\tilde{f} : Y \hookrightarrow \tilde{M}$:

$$\tilde{M} \supset \tilde{M}_\varphi$$

(26)

$$\begin{pmatrix} \tilde{f} \\ \downarrow \\ Y \xrightarrow{f} M \end{pmatrix}$$
and the pull-backs \( f^*\Xi = T(Y) \) and \( f^*V = \nu(Y) \) give the tangent and normal bundles of \( Y \). Furthermore, if \( f \) is an imbedding of an associative submanifold into a \( G_2 \) manifold \((M,\varphi)\), then the image of \( \tilde{f} \) lands in \( \tilde{M}_\varphi \). We will denote this canonical lifting of any 3-manifold \( Y \subset M \) by \( \tilde{Y} \subset \tilde{M} \). Also since we have the dependency \( V = V_\varphi \), we can denote \( \nu(Y) = \nu(Y)_\varphi = \nu_\varphi \) when needed.

\( \tilde{M}_\varphi \) can be thought of as a universal space parameterizing associative submanifolds of \( M \). In particular, if \( \tilde{f} : Y \hookrightarrow \tilde{M}_\varphi \) is the lifting of an associative submanifold, by pulling back we see that the principal \( SO(4) \) bundle \( P(V) \to \tilde{M}_\varphi \) induces an \( SO(4) \)-bundle \( P(Y) \to Y \), and gives the following vector bundles via the representations:

\[
\begin{align*}
\nu(Y) : & \quad y \mapsto qy\lambda^{-1} \\
T(Y) : & \quad x \mapsto qx q^{-1}
\end{align*}
\]

where \([q,\lambda] \in SO(4), \nu = \nu(Y)\) and \( T(Y) = \lambda_+(\nu) \). Also we can identify \( T^*Y \) with \( TY \) by the induced metric. From above we have the action \( T^*Y \otimes \nu \to \nu \) inducing actions \( \Lambda^*(T^*Y) \otimes \nu \to \nu \).

Let \( L = \Lambda^3(\Xi) \to \tilde{M} \) be the determinant (real) line bundle. Recall that the definition (9) implies that \( \chi \) maps every oriented 3-plane in \( T_x(M) \) to its complementary subspace, so \( \chi \) gives a bundle map \( L \to V \) over \( \tilde{M} \), which is a section of \( L^* \otimes V \to \tilde{M} \). Since \( \Xi \) is oriented \( L \) is trivial, so \( \chi \) actually gives a section

\[
\chi = \chi_\varphi \in \Omega^0(\tilde{M},V)
\]

Clearly \( \tilde{M}_\varphi \subset \tilde{M} \) is the codimension 4 submanifold which is the zeros of this section. Associative submanifolds \( Y \subset M \) are characterized by the condition \( \chi|_Y = 0 \), where \( \tilde{Y} \subset \tilde{M} \) is the canonical lifting of \( Y \). Similarly \( \varphi \) defines a map \( \varphi : \tilde{M} \to \mathbb{R} \).

3.1. Pseudo-associative submanifolds.

Here we generalize associative submanifolds to a more flexible class of submanifolds. To do this we first generalize the notion of imbedded submanifolds.

**Definition 6.** A Grassmann-framed 3-manifold in \((M,\varphi)\) is a triple \((Y^3,f,F)\), where \( f : Y \hookrightarrow M \) is an imbedding, \( F : Y \to \tilde{M} \), such that the following commute

\[
\begin{array}{ccc}
\tilde{M} & \xrightarrow{\tilde{f}} & M \\
\downarrow & & \\
Y & \xrightarrow{f} & M
\end{array}
\]

We call \((Y,f,F)\) a pseudo-associative submanifold if in addition \( \text{Image}(F) \subset \tilde{M}_\varphi \). So a pseudo-associative submanifold \((Y,f,F)\) with \( F = \tilde{f} \) is associative.
Remark 2. The bundle $\tilde{M} \to M$ always admits a section, in fact the subbundle $\tilde{M}_\varphi \to M$ has a section. This is because by [1] every orientable 7-manifold admits a non-vanishing linearly independent 2-frame field $\Lambda = \{v_1, v_2\}$. By Graham-Schmidt process with metric $g_\varphi$, we can assume that $\Lambda$ is orthonormal. The cross product assigns $\Lambda$ to an orthonormal 3-frame field $\{v_1, v_2, v_1 \times v_2\}$ on $M$, then 3-plane generated by $\{v_1, v_2, v_1 \times v_2\} := \langle v_1, v_2, v_1 \times v_2 \rangle$ gives a section of $\lambda_\varphi : M \to \tilde{M}_\varphi$. Let $\Phi : \text{Im}(Y, M) \to \mathcal{Z}(Y, M)$ be the canonical sections of a bundle
\begin{equation}
\mathcal{Z}(Y) \xrightarrow{\pi} \text{Im}(Y, M)
\end{equation}
with fibers $\pi^{-1}(f) = \Omega^0(Y, f^*\tilde{M})$. We also have the subbundle $\mathcal{Z}_\varphi(Y) \xrightarrow{\pi} \text{Im}(Y, M)$ with fibers $\pi^{-1}(f) = \Omega^0(Y, f^*\tilde{M}_\varphi)$. So $\mathcal{Z}(Y)$ is the set of triples $(Y, f, F)$ (in short just set of $F$’s), where $F : Y \to \tilde{M}$ is a lifting of the imbedding $f : Y \hookrightarrow M$. Also $\mathcal{Z}_\varphi(Y) \subset \mathcal{Z}(Y)$ is a smooth submanifold, since $\tilde{M}_\varphi \subset \tilde{M}$ is smooth. There is the canonical section $\Phi : \text{Im}(Y, M) \to \mathcal{Z}(Y)$ given by $\Phi(f) = f$. Therefore, $\Phi^{-1}\mathcal{Z}_\varphi(Y) := \text{Im}_\varphi(Y, M)$ is the set of associative imbeddings $Y \subset M$. Also, any 2-frame field $\Lambda$ as above gives to a section $\Phi_\Lambda(f) = \lambda_\varphi \circ f$. To make these definitions parameter free we also have to divide $\text{Im}(Y, M)$ by the diffeomorphism group of $Y$.

\[\text{We thank T.Onder for pointing out [1]}.\]
There are also the vertical tangent bundles of $Z(Y)$ and $Z_{\varphi}(Y)$
\[
\begin{align*}
T^n Z(Y) & \xrightarrow{\pi} Z(Y) \\
T^n Z_{\varphi}(Y) & \xrightarrow{\pi|} Z_{\varphi}(Y)
\end{align*}
\]
with fibers $\pi^{-1}(F) = \Omega^0(Y, F^*(\Xi^* \otimes \mathbb{V}))$. By Lemma 3 the fibers of $T^n(Z_{\varphi})$ can be identified with the kernel of the map induced by the Clifford multiplication
\[
c : \Omega^0(Y, F^*(\Xi^* \otimes \mathbb{V})) \rightarrow \Omega^0(Y, F^*(\mathbb{V}))
\]
One of the nice properties of a pseudo-associative submanifold $(Y, f, F)$ is that there is a Clifford multiplication action (by pull back)
\[
F^*(\Xi^*) \otimes F^*(\mathbb{V}) \rightarrow F^*(\mathbb{V})
\]
If $F$ is close to $\tilde{f}$, by parallel translating the fibers over $F(x)$ and $\tilde{f}(x)$ along geodesics in $\tilde{M}$ we get canonical identifications:
\[
F^*(\Xi) \cong TY, \quad F^*(\mathbb{V}) \cong \nu_f
\]
inducing Clifford multiplication between the tangent and the normal bundles. So if $\forall x \in Y$ the distance between $F(x)$ and $\tilde{f}(x)$ is less then the injectivity radius $j(\tilde{M})$, there is a Clifford multiplication between the tangent and normal bundles of $Y$.

3.2. Dirac operator.

The normal bundle $\nu = \nu(Y)$ of any orientable 3-manifold $Y$ in a $G_2$ manifold $(M, \varphi)$ has a $Spin(4)$ structure (e.g. [32]). Hence we have $SU(2)$ bundles $S$ and $E$ over $Y$ such that $\nu = S \otimes_E E$ (18), with $SO(3)$ reductions $adS = \lambda_+(\nu)$, and $adE = \lambda_- (\nu)$ which is also the bundle of endomorphisms $End(E)$. If $Y$ is associative, then the bundle $ad(S)$ becomes isomorphic to $TY$, i.e. $S$ becomes the spinor bundle of $Y$, so $\nu(Y)$ becomes a twisted spinor bundle.

The Levi-Civita connection of the $G_2$ metric of $(M, \varphi)$ induces connections on the associated bundles $\mathbb{V}$ and $\Xi$ on $M$. In particular, it induces connections on the tangent and normal bundles of any submanifold $Y^3 \subset M$. We will call these connections the background connections. Let $A_0$ be the induced connection on the normal bundle $\nu = S \otimes E$. From the Lie algebra decomposition $so(4) = so(3) \oplus so(3)$, we can write $A_0 = B_0 \oplus A_0$, where $B_0$ and $A_0$ are connections on $S$ and $E$, respectively.

Let $A(E)$ and $A(S)$ be the set of connections on the bundles $E$ and $S$. Hence $A \in A(E)$, $B \in A(S)$ are in the form $A = A_0 + a$, $B = B_0 + b$, where $a \in \Omega^1(Y, ad E)$ and $b \in \Omega^1(Y, ad S)$. So $\Omega^1(Y, \lambda_+(\nu))$ parametrizes connections on $S$ and $E$, and the connections on $\nu$ are in the form $A = B \oplus A$. To emphasize the dependency on $b$ and $a$ we sometimes denote $A = A(b, a)$, and $A_0 = A(0, 0) = A_0$. 


Now, let $Y^3 \subset M$ be any smooth manifold. We can express the covariant
derivative $\nabla_A : \Omega^0(Y, \nu) \to \Omega^1(Y, \nu)$ on $\nu$ by $\nabla_A = \sum e^i \otimes \nabla_{e_i}$, where $\{e_i\}$ and $\{e^i\}$ are orthonormal tangent and cotangent frame fields of $Y$, respectively. Furthermore,
if $Y$ is an associative submanifold, we can use the Clifford multiplication of (25) (i.e. the cross product) to form the twisted Dirac operator $\bar{\mathcal{D}}_A : \Omega^0(Y, \nu) \to \Omega^0(Y, \nu)$

(34) \[ \bar{\mathcal{D}}_A = \sum e^i \cdot \nabla_{e_i} \]
The sections lying in the kernel of this operator are usually called harmonic spinors
twisted by $(E, A)$. Elements of the kernel of $\bar{\mathcal{D}}_A$ are called the harmonic spinors
twisted by $E$, or just the twisted harmonic spinors.

4. Deformations

In [M], McLean showed that the space of associative submanifolds of a $G_2$ manifold $(M, \varphi)$, in a neighborhood of a fixed associative submanifold $Y$, can be identified
with the harmonic spinors on $Y$ twisted by $E$. Since the cokernel of the Dirac op-
erator can vary, the dimension of its kernel is not determined (it has zero index
since $Y$ is odd dimensional). We will remedy this problem by deforming $Y$ in a
larger class of submanifolds. To motivate our approach we will first sketch a proof of
McLean’s theorem (adapting the explanation in [B3]). Let $Y \subset M$ be an associative
submanifold, $Y$ will determine a lifting $\hat{Y} \subset \hat{M}_\varphi$. Let us recall that the $G_2$ structure
$\varphi$ gives a metric connection on $M$, hence it gives a connection
$A_0$ and a covariant
differentiation in the normal bundle $\nu(Y) = T^\ast(Y) \otimes \nu$.
Recall that we identified $T^y_g(Y) \otimes \nu_g(Y)$ by the tangent space of the Grassmannian
of 3-planes $TG(3,7)$ in $T_g(M)$. So the covariant derivative lifts normal vector fields
$v$ of $Y \subset M$ to vertical vector fields $\tilde{v}$ in $T(\hat{M})|_{\tilde{Y}}$. We want the normal vector fields $v$ of $Y$ to move $Y$ in the class of associative submanifolds of $M$, i.e. we want the liftings $\hat{Y}$ of the nearby copies $Y_v$ of $Y$ (pushed off by the vector field $v$) to lie in $\hat{M}_\varphi \subset \hat{M}$ upstairs, i.e. we want the component of $\tilde{v}$ in the direction of the normal bundle $\hat{M}_\varphi \subset \hat{M}$ to vanish. By Lemma 3, this means $\nabla_{A_0}(v)$ should be in the kernel of the Clifford multiplication $c = c_\varphi : \Omega^0(T^\ast(Y) \otimes \nu) \to \Omega^0(\nu)$, i.e. $\bar{\mathcal{D}}_{A_0}(v) = c(\nabla_{A_0}(v)) = 0$, where $\mathcal{D}_{A_0}$ is the Dirac operator induced by the background connection $A_0$, i.e. the composition

(35) \[ \Omega^0(Y, \nu) \xrightarrow{\nabla_{A_0}} \Omega^0(Y, T^\ast(Y) \otimes \nu) \xrightarrow{c} \Omega^0(Y, \nu) \]
The condition $\bar{\mathcal{D}}_{A_0}(v) = 0$ implies $\varphi$ must be integrable at $Y$, i.e. the so(7)-metric connection $\nabla_{A_0}$ on $Y$ coincides with $G_2$-connection (c.f. [B2]).

Now we give a general version of the McLean’s theorem, without integrability
assumption on $\varphi$: Recall from (Section 3.1) that $\Phi^{-1} Z_\varphi(Y)$ is the set of associative
submanifolds $Y \subset M$, where $\Phi : \text{Im}(Y, M) \to \mathcal{Z}(Y)$ is the canonical section (Gauss map) given by $\Phi(f) = \tilde{f}$. Therefore, if $f : Y \hookrightarrow M$ is the above inclusion, then $\Phi(f) \in \mathcal{Z}_\varphi$. So this moduli space is smooth if $\Phi$ was transversal to $\mathcal{Z}_\varphi(Y)$.

**Theorem 4.** Let $(M^7, \varphi)$ be a manifold with a $G_2$ structure, and $Y^3 \subset M$ be an associative submanifold. Then the tangent space of associative submanifolds of $M$ at $Y$ can be identified with the kernel of a Dirac operator $\mathcal{D}_A : \Omega^0(Y, \nu) \to \Omega^0(Y, \nu)$, where $A = A_0 + a$, and $A_0$ is the connection on $\nu$ induced by the metric $g_\varphi$, and $a \in \Omega^1(Y, \text{ad}(\nu))$. In the case $\varphi$ is integrable $a = 0$. In particular, the space of associative submanifolds of $M$ is smooth at $Y$ if the cokernel of $\mathcal{D}_A$ is zero.

**Proof.** Let $f : Y \hookrightarrow M$ denote the imbedding. We consider unparameterized deformations of $Y$ in $\text{Im}(Y, M)$ along its normal directions. Fix a trivialization $TY \cong \text{im}(\mathcal{H})$, by (17) we have an identification $\tilde{f}^*(T^vM) \cong TY^* \otimes \nu + TY + \nu$. We first claim $\Pi \circ d\Phi(v) = \nabla_A(v)$, where $d\Phi$ is the induced map on the tangent space and $\Pi$ is the vertical projection.

$$
\begin{align*}
\Omega^0(Y, \nu) &= T_f \text{Im}(Y, M) \xrightarrow{d\Phi} T_f \mathcal{Z}(Y) = \Omega^0(Y, \tilde{f}^*(T^vM)) \xrightarrow{\Pi} \Omega^0(Y, T^*Y \otimes \nu) \\
&\Downarrow \exp \quad \Downarrow \exp \\
\text{Im}(Y, M) &= \Phi \to \mathcal{Z}(Y)
\end{align*}
$$
The two vertical maps \( v \rightarrow f_v \), and \( w \rightarrow (f)_w \) are exponential projections of tangent vectors, i.e. \( f_v(y) = \exp_{f(y)}(v) \) and \( (f)_w(y) = \exp_{f(y)}(w) \). It suffices to check this claim pointwise. Here for convenience view \( f \) as an inclusion \( Y \subset M \).

Let \( y = (y_1, y_2, y_3) \) be the normal coordinates of \( Y \) centered around \( y_0 \), and \( \{e_j\}_{j=1}^7 \) be an orthonormal frame field of \( M \) defined on \( Y \), with \( e_j(y_0) = \partial/\partial y_j \) for \( j = 1, 2, 3 \). To this data we can associate Fermi coordinates \((y, t)\) around \( f(y_0) \in M \) (they are a version of normal coordinates along a submanifold, see for example [G]):

\[
(y, t) \mapsto f \sum_j t_j e_j(y)
\]

where \( t = (t_1, \ldots, t_7) \). Then we can write \( f(y_0) = e_1 \wedge e_2 \wedge e_3 \). Hence by definition we can express \( d\Phi(v) = (f_v) = (f_v)_*(e_1) \wedge (f_v)_*(e_2) \wedge (f_v)_*(e_3) := e_1(v) \wedge e_2(v) \wedge e_3(v) \).

\[
d\Phi(v)(y_0) = \mathcal{L}_v(e_1 \wedge e_2 \wedge e_3) = \sum_{i=1}^3 (*e_i) \wedge \mathcal{L}_v(e_j)|_Y
\]

where \( \mathcal{L}_v \) denotes Lie derivative along \( v \), and * is the star of \( Y \). The metric connection is torsion free hence \( \mathcal{L}_v(e_j) = \nabla_{e_j}(v) - \nabla_v(e_j) \), where \( \nabla \) is the metric connection of \( M \). In case \((M, \varphi)\) is a \( G_2 \) manifold (i.e. when \( \varphi \) integrable), by (2) and (7), up to quadratic term \( \varphi = \varphi_0 \), therefore we can write:

\[
0 = \nabla_v(\varphi)|_Y = \nabla_v(e_1 \wedge e_2 \wedge e_3)|_Y = \sum_{j=1}^3 (*e_j) \wedge \nabla_v(e_j)|_Y \equiv \text{ which implies }
\]

\[
\Pi \circ d\Phi(v)(y_0) = \sum_j (*e_j) \wedge \nabla_v(e_j)
\]

where \( \{e_j\} \) is the dual coframe, and \( \nabla_v(e_j) \) is the normal component of \( \tilde{\nabla}_v(e_j) \), i.e. it is the induced connection on \( \nu(Y) \). The expression (38) can be viewed as an infinitesimal deformation of the 3-plane \( \tilde{f}(y_0) \). By the identification \( *e_j \leftrightarrow e^j \) we can view it as an element of the tangent space \( T^*Y \otimes \nu \) of the Grassmannian of 3-planes in \( T_{y_0}(M) \)

\[
\Pi \circ d\Phi(v)(y_0) = \sum_j e^j \otimes \nabla_v(e_j)(y_0) = \nabla_{A_0}(v)(y_0)
\]

When \( \varphi \) is not integrable, there is an extra term which we can write

\[
\sum (*e_j) \wedge \nabla_v(e_j) = \sum (*e_j) \wedge \nabla_v(e_j)
\]

where \( \nabla_v(e_j) \) is the normal component of \( \nabla_v(e_j) \). Notice \( \langle \nabla_v(e_k), e_k \rangle = 0 \), which is implied by \( v < e_k, e_k >_v = 0 \). So in this case (39) becomes

\[
\Pi \circ d\Phi(v)(y_0) = \nabla_{A_0}(v) + a(v) = \nabla_{A}(v)
\]

where \( a(v) = \sum e^j \otimes \nabla_v(e_j) \in \Omega^1(Y, ad(\nu)) \) and \( A = A_0 + a \). It easy to check that the expression \( a(v) \) is independent of the choice the orthonormal frame \( \{e_j\} \).
By (31) the vertical tangent space of \( \mathcal{Z}_\varphi(Y) \) is given by the kernel of the Clifford multiplication \( c_\varphi : \Omega^0(T^*Y \otimes \nu) \to \Omega^0(\nu) \). So, locally the moduli space of associative submanifolds of \((M, \varphi)\) is given by the kernel of \( \mathcal{D}_A \), i.e. the condition that \( d\Phi(v) \) lies in \( T^v_\varphi \mathcal{Z}(Y) \) is given by \( \Phi_A(v) = 0 \). The moduli space is smooth if \( \Phi \) is transversal to \( \mathcal{Z}_\varphi(Y) \), i.e. if the cokernel of \( \Phi_A \) is zero. Since \( T^v_\varphi \mathcal{Z}(Y) = \Omega^0(T^*Y \otimes \nu) \) and

\[
T \mathcal{Z}(Y)/T \mathcal{Z}_\varphi(Y) = T^v \mathcal{Z}(Y)/T^v \mathcal{Z}_\varphi(Y)
\]

to check transversality we look at the induced maps, and use \( \Pi \circ d\Phi_j(v) = \nabla_A(v) \)

\[
\Omega^0(\nu) = T_j \text{Im}(Y,M) \xrightarrow{df} T_j \mathcal{Z}(Y) \xrightarrow{\Pi} T^v_\varphi \mathcal{Z}(Y) \supset T^v \mathcal{Z}_\varphi(Y)
\]

**Remark 3.** This theorem can also be proved by generalizing McLean’s proof: The condition that an associative \( Y \subset M \) remains associative, when moved via the exponential map along a normal vector field \( v \in \Omega^0(Y,\nu) \), is \( \mathcal{L}_v(\chi)|_Y = 0 \). We can choose local coordinates \((x_1,\ldots,x_7)\) on \( M \), such that \((x_1,x_2,x_3)\) gives the coordinates of \( Y \). By (13) and (14) \( \chi = \sum a^\alpha_j \, dx^\alpha \otimes \partial / \partial x^\alpha \), with \( \alpha \notin J \) and \( a^\alpha_{123}|_Y = 0 \)

\[
(41) \quad \mathcal{L}_v(\chi)|_Y = \sum a^\alpha_j a^\alpha_{123} \frac{\partial}{\partial x^\alpha} + \sum a^\alpha_j a^\alpha_{123} \mathcal{L}_v(dx|_Y \otimes \frac{\partial}{\partial x^\alpha}) = 0
\]

McLean treated integrable \( \varphi \) case, i.e when \((M, \varphi)\) is a \( G_2 \) manifold. In this case the first term vanishes, and the second becomes \( \mathcal{D}_{A_0}(v) \otimes dx_{123} \). But notice that the first term \( a(v) \) is linear in \( v \) and takes values in \( \Omega^0(Y,\nu) \), hence \( a \in \Omega^0(Y,\text{ad}(\nu)) \). So in the non-integrable case we get a twisted Dirac equation \( \mathcal{D}_A(v) = 0 \), where \( A = A_0 + a \).

If in the proof of Theorem 4 we replace the \( G_2 \) structure \( \varphi \) with another \( G_2 \) structure \( \psi \) inducing the same metric, the identification of the bundle \( TY^\perp = \nu \) doesn’t change but the Clifford action \( c_\varphi \) changes to another one \( c_\psi \), corresponding to another 4-dimensional Clifford representations of \( T^*Y \). These two representations are conjugate by a gauge automorphism \( \gamma \) of \( \nu \).

\[
\begin{align*}
\Omega^0(T^*Y \otimes \nu) & \xrightarrow{c_\varphi} \Omega^0(\nu) \\
1 \otimes \gamma & \downarrow \quad \gamma \downarrow \\
\Omega^0(T^*Y \otimes \nu) & \xrightarrow{c_\psi} \Omega^0(\nu)
\end{align*}
\]

Therefore, if we call the Dirac operator induced by \( \psi \) by \( \mathcal{D}_{A_1} \), we can write

\[
\gamma(\mathcal{D}_{A_1}(w)) = \sum dy^j, \gamma(\nabla_j(w)) = \sum dy^j, \nabla_j \gamma(w) - dy^j, (\nabla_j \gamma)(w)
\]

where the dot "\( \cdot \)" denotes the Clifford product \( c_\varphi \). So, \( D_{A_1}(w) = 0 \) gives a twisted version of the Dirac equation \( \mathcal{D}_{A_0}(v) = 0 \) where \( v = \gamma(w) \), this is because \( \gamma(\mathcal{D}_{A_1}(w)) = \mathcal{D}_{A_0+a}(\gamma(w)) \), where \( a = - \sum dy^j, (\nabla_j \gamma) \gamma^{-1} \). In Theorem 6 we will use the twisting of the Dirac operator, under deformations of \( \varphi \), to obtain its surjectivity.
5. Transversality

We can make the cokernel of Dirac operator $\mathcal{D}_{A_0}$ zero either by deforming the Gauss map $\Phi : Im(Y, M) \to Z(Y)$, or by deforming the $G_2$ structure $\varphi$. Changing $\varphi$ can be realized by deforming $\varphi$ by a gauge transformation: Recall that the $G_2$ structures $\varphi$ on $M$ are the sections $\Omega^3_+ (M)$ of the bundle (5). Also $GL(7, \mathbb{R})$ conjugates $G_2 = G_2^{\chi_0}$ to any other $G_2$ subgroup $G_2^\varphi$ of $GL(7, \mathbb{R})$ where

$$G_2^\varphi = \{ A \in GL(7, \mathbb{R}) \mid A^* \varphi = \varphi \} \xrightarrow{\varphi} SO(7)$$

If we are interested in the $G_2$ structures inducing the same metric, we replace $GL(7, \mathbb{R})$ with $SO(7)$. $SO(7)$ acts on $G(3, 7)$ permuting submanifolds $G^\varphi (3, 7)$, where $\varphi \in \Omega^3_+ (M)$. More generally the gauge group $G(P)$ of $P = P_{SO(7)} \to M$ acts on $\tilde{M}$ permuting $\tilde{M}_p$'s. Recall that $G(P) = \{ P \xrightarrow{\varphi} P \mid s(pg) = s(p)g \}$, which can be identified with sections $\Omega^0(M; Ad(P))$ of the bundle $Ad(P) \to M$ (c.f. [AMR]), where

$$Ad(P) = P \times_{Ad} SO(7) = \{ [p, h] \mid (p, h) \sim (pg, g^{-1}hg) \}$$

One can also identify: $G(P) = \{ s : P \to SO(7) \mid s(pg) = g^{-1}s(p)g \}$

The tangent space of $G(P)$ at the identity $I$ are the sections $g(P) = \Omega^0(M, ad(P))$ of the associated bundle of Lie algebras $ad(P) = P \times_{Ad} so(7) \to M.$ Similarly

$$g(P) = \{ h : P \to so(7) \mid h(pg) = h(p)g - gh(p) \}$$

We can identify $T_s(G_P(M)) \xrightarrow{i_s} g(P)$, by $s \mapsto s^{-1}ds$. There is also an action $G(P) \times M \to M$ given by $(s, [p, L]) \mapsto s[p, L] := [ps(p), s(p)L]$, which we will simply denote it by $(s, L) \mapsto s.L$. There is the pull-back action $G(P) \times \Omega^3_+ (M) \to \Omega^3_+ (M)$ given by $(s, \varphi) \to s^*\varphi$. In particular, $s\tilde{M}_\varphi = \tilde{M}_{s^*\varphi}$. Put another way, if $\chi = \chi_\varphi$ is the 3-form of Definition 3 and $L \in \tilde{M}$, then $\chi|_L = 0 \iff s^*\chi|_{s^{-1}L} = 0$. Hence the 3-plane $sL$ is $\varphi$-associative $\iff L$ is $s^*\varphi$ -associative (similar to the process in the Kleiman transversality, c.f. [AK])

From the above action, we see that the space of $G_2$ structures $\tilde{3}_+(M)$ which induce the same metric on $M$ has the following identification:

**Lemma 5.** Let $G(P_{G_2})$ be the stabilizer of the action of $G(P)$ on $\tilde{3}_+(M)$ (i.e. the gauge transformations fixing $\varphi$) then:

$$\tilde{3}_+(M) = G(P)/G(P_{G_2}) = \Omega^0(M, P \times_{SO(7)} \mathbb{RP}^7)$$

**Proof.** Clearly $G(P)$ acts transitively on $\tilde{3}_+(M)$ with stabilizer $G(P_{G_2})$. To see the second equality, we identify the fibers of the coset space with the fibers of $\tilde{3}_+(M) \to M$ by the map:

$$\mathbb{RP}^7 = SO(7)/G_2^\varphi \to \tilde{3}_+(M)$$
\[ G_2^c s \mapsto s^* \varphi. \] The adjoint action of \( SO(7) \) on \( SO(7) \) moves cosets
\[ G_2^c s \mapsto (g^{-1} G_2^c g) g^{-1} s g = G_2^c \varphi g^{-1} s g \]
Hence by the above identification, on \( \mathbb{R} P^7 \) it induces \( \varphi \mapsto g^* \varphi. \]

Now we can deform the canonical section \( \Phi : \text{Im}(Y, M) \to Z(Y) \) by the map
\[ (s, f) \mapsto \Phi_s(f) = \Phi(f) = s(\tilde{f}), \]
where \( \Phi(f) \) is transversal to \( Z(Y) \). This is because, if \( a(v) \) is a vertical deformation of the 3-plane \( y \) is a section of the pull back of the vertical tangent bundle of \( \tilde{M} \to M \) over \( Y \), i.e. an element \( a(v) \in \text{Im}(\tilde{f}) \). More specifically, if we decompose \( s^{-1} ds(v) \) as an element of \( \text{so}(7) \) on \( T_{\tilde{f}(y)}(M) = T_y Y \oplus \nu_y(Y) \) in block matrices we can write:

\[ (43) \quad s^{-1} ds(v) |_{\tilde{f}(y)} = \begin{pmatrix} \ast & -\alpha(v) \tilde{f} \\ \alpha(v) & \ast \end{pmatrix} \]

Because \( \alpha(v) \) is linear in \( v \), we can view \( \alpha \in \Omega^1(Y, \text{ad} \nu) \), therefore we can express \( s^{-1} \Pi \circ d\Phi_s(v) = \nabla_{A_0}(v) + \alpha(v) = \nabla_{\tilde{A}}(v) \) with \( \tilde{A} = A_0 + \alpha \). So the transversality is measured by the cokernel of the twisted Dirac operator \( c_{\tilde{A}}(\nabla_{\tilde{A}}) = \varnothing_{\tilde{A}} \), where \( c_{\varphi} \) is the Clifford multiplication. Now by choosing \( \alpha(v) \) we show that we can make \( \varnothing_{\tilde{A}} \) onto. This is because, if \( \varnothing_{A_0} \) is not already onto, we choose \( 0 \neq w \in \text{im}(\varnothing_{A_0}) \). By self adjointness of the Dirac operator \( 0 = \langle \varnothing_{A_0}(v), w \rangle = \langle v, \varnothing_{A_0}(w) \rangle \), for all \( v \).

So \( \varnothing_{A_0}(w) = 0 \), by analytic continuation \( w \neq 0 \) on an open set. Then \( w \in \text{im}(\varnothing_{A_0}) \) implies \( c_{\varphi}(\alpha(v)), w \rangle = 0 \) and hence \( w = 0 \), which is a contradiction. The last implication follows from by choosing \( s \) in (43) we can get the full Lie algebra \( \text{so}(7) \), and hence \( v \mapsto a(v) \) is onto, and the Clifford multiplication \( c \) is onto (Lemma 3).

So we obtain a smooth manifold \( \Phi_s^{-1} Z_{\varphi}(Y) \), and by choosing a regular value \( s \) of the projection \( \Phi_s^{-1} Z_{\varphi}(Y) \to G_P(M) \) we get \( \Phi_s^{-1} Z_{\varphi}(Y) \) smooth (note that the derivative of the projection is Fredholm).

Theorem 6. \( \Phi_s \) is transversal to \( Z_{\varphi}(Y) \). Also \( \Phi_s \) is transversal to \( Z_{\varphi}(Y) \) for a generic choice of \( s \), equivalently \( \Phi \) is transversal to \( Z_{s^* \varphi}(Y) \) for a generic \( s \).

Proof. : Let \( \Phi(s, f) \in Z_{\varphi}(Y) \). We can check transversality of \( \Phi \) at \( (s, f) \) by computing its derivative. By the Leibnitz rule and Theorem 4 we can compute
\[ s^{-1} \Pi \circ d\Phi_s(h, v) = g_P(M) \oplus \Omega^0(\nu) \to T_{\tilde{f}(y)}Z(Y) \to T_{\tilde{f}(y)}Z(Y) = \Omega^0(T^* Y \otimes \nu) \]
where \( d\Phi(h, v) = s(f) \nabla_{A_0}(v) + s^{-1} ds(v) \tilde{f} \), and \( v = f_v \) is the perturbation of the inclusion \( f \). Observe that \( \text{ad}(P) = \text{End}(TM) \), and the map \( y \mapsto s^{-1} ds(v) \tilde{f}(y) \) is a vertical deformation of the 3-plane \( y \mapsto \tilde{f}(y) = T_y Y \), hence it is a section of the pull back of the vertical tangent bundle of \( \tilde{M} \to M \) over \( Y \), i.e. an element \( a(v) \in T_{\tilde{f}(y)}Z(Y) = \Omega^0(T^* Y \otimes \nu) \). More specifically, if we decompose \( s^{-1} ds(v) \) as an element of \( \text{so}(7) \) on \( T_{\tilde{f}(y)}(M) = T_y Y \oplus \nu_y(Y) \) in block matrices we can write:

\[ (43) \quad s^{-1} ds(v) |_{\tilde{f}(y)} = \begin{pmatrix} \ast & -\alpha(v) \tilde{f} \\ \alpha(v) & \ast \end{pmatrix} \]

Because \( \alpha(v) \) is linear in \( v \), we can view \( \alpha \in \Omega^1(Y, \text{ad} \nu) \), therefore we can express \( s^{-1} \Pi \circ d\Phi_s(v) = \nabla_{A_0}(v) + \alpha(v) = \nabla_{\tilde{A}}(v) \) with \( \tilde{A} = A_0 + \alpha \). So the transversality is measured by the cokernel of the twisted Dirac operator \( c_{\tilde{A}}(\nabla_{\tilde{A}}) = \varnothing_{\tilde{A}} \), where \( c_{\varphi} \) is the Clifford multiplication. Now by choosing \( \alpha(v) \) we show that we can make \( \varnothing_{\tilde{A}} \) onto. This is because, if \( \varnothing_{A_0} \) is not already onto, we choose \( 0 \neq w \in \text{im}(\varnothing_{A_0}) \). By self adjointness of the Dirac operator \( 0 = \langle \varnothing_{A_0}(v), w \rangle = \langle v, \varnothing_{A_0}(w) \rangle \), for all \( v \).

So \( \varnothing_{A_0}(w) = 0 \), by analytic continuation \( w \neq 0 \) on an open set. Then \( w \in \text{im}(\varnothing_{A_0}) \) implies \( c_{\varphi}(\alpha(v)), w \rangle = 0 \) and hence \( w = 0 \), which is a contradiction. The last implication follows from by choosing \( s \) in (43) we can get the full Lie algebra \( \text{so}(7) \), and hence \( v \mapsto a(v) \) is onto, and the Clifford multiplication \( c \) is onto (Lemma 3).
Theorem 6 says that the space of \( s^*\varphi \) associative deformations of an \( \varphi \) associative submanifold \( Y \subset M \), where \( s \in \mathcal{G}_P(M) \), is a smooth (infinite dimensional) manifold. Infinitesimally these deformations correspond to the kernel of the twisted Dirac operator, twisted by the connections in the normal bundle \( \nu(Y) \). Define

\[
\sigma : \mathcal{G}(P) \to \Omega^0(\tilde{M}, \Xi^* \otimes \lambda(\mathbb{V}))
\]

by \( \sigma(s)(L)(v) = \alpha_s(v, L) \in \Xi^* \otimes \mathbb{V} \), where \( \alpha_s(v, L) \) is obtained by decomposing \( s^{-1}ds(v) \in \Omega^0(M, \text{ad}(P)) \) on \( TM = L \oplus L^\perp \) as an element of \( \text{so}(7) \)

\[
s^{-1}ds(v) \mid_L = \begin{pmatrix} * & -\alpha(v, L)^t \\ \alpha(v, L) & * \end{pmatrix}
\]

We can think of \( \Omega^0(\tilde{M}, \Xi^* \otimes \lambda(\mathbb{V})) \) as an universal space parameterizing connections on \( \nu \to Y \). The Gauss map \( \tilde{f} \) of any imbedding \( f : Y \hookrightarrow M \) pulls back \( \Xi^* \otimes \lambda(\mathbb{V}) \) to the parameter space \( \Omega^1(Y, \lambda(\nu)) \) of the connections on \( \nu(Y) \).

\[
\Omega^0(\tilde{M}, \Xi^* \otimes \lambda(\mathbb{V})) \xrightarrow{\tilde{f}^*} \Omega^1(Y, \lambda(\nu))
\]

Clearly the set \( \Omega^1(\tilde{M}, \lambda(\mathbb{V})) \) can also be used as the universal parameter space. As in Section 3.2, given any imbedding \( f : Y \hookrightarrow M \), we can deform the background connection \( A_0 \to A = \mathcal{A}(b, a) \) in the normal bundle \( \nu(Y) \), with \( b \in \Omega^1(Y, \lambda_+(\nu)) \) and \( a \in \Omega^1(Y, \lambda_-(\nu)) \), and get a perturbed version of (35)

\[
\Omega^0(\nu) \times \Omega^1(\lambda_+(\nu)) \xrightarrow{\mathcal{D}_A} \Omega^0(\nu)
\]

with the twisted Dirac equation \( \mathcal{D}_\mathcal{A}(v) = c(\nabla_\mathcal{A}(v)) = \mathcal{D}_{A_0}(v) + \alpha v = 0 \), where \( \alpha = (b, a) \). Here we prefer perturbing by \( a \) (perturbing \( b \) has the effect of perturbing the metric on \( Y \)). A generic nonzero \( a \) makes the map \( v \mapsto \mathcal{D}_{A_0 + a}(v) \) surjective. We can choose this perturbation term \( a \) universally.

### 6. Complex Associative Submanifolds

Let \( (M, \varphi) \) be a manifold with a \( G_2 \) structure. Here we will study an interesting class of associative submanifolds whose normal bundles come with an almost complex structure. The subgroups \( U(2) \subset SO(4) \subset G_2 = G_2^\mathbb{C} \), more specifically

\[
(S^1 \times SU(2))/\mathbb{Z}_2 \subset (SU(2) \times SU(2))/\mathbb{Z}_2 \subset G_2
\]

give a \( U(2) \)-principal bundle \( \mathcal{P}_{G_2}(M) \to \tilde{M}_\varphi = \mathcal{P}_{G_2}(M)/U(2) \). Also \( \tilde{M}_\varphi \) is the total space of an \( S^2 \) bundle \( \tilde{M}_\varphi \to M_\varphi = \mathcal{P}_{G_2}(M)/SO(4) \), which is just the sphere bundle

\[
\tilde{M}_\varphi = S(\Xi) \to \tilde{M}_\varphi
\]

of the \( \mathbb{R}^3 \)-bundle: \( \lambda_+(\mathbb{V}) = \Xi \to \tilde{M}_\varphi \). We can identify the sections of (48) with almost complex structures on \( \mathbb{V} \). Notice \( \tilde{M}_\varphi \to M \) is a bundle with fibers \( G_2/U(2) \), which we can view as the complex version of the associative Grassmanns \( G_\mathbb{C}^\mathbb{C}(3, 7) \).
In fact if $V_2(M)$ and $G_2(M)$ are the bundle of orthonormal 2-frames and oriented 2-planes in $M$ respectively, the fibration $V_2(M) \to G_2(M)$ can be identified by:

$$\mathcal{P}_{G_2}(M)/SU(2) \to \mathcal{P}_{G_2}(M)/U(2)$$

with its projection \(\{u,v\} \mapsto (\langle u,v,u \times v> , u \times v)\), and also the projection map $\mathcal{P}_{G_2}(M) \to \mathcal{P}_{G_2}(M)/SU(2)$ on the fibers is given by the map $G_2 \to V_2(\mathbb{R}^7)$ defined by $\{v_1,v_2,v_3\} \mapsto \{v_1,v_2\}$ (recall the definition of $G_2$ in (1)). Put another way, $G_2$ acts transitively on $V_2(\mathbb{R}^7)$ with stabilizer $SU(2)$. By summing up above:

**Proposition 7.** $M_\varphi = S(\Xi) = G_2(M)$

More generally, for the Riemannian manifold $(M^7, g_\varphi)$ we can take the sphere bundle $\tilde{M} \to M$ of $\lambda_+ (\mathcal{V}) \to \tilde{M}$, and get codimension 4 inclusion of the smooth manifolds $\tilde{M}_\varphi \subset M$ (of dimensions 17 and 21). The sections of the bundle $\tilde{M} \to M$ gives the parametrization of the almost complex structures on $\mathcal{V}$, and $\tilde{M} \to M$ is a bundle with fibers $G_C(3,7) := SO(7)/U(2) \times SO(3)$. For all $G_2$ structures $\varphi$ inducing the same metric on $M$, we have the inclusions $G_2^\varphi \hookrightarrow SO(7)$ inducing imbeddings $G_2(M) = \tilde{M}_\varphi \hookrightarrow \tilde{M}$, which is fiberwise $\langle u,v \rangle \mapsto \langle u,v,u \times v \rangle$

$$G(2,7) = G_2^\varphi/U(2) \hookrightarrow SO(7)/U(2) \times SO(3)$$

By [1] the bundle $V_2(M) \to M$ has always a section $\Lambda = \{u,v\}$, which induces sections of the bundles $\tilde{M}_\varphi \to \tilde{M}$ and $\tilde{M}_\varphi \to M$ (for simplicity we will abuse notation and denote all these sections by $\Lambda$ also). So $\Lambda$ gives an almost complex structure on $\mathcal{V} \to \tilde{M}_\varphi$. By $\Lambda$, we can pull back $\Xi$ and $\mathcal{V}$ to bundles $\mathcal{E}$ and $\mathcal{V}$ on $M$, respectively, and $\mathcal{V}$ has an almost complex structure (by the discussion following Lemma 2 we can describe this complex structure with the cross product with $u \times v$).

**Definition 7.** From now on, we will denote a 7-manifold with a $G_2$ structure and a nonvanishing 2-frame field $\Lambda$ with $(M, \varphi, \Lambda)$.

Given $(M, \varphi, \Lambda)$, then the induced $U(2)$ structure on $\mathcal{V} \to \tilde{M}_\varphi$ canonically lifts to a $Spin^c(4)$ structure by the diagram:

$$\begin{array}{ccc}
Spin^c(4) & \xrightarrow{\lambda^2} & SO(4) \times S^1 \\
U(2) & \to & SO(4) \times S^1
\end{array}$$

where $U(2) = (S^1 \times S^3)/\mathbb{Z}_2$, $SO(4) = (S^3 \times S^3)/\mathbb{Z}_2$, $Spin^c(4) = (S^3 \times S^3 \times S^1)/\mathbb{Z}_2$, where the horizontal map $[\lambda,A] \mapsto ([\lambda,A], \lambda^2)$ lifts to the map $[\lambda,A] \mapsto (\lambda,A,\lambda)$. This means there is a $\mathbb{C}^2$-bundle $\mathcal{W} \to \tilde{M}_\varphi$ with $V_C = \mathcal{W} \oplus \mathcal{W}$, and transition function $\lambda^2$ gives the determinant line bundle $K = \Lambda^2 \mathcal{W} \to \tilde{M}_\varphi$. Also we can write $V_C = W^+ \oplus W^-$ with $W^+ = K^{-1} + \mathbb{C}$ and $W^- = \mathcal{W}$. Recall $\Xi^* = \lambda_+(\mathcal{V}) = \lambda_+^2(\mathcal{V}) = K + \mathbb{R}$. We have a Clifford action $\mathcal{V} \otimes W^+ \to W^-$ which extends to Clifford action $\Xi^* \otimes W^+ \to W^+$.
The identifications $\mathbb{W}^+ \otimes \bar{\mathbb{W}}^- = \mathbb{C} \oplus \mathbb{C} \oplus K \oplus \bar{K} = (K \oplus \mathbb{R}) \mathbb{C} \oplus \mathbb{C} = \Xi^*_\mathbb{C} \oplus \mathbb{C}$ gives the usual quadratic bundle map of the Seiberg-Witten theory (c.f. [A]):

$$\sigma : \mathbb{W}^+ \otimes \mathbb{W}^+ \to \Xi^*_\mathbb{C}$$

$$\sigma(x, x) = \left( \frac{|z|^2 - |w|^2}{2}, \bar{z}w \right), \text{ where } x = (z, w)$$

**Definition 8.** A submanifold $f : Y^3 \hookrightarrow M$ of $(M^7, \varphi, \Lambda)$ is called $\Lambda$-associative if $\bar{f} = \Lambda \circ f$ where $\bar{f}$ is the Gauss map, and it is called almost $\Lambda$-associative if it comes from transverse section of the bundle $\mathbb{V} \to M$ (recall $\mathbb{V}$ is obtained from $\Lambda$).

An $\Lambda$-associative, more generally almost $\Lambda$-associative submanifold $Y$ of $(M, \varphi, \Lambda)$ induce canonical isomorphisms $TY \cong \bar{f}^*(\Xi)$ and $\nu(Y) \cong \bar{f}^*(\mathbb{V})$ (by transversality).

So normal bundle of any almost $\Lambda$-associative submanifold $Y^3 \subset M^7$ has a $U(2)$ structure, therefore it has a $\text{Spin}^c(4)$ structure, with induced $\mathbb{C}^2$-bundle $W \to Y$ and its determinant line bundle $K \to Y$, and a Clifford action of $T^*Y \otimes W \to W$ (induced from the cross product). An example of a $\Lambda$ associative submanifold is the zero section of the spinor bundle $S \to Y^3$ (the $G_2$ manifold constructed in [BSn]).

In general, the background $SO(4)$ connection $A_0$ on the normal bundle $\nu(Y)$ of a $\Lambda$-associative submanifold $Y$ may not reduce to a $U(2)$ connection if the 1-form whose dual gives the splitting $TY \cong K \oplus \mathbb{R}$ is not parallel. Nevertheless from the $\text{Spin}^c(4)$ structure on $\nu(Y)$ we do get a connection on the complex bundle $W \to Y$ provided we pick a connection on the line bundle $K \to Y$ (from (49)). In the next section we will study the local deformation space of $\Lambda$-associative manifolds, by deforming them in the complex bundle $W$, with the help of the connections on $K$.

**Remark 4.** Associative submanifolds with $\text{Spin}^c(3)$ structure $(Y, c) \hookrightarrow (M, \varphi)$ come equipped with an $U(2)$ structure (hence $\text{Spin}^c(4)$ structure) on their normal bundles (Lemma 2). We can free the deformations space these manifolds from the extra parameter $c$, by picking up a generic $\Lambda$, and studying the deformations of more relax almost $\Lambda$-associative submanifolds $Y \subset (M, \varphi, \Lambda)$. In this case the $\text{Spin}^c(3)$ structure on $TY$ comes from the pull-back. Also, by further deforming the 2-plane field $\Lambda$ on $M$ deforms the the $\text{Spin}^c$ structure on $Y$.

**Remark 5.** We could have considered complex structures on $\mathbb{V}$ corresponding to the right reduction, i.e. the subgroup $(SU(2) \times S^1)/\mathbb{Z}_2 \subset (SU(2) \times SU(2))/\mathbb{Z}_2 \subset G_2$. In this case, they correspond to the sections of the $S^2$ bundle $\lambda_-(\mathbb{V}) \to \tilde{M}_\varphi$. Here we opted to the left reductions since they concretely relate to $\Xi$ by $\lambda_+(\mathbb{V}) = \Xi$. 
7. Deforming $\Lambda$-associative submanifolds

Let $(M, \varphi, \Lambda)$ be a manifold with $G_2$ structure and a non-vanishing 2-plane field, $\mathcal{M}(M, \varphi, \Lambda)$ be the space of $\Lambda$-associative submanifolds. Here we will study the local “complex” deformations of $\mathcal{M}(M, \varphi, \Lambda)$ near a particular $f : Y \hookrightarrow M$. These are the deformations of $Y$ inside its complex normal bundle $W$, with the help of the connections $\mathcal{A}(K)$ on the line bundle $K = \det W$. These deformations are identified with the kernel of a twisted Dirac operator twisted by the connections in $\mathcal{A}(K)$. Introducing new variables $\mathcal{A}(K)$ makes the deformation space smooth. Up to this point this section can be viewed as a version of Theorem 4 for the $\Lambda$-associative submanifolds. But now the connection parameter can be constrained with the natural map (50) to obtain Seiberg-Witten like equations, which gives a compactness result for this more restricted local deformation space of $Y$. Reader should note that these equations are $\text{Spin}^c(4)$ Seiberg-Witten equations on $Y^3$ (which are usually associated to 4-manifolds), as opposed to the usual $\text{Spin}^c(3)$ Seiberg-Witten equations. The Clifford action $T^*Y \otimes \mathbb{C} \rightarrow W$ is induced via the identification $\Lambda^2 + W = T^*Y$, it is also induced by the cross product operation on $M$.

Let $Y \in \mathcal{M}(M, \varphi, \Lambda)$. Let $W \rightarrow Y$ be the complex bundle associated to $\nu(Y)$, and $K \rightarrow Y$ be its determinant line bundle. Let $B_0$ be the background connection on $\nu(Y)$ (induced by $\varphi$), then as discussed in last section $B_0$ along with $A \in \mathcal{A}(K)$ defines a connection on $W \rightarrow Y$, denote by $\mathcal{A} = B_0 \oplus A$. We can write $A = A_0 + a$ with $a \in \Omega^1(Y) = T_{A_0}A(K)$ (tangent space of connections) and $\mathcal{A} = \mathcal{A}(a)$. Then we get a complex version of the map (47) $(v, a) \mapsto \mathcal{D}_{\mathcal{A}}(v) = \mathcal{D}_{\mathcal{A}(0)}(v) + a.v$

\begin{equation}
\Omega^0(Y, W) \times \Omega^1(Y, i\mathbb{R}) \xrightarrow{\mathcal{D}_{\mathcal{A}}} \Omega^0(Y, W)
\end{equation}

which is the derivative of a similarly defined map

\begin{equation}
\Omega^0(Y, W) \times \mathcal{A}(K) \rightarrow \Omega^0(Y, W)
\end{equation}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure3.png}
\caption{Figure 3.}
\end{figure}
In each slice $A(a)$, we are deforming along normal vector fields by the connection $A(a)$, which is a perturbation of the background connection $A(0)$. To get compactness we can cut down this parametrized moduli space with an additional equation (induced from the map (50)) of the Seiberg-Witten theory $Ψ^{-1}(0)$, where

$$Ψ : Ω^0(Y, W) × A(K) → Ω^0(Y, W) × Ω^2(Y, iℝ)$$

induced from the map (50)) of the Seiberg-Witten theory $Ψ^{-1}(0)$, where

$$DΦ(v) = 0$$

$$∗F_A = σ(v, v)$$

where $F_A$ is the curvature of the connection $A = A_0 + a$ in $K$, and $*$ is the star operator on $Y$. Note that $Y$ comes equipped with the natural submanifold metric. Now we proceed exactly as in the Seiberg-Witten theory of 3-manifolds (e.g. [C], [Lin], [Mal], [W]). To obtain smoothness of $Ψ^{-1}(0)$, we perturb the equations by 1-forms $δ ∈ Ω^1(Y)$ and get a new equation $Φ = 0$, where

$$Φ : Ω^0(Y, W) × A(K) × Ω^1(Y) → Ω^0(Y, W) × Ω^1(Y, iℝ)$$

We can choose the perturbation term universally $δ = f^*(Δ)$, where $Δ ∈ Ω^1(M)$. Then $Φ$ has a linearization:

$$DΦ(v, a, δ) = (DΦ_a(v) + a.ν_0, σ(v, v) + iδ − 2σ(v_0, v))$$

We see that $Φ^{-1}(0)$ is smooth and the projection $Φ^{-1}(0) → Ω^1(Y)$ is onto, so by Sard’s theorem for a generic choice of $δ$ we can make $Φ^{-1}(0)$ smooth, where $Φ_δ(v, A) = Φ(v, A, δ)$. The bundle $W$ of $Y$ has a complex structure, so the gauge group $G(K) = Map(Y, S^1)$ acts on the solution set $Φ^{-1}(0)$, and makes the quotient $Φ^{-1}(0)/G(K)$ a smooth zero-dimensional manifold. This is because the infinitesimal action of $G(K)$ on the complex $Φ_δ : Ω^0(Y, W) × A(L) → Ω^0(Y, W) × Ω^1(Y, iℝ)$ is given by the map

$$Ω^0(Y, iℝ) → Ω^0(Y, W) × Ω^1(Y, iℝ)$$

where $G(f) = (f.ν_0, df)$. So after dividing by $G$, tangentially the complex $Φ_δ$ becomes

$$Ω^0(Y, iℝ) → Ω^0(Y, W) × Ω^1(Y, iℝ) → Ω^0(Y, W) × Ω^1(Y, iℝ)/G$$

Hence the index of this complex is the sum of the indices of the Dirac operator $DΦ_0 : Ω^0(Y, W) → Ω^0(Y, W)$ (which is zero), and the index of the following complex

$$Ω^0(Y, iℝ) × Ω^1(Y, iℝ) → Ω^0(Y, iℝ) × Ω^1(Y, iℝ)$$

given by $(f, a) → (d^*a, df + *da)$, which is also zero since $Y^3$ has zero Euler characteristic. Furthermore, $Φ^{-1}(0)/G(K)$ is compact and oriented (the same proof as in the Seiberg-Witten theory). Hence we get a number $SW_Y(M)$. Here we don’t worry about metric dependence of $SW_Y(M)$ since we have a fixed background metric.
induced from the $G_2$ structure. Hence we associated a number to a $\Lambda$-associative submanifold $Y$ of $(M, \varphi, \Lambda)$. In particular, $Y$ moves in an unobstructed way along the parametrized sections the complex normal bundle $\Omega^0(Y, W) \times \mathcal{A}(L)$. Furthermore all these constructions work for almost $\Lambda$-associative submanifolds. So we have:

**Theorem 8.** Let $Y$ be an almost $\Lambda$-associative submanifold of $(M, \varphi, \Lambda)$. By cutting down the space of parametrized complex deformations of $Y$ with an additional equation as in (53) we obtain a zero dimensional compact smooth oriented manifold, hence we can associate a number $\Lambda^\varphi(Y) \in \mathbb{Z}$.

**Remark 6.** Clearly $\Lambda^\varphi(Y)$ is invariant under small isotopies through almost $\Lambda$-associative submanifolds $Y \subset (M, \varphi, \Lambda)$.

The equations (53) can be induced universally from equations on $(M^7, \varphi, \Lambda)$ by restriction: The 2-frame field $<u, v>$ gives a splitting of the tangent bundle $TM = E \oplus V$ with an $SO(3)$ bundle $E = <u, v, u \times v>$ and a $U(2)$-bundle $V = E^\perp$, such that $\lambda_+(V) = E$. Let $W \to M$ be the induced $\mathbb{C}^2$-bundle, and $K \to M$ be the determinant line bundle of $W$. We can define an action $\Lambda^2(T^*M) \otimes W \to W$: For $w = x + y \in TM$, with $x \in E$, $y \in V$ and $z \in W$ with $w.z = xz$. It is easy to check that this is a partial Clifford action, i.e. $w.(w.z) = -|x|^2z \text{Id}$, and it extends to an action $\Lambda^2(T^*M) \otimes W \to W$, and we have the map $\sigma : W \otimes W \to E_C$ of (50).

These bundles inherit connections from the Levi-Civita connection of $(M, g_{\varphi})$. Let $\mathcal{A}(K)$ be the connections on $K$. Let $A_0$ denote the background connections. Then any $A \in \mathcal{A}(K)$ along with $A_0$ determines a connection on $W$. Write $A = A_0 + a$ with $a \in \Omega^1(M)$. Hence for $A \in \mathcal{A}(K)$ we can define a partial Dirac operator $\bar{D}_A(v) = \bar{D}_{A_0}(v) + a.v$ on $W \to M$, which is the composition:

$$\Omega^0(M, W) \xrightarrow{\nabla_A} \Omega^0(M, T^*M \otimes W) \xrightarrow{\sigma} \Omega^0(M, W)$$

We can now write the global version of the equations (54) on $M$ in the usual way

$$\phi : \Omega^0(M, W) \times \mathcal{A}(L) \to \Omega^0(M, W) \times \Omega^1(M)$$

which is

$$\bar{D}_A(v) = 0$$

$$*F_A = \sigma(v, v)$$

where $* : TM \to TM$ is the star operator on $E$ and zero on $V$. We can perturb these equations by 1-forms to $\Phi = 0$, and proceed as before. $W$ has a complex structure. The gauge group $\mathcal{G}(L) = \text{Map}(M, S^1)$ acts on the solution set $\Phi^{-1}(0)$, and the quotient $\Phi^{-1}(0)/\mathcal{G}(L)$ can be formed. To sum up we have:

**Proposition 9.** Any almost $\Lambda$-associative submanifold $f : Y^3 \hookrightarrow (M, \varphi, \Lambda)$ pulls back the equations (54) to the Seiberg-Witten equations (53) on $Y$. 

8. Associative 3-Plane Fields of $G_2$ Manifolds

Recall that, any non-vanishing oriented 2-plane field $\Lambda = \langle u, v \rangle$ on $(M, \varphi)$ determines a section $\Lambda_\varphi : M \to \tilde{M}_\varphi \subset \tilde{M}$. In particular, it gives a non-vanishing associative 3-plane field $E = E_{\Lambda, \varphi} \to M$ on $M$, and a complex structure on the complementary 4-plane field $V = V_{\Lambda, \varphi} \to M$, and a splitting $TM = E \oplus V$, with $\lambda_+(V) = E$. From the construction we get a further splitting $E = \Lambda \oplus \xi$, corresponding to $\langle u, v \rangle \oplus \langle u \times v \rangle$. The orientation of the 2-dimensional bundle $\Lambda$ gives it a complex structure, and we have

\[ TM = \bar{E} \oplus \xi \]

where $E = \Lambda \oplus V$ is a 6-plane bundle with a complex structure and $\xi$ is the line bundle $\langle u \times v \rangle$. Note that if $\varphi$ is integrable and the vector field $u \times v$ is parallel then $M$ would be a Calabi-Yau $\times S^1$ (since $G_2$ holonomy would reduce to $SU(3)$). So non-vanishing oriented 2-plane fields may be thought of objects taming the $G_2$ structure. Any integral submanifold of the corresponding distribution $E$ is an associative submanifold $Y^3 \subset M$ with a Spin$^c$-structure (i.e. the 2-plane field $\xi = \Lambda|_Y$).

By fixing the plane field $\Lambda$, and varying $\varphi \in \tilde{\Omega}_+^3(M)$ (the set of $G_2$ structures inducing the same metric on $M$) has the effect of varying $\xi \in \Lambda^\perp$ (the cross product operation on $\Lambda$) and varying the complex structure on $V = (\Lambda \oplus \xi)^\perp$. These $\xi$’s are the sections of the $S^4$-sphere bundle of $\Lambda^\perp \to M^7$, hence generically any other section will agree with $\Lambda_\varphi$ on some 3-manifold $Y \subset M$. We will show that this 3-manifold is almost $\Lambda$-associative. First consider the parametrized section:

\[ \Lambda : \tilde{\Omega}_+^3(M) \times M \to \tilde{M} \]

$(\lambda, x) \mapsto \Lambda_\lambda(x)$. By Lemma 5 there is an identification $\tilde{\Omega}_+^3(M) = \{ s^*(\varphi) | s \in \mathcal{G}(P) \}$ (the sections of an $\mathbb{RP}^7$ bundle over $M$). We claim $\Lambda$ is transversal to $M_\varphi$.
First we need to recall a few facts: By [12], the deformations of the $G_2$ structure $\varphi$ fixing the metric $g = g_\varphi$, are parametrized by $\varphi_\lambda$ below, where $\lambda = [a, \alpha]$ are the sections $\tilde{\Omega}_\lambda^2(M)$ of the $\mathbb{RP}^7$-bundle, which is the projectivization $P(\mathbb{R} \oplus T^* M) \to M$

$$\varphi_\lambda = (a^2 - |\alpha|^2) \varphi + 2a \ast (\alpha \wedge \varphi) + 2\alpha \wedge \ast (\alpha \wedge \ast \varphi)$$

where $a^2 + |\alpha|^2 = 1$. From the identities $\ast (\alpha \wedge \varphi) = -\alpha \ast \ast \varphi$ and $\ast (\alpha \wedge \ast \varphi) = \alpha \ast \ast \varphi$, where $\alpha \ast$ is the metric dual of $\alpha$, we can also express

$$\varphi_\lambda = \varphi - 2\alpha \ast \ast [a \ast \varphi + \alpha \wedge \varphi]$$

$$\ast \varphi_\lambda = \ast \varphi + 2\alpha \wedge [a \varphi - (\alpha \ast \ast \varphi)]$$

Not to clutter notations, we denote $\Lambda_\lambda = \Lambda_{\varphi_\lambda}$ and use the metric to identify $T^*(M) = E \oplus V$, and identify $M$ with the zero section of the bundle $V \to M$.

**Theorem 10.** For $\alpha \in \Omega^1(M)$ which is a transverse section of $V \to M$, the map $\Lambda_\lambda$, where $\lambda = [a, \alpha]$ and $a \neq 0$, is transversal to $M_{\varphi}$, and $\Lambda^{-1}_\lambda(M_{\varphi}) = \alpha^{-1}(M)$.

**Proof.** The set $\Lambda^{-1}_\lambda(M_{\varphi})$ is given by the solutions of the equation $\Lambda_{\varphi_\lambda}(x) = \Lambda_\varphi(x)$, where $\varphi \mapsto \varphi_\lambda$ is a deformation of $\varphi$. Since $E$ is obtained from the oriented 2-plane field $\Lambda = \langle u, v \rangle$ by association $\langle u, v \rangle \mapsto \langle u, v, u \times v \rangle$, this equation is equivalent to $(u \times v)_\Lambda(x) = (u \times v)(x)$ (up to positive scalar multiple), where $(u \times v)_\lambda$ denotes the cross product corresponding to $\varphi_\lambda$. By using $(u \times v)\# = u \cup v \varphi$, we can calculate the deviation of the cross product operation under the deformation

$$(u \times v)_\lambda = (1 - 2|\alpha|^2)(u \times v) + 2[ - a \chi(u,v,\alpha\#) + \alpha(v)(u \times \alpha\#) - \alpha(u)(v \times \alpha\#) + \varphi(u,v,\alpha\#)\alpha\# ]$$

So the equation $(u \times v)_\lambda(x) = (u \times v)(x)$ is given by the equation $F = 0$ where:

$$F = a \chi(u,v,\alpha\#) - \alpha(v)(u \times \alpha\#) + \alpha(u)(v \times \alpha\#) - \varphi(u,v,\alpha\#)\alpha\# + |\alpha|^2(u \times v)$$

Note that when $\alpha\# \in E$, the equation $F(x) = 0$ holds for all $x \in M$. Let us choose our deformation $\alpha\# \in V$, which is a transverse section of $V \to M$. In this case by (4), Lemma 1, and Lemma 2 the equation $F(x) = 0$ is equivalent to

$$aJ(\alpha\#) = -|\alpha|^2(u \times v)$$

where $J$ is the complex structure defined in Lemma 2. Since $J(\alpha\#) \in V$ and $u \times v \in E$, this equation holds only at points satisfying $\alpha\#(x) = 0$. By taking derivative of $F(a, \alpha)$ we see that $F$ is transversal to $M_\varphi$ when $a \neq 0$. $\square$
9. Cayley Submanifolds of $Spin(7)$

Much of what we have discussed for associative submanifolds of a $G_2$ manifold holds for Cayley submanifolds of a $Spin(7)$ manifold. Let $(N^8, \Psi)$ be a $Spin(7)$ manifold, and $\mathcal{P}_{Spin(7)}(N) \to N$ be its $Spin(7)$ frame bundle, and $G(4, 8)$ be the Grassmannian of oriented 4 planes in $\mathbb{R}^8$. As in the $G_2$ case we can form the bundle

$$\tilde{N} = \mathcal{P}(N) \times_{SO(8)} G(4, 8) \to N.$$  

Similarly we have the universal bundles $\Xi$, $\mathcal{V} \to \tilde{N}$ which are fiberwise extensions of the canonical bundle $\xi \to G(4, 8)$ and its dual $\nu = \xi^\perp \to G(4, 8)$, respectively. $Hom(\Xi, \mathcal{V}) = \Xi^* \otimes \mathcal{V} \to \tilde{N}$ is the vertical subbundle of $T(\tilde{N}) \to N$ with fibers $TG(4, 8)$. Let $G^\Psi(4, 8)$ be the Grassmannian of Cayley 4-planes in $G(4, 8)$ consisting of elements $L \in G(4, 8)$ satisfying $\Psi|L = vol(L)$. The group $Spin(7)$ acts transitively on $G^\Psi(4, 8)$ with the stabilizer $(SU(2) \times SU(2) \times SU(2))/\mathbb{Z}_2$. Therefore, $G^\Psi(4, 8)$ can be identified by the quotient of $Spin(7)$ with the subgroup $(SU(2) \times SU(2) \times SU(2))/\mathbb{Z}_2 \subset Spin(7)$.

The action of $[q_+, q_-, \lambda] \in (SU(2) \times SU(2) \times SU(2))/\mathbb{Z}_2$ on $\mathbb{R}^8 = \mathbb{H} \oplus \mathbb{H}$ is given by $(x, y) \mapsto (q_+ x q_-^\perp, q_y y^\perp)$. As in $G_2$ case there is the Cayley Grassmannian bundle

$$\tilde{N}_\Psi = \mathcal{P}_{Spin(7)}(N) \times_{Spin(7)} G^\Psi(4, 8) \to N$$

which is $\tilde{N}_\Psi = \mathcal{P}(N)/(SU(2) \times SU(2) \times SU(2))/\mathbb{Z}_2 \to \mathcal{P}(N)/Spin(7) = N$. We have restriction of the bundles $\Xi^*, \mathcal{V} \to \tilde{N}_\Psi \subset \tilde{N}$. Furthermore, the principal $(SU(2) \times SU(2) \times SU(2))/\mathbb{Z}_2$ bundle $\mathcal{P}(N) \to \tilde{N}_\Psi$ gives the following associated vector bundles over $\tilde{N}_\Psi$ via the representations (see [HL], [M]).

$$\begin{align*}
\mathbb{W}^+ &= \mathcal{V} : & y &\mapsto q_+ y \lambda^{-1} \\
\mathbb{W}^- &= \mathcal{V} : & y &\mapsto q_- y \lambda^{-1} \\
\Xi^* &= & x &\mapsto q_+ x q_-^{-1} \\
\lambda_+(\Xi^*) &= & x &\mapsto q_+ x q_-^{-1} \\
\lambda_-(\Xi^*) &= & x &\mapsto q_- x q_+^{-1} \\
\lambda_-(\mathbb{W}^+) &= & x &\mapsto \lambda x q_-^{-1}
\end{align*}$$

(58)

where $[q_+, q_-, \lambda] \in SU(2) \times SU(2) \times SU(2))/\mathbb{Z}_2$. We can identify: $\lambda_+(\mathbb{W}^+) = \lambda_+(\Xi^*)$, and we have the usual decomposition $\Lambda^2(\Xi^*) = \lambda_+(\Xi^*) \oplus \lambda_-(\Xi^*)$. We have the Clifford multiplications $\Xi^* \otimes \mathbb{W}^\pm \to \mathbb{W}^\pm$ given by: $x \otimes y \mapsto -xy$ and $x \otimes y \mapsto xy$, on $\mathbb{W}^+$ and $\mathbb{W}^-$ respectively, which extends to $\Lambda^2(\Xi^*) \otimes \mathbb{W}^+ \to W^+$.

The Gauss map of an imbedding $f : X^4 \hookrightarrow N^8$ of any 4-manifold canonically lifts to an imbedding $\tilde{f} : X^4 \hookrightarrow \tilde{N}$, and the pull backs $\tilde{f}^* \Xi^* = T^*(X)$ and $\tilde{f}^* \mathbb{W}^+ = \nu(X)$ give cotangent and normal bundles of $X$. Furthermore, if $X$ is a Cayley submanifold of $N$ then the image of $\tilde{f}$ lands in $\tilde{N}_\Psi$; in this case pulling back the principal $Spin(7)$ frame bundle $\mathcal{P}(N) \to N$ induces an $(SU(2) \times SU(2) \times SU(2))/\mathbb{Z}_2$ bundle $\mathcal{P}(X) \to X$. So by the representations (58) we get associated vector bundles
$W^+ = \nu(X), W^-, T^*X)$ over $X$, i.e. the pull-backs of $\mathbb{W}^+, \mathbb{W}^-, \Xi^*$. So we have the actions $W^+ \otimes \lambda_-(W^+) \to W^+$ and $T^*X \otimes W^\pm \to W^\pm$ and $\Lambda^2(T^*X) \otimes W^+ \to W^+$.

The Levi-Civita connection induced by the $Spin(7)$ metric on $N$, induces connections on tangent and normal bundle of any submanifold $X^4 \subset N$. Call these connections background connections. Let $A_0$ be the induced connection on $\nu(X) = W^+$. Using the Lie algebra decomposition $so(4) = so(3) \oplus so(3)$, we can decompose $A_0 = S_0 \oplus A_0$, where $S_0$ and $A_0$ are connections on $\lambda_+(T^*X)$ and $\lambda_-(W^+)$, respectively. Any connection $A$ of $\lambda_-(W^+)$ is in the form $A = A_0 + a$ where $a \in \Omega^1(X, \lambda_-(W^+))$, and by the association $A \mapsto S_0 \oplus A$ it induces a connection on $\nu(X)$. We will denote this connection by $A = A(a)$, and $A_0 = A(0)$. Later we will consider deformations

$$A_0 \mapsto A.$$  

Let $\nabla_A : \Omega^0(X, W^+) \to \Omega^1(X, W^+)$ by $\nabla_A = \sum e^i \otimes \nabla_{e_i}$, where $\{e_1\}$ and $\{e^1\}$ are orthonormal tangent and cotangent frame fields of $X$, respectively. When $X$ is a Cayley manifold, the Clifford multiplication gives the twisted Dirac operator:

$$\mathcal{D}_A : \Omega^0(X, W^+) \to \Omega^0(X, W^-)$$

The kernel of $\mathcal{D}_A$ gives the infinitesimal deformations of Cayley submanifolds ($\mathbf{M}$). As in the associative case by deforming $A_0 \to A$ we can make cokernel of $\mathcal{D}_A$ zero.

Similar to the case of $\lambda$-associative submanifolds in $G_2$ manifolds, we can study the Cayley submanifolds in $Spin(7)$ manifolds with complex normal bundles. There are several ways of lifting various subbundles to complex bundles, for example

$$Spin^c(4) = (SU(2) \times SU(2) \times S^1)/\mathbb{Z}_2 \subset (SU(2) \times SU(2) \times SU(2))/\mathbb{Z}_2$$

gives a $Spin^c(4)$ bundle $\mathcal{P}(N) \to \tilde{N}_\Psi = \mathcal{P}(N)/Spin^c(4)$, and we have all the corresponding bundles of (58) over $\tilde{N}_\Psi$ (except in this case we have $\lambda \in S^1$). The $S^2$-bundle $\tilde{N}_\Psi \to \tilde{N}_\Psi$ can be identified with the sphere bundle of $\lambda_-(\mathbb{W}^+) \to \tilde{N}_\Psi$, and the sections of this bundle correspond to almost complex structures on $\mathbb{W}^\pm$. Previously, in the case of 7-manifolds, existence of such sections followed from the existence of 2-frame field $\mathbf{I}$, in the 8-dimensional $Spin(7)$ case we don’t have a clean analogue of $\mathbf{I}$, so in this case we will make this an assumption and proceed. So consider a $Spin(7)$ manifold $(N^8, \Psi, \Lambda)$ with a unit section $\Lambda : \tilde{N}_\Psi \to \lambda_-(\mathbb{W}^+)$. Hence $\mathbb{W}^\pm \to \tilde{N}_\Psi$ are $U(2)$ bundles, and $\lambda_-(W^+)$ is a line bundle $L \to \tilde{N}_\Psi$. As in (50) there is a quadratic bundle map $\sigma : \mathbb{W}^+ \otimes \mathbb{W}^+ \to \lambda_-(\Xi^*)$

$$\sigma(x, x) = -\frac{1}{2}(xi\bar{x})i.$$  

Now if $f : X^4 \hookrightarrow N^8$ is a Cayley submanifold, we can pull back these structures onto $X$ by $\Lambda \circ f$. Then we can “perturb” the local Cayley deformations of $X$ by deforming the connection as in (59), i.e. the kernel of the Dirac operator of (60).
Then if we can cut down the solution space $D^{-1}_A(0)$ by a second natural equation (by using "a" as a free variable) we arrive to the Seiberg-Witten equations:

\[
\begin{align*}
D_A(v) &= 0 \\
F_A^+ &= \sigma(v, v)
\end{align*}
\]

As usual, by perturbing these equations by elements of $\Omega^2_+(X)$, i.e. by changing the second equation with $F_A^+ + \delta = \sigma(v, v)$ with $\delta \in \Omega^2_+(X)$ we get smoothness on the zero locus of the parameterized equation $F = 0$ where

\[
F : \Omega^0(X, W^+) \times A(L) \times \Omega^2_+(X) \to \Omega^0(X, W^-) \times \Omega^2_+(X)
\]

and by generic choice of $\delta$ we can make the solution set $F^{-1}_\delta(0)$ smooth. The normal bundle $W^+$ of $X$ has a complex structure, so the gauge group $G(L) = Map(X, S^1)$ acts on $F^{-1}_\delta(0)$, and makes quotient $F^{-1}_\delta(0)/G(L)$ a smooth manifold whose dimension $d$ can be calculated from the index of the elliptic complex:

\[
\begin{align*}
\Omega^0(X) &\to \Omega^0(X, W^+) \times \Omega^1(X) \to \Omega^0(X, W^-) \times \Omega^2_+(X)
\end{align*}
\]

where the first map comes from gauge group action. As in Seiberg-Witten we get:

\[
d = \frac{1}{4} \left[ c_1^2(L) - (2e(X) + 3\sigma(X)) \right]
\]

Here $e$ and $\sigma$ denote the Euler characteristic and the signature. In particular, these parametrized deformations of complex Cayley submanifolds in $\Omega^0(X, W^+) \times A(L)$ are unobstructed.

**Theorem 11.** Given $(X, \Psi, \Lambda)$, to any Cayley submanifold $f : X^4 \to N$ we can assign a number $\Lambda_f(X) \in \mathbb{Z}$. Furthermore, the Seiberg-Witten equations of (61) can be pulled back by $f$ from global equations on $N$ (analogue of Proposition 9).

Note that $SU(3)$ and $G_2$ also act on the corresponding special Lagrangian and coassociative Grassmannians with $SO(3)$ and $SO(4)$ stabilizers, respectively [HL], giving the identifications $G_{SL}(3, 6) = SU(3)/SO(3)$ and $G^\text{coas}(4, 7) = G_2/SO(4)$. As before, one can study special Lagrangians in a Calabi-Yau manifold, and coassociative submanifolds in a $G_2$ manifold, by lifting their normal bundles to $SU(2)$. Their deformation spaces are unobstructed and can be identified with $H^1$ and $H^2_+$, respectively. With a similar approach we can relate them to the reduced Donaldson invariants (as the $\Lambda$-associative and similarly defined Cayley’s are related to Seiberg-Witten invariants). Similarly one can treat the deformations of associative submanifolds whose boundaries lie on coassociative submanifolds, and the Cayley’s in $Spin(7)$ with associative boundaries in $G_2$. Also asymptotically cylindrical associative submanifolds in a $G_2$ manifold with a Calabi-Yau boundary have similar local deformation spaces, their deformations are related to the corresponding holomorphic curves inside the Calabi-Yau boundary.
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