h-CONVEXITY IN METRIC LINEAR SPACES

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ABSTRACT

This paper is devoted to study a new types of convex metric linear spaces. The types of convexity considered here are a strictly h–convex, a uniformly h–convex and a locally uniformly h–convex metric linear spaces.

Key words: Strictly convex, Uniformly convex, Locally uniformly convex, \( h \)–Convex function.

1. INTRODUCTION

In literature three types of convex metric linear spaces are studied, let us recall these types.

A metric linear space \((X,d)\) is said to be :

(1) strictly convex if \( d(x, 0) \leq r, d(y, 0) \leq r \) imply \( d(\frac{x+y}{2}, 0) \leq r \) unless, \( x = y \), where \( x, y \in X \) and \( h(r) \) is any positive number.

(2) uniformly convex if to each pair of positive numbers \((\varepsilon, r)\) there corresponds a \( \delta > 0 \) such that if \( d(x, y) \geq \varepsilon \), \( d(x, 0) \leq r + \delta \) and then \( d(0, 0) \leq \frac{\varepsilon}{3} \).

(3) locally uniformly convex if for each \( r > 0 \) and \( x, y \) \( X \) with \( d(x, 0) < r, \delta (r > 0) \), there exists a \( \delta > 0 \) such that if \( d(x, y) \geq \) and \( d(y, 0) < r \) and \( d(0, 0) \leq r \).

These types of convexities were studied in [2] and [4]. It is known that a uniformly convex metric linear space is locally uniformly convex and a locally uniformly convex metric linear space is strictly convex. Also every totally complete strictly convex metric linear space is uniformly convex.

In the sequel of the paper, I and J are intervals in \( R^+ \) with \((0, 1) \subseteq J \) and functions \( h \) and \( f \) are real non-negative functions defined on \( J \) and \( I \), respectively.

Let \( h : J \rightarrow R^+ \) be a non–zero, non–negative function. We say that \( f : I \rightarrow R \) is an \( h \)–convex function, if \( f \) is non-negative and for all \( x, y \in I, \alpha \in (0, 1) \) we have \( f(\alpha x + (1 - \alpha) y) \leq h(\alpha) f(x) + h(1 - \alpha) f(y) \). This type of convexity was studied in [3].

2. MAIN RESULT

In this section, we introduce a new type of convex metric linear spaces; which so called strictly h–convex, uniformly h–convex and locally uniformly h–convex metric linear spaces and give some related results by using a technique used in [1].

Definition 2.1. Let \( h : J \rightarrow R^+ \) be a non–zero, non–negative function and \( r \in (0, 1) \).

A metric linear space \((X, d)\) is said to be

(1) Strictly h–convex if \( d(x, 0) \leq h(r), d(y, 0) \leq x + y \) and \( d(0, 0) \leq h(r) \) unless \( x = y \), where \( x, y \in X \) and \( h(r) \) is any positive number.
(2) Uniformly h–convex if to each pair of positive numbers \((E, r)\) there corresponds a \(\delta > 0\) such that if \(x, y \in X\) are such that
\[d(x, y) \geq E, \ d(x, 0) \leq h(r) + \delta\]
and
\[d(x, y) \geq E, \ d(x, 0) \leq h(r) + \delta\]
and then
\[d\left(\frac{x + y}{2}, 0\right) \leq h(r) - \delta\]

(3) Locally uniformly convex if for each \(E > 0\) and \(x \in X\) with
\[d(x, 0) < h(r) (h(r) > 0)\], there exists a \(\delta > 0\) such that if \(d(x, y) \geq \varepsilon\) and
\[d(y, 0) < h(r) + \delta\] and
\[d(x, y) \leq h(r)\].

Now, as the previous definitions of convexities of metric linear spaces we can also introduce other types of convexities of metric linear spaces related with the \(h\)–convexity above, as follows:

Let \(0 < s \leq 1\), if \(h(r) = s r\) such that \(r \in (0, 1)\), then a metric linear space \((X, d)\) is said to be:

(1) Strictly \(s\)–convex if \(d(x, 0) \leq s r\), \(d(y, 0) \leq s r\) imply
\[d\left(\frac{x + y}{2}, 0\right) \leq s r - \delta\]
unless \(x = y\), where \(x, y \in X\) and \(r > 0\) where \(s \in (0, 1]\).

(2) Uniformly \(h\)–convex if to each pair of positive numbers \((E, r)\) there corresponds a \(\delta > 0\) such that if \(x, y \in X\) are such that
\[d(x, y) \geq E, \ d(x, 0) \leq h(r) + \delta\]
and
\[d(x, y) \geq E, \ d(x, 0) \leq h(r) + \delta\]
and then
\[d\left(\frac{x + y}{2}, 0\right) \leq r - \delta\]

(3) Locally uniformly convex if for each \(E > 0\) and \(x \in X\) with
\[d(x, 0) < r' (r > 0)\], there exists a \(\delta > 0\) such that if \(d(x, y) \geq \varepsilon\)
and \(d(y, 0) < r' + \delta\) and \(d\left(\frac{x + y}{2}, 0\right) \leq r'\).
\[ d \left( \sum_{i=0}^{k} x_i, 0 \right) \leq (k + 1) h(r) - \delta \]

Conversely, suppose for a given \( \varepsilon > 0 \) and \( h(r) > 0 \), \( x, y \in X \) be such that \( d(x, 0) < h(r) \), \( \delta > 0 \) and \( \sum_{i=0}^{k} \delta_i x_i \leq \varepsilon \) for any \( \delta \in X \). Then, \( x = x_0 < x_1 < x_2 < \ldots < x_k = y \) are \( (k + 1) \) elements in hypothesis

\[ d \left( \sum_{i=0}^{k} x_i, 0 \right) \leq (k + 1) h(r) - \delta \]

This gives

\[ d \left( \frac{(k + 1)(x + y)}{2}, 0 \right) < (k + 1)(h(r) - \delta). \]

**Corollary 2.2.** Let \( 0 < s \leq 1 \). In Theorem 2.1. set \( h(r) = r^s \) such that \( r \in (0, 1) \). A metric linear space \((X, d)\) is uniformly \( s \)-convex if and only if for each \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that if for positive integers \( k \geq 2 \), the \( (k + 1) \) elements \( x_0, x_1, \ldots, x_k \in X \) are such that \( d(x_i, 0) < r^s \), \( 0 < r < 1 \), \( i = 0, 1, 2, \ldots, k \) and \( \min d(x_i, x_j) \geq \varepsilon \), then

\[ d \left( \frac{(k + 1)(x + y)}{2}, 0 \right) < (k + 1)(r^s - \delta). \]

**Proof.** It’s an immediate consequence of Theorem 2.1.

Now, another characterizations of uniformly \( h \)-convex metric linear space are pointed out as follows.

**Theorem 2.3.** Let \( h: J \to R^+ \) be a non–zero, non–negative function and \( r \in (0, 1) \) and suppose \((X, d)\) metric linear space, for \( k \geq 1 \), the following statements are equivalent:

1. \( X \) is uniformly \( h \)-convex.
2. For each \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that if \( \{x_n\} \) is a sequence in \( X \) with \( d(x_n, 0) < h(r) \), \( h(r) > 0, n \in \mathbb{N} \) and \( d(x_i, x_j) \geq \varepsilon \) \( (i \neq j) \). Then, there exists \( a_i \geq 0 \) \( (i = 1, 2, \ldots, k) \) with

\[ \sum_{i=1}^{k} a_i = 1 \]

and

\[ d \left( \sum_{i=1}^{k} a_i x_{n+i}, 0 \right) \leq h(r) - \delta, \quad (n \in \mathbb{N}) \]

3. For each \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that if \( \{x_n\} \) is a sequence in \( X \) with \( d(x_n, 0) < h(r) \), \( h(r) > 0, n \in \mathbb{N} \) and \( d(x_i, x_j) \geq \varepsilon \) \( (i \neq j) \). Then, for each \( n \geq 1 \) there exist \( a_i \geq 0 \) with

\[ \sum_{i=1}^{k} a_i = 1 \]

and

\[ d \left( \sum_{i=1}^{k} a_i x_{n+i}, 0 \right) \leq h(r) - \delta, \quad (n \in \mathbb{N}) \]

**Proof.** (1) \( \Rightarrow \) (2) and (2) \( \Rightarrow \) (3) follows in view of Theorem 2.1. (3) \( \Rightarrow \) (1). Firstly we note that \( x_{n+i} \) \( (i = 1, 2, \ldots, k) \) are \( k \)-vectors in \( X \) satisfying \( d(x_{n+i}, 0) < h(r) \), \( 1 \leq i \leq k \), and \( d(x_i, x_j) \geq \varepsilon \) \( (i \neq j) \). Therefore, by (3)

\[ d \left( \sum_{i=1}^{k} a_i x_{n+i}, 0 \right) = d \left( \sum_{i=1}^{k} (1-a_i)x_{n+i}, \sum_{i=1}^{k} a_i x_{n+i}, 0 \right) < h(r) \sum_{i=1}^{k} (1-a_i) + (h(r) - \delta) = k \left( \frac{h(r) - \delta}{k} \right) = h(r) - \delta \]

Hence, by Theorem 2.1 \( X \) is uniformly \( h \)-convex.

**Corollary 2.4.** Let \( 0 < s \leq 1 \). In Theorem 2.3 set \( h(r) = r^s \) such that \( r \in (0, 1) \) and suppose that \((X, d)\) is a metric linear space. For \( k > 1 \), the following statements are equivalent:

1. \( X \) is uniformly \( h \)-convex.
(2) For each \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that if 
\[ \{ x_n \} \text{ is a sequence in } X \text{ with } d(x_n, 0) < r^s, \quad (r > 0, \quad n \in \mathbb{N}) \] and 
\[ d(x_i, x_j) \geq \varepsilon \quad (i \neq j). \] Then, there exists \( \alpha_i \geq 0 \quad (i = 1, 2, \ldots, k) \) with 
\[ \sum_{i=1}^{k} \alpha_i(x) = 1 \quad \text{and} \quad \sum_{i=1}^{k} d(x_i, 0) \leq r^s - \delta, \quad (n \in \mathbb{N}). \]

(3) For each \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that 
if \( \{ x_n \} \text{ is a sequence in } X \text{ with } d(x_n, 0) < r^s, \quad (r > 0, \quad n \in \mathbb{N}) \) and 
\[ d(x_i, x_j) \geq \varepsilon \quad (i \neq j). \] Then, for each \( \varepsilon \geq 0 \) there exist \( \alpha_i \geq 0 \quad (i = 1, 2, \ldots, k) \) with 
\[ \sum_{i=1}^{k} \alpha_i(x) = 1 \quad \text{and} \quad \sum_{i=1}^{k} d(x_i, 0) \leq r^s - \delta, \quad (n \in \mathbb{N}). \]

**Proof.** It’s an immediate consequence of Theorem 2.3.

**Theorem 2.5.** Let \( h : J \to R^+ \) be a non–zero, non–negative function and \( r \in (0, 1). \) A metric linear space \( (X,d) \) is locally uniformly \( h \)–convex if and only if for each \( \varepsilon > 0 \) and \( x \in X \) with 
\( d(x, 0) < h(r), \) \( (h(r) > 0, \quad \{ i = 1, 2, \ldots, k \} ) , \) imply 
\[ d(\sum_{i=0}^{k} x_i, 0) \leq (k + 1)(h(r) - \delta). \]

**Proof.** The proof can be worked out in the similar fashion.

**Corollary 2.6.** Let \( 0 < s \leq 1. \) In Theorem 2.5 set \( h(r) = r^s \) such that \( r \in (0, 1). \) A metric linear space \( (X,d) \) is locally uniformly \( s \)–convex if and only if for each \( \varepsilon > 0 \) and \( x \in X \) with 
\( d(x, 0) < r^s, \quad (r > 0) , \) there exists a \( \delta > 0 \) such that \( x_0, x_1, \ldots, x_k \in X \) satisfying 
\( d(x, 0) < h(r), \) \( (h(r) > 0, \quad \{ i = 1, 2, \ldots, k \} ) , \) imply 
\[ d(\sum_{i=0}^{k} x_i, 0) \leq (k + 1)r^s. \]

**Proof.** It’s an immediate consequence of Theorem 2.7.

**Theorem 2.7.** Let \( h : J \to R^+ \) be a non–zero, non–negative function and \( r \in (0, 1). \) A metric linear space \( (X,d) \) is strictly \( h \)–convex if and only if for \( (k + 1) \) elements \( x_0, x_1, \ldots, x_k \in X \) satisfying 
\( d(x, 0) < h(r), \) \( (h(r) > 0, \quad \{ i = 1, 2, \ldots, k \} ) , \) imply 
\[ d(\sum_{i=0}^{k} x_i, 0) \leq (k + 1)(h(r) - \delta). \]

Unless \( x_0 = x_1 = x_2 = \ldots = x_k \).

**Proof.** The proof can be worked out in the similar fashion.

**Corollary 2.8.** Let \( 0 < s \leq 1. \) In Theorem 2.7 set 
\( h(r) = r^s \) such that \( r \in (0, 1). \) A metric linear space \( (X,d) \) is strictly \( s \)–convex if and only if for \( (k + 1) \) elements \( x_0, x_1, \ldots, x_k \in X \) satisfying 
\( d(x, 0) < r^s, \) \( (r > 0, \quad \{ i = 1, 2, \ldots, k \} ) , \) imply 
\[ d(\sum_{i=0}^{k} x_i, 0) \leq (k + 1)r^s. \]

Unless \( x_0 = x_1 = x_2 = \ldots = x_k \).

**Proof.** It’s an immediate consequence of Theorem 2.7.

**Theorem 2.9.** For A metric space \( (X,d) \) we have, uniformly \( h \)–convexity \( \Rightarrow \) locally uniformly \( h \)–convexity \( \Rightarrow \) strictly \( h \)–convexity.

**Proof.** It follows from Theorems 2.1–2.3 and Theorem 2.5.
Corollary 2.10. For a metric space \((X,d)\) we have,
uniformly \(s\)-convexity \(\Rightarrow\) locally uniformly \(s\)-convexity \(\Rightarrow\) strictly \(s\)-convexity.

Proof. It follows from Theorem 2.9.

Now, we know that a metric linear space
\((X,d)\) is totally complete if its metrically bounded closed sets are compact. Therefore, we can state the following theorem.

Theorem 2.11. Every totally complete strictly \(h\)-convex metric linear space \((X,d)\) is uniformly \(h\)-convex and thus it’s locally uniformly \(h\)-convex.

Proof. Suppose that \((X,d)\) is totally complete
strictly \(h\)-convex metric linear space.

Let \(U \subseteq X\) be arbitrary compact metrically bounded closed subset of \(X\) and let \(\{x_n\}\) be a sequence in \(U\) . Since \(U\) is bounded closed and compact subset and \((X,d)\) is strictly \(h\)-convex then for \(\varepsilon > 0\) and for a \(k\)-elements \(x_1,..,x_k\) in \(X\) satisfying \(d(x_{n+i}, 0) < h(\varepsilon)\); \((h(\varepsilon) > 0, 1 \leq i \leq k)\) with \(d(x_{n+i},x_{n+j}) \geq \varepsilon (i \neq j)\) there exists \(\delta > 0\) and for \(n > 1\), \(a_i^{(n)} \geq 0, (i = 1, 2, ..., k)\) with
\[
\sum_{i=1}^{k} a_i^{(n)} = 1
\]

such that
\[
\begin{align*}
\sum_{i=1}^{k} d(\sum_{i=1}^{k} a_i^{(n)} x_{n+i}, 0) &= d(\sum_{i=1}^{k} (1-a_i^{(n)}) x_{n+i} + \sum_{i=1}^{k} a_i^{(n)} x_{n+i}, 0) \\
&< h(\varepsilon) \sum_{i=1}^{k} (1-a_i^{(n)}) + h(\varepsilon) \delta \\
&= k \left( \frac{h(\varepsilon) \delta}{k} \right) \\
&= h(\varepsilon) \delta 
\end{align*}
\]

which means that \((X,d)\) is uniformly \(h\)-convex
and thus by Theorem 2.9. \((X,d)\) is locally uniformly \(h\)-convex.

Corollary 2.12. Every totally complete strictly \(s\)-convex metric linear space \((X,d)\) is uniformly \(s\)-convex and thus it’s locally uniformly \(s\)-convex.

Proof. It follows from Theorem 2.11.

Remark 2.13. If \(s = 1\) in the definitions of strictly \(s\)-convex; uniformly \(s\)-convex ; and locally uniformly \(s\)-convex metric linear space \((X,d)\) then \((X,d)\) reduced to strictly convex ; uniformly convex; and locally uniformly convex metric linear space ; respectively.

Remark 2.14. The above results of \(h\)-convexity
holds for strictly convex ; uniformly Convex ; and locally uniformly convex metric linear space \((X,d)\) when \(h(\varepsilon) = \varepsilon\).

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