Diagonal reduction algebras of \( \mathfrak{gl} \) type

S. Khoroshkin\(^\circ\) and O. Ogievetsky\(^\star\)

\(^\circ\)Institute of Theoretical and Experimental Physics, 117259 Moscow, Russia

\(^\star\)Centre de Physique Théorique\(^2\), Luminy, 13288 Marseille, France

Abstract

Several general properties, concerning reduction algebras – rings of definition and algorithmic efficiency of the set of ordering relations – are discussed. For the reduction algebras, related to the diagonal embedding of the Lie algebra \( \mathfrak{gl}_n \) into \( \mathfrak{gl}_n \oplus \mathfrak{gl}_n \), we establish a stabilization phenomenon and list the complete sets of defining relations.

1 Introduction

Reduction algebras were introduced [AST2, M] for a study of representations of a Lie algebra with the help of the restriction to a subalgebra.

Let \( \mathfrak{g} \) be a Lie algebra, \( \mathfrak{k} \subset \mathfrak{g} \) its reductive Lie subalgebra; that is, the adjoint action of \( \mathfrak{k} \) on \( \mathfrak{g} \) is completely reducible (in particular, \( \mathfrak{k} \) is reductive). Suppose \( \mathfrak{k} \) is given with a triangular decomposition

\[ \mathfrak{k} = \mathfrak{n}_- + \mathfrak{h} + \mathfrak{n}_+ . \]

Denote by \( \mathfrak{I}_+ \) the left ideal of \( \mathfrak{A} := U(\mathfrak{g}) \) generated by elements of \( \mathfrak{n}_+ \), \( \mathfrak{I}_+ := A\mathfrak{n}_+ \).

Then the reduction algebra \( S(\mathfrak{g}, \mathfrak{k}) \), related to the pair \( (\mathfrak{g}, \mathfrak{k}) \), is defined as the quotient \( \text{Norm}(\mathfrak{I}_+)/\mathfrak{I}_+ \) of the normalizer of the ideal \( \mathfrak{I}_+ \) over \( \mathfrak{I}_+ \) (one should keep in mind that the notation \( S(\mathfrak{g}, \mathfrak{k}) \) is abbreviated: the data needed for the definiton of the reduction algebra includes, in addition to the pair \( (\mathfrak{g}, \mathfrak{k}) \), the triangular decomposition (1.1)). The space \( S(\mathfrak{g}, \mathfrak{k}) \) is equipped with a natural structure of the associative algebra. By definition, for any \( \mathfrak{g} \)-module \( V \) the space \( V^{n_+} \) of vectors, annihilated by \( \mathfrak{n}_+ \), is a module over \( S(\mathfrak{g}, \mathfrak{k}) \). If \( V \) decomposes, as an \( \mathfrak{k} \)-module, into a direct sum of irreducible \( \mathfrak{k} \)-modules \( V_i \) with finite-dimensional multiplicities, then the \( \mathfrak{g} \)-module structure on \( V \) can be uniquely restored from the \( S(\mathfrak{g}, \mathfrak{k}) \)-module structure on \( V^{n_+} \).

The reduction algebra simplifies after the localization over the multiplicative set generated by elements \( h_\gamma + k \), where \( \gamma \) ranges through the set of roots of \( \mathfrak{k} \), \( k \in \mathbb{Z} \); here \( h_\gamma \) is the coroot corresponding to \( \gamma \). Let \( \mathbb{U}(\mathfrak{h}) \) be the localization of the universal enveloping algebra \( U(\mathfrak{h}) \) of the Cartan sub-algebra \( \mathfrak{h} \) of \( \mathfrak{k} \) over the above multiplicative set. The localized reduction algebra \( Z(\mathfrak{g}, \mathfrak{k}) \) is an algebra over the commutative ring \( \mathbb{U}(\mathfrak{h}) \); the principal

\(^1\)On leave of absence from P.N. Lebedev Physical Institute, Theoretical Department, Leninsky prospekt 53, 119991 Moscow, Russia

\(^2\)Unité Mixte de Recherche (UMR 6207) du CNRS et des Universités Aix–Marseille I, Aix–Marseille II et du Sud Toulon – Var; laboratoire affilié à la FRUMAM (FR 2291)
part of the defining relations is quadratic but the relations may contain linear terms or degree 0 terms, see [Z, KO]. Besides, the reduction algebra admits another description as a (localized) double coset space $A/(A_{n+} + n_{-} A)$ endowed with the multiplication map defined with the help of the extremal projector [KO] of Asherova–Smirnov–Tolstoy [AST].

The general theory of reduction algebras [Z] provides the set of quadratic-linear-constant ordering relations over $D(\mathfrak{h})$, the field of fractions of $U(\mathfrak{h})$, for natural generators of any reduction algebra $Z(\mathfrak{g}, \mathfrak{k})$. However there are two subtle questions concerning these relations. The first question is: are these ordering relations defined over the smaller ring $U(\mathfrak{h})$? Secondly, is it possible to use these ordering relations for an algorithmic ordering of polynomial expressions in the reduction algebra? In the first part of the paper we give affirmative answers to these questions for any reduction algebra $Z(\mathfrak{g}, \mathfrak{k})$.

The main theme of the second part of the paper is the special restriction problem, when $\mathfrak{g}$ is the direct sum of two copies of the Lie algebra $\mathfrak{gl}_{n}$ and $\mathfrak{k}$ is the diagonally embedded $\mathfrak{gl}_{n}$. The resulting reduction algebra we call diagonal reduction algebra of $\mathfrak{gl}_{n}$ and denote by $Z_{n}$. A finite-dimensional irreducible module over $\mathfrak{g} = \mathfrak{gl}_{n} \oplus \mathfrak{gl}_{n}$ is the tensor product of two irreducible $\mathfrak{gl}_{n}$-modules; restricting the $\mathfrak{g}$-module to $\mathfrak{k}$ we obtain the decomposition of the tensor product into the direct sum of irreducible $\mathfrak{gl}_{n}$-modules. One of the main results of the paper is the explicit description of the diagonal reduction algebra $Z_{n}$. Some examples and applications of the diagonal reduction algebras are given in [KO3].

We present a list of defining relations for natural generators of $Z_{n}$. The derivation of these relations uses heavily the Zhelobenko automorphisms [KO] of reduction algebras and is given in the work [KO2]. In the present paper we formulate and prove the stabilization property of the algebras $Z_{n}$. The stabilization phenomenon provides a natural way of extending relations for $Z_{n}$ to relations for $Z_{n+1}$ ($Z_{n}$ is not a subalgebra of $Z_{n+1}$). The stabilization principle is the second essential ingredient for the derivation of the set of defining relations.

We also prove that our list of defining relations is equivalent over $\mathcal{U}(\mathfrak{h})$ to the list of the canonical ordering relations. The proof is not difficult once we treat the algebras over $D(\mathfrak{h})$: the arguments for the equivalence are based on certain asymptotic considerations. The proof of the equivalence over $\mathcal{U}(\mathfrak{h})$ is more delicate, it uses the stabilization phenomenon and calculations of certain determinants of Cauchy type.

2 Reduction algebras related to a reductive pair

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra and $\mathfrak{k} \subset \mathfrak{g}$ its reductive subalgebra. Assume that the embedding $\mathfrak{k} \subset \mathfrak{g}$ is also reductive, that is the adjoint action of $\mathfrak{k}$ in $\mathfrak{g}$ is semi-simple. Let $\mathfrak{p}$ be an ad$\mathfrak{k}$-invariant complement of $\mathfrak{k}$ in $\mathfrak{g}$. Choose a triangular decomposition (1.1) of Lie algebra $\mathfrak{k}$; here $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{k}$ while $\mathfrak{n}_{+}$ and $\mathfrak{n}_{-}$ are nilradicals of two opposite Borel subalgebras $\mathfrak{b}_{\pm} \subset \mathfrak{k}$. Let $\Delta \in \mathfrak{h}^{*}$ be the root system of $\mathfrak{k}$. The subsets of $\Delta$
consisting of the positive and negative roots will be denoted by $\Delta_+$ and $\Delta_-$ respectively. Let $Q$ be the root lattice, $Q := \{ \gamma \in \mathfrak{h}^* | \gamma = \sum_{\alpha \in \Delta_+, n_\alpha \in \mathbb{Z}} n_\alpha \alpha \}$. It contains the positive cone $Q_+$,

$$Q_+ := \{ \gamma \in \mathfrak{h}^* | \gamma = \sum_{\alpha \in \Delta_+, n_\alpha \in \mathbb{Z}, n_\alpha \geq 0} n_\alpha \alpha \}.$$ (2.2)

For $\lambda, \mu \in \mathfrak{h}^*$, the notation

$$\lambda > \mu$$ (2.3)

means that the difference $\lambda - \mu$ belongs to $Q_+$, $\lambda - \mu \in Q_+$. This is a partial order in $\mathfrak{h}^*$.

Let $W$ be the Weyl group of the root system $\Delta$. Let $\sigma_1, \ldots, \sigma_r \in W$ be the reflections in $\mathfrak{h}^*$ corresponding to the simple roots $\alpha_1, \ldots, \alpha_r$. We also use the induced action of the Weyl group $W$ on the vector space $\mathfrak{h}$. It is defined by setting $\lambda(\sigma(H)) = \sigma^{-1}(\lambda)(H)$ for all $\sigma \in W$, $H \in \mathfrak{h}$ and $\lambda \in \mathfrak{h}^*$. We assume that this action is extended to the action of a cover of the group $W$ by automorphisms of the Lie algebra $\mathfrak{g}$. In other words, there are automorphisms $\hat{\sigma}_i : \mathfrak{g} \to \mathfrak{g}$ which satisfy the same braid group relations as $\sigma_i$, preserve the subspaces $\mathfrak{h}$ and $\mathfrak{k}$, and coincide with $\sigma_i$ being restricted to $\mathfrak{h}$. We denote by the same symbols the canonical extensions of $\hat{\sigma}_i$ to automorphisms of $U(\mathfrak{g})$.

Let $\rho$ be the half-sum of the positive roots of $\mathfrak{k}$. Then the \textit{shifted action} $\circ$ of the group $W$ on the vector space $\mathfrak{h}^*$ is defined by setting

$$\sigma \circ \lambda = \sigma(\lambda + \rho) - \rho.$$ (2.4)

With the help of (2.4) we induce the action $\circ$ of $W$ on the commutative algebra $U(\mathfrak{h})$ by regarding the elements of this algebra as polynomial functions on $\mathfrak{h}^*$. In particular, then $(\sigma \circ H)(\lambda) = H(\sigma^{-1} \circ \lambda)$ for $H \in \mathfrak{h}$.

For each $i = 1, \ldots, r$ let $h_{\alpha_i} = \alpha_i^\lor \in \mathfrak{h}$ be the coroot vector corresponding to the simple root $\alpha_i$, so that the value $\alpha_j(H_i)$ equals the $(i, j)$ entry $a_{ij}$ of the Cartan matrix $A$ of $\mathfrak{k}$. Here $h_{\alpha_i}$ belongs to the semi-simple part of $\mathfrak{k}$. Let $e_{\alpha_i} \in \mathfrak{n}_+$ and $e_{-\alpha_i} \in \mathfrak{n}_-$ be the Chevalley generators of that subalgebra corresponding to the roots $\alpha_i$ and $-\alpha_i$ so that

$$[e_{\alpha_i}, e_{-\alpha_j}] = \delta_{ij} h_{\alpha_i}, \quad [h_{\alpha_i}, e_{\alpha_j}] = a_{ij} e_{\alpha_j}, \quad [h_{\alpha_i}, e_{-\alpha_j}] = -a_{ij} e_{-\alpha_j}.$$ 

For each $\alpha \in \Delta$ let $h_\alpha = \alpha^\lor \in \mathfrak{h}$ be the corresponding coroot vector. Denote by $\overline{U(\mathfrak{h})}$ the ring of fractions of the commutative algebra $U(\mathfrak{h})$ relative to the set of denominators

$$\{ h_\alpha + l | \alpha \in \Delta, l \in \mathbb{Z} \}. \quad (2.5)$$

The elements of this ring can also be regarded as rational functions on the vector space $\mathfrak{h}^*$. The elements of $U(\mathfrak{h}) \subset \overline{U(\mathfrak{h})}$ are then regarded as polynomial functions on $\mathfrak{h}^*$. Let $\overline{U(\mathfrak{k})} \subset \overline{U(\mathfrak{g})}$ be the rings of fractions of the algebras $U(\mathfrak{k})$ and $A = U(\mathfrak{g})$ relative to the set of denominators (2.5). These rings are well defined, because both $U(\mathfrak{k})$ and $U(\mathfrak{g})$ satisfy the Ore condition relative to (2.5). Since $\sigma_i$ preserve the set of denominators (2.5), the automorphisms $\hat{\sigma}_i$ admit a natural extension to $\overline{A}$. 

3
Define $Z(\mathfrak{g}, \mathfrak{k})$ to be the double coset space of $\tilde{A}$ by its left ideal $\tilde{I}_+ := \tilde{A}n_+$, generated by elements of $n_+$, and the right ideal $\tilde{I}_- := n_-\tilde{A}$, generated by elements of $n_-$. $Z(\mathfrak{g}, \mathfrak{k}) := \tilde{A}/(\tilde{I}_+ + \tilde{I}_-)$. The space $Z(\mathfrak{g}, \mathfrak{k})$ is an associative algebra with respect to the multiplication map

$$a \circ b := aPb.$$  \hfill (2.6)

Here $P$ is the extremal projector $[AST]$ of the Lie algebra $\mathfrak{k}$ corresponding to the triangular decomposition $(\tilde{I}_+, \tilde{I}_-)$. We call $Z(\mathfrak{g}, \mathfrak{k})$ the reduction algebra associated to the pair $(\mathfrak{g}, \mathfrak{k})$. The assignment $x \mapsto x \mod \tilde{I}_+ + \tilde{I}_-$ establishes an injective homomorphism of the algebra $S(\mathfrak{g}, \mathfrak{k})$ (see Introduction for the definition) to $Z(\mathfrak{g}, \mathfrak{k})$, see $[KO]$. Moreover, the localization of the image of $S(\mathfrak{g}, \mathfrak{k})$ with respect to $\tilde{U}(h)$ coincides with $Z(\mathfrak{g}, \mathfrak{k})$.

The algebra $Z(\mathfrak{g}, \mathfrak{k})$ can be equipped with the action of Zhelobenko automorphisms $[KO]$. Denote by $\hat{\mathfrak{q}}_i$ the Zhelobenko automorphism $\hat{\mathfrak{q}}_i : Z(\mathfrak{g}, \mathfrak{k}) \to Z(\mathfrak{g}, \mathfrak{k})$ corresponding to the simple root $\alpha_i$, $i = 1, \ldots, r$. It is defined as follows $[KO]$. First we define a map $\hat{\mathfrak{q}}_i : A \to \tilde{A}/\tilde{I}_+$ by

$$\hat{\mathfrak{q}}_i(x) := \sum_{k \geq 0} \frac{(-1)^k}{k!} \hat{x}^k \phi_{\alpha_i}(x) \prod_{j=1}^k (h_{\alpha_i} - j + 1)^{-1} \mod \tilde{I}_+. \hfill (2.7)$$

Here $\hat{x}$ stands for the adjoint action of the element $x$, so that $\hat{x}(y) = xy - yx$ for $x \in \mathfrak{k}$ and $y \in \tilde{A}$. The operator $\hat{\mathfrak{q}}_i$ has the property

$$\hat{\mathfrak{q}}_i(hx) = (\sigma_i \circ h)\hat{\mathfrak{q}}_i(x) \hfill (2.8)$$

for any $x \in A$ and $h \in \mathfrak{h}$; $\sigma \circ h$ is defined in (2.4). With the help of (2.8), the map $\hat{\mathfrak{q}}_i$ can be extended to the map (denoted by the same symbol) $\hat{\mathfrak{q}}_i : \tilde{A} \to \tilde{A}/\tilde{I}_+$ by the setting $\hat{\mathfrak{q}}_i(\phi x) = (\sigma_i \circ \phi)\hat{\mathfrak{q}}_i(x)$ for any $x \in A$ and $\phi \in \tilde{U}(\mathfrak{h})$. One can further prove that $\hat{\mathfrak{q}}_i(\tilde{I}_+) = 0$ and $\hat{\mathfrak{q}}_i(\tilde{I}_-) \subset (\tilde{I}_+ + \tilde{I}_+)/\tilde{I}_+$, so that $\hat{\mathfrak{q}}_i$ can be viewed as a linear operator $\hat{\mathfrak{q}}_i : Z(\mathfrak{g}, \mathfrak{k}) \to Z(\mathfrak{g}, \mathfrak{k})$. Due to $[KO$, this is an algebra automorphism, satisfying (2.8). The operators $\hat{\mathfrak{q}}_i$ satisfy the same braid group relations as $\sigma_i$ and the inversion relation $[KO]$:  

$$\hat{\mathfrak{q}}_i^2(x) = (h_{\alpha_i} + 1)^{-1} \hat{x}_i (h_{\alpha_i} + 1), \quad x \in Z(\mathfrak{g}, \mathfrak{k}). \hfill (2.9)$$

Let $\mathfrak{p}$ be an adj-invariant complement of $\mathfrak{k}$ in $\mathfrak{g}$, as above. Choose a linear basis $\{p_K\}$ of $\mathfrak{p}$ and equip it with a total order $\prec$. For an arbitrary element $a \in \tilde{A}$ let $\tilde{a}$ be its image in the reduction algebra; in particular, $\tilde{p}_K$ is the image in $Z(\mathfrak{g}, \mathfrak{k})$ of the basic vector $p_K \in \mathfrak{p}$.

The general theory of reduction algebras, see $[Z]$ for the statements (a)-(c), says:

(a) Since $\mathfrak{h}$ normalizes both $n_+$ and $n_-$, the algebra $Z(\mathfrak{g}, \mathfrak{k})$ is a $\tilde{U}(\mathfrak{h})$-bimodule with respect to the multiplication by elements of $\tilde{U}(\mathfrak{h})$. It is free as a left $\tilde{U}(\mathfrak{h})$-module and as a right $\tilde{U}(\mathfrak{h})$-module. As a generating (over $\tilde{U}(\mathfrak{h})$) subspace one can take a projection of the space $S(\mathfrak{p})$ of symmetric tensors on $\mathfrak{p}$ to $Z(\mathfrak{g}, \mathfrak{k})$, that is a subspace of $Z(\mathfrak{g}, \mathfrak{k})$, formed by linear combinations of images of the powers $p^\nu$, where $p \in \mathfrak{p}$ and $\nu \geq 0$. 


(b) Assignments \( \deg(\widetilde{X}) = l \) for the image of any product of \( l \) elements from \( \mathfrak{p} \), \( X = p_{K_1} p_{K_2} \cdots p_{K_l} \), and \( \deg(Y) = 0 \) for any \( Y \in \mathcal{U}(\mathfrak{h}) \) define the structure of a filtered algebra on \( Z(\mathfrak{g}, \mathfrak{t}) \). The subspace \( Z(\mathfrak{g}, \mathfrak{t})^{(k)} \) of elements of degree not greater than \( k \) is a free left \( \mathcal{U}(\mathfrak{h}) \)-module and a free right \( \mathcal{U}(\mathfrak{h}) \)-module, with a generating subspace formed by linear combinations of images of the powers \( p^\nu \), where \( p \in \mathfrak{p} \) and \( k \geq \nu \geq 0 \).

(c) In the sequel we will choose for \( \{p_K\} \) a weight ordered basis; that is, each \( p_K \) has a certain weight \( \mu_K \),

\[
[h, p_K] = \mu_K(h)p_K \quad (2.10)
\]

for all \( h \in \mathfrak{h} \). The total order \( \prec \) will be compatible with the partial order \( \prec \) on \( \mathfrak{h}^* \), see [2.3], in the sense that \( \mu_K \prec \mu_L \Rightarrow p_K \prec p_L \). Then the images \( \widetilde{p}_L \) of the monomials \( (L \text{ is understood as the multiindex}) \)

\[
p_L := p_{L_1}^{n_1} p_{L_2}^{n_2} \cdots p_{L_m}^{n_m}, \quad p_{L_1} \prec p_{L_2} \prec \cdots \prec p_{L_m}, \quad k = n_1 + \cdots + n_m, \quad (2.11)
\]

in \( Z(\mathfrak{g}, \mathfrak{t})^{(k)} \) are linearly independent over \( \mathcal{U}(\mathfrak{h}) \) and their projections to the quotient \( Z(\mathfrak{g}, \mathfrak{t})^{(k)}/Z(\mathfrak{g}, \mathfrak{t})^{(k-1)} \) form a basis of the left \( \mathcal{U}(\mathfrak{h}) \)-module \( Z(\mathfrak{g}, \mathfrak{t})^{(k)}/Z(\mathfrak{g}, \mathfrak{t})^{(k-1)} \). The structure constants of the algebra \( Z(\mathfrak{g}, \mathfrak{t}) \) in the basis \( \{\widetilde{p}_L\} \) belong to the ring \( \mathcal{U}(\mathfrak{h}) \).

Choosing the PBW basis of \( A \) induced by any ordered basis of \( \mathfrak{t} + \mathfrak{p} \), which starts from a basis in \( \mathfrak{n}_- \) and ends by a basis in \( \mathfrak{n}_+ \), we see that the statement about the monomials \( \{p_L\} \) in (c) is valid without any condition on the order \( \prec \). However, the compatibility of the order \( \prec \) with the partial order \( \prec \) on \( \mathfrak{h}^* \) will be crucial for most of the statements below.

(d) The algebra \( Z(\mathfrak{g}, \mathfrak{t}) \) is the unital associative algebra, generated by \( \mathcal{U}(\mathfrak{h}) \) and all \( \{\widetilde{p}_L\} \),

with the weight relations \( (2.10) \) and the ordering relations

\[
\widetilde{p}_I \circ \widetilde{p}_J = \sum_{K,L: p_K \leq p_L} B_{IJKL} \widetilde{p}_K \circ \widetilde{p}_L + \sum_M C_{IJLM} \widetilde{p}_L + D_{IJ}, \quad p_I \succ p_J, \quad (2.12)
\]

where \( B_{IJKL}, C_{IJLM} \) and \( D_{IJ} \) are certain elements of \( \mathcal{U}(\mathfrak{h}) \).

Let \( \mathcal{D}(\mathfrak{h}) \) be the field of fractions of the ring \( \mathcal{U}(\mathfrak{h}) \). In [Z., sections 4.2.3 - 4.2.4 and 6.1.5, it is proved that the reduction algebra \( Z(\mathfrak{g}, \mathfrak{t}) \) is generated by the elements \( \widetilde{p}_L \) with the defining ordering relations \( (2.12) \) as an algebra over \( \mathcal{D}(\mathfrak{h}) \). We shall now show that the statement (d) holds over the smaller ring \( \mathcal{U}(\mathfrak{h}) \); in other words, the relations \( (2.12) \) are defined over \( \mathcal{U}(\mathfrak{h}) \) and the elements \( \widetilde{p}_L \) generate over \( \mathcal{U}(\mathfrak{h}) \) the algebra \( Z(\mathfrak{g}, \mathfrak{t}) \).

We first prove that the structure constants \( B_{IJKL}, C_{IJLM} \) and \( D_{IJ} \) belong actually to \( \mathcal{U}(\mathfrak{h}) \). This fact can be understood with the help of the factorized formula [AST] for the extremal projector \( P \). Indeed, decomposing the product, we represent the projector \( P \), after some reorderings, as a sum of terms \( \xi e_{-\gamma_1} \cdots e_{-\gamma_m} e_{\gamma_1'} \cdots e_{\gamma_m'} \), where \( \xi \in \mathcal{U}(\mathfrak{h}) \),
\(\gamma_1, \ldots, \gamma_m\) and \(\gamma'_1, \ldots, \gamma'_{m'}\) are positive roots of \(\mathfrak{t}\); the denominator of \(\xi\) is a product of linear factors of the form \(h_\gamma + \rho(h_\gamma) + \ell\), where \(\gamma\) is a positive root of \(\mathfrak{t}\) and \(\ell\) a positive integer, \(\ell > 0\). We calculate the product \(a \circ b\) in the following way. In the summand \(a \xi \cdot \cdots \cdot e_{-\gamma_m} e_{\gamma'_1} \cdots e_{\gamma'_{m'}} b\) of \(a \circ b\), we move \(\xi\) and all \(e_{-\gamma}\)'s to the left through \(a\) by taking multiple commutators with \(a\) and, similarly, all \(e_{\gamma'}\)'s to the right through \(b\). Proceeding this way, we write

\[
\tilde{p}_I \circ \tilde{p}_J = M_{IJKL} \tilde{p}_K \tilde{p}_L
\]

(2.13)

(we recall that \(\tilde{a}\) denotes the image of an element \(a \in \mathfrak{A}\) in the reduction algebra) where the (uniquely defined by the method of calculation) matrix \(M\) with entries in \(\tilde{U}(\mathfrak{h})\) has a triangular structure (even more is true: \(M_{IJKL} \neq 0 \Rightarrow p_I \succ p_K\) with 1's on the diagonal; denominators of entries of the matrix \(M\) are of the form \(h_\gamma + \rho(h_\gamma) + \pi(h_\gamma) + \ell\), where \(\pi\) is the weight, with respect to \(\mathfrak{h}\), of the corresponding \(p_I\) (the summand \(\pi(h_\gamma)\) appeared when, in calculating \(\tilde{p}_I \circ \tilde{p}_J\) as above, we first moved \(\xi \in \tilde{U}(\mathfrak{h})\) to the left through \(\tilde{p}_I\); taking further multiple commutators, we do not change the denominators any more). Take the formal (in the sense that for the moment we do not pay attention to possible dependencies between \(\tilde{p}_I \tilde{p}_J\) or between \(\tilde{p}_K \circ \tilde{p}_L\) in the algebra) inverse: \(\tilde{p}_I \tilde{p}_J = M^{-1}_{IJKL} \tilde{p}_K \circ \tilde{p}_L\); the inverse matrix \(M^{-1}\) is triangular as well, its entries are in \(\tilde{U}(\mathfrak{h})\) and it has 1's on the diagonal; the determinant of \(M\) is thus 1 and it follows that the above described structure of denominators of the entries of the matrix \(M\) remains the same for the matrix \(M^{-1}\). The commutation relation \(p_I p_J = p_J p_I + T, p_I \succ p_J, T \in \mathfrak{g}\), in \(U(\mathfrak{g})\) becomes \(\tilde{p}_I \tilde{p}_J = \tilde{p}_J \tilde{p}_I + \tilde{T}, \tilde{T} \in \mathfrak{p} + \mathfrak{h}\), in the reduction algebra. Translate this into the ordering rule for the product \(\circ\), expressing the projections \(\tilde{p}_I\)’s in terms of the products \(\tilde{p} \circ \tilde{p}\)’s with the help of the matrix \(M^{-1}\) in both, left and right, hand sides: the right hand side, being rewritten in terms of the multiplication \(\circ\), consists of ordered terms only, the left hand side is \(\tilde{p}_I \circ \tilde{p}_J + \ldots\), where dots stand for terms with \(\tilde{p}_I \circ \tilde{p}_L, p_I \succ p_L\); such term is either ordered or, by induction in \(I\), can be rewritten in the ordered form as in (2.12). The coefficient in front of \(\tilde{p}_I \circ \tilde{p}_J\) is from \(\tilde{U}(\mathfrak{h})\), so the reordering of the products \(\tilde{p}_I \circ \tilde{p}_J\) may force the coefficient of degree 1 or degree 0 term in (2.12) to belong to \(\tilde{U}(\mathfrak{h})\).

In the same manner we prove by induction on the filtration (described in the statement (b)) degree, that the algebra \(Z(\mathfrak{g}, \mathfrak{k})\) is generated over \(\tilde{U}(\mathfrak{h})\) by the elements \(\{\tilde{p}_L\}\). To see this, consider the weight basis, described in the statement (c), that is, the basis \(\tilde{p}_\mathcal{T}\) (\(\mathcal{T}\) is the multi-index) of the free \(\tilde{U}(\mathfrak{h})\)-module \(Z(\mathfrak{g}, \mathfrak{k})^{(k)}/Z(\mathfrak{g}, \mathfrak{k})^{(k-1)}\), composed by images in \(Z(\mathfrak{g}, \mathfrak{k})\) of products \(p_{L_1}^a p_{L_2}^b \cdots p_{L_m}^c\), where \(p_{L_1} < p_{L_2} < \ldots < p_{L_m}\) and \(k = n_1 + n_2 + \ldots + n_m\). Equip the set of these basic elements with a total order \(<\) compatible with the partial order \(\prec\) on \(\mathfrak{h}^*\); the compatibility has the same meaning as for the elements \(\{\tilde{p}_L\}\): [\(\mathfrak{h}\)-weight of \(\tilde{p}_\mathcal{T}\)] < [\(\mathfrak{h}\)-weight of \(\tilde{p}_\mathcal{T}\)] \(\Rightarrow\) \(\tilde{p}_\mathcal{T} < \tilde{p}_\mathcal{T}\). By the same, as above, arguments, referring to the structure of the projector \(P\), we have the following generalization of (2.13):

\[
\tilde{p}_I \circ \tilde{p}_\mathcal{T} = M_{IJKL} \tilde{p}_K \tilde{p}_\mathcal{T},
\]

(2.14)

where the matrix \(M\), with entries in \(\tilde{U}(\mathfrak{h})\), has again a triangular structure with 1’s on the diagonal. Therefore, the matrix \(M\) is invertible and its inverse matrix \(M^{-1}\) has entries
in \( \mathcal{U}(\mathfrak{g}) \). The formula \( \tilde{p}_i \tilde{p}_j = \pi_{ij} \tilde{p}_k \tilde{p}_l \) implies the induction step: the subspace \( Z(\mathfrak{g}, \mathfrak{t})^{(k+1)} \) is generated by products in \( Z(\mathfrak{g}, \mathfrak{t}) \) of elements from \( Z(\mathfrak{g}, \mathfrak{t})^{(1)} \).

Note that, before the localization, the algebra \( S(\mathfrak{g}, \mathfrak{t}) = \text{Norm}(\mathfrak{A}_{n+})/\mathfrak{A}_{n+} \), as well as its image in \( Z(\mathfrak{g}, \mathfrak{t}) \), is not generated by the elements of degree 1. The subalgebra of \( S(\mathfrak{g}, \mathfrak{t}) \), generated by the elements of degree 1 ("step algebra"), was the original subject of Mickelsson’s investigation [M].

(e) The following monomials form a basis of the left \( \mathcal{U}(\mathfrak{h}) \)-module \( Z(\mathfrak{g}, \mathfrak{t}) \):

\[
\tilde{p}_{I_1} \circ \tilde{p}_{I_2} \circ \cdots \circ \tilde{p}_{I_a}, \quad p_{I_1} \preceq p_{I_2} \preceq \cdots \preceq p_{I_a}.
\]

(2.15)

Before the proof of (e) we prove a more subtle statement.

**Proposition 1** Any expression in \( Z(\mathfrak{g}, \mathfrak{t}) \) can be written in the ordered form by a repeated application of (2.12) as instructions "replace the left hand side by the right hand side".

**Proof** of Proposition. To save the space in the proof of this proposition we take a liberty to sometimes write \( I \prec J \) instead of \( p_I \prec p_J \) (the same reservation concerns the use of \( \preceq, \succeq \) and \( \succeq \)).

Consider the homogeneous quadratic part of the relations (2.12):

\[
\tilde{p}_{I_1} \circ \tilde{p}_{I_2} = \sum_{I_1', I_2' : I_1' \preceq I_2'} \tilde{p}_{I_1'} \circ \tilde{p}_{I_2'}, \quad I_1 \succ I_2,
\]

(2.16)

where dots stand for coefficients from \( \mathcal{U}(\mathfrak{h}) \). Denote by \( \mathcal{I}(\tilde{p}_{I_1} \circ \tilde{p}_{I_2}) \) the right hand side of (2.16). We understand (2.16) as the set of instructions \( \tilde{p}_{I_1} \circ \tilde{p}_{I_2} \rightsquigarrow \mathcal{I}(\tilde{p}_{I_1} \circ \tilde{p}_{I_2}) \) (\( \rightsquigarrow \) stands for "replace") in the free algebra with the weight generators \( \tilde{p}_I \).

Let us prove the statement for a cubic monomial \( \tilde{p}_{I_1} \circ \tilde{p}_{I_2} \circ \tilde{p}_{I_3} \). For such a monomial one can apply the instructions (2.16) to \( \tilde{p}_{I_1} \circ \tilde{p}_{I_2} \) if \( I_1 \succ I_2 \) and to \( \tilde{p}_{I_2} \circ \tilde{p}_{I_3} \) if \( I_2 \succ I_3 \). Denote the results by \( \mathcal{I}_{12}(\tilde{p}_{I_1} \circ \tilde{p}_{I_2} \circ \tilde{p}_{I_3}) \) and \( \mathcal{I}_{23}(\tilde{p}_{I_1} \circ \tilde{p}_{I_2} \circ \tilde{p}_{I_3}) \) respectively.

For an element \( \psi \in \mathfrak{h}^* \), \( \psi = \sum \lambda_i \alpha_i \), where \( \alpha_i \) are the simple roots, let \( d(\psi) := \sum \lambda_i \). The function \( d \) is compatible with the partial order \( < \) on \( \mathfrak{h}^* \) in the sense that \( d(\alpha) < d(\beta) \) if \( \alpha < \beta \). Denote by the same letter \( d \) the function on the set of indices, labeling the weight base of \( \mathfrak{p} \); it is defined by \( d(I) := d(\mu_I) \), where \( \mu_I \) is the weight of \( \tilde{p}_I \).

We have \( d(I_1') + d(I_2') = d(I_1) + d(I_2) \) for any monomial \( \tilde{p}_{I_1'} \circ \tilde{p}_{I_2'} \) appearing in the right hand side of (2.16) (and the difference \( d(I_1') - d(I_1) \) is an integer). Since \( I_1 \succ I_2 \) and \( I_1' \preceq I_2' \), it follows that \( d(I_1) \geq d(I_2) \) and \( d(I_1') \leq d(I_2') \); therefore, \( d(I_1') \leq d(I_1) \) and \( d(I_2') \geq d(I_2) \).

We have \( \mathcal{I}(\tilde{p}_{I_1} \circ \tilde{p}_{I_2} \circ \tilde{p}_{I_3}) = \mathcal{I}_{12}(\tilde{p}_{I_1} \circ \tilde{p}_{I_2} \circ \tilde{p}_{I_3}) \) if \( I_1' \preceq I_2' \) and \( I_1 \leq I_2 \). If \( I_1 \prec I_2 \) or \( I_1' \succ I_2' \), then \( d(I_1') \leq d(I_2) \) and \( d(I_1') \geq d(I_2) \); therefore, \( d(I_1') \leq d(I_1) \) and \( d(I_1') \geq d(I_2) \).

Associate to a monomial \( \tilde{p}_{I_1} \circ \tilde{p}_{I_2} \circ \tilde{p}_{I_3} \), that is, to an ordered triple \( (I_1, I_2, I_3) \) of indices, the number \( \mathcal{I}(I_1, I_2, I_3) := d(I_1) + d(I_2) \). When we apply the ordering instructions \( \mathcal{I}_{12} \) or \( \mathcal{I}_{23} \) to \( \tilde{p}_{I_1} \circ \tilde{p}_{I_2} \circ \tilde{p}_{I_3} \), the function \( \mathcal{I} \) does not increase; that is, the value of \( \mathcal{I} \) on any of the appearing monomials is not greater than \( \mathcal{I}(I_1, I_2, I_3) \). Indeed, if we replace \( \tilde{p}_{I_1} \circ \tilde{p}_{I_2} \circ \tilde{p}_{I_3} \) with \( \tilde{p}_{I_1} \circ \tilde{p}_{I_2} \circ \tilde{p}_{I_3} \)
by \( \widetilde{p}_{I_1} \circ \widetilde{p}_{I_2} \) then \( 2d(I'_1) + d(I'_2) = d(I'_1) + (d(I'_1) + d(I'_2)) = d(I'_1) + (d(I_1) + d(I_2)) \leq d(I_1) + (d(I_1) + d(I_2)) = 2d(I_1) + d(I_2) \); and if we replace \( \widetilde{p}_{I_2} \circ \widetilde{p}_{I_3} \) by \( \widetilde{p}_{I_2} \circ \widetilde{p}_{I_3} \) then simply \( d(I'_2) \leq d(I_2) \) and \( d(I'_1) = d(I_1) \).

For a linear combination \( X = \sum c_{I_1 I_2 I_3} \widetilde{p}_{I_1} \circ \widetilde{p}_{I_2} \circ \widetilde{p}_{I_3} \) of cubic monomials, with coefficients \( c_{I_1 I_2 I_3} \in \overline{U}(\mathfrak{h}) \), denote the maximal value of \( \mathfrak{d} \) on the monomials, appearing in \( X \), by the same symbol \( \mathfrak{d} \); that is, \( \mathfrak{d}(X) := \max_{(I_1, I_2, I_3) : c_{I_1 I_2 I_3} \neq 0} \mathfrak{d}(I_1, I_2, I_3) \).

Assume that the assertion is false and there exists a cubic monomial which cannot be ordered by the instructions (2.16). Since \( \mathfrak{l} + \mathfrak{p} \) is finite-dimensional, the set of values of the function \( \mathfrak{d} \) on cubic monomials is bounded from below. So the minimal value \( \mathfrak{d}_{\min} \) of the function \( \mathfrak{d} \) on the set of cubic monomials which cannot be ordered is finite, \( \mathfrak{d}_{\min} > -\infty \).

Let \( \widetilde{p}_{I_1} \circ \widetilde{p}_{I_2} \circ \widetilde{p}_{I_3} \) be a monomial, which cannot be ordered, with \( \mathfrak{d}(I_1, I_2, I_3) = \mathfrak{d}_{\min} \). The application of the ordering instructions (2.16) cannot strictly decrease the value of \( \mathfrak{d} \), this would contradict the minimality of \( \mathfrak{d}(I_1, I_2, I_3) \). Therefore, among the appearing monomials, there is at least one monomial \( \widetilde{p}_{I_1} \circ \widetilde{p}_{I_2} \circ \widetilde{p}_{I_3} \) with the same value of \( \mathfrak{d} \).

If \( \widetilde{p}_{I'_1} \circ \widetilde{p}_{I'_2} \circ \widetilde{p}_{I'_3} \) appears in \( \mathcal{I}_{12} (\widetilde{p}_{I_1} \circ \widetilde{p}_{I_2} \circ \widetilde{p}_{I_3}) \) then \( 2d(I'_1) + d(I'_2) = 2d(I_1) + d(I_2) \), \( d(I_1) \geq d(I'_2) \), \( d(I'_1) \leq d(I'_2) \) and \( I'_1 = I_1 \); since the total weight is conserved, \( d(I'_1) = d(I_1) \), and \( d(I'_2) = d(I_2) \), so \( d(I_1) = d(I_2) = d(I'_1) = d(I'_2) \). If \( \widetilde{p}_{I'_1} \circ \widetilde{p}_{I'_2} \circ \widetilde{p}_{I'_3} \) appears in \( \mathcal{I}_{23} (\widetilde{p}_{I_{12}} \circ \widetilde{p}_{I_3}) \) then \( d(I_2) \geq d(I'_3) \), \( d(I'_2) \leq d(I'_3) \) and \( d(I'_3) = d(I_2) \) and \( I'_1 = I_{12} \); by the same arguments we have again, \( d(I_2) = d(I_3) = d(I'_2) = d(I'_3) \).

Due to the structure of the matrix \( M \), defined in (2.13), and the arguments used in the proof of the statement (d), \( \mathcal{I}(\widetilde{p}_{I} \circ \widetilde{p}_{J}) \) with \( d(I) = d(J) \) contains exactly one monomial \( \widetilde{p}_I \circ \widetilde{p}_J \) with \( d(I') = d(I) \) and this monomial is \( \widetilde{p}_I \circ \widetilde{p}_I \). Therefore, up to monomials with the value of \( \mathfrak{d} \) smaller than \( \mathfrak{d}_{\min} \) (they can be ordered by assumption) and up to a coefficient from \( \overline{U}(\mathfrak{h}) \), the operation \( \mathcal{I}_{12} \), \( I_1 \succ I_2 \), is simply \( \widetilde{p}_{I_1} \circ \widetilde{p}_{I_2} \rightsquigarrow \widetilde{p}_{I_2} \circ \widetilde{p}_{I_1} \); the operation \( \mathcal{I}_{23}, I_2 \succ I_3 \), is \( \widetilde{p}_{I_2} \circ \widetilde{p}_{I_3} \rightsquigarrow \widetilde{p}_{I_3} \circ \widetilde{p}_{I_2} \).

The transpositions (12) and (23) of neighbors generate all permutations of three letters. The orbit of \( \widetilde{p}_{I_1} \circ \widetilde{p}_{I_2} \circ \widetilde{p}_{I_3} \) under the group of permutations of three letters \( I_1, I_2 \) and \( I_3 \) contains the ordered monomial, the contradiction.

The degree 0 or 1 terms, contained in the full instructions (2.12), may only cause an appearance of linear or quadratic terms in the process of ordering of a cubic polynomial. So, any cubic polynomial can be ordered by (2.12) as well.

More generally, to a monomial \( X = \widetilde{p}_{I_1} \circ \widetilde{p}_{I_2} \circ \cdots \circ \widetilde{p}_{I_k} \) of an arbitrary degree \( k \) we associate the number \( \mathfrak{d}(I_1, \ldots, I_k) := (k - 1)d(I_1) + (k - 2)d(I_2) + \ldots + d(I_{k-1}) \), and, in the minimal situation, conclude that up to terms smaller than \( X \) in an appropriate sense, the instructions essentially reduce to transpositions \( (i, i + 1) \) of neighbors, which generate the whole symmetric group on \( k \) letters, and thus an ordered expression is in the orbit. \( \square \)

**Proof of statement (e).** By the statement (d) above, the algebra \( Z(\mathfrak{g}, \mathfrak{t}) \) is generated by \( \widetilde{p}_I \) and, due to the form (2.12) of relations, has a filtration by the \( \circ \)-degree. Let \( Z(\mathfrak{g}, \mathfrak{t})^{(\circ k)} \) be the subspace of elements of degree not greater than \( k \) with respect to the product \( \circ \). Since \( \widetilde{p}_{I_1} \circ \widetilde{p}_{I_2} \circ \cdots \circ \widetilde{p}_{I_k} = \widetilde{p}_{I_1} \circ \widetilde{p}_{I_2} \circ \cdots \circ \widetilde{p}_{I_k} \), it follows that \( Z(\mathfrak{g}, \mathfrak{t})^{(\circ k)} \subset Z(\mathfrak{g}, \mathfrak{t})^{(k)} \).

The opposite inclusion holds as well because the algebra \( Z(\mathfrak{g}, \mathfrak{t}) \) is generated by \( \widetilde{p}_I \). We
conclude that the two filtrations coincide.

Therefore, every element \( p_{I_1} \cdots p_{I_k}, I_1 \leq \ldots \leq I_k \), is in \( Z(\fg, \ft)^{(o)} \) and, by proposition above, can be ordered. The cardinalities of the sets \( \{ p_{I_1} \cdots p_{I_k} \mid I_1 \leq \ldots \leq I_k \} \) and \( \{ \bar{p}_{I_1} \cdots \bar{p}_{I_k} \mid I_1 \leq \ldots \leq I_k \} \) are equal, so due to (2.11) the set \( \{ \bar{p}_{I_1} \cdots \bar{p}_{I_k} \mid I_1 \leq \ldots \leq I_k \} \) is a basis of \( Z(\fg, \ft)^{(o)} / Z(\fg, \ft)^{(o(k-1))}. \)

Note that for an order which is not compatible with the partial order \(<\) on \( \fh^* \), the ordering relations of the form (2.12) may exist but the statement (e) does not necessarily hold. For instance, the ordering relations (2.12) can be written for a lexicographical order for the generators \( z_{ij} \) and \( t_i \) (with \( z_{ii} = t_i \)) of the algebra \( Z_n \), defined in the next Section, but the ordering procedure loops for cubic monomials, already for \( n = 2 \) (we don’t give details; it is an explicit calculation).

3 Diagonal reduction algebra of \( \mathfrak{gl}_n \)

Let \( \mathfrak{gl}_n \) be the Lie algebra of the general linear group of \( n \)-dimensional complex linear space. Consider the reductive pair \( (\fg, \ft) \) with \( \fg = \mathfrak{gl}_n \oplus \mathfrak{gl}_n \) and \( \ft = \mathfrak{gl}_n \) diagonally embedded into \( \mathfrak{gl}_n \oplus \mathfrak{gl}_n \). The corresponding reduction algebra we call 'diagonal reduction algebra' and denote it by \( Z_n \).

We fix the following notations for generators of these algebras \( \fg \) and \( \ft \). Let \( E^{(1)}_{ij} \) and \( E^{(2)}_{ij}, i, j = 1, \ldots, n \), be the standard generators of the two copies of the Lie algebra \( \mathfrak{gl}_n \) in \( \mathfrak{gl}_n \oplus \mathfrak{gl}_n \),

\[
[E^{(a)}_{ij}, E^{(b)}_{kl}] = \delta_{ab} \left( \delta_{jk} E^{(a)}_{il} - \delta_{il} E^{(a)}_{kj} \right),
\]

where \( \delta_{ab} \) and \( \delta_{ij} \) are the Kronecker symbols. Set

\[
e_{ij} := \frac{1}{2} (E^{(1)}_{ij} + E^{(2)}_{ij}) , \quad E_{ij} := \frac{1}{2} (E^{(1)}_{ij} - E^{(2)}_{ij}) . \quad \text{(3.1)}
\]

The elements \( e_{ij} \) span the diagonally embedded Lie algebra \( \ft \simeq \mathfrak{gl}_n \), while \( E_{ij} \) form an adjoint \( \ft \)-module. The Lie algebra \( \ft \) and the space \( \fp \) constitute a symmetric pair, that is, \([\ft, \ft] \subset \ft\), \([\ft, \fp] \subset \fp\), and \([\fp, \fp] \subset \ft\):

\[
[e_{ij}, e_{kl}] = \delta_{jk} e_{il} - \delta_{il} e_{kj} , \quad [e_{ij}, E_{kl}] = \delta_{jk} E_{il} - \delta_{il} E_{kj} , \quad [E_{ij}, E_{kl}] = \delta_{jk} e_{il} - \delta_{il} e_{kj} . \quad \text{(3.2)}
\]

In the sequel, \( h_a \) means the element \( e_{aa} \) of the Cartan subalgebra \( \fh \) of the subalgebra \( \ft \in \mathfrak{gl}_n \oplus \mathfrak{gl}_n \) and \( h_{ab} \) the element \( e_{aa} - e_{bb} \).

Let \( \{ \varepsilon_a \} \) be the basis of \( \fh^* \) dual to the basis \( \{ h_a \} \) of \( \fh \), \( \varepsilon_a(h_b) = \delta_{ab} \). We shall use as well the root notation \( h_a, e_a, e_{\alpha} \) for elements of \( \ft \), and \( H_a, E_a, E_{\alpha} \) for elements of \( \fp \). The Lie sub-algebra \( \fn_+ \) in the triangular decomposition is spanned by the root vectors \( e_{ij} \) with \( i < j \) and the Lie sub-algebra \( \fn_- \) by the root vectors \( e_{ij} \) with \( i > j \). Let
Let $\mathfrak{b}_+$ and $\mathfrak{b}_-$ be the corresponding Borel sub-algebras, $\mathfrak{b}_+ = \mathfrak{h} \oplus \mathfrak{n}_+$, $\mathfrak{b}_- = \mathfrak{h} \oplus \mathfrak{n}_-$. The system $\Delta_+$ of positive roots of $\mathfrak{g}$ consists of roots $\varepsilon_i - \varepsilon_j$ with $i < j$ and the system $\Delta_-$ consists of roots $\varepsilon_i - \varepsilon_j$ with $i > j$.

We fix the following action of the cover of the symmetric group $S_n$ (the Weyl group of the diagonal $\mathfrak{t}$) on the Lie algebra $\mathfrak{gl}_n \oplus \mathfrak{gl}_n$ by automorphisms

$$\hat{\sigma}_i(x) := \text{Ad}_{\exp(e_{i+1,i})\exp(-e_{i+1,i})}\text{Ad}_{\exp(e_{i+1,i})}(x) , \quad (3.3)$$

so that $\hat{\sigma}_i(e_{kl}) = (-1)^{\delta_{ik} + \delta_{il}}e_{\sigma_i(k)\sigma_i(l)}$ and $\hat{\sigma}_i(E_{kl}) = (-1)^{\delta_{ik} + \delta_{il}}E_{\sigma_i(k)\sigma_i(l)}$. Here $\sigma_i = (i, i+1)$ is an elementary transposition in the symmetric group. We extend naturally the above action of the cover of $S_n$ to the action by automorphisms on the associative algebra $A \equiv A_n := \mathbb{U}(\mathfrak{g}_n) \otimes \mathbb{U}(\mathfrak{g}_n)$. The restriction of this action to $\mathfrak{h}$ coincides with the natural action $\sigma(h_k) = h_{\sigma(k)}$, $\sigma \in S_n$, of the Weyl group on the Cartan sub-algebra. The shifted action (2.4) of the Weyl group on $\mathfrak{h}$ looks as:

$$\sigma \circ h_k := h_{\sigma(k)} + k - \sigma(k) , \quad k = 1, \ldots, n ; \quad \sigma \in S_n . \quad (3.4)$$

It becomes the usual action for the variables

$$\hat{h}_k := h_k - k , \quad \hat{h}_{ij} := \hat{h}_i - \hat{h}_j ; \quad (3.5)$$

so that for any $\sigma \in S_n$ we have $\sigma \circ \hat{h}_k = \hat{h}_{\sigma(k)}$ and $\sigma \circ \hat{h}_{ij} = \hat{h}_{\sigma(i)\sigma(j)}$. The set of denominators, defining the localizations $\mathbb{U}(\mathfrak{h})$ and $\tilde{A}$ consists of elements

$$h_{ij} + l , \quad l \in \mathbb{Z} , \quad 1 \leq i < j \leq n . \quad (3.6)$$

We choose the set of vectors $E_{ij}$, $i, j = 1, \ldots, n$, as a basis of the space $\mathfrak{p}$. The weight of $E_{ij}$ is $\varepsilon_i - \varepsilon_j$. The compatibility of a total order $<$ with the partial order $<$ on $\mathfrak{h}^*$ means the condition

$$E_{ij} < E_{kl} \quad \text{if} \quad i - j > k - l . \quad (3.7)$$

The order in each subset $\{E_{ij}|i - j = a\}$ with a fixed $a$ can be chosen arbitrarily. For instance, we can set

$$E_{ij} < E_{kl} \quad \text{if} \quad i - j > k - l \quad \text{or} \quad i - j = k - l \quad \text{and} \quad i > k . \quad (3.8)$$

Denote the images of the elements $E_{ij}$ in $\mathbb{Z}_n$ by $z_{ij}$. We use also the notation $t_i$ for the elements $z_{ii}$ and $t_{ij} := t_i - t_j$ for the elements $z_{ii} - z_{jj}$. The order (3.8) induces as well the order on the generators $z_{ij}$ of the algebra $\mathbb{Z}_n$:

$$z_{ij} < z_{kl} \iff E_{ij} < E_{kl} . \quad (3.9)$$

The statement (2.12) implies an existence of structure constants $B_{(ab),(cd),(ij),(kl)} \in \mathbb{U}(\mathfrak{h})$ and $D_{(ab),(cd)} \in \mathbb{U}(\mathfrak{h})$ such that for any $a, b, c, d = 1, \ldots, n$ we have

$$z_{ab} \diamond z_{cd} = \sum_{i,j,k,l: z_{ij}\leq z_{kl}} B_{(ab),(cd),(ij),(kl)} z_{ij} \diamond z_{kl} + D_{(ab),(cd)} . \quad (3.10)$$
Linear terms in the right hand side of (3.10) are absent since here \((g,k)\) form a symmetric pair. The relations (3.10) together with the weight conditions

\[ [h, z_{ab}] = (\varepsilon_a - \varepsilon_b)(h)z_{ab} \]  

are the defining relations for the algebra \(Z_n\).

The structure of denominators of entries of the matrices \(M\) and \(M^{-1}\), mentioned in the proof of (2.12) above, shows that for the algebra \(Z_n\) the denominators of the structure constants \(B_{ab}(e_{ij}), (cd), (ij), (kl)\) and \(D_{ab}(cd)\) are products of linear factors of the form \(\hat{h}_{ij} + \ell, \ i < j\), where \(\ell \geq -1\) is an integer. This is because in our situation the sl2–sub-algebra (of the diagonal \(gl_n\)), corresponding to an arbitrary positive root \(\varepsilon_i - \varepsilon_j, i < j\), has only 1, 2- and 3-dimensional representations in \(p\), so the numbers \(\ell\)'s in the denominators of the summands of the projector can drop at most by 2 due to the presence of the term \((\pi, \gamma)\).

The Chevalley anti-involution \(\varepsilon\) in \(U(gl_n \oplus gl_n)\), \(\varepsilon(e_{ij}) := e_{ji}, \varepsilon(E_{ij}) := E_{ji}\), induces the anti-involution \(\varepsilon\) in the algebra \(Z_n\):

\[ \varepsilon(z_{ij}) = z_{ji}, \quad \varepsilon(h_{k}) = h_{k}. \]  

Besides, the outer automorphism of the Dynkin diagram of \(gl_n\) induces the involutive automorphism \(\omega\) of \(Z_n\),

\[ \omega(z_{ij}) = (-1)^{i+j+1}z_{j'i'}, \quad \omega(h_{k}) = -h_{k'}, \]  

where \(i' = n + 1 - i\). The operations \(\varepsilon\) and \(\omega\) commute, \(\varepsilon\omega = \omega\varepsilon\).

Central elements of the sub-algebra \(U(gl_n) \otimes 1 \subset A\), generated by \(n\) Casimir operators of degrees \(1, \ldots, n\), as well as central elements of the sub-algebra \(1 \otimes U(gl_n) \subset A\) project to central elements of the algebra \(Z_n\). In particular, central elements of degree 1 project to central elements

\[ h_1 + \ldots + h_n \quad \text{and} \quad t_1 + \ldots + t_n \]  

of the algebra \(Z_n\). The difference of central elements of degree two projects to the central element

\[ \sum_{i=1}^{n} (h_i - 2i) t_i \]  

of the algebra \(Z_n\). The images of other Casimir operators are more complicated.

### 3.1 Change of variables

We shall use the following elements of \(U(\hat{h})\):

\[ A_{ij} := \frac{\hat{h}_{ij}}{\hat{h}_{ij} - 1}, \quad A'_{ij} := \frac{\hat{h}_{ij} - 1}{\hat{h}_{ij}}, \quad B_{ij} := \frac{\hat{h}_{ij} - 1}{\hat{h}_{ij} - 2}, \quad B'_{ij} := \frac{\hat{h}_{ij} - 2}{\hat{h}_{ij} - 1}, \quad C'_{ij} := \frac{\hat{h}_{ij} - 3}{\hat{h}_{ij} - 2}. \]
the variables $\hat{h}_{ij}$ are defined in (3.5). Note that $A_{ij}A'_{ij} = B_{ij}B'_{ij} = 1$.

Define elements $\hat{t}_1, \ldots, \hat{t}_n \in \mathbb{Z}_n$ by

$$\hat{t}_1 := t_1, \quad \hat{t}_2 := \hat{q}_1(t_1), \quad \hat{t}_3 := \hat{q}_2\hat{q}_1(t_1), \quad \ldots, \quad \hat{t}_n := \hat{q}_{n-1} \cdots \hat{q}_2\hat{q}_1(t_1).$$

(3.16)

Using (2.7) we find the relations

$$\hat{q}_i(t_i) = -\frac{1}{\hat{h}_{i,i+1} - 1} t_i + \frac{\hat{h}_{i,i+1}}{\hat{h}_{i,i+1} - 1} t_{i+1},$$

$$\hat{q}_i(t_{i+1}) = \frac{\hat{h}_{i,i+1}}{\hat{h}_{i,i+1} - 1} t_i - \frac{1}{\hat{h}_{i,i+1} - 1} t_{i+1},$$

$$\hat{q}_i(t_k) = t_k, \quad k \neq i, i + 1,$$

(3.17)

which can be used to convert the definition (3.16) into a linear over the ring $\mathbb{U}(\mathfrak{h})$ change of variables:

$$\hat{t}_l = t_l \prod_{j=1}^{l-1} A_{jl} - \sum_{k=1}^{l-1} \frac{1}{\hat{h}_{kl} - 1} \prod_{j=1}^{k-1} A_{jl},$$

$$t_l = t_l \prod_{j=1}^{l-1} A'_{jl} + \sum_{k=1}^{l-1} \frac{1}{\hat{h}_{kl}} \prod_{j=1, j\neq k}^{l-1} A'_{jk}.$$

(3.18)

In terms of the new variables $\hat{t}$'s, the linear in $t$ central element (3.14) reads

$$\sum t_i = \sum \hat{t}_i \prod_{a,a\neq i} \frac{\hat{h}_{ia}}{\hat{h}_{ia}}.$$

In the following, we use the notion of coefficient-bounded formulas and relations. It means the following. Given a family of formulas for each $n$ (expressing some action, relations etc.) with coefficients in $\mathbb{U}(\mathfrak{h})$, we say that it is coefficient-bounded if the degrees of the numerators and denominators (in the reduced form, with no common factors) of the coefficients do not grow with $n$.

For example, the set of relations for $\mathbb{Z}_n$, which we shall exhibit, will have coefficient-bounded terms with respect to a certain set of generators. In this sense the action (3.17) is coefficient-bounded while the change of variables (3.18) is however not coefficient-bounded.

### 3.2 Braid group action

Since $\hat{q}_i^2(x) = x$ for any element $x$ of zero weight, the braid group acts as its symmetric group quotient on the space of weight 0 elements. Although the change of variables (3.18)
is not coefficient-bounded in the sense of Section 2, the action of the transformations \( \tilde{q}_i \) on the new variables \( \tilde{t} \)'s is coefficient-bounded: it follows from (3.16) and \( \tilde{q}_i(t_1) = t_1 \) for all \( i > 1 \) that
\[
\tilde{q}_\sigma(\tilde{t}_i) = \tilde{t}_{\sigma(i)} \quad \text{for any} \quad \sigma \in S_n. \tag{3.19}
\]

The action of the Zhelobenko automorphisms on the generators \( z_{kl} \) looks as follows:
\[
\begin{align*}
\tilde{q}_i(z_{ik}) &= -z_{i+1,k} A_{i,i+1}, & \tilde{q}_k(z_{ki}) &= -z_{k,i+1}, & k \neq i, i + 1, \\
\tilde{q}_i(z_{i+1,k}) &= z_{i,k}, & \tilde{q}_k(z_{k,i+1}) &= z_{k,i} A_{i,i+1}, & k \neq i, i + 1, \\
\tilde{q}_i(z_{i,i+1}) &= -z_{i+1,i} A_{i,i+1} B_{i,i+1}, & \tilde{q}_k(z_{i,i+1}) &= -z_{i,i+1}, \\
\tilde{q}_i(z_{j,k}) &= z_{j,k}, & j, k \neq i, i + 1.
\end{align*} \tag{3.20}
\]

Denote \( i' = n + 1 - i \), as before. The braid group action (3.20) is compatible with the anti-involution \( \epsilon \) and the involution \( \omega \) (note that \( \omega(h_{ij}) = h_{j'i'} \)), see (3.12) and (3.13), in the following sense:
\[
\epsilon \tilde{q}_i = \tilde{q}^{-1}_{i-1} \epsilon, \quad \omega \tilde{q}_i = \tilde{q}_{i'-1} \omega. \tag{3.21}
\]

Let \( w_0 \) be the longest element of the Weyl group of \( gl_n \), the symmetric group \( S_n \). Similarly to the squares of the transformations corresponding to the simple roots, see (2.9), the action of \( \tilde{q}_{w_0}^2 \) is the conjugation by a certain element of \( \overline{U}(\mathfrak{h}) \). Moreover, one can observe by a direct calculation, that
\[
\tilde{q}_{w_0}(z_{ij}) = (-1)^{i+j} z_{i'j'} \prod_{a:a<i'} A_{a'i'} \prod_{b:b>j'} A_{j'b}, \quad \tilde{q}_{w_0}(\tilde{t}_i) = \tilde{t}_{i'}. \tag{3.22}
\]

The formula (3.22) implies the existence of the ordering relations for the generators \( z_{ij} \) in the inverse to (3.8)-(3.9) order.

**Corollary 2.** There exist \( B'_{(ab),(cd),(ij),(kl)} \) and \( D'_{(ab),(cd)} \in \overline{U}(\mathfrak{h}) \) such that for any \( z_{ab} \) and \( z_{cd} \) we have
\[
z_{ab} \diamond z_{cd} = \sum_{i,j,k,l:z_{kl} \leq z_{ij}} B'_{(ab),(cd),(ij),(kl)} z_{ij} \diamond z_{kl} + D'_{(ab),(cd)} \cdot \tag{3.23}
\]

Indeed, we apply the transformation \( \tilde{q}_{w_0} \) to the equalities (3.10) and substitute (3.22). This gives the relations (3.23) since the assignment \( (i, j) \mapsto (i', j') \) reverses the order \( < \).

### 3.3 Defining relations

To save space we omit in this section the symbol \( \diamond \) for the multiplication in the algebra \( Z_n \). It should not lead to any confusion since no other multiplication is used in this section.

Each relation which we will derive will be of a certain weight, equal to a sum of two roots. From general considerations the upper estimate for the number of terms in a
quadratic relation of weight \( \lambda = \alpha + \beta \) is the number \(|\lambda|\) of quadratic combinations \( z_{\alpha'}z_{\beta'} \) with \( \alpha' + \beta' = \lambda \). There are several types of relation weights, excluding the trivial one, \( \lambda = 2(\varepsilon_i - \varepsilon_j) \), \(|\lambda| = 1\):

1. \( \lambda = \pm(2\varepsilon_i - \varepsilon_j - \varepsilon_k) \), where \( i, j \) and \( k \) are pairwise distinct. Then \(|\lambda| = 2\).

2. \( \lambda = \varepsilon_i - \varepsilon_j + \varepsilon_k - \varepsilon_l \) with pairwise distinct \( i, j, k \) and \( l \). Then \(|\lambda| = 4\).

3. \( \lambda = \varepsilon_i - \varepsilon_j, \ i \neq j \). For \( z_{\alpha'}z_{\beta'} \), there are \( 2(n - 2) \) possibilities (subtype 3a) with \( \alpha' = \varepsilon_i - \varepsilon_k, \beta' = \varepsilon_k - \varepsilon_j \) or \( \alpha' = \varepsilon_k - \varepsilon_j, \beta' = \varepsilon_i - \varepsilon_k \) with \( k \neq i, j \) and \( 2n \) possibilities (subtype 3b) with \( \alpha' = 0, \beta' = \varepsilon_i - \varepsilon_j \) or \( \alpha' = \varepsilon_i - \varepsilon_j, \beta' = 0 \). Thus \(|\lambda| = 4(n - 1)\).

4. \( \lambda = 0 \). There are \( n^2 \) possibilities (subtype 4a) with \( \alpha' = 0, \beta' = 0 \) and \( n(n - 1) \) possibilities (subtype 4b) with \( \alpha' = \varepsilon_i - \varepsilon_j, \beta' = \varepsilon_j - \varepsilon_i, \ i \neq j \). Here \(|\lambda| = n(2n - 1)\).

Below we write down relations for each type (and subtype) separately. The relations of types 1 and 2 have a simple form in terms of the original generators \( z_{ij} \). To write the relations of types 3 and 4, it is convenient to renormalize the generators \( z_{ij} \) with \( i \neq j \). Namely, we set

\[
\hat{z}_{ij} = z_{ij} \prod_{k=1}^{i-1} A_{ki}.
\] (3.24)

In terms of the generators \( \hat{z}_{ij} \), the formulas (3.20) for the action of the automorphisms \( \tilde{\eta} \) translate as follows:

\[
\tilde{\eta}(\hat{z}_{ik}) = -\hat{z}_{i+1,k}, \quad \tilde{\eta}(\hat{z}_{i+1,k}) = \hat{z}_{i,k} A_{i+1,i}, \quad k \neq i, i + 1,
\]
\[
\tilde{\eta}(\hat{z}_{kj}) = -\hat{z}_{k,i+1}, \quad \tilde{\eta}(\hat{z}_{k,i+1}) = \hat{z}_{k,i} A_{i+1,i} = A'_{i+1,i} \hat{z}_{k,i}, \quad k \neq i, i + 1,
\]
\[
\tilde{\eta}(\hat{z}_{i+1,i}) = -A'_{i+1,i} \hat{z}_{i+1,i}, \quad \tilde{\eta}(\hat{z}_{i+1,i}) = -\hat{z}_{i+1,i} A_{i+1,i},
\]
\[
\tilde{\eta}(\hat{z}_{j,k}) = \hat{z}_{j,k}, \quad j, k \neq i, i + 1.
\]

Although the renormalization (3.24) is not coefficient-bounded, the action of the braid group stays coefficient-bounded.

1. The relations of the type 1 are:

\[
z_{ij}z_{ik} = z_{ik}z_{ij} A_{kj}, \quad z_{ji}z_{ki} = z_{ki}z_{ji} A'_{kj}, \quad \text{for } j < k, \ i \neq j, k.
\] (3.25)

2. Denote \( D_{ijkl} := \hat{h}_{ik}^{-1} - \hat{h}_{jl}^{-1} \). Then, for any four pairwise different indices \( i, j, k \) and \( l \), we have the following relations of the type 2:

\[
[z_{ij}, z_{kl}] = z_{kj} z_{i} D_{ijkl}, \quad i < k, \ j < l,
\]
\[
z_{ij} z_{kl} - z_{kl} z_{ij} A'_{ij} A_{ij} = z_{kj} z_{i} D_{ijkl}, \quad i < k, \ j > l.
\] (3.26)
3a. Let \( i \neq k \neq l \neq i \). Denote

\[
\hat{E}_{ikl} := - \left( (t_i - \hat{t}_k) \frac{\hat{h}_{il} + 1}{h_{ik}\hat{h}_{il}} + (\hat{t}_k - t_l) \frac{\hat{h}_{il} - 1}{h_{kl}\hat{h}_{il}} \right) \hat{z}_{il} + \sum_{a,a \neq i,k,l} \hat{z}_{ia} \frac{B_{ai}}{h_{ka} + 1} .
\]

With this notation the first group of the relations of the type 3 is:

\[
\begin{align*}
\hat{z}_{ik} \hat{z}_{kl} A_{ik}' - \hat{z}_{kl} \hat{z}_{ik} B_{ki} &= \hat{E}_{ikl} , \quad i < k < l , \\
\hat{z}_{ik} \hat{z}_{kl} A_{ik}' B_{lk} - \hat{z}_{kl} \hat{z}_{ik} B_{ki} &= \hat{E}_{ikl} , \quad i < l < k , \\
\hat{z}_{ik} \hat{z}_{kl} A_{ki} - \hat{z}_{kl} \hat{z}_{ik} B_{ki} &= \hat{E}_{ikl} , \quad k < i < l , \\
\hat{z}_{ik} \hat{z}_{kl} A_{kl} B_{il}' - \hat{z}_{kl} \hat{z}_{ik} B_{ki} &= \hat{E}_{ikl} , \quad k < l < i , \\
\hat{z}_{ik} \hat{z}_{kl} A_{lk}' B_{il} - \hat{z}_{kl} \hat{z}_{ik} B_{ki} &= \hat{E}_{ikl} , \quad l < i < k , \\
\hat{z}_{ik} \hat{z}_{kl} A_{lk}' B_{lk} B_{il}' - \hat{z}_{kl} \hat{z}_{ik} B_{ki} &= \hat{E}_{ikl} , \quad l < k < i .
\end{align*}
\]

The relations (3.27) can be written in a more compact way with the help of both systems, \( \hat{z}_{ij} \) and \( \hat{z}_{ij} \), of generators. Let now

\[
E_{ikl} := - \left( (t_i - \hat{t}_k) \frac{\hat{h}_{il} + 1}{h_{ik}\hat{h}_{il}} + (\hat{t}_k - t_l) \frac{\hat{h}_{il} - 1}{h_{kl}\hat{h}_{il}} \right) z_{il} + \sum_{a,a \neq i,k,l} z_{ia} \frac{B_{ai}}{h_{ka} + 1} .
\]

Then

\[
\begin{align*}
\hat{z}_{ik} \hat{z}_{kl} A_{ik}' - \hat{z}_{kl} \hat{z}_{ik} B_{ki} &= E_{ikl} , \quad k < l , \\
z_{ik} \hat{z}_{kl} A_{ik}' B_{lk} - \hat{z}_{kl} z_{ik} B_{ki} &= E_{ikl} , \quad l < k .
\end{align*}
\]

Moreover, after an extra redefinition: \( \hat{z}_{kl} = \hat{z}_{kl} B_{lk} \) for \( k > l \), the left hand side of the second line in (3.28) becomes, up to a common factor, the same as the left hand side of the first line, namely, it reads \( z_{ik} \hat{z}_{kl} A_{ik}' - \hat{z}_{kl} z_{ik} B_{ki} A_{ik}' \).

3b. Let \( l \neq j \). The second group of relations of the type 3 reads:

\[
\begin{align*}
\hat{z}_{ij} \hat{t}_i &= \hat{t}_i \hat{z}_{ij} C_{ji}' - t_j \hat{z}_{ij} \frac{1}{h_{ij} + 2} - \sum_{a,a \neq i,j} \hat{z}_{aj} \frac{1}{h_{ia} + 2} , \\
\hat{z}_{ij} \hat{t}_j &= - \hat{t}_i \hat{z}_{ij} \frac{C_{ji}'}{h_{ij} - 1} + \hat{t}_j \hat{z}_{ij} A_{ij}' B_{ji} + \sum_{a,a \neq i,j} \hat{z}_{aj} \frac{B_{ai}}{h_{ja} + 1} , \\
\hat{z}_{ij} \hat{t}_k &= \hat{t}_i \hat{z}_{ij} \frac{(\hat{h}_{ij} + 3)B_{ji}}{(h_{ik} - 1)(h_{jk} - 1)} + \hat{t}_j \hat{z}_{ij} \frac{(\hat{h}_{ij} + 1)B_{ji}}{(h_{ik} - 1)(h_{jk} - 1)^2} + \hat{t}_k \hat{z}_{ij} A_{ik} A_{kj} B_{jk}' \\
- \hat{z}_{jk} \hat{z}_{ik} \frac{(\hat{h}_{ij} + 1)B_{ki}}{(h_{ik} - 1)(h_{jk} - 1)} - \sum_{a,a \neq i,j,k} \hat{z}_{aj} \frac{\hat{h}_{ij} + 1}{(h_{ik} - 1)(h_{jk} - 1)} \cdot \frac{B_{ai}}{h_{ka} + 1} .
\end{align*}
\]
4a. The relations of the weight zero (the type 4) are also divided into 2 groups. This is the first group of the relations:

\[ [\hat{t}_i, \hat{t}_j] = 0 \]  \hspace{1cm} (3.30)

Note that the relations (3.30) hold for the diagonal reduction algebra for an arbitrary reductive Lie algebra: the images of the generators, corresponding to the Cartan sub-algebra, commute.

4b. The second group of the relations of the type 4 is (here \(i \neq j\))

\[ [\hat{z}_{ij}, \hat{z}_{ji}] = \hat{h}_{ij} - \frac{1}{\hat{h}_{ij}}(\hat{t}_i - \hat{t}_j)^2 + \sum_{a,a \neq i,j} \left( \frac{1}{\hat{h}_{ja}} - 1 \right) \hat{z}_{ai} \hat{z}_{ia} - \frac{1}{\hat{h}_{ia}} \hat{z}_{aj} \hat{z}_{ja} \]  \hspace{1cm} (3.31)

The list of relations is completed.

Denote by \(\mathcal{R}\) the system (3.25) – (3.27), and (3.29) – (3.31) of the relations.

**Theorem 3.** The relations \(\mathcal{R}\) are the defining relations for the weight generators \(z_{ij}\) and \(t_i\) of the algebra \(Z_n\). In particular, the set (3.10) of ordering relations follows over \(U(\mathfrak{h})\) from (and is equivalent to) \(\mathcal{R}\).

The derivation of the relations is given in [KO2]; the proof of Theorem is in Section 4.

The relations (3.25), (3.26) (a straightforward verification), as well as (3.27), (3.29), (3.30) and (3.31), have coefficient-bounded terms with respect to the generators \(\hat{z}_{ij}\) and \(\hat{t}_i\); there is no coefficient-boundedness with respect to the original generators \(z_{ij}\) and \(t_i\). We think that the set (3.10) of ordering relations is not coefficient-bounded.

3.4 Stabilization

Consider an embedding of \(\mathfrak{gl}_n\) to \(\mathfrak{gl}_{n+1}\), given by an assignment \(e_{ij} \mapsto e_{ij}, i, j = 1, \ldots, n\), where \(e_{ij}\) in the source are the generators of \(\mathfrak{gl}_n\) and target \(e_{ij}\) are in \(\mathfrak{gl}_{n+1}\). The same rule \(E_{ij} \mapsto E_{ij}\) defines an embedding of the Lie algebra \(\mathfrak{gl}_n \oplus \mathfrak{gl}_n\) to the Lie algebra \(\mathfrak{gl}_{n+1} \oplus \mathfrak{gl}_{n+1}\) and of the enveloping algebra \(A_n = U(\mathfrak{gl}_n \oplus \mathfrak{gl}_n)\) to \(A_{n+1} = U(\mathfrak{gl}_{n+1} \oplus \mathfrak{gl}_{n+1})\). This embedding clearly maps nilpotent sub-algebras of \(\mathfrak{gl}_n\) to the corresponding nilpotent sub-algebras of \(\mathfrak{gl}_{n+1}\) and thus defines an embedding \(\iota_n : Z_n \rightarrow Z_{n+1}\) of the corresponding double coset spaces. However, the map \(\iota_n\) is not a homomorphism of algebras. This is because the multiplication maps are defined with the help of projectors, which are different for \(\mathfrak{gl}_n\) and \(\mathfrak{gl}_{n+1}\).

Nevertheless, there is an important connection between the two multiplication maps. Namely, let \(V_{n+1}\) be the left ideal of the algebra \(Z_{n+1}\), generated by elements \(z_{i,n+1}\), \(i = 1, \ldots, n\), and \(V'_{n+1}\) be the right ideal of the algebra \(Z_{n+1}\), generated by elements \(z_{n+1,i}\); \(i = 1, \ldots, n\). For a moment denote by \(\circ(n) : Z_n \otimes Z_n \rightarrow Z_n\) and \(\circ(n+1) : Z_{n+1} \otimes Z_{n+1}\) the multiplication maps in \(Z_n\) and \(Z_{n+1}\) (instead of the default notation \(\circ\), see (2.6)).
Let \( \pi_{n+1} : Z_{n+1} \to Z_{n+1} \) be any linear operator in \( Z_{n+1} \), which projects \( Z_{n+1} \) onto \( \iota_n(Z_n) \); assume that either the ideal \( V_{n+1} \) or the ideal \( V'_{n+1} \) is in the kernel of \( \pi_{n+1} \),

\[
\pi_{n+1}(x) = x, \quad x \in \iota_n(Z_n), \quad \text{and} \quad \pi_{n+1}(V_{n+1}) = 0 \quad \text{or} \quad \pi_{n+1}(V'_{n+1}) = 0 .
\]

Define a map \( \tilde{\phi}(n) : \iota_n(Z_n) \otimes \iota_n(Z_n) \to \iota_n(Z_n) \) as a composition

\[
\tilde{\phi}(n) = \pi_{n+1} \circ (n+1) .
\]

**Proposition 4.** We have a commutative diagram of maps

\[
t_n \circ \phi(n) = \tilde{\phi}(n)(t_n \otimes t_n) . \tag{3.32}
\]

More precisely, for \( i, j, k, l \leq n \) the difference \( t_n(z_{ij} \circ (n) z_{kl}) - z_{ij} \circ (n+1) z_{kl} \) in \( Z_{n+1} \) can be written in the form \( \sum_{a=1}^{n} z_{n+1,a} \circ (n+1) z_{i+k-1} j - a, n+1 \xi^{(a)} \), where \( \xi^{(a)} \in \mathcal{U}(\mathfrak{h}) \).

For the proof of Proposition, we need the following

**Lemma 5.** The left ideal of \( Z_n \), generated by all \( z_{in} \), \( i = 1, \ldots, n-1 \), consists of images in \( Z_n \) of sums \( \sum_i X_i E_{in} \) with \( X_i \in \bar{A} \), \( i = 1, \ldots, n-1 \).

The right ideal of \( Z_n \), generated by all \( z_{ni} \), \( i = 1, \ldots, n-1 \), consists of images in \( Z_n \) of sums \( \sum_i E_{ni} Y_i \) with \( Y_i \in \bar{A} \), \( i = 1, \ldots, n-1 \).

**Proof of Lemma.** We follow the arguments used in the proof of the relations (2.12). Present the projector \( P \) as a sum of terms \( \xi e_{-\gamma_1} \cdots e_{-\gamma_m} e_{\gamma_1'} \cdots e_{\gamma_{m'}'} \), where \( \xi \in \mathcal{U}(\mathfrak{h}) \), \( \gamma_1, \ldots, \gamma_m \) and \( \gamma_1', \ldots, \gamma_{m'}' \) are positive roots of \( \mathfrak{t} \). For any \( \lambda \in Q_+ \) denote by \( P_{\lambda} \) the sum of above elements with \( \gamma_1 + \cdots + \gamma_m = \gamma_1' + \cdots + \gamma_{m'}' = \lambda \). Then \( P = \sum_{\lambda \in Q_+} P_{\lambda} \). For any \( X, Y \in \bar{A} \) define an element \( X \circ_{\lambda} Y \) as the image of \( XP_{\lambda}Y \) in the reduction algebra. We have \( X \circ Y = \sum_{\lambda \in Q_+} X \circ_{\lambda} Y \).

For any \( X \in \bar{A} \) and \( i < n \) consider the product \( X \circ_{\lambda} z_{in} \). Let \( \lambda = \sum_{k=1}^{n} \lambda_k e_k \). The product \( X \circ_{\lambda} z_{in} \) is zero if \( \lambda_n \neq 0 \). Indeed, in this case in each summand of \( P_{\lambda} \) one of \( e_{\gamma_k'} \) is equal to some \( e_j \). We can order all the monomials in \( \mathcal{U}(n_+) \) in such a way that all \( e_j \) stand on the right. Since \( [e_j, E_{in}] = 0 \), the product \( e_j E_{in} \) belongs to the left ideal \( \bar{I}_+ \) and thus \( X \circ_{\lambda} z_{in} = 0 \) in \( Z_n \). If \( \lambda_n = 0 \), then by PBW arguments, \( P_{\lambda} \) can be written as a sum of monomials composed of generators \( e_{ij} \), \( 1 \leq i < j < n \), and thus their adjoint action leaves the space, spanned by all \( E_{in}, i < n \), invariant, so \( X \circ_{\lambda} z_{in} \) is presented as an image of the sum \( \sum_j X_j E_{jn} \) with \( X_j \in \bar{A}, j < n \). Thus, the left ideal, generated by \( z_{in} \) is contained in the vector space of images in \( Z_n \) of sums \( \sum_j X_j E_{in} \).

Moreover, \( X \circ z_{in} \) is the image of \( X E_{in} + \sum_{m<i} X^{(m)} E_{mn} \) for some \( X^{(m)} \) and the induction on \( i \) proves the inverse inclusion.

The second part of lemma is proved similarly. \( \square \)
Proof of Proposition 4. It is sufficient to prove the following statement. Suppose $X$ and $Y$ are (non-commutative) polynomials in $E_{ij}$ with $i,j \leq n$. Then the product of $\tilde{X}$ and $\tilde{Y}$ in $\mathbb{Z}_n+1$ coincides with the image in $\mathbb{Z}_n+1$ of $XPY$, where $P$ is the projector for $\mathfrak{gl}_n$, modulo the left ideal in $\mathbb{Z}_n+1$, generated by all $z_{i,n+1}$, $i \leq n$. Again we note that due to the structure of the projector for any $\lambda = \sum_k \lambda_k \varepsilon_k$ with $\lambda_{n+1} = 0$, the product $X \circ \lambda Y$ related to $\mathfrak{gl}_n$ coincides with product $X \circ \lambda Y$ related to $\mathfrak{gl}_{n+1}$. Thus it remains to prove that for any $X$ and $Y$ as above the element $\tilde{X} \circ \lambda \tilde{Y}$ belongs to the ideal in $\mathbb{Z}_n+1$, generated by all $z_{i,n+1}$, $i \leq n$, once $\lambda_{n+1} \neq 0$. But for $\lambda$ with $\lambda_{n+1} \neq 0$ we see, by weight arguments, that $\tilde{X} \circ \lambda \tilde{Y}$ can be presented as an image in $\mathbb{Z}_n+1$ of the sum $\sum X_i Y_i$, such that the $(n+1)$-st component of the weight of each $Y_i$ is not zero. Thus each $Y_i$ necessarily belongs to the left ideal generated by $E_{j,n+1}$, $j = 1, \ldots, n$. Finally we apply Lemma 5 to complete the proof.

The statement of Proposition 4 concerning the ideal $V_{n+1}'$ is proved similarly. □

Corollary 6. The coefficients in the relations (3.25) – (3.27), (3.29), – (3.31) are stable with respect to the above inclusions of $\mathbb{Z}_n$ to $\mathbb{Z}_n+1$.

The stability of the coefficients is understood in the following sense. Let $R$ be a relation for $\mathbb{Z}_{n+1}$ from our defining list $\mathcal{R}$, see Subsection 3.3.

Assume that $R$ does not contain any term with $z_{i,n+1}$, $i = 1, \ldots, n$, as a left factor. Then if we suppress in $R$ terms which contain $z_{i,n+1}$, $i = 1, \ldots, n$, as a right factor (such term automatically contains $z_{n+1,j}$, $j = 1, \ldots, n$, as a left factor), we get a relation in $\mathbb{Z}_n$.

Call "cut" the result of this procedure of getting the relations in $\mathbb{Z}_n$ from the relations in $\mathbb{Z}_{n+1}$ (under the formulated conditions). Then all relations in $\mathbb{Z}_n$ can be obtained by cutting appropriate relations in $\mathbb{Z}_{n+1}$.

Moreover, each relation in $\mathbb{Z}_n$ extends uniquely to a relation in $\mathbb{Z}_{n+1}$ from which it can be obtained by the cut procedure; in other words, there is a bijection between the set of relations in $\mathbb{Z}_n$ and the set of those relations in $\mathbb{Z}_{n+1}$ which do not contain any term with $z_{i,n+1}$, $i = 1, \ldots, n$, as a left factor.

The stabilization rule is certainly not an isolated $\mathfrak{gl}$ phenomenon; it can be generalized to certain other quadruplets of algebras replacing those which participate in the diagram

$$\mathfrak{gl}_n \leftarrow \mathfrak{gl}_n \oplus \mathfrak{gl}_n \rightarrow \mathfrak{gl}_{n+1} \oplus \mathfrak{gl}_{n+1}.$$ 

4 Completeness of relations

1. We first give general arguments, proving the weakened version, in which $\overline{U}(\mathfrak{h})$ is enlarged to $\mathcal{D}(\mathfrak{h})$, of Theorem 3.
As before, denote by $R$ the system (3.25), (3.26), (3.27), (3.29), (3.30) and (3.31) of relations. We shall see that it is equivalent to the system (3.10) of the ordering rules. The system $R$ follows from (3.10) since (3.10) is the set of defining relations for the weight generators; we have to verify the opposite implication. For a moment denote the generators from the set \{\check{z}_{ij}, \check{t}_i\} by symbols $\check{p}_L$, labeled by a single index $L$, $L = 1, 2, \ldots, n^2$. The number of ordering rules for $n^2$ variables $\check{p}_L$ is $n^2(n^2 - 1)/2$. So, to prove the completeness, it is sufficient to show that the dimension of the subspace (over $D(\mathfrak{h})$) spanned by $R$ is at least $n^2(n^2 - 1)/2$. Any relation from $R$ is a sum of products $\check{p}_L \diamond \check{p}_M$ with coefficients in $U(\mathfrak{h})$ plus, possibly, a term of zero degree in $\check{p}$’s. Denote by $R_0$ the system $R$ with degree zero terms dropped. It suffices to show that the system $R_0$ contains $n^2(n^2 - 1)/2$ linearly independent over $D(\mathfrak{h})$ relations. (4.1)

Once the coefficients from $U(\mathfrak{h})$ in all relations from $R_0$ are placed on the same side, say, on the right, from the monomials $\check{p}_L \diamond \check{p}_M$, one can give arbitrary numerical values to the variables $\check{h}_{ij}$ (respecting linear dependencies between them). To check the assertion (4.1) it is enough to find a set of values for which the corresponding system with numerical coefficients has $n^2(n^2 - 1)/2$ linearly independent relations. But when all $\check{h}_{ij}$ tend to $\infty$ (in the following way: $\check{h}_{i,i+1} = c_{i,i+1}h$, $h \to \infty$ and $c_{i,i+1}$ are constants), we directly observe that the system $R_0$ becomes simply $\check{p}_L \diamond \check{p}_M = \check{p}_M \diamond \check{p}_L$, $M > L$. The proof of the completeness over $D(\mathfrak{h})$ is finished.

Note that we did not use in the above arguments the compatibility of the ordering $\prec$ with the partial order $<$ on $\mathfrak{h}^*$.

2. Given an order, let $X$ be a formal vector of all unordered products $\check{p}_L \diamond \check{p}_K$ and $Y$ a formal vector of all ordered products. To rewrite $R$ in the form of ordering relations, one has to solve for $X$ a linear system of equations

$$AX = BY + C,$$  \hspace{1cm} (4.2)

where $C$ is a vector of degree 0 terms; $A$ and $B$ are certain matrices with coefficients in $U(\mathfrak{h})$ (by the above proof, $A$ is a square matrix). The solution of this system may cause an appearance of coefficients from $D(\mathfrak{h})$ (not from $U(\mathfrak{h})$) in the ordering relations. This happens, for example, for the lexicographical order for the generators $z_{ij}$ and $t_i$ (with $z_{ii} = t_i$) of $Z_n$ for $n > 2$ (we don’t give details; it is an explicit calculation). It follows from (2.12) (and statement (e) of Section 2) that for the order (3.8) - (3.9) the solution of the system (4.2) is defined over the ring $U(\mathfrak{h})$. However, this shows only that for this order possible terms from $D(\mathfrak{h})$ in the determinant of $A$ simplify in the combinations $A^{-1}B$ and $A^{-1}C$; the systems $R$ and $R^\prec$ may still be not equivalent over $U(\mathfrak{h})$, in the sense that the elements of the matrix $A^{-1}$ may not belong to $U(\mathfrak{h})$ and we cannot transform the system $R$ to the system $R^\prec$ by composing linear over $U(\mathfrak{h})$ combinations of relations from $R$. 

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3. We now pass to the proof of Theorem 3. Let $\mathcal{F}$ be the free algebra with the weight generators $z_{ij}$ and $t_i$ over $\mathcal{U}(\mathfrak{h})$. Let $\mathfrak{R}^<$ be the set of ordering relations $3.10$. Both $\mathfrak{R}$ and $\mathfrak{R}^<$ are defined over $\mathcal{U}(\mathfrak{h})$ and we have the homomorphism $\varpi : \mathcal{F}/\mathfrak{R} \to \mathcal{F}/\mathfrak{R}^<$. According to the weak form of theorem 3, see paragraph 1 of this subsection, the homomorphism $\varpi$ becomes the isomorphism after taking the tensor product with $\mathcal{D}(\mathfrak{h})$ (over $\mathcal{U}(\mathfrak{h})$). We shall now prove that $\varpi$ itself is the isomorphism.

The proof is done by induction in $n$, with the help of the stabilization law and an explicit calculation of certain determinants (and one can follow the precise structure of appearing denominators at each step). The induction base is $n = 1$, there is nothing to prove for $Z_1$.

All we have to show in general case is that the numerator of the determinant of the matrix $A$, figuring in (4.2), is a product of linear factors of the form (3.6). The relations are weighted so the matrix $A$ has a block structure, blocks $A_\lambda$ are labeled by the relation weights. The determinant of $A$ is the product of the determinants of the blocks $A_\lambda$.

Consider $Z_{n-1}$ as a subspace in $Z_n$ as in section 3.4. Fix a weight $\lambda$ for $Z_{n-1}$. Call $L_\lambda^{(n)}$ the linear subsystem $A_\lambda X_\lambda = B_\lambda Y_\lambda + C_\lambda$ of (3.2), corresponding to the weight $\lambda$ for $Z_n$. The system $L_\lambda^{(n)}$ contains the subsystem $(n)L_\lambda^{(n-1)}$, corresponding to the generators from $Z_{n-1}$ (recall that the relations are labeled by pairs of generators, so the subsystem $(n)L_\lambda^{(n-1)}$ is well defined). Compare $(n)L_\lambda^{(n-1)}$ with the corresponding system $L_\lambda^{(n-1)}$ for $Z_{n-1}$. By the stabilization principle, the system $L_\lambda^{(n-1)}$ is the cut of the system $(n)L_\lambda^{(n-1)}$ in the sense of section 3.4: there is a bijection between the two systems and the relations from $(n)L_\lambda^{(n-1)}$ have, compared to the corresponding relations from $L_\lambda^{(n-1)}$, extra terms with $z_{ni} \circ z_{jn}$ for certain $i, j < n$. By induction, $L_\lambda^{(n-1)}$ is equivalent, over its own $\mathcal{U}(\mathfrak{h})$, to the system of ordering relations. Making the same transformation with the system $(n)L_\lambda^{(n-1)}$ preserves the ordered form since the terms $z_{ni} \circ z_{jn}$ are ordered. This argument shows that we need to consider only the subset of relations labeled by those pairs of generators $\tilde{p}_L, \tilde{p}_M$ for which $\tilde{p}_L$ or $\tilde{p}_M$ do not belong to $Z_{n-1},$

$$ (\tilde{p}_L, \tilde{p}_M) : \tilde{p}_L \notin Z_{n-1} \text{ or } \tilde{p}_M \notin Z_{n-1}. \quad (4.3) $$

Applying the just constructed ordering rules (equivalent to the system $(n)L_\lambda^{(n-1)}$) to these remaining relations, we leave in them only ordered terms $\tilde{p}_L', \tilde{p}_M'$, $L' < M'$, with two generators from $Z_{n-1}, \tilde{p}_L', \tilde{p}_M' \in Z_{n-1}$.

We shall now consider separately each type of weight relations listed in the beginning of Section 3.3. The relations of types 1 and 2 do not cause any difficulty.

The number of relations of the types 3 or 4 grows with $n$. The change of variables (3.18), as well as the renormalization (3.24) and its inverse, have allowed denominators, so we can work with the generators $\hat{t}_i$ and $\hat{z}_{ij}$ instead of $t_i$ and $z_{ij}$.

4. Relations of type 4. For the relations (3.30) and (3.31), the restriction (4.3) shows that we have to consider only the subsystem, corresponding to the pairs $(z_{ni}, z_{in}), i < n,$
of the relations (3.31). By the arguments from the paragraph above, we assume that the
only unordered quadratic monomials in this subsystem are \( \dot{z}_{in} \diamond \dot{z}_{ni}, \ i < n \). Rewrite this
subsystem in the form (4.2):

\[
\dot{z}_{in} \diamond \dot{z}_{ni} + \sum_{a:a<n,a \neq i} \frac{1}{\hat{h}_{ia} + 1} \dot{z}_{an} \diamond \dot{z}_{na} = \ldots ,
\]

(4.4)

where dots stand for ordered terms. Therefore, the matrix \( \hat{A} \), whose determinant we need
to calculate, is simply

\[
\hat{A}_{ij} := \frac{1}{\hat{h}_{ij} + 1} ,
\]

(4.5)

where, we recall, \( \hat{h}_{ij} = \hat{h}_{i} - \hat{h}_{j} \); in particular, \( \hat{h}_{ii} = 0 \). The determinant of such matrix is
well known. The matrix \( \hat{A} \) is the specialization of the matrix

\[
A_{ij} := \frac{1}{x_{i} + y_{j}}
\]

(4.6)

at \( x_{i} = \hat{h}_{i} \) and \( y_{j} = -\hat{h}_{j} + 1 \). The determinant of the matrix \( A \), calculated in [Ca], is
\[
\det A = \prod_{i,j; i < j} \left( (x_{i} - x_{j})(y_{i} - y_{j}) \right) / \prod_{i,j} (x_{i} + y_{j}).
\]

It follows that \( \det \hat{A} = \prod_{i,j; i < j} \hat{h}_{ij}^{2} / (\hat{h}_{ij}^{2} - 1) \). The inverse of \( \hat{A} \) has thus allowed denominators.

5. Relations of type 3. For the relations (3.27) and (3.29) of the type 3, the restriction
(4.3) shows that we have to consider only the relations of the weights \( \varepsilon_{i} - \varepsilon_{n} \) and \( \varepsilon_{n} - \varepsilon_{i}, \ i < n \).

We start with the weight \( \varepsilon_{i} - \varepsilon_{n} \) with a fixed \( i, \ i < n \). The unordered quadratic
monomials of the weight \( \varepsilon_{i} - \varepsilon_{n} \) are

\[
\dot{z}_{an} \diamond \dot{z}_{ia} \quad \text{with} \quad a : 2a < i + n, \ a \neq i ,
\]

(4.7)

\[
\dot{z}_{ij} \diamond \dot{z}_{jn} \quad \text{with} \quad j : i + n \leq 2j ,
\]

(4.8)

\[
\dot{z}_{in} \diamond \hat{t}_{b} .
\]

(4.9)

All relations (3.27) and (3.29) participate in our system. However, the system is block-
triangular and can be analyzed.

Denote by \( r_{ijkl} \) the relation from the list (3.27) whose left hand side starts with \( \dot{z}_{ik} \diamond \dot{z}_{kl} \). Let \( \kappa_{ijkl} \) be the coefficient of the term \( \dot{z}_{ik} \diamond \dot{z}_{kl} \) in \( r_{ijkl} \). The relations \( r_{ijn} \) can be rewritten
in the form (since \( \hat{h}_{jj} = 0 \))

\[
\dot{z}_{ij} \diamond \dot{z}_{jn} \kappa_{ijn} = \sum_{a:a \neq i,a<n} \dot{z}_{an} \diamond \dot{z}_{ia} \frac{B_{ai}}{\hat{h}_{ja} + 1} + \ldots .
\]

(4.10)
Here dots stand for ordered terms with $\hat{t}_b \odot \hat{z}_{jn}$ (the term with $\hat{z}_{jn} \odot \hat{z}_{ij}$ is absorbed into the sum). Among the unordered monomials (1.7)-(1.9) only the monomials (1.7) enter the relations $v_{ijn}$ with $j$ such that $2j < i + n$ and $j \neq i$. Thus the subsystem \{ $v_{ijn} \mid j : 2j < i + n, j \neq i$ \} contains as many relations as unordered monomials. The matrix, whose determinant we have to calculate in order to express, using this subsystem, the unordered monomials (4.7) in terms of ordered monomials is $\hat{A}'_{a} = \frac{B_{ai}}{(\hat{h}_{ja} + 1)}$; the $a$-th row contains $B_{ai}$ as the common factor, so the determinant of the matrix $\hat{A}'$ is the product of $B_{ai}$ (over $a$ such that $2a < i + n$ and $a \neq i$) times the determinant of the matrix of the same form (4.5) as before. Thus the inverse of the determinant of the matrix $\hat{A}'$ belongs to $U(\hat{h})$. We use this subsystem to order the monomials (4.7).

After the monomials (4.7) are ordered, the rest of the relations $v_{ijn}$ (with $j : i + n \leq 2j$) turns into the set of the ordering relations for the monomials (4.8); each relation contains exactly one unordered monomial of the form (4.8) with the coefficient $\kappa_{ijn}$ whose inverse has allowed determinants.

The set of relations (3.29) provides the ordering rules for the monomials $\hat{z}_{in} \odot \hat{t}_k$ once one knows the ordered expressions for all monomials $\hat{z}_{an} \odot \hat{z}_{ia}$.

6. Relations of type 3, weight $\varepsilon_n - \varepsilon_i$. The considerations of Section 2 show that for any two orders on the weight basis of $p$, compatible with the partial order $<$ on $\mathfrak{h}^*$, the ordering relations (2.12) for them are equivalent over $U(\mathfrak{h})$. Define, instead of (3.8)-(3.9), the order $\prec$ by

$$z_{ij} \prec z_{kl} \quad \text{if} \quad i - j > k - l \quad \text{or} \quad \begin{cases} \quad i > k & \text{if} \quad i - j = k - l > 0, \\ \quad i < k & \text{if} \quad i - j = k - l < 0, \\ \quad \text{arbitrarily} & \text{if} \quad i - j = k - l = 0. \end{cases}$$

(4.11)

The peculiarity of the order $\prec$ is that the anti-involution $\epsilon$, see (3.12), transforms the set of quadratic ordered monomials of any non-zero weight $\lambda$, $\lambda \neq 0$, into the set of quadratic ordered monomials of the weight ($-\lambda$).

It is proved in [KO2], that the system $\mathcal{R}$ is closed under the anti-involution $\epsilon$ (that is, $\mathcal{R}$ and $\epsilon(\mathcal{R})$ are equivalent over $\overline{U}(\mathfrak{h})$). For the order $\succ$, the application of the anti-involution $\epsilon$ reduces the question about the equivalence over $\overline{U}(\mathfrak{h})$ of $\mathcal{R}$ and the set (3.10) of the ordering relations for the weight $\varepsilon_n - \varepsilon_i$ to the same question for the weight $\varepsilon_i - \varepsilon_n$. By the preceding paragraph, the equivalence assertion follows for the order $\succ$ and therefore for any other order, compatible with the partial order $<$ on $\mathfrak{h}^*$, for example, the order $\prec$.

The proof of the theorem 3 is completed. $\square$

The set $\mathcal{R}^\prec$ of ordering relations (3.10) is, by construction, closed over $\overline{U}(\mathfrak{h})$ under the involution $\omega$, see (3.13). As a by-product of the equivalence of $\mathcal{R}$ and $\mathcal{R}^\prec$ over $\overline{U}(\mathfrak{h})$ we observe that $\mathcal{R}$ is closed over $\overline{U}(\mathfrak{h})$ under the involution $\omega$ as well.

Note that all denominators, which appeared in the proof, are of the form $\hat{h}_{ij} \pm \varsigma, i < j$, where $\varsigma = 0, 1$ or 2.
7. As the proof shows, essentially the only matrix we have to invert is of the form (4.5). The matrix, inverse to (4.5) reads
\[
(\tilde{A}^{-1})_{ij} = -\frac{1}{h_{ij} - 1} \prod_{a:a \neq i} \frac{\tilde{h}_{ia} - 1}{\tilde{h}_{ia}} \prod_{b:b \neq j} \frac{\tilde{h}_{jb} + 1}{\tilde{h}_{jb}}.
\] (4.12)

The verification of (4.12) in the form \(\sum_j (\tilde{A}^{-1})_{ij} \tilde{A}_{jk} = \delta_{jk}\), where \(\delta_{jk}\) is the Kronecker delta, reduces to the identity
\[
\frac{1}{h_{ik} + 1} \prod_{b,b \neq i} \frac{\tilde{h}_{ib} + 1}{\tilde{h}_{ib}} - \sum_{j,j \neq i} \frac{1}{h_{ij}(\tilde{h}_{jk} + 1)} \prod_{b:b \neq i,j} \frac{\tilde{h}_{jb} + 1}{\tilde{h}_{jb}} = \delta_{ik} \prod_{b,b \neq i} \frac{\tilde{h}_{ib}}{\tilde{h}_{ib} - 1},
\] (4.13)

which is checked by an evaluation of residues and the values at infinity of both sides as functions of \(\tilde{h}_i\).

The inverse of the more general matrix (4.6) reads
\[
(\tilde{A}^{-1})_{ij} = (x_j + y_i) \prod_{a:a \neq j} \frac{x_a + y_i}{x_a - x_j} \prod_{b:b \neq i} \frac{y_b + x_j}{y_b - y_i}.
\] (4.14)

It is demonstrated similarly to (4.12), by an appropriate evaluation of residues and the values at infinity.

The formula (4.12) is equivalent (not directly equal) to the specialization of (4.14) at \(x_i = \tilde{h}_i\) and \(y_j = -\tilde{h}_j + 1\).

The formula (4.12) provides a recursive way to transform the system \(\mathfrak{R}\) into the set of ordering relations.

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