Recurrent Extensions of Real-Valued Self-Similar Markov Processes

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Abstract
Let $X = (X_t, t \geq 0)$ be a self-similar Markov process taking values in $\mathbb{R}$ such that the state 0 is a trap. In this paper, we present a necessary and sufficient condition for the existence of a self-similar recurrent extension of $X$ that leaves 0 continuously. The condition is expressed in terms of the associated Markov additive process via the Lamperti-Kiu representation. Our results extend those of Fitzsimmons (Electron. Commun. Probab. 11, 230–241 2006) and Rivero (Bernoulli 11, 471–509 2005, 13, 1053–1070 2007) where the existence and uniqueness of a recurrent extension for positive self-similar Markov processes were treated. In particular, we describe the recurrent extension of a stable Lévy process which to the best of our knowledge has not been studied before.

Keywords Real self-similar Markov processes · Stable processes · Markov additive processes · Lamperti–Kiu representation · Exponential functional

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1 Introduction and main results
In his seminal work [17], Lamperti studied the structure of positive self-similar Markov processes (pssMp) and posed the problem of determining those pssMp that agree with a given
pssMp up to the time the latter process first hits 0. Lamperti [17] answered this question in the special case of Brownian motion killed at 0 and he found that the class of those extensions which are self-similar consists of the reflecting and absorbing Brownian motions and the extensions which immediately after reaching 0 jump according to the measure $d\lambda(x)/x^{\beta+1}$, $\beta \in (0, 1)$. Voualle-Apiala [22] used Itô’s excursion theory to study the general case and provided a sufficient condition on the resolvent of pssMp for the existence of recurrent extensions that leave 0 continuously. The main contribution of Voualle-Apiala to this problem consist on the existence of a unique entrance law under which there exists a unique recurrent self-similar Markov process which turns out to be an extension of the pssMp after it reaches 0. Motivated by Voualle-Apiala’s result, Rivero [19] provided a simpler sufficient condition for the existence of such recurrent extension and a more explicit description of the entrance law. The sufficient condition found by Rivero was determined in terms of the underlying Lévy process in the so-called Lamperti’s transform of pssMp. Motivated by the aforementioned studies, Fitzsimmons [9] and Rivero [20] provided, independently, a necessary and sufficient condition for the existence of recurrent extensions that leave 0 continuously. To be more precisely, their main result can be stated as follows: A pssMp that hits 0 in a finite time admits a self-similar recurrent extension that leaves 0 continuously if and only if the Lévy process in the Lamperti transformation satisfies the so-called Cramér’s condition.

Recently, Chaumont et al. [6] studied the structure of real valued self-similar Markov processes and established a Lamperti type representation for such class of processes up to their first hitting time of 0 in terms of Markov additive processes (MAP). Hence it is natural to pose the same question of Lamperti for such class of processes, in other words our aim is to determine those real valued self-similar Markov processes that agree with a given real valued self-similar Markov process (rssMp) up to the time the latter process first hits 0. In particular, we would like to describe the recurrent extension of a stable Lévy process with scaling parameter $\alpha \in (1, 2)$ up to its first hitting time of 0, which to the best of our knowledge has not been studied before.

Our arguments follow a similar strategy as in [20], nevertheless the construction of the recurrent extension of a real valued self-similar Markov processes is not straightforward and requires a careful analysis. Indeed, some fluctuation properties of MAPs and real valued self-similar Markov processes are required in order to guarantee the existence of the excursion measure as well as its characterization. For instance, a complete understanding of eigenfunctions of a MAP, and of the moments of exponential functionals of such processes is needed in terms of their characteristics, since they are strongly related with the description of the entrance law of the recurrent extension. We conjecture this strategy can also be applied for the $d$-dimensional case ($d \geq 2$), where a Lamperti type representation has been obtained recently by Alili et al. [1], but it seems that a much deeper analysis and a good understanding of processes behind the Lamperti transform is required, as well as the description of its entrance law, which according to Kyprianou, Rivero, Sengul and Yang [16] is complicated due to the fact that the driving part in a MAP can be essentially any Markov process taking values in the $d$-dimensional sphere, and hence many technical assumptions would be needed.

To state our results precisely, we introduce some notation and recall some of the basic theory of real self-similar Markov processes. Let $\mathbb{D}$ be the space of càdlàg paths defined on $[0, \infty)$ with values in $\mathbb{R}$, endowed with the Skorohod topology and the corresponding Borel $\sigma$-field $\mathcal{D}$. A family of distributions $(\mathbb{P}_x, x \in \mathbb{R})$ on $(\mathbb{D}, \mathcal{D})$ is called strong Markov family on $\mathbb{R}$ if the canonical process $(X_t, t \geq 0)$ is a standard Markov process (in the sense of Blumenthal and Getoor [5]) with respect to $(\mathcal{F}_t)_{t \geq 0}$, the canonical right
continuous filtration. If additionally the process satisfies the so-called scaling property: for all \( c > 0 \),
\[
\{(cX_{t+ct^{-\alpha}}, t \geq 0), P_x\} \overset{\text{Law}}{=} \{(X_t, t \geq 0), P_{cx}\}, \quad \text{for } x \in \mathbb{R},
\]
then, the process is called real self-similar Markov process (rssMp). We denote by \( T_0 \), the first hitting time of 0 for the process \( X \), i.e.
\[
T_0 = \inf\{t > 0 : X_t = 0\},
\]
and we will assume \( T_0 < \infty \), \( P_x \)-a.s. and then the process dies i.e. 0 is a cemetery point, for all \( x \in \mathbb{R} \).

A crucial point in our arguments is the following time change representation of rssMp, due to Chaumont et al. [6], in terms of a Markov additive process taking values in \( \mathbb{R} \times \{-1, 1\} \), here denoted by \((\xi, J)\). For simplicity, we write \( \{\pm 1\} := \{-1, 1\} \) and set \( \mathbb{R}^* := \mathbb{R} \setminus \{0\} \). The so-called Lamperti-Kiu representation can be stated as follows: let \( x \in \mathbb{R}^* \) then, under \( P_x \), the rssMp \( X \) can be represented as follows
\[
X_t 1_{\{t<T_0\}} = x \exp \left\{ \xi\left(\left(|x|^{-\alpha}t\right)\right) J\left(\left(|x|^{-\alpha}t\right)\right) \right\}, \quad t \geq 0,
\]
where
\[
\tau(t) = \inf\left\{ s \geq 0 : \int_0^s \exp[\alpha \xi(u)] \, du \geq t \right\}.
\]
Let \((G_t)_{t \geq 0}\) be a standard filtration. Recall that a Markov additive process, \((\xi, J)\), tacking values in \( \mathbb{R} \times \{\pm 1\} \), with respect to \((G_t)_{t \geq 0}\), is a two states continuous-time Markov chain and the following property is satisfied: for any \( i \in \{\pm 1\}, s, t \geq 0 \), given \( J(t) = i \), the pair \((\xi(t + s) - \xi(t), J(t + s))\) is independent of \( G_t \) and has the same distribution as \((\xi(s) - \xi(0), J(s))\) given \( J(0) = i \).

If the MAP is killed, then \( \xi \) shall be set to \(-\infty\). We let \( P_{z,i} \) be the law of \((\xi, J)\) started from the state \((z, i)\), and if \( \mu \) is a probability distribution on \( \{\pm 1\} \), we write
\[
P_{z,\mu}(\cdot) = \sum_{i \in \{\pm 1\}} \mu(i) P_{z,i}(\cdot).
\]
We adopt a similar convention for expectations. It is well-known that a MAP \((\xi, J)\) can also be described in the following way (see for instance Asmussen [2, §XI.2a] and Ivanovs [12, Proposition 2.5]): for \( i, j \in \{\pm 1\} \), there exists a sequence of iid Lévy processes \((\xi^n_i)_{n \geq 0}\) and a sequence of iid random variables \((U^n_{ij})_{n \geq 0}\), independent of the chain \( J \), such that if \( S_0 = 0 \) and \((S_n)_{n \geq 1}\) are the jump times of \( J \), the process \( \xi \) has the representation
\[
\xi(t) = \begin{cases} 
\xi(S_n) + U^n_{J(S_n),J(S_n)}(t - S_n), & \text{if } t \in [S_n, S_{n+1}), t < k, \\
-\infty, & \text{if } t \geq k,
\end{cases}
\]
where the killing time \( k \) is the first time one of the appearing Lévy processes is killed. Roughly speaking the behaviour of a MAP can be described as follows: if \( J \) is in state 1, then \( \xi \) evolves according to a copy of \( \xi_1 \), a Lévy process. Once \( J \) changes from 1 to \(-1\), which happens at rate \( q_{1,-1} \), \( \xi \) has an additional transitional jump and until the next jump of \( J \), \( \xi \) evolves according to a copy of \( \xi_{-1} \). The MAP is killed as soon as one of the Lévy processes is killed. Consequently, the mechanism behind the Lamperti-Kiu representation is simple: the Markov chain \( J \) governs the sign of the rssMp and on intervals with constant sign the Lamperti-Kiu representation simplifies to the Lamperti representation.

Hence in order to describe a MAP on \( \mathbb{R} \times \{\pm 1\} \), we require five characteristic components which are mutually independent: two possibly killed Lévy processes, say \( \xi = (\xi_1(t), t \geq 0) \) and \( \xi_2 = (\xi_2(t), t \geq 0) \) respectively; a two states continuous-time Markov chain \( J \), such that if \( J(t) = 1 \), then \( \xi_1 \) is the Lévy process and \( \xi_2 \) is a cemetery point, and \( J(t) = -1 \), then \( \xi_2 \) is the Lévy process and \( \xi_1 \) is a cemetery point; a sequence of iid random variables \((U^n_{ij})_{n \geq 0}\), independent of the chain \( J \), with respect to \((G_t)_{t \geq 0}\), which is used to change the sign of \( \xi \). The MAP is killed as soon as one of the Lévy processes is killed. Consequently, the mechanism behind the Lamperti-Kiu representation is simple: the Markov chain \( J \) governs the sign of the rssMp and on intervals with constant sign the Lamperti-Kiu representation simplifies to the Lamperti representation.
and \( \xi_{-1} = (\xi_{-1}(t), t \geq 0) \), two random variables defined on \( \mathbb{R} \), say \( U_{1,-1} \) and \( U_{-1,1} \) and a \( 2 \times 2 \) intensity matrix \( Q = (q_{ij})_{i,j \in \{\pm 1\}} \), which is the transition rate matrix of the chain \( J \).

Before we introduce the matrix exponent of a MAP, we establish the convention that all matrices appearing in this work are written in the following form

\[
A = \begin{pmatrix} a_{11} & a_{1-1} \\ a_{-11} & a_{-1-1} \end{pmatrix}.
\]

We also denote by \( A^T \) for its transpose. Let \( \psi_{-1} \) and \( \psi_1 \) be the Laplace exponent of \( \xi_{-1} \) and \( \xi_1 \), respectively (when they exist). For \( z \in \mathbb{C} \), let \( G(z) \) denote the matrix whose entries are given by \( G_{ij}(z) = E\left[e^{zU_{ij}}\right] \) (when they exist) for \( i \neq j \), and for \( i = j \), \( G_{ii}(z) = 1 \). For \( z \in \mathbb{C} \), when it exist, we define

\[
F(z) = \text{diag}(\psi_1(z), \psi_{-1}(z)) + Q \circ G(z),
\]

where \( \circ \) indicates element-wise multiplication also known as Hadamard multiplication. A straightforward computation yields for \( t \geq 0 \),

\[
E_{0,i}\left[e^{z \xi(t)}; J(t) = j \right] = \left(e^{F(z)t}\right)_{ij}, \quad i, j \in \{\pm 1\},
\]

see Proposition 2.1, section XI.2 in [2]. For this reason, \( F \) is called the matrix exponent of the MAP \((\xi, J)\).

We will also be interested on the dual process of \((\xi, J)\), here denoted by \((\hat{\xi}, \hat{J})\). Whilst the dual of a Lévy process is equal in law to its negative, the situation for MAPs is a little more involved. The dual process is the MAP with probabilities \( \hat{P}_{x,i} \), for \((x, i) \in \mathbb{R} \times \{\pm 1\} \), and whose Matrix exponent, whenever it is well-defined, is given by

\[
\hat{F}(z) = \text{diag}(\psi_1(-z), \psi_{-1}(-z)) + \hat{Q} \circ G(-z)^T,
\]

and \( \hat{Q} \) is the intensity matrix of the modulating Markov chain on \( \{\pm 1\} \) with entries given by

\[
\hat{q}_{ij} = \frac{\pi_j}{\pi_i} q_{ji}, \quad i, j \in \{\pm 1\},
\]

where \( \pi = (\pi_1, \pi_{-1}) \) is the invariant distribution associated to \( J \). Note that the latter can also be written \( \hat{Q} = \Delta_\pi^{-1} Q^T \Delta_\pi \) where \( \Delta_\pi = \text{diag}(\pi, \pi) \), the matrix with diagonal entries given by \( \pi \) and zeros everywhere else. Hence, when it exists,

\[
\hat{F}(z) = \Delta_\pi^{-1} F(-z)^T \Delta_\pi.
\]

Equivalently, we have

\[
\pi_i \hat{E}_{0,i}\left[e^{z \xi(t)}; J(t) = j \right] = \pi_j E_{0,j}\left[e^{-z \xi(t)}; J(t) = i \right].
\]

According to Dereich et al. [8], we have the following time reversal property between \( \xi \) and its dual, which will be relevant for our purposes. For any \( t > 0 \), fixed

\[
\{((\xi(t) - s), J((t) - s)) : s \leq t\}, \mathbb{P}_{0,\pi} \xrightarrow{\text{Law}} \{((\hat{\xi}(s), J(s)) : s \leq t\}, \hat{P}_{0,\pi}\}.
\]

Another important property for our purposes is the construction of exponential martingales of MAP. It is known, see for instance [2, §XI.2c] and [12, Proposition 2.12], that the matrix \( F(z) \), for \( z \) real, when it exists, has a real simple eigenvalue \( \kappa(z) \), which is smooth
and convex on its domain and larger than the real part of all its other eigenvalues. Furthermore, the corresponding right-eigenvector \( v(z) \) may be chosen so that \( v_i(z) > 0 \) for every \( i \in E \), and normalised such that

\[
\pi v(z) = 1. \tag{1.5}
\]

The leading eigenvalue is sometimes also called the \textit{Perron-Frobenius eigenvalue} and it identifies a martingale (also known as the Wald martingale) which allow us to define a change of measure analogous to the Esscher transform for Lévy processes; cf. [2, Proposition XI.2.4, Theorem XIII.8.1]. More precisely, if

\[
M(t, \gamma) = e^{\gamma \xi(t) - \kappa(\gamma) v J(t)(\gamma)} v J(0)(\gamma), \quad t \geq 0,
\]

for some \( \gamma \in \mathbb{R} \) such that the right-hand side is defined, then \( M(\cdot, \gamma) \) is a unit-mean martingale with respect to \((\mathcal{G}_t)_{t \geq 0}\) under any initial distribution of \((\xi(0), J(0))\). Thus, we can define the change of measure

\[
\frac{d\mathbb{P}^{(\gamma)}}{d\mathbb{P}} \bigg|_{\mathcal{G}_t} = M(t, \gamma). \tag{1.6}
\]

Moreover, under the probability measure \( \mathbb{P}^{(\gamma)} \), the process \((\xi, J)\) is a MAP with matrix exponent \( F^{(\gamma)} \) and its leading eigenvalue is given by \( \kappa^{(\gamma)}(z) = \kappa(z + \gamma) - \kappa(\gamma) \).

In the sequel, we consider the following assumption that is also known as \textit{Cramér’s condition}.

\textbf{Assumption 1.1} There exist a \( \theta > 0 \) and a vector \((v_1, v_{-1})\)

\[
\sum_{j \in \{\pm 1\}} \mathbb{E}_{0,i} \left[ e^{\theta \xi(t) \gamma / \alpha}; J(t) = j \right] v_j = v_i, \quad i \in \{\pm 1\}.
\]

The number \( \theta \) is called the \textit{Cramér number}.

The latter condition implies that \( F(\theta) \) exists and \( \kappa(\theta) = 0 \). Furthermore, applying Jensen’s inequality to \( \mathbb{E}_{0,i}[e^{\theta \xi(t)}; J(t) = j] \), for \( z \in (0, \theta) \), allows us to deduce that \( F(z) \) is well defined on \((0, \theta)\), and since \( \kappa \) is a convex function on its domain, it follows \( \kappa(z) \leq 0 \) for \( z \in (0, \theta) \). Conversely, if there exists a number \( \theta > 0 \) such that \( F(z) \) is well defined on \((0, \theta]\) with \( \kappa(\theta) = 0 \), then Assumption 1.1 holds. In other words, another equivalent way to state Assumption 1.1 is as follows: there exists a number \( \theta > 0 \) such that \( F(z) \) is well defined on \((0, \theta]\) and \( \kappa(\theta) = 0 \).

We set \( \mathbb{P}^\theta := \mathbb{P}^{(\theta)} \), where \( \mathbb{P}^{(\theta)} \) is defined by the exponential change of measure introduced in Eq. 1.6 with \( \theta \) satisfying the Cramér condition. Recall that under \( \mathbb{P}^\theta \), the process \((\xi, J)\) is a MAP with \( \kappa^\theta(z) = \kappa(z + \theta) \) and denote for its dual by \((\xi, J, \mathbb{P}^\theta)\) which is a MAP whose leading eigenvalue is such that \( \hat{\kappa}^\theta(z) = \kappa^\theta(-z) \). Furthermore, if \( I \) denotes the exponential functional of \( \xi \), i.e.,

\[
I = \int_0^\infty \exp(\alpha \xi(t)) dt,
\]

then \( \hat{\mathbb{E}}_{0,i} \left[ I^{\theta / \alpha - 1} \right] < \infty \), for \( i \in \{\pm 1\} \), as it will be seen in Lemma 4.5.

We now formally introduce the notion of a \textit{recurrent extension}. Let \((X, \mathbb{P})\) be a rssMp, defined as above, and \( T_0 \) its first hitting time to 0, we will refer to \((X, \mathbb{P}, \mathbb{P}_{X})\) as the minimal process under \( \mathbb{P}_{X} \). We say that a real valued Markov process \((\overline{X}, \overline{\mathbb{P}})\) satisfying the scaling property is a recurrent extension of \((X, \mathbb{P})\) provided that it behaves like the minimal process up to its first hitting time to 0 and for which the state 0 is a regular and recurrent state.
We say that a $\sigma$-finite measure $n$ on $(\mathbb{D}, \mathcal{F}_\infty)$ having infinite mass is an excursion measure compatible with $(X, P)$ if the following are satisfied:

1. $n$ is carried by
   \[
   \{\omega \in \mathbb{D} : T_0(\omega) > 0, X_t(\omega) = 0, \forall t \geq T_0\};
   \]
2. for every bounded $\mathcal{F}_\infty$-measurable $H$ and each $t > 0, \Lambda \in \mathcal{F}_t$
   \[
   n(H \circ \theta_t, \Lambda \cap \{t < T_0\}) = n(E_{X_t}(H), \Lambda \cap \{t < T_0\}),
   \]
   where $\theta_t$ denotes the shift operator;
3. $n(1 - e^{-T_0}) < \infty$.

In the case that the measure $n$ only satisfies properties 1 and 2, then it is called a pseudo-exursion measure. If the excursion measure $n$ is such that $n(1 - e^{-T_0}) = 1$, then it is known as normalized excursion measure. Moreover, we say that $n$ is self-similar if it has the following scaling property: there exists $\gamma \in (0, 1)$ such that for all $a > 0$, it holds
   \[
   H_a n = a^{\gamma a} n,
   \]
   where the measure $H_a n$ is the image of $n$ under the mapping $H_a : \mathbb{D} \to \mathbb{D}$, defined by $H_a(\omega)(t) = a\omega(a^{-\alpha} t), t \geq 0$. The parameter $\gamma$ is called the index of self-similarity of $n$.

We say that the recurrent extension $(X, P)$ for which 0 is a regular and recurrent state leaves continuously (resp., by a jump) the state 0 whenever its excursion measure $n$ is carried by the paths that leave 0 continuously (resp., that leave 0 by a jump), i.e.
   \[
   n(X_{0+} > 0) = 0 \quad \text{(resp., } n(X_{0+} = 0) = 0).\]

We now state our main results. Our first main results claims that the Cramér condition is necessary and sufficient for the existence of a recurrent extension that leaves 0 continuously.

**Theorem 1.2** Let $(X, P)$ be a rssMp with index $\alpha > 0$. Suppose that $(X, P)$ hits its cemetery point 0 in a finite time $P$-a.s., and let $((\xi, J), \mathbb{P})$ be the MAP associated to $(X, P)$ via the Lamperti-Kiu representation. Then the following are equivalent:

(i) there exist a Cramér number $\theta \in (0, \alpha)$;
(ii) there exist a recurrent extension of $(X, P)$ that leaves 0 continuously and such that its associated excursion measure away from 0, say $n$, is such that $n(1 - e^{-T_0}) = 1$.

In this case, the recurrent extension in (ii) is unique, up to normalisation of the local time, and the entrance law associated with the excursion measure $n$ satisfies, for any $f$ bounded and measurable,
   \[
   n(f(X_t), t < T_0) = \frac{1}{C_{\alpha, \theta} t^{\theta/\alpha}} \left( v_1 \pi_1 \hat{E}_{0,1}^{\theta} \left[ f \left( \frac{I^{1/\alpha}}{I^{1/\alpha}} \right)^{\theta/\alpha - 1} \right] + v_{-1} \pi_{-1} \hat{E}_{0,-1}^{\theta} \left[ f \left( -\frac{I^{1/\alpha}}{I^{1/\alpha}} \right)^{\theta/\alpha - 1} \right] \right),
   \]
   where $\theta$ is the Cramér number and
   \[
   C_{\alpha, \theta} = \Gamma(1 - \theta/\alpha) \left( v_1 \pi_1 \hat{E}_{0,1}^{\theta} \left[ I^{\theta/\alpha - 1} \right] + v_{-1} \pi_{-1} \hat{E}_{0,-1}^{\theta} \left[ I^{\theta/\alpha - 1} \right] \right).\]

Our second main result provides necessary and sufficient conditions on the underlying MAP for the existence of recurrent extensions of rssMp that leave 0 by a jump.
Theorem 1.3  For $\beta \in (0, \alpha)$, the following are equivalent:

(i) $\kappa$ is well defined in $\beta$ and $\kappa(\beta) < 0$.
(ii) $E_{0,i}[I^{\beta/\alpha}] < \infty$, for $i \in \{\pm 1\}$.
(iii) The pseudo-excursion measure $n^j = P_\eta$, based on the jumping-in measure

$$\eta(dx) = |x|^{-(\beta+1)}dx,$$

is an excursion measure.
(iv) The minimal process $(X, T_0)$ admits a recurrent extension that leaves $0$ by a jump and whose associated excursion measure $n^\beta$ satisfies

$$n^\beta (X_{0,+} \in dx) = b_{\alpha, \beta}^1 [x] |x|^{-(\beta+1)}dx,$$

where $[x] = \text{sign}(x)$ and $b_{\alpha, \beta}^1$, $b_{\alpha, \beta}^{-1}$ are such that

$$b_{\alpha, \beta}^1 E_0,1[I^{\beta/\alpha}] + b_{\alpha, \beta}^{-1} E_0,-1[I^{\beta/\alpha}] = \frac{\beta}{\Gamma (1 - \beta/\alpha)}.$$
It is well known that for $\alpha \in (0, 1]$, stable processes do not hit points and in particular they do not hit the point 0. On the other hand, for $\alpha \in (1, 2)$, stable processes make infinitely many jumps across a point, say $z$, before the first hitting time of $z$. Moreover, stable processes are transient for $\alpha \in (0, 1)$ and oscillate otherwise. Since we are interested in $\text{rssMp}$ up to its first hitting time of 0, we will assume that $\alpha \in (1, 2)$. Nonetheless, it is important to point out that the computations below holds for any value of $\alpha$.

Let $(X, \mathbb{P})$ be the stable process killed up to its first hitting time of 0 and we denote by $((\xi, J), \mathbb{P})$ its associated MAP via the Lamperti-Kiu representation. According to Kuznetsov et al. [13], one can compute explicitly the matrix exponent of $((\xi, J), \mathbb{P})$ which satisfies

$$F(z) = \begin{pmatrix}
-\frac{\Gamma(\alpha - z)\Gamma(1 + z)}{\Gamma(\alpha \hat{\rho} - z)\Gamma(1 - \alpha \hat{\rho} + z)} & \frac{\Gamma(\alpha - z)\Gamma(1 + z)}{\Gamma(\alpha \hat{\rho})\Gamma(1 - \alpha \hat{\rho})} \\
\frac{\Gamma(\alpha - z)\Gamma(1 + z)}{\Gamma(\alpha \rho)\Gamma(1 - \alpha \rho)} & -\frac{\Gamma(\alpha - z)\Gamma(1 + z)}{\Gamma(\alpha\rho - z)\Gamma(1 - \alpha \rho + z)}
\end{pmatrix},$$

for $\text{Re}(z) \in (-1, \alpha)$. Using the reflection identity twice,

$$\Gamma(1 - z)\Gamma(z) = \frac{\pi}{\sin(\pi z)}, \quad \text{for } z \notin \mathbb{Z},$$

and Ptolemy’s identity,

$$\sin(\delta_1 + \delta_2) \sin(\delta_2 + \delta_3) = \sin(\delta_1) \sin(\delta_3) + \sin(\delta_1 + \delta_2 + \delta_3) \sin(\delta_2),$$

with $\delta_1 = \pi \alpha \rho, \delta_2 = -\pi z$ and $\delta_3 = \pi \alpha \hat{\rho}$; we conclude that

$$\det(F(z)) = -\frac{\Gamma(\alpha - z)\Gamma(1 + z)}{\Gamma((-z))\Gamma(1 - \alpha + z)} \quad \text{for } z \in (-1, \alpha) \setminus \mathbb{Z}.$$ 

The latter identity implies that for $\alpha \in (1, 2)$, the Cramér number of $((\xi, J), \mathbb{P})$ is $\theta = \alpha - 1$. Hence, applying Theorem 1.2, the stable process with scaling index $\alpha \in (1, 2)$ has a unique recurrent extension that leaves 0 continuously.

2.2 The MAP-dual of a Stable Process and the Stable Process Conditioned to be Continuously Absorbed at the Origin

In this example, we consider the case when stable processes are transient and do not hit points i.e. that $\alpha \in (0, 1)$. In this case, the process $(X, \mathbb{P})$ never hits 0 and its radial part $|X|$ drifts to $+\infty$. In other words, its associated MAP $((\xi, J), \mathbb{P})$, via the Lamperti-Kiu representation, drifts to $+\infty$. In this particular example, we are interested in the dual process $((\hat{\xi}, J), \hat{\mathbb{P}})$ which in turn drifts to $-\infty$. We introduce its associated $\text{rssMp} (X, \hat{\mathbb{P}})$, via the Lamperti-Kiu representation, which we refer as the MAP-dual stable process. It is important to observe that the latter process reaches the point zero at finite time.

In order to compute the matrix exponent $\hat{F}$, we first observe that using some explicit computations from [6] and [14], we can get explicitly the stationary distribution of $J$. More precisely, we have

$$\pi_1 = k(\alpha)\Gamma(\alpha \hat{\rho})\Gamma(1 - \alpha \hat{\rho}), \quad \pi_{-1} = k(\alpha)\Gamma(\alpha \rho)\Gamma(1 - \alpha \rho),$$

$$\pi_{-1} = k(\alpha)\Gamma(\alpha \rho)\Gamma(1 - \alpha \rho),$$

$$\pi_{-1} = k \Gamma(\alpha \rho) \Gamma(1 - \alpha \rho)$$
with $k^{-1}(\alpha) = \Gamma(\alpha \rho) \Gamma(1 - \alpha \rho) + \Gamma(\alpha \hat{\rho}) \Gamma(1 - \alpha \hat{\rho})$. Thus, from identity (1.3) and straightforward computations, we deduce

$$
\widehat{F}(z) = \begin{pmatrix}
\frac{\Gamma(\alpha + z) \Gamma(1 - z)}{\Gamma(\alpha \hat{\rho} + z) \Gamma(1 - \alpha \hat{\rho} - z)} & \frac{\Gamma(\alpha + z) \Gamma(1 - z)}{\Gamma(\alpha \hat{\rho}) \Gamma(1 - \alpha \hat{\rho})} \\
\frac{\Gamma(\alpha + z) \Gamma(1 - z)}{\Gamma(\alpha \rho) \Gamma(1 - \alpha \rho)} & \frac{\Gamma(\alpha + z) \Gamma(1 - z)}{\Gamma(\alpha \rho + z) \Gamma(1 - \alpha \rho - z)}
\end{pmatrix},
$$

for $\text{Re}(z) \in (-\alpha, 1)$. Furthermore,

$$
\det(\widehat{F}(z)) = \det(F(-z)) = \frac{\Gamma(\alpha + z) \Gamma(1 - z)}{\Gamma(z) \Gamma(1 - \alpha - z)} \quad \text{for} \quad z \in (-\alpha, 1) \setminus \mathbb{Z},
$$

implying that its Cramér number is $\theta = 1 - \alpha$. Hence, from Theorem 1.2 we observe that the MAP-dual stable process $(X, \widehat{\mathbf{P}})$ has a unique recurrent extension that leaves $0$ continuously whenever $\alpha \in (1/2, 1)$.

It is important to point out that there is a relationship between the former process and the stable process conditioned to be continuously absorbed at the origin is the dual of the MAP associated to stable process conditioned to be continuously absorbed at the origin, for $\alpha \in (0, 1)$. More precisely, according to Kyprianou et al. [15], the law of the stable process conditioned to be continuously absorbed at the origin, here denoted by $(\mathbf{P}_0, x \in \mathbb{R}^*)$, can be defined via the following Doob $h$-transform

$$
\frac{d\mathbf{P}_0}{d\mathbf{P}}_{|\mathcal{F}_t} := \frac{\sin(\pi \alpha \hat{\rho}) \mathbf{1}_{(X_t > 0)} + \sin(\pi \alpha \rho) \mathbf{1}_{(X_t < 0)}}{\sin(\pi \alpha \hat{\rho}) \mathbf{1}_{(x > 0)} + \sin(\pi \alpha \rho) \mathbf{1}_{(x < 0)}} \bigg|_{x | X_t}^{-1} 1_{(t < \tau_0)}, \quad t \geq 0,
$$

where $(\mathcal{F}_t)_{t \geq 0}$, denotes the natural filtration generated by the stable process $X$ satisfying the usual conditions. The MAP associated to the process $(X, \mathbf{P}^0)$, via the Lamperti-Kiu representation has matrix exponent $F^0$ which is similar to $F$ but with the roles of $\rho$ and $\hat{\rho}$ interchanged (see Theorem 3.1 in [15]). In other words, the MAP associated to stable process conditioned to be continuously absorbed at the origin is the dual of the MAP associated to $(-X, \mathbf{P})$.

Considering this, it can be verified that $\det(F^0(z)) = \det(\widehat{F}(z))$, for $z \in (-\alpha, 1)$, and therefore the Cramér number is $\theta = 1 - \alpha$. Thus the stable process conditioned to be continuously absorbed at the origin $(X, \mathbf{P}^0)$ has a unique recurrent extension that leaves $0$ continuously if and only if $\alpha \in (1/2, 1)$.

### 2.3 Spectrally Negative Case

In this example, we suppose that $(X, \mathbf{P})$ is a rssMp with no positive jumps and we will refer to this class as spectrally negative rssMp. From the Lamperti-Kiu representation, it is clear that its associated MAP $((\xi, J), \mathbb{P})$ also has no positive jumps. Therefore the rate matrix of the Markov chain $J$ is given by

$$
Q = \begin{pmatrix}
-q_{1-1} & q_{1-1} \\
0 & 0
\end{pmatrix}.
$$

For simplicity, we write $q_{1-} = q_{1-1}$. Furthermore, since the process $X$ has no positive jumps, then $U_{-1,1} = 0$. Putting all the pieces together, we obtain that the matrix exponent of $((\xi, J), \mathbb{P})$ satisfies

$$
F(z) = \begin{pmatrix}
\psi(z) & q_{1-} \psi_{1-}(z) \\
0 & \psi_{-1}(z)
\end{pmatrix}, \quad \text{for} \quad z \geq 0,
$$
where \( \psi^\dagger(z) = \psi_1(z) - q^+ \), is the Laplace exponent of \( \xi_1 \) a spectrally negative Lévy process killed at exponential time with parameter \( q^+ \) which is associated, via the Lamperti representation, to the process \( X \) killed at the first time it enters \( (-\infty, 0) \). Since \( \xi_1 \) and \( \xi_{-1} \) are spectrally negative, \( \psi_i, i = -1, 1 \) are well defined for \( z \geq 0 \). If \( G_{1-1}(z) \) is finite for \( 0 \leq z < z_0 \), for some \( z_0 > 0 \), then \( F(z) \) is well defined for \( z \in [0, z_0) \). From the form of the matrix exponent \( F \), it is clear that \( \det(F(z)) = 0 \) if and only if \( \psi_1^\dagger(z) = 0 \) or \( \psi_{-1}(z) = 0 \). In other words, \( ((\xi, J), \mathbb{P}) \) satisfies the Cramér condition if and only if some of its associated Lévy processes satisfies the Cramér condition.

On the other hand, it is well known that \( \psi_1^\dagger \) is a convex function with \( \psi_1(0) = 0 \). We denote by \( \Phi_1^\dagger(0) \) for its largest zero. Similarly \( \Phi_{-1}(0) \) denotes the largest zero of \( \psi_{-1} \). Hence, \((X, \mathbb{P})\) has a recurrent extension that leaves 0 continuously whenever \( \Phi_1^\dagger(0) \) or \( \Phi_{-1}(0) \in (0, \alpha) \). Since \( ((\xi, J), \mathbb{P}) \) drifts to \(-\infty\), then \( \xi_{-1} \) drifts to \(-\infty\), so \( \Phi_{-1}(0) \) always exists. Therefore, by Theorem 1.2, \( \Phi_{-1}(0) \in (0, \alpha) \) if and only if \((X, \mathbb{P})\) has a recurrent extension that leaves 0 continuously.

To illustrate this, we consider the spectrally negative stable process, with \( \alpha \in (1, 2) \). In this case \( \rho = 1/\alpha \). Recall that the Lévy process \( \xi_1 \) is associated to the stable process killed at the first time it enters \( (-\infty, 0) \), via the Lamperti representation. The process \( \xi_1 \) is the a Lamperti stable process that appears in [14] (and is denoted as \( \xi^* \)) and its Laplace exponent satisfies
\[
\psi_1^\dagger(z) = \frac{1}{\pi} \Gamma(\alpha - z) \Gamma(1 + z) \sin(\pi(z - \alpha + 1)).
\]
We also observe that the Lévy process \( \xi_{-1} \) is associated to the negative of a spectrally positive stable process killed at the first time it hits 0, which is also the Lamperti stable process that appears in [14] (but with \( \rho = 1 - 1/\alpha \)) and its Laplace exponent satisfies
\[
\psi_{-1}(z) = \frac{1}{\pi} \Gamma(\alpha - z) \Gamma(1 + z) \sin(\pi(z - \alpha)).
\]
From the latter two expression, we have that \( \theta = \alpha - 1 \) since \( \psi_1^\dagger(\theta) = \psi_{-1}(\theta) = 0 \). Moreover since \( \theta \in (0, \alpha) \) we deduce that \((X, \mathbb{P})\) has a recurrent extension that leaves 0 continuously as expected.

### 3 Some Properties of Excursion Measures for \( \text{rssMp} \)

In this section we derive the existence of a recurrent extension for real-valued Markov processes and some properties of their excursion measure that are needed for the sequel. The result established in the first part of this section is an extension to the real-valued case of a result that appears in Rivero [19]. For simplicity, we use the same notation as in [19].

In what follows, we set \( \mathbb{R}^* := \mathbb{R} \setminus \{0\} \). Let \((Y_t, t \geq 0)\) and \((\hat{Y}_t, t \geq 0)\) be two real valued Markov processes having 0 as a cemetery point. We denote by \( \mathbb{Q} \) and \( \mathbb{E}_0 \) (resp. \( \hat{\mathbb{Q}} \) and \( \hat{\mathbb{E}}_0 \)) for the probability and expectation associated to \( Y \) (resp. for \( \hat{Y} \)). Similarly, we introduce \( T_0 \) (resp. \( \hat{T}_0 \)) for the first hitting time of 0 for \( Y \) (resp. \( \hat{Y} \)). Assume that \( Q_x(T_0 < \infty) = \hat{Q}_{\hat{x}}(\hat{T}_0 < \infty) = 1 \), for all \( x \in \mathbb{R}^* \). Let \((Q_t, t \geq 0)\), \( \mathbb{W}_x \), (resp. \((\hat{Q}_t, t \geq 0)\), \( \hat{\mathbb{W}}_x \)) denote the semigroup and \( \lambda \)-resolvent for \( Y \) killed at \( T_0 \), (resp. for \( \hat{Y} \)). For \( \lambda > 0 \), define the functions \( \varphi_\lambda, \hat{\varphi}_\lambda : \mathbb{R}^* \rightarrow [0, 1) \), by
\[
\varphi_\lambda(x) := \mathbb{E}_{Q_x}[e^{-\lambda T_0}] \quad \text{and} \quad \hat{\varphi}_\lambda(x) := \hat{\mathbb{E}}_{\hat{Q}_{\hat{x}}}[e^{-\lambda \hat{T}_0}], \quad x \in \mathbb{R}^*.
\]
3.1 Existence Theorem

We consider the following hypotheses:

H.1. The processes \( Y \) and \( \hat{Y} \) satisfy the basic hypotheses in Blumenthal [4].

H.2. The resolvents \( W_\lambda \) and \( \hat{W}_\lambda \) are in weak duality with respect to a \( \sigma \)-finite measure \( \vartheta \) defined on \( \mathbb{R}^* \).

H.3. The following integral conditions are satisfied,
\[
\int_{\mathbb{R}^*} \varphi_\lambda(x) \vartheta(dx) < \infty \quad \text{and} \quad \int_{\mathbb{R}^*} \hat{\varphi}_\lambda(x) \vartheta(dx) < \infty, \quad \text{for } \lambda > 0.
\]

The main theorem of this section which is established below corresponds to the real-valued version of Theorem 3 given in [19].

**Theorem 3.1** Under hypotheses H.1-H.3, there exist excursion measures \( m \) and \( \hat{m} \) compatible with the semigroups \( (Q_t, t \geq 0) \) and \( (\hat{Q}_t, t \geq 0) \), respectively, such that the Laplace transforms of the entrance laws \( (m_s, s > 0) \) and \( (\hat{m}_s, s > 0) \) associated with \( m \) and \( \hat{m} \), respectively, are determined by
\[
\int_0^\infty e^{-\lambda s} m_s f ds = \int_{\mathbb{R}^*} \varphi_\lambda(x) f(x) \vartheta(dx), \quad \int_0^\infty e^{-\lambda s} \hat{m}_s f ds = \int_{\mathbb{R}^*} \hat{\varphi}_\lambda(x) f(x) \vartheta(dx),
\]
for any continuous, bounded function \( f \) and \( \lambda > 0 \). Furthermore, associated with these excursion measures there exist Markov processes \( Y^* \) and \( \hat{Y}^* \) which are extensions of \( Y \) and \( \hat{Y} \), respectively, and which are still in weak duality with respect to measure \( \vartheta \).

The proof of the previous theorem follows similar arguments as those given in [19], and in particular, it is based on the following three lemmas. For these reasons we will just provide the main ideas of their proofs.

**Lemma 3.2** The family of measures \( (M_\lambda, \lambda > 0) \), defined by
\[
M_\lambda f := \int_{\mathbb{R}^*} f(x) \varphi_\lambda(x) \vartheta(dx),
\]
is such that the following hold:

(i) \( \lim_{\lambda \to \infty} M_\lambda 1 = 0 \);

(ii) for \( \mu, \lambda > 0 \) such that \( \mu \neq \lambda \) and \( f \) continuous and bounded, we have
\[
(\mu - \lambda) M_\lambda W_\mu f = M_\lambda f - M_\mu f.
\]

**Proof** We first observe that the hypothesis H.3 implies that \( M_\lambda 1 \) is finite for all \( \lambda > 0 \). Hence claim (i) follows from the dominated convergence theorem. To prove (ii), we first observe that for \( \lambda > 0 \), \( \hat{W}_\lambda 1 = \lambda^{-1} (1 - \hat{\varphi}_\lambda) \). Hence, using the well-known identity for resolvents \( (\mu - \lambda) \hat{W}_\mu = \hat{W}_\lambda - \hat{W}_\mu \), for \( \mu \neq \lambda \), we deduce
\[
(\mu - \lambda) \hat{W}_\mu \varphi_\lambda = \varphi_\lambda - \varphi_\mu, \quad \text{for } \mu \neq \lambda.
\]
Thus, using the latter identity and the weak duality between the resolvents \( W_\lambda \) and \( \hat{W}_\lambda \), we get for \( \mu \neq \lambda \),
\[
(\mu - \lambda) M_\lambda W_\mu f = M_\lambda f - M_\mu f,
\]
as expected. This completes the proof. \( \square \)
Next, we observe that Lemma 3.2 and Theorem 6.9 in [11] guarantee that there exists a unique entrance law \((m_t, t > 0)\) for the semigroup \((Q_t, t \geq 0)\), such that for \(\lambda > 0\) and \(f\) measurable and bounded,
\[
\int_0^1 m_t \, dt < \infty, \quad \text{and} \quad M_\lambda f = \int_0^\infty e^{-\lambda t} m_t f \, dt.
\]

According with [4], for an entrance law \((m_s, s > 0)\), there exists a unique excursion measure \(\hat{m}\) having this entrance law. The same arguments guarantee the existence of an excursion measure \(\hat{m}\) and an entrance law \((\hat{m}_t, t > 0)\) for the semigroup \((\hat{Q}_t, t \geq 0)\).

Using the results in [4], we obtain that associated with the excursion measure \(m\) there exists a unique Markov process \(\hat{Y}\) extending \(Y\) and the \(\lambda\)-resolvent of \(\hat{Y}\) is determined by the following identities
\[
W_\lambda^* f(0) = \frac{M_\lambda f}{\lambda M_\lambda 1}, \quad \text{and} \quad W_\lambda^* f(x) = W_\lambda f(x) + \varphi_\lambda(x) W_\lambda^* f(0), \quad \text{for} \ x \in \mathbb{R}^*,
\]
for \(f\) measurable and bounded. Similarly, we obtain the existence of \(\hat{Y}\), and its associated \(\lambda\)-resolvent \(\hat{W}_\lambda^*\) is defined in a similar way.

Now, since \(m_s 1\) is decreasing in \(s\) and \(\int_0^1 m_s 1 \, ds\) is finite, we deduce
\[
\mu M_\mu 1 = \lim_{s \to \infty} m_s 1 + \int_0^\infty (1 - e^{-\mu t}) \nu(\, dt),
\]
where \(\nu(\, dt) = -dm_t.1\). A similar identity holds for \(\hat{M}_\mu\) with \(\hat{\nu}(\, dt) = -d\hat{m}_t.1\). On the one hand, using Lemma 3.2 part (ii), we get
\[
(\lambda - \mu) M_\lambda \varphi_\mu = \lambda M_\lambda 1 - \mu M_\mu 1 \quad \text{and} \quad (\lambda - \mu) \hat{M}_\lambda \hat{\varphi}_\mu = \lambda \hat{M}_\lambda 1 - \mu \hat{M}_\mu 1.
\]
The latter identities, together with \(M_\lambda \varphi_\mu = \hat{M}_\mu \hat{\varphi}_\mu\), imply
\[
\lambda M_\lambda 1 - \mu M_\mu 1 = \lambda \hat{M}_\lambda 1 - \mu \hat{M}_\mu 1.
\]
Therefore by letting \(\mu \to 0\), it is clear
\[
\lambda M_\lambda 1 - \lim_{s \to \infty} m_s 1 = \lambda \hat{M}_\lambda 1 - \lim_{s \to \infty} \hat{m}_s 1,
\]
implying
\[
\int_0^\infty (1 - e^{-\lambda s}) \nu(\, ds) = \int_0^\infty (1 - e^{-\lambda s}) \hat{\nu}(\, ds).
\]
On the other hand, since \(m\) is the excursion measure associated to the entrance law \((m_s, s > 0)\), we have
\[
m(1 - e^{-\lambda T_0}) = \lambda M_\lambda 1 = \lim_{s \to \infty} m_s 1 + \int_0^\infty (1 - e^{-\lambda t}) \nu(\, dt).
\]
Thus if we let \(\lambda \to 0\), the dominated convergence theorem implies \(\lim_{s \to \infty} m_s 1 = 0\). Similarly, one can deduce that \(\lim_{s \to \infty} \hat{m}_s 1 = 0\). Putting all the pieces together, the following result can be deduced.

**Lemma 3.3** For every \(\lambda > 0\), we have \(\lambda M_\lambda 1 = \lambda \hat{M}_\lambda 1\).
Lemma 3.4 For every $\lambda > 0$ and every measurable functions $f, g : \mathbb{R} \to \mathbb{R}$, we have
\[
\int_{\mathbb{R}^*} g(y) W_\lambda^* f(y) \vartheta(dy) = \int_{\mathbb{R}^*} f(y) \hat{W}_\lambda^* g(y) \vartheta(dy).
\]

Proof Using the second identity in Eq. 3.1, we obtain
\[
\int_{\mathbb{R}^*} g(y) W_\lambda^* f(y) \vartheta(dy) = \int_{\mathbb{R}^*} g(y) W_\lambda f(y) \vartheta(dy) + W_\lambda^* f(0) \hat{M}_\lambda g.
\]
Now, the first identity in Eq. 3.1 and Lemma 3.3 imply $W_\lambda^* f(0) \hat{M}_\lambda g = \hat{W}_\lambda g(0) M_\lambda f$. Thus, using the weak duality between $W_\lambda$ and $\hat{W}_\lambda$, we conclude
\[
\int_{\mathbb{R}^*} g(y) W_\lambda^* f(y) \vartheta(dy) = \int_{\mathbb{R}^*} f(y) \hat{W}_\lambda g(y) \vartheta(dy) + \hat{W}_\lambda^* g(0) M_\lambda f.
\]
This completes the proof. \hfill \Box

3.2 Self-Similarity Property

For the development of this section, we introduce the following transformation: for $c \in \mathbb{R}$, let $H_c$ be such that $H_c f(x) = f(cx)$. Our first lemma provides equivalences of the self-similarity property of the excursion measure associated to the recurrent extension. Its proof follows the same arguments as those used in Lemma 2 in [19] for positive self-similar Markov processes. So, we omit its proof.

Lemma 3.5 Let $n$ be an excursion measure and $\overline{X}$ the associated recurrent extension of the minimal process. The following are equivalent:

(i) The process $\overline{X}$ satisfies the scaling property.

(ii) There exists $\gamma \in (0, 1)$ such that, for any $c > 0$ and $f \in C_b(\mathbb{R}^*)$,
\[
n \left( \int_0^{T_0} e^{-qs} f(X_s) ds \right) = c^{\alpha(1-\gamma)} n \left( \int_0^{T_0} e^{-qs} H_c f(X_s) ds \right).
\]

(iii) There exists $\gamma \in (0, 1)$ such that, for any $c > 0$ and $f \in C_b(\mathbb{R}^*)$,
\[
n_s f = c^{-\alpha/(1-\gamma)} n_s/c^{\alpha} H_c f, \quad \text{for all} \quad s > 0.
\]

Lemma 3.6 Let $n$ be a normalized excursion measure and $\overline{X}$ the associated extension of the minimal process. Assume that one of the conditions in Lemma 3.5 holds. Then there exist finite constants $C_{\alpha, \gamma}^1, C_{\alpha, \gamma}^{-1}$ different from zero such that
\[
n \left( \int_0^{T_0} 1_{\{X_s \in dy\}} ds \right) = C_{\alpha, \gamma}^1 |y|^{\alpha-1-\gamma} dy, \quad y \in \mathbb{R}^*, \quad (3.2)
\]
with $\gamma$ determined by (ii) of Lemma 3.5. Furthermore, $C_{\alpha, \gamma}^1, C_{\alpha, \gamma}^{-1}$ satisfy
\[
C_{\alpha, \gamma}^1 \mathbb{E}_{0, 1} \left[ I^{-\gamma} \right] + C_{\alpha, \gamma}^{-1} \mathbb{E}_{0, -1} \left[ I^{-\gamma} \right] = \frac{\alpha}{\Gamma(1-\gamma)}, \quad (3.3)
\]
where
\[
I = \int_0^\infty \exp\{\alpha \xi(t)\} dt.
\]
As a consequence, \( \mathbb{E}_{0,1} \left[ I^{-(1-\gamma)} \right] \), \( \mathbb{E}_{0,-1} \left[ I^{-(1-\gamma)} \right] < \infty \).

**Proof** Recall that the sojourn measure

\[
\mathbf{n} \left( \int_0^{T_0} 1_{\{X_s \in dy\}} \, ds \right) = \int_0^{\infty} \mathbf{n}_s(dy) \, ds
\]

is a \( \sigma \)-finite measure on \( \mathbb{R}^* \) and is the unique excessive measure for the semigroup of the process \( X \). Now, using part (iii) from Lemma 3.5 and Fubini’s theorem, we obtain for \( f \geq 0 \) measurable:

\[
\int_0^{\infty} \mathbf{n}_s f \, ds = \int_0^{\infty} s^{-\gamma} \mathbf{n}_1(H_{t/\alpha} f) \, ds = \int_{\mathbb{R}^*} \mathbf{n}_1(dz) \int_0^{\infty} ds \, s^{-\gamma} f(s^{1/\alpha} z) = C_{\alpha,\gamma}^1 \int_0^{\infty} f(u) u^{\alpha-1-\gamma} \, du + C_{\alpha,\gamma}^{-1} \int_{-\infty}^0 f(u)(-u)^{\alpha-1-\gamma} \, du,
\]

where in the last identity we have performed the change of variables \( u = s^{1/\alpha} z \),

\[
C_{\alpha,\gamma}^1 := \alpha \int_0^{\infty} z^{-\alpha(1-\gamma)} \mathbf{n}_1(dz) \quad \text{and} \quad C_{\alpha,\gamma}^{-1} := \alpha \int_{-\infty}^0 (-z)^{-\alpha(1-\gamma)} \mathbf{n}_1(dz).
\]

Thus, it holds the following representation of the sojourn measure,

\[
\mathbf{n} \left( \int_0^{T_0} f(X_s) \, ds \right) = \int_0^{\infty} \mathbf{n}_s f \, ds = \int_{\mathbb{R}^*} f(u) C_{\alpha,\gamma}^{|u|} u^{\alpha-1-\alpha \gamma} \, du,
\]

which implies (3.2).

Next, we observe that the function \( \varphi(x) = \mathbb{E}_x[e^{T_0}] \) is integrable with respect to the sojourn measure since from the Markov property of \( \mathbf{n} \), we have

\[
\mathbf{n} \left( \int_0^{T_0} \varphi(X_s) \, ds \right) = \int_0^{\infty} \mathbf{n}(e^{-(T_0-s)}, s < T_0) \, ds = \mathbf{n}(1-e^{-T_0}) = 1.
\]

On the other hand, using the representation of the sojourn measure, Fubini’s theorem and the fact that \( T_0 \) under \( \mathbb{P}_y \) has the same law as \( |y|^\alpha I \) under \( \mathbb{P}_{0,|y|} \), we deduce

\[
1 = \mathbf{n} \left( \int_0^{T_0} \varphi(X_s) \, ds \right) = \int_{\mathbb{R}^*} \mathbb{E}_y[e^{-T_0}] C_{\alpha,\gamma}^{|y|} |y|^{\alpha-1-\alpha \gamma} \, dy
\]

\[
= C_{\alpha,\gamma}^1 \mathbb{E}_{0,1} \left[ \int_0^{\infty} e^{-\gamma u} u^{\alpha-1-\alpha \gamma} \, dy \right] + C_{\alpha,\gamma}^{-1} \mathbb{E}_{0,-1} \left[ \int_{-\infty}^0 e^{-(\gamma u)} (-\gamma)^{\alpha-1-\alpha \gamma} \, dy \right]
\]

\[
= \Gamma(1-\gamma) \left[ C_{\alpha,\gamma}^1 \mathbb{E}_{0,1}[I^{-(1-\gamma)}] + C_{\alpha,\gamma}^{-1} \mathbb{E}_{0,-1}[I^{-(1-\gamma)}] \right].
\]

Therefore, \( \mathbb{E}_{0,i}[I^{-(1-\gamma)}] < \infty, i = -1, 1 \) and identity (3.3) holds. This ends the proof. \( \square \)

Voullé-Apiala [22] and Rivero [20] proved that any pssMp for which 0 is a regular and recurrent state either leaves 0 continuously or by jumps. The same occurs in the real-valued case. This is stated in the following lemma and its proof is similar to the one provided in [20]. For the sake of completeness, we provide its proof.
Lemma 3.7 Let \( n \) be a self-similar excursion measure compatible with \((X, P)\) and with self-similarity index \( \gamma \in (0, 1) \). Then,

\[
either \quad n(X_0 \neq 0) = 0 \quad or \quad n(X_0^+ = 0) = 0.
\]

Proof We proceed by contradiction. Suppose that our claim is not true. Let

\[
n^c := c^{(i)} n_{|X_0^+ = 0} \quad and \quad n^j := c^{(j)} n_{|X_0^+ \neq 0},
\]

be the restriction of \( n \) to the set of trajectories \( \{X_0^+ = 0\} \) and \( \{X_0^+ \neq 0\} \), respectively, and \( c^{(i)}, c^{(j)} \) be normalizing constants such that

\[
n^c(1 - e^{-T_0}) = n^j(1 - e^{-T_0}) = 1.
\]

The measures \( n^c \) and \( n^j \) are self-similar excursion measures compatible with \((X, P)\) and with the same self-similarity index \( \gamma \). According to Lemma 3.6, the potential measure \( n^c \) and that of \( n^j \) are given by the same purely excessive measure. In other words,

\[
n^c \left( \int_0^{T_0} \mathbf{1}_{\{X_s \in dy\}} ds \right) = C_{a, \gamma}^1 |y|^{\alpha - 1 - \gamma \alpha} dy = n^j \left( \int_0^{T_0} \mathbf{1}_{\{X_s \in dy\}} ds \right),
\]

where \( C_{a, \gamma}^1, C_{a, \gamma}^{-1} \) are constants satisfying \( 0 < C_{a, \gamma}^1 + C_{a, \gamma}^{-1} < \infty \). So, by Theorem 5.25 in Getoor [10] on the uniqueness of purely excessive measures, the entrance laws associated with \( n^c \) and \( n^j \) are equal. Hence, by Theorem 4.7 of Chapter V in [4], the measures \( n^c \) and \( n^j \) are equal. This leads to a contradiction since the supports of the measures \( n^c \) and \( n^j \) are disjoint.

The last result of this section characterizes the form of the jumping measures associated to a self-similar excursion measure of recurrent extension of \( \text{rssMp} \). Its proof follows similar arguments as those used by Voule-Apiala in [22] where this result is established for \( \text{pssMp} \).

Lemma 3.8 The only possible jumping-in measures such that the associated excursion measure satisfies (ii) in Lemma 3.5 are of the type

\[
\eta(dx) = b_1^{-1} a, \beta^x x^{(1+\beta)} dx + b_1^{-1} a, \beta (-x)^{-(\beta+1)} dx, \quad for \quad x \neq 0, 0 < \beta < \alpha.
\]

Proof According to Blumenthal [4], there exists a \( \sigma \)-finite measure \( \eta \) on \( \mathbb{R}^* \), such that, the entrance law \( (n_s, s > 0) \) has the following representation

\[
n_s = \theta_s + \eta P_{s}^0,
\]

where \( \theta_s \) is the entrance law for \((P_t, t \geq 0)\) with the additional property

\[
\theta_s(U^c) \to 0, \quad as \quad s \to 0,
\]

for every neighbourhood \( U \) of 0. Furthermore, \( \eta \) satisfies

\[
\lim_{s \to 0} n_s g = \eta g,
\]

for all function \( g \) with compact support on \( \mathbb{R}^* \). From Lemma 3.5 (iii) and the previous identity, we deduce that for any \( c > 0 \):

\[
\lim_{s \to 0} \int_{\mathbb{R}^*} g(x) \eta(-dx) = \lim_{s \to 0} \int_{\mathbb{R}^*} g(x) c^{\alpha \gamma} \eta(-cdx).
\]

In particular, this shows that for every \( c > 0 \),

\[
\eta(-dx) = c^{\alpha \gamma} \eta(-cdx), \quad x \in \mathbb{R}^*.
\] (3.4)
Let \( \beta = \alpha \gamma \) and define on \( \mathbb{R}^+ \) the measure
\[
\nu(A) := \int_A x^\beta \eta(-dx), \quad A \in \mathcal{B}(\mathbb{R}^+).
\]

By Eq. 3.4 we have that \( \nu \) satisfies
\[
\nu(yA) = \int_{yA} x^\beta \eta(-dx) = \int_A (yx)^\beta \eta(-ydx) = \int_A x^\beta \eta(-dx) = \nu(A).
\]

That is to say, the measure \( \nu \) is left invariant on the group of positive real numbers under multiplication. The uniqueness of left Haar measures implies that there exists a constant \( b_{\alpha,\beta}^{-1} \) such that
\[
\nu(dx) = b_{\alpha,\beta}^{-1} x d\eta, \quad x > 0.
\]

We refer to chapter 9 in Cohn [7] (see Theorem 9.2.3 and Exercise 3) for these details on Haar measures. The latter identity implies
\[
\eta(dx) = b_{\alpha,\beta}^{-1} (-x)^{-(1+\beta)} d\eta, \quad x < 0.
\]

A similar procedure allow us to obtain that for \( x > 0 \),
\[
\eta(dx) = b_{\alpha,\beta} x^{-(1+\beta)} d\eta.
\]

This completes the proof. \( \Box \)

## 4 Proofs

### 4.1 Existence of Recurrent Extensions

The time reversal property (1.4), implies the following duality result between the resolvents of the rssMp \( X \) (associated to \( \xi \)) and its dual. We refer to Theorem 2 in Alili et al. [1] for its proof and whose arguments follows similar ideas to those developed in Lemma 2 of Bertoin and Yor [3] for the positive case (i.e. pssMp).

Recall that for \( q \geq 0 \) and a measurable \( f: \mathbb{R} \to \mathbb{R} \), the resolvent operators are given by
\[
V^q f(x) = \mathbb{E}_x \left[ \int_0^\xi e^{-qt} f(X_t) dt \right], \quad \widehat{V}^q f(x) = \mathbb{E}_x \left[ \int_0^\xi e^{-qt} f(X_t) dt \right].
\]

**Lemma 4.1** For every \( q \geq 0 \) and every measurable functions \( f, g: \mathbb{R} \to \mathbb{R} \), we have
\[
\int_{-\infty}^\infty f(x) V^q g(x) \mu(dx) = \int_{-\infty}^\infty g(x) \widehat{V}^q f(x) \mu(dx), \quad (4.1)
\]
\[
\text{where} \quad \mu(dx) := |x|^{\alpha-1} \pi_{\lfloor x \rfloor} dx.
\]

In order to establish weak duality for rssMp an invariant function for the semigroup of the process killed at its first hitting time of zero is needed. The invariant function is given below and was determined by Kyprianou et al. [15] (see Theorem 2.1). Its proof relies on the Lamperti-Kiu representation for rssMp and the optional stopping theorem.

**Lemma 4.2** Assume that Assumption 1.1 holds and denote by \( \theta \) and \( \nu \) for the Cramér number and its associated vector. Define the function \( h: \mathbb{R} \to [0, \infty) \) by
\[
h(x) = |x|^\theta \nu_{\lfloor x \rfloor}(\theta).
\]
Then $h$ is an invariant function for the semigroup of the rssMp killed at its first hitting time of the point zero, here denoted by $(P^0_t)_{t \geq 0}$.

The proofs of Theorem 1.2 and Theorem 1.3 relies on the following three technical Lemmas. The first one is a linear algebra result whose proof is included for sake of completeness, the second provides a necessary and sufficient condition for the finiteness of $E_{0,i}[I^s]$, while the third establishes the conditions which are required by Theorem 3.1.

Recall that $P^\circ := P(\theta)$, where $P(\theta)$ is defined by the exponential change of measure introduced in Eq. 1.6 with $\theta$ being the Cramér number.

**Lemma 4.3** Let $A$ be a $2 \times 2$ matrix with real eigenvalues $\lambda_1 \leq \lambda_2$. If $\text{tr}(A) \leq 2$ and $\det(I - A) \geq 0$, then $\lambda_2 \leq 1$ and $\lambda_2 < 1$ holds whenever $\det(I - A) > 0$.

**Proof** It is easy to see that $\det(I - A) \geq 0$ if and only if $2 - \text{tr}(A) \geq |2 - \text{tr}(A) - 2\det(A)|$ and since $2 - \text{tr}(A) \geq 0$ by hypothesis, the latter inequality implies

$$\lambda_2 = \frac{\text{tr}(A) + \sqrt{\text{tr}^2(A) - 4\det(A)}}{2} \leq 1.$$ 

The second part follows using the same arguments. \qed

Recall that

$$I = \int_0^\infty \exp\{\alpha \xi(t)\}dt.$$ 

**Lemma 4.4** Let $((\xi, J), P)$ be a MAP and $s \in (0, 1)$. Then, $\kappa(\alpha s) < 0$ if and only if $E_{0,i}[I^s] < \infty$, for $i = -1, 1$.

**Proof** We first consider the MAP $((\alpha \xi, J), P)$ and observe that the same arguments used in the proof of Proposition 3.6 in [13] provides the direct implication. For the reciprocal, we suppose that $E_{0,i}[I^s] < \infty$, $i = -1, 1$. Thus, for $i = -1, 1$, we have by the Markov property,

$$E_{0,i}[I^s] > E_{0,i}\left[\left(\int_1^\infty \exp[\alpha \xi(u)]du\right)^s\right] = E_{0,i}\left[\exp[\alpha s \xi(1)]\left(\int_0^\infty \exp[\alpha(\xi(1 + u) - \xi(1))]du\right)^s\right] = E_{0,i}\left[e^{\alpha s \xi(1)}; J(1) = i\right]E_{0,i}[I^s] + E_{0,i}\left[e^{\alpha s \xi(1)}; J(1) = j\right]E_{0,j}[I^s]. \quad j \neq i.$$

From the latter, and since all quantities are positive, it follows

$$E_{0,1}[I^s]\left(1 - \left(e^{F(\alpha s)}\right)_{11}\right) > \left(e^{F(\alpha s)}\right)_{1-1} E_{0,-1}[I^s] > 0,$$

$$E_{0,-1}[I^s]\left(1 - \left(e^{F(\alpha s)}\right)_{-11}\right) > \left(e^{F(\alpha s)}\right)_{-1-1} E_{0,1}[I^s] > 0.$$
The previous inequalities imply
\[
\begin{align*}
(e^F(\alpha s))_{11} &< 1, \\
(1 - (e^F(\alpha s))_{11}) &< (e^F(\alpha s))_{-1-1} < 1, \\
(1 - (e^F(\alpha s))_{11}) &> (e^F(\alpha s))_{-11}.
\end{align*}
\]

Putting all the pieces together, we deduce
\[
\begin{align*}
\text{tr} \left( e^F(\alpha s) \right) &= (e^F(\alpha s))_{11} + (e^F(\alpha s))_{-1-1} < 2, \\
\text{det} \left( I - e^F(\alpha s) \right) &= \left( 1 - (e^F(\alpha s))_{11} \right) \left( 1 - (e^F(\alpha s))_{-1-1} \right) > 0.
\end{align*}
\]

Using Lemma 4.3 we get the leading eigenvalue is less than 1. In other words, \( e^{\kappa(\alpha s)} < 1 \) and implicitly \( \kappa(\alpha s) < 0 \), completing the proof.

**Lemma 4.5** Assume that Assumption 1.1 holds and denote by \( \theta \) and \( v \) for the Cramér number and its associated vector. Suppose that \( 0 < \theta < \alpha \), then \( E_{0,i}[I_{\theta/\alpha} - 1] \) and \( \hat{E}_{0,i}^\prime[I_{\theta/\alpha}] \) are finite, for \( i = -1, 1 \).

**Proof** Since the MAP \(((\alpha \xi, J), \mathbb{P})\) has Cramér number \( \theta/\alpha \), a direct application of Proposition 3.6 in [13] provides that \( E_{0,i}[I_{\theta/\alpha} - 1] \) is finite, for \( i = -1, 1 \).

For the second expectation, we observe that the MAP \(((\alpha \xi, J), \hat{\mathbb{P}}^\xi)\) satisfies \( \kappa^\prime(\theta/\alpha) = \kappa(0) \). If \( \kappa(0) = 0 \), then \( \theta/\alpha \) is the Cramér number of \(((\alpha \xi, J), \hat{\mathbb{P}}^\xi)\) and by the first part of the proof we have \( \hat{E}_{0,i}^\prime[I_{\theta/\alpha}] \) is finite, for \( i = -1, 1 \). Now, if \( \hat{\kappa}^\prime(\theta/\alpha) = \kappa(0) < 0 \), Lemma 4.4 guarantees that \( \hat{E}_{0,i}^\prime[I_{\theta/\alpha}] \) is finite, for \( i = -1, 1 \), which implies that for any \( t > 0 \),
\[
\hat{E}_{0,i}^\prime \left[ \left( \int_0^t e^{\alpha \xi(u)} du \right)^{\theta/\alpha} \right] < \infty \quad \text{for} \quad i = -1, 1. \tag{4.2}
\]

From the proof of Proposition 3.6 in [13], we know that the following identity holds for all \( s > 0 \) and \( t \geq 0 \),
\[
\left( \int_0^\infty e^{\alpha \xi(u)} du \right)^s - \left( \int_t^\infty e^{\alpha \xi(u)} du \right)^s = s \int_t^\infty e^{s \xi(u)} \left( \int_0^\infty e^{\alpha \xi(v + u - \alpha \xi(u))} dv \right)^{s-1} du.
\]

Hence taking expectations from both sides of the above identity, with \( s = \theta/\alpha \), and applying the Markov property, we obtain
\[
\hat{E}_{0,i}^\prime \left[ \left( \int_0^\infty e^{\alpha \xi(u)} du \right)^{\theta/\alpha} \right] - \left( \int_t^\infty e^{\alpha \xi(u)} du \right)^{\theta/\alpha} = \frac{\theta}{\alpha} \int_0^t \sum_{j \in \{\pm 1\}} \hat{E}_{0,i}^\prime \left[ e^{\theta \xi(u)}; J(u) = j \right] \hat{E}_{0,j}^\prime[I_{\theta/\alpha-1}] du.
\]

Since \( ||x||^s - |y|^s | \leq |x - y|^s \), for any \( x, y \in \mathbb{R} \), and \( 0 < s < 1 \), the left-hand side of the above equation is bounded by Eq. 4.2. Since \( \hat{E}_{0,i}^\prime \left[ e^{\theta \xi(u)}; J(u) = i \right] \neq 0 \), it follows that \( \hat{E}_{0,i}^\prime[I_{\theta/\alpha-1}] \) is finite, for \( i = -1, 1 \), and the proof is now completed. \( \square \)
Proof of Theorem 1.2 The proof relies on Theorem 3.1. Let \((\xi, J), \hat{\mathbb{P}}^\alpha\) the dual process of \((\xi, J), \mathbb{P}^\alpha\). By Proposition 4 in [8], we have that \((\xi, J), \hat{\mathbb{P}}^\alpha\) drifts to \(\infty\) which imply that \((\xi, J), \hat{\mathbb{P}}^\alpha\) drifts to \(-\infty\), and therefore \(I < \infty, \hat{\mathbb{P}}^\alpha\)-a.s.

Let \(\hat{\mathbb{P}}^\alpha\) be the law of the \(\alpha\)-rssMp associated with \((\xi, J), \hat{\mathbb{P}}^\alpha\) via the Lamperti-Kiu transform. The process \((X, \hat{\mathbb{P}}^\alpha)\) hits 0 continuously and in a finite time, \(\hat{\mathbb{P}}^\alpha\)-a.s. Now, by Lemma 4.1, \((X, \mathbb{P}^\alpha)\) and \((X, \hat{\mathbb{P}}^\alpha)\) are in weak duality with respect to the measure \(\mu(dx) = |x|^{\alpha-1} \pi(x)dx\), for \(x \neq 0\). Since the law \(\mathbb{P}^\alpha\) is constructed via a Doob \(h\)-transform of the law \(\mathbb{P}\), with \(h(x) = |x|^\theta v_{|x|}\) (the invariant function for the semigroup of \((X, \mathbb{P})\)), it follows that \((X, \mathbb{P})\) and \((X, \hat{\mathbb{P}}^\alpha)\) are in weak duality with respect to the measure

\[
\nu(dx) = \alpha |x|^{\alpha-1-\theta} v_{|x|} \pi(x) dx.
\]

From Lemma 4.5, we know that \(\mathbb{E}_{0,i}[I^{\theta/\alpha-1}]\) and \(\hat{\mathbb{E}}^\alpha_{0,i}[I^{\theta/\alpha-1}]\) are finite, for \(i = -1, 1\). Thus, for all \(\lambda > 0\), we necessarily have

\[
\int_{\mathbb{R}^+} \mathbb{E}_x[e^{-\lambda T}] \nu(dx) < \infty, \quad \text{and} \quad \int_{\mathbb{R}^+} \hat{\mathbb{E}}^\alpha_x[e^{-\lambda T}] \nu(dx) < \infty.
\]

Indeed, from the Lamperti-Kiu transform and Fubini’s Theorem, we see

\[
\int_{\mathbb{R}^+} \mathbb{E}_x[e^{-\lambda T}] \nu(dx) = \int_{\mathbb{R}^+} \alpha |x|^{\alpha-1-\theta} v_{|x|} \pi(x) \mathbb{E}_{0,i}[^{-\lambda|x|^\theta} I] dx
\]

\[
= \Gamma(1 - \theta/\alpha) \lambda^{\theta/\alpha-1} \left(v_1 \pi_1 \mathbb{E}_{0,1}[I^{\theta/\alpha-1}] + v_{-1} \pi_{-1} \mathbb{E}_{0,-1}[I^{\theta/\alpha-1}]\right),
\]

which is finite. In a similar way, we can deduce

\[
\int_{\mathbb{R}^+} \hat{\mathbb{E}}^\alpha_x[e^{-\lambda T}] \nu(dx) < \infty.
\]

Thus, the conditions of Theorem 3.1 hold and we guarantee that there exists a unique recurrent extension of \((X, \mathbb{P})\) such that the \(\lambda\)-resolvent of its excursion measure \(n\) satisfies

\[
n\left(\int_0^{T_0} e^{-\lambda t} f(X_t) dt\right) = \frac{1}{\hat{C}_{\alpha,\theta}} \int_{-\infty}^{\infty} f(x) \hat{\mathbb{E}}^\alpha_x[e^{-\lambda T}] |x|^{\alpha-1-\theta} v_{|x|} \pi(x) dx,
\]

where

\[
\hat{C}_{\alpha,\theta} = \frac{\Gamma(1 - \theta/\alpha)}{\alpha} \left(v_1 \pi_1 \mathbb{E}_{0,1}[I^{\theta/\alpha-1}] + v_{-1} \pi_{-1} \mathbb{E}_{0,-1}[I^{\theta/\alpha-1}]\right).
\]

Furthermore, \(n(1 - e^{-T_0}) = 1\). The characterization of the entrance law is obtained from the following series of identities

\[
n\left(\int_0^{T_0} e^{-\lambda t} f(X_t) dt\right) = \frac{1}{\hat{C}_{\alpha,\theta}} \left(\int_0^{\infty} f(x) \hat{\mathbb{E}}^\alpha_{0,1}[e^{-\lambda \alpha I}] x^{\alpha-1-\theta} v_{|x|} \pi_1 dx \right.
\]

\[
+ \left. \int_{-\infty}^{0} f(x) \hat{\mathbb{E}}^\alpha_{0,-1}[e^{-\lambda (-x)^\alpha I}] (-x)^{\alpha-1-\theta} v_{-1} \pi_{-1} dx\right)
\]

\[
= \frac{1}{\alpha \hat{C}_{\alpha,\theta}} \int_0^{\infty} e^{-\lambda t} e^{-\theta/\alpha} \left(v_1 \pi_1 \mathbb{E}_{0,1}[f \left(\frac{t^{1/\alpha}}{I^{1/\alpha}}\right) I^{\theta/\alpha-1}]\right)
\]

\[
+ v_{-1} \pi_{-1} \mathbb{E}_{0,-1}[f \left(\frac{t^{1/\alpha}}{I^{1/\alpha}}\right) I^{\theta/\alpha-1}] dt,
\]

where in the first identity we used the Lamperti-Kiu transform and in the second identity, we used Fubini’s theorem and performed a change of variables.
To prove the converse, first we will verify that there exist a $\theta$ in $(0, \alpha)$ such that $\kappa(\theta) \leq 0$. By Lemma 3.6, we can deduce that there exist a $\theta \in (0, \alpha)$ such that potential of the measure $\nu$ is given by

$$\nu(dy) := n \left( \int_0^{T_0} 1_{[X_t \in dy]} \right) = C^{[y]}_{\alpha,\alpha\theta} |y|^\alpha - \theta dy, \quad y \in \mathbb{R}^*,$$

where $C^{[y]}_{\alpha,\alpha\theta}, C^{-1}_{\alpha,\alpha\theta}$ are constants such that $0 < C^{[y]}_{\alpha,\alpha\theta} + C^{-1}_{\alpha,\alpha\theta} < \infty$. Furthermore, $\nu$ is the unique invariant measure for $X$ (the uniqueness holds up to a multiplicative constant). Hence, it follows that $\nu$ is an excessive measure for $(X, \mathbb{P})$. On the other hand, the Revuz measure of the additive functional $B$ defined by $B_t = \int_0^t |X_s|^\alpha ds$, for $0 \leq t < T_0$, relative to $\nu$, is given by

$$\nu_B(dy) = C^{[y]}_{\alpha,\alpha\theta} |y|^{-\theta} dy, \quad y \in \mathbb{R}^*.$$

Indeed, the Revuz measure of the additive functional $B$, relative to $\nu$, is such that

$$\nu_B(f) = \lim_{t \to 0} \frac{1}{t} \mathbb{E}_\nu \left[ \int_0^t f(X_s) dB_s, t < T_0 \right]$$

$$= \lim_{t \to 0} \int_{\mathbb{R}^*} \frac{1}{t} \mathbb{E}_x \left[ \int_0^t f(X_s) |X_s|^{-\alpha} ds, t < T_0 \right] \nu(dx)$$

$$= \int_{\mathbb{R}^*} f(x) |x|^{-\alpha} \nu(dx) = \int_{\mathbb{R}^*} f(x) C^{[x]}_{\alpha,\alpha\theta} |x|^{-\theta} dx,$$

for all positive bounded function $f$, see e.g. [18].

Since the Revuz measure is excessive for the process $(X, \mathbb{P})$, from the Lamperti-Kiu transform it follows

$$\int_{\mathbb{R}^*} \mathbb{E}_{0,i} \left[ f(y e^{\xi(t)} J(t)) \right] C^{[y]}_{\alpha,\alpha\theta} |y|^{-\theta} dy \leq \int_{\mathbb{R}^*} f(y) C^{[y]}_{\alpha,\alpha\theta} |y|^{-\theta} dy,$$

for all positive function $f$ and all $i = -1, 1$. The left hand side of the latter inequality can be written as follows

$$\int_{\mathbb{R}^*} \mathbb{E}_{0,i} \left[ e^{\theta\xi(t)} ; J(t) = 1 \right] C^{[x]}_{\alpha,\alpha\theta} f(x) |x|^{-\theta} dx$$

$$+ \int_{\mathbb{R}^*} \mathbb{E}_{0,i} \left[ e^{\theta\xi(t)} ; J(t) = -1 \right] C^{-1}_{\alpha,\alpha\theta} f(x) |x|^{-\theta} dx,$$

for all $i = -1, 1$. Hence,

$$\int_{\mathbb{R}^*} C^{[x]}_{\alpha,\alpha\theta} f(x) |x|^{-\theta} dx \geq \int_{\mathbb{R}^*} \left[ (e^{F(\theta)t})_{i1} C^{[x]}_{\alpha,\alpha\theta} + (e^{F(\theta)t})_{i-1} C^{-1}_{\alpha,\alpha\theta} \right] f(x) |x|^{-\theta} dx,$$

for $i = -1, 1$. Since the latter inequality holds for all positive function $f$, we deduce

$$\left( 1 - (e^{F(\theta)t})_{i1} \right) - (e^{F(\theta)t})_{i-1} \left[ C^{[x]}_{\alpha,\alpha\theta} + C^{-1}_{\alpha,\alpha\theta} \right] \geq 0,$$

for $i = -1, 1$, which implies the series of inequalities

$$\left( e^{F(\theta)t} \right)_{11} \leq 1, \quad \left( e^{F(\theta)t} \right)_{1-1} \leq 1 - \left( e^{F(\theta)t} \right)_{11},$$

$$\left( e^{F(\theta)t} \right)_{-11} \leq 1, \quad \left( e^{F(\theta)t} \right)_{-1-1} \leq 1 - \left( e^{F(\theta)t} \right)_{-11}.$$
and 
\[ \det \left( I - e^{F(\theta)t} \right) = \left( 1 - (e^{F(\theta)t})_{11} \right) \left( 1 - (e^{F(\theta)t})_{-1-1} \right) - (e^{F(\theta)t})_{1-1} (e^{F(\theta)t})_{-11} \geq 0. \]

Thus the conditions of Lemma 4.3 hold and therefore \( e^{\kappa(\theta)} \leq 1 \), i.e., \( \kappa(\theta) \leq 0 \).

Finally, if \( \kappa(\theta) < 0 \) then Theorem 1.3 (which is proved below) implies that \((X, \mathcal{P})\) admits a recurrent extension that leaves 0 by a jump with jumping-in measure proportional to \( \eta_0(dx) = |x|^{-(\theta + 1)}dx \), for \( x \neq 0 \). In other words, the measure \( m = 2^{-1}n + 2^{-1}c_{\alpha,\theta}P_{\eta_0} \) is a self-similar excursion measure compatible with \((X, \mathcal{P})\) and with index of self-similarity \( \theta/\alpha \); where \( c_{\alpha,\theta} \) is a normalizing constant. Therefore, there exists a recurrent extension of \((X, \mathcal{P})\) with excursion measure \( m \) that may leave 0 by a jump and continuously at the same time, which leads to a contradiction since any recurrent extension of \((X, \mathcal{P})\) either leaves 0 by a jump or continuously (see Lemma 3.7). Therefore, \( \kappa(\theta) = 0 \), that is to say \( \theta \in (0, \alpha) \) is a Cramér number. \( \square \)

**Proof of Theorem 1.3** The equivalence of assertions (i) and (ii) follow from Lemma 4.4 considering the MAP \((\xi, J, \mathbb{P})\) and \( s = \beta/\alpha \).

Now, we prove the equivalence of the assertions (ii) and (iii). Using again that \( T_0 \) under \( \mathcal{P}_\alpha \) has the same law as \( |x|^{-\alpha} \) under \( \mathbb{P}_{0,\{s\}} \), we obtain
\[
\int_{\mathbb{R}^+} \mathbb{E}_\chi[1 - e^{-T_0}] |x|^{-(\beta+1)}dx = \mathbb{E}_{0,1} \left[ \int_0^\infty (1 - e^{-y}) \frac{1}{\alpha} y^{-\beta/\alpha - 1} dy \right]
+ \mathbb{E}_{0,-1} \left[ \int_0^\infty (1 - e^{-y}) \frac{1}{\alpha} y^{-\beta/\alpha - 1} dy \right]
= \frac{\Gamma(1 - \beta/\alpha)}{\beta} \left( \mathbb{E}_{0,1}[I^{\beta/\alpha}] + \mathbb{E}_{0,-1}[I^{\beta/\alpha}] \right).
\]

Thus, if \( \eta(dx) = |x|^{-(\beta+1)}dx \) and \( n^j \) is the pseudo-excursion measure \( n^j = \mathcal{P}_\eta \), then
\[
n^j(1 - e^{-T_0}) = \int_{\mathbb{R}^+} \mathbb{E}_\chi[1 - e^{-T_0}] |x|^{-(\beta+1)}dx
\]
is finite if and only if \( \mathbb{E}_{0,i}[I^{\beta/\alpha}] < \infty \), for \( i = -1, 1 \). This proves the equivalence between assertions in (ii) and (iii).

Finally, if (iii) holds, according with [4] and Lemma 3.8, associated with the normalized excursion \( n^{j'} = \mathcal{P}_\eta \) where \( \eta(dx) = b_{\alpha,\beta}^{[x]} |x|^{-(\beta+1)}dx \) there exists a unique extension of the minimal process \((X, T_0)\) which is a self-similar Markov process and which leaves 0 by a jump, according to the jumping-in measure \( \eta(dx) = b_{\alpha,\beta}^{[x]} |x|^{-(\beta+1)}dx \), implying (iv). Conversely, if (iv) holds the Itô excursion measure of \( \bar{X} \) is \( n^{j'} = \mathcal{P}_\eta \), with \( \eta(dx) = b_{\alpha,\beta}^{[x]} |x|^{-(\beta+1)}dx \) and the statement in (iii) follows. This completes the proof. \( \square \)

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