Property C and applications to inverse problems

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Abstract

Let \( \ell_j := -\frac{d^2}{dx^2} + k^2 q_j(x) \), \( k = \text{const} > 0 \), \( j = 1, 2, 0 < c_0 \leq q_j(x) \leq c_1 \), \( q \) has finitely many discontinuity points \( x_m \in [0, 1] \), and is real-analytic on the intervals \([x_m, x_{m+1}]\) between these points. The set of such functions \( q \) is denoted by \( M \). Only the following property of \( M \) is used: if \( q_j \in M \), \( j = 1, 2 \), then the function \( p(x) := q_2(x) - q_1(x) \) changes sign on the interval \([0, 1]\) at most finitely many times. Suppose that \((\ast)\) implies \( h = 0 \), then the pair \( \{\ell_1, \ell_2\} \) is said to have property \( C \) on the set \( M \). This property is proved for the pair \( \{\ell_1, \ell_2\} \). Applications to some inverse problems for a heat equation are given.

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1 Introduction

Property \( C \) stands for completeness of the set of products of solutions to homogeneous equations. This notion was introduced by the author in [3], [4], [8], and used widely as a powerful tool for proving uniqueness theorems for many inverse problems ([5]-[11]). In [8] Property \( C \) was proved for the pair of operators \( \{\frac{d^2}{dx^2} + k^2 - q_1(x), \frac{d^2}{dx^2} + k^2 - q_2(x)\} \), where \( q_1, q_2 \in L_{1,1}, L_{1,1} := \{q : q = \mathcal{J}_0(1 + x)|q(x)|dx < \infty\} \). The novel point in our paper is the proof of Property \( C \) for the pair of differential operators with a different dependence on the spectral parameter. This new version of Property \( C \) turns to be basic, for example, in the proof of uniqueness theorem for an inverse problem for a heat equation with a discontinuous thermal conductivity.

The aim of this paper is to prove Property \( C \) for the pair \( \{\ell_1, \ell_2\} \), where \( \ell_j := -\frac{d^2}{dx^2} + k^2 q_j(x) \), \( k = \text{const} > 0 \), \( j = 1, 2, 0 < c_0 \leq q_j(x) \leq c_1 \leq \infty \), \( c_0, c_1 \) are constants, \( q_j(x) \in M \), and \( M \) is the set of real-valued integrable functions.
such that if \( q_j \in M, j = 1, 2 \), are arbitrary members of \( M \), then the function \( p(x) := q_2(x) - q_1(x) \) changes sign at most finitely many times on the interval \([0, 1]\). For example, \( M \) can be a set of piecewise-analytic real-valued functions with finitely many discontinuity points on the interval \([0, 1]\) (see [2]).

**Definition 1.** Let \( h \in M \) be an arbitrary fixed function, and

\[
\ell_j u_j(x, k) = 0, \quad u'_j(0, k) = 0, \quad u_j(0, k) = 1, \quad 0 \leq x \leq 1.
\]

If the orthogonality relation

\[
\int_0^1 h(x) u_1(x, k) u_2(x, k) dx = 0 \quad \forall k > 0,
\]

implies \( h = 0 \), then we say that the pair \( \{\ell_1, \ell_2\} \) has Property C for the set \( M \).

Our first result is:

**Theorem 1.** The pair \( \{\ell_1, \ell_2\} \) has Property C for the set \( M \).

Let us give an example of applications of Property C. Consider the problem:

\[
U_t = (a(x) U')', \quad 0 \leq x \leq 1, \quad t > 0; \quad U' := \frac{\partial U}{\partial x}.
\]

\[
U(x, 0) = 0; \quad U(0, t) = 0, \quad U(1, t) = F(t),
\]

\[
a(1)U'(1, t) = G(t).
\]

Assume that

\[
0 < c'_0 \leq a(x) \leq c'_1 < \infty, \quad a(x) \in M, \quad c'_0, c'_1 = \text{const}.
\]

The function \( F(t) \neq 0, F(t) \geq 0, F(t) = 0 \) if \( t > T \), where \( T > 0 \) is an arbitrary fixed number, and \( F \in L^1([0, T]) \). Problem \([3]-[4]\) has a unique solution. The function \( G(t) \) is the measured datum (extra datum), which is the heat flux at the point \( x = 1 \). The inverse problem is:

**IP**: Given \( \{F(t), G(t)\}_{t > 0} \), find \( a(x) \).

The function \( a(x) \in M \) may have finitely many discontinuity points, and the solution to \([3]-[4]\) is understood in the weak sense.

Let us formulate the IP in an equivalent but different form. Let

\[
v(x, \lambda) := \int_0^\infty U(x, t)e^{-\lambda t} dt, \quad f(\lambda) = \int_0^\infty F(t)e^{-\lambda t} dt,
\]

\[
g(\lambda) = \int_0^\infty G(t)e^{-\lambda t} dt.
\]

Take the Laplace of \([3]-[5]\) and get

\[
-(a(x)v')' + \lambda v = 0, \quad 0 \leq x \leq 1, \quad \lambda > 0,
\]
Let $u(x, \lambda) := a(x)u'(x, \lambda)$. Differentiate (5) and get
\begin{equation}
-u'' + \lambda a^{-1}(x)u = 0, \quad 0 \leq x \leq 1, \quad u'(0, \lambda) = 0,
\end{equation}
and the data are $\{g(k^2), k^2f(k^2)\}_{vk>0}$. The IP can be reformulated as follows:
\begin{itemize}
  \item IP2: Given the data $\{g(k^2), k^2f(k^2)\}_{vk>0}$, find $q(x)$.
\end{itemize}
Our second result is:
\begin{theorem}
The IP2 has at most one solution.
\end{theorem}

Theorem 2 implies the uniqueness of the solution to IP1 in the class of the piecewise-analytic strictly positive functions $a(x)$ with finitely many discontinuity points on the interval $[0, 1]$, or, more generally, in a class $M$.

In the literature IP1 has been considered earlier (see, e.g., [9], [10] and references therein) in the case when $a(x) \in H^2([0, 1])$, where $H^2$ is the Sobolev space. For piecewise-constant thermal conductivity coefficients $a(x)$ with finitely many discontinuity points the IP1 was studied recently in [2]. An inverse problem for equation (3) with different extra data, namely $\nu(x, \lambda)$ different discontinuity points on the interval $[0, 1]$, was studied in [1]. In IP1 the extra data are collected at just one point $x = 1$.

Our arguments prove the uniqueness result for Theorem 1 in the case when the data in IP1 are the values $\{F(t), G(t)\}_{t \in [0, T+\epsilon]}$ for an arbitrary small $\epsilon > 0$. These data determine $a(x)$ uniquely because the solution $U(x, t)$ is an analytic function of $t$ in a neighborhood of the set $(T, \infty)$, so the knowledge of $U(x, t)$ on the segment $[0, T + \epsilon]$ determines $U(x, t)$ uniquely for all $t > 0$.

In Section 2 proofs are given.

2 Proofs

\textit{Proof of Theorem 1} The solution to (1) solves the equation
\begin{equation}
u_j(x, k) = 1 + k^2 \int_0^x (x - s)q_j(s)u_j(s, k)ds, \quad x \geq 0, \quad j = 1, 2.
\end{equation}
This is a Volterra equation. It has a unique solution $u_j(x, k)$. This solution has the following properties:
\begin{equation}
u_j(x, k) \geq 1, \quad u_j'(x, k) \geq 0, \quad u_j''(x, k) > 0, \quad 0 \leq x \leq 1,
\end{equation}
\begin{equation}
\frac{\partial^i u_j}{\partial (k^2)^i} \geq 0, \quad i = 1, 2, 3, \ldots,
\end{equation}
\begin{equation}
u(0, \lambda) = 0, \quad v(1, \lambda) = f(\lambda), \quad a(1)u'(1, \lambda) = g(\lambda).
\end{equation}
\text{lim}_{k \to \infty} u_j(y,k) = 0, \quad 0 \leq y < x \leq 1. \quad (16)

Properties (14)-(15) are immediate consequences of (13). Let us prove (16). One has
\[ u_j(x,k) = u_j(y,k) + \int_y^x u_j'(s,k) \, ds. \quad (17) \]

From (13) and (14) one obtains
\[ u_j'(x,k) = k^2 \int_0^x q_j(s)u_j(s,k) \, ds \geq k^2 \int_0^x q_j(s) \, ds. \quad (18) \]

From equations (17) and (18) one gets:
\[ u_j(x,k) = u_j(y,k) + \int_y^x u_j'(s,k) \, ds \geq 1 + k^2 \int_y^x \int_0^z q_j(s) \, ds \, u_j(z,k) \, dz. \quad (19) \]

Thus, (16) is proved.

Since \( h \in M \), the segment \([0, 1]\) is a union of finitely many intervals without common interior points on each of which the function \( h(x) \) keeps sign. Let \([z, 1]\) be such an interval. We want to prove that \( h = 0 \) on this interval. If this is done then similarly, in a finite number of steps, one proves that \( h = 0 \) on the whole interval \([0, 1]\), and then the proof of Theorem 1 is completed.

Let us rewrite relation (2) as
\[ \int_z^1 h(x)u_1(x,k)u_2(x,k) \, dx = - \int_0^z h(x)u_1(x,k)u_2(x,k) \, dx \leq u_1(z,k)u_2(z,k) \int_0^z |h(x)| \, dx, \quad (20) \]

where the monotonicity and the positivity of \( u_j \) was used, see (14). Without loss of generality assume that \( h(x) \geq 0 \) on \([z, 1]\) and fix an arbitrary \( y \in (z, 1) \). Then
\[ \int_z^1 h(x)u_1(x,k)u_2(x,k) \, dx \geq \int_y^1 h(x)dx \, u_1(y,k)u_2(y,k). \quad (21) \]

From (20) and (21) one gets:
\[ \int_y^1 h(x)dx \leq \frac{u_1(z,k)u_2(z,k)}{u_1(y,k)u_2(y,k)} \int_0^z |h(x)| \, dx, \quad y > z. \quad (22) \]

Let \( k \to \infty \) in (22) and use (10) to get \( \int_y^1 h(x) \, dx = 0 \). Since \( h(x) \geq 0 \) on \([z, 1]\), it follows that \( h = 0 \) on \([y, 1]\). Since the point \( y \in (z, 1) \) is arbitrary, it follows that \( h = 0 \) on \([z, 1]\).
Theorem 1 is proved. □

Proof of Theorem 2. Assume the contrary, i.e., there are pairs of functions \( \{ \psi_1, q_1 \} \) and \( \{ \psi_2, q_2 \} \) which solve (12) and (11) with \( \lambda = k^2 \). Let \( w = \psi_1 - \psi_2 \). Then
\[
w'(0, k) = w(1, k) = w'(1, k) = 0,
\]
and
\[
- w'' + k^2 q_1(x) w = k^2 p(x) \psi_2, \quad p(x) := q_2(x) - q_1(x).
\]
Multiply (24) by \( u_1(x, k) \), integrate over \([0, 1]\), and then integrate by parts using (23). The result is:
\[
k^2 \int_0^1 p(x) u_1(x, k) \psi_2(x, k) dx = 0 \quad \forall k > 0.
\]
Since \( \psi_2(x) = c(k) u_2(x, k) \), where \( c(k) = \text{const} \neq 0 \), and \( p \in M \), it follows from (25) and Theorem 1 that \( p = 0 \), so \( q_1 = q_2 \).
Theorem 2 is proved. □

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