Formulating categorical concepts with classes

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July 27, 2018

Abstract

We examine the use of classes to formulate several categorical notions. This leads to two proposals: an explicit structure for working with subobjects, and a hierarchy of $k$-classes. We apply the latter to both ordinary and higher categories.

1 Introduction

The notion of “class” pervades category theory but its role is not always apparent. This article brings together a number of common concepts that it affects, and proposes some ways of formulating them. We begin in Section 2 by reviewing the basic notions of universe, class and category, and taking note of encoding issues for quotients and tuples. In Section 3 we look at the theory of subobjects, leading to a notion of “well-powering”, an explicit structure for well-powered categories. In Section 4 we propose a hierarchy of $k$-classes that is useful for formulating the Yoneda lemma and several other constructions, including higher category theory. To make the general framework more user-friendly, Section 5 proposes a convention—inspired by [Mur06]—for indicating size restrictions. We sum up in Section 6.

Many foundational systems have been proposed to deal with size issues in category theory, see e.g. [EGM17, Fef69, Mul01, Shu08]; an extensive survey is given in [Shu08]. But we shall use the conventional framework of ZFC with universes.

Related work. Size issues have been widely discussed, e.g. in the textbooks [AHS90, ML71]. Dowd [Dow93] considered categorical applications of
a hierarchy of classes in an extended version of ZFC. Categories of classes
have been studied by the “Algebraic set theory” school, e.g. [JM95, ABSS14],
and functors on them by [AM89, AMV04].

2 Preliminaries

2.1 Universes

In many accounts of category theory, a category is taken to have a class of
objects, and there is a category of all sets. However (as stated above) we are
working in ZFC, so we cannot speak of classes. Instead we define a category
\( C \) to consist of a set \( \text{ob} \ C \) and a family of sets \( (C(x,y))_{x,y \in C} \) together with
composition and identities. By Russell’s Theorem, there is no category of all
sets. That is a problem, and the notion of a (Grothendieck) universe provides
a way of dealing with it.

Definition 1 Let \( \mathcal{U} \) be a set. A universe is a set \( \mathcal{U} \) with the following prop-
erties.

- Any set in \( \mathcal{U} \) is a subset of \( \mathcal{U} \).
- \( \emptyset \in \mathcal{U} \).
- If \( x, y \in \mathcal{U} \) then \( \{x, y\} \in \mathcal{U} \).
- If \( I \) is a set in \( \mathcal{U} \) and \( (A_i)_{i \in I} \) is a family of sets in \( \mathcal{U} \) then \( \bigcup_{i \in I} A_i \in \mathcal{U} \).
- If \( A \) is a set in \( \mathcal{U} \) then \( \mathcal{P}A \in \mathcal{U} \).

The least universe is the set \( \text{HF} \) of hereditarily finite sets, which does not
contain \( \mathbb{N} \). All other universes do contain \( \mathbb{N} \), but it cannot be proved in ZFC
that such universes exist (assuming ZFC consistent).

Definition 2 Let \( \mathcal{U} \) be a universe.

- A \( \mathcal{U} \)-small set is a set in \( \mathcal{U} \).
- A \( \mathcal{U} \)-class is a subset of \( \mathcal{U} \).

We write

- \( \text{Set}_\mathcal{U} \) for the category of \( \mathcal{U} \)-small sets and functions
• **Class**_{\mathcal{U}} for the category of \mathcal{U}-classes and functions.

Thus Set_{\mathcal{U}} \subseteq Class_{\mathcal{U}}. In particular, \mathcal{U} itself is a \mathcal{U}-class but not a \mathcal{U}-set.

The construction \mathcal{U} \mapsto Set_{\mathcal{U}} is designed to serve as a kind of substitute for the category of all sets. But the extent to which it succeeds depends on what we assume about the existence of universes. To see why, consider the following statements:

**Proposition 1** In any category \mathcal{C}, a morphism \( f : A \to B \) has at most one inverse.

**Proposition 2** Let \( \mathcal{U} \) be a universe. In Set_{\mathcal{U}}, a morphism \( f : A \to B \) has at most one inverse.

**Proposition 3** For any sets \( A \) and \( B \), a function \( f : A \to B \) has at most one inverse.

Proposition 2 is an instance of Proposition 1, but Proposition 3 (though easy to prove directly) cannot be deduced from Proposition 2, because there might be no universe containing \( A \) and \( B \). So the construction \( \mathcal{U} \mapsto Set_{\mathcal{U}} \) fails in its task of serving as a substitute for the category of all sets. To avoid such difficulties, Grothendieck and Verdier [GV64] proposed the *Universe Axiom*: every set belongs to a universe. Assuming this axiom allows us to deduce Proposition 3 from Proposition 2. (Even so, there remains a mismatch between the construction \( \mathcal{U} \mapsto Set_{\mathcal{U}} \) and the desired category of all sets. See the discussion of reflection principles in [Shu08].)

This article is written both for people who assume the Universe Axiom and for those who do not. Note that the books [AHS90, ML71] assume just one universe containing \( \mathbb{N} \).

Henceforth, let \( \mathcal{U} \) be a universe. We usually leave \( \mathcal{U} \) implicit, e.g. saying “small” for \( \mathcal{U} \)-small, “class” for \( \mathcal{U} \)-class, Set for Set_{\mathcal{U}}, and Class for Class_{\mathcal{U}}.

A set is *essentially small* when it is isomorphic to a small set. Essential smallness may seem a more attractive notion than smallness, but there is no category of all essentially small sets. For example, “essentially HF-small” means finite, and there is no category of all finite sets.

### 2.2 Small, light and moderate categories

We consider the relationships between categories and \( \mathcal{U} \).
Definition 3  A category $\mathcal{C}$ is

- small when $\text{ob} \; \mathcal{C}$ and all the homsets are small
- light when $\text{ob} \; \mathcal{C}$ is a class and all the homsets are small
- moderate when $\text{ob} \; \mathcal{C}$ and all the homsets are classes [Shu12, Str81].

We write

- $\textbf{Cat}$ for the 2-category of small categories
- $\textbf{CAT}$ for the 2-category of light categories
- $\textbf{CAT}$ for the 2-category of moderate categories.

Thus $\textbf{Cat} \subset \textbf{CAT} \subset \textbf{CAT}$. Here are some examples:

1. The category $\textbf{Set}_{\text{HF}}$ is small, assuming $\mathbb{N} \in \mathfrak{U}$.

2. The category $\textbf{Set}$, and the category $\textbf{Rel}$ of small sets and relations, are light but not small.

3. For sets $A, B$ a multirelation $A \overset{p}{\rightarrow} B$ is a family of cardinals $(p_{a,b})_{a\in A, b\in B}$.
   The identity multirelation on a set $A$ is given at $a, a' \in A$ by $1$ if $a = a'$ and $0$ otherwise; the composite of multirelations $A \overset{p}{\rightarrow} B \overset{q}{\rightarrow} C$ is given at $a \in A, c \in C$ by $\sum_{b\in B} p_{a,b} q_{b,c}$. The category $\textbf{Multirel}$ of small sets and small multirelations (i.e. multirelations consisting of small cardinals) is moderate but not light.

4. The category $\textbf{Class}$ and the functor category $[\textbf{Set}, \textbf{Set}]$ are not moderate.

Note that $\textbf{Cat}$ is cartesian closed but $\textbf{CAT}$ and $\textbf{CAT}$ are not. If we want a cartesian closed 2-category containing $\textbf{Set}$, we may use $\textbf{Cat}_{\mathfrak{U}'}$ for some universe $\mathfrak{U}'$ larger than $\mathfrak{U}$, provided it exists (an instance of the Universe Axiom).

The following conditions, weaker than lightness, are sometimes considered.
A category is **locally small** when all its homsets are small. Thus a light category is one that is both moderate and locally small. Some theorems about light categories, such as the adjoint functor theorems, hold more generally for locally small categories. But it is hard to find natural examples of locally small categories that are not light, other than ones arising from a preordered set. Moreover, there is no 2-category of all locally small categories.

A category is **essentially light** when it is equivalent to a light category. For example, given a light category \( \mathcal{C} \), let \( \text{SP}(\mathcal{C}) \) be the full subcategory of \([\mathcal{C}^{\text{op}}, \text{Set}]\) on presheaves that are “small”, i.e. isomorphic to the colimit of some small diagram of representables [DL07]. This category is neither moderate nor locally small, but it is essentially light. Moreover, via the Yoneda embedding, it is a free cocompletion of \( \mathcal{C} \). So we might wish to view the construction \( \mathcal{C} \mapsto \text{SP}(\mathcal{C}) \) as a reflection of a 2-category of categories into a 2-category of cocomplete categories. But we cannot, as there is no 2-category of essentially light categories.

### 2.3 Quotient and tuple classes

When working with classes, one must take care with the encoding of quotients and tuples.

- For an equivalence relation \( R \) on a class \( A \), the usual quotient \( A/R \) is not a class. In order to form quotient classes, we first associate to every inhabited class \( X \) an element \( \theta X \in \mathcal{U} \), in such a way that \( \theta X \neq \theta Y \) whenever \( X \cap Y = \emptyset \). The following are two ways of doing this.

  1. Let \( \theta \) be a choice function on \( \mathcal{U} \), so \( \theta X \in X \).
  2. Scott’s trick: let \( \theta X \) be the set of elements of \( X \) of least rank.

Now we set

\[
A/^{\ast}R \overset{\text{def}}{=} \{[x]_R^\ast \mid x \in A\}
\]

where \([x]_R^\ast \overset{\text{def}}{=} \theta\{y \in A \mid (x, y) \in R\}\). The * superscript indicates a non-standard encoding.
• For classes $A$ and $B$, the Kuratowski pair $(A, B) \overset{\text{def}}{=} \{\{A\}, \{A, B\}\}$ is not a class. In order to form pair classes, following e.g. [AHS90], we may use the encoding

$$(A, B)^1 \overset{\text{def}}{=} \{(0, x) \mid x \in A\} \cup \{(1, y) \mid y \in B\}$$

Likewise, for a class $I$, we may encode an $I$-indexed tuple of classes by

$$(A_i)_{i \in I} \overset{\text{def}}{=} \{(i, x) \mid i \in I, x \in A_i\}$$

A moderate category, encoded in this way, is a class.

## 3 Subobjects

The theory of subobjects is commonly formulated using quotient classes. We shall present this formulation and then propose a slight change. The theory arises in the following situation.

**Definition 4** A wide subcategory $\mathcal{M}$ of a category $\mathcal{C}$ is mono-like when

- every $\mathcal{M}$-morphism is monic in $\mathcal{C}$
- if a composite $a \xrightarrow{f} b \xrightarrow{g} c$ is in $\mathcal{M}$, then so is $f$.

Thus, in particular, all split monos are in $\mathcal{M}$. Given a light category $\mathcal{C}$ with a mono-like subcategory $\mathcal{M}$, we proceed as follows.

**Definition 5** Let $c \in \mathcal{C}$.

1. We form the class $\mathcal{M}/c$ of pairs $(x, f)$ consisting of $x \in \mathcal{C}$ and an $\mathcal{M}$-morphism $f: x \longrightarrow c$, preordered as follows: $(x, f) \preceq (y, g)$ when there is a morphism $h: x \longrightarrow y$, necessarily unique and in $\mathcal{M}$, making $x \xrightarrow{h} y \xleftarrow{g} c$ commute.

2. When $(x, f)$ and $(y, g)$ are mutually related, the two mediating maps are mutually inverse, so we write $(x, f) \cong (y, g)$.

Our task is to represent these pairs $(x, f)$ modulo $(\cong)$. A commonly used formulation is as follows.
Definition 6 For $c \in C$, the class of $\mathcal{M}$-subobjects of $c$ is

$$\text{Sub}^*(c) \overset{\text{def}}{=} (\mathcal{M}/c) /^* (\cong)$$

ordered as follows:

$$[(x, f)]^*_{\cong} \leq [(y, g)]^*_{\cong} \overset{\text{def}}{=} (x, f) \sqsubseteq (y, g)$$

We say $\mathcal{C}$ is $\mathcal{M}$-well-powered when $\text{Sub}^*(c)$ is small for all $c \in \mathcal{C}$.

Note that the isomorphic alternative

$$\text{Sub}(c) \overset{\text{def}}{=} (\mathcal{M}/c) / (\cong)$$

would be unsuitable. For example, $\text{Sub}^*(1)$ is a subobject classifier in $\text{Set}$, but $\text{Sub}(1)$ is not, since it is not even an object.

Definition 6 ingeniously makes $\mathcal{M}$-well-poweredness into a property of $\mathcal{C}$ and $\mathcal{M}$, with no need for additional data. But we propose a slight reformulation that, while it does require additional data, avoids the need for quotient classes.

Definition 7 Let $R$ be an equivalence relation on a set $A$. A family of unique $R$-representatives for $A$ is a set $I$ and family $(a_i)_{i \in I}$ of elements of $A$, such that, for every $a \in A$, there is a unique $i \in I$ for which $(a, a_i) \in R$.

Definition 8 An $\mathcal{M}$-well-powering $\mathcal{W}$ assigns to each $c \in \mathcal{C}$ a small family of unique $^* (\cong)$-representatives for $\mathcal{M}/c$. We write

$$\mathcal{W}: c \mapsto (^0 U, i_U)_{U \in \text{Sub}(c)}$$

We call $\text{Sub}(c)$ the set of $\mathcal{W}$-subobject-indices of $c$, ordered as follows.

$$U \leq V \overset{\text{def}}{=} (^0 U, i_U) \sqsubseteq (^0 V, i_V)$$

Proposition 4

1. There is an $\mathcal{M}$-well-powering $\mathcal{W}$ iff $\mathcal{M}$ is well-powered. Moreover, $\mathcal{W}$ is unique up to unique isomorphism.

2. $\mathcal{M}$ is determined by $\mathcal{W}$. Explicitly, a $\mathcal{C}$-morphism $b \rightarrow c$ is in $\mathcal{M}$ iff it is of the form $b \xrightarrow{g} ^0 U \xrightarrow{i_U} c$ for a (necessarily unique) pair $(U, g)$ consisting of $U \in \text{Sub}(c)$ and an isomorphism $g: b \cong ^0 U$.

Proof (1)(\iff) is by the Axiom of Choice and the rest is straightforward. \qed
In many cases there is a canonical $\mathcal{M}$-well-powering. For example, a well-powering of $\textbf{Set}$ for injections is given by

$$c \mapsto (U, i_U)_{U \in \mathcal{P}c}$$

where $i_U : U \rightarrowtail c$ is the inclusion $x \mapsto x$. Thus the subobject-indices of $c$ are subsets and ordered by inclusion, rather than sets of (set, injection) pairs.

There is an evident dual notion of an $\mathcal{E}$-co-well-powering of $\mathcal{C}$, where $\mathcal{E}$ is an epi-like subcategory. Again, in many cases there is a canonical one. For example, a co-well-powering of $\textbf{Set}$ for surjections is given by

$$c \mapsto (c/r, p_r)_{r \in \text{Eq}(c)}$$

where $\text{Eq}(c)$ is the set of equivalence relations on $c$, and $p_r : c \rightarrow c/r$ sends $x \mapsto [x]_r$. Thus the quotient-indices of $c$ are equivalence relations and ordered by inclusion, rather than sets of (set, surjection) pairs.

The convenience of these notions for categorical writing is illustrated in [Lev15] (though they are not explicitly formulated there). The content of that paper is presented both in the general setting of a category with a factorization system and in special cases involving subsets and equivalence relations. The latter cases are instances of the former—precisely, not just up to isomorphism—because of the use of subobject-indices and quotient-indices.

For another example where a family of unique representatives is used instead of a quotient class, see [AMMS13, Theorem 3.24].

4 A hierarchy of classes

4.1 The target of the Yoneda lemma

In Section 4.2 we shall introduce a new notion of $k$-class. To motivate this, we first discuss the Yoneda lemma. For a light category $\mathcal{C}$, we define in the usual way

- a functor $\mathcal{Y} : \mathcal{C} \rightarrow [\mathcal{C}^{\text{op}}, \textbf{Set}]$

- for $c \in \mathcal{C}$ and $F : \mathcal{C}^{\text{op}} \rightarrow \textbf{Set}$ and $x \in Fc$, a natural transformation $\beta_{c,F}(x) : \mathcal{Y}c \rightarrow F$.

Here is our first attempt to state the Yoneda lemma:
**Proposition 5** Let \( \mathcal{C} \) be a light category. Then \( \beta_{c,F} \) is a bijection \( Fc \cong [\mathcal{C}^{\text{op}}, \text{Set}](\mathcal{Y}c, F) \), natural in \( c \) and \( F \).

Expanding this statement reveals a problem.

**Proposition 6** Let \( \mathcal{C} \) be a light category. Then we have a natural isomorphism

\[
\mathcal{C}^{\text{op}} \times [\mathcal{C}^{\text{op}}, \text{Set}] \xrightarrow{\text{app}} [\mathcal{C}^{\text{op}}, \text{Set}]^{\text{op}} \times [\mathcal{C}^{\text{op}}, \text{Set}]
\]

What should the target category be?

Before answering this, let us note that unpacking Proposition 5 gives a collection of statements that do not mention \([\mathcal{C}^{\text{op}}, \text{Set}]\). For example, the claim that \( \beta_{c,F} \) is natural in \( F \) means that for any natural transformation \( \mathcal{C}^{\text{op}} F \xrightarrow{\alpha} G \) and \( c \in \mathcal{C} \) and \( x \in Fc \), the composite \( \mathcal{Y}c \beta_{c,F}(x) \xrightarrow{\alpha} G \) is \( \beta_{c,G}(\alpha_c x) \). So we might view Proposition 5 as a mere figure of speech, summarizing this collection of statements. But we are going to take it literally. So we need a target category.

One option is to use \( \text{Set}_{\mathcal{U}'} \), where \( \mathcal{U}' \) is a universe greater than \( \mathcal{U} \) such that \( \mathcal{C} \) is \( \mathcal{U}' \)-small. But while such a universe is guaranteed to exist if the Universe Axiom is assumed, it is hardly relevant to the Yoneda lemma. After all, each homset of \([\mathcal{C}^{\text{op}}, \text{Set}]\) is just a set of classes. This suggests using a smaller category than \( \text{Set}_{\mathcal{U}'} \), one that is not cartesian closed.

### 4.2 \( k \)-classes

To summarize our situation, we want to formulate the Yoneda lemma for a light category \( \mathcal{C} \) without mentioning a larger universe. Let us say that our target category will be the category of “2-classes”. What is a 2-class?

We certainly want every set of classes to be a 2-class. So, noting that \((\mathcal{P}^k \mathcal{U})_{k \in \mathbb{N}}\) is an increasing chain, it is reasonable to define a \( k \)-class to be an element of \( \mathcal{P}^k \mathcal{U} \). But if we adopt this definition, then a binary product of 2-classes is not a 2-class, because a pair \((A, B)\) of classes is not a class. Using the pair encoding \((A, B)^1\) from Section 2.3 would only postpone the problem: a pair \((A, B)^1\) of 2-classes is not a 2-class.
One solution would be to adopt a different encoding for each level:

\[
\begin{align*}
(A, B)^0 & \overset{\text{def}}{=} (A, B) \\
(A_i)^0_{i \in I} & \overset{\text{def}}{=} (A_i)_{i \in I} \\
(A, B)^{k+1} & \overset{\text{def}}{=} \{(0, x)^k \mid x \in A\} \cup \{(1, y)^k \mid y \in B\} \\
(A_i)^{k+1}_{i \in I} & \overset{\text{def}}{=} \{(i, x)^k \mid i \in I, x \in A_i\}
\end{align*}
\]

Then \(\mathcal{P}^k\mathcal{U}\) is closed under \((-,-)^k\) and, for \(I \in \mathcal{P}^k\mathcal{U}\), under \((-)_{i \in I}^k\). But having to continually distinguish all these encodings would be inconvenient. Scott and McCarty [SM08] solved this problem by proving\(^1\) that there is a unique binary operation \((-,-)^*\) satisfying

\[
(A, B)^* = \{(0, x)^* \mid x \in A\} \cup \{(1, y)^* \mid y \in B\}
\]

It is an ordered pair operation and every universe is closed under it. They likewise encode indexed tuples:

\[
(A_i)^*_{i \in I} \overset{\text{def}}{=} \{(i, x)^* \mid i \in I, x \in A_i\}
\]

It follows that \(\mathcal{P}^k\mathcal{U}\) is closed under \((-,-)^*\) and, for \(I \in \mathcal{P}^k\mathcal{U}\), under \((-)_{i \in I}^*\).

This is an ingenious solution, but we propose a different approach that avoids the need to replace the Kuratowski encoding. It uses the following construction.

**Definition 9** Let \(A\) be a set of sets. We inductively define the set \(\Psi\mathcal{U}A\), or \(\Psi A\) for short, as follows.

- If \(x \in \mathcal{U}\), then \(x \in \Psi A\).
- If \(I \in A\), then \(I \in \Psi A\).
- If \(x, y \in \Psi A\), then \((x, y) \in \Psi A\).
- If \(I \in A\), and \(x_i \in \Psi A\) for all \(i \in I\), then \((x_i)_{i \in I} \in \Psi A\).

Concisely, \(\Psi A\) is the least prefixpoint of \(X \mapsto \mathcal{U} \cup A \cup \bigcup X \times X \cup \bigcup_{I \in A} X^I\).

Thus any element of \(\Psi A\) can be represented (not necessarily uniquely) by a well-founded tree that has

\(^1\)This is a theorem of NBG class theory.
• leaves labelled by some \( x \in \mathcal{U} \)
• leaves labelled by some \( I \in A \)
• binary nodes
• and nodes labelled by some \( I \in A \), which are \( I \)-ary.

Our key observation is that \( \mathcal{P} \Psi A \) is closed under several constructions.

**Proposition 7** Let \( A \) be a set of sets.

1. If \( B \) and \( C \) are subsets of \( \Psi A \), then so are
\[
B + C \overset{\text{def}}{=} \{(0, b) \mid b \in B\} \cup \{(1, c) \mid c \in C\}
\]
and
\[
B \times C \overset{\text{def}}{=} \{(b, c) \mid b \in B, c \in C\}
\]

2. If \( B, \) and \( C_b \) for all \( b \in B \), are subsets of \( \Psi A \), then so is
\[
\sum_{b \in B} C_b \overset{\text{def}}{=} \{(b, c) \mid b \in B, c \in C_b\}
\]

3. Let \( I \in A \). If \( B_i \), for all \( i \in I \), is a subset of \( \Psi A \), then so is
\[
\prod_{i \in I} B_i \overset{\text{def}}{=} \{(b_i)_{i \in I} \mid \forall i \in I. b_i \in B_i\}
\]

Let us write \( \tilde{\mathcal{U}} \) for the set of sets in \( \mathcal{U} \). (In ZFC, everything is a set so \( \tilde{\mathcal{U}} = \mathcal{U} \).
But in a set theory that allows urelements, \( \mathcal{U} \) might be a proper subset of \( \tilde{\mathcal{U}} \).)
Since \( \Psi \) and \( \mathcal{P} \) are monotone and \( \Psi \tilde{\mathcal{U}} = \mathcal{U} \), we have
\[
\begin{align*}
\tilde{\mathcal{U}} &\subseteq \mathcal{U} \\
\mathcal{P} \Psi \tilde{\mathcal{U}} &\subseteq \mathcal{P} \mathcal{U} &\subseteq \mathcal{P} \mathcal{P} \mathcal{U} \\
\mathcal{P} \Psi \mathcal{P} \tilde{\mathcal{U}} &\subseteq \mathcal{P} \Psi \mathcal{P} \mathcal{U} &\subseteq \mathcal{P} \Psi \mathcal{P} \mathcal{P} \mathcal{U} \\
\mathcal{P} \Psi \mathcal{P} \Psi \mathcal{P} \tilde{\mathcal{U}} &\subseteq \mathcal{P} \Psi \mathcal{P} \Psi \mathcal{P} \mathcal{U} &\subseteq \mathcal{P} \Psi \mathcal{P} \Psi \mathcal{P} \mathcal{P} \mathcal{U} \\
&\vdots &\vdots &\vdots
\end{align*}
\]

This suggests the following definition.
Definition 10

1. A \((\mathfrak{U}, k)\)-entity, or \(k\)-entity for short, is an element of \((\Psi \tilde{U})^k \mathfrak{U}\).

2. A \((\mathfrak{U}, k)\)-class, or \(k\)-class for short, is an element of \((\mathcal{P}\Psi)^k \tilde{\mathfrak{U}}\).

3. The category of \(k\)-classes is called \(\text{Class}_k\).

Thus “0-class” means small set and “\((k+1)\)-class” means set of \(k\)-entities. Moreover, every \(k\)-class is a \(k\)-entity.

Proposition 7 gives the following ways of constructing \(k\)-classes. For \(k = 0\) we read “\(k - 1\)” as 0.

Proposition 8

1. If \(B\) and \(C\) are \(k\)-classes, then so are \(B + C\) and \(B \times C\).

2. If \(B\), and \(C_b\) for all \(b \in B\), are \(k\)-classes, then so is \(\sum_{b \in B} C_b\).

3. Let \(I\) be a \((k - 1)\)-class. If \(B_i\), for all \(i \in I\), is a \(k\)-class, then so is \(\prod_{i \in I} B_i\).

Remark In view of the Ackermann coding \(\mathbb{N} \cong \text{HF}\), perhaps \((\text{HF}, k)\)-classes might constitute a convenient model of higher-order arithmetic, cf. [KW07].

4.3 \(k\)-moderate categories

We shall see that \(k\)-classes provide useful relationships between categories and \(\mathfrak{U}\).

Definition 11 A category \(\mathcal{C}\) is \(k\)-moderate when \(\text{ob } \mathcal{C}\) and all the homsets are \(k\)-classes.

Thus “0-moderate” means small, and “\((k+1)\)-moderate” means that all objects and morphisms are \(k\)-entities.

Proposition 7 implies the following.

Proposition 9 A functor category \([\mathcal{C}, \mathcal{D}]\) is

- small if \(\mathcal{C}\) and \(\mathcal{D}\) are small
- light if \(\mathcal{C}\) is small and \(\mathcal{D}\) light
• *k*-moderate if $\mathcal{C}$ is $(k-1)$-moderate and $\mathcal{D}$ is $k$-moderate.

![](https://via.placeholder.com/150)

**Corollary 10** The category $\text{Class}_k$ is $(k+1)$-moderate.

If $\mathcal{C}$ is light (hence 1-moderate), then $[\mathcal{C}, \text{Set}]$ is 2-moderate, by Proposition 9. So we can formulate the Yoneda lemma as follows.

**Proposition 11** Let $\mathcal{C}$ be a light category. Then we have a natural isomorphism

\[
\begin{array}{ccc}
\mathcal{C}^{\text{op}} \times [\mathcal{C}^{\text{op}}, \text{Set}] & \xrightarrow{\gamma \times [\mathcal{C}^{\text{op}}, \text{Set}]} & [\mathcal{C}^{\text{op}}, \text{Set}]^{\text{op}} \times [\mathcal{C}^{\text{op}}, \text{Set}] \\
\downarrow \text{app} & & \downarrow \text{hom} \\
\text{Set} & \xrightarrow{\beta} & \text{Class}_2
\end{array}
\]

Note, by the way, the requirement for $\mathcal{C}$ to be light, i.e. both moderate and locally small. The statement would not make sense if we weakened the moderateness assumption to essential moderateness, or the local smallness assumption to local essential smallness.

### 4.4 Higher categories

Let us now consider

- the 2-category $\text{Cat}$ of small categories
- the 2-category $\text{CAT}$ of light categories
- the 2-category $\text{CAT}_k$ of $k$-moderate categories.

What is the relationship between these 2-categories and $\mathfrak{U}$? In order to answer this question, let us formulate, more generally, relationships between $n$-categories and $\mathfrak{U}$.

We fix $n$, where $n \in \mathbb{N} \cup \{\infty\}$. For $n = \infty$ we assume $\mathbb{N} \in \mathfrak{U}$ (as there does not appear to be a reasonable notion of HF-small $\infty$-category) and read "$n + 1$" as $\infty$.

An $n$-category (which in this article means weak $n$-category) consists of two parts. Firstly, a collection of $r$-homsets, for $0 \leq r < n+1$. More precisely we have

- the 0-homset $\mathcal{C}(\cdot)$, i.e. set of objects
Figure 1: Properties of an $n$-category, in order of increasing liberality

- for any $a_0, b_0 \in C()$, the 1-homset $C(a_0, b_0)$
- for any $a_0, b_0 \in C()$ and $a_1, b_1 \in C(a_0, b_0)$, the 2-homset $C(a_0, b_0; a_1, b_1)$
- and so forth.

Secondly some structure, which we omit. Many definitions have been proposed (see e.g. [Lei02]) and we shall not adopt any particular one. So the statements in this section are merely proposals that we expect to be true for any reasonable notion of (weak) $n$-category.

We shall now define the properties displayed in Figure 1. In so doing we generalize Definitions 3 and 11.

**Definition 12** Let $C$ be an $n$-category.

1. We say $C$ is small when, for $0 \leq r < n + 1$, each $r$-homset is small.

2. Let $0 \leq k \leq n + 1$. We say $C$ is $k$-light when

- for $0 \leq r < k$, each $r$-homset is a class
- for $k \leq r < n + 1$, each $r$-homset is small.

3. Let $k \in \mathbb{N}$. We say $C$ is $k$-moderate when, for $0 \leq r < n + 1$, each $r$-homset is a $k$-class.

As usual the “1-” prefix may be omitted.

Thus “0-moderate” means small, and “$(k + 1)$-moderate” means that, for $0 \leq r < n + 1$, all $r$-cells are $k$-entities.

We generalize Proposition 9 as follows.

**Proposed Theorem 12** For $n$-categories $C$ and $D$, the functor $n$-category $[C, D]$ is

- small if $C$ and $D$ are small
• $k$-light if $\mathcal{C}$ is small and $\mathcal{D}$ is $k$-light

• $k$-moderate if $\mathcal{C}$ is $(k-1)$-moderate and $\mathcal{D}$ is $k$-moderate.

**Proposed Theorem 13**

1. The $(n+1)$-category $n\text{Cat}$ of small $n$-categories is light.

2. The $(n+1)$-category $n\text{CAT}_k$ of $k$-light $n$-categories is 2-moderate.

3. The $(n+1)$-category $n\text{CAT}_k$ of $k$-moderate $n$-categories is $(k+1)$-moderate.

The case $n = 0$ of Proposed Theorem 13 consists of familiar facts:

• Set is light.

• Class is 2-moderate.

• Class$_k$ is $(k+1)$-moderate.

The case $n = 1$ answers our initial question:

• Cat is light.

• CAT is 2-moderate.

• CAT$_k$ is $(k+1)$-moderate.

Another useful case, for finite $n$, is that $n\text{CAT} \overset{\text{def}}{=} n\text{CAT}_n$ is $(n+1)$-moderate.

As for the notion of $k$-lightness, the following illustrates its significance.

**Proposed Theorem 14** Let $\mathcal{C}$ be an $n$-category. Then the $(n+1)$-category $\text{Span}(\mathcal{C})$ is

• small if $\mathcal{C}$ is small

• $(k+1)$-light if $\mathcal{C}$ is $k$-light

• $k$-moderate if $\mathcal{C}$ is $k$-moderate.
5 Standard By Default

As we have seen, in certain situations where two or more universes are commonly used, one suffices. This simplifies categorical writing: we can work with a single universe parameter and leave it implicit, as we have done. Only when we genuinely want more than one, or to choose an appropriate one using the Universe Axiom, would we mention universes explicitly.

Nonetheless, our terminology is still too verbose. Consider the following passage:

A light category $\mathcal{C}$ consists of a class $\text{ob}\ C$ and family of small sets $(\mathcal{C}(a, b))_{a, b \in \mathcal{C}}$ with composition and identities. An example is the light category of small groups, which has all small limits. Another is given by the well-ordered class of small ordinals. Any small poset or small monoid gives a small category, and any light category $\mathcal{C}$ gives a 2-moderate category $[\mathcal{C}^\text{op}, \text{Set}]$.

Light categories form a 2-moderate 2-category. There is also the 2-light 2-category of small sets and small spans. Finally we may consider the light $(\infty, 1)$-category of small $\infty$-groupoids. It contains the fundamental $\infty$-groupoid of every small topological space.

This passage illustrates the convention we have used so far, which may be called Unrestricted By Default. Every set, category etc. mentioned is unrestricted, unless we specify some relationship with $\mathcal{U}$. This convention has served us well during our exploration of such relationships. But it is unsuitable for ordinary writing, where size issues are not the main subject and should obtrude as little as possible.

To resolve this situation, we introduce the following terminology.

**Definition 13** A mathematical entity is described as $\mathcal{U}$-standard, or standard for short, according to the following rules.

- A set, monoid, topological space, poset, family\(^3\), graph\(^4\), diagram\(^5\), cardinal, ordinal etc. is standard when it is small.

\(^2\)An $(\infty, n)$-category is an $\infty$-category where, for all $k > n$, the $k$-cells are weakly invertible. Several definitions have been proposed; see e.g. [BR13].

\(^3\)In the sense of a pair $(I, (a_i)_{i\in I})$, where $I$ is a set.

\(^4\)A graph (more precisely called a quiver) consists of a set $V$ of vertices, a set $E$ of edges, and source and target functions $s, t: E \rightarrow V$.

\(^5\)In the sense of a pair $(\mathcal{I}, D: \mathcal{I} \rightarrow \mathcal{C})$, where $\mathcal{I}$ is a graph.
• A category, groupoid, multicategory, locally ordered category etc. is standard when it is light.

• For $2 \leq n < \infty$, an $n$-category is standard when it is $n$-moderate.

• An $\infty$-groupoid is standard when it is small.

• An $(\infty, 1)$-category is standard when it is light.

• For $2 \leq n < \infty$, an $(\infty, n)$-category is standard when it is $n$-moderate.

• A function, relation, subset, functor, natural transformation etc. is always standard.

Definition 13 is open-ended and based purely on convenience. It gives rise to a Standard By Default convention: every entity is assumed to be standard, unless specified otherwise. If we want to say that a set is not assumed to be small, we describe it as “unrestricted” or “large”. If we want to say that a category is not assumed to be light, we describe it as “unrestricted” or “heavy”.

Here is a Standard By Default translation of the above passage:

A category $\mathcal{C}$ consists of a class $\text{ob} \mathcal{C}$ and family of sets $(\mathcal{C}(a, b))_{a, b \in \mathcal{C}}$ with composition and identities. An example is the category of groups, which has all limits. Another is given by the well-ordered class of ordinals. Any poset or monoid gives a small category, and any category $\mathcal{C}$ gives a 2-moderate category $[\mathcal{C}^{\text{op}}, \text{Set}]$.

Categories form a 2-category. There is also the 2-light 2-category of sets and spans. Finally we may consider the $(\infty, 1)$-category of $\infty$-groupoids. It contains the fundamental $\infty$-groupoid of every topological space.

Arguably this is close to current practice and not too onerous. But the problem remains of interfacing with ordinary writing about groups, topological spaces, ordinals, $\infty$-groupoids etc. Such writing has no universe parameter and therefore uses the Unrestricted By Default convention. The clash of conventions must be handled carefully, whether or not the Universe Axiom is assumed.

We finish by using Standard By Default to easily formulate an example from [Shu08]. For a monoidal category $\mathcal{V}$, a $\mathcal{V}$-enriched category $\mathcal{C}$ consists
of a class \( \text{ob} \, C \) and a family of \( V \)-objects \( (C(a, b))_{a, b \in C} \) with composition and identities. We write \( \text{MONCAT} \) for the 2-category of monoidal categories, and \( \text{VCAT} \) for that of \( V \)-enriched categories.

**Proposition 15** The construction \( V \mapsto \text{VCAT} \) is a 2-functor from \( \text{MONCAT} \) to \( 2\text{CAT} \).

6 Conclusion

Using families of representatives and \( k \)-classes, we have formulated several categorical concepts in a way that avoids the need for sophisticated encodings of quotients and tuples. All our definitions and statements are given relative to at most one universe. The Standard By Default convention makes this into a reasonably lightweight framework.

Our treatment is robust in the following sense. ZFC assumes that everything is a set and \( \in \)-well-founded—the von Neumann assumptions. Our formulations, unlike Scott’s trick and Scott-McCarty pairing, do not rely on these assumptions. So they are suitable for those who adopt a weaker set theory that, for example, may allow class-many urelements or Quine atoms\(^6\).

The notion of 2-class, i.e. set of 1-entities, has been especially useful. We have made use of \( n\text{Cat}_2 \) and its \( n = 0 \) case \( \text{Class}_2 \), but not of the fact that they are 3-moderate. It would be interesting to know whether any 4-class, or the notions of 3-class or 2-entity, appear in a significant concept or theorem.

**Acknowledgements** I thank Ohad Kammar for helpful discussion. I also thank Eduardo Dubuc, Thomas Streicher and Richard Williamson for explaining a curious claim in [GV64, page 3] that, for a small category \( C \), the functor category \( [C, \text{Set}] \) is neither moderate nor locally small. This arises from the practice of tagging every function with its domain and codomain, and likewise every functor and natural transformation. By not adopting that practice, the problem is avoided.

\(^6\)A Quine atom is a set \( x \) that is equal to \( \{x\} \).
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