A special value of the spectral zeta function of the non-commutative harmonic oscillators

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Abstract: The non-commutative harmonic oscillator is a $2 \times 2$-system of harmonic oscillators with a non-trivial correlation. We write down explicitly the special value at $s = 2$ of the spectral zeta function of the non-commutative harmonic oscillator in terms of the complete elliptic integral of the first kind, which is a special case of a hypergeometric function.

1 Introduction

The non-commutative harmonic oscillator $Q = Q(x, \partial_x)$ is defined to be the second-order ordinary differential operator

$$Q(x, \partial_x) = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} \left( -\frac{\partial_x^2}{2} + \frac{x^2}{2} \right) + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} (x \partial_x + \frac{1}{2}).$$

The first term is two harmonic oscillators, which are mutually independent, with the scaling constant $\alpha > 0$ and $\beta > 0$, while the second term is considered to be the correlation with a self-adjoint manner. The spectral problem is a $2 \times 2$ system of the ordinary differential equations

$$Q(x, \partial_x)u(x) = \lambda u(x)$$

with an eigenstate $u(x) = \begin{bmatrix} u_1(x) \\ u_2(x) \end{bmatrix} \in L^2(\mathbb{R}) \otimes^2$ and a spectrum $\lambda \in \mathbb{R}$. It is known [7] that under the natural assumption $\alpha \beta > 1$ on the positivity, which is also assumed in this paper, the operator $Q$ defines a positive, self-adjoint operator with a discrete spectrum

$$(0 <) \lambda_1 \leq \lambda_2 \leq \cdots \to +\infty$$

The corresponding spectral zeta function is defined to be

$$\zeta_Q(s) = \sum_{n=1}^{\infty} \lambda_n^{-s}.$$ 

An expression of the special value $\zeta_Q(2)$ is obtained in [2] in terms of a certain contour integral using the solution of a singly confluent type Heun differential equation. It would be indicated that these special values are complicated enough and highly transcendental as reflecting the transcendence of the spectra of the non-commutative harmonic oscillator.

* The research of the author is supported in part by a Grant-in-Aid for Scientific Research (B) 15340005 from the Ministry of Education, Culture, Sports, Science and Technology.

Mathematics Subject Classification; Primary 11M36, Secondary 33C20, 33C75.

Keywords and Phrases: Heun’s equation, spectral zeta, special values, harmonic oscillator.

Abbreviated title: A special value of spectral zeta.
However, in this paper, we prove the following simple expression:

\[
\zeta_Q(2) = \frac{\pi^2}{4} \frac{(\alpha^{-1} + \beta^{-1})^2}{(1 - \alpha^{-1} \beta^{-1})} \left( 1 + \left( \frac{\alpha^{-1} - \beta^{-1}}{\alpha^{-1} + \beta^{-1}} \right)^2 \right) \frac{1}{2} \frac{3}{1} \frac{1}{1 - \alpha \beta} 2F_1 \left( \frac{1}{4}, \frac{3}{4}, 1; 1 - \alpha \beta \right).
\]

(1)

where \(2F_1\) is the Gauss hypergeometric series. We also derive the following representation which involves the complete elliptic integral of the first kind as

\[
\zeta_Q(2) = \frac{\pi^2}{4} \frac{(\alpha^{-1} + \beta^{-1})^2}{(1 - \alpha^{-1} \beta^{-1})} \left( 1 + \left( \frac{\alpha^{-1} - \beta^{-1}}{\alpha^{-1} + \beta^{-1}} \right) \int_0^{2\pi} \frac{d\theta}{2\pi \sqrt{1 + (\cos \theta)/\sqrt{1 - \alpha \beta}}} \right)^2.
\]

(2)

In this sense, the special value \(\zeta_Q(2)\) is written in terms of a hypergeometric series, which is more tractable and many of its properties are known. Note that each spectrum is related with the monodromy problem of Heun’s differential equation, which is far from hypergeometric, see [4], [5]. Only the total of spectra has an extra simple form, in some sense.

In Section 2, we recall the expression of \(\zeta_Q(2)\) given in [2], and derive more explicit formula of the generating function appearing in that expression. We prove in Section 3 our main results, the equations (1) and (2). The proof depends on several formulae of hypergeometric series not only for \(2F_1\) but also for \(3F_2\) such as Clausen’s identity.

2 An expression of the generating function

We start from the series-expression of the special value \(\zeta_Q(2)\) of the non-commutative harmonic oscillator given in [2] (4.5a)]

\[
\zeta_Q(2) = Z_1(2) + \sum_{n=0}^{\infty} Z_n'(2).
\]

We introduce notations. Recall that \(\alpha > 0, \beta > 0\) with \(\alpha \beta > 1\). Let us introduce the parameters \(\gamma = 1/\sqrt{\alpha \beta}\) and \(a = \gamma/\sqrt{1 - \gamma^2} = 1/\sqrt{\alpha \beta - 1}\) as in [2] (4.1)]. Note that they satisfy \(0 < \gamma < 1\) and \(a > 0\).

The term \(Z_1(2)\) is given in [2] (4.5b)] and \(Z_n'(2)\) are given in [2] (4.9)] as

\[
Z_1(2) = \frac{(\alpha^{-1} + \beta^{-1})^2}{2(1 - \gamma^2)} - 3\zeta(2),
\]

(3)

\[
Z_n'(2) = (-1)^n \frac{(\alpha^{-1} - \beta^{-1})^2}{(1 - \gamma^2)} \binom{2n - 1}{n} \left( \frac{a}{2} \right)^{2n} J_n.
\]

(4)

The values \(\{J_n\}_{n=1,2,\ldots}\) are specified by the generating function

\[
w(z) := \sum_{n=0}^{\infty} J_n z^n.
\]

The function \(w(z)\) is a solution of the ordinary differential equation

\[
z(1 - z) \frac{d^2 w}{dz^2} + (1 - 3z)(1 - z) \frac{dw}{dz} + \left( z - \frac{3}{4} \right) w = 0
\]

(5)

which is given in [2] Theorem 4.13] and called a singly confluent Heun’s differential equation. The constant term is given by \(w(0) = J_0 = 3\zeta(2) = \pi^2/2\). It is easy to see that there exists a unique power-series
solution of this homogeneous differential equation with the initial condition \( w(0) = \pi^2/2 \). The final target \( \zeta_Q(2) \) involves these \( J_n \)'s with an infinite sum which does not seem to have a closed form.

In this section, we give a simple expression of the generating function \( w(z) \). We denote by \( \partial_z = \partial/\partial z \).

**Lemma 1** The differential equation (5) is equivalent to

\[
4(1-z)\partial_z z \partial_z (1-z)w + w = 0.
\]

Proof: This directly follows from Leibniz rule. QED

**Lemma 2** Let \( t = z/(z-1) \) be a new independent variable, and \( \eta(t) = (1-z)w(z) \) a new unknown function. Then the differential equation (6) is equivalent to

\[
t(1-t)\partial_t^2 \eta + (1-2t)\partial_t \eta - \frac{1}{4} \eta = 0.
\]

Proof: The differential equation (6) is equivalent to

\[
4(z-1)^2 \partial_z z \partial_z (z-1)w + (z-1)w = 0.
\]

Note that \((z-1)(t-1) = 1\) and \( \partial_t := \partial/\partial t = -(z-1)^2 \partial_z \). Then

\[4\partial_t t(t-1)\partial_t \eta + \eta = 0.\]

By Leibniz rule, this is equivalent to (7). QED

**Proposition 3**

\[w(z) = \frac{J_0}{1-z} {}_2F_1 \left( \frac{1}{2} \frac{1}{2} ; 1 ; \frac{z}{z-1} \right).\]

Proof: Since any power-series solution of (7) in \( t \) is a constant multiple of \( {}_2F_1 \left( \frac{1}{2} \frac{3}{4} ; 1 ; -a^2 \right) \), we have the conclusion. QED

### 3 The special value

We introduce the auxiliary series

\[g(a) := \frac{2}{J_0} \sum_{n=0}^{\infty} (-1)^n \left( \frac{2n-1}{n} \right) \left( \frac{a}{2} \right)^{2n} J_n\]

so that

\[
\zeta_Q(2) = \frac{(\alpha^{-1} + \beta^{-1})^2}{2(1-\gamma^2)} 3\zeta(2) + \frac{(\alpha^{-1} - \beta^{-1})^2}{2(1-\gamma^2)} 3\zeta(2) g(a)
\]

\[= \frac{\pi^2}{4} \frac{(\alpha^{-1} + \beta^{-1})^2}{(1-\alpha^{-1}\beta^{-1})^4} \left( 1 + \frac{(\alpha^{-1} - \beta^{-1})^2}{(\alpha^{-1} + \beta^{-1})^2} g(a) \right)\]

**Theorem 4**

\[g(a) = {}_2F_1 \left( \frac{1}{4} \frac{3}{4} ; 1 ; -a^2 \right)^2.\]
Proof: We note that
\[
\binom{2n-1}{n} \left(\frac{1}{2}\right)^{2n} = \frac{1}{2} \times \frac{(2n-1)!!}{(2n)!!} = \frac{1}{2\pi} \int_0^1 \frac{u^n du}{\sqrt{u(1-u)}}.
\]
Then, integration by parts implies that
\[
g(a) = \frac{2}{2\pi J_0} \sum_{n=0}^{\infty} (-1)^n \int_0^1 \frac{u^n du}{\sqrt{u(1-u)}} a^{2n} J_n = \frac{1}{\pi J_0} \int_0^1 w(-a^2 u) du.
\]
By Proposition 3, the function \(w\) is written in terms of hypergeometric series \(2F_1\). Substitute such an expression into the equation (10), to obtain
\[
g(a) = \frac{1}{\pi} \int_0^1 \frac{1}{1 + a^2 u} 2F_1 \left( \frac{1}{2} \frac{1}{2} 1 1; \frac{a^2 u}{a^2 u + 1} \right) \frac{du}{\sqrt{u(1-u)}}.
\]
We introduce a new variable \(v = (1 + a^2)u/(1 + a^2 u)\). Then
\[
g(a) = \frac{1}{\pi} \int_0^1 2F_1 \left( \frac{1}{2} \frac{1}{2} 1 1; a^2 v \right) \frac{dv}{\sqrt{v(1-v)(1 + a^2)}}.
\]
Now we use the formula (2.2.2) of [1]
\[
3F_2(a_1, a_2, a_3; b_1, b_2; x) = \frac{\Gamma(b_2)}{\Gamma(a_3)\Gamma(b_2 - a_3)} \int_0^1 t^{a_3-1} (1 - t)^{b_2-a_3-1} 2F_1(a_1, a_2; b_1; xt) dt.
\]
This shows that
\[
g(a) = \frac{1}{\sqrt{1 + a^2}} 3F_2 \left( \frac{1}{2} \frac{1}{2} 1 1; \frac{1}{1 + a^2} \right).
\]
By Clausen’s identity (in e.g., Exercise 13 of Chapter 2 in [1])
\[
2F_1 \left( a, b; a + b + \frac{1}{2}; x \right)^2 = 3F_2 \left( 2a, 2b, a + b; 2a + 2b, a + b + \frac{1}{2}; x \right),
\]
we find that
\[
g(a) = \frac{1}{\sqrt{1 + a^2}} 2F_1 \left( \frac{1}{4} \frac{1}{4} 1; 1; \frac{a^2}{1 + a^2} \right)^2.
\]
Moreover Pfaff formula’s Theorem 2.2.5 of [1]
\[
2F_1(a, b; c; x) = (1 - x)^{-a} 2F_1(a, c - b; c; x/(x - 1)),
\]
yields
\[
2F_1 \left( \frac{1}{4} \frac{3}{4} 1; -a^2 \right) = (1 + a^2)^{-1/4} 2F_1 \left( \frac{1}{4} \frac{1}{4} 1; \frac{a^2}{a^2 + 1} \right).
\]
This shows that
\[
g(a) = 2F_1 \left( \frac{1}{4} \frac{3}{4} 1; -a^2 \right)^2.
\]
QED
Remark 5 In the earlier version of the paper, it was suggested to make use of the hypergeometric series \( _3F_2 \) with this special parameter \((1/2, 1/2, 1/2; 1, 1)\) by the multi-variable hypergeometric function of type \((3,6)\), especially by its restriction on the stratum called \(X_{1b}\) in [3]. However, we can avoid to use a multi-variable hypergeometric function in the present version as is seen above.

Theorem 4 with the help of the equation (9) shows the equation (1). The equation (2) is shown as follows. By Theorem 3.13 of [1]

\[
\begin{align*}
_2F_1(a, b; 2a; x) &= \left(1 - \frac{x}{2}\right)^{-b} _2F_1\left(\frac{b}{2}, \frac{b+1}{2}; \frac{1}{2} + \frac{1}{2} \left(\frac{x}{2} - \frac{1}{2}\right)^2\right),
\end{align*}
\]

we have

\[
_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{2ia}{ia+1}\right) = (1 + ia)^{1/2} _2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; -a^2\right).
\]

Let us recall the definition of the elliptic integral of the first kind:

\[
K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} = \frac{\pi}{2} _2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; k^2\right).
\]

Then we have

\[
_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; -a^2\right) = \frac{2}{\pi} (1 + ia)^{-1/2} K\left(\frac{2ia}{ia+1}\right) = \frac{2}{\pi} \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 + ia \cos 2\theta}} = \frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{\sqrt{1 + ia \cos \theta}}
\]

and the equation (2).

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