ANALYSIS OF THE MORLEY ELEMENT FOR THE
CAHN-HILLIARD EQUATION AND THE HELE-SHAW FLOW

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Abstract. The paper analyzes the Morley element method for the Cahn-Hilliard equation. The objective is to derive the optimal error estimates and to prove the zero-level sets of the Cahn-Hilliard equation approximate the Hele-Shaw flow. If the piecewise $L^\infty (H^2)$ error bound is derived by choosing test function directly, we cannot obtain the optimal error order, and we cannot establish the error bound which depends on $\frac{1}{\epsilon}$ polynomially either. To overcome this difficulty, this paper proves them by the following steps, and the result in each next step cannot be established without using the result in its previous one. First, it proves some a priori estimates of the exact solution $u$, and these regularity results are minimal to get the main results; Second, it establishes $L^\infty (L^2)$ and piecewise $L^2(H^2)$ error bounds which depend on $\frac{1}{\epsilon}$ polynomially based on the piecewise $L^\infty (H^{-1})$ and $L^2(H^1)$ error bounds; Third, it establishes piecewise $L^\infty (H^2)$ optimal error bound which depends on $\frac{1}{\epsilon}$ polynomially based on the piecewise $L^\infty (L^2)$ and $L^2(H^2)$ error bounds; Finally, it proves the $L^\infty (L^\infty)$ error bound and the approximation to the Hele-Shaw flow based on the piecewise $L^\infty (H^2)$ error bound. The nonstandard techniques are used in these steps such as the generalized coercivity result, integration by part in space, summation by part in time, and special properties of the Morley elements. If one of these techniques is lacked, either we can only obtain the sub-optimal piecewise $L^\infty (H^2)$ error order, or we can merely obtain the error bounds which are exponentially dependent on $\frac{1}{\epsilon}$. The approach used in this paper provides a way to bound the errors in higher norm from the errors in lower norm step by step, which has a profound meaning in methodology. Numerical results are presented to validate the optimal $L^\infty (H^2)$ error order and the asymptotic behavior of the solutions of the Cahn-Hilliard equation.

Key words. Morley element, Cahn-Hilliard equation, generalized coercivity result, $\frac{1}{\epsilon}$ polynomial dependence, Hele-Shaw flow

AMS subject classifications. 65N12, 65N15, 65N30

1. Introduction. Consider the following Cahn-Hilliard equation with Neumann boundary conditions:

(1.1) \[ u_t + \Delta (\epsilon \Delta u - \frac{1}{\epsilon} f(u)) = 0 \quad \text{in} \quad \Omega_T := \Omega \times (0, T], \]

(1.2) \[ \frac{\partial u}{\partial n} = \frac{\partial}{\partial n}(\epsilon \Delta u - \frac{1}{\epsilon} f(u)) = 0 \quad \text{on} \quad \partial \Omega_T := \partial \Omega \times (0, T], \]

(1.3) \[ u = u_0 \quad \text{in} \quad \Omega \times \{t = 0\}, \]

where $\Omega \subseteq \mathbb{R}^2$ is a bounded domain, $f(u) = u^3 - u$ is the derivative of a double well potential $F(u)$ which is defined by

(1.4) \[ F(u) = \frac{1}{4}(u^2 - 1)^2. \]

The Allen-Cahn equation [3, 6, 12, 20, 17, 16, 19, 24] and the Cahn-Hilliard equation [2, 12, 25, 29] are two basic phase field models to describe the phase transition process. They are also proved to be related to geometric flow. For example, the zero-level sets of the Allen-Cahn equation approximate the mean curvature [15, 24] and the zero-level sets of the Cahn-Hilliard equation approximate the Hele-Shaw flow [28, 2]. The Cahn-Hilliard equation was introduced by J. Cahn and J. Hilliard in [11] to describe the

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process of phase separation, by which the two components of a binary fluid separate and form domains pure in each component. It can be interpreted as the $H^{-1}$ gradient flow \cite{2} of the Cahn-Hilliard energy functional

\begin{equation}
J_\epsilon(v) := \int_\Omega \left( \frac{\epsilon}{2} |\nabla v|^2 + \frac{1}{\epsilon} F(v) \right) \, dx.
\end{equation}

There are a few papers \cite{4, 30, 13, 14} discussing the error bounds, which depend on the exponential power of $\frac{1}{\epsilon}$, of the numerical methods for Cahn-Hilliard equation. Such an estimate is clearly not useful for small $\epsilon$, in particular, in addressing the issue whether the computed numerical interfaces converge to the original sharp interface of the Hele-Shaw problem. Instead, the polynomial dependence in $\frac{1}{\epsilon}$ is proved in \cite{21, 22} using the standard finite element method, and in \cite{18, 26} using the discontinuous Galerkin method. Due to the high efficiency of the Morley elements, compared with mixed finite element methods or $C^1$-conforming finite element methods, the Morley finite element method is used to derive the error bound which depends on $\frac{1}{\epsilon}$ polynomially in this paper.

The highlights of this paper are fourfold. First, it establishes the piecewise $L^\infty(L^2)$ and $L^2(H^2)$ error bounds which depend on $\frac{1}{\epsilon}$ polynomially. If the standard technique is used, we can only prove that the error bounds depend on $\frac{1}{\epsilon}$ exponentially, which can not be used to prove our main theorem. To prove these bounds, special properties of the Morley elements are explored, i.e., Lemma 2.3 in \cite{14}, and piecewise $L^\infty(H^{-1})$ and $L^2(H^1)$ error bounds \cite{27} are required. Second, by making use of the piecewise $L^\infty(L^2)$ and $L^2(H^2)$ error bounds above, it establishes the piecewise $L^\infty(H^2)$ error bound which depends on $\frac{1}{\epsilon}$ polynomially. If the standard technique is used, we can only get the error bound in Remark 2, which does not have an optimal order. The crux here is to employ the summation by part in time and integration by part in space techniques simultaneously to handle the nonlinear term, together with the special properties of the Morley elements. Third, the minimal regularity of $u$ is used, i.e., $\|u_{tt}\|_{L^2(L^2)}$ regularity instead of $\|u_{tt}\|_{L^\infty(L^2)}$ regularity is used, and the a priori estimate is derived in Theorem 2.2. Fourth, the $L^\infty(L^\infty)$ error bound is established using the optimal piecewise $L^\infty(H^2)$ error, by which the main result that the zero-level sets of the Cahn-Hilliard equation approximate the Hele-Shaw flow is proved in Section 5.

The organization of this paper is as follows. In Section 2, the standard Sobolev space notation is introduced, some useful lemmas are stated, and a new a priori estimate of the exact solution $u$ is derived. In Section 3, the fully discrete approximation based on the Morley finite element space is presented. In Section 4, first the polynomially dependent piecewise $L^\infty(L^2)$ and $L^2(H^2)$ error bounds are established based on piecewise $L^\infty(H^{-1})$ and $L^2(H^1)$ error bounds, then the polynomially dependent piecewise $L^\infty(H^2)$ error bound is established based on piecewise $L^\infty(L^2)$ and $L^2(H^2)$ error bounds, by which the $L^\infty(L^\infty)$ error bound is proved. In Section 5, the approximation of the zero-level sets of the Cahn-Hilliard equation of the Hele-Shaw flow is proved. In Section 6, numerical tests are presented to validate our theoretical results, including the optimal error orders and the approximation of the Hele-Shaw flow.

2. Preliminaries. In this section, we present some results which will be used in the following sections. Throughout this paper, $C$ denotes a generic positive constant which is independent of interfacial length $\epsilon$, spacial size $h$, and time step size $k$, and it may have different values in different formulas. The standard Sobolev space notation below is used in this paper.
\[\|v\|_{0,p,A} = \left( \int_A |v|^p \, dx \right)^{1/p}, \quad 1 \leq p < \infty,\]
\[\|v\|_{0,\infty,A} = \operatorname{ess sup}_A |v|,\]
\[|v|_{m,p,A} = \left( \sum_{|\alpha|=m} \|D^\alpha v\|_{0,p,A}^p \right)^{1/p}, \quad 1 \leq p < \infty,\]
\[\|v\|_{m,p,A} = \left( \sum_{j=0}^m |v|_{m,p,A}^p \right)^{1/p}.
\]

Here \(A\) denotes some domain, i.e., a single mesh element \(K\) or the whole domain \(\Omega\). When \(A = \Omega\), \(\|\cdot\|_{H^k}, \|\cdot\|_{L^k}\) are used to denote \(\|\cdot\|_{H^k(\Omega)}, \|\cdot\|_{L^k(\Omega)}\) respectively, and \(\|\cdot\|_{0,2}\) is also used to denote \(\|\cdot\|_{L^2(\Omega)}\). Let \(T_h\) be a family of quasi-uniform triangulations of domain \(\Omega\), and \(E_h\) be a collection of edges, then the global mesh dependent semi-norm, norm and inner product are defined below
\[|v|_{j,p,h} = \left( \sum_{K \in T_h} |v|_{j,p,K}^p \right)^{1/p},\]
\[\|v\|_{j,p,h} = \left( \sum_{K \in T_h} \|v\|_{j,p,K}^p \right)^{1/p},\]
\[(w,v)_h = \sum_{K \in T_h} \int_K w(x)v(x) \, dx.\]

Define \(L^2_0(\Omega)\) as the mean zero functions in \(L^2(\Omega)\). For \(\Phi \in L^2_0(\Omega)\), let \(u := -\Delta^{-1}\Phi \in H^1(\Omega) \cap L^2_0(\Omega)\) such that
\[-\Delta u = \Phi \quad \text{in} \quad \Omega,\]
\[\frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial \Omega.\]

Then we have
\[-(\nabla \Delta^{-1}\Phi, \nabla v) = (\Phi, v) \quad \text{in} \quad \Omega \quad \forall v \in H^1(\Omega) \cap L^2_0(\Omega).\]

For \(v \in L^2_0(\Omega)\) and \(\Phi \in L^2_0(\Omega)\), define the continuous \(H^{-1}\) inner product by
\[(\Phi, v)_{H^{-1}} := (\nabla \Delta^{-1}\Phi, \nabla \Delta^{-1}v) = (\Phi, -\Delta^{-1}v) = (v, -\Delta^{-1}\Phi).\]

As in [12, 18, 21, 22, 26, 27], we made the following assumptions on the initial condition. These assumptions were used to derive the a priori estimates for the solution of problem (1.1)–(1.4).

**General Assumption (GA)**

(1) Assume that \(m_0 \in (-1, 1)\) where
\[m_0 := \frac{1}{|\Omega|} \int_{\Omega} u_0(x) \, dx.\]

(2) There exists a nonnegative constant \(\sigma_1\) such that
\[J_c(u_0) \leq C e^{-2\sigma_1}.\]
(3) There exist nonnegative constants $\sigma_2$, $\sigma_3$ and $\sigma_4$ such that
\[ \| -\epsilon \Delta u_0 + \epsilon^{-1} f(u_0) \|_{H^\ell} \leq C\epsilon^{-\sigma_2 + \ell} \quad \ell = 0, 1, 2. \]

Under the above assumptions, the following a priori estimates of the solution were proved in [18, 21, 22, 26].

**Theorem 2.1.** The solution $u$ of problem (1.1)–(1.4) satisfies the following energy estimate:

\[ \text{ess sup} \int_{0}^{T} \left( \frac{\epsilon}{2} \| \nabla u(t) \|_{L^2}^2 + \frac{1}{\epsilon} \| F(u(t)) \|_{L^1} \right) + \int_{0}^{T} \| u_t(t) \|_{H^{-1}}^2 \, dt \leq J_\epsilon(u_0). \]  

Moreover, suppose that GA (1)–(3) hold, $u_0 \in H^4(\Omega)$ and $\partial \Omega \in C^{2,1}$, then $u$ satisfies the additional estimates:

\[ \text{ess sup} \int_{0}^{T} \| \Delta u(t) \|_{L^2} \leq C\epsilon^{-\max\{\sigma_1 + \frac{3}{2}, \sigma_3 + 1\}}, \]
\[ \text{ess sup} \int_{0}^{T} \| \nabla \Delta u(t) \|_{L^2} \leq C\epsilon^{-\max\{\sigma_1 + \frac{5}{2}, \sigma_3 + 1\}}, \]
\[ \epsilon \int_{0}^{T} \| \Delta u_t(t) \|_{L^2} \, dt + \text{ess sup} \int_{0}^{T} \| u(t) \|_{L^2}^2 \leq C\epsilon^{-\max\{2\sigma_1 + \frac{13}{2}, 2\sigma_3 + 2\sigma_2 + 4, 2\sigma_4\}}. \]

Furthermore, if there exists $\sigma_5 > 0$ such that

\[ \lim_{s \to 0^+} \| \nabla u_t(t) \|_{L^2} \leq C\epsilon^{-\sigma_5}, \]

then there hold

\[ \text{ess sup} \int_{0}^{T} \| \nabla u_t(t) \|_{L^2}^2 + \epsilon \int_{0}^{T} \| \nabla \Delta u(t) \|_{L^2}^2 \, dt \leq C\rho_0(\epsilon), \]
\[ \int_{0}^{T} \| u(t) \|_{H^{-1}}^2 \, dt \leq C\rho_1(\epsilon), \]

where

\[ \rho_0(\epsilon) := \epsilon - \frac{1}{4} \max\{2\sigma_1 + 5, 2\sigma_3 + 2\} + \max\{2\sigma_1 + \frac{13}{4}, 2\sigma_3 + 2\sigma_2 + 4\} + \epsilon^{-2\sigma_5}, \]
\[ \rho_1(\epsilon) := \epsilon \rho_0(\epsilon). \]

Besides, an extra a priori estimates of solution $u$ is needed in this paper.

**Theorem 2.2.** Under the assumptions of Theorem 2.1 and if there exists $\sigma_6 > 0$ such that

\[ \| \Delta u_0(0) \|_{L^2} \leq C\epsilon^{-\sigma_6}, \]

then there hold

\[ \text{ess sup} \int_{0}^{T} \| \Delta u(t) \|_{L^2}^2 + \epsilon \int_{0}^{T} \| \Delta^2 u(t) \|_{L^2}^2 \, dt \leq C\rho_2(\epsilon), \]
\[ \epsilon \text{ess sup} \int_{0}^{T} \| u(t) \|_{L^2}^2 \, dt + \int_{0}^{T} \| u(t) \|_{L^2}^2 \, dt \leq C\rho_3(\epsilon), \]
where
\[
\rho_2(\epsilon) := \epsilon^{-\max\{2\sigma_1 + \frac{12}{5}, 2\sigma_3 + \frac{7}{2}, 2\sigma_2 + 4, 2\sigma_4\} - \max\{2\sigma_1 + 5, 2\sigma_3 + 2\} - 3} \\
\quad + \epsilon^{-\max\{\sigma_1 + \frac{5}{2}, \sigma_3 + 1\} - 3} \rho_0(\epsilon) + \epsilon^{-2\sigma_6}.
\]
\[
\rho_3(\epsilon) := \epsilon \rho_2(\epsilon).
\]

**Proof.** Using the Gagliardo-Nirenberg inequalities [1] in two-dimensional space, we have
\[
\|\nabla u\|_{L^\infty} \leq C \left( \|\nabla \Delta u\|_{L^2}^{\frac{3}{2}} \|u\|_{L^\infty}^{\frac{1}{2}} + \|u\|_{L^\infty} \right) \leq C \epsilon^{-\frac{1}{2}} \max\{\sigma_1 + \frac{5}{2}, \sigma_3 + 1\}.
\]
Since \(f'(u) = 3u^2 - 1\), using Sobolev embedding theorem [1], (2.3), (2.5), (2.6), (2.7) and (2.9), we have
\[
\int_0^T \|\Delta(f'(u)u_t)\|_{L^2}^2 \, ds \\
= \int_0^T \|6u_t \Delta u + 12u \nabla u \cdot \nabla u_t + 6u_t \nabla u \cdot \nabla u + (3u^2 - 1) \Delta u_t\|_{L^2}^2 \, ds \\
\leq C \int_0^T \|\Delta u\|_{L^\infty}^2 \|u_t\|_{L^\infty}^2 \, ds + C \int_0^T \|\nabla u_t\|_{L^2}^2 \, ds + C \int_0^T \|\Delta u_t\|_{L^2}^2 \, ds \\
\leq C \|\Delta u\|_{L^\infty(L^\infty)}^2 \int_0^T \|u_t\|_{H^2}^2 \, ds + C \|\nabla u_t\|_{L^\infty(L^\infty)}^2 \|\nabla u\|_{L^\infty(L^\infty)}^2 \\
+ C \|\nabla u_t\|_{L^\infty(L^\infty)}^2 \|u_t\|_{L^\infty(L^\infty)}^2 \, ds + C \int_0^T \|\Delta u_t\|_{L^2}^2 \, ds \\
\leq C \epsilon^{-\max\{2\sigma_1 + \frac{12}{5}, 2\sigma_3 + \frac{7}{2}, 2\sigma_2 + 4, 2\sigma_4\} - \max\{2\sigma_1 + 5, 2\sigma_3 + 2\} - 1} \\
+ C \epsilon^{-\max\{\sigma_1 + \frac{5}{2}, \sigma_3 + 1\} - \max\{2\sigma_1 + 5, 2\sigma_3 + 2\} - 1} \\
\quad + C \epsilon^{-\max\{2\sigma_1 + \frac{12}{5}, 2\sigma_3 + \frac{7}{2}, 2\sigma_2 + 4, 2\sigma_4\} - \max\{2\sigma_1 + 5, 2\sigma_3 + 2\} - 1} \\
\quad + C \epsilon^{-\max\{2\sigma_1 + \frac{12}{5}, 2\sigma_3 + \frac{7}{2}, 2\sigma_2 + 4, 2\sigma_4\} - \max\{2\sigma_1 + 5, 2\sigma_3 + 2\} - 1} \\
\quad + C \epsilon^{-\max\{\sigma_1 + \frac{5}{2}, \sigma_3 + 1\} - \max\{2\sigma_1 + 5, 2\sigma_3 + 2\} - 1} \\
\quad + C \epsilon^{-\max\{\sigma_1 + \frac{5}{2}, \sigma_3 + 1\} - \max\{2\sigma_1 + 5, 2\sigma_3 + 2\} - 1} \\
\quad + C \epsilon^{-\max\{\sigma_1 + \frac{5}{2}, \sigma_3 + 1\} - \max\{2\sigma_1 + 5, 2\sigma_3 + 2\} - 1} \\
\quad + C \epsilon^{-\max\{\sigma_1 + \frac{5}{2}, \sigma_3 + 1\} - \max\{2\sigma_1 + 5, 2\sigma_3 + 2\} - 1} \\
\quad + C \epsilon^{-\max\{\sigma_1 + \frac{5}{2}, \sigma_3 + 1\} - \max\{2\sigma_1 + 5, 2\sigma_3 + 2\} - 1}.
\]

Taking the derivative with respect to \(t\) on both sides of (1.1), we get
\[
u_t + \epsilon \Delta^2 u_t - \frac{1}{\epsilon} \Delta(f'(u)u_t) = 0.
\]
Testing (2.16) with \(\Delta^2 u_t\), and taking the integral over \((0, T)\), we obtain
\[
\frac{1}{2} \|\Delta u_t(T)\|_{L^2}^2 + \epsilon \int_0^T \|\Delta^2 u_t\|_{L^2}^2 \, ds \\
= \frac{1}{\epsilon} \int_0^T \langle \Delta(f'(u)u_t), \Delta^2 u_t \rangle \, ds + \frac{1}{2} \|\Delta u_t(0)\|_{L^2}^2 \\
\leq \frac{C}{\epsilon^3} \int_0^T \|\Delta(f'(u)u_t)\|_{L^2}^2 \, ds + \frac{\epsilon}{2} \int_0^T \|\Delta^2 u_t\|_{L^2}^2 \, ds + C \epsilon^{-2\sigma_6}.
\]
Then (2.12) is obtained by (2.15).

Next we bound (2.13). Testing (2.16) with \(u_{tt}\), taking the integral over \((0, T)\), and using (2.17), we obtain

\[
\int_0^T \|u_{tt}\|^2_{L^2} \, ds + \frac{\epsilon}{2} \|\Delta u_t(T)\|^2_{L^2} \leq \frac{\epsilon}{2} \|\Delta u_t(0)\|^2_{L^2} + \frac{C}{\epsilon^2} \int_0^T \|\Delta (f'(u)u_t)\|^2_{L^2} \, ds + \frac{1}{2} \int_0^T \|u_{tt}\|^2_{L^2} \, ds.
\]

Then (2.13) is obtained by (2.15). \(\square\)

The next lemma gives an \(\epsilon\)-independent lower bound for the principal eigenvalue of the linearized Cahn-Hilliard operator \(L_{CH}\) defined below. The proof of this lemma can be found in [12].

**Lemma 2.3.** Suppose that GA (1)–(3) hold. Given a smooth initial curve/surface \(\Gamma_0\), let \(u_0\) be a smooth function satisfying \(\Gamma_0 = \{x \in \Omega; u_0(x) = 0\}\) and some profile described in [12]. Let \(u\) be the solution to problem (1.1)–(1.4). Define \(L_{CH}\) as

\[
L_{CH} := \Delta \left( \epsilon \Delta - \frac{1}{\epsilon} f'(u) I \right).
\]

Then there exists \(0 < \epsilon_0 \ll 1\) and a positive constant \(C_0\) such that the principle eigenvalue of the linearized Cahn-Hilliard operator \(L_{CH}\) satisfies

\[
\lambda_{CH} := \inf_{0 \neq \psi \in H^1(\Omega) \setminus \{0\}} \frac{\epsilon\|\nabla \psi\|^2_{L^2} + \frac{1}{\epsilon} (f'(u) \psi, \psi)}{\|\nabla w\|^2_{L^2}} \geq -C_0
\]

for \(t \in [0, T]\) and \(\epsilon \in (0, \epsilon_0)\).

### 3. Fully Discrete Approximation.

In this section, the backward Euler is used for time stepping, and the Morley finite element discretization is used for space discretization.

#### 3.1. Morley finite element space.

Define the Morley finite element spaces \(S^h\) below [8, 10, 14]:

\[
S^h := \{v_h \in L^\infty(\Omega); v_h \in P_2(K), v_h\text{ is continuous at the vertices of all triangles, } \frac{\partial v_h}{\partial n}\text{ is continuous at the midpoints of interelement edges of triangles}\}.
\]

We use the following notation

\[
H^j_E(\Omega) := \{v \in H^j(\Omega); \frac{\partial v}{\partial n} = 0 \text{ on } \partial\Omega\} \quad j = 1, 2, 3.
\]

Corresponding to \(H^j_E(\Omega)\), define \(S^h_E\) as a subspace of \(S^h\) below:

\[
S^h_E := \{v_h \in S^h; \frac{\partial v_h}{\partial n} = 0 \text{ at the midpoints of the edges on } \partial\Omega\}.
\]

We also define \(H^j_E(\Omega) = H^j_E(\Omega) \cap L^2_0(\Omega), j = 1, 2, 3\), and \(S^h_E = S^h_E \cap L^2_0(\Omega)\), where \(L^2_0(\Omega)\) denotes the set of mean zero functions.
The enriching operator \( \tilde{E}_h \) is restated \([7, 8, 10]\). Let \( \tilde{S}_E^h \) be the Hsieh-Clough-Tocher macro element space, which is an enriched space of the Morley finite element space \( S_E^h \). Let \( p \) and \( m \) be the internal vertices and midpoints of triangles \( T_h \). Define \( \tilde{E}_h : S_E^h \to \tilde{S}_E^h \) by

\[
(\tilde{E}_h v)(p) = v(p), \quad \frac{\partial (\tilde{E}_h v)}{\partial n}(m) = \frac{\partial v}{\partial n}(m), \quad (\partial^2 (\tilde{E}_h v))(p) = \text{average of } (\partial^2 v_i)(p) \quad |\beta| = 1,
\]

where \( v_i = v|_{T_i} \) and triangle \( T_i \) contains \( p \) as a vertex.

Define the interpolation operator \( I_h : H^2_E(\Omega) \to S_E^h \) such that

\[
(I_h v)(p) = v(p), \quad \frac{\partial (I_h v)}{\partial n}(m) = \frac{1}{|e|} \int_e \frac{\partial v}{\partial n} \, dS,
\]

where \( p \) ranges over the internal vertices of all the triangles \( T \), and \( m \) ranges over the midpoints of all the edges \( e \). It can be proved that \([7, 8, 10, 14]\)

\[
\begin{align*}
(3.1) \quad |v - I_h v|_{j,p,K} & \leq Ch^{3-j} |v|_{3,p,K} \quad \forall K \in \mathcal{T}_h, \quad \forall v \in H^3(K), \quad j = 0, 1, 2, \\
(3.2) \quad \|\tilde{E}_h v - v\|_{j,2,h} & \leq Ch^{2-j} |v|_{2,2,h} \quad \forall v \in S_E^h, \quad j = 0, 1, 2.
\end{align*}
\]

Notice that \( \tilde{E}_h \) and \( I_h \) cannot preserve the mean zero functions. Let \( \hat{S}_E^h := \tilde{S}_E^h \cap L^2_0(\Omega) \). Define \( \hat{E}_h : \hat{S}_E^h \to \hat{S}_E^h \) such that

\[
\hat{E}_h v = \tilde{E}_h v - \frac{1}{|\Omega|} \int_\Omega \tilde{E}_h v \, dx.
\]

Using (3.2), we have

\[
\int_\Omega \tilde{E}_h v \, dx = (\tilde{E}_h v - v, 1) \leq |\Omega|^{1/2} \|\tilde{E}_h v - v\|_{0,2} \leq Ch^{2} |v|_{2,2,h} \quad \forall v \in \hat{S}_E^h.
\]

Then

\[
\|\tilde{E}_h v - v\|_{j,2,h} \leq Ch^{2-j} |v|_{2,2,h} \quad \forall v \in \hat{S}_E^h, \quad j = 0, 1, 2.
\]

Finally the following spaces are needed

\[
\begin{align*}
H^{3,h}(\Omega) &= S^h \oplus H^3(\Omega), & H^{3,h}_E(\Omega) &= S_E^h \oplus H^3_E(\Omega), \\
H^{2,h}(\Omega) &= S^h \oplus H^2(\Omega), & H^{2,h}_E(\Omega) &= S_E^h \oplus H^2_E(\Omega), \\
H^{1,h}(\Omega) &= S^h \oplus H^1(\Omega), & H^{1,h}_E(\Omega) &= S_E^h \oplus H^1_E(\Omega),
\end{align*}
\]

where, for instance,

\[
S_E^h \oplus H^3_E(\Omega) := \{ u + v : u \in S_E^h \text{ and } v \in H^3_E(\Omega) \}.
\]
3.2. Formulation. The weak form of (1.1)–(1.4) is to seek $u(\cdot, t) \in H^2_E(\Omega)$ such that

$$
(u_t, v) + \epsilon a(u, v) + \frac{1}{\epsilon} (\nabla f(u), \nabla v) = 0 \quad \forall v \in H^2_E(\Omega),
$$

$$
(u(\cdot, 0), v) = u_0 \in H^2_E(\Omega),
$$

where the bilinear form $a(\cdot, \cdot)$ is defined as

$$
a(u, v) := \int_\Omega \Delta u \Delta v + \left( \frac{\partial^2 u}{\partial x \partial y} \frac{\partial^2 v}{\partial x \partial y} - \frac{1}{2} \frac{\partial^2 u}{\partial x^2} \frac{\partial^2 v}{\partial y^2} - \frac{1}{2} \frac{\partial^2 u}{\partial y^2} \frac{\partial^2 v}{\partial x^2} \right) dx dy
$$

with Poisson's ratio $\frac{1}{2}$.

Next define the discrete bilinear form

$$
a_h(u, v) := \sum_{K \in T_h} \int_K \Delta u \Delta v + \left( \frac{\partial^2 u}{\partial x \partial y} \frac{\partial^2 v}{\partial x \partial y} - \frac{1}{2} \frac{\partial^2 u}{\partial x^2} \frac{\partial^2 v}{\partial y^2} - \frac{1}{2} \frac{\partial^2 u}{\partial y^2} \frac{\partial^2 v}{\partial x^2} \right) dx dy.
$$

Based on the bilinear form (3.8), a fully discrete Galerkin method is to seek $u^n_h \in S^n_E$ such that

$$
(d_t u^n_h, v_h) + \epsilon a_h(u^n_h, v_h) + \frac{1}{\epsilon} (\nabla f(u^n_h), \nabla v_h)_h = 0 \quad \forall v_h \in S^n_E,
$$

$$
\left. u^n_h = u^n_0 \right|_{h} \in S^n_E,
$$

where the difference operator $d_t u^n_h := \frac{u^n_h - u^{n-1}_h}{k}$ and $u^n_0 := P_h u(t_0)$, where the operator $P_h$ is defined below.

3.3. Elliptic operator $P_h$. We define

$$
R := \{ v \in H^2_E(\Omega) : \Delta v \in H^2_E(\Omega) \}.
$$

Then $\forall v \in R$, define the elliptic operator $P_h$ (cf. [14]) by seeking $P_h v \in S^n_E$ such that

$$
\tilde{b}_h(P_h v, w) := (\epsilon \Delta^2 v - \frac{1}{\epsilon} \nabla \cdot (f'(u) \nabla v) + \alpha v, w) \quad \forall w \in S^n_E,
$$

where

$$
\tilde{b}_h(v, w) := \epsilon a_h(v, w) + \frac{1}{\epsilon} (f'(u) \nabla v, \nabla w)_h + \alpha (v, w),
$$

and $\alpha$ should be chosen as $\alpha = \alpha_0 \epsilon^{-3}$ to guarantee the coercivity of $\tilde{b}_h(\cdot, \cdot)$. More precisely, first we cite some lemmas in [14], which will be used in this paper.

**Lemma 3.1** (Lemma 2.3 in [14]). Let $w, z \in H^2_E(\Omega)$, then

$$
\left| \sum_{K \in T_h} \int_{\partial K} \frac{\partial w}{\partial n} z \ dS \right| \leq C h \| w \|_{2,2,h} \| z \|_{2,2,h} + \| w \|_{1,2,h} \| z \|_{2,2,h} + \| w \|_{2,2,h} \| z \|_{1,2,h}.
$$

**Lemma 3.2** (Lemma 2.5 in [14]). Let $z \in H^2(\Omega)$ and $w \in H^2_E(\Omega) \cap H^3(\Omega)$, and define $B_h(w, z)$ by

$$
B_h(w, z) = \sum_{K \in T_h} \int_{\partial K} \left( \Delta w \frac{\partial z}{\partial n} + \frac{1}{2} \frac{\partial^2 w}{\partial n \partial s} - \frac{1}{2} \frac{\partial^2 w}{\partial s^2} \frac{\partial z}{\partial n} \right) dS,
$$

then we have

$$
|B_h(w, z)| \leq C h \| w \|_{3,2,h} \| z \|_{2,2,h}.
$$
For any $w \in S^h_E$, using Lemma 3.1 and the inverse inequality, we have
\[
|w|^2_{1,2,h} \leq |w|_{2,2,h}||w||_{0,2} + \left| \sum_{K \in T_h} \int_{\partial K} \frac{\partial w}{\partial n} z \, dS \right| \leq C||w||_{2,2,h}||w||_{0,2} \\
\leq C(||w||_{2,2,h}||w||_{0,2} + |w|_{1,2,h}||w||_{0,2} + ||w||^2_{0,2}).
\]
The kick-back argument gives
\[
|w|^2_{1,2,h} \leq C(||w||_{2,2,h}||w||_{0,2} + ||w||^2_{0,2}).
\]
Hence,
\[
(3.14) \quad \tilde{b}_h(w, w) = \epsilon a_h(w, w) + \frac{1}{\epsilon} (f'(u) \nabla w, \nabla w) + \frac{\alpha_0}{\epsilon^3} (w, w) \\
\geq \frac{1}{\epsilon^3} \left( \frac{\epsilon^4}{2} |w|^2_{2,2,h} - C\epsilon^2 |w|_{1,2,h}^2 + \alpha_0 ||w||^2_{0,2} \right) \\
\geq \frac{1}{\epsilon^3} \left( \frac{\epsilon^4}{4} |w|^2_{2,2,h} + (\alpha_0 - C)||w||^2_{0,2} \right),
\]
which implies the coercivity of $b_h(\cdot, \cdot)$ when $\alpha_0$ is large enough but independent of $\epsilon$.

Next we give the properties of $P_h$. Define $b_h(\cdot, \cdot) := \epsilon^3 \tilde{b}_h(\cdot, \cdot)$ and a norm
\[
||v||^2_{2,2,h} := \epsilon^4 |v|^2_{2,2,h} + \epsilon^2 |v|^2_{1,2,h} + ||v||^2_{0,2}.
\]

**Lemma 3.3.** Consider the following problems:

(3.15) \quad b_h(v, \eta) = F_h(\eta) \quad \forall \eta \in H^2_h(\Omega),

(3.16) \quad b_h(v_h, \chi) = F_h(\chi) \quad \forall \chi \in S^h_E.

Then we have
\[
(3.17) \quad ||v - v_h||_{2,2,h} \\
\leq C h \left\{ (\epsilon + h)^2 |v|^2_{3,2} + |v|_{1,2} + \sup_{\chi \in S^h_E} \frac{F_h(\tilde{E}_h \chi) - F_h(\chi) + \alpha_0 (v, \chi - \tilde{E}_h \chi)}{||\chi||_{2,2,h}} \right\}.
\]

**Proof.** Using (3.14) and the Strang Lemma, we have
\[
||v - v_h||_{2,2,h} \\
\leq C \left\{ \inf_{\psi \in S^h_E} ||v - \psi||_{2,2,h} + \sup_{\chi \in S^h_E} \frac{b_h(v, \chi) - F_h(\chi)}{||\chi||_{2,2,h}} \right\} \\
\leq C \left\{ \inf_{\psi \in S^h_E} ||v - \psi||_{2,2,h} + \sup_{\chi \in S^h_E} \frac{b_h(v, \chi - \tilde{E}_h \chi) + b_h(v, \tilde{E}_h \chi) - F_h(\chi)}{||\chi||_{2,2,h}} \right\} \\
\leq C \left\{ \inf_{\psi \in S^h_E} ||v - \psi||_{2,2,h} + \sup_{\chi \in S^h_E} \frac{b_h(v, \chi - \tilde{E}_h \chi) + F_h(\tilde{E}_h \chi) - F_h(\chi)}{||\chi||_{2,2,h}} \right\}.
\]
Using Lemma 3.2 and (3.2), we have
\[
b_h(v, \chi - \tilde{E}_h \chi) = \epsilon^4 a_h(v, \chi - \tilde{E}_h \chi) + \epsilon^2 (f'(u) \nabla v, \nabla (\chi - \tilde{E}_h \chi)) + (\alpha_0 v, \chi - \tilde{E}_h \chi) \\
\leq C h \left\{ \epsilon^4 |v|^2_{3,2} + |v|_{1,2} |\chi|_{2,2,h} + (\alpha_0 v, \chi - \tilde{E}_h \chi) \right\} \\
\leq C (\epsilon^2 |v|^2_{3,2} + |v|_{1,2} ||\chi||_{2,2,h} + (\alpha_0 v, \chi - \tilde{E}_h \chi))
\]
Then we obtain the desired bound (3.17) by the approximation properties of Morley interpolation operator (3.1).

\textbf{Theorem 3.4.} Suppose \( u \) solves the Cahn-Hilliard equation (1.1) - (1.3), then we have

\begin{align*}
e^2|u - P_h u|_{2,2,h} + \epsilon |u - P_h u|_{1,2,h} + \| u - P_h u \|_{0,2} & \leq Ch((\epsilon + h)^2|u|_{3,2} + |u|_{1,2} + ch\|u\|_{0,2}), \\
e^2|u_t - (P_h u)_t|_{2,2,h} + \epsilon |u_t - (P_h u)_t|_{1,2,h} + \| u_t - (P_h u)_t \|_{0,2} & \leq Ch\{(\epsilon + h)^2|u_t|_{3,2} + |u_t|_{1,2} + ch\|u_t\|_{0,2} + \| u_t \nabla u \|_{0,2} \\
& + \epsilon^{-1}|\ln h|^{1/2}\|u_t\|_{0,2}(\epsilon + h)^2|u|_{3,2} + |u|_{1,2} + ch\|u_t\|_{0,2}\}.
\end{align*}

\textit{Proof.} Taking \( v = u \) and \( v_h = P_h u \) in Lemma 3.3, and noticing that

\[ F_h(\psi) = \tilde{F}_h(\psi) = (\epsilon^4 \Delta^2 u - e^2 \Delta f(u) + \alpha_0 u, \psi) = (\epsilon^4 u_t + \alpha_0 u, \psi), \]

we obtain the bound (3.18) from (3.2) and (3.17).

Then we get

\[ F_h(\tilde{E}_h \chi) - \tilde{F}(\chi) + \alpha_0 (u_t, \chi - \tilde{E}_h \chi) \]
\[ = (\epsilon^4 \Delta^2 u_t - e^2 \Delta f(u)_t, \tilde{E}_h \chi - \chi) \\
- (\epsilon^4 f''(u)u_t \nabla u_t \nabla \tilde{E}_h \chi - \chi) - (\epsilon^2 f''(u)u_t \nabla (u - P_h u), \nabla \chi) \]
\[ \leq \epsilon^2 h^2\|u_t\|_{0,2} |\chi|_{2,2,h} + \epsilon^2 h^2\|u_t \nabla u \|_{0,2} |\chi|_{2,2,h} + \epsilon^2 h^2\|u_t\|_{0,2} \| \nabla \chi \|_{0,\infty} |u - P_h u|_{1,2,h} \]
\[ \leq Ch\{\epsilon h\|u_t\|_{0,2} + \| u_t \nabla u \|_{0,2} \\
+ \epsilon^{-1}|\ln h|^{1/2}\|u_t\|_{0,2}(\epsilon + h)^2|u|_{3,2} + |u|_{1,2} + ch\|u_t\|_{0,2}\} \| \chi \|_{2,2,h}, \]

where we use the discrete Sobolev inequality and the fact that \( \nabla \chi \) belongs to the Crouzeix-Raviar finite element space [9]. This implies the bound (3.19).

Combining with the a priori estimates of the bounds given in Section 2, we have the following theorem.

\textbf{Theorem 3.5.} Assume \( h \leq C\epsilon \), then there hold

\begin{align*}
e^4|u - P_h u|_{2,2,h}^2 + \epsilon^2|u - P_h u|_{1,2,h}^2 + \| u - P_h u \|_{0,2}^2 & \leq Ch^2 \rho_4(\epsilon), \\
\int_0^T e^4|u_t - (P_h u)_t|_{2,2,h}^2 + \epsilon^2|u_t - (P_h u)_t|_{1,2,h}^2 + \| u_t - (P_h u)_t \|_{0,2}^2 \, ds & \leq Ch^2 e^4 \rho_5(\epsilon) + Ch^2 |\ln h| \rho_5(\epsilon),
\end{align*}

where

\[ \rho_4(\epsilon) := \epsilon - \max\{2\sigma_1 + 1/2, 2\sigma_3 + 1/2, 2\sigma_2 + 4, 2\sigma_4\} + 1, \]
\[ \rho_5(\epsilon) := \epsilon - 2\max\{2\sigma_1 + 1/2, 2\sigma_3 + 1/2, 2\sigma_2 + 4, 2\sigma_4\} + 1. \]
Proof. Using (2.3), (2.6) and (2.7), we have

\begin{align}
(3.22) \quad (\epsilon + h)^4 |u|_{3,2}^2 + |u_{\epsilon}|_{1,2}^2 + \epsilon^2 h^2 \|u_{\epsilon,\ell}\|_{0,2}^2
\leq C e^{-\max\{2\sigma_1 + 5, 2\sigma_3 + 2\} + 4} + C e^{-2\sigma_1 - 1} + C e^{-\max\{2\sigma_1 + \frac{1}{2}, 2\sigma_3 + \frac{1}{2}, 2\sigma_2 + 4, 2\sigma_4\} + 4}
\leq C \rho_4(\epsilon),
\end{align}

which implies the bound (3.20) by (3.18).

Using (2.7), (2.13), (2.9) and (2.14), we obtain

\begin{align}
&\int_0^T (\epsilon + h)^4 |u_t|_{3,2}^2 + |u_{\epsilon,t}|_{1,2}^2 + \epsilon^2 h^2 \|u_{\epsilon,t,\ell}\|_{0,2}^2 + \|u_{\ell} \nabla u\|_{0,\infty}^2 \, ds
\leq C \int_0^T \epsilon^4 |u_t|_{3,2}^2 + |u_{\epsilon,t}|_{1,2}^2 + \epsilon^4 \|u_{\epsilon,t,\ell}\|_{0,2}^2 + \|u_{\ell} \nabla u\|_{0,\infty}^2 \, ds
\leq C e^3 \rho_0(\epsilon) + C \rho_0(\epsilon) + C e^4 \rho_3(\epsilon)
\end{align}

Further, using (2.7) and (3.22), we obtain

\begin{align}
&\int_0^T \epsilon^{-2} \|u_t\|_{0,2}^2 \,(\epsilon + h)^2 |u|_{3,2} + |u|_{1,2} + \epsilon h \|u_t\|_{0,2}^2 \, ds \leq C \rho_5(\epsilon).
\end{align}

This implies the bound (3.21).

\section{Corollary 3.6.} Under the condition that

\begin{align}
(3.23) \quad h \leq C e^2 \rho_4^{-\frac{1}{2}}(\epsilon), \quad h \leq C \rho_3^{-\frac{3}{2}}(\epsilon), \quad h^\frac{1}{2} \|\nabla h\| \leq C \rho_2^{-\frac{1}{2}}(\epsilon),
\end{align}

there hold

\begin{align}
(3.24) \quad |P_h u|_{j,2,h}^2 \leq C (1 + |u|_{j,2,h}^2) \quad j = 0, 1, 2,

\int_0^T |P_h u|_{j,2,h}^2 \, ds \leq C (1 + \int_0^T |u|_{j,2,h}^2) \quad j = 0, 1, 2,

\|P_h u\|_{0,\infty} \leq C.
\end{align}

Proof. By the Sobolev embedding and (3.20), we have

\begin{align}
\|P_h u\|_{0,\infty} \leq \|u\|_{0,\infty} + \|u - P_h u\|_{2,2,h} \leq C + C e^{-2} \rho_4^{1/2}(\epsilon) \leq C.
\end{align}

The first two bounds are the direct consequences of Theorem 3.5.

\section{4. Error Estimates.} In this section, first we derive the piecewise \(L^\infty(L^2)\) and \(L^2(H^2)\) error bounds which depend on \(\frac{1}{2}\) polynomially based on the generalized coercivity result in Theorem 4.3, and piecewise \(L^\infty(H^{-1})\) and \(L^3(H^1)\) error bounds. Then we prove the piecewise \(L^\infty(H^2)\) error bound based on the piecewise \(L^\infty(L^2)\) and \(L^2(H^2)\) error bounds. Finally, the \(L^\infty(L^\infty)\) error bound is established.

Decompose the error

\begin{align}
(4.1) \quad u - u_h^\rho = (u - P_h u) + (P_h u - u_h^\rho) := \rho^n + \theta^n.
\end{align}

The following two lemmas will be used in this section.
We first cite the generalized coercivity result, piecewise estimates. Then there exists an \( \epsilon \)
\( \rho^n \in \hat{S}^{h} \), \( \theta^n \in \hat{S}^{h} \).

Proof. Testing (1.1) with constant 1, and then taking the integration over \((0, t)\), we can obtain for any \( t \geq 0 \),
\[
\int_{\Omega} u(t) dx = \int_{\Omega} u(0) dx.
\]
Then choosing \( v = u(t), w = 1 \) in (3.11), we have for any \( t \geq 0 \),
\[
\int_{\Omega} P_h u(t) dx = \int_{\Omega} u(t) dx.
\]
Choosing \( v_h = 1 \) in (3.9), then
\[
\int_{\Omega} u^n_h dx = \int_{\Omega} u^{n-1}_h dx = \cdots = \int_{\Omega} u^0_h dx.
\]
Therefore, if choosing \( u^n_h = P_h u(0) \), then
\[
\int_{\Omega} u^n_h dx = \int_{\Omega} u^0_n dx = \int_{\Omega} P_h u(0) dx
\]
\[
= \int_{\Omega} u(0) dx = \int_{\Omega} u(t_n) dx = \int_{\Omega} P_h u(t_n) dx.
\]
Hence, \( P_h u(t_n) - u^n_h \in \hat{S}^{h} \).

4.1. Generalized coercivity result, piecewise \( L^\infty(H^{-1}) \) and \( L^2(H^1) \) error estimates. We first cite the generalized coercivity result, piecewise \( L^\infty(H^{-1}) \) and \( L^2(H^1) \) error estimates established in [27].

Theorem 4.3 (Generalized coercivity). Suppose there exists a positive number \( \gamma_3 > 0 \) such that the solution \( u \) of problem (1.1)–(1.4) and elliptic operator \( P_h \) satisfy
\begin{equation}
\label{4.2}
\|u - P_h u\|_{L^\infty((0, T):L^\infty)} \leq C_1 h \epsilon^{-\gamma_3}.
\end{equation}
Then there exists an \( \epsilon \)-independent and \( h \)-independent constant \( C > 0 \) such that for \( \epsilon \in (0, \epsilon_0), \ a.e. \ t \in [0, T] \), and for any \( \psi \in S^{h}_E \),
\[
(\epsilon - \epsilon^4) (\nabla \psi, \nabla \psi_h) + \frac{1}{\epsilon} (f'(P_h u(t)) \psi, \psi)_h \geq -C \|\nabla \Delta^{-1} \psi\|_{L^2}^2 - C \epsilon^{-2\gamma_2} - h^4,
\]
provided that \( h \) satisfies the constraint
\begin{equation}
\label{4.3}
h \leq (C_1 C_2)^{-1} \epsilon^{\gamma_3 + 3},
\end{equation}
where \( \gamma_2 = 2\gamma_1 + \sigma_1 + 6 \) and \( C_2 \) is determined by
\[
C_2 := \max_{\|\xi\| \leq \|\xi\|_{L^\infty((0, T):L^\infty)}} |f''(\xi)|.
\]
Remark 1. Thanks to the Sobolev embedding theorem and (3.20), we have

\[(4.4) \quad \| u - P_h u \|_{0, \infty} \leq \| u - P_h u \|_{2, 2, h} \leq Ch^2 \rho_2^2(\epsilon), \]

which gives the explicit formulation of $\gamma_3$ in (4.2).

**Theorem 4.4 (Piecewise $L^\infty(H^{-1})$ and $L^2(H^1)$ error estimates).** Assume $u$ is the solution of (1.1)–(1.4), $u_h^k$ is the numerical solution of scheme (3.9)–(3.10). Under the mesh constraints in Theorem 3.15 in [27], we have the following error estimate

\[
\frac{1}{4} \| \nabla \Delta_h^{-1} \theta \|^2_{0,2,h} + \frac{k^2}{4} \sum_{n=1}^{\ell} \| \nabla \Delta_h^{-1} d_i \theta^n \|^2_{0,2,h} + \frac{\epsilon^4 \ell}{16} \sum_{n=1}^{\ell} (\nabla \theta^n, \nabla \theta^n)_h
\]

\[
+ \frac{k}{\ell} \sum_{n=1}^{\ell} \| \theta^n \|^4_{0,4, h} \leq C(\rho_0(\epsilon)) \ln h \| h^2 + \rho_1(\ell) k^2),
\]

where $\rho_0(\epsilon)$ and $\rho_1(\epsilon)$ are polynomial $\frac{1}{\ln h}$-dependent functions and $\Delta_h^{-1}$ is a discrete inverse Laplace operator defined in [27].

**4.2. $L^\infty(L^2)$ and piecewise $L^2(H^2)$ error estimates.** Based on Theorem 4.4, the $L^\infty(L^2)$ and piecewise $L^2(H^2)$ error estimates which depend on $\frac{1}{\ln h}$ polynomially, instead of exponentially, are derived below. Notice that the Theorem 4.4 is used to circumvent the use of interpolation of $\| \cdot \|_{1, 2, h}$ between $\| \cdot \|_{0,2, h}$ and $\| \cdot \|_{2,2, h}$, by which only the exponential dependence can be derived.

**Theorem 4.5. Assume $u$ is the solution of (1.1)–(1.4), $u_h^k$ is the numerical solution of scheme (3.9)–(3.10).** Under the mesh constraints in Theorem 3.15 in [27] and (3.23), the following $L^\infty(L^2)$ and piecewise $L^2(H^2)$ error estimates hold

\[(4.5) \quad ||\theta||^2_{0,2,\Omega} + k \sum_{n=1}^{\ell} ||d_i \theta^n||^2_{0,2,\Omega} + \epsilon k \sum_{n=1}^{\ell} a_h(\theta^n, \theta^n)
\]

\[\leq C\rho_2(\epsilon)|\ln h|^2 h^2 + C\rho_3(\epsilon)|\ln h|k^2,\]

where

\[\rho_2(\epsilon) := \epsilon^4 \rho_3(\epsilon) + \epsilon^{-2\sigma_1} \rho_4(\epsilon) + \rho_5(\epsilon) + \epsilon^{-5} \rho_6(\epsilon) + \epsilon^{-2\gamma_1 - 2\gamma_2} \rho_7(\epsilon),\]

\[\rho_3(\epsilon) := \rho_3(\epsilon) + \epsilon^{-5} \rho_4(\epsilon) + \epsilon^{-2\gamma_1 - 2\gamma_2} \rho_5(\epsilon).
\]

**Proof.** It follows from (3.9), (3.11), and (3.12) that for any $v_h \in S_h^0$, \(d_i \theta^n, v_h) + c a_h(\theta^n, v_h)

\[
= [(d_i P_h u, v_h) + \epsilon a_h(P_h u, v_h)] - [(d_i u_h^n, v_h) + \epsilon a_h(u_h^n, v_h)]
\]

\[= - (d_i \rho^n, v_h) + (ut + \epsilon \Delta u - \frac{1}{\epsilon} \Delta f(u) + \alpha u, v_h) + (R^n(u_{tt}), v_h)
\]

\[- \frac{1}{\epsilon} (f'(u) \nabla P_h u, \nabla v_h)_h - \alpha (P_h u, v_h) + \frac{1}{\epsilon} (\nabla f(u_h^n), \nabla v_h)_h
\]

\[= (-d_i \rho^n + \alpha \rho^n, v_h) - \frac{1}{\epsilon} (f'(u) \nabla P_h u, \nabla v_h)_h + (R^n(u_{tt}), v_h),
\]

where the remainder

\[(4.7) \quad R^n(u_{tt}) := \frac{u(t_n) - u(t_{n-1})}{k} - u_{t}(t_n) = -\frac{1}{k} \int_{t_{n-1}}^{t_n} (s - t_{n-1}) u_{tt}(s) \, ds.
\]
Choosing $v_h = \theta^n$, taking summation over $n$ from 1 to $\ell$, multiplying $k$ on both sides of (4.6), we have

$$
(4.8) \quad \frac{1}{2} \|\theta^\ell\|_{0,2}^2 + \frac{k}{2} \sum_{n=1}^\ell \|d_t\theta^n\|_{0,2}^2 + \epsilon k \sum_{n=1}^\ell a_h(\theta^n, \theta^n) \\
= k \sum_{n=1}^\ell (-d_t\rho^n + \alpha \rho^n, \theta^n) - \frac{k}{\epsilon} \sum_{n=1}^\ell (f'(u)\nabla P_h u - \nabla f(u^n_h), \nabla \theta^n)_h \\
+ k \sum_{n=1}^\ell (R^n(u_{ht}), \theta^n) := I_1 + I_2 + I_3.
$$

Estimate of $I_1$: The first term on the right hand side of (4.6) can be bounded by

$$
(4.9) \quad I_1 = k \sum_{n=1}^\ell (-d_t\rho^n + \alpha \rho^n, \theta^n) \\
\leq Ck \sum_{n=1}^\ell \|d_t\rho^n\|_{0,2}^2 + Ck \sum_{n=1}^\ell \alpha^2 \|\rho^n\|_{0,2}^2 + Ck \sum_{n=1}^\ell \|\theta^n\|_{0,2}^2 \\
\leq C(\epsilon^4 \rho_3(\epsilon) + \epsilon^{-6} \rho_4(\epsilon)) h^2 + C\rho_5(\epsilon) \ln h |h|^2 + Ck \sum_{n=1}^\ell \|\theta^n\|_{0,2}^2,
$$

where by (3.20) and (3.21)

$$
(4.10) \quad k \sum_{n=1}^\ell \|d_t\rho^n\|_{0,2}^2 = \frac{1}{k} \sum_{n=1}^\ell \| \int_{t_{n-1}}^{t_n} \rho_t \, ds \|_{0,2}^2 \\
\leq \sum_{n=1}^\ell \| \rho_t \|_{0,2}^2 \, ds \leq C\epsilon^4 \rho_3(\epsilon) h^2 + C\rho_5(\epsilon) \ln h |h|^2,
$$

$$
(4.11) \quad k \sum_{n=1}^\ell \alpha^2 \|\rho^n\|_{0,2}^2 \leq C\epsilon^{-6} \sup_{1 \leq n \leq \ell} \|\rho^n\|_{0,2}^2 \leq C\epsilon^{-6} \rho_4(\epsilon) h^2.
$$

Estimate of $I_2$: The second term on the right hand side of (4.8) can be written as

$$
(4.12) \quad - \frac{k}{\epsilon} \sum_{n=1}^\ell (f'(u)\nabla P_h u - \nabla f(u^n_h), \nabla \theta^n)_h \\
= - \frac{k}{\epsilon} \sum_{n=1}^\ell (f'(u)\nabla P_h u - f'(P_h u)\nabla P_h u, \nabla \theta^n)_h \\
- \frac{k}{\epsilon} \sum_{n=1}^\ell (\nabla f(P_h u) - f'(P_h u)\nabla u^n_h, \nabla \theta^n)_h \\
- \frac{k}{\epsilon} \sum_{n=1}^\ell (f'(P_h u)\nabla u^n_h - \nabla f(u^n_h), \nabla \theta^n)_h := J_1 + J_2 + J_3.
$$

By (2.3), (3.20) and mesh condition (3.23), we have

$$
\|\nabla P_h u\|_{0,2}^2 \leq \|\nabla u\|_{0,2}^2 + C \leq \epsilon^{-2\sigma_1 - 1}.
$$
Then, using (4.4) and the piecewise $L^2(H^1)$ error estimate given in Theorem 4.4, the first term on the right-hand side of (4.12) can be bounded below

$$J_1 = -\frac{3k}{\epsilon} \sum_{n=1}^{\ell} (\rho^n(u + P_h u) \nabla P_h u, \nabla \theta^n)_h$$

$$\leq Ck \frac{\epsilon}{\ell} \sum_{n=1}^{\ell} \|u + P_h u\|_{0,\infty}^2 \|ho^n\|_{0,\infty}^2 \|\nabla P_h u\|_{0,2}^2 + \frac{Ck}{\epsilon} \sum_{n=1}^{\ell} (\nabla \theta^n, \nabla \theta^n)_h$$

$$\leq C \epsilon^{-2\gamma_1 - 6} \rho_4(\epsilon) h^2 + C \epsilon^{-5} \rho_0(\epsilon) |\ln h| h^2 + C \epsilon^{-5} \hat{\rho}_1(\epsilon) k^2.$$

Again, thanks to the piecewise $L^2(H^1)$ error estimate given in Theorem 4.4, the second term on the right-hand side of (4.12) can be written as

$$J_2 = -\frac{k}{\epsilon} \sum_{n=1}^{\ell} (f(P_h u) \nabla \theta^n, \nabla \theta^n)_h \leq Ck \frac{\epsilon}{\ell} \sum_{n=1}^{\ell} (\nabla \theta^n, \nabla \theta^n)_h$$

$$\leq C \epsilon^{-5} \rho_0(\epsilon) |\ln h| h^2 + C \epsilon^{-5} \hat{\rho}_1(\epsilon) k^2.$$

By the discrete Sobolev inequality and Theorem 3.14 in [27], we have for any $n$,

$$\|u^n_h\|_{1,\infty; h} \leq C |\ln h|^{\frac{1}{2}} \|u^n_h\|_{2,2; h} \leq C \epsilon^{-\gamma_2} |\ln h|^{\frac{1}{2}}.$$  

Then, the third term on the right-hand side of (4.12) can be bounded by

$$J_3 = -\frac{3k}{\epsilon} \sum_{n=1}^{\ell} (\theta^n(P_h u + u_h) \nabla u^n_h, \nabla \theta^n)$$

$$\leq Ck \sum_{n=1}^{\ell} \|\theta^n\|_{0,2}^2 + \frac{Ck}{\epsilon^2} \sum_{n=1}^{\ell} \|P_h u + u_h^n\|_{0,\infty}^2 \|u^n_h\|_{1,\infty; h} \|\nabla \theta^n\|_{0,2}^2$$

$$\leq Ck \sum_{n=1}^{\ell} \|\theta^n\|_{0,2}^2 + C \epsilon^{-2\gamma_1 - 2\gamma_2 - 2} |\ln h| k \sum_{n=1}^{\ell} \|\nabla \theta^n\|_{0,2}^2$$

$$\leq Ck \sum_{n=1}^{\ell} \|\theta^n\|_{0,2}^2 + C \epsilon^{-2\gamma_1 - 2\gamma_2 - 2} (\rho_0(\epsilon) |\ln h| h^2 + \hat{\rho}_1(\epsilon) |\ln h| k^2).$$

Estimate of $I_3$: The third term on the right hand side of (4.6) can be bounded by

$$I_3 = k \sum_{n=1}^{\ell} (R^n(u_{tt}), \theta^n) \leq Ck \sum_{n=1}^{\ell} \|R^n(u_{tt})\|_{0,2}^2 + Ck \sum_{n=1}^{\ell} \|\theta^n\|_{0,2}^2$$

$$\leq C \rho_3(\epsilon) k^2 + Ck \sum_{n=1}^{\ell} \|\theta^n\|_{0,2}^2,$$

where by (2.13) and (4.7),

$$k \sum_{n=1}^{\ell} \|R^n(u_{tt})\|_{0,2}^2 \leq \frac{1}{k} \sum_{n=1}^{\ell} \left( \int_{t_{n-1}}^{t_n} (s - t_{n-1})^2 \, ds \right) \left( \int_{t_{n-1}}^{t_n} \|u_{tt}(s)\|_{0,2}^2 \, ds \right) \leq C \rho_3(\epsilon) k^2.$$
Taking (4.9), (4.13), (4.14), (4.16), (4.16) into (4.8), we have

\[ \frac{1}{2} \| \theta' \|^2_{0,2} + \frac{k}{2} \sum_{n=1}^{\ell} \| d_t \theta^n \|^2_{0,2} + c h_k \sum_{n=1}^{\ell} a_h(\theta^n, \theta^n) \]

\[ \leq C k \sum_{n=1}^{\ell} \| \theta^n \|^2_{0,2} + C(\epsilon^\gamma \rho_4(\epsilon) + \epsilon^{-2\sigma_1-6} \rho_4(\epsilon))h^2 \]

\[ + C(\rho_5(\epsilon) + \epsilon^{-5} \rho_0(\epsilon)) | \ln h^2 + \epsilon^{-2\gamma_1-2\gamma_2-2} \rho_0(\epsilon) | \ln h|2h^2 \]

\[ + C(\rho_3(\epsilon) + \epsilon^{-5} \rho_1(\epsilon))h^2 + C \epsilon^{-2\gamma_1-2\gamma_2-2} \rho_1(\epsilon) | \ln h|2h^2. \]

The desired result (4.5) is therefore obtained by the Gronwall’s inequality.

**4.3. Piecewise \( L^\infty(H^2) \) and \( L^\infty(L^\infty) \) error estimates.** In this subsection, we give the \( \| \theta' \|^2_{2,2,h} \) estimate by taking the summation by parts in time and integrating by parts in space, and using the special properties of the Morley element. The \( \| \theta' \|^2_{2,2,h} \) estimate below is “almost” optimal with respect to time and space.

**Theorem 4.6.** Assume \( u \) is the solution of (1.1)–(1.4), \( u_h^n \) is the numerical solution of scheme (3.9)–(3.10). Under the mesh constraints in Theorem 3.15 in [27] and (3.23), the following piecewise \( L^\infty(H^2) \) error estimate holds

\[ \frac{k}{2} \sum_{n=1}^{\ell} \| d_t \theta^n \|^2_{L^2} + c h_k \sum_{n=1}^{\ell} a_h(\theta^n, \theta^n) + \epsilon \| \theta' \|^2_{2,2,h} \]

\[ \leq C \rho_4(\epsilon) | \ln h^2|2h^2 + C \rho_5(\epsilon) | \ln h|2h^2, \]

where

\[ \rho_4(\epsilon) = \epsilon^{-2\sigma_1-1} \rho_3(\epsilon) + \epsilon^{-2\sigma_1+5} \rho_3(\epsilon) \]

\[ + \left( \epsilon^{-4\gamma_1-3} + \epsilon^{-4\gamma_2-2} + \epsilon^{-\max\{2\sigma_1+5,2\sigma_3+2\}} \right) \rho_2(\epsilon), \]

\[ \rho_5(\epsilon) = \left( \epsilon^{-4\gamma_1-3} + \epsilon^{-4\gamma_2-2} + \epsilon^{-\max\{2\sigma_1+5,2\sigma_3+2\}} \right) \rho_3(\epsilon). \]

**Proof.** Choosing \( v_h = \theta^n - \theta^{n-1} = k d_t \theta^n \) in (4.6), taking summation over \( n \) from 1 to \( \ell \), we get

\[ \frac{k}{2} \sum_{n=1}^{\ell} \| d_t \theta^n \|^2_{L^2} + \frac{\epsilon}{2} a_h(\theta^\ell, \theta^\ell) + \frac{c h_k}{2} \sum_{n=1}^{\ell} a_h(\theta^n, \theta^n) \]

\[ = k \sum_{n=1}^{\ell} (-d_t \theta^n + \alpha \theta^n, d_t \theta^n) - \frac{\epsilon}{c} \sum_{n=1}^{\ell} (f'(u) \nabla P_h u - \nabla f(u_h^n) \cdot \nabla (d_t \theta^n))_h \]

\[ + k \sum_{n=1}^{\ell} (R^n(u_{tt}), d_t \theta^n) := I_1 + I_2 + I_3. \]
Here we use the fact that
\[ e \alpha(h(\theta^n, \theta^n - \theta^{n-1})) = \frac{e k^2}{2} a_h(d_t \theta^n, d_t \theta^n) + \frac{e}{2} a_h(\theta^n, \theta^n) - \frac{e}{2} a_h(\theta^{n-1}, \theta^{n-1}). \]

Estimates of \( I_1 \) and \( I_3 \): Similar to (4.9), using (4.10) and (4.11), we have
\[
I_1 \leq C k \sum_{n=1}^{\ell} \| d_t \rho^n \|_{L_2}^2 + C k \sum_{n=1}^{\ell} \alpha^2 \| \rho^n \|_{L_2}^2 + \frac{k}{8} \sum_{n=1}^{\ell} \| d_t \theta^n \|_{L_2}^2
\]
\[
\leq C(e^4 \rho_2(c) + e^{-6} \rho_4(c)) h^2 + C \rho_3(c) \ln h^2 + \frac{k}{8} \sum_{n=1}^{\ell} \| d_t \theta^n \|_{0,2}^2.
\]

From (4.17) and (4.18), we also obtain the estimate of \( I_3 \) below
\[
I_3 = k \sum_{n=1}^{\ell} (R^n(u_{tt}), d_t \theta^n) \leq C k \sum_{n=1}^{\ell} \| R^n(u_{tt}) \|_{L_2}^2 + \frac{k}{8} \sum_{n=1}^{\ell} \| d_t \theta^n \|_{0,2}^2
\]
\[
\leq C \rho_3(c) k^2 + \frac{k}{8} \sum_{n=1}^{\ell} \| d_t \theta^n \|_{0,2}^2.
\]

Estimate of \( I_2 \): Next we bound the more complicated term \( I_2 \). Using integration by parts, we have
\[
I_2 = -k \epsilon \sum_{n=1}^{\ell} (f'(u)) \nabla P_h u - \nabla f(P_h u), d_t \nabla \theta^n)_h + \frac{k}{\epsilon} \sum_{n=1}^{\ell} (\nabla (f(P_h u) - f(u^n_h)), d_t \nabla \theta^n)_h
\]
\[
- \frac{k}{\epsilon} \sum_{n=1}^{\ell} \sum_{E \in \mathcal{E}_h} \{ f(P_h u) - f(u^n_h) \} [d_t \nabla \theta^n]_E
\]
\[
- \frac{k}{\epsilon} \sum_{n=1}^{\ell} \sum_{E \in \mathcal{E}_h} \{ f(P_h u) - f(u^n_h) \} [\nabla d_t \theta^n]_E := J_1 + J_2 + J_3 + J_4.
\]

Here we adopt the standard DG notation and the DG identity, see [5, Equ. (3.3)].

Next we bound \( J_1 \) to \( J_4 \) respectively.
- **Estimate of \( J_1 \).** Using summation by parts in Lemma 4.1, we have
\[
J_1 = \frac{k}{\epsilon} \sum_{n=1}^{\ell} (d_t (\rho(u + P_h u)\nabla P_h u), \nabla \theta^{n-1})_h - \frac{1}{\epsilon} (\rho^\ell (u^\ell + P_h u^\ell) \nabla P_h u^\ell, \nabla \theta^\ell)_h.
\]
Thanks to (2.3), (2.7), (2.9), (3.20), (3.21), (3.24), and the piecewise $L^2(H^1)$ estimate in Theorem 4.4, the first term on the right hand side of (4.25) can be bounded by

\begin{align}
\frac{k}{\epsilon} \sum_{n=1}^\ell (d_{i}(\rho(u + P_h u) \nabla P_h u), \nabla \theta^{n-1})_h \\
\leq \frac{1}{k} \sum_{n=1}^\ell \int_{t_{n-1}}^{t_n} (\rho(u + P_h u) \nabla P_h u)_t ds_0^2 + C\epsilon^{-2} k \sum_{n=1}^\ell |\theta^{n-1}|^2_{1,2,h} \\
\leq \text{ess sup} \|\nabla P_h u\|_{0,2}^2 \int_0^T \|\rho_t\|_{0,\infty} ds + \text{ess sup} \|\rho\|_{0,\infty} \int_0^T \|\nabla (P_h u)_t\|_{0,2}^2 ds + C\epsilon^{-2} k \sum_{n=1}^\ell |\theta^{n-1}|^2_{1,2,h} \\
\leq C\epsilon^{-2\sigma_1-1}(\rho_3(\epsilon) + \epsilon^{-4}\rho_5(\epsilon) \ln h)|h|^2 + C\epsilon^{-4}\rho_0(\epsilon)\rho_4(\epsilon)|h|^2 \\
+ C\epsilon^{-2\sigma_1-6-(2\sigma_3+\frac{C}{\sigma_1})2}\rho_3(\epsilon)\rho_4(\epsilon)|h|^2 \\
+ C\epsilon^{-6}\rho_0(\epsilon) \ln h|h|^2 + C\epsilon^{-6}\tilde{\rho}_1(\epsilon)k^2.
\end{align}

Thanks to (2.3), (3.20) and the $L^\infty(L^2)$ estimate in Theorem 4.5, the second term on the right hand side of (4.25) can be bounded by

\begin{align}
-\frac{1}{\epsilon} (\rho f(u^\ell + P_h u^\ell) \nabla P_h u^\ell, \nabla \theta^\ell)_h \\
\leq C\epsilon^{-2}\|\rho\|_{0,\infty}^2|P_h u^\ell|_{1,2,h}^2 + C\epsilon^{-1}\|\theta\|_{0,2}^2 + \frac{\epsilon}{8} a_h(\theta^\ell, \theta^\ell) \\
\leq C\epsilon^{-2\sigma_1-7}\rho_4(\epsilon)|h|^2 + C\epsilon^{-1}\tilde{\rho}_2(\epsilon) \ln h|h|^2 + C\epsilon^{-1}\tilde{\rho}_3(\epsilon) \ln h|k|^2 + \frac{\epsilon}{8} a_h(\theta^\ell, \theta^\ell).
\end{align}

Combining (4.26) and (4.27), simplifying the coefficients according to the definition of $\rho_1(\epsilon)$ and $\tilde{\rho}_1(\epsilon)$, we obtain the bound for $J_1$:

\begin{align}
J_1 \leq C(\epsilon^{-2\sigma_1-1}\rho_3(\epsilon) + \epsilon^{-4}\rho_0(\epsilon)\rho_4(\epsilon) + \epsilon^{-2\sigma_1-5}\rho_5(\epsilon) + \epsilon^{-1}\tilde{\rho}_2(\epsilon)) \ln h|\theta|^2 |h|^2 \\
+ C\epsilon^{-1}\tilde{\rho}_3(\epsilon) \ln h|k|^2 + \frac{\epsilon}{8} a_h(\theta^\ell, \theta^\ell).
\end{align}

- **Estimate of $J_2$.** Define $f(P_h u) - f(u_h^n) := M^n \theta^n$, where $M^n$ is given as

$$M^n := (P_h(u(t_n))^2 + P_h(u(t_n))u_h^n + (u_h^n)^2 - 1.$$ 

Using summation by parts in Lemma 4.1, we have

\begin{align}
J_2 = -\frac{k}{\epsilon} \sum_{n=1}^\ell (d_{i}(M^n \theta^n), \Delta \theta^{n-1})_h + \frac{1}{\epsilon} (M^i \theta^\ell, \Delta \theta^\ell)_h \\
\leq \frac{Ck}{\epsilon} \sum_{n=1}^\ell \|d_{i}(M^n \theta^n)\|_{0,2} |\theta|_{2,2,h} + \frac{C}{\epsilon} \|M^i \theta^\ell\|_{0,2} |\theta^\ell|_{2,2,h}.
\end{align}
Since $d_t u_h^n = d_t (P_h u^n) - d_t \theta^n$, a direct calculation shows that
\[
d_t (M^n \theta^n) = \theta^n d_t M^n + M^{n-1} d_t \theta^n
\]
\[
= M^{n-1} d_t \theta^n + \theta^n (P_h u^n + P_h u^{n-1}) d_t (P_h u^n)
+ \theta^n (u_h^n + u_h^{n-1}) d_t (P_h u^n) - \theta^n (u_h^n + u_h^{n-1}) d_t \theta^n
+ (P_h u^n + 2 P_h u^{n-1} + 2 u_h^n + u_h^{n-1}) \theta^n d_t (P_h u^n).
\]

Using the $L^2(H^2)$ error estimate (4.5) and the assumption on the $L^\infty$ bound of $u_h^n$, we get
\[
(3.30) \quad \frac{C_k}{\epsilon} \sum_{n=1}^\ell \| d_t (M^n \theta^n) \|_{0,2} |\theta^n|_{2,2,h}
\leq C \epsilon^{-\gamma_1 - 1} k \sum_{n=1}^\ell \| d_t \theta^n \|_{0,2} |\theta^n|_{2,2,h} + C \epsilon^{\gamma_1 - 1} k \sum_{n=1}^\ell \| \theta^n d_t (P_h u) \|_{0,2} |\theta^n|_{2,2,h}
\leq \frac{k}{8} \sum_{n=1}^\ell \| d_t \theta^n \|_{0,2}^2 + C \epsilon^{\gamma_1 - 3} k \sum_{n=1}^\ell |\theta^n|_{2,2,h} + C \epsilon^{2\gamma_1} k \sum_{n=1}^\ell \| \theta^n d_t (P_h u) \|_{0,2}^2
\leq \frac{k}{8} \sum_{n=1}^\ell \| d_t \theta^n \|_{0,2}^2 + C \epsilon^{\gamma_1 - 3} (\tilde{\rho}_2(\epsilon) |\ln h|^2 h^2 + \tilde{\rho}_3(\epsilon) |\ln h|^2 k^2)
+ C \epsilon^{2\gamma_1 - \max(2\sigma_1 + \frac{1}{2} \frac{1}{2} \sigma_3 + \frac{3}{2} \sigma_2 + 4, 2\sigma_4) - 1} (\tilde{\rho}_2(\epsilon) |\ln h|^2 h^2 + \tilde{\rho}_3(\epsilon) |\ln h|^2 k^2),
\]
where by (2.7) and the $L^\infty(L^2)$ error estimate (4.5),
\[
k \sum_{n=1}^\ell \| \theta^n d_t (P_h u) \|_{0,2}^2
\leq \sup_{1 \leq n \leq \ell} \| \theta^n \|_{0,2}^2 \frac{1}{k} \| \int_{t_{n-1}}^{t_n} (P_h u)_t \|_{0,\infty}^2
\leq \sup_{1 \leq n \leq \ell} \| \theta^n \|_{0,2}^2 \int_0^T \| (P_h u)_t \|_{0,\infty}^2 \text{d}s
\leq C \epsilon^{-\max(2\sigma_1 + \frac{1}{2} \frac{1}{2} \sigma_3 + \frac{3}{2} \sigma_2 + 4, 2\sigma_4) - 1} (\tilde{\rho}_2(\epsilon) |\ln h|^2 h^2 + \tilde{\rho}_3(\epsilon) |\ln h|^2 k^2).
\]

And the second term on the right hand side of (4.29) can be bounded by
\[
(3.31) \quad \frac{C}{\epsilon} \| M^t \theta^t \|_{0,2} |\theta^t|_{2,2,h} \leq C \epsilon^{-\gamma_1 - 3} |\theta^t|_{0,2}^2 + \frac{\epsilon}{8} a_h (\theta^t, \theta^t)
\leq C \epsilon^{-\gamma_1 - 3} (\tilde{\rho}_2(\epsilon) |\ln h|^2 h^2 + \tilde{\rho}_3(\epsilon) |\ln h|^2 k^2) + \frac{\epsilon}{8} a_h (\theta^t, \theta^t).
\]
Combining (3.30) and (3.31), we obtain the bound for $J_2$:
\[
(3.32) \quad J_2 \leq \frac{k}{8} \sum_{n=1}^\ell \| d_t \theta^n \|_{0,2}^2 + \frac{\epsilon}{8} a_h (\theta^t, \theta^t) + C \epsilon^{\gamma_1 - 3} (\tilde{\rho}_2(\epsilon) |\ln h|^2 h^2 + \tilde{\rho}_3(\epsilon) |\ln h|^2 k^2)
+ C \epsilon^{2\gamma_1 - \max(2\sigma_1 + \frac{1}{2} \frac{1}{2} \sigma_3 + \frac{3}{2} \sigma_2 + 4, 2\sigma_4) - 1} (\tilde{\rho}_2(\epsilon) |\ln h|^2 h^2 + \tilde{\rho}_3(\epsilon) |\ln h|^2 k^2).
Using the piecewise Lemma 2.6 in [14], we have

\[
\int_E \| \nabla \theta^n \| \, dS = 0 \quad \forall E \in \mathcal{E}_h.
\]

Hence, \( J_3 \) has the same bound as \( J_2 \).

- **Estimate of \( J_4 \).** Since \( P_h u \) and \( u_h \) are continuous at vertexes of \( T_h \), thanks to Lemma 2.6 in [14], we have

\[
J_4 \leq \frac{Ck}{\epsilon^2} \sum_{n=1}^\ell h |M^n\theta^n|_{2,2,h} |d_t \theta^n|_{1,2,h}
\]

Using the piecewise \( L^2(H^2) \) estimate given in Theorem 4.4, we have

\[
\frac{Ck}{\epsilon^2} \sum_{n=1}^\ell |M^n\theta^n|_{2,2,h}^2 \leq C \frac{1}{\epsilon^2} \sup_{1 \leq n \leq \ell} |M^n|_{2,2,h}^2 + k \sum_{n=1}^\ell \| \theta^n \|_{2,2,h}^2
\]

\[
\leq C(\epsilon^{-4\gamma_2^2} + \epsilon^{-\max\{2\gamma_1 + 5,2\gamma_3 + 2\}}) (\tilde{\rho}_2(\epsilon) \ln h^2 h^2 + \tilde{\rho}_3(\epsilon) \ln h^2 h^2),
\]

where by (2.6) and the fact that \( \| u_h^n \|_{2,2,h} \leq C \epsilon^{-\gamma_2} \) (c.f. [27, Theorem 3.14])

\[
\| M^n \|_{2,2,h} \leq C(\| P_h u^n \|_{2,2,h} + \| u_h^n P_h u^n \|_{2,2,h} + \| (u_h^n)^2 \|_{2,2,h})
\]

\[
\leq C(\| P_h u^n \|_{2,2,h} + \| P_h u^n \|_{2,2,h} + \| u_h^n \|_{0,\infty} \| u_h^n \|_{1,4,h} + \| u_h^n \|_{1,4,h} + \| u_h^n \|_{1,4,h}^2)
\]

**Piecewise \( L^\infty(H^2) \) error estimate:** Taking (4.22), (4.23), (4.28), (4.32) and (4.33)
into (4.21), we obtain

\begin{equation}
\sum_{n=1}^{\ell} \left\| d_i \theta^n \right\|_{2,2,h}^2 + \frac{\epsilon}{8} a_h(\theta^\ell, \theta^\ell) + \frac{ck^2}{2} \sum_{n=1}^{\ell} a_h(d_i \theta^n, d_i \theta^n)
\end{equation}

\begin{equation}
\leq C\left(\epsilon^4 \rho_3(\epsilon) + \epsilon^-6 \rho_4(\epsilon)\right)h^2 + C\rho_5(\epsilon)|\ln h|^2h^2 + C\rho_3(\epsilon)k^2
\end{equation}

\begin{equation}
+ C\left(\epsilon^{-2\sigma_1-1} \rho_3(\epsilon) + \epsilon^{-4} \rho_0(\epsilon) \rho_4(\epsilon) + \epsilon^{-2\sigma_1-5} \rho_5(\epsilon) + \epsilon^{-1} \rho_2(\epsilon)\right)|\ln h|^2h^2
\end{equation}

\begin{equation}
+ C\left(\epsilon^{-4\gamma_1-3} \rho_2(\epsilon) |\ln h|^2h^2 + \rho_3(\epsilon) |\ln h|^2k^2\right)
\end{equation}

\begin{equation}
+ C\left(\epsilon^{-4\gamma_2-2} + \epsilon^{-\max\{2\sigma_1+\frac{7}{2}, 2\sigma_2+\frac{3}{2}, 2\sigma_2+4-2\sigma_1\}-1} \rho_2(\epsilon) |\ln h|^2h^2 + \rho_3(\epsilon) |\ln h|^2k^2\right)
\end{equation}

Then the theorem can be proved by simplifying the coefficients according to the definitions of \(\rho_i(\epsilon)\) and \(\tilde{\rho}_i(\epsilon)\).

**Remark 2.** If the summation by part for time and integration by part for space techniques are not employed simultaneously, one can only obtain a coarse estimate

\begin{equation}
\|\theta^\ell\|_{2,2,h}^2 + \frac{\epsilon}{8} \sum_{n=1}^{\ell} \left\| d_i \theta^n \right\|_{2,2,h}^2 + \frac{ck^2}{2} \sum_{n=1}^{\ell} a_h(d_i \theta^n, d_i \theta^n)
\end{equation}

\begin{equation}
\leq Ck^{-\frac{1}{2}}(\epsilon^{-\gamma_4} |\ln h|^2h^2 + \epsilon^{-\gamma_5} |\ln h|k^2),
\end{equation}

where \(\gamma_4, \gamma_5\) denote some positive constants.

Finally, using (4.4), Theorem 4.6 and the Sobolev embedding theorem, we can prove the desired \(L^\infty(L^\infty)\) error estimate.

**Theorem 4.7.** Assume \(u\) is the solution of (1.1)-(1.4), \(u^n_h\) is the numerical solution of scheme (3.9)-(3.10). Under the mesh constraints in Theorem 3.15 in [27] and (3.23), we have the \(L^\infty(L^\infty)\) error estimate

\begin{equation}
\|u(t_n) - u^n_h\|_{L^\infty} \leq C|\ln h|^{\frac{1}{2}}((\tilde{\rho}_4(\epsilon))^\frac{1}{2} |\ln h|^{\frac{1}{2}}h + (\tilde{\rho}_5(\epsilon))^\frac{1}{2}k) \quad \forall 1 \leq n \leq \ell.
\end{equation}

**Remark 3.** The mesh constraints in Theorem 3.15 in [27] and (3.23) can be achieved by \(h = C\epsilon^{p_1}\) and \(k = C\epsilon^{p_2}\) for certain positive \(p_1, p_2\). Hence, the \(|\ln h|k^2\) decreases asymptotically as \(k^2\) when \(\epsilon\) goes to zero.

5. Convergence of the Numerical Interface. In this section, we prove that the numerical interface defined as the zero level set of the Morley element interpolation of the solution \(U^n\) converges to the moving interface of the Hele-Shaw problem under the assumption that the Hele-Shaw problem has a unique global (in time) classical solution. We first cite the following convergence result established in [2].

**Theorem 5.1.** Let \(\Omega\) be a given smooth domain and \(\Gamma_{00}\) be a smooth closed hypersurface in \(\Omega\). Suppose that the Hele-Shaw problem starting from \(\Gamma_{00}\) has a unique smooth solution \((w, \Gamma) := \bigcup_{0 \leq t \leq T}(\Gamma_t \times \{t\})\) in the time interval \([0, T]\) such that \(\Gamma_t \subseteq \Omega\) for all \(t \in [0, T]\). Then there exists a family of smooth functions \(\{u^n_0\}_{0 < \epsilon \leq 1}\) which are uniformly bounded in \(\epsilon \in (0, 1]\) and \((x, t) \in \Omega_T\), such that if \(u^n\) solves the Cahn-Hilliard problem (1.1)-(1.3), then

\[
\lim_{\epsilon \to 0} u^n(x, t) = \begin{cases} 
1 & \text{if } (x, t) \in O \\
-1 & \text{if } (x, t) \in I
\end{cases}
\]

uniformly on compact subsets, where \(I\) and \(O\) stand for the “inside” and “outside” of \(\Gamma\).
By Theorem 5.2, there exists

\[ A \]

Let \( N \) denote the zero level set of the Hele-Shaw problem and \( U_{e,h,k}(x,t) \) denotes the piecewise linear interpolation in time of the numerical solution \( u_h^n \), namely,

\[
U_{e,h,k}(x,t) := \frac{t-t_{n-1}}{k} u_h^n(x) + \frac{t_n-t}{k} u_h^{n-1}(x),
\]

for \( t_{n-1} \leq t \leq t_n \) and \( 1 \leq n \leq M \). Then, under the mesh and starting value constraints of Theorem 4.6 and \( k = O(h^q) \) with \( 0 < q < 1 \), we have

\[ \begin{align*}
(i) & \quad U_{e,h,k}(x,t) \xrightarrow{\epsilon \downarrow 0} 1 \quad \text{uniformly on compact subset of } \mathcal{O}, \\
(ii) & \quad U_{e,h,k}(x,t) \xrightarrow{\epsilon \downarrow 0} -1 \quad \text{uniformly on compact subset of } \mathcal{I}.
\end{align*} \]

**Proof.** For any compact set \( A \subset \mathcal{O} \) and for any \( (x,t) \in A \), we have

\[
|U_{e,h,k} - 1| \leq |U_{e,h,k} - u^\epsilon(x,t)| + |u^\epsilon(x,t) - 1| \\
\leq |U_{e,h,k} - u^\epsilon(x,t)|_{L^\infty(\Omega_T)} + |u^\epsilon(x,t) - 1|.
\]

Theorem 4.7 infers that

\[
|U_{e,h,k} - u^\epsilon(x,t)|_{L^\infty(\Omega_T)} \leq C(\tilde{\rho}_0(\epsilon))^{1/2} h^q |\ln h|.
\]

where \( \tilde{\rho}_0(\epsilon) = \max\{\tilde{\rho}_4(\epsilon), \tilde{\rho}_5(\epsilon)\} \).

The first term on the right-hand side of (5.2) tends to 0 when \( \epsilon \downarrow 0 \) (note that \( h,k \downarrow 0 \), too). The second term converges uniformly to 0 on the compact set \( A \), which is ensured by (i) of Theorem 5.1. Hence, the assertion (i) holds.

To show (ii), we only need to replace \( \mathcal{O} \) by \( \mathcal{I} \) and 1 by \(-1\) in the above proof. \( \square \)

The second main theorem addresses the convergence of numerical interfaces.

**Theorem 5.3.** Let \( \Gamma_{t}^{e,h,k} := \{ x \in \Omega ; U_{e,h,k}(x,t) = 0 \} \) be the zero level set of \( U_{e,h,k}(x,t) \), then under the assumptions of Theorem 5.2, we have

\[ \sup_{x \in \Gamma_{t}^{e,h,k}} \text{dist}(x, \Gamma_t) \xrightarrow{\epsilon \downarrow 0} 0 \quad \text{uniformly on } [0,T]. \]

**Proof.** For any \( \eta \in (0,1) \), define the tabular neighborhood \( \mathcal{N}_\eta \) of width \( 2\eta \) of \( \Gamma_t \)

\[
\mathcal{N}_\eta := \{ (x,t) \in \Omega_T ; \text{dist}(x,\Gamma_t) < \eta \}.
\]

Let \( A \) and \( B \) denote the complements of the neighborhood \( \mathcal{N}_\eta \) in \( \mathcal{O} \) and \( \mathcal{I} \), respectively,

\[ A = \mathcal{O} \setminus \mathcal{N}_\eta \quad \text{and} \quad B = \mathcal{I} \setminus \mathcal{N}_\eta. \]

Note that \( A \) is a compact subset outside \( \Gamma_t \) and \( B \) is a compact subset inside \( \Gamma_t \).

By Theorem 5.2, there exists \( \epsilon_1 > 0 \), which only depends on \( \eta \), such that for any \( \epsilon \in (0, \epsilon_1) \)

\[ \begin{align*}
(i) & \quad |U_{e,h,k}(x,t) - 1| \leq \eta \quad \forall (x,t) \in A, \\
(ii) & \quad |U_{e,h,k}(x,t) + 1| \leq \eta \quad \forall (x,t) \in B.
\end{align*} \]
Now for any \( t \in [0, T] \) and \( x \in \Gamma_t^{*,h,k} \), from \( U_{\epsilon,h,k}(x,t) = 0 \) we have

\[
\begin{align*}
(5.7) & \quad |U_{\epsilon,h,k}(x,t) - 1| = 1 \quad \forall (x,t) \in A, \\
(5.8) & \quad |U_{\epsilon,h,k}(x,t) + 1| = 1 \quad \forall (x,t) \in B.
\end{align*}
\]

(5.5) and (5.7) imply that \((x,t)\) is not in \( A \), and (5.6) and (5.8) imply that \((x,t)\) is not in \( B \), then \((x,t)\) must lie in the tubular neighborhood \( \mathcal{N}_{\eta} \). Therefore, for any \( \epsilon \in (0, \epsilon_1) \),

\[
(5.9) \quad \sup_{x \in \Gamma_t^{*,h,k}} \text{dist}(x, \Gamma_t) \leq \eta \quad \text{uniformly on } [0, T].
\]

The proof is complete. \( \square \)

6. Numerical experiments. In this section, we present two two-dimensional numerical tests to gauge the performance of the proposed fully discrete Morley finite element method for Cahn-Hilliard equation. The square domain \( \Omega = [-1, 1]^2 \) is used in both tests.

Test 1. Consider the Cahn-Hilliard problem with an ellipse initial interface determined by \( \Gamma_0 : \frac{x^2}{0.36} + \frac{y^2}{0.04} = 0 \). The initial condition is chosen to have the form \( u_0(x,y) = \tanh\left(\frac{d_0(x,y)}{\sqrt{\epsilon}}\right) \), where \( d_0(x,y) \) denotes the signed distance from \((x,y)\) to the initial ellipse interface \( \Gamma_0 \) and \( \tanh(t) = (e^t - e^{-t})/(e^t + e^{-t}) \).

Figure 1 displays four snapshots at four fixed time points of the numerical interface with four different \( \epsilon \)'s. Here time step size \( k = 1 \times 10^{-4} \) and space size \( h = 0.01 \) are used. They clearly indicate that at each time point the numerical interface converges to the sharp interface \( \Gamma_t \) of the Hele-haw flow as \( \epsilon \) tends to zero. Note that this initial condition may not satisfy the General Assumption (GA) due to the singularity of the signed distance function. We will adopt a smooth initial condition in the later test.

Test 2. Consider the following initial condition, which is also adopted in [23],

\[
(5.10) \quad u_0(x,y) = \tanh\left(\left((x-0.3)^2 + y^2 - 0.25^2\right)/\epsilon\right) \tanh\left(\left((x+0.3)^2 + y^2 - 0.3^2\right)/\epsilon\right).
\]

Table 1 and 2 show the errors of spatial \( L^2, H^1 \) and \( H^2 \) semi-norms and the rates of convergence at \( T = 0.0002 \) and \( T = 0.001 \). \( \epsilon = 0.08 \) is used to generate the table. \( k = 1 \times 10^{-5} \) is chosen so that the error in time is relatively small to the error in space. The \( L^\infty(H^2) \) norm error is in agreement with the convergence theorem, but \( L^\infty(L^2) \) and \( L^\infty(H^1) \) norm errors are one order higher than our theoretical results. We note that in [14], the second order convergence for both \( L^\infty(L^2) \) and \( L^\infty(H^1) \) norms are proved, whereas only \( 1/\epsilon \)-exponential dependence can be derived.

| \( h \) | \( L^\infty(L^2) \) error | order | \( L^\infty(H^1) \) error | order | \( L^\infty(H^2) \) error | order |
|---|---|---|---|---|---|
| \( 0.2\sqrt{2} \) | 0.079659 | — | 1.761563 | — | 34.097686 | — |
| \( 0.1\sqrt{2} \) | 0.023142 | 1.7833 | 0.642870 | 1.4534 | 21.604986 | 0.6583 |
| \( 0.05\sqrt{2} \) | 0.007598 | 1.6067 | 0.183600 | 1.8080 | 11.783724 | 0.8746 |
| \( 0.025\sqrt{2} \) | 0.002151 | 1.8201 | 0.048042 | 1.9342 | 6.045416 | 0.9629 |
| \( 0.0125\sqrt{2} \) | 0.000557 | 1.9501 | 0.012167 | 1.9813 | 3.042138 | 0.9908 |

Table 1. Spatial errors and convergence rates of Test 2: \( \epsilon = 0.08, k = 1 \times 10^{-5}, T = 0.0002 \).

Figure 2 displays six snapshots at six fixed time points of the numerical interface with four different \( \epsilon \). Again, they clearly indicate that at each time point the numerical interface converges to the sharp interface \( \Gamma_t \) of the Hele-haw flow as \( \epsilon \) tends to zero.
Fig. 1. Test 1: Snapshots of the zero-level sets of $u^{\epsilon,k}$ at $t = 0, 0.005, 0.015, 0.03$ and $\epsilon = 0.08, 0.04, 0.03, 0.02$.

| $h$ | $L^\infty(L^2)$ error | order | $L^\infty(H^1)$ error | order | $L^\infty(H^2)$ error | order |
|-----|-----------------------|-------|------------------------|-------|------------------------|-------|
| $0.2\sqrt{2}$ | 0.137170 | — | 2.469682 | — | 43.008910 | — |
| $0.1\sqrt{2}$ | 0.032310 | 2.0859 | 0.710340 | 1.7977 | 23.320078 | 0.8831 |
| $0.05\sqrt{2}$ | 0.008830 | 1.8715 | 0.183932 | 1.9493 | 11.774451 | 0.9859 |
| $0.025\sqrt{2}$ | 0.002349 | 1.9103 | 0.046810 | 1.9743 | 5.927408 | 0.9902 |
| $0.0125\sqrt{2}$ | 0.000597 | 1.9746 | 0.011764 | 1.9924 | 2.970322 | 0.9968 |

Table 2: Spatial errors and convergence rates of Test 2: $\epsilon = 0.08$, $k = 1 \times 10^{-5}$, $T = 0.001$.

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Fig. 2. Test 2: Snapshots of the zero-level sets of $u^{\epsilon,k}$ at $t = 0, 0.00005, 0.0002, 0.001, 0.006, 0.015$ and $\epsilon = 0.08, 0.04, 0.03, 0.02$.

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