Long wavelength limit of evolution of cosmological perturbations in the universe where scalar fields and fluids coexist

Takashi Hamazaki

Kamiyugi 3-3-4-606 Hachioji-city
Tokyo 192-0373 Japan

Abstract

We present the LWL formula which represents the long wavelength limit of the solutions of evolution equations of cosmological perturbations in terms of the exactly homogeneous solutions in the most general case where multiple scalar fields and multiple perfect fluids coexist. We find the conserved quantity which has origin in the adiabatic decaying mode, and by regarding this quantity as the source term we determine the correction term which corrects the discrepancy between the exactly homogeneous perturbations and the $k \to 0$ limit of the evolutions of cosmological perturbations. This LWL formula is useful for investigating the evolutions of cosmological perturbations in the early stage of our universe such as reheating after inflation and the curvaton decay in the curvaton scenario. When we extract the long wavelength limits of evolutions of cosmological perturbations from the exactly homogeneous perturbations by the LWL formula, it is more convenient to describe the corresponding exactly homogeneous system with not the cosmological time but the scale factor as the evolution parameter. By applying the LWL formula to the reheating model and the curvaton model with multiple scalar fields and multiple radiation fluids, we obtain the S formula representing the final amplitude of the Bardeen parameter in terms of the initial adiabatic and isocurvature perturbations.

PACS number(s):98.80.Cq

\[1\] email address: yj4f-hmzk@asahi-net.or.jp
§1 Introduction and summary

Recently we come to be required to investigate the evolution of cosmological perturbations in the very early universe [26], [6]. According to the inflationary scenarios and the curvaton scenario, in the early universe the wavelength of cosmological perturbations responsible for the present cosmic structures such as galaxies and clusters of galaxies is much larger than the horizon scales. Therefore the methods for researching the cosmological perturbations on superhorizon scales have been sought. In this context, Nambu and Taruya pointed out that there exists an LWL formula representing the $k \to 0$ limit of the cosmological perturbations in terms of the exactly homogeneous perturbations [32]. Soon later in the multiple scalar fields system [15], [30], the complete LWL formulae were constructed. Since the evolution equations of the corresponding exactly homogeneous universe look simpler than the evolution equations of cosmological perturbations, the LWL formula brought about great simplification. In addition, the viewpoint that the evolutions of the cosmological perturbations on superhorizon scales are governed by the stability and instability of the corresponding exactly homogeneous universe [9], [10] is useful for physical interpretation. In this context, the phase space of the corresponding exactly homogeneous system was investigated in detail and the role of the fixed points in the phase space in the stability and instability of cosmological perturbations was discussed [10]. For these reasons, the LWL formula was used for investigating the evolution of cosmological perturbations on superhorizon scales by several authors [14], [30], [32], [9], [10]. In this paper, in section 2, in order to investigate the evolutionary behaviors of cosmological perturbations during reheating and the curvaton decays, we construct the complete LWL formulae for the most general system where the multiple scalar fields and the multiple perfect fluids coexist.

In the early universe, the cosmological perturbations on superhorizon scales responsible for the cosmic structures and the CMB temperature anisotropies experience the reheating and/or the curvaton decays. In these processes, the multiple scalar fields such as inflatons and curvatons oscillate coherently, gradually decaying into radiation fluid. By replacing the oscillating scalar fields with the dust fluids, the evolution of the cosmological perturbations during reheating after the inflation [8] and in the curvaton decay [20] were investigated. These authors treated the system dominated by dust-like scalar field fluid and radiation and investigated the influence of the entropy perturbation originating from the multicomponent property to the evolution of the total curvature perturbation variables such as the Bardeen parameter. The purpose of these analyses was to determine the initial perturbation of the present Friedmann universe in terms of the early stage seed perturbation. Although it was shown partially that this replacement is physically reasonable [14], [8], we are required to treat the decaying oscillatory scalar fields directly. In fact, the instabilities characteristic to the rapidly oscillating scalar fields were pointed out [3], [5], [37], [10], [36] and investigated [33], [16], [31], [17]. In order to treat the oscillatory scalar fields directly, the action angle variables were introduced and the averaging method representing the averaging over the fast changing angle variables was applied [9], [10]. In this paper, in order to investigate the evolutionary behaviors of cosmological perturbations during reheating and/or the curvaton decays, in section 3 the action angle variables and the action angle perturbation variables are introduced, and in section 5, the averaging method was applied to the decaying oscillatory scalar fields.

In the papers [9], [10], the averaging method was used to investigate the corresponding
exactly homogeneous system, and by the LWL formula the evolution of the cosmological perturbation in the long wavelength limit was constructed from the corresponding exactly homogeneous perturbation. By using the LWL formula and by applying the averaging method, in single oscillatory scalar field system \cite{14} and in nonresonant multiple oscillatory scalar fields system \cite{9} it was shown that the Bardeen parameter is conserved, and in resonant multiple oscillatory scalar fields system \cite{10} it was shown that the cosmological perturbation including the Bardeen parameter can grow. In this paper, in section 6, 7, by using the LWL formula and by applying the averaging method to the decaying oscillatory scalar fields, we construct S formulae representing the final amplitude of the Bardeen parameter in terms of the initial seed adiabatic and entropic perturbations, in the reheating and in the curvaton decays, respectively.

The organization and the summary of the paper is explained as follows. In section 2, we construct the LWL formula as for such scalar-fluid composite system based on the the philosophy of the paper \cite{15}. The discrepancy exists between the evolution equations of the cosmological perturbations in the $k \rightarrow 0$ limit and the evolution equations of the exactly homogeneous perturbations because the former contains $k^2 \Phi = O(1)$ terms and the latter does not, therefore this discrepancy should be corrected by the correction term which contributes the adiabatic decaying mode, but any general methods for determining such correction term have not been presented yet, and only in the multiple scalar fields system such correction term was determined. We show in the $k \rightarrow 0$ limit the existence of the conserved quantity which has origin in the adiabatic decaying mode and which is related with $k^2 \Phi$. By regarding this conserved quantity as the source term, and obtaining the special solution $A^\flat$, we correct the exactly homogeneous perturbation $A^\sharp$ and we obtain the complete LWL formula $A = A^\sharp + A^\flat$ in the most general scalar-fluid composite system.

In section 3, we point out that it is more appropriate to use the scale factor $a$ rather than the cosmological time $t$ as the evolution parameter when we use the LWL formula. As for the scalar quantity $T$, we use the perturbation variable $DT$, $D$ is the operator which maps the exactly homogeneous scalar quantity $T$ to the gauge invariant perturbation variable representing the $T$ fluctuation in the flat slice. $D$ defined in this way can be interpreted as a kind of derivative operator. In fact, the exactly homogeneous part $(DT)^\sharp$ can be expressed as the derivative of $T$ with respect to the solution constant with the scale factor $a$ fixed. In order to investigate the exactly homogeneous system containing oscillatory scalar fields, we use the action angle variables $I_a$, $\theta_a$. By using $D$ defined in this way, we can define the action angle perturbation variables $DI_a$, $D\theta_a$ whose exactly homogeneous parts are given as the derivatives of $I_a$, $\theta_a$ with solution constant $C$ with the scale factor $a$ fixed. When we use the derivative operator $D$ and the LWL formulae, it is essential to use the scale factor $a$ as the evolution parameter. In section 4, we apply the LWL formulae to the non-interacting multicomponents system and discuss the long wavelength limit of the evolution of the Bardeen parameter. In section 5, we apply the averaging method by which the system is averaged over the fast changing angle variables to the decaying scalar fields which have been discussed in the reheating model and the curvaton model. By evaluating the corrections produced by the averaging process and the errors produced by the truncation of the sufficient reduced angle variables dependent part, the validity of the averaging method is established. In section 6, 7, we apply the LWL formula and the averaging method to the interacting multicomponents model such as the reheating model, the curvaton model,
respectively. We assume that the multiple scalar fields and the multiple radiation fluid components exist. In these models, we construct the $S$ formulae representing the final amplitude of the Bardeen parameter in terms of the initial adiabatic and isocurvature perturbations. In our previous paper [10], the evolutionary behaviors of cosmological perturbations in the early universe where multiple oscillatory scalar fields interact with each other have been investigated. This $S$ formula give the information about how the cosmological perturbations which grew in such early era are transmitted into the radiation energy density perturbations through the energy transfer from the scalar fields into the radiation fluids. We present the necessary condition for the initial entropic perturbations produced in the early era to survive until the late radiation dominant universe. Section 8 is devoted to discussions containing non-linear generalization of our LWL formalism and comment of the case where the decay rate depends on other physical quantities. In appendices, the proofs of the propositions presented in section 5 and the evaluations of the useful mathematical formulae used in section 6 are contained.

In this paper, we consider the case where the homogeneous scalar fields obey the phenomenological evolution equations as

$$
\ddot{\phi} + 3H\dot{\phi} + \frac{\partial U}{\partial \phi} + S = 0.
$$

(1.1)

The interactions between scalar fields are described by the interaction potential $U$, while the interaction between scalar fields and fluids are described by $S$. This analysis includes the well known case $S = \Gamma \dot{\phi}$ [11], [21] and the general case where $S$ is an arbitrary analytic function of $\phi, \dot{\phi}$ which was discussed in the paper [35] but whose perturbations have not been investigated yet. Another supplemental purpose of our paper is to present the evolution equations of cosmological perturbations corresponding to the homogeneous system with various source term $S$ especially dependent on $\dot{\phi}$.

The notation used in this paper are based on the the review [12] and the paper [15].

§2 Derivation of the LWL formula

We give the definitions and the evolution equations as for the background and the perturbation variables. Based on these notations, in the most general model where the multiple scalar fields and the multiple perfect fluid components interact, we give the LWL formula representing the evolutions of the perturbations variables in terms of the exactly homogeneous solutions.

We consider perturbations on a spatially flat Robertson-Walker universe given by

$$
\tilde{d}s^2 = -(1 + 2AY)dt^2 - 2aBY_jdx^j
+a^2[(1 + 2H_LY)\delta_{jk} + 2H_TY_{jk}]dx^jdx^k,
$$

(2.1)

where $Y, Y_j$ and $Y_{jk}$ are harmonic scalar, vector and tensor for a scalar perturbation with wave vector $k$ on flat three-space:

$$
Y := e^{ik \cdot x}, \quad Y_j := -i\frac{k_j}{k}Y, \quad Y_{jk} := \left(\frac{1}{3}\delta_{jk} - \frac{k_jk_k}{k^2}\right)Y.
$$

(2.2)
By using the gauge dependent variables $R$ and $\sigma_g$ representing the spatial curvature perturbation and the shear, respectively:

$$R := H_L + \frac{1}{3} H_T, \quad \sigma_g := \frac{a}{k} H_T - B,$$

we can define two independent gauge invariant variables:

$$A := A - \left( \frac{R}{H} \right) \cdot, \quad \Phi := \frac{a}{k} R - \sigma_g.$$  \tag{2.4}

In order to define the matter perturbation variables, we will consider the scalar quantity perturbation variables generally. As for covariant scalar quantity $\tilde{T} = T + \delta T Y$, we define the gauge invariant perturbation variable representing the $T$ fluctuation in the flat slice:

$$DT := \delta T - \frac{\dot{T}}{H} R,$$  \tag{2.5}

Next we consider the covariant scalar quantity $\tilde{T}_2$ whose background quantity is the time derivative of $T$: $\dot{T}$. The extension of $\dot{T}$ into the covariant scalar quantity $\tilde{T}_2$ is not unique. For example,

$$\tilde{T}_2 = \text{sgn} \left( \partial_0 \tilde{T} \right) \left[ -\tilde{g}^{\mu \nu} \tilde{\nabla}_\mu \tilde{\nabla}_\nu \tilde{T} \right]^{1/2},$$

and

$$\tilde{T}_2 = \tilde{n}^\mu \tilde{\nabla}_\mu \tilde{T},$$

where $\tilde{n}^\mu$ is an arbitrary vector field satisfying

$$\tilde{n}^\mu \tilde{n}_\mu = -1,$$  \tag{2.8}

have the same $\dot{T}$ as the background part. But these different $\tilde{T}_2$’s give the unique perturbation part: $DT_2 = (DT)^\cdot - \dot{T} \cdot A$. Therefore we can define $DT$ by

$$D\dot{T} := (DT)^\cdot - \dot{T} \cdot A.$$  \tag{2.9}

We consider the universe where the scalar fields $\phi_a$, ($1 \leq a \leq N_S$) and the fluids $\rho_\alpha, P_\alpha$ ($1 \leq \alpha \leq N_f$) coexist, whose energy momentum tensor is divided into $A = (S, f)$ parts where $S$ represents the multiple scalar fields, $f$ represents the multiple fluids. The energy momentum tensor of $f$ part are further divided into individual fluids parts $\alpha$. On the other hand, the energy momentum tensor of $S$ part cannot be divided into individual scalar fields parts $a$, since the interaction potential $U$ contains the terms consisting of plural scalar fields $\phi_a$:

$$\tilde{T}_\nu^\mu = \left( \tilde{T}_\nu^\mu \right)_S + \left( \tilde{T}_\nu^\mu \right)_f = \left( \tilde{T}_\nu^\mu \right)_S + \sum_\alpha \tilde{T}_\alpha^\mu, \quad \text{(2.10)}$$

$$0 = \left( \tilde{Q}_\mu \right)_S + \left( \tilde{Q}_\mu \right)_f = \left( \tilde{Q}_\mu \right)_S + \sum_\alpha \tilde{Q}_\alpha, \quad \text{(2.11)}$$

where the energy-momentum transfer vector $\tilde{Q}_{A\mu}$ is defined by

$$\tilde{\nabla}_\nu \tilde{T}_\nu^\mu = \tilde{Q}_{A\mu} = \tilde{Q}_{A} \tilde{u}_\mu + \tilde{f}_{A\mu}, \quad \text{(2.12)}$$
where \( \tilde{u}_\mu \) is the four velocity of the whole matter system and the momentum transfer \( \tilde{f}_{A\mu} \) satisfies \( \tilde{u}^\mu \tilde{f}_{A\mu} = 0 \). For the scalar perturbation, the energy-momentum tensor and the energy-momentum transfer vector of each individual component are expressed as

\[
\tilde{T}^0_{A0} = -(\rho_A + \delta \rho_A Y), \quad (2.13)
\]
\[
\tilde{T}^0_{Aj} = a(\rho_A + P_A)(v_A - B)Y_j, \quad (2.14)
\]
\[
\tilde{T}^i_{Ak} = (P_A \delta^i_k + \delta P_A Y \delta^i_k + \Pi_{TA} Y^i_k), \quad (2.15)
\]

and

\[
\tilde{Q}^0_{A0} = -\left[ Q_A + (Q_A A + \delta Q_A) Y \right], \quad (2.16)
\]
\[
\tilde{Q}^0_{Aj} = a\left[ Q_A (v - B) + F_{cA} \right] Y_j, \quad (2.17)
\]

where \( \rho_A, P_A \) and \( Q_A \) are the background quantities of the energy density, the pressure and the energy transfer of the individual component \( A \), respectively. The anisotropic pressure perturbation \( \Pi_{TA} \) and the momentum transfer perturbation \( F_{cA} \) are already gauge invariant. As for the scalar quantities \( T = (\rho_A, P_A, Q_A) \), we use \( DT \) as the gauge invariant perturbation variables. As for the gauge invariant velocity perturbation variable, we use

\[
Z_A := R - \frac{aH}{k} (v_A - B). \quad (2.18)
\]

The energy-momentum tensor of scalar fields part is given by

\[
\left( \tilde{T}_\mu^\nu \right)_S = \tilde{\nabla}_\mu \tilde{\phi} \cdot \tilde{\nabla}_\nu \tilde{\phi} - \frac{1}{2} \left( \tilde{\nabla}^\lambda \tilde{\phi} \cdot \tilde{\nabla}_\lambda \tilde{\phi} + 2 \tilde{U} \right) \delta^\mu_\nu. \quad (2.19)
\]

Since divergence of the energy momentum tensor is given by

\[
\left( \tilde{\nabla}_\mu \tilde{T}_\nu^\mu \right)_S = \left( \tilde{\Box} \tilde{\phi}_a - \frac{\partial \tilde{U}}{\partial \tilde{\phi}_a} \right) \tilde{\nabla}_\nu \tilde{\phi}_a, \quad (2.20)
\]

in order that the phenomenological equations of motion of the scalar fields become

\[
\tilde{\Box} \tilde{\phi}_a - \frac{\partial \tilde{U}}{\partial \tilde{\phi}_a} = \tilde{S}_a, \quad (2.21)
\]

we assume that

\[
\left( \tilde{Q}_\nu \right)_S = \tilde{S}_a \tilde{\nabla}_\nu \tilde{\phi}_a, \quad (2.22)
\]

By using the scalar fields background variables \( \phi_a, \dot{\phi}_a, S_a \) and the corresponding perturbation variables \( D\phi_a, D\dot{\phi}_a, DS_a \), the background part of the fluid variables are given by

\[
\rho_S = \frac{1}{2} (\dot{\phi})^2 + U, \quad (2.23)
\]
\[
P_S = \frac{1}{2} (\dot{\phi})^2 - U, \quad (2.24)
\]
\[
h_S = (\dot{\phi})^2, \quad (2.25)
\]
\[
Q_S = -S \cdot \dot{\phi}, \quad (2.26)
\]
and the perturbation part of fluid variables are given by

\[(D\rho)_S = \frac{\partial \rho_S}{\partial \phi} \cdot D\phi + \frac{\partial \rho_S}{\partial \phi} \cdot D\dot{\phi}, \tag{2.27}\]

\[(DP)_S = \frac{\partial P_S}{\partial \phi} \cdot D\phi + \frac{\partial P_S}{\partial \phi} \cdot D\dot{\phi}, \tag{2.28}\]

\[(hZ)_S = -H \dot{\phi} \cdot D\phi, \tag{2.29}\]

\[(\Pi_T)_S = 0, \tag{2.30}\]

\[(DQ)_S = -S \cdot D\dot{\phi} - \dot{\phi} \cdot DS, \tag{2.31}\]

\[(aF_c)_S = S_a \left( -k D\phi_a - \frac{k}{H} \phi_a Z \right). \tag{2.32}\]

When the source of the scalar field \(\phi_a, S_a\) is given as functions of the covariant scalar quantities \(\tilde{T}\) and \(\tilde{T}_2\) whose background part is \(\bar{T}\), that is \(S_a = \bar{S}_a(\bar{T}, \bar{T}_2), DS_a\) is given by

\[DS_a = \frac{\partial S_a}{\partial T} \cdot DT + \frac{\partial S_a}{\partial \bar{T}} \cdot D\bar{T}. \tag{2.33}\]

In such case, \((DQ)_S\) can be written as

\[((DQ)_S = \frac{\partial Q_S}{\partial T} \cdot DT + \frac{\partial Q_S}{\partial \bar{T}} \cdot D\bar{T}, \tag{2.34}\]

which is assumed from now on. In the same way as the individual components \(\tilde{T}^\mu_{\alpha \nu}\), as for the total energy-momentum tensor \(\tilde{T}^\mu_{\alpha \nu} = \sum_A \tilde{T}^\mu_{\alpha \nu}\), we can define the gauge invariant perturbation variables such as \(D\rho, DP, hZ\) and \(\Pi_T\). From (2.10),(2.11), we obtain the background equations as

\[\rho = \rho_S + \sum\alpha \rho_\alpha, \tag{2.35}\]

\[P = P_S + \sum\alpha P_\alpha, \tag{2.36}\]

\[h = h_S + \sum\alpha h_\alpha, \tag{2.37}\]

\[0 = Q_S + \sum\alpha Q_\alpha. \tag{2.38}\]
and perturbation equations as

\[
D\rho = D\rho_S + \sum_\alpha D\rho_\alpha, \quad \tag{2.39}
\]

\[
DP = DP_S + \sum_\alpha DP_\alpha, \quad \tag{2.40}
\]

\[
hZ = (hZ)_S + \sum_\alpha h_\alpha Z_\alpha, \quad \tag{2.41}
\]

\[
\Pi_T = (\Pi_T)_S + \sum_\alpha \Pi_{T\alpha}, \quad \tag{2.42}
\]

\[
0 = (DQ)_S + \sum_\alpha DQ_\alpha, \quad \tag{2.43}
\]

\[
0 = (F_c)_S + \sum_\alpha F_{c\alpha}. \quad \tag{2.44}
\]

This \( Z \) is known as the Bardeen parameter [2] [12], [23]. In the long wavelength limit, the Bardeen parameter is conserved in the case where the entropy perturbations are negligible. But in various systems it was reported that the entropy perturbations cannot be neglected [3], [5], [6], [10], so in the present paper we will investigate the evolutionary behavior of the Bardeen parameter more carefully. Until now, as for the gauge invariant scalar quantity perturbation variables, we use \( D \). But traditionally most scalar quantity perturbation variables have been written without using \( D \):

\[
Y_a := D\phi_a, \quad \rho_\alpha \Delta_{ga} := D\rho_\alpha, \quad P_\alpha \Pi_{La} := DP_\alpha, \quad Q_\alpha E_{ga} := DQ_\alpha. \quad \tag{2.45}
\]

This \( Y_a \) has been called the Sasaki-Mukhanov variable [28] [22].

In terms of the gauge independent variables defined above, we give the evolution equations of cosmological perturbations. From [2,21], the background and the perturbation parts can be written as

\[
\ddot{\phi} + 3H\dot{\phi} + \frac{\partial U}{\partial \phi} + S = 0, \quad \tag{2.46}
\]

\[
L_1(DT, A) = -\frac{k^2}{a^2} D\phi - \frac{k^2}{a^2 H} \dot{\phi}, \quad \tag{2.47}
\]

where

\[
L_1(DT, A) = (D\phi)^\dagger + 3H(D\phi)^\dagger + \frac{\partial^2 U}{\partial \phi^2 \phi} D\phi + DS - \dot{\phi} A + 2\left(\frac{\partial U}{\partial \phi} + S\right) A. \quad \tag{2.48}
\]

As for the fluid components, \( \tilde{\nabla}_\mu \tilde{T}_{\alpha\nu} = \tilde{Q}_{\alpha\nu} \) gives the background equations as

\[
\dot{\rho}_\alpha = -3H h_\alpha + Q_\alpha, \quad \tag{2.49}
\]

and the perturbation equations as

\[
L_{2\alpha}(DT, A) = -\frac{k^2}{a^2 H} h_\alpha (\Phi - Z_\alpha), \quad \tag{2.50}
\]

\[
\left(\frac{h_\alpha Z_\alpha}{H}\right) + 3h_\alpha Z_\alpha + h_\alpha A + DP_\alpha - \frac{2}{3} \Pi_{T\alpha} = -\frac{a}{k} F_{c\alpha} + \frac{Q_\alpha}{H} Z, \quad \tag{2.51}
\]

7
where
\[ L_{2\alpha}(DT, A) = (D \rho_\alpha)' + 3H D \rho_\alpha + 3H D P_\alpha - Q_\alpha A - D Q_\alpha. \] (2.52)
\[ \tilde{G}^\mu_\nu = \kappa^2 \tilde{T}^\mu_\nu \] gives the background equations as
\[ H^2 = \frac{\kappa^2}{3} \rho, \] (2.53)
\[ \dot{\rho} = -3Hz, \] (2.54)
\[ \dot{H} = -\frac{3}{2}(1 + w)H^2, \] (2.55)
and the perturbation equations as
\[ L_3(DT, A) = 2\rho \frac{k^2}{3a^2H^2} \Phi, \] (2.56)
\[ L_4(DT, A) = -\frac{\kappa^2}{3} \Pi_T - \frac{k^2}{a^2} \Phi, \] (2.57)
\[ A + \frac{3}{2}(1 + w)Z = 0, \] (2.58)
\[ A + \frac{1}{a} \left( \frac{a}{H} \Phi \right) = -\frac{\kappa^2}{k^2 a^2} \Pi_T, \] (2.59)

where
\[ L_3(DT, A) = 2\rho A + D\rho, \] (2.60)
\[ L_4(DT, A) = H \dot{A} + 2\dot{H}A - \frac{\kappa^2}{2} (D\rho + DP). \] (2.61)

The dynamical perturbation variables are classified into two groups, that is, what has analogy with the exactly homogeneous perturbations and what is not related with the exactly homogeneous perturbations at all. The dynamical perturbation variables of the former type are \( DT \) representing the scalar quantity \( T = (\rho, P, \phi, Q, S) \) perturbation in the flat slice, \( D\dot{T} \) and the metric perturbation variable \( A \). The dynamical perturbation variables of the latter type are the Newtonian gravitational potential \( \Phi \) and \( Z_A, F_{cA}, \Pi_{TA} \) which have vector or tensor origin. In the above \( L_i(i = 1, \cdots, 4) \) equations, the former type dynamical perturbation variables are contained in the left hand side while the latter type perturbation variables are collected in the right hand side. The exactly homogeneous perturbations \( DT^\sharp \) and \( A^\sharp \) corresponding to \( DT \) and \( A \), respectively are constructed as
\[ (DT)^\sharp := \left( \frac{\partial T}{\partial C} \right)_t - \frac{T}{H} \mathcal{R}^\sharp, \] (2.62)
\[ A^\sharp := -\left( \frac{\mathcal{R}^\sharp}{H} \right), \] (2.63)
\[ \mathcal{R}^\sharp := \frac{1}{a} \left( \frac{\partial a}{\partial C} \right)_t, \] (2.64)

where \( C \) is the solution constant of the background solution and the subscript \( t \) implies that the derivative with respect to \( C \) is performed with the cosmological time \( t \) fixed. On
the other hand, the dynamical perturbation variables of the latter type such as $\Phi$, $Z_A$, $F_{cA}$ and $\Pi_{TA}$ do not have exactly homogeneous counterparts. The evolution equations of cosmological perturbations containing $L_i(i = 1, \cdots, 4)$ have analogy in the exactly homogeneous perturbation equations. In fact, the variations of the exactly homogeneous equations (2.46), (2.49), (2.53) and (2.55) give

$$L_i(DT^A, A^A) = 0 \quad (i = 1, \cdots, 4),$$

(2.65) respectively. The only difference between the exactly homogeneous perturbation $L_i(i = 1, \cdots, 4)$ equations and the actual $k \neq 0$ cosmological perturbation $L_i(i = 1, \cdots, 4)$ equations is that $k^2\Phi$ terms exist in the latter but $k^2\Phi$ terms do not exist in the former. Then the effect of the source term $k^2\Phi$ is corrected in the following way. In performing the correction process, it is important to notice that the source terms $k^2\Phi$ can be represented in terms of conserved quantity which has origin in the universal adiabatic decaying mode. In fact, as for $f$ defined by

$$f = a^3H \left( A + \frac{1}{2}\Delta_g \right) = \frac{k^2}{3H}a\Phi,$$

(2.66) using (2.50), (2.56), (2.57) yields

$$\frac{df}{dt} = -a^3H^2w\Pi_T + \frac{1}{2}ak^2(1+w)Z.$$  

(2.67) When we assume that for $k \to 0$ limit

$$\Pi_T \to 0, \quad kZ \to 0,$$

(2.68) are satisfied, the quantity $f$ is conserved, whose value is written as $c$. Therefore for $k \to 0$ limit,

$$k^2\Phi \to \frac{3H}{a}c = O(1).$$

(2.69) This expression of $\Phi$ is well known as that of the universal adiabatic decaying mode [13]. In the $L_i(i = 1, \cdots, 4)$ equations containing $DT$, $A$, the Newtonian potential $\Phi$ appears only in the form $k^2\Phi$, that is, accompanied by $k^2$. When we assume that $DT = O(1)$, $A = O(1)$, $k^2\Phi$ behaves as $O(1)$. Since in the linear perturbation, the scale of the perturbation variables is arbitrary, the fact that $\Phi = O(1/k^2)$ does not imply the breakdown of the linear perturbation. If one want to get $\Phi = O(1)$, one simply assumes that $DT = O(k^2)$, $A = O(k^2)$. But as explained later, we cannot assume that $\Phi$ is vanishing, since $c$ defined by (2.69) must satisfy the constraint (2.80). Therefore in the $k \to 0$ limit where (2.68), (2.69) are satisfied, (2.47), (2.50), (2.56), (2.57) can be written as

$$L_{1A}(DT, A) = -\frac{3\phi_a}{a^3}c,$$

(2.70)

$$L_{2A}(DT, A) = -\frac{3h_\alpha}{a^3}c,$$

(2.71)

$$L_3(DT, A) = \frac{2\rho}{a^3H}c,$$

(2.72)

$$L_4(DT, A) = -\frac{3H}{a^3}c.$$  

(2.73)
It can be verified that above four sets of equations (2.70), (2.71), (2.72) and (2.73) are satisfied by

\[ A = \frac{3}{2}(1 + w)g + \frac{\dot{g}}{H}, \]  

\[ DT = \frac{T}{H}g, \]  

where

\[ g = c \int_{t_0}^{t} dt \frac{1}{a^3}. \]  

This special solution for \( A = (DT, A) \) is written as \( A^\flat \). Since the variation of the exactly homogeneous solution \( A^\sharp \) satisfies (2.65), the general solutions of (2.70), (2.71), (2.72), (2.73) \( A = (DT, A) \) can be expressed as

\[ A = A^\sharp + A^\flat. \]  

The perturbation equations except \( L_i (i = 1, \ldots, 4) \) equations have vector origin, that is, they are derived from the space component of the Einstein equations. Therefore these perturbation equations do not have any analogy with the exactly homogeneous perturbation equations. As explained in the paper [15], these perturbation equations determine the evolutions of the dynamical perturbation variables which have vector or tensor origin, that is, which have no correspondence with the exactly homogeneous pertubations, or give the constraint which should be satisfied in order that the exactly homogeneous perturbations become the \( k \to 0 \) limit of evolutions of cosmological perturbations. Therefore (2.51), (2.58) can be interpreted as the decision of the evolution of the variables \( Z_\alpha \) which is not related to the exactly homogeneous solution at all in terms of \( A = (DT, A) \), the constraint to the exactly homogeneous perturbations, respectively. Integrating (2.51) yields

\[ h_\alpha Z_\alpha \to H \int_{t_0}^{t} dt \alpha^3 \left[ -h_\alpha A - DP_\alpha - \frac{a}{k} F_{\alpha} + \frac{Q_\alpha}{H} Z_\alpha \right]. \]  

By summing (2.78) with respect to all the fluid components, we obtain

\[ (hZ)_f = \int_{t_0}^{t} dt \alpha^3 \left[ \sum \alpha C_\alpha \left( -h_\alpha A - (DP)_f - S \cdot D\phi \right) \right]. \]

Therefore (2.58) gives the constraint between \( C_\alpha \), \( c \) defined by (2.69), and \( 2N_S + N_f \) solution constants of the exactly homogeneous perturbation as

\[ \sum \alpha C_\alpha + \frac{2}{k^2} c - \left( \frac{2}{3} a^3 H \rho A + a^3 \phi \cdot D\phi \right)_0 = 0. \]  

Integrating (2.59) gives

\[ \Phi = \frac{H}{a} \left( C_t - \int_{t_0}^{t} dt \alpha A \right), \]
where the first term containing \( C_t \) is well known universal adiabatic decaying mode [15] and by comparing with (2.69) we obtain

\[
C_t = \frac{3}{k^2} c. \tag{2.82}
\]

If we assume \( c = 0 \), since (2.80) gives one constraint relation, we obtain \( 2N_S + 2N_f - 1 \) solutions and the Newtonian potential is obtained by (2.81) with \( C_t = 0 \):

\[
\Phi = -\frac{H}{a} \int_{t_0}^t dt a A, \tag{2.83}
\]

If we assume that \( c \) is nonvanishing, since \( c = O(1) \), \( A \to O(1) \), therefore

\[
\frac{A}{C_t} \to O(k^2), \tag{2.84}
\]

we obtain the \( k \to 0 \) limit of the universal adiabatic decaying mode [15]:

\[
\frac{1}{3} k^2 \Phi \to \frac{H}{a} c, \tag{2.85}
\]

which is consistent with (2.69). Then we have obtained the long wavelength limit of all the solutions to the evolution equations of cosmological perturbations.

§3 Use of the scale factor as the evolution parameter

As the gauge invariant variable representing the fluctuation of the scalar quantity \( T \), we adopt \( DT \) defined by (2.5) which represents the \( T \) fluctuation in the flat slice, since it is the easiest to see the correspondence with the exactly homogeneous perturbation of \( T \). While until now we described the exactly homogeneous variables as functions of \( t, C \) where \( t \) is the cosmological time and \( C \)'s are solution constants, we can describe the exactly homogeneous variables as functions of \( a, C \) where \( a \) is the scale factor. For an arbitrary scalar quantity such as \( \rho, P, S, Q, \phi \), from (2.62), (2.64), \((DT)^\sharp \) can be written as the partial derivative of the corresponding exactly homogeneous scalar quantity \( T \) with respect to solution constant \( C \) with the scale factor \( a \) fixed:

\[
(DT)^\sharp = \left( \frac{\partial T}{\partial C} \right)_a. \tag{3.1}
\]

Since

\[
\frac{1}{a} \left( \frac{\partial}{\partial C} \dot{a} \right)_a = \frac{1}{2\rho} \left( \frac{\partial \rho}{\partial C} \right)_a = \frac{1}{2\rho} (D\rho)^\sharp = -A^\sharp, \tag{3.2}
\]

this property of \( D \) also holds as for the time derivative of the scalar quantity \( \dot{T} \):

\[
\left( D\dot{T} \right)^\sharp = \left( \frac{\partial}{\partial C} \dot{T} \right)_a. \tag{3.3}
\]

Therefore the operator \( D \) defined by (2.5) can be interpreted as a kind of derivative operator, that is, the derivative with respect to the solution constant \( C \) with \( a \) fixed. Because
of this derivative property of $D$, as for the scalar quantities $T = (\rho, P, S, Q)$ which are functions of $\phi$, $\dot{\phi}$, we can understand

$$DT = \frac{\partial T}{\partial \phi} \cdot D\phi + \frac{\partial T}{\partial \dot{\phi}} \cdot D\dot{\phi}, \quad (3.4)$$

easily. The contribution to $D\dot{T}$ from the adiabatic decaying mode is given by the same form as that of $DT$:

$$\left(D\dot{T}\right)^b = \frac{\dot{T}}{H} g, \quad (3.5)$$

where $g$ is defined by (2.76).

As seen from the above discussion, in order to derive the long wavelength limit of cosmological perturbations from the corresponding exactly homogeneous system by using the LWL formula, it is more appropriate to use as the evolution parameter the scale factor $a$ than the cosmological time $t$. For example, as for the scalar-fluid composite system, the corresponding exactly homogeneous expressions are obtained by solving the first order differential equations setting $\phi_a, p_a := a^3 \dot{\phi}_a, \rho_a$ as independent variables and the scale factor $a$ as the evolution parameter:

$$a \frac{d}{da} \phi_a = \frac{1}{H} \frac{p_a}{a^3},$$

$$a \frac{d}{da} p_a = -a^3 \frac{\partial U}{H \partial \phi_a} - \frac{a^3}{H} S_a,$$

$$a \frac{d}{da} \rho_a = -3h_a + \frac{Q_a}{H}, \quad (3.6)$$

replacing $H$ with the right hand side of the Hubble law:

$$H^2 = \frac{\kappa^2}{3} \left[ \frac{1}{2a^6} \sum_a p_a^2 + U(\phi) + \sum_a \rho_a \right]. \quad (3.7)$$

While under use of $t$ as the evolution parameter our system is the constrained system with the Hamiltonian constraint (3.7), under use of $a$ as the evolution parameter our system becomes the unconstrained system with respect to independent variables $\phi_a, p_a := a^3 \dot{\phi}_a, \rho_a$. The corresponding first order perturbation variables are $D\phi_a, P_a := a^3 D\dot{\phi}_a, D\rho_a$.

For some time, we consider the system consisting of multiple scalar fields $\phi_a$ only. Since the evolutions of $\phi_a, p_a := a^3 \dot{\phi}_a$ can be described in terms of the Hamilton equations of motion, the evolutions of the corresponding perturbation variables $Y_a = D\phi_a, P_a := a^3 D\dot{\phi}_a$ can also be written in terms of the Hamilton equations of motion:

$$\frac{dY_a}{dt} = \frac{\partial \bar{H}}{\partial P_a}, \quad \frac{dP_a}{dt} = -\frac{\partial \bar{H}}{\partial Y_a}, \quad (3.8)$$

whose Hamiltonian is given by

$$\bar{H} = \frac{1}{2a^3} P_a P_a + \frac{a^3}{2} \tilde{V}_{ab} Y_a Y_b + \frac{\kappa^2}{2H} \dot{\phi}_a \dot{\phi}_b P_a Y_b, \quad (3.9)$$

$$\tilde{V}_{ab} = \frac{\partial^2 U}{\partial \phi_a \partial \phi_b} + \frac{3\kappa^2}{2} \dot{\phi}_a \dot{\phi}_b + \frac{\kappa^2}{2H} \left( \frac{\partial U}{\partial \phi_a} \dot{\phi}_b + \dot{\phi}_a \frac{\partial U}{\partial \phi_b} \right) + \frac{\kappa^2}{a^2} \delta_{ab}. \quad (3.10)$$
When we discuss the quantization of the fluctuations, the other set of the canonical perturbation variables $\tilde{Y}_a := Y_a, \tilde{P}_a := a^3 \dot{Y}_a$ has been used [23]. But in the viewpoint of the LWL formalism, the set of the canonical variables $Y_a := D\phi_a, P_a := a^3 \dot{D}\phi_a$ is more natural than the set of the canonical variables $\tilde{Y}_a := Y_a, \tilde{P}_a := a^3 \dot{Y}_a$, because the long wavelength limit of the former set is generated from the derivative of the homogeneous variables $\phi_a, p_a := a^3 \dot{\phi}_a$ with respect to the solution constants with the scale factor fixed. The connection between the old canonical variables $\tilde{Y}_a := Y_a, \tilde{P}_a := a^3 \dot{Y}_a$ and the new canonical variables $Y_a := D\phi_a, P_a := a^3 \dot{D}\phi_a$ are given by the canonical transformation defined by the generating function:

$$W = \tilde{Y}_a P_a + \frac{3}{4} a^3 \frac{H}{\rho} \dot{\phi}_a \dot{\phi}_b \tilde{Y}_a \tilde{Y}_b.$$ (3.11)

When we treat the oscillatory scalar fields, the action angle variables $I_a, \theta_a$ are useful [9], [10]:

$$\phi_a = \frac{1}{a^{3/2}} \sqrt{\frac{2 I_a}{m_a}} \cos \theta_a,$$

$$p_a = -a^{3/2} \sqrt{2m_a I_a} \sin \theta_a,$$ (3.12)

where $m_a$ is the mass of the scalar field $\phi_a$. The action angle variables obey the evolution equation as

$$a \frac{d}{da} I_a = -a^3 \frac{\partial U_{\text{int}}}{\partial \theta_a} + a^{3/2} \frac{H}{\sqrt{2I_a}} \sin \theta_a S_a + 3I_a \cos 2\theta_a,$$ (3.13)

$$a \frac{d}{da} \theta_a = \frac{m_a}{H} + a^3 \frac{\partial U_{\text{int}}}{\partial I_a} + a^{3/2} \frac{1}{\sqrt{2m_a I_a}} \cos \theta_a \frac{S_a}{2} - \frac{3}{2} \sin 2\theta_a.$$ (3.14)

In order to investigate the cosmological perturbations in the universe containing oscillatory scalar fields, by using $D$ defined in the above we define the action angle perturbation variables $DI_a, D\theta_a$ starting from $Y_a := D\phi_a, P_a := a^3 \dot{D}\phi_a$. In the LWL formalism, the perturbation variables corresponding with the action angle variables $I_a, \theta_a$ are $DI_a, D\theta_a$ defined by the following expressions:

$$Y_a = D \left[ \frac{1}{a^{3/2}} \sqrt{\frac{2 I_a}{m_a}} \cos \theta_a \right],$$

$$P_a = D \left[ -a^{3/2} \sqrt{2m_a I_a} \sin \theta_a \right].$$ (3.15)

where $D$ in the right hand side is interpreted as

$$D = \sum_a DI_a \frac{\partial}{\partial I_a} + \sum_a D\theta_a \frac{\partial}{\partial \theta_a}.$$ (3.16)

The expressions obtained from variations of $I_a, \theta_a$ with $a$ fixed in the previous papers [9], [10] are the long wavelength limits of $DI_a, D\theta_a$ defined by (3.15). In fact, the LWL formulae

$$DI_a = \left( \frac{\partial I_a}{\partial C} \right)_a + \left( \frac{\dot{I}_a}{H} - 3I_a \right) g,$$ (3.17)

$$D\theta_a = \left( \frac{\partial \theta_a}{\partial C} \right)_a + \left( \frac{\dot{\theta}_a}{H} - \frac{1}{2} \right) g.$$ (3.18)
where $g$ is defined in (2.76) hold. While the third term in the right hand side of (3.17) appears because of the scale factor $a$ dependence of the transformation law from $\phi_a$, $\dot{\phi}_a$ to $I_a$, $\theta_a$, the $\sharp$ parts of (3.17) and (3.18) reflect the fact that $D$ is the derivative operator with respect to the solution constant with the scale factor $a$ fixed.

In order to solve the dynamics of the system containing the oscillatory scalar fields, we are required to perform the averaging over the fast changing angle variables $\theta_a$ [9], [10]. If we use the cosmological time $t$ as the evolution parameter, our system is a constrained system. Therefore we must check that our averaging procedure is consistent with the constraint and this process is rather cumbersome. But if we use the scale factor $a$ as the evolution parameter, our system becomes unconstrained system, so the definition of the averaging procedure becomes rather simple.

We can conclude that use of the scale factor $a$ as the evolution parameter brings about the two merits. One is that it becomes easier to see the correspondence between the exactly homogeneous solution and the long wavelength limit of the first order perturbation and that the LWL formulae become more simple. The other is the more simple definition of the averaging process.

We consider the evolution of the Bardeen parameter $Z$. Following the paper [34], we define $\zeta$ as the gauge invariant variable representing the curvature perturbation in the uniform density slice:

$$\zeta := -\frac{H}{\dot{\rho}}D\rho = \mathcal{R} - \frac{H}{\dot{\rho}}\delta \rho.$$  \hspace{1cm} (3.19)

From (2.56) (2.58) we can see that the Bardeen parameter $Z$ and $\zeta$ are closely connected as

$$Z = \zeta - \frac{2}{9}\frac{k^2}{1 + w\frac{a^2}{H^2}}\Phi,$$  \hspace{1cm} (3.20)

whose $k \to 0$ limit is

$$Z = \zeta - \frac{2}{3}\frac{c}{1 + w\frac{a^3}{H}},$$  \hspace{1cm} (3.21)

where $c$ is constant related with the adiabatic decaying mode defined by (2.69). While the Bardeen parameter $Z$ is expressed as the weighted sum of $Z_S$, $Z_\alpha$ which do not related with the exactly homogeneous quantity at all and whose evolution is written in the rather cumbersome integral form (2.78), $\zeta$ is connected with the corresponding exactly homogeneous quantity and $\zeta^\sharp$ evolution can be written in terms of the derivative of the total energy density $\rho$ with respect to solution constant. Then we consider much easier $\zeta^\sharp$ evolution.

In the paper [19], as the nonlinear generalization of the Bardeen parameter $\zeta$, $\zeta(t, x)$ was introduced. When $P = P(\rho)$, $\zeta(t, x)$ is reduced to

$$\zeta(t, x) = \ln a(t, x) + \frac{1}{3} \int_{\rho(t, x)}^{\rho} \frac{d\rho}{\rho + P(\rho)}.$$  \hspace{1cm} (3.22)

In fact, the first order quantity of $\zeta(t, x)$ is given by

$$\zeta_1(t, x) = \frac{\delta a(t, x)}{a} + \frac{1}{3} \frac{\delta \rho(t, x)}{\rho + P(\rho)}.$$  \hspace{1cm} (3.23)

which agrees with the Bardeen parameter $\zeta$. In the viewpoint of the LWL formalism, we adopt the zero curvature slice $\partial a(t, x)/\partial x^i = 0$. We assume the equation of state $P_r = \rho_r/3$,
since the final state of reheating and of the curvaton decay is radiation dominant. In this case, the above \( \zeta(t, \mathbf{x}) \) is reduced to

\[
\zeta(a, \mathbf{x}) = \frac{1}{4} \ln \rho_r(a, C(\mathbf{x})).
\]  
(3.24)

\( \rho_r(a, C(\mathbf{x})) \) is the expression of \( \rho_r \) obtained by solving the locally homogeneous system (see the separate universe approach \([34]\)) with use of \( a \) as the evolution parameter. The solution constants \( C(\mathbf{x}) \) have spatial dependence. By considering \( C(\mathbf{x}) = C + \delta C(\mathbf{x}) \) and expanding with respect to \( \delta C(\mathbf{x}) \), we can obtain the perturbation of an arbitrary order up to the decaying modes of perturbations. For example, the \( n \)-th order perturbation is given by

\[
\frac{1}{n!} \sum_{a_1} \frac{\partial^n}{\partial C_{a_1} \partial C_{a_2} \cdots \partial C_{a_n}} \left[ \frac{1}{4} \ln \rho_r(a, C) \right] \delta C_{a_1}(\mathbf{x}) \delta C_{a_2}(\mathbf{x}) \cdots \delta C_{a_n}(\mathbf{x}).
\]  
(3.25)

When we obtain the exactly homogeneous expression \( \rho_r = \rho_r(a, C) \), we can know the long wavelength limits of perturbations of arbitrary orders. Later we determine the expressions \( \rho_r = \rho_r(a, C) \) in the reheating and in the curvaton decay.

In the reheating and in the curvaton decay, \( \delta C(\mathbf{x}) \)'s are given by the action angle variables in the initial time:

\[
\delta I_\alpha(\mathbf{x}) := \delta I_\alpha(a = a_0, \mathbf{x}), \quad \delta \theta_\alpha(\mathbf{x}) := \delta \theta_\alpha(a = a_0, \mathbf{x}).
\]  
(3.26)

We discuss how to determine the statistical properties of \( \delta I_\alpha(\mathbf{x}), \delta \theta_\alpha(\mathbf{x}) \). \( \delta I_\alpha(\mathbf{x}), \delta \theta_\alpha(\mathbf{x}) \) are given at the time when the slow rolling phase ends and the coherent oscillation begins. At this time, \( \dot{\phi}_\alpha = 0 \), and as for the perturbations

\[
\delta \phi_\alpha(\mathbf{x}) = \int d^3k e^{i \mathbf{k} \cdot \mathbf{x}} e_a(\mathbf{k}), \quad \dot{\delta \phi_\alpha}(\mathbf{x}) = 0,
\]  
(3.27)

where \( e_a(\mathbf{k}) \) is the Gaussian random variable satisfying

\[
< e_a(\mathbf{k}) e_b(\mathbf{k}') >= P_a(k) \delta_{ab} \delta^3(\mathbf{k} + \mathbf{k}'), \quad k := |\mathbf{k}|.
\]  
(3.28)

By solving \( I_\alpha, \theta_\alpha \) in terms of \( \phi_\alpha, \dot{\phi}_\alpha \) and by Taylor expanding, we obtain

\[
\delta I_\alpha(\mathbf{x}) = a_0^3 m_a \phi_\alpha(\mathbf{x}) + \frac{1}{2} a_0^3 m_a [\delta \phi_\alpha(\mathbf{x})]^2,
\]  
(3.29)

\[
\delta \theta_\alpha(\mathbf{x}) = 0,
\]  
(3.30)

where we use \( \dot{\phi}_\alpha = 0, \dot{\delta \phi_\alpha}(\mathbf{x}) = 0 \). The above fact that \( \delta \theta_\alpha(a = a_0, \mathbf{x}) = 0 \) does not imply \( \delta \theta_\alpha(a, \mathbf{x}) = 0 \), since \( \theta_\alpha(a) \) is a function depending on not only \( \theta_\alpha(a_0) \) but also \( I_\alpha(a_0) \). Therefore it is often important to consider the role of the perturbations of angle variables.

Determining the many point correlation function of fluctuations is reduced to the evaluation of

\[
< e_{a_1}(\mathbf{k}_1) e_{a_2}(\mathbf{k}_2) \cdots e_{a_n}(\mathbf{k}_n) >.
\]  
(3.31)

This quantity can be determined by applying the differential operation defined by

\[
\exp \left[ \sum_a \int d^3k P_a(k) \frac{\delta}{\delta e_a(\mathbf{k})} \frac{\delta}{\delta e_a(-\mathbf{k})} \right]_{\epsilon=0}.
\]  
(3.32)
§4 Application of the LWL formula to the non-interacting multicomponent system

Based on the results obtained in sections 2, 3, we consider the long wavelength limit of the evolutions of cosmological perturbations in the universe consisting of multiple cosmic components. For simplicity, we consider the case where each component do not interact with each other.

We consider the $w_\alpha$ fluid where $w_\alpha$ is constant and which does not interact with other components $Q_\alpha = 0$. $\rho_\alpha$ is solved as

$$\rho_\alpha = \frac{A_\alpha}{a^{3(1+w_\alpha)}}, \quad (4.1)$$

where $A_\alpha$ is a solution constant and by differentiating with respect to $A_\alpha$ with the scale factor $a$ fixed, we obtain

$$(D\rho_\alpha)^\sharp = \frac{\delta A_\alpha}{a^{3(1+w_\alpha)}}, \quad (4.2)$$

where $\delta A_\alpha$ is a perturbation solution constant corresponding with $A_\alpha$. Therefore we obtain

$$\Delta_g^\sharp = \frac{\delta A_\alpha}{A_\alpha} = \text{const.} \quad (4.3)$$

Next we consider the case that the oscillatory scalar field $\phi_a$ does not interact with other cosmic components, that is $U_{\text{int}} = 0$, $S_a = 0$. Since in such case the right hand side of (3.13) is oscillatory function depending on the angle variable $\theta_a$ with vanishing mean value, by taking the averaging over $\theta_a$, we obtain

$$I_a \approx A_a, \quad (4.4)$$

where $A_a$ is constant. The estimate about the effects of the oscillations due to the fast changing angle variables $\theta_a$ was discussed in the previous papers [9] [10]. Therefore by taking the derivative with respect to solution constant $A_a$ with the scale factor $a$ fixed, we obtain

$$D I_a^\sharp \approx \delta A_a, \quad (4.5)$$

where $\delta A_a$ is a perturbation constant corresponding with $A_a$. Since

$$\rho_a = \frac{m_a I_a}{a^3}, \quad D\rho_a = \frac{m_a D I_a}{a^3}, \quad (4.6)$$

where the above second expression is given by the $D$ operation to the above first expression, we obtain

$$\Delta_g^\sharp = \frac{D I_a^\sharp}{I_a} \approx \frac{\delta A_a}{A_a} = \text{const.} \quad (4.7)$$

As seen from the above expressions, as for the oscillatory scalar field $\phi_a$ the energy density $\rho_a$ and the energy density perturbation $\rho_a \Delta_g^{\sharp}$ behave like those of the dust fluid. We can summarize that for non-interacting cosmic components, $\Delta_g^\sharp$, $\Delta_g^{\sharp}$ are conserved.
As for the multicomponent non-interacting fluids system, $\zeta^\#$ evolution is given by

$$\zeta^\# = \left( \sum_{\alpha} \frac{\delta A_\alpha}{a^{3(1+w_\alpha)}} \right) / \left( 3 \sum_{\alpha} \frac{A_\alpha}{a^{3(1+w_\alpha)}} \right).$$

(4.8)

We can see that $\zeta^\#$ is exactly conserved for the adiabatic growing mode defined by

$$\frac{\delta A_\alpha}{A_\alpha} \frac{1}{1+w_\alpha} = \alpha \text{ independent}.$$  

(4.9)

For some time, we consider the two components system consisting of dust and radiation. In this case, (4.8) is reduced to

$$\zeta^\# = \frac{a \delta A_d + \delta A_r}{3aA_d + 4A_r},$$

(4.10)

where the suffix $d$, $r$ imply dust, radiation, respectively. We obtain

$$\zeta^\#_{\text{init}} = \frac{1}{4} \frac{\delta A_r}{A_r}, \quad \zeta^\#_{\text{fin}} = \frac{1}{3} \frac{\delta A_d}{A_d},$$

(4.11)

in the limit $a \to 0, a \to \infty$, respectively. We adopt different, more physical parameterization:

$$\delta A_d = 3\xi A_d - \eta A_d, \quad \delta A_r = 4\xi A_r.$$  

(4.12)

$\xi$ represents the adiabatic growing mode and $\eta$ represents the isocurvature mode defined by

$$\zeta^\#_{\text{init}} = 0, \quad S^\#_{rd} = \frac{3}{4} \Delta^\#_{gr} - \Delta^\#_{gd} = \text{const} =: \eta.$$  

(4.13)

Then we obtain

$$\zeta^\#_{\text{fin}} = \zeta^\#_{\text{init}} - \frac{1}{3} \eta,$$

(4.14)

which is the famous formula.\[13\]

Although in the paper [34], (4.10) has already been derived essentially without using the LWL formula, our result is more rigorous in the point that we treat the contribution from the adiabatic decaying mode characterized by $c$ defined by (2.69) more appropriately, while the paper [34] simply assumes that $k^2 \Phi$ is vanishing.

§5 Application of the averaging method to the decaying scalar fields

We derive the evolution equations of the multiple scalar fields decaying into the multiple radiation fluids. By solving these evolution equations and taking the exactly homogeneous perturbations, we can obtain the information of the evolutionary behaviors of cosmological perturbations during reheating and in the curvaton model. We assume that the source $S_a$ is given by

$$S_a = \Gamma_a \dot{\phi}_a.$$  

(5.1)
We nondimensionalize the dynamical quantities as
\[
\frac{a}{a_0} \to a, \quad \frac{I_a}{I_0} \to I_a, \quad \frac{\rho_\alpha}{\rho_0} \to \rho_\alpha, \quad \frac{U_{\text{int}}}{\rho_0} \to U_{\text{int}} = O(\nu),
\]
and the parameters as
\[
\frac{m_a}{m_0} \to m_a, \quad \frac{\Gamma_a}{\Gamma_0} \to \Gamma_a,
\]
where \( \nu \) is the small parameter implying the ratio of the interaction energy to the free part energy defined by
\[
\rho_0 := \frac{m_0 I_0}{a_0^3}.
\]
Then we obtain the dimensionless parameters
\[
\epsilon := \frac{\kappa \rho_0^{1/2}}{\sqrt{3} m_0}, \quad \gamma := \frac{\sqrt{3} \Gamma_0}{\kappa \rho_0^{1/2}},
\]
which imply the ratio of the Hubble parameter to the mass of the scalar fields \( H/m, \) the ratio of the decay rate to the Hubble parameter \( \Gamma/H \) at the initial time \( a = a_0, \) respectively.

By using the above dynamical variables and parameters, the evolution equations can be expressed as
\[
a \frac{d}{da} I_a = -\frac{1}{\epsilon} \frac{a^3}{\rho_0^{1/2}} \frac{\partial U_{\text{int}}}{\partial \theta_a} - \gamma \Gamma_a \frac{I_a}{\rho_0^{1/2}} (1 - \cos 2 \theta_a) + 3 I_a \cos 2 \theta_a,
\]
\[
a \frac{d}{da} \theta_a = \frac{1}{\epsilon} \frac{m_a}{\rho_0^{1/2}} + \frac{a^3}{\epsilon} \frac{\partial U_{\text{int}}}{\partial I_a} - \frac{1}{2} \frac{\gamma a}{\rho_0^{1/2}} \sin 2 \theta_a - \frac{3}{2} \sin 2 \theta_a,
\]
\[
a \frac{d}{da} \sigma_\alpha = \gamma a \frac{\sigma_\alpha}{\rho_0^{1/2}} \sum_a \Gamma_{\alpha a} m_a I_a (1 - \cos 2 \theta_a),
\]
where \( \sigma_\alpha \) is defined by
\[
\rho_\alpha = \frac{\sigma_\alpha}{a^3},
\]
and \( \Gamma_{\alpha a} \) is the decay rate from the scalar field \( I_a \) to the radiation component \( \sigma_\alpha \) and therefore \( \Gamma_a \) is given by
\[
\Gamma_a = \sum_\alpha \Gamma_{\alpha a}.
\]

In this paper, we investigate the evolutionary behavior of cosmological perturbations during the period when the decay rates from the scalar fields to the radiation fluids are large compared to the interaction between the scalar fields, that is \( \gamma \gg \nu/\epsilon, \) while in the paper [10] the evolutions of cosmological perturbations during the period when the interaction between the scalar fields is dominant \( \nu/\epsilon \gg \gamma \) were discussed, which is thought to give the initial conditions for the present studies.

Next we show that there exists a transformation such that in a system obtained by that transformation, the dynamics of the action variables \( I_a \) and the radiation energy densities \( \sigma_\alpha \) can be determined independently of the angle variables \( \theta_a. \) In order to show this statement, we put several assumptions.
(i) The interaction energy of the scalar fields $U_{\text{int}}$ is analytic with respect to the dynamical variables $I_a, \theta_a$ and is $2\pi$ periodic with respect to $\theta_a$. $U_{\text{int}}$ is bounded as

$$U_{\text{int}} \sim \frac{\nu}{a^{9/2}}|I|,$$

which implies that there exists a positive constant $M$ such that

$$|a^{9/2}U_{\text{int}}| \leq \nu M|I|,$$

where

$$|I| := \sum_a |I_a|.$$

(ii) As for the total energy density

$$\rho = \sum_a \frac{m_a I_a}{a^3} + \sum_{\alpha} \sigma_{\alpha} a^4 + U_{\text{int}},$$

the action variables $I_a$ and the radiation energy densities $\sigma_{\alpha}$ satisfy

$$c_1 \leq a \sum_a m_a I_a + \sum_{\alpha} \sigma_{\alpha} \leq c_2,$$

for some positive constants $c_1, c_2$.

Note that the evolution equations of the system can be written as

$$a \frac{d}{da} I_a = F_a(I, \sigma, \theta, a),$$

$$a \frac{d}{da} \sigma_{\alpha} = F_{\alpha}(I, \sigma, \theta, a),$$

$$a \frac{d}{da} \theta_a = \frac{1}{\epsilon} \omega_a(I, \sigma, a) + G_a(I, \sigma, \theta, a),$$

where $F_a, F_{\alpha}, G_a$ are analytic with respect to the dynamical variables $I_a, \sigma_{\alpha}, \theta_a$ and $2\pi$ periodic with respect to the angle variables $\theta_a$. We say that the evolution equations is of the type $C_k$:

(i) The averaged parts of $F_a, F_{\alpha}, G_a$ are bounded as

$$<F_a> \sim a^2 |I|,$$

$$<F_{\alpha}> \sim a^3 |I|,$$

$$<G_a> \sim a^2,$$

where $<A>$ implies the averaging over the angle variables $\theta_a$:

$$<A> := \frac{1}{(2\pi)^{N_S}} \int_0^{2\pi} d^{N_S} \theta \ A.$$

In case of the resonant case, it is prescribed that the averaging is performed with respect to the fast angle variables only [10].
(ii) The oscillatory parts of $F_a, F_{\alpha}, G_a$ are bounded as
\[
\begin{align*}
\tilde{F}_a &\sim \epsilon^3 a^2 |I|, \\
\hat{F}_a &\sim \epsilon^2 a^3 |I|, \\
\hat{G}_a &\sim \epsilon^2 a^2,
\end{align*}
\]
where $\tilde{A}$ implies the residual part after the averaging over the angle variables $\theta_a$:
\[
\tilde{A} := A - < A > .
\]

Under this notation, the following proposition holds.

**Proposition 1** Let $k$ be some non-negative integer and consider the evolution equations as
\[
\begin{align*}
{\frac{d}{da}} I_a^{(k)} &= F_a^{(k)}(I^{(k)}, \sigma^{(k)}, \theta^{(k)}, a), \\
{\frac{d}{da}} \sigma_a^{(k)} &= F_{\alpha}^{(k)}(I^{(k)}, \sigma^{(k)}, \theta^{(k)}, a), \\
{\frac{d}{da}} \theta_a^{(k)} &= \frac{1}{\epsilon} \omega_a^{(k)}(I^{(k)}, \sigma^{(k)}, a) + G_a^{(k)}(I^{(k)}, \sigma^{(k)}, \theta^{(k)}, a).
\end{align*}
\]

Suppose that this set of evolution equations is of the type $C_k$, there exists a transformation
\[
\begin{align*}
I_a^{(k)} &= I_a^{(k+1)} + u_a^{(k)}(I^{(k+1)}, \sigma^{(k+1)}, \theta^{(k+1)}, a), \\
\sigma_a^{(k)} &= \sigma_a^{(k+1)} + u_a^{(k)}(I^{(k+1)}, \sigma^{(k+1)}, \theta^{(k+1)}, a), \\
\theta_a^{(k)} &= \theta_a^{(k+1)} + \alpha_a^{(k)}(I^{(k+1)}, \sigma^{(k+1)}, \theta^{(k+1)}, a),
\end{align*}
\]
satisfying the following conditions:
\[
\begin{align*}
\text{(i)} \quad u_a^{(k)}, u_{a}^{(k)}, v_a^{(k)} \text{ are analytic with respect to the dynamical variables } I_a^{(k+1)}, \sigma_a^{(k+1)}, \theta_a^{(k+1)}, \\
\text{are } 2\pi \text{ periodic with respect to the angle variables } \theta_a^{(k+1)}, \text{ and are bounded as}\n
\begin{align*}
u_a^{(k)} &\sim \epsilon^{k+1} a |I^{(k+1)}|, \\
u_{a}^{(k)} &\sim \epsilon^{k+1} a |I^{(k+1)}|, \\
v_a^{(k)} &\sim \epsilon^{k+1}.
\end{align*}
\]
\[
\text{(ii)} \quad \text{The evolution equations of the transformed variables } I_a^{(k+1)}, \sigma_a^{(k+1)}, \theta_a^{(k+1)} \text{ are of the type } C_{k+1} \text{ and the changes of } < F_a > < F_{\alpha} > < G_a > \text{ are bounded as}\n
\begin{align*}
\Delta < F_a > &\sim \epsilon^{k+1} a^2 |I^{(k+1)}|, \\
\Delta < F_{\alpha} > &\sim \epsilon^{k+1} a^3 |I^{(k+1)}|, \\
\Delta < G_a > &\sim \epsilon^{k+1} a^2,
\end{align*}
\]
where $\Delta < A >$ is defined by
\[
\Delta < A > := < A^{(k+1)}(I^{(k+1)}, \sigma^{(k+1)}, \theta^{(k+1)}, a) > - < A^{(k)}(I^{(k+1)}, \sigma^{(k+1)}, \theta^{(k+1)}, a) > .
\]
This proposition implies that we can make the part depending on the angle variables arbitrarily small by taking the original set of the evolution equations as the starting point $k = 0$ and applying transformations given in the proposition iteratively. Therefore it can be expected that the evolution of the dynamical variables can be described by the truncated system obtained by discarding the angle variable dependent part with sufficiently good accuracy, if we take sufficiently large $k$. By estimating the errors produced by the truncation, we show that the above expectation is correct. We use the symbol $\Delta$ to represent the difference of a quantity for the exact system and a corresponding quantity for the truncated system. For $A = (I_a, \sigma, \theta)$, the errors of the background variables are written as

$$\Delta A := A - A_{\text{tr}}, \quad (5.40)$$

and the errors of the perturbation variables are written as

$$\Delta \delta A := \delta A - \delta A_{\text{tr}}, \quad (5.41)$$

where $A, \delta A$ represent quantities of the exact system and $A_{\text{tr}}, \delta A_{\text{tr}}$ represent quantities of the corresponding truncated system. For a function $f(a)$, let us write

$$f(a) = E(-a^2), \quad (5.42)$$

when $f(a)$ is bounded as

$$|f(a)| \leq p(a) \exp(-\lambda a^2) \quad (5.43)$$

for a polynomial of $a$: $p(a)$, and for a positive number $\lambda$. For a function $f(a)$, let us define $\|f(a)\|$ by

$$\|f(a)\| := \sup_{1 \leq a' \leq a} |f(a')|. \quad (5.44)$$

When we write all the inequalities, it is prescribed that all the coefficients of order unity are omitted. The truncation error for the $m$-th order system can be estimated as follows.

**Proposition 2A** Let $m$ be an integer larger than or equal to 2. For the $m$-th order system, the truncation errors of the background variables are given by

$$|\Delta I| \leq E(-a^2) \epsilon^m, \quad (5.45)$$

$$\|\Delta \sigma\|(a) \leq \epsilon^m, \quad (5.46)$$

$$|\Delta \theta| \leq a^2 \epsilon^{m-1}, \quad (5.47)$$

and the truncation errors of the perturbation variables are given by

$$|\Delta \delta I| \leq E(-a^2) \epsilon^{m-1} \delta A_1(1), \quad (5.48)$$

$$|\Delta \delta \sigma| \leq \epsilon^{m-1} \delta A_1(1), \quad (5.49)$$

$$|\Delta \delta \theta| \leq a^2 \epsilon^{m-2} \delta A_1(1) + \epsilon^{m-1} \exp(a^2 \epsilon^m) [a^2 \epsilon |\delta \theta(1)| + a^4 \delta A_m(1)], \quad (5.50)$$

where

$$\delta A_m(1) := |\delta I(1)| + |\delta \sigma(1)| + \epsilon^m |\delta \theta(1)|, \quad (5.51)$$
under the initial conditions

\[
\Delta I(1) = \Delta \sigma(1) = \Delta \theta(1) = 0, \quad (5.52)
\]
\[
\Delta \delta I(1) = \Delta \delta \sigma(1) = \Delta \delta \theta(1) = 0. \quad (5.53)
\]

(For the proof, see Appendix A.)

By the transformation laws, the errors of the \(m\)-th order variables affect the original variables as shown by the next proposition.

**Proposition 2B** The difference between \(A^{(0)}\) obtained from \(A^{(m)}\) by the transformation laws and \(A_{tr}^{(0)}\) obtained from \(A_{tr}^{(m)}\) by the same transformation laws has upper bound

\[
|\Delta I^{(0)}| \leq E(-a^2)e^m, \quad (5.54)
\]
\[
|\Delta \sigma^{(0)}| \leq e^m, \quad (5.55)
\]
\[
|\Delta \theta^{(0)}| \leq a^2e^{m-1}, \quad (5.56)
\]

and the corresponding difference as for the perturbation variables has upper bound

\[
|\Delta \delta I^{(0)}| \leq E(-a^2)e^{m-1}\delta A_1^{(m)}(1), \quad (5.57)
\]
\[
|\Delta \delta \sigma^{(0)}| \leq e^{m-1}\delta A_1^{(m)}(1), \quad (5.58)
\]
\[
|\Delta \delta \theta^{(0)}| \leq a^2e^{m-2}\delta A_1^{(m)}(1) + e^{m-1}\exp (a^2e^m) [a^2\epsilon|\delta \theta^{(m)}(1)| + a^4\delta A_1^{(m)}(1)], \quad (5.59)
\]

under the same initial condition as in the previous proposition.

(For the proof, see Appendix A.)

According to the above proposition, we can conclude that we can make the truncation errors as for the original variables arbitrarily small if we truncate the system at the arbitrarily large \(m\)-th order system.

From Proposition 1, we can see that the part independent of the angle variables of the evolution equations are shifted after the transformations reducing the part dependent on the angle variables. By the truncation, our system become much simpler than the original system. But it is still difficult to solve the truncated evolution equations with such correction terms because the evolution equations are complicatedly entangled with each other. In order to solve the evolution equations analytically, we want to discard such correction terms. The errors produced by discarding such corrections are evaluated in the following proposition.

**Proposition 3** The difference between the system with the correction terms produced by the transformation and the system obtained by discarding such correction terms is evaluated in the following way. As for the background variables, the discard errors are evaluated as

\[
|\Delta I| \leq E(-a^2)\epsilon, \quad (5.60)
\]
\[
|\Delta \sigma| \leq \epsilon, \quad (5.61)
\]
\[
|\Delta \theta| \leq a^2, \quad (5.62)
\]
and as for the perturbation variables, the discard errors are evaluated by

\[ |\Delta \delta I| \leq E(-a^2)\epsilon \delta B(1), \quad (5.63) \]
\[ |\Delta \delta \sigma| \leq \epsilon \delta B(1), \quad (5.64) \]
\[ |\Delta \delta \theta| \leq a^2 \delta B(1), \quad (5.65) \]

where

\[ \delta B(1) := |\delta I(1)| + |\delta \sigma(1)|, \quad (5.66) \]

under the initial conditions

\[ \Delta I(1) = \Delta \sigma(1) = \Delta \theta(1) = 0, \quad (5.67) \]
\[ \Delta \delta I(1) = \Delta \delta \sigma(1) = \Delta \delta \theta(1) = 0. \quad (5.68) \]

(For the proof, see Appendix A.)

By the above propositions, it can be understood that we can obtain the information of the original system with sufficiently good accuracy by investigating the evolution equations by simply dropping the part dependent on the angle variables, because the errors produced by dropping are sufficiently small and in particular \( \Delta \sigma, \Delta \delta \sigma \) are bounded. The reason why these errors are mild is that the final state in which all the energy of the scalar fields is completely transferred into that of radiation fluids is the attracting equilibrium around which the perturbations do not grow.

§6 Application of the LWL formula to the multicomponent reheating model

In this section, we apply the LWL formula to the reheating where the energy of the multiple scalar fields is transferred into that of the multiple radiation fluids. The decay rate from the scalar field \( \phi_a \) to the radiation fluid \( \rho_\alpha \) is given by \( \Gamma_{\alpha a} \). When the interactions between the scalar fields \( \phi_a, U_{\text{int}} \) are neglected, the background quantities are solved as

\[ m_a I_a = A_a \exp \left\{ -\gamma \Gamma_a \int_1^a \frac{1}{\rho^{1/2}a} \right\}, \quad (6.1) \]
\[ \sigma_\alpha = B_{\alpha} + \int_1^a \frac{\gamma}{\rho^{1/2}} \sum_a \Gamma_{\alpha a} m_a I_a, \quad (6.2) \]

where \( A_a, B_{\alpha} \) are integration constants. As long as we do not give the expression of the total energy density \( \rho \) in the integrals, the above solutions do not give any physical information of reheating. But it is difficult to solve the evolution equations in the form where the exact expression of \( \rho \) is explicitly described, because the evolution equations of \( I_a, \sigma_\alpha \) are complicated and highly nonlinear. Then we expect that the contribution to the integrations owes mainly to the period when the energy of the scalar fields is dominant, that is \( \rho \) can be approximated as

\[ \rho = \frac{A}{a^3} + \frac{B}{a^4}, \quad (6.3) \]
where
\[ A := \sum_a A_a, \quad B := \sum_a B_a. \tag{6.4} \]

We assume that the initial radiation energy density \( B \) is negligibly small \( B \ll A \). We assume that all \( \Gamma_a := \sum_{\alpha} \Gamma_{aa} (a = 1, 2, \cdots, N_S) \) are of the same order of magnitude. By substituting the above \( \rho \) expression to the solutions (6.1)-(6.2), by expanding the solutions with respect to \( B \) around \( B = 0 \), we obtain

\[ m_a I_a = A_a \exp \left\{ -\frac{2}{3} \frac{\gamma \Gamma_a}{A^{5/2}} a^{3/2} \right\} + \frac{\gamma \Gamma_a}{3 A^{3/2}} a^{1/2} \exp \left\{ -\frac{2}{3} \frac{\gamma \Gamma_a}{3 A^{5/2}} a^{3/2} \right\} + O(B^2), \]

\[ a^4 \rho_r = a^4 \sum_a \rho_a \]

\[ = B \left\{ 1 + \frac{3}{2} G(2) - \frac{1}{2} G(1) \right\} + \left( \frac{3}{2} \right)^{2/3} G(5/3) \frac{A^{1/3}}{\gamma^{2/3}} \sum_a \frac{A_a}{\Gamma_a^{2/3}} + O(B^2), \tag{6.5} \]

where
\[ \Gamma_a := \sum_{\alpha} \Gamma_{aa}, \tag{6.6} \]

and \( G(t) \) is the Gamma function defined by
\[ G(t) := \int_0^\infty dx x^{t-1} e^{-x}, \tag{6.7} \]

which is convergent for \( t > 0 \). By taking the exactly homogeneous perturbation of (6.5) defined by
\[ D := \left( \delta A \cdot \frac{\partial}{\partial A} + \delta B \cdot \frac{\partial}{\partial B} \right)_{a B=0}, \tag{6.8} \]

we can obtain the Bardeen parameter in the final state \( a \to \infty \):

\[ \zeta_{\text{fin}}^2 = \frac{1}{4} \frac{D \rho_r^2}{\rho_r} \]
\[ = \frac{1}{4} \left( \frac{2}{3} \right)^{2/3} \frac{\gamma^{2/3}}{A^{5/3}} \delta B \left\{ 1 + \frac{3}{2} G(2) - \frac{1}{2} G(1) \right\} / \sum_a \frac{A_a}{\Gamma_a^{2/3}} G(5/3) \]
\[ + \frac{1}{4} \left\{ \frac{1}{3} \frac{\delta A}{A} \sum_a \frac{A_a}{\Gamma_a^{2/3}} + \sum_a \frac{\delta A_a}{\Gamma_a^{2/3}} \right\} / \sum_a \frac{A_a}{\Gamma_a^{2/3}}. \tag{6.9} \]

From now on, we name the expressions representing the final amplitude of the Bardeen parameter \( \zeta_{\text{fin}}^2 \) in terms of the initial perturbation amplitudes such as \( \delta A_a \delta B_\alpha \), \( S \) formula after \( S \) matrix in the quantum mechanics. As for the initial energy density perturbations of the scalar fields \( \delta A_a \), by adopting more physical parametrization introduced by
\[ \frac{1}{3} \frac{\delta A}{A} =: \xi, \quad S_{ab} = \frac{\delta A_a}{A_a} - \frac{\delta A_b}{A_b} =: \eta_{ab}, \tag{6.10} \]

\( \delta A_a \) can be written as
\[ \delta A_a = 3 A_a \xi + \sum_b \frac{A_a A_b}{A} \eta_{ab}. \tag{6.11} \]
ξ represents the adiabatic growing mode and \( \eta_{ab} \) represent the isocurvature modes. Since \( \eta_{ab} \) satisfy
\[
\eta_{ab} = -\eta_{ba}, \quad \eta_{ab} + \eta_{bc} = \eta_{ac},
\]
the independent quantities are given by \( \xi, \eta_{12}, \eta_{23}, \ldots, \eta_{N_{S} - 1, N_{S}} \). By using this parametrization, \( \delta A_a \) dependent part of \( \zeta_{\text{fin}}^2 \) is written as
\[
\zeta_{\text{fin}}^2 \supset \xi + \frac{1}{8} \sum_{ab} \left( \frac{1}{\Gamma_a^{2/3}} - \frac{1}{\Gamma_b^{2/3}} \right) \frac{A_a A_b}{A} \eta_{ab} / \sum_a A_a \Gamma_a^{2/3},
\]
where \( A \supset B \) implies that \( B \) is contained by \( A \), that is \( A = B + \cdots \). From this S formula, we can conclude that for the adiabatic growing mode \( \xi \), the Bardeen parameter is conserved, and that the initial entropy perturbations survive in the case that the decay rates are dependent on the scalar field \( \phi_a \) from which the radiation energy comes, that is \( \Gamma_a \neq \Gamma_b \) \((a \neq b)\). In the case where multiple scalar fields exist, there is no reason why the perturbation has only the adiabatic component, and it is natural to think that in the perturbation the adiabatic components and the entropic components coexist. In such mixed cases, so called conservation of the Bardeen parameter does not hold and the above S formula gives useful tool for calculating the final Bardeen parameter.

We can consider the case where the energy transfer rates \( \Gamma_{aa} \) fluctuates \[29\], which is called as the modulated reheating scenario \[4\]. For simplicity, we consider the one scalar field case. By taking the derivative of (6.5) with respect to \( \Gamma \) with \( B \) vanishing, we obtain
\[
\zeta_{\text{fin}}^2 \supset -\frac{1}{6} \frac{\delta \Gamma}{\Gamma},
\]
which is well known formula derived in the paper \[4\], and where the coefficient is successfully determined in this paper.

We consider the influence of the resonant interaction between scalar fields on the final amplitude of the Bardeen parameter. We take the interaction \( U_{\text{int}} \) into account by iteration. As the first order correction from the interaction term as
\[
- \frac{1}{\epsilon^3 a^3} \rho^{1/2} \frac{\partial U_{\text{int}}}{\partial \theta_a} \supset 0 \frac{d}{da} I_a,
\]
we obtain
\[
\sigma_a \supset \int_1^a d\gamma \rho^{1/2} \sum_a \Gamma_{aa} \Delta (m_a I_a),
\]
where
\[
\Delta (m_a I_a) := \exp \left\{ -\gamma \Gamma_a \int_1^a \frac{d\gamma}{\rho^{1/2} a} \right\} \int_1^a \frac{d\gamma}{\rho^{1/2} a} \exp \left\{ \gamma \Gamma_a \int_1^a \frac{d\gamma}{\rho^{1/2} a} \right\} \left( -\frac{1}{\epsilon^3 a^3} \rho^{1/2} \frac{\partial U_{\text{int}}}{\partial \theta_a} \right).
\]
Since as the zeroth order approximation \( I_a \) obeys (6.11), we can write
\[
\frac{\partial U_{\text{int}}}{\partial \theta_a} = \frac{\partial U_{\text{int}}}{\partial \theta_a} \bigg|_{a=1} \frac{1}{a^{3n/2}} \exp \left\{ -\gamma \Gamma \int_1^a \frac{d\gamma}{\rho^{1/2} a} \right\},
\]
where we assumed that $U_{\text{int}}$ contains an $n$-th order interaction term and that $\Gamma$ is the appropriate sum of $\Gamma_a/2$. By substituting the above expression and by expanding the correction term with respect to $B$ around $B = 0$, we obtain

$$
\sigma_a \supset \sum_a \Gamma_{aa} m_a \frac{\partial U_{\text{int}}}{\partial \theta_a} \bigg|_{a=1} \left( -\frac{1}{\epsilon} \right) \left( \frac{2}{3} \right) \frac{\gamma}{A^{1/2}} \frac{n-8/3}{\gamma} \frac{1}{A}
$$

$$
\times \left[ G(5/3, -n + 3, \Gamma_a, \Gamma) + \left( \frac{\gamma}{A^{1/2}} \right)^{2/3} \frac{B}{A} \left\{ \left( \frac{1}{2} \right)^{1/3} (\Gamma - \Gamma_a) G(5/3, -n + 10/3, \Gamma_a, \Gamma)
\right. \\
- \frac{1}{2} \left( \frac{2}{3} \right)^{2/3} G(5/3, -n + 7/3, \Gamma_a, \Gamma) + \left( \frac{3}{2} \right)^{1/3} \Gamma_a G(2, -n + 3, \Gamma_a, \Gamma)
\right. \\
- \frac{1}{2} \left( \frac{2}{3} \right)^{2/3} G(1, -n + 3, \Gamma_a, \Gamma) \right] 
$$

(6.19)

where

$$
G(n_1, n_2, \Gamma_1, \Gamma_2) := \int_{x_0}^{x} dx x^{n_1-1} \exp (-\Gamma_1 x) \int_{x_0}^{x} dy y^{n_2-1} \exp \{ (\Gamma_1 - \Gamma_2) y \},
$$

(6.20)

where

$$
x := \gamma \frac{2}{A^{1/2} 3} a^{3/2}, \quad x_0 := \gamma \frac{2}{A^{1/2} 3}
$$

(6.21)

The evaluation of the double Gamma function defined by (6.20) is treated in Appendix B. We consider the concrete example defined by

$$
U_{\text{int}} = \lambda \phi_1^2 \phi_2^2, \quad m_1 = m_2.
$$

(6.22)

In this case, we use the independent variables defined by

$$
\begin{align*}
\theta_1 &= q_0, \quad \theta_2 = q_0 + q_1 \\
I_1 &= p_0 - p_1, \quad I_2 = p_1,
\end{align*}
$$

(6.23)

where $(q_0, p_0)$ and $(q_1, p_1)$ are called fast and slow action-angle variables, respectively [10]. Because of the resonant relation satisfied by the masses of the scalar fields, the slow angle variable $q_1$ moves much more slowly than the fast angle variable $q_0$. The averaging over the slow angle variable $q_1$ cannot be justified, and therefore the slow action-angle variables $(q_1, p_1)$ can have evolutions. In the previous paper [10], we investigated the influences of the resonant interaction on the evolution of the cosmological perturbations before the energy transfer from the scalar fields to the radiation fluids begins. According to this study, the slow action-angle variables can suffer from the instability near the hyperbolic fixed point in the phase space of the slow action-angle variables. Since the initial adiabatic perturbation $\xi$ and the initial isocurvature perturbation $\eta_{12}$ are given by

$$
\xi = \frac{1}{3} \frac{\delta p_0}{p_0}
$$

(6.24)

and

$$
\eta_{12} = \frac{p_1 \delta p_0 - p_0 \delta p_1}{(p_0 - p_1)p_1},
$$

(6.25)
respectively, the instability of the slow action variable $p_1$ has influence on the isocurvature mode. Therefore from (6.13) in the case $\Gamma_1 \neq \Gamma_2$ the instability of the action-angle variables survives in the final amplitude of the Bardeen parameter. Next we calculate the first order correction term (6.19) in the present model (6.22). The present model has the hyperbolic fixed point at

$$q_1 = \frac{\pi}{2} (2k + 1), \quad 2p_1 = p_0 = c,$$

(6.26)

where $k$ is an integer. At this hyperbolic fixed point, the first order correction to the final amplitude of the Bardeen parameter (6.19) is calculated as

$$\zeta_{\text{fin}} \supset \frac{1}{12} A^{1/2} \nu c^2 \delta q_1(1) \left( \frac{1}{\Gamma_1^{2/3}} - \frac{1}{\Gamma_2^{2/3}} \right) / \sum_a A_a \Gamma_a^{2/3},$$

(6.27)

where

$$\nu := \frac{\lambda I_0}{m_0^3 a_0^3}$$

(6.28)

and the non-dimensional masses are scaled as $m_1 = m_2 = 1$. In the present model, as for the first order correction term also, in order that the slow action-angle variables instability has influence on the final Bardeen parameter, $\Gamma_1 \neq \Gamma_2$ is necessary.

Until now, we evaluate $I_a \sigma_a$ by assuming that $\rho$ is given by (6.3). Now we evaluate the contribution to $I_a \sigma_a$ from the late stage of reheating when $\rho$ is given by

$$\rho_{\text{late}} = \left( \frac{3}{2} \right)^{2/3} G (5/3) \frac{1}{a^4} \frac{A^{1/3}}{\gamma^{2/3}} \sum_a \frac{A_a}{\Gamma_a^{2/3}}.$$  

(6.29)

Such late stage of reheating begins at

$$a_1 = d \left( \frac{3}{2} \right)^{2/3} G (5/3) \frac{1}{\gamma^{2/3} A^{2/3}} \sum_a \frac{A_a}{\Gamma_a^{2/3}},$$

(6.30)

because at this $a_1$, $A/a^3$ is almost equal to $\rho_{\text{late}}$. $d$ is a numerical factor which can be assumed to be larger than unity. By using $\rho_{\text{late}}, a_1$, we can evaluate the contribution to $\rho_r$ from the late stage of reheating as

$$\rho_r \supset \frac{1}{a^4} \left( \frac{3}{2} \right)^{2/3} G (5/3) \sum_a d \exp \left\{ -d^{3/2} G (5/3)^{3/2} \frac{A_a}{A^{3/2}} \left( \sum_b \frac{A_b}{\Gamma_b^{2/3}} \right)^{3/2} \right\} \times \frac{A_a}{\gamma^{2/3} A^{2/3}} \sum_b \frac{A_b}{\Gamma_b^{2/3}},$$

(6.31)

whose size is characterized by

$$r_{l/e} := d \exp \left\{ -d^{3/2} G (5/3)^{3/2} \right\},$$

(6.32)

which is the ratio of the late contribution to the main early contribution to $\rho_r$. The value of $r_{l/e}$ is 0.43, 0.18 and 0.035 for $d = 1, 2$ and 3, respectively. Therefore we can conclude
that the S formula which is derived by using (6.3) is rather good approximation to the real S formula.

We consider the case where the decay rates $\Gamma_a$ depend on the radiation temperature $T$. In the high temperature limit $T \gg m$ where $m$ is the mass scale of the oscillatory scalar fields, the decay rate $\Gamma_a$ depends on the radiation temperature \[35\]. When the decay product is the fermion, $\Gamma_a$ is given by

$$\Gamma_a = \frac{1}{T}. \quad (6.33)$$

When the decay product is the boson, $\Gamma_a$ is given by

$$\Gamma_a = \beta_a T. \quad (6.34)$$

According to the paper \[35\] the reason is following. We consider the case where $\rho_r$ is sufficiently high. In the fermion case, the Pauli exclusion principle inhibits the decay of $\phi_a$ into fermions since the fermions have already occupied the energy levels into which $\phi_a$ would decay. In the boson case, the induced effect promotes the decay of $\phi_a$ into bosons since the bosons occupy the energy levels into which $\phi_a$ decay. For simplicity, we consider the case where the radiation consists of one component. We interpret that the radiation temperature $T$ appearing in (6.33) is the temperature $T_a := [\rho_r(a(\Gamma_a))]^{1/4}$ at the time when the decay process proceeds given by

$$a(\Gamma_a) := \frac{A_1/3}{\gamma^{2/3} \Gamma_a^{2/3}}. \quad (6.35)$$

In the fermion case, by substituting $T_a$ defined above to (6.33) it can be verified that

$$\sum_a \frac{A_a}{\Gamma_a^{2/3}} = \gamma^{2/9} \left( \sum_a \frac{A_a}{\alpha_a^{2/5}} \right)^{10/9}. \quad (6.36)$$

Therefore we obtain

$$a^4 \rho_r \supset \frac{A^{2/9}}{\gamma^{4/9}} \left( \sum_a \frac{A_a}{\alpha_a^{2/5}} \right)^{10/9}, \quad (6.37)$$

$$\zeta^z_{\text{fin}} \supset \xi + \frac{5}{36} \sum_{ab} \left( \frac{1}{\alpha_a^{2/5}} - \frac{1}{\alpha_b^{2/5}} \right) \frac{A_a A_b}{A} \eta_{ab} / \sum_a \frac{A_a}{\alpha_a^{2/5}}. \quad (6.38)$$

In the same way as in the fermion case, in the boson case we can obtain

$$a^4 \rho_r \supset \frac{A^{2/3}}{\gamma^{4/3}} \left( \sum_a \frac{A_a}{\beta_a^{2}} \right)^{2/3}, \quad (6.39)$$

$$\zeta^z_{\text{fin}} \supset \xi + \frac{1}{12} \sum_{ab} \left( \frac{1}{\beta_a^{2}} - \frac{1}{\beta_b^{2}} \right) \frac{A_a A_b}{A} \eta_{ab} / \sum_a \frac{A_a}{\beta_a^{2}}. \quad (6.40)$$

The radiation temperature dependence of the decay rate $\Gamma_a$ affects how the isocurvature modes are transmitted into the final amplitude of the Bardeen parameter.
We consider the non-Gaussianity of perturbations. For simplicity, we assume that all the perturbations are generated by only one Gaussian variable, say locally homogeneous perturbed variable $C(x) = C + \delta C(x)$ where $\delta C(x)$ is spatially dependent Gaussian random variable. By assuming that $\rho_r(a, C(x)) \propto C(x)^{\alpha}$, from (3.24), the non-linearity parameters defined by

$$\zeta = \zeta_1 + \frac{3}{5} f_{NL} \zeta_1^2 + \frac{9}{25} g_{NL} \zeta_1^3 + \cdots$$

(6.41)

where $\zeta_1$ is the first order perturbation of the Bardeen parameter generated by one Gaussian random variable, are given by

$$f_{NL} = -\frac{10}{3\alpha}, \quad g_{NL} = \frac{100}{27\alpha^2}.$$  

(6.42)

In the reheating model with only one scalar field, the final radiation energy density is given by

$$a^4 \rho_r \sim \frac{1}{\gamma^{2/3}} \frac{A^{4/3}}{\Gamma^{2/3}}.$$  

(6.43)

When the initial action variable $A/m$ is the random Gaussian variable, by considering $A(x) \propto C(x)^2$ from (3.29), non-linearity parameters are given by

$$f_{NL} = -\frac{5}{4}, \quad g_{NL} = \frac{25}{12}.$$  

(6.44)

In the modulated reheating scenario, by assuming that the decay rate $\Gamma(x)$ is proportional to $\phi(x)^\beta$, the non-linearity parameters are given by

$$f_{NL} = \frac{5}{\beta}, \quad g_{NL} = \frac{100}{3\beta^2}.$$  

(6.45)

By observing the non-Gaussianity, we can determine whether the nonnegligible Gaussian random variable lies in the action variable $A(x)/m$ or the decay rate $\Gamma(x)$.

§7 Application of the LWL formula to the multicomponent curvaton model

In this section, we apply the LWL formula to the curvaton scenario where multiple weakly coupled massive scalar fields called curvatons decay into multiple radiation fluids some time later after the inflation has ended. In this curvaton scenario, the curvaton fields other than the inflaton fields driving the inflation are responsible for the origin of the cosmic structures. First, we assume that all $\Gamma_a := \sum_{\alpha} \Gamma_{\alpha a} (a = 1, 2, \cdots, N_S)$ are of the same order of magnitude.

First we consider the limit where in the initial time the curvaton fields energy $A$ is small compared to the radiation fluids energy $B$: $A \ll B$. By substituting the $\rho$ expression (6.3) to (6.1) (6.2) and by expanding it with respect to $A$ around $A = 0$, we obtain

$$m_a I_a = A_a \exp \left\{ -\frac{1}{2} \frac{\gamma \Gamma_a}{B^{1/2}} a^2 \right\} + O(A^2),$$

$$a^4 \rho_r = a^4 \sum_\alpha \rho_\alpha = B + \sqrt{2} G(3/2) \frac{B^{1/4}}{\gamma^{1/2}} \sum_a \frac{A_a}{\Gamma_a^{1/2}} + O(A^2).$$  

(7.1)
By taking the exactly homogeneous perturbation, that is $D$ operation around $A = 0$, we obtain
\[
\zeta_{\text{fin}}^2 \simeq \frac{1}{4} \frac{D\rho^2}{\rho} = \frac{1}{4} \frac{\delta B}{B} + \frac{\sqrt{2}}{4} G(3/2) \frac{1}{\gamma^{1/2} B^{3/4}} \sum_a \frac{\delta A_a}{\Gamma_a^{1/2}}. \tag{7.2}
\]

Next we consider the case where the energy densities of the curvatons are large compared with those of radiation fluids when the energy transfer from the curvatons to the radiation fluids proceeds. In this case, the exponent of (6.1) is written as
\[
\gamma \Gamma_a \int_1^\{x(\Gamma_a)-B\}/A \frac{1}{\rho^{1/2} a} \, da = 1, \tag{7.3}
\]
where
\[
x := Aa + B. \tag{7.4}
\]
As for $x(\Gamma_a)$ defined by
\[
\gamma \Gamma_a \int_1^{x(\Gamma_a)/A} \frac{1}{\rho^{1/2} a} \, da = 1, \tag{7.5}
\]
we obtain
\[
x(\Gamma_a) = x_0(\Gamma_a) \left[ 1 + 2 \frac{B}{x_0(\Gamma_a)} - \frac{4}{3} \left( \frac{B}{x_0(\Gamma_a)} \right)^{3/2} + O \left( \frac{B^2}{x_0(\Gamma_a)^2} \right) \right], \tag{7.6}
\]
where
\[
x_0(\Gamma_a) := \left( \frac{3}{2} \right)^{2/3} \left( \frac{A^2}{\gamma \Gamma_a} \right)^{2/3}. \tag{7.7}
\]
The expansion parameter $B/x_0(\Gamma_a)$ implies the ratio of the energy of radiations to the energy of the curvaton $\phi_a$ when the energy transfer proceeds. By using $x(\Gamma_a)$, we approximate the exponential function by the step function:
\[
\exp \left\{ -\gamma \Gamma_a \int_1^{x(\Gamma_a)/A} \frac{1}{\rho^{1/2} a} \, da \right\} \to \theta (x(\Gamma_a) - x). \tag{7.8}
\]
By using this approximation, we obtain the radiation energy density in the $a \to \infty$ limit:
\[
\rho_r = \sum_a \rho_a = \frac{1}{a^4} \left[ B + 2 \left( \frac{3}{2} \right)^{5/3} \frac{A^{1/3}}{\gamma^{2/3}} \sum_a \frac{A_a}{\Gamma_a^{2/3}} + B + \cdots \right]. \tag{7.9}
\]
By taking the exactly homogeneous perturbation, we obtain the S formula:
\[
\zeta_{\text{fin}}^2 = \frac{1}{4} \frac{\delta A_*}{A_*} + \frac{1}{2} \frac{\delta B}{A_*} - \frac{B}{2 A_*^2} \delta A_* + \cdots, \tag{7.10}
\]
where
\[
A_* := \frac{2}{5} \left( \frac{3}{2} \right)^{5/3} \frac{A^{1/3}}{\gamma^{2/3}} \sum_a \frac{A_a}{\Gamma_a^{2/3}}. \tag{7.11}
\]
We define more physical parametrization by

\[
\delta A_a = 3A_a \xi + 3A_a \eta + \sum_b \frac{A_a A_b}{A} \eta_{ab},
\]

(7.12)

\[
\delta B_\alpha = 4B_\alpha \xi + \sum_\beta \frac{B_\alpha B_\beta}{B} \eta_{\alpha\beta}.
\]

(7.13)

In particular, since

\[
\delta A = 3A \xi + 3A \eta, \quad \delta B = 4B \xi,
\]

(7.14)

\(\xi\) implies the adiabatic growing mode and \(\eta\) means the isocurvature mode between the total curvaton and the total radiations. By using this parametrization, the S formula can be rewritten as

\[
\zeta_{\text{fin}}^2 = \xi + \left(1 - 2 \frac{B}{A^*}\right) \eta
\]

\[
+ \left(\frac{1}{8} - \frac{B}{4A^*}\right) \sum_{ab} \left(\frac{1}{\Gamma_a^{2/3}} - \frac{1}{\Gamma_b^{2/3}}\right) \frac{A_a A_b}{A} \eta_{ab} / \sum_a \frac{A_a^{2/3}}{\Gamma_a^{2/3}} + \cdots
\]

(7.15)

In the most simple curvaton scenario of one curvaton field and one radiation fluid, the empirical S formula was obtained from the numerical calculation [7]:

\[
\zeta_{\text{fin}} = r(p) \eta,
\]

(7.16)

\[
r(p) = 1 - (1 + \frac{0.924}{1.24} p)^{-1.24},
\]

(7.17)

where

\[
p := \frac{A}{\Gamma^{1/2} \Gamma^{1/2} B^{3/4}}.
\]

(7.18)

In the limits \(p \ll 1, p \gg 1\), this empirical S formula gives

\[
\zeta_{\text{fin}} = 0.924 p \eta \quad p \ll 1,
\]

(7.19)

\[
\zeta_{\text{fin}} = \left(1 - \frac{1.44}{p^{1.24}}\right) \eta \quad p \gg 1,
\]

(7.20)

respectively. Our analytical results (7.2) (7.15) give

\[
\zeta_{\text{fin}}^2 \supset 0.940 p \eta \quad p \ll 1,
\]

(7.21)

\[
\zeta_{\text{fin}}^2 \supset \left(1 - \frac{2.54}{p^{1.33}}\right) \eta \quad p \gg 1,
\]

(7.22)

respectively. In the case \(p \ll 1\), the empirical formula and our analytic result agree with good accuracy. In the case \(p \gg 1\), our analytic result is obtained by rather rough treatment approximating the exponential function by the step function. But our analytic S formula
agree well with the empirical formula. For reference, for $p \gg 1$, according to our method, a more precise calculation gives

\[ \zeta_{\text{fin}}^2 \supset \left[ 1 - \frac{10}{3} \left( \frac{2}{3} \right)^{2/3} \frac{1}{p^{1/3}} + \frac{10}{3} \frac{1}{p^2} + \frac{10}{9} \left( \frac{2}{3} \right)^{4/3} \frac{1}{p^{8/3}} + O \left( \frac{1}{p^{10/3}} \right) \right] \eta \]

\[ = \left[ 1 - 2.54 \frac{1}{p^{1/3}} + 3.33 \frac{1}{p^2} + 0.647 \frac{1}{p^{2.67}} + O \left( \frac{1}{p^{3.33}} \right) \right] \eta. \] (7.23)

The errors between the above formula and the empirical formula are 1.3 percent, 1.1 percent for $p = 10$, $p = 5$, respectively.

In section 6, until now in section 7, we assumed that all decay rates $\Gamma_{\alpha}$ are of the same order of magnitude. As application example of the formulae (7.1) (7.9), we consider the reheating where the decay rate of $\phi_1$ is much larger than the decay rate $\phi_2$, $\Gamma_1 \gg \Gamma_2$. Just after the scalar field $\phi_1$ decays, by using the result of section 6, we obtain

\[ a^4 \rho_r \sim A^{1/3} \frac{A_1}{\gamma^{2/3} \Gamma_1^{2/3}}, \quad a^3 \rho_S \sim A_2. \] (7.24)

By regarding $a^4 \rho_r$ in the above as $B$, we can use the formulae (7.1) (7.9) of the curvaton scenario. In the case $A_2/A_1 \gg (\Gamma_2/\Gamma_1)^{1/2}$, by using (7.1), the final radiation energy density is calculated as

\[ a^4 \rho_r \sim A^{1/3} \frac{A_1}{\gamma^{2/3} \Gamma_1^{2/3}} + \frac{1}{\gamma^{2/3}} \frac{A^{1/2} A_1^{1/4} A_2}{\Gamma_1^{1/6} \Gamma_1^{1/2}}. \] (7.25)

In the case $A_2^{4/3}/A_1^{1/3} A_1 \gg (\Gamma_2/\Gamma_1)^{2/3}$, by using (7.9), the final radiation energy density is calculated as

\[ a^4 \rho_r \sim A^{1/3} \frac{A_1}{\gamma^{2/3} \Gamma_1^{2/3}} + \frac{1}{\gamma^{2/3}} \frac{A_2^{4/3}}{\Gamma_2^{2/3}}. \] (7.26)

By substituting the above two expressions to (3.24) and by expanding with respect to Gaussian random perturbations, we can obtain the Bardeen parameter of arbitrary order.

**§8 Discussion**

In this paper, we constructed the LWL formula expressing the long wavelength limit of evolution of cosmological perturbations in terms of the corresponding exactly homogeneous perturbations in the most general scalar-fluid composite system. We determined the correction term which corrects the difference between the long wavelength limit of cosmological perturbations and the exactly homogeneous perturbations, and we showed that the correction term contributes the well known adiabatic decaying mode. It was pointed out that when we extract the long wavelength limits of evolutions of cosmological perturbations from the exactly homogeneous variables, the use of the scale factor $a$ as the evolution parameter is more useful. The scalar-fluid composite system whose LWL formula is constructed in this paper can be used to describe the early stage of the universe such as reheating after inflation and the curvaton decay in the curvaton scenario, when the fluid is assumed to be radiation. In this paper, the LWL formula is applied to the most general case of reheating.
and of the curvaton decay containing the multiple scalar fields and the multiple radiation fluids, and the S formulae representing the final amplitude of the Bardeen parameter in terms of the initial adiabatic and entropic perturbations are constructed. In case where for different $a$, the value of the decay rate $\Gamma_a$ is different; that is $\Gamma_a \neq \Gamma_b$ for $a \neq b$, the initial isocurvature modes survive in the final amplitude of the Bardeen parameter.

We discuss the non-linear generalization of the LWL formalism. Recently the gradient expansion has been discussed as the method for investigating the evolutions of non-linear perturbations on superhorizon scales \[27\]. In the lowest order of the gradient expansion, in the zero curvature slice $\partial a(t, x)/\partial x^i = 0$, the evolution equation of the scalar quantity $T$ has exactly the same form as that of the exactly homogeneous equation of $T$ \[25\]. But the coefficients of the evolution equation are spatially dependent, therefore this evolution equations describes the non-linear superhorizon scale inhomogeneities. By using the solution of the exactly homogeneous system with the scale factor as the evolution parameter $T(a, C)$, the solution of the locally homogeneous evolution equation is given by $T(a, C(x))$, where $C(x) = C + \delta C(x)$ is spatially dependent solution constant. The spatially dependent perturbation part of $T(a, C(x))$ is given by Taylor expanding with respect to $C(x) = C + \delta C(x)$;

$$T(a, C(x)) = T(a, C) + \sum_{k=1}^{\infty} \sum_{a_1} \cdots \sum_{a_k} \frac{1}{k!} \frac{\partial^k T}{\partial C_{a_1} \cdots \partial C_{a_k}} \delta C_{a_1}(x) \cdots \delta C_{a_k}(x),$$

(8.1)

whose first order perturbation part agrees with our linear perturbation formula with neglecting the adiabatic decaying mode; $DT^\sharp = (\partial T/\partial C)_a$. Therefore $T(a, C(x))$ is the non-linear generalization of our linear perturbation variable $DT$. When $P_r = \rho_r/3$ as in the final state of reheating or the curvaton decay, $T = \ln \rho_r/4$ is the non-linear generalization of the Bardeen parameter $\zeta$. Since in this paper we determine $\rho_r$ in the final state of reheating or the curvaton decay with the scale factor as the evolution parameter, we can obtain the information of the non-linear evolution and the non-Gaussianity of perturbations which fluctuate spatially on superhorizon scales.

Our evolution equations have arbitrary functions $S$ which describe the energy transfer between scalar fields and perfect fluids. The source functions $S$ can be determined from the microscopic dynamics between the coherently oscillating scalar fields and radiation, concretely speaking, by path integrating out the fields constituting the radiation and interacting with the coherently oscillating scalar fields in the effective in-in action \[35\]. When the scalar fields oscillate coherently, the source term such as $S = \Gamma \dot{\phi}$ is important in order that the energy is transferred from the scalar fields into radiation effectively. As shown in the paper \[33\], $\Gamma$ is given as the function $\Gamma = \Gamma(\phi, \dot{\phi}, \rho_r)$ where $\rho_r$ is the energy density of radiation. As seen in the section 6, when $\Gamma$ is a function of the scalar quantities which are closely related to reheating process, the functional form of $\Gamma$ affects how the initial isocurvature components are converted into the adiabatic component such as the final amplitude of the Bardeen parameter, but it does not affect the evolution of the initial adiabatic growing mode. On the other hand, the scenario where the energy transfer is controlled by the scalar quantities not related to reheating is considered as the modulated reheating scenario \[29\] \[4\]. As such scalar quantities, we can choose flat direction scalar field which does not govern the energy of the universe but fluctuates of the order of the Hubble parameter during the inflationary expansion, or the scalar field written in terms of the slow action
variable which suffers from the hyperbolic instability due to the resonance of the masses of the scalar fields during the oscillatory stage \cite{10}. As seen from the formula (6.14) the fluctuations of such modulating scalar quantities are imprinted on the $\rho_r$ fluctuation.

We consider the system where multiple oscillatory scalar fields and multiple radiation fluids interact. As for the system without radiation, its evolution of cosmological perturbations have been investigated in detail \cite{9}, \cite{10}. The system of multiple scalar fields only can be written in terms of Hamiltonian form. Any Hamiltonian system obeys the Liouville theorem, that is, volume occupied by group of orbits are invariant. Therefore according to the LWL formula, in the stable case the perturbations do not grow and in the unstable case the same numbers of growing modes and decaying modes appear because of the squeezing of the phase space volume. The former case occurs in the case where masses of scalar fields are incommensurable and near the elliptic fixed points in the case where masses of scalar fields are commensurable. The latter case occurs near the hyperbolic fixed points in the case where masses of scalar fields are commensurable. Then we include dissipative interaction with radiation. In this case our system is not a Hamiltonian system and it does not obey the Liouville theorem. In this dissipative system, in addition to two possibilities mentioned above we can expect the third possibility where group of orbits are attracted into the attracting set. The final state where all the energy of the scalar fields is transferred into that of radiation fluids is the attracting equilibrium. Around the attracting set, the adjacent orbits come nearer and nearer, therefore the LWL formula tells us that all the perturbation modes are stable, that is, converge into some constants or decay. It is useful to investigate how the behavior of cosmological perturbations around the hyperbolic fixed points is changed due to the dissipative interaction with radiation under the spirit of the LWL formula. In this line of researches, we will have new explanation of the backreaction which suppresses the instability due to the resonance. In the future publication, we will return to this problem.

Acknowledgments

The author would like to thank Professors H. Kodama, S. Mukohyama, T. Nakamura, M. Sasaki, K. Sato, N. Sugiyama, T. Tanaka, A. Taruya, J. Yokoyama for continuous encouragements. He would like to thank Professor V.I. Arnold for writing his excellent textbook and/or review [11], from which he learned a lot about the dynamical system.

§A Proofs of the propositions in §5

A.1 Technical Lemmas

Technical Lemma 1 Under the assumption (5.15), for $|f| \leq 1$,

$$\left| \frac{\partial f}{\partial I} \right| \leq a, \quad \left| \frac{\partial f}{\partial a} \right| \leq \frac{1}{a},$$

(A.1)

and for $|f| \leq |I|$,

$$\left| \frac{\partial f}{\partial I} \right| \leq 1, \quad \left| \frac{\partial f}{\partial a} \right| \leq \frac{|I|}{a},$$

(A.2)
Technical Lemma 2  For the general physical quantity $A(I, \sigma, \theta)$ with Fourier decomposition as

$$A = A_0(I, \sigma) + \sum_{k \neq 0} A_k(I, \sigma) \exp (i k \cdot \theta), \quad (A.3)$$

the solution to the first order partial differential equation

$$\omega_a \frac{\partial}{\partial \theta_a} S = A - < A >, \quad (A.4)$$

is given by

$$S = \{ A \}, \quad (A.5)$$

where

$$\{ A \} := \sum_{k \neq 0} \frac{A_k}{i(k \cdot \omega)} \exp (i k \cdot \theta). \quad (A.6)$$

A.2 Proof of Proposition 1

In order to make the notation simple, we omit the superscript $(k)$ and replace $(k+1)$ with (1). By substituting the transformation laws of $I \sigma \theta$ to the evolution equations of $I_a \sigma_a \theta_a$, we obtain

$$< F_a > + F_a(I, \sigma, \theta, a) - F_a(I^{(1)}, \sigma^{(1)}, \theta^{(1)}, a) - a \frac{\partial u_a}{\partial a}$$

$$= F_a^{(1)} + \frac{\partial u_a}{\partial I_b^{(1)}} F_b^{(1)} + \frac{\partial u_a}{\partial \sigma_\beta^{(1)}} F_\beta^{(1)} + \frac{\partial u_a}{\partial \theta_b^{(1)}} G_b^{(1)}, \quad (A.7)$$

$$< F_a > + F_a(I, \sigma, \theta, a) - F_a(I^{(1)}, \sigma^{(1)}, \theta^{(1)}, a) - a \frac{\partial u_a}{\partial a}$$

$$= F_a^{(1)} + \frac{\partial u_a}{\partial I_b^{(1)}} F_b^{(1)} + \frac{\partial u_a}{\partial \sigma_\beta^{(1)}} F_\beta^{(1)} + \frac{\partial u_a}{\partial \theta_b^{(1)}} G_b^{(1)}, \quad (A.8)$$

and

$$\frac{1}{\epsilon} \omega_a(I, \sigma, a) - \frac{1}{\epsilon} \omega_a(I^{(1)}, \sigma^{(1)}, a) - \frac{1}{\epsilon} \frac{\partial \omega_a}{\partial I_b^{(1)}} u_b - \frac{1}{\epsilon} \frac{\partial \omega_a}{\partial \sigma_\beta^{(1)}} u_\beta$$

$$+ < G_a > + G_a(I, \sigma, \theta, a) - G_a(I^{(1)}, \sigma^{(1)}, \theta^{(1)}, a) - a \frac{\partial v_a}{\partial a}$$

$$= G_a^{(1)} + \frac{\partial v_a}{\partial I_b^{(1)}} F_b^{(1)} + \frac{\partial v_a}{\partial \sigma_\beta^{(1)}} F_\beta^{(1)} + \frac{\partial v_a}{\partial \theta_b^{(1)}} G_b^{(1)}, \quad (A.9)$$

when we choose $u_a, u_\alpha, v_a$ as

$$u_a = \epsilon \{ F_a \} \sim \epsilon^{k+1} |I|, \quad (A.10)$$

$$u_\alpha = \epsilon \{ F_\alpha \} \sim \epsilon^{k+1} |I|, \quad (A.11)$$

$$v_a = \left\{ \frac{\partial \omega_a}{\partial I_b^{(1)}} u_b + \frac{\partial \omega_a}{\partial \sigma_\beta^{(1)}} u_\beta + \epsilon G_a \right\} \sim \epsilon^{k+1}. \quad (A.12)$$
As for $\Delta F_a$, $\Delta F_\alpha$, $\Delta G_a$ defined by

\[
\begin{align*}
\Delta F_a &:= F_a^{(1)} - < F_a >, \\
\Delta F_\alpha &:= F_\alpha^{(1)} - < F_\alpha >, \\
\Delta G_a &:= G_a^{(1)} - < G_a >,
\end{align*}
\]

(A.13) (A.14) (A.15)

applying the mean value theorem to (A.7) (A.8) (A.9) gives

\[
\begin{align*}
\Delta F_a + \varepsilon^{k+1} \Delta F_b + \varepsilon^{k+1} |I| \Delta F_\beta + \varepsilon^{k+1} |I| \Delta G_b &= \varepsilon^{k+1} a^2 |I|, \\
\Delta F_\alpha + \varepsilon^{k+1} a \Delta F_b + \varepsilon^{k+1} |I| \Delta F_\beta + \varepsilon^{k+1} |I| \Delta G_b &= \varepsilon^{k+1} a^3 |I|, \\
\Delta G_a + \varepsilon^{k+1} a \Delta F_b + \varepsilon^{k+1} \Delta F_\beta + \varepsilon^{k+1} \Delta G_b &= \varepsilon^{k+1} a^2,
\end{align*}
\]

(A.16) (A.17) (A.18)

where all coefficients of order unity are omitted. By solving the above three equations, we obtain

\[
\begin{align*}
\Delta F_a &= \varepsilon^{k+1} a^2 |I|, \\
\Delta F_\alpha &= \varepsilon^{k+1} a^3 |I|, \\
\Delta G_a &= \varepsilon^{k+1} a^2,
\end{align*}
\]

(A.19) (A.20) (A.21)

$\Delta F_a$, $\Delta F_\alpha$, $\Delta G_a$ are decomposed into the angle variables independent parts $\Delta < F_a >$, $\Delta < F_\alpha >$, $\Delta < G_a >$ and the angle variables dependent parts $\tilde{F}_a^{(1)}$, $\tilde{F}_\alpha^{(1)}$, $\tilde{G}_a^{(1)}$.

### A.3 Lemmas and the preparatory propositions

**Lemma 1**  The solution to the differential equation

\[
a \frac{d}{da} A = -\lambda a^2 A + E(-a^2) B(a)
\]

(A.22)

where $\lambda$ is a positive constant, is bounded as

\[
|A(a)| \leq \exp \left( -\frac{\lambda}{2} a^2 \right) |A(1)| + E(-a^2) \|B\|(a)
\]

(A.23)

**Lemma 2**  When $B$ satisfies

\[
\left| \frac{d}{da} B \right| \leq E(-a^2) |B| + E(-a^2) \{ C + \|B\|(a) \}
\]

(A.24)

where $C$ is a positive constant, for an arbitrary $a \geq 1$

\[
\|B\|(a) \leq \frac{a_1}{a_1 - 1} (C + \|B\|(a_1)),
\]

(A.25)

where $a_1$ is a constant satisfying $a_1 > 1$. For example, by putting $a_1 = 2$ we obtain

\[
\|B\|(a) \leq C + \|B\|(2).
\]

(A.26)
Proof  By solving the differential equation, we obtain

\[ |B(a)| \leq |B(1)| + \int_1^a da E(-a^2) \{C + \|B\|(a)\}. \quad (A.27) \]

Since for \(a\) satisfying \(1 \leq a \leq a_1\),

\[ |B(a)| \leq C + \|B\|(a_1), \quad (A.28) \]

and for \(a \geq a_1\)

\[ |B(a)| \leq C + \|B\|(a_1) + E(-a_1^2)\|B\|(a), \quad (A.29) \]

then for an arbitrary \(a \geq 1\)

\[ \|B(a)\| \leq C + \|B\|(a_1) + E(-a_1^2)\|B\|(a), \quad (A.30) \]

whose right hand side is an increasing function of \(a\). Since \(E(-a_1^2) \leq 1/a_1\), we obtain \((A.25)\).

Lemma 3  When \(\delta I, \delta \sigma\) satisfy

\[ \left| \frac{d}{da} \delta I \right| \leq a|\delta I| + |\delta \sigma| + |A|, \quad (A.31) \]

\[ \left| \frac{d}{da} \delta \sigma \right| \leq a^2|\delta I| + a|\delta \sigma| + |B|, \quad (A.32) \]

the following inequality hold:

\[ \|\delta \sigma\|(2) \leq \|\delta I\|(1) + \|\delta \sigma\|(1) + \|A\|(2) + \|B\|(2). \quad (A.33) \]

Proof  We consider the differential equation as for \(a|\delta I| + |\delta \sigma|\):

\[ \frac{d}{da} (a|\delta I| + |\delta \sigma|) \leq a (a|\delta I| + |\delta \sigma|) + a|A| + |B|. \quad (A.34) \]

Then we obtain

\[ \|\delta \sigma\|(a) \leq \exp \left( \frac{1}{2} a^2 \right) \{a|\delta I(1)| + |\delta \sigma(1)| + a^2\|A\|(a) + a\|B\|(a)\}. \quad (A.35) \]

We put \(a = 2\).

Proposition Ap1  For the \(m\)-th order system, for the background quantities, the following inequalities hold:

\[ |I| \leq \exp (-a^2), \quad |\sigma| \leq 1. \quad (A.36) \]

Proof  We solve the evolution equations given by

\[ \left| \frac{d}{da} I \right| \leq -a^2|I| + a^2 e^m |I|, \quad (A.37) \]

\[ \left| \frac{d}{da} \sigma \right| \leq a^3|I| + a^3 e^m |I|. \quad (A.38) \]
**Proposition Ap2**  Let \( m \) be an integer larger than or equal to 2. For the \( m \)-th order system, for the perturbation quantities, the following inequalities hold:

\[
\begin{align*}
|\delta I(a)| & \leq E(-a^2)\delta A_m(1), \\
\|\delta \sigma\|(a) & \leq \delta A_m(1), \\
|\delta \theta(a)| & \leq \exp(a^2\epsilon^m) \left\{ |\delta \theta(1)| + \frac{a^2}{\epsilon}\delta A_m(1) \right\},
\end{align*}
\]

(A.39)  \hspace{1cm} (A.40)  \hspace{1cm} (A.41)

where \( \delta A_m(1) \) is defined by \([5.51]\).

**Proof**  The perturbation variables satisfy the evolution equations as

\[
\begin{align*}
\frac{d}{da} \delta I &= -a\delta I + a|I|\delta \sigma + ae^m|I|\delta \theta, \\
\frac{d}{da} \delta \sigma &= a^2\delta I + a^2|I|\delta \sigma + a^2\epsilon^m|I|\delta \theta, \\
\frac{d}{da} \delta \theta &= \frac{a^2}{\epsilon}\delta I + \frac{a}{\epsilon}\delta \sigma + ae^m\delta \theta,
\end{align*}
\]

(A.42)  \hspace{1cm} (A.43)  \hspace{1cm} (A.44)

where \( |I| \) means a function bounded by \( M|I| \) for a positive constant \( M \). It is important to notice the coefficient of \( \delta I \) in (A.42) is negative. Although in (A.42), terms such as \( a^2|I|\delta I \) is contained, such terms can be neglected because

\[
\int_1^a da a^2|I| \leq 1.
\]

(A.45)

By substituting the estimation obtained from (A.44)

\[
|\delta \theta| \leq \exp(a^2\epsilon^m) \left\{ |\delta \theta(1)| + \frac{a^3}{\epsilon}\|\delta I\|(a) + \frac{a^2}{\epsilon}\|\delta \sigma\|(a) \right\},
\]

(A.46)

into (A.42), by applying Lemma 1, we obtain

\[
|\delta I(a)| \leq E(-a^2) \left\{ |\delta I(1)| + \epsilon^m|\delta \theta(1)| + \|\delta \sigma\|(a) \right\}.
\]

(A.47)

By using the above inequalities in (A.43) and applying Lemma 2, we obtain

\[
\|\delta \sigma\|(a) \leq |\delta I(1)| + \epsilon^m|\delta \theta(1)| + \|\delta \sigma\|(2).
\]

(A.48)

On the other hand, by applying Lemma 3, we obtain

\[
\|\delta \sigma\|(2) \leq \delta A_m(1).
\]

(A.49)

Then we can prove the results.
A.4 Proof of Proposition 2A

The evolution equations of $\Delta A$ where $A = (I, \sigma, \theta)$ are given by

\[
\frac{d}{da} \Delta I = -a \Delta I + a|I|\Delta \sigma +ae^m|I|, \quad (A.50)
\]

\[
\frac{d}{da} \Delta \sigma = a^2 \Delta I + a^2|I|\Delta \sigma +a^2\epsilon|I|, \quad (A.51)
\]

\[
\frac{d}{da} \Delta \theta = \frac{a^2}{\epsilon} \Delta I + \frac{a}{\epsilon} \Delta \sigma + ae^m. \quad (A.52)
\]

In the same way as the proof of the Proposition Ap2, we obtain

\[
|\Delta I| \leq E(-a^2) (|\Delta I(1)| + |\Delta \sigma(1)| + \epsilon^m), \quad (A.53)
\]

\[
\|\Delta \sigma\|(a) \leq |\Delta I(1)| + |\Delta \sigma(1)| + \epsilon^m, \quad (A.54)
\]

\[
|\Delta \theta| \leq |\Delta \theta(1)| + \frac{a^2}{\epsilon} (|\Delta I(1)| + |\Delta \sigma(1)| + \epsilon^m), \quad (A.55)
\]

In the above estimations, we put $\Delta A(1) = 0$.

Next we consider the evolutions of $\Delta \delta A$ where $A = (I, \sigma, \theta)$. We take differences between the upper equations and the lower equations. As for $\Delta \delta I$

\[
\frac{d}{da} \delta I = -a \delta I + a|I|\delta \sigma +ae^m (\delta I + |I|\delta \sigma + |I|\delta \theta), \quad (A.56)
\]

\[
\frac{d}{da} \delta I_{tr} = -a \delta I_{tr} + a|I_{tr}|\delta \sigma_{tr}, \quad (A.57)
\]

as for $\Delta \delta \sigma$

\[
\frac{d}{da} \delta \sigma = a^2 \delta I + a^2|I|\delta \sigma +a^2\epsilon^m (\delta I + |I|\delta \sigma + |I|\delta \theta), \quad (A.58)
\]

\[
\frac{d}{da} \delta \sigma_{tr} = a^2 \delta I_{tr} + a^2|I_{tr}|\delta \sigma_{tr}, \quad (A.59)
\]

and as for $\Delta \delta \theta$

\[
\frac{d}{da} \delta \theta = \frac{a^2}{\epsilon} \delta I + \frac{a}{\epsilon} \delta \sigma + ae^m (a \delta I + \delta \sigma + \delta \theta), \quad (A.60)
\]

\[
\frac{d}{da} \delta \theta_{tr} = \frac{a^2}{\epsilon} \delta I_{tr} + \frac{a}{\epsilon} \delta \sigma_{tr}. \quad (A.61)
\]

By taking into account the fact that in the first two terms on the right hand sides the coefficients depending on $I$, $\sigma$, not on $\theta$ are multiplied, and using the estimations of $\delta A$, $\Delta A$ where $A = (I, \sigma)$, we obtain

\[
\delta I - \delta I_{tr} = \Delta \delta I + \delta I(a \Delta I + \Delta \sigma)
\]

\[
= \Delta \delta I + E(-a^2)\delta A_m(1)\epsilon^m, \quad (A.62)
\]

\[
I \delta \sigma - I_{tr} \delta \sigma_{tr} = I_{tr} \Delta \delta \sigma + \delta \sigma(\Delta I + |I_{tr}|\Delta \sigma)
\]

\[
= E(-a^2) \Delta \delta \sigma + E(-a^2)\delta A_m(1)\epsilon^m, \quad (A.63)
\]

\[
\delta \sigma - \delta \sigma_{tr} = \Delta \delta \sigma + \delta \sigma(a \Delta I + \Delta \sigma)
\]

\[
= \Delta \delta \sigma + \delta A_m(1)\epsilon^m. \quad (A.64)
\]
By using estimations of $\delta A$ where $A = (I, \sigma, \theta)$, we obtain

$$\delta I + |I|\delta \sigma + |I|\delta \theta = \frac{1}{\epsilon}E(-a^2)\delta A_1(1),$$

(A.65)

$$a\delta I + \delta \sigma + \delta \theta = \exp(a^2\epsilon m) \left(|\delta \theta(1)| + \frac{a^2}{\epsilon}\delta A_m(1)\right).$$

(A.66)

Therefore we get

$$\frac{d}{da}\Delta \delta I = -a\Delta \delta I + E(-a^2)\Delta \delta \sigma + \epsilon^{m-1}E(-a^2)\delta A_1(1),$$

(A.67)

$$\frac{d}{da}\Delta \delta \sigma = a^2\Delta \delta I + E(-a^2)\Delta \delta \sigma + \epsilon^{m-1}E(-a^2)\delta A_1(1),$$

(A.68)

$$\frac{d}{da}\Delta \delta \theta = \frac{a^2}{\epsilon}\Delta \delta I + \frac{a}{\epsilon}\Delta \delta \sigma$$

$$+ a\epsilon^{m-1}\exp(a^2\epsilon m) \left(a^2\epsilon|\delta \theta(1)| + a^2\delta A_m(1)\right).$$

(A.69)

In the same way as the proof of the Proposition A5.2, we obtain

$$|\Delta \delta I| \leq E(-a^2) \left(|\Delta \delta I(1)| + |\Delta \delta \sigma(1)| + \epsilon^{m-1}|\Delta A_1(1)|\right),$$

(A.70)

$$\|\Delta \delta \sigma\| (a) \leq |\Delta \delta I(1)| + |\Delta \delta \sigma(1)| + \epsilon^{m-1}|\Delta A_1(1)|,$$

(A.71)

$$|\Delta \delta \theta| \leq |\Delta \delta \theta(1)| + \frac{a^2}{\epsilon} \left(|\Delta \delta I(1)| + |\Delta \delta \sigma(1)| + \epsilon^{m-1}|\Delta A_1(1)|\right)$$

$$+ \epsilon^{m-1}\exp(a^2\epsilon m) \left(a^2\epsilon|\delta \theta(1)| + a^2\delta A_m(1)\right).$$

(A.72)

In the above estimations, we put $\Delta \delta A(1) = 0$ where $A = (I, \sigma, \theta)$. Then we complete the proof.

### A.5 Proof of Proposition 2B

From Proposition 1, we obtain

$$I^{(0)} = I^{(m)} + \epsilon|I^{(m)}|,$$

(A.73)

$$\sigma^{(0)} = \sigma^{(m)} + \epsilon a|I^{(m)}|,$$

(A.74)

$$\theta^{(0)} = \theta^{(m)} + \epsilon,$$

(A.75)

where $|I^{(m)}|$ means the function of $A^{(m)}$ where $A = (I, \sigma, \theta)$ bounded by $M|I^{(m)}|$ for a positive constant $M$. As for $\Delta A$ where $A = (I, \sigma, \theta)$, we obtain

$$|\Delta I^{(0)} - \Delta I^{(m)}| \leq \epsilon \left(|\Delta I^{(m)}| + |I^{(m)}||\Delta \sigma^{(m)}| + |I^{(m)}||\Delta \theta^{(m)}|\right)$$

$$\leq \epsilon^m E(-a^2),$$

(A.76)

$$|\Delta \sigma^{(0)} - \Delta \sigma^{(m)}| \leq \epsilon a \left(|\Delta I^{(m)}| + |I^{(m)}||\Delta \sigma^{(m)}| + |I^{(m)}||\Delta \theta^{(m)}|\right)$$

$$\leq \epsilon^m E(-a^2),$$

(A.77)

$$|\Delta \theta^{(0)} - \Delta \theta^{(m)}| \leq \epsilon \left(a|\Delta I^{(m)}| + |\Delta \sigma^{(m)}| + |\Delta \theta^{(m)}|\right)$$

$$\leq \epsilon^m a^2.$$  

(A.78)
By using the estimations of Proposition 2 and the above evaluations, we obtain the results of the former part.

Next we consider $\Delta \delta A$ where $A = (I, \sigma, \theta)$. By taking the variations of the transformation laws, we obtain

$$\delta I^{(0)} - \delta I^{(m)} = \epsilon \left( \delta I^{(m)} + |I^{(m)}| \delta \sigma^{(m)} + |I^{(m)}| \delta \theta^{(m)} \right), \quad \text{(A.79)}$$

$$\delta \sigma^{(0)} - \delta \sigma^{(m)} = \epsilon a \left( \delta I^{(m)} + |I^{(m)}| \delta \sigma^{(m)} + |I^{(m)}| \delta \theta^{(m)} \right), \quad \text{(A.80)}$$

$$\delta \theta^{(0)} - \delta \theta^{(m)} = \epsilon \left( a \delta I^{(m)} + \delta \sigma^{(m)} + \delta \theta^{(m)} \right), \quad \text{(A.81)}$$

where the coefficients are the functions of $A^{(m)}$ where $A = (I, \sigma, \theta)$. We take the differences of the transformation laws of the exact variables $\delta A$ and those of the truncated variables $\delta A_{tr}$. By using

$$\delta I^{(m)} - \delta I_{tr}^{(m)} = \Delta \delta I^{(m)} + \delta I^{(m)}(a \Delta I^{(m)} + \Delta \sigma^{(m)} + \Delta \theta^{(m)}), \quad \text{(A.82)}$$

$$I \delta \sigma^{(m)} - I_{tr} \delta \sigma_{tr}^{(m)} = I_{tr} \Delta \delta \sigma^{(m)} + \delta \sigma^{(m)}(I \delta I^{(m)} + I_{tr} \Delta \sigma^{(m)} + I_{tr} \Delta \theta^{(m)}), \quad \text{(A.83)}$$

$$\delta \sigma^{(m)} - \delta \sigma_{tr}^{(m)} = \Delta \delta \sigma^{(m)} + \delta \sigma^{(m)}(a \Delta I^{(m)} + \Delta \sigma^{(m)} + \Delta \theta^{(m)}), \quad \text{(A.84)}$$

$$\delta \theta^{(m)} - \delta \theta_{tr}^{(m)} = \Delta \delta \theta^{(m)} + \delta \theta^{(m)}(a \Delta I^{(m)} + \Delta \sigma^{(m)} + \Delta \theta^{(m)}), \quad \text{(A.85)}$$

$$I \delta \theta^{(m)} - I_{tr} \delta \theta_{tr}^{(m)} = I_{tr} \Delta \delta \theta^{(m)} + \delta \theta^{(m)}(I \delta I^{(m)} + I_{tr} \Delta \sigma^{(m)} + I_{tr} \Delta \theta^{(m)}), \quad \text{(A.86)}$$

and by using the estimations of $\Delta A^{(m)}$ and $\delta A^{(m)}$ where $A = (I, \sigma, \theta)$, we obtain

$$\Delta \delta I^{(0)} - \Delta I^{(m)} = \epsilon^{m-1} E(-a^2) \delta A_1(1), \quad \text{(A.87)}$$

$$\Delta \delta \sigma^{(0)} - \Delta \sigma^{(m)} = \epsilon^{m-1} E(-a^2) \delta A_1(1), \quad \text{(A.88)}$$

$$\Delta \delta \theta^{(0)} - \Delta \theta^{(m)} = \exp(a^2 \epsilon^{m-1}) \left( a^2 \epsilon \delta \theta(1) + a^4 \delta A_m(1) \right). \quad \text{(A.89)}$$

By using the estimations of Proposition 2 and the above evaluations, we obtain the results of the latter part. We complete the proof.

### A.6 Proof of Proposition 3

As for $\Delta A$ where $A = (I, \sigma, \theta)$, we can obtain the results by putting $m = 1$ in the proof of Proposition 2.

Next we consider the evolutions of $\Delta \delta A$ where $A = (I, \sigma, \theta)$. We take differences between the upper equations and the lower equations. As for $\Delta \delta I$

$$\frac{d}{da} \delta I_{tr} = -a \delta I_{tr} + a |I_{tr}| \delta \sigma_{tr} + a \epsilon \left( \delta I_{tr} + |I_{tr}| \delta \sigma_{tr} \right), \quad \text{(A.90)}$$

$$\frac{d}{da} \delta I_{n} = -a \delta I_{n} + a |I_{n}| \delta \sigma_{n}, \quad \text{(A.91)}$$

as for $\Delta \delta \sigma$

$$\frac{d}{da} \delta \sigma_{tr} = a^2 \delta I_{tr} + a^2 |I_{tr}| \delta \sigma_{tr} + a^2 \epsilon \left( \delta I_{tr} + |I_{tr}| \delta \sigma_{tr} \right), \quad \text{(A.92)}$$

$$\frac{d}{da} \delta \sigma_{n} = a^2 \delta I_{n} + a^2 |I_{n}| \delta \sigma_{n}, \quad \text{(A.93)}$$
and as for $\Delta \delta \theta$

$$ \frac{d}{da} \delta \theta_{tr} = \frac{a^2}{\epsilon} \delta I_{tr} + \frac{a}{\epsilon} \delta \sigma_{tr} + a \epsilon (a \delta I_{tr} + \delta \sigma_{tr}), \quad (A.94) $$

$$ \frac{d}{da} \delta \theta_n = \frac{a^2}{\epsilon} \delta I_n + \frac{a}{\epsilon} \delta \sigma_n. \quad (A.95) $$

In the above equations, the subscript tr implies that in the present system, the angle variables dependent parts which have been made sufficiently small by the transformations defined in the proof of Proposition 1 have already been truncated, and the subscript n means the further neglection of $\epsilon$-order corrections produced by such transformations. We obtain the evolution equations:

$$ \frac{d}{da} \Delta \delta I = -a \Delta \delta I + E(-a^2) (\Delta \delta \sigma + \epsilon \delta B(1)), \quad (A.96) $$

$$ \frac{d}{da} \Delta \delta \sigma = a^2 \Delta \delta I + E(-a^2) (\Delta \delta \sigma + \epsilon \delta B(1)), \quad (A.97) $$

$$ \frac{d}{da} \Delta \delta \theta = \frac{a^2}{\epsilon} \Delta \delta I + \frac{a}{\epsilon} \Delta \delta \sigma + a \delta B(1). \quad (A.98) $$

In the same way as the proof of the Proposition Ap2, we obtain

$$ |\Delta \delta I| \leq E(-a^2) (|\Delta \delta I(1)| + |\Delta \delta \sigma(1)| + \epsilon \delta B(1)), \quad (A.99) $$

$$ \|\Delta \delta \sigma\|(a) \leq |\Delta \delta I(1)| + |\Delta \delta \sigma(1)| + \epsilon \delta B(1), \quad (A.100) $$

$$ |\Delta \delta \theta| \leq |\Delta \delta \theta(1)| + \frac{a^2}{\epsilon} (|\Delta \delta I(1)| + |\Delta \delta \sigma(1)| + \epsilon \delta B(1)). \quad (A.101) $$

By putting $\Delta \delta A(1) = 0$ where $A = (I, \sigma, \theta)$ in the above inequalities, we obtain the results of the latter part. We complete the proof.

§B Evaluation of the gamma-like function

In the present paper, we often have to evaluate the integrals defined by

$$ G(t, \Gamma) := \int_{x_0}^\infty dx x^{t-1} e^{-\Gamma x}, \quad (B.1) $$

where $x_0$ is defined by

$$ x_0 := \frac{2}{3} \frac{\gamma}{A^{1/2}}, \quad (B.2) $$

where $\gamma$ is assumed to be sufficiently small. By expanding with respect to the small parameter $x_0$ by the partial integration, we obtain the evaluations as follows. For $t > 0$,

$$ G(t, \Gamma) = \frac{1}{\Gamma t} G(t) = O(1), \quad (B.3) $$

for $t = 0$,

$$ G(0, \Gamma) = -\ln x_0 + O(1), \quad (B.4) $$

42
for $t < 0$ and $t$ is an integer,

$$G(t, \Gamma) = -\frac{1}{t}x_0^t + O(x_0^{t+1}, ln x_0), \quad (B.5)$$

and for $t < 1$ and $t$ is not an integer,

$$G(t, \Gamma) = -\frac{1}{t}x_0^t + O(x_0^{t+1}, 1). \quad (B.6)$$

$G(t)$ is the well known Gamma function.

Next by using the above evaluations, we evaluate the integrals defined by (6.20) which appear when we evaluate the effects of the interactions between scalar fields on the final radiation energy density $\rho_\alpha = \sigma_\alpha/a^4$. By expanding with respect to the small parameter $x_0$ by the partial integration, we obtain the evaluations as follows. We assume that $n_1 > 0$. For $n_2 > 0$,

$$G(n_1, n_2, \Gamma_1, \Gamma_2) = O(1), \quad (B.7)$$

for $n_2 = 0$,

$$G(n_1, 0, \Gamma_1, \Gamma_2) = -\frac{1}{\Gamma_1} \ln x_0 G(n_1) + O(1), \quad (B.8)$$

and for $n_2 < 0$,

$$G(n_1, n_2, \Gamma_1, \Gamma_2) = -\frac{1}{n_2} \frac{1}{\Gamma_1} x_0^{n_2} G(n_1) + O(x_0^{n_1+n_2}, x_0^{n_2+1}, \ln x_0). \quad (B.9)$$

Finally we evaluate the incomplete Gamma function defined by

$$G(t; x_1) := \int_{x_1}^{\infty} dx x^{t-1} e^{-x}, \quad (B.10)$$

for large $x_1$. By partial integration, we obtain

$$G(t; x_1) = x_1^{t-1} e^{-x_1} + O(x_1^{t-2} e^{-x_1}), \quad (B.11)$$

for sufficiently large $x_1$.

References

[1] Arnold.V.I., Mathematical Methods of Classical Mechanics, (Springer, New York) (1978); Arnold, V.I. and Avez, A. Problèmes ergodiques de la mécanique classique, (Gauthier-Villars, Paris) (1967)

[2] Bardeen, J.M., Phys. Rev. D 22, 1882 (1980).

[3] Bassett, B.A. and Viniegra, F., Phys. Rev. D 62, 043507 (2000).

[4] Dvali, G., Gruzinov, A. and Zaldarriga, M., Phys. Rev. D 69, 023505 (2004).

[5] Finelli, F. and Brandenberger, R., Phys. Rev. D62, 083502 (2000).
[6] Gordon, C., Wands, D., Bassett, B.A. and Maartens, R., Phys. Rev D63 123506 (2001).

[7] Gupta, S., Malik, K.A. and Wands, D., Phys. Rev D69 063513 (2004).

[8] Hamazaki, T. and Kodama, H., Prog. Theor. Phys. 96,1123–1146 (1996).

[9] Hamazaki, T., Phys. Rev. D 66, 023529 (2002).

[10] Hamazaki, T., Nucl. Phys. B 698,335–385 (2004).

[11] Hosoya, A. and Sakagami, M., Phys. Rev D29 2228 (1984).

[12] Kodama, H. and Sasaki, M., Prog. Theor. Phys. Suppl. 78, 1–166 (1984).

[13] Kodama, H. and Sasaki, M., Int. J. Mod. Phys. A2, 491 (1987).

[14] Kodama, H. and Hamazaki, T., Prog. Theor. Phys. 96,949–970 (1996).

[15] Kodama, H. and Hamazaki, T., Phys. Rev. D57, 7177–7185 (1998).

[16] Kofman, K.A., Linde, A.D. and Starobinsky, A.A. Phys. Rev. Lett.73, 3195 (1994).

[17] Kofman, K.A., Linde, A.D. and Starobinsky, A.A. Phys. Rev. D56, 3258 (1997).

[18] Komatsu, E. and Spergel, D.N. Phys. Rev. D63, 063002 (2001).

[19] Lyth, D.H., Malik, K.A. and Sasaki, M. JCAP. 0505, 004 (2005).

[20] Malik, K.A., Wands, D. and Ungarelli, C., Phys. Rev D67 063516 (2003).

[21] Morikawa, M., Phys. Rev D33 3607 (1986).

[22] Mukhanov, V.F., Sov. phys.—JETP 67, 1297–1302 (1988).

[23] Mukhanov, V.F., Feldman, H.A. and Brandenberger, R.H. Phys. Rep. 215, 203 (1992).

[24] Nambu, Y. and Taruya, A., Prog. Theor. Phys. 97,83–89 (1997).

[25] Nambu, Y. and Araki, Y., Class. Quant. Grav. 23,511 (2006).

[26] Polarski, D. and Starobinsky, A.A., Nucl. Phys. B 385 623 (1992).

[27] Rigopoulos, G.I. and Shellard, E.P.S. Phys. Rev. D68, 123518 (2003).

[28] Sasaki, M. , Prog. Theor. Phys. 76,1036 (1986).

[29] Sasaki, Y. and Yokoyama, J. Phys. Rev. D44, 970 (1991).

[30] Sasaki, M. and Tanaka, T , Prog. Theor. Phys. 99,763–782 (1998).

[31] Shtanov, Y., Traschen, J. and Brandenberger, R.H. Phys. Rev. D51, 5438 (1995).

[32] Taruya, A. and Nambu, Y., Phys. Lett. B428 37–43 (1998).

[33] Traschen, J. and Brandenberger, R.H. Phys. Rev. D42, 2491 (1990).

[34] Wands, D., Malik, K.A., Lyth, D.H. and Liddle, A.R. Phys. Rev. D62, 043527 (2000).
[35] Yokoyama, J. Phys. Rev. D70, 103511 (2004).

[36] Yoshida, J. and Tsujikawa, S., Class. Quant. Grav. 23, 353 (2006).

[37] Zibin, J.P., Brandenberger, R and Scott, D, Phys. Rev. D63, 043511 (2001).