ON ANALYTICAL APPLICATIONS OF STABLE HOMOTOPY
( THE ARNOLD CONJECTURE, CRITICAL POINTS)

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ABSTRACT. We prove the Arnold conjecture for closed symplectic manifolds with
$\pi_2(M) = 0$ and $\text{cat } M = \dim M$. Furthermore, we prove an analog of the Lusternik–
Schnirelmann theorem for functions with “generalized hyperbolicity” property.

Introduction

Here we show that the technique developed in [R98] can be applied to the Arnold
conjecture and to estimation of the number of critical points. For convenience of
the reader, this paper is written independently of [R98].

Given a smooth ($= C^\infty$) manifold $M$ and a smooth function $f : M \to \mathbb{R}$, we
denote by $\text{crit } f$ the number of critical points of $f$ and set $\text{Crit } M = \min \{ \text{crit } f \}$
where $f$ runs over all smooth functions $M \to \mathbb{R}$.

The Arnold conjecture [Ar89, Appendix 9] is a well-known problem in Hamil-
tonian dynamics. We recall the formulation. Let $(M, \omega)$ be a closed symplectic
manifold, and let $\phi : M \to M$ be a Hamiltonian symplectomorphism (see [HZ94],
[MS95] for the definition). Furthermore, let $\text{Fix } \phi$ denote the number of fixed points
of $\phi$. Finally, let

$$\text{Arn}(M, \omega) := \min_{\phi} \text{Fix } \phi$$

where $\phi$ runs over all Hamiltonian symplectomorphisms $M \to M$. The Arnold
conjecture claims that $\text{Arn}(M, \omega) \geq \text{Crit } M$.

It is well known and easy to see that $\text{Arn}(M, \omega) \leq \text{Crit } M$. Thus, in fact, the
Arnold conjecture claims the equality $\text{Arn}(M, \omega) = \text{Crit } M$.

Let $\text{cat } X$ denote the Lusternik–Schnirelmann category of a topological space $X$
(normalized, i.e., $\text{cat } X = 0$ for $X$ contractible).

Given a symplectic manifold $(M^{2n}, \omega)$, we define the homomorphisms

$$I_\omega : \pi_2(M) \to \mathbb{Q}, \quad I_\omega(x) = \langle \omega, h(x) \rangle$$
$$I_c : \pi_2(M) \to \mathbb{Z}, \quad I_c(x) = \langle c, h(x) \rangle$$

where $h : \pi_2(M) \to H_2(M)$ is the Hurewicz homomorphism, $c = c_1(\tau M)$ is the first
Chern class of $M$ and $\langle -, - \rangle$ is the Kronecker pairing.

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Theorem A (see 3.6). Let \((M^n, \omega), n = \dim M\) be a closed connected symplectic manifold such that \(I_\omega = 0 = I_c\) (e.g., \(\pi_2(M) = 0\)) and \(\text{cat} M = n\). Then \(\text{Arn}(M, \omega) \geq \text{Crit} M\), i.e., the Arnold conjecture holds for \(M\).

It is well known that \(\text{Crit} M \geq 1 + \text{cl} M\) for every closed manifold \(M\), where \(\text{cl}\) denotes the cup-length, i.e., the length of the longest non-trivial cup-product in \(\tilde{H}^*(M)\). So, one has the following weaker version of the Arnold conjecture:

\[
\text{Arn}(M, \omega) \geq 1 + \text{cl}(M),
\]

and most known results deal with this weak conjecture, see [CZ83], [S85], [H88], [F89-1], [F89-2], [LO96]. (Certainly, there are lucky cases when \(\text{Crit} M = 1 + \text{cl} M\), e.g., \(M = T^{2n}\), cf. [CZ83].) For example, Floer [F89-1], [F89-2] proved that \(\text{Arn}(M) \geq 1 + \text{cl} M\) provided \(I_\omega = 0 = I_c\), cf. also Hofer [H88]. So, my contribution is the elimination of the clearance between \(\text{Crit} M\) and \(1 + \text{cl} M\). (It is easy to see that there are manifolds \(M\) as in Theorem A with \(\text{Crit} M > 1 + \text{cl} M\), see 3.7 below.)

Actually, we prove that \(\text{Arn}(M, \omega) \geq 1 + \text{cat} M\) and use a result of Takens [T68] which implies that \(\text{Crit} M = 1 + \text{cat} M\) provided \(\text{cat} M = \dim M\).

After submission of the paper the author and John Oprea proved that \(\text{cat} M = \dim M\) for every closed symplectic manifold \((M, \omega)\) with \(I_\omega = 0\), see [RO97]. So, the condition \(\text{cat} M = n\) in Theorem A can be omitted.

Passing to critical points, we prove the following theorem.

Theorem B (see 4.5). Let \(M\) be a closed orientable manifold, and let \(g : M \times \mathbb{R}^{p+q} \to \mathbb{R}\) be a \(C^2\)-function with the following properties:

1. There exist disks \(D_+ \subset \mathbb{R}^p\) and \(D_- \subset \mathbb{R}^q\) centered in origin such that \(\text{int}(M \times D_+ \times D_-)\) contains all critical points of \(g\);
2. \(\nabla g(x)\) points inward on \(M \times \partial D_+\) and outward on \(M \times \text{int} D_+ \times \partial D_-\).

Then \(\text{crit} g \geq 1 + r(M)\).

In particular, if \(M\) is aspherical then \(\text{crit} g \geq 1 + \text{cat} M\).

† Notice that functions \(g\) as in Theorem B are related to the Conley index theory, see [C76]. I remark that Cornea [Co98] have also estimated the number of critical points of functions as in Theorem B.

We reserve the term “map” for continuous functions of topological spaces, and we call a map inessential if it is homotopic to a constant map. The disjoint union of spaces \(X\) and \(Y\) is denoted by \(X \sqcup Y\). Furthermore, \(X^+\) denotes the disjoint union of \(X\) and a point, and \(X^+\) is usually considered as a pointed space where the base point is the added point.

We follow Switzer [Sw75] in the definition of \(CW\)-complexes. A \(CW\)-space is defined to be a space which is homeomorphic to a \(CW\)-complex.

Given a pointed \(CW\)-complex \(X\), we denote by \(\Sigma^\infty X\) the spectrum \(E = \{E_n\}\) where \(E_n = S^nX\) for every \(n \geq 0\) and \(E_n = \text{pt}\) for \(n < 0\); here \(S^nX\) is the \(n\)-fold
reduced suspension over $X$. Clearly, $\Sigma^\infty$ is a functor from pointed CW-complexes to spectra.

Given any (bad) space $X$, the cohomology group $H^n(X; \pi)$ is always defined to be the group $[X, K(\pi, n)]$ where $[-, -]$ denotes the set of homotopy classes of maps.

"Smooth" always means "$C^\infty$".
"Fibration" always means a Hurewicz fibration.
"Connected" always means path connected.
The sign "$\simeq$" denotes homotopy of maps (morphisms) or homotopy equivalence of spaces (spectra).

§1. Preliminaries on the Lusternik–Schnirelmann category

1.1. Definition. (a) Given a subspace $A$ of a topological space $X$, we define $\text{cat}_X A$ to be the minimal number $k$ such that $A = U_1 \cup \cdots \cup U_{k+1}$ where each $U_i$ is open in $A$ and contractible in $X$. We also define $\text{cat}_X A = -1$ if $A = \emptyset$.

(b) Given a map $f : X \to Y$, we define $\text{cat}_f A = -1$ if $A = \emptyset$.

(c) We define the Lusternik–Schnirelmann category $\text{cat} X$ of a space $X$ by setting $\text{cat} X := \text{cat}_X X = \text{cat} 1_X$.

Clearly, $\text{cat} f \leq \min\{\text{cat} X, \text{cat} Y\}$.

The basic information concerning the Lusternik–Schnirelmann category can be found in [Fox41], [J78], [Sv66].

Let $X$ be a connected space. Take a point $x_0 \in X$, set $PX = P(X, x_0) = \{\omega \in X^I \mid \omega(0) = x_0\}$ and consider the fibration $p : PX \to X$, $p(\omega) = \omega(1)$ with the fiber $\Omega X$.

Given a natural number $k$, we use the short notation

$$p_k : P_k(X) \to X.$$  \hspace{1cm} (1.2)

for the map

$$\underbrace{p_X \ast_X \cdots \ast_X p_X}_{k \text{ times}} : \underbrace{PX \ast_X \cdots \ast_X PX}_{k \text{ times}} \longrightarrow X$$

where $\ast_X$ denotes the fiberwise join over $X$, see e.g. [J78]. In particular, $P_1(X) = PX$.

1.3. Proposition. For every connected compact metric space $X$ and every natural number $k$ the following hold:

(i) $p_k : P_k(X) \to X$ is a fibration;

(ii) $\text{cat} P_k(X) < k$;

(iii) The homotopy fiber of the fibration $p_k : P_k(X) \to X$ is the $k$-fold join $(\Omega X)^*k$;

(iv) If $\text{cat} X = k$ and $X$ is $(q-1)$-connected then $p_k : P_k(X) \to X$ is a $(kq - 2)$-equivalence.
(v) If $X$ has the homotopy type of a CW-space then $P_k(X)$ does.

Proof. (i) This holds since a fiberwise join of fibrations is a fibration, see [CP86].

(ii) It is easy to see that $\text{cat}(E_1 * X E_2) \leq \text{cat} E_1 + \text{cat} E_2 + 1$ for every two maps $f_1 : E_2 \to X$ and $f_2 : E_2 \to X$. Now the result follows since $\text{cat} P_1(X) = 0$.

(iii) This holds since the homotopy fiber of $p_1$ is $\Omega X$.

(iv) Recall that $A * B$ is $(a + b + 2)$-connected if $A$ is $a$-connected and $B$ is $b$-connected. Now, $\Omega X$ is $(q - 2)$-connected, and so the fiber $(\Omega X)^*$ of $p_k$ is $(kq - 2)$-connected.

(v) It is a well-known result of Milnor [M59] that $\Omega X$ has the homotopy type of a CW-space. Hence, the space $(\Omega X)^*$ has it. Finally, the total space of any fibration has the homotopy type of a CW-space provided both the base and the fiber do, see e.g. [FP90, 5.4.2]. □

1.4. Theorem ([Sv66, Theorems 3 and 19]). Let $f : X \to Y$ be a map of connected compact metric spaces. Then $\text{cat} f < k$ iff there is a map $g : X \to P_k(Y)$ such that $p_k g = f$. □

§2. AN INVARIANT $r(X)$

Consider the Puppe sequence

$$P_m(X) \xrightarrow{p_m} X \xrightarrow{j_m} C_m(X) := C(p_m)$$

where $p_m : P_m(X) \to X$ is the fibration (1.2) and $C(p_m)$ is the cone of $p_m$.

2.1. Definition. Given a connected space $X$, we set

$$r(X) := \sup\{m | j_m \text{ is stably essential}\}.$$

(Recall that a map $A \to B$ is called stably essential if it is not stably homotopic to a constant map.)

2.2. Proposition. (i) $r(X) \leq \text{cat} X$ for every connected compact metric space $X$.

(ii) Let $X$ be a connected CW-space, let $E$ be a ring spectrum, and let $u_i \in \tilde{E}^*(X)$, $i = 1, \ldots, n$ be elements such that $u_1 \cdots u_n \neq 0$. Then $r(X) \geq n$. In other words, $r(X) \geq c_{E}(X)$ for every ring spectrum $E$.

It makes sense to remark that $r(X) = \text{cat} X$ iff $X$ possesses a detecting element, as defined in [R96].

Proof. (i) This follows from 1.4.

(ii) Because of 1.3(v), without loss of generality we can and shall assume that $C_n(X)$ is a CW-space. We set $u = u_1 \cdots u_n \in \tilde{E}^d(X)$. Because of the cup-length estimation of the Lusternik–Schnirelmann category, and by 1.3(ii), we have $p_n^*(u) = 0$. Hence, there is a homotopy commutative diagram

$$\begin{array}{ccc}
\Sigma^\infty X & \xrightarrow{\Sigma^\infty j_n} & \Sigma^\infty C_n(X) \\
\downarrow u & & \downarrow \\
\Sigma^d E & \xrightarrow{} & \Sigma^d E.
\end{array}$$

Now, if $r(X) < n$ then $\Sigma^\infty j_n$ is inessential, and so $u = 0$. This is a contradiction. □
2.3. Lemma. Let $f : X \to Y$ be a map of compact metric spaces with $Y$ connected, and let $j_m : Y \to C_m(Y)$ be as in 2.1. If the map $j_m f$ is essential then $\text{cat } f \geq m$. In particular, if $r(Y) = r$ and the map

$$f^\# : [Y, C_r(Y)] \to [X, C_r(Y)]$$

is injective then $\text{cat } f \geq r$.

Proof. Consider the diagram

$$
\begin{array}{ccc}
X & \xrightarrow{f} & P_m(Y) \\
\downarrow & & \downarrow p_m \\
Y & \xrightarrow{j_m} & C_m(Y).
\end{array}
$$

If $\text{cat } f < m$ then, by 1.4, there is a map $g : X \to P_m(Y)$ with $f = p_m g$, and hence $j_m f$ is inessential. This is a contradiction. $\square$

2.4. Theorem. Let $M^n$ be a closed oriented connected $n$-dimensional PL manifold such that $\text{cat } M = \dim M \geq 4$. Then $r(M) = \text{cat } M$.

Proof. Let $\text{MSPL}_*(-)$ denote the oriented PL bordism theory. By the definition of $r(X)$, it suffices to prove that $(j_n)_* : \text{MSPL}_*(M) \to \text{MSPL}_*(C_n(M))$ is a non-zero homomorphism. Hence, it suffices to prove that $(p_n)_* : \text{MSPL}_n(P_n(M)) \to \text{MSPL}_n(M)$ is not an epimorphism. Clearly, this will be proved if we prove that $[1_M] \in \text{MSPL}_n(M)$ does not belong to $\text{Im}(p_n)_*$.

Suppose the contrary. Then there is a map $F : W \to M$ with the following properties:

1. $W$ is a compact $(n+1)$-dimensional oriented PL manifold with $\partial W = M \sqcup V$;
2. $F|\partial W = 1_M$, $F|V : V \to M$ lifts to $P_n(M)$ with respect to the map $p_n : P_n(M) \to M$.

Without loss of generality we can assume that $W$ is connected.

Suppose for a moment that $\pi_1(W)$ is a one-point set (i.e., the pair $(W, M)$ is simply connected). Then $(W, M)$ has the handle presentation without handles of indices $\leq 1$, see [St68, 8.3.3, Theorem A]. By duality, the pair $(W, V)$ has the handle presentation without handles of indices $\geq n$. In other words, $W \simeq V \cup e_1 \cup \cdots \cup e_s$ where $e_1, \ldots, e_s$ are cells attached step by step and such that $\dim e_i \leq n - 1$ for every $i = 1, \ldots, s$. However, the fibration $p_n : P_n(M) \to M$ is $n - 2$ connected. Thus, $F : W \to M$ can be lifted to $P_n(M)$. In particular, $p_n$ has a section. But this contradicts 1.4.

So, it remains to prove that, for every membrane $(W, F)$, we can always find a membrane $(U, G)$ with $\pi_1(U, G) = *$ and $G|\partial U = F|\partial W$. Here $\partial U = \partial W = M \sqcup V$ and $G : U \to M$. We start with an arbitrary connected membrane $(W, F)$. Consider a PL embedding $i : S^1 \to \text{int } W$. Then the normal bundle $\nu$ of this embedding is trivial. Indeed, $w_1(\nu) = 0$ because $W$ is orientable.

Since $M$ is a retract of $W$, there is a commutative diagram

$$
\begin{array}{cccc}
0 & \longrightarrow & \pi_1(M) & \longrightarrow & \pi_1(W) & \longrightarrow & \pi_1(W, M) & \longrightarrow & 0 \\
\| & & \| & & \| & & \| \\
(M) & \longrightarrow & (W) & \longrightarrow & (W, M) & \longrightarrow & 0
\end{array}
$$

However, the fibration $p_n : P_n(M) \to M$ is $n - 2$ connected. Thus, $F : W \to M$ can be lifted to $P_n(M)$. In particular, $p_n$ has a section. But this contradicts 1.4.
where the top line is the homotopy exact sequence of the pair \((W, M)\). Clearly, if \(F_*\) is monic then \(\pi_1(W, M) = *\).

Let \(\pi_1(W)\) be generated by elements \(a_1, \ldots, a_k\). We set \(g_i := F_*(a_i)a_i^{-1} \in \pi_1(W)\) where we regard \(\pi_1(M)\) as the subgroup of \(\pi_1(W)\). Then \(\text{Ker} F_*\) is the smallest normal subgroup of \(\pi_1(W)\) contained \(g_1, \ldots, g_k\). Now we realize \(g_1, \ldots, g_k\) by PL embeddings \(S^1 \to \text{int} W\) and perform the surgeries of \((W, F)\) with respect to these embeddings, see [W70]. The result of the surgery establishes us a desired membrane. \(\Box\)

2.5. Corollary. Let \(M\) be as in 2.4, let \(X\) be a compact metric space, and let \(f : X \to M\) be a map such that \(f^* : H^n(M; \pi_n(C_n(M))) \to H^n(X; \pi_n(C_n(M)))\) is a monomorphism. Then \(\text{cat} f \geq \text{cat} M\).

Notice that, in fact, \(\text{cat} f = \text{cat} M\) since \(\text{cat} f \leq \text{cat} M\) for general reasons.

**Proof.** We set \(\pi = \pi_n(C_n(M))\). It is easy to see that \(C_n(M)\) is simply connected. Hence, by 1.3(iv) and the Hurewicz theorem, \(C_n(M)\) is \((n - 1)\)-connected. Thus, \([M, C_n(M)] = H^n(M; \pi)\). Let \(\iota : C_n(M) \to K(\pi, n)\) denote the fundamental class. Then \(f^*\) can be decomposed as

\[
f^* : H^n(M; \pi) = [M, C_n(M)] \xrightarrow{f^*} [X, C_n(M)] \xrightarrow{\iota_*} [X, K(\pi, n)] = H^n(X; \pi).
\]

Since \(f^*\) is a monomorphism, we conclude that \(f^*\) is. Thus, by 2.3 and 2.4,

\[
\text{cat} f \geq r(M) = \text{cat} M. \quad \Box
\]

§3. The invariant \(r(M)\) and the Arnold Conjecture

Recall (see the introduction) that the Arnold conjecture claims that \(\text{Arn}(M, \omega) \geq \text{Crit} M\) for every closed symplectic manifold \((M, \omega)\).

3.1. Recollection. A flow on a topological space \(X\) is a family \(\Phi = \{\varphi_t\}, t \in \mathbb{R}\) where each \(\varphi_t : X \to X\) is a self-homeomorphism and \(\varphi_s \varphi_t = \varphi_{s+t}\) for every \(s, t \in \mathbb{R}\) (notice that this implies \(\varphi_0 = 1_X\)).

A flow is called **continuous** if the function \(X \times \mathbb{R} \to X, (x, t) \mapsto \varphi_t(x)\) is continuous.

A point \(x \in X\) is called a **rest point** of \(\Phi\) if \(\varphi_t(x) = x\) for every \(t \in \mathbb{R}\). We denote by \(\text{Rest} \Phi\) the number of rest points of \(\Phi\).

A continuous flow \(\Phi = \{\varphi_t\}\) is called **gradient-like** if there exists a continuous (Lyapunov) function \(F : X \to \mathbb{R}\) with the following property: for every \(x \in X\) we have \(F(\varphi_t(x)) < F(\varphi_s(x))\) whenever \(t > s\) and \(x\) is not a rest point of \(\Phi\).

3.2. Definition (cf. [H88], [MS95]). Let \(X\) be a topological space. We define an **index function** on \(X\) to be any function \(\nu : 2^X \to \mathbb{N} \cup \{0\}\) with the following properties:

1. (monotonicity) If \(A \subset B \subset X\) then \(\nu(A) \leq \nu(B)\);
2. (continuity) For every \(A \subset X\) there exists an open neighbourhood \(U\) of \(A\) such that \(\nu(A) = \nu(U)\);
3. (subadditivity) \(\nu(A \cup B) \leq \nu(A) + \nu(B)\);
4. (invariance) If \(\{\varphi_t\}, t \in \mathbb{R}\) is a continuous flow on \(X\) then \(\nu(\varphi_t(A)) = \nu(A)\) for every \(A \subset X\) and \(t \in \mathbb{R}\);
5. (normalization) \(\nu(\emptyset) = 0\). Furthermore, if \(A \neq \emptyset\) is a finite set which is contained in a connected component of \(X\) then \(\nu(A) = 1\).
3.3. Theorem. Let $\Phi$ be a gradient-like flow on a compact metric space $X$. Then
\[
\text{Rest } \Phi \geq \nu(X)
\]
for every index function $\nu$ on $X$.

Proof. The proof follows the ideas of Lusternik–Schnirelmann. For $X$ connected see [H88], [MS95, p.346 ff]. Furthermore, if $X = \sqcup X_i$ with $X_i$ connected then
\[
\text{Rest } \Phi = \sum \text{Rest}(\Phi|X_i) \geq \sum \nu(X_i) \geq \nu(X). \quad \Box
\]

3.4. Corollary. Let $\Phi$ be a gradient-like flow on a compact metric space $X$, let $Y$ be a Hausdorff space which admits a covering $\{U_\alpha\}$ such that each $U_\alpha$ is open and contractible in $Y$, and let $f : X \to Y$ be an arbitrary map. Then
\[
\text{Rest } \Phi \geq 1 + \text{cat } f.
\]

Proof. Given a subspace $A$ of $X$, we define $\nu(A)$ to be the minimal number $m$ such that $A \subset U_1 \cup \cdots \cup U_m$ where each $U_i$ is open in $X$ and $f|U_i$ is inessential. It is easy to see that $\nu$ is an index function on $X$ (normalization follows from the properties of $Y$). But $\nu(X) = 1 + \text{cat } f$, and so, by 3.3, we conclude that $\text{Rest } \Phi \geq 1 + \text{cat } f. \quad \Box$

3.5. Theorem. Let $(M, \omega)$ be a closed connected symplectic manifold with $I_\omega = 0 = I_c$, and let $\phi : M \to M$ be a Hamiltonian symplectomorphism. Then there exists a map $f : X \to M$ with the following properties:

(i) $X$ is a compact metric space;
(ii) $X$ possesses a gradient-like flow $\Phi$ such that $\text{Rest } \Phi = \text{Fix } \phi$;
(iii) The homomorphism $f^* : H^n(M; G) \to H^n(X; G)$ is a monomorphism for every coefficient group $G$.

Proof. This can be proved following [F89-2, Theorem 7]. (Note that the formulation of this theorem contains a misprint: there is typed $z^*P = 0$, while it must be typed $z^*[P] \neq 0$. Furthermore, the reference [CE] in the proof must be replaced by [F7].) In fact, Floer denoted by $z : \mathcal{S} \to P$ what we denote by $f : X \to M$, and he showed that the homomorphism $z^* : Z = H^n(P) \to H^n(\mathcal{S})$ is monic. He did it for $\mathbb{Z}$-coefficients, but the proof for arbitrary $G$ is similar.

Also, cf. [H88] and [HZ94, Ch. 6].

In fact, Floer considered Alexander–Spanier cohomology, but for compact metric spaces it coincides with $H^*(-)$. In greater detail, you can find in [Sp66] an isomorphism between Alexander–Spanier and Čech cohomology and in [Hu61] an isomorphism between Čech cohomology and $H^*(-). \quad \Box$

Recall that every smooth manifold turns out to be a PL manifold in a canonical way, see e.g [Mu66].

3.6. Theorem. Let $(M, \omega)$ be a closed connected symplectic manifold with $I_\omega = 0 = I_c$ and such that $\text{cat } M = \dim M$. Then $\text{Arn}(M, \omega) \geq \text{Crit } M$.

Proof. The case $\dim M = 2$ is well known, see [F89-1], [H88], so we assume that $\dim M \geq 4$. Consider any Hamiltonian symplectomorphism $\phi : M \to M$ and the corresponding data $\Phi$ and $f : X \to M$ as in 3.5. Then, by 3.5, $\text{Fix } \phi = \text{Rest } \Phi$.
and hence, by 3.4 and 2.5, $\text{Fix} \, \phi \geq 1 + \text{cat} \, M$, and thus $\text{Arn}(M, \omega) \geq 1 + \text{cat} \, M$. Furthermore, by a theorem of Takens [T68], $\text{Crit} \, M \leq 1 + \dim \, M$. Now,

$$1 + \text{cat} \, M \leq \text{Crit} \, M \leq 1 + \dim \, M = 1 + \text{cat} \, M,$$

and thus $\text{Arn}(M, \omega) \geq \text{Crit} \, M$. □

3.7. Example (cat $M > \text{cl} \, M$). Let $M$ be a four-dimensional aspherical symplectic manifold described in [MS95, Example 3.8]. It is easy to see that $H^1(M) = \mathbb{Z}^3$. Furthermore, $H^*(M)$ is torsion free, and so $a^2 = 0$ for every $a \in H^1(X)$. Hence, $\text{cl} \, M = 3$. However, cat $M = 4$ because $\text{cat} \, V = \dim \, V$ for every closed aspherical manifold $V$, see [EG57]. Moreover, for every closed symplectic manifold $N$ we have $\text{cl}(M \times N) < \text{cat}(M \times N)$ because $\text{cat}(M \times N) = \dim \, N + 4$ according to [RO97].

§4. The invariant $r(M)$ and critical points

Let $X$ be a CW-space and let $A, B$ be two CW-subspaces of $X$. Then for every spectrum $E$ we have the cap-product

$$\cap : E_i(X, A \cup B) \otimes \Pi^j(X, A) \to E_{i-j}(X, B),$$

see [Ad74], [Sw75]. Here $\Pi^*(-)$ denotes stable cohomotopy, i.e., $\Pi^*(-)$ is the cohomology theory represented by the sphere spectrum $S$.

In particular, if $D = D^k$ is the $k$-dimensional disk then for every CW-pair $(X, A)$ we have the cup-product

$$\cap : E_i(X \times D, X \times \partial D \cup A \times D) \otimes \Pi^k(X \times D, X \times \partial D) \to E_{i-k}(X \times D, A \times D).$$

Let $a \in \Pi^k(D, \partial D) = \mathbb{Z}$ be a generator. We set $t = p^* a \in \Pi^k(X \times D, X \times \partial D)$ where $p : (X \times D, X \times \partial D) \to (D, \partial D)$ is the projection.

4.1. Lemma. For every CW-pair $(X, A)$ the homomorphism

$$\cap t : E_i(X \times D, X \times \partial D \cup A \times D) \to E_{i-k}(X \times D, A \times D)$$

is an isomorphism.

In fact, it is a relative Thom–Dold isomorphism.

Proof. If $A = \emptyset$ then $\cap t$ is the standard Thom–Dold isomorphism for the trivial $D^k$-bundle (or the suspension isomorphism, if you want), see e.g. [Sw75]. In other words, for $A = \emptyset$ the homomorphism in question has the form $\cap t : E_i(T \alpha) \to E_{i-k}(X \times D)$ where $T \alpha$ is the Thom space of the trivial $D^k$-bundle $\alpha$. Furthermore, the homomorphism in question has the form

$$E_i(T \alpha, T(\alpha | A)) \to E_{i-k}(X \times D, A \times D).$$

Considering the commutative diagram

$$\cdots \to E_i(T(\alpha | A)) \longrightarrow E_i(T \alpha) \longrightarrow E_i(T \alpha, T(\alpha | A)) \to \cdots$$

$$\downarrow \cap (t | A) \quad \downarrow \cap t \quad \downarrow \cap t$$

$$E_i(A \times D) \quad E_i(X \times D) \quad E_i(X \times D, A \times D)$$
with the exact rows, and using the Five Lemma, we conclude that the homomorphism in question is an isomorphism. □

4.2. Definition ([CZ83], [MO93]). Given a connected closed smooth manifold M, we define $\mathcal{G}H_{p,q}(M)$ to be the set of all $C^2$-functions $g : M \times \mathbb{R}^{p+q} \to \mathbb{R}$ with the following properties:

1. There exist disks $D_+ \subset \mathbb{R}^p$ and $D_- \subset \mathbb{R}^q$ centered in origin such that $\text{int}(M \times D_+ \times D_-)$ contains all critical points of $g$;
2. $\nabla g(x)$ points inward on $M \times \partial D_+ \times \text{int} D_-$ and outward on $M \times \text{int} D_+ \times \partial D_-$.

4.3. Definition ([CZ83], [MO93]). Given $g \in \mathcal{G}H_{p,q}(M)$, consider the gradient flow $\dot{x} = \nabla g(x)$. Let $x \cdot \mathbb{R}$ denote the solution of the flow through $x$. We choose $D_+$ and $D_-$ as in 4.2, set $B := M \times D_+ \times D_-$. and define $S_g = S_{g,B} := \{ x \in B | x \cdot \mathbb{R} \subset B \}$.

4.4. Theorem (cf [MO93, 4.1]). For every function $g \in \mathcal{G}H_{p,q}(M)$, there is a subpolyhedron $K$ of int $B$ such that $S_g \subset K$ and $\text{crit } g \geq 1 + \text{cat}_B K$.

Proof. We set $S = S_g$. Because of 4.2, $S$ is a compact subset of int $B$. Furthermore, $S$ is an invariant set of the gradient flow $\dot{x} = \nabla g(x)$, and $S$ contains all critical points of $g$. Given $A \subset S$, we define $\nu(A) = 1 + \text{cat}_B A$. Clearly, $\nu$ is an index function on $S$. Thus, by 3.3, $\nu(S) \leq \text{crit } g$. Now, let $V_1, \ldots, V_{\nu(S)}$ be a covering of $S$ such that every $V_i$ is open and contractible in $B$. Choose any simplicial triangulation of $B$. Then, by the Lebesgue Lemma, there exists a simplicial subdivision of $B$ with the following property: every simplex $e$ with $e \cap S \neq \emptyset$ is contained in some $V_i$. Now, we set $K$ to be the union of all simplices $e$ with $e \cap S \neq \emptyset$. Clearly, $1 + \text{cat}_B K \leq \nu(S)$, and thus $\text{crit } g \geq 1 + \text{cat}_B K$. Finally, we can find $K \subset \text{int } B$ because of the collar theorem. □

Let $r(M)$ be the invariant defined in 2.1.

4.5. Theorem. For every function $g \in \mathcal{G}H_{p,q}(M)$, the number of critical points of $g$ is at least $1 + r(M)$. In particular, $\text{crit } g \geq 1 + \text{cat } M$ if $M$ is aspherical.

Proof. Here we follow McCord–Oprea [MO93]. However, unlike them, here we use certain extraordinary (co)homology instead of classical (co)homology.

Let $r := r(M)$. We choose $K$ as in 4.4 and prove that $\text{cat}_B K \geq r$. Consider the Puppe sequence

$$P_r(M) \xrightarrow{p_r} M \xrightarrow{j_r} C_r(M).$$

Let $e : M^+ \to C_r(M)$ be a map such that $e|M = j_r$ and $e$ maps the added point to the base point of $C_r(M)$. Let $h : C_r(M) \to C$ be a pointed homotopy equivalence such that $C$ is a $CW$-complex. We set $E = \Sigma^\infty C$ and let $u_r \in E^0(M)$ be the stable homotopy class of the map $he : M^+ \to C$. Then $u_r \neq 0$ since $j_r$ is stably essential.

We define

$$f : K \subset B = M \times \mathbb{R}^{p+q} \xrightarrow{\text{projection}} M.$$

4.6. Lemma. If $f^* u_r \neq 0$ then $\text{cat}_B K \geq r$.

Proof. Since $(p_r, j_r)$ is a Puppe sequence, $p_r^* u_r = 0$. Hence, the map $f$ can’t be lifted to $P_r(M)$, and therefore the inclusion $K \subset B$ can’t be lifted to $P_r(B)$. So, cat$_B K \geq r$. The lemma is proved.
We continue the proof of the theorem. Let \( j : K \subset B \) be the inclusion. By 4.6, it suffices to prove that \( j^* : E^*(B) \to E^*(K) \) is a monomorphism. Notice that if \( Y \) is a CW-subspace of \( \mathbb{R}^N \) then there is a duality isomorphism

\[
E^0(Y) \cong E_{-N}(\mathbb{R}^N, \mathbb{R}^N \setminus Y) := E_{-N}(\mathbb{R}^N \cup C(\mathbb{R}^N \setminus Y))
\]

see e.g. [DP84]. So, it suffices to prove that the dual homomorphism

\[
D(j^*) : E_*(\mathbb{R}^N, \mathbb{R}^N \setminus B) \to E_*(\mathbb{R}^N, \mathbb{R}^N \setminus K)
\]

is monic for a certain (good) embedding \( B \to \mathbb{R}^N \).

We have the following commutative diagram:

\[
\begin{array}{ccc}
E_*(\mathbb{R}^N, \mathbb{R}^N \setminus B) & \xrightarrow{h} & E_*(\mathbb{R}^N, \mathbb{R}^N \setminus B) \\
\downarrow \cong & & \downarrow D(j^*) \\
E_*(\mathbb{R}^N, \mathbb{R}^N \setminus \text{int } B) & \to & E_*(\mathbb{R}^N, \mathbb{R}^N \setminus K) \\
\uparrow \cong & & \uparrow e' \cong \\
E_*(B, \partial B) & \xrightarrow{a^*} & E_*(B, B \setminus K)
\end{array}
\]

where all the homomorphisms except \( D(j^*) \) are induced by the inclusions. Here \( h \) is an isomorphism since the inclusion \( \text{int } B \to B \) is a homotopy equivalence (the space \( B \setminus \{\text{collar}\} \) is a deformation retract of \( \text{int } B \)). Furthermore, \( e \) is an isomorphism since \( (B, \partial B) \) and \( (\mathbb{R}^N, \mathbb{R}^N \setminus \text{int } B) \) are cofibered pairs, while \( e' \) is an isomorphism by Lemma 3.4 from [DP84]. So, \( D(j^*) \) is monic if \( a_* \) is. Since \( B \setminus K \subset B \setminus S \), it suffices to prove that \( E_*(B, \partial B) \to E_*(B, B \setminus S) \) is a monomorphism.

Let \( B_+ = M \times \partial D_+ \times D_- \), and let \( B_- = M \times D_+ \times \partial D_- \). Furthermore, let \( A_+ := \{ x \in B \mid x \cdot \mathbb{R}_- \in B \} \) and let \( A_- := \{ x \in B \mid x \cdot \mathbb{R}_+ \in B \} \). Then \( B_+ \cap A_- = \emptyset = B_- \cap A_+ \), and so there are the inclusions \( i_+ : (B, B_+) \to (B, B \setminus A_-) \) and \( i_- : (B, B_-) \to (B, B \setminus A_+) \). It turns out to be that both \( i_+ \) and \( i_- \) are homotopy equivalences, [CZ83, Lemma 3].

Let \( t \in \Pi^m(B, B_-) \) be the class as in 4.1, and let \( t' := (i_-)^{-1}(t) \). Since \( S = A_+ \cap A_- \), we have the commutative diagram

\[
\begin{array}{ccc}
E_{i}(B, \partial B) & \longrightarrow & E_{i}(B, B \setminus S) \\
\cong \downarrow \cap t & & \downarrow \cap t' \\
E_{i-q}(B, B_+) & \xrightarrow{\cong} & E_{i-q}(B, B \setminus A_-)
\end{array}
\]

where the left map is an isomorphism by 4.1 and the bottom map is the isomorphism \( (i_+)_* \). (Generally, \( (B, B \setminus S) \) is not a CW-pair, but nevertheless in our case the map \( \cap t' \) is defined, see [DP84, 3.5].) Thus, the top homomorphism is injective.

Finally, if \( M \) is aspherical then \( \text{cat } M = \dim M \), [EG57], and so \( r(M) = \text{cat } M \) by 2.4. \( \square \)

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