Aspects of $N = 2$ Super-$W_n$ Strings

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ABSTRACT

We construct $N = 2$ super-$W_{n+1}$ strings and obtain the complete physical spectrum, for arbitrary $n \geq 2$. We also derive more general realisations of the super-$W_{n+1}$ algebras in terms of $k$ commuting $N = 2$ super energy-momentum tensors and $n - k$ pairs of complex superfields, with $0 \leq k \leq \lfloor \frac{n+1}{2} \rfloor$.

Available from hep-th/9205054

* Supported in part by the U.S. Department of Energy, under grant DE-FG05-91ER40633.
1. Introduction

It is well known that two-dimensional gravity is the gauge theory of the Virasoro algebra. There are various extensions of the Virasoro algebra, including the super Virasoro algebras, the $W$ algebras, and the super-$W$ algebras. The gauge theories of these algebras give rise to extensions of two dimensional gravity. Such a theory is anomalous at the quantum level unless the central charge of the algebra takes a specific value which cancels the contribution from the ghosts. Since the ghosts for a bosonic (fermionic) current of spin $s$ contribute $±2(6s^2 - 6s + 1)$ to the central charge, it follows that for an algebra with currents of spin $s$, the central charge for the matter realisation of the corresponding $W$-gravity theory is

$$c = 2 \sum_{\{s\}_B} \left(6s^2 - 6s + 1\right) - 2 \sum_{\{s\}_F} \left(6s^2 - 6s + 1\right).$$

(1.1)

Here $\{s\}_B$ and $\{s\}_F$ denote the set of spins of the bosonic and fermionic currents in the algebra.

If the matter realisation of the $W$ algebra includes scalar fields, these fields may be interpreted as the coordinates of spacetime in a corresponding $W$-string theory. The condition of anomaly freedom (1.1) can be derived from the nilpotency requirement for the BRST operator. The intercepts for the bosonic currents are determined by the requirement that the product of a physical state with the ghost vacuum should be BRST invariant. Owing to the nonlinearity of the $W$ algebra, this is a complicated calculation in general. However, since the Virasoro subalgebra is linear, the intercept for $L_0$ can in fact be determined straightforwardly. The structure of the ghost vacuum was obtained in general in [1]; the resulting spin-2 intercept is

$$L_0 = \frac{1}{2} \sum_{\{s\}_B} s(s - 1) - \frac{1}{2} \sum_{\{s\}_F} (s - \frac{1}{2})^2$$

$$= \frac{1}{12}(c - 2N_B - N_F),$$

(1.2)

where $N_B$ and $N_F$ are the numbers of bosonic and fermionic currents in the $W$ algebra. This result gives the conformal dimension of physical states in the $W$-string theory.

The bosonic $W_{n+1}$ algebra has one current of each spin $s$ in interval $2 \leq s \leq n + 1$, and so

$$c = 2n(2n^2 + 6n + 5),$$

$$L_0 = \frac{1}{6}n(n + 1)(n + 2).$$

(1.3a)

(1.3b)

The $W_{n+1}$ algebra here is based on the Miura transformation for the $A_n$ algebra. In fact any semi-simple Lie algebra $G$ gives rise to an associated $W$ algebra, denoted by $WG$ [2,3]. The $WD_n$ algebra has currents with spins $2, 4, 6, \ldots, 2n - 2$ and $n$. This implies

$$c = 2n(8n^2 - 12n + 5),$$

$$L_0 = \frac{1}{3}n(n - 1)(2n - 1).$$

(1.4a)

(1.4b)

The $WB_n$ algebra has bosonic currents with spins $2, 4, 6, \ldots, 2n$ and a fermionic current with spin $n + \frac{1}{2}$, which implies

$$c = (2n + 1)(8n^2 - 4n + 1),$$

$$L_0 = \frac{1}{6}n(4n^2 - 1).$$

(1.5a)

(1.5b)

The $N = 2$ super-$W_{n+1}$ algebra contains currents which can be grouped into $N = 2$ supermultiplets with the spin content [4]

$$\begin{align*}
\left\{ \begin{array}{c}
1 & \frac{3}{2} \\
\frac{3}{2} & 2
\end{array} \right\} & \left\{ \begin{array}{c}
2 & \frac{5}{2} \\
\frac{5}{2} & 3
\end{array} \right\} \left\{ \begin{array}{c}
3 & \frac{7}{2} \\
\frac{7}{2} & 4
\end{array} \right\} & \cdots & \left\{ \begin{array}{c}
n & \frac{n + 1}{2} \\
\frac{n + 1}{2} & (n + 1)
\end{array} \right\}.
\end{align*}$$

(1.6)
Each lozenge contributes 6 to central charge $c$ and 0 to the intercept $L_0$. So for the $N = 2$ super-$W_{n+1}$ algebra we have

$$c = 6n , \quad \quad (1.7a)$$
$$L_0 = 0 . \quad \quad (1.7b)$$

The $N = 1$ super-$W_{n+1}$ algebra is a subalgebra of the $N = 2$ super-$W_{n+1}$ algebra, whose spin content comprises the even-integer bosonic spin and one of the fermionic spins in each lozenge [4], namely

$$\frac{3}{2}, 2, \frac{5}{2}, \frac{7}{2}, 4, \ldots , \left(\frac{n+1}{2} + \left[\frac{n+1}{2}\right]\right) , \quad (1.8)$$

where $\left[\frac{n+1}{2}\right]$ stands for the integer part of $\frac{n+1}{2}$. For example, when $n = 1$ we have spins $\frac{3}{2}, 2$; when $n = 2$ we have spins $\frac{3}{2}, 2, \frac{5}{2}$; and when $n = 3$ we have spins $\frac{3}{2}, 2, \frac{5}{2}, \frac{7}{2}, 4$. The central charge and the $L_0$ intercept for $N = 1$ super-$W_{n+1}$ algebra are

$$c = 12(-1)^{n+1} \left[\frac{n+1}{2}\right] + 3n , \quad \quad (1.9a)$$
$$L_0 = \frac{1}{2}(-1)^{n+1} \left[\frac{n+1}{2}\right] . \quad \quad (1.9b)$$

Note that when $n$ is odd, both $c$ and $L_0$ are positive, whilst they are both negative when $n$ is even.

So far, $W$ string theories based on the WA$_n$ [5,6,7], WB$_n$ and WD$_n$ [8,9] algebras have been studied in detail. It is interesting to study the supersymmetric extensions, and in this paper we shall look at the $N = 2$ super-$W_{n+1}$ algebras and construct the corresponding string theories. We shall obtain the complete spectrum of the $N = 2$ super-$W_{n+1}$ string, and discuss the relation with the $N = 2$ super-Virasoro minimal models. In section 2, we take the known Miura transformation for the super-$W_{n+1}$ algebra [10,11] and use it to prove that the currents of super-$W_{n+1}$ may be written in terms of those of super-$W_n$ and an extra pair of complex superfield. Applying this recursively, we obtain a realisation in terms of an arbitrary super energy-momentum tensor together with $(n - 1)$ additional pairs of complex superfields. In section 3 we use this realisation to build a super-$W_{n+1}$ string theory, and obtain its physical spectrum. In section 4 we discuss the issue of unitarity. Although some physical states have negative norm, we argue that a truncation to a subspace of positive-norm states can be made. In section 5 we obtain more general realisations, and make concluding remarks.

2. Miura Transformation

A realisation for the $N = 2$ super-$W_{n+1}$ algebra is constructed by the Miura transformation based on the $A(n, n - 1)$ super algebra. It is most conveniently expressed in terms of $N = 1$ superfields [10], and is given by the differential operator

$$\mathcal{M}_n = \left(\prod_{j=1}^{n}[(\alpha_0 D + D\Phi_j - \chi_{n-j}^{(n)})(\alpha_0 D + D\Phi_j - \chi_{n+1-j}^{(n)})]\right)(\alpha_0 D - \chi_{n-1}^{(n)})$$

$$= (\alpha_0 D)^{2n+1} + \sum_{\ell=2}^{2n+1} U_{\ell}^{(n)}(\alpha_0 D)^{2n+1-\ell} , \quad (2.1)$$

where

$$\chi_0^{(n)} = 0$$
$$\chi_{j}^{(n)} = \sum_{k=n-j+1}^{n} D\bar{\Phi}_k(z, \theta) , \quad (2.2)$$
and the $N = 1$ superfields $\Phi_i(z, \theta)$ and $\bar{\Phi}_i(z, \theta)$ are given in terms of components by

$$
\Phi_i(z, \theta) = \phi_i(z) + \theta \psi_i(z), \quad \bar{\Phi}_i(z, \theta) = \bar{\phi}_i(z) + \theta \bar{\psi}_i(z).
$$

(2.3)

Here $D$ is given by

$$
D = \frac{\partial}{\partial \theta} + \theta \partial,
$$

(2.4)

where $z$ and $\theta$ are the bosonic and fermionic coordinates of a 2-dimensional superspace, $\partial \equiv \partial/\partial z$ and the derivative $D$ satisfies $D^2 = \partial$. In equation (2.1), $U^{(n)}_{\ell}$ is a current of super-conformal spin $\ell/2$ in the super-$W_{n+1}$ algebra. In component language, it can be written as

$$
\alpha_0^{-k} U^{(n)}_{2k}(z, \theta) = B_k(z) + \theta F_{k+\frac{1}{2}}(z),
$$

$$
\alpha_0^{-k} U^{(n)}_{2k+1}(z, \theta) = \bar{F}_{k+\frac{1}{2}}(z) + \theta \bar{B}_{k+1}(z).
$$

(2.5)

The components $B$ and $F$ denote the bosonic and fermionic currents with spins indicated by their indices.

By convention, we shall always order products such as the one in (2.1) in decreasing order of $j$, i.e. the largest-$j$ factor sits at the left. As a consequence, we can rewrite (2.1) as

$$
\mathcal{M}_n = \alpha_0 D + D \Phi_n)(\alpha_0 D + D \bar{\Phi}_n - D \Phi_n) \times \left( \prod_{j=1}^{n-1} (\alpha_0 D + D \Phi_j - \chi^{(n-1)}_{n-j-1} - D \Phi_n)(\alpha_0 D + D \Phi_j - \chi^{(n-1)}_{n-j} - D \Phi_n) \right)
$$

$$
\times (\alpha_0 D - \chi^{(n-1)}_{n-1} - D \Phi_n),
$$

(2.6)

where we have used the recursion relation of $\chi^{(n)}_i$:

$$
\chi^{(n)}_i = \chi^{(n-1)}_{i-1} + D \Phi_n,
$$

(2.7)

which follows immediately from (2.2). Since $\partial + (\partial f) = e^{-f} \partial e^f$, one can re-express equation (2.6) in the recursive form

$$
\mathcal{M}_n = \alpha_0 D + D \Phi_n)(\alpha_0 D + D \Phi_n - D \Phi_n)e^{\Phi_n/\alpha_0} \mathcal{M}_{n-1} - \epsilon_n^{-\Phi_n/\alpha_0}.
$$

(2.8)

Using (2.1) with $n \to n-1$, we can therefore write the currents of super-$W_{n+1}$ in terms of those of super-$W_n$ and an additional complex pair of superfields $(\Phi_n, \bar{\Phi}_n)$. Explicitly, we find

$$
U^{(n)}_{2j} = \sum_{i=0}^{j} \binom{n+i-j}{i} \left[ \sum_{i=0}^{j} \binom{n+i-j}{i} \left[ (\alpha_0 D + D \Phi_n)U^{(n-1)}_{2j-2i} \right] + \alpha_0 D \Phi_n(DU^{(n-1)}_{2j-2i}) + \alpha_0^2 \partial U^{(n-1)}_{2j-2i} \right] P_i(\Phi_n),
$$

$$
U^{(n)}_{2j+1} = \sum_{i=0}^{j} \binom{n+i-j}{i} \left[ (\alpha_0 D + D \Phi_n)(\alpha_0 D + D \bar{\Phi}_n) \left( (\alpha_0 D P_i(\Phi_n) - P_i(\bar{\Phi}_n)D \Phi_n)U^{(n-1)}_{2j-2i} \right) + \alpha_0 D \Phi_n U^{(n-1)}_{2j-2i} + \alpha_0 D \bar{\Phi}_n U^{(n-1)}_{2j-2i} \right] P_i(\Phi_n).
$$

(2.9)
where the currents $U^{(n-1)}_j$ with $j < 0$ or $j = 1$ are defined to be zero, and $U^{(n-1)}_0 = 1$. In (2.9) we have defined $P_i(\Phi_n)$, which is a differential polynomial in $\Phi_n$, by

$$P_i(\Phi_n) \equiv e^{\phi_n/\alpha_0} \left( (\alpha_0^2 \partial) e^{-\phi_n/\alpha_0} \right).$$

(2.10)

Equation (2.9) gives a realisation of the super-$W_{n+1}$ currents in terms of those for super-$W_n$, together with an additional pair of complex superfields $(\Phi_n, \bar{\Phi}_n)$. Applying this reduction recursively, one obtains a realisation of the super-$W_{n+1}$ algebra in terms of $(\Phi_1, \bar{\Phi}_1)$, which appear in the currents only via their super energy-momentum tensor, together with $(n - 1)$ pairs of complex superfields $(\Phi_2, \bar{\Phi}_2, \ldots, \Phi_n, \bar{\Phi}_n)$. Since $(\Phi_1, \bar{\Phi}_1)$ commute with the other superfields, their super energy-momentum tensor may be replaced by an arbitrary one that has the same “exterior” characteristics: i.e. commuting with $(\Phi_2, \bar{\Phi}_2, \ldots, \Phi_n, \bar{\Phi}_n)$ and having the same central charge. This construction was given for the super-$W_3$ algebra in [1], where both $N = 2$ super currents and their components of the algebra were given explicitly.

The recursion formulae given in (2.9) are complicated in general; fortunately, as we shall see later, it is unnecessary to solve the explicit forms of the higher-spin super currents. The lower-spin super currents can be easily obtained from (2.9)

$$U^{(n)}_2 = U^{(n-1)}_2 - D\Phi_n D\Phi_n + \alpha_0 \partial \Phi_n - n\alpha_0 \partial \bar{\Phi}_n$$

$$= \sum_{j=1}^{n} \left( - D\Phi_j D\Phi_j + \alpha_0 \partial \Phi_j - j\alpha_0 \partial \bar{\Phi}_j \right),$$

$$U^{(n)}_3 = U^{(n-1)}_3 - \partial \Phi_n D\Phi_n - n\alpha_0 \partial D\Phi_n$$

$$= \sum_{j=1}^{n} \left( - \partial \Phi_j D\Phi_j + j\alpha_0 \partial D\Phi_j \right).$$

(2.11)

Introducing a second anticommuting coordinate $\tilde{\theta}$, one can assemble the the currents $U^{(n)}_2$ and $U^{(n)}_3$ into the $N = 2$ super energy-momentum tensor $T(z, \theta, \tilde{\theta})$ of the super-$W_{n+1}$ algebra

$$T(z, \theta, \tilde{\theta}) = \frac{1}{4} U^{(n)}_2(z, \theta) + \frac{1}{4} \tilde{\theta} \left( 2U^{(n)}_3(z, \theta) - DU^{(n)}_2(z, \theta) \right),$$

(2.12)

which can be expanded in components as

$$T(z, \theta, \tilde{\theta}) = \frac{1}{2} J(z) + \frac{1}{2} \tilde{\theta} G_\theta(z) - \frac{1}{2} \tilde{\theta} \bar{G}_\theta(z) + \tilde{\theta} \theta T(z).$$

(2.13)

The component currents $(J, G_\theta, \bar{G}_\theta, T)$ have conformal spins $(1, \frac{3}{2}, \frac{3}{2}, 2)$ with respect to the energy-momentum tensor $T(z)$. The explicit forms of these component currents are given by

$$J(z) = \sum_{j=1}^{n} \left( - \psi_j \bar{\psi}_j + \alpha_0 \partial \phi_j - j\alpha_0 \partial \bar{\phi}_j \right),$$

$$T(z) = \sum_{j=1}^{n} \left( \frac{1}{2} \psi_j \partial \bar{\psi}_j - \frac{1}{2} \bar{\psi}_j \partial \psi_j - \partial \phi_j \partial \bar{\phi}_j - \frac{1}{2} \alpha_0 \partial^2 \phi_j - \frac{1}{2} j\alpha_0 \partial^2 \bar{\phi}_j \right),$$

$$G(z) \equiv \frac{1}{\sqrt{2}} \left( G_\theta + \bar{G}_\theta \right) = \sum_{j=1}^{n} \sqrt{2} \left( \partial \bar{\phi}_j \psi_j + \alpha_0 \partial \psi_j \right),$$

$$\bar{G}(z) \equiv \frac{1}{\sqrt{2}} \left( G_\theta - \bar{G}_\theta \right) = \sum_{j=1}^{n} \sqrt{2} \left( \partial \phi_j \bar{\psi}_j + j\alpha_0 \partial \bar{\psi}_j \right),$$

(2.14)
where the free fields \((\phi_j, \bar{\phi}_j)\) and \((\psi_j, \bar{\psi}_j)\) are the components of the superfields defined in (2.3). It follows from (2.14) that the super energy-momentum tensor \(T(z, \theta, \bar{\theta})\) given in (2.12) generates the super-Virasoro algebra with central charge
\[
c_n = 3n \left(1 + (n + 1)\alpha_0^2\right). \tag{2.15}
\]

The pair of complex superfields \((\Phi_1, \bar{\Phi}_1)\) appears in (2.9) only via \(U_2^{(1)}\) and \(U_3^{(1)}\), i.e. their super energy-momentum tensor \(T^{\text{eff}}(z, \theta, \bar{\theta}) = \frac{1}{2}U_2^{(1)}(z, \theta) + \frac{1}{4}\bar{\theta}(2U_3^{(1)}(z, \theta) - DU_2^{(1)}(z, \theta))\). The components of this super energy-momentum tensor are given by
\[
\begin{align*}
J^{\text{eff}}(z) &= -\psi_1 \bar{\psi}_1 + \alpha_0 \partial \phi_1 - \alpha_0 \partial \bar{\phi}_1, \\
T^{\text{eff}}(z) &= \frac{1}{2}\psi_1 \partial \bar{\psi}_1 - \frac{1}{2}\partial \psi_1 \bar{\psi}_1 - \frac{1}{2}\alpha_0 \partial^2 \phi_1 - \frac{1}{2}\alpha_0 \partial^2 \bar{\phi}_1, \\
G^{\text{eff}} &= \sqrt{2} \left(\partial \bar{\phi}_1 \psi_1 + \alpha_0 \partial \psi_1\right), \\
\tilde{G}^{\text{eff}} &= \sqrt{2} \left(\partial \phi_1 \bar{\psi}_1 + \alpha_0 \partial \bar{\psi}_1\right).
\end{align*}
\tag{2.16}
\]
Thus the super energy-momentum tensor \(T^{\text{eff}}(z, \theta, \bar{\theta})\) generates the super-Virasoro algebra with central charge \(c^{\text{eff}}\) given by
\[
c^{\text{eff}} = 3 + 6\alpha_0^2 = \frac{2(c_n - 6n)}{n(n + 1)} + 6 - \left(3 - \frac{6}{n + 1}\right). \tag{2.17}
\]

The contribution from \((\Phi_1, \bar{\Phi}_1)\) can then be replaced by an arbitrary super energy-momentum tensor with the same central charge given in (2.17).

In section (3), we shall prove that the solutions to the physical-state conditions for the tachyonic states of the super-\(W_{n+1}\) string form a representation of the Weyl group for the bosonic subalgebra \(A_n \oplus A_{n+1}\) of the super algebra \(A(n, n - 1)\). For this purpose we shall now describe the root space for this super algebra. The Miura transformation (2.1) is a realisation in a particular basis of the general expression
\[
\mathcal{M}_n = \prod_{j=0}^{2n} \left(\alpha_0 D + \bar{H}_j^{(n)} \cdot D \bar{\Phi}^{(n)}\right), \tag{2.18}
\]
where \(\bar{\Phi}^{(n)} \equiv (\bar{\Phi}_1, \bar{\Phi}_2, \bar{\Phi}_3; \ldots; \bar{\Phi}_n, \bar{\Phi}_n)\) which interchanges the components in a particular way from the vector superfield \(\Phi^{(n)} \equiv (\Phi_1, \bar{\Phi}_1; \Phi_2, \bar{\Phi}_2; \ldots; \Phi_n, \bar{\Phi}_n)\). In the conventions of this paper, for any \(2k\)-component vector \(\bar{A} = (a_1, \bar{a}_1; a_2, \bar{a}_2; \ldots; a_k, \bar{a}_k)\), \(\bar{A}^\dagger\) is defined by \(\bar{A}^\dagger = (\bar{a}_1, a_1; \bar{a}_2, a_2; \ldots; \bar{a}_k, a_k)\). Note that
\[
\bar{A} \cdot \bar{B}^\dagger = \bar{A}^\dagger \cdot \bar{B}. \tag{2.19}
\]

The \(\bar{H}_j^{(n)}\) in (2.18) are \(2n\)-component vectors. The Miura transformation (2.18) gives a realisation of super \(W_{n+1}\) provided that they satisfy
\[
\sum_{j=0}^{2n} \frac{1}{(-1)^j \bar{H}_j^{(n)}} = 0.
\tag{2.20}
\]
The Miura transformation (2.1) corresponds to the choice
\[
\begin{align*}
\bar{H}_0^{(n)} &= \left(-1, 0; -1, 0; \ldots; -1, 0\right), \\
\bar{H}_k^{(n)} &= \left(0, 0; \ldots; 0, 0; 0, 1; -1, 0; -1, 0; \ldots; -1, 0\right), \quad 1 \leq k \leq n, \\
\bar{H}_{2k+1}^{(n)} &= \left(0, 0; \ldots, 0, 0; -1, 1; -1, 0; -1, 0; \ldots; -1, 0\right), \quad 0 \leq k \leq n - 1.
\end{align*}
\tag{2.21}
\]
The vectors $\vec{H}_j^{(n)}$ for $0 \leq j \leq (2n-1)$ are the weights of the defining representation of $A(n, n-1)$. The simple roots $\vec{e}_j^{(n)}$ of $A(n, n-1)$ can be given in terms of these weights by

$$\vec{e}_j^{(n)} = (-1)^j (\vec{H}_j^{(n)} - \vec{H}_{j+2}^{(n)}) , \quad 0 \leq j \leq 2n-1 ,$$

(2.22)

where $\vec{H}_{2n+1}^{(n)}$ is defined to be zero. The vectors $\vec{e}_p^{(n)}$, $0 \leq p \leq n-1$, are the simple roots for $A_n$, and $\vec{e}_{2p+1}^{(n)}$, $0 \leq p \leq n-2$, are the simple roots for $A_{n-1}$ in the bosonic $A_n \oplus A_{n-1}$ subalgebra of $A(n, n-1)$. For later purposes we also introduce the Weyl vector $\vec{\rho}^{(n)}$, given by

$$\vec{\rho}^{(n)} = \frac{1}{2} \sum_{j=0}^{2n-1} (-1)^j \left(2n-j\right) \vec{H}_j^{(n)}$$

$$= \frac{1}{2} \sum_{p=0}^{n-1} (p+1)(n-p)\vec{e}_p^{(n)} + \frac{1}{2} \sum_{p=0}^{n-2} (p+1)(n-1-p)\vec{e}_{p+1}^{(n)}$$

(2.23)

$$= \vec{\rho}(A_n) + \vec{\rho}(A_{n-1}) ,$$

where $\vec{\rho}(A_n)$ and $\vec{\rho}(A_{n-1})$ are the Weyl vectors for the $A_n$ and $A_{n-1}$ factors in the $A_n \oplus A_{n-1}$ bosonic subalgebra. In the basis of (2.21), the Weyl vector is

$$\vec{\rho}^{(n)} = \left( \frac{1}{2}, \frac{1}{2}, -1, \frac{1}{2}; \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; \ldots; -\frac{n}{2}, -\frac{n}{2} \right) .$$

(2.24)

In vector notation, one can rewrite the energy-momentum tensor in (2.14) in a nicer form

$$T(z) = \frac{1}{2} \vec{\psi}^{(n)} \cdot \partial \vec{\psi}^{(n)} + \frac{1}{2} \partial \vec{\phi}^{(n)} \cdot \partial \vec{\phi}^{(n)} + \alpha_\theta \vec{\rho}^{(n)} \cdot \partial^2 \vec{\phi}^{(n)} ,$$

(2.25)

where $\vec{\psi}^{(n)} \equiv (\psi_1, \bar{\psi}_1; \psi_2, \bar{\psi}_2; \ldots; \psi_n, \bar{\psi}_n)$ and $\vec{\phi}^{(n)} \equiv (\phi_1, \bar{\phi}_1; \phi_2, \bar{\phi}_2; \ldots; \phi_n, \bar{\phi}_n)$.

Note that the closure of the super-$W_{n+1}$ algebra requires only that the vectors $\vec{H}_j^{(n)}$ in (2.18) satisfy conditions (2.20). There exist many solutions to (2.20). If one only considers the realisation in terms of $n$ pairs of complex superfields, all these solutions are equivalent: an $SO(n, n)$ transformation of $\vec{\Phi}$ will map one solution to another. However, the particular solution given in (2.21) has the nice property that one can express the currents of the super-$W_{n+1}$ algebra in terms of those of super-$W_n$ with an additional pair of complex superfields. This property is essential to construct the string theory since it implies that one eventually obtains a realisation of the super-$W_{n+1}$ algebra in terms of an arbitrary super energy-momentum tensor together with $(n-1)$ additional pairs of complex superfields. We shall see later that the scalar fields in these $(n-1)$ additional pairs of complex superfields do not describe physically-observable coordinates. It is essential that this super energy-momentum tensor can be arbitrary; and therefore, if it contains $D$ pairs of complex superfields $(\Phi_\mu, \bar{\Phi}_\mu)$ one obtains a string theory effectively describing a real $2D$-dimensional target spacetime.

3. The Physical Spectrum of the Super-$W_{n+1}$ String

In this section, we shall construct the complete spectrum of physical states of the super-$W_{n+1}$ string; we shall suppress the superscript index $(n)$ when there is no ambiguity. As we have mentioned in the introduction section, the anomaly-free condition of the string theory requires that the central charge take
its critical value. In our case, that means the central charge given in (2.15) should take the value given in (1.7a). This implies that the background charge parameter $\alpha_0$ takes its critical value $\alpha_0^*$ given by

$$ (\alpha_0^*)^2 = \frac{1}{n + 1}. \quad (3.1) $$

In order to construct a string theory that allows transverse excitations, it is necessary to add extra coordinates as we indicated at the end of section 2. Since these extra coordinates enter only via the effective super energy-momentum tensor whose components are given in (2.16), it is sufficient for the purposes of determining the physical spectrum to work with the basic Miura realisation without additional superfields.

3.1 The physical-state conditions

The physical-state conditions are determined by the requirement that the product of a physical state and the ghost vacuum should be BRST invariant. This leads to the following conditions

$$ (B_k)_m |\text{phys}\rangle = 0, \quad (\bar{B}_{k+1})_m |\text{phys}\rangle = 0, \quad m \geq 1 $$
$$ (F_{k+1/2})_r |\text{phys}\rangle = 0, \quad (\bar{F}_{k+1/2})_r |\text{phys}\rangle = 0, \quad r \geq \frac{1}{2} $$
$$ (B_k)_0 |\text{phys}\rangle = \omega_k |\text{phys}\rangle, \quad (\bar{B}_{k+1})_0 |\text{phys}\rangle = \bar{\omega}_{k+1} |\text{phys}\rangle, \quad (3.2) $$

where $k = 1, 2, 3, \ldots, n$, and the component currents are defined in (2.5). The constants $\omega_k$ and $\bar{\omega}_{k+1}$ are the intercepts for the bosonic currents, which are in principle determined by the BRST invariance condition stated above. Owing to the complexities of the non-linear super-$W_{n+1}$ algebra, it is difficult in general to determine the intercepts by this method. However, since the super-Virasoro algebra forms a linear subalgebra, one can straightforwardly read off from the structure of the ghost vacuum which was described in section 1 that the intercepts for both the spin-1 current and the spin-2 energy-momentum tensor are zero [1]. For the remaining intercepts, we resort to a generalisation of an argument first introduced in [6], and developed in [7]. It was conjectured in [6] that for any $W$ string there is always a particular tachyonic physical state created by the action of the operator $\exp(\lambda \alpha_0^* \vec{\rho} \cdot \vec{\varphi})$ on the vacuum, for some constant $\lambda$, where $\vec{\rho}$ is the Weyl vector of the underlying Lie algebra. The constant $\lambda$ can be determined from the known value of the spin-2 intercept. In [7] it was demonstrated for the $W_3$, $W_4$ and $W_5$ strings that the requirement of unitarity singles out intercept values that correspond to these particular states. It is very plausible that an operator of this form, known as the “cosmological operator,” will always create a physical state, in any $W$-string theory. Assuming this to be the case for the super-$W_{n+1}$ string, i.e. that

$$ e^{\lambda \alpha_0^* \vec{\rho} \cdot \vec{\varphi}} \quad (3.3) $$

satisfies the physical condition (3.2), it follows from (1.7b) that $\lambda$ must satisfy

$$ \lambda (\lambda - 2) = 0. \quad (3.4) $$

Taking the root $\lambda = 0$, it is clear from (3.2) all the intercepts $\omega_k$ and $\bar{\omega}_{k+1}$ are zero. (In fact in any $W$ string theory where there is a physical state of cosmological type, and the $L_0$ intercept is zero, it will always be the case that all other intercepts vanish too.) Note that if the intercepts all vanish in one basis, then they vanish in all bases. In particular, for example, they will vanish in the primary basis. A different method, based on the analysis of null states, was used in [1] to show that all the intercepts in the super-$W_3$ are zero. This provides some evidence in support of the cosmological conjecture.
3.2 Tachyonic Operators and the Weyl Group

Tachyonic, i.e. level-0, states are built from physical operators involving only the bosonic fields complex $\tilde{\phi}$, namely
\[ e^{\tilde{\beta} \tilde{\phi}}. \]

Of the physical-state conditions (3.2), the fermionic ones are therefore empty; the bosonic conditions for positive Laurent modes are automatically satisfied, and the only non-trivial conditions come from the intercept equations. These give a set of polynomial equations for $\tilde{\beta}$. The number of equations, which are non-degenerate, is equal to the number of components of $\tilde{\beta}$, and so the solutions are discrete. The number of such solutions is given by the product of the degrees of the polynomials, i.e. by the product of the spins of the bosonic currents. Thus we have $n!(n+1)!$ discrete solutions for $\tilde{\beta}$. As we shall now show, the polynomial equations are invariant under the action of a discrete symmetry group of dimension $n!(n+1)!$ which acts transitively on the solutions. This is in fact the Weyl group of $A(n, n - 1)$. Thus we can obtain all tachyonic solutions by acting with the Weyl group on any one solution. Since the cosmological solution is assumed to be a physical state, all the other tachyonic physical states can therefore be obtained by acting on it with the Weyl group.

Since the tachyonic physical states do not involve fermions, we may set the fermion fields in the currents to zero when calculating the momentum polynomials coming from the intercept conditions in (3.2). This implies we can just keep $\theta \partial \tilde{\phi}$ in $D \tilde{\Phi}$. The eigenvalues of the zero modes of the bosonic currents acting on the tachyonic state can be read off from the highest order pole of the OPE between the bosonic currents and the tachyonic physical operator. Since
\[ \partial \tilde{\phi}(z) \ e^{\tilde{\beta} \tilde{\phi}}(w) \sim -\frac{\tilde{\beta}}{z - w} e^{\tilde{\beta} \tilde{\phi}}(w), \]
it follows that one can obtain the eigenvalues of the zero modes of the bosonic currents directly from Miura transformation (2.1) and (2.18) by the replacing $D \tilde{\Phi} \rightarrow \frac{-\theta \partial \tilde{\phi}}{z}$, i.e.
\[ \prod_{j=0}^{2n} \left( \alpha_0 D - \theta \tilde{H}_j \cdot \tilde{\beta} \right) = (\alpha_0 D)^{2n+1} + \sum_{p=1}^{n} \alpha_0^p b_p(\tilde{\beta}) (\alpha_0 D)^{2n-2p+1} + \theta \sum_{q=1}^{n} \alpha_0^q \tilde{b}_q(\tilde{\beta}) (\alpha_0 D)^{2n-2q}, \]
where $b_p(\tilde{\beta})$ and $\tilde{b}_q(\tilde{\beta})$, which are polynomials in $\tilde{\beta}$, are the eigenvalues of the zero modes of the bosonic currents $B_p$ and $\tilde{B}_q$. By acting with (3.1) on $z^j$ and $\theta z^j$, for $1 \leq j \leq n$, we have
\[ \alpha_0^{2n+1} \prod_{p=0}^{n} (j - \frac{1}{2}n - \frac{1}{2} \tilde{H}_{2p} \cdot \tilde{\gamma}) = \sum_{s=0}^{j} \frac{j!}{(j-s)!} (b_{n+1-s}(\tilde{\beta}) + \tilde{b}_{n+1-s}(\tilde{\beta})) \alpha_0^{n+s} \]
\[ \alpha_0^{2n+1} \prod_{p=1}^{n} (j - \frac{1}{2}(n+1) - \frac{1}{2} \tilde{H}_{2p-1} \cdot \tilde{\gamma}) = \sum_{s=0}^{j} \frac{j!}{(j-s)!} b_{n-s}(\tilde{\beta}) \alpha_0^{n+s+1}, \]
where $b_{n+1}(\tilde{\beta})$ and $\tilde{b}_1(\tilde{\beta})$ are both equal to zero. The vector $\tilde{\gamma}$ in (3.8a) and (3.8b) is the shifted momentum defined by
\[ \tilde{\beta} = \alpha_0 (\frac{1}{2} \tilde{\gamma} + \tilde{\rho}). \]

It is clear that the equations (3.8a) and (3.8b) are invariant under independent permutations of the $\tilde{H}_{2p}$’s and of the $\tilde{H}_{2p-1}$’s. Thus the eigenvalues $b_p$ and $\tilde{b}_q$ are invariant under a discrete symmetry of order
n!(n+1)!. In fact, this symmetry group is just the Weyl group of \( A_n \oplus A_{n-1} \). To see this, we note that under a Weyl reflection \( S_j \) corresponding to the simple root \( \vec{e}_j \), the simple root \( \vec{e}_i \) transforms as

\[
S_j(\vec{e}_i) = \vec{e}_i - (-1)^j (\vec{e}_i \cdot \vec{e}_j^\dagger) \vec{e}_j = \vec{e}_i - (2\delta_{ij} - \delta_{i,j+2} - \delta_{i+2,j}) \vec{e}_j .
\]  

(3.10)

From (2.22) it follows that \( S_j \) acting on the \( \vec{H}_p \)'s only interchanges \( \vec{H}_j \) with \( \vec{H}_{j+2} \), leaving all others unchanged. Since the scalar product is invariant, a Weyl reflection of \( \vec{\gamma} \)

\[
\vec{\gamma} \rightarrow S_j(\vec{\gamma}) = \vec{\gamma} - (-1)^j (\vec{\gamma} \cdot \vec{e}_j^\dagger) \vec{e}_j
\]

(3.11)

therefore leaves the left-hand sides of (3.8a) and (3.8b) invariant. The dimension of the Weyl group of \( A_n \oplus A_{n-1} \) is \( n!(n+1)! \). All the elements can be generated from the elements \( S_j \) corresponding to the simple roots. For example, they can be generated by choosing one entry from each column of the following

\[
\begin{pmatrix}
1 \\
S_1 \\
S_3 \otimes S_1 \\
S_3 S_1
\end{pmatrix}
\otimes
\begin{pmatrix}
1 \\
S_3 \\
S_5 \\
S_5 S_3
\end{pmatrix}
\otimes \cdots
\begin{pmatrix}
1 \\
S_{2n-3} \\
S_{2n-3} S_{2n-5} \\
S_{2n-3} S_{2n-5} S_{2n-1}
\end{pmatrix}
\]

\[
\otimes \begin{pmatrix}
1 \\
S_0 \\
S_2 \otimes S_0 \\
S_2 S_0
\end{pmatrix}
\otimes \begin{pmatrix}
1 \\
S_4 \\
S_4 S_2 \\
S_4 S_2 S_0
\end{pmatrix}
\otimes \cdots
\begin{pmatrix}
1 \\
S_{2n-2} \\
S_{2n-2} S_{2n-4} \\
S_{2n-2} S_{2n-4} S_0
\end{pmatrix}
\]

(3.12)

Thus the polynomials \( b_p \) and \( \bar{b}_q \) are invariant under the action of the entire Weyl group on the shifted momentum \( \vec{\gamma} \).

The upshot of the above discussion is that the Weyl group maps physical states into physical states. By supposition, the cosmological operator (3.3) with \( \lambda = 0 \) gives a physical state. From (3.9), it follows that the shifted momentum for the cosmological solution is

\[
\vec{\gamma}^{\text{cosmo}} = -2\vec{p}.
\]

(3.13)

The Weyl group acts without fixed points on the Weyl vector \( \vec{p} \), and so we generate all the \( n!(n+1)! \) distinct tachyonic physical states.

3.3 The Spectrum of Physical states

We have seen in section 3.2 that in the Miura realisation for super-\( W_{n+1} \) in terms of \( n \) pairs of complex superfields, all the momentum components of tachyonic states are fixed to a set of discrete values by the physical-state conditions. Clearly the scalars conjugate to these momentum components are not at this stage physically-observable coordinates. When we add additional superfields to the effective super energy-momentum tensor, the momentum components \( \beta_2, \beta_3, \ldots, \beta_n, \bar{\beta}_2, \bar{\beta}_3, \ldots, \bar{\beta}_n \) remain frozen to the same values, whilst \( \beta_1 \) and \( \bar{\beta}_1 \), together with momenta conjugate to the extra coordinates, can now take continuous values subject to constraints imposed by the effective intercepts for \( L_0^{\text{eff}} \) and \( J_0^{\text{eff}} \).
We consider the $T_{\text{eff}}(z, \theta, \bar{\theta})$ with $D$ pairs of complex fields $(\Phi_\mu, \bar{\Phi}_\mu)$, $\mu = 0, 1, \ldots, (D-1)$. In component form, the currents of the effective super energy-momentum tensor given in (2.16) are modified to become

\[
J_{\text{eff}}(z) = -\psi_\mu \bar{\psi}^\mu + Q_\mu \partial \phi^\mu - Q_\mu \partial \bar{\phi}^\mu, \\
T_{\text{eff}}(z) = \frac{i}{2} \psi_\mu \partial \bar{\psi}^\mu - \frac{i}{2} \partial \psi_\mu \bar{\psi}^\mu - \partial \phi_\mu \partial \bar{\phi}^\mu - \frac{i}{2} Q_\mu \partial^2 \phi^\mu - \frac{1}{2} Q_\mu \partial^2 \bar{\phi}^\mu, \\
G_{\text{eff}} = \sqrt{2}(\partial \bar{\psi}_\mu \psi^\mu + Q_\mu \partial \psi^\mu), \\
\bar{G}_{\text{eff}} = \sqrt{2}(\partial \bar{\phi}_\mu \bar{\psi}^\mu + Q_\mu \partial \bar{\psi}^\mu).
\]

(3.14)

The central charge for $T_{\text{eff}}$ must be given by

\[
C_{\text{eff}} = 3 + 6(\alpha_0^*)^2 = 3 + \frac{6}{n+1}.
\]

(3.15)

This implies that the background charge vector $Q_\mu$ must satisfy

\[
Q_\mu Q^\mu = \alpha_0^* \alpha_0 + \frac{1}{2}(1 - D) = \frac{1}{n+1} + \frac{1}{2}(1 - D).
\]

(3.16)

Since the values of the intercepts $\omega_{\text{eff}}$ and $\bar{\omega}_{\text{eff}}$ for $J_{\text{eff}}$ and $L_{\text{eff}}$ are independent of whether or not additional superfields are included in the effective super energy-momentum tensor, it follows from (2.16) that they are given by substituting the discrete solutions for $\beta_1$ and $\bar{\beta}_1$ found in section 3.2 into

\[
\omega_{\text{eff}} = -\alpha_0^* \beta_1 + \alpha_0 \bar{\beta}_1 = \frac{1}{2(n+1)}(\bar{\gamma}_1 - \gamma_1)
\]

(3.17a)

\[
\bar{\omega}_{\text{eff}} = -\beta_1 \bar{\beta}_1 - \frac{1}{2} \alpha_0^* \beta_1 - \frac{1}{2} \alpha_0 \bar{\beta}_1 = -\frac{1}{4(n+1)}(\gamma_1 \bar{\gamma}_1 - 1)
\]

(3.17b)

From the discussion in section 3.2, we can find all the values of $\gamma_1$ and $\bar{\gamma}_1$ by acting on $\Phi^{\text{cosmo}}$ with the Weyl group. We shall discuss these values below.

Turning now to the higher level states, we have to divide them into two categories. The first consists of states where the excitations lie exclusively in the unfrozen directions $(\Phi_\mu, \bar{\Phi}_\mu)$. The second category consists of states that include excitations in the frozen directions. For reasons that we shall discuss later, all the physical states in the second category seem to have zero norm and hence do not appear in the physical spectrum. We shall therefore concentrate for now on physical states in the first category. These can be written as

\[
|\text{phys}\rangle = e^{\beta_2 \Phi_2 + \cdots + \beta_n \Phi_n + \bar{\beta}_2 \bar{\Phi}_2 + \cdots + \bar{\beta}_n \bar{\Phi}_n}|\text{phys}\rangle_{\text{eff}}.
\]

(3.18)

It is clear that (3.17) will satisfy the physical-state conditions (3.2) provided that $\beta_2, \beta_3, \ldots, \beta_n, \bar{\beta}_2, \bar{\beta}_3, \ldots, \bar{\beta}_n$ take their frozen values found in section 3.2, and $|\text{phys}\rangle_{\text{eff}}$ satisfies the physical-state conditions for the effective super energy-momentum tensor

\[
J_{\text{eff}}|\text{phys}\rangle_{\text{eff}} = 0, \quad L_{\text{eff}}|\text{phys}\rangle_{\text{eff}} = 0, \quad m \geq 1 \\
G_{\text{eff}}|\text{phys}\rangle_{\text{eff}} = 0, \quad \bar{G}_{\text{eff}}|\text{phys}\rangle_{\text{eff}} = 0, \quad r \geq \frac{1}{2} \\
J_{\text{eff}}^\ast|\text{phys}\rangle_{\text{eff}} = \omega_{\text{eff}}|\text{phys}\rangle_{\text{eff}}, \quad L_{\text{eff}}^\ast|\text{phys}\rangle_{\text{eff}} = \omega_{\text{eff}}^\ast|\text{phys}\rangle_{\text{eff}}.
\]

(3.19)

Thus the physical spectrum for the super-$W_{n+1}$ string is given by the physical spectra for a set of effective super-Virasoro strings with a non-standard central charge (3.15) and a set of non-standard intercepts given by (3.17a, b).
The complete physical spectrum is now in principle determined, since $\omega_1^{\text{eff}}$ and $\omega_2^{\text{eff}}$ are given by (3.17a, b), and $\gamma_1$ and $\bar{\gamma}_1$ are determined by acting with the Weyl group on $\vec{\gamma}_{\text{cosmo}}$ given by (3.13). We have examined the examples of super-$\text{WA}_n$ for $n = 2, 3, 4, 5, 6$ explicitly, and found that in all these cases the results for the allowed values of $(\gamma_1, \bar{\gamma}_1)$ are as follows: Each of $\gamma_1$ and $\bar{\gamma}_1$ can take its values from the set of integers $\pm 1, \pm 3, \pm 5, \ldots, \pm (2n - 1)$. All possible combinations from these values can occur provided that the following two conditions are satisfied:

$$\begin{align*}
\gamma_1 &\neq -\bar{\gamma}_1, \\
\gamma_1 \bar{\gamma}_1 &\leq (n^2 - 1). 
\end{align*}$$

We expect that these results will apply also for the general case. As we shall now show, equations (3.20a, b) give $n(3n - 1)$ solutions for $(\gamma_1, \bar{\gamma}_1)$. For the case that $\gamma_1 \bar{\gamma}_1 < 0$, only (3.20a) imposes a non-trivial restriction, so we have $2n(n - 1)$ such solutions for $(\gamma_1, \bar{\gamma}_1)$. They may be written as

$$\begin{align*}
\gamma_1 &= \pm (2r - 1), \\
\bar{\gamma}_1 &= \mp (2s - 1),
\end{align*}$$

with $1 \leq r \neq s \leq n$. These give $n(n - 1)$ positive values for the effective spin-2 intercept:

$$\omega_2^{\text{eff}} = \frac{1}{2(n + 1)}(2rs - r - s + 1).$$

Correspondingly, the effective spin-1 intercepts are

$$\omega_1^{\text{eff}} = \pm \frac{r + s - 1}{n + 1}.$$  

When $\gamma_1 \bar{\gamma}_1 > 0$, only (3.20b) imposes a non-trivial restriction, and we have $n(n + 1)$ such solutions for $(\gamma_1, \bar{\gamma}_1)$. They can be expressed as

$$\begin{align*}
\gamma_1 &= m \pm (\ell + 1), \\
\bar{\gamma}_1 &= -m \pm (\ell + 1),
\end{align*}$$

where $\ell = 0, 1, \ldots, (n - 1)$, and $m = -\ell, -\ell + 2, \ldots, \ell$. These give $\frac{1}{2}n(n + 1)$ negative values for the effective spin-2 intercept:

$$\omega_2^{\text{eff}} = -\frac{\ell(\ell + 2) - m^2}{4(n + 1)}.$$  

the corresponding effective spin-1 intercept values are

$$\omega_1^{\text{eff}} = -\frac{m}{n + 1}.$$  

In the case of bosonic $W$ strings, it was found that there is a relation with minimal models [6,7,8,9]. As we shall now show, a similar relation emerges for the case of super $W$ strings. First of all, we note from (3.15) that the effective central charge for $T^{\text{eff}}$ can be written as

$$c^{\text{eff}} = 6 - \frac{3}{n + 1}.$$  

Here, 6 is the critical central charge for the usual $N = 2$ super-Virasoro string, and the term in parentheses is the central charge of the $(n - 1)$th $N = 2$ superconformal minimal model [12]. When the effective spin-2 intercept $\omega_2^{\text{eff}}$ is negative, as given by (3.25), we observe that $-\omega_2^{\text{eff}}$ and $-\omega_1^{\text{eff}}$ take values equal to the
dimensions and $U(1)$ charges of the above superconformal minimal model. As for the bosonic $W$ strings, the significance of these observations is not fully understood.

4. The Issue of Unitarity

Having obtained the spectrum of physical states for the super-$WA_n$ string, we now address the question of whether these states are unitary. As discussed in section 3.3, physical states involve excitations only in the unfrozen directions, and thus take the form (3.18). The effective physical states $|\text{phys}\rangle_{\text{eff}}$ satisfy super-Virasoro-like physical-state conditions (3.19), with a non-standard central charge (3.15), and non-standard bosonic intercepts $\omega_1^{\text{eff}}$ and $\omega_2^{\text{eff}}$ given by (3.23), (3.26) and (3.22), (3.25). Note that for each value of $\omega_2^{\text{eff}}$, $\omega_1^{\text{eff}}$ can take two values, with equal magnitude but opposite sign. In order to investigate the unitarity of the physical states of the super-$W_n$ string, it suffices to investigate the unitarity of the corresponding effective superstring theories.

We shall take the effective super-Virasoro algebra to be realised by the currents (3.14). The background-charge vector $Q_\mu$ will be taken to lie along the $\mu = 0$ direction. Thus (3.14) becomes

$$J_{\text{eff}}^{\ell}(z) = -\psi_\mu \bar{\psi}^\mu + Q \partial \phi_0 + Q \partial \bar{\phi}_0,$$
$$T_{\text{eff}}^{\ell}(z) = \frac{1}{2} \psi_\mu \partial \bar{\psi}^\mu - \frac{1}{2} \partial \psi_\mu \bar{\psi}^\mu - \partial \phi_0 \partial \bar{\phi}_0 + \frac{1}{4} Q \partial^2 \phi_0 + \frac{1}{4} Q \partial^2 \bar{\phi}_0,$$
$$G_{\text{eff}} = \sqrt{2} (\partial \phi_0 \psi_0 - Q \partial \psi_0),$$
$$\bar{G}_{\text{eff}} = \sqrt{2} (\partial \bar{\phi}_0 \bar{\psi}_0 - Q \partial \bar{\psi}_0),$$

with $Q$ given by

$$Q^2 = \frac{1}{2} (D - 1) - \frac{1}{n + 1}. \quad \text{(4.2)}$$

Note that for $n \geq 2$, $Q$ cannot be zero for any choice of (integer) $D$, and so the necessity for background charges cannot be avoided.

Effective physical states can be constructed by acting on an $SL(2, C)$-invariant vacuum $|0\rangle$ with ground-state operators $P(z)$, i.e. $|\text{phys}\rangle_{\text{eff}} = P(0)|0\rangle$. The ground-state operators take the form

$$P(z) = R(z)e^{\beta \phi + \bar{\beta} \bar{\phi}}. \quad \text{(4.3)}$$

The operators $R(z)$ can be classified by their eigenvalues $q$ and $\ell$ under $J_0^{\text{eff}}$ and $L_0^{\text{eff}}$ respectively. The eigenvalue $q$ measures the fermion charge of the operator $R(z)$; each $\psi_\mu$ in a monomial in $R(z)$ contributes +1, each $\bar{\psi}_\mu$ contributes −1, and $\partial \phi_0$ and $\partial \bar{\phi}_0$ contribute 0. The eigenvalue $\ell$ measures the conformal dimension of the operator $R(z)$, i.e. the level number.

At level $\ell = 0$, $R$ is just the identity operator, with $q = 0$, and $P(z)$ is the “tachyon” ground-state operator. At level $\ell = \frac{1}{2}$, $R$ can be $\xi_\mu \psi^\mu$, with $q = +1$; or $\xi_\mu \bar{\psi}^\mu$, with $q = -1$. At level $\ell = 1$, $q$ can be −2, 0, +2. In general, at level $\ell$, $q$ can take the values

$$q = -2\ell, -2\ell + 2, \ldots, 2\ell - 2, 2\ell. \quad \text{(4.4)}$$

For effective physical states with level number $\ell$ and fermion charge $q$, the $J_0^{\text{eff}}$ and $L_0^{\text{eff}}$ constraints in (3.19) give

$$J_0 : \quad \omega_1^{\text{eff}} = q + Q(\beta_0 - \bar{\beta}_0), \quad \text{(4.5a)}$$
$$L_0 : \quad \omega_2^{\text{eff}} = \ell - \beta^\mu \bar{\beta}_\mu + \frac{1}{2} Q(\beta_0 + \bar{\beta}_0). \quad \text{(4.5b)}$$
The hermiticity conditions for $L_0^{\text{eff}}$ and $J_0^{\text{eff}}$ imply that [13]

\begin{align}
\beta_i^* &= -\bar{\beta}_i ; \quad \bar{\beta}_i^* = -\beta_i , \\
\beta_0^* &= -\bar{\beta}_0 - Q ; \quad \bar{\beta}_0^* = -\beta_0 - Q ,
\end{align}

(4.6a)

(4.6b)

where * denotes complex conjugation, and $i = 1, 2, \ldots, D - 1$.

The norm of an effective physical state of the form $|\text{phys}\rangle_{\text{eff}} = P(0)|0\rangle$, with $P(0)$ given by (4.3), is of the form $\mathcal{N}(R) \langle p|p \rangle$, where $\mathcal{N}(R)$ is a function of the scalar products of the polarisation tensors. The term $\langle p|p \rangle$ gives the usual momentum-conservation delta function, and so the requirement of non-negativity of the norm amounts to the requirement that $\mathcal{N}(R)$ be non-negative whenever $\langle p|p \rangle \neq 0$. Actually, the norms of the physical states of the super-$W_n$ string will also involve momentum-conservation delta functions in the frozen directions as well. These will have the form [7,14]

\[
\prod_{i=2}^{n} \delta(\gamma_i + \bar{\gamma}_i + \gamma_i' + \bar{\gamma}_i')(\gamma_i - \bar{\gamma}_i + \gamma_i' - \bar{\gamma}_i') .
\]

(4.7)

We know from the discussion of the Weyl group in section 3 that if $\mathcal{G}$ satisfies the physical-state conditions then so does $-\mathcal{G}$, so there always exist pairs of states for which (4.7) is non-zero. This pair of states have equal $\omega_2^{\text{eff}}$ values and opposite $\omega_1^{\text{eff}}$ values.

The momentum conservation for the effective superstring states can never be satisfied when the fermion number is non-zero. To see this, we observe from (4.5a) that

\[
\beta_0 - \bar{\beta}_0 = Q^{-1}(\omega_1^{\text{eff}} - q).
\]

(4.8)

Thus the momentum-conservation delta function for the real direction $\phi_0 - \bar{\phi}_0$, i.e. $\delta(\beta_0 - \bar{\beta}_0 + \beta_0' - \bar{\beta}_0')$, can be satisfied only if the right-hand side of (4.8) for the unprimed state cancels against the contribution for the primed state. If $q = 0$, this is easily achieved, since the $\omega_1^{\text{eff}}$ values (3.23) or (3.26) always come in equal and opposite pairs, so the frozen $\beta_0 - \bar{\beta}_0$ value for an unprimed state with given $\omega_1^{\text{eff}}$ can be cancelled by the frozen value for a primed state with the opposite sign for $\omega_1^{\text{eff}}$. If $q \neq 0$, the only possibility would be if the "displaced" intercept $\omega_1^{\text{eff}} - q$ in (4.8) for the unprimed state happened to equal the negative of another displaced value for the primed state. For the case (3.26) this cannot happen, since there $|\omega_1^{\text{eff}}| < 1$, whilst the fermion charge must take integer values. For the case (3.23) it could occur. However, the corresponding $\omega_2^{\text{eff}}$ values for the two states would then be unequal, which would mean, in the light of the discussion above, that the frozen-momentum conservation delta functions (4.7) would be zero. Thus the effective physical states with fermion charge $q \neq 0$ have zero norm, and we need not consider them further.

The first non-trivial physical states that we need consider are thus at level $\ell = 1$, with fermion charge $q = 0$. They have the form (4.3) with

\[
R(z) = \varepsilon_{\mu\nu}\psi^\mu\bar{\psi}^\nu + \xi_\mu\partial\phi^\mu + \xi_\mu\bar{\partial}\bar{\phi}^\mu .
\]

(4.9)

In addition to the $J_0^{\text{eff}}$ and $L_0^{\text{eff}}$ constraints, we have three other independent nontrivial constraints, coming from $J_1^{\text{eff}}$, $C_{1/2}^{\text{eff}}$ and $G_{1/2}^{\text{eff}}$. They give, respectively,

\begin{align}
\varepsilon^\mu_{\mu} &= Q(\xi_0 - \bar{\xi}_0) , \quad (4.10a) \\
\xi_\mu &= \beta^\mu_{\nu} \varepsilon^\nu , \quad (4.10b) \\
\bar{\xi}_\mu &= -\bar{\beta}^\mu_{\nu} \varepsilon^\nu . \quad (4.10c)
\end{align}
The norm $N$ for these states is given by

$$N = \varepsilon_{\mu \nu} e^{\mu \nu} + \xi_{\mu} \xi_{\nu} + \xi_{\mu} \xi_{\nu}.$$  \hspace{1cm} (4.11)

From (4.10b) and (4.10c) we may eliminate $\tilde{\xi}^\mu$ and $\xi^\mu$ in (4.11), and express the norm purely in terms of $\varepsilon_{\mu \nu}$, subject to the constraint implied by (4.10a).

By analogy with the bosonic string, one might expect that the subset of states of the form (4.9) that are most likely to have negative norm are those with $\varepsilon_{\mu \nu}$ given by

$$\varepsilon_{\mu \nu} = \lambda \eta_{\mu \nu} + \kappa \beta_{\mu} \beta_{\nu}.$$  \hspace{1cm} (4.12)

After some algebra, we find that (4.11) can then be written as

$$N = |\lambda|^2 \left(2\omega_2^{\text{eff}} (c^{\text{eff}} - 3) - c^{\text{eff}} + 3(\omega_1^{\text{eff}})^2 - 6 \right) \times \left(3(1 - 2\omega_2^{\text{eff}})((Q^2|\beta_0 + \bar{\beta}_0|^2 - 4Q^2 + 4Q^2 \omega_2^{\text{eff}} + (\omega_1^{\text{eff}})^2) + (4(\omega_2^{\text{eff}} - 1)^2 - (\omega_1^{\text{eff}})^2)(c^{\text{eff}} + 6Q^2 - 3) \right).$$  \hspace{1cm} (4.13)

The first factor in parentheses is non-negative for all our effective intercept values. From the $L_0^{\text{eff}}$ intercept condition (4.5b), we have

$$Q^2|\beta_0 + \bar{\beta}_0|^2 - 4Q^2 + 4Q^2 \omega_2^{\text{eff}} + (\omega_1^{\text{eff}})^2 \geq 0.$$  \hspace{1cm} (4.14)

Thus we find from (4.13) that if $\omega_2^{\text{eff}} \leq \frac{1}{2}$, then

$$N \geq |\lambda|^2 \left(2\omega_2^{\text{eff}} (c^{\text{eff}} - 3) - c^{\text{eff}} + 3(\omega_1^{\text{eff}})^2 - 6 \right) \left((4(\omega_2^{\text{eff}} - 1)^2 - (\omega_1^{\text{eff}})^2)(c^{\text{eff}} + 6Q^2 - 3) \right).$$  \hspace{1cm} (4.15)

It is easy to see that when $\omega_2^{\text{eff}}$ is negative, given by (3.25), then $N \geq 0$. If $\ell$ in (3.25) takes less than its maximum value, i.e. $\ell \leq n - 2$, then $N$ is strictly positive. One can also check that for $0 < \omega_2^{\text{eff}} \leq \frac{1}{2}$, corresponding to a subset of the intercept values given by (3.22), $N$ is again strictly positive. However for $\omega_2^{\text{eff}} > \frac{1}{2}$, i.e. the remainder of the intercept values given by (3.22), $N$ can be made arbitrarily negative by taking $|\beta_0 + \bar{\beta}_0|$ to be large. Thus there are negative-norm states.

We have seen above that the complete physical spectrum of the super-$W_{n+1}$ string includes states with negative norm. Interestingly enough, it is amongst the level-1 states with $\omega_2^{\text{eff}} > 0$ that the non-unitarity occurs, whilst all the level-1 states with $\omega_2^{\text{eff}} < 0$ are unitary. In order to obtain a unitary theory, it is necessary to truncate out the negative-norm states in a consistent manner. Since, as may be seen from (3.22) and (3.23), the effective spin-1 intercept $\omega_1^{\text{eff}}$ is non-zero whenever $\omega_2^{\text{eff}}$ is positive, it follows that a possible choice for the truncation would be to discard the physical states associated with all non-zero values of $\omega_1^{\text{eff}}$. Thus we are left with a set of effective $N = 2$ superstring theories with spin-2 intercepts given by

$$\omega_2^{\text{eff}} = -\frac{\ell(\ell + 2)}{4(n + 1)},$$  \hspace{1cm} (4.16)

with $\ell = 0, 1, \ldots, (n - 1)$.

Since we do not yet have a description of interactions for the physical states of the super-$W_{n+1}$ string, it is not possible to prove explicitly that this truncation is consistent. However, since the effective spin-1 intercept $\omega_1^{\text{eff}}$ is a $U(1)$ charge, it seems reasonable to expect, no matter what the detailed form of the interactions may be, that a truncation to the states with zero charge should be consistent. From (4.5a), we see that setting $\omega_1^{\text{eff}} = 0$ implies that the real component of momentum $(\beta_0 - \bar{\beta}_0)$ is constrained to be
zero. Since the background charge in the corresponding real direction is zero, conservation is automatically satisfied for this component of momentum. The fact that momentum in this direction vanishes for all states means that the corresponding coordinate is frozen, and hence physically unobservable. Thus we have the added bonus that although the original theory involved two real time coordinates, one of them is frozen by the physical-state conditions. The occurrence of this phenomenon for \( N = 2 \) superstrings was noted in [13].

5. Realisations of \( N = 2 \) super-\( W_{n+1} \) Algebra

In section 2, we have given the Miura transformation based on the \( A(n, n-1) \) super algebra, from which one can derive a realisation for \( N = 2 \) super-\( W_{n+1} \) in terms of \( n \) pairs of complex superfields. The closure of the algebra requires that the vectors \( \tilde{H}_j^{(n)} \) in the differential operator (2.18) satisfy conditions (2.20). The Miura transformation given in (2.1) corresponds to a particular choice of the vectors \( \tilde{H}_j^{(n)} \) given in (2.21). This choice of \( \tilde{H}_j^{(n)} \) has the nice property that one can express the \( \tilde{H}_j^{(n)} \) in terms of \( \tilde{H}_j^{(n-1)} \), viz.

\[
\tilde{H}_i^{(n)} = \left( \tilde{H}_i^{(n-1)}; -1, 0 \right), \quad 0 \leq i \leq 2(n-1),
\]

\[
\tilde{H}_{2n-1}^{(n)} = \left( 0, 0; \ldots; 0, 0; -1, 1 \right),
\]

\[
\tilde{H}_{2n}^{(n)} = \left( 0, 0; \ldots; 0, 0; 0, 1 \right),
\]

where the \( \tilde{H}_j^{(n-1)} \) are \( 2(n-1) \)-component vectors also satisfying (2.20). It was this property that enabled us to express the currents of the super-\( W_{n+1} \) algebra in terms of those of an arbitrary super-\( W_n \) together with a pair of complex superfields, with explicit formulae given in (2.9). Applying this reduction recursively leads to a realisation of super-\( W_{n+1} \) in terms of an arbitrary \( N = 2 \) energy-momentum tensor together with \( (n-1) \) pairs of complex superfields.

The above reduction can be easily generalised; one can express the currents of super-\( W_{n+1} \) in terms of those of commuting super-\( W_{n-k+1} \) and super-\( W_k \) algebras together with an additional pair of complex superfields. To see this, we first express the vectors \( \tilde{H}_j^{(n)} \) in terms of \( \tilde{H}_j^{(n-k)} \) and \( \tilde{H}_j^{(k-1)} \), viz.

\[
\tilde{H}_i^{(n)} = \left( \tilde{H}_i^{(n-k)}; 0, 0; \ldots; 0, 0; -1, 0 \right), \quad 0 \leq i \leq 2(n-k),
\]

\[
\tilde{H}_{2(n-k)+1}^{(n)} = \left( 0, 0; \ldots; 0, 0; 0, 0; \ldots; 0, 0; -1, 1 \right),
\]

\[
\tilde{H}_j^{(n)} = \left( 0, 0; \ldots; 0, 0; \tilde{H}_j^{(k-1)}; -(k-1); 0, 1 \right), \quad 2(n-k+1) \leq j \leq 2n,
\]

where \( \tilde{H}_j^{(n-k)} \) and \( \tilde{H}_j^{(k-1)} \) are the corresponding vectors for super-\( W_{n-k+1} \) and super-\( W_k \) respectively, both satisfying conditions (2.20). Substituting (5.2) into the differential operator given in (2.18), we have

\[
\mathcal{M}_n = \prod_{j=2(n-k+1)}^{2n} \left( \alpha_0 D + \tilde{H}_j^{(k-1)} \cdot D \tilde{\Phi}^i + D\tilde{\Phi}_n \right) \left( \alpha_0 D + D\tilde{\Phi}_n - D\tilde{\Phi}_n \right)
\]

\[
\times \prod_{j=0}^{2(n-k)} \left( \alpha_0 D + \tilde{H}_j^{(n-k)} \cdot D \tilde{\Phi}^i - D\tilde{\Phi}_n \right),
\]

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where we have grouped the $n$ pairs of superfields $\vec{\Phi}^{(n)}$ into three parts: $\vec{\Phi} \equiv (\Phi_1, \bar{\Phi}_1; \Phi_2, \bar{\Phi}_2; \ldots; \Phi_{n-k}, \bar{\Phi}_{n-k})$, $\vec{\Phi} \equiv (\Phi_{n-k+1}, \bar{\Phi}_{n-k+1}; \Phi_{n-k+2}, \bar{\Phi}_{n-k+2}; \ldots; \Phi_{n-1}, \bar{\Phi}_{n-1})$ and an additional pair of superfields $(\Phi_n, \bar{\Phi}_n)$. It is straightforward to write $M_n$ in terms of $M_{n-k}$ and $M_{k-1}$ as follows

$$M_n = e^{-\Phi_n/\alpha_0}M_{k-1}e^{\Phi_n/\alpha_0}(\alpha_0 D + D\Phi_n - D\bar{\Phi}_n)e^{\bar{\Phi}_n/\alpha_0}M_{n-k}e^{-\bar{\Phi}_n/\alpha_0}.$$  \hspace{1cm} (5.4)

Since $\vec{\Phi}$, $\vec{\Phi}$ and $(\Phi_n, \bar{\Phi}_n)$ commute with each other, we then have a realisation of super-$W_{n+1}$ in terms of the currents of arbitrary super-$W_{n-k}$ and super-$W_{k-1}$ algebras and an extra pair of complex superfields. Similar reductions was also observed in [15] for $WA_n$, $WD_n$ and $WB_n$ algebras. Applying this reduction recursively, we eventually arrive at a realisation of the super-$W_{n+1}$ algebra in terms of $p$ commuting $N = 2$ super energy-momentum tensors and $(n - p)$ pairs of additional complex superfields, with $0 \leq p \leq \lfloor \frac{n+1}{2} \rfloor$. These super energy-momentum tensors could be arbitrary except that they must all have the same central charge. This follows directly from the fact that the vectors $\vec{H}_j^{(1)}$ for super-$W_2 \equiv$ super-Virasoro ($N = 2$) are uniquely determined by the conditions (2.20) and they give the same Weyl vector $\vec{\rho} = (-\frac{1}{2}, -\frac{1}{2})$. Hence all these super energy-momentum tensors must have the same central charge $c_{\text{eff}}$ given by

$$c_{\text{eff}} = 3 + 6\alpha_0^2.$$  \hspace{1cm} (5.5)

In this paper we have studied in detail the case of $p = 1$. We construct the corresponding super-$W_{n+1}$ strings and obtain the complete physical spectrum. Investigating the norm of low-lying physical states, we find that although some physical states are non-unitary, it seems that one can truncate the physical spectrum into a unitary subsector corresponding to vanishing $U(1)$ charges. It will also be interesting to look at the more general cases, namely super-$W_{n+1}$ strings based on the realisations of $p$ commuting $N = 2$ super energy-momentum tensors and $(n - p)$ pairs of complex superfields.
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