Asymptotic Padé Approximants and the SQCD $\beta$-function

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We present a prediction for the four loop $\beta$-function for SQCD based on the method of Asymptotic Padé Approximants.

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Recently a four-loop calculation of the gauge $\beta$-function ($\beta_g$) in an $N = 1$ supersymmetric gauge theory was presented\[1\]. The computation was performed within the usual supersymmetric dimensional regularisation scheme (DRED), which is standard for perturbative calculations in supersymmetry and which consequently would be the most convenient for any future phenomenological applications. The result was derived somewhat indirectly, by starting from a partial calculation for the abelian case. The coupling constant redefinition relating the abelian result to the corresponding exact NSVZ result \[2\] could then be constructed, and this redefinition was then extended to the non-abelian case by imposing the vanishing of the $\beta$-function beyond one loop in the case of $N = 2$ supersymmetry. However, one possible redefinition which was present in general, but vanished for the abelian case, could not be determined, and consequently the four-loop result $\beta_3$ contained one undetermined constant. Our purpose here is to predict the value of this constant by using improved Padé approximation techniques which have recently been developed and applied to the case of standard QCD\[3\]. In that case a prediction for $\beta_3$ was found to be in good agreement with a subsequent analytic calculation\[4\], especially when the contributions involving quartic Casimir group theory structures (which arise for the first time at four loops) are omitted in making the comparison. In fact these structures simply do not arise in the supersymmetric case which concerns us here. This observation, allied with the fact that we have more information about $\beta_3$ than was available to Ellis et al in the QCD case, makes us optimistic regarding the accuracy of our prediction.

The method is as follows: For a perturbative series $P(x) = \sum_{n=0}^{\infty} S_n x^n$, the Padé approximant $P_{[N/M]}(x)$ is a ratio of polynomials $A_N(x)$ and $B_M(x)$, of degree $N$ and $M$ respectively, chosen so that

$$P_{[N/M]} = \frac{A_N(x)}{B_M(x)} = P(x) + O(x^{N+M+1}). \quad (1)$$

It can be argued that in the case of QCD (and presumably also of supersymmetric QCD) the relative error

$$\delta_{N+M+1} \equiv \frac{S_{Pade}^{N+M+1} - S_{N+M+1}}{S_{N+M+1}} \quad (2)$$

has the asymptotic form

$$\delta_{N+M+1} \simeq -\frac{M!A^M}{L^M} \quad (3)$$

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as $N \to \infty$, for fixed $M$, where $A$ is a constant and $L = N + M + aM + b$. We will be concerned with $[0, 1]$ and $[1, 1]$ Padé, so $M = 1$; and we choose $a, b$ so that $a + b = 0$. The Asymptotic-Padé Approximant Prediction (APAP) is then given by:

$$S_{N+M+1}^{\text{APAP}} = \frac{S_{N+M+1}^{\text{Pade}}}{1 + \delta_{N+M+1}}. \quad (4)$$

If $S_0$, $S_1$ and $S_2$ are known, then the prediction for $S_3$ is obtained as follows. Firstly by matching to $P(x)$ up to linear terms as in Eq. (1), we obtain

$$P_{[0,1]} = \frac{S_0^2}{S_0 - S_1 x}, \quad (5)$$

giving as a prediction for the coefficient of the $x^2$ term $S_2^{\text{Pade}} = \frac{S_0^2}{S_0}$. Then from Eq. (2) we find $\delta_2 = \frac{S_0^2}{S_0 S_1} - 1$. From Eq. (3) we deduce $A = -\delta_2$, and thence $\delta_3 = -\frac{1}{2}A = -\frac{1}{2}\delta_2$. The $[1,1]$ Padé prediction for $S_3$ is easily derived to be $S_3^{\text{Pade}} = \frac{S_2^2}{S_1}$, and we then use Eq. (4) to give an improved estimate $S_3^{\text{APAP}}$ incorporating $\delta_3$.

The gauge $\beta$-function is given by

$$\beta_g = g \sum_{n=1}^{\infty} \beta_n x^{n+1}, \quad (6)$$

where $x = \frac{g^2}{16\pi^2}$, and $\beta_n$ corresponds to the $(n+1)$-loop result in perturbation theory. The procedure given above yields a prediction for the four-loop contribution $\beta_3$. The final feature of the method is to perform the above process for a range of values of $N_f$, the number of flavours, and then to match the results to a cubic polynomial in $N_f$. In the QCD case this enabled the authors of Ref. [3] to incorporate as an extra piece of information the known coefficient of $N_f^3$, obtained by large-$N_f$ calculations. In our present case we shall be able to exploit the fact that we know the form of $\beta_3$ as a function of $N_f$ up to a single parameter. The results of Ref. [1] were presented for a general theory; for the present purposes, however, we specialise to supersymmetric QCD, obtaining [5]:

$$\beta_0 = N_f - 3N_c \quad (7a)$$

$$\beta_1 = \left[4N_c - \frac{2}{N_c}\right] N_f - 6N_c^2 \quad (7b)$$

$$\beta_2 = \left[\frac{3}{N_c} - 4N_c\right] N_f^2 + \left[21N_c^2 - \frac{2}{N_c^2} - 9\right] N_f - 21N_c^3 \quad (7c)$$

$$\beta_3 = A + BN_f + CN_f^2 + DN_f^3 \quad (7d)$$
where $N_c$ is the number of colours, and

$$A = -(6 + 36\alpha)N_c^4$$

$$B = 36(1 + \alpha)N_c^3 - (34 + 12\alpha)N_c - \frac{8}{N_c} - \frac{4}{N_c^3}$$

$$C = -\left(\frac{62}{3} + 2\kappa + 8\alpha\right)N_c^2 + \frac{100}{3} + 4\alpha + \frac{6\kappa - 20}{3N_c^2}$$

$$D = -\frac{2}{3N_c}. \quad (8)$$

Here $\kappa = 6\zeta(3)$ and $\alpha$ is the constant which we hope to determine\footnote{It is also possible to apply the APAP method to the cases of QED or supersymmetric QED, for which the complete four-loop results can be extracted from Refs. \[6\] or \[1\] respectively. However, the form of the series in these cases is not conducive to the accuracy of the approximation and the four-loop predictions are somewhat less impressive. We hope to discuss these and other applications of the APAP method elsewhere.}.

Note that $D$ does not involve $\alpha$. It is the leading-$N_f$ contribution at four loops and could, of course, have been obtained from the large-$N_f$ results presented in Refs. \[3\], \[4\]. We may therefore proceed in much the same way as did the authors of Ref. \[3\]. We first compute $S_{3\text{APAP}}$ as described above, and then after determining $A, B, C$ from the fit to Eq. (7d) we can obtain three distinct “predictions” for $\alpha, \alpha(A), \alpha(B)$ and $\alpha(C)$. Naturally a test of the approach is the extent to which these predictions agree and are insensitive to the input value of $N_c$. As in Ref. \[3\] we use input values $N_f = 0, 1, 2, 3, 4$, and used a value for $A, \langle A \rangle$, obtained by averaging $A(N_f)$ for the input values; the outcome is not very sensitive, in fact, to whether we use $\langle A \rangle$ or $A(N_f)$. The results of this procedure are shown in Table 1:

| $N_c$ | $\alpha(A)$ | $\alpha(B)$ | $\alpha(C)$ | $\alpha(C) (\kappa = 0)$ |
|-------|-------------|-------------|-------------|--------------------------|
| 2     | 2.56        | 2.34        | -0.31       | 1.62                     |
| 3     | 2.43        | 2.42        | 0.50        | 2.38                     |
| 4     | 2.46        | 2.47        | 0.63        | 2.48                     |
| 5     | 2.45        | 2.46        | 0.74        | 2.57                     |

Table 1: APAP predictions for the unknown parameter $\alpha$.

The results for $\alpha(A), \alpha(B)$ are remarkable for their consistency and stability. In the fourth column of the table we show the result for $\alpha(C)$ consequent on omitting from $\beta_3$
the terms proportional to \( \kappa \). It could be argued that this is natural since, given that \( \zeta(3) \) occurs for the first time at four loops, these terms cannot be accurately produced by the Padé. (This is similar to the argument made with respect to the quartic Casimir terms in Ref. [3].) In any case, our results clearly suggest a value for \( \alpha \) of around 2.45.

An alternative approach is as follows. The unknown parameter \( \alpha \) in Eq. (8) is one of three parameters that were introduced in Ref. [1], each being the coefficient of a certain coupling constant redefinition. Two of these parameters were in fact determined in Ref. [1]; but we can test the reliability of the APAP method by reintroducing one of them and comparing the APAP prediction with a \( \beta_3 \) now dependent on two parameters. Thus we replace \( \beta_3 \) by \( \tilde{\beta}_3 \) where:

\[
\beta_3 \rightarrow \tilde{\beta}_3 = \tilde{A} + \tilde{B}N_f + \tilde{C}N_f^2 + \tilde{D}N_f^3
\]  

(9)

and

\[
\tilde{A} = A,
\]

\[
\tilde{B} = B + \delta \left[ 24N_c^3 - 12 \left( N_c + \frac{1}{N_c} \right) \right]
\]

\[
\tilde{C} = C + \delta \left[ -20N_c^2 + 16 + \frac{4}{N_c^2} \right]
\]

\[
\tilde{D} = D + \delta \left[ 4N_c - \frac{4}{N_c} \right].
\]

(10)

In the notation of Ref. [1], \( \alpha \equiv \alpha_1 \) and \( \delta \equiv -2\alpha_2 - \alpha_3 \). In Ref. [1] it was shown that \( \alpha_2 = -\frac{2}{3} \), and so we hope to find that \( \delta = 0 \). For fixed \( \alpha \) we determine three distinct “predictions” for \( \delta \) by fitting \( S_3^{\\text{APAP}} \) to a polynomial as before. As an example, for \( N_c = 4 \) we obtain:

\[
\delta(\tilde{B}) = 7.85\alpha - 19.24 \quad (11a)
\]

\[
\delta(\tilde{C}) = 19.38\alpha - 48.16 \quad (11b)
\]

\[
\delta(\tilde{D}) = 51.94\alpha - 126.87. \quad (11c)
\]

If we omit \( \kappa \) terms then Eqs. (11a), (11b) are unaffected, but Eq. (11c) becomes \( \delta(\tilde{C}) = 19.38\alpha - 47.41 \).

When we plot \( \delta(\tilde{B}) \), \( \delta(\tilde{C}) \) and \( \delta(\tilde{D}) \) against \( \alpha \), as in Fig. 1, the results are quite striking. We see that \( \delta(\tilde{B}) \), \( \delta(\tilde{C}) \) and \( \delta(\tilde{D}) \) are zero for very similar values of \( \alpha \), around 2.44–2.49. So the predicted values for \( \delta(\tilde{B}) \), \( \delta(\tilde{C}) \) and \( \delta(\tilde{D}) \) agree rather well with the correct value \( \delta = 0 \) if \( 2.44 \leq \alpha \leq 2.49 \), and this may be interpreted as a prediction for \( \alpha \).
Fig. 1: Graph of $\delta(B)$ (solid line), $\delta(C)$ (dashed line) and $\delta(D)$ (dash-dotted line) against $\alpha$ for $N_c = 4$.

Fig. 2: Graph of $\delta(B)$, $\delta(C)$ and $\delta(D)$ against $\alpha$ for $N_c = 4$ (omitting $\kappa$ terms).
The convergence is again even better if we omit the term in $\kappa$, as depicted in Fig. 2, where each $\delta$ crosses the axis at $\alpha = 2.44$. As $N_c$ is increased, we find that the convergence of the $\delta$s improves in the case where we retain $\kappa$; in fact for large $N_c$ all the $\delta$s are zero for $\alpha = 2.43$ irrespective of whether or not we retain $\kappa$.

What is our final prediction for $\alpha$? The relatively simple relationship between the NSVZ $\beta$-function and the corresponding DRED one as explored in Ref. [1] suggests that $\alpha$ is a simple fraction; given our results here, then we would expect $\alpha = 12/5$ or perhaps $\alpha = 5/2$. ($\alpha = 17/7$ is even closer, but experience suggests that this number is unlikely to emerge from a perturbative calculation.) The result $\alpha = 2\zeta(3)$ is also possible; in which case the apparent better convergence produced by excising $\kappa$ from the comparison would need to be dismissed as coincidental.

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