The β-dual of the Cesàro sequence spaces defined on a generalized Orlicz space

Haryadi,1 Supama2, and Atok Zulijanto2
1 Department of Mathematics, Universitas Muhammadiyah, Palangkaraya, ID
2 Department of Mathematics, Universitas Gadjah Mada, Yogyakarta, ID
E-mail: haryadi_ump@yahoo.co.id

Abstract. In this paper we characterized the β-dual of the Cesàro sequence space with terms in a generalized Orlicz space. Further, we find that the dual is a generalization of the dual of the Cesàro space in the classical Banach space $L_p$ for $1 < p < \infty$.

1. Introduction and Preliminaries
The Kothe-Toeplitz dual is a useful tool in the characterization of class of the matrix transformations between sequence spaces. The recent study of the class of the matrix transformations between sequence spaces can be found in Tripathy and Sen [14]. Khan and Khan [3], and Rahman and Karim [11], examined the characterization of the class of matrix transformations on the scalar-valued Cesàro sequence spaces. Maddox [6, 7], characterized a generalized Kothe-Toeplitz dual of the vector-valued sequence spaces. In [15], Yilmaz and Ozdemir extend the definition of Maddox [6] and examined Kothe-Toeplitz duals of some vector-valued Orlicz sequence spaces. The properties of some new Cesàro sequence spaces were studied by Bala [1], Kubiak [5], Petrot and Suantai [10], and Sever and Altay [13]. Bhardwaj and Gupta [2] studied Kothe-Toeplitz dual and matrix map on the Cesàro summable difference sequence spaces.

In this paper, we characterize the β-dual of the Cesàro sequence spaces constructed by the modular $\rho_\phi$ with terms in a generalized Orlicz space.

We begin by recalling the definition of the Orlicz function. Let $\phi : \mathbb{R} \to [0, \infty)$ be an Orlicz function, that is, $\phi$ is even, $\phi(x) = 0$ if and only if $x = 0$, $\lim_{x \to \infty} \phi(x) = \infty$, continuous, and convex. If in addition, $\phi$ satisfies the conditions $\lim_{x \to 0} \phi(x)/x = 0$ and $\lim_{x \to \infty} \phi(x)/x = \infty$, then it is called $N$-function. The complementary to the Orlicz function $\phi$ is a function $\psi$ such that $|xy| \leq \phi(x) + \psi(y)$, for every $x, y \in \mathbb{R}$. For any Orlicz function $\phi$, the function $\psi$ defined by $\psi(y) = \sup\{|y|x - \phi(x) : x \geq 0\}$ is an Orlicz function complementary to $\phi$. An Orlicz function $\phi$ is said to satisfy the $\Delta_2$-condition if there is a $K > 0$ such that $\phi(2x) \leq K\phi(x)$ for each $x \geq 0$ (see e.g., [4]). We denote by $\phi^{-1}$, the inverse function of the Orlicz function $\phi$ in the non-negative values argument.

Let $S$ denote the collection of all Lebesgue measurable real valued functions on $[a, b]$. Let $(\phi, \psi)$ be a pair of complementary Orlicz functions. The generalized Orlicz space $L_\phi$ is defined
as the space of all \( u \in S \) such that \( |\int_a^b u(x)v(x)dx| < \infty \) for every \( v \) with \( \int_a^b \psi(v(x))dx < \infty \). It is easy to show that

\[
\rho_{\phi}(u) = \int_a^b \phi(u(x))dx
\]

is a modular on \( L_\phi \), i.e. \( \rho_{\phi} \) satisfies:

1. \( \rho_{\phi}(u) = 0 \iff u = \theta \), where \( \theta \) is the zero function in \( L_\phi \).
2. \( \rho_{\phi}(-u) = \rho_{\phi}(u) \) for each \( u \in L_\phi \), and
3. \( \rho_{\phi}(\alpha u + \beta v) \leq \rho_{\phi}(u) + \rho_{\phi}(v) \) for each \( u, v \in L_\phi \) and \( \alpha, \beta \geq 0 \), \( \alpha + \beta = 1 \). In addition, the convexity of \( \phi \) implies the convexity of \( \rho_{\phi} \).

For any Orlicz function \( \phi \), \( L_\phi \) is a Banach space with respect to the Orlicz norm

\[
\|f\|_\phi = \sup \left\{ \left| \int_a^b f(x)g(x)dx \right| : \rho_{\psi}(g) \leq 1 \right\},
\]

where \( \psi \) is complementary to \( \phi \). In addition, it is true that (i) \( \|f\|_\phi \leq \rho_{\phi}(f) + 1 \) for every \( f \in L_\phi \) and (ii) \( \rho_{\phi}(f) \leq 1 \) if and only if \( \|f\|_\phi \leq 1 \) (see, e.g., [4, 12]).

A sequence \( (f_k) \) in \( L_\phi \) is said to be \( \rho_{\phi} \)-convergent if \( \rho_{\phi}(f_k - f_0) \to 0 \) as \( k \to \infty \); it is called \( \rho_{\phi} \)-Cauchy if \( \rho_{\phi}(f_k - f_l) \to 0 \) as \( k, l \to \infty \). It can be shown (see e.g. [4, 9]), if the Orlicz function \( \phi \) satisfies \( \Delta_2 \)-condition, then the norm convergence is equivalent to \( \rho_{\phi} \)-convergence in \( L_\phi \).

Let \( \omega(L_\phi) \) denotes the space of all sequences with terms in \( L_\phi \). The member of \( L_\phi \) is written as \( f = (f_k) \). We modify the definition of Maddox [6], the generalized Cesàro sequence space, as follows:

\[
W_{0,\phi} = \left\{ (f_k) : (\exists \lambda > 0) \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^N \int_a^b \phi(\lambda f_k(x))dx = 0 \right\}.
\]

In relation to the sequence space above, we also defined the following spaces:

\[
W_\phi = \left\{ (f_k) : (f_k - f_0) \in W_{0,\phi} \text{ for some } f_0 \in L_\phi \right\}
\]

\[
W_{0,\|\|} = \left\{ (f_k) \subset L_\phi : \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^N \|f_k\|_\phi = 0 \right\}.
\]

We denote by \( \omega \), the set of all sequence in \( \mathbb{R} \). Further, if the Orlicz function \( \phi \) satisfies the \( \Delta_2 \)-condition, we define

\[
w_{0,\phi} = \left\{ (\lambda_k) \in \omega : \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^N \phi(\lambda_k) = 0 \right\}.
\]

It is clear that \( W_{0,\phi} \subset W_\phi \). Note that any \( \rho_{\phi} \)-convergent sequence in \( L_\phi \) belongs to \( W_\phi \). Especially, for any \( p, 1 \leq p < \infty \), every mean convergent sequence in \( L_p \), belongs to \( W_\phi \) for the Orlicz function \( \phi(x) = |x|^p \).

Using the fact that \( \rho_{\phi} \) is a modular, it is easy to establish that the function \( \rho : \omega(L_\phi) \to [0, \infty] \) defined by

\[
\rho(f) = \sup_{N} \left\{ \frac{1}{N} \sum_{k=1}^N \int_a^b \phi(f_k(x))dx \right\}
\]
is a modular. Furthermore, $\rho(f) < \infty$ for every $f \in \mathcal{W}_\phi$. Note that the convexity of $\phi$ implies the convexity of $\rho$. Further, each of the space $\mathcal{W}_{0,\phi}$ and $\mathcal{W}_\phi$ is a Banach space with respect to the Luxemburg norm

$$\|f\|_\rho = \inf \left\{ t > 0 : \rho \left( \frac{f}{t} \right) \leq 1 \right\}.$$ 

Let $\mathcal{L}_\phi^*$ be the space of bounded linear functionals on $\mathcal{L}_\phi$. We write $u^*, u$ the value of linear functional $u^*$ at $u$. The dual of $X \subset \omega(\mathcal{L}_\phi)$ is defined as

$$[X]^\beta = \{ (f_k) \subset \mathcal{L}_\phi^* : \sum_{k=1}^{\infty} f_k x_k \text{ convergent for each } (f_k) \in X \}.$$ 

2. The Results

Our results mainly concern on the Cesàro sequence space which generated by the Orlicz function satisfying $\Delta_2$-condition.

**Theorem 1.** If the Orlicz function $\phi$ satisfies the $\Delta_2$-condition, then

$$\mathcal{W}_{0,\phi} = \left\{ (f_k) : \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \int_{a}^{b} \phi(f_k(x))dx = 0 \right\} \quad (1)$$

$$\mathcal{W}_\phi = \left\{ (f_k) : \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \int_{a}^{b} \phi(f_k(x) - f_0(x))dx = 0 \text{ for some } f_0 \in \mathcal{L}_\phi \right\} \quad (2)$$

**Proof.** To prove part (1), let we write the right hand side of (1) by $\mathcal{W}_{0,\phi}'$. It is enough to prove that $\mathcal{W}_{0,\phi} \subset \mathcal{W}_{0,\phi}'$. Let $(f_k) \in \mathcal{W}_{0,\phi}$, then there exists $\lambda > 0$ such that $\frac{1}{N} \sum_{k=1}^{\infty} \rho(\lambda f_k) \to 0$ as $N \to \infty$. For $\lambda \geq 1$, the monotonicity of $\phi$ implies $f \in \mathcal{W}_{0,\phi}'$. For $\lambda < 1$, since $\phi$ satisfies the $\Delta_2$-condition, there then exist $K > 0$ and $l_0 \in \mathbb{N}$ such that $\phi(\frac{u}{K}) \leq K^{l_0} \phi(u)$. Hence,

$$\frac{1}{N} \sum_{k=1}^{N} \int_{a}^{b} \phi(f_k(x))dx = \frac{1}{N} \sum_{k=1}^{N} \int_{a}^{b} \phi(\frac{\lambda}{\lambda} f_k(x))dx$$

$$\leq K^{l_0} \frac{1}{N} \sum_{k=1}^{N} \int_{a}^{b} \phi(\lambda f_k(x))dx \to 0$$

as $N \to \infty$, i.e. $(f_k) \in \mathcal{W}_{0,\phi}'$.

The proof of part (2) is similar. \hfill \qed

The Cesàro sequence space constructed by the modular $\rho_\phi$ is contained in the space defined by the Orlicz norm, as stated by the following lemma.

**Lemma 1.** If the Orlicz function $\phi$ satisfying the $\Delta_2$-condition then $\mathcal{W}_{0,\phi} \subset \mathcal{W}_{0,\|\cdot\|}$.

**Proof.** Let $\varepsilon > 0$ be arbitrary and $(f_k) \in \mathcal{W}_{0,\phi}$. Take $n_0 \in \mathbb{N}$ such that $2^{-n_0 + 1} < \varepsilon$, and let $K > 0$ such that $\phi(2^{n_0} f_k(x)) \leq K^{n_0} \phi(f_k(x))$. Take $N_0 \in \mathbb{N}$ such that

$$\frac{1}{N} \sum_{k=1}^{N} \int_{a}^{b} \phi(f_k(x))dx < \frac{1}{K^{n_0}} \text{ for each } N \geq N_0,$$
Then for each $N \geq N_0$,
\[
\frac{1}{N} \sum_{k=1}^{N} \int_{a}^{b} \phi(2^{n_0}(f_k(x)))dx \leq \frac{K^{n_0}}{N} \sum_{k=1}^{N} \int_{a}^{b} \phi(f_k(x))dx \leq 1.
\]
Hence, for each $N \geq N_0$ we have
\[
\frac{1}{N} \sum_{k=1}^{N} \|2^n f_k\|_\phi \leq \frac{1}{N} \sum_{k=1}^{N} \int_{a}^{b} \phi(2^{n_0}(f_k(x)))dx + 1 \leq 2
\]
which implies \(\frac{1}{N} \sum_{k=1}^{N} \|f_k\|_\phi \leq 2^{-n_0+1} < \varepsilon.\)

Note that this lemma implies \(\frac{1}{N} \sum_{k=2^{n_0+1-1}}^{2^n-1} \|f_k\|_\phi \to 0\) as $v \to \infty$ for every $(f_k) \in \mathcal{W}_{0,\phi}$. The converse of Lemma 1 is not necessarily true in general. This is clear from the following example.

Example. Let $\phi(x) = |x|^2$ and $(f_k)$ be a sequence of real function on $[0, 1]$ with $f_k(x) = \frac{1}{k}$ for $k \neq j^3$ and $f_k(x) = \sqrt[3]{k}$ for $k = j^3$. Then \(\frac{1}{N} \sum_{k=1}^{N} \|f_k\|_\phi \to 0\), but \(\frac{1}{N} \sum_{k=1}^{N} \rho_\phi(f_k) \not\to 0\).

**Lemma 2.** If the Orlicz function $\phi$ satisfying the $\Delta_2$-condition and $(f_k) \in \mathcal{W}_{0,\phi}$, then
\[
\sup \left\{ \frac{\|f_k\|_\phi}{\phi^{-1}(\rho_\phi(f_k))} : \|f_k\|_\phi > 1 \right\} < \infty.
\]

**Proof.** Let $\varepsilon > 0$ and $(f_k) \in \mathcal{W}_{0,\phi}$. There exists $N_0 \in \mathbb{N}$ such that \(\frac{1}{N} \sum_{k=1}^{N} \rho_\phi(f_k) < \varepsilon\) for each $N \geq N_0$, which implies
\[
\rho_\phi(f_k) < \varepsilon \cdot k, \quad \forall k \geq N_0. \tag{4}
\]
Suppose (3) does not hold. Then for any natural number $n \geq N_0$ there exists $\|f_k\|_\phi > 1$ such that
\[
n \phi^{-1}(\rho_\phi(f_k)) < \|f_k\|_\phi.
\]
Since $1 < \|f_k\|_\phi \leq \rho_\phi(f_k)$, then
\[
\left(\frac{n}{k}\right) \phi^{-1}(1) \cdot k < \rho_\phi(f_k),
\]
contradict to (4). \hfill \Box

**Corollary 1.** Let the Orlicz function $\phi$ satisfying the $\Delta_2$-condition and $(f_k) \in \mathcal{W}_{0,\phi}$. There exists $c > 0$ such that
\[
\phi(\|f_k\|_\phi) \leq c \rho_\phi(f_k), \quad \forall \|f_k\|_\phi > 1.
\]

From Lemma 1 and Corollary 1, it can be shown the following corollary.

**Corollary 2.** Let the Orlicz function $\phi$ satisfying the $\Delta_2$-condition. If $(f_k) \in \mathcal{W}_{0,\phi}$, then $(\|f_k\|_\phi) \in \mathcal{W}_{0,\phi}$.

A linear space $X \subset \omega(C)$ is said to be normal if for every $f$ with $|f| \leq |g|$ for some $g \in X$, implies $f \in X$. The space $\mathcal{W}_{0,\phi}$ is normal. For, let $g \in \mathcal{W}_{0,\phi}$ and $|f| \leq |g|$ in the sense $|f_k| \leq |g_k|$ a.e. on $[a, b]$ for each $k \in \mathbb{N}$. Obviously $f_k \in \mathcal{S}$ and the monotonicity of $\phi$ implies $\phi \circ f_k \leq \phi \circ g_k$. Hence $f \in \mathcal{W}_{0,\phi}$.
In addition, $W_\phi$ is not a normal space, let $\phi(u) = \|u\|$ and define $f_1(x) = 1$, $f_2(x) = 0$ and

$$f_k(x) = \begin{cases} 1, & 2^{r-1} < k \leq 3 \cdot 2^{r-2}, \ r = 2, 3, \ldots \\ 0, & \text{otherwise} \end{cases}$$

for each $x \in [a, b]$. Then $\rho_\phi(k) \leq 1$ but $(k) \not\in W_\phi$.

**Theorem 2.** Let the Orlicz function $\phi$ satisfies the $\Delta_2$-condition. Then $(k^*_k) \in [W_{0,\phi}]^\beta$ if and only if

$$\|f^*_k\| \in [w_{0,\phi}]^\alpha. \quad (5)$$

**Proof.** Suppose that $\sum_{k=1}^{\infty} f^*_kf_k$ convergent for every $(k) \in W_{0,\phi}$. Since $f^*_k \in L^*_\phi$ then there exists $f_k \in L_\phi$ with $\|f_k\|_\phi \leq 1$ such that

$$\|f^*_k\| \leq 2|f^*_kf_k| \quad (6)$$

for each $k \in \mathbb{N}$. It is easy to see that there exists $c > 0$ such that $\rho_\phi(\lambda f_k) \leq c\phi(\lambda)$ for each $\lambda \in \mathbb{R}$ if $\|f_k\|_\phi \leq 1$. Let $(\lambda_k) \in w_{0,\phi}$, and define $g_k = |\lambda_k|sgn(f^*_kf_k)f_k$. Then

$$\frac{1}{N} \sum_{k=1}^{N} \rho_\phi(g_k) = \frac{1}{N} \sum_{k=1}^{N} \rho_\phi(\lambda_k f_k) \leq \frac{c}{N} \sum_{k=1}^{N} \phi(\lambda_k) \to 0$$

as $N \to \infty$, i.e. $(g_k) \in W_{0,\phi}$. Consequently

$$\sum_{k=1}^{\infty} |\lambda_k| \cdot |f^*_kf_k| = \sum_{k=1}^{\infty} f^*_kg_k$$

is convergent, which implies that $\sum_{k=1}^{\infty} \|f^*_k\| \cdot |\lambda_k| \leq 2 \sum_{k=1}^{\infty} |\lambda_k| \cdot |f^*_kf_k| < \infty$, i.e. $(f^*_k) \in w_{0,\phi}^\alpha$.

The sufficiently, let $(\|f^*_k\|) \in [w_{0,\phi}]^\alpha$ and $f \in W_{0,\phi}$. Since $(\|f_k\|_\phi) \in w_{0,\phi}$, then

$$\sum_{k=1}^{\infty} \|f^*_k\|_\phi \|f_k\|_\phi < \infty.$$ Hence, for each $\varepsilon > 0$, there exist $k_0$ such that

$$\left| \sum_{k=m}^{n} f^*_k f_k \right| \leq \sum_{k=m}^{n} \|f^*_k\|_\phi \|f_k\|_\phi < \varepsilon, \ \forall m, n \geq k_0,$$

i.e. $(\sum_{k=1}^{n} f^*_k f_k)$ is a Cauchy sequence in $\mathbb{R}$. Hence $\sum_{k=1}^{\infty} f^*_k f_k$ convergent.

**Corollary 3.** Let the Orlicz function $\phi$ satisfies the $\Delta_2$-condition. Then

$$[W_{0,\phi}]^\beta = [W_{0,\phi}]^\beta.$$

**Proof.** Since $W_{0,\phi} \subset W_\phi$, it is enough to show that $[W_{0,\phi}]^\beta \subset [W_{0,\phi}]^\beta$.

Let $(k) \in [W_{0,\phi}]^\beta$ and $(f_k) \subset W_\phi$. Let $f_0 \in L_\phi$ such that $(f_k - f_0) \in W_{0,\phi}$. By Theorem 2, $(\|f^*_k\|) \in [w_{0,\phi}]^\alpha$. Since $(\|f_k - f_0\|_\phi) \in w_{0,\phi}$, then

$$\sum_{k=1}^{\infty} \|f^*_k(f_k - f_0)\| \leq \sum_{k=1}^{\infty} \|f^*_k\|_\phi \|f_k - f_0\|_\phi < \infty.$$
Since $\phi$ satisfies $\Delta_2$-condition, then there exists $1 < p < \infty$ such that $\phi(x) \leq c|x|^p$ (see [4]), and hence, $w_{0,p} \subset w_{0,\phi}$. Since $w_{0,\phi} \subset w_{0,p}$, then $\sum_{r=0}^{\infty} 2^{r/p} (\sum_{k=2^r}^{2^{r+1}-1} \|f_k^*\|_q)^{1/q} < \infty$, $1/p + 1/q = 1$ (see [8]). Hence,

$$\sum_{k=1}^{\infty} |f_k^* f_0| \leq \sum_{k=1}^{\infty} \|f_k^*\| \|f_0\|_\phi \leq \sum_{r=0}^{\infty} 2^{r/p} \left( \sum_{k=2^r}^{2^{r+1}-1} \|f_k^*\|_q \right)^{1/q} \left( \frac{1}{2^r} \sum_{k=2^r}^{2^{r+1}-1} \|f_0\|_\phi \right)^{1/p} < \infty.$$ 

Finally, we have

$$\sum_{k=1}^{\infty} |f_k^* f_k| \leq \sum_{k=1}^{\infty} |f_k^* (f_k - f_0)| + \sum_{k=1}^{\infty} |f_k^* f_0| < \infty,$$

and hence, $\sum_{k=1}^{\infty} f_k^* f_k$ converges.

\[\square\]

**Acknowledgments**

The author would like to thank to the reviewer for making some useful corrections which improve the paper.

**References**

[1] Bala I 2012 *Communications in Mathematics and Applications* 3(2) pp 197–204
[2] Bhardwaj V K and Gupta S 2013 *J. Inequalities and App.* 315 pp 1–9
[3] Khan F M and Khan M A 1994 *Indian J. Pure Appl. Math.* 25(6) pp 641–645
[4] Krasnosel’skii MA and Rutickii Y B 1961 *Convex Functions and Orlicz Spaces* (Netherlands: P. Noordhoff Ltd)
[5] Kubiak D 2009 *J. Math. Anal. and App.* 349 pp 291–296
[6] Maddox I J 1980 *J. Math. Math. Sci.* 3 pp 423–432
[7] Maddox I J 1974 *Comp. Math.* 29 pp 35–42
[8] Malkowsky E and Rakcevic V 2000 *An Introduction to the theory of sequence spaces and measures of noncompactness* (Zbornik radova, Matematicki institut SANU, Beograd)
[9] Musielak J 1983 *Orlicz spaces and modular space* (Berlin Heidelberg New York Tokyo: Springer Verlag)
[10] Petrot N and Suantai S 2005 *Sci. Asia* 31 pp 173–177
[11] Rahman MF and Karim A B R M 2015 *Pure and App. Math. J.* 4(6) pp 237–241
[12] Rao M M and Ren Z D 1991 *Theory of Orlicz Spaces* (New York: Marcel Dekker Inc.)
[13] Sever Y and Altay B 2014 *Filomat* 28(7) pp 1417–1424
[14] Tripathy B C and Sen M 2006 *Tamkang J. Math.* 37(2) pp 155–162
[15] Yilmaz Y and Ozdemir M K 2005 *Soochow J. Math.* 31(3) pp 389–402.