On the Existence of a Self-Similar Coarse Graining of a Self-Similar Space

Akihiko Kitada, Tomoyuki Yamamoto, Tsuyoshi Yoshioka and Shousuke Ohmori
Laboratory of mathematical design for materials, Faculty of Science and Engineering, Waseda University, 3-4-1 Okubo, Shinjuku-ku, Tokyo 169-8555

Abstract
A topological space homeomorphic to a self-similar space is demonstrated to be self-similar. There exists a self-similar space $S$ whose coarse graining is homeomorphic to $S$. The coarse graining of $S$ is, therefore, self-similar again. In the same way, the coarse graining of the self-similar coarse graining of $S$ is, furthermore, self-similar. These situations succeed endlessly. Such a self-similar $S$ is generated actually from an intense quadratic dynamics.

Keywords: self-similar set, fractals, dynamical system, Cantor set, coarse graining

1 Introduction

In the fractal sciences, the fine structure of the self-similar space is characterized by the property that every details looks similar with the whole. In the present report, we are oppositely concerned with the coarse structures of a self-similar space, that is, with the problem "what self-similar space can have a coarse graining of it with a self-similarity again?". According to A. Fernández [1], the procedure of the coarse graining or the block construction [2] of a space in the statistical physics corresponds mathematically to that of the construction of a quotient space which is defined by a classification of all points in the space through the identification of the different points based on an equivalence relation.

At first, a sufficient condition for a given topological space to be metrizable and self-similar with respect to the metric is investigated, and, second, the existence of a decomposition space [3] as a coarse graining of a self-similar space $S$ whose self-similarity is defined by a system of weak contractions which is topologically closely related to that defining the self-similarity of $S$
is discussed in a quite elementary way. As a consequence, we are convinced that there exists a sequence of self-similar coarse graining of a self-similar space even for the quadratic dynamics known to be one of the simplest dynamical system. Finally, it is noted that each step of the sequence can equally generate a topological space characteristic of condensed matter such as dendrite [4].

2 A condition for a topological space to be self-similar

An answer of the problem “for what topological space, can we find a system of weak contractions which makes the space self-similar?” is simply stated as follows.

Proposition. The existence of a self-similar space which is homeomorphic to \((Y, \tau)\) is sufficient for a topological space \((Y, \tau)\) to be a metrizable space and self-similar with respect to the metric.

Proof. Let \((X, \tau_d)\) be self-similar based on a system of weak contractions \(p_j : (X, \tau_d) \to (X, \tau_d), d(p_j(x), p_j(x')) \leq \alpha_j(\eta)d(x, x')\) for \(d(x, x') < \eta, 0 \leq \alpha_j(\eta) < 1, j = 1, \ldots, m\) (\(2 \leq m < \infty\)). That is, \(\bigcup_{j=1}^{m} p_j(X) = X\). Using a homeomorphism \(h : (X, \tau_d) \simeq (Y, \tau)\), we can define a metric \(\rho\) on \(Y\) as

\[\rho(y, y') = d(h^{-1}(y), h^{-1}(y')), \quad y, y' \in Y.\]

The metric topology \(\tau_\rho\) is identical with the initial topology \(\tau\). From the relations 1) and 2) below, the metric space \((Y, \tau_\rho)\) is confirmed to be self-similar by a system of weak contractions \(q_j : (Y, \tau_\rho) \to (Y, \tau_\rho), j = 1, \ldots, m\) where \(q_j\) is topologically conjugate to \(p_j\) with the above homeomorphism \(h\), that is, \(q_j = h \circ p_j \circ h^{-1}\).

1) \[\rho(q_j(y), q_j(y')) = d(h^{-1}(q_j(y)), h^{-1}(q_j(y'))) = d(p_j(h^{-1}(y)), p_j(h^{-1}(y'))) \leq \alpha_j(\eta)d(h^{-1}(y), h^{-1}(y')) = \alpha_j(\eta)\rho(y, y') \quad \text{for} \quad \rho(y, y') < \eta.\]

2) \[\bigcup_{j=1}^{m} q_j(Y) = \bigcup_{j=1}^{m} q_j(h(X)) = h(\bigcup_{j=1}^{m} p_j(X)) = h(X) = Y. \quad \square\]
3 Existence of a self-similar decomposition space

As an application of Proposition, we will show the existence of a self-similar decomposition space of a self-similar space.

Let $S$ be a self-similar, perfect \(\mathbb{[6]}\), zero-dimensional (0-dim) \(\mathbb{[7]}\), compact metric space, and \((X, \tau_d)\) be any compact metric space which is self-similar. Then, there exists a continuous map $f$ from $S$ onto $X$ \(\mathbb{[8]}\), and $X$ is homeomorphic to the decomposition space \((D_f, \tau(D_f))\) of $S$ with a homeomorphism $h : (X, \tau_d) \simeq (D_f, \tau(D_f))$, $x \mapsto f^{-1}(x)$ \(\mathbb{[9]}\). Here, $D_f = \{f^{-1}(x) \subset S; \ x \in X\}$ and $\tau(D_f) = \{U \subset D_f; \ \bigcup_{D \in U} \bigcup_{D \in \{D_f\}} \text{is an open set of} \ S\}$. The decomposition topology $\tau(D_f)$ is identical with a metric topology $\tau_{\rho}$ with a metric $\rho(y, y') = d(h^{-1}(y), h^{-1}(y'))$, $y, y' \in D_f$ \(\mathbb{[10]}\). Since the metric space $(X, \tau_d)$ is assumed to be self-similar, from Proposition, the decomposition space $(D_f, \tau_{\rho})$ must be self-similar based on a system of weak contractions each of which is topologically conjugate to each weak contraction which defines the self-similarity of $X$. According to the self-similarity of the selected space $X$, the decomposition space $D_f$ of $S$ can have various types of self-similarity.

Now, let us consider a special case where the system of contructions defining the self-similarity of the decomposition space $D_f$ of $S$ is topologically related to that defining the self-similarity of $S$. Let \(\{S_1, \cdots, S_n\}\) be a partition of $S$ \(\mathbb{[3]}\) such that each $S_i$ is a clopen (closed and open) set of $S$. (Concerning the existence of such partition of $S$, see Appendix.) Since the metric space $S_1$ is perfect, 0-dim, compact, it is homeomorphic to the Cantor’s Middle Third Set (abbreviated to CMTS) \(\mathbb{[11]}\) as well as the space $S$. Therefore, $S_1$ and $S$ are homeomorphic. Let $f : S \rightarrow S_1$ be a not one to one, continuous, onto map. For example, the map $f : S \rightarrow S_1$ defined as $f(x) = x$ for $x \in S_1$, $f(x) \equiv q_2 \in S_1$ for $x \in S_2$, \ldots, $f(x) \equiv q_n \in S_1$ for $x \in S_n$ is a continuous, onto map. It must be noted that $D_f$ is not trivial decomposition space \(\{\{x\} \subset S; x \in S\}\) because the map $f$ is not one to one \(\mathbb{[12]}\). Since the decomposition space $D_f$ of $S$ is homeomorphic to $S_1$ \(\mathbb{[3]}\), $S$ must be homeomorphic to $D_f$. Therefore, from Proposition, $D_f$ is self-similar based on a system of weak contractions each of which is topologically conjugate to each weak contraction which defines the self-similarity of $S$.

Since the metric space $D_f$ is perfect, 0-dim and compact, the same situation as for the initial space $S$ can take place for the decomposition space
Therefore, continuing this process endlessly, we obtain an infinite sequence of self-similar decomposition spaces or self-similar coarse graining starting from the self-similar space $S$, namely, a hierarchic structure of self-similar spaces as shown in Fig. 1. In Fig. 1, the above mentioned decomposition space $\mathcal{D}_f$ of $S$ is denoted by $\mathcal{D}_1$. $\mathcal{D}_1$ is self-similar due to a system of weak contractions $\{f_j^1 = h^1 \circ f_j \circ (h^1)^{-1} : \mathcal{D}_1 \to \mathcal{D}_1; j = 1, \ldots, m\}$. Here, $\{f_j : S \to S; j = 1, \ldots, m\}$ is a system of weak contractions which defines the self-similarity of $S$, and $h^1$ is a homeomorphism from $S$ to $\mathcal{D}_1$. The decomposition space $\mathcal{D}_2$ of $\mathcal{D}_1$ is self-similar based on a system of weak contractions $\{f_j^2 = h^2 \circ f_j^1 \circ (h^2)^{-1} : \mathcal{D}_2 \to \mathcal{D}_2; j = 1, \ldots, m\}$ where $h^2$ is a homeomorphism from $\mathcal{D}_1$ to $\mathcal{D}_2$. We can continue the procedure in this manner.

Statement. \[14, 15, 16\] Let $(Z, \tau_d)$ be a compact metric space. If the system $\{f_j : (Z, \tau_d) \to (Z, \tau_d), j = 1, \ldots, m\}$ of weak contractions $d(f_j(z), f_j(z')) \leq \alpha_j(\eta)d(z, z')$ for $d(z, z') < \eta$, $0 < \alpha_j(\eta) < 1$, $\inf_{\eta > 0} \alpha_j(\eta) > 0$, $j = 1, \ldots, m$ satisfies three conditions

i) Each $f_j$ is one to one,

ii) The set $\bigcup_{j=1}^{m} \{z \in Z; f_j(z) = z\}$ is not a singleton,

iii) $\sum_{j=1}^{m} \inf_{\eta > 0} \alpha_j(\eta) < 1$,

then, there exists a perfect, 0-dim, compact $S (\subset Z)$ such that $\bigcup_{j=1}^{m} f_j(S) = S$.

Concludingly, we are convinced of the existence of a sequence as shown in Fig. 1 of self-similar coarse graining of a self-similar space based on the above quadratic dynamics $F_\mu(x)$ with a sufficiently large rate constant $\mu > 0$.

4 Generation of dendrites from each step of the sequence $S, \mathcal{D}_1, \mathcal{D}_2, \cdots$

Since all of the metric spaces $S, \mathcal{D}_1, \mathcal{D}_2, \cdots$ in Fig. 1 are perfect, 0-dim and compact, there exist continuous maps \[8\], $k$ from $S$ onto the dendrite $\delta$ as a compact metric space, $k^1$ from $\mathcal{D}_1$ onto $\delta$, $k^2$ from $\mathcal{D}_2$ onto $\delta$, \ldots, respectively \[17\]. The decomposition spaces $\delta_S = \{k^{-1}(x) \subset S; x \in \delta\}$

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of $S$ due to $f$, $\delta_{D^1} = \{(k^1)^{-1}(x) \subset D^1; x \in \delta\}$ of $D^1$ due to $k^1$, $\delta_{D^2} = \{(k^2)^{-1}(x) \subset D^2; x \in \delta\}$ of $D^2$ due to $k^2$, $\cdots$ are homeomorphic to the dendrite $\delta$, and therefore, $\delta_S, \delta_{D^1}, \delta_{D^2}, \cdots$ must have the dendritic structure in common (Fig. 3). For example, the self-similar space $S$ generated from a quadratic dynamics $F_\mu(x) = \mu x(1 - x)$ with a sufficiently large $\mu > 0$ is mathematically demonstrated to be able to form a dendrite through the coalescence or the rearrangement of constituents of $S$.

Appendix

Let $S$ be a perfect, 0-dim $T_0$-space. Then, for any $n$, there exist $n$ non-empty clopen (closed and open) sets $S_1, \cdots, S_n$ of $S$ such that $S_i \cap S_{i'} = \emptyset$ for $i \neq i'$ and $\bigcup_{i=1}^n S_i = S$. For any $n$, there exist $n$ non-empty clopen sets $S_{i_1}, \cdots, S_{i_n}$ of $S$ such that $S_{i_j} \cap S_{i_{j'}} = \emptyset$ for $j \neq j'$ and $\bigcup_{j=1}^n S_{i_j} = S_i$. We can continue in this manner endlessly.

**proof**) To use the mathematical induction, let the statement hold for $n - 1$. Since $S$ is perfect, the open set $S_{n-1}$ has at least two distinct points $a$ and $b$. Since $S$ is a $T_0$-space, there exists an open set $u$ containing $a$ such that $b \notin u$ without loss of generality. Since $S$ is 0-dim, there exists a clopen set $v$ which contains the point $a$ and is contained in the open set $u \cap S_{n-1}$. Since $b \in S_{n-1} - v$, the clopen set $S_{n-1} - v$ is not empty. Thus, we obtain a desired $n$-partition $\{S_1, \cdots, S_{n-2}, v, S_{n-1} - v\}$ of $S$. Concerning the subspace $S_i$, it suffices to remember that any non-empty open set in a perfect space is perfect again. □

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References

[1] A. Fernández: J. Phys. A 21 (1988) L295.

[2] S.K. Ma: *Modern theory of critical phenomena* (Benjamin, 1976) p.246.

[3] Let $(A, \tau)$ be a topological space. A partition $\mathcal{D}$ of $A$ is a set $\{\emptyset \neq D \subset A\}$ of nonempty subsets of $A$ such that $D \cap D' = \emptyset$ for $D \neq D'$, $D, D' \in \mathcal{D}$.
and \( \bigcup \mathcal{D}(= \bigcup_{D \in \mathcal{D}} D) = A \). A decomposition space \( (\mathcal{D}, \tau(\mathcal{D})) \) of \( (A, \tau) \) is a topological space whose topology \( \tau(\mathcal{D}) \) on a partition \( \mathcal{D} \) of \( A \) is defined by

\[
\tau(\mathcal{D}) = \{ U \subset \mathcal{D} : \bigcup_{D \in U} D \in \tau \}. 
\]

The space \( (\mathcal{D}, \tau(\mathcal{D})) \) is a kind of quotient space of \( (A, \tau) \). See, for the detailed discussions, S.B. Nadler Jr., *Continuum theory* (Marcel Dekker, 1992) p.36.

[4] A dendrite is a metric space which is locally connected, connected and compact. The reference in [3], p.165.

[5] Topological spaces \( (E, \tau) \) and \( (F, \tau') \) are said to be homeomorphic provided that there exists a continuous, one to one, open (or closed) map from \( E \) onto \( F \). If \( E \) and \( F \) are homeomorphic, all of the topological properties in \( E(F) \) are preserved in \( F(E) \). See, for example, A. Kitada: *Isoukuukan to sono ouyou* (Akakura Shoten, 2007) p.24 [in Japanese].

[6] A topological space \( (A, \tau) \) is said to be perfect provided that any set \( \{x\} \) composed of single point \( x \in A \) is not an open set, that is, \( \{x\} \notin \tau \).

[7] A topological space \( (A, \tau) \) is said to be 0-dim provided that at any point \( x \in A \), and for any open set \( U \) containing \( x \), there exists a closed and open set (so-called a clopen set) \( u \) containing \( x \) such that \( u \subset U \). See, for the detailed discussions, W. Hurewicz and H. Wallman, *Dimension Theory* (Princeton University Press, Princeton, 1941) p.10.

[8] The reference in [3], p.106 and p.109.

[9] The reference in [3], p.44. The decomposition space \( \mathcal{D}_f \) of \( S \) can have various types of topological structure. For example, if \( X \) is a dendrite (the reference in [3], p.165), also \( \mathcal{D}_f \) must be dendrite.

[10] To topologize the set \( \mathcal{D}_f \), we use the metric \( \rho \) defined by means of the homeomorphism \( h \) rather than the Vietoris topology (see, for example, A. Illanes and S.B. Nadler Jr., *Hyperspaces* (Marcel Dekker, 1999) p.9) which has been customarily employed for the topological discussions of the phenomena in the Chaos-Fractal sciences (see, for example, J. Banks: Chaos, Solitons & Fractals 25 (2005) 681). Since \( X \) and \( \mathcal{D}_f \) are homeomorphic, the employment of the metric topology \( \tau_\rho \) which is identical with the decomposition topology \( \tau(\mathcal{D}_f) \) seems to be quite natural.
[11] The reference in [3], p.109.

[12] One of the simplest example of such decomposition $\mathcal{D}_f$ is as follows.
Let the self-similar, perfect, 0-dim, compact metric space $S$ be CMTS itself and let the partition \{S_1, S_2\} of $S$ be the set \{CMTS $\cap$[0, 1/3], CMTS $\cap$[2/3, 1]\}. Then, the set \{${f^{-1}(x)} \subset$ CMTS $\ , x \in S_1$\} where $f^{-1}(x) = \{x\} \subset X$ for $x \in S_1 - \{q\}$ and $f^{-1}(q) = \{q\} \cup S_2$, is a decomposition $\mathcal{D}_f$ of CMTS.

[13] R.L. Devaney: *An introduction to chaotic dynamical systems* (Westview Press, 2003) 2nd ed., p.35.

[14] A. Kitada and Y. Ogasawara: Chaos, Solitons & Fractals 24 (2005) 785; A. Kitada and Y. Ogasawara: Chaos, Solitons & Fractals 25 (2005) 1273.

[15] A. Kitada, T. Konishi and T. Watanabe: Chaos, Solitons & Fractals 13 (2002) 363.

[16] S. Nakamura, T. Konishi and A. Kitada, J. Phys. Soc. Jpn. 64 (1995) 731.

[17] It is noted that continuous maps $k, k^1, k^2, \cdots$ must not be one to one. In fact, if they are one to one, they must be homeomorphisms between 0-dim (i.e., totally disconnected) spaces $S, \mathcal{D}^1, \mathcal{D}^2, \cdots$ and a connected space $\delta$. It is impossible.
Figure 1: A hierarchic structure of self-similar spaces. \( h^i, i = 1, 2, \cdots \) are homeomorphisms. \( f_j, f_j^1, f_j^2, \cdots \) are weak contractions such that \( \bigcup_{j=1}^{m} f_j(S) = S \), \( \bigcup_{j=1}^{m} f_j^1(D^1) = D^1 \), \( \bigcup_{j=1}^{m} f_j^2(D^2) = D^2 \), \cdots \), respectively.
Figure 2: a) $F_\mu(x) = \mu x(1 - x), \quad \mu > 4, \quad x \in [0, 1]$. b) The quadratic dynamics $F_\mu(x)$ defines a system of contractions $\{f_j : [0, 1] \to [0, 1], j = 1, 2\}$ which satisfies three conditions i), ii), iii) in Statement in the text. In fact, $\bigcup_{j=1,2} \{x \in [0, 1]; f_j(x) = x\} = \{0, a\}$. 
Figure 3: Generation of dendrites from each step of the sequence $S, D^1, D^2, \ldots$. $\delta, \delta_S, \delta_{D^1}, \delta_{D^2}, \ldots$ are dendrites.