MINIMAL TRANSLATION SURFACES IN THE HEISENBERG GROUP $\text{Nil}_3$

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Abstract. A translation surface in the Heisenberg group $\text{Nil}_3$ is a surface constructed by multiplying (using the group operation) two curves. We completely classify minimal translation surfaces in the Heisenberg group $\text{Nil}_3$.

1. Introduction

A surface $M$ in the Euclidean space is called a translation surface if it is given by the graph $z(x, y) = f(x) + g(y)$, where $f$ and $g$ are smooth functions on some interval of $\mathbb{R}$. In [15], Scherk proved that, besides the planes, the only minimal translation surfaces are given by

$$z(x, y) = \frac{1}{a} \log \left| \frac{\cos(ax)}{\cos(ay)} \right|,$$

where $a$ is a non-zero constant. These surfaces are now referred as Scherk’s minimal surfaces.

The study of translation surfaces in the Euclidean space was extended when the second fundamental form was considered as a metric on a non-developable surface. A classification is given for translation surfaces for which the second Gaussian curvature and the mean curvature are proportional [14]. When the ambient is the Minkowski 3-space, translation surfaces of Weingarten type are classified [5]. In [5], translation surfaces with vanishing second Gaussian curvature in Euclidean and Minkowski 3-space are studied.

In the last decade, there has been an intensive effort to develop the theory of surfaces in homogeneous Riemannian 3-spaces of non-constant curvature. Since the discovery of holomorphic quadratic differential (called generalized Hopf differential or Abresch-Rosenberg differential) for constant mean curvature surfaces in 3-dimensional homogeneous Riemannian spaces with 4-dimensional isometry group, global geometry of constant mean curvature surfaces in such spaces has been extensively studied. We refer the survey [1], [7] or lecture notes [4] and references therein. In particular, integral representation formulae for minimal surfaces in the Heisenberg group $\text{Nil}_3$ were obtained independently in [3, 6, 10]. Some fundamental examples of minimal surfaces are constructed in [11]. Berdinskiǐ and Taimanov [2] gave a representation formula for minimal surfaces in 3-dimensional Lie groups in terms of spinors and Dirac operators.

Translation surfaces can be defined in any 3-dimensional Lie groups equipped with left invariant Riemannian metric. Some examples of minimal translation surfaces in the Heisenberg group $\text{Nil}_3$ are obtained in [11]. In our previous paper [13], we have classified minimal translation surfaces in the model space $\text{Sol}_3$ of solvgeometry in the sense of Thurston [16].
The purpose of this article is to study and classify minimal translation surfaces of \( \text{Nil}_3 \).

## 2. Preliminaries

### The Heisenberg group
\( \text{Nil}_3 \) is defined as \( \mathbb{R}^3 \) with the group operation
\[
(x, y, z) \ast (\overline{x}, \overline{y}, \overline{z}) = \left( x + \overline{x}, y + \overline{y}, z + \overline{z} + \frac{x\overline{y}}{2} - \frac{y\overline{x}}{2} \right).
\]
The identity of the group is \((0, 0, 0)\) and the inverse of \((x, y, z)\) is given by \((-x, -y, -z)\). It is simply connected and connected 2-step nilpotent Lie group \([16]\). The following metric is left invariant
\[
\tilde{g} = dx^2 + dy^2 + \left( dz + \frac{1}{2}(y \, dx - x \, dy) \right)^2.
\]
The resulting Riemannian manifold \((\text{Nil}_3, \tilde{g})\) is the model space of nilgeometry in the sense of Thurston \([16]\).

The following vector fields form a left invariant orthonormal frame on \( \text{Nil}_3 \):
\[
e_1 = \partial_x - \frac{y}{2} \partial_z, \quad e_2 = \partial_y + \frac{x}{2} \partial_z, \quad e_3 = \partial_z.
\]
The geometry of \( \text{Nil}_3 \) can be described in terms of this frame as follows.

1. These vector fields satisfy the commutation relations
   \[
   [e_1, e_2] = e_3, \quad [e_2, e_3] = 0, \quad [e_3, e_1] = 0.
   \]
2. The Levi-Civita connection \( \tilde{\nabla} \) of \( \text{Nil}_3 \) is given by
   \[
   \tilde{\nabla}_{e_1} e_1 = 0, \quad \tilde{\nabla}_{e_1} e_2 = \frac{1}{2} e_3, \quad \tilde{\nabla}_{e_1} e_3 = -\frac{1}{2} e_2, \\
   \tilde{\nabla}_{e_2} e_1 = -\frac{1}{2} e_3, \quad \tilde{\nabla}_{e_2} e_2 = 0, \quad \tilde{\nabla}_{e_2} e_3 = \frac{1}{2} e_1, \\
   \tilde{\nabla}_{e_3} e_1 = -\frac{1}{2} e_2, \quad \tilde{\nabla}_{e_3} e_2 = \frac{1}{2} e_1, \quad \tilde{\nabla}_{e_3} e_3 = 0.
   \]
3. The Riemann-Christoffel curvature tensor \( \tilde{R} \) of \( \text{Nil}_3 \) is determined by
   \[
   \tilde{R}(X, Y)Z = -\frac{3}{4} \left( \tilde{g}(Y, Z)X - \tilde{g}(X, Z)Y \right) \\
   + \left( \tilde{g}(Y, e_3)\tilde{g}(Z, e_3)X - \tilde{g}(X, e_3)\tilde{g}(Z, e_3)Y \right) \\
   + \tilde{g}(X, e_3)\tilde{g}(Y, Z)e_3 - \tilde{g}(Y, e_3)\tilde{g}(X, Z)e_3,
   \]
   for \( p \in \text{Nil}_3 \) and \( X, Y, Z \in T_p \text{Nil}_3 \).

The Heisenberg group \( \text{Nil}_3 \) has a four-dimensional isometry group generated by left translations \( g \mapsto hg \), \( h \in \text{Nil}_3 \), and by rotations around z-axis. Moreover the action of the isometry group is transitive. The Heisenberg group is represented by \( \text{Nil}_3 = \text{Nil}_3 \times \text{SO}(2)/\text{SO}(2) \) as a homogeneous Riemannian 3-space.

### Minimal surfaces in \( \text{Nil}_3 \)

Let \( r : M \rightarrow \text{Nil}_3 \) be an orientable surface, isometrically immersed in \( \text{Nil}_3 \). Denote by \( g := r^*\tilde{g} \) (resp. \( \tilde{\nabla} \)) the induced metric (resp. Levi-Civita connection) on \( M \). For later use we write down the Gauss and the Weingarten formulae
\[
\text{(G)} \quad \tilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y)N, \quad \sigma(X, Y) = \tilde{g}(\tilde{\nabla}_X Y, N),
\]
\[
\text{(W)} \quad \tilde{\nabla}_X N = -AX,
\]
where $X, Y$ are tangent to $M$ and $N$ is a unit normal to $M$. Here $\sigma$ denotes the scalar valued second fundamental form of the immersion, $A$ is known as the shape operator associated to $N$. The shape operator $A$ is self-adjoint with respect to the metric $g$ on $M$ and it is related to $\sigma$ by $\sigma(X, Y) = g(AX, Y)$. The mean curvature of the immersion is defined as $H = \frac{1}{2} \text{trace}(A)$, in any point of $M$. At each tangent plane $T_p M$ we take a basis $\{r_u, r_v\}$, where $u, v$ are local coordinates on $M$. Denote by $E, F, G$ the coefficients of the first fundamental form on $M$:

$$E = \bar{g}(r_u, r_u), \quad F = \bar{g}(r_u, r_v), \quad G = \bar{g}(r_v, r_v).$$

We find

$$H = \frac{G\langle N, \bar{\nabla}_E E_1 \rangle - 2F\langle N, \bar{\nabla}_E E_2 \rangle + E\langle N, \bar{\nabla}_E E_2 \rangle}{2(EG - F^2)},$$

where $\{E_1, E_2\}$ form an arbitrary basis on the surface $M$. As we are interested in minimal surfaces, i.e. surfaces with $H = 0$, in the above expression of $H$, we may change $N$ by other proportional vector $\bar{N}$. Then $M$ is a minimal surface if and only if

$$(2) \quad G\langle N, \bar{\nabla}_E E_1 \rangle - 2F\langle N, \bar{\nabla}_E E_2 \rangle + E\langle N, \bar{\nabla}_E E_2 \rangle = 0.$$

### 3. Minimal translation surfaces

In this section we define and study translation surfaces in $\text{Nil}_3$. A translation surface $M(\gamma_1, \gamma_2)$ in $\text{Nil}_3$ is a surface parametrized by

$$r : M \rightarrow \text{Nil}_3, \quad r(x, y) = \gamma_1(x) \ast \gamma_2(y),$$

where $\gamma_1$ and $\gamma_2$ are curves situated in the planes of coordinates of $\mathbb{R}^3$. Since the multiplication $\ast$ is not commutative, for each choice of curves $\gamma_1$ and $\gamma_2$ we may construct two translation surfaces, namely $M(\gamma_1, \gamma_2)$ and $M(\gamma_2, \gamma_1)$, which are different. Consequently, we distinguish six types of translation surfaces in $\text{Nil}_3$.

#### 3.1. Surfaces of type 1

Let the curves $\gamma_1$ and $\gamma_2$ be given by $\gamma_1(x) = (x, 0, u(x))$ and $\gamma_2(y) = (0, y, v(y))$. The translation surface $M(\gamma_1, \gamma_2) = \gamma_1(x) \ast \gamma_2(y)$ of type 1 is thus parameterized as

$$r(x, y) = (x, 0, u(x)) \ast (0, y, v(y)) = \left(x, y, u(x) + v(y) + \frac{xy}{2}\right).$$

The minimality condition (2) yields the following ODE

$$(3) \quad u''(x) (1 + v'(y)^2) - (u'(x) + y) v'(y) + v''(y) \left[1 + (u'(x) + y)^2\right] = 0. $$

In order to solve it, divide first by $1 + v'(y)^2 \neq 0$. We get

$$u''(x) - (u'(x) + y) \frac{v'(y)}{1 + v'(y)^2} + \frac{v''(y)}{1 + v'(y)^2} \left[1 + (u'(x) + y)^2\right] = 0,$$

for all $x, y$ in the domain or $r$. Taking the derivative with respect to $x$, we obtain

$$u'''(x) - u''(x) \frac{v'(y)}{1 + v'(y)^2} + 2 \frac{v''(y)}{1 + v'(y)^2} (u'(x) + y) u''(x) = 0.$$
The case \( u''(x) = 0 \) will be treated separately. Let us suppose now that \( u''(x) \neq 0 \) on an open interval, and divide by \( u''(x) \). It follows

\[
\frac{u'''(x)}{u''(x)} - \frac{v'(y)}{1 + v'(y)^2} + 2 \frac{v''(y)}{1 + v'(y)^2} (u'(x) + y) = 0.
\]

Taking the derivative with respect to \( y \), we get

\[
- \frac{d}{dy} \left( \frac{v'(y)}{1 + v'(y)^2} \right) + 2 (u'(x) + y) \frac{d}{dy} \left( \frac{v''(y)}{1 + v'(y)^2} \right) + 2 \frac{v''(y)}{1 + v'(y)^2} = 0.
\]

If \( \frac{d}{dy} \left( \frac{v''(y)}{1 + v'(y)^2} \right) \neq 0 \) it follows that \( u'(x) + y \) depends only on \( y \) yielding \( u'(x) = \text{constant} \), and hence \( u''(x) = 0 \), which is a contradiction.

So \( \frac{v''(y)}{1 + v'(y)^2} = A, \ A \in \mathbb{R} \). Replacing it in (4), one obtains

\[
- \frac{v''(y)}{1 + v'(y)^2} + 2 \frac{v'(y)^2 v''(y)}{(1 + v'(y)^2)^2} + 2A = 0,
\]

equivalently to

\[
A \left( 1 + \frac{2v'(y)^2}{1 + v'(y)^2} \right) = 0.
\]

It follows that \( A = 0 \), and hence \( v''(y) = 0 \).

Summarizing, we proved that for a minimal translation surface of type 1, we should have either \( u'' = 0 \) or \( v'' = 0 \).

Moreover, if \( v'' = 0 \), then \( v'(y) = c, \ c \in \mathbb{R} \). Replacing in (3), one obtains

\[
u''(x)(1 + c^2) - c(u(x) + y) = 0, \ \forall \ x, y.
\]

Hence, \( c = 0 \) and \( u''(x) = 0 \). Consequently, for any minimal translation surface of type 1, we have \( u''(x) = 0 \).

Take \( u(x) = ax + u_0, \ a, u_0 \in \mathbb{R} \). Replacing in (3) we get

\[-(a + y)v'(y) + v''(y) (1 + (a + y)^2) = 0,
\]

which has the solution

\[
v(y) = c \left[ (a + y)\sqrt{1 + (a + y)^2} + \ln(a + y + \sqrt{1 + (a + y)^2}) \right] + v_0, \ c, v_0 \in \mathbb{R}.
\]

We conclude with the following

**Theorem 3.1.** Minimal translation surfaces of type 1 in the Heisenberg group \( \text{Nil}_3 \) are parameterized by

\[
r(x, y) = (x, 0, u(x)) \ast (0, y, v(y)) = \left( x, y, u(x) + v(y) + \frac{xy}{2} \right),
\]

where \( u(x) = ax + u_0, \ a, u_0 \in \mathbb{R} \) and \( v(y) \) is given by (5).
3.2. **Surfaces of type 4.** Considering the same curves as in previous case, define the translation surface of type 4 by \( M(\gamma_2, \gamma_1) = \gamma_2(y) \ast \gamma_1(x) \). Similar computations yield the following equation

\[
(1 + u'(x)^2)v''(y) + (v'(y) - x)u'(x) + (1 + (v'(y) - x)^2)u''(x) = 0.
\]

We may state the next result.

**Theorem 3.2.** Minimal translation surfaces of type 4 in the Heisenberg group \( \text{Nil}_3 \) are parameterized by

\[
(7) \quad r(x, y) = (0, y, v(y)) \ast (x, 0, u(x)) = \left( x, y, u(x) + v(y) - \frac{xy}{2} \right).
\]

where \( u(x) = c \left[ (a + x)\sqrt{1 + (a + x)^2} + \ln(a + x + \sqrt{1 + (a + x)^2}) \right] + u_0, \)

and \( v(y) = -ay + v_0 \), with \( a, c, u_0, v_0 \in \mathbb{R} \).

**Remark 3.3.** If \( a \) and \( u_0 \) (resp. \( a \) and \( v_0 \)) vanish in Theorem 3.1 (resp. Theorem 3.2), then the surface \( M \) is a left cylinder over the curve \( \gamma_2(y) = (0, y, v(y)) \) (resp. over the curve \( \gamma_1(x) = (x, 0, u(x)) \)). See [11].

3.3. **Surfaces of type 2.** Let us take the two curves as follows: \( \gamma_1(x) = (x, 0, u(x)) \) and \( \gamma_2(y) = (v(y), y, 0) \). Then, the translation surface \( M(\gamma_1, \gamma_2) \) of type 2 is parametrized by

\[
r(x, y) = \gamma_1(x) \ast \gamma_2(y) = \left( x + v(y), y, u(x) + \frac{xy}{2} \right).
\]

The minimality condition \([2]\) becomes:

\[
2v(y) \left( y + u'(x) - yv'(y)u''(x) \right) - 2v'(y) \left( 2 + y^2 + yu'(x) \right) - 2v''(y) \left( y + u'(x) \right) \left( 1 + y^2 + 2yu'(x) + u''(x) \right) + u''(x) \left( 4 + v(y)^2 + (4 + y^2)v'(y)^2 \right) = 0.
\]

Let us make some notations:

\[
T_0(y) = 4 + 4v'(y)^2 + (v(y) - yv'(y))^2,
T_1(y) = 2 \left( v(y) - yv'(y) - 2(1 + 2y^2)v''(y) \right),
T_2(y) = -10yv''(y),
T_3(y) = -4v''(y),
T_4(y) = 2vy(y) - 2(2 + y^2)v'(y) - 2y(1 + y^2)v''(y).
\]

The previous equation may be rewritten as

\[
(9) \quad T_0(y)v''(x) + T_1(y)u'(x) + T_2(y)u'(x)^2 + T_3(y)u'(x)^3 + T_4(y) = 0.
\]

Since \( T_0 \) cannot vanish, divide \([9]\) by \( T_0 \) and then take the derivatives with respect to \( x \) and \( y \) respectively. We get

\[
\frac{d}{dy} \left( \frac{T_1}{T_0} \right) u''(x) + 2 \frac{d}{dy} \left( \frac{T_2}{T_0} \right) u'(x)u''(x) + 3 \frac{d}{dy} \left( \frac{T_3}{T_0} \right) u'(x)^2u''(x) = 0.
\]
The case \( u''(x) = 0 \) will be discussed separately. Suppose now that \( u''(x) \neq 0 \). Dividing by \( u''(x) \), and continuing the procedure we obtain
\[
\frac{d}{dy} \left( \frac{T_1}{T_0} \right) = 0, \quad \frac{d}{dy} \left( \frac{T_2}{T_0} \right) = 0, \quad \frac{d}{dy} \left( \frac{T_3}{T_0} \right) = 0.
\]
Looking at the form of \( T_2 \) and \( T_3 \) we conclude that \( v''(y) = 0 \), hence \( v(y) = ay + d \), with \( a, d \in \mathbb{R} \). The minimality condition becomes
\[
-4a + 2dy + 2du'(x) + (4 + 4a^2 + d^2)u''(x) = 0,
\]
for all \( x \) and \( y \). It follows that \( d = 0 \) and \( u(x) = \frac{a}{2(1 + a^2)} x^2 + bx + c \), with \( b, c \in \mathbb{R} \).

Return to the remained case \( u''(x) = 0 \). Then \( u(x) = ax + u_0 \) and the minimality condition becomes
\[
(a + y)v(y) - [2 + y(a + y)]v'(y) - 2a + y[(1 + (a + y)^2)v''(y) = 0.
\]
Denote \( w(y) = (2a + y)v(y) \). The equation above may be written as
\[
(a + y)w'(y) = [1 + (a + y)^2]w''(y).
\]
If \( w = \text{constant} \) then \( v(y) = y + 2a \). If \( w'(y) \neq 0 \) then we may solve the ODE obtaining
\[
w(y) = \frac{b}{2a + y} + \frac{c}{2(2a + y)} \left[ (a + y)\sqrt{1 + (a + y)^2} + \ln(a + y + \sqrt{1 + (a + y)^2}) \right] + b, \quad b, c \in \mathbb{R}.
\]
The case \( w = \text{constant} \) may be included in the previous expression. Hence
\[
(10) \quad v(y) = \frac{b}{2a + y} + \frac{c}{2(2a + y)} \left[ (a + y)\sqrt{1 + (a + y)^2} + \ln(a + y + \sqrt{1 + (a + y)^2}) \right].
\]
We conclude with the following

**Theorem 3.4.** Minimal translation surfaces of type 2 in the Heisenberg group Nil_3 are parameterized by
\[
(11) \quad r(x, y) = (x, 0, u(x)) \ast (v(y), y, 0) = \left( x + v(y), y, u(x) + \frac{xy}{2} \right),
\]
where
(i) either \( u(x) = \frac{a}{2(1 + a^2)} x^2 + bx + c \) and \( v(y) = ay + d \), with \( a, b, c, d \in \mathbb{R} \),
(ii) or \( u(x) = ax + u_0 \) and \( v(y) \) is given by \( (10) \), with \( a, u_0 \in \mathbb{R} \).

**Remark 3.5.** If \( a \) and \( d \) in (i) vanish (resp. \( b \) and \( c \) in (ii)), then \( M \) is a right cylinder over the curve \( \gamma_1(x) = (x, 0, u(x)) \). See also [11].

### 3.4. Surfaces of type 5.

Taking the two curves as in the previous case: \( \gamma_1(x) = (x, 0, u(x)) \) and \( \gamma_2(y) = (v(y), y, 0) \), the translation surface \( M(\gamma_2, \gamma_1) \) of type 5 is parametrized by
\[
r(x, y) = \gamma_2(y) \ast \gamma_1(x) = \left( x + v(y), y, u(x) - \frac{xy}{2} \right).
\]
The two parametrizations of \( M(\gamma_2, \gamma_1) \) and \( M(\gamma_1, \gamma_2) \) look very similar, yet they are quite different.

Straightforward computations yield the following minimality condition
\[
u''(x) \left[ -2y(v(y) + 2x)v'(y) + (y^2 + 4)v'(y)^2 + (v(y) + 2x)^2 + 4 \right] - 4u'(x)^3v''(y)
+ 2yu'(x)^2v''(y) - 2u'(x) [2(v''(y) + x) - yv'(y) + v(y)]
+ 2(yv''(y) + 2v'(y)) = 0.
\]

Let us make some notations:

\[ P_1(x, y) = 2x + v(y) - yv'(y), \]
\[ P_2(y) = 2v'(y), \]
\[ P_3(y) = P_2'(y) = 2v''(y), \]
\[ P_4(y) = 2yv''(y). \]

The equation (12) may be rewritten as

\[ (4 + P_1^2 + P_2^2) u''(x) - 2(P_1 + P_3)u'(x) + P_4u'(x)^2 - 2P_3u'(x)^3 + (2P_2 + P_4) = 0. \]

Taking the derivative with respect to \( x \) we obtain

\[ \frac{d}{dx} \left( 4 + P_1^2 + P_2^2 \right) u''(x) + 2(P_1 - P_3)u''(x) - 4u'(x) + 2P_4u'(x)u''(x) - 6P_3u'(x)^2u''(x) = 0. \]

(i) \( u''(x) = 0 \): From (15) it follows \( u'(x) = 0 \) and hence \( u = u_0, u_0 \in \mathbb{R} \). Combining with (14) we obtain \( 2P_2 + P_4 = 0 \). This ODE has the general solution \( v(y) = \frac{a}{y} + b, a, b \in \mathbb{R} \).

(ii) \( u''(x) \neq 0 \): Dividing in (14) by \( u''(x) \) and then taking successively the derivatives with respect to \( x \) and \( y \) one obtains

\[ 2v'(y)v''(y) \frac{1}{u''(x)} \frac{d}{dx} \left( \frac{u'''(x)}{u''(x)} \right) - yu''(y) \frac{u'''(x)}{u''(x)^2} + (v''(y) + yv'''(y)) - 6v''(y)u'(x) = 0. \]

Taking one more derivative with respect to \( x \) and considering

\[ A(x) = \frac{1}{u''(x)} \frac{d}{dx} \left( \frac{1}{u''(x)} \frac{d}{dx} \left( \frac{u'''(x)}{u''(x)} \right) \right), \]
\[ B(x) = \frac{1}{u''(x)} \frac{d}{dx} \left( \frac{u'''(x)}{u''(x)^2} \right), \]

we get

\[ 2v'(y)v''(y)A(x) - yu''(y)B(x) = 6v''(y). \]

(a) If \( v''(y) = 0 \) then \( v(y) = ay + b, a, b \in \mathbb{R} \). Replacing it in (14) we obtain the following ODE

\[ [4 + 4a^2 + (2x + b)^2]u''(x) - 2(2x + b)u'(x) + 4a = 0, \]

with the solution

\[ u(x) = \frac{-ax^2 + abx + c}{2(1 + a^2)} + c_1 \left( \frac{x}{2} + \frac{b}{4} \right) \sqrt{4(1 + a^2) + (2x + b)^2} \]
\[ + c_1(1 + a^2) \ln \left( 2x + b + \sqrt{4(1 + a^2) + (2x + b)^2} \right), \quad c, c_1 \in \mathbb{R}. \]

(b) If \( v''(y) \neq 0 \), it follows either \( A \) and \( B \) are constants, or \( v'(y) = cy, c \neq 0 \) and \( B = 2cA \).

It is not difficult to show that each of the two situations yields a contradiction.

We conclude with the following
Theorem 3.6. Minimal translation surfaces of type 5 in the Heisenberg group \( \text{Nil}_3 \) are parameterized by
\[
 r(x, y) = (v(y), y, 0) * (x, 0, u(x)) = \left( x + v(y), y, u(x) - \frac{xy}{2} \right),
\]
where
(i) either \( u(x) = u_0 \) and \( v(y) = \frac{a}{y} + b \), with \( u_0, a, b \in \mathbb{R} \)
(ii) or \( v(y) = ay + b \) and \( u(x) \) is given by \([16]\).

Remark 3.7. In the case (i), the curve \( \gamma_1(x) \) is a geodesic. Thus \( M(\gamma_2, \gamma_1) \) is a right translation of \( \gamma_2(y) \) by a geodesic \( \gamma_1(x) \). If \( a \) and \( b \) vanish, then the surface \( M \) is a left cylinder over the curve \( \gamma_1(x) = (x, 0, u(x)) \).

3.5. Surfaces of type 3. Let \( \gamma_1(x) = (0, x, u(x)) \) and \( \gamma_2(y) = (y, v(y), 0) \) be the two curves defining the translation surface \( M(\gamma_1, \gamma_2) \), which is parameterized as
\[
 r(x, y) = \gamma_1(y) * \gamma_2(x) = \left( y, x + v(y), u(x) - \frac{xy}{2} \right).
\]
Using the notations given by \([8]\), the minimality equation may be written as
\[
 T_0(y)u''(x) + T_1(y)u'(x) - T_2(y)u'(x)^2 + T_3(y)u'(x)^3 - T_4(y) = 0.
\]

Applying the same technique as in the case of surfaces of type 2, we obtain

Theorem 3.8. Minimal translation surfaces of type 3 in the Heisenberg group \( \text{Nil}_3 \) are parameterized by
\[
 r(x, y) = (x, 0, u(x)) * (v(y), y, 0) = \left( x + v(y), y, u(x) + \frac{xy}{2} \right)
\]
where
(i) either \( u(x) = \frac{ax^2 + bx + c}{2(1 + a^2)} \) and \( v(y) = -ay + d \), with \( a, b, c, d \in \mathbb{R} \),
(ii) or \( u(x) = ax + u_0 \) and
\[
 v(y) = \frac{b}{2a - y} + \frac{c}{2(2a - y)} \left[ (a - y)\sqrt{1 + (a - y)^2} + \ln(a - y + \sqrt{1 + (a - y)^2}) \right],
\]
with \( a, b, c, u_0 \in \mathbb{R} \).

3.6. Surfaces of type 6. Let us consider the two curves as in previous case and define the translation surface \( M(\gamma_2, \gamma_1) = \gamma_2(y) * \gamma_1(x) \).

We can state the following result.

Theorem 3.9. Minimal translation surfaces of type 6 in the Heisenberg group \( \text{Nil}_3 \) are parameterized by
\[
 r(x, y) = (y, v(y), 0) * (0, x, u(x)) = \left( y, x + v(y), u(x) + \frac{xy}{2} \right),
\]
where
(i) either \( u(x) = u_0 \) and \( v(y) = \frac{a}{y} + b \), with \( u_0, a, b \in \mathbb{R} \)
(ii) or \( v(y) = ay + b \) and
\[
 u(x) = \frac{ax^2 + abx + c}{2(1 + a^2)} + c_1 \left( \frac{x}{2} + \frac{b}{4} \right) \sqrt{4(1 + a^2) + (2x + b)^2}
 + c_1(1 + a^2) \ln \left( 2x + b + \sqrt{4(1 + a^2) + (2x + b)^2} \right),
\]
with \( a, b, c, c_1 \in \mathbb{R} \).
**Remark 3.10.** We may construct minimal translation surfaces which are cylinders also for types 3 and 6.

**Remark 3.11.** Minimal translation surfaces of types 1-4 are also flat.

**Remark 3.12.** Some missing cases:

In general, when we consider a curve in a certain plane of coordinates, parameterized by $s$ (not necessary the arc-length), namely $s \mapsto (\alpha(s), \beta(s))$, with $\alpha'(s)^2 + \beta'(s) \neq 0$, we may write the curve also in the explicit form, as $(t, u(t))$ and this can be done if $\alpha'(s) \neq 0$. So, we have also to consider curves given by $(c, t)$, corresponding to $\alpha'(s) = 0$ and $\beta'(s) \neq 0$.

Having this in mind, we describe the missing cases in our classifications:

| type | parametrization | type | parametrization |
|------|----------------|------|----------------|
| 1    | $(c, y, x + v(y) + \frac{c}{2})$ | 4    | $(c, y, x + v(y) - \frac{c}{2})$ |
| 1    | $(x, c, u(x) + y + \frac{c}{2})$ | 4    | $(x, c, u(x) + y - \frac{c}{2})$ |
| 2*   | $(c + v(y), y, x + \frac{c}{2})$ | 5*   | $(c + v(y), y, x - \frac{c}{2})$ |
| 2    | $(x + y, c, u(x) + \frac{c}{2})$ | 5    | $(x + y, c, u(x) - \frac{c}{2})$ |
| 3*   | $(y, c + v(y), x - \frac{c}{2})$ | 6*   | $(y, c + v(y), x + \frac{c}{2})$ |
| 3    | $(c, x + y, u(x) - \frac{c}{2})$ | 6    | $(c, x + y, u(x) + \frac{c}{2})$ |

Except the cases marked by * symbol, the others are all minimal translation surfaces. The stared cases are minimal if and only if $v$ is an affine function.

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**References**

[1] U. Abresch, H. Rosenberg, *Generalized Hopf differentials*, Mat. Contemp. 28 (2005), 1–28.

[2] D. A. Berdinski˘ı, I. A. Taımanov, *Surfaces in three-dimensional Lie groups*, Siberian Math. J. 46 (2005) 6, 1005–1019.

[3] B. Daniel, *The Gauss map of minimal surfaces in the Heisenberg group*, Int Math Res Notices (2011) 3, 674–695.

[4] B. Daniel, L. Hauswirth, P. Mira, *Constant mean curvature surfaces in homogeneous 3-manifolds*, Lectures Notes of the 4th KIAS Workshop on Diff. Geom. *Constant mean curvature surfaces in homogeneous manifolds*, Seoul, 2009.

[5] F. Dillen, W. Goemans, I. Van de Woestyne, *Translation surfaces of Weingarten type in 3-space*, Proc. Conf. RIGA 2008, Bull. Transilvania Univ. Brasov, 15 (2008) 50, 1–14.

[6] C. B. Figueroa, *On the Gauss map of a minimal surface in the Heisenberg group*, Mat. Contemp. 33 (2007), 139–156.

[7] I. Fernández, P. Mira, *Constant mean curvature surfaces in 3-dimensional Thurston geometries*, to appear in Proceedings on the ICM 2010, Hyderabad, arXiv:1004.4752v1 [math.DG] (2010).

[8] W. Goemans, I. Van de Woestyne, *Translation surfaces with vanishing second Gaussian curvature in Euclidean and Minkowski 3-space*, Proceedings of the conference Pure and Applied Differential Geometry, PADGE 2007, Eds. F. Dillen, I. Van de Woestyne, 123–131.
[9] J. Inoguchi, *Flat translation invariant surfaces in the 3-dimensional Heisenberg group*, J. Geom. 82 (2005), 83–90.
[10] J. Inoguchi, *Minimal surfaces in the 3-dimensional Heisenberg group*, Diff. Geom. Dyn. Syst. 10 (2008), 163–169.
[11] J. Inoguchi, T. Kumamoto, N. Ohsugi, Y. Suyama, *Differential geometry of curves and surfaces in 3-dimensional homogeneous spaces*, II, Fukuoka Univ. Sci. Reports 30 (2000) 1, 17–47.
[12] J. Inoguchi, M. I. Munteanu, *Minimal translation surfaces in the hyperbolic 3-space*, in preparation.
[13] R. López, M. I. Munteanu, *Minimal translation surfaces in Sol3*, to appear in J. Math. Soc. Japan.
[14] M. I. Munteanu, A. I. Nistor, *On the geometry of the second fundamental form of translation surfaces in \( \mathbb{E}^3 \)*, Houston J. Math. 37 (2011) 4, xxx - xxx.
[15] H. F. Scherk, *Bemerkungen über die kleinste Fläche innerhalb gegebener Grenzen*, J. Reine Angew. Math. 13 (1835), 185–208.
[16] W. Thurston, *Three-dimensional geometry and topology*, Princeton Math. Ser. 35, Princeton Univ. Press, Princeton, NJ, 1997.

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