Abstract. The images of Hermite and Laguerre Sobolev spaces under the Hermite and special Hermite semigroups (respectively) are characterised. These are used to characterise the image of Schwartz class of rapidly decreasing functions \(f\) on \(\mathbb{R}^n\) and \(\mathbb{C}^n\) under these semigroups. The image of the space of tempered distributions is also considered and a Paley-Wiener theorem for windowed (short-time) Fourier transform is proved.

1. Introduction

A classical result of Bargmann and Fock states that the image of \(L^2(\mathbb{R}^n)\) under the Gauss-Weierstrass semigroup can be described as a weighted Bergman space of entire functions. Such results are known for the semigroups generated by the Laplace-Beltrami operators on compact Lie groups and more generally on compact symmetric spaces. (See \[6\] and \[9\]). Similar results have been proved in the literature for Hermite and special Hermite semigroups as well. (See \[3\], \[8\] and \[12\] for further details). However, unlike the above cases, it was shown by Kr"{o}tz, Thangavelu and Xu in \[8\] that in the case of Heisenberg group \(\mathbb{H}^n\), the image of \(L^2(\mathbb{H}^n)\) under the heat kernel transform is not a weighted Bergman space but the direct sum of two weighted Bergman spaces. Bargmann in \[1\] obtained a characterization for the image of Schwartz class of rapidly decreasing functions on \(\mathbb{R}^n\) under the Segal-Bargmann transform. He showed that if \(F\) is a holomorphic function on \(\mathbb{C}^n\), then there exists a function \(f \in \mathcal{S}(\mathbb{R}^n)\) with \(F = C_tf\) (\(C_t\) denotes the Segal-Bargmann transform) if and only if \(F\) satisfies

2000 Mathematics Subject Classification. Primary 42B35; Secondary 46E20, 42C05, 33C45.

Key words and phrases. Bargmann transform, Hermite functions, Hermite semigroup, special Hermite functions, Short term Fourier transform, Sobolev space.
for some sequence of constants $A_n, n = 1, 2, 3, \cdots$. Hall and Lewkeeratiyutkul in [7] characterized the image of Sobolev spaces under the Segal-Bargmann transform on a compact Lie group $K$. They used this result to obtain a characterization for the functions in the image of $C^\infty(K)$ under this transform. Using Gutzmer’s formula the images of Sobolev spaces under the Segal-Bargmann transform on compact Riemannian symmetric spaces were characterised by Thangavelu recently in [13] extending the results of [7].

In this paper we characterize the image of Hermite Sobolev spaces under the Hermite semigroup and Laguerre Sobolev spaces under special Hermite semigroup. These results are then used to characterize the images of Schwartz space on $\mathbb{R}^n$ and $\mathbb{C}^n$ respectively under these semigroups. Throughout this paper, if $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$ then $x^2$ will denote $x_1^2 + x_2^2 + \cdots + x_n^2$.

2. Holomorphic Hermite Sobolev spaces

Let $h_k(x) = (2^k k! \pi^\frac{k}{2})^{-\frac{1}{2}}(-1)^k k^\frac{k}{2} (e^{-x^2}) e^{\frac{x^2}{2}}, k = 0, 1, 2, \cdots$ denote the normalised Hermite functions. The multi-dimensional Hermite functions are defined as follows: For $\alpha \in \mathbb{N}^n, x \in \mathbb{R}^n$, let $\Phi_\alpha(x) = \Pi_{i=1}^n h_{\alpha_i}(x_i)$. Then the collection $\{\Phi_\alpha : \alpha \in \mathbb{N}^n\}$ forms an orthonormal basis for $L^2(\mathbb{R}^n)$. These functions $\Phi_\alpha (\alpha \in \mathbb{N}^n)$ are eigenfunctions of the Hermite operator $H = -\Delta + \frac{1}{2} x^2$. Any $f \in L^2(\mathbb{R}^n)$, has the Hermite expansion $f = \sum_\alpha (f, \Phi_\alpha) \Phi_\alpha$. If $P_k$ denotes the orthogonal projection of $L^2(\mathbb{R}^n)$ onto the eigenspace spanned by $\{\Phi_\alpha : |\alpha| = k\}$, then the Hermite expansion can be written as $f = \sum_{k=0}^\infty P_k f$. For $f \in L^2(\mathbb{R}^n)$, the series converges in the norm and for other classes of functions $f$, this denotes the formal sum.

The Hermite operator $H$ defines a semigroup, called the Hermite semigroup denoted by $e^{-tH}, t > 0$ by the expansion

$$e^{-tH} f = \sum_{k=0}^\infty e^{-(2k+n)t} P_k f, f \in L^2(\mathbb{R}^n).$$

On a dense subspace, we can write $e^{-tH}$ as an integral operator with kernel $K_t(x, u)$,

$$e^{-tH} f(x) = \int_{\mathbb{R}^n} f(u) K_t(x, u) du.$$
Using Mehler’s formula (See eqn (1.1.36) in [11]), $K_t(x, u)$ can be explicitly written as

$$K_t(x, u) = (2\pi)^{-\frac{n}{2}}(\sinh(2t))^{-\frac{n}{2}}e^{-\frac{1}{2}\coth(2t)(x^2+u^2)+\frac{1}{2}\text{coth}(2t)}x.u.$$ 

The Hermite semigroup initially defined on $L^2 \cap L^p(\mathbb{R}^n)$ extends to the whole of $L^p(\mathbb{R}^n)$ and $\|e^{-tH}f\|_p \leq c_t\|f\|_p$, $1 \leq p \leq \infty$. It is easy to see that $K_t(x, u)$ can be extended to $\mathbb{C}^n$ as an entire function $K_t(z, w)$ and hence $e^{-tH}f$ can also be extended to $\mathbb{C}^n$ and the entire extension will be simply denoted by $e^{-tH}f(z)$, $z = x + iy$. It is well known that the image of $L^2(\mathbb{R}^n)$ under the Hermite semigroup is a weighted Bergman space. To be more precise, let $H_t(\mathbb{C}^n)$ denote the space of all entire functions on $\mathbb{C}^n$ which are square integrable with respect to the weight function $U_t(x + iy) = 2^n(\sinh 4t)^{-\frac{n}{2}}e^{\tanh(2t)x^2-\coth(2t)y^2}$. Then one has the following result which is due to Byun [3].

**Theorem 2.1.** The Hermite semigroup $e^{-tH} : L^2(\mathbb{R}^n) \rightarrow H_t(\mathbb{C}^n)$ is an isometric isomorphism.

The Hermite functions $\Phi_\alpha(x)$ have extension to $\mathbb{C}^n$ as entire functions $\Phi_\alpha(z)$, called complexified Hermite functions. They satisfy the orthogonality property

$$\int_{\mathbb{C}^n} \Phi_\alpha(z)\overline{\Phi_\beta(z)}U_t(z)dz = e^{2(2|\alpha|+n)t}\delta_{\alpha,\beta}. \tag{2.1}$$

Let $\widetilde{\Phi}_\alpha(z) = e^{-(2|\alpha|+n)t}\Phi_\alpha(z)$. Then $\{\widetilde{\Phi}_\alpha : \alpha \in \mathbb{N}^n\}$ forms an orthonormal basis for $H_t(\mathbb{C}^n)$. Thus any $F \in H_t(\mathbb{C}^n)$ can be written as $F = \sum_{\alpha} (\Phi_\alpha)_{H_t(\mathbb{C}^n)}\widetilde{\Phi}_\alpha$.

Hermite Sobolev spaces were studied by Thangavelu in [10] in connection with regularity properties of twisted spherical means. In order to give the definition, we consider the spectral decomposition of $H$:

$$Hf = \sum_{k=0}^{\infty} (2k + n)P_kf.$$ 

Then the Hermite Sobolev space $W_H^{m,2}(\mathbb{R}^n)$ is defined to be the image of $L^2(\mathbb{R}^n)$ under $H^{-m}$, where $m$ is a non negative integer. In other words, we say that $f \in W_H^{m,2}(\mathbb{R}^n)$ if and only if

$$\|f\|_{W_H^{m,2}}^2 = \sum_{k=0}^{\infty} (2k + n)^{2m}\|P_k f\|_2^2 < \infty.$$ 

The norm can be written as

$$\|f\|_{W_H^{m,2}}^2 = \sum_{\alpha} (2|\alpha| + n)^{2m}|(f, \Phi_\alpha)|^2.$$
The Sobolev space $W^{m,2}_H(\mathbb{R}^n)$ is a Hilbert space under the inner product
\[(f, g)_{W^{m,2}_H(\mathbb{R}^n)} = \sum_{\alpha} (2|\alpha| + n)^{2m} (f, \Phi_\alpha)(g, \Phi_\alpha).\]

As $H^m f = \sum_{\alpha} (2|\alpha| + n)^m (f, \Phi_\alpha)\Phi_\alpha$, the above inner product can also be rewritten as
\[(f, g)_{W^{m,2}_H(\mathbb{R}^n)} = (H^m f, H^m g)_{L^2(\mathbb{R}^n)}.
\]

We shall now define the holomorphic Sobolev space $W^{m,2}_t(\mathbb{C}^n)$ to be the image of $W^{m,2}_H(\mathbb{R}^n)$ under $e^{-tH}$. The space $W^{m,2}_t(\mathbb{C}^n)$ is made into a Hilbert space simply by transferring the Hilbert space structure of $W^{m,2}_H(\mathbb{R}^n)$ to $W^{m,2}_t(\mathbb{C}^n)$ so that the Hermite semigroup $e^{-tH}$ is an isometric isomorphism from $W^{m,2}_H(\mathbb{R}^n)$ onto $W^{m,2}_t(\mathbb{C}^n)$. This means that
\[(F, G)_{W^{m,2}_t(\mathbb{C}^n)} = \sum_{\alpha} (2|\alpha| + n)^{2m} (f, \Phi_\alpha)_{L^2(\mathbb{R}^n)}(g, \Phi_\alpha)_{L^2(\mathbb{R}^n)}
\]
whenever $F = e^{-tH} f$ and $G = e^{-tH} g$.

Let $O(\mathbb{C}^n)$ denote the collection of all holomorphic functions on $\mathbb{C}^n$. Let $F_t^m(\mathbb{C}^n)$ denote the space of all functions in $O(\mathbb{C}^n)$ which are square integrable with respect to the measure $\left| \frac{d^m}{dt^m} U_t(z) \right| dz$. We equip $F_t^m(\mathbb{C}^n)$ with the sesquilinear form
\[\begin{equation}
(F, G)_m = \int_{\mathbb{C}^n} F(z) \overline{G(z)} \frac{d^m}{dt^m} U_t(z) dz.
\end{equation}

We shall show below that this defines a pre-Hilbert space structure on $F_t^m(\mathbb{C}^n)$. Let $B_t^m(\mathbb{C}^n)$ denote the completion of $F_t^m(\mathbb{C}^n)$ with respect to the norm induced by the above inner product. In the following proposition, we also show that $||F||_m$ and $||F||_{W^{m,2}_t}$ coincide up to a constant multiple.

**Proposition 2.2.** The sesquilinear form
\[\begin{equation}
(F, G)_m = \int_{\mathbb{C}^n} F(z) \overline{G(z)} \frac{d^m}{dt^m} U_t(z) dz
\end{equation}

is an inner product on $F_t^m(\mathbb{C}^n)$ and hence induces a norm $||F||_m^2 = (F, F)_m$. We also have $||F||^2_m = 2^{2m}||F||_{W^{m,2}_t}$ for all functions $F = e^{-tH} f$, with $f \in S(\mathbb{R}^n)$.

**Proof.** Consider the integral
\[\begin{equation}
\int_{\mathbb{C}^n} |F(x + iy)|^2 U_t(x + iy) dx dy.
\end{equation}\n
Since the restriction of $F$ to $\mathbb{R}^n$ has an orthogonal expansion in terms of $\Phi_\alpha$, we have $F(x + iy) = \sum_\alpha (F, \Phi_\alpha)\Phi_\alpha(x + iy)$ where $(F, \Phi_\alpha) = \int_{\mathbb{C}^n} F(x + iy) \overline{\Phi_\alpha(x + iy)} dx dy$. Therefore, we have
\[\begin{equation}
\int_{\mathbb{C}^n} (F, G)_m = \sum_\alpha (2|\alpha| + n)^{2m} (F, \Phi_\alpha)(G, \Phi_\alpha)
\end{equation}\nwhich implies the proposition.

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\[\begin{equation}
\int_{\mathbb{C}^n} (F, G)_m = \sum_\alpha (2|\alpha| + n)^{2m} (F, \Phi_\alpha)(G, \Phi_\alpha)
\end{equation}\nwhich implies the proposition.
\[ \int_{\mathbb{R}^n} F(x) \Phi_\alpha(x) dx \]

Using the orthogonality relation (2.1), we can show that the integral (2.3) can be written as

\[ \sum_\alpha \left| (F, \Phi_\alpha) \right|^2 e^{2(2|\alpha|+n)t}. \]

By definition

\[ (F, F)_m = \int_{\mathbb{C}^n} |F(x+iy)|^2 \frac{d^{2m}}{dt^{2m}} U_t(z) \, dz \]

which is the same as

\[ \frac{d^{2m}}{dt^{2m}} \sum_\alpha \left| (F, \Phi_\alpha) \right|^2 e^{2(2|\alpha|+n)t} \]

and hence non-negative. Thus it follows that the sesquilinear form defined in (2.2) is positive definite and induces the norm \( \|F\|_m \). On the other hand if \( F \in H_t(\mathbb{C}^n) \), we have the expansion

\[ F(z) = \sum_\beta (F, \tilde{\Phi}_\beta)_{H_t(\mathbb{C}^n)} \tilde{\Phi}_\beta(z) \]

so that the restriction of \( F \) to \( \mathbb{R}^n \) can be written as

\[ F(x) = \sum_\beta (F, \tilde{\Phi}_\beta)_{H_t(\mathbb{C}^n)} \tilde{\Phi}_\beta(x). \]

Thus

\[ (F, \Phi_\alpha) = \int_{\mathbb{R}^n} \sum_\beta (F, \tilde{\Phi}_\beta)_{H_t(\mathbb{C}^n)} \tilde{\Phi}_\beta(x) \Phi_\alpha(x) dx \]

\[ = \int_{\mathbb{R}^n} \sum_\beta (f, \Phi_\beta) e^{-(2|\beta|+n)t} \Phi_\beta(x) \Phi_\alpha(x) dx \]

\[ = (f, \Phi_\alpha) e^{-(2|\alpha|+n)t} \]

where \( F = e^{-tH} f \). Again using (2.1), we get

\[ \|F\|_m^2 = 2^{2m} \sum_\alpha (2|\alpha| + n)^{2m} \left| (F, \Phi_\alpha) \right|^2 e^{2(2|\alpha|+n)t} \]

\[ = 2^{2m} \sum_\alpha (2|\alpha| + n)^{2m} \left| (f, \Phi_\alpha) \right|^2 \]

\[ = 2^{2m} \sum_\alpha (2|\alpha| + n)^{2m} \left| (F, \tilde{\Phi}_\alpha)_{H_t(\mathbb{C}^n)} \right|^2 \]

\[ = 2^{2m} \|F\|_{W_t^{m,2}}^2. \]
Using this proposition we can easily prove the following result on the image of Hermite Sobolev spaces under the Hermite semigroup.

**Theorem 2.3.** For every non-negative integer $m$, $W_{t}^{m,2}(\mathbb{C}^n)$ coincides with $\mathcal{B}_t^m(\mathbb{C}^n)$ and the Hermite semigroup $e^{-tH}$ is an isometric isomorphism of $W_{H}^{m,2}(\mathbb{R}^n)$ onto $\mathcal{B}_t^m(\mathbb{C}^n)$ up to a constant multiple.

**Proof.** Since $\frac{d^m}{dt^m}U_t(z)$ is a polynomial in $x, y$ times $U_t(z)$, any $F$ which is square integrable with respect to $\frac{d^m}{dt^m}U_t(z)$ belongs to $\mathcal{H}_t(\mathbb{C}^n)$ and hence of the form $e^{-tH}f$. Further, it follows from the above proposition, as the norms $||F||_m$ and $||F||_{W_{H}^{m,2}}$ coincide, $f \in W_{H}^{m,2}(\mathbb{R}^n)$. Consequently, $\mathcal{F}_t^m(\mathbb{C}^n)$ is contained in $W_{t}^{m,2}(\mathbb{C}^n)$. Notice that $\tilde{\Phi}_\alpha \in \mathcal{B}_t^m(\mathbb{C}^n)$. Further if $\langle F, \tilde{\Phi}_\alpha \rangle_{W_{H}^{m,2}} = 0$ for $\alpha \in \mathbb{N}^n$, then it can be easily seen that $\langle F, \tilde{\Phi}_\alpha \rangle_{\mathcal{H}_t(\mathbb{C}^n)} = 0$ for $\alpha \in \mathbb{N}^n$ which forces that $F = 0$. Hence $\mathcal{F}_t^m(\mathbb{C}^n)$ is dense in $W_{t}^{m,2}(\mathbb{C}^n)$.

### 3. Holomorphic Laguerre Sobolev Spaces

The special Hermite functions are defined using Hermite functions by

$$
\Phi_{\alpha\beta}(z) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{i\alpha \xi} \Phi_{\alpha}(\xi + \frac{1}{2}y) \Phi_{\beta}(\xi - \frac{1}{2}y) d\xi
$$

for $\alpha, \beta \in \mathbb{N}^n$, $z = x + iy$. Then $\{\Phi_{\alpha\beta} : \alpha, \beta \in \mathbb{N}^n\}$ forms an orthonormal basis for $L^2(\mathbb{C}^n)$. The functions $\Phi_{\alpha\beta}$ are eigenfunctions of the special Hermite operator $L$ with eigenvalues $(2|\beta| + n)$, where

$$
L = -\Delta_z + \frac{1}{4}|z|^2 - i \sum_{i=1}^{n} (x_i \frac{\partial}{\partial y_j} - y_j \frac{\partial}{\partial x_j})
$$

where $-\Delta_z$ denotes the Laplacian on $\mathbb{C}^n$. Any $f \in L^2(\mathbb{C}^n)$ has the special Hermite expansion given by $f = \sum_{\alpha} \sum_{\beta} (f, \Phi_{\alpha\beta}) L^2(\mathbb{C}^n) \Phi_{\alpha\beta}$.

Using the operation twisted convolution defined by

$$
(f \times g)(z) = \int_{\mathbb{C}^n} f(z - w)g(w)e^{\frac{i}{2}Im(zw)}dw,
$$

the special Hermite expansion can be put in the following compact form: $f = (2\pi)^{-n} \sum_{k=0}^{\infty} f \times \varphi_k$ where $\varphi_k$ stands for the Laguerre function $\varphi_k(z) = L_{k}^{n-1}(\frac{1}{2}|z|^2) e^{-\frac{1}{2}|z|^2}$ in which $L_k^{n-1}$ denotes the $k$th Laguerre polynomial of type $n - 1$. For various results concerning Hermite and special Hermite expansions we refer to Thangavelu [11].
The special Hermite operator $L$ defines a semigroup (called the special Hermite semigroup and denoted by $e^{-tL}$, $t > 0$) by the expansion

$$e^{-tL} f = (2\pi)^{-n} \sum_{k=0}^{\infty} e^{-(2k+n)t} f \times \varphi_k$$

for $f \in L^2(\mathbb{C}^n)$. Again, on a dense subspace, $e^{-tL}$ can be explicitly written as $e^{-tL} f(z) = f \times p_t(z)$, where

$$p_t(z) = (2\pi)^{-n} (\sinh t)^{-n} e^{-\frac{1}{4} \cosh |z|^2}.$$ 

We shall identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ and write $z = x + iy \in \mathbb{C}^n$ as $(x, u) \in \mathbb{R}^{2n}$. For $f \in L^2(\mathbb{C}^n)$, the function $e^{-tL} f(x, u)$ is given by

$$f \times p_t(x, u) = \int_{\mathbb{R}^n} f(x', u') p_t(x - x', u - u') e^{-\frac{1}{2}(x'u - xu')} dx' du'.$$

The special Hermite functions $\Phi_{\alpha\beta}(x, u)$ can be extended to $\mathbb{C}^{2n}$ as entire functions $\Phi_{\alpha\beta}(z, w)$. Moreover, $f \times p_t(x, u)$ also extends to $\mathbb{C}^{2n}$ as an entire function $f \times p_t(z, w)$, where $z = x + iy, w = u + iv$.

Let $\mathcal{B}_t^{*}(\mathbb{C}^{2n})$ denote the set of all entire functions on $\mathbb{C}^{2n}$ which are square integrable with respect to the weight function

$$W_t(z, w) = W_t(x + iy, u + iv) = 4^n e^{(uy - vx)} p_{2t}(2y, 2v).$$

Then the following result is known, see [8]:

**Theorem 3.1.** The special Hermite semigroup $e^{-tL} : L^2(\mathbb{R}^{2n}) \to \mathcal{B}_t^{*}(\mathbb{C}^{2n})$ is an isometric isomorphism.

Let $\Phi_{\alpha\beta}(z, w) = e^{-(2|\beta| + n)t} \Phi_{\alpha\beta}(z, w)$. Then the family $\{\Phi_{\alpha\beta} / \alpha, \beta \in \mathbb{N}^n\}$ forms an orthonormal basis for $\mathcal{B}_t^{*}(\mathbb{C}^{2n})$. For further details, we refer to [12].

Laguerre Sobolev spaces were introduced by Peetre and Sparr in 1975. These Sobolev spaces along with Hermite Sobolev spaces were studied in connection with the regularity of the twisted spherical means in [10].

The Laguerre Sobolev space $W^{m,2}_L(\mathbb{C}^n)$ is defined to be the image of $L^2(\mathbb{C}^n)$ under $L^{-m}$, where $L$ denotes the special Hermite operator mentioned earlier. In other words $f \in W^{m,2}_L(\mathbb{C}^n)$ if and only if

$$||f||_{W^{m,2}_L}^2 = \sum_{\alpha} \sum_{\beta} (2|\beta| + n)^{2m} |(f, \Phi_{\alpha\beta})_{L^2(\mathbb{C}^n)}|^2 < \infty.$$ 

The holomorphic Sobolev space $W^{m,2}_t(\mathbb{C}^{2n})$ is defined as the image of $W^{m,2}_L(\mathbb{C}^n)$ under $e^{-tL}$. Recall that $\tilde{\Phi}_{\alpha\beta} = e^{-tL} \Phi_{\alpha\beta}$. The norm in
$W_t^{*, m, 2}(\mathbb{C}^{2n})$ is given by

$$
||F||_{W_t^{*, m, 2}(\mathbb{C}^{2n})} = \sum_{\alpha} \sum_{\beta} (2|\beta| + n)^{2m} |(F, \tilde{\Phi}_{\alpha\beta} B_t^*(\mathbb{C}^{2n})|^2.
$$

Let $G_t^m(\mathbb{C}^{2n})$ denote the space of all holomorphic functions on $\mathbb{C}^{2n}$ which are square integrable with respect to $|\frac{d^{2m}}{dt^{2m}} W_t(z, w)| dz dw$. As in section 2, we can show that

$$
(F, G_t^m(\mathbb{C}^{2n})) = \int_{\mathbb{C}^{2n}} F(z, w) G(z, w) \frac{d^{2m}}{dt^{2m}} W_t(z, w) dz dw
$$

(3.1)

is an inner product on $G_t^m(\mathbb{C}^{2n})$. Let $\mathcal{B}_t^{*, m}(\mathbb{C}^{2n})$ be the completion of $G_t^m(\mathbb{C}^{2n})$ with this inner product (3.1). We have the following result.

**Theorem 3.2.** For every non negative integer $m$, $W_t^{*, m, 2}(\mathbb{C}^{2n})$ coincides with $\mathcal{B}_t^{*, m}(\mathbb{C}^{2n})$ and the special Hermite semigroup $e^{-tL}$ is an isometric isomorphism of $W_t^{m, 2}(\mathbb{C}^{n})$ onto $\mathcal{B}_t^{*, m}(\mathbb{C}^{2n})$ up to a constant multiple.

This theorem is proved by using the fact that $F$ can be expressed in two ways

$$
(F, \Phi_{\alpha\beta})_{L^2(\mathbb{C}^{n})} = \int_{\mathbb{C}^{n}} F(x + iy) \Phi_{\alpha\beta}(x + iy) dxdy,
$$

the orthogonality relation

$$
\int_{\mathbb{C}^{2n}} \Phi_{\alpha\beta}(z, w) \overline{\Phi_{\mu\nu}(z, w)} W_t(z, w) dz dw = e^{2(2|\beta| + n\nu)t} \delta_{\alpha\mu} \delta_{\beta\nu},
$$

and the fact that $\tilde{\Phi}_{\alpha\beta} \in \mathcal{B}_t^{*, m}(\mathbb{C}^{2n})$, and proceeding as in section 2.

4. **The Image of Schwartz class functions under Hermite and special Hermite semigroups**

First, we shall describe the image of $S(\mathbb{R}^n)$ under $e^{-tH}$. In order to do this, first we shall obtain pointwise estimates for a function $F \in H_t(\mathbb{C}^{n})$; i.e for $F = e^{-tH} f$, $f \in L^2(\mathbb{R}^n)$. Since $F \mapsto F(z)$ is a continuous linear functional on $H_t(\mathbb{C}^{n})$ for each $z \in \mathbb{C}^{n}$, an application of Riesz representation theorem shows that there exists a unique $K_t(z, .) \in H_t(\mathbb{C}^{n})$ such that

$$
F(z) = (F, K_t(z, .)) = \int_{\mathbb{C}^{n}} F(w) K_t(z, w) U_t(w) dw.
$$
The function $K_t(z, w)$ is called the reproducing kernel for $H_t(\mathbb{C}^n)$. By expanding $F$ in terms of $\tilde{\Phi}_\alpha$, we can write
\begin{equation}
F(z) = \int_{\mathbb{C}^n} F(w) \sum_\alpha \tilde{\Phi}_\alpha(w) \Phi_\alpha(z) U_t(w) dw.
\end{equation}
This means that
\begin{equation}
K_t(z, w) = \sum_\alpha e^{-(2|\alpha|+n)2t} \Phi_\alpha(w) \Phi_\alpha(z).
\end{equation}
Cauchy-Schwarz inequality applied to (4.1) gives us
\begin{equation}
|F(z)|^2 \leq ||F||^2 ||K_t(z, .)||^2 = ||F||^2 K_t(z, z).
\end{equation}
By using Mehler’s formula, we can explicitly calculate $K_t(z, z)$. In fact,
\begin{equation}
K_t(z, z) = (2\pi)^{-\frac{n}{2}} (\sinh 4t)^{-\frac{n}{4}} e^{-\frac{x^2}{2} \tanh 2t + y^2 \coth 2t} e^{\frac{x^2}{2} - \frac{y^2}{2} \cosech 4t} ds.
\end{equation}
Since
\begin{equation}
\begin{aligned}
- \coth 4t(x^2 - y^2) + \cosech 4t(x^2 + y^2) = -x^2 \tanh 2t + y^2 \coth 2t
\end{aligned}
\end{equation}
we obtain
\begin{equation}
|F(z)|^2 \leq C_n (\sinh 4t)^{-\frac{n}{4}} e^{-\frac{x^2}{2} \tanh 2t + y^2} ||F||^2.
\end{equation}
Thus we have obtained a pointwise estimate for functions $F \in H_t(\mathbb{C}^n)$.
In order to obtain pointwise estimates for $F \in W^{m,2}_t(\mathbb{C}^n)$ we observe that the reproducing kernel for $W^{m,2}_t(\mathbb{C}^n)$ is given by
\begin{equation}
K_t^{2m}(z, w) = \sum_\alpha (2|\alpha| + n)^{-2m} \Phi_\alpha(z) \Phi_\alpha(w).
\end{equation}
We can write this as
\begin{equation}
K_t^{2m}(z, w) = \frac{1}{(2m-1)!} \int_0^\infty s^{2m-1} K_{s+t}(z, w) ds.
\end{equation}
Using the explicit formula for $K_s(z, z)$ we have
\begin{equation}
\begin{aligned}
K_t^{2m}(z, z) &= \frac{(2\pi)^{-\frac{n}{2}}}{(2m-1)!} \int_0^\infty s^{2m-1} (\sinh 4(t+s))^{-\frac{n}{4}} \\
& \times e^{-x^2 (\coth 4(t+s) - \cosech 4(t+s))} e^{y^2 (\cosech 4(t+s) + \coth 4(t+s))} ds
\end{aligned}
\end{equation}
\begin{equation}
= \frac{(2\pi)^{-\frac{n}{2}}}{(2m-1)!} \int_0^\infty s^{2m-1} (\sinh 4(t+s))^{-\frac{n}{4}} e^{-x^2 \tanh 2(t+s) + y^2 (\coth 2(t+s))} ds.
\end{equation}
From the above expression for the reproducing kernel it is now an easy matter to establish the following pointwise estimates for functions from the holomorphic Sobolev spaces.
Theorem 4.1. (Sobolev-embedding theorem) Let $m$ be a non-negative integer. Then every $F \in W^{m,2}_t(\mathbb{C}^n)$ satisfies the estimate
\[ |F(z)|^2 \leq C(1 + x^2 + y^2)^{-2m} e^{-x^2 \tanh 2t + y^2 \coth 2t} . \]

Proof. In order to prove the theorem we need to estimate the integral appearing in the representation of the reproducing kernel. We rewrite the kernel as
\[ I = \int_0^\infty s^{2m-1} (\sinh 4(t + s))^{-\frac{n}{2}} e^{-y^2 \frac{\sinh 2s}{\sinh 2(t+s) \sinh(2t)}} e^{-x^2 \frac{\sinh 2s}{\cosh 2(t+s) \cosh(2t)}} ds \]
which after some simplification yields
\[ I = \int_0^\infty s^{2m-1} (\sinh 4(t + s))^{-\frac{n}{2}} e^{-y^2 \frac{\sinh 2s}{\sinh 2(t+s) \sinh(2t)}} e^{-x^2 \frac{\sinh 2s}{\cosh 2(t+s) \cosh(2t)}} ds . \]

Thus we only need to show that the above integral is bounded by a constant times $(1 + x^2 + y^2)^{-2m}$.

To prove this estimate we break up the above integral into two parts. Using the fact that sinh and cosh are increasing functions and sinh $s > s$ we see that
\[ \int_0^t s^{2m-1} (\sinh 4(t + s))^{-\frac{n}{2}} e^{-y^2 \frac{\sinh 2s}{\sinh 2(t+s) \sinh(2t)}} e^{-x^2 \frac{\sinh 2s}{\cosh 2(t+s) \cosh(2t)}} ds \]
is bounded by
\[ \int_0^\infty s^{2m-1} e^{-2ns} e^{-2 \left( \frac{x^2}{\cosh^2 4t} + \frac{y^2}{\sinh^2 4t} \right)} ds \leq C_t (1 + x^2 + y^2)^{-2m} . \]

On the other hand the integral
\[ \int_t^\infty s^{2m-1} (\sinh 4(t + s))^{-\frac{n}{2}} e^{-y^2 \frac{\sinh 2s}{\sinh 2(t+s) \sinh(2t)}} e^{-x^2 \frac{\sinh 2s}{\cosh 2(t+s) \cosh(2t)}} ds \]
is bounded by a constant times
\[ e^{-(a_t x^2 + b_t y^2)} \int_0^\infty s^{2m-1} e^{-2ns} ds \]
where $a_t$ and $b_t$ are the infima of $\frac{\sinh 2s}{\cosh 2(t+s) \cosh(2t)}$ and $\frac{\sinh 2s}{\sinh 2(t+s) \sinh(2t)}$ over $s > t$ respectively. The above clearly gives the required estimate. \qed

Now, we are in a position to prove the following result which characterises the image of $\mathcal{S}(\mathbb{R}^n)$ under $e^{-tH}$. 


Theorem 4.2. Let \( t > 0 \) be fixed. Suppose \( F \) is a holomorphic function on \( \mathbb{C}^n \). Then there exists a function \( f \in \mathcal{S}(\mathbb{R}^n) \) such that \( F = e^{-tH}f \) iff \( F \) satisfies

\[
|F(z)|^2 \leq A_m \frac{e^{-(\tanh 2t)x^2 + (\coth 2t)y^2}}{(1 + x^2 + y^2)^{2m}}
\]

for some constants \( A_m, m = 1, 2, 3, \ldots \).

Proof. If \( f \in \mathcal{S}(\mathbb{R}^n) \), then \( f \in \mathcal{W}_t^{m,2}(\mathbb{R}^n) \) \( \forall \) \( m \), which in turn implies that \( F = e^{-tH} \in \mathcal{W}_t^{m,2}(\mathbb{C}^n) \) \( \forall \) \( m \). Then (4.5) follows from theorem 4.1. Conversely, suppose \( F \) satisfies (4.5). Then by choosing \( m \) large we see that

\[
\int_{\mathbb{C}^n} |F(z)|^2 U_t(z) \, dz < \infty
\]

from which it follows that \( F \in \mathcal{H}(\mathbb{C}^n) \). Thus there exists a function \( f \in L^2(\mathbb{R}^n) \) such that \( F = e^{-tH}f \). Since \( \frac{d^{2m}}{dt^{2m}} U_t(z) \) is the sum of \( 2m+1 \) terms, where each term is of the form \( (p(t)x^2 + q(t)y^2 + c)^k \leq C_t(1 + x^2 + y^2)^{2m} \), with \( k \leq 2m \). Thus it follows from (4.5) that \( F \in \mathcal{B}_t^m(\mathbb{C}^n) = \mathcal{W}_t^{m,2}(\mathbb{C}^n) \) using Theorem 2.3. This leads to the fact that \( F \in \mathcal{W}_t^{m,2}(\mathbb{C}^n) \) \( \forall \) \( m \). Consequently \( f \in \mathcal{W}_t^{m,2}(\mathbb{R}^n) \) \( \forall \) \( m \). But since \( \bigcap_m \mathcal{W}_t^{m,2}(\mathbb{R}^n) = \mathcal{S}(\mathbb{R}^n) \), the result follows.

Now we shall characterise the image of \( \mathcal{S}(\mathbb{C}^n) \) under \( e^{-tL} \). In order to do so, first we get pointwise estimates for functions in \( \mathcal{B}_t^1(\mathbb{C}^{2n}) \). Let \( F \in \mathcal{B}_t^1(\mathbb{C}^{2n}) \). Then \( |F(z, w)| = |e^{-tL}f(z, w)| = |(f \times p_t)(z, w)| \) where

\[
p_t(x, u) = (2\pi)^{-n} (\sinh t)^{-n} e^{-\frac{1}{2t} \coth t(x^2 + y^2)}
\]

(see [12] for details.) Recalling the definition of twisted convolution,

\[
f \times p_t(x, u) = \int_{\mathbb{R}^{2n}} f(x', u') p_t(x - x', u - u') e^{-\frac{i}{2t}(x'u - xu)} \, dx' du',
\]

we get

\[
|f \times p_t(z, w)| = |\int_{\mathbb{R}^{2n}} f(x', u') p_t(z - x', w - u') e^{-\frac{i}{2t}(x'u - xu)} \, dx' du'| \\
\leq \int_{\mathbb{R}^{2n}} |f(x', u')| |p_t(z - x', w - u')| e^{\frac{x^2}{2t} + \frac{y^2}{2t}} \, dx' du' \\
\leq ||f||_{L^2(\mathbb{C}^n)} (2\pi)^{-n} (\sinh 2t)^{-n} e^{ex - uy} e^{coth 4t(y^2 + v^2)}
\]

where \( w = u + iv, z = x + iy \). Proceeding as in section 3, we can show that if \( F \in \mathcal{W}_t^{m,2}(\mathbb{C}^{2n}) \), then

\[
|F(z, w)|^2 \leq C_t \frac{e^{ex - uy} e^{coth 4t(y^2 + v^2)}}{(1 + y^2 + v^2)^{2m}}.
\]
But we can do better than this for Schwartz functions. We require the following simple lemma.

**Lemma 4.3.** If $f \in \mathcal{S}(\mathbb{R}^{2n})$, then

\begin{align}
(4.8) \quad e^{-tL} \left( \prod_{j=1}^{n} \left( \frac{\partial}{\partial x_j} - ax_j \right) f \right) &= \prod_{j=1}^{n} (-az_j + bw_j) e^{-tL} f \\
(4.9) \quad e^{-tL} \left( \prod_{j=1}^{n} \left( \frac{\partial}{\partial u_j} + bu_j \right) f \right) &= \prod_{j=1}^{n} (bz_j + aw_j) e^{-tL} f
\end{align}

where $a = -\frac{1}{2} \coth t$, $b = \frac{i}{2}$.

**Proof.** We shall prove the result for $n = 1$. Consider

$$e^{-tL} \left( \frac{\partial}{\partial x'} f \right)(z, w) =$$

$$\frac{1}{4\pi} (\sinh t)^{-1} \int_{\mathbb{R}^2} \left( \frac{\partial}{\partial x'} f \right)(x', u') e^{-\frac{1}{4} \coth t [(z-x')^2 + (w-u')^2]} e^{-\frac{i}{2} (x'w - zu')} \, dx' \, du'.$$

Integration by parts leads to

$$e^{-tL} \left( \frac{\partial}{\partial x'} f \right)(z, w) =$$

$$= \frac{1}{4\pi} (\sinh t)^{-1} \int_{\mathbb{R}^2} \left( \frac{\partial}{\partial x'} f \right)(x', u') \left[ \frac{1}{2} \coth t (z - x') - \frac{i}{2} w \right]$$

$$\times e^{-\frac{1}{4} \coth t [(z-x')^2 + (w-u')^2]} e^{-\frac{i}{2} (x'w - zu')} \, dx' \, du'$$

$$= \left( -\frac{1}{2} \coth t + \frac{i}{2} w \right) e^{-tL} f(z, w) + \frac{1}{8\pi} (\sinh t)^{-1} \coth t$$

$$\times \int_{\mathbb{R}^2} x' \left( \frac{\partial}{\partial x'} f \right)(x', u') e^{-\frac{1}{4} \coth t [(z-x')^2 + (w-u')^2]} e^{-\frac{i}{2} (x'w - zu')} \, dx' \, du'.$$

Thus

$$e^{-tL} \left( \frac{\partial}{\partial x'} f \right)(z, w) =$$

$$\left( -\frac{1}{2} \coth t + \frac{i}{2} w \right) e^{-tL} f(z, w) + \frac{1}{2} \coth t e^{-tL} f(x', f)(z, w).$$

Hence

$$e^{-tL} \frac{\partial}{\partial x'} f = (-az + bw) e^{-tL} f + ae^{-tL} f.$$

So

$$e^{-tL} \left( \frac{\partial}{\partial x'} - ax' \right) f = (-az + bw) e^{-tL} f.$$
In the case of $n$ dimension if $x' = (x'_1, x'_2, \cdots, x'_n)$, then we have to apply the same procedure for each $x'_j$, $j = 1, 2, \cdots, n$ in order to obtain (4.8). In a similar way (4.9) can be proved.

In what follows we use the standard vector notation $(\frac{\partial}{\partial x^j} - ax^j)^\alpha = \prod_{j=1}^n (\frac{\partial}{\partial x^j} - ax^j)^\alpha$, where $\alpha$ is a multi-index.

**Theorem 4.4.** Suppose $F$ is a holomorphic function on $\mathbb{C}^{2n}$. Fix $t > 0$. Then there exists a function $f \in \mathcal{S}(\mathbb{R}^{2n})$ with $F = e^{-tL}f$ iff $F$ satisfies

$$|F(z, w)|^2 \leq B_m \frac{e^{ex-uy}e^{\coth 4t(y^2 + v^2)}}{1 + x^2 + y^2 + u^2 + v^2)^2m}$$

for some constants $B_m, m = 1, 2, 3, \cdots$.

**Proof.** Multiplying (4.8) by $a$, multiplying (4.9) by $b$ and subtracting (4.8) from (4.9) we get

$$-ae^{-tL}[(\frac{\partial}{\partial x} - ax)f] + be^{-tL}[(\frac{\partial}{\partial u} + bu)f] = (a^2 + b^2)ze^{-tL}f.$$  \hspace{1cm} (4.10)

Multiplying (4.8) by $b$, multiplying (4.9) by $a$ and adding (4.8) and (4.9) we get

$$be^{-tL}[(\frac{\partial}{\partial x} - ax)f] + ae^{-tL}[(\frac{\partial}{\partial u} - bu)f] = (a^2 + b^2)we^{-tL}f.$$ \hspace{1cm} (4.11)

Applying the lemma (4.3) iteratively we obtain the following : Let $k = (k_1, k_2, ..., k_n), l = (l_1, l_2, ..., l_n) \in \mathbb{N}^n$ and $x^k = x_1^{k_1}x_2^{k_2}...x_n^{k_n}$. If $f \in \mathcal{S}(\mathbb{R}^{2n})$ then $(\frac{\partial}{\partial x} - ax)^k f, (\frac{\partial}{\partial u} + bu)^l f \in \mathcal{S}(\mathbb{R}^{2n})$ and

$$e^{-tL}[(\frac{\partial}{\partial x} - ax)^k f] = (-az + bw)^k e^{-tL}f$$

$$e^{-tL}[(\frac{\partial}{\partial u} + bu)^l f] = (bz + aw)^l e^{-tL}f.$$  \hspace{1cm} (4.11)

As in the case of (4.11) and (4.12), by carrying out appropriate algebraic manipulation we can find a differential operator $T_{k,l}$ such that

$$e^{-tL}(T_{k,l}f) = z^k w^l e^{-tL}f$$

for any $k, l \in \mathbb{N}^n$. Since $T_{k,l}f \in \mathcal{S}(\mathbb{R}^{2n})$, it follows from (4.7) that

$$|z^{k} w^{l} e^{-tL}f(z, w)|^2 \leq C_t e^{ex-uy}e^{\coth 4t(y^2 + v^2)}$$

Thus

$$|(1 + |z|^2 + |w|^2)^m F(z, w)|^2 \leq C_{t,m} e^{ex-uy}e^{\coth 4t(y^2 + v^2)},$$

where $F = e^{-tL}f$. \hspace{1cm} (4.11)
5. TEMPERED DISTRIBUTIONS AND A PALEY-WIENER THEOREM FOR THE WINDOWED FOURIER TRANSFORM

In this section we consider the image of tempered distributions on \( \mathbb{R}^n \) under the Hermite semigroup. The characterisation obtained leads to a Paley-Wiener theorem for the windowed Fourier transform. We prove the following analogue of Theorem 4.2 for tempered distributions.

**Theorem 5.1.** Suppose \( F \) is a holomorphic function on \( \mathbb{C}^n \). Then there exists a distribution \( f \in S'(\mathbb{R}^n) \) with \( F = e^{-tH}f \) if and only if
\[
|F(z)|^2 \leq C(1 + |z|^2)^{2m} e^{-x^2 \tanh 2t + y^2 \coth 2t}
\]
for some non-negative integer \( m \).

We obtained Theorem 4.2 as a consequence of Theorem 4.1 using the fact that the intersection of all the Hermite Sobolev spaces \( W_H^{m,2}(\mathbb{R}^n) \) is precisely the space of Schwartz functions. Since the union of all \( W^{m,2}_H(\mathbb{R}^n) \) is \( S'(\mathbb{R}^n) \) we only need to prove the following analogue of Theorem 4.1 for functions from \( W_t^{-m,2}(\mathbb{C}^n) \) where \( m \) is a positive integer.

**Theorem 5.2.** Let \( m \) be a positive integer. Then every \( F \in W_t^{-m,2}(\mathbb{C}^n) \) satisfies the estimate
\[
|F(z)|^2 \leq C(1 + |z|^2)^{2m} e^{-x^2 \tanh 2t + y^2 \coth 2t}.
\]
Conversely, if an entire function \( F \) satisfies the above estimate, then \( F \) belongs to \( W_t^{-m-n,2}(\mathbb{C}^n) \).

The necessity of the pointwise estimate is easy to establish. In fact, the reproducing kernel for \( W_t^{-m,2}(\mathbb{C}^n) \) is given by
\[
K_t^{-2m}(z, w) = \sum_{\alpha} (2|\alpha| + n)^{2m} e^{-2(2|\alpha|+n)t} \Phi_{\alpha}(\bar{z}) \Phi_{\alpha}(w).
\]
Therefore, we only need to estimate the \((2m)\)-th derivative of \( K_t(z, z) \) with respect to \( t \). It is easy to see that this leads to the estimate
\[
|F(z)|^2 \leq C(1 + |z|^2)^{2m} e^{-x^2 \tanh 2t + y^2 \coth 2t}.
\]

To prove the converse, we need to make use of duality. As \( W_H^{m,2}(\mathbb{R}^n) \) and \( W_H^{-m,2}(\mathbb{R}^n) \) are dual to each other, it follows from the definition that \( W_t^{m,2}(\mathbb{C}^n) \) and \( W_t^{-m,2}(\mathbb{C}^n) \) are dual to each other. The duality bracket is given by
\[
(F, G) = \int_{\mathbb{C}^n} F(z) \overline{G(z)} U_t(z) dz.
\]
We refer to [13] for details of the duality argument in the context of compact symmetric spaces. As the same proof works in our situation as well we do not give any details but only a brief sketch. If $F$ satisfies the given estimates then for any $G \in W^{m+n,2}_t(C^n)$ the integral defining $(F, G)$ converges and hence $F$ defines a continuous linear functional on $W^{m+n,2}_t(C^n)$. Consequently, $F$ belongs to $W^{-m-n,2}_t(C^n)$ which proves the converse.

We remark that the holomorphic Sobolev spaces $W^{-m,2}_t(C^n)$ are weighted Bergman spaces with a nonnegative weight function which can be given a representation in terms of Riemann-Liouville fractional integrals, see [13] for details.

We now show that the characterisation obtained in Theorem 5.1 leads to a Paley-Wiener theorem for the windowed Fourier transform. This transform, also known as short-time Fourier transform is used very much in Gabor analysis. Given a Schwartz class function $g$ the windowed Fourier transform of $f$ (with window $g$) is defined by

$$V_g(f)(x, y) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} f(u) g(u - y) e^{-ix \cdot u} \, du.$$  

See [4] and [5] for the study of windowed Fourier transforms in connection with Hardy’s theorem and characterisation of Schwartz functions. We are mainly interested in the case where $g(x) = \varphi_a(x) = c_n e^{-\frac{a}{2} |x|^2}$. More precisely, we consider the transform

$$T_a(f)(x) = V_{\varphi_a}(f)(x, 0).$$

Note that $T_a(f)$ is well defined as a function even if $f$ is a tempered distribution.

The classical Paley-Wiener theorem characterises the space of all compactly supported distributions in terms of the holomorphic properties of their Fourier transforms. Even though we can define Fourier transforms of tempered distributions we cannot hope for any such characterisation since the space of all tempered distributions is invariant under the Fourier transform. On the other hand we see that $T_a f$ extends to $\mathbb{C}^n$ as an entire function even when $f$ is only a tempered distribution. This property of the windowed Fourier transform allows us to prove the following analogue of Paley-Wiener theorem.

**Theorem 5.3.** For any $a > 0$ the windowed Fourier transform $T_a f(x)$ of a tempered distribution $f$ on $\mathbb{R}^n$ extends to $\mathbb{C}^n$ as an entire function which satisfies the estimate

$$|T_a f(x + iy)| \leq C(1 + x^2 + y^2)^{\frac{m}{2a - 1}} e^{\frac{1}{2a - 1} y^2}$$
for some non-negative integer \( m \). Conversely, if an entire function \( F \) satisfies such an estimate, then \( F = T_a f \) for a tempered distribution \( f \).

**Proof.** This theorem is easily proved by relating the windowed Fourier transform \( T_a f \) with \( e^{-tH} f \). Indeed, considering the case \( a > 1 \) first and writing \( a = \coth(2t) \) for some \( t > 0 \) we can easily verify that

\[
e^{-tH} f(z) = e^{-\frac{1}{2} \coth(2t) z^2} T_a f \left( \frac{iz}{\sinh(2t)} \right)
\]

for all \( z \in \mathbb{C}^n \). We obtain the required estimate on \( T_a f(z) \) by appealing to Theorem 5.1. Conversely, if \( F \) satisfies the given estimates then again by Theorem 5.1 the function

\[
G(z) = e^{-\frac{1}{2} \coth(2t) z^2} F \left( \frac{iz}{\sinh(2t)} \right)
\]

should be of the form \( e^{-tH} f(z) \) for a tempered distribution \( f \).

When \( a < 1 \) we take \( t > 0 \) so that \( a = \tanh(2t) \). Now the proof requires an analogue of Theorem 5.1 for functions of the form \( e^{-(t+i\frac{\pi}{4})H} f \). But the image of tempered distributions under \( e^{-(t+i\frac{\pi}{4})H} \) can be characterised in a similar way. The final estimates do not depend on the factor \( e^{-i\frac{\pi}{4}H} \) which is just the Fourier transform. This completes the proof of the theorem. \( \square \)

We remark that the case \( a = 1 \) can be read out from the results of \([2]\). This case can be obtained as a limiting case of our results and we think the proof presented here is simpler than the one given in \([2]\). Finally, we remark that we also have the following result which characterises the image of compactly supported distributions under the Hermite semigroup.

**Theorem 5.4.** Let \( f \) be a distribution supported in a ball of radius \( R \) centered at the origin. Then for any \( t > 0 \) the function \( e^{-tH} f \) extends to \( \mathbb{C}^n \) as an entire function which satisfies

\[
|e^{-tH} f(z)| \leq Ce^{-\frac{1}{2} \coth(2t) (x^2 - y^2)} e^{\frac{|z|}{\sinh(2t)}}.
\]

Conversely, any entire function \( F \) satisfying the above will be of the form \( e^{-tH} f \) where \( f \) is supported inside a ball of radius \( R \) centered at the origin.

The proof of this follows from the classical Paley-Wiener theorem for compactly supported distributions. We leave the details to the reader.
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