Stoner-Wohlfarth switching of the condensate magnetization in a dipolar spinor gas and the metrology of excitation damping

Seong-Ho Shinn,1 Daniel Braun,2 and Uwe R. Fischer1

1Department of Physics and Astronomy, Seoul National University, 08826 Seoul, Korea
2Eberhard-Karls-Universität Tübingen, Institut für Theoretische Physik, 72076 Tübingen, Germany

(Dated: June 16, 2021)

We consider quasi-one-dimensional dipolar spinor Bose-Einstein condensates in the homogeneous-local-spin-orientation approximation, that is with unidirectional local magnetization. By analytically calculating the exact effective dipole-dipole interaction, we derive a Landau-Lifshitz-Gilbert equation for the dissipative condensate magnetization dynamics, and show how it leads to the Stoner-Wohlfarth model of a uni-axial ferro-magnetic particle, where the latter model determines the stable magnetization patterns and hysteresis curves for switching between them. For an external magnetic field pointing along the axial, long direction, we analytically solve the Landau-Lifshitz-Gilbert equation. The solution explicitly demonstrates that the magnetic dipole-dipole interaction accelerates the dissipative dynamics of the magnetic moment distribution and the associated dephasing of the magnetic moment direction. Under suitable conditions, dephasing of the magnetization direction due to dipole-dipole interactions occurs within time scales up to two orders of magnitude smaller than the lifetime of currently experimentally realized dipolar spinor condensates, e.g., produced with the large magnetic-dipole-moment atoms 166Er. This enables experimental access to the dissipation parameter Γ in the Gross- Pitaevskii mean-field equation, for a system currently lacking a complete quantum kinetic treatment of dissipative processes and, in particular, an experimental check of the commonly used assumption that Γ is a single scalar independent of spin indices.

I. INTRODUCTION

Ever since a phenomenological theory to describe the behavior of superfluid helium II near the λ point has been developed by Pitaevskii [1], the dynamics of Bose-Einstein condensates (BEC) under dissipation has been intensely studied, see, e.g., [28]. Experimentally, the impact of Bose-Einstein condensation on excitation damping and its temperature dependence has for example been demonstrated in [9][12].

Dissipation in the form of condensate loss is defined by a dimensionless damping rate Γ entering the left-hand side of the Gross-Pitaevskii equation, replacing the time derivative as ∂t → (i − Γ)∂t. While a microscopic theory of condensate damping is comparatively well established in the contact-interaction case, using various approaches, cf., e.g., [13][15], we emphasize the absence of a microscopic theory of damping in dipolar spinor gases. While for scalar dipolar condensates, partial answers as to the degree and origin of condensate-excitation damping have been found see, e.g., Refs. [16–19], in spinor or multicomponent gases the interplay of anisotropic long-range interactions and internal spinor or multicomponent degrees of freedom leads to a highly intricate and difficult-to-disentangle many-body behavior of condensate-excitation damping.

In this paper, we propose a method to experimentally access Γ in a dipolar spinor condensate by using the dynamics of the unidirectional local magnetization in a quasi-one-dimensional (quasi-1D) dipolar spinor BEC in the presence of an external magnetic field. To this end, we first derive an equation of motion for the magnetization of the BEC that has the form of a Landau-Lifshitz-Gilbert (LLG) equation [20][22], with an additional term due to the dipole-dipole interaction between the atoms. The LLG equation is ubiquitous in nanomagnetism, where it describes the creation and dynamics of magnetization. The static limit of this equation is, in the limit of homogeneous local spin-orientation, described by the well-known Stoner-Wolfarth (SW) model [23][25] of a small magnetic particle with an easy axis of magnetization. We then investigate the magnetization switching after flipping the sign of the external magnetic field, and demonstrate the detailed dependence of the switching dynamics on the dissipative parameter Γ.

For a quasi-2D spinor BEC with inhomogeneous local magnetization, Ref. [26] has studied the magnetic domain wall formation process by deriving a LLG type equation. Here, we derive the LLG equation in a quasi-1D spinor BEC with unidirectional local magnetization, in order to establish a most direct connection to the original SW model. In distinction to [27], which studied the effective quasi-1D dipole-dipole interaction resulting from integrating out the two transverse directions within a simple approximation, we employ below an exact analytic form of the dipole-dipole interaction. In Section II we establish the quasi-1D spinor Gross-Pitaevskii (GP) equation with dissipation, and equations of motion for the magnetization direction (unit vector) \( \mathbf{M} \). Section V shows how the LLG equation and the SW model result, and Section VI derives analytical solutions to the equations of motion for \( \mathbf{M} \) when the external magnetic field points along the long, z axis. We summarize our results in section VII.

We defer two longer derivations to Appendices. The analytical form of the effective dipole-dipole interaction energy is deduced in Appendix A, and the quasi-1D GP mean-field equation with dissipation is described in detail.
in Appendix B. Finally, in Appendix C, we briefly discuss to which extent relaxing the usual simplifying assumption that dissipation even in the spinor case is described by a single scalar changes the LLG equation, and whether this affects the SW model and its predictions.

II. GENERAL DESCRIPTION OF DAMPING IN BECS

The standard derivation of the quantum kinetics of Bose-Einstein condensate damping [5] starts from the microscopic Heisenberg equation of motion for the quantum field operator $\hat{\psi}(r,t)$, for a scalar (single component) BEC in the s-wave scattering limit. Using their results, [28] obtained a mean-field equation to describe the dissipation of scalar BEC, whose form is

$$\left(i - \Gamma\right)\hbar \frac{\partial \psi}{\partial t} = H\psi$$

where $\psi$ is the (in the large $N$ limit) dominant mean-field part upon expanding the full bosonic field operator $\hat{\psi}$. In Ref. [1], Pitaevskii obtained a similar but slightly different form of the dissipative mean-field equation based on phenomenological considerations, $i\hbar \frac{\partial \psi}{\partial t} = (1 - i\Gamma) H\psi$, by parametrizing the deviation from exact continuity for the condensate fraction while minimizing the energy [1]. The latter deviation is assumed to be small, which is equivalent to assuming that $\Gamma$ remains small. This provides a clear physical interpretation of the damping mechanism, namely one based on particle loss from the condensate fraction. The version of Pitaevskii can be written as

$$\left(i - \Gamma\right)\hbar \frac{\partial \psi}{\partial t} = (1 + \Gamma^2) H\psi.$$  

(2)

It can thus be simply obtained by rescaling time with a factor $1 + \Gamma^2$ compared to [1]. Hence, as long as one does not predict precisely $\Gamma$, the two dissipative equations [1] and [2] cannot be distinguished experimentally from the dynamics they induce. From the data of [11], [1] estimated typical values of $\Gamma \approx 0.03$ for a scalar BEC of $^{23}$Na atoms (see also [12]), which shows that to distinguish between [1] and [2] experimentally the theoretical predictions of $\Gamma$ would need to be precise to the order of $10^{-4}$.

How eqs. [1] and [2] can be generalized to the dipolar spinor gases is comparatively little investigated. Using a symmetry-breaking mean-field approach by writing the quantum field operator as $\hat{\psi}(r,t)$ as $\hat{\psi}(r,t) = \hat{\psi}(r,t) + \hat{\delta\psi}(r,t)$, with $\langle \psi(r,t) \rangle = \langle \hat{\psi}(r,t) \rangle = 0$, [4] and [28] showed that $\Gamma$ is derived from the three-field correlation function $\langle \delta\hat{\psi}(r,t) \delta\hat{\psi}(r,t) \rangle$ in a basis where $\langle \delta\hat{\psi}(r,t) \rangle = 0$. From this microscopic origin, based on correlation functions, it is clear that in principle $\Gamma$ might depend on the spin indices in a spinor BECs and hence become a tensor (see Appendix C for a corresponding phenomenological generalization). Nevertheless, it is commonly assumed cf.,e.g., [23, 24], that $\Gamma$ does not depend on spin indices, and the scalar value is chosen specifically in [4] for a scalar BEC of $^{23}$Na atoms is commonly used, while a clear justification of this assumption is missing.

Extending the microscopic derivations in [5] and [28] to the spinor case would be theoretically interesting, but is beyond the scope of the present paper. Here, we instead focus on the question whether the standard assumption that the damping of each spinor component can be described by the mean-field equation [28] leads to experimentally falsifiable dynamical signatures. It will turn out that this assumption introduces an additional strong dephasing in the spin-degrees of freedom, amplified by the dipolar interaction. Hence, even on time scales on which the decay of the condensate fraction according to [1] can be neglected, the relaxation of the magnetization of the BEC potentially offers valuable insights whether the scalar-$\Gamma$ assumption is justified. Indeed, in [30] it was shown experimentally that on the time scale of the switching dynamics of the magnetization the number of particles in the condensates remains approximately constant. One might wonder, then, which dissipative mechanism is left. However, as we will show, by assuming the same GP equation for each component of the spinor as for scalar bosons, additional dephasing occurs that is in fact much more rapid than the decay of condensate density due to dephasing accelerated by the dipole-dipole interaction.

III. MEAN-FIELD DYNAMICS OF DAMPING IN DIPOLAR SPINOR BECS

For a spinor BEC, linear and quadratic Zeeman interactions are commonly included in the Hamiltonian. The quadratic Zeeman interaction is related to a second-order perturbation term in the total energy that can be induced by the interaction with an external magnetic field ($q_B$) as well as with the interaction with a microwave field ($q_{MW}$) [31]. Specifically, by applying a linearly polarized microwave field, one can change $q_{MW}$ without changing $q_B$. Following [26], we thus assert that for a dipolar spinor BEC without quadratic Zeeman term, the mean-field equation can be written as
are integers running from $-\alpha, \beta, \gamma, \ldots$ the beginning of the Greek alphabet such as $\alpha, \beta, \gamma, \ldots$ are integers running from $-S$ to $S$). In this expression, $\hbar \hat{f}$ is the spin-$S$ operator where the spin ladder is defined by $f_{\nu} |\alpha\rangle = \alpha |\alpha\rangle$ and $\langle \alpha | \beta\rangle = \delta_{\alpha, \beta}$, while $F_{\nu_1, \nu_2, \ldots, \nu_k} (r, t) := \hat{\psi}^\dagger (r, t) \hat{f}_{\nu_1} \hat{f}_{\nu_2} \cdots \hat{f}_{\nu_k} \hat{\psi} (r, t)$ are the components of the expectation value of $\hat{f}_{\nu_1} \hat{f}_{\nu_2} \cdots \hat{f}_{\nu_k}$. The Larmor frequency vector reads $\mathbf{b} = g_F \mu_B \mathbf{B}/\hbar$ (with Landé g-factor $g_F$, Bohr magneton $\mu_B$, and the external magnetic induction $\mathbf{B}$), $\hbar \alpha_{\nu, \nu'} (r, t) \cdot \mathbf{e}_\nu = c_{\text{d}} \int d^3 r' \sum_{\nu' = x, y, z} Q_{\nu, \nu'} (r - r') F_{\nu'} (r', t)$. Here, $c_{\text{d}} = \mu_0 (g_F \mu_B)^2 / (4\pi)$ and $\mathbf{e}_\nu$ is a unit vector along the $\nu$ axis [31] (by convention, indices from the middle of the Greek alphabet such as $\kappa, \lambda, \nu, \ldots = x, y, z$ denote spatial indices), and $Q_{\nu, \nu'}$ is the spin-space tensor defined in Eq. (A3) of Appendix A. Finally, $m$ is the boson mass, $c_0$ the density-density interaction coefficient, and $c_{2k}$ the interaction coefficient parametrizing the spin-spin interactions, where $k$ is an positive integer running from 1 to $S$ [26]. For example, $c_2$ is the spin-spin interaction coefficient of a spin-1 gas ($S = 1$).

To develop a simple and intuitive physical approach, we consider a quasi-1D gas for which one can perform analytical calculations. We set the trap potential as

$$V_{\text{tr}} (x, y, z) = \frac{1}{2} m \omega_0^2 (x^2 + y^2) + V (z),$$

so that the long axis of our gas is directed along the $z$ axis and the gas is strongly confined perpendicularly.

For a harmonic trap along all directions, i.e. when $V(z) = m \omega_z^2 z^2/2$, we set $\omega_\perp \gg \omega_z$. For a box trap along $z$, i.e. when $V(z) = 0$ for $|z| < L_z$ and $V(z) = \infty$ for $|z| > L_z$, our gas will be strongly confined along $z$ as well as the quasi-1D condition is satisfied, where we will discuss below whether the condition is satisfied, in section VI.A.

Single-domain spinor BECs have been already realized, for example, using spin-1 $^{87}$Rb [34]. This single-domain approximation is common in nanomagnetism, see for example [34], by assuming magnetic particles much smaller than the typical width of a domain wall. The local magnetization is related to the expectation value $\mathbf{d} F (r, t) = g_F \mu_B \mathbf{F} (r, t)$ of the spatial spin density operator by $\mathbf{d} (r, t) = g_F \mu_B \mathbf{F} (r, t)$. An unidirectional local magnetization $\mathbf{d} (z, t)$ is then given by

$$d_x (z, t) = d(z, t) \sin \theta (t) \cos \phi (t),$$
$$d_y (z, t) = d(z, t) \sin \theta (t) \sin \phi (t),$$
$$d_z (z, t) = d(z, t) \cos \theta (t),$$

where $d_{\nu} (z, t) = \mathbf{d} (z, t) \cdot \mathbf{e}_\nu$ is the $\nu$-th component of $\mathbf{d} (z, t)$, $d (z, t) = |\mathbf{d} (z, t)|$, $\theta (t)$ is polar angle of $\mathbf{d} (z, t)$, and $\phi (t)$ is azimuthal angle of $\mathbf{d} (z, t)$. For an illustration of the geometry considered, see Fig. 1. For a single component dipolar BEC, $\mathbf{F} (r, t)$ has a fixed direction. To study the relation of the Stoner-Wohlfarth model, in which $\mathbf{F} (r, t)$ changes its direction, with a dipolar BEC, a multi-component dipolar BEC should therefore be employed.

In the quasi-1D approximation, the order parameter $\hat{\psi}_\alpha (r, t)$ is commonly assumed to be of the form

$$\hat{\psi}_\alpha (r, t) = \frac{e^{-\rho^2/2 \rho_\perp^2}}{\sqrt{l_\perp}} \Psi_\alpha (z, t).$$

where $l_\perp$ is the harmonic oscillator length in the $x-y$ plane and $\rho = \sqrt{x^2 + y^2}$. Assuming our gas is in the homogeneous local spin-orientation limit, we may also apply simple mode approximation in space so that $\Psi_\alpha (z, t) = \Psi_\text{uni} (z, t) \zeta_\alpha (t)$. The time-dependent spinor part is

$$\zeta_\alpha (t) = \langle \alpha | e^{-i \hat{f}_z \phi (t)} e^{-i \hat{f}_\theta \theta (t)} | S \rangle,$$

for spin-$S$ particles [26] and the normalization reads $| \zeta (t) |^2 := \zeta^\dagger (t) \zeta (t) = 1$. Finally, due to the $(i - \Gamma)$ factor on the left-hand side of Eq. (3), for the ease of calculation, we may make the following ansatz for the $\hat{\psi}_\alpha (r, t)$, cf. Ref. [35].

$$\hat{\psi}_\alpha (r, t) = \frac{e^{-\rho^2/2 \rho_\perp^2}}{l_\perp \sqrt{2\pi}} \Psi (z, t) \zeta_\alpha (t) e^{-i(1+\Gamma)\omega_\perp t/(1+\Gamma^2)}.$$
From our ansätze in Eqs. (7) and (8), one concludes that the expectation value of the (spatial) spin-density operator is

\[
\begin{align*}
\hbar F_x (r, t) &= \hbar S e^{-\rho^2/2_l} \Psi (z, t) |^2 e^{-2 \Gamma \omega_z t/(1+\Gamma^2)} \times \sin \theta \left( t \right) \cos \phi \left( t \right), \\
\hbar F_y (r, t) &= \hbar S e^{-\rho^2/2_l} \Psi (z, t) |^2 e^{-2 \Gamma \omega_z t/(1+\Gamma^2)} \times \sin \theta \left( t \right) \sin \phi \left( t \right), \\
\hbar F_z (r, t) &= \hbar S e^{-\rho^2/2_l} \Psi (z, t) |^2 e^{-2 \Gamma \omega_z t/(1+\Gamma^2)} \times \cos \theta \left( t \right).
\end{align*}
\]

The above equations lead to unidirectional local magnetization, which has been assumed in Eqs. (5), in the quasi-1D limit (after integrating out the strongly confining $x$ and $y$ axes). Note however that our ansatz in Eq. (8) is sufficient, but not necessary for the homogeneous-local-spin-orientation limit, and the homogeneous-local-spin-orientation ansatz is thus designed to render our approach as simple as possible.

Because we are not assuming any specific form of $\Psi (z, t)$ in our ansatz in Eq. (8), we cover every possible time behavior of $|\psi (r, t)|^2 := \psi^\dagger (t) \psi (t)$:

\[
|\psi (r, t)|^2 = \frac{e^{-\rho^2/2_l}}{\pi l_{\perp}^2} |\Psi (z, t)|^2 e^{-2 \Gamma \omega_z t/(1+\Gamma^2)}. \tag{10}
\]

Eq. (10) explicitly shows that Eq. (8) does not imply an exponentially decaying wavefunction with time since $|\Psi (z, t)|^2$ can be any physical function of time $t$. However, the ansatz (8) simplifies the resulting equation for $\Psi (z, t)$, Eq. (11) below.

By integrating out the $x$ and $y$ directions, the GP equation for a quasi-1D spin-$S$ BEC can be written as (see for a detailed derivation Appendix B)

\[
(i - \Gamma) \hbar \frac{\partial \Psi (z, t) \zeta_o (t)}{\partial t} = \left\{ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial z^2} + V (z) + \frac{c_0}{2 \pi l_{\perp}^2} n (z, t) \right\} \Psi (z, t) \zeta_o (t)
\]

\[
+ \hbar \left[ -b + S \left\{ M (t) - 3 M_z (t) e_z \right\} P_{dd} (z, t) \right] \left\{ \sum_{\beta = -S}^S \left( \hat{f}_{\alpha,\beta}^\dagger \right) \Psi (z, t) \zeta_\beta (t) \right\}
\]

\[
+ \frac{c_0}{2 \pi l_{\perp}^2} n (z, t) \sum_{\nu_1, \nu_2, \ldots, \nu_k = x, y, z} SM_{\nu_1, \nu_2, \ldots, \nu_k} (t) \left\{ \sum_{\beta = -S}^S \left( \hat{f}_{\nu_1} \hat{f}_{\nu_2} \cdots \hat{f}_{\nu_k} \right) \right\}_{\alpha,\beta} \Psi (z, t) \zeta_\beta (t) \right\}, \tag{11}
\]

where we defined the two functions

\[
M_{\nu_1, \nu_2, \ldots, \nu_k} (t) := \frac{1}{S} \sum_{\alpha,\beta = -S}^S \zeta_\alpha^\dagger (t) \left( \hat{f}_{\nu_1} \hat{f}_{\nu_2} \cdots \hat{f}_{\nu_k} \right)_{\alpha,\beta} \zeta_\beta (t), \tag{12}
\]

\[
P_{dd} (z, t) := \frac{c_{dd}}{2 \hbar l_{\perp}^2} \int_{-\infty}^\infty dz' n (z', t) G \left( \frac{|z - z'|}{l_{\perp}} \right) - \frac{4}{3} \delta \left( \frac{z - z'}{l_{\perp}} \right), \tag{13}
\]

\[
G (\lambda) := \sqrt{\frac{\pi}{2}} (\lambda^2 + 1) e^{\lambda^2/2} \text{Erfc} \left( \frac{\lambda}{\sqrt{2}} \right) - \lambda. \tag{14}
\]

We plot $G (\lambda)$ as a function of $\lambda$ in Fig. 2. Eq. (11) represents our starting point for analyzing the dynamics of magnetization. We will now proceed to show how it leads to the LLG equation and the Stoner-Wolfarth model.

\section{IV. EFFECTIVE LANGRANGIAN DESCRIPTION}

To provide a concise phase space picture of the condensate magnetization dynamics, we discuss in this section a collective coordinate Lagrangian appropriate to our system.

Let $M (t) := d (z, t)/d (z, t)$ where the magnetization $d (z, t)$ is defined in Eq. (5). Explicitly, the local magnetization direction reads $M (t) = (\sin \theta \left( t \right) \cos \phi \left( t \right), \sin \theta \left( t \right) \sin \phi \left( t \right), \cos \theta \left( t \right))$. Then, from Eqs. (9) and (10), $F (r, t) = SM (t) |\psi (r, t)|^2$ and one
obtains (see for a detailed derivation Appendix B)
\[
\frac{\partial M}{\partial t} = M \times \{ b + S \Lambda'_{dd} (t) M_z e_z \} - \Gamma M \times \frac{\partial M}{\partial t}, \quad (15)
\]
where the renormalized interaction function \( \Lambda'_{dd} (t) \) reads
\[
\Lambda'_{dd} (t) = \frac{3}{N(t)} \int_{-\infty}^{\infty} dz \, n(z,t) \, P_{dd}(z,t), \quad (16)
\]
and \( N(t) := \int d^3 r \, |\psi(r,t)|^2 = \int_{-\infty}^{\infty} dz \, n(z,t) \). From Eqs. (A9), (A12), and (13), \( \Lambda'_{dd} (t) \) is connected to the dipole-dipole interaction contribution \( V_{dd} (t) \) by
\[
V_{dd} (t) = \frac{3}{2} \hbar S^2 \left\{ \sin^2 \theta(t) - \frac{2}{3} \right\} \int_{-\infty}^{\infty} dz \, n(z,t) \, P_{dd}(z,t) = \frac{\hbar}{2} S^2 N(t) \Lambda'_{dd} (t) \left\{ \frac{1}{3} - \cos^2 \theta(t) \right\}. \quad (17)
\]
We note that in order to obtain the effective quasi-1D dipolar interaction \[17\], we did not use, in distinction to Ref. [27], any simplifying approximation. A detailed derivation is provided in Appendix A.

Eq. (15) is the LLG equation with the external magnetic field in \( z \)-direction modified by the magnetization in \( z \)-direction due to the dipole-dipole interaction. The corresponding term in units of magnetic field, \( \hbar S^2 \Lambda'_{dd} (t) M_z e_z / (g_F \mu_B) \), can be seen as an additional magnetic field that is itself proportional to the magnetization in \( z \)-direction, and which leads to an additional nonlinearity in the LLG equation.

From Eqs. (13) and (16) to get how \( \Lambda'_{dd} (t) \) depends on time \( t \), one has to calculate the double integral
\[
\int dz \int dz' n(z,t) n(z',t) \left\{ G \left( \frac{|z - z'|}{l_\perp} \right) - \frac{4}{3} \delta \left( \frac{z - z'}{l_\perp} \right) \right\}. \quad (18)
\]

To achieve a simple physical picture, we assume that \( n(z,t) \) does not depend on time \( t \) within the time range we are interested in. Then we may write \( \Lambda'_{dd} (t) = \Lambda''_{dd} \). The lifetime of a typical dipolar BEC with large atomic magnetic dipole moments such as \( ^{164}\text{Dy} \), \( ^{162}\text{Dy} \) and \( ^{166}\text{Dy} \), or \( ^{166}\text{Er} \) is of the order of seconds. Since taking into account the time dependence of \( n(z,t) \) generally requires a numerical solution of Eq. (11), we here consider the case where \( n(z,t) \) is constant in time \( t \) as in [26], to predominantly extract the effect of magnetic dipole-dipole interaction per se.

We also neglect the possible effect of magnetostriction. The latter effect, amounting to a distortion of the aspect ratio of the condensate in a harmonic trap as a function of the angle of the external magnetic field with the symmetry axis of the trap, was measured in a condensate of Chromium atoms [38] (with a magnetic moment of \( 6 \mu_B \)). The magnetostriction effect in that experiment was of the order of 10%. For alkali atoms with spin-1 the effect should be a factor \( \delta^2 \) smaller. In addition, theoretical analyses in the Thomas-Fermi limit show that magnetostriction in harmonic traps becomes particularly small for very small or very large asymmetries of the trap [39, 40].

More specifically, Ref. [41] has shown that magnetostriction is due to the force induced by the dipole-dipole mean-field potential \( \Phi_{dd}(r,t) \). In Appendix D we apply the approach of [41] to a dipolar spinor BEC. From Eqs. (10), (17), (A1), and (D5), \( \Lambda'_{dd} (t) \) contains \( \Phi_{dd}(z,t) \) [the quasi-1D form of \( \Phi_{dd}(r,t) \) defined in Eq. (D5)] by
\[
S^2 \left\{ 1 - 3 \frac{M_z^2 (t)}{N(t) \hbar \Lambda'_{dd} (t)} \right\} = 3 \int_{-\infty}^{\infty} dz \, n(z,t) \, \Phi_{dd}(z,t). \quad (19)
\]

Hence, our LLG-type equation in Eq. (15) effectively contains the dipole-dipole mean-field potential which causes magnetostriction and the form of Eq. (15) itself will not be changed whether the effect of magnetostriction is large or not. Only the value of \( \Lambda'_{dd} (t) \) will be changed because magnetostriction changes the integration domain. Furthermore, we show in Appendix D that for our quasi-1D system, the effect of magnetostriction is smaller in a box trap than in harmonic trap. In fact, for the box trap, this effect can be neglected if \( L_z / l_\perp \) is sufficiently large. Thus, we may neglect the effect of magnetostriction under suitable limits for both box and harmonic traps.

To get a simple physical idea of the dynamical behavior of our system, let us, for now, assume that there is no damping, \( \Gamma = 0 \). When the external magnetic field is chosen to lie in the \( x-z \) plane, \( B = (B_x, 0, B_z) \), Eq. (15) becomes
\[
\frac{d\theta}{dt} = b_x \sin \phi, \quad \frac{d\phi}{dt} = b_z \cot \theta \cos \phi - b_z - S \Lambda'_{dd} \cos \theta. \quad (20)
\]
where we already defined the Larmor frequency vector \( b = g_F \mu_B B / \hbar \), below Eq. (3).

By using the Lagrangian formalism introduced in [42],
the Lagrangian $L$ of this system then fulfills

$$
\frac{L}{\hbar} = \dot{\phi} \cos \theta + b_x \sin \theta \cos \phi + b_z \cos \theta + \frac{S}{4} \mathcal{N}_{dd}' \cos (2\theta),
$$

(21)

where $\dot{\phi} = d\phi/dt$. The equations of motion are

$$
\frac{\partial L}{\partial \dot{\theta}} = -\dot{\phi} \sin \theta + b_x \cos \theta \cos \phi - b_z \sin \theta - \frac{S}{2} \mathcal{N}_{dd}' \sin (2\theta),
$$

$$
\frac{\partial L}{\partial \dot{\phi}} = -b_x \sin \theta \sin \phi, \quad \frac{\partial L}{\partial \dot{\phi}} = \cos \theta.
$$

(22)

One easily verifies that Eq. (21) is indeed the Lagrangian which gives Eqs. (20). Let $p_\phi$ be the conjugate momentum of the coordinate $\xi$. Since $p_\theta = 0$ and $p_\phi = \hbar \cos \theta$ ($\hbar$ times the $z$ component of $\mathbf{M}$), the Hamiltonian $H$ is given by

$$
H = -b_z \sqrt{h^2 - p_\phi^2} \cos \phi - b_z p_\phi + \frac{h^2 - 2 p_\phi}{4 \hbar} \mathcal{N}_{dd}'.
$$

(23)

Note that the energy $\tilde{E} := H - \hbar \mathcal{N}_{dd}' / 4$ is conserved. Hence, if we put $p_\phi = (p_\phi)_{in}$ and $\phi = \pi/2$ at some time $t = t_0$, $\tilde{E} = -b_z (p_\phi)_{in} - \mathcal{N}_{dd}' (p_\phi)_{in}^2 / 2 \hbar$. We can then express $\phi$ as a function of $p_\phi$ as

$$
\cos \phi = -\frac{\tilde{E} + b_z p_\phi + \frac{1}{2} \mathcal{N}_{dd}' p_\phi^2}{b_x \sqrt{h^2 - p_\phi^2}} - \frac{b_z + \mathcal{N}_{dd}' (p_\phi)_{in}^2 + p_\phi}{b_x \sqrt{h^2 - p_\phi^2}} (\phi_{\text{in}} - p_\phi).
$$

(24)

The canonical momentum $p_\phi$ remains the initial $(p_\phi)_{in}$ when $b_x = 0$, implying that $\theta$ does not change when $b_x = 0$, consistent with Eqs. (20). If $|b_z|$ is larger than $|b_z \pm \mathcal{N}_{dd}'|$, we can have $p_\phi \neq (p_\phi)_{in}$ with $|\cos \phi| \leq 1$, which allows for the switching process of the magnetization. Below a threshold value of $|b_z|$ that depends on $b_z$ and $\mathcal{N}_{dd}'$, $p_\phi$ has to remain constant for Eq. (24) to be satisfied, which corresponds to simple magnetization precession about the $z$ axis.

When $p_\phi$ is a function of time, there are two important cases:

(a) $|b_z| \gg \mathcal{N}_{dd}'$ : $\cos \phi = \frac{b_z}{b_x} \frac{(p_\phi)_{in} - p_\phi}{\sqrt{h^2 - p_\phi^2}}$,

(b) $|b_z| \ll \mathcal{N}_{dd}'$ : $\cos \phi = \frac{\mathcal{N}_{dd}' (p_\phi)_{in}^2}{2b_x h} \frac{p_\phi}{\sqrt{h^2 - p_\phi^2}}$.

(25)

We plot the corresponding phase diagrams ($\theta$ vs $\phi$) in Fig. 3

Let $(p_\phi)_{in} = \hbar \cos \theta_{\text{in}}, b_x = \hbar \sin \theta_{\text{in}}$ and $b_z = \hbar \cos \theta_{\text{in}}$. When case (a) holds $|b_z| \gg \mathcal{N}_{dd}'$, one concludes that $\cos \theta \cos \phi + \sin \theta \sin \theta \cos \phi = \cos \theta_0 \cos \theta_{\text{in}}$, which is constant. Since $d \cdot B = db (\cos \theta_0 \cos \phi + \sin \theta_0 \sin \phi \cos \phi)$, in case (a) the magnetization $\mathbf{d}$ precesses around the external magnetic field $\mathbf{B}$, as expected. When (b) holds, SW switching can occur, to the description of which we proceed in the following.

V. CONNECTION TO STONER-WOHLFARTH MODEL

The phenomenological SW model can be directly read off from the equations in the preceding section. From Eq. (23), $\tilde{H} := H - \hbar \mathcal{N}_{dd}' / 4$ is given by

$$
\frac{\tilde{H}}{\hbar} = -b_x \sin \theta \cos \phi - b_z \cos \theta + \frac{\mathcal{N}_{dd}'}{2} \sin^2 \theta.
$$

(26)

Let $(b_{\nu})_{\text{cr}}$ be the value of $b_{\nu}$ at the stability limit where $\partial \tilde{H} / \partial \theta = 0$ and $\partial^2 \tilde{H} / \partial \theta^2 = 0$. Then one obtains the critical magnetic fields

$$
(b_x)_{\text{cr}} \cos \phi = \mathcal{N}_{dd} \sin^3 \theta, \quad (b_z)_{\text{cr}} = -\mathcal{N}_{dd} \cos^3 \theta.
$$

(27)

which satisfy the equation

$$
\{(b_x)_{\text{cr}} \cos \phi\}^{2/3} + (b_z)_{\text{cr}}^{2/3} = \mathcal{N}_{dd}'^{2/3}.
$$

(28)

We coin the curve in the $(b_x, b_z)$-plane described by Eq. (28) the switching curve, in accordance with the terminology established in [43]. Because $\phi$ changes in time [see Eqs. (20) and Fig. 3], the switching curve depends in general on the timing of the applied external magnetic fields. We note that, for $\phi = 0$, Eqs. (26) and (28) are identical to the SW energy functional

$$
\frac{H_{SW}}{\hbar} = -b_x \sin \theta - b_z \cos \theta + K \sin^2 \theta
$$

(29)
and the SW astroid [43], respectively, if we identify \( K = SN_{dd}/2 \).

The LLG equation in Eq. (15) has stationary solutions with \( \mathbf{M} \) parallel to the effective magnetic field \( h (b + \Lambda_{dd} (t) M_z e_z) / (g_\mu_B B) \). Since we set \( b \) to lie in the \( \mathbf{e}_x, \mathbf{e}_y \) plane, \( \phi \) will go to zero for sufficiently large times. Thus Eq. (20) leads to the SW model (29) due to the damping term in (15) if \( \Gamma > 0 \). In Appendix C we demonstrate that a more general tensorial damping coefficient \( \Gamma \) introduces additional terms on the righthand side of the LLG equation (15), which involve time derivatives. While these will thus not affect the SW phenomenology, which results from the steady states as function of the applied magnetic fields, and which is thus governed by the vanishing (in the stationary limit) of the first term on the right-hand side of the LLG equation, they affect the detailed relaxation dynamics of the magnetization and its time scales. These deviations can hence be used to probe deviations from assuming a single scalar \( \Lambda \).

Before we move on to the next section, we show the characteristic behavior of \( N_{dd} \) defined in Eq. (16), for a box-trap scenario defined by \( n(z,t) = N/(2L_z) \) for \(-L_z \leq z \leq L_z \) and \( n (z,t) = 0 \) otherwise (\( N \) is number of particles).

We stress that due to the finite size of the trap along the “long” \( z \) direction, in variance with the Hohenberg-Mermin-Wagner theorem holding for infinitely extended systems in the thermodynamic limit, a quasi-1D BEC can exist also at finite temperatures [43]. This remains true up to a ratio of its proper length to the de-Broglie wavelength [45], beyond which strong phase fluctuations set in [46]. In fact, these strongly elongated quasi-1D BECs at finite temperature have been first realized already long ago, cf., e.g. [47].

For the box trap, \( N_{dd} = \Lambda_{dd} (L_z/l_\perp) \) where

\[
\Lambda_{dd}(\lambda) = \frac{3Nc_{dd}}{2h^2l_\perp} \lambda \left\{ \int_0^{2\lambda} dv \left( 1 - \frac{v}{2\lambda} \right) G(v) - \frac{2}{3} \right\}. \tag{30}
\]

From Eq. (14), \( G(v) \approx 2/v^3 + \mathcal{O}(v^{-5}) \) for \( v \gg 1 \), so that

\[
\Lambda_{dd}(\lambda) \simeq \frac{Nc_{dd}}{2h^2l_\perp} \lambda \quad \text{for} \quad \lambda = \frac{l_z}{l_\perp} \gg 1. \tag{31}
\]

Hence \( \Lambda_{dd}(\lambda) \) is a slowly decreasing function of the cigar’s aspect ratio \( \lambda \) (keeping everything else fixed). We will see below that for the parameters of experiments such as [30], the effective magnetic field due to dipolar interactions greatly exceeds the externally applied magnetic fields (in the range relevant for SW switching to be observed) [48].

VI. ANALYTICAL RESULTS FOR AXIALLY DIRECTED EXTERNAL MAGNETIC FIELD

Without dissipation, when \( b_z = 0 \), \( p_\phi = \hbar \cos \theta = \hbar M_z \) is rendered constant; see Eq. (20). However, in the presence of dissipation, \( M_z \) changes in time even if \( b_z = 0 \). By employing this change, we propose an experimental method to measure \( \Gamma \).

For simplicity, we will assume that the number density is constant in time (also see section IV) and the external magnetic field points along the \( z \) direction, \( B = B_z e_z \). Let a critical (see for a detailed discussion below) value of the magnetization be

\[
(M_z)_c := -\frac{b_z}{SN_{dd}}. \tag{32}
\]

Then Eq. (15) can be written as

\[
\frac{\partial \mathbf{M}}{\partial t} = S\Lambda_{dd} \mathbf{M} \times \mathbf{e}_z \{ (M_z) - (M_z)_c \} - \Gamma \mathbf{M} \times \frac{\partial \mathbf{M}}{\partial t} = \mathbf{M} \times \mathbf{e}_z (b_z + S\Lambda_{dd} M_z) - \Gamma \mathbf{M} \times \frac{\partial \mathbf{M}}{\partial t}. \tag{33}
\]

Since \( \mathbf{M} \cdot \frac{\partial \mathbf{M}}{\partial t} = 0 \), by taking the cross product with \( \mathbf{M} \) on both sides of Eq. (15), one can derive an expression for \( \mathbf{M} \times \frac{\partial \mathbf{M}}{\partial t}:
\]

\[
\frac{\partial M_z}{\partial t} = \frac{\Gamma S\Lambda_{dd}}{1 + \Gamma^2} \{ (M_z) - (M_z)_c \} (M_z^2 - 1)
- \frac{\Gamma}{1 + \Gamma^2} (b_z + S\Lambda_{dd} M_z) (M_z^2 - 1). \tag{34}
\]

Since \( \mathbf{M} \) is the scaled magnetization, \( |\mathbf{M}| = 1 \) with a condensate. Hence, \(-1 \leq M_z \leq 1 \). Also, according to the discussion below Eq. (20), the generally positive SW coefficient (with units of frequency) \( K \) is \( SN_{dd}/2 \).

From Eq. (34), for time-independent \( \Lambda_{dd} \), one concludes that there are three time-independent solutions, \( M_z = (M_z)_c \) and \( M_z = \pm 1 \). For a box-trapped BEC and constant number density, \( \Lambda_{dd} = \Lambda_{dd} \) which is always positive in the quasi-1D limit (cf. Eq. (30) and the discussion following it). For some arbitrary physical quasi-1D trap potential, in which the number density is not constant in space, from Eqs. (13), (16), and Fig. 2 one can infer that \( \Lambda_{dd} > 0 \), due to the fact that the quasi-1D number density \( n(z,t) > 0 \), \( n(z,t) \) has its maximum value near \( z = 0 \) for a symmetric trap centered there, and then \( G(\lambda) \) also has its maximum value near \( \lambda = 0 \). Then, if \( |(M_z)_c| < 1 \), \( M_z = (M_z)_c \) is an unstable solution and \( M_z = \pm 1 \) are stable solutions. When \( |(M_z)_c| < 1 \) and \( |(M_z)_c| > 1 \), \( M_z \) goes to \(-1 \). Likewise, \( M_z \) goes to \( 1 \) when \( (M_z)_c < M_z < 1 \). This bifurcation does not occur if \( |(M_z)_c| > 1 \). For simplicity, we assume that \( |(M_z)_c| > 1 \). This is the more interesting case due to the possibility of a bifurcation of stable solutions leading to SW switching. Let \( (M_z)_c \) be the value of \( M_z \) at \( t = 0 \). The analytic
solution of Eq. (34) satisfies

\[
t = \frac{1 + \Gamma^2}{\Gamma S A'_{dd}} \left[ \frac{1}{(M_z)_{cr}^2} - 1 \right] \ln \left( \frac{(M_z)_{in} - (M_z)_{cr}}{M_z - (M_z)_{cr}} \right) \\
- \frac{1}{2} \left[ (M_z)_{cr}^2 - 1 \right] \ln \left( \frac{(M_z)_{in} - (M_z)_{cr}}{1 + (M_z)_{in}} \right) \\
+ \frac{1}{2} \left[ (1 + (M_z)_{in}) \ln \left( \frac{1 + (M_z)_{in}}{1 + M_z} \right) \right] \\
- \frac{1}{2} \left( b^2 - (S A'_dd)^2 \right) \ln \left( \frac{b_z + S A'_{dd}(M_z)_{in}}{b_z + S A'_{dd}M_z} \right) \\
- \frac{1}{2} \left( b_z + S A'_{dd} \right) \ln \left( \frac{1 - M_z}{1 - (M_z)_{in}} \right) \\
- \frac{1}{2} \left( b_z - S A'_{dd} \right) \ln \left( \frac{1 + (M_z)_{in}}{1 + M_z} \right). \tag{35}\]

The above equation tells us that, if \((M_z)_{in} \neq (M_z)_{cr}\) and \((M_z)_{in} \neq \pm 1\), \(M_z\) goes to its stable time-independent solution \(\left| M_z \right| = 1\) at time \(t = \infty\). Thus, we define a critical switching time \(t_{cr}\) to be the time when \(\left| M_z \right| = 0.99\). Also, note that the form of LLG equation (Eq. (33)) does not change whether BEC is confined in a quasi-1D, quasi-2D, or a three-dimensional geometry. This is because one can find a connection between \(\Lambda'_{dd}\) and the effective dipole-dipole-interaction potential \(V_{eff}\), so one can measure \(\Gamma\) even if the BEC is effectively confined in a space with dimension higher than one, using Eq. (35).

We point out, in particular, that \(t_{cr}\) is inversely proportional to \(\Lambda'_{dd}\). Hence, for a constant density quasi-1D BEC confined between \(-L_z \leq z \leq L_z\), \(\Lambda'_{dd} = \Lambda_{dd}(L_z/l_{\perp})\), and thus \(t_{cr}\) is also inversely proportional to the linear number density along \(z\). This follows from the relation between \(\Lambda_{dd}(L_z/l_{\perp})\) and the linear number density along \(z\) displayed in Eq. (30).

For large dipolar interaction, the asymptotic expression for \(t_{cr}\) is, assuming \(\Gamma \ll 1\)

\[
t_{cr} \approx \frac{1}{\Gamma S A'_{dd}} \ln \left[ \frac{ \sqrt{5} (1 - (M_z)_{in}^2) }{ (M_z)_{in} - (M_z)_{cr} } \right] \tag{36}\]

provided \(S A'_{dd} \gg |b_z| \implies |(M_z)_{cr}| \ll 1\).

The above \(t_{cr}\) diverges at \((M_z)_{in} = (M_z)_{cr}\) or \(\pm 1\), as expected, since \(M_z = (M_z)_{cr}\) and \(M_z = \pm 1\) are time-independent solutions of the LLG equation. We stress that Eq. (36) clearly shows that the magnetic dipole-dipole interaction accelerates the decay of \(M_z\). Hence, by using a dipolar spinor BEC with large magnetic dipole moment such as produced from \(^{164}\)Dy or \(^{166}\)Er one may observe the relaxation of \(M_z\) to the stable state within the BEC lifetime, enabling the measurement of \(\Gamma\).

Before we show how the critical switching time \(t_{cr}\) depends on \((M_z)_{in}\) and \(\Gamma\), we will qualitatively discuss when our quasi-1D assumption and homogeneous-local-spin-orientation assumption are valid. Typically, spin-spin-interaction couplings are much smaller than their density-density-interaction counterparts, by two orders of magnitude. For spin 1 \(^{23}\)Na BEC or spin 1 \(^{87}\)Rb BEC, \(c_0 \simeq 100|c_2| \tag{31} \tag{34}\). Thus we may neglect to a first approximation the \(S^2\) times \(c_{2k}\) terms in Eq. (17) (see the discussion at the end of Appendix [D]). We also require \(|(M_z)_{cr}| < 1\). Thus, we may additionally neglect the \(b\) term compared to the \(P_{dd}(z, t)\) term since, for \(b = b_z e_z\), \(S A'_{dd} > |b|\) should be satisfied to make \(|(M_z)_{cr}| < 1\) (see Eq. (32)) and \(\Lambda'_{dd}\) is related to \(P_{dd}(z, t)\) by Eq. (16). When \(\Gamma = 0\), using our ansatz in Eq. (8) and integrating out the \(x\) and \(y\) directions, Eq. (14) can be approximated by the expression

\[
\mu(t) \Psi(z, t) = \left\{ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial z^2} + V(z) + \frac{c_0}{2m^2 l_{\perp}^2} |\Psi(z, t)|^2 + \Phi_{dd}(z, t) \right\} \Psi(z, t), \tag{37}\]

where, from Eqs. (5), (11), and (17), the dipole-dipole interaction mean-field potential reads

\[
\Phi_{dd}(z, t) = \hbar S^2 \left\{ 1 - 3M_z^2(t) \right\} P_{dd}(z, t) \\
= \frac{c_{dd}}{2\pi l_{\perp}^2} S^2 \left\{ 1 - 3M_z^2(t) \right\} \int_{-\infty}^{\infty} dz' |\Psi(z', t)|^2 \left\{ G \left( \frac{|z' - z|}{l_{\perp}} \right) - \frac{4}{3} \delta \left( \frac{z' - z}{l_{\perp}} \right) \right\} \\
= \frac{c_{dd}}{2\pi l_{\perp}^2} \pi S^2 \left\{ 1 - 3M_z^2(t) \right\} \int_{-\infty}^{\infty} d\tilde{z} \left| \Psi(z + zl_{\perp}, t) \right|^2 G(|\tilde{z}|) - \frac{4}{3} |\Psi(z, t)|^2. \tag{38}\]

From Fig. [2] the function \(G(\lambda)\) is positive and decreases exponentially as \(\lambda\) increases. Thus, if \(l_{\perp}\) is small enough such that \(|\Psi(z + zl_{\perp}, t)|^2\) does not change within the range \(|\tilde{z}| \leq 5\), one may conclude that

\[
\Phi_{dd}(z, t) \simeq \frac{2\pi}{3} S^2 \left\{ 1 - 3M_z^2(t) \right\} \frac{c_{dd}}{2\pi l_{\perp}^2} |\Psi(z, t)|^2, \tag{39}\]

due to the property \(\int_{0}^{\infty} d\lambda G(\lambda) = 1\).

A spinor \((S = 6)\) dipolar BEC has been realized using \(^{166}\)Er [30]. For this BEC, \(c_0 = 4\pi \hbar^2 a/m\) where \(a \simeq 67 a_B\) \((a_B\) is Bohr radius) and \(2\pi S^2 c_{dd}/3 = 0.4911 c_0\). Due to \(|M_z(t)| \leq 1\) from the definition of \(\mathbf{M}(t)\), the maximum value of the chemical potential \(\mu(t)\) is achieved when
\( M_z (t) = 0 \), where

\[
\mu(t) \simeq V(z) + \left( c_0 + \frac{2\pi}{3} S^2 c_{dd} \right) \frac{n(z,t)}{2\pi l_\perp^2}.
\]  

(40)

From above Eq. (40), we may regard the 3D number density as \( n(z,t)/(2\pi l_\perp^2) \). In [39], \( N = 1.2 \times 10^5 \), \( \omega_z/(2\pi) = \sqrt{756} \times 198 \text{ Hz} = 175.75 \text{ Hz} \), \( \omega_z/(2\pi) = 17.2 \text{ Hz} \), \( l_\perp = 0.589 \mu\text{m} \), and the measured peak number density \( n_{\text{peak}} \) is \( 6.2 \times 10^{20} \text{ m}^{-3} \). Using Eq. (37) and (39), by denoting \( L_z \) as the Thomas-Fermi radius along \( z \), \((-L_z \leq z \leq L_z)\) with \( V(z) = m\omega_z^2 z^2/2 \), one derives

\[
L_z = \left\{ \frac{3 (c_0 + 2\pi S^2 c_{dd}/3) N}{4\pi m\omega_z^2 l_\perp^2} \right\}^{1/3},
\]  

(41)

and the mean number density \( \bar{n} = (N/2L_z)/(2\pi l_\perp^2) = 6.721 \times 10^{20} \text{ m}^{-3} \) as well as chemical potential \( \mu/(\hbar\omega_z) = m\omega_z^2 L_z^2/(2\hbar\omega_z) = 23.22 \). Note that \( \bar{n} \simeq 1.1 n_{\text{peak}} \). Because \( \mu \) is not less than \( \hbar\omega_z \), the experiment [30] is not conducted within the quasi-1D limit.

The homogeneous-local-spin-orientation approximation is valid when the system size is on the order of the spin healing length \( \xi_s \) or less, which has been experimentally verified in [34]. Using \( c_0 \simeq 100 |c_2| \), \( \xi_s \approx 10 \xi_d \), where \( \xi_d = \sqrt{\hbar^2/(2mc_0\bar{n})} \) is the density healing length and \( \xi_s = \sqrt{\hbar^2/(2mc_2 \bar{n})} \) is the spin healing length. Thus, if \( L_z \) is on the order of \( 10 \xi_d \), the homogeneous-local-spin-orientation approximation is justified.

Using the \( S = 6 \) element \(^{166}\text{Er} \), we can provide numerical values which satisfy both the quasi-1D and homogeneous-local-spin-orientation limits, as well as they enable us to explicitly show how \( t_{cr} \) depends on \( (M_z)_{in} \) in a concretely realizable setup. We consider below two cases: (A) box trap along \( z \) [49] and (B) harmonic trap along \( z \).

### A. Box traps

We set \( V(z) = 0 \) for \( |z| < L_z \) and \( \infty \) otherwise. Then \( n(z,t) = N/(2L_z) \) and we estimate \( \mu \simeq (c_0 + 2\pi S^2 c_{dd}/3) N/(4\pi l_\perp^2 L_z) \) from Eq. (40). In this case, \( A_{dd}' = A_{dd} (L_z/l_\perp) \) as is calculated in Eq. (30). Fixing \( B_z = -0.03 \text{ mG} \) and \( N = 100 \), we consider the following two cases: (1) \( \omega_z/(2\pi) = 2.4 \times 10^4 \text{ Hz} \) and \( L_z = 3.125 \mu\text{m} \). Then \( L_z/l_\perp = 62.03 \), \( \mu/(\hbar\omega_z) = 0.1692 \), and \( L_z/\xi_d = 29.55 \). Thus, the system is in both the quasi-1D and homogeneous-local-spin-orientation limit. \( SA_{dd}(L_z/l_\perp) = 4.074 \times 10^6 \text{ Hz} \), \( hSA_{dd}(L_z/l_\perp)/(|g_f|B) = 0.3896 \text{ mG} \), and \( \theta_{cr} = \cos^{-1}(M_z)_{cr} = 85.67^\circ \).

(2) \( \omega_z/(2\pi) = 1.2 \times 10^4 \text{ Hz} \) and \( L_z = 6.250 \mu\text{m} \). Then \( L_z/l_\perp = 87.72 \), \( \mu/(\hbar\omega_z) = 0.0846 \), and \( L_z/\xi_d = 29.55 \). Again, the system is in both the quasi-1D and homogeneous-local-spin-orientation limits. \( SA_{dd}(L_z/l_\perp) = 1.028 \times 10^4 \text{ Hz} \), \( hSA_{dd}(L_z/l_\perp)/(|g_f|B) = 0.1002 \text{ mG} \), and \( \theta_{cr} = 72.57^\circ \).

Fig. 3 shows the relation between \( t_{cr} \) and \( (M_z)_{in} \).

### B. Harmonic traps

We set \( V(z) = m\omega_z^2 z^2/2 \). Using the Thomas-Fermi approximation, from Eq. (40), \( \mu = m\omega_z^2 L_z^2/2 \) where \( L_z \) is given by Eq. (41). \( c_0 + 2\pi S^2 c_{dd}/3 n(z,t)/(\pi l_\perp^2) = m\omega_z^2 (L_z^2 - z^2) \) for \( |z| \leq L_z \) and \( n(z,t) = 0 \) for \( |z| > L_z \). From this \( n(z,t) \), we performed a numerical integration to calculate \( A_{dd}' \) in Eq. (10). Fixing \( B_z = -0.03 \text{ mG} \), we consider the following two cases:

(1) \( N = 240 \), \( \omega_z/(2\pi) = 2000 \text{ Hz} \), and \( \omega_z/(2\pi) = 50 \text{ Hz} \), for which \( L_z = 5.703 \mu\text{m} \) and \( L_z/l_\perp = 32.68 \). We obtain again the quasi-1D and homogeneous-local-spin-orientation limits since \( \mu/(\hbar\omega_z) = 0.3337 \) and \( L_z/l_\perp = 17.85 \). Furthermore, \( SA_{dd}/(|g_f|B) = 1.602 \times 10^{-1} \text{ mG} \), and \( \theta_{cr} = 79.21^\circ \).

(2) \( N = 340 \), \( \omega_z/(2\pi) = 1000 \text{ Hz} \), and \( \omega_z/(2\pi) = 25 \text{ Hz} \), where \( L_z = 8.070 \mu\text{m} \) and \( L_z/l_\perp = 32.70 \). Again, we have the quasi-1D and with homogeneous-local-spin-orientation limits fulfilled due to \( \mu/(\hbar\omega_z) = 0.3341 \) and

\( \omega_z/(2\pi) = 2.4 \times 10^4 \text{ Hz} \), \( L_z = 3.125 \mu\text{m} \), and \( l_\perp = 0.0504 \mu\text{m} \) where \( N/(4\pi l_\perp^2) = 10.03 \times 10^{20} \text{ m}^{-3} \) ((Mz)cr = 0.0756).

FIG. 4. \( t_{cr} \) as a function of \( (M_z)_{in} \) when \( B = B_z e_z \), where \( B_z = -0.03 \text{ mG} \) and particle number \( N = 100 \). From top to bottom: Red for \( \Gamma = 0.01 \), black for \( \Gamma = 0.03 \), and blue for \( \Gamma = 0.09 \). Lines are from exact analytic formula in Eq. (35), and dot-dashed are from asymptotic expression in Eq. (36). Generally, \( t_{cr} \) decreases as \( \Gamma \) increases. Also, note that \( t_{cr} \) diverges as \( (M_z)_{in} \rightarrow (M_z)_{cr} \). For larger mean number density \( N/(4\pi l_\perp^2) \) (top), the asymptotic expression of \( t_{cr} \) is essentially indistinguishable from the exact analytic formula of \( t_{cr} \).
eral seconds \( \text{[30]} \) and thus, by measuring time between \( \tau \), may depend on spin indices. Conversely, if one provided \( \Gamma \) indeed does not depend on spin indices as for \( \text{[29]} \), the dependence of \( \gamma \) and the mean number density \( \rho / (4\pi L^2) \) is not immediately apparent. Thus, at least for harmonic traps, and in the Thomas-Fermi approximation, one may use the box trap results of Eq. \( \text{[30]} \) for provide an approximate estimate of the behavior of \( \tau \).

\[ N = 240, \omega_{1}/(2\pi) = 2000 \text{ Hz}, \omega_{2}/(2\pi) = 50 \text{ Hz}. \]
\[ L_{z} = 5.703 \mu \text{m and } l_{z} = 0.1745 \mu \text{m where } N/(4\pi L^2) = 1.010 \times 10^{20} \text{ m}^{-3} \text{ (}(M_{z})_{\text{cr}} = 0.1873). \]

FIG. 5. \( \tau \) as a function of \((M_{z})_{\text{in}}\) when \( B = B_{z} e_{z} \) where \( B_{z} = -0.03 \text{ mG} \), for two particle numbers \( N \) as shown. From top to bottom: Red for \( \Gamma = 0.01 \), black for \( \Gamma = 0.03 \), and blue for \( \Gamma = 0.09 \). Lines are from exact analytic formula in Eq. \( \text{[35]} \), and dot-dashed are from asymptotic expression in Eq. \( \text{[36]} \). Generally, \( \tau \) decreases as \( \Gamma \) increases. Also, note that \( \tau \) diverges as \( (M_{z})_{\text{in}} \to (M_{z})_{\text{cr}} \). For larger mean number density \( N/(4\pi L^2) \) (top), the asymptotic expression of \( \tau \) is essentially indistinguishable from the exact analytic formula of \( \tau \).

\[ L_{z}/\xi_{d} = 17.87. \] In addition, \( SN_{\text{dd}} = 8.230 \times 10^{2} \text{ Hz}, hSN_{\text{dd}}(q_{F} \mu_{B}) = 8.019 \times 10^{-2} \text{ mG}, \text{ and } \theta_{c} = 68.03^\circ. \]

Fig. 2 shows for the harmonic traps the relation between \( \tau \) and \((M_{z})_{\text{in}}\).

C. Measurability of critical switching time

Figs. 4 and 5 demonstrate that the critical switching time \( \tau \) is much smaller than the lifetime of BEC (several seconds \( \text{[30]} \) and thus, by measuring \( \tau \) by varying \((M_{z})_{\text{in}}\), one will be able to obtain the value of \( \Gamma \), provided \( \Gamma \) indeed does not depend on spin indices as for example Refs. \( \text{[29]} \) have assumed. Conversely, if one obtains from the measurements a different functional relation which does not follow Eq. \( \text{[35]} \), this implies that \( \Gamma \) may depend on spin indices.

Note that both figures, Figs. 4 and 5, show that \( \tau \) is inversely proportional to \((M_{z})_{\text{cr}}\).

VII. CONCLUSION

For a quasi-1D dipolar spinor condensate with unidirectional local magnetization (that is in the homogeneous-local-spin-orientation limit), we provided an analytical derivation of the Landau-Lifshitz-Gilbert equation and the Stoner-Wohlfarth model. For an external magnetic field along the long axis, we obtained an exact solution of the quasi-1D Landau-Lifshitz-Gilbert equation. Our analytical solution demonstrates that the magnetic dipole-dipole interaction accelerates the relaxation of the magnetization to stable states and hence strongly facilitates observation of this process within the lifetime of typical dipolar spinor BECs. Employing this solution, we hence propose a method to experimentally access the dissipative parameter(s) \( \Gamma \).

We expect, in particular, that our proposal provides a viable tool to verify in experiment whether \( \Gamma \) is indeed independent of spin indices, as commonly assumed, and does not have to be replaced by a tensorial quantity for spinor gases. We hope that this will stimulate further investigations of the dissipative mechanism in dipolar BECs with internal degrees of freedom.

We considered that the magnetization along \( z, M_{z} \), has contributions solely from the atoms residing in the condensate, an approximation valid at sufficiently low temperatures. When the magnetization from noncondensed atoms is not negligible, as considered by Ref. \( \text{[6]} \) for a contact interacting scalar BEC, correlation terms mixing the noncondensed part and the mean field, such as \( \sum_{\beta=-S} \psi_{\beta}^{\dagger}(r,t) (\delta\psi_{\alpha}(r,t) \delta\psi_{\beta}(r,t)) \) will appear on the right-hand side of Eq. \( \text{[3]} \). Here, \( \delta\psi_{\alpha}(r,t) \) is the \( \alpha \)-th component of quantum field excitations above the mean-field ground state in the spinor basis. Considering the effect of these terms is a subject of future studies.

ACKNOWLEDGMENTS

The work of SHS was supported by the National Research Foundation of Korea (NRF), Grant No. NRF-2015-033908 (Global PhD Fellowship Program). SHS also acknowledges the hospitality of the University of Tübingen during his stay in the summer of 2019. URF has been supported by the NRF.
Appendix A: Derivation of the effective potential $V_{\text{eff}}$

The dipole-dipole interaction term $V_{d\!d}(t)$ in the total energy is given by [31]

$$V_{d\!d}(t) = \frac{c_{d\!d}}{2} \int d^3 r \int d^3 r' \sum_{\nu,\nu' = x,y,z} F_\nu(r,t) Q_{\nu,\nu'}(r - r') F_{\nu'}(r',t), \quad (A1)$$

where $c_{d\!d}$ is dipole-dipole interaction coefficient, $F_\nu(r,t) = \psi^\dagger(r,t) \hat{f}_\nu \psi(r,t)$, and $Q_{\nu,\nu'}(r)$ is defined as the tensor

$$Q_{\nu,\nu'}(r) = \frac{r^2 \delta_{\nu,\nu'} - 3 r_{\nu} r_{\nu'}}{r^5} \quad (A2)$$

in spin space, where $r = |r|$ and $r_{\nu} = r \cdot e_{\nu}$, with $e_{\nu}$ being the unit vector along the $\nu$ axis. From now on, we define $\rho = (x,y)$ such that $dxdy = d^2 \rho = d\varphi d\rho \rho$ where $\tan \varphi = y/x$.

Using the convolution theorem, the dipole-dipole interaction term $V_{d\!d}(t)$ can be expressed by

$$V_{d\!d}(t) = \frac{c_{d\!d}}{2} (2\pi)^{D/2} \int d^3 k \, \tilde{n}(k,t) \tilde{n}(-k,t) \tilde{U}_{d\!d}(k,t) \quad (A3)$$

with the Fourier transform

$$\tilde{U}_{d\!d}(\eta,t) = \frac{1}{n(r,t) n(r',t)} \sum_{\nu,\nu' = x,y,z} F_\nu(r,t) Q_{\nu,\nu'}(\eta) F_{\nu'}(r',t), \quad (A4)$$

where $\tilde{g}(k,t) = (2\pi)^{-D/2} \int dr \, g(r,t) e^{i k \cdot r}$ is the Fourier transform of the function $g(r,t)$ in $D$-dimensional space $r$ (in our case, $D = 3$), $\eta = r - r'$, and $n(r,t) = |\psi(r,t)|^2$.

By denoting $k = (k_\rho, k_z)$, where $k_\rho = (k_x, k_y)$ with $k_\rho = \sqrt{k_x^2 + k_y^2}$ and $\tan \varphi_{k_\rho} = k_y/k_x$, with our mean-field wavefunction in Eq. [8], one derives

$$\tilde{n}(k,t) = \frac{1}{n^2 k} \int d^2 \rho \int_0^{\infty} dz \, e^{-\rho^2} n(z,t) e^{i k \rho} = \frac{1}{2\pi} \tilde{n}(k,z,t) e^{-k^2 \rho^2/4}, \quad (A5)$$

where $n(z,t) = |\Psi(z,t)|^2 e^{-2\rho^2 z/t(1+t^2)}$. Note the factor of $(2\pi)^{-1}$ appearing, when compared to Eq. (12) in Ref. [27], which is stemming from our definition of Fourier transform.

Denoting $\eta = |\eta|$, by writing $e_\eta$ for the unit vector along $\eta$, we obtain

$$U_{d\!d}(\eta,t) = \frac{1}{n^2} \sqrt{\frac{6\pi}{5}} \left[ \left\{ Y_2^0 (e_\eta) e^{-2i\phi(t)} + Y_2^{-2} (e_\eta) e^{2i\phi(t)} \right\} S^2 \sin^2 \theta(t) 
\quad - \left\{ Y_2^1 (e_\eta) e^{-i\phi(t)} - Y_2^{-1} (e_\eta) e^{i\phi(t)} \right\} S^2 \sin \{2\theta(t)\} \right]$$

$$\quad + \frac{1}{n^2} \sqrt{\frac{6\pi}{5}} Y_2^0 (e_\eta) \sqrt{\frac{2}{3}} S^2 \{3 \sin^2 \theta(t) - 2\}, \quad (A6)$$

where $Y_{l}^{m}(e_\eta)$ are the usual spherical harmonics. Its Fourier transform $\tilde{U}_{d\!d}(k,t)$ is

$$\tilde{U}_{d\!d}(k,t) = \frac{1}{(2\pi)^{3/2}} \frac{4\pi}{3} S^2 \left\{ 1 - \frac{3}{2} \sin^2 \theta(t) \right\} \left( \frac{3}{k_\rho^2 + k_z^2} - 1 \right)$$

$$\quad + \frac{1}{\sqrt{2\pi} k_\rho} \frac{k_\rho}{k_\rho^2 + k_z^2} S^2 \sin^2 \theta(t) \cos \left\{ 2\varphi_{k_\rho} - 2\phi(t) \right\} + \frac{\sqrt{2}}{\pi} \frac{k_\rho k_z}{k_\rho^2 + k_z^2} S^2 \sin \{2\theta(t)\} \cos \left\{ \varphi_{k_\rho} - \phi(t) \right\}. \quad (A7)$$

By plugging Eq. (A5) and Eq. (A7) into Eq. (A3), we finally obtain $V_{d\!d}(t)$ as

$$V_{d\!d}(t) = \frac{c_{d\!d}}{2} \sqrt{2\pi} \int_{-\infty}^{\infty} dk_z \, \tilde{n}(k_z,t) \tilde{n}(-k_z,t) \frac{2S^2}{l_z^2 \sqrt{2\pi}} \left\{ 1 - \frac{3}{2} \sin^2 \theta(t) \right\} \left\{ \frac{k_z^2 l_z^2}{2} e^{k_z^2 l_z^2/2} E_1 \left( k_z^2 l_z^2/2 \right) - \frac{1}{3} \right\}, \quad (A8)$$
where $E_1(x) = \int_{-\infty}^{\infty} du \ e^{-u}/u$ is exponential integral.

Note that Eq. (A8) can be also written as

$$V_{dd} (t) = \frac{g_{dd}}{2} \sqrt{2 \pi} \int_{-\infty}^{\infty} dk z \ (k_z (t) \ n (-k_z, t) \ V_{eff} (k_z, t) = \frac{g_{dd}}{2} \int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} dz' n (z, t) n (z', t) V_{eff} (z - z', t). \ (A9)$$

Due to the fact that $\tilde{V}_{eff} (k_z, t)$ can be obtained by Eq. (A8), we can get $V_{eff} (z, t)$ by inverse Fourier transform. As a preliminary step, we first write down some integrals of $E_1 (x)$ as follows:

$$\int_{-\infty}^{\infty} dx \ ex^2 E_1 (x^2) e^{-ikx} = \int_{-\infty}^{\infty} dx \ ex^{-ikx} \int_{-\infty}^{\infty} dt \ e^{-(t-x^2)/t} = (\pi)^{3/2} e^{k^2/4} \text{Erfc} (|k|/2). \ (A10)$$

Differentiating Eq. (A10) with respect to $k$ two times results in

$$\int_{-\infty}^{\infty} dx \ x^2 e^{-ikx} E_1 (x^2) e^{-ikx} = - (\pi)^{3/2} \left\{ \frac{1}{2} \left( \frac{k^2}{2} + 1 \right) e^{k^2/4} \text{Erfc} (|k|/2) - \frac{|k|}{2\sqrt{\pi}} - \frac{2}{\sqrt{\pi}} \delta (k) \right\}. \ (A11)$$

Therefore, $V_{eff} (z, t)$ can be calculated as

$$V_{eff} (z, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk z \ \frac{2S^2}{l_1^2 \sqrt{2\pi}} \left\{ 1 - \frac{3}{2} \sin^2 \theta (t) \right\} \left\{ (k_z l_1^2/2) e^{k_z l_1^2/2} E_1 (k_z l_1^2/2) - \frac{1}{3} \right\} e^{-ik z}$$

$$= \frac{S^2}{l_1^2} \left\{ 3 \sin^2 \theta (t) - 1 \right\} \left\{ G (|z|/l_\perp) - \frac{4}{3} \delta (z/l_\perp) \right\}, \ (A12)$$

where $G (x)$ is defined in Eq. (14), and $\delta (x)$ is the Dirac delta function.

The Fourier transform of Eq. (A12) acquires the form

$$\tilde{V}_{eff} (k_z, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dz \ V_{eff} (z, t) e^{ik_z z} = \sqrt{\frac{2}{\pi}} \frac{S^2}{l_1^2} \left\{ \frac{3}{2} \sin^2 \theta (t) - 1 \right\} \left\{ \int_0^{\infty} dv \ G (v) \cos (k_z l_\perp v) - \frac{2}{3} \right\}$$

$$= \sqrt{\frac{2}{\pi}} \frac{S^2}{l_1^2} \left\{ \frac{3}{2} \sin^2 \theta (t) - 1 \right\} \left\{ \int_0^{\infty} du \ \left\{ \sqrt{\pi} (2u^2 + 1) e^{u^2} \text{Erfc} (u) - 2u \right\} \cos \left( \sqrt{2} k_z l_\perp u \right) - \frac{2}{3} \right\}. \ (A13)$$

From [50], the following integral involving the complementary error function is

$$\int_0^{\infty} du \ e^{u^2} \text{Erfc} (u) \cos (bu) = \frac{1}{2\sqrt{\pi}} e^{b^2/4} E_1 (b^2/4). \ (A14)$$

By differentiating Eq. (A14) two times with respect to $b$, we get

$$\int_0^{\infty} du \ u^2 e^{u^2} \text{Erfc} (u) \cos (bu) = - \frac{1}{2\sqrt{\pi}} \left\{ \frac{1}{2} \left( \frac{b^2}{2} + 1 \right) e^{b^2/4} E_1 (b^2/4) - 1 + \frac{2}{b^2} \right\}. \ (A15)$$

Hence, Eq. (A13) becomes

$$\tilde{V}_{eff} (k_z, t) = \sqrt{\frac{2}{\pi}} \frac{S^2}{l_1^2 \sqrt{2\pi}} \left\{ \frac{3}{2} \sin^2 \theta (t) - 1 \right\} \left\{ - \left\{ \frac{1}{2} (k_z l_1^2 + 1) e^{k_z l_1^2/2} E_1 (k_z l_1^2/2) - 1 + \frac{1}{k_z l_1^2} \right\} \right.$$  

$$+ \frac{1}{2} e^{k_z l_1^2/2} E_1 (k_z l_1^2/2) + \frac{1}{k_z l_1^2} - \frac{2}{3} \right\} \left\{ (k_z l_1^2/2) e^{k_z l_1^2/2} E_1 (k_z l_1^2/2) - \frac{1}{3} \right\}. \ (A16)$$

Comparing Eq. (A8) with Eq. (A16), one verifies that Eq. (A12) is the correct result for the effective interaction of the quasi-1D dipolar spinor gas.
Appendix B: Quasi-1D Gross-Pitaevskii equation with dissipation

By introducing an identical damping coefficient for each component of the spinor, cf., e.g. Refs. [26, 29] (i.e. as if each component effectively behaves as a scalar BEC [28]), and neglecting a possible quadratic Zeeman term, the GP equation for a spin-$S$ BEC can be written as [26]

\[
(i - \Gamma) \hbar \frac{\partial \hat{\psi}_\alpha(r,t)}{\partial t} = \left\{ -\frac{\hbar^2}{2m} \nabla^2 + V_\text{tr}(r) + c_0 |\hat{\psi}(r,t)|^2 \right\} \hat{\psi}_\alpha(r,t) - \hbar \sum_{\beta=-S}^{S} \{ b - b_{d\beta}(r,t) \} \cdot \left( \hat{f} \right)_{\alpha,\beta} \hat{\psi}_\beta(r,t) \\
+ \sum_{k=1}^{S} c_{2k} \sum_{\nu_1,\nu_2,\ldots,\nu_k=x,y,z} F_{\nu_1,\nu_2,\ldots,\nu_k}(r,t) \sum_{\beta=-S}^{S} \left( \hat{f}_{\nu_1} \hat{f}_{\nu_2} \cdots \hat{f}_{\nu_k} \right)_{\alpha,\beta} \hat{\psi}_\beta(r,t),
\]

(B1)

where \( \hat{\psi}_\alpha(r,t) \) is the \( \alpha \)-th component of the mean-field wavefunction \( \psi(r,t) \) (the spin-space index \( \alpha \) is an integer taking \( 2S + 1 \) values running from \( -S \) and \( S \)), \( F_{\nu_1,\nu_2,\ldots,\nu_k}(r,t) := \psi_\nu(r,t) \hat{f}_{\nu_1} \hat{f}_{\nu_2} \cdots \hat{f}_{\nu_k} \psi(r,t) \), \( \hbar \hat{f}_\nu \) is the spin-\( S \) operator, \( b = g_F \mu_B B / \hbar \) (\( g_F \) is the Landé g-factor, \( \mu_B \) is the Bohr magneton, and \( B \) the external magnetic field). Finally, \( \hbar b_{d\beta}(r,t) \cdot e_\nu = c_{d\beta} \int \hbar \psi_{\nu,n}^* \left( \sum_{\nu=\nu_1,\nu_2,\ldots,\nu_k=x,y,z} \Psi_{\nu}(r - r') \right) \cdot e_\nu \), where \( e_\nu \) is the unit vector along the \( \nu \) axis \( (\nu = x, y, z) \) [31]. Applying the formalism of Ref. [1] to a spinor BEC assuming that \( \Gamma \) does not depend on spin indices, one just needs to transform \( t \to (1 + \Gamma)^{-1} t \) in Eq. (B1) and (8). We then integrate out the \( x \) and \( y \) directions in Eq. (B1) to obtain the quasi-1D GP equation.

From Eq. (8) in the main text, we have

\[
\int d^2\rho \sum_{\beta=-S}^{S} e^{-\rho^2/(2l_{\perp}^2)} \left\{ \hbar b_{d\beta}(r) \cdot \left( \hat{f} \right)_{\alpha,\beta} \right\} \hat{\psi}_\beta(r,t) = \frac{c_{d\beta}}{2l_{\perp}^2} \int_{-\infty}^{\infty} dz' n(z',t) \left\{ G \left( \frac{|z - z'|}{l_{\perp}} \right) - \frac{4}{3} \delta \left( \frac{|z - z'|}{l_{\perp}} \right) \right\} \Psi(z,t) e^{-i\omega_{\perp}t/2S} \left[ M(t) - 3M_z(t) e_z \right] \sum_{\beta=-S}^{S} \left( \hat{f} \right)_{\alpha,\beta} \zeta_{\beta}(t),
\]

(B2)

where \( \int d^2\rho := \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \) and \( n(z,t) := \int d^2\rho \ |\psi(r,t)|^2 = |\Psi(z,t)|^2 e^{-2i\omega_{\perp}t/(1 + r^2)} \).

For a spin-$S$ BEC, from Eq. (B1), for the trap potential given in Eq. (4) and if we use Eq. (8), by integrating out the \( x \) and \( y \) directions, one acquires the expression

\[
\begin{align*}
(i - \Gamma) \hbar & \frac{\partial \{ \Psi(z,t) \zeta_{\alpha}(t) \}}{\partial t} = \left\{ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial z^2} + V(z) + \frac{c_0}{2\pi l_{\perp}^2} n(z,t) \right\} \Psi(z,t) \zeta_{\alpha}(t) \\
& \quad + \frac{\hbar b}{\hbar} + \hbar S \left\{ M(t) - 3M_z(t) e_z \right\} P_{d\beta}(z,t) \cdot \left\{ \sum_{\beta=-S}^{S} \left( \hat{f} \right)_{\alpha,\beta} \Psi(z,t) \zeta_{\beta}(t) \right\} \\
& \quad + \sum_{k=1}^{S} \frac{c_{2k}}{2\pi l_{\perp}^2} n(z,t) \sum_{\nu_1,\nu_2,\ldots,\nu_k=x,y,z} SM_{\nu_1,\nu_2,\ldots,\nu_k}(t) \left\{ \sum_{\beta=-S}^{S} \left( \hat{f}_{\nu_1} \hat{f}_{\nu_2} \cdots \hat{f}_{\nu_k} \right)_{\alpha,\beta} \Psi(z,t) \zeta_{\beta}(t) \right\},
\end{align*}
\]

(B3)

where \( M_{\nu_1,\nu_2,\ldots,\nu_k}(t) \) is defined in Eq. (12) and

\[
P_{d\beta}(z,t) = \frac{c_{d\beta}}{2\hbar \epsilon_{\perp}} \int_{-\infty}^{\infty} dz' n(z',t) \left\{ G \left( \frac{|z - z'|}{l_{\perp}} \right) - \frac{4}{3} \delta \left( \frac{|z - z'|}{l_{\perp}} \right) \right\} = \frac{c_{d\beta}}{2\hbar \epsilon_{\perp} S} \left\{ 3 \sin^2 \theta(t) - 2 \right\} \int_{-\infty}^{\infty} dz' n(z',t) V_{\text{eff}}(z - z',t),
\]

(B4)

with \( V_{\text{eff}} \) defined in. It is already clear from Eq. (B3) that, besides particle loss from the condensate encoded in a decaying \( |\Psi(z,t)|^2 \), dissipation also leads to a dephasing, i.e. the decay of \( \zeta(t) \) due to the term \(-\Gamma \partial \zeta(t)/\partial t\).

From now on, if there is no ambiguity, and for brevity, we drop the arguments such as \( x, y, z, t \) from the functions.
From Eq. (B3), we then get
\[
\hbar \frac{\partial \zeta_\alpha}{\partial t} = -\frac{\hbar}{\Psi} \frac{\partial \Psi}{\partial t} + \frac{\Gamma + i}{1 + \Gamma^2} \left( -\frac{\hbar^2}{2m \Psi} \frac{\partial^2 \Psi}{\partial z^2} + V + \frac{c_0}{2 \pi l_\perp^2} n \right) \zeta_\alpha + \frac{\Gamma + i}{1 + \Gamma^2} \left\{ \hbar b - S (M - 3M_z e_z) \right\} h P_{dd} \sum_{\beta = -S}^{S} \left( \hat{f}_{\alpha \beta} \zeta_\beta \right).
\]
(B5)
Since \( \frac{\partial |\zeta|^2}{\partial t} = 0 \) due to the normalization \( |\zeta|^2 = 1 \), we then have
\[
0 = 2 \text{Re} \left\{ -\frac{\hbar}{\Psi} \frac{\partial \Psi}{\partial t} + \frac{1}{1 + \Gamma^2} \left( -\frac{\hbar^2}{2m \Psi} \frac{\partial^2 \Psi}{\partial z^2} + V + \frac{c_0}{2 \pi l_\perp^2} n \right) \right\} + \frac{\Gamma}{1 + \Gamma^2} \left\{ \frac{\hbar^2}{2m \Psi} \frac{\partial^2 \Psi}{\partial z^2} - \frac{1}{\Psi^*} \frac{\partial \Psi^*}{\partial z} \right\}
+ \frac{2\Gamma}{1 + \Gamma^2} \left( \hbar b - S (M - 3M_z e_z) \right) h P_{dd} \cdot S M - \frac{2\Gamma}{1 + \Gamma^2} \sum_{k=1}^{S} \frac{c_{2k}}{2 \pi l_\perp^2} \sum_{\nu_1, \nu_2, \ldots, \nu_k = x, y, z} S^2 M^2_{\nu_1, \nu_2, \ldots, \nu_k}. \tag{B6}
\]
Hence the dynamics of the magnetization direction follows the equation
\[
\hbar S \frac{\partial M_{\nu}}{\partial t} = 2 \text{Re} \left\{ \sum_{\alpha, \beta = -S}^{S} \zeta_\alpha^* \left( \hat{f}_{\nu} \right)_{\alpha, \beta} \left( \hbar \frac{\partial \zeta_\beta}{\partial t} \right) \right\}
= -\frac{2\Gamma}{1 + \Gamma^2} S^2 M_{\nu} \left\{ \hbar b - S (M - 3M_z e_z) \right\} h P_{dd} \cdot M + \frac{2\Gamma}{1 + \Gamma^2} S M_{\nu} \sum_{k=1}^{S} \frac{c_{2k}}{2 \pi l_\perp^2} \sum_{\nu_1, \nu_2, \ldots, \nu_k = x, y, z} S^2 M^2_{\nu_1, \nu_2, \ldots, \nu_k}
+ \frac{\Gamma}{1 + \Gamma^2} \sum_{\mu, x, y, z} \left\{ \hbar b_{\mu} - S (M_{\mu} - 3M_{\mu} e_{\mu, z}) \right\} h P_{dd} \cdot M \delta_{\mu, \nu} + (2S - 1) M_{\mu} M_{\nu}
- \frac{1}{1 + \Gamma^2} \sum_{\mu, x, y, z} \left\{ \hbar b_{\mu} - S (M_{\mu} - 3M_{\mu} e_{\mu, z}) \right\} e_{\nu, \mu, \kappa} S M_{\kappa}
-2 \text{Re} \left\{ \frac{\Gamma + i}{1 + \Gamma^2} \sum_{k=1}^{S} \frac{c_{2k}}{2 \pi l_\perp^2} \sum_{\nu_1, \nu_2, \ldots, \nu_k = x, y, z} S M_{\nu_1, \nu_2, \ldots, \nu_k} \sum_{\alpha, \beta = -S}^{S} \zeta_\alpha^* \left( \hat{f}_{\nu} \hat{f}_{\nu_1} \hat{f}_{\nu_2} \ldots \hat{f}_{\nu_k} \right)_{\alpha, \beta} \right\}, \tag{B7}
\]
since the scalar product \( \zeta^* \left( \hat{f}_{\alpha} \hat{f}_{\beta} + \hat{f}_{\beta} \hat{f}_{\alpha} \right) \zeta = S \left\{ \delta_{\alpha, \beta} + (2S - 1) M_{\alpha} M_{\beta} \right\} \). \( \text{[26]} \)

By direct comparison, we can identify Eq. (B8) below as being identical to Eq. (B21) in \( \text{[26]} \), the only difference consisting in the definition of \( M_{\nu_1, \nu_2, \ldots, \nu_k} \). We employ a scaled version of \( M_{\nu_1, \nu_2, \ldots, \nu_k} \), which is normalized to \( S \) in \( \text{[26]} \). From Eq. (7) in the main text,
\[
\sum_{\nu_1, \nu_2, \ldots, \nu_k = x, y, z} M_{\nu_1, \nu_2, \ldots, \nu_k} \sum_{\alpha, \beta = -S}^{S} \zeta_\alpha^* \left( \hat{f}_{\nu_1} \hat{f}_{\nu_2} \ldots \hat{f}_{\nu_k} \right)_{\alpha, \beta} = \sum_{\nu_1, \nu_2, \ldots, \nu_k = x, y, z} M^2_{\nu_1, \nu_2, \ldots, \nu_k} S^2 M_{\nu}, \tag{B8}
\]
which is real. Therefore, Eq. (B7) can be written in the following form
\[
\frac{\partial M}{\partial t} = -\frac{\Gamma}{1 + \Gamma^2} M \times \left[ M \times \left\{ b - S (M - 3M_z e_z) P_{dd} \right\} \right] + \frac{1}{1 + \Gamma^2} M \times \left\{ b - S (M - 3M_z e_z) \right\} \right\} P_{dd}
- \frac{\Gamma}{1 + \Gamma^2} M \times \left( b + 3SP_{dd} M_{zz} e_z \right) - \frac{\Gamma}{1 + \Gamma^2} M \times \left( b + 3SP_{dd} M_{zz} e_z \right)
= M \times \left( b + 3SP_{dd} M_{zz} e_z \right) - \Gamma M \times \frac{\partial M}{\partial t}, \tag{B9}
\]
since \( M \cdot \frac{\partial M}{\partial t} = 0 \) holds.

As \( P \) is a function of \( z \) and \( t \), but \( M \) is independent of \( z \) [\( M \) is the scaled local magnetization and our aim is to study a dipolar spinor BEC with unidirectional local magnetization (the homogeneous-local-spin-orientation limit)], by multiplying with \( n (z, t) \) both sides of Eq. (B9) and integrating along \( z \), we finally get the LLG equation
\[
\frac{\partial M}{\partial t} = M \times \left( b + S \Lambda_{dd} M_{zz} e_z \right) - \Gamma M \times \frac{\partial M}{\partial t}, \tag{B10}
\]
where \( \Lambda_{dd} \) is defined in Eq. (16). Note here that \( \Lambda_{dd} \) becomes \( \Lambda_{dd} (L_z / l_\perp) \) defined in Eq. (30) when \( n (z, t) = N / (2L_z) \) for \( -L_z \leq z \leq L_z \) and \( n (z, t) = 0 \) otherwise.
Appendix C: Modification of the LLG equation for $\Gamma$ a spin-space tensor

When $\Gamma$ depends on spin indices, i.e. is a tensor, Eq. \((\text{B3})\) can be generalized to read

$$\sum_{\beta=-S}^{S} (i\delta_{\alpha,\beta} - \Gamma_{\alpha,\beta}) \hbar \frac{\partial \{ \Psi(z,t) \zeta_{\beta}(t) \}}{\partial t} = \sum_{\beta=-S}^{S} H_{\alpha,\beta} \Psi(z,t) \zeta_{\beta}(t). \tag{C1}$$

The spinor part of the wavefunction is normalized to unity, $|\zeta|^2 = 1$. Hence, we know that $\frac{\partial \zeta^2}{\partial t} = 0$. Therefore, from Eq. \((\text{C1})\), we derive the expression

$$\sum_{\alpha,\beta=-S}^{S} \text{Re} \left[ -i \zeta^*_{\alpha} \Gamma_{\alpha,\beta} \frac{\partial \zeta_{\beta}}{\partial t} - i \zeta_{\alpha} \Gamma_{\alpha,\beta} \frac{1}{\Psi} \frac{\partial \Psi}{\partial t} - i \frac{1}{\hbar \Psi} \zeta^*_{\alpha} H_{\alpha,\beta} \Psi \zeta_{\beta} \right] - \text{Re} \left[ \frac{1}{\Psi} \frac{\partial \Psi}{\partial t} \right] = 0. \tag{C2}$$

This then leads us to

$$\frac{\partial M_{\nu}}{\partial t} = \frac{2}{S} \sum_{\alpha,\beta,\gamma=-S}^{S} \text{Re} \left[ -i \zeta^*_{\alpha} \Gamma_{\alpha,\gamma} \frac{\partial \zeta_{\gamma}}{\partial t} - i \zeta_{\alpha} \Gamma_{\alpha,\gamma} \frac{1}{\Psi} \frac{\partial \Psi}{\partial t} - i \frac{1}{\hbar \Psi} \zeta^*_{\alpha} \Gamma_{\beta,\gamma} \Psi \zeta_{\beta} \right] - 2 \text{Re} \left[ M_{\nu} \frac{1}{\Psi} \frac{\partial \Psi}{\partial t} \right]. \tag{C3}$$

For scalar $\Gamma$, $\Gamma_{\alpha,\beta} \rightarrow \Gamma \delta_{\alpha,\beta}$, the equation above becomes Eq. \((\text{B7})\).

From Eqs. \((\text{C2})\) and \((\text{C3})\), one concludes that the stationary solution $M_{\nu}$ of Eq. \((\text{C3})\) is independent of $\Gamma$. In other words, whether $\Gamma$ depends on spin indices or not, the SW model \((\text{29})\) is left unaffected, also see the discussion in Section \(\text{V}\) of the main text.

Appendix D: Description of magnetostriction

For a dipolar spinor BEC without quadratic Zeeman term, when there is no dissipation $(\Gamma = 0)$, the mean-field equation in Eq. \((3)\) can be written as

$$\mu_{\alpha} \psi_{\alpha}(r,t) = \left\{ -\frac{\hbar^2}{2m} \nabla^2 + V_{\text{tr}}(r) + c_0 \sum_{\beta=-S}^{S} |\psi_{\beta}(r,t)|^2 \right\} \psi_{\alpha}(r,t) - \hbar \{ \mathbf{b} - b_{\text{dd}}(r,t) \} \cdot \sum_{\beta=-S}^{S} \left\langle \hat{f} \right\rangle_{\alpha,\beta} \psi_{\beta}(r,t)$$

$$+ \sum_{k=1}^{S} c_{2k} \sum_{\nu_1,\nu_2,\cdots,\nu_k = x,y,z} \sum_{\alpha,\beta,\gamma=-S}^{S} \left\langle \hat{f}_{\nu_1} \hat{f}_{\nu_2} \cdots \hat{f}_{\nu_k} \right\rangle_{\alpha,\beta,\gamma} \psi^*_{\alpha,\beta,\gamma}(r,t) \psi_{\beta_1}(r,t) \psi_{\beta_2}(r,t). \tag{D1}$$

where we have substituted $i\hbar \frac{\partial \psi_{\alpha}(r,t)}{\partial t} = \mu_{\alpha}(t) \psi_{\alpha}(r,t)$.

Since we consider the homogeneous-local-spin-orientation limit, we may write $\psi_{\alpha}(r,t) = \psi_{\text{uni}}(r,t) \zeta_{\alpha}(t)$. In this limit, we have

$$|\psi(r,t)|^2 := \psi^\dagger(r,t) \psi(r,t) = \sum_{\alpha=-S}^{S} \psi^\dagger_{\alpha}(r,t) \psi_{\alpha}(r,t) = |\psi_{\text{uni}}(r,t)|^2, \tag{D2}$$

since $\sum_{\alpha=-S}^{S} |\zeta_{\alpha}(t)|^2 = 1$ from the definition of $\zeta_{\alpha}(t)$ in Eq. \((7)\). Thus $|\psi_{\text{uni}}(r,t)|^2$ is equal to the number density. Then Eq. \((\text{D1})\) can be written as

$$\mu_{\alpha} \zeta_{\alpha}(t) \Psi_{\text{uni}}(r,t) = \left\{ -\frac{\hbar^2}{2m} \nabla^2 + V_{\text{tr}}(r) + c_0 |\psi_{\text{uni}}(r,t)|^2 \right\} \zeta_{\alpha}(t) \Psi_{\text{uni}}(r,t)$$

$$- \hbar \{ \mathbf{b} - b_{\text{dd}}(r,t) \} \cdot \sum_{\beta=-S}^{S} \left\langle \hat{f} \right\rangle_{\alpha,\beta} \zeta_{\beta}(t) \Psi_{\text{uni}}(r,t)$$

$$+ \sum_{k=1}^{S} c_{2k} \sum_{\nu_1,\nu_2,\cdots,\nu_k = x,y,z} \sum_{\beta=-S}^{S} M_{\nu_1,\nu_2,\cdots,\nu_k}(t) \left\langle \hat{f}_{\nu_1} \hat{f}_{\nu_2} \cdots \hat{f}_{\nu_k} \right\rangle_{\alpha,\beta} \zeta_{\beta}(t) |\psi_{\text{uni}}(r,t)|^2 \Psi_{\text{uni}}(r,t). \tag{D3}$$
Now, we decompose the chemical potential $\mu(t)$ as $\mu(t) := \sum_{\alpha=-S}^{S} \mu_{\alpha}(t) |\zeta_{\alpha}(t)|^{2}$. Then one obtains

$$
\mu(t) \Psi_{\text{uni}}(r,t) = \left[ -\frac{\hbar^{2}}{2m} \nabla^{2} + V_{t_{\text{e}}} (r) + \left\{ c_{0} + S^{2} \sum_{k=1}^{S} c_{2k} \sum_{\nu_{1},\nu_{2},\cdots,\nu_{k} = x,y,z} M_{\nu_{1},\nu_{2},\cdots,\nu_{k}}^{2} (t) \right\} |\Psi_{\text{uni}}(r,t)|^{2} \right] \Psi_{\text{uni}}(r,t) \\
+ [\Phi_{dd}(r,t) - \hbar \{ b \cdot M(t) \}] |\Psi_{\text{uni}}(r,t)|^{2},
$$

where

$$
\Phi_{dd}(r,t) := S^{2}c_{dd} \left[ \int d^{3}r' \left\{ \sum_{\nu,\nu' = x,y,z} M_{\nu}(t) Q_{\nu,\nu'} (r - r') M_{\nu'} (t) \right\} |\Psi_{\text{uni}}(r',t)|^{2} \right],
$$

is the dipole-dipole mean-field potential following from the definition of $b_{dd}$ below Eq. (3) in the main text. Due to $M_{x}(t) = \sin \theta(t) \cos \phi(t)$, $M_{y}(t) = \sin \theta(t) \sin \phi(t)$, and $M_{z} = \cos \theta(t)$, from Eqs. (A4) and (A6), we have

$$
\sum_{\nu,\nu' = x,y,z} M_{\nu}(t) Q_{\nu,\nu'} (\eta) M_{\nu'} (t) = -\frac{1}{\eta^{3}} \sqrt{\frac{6\pi}{5}} \left[ \left\{ Y^{2}_{2} (e_{\eta}) e^{-2i\phi(t)} + Y^{2}_{2} (e_{\eta}) e^{2i\phi(t)} \right\} \sin^{2} \theta(t) \\
- \left\{ Y^{1}_{2} (e_{\eta}) e^{-i\phi(t)} - Y^{1}_{2} (e_{\eta}) e^{i\phi(t)} \right\} \sin \{ 2\theta(t) \} \right] \\
+ \frac{1}{\eta^{3}} \sqrt{\frac{6\pi}{5}} Y^{2}_{2} (e_{\eta}) \sqrt{\frac{2}{3}} \left\{ 3 \sin^{2} \theta(t) - 2 \right\},
$$

where $Y^{m}_{l} (e_{\eta})$ are the usual spherical harmonics.

By using Eq. (A2), an alternative form of Eq. (D6) can be obtained:

$$
\sum_{\nu,\nu' = x,y,z} M_{\nu}(t) Q_{\nu,\nu'} (\eta) M_{\nu'} (t) = \sum_{\nu,\nu' = x,y,z} \frac{\eta^{2} \delta_{\nu,\nu'} - 3 \eta_{l} e_{\eta} e_{\nu}^{*} M_{\nu}(t) M_{\nu'}(t)}{\eta^{5}} = \frac{\eta^{2} | M(t) |^{2} - 3 \{ \eta \cdot M(t) \}^{2}}{\eta^{5}}.
$$

Thus, $\Phi_{dd}(r,t)$ can be written as

$$
\Phi_{dd}(r,t) = S^{2}c_{dd} \left[ \int d^{3}r' \left\{ \frac{|r - r'|^{2}}{|r - r'|^{5}} - 3 \{ (r - r') \cdot M(t) \}^{2} \right\} |\Psi_{\text{uni}}(r',t)|^{2} \right] \\
= S^{2}c_{dd} \left[ \int d^{3}r' \left\{ \frac{|r - r'|^{2}}{|r - r'|^{5}} - 3 \{ (r - r') \cdot M(t) \}^{2} \right\} |\Psi_{\text{uni}}(r',t)|^{2} \right] \\
= -\frac{3}{2} S^{2}c_{dd} \sin^{2} \theta(t) \int d^{3}\bar{\eta} \ |\Psi_{\text{uni}}(\bar{\eta} + \bar{r},t)|^{2} \frac{1}{\eta^{6}} \left[ 3 \bar{\eta}^{2} - 3 \bar{\eta}_{z}^{2} - 2 \{ \bar{\eta}_{x} \sin \phi(t) - \bar{\eta}_{y} \cos \phi(t) \} \right] \\
- 3 S^{2}c_{dd} \sin \{ 2\theta(t) \} \int d^{3}\bar{\eta} \ |\Psi_{\text{uni}}(\bar{\eta} + \bar{r},t)|^{2} \frac{2}{\eta^{5}} \left[ \bar{\eta}_{z} \cos \phi(t) + \bar{\eta}_{y} \sin \phi(t) \right] \\
+ \frac{1}{2} S^{2}c_{dd} \left\{ 1 - 3 \cos^{2} \theta(t) \right\} \int d^{3}\bar{\eta} \ |\Psi_{\text{uni}}(\bar{\eta} + \bar{r},t)|^{2} \frac{1}{\eta^{6}} \left( 3 \bar{\eta}^{2}_{z} - \bar{\eta}^{2} \right).
$$

where $\bar{r} := r/L$ with $L$ being some length which scales $r$ (so that $\bar{r}$ is a dimensionless vector). For example, in quasi-1D with trap potential being Eq. (3), $L = l_{\perp}$. Note that, in the special case where $M(t) = M_{z}(t) e_{z}$, the form of Eq. (D8) becomes identical to Eq. (6) in Ref. [40].

Since we concentrate on quasi-1D gases, with trap potential given by Eq. (3) in the main text, we will explicitly compute the form of $\Phi_{dd}(r,t)$ for the quasi-1D setup. By writing

$$
|\Psi_{\text{uni}}(r,t)|^{2} = \frac{e^{-r^{2}/l_{\perp}^{2}}}{\pi l_{\perp}^{2}} |\Psi(z,t)|^{2},
$$

(D9)
and integrating out $x$ and $y$ directions, one can get the quasi-1D dipole-dipole-interaction mean-field potential $\Phi_{dd}(z,t)$ as follows (which is in Eq. (38)):

$$\Phi_{dd}(z,t) = \frac{c_{dd}}{2l^2} S^2 \left\{ 1 - 3M_z^2(t) \right\} \left\{ \int_{-\infty}^{\infty} \frac{d\tilde{z}}{\bar{z}} \frac{1}{|\tilde{z}|} \left| \frac{1}{3} \Psi(z+\bar{z}l_{\perp},t) \right|^2 G(|\tilde{z}|) \right\}. \quad (D10)$$

Now, let us consider box trap in quasi-1D case, i.e. $V(z) = 0$ for $|z| \leq L_z$ and $V(z) = \infty$ for $|z| > L_z$ where $V(z)$ is in Eq. (4). Then we may write

$$|\Psi(z,t)|^2 = \begin{cases} \frac{N}{2L_z} & \text{for } |z| \leq L_z, \\ 0 & \text{for } |z| > L_z, \end{cases} \quad (D11)$$

since $V(z) = 0$ for $-L_z \leq z \leq L_z$. Thus, $\Phi_{dd}(z,t)$ can be written as

$$\Phi_{dd}(z,t) = \begin{cases} \bar{\Phi}_{dd}(t) \left\{ \int_{-(L_z+z)/l_{\perp}}^{(L_z-z)/l_{\perp}} d\tilde{z} G(|\tilde{z}|) - \frac{4}{3} |\Psi(z,t)|^2 \right\} & \text{for } |z| \leq L_z, \\ \bar{\Phi}_{dd}(t) \left\{ \int_{-(L_z+z)/l_{\perp}}^{(L_z-z)/l_{\perp}} d\tilde{z} G(|\tilde{z}|) \right\} & \text{for } |z| > L_z, \end{cases} \quad (D12)$$

where $\bar{\Phi}_{dd}(t) := Nc_{dd}S^2 \left\{ 1 - 3M_z^2(t) \right\} / (2L_zl_{\perp}^2)$. $\Phi_{dd}(z,t)$ is discontinuous at $z = \pm L_z$ because of the sudden change of the density at the boundary ($z = \pm L_z$) due to box trap potential.

Defining the scaled density-density mean-field potential $\bar{\Phi}_{dd}(z) := \Phi_{dd}(z,t)/\bar{\Phi}_{dd}(t)$, we obtain Fig. 6 for two different axial extensions, $L_z/l_{\perp} = 10$ and 30. As Fig. 6 clearly illustrates, in a box-trapped quasi-1D gas, $\Phi_{dd}(z,t)$ becomes approximately constant for $|z| < L_z$ and $L_z \rightarrow \infty$ for $L_z/l_{\perp} \gg 1$. Depending on the value of $M(t)$, $\Phi_{dd}(r,t)$ will introduce either a repulsive or an attractive force. This force will however exist only near the boundary for a box trap, where it can lead to a slight modification of the density of atoms. Its relative influence decreases with increasing extension of the trapped gas along the $z$ axis, and can therefore be consistently neglected in the approximation of constant particle-density.

However, to assess whether significant magnetostriction occurs, one has to consider, in addition to $\Phi_{dd}$, the trap potential $V_{tr}$ and the ‘quasi’ density-density interaction mean field potential $\Phi_0$ defined as

$$\Phi_0(r,t) := \left\{ c_0 + S^2 \sum_{k=1}^{S} c_{2k} \sum_{\nu_1,\nu_2,\cdots,\nu_k} M_{\nu_1,\nu_2,\cdots,\nu_k}(t) \right\} |\Psi_{uni}(r,t)|^2. \quad (D13)$$

We can coin $\Phi_0(r,t)$ a ‘quasi’ density-density interaction mean field potential because only $c_0$ is a density-density interaction coefficient ($c_{2k}$ are interaction coefficients parametrizing the spin-spin interactions for spin-$S$ gas where $k$ is an integer with $1 \leq k \leq S$). For example, $c_2$ is the spin-spin interaction coefficient of a spin-1 gas). In our quasi-1D case, this $\Phi_0(r,t)$ potential is $\Phi_0(z,t)$ where

$$\Phi_0(z,t) := \left\{ \frac{c_0}{2\pi l_{\perp}^2} + S^2 \sum_{k=1}^{S} \frac{c_{2k}}{2\pi l_{\perp}^2} \sum_{\nu_1,\nu_2,\cdots,\nu_k} M_{\nu_1,\nu_2,\cdots,\nu_k}(t) \right\} |\Psi(z,t)|^2. \quad (D14)$$
In the main text, we assume that $c_0 \gg S^2 \sum_{k=1}^{S} c_{2k} \sum_{\nu_1, \nu_2, \cdots, \nu_k=1}^{S} M^2_{\nu_1, \nu_2, \cdots, \nu_k}(t)$. For spin-1 $^{23}$Na or $^{87}$Rb, $S = 1$ and $c_0 \approx 100 |c_2|$ [31, 34], so this is an appropriate assumption (note that $\sum_{k=1}^{S} c_{2k} \sum_{\nu_1, \nu_2, \cdots, \nu_k=1}^{S} M^2_{\nu_1, \nu_2, \cdots, \nu_k}(t) = 1$).

The values of the $c_{2k}$ are not yet established for $^{166}$Er. We therefore tacitly assume in the main text, when calculating concrete numerical examples for $^{166}$Er, that the above condition also still holds, despite the prefactor $S^2$ enhancing the importance of spin-spin interactions in $\Phi_0(z, t)$. When this assumption is not applicable, one is required to take into account the time dependence of $\Phi_0(z, t)$ due to $M(t)$ together with magnetostriction due to $\Phi_{dd}(z, t)$, which will change the system size $L_z$ as a function of $t$. This will in turn change the integration domain and quasi-1D density $n(z, t) = |\Psi(z, t)|^2$ in Eq. (19), and incur also a changed time dependence of $\lambda_{dd}(t)$, and the solution of the coupled system of equations [110] and [D4] needs to be found self-consistently.

For a harmonic trap, due to the resulting inhomogeneity of $|\Psi(t)|^2$, $\Phi_{dd}(z, t)$ will have more significant spatial dependence than its box trap counterpart shown in Fig. 6. Here, we note that Ref. [10] has already shown, for a spin-polarized gas, that magnetostriction occurs in a harmonic trap. The effect of magnetostriction is generally expected to be larger in a harmonic trap when compared to a box trap with similar geometrical and dynamical parameters for large relative system size $L_z/I_\perp \gg 1$, at least under the above condition that the $S^2 c_{2k}/c_0$ are sufficiently small.

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