q-Bernstein Polynomials Associated with q-Stirling Numbers and Carlitz’s q-Bernoulli Numbers

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Abstract Recently, T. Kim([4]) introduced q-Bernstein polynomials which are different q-Bernstein polynomials of Phillips([12]). In this paper, we give p-adic q-integral representation for Kim’s q-Bernstein polynomials and investigate some interesting identities of q-Bernstein polynomials associated with q-extension of binomial distribution, q-Stirling numbers and Carlitz’s q-Bernoulli numbers.

1. Introduction

Let \( p \) be a fixed prime number. Throughout this paper, \( \mathbb{Z}_p, \mathbb{Q}_p, \mathbb{C} \) and \( \mathbb{C}_p \) denote the ring of \( p \)-adic integers, the field of \( p \)-adic rational numbers, the complex number field and the completion of algebraic closure of \( \mathbb{Q}_p \), respectively. Let \( \mathbb{N} \) be the set of natural numbers and \( \mathbb{Z}_+ = \mathbb{N} \cup \{ 0 \} \). Let \( \nu_p \) be the normalized exponential valuation of \( \mathbb{C}_p \) with \( |p|_p = p^{-\nu_p(p)} = \frac{1}{p} \).

When one talks of q-extension, \( q \) is variously considered as an indeterminate, a complex number \( q \in \mathbb{C} \) or \( p \)-adic number \( q \in \mathbb{C}_p \). If \( q \in \mathbb{C} \), one normally assumes \( |q| < 1 \), and if \( q \in \mathbb{C}_p \), one normally assumes \( |1 - q|_p < 1 \).

The q-bosonic natural numbers are defined by
\[
[n]_q = \frac{1-q^n}{1-q} = 1 + q + q^2 + \cdots + q^{n-1}
\]
for \( n \in \mathbb{N} \), and the q-factorial is defined by \( [n]_q! = [n]_q [n-1]_q \cdots [2]_q [1]_q \). For the q-extension of binomial coefficient, we use the following notation in the form of
\[
\binom{n}{k}_q = \frac{[n]_q!}{[n-k]_q! [k]_q!} = \frac{[n]_q [n-1]_q \cdots [n-k+1]_q}{[k]_q!}.
\]

Let \( C[0,1] \) denote the set of continuous functions on \([0,1] \subset \mathbb{R} \). Then Bernstein operator for \( f \in C[0,1] \) is defined by
\[
\mathbb{B}_n(f|x) = \sum_{k=0}^{n} f(k) \binom{n}{k} x^k (1-x)^{n-k} = \sum_{k=0}^{n} f(k) B_{k,n}(x),
\]
where \( n, k \in \mathbb{Z}_+ \). The polynomials \( B_{k,n}(x) = \binom{n}{k} x^k (1-x)^{n-k} \) are called Bernstein polynomials of degree \( n \) (see [1]). For \( f \in C[0,1] \), Kim’s q-Bernstein operator of order \( n \) for \( f \) is defined by
\[
\mathbb{B}_{n,q}(f|x) = \sum_{k=0}^{n} f(k) \binom{n}{k} [x]_q^k (1-x)^{n-k} = \sum_{k=0}^{n} f(k) B_{k,n}(x, q),
\]

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where \( n, k \in \mathbb{Z}_+ \). Here \( B_{k,n}(x, q) = \binom{n}{k} [x]^k_q [1 - x]^{n-k}_q \) are called the Kim’s \( q \)-Bernstein polynomials of degree \( n \) (see [4]).

We say that \( f \) is uniformly differentiable function at a point \( a \in \mathbb{Z}_p \), and write \( f \in UD(\mathbb{Z}_p) \), if the difference quotient \( F_l(x, y) = \frac{f(x) - f(y)}{x - y} \) has a limit \( f'(a) \) as \( (x, y) \to (a, a) \). For \( f \in UD(\mathbb{Z}_p) \), the \( p \)-adic \( q \)-integral on \( \mathbb{Z}_p \) is defined by

\[
I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N \to \infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x) q^x, \quad \text{(see [6]).}
\]

Carlitz’s \( q \)-Bernoulli numbers can be represented by \( p \)-adic \( q \)-integral on \( \mathbb{Z}_p \) as follows:

\[
\int_{\mathbb{Z}_p} [x]^n_q d\mu_q(x) = \lim_{N \to \infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} [x]^n_q q^x = \beta_{n,q}, \quad \text{(see [6, 7]).}
\]

The \( k \)-th order factorial of the \( q \)-number \( [x]_q \), which is defined by

\[
[x]_{k,q} = [x]_q [x-1]_q \cdots [x-k+1]_q = \frac{(1-q^x)(1-q^{x-1}) \cdots (1-q^{x-k+1})}{(1-q)^k},
\]

is called the \( q \)-factorial of \( x \) of order \( k \) (see [6]).

In this paper, we give \( p \)-adic \( q \)-integral representation for Kim’s \( q \)-Bernstein polynomials and derive some interesting identities for the Kim’s \( q \)-Bernstein polynomials associated with \( q \)-extension of binomial distribution, \( q \)-Stirling numbers and Carlitz’s \( q \)-Bernoulli numbers.

### 2. \( q \)-Bernstein Polynomials

In this section, we assume that \( 0 < q < 1 \). Let \( \mathbb{P}_q = \{ \sum_i a_i [x]^i_q \mid a_i \in \mathbb{R} \} \) be the space of \( q \)-polynomials of degree less than or equal to \( n \).

For \( f \in C[0, 1] \) and \( n, k \in \mathbb{Z}_+ \), Kim’s \( q \)-Bernstein operator of order \( n \) for \( f \) is defined by

\[
\mathbb{B}_{n,q}(f|x) = \sum_{k=0}^{n} f\left( \frac{k}{n} \right) B_{k,n}(x, q). \quad \text{(2)}
\]

Here \( B_{k,n}(x, q) = \binom{n}{k} [x]^k_q [1 - x]^{n-k}_q \) are the Kim’s \( q \)-Bernstein polynomials of degree \( n \) (see [4]).

Kim’s \( q \)-Bernstein polynomials of degree \( n \) is a basis for the space of \( q \)-polynomials of degree less than or equal to \( n \). That is, Kim’s \( q \)-Bernstein polynomials of degree \( n \) is a basis for \( \mathbb{P}_q \).

We see that Kim’s \( q \)-Bernstein polynomials of degree \( n \) span the space of \( q \)-polynomials. That is, any \( q \)-polynomials of degree less than or equal to \( n \) can be written as a linear combination of the Kim’s \( q \)-Bernstein polynomials of degree \( n \). For \( n, k \in \mathbb{Z}_+ \) and \( x \in [0, 1] \), we have

\[
B_{k,n}(x, q) = \sum_{l=k}^{n} \binom{n}{l} \binom{l}{k} (-1)^{l-k} [x]^l_q, \quad \text{(3)}
\]
If there exist constants $C_0, C_1, \ldots, C_n$ such that $C_0 B_{0,n}(x, q) + C_1 B_{1,n}(x, q) + \cdots + C_n B_{n,n}(x, q) = 0$ holds for all $x$, then we can derive the following equation from (3):

$$
0 = C_0 B_{0,n}(x, q) + C_1 B_{1,n}(x, q) + \cdots + C_n B_{n,n}(x, q)
$$

$$
= C_0 \sum_{i=0}^{n} (-1)^i \binom{n}{i} \binom{i}{0} [x]^i_q + C_1 \sum_{i=1}^{n} (-1)^{i-1} \binom{n}{i} \binom{i}{1} [x]^i_q
$$

$$
+ \cdots + C_n \sum_{i=n}^{n} (-1)^{i-n} \binom{n}{i} \binom{i}{n} [x]^i_q
$$

$$
= C_0 + \{ \sum_{i=0}^{1} C_i (-1)^{i-1} \binom{n}{1} \binom{1}{i} \} [x]^1_q + \cdots + \{ \sum_{i=0}^{n} C_i (-1)^{i-n} \binom{n}{n} \binom{n}{i} \} [x]^n_q.
$$

Since the power basis is a linearly independent set, it follows that

$$
C_0 = 0,
$$

$$
\sum_{i=0}^{1} C_i (-1)^{i-1} \binom{n}{1} \binom{1}{i} = 0,
$$

$$
\vdots
$$

$$
\sum_{i=0}^{n} C_i (-1)^{i-n} \binom{n}{n} \binom{n}{i} = 0,
$$

which implies that $C_0 = C_1 = \cdots = C_n = 0$ ($C_0$ is clearly zero, substituting this in the second equation gives $C_1 = 0$, substituting these two into the third equation gives $C_2 = 0$, and so on).

Let us consider a $q$-polynomial $P_q(x) \in P_q$ which is written by a linear combination of Kim’s $q$-Bernstein basis functions as follows:

$$
P_q(x) = C_0 B_{0,n}(x, q) + C_1 B_{1,n}(x, q) + \cdots + C_n B_{n,n}(x, q).
$$

(4)

It is easy to write (4) as a dot product of two values.

$$
P_q(x) = (B_{0,n}(x, q), B_{1,n}(x, q), \ldots, B_{n,n}(x, q)) \left( \begin{array}{c} C_0 \\ C_1 \\ \vdots \\ C_n \end{array} \right).
$$

(5)

From (5), we can derive the following equation:

$$
P_q(x) = (1, [x]^1_q, \ldots, [x]^n_q) \left( \begin{array}{cccc} b_{00} & 0 & 0 & \cdots \\ b_{10} & b_{11} & 0 & \cdots \\ \vdots & \vdots & \ddots & \vdots \\ b_{n0} & b_{n1} & b_{n2} & \cdots & b_{nn} \end{array} \right) \left( \begin{array}{c} C_0 \\ C_1 \\ \vdots \\ C_n \end{array} \right),
$$

where the $b_{ij}$ are the coefficients of the power basis that are used to determine the respective Kim’s $q$-Bernstein polynomials. We note that the matrix in this case is lower triangular.
From (2) and (3), we note that
\[ B_{0,2}(x, q) = [1 - x]^2_q = \sum_{l=0}^{2} \binom{2}{l} (-1)^l [x]_q^l = 1 - 2[x]_q + [x]^2_q, \]
\[ B_{1,2}(x, q) = \binom{2}{1} [x]_q [1 - x]^1_q = 2[x]_q - 2[x]^2_q, \]
\[ B_{2,2}(x, q) = \binom{2}{2} [x]_q^2 = [x]^2_q. \]

In the quadratic case \((n = 2)\), the matrix representation is
\[ P_q(x) = (1, [x]_q, [x]^2_q) \begin{pmatrix} 1 & 0 & 0 \\ -2 & 2 & 0 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} C_0 \\ C_1 \\ C_2 \end{pmatrix}. \]

In the cubic case \((n = 3)\), the matrix representation is
\[ P_q(x) = (1, [x]_q, [x]^2_q, [x]^3_q) \begin{pmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{pmatrix} \begin{pmatrix} C_0 \\ C_1 \\ C_2 \\ C_3 \end{pmatrix}. \]

In many applications of \(q\)-Bernstein polynomials, a matrix formulation for the Kim’s \(q\)-Bernstein polynomials seems to be useful.

3. \(q\)-BERNSTEIN POLYNOMIALS, \(q\)-STIRLING NUMBERS AND \(q\)-BERNOULLI NUMBERS

In this section, we assume that \(q \in C_p\) with \(|1 - q|_p < 1\).

For \(f \in UD(Z_p)\), let us consider the \(p\)-adic analogue of Kim’s \(q\)-Bernstein type operator of order \(n\) on \(Z_p\) as follows:
\[ B_{n,q}(f|x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} [x]_q^k [1 - x]_q^{n-k} = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) B_{k,n}(x, q). \]

Let \((Eh)(x) = h(x + 1)\) be the shift operator. Then the \(q\)-difference operator is defined by
\[ \Delta^n_q := (E - I)_q^n = \prod_{i=1}^{n} (E - q^{-1} I), \quad (6) \]
where \((Ih)(x) = h(x)\). From (6), we derive the following equation:
\[ \Delta^n_q f(0) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k q^{(k)} f(n - k), \quad (see [7]). \quad (7) \]

By (7), we easily see that
\[ f(x) = \sum_{n \geq 0} \binom{x}{n}_q \Delta^n_q f(0), \quad (see [6, 7]). \]
The \( q \)-Stirling number of the first kind is defined by
\[
\prod_{k=1}^{n} (1 + [k]_q z) = \sum_{k=0}^{n} S_{1,q}(n, k) z^k, \quad (\text{see [5, 6]}),
\]
and the \( q \)-Stirling number of the second kind is also defined by
\[
\prod_{k=1}^{n} \left( \frac{1}{1 + [k]_q z} \right) = \sum_{k=0}^{n} S_{2,q}(n, k) z^k, \quad (\text{see [5]}).
\]
By (6), (7), (8) and (9), we get
\[
S_{2,q}(n, k) = q^{-\binom{n}{2}} \sum_{j=0}^{\min(n, k)} (-1)^{\binom{n}{2}} \binom{k}{j} \binom{n}{k-j} q^{k-j},
\]
for \( n, k \in \mathbb{Z}_+ \) (see [6]).

Let us consider Kim’s \( q \)-Bernstein polynomials of degree \( n \) on \( \mathbb{Z}_p \) as follows:
\[
B_{k,n}(x, q) = \binom{n}{k} [x]_q^k [1 - x]_q^{n-k},
\]
for \( n, k \in \mathbb{Z}_+ \) and \( x \in \mathbb{Z}_p \). Thus, we easily see that
\[
\int_{\mathbb{Z}_p} B_{k,n}(x, q) d\mu_q(x) = \sum_{l=0}^{n-k} \binom{n-k}{l} \binom{n}{k} (-1)^l \int_{\mathbb{Z}_p} [x]_q^{l+k} d\mu_q(x).
\]
By (1) and (10), we obtain the following proposition.

**Proposition 1.** For \( n, k \in \mathbb{Z}_+ \), we have
\[
\int_{\mathbb{Z}_p} B_{k,n}(x, q) d\mu_q(x) = \sum_{l=0}^{n-k} \binom{n-k}{l} \binom{n}{k} (-1)^l \beta_{l+k,q},
\]
where \( \beta_{l+k,q} \) are the \( (l+k) \)-th Carlitz’s \( q \)-Bernoulli numbers.

From the definition of Kim’s \( q \)-Bernstein polynomial, we note that
\[
\sum_{k=1}^{n} \binom{k}{i} B_{k,n}(x, q) = \sum_{k=0}^{i} q^{\binom{k}{2}} \binom{x}{k} [k]_q! S_{2,q}(k, i-k),
\]
where \( i \in \mathbb{N} \). From the definition of \( q \)-binomial coefficient, we have
\[
\binom{n+1}{k}_q = \binom{n}{k-1}_q + q^k \binom{n}{k}_q = q^{n-k} \binom{n}{k-1}_q + \binom{n}{k}_q.
\]
By (12), we see that
\[
\int_{\mathbb{Z}_p} \frac{x^i}{n} d\mu_q(x) = \frac{(-1)^{\binom{n}{2}} q^{\binom{n+1}{2} - \binom{n}{2}}}{[n+1]_q}, \quad (\text{see [6, 7]}).
\]
From (1), (11) and (13), we obtain the following theorem.
Theorem 2. For \( n, k \in \mathbb{Z}_+ \) and \( i \in \mathbb{N} \), we have

\[
\sum_{k=0}^{n} \sum_{l=0}^{n-k} \binom{k}{l} \binom{n-k}{l} (-1)^l \beta_{l+k,q} \sum_{k=0}^{n} q^k [k]_q ! S_{2,q}(k, i - k) \frac{(-1)^k}{[k+1]_q} q^{(k+1) - (k+1)}.
\]

It is easy to see that for \( i \in \mathbb{N} \),

\[
\sum_{k=0}^{n} \binom{k}{i} B_{k,n}(x, q) = [x]^i_q. \tag{14}
\]

By (11) and (14), we easily get

\[
[x]_q^i = \sum_{k=0}^{i} q^k \binom{x}{k}_q [k]_q ! S_{2,q}(k, i - k), \quad \text{(see [6])}.
\]

Thus, we have

\[
\int_{\mathbb{Z}_p} [x]_q^i d\mu_q(x) = \sum_{k=0}^{i} q^k [k]_q ! S_{2,q}(k, i - k) \int_{\mathbb{Z}_p} \binom{x}{k}_q d\mu_q(x) \tag{15}
\]

By (1) and (15), we obtain the following corollary.

Corollary 3. For \( n, k \in \mathbb{Z}_+ \) and \( i \in \mathbb{N} \), we have

\[
\beta_{i,q} = q \sum_{k=0}^{i} [k]_q ! S_{2,q}(k, i - k) \frac{(-1)^k}{[k+1]_q}.
\]

It is known that

\[
S_{2,q}(n, k) = \frac{1}{(1-q)^k} \sum_{j=0}^{k} (-1)^{k-j} \binom{k+n}{j} \binom{j}{j}_q \tag{16}
\]

and

\[
\binom{n}{k}_q = \sum_{j=0}^{n} \binom{n}{j}_q (q - 1)^{j-k} S_{2,q}(k, j - k).
\]

By simple calculation, we have that

\[
q^{nx} = \sum_{k=0}^{n} (q - 1)^k \binom{n}{k}_q [x]_{k,q} \tag{17}
\]

and

\[
q^{nx} = \sum_{m=0}^{n} \left\{ \sum_{k=m}^{n} (q - 1)^k \binom{n}{k}_q S_{1,q}(k, m) \right\} [x]_q^m \tag{18}
\]
From (17) and (18), we note that
\[ \binom{n}{m} = \sum_{k=m}^{n} (q-1)^{-m+k} \binom{n}{k} S_{1,q}(k,m), \quad \text{(see [6])}. \]

Thus, we obtain the following proposition.

**Proposition 4.** For \( n, k \in \mathbb{Z}^+ \), we have
\[
B_{k,n}(x, q) = \binom{n}{k} [x]^k [1 - x]^{n-k} = \sum_{m=k}^{n} (q-1)^{-k+m} \binom{n}{m} S_{1,q}(m,k) [x]^k [1 - x]^{n-k}.
\]

From the definition of the \( q \)-Stirling numbers of the first kind, we get
\[
q\binom{n}{2}(x)_n q^n = \sum_{k=0}^{n} S_{1,q}(n,k) [x]^k.
\]

By (11) and (19), we obtain the following theorem.

**Theorem 5.** For \( n, k \in \mathbb{Z}^+ \) and \( i \in \mathbb{N} \), we have
\[
\sum_{k=i}^{n} \sum_{l=0}^{k} S_{1,q}(k,l) S_{2,q}(k,i-k) [x]^l = \sum_{k=0}^{n} S_{1,q}(n,k) [x]^k.
\]

By (14) and Theorem 5, we obtain the following corollary.

**Corollary 6.** For \( i \in \mathbb{Z}^+ \), we have
\[
\beta_{1,q} = \sum_{k=0}^{i} \sum_{l=0}^{k} S_{1,q}(k,l) S_{2,q}(k,i-k) \beta_{l,q}.
\]

The \( q \)-Bernoulli polynomials of order \( k \in \mathbb{Z}^+ \) are defined by
\[
\beta_{n,q}^{(k)}(x) = \frac{1}{(1-q)^n} \sum_{i=0}^{n} \binom{n}{i} (-1)^i q^i \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{\sum_{i=1}^{k} (k-i+1)x_i} d\mu_q(x_1) \cdots d\mu_q(x_k),
\]

Thus, we have
\[
\beta_{n,q}^{(k)}(x) = \frac{1}{(1-q)^n} \sum_{i=0}^{n} (-1)^i \binom{n}{i} (i+k) \cdots (i+1) \cdot \frac{1}{[i+k]_q \cdots [i+1]_q} q^i x^i, \quad \text{(see [6])}.
\]

The inverse \( q \)-Bernoulli polynomials of order \( k \) are defined by
\[
\beta_{n,q}^{(-k)}(x) = \frac{1}{(1-q)^n} \sum_{i=0}^{n} \frac{(-1)^i \binom{n}{i} q^i x^i}{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{\sum_{i=1}^{k} (k-i+1)x_i} d\mu_q(x_1) \cdots d\mu_q(x_k)}, \quad \text{(see [6])}. \]

In the special case \( x = 0 \), \( \beta_{n,q}^{(k)}(0) = \beta_{n,q}^{(k)} \) are called the \( n \)-th \( q \)-Bernoulli numbers of order \( k \) and \( \beta_{n,q}^{(-k)}(0) = \beta_{n,q}^{(-k)} \) are also called the inverse \( q \)-Bernoulli numbers of order \( k \).
From (21), we have
\[ \beta_k^{(-n)} = \frac{1}{(1 - q)^k} \sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{[j+n]_q \cdots [j+1]_q}{(j+n) \cdots (j+1)} q^j \]
\[ = \frac{1}{(1 - q)^k} \sum_{j=0}^{k} (-1)^j \binom{k+n}{j} \frac{[n]_q!}{n!}. \quad (22) \]

By (16) and (22), we get
\[ \frac{n!}{[n]_q!} \beta_k^{(-n)} = S_{2,q}(n,k). \quad (23) \]

Therefore, by (11) and (23), we obtain the following theorem.

**Theorem 7.** For \( i, n, k \in \mathbb{Z}_+ \), we have
\[ \sum_{k=i}^{n} \frac{k}{[n]_i} B_{k,n}(x,q) = \sum_{k=0}^{i} q^k \frac{i}{k} \binom{x}{k} \beta_k^{(-k)} q^{i-k,q}. \]

It is easy to show that
\[ \frac{q^i}{[n]_q} \binom{x}{n}_q = \frac{1}{[n]_q!} \prod_{k=0}^{n-1} ([x]_q - [k]_q) \]
\[ = \frac{1}{[n]_q!} \sum_{k=0}^{n} (-1)^k [x]_q^{n-k} S_{1,q}(n-1,k). \]

Thus, we have that
\[ \sum_{k=i}^{n} \frac{k}{[n]_i} B_{k,n}(x,q) = \sum_{k=0}^{i} \sum_{j=0}^{k} (-1)^j [x]_q^{k-j} S_{1,q}(k-1,j) \frac{k!}{[k]_q!} \frac{i}{k} \beta_k^{(-k)} q^{i-k,q}, \]
where \( n, k, i \in \mathbb{Z}_+ \).

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