Modelling and simulation for the vibration of a plate with three axes maneuvering

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Abstract. In super-maneuvering flight, aircraft often bears high accelerations and high jerks. To ensure the safety of the aircraft, it is necessary to carry out the ground test to simulate the flight loads and the responses of the aircraft. In this paper, mechanical model of the accelerations are established, the distribution characteristic of the accelerations on the plate is explored and the consistency conditions for the centrifugal load and flight load are studied. The dynamic equations of a rectangular thin plate in three-axis centrifugal environment are established by Kane method, and the responses of the plate are calculated. The model established and the results obtained can be used to reflect the responses of the plate in the flight environment.

Keywords: three-axis centrifugal environment; rigid-flexible coupling dynamics model; Kane-method; distribution characteristic of the accelerations

1. Introduction
With the rapid development of aerospace technology, thrust vector technology has become a key technology for modern fighters, which means the increasing demand for the maneuverability. Some flight accelerations is up to 20 g (g is gravity acceleration), and jerks maybe as high as 15 g/s [1-3]. During a flight, the high acceleration with high change rate (jerk) have a great impact on the performance of aircraft structures. It is necessary to conduct ground tests which can simulate the flight responses of an aircraft to guarantee the safety of the aircraft. Three-dimensional variable acceleration simulations with high acceleration and jerk are generally carried out on a three-axis centrifuge [4]. Steady-state centrifuge test technology is presently a mature technology to simulate the flight acceleration in China and is mainly used in aeronautics and astronautics. However, this technology cannot be used in variable acceleration simulations [4].

In the previous work, the research work on three-axis centrifugal test mostly focused on the kinematics, while few studies on the consistency between centrifugal environment and actual flight environment [5,6]. The effect of the inhomogeneous centrifugal force in steady-state centrifuge tests and the conditions that minimized the difference between experimental stress and actual stress are discussed in references [7,8]. Our research group established a kinematics model of the three-axis centrifuge and analyzed the effect of kinematic parameters on accelerations [9]. Liu Gege et al, have carried out the research on the equivalence relationship between three-axis load and flight load acting on the cantilever beam [10]. Based on the previous studies, the distribution of acceleration load in three-axis centrifugal environment is explored in this paper, which is important for the subsequent research on the equivalence relationship between flight load and three-axis centrifugal load. The rigid-flexible coupling model of elastic plate in large-scale space motion is established by Kane method and Rayleigh-Ritz method [11-14]. The response of rectangular plate mounted on a three-axis centrifuge...
under expected load is studied. Finally, the research on the consistency of two mechanical environments based on equivalent displacement response is conducted, and then the equivalence relationship between flight load and centrifugal load is established. These above researches are an important basis for load design of centrifugal test.

2. The distribution of the acceleration
The schematic diagram of the three-axis centrifuge model is shown in figure 1 [9]. The centrifuge has three stage rotors. The No. 1 axis is parallel to the No. 2 axis, and the axis of No. 2 and No. 3 are vertical. The rotational angles of the No. 1, No. 2 and No. 3 axis are represented by \( \varphi \), \( \theta \) and \( \alpha \), respectively. The angular velocity are \( \omega_1 \), \( \omega_2 \) and \( \omega_3 \), respectively, and angular acceleration are \( \varepsilon_1 \), \( \varepsilon_2 \) and \( \varepsilon_3 \), respectively.

![Figure 1. Three-axis centrifuge model](image)

2. The distribution of the acceleration
The test pieces are mounted on the No. 3 rotor when three-axis centrifugal tests are conducted. In order to describe the acceleration on the test pieces conveniently, the body coordinate system \( O_{xyz} \) attached to No. 3 rotor is established. Acceleration analytical model of any point \( M (x_m, y_m, z_m) \) on the test pieces in body coordinates are obtained [10].

\[
\begin{align*}
a_x &= -\omega_1^2 R_1 \cos \theta - (\omega_1 + \omega_2) \omega_3 (z_m \cos \alpha + y_m \sin \alpha) + (\varepsilon_1 + \varepsilon_2)(z_m \sin \alpha - y_m \cos \alpha) + \varepsilon_1 R_1 \sin \theta \\
a_y &= \omega_2^2 R_1 \sin \theta \cos \alpha - (\omega_1 + \omega_2) \omega_3 (y_m \cos \alpha + (\omega_1 + \omega_2)^2 z_m \sin \alpha \cos \alpha - \omega_3^2 y_m - \varepsilon_2 z_m + \varepsilon_1 R_1 \cos \alpha \cos \theta + (\varepsilon_1 + \varepsilon_2) x_m \cos \alpha \\
a_z &= -\omega_3^2 R_1 \sin \theta \sin \alpha + (\omega_1 + \omega_2) \omega_2 (z_m \sin \alpha - (\omega_1 + \omega_2) x_m \sin \alpha - \omega_3^2 x_m + \varepsilon_3 y_m - \varepsilon_2 R_1 \sin \alpha \cos \theta - (\varepsilon_1 + \varepsilon_2) x_m \sin \alpha
\end{align*}
\]

Equation (1) expresses that the accelerations in all directions are linear functions of \( x \), \( y \) and \( z \). During a three-axis centrifugal test, the rectangular thin plate is installed symmetrically on the No. 3 rotor, and the neutral surface of the thin plate coincides with \( xOxy \). The accelerations on the plate vary linearly along the direction of length (\( x \)) and width (\( y \)) and thickness (\( z \)), according to the acceleration analytical model. Considering that the thickness of plate is thin, the change of acceleration in the thickness direction is neglected. The acceleration in the neutral plane is considered as the acceleration throughout the thickness, thus, the acceleration model is simplified as following:

\[
\begin{align*}
a_x &= -\omega_1^2 R_1 \cos \theta + \varepsilon_1 R_1 \sin \theta - (\omega_1 + \omega_2) \omega_3 y - \cos \alpha (\varepsilon_1 + \varepsilon_2) y \\
a_y &= \omega_2^2 R_1 \sin \theta \cos \alpha + \varepsilon_1 R_1 \cos \alpha \cos \theta + \cos \alpha (\varepsilon_1 + \varepsilon_2) x - (\omega_1 + \omega_2) \omega_3 \cos \alpha \cos \theta + y \omega_3^2 \cos \alpha \omega_3^2 y \\
a_z &= -\omega_3^2 R_1 \sin \theta \sin \alpha - \varepsilon_3 \omega_1 \sin \alpha \cos \theta - \sin \alpha (\varepsilon_1 + \varepsilon_2) x + (\omega_1 + \omega_2)^2 \sin \alpha \cos \alpha + \varepsilon_3 \omega_3 \sin \alpha \cos \alpha + \varepsilon_3 \omega_3 \sin \alpha
\end{align*}
\]
3. Consistency of Simply Supported Plates in Centrifugal and Flight Environments

In the actual flight process, the flight load of the aircraft is a uniform field in the spatial domain, while the load of the aircraft is a non-uniform field varying as time and position in the centrifugal environment. In order to reflect the aircraft response by carrying out centrifuge tests, the displacement responses of the plate in the two load environments are calculated, based on the distribution of acceleration in the centrifugal environment and the actual flight environment. Then, the relationship between the flight load and centrifugal load is studied based on the displacement response equivalence. According to the acceleration expression in equation (2), the load induced by the acceleration on the plate along the thickness direction, the corresponding load means the pressure acting on the plate, and it can be expressed as:

\[ q(x, y) = \rho h \cdot a(x, y) \]  

The length, width and thickness of the plate are \( a, b \) and \( h \), respectively, and \( \rho \) is the density. For the convenience of the following description, according to equation (2) and (3), the pressure acting on the plate in the centrifugal test is expressed by a linear function.

\[ q(x, y) = c_0 + k_1x + k_2y \]  

Where: 
\[ c_0 = \rho h(-\alpha_s^2R_i \sin \theta \sin \alpha - \epsilon_r \sin \alpha \cos \theta); \quad k_1 = \rho h(-\sin \alpha (\epsilon_1 + \epsilon_3)); \quad k_2 = \rho h[(\alpha_1 + \alpha_3)^2 \sin \alpha \cos \alpha + \epsilon_5]. \]

3.1. Study on the Consistency of Displacement response

In this paper, the displacement responses of the plates in centrifugal and flight environments are solved using the double trigonometric series method proposed by Navier. The plate is subjected to transversely distributed loads as equation (4). The solution of bending differential equation of the simply supported Plates is following [17].

\[ w = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \]  

Equation (5) satisfies that: \( \nabla^2 w = \frac{q}{D} \). In equation (5), \( m \) and \( n \) are wave numbers in the direction of length and width, respectively, and they are all positive integers.

- The displacement response in flight environment.

Firstly, the displacement response of the simply supported rectangular plate in flight environment is analyzed, in which the load is uniform and is expressed as \( q(x, y) = q_0 \), where \( q_0 \) is a constant. Based on the theory of plates [15], the deflection under uniform load is obtained.

\[ w_1 = \frac{16q_0}{\pi^6D} \sum_{m=1,3,\ldots}^{\infty} \sum_{n=1,3,\ldots}^{\infty} \frac{1}{mn} \left( \frac{m^2 + n^2}{a^2 + b^2} \right) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \]  

The maximum deflection occurs in the centre of the simply supported plate.

- The displacement response in centrifugal environment.

In the centrifugal environment, the transverse load \( q(x, y) = c_0 + k_1x + k_2y \) is non-uniform as shown in equation (4). The displacement response of the plate is calculated, based on the theory of plates [15].

\[ w_2 = \sum_{m=1,3,\ldots}^{\infty} \sum_{n=1,3,\ldots}^{\infty} \frac{16c_0 + 8ak_1 + 8bk_2}{Dmn\pi^6} \left( \frac{m^2 + n^2}{a^2 + b^2} \right)^2 \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \]  


In order to reflect the responses of the flight environment through the centrifuge test, it is necessary to ensure that the equation (6) is equal to the equation (7). Based on the expressions of the responses in equation (6) and (7), the equivalence relationship of load in the two environments is derived as follows.

\[
\frac{16c_0 + 8ak_1 + 8bk_2}{Dmn\pi^2 \left( \frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^2} = \frac{16q_0}{Dmn\pi^2 \left( \frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^2}
\]

(8)

Where \( q_0 = c_0 + \frac{1}{2} (k_1a + k_2b) \), which means that when the load at the midpoint in the three-axis centrifugal environment is equal to that in the flight environment, the responses at the midpoint in the two load environments are equal.

3.2. Example verification

In order to verify the equivalence relationship derived in equation (8), the responses of a simply supported plate is calculated. The geometric and material parameters of the plate are listed in Table 1. The angular velocity of each axis is set to 8 rad/s, the angular acceleration is set to 8 rad/s\(^2\), and the arm length \( R_1 \) is 8 m. The acceleration load is applied to the simply supported plate, and the simulation analysis is carried out by using ANSYS.

| parameters          | parameters                  |
|---------------------|-----------------------------|
| Length/m            | 0.5                         |
| Width/m             | 0.4                         |
| Thickness/m         | 0.002                       |
| Density/kgm\(^{-3}\) | 2700                        |
| Young’s modulus /GPa | 70                          |
| Poisson ratio       | 0.3                         |
| Coordinates of the load control point(LCP) | (0.5a,0.5b,0) |

Based on the acceleration model expressed in equation (2), the corresponding load (distribution pressure) caused by the acceleration is calculated and the displacement responses of midpoint is calculated. Then, according to equation (8), the midpoint acceleration of the plate is set as the acceleration of the flight environment and the displacement responses are calculated.

Figure 2 shows the displacement response curves under centrifugal and flight environments, and the difference is shown in Figure 3. It can be seen that when the accelerations of the midpoint in the flight environment and centrifuge test are set to be equal, the displacement responses of the midpoint agree well with each other.

![Figure 2](image1.png)

**Figure 2.** Displacement response of the plate in centrifugal and flight Environments

![Figure 3](image2.png)

**Figure 3.** the difference of displacement responses
From the above results obtained, it can be seen that for a simply supported plate, it is reasonable to set the midpoint as the load control point (LCP). In other words, when the acceleration of the midpoint equals the flight acceleration, the three-axis centrifugal tests can effectively reflect the displacement response in the actual flight environment.

4. Rigid-Flexible Coupling Model of Plates in Large-Scale Space Motion

4.1. Derivation of the dynamic equation

The rigid-flexible coupling dynamic model of the plates under centrifugal environment is established considering the elastic vibration. Then, the dynamic response in the three-axis centrifugal environment and flight environment is studied. The body coordinate system of the plate is $XYZ$, whose basis vectors are $e_1, e_2, e_3$.

![Figure 4. Deformation and Displacement Field of Rectangular Plate.](image)

The position vector from $O$ to $P_0$ is $r = x e_1 + y e_2$, where $P_0$ is a generic point on the neutral plane of the plate, and after deformation, the generic point is $P$. The vector of point $P_0$ to $P$ is expresses as:

$$u = u_1 e_1 + u_2 e_2 + u_3 e_3$$  \hspace{1cm} (9)

The absolute velocity and absolute acceleration of the reference point $O$ are respectively expressed as:

$$v^o = v_1 e_1 + v_2 e_2 + v_3 e_3$$  \hspace{1cm} (10)

$$a^o = a_1 e_1 + a_2 e_2 + a_3 e_3$$  \hspace{1cm} (11)

The angular velocity of the plate in the inertial reference system is as following:

$$\omega = \omega_1 e_1 + \omega_2 e_2 + \omega_3 e_3$$  \hspace{1cm} (12)

The corresponding velocity and acceleration of point $P$ are obtained as:

$$v = v^o + \omega \times (r + u) + \dot{\omega} \times v^P$$  \hspace{1cm} (13)

$$a = a^o + \ddot{\omega} \times (r + u) + \omega \times (\omega \times (r + u)) + \dot{\omega} \times v^P + 2\omega \times \dot{v}^P$$  \hspace{1cm} (14)

In equation (13), $v^P$ represents the velocity vector of point $P$ relative to the base point $O$. The Rayleigh-Ritz method is employed to describe the displacement caused by the elastic deformation. The arc length variables of point $P$ on plate neutral surface are $s$ and $r$, and the variable along $e_3$ is $u_3$. Thus, $s$, $r$ and $u_3$ are assumed to be in the following form.
\[
\begin{align*}
    s(x, y, t) &= \sum_{i=1}^{N_i} \phi_i(x, y) q_{si}(t) \\
    r(x, y, t) &= \sum_{i=1}^{N_i} \phi_i(x, y) q_{ri}(t) \\
    u_s(x, y, t) &= \sum_{i=1}^{N_i} \phi_i(x, y) q_{ui}(t)
\end{align*}
\]

(15)

Where: \( \phi_i \) and \( q_{ki} \) \( (k=1,2,3) \) are the basis functions and modal coordinates, respectively. Subscripts \( k = 1, 2, 3 \) correspond to the three directions of the plate and \( N_1, N_2 \) and \( N_3 \) are the modal truncation numbers of the three directions respectively [16].

The deformation geometry of the plate shown in figure 4 is roughly as follows.

\[
\begin{align*}
    x + s &= \int_0^1 \left[ \left( \frac{\partial u_s}{\partial \xi} \right)^2 + \left( \frac{\partial u_s}{\partial \eta} \right)^2 \right]^{1/2} d\xi \\
    y + r &= \int_0^1 \left[ \left( \frac{\partial u_r}{\partial \eta} \right)^2 + \left( \frac{\partial u_r}{\partial \eta} \right)^2 \right]^{1/2} d\eta
\end{align*}
\]

(16)

The deformation potential energy of plates consists of two parts.

\[
U = U_m + U_b
\]

(17)

Where \( U_m \) is deformation energy in the plane, and \( U_b \) is bending deformation energy. \( U_m \) and \( U_b \) are described by \( s, r \) and \( u_i \).

\[
\begin{align*}
    U_m &= \frac{1}{2} \int_0^b \int_0^a \left\{ \beta_1 \left( \left( \frac{\partial s}{\partial \xi} \right)^2 + \left( \frac{\partial r}{\partial \eta} \right)^2 + 2\nu \left( \frac{\partial s}{\partial \xi} \right) \left( \frac{\partial r}{\partial \eta} \right) \right) + \beta_2 \left( \frac{\partial s}{\partial \xi} + \frac{\partial r}{\partial \eta} \right) \right\} d\xi dy \\
    U_b &= \frac{1}{2} \int_0^b \int_0^a \beta_3 \left[ \left( \frac{\partial^2 u_1}{\partial \xi^2} \right)^2 + \left( \frac{\partial^2 u_3}{\partial \eta^2} \right)^2 + 2\nu \left( \frac{\partial^2 u_1}{\partial \xi^2} \right) \left( \frac{\partial^2 u_3}{\partial \eta^2} \right) \right] + 2(1-\nu) \left( \frac{\partial^2 u_1}{\partial \xi \partial \eta} \right) d\xi dy
\end{align*}
\]

(18)

Where \( E \) is Young's modulus, \( G \) is shear modulus, \( h \) is the thickness of the plate, \( \beta_1 = \frac{Eh}{(1-\nu^2)} \); \( \beta_2 = G h \); \( \beta_3 = \frac{Eh^3}{12(1-\nu^2)} \).

The dynamic equations of the system are derived by Kane method, which means the generalized active force and generalized inertial force keep balance.

\[
\begin{align*}
    F_i + F_i^* &= 0 \quad (i=1,2,3)
\end{align*}
\]

(19)

The deduced models of velocity, acceleration and deformation energy are introduced into the expressions of generalized force and generalized inertial force [17].

\[
\begin{align*}
    F_i^* &= -\int_0^b \int_0^a \rho \frac{\partial V}{\partial q_i} \cdot a d\xi dy \\
    F_i &= -\frac{\partial U}{\partial q_i} = -\frac{\partial (U_m + U_b)}{\partial q_i}
\end{align*}
\]

(20) (21)
Where \( q \) and \( q \) are generalized coordinates and generalized velocity, \( v \) denotes the absolute velocity, \( a \) denotes the absolute acceleration. Substituting the generalized active force and generalized inertial force into equation (19) yields the dynamic equation of the system:

\[
M\ddot{q} + G\dot{q} + Kq = F
\]  
(22)

Where \( M, G, K \) are \( N \times 1 \) matrix, and \( F \) is a \( N \times 1 \) column matrix, and \( N = N_1 + N_2 + N_3 \).

4.2. Solution of dynamics equation and calculation of response

In this section, the displacement responses of a simply supported rectangular plate are calculated. The geometric parameters and material parameters of the plate are shown in Table 1, and the plate is installed symmetrically in the plane of \( xO_3y \). The reference point \( O_3 \) is the intersection of axes of the No. 2 and No. 3 rotors. The intersection point is set as the load control point (LCP), which means that the acceleration of the LCP along the three directions of the plate is regarded as the target acceleration. Three-dimensional overload is applied in the three directions of the simply supported plate as shown in figure 5. The overload is the ratio of acceleration of LCP to \( g \) (Gravity acceleration). The displacement responses of the plate in the three-axis centrifugal environments and flight environments are simulated. Compared with the deformations along the thickness direction, the deformations in the length and width directions are very small and can be ignored. Thus, only the deformation in the thickness direction are studied. The deformation responses in the midpoint of the plate in centrifugal and flight environments are shown in figure 6. It can be seen that the response is the same trend with the change of the overload. The response curves in the two different environments are basically consistent, which indicates that the results obtained in the centrifugal tests can be used to simulate the responses in the flight cases.

![Figure 5. Three-dimensional overload](image)

![Figure 6. Displacement response in centrifugal environment and flight environment](image)

5. Conclusion

In this paper, the distribution of the acceleration on the plate in the three-axis centrifugal environment is analyzed. For a simply supported plate, the equivalence relationship of load in the centrifugal test and flight environments is established to ensure that the results in the centrifugal tests can simulate the responses in the flight cases. The rigid-flexible dynamic equations of the plate is derived by Kane method. It is concluded that the midpoint of the simply supported plate can be used as the load control point (LCP) to calculate the test accelerations according to the flight accelerations. When the test accelerations of the LCP are constant with the flight accelerations, the displacement responses obtained in the centrifugal test can be used to simulate the response of the flight environment effectively.
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