Remarks on derived complete modules and complexes

Leonid Positselski\textsuperscript{1,2}

\textbf{Abstract}

Let \( R \) be a commutative ring and \( I \subset R \) a finitely generated ideal. We discuss two definitions of derived \( I \)-adically complete (also derived \( I \)-torsion) complexes of \( R \)-modules, which appear in the literature: the idealistic and the sequential ones. The two definitions are known to be equivalent for a weakly proregular ideal \( I \); we show that they are different otherwise. We argue that the sequential approach works well, but the idealistic one needs to be reinterpreted or properly understood. We also consider \( I \)-adically flat \( R \)-modules.

\textbf{INTRODUCTION}

\textbf{0.0: How many categories are there?}

Let \( I \) be a finitely generated ideal in a commutative ring \( R \). How many abelian categories of \( I \)-adically complete (in some sense) \( R \)-modules are there? An unsuspecting reader would probably guess that there are none. In fact, generally speaking, there are two such abelian categories, one of them a full subcategory in the other. For a Noetherian ring \( R \), the two categories coincide.

Furthermore, how many triangulated categories of derived \( I \)-adically complete complexes are there? We argue that the correct answer is “three,” two of which are the important polar cases and the third one is kind of intermediate. There is also one abelian category of \( I \)-torsion \( R \)-modules and two triangulated categories of derived \( I \)-torsion complexes. These triangulated categories are connected by natural triangulated functors. For a Noetherian ring \( R \) (or more generally, for a weakly proregular ideal \( I \)), these functors are equivalences; so the answer to all the “how many” questions reduces to “one.”

To be precise, the definitions of derived complete and torsion complexes that can be found in the literature do not always agree with our suggested definitions. We discuss both, and explain why we view some of our constructions of the triangulated categories of derived complete and torsion complexes as improvements upon the ones previously considered by other authors.

This paper is inspired by Yekutieli’s paper [45] (as well as his earlier paper [44]), where some of the results of the present author have been mentioned. The credit is due to Yekutieli for posing several questions to which this paper provides the answers.
0.1: Idealistic derived completion

Let us briefly discuss the substantial issue involved, starting for simplicity with the particular case of a principal ideal \( I = (s) \subset R \) generated by an element \( s \in R \). Let \( M \) be an \( R \)-module. What is the derived \( s \)-adic completion of \( M \)?

A more naïve approach is to start with the underived \( I \)-adic completion,

\[
\Lambda_s(M) = \lim_{n \geq 1} M/s^nM.
\]

The problem with the functor \( \Lambda_s \) is that it is the composition of a right exact functor assigning to \( M \) the system of its quotient modules \( \cdots \to M/s^3M \to M/s^2M \to M/sM \) with the left exact functor of projective limit. As such, the functor \( \Lambda_s \) is neither left nor right exact, and in fact not even exact in the middle [43].

Nevertheless, one can consider the left derived functor \( \mathbb{L}\Lambda_s \) of \( \Lambda_s \), computable by applying \( \Lambda_s \) to a projective resolution \( P \cdot \) of an \( R \)-module \( M \). This is equivalent to replacing \( \Lambda_s \) with its better behaved (right exact) 0-th left derived functor \( \mathbb{L}^0\Lambda_s \) and computing the left derived functor of \( \mathbb{L}^0\Lambda_s \). Then, one can say that the derived \( I \)-adic completion of \( M \) is the complex

\[
\mathbb{L}\Lambda_s(M) = \mathbb{L}(\mathbb{L}^0\Lambda_s)(M) = \Lambda_s(P \cdot),
\]

where \( P \cdot \) is a projective resolution of \( M \). In fact, it suffices to require \( P \cdot \) to be a flat resolution. This approach is taken, in the generality of finitely generated ideals \( I \subset R \), by Porta–Shaul–Yekutieli in the paper [22] (see also the much earlier [11, Section 1]). It is called the “idealistic derived adic completion” in [45].

0.2: Sequential derived completion

A more sophisticated approach is to construct the derived \( I \)-adic completion as the derived functor of projective limit of the derived functors of the passage to the quotient module \( M \to M/s^nM \). Furthermore, the latter derived functors are interpreted as taking an \( R \)-module \( M \) to the two-term complexes \( M \to M/s^nM \), concentrated in the cohomological degrees \(-1\) and 0. Then, the complex \( \mathbb{R}\lim_{n \geq 1} (M \to M/s^nM) \) can be computed as

\[
\mathbb{R}\lim_{n \geq 1} (M \to M/s^nM) = \mathbb{R}\text{Hom}_R(R \to R[s^{-1}], M).
\]

This construction can be found in the paper [2, Section 1.1]; see also [18, Section 1.5] and [3, Section 3.4] (once again, the much earlier paper [11, Section 2] is relevant). This approach also extends naturally to finitely generated ideals \( I \subset R \) [3, 11], for which it is helpful to choose a finite sequence of generators \( s_1, \ldots, s_m \in I \), and then show that the construction does not depend on a chosen set of generators. For this reason, it is called the “sequential derived completion” in [45].

0.3: Change-of-ring question

One important difference between the two approaches is that the idealistic derived completion is sensitive to the choice of a base ring \( R \). Given a subring \( R' \subset R \) such that \( s \in R' \), the derived functors \( L_{-R}A_s \) and \( L_{-R'}A_s \) computed in the categories of \( R \)-modules and \( R' \)-modules do not agree with each other in any meaningful way, generally speaking. The sequential derived completion construction, on the other hand, essentially happens over the ring \( \mathbb{Z}[s] \). In particular, for any commutative ring homomorphism \( R' \to R \) and an element \( s' \in R' \), denoting by \( s \) the image of \( s' \) in \( R \), for any \( R \)-module \( M \), one has a natural isomorphism

\[
\mathbb{R}\text{Hom}_R(R \to R[s^{-1}], M) \cong \mathbb{R}\text{Hom}_{R'}(R' \to R'[s'^{-1}], M)
\]

in the derived category of \( R' \)-modules.
0.4: Idealistic and sequential derived torsion

Torsion is easier to think of than completion; but the same difference between two approaches arises. Denote by $\Gamma_s(M) \subset M$ the submodule consisting of all the $s$-torsion elements, that is, elements $m \in M$ for which there exists an integer $n \geq 1$ such that $s^n m = 0$. The functor $\Gamma_s(M)$ is left exact, so there is no problem involved in considering its right derived functor $R\Gamma_s$, computable as

$$R\Gamma_s(M) = \Gamma_s(J^\cdot),$$

where $J^\cdot$ is an injective resolution of $M$. Following the terminology of [45], this is the “idealistic derived torsion functor.”

Alternatively, one can consider the functors $M \mapsto s^n M$, assigning to an $R$-module $M$ its submodule of all elements annihilated by $s^n$. Then, one can interpret the right derived functor of $M \mapsto s^n M$ as taking an $R$-module $M$ to the two-term complex $M \rightarrow M$, concentrated in the cohomological degrees 0 and 1. Finally, the derived $I$-torsion of $M$ is defined as the inductive limit

$$\lim_{n \geq 1} (M \rightarrow M) = (R \rightarrow R[s^{-1}]) \otimes_R M.$$

This is called the “sequential derived torsion functor” in [45].

Similarly to the two derived completions, the two derived torsions differ in how they behave with respect to the ring changes. The idealistic derived torsion is sensitive to the choice of a base ring $R$, while the sequential derived torsion is indifferent to it.

0.5: Weak proregularity and bounded torsion

Let us say a few words about the weak proregularity condition, which plays a central role in our discussion. A principal ideal $I = (s) \subset R$ is weakly proregular if and only if the $s$-torsion in $R$ is bounded, that is, there exists an integer $n_0 \geq 1$ such that $s^n r = 0$ for some $n \geq 1$ and $r \in R$ implies $s^{n_0} r = 0$.

For any $R$-module $M$, there is a natural morphism

$$R\Gamma_s(M) \rightarrow (R \rightarrow R[s^{-1}]) \otimes_R M$$

in the derived category of $R$-modules. It turns out that this morphism is an isomorphism for all $R$-modules $M$ if and only if the $s$-torsion in $R$ is bounded. It suffices to check this condition for injective $R$-modules $M = J$. A version of this result mentioning the weak proregularity in place of the bounded torsion holds for any finitely generated ideal $I \subset R$ [14, Lemma 2.4], [35, Theorem 3.2], [22, Theorem 4.24].

Similarly, for any $R$-module $M$, there is a natural morphism

$$R\text{Hom}_R(R \rightarrow R[s^{-1}], M) \rightarrow L\Lambda_s(M)$$

in the derived category of $R$-modules. Once again, this morphism is an isomorphism for all $R$-modules $M$ if and only if the $s$-torsion in $R$ is bounded. It suffices to check this condition for the free $R$-module with a countable set of generators $M = \bigoplus_{n=1}^{\infty} R$. A version of this result with the bounded torsion replaced by weak proregularity holds for any finitely generated ideal $I \subset R$ [15, Remark 7.8].

0.6: Categories arise from derived completion and torsion functors

Various triangulated categories of derived complete and torsion modules discussed in this paper are related, in one way or another, to the two derived completion and two derived torsion functors mentioned in Sections 0.1, 0.2, and 0.4.
0.7: Locally presentable abelian categories with enough projective objects

Before we finish this introduction, let us say a few words about our motivation. The objects of the two abelian categories of derived $I$-adically complete $R$-modules mentioned in Section 0.0 are called $I$-contramodule $R$-modules and quotseparated $I$-contramodule $R$-modules in this paper. The following example is illuminating.

Let $R = k[x_1, ..., x_m]$ be the ring of polynomials in a finite number of variables over a field $k$ and $I = (x_1, ..., x_m) \subset R$ be the ideal generated by the elements $x_j$. Let $C$ be the coassociative, cocommutative, counital coalgebra over $k$ such that the dual topological algebra $C^e = k[[x_1, ..., x_m]]$ is the algebra of formal Taylor power series in the variables $x_j$. Then, the abelian category of $I$-torsion $R$-modules is equivalent to the category of $C$-comodules. Furthermore, the abelian category of $I$-contramodule $R$-modules (which coincides with the abelian category of quotseparated $I$-contramodule $R$-modules in this case, as the ring $R$ is Noetherian) is equivalent to the abelian category of $C$-contramodules. The latter category was defined by Eilenberg and Moore in [6, Section III.5]; we refer to our overview [26] for a discussion.

For any finitely generated ideal $I$ in a commutative ring $R$, the category of $I$-torsion $R$-modules is a Grothendieck abelian category. On the other hand, both the abelian categories of $I$-contramodule $R$-modules and quotseparated $I$-contramodule $R$-modules are locally presentable abelian categories with enough projective objects. We believe that the latter class of abelian categories, which are dual-analogous or “covariantly dual” to Grothendieck abelian categories [30, 31], is not receiving the attention that it deserves. Thus, we use this opportunity to point out and discuss two classes of examples of locally presentable abelian categories with enough projectives appearing naturally in commutative algebra.

0.8: Notation

Let us make some general notational conventions. Throughout this paper, $R$ is a commutative ring and $I \subset R$ is a finitely generated ideal. When we need to choose a finite set of generators of the ideal $I$, we denote them by $s_1, ..., s_m \in I$. The sequence of elements $s_1, ..., s_m$ is denoted by $s$ for brevity.

Given an abelian (or Quillen exact) category $A$, we denote its bounded and unbounded derived categories by $D^b(A)$, $D^+(A)$, $D^-(A)$, and $D(A)$, as usual. The abelian category of (arbitrarily large) $R$-modules is denoted by $R\text{-mod}$.

1 DERIVED COMPLETE MODULES

The definition of an $Ext$-$p$-$complete$ (abelian, or more generally, nilpotent) group for a prime number $p$ goes back to the book of Bousfield and Kan [4, Sections VI.2–4]. Under the name of $weakly$ $l$-$complete$ abelian groups (where $l$ is still a prime number), they were discussed by Jannsen in [16, Section 4]. Even earlier, the abelian groups decomposable as products of $Ext$-$p$-$complete$ abelian groups over the prime numbers $p$ were studied by Harrison in [12, Section 2] under the name of “co-torsion abelian groups”; this approach was generalized by Matlis in [19]. In our terminology, $Ext$-$p$-$complete$ abelian groups are called $p$-$contramodule$ $\mathcal{Z}$-$modules$ (we refer to the introductions to the papers [28, 29] for a discussion).

Given an element $s \in R$, consider the ring $R[s^{-1}]$ obtained by adjoining to $R$ an element inverse to $s$. Denoting by $S \subset R$ the multiplicative subset $S = \{1, s, s^2, \ldots\}$, one has $R[s^{-1}] = S^{-1}R$. One can easily see that the projective dimension of the $R$-module $R[s^{-1}]$ does not exceed 1.

An $R$-module $C$ is said to be an $s$-$contramodule$ if

$$\text{Hom}_R(R[s^{-1}], C) = 0 = \text{Ext}^1_R(R[s^{-1}], C).$$

A weaker condition is sometimes useful: An $R$-module $C$ is said to be $s$-$contraadjusted$ if $\text{Ext}^1_R(R[s^{-1}], C) = 0$.

An $R$-module $C$ is said to be an $I$-$contramodule$ (or an $I$-$contramodule$ $R$-$module$) if $C$ is an $s$-$contramodule$ for every element $s \in I$. It suffices to check this condition for any chosen set of generators $s = s_1, s_2, \ldots, s_m$ of the ideal $I \subset R$ [28, Theorem 5.1].

**Lemma 1.1.** The full subcategory of $I$-$contramodule$ $R$-$modules$ $R\text{-mod}_{I\text{-ltra}}$ is closed under the kernels, cokernels, extensions, and infinite products (hence also under all limits) in the category of $R$-modules $R\text{-mod}$. Consequently, the category $R\text{-mod}_{I\text{-ltra}}$ is abelian and the inclusion functor $R\text{-mod}_{I\text{-ltra}} \longrightarrow R\text{-mod}$ is exact.

**Proof.** This is [10, Proposition 1.1] or [28, Theorem 1.2(a)].
Denoting by $\bar{R} \subset R$ the subring in $R$ generated by the elements $s_1, \ldots, s_m$ over $\mathbb{Z}$ and by $\bar{I} \subset \bar{R}$ the ideal in $\bar{R}$ generated by the same elements, one observes that an $R$-module $C$ is an $I$-contramodule if and only if its underlying $\bar{R}$-module $C$ is an $\bar{I}$-contramodule. In this specific sense, the definition of $I$-contramodule $R$-modules reduces to the case of Noetherian rings. Moreover, one can replace $\bar{R}$ by the polynomial ring $\bar{R} = \mathbb{Z}[s_1, \ldots, s_m]$ with the ideal $\bar{I} = (s_1, \ldots, s_m)$; an $R$-module is an $I$-contramodule if and only if its underlying $\bar{R}$-module is an $I$-contramodule.

The interpretation of the $s$-contramodule $R$-modules as the $R$-modules with $s$-power infinite summation operation and the $I$-contramodule $R$-modules as the $R$-modules with $[s_1, \ldots, s_m]$-power infinite summation operation is discussed in [28, Sections 3 and 4] (see also [24, Appendix B] for the Noetherian case).

The $I$-adic completion of an $R$-module $C$ is defined as

$$\Lambda_I(C) = \lim_{n \geq 1} C/I^n C.$$  

An $R$-module $C$ is said to be $I$-adically separated if the canonical morphism $C \to \Lambda_I(C)$ is injective, and we say that $C$ is $I$-adically complete if the morphism $C \to \Lambda_I(C)$ is surjective.

Any $I$-adically complete and separated $R$-module is an $I$-contramodule. Any $I$-contramodule $R$-module is $I$-adically complete [28, Theorem 5.6], but it need not be $I$-adically separated.

An $I$-contramodule $R$-module is said to be quot-separated if it is a quotient $R$-module of an $I$-adically separated and complete $R$-module. We denote the full subcategory of $I$-adically separated and complete $R$-modules by $R\text{-mod}_{I-\text{tra}}^{\text{sep}} \subset R\text{-mod}$ and the full subcategory of quot-separated $I$-contramodule $R$-modules by $R\text{-mod}_{I-\text{tra}}^{\text{qs}} \subset R\text{-mod}$. So there are inclusions of full subcategories

$$R\text{-mod}_{I-\text{tra}}^{\text{sep}} \subset R\text{-mod}_{I-\text{tra}}^{\text{qs}} \subset R\text{-mod}_{I-\text{tra}} \subset R\text{-mod}.$$  

An example of a quot-separated, but not separated $I$-contramodule $R$-module can be found already in [43, Example 3.20] or [39, Example 2.5] (see also [22, Example 4.33]). The additive category $R\text{-mod}_{I-\text{tra}}^{\text{sep}}$ is not abelian [28, Example 2.7 (1)]; but the category $R\text{-mod}_{I-\text{tra}}^{\text{qs}}$ is, as the next lemma tells.

**Lemma 1.2.** The full subcategory of quot-separated $I$-contramodule $R$-modules $R\text{-mod}_{I-\text{tra}}^{\text{qs}}$ is closed under subobjects, quotient objects, and infinite products in $R\text{-mod}_{I-\text{tra}}$, and closed under the kernels, cokernels, and infinite products (hence also under all limits) in $R\text{-mod}$. Consequently, the category $R\text{-mod}_{I-\text{tra}}^{\text{qs}}$ is abelian and the inclusion functors $R\text{-mod}_{I-\text{tra}}^{\text{qs}} \to R\text{-mod}_{I-\text{tra}} \to R\text{-mod}$ are exact.

**Proof.** The basic observation is that any submodule of an $I$-adically separated $R$-module is $I$-adically separated. Hence, the class $C$ of all $I$-adically separated $I$-contramodules ($= I$-adically separated and complete $R$-modules) is closed under subobjects in $R\text{-mod}_{I-\text{tra}}$. Moreover, the class of all $I$-adically separated $R$-modules is closed under products in $R\text{-mod}$; hence the class $C$ is closed under products in the abelian category $A = R\text{-mod}_{I-\text{tra}}$.

Let $A$ be an abelian category with an exact inclusion functor $B \to A$. If, moreover, the class $C$ is closed under products in $A$ and the product functors in $A$ are exact, then the full subcategory $B$ is closed under products in $A$.

In the situation at hand, these observations are applicable to the abelian category $A = R\text{-mod}_{I-\text{tra}}$, producing the abelian category $B = R\text{-mod}_{I-\text{tra}}^{\text{qs}}$. Then, the remaining assertions of the lemma follow from Lemma 1.1.  

Both the full subcategories $R\text{-mod}_{I-\text{tra}}$ and $R\text{-mod}_{I-\text{tra}}^{\text{qs}}$ are reflective in $R\text{-mod}$ (i.e., their inclusion functors have left adjoints). The reflector $\Delta_I : R\text{-mod} \to R\text{-mod}_{I-\text{tra}}$ is constructed and discussed at length in [28, Sections 6 and 7] (see formula (3.2) in Section 3 below). The functor $\Delta_I : R\text{-mod} \to R\text{-mod}_{I-\text{tra}}$ is right exact, because it is a left adjoint; since the inclusion $R\text{-mod}_{I-\text{tra}} \subset R\text{-mod}$ is an exact functor, the composition $R\text{-mod} \to R\text{-mod}_{I-\text{tra}} \to R\text{-mod}$ is also right exact. As any reflector onto a full subcategory, the functor $\Delta_I : R\text{-mod} \to R\text{-mod}$ is idempotent, and its essential image is the whole full subcategory $R\text{-mod}_{I-\text{tra}} \subset R\text{-mod}$.

As pointed out in [43, Section 1] and [45, Section 1], the $I$-adic completion functor $\Lambda_I : R\text{-mod} \to R\text{-mod}$ is neither left, nor right exact (even though, by [28, Theorem 5.8], $\Lambda_I : R\text{-mod} \to R\text{-mod}_{I-\text{tra}}^{\text{sep}}$ is the reflector onto the full subcategory of $I$-adically separated and complete modules in $R\text{-mod}$). The next proposition describes the reflector $R\text{-mod} \to R\text{-mod}_{I-\text{tra}}^{\text{qs}}$ as the $0$-th left derived functor of the functor $\Lambda_I$, that is, $L_0\Lambda_I : R\text{-mod} \to R\text{-mod}_{I-\text{tra}}^{\text{qs}}$. 


Proposition 1.3. (a) For any $R$-module $C$, there are two natural surjective $R$-module morphisms $\Delta_f(C) \rightarrow L_0\Lambda_f(C) \rightarrow \Lambda_f(C)$.

(b) For any $R$-module $C$, the $R$-module $L_0\Lambda_f(C)$ is a quotseparated $I$-contramodule.

(c) The functor $L_0\Lambda_f : R\text{-mod} \rightarrow R\text{-mod}_{I\text{-contr}}$ is left adjoint to the inclusion $R\text{-mod}_{I\text{-contr}} \rightarrow R\text{-mod}$.

(d) Consequently, the functor $L_0\Lambda_f : R\text{-mod} \rightarrow R\text{-mod}_{I\text{-contr}}$ is idempotent, and its essential image is the full subcategory $R\text{-mod}_{I\text{-contr}} \subset R\text{-mod}$.

Proof. Part (a): A natural surjective $R$-module morphism $b_{I,C} : \Delta_i(C) \rightarrow \Lambda_f(C)$ is constructed in [28, Lemma 7.5], and its kernel is also computed in [28, Lemma 7.5]. Let us show that the map $b_{I,C}$ factorizes naturally as the composition of two surjective morphisms $\Delta_i(C) \rightarrow L_0\Lambda_f(C) \rightarrow \Lambda_f(C)$.

Let $P_1 \rightarrow P_0 \rightarrow C \rightarrow 0$ be a right exact sequence of $R$-modules with projective $R$-modules $P_1$ and $P_0$. By the definition, $L_0\Lambda_f(C)$ is the cokernel of the induced morphism $\Lambda_f(P_1) \rightarrow \Lambda_f(P_0)$. We will construct a commutative diagram of $R$-module morphisms of the following form:

\[
\begin{array}{c}
\Delta_f(P_1) \rightarrow \Delta_f(P_0) \rightarrow \Delta_f(C) \rightarrow 0 \\
\downarrow b_{I,P_1} \downarrow b_{I,P_0} \downarrow \downarrow \\
\Lambda_f(P_1) \rightarrow \Lambda_f(P_0) \rightarrow L_0\Lambda_f(C) \rightarrow 0 \\
\downarrow \downarrow \downarrow \\
\Lambda_f(P_1) \rightarrow \Lambda_f(P_0) \rightarrow \Lambda_f(C)
\end{array}
\] (1.1)

The upper and lower rows are obtained by applying the functors $\Delta_f$ and $\Lambda_f$, respectively, to the right exact sequence $P_0 \rightarrow P_1 \rightarrow C \rightarrow 0$. The upper leftmost square is obtained by applying the natural transformation $b_I$ to the $R$-module morphism $P_1 \rightarrow P_0$. The upper and middle rows are right exact sequences (as the functor $\Delta_f$ is right exact by the above discussion). The upper dotted vertical arrow is uniquely defined by the condition of commutativity of the upper rightmost square. The morphism $\Delta_f(C) \rightarrow L_0\Lambda_f(C)$ is surjective because the morphism $b_{I,P_0}$ is.

The lower row is not exact in the middle, generally speaking; but the morphism $\Lambda_f(P_0) \rightarrow \Lambda_f(C)$ is still an epimorphism, because the functor $\Lambda_f$ takes epimorphisms to epimorphisms [43, Proposition 1.2]. The lower dotted vertical arrow is uniquely defined by the condition of commutativity of the lower rightmost square. It follows that the morphism $L_0\Lambda_f(C) \rightarrow \Lambda_f(C)$ is surjective.

Finally, the composition $\Delta_f(C) \rightarrow L_0\Lambda_f(C) \rightarrow \Lambda_f(C)$ is equal to the morphism $b_{I,C}$, since both of them make the square diagram $\Delta_f(P_0) \rightarrow \Delta_f(C) \rightarrow \Lambda_f(C), \Delta_f(P_0) \rightarrow \Lambda_f(P_0) \rightarrow \Lambda_f(C)$ commutative.

Part (b): $L_0\Lambda_f(C)$ is the cokernel of a morphism of $I$-adically separated and complete $R$-modules $\Lambda_f(P_1) \rightarrow \Lambda_f(P_0)$. Any such cokernel is a quotseparated $I$-contramodule.

Part (c): In order to establish the adjunction, it remains to show that, for any quotseparated $I$-contramodule $R$-module $K$ and any $R$-module morphism $C \rightarrow K$, the induced morphism of $I$-contramodule $R$-modules $\Delta_f(C) \rightarrow K$ factorizes through the epimorphism $\Delta_f(C) \rightarrow L_0\Lambda_f(C)$. The upper rightmost square in the diagram (1.1) is cocartesian, because the map $b_{I,P_0}$ is surjective. Therefore, it suffices to check that the composition $\Delta_f(P_0) \rightarrow \Delta_f(C) \rightarrow K$ factorizes through the epimorphism $\Delta_f(P_0) \rightarrow \Lambda_f(P_0)$.

Let $L \rightarrow K$ be a surjective $R$-module morphism onto $K$ from an $I$-adically separated and complete $R$-module $L$. The object $\Delta_f(P_0)$ is projective in the abelian category $R\text{-mod}_{I\text{-contr}}$, because the functor $\Delta_f$, being left adjoint to the exact inclusion functor, takes projectives to projectives. Both the $R$-modules $L$ and $K$ are $I$-contramodules; consequently, the $R$-module morphism $\Delta_f(P_0) \rightarrow K$ factorizes through the epimorphism $L \rightarrow K$:

\[
\begin{array}{c}
\Delta_f(P_0) \rightarrow \Delta_f(C) \\
\downarrow \\
L \rightarrow K
\end{array}
\]

Finally, since $L \in R\text{-mod}_{I\text{-contr}}$ and $\Lambda_f$ is the reflector onto $R\text{-mod}_{I\text{-contr}}$, any $R$-module morphism $\Delta_f(P) \rightarrow L$ factorizes through the epimorphism $b_{I,P} : \Delta_f(P) \rightarrow \Lambda_f(\Delta_f(P)) = \Lambda_f(P)$ (for any $R$-module $P$). Simply put, $L$ is $I$-adically separated and $\Lambda_f(P)$ is the maximal $I$-adically separated quotient $R$-module of $\Delta_f(P)$.

Part (d) follows from part (c).

□
For any set $X$, let us denote by $R[X] = R^X$ the free $R$-module with generators indexed by $X$. Then, the $R$-module $\Delta_I(R[X])$ is called the free $I$-contramodule $R$-module with $X$ generators, while $\text{L}_0\Delta_I(R[X]) = \Delta_I(R[X])$ is the free quotseparated $I$-contramodule $R$-module with $X$ generators.

The $R$-module $\Lambda_I(R)$ is what Yekutieli calls the $R$-module of decaying functions $X \rightarrow \Lambda_I(R)$ ([43, Section 2]) (see also the much earlier [34, Section II.2.4.2]). The notation in [43] is $F_{\text{fin}}(X, R) = R[X]$ (for the finitely supported functions $X \rightarrow R$) and $F_{\text{dec}}(X, \Lambda_I(R)) = \Lambda_I(R[X])$ (for the decaying functions $X \rightarrow \Lambda_I(R)$).

There are enough projective objects in both the abelian categories $R\text{-mod}_{I\text{-ctra}}$ and $R\text{-mod}_{I\text{-qts}}$. The projective objects of the category $R\text{-mod}_{I\text{-ctra}}$ are precisely the direct summands of the $R$-modules $\Delta_I(R[X])$. The projective objects of the category $R\text{-mod}_{I\text{-qts}}$ are precisely the direct summands of the $R$-modules $\Lambda_I(R[X])$.

**Lemma 1.4.** (a) Let $Q$ be a projective object of the abelian category $R\text{-mod}_{I\text{-ctra}}$ (i.e., the $R$-module $Q$ is a direct summand of the $R$-module $\Delta_I(R[X])$ for some set $X$). Then, $Q$ is a quotseparated $I$-contramodule $R$-module if and only if it is an $I$-adically separated (and complete) $R$-module.

(b) For any fixed set $X$, the $R$-module $\Delta_I(R[X])$ is quotseparated if and only if the natural map $b_{I,R[X]} : \Delta_I(R[X]) \rightarrow \Lambda_I(R[X])$ is an isomorphism.

**Proof.** Part (a): Suppose $Q$ is quotseparated; then it is a quotient module of some $I$-adically separated and complete $R$-module $C$. Since $C \rightarrow Q$ is an epimorphism in the abelian category $R\text{-mod}_{I\text{-ctra}}$ and $Q$ is a projective object in $R\text{-mod}_{I\text{-ctra}}$, it follows that $Q$ is a direct summand of $C$. Hence, $Q$ is $I$-adically separated. Part (b): Since $\Lambda_I(\Delta_I(P)) = \Lambda_I(P)$ for any $R$-module $P$, the map $b_{I,P} : \Delta_I(P) \rightarrow \Lambda_I(P)$ is an isomorphism if and only if the $R$-module $\Delta_I(P)$ is $I$-adically separated.

The $I$-adic completion $\hat{R} = \Lambda_I(R) = \varprojlim_{n \geq 1} R/I^n$ of the ring $R$ is a topological ring in the topology of projective limit of discrete rings $R/I^n$ (which coincides with the $I$-adic topology of the $R$-module $\hat{R}$). Following the memoir [24, Section 1.2], the paper [31, Sections 1.1–1.2 and 5], or the paper [33, Section 6], one can assign to a topological ring $\hat{R}$ the abelian category of $\text{R}$-contramodules $\hat{R}\text{-contra}$. These are modules with infinite summation operations with families of coefficients converging to zero in $\hat{R}$.

**Proposition 1.5.** The forgetful functor $\hat{R}\text{-contra} \rightarrow R\text{-mod}$ (induced by the canonical ring homomorphism $R \rightarrow \hat{R}$) is fully faithful, and its essential image is the full subcategory of quotseparated $I$-contramodule $R$-modules $R\text{-mod}_{I\text{-qts}}$. So we have an equivalence of abelian categories $\hat{R}\text{-contra} \simeq R\text{-mod}_{I\text{-qts}}$.

**Proof.** This is a straightforward generalization of [32, Theorem 5.20], based on the discussion in [30, Example 3.6 (3)]. One can also observe that the forgetful functor $\hat{R}\text{-contra} \rightarrow R\text{-mod}$ takes the free $\hat{R}$-contramodules $\hat{R}[[X]]$ to the free quotseparated $I$-contramodule $R$-modules $\Lambda_I(R[[X]])$. Simply put, by the definition of an $\hat{R}$-contramodule and in view of Proposition 1.3, both the abelian categories $\hat{R}\text{-contra}$ and $R\text{-mod}_{I\text{-qts}}$ are equivalent to the category of modules over the same monad $X \mapsto \hat{R}[[X]] = \Lambda_I(R[[X]])$ on the category of sets.

In particular, it follows from Proposition 1.5 that every quotseparated $I$-contramodule $R$-module has a natural, functorially defined $\hat{R}$-module structure (extending the $R$-module structure). Analogously, one can define a commutative $R$-algebra structure on the $R$-module $\Lambda_I(R)$ and show that every $I$-contramodule $R$-module has a functorially defined $\Delta_I(R)$-module structure (extending its $R$-module structure) [30, Example 5.2 (3)].

Notice that the full subcategory of quotseparated $I$-contramodule $R$-modules does not need to be closed under extensions in $R\text{-mod}_{I\text{-qts}}$ or in $R\text{-mod}$. In fact, the opposite is true.

**Proposition 1.6.** Every $I$-contramodule $R$-module is an extension of two quotseparated $I$-contramodule $R$-modules.

**Proof.** This is explained in [30, Example 5.2 (6)]. The argument is based on the computation of the kernel of the natural surjective morphism $b_{I,C} : \Delta_I(C) \rightarrow \Lambda_I(C)$ in [28, Lemma 7.5]. Basically, any $I$-contramodule $R$-module $C$ is naturally isomorphic to the $R$-module $\Delta_I(C)$; and for any $R$-module $C$, the $R$-module $\Lambda_I(C)$ is $I$-adically separated and complete, while the kernel of the map $b_{I,C}$ can be described as the cokernel of a natural map of $I$-adically separated and complete $R$-modules.
**Lemma 1.7.** The full subcategories $R-\text{mod}^{qs}_{I-\text{ctr}}$ and $R-\text{mod}_{I-\text{ctr}}$ coincide in $R-\text{mod}$ (i.e., in other words, every $I$-contramodule $R$-module is quotseparated) if and only if the map $b_{I,R[X]} : \Delta_{I}(R[X]) \rightarrow \Lambda_{I}(R[X])$ is an isomorphism for every set $X$. It suffices to check the latter condition for the countable set $X = \omega$.

**Proof.** The first assertion is essentially a tautology; see [30, Proposition 2.1] for the details. The second assertion holds because both the functors $\Delta_{I}$ and $\Lambda_{I}$ preserve countably filtered direct limits; so, for any infinite set $X$, one has $\Delta_{I}(R[X]) = \lim_{Z \subseteq X} \Delta_{I}(R[Z])$ and $\Lambda_{I}(R[X]) = \lim_{Z \subseteq X} \Lambda_{I}(R[Z])$, where $Z$ ranges over all the countably infinite subsets of $X$.

**Examples 1.8.** Let us mention an explicit example of an $I$-contramodule $R$-module that is not quotseparated. According to Lemma 1.4, for this purpose it suffices to come up with an example of a commutative ring $R$ with an ideal $I$ and a set $X$ such that the $R$-module $\Delta_{I}(R[X])$ is not $I$-adically separated, or equivalently, the natural morphism $b_{I,R[X]} : \Delta_{I}(R[X]) \rightarrow \Lambda_{I}(R[X])$ is not an isomorphism. The construction of [27, Example 2.6] produces, for any field $k$, a commutative $k$-algebra $R$ with a principal ideal $I = (p)$ for which the map $b_{I,R : \Delta_{I}(R) \rightarrow \Lambda_{I}(R)}$ is not an isomorphism. Hence, $\Delta_{I}(R)$ is a nonquotseparated $I$-contramodule $R$-module.

Moreover, let $\hat{R} = \Lambda_{I}(R)$ be the $I$-adic completion of the same ring $R$, and let $\hat{I} = s\hat{R} = \hat{I}R = \Lambda_{I}(I) \subseteq \hat{R}$ be the extension of the ideal $I \subseteq R$ in $\hat{R}$, or equivalently, the $I$-adic completion of the ideal $I$. Then, the $\hat{R}$-module $\hat{R}$ is $I$-adically separated and complete; so $\Delta_{I}(\hat{R}) = \Lambda_{I}(\hat{R}) = \hat{R}$. Still it follows from the discussion in Remark 3.8 below that, for any infinite set $X$, the map $b_{I,R[X]} : \Delta_{I}(R[X]) \rightarrow \Lambda_{I}(R[X])$ is not an isomorphism (since all the $s$-torsion in $\hat{R}$ is nondivisible, and it is unbounded, as one can easily see). So $\Delta_{I}(\hat{R}[X])$ is a nonquotseparated $\hat{I}$-contramodule $\hat{R}$-module.

The latter example also shows that the forgetful functor $\hat{R} \rightarrow \text{contra} \rightarrow \hat{R}-\text{mod}_{I-\text{ctr}}$ need not be an equivalence of categories for a commutative ring $\hat{R}$ that is separated and complete with respect to its finitely generated ideal $\hat{I}$. In fact, Proposition 1.5 applied to the ring $\hat{R}$ with the ideal $\hat{I}$ tells that $\hat{R} \rightarrow \text{contra} \simeq \hat{R}-\text{mod}^{qs}_{I-\text{ctr}}$; but $\hat{R}-\text{mod}^{qs}_{I-\text{ctr}}$ may still be a proper full subcategory in $\hat{R}-\text{mod}_{I-\text{ctr}}$. We refer to [30, Examples 5.2 (7–8)] for further discussion.

Let us say a few words about category-theoretic properties of the abelian categories of $I$-contramodule $R$-modules and quotseparated $I$-contramodule $R$-modules. *Neither* one of these abelian categories is Grothendieck. Indeed, let $I = (p) \subseteq Z = R$ be a maximal ideal in the ring of integers, generated by a prime number $p$; then the direct limit of the sequence of monomorphisms $\mathbb{Z}/p\mathbb{Z} \rightarrow \mathbb{Z}/p^{2}\mathbb{Z} \rightarrow \mathbb{Z}/p^{3}\mathbb{Z} \rightarrow \cdots$ vanishes in the abelian category $\mathbb{Z}_{p-\text{ctr}} = \mathbb{Z}^{qs}_{p-\text{ctr}}$. So does the direct limit of the sequence of monomorphisms $\mathbb{Z}_{p} \rightarrow \mathbb{Z}_{p} \rightarrow \mathbb{Z}_{p} \rightarrow \cdots$ (where $\mathbb{Z}_{p}$ denotes the abelian group of $p$-adic integers) [31, Examples 4.4]. See [21] and the appendix to [21] for other examples of similar behavior known in the literature.

This also shows that contramodule categories have a different nature than the categories of “condensed abelian groups” or “solid abelian groups” [38], where filtered direct limits are exact [38, Theorems 2.2 and 5.8]. Infinite coproducts are exact in the category $\mathbb{Z}_{p-\text{ctr}}$, because it is a contramodule category of homological dimension 1 (see [24, Remark 1.2.1]), but not in the abelian categories $R-\text{mod}_{I-\text{ctr}}$ or $R-\text{mod}^{qs}_{I-\text{ctr}}$, in general (not even when the ring $R$ is Noetherian).

On the other hand, both the abelian categories $R-\text{mod}_{I-\text{ctr}}$ and $R-\text{mod}^{qs}_{I-\text{ctr}}$ are always locally presentable. More precisely, they are locally $\mathbb{N}_{1}$-presentable (see [1] for the terminology). In the case of the category $R-\text{mod}_{I-\text{ctr}}$, this is explained in [31, Example 4.1 (3)] and [30, Examples 1.3 (4) and 2.2 (1)]. In the case of the category $R-\text{mod}^{qs}_{I-\text{ctr}}$, one can use Proposition 1.5 in order to reduce the question to the general assertion about categories of contramodules over topological rings [31, Sections 1.2 and 5], [30, Example 1.3 (2)].

Simply put, the abelian category $R-\text{mod}^{qs}_{I-\text{ctr}}$ is the category of modules over the monad $X \mapsto \Lambda_{I}(R[X])$, and the abelian category $R-\text{mod}_{I-\text{ctr}}$ is the category of modules over the monad $X \mapsto \Delta_{I}(R[X])$ on the category of sets. Both the abelian categories are locally $\mathbb{N}_{1}$-presentable, because both the functors $X \mapsto \Lambda_{I}(R[X])$ and $X \mapsto \Delta_{I}(R[X])$ preserve countably filtered direct limits. We refer to [31, Section 1.1], [33, Section 6], and [30, Section 1] for a discussion of accessible additive monads on the category of sets, which describe locally presentable abelian categories with a projective generator.

## 2 SEQUENTIALLY DERIVED TORSION AND COMPLETE COMPLEXES

What Yekutieli [45] calls “derived complete complexes in the sequential sense” are best understood geometrically.

Let us first introduce some notation and recall the definitions. Let $s = (s_{1,}, \ldots, s_{m})$ be a finite sequence of generators of the ideal $I \subseteq R$ (see Section 0.8). For every integer $n \geq 1$, we denote by $s^{n}$ the sequence of elements $s_{1}^{n}, \ldots, s_{m}^{n}$.
Let \( K(R; s) \) denote the Koszul complex
\[
K(R; s) = (R \overset{s_1}{\longrightarrow} R) \otimes_R \cdots \otimes_R (R \overset{s_m}{\longrightarrow} R),
\]
which is a finite complex of finitely generated free \( R \)-modules concentrated in the cohomological degrees \(-m, \ldots, 0\). The infinite dual Koszul complex
\[
K^\vee_{\infty}(R; s) = \lim_{\longleftarrow n} \text{Hom}_R(K(R; s^n), R)
\]
is a finite complex of flat \( R \)-modules concentrated in the cohomological degrees \( 0, \ldots, m \). It can be computed as the tensor product
\[
K^\vee_{\infty}(R; s) = (R \overset{s_1^{-1}}{\longrightarrow} R) \otimes_R \cdots \otimes_R (R \overset{s_m^{-1}}{\longrightarrow} R).
\]
The Čech complex \( \check{C}(R; s) \) is constructed as the kernel of the natural surjective morphism of complexes \( K^\vee_{\infty}(R; s) \twoheadrightarrow R \).

There is an explicit construction of a bounded \((m + 1)\)-term complex of countably generated free \( R \)-modules quasi-isomorphic to \( K^\vee_{\infty}(R; s) \); it is called the “telescope complex” in [22, Section 5] (see also [27, Section 2]).

One can show that the complexes \( K^\vee_{\infty}(R; s) \) and \( \check{C}(R; s) \) do not depend on the choice of a particular sequence of generators of a given ideal \( I \subset R \) up to a natural isomorphism in \( \mathcal{J}(R-\mathcal{M}) \). In fact, the following lemma holds.

**Lemma 2.1.** Let \( I' = (s_1', \ldots, s_m') \) and \( I'' = (s_1'', \ldots, s_n'') \subset R \) be two finitely generated ideals with the same radical \( \sqrt{I'} = \sqrt{I''} \). Put \( s' = (s_1', \ldots, s_m') \) and \( s'' = (s_1'', \ldots, s_n'') \), and let \((s', s'')\) denote the concatenation \((s_1', \ldots, s_m', s_1'', \ldots, s_n'')\) of the two sequences. Then, the natural morphisms of complexes of \( R \)-modules
\[
K^\vee_{\infty}(R; s') \longleftarrow K^\vee_{\infty}(R; (s', s'')) \longrightarrow K^\vee_{\infty}(R; s'')
\]
are quasi-isomorphisms.

**Proof.** This is [22, Corollary 6.2] and [45, Proposition 2.20]. Alternatively, one can observe that all the three Čech complexes in question compute the quasi-coherent sheaf cohomology \( H^*(U, \mathcal{O}_U) \) of the structure sheaf \( \mathcal{O}_U \) on the quasi-compact open complement \( U \subset \text{Spec } R \) of the closed subset \( Z \subset \text{Spec } R \) corresponding to the ideal \( I' + I'' \) (or which is the same, the closed subset corresponding to \( I' \) or \( I'' \)).

Given an element \( s \in R \), an \( R \)-module \( M \) is said to be \( s \)-torsion if for every \( m \in M \), there exists \( n \geq 1 \) such that \( s^n m = 0 \) in \( M \). An \( R \)-module \( M \) is said to be \( I \)-torsion if it is \( s \)-torsion for every \( s \in I \). We denote the Serre subcategory of \( I \)-torsion \( R \)-modules by \( \mathcal{J}(R-\mathcal{M})_{I-\mathcal{M}} \subset \mathcal{J}(R-\mathcal{M}) \).

A complex of \( R \)-modules \( M^* \) is said to be derived \( I \)-torsion in the sequential sense [45, Section 2] if the complex \( \check{C}(R; s) \otimes_R M^* \) is acyclic, or equivalently, the canonical morphism of complexes \( K^\vee_{\infty}(R; s) \otimes_R M^* \twoheadrightarrow M^* \) is a quasi-isomorphism.

**Lemma 2.2.** The following three conditions are equivalent for a complex of \( R \)-modules \( M^* \):

1. \( M^* \) is derived \( I \)-torsion in the sequential sense;
2. the complex \( R[s_j^{-1}] \otimes_R M^* \) is acyclic for every \( j = 1, \ldots, m \);
3. the cohomology \( R \)-module \( H^n(M^*) \) is \( I \)-torsion for every \( n \in \mathbb{Z} \).

**Proof.** (1) \( \iff \) (2) The finite complex of flat \( R \)-modules \( R[s_j^{-1}] \otimes_R K^\vee_{\infty}(R; s) \) is contractible for every \( 1 \leq j \leq m \). Hence, the complex \( R[s_j^{-1}] \otimes_R K^\vee_{\infty}(R; s) \otimes_R M^* \) is acyclic for any complex of \( R \)-modules \( M^* \).

(2) \( \iff \) (1) Every term of the finite complex of \( R \)-modules \( \check{C}(R; s) \) is a finite direct sum of \( R \)-modules of the form \( R[(s_j t)^{-1}] \approx R[s_j^{-1}] \otimes_R R[t^{-1}] \) for some \( 1 \leq j \leq m \) and \( t \in R \). Hence, acyclicity of the complexes \( R[s_j^{-1}] \otimes_R M^* \) implies acyclicity of the complex \( \check{C}(R; s) \otimes_R M^* \).
(2) $\iff$ (3) Clearly, an $R$-module $N$ is $I$-torsion if and only if it is $s_j$-torsion for every $1 \leq j \leq m$. Equivalently, the latter condition means that $R[s_j^{-1}] \otimes_R N = 0$. It remains to apply these observations to the $R$-modules $N = H^n(M^*)$ and recall that the $R$-module $R[s_j^{-1}]$ is flat.

In other words, Lemma 2.2 describes the category of derived $I$-torsion complexes (of $R$-modules) in the sequential sense as the full subcategory $D_{I\text{-tors}}(R\mod) \subset D(R\mod)$ of all complexes of $R$-modules with $I$-torsion cohomology modules in the derived category of $R$-modules.

Dually, a complex of $R$-modules $C^*$ is said to be derived $I$-adically complete in the sequential sense [45, Section 2] if $R\text{Hom}_R(C; s), C^*) = 0$, or equivalently, the canonical morphism $C^* \to R\text{Hom}_R(C; s), C^*)$ is an isomorphism in $D(R\mod)$. An equivalent definition can be found in [17, Definition tag091S]: A complex $C^*$ is “derived complete with respect to $I$” if $R\text{Hom}_R(R[s^{-1}], C^*) = 0$ for all $s \in I$.

Lemma 2.3. The following three conditions are equivalent for a complex of $R$-modules $C^*$:

1. $C^*$ is derived $I$-adically complete in the sequential sense;
2. the object $R\text{Hom}_R(R[s_j^{-1}], C^*) \in D(R\mod)$ vanishes for all $j = 1, \ldots, m$;
3. the cohomology $R$-module $H^n(C^*)$ is an $I$-contramodule for every $n \in \mathbb{Z}$.

Proof. The equivalence (1) $\iff$ (2) is provable in the same way as in the previous lemma. The equivalence (2) $\iff$ (3) holds because every $R$-module $B$ satisfying $Ext^n_B(R[s_j^{-1}], B) = 0$ for all $1 \leq j \leq m$ is an $I$-contramodule [28, Theorem 5.1] and the projective dimension of the $R$-modules $R[s_j^{-1}]$ does not exceed 1.

Lemma 2.3 describes the category of derived $I$-adically complete complexes (of $R$-modules) in the sequential sense as the full subcategory $D_{I\text{-ctr}}(R\mod) \subset D(R\mod)$ of all complexes of $R$-modules with $I$-contramodule cohomology modules in the derived category of $R$-modules.

Now that we are finished with the definitions and basic lemmas, we can have the geometric discussion promised in the beginning of this section. The basic concepts are the quasi-coherent sheaves and the contraherent cosheaves on a scheme $X$. The former is well known, and the latter was introduced in the preprint [25].

The definition of a contraherent cosheaf is obtained by dualizing a suitably formulated definition of a quasi-coherent sheaf. Let us recall the Enochs–Estrada interpretation of quasi-coherent sheaves [9, Section 2], which is convenient for dualization. A quasi-coherent sheaf $\mathcal{F}$ on a scheme $X$ is the same thing as a rule assigning to every affine open subscheme $U \subset X$ an $\mathcal{O}(U)$-module $\mathcal{F}(U)$ and to every pair of affine open subschemes $V \subset U \subset X$ a morphism of $\mathcal{O}(U)$-modules $\mathcal{F}(U) \to \mathcal{F}(V)$ in such a way that the induced morphism of $\mathcal{O}(V)$-modules $\mathcal{F}(V) \otimes_{\mathcal{O}(U)} \mathcal{F}(U) \to \mathcal{F}(V)$ is an isomorphism, and the triangle diagrams $\mathcal{F}(U) \to \mathcal{F}(V) \to \mathcal{F}(W)$ are commutative for all triples of affine open subschemes $W \subset V \subset U \subset X$. One can check that any such set of data extends uniquely to a quasi-coherent sheaf of $\mathcal{O}_X$-modules defined on all (and not only affine) open subsets in $X$.

Dually, a contraherent cosheaf $\mathcal{P}$ on a scheme $X$ is a rule assigning to every affine open subscheme $U \subset X$ an $\mathcal{O}(U)$-module $\mathcal{P}(U)$ and to every pair of affine open subschemes $V \subset U \subset X$ a morphism of $\mathcal{O}(U)$-modules $\mathcal{P}(V) \to \mathcal{P}(U)$ such that the induced morphism of $\mathcal{O}(V)$-modules $\mathcal{P}(V) \to \text{Hom}_{\mathcal{O}(U)}(\mathcal{O}(V), \mathcal{P}(U))$ is an isomorphism, $\text{Ext}^1(\mathcal{O}(U), \mathcal{P}(U)) = 0$, and the triangle diagrams $\mathcal{P}(W) \to \mathcal{P}(V) \to \mathcal{P}(U)$ are commutative. Here, the Ext vanishing condition only needs to be imposed for $\text{Ext}^1$, as the projective dimension of the $\mathcal{O}(U)$-module $\mathcal{O}(V)$ never exceeds 1 [25, beginning of Section 1.1 and Lemma 1.2.4] (no such condition was needed in the quasi-coherent context, as the $\mathcal{O}(U)$-module $\mathcal{O}(V)$ is flat). Furthermore, any such set of data extends uniquely to a cosheaf of $\mathcal{O}_X$-modules defined on all (and not only affine) open subsets in $X$ [25, Theorem 2.2.1].

We will not reproduce here the constructions of the direct and inverse image functors of contraherent cosheaves (denoted by $f_!$ and $f^!$, respectively, for a nice enough scheme morphism $f$). These are dual to the familiar constructions for quasi-coherent sheaves, and their properties are dual. We refer to [25, Section 2.3] for the details.

For any quasi-compact semiseparated scheme $Y$, the derived category of the abelian category of quasi-coherent sheaves on $Y$ is equivalent to the derived category of the exact category of contraherent cosheaves on $Y$ [25, Theorem 4.6.6]
for every bounded or unbounded derived category symbol \( \star = \bullet, +, -, \) or \( \emptyset \). Furthermore, for any morphism of quasi-compact semiseparated schemes \( f : Y \to X \), the equivalences of categories \( D^\star(Y \text{-qcoh}) \cong D^\star(Y \text{-ctrh}) \) and \( D^\star(X \text{-qcoh}) \cong D^\star(X \text{-ctrh}) \) transform the right derived direct image functor of quasi-coherent sheaves

\[
Rf_* : D^\star(Y \text{-qcoh}) \to D^\star(X \text{-qcoh})
\]

into the left derived direct image functor of contraherent cosheaves

\[
Lf_! : D^\star(X \text{-ctrh}) \to D^\star(X \text{-ctrh}),
\]

so \( Rf_* = Lf_! \) [25, Theorem 4.8.1].

In particular, for an affine scheme \( X = \text{Spec} \, R \), the abelian category \( X \text{-qcoh} \) is equivalent to the category of \( R \)-modules. The exact category \( X \text{-ctrh} \) is a full subcategory in \( R \text{-mod} \) consisting of all the contraadjusted \( R \)-modules (i.e., \( R \)-modules that are \( s \)-contraadjusted for all \( s \in R \), in the sense of the definition in Section 1). This restriction does not affect the derived category: The inclusion functor \( X \text{-ctrh} \to R \text{-mod} \) induces an equivalence \( D^\star(X \text{-ctrh}) \cong D^\star(R \text{-mod}) \).

Let \( Z \subset X \) be the closed subscheme \( Z = \text{Spec} \, R/I \), and let \( U = X \setminus Z \) be its open complement. Let \( j : U \to X \) denote the open embedding morphism. Then, the functor \( Rj_* \) has a left adjoint functor

\[
j^* : D^\star(X \text{-qcoh}) \to D^\star(U \text{-qcoh}),
\]

while the functor \( Lj_! \) has a right adjoint functor

\[
j^! : D^\star(X \text{-ctrh}) \to D^\star(U \text{-ctrh}).
\]

Moreover, both the compositions \( j^* \circ Rj_* \) and \( j^! \circ Lj_! \) are identity endofunctors. Hence, one obtains a recollement, which is described purely algebraically in [27, Section 3] (in some form, these results go back to [5, Section 6]).

Then, the full triangulated subcategory of derived \( I \)-torsion complexes in the sequential sense \( D^*_{I \text{-tors}}(R \text{-mod}) \subseteq D(R \text{-mod}) \) can be described as the kernel of the quasi-coherent open restriction functor \( j^* \),

\[
D^*_{I \text{-tors}}(R \text{-mod}) = \ker(j^* : D^\star(X \text{-qcoh}) \to D^\star(U \text{-qcoh})).
\]

Dually, the full triangulated subcategory of derived \( I \)-adically complete complexes in the sequential sense \( D^*_{I \text{-ctrh}}(R \text{-mod}) \subseteq D^\star(R \text{-mod}) \) can be described as the kernel of contraherent open restriction functor \( j^! \),

\[
D^*_{I \text{-ctrh}}(R \text{-mod}) = \ker(j^! : D^\star(X \text{-ctrh}) \to D^\star(U \text{-ctrh})).
\]

In fact, it follows from the existence of the recollement that the categories of complexes with \( I \)-torsion and with \( I \)-contramodule cohomology modules are equivalent [27, Proposition 3.3 and Theorem 3.4]

\[
D^*_{I \text{-tors}}(R \text{-mod}) \cong D^\star(R \text{-mod})/D^\star(U) \cong D^*_{I \text{-ctrh}}(R \text{-mod}). \quad (2.2)
\]

Here, \( D^\star(U) \) is a notation for the category \( D^\star(U \text{-qcoh}) \cong D^\star(U \text{-ctrh}) \), embedded into \( D(R \text{-mod}) \) by the functor \( Rj_* = Lj_! \). The two equivalences in (2.2) are provided by the compositions of the identity inclusions \( D^*_{I \text{-tors}}(R \text{-mod}) \to D^\star(R \text{-mod}) \) and \( D^*_{I \text{-ctrh}}(R \text{-mod}) \to D^\star(R \text{-mod}) \) with the Verdier quotient functor \( D^\star(R \text{-mod}) \to D^\star(R \text{-mod})/D^\star(U) \). So the recollement takes the form

\[
\begin{array}{cccccc}
D^\star(U \text{-qcoh}) & \hookrightarrow & D^\star(X \text{-qcoh}) & \hookrightarrow & D^*_{I \text{-tors}}(R \text{-mod}) \\
\downarrow Rj_* = Lj_! & & \downarrow & & \downarrow \\
D^\star(U) & \to & D^\star(R \text{-mod}) & \to & D^\star(R \text{-mod})/D^\star(U) \\
\downarrow & & \downarrow & & \downarrow \\
D^\star(U \text{-ctrh}) & \hookleftarrow & D^\star(X \text{-ctrh}) & \hookleftarrow & D^*_{I \text{-ctrh}}(R \text{-mod})
\end{array}
\]  
(2.3)
where two-headed arrows $\rightarrowtail$ denote triangulated Verdier quotient functors and arrows with a tail $\rightarrow$ denote triangulated fully faithful embeddings. After the identifications in the vertical equation signs (explained in the discussion above), the functors in the upper row are left adjoint to those in the middle row, which are left adjoint to those in the lower row. In every row, the image of the fully faithful embedding is equal to the kernel of the Verdier quotient functor.

The right adjoint functor to the inclusion (the coreflector) $D^+(R\text{-mod}) \rightarrow D^*_{I\text{-tors}}(R\text{-mod})$ is computable algebraically as the triangulated functor $M^\bullet \mapsto \bigwedge C(R; s) \otimes R M^\bullet$. This is called the sequential derived $I$-torsion functor in [45]. The sequential derived $I$-torsion functor is idempotent because it is a coreflector. One can also see it directly from the quasi-isomorphism

$$K^\vee \bigotimes_R K^\vee \rightarrow K^\vee$$ (2.4)

which is a particular case of Lemma 2.1. In fact, there are two quasi-isomorphisms of complexes in (2.4), given by the two arrows in (2.1) for $s' = s = s''$. The two maps of complexes induce the same isomorphism in $D(r\text{-mod})$.

Dually, the left adjoint functor to the inclusion (the reflector) $D^+(R\text{-mod}) \rightarrow D^*_{I\text{-tra}}(R\text{-mod})$ is computable algebraically as the functor $K^\vee \rightarrow \bigwedge C(R; s) \otimes R K^\vee$. This is called the sequential derived $I$-adic completion functor in [45]. The sequential derived $I$-adic completion functor is idempotent because it is a reflector; one can also see it directly from the quasi-isomorphism (2.4) (cf. [45, Section 1 and Remark 2.23]).

The restrictions of the functors $\mathbb{R}\text{Hom}_R(K^\vee_R (R; s), -)$ and $K^\vee_R (R; s) \otimes_R -$ onto the full subcategories $D^*_{I\text{-tors}}(R\text{-mod})$ and $D^*_{I\text{-tra}}(R\text{-mod}) \subset D^+(R\text{-mod})$ provide the mutually inverse equivalences $D^*_{I\text{-tors}}(R\text{-mod}) \simeq D^*_{I\text{-tra}}(R\text{-mod})$.

Somewhat similarly, the reflector $j^*: D(X\text{-qcoh}) \rightarrow D(U\text{-qcoh})$ onto the full triangulated subcategory $j_!(D(U\text{-qcoh})) \subset D(X\text{-qcoh})$ is computable algebraically as the functor $M^\bullet \mapsto \bigwedge C(R; s) \otimes_R M^\bullet$. The coreflector $j^!: D(X\text{-crr}) \rightarrow D(U\text{-crr})$ onto (the same, under the identification $D(X\text{-crr}) \simeq D(X\text{-crr})$) full triangulated subcategory $j_!(D(U\text{-crr})) \subset D(X\text{-crr})$ is computable algebraically as the functor $C^\bullet \mapsto \mathbb{R}\text{Hom}_R(\bigwedge C(R; s), C^\bullet)$.

**Remark 2.4.** The reader should be warned that the $!$-notation (as in $f_!$ and $f_!$, or $j_!$ and $j_!$) in the above exposition, as well as generally in [25], refers to what is called Neeman’s extraordinary inverse image functor in [25], with the reference to [20]. It should be distinguished from Deligne’s extraordinary inverse image functor constructed by Hartshorne in [13] and by Deligne in the appendix to [13]. In particular, the $!$-notation in [38, Lecture XI] stands for an extraordinary inverse image functor in the sense of Deligne. We refer to [13, Appendix] or [25, Introduction and Section 5.16] for a discussion.

### 3 Weak Proregularity

Let $C$ be an $R$-module. The object $\mathbb{R}\text{Hom}_R(K^\vee_R (R; s), C)$ of the derived category of $R$-modules $D(R\text{-mod})$ plays an important role in our considerations. Its cohomology modules appear in the natural short exact sequences of $R$-modules

$$0 \rightarrow \lim_{n \geq 1} H^{q-1}(K(R; s^n) \otimes_R C) \rightarrow H^q \mathbb{R}\text{Hom}_R(K^\vee_R (R; s), C) \rightarrow \lim_{n \geq 1} H^q(K(R; s^n) \otimes_R C) \rightarrow 0,$$ (3.1)

where $\lim_{n \geq 1}$ denotes the first derived countable projective limit. The sequence (3.1) may be nontrivial for integers $q$ in the interval $-m \leq q \leq 0$. (We recall that, following the notation in Section 0.8, $m$ is the length of the sequence of generators $s = (s_1, \ldots, s_m)$ of the ideal $I \subset R$.)

In particular, the $R$-module $\Delta_I(C)$ is computable [28, Theorem 7.2(iii)] as

$$\Delta_I(C) = H^0 \mathbb{R}\text{Hom}_R(K^\vee_R (R; s), C),$$ (3.2)

while the $I$-adic completion of the $R$-module $C$ is

$$\Lambda_I(C) = \lim_{n \geq 1} H^0(K(R; s^n) \otimes_R C).$$ (3.3)

A countable projective system of abelian groups $(E_n)_{n \geq 1}$ is said to satisfy the Mittag–Leffler condition if for every $i \geq 1$, there exists $j \geq i$ such that the images of the transition maps $E_k \mapsto E_j$ coincide (as subgroups in $E_j$) for all $k \geq j$. A
A countable projective system of abelian groups \((E_n)_{n \geq 1}\) is prozero if and only if the following two conditions hold:

1. The projective system \((E_n)_{n \geq 1}\) satisfies the Mittag-Leffler condition and
2. \(\lim \leftarrow_{n \geq 1} E_n = 0\).

Proof. The “only if” implication is obvious. To prove the “if,” suppose that a projective system \((E_n)_{n \geq 1}\) satisfies the Mittag-Leffler condition, and, for every \(i \geq 1\), denote by \(E'_i \subseteq E_i\) the image of the transition map \(E_j \rightarrow E_i\) for \(j\) large enough. Then, \((E'_n)_{n \geq 1}\) is a projective system of surjective maps of abelian groups such that \(\lim \leftarrow_{n \geq 1} E'_n = \lim \leftarrow_{n \geq 1} E_n\). Hence, the projection maps \(\lim \leftarrow_{n \geq 1} E'_n \rightarrow E'_i\) are surjective, and \(\lim \leftarrow_{n \geq 1} E_n = 0\) implies \(E'_i = 0\) for all \(i \geq 1\). □

For any countable projective system of abelian groups \((E_n)_{n \geq 1}\) satisfying the Mittag-Leffler condition, one has \(\lim \leftarrow_{n \geq 1} E_n = 0\). The following lemma provides a converse implication.

Lemma 3.2. For any countable projective system of abelian groups \((E_n)_{n \geq 1}\), the following two conditions are equivalent:

1. The projective system \((E_n)_{n \geq 1}\) satisfies the Mittag-Leffler condition;
2. \(\lim \leftarrow_{n \geq 1} E_n^{(X)} = 0\) for some (equivalently, every) infinite set \(X\).

Here, we denote by \(E^{(X)}\) the \(X\)-indexed direct sum of copies of an abelian group \(E\).

Proof. This is the result of the paper [8] (see [8, Corollary 6 (i) ⇔ (iii)]). □

We recall the following definition, which plays a key role, and refer to [45, Section 3] for a discussion of its history. The ideal \(I \subset R\) is said to be weakly proregular if, for every fixed \(q < 0\), the projective system

\[
(H^q(K(R; s^n)))_{n \geq 1}
\]

is prozero. This property does not depend on the choice of a particular finite system of generators \(s_1, \ldots, s_m\) of the ideal \(I\) [22, Corollary 6.3]. Furthermore, the following assertion holds.

Theorem 3.3. The ideal \(I \subset R\) is weakly proregular if and only if, for every injective \(R\)-module \(J\) and every integer \(k > 0\), one has \(H^k(K_{\infty}(R; s)) \otimes_R J = 0\).

Proof. This result goes back to Grothendieck [14, Lemma 2.4]; see also [35, Theorem 3.2] or [22, Theorem 4.24]. □

For any \(R\)-module \(M\), we denote by \(\Gamma_I(M) \subset M\) the maximal \(I\)-torsion submodule in \(M\). The functor \(\Gamma_I : R\text{-mod} \rightarrow R\text{-mod}_{I\text{-tors}}\) is right adjoint to the exact, fully faithful inclusion functor \(R\text{-mod}_{I\text{-tors}} \rightarrow R\text{-mod}\).

Corollary 3.4. The ideal \(I \subset R\) is weakly proregular if and only if for every injective \(R\)-module \(J\) the canonical morphism of complexes of \(R\)-modules

\[
\Gamma_I(J) \rightarrow K_{\infty}(R; s) \otimes_R J
\]

is a quasi-isomorphism.

Proof. In fact, for any \(R\)-module \(M\) there is a natural isomorphism of \(R\)-modules \(\Gamma_I(M) \simeq H^0(K_{\infty}(R; s) \otimes_R M)\). Hence, the corollary follows from Theorem 3.3. □
The next proposition and theorem, providing dual versions of Theorem 3.3 and Corollary 3.4, are the main results of this section.

Proposition 3.5. The ideal \( I \subset R \) is weakly proregular if and only if, for every \( q < 0 \), the following two conditions hold:

i. \[ \lim_{\to n \geq 1} H^q(K(R; s^n) \otimes_R R[X]) = 0 \]
for some (equivalently, every) infinite set \( X \);

ii. \[ \lim_{\to n \geq 1} H^q(K(R; s^n) \otimes_R R[X]) = 0 \]
for some (equivalently, every) nonempty set \( X \).

Here, \( R[X] = R^{(X)} \) denotes the free \( R \)-module with \( X \) generators.

Proof. Clearly, for any projective system \((E_n)_{n \geq 1}\) and a nonempty set \( X \), one has

\[ \lim_{\to n \geq 1} E_n = 0 \]
if and only if \( \lim_{\to n \geq 1} E_n^{(X)} = 0 \).

So the conditions in (ii) are equivalent for all nonempty sets \( X \), and they are equivalent to the condition (ii) of Lemma 3.1 for the projective system \( E_n = H^q(K(R; s^n)) \).

Furthermore, by Lemma 3.2, the conditions in (i) are equivalent for all infinite sets \( X \), and they are equivalent to the condition (i) of Lemma 3.1 for the same projective system \((E_n = H^q(K(R; s^n)))_{n \geq 1}\).

The assertion of Lemma 3.1 now provides the desired result. \( \square \)

The following theorem can be also found in [15, Remark 7.8].

Theorem 3.6. The ideal \( I \subset R \) is weakly proregular if and only if for some (equivalently, for every) infinite set \( X \), the canonical morphism

\[ R \text{Hom}_R(K^\omega(R; s), R[X]) \to \Lambda_I(R[X]) \]
is an isomorphism in the derived category of \( R \)-modules \( \mathcal{D}(R\text{-mod}) \).

Proof. Follows immediately from the short exact sequences (3.1) together with the isomorphism (3.3) and Proposition 3.5. \( \square \)

Corollary 3.7. If the ideal \( I \subset R \) is weakly proregular, then the full subcategories \( R\text{-mod}_{I-\text{ctr}}^{qs} \) and \( R\text{-mod}_{I-\text{ctr}} \) coincide in \( R\text{-mod} \).

Proof. Follows from Lemma 1.7, the isomorphism (3.2), and Theorem 3.6. \( \square \)

Remark 3.8. The converse assertion to Corollary 3.7 is not true: The condition that the two full subcategories \( R\text{-mod}_{I-\text{ctr}}^{qs} \) and \( R\text{-mod}_{I-\text{ctr}} \) in \( R\text{-mod} \) coincide is weaker than the weak proregularity of the ideal \( I \). In fact, Proposition 3.5 splits the weak proregularity condition for a sequence of elements \( s_1, \ldots, s_m \in R \) into \( 2m \) pieces. Precisely one of these \( 2m \) conditions, namely, condition (i) for \( q = -1 \), is equivalent to every \( I \)-contramodule \( R \)-module being quotient separable.

So, do the remaining \( 2m - 1 \) conditions do? As we will see in Section 4, the answer is that condition (ii) for all \( q \leq -1 \) and condition (i) for all \( q \leq -2 \) hold together if and only if the triangulated functor \( \mathcal{D}(R\text{-mod}_{I-\text{ctr}}) \to \mathcal{D}(R\text{-mod}) \) induced by the inclusion of abelian categories \( R\text{-mod}_{I-\text{ctr}} \to R\text{-mod} \) is fully faithful. It follows that the ideal \( I \subset R \) is weakly proregular (i.e., all the \( 2m \) conditions hold) if and only if the triangulated functor \( \mathcal{D}(R\text{-mod}_{I-\text{ctr}}^{qs}) \to \mathcal{D}(R\text{-mod}) \) induced by the inclusion of abelian categories \( R\text{-mod}_{I-\text{ctr}}^{qs} \to R\text{-mod} \) is fully faithful.

The special case of \( m = 1 \) is instructive. A principal ideal \( I = (s) \) is weakly proregular if and only if the \( s \)-torsion in the ring \( R \) is bounded, that is, there exists an integer \( n_0 \geq 1 \) such that for any integer \( n \geq 1 \) and any element \( r \in R \), the equation \( s^n r = 0 \) implies \( s^{n_0} r = 0 \). This condition splits into two conditions (i) and (ii) of Proposition 3.5 in the following way.

Given an \( R \)-module \( M \), the \( s \)-torsion submodule of \( M \) consists of all the elements \( m \in M \) for which there exists \( n \geq 1 \) such that \( s^n m = 0 \) in \( M \). We say that an \( R \)-module \( M \) has no divisible \( s \)-torsion if the module \( \text{Hom}_R(R[s^{-1}]/R, M) \) vanishes (where \( R[s^{-1}]/R \) is a simplified notation for the cokernel of the localization map \( R \to R[s^{-1}] \)). The divisible \( s \)-torsion submodule of \( M \) consists of all elements belonging to (the union of) the images of \( R \)-module maps \( R[s^{-1}]/R \to M \). Finally, we say that the nondivisible \( s \)-torsion in \( M \) is bounded if there exists \( n_0 \geq 1 \) such that the \( s \)-torsion submodule in \( M \) is the sum of the divisible \( s \)-torsion submodule and the kernel of the map \( s^{n_0} : M \to M \).
Then condition (i) means that the nondivisible $s$-torsion in the $R$-module $R$ is bounded, while condition (ii) is equivalent to $R$ having no divisible $s$-torsion. So one has $R{-\text{mod}}_{s{-\text{tors}}} = R{-\text{mod}}^{qs}_{s{-\text{tors}}}$ if and only if the nondivisible $s$-torsion in $R$ is bounded, while the functor $D(R{-\text{mod}}_{s{-\text{tors}}}) \to D(R{-\text{mod}})$ is fully faithful if and only if $R$ has no divisible $s$-torsion. The functor $D(R{-\text{mod}}^{qs}_{s{-\text{tors}}}) \to D(R{-\text{mod}})$ is fully faithful if and only if $R$ has bounded $s$-torsion.

4 | FULL-AND-FAITHFULNESS OF TRIANGULATED FUNCTORS AND WEAK PROREGULARITY

We prove several basic theorems in this section before proceeding to discuss derived $I$-adically complete complexes in the idealistic sense in the next one.

To begin with, here is the $I$-torsion version of the main theorem. Let us introduce some notation. Let $\star = b, +, -, or \emptyset$ be a bounded or unbounded derived category symbol. Denote the triangulated functor $D^*(R{-\text{mod}}_{I{-\text{tors}}}) \to D^*(R{-\text{mod}})$ induced by the inclusion of abelian categories $R{-\text{mod}}_{I{-\text{tors}}} \to R{-\text{mod}}$ by

$$\mu^* : D^*(R{-\text{mod}}_{I{-\text{tors}}}) \to D^*(R{-\text{mod}}).$$

The functor $\mu^*$ obviously factorizes into a composition

$$D^*(R{-\text{mod}}_{I{-\text{tors}}}) \xrightarrow{\chi^*} D^*_{I{-\text{tors}}}(R{-\text{mod}}) \xrightarrow{\nu^*} D(R{-\text{mod}}),$$

where $\nu^*$ is the canonical fully faithful inclusion.

In the proofs of the theorems in this section, we will use the concept of a partially defined adjoint functor. Given two categories $C, D$ and a functor $F : C \to D$, we say that a partial functor $G$ right adjoint to $F$ is defined on an object $D \in D$ if there exists an object $G(D) \in C$ such that for every object $C \in C$, there is a bijection of sets $\text{Hom}_C(C, G(D)) \cong \text{Hom}_D(F(C), D)$ functorial in the object $C \in C$. Since an object representing a functor is unique up to a unique isomorphism, the object $G(D)$ is unique if it exists. Partial left adjoint functors are defined similarly.

Theorem 4.1. If the ideal $I \subset R$ is weakly proregular, then for every symbol $\star = b, +, -, or \emptyset$, the functor $\chi^* : D^*(R{-\text{mod}}_{I{-\text{tors}}}) \to D^*_{I{-\text{tors}}}(R{-\text{mod}})$ is an equivalence of categories and the functor $\mu^* : D^*(R{-\text{mod}}_{I{-\text{tors}}}) \to D^*(R{-\text{mod}})$ is fully faithful.

Conversely, if for one of the symbols $\star = b, +, -, or \emptyset$ the functor $\mu^*$ is fully faithful, then the ideal $I \subset R$ is weakly proregular.

Proof. The direct assertion is the result of [27, Theorem 1.3 and Corollary 1.4]. Let us prove the converse. Clearly, if the functor $\mu^*$ is fully faithful for one of the symbols $\star = b, +, -, or \emptyset$, then it is fully faithful for $\star = b$. It is also clear that if the functor $\mu^b$ is fully faithful, then its essential image coincides with the full subcategory $D^b_{I{-\text{tors}}}(R{-\text{mod}}) \subset D^b(R{-\text{mod}})$; so the functor $\chi^b$ is a category equivalence in this case.

Now the functor of inclusion of abelian categories $R{-\text{mod}}_{I{-\text{tors}}} \to R{-\text{mod}}$ has a right adjoint functor $\Gamma_I : R{-\text{mod}} \to R{-\text{mod}}_{I{-\text{tors}}}$. Being right adjoint to an exact functor, the functor $\Gamma_I$ takes injective $R$-modules to injective objects of the category of $I$-torsion $R$-modules. One easily concludes that the functor $\mu^+ : D^+(R{-\text{mod}}_{I{-\text{tors}}}) \to D^+(R{-\text{mod}})$ has a right adjoint functor $\gamma^+ : D^+(R{-\text{mod}}) \to D^+(R{-\text{mod}}_{I{-\text{tors}}})$, which is computable as the right derived functor of the functor $\Gamma_I$. So the functor $\gamma^+$ assigns to a bounded below complex of injective $R$-modules $I^*$ the bounded below complex of injective $I$-torsion $R$-modules $\Gamma(I^*)$. Similarly, for $\star = \emptyset$, the functor $\mu : D(R{-\text{mod}}_{I{-\text{tors}}}) \to D(R{-\text{mod}})$ has a right adjoint functor $\gamma : D(R{-\text{mod}}) \to D(R{-\text{mod}}_{I{-\text{tors}}})$, which can be computed by applying the functor $\Gamma_I$ to homotopy injective complexes of $R$-modules (known also as “$K$-injective complexes” [41]).

For $\star = b$, the functor $\mu^b : D^b(R{-\text{mod}}_{I{-\text{tors}}}) \to D^b(R{-\text{mod}})$ does not seem to necessarily have a right adjoint, in general; but we are interested in its partially defined right adjoint functor $\gamma^b$. It is only important for us that the partial functor $\gamma^b$ is defined on injective $R$-modules $J \in R{-\text{mod}}_{\text{inj}} \subset R{-\text{mod}} \subset D^b(R{-\text{mod}})$; indeed, one has $\gamma^b(J) = \Gamma_I(J) \in R{-\text{mod}}_{I{-\text{tors}}}$, as can be easily seen.

On the other hand, for any symbol $\star = b, +, -, or \emptyset$, the fully faithful inclusion functor $\nu^* : D^*_{I{-\text{tors}}}(R{-\text{mod}}) \to D^*(R{-\text{mod}})$ has a right adjoint functor $\delta^* : D^*(R{-\text{mod}}) \to D^*_{I{-\text{tors}}}(R{-\text{mod}})$. This holds quite generally for any finitely
generated ideal $I \subset R$; following the discussion in Section 2 and [27, Section 3], the functor $\theta^*$ is computable as
\[
\theta^*(M^*) = K^\vee_\infty(R; s) \otimes_R M^* \quad \text{for all } M^* \in D^*(R-\text{mod}).
\]

Finally, if the functor $\chi^b : D^b(R-\text{mod}_{I-\text{tors}}) \to D^b_{I-\text{tors}}(R-\text{mod})$ is a category equivalence, then it identifies the functor $\rho^b$ with the functor $\nu^b$; and consequently it also identifies their adjoint functors $\gamma^b$ and $\vartheta^b$. It follows that the functor $\gamma^b$ is everywhere defined in this case; but this is not important for us. The key observation is that the objects $\gamma^b(J) \in D^b(I-\text{mod}_{I-\text{tors}})$ and $\vartheta^b(J) \in D^b_{I-\text{tors}}(R-\text{mod})$ are identified by the functor $\chi^b$, for any injective $R$-module $J$. In other words, we have an isomorphism of complexes of $R$-modules
\[
\Gamma_J(J) \cong K^\vee_\infty(R; s) \otimes_R J.
\]

Hence, $H^k(K^\vee^\infty_\infty(R; s) \otimes_R J) = 0$ for all $k > 0$. According to Theorem 3.3, it follows that the ideal $I \subset R$ is weakly proregular.

The following quotient separated $I$-contramodule theorem is the main result of this section. In order to formulate and prove it, we need some further notation. Denote the triangulated functor $D^*(R-\text{mod}^I_{I-\text{tra}}) \to D^*(R-\text{mod})$ induced by the inclusion of the abelian categories $R-\text{mod}^I_{I-\text{tra}} \to R-\text{mod}$ by
\[
\rho^* : D^*(R-\text{mod}^I_{I-\text{tra}}) \to D^*(R-\text{mod}).
\]
The functor $\rho^*$ obviously factorizes into a composition
\[
D^*(R-\text{mod}^I_{I-\text{tra}}) \xrightarrow{\xi^*} D^*_I(R-\text{mod}) \xrightarrow{\iota^*} D^*(R-\text{mod}),
\]
where $\iota^*$ is the canonical fully faithful inclusion.

**Theorem 4.2.** If the ideal $I \subset R$ is weakly proregular, then for every symbol $\star = b, +, -, \emptyset$, the functor $\xi^* : D^*(R-\text{mod}^I_{I-\text{tra}}) \to D^*_I(R-\text{mod})$ is an equivalence of categories and the functor $\rho^* : D^*(R-\text{mod}^I_{I-\text{tra}}) \to D^*(R-\text{mod})$ is fully faithful.

Conversely, if for one of the symbols $\star = b, +, -, \emptyset$, the functors $\rho^*$ is fully faithful, then the ideal $I \subset R$ is weakly proregular.

**Proof.** If the ideal $I$ is weakly proregular, then we have $R-\text{mod}^I_{I-\text{tra}} = R-\text{mod}_{I-\text{tra}}$ by Corollary 3.7. Having made this observation, it remains to refer to [27, Theorem 2.9 and Corollary 2.10] for the proof of the direct assertion.

To prove the converse, we argue similarly (or rather, dual-analogously) to the proof of Theorem 4.1. Clearly, if the functor $\rho^*$ is fully faithful for one of the symbols $\star = b, +, -, \emptyset$, then it is fully faithful for $\star = b$. Furthermore, if the functor $\rho^b$ is fully faithful, then its essential image coincides with the full subcategory $D^b_I(R-\text{mod}) \subset D^b(R-\text{mod})$, because the triangulated category $D^b(I-\text{mod}^I_{I-\text{tra}})$ is generated by its abelian subcategory $R-\text{mod}^I_{I-\text{tra}}$ and the triangulated subcategory in $D^b(R-\text{mod})$ generated by $R-\text{mod}^I_{I-\text{tra}}$ is precisely $D^b_I(R-\text{mod})$ (in view of Lemma 1.1 and Proposition 1.6). So the functor $\xi^b$ is an equivalence of categories in this case.

Now the functor of inclusion of abelian categories $R-\text{mod}^I_{I-\text{tra}} \to R-\text{mod}$ has a left adjoint functor $L_0\Lambda_I : R-\text{mod} \to R-\text{mod}^I_{I-\text{tra}}$ (see Proposition 1.3(c)). The functor $L_0\Lambda_I$ is left adjoint to an exact functor, so it takes projective $R$-modules to projective objects of the category $R-\text{mod}^I_{I-\text{tra}}$. One easily concludes that the functor $\rho^- : D^-(R-\text{mod}^I_{I-\text{tra}}) \to D^-(R-\text{mod})$ has a left adjoint functor $\lambda^- : D^-(R-\text{mod}) \to D^-(R-\text{mod}^I_{I-\text{tra}})$, which is computable as the left derived functor of the functor $L_0\Lambda_I$, or which is the same, the left derived functor of the functor $\Lambda_I$. So the functor $\lambda^-$ assigns to a bounded above complex of projective $R$-modules $P^*$ the bounded above complex of projective quotient separated $I$-contramodule $R$-modules $\Lambda_I(P^*)$.

More generally, for $\star = \emptyset$, the functor
\[
\rho : D(R-\text{mod}^I_{I-\text{tra}}) \to D(R-\text{mod})
\]
also has a left adjoint functor
\[ \lambda : D(R-\text{mod}) \rightarrow D\left( R-\text{mod}^{\ell}_{R-\text{ctra}} \right), \]
which can be computed by applying the functor \( \Lambda_I \) to homotopy projective complexes of \( R \)-modules (known also as “K-projective complexes” [41]). Following [22, Proposition 3.6], one can also compute the functor \( \lambda \) by applying the functor \( \Lambda_I \) to homotopy flat (“K-flat”) complexes of \( R \)-modules.

For \( \star = b \), the functor \( \rho^b : D^b(R-\text{mod}_{I-tors}) \rightarrow D^b(R-\text{mod}) \) does not seem to necessarily have a left adjoint, in general; so we are interested in its partially defined left adjoint functor \( \lambda^b \). It is only important for us that the partial functor \( \lambda^b \) is defined on projective \( R \)-modules \( P \in R-\text{mod}_{I-\text{ctra}} \subset D^b(R-\text{mod}) \), as can be easily seen.

On the other hand, for any symbol \( \star = b, +, - \), or \( \emptyset \), the fully faithful inclusion functor \( \iota^\star : D^\star(I-\text{ctra})(R-\text{mod}) \rightarrow D^\star(R-\text{mod}) \) has a left adjoint functor \( \eta^\star : D^\star(R-\text{mod}) \rightarrow D^\star(I-\text{ctra})(R-\text{mod}) \). This holds quite generally for any finitely generated ideal \( I \subset R \); following the discussion in Section 2 and [27, Section 3], the functor \( \eta^\star \) is computable as
\[ \eta^\star(C^\star) = \mathbb{R} \text{Hom}_R(K_{\infty}^C(R; s), C^\star) \quad \text{for all } C^\star \in D^\star(R-\text{mod}). \]

Finally, if the functor \( \varepsilon^b : D^b(R-\text{mod}_{I-\text{ctra}}) \rightarrow D^b(I-\text{ctra})(R-\text{mod}) \) is a category equivalence, then it identifies the functor \( \rho^b \) with the functor \( \iota^b \); and consequently, it also identifies their adjoint functors \( \lambda^b \) and \( \eta^b \). It follows that the functor \( \lambda^b \) is everywhere defined in this case; but this is not important for us. The key observation is that the objects \( \lambda^b(P) \in D^b(R-\text{mod}_{I-\text{ctra}}) \) and \( \eta^b(P) \in D^b(I-\text{ctra})(R-\text{mod}) \) are identified by the functor \( \varepsilon^b \), for any projective \( R \)-module \( P \). In other words, we have an isomorphism of complexes of \( R \)-modules
\[ \lambda_I(P) \cong \mathbb{R} \text{Hom}_R(K_{\infty}^C(R; s), P). \]

Hence, \( H^k \mathbb{R} \text{Hom}_R(K_{\infty}^C(R; s), P) = 0 \) for \( k < 0 \) and \( \Lambda_I(P) \cong \Delta_I(P) \) (see isomorphism (3.2)). It follows that the \( R \)-module \( \Delta_I(P) \) is \( I \)-adically separated, and therefore the canonical surjective morphism \( b_{I,P} : \Delta_I(P) \rightarrow \Lambda_I(P) \) is an isomorphism. According to Theorem 3.6, we can conclude that the ideal \( I \subset R \) is weakly proregular.

Alternatively, for the proof of the converse assertion, one can first observe that if the functor \( \rho^b \) is fully faithful, then \( R-\text{mod}_{I-\text{ctra}}^{\ell} = R-\text{mod}_{I-\text{ctra}} \) (in view of Proposition 1.6). This translates, via Lemma 1.7, into the map \( b_{I,R[X]} : \Delta_I(R[X]) \rightarrow \Lambda_I(R[X]) \) being an isomorphism. Second, one can apply the converse assertion of the next Theorem 4.3; and finally use the isomorphism (3.2) and Theorem 3.6. This is the argument hinted at in Remark 3.8.

Our last theorem in this section applies to the category of (not necessarily quotseparated) \( I \)-contramodules. It is the result promised in Remark 3.8. The cohomology vanishing condition appearing in this theorem corresponds, in terms of Proposition 3.5, to the combination of conditions (i) for all \( q \leq -2 \) and conditions (ii) for all \( q \leq -1 \). So it is a bit weaker than weak proregularity of the ideal \( I \), in that condition (i) for \( q = -1 \) is not required. We refer to Remark 3.8 for the discussion.

Once again, we need to introduce some notation. Denote the triangulated functor \( D^\star(R-\text{mod}_{I-\text{ctra}}) \rightarrow D^\star(R-\text{mod}) \) induced by the inclusion of abelian categories \( R-\text{mod}_{I-\text{ctra}} \rightarrow R-\text{mod} \)
\[ \pi^\star : D^\star(R-\text{mod}_{I-\text{ctra}}) \rightarrow D^\star(R-\text{mod}). \]

The functor \( \pi^\star \) obviously factorizes into a composition
\[ D^\star(R-\text{mod}_{I-\text{ctra}}) \xrightarrow{\varepsilon^\star} D^\star(I-\text{ctra})(R-\text{mod}) \xrightarrow{\iota^\star} D^\star(R-\text{mod}). \]

**Theorem 4.3.** If the cohomology vanishing condition \( H^k \mathbb{R} \text{Hom}_R(K_{\infty}^C(R; s), R[X]) = 0 \) holds for some infinite set \( X \) and all \( k < 0 \), then for every symbol \( \star = b, +, - \), or \( \emptyset \), the functor \( \varepsilon^\star : D^\star(R-\text{mod}_{I-\text{ctra}}) \rightarrow D^\star(I-\text{ctra})(R-\text{mod}) \) is an equivalence of categories and the functor \( \pi^\star : D^\star(R-\text{mod}_{I-\text{ctra}}) \rightarrow D^\star(R-\text{mod}) \) is fully faithful.

Conversely, if for one of the symbols \( \star = b, +, - \), or \( \emptyset \), the functor \( \pi^\star \) is fully faithful, then \( H^k \mathbb{R} \text{Hom}_R(K_{\infty}^C(R; s), R[X]) = 0 \) for all sets \( X \) and all \( k < 0 \).
Proof. For the direct assertion, one has to follow the proofs of [27, Theorem 2.9 and Corollary 2.10] and convince oneself that our present cohomology vanishing condition is sufficient for the purposes of those proofs in lieu of the full weak proregularity. Indeed, one observes that the assertions of [27, Lemma 2.7], and consequently of [27, Lemma 2.8], remain valid under our cohomology vanishing condition (while [27, Lemma 2.5] requires condition (i) for \( q = -1 \), as [27, Example 2.6] shows; but this lemma is not needed for the proofs of [27, Theorem 2.9 and Corollary 2.10]). In the notation of [27, proof of Lemma 2.7(a)], one can consider projective or free \( R \)-modules \( F \), which is enough.

The proof of the converse is similar to that of Theorem 4.2. Clearly, if the functor \( \pi^\star \) is fully faithful for one of the symbols \( \star = b, +, - \), or \( \emptyset \), then it is fully faithful for \( \star = b \). It is also clear that if the functor \( \pi^b \) is fully faithful, then its essential image coincides with the full subcategory \( D^b_{I-\text{ctr}}(R-\text{mod}) \subset D^b(R-\text{mod}) \); so the functor \( \zeta^b \) is a category equivalence in this case.

Now the functor of inclusion of abelian categories \( R-\text{mod}_{I-\text{ctr}} \longrightarrow R-\text{mod} \) has a left adjoint functor \( \Delta_I : R-\text{mod} \longrightarrow R-\text{mod}_{I-\text{ctr}} \). The functor \( \Delta_I \) is left adjoint to an exact functor, so it takes projective \( R \)-modules to projective objects of the category \( R-\text{mod}_{I-\text{ctr}} \). One easily concludes that the functor \( \pi^- : D^-(R-\text{mod}_{I-\text{ctr}}) \longrightarrow D^-(R-\text{mod}) \) has a left adjoint functor \( \delta^- : D^-(R-\text{mod}) \longrightarrow D^-(R-\text{mod}_{I-\text{ctr}}) \), which is computable as the left derived functor of the functor \( \Delta_I \). So the functor \( \delta^- \) assigns to a bounded above complex of projective \( R \)-modules \( \mathcal{P} \) the bounded above complex of projective \( I \)-contramodule \( R \)-modules \( \Delta_I(\mathcal{P}) \).

Similarly, for \( \star = \emptyset \), the functor \( \pi : D(R-\text{mod}) \longrightarrow D(R-\text{mod}_{I-\text{ctr}}) \) also has a left adjoint functor \( \delta : D(R-\text{mod}) \longrightarrow D(R-\text{mod}_{I-\text{ctr}}) \), which can be computed by applying the functor \( \Delta_I \) to homotopy projective complexes of \( R \)-modules.

For \( \star = b \), the functor \( \pi_b : D^b(R-\text{mod}_{I-\text{ctr}}) \longrightarrow D^b(R-\text{mod}) \) does not seem to necessarily have a left adjoint, in general; but we are interested in its partially defined left adjoint functor \( \delta^b \). It is only important for us that the partial functor \( \delta^b \) is defined on projective \( R \)-modules \( \mathcal{P} \in R-\text{mod}_{I-\text{ctr}} \subset R-\text{mod} \subset D^b(R-\text{mod}) \); indeed, one has \( \delta^b(\mathcal{P}) = \Delta_I(\mathcal{P}) \in R-\text{mod}_{I-\text{ctr}} \subset D^b(R-\text{mod}_{I-\text{ctr}}) \).

On the other hand, the fully faithful functor \( \iota^* : D_{I-\text{ctr}}(R-\text{mod}) \longrightarrow D^*(R-\text{mod}) \) has a left adjoint functor \( \eta^* \), as it was explained in the proof of Theorem 4.2. Now if the functor \( \iota^* : D^b(R-\text{mod}_{I-\text{ctr}}) \longrightarrow D^b_{I-\text{ctr}}(R-\text{mod}) \) is a category equivalence, then it identifies the functor \( \pi^b \) with the functor \( \iota^b \); and consequently, it also identifies their adjoint functors \( \delta^b \) and \( \eta^b \). Thus, the objects \( \delta^b(\mathcal{P}) \in D^b(R-\text{mod}_{I-\text{ctr}}) \) and \( \eta^b(\mathcal{P}) \in D^b_{I-\text{ctr}}(R-\text{mod}) \) are identified by the functor \( \delta^b \), for any projective \( R \)-module \( \mathcal{P} \). In other words, we have an isomorphism of complexes of \( R \)-modules

\[
\Delta_I(\mathcal{P}) \cong R \text{Hom}_R(K_{c\Omega}(\mathcal{K}; s), \mathcal{P}).
\]

Hence, \( H^k R \text{Hom}_R(K_{c\Omega}(\mathcal{K}; s), \mathcal{P}) = 0 \) for \( k < 0 \), as desired.

5  IDEALISTICALLY DERIVED COMPLETE COMPLEXES

Let us recall the notation of Theorem 4.2 and its proof. The inclusion of abelian categories \( R-\text{mod}_{I-\text{ctr}} \longrightarrow R-\text{mod} \) induces a triangulated functor

\[
\rho : D(R-\text{mod}_{I-\text{ctr}}) \longrightarrow D(R-\text{mod}),
\]

which has a left adjoint functor

\[
\lambda : D(R-\text{mod}) \longrightarrow D\left( R-\text{mod}_{I-\text{ctr}} \right).
\]

Porta–Shaul–Yekutieli in [22, Section 3] and Yekutieli in [45, Section 1] consider the functor

\[
L\Lambda_I : D(R-\text{mod}) \longrightarrow D(R-\text{mod}),
\]

which is called the idealistic derived \( I \)-adic completion functor in [45]. Following the construction of the functor \( \lambda \) discussed in the proof of Theorem 4.2, the functor \( L\Lambda_I \) is the composition of the two adjoint functors, \( L\Lambda_I = \rho \circ \lambda \),

\[
D(R-\text{mod}) \longrightarrow D\left( R-\text{mod}_{I-\text{ctr}} \right) \longrightarrow D(R-\text{mod}).
\]
The point is that applying the functor $\Lambda_I$ to every term of a homotopy projective (or homotopy flat) complex of $R$-modules $P'$ produces a complex of $I$-adically separated and complete $R$-modules $\Lambda_I(P')$. All such modules are quot-separated $I$-contramodules; so a complex of such modules can be naturally considered as an object of the derived category of quot-separated $I$-contramodules, $\lambda(P') = \Lambda_I(P') \in \mathcal{D}(\text{mod}^{qs}_{I-\text{ctr}})$. One can also consider $\Lambda_I(P')$ as an object of the derived category of $R$-modules; this means applying the functor $\rho$ to the object $\lambda(P')$.

Let us recall the definition from [45, Section 1]. A complex of $R$-modules $C'$ is said to be derived $I$-adically complete in the idealistic sense if the adjunction morphism

$$C' \longrightarrow \rho \lambda(C') = \Lambda_I(C')$$

is an isomorphism in $\mathcal{D}(R\text{-mod})$. Let us denote the full subcategory of derived $I$-adically complete complexes in the idealistic sense by

$$\mathcal{D}(R\text{-mod})_{\text{ideal}}^{I-\text{con}} \subset \mathcal{D}(R\text{-mod}).$$

Similarly, one can denote by $\mathcal{D}(R\text{-mod})_{\text{ideal}}^{I-\text{com}} \subset \mathcal{D}(R\text{-mod})$ the full subcategory of all complexes of quot-separated $I$-contramodule $R$-modules $B'$ for which the adjunction morphism $\lambda\rho(B') \longrightarrow B'$ is an isomorphism in $\mathcal{D}(R\text{-mod})_{I-\text{com}}^{\text{ideal}}$. Then, $\mathcal{D}(R\text{-mod})_{I-\text{com}}^{\text{ideal}} \subset \mathcal{D}(R\text{-mod})$ and $\mathcal{D}(R\text{-mod})_{I-\text{com}}^{\text{ideal}} \subset \mathcal{D}(R\text{-mod})$ are the maximal two full (triangulated) subcategories in the respective triangulated categories in restriction to which the functors $\rho$ and $\lambda$ are mutually inverse equivalences,

$$\mathcal{D}(R\text{-mod})_{I-\text{com}}^{ideal} \supset \mathcal{D}(R\text{-mod})_{I-\text{com}}^{ideal} \cong \mathcal{D}(R\text{-mod})_{I-\text{com}}^{ideal} \subset \mathcal{D}(R\text{-mod}).$$

Lemma 5.1. For any finitely generated ideal $I \subset R$, all derived $I$-adically complete complexes in the idealistic sense are derived $I$-adically complete in the sequential sense.

Proof. All derived $I$-adically complete complexes in the idealistic sense belong to the essential image of the functor $\rho$, which is contained in the full subcategory $\mathcal{D}_{I-\text{ctr}}(R\text{-mod}) \subset \mathcal{D}(R\text{-mod})$ in view of the factorization (4.2). To put it simply, a derived $I$-adically complete complex $C'$ in the idealistic sense is quasi-isomorphic to a complex of $I$-adically separated and complete $R$-modules $\Lambda_I(C')$, while the cohomology modules of any complex of $I$-adically separated and complete $R$-modules are (quot-separated) $I$-contramodule $R$-modules. It remains to recall that, by Lemma 2.3, $\mathcal{D}_{I-\text{ctr}}(R\text{-mod}) \subset \mathcal{D}(R\text{-mod})$ is precisely the full subcategory of derived $I$-adically complete complexes in the sequential sense. □

Proposition 5.2. Let $X$ be an infinite set and $B'$ be the complex (derived category object) $B' = R \text{Hom}_R(K^\vee_\infty(R; s), R[X]) \in \mathcal{D}(R\text{-mod}).$ Then,

(a) the complex $B'$ is derived $I$-adically complete in the sequential sense;
(b) the ideal $I \subset R$ is weakly proregular if and only if the complex $B'$ is derived $I$-adically complete in the idealistic sense.

Proof. Part (a): For any complex of $R$-modules $C'$, the complex/derived category object $R \text{Hom}_R(K^\vee_\infty(R; s), C')$ is derived $I$-adically complete in the sequential sense. In fact, $\eta = R \text{Hom}_R(K^\vee_\infty(R; s), -)$ is the reflector $\eta : \mathcal{D}(R\text{-mod}) \longrightarrow \mathcal{D}_{I-\text{ctr}}(R\text{-mod})$; see the discussion in the proof of Theorem 4.2 and in [27, Section 3]. In other words, it suffices to recall that the sequential derived $I$-adic completion functor is idempotent; see [45, Remark 2.23] and the end of Section 2.

Part (b): If the ideal $I$ is weakly proregular, then a complex of $R$-modules is derived $I$-adically complete in the idealistic sense if and only if it is derived $I$-adically complete in the sequential sense (see [45, Corollary 3.12], which is based on [45, Theorem 3.11 (i) $\Rightarrow$ (iii)] and the results of [22], or alternatively Theorem 4.2 above). Keeping part (a) in mind, this proves the “only if” assertion.

“If”: in the notation of Theorem 4.2 and its proof, we have $B' = \eta(R[X]) = \eta(R[X])$. Notice that the functor left adjoint to the functor $\xi : \mathcal{D}(R\text{-mod})_{I-\text{ctr}}^{\text{ideal}} \longrightarrow \mathcal{D}_{I-\text{ctr}}(R\text{-mod})$ is computable as the composition $\mathcal{D}_{I-\text{ctr}}(R\text{-mod}) \longrightarrow \mathcal{D}(R\text{-mod}) \longrightarrow \mathcal{D}(R\text{-mod})_{I-\text{ctr}}^{\text{ideal}}$, since the functor $\iota$ is fully faithful. It follows that the composition of functors $(\lambda \iota) \eta$ is left adjoint to the functor $\iota \xi = \rho$; hence $\lambda \iota \eta = \lambda$. 

Now assume that the complex $B^*$ is derived $I$-adically complete in the idealistic sense. Then, the adjunction morphism

$$\iota_!(R[X]) = B^* \longrightarrow \rho \lambda(B^*) = \rho \lambda \iota_!(R[X]) = \rho \lambda(R[X])$$

is an isomorphism in $D(R-\text{mod})$. In other words, this means that the canonical morphism

$$\mathbb{R} \text{Hom}_R(K^*_X(R; s), R[X]) = \iota_!(R[X]) \longrightarrow \rho \lambda(R[X]) = \Lambda_I(R[X])$$

is an isomorphism in $D(R-\text{mod})$. According to Theorem 3.6, it follows that the ideal $I \subset R$ is weakly proregular. □

**Proposition 5.3.** Let $X$ be an infinite set and $Q_X = \hat{R}[X] = \Lambda_I(R[X])$ be the $R$-module of decaying functions $X \longrightarrow \hat{R} = \Lambda_I(R)$ (in the sense of [43]). Then,

(a) the $R$-module $Q_X$ is a derived $I$-adically complete complex in the sequential sense;

(b) the ideal $I \subset R$ is weakly proregular if and only if the $R$-module $Q_X$ is a derived $I$-adically complete complex in the idealistic sense.

**Proof.** Part (a): The $R$-module $Q_X$ is a quot separated $I$-contramodule; in fact, $Q_X$ is a projective (in some sense, free) object in the abelian category $R-\text{mod}^{\text{cts}}_{I-\text{tra}}$ (see the discussion in Section 1). Hence, $Q_X \in R-\text{mod}^{\text{cts}}_{I-\text{tra}} \subset R-\text{mod}_{I-\text{tra}} \subset D_{I-\text{tra}}(R-\text{mod})$ is a derived $I$-adically complete complex in the sequential sense by Lemma 2.3.

Part (b): The “only if” assertion follows from part (a) for the same reason as in the proof of Proposition 5.2. To prove the “if,” let us consider the $R$-module $Q_X$ as an object of the derived category $D(R-\text{mod}^{\text{cts}}_{I-\text{tra}})$. Then, the same $R$-module viewed as an object of the derived category $D(R-\text{mod})$ is denoted by $\rho(Q_X)$. Assume that $\rho(Q_X)$ is a derived $I$-adically complete complex in the idealistic sense; then the adjunction morphism $\rho(Q_X) \longrightarrow \rho \lambda \rho(Q_X)$ is an isomorphism.

For any pair of adjoint functors $\rho$ and $\lambda$ and any object $Q$ in the relevant category, the composition of natural morphisms $\rho(Q) \longrightarrow \rho \lambda \rho(Q) \longrightarrow \rho(Q)$ is the identity morphism. If the morphism $\rho(Q) \longrightarrow \rho \lambda \rho(Q)$ is an isomorphism, then the morphism $\rho \lambda \rho(Q) \longrightarrow \rho(Q)$ is an isomorphism, too.

The latter morphism is obtained by applying the functor $\rho$ to the adjunction morphism $\lambda \rho(Q) \longrightarrow Q$. In the situation at hand, the functor $\rho$ is conservative: If $f : B^* \longrightarrow C^*$ is a morphism in the derived category $D(R-\text{mod}^{\text{cts}}_{I-\text{tra}})$ and $\rho(f) : \rho(B^*) \longrightarrow \rho(C^*)$ is an isomorphism in $D(R-\text{mod})$, then the morphism $f$ is an isomorphism (since the inclusion of abelian categories $R-\text{mod}^{\text{cts}}_{I-\text{tra}} \longrightarrow R-\text{mod}$ is an exact functor taking nonzero objects to nonzero objects). We conclude that the adjunction morphism $\lambda \rho(Q_X) \longrightarrow Q_X$ is an isomorphism in $D(R-\text{mod}^{\text{cts}}_{I-\text{tra}})$.

At this point, we have to recall that the object $Q_X$, by definition, depends on the choice of an infinite set $X$. We claim that if the morphism $\lambda \rho(Q_X) \longrightarrow Q_X$ is an isomorphism for one particular infinite set $X$, then so is the adjunction morphism $\lambda \rho(Q_Y) \longrightarrow Q_Y$ for every set $Y$. Indeed, passing to a direct summand, one can assume the set $X$ to be countable. The key observation is that both the inclusion functor $R-\text{mod}^{\text{cts}}_{I-\text{tra}} \longrightarrow R-\text{mod}$ and the $I$-adic completion functor $\Lambda_I : R-\text{mod}^{\text{cts}}_{I-\text{tra}} \longrightarrow R-\text{mod}$ preserve countably filtered direct limits. Using functorial (homotopy) flat resolutions together with the observation that the derived functor $\lambda$ can be computed with (homotopy) flat resolutions [22, Lemma 3.5 and Proposition 3.6], one shows that both the functors $\rho$ and $\lambda$ preserve countably filtered direct limits of complexes of modules (in an appropriate sense). It remains to observe that $Q_Y = \lim Z_{CY} Z$, where the direct limit is taken over all the countable subsets $Z$ of a given infinite set $Y$. We suggest [1] as a background reference on $\aleph_1$-filtered colimits; and leave it to the reader to fill the details of the above argument.

In the remaining last paragraph of this proof, the argument is based on the premise, justified above, that the morphism $\lambda \rho(Q_Y) \longrightarrow Q_Y$ is an isomorphism for every set $Y$. Denote the abelian categories involved by $A = R-\text{mod}$ and $B = R-\text{mod}^{\text{cts}}_{I-\text{tra}}$; and let $B \in B$ be an arbitrary object. Viewing $B$ as an object of the derived category $D(B)$, we can compute

$$\text{Hom}_{D(A)}(\rho(Q_Y), \rho(B)[k]) \cong \text{Hom}_{D(B)}(\lambda \rho(Q_Y), B[k]) \cong \text{Hom}_{D(B)}(Q_Y, B[k]) = 0$$

for all $B \in B$ and $k > 0$. The next lemma allows to conclude that the triangulated functor $\rho^B : D^b(R-\text{mod}^{\text{cts}}_{I-\text{tra}}) \longrightarrow D^b(R-\text{mod})$ is fully faithful. By Theorem 4.2, it follows that the ideal $I \subset R$ is weakly proregular. □
Lemma 5.4. Let \( \rho^a : B \rightarrow A \) be a fully faithful exact functor between two abelian categories, and let \( \rho^b : D^b(B) \rightarrow D^b(A) \) be the induced triangulated functor between the bounded derived categories. Then, the following two conditions are equivalent:

1. The functor \( \rho^b \) is fully faithful;
2. for any two objects \( C \) and \( B \in B \) and all integers \( k > 0 \), the functor \( \rho^a \) induces isomorphisms of the Ext groups
   \[
   \text{Ext}^k_{B}(C, B) \simeq \text{Ext}^k_b(\rho^a(C), \rho^b(B)).
   \]

Furthermore, if there are enough projective objects in the abelian category \( B \), then conditions (1 and 2) are equivalent to

3. for any projective object \( Q \in B \), any object \( B \in B \), and all integers \( k > 0 \), one has
   \[
   \text{Ext}^k_{B}(\rho^a(Q), \rho^b(B)) = 0.
   \]

Proof. The equivalence (1) \( \iff \) (2) holds because the triangulated category \( D^b(B) \) is generated by its full subcategory \( B \subset D(B) \). To check the implication (2) \( \implies \) (3), it suffices to take \( C = Q \). In order to prove (3) \( \implies \) (2), one can choose a projective resolution \( Q \) of the object \( C \) in the category \( B \) and compute that \( \text{Ext}^k_{B}(C, B) = H^k \text{Hom}_B(Q, B) \simeq H^k \text{Hom}_A(\rho^a(Q), \rho^b(B)) \simeq \text{Ext}^k_b(\rho^a(C), \rho^b(B)) \), where the latter isomorphism holds since \( \rho^a(Q) \) is a resolution of the object \( \rho^a(C) \) in \( A \) by objects \( \rho^a(Q_i) \), \( i \geq 0 \), satisfying \( \text{Ext}^k_b(\rho^a(Q_i), \rho^b(B)) = 0 \) for \( k > 0 \).

Corollary 5.5. The idealistic derived \( I \)-adic completion functor \( \Lambda_I : D(R-\text{mod}) \rightarrow D(R-\text{mod}) \) is idempotent if and only if the ideal \( I \subset R \) is weakly proregular.

Proof. “If”: for a weakly proregular ideal \( I \), the functor \( \Lambda_I \) is isomorphic to the sequential derived \( I \)-adic completion functor \( R \text{Hom}_R(K^\gamma(R; s, -)) : D(R-\text{mod}) \rightarrow D(R-\text{mod}) [22, Corollary 5.25], [45, Theorem 3.11 (i) \( \Rightarrow \) (iii)]. The sequential derived \( I \)-adic completion functor is always idempotent (see [45, Remark 2.23] or the discussion at the end of Section 2).

“Only if”: for any \( R \)-module \( C \), the adjunction morphism \( C \rightarrow \Lambda_I(C) \) induces the adjunction morphism \( C \rightarrow \Lambda_0(C) \) after the passage to the degree-zero cohomology modules (cf. Proposition 1.3(c)). In particular, the \( R \)-module \( C \) is a quotseparated \( I \)-contramodule if and only if the induced morphism \( C \rightarrow H^0(\Lambda_I(C)) = \Lambda_0(C) \) is an isomorphism. Hence, the object \( C \in D(R-\text{mod}) \) is derived \( I \)-adically complete in the idealistic sense if and only if \( C \) is a quotseparated \( I \)-contramodule and the \( R \)-modules \( \Lambda_I(C) = H^{-i}(\Lambda_I(C)) \) vanish for all \( i > 0 \). It follows that existence of an isomorphism \( C \simeq \Lambda_I(C) \) in the derived category \( D(R-\text{mod}) \) implies that \( C \) is derived \( I \)-adically complete in the idealistic sense.

Now the functor \( \Lambda_I \) takes the free \( R \)-module with a countable set of generators \( P_X = R[X] \in D(R-\text{mod}) \) to the \( R \)-module of decaying functions \( Q_X = R[[X]] \in D(R-\text{mod}) \). According to the previous paragraph, if the object \( \Lambda_I(Q_X) \in D(R-\text{mod}) \) is isomorphic to \( Q_X \) (i.e., some isomorphism exists), then the object \( Q_X \in D(R-\text{mod}) \) is derived \( I \)-adically complete in the idealistic sense. According to Proposition 5.3(b), it follows that the ideal \( I \subset R \) is weakly proregular.

Proposition 5.3 appears to confirm the feeling that the class of derived \( I \)-adically complete complexes in the idealistic sense, as defined in the paper [45], may be too small (when the ideal \( I \subset R \) is not weakly proregular). Is there any example of a nonzero derived \( I \)-adically complete complex in the idealistic sense, for an arbitrary finitely generated ideal \( I \neq R \) in a commutative ring \( R \)?

We would suggest the derived category \( D(R-\text{mod}^{\text{qf}}_{I-\text{ctr}a}) \) as a proper replacement of the category of derived \( I \)-adically complete complexes in the idealistic sense. At least, the category \( D(R-\text{mod}^{\text{qf}}_{I-\text{ctr}a}) \) contains the module of decaying functions \( R[[X]] = \Lambda_I(R[X]) \) as an object.

Remark 5.6. The complicated behavior of the idealistic derived completion functor \( \Lambda_I \) can be conceptualized in the following way. Suppose that we have chosen a finite set of generators \( s_1, ..., s_m \) for the ideal \( I \). Then, the \( I \)-adic completion \( \Lambda_I(M) \) of an \( R \)-module \( M \), viewed as a \( \mathbb{Z}[s_1, ..., s_m] \)-module, is determined by the underlying \( \mathbb{Z}[s_1, ..., s_m] \)-module structure on \( M \), and does not depend on the \( R \)-module structure. Still the ring \( R \) dictates what kind of resolutions (namely, homotopy flat or homotopy projective complexes of \( R \)-modules, or if one wishes, homotopy projective complexes of projective \( R \)-modules) are to be used in the construction of the derived functor \( \Lambda_I \). When the \( \mathbb{Z}[s_1, ..., s_m] \)-module structure of the ring \( R \) is complicated, so is the resulting derived functor.
One can avoid this problem by resolving the ring $R$ itself. This approach was developed in the paper [40]. Following [40], let $B^*$ be a graded commutative, nonnegatively cohomologically graded DG-ring; so in particular, $B^0$ is a commutative ring, $B^*$ is a complex of $B^0$-modules, and $H^0(B^*)$ is a quotient ring of $B^0$. Assume that $B^*$ is a homotopy flat complex of $B^0$-modules, and let $c_1, \ldots, c_m \in B^0$ be a finite sequence of elements generating a weakly preregular ideal $E \subset B^0$. Let $D(B^*-\text{mod})$ denote the derived category of (unbounded) DG-modules over $B^*$. The derived $E$-adic completion functor $\Lambda_E : D(B^*-\text{mod}) \rightarrow D(B^*-\text{mod})$ is constructed by applying the $E$-adic completion functor $\Lambda : M^* \rightarrow \lim_{n \geq 1} M^*/E^nM^*$ to homotopy projective or homotopy flat DG-modules over $B^*$.

According to [40, Proposition 2.4], under the assumptions above, the (idealistic) derived $E$-adic completion functor $\Lambda_E$ is isomorphic to the sequential derived $E$-adic completion functor $\mathbb{R}\text{Hom}_{B^0}(K\Gamma_{\infty}(B^0; c), -) : D(B^*-\text{mod}) \rightarrow D(B^*-\text{mod})$ where $c$ is a notation for the sequence $c_1, \ldots, c_m$. It follows that $\Lambda_E$ is an idempotent functor $\mathbb{R}\text{Hom}_{B^0}(K\Gamma_{\infty}(B^0; c), -) : D(B^*-\text{mod}) \rightarrow D(B^*-\text{mod})$ [40, Proposition 2.10]. The essential image of $\Lambda_E$ is the full subcategory $\mathbb{D}_{E-\text{ctr}}(B^*-\text{mod}) \subset D(B^*-\text{mod})$ of all DG-modules whose cohomology $H^0(B^*)$-modules are $E$-contramodules, where $E \subset H^0(B^*)$ is the image of the ideal $E \subset B^0$ under the surjective ring homomorphism $B^0 \rightarrow H^0(B^*)$.

Now given a ring $R$ with a finitely generated ideal $I$, one can construct a DG-ring $B^*$ with a finitely generated ideal $E \subset B^0$ satisfying the assumptions above, together with a quasi-isomorphism of DG-rings $B^* \rightarrow R$ such that $I = \bar{E}$ (this is a particular case of [40, Proposition 2.2]). Then, the derived category $D(R-\text{mod})$ is equivalent to $D(B^*-\text{mod})$, so one can view $\Lambda_E$ as a functor $D(R-\text{mod}) \rightarrow D(R-\text{mod})$. This functor is isomorphic to the sequential derived completion functor $\mathbb{R}\text{Hom}_{R}(K\Gamma_{\infty}(R, s), -) : D(R-\text{mod}) \rightarrow D(R-\text{mod})$ where $s_1, \ldots, s_m \in R$ are the images of the elements $c_1, \ldots, c_m \in B^0$ under the surjective ring homomorphism $B^0 \rightarrow H^0(B^*) = R$. The triangulated equivalence $D(B^*-\text{mod}) \simeq D(R-\text{mod})$ identifies the full subcategory $\mathbb{D}_{E-\text{ctr}}(B^*-\text{mod}) \subset D(B^*-\text{mod})$ with the full subcategory $\mathbb{D}_{I-\text{ctr}}(R-\text{mod}) \subset D(R-\text{mod})$.

6 IDEALISTICALLY DERIVED TORSION COMPLEXES

The results of this section are dual-analogous to those of the previous one. To begin with, let us recall the notation of Theorem 4.1. The inclusion of abelian categories $R-\text{mod}_{I-\text{tors}} \rightarrow R-\text{mod}$ induces a triangulated functor

$$\mu : D(R-\text{mod}_{I-\text{tors}}) \rightarrow D(R-\text{mod}),$$

which has a right adjoint functor

$$\gamma : D(R-\text{mod}) \rightarrow D(R-\text{mod}_{I-\text{tors}}).$$

Porta–Shaul–Yekutieli in [22, Section 3] and Yekutieli in [45, Section 1] consider the functor

$$\mathbb{R}\Gamma_I : D(R-\text{mod}) \rightarrow D(R-\text{mod}),$$

which is called the idealistic derived $I$-torsion functor in [45]. Following the construction of the functor $\gamma$ discussed in the proof of Theorem 4.1, the functor $\mathbb{R}\Gamma_I$ is the composition of the two adjoint functors, $\mathbb{R}\Gamma_I = \mu \circ \gamma$,

$$D(R-\text{mod}) \xrightarrow{\gamma} D(R-\text{mod}_{I-\text{tors}}) \xrightarrow{\mu} D(R-\text{mod}).$$

The point is that applying the functor $\Gamma_I$ to every term of a homotopy injective complex of $R$-modules $J^*$ produces a complex of $I$-torsion $R$-modules. Such a complex can be naturally considered as an object of the category $D(R-\text{mod}_{I-\text{tors}})$. One can also view $\Gamma_I(J^*)$ as an object of the derived category of $R$-modules; this means applying the functor $\mu$ to the object $\gamma(J^*)$.

Remark 6.1. The homotopy projective (“K-projective”), homotopy injective, and suchlike resolutions are quintessentially the technique for constructing unbounded derived functors of infinite homological dimension.

The sequential derived torsion and completion functors clearly have finite homological dimension, not exceeding the minimal number of generators of the ideal $I \subset R$. Hence, in the weakly preregular case, the homological dimension of the idealistic derived torsion and completion functors is also finite [22, Corollaries 4.28 and 5.27]. We do not know whether the
homological dimensions of the idealistic derived torsion and completion functors need to be finite for an arbitrary finitely generated ideal $I \subset R$, but we do not expect them to be. It is precisely for this reason that we (following [22] and [45]) are using homotopy projective/flat/injective resolutions in the constructions of the idealistic derived functors.

If we knew these functors to have finite homological dimension, homotopy adjusted resolutions would not be needed for their construction, as the discussion in [27, Sections 1 and 2] illustrates (see [42, Lemma 1.5] for a precise statement).

Following [45, Section 1], a complex of $R$-modules $M^*$ is said to be *derived $I$-torsion in the idealistic sense* if the adjunction morphism

$$\mathbb{R}\Gamma_I(M^*) = \mu\gamma(M^*) \longrightarrow M^*$$

is an isomorphism in $D(R{-\text{mod}})$. Let us denote the full subcategory of derived $I$-torsion complexes in the idealistic sense by

$$D(R{-\text{mod}})^{\text{ideal}}_{I{-\text{tors}}} \subset D(R{-\text{mod}}).$$

Similarly, one can denote by $D(R{-\text{mod}})^{\text{ideal}}_{I{-\text{tors}}}$ the full subcategory of all complexes of $I$-torsion $R$-modules $N^*$ for which the adjunction morphism $N^* \longrightarrow \gamma\mu(N^*)$ is an isomorphism in $D(R{-\text{mod}}_{I{-\text{tors}}})$. Then $D(R{-\text{mod}})^{\text{ideal}}_{I{-\text{tors}}} \subset D(R{-\text{mod}})$ and $D(R{-\text{mod}}_{I{-\text{tors}}})^{\text{ideal}} \subset D(R{-\text{mod}}_{I{-\text{tors}}})$ are the maximal two full (triangulated) subcategories in the respective triangulated categories in restriction to which the functors $\mu$ and $\gamma$ are mutually inverse equivalences,

$$D(R{-\text{mod}}_{I{-\text{tors}}}) \supset D(R{-\text{mod}}_{I{-\text{tors}}})^{\text{ideal}} \simeq D(R{-\text{mod}}_{I{-\text{tors}}})^{\text{ideal}} \subset D(R{-\text{mod}}).$$

**Lemma 6.2.** For any finitely generated ideal $I \subset R$, all derived $I$-torsion complexes in the idealistic sense are derived $I$-torsion in the sequential sense.

**Proof.** All derived $I$-torsion complexes in the idealistic sense belong to the essential image of the functor $\mu$, which is contained in the full subcategory $D_{I{-\text{tors}}}(R{-\text{mod}}) \subset D(R{-\text{mod}})$ in view of the factorization (4.1). Simply put, a derived $I$-torsion complex $M^*$ in the idealistic sense is quasi-isomorphic to a complex of $I$-torsion $R$-modules $\mathbb{R}\Gamma_I(M^*)$, whose cohomology modules are obviously $I$-torsion, too. It remains to recall that, by Lemma 2.2, $D_{I{-\text{tors}}}(R{-\text{mod}}) \subset D(R{-\text{mod}})$ is precisely the full subcategory of derived $I$-torsion complexes in the sequential sense. □

**Proposition 6.3.** Given an injective $R$-module $J$, consider the complex of $R$-modules $N^* = K^\vee_{\infty}(R; s) \otimes_R J$. Then,

(a) the complex $N^*$ is derived $I$-torsion in the sequential sense;

(b) the ideal $I \subset R$ is weakly proregular if and only if the complex $N^*$ is derived $I$-torsion in the idealistic sense for every injective $R$-module $J$.

**Proof.** Part (a): For any complex of $R$-modules $M^*$, the complex $K^\vee_{\infty}(R; s) \otimes_R M^*$ is derived $I$-torsion in the sequential sense, because the sequential derived $I$-torsion functor $K^\vee_{\infty}(R; s) \otimes_R$ is idempotent (see formula (2.4) in the end of Section 2).

If the ideal $I$ is weakly proregular, then a complex of $R$-modules is derived $I$-torsion in the idealistic sense if and only if it is derived $I$-torsion in the sequential sense (see [27, Corollary 4.26] and [45, Section 3], or Theorem 4.1 above). Hence, the “only if” implication in part (b) follows from part (a).

Part (b), “if”: In the notation of Theorem 4.1 and its proof, we have $N^* = \theta(J) = \nu\theta(J)$. Notice that the right adjoint functor to the functor $\chi : D(R{-\text{mod}}_{I{-\text{tors}}}) \longrightarrow D_{I{-\text{tors}}}(R{-\text{mod}})$ is computable as the composition $D_{I{-\text{tors}}}(R{-\text{mod}}) \overset{\nu}{\longrightarrow} D(R{-\text{mod}}) \overset{\gamma}{\longrightarrow} D(R{-\text{mod}}_{I{-\text{tors}}})$, since the functor $\nu$ is fully faithful. It follows that the composition of functors $(\gamma\nu)\theta$ is right adjoint to the functor $\nu\chi = \mu$; hence $\gamma\nu\theta = \gamma$.

Now assume that the complex $N^*$ is derived $I$-torsion in the idealistic sense for every injective $R$-module $J$. Then, the adjunction morphism

$$\mu\gamma(J) = \mu\gamma\theta(J) = \mu\gamma(N^*) \longrightarrow N^* = \nu\theta(J)$$
is an isomorphism in \(D(R - \text{mod})\). In other words, this means that the canonical morphism

\[
\Gamma_I(J) = \mu \gamma(J) \longrightarrow \nu \delta(J) = K^\wedge_r(R; s) \otimes_R J
\]

is an isomorphism in \(D(R - \text{mod})\). According to Corollary 3.4, it follows that the ideal \(I \subset R\) is weakly proregular. \qed

The following result, which is dual-analogous to Proposition 5.3, was obtained by Vyas and Yekutieli in [42, Theorem 0.3]. We provide a sketch of proof using our methods.

**Proposition 6.4.** Given an injective \(R\)-module \(J\), consider the \(R\)-module \(E = \Gamma_I(J)\). Then,

(a) the \(R\)-module \(E\) is a derived \(I\)-torsion complex in the sequential sense;

(b) the ideal \(I \subset R\) is weakly proregular if and only if the \(R\)-module \(E\) is a derived \(I\)-torsion complex in the idealistic sense for every injective \(R\)-module \(J\).

**Sketch of proof.** Part (a): The \(R\)-module \(E\) is \(I\)-torsion by definition; in fact, \(E\) is an injective object in the abelian category \(R - \text{mod}_{I\text{-tors}}\). Hence, \(E \in R - \text{mod}_{I\text{-tors}} \subset D_{I\text{-tors}}(R - \text{mod})\) is a derived \(I\)-torsion complex in the sequential sense by Lemma 2.2. The “only if” assertion in part (b) follows from part (a).

Part (b), “if”: Let us consider the \(R\)-module \(E\) as an object of the derived category \(D(R - \text{mod}_{I\text{-tors}})\). Then, the same \(R\)-module viewed as an object of the derived category \(D(R - \text{mod})\) is denoted by \(\mu(E)\). Assume that \(\mu(E)\) is a derived \(I\)-torsion complex in the idealistic sense, for every injective \(R\)-module \(J\). Then, the adjunction morphism \(\mu \gamma \mu(E) \longrightarrow \mu(E)\) is an isomorphism.

It follows that the morphism \(\mu(E) \longrightarrow \mu \gamma \mu(E)\) obtained by applying \(\mu\) to the adjunction morphism \(E \longrightarrow \gamma \mu(E)\) is an isomorphism, too. Since the triangulated functor \(\mu : D(I\text{-mod}_{I\text{-tors}}) \longrightarrow D(R - \text{mod})\) is conservative (taking complexes with nonzero cohomology to complexes with nonzero cohomology), we can conclude that the adjunction morphism \(E \longrightarrow \gamma \mu(E)\) is an isomorphism in \(D(R - \text{mod}_{I\text{-tors}})\).

Denote the abelian categories involved by \(A = R - \text{mod}\) and \(T = R - \text{mod}_{I\text{-tors}}\); and let \(T \in T\) be an arbitrary object. Viewing \(T\) as an object of the derived category \(D(T)\), we can compute

\[
\text{Hom}_{D(A)}(\mu(T), \mu(E)[k]) \simeq \text{Hom}_{D(T)}(T, \gamma \mu(E)[k]) \simeq \text{Hom}_{D(T)}(T, E[k]) = 0
\]

for all \(T \in T\) and \(k > 0\).

There are enough injectives in the abelian category \(T = R - \text{mod}_{I\text{-tors}}\), and any injective object \(L\) in \(T\) is a direct summand of an object \(E = \Gamma_I(J)\) for some injective \(R\)-module \(J\). Hence, we have \(\text{Ext}^k_T(T, L) = 0\) for all objects \(T \in T\), all injective objects \(L \in T\), and all \(k > 0\). By the dual version of Lemma 5.4, we can conclude that the triangulated functor \(\mu^\wedge : D^b(R - \text{mod}_{I\text{-tors}}) \longrightarrow D^b(R - \text{mod})\) is fully faithful. According to Theorem 4.1, it follows that the ideal \(I \subset R\) is weakly proregular. \qed

**Corollary 6.5.** The idealistic derived \(I\)-torsion functor \(\mathbb{R}\Gamma_I : D(R - \text{mod}) \longrightarrow D(R - \text{mod})\) is idempotent if and only if the ideal \(I \subset R\) is weakly proregular.

**Proof.** Dual-analogous to Corollary 5.5. \qed

Once again, the feeling is that the class of derived \(I\)-torsion complexes in the idealistic sense, as defined in the paper [45], may be too small. We ask the same question: Is there any example of a nonzero derived \(I\)-torsion complex in the idealistic sense, for an arbitrary finitely generated ideal \(I \neq R\) in a commutative ring \(R\)?

We would suggest the derived category \(D(R - \text{mod}_{I\text{-tors}})\) as a proper replacement of the category of derived \(I\)-torsion complexes in the idealistic sense. At least, the category \(D(R - \text{mod}_{I\text{-tors}})\) contains all \(I\)-torsion \(R\)-modules as objects.

### 7 DIGRESSION: ADIC FLATNESS AND WEAK PROREGULARITY

In this section, we use the occasion to provide the precise formulation and a proof of a result of the present author mentioned by Yekutieli in [44, Remark 4.12]. This is closely related to the main results of this paper on the technical level.
Following [44, Definition 4.2], we say that an $R$-module $F$ is $I$-adically flat if $\text{Tor}_k^R(N, F) = 0$ for all $I$-torsion $R$-modules $N$ and all $k > 0$. (In the terminology of [36, Definition 2.6.1], such modules are called “relatively-$I$-flat.”)

In the same spirit, the $R$-modules satisfying the equivalent conditions of the next proposition could be called “$I$-adically projective.”

**Proposition 7.1.** Let $F$ be an $R$-module. Then, the following three conditions are equivalent:

1. The $R$-module $F$ is $I$-adically flat and the $R/I$-module $F/IF$ is projective;
2. $\text{Ext}^k_R(F, D) = 0$ for all $R/I$-modules $D$ and all $k > 0$;
3. $\text{Ext}^k_R(F, C) = 0$ for all $I$-contramodule $R$-modules $C$ and all $k > 0$.

**Proof.** $(1) \iff (2)$ Let us first prove that $(2)$ implies the $I$-adic flatness of $F$. First of all, since the functor $\text{Tor}$ preserves direct limits, it suffices to check that $\text{Tor}^k_F(N, F) = 0$ for all $k > 0$ and all $R/I^n$-modules $N$, where $n$ ranges over the positive integers. Next, one easily reduces to the case $n = 1$; so we can assume that $N$ is an $R/I$-module. It remains to consider the character module $D = N^+ = \text{Hom}_Z(N, \mathbb{Q}/\mathbb{Z})$ and use the natural isomorphism $\text{Tor}_k^R(N, F)^+ \cong \text{Ext}_k^R(F, N^+)$. Now we can assume that the $R$-module $F$ is $I$-adically flat; in particular, $\text{Tor}_i^R(R/I, F) = 0$ for all $i > 0$. Then, for any $R/I$-module $D$ and all $k \geq 0$, there is a natural isomorphism of Ext modules:

$$\text{Ext}^k_R(F, D) \cong \text{Ext}^k_{R/I}(F/IF, D).$$

Hence, condition $(2)$ holds if and only if the $R/I$-module $F/IF$ is projective.

$(2) \implies (3)$ is an “obtainability” argument going back to [28, proof of Theorem 9.5] and subsequently utilized in the paper [32]. One proves that all the $I$-contramodule $R$-modules can be obtained from $R/I$-modules, in the relevant sense. Essentially, the class of all $R$-modules $C$ satisfying $(3)$ for a fixed $R$-module $F$ is closed under certain operations, which are listed in [32, Lemma 3.2 or Definition 3.3]. One shows that all $I$-contramodule $R$-modules can be “obtained” from $R/I$-modules using these operations; this is the assertion of [32, Lemma 8.2].

To spell out a specific argument, one can start by observing that all $I$-contramodule $R$-modules are obtainable as extensions of quotiented $I$-contramodule $R$-modules (by Proposition 1.6). Furthermore, any quotiented $I$-contramodule is obtainable as the cokernel of an injective morphism of separated $I$-contramodules. The latter are the same thing as $I$-adically separated and complete $I$-modules.

**Theorem 7.2.** Let $X$ be an infinite set and $Q_X = \tilde{\mathbb{R}}([X]) = \Lambda_I(R(X))$ be the $R$-module of decaying functions $X \to \tilde{\mathbb{R}} = \Lambda_I(R)$. Then, the $R$-module $Q_X$ is $I$-adically flat if and only if the ideal $I \subset R$ is weakly preresgular.

**Proof.** The “if” assertion is a particular case of [44, Theorem 1.6(1) or Theorem 6.9]. One can also obtain it by reversing the arguments in the proof of the “only if” assertion that follows below.

The argument is somewhat similar to the proof of Proposition 5.3. Assume that, for one particular infinite set $X$, the $R$-module $Q_X$ is $I$-adically flat. Passing to a direct summand, we can assume the set $X$ to be countable. For every infinite set $Y$, the $R$-module $Q_Y$ is a direct limit of $R$-modules isomorphic to $Q_X$. Since the class of $I$-adically flat modules is closed under direct limits, it then follows that the $R$-module $Q_Y$ is $I$-adically flat as well. Passing to the direct summands again, we see that all the projective objects of the category of quotseparated $I$-contramodules $B = R - \text{mod}^{\text{qs}}_{I-\text{ctra}}$ are $I$-adically flat $R$-modules.

Furthermore, the $R/I$-module $Q_Y/IQ_Y \cong (R/I)[Y]$ is obviously free, hence projective, for any module of decaying functions $Q_Y$. Therefore, the $R/I$-module $Q/IQ$ is projective for any projective object $Q$ of the category $R - \text{mod}^{\text{qs}}_{I-\text{ctra}}$.

By Proposition 7.1 $(1) \Rightarrow (3)$, we can conclude that $\text{Ext}^k_Q(Q, B) = 0$ for all projective quotseparated $I$-contramodules $Q$, all (quotseparated) $I$-contramodule $R$-modules $B$, and all $k > 0$. According to Lemma 5.4, it then follows that the triangulated functor $\varrho^B : D^b(R - \text{mod}^{\text{qs}}_{I-\text{ctra}}) \to D^b(R - \text{mod})$ is fully faithful. Consequently, the ideal $I \subset R$ is weakly preresgular by Theorem 4.2.

**Remark 7.3.** The notion of $I$-adic projectivity provided by the equivalent conditions of Proposition 7.1 is modeled after the definition of $I$-adic flatness in [44, Definition 4.2]. It is designed to be relevant for the purposes of Theorem 7.2.
A quite different property was discussed under the name of “adic projectivity” in [43, Definition 3.16] and [23, Definition 1.4]. What would be called “I-adically projective R-modules” in the terminology of [23, 43] are called projective quotseparated I-contramodule R-modules in this paper. These are the projective objects of the abelian category $R\text{-mod}_{I-\text{ctr}}^\delta$.

The related flatness notion is that of a flat quotseparated I-contramodule R-module [32, Sections 5.3 and 5.6], [31, Sections 5–7], [25, Section D.1]. A quotseparated I-contramodule R-module $F$ is said to be flat if the $R/I^n$-module $F/I^nF$ is flat for every $n \geq 1$. Unless the ring $R$ is Noetherian (cf. [28, Corollary 10.3], [44, Theorem 1.6 (2)]), a flat quotseparated I-contramodule R-module need not be a flat R-module [44, Theorem 7.2].

Any flat quotseparated I-contramodule R-module is I-adically separated [25, Corollary D.1.7], [31, Corollary 6.15] (these results are applicable in view of Proposition 1.5 above). For any short exact sequence of quotseparated I-contramodule R-modules $0 \rightarrow B \rightarrow C \rightarrow F \rightarrow 0$ with a flat quotseparated I-contramodule R-module $F$, and for any I-torsion R-module $N$, the tensor product sequence $0 \rightarrow N \otimes_R B \rightarrow N \otimes_R C \rightarrow N \otimes_R F \rightarrow 0$ is exact [31, Lemma 6.10].

The projective quotseparated I-contramodule R-modules can be characterized as follows [25, Corollary D.1.10], [24, Lemma B.10.2] (see [23, Theorem 1.10] for the Noetherian case). For any quotseparated I-contramodule R-module $F$, the following conditions are equivalent:

1. $F$ is a projective object in $R\text{-mod}_{I-\text{ctr}}^\delta$;
2. the $R/I^n$-module $F/I^nF$ is projective for every $n \geq 1$;
3. $F$ is a flat quotseparated I-contramodule R-module and the $R/I$-module $F/IF$ is projective.

Any quotseparated I-contramodule R-module that is I-adically flat in the sense of [44, Definition 4.2], is flat in the sense of the definition above in this remark; and any quotseparated I-contramodule R-module that is I-adically projective in the sense of the equivalent conditions in Proposition 7.1, is a projective object in $R\text{-mod}_{I-\text{ctr}}^\delta$. But the converse implications only hold for a weakly proregular ideal $I \subset R$ [44, Theorem 1.6 (1)]. Otherwise, a flat quotseparated I-contramodule R-module need not be I-adically flat, and a projective object of $R\text{-mod}_{I-\text{ctr}}^\delta$ need not satisfy the equivalent conditions of Proposition 7.1, as Theorem 7.2 shows.

Yet another relevant projectivity notion is that of a projective I-contramodule R-module, that is, a projective object in $R\text{-mod}_{I-\text{ctr}}$. The classes of projective objects in $R\text{-mod}_{I-\text{ctr}}$ and in $R\text{-mod}_{I-\text{ctr}}^\delta$ coincide if and only if the two abelian categories coincide (see Lemmas 1.4 and 1.7).

Here are the dual-analogous definition and assertion. We say that an R-module $E$ is I-adically injective if $\mathrm{Ext}^k_R(N, E) = 0$ for all I-torsion R-modules $N$ and all $k > 0$. In view of the Eklof lemma [7, Lemma 1], [32, Lemma 3.6(a)], it suffices to check this condition for finitely generated I-torsion R-modules $N$. (In the terminology of [36, Definition 2.6.1], such R-modules $E$ would be called “relatively-I-injective.”)

The following proposition is essentially equivalent to [42, Theorem 0.3] (cf. Proposition 6.4 above); see also [37, Proposition 2.6 and Corollary 3.4].

**Proposition 7.4.** The ideal $I \subset R$ is weakly proregular if and only if the R-module $E = \Gamma_I(J)$ is I-adically injective for every injective R-module $J$.

**Proof.** Put $A = R\text{-mod}$ and $T = R\text{-mod}_{I-\text{t}}$. If the ideal $I \subset R$ is weakly proregular, then, by [27, Theorem 1.3], the map $\mathrm{Ext}^k_A(N, E) \rightarrow \mathrm{Ext}^k_A(N, E)$ induced by the exact inclusion of abelian categories $R\text{-mod}_{I-\text{t}} \rightarrow R\text{-mod}$ is an isomorphism for all objects $N, E \in R\text{-mod}_{I-\text{t}}$ and all $k \geq 0$. In the situation at hand with $E = \Gamma_I(J)$, the object $E$ is injective in $R\text{-mod}_{I-\text{t}}$; so $\mathrm{Ext}^k_A(N, E) = \mathrm{Ext}^k_A(N, E) = 0$ for $k > 0$.

Conversely, if $\mathrm{Ext}^k_A(N, E) = 0$ for all $N \in T, E = \Gamma_I(J)$, and $k > 0$, where $J$ ranges over all the injective R-modules, then the dual version of Lemma 5.4 tells us that the functor $\mu_I^k : \mathcal{D}(R\text{-mod}_{I-\text{t}}) \rightarrow \mathcal{D}(R\text{-mod})$ is fully faithful. Applying Theorem 4.1, we conclude that the ideal $I \subset R$ is weakly proregular.

**8 | CONCLUSION**

For any finitely generated ideal $I$ in a commutative ring $R$, $R\text{-mod}_{I-\text{ctr}}^\delta \subset R\text{-mod}_{I-\text{ctr}} \subset R\text{-mod}$ are two abelian full subcategories in the category of R-modules $R\text{-mod}$. Trying to follow Yekutieli’s suggested terminology [45], one could
say that the category of \( I \)-contramodule \( R \)-modules \( \mathcal{D}_{I-\text{contra}}(R-\text{mod}) \) is “the category of derived \( I \)-adically complete modules in the sequential sense,” while the category of quotseparated \( I \)-contramodule \( R \)-modules \( \mathcal{D}_{I-\text{contra}}^{\text{des}}(R-\text{mod}) \) is “the category of derived \( I \)-adically complete modules in the idealistic sense.”

What Yekutieli \([45]\) calls “the category of derived \( I \)-adically complete complexes in the sequential sense” is, in our language, the full subcategory \( \mathcal{D}_{I-\text{contra}}(R-\text{mod}) \subset \mathcal{D}(R-\text{mod}) \) of complexes of \( R \)-modules with \( I \)-contramodule cohomology modules.

What Yekutieli \([45]\) calls “the category of derived \( I \)-adically complete complexes in the idealistic sense” is likely to be too small. We would suggest to modify the definition by considering the derived category \( \mathcal{D}(R-\text{mod})^{\text{des}}_{I-\text{contra}} \) as the proper version/replacement of the category of derived \( I \)-adically complete complexes in the idealistic sense. Then, there is the triangulated functor \( \rho : \mathcal{D}(R-\text{mod})^{\text{des}}_{I-\text{contra}} \to \mathcal{D}(R-\text{mod}) \), but it is only fully faithful when the ideal \( I \) is weakly proregular.

As a middle ground between the above two versions of “the category of derived \( I \)-adically complete complexes,” one can also consider the derived category \( \mathcal{D}(R-\text{mod})^{\text{des}}_{I-\text{contra}} \). Then, there is the triangulated functor \( \rho : \mathcal{D}(R-\text{mod})^{\text{des}}_{I-\text{contra}} \to \mathcal{D}(R-\text{mod}) \), which is also, generally speaking, not fully faithful.

To summarize the discussion of derived \( I \)-adically complete complexes and derived \( I \)-adic completions, there is a diagram of triangulated functors

\[
\begin{array}{ccc}
\mathcal{D}(R-\text{mod})^{\text{des}}_{I-\text{contra}} & \xrightarrow{\beta} & \mathcal{D}(R-\text{mod})_{I-\text{contra}} \\
\quad & \quad & \downarrow \rho \\
\mathcal{D}(R-\text{mod}) & \xrightarrow{\delta} & \mathcal{D}_{I-\text{contra}}(R-\text{mod}) \\
\end{array}
\]

Here, straight arrows form a commutative diagram. The curvilinear arrows show left adjoint functors. Only the functor shown by the rightmost diagonal arrow (with a tail) is fully faithful in general. The diagonal arrow with two heads on the right-hand side of the diagram is a Verdier quotient functor.

The sequential derived \( I \)-adic completion functor is the composition of the two adjoint functors on the right-hand side,

\[
\iota \eta = \mathbb{R} \text{Hom}_R(K^\bullet_\infty(R; s), -) : \mathcal{D}(R-\text{mod}) \to \mathcal{D}(R-\text{mod}).
\]

The idealistic derived \( I \)-adic completion functor is the composition of the two adjoint functors on the left-hand side,

\[
\rho \iota \lambda = \mathbb{L} \Lambda_I : \mathcal{D}(R-\text{mod}) \to \mathcal{D}(R-\text{mod}).
\]

The leftmost horizontal functor \( \beta \) in (8.1) is fully faithful if and only if it is a triangulated equivalence (and if and only if \( R-\text{mod}^{\text{des}}_{I-\text{contra}} = R-\text{mod}_{I-\text{contra}} \)). Similarly, the rightmost horizontal functor \( \zeta \) is fully faithful if and only if it is a triangulated equivalence. The composition of the two horizontal functors (denoted by \( \xi \) in Section 4) is fully faithful if and only if it is a triangulated equivalence.

If the ideal \( I \) is weakly proregular, then both horizontal functors in (8.1) are triangulated equivalences. If the ideal \( I \) is not weakly proregular, then the composition \( \xi = \zeta \circ \beta \) of the two horizontal functors is not an equivalence (but one of them can be). See Remark 3.8 for further discussion.

The situation with derived torsion complexes is similar but simpler. There is only one abelian category of \( I \)-torsion \( R \)-modules, \( R-\text{mod}^{\text{des}}_{I-\text{tors}} \subset R-\text{mod} \).

What is called “the category of derived \( I \)-torsion complexes in the sequential sense” in \([45]\) is, in our language, the full subcategory \( \mathcal{D}_{I-\text{tors}}(R-\text{mod}) \subset \mathcal{D}(R-\text{mod}) \) of complexes of \( R \)-modules with \( I \)-torsion cohomology modules.

What is called “the category of derived \( I \)-torsion complexes in the idealistic sense” in \([45]\) is likely to be too small. We would suggest to modify the definition by considering the derived category \( \mathcal{D}(R-\text{mod})^{\text{des}}_{I-\text{tors}} \) as the proper version/replacement of the category of derived \( I \)-torsion complexes in the idealistic sense. Then, there is the triangulated functor \( \mu : \mathcal{D}(R-\text{mod})^{\text{des}}_{I-\text{tors}} \to \mathcal{D}(R-\text{mod}) \), but it is only fully faithful when the ideal \( I \) is weakly proregular.

To summarize the discussion of derived \( I \)-torsion complexes and derived \( I \)-torsion functors, there is a diagram of triangulated functors:

\[
\begin{array}{ccc}
\mathcal{D}(R-\text{mod})^{\text{des}}_{I-\text{tors}} & \xrightarrow{\beta} & \mathcal{D}(R-\text{mod})_{I-\text{tors}} \\
\quad & \quad & \downarrow \rho \\
\mathcal{D}(R-\text{mod}) & \xrightarrow{\delta} & \mathcal{D}_{I-\text{tors}}(R-\text{mod}) \\
\end{array}
\]
Here, straight arrows form a commutative triangular diagram. The curvilinear arrows show right adjoint functors. The diagonal arrow with a tail on the right-hand side shows a fully faithful functor. The rightmost diagonal arrow with two heads shows a Verdier quotient functor.

The sequential derived $I$-torsion functor is the composition of the two adjoint functors on the right-hand side,

$$\mu \circ \gamma = \mathbb{R}\Gamma_I : \mathcal{D}(R-\text{mod}) \longrightarrow \mathcal{D}(R-\text{mod}).$$

The idealistic derived $I$-torsion functor is the composition of the two adjoint functors on the left-hand side,

$$\nu \circ \theta = K^\vee_{\infty}(R; \mathfrak{s}) \otimes_R - : \mathcal{D}(R-\text{mod}) \longrightarrow \mathcal{D}(R-\text{mod}).$$

The horizontal functor $\chi$ in (8.2) is fully faithful if and only if it is a triangulated equivalence, and if and only if the ideal $I \subset R$ is weakly proregular.

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**ORCID**

Leonid Positselski https://orcid.org/0000-0001-8836-3911

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