Fuglede’s conjecture holds on $\mathbb{Z}_p^2 \times \mathbb{Z}_q$

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Abstract

The study of Fuglede’s conjecture on the direct product of elementary abelian groups was initiated by Iosevich et al. For the product of two elementary abelian groups the conjecture holds. For $\mathbb{Z}_3^2$ the problem is still open if $p \geq 11$. In connection we prove that Fuglede’s conjecture holds on $\mathbb{Z}_p^2 \times \mathbb{Z}_q$ by developing a method based on ideas from discrete geometry.

1 Introduction

Let $\Omega$ be a bounded measurable set with positive Lebesgue measure in $\mathbb{R}^n$. We say that $\Omega$ is a tile if there exists $T \subset \mathbb{R}^n$ such that for almost every element $x$ of $\mathbb{R}^n$ we can uniquely write $x$ as the sum of an element of $\Omega$ and $T$. In this case $T$ is called the tiling complement of $\Omega$. We say that $\Omega$ is spectral if there is a base of $L^2(\Omega)$ consisting only of exponential functions, $\{f(x) = e^{2\pi i <x, \lambda>} | \lambda \in \Lambda\}$. In this case $\Lambda$ is called a spectrum for $\Omega$. The classical Fuglede conjecture [9] states that the spectral sets are the tiles in $\mathbb{R}^n$.

The conjecture was motivated by the result of Fuglede [9] that if $\Omega$ is a tile with a tiling complement, which is a lattice, then $\Omega$ is spectral. After some valuable positive results, Tao [17] disproved the conjecture by showing a spectral set which is not a tile in $\mathbb{R}^n$ for $n \geq 5$. This was improved in two ways. Firstly, there were found some non-tiling spectral sets in $\mathbb{R}^n$ for $n \geq 4$ in [15] and later $n \geq 3$ in [12]. Secondly, there were shown non-spectral tiles in $\mathbb{R}^n$ for $n \geq 3$ [5] (for further references see [6, 13]). All of these counterexamples are based on Fuglede’s conjecture on finite Abelian groups. We remarkably note that both directions of the conjecture are still open in $\mathbb{R}$ and $\mathbb{R}^2$.

Let $G$ be a finite Abelian group and $\hat{G}$ the set of irreducible representations of $G$, which can be considered as a group and it is isomorphic to $G$. The elements of $\hat{G}$ are indexed by the elements of $G$. Then $S \subset G$ is spectral if and only if there exists a $\Lambda \in G$ such that $(\chi_l(x))_{l \in \Lambda}$ is an orthogonal base of complex valued functions defined on $S$. It is worth to note that if $\Lambda$ is a spectrum for $S$, then $S$ is a spectrum for $\Lambda$, and we say that $(S, \Lambda)$ is a spectral pair. It simply follows that $|S| = |\Lambda|$. For a finite group $G$ and a subset $S$ of $G$ we say that $S$ is a tile if there is a $T \subset G$ such that $S + T = G$ and $|S| \cdot |T| = |G|$. The discrete version of the problem is the following: which finite abelian groups satisfy that the spectral sets and the tiles coincide.

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The case of finite cyclic groups is particularly interesting since Dutkay and Lai [3] showed that the tile-spectral direction of Fuglede’s conjecture on $\mathbb{R}$ holds if and only if the discrete version holds for every finite cyclic group. Recently, it was showed by Lai and Wang [14] that a similar statement holds for the spectral-tile direction as well. Hence in order to prove the one dimensional Fuglede’s conjecture we have to verify it for every finite cyclic group.

Tao’s [17] example for a spectral set which does not tile comes from a non-tiling spectral set in $\mathbb{Z}_2^2$ so it makes sense to investigate elementary abelian $p$-groups. It was proved in [10] that for $\mathbb{Z}_p^2$ Fuglede’s conjecture holds for every prime $p.$

On the other hand it was shown in [1] that the spectral tile direction of the conjecture does not hold for $\mathbb{Z}_p^2$ if $p$ is an odd prime and for $\mathbb{Z}_4^d$ if $q \equiv 3 \pmod{4}$ is a prime. This was strengthened by Ferguson and Sothanaphan by exhibiting a non-spectral tile for $\mathbb{Z}_p^d,$ see [7]. If $p = 2,$ then the situation is slightly different. It was shown that Fuglede’s conjecture fails for $\mathbb{Z}_2^d$ if $d \geq 10$ [7], and holds for if $d \leq 6$ [7, 8]. For $7 \leq d \leq 9$ the answer is not known.

The question whether Fuglede’s conjecture holds for $\mathbb{Z}_3^d$ is still widely open, although partial results have been obtained recently for $p \leq 5$ in [2] and for $p \leq 7$ in [1]. Note that the tile-spectral conjecture holds for $\mathbb{Z}_p^3,$ see [1].

R. Shi investigated the mixed direct products. He verified the conjecture for $\mathbb{Z}_p^2 \times \mathbb{Z}_q,$ recently. Reaching closer to decide the validity of Fuglede’s conjecture for $\mathbb{Z}_p^3,$ in our paper we prove the following.

**Theorem 1.1.** Fuglede’s conjecture holds on $\mathbb{Z}_p^2 \times \mathbb{Z}_q.$

2 Representations of $\mathbb{Z}_p^2 \times \mathbb{Z}_q$

Let $G = \mathbb{Z}_p^2 \times \mathbb{Z}_q.$ We can consider the elements of $G$ as pairs of vectors $(u, v),$ where $u \in \mathbb{Z}_p^2$ and $v \in \mathbb{Z}_q.$ The representations of $G$ are indexed by the elements of $G$ so

$$\chi_{(a,b)}(u,v) = e^{2\pi i \langle u,a \rangle} e^{2\pi i v \cdot b},$$

where $\langle u,a \rangle$ denotes the scalar product of $u$ and $a.$

Denote the order of $g \in G$ by $o(g).$ We can distinguish 4 different types of representations according to their order. Namely, $o(\chi) \in \{1, p, q, pq\}.$ The trivial representation is the only one of order 1. Now we describe the kernel of the nontrivial representations.

**Equidistributed property:**

Let $M = \{m_i = (u_i, v_i) \mid i \in I\}$ be a multiset (i.e, some of the pairs $(u_i, v_i)$ can coincide) and $\chi_{(a,0)} \in G$ of order $p.$ Then

$$\chi_{(a,0)}(M) := \sum_{i \in I} \chi_{(a,0)}(m_i) = \sum_{i \in I} e^{2\pi i \langle u_i, a \rangle} = 0.$$ 

Since the minimal polynomial of $e^{2\pi i / p}$ over $\mathbb{Q}$ is $\sum_{j=0}^{p-1} x^j,$ we have that $|M_k| = |\{m = (u, v) \in M \mid \langle u, a \rangle \equiv k \pmod{p}\}| = \frac{|M|}{p}$ for each $k = 0, \ldots, p - 1.$ In this case we say that $M$ is equidistributed on the cosets of a subgroup of index $p,$ where this subgroup is determined by equation $\langle x, a \rangle = 0.$

A similar argument shows that $\chi_{(0,b)}(M) = 0,$ where $b \neq 0$ if and only if $M$ is equidistributed on the cosets of $\mathbb{Z}_p^2,$ which is a unique subgroup of $\mathbb{Z}_p^2 \times \mathbb{Z}_q.$ Hence $|\{m = (u, v) \in M \mid v = k\}| = \frac{|M|}{q}$ for every $k = 0, \ldots, q - 1.$
Corollary 2.1. If $\chi(M) = 0$, where $o(\chi) = p$ or $o(\chi) = q$, then $p \mid |M|$ or $q \mid |M|$, respectively.

Finally, if $o(\chi_{(a,b)}) = pq$, then we project $M$ to the subgroup isomorphic to $\mathbb{Z}_{pq}$ generated by $(a,0)$ and $(0,b)$ and obtain a multiset $M_{(a,b)}$. More precisely, $M_{(a,b)}((u,v)) = |\{(x,v) \in M \mid (x-u,a) = 0\}|$, where $u$ is a multiple of $a$ (and clearly $v$ is a multiple of $b$). The multiset $M_{(a,b)}$ is a nonnegative integer valued function on $\mathbb{Z}_{pq}$. Proposition 3.8 (b) in [11] claims that $\chi_{(a,b)}(M) = 0$ if and only if $M_{a,b}$ is the sum of $\mathbb{Z}_p$-cosets and $\mathbb{Z}_q$-cosets.

3 Tile-Spectral

Let $A \bigoplus B = G$ be a tiling of $G$. Without loss of generality we assume $0 \in A \cap B$. We denote by $1_H$ the characteristic function of the set $H$ and by $\hat{f}$ the Fourier transform of $f$. Then $\hat{1}_A \cdot \hat{1}_B = |G| \cdot \delta_1$, where $\delta$ denotes the Dirac delta and $1$ denotes the trivial representation of $G$. The following is well-known.

Lemma 3.1. Let $C$ be a multiset on an abelian group $G$. Then $\chi(C) = 0$ for every $1 \neq \chi \in \hat{G}$ if and only if $C$ is constant on $G$.

We may assume $A \neq G$, $B \neq G$. Hence there are $1 \neq \chi, \chi' \in \hat{G}$ with $\chi(A) \neq 0$ and $\chi'(B) \neq 0$. Then $\chi(B) = 0$ and $\chi'(A) = 0$.

Lemma 3.2. Let $A \subset G$ such that one of its projection, which we denote by $P$, is a subgroup of $G$. Suppose there exists a subgroup $B \leq G$ such that $P \bigoplus B = G$. Then $A$ is spectral and its spectrum is $P$.

Proof. Since $|\hat{P}| = |A|$ it is enough to prove that the elements of $\hat{P}$ are orthogonal on $A$. We write the elements of $G$ as $(a,b)$, where $a \in P$ and $b \in B$ since $P \bigoplus B = G$. Indeed,

$$\sum_{(a,b) \in A} \chi_{(a,0)}(a,b)\hat{\chi}_{(u',0)}(a,b) = \sum_{(a,0) \in P} \chi_{(u'-0)}(a,0) = 0,$$

where $a, u \neq u' \in P$ and $b \in B$. The last equality follows from Lemma 3.1. $\blacksquare$

Note that $\mathbb{Z}_p^2 \times \mathbb{Z}_q$ is a complemented group. Proof of tile-spectral direction of Theorem [11].

Case 1: $|A| = q, \ |B| = p^2$

It is clear from Corollary 2.1 that $\chi(A) \neq 0$ if $o(\chi) = p$. Thus $\chi(A) = 0$ for some irreducible representation $\chi$ of order $q$ or $pq$. If $o(\chi) = q$, then we are done since $\mathbb{Z}_q \leq \hat{G}$ is a spectrum for $A$.

If $o(\chi) = pq$, then we project $A$ to the subgroup of order $pq$ determined by $\chi$. The projection of $A$ is a multiset that is the sum of $\mathbb{Z}_p$ and $\mathbb{Z}_q$-cosets (see Proposition 3.8 (b) in [11]). Hence $|A| = kp + lq$ for some $k, l \in \mathbb{N}$. Then we have $k = 0, l = 1$ since $|A| = q$. Thus the projection of $A$ is a coset of $\mathbb{Z}_q$. We may assume $0 \in A$ so the projection is a subgroup isomorphic to $\mathbb{Z}_q$. By Lemma 3.2, $A$ is spectral with spectrum $\mathbb{Z}_q$.

It is clear by Corollary 2.1 that $\chi(B) = 0$ if $o(\chi) = p$. Then by Lemma 3.1 the projection of $B$ to $\mathbb{Z}_p^2$ denoted by $B_{p^2}$ is constant. Since $|B| = p^2$ we have $B_{p^2} = \mathbb{Z}_p^2$ so its spectrum is $\mathbb{Z}_p$ using Lemma 3.2.

Case 2: $|A| = p, \ |B| = pq$
First, we prove that \( A \) is spectral. If \( \chi(A) = 0 \), with \( o(\chi) = p \), then as before, the subgroup generated by \( \chi \) is a spectrum for \( A \). In this case we are done.

It is clear from Corollary 2.1 that \( \chi(A) \neq 0 \) if \( o(\chi) = q \). Therefore there is an irreducible representation \( \chi(a,b) \) of \( G \) of order \( pq \) with \( \chi(a,b)(A) = 0 \). Then the projection \( A_{(a,b)} \) of \( A \) onto the corresponding \( \mathbb{Z}_{pq} \)-coset is the sum of \( \mathbb{Z}_p \)-cosets and \( \mathbb{Z}_q \)-cosets again. Since \(|A| = p\), the projection is a \( \mathbb{Z}_p \)-coset so it is spectral by Lemma 3.2.

Now we show that \( B \) is spectral. Since \( q \nmid |A| \) we have \( \chi(B) = 0 \) for every \( \chi \in \hat{G} \) with \( o(\chi) = q \). Assume that \( \chi(B) = 0 \) whenever \( o(\chi) = pq \). Since \( p^2 \neq |A| \) by Lemma 3.1 there exists \( \chi' \in \hat{G} \) of order \( p \) such that \( \chi'(A) \neq 0 \) so \( \chi'(B) = 0 \). The union of the set of representations of order \( pq \), \( q \) and subgroup generated by \( \chi' \) contains a subgroup isomorphic to \( \mathbb{Z}_{pq} \). Thus \( B \) is spectral.

Assume \( \chi(a,b)(B) \neq 0 \) for some \( \chi(a,b) \in \hat{G} \) with \( o(\chi(a,b)) = pq \). Then \( \chi(a,b)(A) = 0 \). We have that the projection \( A_{(a,b)} \) of \( A \) on a subgroup isomorphic to \( \mathbb{Z}_{pq} \), which is generated by \( (a,0) \) and \( (0,b) \). As usual \( A_{(a,b)} \) is the sum of \( \mathbb{Z}_p \)-cosets and \( \mathbb{Z}_q \)-cosets. Plainly \( A_{(a,b)} \) is a \( \mathbb{Z}_p \)-coset since \(|A| = p\). Using \( 0 \in A \) this is subgroup isomorphic to \( \mathbb{Z}_p \) so it is generated by \( (a,0) \).

As \( 0 \in A \) all elements of \( A \) are contained in \( \mathbb{Z}_p^2 \). Let \( B_i = \{(x,i) \in B \mid x \in \mathbb{Z}_p^2 \} \), where \( i \in \mathbb{Z}_q \). Then for \( i = 0, \ldots, q-1 \) we have \( A \ominus (B_i - i) = \mathbb{Z}_p^2 \). For \( (a,0) \in A - A \subset \mathbb{Z}_p^2 \) let \( (a',0) \in \mathbb{Z}_p^2 \) be an element of \( \mathbb{Z}_p^2 \) orthogonal to \( (a,0) \). Then \( \chi(a',0)(A) \neq 0 \) and hence \( \chi(a',0)(B_i) = 0 \) for all \( i = 0, \ldots, q-1 \). This means that all \( B_i - i \) as subsets of \( \mathbb{Z}_p^2 \) are equidistributed on the cosets of \( \langle (a,0) \rangle \), i.e. \( B_i \) has one element on each \( \langle (a,0) \rangle \)-coset. We have seen that \( (a,0) \) is independent from \( i \) we have that \( B \) is a tile on \( \mathbb{Z}_p^2 \times \mathbb{Z}_q \). Moreover, \( B \) projects bijectively onto a certain \( \mathbb{Z}_{pq} \)-coset. Hence \( B \) is spectral by Lemma 3.2.

### 4 Spectral-Tile

Let \( (S,\Lambda) \) be a spectral pair. Assume \(|S| = |\Lambda| > 1\). It is proved in [10,16] that Fuglede’s conjecture holds for every proper subgroup of \( \mathbb{Z}_p^2 \times \mathbb{Z}_q \). Hence we may assume neither \( S \) nor \( \Lambda \) is contained in a proper subgroup of \( \mathbb{Z}_p^2 \times \mathbb{Z}_q \), see Lemma 4.3 and Lemma 4.4 in [11]. We distinguish several cases as follows.

**Case 1:** \( p^2 \mid |S| \)

In this case either \( S \) is a tile or there are two elements of \( S \) that only differ in their \( q \)-coordinate. Then \( \chi(\Lambda) = 0 \) for some \( o(\chi) = q \). By Corollary 2.1 \( q \mid |\Lambda| = |S| \), a contradiction.

**Case 2:** \( pq \mid |S| \)

If \( p+1 \) elements of \( S \) are contained in a \( \mathbb{Z}_p^2 \)-coset, then every nonzero element in \( \mathbb{Z}_p^2 \) has a nonzero multiple which is in \( S - S \). In this case we say that every direction of \( \mathbb{Z}_p^2 \) appears in \( S \). Hence for all \( \chi \in \mathbb{Z}_p^2 = \hat{H} \leq \hat{G} \) we have \( \chi(\Lambda) = 0 \). Thus \( \Lambda \) vanishes in every element of \( H \), so does the projection of \( \Lambda \) onto \( \mathbb{Z}_p^2 \). Then by Lemma 3.1 it follows that \( p^2 \mid |\Lambda| \), which contradicts our assumption.

Therefore we can assume that \( S \) has exactly \( p \) elements in every \( \mathbb{Z}_p^2 \)-coset. Further this argument implies that there is \( a \in \mathbb{Z}_p^2 \) whose nonzero multiples cannot appear in the set \( S - S \) on any \( \mathbb{Z}_p^2 \)-coset. Since \( S = pq \), this implies that each \( \langle a \rangle \)-coset contains exactly one element of \( S \), which shows that \( S \) tiles \( G \) with tiling complement \( \langle a \rangle \).

**Case 3:** \( pq \geq |S| \)

\(^1\mathbb{Z}_p^2 \) embeds uniquely and naturally in \( \mathbb{Z}_p^2 \times \mathbb{Z}_q \).
Following the argument of the previous case we have that $p^2 \mid |S|$ or $S$ is a tile. The former case is handled in Case 1.

Case 4: $1 \mid |S| \text{ or } q \mid |S|$  
As $p \mid |S|$ there cannot be two elements of $S$ in the same $\mathbb{Z}_p^2$-coset. Then either $S$ has $q$ elements, and it is a tile with tiling complement $\mathbb{Z}_p^2 \setminus \Lambda$ or $|S| = |\Lambda| < q$. Hence we may assume $q \mid |S|$. By Corollary 2.1 we have $\chi(S) \neq 0$, if $o(\chi) \in \{1, p, q\}$. As $|S| > 1$, there exist a $\chi(a, b)$ with $o(\chi(a, b)) = pq$ satisfying $\chi(a, b)(S) = 0$. Now the projection of $S$ denoted by $S(a, b)$ is the sum of $\mathbb{Z}_p$-cosets and $\mathbb{Z}_q$-cosets. This implies that $|S| = |S(a, b)| = kp + lq$, where $k, l \geq 0$. As $|S| < q$ we have $l = 0$ and then $p \mid |S|$, a contradiction.

5. case: $p \mid |S|$ and $|S| < pq$
We have already assumed $\Lambda$ is not contained in a $\mathbb{Z}_p^2$-coset or a $\mathbb{Z}_q$-coset. Then there is $\chi(a, b) \in \hat{G}$, $(a \in \mathbb{Z}_p^2, b \in \mathbb{Z}_q)$ such that $\chi(a, b)(S) = 0$ and $o(\chi(a, b)) = pq$.

Using the same argument as in the previous case we have $|S| = kp + lq$ with $k, l \in \mathbb{N}$. Then $l = 0$ since $p \mid |S|$ and $|S| < pq$. Thus we have $a_i \cdot p$ elements of $S$ on the coset $i + \mathbb{Z}_p^2$ for $i = 0, \ldots, q − 1$.

Similar argument holds if we change the role of $S$ and $\Lambda$ with possibly different sequence of $b_i$ satisfying that $b_i \cdot p$ elements of $\Lambda$ are contained in $\mathbb{Z}_p^2 + i$ for $i = 0, \ldots, q − 1$.

If $a_i > 1$ (resp. $b_i > 1$) for some $i = 0, \ldots, q − 1$, then we have $p + 1$ elements of $S$ (resp. $\Lambda$) in a $\mathbb{Z}_p^2$-coset. Then every direction of $\mathbb{Z}_p^2$ appears in $S$ (resp. $\Lambda$). Thus may apply Lemma 3.1 to the projection of $S$ (resp. $\Lambda$) to $\mathbb{Z}_p^2$ so we have $p^2 \mid |S| = |\Lambda|$, a contradiction. From now on we assume $\mathbb{Z}_p^2$-coset contains either 0 or $p$ elements of $S$ and $\Lambda$.

Let $k$ denote the number of $\mathbb{Z}_p^2$-cosets containing $p$ elements of $S$. Then $2 \leq k < q$ since $S$ is not contained in any proper coset of $G$ and $|S| < pq$. Let $S_{p^2}$ denote the projection of $S$ to $\mathbb{Z}_p^2$. Then $S_{p^2}$ is a set since $p \mid |S|$. Otherwise $S − S$ contains an element of order $p$ and the fact that $(\Lambda, S)$ is also a spectral pair implies $q \mid |\Lambda| = |S|$. Further $|S_{p^2}| > p$ so every direction of $\mathbb{Z}_p^2$ appears in $S_{p^2}$.

We claim that if $S − S$ contains an element $(a, 0)$ of order $p$, then it does not contain $(a, b)$, where $b \in \mathbb{Z}_q \setminus \{0\}$ and vica versa and the same holds for $\Lambda − \Lambda$ as well. Indeed, suppose $(a, 0) \in \mathbb{Z}_p^2 \cap (S − S)$. Then if $(a, a') = 0$ for some $0 \neq a' \in \mathbb{Z}_p^2$, then $\Lambda − \Lambda$ cannot contain $(a', b)$ with $0 \neq b \in \mathbb{Z}_q$ since otherwise $\chi(a', b)(S) = 0$. Our assumptions show that two elements of $S$ would project to the same element in $S(a', b)$, contradicting $a_i \leq 1$. We have seen $\Lambda_{p^2}$ contains every direction of $\mathbb{Z}_p^2$ so $(a', 0) \in \Lambda − \Lambda$. Repeating the argument above for the spectral pair $(\Lambda, S)$ we obtain that $(a, b) \in S − S$ for $b \in \mathbb{Z}_q \setminus \{0\}$.

Now take two elements of $\Lambda$ in a $\mathbb{Z}_p^2$-coset, whose difference is $a' \neq 0$. Then for $0 \neq a \in \mathbb{Z}_p^2$ with $a \cdot a' = 0$ we have that each $(a)$-coset contains $k$ elements of $S_{p^2}$. If we take $k$ elements from $S_{p^2} \cap (a)$, then their preimages are contained in a $(\langle a, 0 \rangle)$-coset using the same argument as above. Thus the number of elements of $S$ contained in a $\mathbb{Z}_p^2$-coset is divisible by $k$ so $k \mid p$ We have $2 \leq k$ thus $k = p$. Then $|S| = p^2$, a contradiction.

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