NILPOTENT COMMUTING VARIETIES OF THE WITT ALGEBRA

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Abstract. Let \( g \) be the \( p \)-dimensional Witt algebra over an algebraically closed field \( k \) of characteristic \( p > 3 \). Let \( \mathcal{N} = \{ x \in g \mid x^{[p]} = 0 \} \) be the nilpotent variety of \( g \), and \( \mathcal{C}(\mathcal{N}) := \{(x, y) \in \mathcal{N} \times \mathcal{N} \mid [x, y] = 0\} \) the nilpotent commuting variety of \( g \). As an analogue of Premet’s result in the case of classical Lie algebras [A. Premet, Nilpotent commuting varieties of reductive Lie algebras. Invent. Math., 154, 653-683, 2003.], we show that the variety \( \mathcal{C}(\mathcal{N}) \) is reducible and equidimensional. Irreducible components of \( \mathcal{C}(\mathcal{N}) \) and their dimension are precisely given. Furthermore, the nilpotent commuting varieties of Borel subalgebras are also determined.

1. Introduction

Let \( k \) be an algebraically closed field of characteristic \( p > 0 \). For a restricted Lie algebra \( g \) over \( k \), let \( \mathcal{N} = \{ x \in g \mid x^{[p]^s} = 0 \text{ for } s \gg 0\} \) be the nilpotent variety of \( g \). The nilpotent commuting variety \( \mathcal{C}(\mathcal{N}) \) of \( g \) is defined as the collection of all 2-tuples of pairwise commuting elements in \( \mathcal{N} \). It is a closed subvariety of \( \mathcal{N} \times \mathcal{N} \). For \( g = \text{Lie}(G) \) where \( G \) is a connected reductive algebraic group and \( p \) is good for \( G \), Premet [5] showed that the nilpotent commuting variety \( \mathcal{C}(\mathcal{N}) \) is equidimensional, and the irreducible components are in correspondence with the distinguished nilpotent \( G \)-orbits in \( \mathcal{N} \). The nilpotent commuting variety plays an important role for the study of support varieties of modules over reduced enveloping algebras of \( g \) and cohomology theory of the second Frobenius kernel \( G_2 \) of \( G \). Premet’s theorem was also proved in characteristic zero. Quite recently, Goodwin and Röhrle [2] gave an analogue of Premet’s theorem on the nilpotent commuting varieties of Borel subalgebras of \( g \) in the case of characteristic zero. In this paper, we initiate the study of nilpotent commuting varieties of Lie algebras of Cartan type over \( k \).

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Let \( \mathfrak{g} = W_1 \) be the Witt algebra which was found by E. Witt as the first example of non-classical simple Lie algebra in 1930s. As is known to all, \( \mathfrak{g} \) is a restricted Lie algebra, and has a natural \( \mathbb{Z} \)-grading \( \mathfrak{g} = \sum_{i=-1}^{p-2} \mathfrak{g}[i] \). Associated with this grading, one has a filtration \( (\mathfrak{g}_i)_{i \geq -1} \) with \( \mathfrak{g}_i = \sum_{j \geq i} \mathfrak{g}[j] \) for \( i \geq -1 \). Let \( \mathcal{N} = \{ x \in \mathfrak{g} \mid x^{[p]} = 0 \} \) be the nilpotent variety of \( \mathfrak{g} \), which is a closed subvariety in \( \mathfrak{g} \). Set \( \mathcal{C}(\mathcal{N}) = \{(x,y) \in \mathcal{N} \times \mathcal{N} \mid [x,y] = 0\} \), the nilpotent commuting variety of \( \mathfrak{g} \). It is showed that the variety \( \mathcal{C}(\mathcal{N}) \) is reducible and equidimensional. There are \( \frac{p-1}{2} \) irreducible components of the same dimension \( p \) (see Theorem 3.6). Consequently, the variety \( \mathcal{C}(\mathcal{N}) \) is not normal (see Corollary 3.7). Furthermore, let \( \mathcal{B}^+ = \mathfrak{g}_0 \) be the standard Borel subalgebra of \( \mathfrak{g} \), and \( \mathcal{N}(\mathcal{B}^+) = \{ x \in \mathcal{B}^+ \mid x^{[p]} = 0 \} = \mathfrak{g}_1 \) the nilpotent variety of \( \mathcal{B}^+ \). Set \( \mathcal{C}(\mathcal{N}(\mathcal{B}^+)) = \{(x,y) \in \mathcal{N}(\mathcal{B}^+) \times \mathcal{N}(\mathcal{B}^+) \mid [x,y] = 0\} \), the nilpotent commuting variety of the Borel subalgebra \( \mathcal{B}^+ \). The variety \( \mathcal{C}(\mathcal{N}(\mathcal{B}^+)) \) is showed to be reducible and equidimensional. There are \( \frac{p^2-1}{2} \) irreducible components of the same dimension \( p \) (see Theorem 4.3). Moreover, the variety \( \mathcal{C}(\mathcal{N}(\mathcal{B}^+)) \) is not normal (see Corollary 4.5). As a motivation for further study, it should be mentioned that the nilpotent commuting variety \( \mathcal{C}(\mathcal{N}(\mathcal{B}^+)) \) of the Borel subalgebra \( \mathcal{B}^+ \) plays a very important role in the cohomology theory of the second Frobenius kernel \( G_2 \) of \( G \), where \( G \) is the automorphism group of \( \mathfrak{g} \). To be more precise, it was proved in [9] that \( \mathcal{C}(\mathcal{N}(\mathcal{B}^+)) \) is homeomorphic to the spectrum of maximal ideals of the Yoneda algebra \( \bigoplus_{i \geq 0} H^2(\mathcal{N}(\mathcal{B}^+), k) \) of the second Frobenius kernel \( G_2 \) of \( G \) whenever \( p \) is sufficiently large.

2. Preliminaries

Throughout this paper, we assume that the ground field \( k \) is algebraically closed, and of characteristic \( p > 3 \). Let \( \mathfrak{A} = k[X]/(X^p) \) be the truncated polynomial algebra of one indeterminate, where \( (X^p) \) denotes the ideal of \( k[X] \) generated by \( X^p \). For brevity, we also denote by \( X \) the coset of \( X \) in \( \mathfrak{A} \). There is a canonical basis \( \{1, X, \cdots, X^{p-1}\} \) in \( \mathfrak{A} \). Let \( D \) be the linear operator on \( \mathfrak{A} \) subject to the rule \( DX^i = iX^{i-1} \) for \( 0 \leq i \leq p-1 \). Denote by \( W_1 \) the derivation algebra of \( \mathfrak{A} \), namely the Witt algebra. In the following, we always assume \( \mathfrak{g} = W_1 \) unless otherwise stated. By [7, §4.2], \( \mathfrak{g} = \text{span}_k\{X^i D \mid 0 \leq i \leq p-1\} \). There is a natural \( \mathbb{Z} \)-grading on \( \mathfrak{g} \), i.e., \( \mathfrak{g} = \sum_{i=-1}^{p-2} \mathfrak{g}[i] \), where \( \mathfrak{g}[i] = kX^{i+1}D, -1 \leq i \leq p-2 \). Associated with this grading, one has the following natural filtration:

\[
\mathfrak{g} = \mathfrak{g}_{-1} \supset \mathfrak{g}_0 \supset \cdots \supset \mathfrak{g}_{p-2} \supset 0,
\]

where

\[
\mathfrak{g}_i = \sum_{j \geq i} \mathfrak{g}[j], -1 \leq i \leq p-2.
\]
This filtration is preserved under the action of the automorphism group $G$ of $g$ (cf. [1, 6, 9]). Furthermore, $g$ is a restricted Lie algebra with the $[p]$-mapping defined as the $p$-th power as usual derivations. Precisely speaking,

$$(X^iD)^{[p]} = \begin{cases} 0, & \text{if } i \neq 1, \\ XD, & \text{if } i = 1. \end{cases}$$

We need the following result on the automorphism group of $g$.

**Lemma 2.1.** (cf. [1, 3], see also [6, Theorem 12.8]) Let $g = W_1$ be the Witt algebra over $k$ and $G = \text{Aut}(g)$. Then the following statements hold.

(i) $G$ is a connected algebraic group of dimension $p - 1$.

(ii) $\text{Lie}(G) = g_0$.

**Remark 2.2.** Lemma 2.1 is not valid for $p = 3$. In fact, when $p = 3$, the Witt algebra $W_1 \cong \mathfrak{sl}_2$, and $\text{Aut}(\mathfrak{sl}_2)$ has dimension 3.

Based on [11, Proposition 3.3 and Proposition 3.4], we get the following useful result by a direct computation.

**Lemma 2.3.** Let $g = W_1$ be the Witt algebra. For $x \in g$, let $\mathfrak{z}_g(x) = \{y \in g \mid [x, y] = 0\}$ be the centralizer of $x$ in $g$. Then

$$\mathfrak{z}_g(x) = \begin{cases} kx, & \text{if } x \in G \cdot D, \\ kx \oplus g_{p-1-i}, & \text{if } x \in g_i \setminus g_{i+1}, 1 \leq i < \frac{p-1}{2}, \\ g_{p-1-i}, & \text{if } x \in g_i \setminus g_{i+1}, i \geq \frac{p-1}{2}. \end{cases}$$

**Remark 2.4.** For $x \in g_1$, let $\mathfrak{z}_{g_1}(x) = \{y \in g_1 \mid [x, y] = 0\}$ be the centralizer of $x$ in $g_1$, then $\mathfrak{z}_{g_1}(x) = \mathfrak{z}_g(x)$.

### 3. Nilpotent commuting variety of the Witt algebra

Keep in mind that $g = W_1$ is the Witt algebra over $k$. Set $\mathcal{N} = \{x \in g \mid x^{[p]} = 0\}$, which is a closed subvariety of $g$. Then $\mathcal{N}$ is just the set of all nilpotent elements in $g$. In the literature, $\mathcal{N}$ is usually called the nilpotent cone or nilpotent variety of $g$. The variety $\mathcal{N}$ was extensively studied by Premet in [4]. The following result is due to Premet.

**Lemma 3.1.** (cf. [4, Theorem 2 and Lemma 4] or [11, Lemma 3.1]) Keep notations as above, then the following statements hold.

(i) The orbit $G \cdot D$ is open and dense in $\mathcal{N}$. Moreover, it coincides with $(g \setminus g_0) \cap \mathcal{N}$. 

(ii) We have decomposition $\mathcal{N} = G \cdot D \cup g_1$.

(iii) $\dim \mathcal{N} = p - 1$.

Let $\mathcal{C}(\mathcal{N}) := \{ (x, y) \in \mathcal{N} \times \mathcal{N} \mid [x, y] = 0 \}$, the nilpotent commuting variety of $g$. Obviously, the Zariski closed set $\mathcal{C}(\mathcal{N})$ is preserved by the diagonal action of $G$ on $\mathcal{N} \times \mathcal{N}$. In this section, we study the structure of the variety $\mathcal{C}(\mathcal{N})$.

For $i \in \{1, \cdots, p - 2\}$, set

$$C(i) := \{ (x, y) \in \mathcal{N} \times \mathcal{N} \mid x \in g_i \setminus g_{i+1}, [x, y] = 0 \}.$$ 

Let

$$C(0) = \{ (x, ax) \mid x \in \mathcal{N}, a \in k \}$$ 

and

$$C(p - 1) = \{ (0, x) \mid x \in \mathcal{N} \}.$$ 

It is obvious that $C(p - 1)$ is a closed subvariety of dimension $p - 1$. Set

$$\mathcal{C}(i) = \overline{C(i)}$$

for $0 \leq i \leq p - 1$.

We have the following preliminary result describing the nilpotent commuting variety $\mathcal{C}(\mathcal{N})$ of $g$, the proof of which is straightforward.

**Lemma 3.2.** Let $g$ be the Witt algebra, $\mathcal{N}$ the nilpotent variety. Then $\mathcal{C}(\mathcal{N}) = \bigcup_{i=0}^{p-1} C(i)$.

Henceforth, $\mathcal{C}(\mathcal{N}) = \bigcup_{i=0}^{p-1} \mathcal{C}(i)$.

**Lemma 3.3.** $\mathcal{C}(i)$ is irreducible for any $0 \leq i \leq p - 1$, and

$$\dim \mathcal{C}(i) = \begin{cases} p, & \text{if } 0 \leq i < \frac{p-1}{2}, \\ p - 1, & \text{if } \frac{p-1}{2} \leq i \leq p - 1. \end{cases}$$

Moreover, $\mathcal{C}(p - 1) \subseteq \mathcal{C}(0)$.

**Proof.** Obviously, $\mathcal{C}(0)$ and $\mathcal{C}(p - 1)$ are irreducible varieties of dimension $p$ and $p - 1$, respectively. For $1 \leq i < \frac{p-1}{2}$, let

$$\varphi : (g_i \setminus g_{i+1}) \times g_{p-1-i} \times \mathbb{A}^1 \longrightarrow C(i)$$

$$(x, z, a) \longmapsto (x, ax + z)$$

be the canonical morphism. It follows from Lemma 2.3 that $\varphi$ is bijective, so that $\mathcal{C}(i)$ is irreducible, and

$$\dim \mathcal{C}(i) = \dim(g_i \setminus g_{i+1}) + \dim g_{p-1-i} + 1 = (p - 1 - i) + i + 1 = p.$$
For $\frac{p-1}{2} \leq i \leq p-2$, let
\[
\psi: (g_i \setminus g_{i+1}) \times g_{p-1-i} \longrightarrow C(i)
\]
\[(x, y) \longmapsto (x, y)
\]
be the canonical morphism. It follows from Lemma 2.3 that $\psi$ is an isomorphism, so that $C(i)$ is irreducible, and
\[
\dim C(i) = \dim(g_i \setminus g_{i+1}) + \dim g_{p-1-i} = (p-1-i) + i = p-1.
\]
Fix $x \in \mathcal{N}$, then
\[
\{(\lambda x, x) \mid \lambda \in k^\times\} \subseteq C(0).
\]
Since
\[
\{(\lambda x, x) \mid \lambda \in k^\times\} \cong k^\times,
\]
it follows that
\[
\{(ax, x) \mid a \in k\} = \{(\lambda x, x) \mid \lambda \in k^\times\} \subseteq C(0) = \mathfrak{C}(0).
\]
In particular, $(0, x) \in \mathfrak{C}(0)$ for any $x \in \mathcal{N}$, i.e.,
\[
\mathfrak{C}(p-1) = \{(0, x) \mid x \in \mathcal{N}\} \subseteq \mathfrak{C}(0).
\]
\[\square\]
As a direct consequence, we have

**Corollary 3.4.** Let $\mathfrak{g} = W_1$ be the Witt algebra, $\mathcal{N}$ the nilpotent variety of $\mathfrak{g}$, and $\mathfrak{C}(\mathcal{N})$ the nilpotent commuting variety of $\mathfrak{g}$. Then $\dim \mathfrak{C}(\mathcal{N}) = p$.

Combining Lemma 3.2 with Lemma 3.3, we get the following result which determines the possible irreducible components of the nilpotent commuting variety $\mathfrak{C}(\mathcal{N})$.

**Proposition 3.5.** Let $\mathfrak{g} = W_1$ be the Witt algebra, $\mathcal{N}$ the nilpotent variety of $\mathfrak{g}$. Let $\mathfrak{C}(\mathcal{N})$ be the nilpotent commuting variety of $\mathfrak{g}$. Then each irreducible component of $\mathfrak{C}(\mathcal{N})$ is of the form $\mathfrak{C}(i)$ for some $i \in \{0, 1, \cdots, p-2\}$.

Now we are ready for the main result of this section.

**Theorem 3.6.** Let $\mathfrak{g} = W_1$ be the Witt algebra, $\mathcal{N}$ the nilpotent variety of $\mathfrak{g}$. Then the nilpotent commuting variety $\mathfrak{C}(\mathcal{N})$ of $\mathfrak{g}$ is reducible and equidimensional. More precisely, $\mathfrak{C}(\mathcal{N}) = \bigcup_{i=0}^{(p-3)/2} \mathfrak{C}(i)$ is the decomposition of $\mathfrak{C}(\mathcal{N})$ into irreducible components.
Proof. We divide the proof into several steps.

**Step 1:** The group $GL(2, k)$ acts on $g \times g$ via

\[
\begin{pmatrix}
\alpha & \beta \\
\gamma & \delta
\end{pmatrix} \cdot (x, y) = (\alpha x + \beta y, \gamma x + \delta y).
\]

Since any linear combination of two commuting elements in $N$ is again in $N$, the nilpotent commuting variety $C(N)$ is $GL(2, k)$-invariant. As $GL(2, k)$ is a connected group, it fixes each irreducible component of $C(N)$. In particular, each irreducible component of $C(N)$ is invariant under the involution $\sigma: (x, y) \mapsto (y, x)$ on $N \times N$.

**Step 2:** Let \( \pi_1: N \times N \to N \) be the canonical projection. Then

\[
\pi_1(C(i)) = g_i \setminus g_{i+1}, \ 1 \leq i \leq p-2,
\]

and

\[
\pi_1(C(0)) = N,
\]

so that

\[
\pi_1(\mathcal{C}(i)) = \pi_1(\overline{C(i)}) = \overline{g_i \setminus g_{i+1}} = g_i
\]

and

\[
\pi_1(\mathcal{C}(0)) = \pi_1(\overline{C(0)}) = \overline{N} = N.
\]

It follows that $\mathcal{C}(i) \neq \mathcal{C}(j)$ for distinct $i, j \in \{0, \cdots, p-2\}$.

**Step 3:** If $\mathcal{C}(i)$ is an irreducible component of $C(N)$ for some $i \geq 1$, we aim to show that $i \leq \frac{p-1}{2}$. For any $x \in g_i \setminus g_{i+1}$ and $y \in g_0(x)$, since $(x, y) \in \mathcal{C}(i)$, it follows from Step 1 that $(y, x) \in \mathcal{C}(i)$. Consequently,

\[
y = \pi_1(y, x) \in \pi_1(\mathcal{C}(i)) = g_i.
\]

Hence, $g_0(x) \subseteq g_i$. It follows from Lemma 2.3 that $g_{p-1-i} \subseteq g_0(x) \subseteq g_i$. Hence, $p-1-i \geq i$, i.e., $i \leq \frac{p-1}{2}$.

In conclusion, the set of possible irreducible components in $C(N)$ is $\{C(i) \mid 0 \leq i \leq \frac{p-1}{2}\}$.

**Step 4:** $\mathcal{C}(i)$ is an irreducible component of $C(N)$ for $0 \leq i \leq \frac{p-3}{2}$. Indeed, if $\mathcal{C}(i)$ is not an irreducible component, it must be contained in $\mathcal{C}(j)$ for some $0 \leq j \leq \frac{p-1}{2}$ and $j \neq i$ by Step 3. Moreover, we get $\mathcal{C}(i) = \mathcal{C}(j)$ by comparing the dimension. This contradicts the assertion in Step 2.
Step 5: By Lemma 2.3,

\[ C\left(\frac{p-1}{2}\right) = \{(x, y) \mid x \in g_{\frac{p-1}{2}} \setminus g_{\frac{p+1}{2}}, [x, y] = 0\} \]

\[ = \{(x, y) \mid x \in g_{\frac{p-1}{2}} \setminus g_{\frac{p+1}{2}}, y \in g_{\frac{p-1}{2}}\}. \]

It follows that

\[ C\left(\frac{p-1}{2}\right) = C\left(\frac{p-1}{2}\right) = g_{\frac{p-1}{2}} \times g_{\frac{p-1}{2}}. \]

Moreover,

\[ C\left(\frac{p-1}{2}\right) \subseteq \bigcup_{i=0}^{(p-1)/2} C(i). \]

In fact, for any \((x, y) \in g_{\frac{p-1}{2}} \times g_{\frac{p-1}{2}}\), we claim that \((x, y) \in C(i)\) for some \(i \in \{0, \cdots, \frac{p-3}{2}\}\). We divide the discussion into the following cases.

**Case 1:** \(x = 0\) or \(y = 0\).

In this case, it is obvious that \((x, y) \in C(0)\).

**Case 2:** \(y \in g_j \setminus g_{j+1}\) for some \(j > \frac{p-1}{2}\).

In this case, set \(i = p - 1 - j < \frac{p-1}{2}\), then

\[ \{(u, y) \in N \times N \mid u \in g_i \setminus g_{i+1}\} \subseteq C(i). \]

It follows from Lemma 2.3 that

\[ (x, y) \in \{(v, y) \in N \times N \mid v \in g_i\} = \{(u, y) \in N \times N \mid u \in g_i \setminus g_{i+1}\} \subseteq C(i). \]

**Case 3:** \(x \in g_j \setminus g_{j+1}\) for some \(j > \frac{p-1}{2}\).

According to Case 2, \((y, x) \in C(i)\) for some \(i < \frac{p-1}{2}\). Since

\[ (x, y) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot (y, x), \]

it follows from Step 1 and Step 4 that \((x, y) \in C(i)\).

**Case 4:** \(x, y \in g_{\frac{p-1}{2}} \setminus g_{\frac{p+1}{2}}\).

In this case, \(y = ax + z\) for some \(a \in k^\times\) and \(z \in g_j\) with \(j > \frac{p-1}{2}\). Since

\[ (x, y) = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \cdot (x, z), \]

it follows from Step 1, Step 4, Case 1 and Case 2 that \((x, y) \in C(i)\) for \(i = p - 1 - j < \frac{p-1}{2}\) or \(i = 0\).

In conclusion, \((x, y) \in C(i)\) for some \(i \in \{0, \cdots, \frac{p-3}{2}\}\).
Step 6: It follows from Step 4 and Step 5 that the set of irreducible components of $\mathcal{C}(\mathcal{N})$ is exactly $\{\mathcal{C}(i) \mid 0 \leq i \leq \frac{p-3}{2}\}$, so that $\mathcal{C}(\mathcal{N}) = \bigcup_{i=0}^{(p-3)/2} \mathcal{C}(i)$ is the decomposition of $\mathcal{C}(\mathcal{N})$ into irreducible components.

The proof is completed. □

Since $(0,0) \in \bigcap_{i=0}^{(p-3)/2} \mathcal{C}(i)$, the following result is a direct consequence of Theorem 3.6.

Corollary 3.7. Let $\mathfrak{g} = W_1$ be the Witt algebra, $\mathcal{N}$ the nilpotent variety. Then the nilpotent commuting variety $\mathcal{C}(\mathcal{N})$ is not normal.

4. Nilpotent commuting varieties of Borel subalgebras in the Witt algebra

Let $\mathfrak{g} = W_1$ be the Witt algebra and $\mathcal{B}$ be a Borel subalgebra. Let $\mathcal{N}(\mathcal{B})$ be the nilpotent variety of $\mathcal{B}$, and

$$\mathcal{C}(\mathcal{N}(\mathcal{B})) = \{ (x,y) \in \mathcal{N}(\mathcal{B}) \times \mathcal{N}(\mathcal{B}) \mid [x,y] = 0 \}$$

the nilpotent commuting variety of $\mathcal{B}$. According to [10], $\mathcal{B}$ is conjugate to $\mathcal{B}^+$ or $\mathcal{B}^-$ under the automorphism group $G = \text{Aut}(\mathfrak{g})$ of $\mathfrak{g}$, where $\mathcal{B}^+ = \mathfrak{g}_0$ and $\mathcal{B}^- = \text{span}_k \{ D, XD \}$ are the so-called standard Borel subalgebras. It is easy to check that $\mathcal{N}(\mathcal{B}^-) = kD$ and $\mathcal{C}(\mathcal{N}(\mathcal{B}^-)) = \mathcal{N}(\mathcal{B}^-) \times \mathcal{N}(\mathcal{B}^-)$. In the following, we always assume $\mathcal{B} = \mathcal{B}^+$. In this case, $\mathcal{N}(\mathcal{B}) = \mathfrak{g}_1$. We will determine the structure of the nilpotent commuting variety $\mathcal{C}(\mathfrak{g}_1)$ of the Borel subalgebra $\mathcal{B} = \mathcal{B}^+$.

Set

$$C(p) = \{ (0,x) \mid x \in \mathfrak{g}_1 \}, \quad \mathcal{C}(p) = \overline{C(p)}.$$

We have the following preliminary result describing the nilpotent commuting variety $\mathcal{C}(\mathfrak{g}_1)$ of the Borel subalgebra $\mathcal{B}^+$, the proof of which is straightforward.

Lemma 4.1. Let $\mathfrak{g}$ be the Witt algebra. Then $\mathcal{C}(\mathfrak{g}_1) = C(p) \cup \left( \bigcup_{i=1}^{p-2} \mathcal{C}(i) \right)$. Henceforth, $\mathcal{C}(\mathfrak{g}_1) = \mathcal{C}(p) \cup \left( \bigcup_{i=1}^{p-2} \mathcal{C}(i) \right)$.

The following result describes the possible irreducible components of $\mathcal{C}(\mathfrak{g}_1)$.

Proposition 4.2. Let $\mathfrak{g}$ be the Witt algebra. Then each irreducible component of the nilpotent commuting variety $\mathcal{C}(\mathfrak{g}_1)$ is of the form $\mathcal{C}(i)$ for some $i \in \{1, \cdots, p-2\}$.

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Proof. It follows from Lemma 3.3 and Lemma 4.1 that each irreducible component of $C(g_1)$ is of the form $C(i)$ for some $i \in \{1, \ldots, p - 2, p\}$. We claim that

$$C(p) \subseteq \bigcup_{i=1}^{p-2} C(i),$$

from which the assertion follows.

Let $x \in g_1$, then either $x = 0$ or there exists a unique $i \in \{1, \ldots, p - 2\}$ such that $x \in g_i \setminus g_{i+1}$.

**Case 1:** $x = 0$.

In this case, it is obvious that $(0, 0) \in C(j)$ for any $1 \leq j \leq p - 2$.

**Case 2:** $x \in g_i \setminus g_{i+1}$.

In this case,

$$(0, x) \in \{(ax, x) \mid a \in k\} = \{(ax, x) \mid a \in k^\times\} \subseteq \overline{C(i)} = C(i).$$

Therefore,

$$C(p) \subseteq \bigcup_{i=1}^{p-2} C(i).$$

We are done. \qed

We are now in the position to present the main result of this section.

**Theorem 4.3.** Let $g = W_1$ be the Witt algebra. Then the nilpotent commuting variety $C(g_1)$ of the Borel subalgebra $B^+$ is reducible and equidimensional. More precisely, $C(g_1) = \bigcup_{i=1}^{(p-3)/2} C(i)$ is the decomposition of $C(g_1)$ into irreducible components. In particular, $\dim C(g_1) = p$.

**Proof.** The proof is similar to that of Theorem 3.6. \qed

**Remark 4.4.** Let $G = \text{Aut}(g)$ be the automorphism group of $g$. Since $\text{Lie}(G) = g_0 = B^+$, it follows from [8] that the nilpotent commuting variety $C(g_1)$ of the Borel subalgebra $B^+$ is homeomorphic to the spectrum of maximal ideals of the Yoneda algebra $\bigoplus_{i \geq 0} H^2(G_2, k)$ of the second Frobenius kernel $G_2$ of $G$ provided that $p$ is sufficiently large.

Since $(0, 0) \in \bigcap_{i=1}^{(p-3)/2} C(i)$, the following result is a direct consequence of Theorem 4.3.

**Corollary 4.5.** Let $g = W_1$ be the Witt algebra. Then the nilpotent commuting variety $C(g_1)$ is not normal.
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