PVC Polyhedra

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May 2, 2017

Abstract

We describe how to construct a dodecahedron, tetrahedron, cube, and octahedron out of pvc pipes using standard fittings.

1 Introduction

I wanted to build a huge dodecahedron for a museum exhibit. What better way to draw interest than a huge structure that you can walk around and through? The question was how to fabricate such an object? The dodecahedron is a fairly simple solid, made up of edges meeting in threes at certain angles that form pentagon faces. We could have them 3D printed. But could we do this with “off-the-shelf” parts?

The answer is yes, as seen in Figure 1. The key fact to consider is that each vertex corner consists of three edges coming together in a fairly symmetric way. Therefore, we can take a connector with three pipe inputs and make the corner a graph over it. In particular, if we take a connector that takes three pipes each at 120 degree angles from the others (this is called a “true wye”) and we take elbows of the appropriate angle, we can make the edges come together below the center at exactly the correct angles.

2 What is the correct elbow angle?

Suppose the wye has three length 1 pipes connected to it and the actual vertex \( v \) is below the vertex \( \nu W \) of the wye. Let \( X \) and \( Y \) be the endpoints of two of the wye segments. Then the segment \( XY \) is of length \( \sqrt{3} \) since it is opposite a 120 degree angle in the triangle \( XY\nu W \). We know that the angle \( XvY \) at vertex \( v \) has the angle of a regular pentagon, which is 108 degrees. Thus the length of segment \( Xv \) is

\[
\frac{\sqrt{3}}{2}/\sin\left(108^\circ \cdot \frac{1}{2}\right) = \frac{1}{2}\sqrt{15} - \frac{1}{2}\sqrt{3} = 1.071.
\]

The needed angle at the elbow (in degrees) is

\[
\arccos\left(\frac{1}{1.071}\right) = 20.98^\circ
\]

We can get an elbow that is 22.5 degrees (which is half of 45 degrees). That would be off by about 1.5 degrees, so the angles for the pentagons are actually off by 3 degrees, which is \( 3/108 = 2.8\% \). The length of \( \nu W \) is then

\[
\sqrt{\left(\frac{1}{2}\sqrt{15} - \frac{1}{2}\sqrt{3}\right)^2 - 1} = \frac{3}{2} - \frac{1}{2}\sqrt{5} = 0.382.
\]

Thus the actual vertices are 0.382 times the distance from the elbow to the center of the wye. Ours is 2.5 inches, and so the actual vertices are \( 0.382 \times 2.5 = 0.955 \) inches from the center of the wye. Furthermore, given whatever length we have of the tube, the actual vertices will be an additional \( 1.071 \times 2.5 = 2.678 \) inches on each side. A close-up of the vertex joint can be seen in Figure 2.

∗Partially funded by NSF DMS 0748283.
Figure 1: The PVC Dodecahedron. Picture courtesy of Bruce Bayly.

Figure 2: A close-up of the vertex joint for the dodecahedron.
3 More platonic solids

There are actually four platonic solids: the tetrahedron, the cube, the octahedron, the dodecahedron, and the icosahedron. The vertices of a cube can be purchased at a plumbing supply shop. We can also make the octahedron and a tetrahedron with careful consideration.

For a regular tetrahedron, we compute similarly

\[ |Xv| = \frac{\sqrt{3}}{2} / \sin(30^\circ) = \sqrt{\frac{\sqrt{6}}{2}} - 3 \]

and the needed elbow is \( \arccos \left( \frac{1}{\sqrt{3}} \right) = 54.7^\circ \). If we use a three-way cube coupling, the angle with the vertex is \( \arccos \left( \frac{2}{\sqrt{6}} \right) = 35.3^\circ \).

So with this we just need an elbow of \( 54.7 - 35.3 = 19.4^\circ \).

This is not too far from 22.5 degrees.

For an octahedron, it can be verified that what is needed is simply a four-way plus connector and 45 degree elbows. These can actually be purchased at most large hardware/home improvement stores. The tetrahedron and octahedron can be seen in Figure 3.

4 Make your own dodecahedron

If you want to make your own dodecahedron, here is a quick inventory and size calculator. Wikipedia is a good resource for some of these numbers \([4]\). Since the dodecahedron has 20 vertices, you will need 20 true wyes and 60 twenty-two and half degree elbows. You will then need 60 small lengths of PVC to connect the wyes and elbows, and you will want to use PVC glue to make the connections strong. Then you will need 30 edges of roughly the same length (it is very forgiving). To calculate the lengths, you need to decide how large you want your dodecahedron to be.

Consider a dodecahedron with edge length 1. We have the following:
Figure 4: A birds-eye view of the exhibit Proofs, Puzzles, and Patterns: Explore the World of Mathematics at Flandrau Science Center in Tucson, AZ. Picture courtesy of Flandrau Science Center.

|                           |                                                          |
|---------------------------|----------------------------------------------------------|
| inradius                  | $\frac{1}{20}\sqrt{250 + 110\sqrt{5}} \approx 1.114$    |
| circumradius              | $\frac{1}{4}(\sqrt{15} + \sqrt{3}) \approx 1.401$      |
| inradius of face          | $\frac{1}{10}\sqrt{25 + 10\sqrt{5}} \approx 0.688$    |
| circumradius of face      | $\frac{1}{10}\sqrt{50 + 10\sqrt{5}} \approx 0.851$    |
| dihedral angle            | $\arccos\left(-\frac{1}{\sqrt{5}}\right) \approx 2.034 \approx 116.56^\circ$ |
| height of second row of vertices | $\frac{1}{5}\sqrt{5} + 2\sqrt{5}\sin\left(\pi - \arccos\left(-\frac{1}{\sqrt{5}}\right)\right) \approx 1.376$ |

So to have a dodecahedron that sits 6 ft. tall, we need edge length of $6/2.228 = 2.693$ ft. Also, the height of the second row of vertices (highest height to enter from) would be about $(1.376)(2.693) = 3.706$ ft. Inside the dodecahedron we would have around $(0.688)(2.693)(2) = 3.706$ feet in diameter to stand. The dodecahedron itself takes up a diameter of $(1.401)(2.693)(2) = 7.546$ ft. The base of the dodecahedron takes up a diameter of $(0.851)(2.693)(2) = 4.584$ ft.

5 Epilogue

The exhibit Proofs, Puzzles, and Patterns: Explore the World of Mathematics [3][2] opened at Flandrau Science Center in October 2015, as seen in Figure 4. The PVC dodecahedron was a hit, but in the end it was not sturdy enough for the exhibit floor. The dodecahedron and other polyhedra have found use in traveling exhibits by Bruce Bayly and his Mathematics Road Show [1] as well as at some other enrichment locations. It is a great group activity to put together the dodecahedron, and takes only around 10 minutes.

Acknowledgement 1 The author would like to thank the staff at Flandrau Science Center for their help and support in the construction of the Proofs, Puzzles, and Patterns exhibit, especially Bill Plant, Shiloe Fontes, and Kellee Campbell, Neil McSweeney, and Shiperd Reed; Greg McNamee for all of his work on the panels for the exhibit; Marta Civil for her providing most of the puzzles aspect of the exhibit; and Bruce Bayly, who is credited with the first picture and who first assembled the PVC dodecahedron for the author in his own back yard, and who has been a huge encouragement throughout.

References

[1] Arizona Math Road Show. http://math.arizona.edu/outreach/programs/az-math-roadshow
[2] R. Peiffer. Flandrau Exhibition Goes Beyond the Mere Facts in Math. UA News, May 23, 2016.

[3] Puzzles, Proofs, and Patterns: Experience the World of Mathematics. Flandrau Science Center, Tucson, AZ. [http://flandrau.org/exhibits/puzzles](http://flandrau.org/exhibits/puzzles)

[4] Dodecahedron. [http://en.wikipedia.org/wiki/Dodecahedron](http://en.wikipedia.org/wiki/Dodecahedron)