Three-dimensional Chiral Lattice Fermion in Floquet Systems

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We show that the Nielsen-Ninomiya no-go theorem still holds on Floquet lattice: there is an equal number of right-handed and left-handed Weyl points in 3D Floquet lattice. However, in the adiabatic limit, where the time evolution of low-energy subspace is decoupled from the high-energy subspace, we show that the bulk dynamics in the low-energy subspace can be described by Floquet bands with purely left/right-handed Weyl points, despite the no-go theorem. For the adiabatic evolution of two bands, we show that the difference of the number of right-handed and left-handed Weyl points equals twice the winding number of the Floquet operator of the low-energy subspace over the Brillouin zone, thus guaranteeing the number of Weyl points to be even. Based on this observation, we propose to realize purely left/right-handed Weyl points in the adiabatic limit using a Hamiltonian obtained through dimensional reduction of four-dimensional quantum Hall system. We address the breakdown of the adiabatic limit on the surface due to the presence of gapless boundary states. This effect induces a circular motion of a wave packet in an applied magnetic field, travelling alternatively in the low-energy and high-energy subspace of the system.

Introduction.— In 1981, Nielsen and Ninomiya proved a theorem [1, 2] implying the absence of neutrinos on a lattice: there are equal number of left-handed and right-handed Weyl particles appearing in the continuum limit. In solid state physics, where there is a natural lattice, the generic nodes of electron bands are Weyl points [3, 4], which carry a chirality ±1, depending on the net Berry flux pierced through a sphere enclosing the node. There have been extensive studies of the physics of Weyl points, including research of phenomena linked to the chiral anomaly [5–17], surface Fermi-arc states [18–25], and anomalous transport properties [26, 27]. It has become a well established knowledge in the field of topological semimetals that the net chirality of all the Weyl points has to be zero.

Recently, periodically driven systems have attracted interest from condensed matter [28–63], photonics [64–73] and cold atoms [74–77] communities. In periodically driven lattices, an important quantity is the time-evolution operator over the period of one cycle (the Floquet operator $U_k$), whose eigenvalues $\exp[-i\epsilon(k)T]$ constitute quasi-energy bands $\epsilon(k)$. Given such novel platforms, it is natural to revisit the derivation of the Nielsen-Ninomiya theorem for quasi-energy bands. Especially, one of the assumptions made by Refs. [1, 2] is that the energy spectrum can be ordered at each momentum $k$ as $E_1(k) < E_2(k) < ... < E_n(k)$. Such a premise does not apply to quasi-energy bands because quasi energy is determined up to an ambiguity of a multiple of $2\pi/T$. Furthermore, if one only considers the periodicity of Berry curvature on the Brillouin zone (BZ) boundary, one can easily find a “counterexample” of the theorem as schematically illustrated in Fig. 1(b). Even more simply, one can find a one-dimensional (1D) quasi-energy band with a single chiral mode[55] as shown in Fig. 1(a), which presents a “counterexample” of the analogous no-go theorem in 1D.

In this letter, we present a topological argument proving that the Nielsen-Ninomiya no-go theorem generalizes to periodically driven lattices. However, we also show that the mentioned “counterexamples” become physically meaningful in the adiabatic limit (i.e. when the rate of changing the Hamiltonian is slow compared to the energy separation of the utilized eigenstates of the instantaneous Hamiltonian from the rest of the spectrum.) In the latter case, the dynamics of the low energy states (the states below the gap of the Hamiltonian at $t = 0$) is decoupled from the dynamics of the high energy states, i.e. $U_k$ becomes block diagonal. Although the spectrum of $U_k$ obeys the no-go theorem, the spectrum of the individual low/high-energy blocks is allowed to exhibit Floquet bands with purely left- or right-handed Weyl points. This discovery opens an opportunity to experimentally observe the dynamics of chiral Weyl particles (neutrinos) on a lattice. For this purpose, we develop a lattice model exhibiting chiral Weyl particles, which is obtained from a 4D quantum Hall state [78] by interpreting one momentum as the adiabatic parameter. Analogous constructions were previously considered for two adiabatic parameters in two-dimensional space both theoretically [79] and experimentally [80, 81]. Using this analogy, we infer the breakdown of the adiabatic limit on the surface due to the presence of topologically protected boundary states. Ultimately, we predict that this facilitates a circular motion of a wave packet in an applied magnetic field, travelling alternatively in the low-energy and high-energy sectors of the Floquet operator.

The no-go theorem.— Similar to static electron bands, the generic nodal structure of a 3D Floquet lattice is still a Weyl point [82], protected by Chern number on a sphere enclosing the node. Assuming that the translational symmetry is preserved, a Weyl point can be removed only through a pairwise annihilation with a Weyl point of opposite chirality. Therefore, even for the Floquet bands, the difference of the number of right-handed and left-handed Weyl points $n_R - n_L$ is a topologically stable quantity, i.e. a topological invariant of $U_k$. If we allow the unitary matrix $U_k$ to be an arbitrary (but continuous) function of $k$, then $n_R - n_L$ can indeed be non-zero. However, Floquet operators are subject to the no-go theorem
because of the following observation: It is possible to continuously deform all the legitimate Floquet operators $U_k$ to identity matrix $I_{NNN}$ through backward time evolution to $t = 0$ while keeping $n_R - n_L$ invariant. Formally, the time evolution at momentum $k$: $k \mapsto T \exp[-i \int_{0}^{t} H_k(t')dt']$ continuously interpolates $k \mapsto I_{NNN}$ at $t = 0$ and $k \mapsto U_k$ at $t = T$. Since $n_R - n_L = 0$ for the spectrum of identity matrices at $t = 0$, the same must hold for the Floquet operator $U_k$.

It is germane to emphasize the following observation: assuming general continuous deformation without further constraints, the Floquet operator over one cycle $U_k$ can always retract to topologically trivial identity matrices (the validation of the no-go theorem serving as an example). Therefore, to define a non-trivial topological property, one has to impose certain restrictions on the admissible deformations. A rather popular choice is to permit only those that keep a finite gap in the quasi-energy spectrum. Such a choice, akin to the tenfoldway classification of static systems [83, 84], defines topological invariants of a gap, and usually determines a boundary state inside the gap [56, 57]. In this letter, we consider another type of constraint [55], namely that of the adiabatic limit. This requires the presence of a finite gap between the low-energy and the high-energy subspace of the instantaneous Hamiltonian as shown in Fig. 2(a), and a time evolution slow compared to the energy separation of the two sectors. In the next section, we show that the argument based on the backward time evolution does not apply to the low-energy sector in the adiabatic limit, allowing us to find the “counterexamples” suggested in the introduction. Nevertheless, we remark that the “counterexamples” are consistent with the no-go theorem in the sense that there are always modes in the high-energy sector, which compensate the difference $n_R - n_L$ in the low-energy sector.

**Adiabatic limit.—** In the adiabatic limit, the time evolution operator $\tilde{U}_k$ of the low/high-energy subspace over one cycle corresponds to a Wilson loop in the parameter space,

$$\tilde{U}_k = \mathcal{P} e^{i \frac{k}{\hbar_0} a_k(\mathbf{R}) \cdot d \mathbf{R}},$$

where the closed path $\mathbf{R}(t)$ represents the variation of the adiabatic parameters $\mathbf{R}$ over one cycle $t \in [0, T]$ (for simplicity, we set the cycle period to $T = 1$), and $\mathcal{P}$ indicates path-ordering. Finally, $a_k(\mathbf{R})$ is the non-Abelian Berry connection [85–87]

$$[a_k(\mathbf{R})]_{mn} = i \langle k | \mathbf{R}, m | \nabla_{\mathbf{R}} | k, n \rangle,$$

where $|m\rangle, |n\rangle$ label the low-energy (or high-energy) eigenstates of the instantaneous Hamiltonian. The Wilson loop is a geometric property of the path $\mathbf{R}(t)$. Importantly, if the path is not contractible to a point [see Fig. 2(b)] in the parameter space (assuming we avoid singularities of $a_k(\mathbf{R})$ where the adiabatic approximation fails), then the function $k \mapsto \tilde{U}_k$ may fail to be continuously deformable to the identity $k \mapsto I_{NNN}$, thus possibly exhibiting a non-trivial topology [88].

We first illustrate such a topological property for a 1D system with momentum $k$ and adiabatic Floquet operator $\tilde{U}_k = \exp(-ik)$. The eigenvalue $\exp(-ie(k))$ has a chiral dispersion of quasi-energy, $\epsilon(k) = k \bmod 2\pi$. We display the chiral dispersion of $\epsilon(k)$ in Fig. 1(a) (blue line). Counting the number of right movers $n_R^{1D}(\epsilon)$ and the number of left movers $n_L^{1D}(\epsilon)$ on each quasi-energy cut $\epsilon$ reveals that $n_R^{1D}(\epsilon) - n_L^{1D}(\epsilon)$ is the same for all $\epsilon$. Furthermore, this difference does not change upon continuous deformation of the dispersion, nor upon adding a trivial band [i.e. one with $n_R^{1D}(\epsilon) - n_L^{1D}(\epsilon) = 0$], therefore suggesting a topological character. It is easily checked [55, 88] that the difference equals to the winding number of $\tilde{U}_k$,

$$n_R^{1D}(\epsilon) - n_L^{1D}(\epsilon) = n_1 \equiv \frac{i}{2\pi} \int_{-\pi}^{\pi} \text{Tr} \left[ \tilde{U}_k^{-1} \partial_\epsilon \tilde{U}_k \right]$$

over the 1D BZ of the system.

Inspired by the 1D case summarized by Eq. (3), we expect the difference $n_R - n_L$ between the number of right-handed
and left-handed Weyl points in a 3D system to be related to the winding number \( \nu_3 \) of \( \tilde{U}(k) \) over a 3D BZ, defined by [55]

\[
\nu_3 = \frac{1}{24\pi^2} \int d^3k \varepsilon^{\alpha\beta\gamma} \times \nabla [\tilde{U}_k^{-1} \partial_\alpha \tilde{U}_k] (\tilde{U}_k^{-1} \partial_\beta \tilde{U}_k) (\tilde{U}_k^{-1} \partial_\gamma \tilde{U}_k),
\]

where \( \varepsilon^{\alpha\beta\gamma} \) is the anti-symmetric tensor and \( \alpha, \beta, \gamma \in \{x, y, z\} \) are spatial indices. In the next section, we inspect the relation between topological quantities \( \nu_3 \) and \( n_R - n_L \) for a class of two-band models.

**Two-band model.**— The presence of a Weyl point requires a minimum of two bands. We thus consider a pair of bands in the adiabatic limit, and decompose the Floquet operator into

\[
\tilde{U}_k \in U(2) \cong S^1 \times SU(2),
\]

where \( S^1 \equiv U(1) \) part refers to matrices of the form \( \text{diag}(\text{det}(\tilde{U}_k), 1) \), while the \( SU(2) \) part has unit determinant. The \( \nu_3 \) invariant comes from a non-trivial third homotopy group, which is independent of the \( S^1 \) part. For simplicity, we narrow our discussion to systems with \( v_1 = 0 \) on all closed paths inside the BZ, such that the image in the \( S^1 \) component can be continuously deformed to identity. We decompose

\[
\tilde{U}_k = n_0(k) \sigma_0 + i[n_1(k) \sigma_1 + n_2(k) \sigma_2 + n_3(k) \sigma_3],
\]

where \( \sigma_0 \) is the identity and \( \sigma_{1,2,3} \) are the Pauli matrices. The condition on unit determinant requires \( \tilde{n}(k) = (n_0(k), n_1(k), n_2(k), n_3(k)) \) to be a real unit vector on a three-dimensional sphere \( S^3 \). The number of times that the image of \( T^3 \) “wraps” around the \( S^3 \) is given by the winding number

\[
\nu_3 = \frac{1}{2\pi} \int d^3k \varepsilon^{abcd} \partial_a (n_b \partial_c n_d) (\partial_b n_c) (\partial_c n_d),
\]

where \( \varepsilon^{abcd} \) is the anti-symmetric tensor and indices \( a, b, c, d \in \{0, 1, 2, 3\} \) pertain to components of \( \tilde{n} \). Geometrically, the winding number density (i.e. the integrand) represents the oriented area that \( \tilde{n}(k) \) swipes when we vary \( k \) over an infinitesimal cube \( \partial^3k \) in BZ. A heuristic picture is that the image in \( \partial^3k \) is “covering” the \( S^3 \) at \( k \) if the oriented area is positive, while it is “uncovering” the \( S^3 \) if the oriented area is negative. We illustrate this concept on a pair of simple examples in Fig. 3, where we partition BZ into a family of submanifolds labelled by \( \lambda \in \{0, 1\} \) for easier visualization.

A generic point of \( S^3 \) is covered (uncovered) \( n_+ \) (\( n_- \)) times by \( \tilde{U}_k \). The geometric meaning implies that for all points

\[
\nu_3 = n_+ - n_-.
\]

Especially, Eq. (8) also applies to the “north pole” and “south pole”, \( \pm \sigma_0 \in S^3 \), which correspond to degeneracies of the Floquet bands at quasi-energy 0 vs. \( \pi \). The Floquet operator in the vicinity of a right-handed (+) and left-handed (−) Weyl point takes the form \( \tilde{U}_k = e^{i(k-Q_{\lambda})} \sigma_0 \) at the north pole \( \tilde{U}_k = e^{i(k-Q_{\lambda})} \sigma_0 \) at the south pole, where \( Q_{N/S} \) is the momentum of the Weyl point. The integrand of Eq. (7) is positive at right-handed Weyl points, and negative at left-handed Weyl points. Therefore, we find using Eq. (8) that

\[
\nu_3 = n_R^N - n_L^N = n_S^S - n_L^S,
\]

where the superscript indicates the quasi-energy of the Weyl points (i.e. the corresponding pole of the \( S^3 \)). This implies that for the case of two bands in the adiabatic limit \( n_R - n_L = 2\nu_3 \). The result in Eq. (9) further means that Weyl points of opposite chirality but corresponding to opposite poles are not able to annihilate. Finally, we find that the number of Weyl points has to be even for the adiabatic evolution of two bands. (More generally, \( \nu_3 \) counts the number of Berry phase quanta flowing through the Floquet bands in the quasi-energy direction, so we expect \( N \geq 2 \) bands to exhibit a minimum of \( N\nu_3 \) Weyl points.) We conclude that the relation in Eq. (9) opens an opportunity to study Floquet bands with a non-vanishing \( n_R - n_L \). In the next section, we present a way of constructing such chiral models with the help of a 4D quantum Hall systems.

**Four-dimensional (4D) quantum Hall model.**— A Floquet lattice with a non-trivial winding number \( \nu_4 \) is related to 4D quantum Hall system [90, 91] if we identify the adiabatic parameter as the momentum \( k_w \) along the fourth dimension. It was shown by Ref. [55] that \( \nu_3 \) of a Floquet operator of the occupied bands in the adiabatic evolution is equal to the second Chern number of the corresponding 4D model. This relation provides a practical way for developing Floquet models with a non-trivial \( \nu_3 \) and thus, according to Eq. (9), with non-zero \( n_R - n_L \). For example, one such a simple Hamiltonian [88] is

\[
H(k, k_w) = (\cos k_x + \cos k_y + \cos k_z + \cos k_w - m) \Gamma_1 + \sin k_x \Gamma_2 + \sin k_y \Gamma_3 + \sin k_z \Gamma_4 + \sin k_w \Gamma_5,
\]
where the Dirac matrices \( \Gamma_i \) obey the anti-commutation relation \( \{ \Gamma_i, \Gamma_j \} = \delta_{ij} \).

The non-linear 4D quantum Hall response implies that the chiral Floquet systems produce a current \( j \propto v_3(\partial_\nu k_w)B \) in an applied magnetic field, where \( \partial_\nu k_w \) is analogous to electric field in the \( \nu \) direction. Taking the case of Fig. 1(b) as a simple example, the appearance of current follows easily by sketching the Landau level spectrum, which contains chiral modes [see Fig. 4(a)] traveling in the direction of the applied magnetic field. If the material has no boundary in the direction of the applied field, this phenomenon corresponds to chiral magnetic effect [5].

On the other hand, we expect the presence of a boundary to facilitate a circular motion of a wave-packet through the system. To see the origin of such a phenomenon, first note that the adiabatic approximation breaks down on the boundary since 4D quantum Hall Hamiltonian exhibits gapless boundary states for certain \( k_w \). This allows the low-energy and the high-energy subspaces to couple at the boundary. Since each sector has a non-vanishing total (and mutually opposite) chirality, we expect the coupling to take the form of Fermi arcs connecting the two sectors. To complete the argument, let us consider a wave-packet with momentum near the Weyl point of the low-energy sector. In an applied magnetic field, the wave-packet moves upward along the system via the bulk chiral Landau level, until it reaches the system boundary. Then it evolves along the surface Fermi arc under the influence of Lorentz force while reaching the high-energy sector. The new setting allows the wave-packet to descend through the system along the Landau level of opposite chirality, until it finally completes the cycle by returning to the low-energy sector along the Fermi arc on the bottom of the system [see Fig. 4(b)]. This circular motion [92–94] of wave-packet dynamics under magnetic field is an experimental signature once such a model is realized not only in electron system, but in photonic and cold atom system, where an effective magnetic field can be applied in a synthetic way [95–99].

**Experimental realization proposals.**— There has been experiments [81, 100] or proposals [70] to realize 4D quantum Hall effect(QHE). As experimental systems are limited to three spatial dimensions, the key integrant of realizing 4D QHE is to introduce additional synthetic dimensions. The idea of synthetic dimension has been discussed in the context of superconducting qubits [101], ultra-cold atoms [102] and optics [103]. Due to the highly controllability and tunability, ultra-cold atoms and photonic wave guides can serve as ideal platforms for realizing such ideas. In fact, the very recent realizations of 4D quantum Hall physics are on these two platforms [70, 80, 81, 100]. A suitable platform for investigating the Hamiltonian of our system as shown in Eq. (10) is a 3D array of ring resonators. 4D QHE in such a system has already been studied theoretically in Ref. [70]. One can construct a 4D system with three spatial dimensions and a synthetic frequency dimension. Here the synthetic dimension consists of resonance modes of a ring resonator which is equally spaced in frequency [69, 70]. The coupling in three spatial dimensions can be achieved with delay line waveguides, and the coupling in frequency dimension is realized by dynamic modulation. Furthermore, the synthetic frequency dimension can be treated as the adiabatic pumping direction, where the pumping process can be achieved by tuning the modulation phase simultaneously on all the ring resonators that effectively evolves \( k_w \) from 0 to \( 2\pi \) in one cycle. The dynamics of waves under such periodic pumping can practically simulate that of chiral Weyl particles.

**Conclusion.**— We have shown the validation of Nielsen-Ninomiya no-go theorem in Floquet lattice and demonstrated the possibility of getting purely left/right-handed Weyl points in the adiabatic limit, if only the dynamics of the low-energy states is considered. We have also studied the adiabatic dynamics of two bands and proven the sum of the chirality of Weyl points is equal to twice the 3D winding number of the Floquet operator. We have made analogy of such a system to four-dimensional quantum Hall system and proposed circular motion of wave packet alternately in the low and high energy subspace as a signature. The four-dimensional quantum Hall physics has been demonstrated experimentally by adiabatic pumping in both cold atom and photonic systems. Our work will shed light on those experiments and serve as a theoretical ground work of observing features of purely left/right-handed Weyl points in those systems.

**Note added.**— After finishing this manuscript, we became aware of a related eprint by Higashikawa et al. [104], in which a Floquet band with non-vanishing total chirality of Weyl points is constructed without the analogy of four-dimensional quantum Hall system and the argument that no-go theorem still holds.

**Acknowledgments.**— X.-Q. S and S.-C. Z acknowledge...
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In this manuscript, we only discuss generic nodal structure corresponding to Weyl points. Fine-tuned models and models with additional symmetry can exhibit more elaborate band degeneracies, such as nodal lines/surfaces as well as double Weyl points. In those cases, the \( n_R = n_L \) that we mentioned throughout the text should instead be replaced by summing over the Chern number on the surface enclosed each node instead.

Because of van-Hove singularities, some of the submanifolds have more complicated topology. This poses some constraints on their images, but it does not interfere with the notion of positive/negative orientation.

Due to the engineering techniques related in photonics.

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