Abstract. We establish the strictness of several set quantifier alternation hierarchies that are based on modal logic, evaluated on various classes of finite graphs. This extends to the modal setting a celebrated result of Matz, Schweihardt and Thomas (2002), which states that the analogous hierarchy of monadic second-order logic is strict.

Thereby, the present paper settles a question raised by van Benthem (1983), revived by ten Cate (2006), and partially answered by Kuusisto (2008, 2015).

1 Introduction

One of the central concerns in theoretical computer science is to understand which components of any given formal system are responsible for its expressive power. A typical sort of question is thus whether the power of a particular model decreases when it is deprived of a certain feature. In the present article, we investigate two questions of that very sort. As they have emerged independently in separate areas of research, with different perspectives, their relationship may not be obvious at first sight.

1.1 Motivation and Related Work

Our first question is an old problem in the field of modal logic. Formulas in that system are usually evaluated on labeled graphs from the point of view of a distinguished vertex, which can only see the labels of its neighbors within a small radius. A natural generalization of this model, investigated in various places of the literature, such as [Bul69] and [Fin70], is to introduce quantifiers over sets, by means of which a formula can, so to speak, extend the vertex labeling of a given graph. We shall refer to a variant of the resulting formalism as hybrid logic with set quantifiers (HS). Already in 1983, van Benthem asked in [Ben83] whether the syntactic hierarchy obtained by alternating between existential and universal set quantifiers induces a corresponding hierarchy on the semantic side. Remaining unanswered, the question was raised again by ten Cate in [Cat06], and finally a partial answer was provided by Kuusisto in [Kuu08, Kuu15]: He could show that HS induces an infinite hierarchy over finite directed graphs. This tells us...
that the hierarchy does not completely collapse at some level, but leaves open whether or not each number of quantifier alternations corresponds to a separate semantic level. Kuusisto’s proof builds upon the work of Matz, Schwikardt and Thomas in [MST02] (elaborating on their previous results in [MT97] and [Sch97]), where they have shown that, in the case of monadic second-order (MSO) logic, the analogous hierarchy is strict. Thus, each additional alternation between the two types of set quantifiers properly extends the according family of definable graph properties. Significantly, this result also holds for a more restrictive class of structures called grids.

The second problem, and the original motivation for the present article, stems from the field of automata theory on graphs. There, the author has introduced in [Rei15] a notion of graph automaton, dubbed alternating distributed graph automaton (ADGA), which operates in a manner similar to a distributed algorithm. Equipped with the ability to alternate between nondeterministic decisions of the individual processors and the creation of parallel computation branches, this model has been shown equivalent to MSO on graphs. As a natural follow-up question, one can ask whether successive restrictions on the capability of alternating between the two operation modes always lead to a decrease in expressive power. This seems particularly relevant in light of the corresponding separation results on MSO obtained by Matz, Schwikardt and Thomas.

As it turns out, the two problems above are strongly related. The reason is that ADGA can be viewed as the automata-theoretic counterpart of HGS, a logic extending HS with an additional operator, called global modality, that allows quantification over all evaluation points. Two corresponding levels of alternation in the frameworks of ADGA and HGS characterize exactly the same graph properties. Although this observation has so far not been formally published, it is relatively easy to verify, once pointed out. Besides, a similar correspondence between local distributed algorithms and modal logic has already been established by Hella et al. in [H+12, H+15], and the equivalence of HGS and MSO that can be indirectly inferred from [Rei15] has previously been shown in [Kuu08, Kuu15].

Hereafter, we shall work with HGS instead of the ADGA model, because compared to state diagrams, logical formulas take up less space and are usually easier to manipulate.

1.2 Contribution
The present work gives a complete answer to both of the questions mentioned above and to some variants thereof. In particular, the set quantifier alternation hierarchies induced by HS and HGS are shown to be strict over finite directed graphs. Just as Kuusisto has done in [Kuu08, Kuu15], we will use as a starting point the strictness result of [MST02] for MSO on grids. But from there on, the two proof methods diverge considerably.

Kuusisto’s approach is mainly based on the fact that one can simulate first-order quantifiers by means of set quantifiers, combined with a formula stating that a set is a singleton. For HGS, this results in the mentioned equivalence with MSO, which immediately implies that the hierarchy of HGS must be infinitely ascending. The spirit of his proof remains the same for HS, although the details are much more technical. It is precisely this use of additional second-order quantifiers that leads to the loss of the specific separation results provided by [MST02].

In contrast, one crucial insight will enable us to take full advantage of those results: When restricted to the class of grids, HGS and MSO are more than just equivalent — they are levelwise equivalent, and consequently all the separation results shown for MSO also hold for HGS on grids. This is based on the observation that the existential fragment of HGS can simulate another model, called tiling systems, which has been shown to be equivalent to the existential fragment of MSO in [GRST96]. On the basis of this new finding, we can then transfer the given separation results from HGS on grids to other classes of graphs and other extensions of modal logic, such as HS. While this works along the same general principle as the strong first-order reductions used in [MST02], the additional limitations imposed by modal logic force us to introduce custom encoding techniques that cope with the lack of resources.

1.3 Outline
The remainder of this paper is organized in a top-down manner. After introducing the necessary terminology in Section 2, we present the main results in Section 3, and almost immediately get to the central proof in Section 4. The latter relies on several other propositions, but since those are treated as “black boxes”, the main line of reasoning should be comprehensible without reading any further. We then provide all the missing details in the last two sections, which are independent of each other. Section 5 establishes the levelwise equivalence of three different alternation hierarchies on grids, and may thus be interesting on its own. On the other hand, Section 6 is dedicated to encoding functions, which constitute the more technical part of our demonstration.

2 Notation and Terminology
We begin by defining the basic vocabulary used throughout this paper. It is mostly standard, with one noteworthy exception: We shall not make the usual distinction between variables and (non-logical) constants. Instead,
there is simply a fixed supply of symbols, which can serve both as variables and as constants. The set $S_0$ contains our element symbols, among which there is a special position symbol @. Furthermore, for every positive integer $k$, we let $S_k$ be the set of $k$-ary relation symbols. All of these sets are infinite and pairwise disjoint. We also denote the set of all symbols by $S$, i.e., $S := \bigcup_{k \geq 0} S_k$, and shall often refer to the unary relation symbols in $S_1$ as set symbols.

### 2.1 Structures

Let $\sigma$ be any subset of $S$. A structure $\mathfrak{A}$ of signature $\sigma$ consists of a nonempty set $\text{dom}(\mathfrak{A})$, called the domain of $\mathfrak{A}$, an element $q^\mathfrak{A}$ of $\text{dom}(\mathfrak{A})$ for each element symbol $q$ in $\sigma$, and a $k$-ary relation $R^\mathfrak{A}$ on $\text{dom}(\mathfrak{A})$ for each $k$-ary relation symbol $R$ in $\sigma$. Here, $q^\mathfrak{A}$ and $R^\mathfrak{A}$ are called $\mathfrak{A}$’s interpretations of the symbols $q$ and $R$. We may also say that $\mathfrak{A}$ is a structure over $\sigma$, or that $\sigma$ is the underlying signature of $\mathfrak{A}$, and we denote $\sigma$ by $\text{sig}(\mathfrak{A})$. In case the position symbol @ lies in $\text{sig}(\mathfrak{A})$, we call $\mathfrak{A}$ a pointed structure.

As is customary, we are only interested in structures up to isomorphism. That is, two structures over $\sigma$ are considered to be equal if there is a bijection between their domains that preserves the interpretations of all symbols in $\sigma$.

For convenience, we will often neglect the notational distinction between a structure and its domain. Hence, when we write $A \in \mathfrak{A}$ and $A \subseteq \mathfrak{A}^k$, we mean $A \in \text{dom}(\mathfrak{A})$ and $A \subseteq (\text{dom}(\mathfrak{A}))^k$, respectively.

We use the notation $\mathfrak{A}[S \mapsto \alpha]$ to designate the structure $\mathfrak{A}'$ obtained from $\mathfrak{A}$ by interpreting the symbol $S$ as $\alpha$, where $\alpha \in \mathfrak{A}$ if $S \in S_0$, and $\alpha \subseteq \mathfrak{A}^k$ if $S \in S_k$. More precisely, $\text{dom}(\mathfrak{A}') = \text{dom}(\mathfrak{A})$, $\text{sig}(\mathfrak{A}') = \text{sig}(\mathfrak{A}) \cup \{S\}$, $S^\mathfrak{A'} = \alpha$, and $T^\mathfrak{A'} = T^\mathfrak{A}$ for $T \in \text{sig}(\mathfrak{A}) \setminus \{S\}$. This is generalized to multiple symbols in the obvious way. If the interpretation of $S$ is clear from context, we may refer to the structure above as the S-extended variant of $\mathfrak{A}$.

### 2.2 Different Kinds of Graphs

Our main focus will be on several types of structures with finite domains and relations of arity at most 2. In the following definitions, let $t$ and $u$ be non-negative integers.

A t-bit labeled, u-relational directed graph $\mathcal{D}$ is a finite structure of signature $\{P_1, \ldots, P_t, R_1, \ldots, R_u\}$, where $P_1, \ldots, P_t$ are set symbols, and $R_1, \ldots, R_u$ are binary relation symbols. The class of all such structures is denoted by $\text{DiGraph}[t, u]$. We sometimes refer to $P_1^D, \ldots, P_t^D$ as labeling sets and to $R_1^D, \ldots, R_u^D$ as edge relations. If the latter are all irreflexive and symmetric, then $\mathcal{D}$ is called a t-bit labeled, u-relational undirected graph, and the corresponding class is $\text{Graph}[t, u]$. Furthermore, if $t = 0$ and $u = 1$, we say that $\mathcal{D}$ is simply an (un)directed graph, and use the shorthands $\text{DiGraph} := \text{DiGraph}[0, 1]$ and $\text{Graph} := \text{Graph}[0, 1]$. We shall also drop the subscripts, and just write $P$ or $R$, if there is only one symbol of a given arity.

As can be easily guessed from the previous definitions, a pointed directed graph is a directed graph in which some element has been marked by the position symbol @, i.e., a structure of the form $\mathcal{D}[@ \mapsto d]$, with $\mathcal{D} \in \text{DiGraph}$ and $d \in \mathcal{D}$. We write $\text{PDigraph}$ for the set of all pointed directed graphs.

Finally, we also consider an important subclass of $\text{DiGraph}[t, 2]$, whose members represent rectangular labeled grids (also called pictures). In such a structure $\mathcal{C}$, each element is identified with a grid cell, and the edge relations $R_1^C$ and $R_2^C$ are interpreted as the “vertical” and “horizontal” successor relations, respectively. The unique element that has no predecessor at all is regarded as the “upper-left corner”, and all the usual terminology of matrices applies. Formally, $\mathcal{C}$ is a t-bit labeled grid if, for some $m, n \geq 1$, it is isomorphic to a structure with domain $\{1, \ldots, m\} \times \{1, \ldots, n\}$ and edge relations

$$R_1^C = \{(i, j), (i + 1, j)\} \mid 1 \leq i < m, 1 \leq j \leq n\},$$
$$R_2^C = \{(i, j), (i, j + 1)\} \mid 1 \leq i \leq m, 1 \leq j < n\}.$$

If $t = 0$, we refer to $\mathcal{C}$ simply as a grid. In alignment with the previous nomenclature, we let $\text{Grid}$ and $\text{Grid}[t]$ denote the classes of grids and t-bit labeled grids.

### 2.3 The Considered Logics

As we shall contemplate both classical logic and numerous variants of modal logic, we introduce them all in a common framework. First we define the syntax and semantics of a generalized language, and then we specify which particular syntactic fragments we are interested in.

Table 1 shows how formulas are built up, and what they mean. Furthermore, it indicates how to obtain the set free($\varphi$) of symbols that occur freely in a given formula $\varphi$, i.e., outside the scope of a binding operator. If free($\varphi$) $\subseteq \sigma$, we say that $\varphi$ is a sentence over $\sigma$. The relation $\models$ defined in Table 1 specifies in which cases a structure $\mathfrak{A}$ satisfies $\varphi$, written $\mathfrak{A} \models \varphi$, assuming that $\varphi$ is a sentence over sig($\mathfrak{A}$). Otherwise, we stipulate that $\mathfrak{A} \not\models \varphi$.

Of particular interest for this paper are those formulas in which the element symbol @ is considered to be free, although it might not occur explicitly. They are evaluated on a pointed structure $\mathfrak{A}$ from the perspective of the element @. Atomic formulas of the form $P$ or $R$, with $q \in S_0$ and $P \in S_1$, are satisfied if @ is labeled by the corresponding symbol. Using the operator $\Diamond$, which is called the $R$-diamond, we can remap the symbol @ through existential quantification over the elements in $\mathfrak{A}$ that are reachable from @ through the relation $R$. If we want to do the same with respect to the inverse relation of $R$, we can use the backward $R$-diamond $\Diamond$. In addition,
Here, \( p, q, q_0, \ldots, q_k \in S_0, P \in S_1, R \in S_{k+1} \), and \( \varphi, \varphi_1, \ldots, \varphi_k \) are formulas, for \( k \geq 1 \).

**Table 1.** Syntax and semantics of the considered logics.

| Syntax | Free symbols | Semantics |
|--------|--------------|------------|
| Formula \( \psi \) | Symbol set \( \text{free}(\psi) \) | Necessary and sufficient condition for \( \mathcal{A} \models \psi \) |
| \( q \) | \{ @, q \} | \( @^a = q^a \) |
| \( (p \models q) \) | \{ p, q \} | \( p^a = q^a \) |
| \( p \) | \{ @, P \} | \( @^a \in P^a \) |
| \( P(q) \) | \{ q, P \} | \( q^a \in P^a \) |
| \( R(q_0, \ldots, q_k) \) | \{ q_0, \ldots, q_k, R \} | \( (q_0^a, \ldots, q_k^a) \in R^a \) |
| \( \neg \varphi \) | \text{free}(\varphi) \) | \( \mathcal{A} \models \varphi \) |
| \( (\varphi_1 \lor \varphi_2) \) | \text{free}(\varphi_1) \cup \text{free}(\varphi_2) \) | \( \mathcal{A} \models \varphi_1 \) or \( \mathcal{A} \models \varphi_2 \) |
| \( \Box (\varphi_1, \ldots, \varphi_k) \) | \{ @, R \} \cup \bigcup_{1 \leq i \leq k} \text{free}(\varphi_i) \) | For some \( a_1, \ldots, a_k \in \mathcal{A} \) such that \( (@^a, a_1, \ldots, a_k) \in R^a \), we have \( \mathcal{A}[\@ \mapsto a_i] \models \varphi_i \) for each \( i \in \{1, \ldots, k\} \). |
| \( \Box (\varphi_1, \ldots, \varphi_k) \) same as above | As above, except for the condition \( (a_k, \ldots, a_1, c^a) \in R^a \). |
| \( \Diamond \varphi \) | \text{free}(\varphi) \setminus \{ @ \} \) | \( \mathcal{A}[\@ \mapsto a] \models \varphi \) for some \( a \in \mathcal{A} \) |
| \( \exists q \varphi \) | \text{free}(\varphi) \setminus \{ q \} \) | \( \mathcal{A}[q \mapsto a] \models \varphi \) for some \( a \in \mathcal{A} \) |
| \( \exists p \varphi \) | \text{free}(\varphi) \setminus \{ P \} \) | \( \mathcal{A}[P \mapsto A] \models \varphi \) for some \( A \in \mathcal{A} \) |

Here, \( p, q, q_0, \ldots, q_k \in S_0, P \in S_1, R \in S_{k+1} \), and \( \varphi, \varphi_1, \ldots, \varphi_k \) are formulas, for \( k \geq 1 \).

**Table 2.** Languages of the considered logics.

| Language | Generating grammar |
|----------|--------------------|
| \( H \) basic Hybrid | \( \varphi ::= q \mid P \mid \neg \varphi \mid (\varphi_1 \lor \varphi_2) \mid \Box (\varphi_1, \ldots, \varphi_k) \) |
| \( HB \) Hybrid with Backward modalities | \( \varphi ::= q \mid P \mid \neg \varphi \mid (\varphi_1 \lor \varphi_2) \mid \Box (\varphi_1, \ldots, \varphi_k) \mid \Box (\varphi_1, \ldots, \varphi_k) \) |
| \( HG \) Hybrid with Global modalities | \( \varphi ::= q \mid P \mid \neg \varphi \mid (\varphi_1 \lor \varphi_2) \mid \Box (\varphi_1, \ldots, \varphi_k) \mid \Diamond \varphi \) |
| \( HS \) Hybrid with Set quantifiers | \( \varphi ::= q \mid P \mid \neg \varphi \mid (\varphi_1 \lor \varphi_2) \mid \Box (\varphi_1, \ldots, \varphi_k) \mid \exists p \varphi \) |
| \( HBG, HBS, HGS \) and \( HBGS \) analogous to the preceding grammars |
| \( FO \) First-Order | \( \varphi ::= (p \models q) \mid P(q) \mid R(q_0, \ldots, q_k) \mid \neg \varphi \mid (\varphi_1 \lor \varphi_2) \mid \exists q \varphi \) |
| \( MSO \) Monadic Second-Order | \( \varphi ::= (p \models q) \mid P(q) \mid R(q_0, \ldots, q_k) \mid \neg \varphi \mid (\varphi_1 \lor \varphi_2) \mid \exists q \varphi \mid \exists p \varphi \) |

Here, \( p, q, q_0, \ldots, q_k \in S_0, P \in S_1, \) and \( R \in S_{k+1} \), for \( k \geq 1 \).
there is also the *global diamond* ◊ (unfortunately often called “universal modality”), which ranges over all elements of $\mathfrak{A}$. It can be considered as the diamond operator corresponding to the edge relation of the complete graph over $\text{dom}(\mathfrak{A})$. To facilitate certain descriptions, we shall sometimes treat $\otimes$ and ◊ as (extended) special cases of $\otimes$, assuming that they are implicitly associated with $\otimes$ and ◊ as (extended) special cases of $\otimes$, respectively. These symbols do not belong to $\mathfrak{S}$, and therefore cannot be interpreted by any structure.

Allowing a bit of syntactic sugar, we will make liberal use of the remaining operators of predicate logic, i.e., $\land$, $\to$, $\leftrightarrow$, $\forall$, and we may leave out some parentheses, assuming that $\lor$ and $\land$ take precedence over $\to$ and $\leftrightarrow$. Furthermore, we define the abbreviations

$$
\top := @, \bot := \neg @ \text{ and } \mathbb{R}(\varphi_1, \ldots, \varphi_k) := \neg \Box (\neg \varphi_1, \ldots, \neg \varphi_k).
$$

Note that the first line makes sense because, by definition, the atomic formula $@$ is always satisfied at the point of evaluation. Also, the second line remains applicable if one substitutes $\mathbb{R}^{-1}$ or $\mathbb{T}$, for $\mathbb{R}$. The resulting operators $\mathbb{R}$, $\mathbb{R}^{-1}$ and ◊ provide universal quantification and are called *boxes* (using the same attributes as for diamonds). Diamonds and boxes are collectively referred to as *modalities* or *modal operators*. In case we restrict ourselves to structures that only have a single relation, we may omit the relation symbol $\mathbb{R}$, and just use empty modalities such as ◊. Similarly, if the relation symbols involved are indexed, like $R_1, \ldots, R_u$, we associate them with modalities of the form $\Box_i$, for $1 \leq i \leq u$.

Let us now turn to the specific classes of formulas considered in this article, which are presented in Table 2. The first-order (FO) and monadic second-order (MSO) languages are defined in the usual way. The remaining classes can all be qualified as modal languages, insofar as they include modal operators, but not the classical first-order quantifiers. We refer to them as *hybrid* languages because, unlike basic modal logic, they also provide element symbols (or “nominals”). This is consistent with the terminology of the modal logic community (see, e.g., [AC06] or [BRV02]). However, there does not seem to be any established alphabetical nomenclature that is both concise and easily extensible to fit our purposes. Therefore, we introduce the following system: Starting with the letter $\mathbb{H}$, for “hybrid”, we add $B$ or $G$ if we want to include backward or global modalities, respectively. In the same manner, the letter $S$ gives us the key ingredient investigated in this paper — namely, set quantifiers. With this, the classes SOPML and SOPMLE of [Kuu15] become HS and HGS.

For any set of formulas $\Phi$ (e.g., HGS), we shall refer to its members as $\Phi$-*formulas*. Given such a $\Phi$-formula $\varphi$ and a class of structures $\mathcal{C}$ (e.g., Digraph), we use the semantic bracket notations $[\varphi]_c$ and $[\Phi]_c$ to denote the set of structures defined by $\varphi$ over $\mathcal{C}$, and the family of sets definable in $\Phi$ over $\mathcal{C}$. More formally,

$$
[\varphi]_c := \{ \mathfrak{A} \in \mathcal{C} \mid \mathfrak{A} \models \varphi \}, \text{ and } [\Phi]_c := \{ [\varphi]_c \mid \varphi \in \Phi \}.
$$

If $\mathcal{C}$ is equal to the set of all structures, we omit the subscript and simply write $[\varphi]$ and $[\Phi]$. Similarly, we use $[\varphi]_c := \{ \psi \mid [\psi]_c = [\varphi]_c \}$ for the equivalence class of $\varphi$ over $\mathcal{C}$, and $[\Phi]_c := \bigcup_{\varphi \in \Phi} [\varphi]_c$ for the set of all formulas that are equivalent over $\mathcal{C}$ to some formula in $\Phi$. We may again drop the subscript if we do not want to restrict to a particular class of structures.

### 2.4 A Useful Example

As we do not allow first-order quantification in our hybrid formulas, some properties that seem very natural in FO become rather cumbersome to express. Nevertheless, translation from FO to HGS is always possible because we can simulate first-order quantifiers by set quantifiers relativized to singletons, which, by extension, also leads to the equivalence between SO and HGS. With this in mind, let us consider the following formula schema, where $X \in S_1$, $R \in S_2$, and $\varphi$ can be any HBG-formula:

$$
\text{see1}_R(\varphi) := \Box \varphi \land \forall X (\Box (\varphi \land X) \to \Box (\varphi \to X)).
$$

When evaluated on a pointed structure $\mathfrak{A}$ whose signature includes $\{@, R \} \cup \text{free}(\varphi)$, the formula $\text{see1}_R(\varphi)$ states that there is exactly one element $a \in \mathfrak{A}$ reachable from $@\mathfrak{A}$ through an $R^3$-edge, such that $\varphi$ is satisfied at $a$ (i.e., by the structure $\mathfrak{A}[@ \to a]$). In the context of 1-relational graphs, we may use the shorthand $\text{see1}(\varphi)$ to invoke this schema. Coming back to the original motivation, we also define

$$
\text{tot1}(\varphi) := \text{see1}_\mathbb{T}(\varphi),
$$

which states that there is precisely one element in the entire structure $\mathfrak{A}$ at which $\varphi$ is satisfied. Here, $\mathfrak{A}$ does not necessarily have to be pointed, and, of course, $\text{sig}(\mathfrak{A})$ never contains $\mathbb{T}$.

Anticipating the notation of the next subsection, the formulas obtained by this construction can be classified as $[\Pi_1(\Phi)]$-formulas, where $\Phi \in \{H, HB, HG, HBG\}$ depends on the specific modalities that we use.

### 2.5 Alternation Hierarchies

We now come to our primary objects of interest. Assume we are given some set of formulas $\Phi$, referred to as *kernel*, which is free of set quantifiers and closed under negation (e.g., HG). Then, for $\ell \geq 0$, the class $\Sigma_\ell(\Phi)$ consists of those formulas that one can construct by taking a
member of Φ and prepending to it at most ℓ consecutive blocks of set quantifiers, alternating between existential and universal blocks, such that the first block is existential. Reformulating this solely in terms of existential quantifiers and negations, we get

\[
\Sigma_0(\Phi) := \Phi \quad \text{and} \quad \Sigma_{\ell+1}(\Phi) := \{ \exists \varphi \mid P \in \mathbb{S}_1 \} \cdot \{ \neg \varphi \mid \varphi \in \Sigma_{\ell}(\Phi) \},
\]

where the second line uses set concatenation and the Kleene star. We define Π_ℓ(Φ) as the corresponding dual class, i.e., the set of all negations of formulas in Σ_ℓ(Φ). Generalizing this to arbitrary Boolean combinations, let BC Σ_ℓ(Φ) denote the smallest superclass of Σ_ℓ(Φ) that is closed under negation and disjunction.

The formulas in Σ_ℓ(Φ) and Π_ℓ(Φ) are said to be in \textit{prenex normal form} with respect to the kernel Φ. It is well known that every MSO-formula can be transformed into prenex normal form with kernel class FO. This is based on the observation that first-order quantifiers can be replaced by second-order ones. Using the construction of Section 2.4, it is not difficult to see that the analogue holds for HS, HBS, HGS and HBGS with respect to their corresponding kernel classes. A more elaborate explanation can be found in [Cat06, Prp. 3].

For the sake of clarity, we break with the tradition of implicit quantification that is customary in modal logic. Instead of evaluating HS-formulas on non-pointed structures by means of “hidden” universal quantification, we shall explicitly put a global box in front of our formulas. This leads to the class

\[ \square \Sigma_\ell(\mathcal{H}) := \{ \square \cdot \Sigma_\ell(\mathcal{H}) \}. \]

Analogously, we also define \[ \square \Pi_\ell(\mathcal{H}). \]

All of our results will be stated in terms of the semantic classes that one obtains by evaluating the preceding formula classes on some set of structures \( \mathcal{E} \). On the semantic side, we will additionally consider the class

\[ [\Delta_\ell(\Phi)]_{\mathcal{E}} := [\Sigma_\ell(\Phi)]_{\mathcal{E}} \cap [\Pi_\ell(\Phi)]_{\mathcal{E}}. \]

Since it is not based on any syntactic counterpart, there is no meaning attributed to the notation \( \Delta_\ell(\Phi) \) by itself (without the brackets).

### 3 Main Results

With the notation in place, we are ready to formally enunciate the main theorem, whose complete proof will be the subject of the remainder of this paper. It is an extension to modal kernel formulas of the following result of Matz, Schweikardt and Thomas, which can be obtained by combining [MST02, Thm. 1] and [Mat02, Thm. 2.26]¹:

\[ \begin{align*}
\text{Theorem 1 (Matz, Schweikardt, Thomas).} \\
\quad & \text{The set quantifier alternation hierarchy of MSO is strict over the classes of grids, directed graphs and undirected graphs.} \\
\quad & \text{A more precise statement of this theorem, referred to as Theorem 1 (a) and (b), is given in Table 3.} \quad \blacktriangleleft
\end{align*} \]

Roughly speaking, the extension provided in the present paper tells us that the preceding separations are largely maintained if we replace the first-order kernel by certain classes of modal formulas. To facilitate comparisons, the formal statements of both theorems are presented together in the same table.

\[ \text{Theorem 2 (Main Results).} \quad \blacktriangleleft \]

The set quantifier alternation hierarchies of HBGS and HGS are strict over the classes of grids, directed graphs and 1-bit labeled undirected graphs.

Furthermore, the corresponding hierarchies of HS and \( \Box \text{HS} \) are (mostly) strict over the classes of pointed directed graphs and directed graphs, respectively.

\[ \text{A more precise statement of this theorem, referred to as Theorem 2 (a), (b), (c) and (d), is given in Table 3.} \quad \blacktriangleleft \]

By basic properties of predicate logic² and the transitivity of set inclusion, it is easy to infer from Theorem 2 the hierarchy diagrams represented in Figures 1 and 2.

If we take into account all the depicted relations, the diagram in Figure 1 is the same as in [MST02] and [Mat02]. Hence, when switching to one of the modal kernels that include global modalities, i.e., HBG or HG, the separations of Theorem 1 are completely preserved on grids and directed graphs. Our proof method also allows us to easily transfer this result to undirected graphs, as long as we admit that the vertices may be labeled with at least one bit. Additional work would be required to eliminate this condition.

As a spin-off, Theorem 2 also provides an extension of some of these separations to \( H \), a kernel class without global modalities. Following [Kuu08, Kuu15], we consider the alternation hierarchies of both HS and \( \Box \text{HS} \). For the former, which is evaluated on pointed directed graphs, Figure 1 gives a detailed picture, leaving open only whether the inclusion \( [\Box \Sigma_\ell(\Phi)]_{\mathcal{E}} \subseteq [\Delta_\ell+1(\Phi)]_{\mathcal{E}} \) is proper. Inferring the strictness of this inclusion from the preceding results does not seem very difficult, but would call for a generalization of our framework. In contrast, the second hierarchy based on H is arguably less natural, since every \( \Box \text{HS} \)-formula is prefixed by a global box, regardless of the occurring set quantifiers. This creates a certain asymmetry between the \( \Sigma_{\ell-1} \) and \( \Pi_{\ell} \)-levels, which becomes apparent.

¹ [Mat02, Thm. 2.26] states that \( [\Sigma_\ell(\Phi)]_{\mathcal{E}} \not\subseteq [\Pi_\ell(\Phi)]_{\mathcal{E}} \), which, by duality, also implies \( [\Sigma_\ell(\Phi)]_{\mathcal{E}} \not\subseteq [\Pi_\ell(\Phi)]_{\mathcal{E}} \).

² In particular, the inclusion \( [\Box \Sigma_\ell(\Phi)]_{\mathcal{E}} \subseteq [\Delta_\ell+1(\Phi)]_{\mathcal{E}} \) follows from the fact that, when transforming a Boolean combination of \( \Sigma_\ell(\Phi) \)-formulas into prenex normal form, one is free to choose whether the resulting formula (with up to \( \ell + 1 \) quantifier alternations) should start with an existential or a universal quantifier.
Theorem 2 (d) for Figure 2. The set quantifier alternation hierarchies established by Theorem 2 (a), (b) and (c). If we include the noninclusion in parentheses, this diagram holds for \( \Phi \in \{ \text{HBG, HG} \} \) and \( \mathcal{C} \in \{ \text{GRID, DIGRAPH, GRAPH[1, 1]} \} \). If we ignore that noninclusion, it is also verified for \( \Phi = H \) and \( \mathcal{C} = \text{PDIGRAPH} \). In both cases, we assume \( \ell \geq 1 \).

when considering the missing relations in Figure 2. Unlike for the other hierarchies, one cannot simply argue by duality to deduce from \( [\square \Sigma_\ell(\Phi)]_e \not\subseteq [\square \Pi_\ell(\Phi)]_e \) that the converse noninclusion also holds. Nevertheless, the presented result is strong enough to completely settle the specific strictness question mentioned in [Kuu08], and left unanswered in [Kuu15]: For arbitrarily high \( \ell \), we have

\[
[\square \Sigma_\ell(H)]_{\text{DIGRAPH}} \not\supseteq [\square \Sigma_{\ell+1}(H)]_{\text{DIGRAPH}}.
\]

### Table 3. The specific separation results of Theorems 1 and 2.

| Separation result | Kernel Class | Structures | Levels | Theorem |
|-------------------|--------------|------------|--------|---------|
| \( [\Delta_{\ell+1}(\Phi)]_e \not\subseteq [\text{BC} \Sigma_\ell(\Phi)]_e \) | FO | GRID, DIGRAPH, GRAPH | 1 | 1 (a) * |
| \( [\Sigma_\ell(\Phi)]_e \not\subseteq [\Pi_\ell(\Phi)]_e \) | FO | GRID, DIGRAPH, GRAPH[1, 1] | 2 | 1 (b) * |
| \( [\square \Sigma_\ell(\Phi)]_e \not\subseteq [\square \Pi_\ell(\Phi)]_e \) | H | DIGRAPH | 2 | 2 (d) |

Theorem 1 (marked by asterisks for better visibility) is due to Matz, Schweikardt and Thomas.

### Figure 2. The set quantifier alternation hierarchy implied by Theorem 2 (d) for \( \Phi = H \), \( \mathcal{C} = \text{DIGRAPH} \) and \( \ell \geq 2 \).

### 4 Top-Level Proofs

In accordance with our top-down approach, the present section already provides the proof of our main theorem, where everything comes together. It therefore acts as a gateway to the sections with the technical parts, especially Section 6.

#### 4.1 Figurative Inclusions

First of all, we need to introduce the primary tool with which we will transfer separation results from one setting to another. It can be seen as an abstraction of the strong first-order reductions used in [MST02]. Unlike the latter, it is formulated independently of any logical language, which allows us to postpone the technical details to the end of the paper.

**Definition 3** (Figurative Inclusion).

- Consider two sets \( \mathcal{C} \) and \( \mathcal{D} \) and a partial injective function \( \mu: \mathcal{C} \rightarrow \mathcal{D} \). For any two families of subsets \( \mathcal{L} \subseteq 2^\mathcal{C} \) and \( \mathcal{M} \subseteq 2^\mathcal{D} \), we say that \( \mathcal{L} \) is forward included in \( \mathcal{M} \) if, for every set \( L \in \mathcal{L} \), there is a set \( M \in \mathcal{M} \) such that \( \mu(M) = M \cap \mu(\mathcal{C}) \).

Figuratively speaking, the partial bijection \( \mu \) creates a tunnel between \( \mathcal{C} \) and \( \mathcal{D} \), and all the sets in \( \mathcal{L} \) and \( \mathcal{M} \) are cropped to fit through that tunnel. Two original sets
are considered to be equal if their cropped versions are mapped onto each other by $\mu$.

We also define the shorthands $\overrightarrow{\subseteq}_{D\mu}$ and $\equiv_{D\mu}$ as natural extensions of the previous notation: $L \overrightarrow{\subseteq}_{D\mu} M$, which is defined as $M \subseteq_{D\mu-1} L$, means that $M$ is \textit{backward included} in $L$ in the figure $D\mu$, and $L \equiv_{D\mu} M$, an abbreviation for the conjunction of $L \overrightarrow{\subseteq}_{D\mu} M$ and $L \overrightarrow{\supseteq}_{D\mu} M$, states that $L$ is \textit{forward equal} to $M$ in the figure $D\mu$. All of these relations are referred to as \textit{figurative inclusions}.

Note that ordinary inclusion is a special case of figurative inclusion, i.e., for $\mathcal{C} = D$, $L \subseteq M$ if and only if $L \overrightarrow{\subseteq}_{\text{id}_D} M$.

Furthermore, figurative inclusion is transitive in the sense that

\[ L \overrightarrow{\subseteq}_{D\mu} M \subseteq_{D\nu} N \quad \text{implies} \quad L \overrightarrow{\subseteq}_{D\nu\mu} N. \]

(This depends crucially on the fact that $\nu$ is injective.)

In our specific context, given a noninclusion $[\Phi_2]_{\mathcal{C}} \not\subseteq [\Phi_1]_{\mathcal{C}}$, we shall use the concept of figurative inclusion to infer from it another noninclusion $[\Psi_2]_{\mathcal{D}} \not\subseteq [\Psi_1]_{\mathcal{D}}$. Here, $\Phi_1, \Phi_2, \Psi_1, \Psi_2$ and $\mathcal{C}, \mathcal{D}$ refer to some classes of formulas and structures, respectively. The key part of the argument will be to construct an appropriate encoding function $\mu: \mathcal{C} \to \mathcal{D}$, in order to apply the following lemma.

**Lemma 4.**

- Let $L_1, L_2 \subseteq 2^c$ and $M_1, M_2 \subseteq 2^d$ be families of subsets of some sets $\mathcal{C}$ and $\mathcal{D}$. If there is a total injective function $\mu: \mathcal{C} \to \mathcal{D}$ such that $L_2 \overrightarrow{\subseteq}_{D\mu} M_2$ and $L_1 \overrightarrow{\subseteq}_{D\mu} M_1$, then

  \[ L_2 \not\subseteq L_1 \quad \text{implies} \quad M_2 \not\subseteq M_1. \]

**Proof.** To show the contrapositive, let us suppose that $M_2 \subseteq M_1$, or, equivalently, $M_2 \overrightarrow{\subseteq}_{\text{id}_D} M_1$. Then the chain of figurative inclusions

\[ L_2 \overrightarrow{\subseteq}_{D\mu} M_2 \overrightarrow{\subseteq}_{\text{id}_D} M_1 \subseteq_{D\mu-1} L_1 \]

yields $L_2 \overrightarrow{\subseteq}_{\text{id}_D} L_1$, since $(\mu^{-1} \circ \text{id}_D \circ \mu) = \text{id}_c$. (This depends on $\mu$ being total and injective.) Consequently, we have $L_2 \subseteq L_1$.

In some cases, we can combine two given figurative inclusions in order to obtain a new one that relates the corresponding intersection classes. This property will be very useful for establishing figurative inclusions between classes of the form $[\Delta_1(\Phi)]_{\mathcal{C}}$.

**Lemma 5.**

- Consider two sets $\mathcal{C}$ and $\mathcal{D}$, a partial injective function $\mu: \mathcal{C} \to \mathcal{D}$, and four families of subsets $L_1, L_2 \subseteq 2^c$ and $M_1, M_2 \subseteq 2^d$. If $\mu(\mathcal{C})$ is a member of $M_1 \cap M_2$, and $M_1, M_2$ are both closed under intersection, then

  \[ L_1 \overrightarrow{\subseteq}_{D\mu} M_1 \quad \text{and} \quad L_2 \overrightarrow{\subseteq}_{D\mu} M_2 \]

  imply

  \[ L_1 \cap L_2 \overrightarrow{\subseteq}_{D\mu} M_1 \cap M_2. \]

**Proof.** Let $L$ be any set in $L_1 \cap L_2$. Since $L_1 \overrightarrow{\subseteq}_{D\mu} M_1$, there is, by definition, a set $M$ in $M_1$ such that $\mu(L) = M \cap \mu(\mathcal{C})$. Furthermore, we also know that $\mu(\mathcal{C})$ lies in $M_1$, and that the latter is closed under intersection. Hence, $\mu(L) \in M_1$. Analogously, we also get that $\mu(L) \in M_2$. Finally, knowing that for all $L$ in $L_1 \cap L_2$, $\mu(L)$ lies in $M_1 \cap M_2$, we obviously have a sufficient condition for $L_1 \cap L_2 \overrightarrow{\subseteq}_{D\mu} M_1 \cap M_2$.

\[ \square \]

### 4.2 Proving the Main Theorem

We are now ready to give the central proof of this paper. Although it makes references to many statements of Sections 5 and 6, it is formulated in a way that can be understood without having read anything beyond this point.

**Proof of Theorem 2.** The basis of our proof shall be laid in Section 5, where the case $\ell = 0$ of Theorem 9 will state the following: When restricted to the class of grids, the set quantifier alternation hierarchies of MSO, HBGS and HGS are equivalent. More precisely, for every $\ell \geq 1$ and $\Xi \in \{\Sigma, \Pi, \text{BC} \Sigma, \Delta\}$, it holds that

\[ [\Xi(\text{FO})]_{\text{Grid}} = [\Xi(\text{HBG})]_{\text{Grid}} = [\Xi(\text{HG})]_{\text{Grid}}. \]

Hence, if we consider only the case $\mathcal{C} = \text{Grid}$, the separation results for the kernel class FO stated in Theorem 1 (a) and (b) immediately imply those for HBG and HG in Theorem 2 (a) and (b).

The remainder of the proof now consists of establishing suitable figurative inclusions, in order to transfer these results to other classes of structures and, to some extent, to weaker classes of kernel formulas. For this purpose, we shall introduce in Section 6 a notion of translatability between two classes of kernel formulas $\Phi$ and $\Psi$, with respect to a given total injective function $\mu$ that encodes structures from a class $\mathcal{C}$ into structures of some class $\mathcal{D}$. As will be shown in Lemma 14, bidirectional translatability implies

\[ [\Xi(\Phi)]_{\mathcal{C}} \equiv_{\mu} [\Xi(\Psi)]_{\mathcal{D}} \]

for all $\Xi \in \{\Sigma, \Pi, \text{BC} \Sigma, \Delta\}$ with $\ell \geq 0$. If we can additionally show that $\mu(\mathcal{C})$ is (at most) $\Delta_2(\Psi)$-definable over $\mathcal{D}$, then, by Lemma 5, the figurative equality (*) also holds for $\Xi = \Delta_{\ell+1}$ with $\ell \geq 1$. Note that the backward part “$\overrightarrow{\subseteq}_{D\mu}$” is always true, since $\mu^{-1}(\mathcal{D})$ is trivially $\Delta_2(\Phi)$-definable over $\mathcal{C}$.

The groundwork being in place, we proceed by applying Lemma 4 as follows:

- If we have established (*) for $\Xi \in \{\Sigma, \Pi, \text{BC} \}$, then we can transfer the separation

  \[ [\Sigma(\ell)(\Phi)]_{\mathcal{C}} \equiv_{\mu} [\Pi(\ell)(\Phi)]_{\mathcal{D}} \]

  to the kernel class $\Psi$ evaluated on the class of structures $\mathcal{D}$.
The injective function $\mu_5$, which satisfies the translatability property required to obtain this figurative equality via Lemma 14, will be provided by Proposition 19. Its image $\mu_5(\text{DIGRAPH})$ is not HS-definable, for the simple reason that an HS-formula is unable to distinguish between two structures that are isomorphic when restricted to the connected component containing the position marker @. Hence, we cannot merely apply Lemma 5 to show (2). Our approach would have to be refined to take into account equivalence classes of structures, which we shall not do in this paper.

(d) Finally, Proposition 19 will also state that $\mu_5$ can be converted into an encoding $\mu'_5$, from DIGRAPH back into DIGRAPH, that satisfies the following figurative inclusions for all $\ell > 2$:

\[
\begin{align*}
\llbracket \Sigma_\ell(HG) \rrbracket_{\text{DIGRAPH}} & \subseteq \mu'_5 \llbracket \llbracket \square \Sigma_\ell(H) \rrbracket_{\text{DIGRAPH}} \rrbracket_{\text{DIGRAPH}} \\
\llbracket \Pi_\ell(HG) \rrbracket_{\text{DIGRAPH}} & \subseteq \mu'_5 \llbracket \llbracket \square \Pi_\ell(H) \rrbracket_{\text{DIGRAPH}} \rrbracket_{\text{DIGRAPH}}.
\end{align*}
\]

Using (1) for HG on directed graphs, and applying Lemma 4, we can infer from this that

\[
\llbracket \llbracket \square \Sigma_\ell(H) \rrbracket_{\text{DIGRAPH}} \rrbracket_{\text{DIGRAPH}} \not\subseteq \llbracket \llbracket \square \Pi_\ell(H) \rrbracket_{\text{DIGRAPH}} \rrbracket_{\text{DIGRAPH}}.
\]

5 Grids as a Starting Point

In this section, we establish that the set quantifier alternation hierarchies of MSO, HBGS and HGS are equivalent on labeled grids. In addition, we give a $\llbracket \Pi_\ell(HBG) \rrbracket$-formula that characterizes the class of grids.

5.1 The Standard Translation

Our first building block is a well-known property of modal logic, which holds even if we do not confine ourselves to the setting of grids.

Proposition 6.

For every HBG-formula, there is an equivalent FO-formula, i.e.,

\[
\llbracket \text{HBG} \rrbracket \subseteq \llbracket \text{FO} \rrbracket.
\]

Proof. Given an HBG-formula $\varphi$, we have to construct an FO-formula $\psi_\varphi$ such that $\mathcal{A} \models \varphi$ if and only if $\mathcal{A} \models \psi_\varphi$, for every structure $\mathcal{A}$. This is simply a matter of transcribing the semantics of HBG given in Table 1 to the language of first-order logic, a method known as the standard translation in modal logic (see, e.g., [BRV02, Def. 2.45]). The following table gives a recursive specification of this translation.
Here, \( q \in S_0 \), \( Q \in S_1 \), \( R \in S_{k+1} \), \( \varphi_1 \ldots \varphi_k \in \text{HBG} \), for \( k \geq 1 \), and \( x_1 \ldots x_k \) are element symbols, chosen such that \( x_i \notin \text{free}(\psi_{\varphi_i}) \). The notation \( \psi_{\varphi_i}(\@ \rightarrow x_i) \) designates the formula obtained by substituting each free occurrence of \( @ \) in \( \psi_{\varphi_i} \) by \( x_i \).

\[ \varphi \in \text{HBG} \quad \text{Equivalent formula } \psi_{\varphi} \in \text{FO} \]

\[
\begin{array}{ll}
q & @ \equiv q \\
\neg \varphi & \neg \psi_{\varphi} \\
\varphi_1 \lor \varphi_2 & \psi_{\varphi_1} \lor \psi_{\varphi_2} \\
\Theta(\varphi_1, \ldots, \varphi_k) & \exists_{x_1 \ldots x_k} (R(\@, x_1 \ldots x_k) \land \bigwedge_{1 \leq i \leq k} \psi_{\varphi_i}(\@ \rightarrow x_i)) \\
\bigotimes(\varphi_1, \ldots, \varphi_k) & \text{as above, except } R(x_k \ldots x_1, \@) \\
\otimes \varphi_1 & \exists_{\psi_{\varphi_1}}
\end{array}
\]

5.2 A Detour through Tiling Systems

By restricting our focus to the class of labeled grids, we can take advantage of a well-studied automaton model introduced by Giammarresi and Restivo in [GR92], which is closely related to MSO. A “machine” in this model, called a tiling system, is defined as a tuple \( \mathcal{T} = (\Gamma, \Omega, \Theta) \), where

- \( \Gamma = \{0, 1\}^4 \) is seen as an alphabet, with \( t \geq 0 \),
- \( \Omega \) is a finite set of states, and
- \( \Theta \subseteq ((\Gamma \times \Omega) \cup \{\#\})^4 \) is a set of 2×2-tiles that may use a fresh symbol \# not contained in \((\Gamma \times \Omega)\).

For a fixed number of bits \( t \), we denote by \( \text{TS}_t \) the set of all tiling systems with alphabet \( \Gamma = \{0, 1\}^t \).

Given a \( t \)-bit labeled grid \( \mathcal{C} \), a tiling system \( \mathcal{T} \in \text{TS}_t \) operates similarly to a nondeterministic finite automaton generalized to two dimensions. A run of \( \mathcal{T} \) on \( \mathcal{C} \) is an extended labeled grid \( \mathcal{C}^{\#} \), obtained by nondeterministically labeling each cell of \( \mathcal{C} \) with some state \( \omega \in \Omega \) and surrounding the entire grid with a border consisting of new \#-labeled cells. We consider \( \mathcal{C}^{\#} \) to be a valid run if each of its 2×2-subgrids can be identified with some tile in \( \Theta \). The set recognized by \( \mathcal{T} \) consists precisely of those labeled grids for which such a run exists. By analogy with our existing notation, we write \([TS_t] \) for the class formed by the sets of \( t \)-bit labeled grids that are recognized by some tiling system in \( TS_t \).

Exploiting a locality property of first-order logic, Giammarresi, Restivo, Seibert and Thomas have shown in [GRST06] that tiling systems capture precisely the existential fragment of MSO on labeled grids:

**Theorem 7** (Giammarresi, Restivo, Seibert, Thomas).

- For arbitrary \( t \geq 0 \), a set of \( t \)-bit labeled grids is \( \text{TS}-\)recognizable if and only if it is \( \Sigma_1(\text{FO}) \)-definable over \( \text{Grid}[t] \), i.e.,
  \[
  [TS_t] = [\Sigma_1(\text{FO})]_{\text{Grid}[t]}.
  \]

The preceding result is extremely useful for our purposes, because, from the perspective of modal logic, it provides a much easier access to MSO. This brings us to the following proposition.

**Proposition 8.**

- For arbitrary \( t \geq 0 \), if a set of \( t \)-bit labeled grids is \( \text{TS}-\)recognizable, then it is also \( \Sigma_1(\text{HG}) \)-definable over \( \text{Grid}[t] \), i.e.,
  \[
  [TS_t] \subseteq [\Sigma_1(\text{HG})]_{\text{Grid}[t]}.
  \]

**Proof.** Let \( \mathcal{T} = (\Gamma, \Omega, \Theta) \) be a tiling system with alphabet \( \Gamma = \{0, 1\}^t \). We have to construct a \( \Sigma_1(\text{HG}) \)-sentence \( \varphi_T \) over the signature \( \{P_1, \ldots, P_5, R_1, R_2\} \), such that each labeled grid \( \mathcal{C} \in \text{Grid}[t] \) satisfies \( \varphi_T \) if and only if it is accepted by \( \mathcal{T} \).

The idea is standard: We represent the states of \( \mathcal{T} \) by additional set symbols \( (X_\omega)_{\omega \in \Omega} \), and our formula asserts that there exists a corresponding partition of \( \text{dom}(\mathcal{C}) \) into \( |\Omega| \) subsets that represent a run \( \mathcal{C}^{\#} \) of \( \mathcal{T} \) on \( \mathcal{C} \). To verify that it is indeed a valid run, we have to check that each 2×2-subgrid of \( \mathcal{C}^{\#} \) corresponds to some tile

\[
\theta = \begin{bmatrix}
\theta_1 & \theta_2 \\
\theta_3 & \theta_4
\end{bmatrix}
\]

in \( \Theta \). If the entry \( \theta_1 \) is different from \( \# \), we can easily write down an H-formula \( \varphi_0 \) that checks at a given position \( c \in \mathcal{C} \), whether the 2×2-subgrid of \( \mathcal{C}^{\#} \) with upper-left corner \( c \) matches \( \theta \). Here, \( \theta_1 \) is chosen as the representative entry of \( \theta \), because the upper-left corner of the tile can “see” the other elements by following the directed \( R_1 \)- and \( R_2 \)-edges. Otherwise, if \( \theta_1 \) is equal to \( \# \), there is no such element \( c \), since \( \mathcal{C} \) does not contain special border elements. However, we can always choose some other entry \( \theta_1 \), different from \( \# \), to be the representative of \( \theta \), and write a formula \( \varphi_0 \) describing the tile from the point of view of an element corresponding to \( \theta_1 \). This choice is never arbitrary, because the representative must be able to “see” the other non-\( \# \) entries of the tile. Consequently, we divide \( \Theta \) into four disjoint sets \( \Theta_1, \Theta_2, \Theta_3, \Theta_4 \), such that \( \Theta_1 \) contains those tiles \( \theta \) that are represented by their entry \( \theta_1 \). In order to facilitate the subsequent formalization, we further subdivide each set into partitions according to the \#-borders that occur within the tiles: \( \Theta_M \) contains the “middle tiles” (all entries different from \( \# \), \( \Theta_L \) the “left tiles” (with \( \theta_1 \) and \( \theta_3 \) equal to \( \# \)), \( \Theta_R \) the “bottom-right tiles”, and so forth . . . Altogether, \( \Theta \) is partitioned into nine subsets, grouped into four types:

- \( \Theta_1 = \Theta_M \cup \Theta_B \cup \Theta_L \cup \Theta_BL \)
- \( \Theta_2 = \Theta_L \cup \Theta_SL \)
- \( \Theta_3 = \Theta_T \cup \Theta_TR \)
- \( \Theta_4 = \Theta_TL \)
We now construct the formula $\varphi_\mathcal{T}$ in a bottom-up manner, starting with a subformula $\varphi_{\theta_1}$ for each entry $\theta_1$ other than $\#$, for every tile $\theta \in \Theta$. Letting $\theta_1$ be equal to $(\gamma, \omega) \in \Gamma \times \Omega$, with $\gamma = (\gamma_j)_{1 \leq j \leq 1}$, the formula $\varphi_{\theta_1}$ checks at a given position $c \in \mathcal{C}$ if the labeling of $c$ matches $\theta_1$.

$$\varphi_{\theta_1} = \bigwedge_{\gamma_1 = 1}^{\gamma_1} \bigwedge_{\gamma_1 = 0}^{\gamma_1} \neg P_j \land X_\omega \land \bigwedge_{\omega' \neq \omega} \neg X_{\omega'}$$

Building on this, we can define for each tile $\theta \in \Theta$ the formula $\varphi_\theta$ mentioned above. Since HG does not have backward modalities, there is a certain asymmetry between tiles in $\Theta_1$, where the representative can "see" the entire $2 \times 2$-subgrid, and the remaining tiles, where the representative must "know" that it lies in the leftmost column or the uppermost row of the grid $\mathcal{C}$. We shall address this issue shortly, and just assume that information not accessible to the representative is verified by another part of the ultimate formula $\varphi_\mathcal{T}$. For tiles in $\Theta_B$, $\Theta_R$, $\Theta_{BL}$, $\Theta_T$, $\Theta_{TR}$, the method is completely analogous.

$$\Theta_M \ni [\theta_1, \theta_2, \theta_3, \theta_4] \quad \varphi_{\theta_1} \land \bigodot \varphi_{\theta_2} \land \bigodot \varphi_{\theta_3} \land \bigotimes \varphi_{\theta_4}$$

$$\Theta_{BR} \ni [\theta_1 \neq \theta_2, \theta_3, \theta_4] \quad \varphi_{\theta_2} \land \bigotimes \\ \varphi_{\theta_4}$$

$$\Theta_L \ni [\theta_1 \neq \theta_3, \theta_2, \theta_4] \quad \varphi_{\theta_2} \land \bigotimes \varphi_{\theta_4}$$

$$\Theta_{TL} \ni [\theta_1 \neq \theta_4, \theta_3, \theta_2] \quad \varphi_{\theta_1}$$

It remains to mark the top and left borders of $\mathcal{C}$, using two additional predicates $Y_T$ and $Y_L$, over which we will quantify existentially. To this end, we write an HG-formula $\varphi_{\text{border}}$, checking that top [resp. left] elements have no $R_1$-[resp. $R_2$-] predecessor, that there is a top-left element, and that being top [resp. left] is passed on to the $R_2$-[resp. $R_1$-] successor, if it exists.

$$\varphi_{\text{border}} = \neg \bigodot (Y_T \lor \bigotimes Y_L) \land \bigotimes (Y_T \land Y_L) \land \bigodot (Y_T \rightarrow \bigotimes Y_T) \land \bigotimes (Y_L \rightarrow \bigotimes Y_L)$$

Finally, we can put everything together to describe the acceptance condition of $\mathcal{T}$. Every element $c \in \mathcal{C}$ has to ensure that it corresponds to the upper-left corner of some tile in $\Theta_1$. Furthermore, elements in the leftmost column or uppermost row of $\mathcal{C}$ must additionally check that the assignment of states is compatible with the tiles in $\Theta_2$, $\Theta_3$, $\Theta_4$. This leads to the desired formula $\varphi_\mathcal{T}$:

$$\exists (X_\omega)_{\omega \in \Omega}. Y_T, Y_L \left( \varphi_{\text{border}} \land \bigotimes (Y_T \rightarrow \bigotimes Y_T) \land \bigotimes (Y_L \rightarrow \bigotimes Y_L) \right)$$

Note that we do not need a separate subformula to check that the interpretations of $(X_\omega)_{\omega \in \Omega}$ form a partition of dom($\mathcal{C}$), since this is already done implicitly in the conjunct $\bigotimes (Y_T \rightarrow \bigotimes Y_T)$.

5.3 Equivalent Hierarchies on Grids

Now we have everything at hand to prove the levelwise equivalence of MSO, HBGS and HGS on labeled grids.

**Theorem 9.**

Let $t \geq 0$, $\ell \geq 1$ and $\Xi \in \{\Sigma_\ell, \Pi_\ell, BC \Sigma_\ell, \Delta_\ell\}$. When restricted to the class of $t$-bit labeled grids, $\Xi(FO)$, $\Xi(HBG)$ and $\Xi(HG)$ are equivalent, i.e.,

$$\Xi(FO)_{\text{Grid}}[t] = \Xi(HBG)_{\text{Grid}}[t] = \Xi(HG)_{\text{Grid}}[t].$$

**Proof.** First, we show that the claim holds for the case $\Xi = \Sigma_1$ (with arbitrary $t \geq 0$). This can be seen from the following circular chain of inclusions:

$$\Sigma_1(FO)_{\text{Grid}}[t] \subseteq \Sigma_1(HBG)_{\text{Grid}}[t] \subseteq \Sigma_1(HG)_{\text{Grid}}[t]$$

(a) The first inclusion follows from the fact that $\Sigma_1(\mathcal{H})$ is a syntactic fragment of $\Sigma_1(\mathcal{H})$.

(b) For the second inclusion, consider any $\Sigma_1(\mathcal{H})$-formula $\hat{\psi} = \exists Q_1, \ldots, Q_n(\psi)$, where $Q_1, \ldots, Q_n$ are set symbols and $\varphi$ is an HBG-formula. By Proposition 6, we can replace $\varphi$ in $\hat{\psi}$ by an equivalent FO-formula $\psi_\varphi$. This results in the $\Sigma_1(FO)$-formula $\psi_\hat{\psi} = \exists Q_1, \ldots, Q_n(\psi_\varphi)$, which is equivalent to $\hat{\psi}$ on arbitrary structures, and thus, in particular, on $t$-bit labeled grids.

(c) The translation from $\Sigma_1(FO)$ on labeled grids to tiling systems corresponds to the more challenging direction of Theorem 7, which is the main result of [GRST96].

(d) The last inclusion is given by Proposition 8.

The general version of the theorem can now be obtained by induction on $\ell$. This is straightforward, because the classes $\Pi_\ell(\mathcal{F})$, $BC \Sigma_\ell(\mathcal{F})$ and $\Sigma_{\ell+1}(\mathcal{F})$ are defined syntactically in terms of $\Sigma_\ell(\mathcal{F})$, for any set of kernel formulas $\mathcal{F}$ (see Section 2.5), and if the claim holds for $\Xi \in \{\Sigma_\ell, \Pi_\ell\}$, then it also holds for the intersection classes of the form $[\Delta_\ell(\mathcal{F})]_{\text{Grid}}[t]$. 

$\square$
5.4 A Logical Characterization of Grids

We conclude this section by showing that a single layer of universal set quantifiers is enough to describe grids in HBGS.

Proposition 10.

The set of all grids is \( \Pi_1(\text{HBG}) \)-definable over 2-relational directed graphs, i.e.,

\[
\text{GRID} \in \left[ \Pi_1(\text{HBG}) \right]_{\text{DIGRAPH}[0,2]}.\]

Proof. In the course of this proof, we give a list of properties, items (a) to (f), which are obviously necessary for a 2-relational directed graph \( \mathcal{D} \) to be a grid, and show how to express them as \( \Pi_1(\text{HBG}) \)-formulas. We argue that the conjunction of all of these properties also constitutes a sufficient condition for being a grid, which immediately provides us with the required formula, since \( \left[ \Pi_1(\text{HBG}) \right] \) is closed under intersection.

(a) For each relation symbol \( R \in \{ R_1, R_2 \} \), every element has at most one \( R \)-predecessor and at most one \( R \)-successor; in other words, \( R^\mathcal{D}_1 \) and \( R^\mathcal{D}_2 \) are partial injective functions.

\[
\forall X \bigwedge_{R \in \{ R_1, R_1^\bot, R_2, R_2^\bot \}} \forall X (\nabla X \rightarrow R X).
\]

(b) Again considering each \( R \in \{ R_1, R_2 \} \) separately, there is a directed \( R \)-path from every element to an \( R \)-sink, i.e., to some element without \( R \)-successor.

\[
\forall X \bigwedge_{R \in \{ R_1, R_2 \}} \forall X (\Diamond X \land \nabla X \rightarrow \Diamond (X \land R \bot)).
\]

Taken together, properties (a) and (b) state that \( R^\mathcal{D}_1 \) and \( R^\mathcal{D}_2 \) each form a collection of directed, acyclic, pairwise vertex-disjoint paths. Let us refer to the first elements of those paths as \( R_1 \)- and \( R_2 \)-sources, respectively.

(c) There is precisely one element that is both an \( R_1 \)- and an \( R_2 \)-source.

\[
\text{tot}(\top_1 \land \top_2 \bot)
\]

(Here, \( \text{tot} \) is the schema from Section 2.4.)

(d) The \( R_1 \)-predecessors and \( R_1 \)-successors of \( R_2 \)-sources must be \( R_2 \)-sources themselves.

\[
\bigwedge_{R \in \{ R_2 \}} (\top_1 \rightarrow \top_2 \land \top_1)
\]

By adding (c) and (d) to our list of conditions, we ensure that there is an \( R_1 \)-path consisting precisely of the \( R_2 \)-sources, thereby also forcing the graph \( \mathcal{D} \) to be connected.

(e) If an element has both an \( R_1 \)- and an \( R_2 \)-successor, then it also has a descendant reachable by first taking an \( R_1 \)-edge and then an \( R_2 \)-edge.

\[
\bigwedge_{R \in \{ R_1, R_2 \}} (\nabla X \land \nabla X \rightarrow \nabla X)
\]

(f) The relations \( R^\mathcal{D}_1 \) and \( R^\mathcal{D}_2 \) commute. This means that following an \( R_1 \)-edge and then an \( R_2 \)-edge leads to the same element as first taking an \( R_2 \)-edge and then an \( R_1 \)-edge.

\[
\forall X (\Diamond X \land \nabla X \rightarrow \Diamond X)
\]

Considered in conjunction with condition (a), there are only two ways to satisfy (e) and (f) from the point of view of two elements \( d_1, d_2 \in \mathcal{D} \) that are connected by an \( R_1 \)-edge from \( d_1 \) to \( d_2 \); either both elements are \( R_2 \)-sinks, or they have \( R_2 \)-successors \( d_1' \) and \( d_2' \), respectively, with an \( R_1 \)-edge from \( d_1' \) to \( d_2' \). Moreover, \( d_2' \) only possesses an \( R_1 \)-successor if \( d_2 \) does. Now, imagine we start from the left border, i.e., from the \( R_1 \)-path that consists of all the \( R_2 \)-sources, which is provided by properties (a) to (d), and iteratively enforce the requirements just mentioned. Then, in doing so, we propagate the grid topology through the entire graph. More specifically, the additional requirements of (e) and (f) entail that all the \( R_2 \)-paths have the same length, and that the elements lying at a fixed (horizontal) position of those \( R_2 \)-paths constitute an independent \( R_1 \)-path, ordered in the same way as their respective \( R_2 \)-predecessors. \( \square \)

6 A Toolbox of Encodings

In this section, we provide all the encoding functions used in the proof of Theorem 2 (see Section 4.2), and show that they satisfy suitable translatability properties, allowing us to establish the required figurative inclusions. With a view to modularity and reusability, some of our constructions are more general than needed.

From here on, integer intervals of the form \( \{ i \mid m \leq i \leq n \} \) will be denoted by \( [m:n] \). We may also use the shorthand \( [n] := [1:n] \), and, by analogy with the Bourbaki notation for real intervals, \([m:n] \) indicates that we exclude the endpoint \( m \). Furthermore, given a set of symbols \( \sigma \), the extension \( \sigma \cup \{ \varnothing \} \) will be abbreviated to \( \sigma_{\varnothing} \).

6.1 Encodings that Allow for Translation

We shall only consider encoding functions that are linear in the following sense:

Definition 11 (Linear Encoding).

Let \( \mathcal{C}, \mathcal{D} \) be two classes of structures, and \( m, n \) be integers such that \( 1 \leq m \leq n \). A linear encoding from \( \mathcal{C} \) into \( \mathcal{D} \) with parameters \( m, n \) is a total injective function \( \mu : \mathcal{C} \rightarrow \mathcal{D} \) that assigns to each structure \( \mathfrak{A} \in \mathcal{C} \) a structure \( \mu(\mathfrak{A}) \in \mathcal{D} \), whose domain is composed of \( m \) disjoint copies of the domain of \( \mathfrak{A} \) and \( n - m \) additional elements, i.e.,

\[
\text{dom}(\mu(\mathfrak{A})) = ([1:m] \times \text{dom}(\mathfrak{A})) \cup [m:n].
\]
Given such a linear encoding \( \mu \) and some HBG-formula \( \varphi \), we want to be able to construct a new formula \( \psi_\varphi \), such that evaluating \( \varphi \) on \( \varepsilon \) is equivalent to evaluating \( \psi_\varphi \) on \( \mu(\varepsilon) \). Conversely, we also desire a way of constructing a formula \( \varphi_\psi \) that is equivalent on \( \varepsilon \) to a given formula \( \psi \) on \( \mu(\varepsilon) \). The following two definitions formalize this translatability property for both directions. We then show in Lemma 14 that they adequately capture our intended meaning. Although the underlying idea is very simple, the presentation is a bit lengthy because we have to exhaustively cover the structure of HBG-formulas.

**Definition 12** (Forward Translation).

Consider two classes of structures \( \mathcal{C} \) and \( \mathcal{D} \) over signatures \( \sigma \) and \( \tau \), respectively, two classes of formulas \( \Phi, \Psi \in \{H, HB, HG, HBG\} \), and a linear encoding \( \mu: \mathcal{C} \to \mathcal{D} \) with parameters \( m, n \). We say that \( \mu \) allows for forward translation from \( \Phi \) to \( \Psi \) if the following properties are satisfied:

(a) For each element symbol or set symbol \( P \) in \( \sigma \), there is a \( \Psi \)-sentence \( \psi_P \) over \( \tau_\sigma \), such that
\[
\mathcal{A}[\sigma \mapsto a] \models P \iff \mu(\mathcal{A})[\sigma \mapsto (1, a)] \models \psi_P,
\]
for all \( \mathcal{A} \in \mathcal{C} \) and \( a \in \mathcal{A} \).

(b) For each relation symbol \( R \) in \( \sigma \) of arity \( k+1 \geq 2 \), there is a \( \Psi \)-sentence \( \varphi_R \) over \( \tau_\sigma \) enriched with additional set symbols \( (Y_i)_{1 \leq i \leq k} \), such that
\[
\mathcal{A}[\sigma, (X_i)_{1 \leq i \leq k} \mapsto a, (A_i)_{1 \leq i \leq k}] \models (X_i)_{1 \leq i \leq k}
\]
if and only if
\[
\mu(\mathcal{A})[\sigma, (Y_i)_{1 \leq i \leq k} \mapsto (1, a), (B_i)_{1 \leq i \leq k}] \models \varphi_R,
\]
assuming \( A_i, B_i \) satisfy \( a' \in A_i \iff (1, a') \in B_i \),
for all \( \mathcal{A} \in \mathcal{C} \), \( a \in \mathcal{A} \), sets \( (A_i)_{1 \leq i \leq k} \subseteq \mathcal{A} \) and \( (B_i)_{1 \leq i \leq k} \subseteq \mu(\mathcal{A}) \), and set symbols \( (X_i)_{1 \leq i \leq k} \).

(c) If \( \Phi \) includes backward modalities, then for each relation symbol \( R \) in \( \sigma \) of arity at least 2, there is a \( \Psi \)-formula \( \varphi_{\neg R} \) that satisfies the property of item (b) for \( R^{-1} \) instead of \( R \).

(d) If \( \Phi \) includes global modalities, then there is a \( \Psi \)-formula \( \psi_{\neg 1} \) that satisfies the property of item (b) for \( \neg 1 \) instead of \( 1 \).

(e) There is a \( \Psi \)-sentence \( \psi_{\neg 1} \) over \( \tau \) enriched with an additional set symbol \( Y \), such that
\[
\mathcal{A}[X \mapsto A] \models_X \iff \mu(\mathcal{A})[Y \mapsto B] \models \psi_{\neg 1},
\]
assuming \( A, B \) satisfy \( a \in A \iff (1, a) \in B \),
where \( \models_X \) is \( \models \) if \( \sigma \in \sigma \), and otherwise, for all \( \mathcal{A} \in \mathcal{C} \), \( A \subseteq \mathcal{A} \), \( B \subseteq \mu(\mathcal{A}) \) and \( X \in S_1 \).

**Definition 13** (Backward Translation).

Consider two classes of structures \( \mathcal{C} \) and \( \mathcal{D} \) over signatures \( \sigma \) and \( \tau \), respectively, two classes of formulas \( \Phi, \Psi \in \{H, HB, HG, HBG\} \), and a linear encoding \( \mu: \mathcal{C} \to \mathcal{D} \) with parameters \( m, n \). We say that \( \mu \) allows for backward translation from \( \Psi \) to \( \Phi \) if the following properties are satisfied:

(a) For each element symbol or set symbol \( Q \) in \( \tau \) and all \( h \in [n] \), there is a \( \Phi \)-sentence \( \varphi_Q^h \) over \( \sigma_\sigma \), such that
\[
\mathcal{A}[\sigma \mapsto a] \models \varphi_Q^h \iff \mu(\mathcal{A})[\sigma \mapsto b] \models Q,
\]
where \( b \) is \( (h, a) \) if \( h \leq m \), and \( h \) otherwise, for all \( \mathcal{A} \in \mathcal{C} \) and \( a \in \mathcal{A} \).

(b) For each relation symbol \( S \) in \( \tau \) of arity \( k+1 \geq 2 \), and all \( h \in [n] \), there is a \( \Phi \)-sentence \( \varphi_S^h \) over \( \sigma_\sigma \) enriched with additional set symbols \( (X_i)_{1 \leq i \leq n} \), such that
\[
\mathcal{A}[\sigma, (X_i)_{1 \leq i \leq n} \mapsto a, (A_i)_{1 \leq i \leq n}] \models \varphi_S^h
\]
if and only if
\[
\mu(\mathcal{A})[\sigma, (Y_i)_{1 \leq i \leq k} \mapsto b, (B_i)_{1 \leq i \leq k}] \models (X_i)_{1 \leq i \leq k},
\]
where \( b \) is \( (h, a) \) if \( h \leq m \), otherwise \( h \), and
\[
B_i = \bigcup_{1 \leq j < m} \{ j \times A_j \} \cup \bigcup_{m < j \leq n} \{ j \mid A_j = \text{dom}(\mathcal{A}) \},
\]
for all structures \( \mathcal{A} \in \mathcal{C} \), elements \( a \in \mathcal{A} \), sets \( (A_i)_{1 \leq i \leq m} \subseteq \mathcal{A} \) and \( (A_i)_{m < j \leq n} \in \{ \emptyset, \text{dom}(\mathcal{A}) \} \), and set symbols \( (Y_i)_{1 \leq i \leq k} \).

(c) If \( \Psi \) includes backward modalities, then for each relation symbol \( S \) in \( \tau \) of arity at least 2, and all \( h \in [n] \), there is a \( \Phi \)-formula \( \varphi_{\neg S}^h \) that satisfies the property of item (b) for \( S^{-1} \) instead of \( S \).

(d) If \( \Psi \) includes global modalities, then for all \( h \in [n] \), there is a \( \Phi \)-formula \( \varphi_{\neg 1}^h \) that satisfies the property of item (b) for \( \neg 1 \) instead of \( 1 \).

(e) There is a \( \Phi \)-sentence \( \varphi_{\neg 1} \) over \( \sigma \) enriched with additional set symbols \( (X_i)_{1 \leq i \leq n} \), such that
\[
\mathcal{A}[(X_i)_{i \leq n} \mapsto (A_i)_{i \leq n}] \models \varphi_{\neg 1}
\]
if and only if
\[
\mu(\mathcal{A})[Y \mapsto B] \models Y_{\neg 1},
\]
where \( Y \) is \( Y \) if \( \sigma \in \sigma \), otherwise \( Y \), and
\[
B = \bigcup_{1 \leq j \leq m} \{ j \times A_j \} \cup \bigcup_{m < j \leq n} \{ j \mid A_j = \text{dom}(\mathcal{A}) \},
\]
for all structures \( \mathcal{A} \in \mathcal{C} \), sets \( (A_i)_{1 \leq i \leq m} \subseteq \mathcal{A} \) and \( (A_i)_{m < j \leq n} \in \{ \emptyset, \text{dom}(\mathcal{A}) \} \), and \( Y \in S_1 \).

To simplify matters slightly, we shall say that a linear encoding \( \mu \) allows for bidirectional translation between \( \Phi \) and \( \Psi \)
and $\Psi$, if it allows for both forward translation from $\Phi$ to $\Psi$ and backward translation from $\Psi$ to $\Phi$. Furthermore, in case $\Phi = \Psi$, we may say "within $\Phi$" instead of "between $\Phi$ and $\Phi$".

Let us now prove that our notion of translatability is indeed sufficient to imply figurative inclusion on the semantic side, even if we bring set quantifiers into play.

**Lemma 14.**

- Consider two classes of structures $\mathcal{C}$ and $\mathcal{D}$, a linear encoding $\nu: \mathcal{C} \to \mathcal{D}$, two classes of formulas $\Phi, \Psi \in \{H, HB, HG, HBG\}$, and let $\Xi \in \{\Sigma_\ell, \Pi_\ell, BC \Sigma_\ell\}$, for some arbitrary $\ell \geq 0$.

(a) If $\mu$ allows for forward translation from $\Phi$ to $\Psi$, then we have

$$\llbracket \Xi(\Phi) \rrbracket_\nu \subseteq \llbracket \Xi(\Psi) \rrbracket_\nu.$$

(b) Similarly, if $\mu$ allows for backward translation from $\Psi$ to $\Phi$, then we have

$$\llbracket \Xi(\Phi) \rrbracket_\mu \subseteq \llbracket \Xi(\Psi) \rrbracket_\mu.$$  \hfill \Box

**Proof.** Let $\sigma$ and $\tau$ be the signatures underlying $\mathcal{C}$ and $\mathcal{D}$, respectively. Parts (a) and (b) of the lemma are treated separately in the following proof.

In several places, given some HBGS-formula $\phi$, the need will arise to substitute newly created HBG-formulas $\phi_1, \ldots, \phi_k$ for set symbols $X_1, \ldots, X_k$. We shall write $\phi[(X_1)_{i \leq k} \mapsto (\phi_i)_{i \leq k}]$ to denote the HBGS-formula that one obtains by simultaneously replacing each free occurrence of each $X_i$ in $\phi$ by the formula $\phi_i$.

(a) For every $\Xi(\Phi)$-sentence $\phi$ over $\sigma$, we must construct a $\Xi(\Psi)$-sentence $\psi_\phi$ over $\tau$, such that $\psi_\phi$ says about $\mu(\mathcal{A})$ the same as $\phi$ says about $\mathcal{A}$, for all structures $\mathcal{A} \in \mathcal{C}$.

We start by focusing on the kernel classes $\Phi, \Psi$, and show the following by induction on the structure of $\Phi$-formulas: For every $\Phi$-sentence $\phi$ over $\sigma_\Phi \cup \tau_\Psi$, with $\gamma = \{Z_1, \ldots, Z_t\}$ being any collection of set symbols disjoint from $\sigma$ and $\tau$ (i.e., "free set variables"), there is a $\Psi$-sentence $\psi_\phi$ over $\tau_\Psi \cup \gamma$ such that

$$\mathcal{A}[\tau_\Psi, (Z_r)_{r \leq t} \mapsto a, (A_r)_{r \leq t}] \models \psi_\phi$$

if and only if

$$\mu(\mathcal{A})[\tau_\Psi, (Z_r)_{r \leq t} \mapsto (1, a), (B_r)_{r \leq t}] \models \psi_\phi^*,$$

assuming $A_r, B_r$ satisfy $a \in A_r \iff (1, a') \in B_r$,

for all structures $\mathcal{A} \in \mathcal{C}$, elements $a \in \mathcal{A}$, and sets $(A_r)_{1 \leq r \leq t} \subseteq \mathcal{A}$ and $(B_r)_{1 \leq r \leq t} \subseteq \mu(\mathcal{A})$.

- If $\phi = \lnot \phi$ or $\phi = Z$, for some $Z \in \gamma$, it suffices to set $\psi_\phi^* = \phi$.
- If $\phi = P$, for some element symbol or set symbol $P$ in $\sigma$, we exploit that $\mu$ allows for forward translation from $\Phi$ to $\Psi$, and choose $\psi_\phi^* = \psi_P$. Here, $\psi_P$ is the formula postulated by Definition 12 (a); it fulfills the induction hypothesis, since adding interpretations of the symbols $Z_1, \ldots, Z_t$ to a structure has no influence on whether or not that structure satisfies a sentence over a signature that does not contain these symbols.
  - If $\phi = \lnot \phi_1$ or $\phi = \phi_1 \lor \phi_2$, where $\phi_1$ and $\phi_2$ are formulas that satisfy the induction hypothesis, we set $\psi_\phi^* = \lnot \psi_\phi^*_1$ or $\psi_\phi^* = \psi_\phi^*_1 \lor \psi_\phi^*_2$, respectively.
  - If $\phi = \bigotimes (\phi_i)_{i \in k}$, where $R$ is a relation symbol in $\sigma$ of arity $k + 1 \geq 2$, and $(\phi_i)_{i \in k}$ are $\Phi$-sentences over $\sigma_\Phi \cup \tau_\Psi$ satisfying the induction hypothesis, we again use the fact that $\mu$ allows for forward translation from $\Phi$ to $\Psi$. The desired formula $\psi_\phi^*$ is obtained by substituting $(\psi_\phi^*_i)_{i \in k}$ for the symbols $(Y_i)_{i \in k}$ in the formula $\psi_R$, whose existence is asserted by Definition 12 (b), i.e.,

$$\psi_\phi^* = \psi_R[(Y_i)_{i \in k}] \mapsto ((\psi_\phi^*_i)_{i \in k}).$$

For $1 \leq i \leq k$, let $A'_i$ be the set of elements $a' \in \mathcal{A}$ that satisfy $\phi_i$ in $\mathcal{A}[(Z_r)_{r \leq t} \mapsto (A_r)_{r \leq t}]$, and let $B'_i$ be the set of elements $b' \in \mu(\mathcal{A})$ that satisfy $\psi_\phi^*_i$ in $\mu(\mathcal{A})[(Z_r)_{r \leq t} \mapsto (B_r)_{r \leq t}]$. By induction hypothesis, we are guaranteed that all the sets $A'_{i}, B'_{i}$ are such that an element $a'$ lies in $A'_{i}$ if and only if $(1, a')$ lies in $B'_{i}$. Thus, we have

$$\mathcal{A}[\tau_\Psi, (Z_r)_{r \leq t} \mapsto a, (A_r)_{r \leq t}] \models \phi$$

iff

$$\mathcal{A}[\tau_\Psi, (X_i)_{i \in k} \mapsto (\phi_i)_{i \in k} \Rightarrow \bigotimes (X_i)_{i \in k}] \models \psi_R$$

iff

$$\mu(\mathcal{A})[\tau_\Psi, (Y_i)_{i \in k} \mapsto (\phi_i)_{i \in k} \Rightarrow (1, a'), (B'_i)_{i \in k}] \models \psi_R$$

iff

$$\mu(\mathcal{A})[\tau_\Psi, (Z_r)_{r \leq t} \mapsto (1, a), (B_r)_{r \leq t}] \models \psi_\phi^*.$$
• If \( \psi \) lies in the kernel \( \Phi \), we make use of the claim just proven, together with the formula \( \psi_{\text{ini}} \) described in Definition 12 (e). We set \( \psi_{\text{f}} = \psi_{\text{ini}}[Y \mapsto \psi_{\text{f}}] \).
  - If \( \sigma \) belongs to \( \sigma \), the asserted property of \( \psi_{\text{ini}} \) guarantees that \( \psi \) holds at the initial position \( @^A \) in the \( \mathcal{Z} \)-extended variant of \( \mathfrak{A} \) if and only if \( \psi_{\text{f}} \) is satisfied by the \( \mathcal{Z} \)-extended variant of \( \mu(\mathfrak{A}) \).

• Otherwise, \( @ \) cannot be free in \( \phi \), since \( \phi \) is a sentence over \( \sigma \cup \mathcal{Z} \), which also implies that \( \Phi \) incorporates global modalities. It follows that \( \phi \) is equivalent to \( \bigotimes^\phi \). Again applying the definition of \( \psi_{\text{ini}} \), we obtain that the \( \mathcal{Z} \)-extended variant of \( \mathfrak{A} \) satisfies \( \bigotimes^\phi \), and thus \( \phi \), if and only if the \( \mathcal{Z} \)-extended variant of \( \mu(\mathfrak{A}) \) satisfies \( \psi_{\text{f}} \).

• If \( \phi \) is a Boolean combination of formulas that satisfy the induction hypothesis, the translation is straightforward, just as in the previous part of the proof.

• If \( \phi = \exists_{Z_{i+1}} \phi_1 \), where \( \phi_1 \) is a \( \exists(\Phi) \)-sentence over \( \sigma \cup \{Z_1, \ldots, Z_{i+1}\} \) that satisfies the hypothesis, we choose \( \psi_{\text{f}} = \exists_{Z_{i+1}} \psi_{\text{f}} \). To justify this choice, let \( \mathfrak{A}' \) and \( \mu(\mathfrak{A}') \) denote the \( \mathcal{Z} \)-extended variants of \( \mathfrak{A} \) and \( \mu(\mathfrak{A}) \), respectively. We get the following by induction:
  - If choosing \( Z_{i+1} \rightarrow A_{i+1} \) leads to satisfaction of \( \phi_1 \) in \( \mathfrak{A}' \), then choosing \( Z_{i+1} \rightarrow \{1 \} \times A_{i+1} \) does the same for \( \psi_{\text{f}} \) in \( \mu(\mathfrak{A}) \).
  - Conversely, if \( Z_{i+1} \rightarrow B_{i+1} \) is a satisfying choice for \( \psi_{\text{f}} \) in \( \mu(\mathfrak{A}) \), then so is \( Z_{i+1} \rightarrow \{1 \} \times A_{i+1} \) for \( \psi_{\text{f}} \) in \( \mathfrak{A}' \).

(b) The proof of the reverse direction of the lemma is very similar to the previous one, but a bit more cumbersome, because each element of a structure \( \mathfrak{A} \) has to play the role of several different elements in \( \mu(\mathfrak{A}) \). Given any \( \exists(\Psi) \)-sentence \( \psi \) over \( \tau \), we need to construct a \( \exists(\Phi) \)-sentence \( \phi_{\text{f}} \) over \( \sigma \), such that evaluating \( \phi_{\text{f}} \) on \( \mathfrak{A} \) is equivalent to evaluating \( \psi \) on \( \mu(\mathfrak{A}) \), for all \( \mathfrak{A} \in \mathcal{E} \).

For the remainder of this proof, let \( m \), \( n \) be the parameters of the linear encoding \( \mu \).

Again, we first deal with the kernel classes \( \Phi, \Psi, \) and show the following claim by induction on the structure of \( \Psi \)-formulas: For every \( \Psi \)-sentence \( \psi \) over \( \tau \cup \mathcal{Z} \) and all \( h \in [n] \), with \( \mathcal{Z} = \{Z_1, \ldots, Z_N\} \subseteq S_1 \setminus \tau \), there is a \( \Phi \)-sentence \( \phi_{h} \) over \( \sigma \cup \mathcal{Z} \), with \( \mathcal{Z} = \{Z_1, \ldots, Z_N\} \subseteq S_1 \setminus \tau \), such that

\[
\exists \{a \mid (a, A_1) \in \mathfrak{A}\} \exists \{a \mid (a, A_1) \in \mathfrak{A}\} \implies \exists \{a \mid (a, A_1) \in \mathfrak{A}\} \forall \{a \mid (a, A_1) \in \mathfrak{A}\} \implies \exists \{a \mid (a, A_1) \in \mathfrak{A}\} \forall \{a \mid (a, A_1) \in \mathfrak{A}\}
\]

and only if

\[
\mu(\mathfrak{A})[@, (\mathcal{Z})_{\tau} \in \mathfrak{A}] = b, \text{ where } b \text{ is } (h, a) \text{ if } h \leq m, \text{ otherwise } h, \text{ and } B_{\tau} = \bigcup_{1 \leq \tau < i} (\{1\} \times A_{\tau}) \cup \bigcup_{1 \leq \tau < j \leq n} \{1 \} \times A_{\tau} = \text{dom}(\mathfrak{A}) \text{ for all } \mathfrak{A} \in \mathcal{E}, a \in \mathfrak{A} \text{, and sets } (A_{1})_{1 \leq \tau < i} \subseteq \mathfrak{A} \text{ and } (A_{1})_{1 \leq \tau < i} \subseteq \mathfrak{A} \text{.}
\]

Hence, we obtain the required equivalence as follows:

\[
\exists \{a \mid (a, A_1) \in \mathfrak{A}\} \forall \{a \mid (a, A_1) \in \mathfrak{A}\} \implies \exists \{a \mid (a, A_1) \in \mathfrak{A}\} \forall \{a \mid (a, A_1) \in \mathfrak{A}\}
\]

\[
\exists \{a \mid (a, A_1) \in \mathfrak{A}\} \forall \{a \mid (a, A_1) \in \mathfrak{A}\} \implies \exists \{a \mid (a, A_1) \in \mathfrak{A}\} \forall \{a \mid (a, A_1) \in \mathfrak{A}\}
\]

• If \( \psi = @, \) it suffices to set \( \phi_{h} = @ \).

• If \( \psi = Z_r, \) for some \( Z_r \in \mathcal{Z}, \) the translation is given by \( \phi_{h} = Z_r^h. \)

• If \( \psi = Q, \) for some element symbol or set symbol \( Q \) in \( \tau, \) we use the fact that \( \mu \) allows for backward translation from \( \Psi \) to \( \Phi, \) and choose \( \phi_{h} \) to be the formula \( \phi_{h} \), which is provided by Definition 13 (a). The definition asserts that this formula fulfills the induction hypothesis for the case where \( \mathfrak{A} \) and \( \mu(\mathfrak{A}) \) are not extended using additional set symbols from \( \mathcal{Z} \) and \( \mathcal{X}. \)

Similarly to the proof of part (a), we now extend the previous property to cover formulas with set quantifiers,
evaluated on structures that may interpret the position symbol @ arbitrarily. Our induction hypothesis is the following: For every $\Sigma(\forall)$-sentence $\psi$ over $\tau \cup \mathcal{Z}$, with $\mathcal{Z} = \{Z_1, \ldots, Z_t\} \subseteq S_1(\forall \tau)$ (possibly empty), there is a $\Sigma(\forall)$-sentence $\varphi_\psi$ over $\sigma \cup \mathcal{Z}$, with $\hat{\mathcal{Z}} = \{Z_1, \ldots, Z_t^n\} \subseteq S_1(\forall \sigma)$, such that

$$\forall (Z_1^t)_{1 \leq t \leq s} \left( (Z_1^t)_{1 \leq t \leq s} \mapsto (A_1^t)_{1 \leq t \leq s} \right) \models \varphi_\psi$$

if and only if

$$\mu(\mathfrak{A})(Z_1^t)_{1 \leq t \leq s} \mapsto (B_r)_{r \leq \tau} \models \psi,$$

where

$$B_r = \bigcup_{1 \leq i \leq m} \{j \mid A_i^t = \text{dom}(\mathfrak{A})\},$$

for all structures $\mathfrak{A} \in \mathcal{C}$, and sets $(A_i^t)_{1 \leq r \leq t} \subseteq \mathfrak{A}$ and $(A_i^t)_{1 \leq t \leq s} \subseteq \mathfrak{A}$.

- If $\psi$ belongs to the kernel class $\Psi$, we apply the claim just proven, and construct $\varphi_\psi$ by substituting into the formula $\varphi_{\text{ini}}$ provided by Definition 13 (e): $\varphi_\psi = \varphi_{\text{ini}}(\psi)^1 \leq n \mapsto (\psi_\psi)^1 \leq n]$. Proceeding analogously to the proof of part (a), we have to distinguish whether or not the position symbol @ belongs to $\tau$. If it does not, $\psi$ is necessarily equivalent to $\Diamond \psi$. In both cases, the definition of $\varphi_{\text{ini}}$ guarantees that the $\hat{\mathcal{Z}}$-extended variant of $\mathfrak{A}$ satisfies $\varphi_\psi$ if and only if the $\mathcal{Z}$-extended variant of $\mu(\mathfrak{A})$ satisfies $\psi$.

- If $\psi$ is a Boolean combination of subformulas that satisfy the induction hypothesis, then $\varphi_\psi$ is simply the corresponding Boolean combination of the translated subformulas.

- If $\psi = \exists_{Z_{t+1}} \psi_1$, where $\psi_1$ is a $\Sigma(\forall)$-sentence over $\tau \cup \{Z_1, \ldots, Z_{t+1}\}$ that satisfies the induction hypothesis, we choose $\varphi_\psi$ to be the formula

$$\exists_{(Z_{t+1})^{1 \leq m}} \left( \bigvee_{N \subseteq [m:n]} \psi_1 \left( (Z_{t+1})^{1 \leq m} \mapsto (N(j))^{1 \leq m} \right) \right),$$

with $N(j) = T$ if $j \in N$, and $N(j) = \bot$ otherwise. For each set $N \subseteq [m:n]$, let $\psi_1^N$ denote the disjunct corresponding to $N$ in the formula above. By induction, we have the following equivalence: the interpretation $(Z_{t+1})^{1 \leq m} \mapsto (A_{t+1}^t)^{1 \leq m}$ leads to satisfaction of $\psi_1^N$ in the $\hat{\mathcal{Z}}$-extended variant of $\mathfrak{A}$ if and only if

$$Z_{t+1} \mapsto \bigcup_{1 \leq i \leq m} \{j \mid A_i^{t+1} \} \cup \mathcal{N}$$

is a satisfying choice for $\psi_1$ in the $\hat{\mathcal{Z}}$-extended variant of $\mu(\mathfrak{A})$.

$\square$

### 6.2 Getting Rid of Multiple Edge Relations

We now show how to encode a multi-relational graph into a 1-relational one, by inserting additional labeled elements that represent the different edge relations.

**Proposition 15.**

- For all $t, u \geq 0$ and $\Phi \in \{HG, HBG\}$, there is a linear encoding $\mu$ from $\text{DIGRAPH}[t, u]$ into $\text{DIGRAPH}[t+u, 1]$ that allows for bidirectional translation within $\Phi$.

Moreover, $\mu(\text{DIGRAPH}[t, u])$ is $\Pi_1(HG)$-definable over $\text{DIGRAPH}[t+u, 1]$.

**Proof.** We choose $\mu$ to be the linear encoding that assigns to each $t$-bit labeled, $u$-related directed graph $\mathcal{D}$ the $(t+u)$-bit labeled (1-relational) directed graph $\mu(\mathcal{D}) = \mathcal{E}$ with domain $[u+1] \times \text{dom}(\mathcal{D})$, labeling sets $P_{t}^\mathcal{E} = \{1\} \times P_{t}$, for $1 \leq i \leq t$, and $P_{t+u}^\mathcal{E} = \{1\} \times \text{dom}(\mathcal{D})$, for $1 \leq i \leq u$, and edge relation

$$R^\mathcal{E} = \bigcup_{1 \leq i \leq u} \left\{ ((i, d), (i+1, d')) \mid d \in \mathcal{D} \right\} \cup \left\{ ((1, d), (1+1, d')) \mid d \in \mathcal{D} \right\} \cup \left\{ ((1, d), (1+1, d')) \mid (d, d') \in R^\mathcal{D} \right\}.$$ 

That is, for each element $d \in \mathcal{D}$ and for $1 \leq i \leq u$, we introduce an additional element representing the “$R_t$-port” of $d$, and connect everything accordingly.

Our forward translation, from $\Phi$ on $\text{DIGRAPH}[t, u]$ to $\Phi$ on $\mu(\text{DIGRAPH}[t, u])$ on $\mu(\text{DIGRAPH}[t, u])$ to $\Phi$ on $\text{DIGRAPH}[t, u]$, is given by

$$\psi_{R_{t+1}} = P_{t+1}^\mathcal{E} \text{ for } 1 \leq i \leq t,$$

$$\psi_{R_{t+1}} = \bigotimes (\psi_{t+1} \wedge \psi_1 \wedge \gamma) \text{ for } 1 \leq i \leq u,$$

$$\psi_{R_{t+1}} = \bigotimes (\psi_{t+1} \wedge \gamma),$$

where $\psi_1 = \neg \bigvee_{1 \leq i \leq u} (\psi_{t+1})$ ("regular"),

$$\psi_{R_{t+1}} = P_{t+1}^\mathcal{E} \text{ for } 1 \leq i \leq u \text{ ("R_t").}$$

Our translation in the other direction, from $\Phi$ on $\mu(\text{DIGRAPH}[t, u])$ to $\Phi$ on $\text{DIGRAPH}[t, u]$, is given by

$$\psi_{P_{t+1}} = \begin{cases} P_{t+1} & \text{for } h = 0 \text{ and } 1 \leq i \leq t, \\ \bot & \text{for } h = 0 \text{ and } t + 1 \leq i \leq t + u, \\ T & \text{for } 1 \leq h \leq u \text{ and } i = t + h, \\ \bot & \text{for } 1 \leq h \leq u \text{ and } i \neq t + h, \end{cases}$$

$$\psi_{P_{t+1}} = \begin{cases} \bigvee_{1 \leq i \leq u} X_i^{t+1} & \text{for } h = 0, \\ X_i \bigotimes \psi_{h^{t+1}} & \text{for } 1 \leq h \leq u, \end{cases}$$

$$\psi_{P_{t+1}} = \begin{cases} \bigvee_{1 \leq i \leq u} X_i^{t+1} & \text{for } h = 0, \\ X_i \bigotimes \psi_{h^{t+1}} & \text{for } 1 \leq h \leq u, \end{cases}$$

$$\psi_{P_{t+1}} = \bigotimes (X_i \bigvee \ldots \bigvee X_i^{u+1}) \quad \text{for } 0 \leq h \leq u,$$

$$\varphi_{\text{ini}} = \psi_{R_{t+1}}.$$
We construct the linear encoding of each labeling set (see Proposition 15), our natural
6.3 Getting Rid of Vertex Labels
Every "regular element" is connected to its \( u \) different "ports" and nothing else. The uniqueness of each "\( R_i \)-port" can be expressed by the \([\Pi_1(HG)]\)-formula see\(1(\psi_{i+1})\), using the construction from Section 2.4.
\[
\Box\Bigl(\psi_1 \rightarrow \neg \Box \psi_1 \land \bigwedge_{1 \leq i \leq u} \text{see}\(1(\psi_{i+1})\)\Bigr)
\]
Similarly, each "port" is connected to precisely one "regular element" and to an arbitrary number of "ports" of the same relation symbol, but not to any other ones.
\[
\bigwedge_{1 \leq i \leq u} \Box\Bigl( \psi_{i+1} \rightarrow \text{see}\(1(\psi_1) \land \neg \bigvee_{1 \leq j \leq u, j \neq i} \Box \psi_{j+1}\)\Bigr)
\]
Finally, the links between "regular elements" and "ports" have to be bidirectional: for each edge from an element of one type to an element of a different type, the corresponding inverse edge must also exist.
\[
\bigwedge_{1 \leq i \leq u+1} \forall X \Box(\psi_1 \land X \rightarrow \Box(\neg \psi_1 \rightarrow \Box(X)))
\]
Note that, in combination with the previous properties, this ensures that we have the same total number of elements for each type \( i \in [1:t+3] \).

6.3 Getting Rid of Vertex Labels
Being able to eliminate multiple edge relations at the cost of additional labeling sets (see Proposition 15), our natural next step is to encode labeled graphs into unlabeled ones.

**Proposition 16.**
For all \( t \geq 1 \) and \( \Phi \in \{HG,HBG\} \), there is a linear encoding \( \mu \) from \( \text{Digraph}[t,1] \) into \( \text{Digraph} \) that allows for bidirectional translation within \( \Phi \).

Moreover, \( \mu(\text{Digraph}[t,1]) \) is \( \Pi_1(HG) \)-definable over \( \text{Digraph} \).

**Proof.** We construct the linear encoding \( \mu \) that assigns to each \( t \)-bit labeled directed graph \( D \) (the (unlabeled) directed graph \( \mu(D) = \mathcal{E} \) with domain \( \{\{1\} \times \text{dom}(D)\} \cup [2:t+3] \) and edge relation
\[
\mathcal{R}^E = \{((1,d),(1,d')) | (d,d') \in \mathcal{R}^D\} \\
\cup \{((1,d),3) | d \in \mathcal{D}\} \\
\cup \bigcup_{1 \leq i \leq t} \{((1,d),i+3) | d \in P_i^D\} \\
\cup \{(i+3,i+2) | 1 \leq i \leq t\} \\
\cup \{(i+3,2) | 0 \leq i \leq t\}.
\]

The idea is to introduce a gadget that contains a separate element for each labeling set of the original graph, and then connect the "regular elements" to this gadget in a way that corresponds to their respective labeling. The gadget is easily identifiable because it contains the only element in the graph that has no outgoing edge (namely, element 2).

We ensure this by connecting all the "regular elements" to element 3.

Our forward translation, from \( \Phi \) on \( \mu(\text{Digraph}[t,1]) \) to \( \Phi \) on \( \mu(\text{Digraph}[t,1]) \), is given by
\[
\psi_{P_i} = \Box(\psi_{i+3}) \quad \text{for } 1 \leq i \leq t,
\]
\[
\psi_R = \Box(\psi_1 \land Y),
\]
\[
\psi_{R-1} = \Box Y,
\]
\[
\psi_{T_1} = \Box(\psi_1 \land Y),
\]
\[
\psi_{ini} = \psi_{T_1}.
\]

where
\[
\psi_1 = \neg \bigvee_{2 \leq t \leq t+3}(\psi_i),
\]
\[
\psi_2 = \bigwedge \downarrow,
\]
\[
\psi_3 = \Box \psi_2 \land \Box \psi_2,
\]
\[
\psi_{i+3} = \Box \psi_2 \land \Box \psi_{i+2} \land \Box(\psi_2 \lor \psi_{i+2})
\]
for \( 1 \leq i \leq t \).

Our translation in the other direction, from \( \Phi \) on \( \mu(\text{Digraph}[t,1]) \) to \( \Phi \) on \( \mu(\text{Digraph}[t,1]) \), is given by
\[
\varphi^h_R = \begin{cases} 
\Box X^1 \lor X^3 \lor \bigvee_{1 \leq i \leq t}(P_i \lor X^{i+3}) & \text{for } h = 1, \\
\downarrow & \text{for } h = 2, \\
X^2 \lor X^{h-1} & \text{for } h = 3, \\
\Box X^1 & \text{for } h = 4 \leq h \leq t+3, \\
\bigvee_{0 \leq i \leq t} X^i & \text{for } h = 5, \\
\Box X^1 & \text{for } h = 6, \\
\Box(\psi_{h-3} \land X^1) \lor X^h & \text{for } 4 \leq h \leq t+2, \\
\Box(\psi_{h-3} \land X^1) & \text{for } h = t+3,
\end{cases}
\]
\[
\varphi^h_{R-1} = \begin{cases} 
\Box X^1 \lor X^{h+1} & \text{for } h = 1, \\
\bigvee_{0 \leq i \leq t} X^i & \text{for } h = 2, \\
\Box(\psi_{h-3} \land X^1) \lor X^{h+1} & \text{for } 4 \leq h \leq t+2, \\
\Box(\psi_{h-3} \land X^1) & \text{for } h = t+3,
\end{cases}
\]
\[
\varphi^h_{ini} = \varphi^h_{T_1}.
\]

Using the helper formulas \( \psi_{i+1} \) from the forward translation, we can characterize \( \mu(\text{Digraph}[t,1]) \) over \( \text{Digraph} \) as
\[
\psi_1 \land \bigwedge_{2 \leq t \leq t+3} \text{tot}\(1(\psi_1) \land \Box(\psi_{i+3} \land \neg \Box \psi_2)\).
\]
Here, each \( [\Pi_1(HG)] \)-subformula \( \text{tot}\(1(\psi_1) \) is obtained through the singleton construction from Section 2.4.

6.4 Getting Rid of Backward Modalities
For the sake of completeness, we also describe the encoding that lets us simulate backward modalities by means of an additional edge relation.
Proposition 17.

There is a linear encoding $\mu$ from Digraph into Digraph$[0,2]$ that allows for bidirectional translation between HBG and HG.

Moreover, $\mu(Digraph)$ is $\Pi_1(HG)$-definable over Digraph$[0,2]$.

Proof. The encoding is straightforward: to each directed graph $D$, we assign a copy $\mu(D) = \mathcal{G}$ that is enriched with a second edge relation, which coincides with the inverse of the first. Formally, $\text{dom}(\mathcal{G}) = \{1\} \times \text{dom}(D)$,

$$R_1^\mathcal{G} = \{(1,d),(1,d')\} \mid (d,d') \in R^D,$$

and

$$R_2^\mathcal{G} = \{(e',e) \mid (e,e') \in R^D\}.$$  

With this, in order to translate between HBG on Digraph and HG on $\mu$-Digraph, we merely have to replace backward modalities by $R_2$-modalities, and vice versa. Hence, when we fix our forward translation, we choose $\psi_R = \bigodot Y$ and $\psi_{R^{-1}} = \bigodot Y$, and for the backward translation we set $\psi_{R1} = \bigotimes X^1$ and $\psi_{R1^{-1}} = \bigotimes X^1$.

To define $\mu(Digraph)$ over Digraph$[0,2]$, we can use the following $\Pi_1(HG)$-formula:

$$\forall_X \exists_X (X \rightarrow \bigodot X \land \exists_X \bigodot X) \quad \square$$

6.5 Getting Rid of Directed Edges

In order to encode a directed graph into an undirected one, we proceed in a similar manner to the elimination of multiple edge relations in Proposition 15. Using an ad-hoc trick, we can do this by introducing only one additional labeling set.

Proposition 18.

There is a linear encoding $\mu$ from Digraph into Graph$[1,1]$ that allows for bidirectional translation between HBG and HG.

Moreover, $\mu(Digraph)$ is $\Pi_1(HG)$-definable over Graph$[1,1]$.

Proof. A suitable choice for $\mu$ is to take the function that assigns to every directed graph $D$ the 1-bit labeled undirected graph $\mu(D) = \mathcal{G}$ with domain $([3] \times \text{dom}(D)) \cup [4:6]$, labeling set $P = [4:6]$, and edge relation $R^\mathcal{G} = \{(g,g') \mid (g,g') \in \mathcal{G}\}$, where

$$G = \{(1,d),(2,d)\} \mid d \in D \}
\cup \{(1,d),(3,d)\} \mid d \in D \}
\cup \{(2,d),(3,d)\} \mid (d,d') \in R^D \}
\cup \{(2,d),4\} \mid d \in D \}
\cup \{(3,d),5\} \mid d \in D \}
\cup \{(5,6)\}.$$

The idea is that we connect each original element $d \in D$ to two new elements, which represent the "outgoing port" and "incoming port" of $d$, and use undirected edges between "ports" to simulate directed edges between "regular elements". In order to distinguish the different types of elements, we connect them in different ways to the additional $P$-labeled elements.

Our forward translation, from HBG on Digraph to $\mu$-Digraph, is given by

$$\psi_R = \bigodot (\psi_2 \land \bigodot (\psi_1 \land Y)),$$

$$\psi_{R^{-1}} = \bigodot (\psi_3 \land \bigodot (\psi_1 \land Y)),$$

$$\psi_{\text{init}} = \psi_{\text{fin}}.$$  

where $\psi_1 = \neg(\psi_2 \lor \cdots \lor \psi_6)$ ("regular"), $\psi_2 = \bigodot \psi_4$ ("outgoing"), $\psi_3 = \neg P \land \bigodot \psi_5$ ("incoming"), $\psi_4 = P \land \neg \bigodot P \land \bigotimes \neg P$, $\psi_5 = P \land \bigodot P \land \bigotimes \neg P$, $\psi_6 = P \land \bigotimes P \land \neg \bigotimes P$.

Our backward translation, from HG on $\mu$-Digraph to HBG on Digraph, is given by

$$\phi^b_1 = \bigodot (X^1 \lor \cdots \lor X^6) \quad \text{for } 1 \leq h \leq 6,$$

$$\phi^b_2 = \bigotimes X^2 \quad \text{for } h = 1,$$

$$\phi^b_3 = \bigotimes X^3 \lor X^4 \quad \text{for } h = 2,$$

$$\phi^b_4 = \bigotimes X^3 \lor X^5 \quad \text{for } h = 3,$$

$$\phi^b_5 = \bigotimes X^3 \lor X^6 \quad \text{for } h = 4,$$

$$\phi^b_6 = \bigotimes X^3 \lor X^6 \quad \text{for } h = 5,$$

$$\phi^b_{\text{init}} = \phi^b_{\text{fin}}.$$  

We can define $\mu(Digraph)$ over Graph$[1,1]$ with the following $[\Pi_1(HG)]$-formula. It makes use of our helper formulas $(\psi_i)_{1 \leq i \leq 6}$ from the forward translation and the constructions $\text{see1}(\psi_i)$ and $\text{tot1}(\psi_i)$ from Section 2.4.

$$\bigwedge_{4 \leq i \leq 6} (\psi_2 \rightarrow \text{see1}(\psi_i) \land \bigotimes (\psi_1 \lor \psi_3 \lor \psi_4)) \land$$

$$\bigotimes (\psi_3 \rightarrow \text{see1}(\psi_i) \land \bigotimes (\psi_1 \lor \psi_2 \lor \psi_5)) \land$$

$$\bigotimes (\psi_1 \rightarrow \text{see1}(\psi_2) \land \text{see1}(\psi_3) \land \bigotimes (\psi_2 \lor \psi_3)).$$

The first line states that the three $P$-labeled elements are unique, which forces 5 and 6 to be connected. The remaining lines ensure that each "port" is connected to exactly one "regular element", and, conversely, that every "regular element" is linked to precisely one "outgoing port" and one "incoming port". As a consequence, the number of "regular elements" must be the same as the number of
“ports” of each type. Furthermore, the formula restricts the types of neighbors each element can have, while the usage of the helper formulas $\psi_2$ and $\psi_3$ makes sure that the required connections to the $P$-labeled elements are established. Finally, the fact that $\psi_1$ characterizes the “regular elements” as the “remaining ones” guarantees that there are no unaccounted-for elements.

6.6 Getting Rid of Global Modalities

Our last encoding function lets us simulate global modalities by inserting a new element that is bidirectionally connected to all the “regular elements”.

**Proposition 19.**

There is a linear encoding $\mu$ from $\text{DIGRAPH}$ into $\text{PDIGRAPH}$ that allows for bidirectional translation between $\text{HG}$ and $\text{H}$.

Furthermore, $\mu$ can be easily adapted into a linear encoding $\mu'$ from $\text{DIGRAPH}$ into $\text{DIGRAPH}$ that satisfies the following figurative inclusions, for arbitrary $\ell \geq 2$:

\[
\begin{align*}
[\Sigma(\text{HG})]_{\text{DIGRAPH}} & \subseteq \perp_{\mu'} [\Sigma(\text{H})]_{\text{DIGRAPH}}, \\
[\Pi(\text{HG})]_{\text{DIGRAPH}} & \subseteq \perp_{\mu'} [\Pi(\text{H})]_{\text{DIGRAPH}}.
\end{align*}
\]

**Proof.** We choose $\mu$ to be the linear encoding that maps each directed graph $\mathcal{D}$ to the pointed directed graph $\mu(\mathcal{D}) = \Sigma$ with domain $\{1 \times \text{dom}(\mathcal{D})\} \cup \{2, 3\}$, position $\mu(\mathcal{D}) = 2$, and edge relation

\[
R^\Sigma = \{(1, d, (1, d')) \mid (d, d') \in R^D\} \\
\cup \{(1, d, 2) \mid d \in D\} \\
\cup \{(2, (1, d)) \mid d \in D\} \\
\cup \{(2, 3)\}.
\]

One can distinguish element 2 from the others because it is connected to 3, which is the only element without any outgoing edge.

Our forward translation, from $\text{HG}$ on $\mu(\text{DIGRAPH})$ to $\text{H}$ on $\mu(\text{DIGRAPH})$, is given by

\[
\begin{align*}
\psi_R &= \bigcirc(\psi_1 \land Y), \\
\psi_T &= \bigcirc(\psi_2 \land \bigcirc(\psi_1 \land Y)), \\
\psi_{\text{ini}} &= \bigcirc(\psi_1 \land Y),
\end{align*}
\]

where $\psi_1 = \Box \Box \Box$ and $\psi_2 = \Box \Box \perp$.

Our backward translation, from $\text{H}$ on $\mu(\text{DIGRAPH})$ to $\text{HG}$ on $\text{DIGRAPH}$, is given by

\[
\begin{align*}
\varphi^R_h &= \begin{cases} 
\Box X^1 \lor X^2 & \text{for } h = 1, \\
\Box X^1 \lor X^3 & \text{for } h = 2, \\
\perp & \text{for } h = 3,
\end{cases} \\
\varphi_{\text{ini}} &= \Box X^2.
\end{align*}
\]

Turning to the second claim of the proposition, we obtain $\mu'(\mathcal{D})$ by simply removing the position marker from $\mu(\mathcal{D})$, i.e., for every directed graph $\mathcal{D}$, $\mu'(\mathcal{D})$ is such that $\mu'(\mathcal{D})[\rightarrow \rightarrow] = \mu(\mathcal{D})$.

For the forward figurative inclusions, let $\Xi \in \{\Sigma, \Pi\}$, for some arbitrary $\ell \geq 0$. By applying Lemma 14 (a) on $\mu$, we get that for every $\Xi(\text{HG})$-sentence $\varphi$ over $\{R\}$, there is a $\Xi(\text{H})$-sentence $\psi_\varphi$ over $\{\Xi\}$ such that, for all $\mathcal{D} \in \text{DIGRAPH}$,

\[
\mathcal{D} \models \varphi \iff \mu'(\mathcal{D})[\rightarrow \rightarrow] \models \psi_\varphi.
\]

Hence, $\Xi(\text{HG})_{\text{DIGRAPH}} \subseteq \mu'(\Xi(\text{H}))_{\text{DIGRAPH}}$.

For the backward figurative inclusion, we require that $\ell \geq 2$. Slightly adapting the proof of Lemma 14 (b) to discard the part where we make use of the formula $\varphi_{\text{ini}}$ from Definition 13 (e) (incidentally allowing us to merge the two consecutive induction proofs), it is easy to show the following: Given $h \in \mathbb{Z}$ and any $\Pi(\text{H})$-sentence $\psi$ over $\{\Xi\}$, we can construct a $\Pi(\text{HG})$-sentence $\psi^h_\psi$ over $\{\Xi\}$ such that, for all $\mathcal{D} \in \text{DIGRAPH}$ and $d \in \mathcal{D}$,

\[
\mathcal{D}[\rightarrow \rightarrow d] \models \psi^h_\psi \iff \mu'(\mathcal{D})[\rightarrow \rightarrow e] \models \psi,
\]

where $e$ is $(h, d)$, if $h = 1$, and $h$ otherwise.

This immediately gives us a way of translating $\Box \psi$:

\[
\mathcal{D} \models \Box(\psi^1_\psi \land \psi^2_\psi \land \psi^3_\psi) \iff \mu'(\mathcal{D}) \models \Box \psi.
\]

The left-hand side sentence can be transformed into prenex normal form by simulating the global box with a universal set quantifier. Checking that a given set is not a singleton can be done in $\Sigma(\text{HG})$, since the negation is $\Pi(\text{HG})$-expressible (see Section 2.4). Thus, the given formula is equivalent to a $\Pi(\text{HG})$-formula, and we obtain that $\Pi(\text{HG})_{\text{DIGRAPH}} \subseteq \mu'(\Pi(\text{H}))_{\text{DIGRAPH}}$.

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