ON THE DIFFERENCE OF SPECTRAL PROJECTIONS

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ABSTRACT. Consider a self-adjoint operator \( T \) and a self-adjoint operator \( S = \langle \cdot, \varphi \rangle \varphi \) of rank one acting on a complex separable Hilbert space of infinite dimension. Denote by \( D(\lambda) := E_{(-\infty, \lambda)}(T + S) - E_{(-\infty, \lambda)}(T) \) the difference of the spectral projections with respect to the interval \((-\infty, \lambda)\). If \( T \) is bounded, then we show that:

- The operator \( D(\lambda) \) is unitarily equivalent to a self-adjoint Hankel operator of finite rank for all \( \lambda \) in \( \mathbb{R} \setminus [\min \sigma_{ess}(T), \max \sigma_{ess}(T)] \), where \( \sigma_{ess}(T) \) denotes the essential spectrum of \( T \).
- If \( \varphi \) is cyclic for \( T \), then \( D(\lambda) \) is unitarily equivalent to a bounded self-adjoint Hankel operator for all \( \lambda \) in \( \mathbb{R} \) except for at most countably many \( \lambda \) in \( \sigma_{ess}(T) \).

Furthermore, we prove a version of the second statement in the case when \( T \) is semibounded but not bounded.

1. INTRODUCTION AND MAIN RESULTS

When a self-adjoint operator \( T \) is perturbed by a bounded self-adjoint operator \( S \), it is important to investigate the (spectral) properties of the difference

\[ f(T + S) - f(T), \]

where \( f \) is a real-valued Borel function on \( \mathbb{R} \). It is also of interest to predict the smoothness of the mapping \( S \mapsto f(T + S) - f(T) \) with respect to the smoothness of \( f \). There is a vast amount of literature dedicated to these problems, see, e.g., Krein, Farforovskaja, Peller, Birman, Solomyak, Pushnitski, Yafaev [2, 6, 12, 13, 19, 20, 22–25], and the references therein.

In the context of compact and Schatten class operators in the space of bounded operators, there are two well-known situations which may be called "extreme" (and both of them will appear in this paper, see Sections 4, 5, and 7 below).

First, if \( f \) is an infinitely differentiable function with compact support and \( S \) is trace class, then \( f(T + S) - f(T) \) is a trace class operator; a precise reference for this result by Peller is given below.

Second, if \( f = 1_{(-\infty, \lambda)} \) is the characteristic function of the interval \((-\infty, \lambda)\) with \( \lambda \) in the essential spectrum of \( T \), then it may occur that

\[ f(T + S) - f(T) \]

is not compact, see Krein’s example [11,12]. In the latter example, \( S \) is a rank one operator, and the difference \( 1_{(-\infty, \lambda)}(T + S) - 1_{(-\infty, \lambda)}(T) \) is a bounded self-adjoint Hankel operator that can be computed explicitly for all \( 0 < \lambda < 1 \). Formally, a
bounded Hankel operator $\Gamma$ on $L^2(0, \infty)$ is a bounded integral operator

$$\Gamma : L^2(0, \infty) \to L^2(0, \infty)$$

such that the kernel function $k$ of $\Gamma$ depends only on the sum of the variables:

$$(\Gamma g)(x) = \int_0^\infty k(x+y)g(y)\,dy.$$ 

For an introduction to the theory of Hankel operators, we refer to Peller’s book [21].

Inspired by Krein’s example, we deal with the following question.

**Question 1.** Let $\lambda \in \mathbb{R}$. Is it true that

$$D(\lambda) := E_{(-\infty, \lambda)}(T + S) - E_{(-\infty, \lambda)}(T),$$

the difference of the spectral projections, is unitarily equivalent to a bounded self-adjoint Hankel operator, provided that $S$ is a rank one operator?

Pushnitski [22, 25] and Yafaev [25] have been studying the spectral properties of the operator $D(\lambda)$ in connection with scattering theory. If the absolutely continuous spectrum of $T$ contains an open interval and under some smoothness assumptions, the results of Pushnitski and Yafaev are applicable, see Sections 5 and 7 below. In this case, one has $\sigma_{\text{ess}}(D(\lambda)) = [-a, a]$, where $a > 0$ depends on $\lambda$ and can be expressed in terms of the scattering matrix for the pair $T, T + S$, see [22] Formula (1.3).

Here and for the rest of this paper, we consider a self-adjoint operator $T$ and a self-adjoint operator $S_\alpha = \alpha \langle \cdot, \varphi \rangle : \mathcal{H} \to \mathcal{H}$ of rank one acting on a complex separable Hilbert space $\mathcal{H}$ of infinite dimension, where $\alpha \in \mathbb{R}$ and $\varphi \in \mathcal{H}$ is a normed vector. We denote the spectrum and the essential spectrum of $T$ by $\sigma(T)$ and $\sigma_{\text{ess}}(T)$, respectively. Throughout the paper, except for Section 8, $T$ is assumed to be bounded.

If $F$ is a self-adjoint operator of finite rank and $\lambda$ is a real number, then we write

$$D(\lambda) = E_{(-\infty, \lambda)}(T + S_\alpha) - E_{(-\infty, \lambda)}(T),$$

$$\tilde{D}(\lambda) := E_{(-\infty, \lambda)}(T + F) - E_{(-\infty, \lambda)}(T).$$

Furthermore, we denote by $\text{span}\{x_i \in \mathcal{H} : i \in \mathcal{I}\}$ the linear span generated by the vectors $x_i, i \in \mathcal{I}$, where $\mathcal{I}$ is some index set. If there exists a vector $x$ such that

$$\text{span}\{E_{\Omega}(T)x : \Omega \in \mathcal{B}(\mathbb{R})\} := \text{span}\{E_{\Omega}(T)x : \Omega \in \mathcal{B}(\mathbb{R})\} = \mathcal{H},$$

then $x$ is called cyclic for $T$. Here $\mathcal{B}(\mathbb{R})$ denotes the sigma-algebra of Borel sets of $\mathbb{R}$.

The following theorem is the main result of this paper.

**Theorem 2.**

1. The operator $D(\lambda)$ is unitarily equivalent to a self-adjoint Hankel operator of finite rank for all $\lambda$ in $\mathbb{R} \setminus [\min \sigma_{\text{ess}}(T), \max \sigma_{\text{ess}}(T)]$.

2. Suppose that $\varphi$ is cyclic for $T$. Then $D(\lambda)$ is unitarily equivalent to a bounded self-adjoint Hankel operator for all $\lambda$ in $\mathbb{R} \setminus \sigma_{\text{ess}}(T)$ and for all but at most countably many $\lambda$ in $\sigma_{\text{ess}}(T)$.

3. If $\varphi$ is not cyclic for $T$, then for all $\lambda \in \mathbb{R}$ there exists a Hankel operator $\Gamma$, depending on $\lambda$, such that the spectra of $D(\lambda)$ and $\Gamma$ coincide.

The theory of bounded self-adjoint Hankel operators has been studied intensively by Rosenblum, Howland, Megretskii, Peller, Treil, and others, see [7, 17].
In their 1995 paper [17], Megretski˘ı, Peller, and Treil have shown that every bounded self-adjoint Hankel operator can be characterized by three properties concerning the spectrum and the multiplicity in the spectrum, see [17, Theorem 1].

We present a version of [17, Theorem 1] for differences of two orthogonal projections in Section 2 below.

The following theorem is an important ingredient in the proof of Theorem 2. Likewise, it is of independent interest.

**Theorem 3.** Suppose that the normed vector \( \varphi \) is cyclic for \( T \).

Let \( \lambda \in \mathbb{R} \setminus \{ \min \sigma_{\text{ess}}(T), \max \sigma_{\text{ess}}(T) \} \). Then the kernel of \( D(\lambda) \) is

1. infinite dimensional if and only if \( \lambda \in \mathbb{R} \setminus [\min \sigma_{\text{ess}}(T), \max \sigma_{\text{ess}}(T)] \).
2. trivial if and only if \( \lambda \in (\min \sigma_{\text{ess}}(T), \max \sigma_{\text{ess}}(T)) \).

In particular, one has

\[
\text{either } \text{Ker} D(\lambda) = \{0\} \text{ or } \dim \text{Ker} D(\lambda) = \infty.
\]

Note that, according to Theorem 3, the kernel of \( D(\lambda) \) is trivial for all \( \lambda \) between \( \min \sigma_{\text{ess}}(T) \) and \( \max \sigma_{\text{ess}}(T) \), no matter if the interval \( (\min \sigma_{\text{ess}}(T), \max \sigma_{\text{ess}}(T)) \) contains points from the resolvent set of \( T \), isolated eigenvalues of \( T \), etc.

In Section 3, we show that \( |\dim \text{Ker} (D(\lambda) - I) - \dim \text{Ker} (D(\lambda) + I)| \leq 1 \) for all \( \lambda \in \mathbb{R} \), where \( I \) denotes the identity operator.

The proof of Theorem 3 in Section 4 below is based on a result by Liaw and Treil [15] and some harmonic analysis.

In Theorem 3, we assumed that the vector \( \varphi \) is cyclic for \( T \). In the case when \( \varphi \) is not cyclic for \( T \), Example 4.4 below shows that Question 1 has to be answered negatively, while Proposition 4.3 provides a list of sufficient conditions so that Question 1 has a positive answer.

In Section 3, we show that the operator \( \tilde{D}(\lambda) \) is non-invertible for all \( \lambda \) in \( \mathbb{R} \) except for \( \lambda \) in an at most countable subset of \( \sigma_{\text{ess}}(T) \), see Theorem 5.1.

Section 6 completes the proof of Theorem 2.

Some examples, including the almost Mathieu operator, are discussed in Section 7 below.

In Section 8, we prove an analogue of Theorem 2 (2) in the case when \( T \) is semibounded but not bounded.

The results of this paper will be part of the author’s Ph.D. thesis at Johannes Gutenberg University Mainz.

## 2. The Main Tool

In this section, we present the main tool for the proof of Theorem 2. First, we state a lemma which follows immediately from [4, Theorem 6.1].

**Lemma 2.1.** Let \( \Gamma \) be the difference of two orthogonal projections. Then \( \sigma(\Gamma) \subset [-1, 1] \). Moreover, the restricted operators \( \Gamma|_{\mathcal{H}_0} \) and \( (-\Gamma)|_{\mathcal{H}_0} \) are unitarily equivalent, where the closed subspace \( \mathcal{H}_0 \) of \( \mathcal{H} \) is defined by

\[
\mathcal{H}_0 := [\text{Ker} (\Gamma - I) \oplus \text{Ker} (\Gamma + I)]^\perp.
\]

In [17], Megretski˘ı, Peller, and Treil solved the inverse spectral problem for self-adjoint Hankel operators. In our situation, [17, Theorem 1] reads as follows:
**Theorem 2.2.** The difference \( \Gamma \) of two orthogonal projections is unitarily equivalent to a bounded self-adjoint Hankel operator if and only if the following three conditions hold:

1. \((C1)\) either \( \text{Ker} \Gamma = \{0\} \) or \( \dim \text{Ker} \Gamma = \infty \);
2. \((C2)\) \( \Gamma \) is non-invertible;
3. \((C3)\) \( |\dim \text{Ker}(\Gamma - I) - \dim \text{Ker}(\Gamma + I)| \leq 1 \).

Note that if one of the dimensions in condition \((C3)\) is infinite, then \((C3)\) holds if and only if the other dimension is also infinite.

**Proof of Theorem 2.2.** Combine Lemma 2.1 and [17, Theorem 1].

As will be shown in Section 3, the operator \( D(\lambda) \) satisfies condition \((C3)\) for all \( \lambda \in \mathbb{R} \). Therefore, a sufficient condition for \( D(\lambda) \) to be unitarily equivalent to a bounded self-adjoint Hankel operator is given by:

The kernel of \( D(\lambda) \) is infinite dimensional.

In Proposition 4.3 below, we present a list of sufficient conditions such that the kernel of \( D(\lambda) \) is infinite dimensional.

### 3. ON THE DIMENSION OF \( \text{Ker}(D(\lambda) \pm I) \)

The main purpose of this section is to show that condition \((C3)\) in Theorem 2.2 is fulfilled for all \( \lambda \in \mathbb{R} \).

Let \( \lambda \in \mathbb{R} \). Since \( T \) and \( S_\alpha \) are fixed, we write \( P_\lambda := E_{(-\infty, \lambda)}(T + S_\alpha) \) and \( Q_\lambda := E_{(-\infty, \lambda)}(T) \).

**Lemma 3.1.** Suppose that \( \alpha > 0 \). Let \( \lambda \in \mathbb{R} \). Then \( \text{Ker}(P_\lambda - Q_\lambda - I) = \{0\} \) and \( \dim \text{Ker}(P_\lambda - Q_\lambda + I) \leq 1 \).

**Proof.** Assume that \( \text{Ker}(P_\lambda - Q_\lambda - I) \) is not trivial. Then there exists a normed vector \( x \in H \) such that \( (P_\lambda - Q_\lambda - I)x = x \). Hence \( P_\lambda x = x \) and \( Q_\lambda x = 0 \) and this implies

\[
\langle (T + S_\alpha)x, x \rangle < \lambda \quad \text{and} \quad \langle Tx, x \rangle \geq \lambda
\]

so that

\[
\lambda > \langle (T + S_\alpha)x, x \rangle = \langle Tx, x \rangle + \alpha |\langle x, \varphi \rangle|^2 \geq \lambda,
\]

which is a contradiction. This proves \( \text{Ker}(P_\lambda - Q_\lambda - I) = \{0\} \).

Now assume that there exists an orthonormal system of two vectors \( x_1, x_2 \in H \) such that

\[
(P_\lambda - Q_\lambda) x_1 = -x_1 \quad \text{and} \quad (P_\lambda - Q_\lambda) x_2 = -x_2.
\]

If \( a := \langle x_1, \varphi \rangle \neq 0 \) and \( b := \langle x_2, \varphi \rangle \neq 0 \), then we put

\[
x_3 := \frac{x_1}{a} - \frac{x_2}{b}.
\]

For at least one \( j \in \{1, 2, 3\} \), one has

\[
\lambda > \langle Tx_j, x_j \rangle = \langle (T + S_\alpha)x_j, x_j \rangle \geq \lambda,
\]

which is a contradiction. This concludes the proof.

The following lemma is proved analogously to Lemma 3.1.
Lemma 3.2. Suppose that $\alpha < 0$. Let $\lambda \in \mathbb{R}$. Then $\ker (P_{\lambda} - Q_{\lambda} + I) = \{0\}$ and $\dim \ker (P_{\lambda} - Q_{\lambda} - I) \leq 1$.

Taken together, Lemmas 3.1 and 3.2 imply:

Lemma 3.3. The operator $D(\lambda)$ fulfills Theorem 2.2 (C3) for all $\lambda \in \mathbb{R}$.

Remark. If we consider an unbounded self-adjoint operator $T$ and an eigenvector $x$ of $P_{\lambda} - Q_{\lambda}$, then the proof of Lemma 3.1 does not work, because $x$ might not belong to the domain of $T$.

The following remark shows that we cannot expect $\tilde{D}(\lambda)$ to be unitarily equivalent to a bounded self-adjoint Hankel operator for perturbations of rank $\geq 2$.

Remark (Perturbations of higher rank). Let $k \in \mathbb{N}$, $k \geq 2$. Denote by $\ell^2(\mathbb{N})$ the space of all complex square summable one-sided sequences $x = (x_1, x_2, \ldots)$. Consider the bounded self-adjoint diagonal operator

$$T = \text{diag} (-1, -3/4, -3/8, -3/16, \ldots) : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$$

and the self-adjoint diagonal operator

$$F = \text{diag} (-1, -1/2, \ldots, -1/k, 0, \ldots) : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$$

Then $F$ is of rank $k$, and we see that

$$\dim \ker (E_{(-\infty, \lambda)}(T + F) - E_{(-\infty, \lambda)}(T) - I) = 2$$

and $\ker (E_{(-\infty, \lambda)}(T + F) - E_{(-\infty, \lambda)}(T) + I) = \{0\}$

for all $\lambda \in (-5/4, -1)$. This violates condition (C3) in Theorem 2.2.

4. On the dimension of $\ker D(\lambda)$

In this section, we deal with the question whether the operator $D(\lambda)$ fulfills condition (C1) in Theorem 2.2. It is easy to see that the kernel of $D(\lambda)$ is nontrivial for all $\lambda \in \mathbb{R}$ if the vector $\varphi$ is not cyclic for $T$. Therefore, it is necessary to distinguish the two cases

- $\varphi$ is cyclic for $T$.
- $\varphi$ is not cyclic for $T$.

First, let us recall the well-known fact (see, e.g., [28, Lemma 9.8]) that $\varphi$ is cyclic for $T$ if and only if $\varphi$ is cyclic for $T + S_{\alpha}$.

4.1. The case when $\varphi$ is cyclic for $T$. This subsection is devoted to the proof of Theorem 3.

Let $\lambda \in \mathbb{R}$. Again, we write

$$P_{\lambda} = E_{(-\infty, \lambda)}(T + S_{\alpha}) \quad \text{and} \quad Q_{\lambda} = E_{(-\infty, \lambda)}(T).$$

Observe that the kernel of $P_{\lambda} - Q_{\lambda}$ is equal to the orthogonal sum of $(\text{Ran} \, P_{\lambda}) \cap (\text{Ran} \, Q_{\lambda})$ and $(\ker P_{\lambda}) \cap (\ker Q_{\lambda})$. Therefore, we will investigate the dimensions of $(\text{Ran} \, P_{\lambda}) \cap (\text{Ran} \, Q_{\lambda})$ and $(\ker P_{\lambda}) \cap (\ker Q_{\lambda})$ separately.

Now we follow [15, pp. 1948–1949] in order to represent the operators $T$ and $T + S_{\alpha}$ such that [15, Theorem 2.1] is applicable.

Define Borel probability measures $\mu$ and $\mu_{\alpha}$ on $\mathbb{R}$ by

$$\mu(\Omega) := \langle E_{\Omega}(T) \varphi, \varphi \rangle \quad \text{and} \quad \mu_{\alpha}(\Omega) := \langle E_{\Omega}(T + S_{\alpha}) \varphi, \varphi \rangle, \quad \Omega \in \mathcal{B}(\mathbb{R}),$$
respectively. According to [28, Proposition 5.18], there exist unitary operators $U : \mathcal{S} \to L^2(\mu)$ and $U_\alpha : \mathcal{S} \to L^2(\mu_\alpha)$ such that $UTU^* = M_t$ is the multiplication operator by the independent variable on $L^2(\mu)$, $U_\alpha(T + S_\alpha)U_\alpha^* = M_s$ is the multiplication operator by the independent variable on $L^2(\mu_\alpha)$, and one has both $(U \varphi)(t) = 1$ on $\mathbb{R}$ and $(U_\alpha \varphi)(s) = 1$ on $\mathbb{R}$. Clearly, the operators $U$ and $U_\alpha$ are uniquely determined by these properties. By [15, Theorem 2.1], the unitary operator $V_\alpha := U_\alpha U^* : L^2(\mu) \to L^2(\mu_\alpha)$ is given by

$$(V_\alpha f)(x) = f(x) - \alpha \int \frac{f(x) - f(t)}{x - t} \, d\mu(t)$$

for all continuously differentiable functions $f : \mathbb{R} \to \mathbb{C}$ with compact support. For the rest of this subsection, we suppose that $V_\alpha$ satisfies (4.1). Without loss of generality, we may further assume that $T$ is already the multiplication operator by the independent variable on $L^2(\mu)$, i.e., we identify $\mathcal{S}$ with $L^2(\mu)$, $T$ with $UTU^*$, and $T + S_\alpha$ with $U(T + S_\alpha)U^*$.

In order to prove Theorem 3 we need the following lemma.

**Lemma 4.1.** Let $\lambda \in \mathbb{R} \setminus \{\max_{\text{ess}} \sigma(T)\}$.

Then one has that the dimension of

$$(\text{Ran} \, P_\lambda) \cap (\text{Ran} \, Q_\lambda)$$

is

1. infinite if and only if $\lambda > \max_{\text{ess}} \sigma(T)$,
2. zero if and only if $\lambda < \min \sigma(T)$.

**Proof.** The idea of this proof is essentially due to the author’s supervisor, Vadim Kostrykin.

The well-known fact (see, e.g., [28, Example 5.4]) that $\text{supp} \mu_\alpha = \sigma(T + S_\alpha)$ implies that the cardinality of $(\lambda, \infty) \cap \text{supp} \mu_\alpha$ is infinite [resp. finite] if and only if $\lambda < \max_{\text{ess}} \sigma(T)$ [resp. $\lambda > \max_{\text{ess}} \sigma(T)$].

**Case 1:** The cardinality of $(\lambda, \infty) \cap \text{supp} \mu_\alpha$ is finite.

Since $\lambda > \max_{\text{ess}} \sigma(T)$, it follows that

$$\dim \text{Ran} \, E_{(\lambda, \infty)}(T + S_\alpha) < \infty \quad \text{and} \quad \dim \text{Ran} \, E_{[\lambda, \infty)}(T) < \infty.$$ 

Therefore, $\text{Ran} \, E_{(-\infty, \lambda)}(T) \cap \text{Ran} \, E_{(-\infty, \lambda)}(T + S_\alpha)$ is infinite dimensional.

**Case 2:** The cardinality of $(\lambda, \infty) \cap \text{supp} \mu_\alpha$ is infinite.

If $\lambda \leq \min \sigma(T)$ or $\lambda \leq \min \sigma(T + S_\alpha)$, then $(\text{Ran} \, P_\lambda) \cap (\text{Ran} \, Q_\lambda) = \{0\}$, as claimed. Now suppose that $\lambda > \min \sigma(T)$ and $\lambda > \min \sigma(T + S_\alpha)$.

Let $f \in (\text{Ran} \, P_\lambda) \cap (\text{Ran} \, Q_\lambda)$. Then one has

$$f(x) = 0 \text{ for } \mu_\alpha \text{-almost all } x \geq \lambda \quad \text{and} \quad (V_\alpha f)(x) = 0 \text{ for } \mu_\alpha \text{-almost all } x \geq \lambda.$$

Choose a representative $\tilde{f}$ in the equivalence class of $f$ such that $\tilde{f}(x) = 0$ for all $x \geq \lambda$. Let $r \in \left(0, \frac{\max_{\text{ess}} \sigma(T) - \lambda}{3}\right)$. According to [10, Corollary 6.4 (a)] and the fact that $\mu$ is a finite Borel measure on $\mathbb{R}$, we know that the set of continuously differentiable functions $\mathbb{R} \to \mathbb{C}$ with compact support is dense in $L^2(\mu)$ with respect to $\| \cdot \|_{L^2(\mu)}$. Thus, a standard mollifier argument shows that we can choose continuously differentiable functions $\tilde{f}_n : \mathbb{R} \to \mathbb{C}$ with compact support such that

$$\|\tilde{f}_n - \tilde{f}\|_{L^2(\mu)} < 1/n \quad \text{and} \quad \tilde{f}_n(x) = 0 \text{ for all } x \geq \lambda + r, \quad n \in \mathbb{N}.$$ 

In particular, we may insert $\tilde{f}_n$ into Formula (4.1) and obtain

$$\left(V_\alpha \tilde{f}_n \right)(x) = \alpha \int_{(-\infty, \lambda + r)} \frac{\tilde{f}_n(t)}{x - t} \, d\mu(t) \quad \text{for all } x \geq \lambda + 2r.$$
It is readily seen that
\[
(Bg)(x) := \int_{(\lambda+r)^2} \frac{g(t)}{t} \, d\nu(t), \quad x \geq \lambda + 2r,
\]
defines a bounded operator \( B : L^2 \left( \mathbb{1}_{(\lambda+r)} \right) \rightarrow L^2 \left( \mathbb{1}_{(\lambda+2r)} \right) \) with operator norm \( \leq 1/r \). It is now easy to show that
\[
(*) \quad \int_{(\lambda+r)} \frac{f(t)}{t} \, d\nu(t) = 0 \quad \text{for } \nu_\alpha \text{-almost all } x \geq \lambda + 2r.
\]
As \( r \in \left( 0, \max \sigma_{\text{ess}}(T) - \lambda \right) \) in \( (*) \) was arbitrary, we get that
\[
\int_{(\lambda+\alpha)} \frac{f(t)}{t} \, d\nu(t) = 0 \quad \text{for } \nu_\alpha \text{-almost all } x > \lambda.
\]
From now on, we may assume without loss of generality that \( \tilde{f} \) is real-valued.

Consider the holomorphic function \( C \setminus (\infty, \lambda) \rightarrow \mathbb{C} \) defined by
\[
z \mapsto \int_{(\infty, \lambda)} \frac{\tilde{f}(t)}{t} \, d\nu(t).
\]
Since \( \lambda < \max \sigma_{\text{ess}}(T) \), the identity theorem for holomorphic functions implies that
\[
\int_{(\infty, \lambda)} \frac{\tilde{f}(t)}{t} \, d\nu(t) = 0 \quad \text{for all } z \in C \setminus (\infty, \lambda).
\]
This yields
\[
(**) \quad \int_{(\infty, \lambda)} \frac{\tilde{f}(t)}{t} \, d\nu(t) = 0 \quad \text{for all } x \in \mathbb{R}, \; y > 0.
\]
Consider the positive finite Borel measure \( \nu_1 : \mathcal{B} \rightarrow [0, \infty) \) and the finite signed Borel measure \( \nu_2 : \mathcal{B} \rightarrow \mathbb{R} \) defined by
\[
\nu_1(\Omega) := \int_{\mathbb{R} \cap (\infty, \lambda)} \nu(t) \, dt, \quad \nu_2(\Omega) := \int_{\mathbb{R} \cap (\infty, \lambda)} \tilde{f}(t) \, dt;
\]
note that \( \tilde{f} \) belongs to \( L^1(\mu) \).

Define Poisson transforms \( P_{\nu_j} : \{ x + iy : x \in \mathbb{R}, y > 0 \} \rightarrow \mathbb{C} \) by
\[
P_{\nu_j}(x + iy) := y \int_{\mathbb{R}} \frac{\nu_j(t)}{(x-t)^2 + y^2} \, dt, \quad x \in \mathbb{R}, \; y > 0, \; j = 1, 2.
\]
It follows from \( (** \) that
\[
P_{\nu_j}(x + iy) = 0 \quad \text{for all } x \in \mathbb{R}, \; y > 0.
\]
Furthermore, since \( \nu_1 \) is not the trivial measure, one has
\[
P_{\nu_1}(x + iy) > 0 \quad \text{for all } x \in \mathbb{R}, \; y > 0.
\]
Now [R Proposition 2.2] implies that
\[
0 = \lim_{y \searrow 0} P_{\nu_j}(x + iy) = \tilde{f}(x) \quad \text{for } \nu_\alpha \text{-almost all } x \leq \lambda.
\]
Hence \( \tilde{f}(x) = 0 \) for \( \nu_\alpha \)-almost all \( x \in \mathbb{R} \). We conclude that \( \text{Ran } P_\lambda \cap \text{Ran } Q_\lambda \) is trivial. This finishes the proof. \( \square \)
Analogously, one shows that the following lemma holds true.

**Lemma 4.2.** Let \( \lambda \in \mathbb{R} \setminus \{ \min \sigma_{\text{ess}}(T) \} \). Then one has that the dimension of \((\text{Ker } P_\lambda) \cap (\text{Ker } Q_\lambda)\) is

1. infinite if and only if \( \lambda < \min \sigma_{\text{ess}}(T) \).
2. zero if and only if \( \lambda > \min \sigma_{\text{ess}}(T) \).

**Remark.** The proof of Lemma 4.1 does not work if \( T \) is unbounded. To see this, consider the case where the essential spectrum of \( T \) is bounded from above and \( T \) has infinitely many isolated eigenvalues greater than \( \max \sigma_{\text{ess}}(T) \).

**Proof of Theorem 3.** Taken together, Lemmas 4.1 and 4.2 imply Theorem 3. \( \square \)

### 4.2. The case when \( \varphi \) is not cyclic for \( T \)

Let \( \lambda \in \mathbb{R} \). If the kernel of \( D(\lambda) = D(-\infty, \lambda)T + S_\alpha - D(-\infty, \lambda)T \) is infinite dimensional, then \( D(\lambda) \) fulfills conditions (C1) and (C2) in Theorem 2.2. The following proposition provides a list of sufficient conditions such that the kernel of \( D(\lambda) \) is infinite dimensional.

**Proposition 4.3.** If at least one of the following four cases occurs for \( X = T + S_\alpha \), then the operator \( D(\lambda) \) is unitarily equivalent to a bounded self-adjoint Hankel operator with infinite dimensional kernel for all \( \lambda \in \mathbb{R} \).

1. The spectrum of \( X \) contains an eigenvalue of infinite multiplicity. In particular, this pertains to the case when the range of \( X \) is finite dimensional.
2. The spectrum of \( X \) contains infinitely many eigenvalues with multiplicity at least two.
3. The restricted operator \( X|_{\mathcal{E}} \) does not have a simple spectrum, where \( \mathcal{E} := \{ x \in \mathcal{H} : x \text{ is an eigenvector of } X \} \).
4. The vector \( \varphi \) is an eigenvector of \( X \).

**Proof.** By Lemma 3.3, we know that condition (C3) in Theorem 2.2 holds true for all \( \lambda \in \mathbb{R} \).

First, suppose that there exists an eigenvalue \( \lambda_0 \) of \( X = T \) with multiplicity \( m \geq 2 \), i.e. \( m \in \{ 2, 3, 4, \ldots \} \cup \{ \infty \} \). Define \( \mathcal{M} := (E_{\{\lambda_0\}}(T)\mathcal{H}) \cap \{ \varphi \}^\perp \neq \{ 0 \} \).

It is easy to show that \( \mathcal{M} \) is a closed subspace of \( \mathcal{H} \) such that

- \( \dim \mathcal{M} \geq m - 1 \),
- \( T|_{\mathcal{M}} = (T + S_\alpha)|_{\mathcal{M}} \),
- \( T(\mathcal{M}) \subset \mathcal{M} \) and \( T(\mathcal{M}^\perp) \subset \mathcal{M}^\perp \),
- \( (T + S_\alpha)(\mathcal{M}) \subset \mathcal{M} \) and \( (T + S_\alpha)(\mathcal{M}^\perp) \subset \mathcal{M}^\perp \).

Therefore, \( \mathcal{M} \) is contained in the kernel of \( D(\lambda) \) for all \( \lambda \in \mathbb{R} \).

This shows that the kernel of \( D(\lambda) \) is infinite dimensional for all \( \lambda \in \mathbb{R} \) whenever case (1) or case (2) occur for the operator \( X = T \); in the case when \( X = T + S_\alpha \), the proof runs analogously.

Now suppose that case (3) occurs for \( X = T \). Define the closed subspace \( \mathcal{N} := \overline{\text{span}} \{ T^j \varphi : j \in \mathbb{N}_0 \} \) of \( \mathcal{H} \). It is well known (see, e.g., [28 p. 195]) that

- \( T|_{\mathcal{N}^\perp} = (T + S_\alpha)|_{\mathcal{N}^\perp} \),
- \( T(\mathcal{N}) \subset \mathcal{N} \) and \( T(\mathcal{N}^\perp) \subset \mathcal{N}^\perp \),
- \( (T + S_\alpha)(\mathcal{N}) \subset \mathcal{N} \) and \( (T + S_\alpha)(\mathcal{N}^\perp) \subset \mathcal{N}^\perp \).
Therefore, $\mathfrak{N}^\perp$ is contained in the kernel of $D(\lambda)$ for all $\lambda \in \mathbb{R}$. By assumption, it follows that there exists a normed vector $\psi$ in

$$\mathfrak{N}^\perp \cap \{ x \in \mathfrak{N} : x \text{ is an eigenvector of } T \}^\perp.$$ 

According to [30, pp. 248–249], the closed subspace $\mathfrak{M}(\psi) := \{ f(T)\psi : f \in L^2(\rho) \}$ is contained in $\mathfrak{N}^\perp$, where the Borel probability measure $\rho$ is defined by $\Omega \mapsto \langle E_\Omega(T)\psi, \psi \rangle =: \rho(\Omega)$. By standard arguments, it follows that $\mathfrak{M}(\psi)$ and hence $\mathfrak{N}^\perp$ is infinite dimensional.

If $X = T + S_0$, then the proof runs analogously.

(4) This is obvious.

Now the proof is complete. \qed

The following example illustrates that $\dim \ker D(\lambda)$ may attain every value in $\mathbb{N}$, provided that $\varphi$ is not cyclic for $T$. Recall that when $\dim \ker D(\lambda)$ is neither zero nor infinity, Theorem 2.2 shows that Question 1 has to be answered negatively.

**Example 4.4.** Essentially, this is an application of Krein’s example from [12, pp. 622–624].

Let $0 < \lambda < 1$. Consider the integral operators $A_j$, $j = 0, 1$, with kernel functions

$$a_0(x, y) = \begin{cases} \sinh(x)e^{-y} & \text{if } x \leq y \\ \sinh(y)e^{-x} & \text{if } x \geq y \end{cases}$$

and

$$a_1(x, y) = \begin{cases} \cosh(x)e^{-y} & \text{if } x \leq y \\ \cosh(y)e^{-x} & \text{if } x \geq y \end{cases}$$

on the Hilbert space $L^2(0, \infty)$. By [12, pp. 622–624], we know that $A_0 - A_1$ is of rank one and that the difference $E_{(-\infty, \lambda)}(A_0) - E_{(-\infty, \lambda)}(A_1)$ is a Hankel operator. Furthermore, it was shown in [11, Theorem 1] that $E_{(-\infty, \lambda)}(A_0) - E_{(-\infty, \lambda)}(A_1)$ has a simple purely absolutely continuous spectrum filling in the interval $[-1, 1]$. In particular, the kernel of

$$E_{(-\infty, \lambda)}(A_0) - E_{(-\infty, \lambda)}(A_1)$$

is trivial. Let $k \in \mathbb{N}$. Now consider operators

$$\tilde{A}_j := A_j \oplus M : L^2(0, \infty) \oplus \mathbb{C}^k \to L^2(0, \infty) \oplus \mathbb{C}^k, \quad j = 0, 1,$$

where $M \in \mathbb{C}^{k \times k}$ is an arbitrary fixed self-adjoint matrix. Then one has

$$\dim \ker \left( E_{(-\infty, \lambda)}(\tilde{A}_0) - E_{(-\infty, \lambda)}(\tilde{A}_1) \right) = k.$$ 

5. ON NON-INVERTIBILITY OF $\tilde{D}(\lambda)$

Recall that $\tilde{D}(\lambda) = E_{(-\infty, \lambda)}(T + F) - E_{(-\infty, \lambda)}(T)$ for $\lambda \in \mathbb{R}$.

In this section, we prove the following theorem.

**Theorem 5.1.** Let $F : \mathfrak{N} \to \mathfrak{N}$ be a self-adjoint operator of finite rank. Let $\mathfrak{M}$ be the set of all points in $\sigma_{\text{ess}}(T)$ which can be approximated from above and from below by sequences in the spectrum of $T$. Then the following assertions hold true.

(1) If $\lambda \in \mathbb{R} \setminus \mathfrak{M}$, then $\tilde{D}(\lambda)$ is a trace class operator. In particular, zero belongs to the essential spectrum of $\tilde{D}(\lambda)$.

(2) Zero belongs to the essential spectrum of $\tilde{D}(\lambda)$ for all but at most countably many $\lambda$ in $\mathfrak{M}$.
Theorem 5.1 is proved below. Note that we cannot exclude the case that the exceptional set is dense in $\sigma_{\text{ess}}(T)$.

Throughout this section, we consider a self-adjoint finite rank operator

$$F = \sum_{j=1}^{N} \alpha_j \langle \cdot, \varphi_j \rangle \varphi_j : \mathcal{H} \to \mathcal{H}, \quad N \in \mathbb{N},$$

where $\varphi_j \in \mathcal{H}$ form an orthonormal system in $\mathcal{H}$ and $\alpha_j$ are nonzero real numbers.

It is easy to show that if there exists $\lambda_0$ in $\mathbb{R}$ such that

$$\dim E_{\{\lambda_0\}}(T) = \infty \quad \text{or} \quad \dim E_{\{\lambda_0\}}(T + F) = \infty,$$

then $\dim \ker \tilde{D}(\lambda) = \infty$ (cf. the proof of Proposition 4.3 (1) above) and hence $0 \in \sigma_{\text{ess}}(\tilde{D}(\lambda))$ for all $\lambda \in \mathbb{R}$. Therefore, we henceforth consider the case where

$$\dim E_{\{\lambda\}}(T) < \infty \quad \text{and} \quad \dim E_{\{\lambda\}}(T + F) < \infty \quad \text{for all} \quad \lambda \in \mathbb{R}.$$

Define the sets $\mathcal{M}$, $\mathcal{M}_-$, and $\mathcal{M}_+$ by

$$\mathcal{M} := \{ \lambda \in \sigma_{\text{ess}}(T) : \text{there exist } \lambda_{\pm}^k \in \sigma(X) \text{ such that } \lambda_{\pm}^k \not\to \lambda, \lambda_{\pm}^k \not\to \lambda \},$$

$$\mathcal{M}_- := \{ \lambda \in \sigma_{\text{ess}}(T) : \text{there exist } \lambda_{\pm}^k \in \sigma(X) \text{ such that } \lambda_{\pm}^k \not\to \lambda \} \setminus \mathcal{M},$$

$$\mathcal{M}_+ := \{ \lambda \in \sigma_{\text{ess}}(T) : \text{there exist } \lambda_{\pm}^k \in \sigma(X) \text{ such that } \lambda_{\pm}^k \not\to \lambda \} \setminus \mathcal{M},$$

where $X = T$ or $X = T + F$. We have to show that these sets do not depend on whether $X = T$ or $X = T + F$. We only show that

$$(*) \quad \{ \lambda \in \sigma_{\text{ess}}(T) : \text{there exist } \lambda_{\pm}^k \in \sigma(T) \text{ s.t. } \lambda_{\pm}^k \not\to \lambda, \lambda_{\pm}^k \not\to \lambda \} = \{ \lambda \in \sigma_{\text{ess}}(T) : \text{there exist } \lambda_{\pm}^k \in \sigma(T + F) \text{ s.t. } \lambda_{\pm}^k \not\to \lambda, \lambda_{\pm}^k \not\to \lambda \};$$

the other cases are proved analogously. $(*)$ follows immediately from the following general fact which is probably well known to specialists.

**Lemma 5.2.** Let $X : \mathcal{H} \to \mathcal{H}$ be a bounded self-adjoint operator such that the range of $E_{\{\lambda\}}(X)$ is finite dimensional for all $\lambda \in \mathbb{R}$. Suppose that $\lambda_0$ is in $\mathcal{M}$ with respect to $X$. Then $\lambda_0$ is in $\mathcal{M}$ with respect to $X + F'$ for all self-adjoint finite rank operators $F'$.

**Proof.** We show the claim by induction on $N := \text{rank}(F') \in \mathbb{N}$.

Let $N = 1$. It follows easily from the assumptions that $E_{\{\lambda\}}(X + F') \mathcal{H}$ is finite dimensional for all $\lambda \in \mathbb{R}$. Thus, according to the invariance of the essential spectrum under compact perturbations, we know that $\lambda_0$ belongs to one of the three disjoint sets $\mathcal{M}$, $\mathcal{M}_-$, $\mathcal{M}_+$ with respect to $X + F'$. Assume for contradiction that $\lambda_0$ is in $\mathcal{M}_-$ with respect to $X + F'$; the case $\lambda_0 \in \mathcal{M}_+$ with respect to $X + F'$ runs analogously. Without loss of generality, we may assume that $\varphi'$ is cyclic for $X + F'$, and

$$F' = \alpha' \langle \cdot, \varphi' \rangle \varphi'.$$

Choose $\epsilon' > 0$ such that $(\lambda_0, \lambda_0 + \epsilon')$ is contained in the resolvent set of $X + F'$. Then there exist infinitely many simple isolated eigenvalues of $X$ in $(\lambda_0, \lambda_0 + \epsilon')$. By standard Aronszajn-Donoghue theory of rank one perturbations (see, e.g., [28, Theorem 9.10]), it easily follows that there exist infinitely many eigenvalues of $X + F'$ in $(\lambda_0, \lambda_0 + \epsilon')$, which is a contradiction.

If $\text{rank}(F') \geq 2$, then we may write

$$X + F' = \left( X + \sum_{j=1}^{N-1} \alpha_j \langle \cdot, \varphi_j \rangle \varphi_j \right) + \alpha'_N \langle \cdot, \varphi'_N \rangle \varphi'_N, \quad N \geq 2.$$

Now the claim follows by induction. \qed
Proposition 5.4. One has

\[ \nu \rightarrow \tilde{\nu} \text{ follows that} \]

Lemma 5.3. Let \( \lambda \in \mathbb{R} \setminus \mathcal{M} \). Then \( \tilde{D}(\lambda) \) is a trace class operator.

Proof. First, let \( \lambda \) be in \( (\mathbb{R} \setminus \sigma_{\text{ess}}(T)) \cup \mathcal{M}_+ \). Then there exists an infinitely differentiable function \( \psi : \mathbb{R} \rightarrow \mathbb{R} \) with compact support such that

\[ E_{(\infty, \lambda)}(T + F) - E_{(\infty, \lambda)}(T) = \psi(T + F) - \psi(T). \]

Combine [2], p. 156, Equation (8.3) with [20], p. 532] and [20], Theorem 2, and it follows that \( \tilde{D}(\lambda) \) is a trace class operator.

The case where \( \lambda \) is in \( \mathcal{M}_- \) runs analogously; note that, by (5.1), we know that \( E_{(\lambda)}(T + F) - E_{(\lambda)}(T) \) is a finite rank operator. \( \Box \)

Proposition 5.4. One has \( 0 \in \sigma_{\text{ess}}(\tilde{D}(\lambda)) \) for all but at most countably many \( \lambda \in \mathbb{R} \).

In the proof of Proposition 5.4 we will use the notion of weak convergence for sequences of probability measures.

Definition 5.5. Let \( \mathcal{E} \) be a metric space. A sequence \( \nu_1, \nu_2, ... \) of Borel probability measures on \( \mathcal{E} \) is said to converge weakly to a Borel probability measure \( \nu \) on \( \mathcal{E} \) if

\[ \lim_{n \rightarrow \infty} \int f \, d\nu_n = \int f \, d\nu \text{ for every bounded continuous function } f : \mathcal{E} \rightarrow \mathbb{R}. \]

If \( \nu_1, \nu_2, ... \) converges weakly to \( \nu \), then we shall write \( \nu_n \overset{\text{w}}{\rightarrow} \nu, \ n \rightarrow \infty \).

Proof of Proposition 5.4 First, we note that if \( \lambda < \min (\sigma(T) \cup \sigma(T + F)) \) or \( \lambda > \max (\sigma(T) \cup \sigma(T + F)) \), then \( \tilde{D}(\lambda) \) is the zero operator, and there is nothing to show. So let us henceforth assume that \( \lambda \geq \min (\sigma(T) \cup \sigma(T + F)) \) and \( \lambda \leq \max (\sigma(T) \cup \sigma(T + F)) \).

The idea of the proof is to apply Weyl’s criterion (see, e.g., [28], Proposition 8.11) to a suitable sequence of normed vectors. In this proof, we denote by \( ||g||_{\sup} \) the supremum norm of a function \( g : \mathcal{K} \rightarrow \mathbb{R} \), where \( \mathcal{K} \) is a compact subset of \( \mathbb{R} \), and by \( ||A||_{\text{op}} \) the usual operator norm of an operator \( A : \mathcal{K} \rightarrow \mathcal{K} \).

Choose a sequence \((x_n)_{n \in \mathbb{N}}\) of normed vectors in \( \mathcal{K} \) such that

\[ x_1 \perp \{ \varphi_k : k = 1, ..., N \}, \quad x_2 \perp \{ x_1, \varphi_k, T \varphi_k : k = 1, ..., N \}, \quad ..., \]

\[ x_n \perp \{ x_1, ..., x_{n-1}, T^j \varphi_k : j \in \mathbb{N}_0, \ j \leq n - 1, \ k = 1, ..., N \}, \quad ..., \]

Consider sequences of Borel probability measures \((\nu_n)_{n \in \mathbb{N}}\) and \((\tilde{\nu}_n)_{n \in \mathbb{N}}\) that are defined as follows:

\[ \nu_n(\Omega) := \langle E_{\Omega}(T)x_n, x_n \rangle, \quad \tilde{\nu}_n(\Omega) := \langle E_{\Omega}(T + F)x_n, x_n \rangle, \quad \Omega \in \mathcal{B}(\mathbb{R}). \]

It is easy to see that by Prohorov’s theorem (see, e.g., [18], Proposition 7.2.3), there exist a subsequence of a subsequence of \((x_n)_{n \in \mathbb{N}}\) and Borel probability measures \( \nu \) and \( \tilde{\nu} \) with support contained in \( \sigma(T) \) and \( \sigma(T + F) \), respectively, such that

\[ \nu_{n_k} \overset{\text{w}}{\rightarrow} \nu \text{ as } k \rightarrow \infty \quad \text{and} \quad \tilde{\nu}_{n_k} \overset{\text{w}}{\rightarrow} \tilde{\nu} \text{ as } \ell \rightarrow \infty. \]

Due to this observation, we can consider the sequences \((x_{n_k})_{\ell \in \mathbb{N}}\), \((\nu_{n_k})_{\ell \in \mathbb{N}}\), and \((\tilde{\nu}_{n_k})_{\ell \in \mathbb{N}}\) which will be denoted again by \((x_{n_k})_{\ell \in \mathbb{N}}\), \((\nu_{n_k})_{\ell \in \mathbb{N}}\), and \((\tilde{\nu}_{n_k})_{\ell \in \mathbb{N}}\).

Put \( \mathcal{N}_T := \{ \mu \in \mathbb{R} : \nu(\{ \mu \}) > 0 \} \) and \( \mathcal{N}_{T + F} := \{ \mu \in \mathbb{R} : \tilde{\nu}(\{ \mu \}) > 0 \} \). Then the set \( \mathcal{N}_T \cup \mathcal{N}_{T + F} \) is at most countable. Consider the case where \( \lambda \) does
not belong to $N_T \cup N_{T,F}$. Define $\xi := \min \{ \min \sigma(T), \min \sigma(T + F) \} - 1$. Consider the continuous functions $f_m : \mathbb{R} \to \mathbb{R}$, $m \in \mathbb{N}$, that are defined by

$$f_m(t) := (1 + m(t - \xi)) \cdot 1_{[\xi - 1/m, \xi]}(t) + 1_{(\xi, \lambda)}(t) + (1 - m(t - \lambda)) \cdot 1_{[\lambda, \lambda + 1/m]}(t).$$

The figure below shows (qualitatively) the graph of $f_m$.

**FIGURE 1.** The graph of $f_m$.

For all $m \in \mathbb{N}$, choose polynomials $p_{m,k}$ such that

$$\|f_m - p_{m,k}\|_{\infty, \mathcal{K}} \to 0 \quad \text{as} \quad k \to \infty,$$

where $\mathcal{K} := \left[ \min \left( \min \sigma(T) \cup \sigma(T + F) \right) - 10, \max \left( \min \sigma(T) \cup \sigma(T + F) \right) + 10 \right]$.

By construction of $(x_n)_{n \in \mathbb{N}}$, one has

$$p_{m,k}(T + F)x_n = p_{m,k}(T)x_n \quad \text{for all} \quad n > \text{degree of} \quad p_{m,k}.$$

For all $m \in \mathbb{N}$, the function $|1_{(-\infty,\lambda)} - f_m|^2$ is bounded, measurable, and continuous except for a set of both $\psi$-measure zero and $\hat{\psi}$-measure zero.

Now (5.3) and the Portmanteau theorem (see, e. g., [9, Theorem 13.16 (i) and (iii)]) imply

$$\limsup_{n \to \infty} \left\| \left( E_{(-\infty,\lambda)}(T + F) - E_{(-\infty,\lambda)}(T) \right) x_n \right\|$$

$$\leq \left( \int_{\mathbb{R}} |1_{(-\infty,\lambda)}(t) - f_m(t)|^2 d\psi(t) \right)^{1/2}$$

$$+ \left( \int_{\mathbb{R}} |1_{(-\infty,\lambda)}(s) - f_m(s)|^2 d\hat{\psi}(s) \right)^{1/2}$$

$$+ \|f_m(T) - p_{m,k}(T)\|_{\text{op}}$$

$$+ \|f_m(T + F) - p_{m,k}(T + F)\|_{\text{op}}$$

for all $m \in \mathbb{N}$ and all $k \in \mathbb{N}$. First, we send $k \to \infty$ and then we take the limit $m \to \infty$. As $m \to \infty$, the sequence $|1_{(-\infty,\lambda)} - f_m|^2$ converges to zero pointwise almost everywhere with respect to both $\psi$ and $\hat{\psi}$. Now (5.2) and the dominated convergence theorem imply

$$\lim_{n \to \infty} \left\| \left( E_{(-\infty,\lambda)}(T + F) - E_{(-\infty,\lambda)}(T) \right) x_n \right\| = 0.$$

Recall that $(x_n)_{n \in \mathbb{N}}$ is an orthonormal sequence. Thus, an application of Weyl’s criterion (see, e. g., [28, Proposition 8.11]) concludes the proof. \qed

**Remark.** If $T$ is unbounded, then the spectrum of $T$ is unbounded, so that the proof of Proposition 5.4 does not work. For instance, we used the compactness of the spectra in order to uniformly approximate $f_m$ by polynomials.

Moreover, it is unclear whether an orthonormal sequence $(x_n)_{n \in \mathbb{N}}$ as in the proof of Proposition 5.4 can be found in the domain of $T$. 
Proof of Theorem 5.1. Taken together, Lemma 5.3 and Proposition 5.4 imply Theorem 5.1. □

In order to apply a result of Pushnitski [22] to $\tilde{D}(\lambda)$, we check the corresponding assumptions stated in [22, p. 228].

First, define the finite rank operators $G, F_0 : H \to H$ by

$$G = |F|^{1/2} = \sum_{j=1}^{N} |\alpha_j|^{1/2} \langle \cdot, \varphi_j \rangle \varphi_j$$

and

$$F_0 = \text{sign}(F) = \sum_{j=1}^{N} \text{sign}(\alpha_j) \langle \cdot, \varphi_j \rangle \varphi_j,$$

respectively. Obviously, one has $F = G^*F_0G$. Define the operator-valued functions $h_0$ and $h$ on $\mathbb{R}$ by

$$h_0(\lambda) = GE_{(-\infty, \lambda)}(T)G^*, \quad h(\lambda) = GE_{(-\infty, \lambda)}(T+F)G^*, \quad \lambda \in \mathbb{R}.$$ 

In order to fulfill [22, Hypothesis 1.1], we need the following assumptions.

**Hypothesis.** Suppose that there exists an open interval $\delta$ contained in the absolutely continuous spectrum of $T$. Next, we assume that the derivates $\hat{h}_0(\lambda) = \frac{d}{d\lambda}h_0(\lambda)$ and $\hat{h}(\lambda) = \frac{d}{d\lambda}h(\lambda)$ exist in operator norm for all $\lambda \in \delta$, and that the maps $\delta \ni \lambda \mapsto \hat{h}_0(\lambda)$ and $\delta \ni \lambda \mapsto \hat{h}(\lambda)$ are Hölder continuous (with some positive exponent) in the operator norm.

Now [22, Theorem 1.1] yields that for all $\lambda \in \delta$, there exists a nonnegative real number $a$ such that

$$\sigma_{\text{ess}}(\tilde{D}(\lambda)) = [-a, a].$$

The number $a$ depends on $\lambda$ and can be expressed in terms of the scattering matrix for the pair $T, T+F,$ see [22, Formula (1.3)].

**Example 5.6.** Again, consider Kreǐn’s example [12, pp. 622–624]. That is, $\mathfrak{H} = L^2(0, \infty)$, the initial operator $T = A_0$ is the integral operator from Example 4.4, $\varphi(x) = \sqrt{2} e^{-x}$, and $\alpha = 1/2$. Put $\delta = (0, 1)$. Then Pushnitski has shown in [22, Section 1.3] that, by [22, Theorem 1.1], one has $\sigma_{\text{ess}}(D(\lambda)) = [-1, 1]$ for all $0 < \lambda < 1$.

In particular, the operator $D(\lambda)$ fulfills condition (C2) in Theorem 2.2 for all $0 < \lambda < 1$.

6. Proof of Theorem 2

(1) follows easily from Lemma 3.3 and Theorem 2.2, because

$$D(\lambda) = E_{(-\infty, \lambda)}(T + S_{\alpha}) - E_{(-\infty, \lambda)}(T)$$

$$= E_{[\lambda, \infty)}(T) - E_{[\lambda, \infty)}(T + S_{\alpha})$$

is a finite rank operator for all $\lambda$ in $(-\infty, \min \sigma_{\text{ess}}(T)) \cup (\max \sigma_{\text{ess}}(T), \infty)$.

(2) is a direct consequence of Lemma 3.3, Theorem 3, Theorem 5.1, and Theorem 2.2.

(3) If $\varphi$ is not cyclic for $T$, then, obviously, zero is an eigenvalue of $D(\lambda)$ for all $\lambda \in \mathbb{R}$. Hence, according to [16, Theorem 1.1], we know that there exists a Hankel operator $\Gamma$, depending on $\lambda$, such that $\sigma(\Gamma) = \sigma(D(\lambda))$. This concludes the proof of Theorem 2.
7. Some Examples

First, we consider the case when \( \phi \) is cyclic for \( T \).

**Example 7.1.** Once again, consider Krein’s example [12, pp. 622–624].

The operators \( T = A_0 \) and \( T + \frac{1}{2} \langle \cdot, \phi \rangle \phi = A_1 \), where \( \phi(x) = \sqrt{2} \, e^{-x} \), from Example 4.4 both have a simple purely absolutely continuous spectrum filling in the interval \([0, 1]\). Therefore, \( D(\lambda) \) is the zero operator for all \( \lambda \in \mathbb{R} \setminus (0, 1) \).

In Example 5.6 above, we have seen that the operator \( D(\lambda) \) fulfills condition (C2) in Theorem 2.2 for all \( 0 < \lambda < 1 \).

\( \ast \) The function \( \phi \) is cyclic for \( T \).

Hence, Theorem 3 implies that the kernel of \( D(\lambda) \) is trivial for all \( 0 < \lambda < 1 \).

Recall that, by Lemma 3.3, we know that \( D(\lambda) \) fulfills condition (C3) in Theorem 2.2 for all \( \lambda \in \mathbb{R} \).

Therefore, an application of Theorem 2.2 yields that \( D(\lambda) \) is unitarily equivalent to a bounded self-adjoint Hankel operator for all \( \lambda \in \mathbb{R} \).

Note that, in this example, there are no exceptional points.

**Proof of \( \ast \).** Let \( k \) be in \( \mathbb{N}_0 \). Define the \( k \)th Laguerre polynomial \( L_k \) on \((0, \infty)\) by

\[
L_k(x) := e^x \frac{d^k}{dx^k} (x^k e^{-x}).
\]

Furthermore, define \( \psi_k \) on \((0, \infty)\) by \( \psi_k(x) := x^k e^{-x} \).

A straightforward computation shows that

\[
(A_0 \psi_k)(x) = \frac{1}{2} e^{-x} \left\{ \frac{x^{k+1}}{k+1} + \frac{1}{2^{k+1}} \sum_{\ell=0}^{k-1} (2x)^{k-\ell} \frac{k!}{(k-\ell)!} \right\}.
\]

By induction on \( n \in \mathbb{N}_0 \), it easily follows that \( p \cdot \phi \) belongs to the linear span of \( A_0^l \phi, \ell \in \mathbb{N}_0, \ell \leq n \), for all polynomials \( p \) of degree \( \leq n \).

In particular, the functions \( \phi_j \) on \((0, \infty)\) defined by \( \phi_j(x) := \sqrt{2} \, L_j(2x) e^{-x} \)

are elements of \( \text{span}\{ A_0^l \phi : \ell \in \mathbb{N}_0, \ell \leq n \} \) for all \( j \in \mathbb{N}_0 \) with \( j \leq n \).

Since \((\phi_j)_{j \in \mathbb{N}_0}\) is an orthonormal basis of \( L^2(0, \infty) \), it follows that \( \phi \) is cyclic for \( T \).

Example 7.1 suggests the conjecture that Theorem 2.2 can be strengthened to hold up to a finite exceptional set.

Next, we consider different examples where the multiplicity in the spectrum of \( T \) is such that we can apply Proposition 4.3 so that \( D(\lambda) \) is unitarily equivalent to a bounded self-adjoint Hankel operator for all \( \lambda \in \mathbb{R} \). Since, in these situations, \( S_\alpha \) can be an arbitrary self-adjoint operator of rank one, we do not mention it in the following example.
Example 7.2.  
(1) Let $T$ be an arbitrary orthogonal projection on $H$. Then zero or one is an eigenvalue of $T$ with infinite multiplicity, and we can apply Proposition 4.3.

(2) Put $H = L^2(0, \infty)$ and let $T$ be the Carleman operator, i.e., the bounded Hankel operator such that 

$$(Tg)(x) = \int_0^\infty \frac{g(y)}{x+y} \, dy$$

for all continuous functions $g : (0, \infty) \to \mathbb{C}$ with compact support.

It is well known (see, e.g., [21, Chapter 10, Theorem 2.3]) that the Carleman operator has a purely absolutely continuous spectrum of uniform multiplicity two filling in the interval $[0, \pi]$. Therefore, Proposition 4.3 can be applied.

Jacobi operators. Consider a bounded self-adjoint Jacobi operator $H$ acting on the Hilbert space $\ell^2(\mathbb{Z})$ of complex square summable two-sided sequences. More precisely, suppose that there exist bounded real-valued sequences $a = (a_n)_n$ and $b = (b_n)_n$ with $a_n > 0$ for all $n \in \mathbb{Z}$ such that 

$$(Hx)_n = a_n x_{n+1} + a_{n-1} x_{n-1} + b_n x_n, \quad n \in \mathbb{Z},$$

cf. [31] Theorem 1.5 and Lemma 1.6. The following result is well known.

Proposition 7.3 (see [31], Lemma 3.6). Let $H$ be a bounded self-adjoint Jacobi operator on $\ell^2(\mathbb{Z})$. Then the singular spectrum of $H$ has spectral multiplicity one, and the absolutely continuous spectrum of $H$ has multiplicity at most two.

In the case where $H$ has a simple spectrum, there exists a normed cyclic vector for $H$, and we can apply Theorem 2. Otherwise, $H$ fulfills condition (3) in Proposition 4.3. Let us discuss some examples in the latter case with $T = H$. Again, $S_\alpha$ can be an arbitrary self-adjoint operator of rank one, and we do not mention it in the following.

Example 7.4. Consider the discrete Schrödinger operator $H = H_V$ on $\ell^2(\mathbb{Z})$ with bounded potential $V : \mathbb{Z} \to \mathbb{R}$, 

$$(H_V x)_n = x_{n+1} + x_{n-1} + V_n x_n, \quad n \in \mathbb{Z}.$$ 

If the spectrum of $H_V$ contains only finitely many points outside of the interval $[-2, 2]$, then [21, Theorem 2] implies that $H_V$ has a purely absolutely continuous spectrum of multiplicity two on $[-2, 2]$.

It is well known that the free Jacobi operator $H_0$ with $V = 0$ has a purely absolutely continuous spectrum of multiplicity two filling in the interval $[-2, 2]$.

Last, we consider the almost Mathieu operator $H = H_{\kappa, \beta, \theta} : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ defined by 

$$(H x)_n = x_{n+1} + x_{n-1} + 2\kappa \cos \left(2\pi(\theta + n\beta)\right) x_n, \quad n \in \mathbb{Z},$$

where $\kappa \in \mathbb{R} \setminus \{0\}$ and $\beta, \theta \in \mathbb{R}$. In fact, it suffices to consider $\beta, \theta \in \mathbb{R}/\mathbb{Z}$.

The almost Mathieu operator plays an important role in physics, see, for instance, the review [14] and the references therein.

Here, we are interested in cases where $H$ fulfills at least one condition in Proposition 4.3 above.
Proposition 7.5.  

(1) If $\beta$ is rational, then for all $\kappa$ and $\theta$ the Jacobi operator $H_{\kappa,\beta,\theta}$ is periodic and has a purely absolutely continuous spectrum of uniform multiplicity two.

(2) If $\beta$ is irrational and $|\kappa| < 1$, then for all $\theta$ the almost Mathieu operator $H_{\kappa,\beta,\theta}$ has a purely absolutely continuous spectrum of uniform multiplicity two.

Proof. (1) If $\beta$ is rational, then $H_{\kappa,\beta,\theta}$ is a periodic Jacobi operator. Hence, it is well known (see, e.g., [31, p. 122]) that the spectrum of $H_{\kappa,\beta,\theta}$ is purely absolutely continuous. According to [5, Theorem 9.1], we know that the absolutely continuous spectrum of $H_{\kappa,\beta,\theta}$ is uniformly of multiplicity two. This proves (1).

(2) Suppose that $\beta$ is irrational. Avila has shown (see [1, Main Theorem]) that the almost Mathieu operator $H_{\kappa,\beta,\theta}$ has a purely absolutely continuous spectrum if and only if $|\kappa| < 1$. Again, [5, Theorem 9.1] implies that the absolutely continuous spectrum of $H_{\kappa,\beta,\theta}$ is uniformly of multiplicity two. This finishes the proof. \hfill $\square$

Problems 4–6 of Simon’s list [29] are concerned with the almost Mathieu operator. Avila’s result [1, Main Theorem], which we used in the above proof, is a solution for Problem 6 in [29].

8. The semibounded case

Up to now, the self-adjoint operator $T$ was assumed to be bounded. In this section, which is based on Krien’s approach in [12, pp. 622–623], we will deal with the case when $T$ is semibounded but not bounded. As before, $S_\alpha$ is a self-adjoint operator of rank one and $D(\lambda) = E_{(-\infty,\lambda)}(T + S_\alpha) - E_{(-\infty,\lambda)}(T)$ for $\lambda \in \mathbb{R}$.

First, consider the case when $T$ is bounded from below. Choose $c \in \mathbb{R}$ such that

$$T + cI \geq 0 \quad \text{and} \quad T + S_\alpha + cI \geq 0.$$  

It suffices to consider $D(\lambda)$ for $\lambda \geq -c$. Compute

$$D(\lambda) = E_{[\lambda,\infty)}(T) - E_{[\lambda,\infty)}(T + S_\alpha)$$
$$= E_{(-\infty,\mu)}((T + (1 + c)I)^{-1}) - E_{(-\infty,\mu)}((T + S_\alpha + (1 + c)I)^{-1})$$

and $(T + S_\alpha + (1 + c)I)^{-1} = (T + (1 + c)I)^{-1} + \alpha'(\cdot, \varphi')\varphi'$. Here $\mu = \frac{1}{\lambda + 1 + c}$, $\varphi' := \frac{T + (1 + c)I}{T + (1 + c)I}^{-1}\varphi$, and $\alpha' \in \mathbb{R}$ is such that

$$\alpha'(\cdot, \varphi')\varphi' = -(T + S_\alpha + (1 + c)I)^{-1}S_\alpha(T + (1 + c)I)^{-1}. $$

We have shown:

Lemma 8.1. Let $T$ be a self-adjoint operator which is bounded from below. Let $c$ be such that (8.1) holds. Then $D(\lambda) = 0$ for all $\lambda < -c$ and

$$D(\lambda) = E_{(-\infty,\mu)}(T') - E_{(-\infty,\mu)}(T' + \alpha'(\cdot, \varphi')\varphi') \quad \text{for all } \lambda \geq -c.$$ 

Here $\mu = \frac{1}{\lambda + 1 + c}$, $T' = (T + (1 + c)I)^{-1}$, $\varphi' = \frac{T'\varphi}{\|T'\varphi\|}$, and $\alpha'$ is such that (8.2) holds.

The right-hand side of (8.3) is of the form investigated in Sections 1–7 above, except that here we consider closed intervals $(-\infty, \mu]$. 


Remark. If we consider the difference of the spectral projections associated with the closed interval \((-\infty, \mu]\) instead of the open interval \((-\infty, \mu)\), then all assertions in Theorem 2, Theorem 3, Lemma 3.3, Proposition 4.3, and Theorem 5.1 remain true. All proofs can easily be modified.

In particular, we obtain the following corollary.

Corollary 8.2. Let \(T'\) be a bounded self-adjoint operator on \(\mathcal{H}\) and define \(B(\mu) := E_{(-\infty, \mu]}(T') - E_{(-\infty, \mu]}(T' + S_\alpha)\) for \(\mu \in \mathbb{R}\). Then one has:

1. The operator \(B(\mu)\) is unitarily equivalent to a self-adjoint Hankel operator of finite rank for all \(\mu \in \mathbb{R} \setminus [\min \sigma_{\text{ess}}(T'), \max \sigma_{\text{ess}}(T')]\).

2. Suppose that \(\varphi\) is cyclic for \(T'\). Then \(B(\mu)\) is unitarily equivalent to a bounded self-adjoint Hankel operator for all \(\mu \in \mathbb{R} \setminus \sigma_{\text{ess}}(T')\) and for all but at most countably many \(\mu \in \sigma_{\text{ess}}(T')\).

3. If \(\varphi\) is not cyclic for \(T'\), then for all \(\mu \in \mathbb{R}\) there exists a Hankel operator \(\Gamma\), depending on \(\mu\), such that the spectra of \(B(\mu)\) and \(\Gamma\) coincide.

The case when \(T\) is bounded from below can now be pulled back to the bounded case.

Proposition 8.3. Suppose that \(T\) is a self-adjoint operator which is bounded from below but not bounded. Assume further that the spectrum of \(T\) is not purely discrete and that the vector \(\varphi\) is cyclic for \(T\). Then the kernel of \(D(\lambda)\) is trivial for all \(\lambda > \min \sigma_{\text{ess}}(T)\).

Furthermore, \(D(\lambda)\) is unitarily equivalent to a bounded self-adjoint Hankel operator for all \(\lambda \in \mathbb{R} \setminus \sigma_{\text{ess}}(T)\) and for all but at most countably many \(\lambda \in \sigma_{\text{ess}}(T)\).

Proof. Let \(c\) be such that (8.1) holds.

It is easy to show that \((T + (1 + c)I)^{-1}\varphi\) is cyclic for \((T + (1 + c)I)^{-1}\) if \(\varphi\) is cyclic for \(T\).

Furthermore, it is easy to show that the function \(x \mapsto \frac{1}{x + 1 + c}\) is one-to-one from \(\sigma_{\text{ess}}(T)\) onto \(\sigma_{\text{ess}}((T + (1 + c)I)^{-1}) \setminus \{0\}\).

One has that \(\min \sigma_{\text{ess}}((T + (1 + c)I)^{-1}) = 0\) and, since the spectrum of \(T\) is not purely discrete, \(\max \sigma_{\text{ess}}((T + (1 + c)I)^{-1}) = \frac{1}{\lambda_0 + 1 + c}\), where \(\lambda_0 := \min \sigma_{\text{ess}}(T)\).

Therefore, \(\mu = \frac{1}{\lambda_0 + 1 + c}\) belongs to the open interval \((0, \frac{1}{\lambda_0 + 1 + c})\) if and only if \(\lambda > \lambda_0\).

In view of Lemma 8.1 and the subsequent remark, the claims follow.

Moreover, standard computations show:

Corollary 8.4. We obtain the same list of sufficient conditions for \(D(\lambda)\) to be unitarily equivalent to a bounded self-adjoint Hankel operator as in Proposition 4.3 above.

Proof. Let \(X = T\) or \(X = T + S_\alpha\) and let \(c\) be such that (8.1) holds. One has:

- The real number \(\lambda\) is an eigenvalue of \(X\) with multiplicity \(k \in \mathbb{N} \cup \{\infty\}\) if and only if \(\frac{1}{\lambda_0 + 1 + c}\) is an eigenvalue of \((X + (1 + c)I)^{-1}\) with the same multiplicity \(k\).

- The restricted operator \(X_{\mathcal{E}}\) does not have a simple spectrum if and only if the restricted operator \((X + (1 + c)I)^{-1}_{\mathcal{E}}\) does not have a simple spectrum, where \(\mathcal{E} := \{x \in \mathcal{H} : x\) is an eigenvector of \(X\}\).
In view of Lemma [8] and the subsequent remark, the claim follows.

Now, consider the case when $T$ is bounded from above. Choose $c \in \mathbb{R}$ such that

$$T - cI \leq 0 \quad \text{and} \quad T + S_\alpha - cI \leq 0.$$ 

It suffices to consider $D(\lambda)$ for $\lambda \leq c$. Compute

$$D(\lambda) = E_{(\mu, \infty)}((T + S_\alpha - (1 + c)I)^{-1}) - E_{(\mu, \infty)}((T - (1 + c)I)^{-1})$$

$$= E_{(-\infty, \mu)}((T - (1 + c)I)^{-1}) - E_{[-\infty, \mu]}((T + S_\alpha - (1 + c)I)^{-1})$$

and $(T + S_\alpha - (1 + c)I)^{-1} = (T - (1 + c)I)^{-1} + \alpha''(\cdot, \varphi'').$ Here $\mu = \frac{1}{\lambda - 1 - c},$

$$\varphi'' := \frac{(T - (1 + c)I)^{-1}\varphi}{\|{(T - (1 + c)I)^{-1}\varphi}\|}, \quad \text{and} \quad \alpha'' \in \mathbb{R}$$

is such that

$$\alpha''(\cdot, \varphi'')\varphi'' = - (T + S_\alpha - (1 + c)I)^{-1}S_\alpha(T - (1 + c)I)^{-1}.$$ 

Now proceed analogously to the case when $T$ is bounded from below.

It follows that Proposition [8] holds in the case when $T$ is bounded from above but not bounded if we replace $\lambda > \min \sigma_{\text{ess}}(T)$ by $\lambda < \max \sigma_{\text{ess}}(T)$.

Furthermore, we obtain the same list of sufficient conditions for $D(\lambda)$ to be unitarily equivalent to a bounded self-adjoint Hankel operator as in Proposition [4] above.

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