IDENTIFICATION AND ESTIMATION OF NONIGNORABLE MISSING OUTCOME MEAN WITHOUT IDENTIFYING THE FULL DATA DISTRIBUTION

BY WEI LI 1, WANG MIAO 2 AND ERIC TCHETGEN TCHETGEN 3

1Center for Applied Statistics and School of Statistics, Renmin University of China, weilistat@ruc.edu.cn
2School of Mathematical Sciences, Peking University, mwfy@pku.edu.cn
3Department of Statistics, University of Pennsylvania, ett@wharton.upenn.edu

We consider the problem of making inference about the population outcome mean of an outcome variable subject to nonignorable missingness. By leveraging a so-called shadow variable for the outcome, we propose a novel condition that ensures nonparametric identification of the outcome mean, although the full data distribution is not identified. The identifying condition requires the existence of a function as a solution to a representer equation that connects the shadow variable to the outcome mean. Under this condition, we use sieves to nonparametrically solve the representer equation and propose an estimator which avoids modeling the propensity score or the outcome regression. We establish the asymptotic properties of the proposed estimator. We also show that the estimator is locally efficient and attains the semiparametric efficiency bound for the shadow variable model under certain regularity conditions. We illustrate the proposed approach via simulations and a real data application on home pricing.

1. Introduction. Missing response data are frequently encountered in social science and biomedical studies, due to reluctance to answer sensitive survey questions or dropout of follow-up in clinical trials. Certain characteristics of the missing data mechanism is used to define a taxonomy to describe the missingness process (Rubin, 1976; Little and Rubin, 2002). The latter is called missing at random (MAR) if the propensity of missingness conditional on all study variables is unrelated to the missing values. Otherwise, it is called missing not at random (MNAR) or nonignorable. MAR has been commonly used for statistical analysis in the presence of missing data; however, in many fields of study, suspicion that the missing data mechanism may be nonignorable is often warranted (Scharfstein et al., 1999; Robins et al., 2000; Rotnitzky and Robins, 1997; Rotnitzky et al., 1998; Ibrahim et al., 1999; Zhao and Shao, 2015). For example, nonresponse rates in surveys about income tend to be higher for low socio-economic groups (Kim and Yu, 2011). In another example, efforts to estimate HIV prevalence in developing countries via household HIV survey and testing such as the well-known Demographic and Health Survey, are likewise subject to nonignorable missing data on participants’ HIV status due to highly selective non-participation in the HIV testing component of the survey study (Tchetgen Tchetgen and Wirth, 2017). There currently exist a variety of methods for the analysis of MAR data, such as likelihood based inference (Dempster et al., 1977), multiple imputation (Rubin, 2004), inverse probability weighting (Horvitz and Thompson, 1952; Robins et al., 1994), and doubly robust methods (Van der Laan and Robins, 2003; Bang and Robins, 2005; Tsiatis, 2006). However, these methods can result in severe bias and invalid inference in the presence of nonignorable missing data.

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In this paper, we focus on estimation of the population mean of an outcome variable subject to nonignorable missingness. Estimating an outcome mean is a goal common in sampling survey and causal inference, and thus is of significant practical importance. However, there are several difficulties for analysis of nonignorable missing data. The first challenge is identification, which means that the parameter of interest is uniquely determined from observed data distribution. Identification is straightforward under MAR as the conditional outcome distribution in complete-cases equals that in incomplete cases given fully observed covariates, whereas it becomes difficult under MNAR because the selection bias due to missing values is no longer negligible. Even if stringent fully-parametric models are imposed on both the propensity score and the outcome regression, identification may not be achieved; for counterexamples, see Miao et al. (2016); Wang et al. (2014). To resolve the identification difficulty, previous researchers (Robins et al., 2000; Kim and Yu, 2011) have assumed that the selection bias is known or estimated from external studies, but this approach should be used rather as a sensitivity analysis, until the validity of the selection bias assumption is assessed. Without knowing the selection bias, identification can be achieved by leveraging fully observed auxiliary variables that are available in many empirical studies. For instance, instrumental variables, which are related to the nonresponse propensity but not related to the outcome given covariates, have been used in missing data analysis since Heckman (1979). The corresponding semiparametric theory and inference are recently established by Liu et al. (2020); Sun et al. (2018); Tchetgen Tchetgen and Wirth (2017); Das et al. (2003). Recently, an alternative approach called the shadow variable approach has grown in popularity in sampling survey and missing data analysis. In contrast to the instrumental variable, this approach entails a shadow variable that is associated with the outcome but independent of the missingness process given covariates and the outcome itself. Shadow variable is increasingly popular in sampling designs and is available in many applications (Kott, 2014). For example, Zahner et al. (1992) and Ibrahim et al. (2001) considered a study of the children’s mental health evaluated through their teachers’ assessments in Connecticut. However, the data for the teachers’ assessments are subject to nonignorable missingness. As a proxy of the teacher’s assessment, a separate parent report is available for all children in this study. The parent report is likely to be correlated with the teacher’s assessment, but is unlikely to be related to the teacher’s response rate given the teacher’s assessment and fully observed covariates. Hence, the parental assessment is regarded as a shadow variable in this study. The shadow variable design is quite general. In health and social sciences, an expensive outcome is routinely available only for a subset of patients, but one or more surrogates may be fully observed. For instance, Robins et al. (1994) considered a cardiovascular disease setting where, due to high cost of laboratory analyses, and to the small amount of stored serum per subject (about 2% of study subjects had stored serum thawed and assayed for antioxidants serum vitamin A and vitamin E), error prone surrogate measurements of the biomarkers derived from self-reported dietary questionnaire were obtained for all subjects. For identification, the authors assumed the gold standard biomarker measurements were missing at random given surrogate measurements. In such a setting, an alternative more realistic shadow variable assumption would entail that the assaying selection process is nonignorable, however, that the surrogate measurements can be rendered non-informative about the assaying process upon conditioning on the biomarker’s value, as the former is merely a proxy for the latter. Other important related settings include the semi-supervised set up in comparative effectiveness research where the true outcome is measured only for a small fraction of the data, e.g. diagnosis requiring a costly panel of physicians while surrogates are obtained from databases including ICD-9 codes for certain comorbidities. Instead of assuming MAR or nondifferential measurement error, shadow variable conditions may be more appropriate in presence of informative selection bias. By leveraging a shadow variable, D’Haultfoeuille (2010); Miao et al. (2019) established identification conditions for nonparametric models. In related work, Wang et al. (2014); Shao and
Wang (2016); Tang et al. (2003); Zhao and Shao (2015); Zhao and Ma (2018) proposed identification conditions for a suite of parametric and semiparametric models that requires either the propensity score or the outcome regression, or both to be parametric. All existing shadow variable approaches impose sufficiently strong conditions to identify the full data distribution, although in practice one may only be interested in a parameter or a functional of the outcome which may be identifiable even when the full data is not.

The second challenge for MNAR data is the threat of bias due to model misspecification in estimation, after identification is established. Likelihood-based inference (Greenlees et al., 1982; Tang et al., 2014, 2003; Zhao and Shao, 2015), imputation-based methods (Kim and Yu, 2011; Zhao et al., 2017), inverse probability weighting (Wang et al., 2014), and doubly robust estimation (Robins and Rotnitzky, 2001; Miao and Tchetgen Tchetgen, 2016; Miao et al., 2019; Liu et al., 2020) have been developed for analysis of MNAR data. These existing estimation methods require correct model specification of either the propensity score or the outcome regression, or both. However, bias may arise due to specification error of parametric models as they have limited flexibility, and moreover, model misspecification is more likely to appear in the presence of missing values.

In this paper, we develop a novel strategy to nonparametrically identify and estimate the outcome mean. In contrast to previous approaches in the literature, this work has the following distinctive features and makes several contributions to nonignorable missing data literature. First, given a shadow variable, we are the first to directly work on the identification of the outcome mean without identifying the full data distribution. Our identifying condition involves the existence of a function as a solution to a representer equation that relates the outcome mean and the shadow variable. Second, under the identifying condition, we propose nonparametric estimation that no longer involves parametrically modeling the propensity score or the outcome regression. For estimation, since the solution to the representer equation may not be unique, we first construct a consistent estimator of the solution set. We use the method of sieves to approximate unknown smooth functions as possible solutions and estimate corresponding coefficients by applying a minimum distance procedure, inspired by semiparametric and nonparametric econometric literature (Newey and Powell, 2003; Ai and Chen, 2003; Santos, 2011; Chen and Pouzo, 2012). After the solution set is consistently estimated, we adapt the theory of extremum estimators to find from the estimated set a consistent estimator for an appropriately chosen solution. Based on such an estimator, we propose a representer-based estimator for the outcome mean. Under certain regularity conditions, we establish the asymptotic properties for the proposed estimator, including consistency and asymptotic normality. The proposed estimator is then shown to be semiparametric locally efficient for the outcome mean under our shadow variable model. To the best of our knowledge, the proposed procedure is the first to provide the √n-estimation of the outcome mean without requiring the underlying data distribution to be identified in the nonignorable missing data literature.

The remainder of this paper is organized as follows. In Section 2, we establish nonparametric identification of the outcome mean under a representer-based condition. In Section 3, we develop a consistent estimator for the solution set of the representer equation, and identify from the estimated set a consistent estimator for a well-defined fixed solution, which we employ to construct an estimator for the outcome mean. We then establish the asymptotic theory of the proposed estimator, and its semiparametric local efficiency. In Section 4, we study the finite-sample performance of the proposed approach via both simulation studies and a home pricing real data example. We conclude with a discussion in Section 5 and relegate proofs to the Appendix.
2. Identification. Let $X$ denote a vector of fully observed covariates, $Y$ the outcome variable that is subject to missingness, and $R$ the missingness indicator with $R = 1$ if $Y$ is observed and $R = 0$ otherwise. The missingness process may depend on the missing values. We let $f(\cdot)$ denote the probability density or mass function of a random variable (vector). The observed data contain $n$ independent and identically distributed realizations of $(R, X, Y, Z)$ with the values of $Y$ missing for $R = 0$. We are interested in making inference about the outcome mean, $\mu = E(Y)$. Suppose we observe an additional shadow variable $Z$ that meets the following assumption.

**ASSUMPTION 1 (Shadow variable).** (i) $Z \perp R \mid (X,Y)$; (ii) $Z \not\perp Y \mid X$.

Assumption 1 reveals that the shadow variable does not affect the missingness process given the covariates and outcome, and it is associated with the outcome given the covariates. This assumption has been used for adjustment of selection bias in sampling surveys (Kott, 2014) and in missing data literature (D’Haultfœuille, 2010; Wang et al., 2014; Zhao and Shao, 2015; Miao and Tchetgen Tchetgen, 2016). Examples and extensive discussions about the assumption can be found in Zahner et al. (1992); Ibrahim et al. (2001); Miao and Tchetgen Tchetgen (2016); Miao et al. (2019); Zhao and Ma (2018, 2021).

We further make the following representer assumption for identification of $\mu$.

**ASSUMPTION 2 (Representer).** There exist a function $\delta_0(X, Z)$ such that

\begin{equation}
E\{\delta_0(X, Z) \mid R = 1, X, Y\} = Y.
\end{equation}

Assumption 2 is a novel identification condition in the missing data literature. This assumption is motivated by the condition for identification and $\sqrt{n}$-estimability of linear functionals of nonparametric regression models with endogenous regressors in Severini and Tripathi (2012). Equation (1) relates the outcome mean and the shadow variable via the representer function $\delta_0(x, z)$. The equation is a Fredholm integral equation of the first kind, and the technical conditions for existence of solutions to the Fredholm integral equation of the first kind are discussed in Miao et al. (2018); Carrasco et al. (2007a); Cui et al. (2020). If there exists some transformation of $Z$ such that $E\{\lambda(Z) \mid X, Y\} = \alpha(X) + \beta(X)Y$ and $\beta(x) \neq 0$, then Assumption 2 is met with $\delta_0(X, Z) = (\lambda(Z) - \alpha(X))/\beta(X)$. As a special case, $\lambda(Z) = Z$ when $E(Z \mid X, Y)$ is linear in $Y$. For simplicity, we may drop the arguments in $\delta_0(X, Z)$ and directly use $\delta_0$ in what follows, and notation for other functions are treated in a similar way.

Note that Assumption 2 only requires the existence of solutions to equation (1), but not uniqueness. For instance, if both $Z$ and $Y$ are binary, then $\delta_0$ is unique and

$$
\delta_0(X, Z) = \frac{Z - f(Z = 1 \mid R = 1, X, Y = 0)}{f(Z = 1 \mid R = 1, X, Y = 1) - f(Z = 1 \mid R = 1, X, Y = 0)}.
$$

However, if $Z$ has more levels than $Y$, $\delta_0$ may not be unique.

**THEOREM 2.1.** Under Assumptions 1 and 2, $\mu$ is identifiable, and

$$
\mu = E\{RY + (1 - R)\delta_0(X, Z)\}.
$$

The proof of Theorem 2.1 is given in the Appendix. From Theorem 2.1, even if $\delta_0$ is not uniquely determined, all solutions to Assumption 2 must result in an identical value of $\mu$. Moreover, this identification result does not require identification of the full data distribution $f(R, X, Y, Z)$. In fact, identification of $f(R, X, Y, Z)$ is not ensured under Assumptions 1...
and 2 only; see Example 2. In contrast to previous approaches (D’Haultfœuille, 2010; Miao et al., 2019; Zhao and Ma, 2021) that have to identify the full data distribution, our identification strategy allows for a larger class of models where only the parameter of interest is uniquely identified even though the full data law may not be. For example, Zhao and Ma (2021) imposed the completeness condition given in Example 1, to guarantee identifiability of the full data distribution. To our knowledge, the identifying Assumptions 1–2 are so far the weakest for the shadow variable approach. We further illustrate Assumption 2 with the following two examples.

**Example 1.** Suppose that the completeness condition holds; that is, for any square-integrable function \( g \),  
\[
E\{g(X,Y) \mid X,Z\} = 0 \quad \text{almost surely if and only if} \quad g(X,Y) = 0 \quad \text{almost surely.}
\]
Then under Conditions A1–A3 in the Appendix, the solution to (1) exists.

**Example 2.** Consider the following two models:

Model 1: \( Y \sim U(0,1) \), \( Z \mid y \sim \text{Bern}(y) \), and \( f(R = 1 \mid y, z) = 4y^2(1 - y) \), where \( \text{Bern}(y) \) denotes Bernoulli distribution with probability \( y \).

Model 2: \( Y \sim \text{Be}(2, 2) \), \( Z \mid y \sim \text{Bern}(y) \), and \( f(R = 1 \mid y, z) = 2y/3 \), where \( \text{Be}(2, 2) \) denotes Beta distribution with parameters 2 and 2.

It is easy to verify that the above two models satisfy Assumption 2 by choosing \( h_0(X,Z) = Z \). These two models imply the same outcome mean \( E(Y) = 1/2 \) and the same observed data distribution, because  
\[
f(R = 1, y, z) = 1 \cdot 1 \cdot \{zy + (1 - z)(1 - y)\} \cdot 4y^2(1 - y)
\]
\[
= 1 \cdot 6y(1 - y) \cdot \{zy + (1 - z)(1 - y)\} \cdot \frac{2}{3}y
\]
\[
= 4y^2(1 - y)\{zy + (1 - z)(1 - y)\}, \quad \text{and}
\]
\[
f(z) = \int_0^1 \{zy + (1 - z)(1 - y)\} dy
\]
\[
= \int_0^1 \{zy + (1 - z)(1 - y)\} \cdot 6y(1 - y) dy = \frac{1}{2}.
\]

However, the full data distributions of these two models are different.

3. Estimation and inference. In this section, we provide a novel estimation procedure without modeling the propensity score or outcome regression. Previous approaches often require fully or partially parametric models for at least one of them. For example, Qin et al. (2002) and Wang et al. (2014) assumed a fully parametric model for the propensity score; Kim and Yu (2011) and Shao and Wang (2016) relaxed their assumption and considered a semiparametric exponential tilting model for the propensity; Miao and Tchetgen Tchetgen (2016) proposed doubly robust estimation methods by either requiring a parametric propensity score or an outcome regression to be correctly specified. Our approach aims to be more robust than existing methods by avoiding (i) point identification of the full data law under more stringent conditions, and (ii) over-reliance on parametric assumptions either for identification or for estimation.

As implied by Theorem 2.1, any solution to (1) provides a valid \( \delta_0 \) for recovering the parameter \( \mu \). Suppose that all such solutions belong to a set \( \Delta \) of smooth functions, with specific requirements for smooth functions given in Definition 3.1. Then the set of solutions to (1) is denoted by  
\[
\Delta_0 = \left\{ \delta \in \Delta : E\{\delta(X,Z) \mid R = 1, X,Y\} = Y \right\}.
\]
For estimation and inference about \( \mu \), we need to construct a consistent estimator for some fixed \( \delta_0 \in \Delta_0 \). If \( \Delta_0 \) were known, then we would simply select one element \( \delta_0 \) from the set and use this element to estimate \( \mu \). Unfortunately, the solution set \( \Delta_0 \) is unknown, and the lack of identification of \( \delta_0 \) presents important technical challenges. Directly solving (1) does not generally yields a consistent estimator for some fixed \( \delta_0 \). Instead, by noting that the solution set \( \Delta_0 \) is identified, we aim to obtain an estimator \( \hat{\delta}_0 \) in the following two steps: first, construct a consistent estimator \( \hat{\Delta}_0 \) for the set \( \Delta_0 \); second, carefully select \( \hat{\delta}_0 \in \hat{\Delta}_0 \) such that it is a consistent estimator for a fixed element \( \delta_0 \in \Delta_0 \).

### 3.1. Estimation of the solution set \( \Delta_0 \)

Define the criterion function

\[
Q(\delta) = E[R\{E(Y - \delta(X,Z) \mid R = 1, X,Y)\}^2].
\]

Then the solution set \( \Delta_0 \) in (2) is equal to the set of zeros of \( Q(\delta) \), i.e.,

\[
\Delta_0 = \{ \delta \in \Delta : Q(\delta) = 0 \},
\]

and hence, estimation of \( \Delta_0 \) is equivalent to estimation of zeros of \( Q(\delta) \). This can be accomplished with the approximate minimizers of a sample analogue of \( Q(\delta) \) (Chernozhukov et al., 2007).

We adopt a method of sieves approach to construct a sample analogue function \( Q_n(\delta) \) for \( Q(\delta) \) and a corresponding approximation \( \Delta_n \) for \( \Delta_0 \). Let \( \{\psi_q(x,z)\}_{q=1}^{\infty} \) denote a sequence of known approximating functions of \( x \) and \( z \), and

\[
(3) \quad \Delta_n = \left\{ \delta \in \Delta : \delta(x,z) = \sum_{q=1}^{q_n} \beta_q\psi_q(x,z) \right\}
\]

for some known \( q_n \) and unknown parameters \( \{\beta_q\}_{q=1}^{\infty} \). The construction of \( Q_n \) entails a nonparametric estimator of conditional expectations. Let \( \{\phi_k(x, y)\}_{k=1}^{\infty} \) be a sequence of known approximating functions of \( x \) and \( y \). Denote the vector of the first \( k_n \) terms of the basis functions by

\[
\phi(x, y) = \{\phi_1(x, y), \ldots, \phi_{k_n}(x, y)\}^T,
\]

and let

\[
\Phi = \{\phi(X_1, Y_1), \ldots, \phi(X_n, Y_n)\}^T, \quad \Lambda = \text{diag}(R_1, \ldots, R_n).
\]

For a generic random variable \( B = B(X,Y,Z) \) with realizations \( \{B_i = B(X_i, Y_i, Z_i)\}_{i=1}^{n} \), the nonparametric sieve estimator of \( E(B \mid R = 1, x, y) \) is obtained by the linear regression of \( B \) on the vector \( \phi(X,Y) \) with observed data, i.e.,

\[
(4) \quad \hat{E}(B \mid R = 1, X,Y) = \phi^T(X,Y)(\Phi^T \Lambda \Phi)^{-1}\sum_{i=1}^{n} R_i \phi(X_i, Y_i)B_i.
\]

Then the sample analogue \( Q_n \) of \( Q \) is

\[
(5) \quad Q_n(\delta) = \frac{1}{n} \sum_{i=1}^{n} R_i \hat{e}^2(X_i, Y_i, \delta),
\]

with

\[
(6) \quad \hat{e}(X_i, Y_i, \delta) = \hat{E}\{Y - \delta(X,Z) \mid R = 1, X_i, Y_i\},
\]

where the explicit expression of \( \hat{e}(X_i, Y_i, \delta) \) is obtained from (4).
Finally, the proposed estimator of $\Delta_0$ is

$$\hat{\Delta}_0 = \{ \delta \in \Delta_n : Q_n(\delta) \leq c_n \},$$

where $\Delta_n$ and $Q_n(\delta)$ are given in (3) and (5), respectively, and $\{c_n\}_{n=1}^\infty$ is a sequence of small positive numbers converging to zero at an appropriate rate. The requirement on the rate of $c_n$ will be discussed later for theoretical analysis.

3.2. Set consistency. We establish the set consistency of $\hat{\Delta}_0$ for $\Delta_0$ in terms of Hausdorff distances. For a given norm $\| \cdot \|$, the Hausdorff distance between two sets $\Delta_1, \Delta_2 \subseteq \Delta$ is

$$d_H(\Delta_1, \Delta_2, \| \cdot \|) = \max \{ d(\Delta_1, \Delta_2), d(\Delta_2, \Delta_1) \},$$

where $d(\Delta_1, \Delta_2) = \sup_{\delta_1 \in \Delta_1} \inf_{\delta_2 \in \Delta_2} \| \delta_1 - \delta_2 \|$ and $d(\Delta_2, \Delta_1)$ is defined analogously. Thus, $\hat{\Delta}_0$ is consistent under the Hausdorff distance if both the maximal approximation error of $\hat{\Delta}_0$ by $\Delta_0$ and of $\Delta_0$ by $\hat{\Delta}_0$ converge to zero in probability.

We consider two different norms for the Hausdorff distance: the pseudo-norm $\| \cdot \|_w$ defined by

$$\| \delta \|_w^2 = E \left[ R \{ E(\delta(X, Z) \mid R = 1, X, Y) \}^2 \right],$$

and the supremum norm $\| \cdot \|_\infty$ defined by

$$\| \delta \|_\infty = \sup_{x,z} |\delta(x, z)|.$$

From the representer equation (1), we have that for any $\hat{\delta}_0 \in \hat{\Delta}_0$ and $\delta_0, \delta \in \Delta_0$, $\| \hat{\delta}_0 - \delta_0 \|_w = \| \hat{\delta}_0 - \delta \|_w$. Hence,

$$\| \hat{\delta}_0 - \delta \|_w = \inf_{\delta \in \Delta_0} \| \hat{\delta}_0 - \delta \|_w \leq d_H(\hat{\Delta}_0, \Delta_0, \| \cdot \|_w).$$

This result implies that we can obtain the convergence rate of $\| \hat{\delta}_0 - \delta_0 \|_w$ by deriving that of $d(\hat{\Delta}_0, \Delta_0, \| \cdot \|_w)$. However, the identified set $\Delta_0$ is an equivalence class under the pseudo-norm, and the convergence under $d_H(\hat{\Delta}_0, \Delta_0, \| \cdot \|_w)$ does not suffice to consistently estimate a given element $\delta_0 \in \Delta_0$. Whereas the supremum norm $\| \cdot \|_\infty$ is able to differentiate between elements in $\Delta_0$, and $d_H(\hat{\Delta}_0, \Delta_0, \| \cdot \|_\infty) = o_p(1)$ under certain regularity condition as we will show later.

We make the following assumptions to guarantee that $\hat{\Delta}_0$ is consistent under the metric $d_H(\hat{\Delta}_0, \Delta_0, \| \cdot \|_\infty)$ and to obtain the rate of convergence for $\hat{\Delta}_0$ under the weaker metric $d_H(\hat{\Delta}_0, \Delta_0, \| \cdot \|_w)$.

Assumption 3. The vector of covariates $X \in \mathbb{R}^d$ has support $[0, 1]^d$, and the outcome $Y \in \mathbb{R}$ and the shadow variable $Z \in \mathbb{R}$ have compact supports.

Assumption 3 requires $(X, Y, Z)$ to have compact supports, and without loss of generality, we assume that $X$ has been transformed such that the support is $[0, 1]^d$. These are standard conditions that are usually required in the semiparametric literature. Although $Y$ and $Z$ are also required to have compact support, the proposed approach may still be applicable if the supports are infinite with sufficiently thin tails. For instance, in our simulation studies where the variables $Y$ and $Z$ are drawn from a normal distribution in Section 4, the proposed approach continues to perform quite well.

We next impose restrictions on the smoothness of functions in the set $\Delta$. We use the following Sobolev norm to characterize the smoothness of functions.
For a generic function \( \rho(w) \) defined on \( w \in \mathbb{R}^d \), we define
\[
\| \rho \|_{\infty, \alpha} = \max_{|\lambda| \leq \underline{\alpha}} \sup_w |D^\lambda \rho(w)| + \max_{\lambda = \underline{\alpha}} \sup_{w \neq w'} \frac{D^\lambda \rho(w) - D^\lambda \rho(w')}{\| w - w' \|^{\alpha - \underline{\alpha}}},
\]
where \( \lambda \) be a \( d \)-dimensional vector of nonnegative integers, \( |\lambda| = \sum_{i=1}^d \lambda_i \), \( \alpha \) denotes the largest integer smaller than \( \alpha \), \( D^\lambda \rho(w) = \partial^{|\lambda|} \rho(w)/\partial w_1^{\lambda_1} \ldots \partial w_d^{\lambda_d} \), and \( D^0 \rho(w) = \rho(w) \).

A function \( \rho \) with \( \| \rho \|_{\infty, \alpha} < \infty \) has uniformly bounded partial derivatives up to order \( \underline{\alpha} \); besides, the \( \alpha \)th partial derivative of this function is Lipschitz of order \( \alpha - \underline{\alpha} \).

**Assumption 4.** The following conditions hold:

(i) \( \sup_{\delta \in \Delta} \| \delta \|_{\infty, \alpha} < \infty \) for some \( \alpha > (d+1)/2 \); in addition, \( \Delta_0 \neq \emptyset \), and both \( \Delta_n \) and \( \Delta \) are closed;

(ii) for every \( \delta \in \Delta \), there is \( \Pi_n \delta \in \Delta_n \) such that \( \sup_{\delta \in \Delta} \| \delta - \Pi_n \delta \|_{\infty} = O(\eta_n) \) for some \( \eta_n = o(1) \).

Assumption 4(i) requires that each function \( \delta \in \Delta \) is sufficiently smooth and bounded. The closedness condition in this assumption and Assumption 3 together imply that \( \Delta \) is compact under \( \| \cdot \|_{\infty} \). It is well known that solving integral equations as in (1) is an ill-posed inverse problem. The ill-posedness due to noncontinuity of the solution and difficulty of computation can have a severe impact on the consistency and convergence rates of estimators. The compactness condition is imposed to ensure that the consistency of the proposed estimator under \( \| \cdot \|_{\infty} \) is not affected by the ill-posedness. Such a compactness condition is commonly made in the nonparametric and semiparametric literature; e.g., Newey and Powell (2003), Ai and Chen (2003), and Chen and Pouzo (2012). Alternatively, it is possible to address the ill-posed problem by employing a regularization approach as in Horowitz (2009) and Darolles et al. (2011).

Assumption 4(ii) quantifies the approximation error of functions in \( \Delta \) by the sieve space \( \Delta_n \). This condition is satisfied by many commonly-used function spaces (e.g., Hölder space), whose elements are sufficiently smooth, and by popular sieves (e.g., power series, splines). For example, consider the function set \( \Delta \) with \( \sup_{\delta \in \Delta} \| \delta \|_{\infty, \alpha} < \infty \). If the sieve functions \( \{ \psi_q(x, z) \} \) are polynomials or tensor product univariate splines, then uniformly on \( \delta \in \Delta \), the approximation error of \( \delta \) by functions of the form \( \sum_{q=1}^{q_n} \beta_q \psi_q(x, z) \) is of the order \( O\{q_n^{-\alpha/(d+1)}\} \). Thus, Assumption 4(ii) is met with \( \eta_n = q_n^{-\alpha/(d+1)} \); see Chen (2007) for further discussion.

**Assumption 5.** The following conditions hold:

(i) the smallest and largest eigenvalues of \( E\{ R\phi(X, Y)\phi(X, Y)^\top \} \) are bounded above and away from zero for all \( k_n \);

(ii) for every \( \delta \in \Delta \), there is a \( \pi_n(\delta) \in \mathbb{R}^{k_n} \) such that
\[
\sup_{\delta \in \Delta} \| E\{ R\phi(X, Z) \ | \ r = 1, x, y \} - \phi^\top(x, y) \pi_n(\delta) \|_{\infty} = O\left( k_n^{-\frac{\alpha}{d+1}} \right);
\]

(iii) \( \xi_n^2 k_n = o(n) \), where \( \xi_n = \sup_x \| \phi(x, y) \|_2 \).

Assumption 5 bounds the second moment matrix of the approximating functions away from singularity, presents a uniform approximation error of the series estimator to the conditional mean function, and restricts the magnitude of the series terms. These conditions are standard for series estimation of conditional mean functions; see, e.g., Newey
(1997), Ai and Chen (2003), and Huang (2003). Primitive conditions are discussed below so that the rate requirements in this assumption hold. Consider any $\delta \in \Delta$ satisfying Assumption 4, i.e., $\sup_{\delta \in \Delta} \| \delta \|_{\infty, \alpha} < \infty$. If the partial derivatives of $f(z \mid r = 1, x, y)$ with respect to $(x, y)$ are continuously differentiable up to order $q + 1$, then under Assumption 3, we have $\sup_{\delta} \| E \{ \delta(X, Z) \mid R = 1, x, y \} \|_{\infty, \alpha} < \infty$. In addition, if the sieve functions $\{ \phi_k(x, y) \}_{k=1}^{k_n}$ are polynomials or tensor product univariate splines, then by similar arguments after Assumption 4, we conclude that the approximation error under $\| \cdot \|_{\infty}$ is of the order $O(k_n^{-\alpha/(d+1)})$ uniformly on $\delta \in \Delta$. Verifying Assumption 5(iii) depends on the relationship between $\xi_n$ and $k_n$. For example, if $\{ \phi_k(x, y) \}_{k=1}^{k_n}$ are tensor product univariate splines, then $\xi_n = O(k_n^{(d+1)/2})$.

Write $c_n$ in (7) by $b_n/a_n$ with appropriate sequences $a_n$ and $b_n$, and define $\lambda_n = k_n/n + k_n^{-2\alpha/(d+1)} + \eta_n^2$.

**Theorem 3.2.** Suppose that Assumptions 3–5 hold. If $a_n = O(\lambda_n^{-1})$, $b_n \to \infty$ and $b_n = o(a_n)$, Then

$$
\Delta \left( \hat{\Delta}_0, \Delta_0, \| \cdot \|_{\infty} \right) = o_p(1), \quad \text{and} \quad \Delta \left( \hat{\Delta}_0, \Delta_0, \| \cdot \|_w \right) = O_p(\epsilon_n^{1/2}).
$$

The proof of Theorem 3.2 is given in the Appendix. Theorem 3.2 shows the consistency of $\hat{\Delta}_0$ under the supremum-norm metric $d_H(\hat{\Delta}_0, \Delta_0, \| \cdot \|_{\infty})$ and establishes the rate of convergence of $\hat{\Delta}_0$ under the weaker pseudo-norm metric $d_H(\hat{\Delta}_0, \Delta_0, \| \cdot \|_w)$. In particular, if we let $k_n^2 = o(n)$, $k_n^{-3\alpha/(d+1)} = o(n^{-1})$, and $\eta_n = o(n^{-1/3})$ as imposed in Assumption 7 in the next subsection, then $\lambda_n = o(n^{-2/3})$ or $\lambda_n^{-1} n^{-2/3} \to \infty$. We take $a_n = \lambda_n^{-1/2} n^{1/3} \to \infty$ and $b_n = a_n^{-1/2} n^{1/3}$. Thus, $a_n = \lambda_n^{-1} (\lambda_n n^{2/3})^{1/2} = o(\lambda_n^{-1})$, $b_n = (\lambda_n^{-1} n^{-2/3})^{1/4} \to \infty$, and $b_n = a_n^{-1/2} n^{-1/3} = o(a_n)$. In fact, under such rate requirements, we have $n^{2/3} b_n = o(a_n)$ and $d_H(\hat{\Delta}_0, \Delta_0, \| \cdot \|_w) = o_p(n^{-1/4})$, which are sufficient to establish the asymptotic normality of the proposed estimator given in subsection 3.4.

### 3.3. A representer-based estimator

After we have obtained a consistent estimator $\hat{\Delta}_0$ for $\Delta_0$, we remain to select an estimator from $\hat{\Delta}_0$ such that it converges to a unique element belonging to $\Delta_0$. We adapt the theory of extremum estimators to achieve this goal. Let $M: \Delta \to \mathbb{R}$ be a population criterion functional that attains a unique minimum $\delta_0$ on $\Delta_0$ and $M_n(\delta)$ be its sample analogue. We then choose the minimizer of $M_n(\delta)$ over the estimated solution set $\hat{\Delta}_0$, denoted by

$$
\hat{\delta}_0 \in \arg\min_{\delta \in \Delta_0} M_n(\delta),
$$

which is expected to converge to the unique minimum $\delta_0$ of $M(\delta)$ on $\Delta_0$.

**Assumption 6.** The function set $\Delta$ is convex; the functional $M: \Delta \to \mathbb{R}$ is strictly convex and attains a unique minimum at $\delta_0$ on $\Delta_0$; its sample analogue $M_n: \Delta \to \mathbb{R}$ is continuous and $\sup_{\delta \in \Delta} | M_n(\delta) - M(\delta) | = o_p(1)$.

One example of particular interest is

$$
M(\delta) = E \left[ \left\{ (1 - R) \delta(X, Z) \right\}^2 \right].
$$

This is a convex functional with respect to $\delta$. In addition, since $E\{(1 - R)\delta_0(X, Z)\} = E\{(1 - R)Y\}$ for any $\delta_0 \in \Delta_0$, the minimizer of $M(\delta)$ on $\Delta_0$ in fact minimizes the variance.
of \((1 - R)\delta_0(X, Z)\) among \(\delta_0 \in \Delta_0\). Its sample analogue is

\[
M_n(\delta) = \frac{1}{n} \sum_{i=1}^{n} (1 - R_i) \delta^2(X_i, Z_i).
\]

Under Assumptions 3–4, one can show that the function class \(\{(1 - R)\delta : \delta \in \Delta\}\) is a Glivenko-Cantelli class, and thus \(\sup_{\delta \in \Delta} |M_n(\delta) - M(\delta)| = o_p(1)\).

**Theorem 3.3.** Suppose that Assumptions 3–6 hold. Then

\[
\|\hat{\delta}_0 - \delta_0\|_\infty = o_p(1),
\]

where \(\hat{\delta}_0\) is defined through (9) and \(\delta_0\) is defined in Assumption 6. In addition, if \(a_n = O(\lambda_n^{-1})\), \(b_n \rightarrow \infty\) and \(b_n = o(a_n)\), we then have

\[
\|\hat{\delta}_0 - \delta_0\|_w = O_p(c_n^{1/2}).
\]

The proof of Theorem 3.3 is given in the Appendix. Theorem 3.3 implies that by choosing an appropriate function \(M(\delta)\), it is possible to construct a consistent estimator \(\hat{\delta}_0\) for some unique element \(\delta_0 \in \Delta_0\) in terms of supremum norm \(\|\cdot\|_\infty\) and further obtain its rate of convergence under the weaker pseudo-norm \(\|\cdot\|_w\).

Based on the estimator \(\hat{\delta}_0\) given in (9), we obtain the following representer-based estimator \(\hat{\mu}_{\text{rep}}\) of \(\mu\):

\[
\hat{\mu}_{\text{rep}} = \frac{1}{n} \sum_{i=1}^{n} \left\{ R_i Y_i + (1 - R_i) \hat{\delta}_0(X_i, Z_i) \right\}.
\]

Below we discuss the asymptotic expansion of the estimator \(\hat{\mu}_{\text{rep}}\).

Let \(\bar{\Delta}\) be the closure of the linear span of \(\Delta\) under \(\|\cdot\|_w\), which is a Hilbert space with inner product:

\[
\langle \delta_1, \delta_2 \rangle_w = E \left[ RE \{ \delta_1(X, Z) \mid R = 1, X, Y \} E \{ \delta_2(X, Z) \mid R = 1, X, Y \} \right]
\]

for any \(\delta_1, \delta_2 \in \bar{\Delta}\).

**Assumption 7.** The following conditions hold:

(i) there exists a function \(h_0 \in \Delta\) such that

\[
\langle h_0, \delta \rangle_w = E \{ (1 - R)\delta(X, Z) \} \text{ for all } \delta \in \bar{\Delta}.
\]

(ii) \(\eta_n = o(n^{-1/3})\), \(k_n^{-3\alpha/(d+1)} = o(n^{-1})\), \(k_n^3 = o(n)\), \(\xi_n^2 k_n^2 = o(n)\), and \(\xi_n^2 k_n^{-2\alpha/(d+1)} = o(1)\).

Note that the linear functional \(\delta \mapsto E \{ (1 - R)\delta(X, Z) \}\) is continuous under \(\|\cdot\|_w\). Hence, by the Riesz representation theorem, there exists a unique \(h_0 \in \bar{\Delta}\) (up to an equivalence class in \(\|\cdot\|_w\)) such that \(\langle h_0, \delta \rangle_w = E \{ (1 - R)\delta(X, Z) \} \) for all \(\delta \in \bar{\Delta}\). However, Assumption 7(i) further requires that this equivalence class must contain at least one element that falls in \(\Delta\). A primitive condition for Assumption 7(i) is that the inverse probability weight also has a smooth representer: if

\[
E \left[ h_0(X, Z) + 1 \mid R = 1, X, Y \right] = \frac{1}{f(R = 1 \mid X, Y)}
\]

then \(h_0\) satisfies Assumption 7(i).

Assumption 7(ii) imposes some rate requirements, which can be satisfied as long as the function classes being approximated in Assumptions 4 and 5 are sufficiently smooth.
Then define the estimator for $\Pi$ and its sample analogue, the de-biased estimator which is regular and asymptotically normal. We further establish that the de-biased estimator is semiparametric locally efficient under a shadow variable model at a given submodel where a key completeness condition holds.

Theorem 3.4. Suppose that Assumptions 3–7 hold. We have that
\[
\sqrt{n}(\hat{\mu}_{rep} - \mu) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left[ (1 - R)\delta_0(X_i, Z_i) + R_iY_i - \mu + R_iE\{h_0(X, Z) \mid R = 1, X, Y_i \} \times \{Y_i - \delta_0(X_i, Z_i)\} \right] - \sqrt{nr_n(\hat{\delta}_0)} + o_p(1),
\]
with
\[ r_n(\hat{\delta}_0) = \frac{1}{n} \sum_{i=1}^{n} R_i\hat{E}\{\Pi_n h_0(X, Z) \mid R = 1, X, Y_i \} \tilde{e}(X_i, Y_i, \hat{\delta}_0), \]
where $\Pi_n h_0 \in \Delta_n$ approximates $h_0$ as given in Assumption 4(ii), $\hat{E}(\cdot)$ and $\tilde{e}(\cdot)$ are defined in (4) and (6), respectively.

The proof of Theorem 3.4 is given in the Appendix. Theorem 3.4 reveals an asymptotic expansion of $\hat{\mu}_{rep}$. However, the estimator $\hat{\mu}_{rep}$ is not necessarily asymptotically normal as the bias term $\sqrt{nr_n(\hat{\delta}_0)}$ may not be asymptotically negligible. In the next subsection, we propose a de-biased estimator which is regular and asymptotically normal. We further establish that the de-biased estimator is semiparametric locally efficient under a shadow variable model at a given submodel where a key completeness condition holds.

3.4. A debiased semiparametric locally efficient estimator. Note that only $\Pi_n h_0$ is unknown in the bias term $r_n(\hat{\delta}_0)$ in (12). We propose to construct an estimator of $\Pi_n h_0$ and then subtract the bias to obtain an estimator of $\mu$ that is asymptotically normal. We define the criterion function:
\[
C(\delta) = E\left[ R\{E(\delta(X, Z) \mid R = 1, X, Y)\}^2 \right] - 2E\{(1 - R)\delta(X, Z)\}, \quad \delta \in \Delta
\]
and its sample analogue,
\[
C_n(\delta) = \frac{1}{n} \sum_{i=1}^{n} R_i\hat{E}\{\delta(X, Z) \mid R = 1, X, Y_i\}^2 - 2\sum_{i=1}^{n} (1 - R_i)\delta(X_i, Z_i), \quad \delta \in \Delta.
\]
Since $E\{(1 - R)\delta(X, Z)\} = \langle h_0, \delta \rangle_w$ by Assumption 7, it follows that $C(\delta) = \|\delta - h_0\|_w^2 - \|h_0\|_w^2$. Thus, $h_0$ is the unique minimizer of $\delta \mapsto C(\delta)$ up to the equivalence class in $\|\cdot\|_w$. In addition, since $h_0$ and $\Pi_n h_0$ are close under the metric $\|\cdot\|_{\infty}$ by Assumption 4(ii), we then define the estimator for $\Pi_n h_0$ by:
\[ \hat{h} \in \arg\min_{\delta \in \Delta_n} C_n(\delta), \]
Given the estimator $\hat{h}$, the approximation to the bias term $r_n(\hat{\delta}_0)$ is
\[
\hat{r}_n(\hat{\delta}_0) = \frac{1}{n} \sum_{i=1}^{n} R_i\hat{E}\{\hat{h}(X, Z) \mid R = 1, X, Y_i\} \tilde{e}(X_i, Y_i, \hat{\delta}_0).
\]

Lemma 3.5. Suppose that Assumptions 3–5 and 7 hold. Then it follows that
\[
\sup_{\hat{\delta}_0 \in \Delta_0} \left| \hat{r}_n(\hat{\delta}_0) - r_n(\hat{\delta}_0) \right| = O_p\left[ \epsilon_n^{1/2} \left\{ \left( \frac{k_n}{n} \right)^{1/4} + k_n \right\} \right].
\]
The proof of this lemma is given in the Appendix. This lemma establishes the rate of convergence of $\widehat{\tau}_n(\delta_0)$ to $r_n(\delta_0)$ uniformly on $\Delta_0$. If $c_n$ converges to zero sufficiently fast, then $\sup_{\delta \in \Delta} \sqrt{n} |r_n(\delta) - c_n(\delta)| = o_p(1)$. The rate conditions imposed in Assumption 7(ii) guarantee that such a choice of $c_n$ is feasible. As a result, Theorem 7 and Lemma 3.5 imply that it is possible to construct a debiased estimator that is $\sqrt{n}$-consistent and asymptotically normal by subtracting the estimated bias $\widehat{\tau}_n(\delta_0)$ from $\widehat{\mu}_{db}$:

$$\widehat{\mu}_{db} = \widehat{\mu} + \widehat{\tau}_n(\delta_0).$$

**Theorem 3.6.** Suppose that Assumptions 3–7 hold. If $a_n = O(\lambda_n^{-1})$, $b_n \to \infty$ and $n^{2/3} b_n = o(a_n)$, then $\sqrt{n}(\widehat{\mu}_{db} - \mu)$ converges in distribution to $N(0, \sigma^2)$, where $\sigma^2$ is the variance of

$$\sigma^2 = \frac{1}{n} \sum_{i=1}^{n} \left[ (1 - R_i)\widehat{\tau}_0^2(X_i, Z_i) + R_i Y_i^2 - \left\{ \frac{1}{n} \sum_{i=1}^{n} (1 - R_i)\widehat{\tau}_0(X_i, Z_i) + R_i Y_i \right\}^2 \right]$$

$$+ R_i \left\{ \widehat{E}(\delta_0(X, Z) | R = 1, X, Y) \right\}^2 \left\{ Y_i - \widehat{\tau}_0(X_i, Z_i) \right\}^2.$$  

The formula (16) presents the influence function for $\widehat{\mu}_{db}$. The influence function is locally efficient in the sense that it attains the semiparametric efficiency bound for the outcome mean under certain conditions in the semiparametric model $\mathcal{M}_{\text{np}}$ defined below.

$$\mathcal{M}_{\text{np}} = \left\{ f(R, X, Y, Z) : E\{\gamma(X, Y) | R = 1, X, Z\} = \beta(X, Z) \right\},$$

with

$$\gamma(X, Y) = \frac{f(R = 0 | X, Y)}{f(R = 1 | X, Y)}, \quad \text{and} \quad \beta(X, Z) = \frac{f(R = 0 | X, Z)}{f(R = 1 | X, Z)}.$$  

When the shadow variable assumption holds, the equation in $\mathcal{M}_{\text{np}}$ is automatically satisfied. Hence, the model $\mathcal{M}_{\text{np}}$ is possibly larger than the model restricted by shadow variable assumption.

**Assumption 8.** The following conditions hold:

(i) Completeness: (1) for any square-integrable function $\xi(x, y)$, $E\{\xi(X, Y) | R = 1, X, Z\} = 0$ almost surely if and only if $\xi(X, Y) = 0$ almost surely; (2) for any square-integrable function $\eta(x, z)$, $E\{\eta(X, Z) | R = 1, X, Y\} = 0$ almost surely if and only if $\eta(X, Z) = 0$ almost surely.

(ii) Denote $\Omega(x, z) = E\{[\gamma(X, Y) - \beta(X, Z)]^2 | R = 1, X = x, Z = z\}$. Suppose that $0 < \inf_{x,z} \Omega(x, z) \leq \sup_{x,z} \Omega(x, z) < \infty$.

(iii) Let $T : L_2(X, Y) \rightarrow L_2(X, Z)$ be the bounded linear operator given by $T(\xi) = E\{\xi(X, Y) | R = 1, X, Z\}$. Its adjoint $T' : L_2(X, Z) \rightarrow L_2(X, Y)$ is the bounded linear map $T'(\eta) = E\{\eta(X, Z) | R = 1, X, Y\}$.

Under the completeness condition in Assumption 8(i), $\gamma(X, Y)$ is identifiable, $\delta_0(X, Z)$ and $h_0(X, Z)$ that respectively solve (1) and (11) are also uniquely identified. Assumption 8(ii) bounds $\Omega(x, z)$ away from zero and infinity. Note that conditional expectation operators can be shown to be bounded under weak conditions on the joint density (Carrasco et al., 2007b).
Corollary 3.1. The influence function (16) attains the efficiency bound of \( \mu \) in \( \mathcal{M}_{np} \) at the submodel where \( h_0(x,z) \) solves (11) and Assumptions 1–8 hold.

4. Numerical studies.

4.1. Simulation. In this subsection, we conduct simulation studies to evaluate the performance of the proposed estimators in finite samples. We consider two different cases. In the first case, the data are generated under models where the full data distribution is identified. In the second case, the full data distribution is not identified but Assumption 2 holds.

For the first case, we generate four covariates \( X = (X_1, X_2, X_3, X_4)^\top \) according to \( X_j \sim U(0, 1) \) for \( j = 1, \ldots, 4 \). We consider four data generating settings, including combinations of two choices of outcome models and two choices of propensity score models.

\[
\begin{align*}
f(Y = y \mid X = x) &\sim \begin{cases} N(1 + 2x_1 + 4x_2 + x_3 + 3x_4, 1), & \text{Linear} \\ N\{1 + 2x_1^2 + 2\exp(x_2) + \sin(x_3) + x_4, 1\}, & \text{Nonlinear} \end{cases} \\
\logit f(R = 1 \mid x, y) &\sim \begin{cases} 3 + 2x_1 + x_2 + x_3 - 0.5x_4 - 0.8y, & \text{Linear} \\ 3.5 + 3x_1^2 + 4\exp(x_2) + \sin(x_3) + 0.5x_4 - 2y, & \text{Nonlinear} \end{cases} \\
f(Z = z \mid X = x, Y = y) &\sim N(3 - 2x_1 + x_1^2 + 4x_2 + x_3 - 2x_4 + 3y, 1).
\end{align*}
\]

The missing data proportion in each of these settings is about 50%. For each setting, we replicate 1000 simulations at sample sizes 500 and 1000. We apply the proposed estimators \( \hat{\mu}_{\text{rep-db}} (\text{REP-DB}) \) and \( \hat{\mu}_{\text{rep}} (\text{REP}) \) to estimate the population outcome mean \( \mu \). For comparison, we also use an inverse probability weighted estimator (IPW) with a linear-logistic propensity score model assuming MNAR and a regression-based estimator (marREG) assuming MAR to estimate \( \mu \).

Simulation results are reported in Figure 1. In all four settings, the proposed estimators REP-DB and REP have negligible bias. In contrast, the IPW estimator can have comparable bias with ours only when the propensity score model is correctly specified; see settings (a) and (c). If the propensity score model is incorrectly specified as in settings (b) and (d), the IPW estimator exhibits an obvious downward bias and does not vanish when the sample size increases. As expected, the marREG estimator has non-negligible bias in all settings.

We also calculate the 95% confidence interval based on the proposed estimator REP-DB and the IPW estimator. Coverage probabilities of these two approaches are shown in Table 1. The REP-DB estimator based confidence intervals have coverage probabilities close to the nominal level of 0.95 in all scenarios even under small sample size \( n = 500 \). In contrast, the IPW estimator based confidence intervals have coverage probabilities well below the nominal value if the propensity score model is incorrectly specified.

| n   | Methods | LL  | NL  | LN  | NN  |
|-----|---------|-----|-----|-----|-----|
| 500 | REP-DB  | 0.940 | 0.932 | 0.942 | 0.939 |
|     | IPW     | 0.930 | 0.635 | 0.928 | 0.491 |
| 1000| REP-DB  | 0.945 | 0.933 | 0.948 | 0.951 |
|     | IPW     | 0.943 | 0.381 | 0.951 | 0.177 |

For the second case, we generate data according to Model 1 in Example 2. As with case one, we consider two different sample sizes \( n = 500 \) and \( n = 1000 \). We calculate the bias
Fig 1: Comparisons in the first case between the proposed two estimators \(\text{REP-DB}\) and \(\text{REP}\) and existing estimators \(\text{IPW}\) and \(\text{marREG}\) under sample sizes \(n = 500\) and \(n = 1000\). The abbreviation \(\text{LL}\) stands for Linear-logistic propensity score model with Linear outcome model, and the other three scenarios are analogously defined. The horizontal line marks the true value of the outcome mean.

(Bias), Monte Carlo standard deviation (SD) and 95% coverage probabilities (CP) based on 1000 replications in each setting. For comparison, we also apply the \(\text{IPW}\) estimator with a correct propensity score model to estimate \(\mu\). Since the full data distribution is not identified, the performance of \(\text{IPW}\) estimator depends on initial values during the optimization process. We consider two different settings of initial values for optimization parameters: true values and random values from the uniform distribution \(U(0, 1)\). The results are summarized in Table 2.

| \(n\)   | REP-DB Bias | REP-DB SD | REP-DB CP | IPW-true Bias | IPW-true SD | IPW-true CP | IPW-uniform Bias | IPW-uniform SD | IPW-uniform CP |
|---------|-------------|-----------|-----------|---------------|-------------|-------------|-------------------|----------------|----------------|
| 500     | 0.008       | 0.033     | 0.917     | −0.003        | 0.050       | 0.923       | −0.113            | 0.206          | 0.709          |
| 1000    | 0.003       | 0.024     | 0.942     | −0.004        | 0.035       | 0.933       | −0.134            | 0.216          | 0.667          |
We observe from Table 2 that the proposed estimator REP-DB has negligible bias, small standard deviation and satisfactory coverage probability even under sample size $n = 500$. As sample size increases to $n = 1000$, the 95% coverage probability is close to the nominal level. For the IPW estimator, only when the initial values for optimization parameters are set to be true values, it has comparable performance with REP-DB. However, if the initial values are randomly drawn from $U(0, 1)$, the IPW estimator has non-negligible bias, large standard deviation and low coverage probability. As sample size increases, the situation becomes worse. We also calculate the IPW estimator when initial values are drawn from other distributions, e.g., standard normal distribution. The performance is even worse and we do not report the results here. The simulations in this case demonstrate the superiority of the proposed estimator over existing estimators which require identifiability of the full data distribution.

4.2. Empirical example. We apply the proposed methods to the China Family Panel Studies, which was previously analyzed in Miao et al. (2019). The dataset includes 3126 households in China. The outcome $Y$ is the log of current home price (in $10^{4}$ RMB yuan), and it has missing values due to the nonresponse of house owner and the non-availability from the real estate market. The missingness process of home price is likely to be not at random, because subjects having expensive houses may be less likely to disclose their home prices. The missing data rate of current home price is 21.8%. The completely observed covariates $X$ includes 5 continuous variables: travel time to the nearest business center, house building area, family size, house story height, log of family income, and 3 discrete variables: province, urban (1 for urban households, 0 rural), refurbish status. The shadow variable $Z$ is chosen as the construction price of a house, which is also completely observed. The construction price is related to the current price of a house, and it the shadow variable assumption that nonresponse is independent of the construction price conditional on the current price and fully observed covariates is a reasonable assumption as the construction price can be viewed as error prone proxy for the current home value, and as such is no longer predictive of the missingness mechanism once the current home value has been accounted for.

We apply the proposed estimator REP-DB to estimate the outcome mean and the 95% confidence interval. We also use the competing IPW estimator and two estimators assuming MAR (marREG and marIPW) for comparison. The results are shown in Table 3. We observe that the results from the proposed estimator are similar to those from the IPW estimator, both yielding lower estimates of home price on the log scale than those obtained from the standard MAR estimators. These analysis results are generally consistent with those in Miao et al. (2019).

| Methods | Estimate | 95% confidence interval |
|---------|----------|------------------------|
| REP-DB  | 2.591    | (2.520, 2.661)         |
| IPW     | 2.611    | (2.544, 2.678)         |
| marREG  | 2.714    | (2.661, 2.766)         |
| marIPW  | 2.715    | (2.659, 2.772)         |

5. Discussion. With the aid of a shadow variable, we have proposed a novel condition for nonparametric identification of the nonignorable missing outcome mean even if the joint distribution is not identified. The identifying condition involves the existence of solutions to
a representer equation, which is a Fredholm integral equation of the first kind and can be satisfied under mild requirements. Based on the representer equation, we propose a sieve-based estimator for the outcome mean, which bypasses the difficulties of correctly specifying and estimating the unknown missingness mechanism and the outcome regression. Although the joint distribution is not identifiable, the proposed estimator is shown to be consistent for the outcome mean. In addition, we establish conditions under which the proposed estimator is asymptotically normal and attains the semiparametric efficiency bound for the shadow variable model at a key submodel where the representer is uniquely identified. The availability of a valid shadow variable is crucial for the proposed approach. Although it is generally not possible to test the shadow variable assumption via observed data without making another untestable assumption, the existence of such a variable is practically reasonable in the empirical example presented in this paper and similar situations where one or more proxies or surrogates of a variable prone to missing data may be available. In fact, it is not uncommon in survey studies and/or cohort studies in the health and social sciences, that certain outcomes may be sensitive and/or expensive to measure accurately, so that a gold standard measurement is obtained only for a select subset of the sample, while one or more proxies or surrogate measures may be available for the remaining sample. Instead of a standard measurement error model often used in such settings which requires stringent identifying conditions, the more flexible shadow variable approach proposed in this paper provides a more robust alternative to incorporate surrogate measurement in a nonparametric framework, under minimal identification conditions. Still, the validity of the shadow variable assumptions generally requires domain-specific knowledge of experts and needs to be investigated on a case-by-case basis. As advocated by Robins et al. (2000), in principle, one can also conduct sensitivity analysis to assess how results would change if the shadow variable assumption were violated by some pre-specified amount.

The proposed methods may be improved or extended in several directions. Firstly, the proposed identification and estimation framework may be extended to handle nonignorable missing outcome regression or missing covariate problems. Secondly, one can use modern machine learning techniques to solve the representer equation so that an improved estimator may be achieved that adapts to sparsity structures in the data. Thirdly, it is of great interest to extend our results to handling other problems of coarsened data, for instance, unmeasured confounding problems in causal inference. We plan to pursue these and other related issues in future research.

APPENDIX

This appendix contains two parts. We will sequentially present regularity conditions for Example 1, and proofs of lemmas and theorems.

A.1. Conditions for Example 1. We adopt the singular value decomposition (Carrasco et al. (2007b), Theorem 2.41) of compact operators to characterize conditions for existence of a solution to (1). Let \( L^2 \{ F(t) \} \) denote the space of all square-integrable functions of \( t \) with respect to a cumulative distribution function \( F(t) \), which is a Hilbert space with inner product \( \langle g, h \rangle = \int g(t) h(t) dF(t) \). Let \( K_x \) denote the conditional expectation operator \( L^2 \{ F(z \mid x) \} \to L^2 \{ F(y \mid x) \} \), \( K_x h = E \{ h(Z) \mid x, y \} \) for \( h \in L^2 \{ F(z \mid x) \} \), and let \( (\lambda_n, \varphi_n, \psi_n)_{n=1}^{+\infty} \) denote a singular value decomposition of \( K_x \). We assume the following regularity conditions:

\begin{align*}
\text{CONDITION A1.} & \quad \int \int f(z \mid x, y) f(y \mid x, z) dy dz < +\infty \\
\text{CONDITION A2.} & \quad \int y^2 f(y \mid x) dy < +\infty.
\end{align*}
Given \( f(z \mid x, y) \), the solution to (1) must exist if the completeness condition in Example 1 and Conditions A1–A3 all hold. The proof follows immediately from Picard’s theorem (Kress (1989), Theorem 15.18) and Lemma 2 of Miao et al. (2018).

### A.2. Proofs and additional lemmas.

**Proof of Theorem 2.1.** Suppose that there exist two sets of distributions \( f_1(r, x, y, z) \) and \( f_2(r, x, y, z) \) satisfying the same observed likelihood function:

\[
f_1(r = 1, x, y, z)^r = 1 \cdot f_1(r = 0, x, y, z)^r = f_2(r = 1, x, y, z)^r = 1 \cdot f_2(r = 0, x, y, z)^r = 0.
\]

Let \( E_j(\cdot) \) denote the expectation with respect to the distribution \( f_j(r, x, y, z) \) for \( j = 1, 2 \). Define \( \nu = E(Y \mid R = 0)f(R = 0) = E\{1 - R)Y\}. \) Since \( \mu = E(Y) = E(RY) + \nu \), we only need to show the identifiability of \( \nu \). Let \( \nu_j = E_j\{(1 - R)Y\} \) for \( j = 1, 2 \). It suffices to show that \( \nu_1 = \nu_2 \). Suppose that there exists a function \( \delta_0(X, Z) \) such that the quality in (1) holds with respect to \( E_1 \). Then under Assumption 1, we have

\[
\nu_1 = E_1\{(1 - R)Y\} = E_1\left[E_1\{(1 - R)Y \mid X\}\right] = E_1\left[E_1\left[E_1\left(E_1(\delta_0(X, Z) \mid R = 1, Y, X) \mid X\right) \mid X\right]\right] = E_1\left[E_1\left(E_1((1 - R)\delta_0(X, Z) \mid R, Y, X) \mid X\right) \mid X\right] = E_1\left\{E_1((1 - R)\delta_0(X, Z)) \mid R, Y, X\right\}.
\]

Because \( f_1(r = 1, x, y, z) = f_2(r = 1, x, y, z) \) according to (17), then \( E_2\{\delta_0(X, Z) \mid R = 1, X, Y\} = Y \) also holds. Analogous to the above derivation, we have \( \nu_2 = E_2\{(1 - R)\delta_0(X, Z)\} \). Again, according to (17), we have \( f_1(r = 0, x, z) = f_2(r = 0, x, z) \). Therefore, \( \nu_1 = \nu_2 \); that is, \( \nu \) is identifiable. Consequently, \( \mu \) is identifiable, and \( \mu = E\{RY + (1 - R)\delta_0(X, Z)\} \).

Here are the convergence results for some general objective functional \( Q(\delta) \) and its sample analogue \( Q_n(\delta) \).

**Lemma A.1.** Assume

1. \( Q(\delta) \geq 0 \) for any \( \delta \in \Delta \) with \( \Delta \) compact in some norm \( \| \cdot \| \);
2. \( \Delta_n \subseteq \Delta \) are closed and the distance \( d(\Delta, \Delta_n) = O(c_{1n}) \);
3. uniformly on \( \Delta_n \), \( Q_n(\delta) \leq C_1Q(\delta) + O_p(c_{2n}) \) and \( Q(\delta) \leq C_1Q_n(\delta) + O_p(c_{2n}) \) with probability approaching one for some constant \( C_1 > 0 \) and sequences \( c_{1n} \to 0, c_{2n} \to 0 \);
4. \( Q(\delta) \leq C_2 \inf_{\delta_0 \in \Delta_0} \| \delta - \delta_0 \|_{\kappa_1} \) for some \( \kappa_1 > 0 \) and \( C_2 > 0 \).

Then for \( a_n = O\{\max\{c_{1n}^\kappa, c_{2n}\}\}^{-1} \) and \( b_n \to \infty \) with \( b_n = o(a_n) \), the set \( \Delta_0 \) defined in (7) with \( c_n = b_n/a_n \) satisfies \( d_H(\Delta_0, \Delta_0, \| \cdot \|) = o_p(1) \). If in addition

5. \( Q(\delta) \geq \inf_{\delta_0 \in \Delta_0} C_2\| \delta - \delta_0 \|_{\kappa_2} \) for some \( \kappa_2 > 0 \),

then \( d_H(\Delta_0, \Delta_0, \| \cdot \|) = O_p\{\max\{\min\{b_n/a_n\}^{1/\max\{\kappa_2, 1\}}, c_{2n}\}\} \).
The proof of this lemma follows along the same lines as the proof of Theorem B.1 in Santos (2011).

**Lemma A.2.** For $\omega : [0, 1]^d \times Y \times Z \to \mathbb{R}$, let $E \{ \omega(X, Y, Z) \ | \ r = 1, x, y \} = \phi^r(x, y)(\Phi^r \Lambda \Phi)^{-1}$ $\sum_{i=1}^{n} R_i \phi(X_i, Y_i) E \{ \omega(X, Y, Z) \ | \ R = 1, X_i, Y_i \}$. If Assumptions 3–5 hold, then

\( \begin{align*}
(i) \quad & \sup_{\delta \in \Delta} \frac{1}{n} \sum_{i=1}^{n} R_i \left[ E \{ \delta(X, Z) \ | \ R = 1, X_i, Y_i \} - E \{ \delta(X, Z) \ | \ R = 1, X_i, Y_i \} \right]^2 = O_p \left( \frac{k_n}{n} + k_n^{-\frac{2\alpha}{\alpha+1}} \right), \\
(ii) \quad & \sup_{\delta \in \Delta} E \left[ RE \{ Y - \delta(X, Z) \ | \ R = 1, X, Y \} - RE \{ Y - \delta(X, Z) \ | \ R = 1, X, Y \} \right]^2 = O_p \left( k_n^{-\frac{2\alpha}{\alpha+1}} \right).
\end{align*} \)

**Proof.** (i) In order to establish the first claim of this lemma, we note that

\[
\sup_{\delta \in \Delta} \frac{1}{n} \sum_{i=1}^{n} R_i \left[ E \{ \delta(X, Z) \ | \ R = 1, X_i, Y_i \} - E \{ \delta(X, Z) \ | \ R = 1, X_i, Y_i \} \right]^2 \\
\leq \sup_{\delta \in \Delta} \frac{1}{n} \sum_{i=1}^{n} R_i \left[ E \{ \delta(X, Z) \ | \ R = 1, X_i, Y_i \} - E \{ \delta(X, Z) \ | \ R = 1, X_i, Y_i \} \right]^2 \\
+ \sup_{\delta \in \Delta} \frac{1}{n} \sum_{i=1}^{n} R_i \left[ E \{ \delta(X, Z) \ | \ R = 1, X_i, Y_i \} - E \{ \delta(X, Z) \ | \ R = 1, X_i, Y_i \} \right]^2 \\
\equiv D_1 + D_2,
\]

where $a \lesssim b$ means $a \leq Mb$ for a universal constant $M > 0$. We examine $D_1$ and $D_2$ separately. Let $\epsilon_i(\delta) = \delta(X_i, Z_i) - E \{ \delta(X, Z) \ | \ R = 1, X_i, Y_i \}$ and $\epsilon(\delta) = \{ \epsilon_1(\delta), \ldots, \epsilon_n(\delta) \}^T$. Then we have

**Proof.** (i) In order to establish the first claim of this lemma, we note that

\[
\sup_{\delta \in \Delta} \frac{1}{n} \sum_{i=1}^{n} R_i \left[ E \{ \delta(X, Z) \ | \ R = 1, X_i, Y_i \} - E \{ \delta(X, Z) \ | \ R = 1, X_i, Y_i \} \right]^2
\]

where $a \lesssim b$ means $a \leq Mb$ for a universal constant $M > 0$. We examine $D_1$ and $D_2$ separately. Let $\epsilon_i(\delta) = \delta(X_i, Z_i) - E \{ \delta(X, Z) \ | \ R = 1, X_i, Y_i \}$ and $\epsilon(\delta) = \{ \epsilon_1(\delta), \ldots, \epsilon_n(\delta) \}^T$. Then we have

\[
D_1 = \sup_{\delta \in \Delta} \frac{1}{n} \sum_{i=1}^{n} \left\{ R_i \phi^r(X_i, Y_i)(\Phi^r \Lambda \Phi)^{-1} \sum_{j=1}^{n} R_j \phi(X_j, Y_j) \epsilon_j(\delta) \right\}^2.
\]

Before proceeding further, we derive an intermediate result. Let $G_n$ denote the linear span of $r \phi(x, y)$; that is, for any $g \in G_n$, we have $g(r, x, y) = r \phi^r(x, y) a$ for some $a \in \mathbb{R}^{k_n}$. Similar with Newey (1997), under Assumption 5(i) we may assume without loss of generality that $E \{ R \phi(X, Y) \phi^r(X, Y) \} = I$. Then we have

\[
\left\| g \right\|_{\infty} = \frac{1}{\left\| g \right\|_{L^2}} \sup_{r, x, y} \left\{ r \sum_{j=1}^{k_n} a_j \phi^r_j(x, y) \right\} \left\{ \sum_{j=1}^{k_n} a_j^2 \right\}^{1/2} \leq \sup_{x, y} \left\| \phi(x, y) \right\|.
\]

Let $A_n = \sup_{G_n} \| g \|_{\infty} / \| g \|_{L^2}$. Then Assumption 5(ii) implies that $A_n^2 k_n/n \to 0$. According to Lemma 2.3(i) in Huang (2003), we have that the following holds uniformly in $G_n$

\[
\frac{1}{2} E \{ g^2(R, X, Y) \} \leq \frac{1}{n} \sum_{i=1}^{n} g^2(R_i, X_i, Y_i) \leq 2E \{ g^2(R, X, Y) \}
\]
with probability tending to one. Applying this result to (18) yields that

\[ D_1 \leq 2 \sup_{\delta \in \Delta} E \left[ \left\{ R\phi^T(X,Y)(\Phi^T\Lambda \Phi)^{-1} \sum_{j=1}^{n} R_j \phi(X_j,Y_j) \epsilon_j(\delta) \right\}^2 \right] \]

\[ = 2 \sup_{\delta \in \Delta} E \left[ \frac{1}{n} \sum_{i=1}^{n} \left\{ R_i \phi^T(X_i,Y_i)(\Phi^T\Lambda \Phi)^{-1} \sum_{j=1}^{n} R_j \phi(X_j,Y_j) \epsilon_j(\delta) \right\}^2 \right] \]

\[ = 2 \sup_{\delta \in \Delta} E \left\{ \frac{1}{n} \sum_{i=1}^{n} R_i \phi^T(X_i,Y_i)(\Phi^T\Lambda \Phi)^{-1} \Phi^T \Lambda \epsilon(\delta) \epsilon(\delta)^T \Lambda \Phi (\Phi^T \Lambda \Phi)^{-1} \phi(X_i,Y_i) \right\} \]

\[ = 2 \sup_{\delta \in \Delta} E \left[ \frac{1}{n} \sum_{i=1}^{n} R_i \phi^T(X_i,Y_i)(\Phi^T \Lambda \Phi)^{-1} \phi(X_i,Y_i) \right] \]

\[ \leq 2 \sup_{\delta \in \Delta} \|\delta\|_\infty^2 \cdot E \left\{ \frac{1}{n} \sum_{i=1}^{n} R_i \phi^T(X_i,Y_i)(\Phi^T \Lambda \Phi)^{-1} \phi(X_i,Y_i) \right\} \]

\[ = 2 \sup_{\delta \in \Delta} \|\delta\|_\infty^2 \cdot \frac{1}{n} E \left[ \sum_{i=1}^{n} \text{tr}\left\{ R_i \phi(X_i,Y_i) \phi(X_i,Y_i)(\Phi^T \Lambda \Phi)^{-1} \right\} \right] = O\left( \frac{k_n}{n} \right) \]

with probability tending to one, i.e., \( D_1 = O_p(k_n/n) \).

Now we consider \( D_2 \). Let \( \hat{\pi}_n(\delta) = (\Phi^T \Lambda \Phi)^{-1} \sum_{j=1}^{n} R_j \phi(X_j,Y_j) E\{\delta(X,Z) \mid R = 1, X_i, Y_i\} \in \mathbb{R}^{k_n} \). Then

\[ D_2 = \sup_{\delta \in \Delta} \frac{1}{n} \sum_{i=1}^{n} R_i \left[ \phi^T(X_i,Y_i) \hat{\pi}_n(\delta) - E\{\delta(X,Z) \mid R = 1, X_i, Y_i\} \right]^2 \]

\[ \leq \sup_{\delta \in \Delta} \frac{2}{n} \sum_{i=1}^{n} R_i \left[ \phi^T(X_i,Y_i) \hat{\pi}_n(\delta) - \phi^T(X_i,Y_i) \pi_n(\delta) \right]^2 + O_p\left( \frac{k_n^{-2/(a + b)} \pi_n}{n} \right) \]

\[ \leq 4 \sup_{\delta \in \Delta} E \left[ R \left\{ \phi^T(X,Y) \hat{\pi}_n(\delta) - \phi^T(X,Y) \pi_n(\delta) \right\}^2 \right] + O_p\left( k_n^{-2/(a + b)} \right) \]

\[ = 4 \sup_{\delta \in \Delta} \|\hat{\pi}_n(\delta) - \pi_n(\delta)\|^2 + O_p\left( k_n^{-2/(a + b)} \right), \]

where the second line holds because of the basic inequality with \((a - b)^2 \leq 2(a^2 + b^2)\) and Assumption 5(ii), the third line follows from (19), and the last line holds because \( E\{R \phi(X,Y) \phi^T(X,Y)\} = I \). Let \( \delta_i = E\{\delta(X,Z) \mid R = 1, X_i, Y_i\} \) and \( \delta_E = (\delta_1, \ldots, \delta_n)^T \). Let \( \gamma_n \) be the largest eigenvalue of \( n(\Phi^T \Lambda \Phi)^{-1} \). Following the proof of Theorem 1 in Newey...
(1997), one can obtain that \( \gamma_n = O_p(1) \) under Assumption 5(i). Then
\[
\sup_{\delta \in \Delta} \left\| \Pi_n(\delta) - \pi_n(\delta) \right\|^2 = \sup_{\delta \in \Delta} \left\| (\Phi^T \Lambda \Phi)^{-1} \Phi^T \Lambda \{ \delta_E - \Phi \pi_n(\delta) \} \right\|^2
\leq \left\| (\Phi^T \Lambda \Phi)^{-1/2} \right\|^2 \cdot \sup_{\delta \in \Delta} \left\| (\Phi^T \Lambda \Phi)^{-1/2} \Phi^T \Lambda \{ \delta_E - \Phi \pi_n(\delta) \} \right\|^2
= \frac{\gamma_n}{n} \cdot \sup_{\delta \in \Delta} \left\{ \delta_E - \Phi \pi_n(\delta) \right\}^T \Lambda \Phi (\Phi^T \Lambda \Phi)^{-1} \Phi^T \Lambda \left\{ \delta_E - \Phi \pi_n(\delta) \right\}
= O_p\left(k_n^{-\frac{2n}{4n+1}}\right),
\]
where the last result follows from Assumption 5(ii) and \( \Lambda \Phi (\Phi^T \Lambda \Phi)^{-1} \Phi^T \Lambda \) being idempotent.

For the second claim of this lemma, we define \( \eta_n(\delta) = E\{Y - \delta(X, Z) \mid R = 1, X_1, Y_1\} - \phi^*(X_1, Y_1) \} (\pi_n(\delta) - \pi_n(\delta)) \) and \( \eta(\delta) = \{ \eta_1(\delta), \ldots, \eta_n(\delta) \}^T \). Let \( D_3 = \bar{E} \{Y - \delta(X, Z) \mid R = 1, X, Y\} - E\{Y - \delta(X, Z) \mid R = 1, X, Y\} \). It follows from Assumption 5(iii) that
\[
D_3 = \phi^*(X, Y)(\Phi^T \Lambda \Phi)^{-1} \sum_{j=1}^n R_j \phi(X_j, Y_j) \left[ \phi^*(X_j, Y_j) \{ \pi_n(\delta_0) - \pi_n(\delta) \} + \eta_j(\delta) \right]
- E\{Y - \delta(X, Z) \mid R = 1, X, Y\}
= \phi^*(X, Y) \{ \pi_n(\delta_0) - \pi_n(\delta) \} + \phi^*(X, Y)(\Phi^T \Lambda \Phi)^{-1} \Phi^T \Lambda \eta(\delta)
- E\{Y - \delta(X, Z) \mid R = 1, X, Y\}
= \phi^*(X, Y)(\Phi^T \Lambda \Phi)^{-1} \Phi^T \Lambda \eta(\delta) + O\left(k_n^{-\frac{2n}{4n+1}}\right).
\]
Then by similar arguments as in \eqref{22} we have
\[
\sup_{\delta \in \Delta} E(RD_3^2) \leq 2 \sup_{\delta \in \Delta} \left[ \bar{R} \left\{ \phi^*(X, Y)(\Phi^T \Lambda \Phi)^{-1} \Phi^T \Lambda \eta(\delta) \right\}^2 \right] + O\left(k_n^{-\frac{2n}{4n+1}}\right)
= 2 \sup_{\delta \in \Delta} \left\| (\Phi^T \Lambda \Phi)^{-1} \Phi^T \Lambda \eta(\delta) \right\|^2 + O\left(k_n^{-\frac{2n}{4n+1}}\right)
= O_p\left(k_n^{-\frac{2n}{4n+1}}\right).
\]
This completes proof of Lemma A.2. \( \square \)

**Corollary A.1.** If Assumptions 3–5 hold, then uniformly in \( \Delta_n \), the following two claims hold with probability tending to one: (i) \( Q_n(\delta) \leq 8Q(\delta) + O_p(\tau_n) \), and (ii) \( Q(\delta) \leq 8Q_n(\delta) + O_p(\tau_n) \) for \( \tau_n = k_n^{-\frac{1}{2n}} + k_n^{-\frac{2n}{4n+1}} \).
This completes the proof of Corollary A.1.
PROOF OF THEOREM 3.2. We prove Theorem 3.2 by verifying conditions in Lemma A.1. First, conditions (i) and (ii) hold with \( \| \cdot \| = \| \cdot \|_\infty \) and \( c_{1n} = \eta_n \) by Assumption 4. Next, condition (iii) holds with \( c_{2n} = k_n/n + k_n^{-2n/(d+1)} \) and \( C_1 = 8 \) by Corollary A.1. Finally, we note that

\[
\begin{align*}
Q(\delta) &= E \left[ R \left\{ E(Y - \delta(X, Z) | R = 1, X, Y) \right\}^2 \right] \\
&= E \left[ R \left\{ E(\delta_0(X, Z) - \delta(X, Z) | R = 1, X, Y) \right\}^2 \right] \\
&\leq \inf_{\Delta_n} \| \delta_0 - \delta \|_\infty^2.
\end{align*}
\]

Condition (iv) holds with \( C_2 = 1 \) and \( \kappa_1 = 2 \). The first claim of Theorem 3.2 follows by Lemma A.1.

For the second claim of this theorem, note that \( \| \cdot \|_w \leq \| \cdot \|_\infty \). Similar to the above discussions, conditions (i)-(iii) can be verified with \( \| \cdot \| = \| \cdot \|_w \). In addition, since \( Q(\delta) = \| \delta - \delta_0 \|_w^2 \) as derived above, conditions (iv) and (v) holds with \( C_1 = C_2 = 1 \) and \( \kappa_1 = \kappa_2 = 2 \). According to Lemma A.1, it follows that

\[
d_H(\hat{\Delta}_0, \Delta_0, \| \cdot \|_w) = O_p(c_n^{1/2}/\eta_n).
\]

For the \( a_n \) and \( b_n \) defined in this theorem, we have \( c_n^{1/2}/\eta_n \to \infty \). Thus,

\[
d_H(\hat{\Delta}_0, \Delta_0, \| \cdot \|_w) = O_p(c_n^{1/2}).
\]

This completes proof of Theorem 3.2. \( \Box \)

**LEMMA A.3.** Suppose that the following conditions hold: (i) \( \Delta_0 \subseteq \Delta \) is closed with \( \Delta \) compact and \( M : \Delta \to R \) has a unique minimum on \( \Delta_0 \) at \( \delta_0 \); (ii) \( \Delta_0 \subseteq \Delta \) satisfies \( d_H(\hat{\Delta}_0, \Delta_0, \| \cdot \|) = o_p(1) \); (iii) \( M_n : \Delta \to R \) and \( M : \Delta \to R \) are continuous; (iv) \( \sup_{\delta \in \Delta} |M_n(\delta) - M(\delta)| = o_p(1) \). If \( \hat{\delta}_0 \in \arg\min_{\delta \in \Delta_0} M_n(\delta) \), then \( \| \hat{\delta}_0 - \delta_0 \| = o_p(1) \).

The proof of Lemma A.3 follows the same lines as those in the proof of of Theorem 3.2 in Santos (2011).

PROOF OF THEOREM 3.3. We show the first claim of Theorem 3.3 by verifying conditions of Lemma A.3. First, by Assumption 4, both \( \Delta_0 \) and \( \Delta \) are compact under \( \| \cdot \|_\infty \), and hence condition (i) of Lemma A.3 holds. Second, the convexity of \( \Delta_0 \) and strict convexity of \( M \) implies that \( M \) has a unique minimum on \( \Delta_0 \). With additional conditions in Theorem 3.3, Lemma A.3 implies that

\[
\| \hat{\delta}_0 - \delta_0 \|_\infty = o_p(1).
\]

For the second claim of Theorem 3.3, we note that

\[
\| \hat{\delta}_0 - \delta_0 \|_w \leq d_H(\hat{\Delta}_0, \Delta_0, \| \cdot \|_w).
\]

This, combined with Theorem 3.2, shows the second claim. \( \Box \)

**LEMMA A.4.** Suppose that Assumptions 3 and 4(i) hold. Let \( A_n \) be the \( \sigma \)-field generated by \( \{R_i, X_i, Y_i\}_{i=1}^n \) and let \( \{W_{in}\}_{i=1}^n \) be a triangle array of random variables that is measurable with respect to \( A_n \). If \( n^{-1} \sum_{i=1}^n R_i W_{in}^2 = O_p(1) \) and \( \| \delta_0 - \delta_0 \|_\infty = o_p(1) \), then it follows that

\[
\frac{1}{\sqrt{n}} \sum_{i=1}^n R_i W_{in} \left[ \delta_0(X_i, Z_i) - \delta_0(X_i, Z_i) - E \left\{ \hat{\delta}_0(X, Z) - \delta_0(X, Z) | R = 1, X_i, Y_i \right\} \right] = o_p(1).
\]
PROOF. Define the function class
$$\mathcal{F}_n^{\zeta_n} = \left\{ \delta(x, z) - \delta_0 (x, z) - E \left\{ \delta(X, Z) - \delta_0 (X, Z) \mid r = 1, x, y \right\} : \delta \in \Delta \text{ and } \| \delta - \delta_0 \|_{\infty} \leq \zeta_n \right\},$$
where $\{ \zeta_n \}_{n=1}^{\infty}$ is a sequence of real numbers decreasing to zero and satisfies $\| \hat{\delta}_0 - \delta_0 \|_{\infty} = o_p(\zeta_n)$. Since $W_{in}$ is measurable with respect to $\mathcal{A}_n$, we have
$$E \left\{ R_i W_{in} f(X_i, Y_i, Z_i) \right\} = 0$$
for any $f \in \mathcal{F}_n^{\zeta_n}$. Then apply Markov’s inequality for conditional expectations and Lemma 2.3.6 in Van Der Vaart and Wellner (1996) to get that
$$\operatorname{pr} \left[ \frac{1}{\sqrt{n}} \sum_{i=1}^{n} R_i W_{in} \left\{ \hat{\delta}_0 (X_i, Z_i) - \delta_0 (X_i, Z_i) \right\} - E \left( \hat{\delta}_0 (X_i, Z_i) - \delta_0 (X_i, Z_i) \mid R = 1, X_i, Y_i \right) \right] > \eta | \mathcal{A}_n \right\]$$
(23)
$$\leq \frac{1}{\eta} E \left[ \sup_{f \in \mathcal{F}_n^{\zeta_n}} \left| \frac{2}{\sqrt{n}} \sum_{i=1}^{n} R_i W_{in} f(X_i, Y_i, Z_i) \epsilon_i \right| \right] \times o(1),$$
where $\{ \epsilon_i \}$ are iid Rademacher random variables independent of $\{ R_i, X_i, Y_i, Z_i \}_{i=1}^{n}$. Define the semimetric on $\mathcal{F}_n^{\zeta_n}$:
$$\| f \|_n = \sqrt{d_n} \times \| f \|_{\infty},$$
where $d_n = n^{-1} \sum_{i=1}^{n} R_i W_{in}^2$. By definition, the diameter of $\mathcal{F}_n^{\zeta_n}$ under $\| \cdot \|_n$ is less than or equal to $2 \zeta_n \sqrt{d_n}$. Then according to Lemma 2.2.8 in Van Der Vaart and Wellner (1996), we have
$$E_{\epsilon} \left\{ \sup_{f \in \mathcal{F}_n^{\zeta_n}} \left| \frac{1}{\sqrt{n}} \sum_{i=1}^{n} R_i W_{in} f(X_i, Y_i, Z_i) \epsilon_i \right| \right\} \leq \int_{0}^{\infty} \sqrt{\log N(\epsilon, \mathcal{F}_n^{\zeta_n}, \| \cdot \|_n)} d\epsilon \leq \int_{0}^{2 \zeta_n \sqrt{d_n}} \sqrt{\log N(\epsilon/\sqrt{d_n}, \mathcal{F}_n^{\zeta_n}, \| \cdot \|_{\infty})} d\epsilon,$$
(24)
where $N(\epsilon, \mathcal{F}_n^{\zeta_n}, \| \cdot \|_n)$ denotes the covering number. Note that
$$\log N(\epsilon/\sqrt{d_n}, \mathcal{F}_n^{\zeta_n}, \| \cdot \|_{\infty}) \leq \log N_{\|}(2\epsilon/\sqrt{d_n}, \mathcal{F}_n^{\zeta_n}, \| \cdot \|_{\infty}) \leq \log N_{\|}(\epsilon/\sqrt{\Delta}, \| \cdot \|_{\infty}) \leq \left( \frac{\sqrt{d_n}}{\epsilon} \right)^{(d+1)/(2\alpha)},$$
(25)
where $N_{\|}(\epsilon, \mathcal{F}_n^{\zeta_n}, \| \cdot \|_{\infty})$ denotes the bracketing number and the last inequality follows from Corollary 2.7.2 in Van Der Vaart and Wellner (1996). This equation, combined with (24), yields that
$$E_{\epsilon} \left\{ \sup_{f \in \mathcal{F}_n^{\zeta_n}} \left| \frac{1}{\sqrt{n}} \sum_{i=1}^{n} R_i W_{in} f(X_i, Y_i, Z_i) \epsilon_i \right| \right\} \leq \int_{0}^{2 \zeta_n \sqrt{d_n}} \left( \frac{\sqrt{d_n}}{\epsilon} \right)^{(d+1)/(2\alpha)} d\epsilon \leq \sqrt{d_n} (2 \zeta_n)^{1-(d+1)/(2\alpha)}.$$
Thus, since $\zeta_n \to 0$, $\alpha > (d + 1)/2$ by Assumption 4(i) and $d_n = O_p(1)$ by hypothesis, the above equation together with (23) imply the desired result.  

\[ \text{LEMMA A.5. Suppose that Assumptions 3 and 4(i) hold. Let } \mathcal{A}_n \text{ be the } \sigma \text{-field generated by } \{R_i, X_i, Y_i\}_{i=1}^n \text{ and } \{W_{in}\}_{i=1}^n, \{\tilde{W}_{in}\}_{i=1}^n \text{ be triangular arrays of random variables that are measurable with respect to } \mathcal{A}_n \text{ such that } n^{-1/2} \sum_i R_i(W_{in} - \tilde{W}_{in})^2 = O_p(1). \text{ If } \tilde{\delta}_0 \in \Delta_n \text{ satisfies } \Vert \tilde{\delta}_0 - \delta_0 \Vert_\infty = o_p(1) \text{ and } \Vert \tilde{\delta}_0 - \delta_0 \Vert_w = o_p(n^{-1/4}), \text{ then it follows that}
\]

\[ \frac{1}{\sqrt{n}} \sum_{i=1}^n R_i W_{in} \left\{ Y_i - \tilde{\delta}_0(X_i, Z_i) \right\} = \frac{1}{\sqrt{n}} \sum_{i=1}^n R_i \tilde{W}_{in} \left\{ Y_i - \tilde{\delta}_0(X_i, Z_i) \right\} + o_p(1). \]

\[ \text{PROOF. First notice that since } n^{-1/2} \sum_i R_i(W_{in} - \tilde{W}_{in})^2 = O_p(1), \text{ Lemma A.4 implies: (26)}
\]

\[ \frac{1}{\sqrt{n}} \sum_{i=1}^n R_i(W_{in} - \tilde{W}_{in}) \tilde{\delta}_0(X_i, Z_i)
\]

\[ = \frac{1}{\sqrt{n}} \sum_{i=1}^n R_i(W_{in} - \tilde{W}_{in}) \left[ \tilde{\delta}_0(X_i, Z_i) - \delta_0(X_i, Z_i) - E \left\{ \tilde{\delta}_0(X, Z) - \delta_0(X, Z) \mid R = 1, X_i, Y_i \right\} \right]
\]

\[ + \delta_0(X_i, Z_i) + E \left\{ \tilde{\delta}_0(X, Z) - \delta_0(X, Z) \mid R = 1, X_i, Y_i \right\}
\]

\[ = \frac{1}{\sqrt{n}} \sum_{i=1}^n R_i(W_{in} - \tilde{W}_{in}) \left[ \tilde{\delta}_0(X_i, Z_i) + E \left\{ \tilde{\delta}_0(X, Z) - \delta_0(X, Z) \mid R = 1, X_i, Y_i \right\} \right] + o_p(1).
\]

Further observe that since $\sup_{\delta \in \Delta} \Vert \delta \Vert_\infty < \infty$ by Assumption 4(i), we obtain from Cauchy-Schwartz inequality that

\[ \left( \frac{1}{\sqrt{n}} \sum_{i=1}^n R_i(W_{in} - \tilde{W}_{in}) E \left\{ \tilde{\delta}_0(X, Z) - \delta_0(X, Z) \mid R_i = 1, X_i, Y_i \right\} \right)^2
\]

\[ = \left( \frac{1}{n} \sum_{i=1}^n R_i(W_{in} - \tilde{W}_{in})^2 \right)^{1/2} \left[ \frac{1}{n} \sum_{i=1}^n R_i \left\{ E(\tilde{\delta}_0(X, Z) - \delta_0(X, Z) \mid R_i = 1, X_i, Y_i) \right\}^2 \right]^{1/2}
\]

\[ = O_p(n^{-1/4}) \times \Vert \tilde{\delta}_0 - \delta_0 \Vert_w = o_p(n^{-1/2}).
\]

Combining this equation with (26) yields that

\[ \frac{1}{\sqrt{n}} \sum_{i=1}^n R_i(W_{in} - \tilde{W}_{in}) \left\{ Y_i - \tilde{\delta}_0(X_i, Z_i) \right\}
\]

\[ = \frac{1}{\sqrt{n}} \sum_{i=1}^n R_i(W_{in} - \tilde{W}_{in}) \left\{ Y_i - \delta_0(X_i, Z_i) \right\} + o_p(1).
\]

\[ (28)\]
In addition, since $E\{\delta_0(X, Z) \mid R = 1, X_i, Y_i\} = Y_i$, and $\{W_{in}\}, \{\tilde{W}_{in}\}$ are both measurable with respect to $A_n$, we have

$$E \left[ \frac{1}{\sqrt{n}} \sum_{i=1}^{n} R_i(W_{in} - \tilde{W}_{in})(Y_i - \delta_0(X_i, Z_i)) \right]^2 \mid A_n$$

$$= \frac{1}{n} \sum_{i=1}^{n} R_i(W_{in} - \tilde{W}_{in})^2 E \left[ \{Y - \delta_0(X, Z)\}^2 \mid R = 1, X_i, Y_i\right] = o_p(1),$$

where the last equality follows from the hypothesis $n^{-1/2} \sum_i R_i(W_{in} - \tilde{W}_{in})^2 = O_p(1)$ and the boundedness conditions in Assumptions 3 and 4(i). Then the desired claim can be obtained by the above equation and (28).

\[\Box\]

**Lemma A.6.** Suppose that Assumptions 3, 4(i), 5, and 7(ii) hold. Let the function class $G$ satisfy: for all $g \in G$, $\|g\|_\infty \leq K$ for some constant $K > 0$, $E\{g(X, Z) \mid R = 1, X, Y\} = 0$, and $\int_0^{\infty} \sqrt{\log N(\epsilon, G, \|\cdot\|_\infty)}d\epsilon < \infty$. If $E\{\delta(X, Z) \mid r = 1, x, y\} = \phi^T(x, y)(\Phi\Lambda\Phi)^{-1} \sum_i R_i \phi(X_i, Y_i) E\{\delta(X, Z) \mid R = 1, X_i, Y_i\}$, then it holds that

$$\sup_{g, \Delta_n} \left| \frac{1}{\sqrt{n}} \sum_{i=1}^{n} R_i \left[ \tilde{E}\{\delta(X, Z) \mid R = 1, X_i, Y_i\} - E\{\delta(X, Z) \mid R = 1, X_i, Y_i\} \right] g(X_i, Z_i) \right| = o_p(1).$$

**Proof.** Define $\epsilon_i(\delta) = \delta(X_i, Z_i) - E\{\delta(X, Z) \mid R = 1, X_i, Y_i\}, \epsilon(\delta) = \{\epsilon_1(\delta), \ldots, \epsilon_n(\delta)\}$, and $\eta(\delta) = (\Phi^T\Lambda\Phi)^{-1}\Phi^T\Lambda\epsilon(\delta)$. Note that (29)

$$\sup_{g \in G, \delta \in \Delta_n} \left| \frac{1}{\sqrt{n}} \sum_{i=1}^{n} R_i \left[ \tilde{E}\{\delta(X, Z) \mid R = 1, X_i, Y_i\} - E\{\delta(X, Z) \mid R = 1, X_i, Y_i\} \right] g(X_i, Z_i) \right|$$

$$= \sup_{g \in G, \delta \in \Delta_n} \left| \frac{1}{\sqrt{n}} \sum_{i=1}^{n} R_i \phi^T(X_i, Y_i) g(X_i, Z_i) \eta(\delta) \right|.$$

By Assumption 5(i), the smallest eigenvalue of $E\{R\phi(X, Y)\phi^T(X, Y)\}$ is bounded away from zero. Then we have

$$\sup_{\delta \in \Delta_n} \|\eta(\delta)\|^2$$

$$= \sup_{\delta \in \Delta_n} \epsilon(\delta)^T\Lambda\Phi(\Phi^T\Lambda\Phi)^{-2}\Phi^T\Lambda\epsilon(\delta)$$

$$\lesssim E\left\{ \epsilon(\delta)^T\Lambda\Phi(\Phi^T\Lambda\Phi)^{-1}R\phi(X, Y)\phi^T(X, Y)(\Phi^T\Lambda\Phi)^{-1}\Phi^T\Lambda\epsilon(\delta) \right\}$$

$$\lesssim \sup_{\delta \in \Delta_n} \frac{1}{n} \sum_{i=1}^{n} \left\{ R_i \phi^T(X_i, Y_i) \eta(\delta) \right\}^2$$

$$= \sup_{\delta \in \Delta_n} \frac{1}{n} \sum_{i=1}^{n} R_i \left[ \tilde{E}\{\delta(X, Z) \mid R = 1, X_i, Y_i\} - E\{\delta(X, Z) \mid R = 1, X_i, Y_i\} \right]^2$$

$$= O_p\left( \frac{k_n}{n} \right),$$

where the third line holds because of (19), and the last lines holds because of (20). Let $\{\zeta_n\}$ be a sequence of real numbers decreasing to zero such that $\zeta_n \sqrt{n} / \sqrt{k_n} \to \infty$. Define $H^{(n)}_{\zeta_n} =$
\{ r \phi^2(x, y) \varepsilon : \| \varepsilon \| \leq \zeta_n \}. The equations (29) and (30), together with Markov’s inequality imply that

\begin{equation}
\Pr \left[ \sup_{g, \Delta_n} \left\{ \frac{1}{\sqrt{n}} \sum_{i=1}^{n} R_i \left\{ \hat{E} \left( \delta(X, Z) \mid R = 1, X_i, Y_i \right) - \overline{E} \left( \delta(X, Z) \mid R = 1, X_i, Y_i \right) \right\} g(X_i, Z_i) \right\} > \kappa \right] 
\end{equation}

\begin{equation}
\leq \frac{1}{\kappa} E \left\{ \sup_{g \in \mathcal{G}, h \in \mathcal{H}_{\infty}} \left\{ \frac{1}{\sqrt{n}} \sum_{i=1}^{n} h(R_i, X_i, Y_i) g(X_i, Z_i) \right\} \right\} + o(1)
\end{equation}

\begin{equation}
\lesssim K \zeta_n \xi_n \int_0^1 \sqrt{1 + \log N(e K \zeta_n \xi_n, \mathcal{H}_{\infty} \times \mathcal{G}, \| \cdot \|_{\infty})} de,
\end{equation}

where the last inequality follows from Theorem 2.14.1 in Van Der Vaart and Wellner (1996) and that \( \|h\|_{\infty} \leq K \zeta_n \xi_n \) uniformly in \( h \in \mathcal{H}_{\infty} \) and \( g \in \mathcal{G} \). We note that for every \( h_j \in \mathcal{H}_{\infty} \) and \( g_j \in \mathcal{G} \), the triangle and Cauchy-Schwartz inequality imply that

\begin{equation}
\|g_1 h_1 - g_2 h_2\|_{\infty} \leq K \zeta_n \xi_n \|e_1 - e_2\| + \zeta_n \xi_n \|g_1 - g_2\|_{\infty}.
\end{equation}

Let \( \mathcal{B}^n_{\zeta_n} \) be a sphere of radius \( \zeta_n \) in \( \mathbb{R}^{k_n} \). We then obtain from equation (32) that

\begin{equation}
N \left( \varepsilon, \mathcal{H}_{\infty} \times \mathcal{G}, \| \cdot \|_{\infty} \right) \leq N \left( \frac{\varepsilon}{K \zeta_n}, \mathcal{B}^n_{\zeta_n}, \| \cdot \| \right) \times N \left( \frac{\varepsilon}{\zeta_n \xi_n}, \mathcal{G}, \| \cdot \|_{\infty} \right).
\end{equation}

Since \( N(\varepsilon, \mathcal{B}^n_{\zeta_n}, \| \cdot \|) \leq (2\zeta_n/\varepsilon)^{k_n} \), the above equation together with (31) implies that

\begin{equation}
\Pr \left[ \sup_{g, \Delta_n} \left\{ \frac{1}{\sqrt{n}} \sum_{i=1}^{n} R_i \left\{ \hat{E} \left( \delta(X, Z) \mid R = 1, X_i, Y_i \right) - \overline{E} \left( \delta(X, Z) \mid R = 1, X_i, Y_i \right) \right\} g(X_i, Z_i) \right\} > \kappa \right] 
\end{equation}

\begin{equation}
\lesssim \zeta_n \xi_n \int_0^1 \sqrt{1 + k_n \log(2/\varepsilon) + \log N(e K, \mathcal{G}, \| \cdot \|_{\infty})} de
\end{equation}

\begin{equation}
\lesssim \sqrt{k_n \zeta_n \xi_n}.
\end{equation}

By Assumption 7(ii), \( \xi_n k_n / \sqrt{n} = o(1) \). Then we can choose some \( \zeta_n \) such that \( \zeta_n \sqrt{k_n} \xi_n = o(1) \), e.g., \( \zeta_n = n^{-1/4} \xi_n^{-1/2} \). It can be verify that \( \zeta_n \) converges to zero and satisfies \( \zeta_n \sqrt{n} / \sqrt{k_n} \to \infty \). This completes the proof of Lemma A.6.

\[ \square \]

**Lemma A.7.** Suppose that Assumptions 3–5 and 7 hold. Let \( u = \pm h_0 \) and \( u_n = \pm \Pi_n u \).

If \( \tilde{\delta}_0 \in \Delta_n \) satisfies \( \| \tilde{\delta}_0 - \delta_0 \|_w = o_p(n^{-1/4}) \) and \( \| \tilde{\delta}_0 - \delta_0 \|_{\infty} = o_p(1) \), then:

\begin{enumerate}
  \item \( \frac{1}{\sqrt{n}} \sum_{i=1}^{n} R_i E \{ u(X, Z) \mid R = 1, X_i, Y_i \} \left[ \hat{\epsilon}(X_i, Y_i, \tilde{\delta}_0) - \{ Y_i - \hat{\delta}_0(X_i, Z_i) \} \right] = o_p(1), \)
  \item \( \sqrt{n} \langle u, \delta_0 - \tilde{\delta}_0 \rangle_w = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} R_i E \{ u(X, Z) \mid R = 1, X_i, Y_i \} \left\{ \hat{\epsilon}(X_i, Y_i, \tilde{\delta}_0) - \hat{\epsilon}(X_i, Y_i, \delta_0) \right\} + o_p(1). \)
\end{enumerate}
Then according to equations (21), (22) and Assumption 7(ii), we have

\[ \frac{1}{\sqrt{n}} \sum_{i=1}^{n} R_i E \left\{ u(X, Z) \mid R = 1, X_i, Y_i \right\} \left[ \epsilon(X_i, Y_i, \hat{\theta}_0) - \left\{ Y_i - \hat{\theta}_0(X_i, Z_i) \right\} \right] \]

\[ = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} R_i E \left\{ u(X, Z) \mid R = 1, X_i, Y_i \right\} \left[ \phi(X_i, Y_i)(\Phi^T \Lambda \Phi)^{-1} \sum_{j=1}^{n} R_j \phi(X_j, Y_j) \times \left\{ Y_j - \hat{\theta}_0(X_j, Y_j) \right\} - \left\{ Y_i - \hat{\theta}_0(X_i, Z_i) \right\} \right] \]

\[ = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} R_j \left\{ Y_j - \hat{\theta}_0(X_j, Z_j) \right\} \phi(X_j, Y_j)(\Phi^T \Lambda \Phi)^{-1} \sum_{i=1}^{n} R_i \phi(X_i, Y_i) E \left\{ u(X, Z) \mid R = 1, X_i, Y_i \right\} \]

\[ - \frac{1}{\sqrt{n}} \sum_{i=1}^{n} R_i E \left\{ u(X, Z) \mid R = 1, X_i, Y_i \right\} \left\{ Y_i - \hat{\theta}_0(X_i, Z_i) \right\} \]

\[ = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} R_i \left[ E \left\{ u(X, Z) \mid R = 1, X_i, Y_i \right\} - E \left\{ u(X, Z) \mid R = 1, X_i, Y_i \right\} \right] \left\{ Y_i - \hat{\theta}_0(X_i, Z_i) \right\}. \]

Then according to equations (21), (22) and Assumption 7(ii), we have

\[ \sup_{\delta \in \Delta} \frac{1}{n} \sum_{i=1}^{n} R_i \left[ E \left\{ \delta(X, Z) \mid R = 1, X_i, Y_i \right\} - E \left\{ \delta(X, Z) \mid R = 1, X_i, Y_i \right\} \right]^2 = o_p(n^{-2/3}). \]

Consequently, by Lemma A.5, we obtain the desired claim (i).

We next prove claim (ii). First, similar to the derivation in (33), we have

\[ \frac{1}{\sqrt{n}} \sum_{i=1}^{n} R_i E \left\{ u(X, Z) \mid R = 1, X_i, Y_i \right\} \left\{ \epsilon(X_i, Y_i, \hat{\theta}_0) - \epsilon(X_i, Y_i, \theta_0) \right\} \]

\[ (34) \]

\[ = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} R_i E \left\{ u(X, Z) \mid R = 1, X_i, Y_i \right\} \left\{ \delta_0(X_i, Z_i) - \hat{\theta}_0(X_i, Z_i) \right\}. \]

We then show that \( \sup_{x,y} \left| E \left\{ u(X, Z) \mid r = 1, x, y \right\} \right| = O_p(1) \). Let \( \epsilon(X, \delta) = E \{ \delta(X, Z) \mid R = 1, X_i, Y_i \} - \phi(X_i, Y_i)^T \pi_n(\delta), \epsilon(\delta) = \{ \epsilon_1(\delta), \ldots, \epsilon_n(\delta) \}^T \). By Assumption 5, we have

\[ \sup_{\delta \in \Delta} \left\{ E \left\{ \delta(X, Z) \mid r = 1, x, y \right\} - E \left\{ \delta(X, Z) \mid r = 1, x, y \right\} \right\}^2 \]

\[ \leq \sup_{\delta \in \Delta, x, y} \left\{ \phi^T(x, y)(\Phi^T \Lambda \Phi)^{-1} \Phi^T \Lambda \epsilon(\delta) \right\}^2 + O \left\{ k_n^{-2\alpha/(d+1)} \right\}. \]

Then similar to the derivation in (22), we have

\[ \sup_{\delta \in \Delta, x, y} \left\{ \phi^T(x, y)(\Phi^T \Lambda \Phi)^{-1} \Phi^T \Lambda \epsilon(\delta) \right\}^2 \]

\[ \leq \sup_{\delta \in \Delta, x, y} \left\| \phi(x, y) \right\|^2 \times \left\| \phi^T(x, y)(\Phi^T \Lambda \Phi)^{-1} \Phi^T \Lambda \epsilon(\delta) \right\|^2 = O_p \left\{ \xi_n^2 \times k_n^{-2\alpha/(d+1)} \right\} = o_p(1), \]

where the last equality follows from Assumption 7(ii). Finally, since \( \delta(x, z) \) is bounded by Assumption 4(i), the triangle inequality implies that \( \sup_{x,y} \left| E \{ \delta(X, Z) \mid r = 1, x, y \} \right| = O_p(1) \), and consequently, \( \sup_{x,y} \left| E \{ u(X, Z) \mid r = 1, x, y \} \right| = O_p(1) \).
Thus, by Lemma A.4 and equation (34), we have

\[ \frac{1}{\sqrt{n}} \sum_{i=1}^{n} R_i E \left\{ u(X, Z) \mid R = 1, X_i, Y_i \right\} \left\{ \tilde{e}(X_i, Y_i, \hat{\delta}) - \hat{e}(X_i, Y_i, \delta_0) \right\} \]

\[ = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} R_i E \left\{ u(X, Z) \mid R = 1, X_i, Y_i \right\} E \left\{ \delta_0(X_i, Z_i) - \hat{\delta}_0(X_i, Z_i) \mid R = 1, X_i, Y_i \right\} + o_p(1). \]

Apply the Cauchy-Schwartz inequality to get that

\[ \left\| \frac{1}{\sqrt{n}} \sum_{i=1}^{n} R_i \left[ E \left\{ u(X, Z) \mid R = 1, X_i, Y_i \right\} - E \left\{ u(X, Z) \mid R = 1, X_i, Y_i \right\} \right] \right\| \]

\[ \times E \left\{ \delta_0(X_i, Z_i) - \hat{\delta}_0(X_i, Z_i) \mid R = 1, X_i, Y_i \right\} \]

\[ \leq \left( \frac{1}{\sqrt{n}} \sum_{i=1}^{n} R_i \left\{ E \left\{ u(X, Z) \mid R = 1, X_i, Y_i \right\} - E \left\{ u(X, Z) \mid R = 1, X_i, Y_i \right\} \right\} \right)^{1/2} \]

\[ \times \left( \frac{1}{\sqrt{n}} \sum_{i=1}^{n} R_i \left\{ \delta_0(X_i, Z_i) - \hat{\delta}_0(X_i, Z_i) \mid R = 1, X_i, Y_i \right\} \right)^{1/2} \]

\[ = o_p \left( n^{1/4} k_n^{-\alpha/(d+1)} \right) \times o_p \left( n^{1/2} \right) = o_p(1), \]

where the last equality follows from Assumption 7(ii). This equation, combined with (35), implies that

\[ \frac{1}{\sqrt{n}} \sum_{i=1}^{n} R_i E \left\{ u(X, Z) \mid R = 1, X_i, Y_i \right\} \left\{ \tilde{e}(X_i, Y_i, \hat{\delta}) - \hat{e}(X_i, Y_i, \delta_0) \right\} \]

\[ = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} R_i E \left\{ u(X, Z) \mid R = 1, X_i, Y_i \right\} E \left\{ \delta_0(X_i, Z_i) - \hat{\delta}_0(X_i, Z_i) \mid R = 1, X_i, Y_i \right\} + o_p(1). \]

Define the class

\[ F = \left\{ r E \left\{ u(X, Z) \mid r = 1, x, y \right\} E \left\{ \delta_0(X, Z) - \delta(X, Z) \mid r = 1, x, y \right\} : \delta \in \Delta \right\}. \]

Since for any \( \delta_1, \delta_2 \in \Delta \),

\[ \sup_{r, x, y} \left| r E \left\{ u(X, Z) \mid r = 1, x, y \right\} E \left\{ \delta_0(X, Z) - \delta_1(X, Z) \mid r = 1, x, y \right\} \right| \]

\[ - r E \left\{ u(X, Z) \mid r = 1, x, y \right\} E \left\{ \delta_0(X, Z) - \delta_2(X, Z) \mid r = 1, x, y \right\} \right| \lesssim \| \delta_1 - \delta_2 \|_{\infty}, \]

we thus have \( N \left[ \epsilon, F, \| \cdot \|_{\infty} \right] \leq N \left[ \epsilon/K, \Delta, \| \cdot \|_{\infty} \right] \) for some \( K > 0 \). Since in addition \( \alpha > (d+1)/2 \) by hypothesis, Theorems 2.7.1 and 2.5.6 in Van Der Vaart and Wellner (1996) imply that \( F \) is a Donsker class. Then because \( \| \delta_0 - \delta_0 \|_{\infty} = o_p(1) \), it implies that

\[ \sup_{r, x, y} \left| r E \left\{ u(X, Z) \mid r = 1, x, y \right\} E \left\{ \delta_0(X, Z) - \delta_0(X, Z) \mid r = 1, x, y \right\} \right| = o_p(1). \]
Consequently, we have
\[
\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left[ R_i E \left\{ u(X, Z) \mid R = 1, X_i, Y_i \right\} E \left\{ \hat{\delta}_0(X, Z) - \delta_0(X, Z) \mid R = 1, X_i, Y_i \right\} - \langle u, \hat{\delta}_0 - \delta_0 \rangle_w \right] = o_p(1).
\]
This equation, combined with (37), implies that
\[
\sqrt{n}(\hat{u}, \delta_0 - \delta_0)_w = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} R_i E \left\{ u(X, Z) \mid R = 1, X_i, Y_i \right\} \left\{ \hat{\delta}(X_i, Y_i, \hat{\delta}_0) - \hat{\delta}(X_i, Y_i, \delta_0) \right\} + o_p(1).
\]
This completes the proof of Lemma A.7.

**Proof of Theorem 3.4.** Define the function class \( G = \{ g(r, x, z) = (1 - r)\delta(x, z) : \delta \in \Delta \} \). Since \( \|g\|_{L^2} \leq \|\delta\|_{\infty} \) for any \( g \in G \), we have
\[
\int_0^{\infty} \sqrt{\log N_{\|\cdot\|_{L^2}}(\epsilon, G, \|\cdot\|_{L^2})} d\epsilon \leq \int_0^{\infty} \sqrt{\log N_{\|\cdot\|_{\Delta}, \|\cdot\|_{\infty}}(\epsilon, \Delta, \|\cdot\|_{\infty})} d\epsilon < \infty,
\]
where the last inequality follows from Assumption 4(i) and Theorem 2.7.1 in Van Der Vaart and Wellner (1996). Then apply Theorem 2.5.6 in Van Der Vaart and Wellner (1996) to conclude that \( G \) is a Donsker class. In addition, because \( \|\hat{\delta}_0 - \delta_0\|_{\infty} = o_p(1) \), it implies that \( \sup_{r, x, z} |\hat{\delta}_0(x, z) - \delta_0(x, z)| = o_p(1) \). Consequently, we have
\[
\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left[ (1 - R_i) \left\{ \hat{\delta}_0(X_i, Z_i) - \delta_0(X_i, Z_i) \right\} - E \left\{ (1 - R)(\hat{\delta}_0(X, Z) - \delta_0(X, Z)) \right\} \right] = o_p(1).
\]
Thus,
\[
\sqrt{n}(\hat{\mu}_{rep} - \mu) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left\{ (1 - R_i)\delta_0(X_i, Z_i) + R_i Y_i - \mu \right\}
\]
\[
+ \sqrt{n}E \left\{ (1 - R)\left\{ \hat{\delta}_0(X, Z) - \delta_0(X, Z) \right\} \right\} + o_p(1).
\]
By definition of \( \delta_0 \), we have that
\[
\sqrt{n}E \left\{ (1 - R)\left\{ \hat{\delta}_0(X, Z) - \delta_0(X, Z) \right\} \right\} = \sqrt{n}(\hat{\delta}_0 - \delta_0)_w.
\]
Apply Lemma A.7 to get that
\[
\sqrt{n}E \left\{ (1 - R)\left\{ \hat{\delta}_0(X, Z) - \delta_0(X, Z) \right\} \right\}
\]
\[
= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} R_i E \left\{ h_0(X, Z) \mid R = 1, X_i, Y_i \right\} \left\{ \hat{\delta}(X_i, Y_i, \delta_0) - \hat{\delta}(X_i, Y_i, \hat{\delta}_0) \right\} + o_p(1)
\]
\[
= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} R_i E \left\{ h_0(X, Z) \mid R = 1, X_i, Y_i \right\} \left\{ Y_i - \delta_0(X_i, Z_i) \right\} - \frac{1}{\sqrt{n}} \sum_{i=1}^{n} R_i \times
\]
\[
E \left\{ h_0(X, Z) \mid R = 1, X_i, Y_i \right\} \hat{\delta}(X_i, Y_i, \delta_0) + o_p(1).
\]
By Cauchy-Schwartz inequality, we have
\[
\frac{1}{\sqrt{n}} \sum_{i=1}^{n} R_i E \left\{ h_0(X, Z) - \Pi_n h_0(X, Z) \mid R = 1, X_i, Y_i \right\} \tilde{e}(X_i, Y_i, \hat{\delta}_0) \leq \left[ \frac{1}{\sqrt{n}} \sum_{i=1}^{n} R_i \left\{ E \left( h_0(X, Z) - \Pi_n h_0(X, Z) \mid R = 1, X_i, Y_i \right) \right\}^2 \right]^{1/2} \times \left\{ \frac{1}{\sqrt{n}} \sum_{i=1}^{n} R_i \tilde{e}(X_i, Y_i, \hat{\delta}_0)^2 \right\}^{1/2} \leq n^{1/4} \| h_0 - \Pi_n h_0 \|_\infty \times n^{1/4} \left\{ Q_n(\hat{\delta}_0) \right\}^{1/2}.
\]
(40)

Then by Corollary A.1 and Assumptions 4(ii), 7(ii), we obtain that
\[
\frac{1}{\sqrt{n}} \sum_{i=1}^{n} R_i E \left\{ h_0(X, Z) - \Pi_n h_0(X, Z) \mid R = 1, X_i, Y_i \right\} \tilde{e}(X_i, Y_i, \hat{\delta}_0) = o_p(1).
\]
(41)

Similarly, apply Cauchy-Schwartz inequality, Lemma A.2, Corollary A.1 and Assumption 7(ii) to get that
\[
\frac{1}{\sqrt{n}} \sum_{i=1}^{n} R_i \left[ E \{ \Pi_n h_0(X, Z) \mid R = 1, X_i, Y_i \} - \hat{E} \{ \Pi_n h_0(X, Z) \mid R = 1, X_i, Y_i \} \right] \tilde{e}(X_i, Y_i, \hat{\delta}_0) = o_p(1).
\]

This equation, combined with (38)–(41), yields that
\[
\sqrt{n}(\hat{\mu}_{\text{tep}} - \mu) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left[ (1 - R_i)\delta_0(X_i, Z_i) + R_i Y_i - \mu + R_i \left\{ E \left( h_0(X, Z) \mid R = 1, X_i, Y_i \right) \right\} \right] \times \left\{ Y_i - \delta_0(X_i, Z_i) \right\} - \sqrt{n} r_n(\hat{\delta}_0) + o_p(1),
\]
where
\[
r_n(\hat{\delta}_0) = n^{-1} \sum_{i=1}^{n} R_i \hat{E} \{ \Pi_n h_0(X, Z) \mid R = 1, X_i, Y_i \} \tilde{e}(X_i, Y_i, \hat{\delta}_0).
\]
This completes proof of Theorem 3.4.

**Proof of Lemma 3.5.** First, by Assumption 4(ii) and Cauchy-Schwartz inequality, we have
\[
\| \hat{h} - \Pi_n h_0 \|_w^2 \leq 2 \| \hat{h} - h_0 \|_w^2 + O(\eta_n^2).
\]
Observe that
\[
C(\hat{h}) - C(\Pi_n h_0) = \| \hat{h} - h_0 \|_w^2 - \| \Pi_n h_0 - h_0 \|_w^2.
\]
Thus, we have
\[
\| \hat{h} - \Pi_n h_0 \|_w^2 \leq 2 \{ C(\hat{h}) - C(\Pi_n h_0) \} + O(\eta_n^2)
\]
\[
\leq 2 \{ C(\hat{h}) - C_n(\hat{h}) + C_n(\hat{h}) - C_n(\Pi_n h_0) + C_n(\Pi_n h_0) - C(\Pi_n h_0) \} + O(\eta_n^2)
\]
\[
\leq 4 \sup_{\delta \in \Delta_n} | C(\delta) - C_n(\delta) | + O(\eta_n^2).
\]
Next, we derive the convergence rate of \( \sup_{\delta \in \Delta_n} |C(\delta) - C_n(\delta)| \).

\[
\sup_{\delta \in \Delta_n} |C(\delta) - C_n(\delta)|
\lesssim \sup_{\delta \in \Delta_n} \left| E \left[ R \{ E(\delta(Z, X) \mid R = 1, X, Y) \} \right]^2 - \frac{1}{n} \sum_{i=1}^{n} R_i \left[ \hat{E} \{ \delta(Z, X) \mid R = 1, X_i, Y_i \} \right]^2 \right|
+ \sup_{\delta \in \Delta_n} \left| \frac{1}{n} \sum_{i=1}^{n} (1 - R_i) \delta(X_i, Z_i) - E \left\{ (1 - R) \delta(X, Z) \right\} \right|
\equiv \sup_{\delta \in \Delta_n} \left| E \left[ R \{ E(\delta(Z, X) \mid R = 1, X, Y) \} \right]^2 - \frac{1}{n} \sum_{i=1}^{n} R_i \left[ \hat{E} \{ \delta(Z, X) \mid R = 1, X_i, Y_i \} \right]^2 \right|
+ O_P(n^{-1/2}),
\]

where the last line holds because the function class \( G = \{ g(r, z, x) = (1 - r)\delta(x, z) : \delta \in \Delta \} \) is a Donsker-class as shown in the proof of Theorem 3.4. Define another function class
\[
\mathcal{F} = \left\{ f(r, x, y) = r \left\{ E(\delta(X, Z) \mid r = 1, x, y) \right\} \right)^2 : \delta \in \Delta \right\}.
\]

Note that for every \( \delta_j \in \Delta \), we have
\[
\left| r \left[ E \left\{ \delta_1(X, Z) \mid r = 1, x, y \right\} \right]^2 - r \left[ E \left\{ \delta_2(X, Z) \mid r = 1, x, y \right\} \right]^2 \right|
\leq \left| E \left\{ \delta_1(X, Z) - \delta_2(X, Z) \mid r = 1, x, y \right\} \times E \left\{ \delta_1(X, Z) + \delta_2(X, Z) \mid r = 1, x, y \right\} \right|
\lesssim \| \delta_1 - \delta_2 \|_{\infty}.
\]

Then apply Theorems 2.7.11, 2.7.1, and 2.5.6 in Van Der Vaart and Wellner (1996) to conclude that \( \mathcal{F} \) is also a Donsker-class. Thus,

\[
\sup_{\delta \in \Delta_n} \left| \frac{1}{n} \sum_{i=1}^{n} R_i \left[ E \{ \delta(X, Z) \mid R = 1, X_i, Y_i \} \right]^2 - E \left[ R \{ E(\delta(Z, X) \mid R = 1, X, Y) \} \right]^2 \right| = O_P(n^{-1/2}).
\]

We then aim to bound the following expression:

\[
\sup_{\delta \in \Delta_n} \left| \frac{1}{n} \sum_{i=1}^{n} R_i \left[ E \{ \delta(X, Z) \mid R = 1, X_i, Y_i \} \right]^2 - \frac{1}{n} \sum_{i=1}^{n} R_i \left[ \hat{E} \{ \delta(Z, X) \mid R = 1, X_i, Y_i \} \right]^2 \right|
\equiv \sup_{\delta \in \Delta_n} \left| \frac{1}{n} \sum_{i=1}^{n} (B_{11}^2 - B_{12}^2) \right|
\]

Note that

\[
\sup_{\delta \in \Delta_n} \left| \frac{1}{n} \sum_{i=1}^{n} (B_{11}^2 - B_{12}^2) \right|
\lesssim \sup_{\delta \in \Delta_n} \left| \frac{1}{n} \sum_{i=1}^{n} (B_{12} - B_{11})^2 \right| + \sup_{\delta \in \Delta_n} \left| \frac{1}{n} \sum_{i=1}^{n} B_{11} (B_{12} - B_{11}) \right|
\lesssim \sup_{\delta \in \Delta_n} \left| \frac{1}{n} \sum_{i=1}^{n} (B_{12} - B_{11})^2 \right| + \sup_{\delta \in \Delta_n} \left( \frac{1}{n} \sum_{i=1}^{n} B_{11}^2 \right)^{1/2} \left\{ \frac{1}{n} \sum_{i=1}^{n} (B_{12} - B_{11})^2 \right\}^{1/2}.
\]
Then by Assumption 4(i) and Lemma A.2, we have
\[
\sup_{\delta \in \Delta_n} \left| \frac{1}{n} \sum_{i=1}^{n} R_i \left[ E \left\{ \delta(X, Z) \mid R = 1, X_i, Y_i \right\} \right] ^2 - \frac{1}{n} \sum_{i=1}^{n} R_i \left[ \hat{E} \left\{ \delta(X, Z) \mid R = 1, X_i, Y_i \right\} \right] ^2 \right| = O_p \left\{ \left( k_n/n \right)^{1/2} + k_n^{-\alpha/(d+1)} \right\}.
\]
This equation together with (42) and (43) imply that
\[
\sup_{\delta \in \Delta_n} \left| C(\delta) - C_n(\delta) \right| = O_p \left\{ \left( k_n/n \right)^{1/2} + k_n^{-\alpha/(d+1)} \right\},
\]
and consequently,
\[
\sup_{\delta \in \Delta_n} \left\| \hat{h} - \Pi_n h_0 \right\|_w = O_p \left\{ \left( k_n/n \right)^{1/2} + k_n^{-\alpha/(d+1)} \right\}.
\]
Finally, by Cauchy-Schwartz inequality, we have
\[
\sup_{\delta \in \Delta_n} \left\| \tilde{r}_n(\delta) - r_n(\delta) \right\|
\leq \left[ \left( \frac{k_n}{n} \right)^{1/2} + k_n^{-\alpha/(d+1)} \right] \times c_n^{1/2}
\leq \sup_{\delta \in \Delta_n} \left\| \hat{h} - \Pi_n h_0 \right\|_w + O_p \left\{ \sqrt{k_n/n} + k_n^{-\alpha/(d+1)} \right\} \times c_n^{1/2}
= O_p \left\{ c_n^{1/2} \left\{ \left( \frac{k_n}{n} \right)^{1/4} + k_n^{-\alpha/(d+1)} \right\} \right\},
\]
where the second line follows from Lemma A.2. This completes the proof of Lemma 3.5. \( \square \)

**Proof of Theorem 3.6.** Under Assumption 7, we have \( k_n/n \) \( = o(n^{-1/6}) \), and \( k_n^{-\alpha/(2d+1)} = o(n^{-1/6}) \). In addition, since \( n^{2/3} b_n = o(a_n) \) as imposed in Theorem 3.6, it yields from Lemma A.5 that
\[
\sup_{\delta \in \Delta} \sqrt{n} |r_n(\delta) - \delta_0(\delta)| = o_p(1).
\]
This result, combined with Theorem 3.4, implies that
\[
\sqrt{n} (\hat{\mu}_{rep-db} - \mu) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left[ (1 - R_i) \delta_0(X_i, Z_i) + R_i Y_i - \mu + R_i \left\{ E(h_0(X, Z) \mid R = 1, X_i, Y_i) \right\} \right] \times \left\{ Y_i - \delta_0(X_i, Z_i) \right\} + o_p(1).
\]
Note that
\[
E \left[ R \left\{ (1 - R)\delta_0(X, Z) + RY - \mu \right\} E \left\{ h_0(X, Z) \mid R = 1, X, Y \right\} \left\{ Y - \delta_0(X, Z) \right\} \right]
\]
\[
= E \left[ R(RY - \mu)E \left\{ h_0(X, Z) \mid R = 1, X, Y \right\} \left\{ Y - \delta_0(X, Z) \right\} \right]
\]
\[
= E_{R=1, X, Y} \left[ (Y - \mu)E \left\{ h_0(X, Z) \mid R = 1, X, Y \right\} \left\{ Y - E(\delta_0(X, Z) \mid R = 1, X, Y) \right\} \right]
\]
\[
= 0,
\]
where the last equality holds because of the representer equation in Assumption 2. Thus, 
\[
\sqrt{n}(\hat{\mu}_{\text{rep-db}} - \mu) \text{ converges in distribution to } N(0, \sigma^2),
\]
where
\[
\sigma^2 = E \left\{ (1 - R)\delta_0(X, Z) + RY - \mu \right\}^2 + R \left\{ E(h_0(X, Z) \mid R = 1, X, Y) \right\}^2 \left\{ Y - \delta_0(X, Z) \right\}^2.
\]
This completes proof of Theorem 3.6. □

**Proof of Corollary 3.1.** Consider a regular parametric submodel in \( \mathcal{M}_{np} \) indexed by \( t \) that includes the true data generating mechanism at \( t = 0 \). The distribution of observed data \( O = (R, RY, X, Z) \) is depicted by
\[
f(O) = Rf(Y \mid R = 1, X, Z)f(R = 1 \mid X, Z)f(X, Z) + (1 - R)f(R = 0 \mid X, Z)f(X, Z).
\]
Thus, the observed data score is
\[
S(O) = R \left\{ S(Y \mid R = 1, X, Z) + S(R = 1 \mid X, Z) + S(X, Z) \right\}
\]
\[
+ (1 - R) \left\{ -\frac{1}{\beta(X, Z)} S(R = 1 \mid X, Z) + S(X, Z) \right\},
\]
where \( S(A \mid B) = \partial \log \{ f_t(A \mid B) \} \partial t \mid_{t=0} \) for generic random variables (vectors) \( A \) and \( B \). In addition, recall from the restriction of \( \mathcal{M}_{np} \) that
\[
E \left\{ \gamma(X, Y) \mid R = 1, X, Z \right\} = \beta(X, Z).
\]
Then we have
\[
\partial E_t \left\{ \gamma_t(X, Y) - \beta_t(X, Z) \mid R = 1, X, Z \right\} / \partial t \mid_{t=0} = 0.
\]
Equivalently,
\[
\int \frac{\partial \left[ \{ \gamma_t(X, y) - \beta_t(X, Z) \} f_t(y \mid R = 1, X, Z) \right]}{\partial t} \mid_{t=0} dy = 0.
\]
It then follows that
\[
E \left\{ \gamma(X, Y)S(Y \mid R = 1, X, Y) \mid R = 1, X, Z \right\} + \frac{S(R = 1 \mid X, Z)}{f(R = 1 \mid X, Z)}
\]
\[
= E \left\{ \partial \gamma_t(X, Y) / \partial t \mid_{t=0} R = 1, X, Z \right\}.
\]
Based on the above results and Assumption 8(ii), we then follow from the derivations in Severini and Tripathi (2012) to obtain the tangent space as follows:

\[ \Lambda_1 + \Lambda_2 \]

\[ \equiv \left\{ S(X, Z) \in L_2(X, Z) : E\{S(X, Z)\} = 0 \right\} \]

\[ + \left\{ RS(Y \mid R = 1, X, Z) + \left\{ R - \frac{1 - R}{\beta(X, Z)} \right\} S(R = 1 \mid X, Z) : \right\} \]

\[ E\left\{ S(Y \mid R = 1, X, Z) \mid R = 1, X, Z \right\} = 0, \text{ and} \]

\[ E\left\{ \gamma(X, Y)S(Y \mid R = 1, X, Z) \mid R = 1, X, Z \right\} + \frac{S(R = 1 \mid X, Z)}{f(R = 1 \mid X, Z)} \in cl(\mathcal{R}(T)) \],

where \( \mathcal{R}(T) \) denotes the range space of \( T \). According to Theorems 3.4 and 3.6, the influence function of \( \mu \) is given in (16). In order to show that it is efficient, we only need to show that it belongs to the above tangent space. For simplicity, below we may use \( \delta_0, \gamma, \) and \( \beta \) to denote \( \delta_0(X, Z), \gamma(X, Y), \) and \( \beta(X, Z), \) respectively.

As assumed, \( h_0(X, Z) \) uniquely solves (7). Then we have the following decomposition of (16):

\[ (1 - R)\delta_0(X, Z) + RY + RE\{ h_0(X, Z) \mid R = 1, X, Y \} \{ Y - \delta_0(X, Z) \} - \mu \]

\[ = (1 - R)\delta_0 + RY + R\gamma(Y - \delta_0) - \mu \]

\[ = \frac{1}{\beta + 1} \left\{ E(Y + Y\gamma - \gamma\delta_0 \mid R = 1, X, Z) - \delta_0 \right\} + \delta_0 - \mu \]

\[ + R\left\{ Y + Y\gamma - \gamma\delta_0 - E(Y + Y\gamma - \gamma\delta_0 \mid R = 1, X, Z) \right\} \]

\[ + \left( R - \frac{1 - R}{\beta} \right) \frac{\beta}{\beta + 1} \left\{ E(Y + Y\gamma - \gamma\delta_0 \mid R = 1, X, Z) - \delta_0 \right\} \]

Note that

\[ \frac{1}{\beta + 1} = f(R = 1 \mid X, Z) \]

\[ E(Y\gamma \mid R = 1, X, Z) = \beta(X, Z)E(Y \mid R = 0, X, Z) \]

\[ E(\gamma \mid R = 1, X, Z) = \beta(X, Z). \]

Then, we have

\[ E\left[ \frac{1}{\beta + 1} \left\{ E(Y + Y\gamma - \gamma\delta_0 \mid R = 1, X, Z) - \delta_0 \right\} + \delta_0 - \mu \right] = 0. \]

Consequently,

\[ \frac{1}{\beta + 1} \left\{ E(Y + Y\gamma - \gamma\delta_0 \mid R = 1, X, Z) - \delta_0 \right\} + \delta_0 - \mu \in \Lambda_1. \]

According to Assumption 8(i) and (iii), we have \( \mathcal{N}(T') = \{0\} \) and \( cl(\mathcal{R}(T)) = \mathcal{N}(T')^\perp = L_2(X, Z) \). Take

\[ S(Y \mid R = 1, X, Z) = Y + Y\gamma - \gamma\delta_0 - E(Y + Y\gamma - \gamma\delta_0 \mid R = 1, X, Z), \]

\[ S(R = 1 \mid X, Z) = \frac{\beta}{\beta + 1} \left\{ E(Y + Y\gamma - \gamma\delta_0 \mid R = 1, X, Z) - \delta_0 \right\}. \]
Apparently,

\[
E \left\{ S(Y \mid R = 1, X, Z) \mid R = 1, X, Z \right\} = 0,
\]

\[
E \left\{ \gamma(X, Y) S(Y \mid R = 1, X, Z) \mid R = 1, X, Z \right\} + \frac{S(R = 1 \mid X, Z)}{f(R = 1 \mid X, Z)} \in L_2(X, Z).
\]

Thus,

\[
R \left\{ Y + Y \gamma - \gamma \delta_0 - E\{Y + Y \gamma - \gamma \delta_0 \mid R = 1, X, Z\} \right\} + \left( R - \frac{1 - R}{\beta} \right) \frac{\beta}{\beta + 1} \left\{ E\{Y + Y \gamma - \gamma \delta_0 \mid R = 1, X, Z\} - \delta_0 \right\} \in \Lambda_2
\]

This completes the proof of Corollary 3.1. \(\square\)

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