Quantum Energy Teleportation with Trapped Ions

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Abstract

We analyse a protocol of quantum energy teleportation that transports energy from the left edge of a linear ion crystal to the right edge by local operations and classical communication at a speed considerably greater than the speed of a phonon in the crystal. A probe qubit is strongly coupled with phonon fluctuation in the ground state for a short time, and it is projectively measured in order to obtain information about this phonon fluctuation. During the measurement process, phonons are excited by the time-dependent measurement interaction, and the energy of the excited phonons must be infused from outside the system. The obtained information is transferred to the right edge of the crystal through a classical channel. Even though the phonons excited at the left edge do not arrive at the right edge at the same time as when the information arrives at the right edge, we are able to soon extract energy from the ions at the right edge by using the transferred information. Because the intermediate ions of the crystal are not excited during the execution of the protocol, energy is transmitted in the energy transfer channel without heat generation.
1 Introduction

A trapped ion system is expected to be a promising candidate for developing quantum computers [1]. Recently, the experimental development of trapped ions has undergone a significant technical advancement [2]. In addition, this technology has been applied to quantum teleportation [3] and quantum error correction [4]. The successful application of trapped ion systems to the abovementioned tasks proves that these systems have a great potential for use in other fascinating quantum tasks, which have not yet been experimentally executed. In this study, we analyse quantum energy teleportation with cold ions in a linear trap, that is, a linear ion crystal. Recently, quantum energy teleportation (QET) has been proposed [5]-[8]; this protocol is able to transport energy simply by local operations and classical communication. Energy can be effectively teleported without breaking any physical laws including causality and local energy conservation by measuring zero-point oscillation at one site in the entangled ground state of a many-body quantum system and providing the measurement result to distant sites. The key point is that there exists quantum correlation between local fluctuations of different sites in the ground state. Therefore, the measurement result of local fluctuation in some site includes information about fluctuation in other sites. By selecting and performing a suitable local operation based on the transferred information, the zero-point oscillation of a site away from the measurement site can be more suppressed than that of the ground state, yielding negative energy density. Here, the origin of energy density is fixed such that the expectation value vanishes for the ground state. It is well known that such regions with negative energy density are created by superposing energy eigenstates quantum mechanically. Even if a system comprises a region with negative energy density, other regions with positive energy density still exist, and therefore, the total energy of the system remains non-negative. Discussions on negative energy density of non-relativistic systems are given in [7] and those for negative energy density of relativistic systems are given in [9]. While performing the above local operation to create negative energy density in the system, the surplus energy is transferred from quantum fluctuations to external systems and can be harnessed. Though the protocols of energy teleportation can be implemented for other systems such as quantum fields [5] and spin chains [6, 7, 8], linear ion crystals can be effectively used for
QET. Due to the Coulomb interaction and the harmonic potential for linear trapping, the ions interact with each other so strongly that entanglement for energy teleportation is easily generated in the phonon ground state. It is also possible to POVM measure the local fluctuation of phonons in the linear ion crystal by coupling the phonon modes with the internal energy levels of each ion via a laser field [2].

In this study, we use the linear trap models that are extensively analysed by James [10] and consider a protocol of QET. In what follows, \( N \) cold ions, which are strongly bound in the \( y \) and \( z \) directions but weakly bound in a harmonic potential in the \( x \) direction, form a linear ion crystal called the QET channel. The first ion that stays at the left edge of the crystal is the gateway of the QET channel where energy is input. The \( N \)-th ion that stays at the right edge of the crystal is the exit of the QET channel where the teleported energy is output. We select two suitable internal energy levels of the gateway ion and regard them as the energy levels of a probe qubit in order to measure the local phonon fluctuation. The probe qubit is strongly coupled with the phonon fluctuation in the ground state for a short period of time via a laser field, and it is projectively measured in order to obtain 1-bit information about phonon fluctuation. During the measurement process at the gateway, phonons are excited in the system by the time-dependent measurement interaction, and hence, the energy of the excited phonons must be infused from outside the system. In the measurement models used in this study, the kinetic energy of the gateway ion increases after the measurement; however, the kinetic energy of other ions and the potential energy of all the ions remain unchanged. This infusion of energy is regarded as energy input at the gateway of the QET channel. The obtained information is transferred through a classical channel from the gateway point to the exit point. In principle, the speed at which this information is transferred is equal to the speed of light, which is considerably greater than that of phonon propagation in the ion crystal. It is emphasized that even when the phonons excited at the QET gateway do not arrive at the exit point, the information still arrives at the exit point. Because the measurement process at the gateway is local, the state of the exit ion before the phonons arrive is locally the same as the ground state. Surprisingly, however, we are able to soon extract energy from the exit ion by using the transferred information. Performing a local operation that depends on the measurement result suppresses the zero-point oscillation of the phonon fluctuation at the exit ion, yielding negative energy density around the exit. The surplus energy of the fluctuation is transferred
from the phonon system to the external systems including the device, in order to perform this operation. Here, it should also be emphasized that without the measurement result, we cannot extract energy from the exit ion in the local ground state by performing an arbitrary local quantum operation on the ion. Because the intermediate ions of the crystal are not excited during the protocol execution, this QET ensures energy transportation in the channel without the generation of heat. The non-negativity of the total Hamiltonian of phonons ensures that the amount of the teleported energy is less than the amount of the input energy. The amount of energy required for transferring the information is negligibly small in principle as compared to the amount of the teleported energy, because the information can be simply transported by low-energy physical carriers such as electromagnetic waves with long wavelengths.

The paper is organized as follows. In section 2, the linear ion crystal model developed by James [10] is reviewed. In section 3, QET for the model is analysed. In the last section, the summary and discussion are given. In this study, we adopt a unit $\hbar = 1$.

2 Linear Ion Crystal Model

Following James [10], let us consider $N$ ions with charge $Ze$ and mass $m$ that are strongly bound in the $y$ and $z$ directions but weakly bound in a harmonic potential in the $x$ direction. The position of the $n$-th ion is denoted by $x_n(t)$, where the ions are numbered from left to right. The Hamiltonian is expressed as

$$H_{\text{ion}} = \sum_{n=1}^{N} \frac{m}{2} \dot{x}_n(t)^2 + \sum_{n=1}^{N} \frac{m}{2} \nu^2 x_n(t)^2 + \frac{1}{2} \sum_{n,n' = 1, n \neq n'}^{N} \frac{Z^2 e^2}{|x_n(t) - x_{n'}(t)|},$$

where the dot denotes time derivative, and $\nu$ is the trap frequency, which characterizes the strength of the trapping harmonic potential in the $x$ direction. In the case of cold ions, we can approximate the position of the $n$-th ion by

$$x_n(t) \approx x_n^{(0)} + q_n(t),$$

where $x_n^{(0)}$ is the position of the ion in the ground state, and $q_n(t)$ is a small displacement describing the phonon modes in the ion crystal. The
equilibrium positions in the ground state are determined by the following equation derived from Eq. (1):

$$mν^2x_n^{(0)} - \sum_{n' = 1}^{n-1} \frac{Z^2e^2}{\left(x_n^{(0)} - x_{n'}^{(0)}\right)^2} + \sum_{n' = n+1}^N \frac{Z^2e^2}{\left(x_n^{(0)} - x_{n'}^{(0)}\right)^2} = 0.$$  \hspace{1em} (2)

Let us introduce a scale length given by

$$l = \left(\frac{Z^2e^2}{mν^2}\right)^{1/3}$$

and rescale the position variables as

$$u_n = \frac{x_n^{(0)}}{l}.$$  \hspace{1em}

Then, Eq. (2) is rewritten in a dimensionless form as

$$u_n - \sum_{n' = 1}^{n-1} \frac{1}{(u_n - u_{n'})^2} + \sum_{n' = n+1}^N \frac{1}{(u_n - u_{n'})^2} = 0.$$  \hspace{1em} (3)

First, it is pointed out [10] that Eq. (3) can be analytically solved for the cases with $N = 2, 3$. When $N = 2$, the solution of the above equation can be expressed as

$$u_1 = -\left(\frac{1}{2}\right)^\frac{2}{3}, \quad u_2 = \left(\frac{1}{2}\right)^\frac{2}{3}.$$  \hspace{1em}

When $N = 3$, the solution can be expressed as

$$u_1 = -\left(\frac{5}{4}\right)^\frac{4}{3}, \quad u_2 = 0, \quad u_3 = \left(\frac{5}{4}\right)^\frac{4}{3}.$$  \hspace{1em}

For $N$ greater than 3, $u_n$ were numerically solved by James. The Hamiltonian of phonons in the crystal is derived by expanding Eq. (1) in terms of $q_n(t)$ and by considering only bilinear terms; it is expressed as follows:

$$H = \sum_{n = 1}^N \frac{1}{2m} p_n^2 + \sum_{n,n' = 1}^N \frac{1}{2} mν^2 A_{nn'} q_n q_{n'} - E_g.$$  \hspace{1em} (4)
where the zero-point energy $E_g$ is subtracted from the original form in order
to make the lowest eigenvalue of $H$ zero, and a real symmetric matrix $A_{nn'}$
is defined by

$$A_{nn} = 1 + 2 \sum_{n''=1 \atop n'' \neq n}^{N} \frac{1}{|u_n - u_{n''}|^3}$$

for $n = n'$ and

$$A_{nn'} = -2 \frac{1}{|u_n - u_{n'}|^3}$$

for $n \neq n'$. The eigenvectors $b^{(k)}_n (k = 1, 2, \cdots, N)$ of $A_{nn'}$
determined by

$$\sum_{n'=1}^{N} A_{nn'} b^{(k)}_{n'} = \mu_k b^{(k)}_n$$

are real. In addition, the eigenvalues $\mu_k$ of $A_{nn'}$ are real and non-negative
[10]. The eigenvectors are numbered in the order of increasing eigenvalues
and normalized as

$$\sum_{n=1}^{N} b^{(k)}_{n} b^{(k')}_{n} = \delta_{kk'}.$$ 

The first and second eigenvectors are analytically obtained. For $k = 1$, the
eigenvector is given by

$$b^{(1)}_n = \frac{1}{\sqrt{N}},$$

with its eigenvalue $\mu_1 = 1$. For $k = 2$, the eigenvector is given by

$$b^{(2)}_n = \frac{u_n}{\sqrt{\sum_{n'=1}^{N} u_{n'}^2}},$$

with its eigenvalue $\mu_2 = 3$. Higher eigenvalues and eigenvectors are numerically solved and listed in table 2 of [10]. Next, let us introduce the normal modes of phonons as

$$Q_k = \sum_{n=1}^{N} b^{(k)}_n q_n.$$ 

Then, $H$ is diagonalized for the modes as

$$H = \sum_{k=1}^{N} \left( \frac{1}{2m} p^2_k + \frac{1}{2} m v^2 \mu_k Q_k^2 \right) - E_g,$$
where $P_k$ is the conjugate momentum of $Q_k$ and $P_k(t) = m\dot{Q}_k(t)$. Let us introduce phonon creation and annihilation operators as

$$a_k^\dagger = \sqrt{\frac{1}{2m\nu \sqrt{\mu_k}}} P_k + i\sqrt{\frac{m\nu \sqrt{\mu_k}}{2}} Q_k,$$

$$a_k = \sqrt{\frac{1}{2m\nu \sqrt{\mu_k}}} P_k - i\sqrt{\frac{m\nu \sqrt{\mu_k}}{2}} Q_k.$$

These operators satisfy the following commutation relation:

$$[a_k, a_{k'}^\dagger] = \delta_{kk'}.$$

The ground state $|g\rangle$ of the phonon system is defined by

$$a_k |g\rangle = 0,$$  \hfill (6)

and simultaneously satisfies

$$H |g\rangle = 0$$  \hfill (7)

due to the subtraction of the zero-point energy $E_g$. From Eq. (7), it is verified that $H$ is a non-negative operator. i.e. $H \geq 0$. By using the normal modes, it is easy to solve the Heisenberg operator for the displacement of the $n$-th ion from the equilibrium position as

$$q_n(t) = \sum_{k=1}^{N} b_n^{(k)} Q_k(t)$$

$$= \sum_{k=1}^{N} b_n^{(k)} \frac{i}{\sqrt{2m\nu \sqrt{\mu_k}}} \left( a_k e^{-i\nu \sqrt{\mu_k} t} - a_k^\dagger e^{i\nu \sqrt{\mu_k} t} \right).$$  \hfill (8)

Its corresponding conjugate momentum operator is solved as

$$p_n(t) = \sum_{k=1}^{N} b_n^{(k)} P_k(t)$$

$$= \sum_{k=1}^{N} b_n^{(k)} \sqrt{\frac{m\nu \sqrt{\mu_k}}{2}} \left( a_k e^{-i\nu \sqrt{\mu_k} t} + a_k^\dagger e^{i\nu \sqrt{\mu_k} t} \right).$$  \hfill (9)
These Heisenberg operators can be computed from their Schrödinger operators $q_n (= q_n(0))$ and $p_n (= p_n(0))$ as follows:

$$q_n(t) = \sum_{n'=1}^{N} W_{nn'}^{(1)}(t) q_{n'} + \frac{1}{m\nu} \sum_{n'=1}^{N} W_{nn'}^{(2)}(t) p_{n'},$$

$$p_n(t) = -m\nu \sum_{n'=1}^{N} W_{nn'}^{(3)}(t) q_{n'} + \sum_{n'=1}^{N} W_{nn'}^{(1)}(t) p_{n'},$$

where $W_{nn'}^{(r)}(t)$ ($r = 1, 2, 3$) are real symmetric matrices defined by

$$W_{nn'}^{(1)}(t) = \sum_{k=1}^{N} \cos (\nu \sqrt{\mu_k} t) b_{n}^{(k)} b_{n'}^{(k)},$$

$$W_{nn'}^{(2)}(t) = \sum_{k=1}^{N} \frac{1}{\sqrt{\mu_k}} \sin (\nu \sqrt{\mu_k} t) b_{n}^{(k)} b_{n'}^{(k)},$$

$$W_{nn'}^{(3)}(t) = \sum_{k=1}^{N} \sqrt{\mu_k} \sin (\nu \sqrt{\mu_k} t) b_{n}^{(k)} b_{n'}^{(k)}.$$

The operators $q_n(t)$ and $p_n(t)$ describe the quantum motion of phonons in the crystal. Under real experimental conditions, the typical order of the largest frequency $\nu \sqrt{\mu_N}$ of the phonon oscillation is $O(10^6)$ Hz for $N = 2 \sim 10$, and the typical order of the ion crystal size is $O(10^{-6})$ m. Therefore, the ratio of the phonon velocity to the light velocity is estimated to be $O(10^{-8}) \ll 1$. This estimation ensures that the system can be treated non-relativistically.

In the later discussion, it is convenient to introduce phonon coherent states. Let us consider a unitary operator given by

$$U(\alpha, \beta) = \exp \left[ i \sum_{n=1}^{N} (\alpha_n q_n - \beta_n p_n) \right],$$

where $\alpha = (\alpha_1, \ldots, \alpha_N)$ and $\beta = (\beta_1, \ldots, \beta_N)$ are real vectors. By performing $U(\alpha, \beta)$ on the ground state, a coherent state is obtained as follows:

$$| (\alpha, \beta) \rangle = U(\alpha, \beta) |g \rangle. \quad (10)$$

This state is an eigenstate of the operator $a_k$ expressed as

$$a_k | (\alpha, \beta) \rangle = \left( \frac{1}{\sqrt{2m\nu \sqrt{\mu_k}}} A_k - i \sqrt{\frac{m\nu \sqrt{\mu_k}}{2} B_k} \right) | (\alpha, \beta) \rangle,$$
where

\[ A_k = \sum_{n=1}^{N} b^{(k)}_n \alpha_n, \]

\[ B_k = \sum_{n=1}^{N} b^{(k)}_n \beta_n. \]

Solving Eqs. (8) and (9) with \( t = 0 \), \( |(\alpha, \beta)\rangle \) can be explicitly written as

\[
| (\alpha, \beta) \rangle = \exp \left[ -\frac{1}{4} \sum_{k'=1}^{N} \left| \frac{1}{\sqrt{m\nu\sqrt{\mu_k}}} A_{k'} - i \sqrt{m\nu\sqrt{\mu_k}} B_{k'} \right|^2 \right] 
\times \exp \left[ \sum_{k=1}^{N} \left( \frac{1}{\sqrt{2m\nu\sqrt{\mu_k}}} A_k - i \sqrt{\frac{m\nu}{2\mu_k}} B_k \right) a_k^\dagger \right] |g\rangle. \tag{11}
\]

In addition, the inner product between two coherent states is calculated as

\[
\langle (\alpha, \beta) | (\alpha', \beta') \rangle = \exp \left[ \frac{i}{2} \sum_{n=1}^{N} (\alpha_n \beta'_n - \beta_n \alpha'_n) \right] 
\times \exp \left[ -\frac{1}{4} \sum_{k=1}^{N} \left| \frac{1}{\sqrt{m\nu\sqrt{\mu_k}}} (A_k - A'_k) - i \sqrt{m\nu\sqrt{\mu_k}} (B_k - B'_k) \right|^2 \right]. \tag{12}
\]

These formulas are used in the next section.

### 3 Quantum Energy Teleportation

In this section, we analyse a QET protocol by treating the linear ion crystal discussed in the previous section as a QET channel. The first ion at \( x = x_1^{(0)} \) is the gateway of the QET channel where energy is input. The \( N \)-th ion at \( x = x_N^{(0)} \) is the exit of the QET channel where the teleported
energy is output. First, let us perform a local POVM measurement of the phonon fluctuation in the ground state $|g\rangle$ at the gateway. The probe qubit used to measure the fluctuation is composed of two internal energy levels of the gateway ion in the same way as [2]. The measurement interaction in our model is given by

$$H_m = g(t)\sigma_y G_1,$$  \hspace{1cm} (13)

where $g(t)$ is a time-dependent real coupling constant, $\sigma_y$ is the $y$ component of the Pauli matrix, and $G_1$ is a Hermitian local operator on the first ion defined by

$$G_1 = \phi + \lambda q_1.$$ \hspace{1cm} (14)

Here, $\phi$ and $\lambda$ are time-independent real coupling constants. Let us assume that the coupling constant $g(t)$ does not vanish only during a very short period of time via the laser pulse field and that it can be approximated as

$$g(t) = \delta(t).$$

The initial state of the probe qubit is assumed to be the up eigenstate of the $z$ component of the Pauli matrix, $\sigma_z,$ given by

$$|+\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$  

After the switch off of the measurement interaction in Eq. (13), we projectively measure $\sigma_z$ of the probe qubit. This POVM measurement can be described by two measurement operators [11] given by

$$M_\pm = \langle \pm | \exp [-i\sigma_y G_1] | + \rangle,$$ \hspace{1cm} (15)

where $|-\rangle$ is the down eigenstate of $\sigma_z$ given by

$$|-\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$  

The operators $M_\pm$ are computed explicitly as

$$M_+ = \cos G_1,$$ \hspace{1cm} (16)

$$M_- = \sin G_1.$$ \hspace{1cm} (17)
In addition, they satisfy the following relations:

\[ \sum_{s=\pm} M_s^\dagger M_s = 1, \tag{18} \]
\[ \sum_{s=\pm} sM_s^\dagger M_s = \cos (2G_1). \tag{19} \]

The average state of the phonon system after the measurement is given by

\[ \rho_M = \sum_{s=\pm} M_s|g\rangle\langle g| M_s^\dagger. \]

Before the measurement, the phonon system is in the ground state \( |g\rangle \) and has no energy such that

\[ \langle g| H |g\rangle = 0. \]

After the measurement, the average energy of the phonon system is given by

\[ E_{in} = \text{Tr} [H \rho_M] = \sum_{s=\pm} \langle g| M_s^\dagger H M_s |g\rangle. \tag{20} \]

This can be easily evaluated using Eq. (18). Taking into account that \( M_s \) commutes with the kinetic energy of the ions for \( n = 2 \sim N \) and the potential energy of all the ions, it can be proven that only the kinetic energy \( p_1^2/(2m) \) of the gateway ion changes during the measurement. In order to evaluate this change, the following relation can be used:

\[ [p_1, M_\pm] = \pm i\lambda M_\mp \tag{21} \]

From Eq. (21), we can obtain the following relation given by

\[ M_\pm^\dagger p_1^2 M_\pm = M_\mp^\dagger M_\pm (p_1^2 + \lambda^2) \mp 2i\lambda M_+ M_- p_1. \tag{22} \]

Using Eqs. (18) and (22), \( E_{in} \) is computed as

\[ E_{in} = \sum_{s=\pm} \langle g| M_s^\dagger \frac{p_1^2}{2m} M_s |g\rangle - \langle g| \frac{p_1^2}{2m} |g\rangle = \frac{\lambda^2}{2m}. \tag{23} \]

This energy of the excited phonons must be infused from outside the system and is regarded as the energy inputted at the gateway of the QET channel. The obtained information \( s \) is transferred through a classical channel from
the gateway point at $x = x_1^{(0)}$ to the exit point at $x = x_N^{(0)}$. In principle, the speed at which the information is transferred is equal to the speed of light, which is considerably greater than that of phonon propagation in the ion crystal. It is emphasized that the phonons excited at the QET gateway do not arrive at the exit point when the information arrives at the exit point. Because the measurement process at the gateway is local, the state of the exit ion before the arrival of the phonons is locally the same as the ground state. Interestingly, however, we are able to soon extract energy from the exit ion by using the transferred $s$. Let us perform a local unitary operation dependent on $s$ given by

$$U_s = \exp \left( is\theta p_N \right), \quad (24)$$

where $\theta$ is a real parameter fixed later. It is possible to suppress the zero-point oscillation of the phonon fluctuation at the exit ion by performing the above operation. Neglecting time evolution of the system during the high-speed transfer of $s$, the average state of the system after the operation is given by

$$\rho_F = \sum_{s=\pm} U_s M_s |g\rangle \langle g| M_s^\dagger U_s^\dagger. \quad (25)$$

Using Eq. (20) and a formula given by

$$U_s^\dagger H U_s = H - s\theta m\nu^2 \sum_{n=1}^N A_{Nn} q_n + \theta^2 \frac{m\nu^2}{2} A_{NN},$$

the average energy after the operation can be evaluated as follows:

$$E_F = \text{Tr} [H \rho_F] = \sum_{s=\pm} \langle g| M_s^\dagger U_s^\dagger H U_s M_s |g\rangle$$

$$= E_{in} - \theta m\nu^2 \sum_{n=1}^N A_{Nn} \langle g| q_n \left( \sum_{s=\pm} s M_s^\dagger M_s \right) |g\rangle + \theta^2 \frac{m\nu^2}{2} A_{NN}. \quad (26)$$

Here, we have used the fact that $[M_s^\dagger, q_n] = 0$. Substituting Eq. (19) into Eq. (26) yields

$$E_F = E_{in} - \theta \eta + \theta^2 \xi, \quad (27)$$
where $\eta$ and $\xi$ are real coefficients given by

$$\eta = m\nu^2 \sum_{n=1}^{N} A_{Nn} \langle g| q_n \cos(2G_1) |g \rangle, \quad (28)$$

$$\xi = \frac{m\nu^2}{2} A_{NN}. \quad (29)$$

The expression of $\eta$ in Eq. (28) can be simplified further as follows. First, it is pointed out that the following relation holds:

$$\cos(2G_1)|g\rangle = \frac{1}{2} \left[ e^{2i\phi} e^{2i\lambda q_1} |g\rangle + e^{-2i\phi} e^{-2i\lambda q_1} |g\rangle \right]. \quad (30)$$

It is observed that the two states $|\pm 2\lambda\rangle = e^{\pm 2i\lambda q_1}|g\rangle$ in the above equation are the following phonon coherent states in Eq. (10):

$$|\pm 2\lambda\rangle = |((\pm 2\lambda, 0, \cdots, 0), (0, 0, \cdots, 0))\rangle.$$

Therefore, the states $|\pm 2\lambda\rangle$ are the eigenstates of $a_k$ such that

$$a_k |\pm 2\lambda\rangle = \pm \frac{2\lambda b^{(k)}}{\sqrt{2m\nu \sqrt{\mu_k}}} |\pm 2\lambda\rangle. \quad (31)$$

By introducing a matrix $\Delta_{nn'}$ given by

$$\Delta_{nn'} = W_{nn'}^{(1)}(0) = \sum_{k=1}^{N} \frac{1}{\sqrt{\mu_k}} b_n^{(k)} b_{n'}^{(k)},$$

and using Eq. (6), Eq. (8) with $t = 0$, and Eq. (31), the following relation is proved.

$$\langle 0| q_n |\pm 2\lambda\rangle = \pm i \frac{\lambda}{m\nu} \langle 0| \pm 2\lambda\rangle \Delta_{1n}. \quad (32)$$

In addition, Eq. (12) yields the following relation:

$$\langle 0| \pm 2\lambda\rangle = \exp \left[ -\frac{\lambda^2}{m\nu} \Delta_{11} \right]. \quad (33)$$
From Eq. (30), Eq. (32), and Eq. (33), the following relation is obtained:

$$\langle g | q_n \cos (2G_1) | g \rangle = -\frac{\lambda \sin (2\phi)}{m\nu} \exp \left[ -\frac{\lambda^2}{m\nu} \Delta_{11} \right] \Delta_{1n}.$$ 

Substituting this relation into Eq. (28) yields the final expression of $\eta$ as follows:

$$\eta = -\lambda \nu \sin (2\phi) \exp \left[ -\frac{\lambda^2}{m\nu} \Delta_{11} \right] \sum_{n=1}^{N} \Delta_{1n} A_{nN}.$$ 

From this result, we can show that the coefficient $\eta$ does not vanish as long as the factor $\sin (2\phi)$ does not vanish. This observation indicates that the operation in Eq. (24) ensures that $E_F$ is smaller than $E_{in}$. In fact, $E_F$ in Eq. (27) can be minimized by considering the parameter $\theta$ such that

$$\theta = \frac{\eta}{2\xi}.$$ 

Then, Eq. (27) can be rewritten as

$$E_F = E_{in} - \frac{\eta^2}{4\xi}. \quad (34)$$

Because $A_{NN}$ in Eq. (5) is positive, $\xi$ is also positive. This implies that during the operation $U_s$, a positive amount of energy given by

$$E_{out} = \frac{\eta^2}{4\xi} = \left( \frac{\lambda^2 \sin^2 (2\phi)}{2m A_{NN}} \right) \exp \left[ -\frac{2\lambda^2}{m\nu} \Delta_{11} \right] \left| \sum_{n=1}^{N} \Delta_{1n} A_{nN} \right|^2 \quad (35)$$

is transferred from the phonon system to the external systems including the device system executing the operation $U_s$. For example, $E_{out}$ for $N = 2$ is analytically evaluated as

$$E_{out} = \frac{2 - \sqrt{3}}{4} \cdot \frac{\lambda^2 \sin^2 (2\phi)}{2m} \exp \left[ -\left( 1 + \frac{1}{\sqrt{3}} \right) \frac{\lambda^2}{m\nu} \right].$$

This energy extraction is regarded as outputting the teleported energy from the exit of the QET channel. Figure 1 shows the QET protocol. Eq. (35) can be expressed in terms of the input energy $E_{in} = \frac{\lambda^2}{2m}$ as follows:

$$E_{out} = \gamma_N E_{in} \exp \left[ -\zeta_N \frac{E_{in}}{\nu} \right] \sin^2 (2\phi), \quad (36)$$

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where the coefficients $\gamma_N$ and $\zeta_N$ are defined as

$$
\gamma_N = \frac{1}{A_{NN}} \left| \sum_{n=1}^{N} \Delta_{1n} A_{nN} \right|^2, \tag{37}
$$

$$
\zeta_N = 4\Delta_{11}. \tag{38}
$$

The maximum value of the output energy $E_{\text{out}}$ with respect to $\phi$ is obtained by setting $\phi = \pm \frac{\pi}{4}$. From Eq. (36), it is observed that $E_{\text{out}}$ exponentially decays when the input energy $E_{\text{in}}$ is considerably greater than the typical energy $\nu$ of one phonon in the ion crystal. Therefore, $E_{\text{in}}$ should be of the same order as $\nu$; this is ensured by choosing $\lambda = O(\sqrt{m\nu})$ in Eq. (14). Further, it is observed that $E_{\text{out}}$ rapidly decays when $N$ becomes large. In Figures 2 and 3, numerical results of $\ln \gamma_N$ and $\zeta_N$ are plotted as functions of $N$. Though $\zeta_N$ does not have a drastic $N$-dependence, $\gamma_N$ decays approximately exponentially ($\propto e^{-1.1N}$). Hence, the output energy $E_{\text{out}}$ behaves as $e^{-1.1N} O(\nu)$. It is emphasized that without using the transferred $s$, we cannot extract energy from the exit ion on an average by performing an arbitrary local quantum operation on the ion. This is because the state of the exit ion is locally the same as the ground state. Due to the passivity property of the ground state, any local quantum operation independent of $s$ on the exit ion gives energy to the crystal, generating an excited state, or it ensures that the state is unchanged with no energy gain. The non-negativity of $H$ ensures that $E_F \geq 0$ in Eq. (34). Thus, $E_{\text{out}}$ is not greater than $E_{\text{in}}$.

4 Summary and Discussion

In this study, we have analysed a QET protocol for the linear ion crystal model developed by James [10]. At the gateway of the QET channel (at $x = x_1^{(0)}$), the phonon fluctuation in the ground state is POVM measured. The measurement operators are given by Eqs. (16) and (17). During the measurement process, the input energy $E_{\text{in}}$ in Eq. (23) is infused from outside the system. The kinetic energy of the gateway ion increases after the measurement; however, the kinetic energy of the other ions and the potential energy of all the ions remain unchanged. The obtained information $s$ by the POVM measurement is transferred through a classical channel to the exit point at $x = x_N^{(0)}$. In principle, the speed at which the information is
transferred is equal to the speed of light, which is considerably greater than that of the phonon propagation in the ion crystal. Even when the phonons excited at the gateway point do not arrive at the exit point, we are able to soon extract the energy $E_{\text{out}}$ in Eq. (35) from the exit ion by performing $U_s$ in Eq. (24). Because the intermediate ions of the crystal are not excited during all the operations of the protocol, this QET involves the transportation of energy without the generation of heat in the channel.

Thus far, even though the QET mechanism is an interesting phenomenon, it has not yet been experimentally verified. Clearly, the experimental verification of QET requires fast performance of local operations and classical communication. Therefore, it may be better to perform these operations collectively using one physical carrier of information. To realize such a situation, polarization of a laser pulse in a fibre as a probe qubit can be carried out instead of using the internal energy levels of the ion as the qubit. The laser pulse is coupled sequentially with the first and $N$-th ions in the crystal during its propagation in the fibre. This situation is depicted in Figure 4. To make the discussion more concrete, let us consider the creation and annihilation bosonic operators $\Psi_s^\dagger(\zeta)$ and $\Psi_s(\zeta)$ for one photon of the laser field with polarization $s = \pm$ in the fibre parametrized by a coordinate $\zeta$. The fibre is connected between the initial point ($\zeta = \zeta_i$) and the final point ($\zeta = \zeta_f$) via the first ion of the crystal at $\zeta = \zeta_1$ and the $N$-th ion of the crystal at $\zeta = \zeta_N$, where $\zeta_i < \zeta_1 < \zeta_N < \zeta_f$. The operators $\Psi_s^\dagger(\zeta)$ and $\Psi_s(\zeta)$ satisfy the following commutation relations:

$$
[\Psi_s(\zeta), \Psi_s^\dagger(\zeta')] = \delta_{ss'}\delta(\zeta - \zeta'),
$$

$$
[\Psi_s(\zeta), \Psi_s(\zeta')] = 0,
$$

$$
[\Psi_s^\dagger(\zeta), \Psi_s^\dagger(\zeta')] = 0.
$$

The vacuum state $|0\rangle$ of the laser field is defined by

$$
\Psi_s(\zeta)|0\rangle = 0.
$$

Let us assume that the initial state of the laser field is a pulse-wave coherent state with polarization $s = +$ given by $|f_i\rangle \propto \exp\left(\int f_i(\zeta)\Psi_s^\dagger(\zeta)d\zeta\right)|0\rangle$, where $f_i(\zeta)$ is a function with a support localized around $\zeta = \zeta_i$. Let us consider a free Hamiltonian of the fibre photon as

$$
H_\Psi = -\frac{ic}{2} \int_{-\infty}^{\infty} \left[\Psi(\zeta)^\dagger \partial_\zeta \Psi(\zeta) - \partial_\zeta \Psi(\zeta)^\dagger \Psi(\zeta)\right] d\zeta,
$$
where \( c \) is the light velocity and \( \Psi(\zeta) \) is given by

\[
\Psi(\zeta) = \begin{bmatrix} \Psi_+(\zeta) \\ \Psi_- (\zeta) \end{bmatrix}.
\]

The free evolution of the photon field is given by

\[
e^{itH_\Psi} \Psi^\dagger_\Psi(\zeta)e^{-itH_\Psi} = \Psi^\dagger_\Psi(\zeta - ct).
\]

In order to simulate the protocol discussed in section 3, this laser field couples with the first and the \( N \)-th ions by interactions given by

\[
H_M = \frac{c}{d} G_1 \int_{\zeta_1-d/2}^{\zeta_1+d/2} \Psi^\dagger(\zeta)\sigma_y \Psi(\zeta) d\zeta,
\]

\[
H_{LO} = -\frac{\theta c}{d} pN \int_{\zeta_N-d/2}^{\zeta_N+d/2} \Psi^\dagger(\zeta)\sigma_z \Psi(\zeta) d\zeta,
\]

where \( d \) is the length of the interaction region. The total Hamiltonian of the composite system is expressed as

\[
H_{tot} = H + H_M + H_{LO} + H_C,
\]

and it is independent of time. Hence, \( H_{tot} \) is conserved in time. The initial state of the composite system is given by \( |g\rangle_q \otimes |f\rangle_\Psi \). In this model, the evolution of the laser pulse induces effective switching of the interactions for the QET operations on the phonon system. The interactions in Eqs. \((39)\) and \((40)\) are active only when the pulses exist at their corresponding interaction regions. Based on the interaction in Eq. \((39)\), the information \( s \) about the phonon fluctuation is imprinted into the polarization of the laser pulse. The interaction in Eq. \((40)\) gives a controlled operation gate of the exit ion by the value \( s \) of \( \sigma_z \) of the laser pulse. The energy of the laser pulse changes when the pulse passes through the ion regions. The initial energy of the pulse is denoted by \( E_1 \). In principle, the photon energy \( E_1 \) is chosen to be independent of \( E_{out} \). In order to perform a sensitive experiment for the detection of the QET effect, it is better to consider that the order of \( E_1 \) is the same as that of \( E_{out} (= O(\gamma N \nu)) \). After the interaction with the first ion in Eq. \((39)\), the pulse energy is decreased to \( E_2 \) by the exciting phonons with energy \( E_{in} = E_1 - E_2 \) in the crystal. After the interaction with the \( N \)-th ion in Eq. \((40)\), the pulse energy is increased to \( E_3 \) by extracting the
energy from the ion as the QET effect. The amount of the extracted energy is $E_{\text{out}} = E_3 - E_2$. Experimental observation of this change in the pulse energy leads to the verification of the QET mechanism. Detailed analysis of this model will be reported elsewhere.

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Figure Captions

Figure 1: Schematic diagram of QET for the linear ion crystal. The trapped cold ions are denoted by circles. The POVM measurement defined by Eqs. (16) and (17) is performed on the ion at the left edge of the crystal to obtain the information $s$ about phonon fluctuation in the ground state $|g\rangle$. During the measurement, energy $E_{\text{in}}$ is infused into the crystal. The measurement result $s$ is transferred to the right edge of the crystal through a classical channel. The energy $E_{\text{out}}$ is extracted by performing $U_s$ in Eq. (24) on the ion at the right edge.

Figure 2: $\ln \gamma_N$ is plotted as a function of $N$.

Figure 3: $\zeta_N$ is plotted as a function of $N$.

Figure 4: Schematic diagram of the verification experiment of QET. The black line denotes a photonic fibre for a laser pulse that controls the switching of the interactions for the QET protocol. The fibre is parametrized by a coordinate $\zeta$ and connected between the initial point ($\zeta = \zeta_i$) and the final point ($\zeta = \zeta_f$) via the first ion of the crystal at $\zeta = \zeta_1$ and the $N$-th ion of the crystal at $\zeta = \zeta_N$. The initial energy of the laser pulse is denoted by $E_1$. After the interaction with the first ion in Eq. (39), the pulse energy is decreased to $E_2$ by the exciting phonons with energy $E_{\text{in}} = E_1 - E_2$ in the crystal. After the interaction with the $N$-th ion in Eq. (40), the pulse energy is increased to $E_3$ by extracting the energy from the ion as the QET effect. The amount of energy extracted is given by $E_{\text{out}} = E_3 - E_2$. 
Figure 1
$\ln \gamma_N$ vs $N$

Figure 2
Figure 3
Figure 4