Bound states of bosons and fermions in a mixed vector–scalar coupling with unequal shapes for the potentials

Luis B Castro and Antonio S de Castro

UNESP—Campus de Guaratinguetá, Departamento de Física e Química, 12516-410 Guaratinguetá SP, Brazil
E-mail: benito@feg.unesp.br and castro@pesquisador.cnpq.br

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Abstract
The Klein–Gordon and the Dirac equations with vector and scalar potentials are investigated under a more general condition, $V_v + V_s = \text{constant}$. These intrinsically relativistic and isospectral problems are solved in the case of squared hyperbolic potential functions and bound states for either particles or antiparticles are found. The eigenvalues and eigenfunctions are discussed in some detail and the effective Compton wavelength is revealed to be an important physical quantity. It is revealed that a boson is better localized than a fermion when they have the same mass and are subjected to the same potentials.

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There has been a continuous interest in solving the Klein–Gordon (KG) and the Dirac equations in the four-dimensional (4D) space-time, as well as in lower dimensions for a variety of potentials. It is well known from the quarkonium phenomenology that the best fit for meson spectroscopy is found for a convenient mixture of vector and scalar potentials put by hand in the equations (see, e.g. [1] and references therein). The same can be said about the treatment of the nuclear phenomena describing the influence of the nuclear medium on the nucleons [2]. The mixed vector–scalar potential has also been analyzed in $1+1$ dimensions. In this mixed 2D context, all the works have been devoted to the investigation of the solutions of the relativistic equations by assuming that the vector and scalar potential functions are proportional [3]. Recently the complete set of bound states of fermions and bosons with mixed vector–scalar potentials satisfying the constraint $V_v - V_s = \text{constant}$, in the case of squared trigonometric potential functions, has been addressed in [4]. In this last work it was concluded that changing the sign of coupling constant allows us to migrate from the particle sector to the antiparticle sector.

In the present work, the problem of relativistic particles is considered with a mixing of vector and scalar Lorentz structures with unequal potential functions. The mixing for this enlarged class of problems is chosen in such a way that the sum of the vector and scalar potential functions is a constant, a case which does not permit bound-state solutions in the nonrelativistic regime. Except for a possible isolated solution for the Dirac equation, the KG equation and the Dirac equation for the lower component of the Dirac spinor are both mapped into a Schrödinger-like equation, a phenomenon discovered recently [5]. Squared hyperbolic potential functions are chosen in such a way that the relativistic problem is mapped into a Sturm–Liouville problem with the effective symmetric modified Pöschl–Teller potential [6]. The process of solving the KG and the Dirac equations for the eigenenergies has been transmuted into solving an irrational algebraic equation. Then the whole relativistic spectrum is found, if the particle is massless or not. These solutions do not manifest in a nonrelativistic approach even though one can find $E \simeq mc^2$.

Although relativistic equations can give relativistic corrections to the nonrelativistic quantum mechanics, in the circumstance explored in this work they do not present solutions found in a nonrelativistic scheme. Undoubtedly such a circumstance appears to be a powerful tool to obtain a deeper insight into the nature of the relativistic equations and their solutions. Apart from the intrinsic interest as new solutions of fundamental equations in physics, the bound-state solutions of these systems are important in condensed matter mainly because of their potential applications ranging from ferroelectric domain walls in solids, magnetic chains and Josephson junctions [7].
In the presence of vector and scalar potentials the 1 + 1-dimensional time-independent KG equation for a particle of rest mass $m$ reads

$$-\hbar^2 c^2 \frac{d^2 \phi}{dx^2} + (mc^2 + V_x)\phi = (E - mc^2)\phi,$$

where $E$ is the energy of the particle, $c$ is the velocity of light and $\hbar$ is the Planck constant. The subscripts for the terms of potential denote their properties under a Lorentz transformation: $v$ for the time component of the 2-vector potential and $s$ for the scalar term. In the presence of time-independent vector and scalar potentials the 1 + 1-dimensional time-independent Dirac equation for a fermion of rest mass $m$ reads

$$[\sigma p + \beta (mc^2 + V_s) + V_x] \psi = E \psi,$$

where $p$ is the momentum operator, $\alpha$ and $\beta$ are Hermitian square matrices satisfying the relations $\alpha^2 = \beta^2 = 1, [\alpha, \beta] = 0$. From the last two relations it follows that both $\alpha$ and $\beta$ are traceless and have eigenvalues equal to ±1, so that one can conclude that $\alpha$ and $\beta$ are even-dimensional matrices. One can choose the $2 \times 2$ Pauli matrices satisfying the same algebra as $\alpha$ and $\beta$, resulting in a 2-component spinor $\psi$. We use $\alpha = \sigma_1$ and $\beta = \sigma_2$. Provided that the spinor is written in terms of the upper and the lower components, $\psi_+$ and $\psi_-$ respectively, the Dirac equation decomposes into:

$$i\hbar c \psi_\pm = (V_x - E \mp (mc^2 + V_s)) \psi_\pm,$$  

where the prime denotes differentiation with respect to $x$.

In the nonrelativistic approximation (potential energies are small as compared to $mc^2$ and $E \simeq mc^2$) equation (1) becomes

$$-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \phi = (E - mc^2)\phi,$$

so that $\phi$ obeys the Schrödinger equation with binding energy equal to $E - mc^2$ without distinguishing the contributions of vector and scalar potentials. In this approximation equation (3) becomes

$$\psi_+ = \frac{p}{2mc}\psi_-, \quad \psi_- = \frac{E - mc^2 - E_{eff} - m_{eff} c^2}{2m_{eff}} \psi_+,$$

and because of this $\psi_-$ obeys the same equations as $\phi$. Equation (5) shows that $\psi_-$ is of order $v/c \ll 1$ relative to $\psi_+$.

It is remarkable that the KG and the Dirac equations with a scalar potential, or a vector potential contaminated with some scalar coupling, is not invariant under the simultaneous changes $V \rightarrow V + \text{const.}$ and $E \rightarrow E + \text{const.}$, this is so because only the vector potential couples to the positive energies in the same way it couples to the negative ones, whereas the scalar potential couples to the mass of the particle. Therefore, if there is any scalar coupling the energy itself has physical significance and not just the energy difference. It is well known that a confining potential in the nonrelativistic approach is not confining in the relativistic approach when it is considered as a Lorentz vector. It is surprising that relativistic confining potentials may result in nonconfinement in the nonrelativistic approach. The case $V_x = -V_s + \text{constant}$ investigated in this work, for instance, presents bounded solutions in the relativistic approach, although it reduces to the problem of a particle subject to a uniform background potential in the nonrelativistic limit. This last phenomenon is a consequence of the fact that vector and scalar potentials couple differently in the relativistic equations whereas there is no such distinction among them in the Schrödinger equation. Regarding the structure of the wavefunctions under the simultaneous changes $V_x \rightarrow -V_x$ and $E \rightarrow -E$, from the charge-conjugation operation, one can see that if $\psi$ is a solution with energy $E$ for the potential $V_x$, then $\sigma_1 \psi^\ast$ is also a solution with energy $-E$ for the potential $-V_x$. Thus, one has $(\psi_{\pm})^\ast = \psi_{\mp}$ and that means that the upper and lower components of the Dirac spinor have their roles changed. As for the KG wavefunction, its nodal structure is trivially preserved in such a way that particle and antiparticle can be distinguished only by the eigenenergies.

Supposing that the vector and scalar potentials are constrained by the relation $V_x + V_s = V_0$, where $V_0$ is a constant, and defining

$$m_{eff} = \sqrt{m^2 + V_0^2}, \quad E_{eff} = \frac{E^2 - m_{eff}^2 c^4}{2m_{eff} c^2},$$

$$V_{eff} = -\frac{\hbar^2}{2m_{eff}} \frac{d^2 \phi}{dx^2} + m_{eff} c^2 \frac{d^2 \phi}{dx^2} = E - m_{eff} c^2 \sigma_1 \phi = E_{eff} \phi,$$

the KG equation can be written as

$$-\frac{\hbar^2}{2m_{eff}} \frac{d^2 \phi}{dx^2} + V_{eff} \phi = E_{eff} \phi.$$

On the other hand, for $E \neq m_{eff} c^2 \sigma_1 \phi = E_{eff} \phi$ the same Sturm–Liouville equation for $\phi$ is obeyed by $\psi_-$ whereas

$$\psi_+ = \frac{2i\hbar c}{E - E_{eff} - m_{eff} c^2 \sigma_1} \psi_-, \quad \psi_- = \frac{E - E_{eff} - m_{eff} c^2 \sigma_1}{2m_{eff} c^2} \psi_+,$$

Otherwise, for $E = m_{eff} c^2 \sigma_1 \phi = E_{eff} \phi$, it might be possible the existence of an isolated solution is given by

$$\psi_+ = \text{const.}, \quad \psi_- = \frac{2i\hbar c}{E - E_{eff} - m_{eff} c^2 \sigma_1} \int_{-\infty}^{x} dx (V_s + mc^2).$$

Of course, this solution does not exist if the domain is infinity because $\psi_-$ would not be square integrable. Note that apart from the possible isolated solution, $\psi_-$ satisfies the KG equation. An equally interesting result in the case of vanishing mass is that the spectrum just changes sign when $V_0$ does. For the eigenfunctions, $\phi$ and $\psi_-$ are invariant under the change of the sign of $V_0$ whereas $\psi_+$ changes sign.

For the specific case of the two-parameter potential functions $V_x = V_0 \sech^2 \alpha x$ and $V_s = V_0 \tanh^2 \alpha x$, the isolated solution of the Dirac equation is not normalizable and the effective potential of the Sturm–Liouville problem for both $\phi$ and $\psi_-$ can be expressed as

$$V_{eff} = -U_0 \sech^2 \alpha x,$$

$$U_0 = \left[ m_{eff}^2 \sigma_1 \phi = E_{eff} \phi \right] \frac{V_0}{m_{eff} c^2}.$$
Notice that $V_{\text{eff}}$ is invariant under the change $\alpha \to -\alpha$ so that the results can depend only on $|\alpha|$. Furthermore, the effective potential is an even function under $x \to -x$ in such a way that $\phi$ and $\psi_-$ can be taken to be even or odd. Note also that when $E > m_{\text{eff}}c^2$ for $V_0 > 0$, $E < m_{\text{eff}}c^2$ for $-mc^2 < V_0 < 0$ and $E < -m_{\text{eff}}c^2$ for $V_0 < -mc^2$ one has $V_0 < 0$. In this case the effective potential is a potential barrier and only scattering states are allowed with energies restricted to $E > m_{\text{eff}}c^2$ for $V_0 > 0$ and $E < -m_{\text{eff}}c^2$ for $V_0 < 0$. Contrariwise, $U_0 > 0$ and the effective potential is identified as the exactly solvable symmetric modified Pöschl–Teller potential [6, 8, 9].

In this last case there is also a continuum for $E < -m_{\text{eff}}c^2$ for $V_0 > 0$ and $E > m_{\text{eff}}c^2$ for $V_0 < 0$, and finite sets of discrete energies are allowed in the ranges $-m_{\text{eff}}c^2 < E < 2V_0 - m_{\text{eff}}c^2$ for $V_0 > 0$ and $|E| < m_{\text{eff}}c^2$ for $V_0 < -mc^2$.

Focusing attention on the bound-state solutions, one can see that the normalizable eigenfunctions are subjected to the boundary conditions $\phi = \psi_0 = 0$ as $|x| \to \infty$ in such a manner that the solution of our relativistic problem can be developed by taking advantage from the knowledge of the exact solution for the symmetric modified Pöschl–Teller potential. The corresponding effective eigenenergy is given by [6, 8, 9]

$$E_{\text{eff}} = -\frac{\hbar^2 \alpha^2 a_n^2}{2m_{\text{eff}}},$$

where

$$a_n = s - n, \quad n = 0, 1, 2, \ldots < s$$

and

$$s = \frac{1}{2} \left( -1 + \sqrt{1 + \frac{8m_{\text{eff}}U_0}{\hbar^2 \alpha^2}} \right).$$

Note that the number of allowed bound states increases with $|V_0|$ and decreases with $|\alpha|$, and that there is always at least one bound-state solution. Now, equations (13)–(15) lead to the quantization condition

$$\sqrt{\hbar^2 c^2 \alpha^2 + 8V_0 \left[ m_{\text{eff}}c^2 \text{sgn}(mc^2 + V_0) - E \right]} - 2\sqrt{m_{\text{eff}}^2c^4 - E^2} = \hbar c |\alpha| (2n + 1).$$

Once one makes sure that one meets the requirement dictated by (14), the solutions of (16) determine the eigenvalues of the relativistic problem. This equation can be solved easily with a symbolic algebra program by searching eigenenergies in the range $-m_{\text{eff}}c^2 < E < 2V_0 - m_{\text{eff}}c^2$ for $V_0 > 0$ and $|E| < m_{\text{eff}}c^2$ for $V_0 < -mc^2$, as foreseen by the preceding qualitative arguments. Of course, for $V_0 > 0$ one obtains $E \approx -mc^2$ when $V_0 \ll mc^2$ and $-V_0 < E < 0$ when $V_0 \gg mc^2$. On the other hand, for $V_0 < -mc^2$ one finds $E \approx 0$ when $V_0 \approx -mc^2$, and $-|V_0| < E < |V_0|$ when $|V_0| > mc^2$.

It happens that there is at most one solution of (16) for a given quantum number. Are these energies related to particle or antiparticle energy levels? To answer this question we plot the energy levels in terms of the parameters of the potential. Figures 1 and 2 show the behavior of the energies as a function of $V_0$ and $\alpha$, respectively. From figure 1 one sees that all the energy levels emerge from the negative-energy continuum so that it is plausible to identify them with antiparticle levels, although for a given $V_0$ some of the levels can have positive energies. Meanwhile, from figure 2 one sees that the energy levels tend to disappear one after another as $\alpha$ increases and just the ground-state energy level survives as $\alpha \to \infty$. The energy levels passing out of the picture as $\alpha$ increases (or $V_0$ decreases as in figure 1) sink into the negative continuum but this does not menace the single-particle interpretation of either KG or Dirac equations since one has antiparticle levels plunging into the antiparticle continuum. It is also noticeable from both of these figures that for a given set of potential parameters one finds that the lowest quantum numbers correspond to the highest eigenenergies, as it should be for antiparticle energy levels.

For $V_0 < -mc^2$ the spectrum presents a similar behavior but the energy levels emerge from the upper continuum and are to be identified with particle levels. If we had plotted the spectra for a massless particle, we would encounter, up to the sign of $E$, identical spectra for both signs of $V_0$. At any circumstance, the spectrum contains either particle-energy levels or antiparticle-energy levels. This conclusion confirms what has already been analyzed in [4]: the spectrum contains either particle-energy levels or antiparticle-energy levels depending on the sign of the coupling constant.
The KG eigenfunction, as well as the lower component of the Dirac spinor can be given by [9]

$$
\phi = \psi_- = N 2^{\alpha} \Gamma \left( a_n + 1 \right) \frac{|\alpha| d_n}{\pi} \Gamma(n + 1) \Gamma(n + 1 + 2a_n)
\times \left( 1 - z^2 \right)^{n\alpha+1/2} C_n^{(\alpha+1/2)}(z),
$$

(17)

where \( z = \tanh \alpha x \) and \( C_n^{(\alpha)}(z) \) is the Gegenbauer (ultraspherical) polynomial of degree \( n \). Since \( C_n^{(\alpha)}(-z) = (-1)^n C_n^{(\alpha)}(z) \) and \( C_n^{(\alpha)}(z) \) has \( n \) distinct zeros (see e.g. [10]), it becomes clear that \( \psi_+ \) and \( \psi_- \) have definite and opposite parities. The constant \( N \) is the unit in the KG problem and it is chosen such that \( \int_{-\infty}^{\infty} dx \left( |\psi_+|^2 + |\psi_-|^2 \right) = 1 \) in the Dirac problem. Figure 3 illustrates the behavior of the upper and lower components of the Dirac spinor \( |\psi_+|^2 \) and \( |\psi_-|^2 \), and the position probability densities \( |\psi|^2 = |\psi_+|^2 + |\psi_-|^2 \) and \( |\phi|^2 \) for \( n = 0 \). The relative normalization constant was calculated numerically. Comparison of \( |\psi_+|^2 \) and \( |\psi_-|^2 \) shows that \( \psi_+ \) is suppressed relative to \( \psi_- \) This result is expected since we have here an antiparticle eigenstate. Nevertheless, the same behavior shows its face for the particle eigenstates (for \( V_0 < -mc^2 \)). One might say that this kind of effect is because we are dealing with a quintessential relativistic potential. In addition, comparison of \( |\phi|^2 \) and \( |\psi|^2 \) shows that a KG particle tends to be better localized than a Dirac particle. As a matter of fact, a numerical calculation of the uncertainty in the position (with \( m = c = \hbar = 1 \) and \( V_0 = a = 5 \)) furnishes 0.160 and 0.179, respectively. Here we have purposely shown an odd fact. It seems that the uncertainty principle dies away provided such a principle implies that it is impossible to localize a particle into a region of space less than half of its Compton wavelength (see e.g. [11]). This apparent contradiction can be remedied by recurring to the concept of effective Compton wavelength, as has been done previously in connection with pseudoscalar couplings in the Dirac equation [12]. Indeed, equation (6) suggests that we can define the effective Compton wavelength as \( \lambda_{\text{eff}} = h/(m_{\text{eff}} c) \) so that the minimum uncertainty consonant with the uncertainty principle is given by \( \lambda_{\text{eff}}/2 \) whereas the maximum uncertainty in the momentum is given by \( m_{\text{eff}} c \). The appropriateness of the concept of effective Compton wavelength has been checked for a large range of the potential parameters.

In summary, the methodology for finding solutions of the KG and the Dirac equations for the enlarged class of mixed vector–scalar potentials satisfying the constraint \( V_r + V_l = V_0 \) have been put forward. Although the KG and the Dirac equations exhibit the very same spectrum their eigenfunctions make all the difference. With the two-parameter potential functions \( V_r = V_0 \text{sech}^2 \alpha x \) and \( V_l = V_0 \tanh^2 \alpha x \), the KG and the Dirac equations have been mapped into a Schrödinger-like equation with the symmetric modified Pöschl–Teller potential and we have shown that a KG particle tends to be better localized than a Dirac particle. In both cases, the spectrum consists of either particles or antiparticles, depending on the sign of \( V_0 \). An apparent contradiction with the uncertainty principle was cured by introducing the effective Compton wavelength.

References

[1] Lucha W et al 1991 Phys. Rep. 200 127
[2] Serot B D and Walecka J D 1986 Advances in Nuclear Physics vol 16 ed Negele J W and Vogt E (New York: Plenum)
[3] Ginocchio J N 1997 Phys. Rev. Lett. 78 436
[4] Ginocchio J N and Levitian A 1998 Phys. Lett. B 425 1
[5] Ginocchio J N 1999 Phys. Rep. 315 231
[6] Alberto P et al 2002 Phys. Rev. C 65 034307
[7] Chen T-S et al 2003 Chin. Phys. Lett. 20 358
[8] Mao G 2003 Phys. Rev. C 67 044318
[9] Lisboa R et al 2004 Phys. Rev. C 69 024319
[10] Gumbs G and Kiang D 1986 Am. J. Phys. 54 462
[11] Dominguez-Adame F 1990 Am. J. Phys. 58 886
[12] de Castro A S 2002 Phys. Lett. A 305 100
[13] Domínguez-Adame F 1990 Am. J. Phys. 58 886
[14] de Castro A S 2005 Phys. Lett. A 338 81
[15] de Castro A S 2005 Phys. Lett. A 346 71
[16] de Castro A S 2005 Ann. Phys. (N.Y.) 316 414
[17] de Castro A S 2006 Int. J. Mod. Phys. A 21 5141
[18] de Castro A S 2007 Int. J. Mod. Phys. A 22 2609
[19] de Castro A S et al 2006 Phys. Rev. C 73 054309
[20] Castro L B and de Castro A S 2007 Phys. Scr. 75 170
[21] Alberto P, de Castro A S and Malheiro M 2007 Phys. Rev. C 75 047303
[22] Pöschl G and Teller E 1933 Z. Phys. 83 143
[23] Braun O M and Kivshar Y S 2004 The Frenkel-Kontorova Model: Concepts, Methods and Applications (Berlin: Springer)
[24] Landau L D and Lifshitz E M 1958 Quantum Mechanics (New York: Pergamon)
[25] Flügge S 1999 Practical Quantum Mechanics (Berlin: Springer)
[26] Nieto M M and Simmons Jr L M 1978 Phys. Rev. A 17 1273
[27] Abramowitz M and Stegun I A 1965 Handbook of Mathematical Functions (Toronto: Dover)
[28] Greiner W 1990 Relativistic Quantum Mechanics: Wave Equations (Berlin: Springer)
[29] Strange P 1998 Relativistic Quantum Mechanics (Cambridge: Cambridge University Press)
[30] de Castro A S and Pereira W G 2003 Phys. Lett. A 308 131
[31] de Castro A S 2005 Ann. Phys. (N.Y.) 320 56
[32] de Castro A S and Hott M 2006 Phys. Lett. A 351 379
[33] de Castro A S 2006 Int. J. Mod. Phys. A 21 2321