Clapeyron equation and phase equilibrium properties in higher dimensional charged topological dilaton AdS black holes with a nonlinear source

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Abstract

Using Maxwell’s equal area law, we discuss the phase transition of higher dimensional charged topological dilaton AdS black holes with a nonlinear source. The coexisting region of the two phases is found and we depict the coexistence region in $P - v$ diagrams. The two-phase equilibrium curves in $P - T$ diagrams are plotted, and we take the first order approximation of volume $v$ in the calculation. To better compare with a general thermodynamic system, the Clapeyron equation is derived for higher dimensional charged topological black hole with a nonlinear source. The latent heat of isothermal phase transition is investigated. We also study the effect of the parameters of the black hole on the region of two-phases coexistence. The results show that the black hole may go through a small-large phase transition similar to those of usual non-gravity thermodynamic systems.

Keywords: two-phase equilibrium; Clapeyron equation; charged topological dilaton black hole; non-linear source

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I. INTRODUCTION

Black holes have been used as the laboratory of many kinds of theories, specially the thermodynamics of black holes plays an important roles. It has been found that a black hole possesses not only standard thermodynamic quantities but abundant phase structures

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and critical phenomena, such as those involved in the Hawking-Page phase transition, similar
to the ones of a general thermodynamic system. Even more interesting is that the studies on
the charged black holes show they may have an analogous phase transition with that of van
der Walls-Maxwell’s liquid-gas. In recent years, the cosmological constant in $n$-dimensional
AdS and dS spacetime has been regarded as pressure of black hole thermodynamic system with
\[ P = \frac{n(n - 1)}{16\pi l^2}, \quad \Lambda = -\frac{n(n - 1)}{2l^2} \] (1.1)
and the corresponding conjugate quantity, thermodynamic volume
\[ V = \left(\frac{\partial M}{\partial P}\right)_{S,Q,J_k}. \] (1.2)
The $(P \sim V)$ critical behaviors in AdS and dS black holes have been extensively studied\[1-42\]. A completely simulated gas-liquid system has been put forward.

Using Ehrenfest scheme, Ref.\[20-24\] studied the critical phenomena in a series of black
holes in AdS spacetime, and proved the phase transition at critical point is the second
order one, which has also been confirmed in Ref.\[43-47\] by studying thermodynamics and
state space geometry of black holes. Although some encouraging results about black hole
thermodynamic properties in AdS and dS spacetimes have been achieved and the problems
about phase transition of black holes have been extensively discussed, an unified recognition
about the phase transition of black hole has not been put forward. It is significant to further
explore phase equilibrium and phase structure in black holes, which can help to recognize the
evolution of black holes. We also expect to provide some relevant information for exploring
quantum gravity properties by studying the phase transition of charged topological dilaton
AdS black holes.

The motivation for studying higher dimensional black holes comes from developments in
string/M-theory, which is believed to be the most consistent approach to quantum theory of
gravity. It was argued that black holes may play a crucial role in the analysis of dynamics in
higher dimensions as well as in the compactification mechanisms. In particular, to test novel
predictions of string/M-theory black holes may serve as good theoretical laboratories. It has
been thought that the statistical-mechanical calculation of the Bekenstein-Hawking entropy
for a class of supersymmetric black holes in five dimensions is one of the remarkable results
in string theory \[48, 49\]. Another motivation on studying higher dimensional black holes
originates from the braneworld scenarios, as a new fundamental scale of quantum gravity.
An interesting consequence of these models is the possibility of mini black hole production at future colliders.

The theory of nonlinear electrodynamics was first introduced in 1930’s by Born and Infeld to obtain a classical theory of charged particles with finite self-energy. Born-Infeld(BI) theory has received renewed attentions since it turns out to play an important role in string theory. The BI action, including a dilaton and an axion field, appears in the coupling of an open superstring and an Abelian gauge field theory. Here we would like to consider another type of nonlinear electrodynamics, namely, the exponential form of the nonlinear electrodynamics in the setup of dilaton gravity. These serve as our main motivation to explore the effects of dilaton field on the properties of higher dimensional charged black holes with a nonlinear source.

The isotherms in $P \sim v$ diagrams of charged topological dilaton AdS black hole in Ref.[10] show there exists thermodynamic unstable region with $\partial P / \partial v > 0$ when temperature is below critical temperature and the negative pressure emerges when temperature is below a certain value. This situation also exists in van der Waals-Maxwell gas-liquid system, which has been resolved by Maxwell equal area law. In this paper, using the Maxwell equal area law, we establish a phase transition process in charged topological dilaton AdS black holes, where the issues about unstable states and negative pressure are resolved. By studying the phase transition process, we acquire the two-phase equilibrium properties including the $P-T$ phase diagram, Clapeyron equation and latent heat of phase change. The results show the phase transition below critical temperature is of the first order but phase transition at critical point belongs to the continuous one.

Outline of this paper is as follows: The higher dimensional charged topological dilaton AdS black hole as a thermodynamic system is briefly introduced in section 2. In section 3, by Maxwell equal area law the phase transition processes at certain temperatures are obtained and the boundary of two phase equilibrium region are depicted in $P-v$ diagram for a higher dimensional charged topological dilaton AdS black hole. Then some parameters of the black hole are analyzed to find the relevance with the two-phase equilibrium. In section 4, the $P-T$ phase diagrams are plotted and the Clapeyron equation and latent heat of the phase change are derived. We make some discussions and conclusions in section 5. we use the units $G_d = \hbar = k_B = c = 1$ in this paper)
II. HIGHER-DIMENSIONAL CHARGED DILATON BLACK HOLES

We consider the $n$-dimensional ($n \geq 4$) action in which gravity is coupled to dilaton and nonlinear electrodynamic field. The Einstein-Maxwell-Dilaton (EMD) action in $(n + 1)$-dimensional ($n \geq 3$) spacetime is

$$S = \frac{1}{16\pi} \int d^n x \sqrt{-g} \left( R - \frac{4}{n - 2} (\nabla \Phi)^2 - V(\Phi) - L(F, \Phi) \right),$$

(2.1)

where the dilaton potential contains two Liouville terms:

$$V(\Phi) = 2\Lambda_0 e^{2\varsigma_0 \Phi} + 2\Lambda e^{2\varsigma \Phi},$$

(2.2)

and

$$L(F, \Phi) = 4\beta^2 e^{4\alpha \Phi/(n-2)} \left[ \exp \left( -\frac{e^{-8\alpha \Phi/(n-2)} F^2}{4\beta^2} - 1 \right) \right]$$

(2.3)

where $R$ is the Ricci scalar curvature, $\Phi$ is the dilaton field and $V(\Phi)$ is a potential for $\Phi$, $\Lambda_0$, $\Lambda$, $\varsigma_0$ and $\varsigma$ are constants, $\alpha$ is a constant determining the strength of coupling of the scalar and electromagnetic field. This kind of potential was previously investigated in the context of BI-dilaton (BId) black holes [15, 16] as well as EMD gravity [22–27].

The metric of such a spacetime can be written

$$ds^2 = -f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 R^2(r) d\Omega_{n-2}^2,$$

(2.4)

where

$$f(r) = -\frac{(n-3)(\alpha^2 + 1)^2 b^{-\gamma} r^\gamma}{(\alpha^2 - 1)(\alpha^2 + n - 3)} - \frac{m}{r^{n-3-(n-2)\gamma/2}} + \frac{2(2\Lambda + 2\beta^2)(\alpha^2 + 1)^2 b^{-\gamma}}{(n-2)(\alpha^2 - n + 1)} r^{2-\gamma} \times \left( \frac{1 - \alpha^2}{2n - 4} \right)^{2-2n+\gamma \over (\gamma-2)(2n-4)} \times \left\{ -(n-2)^2(\gamma - 2)^2 \left[ \Gamma \left( \frac{\alpha^2 + 3n - 7}{2n - 4}, \frac{1 - \alpha}{2n - 4} L_W(\eta) \right) - \Gamma \left( \frac{\alpha^2 + 3n - 7}{2n - 4} \right) \right] + (\gamma - 1)^2 \left[ \Gamma \left( \frac{\alpha^2 - n + 1}{2n - 4}, \frac{1 - \alpha^2}{2n - 4} L_W(\eta) \right) - \Gamma \left( \frac{\alpha^2 - n + 1}{2n - 4} \right) \right] \right\},$$

(2.5)

and

$$\Phi(r) = \frac{(n-2)\alpha}{2(\alpha^2 + 1)} \ln \left( \frac{b}{r} \right).$$

(2.6)
with $b$ is an arbitrary constant, $\gamma = \alpha^2/(\alpha^2 + 1)$, and

$$
\eta \equiv \frac{q^2 b^{(2-n)\gamma}}{\beta^2 r^{(n-2)(2-\gamma)}}
$$

(2.7)

$m$ appears as an integration constant and is related to the mass of the black hole, $q$ is an integration constant which is related to the electric charge of the black hole. The electric charge is

$$
Q = \frac{q \omega_{n-2}}{4\pi},
$$

(2.8)

where $\omega_{n-2}$ represents the volume of constant curvature hypersurface described by $d\Omega_{n-2}^2$. $L_W(x)$ is the Lambert function which satisfies the identity

$$
L_W(x)e^{L_W(x)} = x,
$$

(2.9)

and has the following series expansion

$$
L_W(x) = x - x^2 + \frac{3}{2} x^3 - \frac{8}{3} x^4 + \cdots.
$$

(2.10)

$\Gamma(a, z)$ and $\Gamma(a)$ are gamma functions and they are related to each other via

$$
\Gamma(a, z) = \Gamma(a) - \frac{z^a}{a} F(a, 1 + a, -z),
$$

(2.11)

where $F(a, a, z)$ is hypergeometric function.

Using the fact that $L_W(x)$ has a convergent series expansion for $|x| < 1$ as given in (2.10), we can expand (2.5) for large $\beta$. The result is

$$
f(r) = -\frac{(n - 3)(\alpha^2 + 1)^2 b^{-\gamma} r^{-\gamma}}{(\alpha^2 - 1)(\alpha^2 + n - 3)} - \frac{m}{r^{n-3-(n-2)\gamma/2}} + \frac{2\Lambda(\alpha^2 + 1)^2 b^\gamma}{(n - 2)(\alpha^2 - n + 1)} r^{2-\gamma} + \frac{2q^2(\alpha^2 + 1)^2 b^{-(n-3)\gamma}}{(n-2)(\alpha^2 + n - 3)r^{-(n-3)\gamma+2n+6}} - \frac{q^4(\alpha^2 + 1)^2 b^{-(2n-5)\gamma}}{2\beta^2(n-2)(\alpha^2 + 3n - 7)r^{(2n-5)(2-\gamma)}} + o\left(\frac{1}{\beta^4}\right).
$$

(2.12)

The location of black horizon satisfies the equation $f(r_+) = 0$, from which we obtain

$$
m = -\frac{(n - 3)(\alpha^2 + 1)^2 b^{-\gamma}}{(\alpha^2 - 1)(\alpha^2 + n - 3)} r^{n-3-(n-4)\gamma/2} + \frac{2\Lambda(\alpha^2 + 1)^2 b^\gamma}{(n - 2)(\alpha^2 - n + 1)} r^{n-1-n\gamma/2} + \frac{2q^2(\alpha^2 + 1)^2 b^{-(n-3)\gamma}}{(n-2)(\alpha^2 + n - 3)r^{-(n-4)\gamma/2+n-3}} - \frac{q^4(\alpha^2 + 1)^2 b^{-(2n-5)\gamma}}{2\beta^2(n-2)(\alpha^2 + 3n - 7)r^{3n-7+(8-3n)\gamma/2}} + o\left(\frac{1}{\beta^4}\right).
$$

(2.13)

The cosmological constant is related to spacetime dimension $n$ by

$$
\Lambda = -\frac{n(n - 1)}{2l^2},
$$

(2.14)
where $l$ denotes the AdS length scale. In (2.13), $m$ appears as an integration constant and is related to the (Arnowitt-Deser-Misner)ADM mass of the black hole. According to the definition of mass due to Abbott and Deser [54, 55], the ADM mass of the solution (2.12) is

$$M(S, Q, P) = \frac{b^{(n-2)\gamma/2}(n-2)\omega_{n-2}}{16\pi(\alpha^2 + 1)}m$$

$$= -\frac{(n-3)(n-2)\omega_{n-2}(\alpha^2 + 1)b^{(n-4)\gamma/2}}{16\pi(\alpha^2 - 1)(\alpha^2 + n - 3)}r^{n-3-(n-4)\gamma/2} + \frac{2\Lambda(\alpha^2 + 1)\omega_{n-2}b^{n\gamma/2}}{16\pi(\alpha^2 - n + 1)}r^{n-1-n\gamma/2}$$

$$+ \frac{2q^2(\alpha^2 + 1)\omega_{n-2}b^{-(n-4)\gamma/2}}{16\pi(\alpha^2 + n - 3)r^{-(n-4)\gamma/2+n-3} - \frac{q^4(\alpha^2 + 1)\omega_{n-2}b^{-(3n-8)\gamma/2}}{32\pi\beta^2(\alpha^2 + 3n - 7)r^{3n-7+(8-3n)\gamma/2} + o \left( \frac{1}{\beta^4} \right)}.$$  (2.15)

The entropy of the EMD black hole still obeys the so-called area law of the entropy which states that the entropy of the black hole is a quarter of the event horizon area [26]. This universal law applies to almost all kinds of black holes, including dilaton black holes, in Einstein gravity [27]. It is easy to show

$$S = \frac{b^{(n-2)\gamma/2}\omega_{n-2}r_+^{(n-2)(1-\gamma/2)}}{4}.$$  (2.16)

Recent development on the thermodynamics of black holes in extended phase space shows that the cosmological constant can be interpreted as the thermodynamic pressure and treated as a thermodynamic variable in its own right,

$$P = -\frac{1}{8\pi}\Lambda.$$  (2.17)

One may then regard the parameters $S$, $Q$ and $P$ as a complete set of parameters for the mass $M(S, Q, P)$ and define the parameters conjugate to $S$, $Q$ and $P$. These quantities are the temperature, the electric potential and volume

$$T = \left( \frac{\partial M}{\partial S} \right)_{Q,P}, \quad U = \left( \frac{\partial M}{\partial Q} \right)_{S,P}, \quad V = \left( \frac{\partial M}{\partial P} \right)_{Q,S}.$$  (2.18)

where the temperature of Hawking radiation

$$T = -\frac{(n-3)(1 + \alpha^2)b^{-\gamma}}{4\pi(\alpha^2 - 1)}r_+^{\gamma-1} - \frac{\Lambda(1 + \alpha^2)b^{\gamma}}{2\pi(n-2)}r_+^{1-\gamma}$$

$$- \frac{q^2(1 + \alpha^2)b^{-(n-3)\gamma}}{2\pi(n-2)}r_+^{(n\gamma+5-3\gamma-2n)} + \frac{q^4(1 + \alpha^2)b^{-(2n-5)\gamma}}{8\pi(n-2)\beta^2}r_+^{9+2n\gamma-4n-5\gamma} + o \left( \frac{1}{\beta^4} \right).$$  (2.19)
The electric potential

\[ U = \frac{4\pi Q(\alpha^2 + 1)b^{(4-n)\gamma/2}}{(\alpha^2 + n - 3)\omega_{n-2}r_+^{n-3+3\gamma-n\gamma/2}} - \frac{(4\pi)^3Q^3(\alpha^2 + 1)b^{-(3n-8)\gamma/2}}{2\beta^2\omega_{n-2}^3(\alpha^2 + 3n - 7)r^{3n-7+(8-3n)\gamma/2}} + O\left(\frac{1}{\beta^4}\right), \]  

(2.20)

and the volume

\[ V = -\frac{(\alpha^2 + 1)b^{\gamma n/2}\omega_{n-2}r_+^{n-1-\gamma n/2}}{(\alpha^2 - n + 1)}. \]  

(2.21)

Note that in the limiting case where \( \beta \to \infty \), expression (2.19) reduces to the temperature of higher dimensional EMD black holes [24]. Thus, the thermodynamics quantities satisfy the first law of thermodynamics

\[ dM = TdS + UdQ + VdP. \]  

(2.22)

For the electric potential satisfies the superposition principle, we can rewrite the equation (2.20),

\[ U = U_1 + U_2, \]  

(2.23)

where

\[ U_1 = \frac{4\pi Q(\alpha^2 + 1)b^{(4-n)\gamma/2}}{(\alpha^2 + n - 3)\omega_{n-2}r_+^{n-3+3\gamma-n\gamma/2}}, \]

\[ U_2 = -\frac{(4\pi)^3Q^3(\alpha^2 + 1)b^{-(3n-8)\gamma/2}}{2\beta^2\omega_{n-2}^3(\alpha^2 + 3n - 7)r^{3n-7+(8-3n)\gamma/2}} + O\left(\frac{1}{\beta^4}\right). \]  

(2.24)

So the Smarr formula:

\[ M = \frac{(n-2)}{\alpha^2 + n - 3}TS + U_1Q + \frac{(\alpha^2 + 2n - 5)}{2(\alpha^2 + n - 3)}U_2Q + \frac{2(\alpha^2 - 1)}{\alpha^2 + n - 3}VP. \]  

(2.25)

we can rewrite the eq. (2.19),

\[ P = \frac{T}{v} + \frac{k(n-2)(\alpha^2 + 1)^2}{\pi(n-1)(\alpha^2 - 1)v^2} + \frac{Q^2b^{2(1-n)\gamma/2}2\pi}{\omega_{n-1}^2} \left( \frac{v(n-1)}{4(\alpha^2 + 1)b^{2\gamma}} \right)^{(2-n)(2-\gamma)}/(1-2\gamma) \]

\[ -\frac{Q^4(4\pi)^3(1 + \alpha^2)\omega_{n-2}^2\beta^2}{(n-2)^2}r^{9+2n\gamma-4n-5\gamma} \]

\[ = \frac{T}{v} - \frac{A}{v^2} + \frac{BQ}{v} - \frac{CQ}{v^2}, \]  

(2.26)

with the specific volume [10],

\[ v = \frac{4(\alpha^2 + 1)b^{\gamma}}{(n-2)}r^{1-\gamma}, \]  

(2.27)
and
\[ d = \frac{(n - 2)(2 - \gamma)}{1 - \gamma}, \quad A = \frac{(n - 3)(\alpha^2 + 1)^2}{\pi(n - 2)(1 - \alpha^2)}, \]
\[ B = \frac{b^{2(n-1)2\pi}}{\omega_{n-1}^2} \left( \frac{4(\alpha^2 + 1)b^{2\gamma}}{(n - 2)} \right)^{(n-2)(2-\gamma)/(1-\gamma)}, \]
\[ C = \frac{b^{2(n-1)3}}{8\omega_{n-1}^4 b^2} \left( \frac{4(\alpha^2 + 1)b^{2\gamma}}{(n - 2)} \right)^{2(n-2)(2-\gamma)/(1-\gamma)} \]
(2.28)

In Fig.1 we plot the isotherms in \( P - v \) diagrams in terms of state equation Eq. (2.26) at different dimension \( n \), charge \( Q \), and parameters \( b \) and \( \alpha \). One can see from Fig.1 that there are thermodynamic unstable segments with \( \partial P/\partial v > 0 \) on the isotherms when temperature \( T < T_c \), where \( T_c \) is critical temperature. When the temperature \( T = T_0 \), there is a point of intersection between the isotherms and the horizontal \( v \) axis. And the negative pressure emerges when temperature is below certain value \( T_0 \). \( T_0 \) and the corresponding specific volume \( v_0 \) can be derived

\[ A v_0^{2d-1} = BQ^2(d - 1)v_0^{d} - CQ^4(2d - 1), \quad T_0 = \frac{A}{v_0} - \frac{BQ^2}{v_0^{2d-1}} + \frac{CQ^4}{v_0^{2d-1}}. \]  (2.29)

(a) \( n = 4, b = 0.8, Q = 1.5, \alpha = 0 \)  (b) \( n = 5, b = 1, Q = 1.2, \alpha = \frac{1}{\sqrt{2}} \)  (c) \( n = 6, b = 1.2, Q = 1, \alpha = \frac{1}{\sqrt{3}} \)

FIG. 1: Isotherms in \( P - v \) diagrams of higher-dimensional charged topological dilaton black holes in \( n \) dimensional AdS spacetime

III. TWO-PHASE EQUILIBRIUM AND MAXWELL EQUAL AREA LAW

The state equation of the charged topological black hole is exhibited by the isotherms in Fig.1, in which the thermodynamic unstable states with \( \partial P/\partial v > 0 \) will lead to the system automatically expansion or contraction unrestrictedly and the negative pressure situation
have no physical meaning. The cases occur also in van der Waals equation but they have been resolved by Maxwell equal area law.

We extend the Maxwell equal area law to \(n\)-dimensional charged topological dilaton AdS black hole to establish an phase transition process of the black hole thermodynamic system. On the isotherm with temperature \(T_0\) in \(P-v\) diagram, the two points \((P_0, v_1)\) and \((P_0, v_2)\) meet the Maxwell equal area law,

\[
P_0(v_2 - v_1) = \int_{v_1}^{v_2} P \, dv, \tag{3.1}
\]

which results in

\[
P_0(v_2 - v_1) = T_0 \ln \left( \frac{v_2}{v_1} \right) - A \left( \frac{1}{v_1} - \frac{1}{v_2} \right) + \frac{BQ^2}{d-1} \left( \frac{1}{v_1^{d-1}} - \frac{1}{v_2^{d-1}} \right) - CQ^4 \left( \frac{1}{v_1^{2d-1}} - \frac{1}{v_2^{2d-1}} \right), \tag{3.2}
\]

where the two points \((P_0, v_1)\) and \((P_0, v_2)\) are seen as endpoints of isothermal phase transition. Considering \(P_0 = T_0 \frac{v_1}{v_2} - A \left( \frac{1}{v_1} - \frac{1}{v_2} \right) + BQ^2 \left( \frac{1}{v_1^d} - \frac{1}{v_2^d} \right) - CQ^4 \left( \frac{1}{v_1^{2d}} - \frac{1}{v_2^{2d}} \right), \tag{3.3}\)

From the eq.(3.3), we can get

\[
0 = T_0 \left( \frac{1}{v_1} - \frac{1}{v_2} \right) - A \left( \frac{1}{v_1^2} - \frac{1}{v_2^2} \right) + BQ^2 \left( \frac{1}{v_1^d} - \frac{1}{v_2^d} \right) - CQ^4 \left( \frac{1}{v_1^{2d}} - \frac{1}{v_2^{2d}} \right), \tag{3.4}
\]

\[
2P_0 = T_0 \left( \frac{1}{v_1} + \frac{1}{v_2} \right) - A \left( \frac{1}{v_1^2} + \frac{1}{v_2^2} \right) + BQ^2 \left( \frac{1}{v_1^d} + \frac{1}{v_2^d} \right) - CQ^4 \left( \frac{1}{v_1^{2d}} + \frac{1}{v_2^{2d}} \right), \tag{3.5}
\]

Using the eqs.(3.2),(3.3) and (3.4) we can get

\[
Av_2^{2d-2}x^{2d-2} \left( 2(1-x) + (1+x) \ln x \right) = BQ^2v_2^{d^2}x^d \left( \frac{d(1-x^{d-1})}{(d-1)} + \frac{(1-x^d) \ln x}{(1-x)} \right) - CQ^4 \left( \frac{2d(1-x^{2d-1})}{(2d-1)} + \frac{(1-x^{2d}) \ln x}{(1-x)} \right), \tag{3.6}
\]

and \(x = v_1/v_2\) (\(0 < x < 1\)). When \(x\) is given, we can obtain the \(v_2\) and \(v_1\) correspond to the \(x\) from (3.6).

For \(C\) is little, taking the zero order approximation, we obtain

\[
v_2^{d-2} = \frac{B}{A} \frac{d(1-x^{d-1})(1-x) + (d-1)(1-x^d) \ln x}{x^{d-2}(d-1)(1-x)(2(1-x) + (1+x) \ln x)} = f^{d-2}(x), \tag{3.7}
\]
we can obtain the first order approximation solution \( v_{2,1}^{d-2} \) by substituting the \( v_{2,0}^{d-2} \) to the eq. (3.6). Similarly, we can obtain the second order approximation solution \( v_{2,2}^{d-2} \) by substituting the \( v_{2,2}^{d-2} \) to the eq. (3.6). And so on, we can obtain the arbitrary order approximation solution. For convenience, we take

\[
v_2 = F(x).
\]

In the first order approximation, we have

\[
v_{2,1}^{2d-2} = \left( \frac{d(1 - x^{d-1})}{(d - 1)} + \frac{(1 - x^d) \ln x}{(1 - x)} \right) \frac{BQ^2 f_d(x)}{Ax^{d-2}(2(1 - x) + (1 + x) \ln x)}
\]

\[
- \frac{CQ^4}{Ax^{2d-2}(2(1 - x) + (1 + x) \ln x)} \left( \frac{2d(1 - x^{2d-1})}{(2d - 1)} + \frac{(1 - x^{2d}) \ln x}{(1 - x)} \right) \approx F^{2d-2}(x).
\]

When \( x \to 1 \), the corresponding state is critical point state. From (3.7) and (3.9)

\[
v_c^{d-2} = v_c^d \lim_{x \to 1} \frac{BQ^2}{Ax^{d-2}(2(1 - x) + (1 + x) \ln x)} \left( \frac{d(1 - x^{d-1})}{(d - 1)} + \frac{(1 - x^d) \ln x}{(1 - x)} \right)
\]

\[- \lim_{x \to 1} \frac{CQ^4}{Ax^{2d-2}(2(1 - x) + (1 + x) \ln x)} \left( \frac{2d(1 - x^{2d-1})}{(2d - 1)} + \frac{(1 - x^{2d}) \ln x}{(1 - x)} \right).
\]

From (3.4),

\[
Tv_2^{2d-1}x^{2d-1} = Av_2^{2d-2}x^{2d-2}(1 + x) - BQ^2 \frac{v_c^d x^d(1 - x^d)}{1 - x} + CQ^4 \frac{(1 - x^{2d})}{1 - x},
\]

Substituting (3.8) into (3.11), we can obtain

\[
\chi T_c x^{2d-1} F^{2d-1}(x) = Ax^{2d-2}F^{2d-2}(x)(1 + x) - BQ^2 \frac{x^d F_d(x)(1 - x^d)}{1 - x} + CQ^4 \frac{(1 - x^{2d})}{1 - x},
\]

with \( T = \chi T_c \). Because we take account of the case that the temperature \( T \) below the critical temperature \( T_c \), the value of \( \chi = \frac{T}{T_c} \) is form 0 to 1. When \( x \to 1 \) and \( \chi \to 1 \), the corresponding state is critical state. For a fixed \( \chi \), i.e. a fixed \( T_0 \), we can get a certain \( x \) from Eq. (3.12), and then according to Eqs. (3.3) and (3.6), the \( v_2 \) and \( P_0 \) are solved.

To analyze the effect of parameters \( \alpha \) and \( b \) on the phase transition processes, we take \( \chi = 0.9, 0.7, 0.5, 0.3, 0.1 \), and calculate the quantities \( x, v_2, P_0 \) as \( n = 4, 5, 6 \) and \( Q = 1, 1.2, 1.5 \) when \( \alpha = 1/\sqrt{3}, b = 1.2 \) respectively. The results are shown in Table 1.

From Table 1, we can see that \( x \) is incremental with \( \chi \) and \( \alpha \). \( v_2 \) decreases with increasing \( n, Q \) and \( \chi \). \( P_0 \) is incremental with \( \chi \) and \( n \), but decreases with increasing \( Q \). So phase transition process become shorter with increasing electric charge \( Q \) and spacetime dimension \( n \).
FIG. 2: The simulated isothermal phase transition by isobars and the boundary of two phase coexistence region for the topological dilaton black hole as $n = 5$, $b = 1.2$, $Q = 1.2$, $\alpha = \frac{1}{\sqrt{3}}$.

IV. TWO-PHASE COEXISTENT CURVES AND THE PHASE CHANGE LATENT

Due to lack of knowledge of chemical potential, the $P-T$ curves of two-phase equilibrium coexistence for general thermodynamic system are usually obtained by experiment. However, the slope of the curves can be calculated by Clapeyron equation in theory,

$$\frac{dP}{dT} = \frac{L}{T(v^\beta - v^\alpha)},$$

where the latent heat of phase change $L = T(s^\beta - s^\alpha)$, $v^\alpha$, $s^\alpha$ and $v^\beta$, $s^\beta$ are the molar volumes and molar entropy of phase $\alpha$ and phase $\beta$ respectively. So Clapeyron equation provides a direct experimental verification for some phase transition theories.

Here we investigate the two phase equilibrium coexistence $P-T$ curves and the slope of them for the topological dilaton AdS black hole. From the (3.12), we can obtain the temperature

$$T = A \frac{(1 + x)}{xF(x)} - BQ^2 \frac{(1 - x^d)}{(1 - x)x^{d-1}F^{d-1}(x)} + CQ^4 \frac{(1 - x^{2d})}{1 - x}$$

$$+ CQ^4 \frac{(1 - x^{2d})}{(1 - x)x^{2d-1}F^{2d-1}(x)} = G(x),$$

Meanwhile, from the eq.(2.20), we obtain

$$P = \frac{T}{F(x)} - \frac{A}{F^2(x)} + \frac{BQ^2}{F^d(x)} - \frac{CQ^4}{F^{2d}(x)}.$$
TABLE I: State quantities at phase transition endpoints with different parameters $Q$ and spacetime dimensional $n$ as $b = 1.2, \alpha = 1/\sqrt{3}$

| $Q$ | $\chi$ | $x$  | $v_2$ | $P_0$   | $x$  | $v_2$ | $P_0$   | $x$  | $v_2$ | $P_0$   |
|-----|--------|------|-------|---------|------|-------|---------|------|-------|---------|
| 1   | 0.9    | 0.4755 | 9.4245 | 0.0062 | 0.5500 | 3.7338 | 0.0528 | 0.5954 | 2.6538 | 0.1478  |
|     | 0.7    | 0.2182 | 17.5582 | 0.0032 | 0.2826 | 6.4635 | 0.0284 | 0.3240 | 3.9383 | 0.0811  |
|     | 0.5    | 0.0859 | 40.681  | 0.0011 | 0.1235 | 13.8096 | 0.0111 | 0.1491 | 8.0946 | 0.0327  |
|     | 0.3    | 0.0166 | 196.503 | 0.0002 | 0.0286 | 56.5838 | 0.0019 | 0.0378 | 30.6323 | 0.0061 |
|     | 0.1    | 0.00002 | 155970 | 7.31E-8 | 0.0001 | 17596.8 | 2.20E-6 | 0.0002 | 6047.54 | 0.00001 |
| 1.2 | 0.9    | 0.5210 | 9.1578 | 0.0052 | 0.5803 | 3.6715 | 0.0479 | 0.6196 | 2.3264 | 0.1382  |
|     | 0.7    | 0.2787 | 14.8187 | 0.0029 | 0.3294 | 5.7875 | 0.0274 | 0.3649 | 3.6026 | 0.0794  |
|     | 0.5    | 0.1473 | 25.735  | 0.0014 | 0.1763 | 10.1328 | 0.0126 | 0.1980 | 6.2952 | 0.0364  |
|     | 0.3    | 0.0646 | 55.1282 | 0.0004 | 0.0736 | 23.1812 | 0.0037 | 0.0819 | 14.6547 | 0.0106 |
|     | 0.1    | 0.0173 | 195.784 | 0.0004 | 0.0153 | 107.47 | 0.0003 | 0.0151 | 76.8399 | 0.0008 |
| 1.5 | 0.9    | 0.5795 | 8.9024 | 0.0037 | 0.6190 | 3.6036 | 0.0410 | 0.6506 | 2.2873 | 0.1246  |
|     | 0.7    | 0.3579 | 12.6653 | 0.0022 | 0.3894 | 5.1657 | 0.0246 | 0.4172 | 3.2715 | 0.0744  |
|     | 0.5    | 0.2327 | 18.0221 | 0.0011 | 0.2462 | 7.6850 | 0.0124 | 0.2620 | 4.9546 | 0.0372  |
|     | 0.3    | 0.1435 | 27.0342 | 0.0004 | 0.1430 | 12.6537 | 0.0045 | 0.1469 | 8.5204 | 0.0133  |
|     | 0.1    | 0.0847 | 44.6749 | -0.00001 | 0.0695 | 25.1324 | 0.0004 | 0.0650 | 18.7034 | 0.0015 |

Substituting the eq.(4.2) to eq.(4.3), we can obtain

$$P = \frac{G(x)}{F(x)} - \frac{A}{F^2(x)} + \frac{BQ^2}{F^d(x)} - \frac{CQ^4}{F^{2d}(x)} = H(x).$$

we plot the $P-T$ curves with $0 < x \leq 1$ in Fig.3 when the parameters $b, \alpha, Q$ take different values respectively. The curves represent two-phase equilibrium condition for the topological dilaton AdS black hole and the terminal points of the curves represent corresponding critical points. Fig.3 shows that for fixed $b$ and $Q$, both the critical temperature and critical pressure increases as $\alpha$ increases. Both critical pressure and temperature are incremental with $\alpha$, but two-phase equilibrium pressure decreases with increasing $b$ at certain temperature. The change of two-phase equilibrium curve with parameter $Q$ is similar to that with parameter
FIG. 3: Two phase equilibrium coexistence curves in $P-T$ diagrams for the topological dilaton black hole in 5-dimensional AdS spacetime. In the diagram, the first line with $b = 0.8$, the Second line with $b = 1.0$, the final line with $b = 1.2$ and the first column with $Q = 1.0$, the second column with $Q = 1.2$, the final column with $Q = 1.5$.

b. As $Q$ becomes larger the critical pressure and critical temperature become smaller.

From Eq. (4.4), we obtain

$$\frac{dP}{dT} = \frac{H'(x)}{G'(x)},$$

(4.5)

where $H'(x) = \frac{dH}{dx}$. The Eq. (4.5) represents the slope of two-phase equilibrium $P-T$ curve as function of $x$. From Eqs. (4.1) and (4.5) we can get the latent heat of phase change as
function of \( x \) for \( n \)-dimensional charged topological dilaton AdS black hole,

\[
L = T(1 - x) \frac{H'(x)}{G'(x)} F(x) = (1 - x) \frac{H'(x)}{G'(x)} G(x) F(x).
\] (4.6)

We plot the \( L - x \) curves with \( 0 < x \leq 1 \) in Fig.4, as the parameters \( b, \alpha \) and \( Q \) take certain values. From Fig.5 we can see that the effects of \( x \) and the parameters \( n \) and \( Q \) on phase change latent heat \( L \).

The rate of change of latent heat of phase change with temperature for some usual thermodynamic systems

\[
\frac{dL}{dT} = C^\beta_P - C^\alpha_P + \frac{L}{T} - \left[ \left( \frac{\partial v^\beta}{\partial T} \right)_P - \left( \frac{\partial v^\alpha}{\partial T} \right)_P \right] \frac{L}{v^\beta - v^\alpha}.
\] (4.7)

where \( C^\beta_P \) and \( C^\alpha_P \) are molar heat capacity of phase \( \beta \) and phase \( \alpha \). For \( n + 1 \)-dimensional charged topological dilaton AdS black hole, the rate of change of latent heat of phase transition with temperature can be obtained from Eqs. (4.6) and (4.2),

\[
\frac{dL}{dT} = \frac{dL}{dx} \frac{dx}{dT} = \frac{dL}{dx} \frac{1}{G'(x)}.
\] (4.8)

Using Eqs. (4.6) and (4.2) we plot \( L - T \) curves in Fig.5 as the parameters \( b, \alpha \) and \( Q \) take certain values. From Fig.5 we can see that the effects of \( T \) and the parameters \( \alpha, b, \) and \( Q \) on latent heat of phase change \( L \). When \( T \) increases, \( L \) is not monotonous but increases firstly and then decreases to zero as \( T \to T_c \). \( L \) decreases with increasing \( \alpha \) as other parameters \( b \) and \( Q \) are fixed. Similarly \( L \) decreases with increasing \( Q \) for fixed \( b \) and \( \alpha \).
FIG. 5: \(L - T\) curves for the topological dilaton black hole in \(n\)-dimensional AdS spacetime as \(n = 5\). In each diagram, the highest curves (red) correspond to \(b = 0.01\), the middle curves (green) meet \(b = 0.02\), and the lowest curves (blue) are with \(b = 0.05\).

V. CONCLUDING REMARKS

Investigation on the phase transition of the black holes is important and necessary. On one hand, it is helpful for us to understand the structure and nature of the space time. On the other hand, it may uncover some phase transitions of the realistic physics in the conformal field theory according to the AdS/CFT correspondence. The higher-dimensional
charged topological dilaton AdS black hole with a non-linear source is regarded as a thermodynamic system, and its state equation has been derived. But when temperature is below critical temperature, thermodynamic unstable situation appears on isotherms, and when temperature reduces to a certain value the negative pressure emerges, which can be seen in Fig.1 and Fig.2. However, by Maxwell equal law we established a phase transition process and the problems can be resolved. The phase transition process at a defined temperature happens at a constant pressure, where the system specific volume changes along with the ratio of the two coexistent phases. According to Ehrenfest scheme the phase transition belongs to the first order one. We draw the isothermal phase transition process and depict the boundary of two-phase coexistence region in Fig.2.

Appropriate theoretical interpretation to the phase structure of the AdS black hole thermodynamic system can help to know more about black hole thermodynamic properties, such as entropy, temperature, heat capacity and so on, of black hole and that is significant for improving self-consistent thermodynamic theory of black holes. The Clapeyron equation of usual thermodynamic system agrees well with experiment result. In this paper we have plotted the two-phase equilibrium curves in $P - T$ diagrams, derived the slope of the curves, and acquired information on latent heat of phase change by Clapeyron equation, which could create condition for finding some usual thermodynamic systems similar to black holes in thermodynamic properties and provide theoretical basis for experimental research on analogous black holes.

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