Integrable structure of melting crystal model with two $q$-parameters

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Abstract
This paper explores integrable structures of a generalized melting crystal model that has two $q$-parameters $q_1, q_2$. This model, like the ordinary one with a single $q$-parameter, is formulated as a model of random plane partitions (or, equivalently, random 3D Young diagrams). The Boltzmann weight contains an infinite number of external potentials that depend on the shape of the diagonal slice of plane partitions. The partition function is thereby a function of an infinite number of coupling constants $t_1, t_2, \ldots$ and an extra one $Q$. There is a compact expression of this partition function in the language of a 2D complex free fermion system, from which one can see the presence of a quantum torus algebra behind this model. The partition function turns out to be a tau function (times a simple factor) of two integrable structures simultaneously. The first integrable structure is the bigraded Toda hierarchy, which determine the dependence on $t_1, t_2, \ldots$. This integrable structure emerges when the $q$-parameters $q_1, q_2$ take special values. The second integrable structure is a $q$-difference analogue of the 1D Toda equation. The partition function satisfies this $q$-difference equation with respect to $Q$. Unlike the bigraded Toda hierarchy, this integrable structure exists for any values of $q_1, q_2$.

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1 Introduction

The melting crystal model is a model of statistical physics and describes a melting corner of a crystal that fills the first quadrant of the 3D Euclidean space. The complement of the crystal in the first quadrant may be thought of as a 3D analogue of Young diagrams. These 3D Young diagram can be represented by plane partitions. Thus the melting crystal model can be formulated as a model of random plane partitions.

This model has been applied to string theory [1] and gauge theory [2, 3]. From the point of view of gauge theory, the partition function of the melting crystal model is a 5D analogue of the instanton sum of 4D $\mathcal{N} = 2$ supersymmetric Yang-Mills theory [4, 5, 6]. (Curiously, the 4D instanton sum also resembles a generating function the Gromov-Witten invariants of the Riemann sphere [7, 8].) This analogy will need further explanation, because the 4D instanton sum is a sum over ordinary partitions rather than plane partitions. The fact is that one can use the idea of diagonal slicing [9] to rewrite the partition function of the melting crystal model to a sum over ordinary partitions [2]. Comparing these two models of random partitions, one can consider the melting crystal model as a kind of $q$-deformation of the 4D instanton sum. Here $q$ is a parameter of the melting crystal model related to temperature.

In our previous work [10] (see also the review [11]), we introduced a set of external potentials into this model, and identified an integrable structure that lies behind this partition function. Namely, the partition function, as a function of the coupling constants $t_1, t_2, \ldots$ of potentials, turns out to be equal to a tau function (times a simple factor) of the Toda hierarchy [12, 13]. Moreover, the tau function satisfy a set of constraints that reduces the full Toda hierarchy to the so called 1D Toda hierarchy. Though a similar fact was known for the 4D instanton sum [14, 15, 16], we found that the partition function of the melting crystal model can be treated in a more direct manner. We derived these results on the basis of a fermionic formula of the partition function [14]. A technical clue is a set of algebraic relations among the basis of a quantum torus (or cylinder) algebra realized by fermions. These relations enabled us to rewrite the partition function to a tau function of the Toda hierarchy.

In the present paper, we generalize these results to a melting crystal model with two $q$-parameters $q_1, q_2$ [17]. Actually, since the potentials have another $q$-parameter $q$, this model has altogether three $q$-parameters $q_1, q_2$ and $q$; letting $q_1 = q_2 = q$, we can recover the previous model.

Our goal is two-fold. Firstly, we elucidate an integrable structure that emerges when $q_1$ and $q_2$ satisfy the relations $q_1 = q^{1/N_1}$ and $q_2 = q^{1/N_2}$ for
a pair of positive integers \( N_1 \) and \( N_2 \). The partition function in this case turns out to be, up to a simple factor, a tau function of (a variant of) the bigraded Toda hierarchy of type \((N_1, N_2)\) \[18\], which is also a reduction of the Toda hierarchy. Secondly, without such condition on the parameters \( q_1, q_1 \) and \( q \), we show that the partition function satisfies a \( q \)-difference analogue \[19, 20, 21, 22\] of the Toda equation with respect to yet another coupling constant \( Q \). In the gauge theoretical interpretation, \( Q \) is related to the energy scale \( \Lambda \) of supersymmetric Yang-Mills theory.

This paper is organized as follows. Section 2 is a review of combinatorial aspects of the usual melting crystal model. The model with two \( q \)-parameters is introduced in the end of this section. Section 3 is an overview of the fermionic formula of the partition function. After reviewing these basic facts, we present our results on integrable structures in Sections 4 and 5. Section 4 deals with the bigraded Toda hierarchy, and Section 5 the \( q \)-difference Toda equation. We conclude this paper with Section 6.

## 2 Melting crystal model

In the following, we shall use a number of notions and results on partitions, Young diagrams and Schur functions. For details of those combinatorial tools, we refer the reader to Macdonald’s book \[23\]. See also Bressoud’s book \[24\] for related issues and historical backgrounds.

### 2.1 Simplest model

Let us start with a review of the ordinary melting crystal model with a single parameter \( q \) \((0 < q < 1)\). As a model of statistical physics, this system can take various states with some probabilities, and these states are represented by plane partitions.

Plane partitions are 2D analogues of ordinary (one-dimensional) partitions \( \lambda = (\lambda_1, \lambda_2, \ldots) \), and denoted by 2D arrays

\[
\pi = (\pi_{ij})_{i,j=1}^\infty = \begin{pmatrix}
\pi_{11} & \pi_{12} & \cdots \\
\pi_{21} & \pi_{22} & \cdots \\
\vdots & \vdots & \ddots
\end{pmatrix}
\]

of nonnegative integers \( \pi_{ij} \) (called parts) such that only a finite number of parts are non-zero and the inequalities

\[
\pi_{ij} \geq \pi_{i,j+1}, \quad \pi_{ij} \geq \pi_{i+1,j}
\]
are satisfied. Let $|\pi|$ denote the sum

$$
|\pi| = \sum_{i,j=0}^{\infty} \pi_{ij}
$$

of all parts $\pi_{ij}$.

Such a plane partition $\pi$ represents a 3D Young diagram in the first quadrant $x, y, z \geq 0$ of the $(x, y, z)$ space. In this geometric interpretation, $\pi_{ij}$ is equal to the height of the stack of cubes over the $(i, j)$-th position of the base $(x, y)$ plane. Therefore $|\pi|$ is equal to the volume of the 3D Young diagram.

In the formulation of the melting crystal model, the complement of the 3D Young diagram in the first quadrant embodies the shape of a partially melted crystal. We assume that such a crystal has energy proportional to $|\pi|$. Consequently, the partition function of this system is given by the sum

$$
Z = \sum_{\pi} q^{|\pi|}
$$

of the Boltzmann weight $q^{|\pi|}$ over all plane partitions $\pi$.

### 2.2 Diagonal slicing

We can convert this model of random plane partitions to a model of random partitions by diagonal slicing. This idea originates in the work of Okounkov and Reshetikhin [9] on a model of stochastic process (Schur process).

Let $\pi(m)$ $(m \in \mathbb{Z})$ denotes the partition that represents the Young diagram obtained by slicing the 3D Young diagram along the diagonal plane $x - y = m$ in the $(x, y, z)$ space. In terms of the parts $\pi_{ij}$ of the plane partition, these diagonal slices can be defined as

$$
\pi(m) = \begin{cases} 
(\pi_{i,i+m})_{i=1}^{\infty} & \text{if } m \geq 0 \\
(\pi_{j-j,m})_{j=1}^{\infty} & \text{if } m < 0
\end{cases}
$$

The plane partition can be recovered from this sequence $\{\pi(m)\}_{m=-\infty}^{\infty}$ of partitions. To be diagonal slices of a plane partition, however, these partitions cannot be arbitrary, and have to satisfy the condition

$$
\cdots \prec \pi(-2) \prec \pi(-1) \prec \pi(0) \succ \pi(1) \succ \pi(2) \succ \cdots .
$$

Here “$\prec$” denotes the interlacing relation

$$
\lambda = (\lambda_1, \lambda_2, \ldots) \succ \mu = (\mu_1, \mu_2, \ldots) \iff \lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots ,
$$
namely, \( \lambda \succ \mu \) means that the skew diagram \( \lambda/\mu \) is a horizontal strip.

These diagonal slices, in turn, determine two Young tableaux

\[
T = (T_{ij})_{(i,j) \in \lambda}, \quad T' = (T'_{ij})_{(i,j) \in \lambda}
\]
on the main diagonal slice

\[
\lambda = \pi(0)
\]
as

\[
T_{ij} = m \quad \text{if} \quad (i, j) \in \pi(-m)/\pi(-m - 1)
\]
\[
T'_{ij} = m \quad \text{if} \quad (i, j) \in \pi(m)/\pi(m + 1).
\]

Since the skew diagrams \( \pi(\pm m)/\pi(\pm (m+1)) \) are horizontal strips, these Young tableaux turn out to be semi-standard tableaux, namely, satisfy the inequalities

\[
T_{ij} > T_{i+1,j}, \quad T_{ij} \geq T_{i,j+1}.
\] (2)

These semi-standard tableaux \( T, T' \) encode the left and right halves of the plane partition. The Boltzmann weight \( q^{[\pi]} \) thereby factorizes as

\[
q^{[\pi]} = q^T q^{T'},
\] (3)

where

\[
q^T = \prod_{m=0}^{\infty} q^{(m+1/2)|\pi(-m)/\pi(-m-1)|}, \quad q^{T'} = \prod_{m=0}^{\infty} q^{(m+1/2)|\pi(m)/\pi(m+1)|}.
\]

Since the triple \((\lambda, T, T')\) is in one-to-one correspondence with the plane partition \(\pi\), the partition function \(Z\) can be reorganized into the sum over the partition \(\lambda = \pi(0)\) and the sum over the pair \((T, T')\) of semi-standard tableaux of shape \(\lambda\):

\[
Z = \sum_{\lambda} \sum_{T\text{; }T\text{-shape } \lambda} q^T q^{T'}. \quad (4)
\]

By the well known combinatorial definition of the Schur functions, the partial sum over semi-standard tableaux becomes a special value of the Schur function \(s_\lambda(x)\) of infinite variables \(x = (x_1, x_2, \ldots)\) as

\[
\sum_{T\text{; }T\text{-shape } \lambda} q^T = \sum_{T'\text{; }T'\text{-shape } \lambda} q^{T'} = s_\lambda(q^\rho), \quad (5)
\]

\footnote{Note that the entries of the tableaux are arrayed in decreasing order rather than usual increasing order. This difference is immaterial as far as we consider Young tableaux on a fixed Young diagram \(\lambda\) to describe the associated Schur function \(s_\lambda(x)\).}
where
\[ q^\rho = (q^{1/2}, q^{3/2}, \ldots, q^{m+1/2}, \ldots). \]

Note that the q-weights \(q^T\) and \(q^{T'}\) are identified with the monomials
\[ x^T = \prod_{(i,j) \in \lambda} x_{T_{ij}} \]
in the combinatorial definition
\[ s_\lambda(x) = \sum_{T: \text{shape } \lambda} x^T \]
of Schur functions. We thus obtain the Schur function expansion
\[ Z = \sum_\lambda s_\lambda(q^\rho)^2 \quad (6) \]
of the partition function.

### 2.3 Calculation of partition function

We can calculate the sum (6) by the Cauchy identity
\[ \sum_\lambda s_\lambda(x)s_\lambda(y) = \prod_{i,j=1}^\infty (1 - x_iy_j)^{-1} = \exp \left( \sum_{k=1}^\infty kt_k \bar{t}_k \right), \quad (7) \]
where
\[ t_k = \frac{1}{k} \sum_{k=1}^\infty x_i^k, \quad \bar{t}_k = \frac{1}{k} \sum_{k=1}^\infty y_i^k. \]
Letting \(x = y = q^\rho\) amounts to setting
\[ t_k = \bar{t}_k = \frac{1}{k} \sum_{m=0}^\infty (q^{m+1/2})^k = \frac{q^{k/2}}{k(1 - q^k)}. \]
Consequently,

\[ \sum_{\lambda} s_\lambda(q^\rho)^2 = \exp\left( \sum_{k=1}^{\infty} \frac{q^k}{k(1 - q^k)^2} \right) = \exp\left( \sum_{k=1}^{\infty} \sum_{m,n=0}^{\infty} \frac{q^{mk+nk+k}}{k} \right) = \exp\left( - \sum_{m,n=0}^{\infty} \log(1 - q^{m+n+1}) \right) = \prod_{m,n=0}^{\infty} (1 - q^{m+n+1})^{-1}. \]

Grouping the terms in the last infinite product with respect to the value of \( l = m + n + 1 \), we find that the partition function becomes the so called MacMahon function:

\[ Z = \prod_{l=1}^{\infty} (1 - q^l)^{-l}. \]  \( \tag{8} \)

### 2.4 Models with external potentials

The foregoing melting crystal model can be deformed by external potentials that depend on the main diagonal slice \( \lambda = \pi(0) \).

For example, we can insert the term \( Q|\lambda| (Q > 0) \), namely, the potential \(|\lambda| \) with coupling constant \( \log Q \). The partition function

\[ Z(Q) = \sum_{\pi} q^{[\pi]} Q^{[\pi(0)]} \]

can be calculated in much the same way and becomes the deformed MacMahon function

\[ Z(Q) = \prod_{l=1}^{\infty} (1 - Qq^l)^{-l}. \]  \( \tag{9} \)

We can further consider the potentials

\[ \Phi_k(\lambda, p) = \sum_{i=1}^{\infty} q^{k(p+\lambda_i-i+1)} - \sum_{i=1}^{\infty} q^{k(-i+1)}, \]

which depend on an integer parameter \( p \) as well. These potentials originate in 5D supersymmetric \( U(1) \) Yang-Mills theory \([2,3]\). Note that this definition
is rather heuristic; the infinite sums on the right hand side are divergent for the parameter $q$ in the range $0 < q < 1$. A true definition is obtained by pairing the term $e^{p+\lambda_{i-1}+1}$ in the first sum with the term $e^{p-i+1}$ as

$$\Phi_k(\lambda, p) = \sum_{i=1}^\infty \left(q^k(p+\lambda_{i-1}+1) - q^k(p-i+1)\right) + q^k \frac{1 - q^{pk}}{1 - q^k}. \quad (10)$$

The second term on the right hand side amounts to the terms that cannot be paired. Since $\lambda_i = i$ for all but a finite number of $i$'s, the sum on the right hand side is a finite sum. Thus $\Phi_i(\lambda, p)$ turns out to be well defined.

Our previous work [10] deals with the model deformed by the linear combination

$$\Phi(t, \lambda, p) = \sum_{k=1}^\infty t_k \Phi_k(\lambda, p)$$

of these potentials with an infinite number of coupling constants $t = (t_1, t_2, \ldots)$. Its partition function reads

$$Z(t, p, Q) = \sum_{\pi} q^{\left|\pi\right|} Q^{|\pi(0)|+p(p+1)/2} e^{\Phi(t,\pi(0),p)}.$$  

Note that the potential $|\lambda|$ of the weight $Q^{\lambda}$ is also modified to $|\lambda| + p(p+1)/2$. Unlike $Z$ and $Z(Q)$, this partition function cannot be calculated in a closed form. We could, however, show that it coincides, up to a simple factor, with a tau function of the 1D Toda hierarchy.

### 2.5 Models with two $q$-parameters

We now turn to the models with two $q$-parameters [17]. These models are obtained by substituting the Schur function factors as

$$s_\lambda(q^p)^2 \rightarrow s_\lambda(q_1^p) s_\lambda(q_2^p).$$

The new $q$-parameters $q_1, q_2$ are assumed to be in the range $0 < q_1 < 1$ and $0 < q_2 < 1$. This substitution amounts to modifying the $q$-weights $q^T, q^{T'}$ of semi-standard tableaux $T, T'$ on the main diagonal slice as

$$q^T \rightarrow q_1^T, \quad q^{T'} \rightarrow q_2^{T'}.$$
The previous partition functions $Z$, $Z(Q)$ and $Z(t, Q, p)$ are thereby replaced by

$$Z(q_1, q_2) = \sum_{\lambda} s_\lambda(q_1^\rho)s_\lambda(q_2^\rho),$$

$$Z(Q; q_1, q_2) = \sum_{\lambda} s_\lambda(q_1^\rho)s_\lambda(q_2^\rho)Q^{\lambda|},$$

$$Z(t, Q, p; q_1, q_2) = \sum_{\lambda} s_\lambda(q_1^\rho)s_\lambda(q_2^\rho)Q^{\lambda|} e^{\Phi(t, \lambda, p)}.$$

Remember that the potentials $\Phi(t, \lambda, p)$ contain the third $q$-parameter $q$ as well. It is these partition functions that we shall consider in detail. As regards the first two, we can apply the previous method to give an infinite product formula:

$$Z(q_1, q_2) = \prod_{m,n=0}^{\infty} (1 - q_1^{m+1/2}q_2^{m+1/2})^{-1},$$

$$Z(Q; q_1, q_2) = \prod_{m,n=0}^{\infty} (1 - Qq_1^{m+1/2}q_2^{n+1/2})^{-1}. \quad (11)$$

3 Fermionic formula of partition function

We use a 2D complex free fermion system to reformulate the partition functions. Notations and conventions are the same as those in our previous work [10].

3.1 Complex fermions

Let $\psi_n, \psi_n^\star$ ($n \in \mathbb{Z}$) denote the Fourier modes of the free fermion fields

$$\psi(z) = \sum_{n=-\infty}^{\infty} \psi_n z^{-n-1}, \quad \psi^\star(z) = \sum_{n=-\infty}^{\infty} \psi_n^\star z^{-n}.$$

They satisfy the anti-commutation relations

$$\{\psi_m, \psi_n^\star\} = \delta_{m+n,0}, \quad \{\psi_m, \psi_n\} = \{\psi_m^\star, \psi_n^\star\} = 0.$$

The Fock space $\mathcal{F}$ splits into charge $p$ subspaces $\mathcal{F}_p$ ($p \in \mathbb{Z}$). $\mathcal{F}_p$ has a normalized ground state (charge $p$ vacuum) $|p\rangle$, which is characterized by the vacuum condition

$$\psi_n|p\rangle = 0 \quad \text{for} \quad n \geq -p, \quad \psi_n^\star|p\rangle = 0 \quad \text{for} \quad n \geq p + 1.$$
The dual Fock space $\mathcal{F}^*$, too, splits into charge $p$ subspaces $\mathcal{F}_p^*$ ($p \in \mathbb{Z}$). The vacuum condition for the normalized ground state $\langle p \rangle$ of $\mathcal{F}_p^*$ reads

$$\langle p | \psi_n = 0 \quad \text{for} \quad n \leq -p - 1, \quad \langle p | \psi^* = 0 \quad \text{for} \quad n \leq p.$$ 

$\mathcal{F}_p$ and $\mathcal{F}_p^*$ have excited states $|\lambda, p\rangle$ and $\langle \lambda, p |$ labeled by partition $\lambda$, which altogether form a basis of $\mathcal{F}_p$ and $\mathcal{F}_p^*$.

If $\lambda$ is of length $\leq n$, namely, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n, 0, 0, \ldots)$, these excited states can be obtained from $|p\rangle$ as

$$|\lambda, p\rangle = \psi_{-(p+\lambda_1-1)} \cdots \psi_{-(p+\lambda_n-n)} - 1 \psi^*_{(p-n)+1} \cdots \psi^*_{(p-1)+1} |p\rangle,$$

$$\langle \lambda, p | = \langle \lambda, p | \psi_{-(p-1)} \cdots \psi_{-(p-n)} - 1 \psi^*_{(p+n-\lambda_n-n)+1} \cdots \psi^*_{(p+\lambda_1-1)+1}.$$

These states are mutually orthonormal:

$$\langle \lambda, p | \mu, q \rangle = \delta_{\lambda, \mu} \delta_{pq}.$$

### 3.2 Partition functions of simplest model

Let us introduce the special fermion bilinears

$$J_m = \sum_{n=-\infty}^{\infty} : \psi_{m-n} \psi^*_n :, \quad L_0 = \sum_{n=-\infty}^{\infty} n : \psi_{-n} \psi^*_n :, \quad H_k = \sum_{n=-\infty}^{\infty} q^{kn} : \psi_{-n} \psi^*_n :,$$

where $: :$ stands for normal ordering, namely,

$$: \psi_m \psi^*_n : = \psi_m \psi^*_n - \langle 0 | \psi_m \psi^*_n | 0 \rangle.$$

$J_m$’s are the Fourier modes of the $U(1)$ current

$$J(z) = : \psi(z) \psi^*(z) : = \sum_{m=-\infty}^{\infty} J_m z^{-m-1},$$

and satisfy the commutation relations

$$[J_m, J_n] = m \delta_{mn}$$

of the Heisenberg algebra. $L_0$ is one of the basis $\{ L_m \}_{m=-\infty}^{\infty}$ of the Virasoro algebra. $H_k$’s are the same “Hamiltonians” as used in the case of $q_1 = q_2 = 1$ [10]. $H_k$’s and $L_0$ commutate with each other, and have the excited states $\langle \lambda, p |$ and $| \lambda, p \rangle$ as joint eigenstates, namely,

$$\langle \lambda, p | H_k = \langle \lambda, p | \Phi_k (\lambda, p) |, \quad \langle \lambda, p | L_0 = \langle \lambda, p | \left( | \lambda \rangle + \frac{p(p+1)}{2} \right)$$

(13)
and

\[ H_k |\lambda, p\rangle = \Phi_k(\lambda, p) |\lambda, p\rangle, \quad L_0 |\lambda, p\rangle = \left(|\lambda| + \frac{p(p+1)}{2}\right) |\lambda, p\rangle. \tag{14} \]

In particular, the potentials \( \Phi_k(\lambda, p) \) and their linear combination \( \Phi(t, \lambda, p) \) show up here as the eigenvalues of \( H_k \) and their linear combinations

\[ H(t) = \sum_{k=1}^{\infty} t_k H_k. \]

We further introduce the exponential operators \([1]\)

\[ G_\pm = \exp \left( \sum_{k=1}^{\infty} \frac{q^k/2}{k(1-q^k)} J_{\pm k} \right), \]

which belong to the Clifford group \( GL(\infty) \). These operators can be factorized as

\[ G_\pm = \prod_{m=0}^{\infty} V_\pm(q^{m+1/2}), \tag{15} \]

where \( V_\pm(z) \) denote the familiar vertex operators

\[ V_\pm(z) = \exp \left( \sum_{k=1}^{\infty} \frac{z^k}{k} J_{\pm k} \right). \]

These vertex operators act on \( |\lambda, p\rangle \) and \( |\lambda, p\rangle \) as

\[ \langle \lambda, p | V_\pm(z) = \sum_{\mu > \lambda} z^{||\mu| - |\lambda|} \langle \mu, p |, \quad V_\pm(z) |\lambda, p\rangle = \sum_{\mu > \lambda} z^{||\mu| - |\lambda|} |\mu, p\rangle. \tag{16} \]

One can thereby deduce \([1]\) that the action of \( G_\pm \) on the ground states \( |p\rangle \) and \( |p\rangle \) yields a linear combination of \( \langle \lambda, p | \) and \( |\lambda, p\rangle \) with coefficients \( s_\lambda(q^\rho) \):

\[ \langle p | G_+ = \sum_{\lambda} \sum_{T\text{-shape} \lambda} q^T \langle \lambda, p | = \sum_{\lambda} s_\lambda(q^\rho) \langle \lambda, p |, \]

\[ G_- |p\rangle = \sum_{\lambda} \sum_{T^\prime\text{-shape} \lambda} q^{T^\prime \langle \lambda, p | = \sum_{\lambda} s_\lambda(q^\rho) |\lambda, p\rangle. \tag{17} \]

\( G_\pm \) thus play the role of “transfer (or transition) matrices” in lattice models.
Since $\langle \lambda, p |$ and $| \lambda, p \rangle$ are orthonormal, the inner product of $\langle p | G_+ \text{ and } G_- | p \rangle$ becomes a sum of $s_\lambda(q^\rho)^2$,

$$\langle p | G_+ G_- | p \rangle = \sum_{\lambda, \mu} s_\lambda(q^\rho) s_\mu(q^\rho) \delta_{\lambda \mu} = \sum_{\lambda} s_\lambda(q^\rho)^2,$$

which is independent of $p$. We thus obtain the fermionic formula

$$Z = \langle 0 | G_+ G_- | 0 \rangle \quad (18)$$

of the partition function $Z$. The MacMahon function formula $Z$ can be also derived from this formula and the commutation relation

$$G_+ G_- = G_- G_+ \exp \left( \sum_{k=1}^\infty \frac{q^k}{k(1-q^k)^2} \right) \quad (19)$$

of $G_\pm$.

### 3.3 Partition functions of deformed models

A fermionic formula of $Z(Q)$ and $Z(t, Q, p)$ can be obtained by inserting a combination of $L_0$ and $H_k$’s with coupling constants into $Z$. As regards $Z(Q)$, the operator to be inserted is $Q L_0$, which acts on $\langle \lambda, 0 |$ and $| \lambda, 0 \rangle$ as

$$\langle \lambda, 0 | Q L_0 = \langle \lambda, 0 | Q L_0 | \lambda, 0 \rangle = Q^{L_0} | \lambda, 0 \rangle.$$ 

$\langle 0 | G_+ Q L_0 G_- | 0 \rangle$ can be thereby expanded as

$$\langle 0 | G_+ Q L_0 G_- | 0 \rangle = \sum_{\lambda} \langle 0 | G_+ Q L_0 | \lambda, 0 \rangle \langle \lambda, 0 | G_- | 0 \rangle = \sum_{\lambda} s_\lambda(q^\rho)^2 Q^{L_0}. $$

This is nothing but $Z(Q)$. Thus we obtain the fermionic formula

$$Z(Q) = \langle 0 | G_+ Q L_0 G_- | 0 \rangle. \quad (20)$$

In the same way, we can derive the fermionic formula

$$Z(t, Q, p) = \langle p | G_+ Q L_0 e^{H(t)} G_- | p \rangle \quad (21)$$

for $Z(t, Q, p)$.

These fermionic formulae can be readily generalized to $Z(q_1, q_2)$, $Z(Q; q_1, q_2)$ and $Z(t, Q; p, q_1, q_2)$. Let us introduce the operators

$$G_+(q_1) = \exp \left( \sum_{k=1}^\infty \frac{q_1^{k/2}}{k(1-q_1^k)} J_k \right),$$

$$G_-(q_2) = \exp \left( \sum_{k=1}^\infty \frac{q_2^{k/2}}{k(1-q_2^k)} J_{-k} \right).$$
in place of $G_{\pm}$. As shown in (17), these operators generate a linear combination of $\langle \lambda, p |$ and $| \lambda, p \rangle$ with coefficients $s_{\lambda}(q_1^{\rho})$ and $s_{\lambda}(q_2^{\rho})$. The rest of calculations in the previous case applies to this case without modification. Thus we obtain the following fermionic formulae of $Z(q_1, q_2)$, $Z(Q; q_1, q_2)$ and $Z(t, Q, p; q_1, q_2)$:

$$Z(q_1, q_2) = \langle 0 | G_+(q_1) G_- (q_2) | 0 \rangle,$$

$$Z(Q; q_1, q_2) = \langle 0 | G_+(q_1) Q^{L_0} G_- (q_2) | 0 \rangle,$$

$$Z(t, Q, p; q_1, q_2) = \langle p | G_+(q_1) Q^{L_0} e^{H(t)} G_- (q_2) | p \rangle.$$

(22)

4 Relation to Toda hierarchy

We now turn to the issues of integrable structures. It is shown in our previous paper [10] that the partition function $Z(t, Q, p)$ coincides, up to a simple factor, with a special tau function of the 1D Toda hierarchy. The goal of this section is to generalize this result to the case where $q_1$ and $q_2$ are related to $q$ as

$$q_1 = q^{1/N_1}, \quad q_2 = q^{1/N_2}$$

for some positive integers $N_1$ and $N_2$. We assume this condition throughout this section.

4.1 Intertwining relations in quantum torus algebra

Let us recall yet another set of fermion bilinears

$$W_0 = \sum_{n=-\infty}^{\infty} n^2 \psi_n \psi_n^*; \quad V_m^{(k)} = q^{-km/2} \sum_{n=-\infty}^{\infty} q^{kn} \psi_{m-n} \psi_n^*;$$

from our previous work [10]. $W_0$ is one of the basis $\{ W_m \}_{m=-\infty}^{\infty}$ of the so-called $W_3$ subalgebra in the $W_\infty$ algebra of complex fermions. $V_m^{(k)}$'s satisfy the commutation relations

$$[V_m^{(k)}, V_n^{(l)}] = (q^{(lm-kn)/2} - q^{(kn-lm)/2}) (V_{m+n}^{(k+l)} - \delta_{m+n,0} \frac{q^{k+l}}{1-q^{k+l}})$$

(24)

of the quantum torus algebra\footnote{Substantially the same realization of this algebra is considered in different contexts by Gao [25] and Okounkov and Pandharipande [7, 8].} Actually, we need just a half of this algebra (so to speak, a quantum cylinder algebra) spanned by $V_{m}^{(k)}$ with $k \geq 1$ and $m \in \mathbb{Z}$. $J_m$ and $H_k$ are part of this basis:

$$J_m = V_m^{(0)}, \quad H_k = V_0^{(k)}.$$  

(25)
This Lie algebra has an inner symmetry ("shift symmetry") \[1 0\]. A consequence of this symmetry is the following intertwining relations among \(J_m\)'s and \(H_k\)'s, which play a fundamental role in identifying the integrable structure of \(Z(t, Q, p)\).

Lemma 1. \(J_m\)'s and \(H_k\)'s satisfy the intertwining relations

\[
q^{W_0/2}G_+H_k = \left((-1)^kJ_k + \frac{q^k}{1 - q^k}\right)q^{W_0/2}G_+, \tag{26}
\]

\[
H_kG_+q^{W_0/2} = G_-q^{W_0/2}\left((-1)^kJ_k + \frac{q^k}{1 - q^k}\right). \tag{26}
\]

We now use these intertwining relations replacing \(q \to q_1, q_2\). \[23\] implies that \(H_k\) may be thought of as elements of the quantum torus algebras with \(q\)-parameters \(q = q_1\) and \(q = q_2\), namely,

\[
H_k = V_0^{(N_1k)}(q_1) = V_0^{(N_2k)}(q_2), \tag{27}
\]

where \(V_m^{(k)}(q_1)\) and \(V_m^{(k)}(q_2)\) denote the counterparts of \(V_m^{(k)}\) for \(q = q_1\) and \(q = q_2\). Therefore, applying \[26\] to the cases where \(q = q_1\) and \(q = q_2\), we obtain the relations

\[
q_1^{W_0/2}G_-(q_1)G_+(q_1)H_k = \left((-1)^{N_1k}J_{N_1k} + \frac{q^k}{1 - q^k}\right)q_1^{W_0/2}G_-(q_1)G_+(q_1) \tag{28}
\]

interchanging \(J_{N_1k}\)'s and \(H_k\)'s and

\[
H_kG_-(q_2)G_+(q_2)q_2^{W_0/2} = G_-G_+q_2^{W_0/2}\left((-1)^{N_2k}J_{-N_2k} + \frac{q^k}{1 - q^k}\right) \tag{29}
\]

interchanging \(J_{-N_2k}\)'s and \(H_k\)'s. Note that the c-number terms in the parentheses have been rewritten as

\[
\frac{q_1^{N_1k}}{1 - q^{N_1k}} = \frac{q_2^{N_2k}}{1 - q^{N_2k}} = \frac{q^k}{1 - q^k}. \tag{30}
\]

4.2 Partition function as tau function

Armed with these intertwining relations, we can generalize our previous result \[10\] to \(Z(t, Q, p; q_1, q_2)\). Let us recall that a general tau function \(\tau(T, \bar{T}, p)\) of the Toda hierarchy depends on two sets of continuous variables \(T = (T_1, T_2, \ldots)\) and \(\bar{T} = (\bar{T}_1, \bar{T}_2, \ldots)\) and has a fermionic formula of the form \[26\] \[27\]

\[
\tau(T, \bar{T}, p) = \langle p | \exp \left(\sum_{k=1}^{\infty} T_k J_k\right) g \exp \left(-\sum_{k=1}^{\infty} \bar{T}_k J_{-k}\right) | p \rangle, \tag{31}
\]
where $g$ is an element of $GL(\infty)$. We now show that $Z(t, Q, p; q_1, q_2)$ coincides, up to a simple factor, with such a tau function.

**Theorem 1.** $Z(t, Q, p; q_1, q_2)$ can be rewritten in two different forms as

$$Z(t, Q, p; q_1, q_2) = (q_1q_2)^{-p(p+1)(2p+1)/12} \exp \left( \sum_{k=1}^{\infty} \frac{q^k t_k}{1 - q^k} \right) \times \langle p | \exp \left( \sum_{k=1}^{\infty} (-1)^N_1 t_k J_{N_1 k} \right) g | p \rangle \quad (32)$$

and

$$Z(t, Q, p; q_1, q_2) = (q_1q_2)^{-p(p+1)(2p+1)/12} \exp \left( \sum_{k=1}^{\infty} \frac{q^k t_k}{1 - q^k} \right) \times \langle p | g \exp \left( \sum_{k=1}^{\infty} (-1)^N_2 t_k J_{-N_2 k} \right) | p \rangle, \quad (33)$$

where $g$ is an element of $GL(\infty)$ of the form

$$g = q_1^{W_0/2} G_-(q_1) G_+(q_1) Q^{L_0} G_-(q_2) G_+(q_2) q_2^{W_0/2}. \quad (34)$$

**Proof.** To derive (32), let us think of the right hand side of the fermionic formula (22) as the inner product of $\langle p | G_+(q_1) \in \mathcal{F}^s$ and $G_-(q_2) | p \rangle \in \mathcal{F}$ in which $Q^{L_0} e^{H(t)}$ is inserted. Also remember that the order of $Q^{L_0}$ and $e^{H(t)}$ is immaterial, because $L_0$ and $H(t)$ commute with each other. Since $G_-(q_1)$ and $q_1^{W_0/2}$ act on $\langle p |$ almost trivially as

$$\langle p | G_-(q_1) = \langle p |, \quad \langle p | q_1^{W_0/2} = q_1^{p(p+1)(2p+1)/12} \langle p |,$$

we can rewrite $\langle p | G_+(q_1)$ in (22) as

$$\langle p | G_+(q_1) = q_1^{-p(p+1)(2p+1)/12} \langle p | q_1^{W_0/2} G_-(q_1) G_+(q_1).$$

In the same way,

$$G_-(q_2) | p \rangle = q_2^{-p(p+1)(2p+1)/12} G_-(q_2) G_+(w_2) q_2^{W_0/2} | p \rangle.$$

Thus $Z(t, Q, p; q_1, q_2)$ can be cast into such a form as

$$Z(t, Q, p; q_1, q_2) = (q_1q_2)^{-p(p+1)(2p+1)/12} \times \langle p | q_1^{W_0/2} G_-(q_1) G_+(q_1) e^{H(t)} Q^{L_0} G_-(q_2) G_+(q_2) q_2^{W_0/2} | p \rangle.$$
Since the intertwining relation (28) implies the identity
\[ q_1^{W_0/2}G_-(q_1)G_+(q_1)e^{H(t)} = \exp \left( \sum_{k=1}^{\infty} \frac{q^k t_k}{1 - q^k} \right) \exp \left( \sum_{k=1}^{\infty} (-1)^{N_1} t_k J_{N_1} \right) q_1^{W_0/2}G_-(q_1)G_+(q_1), \]
we can move \( e^{H(t)} \) to the right of \( \langle p \rangle \) and obtain the first formula (32). The second formula (33) can be derived in much the same way. \( \Box \)

Thus \( Z(t, Q, p; q_1, q_2) \) turns out to coincide, up to a simple factor, with the tau function \( \tau(T, \bar{T}, p) \) determined by the \( GL(\infty) \) element \( g \) of (34). Note that the time variables \( T, \bar{T} \) of the full Toda hierarchy are now restricted to a subspace. In particular, unlike the case where \( N_1 = N_2 = 1 \), not all of these variables join the game. Namely, it is only \( T_{N_1}, \bar{T}_{N_2}, k = 1, 2, \ldots \), that correspond to the coupling constants \( t \) of this generalized melting crystal model.

### 4.3 Bigraded Toda hierarchy

Since the same partition function \( Z(t, Q, p; q_1, q_2) \) is expressed in two apparently different forms as (32) and (33), we find that the identity
\[ \langle p| \exp \left( \sum_{k=1}^{\infty} (-1)^{N_1} t_k J_{N_1} \right) g |p \rangle = \langle p| g \exp \left( \sum_{k=1}^{\infty} (-1)^{N_2} t_k J_{-N_2} \right) |p \rangle \]
holds. This is a manifestation of the following more fundamental fact.

**Theorem 2.** \( J_k \)'s and the \( GL(\infty) \) element \( g \) of (34) satisfy the intertwining relations
\[ (-1)^{N_1} J_{N_1} g = (-1)^{N_2} J_{-N_2} \]
for \( k = 1, 2, \ldots \).

**Proof.** Using the intertwining relations (28) and (29), we can derive (36) as follows:
\[
(-1)^{N_1} J_{N_1} g = (-1)^{N_1} J_{N_1} q_1^{W_0/2} G_-(q_1) G_+(q_1) Q L_0 G_-(q_2) G_+(q_2) q_2^{W_0/2} \\
= q_1^{W_0/2} G_-(q_1) G_+(q_1) \left( H_k - \frac{q^k}{1 - q^k} \right) Q L_0 G_-(q_2) G_+(q_2) q_2^{W_0/2} \\
= q_1^{W_0/2} G_-(q_1) G_+(q_1) Q L_0 \left( H_k - \frac{q^k}{1 - q^k} \right) G_-(q_2) G_+(q_2) q_2^{W_0/2} \\
= q_1^{W_0/2} G_-(q_1) G_+(q_1) Q L_0 G_-(q_2) G_+(q_2) q_2^{W_0/2} (-1)^{N_2} J_{-N_2} \\
= g (-1)^{N_2} J_{-N_2}.
\]
By these intertwining relations, we can freely move $J_{N_1 k}$ and $J_{-N_2 k}$ to the far side of $g$. This implies that $\tau(T, \bar{T}, p)$ depends on $T_{N_1 k}$ and $\bar{T}_{N_2 k}$ only through the linear combination $(-1)^{N_1 k}T_{N_1 k} - (-1)^{N_2 k}\bar{T}_{N_2 k}$. In other words, $\tau(T, \bar{T}, p)$ satisfies the constraints

$$(-1)^{N_1 k} \frac{\partial \tau}{\partial T_{N_1 k}} + (-1)^{N_2 k} \frac{\partial \tau}{\partial \bar{T}_{N_2 k}} = 0$$

for $k = 1, 2, \ldots$. Apart from the presence of the signature factors $(-1)^{N_1 k}$ and $(-1)^{N_2 k}$, these constraints are the same as those that characterize the bigraded Toda hierarchy of type $(N_1, N_2)$ [18] as a reduction of the full Toda hierarchy.

In the language of the Lax operators [12, 13]

$$L = e^{\partial_p} + u_1 + u_2 e^{-\partial_p} + \cdots, \quad \bar{L} = \bar{u}_0 e^{\partial_p} + \bar{u}_1 e^{2\partial_p} + \cdots$$

of the Toda hierarchy, the reduction to the (slightly modified) bigraded Toda hierarchy can be characterized by the constraint

$$(-L)^{N_1} = (-\bar{L})^{-N_2}.$$  

Let $\mathcal{L}$ denote the difference operator defined by both hand sides of this constraint. This reduced Lax operator is a difference operator of the form

$$\mathcal{L} = (-e^{\partial_p})^{N_1} + b_1(-e^{\partial_p})^{N_1-1} + \cdots + b_{N_1+N_2}(-e^{\partial_p})^{-N_2},$$

and satisfies the Lax equations

$$\frac{\partial \mathcal{L}}{\partial T_k} = [B_k, \mathcal{L}], \quad \frac{\partial \mathcal{L}}{\partial \bar{T}_k} = [\bar{B}_k, \mathcal{L}],$$

where $B_k$ and $\bar{B}_k$ are given by

$$B_k = (L^k)_{\geq 0}, \quad \bar{B}_k = (L^{-k})_{<0},$$

$(\quad)_{\geq 0}$ and $(\quad)_{<0}$ standing for the projection onto the part of nonnegative and negative powers of $e^{\partial_p}$. More precisely, since the $(\quad)_{\geq 0}$ and the $(\quad)_{<0}$ parts of $\mathcal{L} = (-L)^{N_1} = (-\bar{L})^{-N_2}$ are given by

$$(\mathcal{L})_{\geq 0} = (-1)^{N_1}B_{N_1}, \quad (\mathcal{L})_{<0} = (-1)^{N_2}\bar{B}_{N_2},$$

we can express $\mathcal{L}$ itself as

$$\mathcal{L} = (-1)^{N_1}B_{N_1} + (-1)^{N_2}\bar{B}_{N_2}.$$  

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The powers $\mathcal{L}^k$, $k = 1, 2, \ldots$, of $\mathcal{L}$ can be likewise expressed as

$$\mathcal{L}^k = (-1)^{N_{1k}}B_{N_{1k}} + (-1)^{N_{2k}}\bar{B}_{N_{2k}}.$$ 

This readily implies that the stationary equations

$$(−1)^{N_{1k}}\frac{∂\mathcal{L}}{∂T_{N_{1k}}} + (−1)^{N_{2k}}\frac{∂\mathcal{L}}{∂\bar{T}_{N_{2k}}} = [\mathcal{L}^k, \mathcal{L}] = 0,$$ \hspace{1cm} (42)

hold for $\mathcal{L}$. These stationary equations are counterparts of (37) in the Lax formalism.

5 Relation to $q$-difference Toda equation

In this section, we consider another integrable structure, which is hidden in $Q$-dependence of the partition function. We no longer have to assume (23), namely, $q_1, q_2$ and $q$ are independent parameters throughout this section.

5.1 Variables for $q$-difference analogue

It is known [20, 21, 22] that tau functions of the Toda hierarchy can be converted to tau functions of a $q$-difference analogue of the 2D Toda equation by changing variables as

$$T_k = \frac{x^k}{k(1-q^k)}, \quad \bar{T}_k = -\frac{y^k}{k(1-q^2)}.$$ \hspace{1cm} (43)

Let $\sigma(x, y, p)$ denote such a transformed tau function, namely,

$$\sigma(x, y, p) = \tau([x]_{q_1}, -[y]_{q_2}, p),$$ \hspace{1cm} (44)

where $\tau(T, \bar{T}, p)$ is a tau function of the Toda hierarchy and $[x]_q$ denotes the $q$-difference analogue

$$[x]_q = \left(\frac{x}{1-q}, \frac{x^2}{2(1-q^2)}, \ldots, \frac{x^k}{k(1-q^k)}, \ldots\right)$$

of the notation

$$[x] = \left(\frac{x}{2}, \ldots, \frac{x^k}{k}, \ldots\right).$$

\hspace{1cm} \textsuperscript{3}This definition is slightly modified from the common one in the literature [20, 21, 22]. We shall make a remark on this issue later on.
that play a fundamental role in the study of KP and Toda hierarchies.

By this change of variables (43), the fermionic formula (31) of tau functions of the Toda hierarchy turns into the formula

$$\sigma(x,y,p) = \langle p | \exp \left( \sum_{k=1}^{\infty} \frac{x^k}{k(1-q_1^k)} J_k \right) g \exp \left( \sum_{k=1}^{\infty} \frac{y^k}{k(1-q_2^k)} J_{-k} \right) | p \rangle$$

of $\sigma(x,y,p)$. In particular, when $x,y$ and $g$ are specialized as

$$x = q_1^{k/2}, \quad y = q_2^{k/2}, \quad g = Q L_0 e^{H(t)},$$

$\sigma(x,y,p)$ coincides with $Z(t,Q,p;q_1,q_2)$. The goal of this section is to derive a $q$-difference equation that $Z(t,Q,p;q_1,q_2)$ satisfies with respect to $Q$.

### 5.2 $q$-difference 2D Toda equation for $\sigma(x,y,p)$

As a preliminary step towards a $q$-difference equation for $Z(t,Q,p;q_1,q_2)$, we now show that the transformed tau function $\sigma(x,y,t)$ satisfies a $q$-difference analogue of the 2D Toda equation (in a bilinear form)

$$\frac{\partial^2 \tau(x,y,p)}{\partial x \partial y} \tau(x,y,p) - \frac{\partial \tau(x,y,p)}{\partial x} \frac{\partial \tau(x,y,p)}{\partial y} = \tau(x,y,p+1) \tau(x,y,p-1).$$

Now that this is the lowest equation satisfied by the tau function of the Toda hierarchy with respect to $x = T_1$ and $y = -\bar{T}_1$.

**Lemma 2.** For any tau function of the Toda hierarchy, the function $\sigma(x,y,p)$ defined by (44) satisfies the $q$-difference 2D Toda equation

$$\sigma(q_1 x, q_2 y, p) \sigma(x,y,p) - \sigma(x, q_2 y, p) \sigma(q_1 x, y, p)$$

$$= x y \sigma(x,y,p+1) \sigma(q_1 x, q_2 y, p-1).$$

**Proof.** This is a consequence of the difference analogue

$$\tau(T - [x], \bar{T}, p) \tau(T, \bar{T} - [y], p) - \tau(T, \bar{T}, p) \tau(T - [x], \bar{T} - [y], p)$$

$$= x y \tau(T, \bar{T} - [y], p + 1) \tau(T - [x], T, p - 1)$$

of the 2D Toda equation (47). This equation is one of “Fay-type identities” [28, 29] that hold for any tau function of the Toda hierarchy. We shift $\bar{T}$ as
\( T \to \bar{T} + [y] \) and substitute \( T = [x]_{q_1} \) and \( \bar{T} = -[y]_{q_2} \) in this equation. The outcome is the equation

\[
\tau([x]_{q_1} - [x], -[y]_{q_2} + [y], p) \tau([x]_{q_1}, -[y]_{q_2}, p) \\
- \tau([x]_{q_1}, -[y]_{q_2} + [y], p) \tau([x]_{q_1} - [x], -[y]_{q_2}, p) \\
= xy \tau([x]_{q_1} - [x], -[y]_{q_2} + [y], p - 1),
\]

which we can further rewrite as

\[
\tau([q_1,x]_{q_1}, -[q_2,y]_{q_2}, p) \tau([x]_{q_1}, -[y]_{q_2}, p) \\
- \tau([x]_{q_1}, -[q_2,y]_{q_2}, p) \tau([q_1,x]_{q_1}, -[y]_{q_2}, p) \\
= xy \tau([x]_{q_1} - [x], -[y]_{q_2} + [y], p - 1)
\]

by the \( q \)-shift property

\[
[q x]_q = [x]_q - [x]
\]

of \([x]_q\). The last equation is nothing but (48).

\[\square\]

A few remarks are in order.

1. One can rewrite the \( q \)-difference equation (48) as

\[
D_{q_1,x} D_{q_2,y} \sigma(x, y, p) - \sigma(x, y, p) - D_{q_1,x} \sigma(x, y, p) \cdot D_{q_2,y} \sigma(x, y, p) \\
= \sigma(x, y, p + 1) \sigma(q_1, q_2 y, p - 1),
\]

where \( D_{q_1,x} \) and \( D_{q_2,y} \) stand for the \( q \)-difference operators

\[
D_{q_1,x} \sigma(x, y, p) = \frac{\sigma(x, y, p) - \sigma(q_1 x, y, p)}{(1 - q_1)x}, \\
D_{q_2,y} \sigma(x, y, p) = \frac{\sigma(x, y, p) - \sigma(x, q_2 y, p)}{(1 - q_2)y}.
\]

As \( q_1, q_2 \to 0 \), this equation turns into the 2D Toda equation (47).

2. One can modify (43) as

\[
T_k = \frac{x^k}{k(1 - q_1^k)}, \quad \bar{T}_k = \frac{y^k}{k(1 - q_2^k)}.
\]

The transformed tau function

\[
\rho(x, y, p) = \tau([x]_{q_1}, [y]_{q_2}, p)
\]

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has a fermionic formula of the form
\[ \rho(x, y, p) = \langle p | \exp \left( \sum_{k=1}^{\infty} \frac{x^k}{k(1 - q_1^k)} J_k \right) g \exp \left( - \sum_{k=1}^{\infty} \frac{y^k}{k(1 - q_2^k)} J_{-k} \right) | p \rangle \] (54)

and satisfies the equation
\[ \rho(q_1 x, y, p) \rho(x, q_2 y, p) - \rho(x, y, p) \rho(q_1 x, q_2 y, p) = xy \rho(x, q_2 y, p + 1) \rho(q_1 x, y, p - 1). \] (55)

This is another version of the \( q \)-difference Toda equation, which is rather common in the literature [20, 21, 22].

3. One can consider the functions
\[ \tilde{\sigma}(x, y, p) = \sigma(x, y, p) \prod_{m,n=0}^{\infty} (1 - xy q_1^m q_2^n), \]
\[ \tilde{\rho}(x, y, p) = \rho(x, y, p) \prod_{m,n=0}^{\infty} (1 - xy q_1^m q_2^n)^{-1} \] (56)
in place of \( \sigma(x, y, p) \) and \( \rho(x, y, p) \). In the usual formulation of the Toda hierarchy, these modified tau functions amounts to
\[ \tilde{\tau}(\mathbf{T}, \mathbf{\bar{T}}, p) = \tau(\mathbf{T}, \mathbf{\bar{T}}, p) \exp \left( - \sum_{k=1}^{\infty} k T_k \bar{T}_k \right), \] (57)

which is suited for the construction of the so called Wronskian solutions of the Toda hierarchy. \( \tilde{\sigma}(x, y, p) \) and \( \tilde{\rho}(x, y, p) \) satisfy the \( q \)-difference equations
\[ \tilde{\sigma}(q_1 x, q_2 y, p) \tilde{\sigma}(x, y, p) - (1 - xy) \tilde{\sigma}(x, q_2 y, p) \tilde{\sigma}(q_1 x, y, p) = xy \tilde{\sigma}(x, y, p + 1) \tilde{\sigma}(q_1 x, q_2 y, p - 1) \] (58)

and
\[ \tilde{\rho}(q_1 x, y, p) \tilde{\rho}(x, q_2 y, p) - (1 - xy) \tilde{\rho}(x, y, p) \tilde{\rho}(q_1 x, q_2 y, p) = xy \tilde{\rho}(x, q_2 y, p + 1) \tilde{\rho}(q_1 x, y, p - 1). \] (59)

The equation for \( \tilde{\rho}(x, y, p) \) coincides with the \( q \)-difference Toda equation of Kajiwara and Satsuma [19].
5.3 $q$-difference equation for $Z(t, Q, p; q_1, q_2)$

As already mentioned in the beginning of this section, $Z(t, Q, p; q_1, q_2)$ is equal to a special value of the function $\sigma(x, y, p)$ defined by the fermionic formula (15) with $g = Q^{L_0}e^{H(t)}$, namely,

$$Z(t, Q, p; q_1, q_2) = \sigma(q_1^{1/2}, q_2^{1/2}, p). \quad (60)$$

To derive a $q$-difference equation with respect to $Q$, it is more convenient to consider this relation in a slightly more general form as follows.

**Lemma 3.** This special solution $\sigma(x, y, p)$ of the $q$-difference Toda equation is related to the partition function $Z(t, Q, p; q_1, q_2)$ as

$$\sigma(x, y, p) = (q_1^{-1/2}q_2^{-1/2}xy)^{-p(p+1)/2}Z(t, q_1^{-1/2}q_2^{-1/2}xyQ, p; q_1, q_2). \quad (61)$$

**Proof.** Since $L_0$ and $J_k$’s satisfy the well known commutation relations

$$[L_0, J_k] = -J_k,$$

we can rewrite the exponential operators in (15) as

$$\exp\left(\sum_{k=1}^{\infty} \frac{x^k}{k(1-q_1^k)}J_k\right) = (q_1^{-1/2}x)^{-L_0}\exp\left(\sum_{k=1}^{\infty} \frac{q_1^{1/2}}{k(1-q_1^k)}J_k\right)(q_1^{-1/2}x)^{L_0},$$

$$\exp\left(\sum_{k=1}^{\infty} \frac{y^k}{k(1-q_2^k)}J_{-k}\right) = (q_2^{-1/2}y)^{L_0}\exp\left(\sum_{k=1}^{\infty} \frac{q_2^{1/2}}{k(1-q_2^k)}J_{-k}\right)(q_2^{-1/2}y)^{-L_0}.$$

Consequently,

$$\sigma(x, y, p) = |p|(q_1^{-1/2}x)^{-L_0}\exp\left(\sum_{k=1}^{\infty} \frac{q_1^{1/2}}{k(1-q_1^k)}J_k\right)(q_1^{-1/2}x)^{L_0}Q^{L_0}e^{H(t)}$$

$$\times (q_2^{-1/2}y)^{L_0}\exp\left(\sum_{k=1}^{\infty} \frac{q_2^{1/2}}{k(1-q_2^k)}J_{-k}\right)(q_2^{-1/2}y)^{-L_0}|p|.$$ 

Since

$$|p|(q_1^{-1/2}x)^{-L_0} = (q_1^{-1/2}x)^{-p(p+1)/2}|p|,$$

$$(q_2^{-1/2}y)^{-L_0}|p| = (q_2^{-1/2}y)^{-p(p+1)/2}|p|$$

and

$$(q_1^{-1/2}x)^{L_0}Q^{L_0}e^{H(t)}(q_2^{-1/2}y)^{L_0} = (q_1^{-1/2}q_2^{-1/2}xyQ)^{L_0}e^{H(t)},$$

this expression of $\sigma(x, y, p)$ boils down to (61). \qed

We can now derive a $q$-difference equation for $Z(t, Q, ; q_1, q_2)$ from the $q$-difference 2D Toda equation (15) as follows.
Theorem 3. \( Z(t, Q, p; q_1, q_2) \) satisfies the \( q \)-difference equation

\[
Z(t, q_1 q_2 Q; q_1, q_2) Z(t, Q; q_1, q_2) = Z(t, q_1 Q; q_1, q_2) Z(t, q_2 Q; q_1, q_2)
= (q_1 q_2)^{p+1/2} Z(t, Q, p+1; q_1, q_2) Z(t, q_1 q_2 Q, p-1; q_1, q_2). \tag{62}
\]

Proof. Plugging \( \sigma(x, y, p) \) of (61) into (48) yields the equation

\[
Z(t, q_1^{1/2} q_2^{1/2} x y Q; q_1, q_2) Z(t, q_1^{-1/2} q_2^{-1/2} x y Q; q_1, q_2)
= Z(t, q_1^{1/2} q_2^{1/2} x y Q; q_1, q_2) Z(t, q_1^{-1/2} q_2^{-1/2} x y Q; q_1, q_2)
= (q_1 q_2)^{p+1/2} Z(t, q_1^{-1/2} q_2^{-1/2} x y Q, p+1; q_1, q_2) Z(t, q_1^{1/2} q_2^{1/2} x y Q, p-1; q_1, q_2).
\]

Upon rescaling \( Q \) as \( Q \to x^{-1} y^{-1} q_1^{1/2} q_2^{1/2} Q \), this equation turns into (62). \( \square \)

It will be instructive to compare this result with the reduction process from the 2D Toda equation (47) to the 1D Toda equation

\[
\frac{\partial^2 \tau(t, p)}{\partial t^2} \tau(t, p) - \left( \frac{\partial \tau(t, p)}{\partial t} \right)^2 = \tau(t, p+1) \tau(t, p - 1). \tag{63}
\]

(63) is obtained from (47) by assuming the condition that

\[
\tau(x, y, p) = \tau(x+y, p),
\]

namely, \( \tau(x, y, p) \) be a function of \( t = x+y \) and \( p \). In the same sense, if \( \sigma(x, y, p) \) is a function of \( t = x y \) and \( p \), namely,

\[
\sigma(x, y, p) = \sigma(x y, p),
\]

then the \( q \)-difference 2D Toda equation (48) reduces to the \( q \)-difference analogue

\[
\sigma(q_1 q_2 t, p) \sigma(t, p) - \sigma(q_1 t, p) \sigma(q_2 t, p) = t \sigma(t, p+1) \sigma(q_1 q_2 t, p - 1) \tag{64}
\]

of the 1D Toda equation (63). The \( q \)-difference equation (62) stems from substantially the same idea. The apparent discrepancy of (62) and (64) is due to the prefactor \((q_1^{-1/2} q_2^{-1/2} xy)^{-p(p+1)}\) on the right hand side of (61).

6 Conclusion

We have found two kinds of integrable structures hidden in the partition function \( Z(t, Q, p; q_1, q_2) \) of the generalized melting crystal model. The first
integrable structure is the bigraded Toda hierarchy that determines the \( t \)-dependence of the partition function. This integrable structure emerges when \( q_1, q_2 \) and \( q \) satisfy the algebraic relations (23) (Theorems 1 and 2). This is a natural generalization of the main result of our previous work [10]. The second integrable structure is a \( q \)-difference analogue of the 1D Toda equation (Theorem 3). The role of time variable therein is played by \( Q \), and this equation holds without any condition on \( q_1, q_2 \) and \( q \).

It is easy to see that these results still hold if a \( \text{GL}(\infty) \) element of the form

\[
h = \exp \left( \sum_{n=-\infty}^{\infty} c_n \psi_{-n} \psi_n^* \right)
\]

is inserted in front of \( Q^{L_0} \) (though the ordering of \( h \), \( Q^{L_0} \) and \( e^{H(t)} \) can be changed arbitrarily). Of particular interest is the case where

\[
h = e^{\beta W_0} \quad (\beta \text{ is a constant}).
\]

This amounts to introducing a new potential of the form\(^4\)

\[
\sum_{i=1}^{\infty} (p + \lambda_i - i + 1)^2 - \sum_{i=1}^{\infty} (-i + 1)^2 = \sum_{i=1}^{\infty} ((p + \lambda_i - i + 1)^2 - (p - i + 1)^2) + \frac{p(p+1)(2p+1)}{6} \tag{65}
\]

with coupling constant \( \beta \). The partition function

\[
Z(t, \beta, Q, p; q_1, q_2) = \langle p | G_+(q_1)e^{\beta W_0}Q^{L_0}e^{H(t)}G_-(q_2)|p \rangle. \tag{66}
\]

thus obtained can be found in 5D supersymmetric Yang-Mills theory [2, 3], and also related to topological strings on a class of toric Calabi-Yau threefolds and Huwitz numbers of the Riemann sphere [30].

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\(^4\)Note that each sum on the left hand side of this definition is divergent. The right hand side shows its regularized form.
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