Nuclearity and Split for Thermal Quantum Field Theories

Dedicated to Prof. Walter Thirring on the occasion of his 70th birthday

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Abstract. We review the heuristic arguments suggesting that any thermal quantum field theory, which can be interpreted as a quantum statistical mechanics of (interacting) relativistic particles, obeys certain restrictions on its number of local degrees of freedom. As in the vacuum representation, these restrictions can be expressed by a ‘nuclearity condition’. If a model satisfies this nuclearity condition, then the net of von Neumann algebras representing the local observables in the thermal representation has the split property.

1. Introduction

Haag and Swieca [HS] suggested that a quantum field theory, which allows a particle interpretation, should have specific phase–space properties in the vacuum sector. This idea motivated Buchholz and Wichmann [BW] to investigate the restrictions on the energy level density in the vacuum sector imposed by the existence of thermal equilibrium states. The result of their careful analysis is a ‘nuclearity condition’ which on one hand is satisfied in all models of physical relevance and on the other hand tightens up the axiomatic structure considerably. Numerous results in algebraic QFT (e.g., the existence of KMS states [BJb], a local version of the Noether theorem [BDL], etc.) emerged from this refinement of the axiomatic structure.

In this article we formulate a nuclearity condition for thermal field theories (TFTs) and investigate its consequences in the axiomatic framework. Thermal representations are always reducible. Therefore their structural properties are somehow complementary to the ones known from zero temperature quantum field theory. Nevertheless a number of basic physical properties like the Reeh–Schlieder property, the Schlieder property and the Borchers property hold; in fact, they can be established without taking recourse to results from the vacuum sector [Jä a,b]. What is known so far concerning the statistical independence of local observables can be summarized as follows:

Theorem 1.1. Assume a TFT is specified by a net

\[ \mathcal{O} \rightarrow \mathcal{R}_\beta(\mathcal{O}), \quad \mathcal{O} \subset \mathbb{R}^4, \]  

\[ (1) \]

\textit{a)} The author apologizes for the substantial delay of the present version of this article.
of von Neumann algebras, subject to the standard assumptions stated explicitly in the next section (see p.6). Now let \( \mathcal{O}, \hat{\mathcal{O}} \) denote a pair of space–time regions in Minkowski space such that the closure of the open (not necessarily bounded) region \( \mathcal{O} \) is contained the interior of \( \hat{\mathcal{O}} \). (This geometrical situation will be denoted by \( \mathcal{O} \subset \subset \hat{\mathcal{O}} \) in the sequel.) It follows that

(i) for every normal state \( \omega_1 \) on \( \mathcal{R}_\beta(\mathcal{O}) \) and every normal state \( \omega_2 \) on \( \mathcal{R}_\beta(\hat{\mathcal{O}})' \) there exists a normal state \( \omega \) on \( \mathcal{B}(\mathcal{H}_\beta) \) such that

\[
\omega|_{\mathcal{R}_\beta(\mathcal{O})} = \omega_1 \quad \text{and} \quad \omega|_{\mathcal{R}_\beta(\hat{\mathcal{O}})'} = \omega_2.
\]  

(ii) for every state \( \phi_1 \) on \( \mathcal{R}_\beta(\mathcal{O}) \) and every state \( \phi_2 \) on \( \mathcal{R}_\beta(\hat{\mathcal{O}})' \) there exists a state \( \phi \) on \( \mathcal{B}(\mathcal{H}_\beta) \) such that

\[
\phi(AB) = \phi_1(A)\phi_2(B)
\]  

for all \( A \in \mathcal{R}_\beta(\mathcal{O}) \) and all \( B \in \mathcal{R}_\beta(\hat{\mathcal{O}})' \).

As usual, the Hermitian elements of \( \mathcal{R}_\beta(\mathcal{O}) \) are interpreted as the observables which can be measured at times and locations in \( \mathcal{O} \).

Remark. As it turned out, the two statements (i) and (ii) are equivalent [FS]. As we will show in Section 5, there remain only two possibilities (see also [Bu][Su][FS]):

(i) if there exists at least one normal state \( \phi \) on \( \mathcal{B}(\mathcal{H}_\beta) \), which is a product state for \( \mathcal{R}_\beta(\mathcal{O}) \) and \( \mathcal{R}_\beta(\hat{\mathcal{O}})' \), then there exist sufficiently many. More precisely, there exists, for any pair of normal states \( \omega_1 \) of \( \mathcal{R}_\beta(\mathcal{O}) \) and \( \omega_2 \) of \( \mathcal{R}_\beta(\hat{\mathcal{O}})' \), a normal state \( \omega_{1,2} \) on \( \mathcal{B}(\mathcal{H}_\beta) \), which is a normal extension of \( \omega_1 \) and \( \omega_2 \) and a product state for \( \mathcal{R}_\beta(\mathcal{O}) \) and \( \mathcal{R}_\beta(\hat{\mathcal{O}})' \). The existence of these normal product states is equivalent to the existence of a type I factor \( \mathcal{N}_\beta \) such that

\[
\mathcal{R}_\beta(\mathcal{O}) \subset \mathcal{N}_\beta \subset \mathcal{R}_\beta(\hat{\mathcal{O}});
\]  

in this case the inclusion \( \mathcal{R}_\beta(\mathcal{O}) \subset \mathcal{R}_\beta(\hat{\mathcal{O}}) \) is called split. (For a general discussion of split inclusions see [DL].)

(ii) all normal partial states have normal extensions, none of which is a product state, and also all partial states have extensions to product states, none of which is normal.

Just as in the vacuum sector, the missing piece of information in order to favour one of the two possibilities (i) or (ii) stated in the previous Remark is encoded in the phase–space properties of a given TFT: In Section 4 we prove that the split property (4) can be derived from an appropriate nuclearity condition, which we expect to be satisfied in all physically
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relevant TFTs; thus we can rule out possibility (ii) for those theories. In Section 3 we give a self-contained, heuristic derivation of the nuclearity condition in the thermal sector, based on the work of several authors (see for instance [BW][BD’AL b][BY], etc.). Section 5 exploits several equivalent formulations of the split property; whereas in the final section we list some of its implications.

2. Preliminary Definitions and Results

Let us briefly recall the standard setup: In the Araki–Haag–Kastler framework [H] a quantum field theory (QFT) is specified by a net

\[ \mathcal{O} \to \mathcal{A}(\mathcal{O}), \quad \mathcal{O} \subset \mathbb{R}^4, \]  

(5)

of \( C^* \)-algebras. \( \mathcal{A}(\mathcal{O}) \) represents the algebra generated by the observables which can be measured in the space–time region \( \mathcal{O} \).

2.1. Representation Independent Properties

The net \( \mathcal{O} \to \mathcal{A}(\mathcal{O}) \) has certain properties irrespective of the (global) properties of the (initial) physical state under consideration:

i.) The net \( \mathcal{O} \to \mathcal{A}(\mathcal{O}) \) is isotonous, i.e., there exists a unital embedding

\[ \mathcal{A}(\mathcal{O}_1) \hookrightarrow \mathcal{A}(\mathcal{O}_2) \quad \text{if} \quad \mathcal{O}_1 \subset \mathcal{O}_2. \]  

(6)

Isotony allows us to consider the quasi-local algebra

\[ \mathcal{A} := \overline{\bigcup_{\mathcal{O} \subset \mathbb{R}^4} \mathcal{A}(\mathcal{O})}^{C^*}, \]  

(7)

which is defined as the \( C^* \)-inductive limit of the local algebras. The elements of \( \mathcal{A} \) are called quasi-local observables; they can be approximated in norm topology by strictly local elements; the total energy, total charge, etc., are considered to be unobservable; these quantities refer to infinitely extended regions and can not be controlled by local measurements.

ii.) Observables localized in spacelike separated space–time regions commute:

\[ \mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}^c(\mathcal{O}_2) \quad \text{if} \quad \mathcal{O}_1 \subset \mathcal{O}_2'. \]  

(8)

Here \( \mathcal{O}' \) denotes the spacelike complement of \( \mathcal{O} \) and \( \mathcal{A}^c(\mathcal{O}) \) denotes the set of operators in \( \mathcal{A} \) which commute with all operators in \( \mathcal{A}(\mathcal{O}) \).
iii.) The space–time symmetry of Minkowski space manifests itself in the existence of a representation
\[ \alpha: (\Lambda, x) \mapsto \alpha_{\Lambda, x} \in Aut(\mathcal{A}), \quad (\Lambda, x) \in \mathcal{P}_+^+, \tag{9} \]
of the (orthochronous) Poincaré group \( \mathcal{P}_+^+ \). Lorentz transformations \( \Lambda \) and space–time translations \( x \) act geometrically:
\[ \alpha_{\Lambda, x}(\mathcal{A}(O)) = \mathcal{A}(\Lambda O + x) \quad \forall (\Lambda, x) \in \mathcal{P}_+^+. \tag{10} \]

**Remark.** Without loss of generality, we may assume that the space–time translations \( \alpha: \mathbb{R}^4 \to Aut(\mathcal{A}) \) are strongly continuous. In this case the energy–momentum transfer of an element \( a \in \mathcal{A} \) has a representation independent meaning. One can define the Fourier transforms of the operator valued functions \( x \mapsto \alpha_x(a), a \in \mathcal{A} \), in the sense of distributions: for each \( f \in L^1(\mathbb{R}^4, d^4x) \) the expression
\[ \alpha_f(a) := \int d^4x f(x)\alpha_x(a), \quad a \in \mathcal{A}, \tag{11} \]
exists as a Bochner integral in \( \mathcal{A} \), since \( \|\alpha_f(a)\| \leq \|f\|_1\|a\| \). The energy–momentum transfer of an element \( a \in \mathcal{A} \) is defined as the smallest closed subset \( \bar{O} \subset \mathbb{R}^4 \) such that
\[ \alpha_f(a) = 0 \quad \forall f \in L^1(\mathbb{R}^4) \quad \text{with} \quad \text{supp } \tilde{f} \subset \mathbb{R}^4 \setminus \bar{O}, \tag{12} \]
where \( \tilde{f} \) denotes the Fourier transform of \( f \) (cf. [BV]).

**Remark.** For the present article we may restrict our attention to the (strongly continuous) one-parameter subgroup of time translations \( \tau: \mathbb{R} \to Aut(\mathcal{A}) \). Of course, it acts geometrically, i.e.,
\[ \tau_t(\mathcal{A}(O)) = \mathcal{A}(O + te) \quad \forall t \in \mathbb{R}. \tag{13} \]
Here \( e \) is a unit vector denoting the time direction with respect to a given Lorentz frame.

### 2.2. Representation Dependent Properties

The relevant states describing thermal equilibrium are distinguished within the set of all time invariant normalized, positive linear functionals of \( \mathcal{A} \) by their stability properties with respect to timelike translations. They are conveniently characterised by the KMS condition [HHW]:
Definition. A state $\omega_\beta$ over $\mathcal{A}$ is called a $(\tau, \beta)$-KMS state for some $\beta \in \mathbb{R} \cup \{\pm \infty\}$, if
\[ \omega_\beta(a_{\tau \beta}(b)) = \omega_\beta(ba) \] (14)
for all $a, b$ in a norm dense, $\tau$-invariant $*$-subalgebra of $\mathcal{A}_\tau$. ($\mathcal{A}_\tau \subset \mathcal{A}$ denotes the set of analytic elements for $\tau$.)

Given a KMS state $\omega_\beta$, the GNS construction gives rise to a Hilbert space $\mathcal{H}_\beta$ and a representation $\pi_\beta$, called a thermal representation, of $\mathcal{A}$. The algebra $\mathcal{R}_\beta := \pi_\beta(\mathcal{A})''$ possesses a cyclic (due to the GNS construction) and separating (due to the KMS condition) vector $\Omega_\beta$ such that
\[ \omega_\beta(a) = (\Omega_\beta, \pi_\beta(a)\Omega_\beta) \quad \forall a \in \mathcal{A}. \] (15)

Notation. The state vector $\Omega_\beta$ induces a natural extension of $\omega_\beta$ to $\mathcal{B}(\mathcal{H}_\beta)$. By abuse of notation the same symbol, namely $\omega_\beta$, will be used to denote both the extension and the original state.

A KMS state is time invariant. Therefore the one-parameter group of unitaries implementing the time translations $\tau : \mathbb{R} \to \text{Aut}(\mathcal{A})$ in the representation $\pi_\beta$ is uniquely specified by putting
\[ e^{iH_\beta t}\pi_\beta(a)\Omega_\beta := \pi_\beta(\tau_t(a))\Omega_\beta \quad \forall a \in \mathcal{A}. \] (16)

Remark. In order to support our heuristic argumentation later on, let us assume, just for a moment, that the time evolution $\tau$ is inner, i.e., $\tau$ is generated by an element $h$ of the $C^*$-algebra $\mathcal{A}$:
\[ \tau_t(a) = e^{iht}ae^{-iht} \quad \forall a \in \mathcal{A}. \] (17)
It follows that the generator $H_\beta$ can be identified as
\[ H_\beta = \pi_\beta(h) - J_\beta \pi_\beta(h)J_\beta, \] (18)
where $J_\beta$ is be the modular conjugation associated with the pair $(\mathcal{R}_\beta, \Omega_\beta)$. Note that even in this case $H_\beta$ and $\pi_\beta(h)$ differ from each other not only by the thermal expectation value of the energy $\omega_\beta(h)$, but in the removal of an operator of $\mathcal{R}'_\beta$. If one withdraws the (spatial or/and momentum) cut-offs which are implicitly enforced by requiring that $h \in \mathcal{A}$, then the decomposition (18) of $H_\beta$ is no longer possible.

We will not require that spacelike translations can be unitarily implemented in the representation $\pi_\beta$, since spatial translation invariance may be spontaneously broken in a KMS state.

We could now continue to derive more specific properties of the net
\[ \mathcal{O} \to \mathcal{R}_\beta(\mathcal{O}) := \pi_\beta(\mathcal{A}(\mathcal{O})))'' \] (19)
from first principles†, but we rather prefer to conclude our outline of the general setting at this point; a more detailed description will be presented elsewhere. Instead we emphasize that in the rigorous part of this article, which starts in Section 4, we will exclusively rely on the following

**Standard Assumptions of Thermal Field Theory.** A TFT is specified by a von Neumann algebra $\mathcal{R}_\beta$ with a cyclic and separating vector $\Omega_\beta$ together with a net of subalgebras

$$\mathcal{O} \to \mathcal{R}_\beta(\mathcal{O}), \quad (20)$$

which is subject to the following conditions:

i.) the subalgebras associated with spacelike separated space–time regions commute, i.e.,

$$\mathcal{R}_\beta(\mathcal{O}_1) \subseteq \mathcal{R}_\beta(\mathcal{O}_2)' \quad \text{if} \quad \mathcal{O}_1 \subseteq \mathcal{O}_2'. \quad (21)$$

ii.) the modular group $t \mapsto \Delta^t$ associated with the pair $(\mathcal{R}_\beta, \Omega_\beta)$ coincides — up to rescaling — with the time evolution and therefore acts geometrically, i.e.,

$$e^{i\beta t} \mathcal{R}_\beta(\mathcal{O}) e^{-i\beta t} = \mathcal{R}_\beta(\mathcal{O} + te) \quad \forall t \in \mathbb{R}. \quad (22)$$

Here $e$ is the unit vector denoting the time direction w.r.t. the distinguished rest frame and the modular operator $\Delta_\beta = \exp(-\beta H_\beta)$.

iii.) $\mathcal{H}_\beta$ is separable and $\Omega_\beta$ is the unique — up to a phase — time invariant vector in $\mathcal{H}_\beta$.

iv.) $\Omega_\beta$ is cyclic for the local algebra $\mathcal{R}_\beta(\mathcal{O})$, where $\mathcal{O}$ is an open subset of $\mathbb{R}^4$; i.e.,

$$\overline{\mathcal{R}_\beta(\mathcal{O})}\Omega_\beta = \mathcal{H}_\beta. \quad (23)$$

This property is called the Reeh–Schlieder property.

**Remark.** The Reeh–Schlieder property follows from the relativistic KMS condition of Bros and Buchholz [BB] provided the net $\mathcal{O} \to \mathcal{R}_\beta(\mathcal{O})$ satisfies additivity [Jäb]. As has been shown by Junglas [Ju] the Reeh–Schlieder property can as well be derived from the standard KMS condition, as long as $\omega_\beta$ is locally normal w.r.t. the vacuum representation.

† For instance, if the KMS state $\omega$ is extremal and the time evolution is asymptotically abelian, i.e.,

$$\lim_{t \to \infty} \| [a, \tau_t(b)] \| = 0 \quad \text{for all} \ a, b \in \mathcal{A},$$

then $\Omega_\beta$ is the unique — up to a phase — time invariant vector in $\mathcal{H}_\beta$. 

3. The Nuclearity Condition in the Thermal Sector

In quantum mechanics the number of states in a finite phase–space volume is finite (it is of the order $\text{phase–space volume} / 2\pi \hbar$). For QFTs the situation is — due to imperfect localization properties — more delicate. Here we claim that even for thermal field theories the set of normal states representing excitations which are ‘well-localized in phase–space’ is ‘small’ (although not finite dimensional). More precisely, we propose to use — for $\lambda > 0$ and $\mathcal{O}$ bounded —

$$S_\beta(\mathcal{O}, \lambda) = \{ e^{-\lambda |H_\beta|} A \Omega_\beta \in \mathcal{H}_\beta : A \in \mathcal{R}_\beta(\mathcal{O}), \|A\| \leq 1 \}$$

(24)

as an appropriate set of (not normalized) state vectors describing excitations of the KMS state which are well localized both in momentum and coordinate space. (Recall that all normal states are vector states in a thermal representation. Therefore $S_\beta(\mathcal{O}, \lambda)$ specifies a set of normal states.) It is the aim of the following two subsections to make precise what we mean by claiming that these normal states are well-localized in phase–space and in which sense $S_\beta(\mathcal{O}, \lambda)$ is small.

3.1. Excitations of a Thermal State

A normal state will be called a strictly localized excitation of the KMS state, if it can not be distinguished from the thermal equilibrium by measurements in the spacelike complement $\mathcal{O}'$ of $\mathcal{O}$. Identifying state vectors and normal states, the strictly localized excitations can be described by the following set of vectors:

$$L_\beta(\mathcal{O}) := \{ \Psi \in \mathcal{H}_\beta : (\Psi, B \Psi) = \omega_\beta(B) \quad \forall B \in \mathcal{R}_\beta(\mathcal{O}') \} \subset \mathcal{H}_\beta.$$

(25)

Strict localization is a rather cumbersome notion: in general, not even linear combinations of elements of $L_\beta(\mathcal{O})$ will belong to $L_\beta(\mathcal{O})$. This problem can be circumvented by relaxing the localization criterion: for extremal KMS states decent infrared properties of $H_\beta$ — as specified in (30) below — ensure that we can as well use

$$S_\beta(\mathcal{O}) := \{ A \Omega_\beta \in \mathcal{H}_\beta : A \in \mathcal{R}_\beta(\mathcal{O}), \|A\| \leq 1 \}$$

(26)

as a suitable set of state vectors with good localization properties in coordinate space. This argumentation is supported by the cluster theorem for KMS states presented below [Jä c].

Notation. The state vector $A \Omega_\beta$, $A \in \mathcal{R}_\beta(\mathcal{O})$, induces a state $\omega^A_\beta$, specified by

$$\mathcal{R}_\beta \ni C \mapsto \omega^A_\beta(C) := \frac{(A \Omega_\beta, C A \Omega_\beta)}{\|A \Omega_\beta\|^2}.$$

(27)

Since the KMS state distinguishes a restframe, there exists a distinguished time direction $e = (1,0,0,0)$. Let $e_\perp = (0,r,s,t)$, $r,s,t \in \mathbb{R}$, $\|e_\perp\| = 1$, be a spatial vector w.r.t. the distinguished restframe. Finally, consider the double cone

$$\mathcal{O} := (V_+ - \lambda e) \cap (V_- + \lambda e), \quad \lambda \in \mathbb{R}^+, \quad (28)$$

where the forward (resp. backward) light cone is $V_\pm = \{ x \in \mathbb{R}^4 : x^0 > \pm |\vec{x}| \}$.
Proposition 3.1. Let $O$ be the double cone introduced in (28). Furthermore, let $A \in R_{\beta}(O)$, let $\delta > 0$ be a real number and let $B \in R_{\beta}(O + \delta e_\perp)$. Assume there exist positive constants $m > 0$ and $C(O)$ such that

$$\left\| e^{-\frac{\lambda}{2}|H_{\beta}|}(A - \omega_{\beta}(A)) \Omega_{\beta} \right\| \leq C(O) \lambda^{-m} \|A\|. \quad (29)$$

It follows that (for $\delta$ large compared to $\beta$ and the diameter of $O$) the expectation values in the state $\omega^A_{\beta}$ converge to the thermal expectation values as the spacelike distance $\delta$ of the regions of $O$ and $O + \delta e_\perp$ increases:

$$\left| \omega^A_{\beta}(B) - \omega_{\beta}(B) \right| \leq \text{const.} \delta^{-2m} \frac{\|A\|^2}{\|A\Omega_{\beta}\|^2} \|B\|. \quad (30)$$

(The const. $\in \mathbb{R}^+$ is independent of $\delta$, $A$ and $B$.)

**Remark.** Due to the Reeh–Schlieder property $S_{\beta}(O)$ is dense in $H_{\beta}$. But in order to recognize the deviations from the thermal expectation values in the region $O + \delta e_\perp$ (whose spacelike distance to $O$ — neglecting the diameter of $O$ — may be only several times the thermal wavelength), it is necessary to increase the ratio between ‘cost and effort’ $\|A\|/\|A\Omega_{\beta}\|$ on the r.h.s. of (30) or the sensitivity of the measurement, i.e., the norm of $B \in R_{\beta}(O + \delta e_\perp)$. Thus the essential point in the definition of $S_{\beta}(O)$ is that the requirement $\|A\| \leq 1$. It implies that a vector $A\Omega_{\beta}$, which describes an excitation that is not essentially localized in $O$, has a rather small norm.

In order to specify normal states, which are also well-localized in momentum space, it is sufficient to restrict the energy transferred by the element $A \in R_{\beta}(O)$ onto the KMS state. (As we have pointed out the energy momentum transfer has a representation independent meaning.) This can be achieved by taking time averages

$$\frac{1}{\sqrt{2\pi}} \int dt f(t)e^{iH_{\beta}t}A\Omega_{\beta} = \tilde{f}(H_{\beta})A\Omega_{\beta}, \quad A \in R_{\beta}(O), \quad (31)$$

with suitable testfunctions $f(t)$, whose Fourier transforms $\tilde{f}(\nu)$ decrease exponentially [BD'AL b]. A convenient choice is $\tilde{f}(\nu) := e^{-\lambda|\nu|}$ with $\lambda > 0$. We conclude that if $\lambda > 0$, then the elements of

$$S_{\beta}(O, \lambda) = \{ e^{-\lambda|H_{\beta}|}A\Omega_{\beta} \in H_{\beta} : A \in R_{\beta}, \|A\| \leq 1 \} \quad (32)$$

induce vector states with good localization properties in coordinate and momentum space.
3.2. Finite Volume Gibbs States

Let us, for simplicity, consider massive particles in a finite volume $V$ in the grand canonical ensemble. The energy spectrum will then be discrete and the theory can be conveniently described in terms of energy eigenfunctions $\Psi_i$ in Fockspace $H_F$:

$$H_F \Psi_i = E_i \Psi_i \quad \text{with} \quad E_i \in \mathbb{R}^+ \cup \{0\}. \quad (33)$$

$H_F \geq 0$ denotes the Hamilton operator acting on a dense domain in $H_F$. The grand canonical equilibrium state (at zero chemical potential) is described by a density matrix $\rho_\beta \in \mathcal{B}(H_F)$:

$$\mathcal{A} \ni a \mapsto \text{Tr} \rho_\beta \pi_F(a). \quad (34)$$

Here $\pi_F(a)$ denotes the Fock space representation of an element $a \in \mathcal{A}$. For a given inverse temperature $\beta$ the grand canonical equilibrium state is unique, once the boundary conditions for the Hamiltonian $H_F$ are fixed. As long as the volume $V$ of the ‘box’ is finite, it is reasonable to assume that $e^{-\beta H_F}$ is traceclass. In this case the Gibbs density matrix is just

$$\rho_\beta = \frac{e^{-\beta H_F}}{\text{Tr} e^{-\beta H_F}}, \quad \beta > 0. \quad (35)$$

However, as the volume $V$ of the box increases, the spacing of the eigenvalues decreases drastically. In the thermodynamic limit the spectrum of the Hamiltonian becomes continuous and $e^{-\beta H_F}$ can no longer be traceclass. In order to characterize the phase–space properties of an infinite system it is therefore necessary to look for more decent properties, which may survive the thermodynamic limit. We start with a rather general classification:

**Definition.** A continuous linear mapping $\Theta$ from a Banach space $\mathcal{E}$ to another Banach space $\mathcal{F}$ is said to be of type $l^p$, $p > 0$, if there exists a sequence of linear mappings $\Theta_k$ of rank $k$ such that

$$\sum_{k=0}^{\infty} \|\Theta - \Theta_k\|^p < \infty. \quad (36)$$

$\Theta$ is said to be nuclear, if $\Theta$ is of type $l^p$ for $p = 1$. $\Theta$ is said to be of type $s$, if $\Theta$ is of type $l^p$ for all $p > 0$. The order $q$ of the map $\Theta$ is defined as the nonnegative number (if it exists)

$$q = \limsup_{\epsilon \searrow 0} \frac{\ln \ln N(\epsilon)}{\ln 1/\epsilon}, \quad (37)$$

where $N(\epsilon)$, the $\epsilon$-content of $\Theta$, is the maximal number of elements $E_i$ in the unit ball of $\mathcal{E}$ such that $\|\Theta(E_i - E_k)\| > \epsilon$ if $i \neq k$.

**Remark.** The maps of fixed type form an ideal in the space of all bounded maps between Banach spaces $[P]$. 
A delicate part of the argument concerns the relation between the Fock representation and the GNS representation induced by
\[ \rho_\beta = \frac{1}{\sum_k e^{-\beta E_k}} \sum_i e^{-\beta E_i} |\Psi_i\rangle\langle\Psi_i|, \quad \beta > 0. \] (38)

The GNS representation is unitarily equivalent to the representation \( \pi_{\beta,V} : a \mapsto \pi_{\beta,V}(a) := \pi_F(a) \otimes \mathbb{1} \) constructed with the cyclic vector
\[ |\sqrt{\rho_\beta}\rangle = \sqrt{\sum_k e^{-\beta E_k}} \sum_i e^{-\beta E_i/2} \Psi_i \otimes \Psi_i \in \mathcal{H}_F \otimes \mathcal{H}_F. \] (39)

One finds
\[ J_{\beta,V} \pi_{\beta,V}(a) J_{\beta,V} = \mathbb{1} \otimes \pi_F(a) \quad \forall a \in \mathcal{A}. \] (40)

Here \( J_{\beta,V} \) denotes the modular conjugation associated with the pair \( (\pi_{\beta,V}(A)'', |\sqrt{\rho_\beta}\rangle) \).

Corroborating the insights gotten from inner time evolutions (18) we conclude that in the representation \( \pi_{\beta,V} \) the time evolution is generated by
\[ H_F \otimes \mathbb{1} - \mathbb{1} \otimes H_F. \] (41)

We can now estimate

3.3. The Size of \( S_\beta(\mathcal{O}, \lambda) \)

Let us consider the map \( \Theta_V : \pi_{\beta,V}(\mathcal{A}(\mathcal{O})) \to \mathcal{H}_F \otimes \mathcal{H}_F, \)
\[ A \mapsto \exp\left(-\lambda |H_F \otimes \mathbb{1} - \mathbb{1} \otimes H_F|\right) (A \otimes \mathbb{1}) |\sqrt{\rho_\beta}\rangle. \] (42)

A straight forward computation yields
\[ \Theta_V(A) = \sum_{i,j} e^{-\lambda |E_i - E_j|} (A_{i,j} \otimes \mathbb{1}) \frac{e^{-\beta E_j/2}}{\sqrt{\sum_k e^{-\beta E_k}}} \Psi_j \otimes \Psi_j \]
\[ = \frac{1}{\sqrt{\sum_k e^{-\beta E_k}}} \sum_{i,j} e^{-\lambda |E_i - E_j| - \beta E_j/2} (A_{i,j} \Psi_j \otimes \Psi_j) \] (43)
where
\[ A_{i,j} := |\Psi_i\rangle\langle\Psi_i| A |\Psi_j\rangle\langle\Psi_j| \] (44)
is a rank 1 operator. Moreover, the sum in (43) is convergent for \( \lambda > 0 \); thus \( \Theta_V \) is a nuclear map; in fact since \( \Theta_V \) is nuclear for all \( \lambda > 0 \) it is even an element of all Schatten–von Neumann classes, thus it is of type \( s \) (order 0).
As long as long–range correlations play no significant role\footnote{If long-range correlations are not negligible, then boundary effects may spoil this part of our argument. But as long as \( H_\beta \) has decent infrared properties, the cluster theorem indicates that this should not be the case.}, we may compare the theory in a large (compared to the spatial extension of the bounded space–time region \( O \subset \mathbb{R}^4 \)) but finite volume \( V \) with the infinite volume theory. Disregarding boundary effects, there should exist a similarity transformation (i.e., a bounded, invertible map) \( S \) from the finite volume Hilbert space
\[
\mathcal{H}_\mathcal{F} \otimes \mathcal{H}_\mathcal{F} = \pi_{\beta,V}(A)^\prime \sqrt{\rho_\beta}
\]
onto \( \mathcal{H}_\beta \) such that
\[
e^{-\lambda |H_\beta|} \pi_\beta(A_1(O)) \Omega_\beta \subset S \cdot \exp\left(-\lambda |H_\mathcal{F} \otimes 1 - 1 \otimes H_\mathcal{F}|\right) (\pi_{\mathcal{F}}(A_1(O)) \otimes 1) _{\sqrt{\rho_\beta}},
\]
where \( A_1(O) \) denotes the unit ball in \( A(O) \). If this is the case, then the map
\[
\Phi_V: \pi_\beta(A(O)) \rightarrow \mathcal{H}_\mathcal{F}
\]
specified by
\[
\Phi_V(A) = \exp\left(-\lambda |H_\mathcal{F} \otimes 1 - 1 \otimes H_\mathcal{F}|\right) S^{-1} e^{-\lambda |H_\beta|} A \Omega_\beta,
\]
is bounded by 1 if \( \|A\| = 1 \). Hence — for \( O \) bounded — the map \( \Theta_{\lambda,O}: \pi_\beta(A(O)) \rightarrow \mathcal{H}_\beta \)
\[
A \mapsto e^{-\lambda |H_\beta|} A \Omega_\beta
\]
which is obtained by composing \( \exp\left(-\lambda |H_\mathcal{F} \otimes 1 - 1 \otimes H_\mathcal{F}|\right) \) with the bounded maps \( \Phi_V \) and \( S \), respectively, is of type \( s \) (order 0) too, for any \( \lambda > 0 \) and any \( \beta > 0 \).

4. The Split Property in the Thermal Sector

To summarize the previous section, we propose a nuclearity condition, which should be checked in models: for fixed \( \beta > 0 \) and any bounded space–time region \( O \subset \mathbb{R}^4 \) the maps
\[
\Theta_{\lambda,O}: \mathcal{R}_\beta(O) \rightarrow \mathcal{H}_\beta
\]
\[
A \mapsto e^{-\lambda |H_\beta|} A \Omega_\beta
\]
should be of type \( s \) (order 0) for any \( \lambda > 0 \). This condition will now serve as the starting point for our derivation of the split property in the thermal sector.

We start with a reformulation of this condition, which will be more convenient in the sequel. The following (simplified) Lemma is due to Buchholz, D’Antoni and Longo [BD’AL b]. For the sake of completeness we reproduce their proof, adjusting the notation such that it confirms with our conventions.
Lemma 4.1. If the maps $\Theta_{\lambda, \mathcal{O}}$ are of order $q = 0$ for all $\lambda > 0$, then the maps
\[ A \mapsto e^{-\lambda H_{\beta}} A \Omega_{\beta}, \quad A \in \mathcal{R}_{\beta}(\mathcal{O}), \] (51)
are of order $q = 0$ for all $0 < \lambda < \beta/2$.

Proof. Let $A \in \mathcal{R}_{\beta}(\mathcal{O})$ and let $P^\pm$ denote the projections onto the (strictly) positive and negative spectrum of $H_{\beta}$, respectively. If the map $\Theta_{\lambda, \mathcal{O}}$ is of order 0, then the map $A \mapsto e^{-\lambda H_{\beta}} P^+ A \Omega_{\beta}$ is also of order 0, since $e^{-\lambda H_{\beta}} P^+ = P^+ e^{-\lambda |H_{\beta}|}$. The modular group $t \mapsto \Delta^t$ associated with the pair $(\mathcal{R}_{\beta}, \Omega_{\beta})$ coincides, up to the rescaling $t \mapsto -t\beta$, with the time evolution $t \mapsto e^{itH_{\beta}}$. Taking advantage of the associated modular conjugation $J$ we find:
\[ e^{-\lambda H_{\beta}} P^- A \Omega_{\beta} = P^- e^{-\lambda H_{\beta}} J e^{-\frac{\beta}{2} H_{\beta}} A^* \Omega_{\beta} = JP^+ e^{-\left(\beta/2 - \lambda\right) H_{\beta}} A^* \Omega_{\beta}, \quad 0 \leq \lambda \leq \beta/2 \] (52)
Since $J$ is bounded, this equality implies that the map $A \mapsto P^- e^{-\left(\beta/2 - \lambda\right) H_{\beta}} A \Omega_{\beta}$ is, for $0 < \lambda < \beta/2$, of order 0, too. The maps of order 0 form a linear space. It follows that the maps $A \mapsto e^{-\lambda H_{\beta}} A \Omega_{\beta}$ are of order 0 for the given range of $\lambda$. \qed

Given an inclusion $\mathcal{O} \subset \subset \hat{\mathcal{O}}$ of two space–time regions, our task is to show that the von Neumann algebra generated by $\mathcal{R}_{\beta}(\mathcal{O})$ and $\mathcal{R}_{\beta}(\hat{\mathcal{O}})'$ is isomorphic to the $W^*$-tensor product of the two algebras, i.e.,
\[ \mathcal{R}_{\beta}(\mathcal{O}) \vee \mathcal{R}_{\beta}(\hat{\mathcal{O}})' \cong \mathcal{R}_{\beta}(\mathcal{O}) \otimes \mathcal{R}_{\beta}(\hat{\mathcal{O}})' \] (53)
We will show later on that the split property (4) is a direct consequence of (53). The first step is to insert two bounded space–time regions $\mathcal{O}_1, \mathcal{O}_2$ in between $\mathcal{O}$ and $\hat{\mathcal{O}}$ such that
\[ \mathcal{O} \subset \subset \mathcal{O}_1 \subset \subset \mathcal{O}_2 \subset \subset \hat{\mathcal{O}}. \] (54)
Following Buchholz and Wichmann [BW], we consider two representations of
\[ C_{\beta}(\mathcal{O}_1, \mathcal{O}_2) := \mathcal{R}_{\beta}(\mathcal{O}_1) \circ \mathcal{R}_{\beta}(\mathcal{O}_2)', \] (55)
the algebraic tensor product of $\mathcal{R}_{\beta}(\mathcal{O}_1)$ and $\mathcal{R}_{\beta}(\mathcal{O}_2)'$: the first one acts on $\mathcal{H}_{\beta}$ and is given by
\[ \pi \left( \sum_k A_k \otimes B_k \right) = \sum_k A_k B_k \quad \text{for} \quad A_k \in \mathcal{R}_{\beta}(\mathcal{O}_1), \quad B_k \in \mathcal{R}_{\beta}(\mathcal{O}_2)'. \] (56)
The operators in $\mathcal{R}_{\beta}(\mathcal{O}_1)$ and $\mathcal{R}_{\beta}(\mathcal{O}_2)'$ commute, so $\pi$ defines a $\ast$-representation of the algebraic tensor product. The second representation, denoted by $\pi_p$, acts on $\mathcal{H}_{\beta} \otimes \mathcal{H}_{\beta}$ and is determined by
\[ \pi_p \left( \sum_k A_k \otimes B_k \right) = \sum_k A_k \otimes B_k \quad \text{for} \quad A_k \in \mathcal{R}_{\beta}(\mathcal{O}_1), \quad B_k \in \mathcal{R}_{\beta}(\mathcal{O}_2)'. \] (57)
As recently shown by the author [Jä a], the Schlieder property holds for the pair \( \mathcal{R}_\beta(\mathcal{O}_1) \) and \( \mathcal{R}_\beta(\mathcal{O}_2)' \); i.e.,

\[
AB = 0 \implies A = 0 \quad \text{or} \quad B = 0 \quad \forall A \in \mathcal{R}_\beta(\mathcal{O}_1), \quad B \in \mathcal{R}_\beta(\mathcal{O}_2)'.
\]

It follows that \( \pi_p \) is well defined: \( \sum_k A_k \otimes B_k = 0 \implies \sum_k A_k \otimes B_k = 0 \).

The next step is to show that the representations \( \pi \) and \( \pi_p \) of \( \mathcal{C}_\beta(\mathcal{O}_1, \mathcal{O}_2) \) are not disjoint. This follows — up to minor adjustments — from a result of Buchholz and Yngvason [BuY]:

**Proposition 4.2.** Let \( \mathcal{O}_1 \) be a bounded space–time region and assume there exists some \( \delta > 0 \) such that \( \mathcal{O}_1 + t e \subset \mathcal{O}_2 \) for all \( |t| < \delta \). Let \( \mathcal{R}_\beta(\mathcal{O}_2)' \) denote the predual of \( \mathcal{R}_\beta(\mathcal{O}_2)' \). It follows that the map \( \Xi_{\beta,*}: \mathcal{R}_\beta(\mathcal{O}_1) \to \mathcal{R}_\beta(\mathcal{O}_2)' \) given by

\[
A \mapsto (\Omega_\beta, A \cdot \Omega_\beta)
\]

is nuclear.

**Proof.** Let \( A \in \mathcal{R}_\beta(\mathcal{O}_1) \) and \( B \in \mathcal{R}_\beta(\mathcal{O}_2)' \). The function

\[
z \mapsto (\Omega_\beta, Be^{i z H_\beta} A \Omega_\beta)
\]

is analytic in the strip \( 0 < \Im z < \beta/2 \), while the function

\[
z \mapsto (e^{i z H_\beta} A^* \Omega_\beta, B \Omega_\beta)
\]

is analytic in the strip \(-\beta/2 < \Im z < 0 \). Both functions are bounded and have continuous boundary values for \( \Im z \searrow 0 \) and \( \Im z \nearrow 0 \), respectively. Locality implies

\[
\lim_{\Im z \searrow 0} (\Omega_\beta, Be^{i z H_\beta} A \Omega_\beta) = \lim_{\Im z \nearrow 0} (e^{i z H_\beta} A^* \Omega_\beta, B \Omega_\beta) \quad \forall |t| < \delta.
\]

Applying the Edge-of-the-Wedge Theorem [SW] we conclude that there exists a function

\[
f_{A,B}: G_\delta \to \mathbb{C},
\]

analytic on the doubly cut strip \( G_\delta = \{ z \in \mathbb{C} : |\Im z| < \beta/2 \} \setminus \{ t \in \mathbb{R} : |t| \geq \delta \} \) such that

\[
f_{A,B}(z) = \begin{cases} 
(\Omega_\beta, Be^{i z H_\beta} A \Omega_\beta) & \text{for } 0 < \Im z < \beta/2, \\
(e^{i z H_\beta} A^* \Omega_\beta, B \Omega_\beta) & \text{for } -\beta/2 < \Im z < 0.
\end{cases}
\]

The absolute value of \( f_{A,B} \) at the origin can be estimated from the values \( f_{A,B} \) takes at the boundaries:

\[
|(\Omega_\beta, BA \Omega_\beta)| \leq \inf_{0 < \lambda < \beta/2} \left( \sup_{|t| \geq \delta} |f_{A,B}(t \pm i 0)|^{1-k} \cdot \sup_{t \in \mathbb{R}} |f_{A,B}(t + i \lambda)|^{\frac{1}{2}} \cdot \sup_{t \in \mathbb{R}} |f_{A,B}(t - i \lambda)|^{\frac{1}{2}} \right) \times \\
\times \left( |\Omega_\beta|^2 \cdot |A| \cdot |B| \right)^{1-k} \times \\
\times \left( |\Omega_\beta|^2 \cdot |B|^2 \cdot \|e^{-\lambda H_\beta} A \Omega_\beta\| \cdot \|e^{-\lambda H_\beta} A^* \Omega_\beta\| \right)^{k/2}
\]

(65)
where \( k = \frac{2}{\pi} \arctan \left( 2 \sinh \frac{\pi \delta}{2} \right) \). Taking the supremum over the unit ball for \( B \in \mathcal{R}_\beta(\mathcal{O}_2)' \) and putting \( \lambda = \beta/4 \) we obtain, for \( \| A \| \leq 1 \),

\[
\| \Xi_{\beta,*}(A \pm A^*) \| \leq \text{const} \cdot \| e^{-\frac{2}{\pi} H_\beta (A \pm A^*)} \Omega_\beta \| \frac{2}{\pi} \arctan \left( 2 \sinh \frac{2 \pi \delta}{\pi} \right). \tag{66}
\]

By assumption, \( \Theta_{\lambda, \mathcal{O}_1} \) is of order \( q = 0 \), thus (66) implies that \( \Xi_{*,\beta} \) is of order \( q_* = 0 \), too [BD'AL b][BY]. Since the real linear maps \( A \mapsto (A \pm A^*) \) are bounded, we conclude that \( \Xi_{\beta,*} \) is nuclear.

\[\boxdot\boxdot\]

**Corollary 4.3.** There exist non-trivial subrepresentations \( \hat{\pi} \) of \( \pi \) and \( \hat{\pi}_p \) of \( \pi_p \), respectively, which are unitarily equivalent.

**Proof.** As noted in [BD'AL a], the nuclearity of the map \( \Xi_{\beta,*} \) simply means that there exist sequences \( \phi_i \in \mathcal{R}_\beta(\mathcal{O}_1)_* \) and \( \psi_i \in \mathcal{R}_\beta(\mathcal{O}_2)'_* \) with \( \sum \| \phi_i \| \| \psi_i \| < \infty \) such that

\[
(\Omega_\beta, \pi(A \otimes B)\Omega_\beta) = \sum_i \phi_i(A)\psi_i(B) \quad \forall A \in \mathcal{R}_\beta(\mathcal{O}_1), \quad B \in \mathcal{R}_\beta(\mathcal{O}_2)'.
\]

As an absolutely convergent sum of normal functionals

\[
\sum_i \phi_i \otimes \psi_i(\cdot) : \mathcal{C}(\mathcal{O}_1, \mathcal{O}_2) \to \mathbb{C}
\]

itself is, w.r.t. the representation \( \pi_p \), a normal† functional on the algebraic tensor product \( \mathcal{C}_\beta(\mathcal{O}_1, \mathcal{O}_2) \). Now the algebraic tensor product is weakly dense in the \( W^* \)-tensor product. It follows that the functional \( (\Omega_\beta, \pi(\cdot)\Omega_\beta) \) allows a unique continuous extension to a normal state on the \( W^* \)-tensor product \( \mathcal{R}_\beta(\mathcal{O}_1) \otimes \mathcal{R}_\beta(\mathcal{O}_2)' \), which will be denoted

\[
\omega_{\otimes}(\cdot) := \sum_i \phi_i(\cdot) \otimes \psi_i(\cdot). \tag{69}
\]

Consequently, the representations \( \pi \) and \( \pi_p \) can not be disjoint. \(\boxdot\boxdot\)

**Theorem 4.4.** Let \( \hat{\pi} \) and \( \hat{\pi}_p \) be two arbitrary subrepresentations of \( \pi \) and \( \pi_p \), respectively. It follows that

(i) the restrictions of \( \hat{\pi} \) and \( \pi \) to \( \mathcal{C}_\beta(\mathcal{O}, \hat{\mathcal{O}}) \) are unitarily equivalent;

(ii) the restrictions of \( \hat{\pi}_p \) and \( \pi_p \) to \( \mathcal{C}_\beta(\mathcal{O}, \hat{\mathcal{O}}) \) are unitarily equivalent.

† A linear functional on \( \mathcal{R}_\beta(\mathcal{O}_1) \otimes \mathcal{R}_\beta(\mathcal{O}_2)' \) is said to be normal relative to \( \pi_p \), if it is continuous with respect to the ultra-weak topology determined by \( \pi_p \).
Combining this theorem with Corollary 3.3 we arrive at

**Theorem 4.5.** The restrictions of \( \pi \) and \( \pi_p \) to \( \mathcal{C}_\beta(\mathcal{O}, \hat{\mathcal{O}}) \) are unitarily equivalent.

In order to prove (i) of Theorem 4.4, we need the following

**Lemma 4.6.** Let \( E \) denote the projection onto the subspace \( \mathcal{K}_\beta \subset \mathcal{H}_\beta \) reducing \( \pi \) to \( \hat{\pi} \). It follows that \( \Omega_\beta \) is cyclic for \( \pi(\mathcal{C}_\beta(\mathcal{O}, \hat{\mathcal{O}}))'' \) and \( E\Omega_\beta \) is separating for \( \pi(\mathcal{C}_\beta(\mathcal{O}, \hat{\mathcal{O}}))'' \).

**Proof.** Since \( \mathcal{O} \subset \mathbb{R}^4 \) is by assumption open and \( \mathcal{R}_\beta(\mathcal{O}) \subset \pi(\mathcal{C}_\beta(\mathcal{O}, \hat{\mathcal{O}}))'' \), the first part of the statement is a direct consequence of the Reeh–Schlieder property. The second part can be seen as follows: From the inclusions

\[
\mathcal{O} \subset \subset \mathcal{O}_1 \subset \subset \mathcal{O}_2 \subset \subset \hat{\mathcal{O}}
\]

it follows that

\[
\pi(\mathcal{C}_\beta(\mathcal{O}, \hat{\mathcal{O}}))' \cap \pi(\mathcal{C}_\beta(\mathcal{O}_1, \mathcal{O}_2)) \supset \mathcal{R}_\beta(\mathcal{O})' \cap \mathcal{R}_\beta(\mathcal{O}_1).
\]

By the Reeh–Schlieder theorem \( \Omega_\beta \) is cyclic for \( \mathcal{R}_\beta(\mathcal{O})' \cap \mathcal{R}_\beta(\mathcal{O}_1) \), since by assumption the closure of the open and bounded region \( \mathcal{O} \) lies inside the interior of the region \( \mathcal{O}_1 \). Thus \( \Omega_\beta \) is separating for

\[
\pi(\mathcal{C}_\beta(\mathcal{O}, \hat{\mathcal{O}}))'' \cap \pi(\mathcal{C}_\beta(\mathcal{O}_1, \mathcal{O}_2))'.
\]

By definition, the subspace \( \mathcal{K}_\beta := \mathcal{E}\mathcal{H}_\beta \) and its orthogonal complement \( \mathcal{K}_\beta^\perp \) are invariant under the action of \( \pi(\mathcal{C}_\beta(\mathcal{O}_1, \mathcal{O}_2)) \). By standard arguments it follows that \( E \in \pi(\mathcal{C}_\beta(\mathcal{O}_1, \mathcal{O}_2))' \). Hence, if \( ZE \Omega_\beta = 0 \) for some projection\(^\dagger\) \( Z \in \pi(\mathcal{C}_\beta(\mathcal{O}, \hat{\mathcal{O}}))'' \), then (72) implies \( ZE = 0 \). Because of locality

\[
[e^{iH_\beta t}Z e^{-iH_\beta t}, E] = 0 \quad \forall t \in \mathcal{U},
\]

where \( \mathcal{U} \) denotes some open neighborhood of the origin in \( \mathbb{R} \). Since \( \pi(\mathcal{C}_\beta(\mathcal{O}_1, \mathcal{O}_2))' \subset \mathcal{R}_\beta \) the thermal version [Jä a] of a classical Lemma by Borchers [Bo] applies and yields

\[
E e^{iH_\beta t} Z = 0 \quad \forall t \in \mathbb{R}.
\]

By assumption, \( \Omega_\beta \) is the unique — up to a phase — normalized eigenvector for the only discrete eigenvalue \( \{0\} \) of \( H_\beta \), thus

\[
0 = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} dt \left( \Omega_\beta, E e^{itH_\beta} Z \Omega_\beta \right) = \left( \Omega_\beta, E \Omega_\beta \right) \left( \Omega_\beta, Z \Omega_\beta \right) = \|E \Omega_\beta\|^2 \|Z \Omega_\beta\|^2.
\]

By definition, \( E \Omega_\beta \neq 0 \), thus (75) implies \( Z \Omega_\beta = 0 \). \( \Omega_\beta \) is separating for \( \pi(\mathcal{C}_\beta(\mathcal{O}, \hat{\mathcal{O}}))'' \), thus \( Z = 0 \). This proves that the vector \( E \Omega_\beta \) is separating for \( \pi(\mathcal{C}_\beta(\mathcal{O}, \hat{\mathcal{O}}))'' \). \( \square \n
\(^\dagger\) Given an arbitrary element \( C \in \pi(\mathcal{C}_\beta(\mathcal{O}, \hat{\mathcal{O}}))'' \) one can use the spectral decomposition of \( C^\ast C \) in order to reduce the general case to the case of projections: With \( C^\ast C \) also the spectral projections of \( C^\ast C \) belong to \( \pi(\mathcal{C}_\beta(\mathcal{O}, \hat{\mathcal{O}}))'' \); and obviously \( C^\ast C = 0 \) implies \( C = 0 \).
Corollary 4.7. Let $E$ denote the projection onto the subspace $\mathcal{K}_\beta \subset \mathcal{H}_\beta$ reducing $\pi$ to $\hat{\pi}$. It follows that $E \in \pi(\mathcal{C}_\beta(\mathcal{O}_1, \mathcal{O}_2))'$ can be represented in the form

$$E = VV^*, \quad \text{where} \quad V \in \pi(\mathcal{C}_\beta(\mathcal{O}, \hat{\mathcal{O}}))' \quad (76)$$

is an isometry, i.e., $V^*V = 1$.

Proof. $\Omega_\beta$ is cyclic for $\pi(\mathcal{C}_\beta(\mathcal{O}, \hat{\mathcal{O}}))''$ and in addition has the property that $E\Omega_\beta$ is separating for $\pi(\mathcal{C}_\beta(\mathcal{O}, \hat{\mathcal{O}}))''$. It follows that $(E\Omega_\beta, E\Omega_\beta)$ defines a faithful normal state on $\pi(\mathcal{C}_\beta(\mathcal{O}, \hat{\mathcal{O}}))''$. Moreover, $\pi(\mathcal{C}_\beta(\mathcal{O}, \hat{\mathcal{O}}))''$ has a cyclic and separating vector, namely $\Omega_\beta$. We conclude (see e.g. [Sa, 2.7.9] or [BR, 2.5.31]) that there exists another vector $\Psi \in \mathcal{H}_\beta$, cyclic and separating for $\pi(\mathcal{C}_\beta(\mathcal{O}, \hat{\mathcal{O}}))''$, which satisfies

$$(\Psi, \pi(C^*C)\Psi) = (E\Omega_\beta, \pi(C^*C)\Omega_\beta) \quad \forall C \in \mathcal{C}_\beta(\mathcal{O}, \hat{\mathcal{O}}). \quad (77)$$

Taking into account the properties of $\Omega_\beta$ and $\Psi$ and

$$E \in \pi(\mathcal{C}_\beta(\mathcal{O}_1, \mathcal{O}_2))' \subset \pi(\mathcal{C}_\beta(\mathcal{O}, \hat{\mathcal{O}}))' \quad (78)$$

it follows that

$$V\pi(C)\Psi = \pi(C)E\Omega_\beta \quad \text{for} \quad C \in \mathcal{C}_\beta(\mathcal{O}, \hat{\mathcal{O}}) \quad (79)$$

defines an isometry $V$ with the desired properties. \qed

Remark. The isometry $V: \mathcal{H}_\beta \to \mathcal{K}_\beta$ satisfies $V^*V = 1_{\mathcal{H}_\beta}$ and $VV^* = 1_{\mathcal{K}_\beta}$. It therefore establishes the unitary equivalence between the restrictions of $\pi$ and $\hat{\pi}$ to $\mathcal{C}_\beta(\mathcal{O}, \hat{\mathcal{O}})$.

The proof of part (ii) of Theorem 3.4 follows the same line of arguments: We show that $\Omega_\beta \otimes \Omega_\beta \in \mathcal{H}_\beta \otimes \mathcal{H}_\beta$ is cyclic for $\pi_p(\mathcal{C}_\beta(\mathcal{O}, \hat{\mathcal{O}}))''$ and in addition has the property that $E_p(\Omega_\beta \otimes \Omega_\beta)$ is separating for $\pi_p(\mathcal{C}_\beta(\mathcal{O}, \hat{\mathcal{O}}))''$, where $E_p$ denotes the projection onto the subspace $\mathcal{K}_p \subset \mathcal{H}_\beta \otimes \mathcal{H}_\beta$ reducing $\pi_p$ to $\hat{\pi}_p$. In order to do so, we adapt the classical lemma of Borchers cited above to the tensor product representation.

Lemma 4.8. Let $P \in \mathcal{R}_\beta \otimes \mathcal{B}(\mathcal{H}_\beta)$ and let $Q \in \mathcal{B}(\mathcal{H}_\beta) \otimes \mathcal{R}_\beta$ be a (self-adjoint) projection operator such that

$$QP = 0 \quad \text{and} \quad [U_p(t)QU_p(-t), P] = 0 \quad \forall |t| < \delta, \quad (80)$$

where $U_p: \mathbb{R} \to \mathcal{B}(\mathcal{H}_\beta) \otimes \mathcal{B}(\mathcal{H}_\beta)$ is given by $t \mapsto e^{it\mathcal{H}_\beta} \otimes e^{it\mathcal{H}_\beta}$ and $\delta > 0$. It follows that

$$\left(\Omega_\beta \otimes \Omega_\beta, QU_p(t)P(\Omega_\beta \otimes \Omega_\beta)\right) = 0 \quad \forall t \in \mathbb{R}. \quad (81)$$
Proof. Due to the KMS relation, the function
\[ f^+(z) := \left( (1 \otimes e^{-izH_\beta}) Q^* (\Omega_\beta \otimes \Omega_\beta) , (e^{izH_\beta} \otimes 1) P (\Omega_\beta \otimes \Omega_\beta) \right) \] (82)
is analytic in the strip \( S(0, \beta/2) := \{ z \in \mathbb{C} : 0 < \Im z < \beta/2 \} \), while the function
\[ f^-(z) := \left( (e^{izH_\beta} \otimes 1) P^* (\Omega_\beta \otimes \Omega_\beta) , (1 \otimes e^{-izH_\beta}) Q (\Omega_\beta \otimes \Omega_\beta) \right) \] (83)
is analytic in the strip \( S(-\beta/2, 0) := \{ z \in \mathbb{C} : -\beta/2 < \Im z < 0 \} \). Both functions are bounded and have continuous boundary values for \( \Im z \searrow 0 \) and \( \Im z \nearrow 0 \), respectively. Now (80) implies
\[ \lim_{\Im z \searrow 0} \left( (1 \otimes e^{-izH_\beta}) Q^* (\Omega_\beta \otimes \Omega_\beta) , (e^{izH_\beta} \otimes 1) P (\Omega_\beta \otimes \Omega_\beta) \right) = \left( (\Omega_\beta \otimes \Omega_\beta) , QU_p(\Re z) P (\Omega_\beta \otimes \Omega_\beta) \right) = \left( (\Omega_\beta \otimes \Omega_\beta) , PU_p(-\Re z) Q (\Omega_\beta \otimes \Omega_\beta) \right) \]
\[ = \lim_{\Im z \searrow 0} \left( (e^{izH_\beta} \otimes 1) P^* (\Omega_\beta \otimes \Omega_\beta) , (1 \otimes e^{-izH_\beta}) Q (\Omega_\beta \otimes \Omega_\beta) \right) \quad \forall |\Re z| < \delta. \] (84)

Using the Edge-of-the-Wedge Theorem one concludes that there exists a function
\[ f_{P,Q} : G_\delta \to \mathbb{C} \] (85)
which is analytic on the doubly cut strip
\[ G_\delta = \{ z \in \mathbb{C} : -\beta/2 < \Im z < \beta/2 \} \setminus \{ z \in \mathbb{C} : \Im z = 0, |\Re z| \geq \delta \} \] (86)
and satisfies
\[ f_{P,Q}(z) = \begin{cases} f^+(z) & \text{for } 0 < \Im z < \beta/2, \\ f^-(z) & \text{for } -\beta/2 < \Im z < 0. \end{cases} \] (87)

By assumption \(QP = 0\), hence \( f_{P,Q}(0) = 0 \). According to Lagrange’s theorem \( f_{P,Q} \) vanishes identically, if 0 is a zero of infinite order. This follows from the original arguments of Borchers: set
\[ t_j^{(i)} := \frac{\delta j}{2in}, \quad i \in \mathbb{N}, \quad j = \{1, \ldots, n\}, \] (88)
and \( Q(t_j^{(i)}) := U_p(t_j^{(i)}) QU_p(-t_j^{(i)}) \). It follows that
\[ [P, U_p(t) Q(t_1^{(i)}) \ldots Q(t_j^{(i)}) U_p(-t)] = 0 \quad \forall |t| < \delta/2. \] (89)
The functions \( f_{\delta\delta}^{(i)} : S(0, \beta/2) \to \mathbb{C}, \)
\[ z \mapsto \left( (1 \otimes e^{-izH_\beta}) Q^*(t_1^{(i)}) \ldots Q^*(t_j^{(i)}) (\Omega_\beta \otimes \Omega_\beta) , (e^{izH_\beta} \otimes 1) P (\Omega_\beta \otimes \Omega_\beta) \right) \] (90)
and $f_{t_1^{(i)},...,t_n^{(i)}}^-(z) : S(-\beta/2, 0) \to \mathbb{C}$,

$$z \mapsto \left( (e^{izH_\beta} \otimes 1) P^* (\Omega_\beta \otimes \Omega_\beta), \ (1 \otimes e^{-izH_\beta}) Q(t_1^{(i)} \ldots t_n^{(i)}) (\Omega_\beta \otimes \Omega_\beta) \right)$$

(91)

are bounded, analytic in the interior of their domains and continuous at the boundary. The boundary values for $\Im z \searrow 0$ resp. $\Im z \nearrow 0$ coincide for $|\Re z| < \delta/2$. Applying the Edge-of-the-Wedge Theorem [SW] one concludes that the functions defined in (90) and (91) are the restrictions to the upper (resp. lower) half of the doubly cut strip $G_{\delta/2}$ of a function

$$f_{t_1^{(i)},...,t_n^{(i)}}(z) := \begin{cases} f_{t_1^{(i)},...,t_n^{(i)}}^+(z) & \text{for } 0 < \Im z < \beta/2, \\ f_{t_1^{(i)},...,t_n^{(i)}}^-(z) & \text{for } -\beta/2 < \Im z < 0, \end{cases}$$

(92)

defined and analytic for $z \in G_{\delta/2}$. The function $f_{t_1^{(i)},...,t_n^{(i)}}$ has continuous boundary values for $z \to \partial G_{\delta/2}$, uniformly bounded by one: For example,

$$\sup_{s \in \mathbb{R}} f_{t_1^{(i)},...,t_n^{(i)}}(s + i\beta/2) \leq \left\| (1 \otimes J e^{-\frac{\beta}{2}H_\beta}) Q^* (t_1^{(i)} \ldots t_n^{(i)}) (\Omega_\beta \otimes \Omega_\beta) \right\| \times \left\| (J e^{-\frac{\beta}{2}H_\beta} \otimes 1) P (\Omega_\beta \otimes \Omega_\beta) \right\|.$$

(93)

Note that $P = P^*$ implies

$$(J e^{-\frac{\beta}{2}H_\beta} \otimes 1) P (\Omega_\beta \otimes \Omega_\beta) = \sum_i J e^{-\frac{\beta}{2}H_\beta} P_i^{(1)} (\Omega_\beta \otimes P_i^{(2)} (\Omega_\beta)

= \sum_i (P_i^{(1)})^* (\Omega_\beta \otimes P_i^{(2)} (\Omega_\beta) = P (\Omega_\beta \otimes \Omega_\beta).$$

(94)

The same argument applies to the first term on the r.h.s. in (93). Moreover, $\|\Omega_\beta\| = 1$, $\|P\| = 1$, $\|Q\| = 1$, and $\|U_p(s)\| = 1$ for all $s \in \mathbb{R}$. By an application of the maximum modulus principle we obtain

$$|f_{t_1^{(i)},...,t_n^{(i)}}(z)| \leq 1 \quad \forall z \in G_{\delta/2}.$$  

(95)

By assumption $QP = 0$, hence

$$f_{t_1^{(i)},...,t_n^{(i)}}(-t_j^{(i)}) = 0.$$  

(96)

We conclude that inside the circle $|z| < \delta/2$ each of the functions $f_{t_1^{(i)},...,t_n^{(i)}}$ possesses $n$ zeros for pairwise different values of $t_j^{(i)}$. Thus all of the functions

$$g_{t_1^{(i)},...,t_n^{(i)}}(z) := \frac{f_{t_1^{(i)},...,t_n^{(i)}}(z)}{\prod_{j=1}^n (z + t_j^{(i)})}, \quad i \in \mathbb{N}, \quad j = \{1, \ldots, n\},$$

(97)
are analytic in the open disc $D_{\delta/2}$ of radius $\delta/2$ and centered at the origin. Note that by
definition $D_{\delta/2} \subset G_{\delta/2}$. Yet the number of zeros does not change in the limit $t_j^{(i)} \to 0$ and
consequently, for $i > 1$,
\[
\left| g_{t_1^{(i)}, \ldots, t_n^{(i)}}(z) \right| \leq \sup_{w \in \partial D_{\delta/2}} \frac{|f_{t_1^{(i)}, \ldots, t_n^{(i)}}(w)|}{\prod_{j=1}^n|w + t_j^{(i)}|} \leq \left(\frac{4}{\delta}\right)^n \quad \forall z \in D_{\delta/2}.
\] (98)

In the last inequality we used $|w + t_j^{(i)}| \geq |w| - |t_j^{(i)}|$ and $|w| = \delta/2$ together with $|t_j^{(i)}| < \delta/4$ for $i > 1$. Hence,
\[
|f_{t_1^{(i)}, \ldots, t_n^{(i)}}(z)| \leq \left(\frac{4}{\delta}\right)^n \prod_{j=1}^n|z + t_j^{(i)}| \leq \text{const} \cdot |z|^n \quad \forall z \in D_{\delta/2}.
\] (99)

Because of $Q^2 = Q$, $f_{0,\ldots,0}$ coincides with $f_{P,Q}$. The group $t \mapsto U_p(t)$ is strongly continuous, thus
\[
|f_{t_1^{(i)}, \ldots, t_n^{(i)}}(z) - f_{0,\ldots,0}(z)| \to 0 \quad \text{for} \quad t_j^{(i)} \to 0, \quad j = 1, \ldots, n,
\] (100)
uniformly in $z \in G_{\delta/2}$. Hence 0 is a zero of $n$-th order:
\[
|f_{P,Q}(z)| \leq \text{const} \cdot |z|^n \quad \forall z \in D_{\delta/2}.
\] (101)

Since $n \in \mathbb{N}$ was arbitrary, we conclude that $f_{P,Q}$ vanishes identically. \qed

**Lemma 4.9.** The vector $\Omega_\beta \otimes \Omega_\beta \in \mathcal{H}_\beta \otimes \mathcal{H}_\beta$ is cyclic for $\pi_p(\mathcal{C}_\beta(\mathcal{O}, \hat{\mathcal{O}}))''$ and $E_p(\Omega_\beta \otimes \Omega_\beta)$
is separating for $\pi_p(\mathcal{C}_\beta(\mathcal{O}, \hat{\mathcal{O}}))''$, where $E_p \in \pi_p(\mathcal{C}_\beta(\mathcal{O}_1, \mathcal{O}_2))'$
denotes the projection onto the subspace $\mathcal{K}_p \subset \mathcal{H}_\beta \otimes \mathcal{H}_\beta$ reducing $\pi_p$.

**Proof.** By the Reeh–Schlieder theorem $\Omega_\beta \otimes \Omega_\beta$ is cyclic for
\[
\mathcal{R}_\beta(\mathcal{O})' \otimes \mathcal{R}_\beta(\hat{\mathcal{O}}) \subset \pi_p(\mathcal{C}_\beta(\mathcal{O}, \hat{\mathcal{O}}))''.
\] (102)

By assumption
\[
\mathcal{O} \subset \subset \mathcal{O}_1 \subset \subset \mathcal{O}_2 \subset \subset \hat{\mathcal{O}}.
\] (103)

It follows from the general theory of intersections of $W^*$-tensor products [Ta] that
\[
\pi_p(\mathcal{C}_\beta(\mathcal{O}, \hat{\mathcal{O}}))' \cap \pi_p(\mathcal{C}_\beta(\mathcal{O}_1, \mathcal{O}_2))'' \supset \left(\mathcal{R}_\beta(\mathcal{O}) \otimes \mathcal{R}_\beta(\hat{\mathcal{O}})\right)' \cap \left(\mathcal{R}_\beta(\mathcal{O}_1) \otimes \mathcal{R}_\beta(\mathcal{O}_2)'\right)
\[
= \left(\mathcal{R}_\beta(\mathcal{O})' \cap \mathcal{R}_\beta(\mathcal{O}_1)\right) \otimes \left(\mathcal{R}_\beta(\hat{\mathcal{O}}) \cap \mathcal{R}_\beta(\mathcal{O}_2)'\right)
\[
\supset \left(\mathcal{R}_\beta(\mathcal{O})' \cap \mathcal{R}_\beta(\mathcal{O}_1)\right) \otimes \left(\mathcal{R}_\beta(\hat{\mathcal{O}}) \cap \mathcal{R}_\beta(\mathcal{O}_2)'\right).
\] (104)
By assumption, both
\[ \mathcal{O}' \cap O_1 \quad \text{and} \quad \hat{O} \cap O'_2 \] (105)
contain open subsets. Thus, due to the Reeh–Schlieder property, \( \Omega_\beta \otimes \Omega_\beta \) is cyclic for
\[ \pi_p(\mathcal{C}_\beta(\mathcal{O}, \hat{\mathcal{O}}))' \cap \pi_p(\mathcal{C}_\beta(\mathcal{O}_1, \mathcal{O}_2))'' \] and therefore separating for
\[ \pi_p(\mathcal{C}_\beta(\mathcal{O}, \hat{\mathcal{O}}))'' \lor \pi_p(\mathcal{C}_\beta(\mathcal{O}_1, \mathcal{O}_2))'. \] (106)

Now let \( Z_p \in \pi_p(\mathcal{C}_\beta(\mathcal{O}, \hat{\mathcal{O}}))'' = R_\beta(\mathcal{O}) \otimes R_\beta(\hat{\mathcal{O}})' \) be some projection such that
\[ Z_p E_p (\Omega_\beta \otimes \Omega_\beta) = 0. \] (107)
Since \( E_p \in \pi_p(\mathcal{C}_\beta(\mathcal{O}_1, \mathcal{O}_2))' \), it follows that \( Z_p E_p \in \pi_p(\mathcal{C}_\beta(\mathcal{O}, \hat{\mathcal{O}}))'' \lor \pi_p(\mathcal{C}_\beta(\mathcal{O}_1, \mathcal{O}_2))' \) and consequently \( Z_p E_p = 0 \). Moreover,
\[ [U_p(t)Z_pU_p(-t), E_p] = 0 \quad \forall t \in U, \] (108)
where \( U \) denotes some open neighborhood of the origin in \( \mathbb{R} \). According to Lemma 4.8
\( Z_p E_p = 0 \) now implies
\[ \left( \Omega_\beta \otimes \Omega_\beta, Z_p U_p(t)E_p(\Omega_\beta \otimes \Omega_\beta) \right) = 0 \quad \forall t \in \mathbb{R}. \] (109)

Now \( \Omega_\beta \otimes \Omega_\beta \) is the unique — up to a phase — normalized, invariant eigenvector for the
one-parameter group \( t \mapsto U_p(t) \). Thus, by the same argument as in the proof of Lemma 3.6,
\[ E_p(\Omega_\beta \otimes \Omega_\beta) \neq 0 \Rightarrow Z_p(\Omega_\beta \otimes \Omega_\beta) = 0, \] (110)
which is only possible if \( Z_p = 0 \). It follows that the vector \( E_p(\Omega_\beta \otimes \Omega_\beta) \) is separating
for \( \pi_p(\mathcal{C}_\beta(\mathcal{O}, \hat{\mathcal{O}}))'' \). \( \square \)

**Corollary 4.10.** Let \( E_p \) denote the projection onto the subspace \( \mathcal{K}_p \subset \mathcal{H}_\beta \otimes \mathcal{H}_\beta \) reduc-
ing \( \pi_p \) to \( \hat{\pi}_p \). It follows that \( E_p \in \pi_p(\mathcal{C}_\beta(\mathcal{O}_1, \mathcal{O}_2))' \) can be represented in the form
\[ E_p = V_p V_p^*, \quad \text{where} \quad V_p \in \pi_p(\mathcal{C}_\beta(\mathcal{O}, \hat{\mathcal{O}}))' \] (111)
is an isometry, i.e., \( V_p^* V_p = 1_{\mathcal{H}_\beta \otimes \mathcal{H}_\beta} \) and \( V_p V_p^* = 1_{\mathcal{K}_p} \).

The proof of this result is — up to notation — identical with the proof of Corollary 3.7, therefore we do not repeat the argument.

We summarize our result in the following
Theorem 4.11. Assume a TFT is specified by a net
\[ \mathcal{O} \to \mathcal{R}_\beta(\mathcal{O}), \quad \mathcal{O} \subset \mathbb{R}^4, \tag{112} \]
of von Neumann algebras, subject to the standard assumptions stated explicitly on p.6. Furthermore assume that for any bounded space–time region \( \mathcal{O} \) the maps \( \Theta_{\lambda, \mathcal{O}}: \mathcal{R}_\beta(\mathcal{O}) \to \mathcal{H}_\beta, \)
\[ A \mapsto e^{-\lambda |H_\beta|} A \Omega_\beta, \tag{113} \]
are of type \( s \) (order 0) for any \( \lambda > 0 \). It follows that for any inclusion of open bounded space–time regions \( \mathcal{O} \subset \subset \hat{\mathcal{O}} \), there exists a type I factor \( \mathcal{N}_\beta(\mathcal{O}, \hat{\mathcal{O}}) \) such that
\[ \mathcal{R}_\beta(\mathcal{O}) \subset \mathcal{N}_\beta(\mathcal{O}, \hat{\mathcal{O}}) \subset \mathcal{R}_\beta(\hat{\mathcal{O}}), \tag{114} \]
provided the closure of \( \mathcal{O} \) is contained in the interior of \( \hat{\mathcal{O}} \).

Proof. Theorem 4.5 ensures that there exists a unitary operator \( W \) mapping \( \mathcal{H}_\beta \) onto \( \mathcal{H}_\beta \otimes \mathcal{H}_\beta \) such that
\[ WABW^{-1} = A \otimes B \tag{115} \]
for all \( A \in \mathcal{R}_\beta(\mathcal{O}_1) \) and all \( B \in \mathcal{R}_\beta(\mathcal{O}_2)' \). Set \( \mathcal{N}_\beta := W^*\left( B(\mathcal{H}_\beta) \otimes 1 \right) W \). Clearly \( \mathcal{N}_\beta \) is a type I factor and since there holds the trivial inclusion
\[ W^{-1}\left( \mathcal{R}_\beta(\mathcal{O}) \otimes 1 \right) W \subset W^{-1}\left( B(\mathcal{H}_\beta) \otimes 1 \right) W \subset \left( W^{-1}\left( 1 \otimes \mathcal{R}_\beta(\mathcal{O}_2) \right) W \right)' \tag{116} \]
we arrive at (114).

5. Equivalent Formulations of the Split Property

We start with the following

Theorem 5.1. Let \( \mathcal{O} \) be a bounded space–time region such that the closure of \( \mathcal{O} \) is contained in the interior of \( \hat{\mathcal{O}} \). Then the following five conditions are equivalent:

(i) (Split property). There exists a type I factor \( \mathcal{N}_\beta \) such that
\[ \mathcal{R}_\beta(\mathcal{O}) \subset \mathcal{N}_\beta \subset \mathcal{R}_\beta(\hat{\mathcal{O}}). \tag{117} \]

(ii) (Existence of normal product state extensions for partial states). For any pair of normal states \( \phi_1 \) of \( \mathcal{R}_\beta(\mathcal{O}) \) and \( \phi_2 \) of \( \mathcal{R}_\beta(\hat{\mathcal{O}})' \) there exists a normal state \( \phi_{1,2} \) on \( B(\mathcal{H}_\beta) \) which is an extension of \( \phi_1 \) and \( \phi_2 \) and a product state for the von Neumann algebras \( \mathcal{R}_\beta(\mathcal{O}) \) and \( \mathcal{R}_\beta(\hat{\mathcal{O}})' \).
(iii) (Existence of a normal product state). There exists a normal state $\varphi$ on $B(H_\beta)$ which is a product state for the von Neumann algebras $R_\beta(\mathcal{O})$ and $R_\beta(\hat{\mathcal{O}})'$.

(iv) (Existence of a faithful normal product state extension of the KMS state). There exists a normal product state $\omega_p$ on $R_\beta(\mathcal{O}) \vee R_\beta(\hat{\mathcal{O}})'$ such that

$$\omega_p(AB) = (\Omega_\beta, A\Omega_\beta)(\Omega_\beta, B\Omega_\beta) \quad (118)$$

for all $A \in R_\beta(\mathcal{O})$ and $B \in R_\beta(\hat{\mathcal{O}})'$. Moreover, $\omega_p$ is faithful on the von Neumann algebra $R_\beta(\mathcal{O}) \vee R_\beta(\hat{\mathcal{O}})'$.

(v) (Canonical cyclic and separating product vector). There exists a unique vector $\eta \in H_\beta$ in the natural positive cone $\mathcal{P}_{\Omega_\beta}(R_\beta(\mathcal{O}) \vee R_\beta(\hat{\mathcal{O}})')$ such that

a.) $(\eta, AB\eta) = (\Omega_\beta, A\Omega_\beta)(\Omega_\beta, B\Omega_\beta)$ for all $A \in R_\beta(\mathcal{O})$ and $B \in R_\beta(\hat{\mathcal{O}})'$.

b.) $\eta$ is cyclic and separating for $R_\beta(\mathcal{O}) \vee R_\beta(\hat{\mathcal{O}})'$.

(vi) (Statistical independence). The von Neumann algebra generated by $R_\beta(\mathcal{O})$ and $R_\beta(\hat{\mathcal{O}})'$ is isomorphic to the $W^*$-tensor product of the two algebras. This means that there exists a unitary operator $W: H_\beta \to H_\beta \otimes H_\beta$ such that

$$WABW^* = A \otimes B \quad (119)$$

for all $A \in R_\beta(\mathcal{O})$ and $B \in R_\beta(\hat{\mathcal{O}})'$ and, hence, locality is reflected in an especially simple algebraic structure of the net $\mathcal{O} \to R_\beta(\mathcal{O})$.

\textbf{Proof.} i) $\Rightarrow$ ii) The KMS vector $\Omega_\beta$ is cyclic and separating for $R_\beta(\mathcal{O})$, $R_\beta(\hat{\mathcal{O}})$ and $R_\beta(\mathcal{O})' \cap R_\beta(\hat{\mathcal{O}})$; and therefore this $W^*$-Split-inclusion is standard. Consequently, the underlying Hilbert space $H_\beta$ is separable and infinite dimensional [DL, Prop. 1.6]: The KMS state is faithful w.r.t. $R_\beta$, $N_\beta$ is countably decomposable, hence separable in the ultraweak topology (being of type I). It follows that

$$H_\beta = N_\beta\Omega_\beta \quad (120)$$

is separable. All infinite type I factors with infinite commutant on a separable Hilbert space are unitarily equivalent to $B(H_\beta) \otimes \mathbb{1}$ [KR, Ch. 9.3]. It follows that there exists a unitary operator $W: H_\beta \to H_\beta \otimes H_\beta$ such that

$$N_\beta = W^*(B(H_\beta) \otimes \mathbb{1})W \quad (121)$$

The split property (117) implies

$$WR_\beta(\mathcal{O})W^* \subset B(H_\beta) \otimes \mathbb{1} \subset WR_\beta(\hat{\mathcal{O}})W^* \quad (122)$$

and $R_\beta(\mathcal{O})' \supset N_\beta' \supset R_\beta(\hat{\mathcal{O}})'$. It follows that

$$WR_\beta(\mathcal{O})'W^* \supset \mathbb{1} \otimes B(H_\beta) \supset WR_\beta(\hat{\mathcal{O}})'W^* \quad (123)$$
Let $\phi_1$ and $\phi_2$ denote two normal states over $\mathcal{R}_\beta(\mathcal{O})$ and $\mathcal{R}_\beta(\hat{\mathcal{O}})'$, respectively. Set

$$\phi_{1,2} := (\phi_1 \otimes \phi_2)^W,$$

(124)

where $\phi^W(C) := \phi(WCW^*)$ for all $C \in \mathcal{R}_\beta(\mathcal{O}) \lor \mathcal{R}_\beta(\hat{\mathcal{O}})'$. The state $\phi_{1,2}$ is normal and satisfies

$$\phi_{1,2}(AB) = \phi_1(A)\phi_2(B)$$

(125)

for all $A \in \mathcal{R}_\beta(\mathcal{O})$ and $B \in \mathcal{R}_\beta(\hat{\mathcal{O}})'$.

(ii) $\Rightarrow$ (iii) is trivial.

(iii) $\Rightarrow$ (iv) Let $\mathcal{O}_0$, $\mathcal{O}_1$, $\mathcal{O}_2$ and $\mathcal{O}_3$ denote space–time regions such that

$$\mathcal{O} + te \subset \mathcal{O}_0 \subset \mathcal{O}_1 \subset \mathcal{O}_2 \subset \mathcal{O}_3 + te \subset \hat{\mathcal{O}} \text{ for } |t| < \delta/3.$$

From (iii) we conclude that there exists a normal product state $\hat{\phi}$ for the pair $\mathcal{R}_\beta(\mathcal{O}_1)$ and $\mathcal{R}_\beta(\mathcal{O}_2)'$. Moreover, $\mathcal{R}_\beta(\mathcal{O}_1) \lor \mathcal{R}_\beta(\mathcal{O}_2)'$ has a cyclic and separating vector, namely $\Omega_\beta$. It follows (see [BR, 2.5.31]) that there exists a vector $\hat{\xi} \in \mathcal{H}_\beta$ such that

$$\hat{\phi}(C) = (\hat{\xi}, C\hat{\xi}) \quad \forall C \in \mathcal{R}_\beta(\mathcal{O}_1) \lor \mathcal{R}_\beta(\mathcal{O}_2)'.$$ 

(127)

The following argument is due to Buchholz [Bu]: Let $P_1$, $P_2$ be the projections onto the closed subspaces $\mathcal{R}_\beta(\mathcal{O}_1)\hat{\xi}$ and $\mathcal{R}_\beta(\mathcal{O}_2)'\hat{\xi}$ of $\mathcal{H}_\beta$. It is obvious that $P_1 \in \mathcal{R}_\beta(\mathcal{O}_1)'$ and $P_2 \in \mathcal{R}_\beta(\mathcal{O}_2)$. From the factorization property of $\hat{\xi}$ it follows that

$$P_1BP_1 = (\hat{\xi}, B\hat{\xi}) \cdot P_1 \quad \forall B \in \mathcal{R}_\beta(\mathcal{O}_2)'$$

(128)

and

$$P_2AP_2 = (\hat{\xi}, A\hat{\xi}) \cdot P_2 \quad \forall A \in \mathcal{R}_\beta(\mathcal{O}_1).$$

(129)

Therefore the state

$$\omega_1(.) := \frac{(P_1\Omega_\beta, P_1\Omega_\beta)}{\|P_1\Omega_\beta\|^2}$$

is again a product state for $\mathcal{R}_\beta(\mathcal{O}_1)$ and $\mathcal{R}_\beta(\mathcal{O}_2)'$. Now assume

$$\omega_1(A^*A) = 0 \quad \text{for} \quad A \in \mathcal{R}_\beta(\mathcal{O}_0).$$

(131)

The KMS vector $\Omega_\beta$ is separating for $\mathcal{R}_\beta(\mathcal{O}_0) \lor \mathcal{R}_\beta(\mathcal{O}_1)'$, thus

$$AP_1\Omega_\beta = 0 \Rightarrow AP_1 = 0.$$ 

(132)

The Schlieder property for $\mathcal{R}_\beta(\mathcal{O}_0)$ and $\mathcal{R}_\beta(\mathcal{O}_1)'$ implies $A = 0$ or $P_1 = 0$. We conclude that $\omega_1$ is faithful for $\mathcal{R}_\beta(\mathcal{O}_0)$. It follows (see [BR, 2.5.31]) that there exists a vector $\xi_1 \in$
$\mathcal{H}_\beta$, cyclic and separating for $\mathcal{R}_\beta(\mathcal{O}_c)$, which represents the restriction of $\omega_1$ to $\mathcal{R}_\beta(\mathcal{O}_c)$. Consequently, we can construct in a canonical way an isometric operator $U_1 \in \mathcal{R}_\beta(\mathcal{O}_1)'$:

$$U_1 A\xi_1 := A \cdot \frac{P_1 \Omega_\beta}{\|P_1 \Omega_\beta\|} \quad \text{for} \quad A \in \mathcal{R}_\beta(\mathcal{O}_c). \quad (133)$$

It is evident that the range of $U_1$ equals $P_1 \mathcal{H}_\beta$; thus

$$U_1 U_1^* = P_1 \quad \text{and} \quad U_1^* U_1 = 1. \quad (134)$$

From (134) and the relation $P_1 B P_1 = (\hat{\xi}, B \hat{\xi}) \cdot P_1$ we get

$$(A\xi_1, U_1^* B U_1 A\xi_1) = (U_1 A\xi_1, P_1 B P_1 U_1 A\xi_1)
= (\hat{\xi}, B \hat{\xi})(\xi_1, A^* A\xi_1) \quad (135)$$

The cyclicity of $\xi_1$ w.r.t. $\mathcal{R}_\beta(\mathcal{O}_c)$ implies

$$U_1^* B U_1 = (\hat{\xi}, B \hat{\xi}) \cdot 1 \quad \text{for} \quad B \in \mathcal{R}_\beta(\mathcal{O}_2)'. \quad (136)$$

Therefore the state

$$\omega_1(\cdot) := (U_1 \Omega_\beta, .U_1 \Omega_\beta) \quad (137)$$

is a product state for $\mathcal{R}_\beta(\mathcal{O}_c)$ and $\mathcal{R}_\beta(\mathcal{O}_2)'$ and the restriction of $\omega_1$ to $\mathcal{R}_\beta(\mathcal{O}_c)$ coincides with the restriction of the KMS state $\omega_\beta$ to this algebra. If one carries through the whole construction once more starting with $\hat{\phi}$ instead of $\hat{\phi}$, then one gets a product state $\hat{\omega}_p$ for $\mathcal{R}_\beta(\mathcal{O}_c)$ and $\mathcal{R}_\beta(\mathcal{O}_3)'$ which coincides with the vector state induced by $\Omega_\beta$ on each algebra separately.

By a suitable smoothing procedure in the time variable we can now construct a faithful normal product state $\omega_h$ for $\mathcal{R}_\beta(\mathcal{O}_c) \lor \mathcal{R}_\beta(\hat{\mathcal{O}})'$ such that $\omega_h$ coincides with the vector state induced by $\Omega_\beta$ on both algebras: Let $\hat{\chi}$ denote the vector in the natural positive cone

$$\mathcal{P}^\natural_{\Omega_\beta}(\mathcal{R}_\beta(\mathcal{O}_c) \lor \mathcal{R}_\beta(\mathcal{O}_3)') \quad (138)$$

which induces $\hat{\omega}_p$ on $\mathcal{R}_\beta(\mathcal{O}_c) \lor \mathcal{R}_\beta(\mathcal{O}_3)'$ (see once again [BR, 2.5.31]). It follows that there exists an isometry $I$ which satisfies

$$I T \Omega_\beta = T \hat{\chi}, \quad \forall T \in \mathcal{R}_\beta(\mathcal{O}_3)'. \quad (139)$$

Thus $I \in \mathcal{R}_\beta(\mathcal{O}_3)$ and $I \Omega_\beta \in \mathcal{D}(e^{-\lambda H_3})$ for all $0 \leq \lambda \leq \beta/2$. This property implies that for any non-zero operator $C \in \mathcal{R}_\beta(\mathcal{O}_c) \lor \mathcal{R}_\beta(\mathcal{O}_3)'$ the set

$$\{ t \in \mathbb{R} : Ce^{itH_3} I \Omega_\beta \neq 0 \} \quad (140)$$

is dense in $\mathbb{R}$. The details are as follows: assume there exists some interval $]t_1, t_2[$ such that

$$Ce^{itH_3} I \Omega_\beta = 0 \quad \forall t \in ]t_1, t_2[. \quad (141)$$
The vector-valued function

\[ z \mapsto C e^{izH_\beta} I \Omega_\beta, \quad 0 < \Im z < \beta/2, \]  

is analytic in the strip \( 0 < \Im z < \beta/2 \) and continuous for \( \Im z \downarrow 0 \). Thus (141) implies that the function defined in (142) vanishes identically. By assumption, \( \Omega_\beta \) is the unique — up to a phase — time invariant vector in \( \mathcal{H}_\beta \). Taking an appropriate mean over the real axis we find

\[ 0 = C \Omega_\beta \cdot (\Omega_\beta, I \Omega_\beta). \]  

(143)

Now \( I \in \mathcal{R}_\beta(\mathcal{O}_3) \), \( \omega_\beta \) is faithful for \( \mathcal{R}_\beta(\mathcal{O}_3) \) and \( \Omega_\beta \) is separating for \( \mathcal{R}_\beta(\mathcal{O}_3) \lor \mathcal{R}_\beta(\mathcal{O}_3)' \). Therefore (143) implies \( C = 0 \) in contradiction to the assumption that \( C \) is non-zero. Therefore the set (140) is dense in \( \mathbb{R} \). Now let \( h \in L_1(\mathbb{R}) \) be a smooth positive function with support \( ] - \delta/3, \delta/3[ \) and \( \|h\|_1 = 1 \). Locality together with (126) implies that

\[ \mathcal{R}_\beta(\mathcal{O}) \lor \mathcal{R}_\beta(\hat{\mathcal{O}})' \ni C \mapsto \omega_h(C) = \int_{-\delta/3}^{\delta/3} dt \, h(t)(\hat{\chi}, e^{-itH_\beta} C e^{itH_\beta} \hat{\chi}) \]  

(144)

defines a product state for the pair \( \mathcal{R}_\beta(\mathcal{O}) \) and \( \mathcal{R}_\beta(\hat{\mathcal{O}})' \). In fact,

\[ \omega_h(AB) = \int_{-\delta/3}^{\delta/3} dt \, h(t)(\Omega_\beta, e^{-itH_\beta} A e^{itH_\beta} \Omega_\beta)(\Omega_\beta, e^{-itH_\beta} B e^{itH_\beta} \Omega_\beta) = (\Omega_\beta, A \Omega_\beta)(\Omega_\beta, B \Omega_\beta), \quad \text{for} \quad A \in \mathcal{R}_\beta(\mathcal{O}), \quad B \in \mathcal{R}_\beta(\hat{\mathcal{O}})' \]  

(145)

Thus the restriction \( \omega_p \) of \( \omega_h \) to the algebra \( \mathcal{R}_\beta(\mathcal{O}) \lor \mathcal{R}_\beta(\hat{\mathcal{O}})' \) is independent of \( h \) and coincides with the vector state induced by \( \Omega_\beta \) on both algebras. Moreover, combining (140) and (144) we conclude that \( \omega_p \) is faithful on \( \mathcal{R}_\beta(\mathcal{O}) \lor \mathcal{R}_\beta(\hat{\mathcal{O}})' \).

(iv) \( \Rightarrow \) (v) From [BR, 2.5.31] we infer that there exists a unique vector \( \eta \) in the natural positive cone \( \mathcal{P}^2_{\Omega_\beta}(\mathcal{R}_\beta(\mathcal{O}) \lor \mathcal{R}_\beta(\hat{\mathcal{O}})') \) such that

\[ (\eta, C \eta) = \omega_p(C) \quad \forall C \in \mathcal{R}_\beta(\mathcal{O}) \lor \mathcal{R}_\beta(\hat{\mathcal{O}})' \]  

(146)

Moreover, \( \omega_p \) is faithful on \( \mathcal{R}_\beta(\mathcal{O}) \lor \mathcal{R}_\beta(\hat{\mathcal{O}})' \). Thus \( \eta \) is cyclic and separating for \( \mathcal{R}_\beta(\mathcal{O}) \lor \mathcal{R}_\beta(\hat{\mathcal{O}})' \).

(v) \( \Rightarrow \) (vi) Let \( W_\eta \) be given by linear extension of

\[ W_\eta AB \eta = A \Omega_\beta \otimes B \Omega_\beta. \]  

(147)

Because of (v) (b) \( W_\eta \) is densely defined and isometric. Due to the Reeh–Schlieder property of the KMS vector \( \Omega_\beta \) the range of \( W_\eta \) is dense in \( \mathcal{H}_\beta \otimes \mathcal{H}_\beta \) too. Thus \( W_\eta \) can be extended to a unitary operator \( W: \mathcal{H}_\beta \rightarrow \mathcal{H}_\beta \otimes \mathcal{H}_\beta \). From (147) we infer

\[ WABW^* = A \otimes B \]  

(148)

for all \( A \in \mathcal{R}_\beta(\mathcal{O}) \) and \( B \in \mathcal{R}_\beta(\hat{\mathcal{O}})' \).

vi) \( \Rightarrow \) i) This part has been provided in the proof of Theorem 4.11. \( \square \)
Remark. Property (vi) implies that the state $\omega_p$ specified in (iv) is uniquely fixed by the factorization property

$$\omega_p(AB) = (\Omega_\beta, A\Omega_\beta)(\Omega_\beta, B\Omega_\beta), \quad \forall A \in \mathcal{R}_\beta(\mathcal{O}), \quad B \in \mathcal{R}_\beta(\hat{\mathcal{O}})'.$$  \hfill (149)

The split property has many interesting implications which will be discussed in our next paper; here we will only quote one more result of Buchholz [Bu]:

**Corollary 5.2.** Assume the inclusion $\mathcal{R}_\beta(\mathcal{O}) \subset \mathcal{R}_\beta(\hat{\mathcal{O}})$ is split and let $\mathcal{O}_a$ and $\mathcal{O}_b$ denote two regions contained in $\mathcal{O}$. If $\Phi$ is an isomorphism which maps $\mathcal{R}_\beta(\mathcal{O}_a)$ onto $\mathcal{R}_\beta(\mathcal{O}_b)$, then $\Phi$ can be implemented by a unitary operator $U \in \mathcal{N}_\beta$:

$$\Phi(A) = UAU^{-1}. \quad \hfill (150)$$

Hence $\Phi$ acts trivially on $\mathcal{R}_\beta(\hat{\mathcal{O}})'$.

**Proof.** Once the existence of a cyclic and separating product vector has been shown for $\mathcal{R}_\beta(\mathcal{O})$ and $\mathcal{R}_\beta(\hat{\mathcal{O}})'$, Buchholz’s result follows by the original arguments. We present them here for completeness only. Let $\eta$ denote the product vector specified in Theorem 5.1. v.) and $P_1$ the projection onto $\overline{\mathcal{R}_\beta(\mathcal{O})}\eta \subset \mathcal{H}_\beta$. Clearly, $\overline{\mathcal{R}_\beta(\mathcal{O})}\eta$ is invariant under the action of $\mathcal{N}_\beta$. Thus we can consider the induced representation $\pi_{P_1}$ of $\mathcal{N}_\beta$ on $\overline{\mathcal{R}_\beta(\mathcal{O})}\eta$. Since $\eta \in P_1\mathcal{H}_\beta$, this representation is faithful and it is easy to verify that

$$\pi_{P_1}(\mathcal{N}_\beta) = B(P_1\mathcal{H}_\beta). \quad \hfill (151)$$

Now $\pi_{P_1}(\mathcal{R}_\beta(\mathcal{O}_a)) \subset \pi_{P_1}(\mathcal{N}_\beta)$ and $\pi_{P_1}(\mathcal{R}_\beta(\mathcal{O}_b)) \subset \pi_{P_1}(\mathcal{N}_\beta)$ both have a cyclic vector, namely $P_1\Omega_\beta \in P_1\mathcal{H}_\beta$ and a separating vector, namely $\eta \in P_1\mathcal{H}_\beta$. Hence every isomorphism which maps $\pi_{P_1}(\mathcal{R}_\beta(\mathcal{O}_a)))$ onto $\pi_{P_1}(\mathcal{R}_\beta(\mathcal{O}_b))$ is spatial [Di p.222, Theorem 3]. Thus there exists a unitary operator $U \in \pi_{P_1}(\mathcal{N}_\beta)$ such that

$$\pi_{P_1} \circ \Phi(A) = U\pi_{P_1}(A)U^{-1} \quad \forall A \in \mathcal{R}_\beta(\mathcal{O}_a), \quad \hfill (152)$$

and from this relation the statement follows immediately. 

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6. Some more Remarks

Let $\eta$ denote the product vector constructed in Theorem 5.1. v.). The set

$$\mathcal{L}_\beta(\mathcal{O}, \hat{\mathcal{O}}) := \overline{\mathcal{R}_\beta(\mathcal{O})}\eta$$

is a convenient linear subset of the set of strictly localized thermal excitations $\mathcal{L}_\beta(\hat{\mathcal{O}})$ defined in (25). In fact (see [BJ b]),
(i) \( \mathcal{L}_\beta(\mathcal{O}, \hat{\mathcal{O}}) \) is a closed subspace of \( \mathcal{H}_\beta \);
(ii) \( \mathcal{L}_\beta(\mathcal{O}, \hat{\mathcal{O}}) \) is invariant under the action of \( \mathcal{R}_\beta(\mathcal{O}) \);
(iii) The vectors of \( \mathcal{L}_\beta(\mathcal{O}, \hat{\mathcal{O}}) \) induce product states on \( \mathcal{R}_\beta(\mathcal{O}) \vee \mathcal{R}_\beta(\hat{\mathcal{O}})' \), which coincide with the vector state induced by the KMS vector \( \Omega_\beta \) on \( \mathcal{R}_\beta(\hat{\mathcal{O}})' \): if \( \Psi \in \mathcal{L}_\beta(\mathcal{O}, \hat{\mathcal{O}}) \), then

\[
(\Psi, AB\Psi) = (\Psi, A\Psi)(\Omega_\beta, B\Omega_\beta)
\]

for all \( A \in \mathcal{R}_\beta(\mathcal{O}) \) and \( B \in \mathcal{R}_\beta(\hat{\mathcal{O}})' \).
(iv) \( \mathcal{L}_\beta(\mathcal{O}, \hat{\mathcal{O}}) \) is complete in the following sense: to every normal state \( \phi \) on \( \mathcal{R}_\beta(\mathcal{O}) \) there exists a \( \Phi \in \mathcal{L}_\beta(\mathcal{O}, \hat{\mathcal{O}}) \) such that

\[
(\Phi, A\Phi) = \phi(A)
\]

for all \( A \in \mathcal{R}_\beta(\mathcal{O}) \).

Property (iv) can be seen as follows: Since \( \mathcal{R}_\beta(\mathcal{O}) \) has a cyclic and separating vector, there exists a vector \( \tilde{\Phi} \in \mathcal{H}_\beta \) which induces the given normal state \( \phi \) on \( \mathcal{R}_\beta(\mathcal{O}) \). Using the isomorphism specified in (119) we find that \( \Phi := W^*(\tilde{\Phi} \otimes \Omega_p) \in \mathcal{H}_\Lambda \) satisfies (155).

It was noticed by Buchholz, D’Antoni and Longo that the split property imposes certain restrictions on the energy level density of excitations of the KMS state described by the state vectors of \( \mathcal{S}_\beta(\mathcal{O}, \lambda) \) [BD’AL b]:

**Theorem 6.1.** Consider a TFT, specified by a von Neumann algebra \( \mathcal{R}_\beta \) with a cyclic and separating vector \( \Omega_\beta \) and a net of subalgebras \( \mathcal{O} \rightarrow \mathcal{R}_\beta(\mathcal{O}) \), subject to the conditions (i) and (ii) stated on p. 6. Assume the inclusion \( \mathcal{R}_\beta(\mathcal{O}) \subset \mathcal{R}_\beta(\hat{\mathcal{O}})' \) is split. Then the maps

\[
\Theta_{\lambda, \mathcal{O}} : \mathcal{R}_\beta(\mathcal{O}) \rightarrow \mathcal{H}_\beta \quad A \mapsto e^{-\lambda H_\beta}A\Omega_\beta,
\]

are compact for \( 0 < \lambda < \beta/2 \). I.e., the set

\[
\mathcal{S}_\beta(\mathcal{O}, \lambda) := \{e^{-\lambda|H_\beta|}A\Omega_\beta : A \in \mathcal{R}_\beta(\mathcal{O}), \|A\| \leq 1\}
\]

is relatively compact in the norm topology for all \( \lambda > 0 \).

**Proof.** The first statement is a consequence of [BD’AL b, Proposition 4.2.] and [BD’AL b, Lemma 3.1.i.]. The second statement follows from an arguments, which we have already reproduced in the proof of Lemma 4.1., and which is also due to Buchholz, D’Antoni and Longo. \( \Box \)
Remark. As pointed out in [BD’AL b], it is clear that these limitations cannot be relaxed. Since
\[ e^{-\frac{\hat{\beta}}{2} H_\beta} A \Omega_\beta = J A^* \Omega_\beta, \] (158)
the map \( \Theta_{\lambda, \mathcal{O}} \) is not even compact for \( \lambda = \beta/2 \).

Our nuclearity condition relies on decent infrared properties of the generator of the time evolution. Our arguments are less conclusive, if \( \omega_\beta \) describes a physical system at a critical point. But if the split property holds in the vacuum sector, then it holds also in the GNS representation associated with any thermal state which is locally normal w.r.t. the vacuum representation. Thus even at a critical point the maps \( \Theta_{\lambda, \mathcal{O}} \) should at least be compact for \( 0 < \lambda < \beta/2 \), as long as the corresponding KMS state is locally normal w.r.t. the vacuum representation. However, there is the possibility that infrared divergencies might destroy local normality (see e.g. [BJ a][BR, Ex. 5.4.15]). Despite the general belief that in 3+1 space–time dimensions all states of physical interest should be locally normal to each other, we can not rule out this possibility.

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