Spectral extremal results on the $\alpha$-index of graphs without minors and star forests

Ming-Zhu Chen, A-Ming Liu,  
School of Science, Hainan University, Haikou 570228, P. R. China,  
Xiao-Dong Zhang†  
School of Mathematical Sciences, MOE-LSC, SHL-MAC  
Shanghai Jiao Tong University, Shanghai 200240, P. R. China

Dedicated to Professor Fan Chung, with admiration and thanks

Abstract

Let $G$ be a graph of order $n$, and let $A(G)$ and $D(G)$ be the adjacency matrix and the degree matrix of $G$ respectively. Define the convex linear combinations $A_{\alpha}(G)$ of $A(G)$ and $D(G)$ by

$$A_{\alpha}(G) = \alpha D(G) + (1 - \alpha)A(G)$$

for any real number $0 \leq \alpha \leq 1$. The $\alpha$-index of $G$ is the largest eigenvalue of $A_{\alpha}(G)$. In this paper, we determine the maximum $\alpha$-index and characterize all extremal graphs for $K_r$ minor-free graphs, $K_{s,t}$ minor-free graphs, and star-forest-free graphs for any $0 < \alpha < 1$ by unified eigenvector approach, respectively.

AMS Classification: 05C50, 05C83, 05C35

Key words: Spectral radius; $\alpha$-index; extremal graphs; star forests; minors

1 Introduction

Let $G$ be an undirected simple graph with vertex set $V(G) = \{v_1, \ldots, v_n\}$ and edge set $E(G)$, where $n$ is called the order of $G$. The adjacency matrix $A(G)$ of $G$ is the $n \times n$ matrix $(a_{ij})$, where $a_{ij} = 1$ if $v_i$ is adjacent to $v_j$, and 0 otherwise. The spectral radius of $G$, denoted by $\rho(G)$, is the largest eigenvalue of $A(G)$. The signless Laplacian spectral radius of $G$, denoted by $q(G)$, is the largest eigenvalue of $Q(G)$, where $Q(G) = A(G) + D(G)$ and $D(G)$ is the degree diagonal matrix. For $v \in V(G)$, the degree $d_G(v)$ of $v$ is the number of vertices adjacent to $v$ in $G$. We write $d(v)$ for $d_G(v)$ if there is no ambiguity. Denote by $\Delta(G)$ the maximum degree of $G$ and $\overline{G}$ the complement graph of $G$. Let $S_{n-1}$ be a star of order $n$. The center of a star is the vertex of maximum degree in the star. A star forest is a forest whose components are stars. The centers of a star forest are the centers of the stars in the star forest. A graph $G$ is $H$-free if it does not contain $H$ as a subgraph. A graph $H$ is called a minor of a graph $G$ if it can be obtained from $G$ by deleting edges,
contracting edges or deleting vertices. A graph $G$ is $H$ minor-free if it does not contain $H$ as a minor. For $X, Y \subseteq V(G)$, $e(X)$ denotes the number of edges in $G$ with two ends in $X$ and $e(X, Y)$ denotes the number of edges in $G$ with one end in $X$ and the other in $Y$. For two vertex disjoint graphs $G$ and $H$, we denote by $G \cup H$ and $G \nabla H$ the union of $G$ and $H$, and the join of $G$ and $H$ which is obtained by joining every vertex of $G$ to every vertex of $H$, respectively. Denote by $kG$ the union of $k$ disjoint copies of $G$. For graph notation and terminology undefined here, readers are referred to [I].

To track the gradual change of $A(G)$ into $Q(G)$, Nikiforov [10] proposed and studied the convex linear combinations $A_\alpha(G)$ of $A(G)$ and $D(G)$ defined by

$$A_\alpha(G) = \alpha D(G) + (1 - \alpha)A(G)$$

for any real number $0 \leq \alpha \leq 1$. Note that $A_0(G) = A(G)$, $2A_{1/2}(G) = Q(G)$, and $A_1(G) = D(G)$. The $\alpha$-index of $G$ is the largest eigenvalue of $A_\alpha(G)$, denoted by $\rho_\alpha(G)$.

Clearly, $\rho_0(G) = \rho(G)$ and $2\rho_{1/2}(G) = q(G)$.

Let $x = (x_u)_{u \in V(G)}$ be an eigenvector to $\rho_\alpha(G)$. By eigenequations of $A_\alpha(G)$ on any vertex $u \in V(G)$,

$$\rho_\alpha(G)x_u = \alpha d(u)x_u + (1 - \alpha) \sum_{v \in E(G)} x_v.$$}

Since $A_\alpha(G)$ is a real symmetric matrix, Rayleigh’s principle implies that

$$\rho_\alpha(G) = \max_{|x| = 1} \sum_{u \in E(G)} (\alpha x_u^2 + 2(1 - \alpha)x_u + \alpha x_v^2),$$

also see [10]. Note that $A_\alpha(G)$ is nonnegative. By Perron-Frobenius theory of nonnegative matrices, if $G$ is connected then $A_\alpha(G)$ has a positive eigenvector corresponding to $\rho_\alpha(G)$, also see [10]. In addition, if $G$ is connected and $H$ is a proper subgraph of $G$, then

$$\rho_\alpha(G) > \rho_\alpha(H).$$

In spectral extremal graph theory, one of the central problems, which is called spectral Turán problem, is to find the maximum $\rho(G)$ or $q(G)$ of a graph $G$ of order $n$ without $H$ as a subgraph or as a minor? This problem is intensively investigated in the literature for many classes of graphs. For example, Tait [12] determined the maximum spectral radius for $K_r$ minor-free graphs and $K_{s,t}$ minor free-graphs by using eigenvector. They pointed out the extremal graphs for maximizing number of edges and spectral radius are the same for small values of $r$ and $s$ and then differed significantly. Chen, Liu and Zhang [2, 3] determined the maximum (signless Laplacian) spectral radius for $kP_3$-free graphs. They [4] also determined the maximum signless Laplacian spectral radius for $K_{2,t}$ minor-free graphs. In addition, Nikiforov [9] gave an excellent survey on this topic. For more results, see [6, 11, 12, 13, 15].

Motivated by above results, we investigate the the maximum $\alpha$-index for $K_r$ minor-free graphs, $K_{s,t}$ minor-free graphs, and star-forest-free graphs. We show the extremal graphs of $K_r$ minor-free graphs and $K_{s,t}$ minor-free graphs for maximizing $\alpha$-index for any $0 < \alpha < 1$ and sufficiently large $n$. Furthermore, we determine the maximum $\alpha$-index and characterize all extremal graphs for star-forest-free graphs for any $0 < \alpha < 1$. The main results of this paper are stated as follows.

**Theorem 1.1.** Let $r \geq 3$ and $G$ be a $K_r$ minor-free graph of sufficiently large order $n$. Then for any $0 < \alpha < 1$,

$$\rho_\alpha(G) \leq \rho_\alpha(K_{r-2}\nabla K_{n-r+2})$$

with equality if and only if $G = K_{r-2}\nabla K_{n-r+2}$.

**Theorem 1.2.** Let $t \geq 2$ and $G$ be a $K_{s,t}$ minor-free graph of sufficiently large order $n$. Then for any $0 < \alpha < 1$, $\rho_\alpha(G)$ is no more than the largest root of $f_\alpha(x) = 0$, and equality holds if and only if $n - s + 1 = pt$ and $G = K_{s-1}\nabla pK_t$, where

$$f_\alpha(x) = x^2 - \left(\alpha n + s + t - 3\right)x + \left(\alpha(n - s + 1) + s - 2\right)(\alpha(s - 1) + t - 1) - (1 - \alpha)^2(s - 1)(n - s + 1).$$
Theorem 1.3. Let $F = \bigcup_{i=1}^{k} S_{d_i}$ be a star forest with $k \geq 2$ and $d_1 \geq \cdots \geq d_k \geq 1$. If $G$ is an $F$-free graph of order $n \geq 4(\sum_{i=1}^{k} d_i + k(2 - 2\alpha \sum_{i=1}^{k} d_i + 3k - 5))$ for any $0 < \alpha < 1$, then $\rho_\alpha(G)$ is no more than the largest root of $f_\alpha(x) = 0$, and equality holds if and only if $G = K_{k-1} \nabla H$ and $H$ is a $(d_k - 1)$-regular graph of order $n - k + 1$, where

$$f_\alpha(x) = x^2 - \left(\alpha n + k + d_k - 3\right)x + \left(\alpha(n - k + 1) + k - 2\right)\left(\alpha(k - 1) + d_k - 1\right) - (1 - \alpha)^2(k - 1)(n - k + 1).$$

The rest of this paper is organized as follows. In Section 2, some lemmas are presented. In Section 3, we give the proofs of Theorem 1.1 and Theorem 1.2. In Section 4, we give the proof of Theorem 1.3 and some corollaries.

2 Preliminary

Lemma 2.1. Let $0 < \alpha < 1$, $k \geq 2$, and $n \geq k - 1$. If $G = K_{k-1} \nabla K_{n-k+1}$, then $\rho_\alpha(G) \geq \alpha(n - 1) + (1 - \alpha)(k - 2).$ In particular, if $n \geq \frac{(2k-3)^2}{2\alpha} - \frac{8k^2 - 18k + 9}{2\alpha} + 2(k - 1)$, then $\rho_\alpha(G) \geq \alpha n + \frac{2k-3-(2k-1)\alpha}{\alpha}.$

Proof. Set for short $\rho_\alpha = \rho_\alpha(G)$ and let $\mathbf{x}_\alpha = (x_v)_{v \in V(G)}$ be a positive eigenvector to $\rho_\alpha$. By symmetry, all vertices corresponding to $K_{k-1}$ in the representation $G := K_{k-1} \nabla K_{n-k+1}$ have the same eigenvector entries, denoted by $x_1$. Similarly, all remaining vertices have the same eigenvector entries, denoted by $x_2$. By eigenequations of $A_\alpha(G)$, we have

$$(\rho_\alpha - \alpha(n - 1) - (1 - \alpha)(k - 2))x_1 = (1 - \alpha)(n - k + 1)x_2$$

$$(\rho_\alpha - \alpha(k - 1))x_2 = (1 - \alpha)(k - 1)x_1.$$ 

Then $\rho_\alpha(G)$ is the largest root of $g(x) = 0$, where

$$g(x) = x^2 - (\alpha n + k - 2) x + (k - 1)(2\alpha - 1)n + (k - 1)(k - k\alpha - 1) = 0.$$ 

Clearly,

$$\begin{align*}
\rho_\alpha(G) &= \frac{\alpha n + k - 2 + \sqrt{(\alpha n + k - 2)^2 - 4(k - 1)(2\alpha - 1)n - 4(k - 1)(k - k\alpha - 1)}}{2} \\
&\geq \frac{\alpha n + k - 2 - (k - 1)\alpha}{2} \\
&= \alpha n + k - 2 - (k - 1)\alpha \\
&= \alpha(n - 1) + (1 - \alpha)(k - 2).
\end{align*}$$

In addition,

$$g\left(\frac{\alpha n + 2k - 3 - (2k - 1)\alpha}{2\alpha}\right) = -\frac{(1 - \alpha)}{2}\left(n - \frac{(2k - 3)^2}{2\alpha^2} + \frac{8k^2 - 18k + 9}{2\alpha} - 2k(k - 1)\right) \leq 0,$$

we have

$$\rho_\alpha(G) \geq \alpha n + \frac{2k - 3 - (2k - 1)\alpha}{2\alpha}.$$

Next we compare two lower bounds of $\rho_\alpha(G)$ in Lemma 2.1.
**Remark.** Note that
\[
\alpha n + \frac{2k - 3 - (2k - 1)\alpha}{2\alpha} - (\alpha(n - 1) + (1 - \alpha)(k - 2)) = \frac{(2k - 2)\alpha - (2k - 3)\alpha}{2\alpha} (\alpha - 1).
\]
If \(0 < \alpha \leq \frac{2k - 3}{2k - 2}\), then
\[
\alpha n + \frac{2k - 3 - (2k - 1)\alpha}{2\alpha} \geq \alpha(n - 1) + (1 - \alpha)(k - 2).
\]
If \(\frac{2k - 3}{2k - 2} < \alpha < 1\), then
\[
\alpha n + \frac{2k - 3 - (2k - 1)\alpha}{2\alpha} < \alpha(n - 1) + (1 - \alpha)(k - 2).
\]

**Lemma 2.2.** Let \(0 < \alpha < 1\), \(d \geq 2\), \(k \geq 1\), \(n \geq \max\{k - 1, 2k - 2 + \frac{d - k + 1}{\alpha}\}\), and \(H\) be a graph of order \(n - k + 1\). If \(G = K_{k-1} \nabla H\) and \(\Delta(H) \leq d - 1\), then \(\rho_\alpha(G)\) is no more than the largest root of \(f_\alpha(x) = 0\), and equality holds if and only if \(H\) is a \((d - 1)\)-regular graph, where
\[
f_\alpha(x) = x^2 - (\alpha n + k + d - 3)x + \left(\alpha(n - k + 1) + k - 2\right)\left(\alpha(k - 1) + d - 1\right) - (1 - \alpha)^2(k - 1)(n - k + 1).
\]

**Proof.** Let \(u_1, u_2, \ldots, u_{k-1}\) be the vertices of \(G\) corresponding to \(K_{k-1}\) in the representation \(G := K_{k-1} \nabla H\). Set for short \(\rho_\alpha = \rho_\alpha(G)\) and let \(x_\alpha = (x_v)_{v \in V(G)}\) be a positive eigenvector to \(\rho_\alpha\). By symmetry, \(x_{u_1} = \cdots = x_{u_{k-1}}\). Choose a vertex \(v \in V(H)\) such that
\[
x_v = \max_{z \in V(H)} x_z.
\]
Since \(\Delta(H) \leq d - 1\) and \(G = K_{n-1} \nabla H\), we have \(d(v) \leq k - 1 + d - 1 = k + d - 2\). By eigenequations of \(A_\alpha(G)\) on \(u_1\) and \(v\), we have
\[
\begin{align*}
(\rho_\alpha - \alpha(n - 1))x_{u_1} &= (1 - \alpha)(k - 2)x_{u_1} + (1 - \alpha) \sum_{u \in V(H)} x_u \\
&\leq (1 - \alpha)(k - 2)x_{u_1} + (1 - \alpha)(n - k + 1)x_v \\
(\rho_\alpha - \alpha(k + d - 2))x_v &\leq (\rho_\alpha - \alpha d(v))x_v = (1 - \alpha)(k - 1)x_{u_1} + (1 - \alpha) \sum_{uv \in E(H)} x_u \\
&\leq (1 - \alpha)(k - 1)x_{u_1} + (1 - \alpha)(d - 1)x_v,
\end{align*}
\]
which implies that
\[
\begin{align*}
(\rho_\alpha - \alpha(n - 1) - (1 - \alpha)(k - 2))x_{u_1} &\leq (1 - \alpha)(n - k + 1)x_v \\
(\rho_\alpha - \alpha(k + d - 2) - (1 - \alpha)(d - 1))x_v &\leq (1 - \alpha)(k - 1)x_{u_1}.
\end{align*}
\]
Note that \(K_{k-1} \nabla K_{n-k+1}\) is a subgraph of \(G\). By Lemma 2.1, we have
\[
\rho_\alpha \geq \rho_\alpha(K_{k-1} \nabla K_{n-k+1}) \geq \alpha(n - 1) + (1 - \alpha)(k - 2) \geq \alpha(k + d - 2) + (1 - \alpha)(d - 1).
\]
Let
\[
f_\alpha(x) = x^2 - (\alpha n + k + d - 3)x + \left(\alpha(n - k + 1) + k - 2\right)\left(\alpha(k - 1) + d - 1\right) - (1 - \alpha)^2(k - 1)(n - k + 1).
\]
Then \(\rho_\alpha\) is no more than the largest root of \(f_\alpha(x) = 0\). If \(\rho_\alpha\) is equal to the largest root of \(f_\alpha(x) = 0\), then all equalities in (1) and (2) hold. So \(d(v) = k + d - 2\) and \(x_z = x_v\) for any vertex \(z \in V(H)\). Since for any \(z \in V(H)\),
\[
(\rho_\alpha - \alpha d(z))x_z = (1 - \alpha)(k - 1)x_{u_1} + (1 - \alpha) \sum_{u \in E(H)} x_u \\
\leq (1 - \alpha)(k - 1)x_{u_1} + (1 - \alpha)(d - 1)x_v = (\rho_\alpha - \alpha d(v))x_v,
\]
we have \(d(z) = d(v) = d + k - 2\). So \(H\) is \((d - 1)\)-regular. \(\square\)
3 Graphs without minors

We first present some structural lemmas for $K_r$ minor-free graphs and $K_{s,t}$ minor-free graphs respectively.

**Lemma 3.1.** Let $r \geq 3$ and $G$ be a bipartite $K_r$ minor-free graph of order $n$ with vertex partition $K$ and $T$. Let $|K| = k$ and $|T| = n - k$. Then there is an absolute constant $C$ depending only on $r$ such that

$$e(G) \leq Ck + (r - 2)n.$$ 

In particular, if $|K| = o(n)$, then $e(G) \leq (r - 2 + o(1))n$.

**Lemma 3.2.** Let $G$ be a $K_r$ minor-free graph of order $n$. Assume that $(1 - 2\delta)n > r$, and $(1 - \delta)n > \binom{r - 2}{2} + 2$, and that there is a set $K$ with $|K| = r - 2$ and a set $T$ with $|T| = (1 - \delta)n$ such that every vertex in $K$ is adjacent to every vertex in $T$. Then we may add edges to $K$ to make it a clique and the resulting graph is still $K_r$ minor-free.

**Lemma 3.3.** Let $t \geq s \geq 2$ and $G$ be a bipartite $K_{s,t}$ minor-free graph of order $n$ with vertex partition $K$ and $T$. Let $|K| = k$ and $|T| = n - k$. Then there is an absolute constant $C$ depending only on $s$ and $t$ such that

$$e(G) \leq Ck + (s - 1)n.$$ 

In particular, if $|K| = o(n)$, then $e(G) \leq (s - 1 + o(1))n$.

**Lemma 3.4.** For any graph $H$, there is a constant $C$ such that if $G$ is an $H$ minor-free graph of order $n$ then

$$e(G) \leq Cn.$$ 

**Proof of Theorem 1.1.** Let $G$ be a $K_r$ minor-free graph of sufficiently large order $n$ with the maximum $\alpha$-index.

**Claim 1.** $G$ is connected.

If $G$ is not connected, then we can add an edge to two components of $G$ to get a $K_r$-minor free graph with larger $\alpha$-index, a contradiction. This proves Claim 1.

Next let $\rho_\alpha = \rho_\alpha(G)$ and $x = (x_v)_{v \in V(G)}$ with the maximum entry 1 be a positive eigenvector to $\rho_\alpha$. Choose an arbitrary $w \in V(G)$ with

$$x_w = \max\{x_v : v \in V(G)\} = 1.$$ 

Set $L = \{v \in V(G) : x_v > \epsilon\}$ and $S = \{v \in V(G) : x_v \leq \epsilon\}$, where $\epsilon$ will be chosen later.

Since $K_{r-2}\sqrt{K_{n-r+2}}$ is $K_r$-minor free, by Lemma 2.1,

$$\rho_\alpha \geq \rho_\alpha(K_{r-2}\sqrt{K_{n-r+2}}) \geq \max \left\{ \alpha n + \frac{2r - 5 - (2r - 3)\alpha}{2\alpha}, \alpha(n - 1) \right\}. \quad (3)$$

By Lemma 3.3, there is a constant $C_1$ such that

$$2e(S) \leq 2e(G) \leq C_1n. \quad (4)$$

**Claim 2.** There exists a constant $C_2$ such that

$$|L| \leq \frac{C_2(1 - \alpha + \alpha\epsilon)}{\epsilon}.$$ 

In addition, $\epsilon$ can be chosen small enough that

$$e(L, S) \leq (k - 1 + \epsilon)n.$$
By eigenequations of $A$, on any vertex $u \in L$, we have 

$$(\rho_\alpha - \alpha d(u))c < (\rho_\alpha - \alpha d(u))x_u = (1 - \alpha) \sum_{u \in E(G)} x_v \leq (1 - \alpha) d(u),$$

which implies that

$$d(u) > \frac{\rho_\alpha \epsilon}{1 - \alpha + \alpha \epsilon}.$$

Thus

$$2e(G) = \sum_{u \in V(G)} d(u) \geq \sum_{u \in L} d(u) \geq \frac{|L| \rho_\alpha \epsilon}{1 - \alpha + \alpha \epsilon},$$

which implies that

$$|L| \leq \frac{2e(G)(1 - \alpha + \alpha \epsilon)}{\rho_\alpha \epsilon}. \tag{5}$$

For sufficiently large $n$, there is a constant $C_2$ such that

$$C_2 \geq \frac{2C_1}{2\alpha^2 + \frac{2r - 5 - (2r - 3)\alpha}{n}}.$$

Hence by (3)-(5),

$$|L| \leq \frac{2e(G)}{\rho_\alpha . \frac{(1 - \alpha + \alpha \epsilon)}{\epsilon}} \leq \frac{C_1 n}{\alpha n + \frac{2r - 5 - (2r - 3)\alpha}{2\alpha}} \frac{(1 - \alpha + \alpha \epsilon)}{\epsilon}$$

$$= \frac{2\alpha C_1}{2\alpha^2 + \frac{2r - 5 - (2r - 3)\alpha}{n}} \frac{\frac{1 - \alpha + \alpha \epsilon}{\epsilon}}{\epsilon} \leq \frac{C_2(1 - \alpha + \alpha \epsilon)}{\epsilon}.$$

Choose $\epsilon$ small enough such that $|L| \leq \epsilon n$. By Lemma 3.1 $e(L, S) \leq (r - 2 + \epsilon)n$. This proves Claim 2.

By Claim 2, we can choose $\epsilon$ small enough such that

$$2e(L) \leq C_1 |L| \leq \frac{C_1 C_2(1 - \alpha + \alpha \epsilon)}{\epsilon} \leq \epsilon n.$$

**Claim 3.** Let $u \in L$. Then for any $u \in L$, there is a constant $C_3$ such that

$$d(u) \geq (1 - C_3(1 - x_u + \epsilon))n.$$

Since

$$\rho_\alpha \sum_{v \in V(G)} x_v$$

$$= \sum_{v \in V(G)} \rho_\alpha x_v = \sum_{v \in V(G)} \left( \alpha d(v)x_v + (1 - \alpha) \sum_{v \in E(G)} x_z \right)$$

$$= \alpha \sum_{v \in V(G)} d(v)x_v + (1 - \alpha) \sum_{v \in E(G)} (x_v + x_z)$$

$$= \alpha \left( \sum_{v \in L} d(v)x_v + \sum_{v \in S} d(v)x_v \right) + (1 - \alpha) \left( \sum_{v \in E(L)} (x_v + x_z) + \sum_{v \in E(L, S)} (x_v + x_z) + \sum_{v \in S} (x_v + x_z) \right)$$

$$\leq \alpha (2e(L) + e(L, S) + \alpha(2e(S) + e(L, S))) + (1 - \alpha)(2e(L) + (1 + \epsilon)e(L, S) + 2\epsilon e(S))$$

$$= 2e(L) + 2e(S) + (1 + \epsilon) e(L, S),$$
Multiplying both sides of the above inequality by \( \rho \), we have
\[
\sum_{v \in V(G)} x_v \leq \frac{2e(L) + 2e(S) + (1 + \epsilon)e(L, S)}{\rho} \leq \frac{2e(L, S) + (1 + \epsilon)(r - 2 + \epsilon)n}{\rho} = \frac{(1 + C_1)\epsilon + (1 + \epsilon)(r - 2 + \epsilon)n}{\rho}.
\] (6)

By eigenequations of \( A_\alpha \) on \( u \), we have
\[
(\rho_\alpha - ad(u))x_u = (1 - \alpha) \sum_{uv \in E(G)} x_v \leq (1 - \alpha) \sum_{v \in V(G)} x_v.
\] (7)

By (3), (6), and (7), we have
\[
d(u) \geq \frac{\rho_\alpha}{\alpha} - \frac{(1 - \alpha)\sum_{v \in V(G)} x_v}{\alpha x_u} \geq \frac{\rho_\alpha}{\alpha} \frac{(1 - \alpha)((1 + C_1)\epsilon + (1 + \epsilon)(r - 2 + \epsilon))n}{\rho_\alpha \alpha x_u} \geq n - 1 - \frac{(1 - \alpha)((1 + C_1)\epsilon + (1 + \epsilon)(r - 2 + \epsilon))}{\alpha^2(1 - \frac{1}{n})x_u}.
\]

Since \( n \) is sufficiently large and \( \epsilon \) is small enough, there is a constant \( C_3 \) such that
\[
d(u) \geq n - 1 - \frac{(1 - \alpha)((1 + C_1)\epsilon + (1 + \epsilon)(r - 2 + \epsilon))}{\alpha^2(1 - \frac{1}{n})x_u} \geq (1 - C_3(1 - x_u + \epsilon))n.
\]

**Claim 4.** Let \( 1 \leq s < r - 2 \). Suppose that there is a set \( X \) of \( s \) vertices such that \( X = \{ v \in V(G) : x_v \geq 1 - \eta \} \), where \( \eta \) is much smaller than 1. Then there is a constant \( C_4 \) and a vertex \( v \in L \setminus X \) such that \( x_v \geq 1 - C_4(\eta + \epsilon) \) and \( d(v) \geq (1 - C_4(\eta + \epsilon))n \).

By eigenequations of \( A_\alpha \) on \( w \), we have
\[
\rho_\alpha - ad(w) = (\rho_\alpha - ad(w))x_w = (1 - \alpha) \sum_{vw \in E(G)} x_v.
\]

Multiplying both sides of the above inequality by \( \rho_\alpha \), we have
\[
\rho_\alpha(\rho_\alpha - ad(w)) = (1 - \alpha) \sum_{vw \in E(G)} \rho_\alpha x_v
\]
\[
= (1 - \alpha) \sum_{vw \in E(G)} \left( ad(v)x_v + (1 - \alpha) \sum_{uv \in E(G)} x_u \right)
\]
\[
= (1 - \alpha) \sum_{vw \in E(G)} ad(v)x_v + (1 - \alpha)^2 \sum_{vw \in E(G)} \sum_{uv \in E(G)} x_u
\]
\[
\leq (1 - \alpha) \left( \sum_{v \in V(G)} ad(v)x_v - ad(w) \right) + (1 - \alpha)^2 \sum_{uv \in E(G)} (x_u + x_v) -
\]
\[
(1 - \alpha)^2 \sum_{vw \in E(G)} x_v
\]
\[
= \alpha(1 - \alpha) \sum_{uv \in E(G)} (x_u + x_v) - \alpha(1 - \alpha)d(w) + (1 - \alpha)^2 \sum_{uv \in E(G)} (x_u + x_v) -
\]
\[
(1 - \alpha)(\rho_\alpha - ad(w))
\]
\[= (1 - \alpha) \sum_{uv \in E(G)} (x_u + x_v) - (1 - \alpha) \rho_\alpha,\]

which implies that

\[\sum_{uv \in E(G)} (x_u + x_v) \geq \frac{\rho_\alpha (\rho_\alpha + 1 - \alpha - \alpha d(w))}{1 - \alpha}.\]

On the other hand,

\[\sum_{uv \in E(G)} (x_u + x_v) = \sum_{uv \in E(L, S)} (x_u + x_v) + \sum_{uv \in E(S)} (x_u + x_v) + \sum_{uv \in E(L)} (x_u + x_v) \leq \sum_{uv \in E(L, S)} (x_u + x_v) + 2\epsilon e(S) + 2\epsilon(L) \leq \epsilon e(L, S) + \sum_{uv \in E(L\setminus X, S)} x_u + \sum_{uv \in E(L\setminus X, S)} x_u + 2\epsilon e(S) + 2\epsilon(L).

Let \( t = |L \cap X| \). Combining with (3), we have

\[
\sum_{uv \in E(L\setminus X, S)} x_u \geq \rho_\alpha (\rho_\alpha + 1 - \alpha - \alpha d(w)) - 2\epsilon e(S) - 2\epsilon(L) - \epsilon e(L, S) - \sum_{uv \in E(L\setminus X, S)} x_u
\]

\[
\geq \left( \frac{\alpha n}{1 - \alpha} + \frac{2r - 5 - (2r - 3)\alpha}{2\alpha(1 - \alpha)} \right) \left( \alpha n + \frac{2r - 5 - (2r - 3)\alpha}{2\alpha} + 1 - \alpha - \alpha n + \alpha \right) - \epsilon C1n - cn - \epsilon(r - 2 + \epsilon)n - tn
\]

\[
= \left( \frac{\alpha n}{1 - \alpha} + \frac{2r - 5 - (2r - 3)\alpha}{2\alpha(1 - \alpha)} \right) \left( \frac{2r - 5 - (2r - 3)\alpha}{2\alpha} \right) - (\epsilon C1 + \epsilon + \epsilon(r - 2 + \epsilon) + t)n
\]

\[
= \left( r - \frac{5}{2} - t - \epsilon(C1 + \epsilon + r - 1) \right) n + \frac{(2r - 5)^2 - (2r - 3)(2r - 5)\alpha}{4\alpha^2}
\]

In addition,

\[e(L\setminus X, S) = e(L, S) - e(L \cap X, S) \leq (r - 2 + \epsilon)n - t(1 - \eta)n + t(t - 1) + t(|L| - t) \leq (r - 2 + \epsilon)n - t(1 - \eta)n + t(t - 1) + t(cn - t) \leq (r - 2 + 2\epsilon - t(1 - \eta - \epsilon))n\]

Note that for any \( \eta > 0 \), there exists a constant \( C_4' \) such that \( C_4' \eta \geq \frac{1}{2} \). Then there is a vertex \( v \in L\setminus X \) such that

\[x_v \geq \frac{\sum_{uv \in E(L\setminus X, S)} x_u}{e(L\setminus X, S)} \geq \frac{(r - \frac{5}{2} - t - \epsilon(C1 + \epsilon + r))n}{(r - 2 + \epsilon - t(1 - \eta - \epsilon))n} \geq \frac{r - \frac{5}{2} - t - \epsilon(C1 + \epsilon + r)}{r - 2 + \epsilon - t(1 - \eta - \epsilon)}\]
In addition, by Claim 5, 

\[ T \]

By the definition of \( R \)

By eigenequations of \( A \)

By Lemma 3.2 and (8)-(10), we have

By Claim 3, Claim 4 follows directly.

by \( K \)

enough that \( G \)

of size is at least \((1 - \epsilon)\)\( n \) and each eigenvector entry is at least \(1 - \epsilon\). From now on, denote by \( K \) the set \( X \) with \( r - 2 \) vertices mentioned above. Let \( T \) be the common neighborhood of \( K \) and \( R = V(G) \setminus (K \cup T) \). Clearly, \(|K| = r - 2, |T| \geq (1 - \epsilon)n\), and \(|R| \leq \epsilon n\).

Claim 5. \( K \) induces a clique and \( T \) is an independent set.

If \( K \) does not induce a clique, then we can add all possible edges to make it a clique. By Lemma 3.2, the resulting graph \( G' \) is still \( K_r \)-minor free. Since \( G \) is connected, \( \rho_{\alpha}(G') > \rho_{\alpha}(G) \), a contradiction. Hence \( K \) induces a clique. If there is an edge in \( T \), then there is a \( K_r \) minor in \( G \), a contradiction. Thus \( T \) is an independent set. This proves Claim 5.

Claim 6. For any \( v \in V(G) \setminus K \), we have \( x_v \leq \frac{\sqrt{\epsilon}}{C_1} \), where \( C_1 \) is the constant in (4).

Since \( G \) is \( K_r \)-minor free, any vertex in \( R \) can be adjacent to at most one vertex in \( T \). By the definition of \( R \), every vertex in \( R \) can be adjacent to at most \( r - 3 \) vertices in \( K \).

In addition, by Claim 5, \( T \) is an independent set and thus any vertex in \( T \) has at most \( r - 2 + |R| \) neighbors. Hence for any vertex \( v \in V(G) \setminus K \),

\[
d(v) \leq r - 2 + |R| \leq r - 2 + \epsilon n. \tag{8}
\]

Since \( R \) is also \( K_r \)-minor free, we have

\[ 2e(R) \leq C_1 |R| \leq C_1 \epsilon n. \]

By eigenequations of \( A_{\alpha}(G) \), we have

\[
\alpha(n - 1) \sum_{u \in R} x_u \leq \rho_{\alpha} \sum_{u \in R} x_u = \sum_{u \in R} \left( \alpha d(u)x_u + (1 - \alpha) \sum_{u \in E(G)} x_v \right) \\
\leq \sum_{u \in R} (\alpha d(u) + (1 - \alpha)d(u)) = \sum_{u \in R} d(u) \\
\leq 2e(R) + (r - 2)|R| \leq C_1 \epsilon n + (r - 2)\epsilon n \\
= (C_1 + r - 2)\epsilon n,
\]

which implies that

\[
\sum_{u \in R} x_u \leq \frac{(C_1 + r - 2)\epsilon n}{\alpha(n - 1)}. \tag{9}
\]

By eigenequations of \( A_{\alpha}(G) \) on any vertex \( v \in V(G) \setminus K \), we have

\[
(\rho_{\alpha} - \alpha d(v))x_v = (1 - \alpha) \sum_{u \in E(G)} x_u \leq (1 - \alpha) \left( r - 2 + \sum_{u \in R} x_u \right). \tag{10}
\]

By Lemma 3.2 and (8)-(10), we have

\[
x_v \leq \frac{(1 - \alpha)(r - 2 + \sum_{u \in R} x_u)}{\rho_{\alpha} - \alpha d(v)}
\]
Then we can choose \( \epsilon \) small enough to make \( \delta \) small enough to get the result.

**Claim 7.** \( R \) is empty.

If \( R \) is not empty, then there exists a vertex \( v \in R \) such that \( v \) has at most \( C_1 \) neighbors in \( R \). Let \( H \) be a graph obtained from \( G \) by removing all edges incident with \( v \) and then connecting \( v \) to each vertex in \( K \). Since \( K \) induces a clique, \( H \) is still \( K_r \) minor-free. Let \( u \in K \) be the vertex not adjacent to \( v \). Then

\[
\rho_\alpha(H) - \rho_\alpha \geq \frac{1}{x^T A_\alpha(H)x} - \frac{1}{x^T A_\alpha x} \geq \frac{1}{x^T x} \left( 2\alpha(1-\delta)^2 - \frac{\alpha(C_1+1)(\alpha + 2(1-\alpha) + \alpha)}{C_1^2} \right) = \frac{2\alpha}{x^T x} \left( 1-\delta^2 - \frac{C_1+1}{C_1^2} \right)
\]

Choose \( \epsilon \) small enough so that \( (1-\delta)^2 > \frac{C_1+1}{C_1^2} \). Then \( \rho_\alpha(H) > \rho_\alpha \), a contradiction. This proves Claim 7.

By Claims 6 and 7, \( G = K_{r-2} \bigtriangledown K_{n-r+2} \). This completes the proof. \( \square \)

**Proof of Theorem 1.2.** Let \( G \) be a \( K_{s,t} \) minor-free graph of order \( n \) with the maximum \( \alpha \)-index.

Similarly to the proof of Claim 1 in Theorem 1.1, \( G \) is connected. Next let \( \rho_\alpha = \rho_\alpha(G) \) and \( x = (x_v)_{v \in V(G)} \) be a positive eigenvector to \( \rho_\alpha \) such that \( w \in V(G) \) and

\[
x_w = \max \{ x_u : u \in V(G) \} = 1.
\]

Set \( L = \{ v \in V(G) : x_v > \epsilon \} \) and \( S = \{ v \in V(G) : x_v < \epsilon \} \), where \( \epsilon \) will be chosen later.

**Claim 1.** For any \( \delta > 0 \), if we choose \( \epsilon \) small enough, then \( G \) contains a set \( K \) with \( s-1 \) vertices such that their common neighborhood of size is at least \( (1-\delta)n \) and each eigenvector entry is at least \( 1-\delta \).

We omit the proof of Claim 1 as it is similar to the proofs of of Claims 2-4 in Theorem 1.1.

Let \( T \) be the common neighborhood of \( K \) and \( R = V(G) \setminus (K \cup T) \).

**Claim 2.** \( R \) is empty.

Noting any vertex in \( R \cup T \) has at most \( t-1 \) neighbors in \( R \cup T \) as \( G \) is \( K_{s,t} \) minor-free. In addition, noting the graph obtained from \( G \) by adding a vertex adjacent to every vertex in \( K \) is still \( K_{s,t} \) minor-free. The proof of Claim 2 is similar to the proofs of Claims 6 and 7. Hence it is omitted here.

Now \( |K| = s-1 \) and \( |T| = n-s+1 \). Let \( H \) be the subgraph of \( G \) induced by \( T \). Now \( G = G[K] \bigtriangledown H \). Since \( G \) is \( K_{s,t} \) minor-free, \( \Delta(H) \leq t-1 \).

First suppose that \( K \) induces a clique. By Lemma 2.2, \( \rho_\alpha(G) \) is no more than the largest root of \( f_\alpha(x) = 0 \), where

\[
x^2 - (\alpha n + s + t - 3)x + (\alpha(n - s + 1) + s - 2))((\alpha(s - 1) + t - 1) - (1 - \alpha)^2(s - 1)(n - s + 1) = 0
\]
and equality holds if and only if \( G = K_{s-1} \setminus H \), where \( H \) is a \((t-1)\)-regular graph. It suffices to prove that equality can hold if and only if \( G = K_{s-1} \setminus pK_t \), where \( n-s+1 = pt \). Suppose that \( H \) has a connected component \( H_1 \) that is not isomorphic to \( K_t \) and set \( h := |V(H_1)| \). Clearly \( H_1 \) is a \((t-1)\)-regular graph of order \( h \geq t+1 \). If \( h = t+1 \), then any two nonadjacent vertices in \( H \) have \( t-1 \) common neighbors, which combining with clique \( K_{s-1} \) yields \( K_{s,t} \), a contradiction. Thus \( h \geq t+2 \). Note that \( G \) is \( K_{s,t} \) minor-free, we have \( H_1 \) is \( K_{1,t} \) minor-free. Hence 
\[
    e(H_1) \leq h + \frac{t(t-3)}{2},
\]
see [5]. However, since \( H_1 \) is a \((t-1)\)-regular graph of order \( h \), we have 
\[
    e(H_1) = \frac{h(t-1)}{2} > h + \frac{t(t-3)}{2},
\]
a contradiction. Hence \( H \) is the union of disjoint complete graphs of order \( t \), i.e., \( G = K_{s-1} \setminus pK_t \), where \( n-s+1 = pt \).

Next suppose that \( K \) does not induce a clique. Let \( G' \) be the graph obtained from \( G \) by adding edges to \( K \) to make it a clique. Then \( \rho_\alpha(G) < \rho_\alpha(G') \). By Lemma 2.2, \( \rho_\alpha(G') \) is no more than the largest root of \( f_\alpha(x) = 0 \), and thus \( \rho_\alpha(G) \) is less than the largest root of \( f_\alpha(x) = 0 \). This completes the proof. \( \square \)

Let \( \alpha = \frac{1}{2} \). It is easy to get the following corollary for \( q(G) \).

**Corollary 3.5.** Let \( t \geq s \geq 2 \) and \( G \) be a \( K_{s,t} \) minor-free graph of sufficiently large order \( n \). Then
\[
    q(G) \leq \frac{n + 2s + 2t - 6 + \sqrt{(n + 2s - 2t - 2)^2 + 8(s-1)(t-s+1)}}{2}
\]
with equality if and only if \( n-s+1 = pt \) and \( G = K_{s-1} \setminus pK_t \).

## 4 Graphs without star forests

In this section, we present the proof of Theorem 1.3 and some corollaries.

**Lemma 4.1.** Let \( F = \cup_{i=1}^k S_{d_i} \) be a star forest with \( k \geq 2 \) and \( d_1 \geq \cdots \geq d_k \geq 1 \). If \( G \) is an \( F \)-free graph of order \( n \geq \sum_{i=1}^k d_i + k \), then
\[
    e(G) \leq \frac{1}{2} \left( \sum_{i=1}^k d_i + 2k - 3 \right) n - \frac{1}{2} (k-1) \left( \sum_{i=1}^k d_i + k - 1 \right).
\]

**Proof.** Let \( C = \{ v \in V(G) : d(v) \geq \sum_{i=1}^k d_i + k - 1 \} \). Since \( G \) is \( F \)-free, \( |C| \leq k-1 \), otherwise we can embed an \( F \) in \( G \) by the definition of \( C \). Hence
\[
    2e(G) = \sum_{v \in C} d(v) + \sum_{v \in V(G) \setminus C} d(v)
\]
\[
    \leq (n-1)|C| + (n-|C|) \left( \sum_{i=1}^k d_i + k - 2 \right)
\]
\[
    = \left( n - \sum_{i=1}^k d_i + k + 1 \right) |C| + \left( \sum_{i=1}^k d_i + k - 2 \right) n
\]
\[
    \leq (k-1) \left( n - \sum_{i=1}^k d_i + k + 1 \right) + \left( \sum_{i=1}^k d_i + k - 2 \right) n
\]
\[
    = \left( \sum_{i=1}^k d_i + 2k - 3 \right) n - (k-1) \left( \sum_{i=1}^k d_i + k - 1 \right)
\]
This completes the proof. □

Next we prove the following result for star-forest-free connected graphs, which plays an important role in the proof of Theorem 1.3.

**Theorem 4.2.** Let $F = \cup_{i=1}^{k} S_{d_i}$ be a star forest with $k \geq 2$ and $d_1 \geq \cdots \geq d_k \geq 2$. If $G$ is an $F$-free connected graph of order $n \geq \frac{4(\sum_{i=1}^{k} d_i + k - 2)(\sum_{i=1}^{k} d_i + 3k - 5)}{\alpha}$ for any $0 < \alpha < 1$, then $\rho_\alpha(G)$ is no more than the largest root of $f_\alpha(x) = 0$ and equality holds if and only if $G = K_{k-1} \nabla H$ and $H$ is a $(d_k - 1)$-regular graph of order $n - k + 1$, where

$$f_\alpha(x) = x^2 - \left(\alpha n + k + d_k - 3\right)x +$$

$$\left(\alpha(n - k + 1) + k - 2\right)\left(\alpha(k - 1) + d_k - 1\right) - (1 - \alpha)^2(k - 1)(n - k + 1).$$

**Proof.** Let $G$ be an $F$-free connected graph of order $n$ with the maximum $\alpha$-index. Set for short $A_\alpha = A_\alpha(G)$ and $\rho_\alpha = \rho_\alpha(G)$. Let $x_\alpha = (x_v)_{v \in V(G)}$ be a positive eigenvector to $\rho_\alpha$ such that $w \in V(G)$ and

$$x_w = \max\{x_u : u \in V\} = 1.$$

Since $K_{k-1} \nabla \nabla_{n-k+1}$ is $F$-free, it follows from Lemma 2.1 that

$$\rho_\alpha \geq \rho_\alpha(K_{k-1} \nabla \nabla_{n-k+1}) \geq \alpha n + \frac{2k - 3 - (2k - 1)\alpha}{2\alpha}.$$

Let $L = \{u \in V(G) : x_u > \epsilon\}$ and $S = \{u \in V(G) : x_u \leq \epsilon\}$, where

$$\epsilon = \frac{1}{4(\sum_{i=1}^{k} d_i + 3k - 5)}.$$

**Claim.** $|L| = k - 1$.

First suppose that $|L| \geq k$. By eigenequations of $A_\alpha$ on any vertex $u \in L$, we have

$$(\rho_\alpha - \alpha d(u))\epsilon < (\rho_\alpha - \alpha d(u))x_u = (1 - \alpha) \sum_{uv \in E(G)} x_v \leq (1 - \alpha)d(u),$$

which implies that

$$d(u) > \frac{\rho_\alpha \epsilon}{1 - \alpha + \alpha \epsilon} \geq \left(\alpha n + \frac{2k - 3 - (2k - 1)\alpha}{2\alpha}\right)\frac{\epsilon}{1 - \alpha + \alpha \epsilon} \geq \sum_{i=1}^{k} d_i + k - 2,$$

where the last inequality holds as $\epsilon \geq \frac{2\alpha(1 - \alpha)(\sum_{i=1}^{k} d_i + k - 2)}{2\alpha^2(n - \sum_{i=1}^{k} d_i + k - 2) - (2k - 1)\alpha + 2k - 3}$. Thus

$$d(u) \geq \sum_{i=1}^{k} d_i + k - 1.$$

Then we can embed an $F$ in $G$ with all centers in $L$, a contradiction.

Next suppose that $|L| \leq k - 2$. Then

$$e(L) \leq \binom{|L|}{2} \leq \frac{1}{2}(k - 2)(k - 3)$$

and

$$e(L, S) \leq (k - 2)(n - k + 2).$$

In addition, by Lemma 4.1

$$e(S) \leq e(G) \leq \frac{1}{2}\left(\sum_{i=1}^{k} d_i + 2k - 3\right)n.$$
By eigenequations of $A$, we have

$$\rho_\alpha - \alpha d(w) = (\rho_\alpha - \alpha d(w))x_w = (1 - \alpha) \sum_{vw \in E(G)} x_v.$$  

Multiplying both sides of the above equality by $\rho_\alpha$, we have

\[
\rho_\alpha (\rho_\alpha - \alpha d(w)) = (1 - \alpha) \sum_{vw \in E(G)} \rho_\alpha x_v
\]

\[
= (1 - \alpha) \sum_{vw \in E(G)} (\alpha d(v)x_v + (1 - \alpha) \sum_{uw \in E(G)} x_u)
\]

\[
= (1 - \alpha) \sum_{vw \in E(G)} \alpha d(v)x_v + (1 - \alpha)^2 \sum_{uw \in E(G)} \sum_{uv \in E(G)} x_u
\]

\[
\leq (1 - \alpha) \left( \sum_{v \in V(G)} \alpha d(v)x_v - \alpha d(w) \right) + (1 - \alpha)^2 \sum_{uv \in E(G)} (x_u + x_v) - (1 - \alpha)^2 \sum_{vw \in E(G)} x_v
\]

\[
= \alpha (1 - \alpha) \sum_{uv \in E(G)} (x_u + x_v) + \alpha (1 - \alpha) d(w) + (1 - \alpha)^2 \sum_{uv \in E(G)} (x_u + x_v) - (1 - \alpha) (\rho_\alpha - \alpha d(w))
\]

\[
= (1 - \alpha) \sum_{uv \in E(G)} (x_u + x_v) - (1 - \alpha) \rho_\alpha,
\]

which implies that

$$\sum_{uv \in E(G)} (x_u + x_v) \geq \frac{\rho_\alpha (\rho_\alpha + 1 - \alpha - \alpha d(w))}{1 - \alpha}.$$  

On the other hand,

\[
\sum_{uv \in E(G)} (x_u + x_v) = \sum_{uv \in E(L,S)} (x_u + x_v) + \sum_{uv \in E(S)} (x_u + x_v) + \sum_{uv \in E(L)} (x_u + x_v)
\]

\[
\leq \sum_{uv \in E(L,S)} (x_u + x_v) + 2\epsilon e(S) + 2\epsilon e(L)
\]

\[
\leq \sum_{uv \in E(L,S)} (x_u + x_v) + \epsilon \left( \sum_{i=1}^{k} d_i + 2k - 3 \right) n + (k - 2)(k - 3).
\]

Hence

\[
\sum_{uv \in E(L,S)} (x_u + x_v) \geq \frac{\rho_\alpha (\rho_\alpha + 1 - \alpha - \alpha d(w))}{1 - \alpha} - \epsilon \left( \sum_{i=1}^{k} d_i + 2k - 3 \right) n - (k - 2)(k - 3)
\]

\[
\geq \frac{\alpha n + \frac{2k - 3 - (2k - 1)\alpha}{2\alpha(1 - \alpha)}}{1 - \alpha} \left( \alpha n + \frac{2k - 3 - (2k - 1)\alpha}{2\alpha} + 1 - \alpha - \alpha(n - 1) \right) - \epsilon \left( \sum_{i=1}^{k} d_i + 2k - 3 \right) n - (k - 2)(k - 3)
\]

\[
= \left( k - \frac{3}{2} - \epsilon \left( \sum_{i=1}^{k} d_i + 2k - 3 \right) \right) n + \frac{(2k - 3)^2 - (2k - 3)(2k - 1)\alpha}{4\alpha^2} - (k - 2)(k - 3),
\]
where the second inequality holds as \( d(u) \leq n - 1 \). On the other hand, by the definition of \( L \) and \( S \), we have

\[
\sum_{uv \in E(L, S)} (x_u + x_v) \leq (1 + \epsilon)e(L, S) \leq (1 + \epsilon)(k - 2)(n - k + 2).
\]

Thus

\[
(1 + \epsilon)(k - 2)(n - k + 2) \geq \left( k - \frac{3}{2} - \epsilon \left( \sum_{i=1}^{k} d_i + 2k - 3 \right) \right)n + \frac{(2k - 3)^2 - (2k - 3)(2k - 1)\alpha}{4\alpha^2} - (k - 2)(k - 3),
\]

which implies that

\[
\left( -\frac{1}{2} + \epsilon \left( \sum_{i=1}^{k} d_i + 3k - 5 \right) \right)n \geq \frac{(2k - 3)^2 - (2k - 3)(2k - 1)\alpha}{4\alpha^2} + (k - 2)(1 + \epsilon(k - 2)).
\]

Since \( \epsilon = \frac{1}{4(\sum_{i=1}^{k} d_i + 3k - 5)} \), we have

\[
n \leq -\frac{(2k - 3)^2 - (2k - 3)(2k - 1)\alpha}{\alpha^2} - 4(k - 2) \left( 1 + \frac{k - 2}{4(\sum_{i=1}^{k} d_i + 3k - 5)} \right)
\]

\[
< \frac{(k - 3)(2k - 1)}{\alpha}
\]

\[
< \frac{4(\sum_{i=1}^{k} d_i + k - 2)(\sum_{i=1}^{k} d_i + 3k - 5)}{\alpha^2},
\]

a contradiction. This proves the Claim.

By Claim, \( |L| = k - 1 \) and thus \( |S| = n - k + 1 \). Then the subgraph \( H \) induced by \( S \) in \( G \) is \( S_{d_k} \)-free. Otherwise, we can embed an \( F \) in \( G \) with \( k - 1 \) centers in \( L \) and a center in \( S \) as \( d(u) \geq \sum_{i=1}^{k} d_i + k - 1 \) for any \( u \in L \), a contradiction. Now \( \Delta(H) \leq d_k - 1 \). Note that the resulting graph obtained from \( G \) by adding all edges in \( L \) and all edges with one end in \( L \) and the other in \( S \) is also \( F \)-free and its spectral radius increase strictly. By the extremality of \( G \), we have \( G = K_{k-1} \nabla H \). By Lemma 2.2 and the extremality of \( G \), \( \rho_\alpha \) is no more than largest root of \( f_\alpha(x) = 0 \), and \( \rho_\alpha \) is equal to the largest root of \( f_\alpha(x) = 0 \) if and only if \( H \) is a \((d_k - 1)\)-regular graph of order \( n - k + 1 \), where

\[
f_\alpha(x) = x^2 - \left( \alpha n + k + d_k - 3 \right)x + \left( \alpha(n - k + 1) + k - 2 \right)(\alpha(k - 1) + d_k - 1) - (1 - \alpha)^2(k - 1)(n - k + 1).
\]

This completes the proof. \( \square \)

**Proof of Theorem 1.3.** Let \( G \) be an \( F \)-free graph of order \( n \) with the maximum \( \alpha \)-index.

If \( G \) is connected, then the result follows directly from Theorem 1.2. Next we suppose that \( G \) is not connected. Since \( K_{n-1} \nabla K_{n-k+1} \) is \( F \)-free,

\[
\rho_\alpha(G) \geq \rho_\alpha(K_{n-1} \nabla K_{n-k+1}) \geq \alpha n + \frac{2k - 3 - (2k - 1)\alpha}{2\alpha}.
\]

Let \( G_1 \) be a component of \( G \) such that \( \rho_\alpha(G_1) = \rho_\alpha(G) \). Set \( n_1 = |V(G_1)| \). Then

\[
n_1 - 1 \geq \rho_\alpha(G_1) = \rho_\alpha(G) \geq \alpha n + \frac{2k - 3 - (2k - 1)\alpha}{2\alpha}
\]

\[
\geq \frac{4(\sum_{i=1}^{k} d_i + k - 2)(\sum_{i=1}^{k} d_i + 3k - 5)}{\alpha^2} + \frac{2k - 3 - (2k - 1)\alpha}{2\alpha},
\]
which implies that
\[ n_1 \geq \frac{4(\sum_{i=1}^{k} d_i + k - 2)(\sum_{i=1}^{k} d_i + 3k - 5)}{\alpha^2} + \frac{(2k - 3)(1 - \alpha)}{2\alpha} \]

By Theorem 4.2 again, \( \rho_\alpha(G_1) \) is no more than the largest root of
\[ x^2 - \left( \alpha n_1 + k + d_k - 3 \right) x + \left( \alpha(n_1 - k + 1) + k - 2 \right) \left( \alpha(k-1) + d_k - 1 \right) - (1 - \alpha)^2(k-1)(n_1 - k + 1) = 0. \]
Hence \( \rho_\alpha(G_1) \) is less than the largest root of
\[ x^2 - \left( \alpha n + k + d_k - 3 \right) x + \left( \alpha(n - k + 1) + k - 2 \right) \left( \alpha(k-1) + d_k - 1 \right) - (1 - \alpha)^2(k-1)(n - k + 1) = 0. \]
This completes the proof.

\[ \square \]

Let \( F_{n,k} = K_{k-1} \setminus (pK_2 \cup qK_1) \), where \( n - (k - 1) = 2p + q \) and \( 0 \leq q < 2 \).

**Corollary 4.3.** Let \( F = \cup_{i=1}^{k} S_{d_i} \) be a star forest with \( k \geq 2 \) and \( d_1 \geq \cdots \geq d_k = 2 \). If \( G \) is an \( F \)-free graph of order \( n \geq \frac{4(\sum_{i=1}^{k} d_i + k - 2)(\sum_{i=1}^{k} d_i + 3k - 5)}{\alpha^2} \), then
\[ \rho_\alpha(G) \leq \rho_\alpha(F_{n,k}) \]
with equality if and only if \( G = F_{n,k} \).

**Proof.** Let \( G \) be a graph having the maximum \( \alpha \)-index among all \( F \)-free graphs of order \( n \). It suffices to show that \( G = F_{n,k} \). If \( G \) is connected, then by the proof of Theorem 4.2, we have \( G = K_{k-1} \setminus H \), where \( H \) is a graph of order \( n - k + 1 \) with \( \Delta(H) \leq 1 \). So \( H \) is the union of some edges and isolated vertices. Hence \( G = F_{n,k} \). If \( G \) is not connected, then by the similar proof of Theorem 4.3, there is a component \( G_1 \) of \( G \) such that \( |V(G_1)| \geq \frac{4(\sum_{i=1}^{k} d_i + k - 2)(\sum_{i=1}^{k} d_i + 3k - 5)}{\alpha^2} \) and \( \rho(G) = \rho(G_1) \). By above case,
\[ \rho_\alpha(G) = \rho_\alpha(G_1) = \rho_\alpha(F_{n_1,k}) < \rho_\alpha(F_{n,k}). \]
Hence the result follows. \[ \square \]

Let \( \alpha = \frac{1}{2} \). By Theorem 4.3 and Corollary 4.3 we have the following corollary.

**Corollary 4.4.** Let \( F = \cup_{i=1}^{k} S_{d_i} \) be a star forest with \( k \geq 2 \) and \( d_1 \geq \cdots \geq d_k \geq 2 \) and \( G \) be an \( F \)-free graph of order \( n \geq 32(\sum_{i=1}^{k} d_i + k - 2)(\sum_{i=1}^{k} d_i + 3k - 5) \).
(i) If \( d_k = 2 \), then
\[ q(G) \leq q(F_{n,k}) \]
with equality if and only if \( G = F_{n,k} \).
(ii) If \( d_k \geq 3 \), then
\[ q(G) \leq \frac{n + 2k + 2d_k - 6 + \sqrt{(n + 2k - 2d_k - 2)^2 + 8(k - 1)(d_k - k + 1)}}{2} \]
with equality if and only if \( G = K_{k-1} \setminus H \), where \( H \) is a \((d_k - 1)\)-regular graph of order \( n - k + 1 \).
References

[1] J.A. Bondy, U.S.R. Murty, Graph Theory, Springer, New York (2007).

[2] M.-Z. Chen, A-M. Liu, X.-D. Zhang, Spectral extremal results with forbidding linear forests, Graphs Combin. 35 (2019) 335–351.

[3] M.-Z. Chen, A-M. Liu, X.-D. Zhang, The signless Laplacian spectral radius of graphs with forbidding linear forests. Linear Algebra Appl. 591 (2020) 25–43.

[4] M.-Z. Chen, X.-D. Zhang, On the signless Laplacian spectral radius of $K_{s,t}$-minor free graphs, Linear Multilinear Algebra https://doi.org/10.1080/03081087.2019.1650880.

[5] G. Ding, T. Johnson, P. Seymour, Spanning trees with many leaves, J. Graph Theory 37 (2001) 189–197.

[6] B. He, Y.-L. Jin, X.-D. Zhang, Sharp bounds for the signless Laplacian spectral radius in terms of clique number, Linear Algebra Appl. 438 (2013) 3851–3861.

[7] W. Mader, Homomorphieeigenschaften und mittlere Kantendichte von Graphen, Math. Ann. 174 (1967) 265–268.

[8] V. Nikiforov, Bounds on graph eigenvalues II, Linear Algebra Appl. 427 (2007) 183–189.

[9] V. Nikiforov, Some new results in extremal graph theory. Surveys in combinatorics 2011, 141-181, London Math. Soc. Lecture Note Ser., 392, Cambridge Univ. Press, Cambridge, 2011.

[10] V. Nikiforov, Merging the $A$- and $Q$-spectral theories. Appl. Anal. Discrete Math. 11 (1) (2017) 81–107.

[11] V. Nikiforov, X.Y. Yuan, Maxima of the $Q$-index: Forbidden even cycles, Linear Algebra Appl. 471 (2015) 636–653.

[12] M. Tait, The Colin de Verdière parameter, excluded minors, and the spectral radius, J. Combin. Theory Ser. A 166 (2019) 42–58.

[13] M. Tait, J. Tobin, Three conjectures in extremal spectral graph theory, J. Combin. Theory Ser. B 126 (2017) 137–161.

[14] A. Thomason, Disjoint complete minors and bipartite minors, European J. Combin. 28 (2007) 1779–1783.

[15] X.Y. Yuan, Maxima of the $Q$-index: forbidden odd cycles, Linear Algebra Appl. 458 (2014) 207–216.