POSITIVITY OF THE RENORMALIZED VOLUME OF
ALMOST-FUCHSIAN HYPERBOLIC 3-MANIFOLDS

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ABSTRACT. We prove that the renormalized volume of almost-Fuchsian hyperbolic
3-manifolds is non-negative, with equality only for Fuchsian manifolds.

1. INTRODUCTION

The renormalized volume $\text{Vol}_R$ is a real number extracted from an infinite-volume Rie-
mannian manifold with some special structure near infinity. For convex co-compact
hyperbolic 3-manifolds, $\text{Vol}_R$ is a Kähler potential for the Weil–Petersson symplectic
form on the Teichmüller space of conformal structures on the ideal boundary compo-
nents:

$$\partial\bar{\partial}\text{Vol}_R = \frac{1}{8i} \omega_{WP}.$$ 

Converting divergent integrals into numerical invariants, such as $\text{Vol}_R$, seems to be self-
evident for physicists. It is not surprising that the earliest instances of renormalized
volumes appear in Henningson–Skenderis [4], for asymptotically hyperbolic Einstein
metrics, and in the paper of Krasnov [6], for Schottky hyperbolic 3-manifolds. This
was followed by works of Takhtajan–Teo [9] and Krasnov–Schlenker [7] in the context
of quasi-Fuchsian hyperbolic 3-manifolds. In [2], the renormalized volume of geometric-
ally finite hyperbolic 3-manifolds without rank-1 cusps appears as the log-norm of a
holomorphic section in the Chern–Simons line bundle over the Teichmüller space. Re-
cently, Huang–Wang [5] started a study of the renormalized volume for almost-Fuchsian
hyperbolic 3-manifolds (see Definition 1). However, their renormalization procedure
does not involve uniformization of the surfaces at infinity, hence the invariant $\text{RV}$ thus
obtained is constant on the moduli space of almost-Fuchsian metrics.

There is a superficial analogy between $\text{Vol}_R$ and the mass of asymptotically Euclidean
manifolds. Like in the positive mass conjecture, one may ask if $\text{Vol}_R$ is positive for all
convex co-compact hyperbolic 3-manifolds, or at least for quasi-Fuchsian hyperbolic
3-manifolds. One piece of supporting evidence is the computation by Krasnov and
Schlenker [7] of the Hessian of $\text{Vol}_R$ at the Fuchsian locus. The functional $\text{Vol}_R$ van-
ishes for Fuchsian hyperbolic 3-manifolds, it has critical points there, and its Hessian
$\text{Hess}(\text{Vol}_R)$ is the Weil–Petersson metric on Teichmüller space. Therefore, at least in a

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neighborhood of the Fuchsian locus, we do have positivity. We emphasize that to ensure vanishing at the Fuchsian locus, the renormalization procedure used in \cite{7} differs from \cite{2} or from \cite{5} by an universal constant. It is the definition from \cite{7} which we use below. Moreover, when one component of the boundary is kept fixed, the only critical point of \( \text{Vol}_R \) is at the Fuchsian metric. This is nevertheless not sufficient to conclude that \( \text{Vol}_R \) is positive since the Teichmüller space is not compact and \( \text{Vol}_R \) does not seem to be proper. Another piece of evidence was recently found by Schlenker \cite{8}, who proved that \( \text{Vol}_R \) is bounded from below by some explicit constant.

In this note we prove the positivity of \( \text{Vol}_R \) on the almost-Fuchsian space, which is an open subset of the space of quasi-Fuchsian metrics. While this improves the result of Krasnov and Schlenker, it does of course not prove positivity for every quasi-Fuchsian metric, that is therefore left for further studies.

2. Almost-Fuchsian hyperbolic 3-manifolds

\textbf{Definition 1.} A quasi-Fuchsian hyperbolic 3-manifold \((X, g)\) is a complete hyperbolic 3-manifold diffeomorphic to \(X = \mathbb{R} \times \Sigma_0\), where \(\Sigma_0\) is a compact Riemann surface of genus \(\geq 2\). The Riemannian metric \(g\) on \(X\) is described as follows. There exist \(t_0^- \leq t_0^+ \in \mathbb{R}\) such that the metric \(g\) on \([t_0^+, \infty) \times \Sigma_0\), respectively on \((-\infty, t_0^-) \times \Sigma_0\), is given by

\[
g = dt^2 + g_t^\pm, \quad g_t^\pm = g_0^\pm((\cosh(t) + A^\pm \sinh(t))^2, \cdot),
\]

where \(g_0^\pm\) is a metric on \(\Sigma_0^\pm = \{t_0^\pm\} \times \Sigma_0\) and \(A^\pm\) is a symmetric endomorphism of \(T\Sigma_0^\pm\) satisfying the Gauss and the Codazzi–Mainardi equations:

\[
d\nabla \Pi^\pm = 0.
\]

Here, \(\kappa^\pm\) is the Gaussian curvature of \((\Sigma_0^\pm, g_0^\pm)\) and \(d\nabla\) represents the de Rham differential twisted by the Levi–Civita connection acting on 1-forms with values in \(T^*\Sigma_0^\pm\). By definition, \(\Pi^\pm := g_0^\pm(A^\pm, \cdot, \cdot)\), called the second fundamental form of the embedding \(\Sigma_0^\pm \hookrightarrow X\), is the bilinear form associated to \(A^\pm\). Notice that the eigenvalues of \(A^\pm\) should be less than 1 in absolute value for the expression \(\Pi\) to be a well-defined metric for all \(t \in \mathbb{R}\).

An almost-Fuchsian hyperbolic 3-manifold is obtained as a small deformation of a Fuchsian hyperbolic 3-manifold, which, by definition, is the quotient of \(\mathbb{H}^3\) by the action of a co-compact Fuchsian group. In particular, Fuchsian hyperbolic 3-manifolds are almost-Fuchsian. More precisely, by \cite{10}, we have:

\textbf{Definition 2.} An almost-Fuchsian hyperbolic 3-manifold \(X\) is a quasi-Fuchsian hyperbolic 3-manifold such that \(X\) contains a closed minimal surface \(\Sigma\) whose principal curvatures belong to \((-1, 1)\).

\textbf{Remark 3.} By \cite{10} Theorem 3.3], an almost-Fuchsian hyperbolic 3-manifold \(X\) admits a unique minimally embedded surface \(\Sigma\), whose principal curvatures are thus in \((-1, 1)\). By taking \(\Sigma_0^\pm = \Sigma\), the expression \(\Pi\) is well-defined for all \(t \in \mathbb{R}\).
2.1. Funnel ends. In this subsection, consider the more general case of $X$ being a geometrically finite hyperbolic 3-manifold without rank-1 cusps. This means in particular that, outside a submanifold $K$ with compact boundary and finite volume, $X$ is isometric to a finite union of funnels, which are described as follows. A funnel is a cylinder $[t_0, \infty) \times \Sigma$ endowed with a hyperbolic metric $g$ of the form $[\Pi]$, where $A$ satisfies the Gauss and the Codazzi–Mainardi equations as above. A funnel has an obvious smooth compactification to a manifold with boundary, namely $\overline{X}$, which is isometric to a finite union of funnels, which are described as follows. A funnel has an asymptotic behavior at infinity, both depending at first sight on the product decomposition of each of the funnels of $X$. However, the surface at infinity is canonically identified with the space of half-infinite geodesics rays escaping from every compact, modulo the equivalence relation of being asymptotically close to each other.

Notice that the gradient of the function $t$, on a given funnel of $X$, is a geodesic vector field of length 1. In addition, if on a given funnel of $X$ we have a second function $t'$ with respect to which the metric $\tilde{g}$ takes the form $[\Pi]$, we obtain that the gradient flow of $t'$ defines another foliation $[\Sigma']$ of the funnel $[t_0, \infty) \times \Sigma$. Up to some compact subset of $X$, we can consider that $[t_0, \infty) \times \Sigma' \hookrightarrow [t_0, \infty) \times \Sigma$. The set $\{ p \in X; t'(p) \leq t_0 \} \subset X$ is geodesically convex, hence its boundary surface $\{ t_0 \} \times \Sigma'$ intersects each geodesic along the $t$ flow in a unique point; therefore, $\Sigma'$ is diffeomorphic to $\Sigma$. The identity map of $X$ extends smoothly on the corresponding compactifications induced by the two foliation structures; so the smooth compactification of $X$ is canonical (i.e., independent of the choice of the function $t$). Moreover, the induced metrics $h_0, h'_0$ with respect to the two foliations are conformal to each other. It follows that the metric $g$ induces a conformal class $[h_0]$ on $\{ \infty \} \times \Sigma \subset \partial_{\infty}X$. In particular, if we consider $(X, g)$ to be an almost-Fuchsian hyperbolic 3-manifold, we obtain that $g$ induces, respectively, two conformal classes $[h_0]$ on the boundary at infinity $\partial_{\infty}X = \{ \pm \infty \} \times \Sigma$.

2.2. The renormalized volume. Let $(X, g)$ be a geometrically finite hyperbolic 3-manifold without rank-1 cusps, with $n$ funnels. For each funnel $F_j$ of $X$ we choose a metric $h_j^0$ in the corresponding conformal class $[h_0^j]$ on the boundary at infinity of $F_j$. Recall that every metric $h_j^0 \in [h_0]$ is realized (near infinity) by a unique function $t_j$, which decomposes that funnel in the product $[t_j, \infty) \times \Sigma_j$ presented in Section 2.1.

We denote by $h_0$ the set of the above chosen metrics $\{h_0^j\}_{j=1,...,n}$ on the boundary at infinity of $X$. The renormalized volume of $X$ with respect to the metrics $h_0$ (or equivalently, with respect to the corresponding functions $t_j$) is defined via the so-called Riesz regularization.

**Definition 4.** Let $(X, g)$ be a hyperbolic 3-manifold which can be decomposed into a finite-volume open set $K$ and a finite number of funnels. As explained above, let $h_0$ be
a metric in the induced conformal class at infinity of $X$ and corresponding to $g$. The renormalized volume with respect to $h_0$ is defined by

$$\text{Vol}_R(X, g; h_0) := \text{Vol}(K) + \text{FP}_{z=0} \int_{X \setminus K} e^{-z|t|} dg,$$

where by FP we denote the finite part of a meromorphic function.

In Definition [4], we implicitly have used the fact, which follows from the proof of Proposition [5] below, that the integral in the right-hand side is meromorphic in $z$. In Krasnov–Schlenker [7], the renormalized volume is defined by integrating the volume form on increasingly large bounded domains and discarding some explicit terms which are divergent in the limit. We refer, for example, to [1] for a discussion of the link between these two types of renormalizations. For the sake of completeness, we include here a proof of the equality between these two definitions. Some care is needed since the addition in the definition of an universal constant, harmless in [2] or [5], drastically alters the positivity properties of $\text{Vol}_R$.

Assume that $X$ has $n$ funnels with fixed foliation structure $[t_j, \infty) \times \Sigma_j$, where $j \in \{1, \ldots, n\}$. Choose $t_0 := \max\{t_j; j \in \{1, \ldots, n\}\}$ and set $\Sigma = \bigsqcup_{j=1}^n \Sigma_j$, so that $[t_0, \infty) \times \Sigma$ is embedded in $X$. For $t \geq t_0$, let $H^t : \{t\} \times \Sigma \to \mathbb{R}$ denote the mean curvature function of the equidistant surfaces in the foliation at time $t$. Let $K_t$ denote the complement in $X$ of the funnels $[t, \infty) \times \Sigma$.

**Proposition 5.** The quantity

$$\text{Vol}_{KS}(X, g; h_0) := \text{Vol}(K_t) - \frac{1}{4} \int_{\Sigma_t} H^t dg_t + t\pi\chi(\Sigma),$$

called the (Krasnov–Schlenker) renormalized volume $\text{Vol}_{KS}$, is independent of $t \in [t_0, \infty)$, and coincides with the renormalized volume $\text{Vol}_R(X, g; h_0)$.

**Proof.** Let

$$g = dt^2 + g_t, \quad g_t = g_0((\cosh(t) + A\sinh(t))^{2-}, \cdot)$$

be the expression of the metric $g$ in the fixed product decomposition of the funnels $[t, \infty) \times \Sigma$, as in [11]. To fix the notations, let $\Pi^t$ denote the second fundamental form of the surface $\Sigma_t = \{t\} \times \Sigma$. Recall that $\Pi^t = g_t(A_t, \cdot) = \frac{1}{2}g_t'$ and $H^t = \text{Tr}(A_t)$. For every $t \in [t_0, \infty)$, one obtains

$$dg_t = [\cosh^2 t + \det(A)\sinh^2(t) + \text{Tr}(A)\cosh(t)\sinh(t)]dg_0$$

$$= [\cosh^2 t + (\kappa_{g_0} + 1)\sinh^2(t) + H^0\cosh(t)\sinh(t)]dg_0$$

(in the second line we used [2] and the definition of $H^0$), and

$$\frac{1}{2}g_t' = g_0((\cosh(t) + A\sinh(t))(\cosh(t) + A\sinh(t))'\cdot, \cdot)$$

$$= g_t((\cosh(t) + A\sinh(t))^{-2}(\cosh(t) + A\sinh(t))(\cosh(t) + A\sinh(t))'\cdot, \cdot)$$

$$= g_t((\cosh(t) + A\sinh(t))^{-1}(\sinh(t) + A\cosh(t))\cdot, \cdot),$$
so

\[ A_t = (\cosh(t) + A \sinh(t))^{-1}(\sinh(t) + A \cosh(t)). \]

Denote by \( \lambda_1, \lambda_2 \) the eigenvalues of the symmetric endomorphism \( A \), so \( \lambda_1 + \lambda^2 = H^0 \) and \( \lambda_1 \lambda_2 = \kappa_{g_0} + 1 \). We deduce

\[ H^t dg_t = (\cosh(2t)H^0 + \sinh(2t)(\kappa_{g_0} + 2))dg_0. \]

Let us prove first that \( \text{Vol}_{\text{KS}}(X, g; h_0) \) is independent of \( t \). For this it is enough to calculate the derivative of \( \text{Vol}_{\text{KS}}(X, g; h_0) \) with respect to \( t \) and to see that this is zero. We have:

\[
\frac{d}{dt} \left( \text{Vol}(K_t) - \frac{1}{4} \int_{\Sigma_t} H^t dg_t \right) \\
= \frac{d}{dt} \left( \text{Vol}(K_{t_0}) + \int_{t_0}^t \int_{\Sigma_x} dg_x dx - \frac{1}{4} \int_{\Sigma_t} H^t dg_t \right) \\
= \int_{\Sigma_t} dg_t - \frac{1}{4} \frac{d}{dt} \left( \int_{\Sigma_t} H^t dg_t \right) \\
= \int_{\Sigma} \left[ \cosh^2(t) + (\kappa_{g_0} + 1)(\sinh(t))^2 + H^0 \cosh(t) \sinh(t) \right] dg_0 \\
- \frac{1}{4} \frac{d}{dt} \left( \int_{\Sigma} (\cosh(2t)H^0 + \sinh(2t)(\kappa_{g_0} + 2))dg_0 \right) \\
= \cosh(2t) \int_{\Sigma} dg_0 + \sinh^2(t)2\pi\chi(\Sigma) + \frac{1}{2} \sinh(2t) \int_{\Sigma} H^0 dg_0 \\
- \frac{1}{2} \sinh(2t) \int_{\Sigma} H^0 dg_0 - \cosh(2t)\pi\chi(\Sigma) - \cosh(2t) \int_{\Sigma} dg_0 \\
= -\pi\chi(\Sigma).
\]

From here it is immediate that, for every \( t \in [t_0, \infty), \)

\[
\text{Vol}(K_t) - \frac{1}{4} \int_{\Sigma_t} H^t dg_t + t\pi\chi(\Sigma) = \text{Vol}(K_{t_0}) - \frac{1}{4} \int_{\Sigma} H^0 dg_0 + t_0\pi\chi(\Sigma).
\]

We thus recover Lemma 3.6 from \[8\]. Let us now prove the second part of the proposition. By the Gauss equation and the expressions of \( dg_t \) and \( H^t g_t \) with respect to \( dg_0 \) we have:
\[
\begin{align*}
\text{FP}_{z=0} \int_{X \setminus K_t} e^{-z|x|} dg &= \text{FP}_{z=0} \int_{\Sigma_t} e^{-z|x|} d\gamma_t \, dx \\
&= \text{FP}_{z=0} \int_{\Sigma_t} \left( e^{2z} \left( \frac{\kappa_{g_{t}}}{4} + \frac{1}{2} + \frac{H^t}{4} \right) + e^{-2z} \left( \frac{\kappa_{g_{t}}}{4} + \frac{1}{2} - \frac{H^t}{4} \right) - \frac{\kappa_{g_{t}}}{2} \right) d\gamma_t \, dx \\
&= \text{FP}_{z=0} \left[ \frac{1}{z - 2} e^{(2-z)t} \int_{\Sigma_t} \left( \frac{\kappa_{g_{t}}}{4} + \frac{1}{2} + \frac{H^t}{4} \right) d\gamma_t \\
&\quad + \frac{1}{2 + z} e^{-(z+2)t} \int_{\Sigma_t} \left( \frac{\kappa_{g_{t}}}{4} + \frac{1}{2} - \frac{H^t}{4} \right) d\gamma_t + \frac{e^{zt}}{z} \int_{\Sigma_t} \frac{\kappa_{g_{t}}}{2} d\gamma_t \right] \\
&= -\frac{1}{2} e^{2t} \int_{\Sigma_t} \left( \frac{\kappa_{g_{t}}}{4} + \frac{1}{2} + \frac{H^t}{4} \right) d\gamma_t + \frac{1}{2} e^{-2t} \int_{\Sigma_t} \left( \frac{\kappa_{g_{t}}}{4} + \frac{1}{2} - \frac{H^t}{4} \right) d\gamma_t + t\pi \chi(\Sigma) \\
&= -\sinh(2t) \int_{\Sigma_t} \frac{\kappa_{g_{t}}}{4} + \frac{2}{4} d\gamma_t - \cosh(2t) \int_{\Sigma_t} \frac{H^t}{4} d\gamma_t + t\pi \chi(\Sigma) \\
&= -\frac{1}{4} \int_{\Sigma_t} H^t d\gamma_t + t\pi \chi(\Sigma).
\end{align*}
\]

This proves that \( \text{Vol}_R(X, g; h_0) = \text{Vol}_{KS}(X, g; h_0) \). \( \square \)

Given a geometrically finite hyperbolic 3-manifold \((X, g)\) without rank-1 cusps, we would like to have a canonical definition of the renormalized volume, which does not depend on the additional choices of the metrics at infinity of \( X \).

**Definition 6.** The renormalized volume \( \text{Vol}_R(X, g) \) is defined as \( \text{Vol}_R(X, g; h_{\text{F}}) \), where the metrics \( h_{\text{F}} \) at infinity of \( X \) that are used for the renormalization procedure are the unique metrics in the conformal class \([h_0]\) having constant Gaussian curvature \( -4 \).

Notice that, by the Gauss–Bonnet formula, the area, with respect to \( h_{\text{F}} \), of the boundary at infinity \( \{\infty\} \times \Sigma \) of each funnel of \( X \) equals \( -\frac{\pi \chi(\Sigma)}{2} \). The following lemma appears in [7, Section 7]; for completion we include below a proof using our current definition of renormalized volume.

**Lemma 7.** Let \((X, g)\) be a geometrically finite hyperbolic 3-manifold without rank-1 cusps. Among all metrics \( h_0 \in [h_0] \) of area equal to \( -\frac{\pi \chi(\Sigma)}{2} \), the renormalized volume \( \text{Vol}_R(X, g; h_0) \) attains its maximum for \( h_0 = h_{\text{F}} \).

The lemma holds evidently for every metric with constant Gaussian curvature \( \kappa < 0 \), when we maximize among metrics of area \( -\frac{2\pi \chi(\Sigma)}{\kappa} \).
Proof. From [3], recall the conformal change formula of the renormalized volume: Let \( h \) be a metric at infinity of \((X, g)\) and multiply \( h \) by \( e^{2\omega} \), for some smooth function \( \omega : \{\infty\} \times \Sigma \to \mathbb{R} \). We have that
\[
\text{Vol}_R(X, g; e^{2\omega}h) = \text{Vol}_R(X, g; h) - \frac{1}{4} \int_\Sigma (|d\omega|^2_h + 2\kappa_h\omega) dh.
\]
In particular, for \( h = h_\tau \) we obtain
\[
\text{Vol}_R(X, g; e^{2\omega}h_\tau) - \text{Vol}_R(X, g; h_\tau) \leq 2 \int_\Sigma \omega dh_\tau.
\]
Now, we assume that \( e^{2\omega}h_\tau \) has the same area as \( h_\tau \); so \( \int_\Sigma e^{2\omega}dh_\tau = \int_\Sigma dh_\tau \). Write \( \omega = c + \omega^\perp \), with \( c \) being a constant and \( \int_\Sigma \omega^\perp dh_\tau = 0 \). Using the inequality \( e^x \geq 1 + x \), valid for all real numbers \( x \), we get
\[
\int_\Sigma dh_\tau = \int_\Sigma e^{2\omega}dh_\tau = e^{2c} \int_\Sigma (1 + 2\omega^\perp)dh_\tau = e^{2c} \int_\Sigma dh_\tau,
\]
implying that \( c \leq 0 \). Hence \( \int_\Sigma \omega dh_\tau = \int_\Sigma c dh_\tau \leq 0 \), proving the assertion of the lemma, in light of (4).

Moreover, when dilating \( h_0 \) by a constant greater than 1, the renormalized volume increases. More precisely, we have:

**Lemma 8.** Let \((X, g)\) be a geometrically finite hyperbolic 3-manifold without rank-1 cusps. Let \( c > 0 \) and let \([h_0]\) be the induced conformal class on the boundary at infinity of \( X \), which by abuse of notation is denoted \( \Sigma \). Let \( h_0 \) be a metric in \([h_0]\). Then
\[
\text{Vol}_R(X, g; c^2h_0) = \text{Vol}_R(X, g; h_0) - \pi \chi(\Sigma) \ln c.
\]

**Proof.** This is a particular case of the formula (3) for the conformal change of the renormalized volume, in which \( \omega \) is constant:
\[
\text{Vol}_R(X, g; e^{2\omega}h_0) = \text{Vol}_R(X, g; h_0) - \frac{1}{4} \omega \int_\Sigma 2\kappa h_0 dh_0 = \text{Vol}_R(X, g; h_0) - \omega \pi \chi(\Sigma)
\]
(in the last equality we have used the Gauss–Bonnet formula). \(\square\)

### 2.3. The renormalized volume of almost-Fuchsian hyperbolic 3-manifolds.

Let \( X \) be an almost-Fuchsian hyperbolic 3-manifold. By [10], recall that \( X \) contains a unique embedded minimal surface, which we denote \( \Sigma \) in what follows. By considering the decomposition \( X = \Sigma \times \mathbb{R} \) (see Remark [3] we obtain two metrics \( h^+_0, h^-_0 \) in the corresponding conformal classes at \( \pm\infty \) of \((X, g)\). Those are defined by \( h^\pm_0 := (e^{-2|t|}g)|_{t=\pm\infty} \). Using the Krasnov–Schlenker definition of the renormalized volume from Proposition [5] it is easy to see that, with respect to the globally defined function \( t \) on the almost-Fuchsian hyperbolic 3-manifold \( X \), we have
\[
\text{Vol}_R(X, g; h^\pm_0) = 0.
\]
This quantity is therefore not very interesting, but it will prove helpful when examining \( \text{Vol}_R(X, g) \). (Note that the vanishing of \( \text{Vol}_R(X, g; h^\pm_0) \) is essentially the content of
Proposition 3.7 in [5], where a slightly different definition is used for the renormalized volume. In Huang–Wang [5] the renormalized volume $\text{Vol}_R(X, g; h^\pm_0)$ equals $-2\pi(g-1)$ independently of the metric on $X$, and its sign is interpreted as some sort of “negativity of the mass”. We defend here the view that the Krasnov–Schlenke definition seems to be the most meaningful, as opposed to [3] or [5], and that with this definition the sign of the volume seems to be positive.)

Our goal is to control the renormalized volume of $(X, g)$ when the metric at $\pm\infty$ is $h^\pm_0$, the unique metrics of Gaussian curvature $-4$ inside the corresponding conformal class $[h^\pm_0]$ at infinity $X$. Recall from Definition 6 that, for this canonical choice (with non-standard constant $-4$) we obtain “the” renormalized volume of the almost-Fuchsian hyperbolic 3-manifold $(X, g)$:

$$\text{Vol}_R(X, g) = \text{Vol}_R(X, g; h^\pm_0).$$

**Theorem 9.** The renormalized volume $\text{Vol}_R(X, g)$ of an almost-Fuchsian hyperbolic 3-manifold $(X, g)$ is non-negative, being zero only at the Fuchsian locus, i.e., for $g$ as in Definition 4 with $A = 0$ and $g_0$ hyperbolic.

**Proof.** Denote the principal curvatures of the unique embedded minimal surface $\Sigma$ of $X$ by $\pm\lambda$ for some continuous function $\lambda : \Sigma \to [0, \infty)$. Recall that $\sup_{x \in \Sigma} |\lambda(x)| < 1$ and that the decomposition of the metric $g$ takes the form (1), for all $t \in \mathbb{R}$.

**Lemma 10.** The Gaussian curvature of $h^\pm_0$ is bounded above by $-4$.

**Proof.** Let $\Sigma_t$ be the leaf of the foliation at time $t$. We compute the Gaussian curvature $\kappa_{h^+_0}$ as the limit of the curvature of $e^{-2t} g_t$ as $t \to +\infty$. The shape operator of $\Sigma_t$ (i.e., at time $t$) is $W_t = \frac{1}{2} g_t^{-1} \dot{g}_t = (A \cosh t + \sinh t)(\cosh t + A \sinh t)^{-1}$. Let $\lambda : \Sigma \to [0, 1)$ be the non-negative principal curvature of $\Sigma \hookrightarrow X$. By the hyperbolic Gauss equation (2), we get

$$\kappa_{g_t} = \det W_t - 1 = \det[(1 + A - e^{-2t}(1 - A))(1 + A + e^{-2t}(1 - A))^{-1}] - 1 = \frac{(1 - e^{-2t(1-\lambda)})(1 - e^{-2t(1+\lambda)})}{(1 + e^{-2t(1+\lambda)})(1 + e^{-2t(1-\lambda)})} - 1$$

so $\kappa_{e^{-2t}g_t} = e^{2t}\kappa_{g_t}$ converges to $-2 \left(\frac{1-\lambda}{1+\lambda} + \frac{1+\lambda}{1-\lambda}\right) \leq -4$ as $t \to \infty$. The inequality for $h^\pm_0$ is proved similarly.

Using the Gauss–Bonnet formula, Lemma 10 implies that the area of $(\{\pm\infty\} \times \Sigma, h^\pm_0)$ is at most equal to $\pi\chi(\Sigma)/2$, which, again by Gauss–Bonnet, is the area of $h^\pm_0$:

$$-4 \int_\Sigma dh^\pm_0 \geq \int_\Sigma \kappa_{h^\pm_0} dh^\pm_0 = 2\pi\chi(\Sigma) = -4 \int_\Sigma dh^\pm_0.$$

So

$$\text{Vol}(\Sigma, h^\pm_0) \leq \text{Vol}(\Sigma, h^\pm_0) = -\pi\chi(\Sigma)/2.$$
Let $c^2 := \frac{\text{Vol}(\Sigma, h_{\pm}^F)}{\text{Vol}(\Sigma, h_0^\pm)}$. By (5), $c \geq 1$. Applying Lemma 8 we obtain

$$\text{Vol}_R(X, g; h_0^\pm) \leq \text{Vol}_R(X, g; c^2 h_0^\pm).$$

Since, by definition $\text{Vol}(\Sigma, c^2 h_0^\pm) = \text{Vol}(\Sigma, h_0^\pm)$, Lemma 7 implies

$$\text{Vol}_R(X, g; c^2 h_0^\pm) \leq \text{Vol}_R(X, g; h_0^\pm) = \text{Vol}_R(X, g).$$

These inequalities are enough to conclude that $\text{Vol}_R(X, g) \geq \text{Vol}_R(X, g; h_0^\pm) = 0$. □

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