IRREDUCIBLE POLYNOMIALS WITH VARYING CONSTRAINTS ON COEFFICIENTS

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Abstract

We study the number of prime polynomials of degree $n$ over $\mathbb{F}_q$ in which the $i^{th}$ coefficient is either preassigned to be $a_i \in \mathbb{F}_q$ or outside a small set $S_i \subset \mathbb{F}_q$. This serves as a function field analogue of a recent work of Maynard, which counts integer primes that do not have specific digits in their base-$q$ expansion. Our work relates to Pollack’s and Ha’s work, which count the amount of prime polynomials with $\ll \sqrt{n}$ and $\ll n$ preassigned coefficients, respectively. Our result demonstrates how one can prove asymptotics of the number of prime polynomials with different types of constraints to each coefficient.

1 Introduction

The number of prime polynomials $P$ of degree $n$ over a finite field $\mathbb{F}_q$ has been known to be approximately $q^n/n$ for quite some time. A lot of work has been done on finding the distribution of primes that satisfy a particular condition. Writing

$$P = T^n + \sum_{i=0}^{n-1} b_i T^i,$$

we might ask how many primes $P$ exist for which the tuple $(b_0, b_1, ..., b_{n-1})$ satisfies given conditions. A natural condition is one of the form $b_i = a$ for some $0 \leq i \leq n - 1$, $a \in \mathbb{F}_q$. More generally, let $\mathcal{I} \subset \{0, \ldots, n - 1\}$ be a set of indices, and denote $I = \#\mathcal{I}$. Let $\{a_i\}_{i \in \mathcal{I}}$ be corresponding coefficients in $\mathbb{F}_q$, satisfying $a_0 \neq 0$ if $0 \in \mathcal{I}$. Denote by $\mathcal{A}$ the set of all monic, degree $n$ polynomials such that the coefficient of $T^i$ is $a_i$ for all $i \in \mathcal{I}$:

$$\mathcal{A} = \left\{ T^n + \sum_{j=0}^{n-1} b_j T^j \in \mathbb{F}_q[T] : b_i = a_i \ \forall i \in \mathcal{I} \right\}$$

A topic of investigation has been to understand the distribution of prime polynomials in $\mathcal{A}$. Hansen and Mullen conjectured that whenever $n \geq 3$ and $\#\mathcal{I} = 1$, the set $\mathcal{A}$ contains prime polynomials. This conjecture was proven by Wan [6] if $n$ or $q$ are sufficiently large, and the remaining cases were later solved by Ham and Mullen in [2].

Further work went into finding the asymptotic behavior of the number of primes in $\mathcal{A}$. One might expect the number of primes in the set $\mathcal{A}$ to be $\frac{1}{q} \pi_q(n)$ in case $0 \notin \mathcal{I}$, and $\frac{1}{q^\nu(q-1)} \pi_q(n)$ if $0 \in \mathcal{I}$. Pollack [4] proves that the asymptotics are indeed so, provided that $\#\mathcal{I} = I < c\sqrt{n}$ with $c < 1$. Further progress has been made by Ha in [1], where existence of primes in $\mathcal{A}$ is proven given that $q$ is sufficiently large with respect to $\#\mathcal{I}/n$. We prove a weaker result than in [1], which generalizes Pollack’s theorem in [4]:
Theorem 1.1. Given $\mathcal{A}, \mathcal{I}, \{a_i\}_{i \in \mathcal{I}} \subset \mathbb{F}_q$ as before, write $I = \# \mathcal{I}$, $m_{n, I} = \min\{n/I, \sqrt{n}\}$, and $\rho = I/n$. Denote $G = q^{-I}$ if $0 \notin \mathcal{I}$, and $G = q^{-\left(I-1\right)(q-1)}$ otherwise. If $I = o\left(n/\log(n)\right)$, then

$$\left| \left( \sum_{P \in \mathcal{A}} 1 \right) - G \cdot \pi_q(n) \right| \leq (1 + o(1)) q^{\frac{-1}{2} \left\lfloor \frac{n}{I} \right\rfloor} + q^{-I} q^{-\left(1+o(1)\right)m_{n, I}}.$$

If $I > 2\sqrt{n}$, the following bound holds:

$$\left| \left( \sum_{P \in \mathcal{A}} 1 \right) - G \cdot \pi_q(n) \right| \leq (1 + o(1)) q^{\frac{-1}{2} \left\lfloor \frac{n}{I} \right\rfloor} + q^{-I} q^{-n/\left(4I + B_{q, \rho}\right)}.$$

with $B_{q, \rho}$ tending to zero as $q$ grows to infinity provided that $n$ is sufficiently large in terms of $\rho$.

Another natural question one might ask is how many primes satisfy the condition $b_i \neq a$ for all $0 \leq i < n$, given $a \in \mathbb{F}_q$. A surprising result by Maynard [5] shows that the number of rational primes without a specific digit in its decimal expansion is of the correct asymptotic. Maynard further proves that the correct asymptotic is kept when taking coefficients in the $q$-basis outside of a set $S \subset \{0, \ldots, q-1\}$ of size $\#S < q^{23/80}$. We adopt the method of Maynard in [5] to obtain an analogous result in function fields. For $a \in \mathbb{F}_q$, denote

$$\mathcal{B} = \left\{ T^n + \sum_{j=0}^{n-1} b_j T^j \in \mathbb{F}_q[T] : b_j \neq a \ \forall \ 0 \leq j < n \right\} \quad (1.2)$$

The expected number of prime polynomials in $\mathcal{B}$ is $G_a \cdot \pi_q(n)$, with $G_a = \frac{(q-1)^{n-1}}{q^n}$ if $a = 0$, and $G_a = \frac{(q-1)^{n-1}}{q^n} \cdot \frac{2q-2}{q-1}$ otherwise.

Theorem 1.2. Let $q \geq 5$, and let $\mathcal{B}$, $a$, and $G_a$ be defined as before. Then

$$\left| \left( \sum_{P \in \mathcal{B}} 1 \right) - G_a \cdot \pi_q(n) \right| \leq \left( 2^n + o(1) \right) q^{n-\frac{1}{2} \left\lfloor \frac{n}{I} \right\rfloor} + nq^{n-c\sqrt{n}} + O(q^{n/2} + 1),$$

with $c = \sqrt{\left(1 - \log_q 2\right)\left(1 - 2 \log_q 2\right)}$.

Note that the result is valid whenever $q \geq 5$, but it is only useful when $2^n q^{n-\frac{1}{2} \left\lfloor \frac{n}{I} \right\rfloor} \ll q^n$, i.e. when $q \geq 17$.

In this thesis, we consider sets $\mathcal{C}$ that combine the two constraints, and prove a theorem that generalizes both Theorem 1.1 and Theorem 1.2. Let $\mathcal{I} \cup \mathcal{J}$ be a partition of $\{0, \ldots, n-1\}$, and write $I = \# \mathcal{I}$. Let $\{a_i\}_{i \in \mathcal{I}} \subset \mathbb{F}_q$ be such that $a_0 \neq 0$ if $0 \notin \mathcal{I}$. Consider sets $S_i \subset \mathbb{F}_q$ for every $i \in \mathcal{J}$, and write $N_i = \# S_i$. Moreover, assume that $0 \notin S_0$ if $0 \notin \mathcal{J}$. Denote

$$\mathcal{C} = \left\{ T^n + \sum_{i=0}^{n-1} b_i T^i : b_i = a_i \ \forall i \in \mathcal{I}, b_j \notin S_j \ \forall j \in \mathcal{J} \right\}. \quad (1.3)$$
Moreover, \( \alpha \) essentially is prescribed to be \( a_i \), and the rest of the coefficients are outside small sets \( S_i \).

The number of primes to be expected in \( C \) is \( \mathcal{G} \cdot \pi_q(n) \), with

\[
\mathcal{G} = \begin{cases} 
\frac{\prod_{i \in I}(q - N_i)}{q^{n/I}(q - 1)} & \text{if } 0 \in I \\
\frac{(q - N_0) \prod_{i \in I}(q - N_i)}{q^{m - N_1}(q - 1)} & \text{if } 0 \in J.
\end{cases}
\]

(1.4)

In section 5 we give a brief explanation why this is indeed the asymptotic one might expect.

For convenience, we define

\[
\alpha(m) = \sup_{i_j < e < i_m, j = 1}^m (N_{i_j} + 1).
\]

(1.5)

Essentially, \( \alpha(m) \) is “small” if the averages of all subsets of \( \{N_j\}_{j \in J} \) of size \( m \) are “small”. Moreover, \( \alpha(n - I) = \prod_{j \in J}(N_j + 1) \). We are now ready to state our main theorem.

**Theorem 1.3.** Let \( n \geq 2 \), and let \( I \cup J = \{0, \ldots, n - 1\} \). To each \( i \in I \) assign \( a_i \in \mathbb{F}_q \), and for every \( j \in J \) assign a set \( S_j \subset \mathbb{F}_q \). Denote \( I = \#I \), \( N_j = \#S_j \) and let \( \alpha, \mathcal{C} \) be defined as in (1.3) and (1.4). Assume that for all \( j \in J \), we have \( N_j < q^\varepsilon \) with \( \varepsilon < 1 \). Assume further that \( \frac{1}{n} < \frac{1 - \varepsilon}{4(1 - \varepsilon)} (1 - \tau) \) for some \( \tau > 0 \). Denote \( s = \sqrt{(1 - \varepsilon)/(1 - 2\varepsilon)} \sqrt{n} \).

If \( I = o(n/\log(n)) \), then

\[
\left| \sum_{P \in \mathcal{C}} 1 - \mathcal{G} \cdot \pi_q(n) \right| \leq (\alpha(n - I) + o(1)) q^{n - \frac{1}{4} \log_q(n)} + q^{n - I} q^{-(1 - 2\varepsilon) o(1)n/I + 4 - 3\varepsilon + B_{q, \varepsilon, \tau, y}},
\]

where \( P \) ranges only over prime polynomials, with \( m_{n, I, \varepsilon} = \min\{n/I, s\} \), and \( \mathcal{G} \) is given in (1.4). If \( y = \frac{i_m}{n} > 1 \) and \( n \) is sufficiently large in terms of \( \varepsilon \) and \( \tau \), the following bound holds:

\[
\left| \sum_{P \in \mathcal{C}} 1 - \mathcal{G} \cdot \pi_q(n) \right| \leq (\alpha(n - I) + o(1)) q^{n - \frac{1}{4} \log_q(n)} + q^{n - I} q^{-(1 - 2\varepsilon) n/I + 4 - 3\varepsilon + B_{q, \varepsilon, \tau, y}},
\]

with \( B_{q, \varepsilon, \tau, y} \) tending to zero as \( q \) grows to infinity.

Note that these bounds are only useful when \( \alpha(n - I) < q^{n/4} \), since the first term in the error term is about \( q^{-\frac{1}{4} \log_q(n)} \). For example, as stated before, when \( N_i = 1 \) for all \( 0 \leq i < n \), the result is interesting for \( q \geq 17 \).

Note that Theorem 1.3 is indeed a generalization of Theorems 1.1 and 1.2, since the definition of the set \( C \) in (1.3) is more general than (1.1) and (1.2). Indeed, if \( S_i = \emptyset \), then the condition \( b_i \notin S_i \) is trivially satisfied. Therefore, if we choose \( S_i = \emptyset \) for all \( i \in J \), then \( C \) will be a set in the form of (1.1). On the other hand, if \( I = \emptyset \), and \( S_0 = \cdots = S_{n-1} = \{a\} \), then the set \( C \) will be in the form of (1.2).
1.1 Notation and definitions.

Denote by
\[ F_q(T)_\infty = \left\{ \sum_{l<k} a_l T^l : a_l \in F_q, k \in \mathbb{Z} \right\}, \]
the completion of \( F_q(T) \) with respect to \( 1/T \). We define the unit interval
\[ \mathcal{U} := \left\{ \sum_{l<0} a_l T^l : a_l \in F_q \right\} \subset F_q(T)_\infty. \]

Denote by \( \psi(\cdot) \) the additive character on \( F_q \) defined by
\[ \psi(a) = \exp \left( \frac{2\pi i}{p} \text{Tr}(a) \right), \quad (1.6) \]
where the trace is taken from \( F_q \) down to its prime field \( F_p \). The Euler totient function is denoted by \( \varphi \). We define the map \( e : F_q(T)_\infty \to \mathbb{C} \) by
\[ e \left( \sum_{i=-\infty}^n a_i T^i \right) = \psi(a_{-1}). \quad (1.7) \]

We denote the set of monic polynomials of degree \( k \) by \( M_k \). We also denote the function field analogue of the usual exponential sum over primes by
\[ f(\theta) := \sum_{P \in M_n} e(\theta P), \quad (1.8) \]
where the sum ranges over the monic irreducible polynomials of degree \( n \). For \( \beta = \sum_{l<k} \beta_l T^l \in F_q(T)_\infty \), denote by \( \{\beta\} \) the fractional part given by
\[ \{\beta\} = \sum_{l<0} \beta_l T^l \in \mathcal{U}. \]

We define \( K_{x,m} = \{x, \ldots, x+m-1\} \) for \( x \geq 0 \). We use \( 1 \) to denote the indicator function, so for a set \( A \) we define
\[ 1_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}. \quad (1.9) \]

We let \( \mathcal{P} \) denote the set of prime polynomials in \( F_q[T] \).
1.2 Outline of Proof

Defining the set $C$ as in (1.3), we perform a Fourier transform on the indicator function $1_C : \mathcal{M}_n \to \{0, 1\}$:

$$\hat{F}_{q,n}(\theta) = \sum_{G \in \mathcal{M}_n} 1_C(G)e(G\theta). \quad (1.10)$$

The Fourier Inversion Theorem then gives

$$1_C(F) = \frac{1}{q^n} \sum_{G \in \mathcal{M}_n} \hat{F}_{q,n}(T^{-n}G)e(-T^{-n}GF), \quad (1.11)$$

and by Parseval’s Formula and (1.8), we get an analytic expression to the prime counting function in Theorem 1.3:

$$\sum_{P \in \mathcal{C}} 1 = \sum_{G \in \mathcal{M}_n} 1_P(G)1_C(G) = \frac{1}{q^n} \sum_{F \in \mathcal{M}_n} \hat{F}_{q,n}(T^{-n}F)f(-T^{-n}F). \quad (1.12)$$

We prove (1.11) and (1.12) in the beginning of Section 3.2.

The main term of (1.12) comes from polynomials $F \in \mathcal{M}_n$ of the form $F = T^n + aT^{n-1}$ with $a \in \mathbb{F}_q$. This is shown in Section 4.4. For the rest of the polynomials $F \in \mathcal{M}_n$ we bound $|f(T^{-n}F)|$ and $|\hat{F}_{q,n}(T^{-n}F)|$.

Section 2 states circle-method bounds from the literature for $|f|$, which are due to Hayes [3] in the setting of Pollack [4].

Section 3, which is the main part of the work, gives bounds for $|\hat{F}_{q,n}|$.

Section 4 derives the proof of Theorem 1.3.

2 Circle Method Bounds

Lemma 2.1. For each $\theta \in \mathcal{U}$, there is a unique pair of coprime polynomials $G, H \in \mathbb{F}_q[T]$ with $H$ monic, $\deg G < \deg H \leq n/2$, and

$$\left|\left\{\theta - \frac{G}{H}\right\}\right| < \frac{1}{q^{\deg H + n/2}}.$$

This is an analogue of a well known result of Dirichlet’s theorem proven by Hayes in 1966. From now on, we will use the notation $|\theta - G/H|$ as an abbreviation of $|\left\{\theta - G/H\right\}|$.

Lemma 2.2. Let $n \geq 2$. Let $\theta \in \mathbb{F}_q(T)_\infty$ and choose $G, H$ as in Lemma 2.1. Then if $1, T \neq H$ is squarefree and $|\theta - G/H| < q^{-n}$, then

$$|f(\theta)| \leq q^{-\frac{1}{2} \lfloor \frac{n}{2} \rfloor} + q^{n - \deg H}.$$

If $|\theta - G/H| \geq q^{-n}$ or $H$ is not squarefree,

$$|f(\theta)| \leq q^{-\frac{1}{2} \lfloor \frac{n}{2} \rfloor}.$$
Proof. The case where $|\theta - G/H| \geq q^{-n}$ or $H$ is not squarefree is immediate from the statement of Pollack [3 Lemma 5]. The assertion of the lemma in the case where $1, T \neq H$ is squarefree and $|\theta - G/H| < q^{-n}$ is proven in Pollack [4 Lemma 6].

3 Bounds on Fourier Coefficients

In this section we give bounds for $|\hat{F}_{q,n}|$. They are later used in the setting of (1.12), where the bounds of $f$ are taken from Lemma 2.2. From this reason, we need bounds for general $\theta$, and bounds for $\theta$ of the form $\theta = \{G/H\} \in \mathcal{U}$, with $H \neq 1, T$ squarefree and $G, H$ coprime. We call the latter part ”fractions”. The bounds for general $\theta$ are obtained in Subsection 3.2, while the bounds for $\theta$ that are fractions are obtained in Subsection 3.3.

3.1 Auxiliary Results

Lemma 3.1. Let $1, T \neq H \in \mathbb{F}_q[T]$ be a monic squarefree polynomial of degree $h$, and $G \in \mathbb{F}_q[T]$ coprime to $H$. Write $\theta = \sum_{j<0} \theta_j T^j = \{G/H\}$. There is no $i < 0$ such that $\theta_j = 0$ for all $i-h < j \leq i$. Equivalently, there are no $h$ consecutive zeros in the coefficients of $\theta$.

Proof. Write $H = T^h + \sum_{j=0}^{h-1} h_j T^j$. Since $H \neq 1, T$ is squarefree, there is a polynomial $H_1 \mid H$ such that $\deg H_1 \geq 1$, and $H_1$ is coprime to $T$. Since $G$ is coprime to $H$, $G$ is also coprime to $H_1$. Hence for every $k \in \mathbb{N}$, we have that $T^k G$ is coprime to $H_1$, hence $T^k \theta \notin \mathbb{F}_q[T]$. In other words, for every $k \in \mathbb{N}$ there exists $i < -k$ such that $\theta_i \neq 0$. Assume by way of contradiction that there is $i_0 < 0$ for which $\theta_{i_0} = \theta_{i_0-1} = \ldots = \theta_{i_0-h+1} = 0$. Since we know there are infinitely many $i < 0$ for which $\theta_i \neq 0$, we can assume without loss of generality that $\theta_{i_0-h} \neq 0$. Denote $\tau = H \theta$, and write $\tau = \sum_{i<h} \tau_i T^i$. Observing $\tau_{i_0}$, we get

$$\tau_{i_0} = \theta_{i_0-h} + \sum_{j=0}^{h-1} h_j \theta_{i_0-j}.$$ 

Since $\theta_i = 0$ for all $i-h+1 \leq i \leq i_0$, we get that $\tau_{i_0} = \theta_{i_0-h} \neq 0$. This is a contradiction to the choice of $\theta$ which implies that $\tau = H\theta = H \cdot \{G/H\} \in \mathbb{F}_q[T]$. 

Lemma 3.2. Take $h \in \mathbb{N}$, $0 \leq x \in \mathbb{Z}$, and $\theta \in \mathcal{U}$. Then there are at most $q$ distinct pairs $G, H \in \mathbb{F}_q[T]$ such that $H \neq 1, T$ is squarefree of degree $h$, $G$ is coprime to $H$ and of smaller degree, and $|T^x G/H - \theta| < q^{-2h}$. Moreover, if $x = 0$, then there is at most one such pair.

Proof. Assume $(G_1, H_1)$ is a pair that satisfies the conditions of the lemma, and define

$$H'_1 = H_1 / \gcd(H_1, T^x),$$  

$$G'_1 = (G_1 T^x / \gcd(H_1, T^x)) \mod H'_1.$$
Then $\deg H'_1 \leq h$, $G'_1$ is coprime to $H'_1$ of smaller degree and $|\theta - G'_1/H'_1| < q^{-2h}$. Thus $(G'_1, H'_1)$ is the unique pair that corresponds to $\theta$ in the sense of Lemma 2.1. Let $(G'_2, H'_2)$ be a different pair that satisfies the conditions of the lemma, and define $G'_2$ and $H'_2$ in a similar manner. From the uniqueness property of Lemma 2.1, we have $G'_1 = G'_2$ and $H'_1 = H'_2$. Since both $\deg H'_1 = \deg H'_2$ and $\deg H_1 = \deg H_2$, we arrive at $\deg \gcd(H_1, T^x) = \deg \gcd(H_2, T^x)$. From this we know that $\gcd(H_1, T^x) = \gcd(H_2, T^x)$, and $H_1 = H'_1 \cdot \gcd(H_1, T^x) = H'_2 \cdot \gcd(H_2, T^x) = H_2$. For convenience sake we now denote $H = H_1$, $H' = H'_1$. Since $H$ is squarefree, $T^2 \not| H$ and thus $\gcd(H, T^x) \in \{1, T\}$. We know that $G'_1 = G'_2$ but $G_1 \neq G_2$, hence $\gcd(H, T^x) \neq 1$ and thus $\gcd(H, T^x) = T$. This serves as a contradiction when $x = 0$, and thus we have proven the second assertion of the lemma. When $x > 0$, we know that $T^x G_1 \equiv TG'_1 \equiv T^x G_2 \mod H$, thus

$$T^x(G_1 - G_2) \equiv 0 \mod H \tag{3.1}$$

This means that $H' \mid G_1 - G_2$, but

$$\deg(G_1 - G_2) \leq \max\{\deg G_1, \deg G_2\} \leq h - 1 = \deg H'. \tag{3.2}$$

So $G_1 - G_2 = cH'$ for some $c \in \mathbb{F}_q$. This completes the proof, since there are exactly $q$ polynomials of the form $G_2 = G_1 - cH'$.

**Lemma 3.3.** Let $1 \leq m \leq n$ and $\mathcal{I} \subset \{0, \ldots, n - 1\}$, and denote $I = \#\mathcal{I}$. For $0 \leq x \leq n - m$, denote $K_{x,m} = \{x, \ldots, x + m - 1\}$. Then there exists $0 \leq y \leq n - m$ such that

$$\#(K_{y,m} \cap \mathcal{I}) < 2m \cdot \frac{I}{n}. \tag{3.3}$$

Moreover, if $m \leq n/2$, there exists $0 \leq y \leq n - m$ such that

$$\#(K_{y,m} \cap \mathcal{I}) < \frac{3m}{2} \cdot \frac{I}{n}. \tag{3.4}$$

**Proof.** Denote $x_i = m \cdot i$, for $0 \leq i < \lfloor n/m \rfloor$. The sets $K_{x_i,m}$ are pairwise disjoint, so

$$\sum_{i=0}^{n/m-1} \#(K_{x_i,m} \cap \mathcal{I}) \leq I. \tag{3.5}$$

Assume by way of contradiction that $\#(K_{x_i,m} \cap \mathcal{I}) \geq \frac{(\lfloor n/m \rfloor + 1)m}{n} \cdot I$ for all $0 \leq i < \lfloor n/m \rfloor$. Since

$$(\lfloor n/m \rfloor + 1) \cdot m > n,$$

we get

$$\sum_{i=0}^{n/m-1} \#(K_{x_i,m} \cap \mathcal{I}) \geq \lfloor n/m \rfloor \cdot \frac{(\lfloor n/m \rfloor + 1)m}{n} \cdot I > n \cdot \frac{I}{n} = I,$$

which is a contradiction.
which contradicts (3.3). So there exists $0 \leq i < \lfloor n/m \rfloor$ such that

$$\#(K_{x_i} \cap I) < \left(\frac{\lfloor n/m \rfloor + 1}{n/m}\right) \cdot \frac{I}{n},$$

from which it is easy to see that there exists $0 \leq y \leq n - m$ that satisfies (3.3) for all $m \leq n$, and that there exists $0 \leq y \leq n - m$ that satisfies (3.4) for all $m \leq n/2$. This completes the proof.

3.2 General Bound

Recall the definitions of $\alpha$ and $\hat{F}_{q,n}$ given in (1.5) and (1.10), respectively. Our goal in this subsection is to establish the following bound:

**Proposition 3.4.** Let $n \geq 2$. Then

$$\sum_{F \in M_n} \left| \hat{F}_{q,n}(T^{-n}F) \right| \leq \alpha(n - I)q^n.$$

We start by proving the Fourier Inverse Formula (1.11) and Parseval’s Formula (1.12): Recall that

$$\hat{F}_{q,n}(\theta) = \sum_{G \in M_n} \mathbb{1}_C(G)e(G\theta).$$

Developing the right-hand side of (1.11) gives

$$\frac{1}{q^n} \sum_{G \in M_n} \hat{F}_{q,n}(T^{-n}G)e(-T^{-n}GF) = \frac{1}{q^n} \sum_{G \in M_n} \sum_{S \in M_n} \mathbb{1}_C(S)e(T^{-n}GS)e(-T^{-n}GF) = \frac{1}{q^n} \sum_{S \in M_n} \mathbb{1}_C(S) \sum_{G \in M_n} e(T^{-n}(F - S)G),$$

and by orthogonality relations we get

$$\frac{1}{q^n} \sum_{G \in M_n} \hat{F}_{q,n}(T^{-n}G)e(-T^{-n}GF) = \frac{1}{q^n} \sum_{S \in M_n} \mathbb{1}_C(S)q^n \mathbb{1}_{F=S} = \mathbb{1}_C(F).$$

So we have explicitly shown (1.11).

Parseval’s Formula in (1.12) is as easy to derive: by (1.11) we may substitute $\frac{1}{q^n} \sum_{G \in M_n} \hat{F}_{q,n}(T^{-n}G)e(-T^{-n}GF)$ for $\mathbb{1}_C(F)$, to get

$$\sum_{P \in \mathcal{C}} 1 = \sum_{G \in M_n} \mathbb{1}_P(G)\mathbb{1}_C(G) = \sum_{G \in M_n} \mathbb{1}_P(G) \left( \frac{1}{q^n} \sum_{F \in M_n} \hat{F}_{q,n}(T^{-n}F)e(-T^{-n}FG) \right).$$
Changing order of summation and noting (1.8) gives

\[ \sum_{P \in \mathcal{C}} 1 = \frac{1}{q^n} \sum_{F \in \mathcal{M}_n} \hat{F}_{q,n}(T^{-n}F) \sum_{G \in \mathcal{M}_n} \mathbb{1}_P(G)e(-T^{-n}FG) \]

\[ = \frac{1}{q^n} \sum_{F \in \mathcal{M}_n} \hat{F}_{q,n}(T^{-n}F)f(-T^{-n}F), \]

so (1.12) is established.

Take \( \theta = \sum_{l<k} \lambda_l T^l \in \mathbb{F}_q(T)_{\infty} \). In order to bound \( |\hat{F}_{q,n}(\theta)| \), we introduce new notation. Define

\[ Z(\theta) = \{ i \in J : \theta_{-i-1} = 0 \}, \quad N(\theta) = \{ i \in J : \theta_{-i-1} \neq 0 \}. \quad (3.6) \]

Essentially, \( Z(\theta) \) is the zero set of \( \theta \) between \(-1\) and \(-n\), and \( N(\theta) \) is the nonzero set of \( \theta \) in the same range. As we can see in Lemma 3.5, these sets hold most of the information on our bound on \( |\hat{F}_{q,n}(\theta)| \).

**Lemma 3.5.** For \( \theta = \sum_{l<k} \lambda_l T^l \), let \( Z(\theta) \), \( N(\theta) \) be defined as in (3.6). Then

\[ |\hat{F}_{q,n}(\theta)| \leq \prod_{i \in N(\theta)} N_i \prod_{i \in Z(\theta)} q. \]

**Proof.** Recall the definition of \( \mathcal{C} \) in (1.3). Using the notation of Theorem 1.3 define \( \mathcal{C}_i = \{ a_i \} \) for \( i \in \mathcal{I} \) and \( \mathcal{C}_i = \mathbb{F}_q \setminus S_i \) for \( i \in J \). Define \( \mathcal{C}_n = \{ 1 \} \). We denote the \( i \)th coefficient of a polynomial \( G \in \mathbb{F}_q[T] \) by \( g_i \). Then

\[ \hat{F}_{q,n}(\theta) = \sum_{G \in \mathcal{M}_n} \mathbb{1}_{\mathcal{C}}(G)e(G\theta) = \sum_{G \in \mathcal{M}_n} \prod_{i=0}^{n} \mathbb{1}_{\mathcal{C}_i}(g_i)e(T^i g_i \theta) \]

\[ = e(T^n \theta) \prod_{i=0}^{n-1} \sum_{g_i \in \mathbb{F}_q} \mathbb{1}_{\mathcal{C}_i}(g_i)e(T^i g_i \theta). \]

More explicitly, by the definition of \( e \) given in (1.7) we may write

\[ \hat{F}_{q,n}(\theta) = e(T^n \theta) \prod_{i=0}^{n-1} \sum_{g_i \in \mathbb{F}_q} \mathbb{1}_{\mathcal{C}_i}(g_i)\psi(g_i \theta_{-i-1}), \]

and taking absolute value gives

\[ |\hat{F}_{q,n}(\theta)| = \prod_{i=0}^{n-1} \sum_{g_i \in \mathbb{F}_q} |\mathbb{1}_{\mathcal{C}_i}(g_i)|\psi(g_i \theta_{-i-1}) \quad (3.7) \]
Denote \( X = \sum_{g_i \in F_q} \mathbb{1}_{C_i}(g_i) \psi(g_i \theta_{-i-1}) \). Note that if \( i \in I \), then \( |X| = 1 \). If \( i \in J \), we divide into two cases: If \( \theta_{-i-1} = 0 \), then \( |X| = q - N_i \). If \( \theta_{-i-1} \neq 0 \), then when \( \mathbb{1}_{C_i}(g_i) = 1 \), \( g_i \) ranges over \( F_q \). From orthogonality relations

\[
X = \sum_{g_i \in F_q} \mathbb{1}_{C_i}(g_i) \psi(g_i \theta_{-i-1}) = - \sum_{b \in S_i} \psi(b \theta_{-i-1}),
\]

thus in this case \( |X| \leq N_i \). Inserting these bounds on \( |X| \) into (3.7) yields

\[
\left| \hat{F}_{q,n}(\theta) \right| \leq \prod_{i \in N(\theta)} N_i \prod_{i \in Z(\theta)} (q - N_i) \leq \prod_{i \in N(\theta)} N_i \prod_{i \in Z(\theta)} q,
\]

which completes the proof of the lemma.

**Lemma 3.6.** For every \( \theta, \eta \in F_q(T)_\infty \) such that \( |\theta - \eta| < q^{-n} \), we have \( \left| \hat{F}_{q,n}(\theta) \right| = \left| \hat{F}_{q,n}(\eta) \right| \).

**Proof.** For every \( \theta \in F_q(T)_\infty \), by (3.7) we have

\[
\left| \hat{F}_{q,n}(\theta) \right| = \prod_{i = 0}^{n-1} \sum_{g_i \in F_q} \mathbb{1}_{C_i}(g_i) \psi(g_i \theta_{-i-1}).
\]

From this, it is easy to see that \( \left| \hat{F}_{q,n}(\theta) \right| \) depends only on \( \theta_{-1}, \ldots, \theta_{-n} \). Let \( \eta \in F_q(T)_\infty \) be such that \( |\theta - \eta| < q^{-n} \). Then \( \theta_i = \eta_i \) for every \( -n \leq i \leq -1 \), hence \( \left| \hat{F}_{q,n}(\theta) \right| = \left| \hat{F}_{q,n}(\eta) \right| \).

**Proof of Proposition 3.4** We turn to prove that

\[
\sum_{F \in M_n} \left| \hat{F}_{q,n}(T^{-n}F) \right| \leq \alpha(n - I)q^n.
\]

Note first that \( \#J = n - I \), thus

\[
\alpha(n - I) = \sup_{i_1, \ldots, i_{n-1}} \prod_{j=1}^{n-I} (N_{i_j} + 1) = \prod_{i \in J} (N_i + 1).
\]

Writing \( F = T^n + \sum_{i=0}^{n-1} f_i T^i \) and \( \theta_F = T^{-n}F \), define \( Z(\theta_F) \), \( N(\theta_F) \) as in (3.6). By Lemma 3.5

\[
\sum_{F \in M_n} \left| \hat{F}_{q,n}(\theta_F) \right| \leq \sum_{F \in M_n} \prod_{i \in N(\theta_F)} N_i \prod_{i \in Z(\theta_F)} q = \sum_{f_0 \in F_q} \cdots \sum_{f_{n-1} \in F_q} \prod_{f_i \neq 0} N_i \prod_{f_i = 0} q.
\]
Changing order of summation and product, we have
\[
\sum_{F \in \mathcal{M}_n} \left| \hat{F}_{q,n}(\theta_F) \right| \leq \prod_{i \in I} \left( \sum_{j \in \mathbb{F}_q} 1 \right) \times \prod_{i \in J} \left( q + \sum_{0 \neq j \in \mathbb{F}_q} N_i \right),
\]
thus
\[
\sum_{F \in \mathcal{M}_n} \left| \hat{F}_{q,n}(\theta_F) \right| \leq \prod_{i \in I} q \prod_{i \in J} (q + N_i \cdot (q - 1)) \leq q^n \prod_{i \in J} (N_i + 1) = \alpha(n - I) q^n,
\]
as claimed. \qed

### 3.3 Bound for Fractions

Let \( n \geq 2 \), and \( 0 \leq h \leq n/2 \). Having obtained a bound for \( \sum_{F \in \mathcal{M}_n} \left| \hat{F}_{q,n}(T^{-n}F) \right| \), we now turn to bound
\[
Y_h = \sum_{G, H \text{deg } H = h} \left| \hat{F}_{q,n}(G/H) \right|, \tag{3.9}
\]
where the sum ranges over \( H \neq 1, T \) squarefree, and \( G \) coprime to \( H \). At the end of the section, we incorporate some of the assumptions of Theorem 1.3 in order to prove

**Proposition 3.7.** Let \( n \geq 2 \), and assume that some \( \varepsilon > 0 \) satisfies that \( \alpha(m) < q^{m} \) for all \( 0 < m \leq n \). For every \( 0 \leq h \leq \min\{n/2, n/\ell\} \) we have
\[
Y_h \leq q^{n+3(1-\varepsilon)-(1-\varepsilon)n/h + 2\varepsilon h - \varepsilon I}, \tag{3.10}
\]
for \( h \leq n/2 \), we have
\[
Y_h \leq q^{n-\ell+1} q^{(2\varepsilon + (1-\varepsilon)\ell / n)h}, \tag{3.11}
\]
and for \( h < n/4 \), we have
\[
Y_h \leq q^{n-\ell+1} q^{(2\varepsilon + (1-\varepsilon)3\ell / n)h}. \tag{3.12}
\]

Note that for \( h < n/4 \) the bound (3.12) is strictly better than (3.11), but sometimes it is more convenient to use (3.11). In the following lemma, we do not use specific properties of fractions \( G/H \), but instead give a bound to \( \left| \hat{F}_{q,n}(\theta) \right| \) that will later be useful when \( \theta = G/H \) with \( \text{deg } H = h \).

**Lemma 3.8.** Let \( 1 \leq l \leq n, \theta \in \mathcal{U}, \) and define \( Z(\theta), N(\theta) \) as in (3.6). For \( 0 \leq x \leq n-l \), denote \( K = K_{x,l} = \{x, \ldots, x + l - 1\} \). Define \( \overline{K} = \{0, \ldots, n - 1\} \setminus K \) and denote \( Z_{\ell} K(\theta) = Z(\theta) \cap \overline{K}, N_{\ell} K(\theta) = N(\theta) \cap \overline{K} \). Then
\[
\left| \hat{F}_{q,n}(\theta) \right| \leq \alpha_{\ell} (\# N_{\ell} K(\theta)) q^{\# Z_{\ell} K(\theta)} \left| \hat{F}_{q,l}(T^x \theta) \right| \tag{3.13}
\]
Proof. As in (3.7), for every \( \theta \) we have

\[
\left| \hat{F}_{q,n}(\theta) \right| = \prod_{i=0}^{n-1} \sum_{g_i \in F_q} 1_{C_i}(g_i) \psi(g_i \theta_{-i-1})
\]

Splitting the product into \( K \) and \( \overline{K} \), we get

\[
\left| \hat{F}_{q,n}(\theta) \right| = \prod_{i \in K} \left| \sum_{g_i \in F_q} 1_{C_i}(g_i) \psi(g_i \theta_{-i-1}) \right| \cdot \prod_{i \in \overline{K}} \left| \sum_{g_i \in F_q} 1_{C_i}(g_i) \psi(g_i \theta_{-i-1}) \right|.
\]

Since \( K = K_{x,l} \), the left-hand element of the product is exactly \( \left| \hat{F}_{q,l}(T^x \theta) \right| \). We use similar arguments to those of Lemma 3.5 in order to bound \( \prod_{i \in \overline{K}} \left| \sum_{g_i \in F_q} 1_{C_i}(g_i) \psi(g_i \theta_{-i-1}) \right| \). This gives

\[
\left| \hat{F}_{q,n}(\theta) \right| \leq \left| \hat{F}_{q,l}(T^x \theta) \right| \prod_{i \in Z_{\hat{q},K}(\theta)} q \prod_{i \in N_{\hat{q},K}(\theta)} N_i.
\]

(3.14) Considering the definition of \( Z_{\hat{q},K}(\theta) \) and \( N_{\hat{q},K}(\theta) \), (3.14) translates to

\[
\left| \hat{F}_{q,n}(\theta) \right| \leq \left| \hat{F}_{q,l}(T^x \theta) \right| \prod_{i \in Z_{\hat{q},K}(\theta)} q \prod_{i \in N_{\hat{q},K}(\theta)} N_i.
\]

Considering the definition of \( \alpha_{\hat{q},K}(m) \), we see that

\[
\left| \hat{F}_{q,n}(\theta) \right| \leq \alpha_{\hat{q},K}(\#N_{\hat{q},K}(\theta)) q^{\#Z_{\hat{q},K}(\theta)} \left| \hat{F}_{q,l}(T^x \theta) \right|.
\]

This completes the proof. \( \square \)

**Lemma 3.9.** Let \( 1 \leq l \leq n \), \( 0 \leq x \leq n-l \), \( \theta \in U \). Denote \( K = K_{x,l} \), and let \( \overline{K} \), \( Z_{\hat{q},K}(\theta) \), \( N_{\hat{q},K}(\theta) \), and \( \alpha_{\hat{q},K} \) be defined as in Lemma 3.8. Write \( I_{\hat{q},K} = #(I \cap K) \), \( I_{\hat{q},K} = I - I_{\hat{q},K} \). For an integer \( t \) in the range \( \#Z_{\hat{q},K}(\theta) \leq t \leq n-l - I_{\hat{q},K} \), the inequality

\[
\left| \hat{F}_{q,n}(\theta) \right| \leq \alpha_{\hat{q},K}(n-l - I_{\hat{q},K} - t) q^t \left| \hat{F}_{q,l}(T^x \theta) \right|
\]

holds.

**Proof.** Since every index \( i \in \overline{K} \) is either in \( Z_{\hat{q},K}(\theta) \), \( N_{\hat{q},K}(\theta) \) or \( I \cap \overline{K} \), it is easy to see that

\[
\#Z_{\hat{q},K}(\theta) + \#N_{\hat{q},K}(\theta) + I_{\hat{q},K} = n - l.
\]

(3.15)
In particular, we have
\[ \#N_{\xi K}(\theta) = n - l - I_{\xi K} - \#Z_{\xi K}(\theta). \] (3.16)
Inserting this into (3.13), we have
\[ \left| \hat{F}_{q,n}(\theta) \right| \leq \alpha_{\xi K} (n - l - I_{\xi K} - \#Z_{\xi K}(\theta)) q^{\#Z_{\xi K}(\theta)} \left| \hat{F}_{q,l}(T^x \theta) \right|. \] (3.17)
Note that by definition we have \( \alpha_{\xi K}(v + u) \leq \alpha_{\xi K}(v)q^u \) for every \( v, u \geq 0 \). Since by assumption \( t \) satisfies \( \#Z_{\xi K}(\theta) \leq t \leq n - l - I_{\xi K} \), using this monotonicity argument with \( u = t - \#Z_{\xi K}(\theta) \) and \( v = n - l - I_{\xi K} - t \) yields
\[ \alpha_{\xi K} (n - l - I_{\xi K} - \#Z_{\xi K}(\theta)) \leq \alpha_{\xi K} (n - l - I_{\xi K} - t) q^{t - \#Z_{\xi K}(\theta)}. \]
Inserting this bound into (3.17) gives
\[ \left| \hat{F}_{q,n}(\theta) \right| \leq \alpha_{\xi K} (n - l - I_{\xi K} - t) q^t \left| \hat{F}_{q,l}(T^x \theta) \right|, \]
as claimed. 

**Lemma 3.10.** Let \( 1 \leq h \leq n/2 \), and \( 0 \leq x < n - 2h \). Define \( K = K_{x,2h} = \{x,...,x+2h\} \), and \( \overline{K} \) as in Lemma 3.8. Write \( I_{\xi K} = \#(I \cap K) \), \( I_{\xi K} = I - I_{\xi K} \). Let \( \alpha \) be defined as in (3.9), and \( Y_h \) be defined as in (3.9). The inequality
\[ Y_h \leq \alpha (2h - I_{\xi K}) q^{n+1 - I_{\xi K}} \]
holds.

**Proof.** We define \( Z_{\xi K}(\theta), N_{\xi K}(\theta), \) and \( \alpha_{\xi K} \) as in Lemma 3.8. It is easy to see from (3.15) that for every \( \theta \in \mathcal{U} \)
\[ \#Z_{\xi K}(\theta) = n - 2h - I_{\xi K} - \#N_{\xi K}(\theta) \leq n - 2h - I_{\xi K}. \] (3.18)
Applying Lemma 3.9 with \( t = n - 2h - I_{\xi K} \), we get
\[ \left| \hat{F}_{q,n}(\theta) \right| \leq \alpha(0) q^{n-2h-I_{\xi K}} \left| \hat{F}_{q,2h}(T^x \theta) \right| = q^{n-2h-I_{\xi K}} \left| \hat{F}_{q,2h}(T^x \theta) \right|. \]

For every pair \( G, H \) with \( H \neq 1 \), \( T \) squarefree of degree \( H \) and \( G \) coprime to \( H \) and of smaller degree, we associate \( F_{G,H,x} \in \mathcal{M}_{2h} \) such that \( |T^{-2h}F_{G,H,x} - T^x G/H| < q^{-2h} \). By Lemma 3.2 we know that each \( F \in \mathcal{M}_{2h} \) corresponds to at most \( q \) such pairs. Using this fact and Lemma 3.6, we obtain the inequality
\[ \sum_{G,H \text{deg } H = h} |\hat{F}_{q,2h}(T^x G/H)| \leq q \sum_{F \in \mathcal{M}_{2h}} \left| \hat{F}_{q,2h}(T^{-2h} F) \right|. \]
Finally, we have
\[
Y_h = \sum_{G,H \text{ deg } H = h} \left| \hat{F}_{q,n}(G/H) \right| \leq q^{n-2h-I_{\not\equiv K}} \sum_{G,H \text{ deg } H = h} \left| \hat{F}_{q,2h}(T^*G/H) \right| \leq q^{n-2h-I_{\not\equiv K}} q \sum_{F \in \mathcal{M}_{2h}} \left| \hat{F}_{q,2h}(T^{-2h} F) \right|,
\]
and applying Lemma 3.4 on the sum gives us
\[
Y_h \leq q^{n+1-2h-I_{\not\equiv K}} \prod_{i \in J \cap K} (N_i + 1) \cdot q^{2h} \leq \alpha (2h + I_{\not\equiv K} - I) q^{n+1-I_{\not\equiv K}},
\]
as claimed.

Lemma 3.11. Let \(1 \leq h \leq n/2\), and choose \(x = 0\). Define \(K_{0,2h}\) as in Lemma 3.8. Write \(I_{\not\equiv K} = (I \setminus K_{0,2h})\). Denote \(t_h = \max\{I_{\not\equiv K}, \left\lfloor \frac{n}{h} \right\rfloor - 2\}\), and let \(\alpha\) and \(Y_h\) be defined as before. The inequality
\[
Y_h \leq \alpha (2h + t_h - I) q^{n-I_{\not\equiv K}}
\]
holds.

Proof. We define \(Z_{\not\equiv K}, N_{\not\equiv K}, \text{ and } \alpha_{\not\equiv K}\) as in Lemma 3.8. Assume \(\theta = G/H\), with \(H \not\in \{1, T\}\) squarefree of degree \(h\), and \(G\) coprime to \(H\) of degree \(< h\). We give two different bounds for \(#Z_{\not\equiv K}(G/H)\). First, it is easy to see from (3.15) that
\[
#Z_{\not\equiv K}(G/H) = n - 2h - I_{\not\equiv K} - #N_{\not\equiv K}(G/H) \leq n - 2h - I_{\not\equiv K}.
\]
(3.19)

Second, by Lemma 3.1 we know that there are no \(h\) consecutive zeros in \(G/H\), so in particular there are at most \(n - 2h - \left\lfloor \frac{n-2h}{h} \right\rfloor\) zero coefficients between \(-2h - 1\) and \(-n\). Hence
\[
#Z_{\not\equiv K}(G/H) \leq n - 2h - \left\lfloor \frac{n-2h}{h} \right\rfloor = n - 2h - \left\lfloor \frac{n}{h} \right\rfloor + 2.
\]
(3.20)

Considering the definition of \(t_h\), (3.19) and (3.20) give
\[
#Z_{\not\equiv K}(G/H) \leq n - 2h - t_h.
\]
(3.21)

Inserting this into Lemma 3.4 yields
\[
\left| \hat{F}_{q,n}(G/H) \right| \leq \alpha_{\not\equiv K} (t_h - I_{\not\equiv K}) q^{n-2h-t_h} \left| \hat{F}_{q,2h}(G/H) \right|.
\]

In a similar manner to Lemma 3.10, \(\left| T^{-2h} F_{G,H} - G/H \right| < q^{-2h}\) for some \(F_{G,H} \in \mathcal{M}_{2h}\). By Lemma 3.2, we know that each \(F \in \mathcal{M}_{2h}\) corresponds to at most one pair \(G, H\) as described.
We can now use Lemma 3.6 to obtain

\[ Y_h \leq \sum_{G,H \atop \deg H = h} |\hat{F}_{q,2h}(G/H)| \leq \sum_{F \in \mathcal{M}_{2h}} |\hat{F}_{q,2h}(T^{-2h}F)|. \]

Finally, we have

\[ Y_h = \sum_{G,H \atop \deg H = h} |\hat{F}_{q,n}(G/H)| \leq \alpha_{gK} (t_h - I_{\bar{g}K}) q^{n-2h-t_h} \sum_{G,H \atop \deg H = h} |\hat{F}_{q,2h}(T^{x}G/H)| \leq \alpha_{gK} (t_h - I_{\bar{g}K}) q^{n-2h-t_h} \sum_{F \in \mathcal{M}_{2h}} |\hat{F}_{q,2h}(T^{-2h}F)|, \]

and applying Lemma 3.4 on the sum gives us

\[ Y_h \leq \alpha_{gK} (t_h - I_{\bar{g}K}) q^{n-2h-t_h} \prod_{i \in J_{\bar{g}K} \cap \{2h, \ldots, n-1\}} (N_i + 1) \cdot q^{2h} \leq \alpha (2h + t_h - I) q^{n-t_h}, \]

as claimed. \(\square\)

In Proposition 3.7, we give our final bound on sums over fractions of the form \(G/H\) with \(\deg H = h\). Having established Lemma 3.10 and Lemma 3.11, most of the work is already accomplished. In order to establish Proposition 3.7, we add most of the assumptions of Theorem 1.3. We assume that \(N_i < q^\varepsilon\) for all \(i \in J\), with \(\varepsilon < 1\).

**Proof of Proposition 3.7.** Recall that, by assumption, \(\alpha(m) \leq q^{\varepsilon m}\) for all \(m \geq 0\). Let \(1 \leq h \leq \min\{n/2, n/I\}\). Write \(I_{\bar{g}2h} = \#(I \cap \{2h, \ldots, n-1\})\), and \(t_h = \max\{I_{\bar{g}2h}, \left\lfloor \frac{n}{h} \right\rfloor - 2\}\). Lemma 3.11 gives us that

\[ \sum_{G,H \atop \deg H = h} |\hat{F}_{q,n}(G/H)| \leq \alpha (2h + t_h - I) q^{n-t_h}. \]

Thus from the assumption on \(\alpha\) we get

\[ \sum_{G,H \atop \deg H = h} |\hat{F}_{q,n}(G/H)| \leq q^{n-t_h+\varepsilon(2h+t_h-I)} = q^{n-t_h(1-\varepsilon)+\varepsilon(2h-I)}. \]

Since \(t_h \geq \lfloor n/h \rfloor - 2\) and \(\varepsilon < 1\), we get

\[ \sum_{G,H \atop \deg H = h} |\hat{F}_{q,n}(G/H)| \leq q^{n-(\lfloor n/h \rfloor - 2)(1-\varepsilon)+\varepsilon(2h-I)} \leq q^{n-(n/h-3)(1-\varepsilon)+\varepsilon(2h-I)} = q^{n+3(1-\varepsilon)-(1-\varepsilon)n/h+2\varepsilon-I}, \]

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which is the first part of the proposition.

We now move to the case where \( n/I \leq h \leq \min\{n/2, n/I\} \). For every choice of \( 0 \leq x \leq n - 2h \), write \( K = K_{x,l} \) and \( I_{\xi K} = \#(I \setminus K) \). Lemma 3.10 gives us that

\[
Y_h \leq \alpha (2h + I_{\xi K} - I) q^{n+1-I_{\xi K}}.
\]

Bounding \( \alpha (2h + I_{\xi K} - I) \) by \( q^{(2h+I_{\xi K}-I)} \) yields

\[
Y_h \leq q^{n+1-I_{\xi K}+\epsilon(2h+I_{\xi K}-I)},
\]

and simplifying yields

\[
Y_h \leq q^{n-I+\epsilon(1-\epsilon)(I-I_{\xi K})+2h}.
\]

Choosing the optimal \( x \) in the sense of Lemma 3.3, we can assume \( I_{\xi K} \geq I - 4hI/n \) in the case where \( h \leq n/2 \) and \( I_{\xi K} \geq I - 3hI/n \) when \( h < n/4 \), thus

\[
Y_h \leq q^{n-I+1} q^{(1-\epsilon)(4h/n+2\epsilon)} = q^{n-I+1} q^{(2\epsilon+(1-\epsilon)4I/n)}
\]

for all \( h \leq n/2 \), and similarly

\[
Y_h \leq q^{n-I+1} q^{(2\epsilon+(1-\epsilon)3I/n)}
\]

when \( h < n/4 \). This completes the proof of the proposition.

\[
\square
\]

4 Proof of the Theorem 1.3

4.1 Auxiliary Results

The following lemmas apply Proposition 3.7 to the setting of Theorem 1.3. In all the lemmas in this section we assume the setting of Proposition 3.7. We have results of two types - when \( I < o(n/\log(n)) \), we get a strong bound on the error term. When \( I \) is larger, we take more care to derive the most from our methods.

**Lemma 4.1.** Assume that \( \epsilon < 1/2 \). Denote \( l = \min\{n/I, n/2\} \), \( s = \sqrt{(1-\epsilon)/(1-2\epsilon)} \sqrt{n} \), \( m_{n,l,\epsilon} = \min\{n/I, s\} \), and \( y = \frac{I^2}{\pi} \). Define \( Y_h \) as in (3.9). The inequality

\[
\sum_{h=1}^{l} q^{-h} Y_h \leq n q^{n-I} q^{3(1-\epsilon)-(1-2\epsilon)m_{n,l,\epsilon}}
\]

holds. If in addition \( y > 1 \), we have

\[
\sum_{h=1}^{n/I} q^{-h} Y_h \leq C_{q,\epsilon,y} q^{n-I} q^{3(1-\epsilon)-(1-2\epsilon)n/I},
\]

with \( C_{q,\epsilon,y} \) tending to 1 as \( q \) tends to infinity.
Proof. By (3.10), we have
\[ X := \sum_{h=1}^{l} q^{-h} Y_h \leq \sum_{h=1}^{l} q^{-h} q^{n+3(1-\varepsilon)-(1-\varepsilon)n/h+2\varepsilon h} . \]

By simplifying we get
\[ X \leq q^{n+3(1-\varepsilon)-\varepsilon l} \sum_{h=1}^{l} q^{-(1-2\varepsilon)h-(1-\varepsilon)n/h} . \] (4.3)

For ease of exposition, we denote the term \(-(1-2\varepsilon)h-(1-\varepsilon)n/h\) by \(r_{n,\varepsilon}(h)\). By (4.3), in order to bound \(X\) it suffices to bound \(\sum_{h=1}^{l} q^{r_{n,\varepsilon}(h)}\).

By deriving \(r_{n,\varepsilon}\), we obtain that its maximum is attained at \(h_{\text{max}} = s = \sqrt{(1-\varepsilon)/(1-2\varepsilon)}/\sqrt{n}\). In the case where \(y = \frac{I \cdot s}{n} \leq 1\), we use the union bound and get
\[ \sum_{h=1}^{l} q^{r_{n,\varepsilon}(h)} \leq l q^{r_{n,\varepsilon}(s)} \leq n q^{r_{n,\varepsilon}(s)} - (1-2\varepsilon) s q^{-1} \leq n q^{-1+2\varepsilon (1-\varepsilon)} . \]

Noting that in this case \(n/s \geq I\), we get
\[ \sum_{h=1}^{l} q^{r_{n,\varepsilon}(h)} \leq n q^{-1+2\varepsilon (1-\varepsilon)} . \] (4.4)

We now bound \(\sum_{h=1}^{l} q^{r_{n,\varepsilon}(h)}\) when \(y > 1\). For all \(h < n/I\) we have
\[ \frac{n}{h-1} - \frac{n}{h} = \frac{n}{h(h-1)} > \frac{I^2}{n} = \frac{y^2 n}{s^2} = \frac{1-2\varepsilon}{1-\varepsilon} y^2 , \] (4.5)

thus
\[ r_{n,\varepsilon}(h) - r_{n,\varepsilon}(h-1) = -1 + 2\varepsilon + (1-\varepsilon) \left( \frac{n}{h-1} - \frac{n}{h} \right) \geq -1 + 2\varepsilon + (1-\varepsilon) \frac{1-2\varepsilon}{1-\varepsilon} y^2 \]
\[ = (1-2\varepsilon)(y^2 - 1) . \]

Write \(m = \lfloor n/I \rfloor\). By induction we obtain that for every \(0 \leq j < m\) we have
\[ r_{n,\varepsilon}(m-j) \leq r_{n,\varepsilon}(m) - j(1-2\varepsilon)(y^2 - 1) \leq r_{n,\varepsilon}(n/I) - j(1-2\varepsilon)(y^2 - 1) , \]

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where the latter inequality is due to monotonicity of $r_{n,\epsilon}$ in the range $1 \leq h \leq n/I$ when $y > 1$. This means that we can bound $\sum_{h=1}^{l} q^{r_{n,\epsilon}(h)}$ by a geometric sum:

$$
\sum_{h=1}^{l} q^{r_{n,\epsilon}(h)} = \sum_{j=0}^{m-1} q^{r_{n,\epsilon}(n/I) - j(1-2\epsilon)(y^2-1)} \leq q^{r_{n,\epsilon}(n/I)} \sum_{j=0}^{m-1} q^{-j(1-2\epsilon)(y^2-1)} = q^{-(1-2\epsilon)n/I} q^{-(1-\epsilon)I} \sum_{j=0}^{m-1} q^{-j(1-2\epsilon)(y^2-1)}. \quad (4.6)
$$

We now turn to bound the right-hand sum $S = \sum_{j=0}^{m-1} q^{-j(1-2\epsilon)(y^2-1)}$ in two ways. First, we note that $(1-2\epsilon)(y^2-1) > 0$, hence a simple union bound gives us

$$
S = \sum_{j=0}^{m-1} q^{-j(1-2\epsilon)(y^2-1)} \leq m - 1 < n,
$$

which combined with (4.6) gives

$$
\sum_{h=1}^{l} q^{r_{n,\epsilon}(h)} < nq^{-(1-2\epsilon)n/I} q^{-(1-\epsilon)I}. \quad (4.7)
$$

Second, we treat $S$ as a geometric series, in which case we bound it by the infinite series

$$
S = \sum_{j=0}^{m-1} q^{-j(1-2\epsilon)(y^2-1)} \leq \frac{1}{1 - q^{-(1-2\epsilon)(y^2-1)}}. \quad (4.8)
$$

Writing

$$
C_{q,\epsilon,y} = \left(1 - q^{-(1-2\epsilon)(y^2-1)}\right)^{-1},
$$

we note that $C_{q,\epsilon,y}$ tends to 1 as $q$ tends to infinity. Inserting (4.8) into (4.6) then gives

$$
\sum_{h=1}^{l} q^{r_{n,\epsilon}(h)} \leq C_{q,\epsilon,y} q^{-(1-2\epsilon)n/I} q^{-(1-\epsilon)I}, \quad (4.9)
$$

and using this in (4.3) yields that when $y > 1$,

$$
X \leq C_{q,\epsilon,y} q^{n-I} q^{3(1-\epsilon)-(1-2\epsilon)m_{n,I,\epsilon}}, \quad (4.10)
$$

with $C_{q,\epsilon,y}$ tending to 1 as $q$ tends to infinity. This concludes the second part of the lemma.
For the first part, recall that $m_{n,I,\epsilon} = \min\{n/I, s\}$. Inserting (4.4) in the case where $y \leq 1$ and (4.7) in the case where $y > 1$ into (4.3) gives

$$X = q^{n+3(1-\epsilon)-\epsilon I} \sum_{h=1}^{l} q^{r_{n,\epsilon}(h)} \leq nq^{n-I}q^{3(1-\epsilon)-(1-2\epsilon)m_{n,I,\epsilon}}$$  \hspace{1cm} (4.11)

for all $y$. This completes the proof of the lemma. 

Lemma 4.2. Let $n/I \leq k \leq n/2$. Assuming $\frac{1}{n} < \frac{1}{4} \cdot \frac{1-2\epsilon}{1-\epsilon} (1 - \tau)$ for some $\tau > 0$, we have

$$\sum_{h=k}^{n/2} q^{-h}Y_h < C_{q,\epsilon,\tau} q^{n-I+1} q^{-k(1-2\epsilon-4(1-\epsilon)I/n)} ,$$  \hspace{1cm} (4.12)

with $C_{q,\epsilon,\tau}$ tending to 1 as $q$ tends to infinity.

Proof. By the (3.11), we know that $Y_h \leq q^{n-I+1} q^{(2\epsilon+(1-\epsilon)4I/n)h}$ for all $k \leq h \leq n/2$. Thus

$$\sum_{h=k}^{n/2} q^{-h}Y_h \leq \sum_{h=k}^{n/2} q^{-h} q^{n-I+1} q^{(2\epsilon+(1-\epsilon)4I/n)h} = q^{n-I+1} \sum_{h=k}^{n/2} q^{-h(1-2\epsilon-4(1-\epsilon)I/n)}$$

Substituting $r = q^{-(1-2\epsilon-4(1-\epsilon)I/n)}$, we get

$$\sum_{h=k}^{n/2} q^{-h}Y_h \leq q^{n-I+1} \sum_{h=k}^{n/2} r^h .$$

Note that from the assumption, $1 - 2\epsilon - 4(1 - \epsilon)I/n > (1 - 2\epsilon)\tau > 0$. So

$$r < q^{-(1-2\epsilon)\tau} < 1 ,$$  \hspace{1cm} (4.13)

and the sum is geometric. Thus we can bound it as

$$\sum_{h=k}^{n/2} q^{-h} \sum_{G,H \deg H=h} |\hat{F}_{q,n}(G/H)| \leq q^{n-I+1} . r^k \cdot \frac{1}{1-r}$$

$$\leq \frac{1}{1-r} q^{n-I+1} q^{-k(1-2\epsilon-4(1-\epsilon)I/n)} .$$

Write

$$C_{q,\epsilon,\tau} = (1 - q^{-(1-2\epsilon)\tau})^{-1} .$$
Then $C_{q,\varepsilon,\tau}$ tends to 1 as $q$ tends to infinity, and from (4.13) we get that $1/(1 - r) < C_{q,\varepsilon,\tau}$. Thus

$$\sum_{h=n/4}^{n/2} q^{-h} \sum_{G,H \text{ deg } H = h} |\hat{F}_{q,n}(G/H)| < C_{q,\varepsilon,\tau} q^{n-I+1} q^{-k(1-2\varepsilon - 4(1-\epsilon)I/n)} ,$$

as needed.

**Lemma 4.3.** Assuming $\frac{I}{n} < \frac{1}{4} \cdot \frac{1-2\varepsilon}{1-\tau} (1-\tau)$, we have

$$\sum_{n/I \leq h < n/4} q^{-h} \sum_{G,H \text{ deg } H = h} |\hat{F}_{q,n}(G/H)| < C_{q,\varepsilon,\tau} q^{n-I+1} q^{-(1-2\varepsilon - 3(1-\varepsilon)I/n)n/I} ,$$

(4.14)

where $C_{q,\varepsilon,\tau}$ is given in Lemma 4.2.

**Proof.** The proof is essentially the same as that of Lemma 4.2, the difference being that we use (3.12) instead of (3.11). Writing $r_2 = q^{-(1-2\varepsilon - 3(1-\varepsilon)I/n)}$, we have $r_2 < q^{-(1-2\varepsilon)\tau} < 1$, and

$$\sum_{h=n/I}^{n/4} q^{-h} \sum_{G,H \text{ deg } H = h} |\hat{F}_{q,n}(G/H)| \leq \frac{1}{1-r_2} q^{n-I+1} r_2^{n/I}$$

$$\leq C_{q,\varepsilon,\tau} q^{n-I+1} q^{-(1-2\varepsilon - 3(1-\varepsilon)I/n)n/I} ,$$

as required.

**Lemma 4.4.** Assume that $\frac{I}{n} < \frac{1}{4} \cdot \frac{1-2\varepsilon}{1-\tau} (1-\tau)$. Denote $s = \sqrt{(1-\varepsilon)/(1-2\varepsilon)\sqrt{n}}$, $y = \frac{I \cdot s}{n}$, and $m_{n,I,\varepsilon} = \min\{n/I, s\}$. Define $Y_h$ as in (3.9). If $I = o(n/\log(n))$, then

$$\sum_{h=1}^{n/2} q^{-h} Y_h \leq q^{n-I} q^{-(1-2\varepsilon + o(1))m_{n,I,\varepsilon}} .$$

**Proof.** We begin by partitioning the sum $\sum_{h=1}^{n/2} q^{-h} Y_h$ into two parts:

$$\sum_{h=1}^{n/2} q^{-h} D_h = \sum_{h=1}^{\min\{n/2,n/I\}} q^{-h} Y_h + \sum_{h=n/I}^{n} q^{-h} Y_h .$$

Since $I = o(n/\log(n))$, it follows that $\log_q(n) = o(n/I)$. Applying Lemma 4.1 gives

$$X_1 \leq \frac{n}{2} q^{n-I} q^{3(1-\varepsilon)-(1-2\varepsilon)m_{n,I,\varepsilon}}$$

$$\leq q^{n-I} q^{-(1-2\varepsilon)m_{n,I,\varepsilon} + \log_q(n) + 3(1-\varepsilon)} .$$

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Since \( \log_q(n) = o(m_{n,I,\varepsilon}) \) we obtain

\[
X_1 \leq q^{n-I}q^{-(1-2\varepsilon+o(1))m_{n,I,\varepsilon}}. \tag{4.15}
\]

Using Lemma 4.2 with \( k = n/I \) gives

\[
X_2 \leq C_{q,\varepsilon,\tau}q^{n-I}q^{-(1-2\varepsilon+4(1-\varepsilon)I/n)n/I} = q^{n-I}q^{-(1-2\varepsilon)n/I+1+4(1-\varepsilon)+\log_q(C_{q,\varepsilon,\tau})} = q^{n-I}q^{-(1-2\varepsilon+o(1))n/I},
\]

and since \( n/I \leq m_{n,I,\varepsilon} \) we get

\[
\sum_{h=1}^{n/2} q^{-h}Y_h = X_1 + X_2 \leq 2q^{n-I}q^{-(1-2\varepsilon+o(1))m_{n,I,\varepsilon}} = q^{n-I}q^{-(1-2\varepsilon+o(1))m_{n,I,\varepsilon}}.
\]

This concludes the proof of the lemma.

Lemma 4.5. Assume that \( \frac{1}{n} < \frac{1-2\varepsilon}{1-\varepsilon} (1-\tau) \). Denote \( s = \sqrt{(1-\varepsilon)/(1-2\varepsilon)} \sqrt{n} \), \( y = \frac{I \cdot s}{n} \). Define \( Y_h \) as in (3.9). Assume that \( y > 1 \) and \( n \) is sufficiently large in terms of \( \varepsilon \) and \( \tau \). The inequality

\[
\sum_{h=1}^{n/2} q^{-h}Y_h \leq q^{n-I}q^{-(1-2\varepsilon)n/I+4-3\varepsilon+B_{q,\varepsilon,\tau,y}}
\]

holds, with \( B_{q,\varepsilon,\tau,y} \) tending to zero as \( q \) grows to infinity.

Proof. We give a partition to \( \sum_{h=1}^{n/2} q^{-h}Y_h \):

\[
\sum_{h=1}^{n/2} q^{-h}Y_h = \sum_{h=1}^{\min n/2,n/I} q^{-h}Y_h + \sum_{n/I \leq h < n/4} q^{-h}Y_h + \sum_{h=n/4}^{n/2} q^{-h}Y_h \tag{4.16}
\]

Under the assumptions of the lemma we know that \( I > \sqrt{(1-2\varepsilon)/(1-\varepsilon)} \sqrt{n} \), hence by considering \( n \) that are sufficiently large in terms of \( \varepsilon \) we may assume that \( n/I < n/4 \). We bound \( X_1 \) using the second part of Lemma 4.1:

\[
X_1 \leq C_{q,\varepsilon,y}q^{n-I}q^{3(1-\varepsilon)-(1-2\varepsilon)n/I},
\]

with \( C_{q,\varepsilon,y} \) tending to 1 as \( q \) tends to infinity. We use Lemma 4.3 in order to bound \( X_2 \), and that gives

\[
X_2 \leq C_{q,\varepsilon,\tau}q^{n-I+1}q^{-(1-2\varepsilon-3(1-\varepsilon)I/n)n/I},
\]

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with $C_{q,\varepsilon,\tau}$ also tending to 1 as $q$ tends to infinity. For $X_3$, we use Lemma 4.2 with $k = n/4$. From that we have

$$X_3 \leq q^{n-I+1}q^{-(1-2\varepsilon-3(1-\varepsilon)I/n)n/4}.$$ 

Now we show that if $n$ is sufficiently large we can guarantee that the bound on $X_2$ dominates the bound on $X_3$. This happens when

$$-(1-2\varepsilon-3(1-\varepsilon)I/n)n/I \geq -(1-2\varepsilon-4(1-\varepsilon)I/n)n/4.$$ 

Writing $\beta = 1-2\varepsilon-4(1-\varepsilon)I/n$, this inequality translates into

$$-(\beta + (1-\varepsilon)I/n)n/I \geq -\beta n/4.$$ 

Simplifying this inequality gives

$$-(\beta + (1-\varepsilon)I/n)n/I \geq -\beta n/4 \geq (1-\varepsilon)/\beta.$$ 

Note that the assumption $\frac{I}{n} < \frac{1}{4} \frac{1-2\varepsilon}{1-\varepsilon}(1-\tau)$ gives us that $\beta = 1-2\varepsilon-4(1-\varepsilon)I/n \geq \tau$, hence $(1-\varepsilon)/\beta < (1-\varepsilon)/\tau$. Since $I \gg \varepsilon \sqrt{n}$, the left-hand side of (4.19) behaves asymptotically like $n$, so for large $n$ we get

$$n/4 - n/I \geq (1-\varepsilon)/\tau \geq (1-\varepsilon)/\beta.$$ 

This implies that

$$X_2, X_3 \leq C_{q,\varepsilon,\tau}q^{n-I+1}q^{-(1-2\varepsilon-3(1-\varepsilon)I/n)n/I}q^{-(1-2\varepsilon)n/I+4-3\varepsilon}.$$ 

Thus

$$X_1 + X_2 + X_3 \leq C_{q,\varepsilon,y}q^{n-I}q^{3(1-\varepsilon)-(1-2\varepsilon)I/n} + C_{q,\varepsilon,\tau}q^{n-I}q^{-(1-2\varepsilon)n/I+4-3\varepsilon} \leq q^{n-I}q^{-(1-2\varepsilon)n/I+4-3\varepsilon + \log_q(C_{q,\varepsilon,y} + 2C_{q,\varepsilon,\tau})}.$$ 

Writing $B_{q,\varepsilon,\tau,y} = \log_q(C_{q,\varepsilon,y} + 2C_{q,\varepsilon,\tau})$ and recalling that $C_{q,\varepsilon,y}$ and $C_{q,\varepsilon,\tau}$ are bounded with respect to $q$, it is clear that $B_{q,\varepsilon,\tau,y}$ tends to zero as $q$ tends to infinity. Inserting this into (4.16) gives

$$\sum_{h=1}^{n/2} q^{-h} \sum_{G,H : \deg H = h} \left| \hat{F}_{q,n}(G/H) \right| \leq q^{n-I}q^{-(1-2\varepsilon)n/I+4-3\varepsilon + B_{q,\varepsilon,\tau,y}},$$

as required. \qed

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4.2 A partition

Recall that as in (1.12)

\[
\sum_{P \in C} 1 = \frac{1}{q^n} \sum_{F \in M_n} \hat{F}_{q,n}(T^{-n}F)f(-T^{-n}F),
\]

(4.20)

with \( f(\theta) = \sum_{P \in M_n} e(\theta P) \). For every polynomial \( F \in M_n \) we denote \( \theta_F = \{T^{-n}F\} \). We denote by \( G_F, H_F \) the corresponding polynomials to \( \theta_F \) as in Lemma 2.1. We divide \( M_n \) into three sets, with relation to Lemma 2.2:

\[
S_1 = \{ F \in M_n : \theta_F = c/T \text{ for some } c \in \mathbb{F}_q \},
\]

\[
S_2 = \{ F \in M_n : |\theta_F - G_F/H_F| < q^{-n} \text{ and } 1, T \not\equiv H_F \text{ is squarefree} \},
\]

(4.21)

\[
S_3 = M_n \setminus (S_1 \cup S_2).
\]

Note that this is indeed a partition of \( M_n \). The sum (4.20) decomposes into three sums accordingly. The sum over the polynomials in \( S_1 \) will give us the main term, which we compute in Subsection 4.4. We use the bounds obtained in Section 3 in order to show that the sums over \( S_2, S_3 \) are of small size in Subsection 4.3. The conclusion of the proof is given in Section 4.5.

4.3 Error Term Bound

Our aim is to bound

\[
Y = \left| \frac{1}{q^n} \sum_{F \in S_3} \hat{F}_{q,n}(\theta_F)f(-\theta_F) + \frac{1}{q^n} \sum_{F \in S_2} \hat{F}_{q,n}(\theta_F)f(-\theta_F) \right|.
\]

(4.22)

First, one can easily check by the definition of \( f \) in (1.8) that for every \( \theta \in \mathcal{U} \) one has \( |f(-\theta)| = |f(\theta)| \). We now apply circle-method bounds on \( f \) given in Lemma 2.2. For \( F \in S_2 \), we have

\[
|f(-\theta_F)| = |f(\theta_F)| \leq q^{n-\frac{1}{2}\lfloor \frac{|F|}{2} \rfloor} + q^{n-\deg H_F}.
\]

(4.23)

For \( F \in S_3 \) we have

\[
|f(-\theta_F)| = |f(\theta_F)| \leq q^{n-\frac{1}{2}\lfloor \frac{|F|}{2} \rfloor}.
\]

(4.24)

Applying the triangle inequality to (1.22) yields

\[
Y \leq \frac{1}{q^n} \sum_{F \in S_3} \hat{F}_{q,n}(\theta_F) \cdot |f(-\theta_F)| + \frac{1}{q^n} \sum_{F \in S_2} \hat{F}_{q,n}(\theta_F) \cdot |f(-\theta_F)|.
\]

By (4.23) and (4.24) we obtain

\[
Y \leq \frac{1}{q^n} \sum_{F \in S_3} \hat{F}_{q,n}(\theta_F) \left( q^{n-\frac{1}{2}\lfloor \frac{|F|}{2} \rfloor} + q^{n-\deg H_F} \right).
\]
Simplifying this gives

\[
Y \leq \frac{q^{n-\frac{1}{2}\left\lVert \frac{q}{2} \right\rVert}}{q^n} \sum_{F \in S_{2} \cup S_{3}} \left| \hat{F}_{q,n}(\theta_{F}) \right| + \frac{1}{q^n} \sum_{F \in S_{2}} q^{n - \deg H_{F}} \left| \hat{F}_{q,n}(\theta_{F}) \right|
\]

\[
\leq \frac{q^{n-\frac{1}{2}\left\lVert \frac{q}{2} \right\rVert}}{q^n} \sum_{F \in M_{n}} \left| \hat{F}_{q,n}(\theta_{F}) \right| + \frac{1}{q^n} \sum_{G,H} q^{n - \deg H_{F}} \left| \hat{F}_{q,n}(G/H) \right|
\]

where in the latter sum \( H \) ranges over squarefree polynomials other than 1, \( T \), and \( G \) is coprime to \( H \) of smaller degree. Note that we replace \( \left| \hat{F}_{q,n}(\theta_{F}) \right| \) by \( \left| \hat{F}_{q,n}(G/H) \right| \) in the right-hand sum due to Lemma 3.6. Applying Proposition 3.4 to the first sum and changing order of summation in the second yields

\[
Y \leq \alpha(n - I)q^{n-\frac{1}{2}\left\lVert \frac{q}{2} \right\rVert} + \frac{1}{q^n} \sum_{h=1}^{n/2} q^{-h} \sum_{G,H} q^{n - \deg H_{F}} \left| \hat{F}_{q,n}(G/H) \right|
\]

where again \( H \) ranges over squarefree polynomials other than 1, \( T \). We assume that \( N_{i} < q^{s_{m}} \) for all \( 0 \leq i < n \), and that \( \frac{1}{n} < \frac{1}{4} \cdot \frac{1 - 2\epsilon}{1 - \epsilon}(1 - \tau) \) for some \( \tau > 0 \). Denote \( s = \sqrt{(1 - \epsilon)/(1 - 2\epsilon)} \sqrt{n} \), and \( m_{n,I,\epsilon} = \min\{n/I, s\} \). By Lemma 4.4 if \( I = o(n/\log(n)) \) then

\[
\sum_{h=1}^{n/2} q^{-h} \sum_{G,H} q^{n - \deg H_{F}} \left| \hat{F}_{q,n}(G/H) \right| \leq q^{n-I}q^{-(1-2\epsilon+o(1))m_{n,I,\epsilon}}.
\]

In this case by (4.25) we obtain a bound on the error term:

\[
\left| \sum_{P \in \mathcal{C}} \frac{1}{q^n} \sum_{F \in S_{1}} \hat{F}_{q,n}(\theta_{F}) f(-\theta_{F}) \right| \leq \alpha(n - I)q^{n-\frac{1}{2}\left\lVert \frac{q}{2} \right\rVert} + q^{-I}q^{-(1-2\epsilon+o(1))m_{n,I,\epsilon}}.
\]

(4.26)

By Lemma 4.5, if we keep our notation and assume that \( y = \frac{1}{n} \delta > 1 \) and that \( n \) is sufficiently large in terms of \( \epsilon \) and \( \tau \), we obtain

\[
\sum_{h=1}^{n/2} q^{-h} \sum_{G,H} q^{n - \deg H_{F}} \left| \hat{F}_{q,n}(G/H) \right| \leq q^{n-I}q^{-(1-2\epsilon)n/I+4-3\epsilon+B_{q,\epsilon,\tau,y}}
\]

with \( B_{q,\epsilon,\tau,y} \) tending to zero as \( q \) grows to infinity. Substituting this bound into (4.25) gives an error term bound of

\[
\left| \sum_{P \in \mathcal{C}} \frac{1}{q^n} \sum_{F \in S_{1}} \hat{F}_{q,n}(\theta_{F}) f(-\theta_{F}) \right| \leq \alpha(n - I)q^{n-\frac{1}{2}\left\lVert \frac{q}{2} \right\rVert} + q^{-I}q^{-(1-2\epsilon)n/I+4-3\epsilon+B_{q,\epsilon,\tau,y}}.
\]

(4.27)
4.4 Main Term Computation

For the main term, we have

\[ X = \frac{1}{q^n} \sum_{F \in S_1} \hat{F}_{q,n}(\theta_F) f(-\theta_F) = \frac{1}{q^n} \sum_{a \in \mathbb{F}_q} \hat{F}_{q,n}(a/T) f(-a/T) \]

Expanding out the definition of \( f \) and \( \hat{F}_{q,n} \) given in (1.8) and (1.10), we get

\[ X = \frac{1}{q^n} \sum_{a \in \mathbb{F}_q} \sum_{F \in M_n} 1_C(F) e \left( F \cdot \frac{a}{T} \right) \sum_{P \in M_n} e \left( -\frac{a}{T} P \right), \]

and changing order of summation gives

\[ X = \frac{1}{q^n} \sum_{F, P \in M_n} 1_C(F) e \left( \frac{(F - P)a}{T} \right) \sum_{a \in \mathbb{F}_q} e \left( -\frac{a}{T} P \right), \]

By the orthogonality relations we have

\[ X = \frac{1}{q^n} \sum_{F, P \in M_n} 1_C(F) q \mathbbm{1}_{f_0 = P_0} = \frac{1}{q^{n-1}} \sum_{F, P \in M_n} 1_C(F) \mathbbm{1}_{f_0 = P_0}. \]

Summing over \( c = f_0 = p_0 \in \mathbb{F}_q \), we see that

\[ X = \frac{1}{q^{n-1}} \sum_{c \in \mathbb{F}_q} \left( \sum_{F \in M_n} 1_C(F) \right) \left( \sum_{P \in M_n} \mathbbm{1}_{P_0 = c} 1 \right) \]  \hspace{1cm} (4.28)

Note that the middle sum is exactly

\[ \sum_{F \in M_n} 1_C(F) = 1_{\mathcal{C}_0}(c) \prod_{1 \leq i \in J} (q - N_i), \]  \hspace{1cm} (4.29)

so substituting this into (4.28) gives

\[ X = \frac{1}{q^{n-1}} \prod_{1 \leq i \in J} (q - N_i) \sum_{c \in \mathbb{F}_q} 1_{\mathcal{C}_0}(c) \sum_{P_0 = c} 1 \]  \hspace{1cm} (4.30)

When \( c = 0 \), we have \( \sum_{P_0 = c} 1 = 0 \). Otherwise, by the Prime Polynomial Theorem in arithmetic progressions we have \( \sum_{P_0 = c} 1 = \frac{1}{q-1} \pi_q(n) + O \left( q^{n/2} \right) \). If \( 0 \in \mathcal{I} \), then the only
c for which \(1_{C_0}(c) \neq 0\) is \(c = a_0\). Thus in this case

\[
X = \frac{1}{q^{n-1}} \prod_{1 \leq i \in J} (q - N_i) \sum_{P \in M_n \atop p_0 = c} 1
\]

\[
= \frac{1}{q^{n-1}} \prod_{1 \leq i \in J} (q - N_i) \left( \frac{1}{q - 1} \pi_q(n) + O(q^{n/2}) \right)
\]

So we have shown that in this case

\[
X = \left( \frac{1}{q^{n-1}(q - 1)} \prod_{i \in J} (q - N_i) \right) \pi_q(n) + O(q^{n/2})
\]  

(4.31)

In the case where \(0 \in J\), we have that \(1_{C_0}(c) \neq 0\) for \(c \in \mathbb{F}_q \setminus S_0\). So in this case

\[
\sum_{c \in \mathbb{F}_q} 1_{C_0}(c) = \sum_{c \in \mathbb{F}_q \setminus S_0} 1 = \pi_q(n) - \sum_{c \in S_0} \sum_{P \in M_n \atop p_0 = c} 1
\]  

(4.32)

By assumption \(0 \notin S_0\), thus

\[
\sum_{c \in S_0} \sum_{P \in M_n \atop p_0 = c} 1 = \frac{N_0}{q - 1} \pi_q(n) + O(N_0 q^{n/2})
\]

Plugging this into (4.32) gives

\[
\sum_{c \in \mathbb{F}_q} 1_{C_0}(c) \sum_{P \in M_n \atop p_0 = c} 1 = \frac{q - N_0 - 1}{q - 1} \pi_q(n) + O(N_0 q^{n/2})
\]  

(4.33)

and combining (4.33) and (4.31) results in

\[
X = \left( \frac{1}{q^{n-1}} \prod_{1 \leq i \in J} (q - N_i) \right) \left( \frac{q - N_0 - 1}{q - 1} \pi_q(n) + O(N_0 q^{n/2}) \right)
\]  

(4.34)

Writing

\[
\mathcal{S} = \begin{cases} 
q^{n-i_q-1} \prod_{i \in J} (q - N_i) & \text{if } 0 \in \mathcal{I} \\
q^{n-i_q-1} \prod_{1 \leq i \in J} (q - N_i) & \text{if } 0 \in \mathcal{J}
\end{cases}
\]  

(4.35)

we have shown in (4.31) and (4.34) that

\[
\left| \frac{1}{q^n} \sum_{a \in \mathbb{F}_q} \hat{F}_{q,n}(a/T) f(-a/T) - \mathcal{S} \cdot \pi_q(n) \right| = |X - \mathcal{S} \cdot \pi_q(n)| = O\left(q^{n/2+1}\right)
\]  

(4.36)

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4.5 Conclusion

Recall that

\[
\sum_{P \in \mathcal{C}} 1 = \frac{1}{q^n} \sum_{F \in \mathcal{M}_n} \hat{F}_{q,n}(\theta_F)f(-\theta_F)
\]

\[
= \frac{1}{q^n} \sum_{F \in S_1 \cup S_2 \cup S_3} \hat{F}_{q,n}(\theta_F)f(-\theta_F)
\]

where \(S_1, S_2, S_3\) provide a partition of \(\mathcal{M}_n\) defined in (4.21). Thus

\[
\left| \left( \sum_{P \in \mathcal{C}} 1 \right) - \frac{1}{q^n} \sum_{F \in S_1} \hat{F}_{q,n}(\theta_F)f(-\theta_F) \right| = \left| \frac{1}{q^n} \sum_{F \in S_2} \hat{F}_{q,n}(\theta_F)f(-\theta_F) + \frac{1}{q^n} \sum_{F \in S_3} \hat{F}_{q,n}(\theta_F)f(-\theta_F) \right| .
\]

Bounds for

\[
\left| \left( \sum_{P \in \mathcal{C}} 1 \right) - \frac{1}{q^n} \sum_{F \in S_1} \hat{F}_{q,n}(\theta_F)f(-\theta_F) \right|
\]

are given in (4.26) and (4.27). For the main term, we have shown in (4.36) that

\[
\left| \frac{1}{q^n} \sum_{F \in S_1} \hat{F}_{q,n}(\theta_F)f(-\theta_F) - \mathcal{S} \cdot \pi_q(n) \right| = O\left(q^{n/2+1}\right),
\]

with \(\mathcal{S}\) defined as in (4.35). Thus, by the triangle inequality,

\[
\left| \left( \sum_{P \in \mathcal{C}} 1 \right) - \mathcal{S} \cdot \pi_q(n) \right| \leq \left| \left( \sum_{P \in \mathcal{C}} 1 \right) - \frac{1}{q^n} \sum_{F \in S_1} \hat{F}_{q,n}(\theta_F)f(-\theta_F) \right| + O\left(q^{n/2+1}\right). \tag{4.37}
\]

Writing \(s = \sqrt{(1 - \varepsilon)/(1 - 2\varepsilon)}\sqrt{n}\) and \(m_{n,t,\varepsilon} = \min\{n/I, s\}\), plugging (4.26) into (4.37) yields

\[
\left| \left( \sum_{P \in \mathcal{C}} 1 \right) - \mathcal{S} \cdot \pi_q(n) \right| \leq \alpha(n - I)q^{n-\frac{1}{2}\left[\frac{s}{I}\right]} + q^{n-I}q^{-(1-2\varepsilon+o(1))m_{n,t,\varepsilon}} + O\left(q^{n/2+1}\right)
\]

\[
= (\alpha(n - I) + o(1))q^{n-\frac{1}{2}\left[\frac{s}{I}\right]} + q^{n-I}q^{-(1-2\varepsilon+o(1))m_{n,t,\varepsilon}},
\]

when \(I = o(n/\log(n))\). If we have larger \(I\), we assume that \(y = \frac{Ls}{n} > 1\) and that \(n\) is sufficiently large in terms of \(\varepsilon\) and \(\tau\). In this case, we use the bound given in (4.27) together with (4.37) in order to obtain

\[
\left| \left( \sum_{P \in \mathcal{C}} 1 \right) - \mathcal{S} \cdot \pi_q(n) \right| \leq \alpha(n - I)q^{n-\frac{1}{2}\left[\frac{s}{I}\right]} + q^{n-I}q^{-(1-2\varepsilon)n/I+4-3\varepsilon+B_{q,\varepsilon,\tau,y}} + O\left(q^{n/2+1}\right)
\]

\[
= (\alpha(n - I) + o(1))q^{n-\frac{1}{2}\left[\frac{s}{I}\right]} + q^{n-I}q^{-(1-2\varepsilon)n/I+4-3\varepsilon+B_{q,\varepsilon,\tau,y}},
\]

with \(B_{q,\varepsilon,\tau,y}\) tending to zero as \(q\) grows to infinity. This completes the proof of the theorem.
5 Discussion

In the introduction we defined a set \( C \) and stated how many primes one might expect \( C \) to contain. For convenience, we defined \( C \) to be

\[
C = \left\{ T^n + \sum_{i=0}^{n-1} b_i T^i : b_i = a_i \forall i \in I, b_j \not\in S_j \forall j \in J \right\},
\]

where \( I \uplus J \) be a partition of \( \{0, \ldots, n-1\} \), \( a_0 \neq 0 \) if \( 0 \in I \), and \( 0 \not\in S_0 \) if \( 0 \in J \). Write \( I = \#I \), and for every \( j \in J \) write \( N_j = \#S_j \). The number of primes to be expected in \( C \) is \( \mathcal{S} \cdot \pi_q(n) \), with

\[
\mathcal{S} = \left\{ \begin{array}{ll}
\frac{\prod_{i \in I} (q-N_i)}{q^n-1(q-1)} & \text{if } 0 \in I \\
\frac{(q-1-N_0) \prod_{i \in J} (q-N_i)}{q^{n-1}(q-1)} & \text{if } 0 \in J.
\end{array} \right.
\]

The asymptotics are indeed what one might expect: Note that there are \( q - N_i \) options for every index \( i \in J \), so \( \#C = \prod_{i \in J} (q - N_i) \). We expect the proportion of primes in \( C \) to be similar to that in all of the monic polynomials, up to a correction factor due to the coefficient \( b_0 \). We think of \( \mathcal{S} \) as \( \mathcal{S} = \frac{\#C}{q^n} \cdot R \), with \( R \) being a correction factor. If \( 0 \in I \), then since \( a_0 \neq 0 \), the probability of being prime increases by a factor of \( \frac{q}{q-1} \), hence in this case

\[
\mathcal{S} = \frac{\#C}{q^n} \cdot R = \frac{\#C}{q^n} \cdot \frac{q}{q-1} = \frac{1}{q^{n-1}(q-1)} \prod_{i \in J} (q - N_i).
\]

In the other case where \( 0 \in J \), having assumed \( 0 \not\in S_0 \) the correction factor will be \( R = \frac{q-N_0-1}{q-1} \cdot \frac{q}{q-1} \), so in this case

\[
\mathcal{S} = \frac{\#C}{q^n} \cdot R = \frac{\#C}{q^n} \cdot \frac{q-N_0-1}{q-1} \cdot \frac{q}{q-1} = \frac{q-N_0-1}{q^{n-1}(q-1)} \prod_{1 \geq i \in J} (q - N_i).
\]

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