Evolutionary Problems Involving Sturm-Liouville Operators.

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The purpose of this paper is to further exemplify an approach to evolutionary problems originally developed in [3], [4] for a special case and extended to more general evolutionary problems, see [7], compare the survey article [5]. The ideas presented in there are utilized for \((1+1)\)-dimensional evolutionary problem, which in a particular case results in a hyperbolic partial differential equation with a Sturm-Liouville type spatial operator constrained by an impedance type boundary condition.

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A canonical form of many linear evolutionary\textsuperscript{1} equations of mathematical physics is given by a dynamic system of equations

\[ \partial_0 V + AU = F \]

completed by a so-called material law

\[ V = MU. \]

Here \( \partial_0 \) denotes derivation with respect to the time variable, \( A \) is a usually unbounded operator containing spatial derivatives and \( M \) is a continuous linear operator. Here we would like to inspect more closely a very specific situation, where the space dimension is 1. Although this is a rather particular case it has the advantage that an impedance type boundary condition, which we wish to consider, can be considered in a more “tangible” way without incurring regularity assumptions on coefficients and data. In the higher dimensional case for the acoustic equations, which can also be discussed with no further regularity requirements, the constraints on the impedance type boundary condition are much less explicit, compare [7], [5]. Moreover, we hope to gain a different access to a class of problems, which are closely linked to Sturm-Liouville operators, in fact yielding a generalization of such operators. That we are discussing the direct time-dependent problem rather than an associated spectral problem will actually be advantageous, since it provides a simpler access to the discussion of well-posedness.

More specifically we want to consider

\[ A = \begin{pmatrix} 0 & 0 & \partial \\ 0 & 0 & 0 \\ \partial & 0 & 0 \end{pmatrix} \]

as a differential operator on the unit interval \([-1/2, 1/2[\) with an impedance type boundary condition of the form

\[ \partial_0 \alpha \left( \pm 1/2 \mp 0 \right) s(\cdot, \pm 1/2 \mp 0) - \nu(\cdot, \pm 1/2 \mp 0) = 0 \]

holding on the real time-line \( \mathbb{R} \) as a constraint characterizing \( \begin{pmatrix} s \\ w \end{pmatrix} \) in the domain \( D(A) \).

\textsuperscript{1}We prefer the term \textit{evolutionary equations}, since the term \textit{evolution equations} is usually reserved for a special case of the class of evolutionary equations considered here.
later. We shall focus here on the time-translation invariant, i.e. autonomous, case. This means that time-translation and consequently time-differentiation commutes with $\alpha$, $\mathcal{M}$ and $A$.

Our discussion is embedded into an abstract setting, which we will develop in Section 1 first. In Section 2 we will then discuss our problem of interest as an application of the solution theory in the abstract setting.

1 The Abstract Solution Framework

1.1 Sobolev chains associated with the time-derivative

A particular instance of the construction of Sobolev chains is the one based on the time-derivative $\partial_0$. We recall, e.g. from [4, 5], that differentiation considered in the complex Hilbert space $H_{\nu,0}(\mathbb{R}) := \{ f \in L^2_{\text{loc}}(\mathbb{R}) | (x \mapsto \exp(-\nu x) f(x)) \in L^2(\mathbb{R}) \}$, $\nu \in \mathbb{R}\setminus \{ 0 \}$, with inner product

$$
(f,g) \mapsto \langle f, g \rangle_{\nu,0} := \int_{\mathbb{R}} f(x)^* g(x) \exp(-2\nu x) \, dx
$$

can indeed be established as a normal operator, which we denote by $\partial_{0,\nu}$, with

$$
\Re \partial_{0,\nu} = \nu.
$$

For $\Im \partial_{0,\nu}$ we have as a spectral representation the Fourier-Laplace transform $\mathcal{L}_\nu : H_{\nu,0}(\mathbb{R}) \to L^2(\mathbb{R})$ given by the unitary extension of

$$
\hat{C}_\nu(\mathbb{R}) \subseteq H_{\nu,0}(\mathbb{R}) \to L^2(\mathbb{R})
$$

$$
\phi \mapsto \left( x \mapsto \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp(-i x y) \exp(-\nu y) \phi(y) \, dy \right).
$$

In other words, we have the unitary equivalence

$$
\Im \partial_{0,\nu} = \mathcal{L}_\nu^{-1} m \mathcal{L}_\nu,
$$

where $m$ denotes the selfadjoint multiplication-by-argument operator in $L^2(\mathbb{R})$. Since 0 is in the resolvent set of $\partial_{0,\nu}$ we have that $\partial_{0,\nu}^{-1}$ is an element of the Banach space $L(H_{\nu,0}(\mathbb{R}), H_{\nu,0}(\mathbb{R}))$ of continuous (left-total) linear mappings in $H_{\nu,0}(\mathbb{R})$. Denoting generally the operator norm of the Banach space $L(X,Y)$ by $\| \cdot \|_{L(X,Y)}$, we get for $\partial_{0,\nu}^{-1}$

$$
\| \partial_{0,\nu}^{-1} \|_{L(H_{\nu,0}(\mathbb{R}), H_{\nu,0}(\mathbb{R}))} = \frac{1}{|\nu|}.
$$

Not too surprisingly, we find for $\nu > 0$

$$
(\partial_{0,\nu}^{-1} \varphi)^{'}(x) = \int_{-\infty}^{x} \varphi^{'}(t) \, dt
$$


and for \( \nu < 0 \)

\[ (\hat{c}_{0,\nu}^{-1} \varphi)(x) = -\int_{x}^{\infty} \varphi(t) \, dt \]

for all \( \varphi \in H_{\nu,0}(\mathbb{R}) \) and \( x \in \mathbb{R} \). Since we are interested in the forward causal situation, we assume \( \nu > 0 \) throughout. Moreover, in the following we shall mostly write \( \hat{c}_0 \) for \( \hat{c}_{0,\nu} \) if the choice of \( \nu \) is clear from the context.

Thus, we obtain a chain \( (H_{\nu,k}(\mathbb{R}))_{k \in \mathbb{Z}} \) of Hilbert spaces, where \( H_{\nu,k}(\mathbb{R}) \) is the completion of the inner product space \( D(\hat{c}_0^k) \) with norm \( | \cdot |_{\nu,k} \) given by

\[ \phi \mapsto |\hat{c}_0^k \phi|_{\nu,0}. \]

Similarly, for \( im + \nu \) as a normal operator in \( L^2(\mathbb{R}) \) we construct the chain of polynomially weighted \( L^2(\mathbb{R}) \)-spaces

\[ \left( L^2_k(\mathbb{R}) \right)_{k \in \mathbb{Z}} \]

with

\[ L^2_k(\mathbb{R}) := \left\{ f \in L^2_{loc}(\mathbb{R}) \mid (im + \nu)^k f \in L^2(\mathbb{R}) \right\} = H_k(im + \nu) \]

for \( k \in \mathbb{Z} \).

Since the unitarily equivalent operators \( \hat{c}_{0,\nu} \) and \( im + \nu \) (via the Fourier-Laplace transform) can canonically be lifted to the \( X \)-valued case, \( X \) an arbitrary complex Hilbert space, we are lead to a corresponding chain \( (H_{\nu,k}(\mathbb{R},X))_{k \in \mathbb{Z}} \) and \( (L^2_k(\mathbb{R},X))_{k \in \mathbb{Z}} \) of \( X \)-valued generalized functions. The Fourier-Laplace transform can also be lifted to the \( X \)-valued case yielding

\[ H_{\nu,k}(\mathbb{R},X) \to L^2_k(\mathbb{R},X) \]

\[ f \mapsto \mathcal{L}_\nu f \]

as a unitary mapping for \( k \in \mathbb{N} \) and by continuous extension, keeping the notation \( \mathcal{L}_\nu \) for the extension, also for \( k \in \mathbb{Z} \). Since \( \mathcal{L}_\nu \) has been constructed from a spectral representation of \( \mathfrak{Im} \hat{c}_{0,\nu} \), we can utilize the corresponding operator function calculus for functions of \( \mathfrak{Im} \hat{c}_{0,\nu} \). Noting that \( \hat{c}_0 = i \mathfrak{Im} \hat{c}_{0,\nu} + \nu \) is a function of \( \mathfrak{Im} \hat{c}_{0,\nu} \) we can define operator-valued functions of \( \hat{c}_0 \).

**Definition 1.1.** Let \( r > \frac{1}{2\nu} > 0 \) and \( M : B_C(r,r) \to L(H,H) \) be bounded and analytic, \( H \) a Hilbert space. Then define

\[ M\left( \hat{c}_0^{-1} \right) := \mathcal{L}_\nu^* M \left( \frac{1}{im + \nu} \right) \mathcal{L}_\nu, \]

where

\[ M\left( \frac{1}{im + \nu} \right) \phi(t) := M \left( \frac{1}{\bar{\nu} + \nu} \right) \phi(t) \quad (t \in \mathbb{R}) \]

for \( \phi \in \mathcal{C}_\infty(\mathbb{R},H) \).
1.2 Abstract Solution Theory

Remark 1.2. The definition of \( M(\tilde{c}_0^{-1}) \) is largely independent of the choice of \( \nu \) in the sense that the operators for two different parameters \( \nu_1, \nu_2 \) coincide on the intersection of the respective domains.

Simple examples are polynomials in \( \tilde{c}_0^{-1} \) with operator coefficients. A more exotic example of an analytic and bounded function of \( \tilde{c}_0^{-1} \) is the delay operator, which itself is a special case of the time translation:

**Examples:** Let \( r > 0, \; \nu > \frac{1}{2r} \), \( h \in \mathbb{R} \) and \( u \in H_{\nu,0}(\mathbb{R}, X) \). We define

\[
\tau_h u := u(\cdot + h).
\]

The operator \( \tau_h \in L(H_{\nu,0}(\mathbb{R}, X), H_{\nu,0}(\mathbb{R}, X)) \) is called a *time-translation operator*. If \( h < 0 \) the operator \( \tau_h \) is also called a *delay operator*. In the latter case the function

\[
B_{\mathbb{C}}(r, r) \ni z \mapsto M(z) := \exp(z^{-1}h)
\]

is analytic and uniformly bounded for every \( r \in \mathbb{R}_{>0} \) (considered as an \( L(X, X) \)-valued function). An easy computation shows for \( u \in H_{\nu,0}(\mathbb{R}, X) \) that

\[
u M_0 + \Re M_1 \geq c_0 > 0
\]

Another class of interesting bounded analytic functions of \( \tilde{c}_0^{-1} \) are mappings produced by a temporal convolution with a suitable operator-valued integral kernel.

1.2 Abstract Solution Theory

We shall discuss equations of the form

\[
(\tilde{c}_0 M(\tilde{c}_0^{-1}) + A) U = \mathcal{J}.
\]

where we shall assume that \( A \) and \( A^* \) are commuting with \( \tilde{c}_0 \) and non-negative in the Hilbert space \( H_{\nu,0}(\mathbb{R}, H) \), \( H \) a given Hilbert space, in the sense that

\[
\Re \langle U | AU \rangle_{\nu,0} \geq 0, \; \Re \langle V | A^* V \rangle_{\nu,0} \geq 0
\]

for all \( U \in D(A), \; V \in D(A^*) \), and \( M \) is a material law in the sense of [3, 6]. More specifically we assume that \( M \) is of the form

\[
M(z) = M_0 + zM_1 + z^2M^{(2)}(z)
\]

where \( M^{(2)} \) is an analytic and bounded \( L(H, H) \)-valued function in a ball \( B_{\mathbb{C}}(r, r) \) for some \( r \in \mathbb{R}_{>0} \) and \( M_0 \) is a continuous, selfadjoint and non-negative operator in \( H \). The operator \( M_1 \in L(H, H) \) is such that

\[
\nu M_0 + \Re M_1 \geq c_0 > 0
\]
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for all sufficiently large $\nu \in \mathbb{R}_{>0}$. The operator $M(\delta_0^{-1})$ is then to be understood in the sense of the operator-valued function calculus associated with the selfadjoint operator $\Im(\delta_0) = \frac{1}{2i}(\delta_0 - \delta_0^*)$.

The appropriate setting turns out to be the Sobolev chain

$$(H_{\nu,k} (\mathbb{R}, H))_{k \in \mathbb{Z}}.$$

From [3, 6] we paraphrase the following solution result.

**Theorem 1.3.** For $J \in H_{\nu,k} (\mathbb{R}, H)$ the problem (1) has a unique solution $U \in H_{\nu,k} (\mathbb{R}, H)$. Moreover,

$$F \mapsto (\delta_0 M(\delta_0^{-1}) + A)^{-1} F$$

is a linear mapping in $L(H_{\nu,k} (\mathbb{R}, H), H_{\nu,k} (\mathbb{R}, H))$, $k \in \mathbb{Z}$. These mappings are causal in the sense that if $F \in H_{\nu,k} (\mathbb{R}, H)$ vanishes on the time interval $] - \infty, a]$, then so does $(\delta_0 M(\delta_0^{-1}) + A)^{-1} F$, $a \in \mathbb{R}$, $k \in \mathbb{Z}$.

**Remark 1.4.** Note that if $U \in H_{\nu,k} (\mathbb{R}, H)$ and $J \in H_{\nu,k} (\mathbb{R}, H)$ equation (1) actually makes sense in $H_{\nu,k-1} (\mathbb{R}, H)$. Initially the solution theory is for the closure of $(\delta_0 M(\delta_0^{-1}) + A)$ as a closed operator in $H_{\nu,k-1} (\mathbb{R}, H)$, but it is

$$(\delta_0 M(\delta_0^{-1}) + A) U = \overline{(\delta_0 M(\delta_0^{-1}) + A) U}$$

with equality holding in $H_{\nu,k-1} (\mathbb{R}, H)$. Indeed, for $\phi \in H_{\nu,k} (\mathbb{R}, H) \cap D(A^*)$ we have

$$\left\langle \phi | (\delta_0 M(\delta_0^{-1}) + A) U \right\rangle_{\nu,k-1,0} = \left\langle (\delta_0^* M(\delta_0^{-1})^* + A^*) \phi | U \right\rangle_{\nu,k-1,0}$$

$$= \left\langle \phi | M(\delta_0^{-1}) \delta_0 U \right\rangle_{\nu,k-1,0} + \left\langle A^* \phi | U \right\rangle_{\nu,k-1,0}$$

and we read off that $U \in D(A)$ if $A$ is considered in $H_{\nu,k-1} (\mathbb{R}, H)$ (rather than $H_{\nu,k} (\mathbb{R}, H)$) giving

$$AU = (\delta_0 M(\delta_0^{-1}) + A) U - M(\delta_0^{-1}) \delta_0 U,$$

$$= (\delta_0 M(\delta_0^{-1}) + A) U - \delta_0 M(\delta_0^{-1}) U.$$

The rigorous argument is somewhat more involved, see [5, 4]. This observation, however, motivates dropping the closure bar throughout.

2 An Application: An Evolutionary Problem Involving a Sturm-Liouville Type Operator with an Impedance Type Boundary Condition

We exemplify the outlined theory by a $(1+1)$-dimensional example.
Consider
\[
A = \begin{pmatrix} 0 & 0 & \partial \\ 0 & 0 & 0 \\ \partial & 0 & 0 \end{pmatrix}
\]
with an impedance type boundary condition implemented in the domain of \( A \) given by
\[
\left\{ \begin{pmatrix} s \\ w \\ v \end{pmatrix} \in H_{\nu,0} (\mathbb{R}, H (\partial, I) \oplus L^2 (I) \oplus H (\partial, I)) \mid a (\partial_0^{-1}) s - \partial_0^{-1} v \in H_{\nu,0} (\mathbb{R}, H (\partial, I)) \right\},
\]
where \( H (\partial, I) \) denotes the completion of the space of smooth function with compact support in \( I = ] - 1/2, 1/2 [ \) with respect to the graph norm of the derivative operator \( \partial \).
The space \( H (\partial, I) \) is the domain of the adjoint of \( \partial \) also equipped with the corresponding graph norm. We focus here on the finite interval case. It should be noted, however, that the same reasoning would likewise work for the half infinite interval case, say \( I = \mathbb{R}_{>0} \).
Indeed this case would be in a sense simpler since only one boundary point would need to be considered. We assume
\[
x \mapsto a (x, \cdot)
\]
to be uniformly continuous bounded-analytic-function-valued mapping. As a matter of simplification we shall assume that \( (z \mapsto a (x, z))_{x \in I} \) is a uniformly bounded family of bounded functions, which are analytic in a ball \( B_C (0, 2r) \) centred at 0 with radius \( r \in \mathbb{R}_{>0} \). Then surely \( (z \mapsto a (x, z))_{x \in I} \) is also a bounded family of analytic functions in \( B_C (r, r) \) and for \( \nu > \frac{1}{2r} \) we have a continuous linear and causal mapping:
\[
a (\partial_0^{-1}) : H_{\nu,0} (\mathbb{R}, L^2 (I)) \to H_{\nu,0} (\mathbb{R}, L^2 (I))
\]
\[
\varphi \mapsto (t \mapsto (x \mapsto a (x, \partial_0^{-1}) \varphi (t, x))).
\]
Assuming further that the distributional derivative \( a' = (x \mapsto a (x, \cdot))' \) is such that
\[
(z \mapsto a' (x, z))_{x \in I \setminus N}
\]
is also a uniformly bounded family of bounded functions, which are analytic in \( B_C (0, 2r) \), \( N \) a Lebesgue null set, we get a bounded linear and causal mapping
\[
a' (\partial_0^{-1}) : H_{\nu,0} (\mathbb{R}, L^2 (I)) \to H_{\nu,0} (\mathbb{R}, L^2 (I))
\]
and the product rule
\[
\partial (a (\partial_0^{-1}) s) = a' (\partial_0^{-1}) s + a (\partial_0^{-1}) \partial s
\]
holds for \( s \in D (\partial) \). As another simplification we assume that \( a \) is real in the sense that
\[
a (x, z)^* = a (x, z^*)
\]

This 3 \( \times \) 3-system has been chosen, rather than an alternative 2 \( \times \) 2-system formulation, since it shows conservativity in a more obvious way, see Footnote 3.
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for $x \in I$ and $z \in B_C (r, r)$.

Such an operator $A$ combined with a suitable material law yields an evolutionary problem of the form

$$\left( \partial_t M \left( \partial_t^{-1} \right) + A \right) U = \mathcal{J}. \tag{3}$$

Indeed, we consider a material law operators of the form $\varepsilon, \eta$ operators are such that (2) is satisfied, i.e.

$$\nu \varepsilon + \Re \eta \geq c_0 > 0$$

for some $c_0 \in \mathbb{R}$ and all sufficiently large $\nu \in \mathbb{R}_{>0}$. Such material laws are suggested by models of linear acoustics, see e.g. [2], or by the so-called Maxwell-Cattaneo-Vernotte law [1, 3] describing heat propagation. In the 1-dimensional case, focused on here, this special material law operator can be reduced to a wave or heat equation type partial differential operator with a Sturm-Liouville type operator as spatial part. Indeed, assuming

This implies a polynomial type material law operator of the form

$$M \left( \partial_t^{-1} \right) = \begin{pmatrix} \kappa_0 & 0 & 0 \\ 0 & \kappa_1 & -\mu_0 \partial_t^{-1} \\ 0 & \mu_0 \partial_t^{-1} & \varepsilon + \eta \partial_t^{-1} + \mu_1 \partial_t^{-2} \end{pmatrix},$$

where $\varepsilon: L^2 (I) \to L^2 (I)$, $\kappa_0: L^2 (I) \to L^2 (I)$, $\kappa_1: L^2 (I) \to L^2 (I)$ are suitable continuous, selfadjoint, non-negative mappings and $\mu_0: L^2 (I) \to L^2 (I)$, $\mu_1: L^2 (I) \to L^2 (I)$ are continuous and linear. We assume of course that the coefficient operators are such that (2) is satisfied, i.e.

$$\nu \varepsilon + \Re \eta \geq c_0 > 0$$

We see that for $\kappa_0, \kappa_1, \varepsilon$ strictly positive continuous linear operators, $\mu_1 = 0$ and $\eta = 0$ and $A$ skew-selfadjoint, e.g. if $a \left( \partial_t^{-1} \right) = 0$, we have a conservative system since then $M^{(2)} \left( \partial_t^{-1} \right) = 0$ and

$$M_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

is skew-selfadjoint making $A + M_1$ and so also $\sqrt{M_0^{-1}} \left( A + M_1 \right) \sqrt{M_0^{-1}}$ skew-selfadjoint. Consequently, $\sqrt{M_0^{-1}} \left( A + M_1 \right) \sqrt{M_0^{-1}}$ generates a unitary 1-parameter group and “energy” conservation holds in the sense that for the solution $U$ of a pure initial value problem we have for $t \in \mathbb{R}_{>0}$

$$\left| \sqrt{M_0} U \left( t \right) \right|_H = \left| \sqrt{M_0} U \left( 0+ \right) \right|_H$$

or if one prefers to underscore the “energy” metaphor

$$E \left( t \right) = \frac{1}{2} \left| \sqrt{M_0} U \left( t \right) \right|_H^2 = \frac{1}{2} \left| \sqrt{M_0} U \left( 0+ \right) \right|_H^2 = E \left( 0+ \right).$$
additionally that $\kappa_0$ and $\kappa_1$ are strictly positive, two elementary row operations\(^4\) applied to
\[
\begin{pmatrix}
\kappa_0 \hat{c}_0 & 0 & \hat{c} \\
0 & \kappa_1 \hat{c}_0 & -\mu_0^* \\
\hat{c} & \mu_0 & \varepsilon \hat{c}_0 + \eta + \mu_1 \hat{c}_0^{-1}
\end{pmatrix}
\]
yield formally
\[
\begin{pmatrix}
\kappa_0 \hat{c}_0 & 0 & \hat{c} \\
0 & \kappa_1 \hat{c}_0 & -\mu_0^* \\
0 & 0 & \hat{c}_0^{-1} (\varepsilon \hat{c}_0^2 + \eta \hat{c}_0 + (\mu_0 \kappa_1^{-1} \mu_0 + \mu_1) - \hat{c} \kappa_0^{-1} \hat{c})
\end{pmatrix}.
\]

Clearly, applying the Fourier-Laplace transform to $(\varepsilon \hat{c}_0^2 + \eta \hat{c}_0 + (\mu_0 \kappa_1^{-1} \mu_0 + \mu_1) - \hat{c} \kappa_0^{-1} \hat{c})$ we obtain point-wise, writing $\sqrt{\lambda}$ instead of $(im + \nu)$,
\[
(\varepsilon \lambda + \eta \sqrt{\lambda} + (\mu_0 \kappa_1^{-1} \mu_0 + \mu_1) - \hat{c} \kappa_0^{-1} \hat{c}),
\]
which for vanishing “damping” $\eta$ is indeed a Sturm-Liouville operator\(^5\):
\[
r \lambda + q - \hat{c} p \hat{c}
\]
\(^4\) A more common point of view for this operation would be to think of new unknowns being introduced.

Indeed, if the system unknowns are \(\begin{pmatrix} s \\ w \\ v \end{pmatrix}\) then letting
\[
y = \hat{c}_0^{-1} v
\]
we would get from a line by line inspection of the system
\[
\begin{pmatrix}
\kappa_0 \hat{c}_0 & 0 & \hat{c} \\
0 & \kappa_1 \hat{c}_0 & -\mu_0^* \\
\hat{c} & \mu_0 & \varepsilon \hat{c}_0 + \eta + \mu_1 \hat{c}_0^{-1}
\end{pmatrix}
\begin{pmatrix} s \\ w \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ f \end{pmatrix}
\]
that
\[
\begin{align*}
s &= -\kappa_0^{-1} \hat{c} y \\
w &= \kappa_1^{-1} \mu_0 y
\end{align*}
\]
and so that
\[
\varepsilon \hat{c}_0^2 y + \eta \hat{c}_0 y + \mu_1 y + (\mu_0 \kappa_1^{-1} \mu_0 + \mu_1) - \hat{c} \kappa_0^{-1} \hat{c} y = f.
\]

\(^5\) If $\varepsilon = 0$ and $\eta$ has a strictly positive definite symmetric part, i.e. the selfadjoint $\Re \eta$ is strictly positive, we arrive at the parabolic type operator
\[
\eta \hat{c}_0 + q - \hat{c} p \hat{c}
\]
and writing $\lambda$ instead of $(im + \nu)$ we get again a Sturm-Liouville type operator
\[
\eta \lambda + q - \hat{c} p \hat{c},
\]
where now $\eta$ plays the role of $r$. 

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with

\[ r := \varepsilon, \quad q := \mu_0 \kappa^{-1} \mu_0 + \mu_1, \quad p := \kappa^{-1}. \]

For our purposes we may allow for general material laws in the problem (3).

Denoting the inner product and norm of \( H_{\nu,0} (\mathbb{R}, L^2 (I) \oplus L^2 (I) \oplus L^2 (I)) \) by \( \langle \cdot | \cdot \rangle_{\nu,0,0} \) and \( | \cdot |_{\nu,0,0} \), respectively, we calculate

\[
\Re \left\langle \chi_{1-x,0} (m_0) \begin{pmatrix} s \\ w \\ v \end{pmatrix} \right| A \begin{pmatrix} s \\ w \\ v \end{pmatrix} \right\rangle_{\nu,0,0} = 
\]

\[
\Re \left\langle \chi_{1-x,0} (m_0) s \frac{d}{dt} v \right\rangle_{\nu,0,0} + \Re \left\langle \chi_{1-x,0} (m_0) v \right\rangle_{\nu,0,0}
\]

\[
= \Re \left\langle \chi_{1-x,0} (m_0) s \frac{d}{dt} \left( v - \tilde{c}_0 a \left( \tilde{c}_0^{-1} \right) s \right) \right\rangle_{\nu,0,0} + 
\]

\[
+ \Re \left\langle \chi_{1-x,0} (m_0) s \frac{d}{dt} \left( \tilde{c}_0 a \left( \tilde{c}_0^{-1} \right) s \right) \right\rangle_{\nu,0,0} + \Re \left\langle \chi_{1-x,0} (m_0) v \right\rangle_{\nu,0,0}
\]

\[
= - \Re \left\langle \frac{d}{dt} \left( v - \tilde{c}_0 a \left( \tilde{c}_0^{-1} \right) s \right) \right\rangle_{\nu,0,0} + \Re \left\langle \chi_{1-x,0} (m_0) s \frac{d}{dt} \left( \tilde{c}_0 a \left( \tilde{c}_0^{-1} \right) s \right) \right\rangle_{\nu,0,0} + 
\]

\[
+ \Re \left\langle \chi_{1-x,0} (m_0) s \frac{d}{dt} \left( \tilde{c}_0 a \left( \tilde{c}_0^{-1} \right) s \right) \right\rangle_{\nu,0,0}.
\]

(4)

For this to be non-negative we assume that \( a \) is such that

\[
\Re \left\langle \chi_{1-x,0} (m_0) \varphi | \pm \tilde{c}_0 a \left( \pm 1/2, \tilde{c}_0^{-1} \right) \varphi \right\rangle_{\nu,0} \geq 0
\]

(5)

for every \( \varphi \in H_{\nu,1} (\mathbb{R}) \). Due to its analyticity at 0 the operators \( a \left( x, \tilde{c}_0^{-1} \right) \) are of the form

\[
a \left( \tilde{c}_0^{-1} \right) = a_0 + a_1 \tilde{c}_0^{-1} + a_2 \tilde{c}_0^{-2} + a_3 \tilde{c}_0^{-3},
\]

where \( a^{(3)} \left( \tilde{c}_0^{-1} \right) \) is bounded. With this we can analyse (5) further. It is

\[
\Re \left\langle \chi_{1-x,0} (m_0) \varphi | \pm \tilde{c}_0 a_0 \left( \pm 1/2 \right) \varphi \right\rangle_{\nu,0} =
\]

\[
= \pm \int_{-\infty}^{0} \varphi(t)^* (\tilde{c}_0 a_0 (\pm 1/2) \varphi)(t) \exp(-2\nu t) \, dt
\]

\[
= \pm \nu \int_{-\infty}^{0} a_0 (\pm 1/2) |\varphi(t)|^2 \exp(-2\nu t) \, dt \pm \frac{1}{2} a_0 (\pm 1/2) |\varphi(0)|^2
\]

which is non-negative if we assume

\[
\pm a_0 (\pm 1/2) \geq 0.
\]

(6)
Similarly
\[
\Re \left< \chi_{1-x,0}(m_0) \varphi \right| \pm a_1 (\pm 1/2) \varphi \right>_{\nu,0} =
= \pm \int_{-\infty}^{0} \varphi(t)^* a_1 (\pm 1/2) \varphi(t) \exp(-2\nu t) \, dt.
\]

Assuming that for \(\nu\)

\[
\pm \nu a_0 (\pm 1/2) \pm a_1 (\pm 1/2) \geq c_0 > 0
\]

for \(\nu \in \mathbb{R}_{>0}\) sufficiently large we obtain
\[
\Re \left< \chi_{1-x,0}(m_0) \varphi \right| \pm \tilde{c}_0 a (\pm 1/2, \tilde{c}_0^{-1}) \varphi \right>_{\nu,0} \geq
\geq (\pm \nu a_0 (\pm 1/2) \pm a_1 (\pm 1/2)) \left| \chi_{1-x,0}(m_0) \varphi \right|_{\nu,0}^2 +
+ \Re \left< \chi_{1-x,0}(m_0) \varphi \right| \pm \tilde{c}_0^{-1} a^{(2)} (\pm 1/2, \tilde{c}_0^{-1}) \varphi \right>_{\nu,0}.
\]

Due to causality we have
\[
\left| \Re \left< \chi_{1-x,0}(m_0) \varphi \right| \pm \tilde{c}_0^{-1} a^{(2)} (\pm 1/2, \tilde{c}_0^{-1}) \varphi \right>_{\nu,0} \right|
= \left| \Re \left< \chi_{1-x,0}(m_0) \varphi \right| \pm a (\pm 1/2, \tilde{c}_0^{-1}) \chi_{1-x,0}(m_0) \tilde{c}_0^{-1} \varphi \right>_{\nu,0} \right|
\leq C_1 \left| \chi_{1-x,0}(m_0) \varphi \right|_{\nu,0} \left| \chi_{1-x,0}(m_0) \tilde{c}_0^{-1} \varphi \right|_{\nu,0}
\leq C_1 \left| \chi_{1-x,0}(m_0) \varphi \right|_{\nu,0} \left| \tilde{c}_0^{-1} \chi_{1-x,0}(m_0) \varphi \right|_{\nu,0}
\leq C_1 \nu^{-1} \left| \chi_{1-x,0}(m_0) \varphi \right|_{\nu,0}^2.
\]

Under this assumption we have
\[
\Re \left< \chi_{1-x,0}(m_0) U |AU\right>_{\nu,0,0} \geq 0
\]
for all \(U \in D(A)\) if \(\nu \in \mathbb{R}_{>0}\) is sufficiently large. Note that by the time-translation invariance this is the same as saying
\[
\Re \left< \chi_{1-x,0}(m_0) U |AU\right>_{\nu,0,0} \geq 0
\]
for all \(U \in D(A)\) and all \(a \in \mathbb{R}\). Letting \(a \to \infty\) we obtain from this
\[
\Re \left< U |AU\right>_{\nu,0,0} \geq 0
\]

If \(a (\tilde{c}_0^{-1}) = a_0 + a_1 \tilde{c}_0^{-1}\) it is sufficient to require
\[
\pm \nu a_0 (\pm 1/2) \pm a_1 (\pm 1/2) \geq 0
\]
for all sufficiently large \(\nu \in \mathbb{R}_{>0}\).
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for all \( U \in D(A) \).

We need to find the adjoint of \( A \). It must satisfy

\[
- \begin{pmatrix} 0 & 0 & \hat{\vartheta} \\ 0 & 0 & 0 \\ \hat{\vartheta} & 0 & 0 \end{pmatrix} \subseteq A^* \subseteq - \begin{pmatrix} 0 & 0 & \hat{\vartheta} \\ 0 & 0 & 0 \\ \hat{\vartheta} & 0 & 0 \end{pmatrix}
\]

in the sense of extensions. We now show that \( D(A^*) \) is given by

\[
\left\{ \begin{pmatrix} s \\ w \\ v \end{pmatrix} \in H_{\nu,0}(\mathbb{R}, H(\hat{\vartheta}, I) \oplus L^2(\mathcal{I}) \oplus H(\hat{\vartheta}, I)) \mid a\left((\hat{\vartheta}_0^{-1})^*\right)s + (\hat{\vartheta}_0^{-1})^*v \in H_{\nu,0}(\mathbb{R}, H(\hat{\vartheta}, I)) \right\}.
\]

Indeed, for

\[
\begin{pmatrix} s \\ w \\ v \end{pmatrix} \in D(A)
\]

we have

\[
\begin{pmatrix} 1 \\ -a(\hat{\vartheta}_0^{-1}) \\ \hat{\vartheta}_0^{-1} \end{pmatrix} \begin{pmatrix} s \\ w \\ v \end{pmatrix} \in H_{\nu,0}(\mathbb{R}, H(\hat{\vartheta}, I) \oplus H(\hat{\vartheta}, I)).
\]

Direct computation gives

\[
\begin{pmatrix} 0 & \hat{\vartheta} \\ \hat{\vartheta} & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -a(\hat{\vartheta}_0^{-1}) \end{pmatrix} = \begin{pmatrix} 0 & \hat{\vartheta} \\ \hat{\vartheta} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \hat{\vartheta}_0^{-1} \end{pmatrix} + \begin{pmatrix} 0 & \hat{\vartheta} \\ \hat{\vartheta} & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ -a(\hat{\vartheta}_0^{-1}) & 0 \end{pmatrix}
\]

\[
= \begin{pmatrix} \hat{\vartheta}_0^{-1} & 0 \\ 0 & \hat{\vartheta}_0^{-1} \end{pmatrix} + \begin{pmatrix} a(\hat{\vartheta}_0^{-1}) & 0 \\ 0 & 0 \end{pmatrix}
\]

\[
= \begin{pmatrix} \hat{\vartheta}_0^{-1} - a(\hat{\vartheta}_0^{-1}) & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & \hat{\vartheta} \\ \hat{\vartheta} & 0 \end{pmatrix} - \begin{pmatrix} a' & 0 \\ 0 & 0 \end{pmatrix}.
\]

Thus, we have

\[
\begin{pmatrix} 0 & \hat{\vartheta} \\ \hat{\vartheta} & 0 \end{pmatrix} \hat{\vartheta}_0^{-1} \begin{pmatrix} s \\ v \end{pmatrix} = \begin{pmatrix} 1 & a(\hat{\vartheta}_0^{-1}) \\ 0 & \hat{\vartheta}_0^{-1} \end{pmatrix} \begin{pmatrix} 0 & \hat{\vartheta} \\ \hat{\vartheta} & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -a(\hat{\vartheta}_0^{-1}) \end{pmatrix} \begin{pmatrix} s \\ v \end{pmatrix} + \begin{pmatrix} 1 & a(\hat{\vartheta}_0^{-1}) \\ 0 & \hat{\vartheta}_0^{-1} \end{pmatrix} \begin{pmatrix} a' & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} s \\ v \end{pmatrix}
\]

\[
= \begin{pmatrix} 1 & a(\hat{\vartheta}_0^{-1}) \\ 0 & \hat{\vartheta}_0^{-1} \end{pmatrix} \begin{pmatrix} 0 & \hat{\vartheta} \\ \hat{\vartheta} & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -a(\hat{\vartheta}_0^{-1}) \end{pmatrix} \begin{pmatrix} s \\ v \end{pmatrix} + \begin{pmatrix} a' & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} s \\ v \end{pmatrix}.
\]
We have not only (9) but also, indeed more straightforwardly, for all

\[ A \] is continuously invertible with causal inverse \( \tilde{\alpha}_0 M (\tilde{\alpha}_0^{-1} + A)^{-1} : H_{\nu,k} (\mathbb{R}, H) \to H_{\nu,k} (\mathbb{R}, H) \] for every \( k \in \mathbb{Z} \) and any sufficiently large \( \nu \in \mathbb{R}_{>0} \). We summarize our findings in the following theorem.
Theorem 2.1. Under assumptions (2) and (5) we have that for every $J \in H_{\nu,k}(\mathbb{R}, H)$ the problem (3) has a unique solution $U \in H_{\nu,k}(\mathbb{R}, H)$. The solution operator $(\hat{c}_0 M (\hat{c}_0^{-1}) + A)^{-1} : H_{\nu,k}(\mathbb{R}, H) \to H_{\nu,k}(\mathbb{R}, H)$ is continuous and causal for every $k \in \mathbb{Z}$ and any sufficiently large $\nu \in \mathbb{R}_{>0}$.

Proof. Under the stated constraints the assumptions of Theorem 1.3 are satisfied and the result follows. \qed

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