Dynamics of Characteristic and One-Point Correlation Functions of Multi-Mode Bosonic Systems: Exactly Solvable Model

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Abstract: In this communication we study dynamics of the open quantum bosonic system governed by the generalized Lindblad equation with both dynamical and environment induced intermode couplings taken into account. By using the method of characteristics we deduce the analytical expression for the normally ordered characteristic function. Analytical results for one-point correlation functions describing temporal evolution of the covariance matrix are obtained.

Keywords: open quantum system; Lindblad equation; characteristic function; one-point correlation function

1. Introduction

Temporal evolution of open quantum systems is generally governed by completely positive and trace-preserving dynamical linear maps known as the quantum channels parameterized by time [1]. Under certain conditions, these maps can be described by master equations for the reduced density matrix derived using different assumptions and approximations [2–6]. It is well known that, within the Markov approximation, the master equations can be written in the so-called Lindblad form [7–9]. Such equation is also sometimes referred to as the Gorini–Kossakowski–Sudarshan–Lindblad (GKSL) equation and its form preserves complete positivity of the dynamics.

Though the Lindblad-type master equations has been extensively discussed [10–13], the bulk of the mathematical methods developed for analysis of bosonic systems are applicable to the special case of single-mode Lindblad Equations [14–19]. These techniques cannot be employed to treat its multi-mode generalizations. In this paper, the most general form of the multi-mode Lindblad equation which is described, e.g., in [20] will be our primary concern.

So we deal with a generalization of the model of a linear chain of coupled harmonic oscillators experiencing Markovian losses recently studied using Lie algebras [27]. Description of quantum dynamics of such quantum systems is important for understanding the
quantum effects that occur in non-Hermitian systems which might pave the way to explore and apply quantum parity-time (PT) symmetry [28] (for recent reviews see, e.g., [29]) and quantum exceptional points [30,31].

The structure of this communication is as follows. In Section 2, after introducing generalized GKSL master equation for multi-mode bosonic systems interacting with the thermal bath we solve the dynamical equation for the normally ordered characteristic function $\chi_N$ using the method of characteristics. The formula for $\chi_N$ is then utilized to deduce the expressions for the one-point correlators describing dynamics of the second order moments that enter the covariance matrix. In Section 3 we discuss the results and make some concluding remarks.

2. Results

Our starting point is the Lindblad equation for the density matrix describing the Markovian quantum dynamics of the $N$-mode open bosonic system of the general form [20]:

$$\frac{\partial \hat{\rho}}{\partial t} = \mathcal{L}\hat{\rho} = \sum_{n,m=1}^{N} \left( -i\Omega_{nm} C_{\hat{a}^\dagger \hat{a}} \hat{\rho} + \Delta^*_nm C_{\hat{a} \hat{a}^\dagger} \hat{\rho} - \Delta_{nm} C_{\hat{a}^\dagger \hat{a}^\dagger} \hat{\rho} \right)$$

$$+ \sum_{n,m=1}^{N} \left( K_{nm} D_{\hat{a}^\dagger \hat{a}} \hat{\rho} + L_{nm} D_{\hat{a} \hat{a}^\dagger} \hat{\rho} + M^*_nm D_{\hat{a}^\dagger \hat{a}^\dagger} \hat{\rho} + M_{nm} D_{\hat{a}^\dagger \hat{a}^\dagger} \hat{\rho} \right)$$  \hspace{1cm} (1)

written in terms of two superoperators given by

$$C_{\hat{A}\hat{B}} : \hat{\rho} \mapsto C_{\hat{A}\hat{B}} \hat{\rho} = [\hat{A}\hat{B}, \hat{\rho}],$$

$$D_{\hat{A}\hat{B}} : \hat{\rho} \mapsto D_{\hat{A}\hat{B}} \hat{\rho} = 2\hat{A}\hat{B} \hat{\rho} - \hat{B}\hat{A} \hat{\rho} - \hat{\rho} \hat{B}\hat{A} = [\hat{A}, \hat{\rho} \hat{B}] + [\hat{A}\hat{\rho}, \hat{B}],$$

where the dagger denotes Hermitian conjugation, $\hat{\rho}$ is the density matrix representing the quantum state; $\hat{a}_n^\dagger$ ($\hat{a}_n$) is the creation (annihilation) operator of the $n$th mode; $[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$ stands for the commutator; $\Omega_{nm}$, $K_{nm}$ and $L_{nm}$ are the elements of the frequency and relaxation matrices, respectively. All these matrices are assumed to be Hermitian: $\Omega = \Omega^\dagger$, $K = K^\dagger$ and $L = L^\dagger$.

From Equation (1) combined with the algebraic identities

$$\text{Tr}(\hat{S}C_{\hat{A}\hat{B}} \hat{\rho}) = (\langle \hat{S}, \hat{A} \rangle \hat{B} + \langle \hat{A}, \hat{S} \rangle \hat{B}),$$

$$\text{Tr}(\hat{S}D_{\hat{A}\hat{B}} \hat{\rho}) = (\langle \hat{B}, \hat{A} \rangle \hat{S} - \langle \hat{S}, \hat{B} \rangle \hat{A})$$

we have

$$\frac{\partial \chi_N(\mathbf{\hat{a}}, t)}{\partial t} = \hat{L}_N(\mathbf{\hat{a}}, t),$$

where the operator $\hat{L}$ on the right hand side of Equation (6)

$$\hat{L} = \sum_{n,m=1}^{N} i\Omega_{nm} D_{nm}^{(-)} + \sum_{n,m=1}^{N} \left( \Delta_{nm} (\hat{c}^{(+)\dagger} - \hat{a}_n^\dagger \hat{a}_m^\dagger) + \Delta^*_{nm} (\hat{c}^{(+)\dagger} - \hat{a}_n^\dagger \hat{a}_m^\dagger) \right)$$

$$- \sum_{n,m=1}^{N} \left( (K_{nm} - L_{nm}) \hat{D}_{nm}^{(+)\dagger} + 2L_{nm} \hat{a}_m \hat{a}_n^\dagger \right)$$

$$- \sum_{n,m=1}^{N} \left( M_{nm} (\hat{a}_n^\dagger \hat{a}_m^\dagger + \hat{a}_n^\dagger \hat{a}_m^\dagger) + M^*_{nm} (\hat{c}^{(+)\dagger} + \hat{a}_n^\dagger \hat{a}_m^\dagger) \right)$$  \hspace{1cm} (7)
is expressed in terms of the differential operators given by

\[
\hat{D}_{mn}^{(\pm)} = \alpha_m \frac{\partial}{\partial \alpha_n} \pm \alpha_n^* \frac{\partial}{\partial \alpha_m}, \quad (8)
\]

\[
\hat{C}_{mn}^{(\pm)} = \alpha_m^* \frac{\partial}{\partial \alpha_n} \pm \alpha_n^* \frac{\partial}{\partial \alpha_m}, \quad (9)
\]

Temporal evolution of the characteristic function \( \chi_N \) is governed by the dynamical Equation (6) supplemented with the initial condition

\[
\chi_N(\tilde{\alpha}, 0) \equiv \chi_{\text{ini}}(\tilde{\alpha}) = \int d\mu(\beta) e^{i(\alpha^* - \alpha^*)} P(\hat{\beta}, 0), \quad (10)
\]

\[
d\mu(\beta) = \prod_{n=1}^N d\mu(\beta_i), \quad d\mu(\beta_i) = \frac{d^2 \beta_i}{\pi}, \quad (11)
\]

where

\[
(\alpha^*, \beta) \equiv \sum_{i=1}^N \alpha_i^* \beta_i \quad (12)
\]

and \( \chi_N \) is related to the Glauber–Sudarshan \( P \) function (quasidistribution), \( P(\hat{\beta}, t) \), that enters the \( P \)-representation of the density matrix:

\[
\hat{\rho}(t) = \int d\mu(\beta) P(\hat{\beta}, t) |\beta\rangle \langle \beta|. \quad (13)
\]

We can now employ the method of characteristics \([32]\) to solve the above initial value problem. According to this method, we begin with the system of characteristic equations

\[
\frac{\partial \alpha_n}{\partial t} = -\sum_{m=1}^N Q_{nm}^{(11)} \alpha_m - \sum_{m=1}^N Q_{nm}^{(12)} \alpha_m^*, \quad (14)
\]

that can be conveniently put into the matrix form

\[
\frac{\partial \tilde{\alpha}}{\partial t} = \frac{\partial}{\partial t} \left( \begin{array}{c} \alpha \\ \alpha^* \end{array} \right) = -Q \left( \begin{array}{c} \alpha \\ \alpha^* \end{array} \right), \quad Q = \left( \begin{array}{cc} Q_{11} & Q_{12} \\ Q_{12}^* & Q_{11}^* \end{array} \right), \quad (15)
\]

where the block structure of \( Q \) is defined by the submatrices given by

\[
Q_{11} = i\Omega - \Gamma, \quad Q_{12} = \Delta_s - M_s, \quad (16)
\]

\[
\Gamma = K - L, \quad \Delta_{s,a} = \Delta \pm \Delta^T, \quad M_{s,a} = M \pm M^T. \quad (17)
\]

Solution of the system (15) written in the matrix form as follows

\[
\tilde{\alpha} = \left( \begin{array}{c} \alpha \\ \alpha^* \end{array} \right) = U(-t) \left( \begin{array}{c} \alpha_0 \\ \alpha_0^* \end{array} \right), \quad (18)
\]

\[
U(t) = e^{Qt} = \left( \begin{array}{cc} U_{11}(t) & U_{12}(t) \\ U_{12}^*(t) & U_{11}^*(t) \end{array} \right), \quad (19)
\]

where \( \alpha_0 = \alpha(0) \). It is not difficult to obtain the solution along the characteristic curves (18) given by

\[
\chi_N(\tilde{\alpha}_0, t) = \exp \left[ -\int_0^t (\tilde{\alpha}_0^*, G(-\tau) \tilde{\alpha}_0) d\tau \right] \chi_{\text{ini}}(\tilde{\alpha}_0), \quad (20)
\]

\[
G(t) = U^t(t) G U(t), \quad G = \left( \begin{array}{cc} L & \frac{1}{2} (\Delta_s - M_s) \\ \frac{1}{2} (\Delta_s^* - M_s^*) & L^* \end{array} \right). \quad (21)
\]
We can now express $\hat{a}_0$ in terms of $\hat{a}$ with the help of Equation (18): $\hat{a}_0 = U(t)\hat{a}$ to transform Formula (20) into the final expression for the characteristic function:

$$\chi_N(\hat{a}, t) = e^{-(\hat{a}^\dagger B(t)\hat{a})}\chi_{\text{ini}}(U(t)\hat{a}),$$

(22) \hspace{1cm} \chi_{\text{ini}}(U(t)\hat{a}) = \int_0^t G(t - \tau)d\tau.

(23)

We can now use formula for the characteristic function (22) to compute time dependence of averages that can be regarded as one-point correlation functions. Derivatives of this function can be easily evaluated giving the expressions for mean values of normally ordered operators. In particular, analytical expressions for second order moments can be readily derived in the following general form:

$$\langle \hat{a}_i^\dagger \hat{a}_j \rangle(t) = -\left. \frac{\partial^2 \chi_N(\alpha, t)}{\partial \hat{a}_i^\dagger \partial \hat{a}_j^\dagger} \right|_{\alpha = 0}$$

$$= 2B_{j i}^{(11)}(t) - \sum_{n, m} U_{ni}^{(11)}(t) U_{mj}^{(12)}(t) \langle \hat{a}_i^\dagger \hat{a}_m \rangle(0) - \sum_{n, m} [U_{ni}^{(12)}(t) U_{mj}^{(11)}(t)]^* \langle \hat{a}_n \hat{a}_m \rangle(0)$$

$$+ \sum_{n, m} U_{ni}^{(11)}(t) [U_{mj}^{(12)}(t)]^* \langle \hat{a}_i^\dagger \hat{a}_m \rangle(0) + \sum_{n, m} [U_{ni}^{(12)}(t) U_{mj}^{(11)}(t)]^* \langle \hat{a}_n \hat{a}_m \rangle(0),$$

(24) \hspace{1cm} B_{j i}^{(11)}(t) = \sum_{n, m} U_{ni}^{(11)}(t) U_{mj}^{(12)}(t) \langle \hat{a}_i^\dagger \hat{a}_m \rangle(0) + \sum_{n, m} [U_{ni}^{(12)}(t) U_{mj}^{(11)}(t)]^* \langle \hat{a}_n \hat{a}_m \rangle(0)

$$- \sum_{n, m} U_{ni}^{(12)}(t) [U_{mj}^{(11)}(t)]^* \langle \hat{a}_i^\dagger \hat{a}_m \rangle(0) - \sum_{n, m} [U_{ni}^{(11)}(t) U_{mj}^{(12)}(t)]^* \langle \hat{a}_n \hat{a}_m \rangle(0).$$

(25)

Relations (24) and (25) govern time dependence of the covariance matrix for our bosonic system [22].

3. Conclusions

In this communication we have obtained the expression for the normally ordered characteristic function of the open multi-mode bosonic system governed by the Lindblad equation taken in the general form. This equation accounts for the effects of dynamical and environment induced intermode couplings and squeezing. The analytical results are applied to obtain formulas describing time dependence of the averages representing the one-point correlation functions that enter the covariance matrix.

Our technique being directly applicable to quantum dynamics of continuous variables [22] might also provide a useful method to investigate Lindblad dynamics of the systems of coupled oscillators with $\mathcal{PT}$ symmetry previously studied in both classical [33,34] and quantum [35,36] regimes. The analytical approach can be readily extended to study a number of problems such as quantum dynamics of mixed polarization states transmitted in lossy quantum communication channels [37–39] and the problems related to controlled quantum dynamics in a realistic setup involving open environments.

Our concluding remark concerns quantum navigation which is an important class of controlled quantum dynamics whereby the objective is to transport one quantum state into another, or to generate quantum gates, in the shortest possible time under the influence of an uncontrollable external field. Problems of this kind can be thought of as representing the quantum counterpart of the classical Zermelo navigation problem of finding the time-optimal control that takes a ship from one location to another, under the influence of external wind or currents [40,41]. In a forthcoming publication we will apply our results to the Zermelo navigation problem for open multimode bosonic systems.
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References

1. Caruso, F.; Giovannetti, V.; Lupo, C.; Mancini, S. Quantum channels and memory effects. Rev. Mod. Phys. 2014, 86, 1203. [CrossRef]

2. Carmichael, H. An Open Systems Approach to Quantum Optics; Springer: Berlin/Heidelberg, Germany, 1993; p. 179.

3. Breuer, H.-P.; Petruccione, F. The Theory of Open Quantum Systems; Oxford University Press: Oxford, UK, 2002; p. 625.

4. Rivas, A.; Huelga, S.F. Open Quantum Systems: An Introduction; SpringerBriefs in Physics; Springer: Berlin/Heidelberg, Germany, 2012; p. 97.

5. Rivas, A.; Huelga, S.F.; Plenio, M.B. Quantum non-Markovianity: Characterization, quantification and detection. Rep. Prog. Phys. 2014, 77, 094001. [CrossRef] [PubMed]

6. de Vega, I.; Alonso, D. Dynamics of non-Markovian open quantum systems. Rev. Mod. Phys. 2017, 89, 015001. [CrossRef]

7. Kossakowski, A. On quantum statistical mechanics of non-Hamiltonian systems. Rep. Math. Phys. 1972, 3, 247–274. [CrossRef]

8. Lindblad, G. On the generators of quantum dynamical semigroups. Commun. Math. Phys. 1976, 48, 119–130 [CrossRef]

9. Gorini, V.; Kossakowski, A.; Sudarshan, E.C.G. Completely positive dynamical semigroups of N-level systems. Commun. Math. Phys. 1976, 17, 821. [CrossRef]

10. Pearle, P. Simple derivation of the Lindblad equation. Eur. J. Phys. 2012, 33, 805–822. [CrossRef]

11. Albash, T.; Boixo, S.; Lidar, D.A.; Zanardi, P. Quantum adiabatic Markovian master equations. New J. Phys. 2012, 14, 123016. [CrossRef]

12. McCauley, G.; Cruikshank, B.; Bondar, D.I.; Jacobs, K. Accurate Lindblad-form master equation for weakly damped quantum systems across all regimes. npj Quantum Inf. 2020, 6, 74. [CrossRef]

13. Manzano, D. A short introduction to the Lindblad master equation. AIP Adv. 2020, 10, 025106. [CrossRef]

14. Arevalo-Aguilar, L.M.; Moya-Cessa, H. Solution to the master equation for a quantized cavity mode. Quantum Semiclassical Opt. J. Eur. Opt. Soc. Part B 1998, 10, 671–674. [CrossRef]

15. Klimov, A.B.; Romero, J.L. An algebraic solution of Lindblad-type master equations. J. Opt. B Quantum Semiclassical Opt. 2003, 5, S316–S321. [CrossRef]

16. Lu, H.-X.; Yang, J.; Zhang, Y.-D.; Chen, Z.-B. Algebraic approach to master equations with superoperator generators of su(1, 1) and su(2) Lie algebras. Phys. Rev. A 2003, 67, 024101. [CrossRef]

17. Tay, B.A.; Petrosky, T. Biorthonormal eigenbasis of a Markovian master equation for the quantum Brownian motion. J. Math. Phys. 2008, 49, 113301. [CrossRef]

18. Honda, D.; Nakazato, H.; Yoshida, M. Spectral resolution of the Liouvillian of the Lindblad master equation for a harmonic oscillator. J. Math. Phys. 2010, 51, 072107. [CrossRef]

19. Tay, B. Eigenvalues of the Liouvillians of quantum master equation for a harmonic oscillator. Phys. A Stat. Mech. Its Appl. 2020, 556, 124768. [CrossRef]

20. Benatti, F.; Floreanini, R. Entangling oscillators through environment noise. J. Phys. A Math. Gen. 2006, 39, 2689–2699. [CrossRef]

21. Braunstein, S.L.; van Loock, P. Quantum information with continuous variables. Rev. Mod. Phys. 2005, 77, 513. [CrossRef]

22. Serafini, A. Quantum Continuous Variables: A Primer of Theoretical Methods; CRC Press, Taylor & Francis Group: Boca Raton, FL, USA, 2017; p. 350.

23. Hiroshima, T. decoherence and entanglement in two-mode squeezed vacuum states. Phys. Rev. A 2001, 63, 022305. [CrossRef]

24. Paz, J.P.; Roncaglia, A.J. Dynamics of the Entanglement between Two Oscillators in the Same Environment. Phys. Rev. Lett. 2008, 100, 220401. [CrossRef] [PubMed]

25. Linowski, T.; Gneiting, C.; Rudnicki, L. Stabilizing entanglement in two-mode Gaussian states. Phys. Rev. A 2020, 102, 042405. [CrossRef]

26. Vendrromin, C.; Dignam, M.M. Continuous-variable entanglement in a two-mode lossy cavity: An analytic solution. Phys. Rev. A 2021, 103, 022418. [CrossRef]

27. Teuber, L.; Scheel, S. Solving the quantum master equation of coupled harmonic oscillators with Lie-algebra methods. Phys. Rev. A 2020, 101, 042124. [CrossRef]

28. Bender, C.M.; Boettcher, S. Real spectra in non-hermitian hamiltonians having symmetry. Phys. Rev. Lett. 1998, 80, 5243. [CrossRef]
29. Christodoulides, D.; Yang, J. (Eds.) *Parity-Time Symmetry and Its Applications, Springer Tracts in Modern Physics*; Springer: Singapore, 2018; Volume 280, p. 578.

30. Heiss, W.D. The physics of exceptional points. *J. Phys. A Math. Theor.* 2012, 45, 444016. [CrossRef]

31. Özdemir, Ş.K.; Rotter, S.; Nori, F.; Yang, L. Parity-time symmetry and exceptional points in photonics. *Nat. Mater.* 2019, 18, 783–798. [CrossRef]

32. Shearer, M.; Levy, R. *Partial Differential Equations: An Introduction to Theory and Applications*; Princeton University Press: New Jersey, NJ, USA, 2015; p. 300.

33. Bender, C.M.; Gianfreda, M.; Klevansky, S.P. Systems of coupled PT-symmetric oscillators. *Phys. Rev. A* 2014, 90, 022114. [CrossRef]

34. Tsoy, E.N. Coupled oscillators with parity-time symmetry. *Phys. Lett. A* 2017, 381, 462–466. [CrossRef]

35. Xu, X.-W.; Liu, Y.-X.; Sun, C.-P.; Li, Y. Mechanical symmetry in coupled optomechanical systems. *Phys. Rev. A* 2015, 92, 013852. [CrossRef]

36. Li, W.; Li, C.; Song, H. Theoretical realization and application of parity-time-symmetric oscillators in a quantum regime. *Phys. Rev. A* 2017, 95, 023827. [CrossRef]

37. Gaidash, A.; Kozubov, A.; Miroshnichenko, G. Dissipative dynamics of quantum states in the fiber channel. *Phys. Rev. A* 2020, 102, 023711. [CrossRef]

38. Kozubov, A.V.; Gaidash, A.A.; Miroshnichenko, G.P.; Kiselev, A.D. Quantum dynamics of mixed polarization states: Effects of environment-mediated intermode coupling. *J. Opt. Soc. Am. B* 2021, 38, 2603.

39. Kiselev, A.D.; Ali, R.; Rybin, A.V. Lindblad Dynamics and Disentanglement in Multi-Mode Bosonic Systems. *Entropy* 2021, 23, 1409. [CrossRef] [PubMed]

40. Brody, D.C.; Gibbons, G.W.; Meier, D.M. Time-optimal navigation through quantum wind. *New J. Phys.* 2015, 17, 033048. [CrossRef]

41. Brody, D.C.; Longstaff, B. Evolution speed of open quantum dynamics. *Phys. Rev. Res.* 2019, 1, 033127. [CrossRef]