2d quantum gravity coupled to renormalizable matter fields

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Abstract

We consider two-dimensional quantum gravity coupled to matter fields which are renormalizable, but not conformal invariant. Questions concerning the $\beta$ function and the effective action are addressed, and the effective action and the dressed renormalization group equations are determined for various matter potentials.

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1 Introduction

In dimensions higher than two the quantum theory of gravity poses a major problem. The problem of gravitational dressing of renormalizable field theories seems minor in comparison. In two dimensions we understand by now how to quantize gravity in the absence of matter, and we understand the gravitational dressing of conformal field theories (at least if \( c \leq 1 \)). However, the gravitational dressing of a general renormalizable two-dimensional field theory is still not well understood. This problem has become increasingly important in the study of two-dimensional black holes, which again serve as interesting toy models for higher dimensional theories.

In the next section we briefly summarize the present knowledge of how to perturb away from a conformal field theory coupled to matter. If we work in conformal gauge the quantized theory of gravity and matter must be conformal invariant with respect to the fiducial metric introduced in the decomposition \( g_{\alpha\beta} \rightarrow \hat{g}_{\alpha\beta} e^{2\rho} \). This is the statement that the \( \beta \)-function must be zero when the free energy is viewed as a function of the coupling constants and the cut-off associated with the fiducial metric. The requirement of a vanishing \( \beta \)-function allows us to determine the back-reaction of quantum gravity on the matter fields to second order in the coupling constants when we perturb the matter theory away from a conformal fixed point by marginal perturbations.

We can view the combined theory of Liouville and matter as a generalized non-linear sigma model with the constraint imposed that the \( \beta \)-function has to vanish. As is well known from the usual critical string theory the vanishing of the \( \beta \)-function of this generalized sigma model can be formulated as a set of equations in target space. These can be viewed as the classical equations of motion for a certain action in target space. Formulated this way our task is to solve the classical equations, and the solution will determine the interaction between Liouville and matter fields. This approach is in certain respects to be preferred to more direct approach outlined in sec. 2 since it is better suited for making a non-perturbative ansatz. It is however plagued by ambiguities due to the choice of boundary conditions for the partial differential equations. These ambiguities reflect that infinite many counter terms can be added to the non-linear sigma model. Several attempts have made to restrict the allowed boundary conditions of the classical equations [10]. In this article we propose to solve the equations perturbatively with respect to the coupling constants which takes the matter theory away from a conformal field theory. Since we know the solution at the conformal point we will have no ambiguity in the solutions of the classical equations if we impose that they should go smoothly over into the corresponding solutions at the conformal point. In this way we can calculate the
effective action in concrete models and verify the “gravitational dressing” of the 
$\beta$-function mentioned above.

The rest of the paper is organized as follows: In sect. 2 we discuss the gravita-
tional back-reaction on matter when we move away from a conformal matter theory. 
The universal change of the $\beta$-function is found. In sect. 3 we review how to re-
formulate the problem as classical equations of motion of a certain action in target 
space, and in sections 4 to 6 we solve perturbatively these equations in specific 
models. Finally section 7 contains a discussion of the results.

2 Matter coupled to gravity

2.1 First order corrections to DDK

We know at the formal level how to couple conformal invariant field theories to 
2d quantum gravity. The results of KPZ [1] and DDK [2] solves the problem for 
c $< 1$ and (at a formal level) for $c > 25$, where $c$ denotes the conformal charge 
of the matter theory before coupling to gravity. We even know the correct way to 
discretize the coupled model [3, 4, 5] and to look for critical points in the corre-
sponding statistical system, thereby reaching contact with the continuum results of 
KPZ and DDK. However, in a statistical mechanics context the critical point and 
the corresponding conformal field theory are singular, albeit interesting points. The 
interest is usually centered around the approach to the critical point, as governed by 
the renormalization group. This approach allows us to identify masses and coupling 
constants of the class of theories associated with the critical point. The concept 
of distance plays an important role when discussing the renormalization group and 
masses. In a theory where we integrate over all metrics the most naive definition 
of correlation functions involves the use of the geodesic distance $r$ for each metric 
g, the calculation of the correlation function $\Delta_g(r)$ (this involves both angular and 
translational average) and the integration over all metrics with the weight dictated 
by quantum gravity. Defined this way the exponential decay of $\langle \Delta_g \rangle$ should define 
the massive excitations and the short distance behaviour should define the $\beta$- and 
$\gamma$-functions, i.e. the renormalization group equations.

Unfortunately the above program is rather inconvenient to implement (see how-
ever [6] for a number of interesting observations). If we only want the $\beta$-function 
we can use the action itself. This has the advantage that we work with quantities 
integrated over space-time and we thereby avoid addressing explicitly the concept 
of geodesic distances. Let us here discuss how to obtain the $\beta$-function for matter 
coupled to gravity. The starting point is the simplest derivation of the $\beta$-function in
the context of conformal field theory \[7\]. Let \( S_0 \) denote the action of the conformal invariant theory, and let \( V \) denote a marginal operator. The total theory will be given by:

\[
S = S_0 + \lambda \int d^2 z V \tag{1}
\]

and will in general describe a perturbation away from the conformal fixed point defined by \( S_0 \). Denote the short distance cut-off \( \hat{a} \). The \( \beta \)-function describes the change in coupling constants needed to compensate a change in cut-off for the partition function \( Z(\hat{a}, \lambda) \) or for the free energy \( F(\hat{a}, \lambda) = -\log Z(\hat{a}, \lambda) \):

\[
\left(-\hat{a} \frac{\partial}{\partial \hat{a}} + \beta(\lambda) \frac{\partial}{\partial \lambda}\right) F(\hat{a}, \lambda) = 0. \tag{2}
\]

The free energy associated with \( S \) can be written as a power series in the coupling constant \( \lambda \) and in this perturbation theory the dependence on the cut-off \( \hat{a} \) due to ultra-violet divergences can be found by assuming an operator product expansion. Let us for simplicity of the argument assume that \( V \) has the operator product expansion

\[ V \cdot V = [V], \]

i.e.

\[
V(r)V(0) \sim \frac{c}{r^2} V(0), \quad r \to 0. \tag{3}
\]

A term like

\[
\lambda^n \int d^2 z_1 \cdots d^2 z_n \langle V(z_1) \cdots V(z_n) \rangle_{S_0} \tag{4}
\]

will have logarithmic singularities at coinciding arguments \( z_i \to z_j \). Using (3) we get

\[
\int d^2 z_1 d^2 z_2 \langle \cdots V(z_1) V(z_2) \cdots \rangle_{S_0} \sim -2\pi c \cdot \log \hat{a} \int d^2 z \langle \cdots V(z) \cdots \rangle_{S_0}. \tag{5}
\]

A change in cut-off \( \hat{a} \to \hat{a}(1 + dl) \) can be compensated by a change \( \lambda \to \lambda - dl \pi c \lambda^2 \) in the lower order term in the perturbation expansion and this leads to the \( \beta \)-function

\[
\beta(\lambda) = -\frac{d\lambda}{dl} = \pi c \lambda^2 + O(\lambda^3). \tag{6}
\]

The argument is easily generalized to the situation where the operators \( V \) are almost marginal. If the scaling dimension of \( V \) is assumed to be \( 2 - y \), and we still insist that the coupling constant \( \lambda \) is dimensionless, there will be an explicit cut-off dependence in the interaction term in (1):

\[
\lambda \hat{a}^y \int d^2 z V, \tag{7}
\]

and (3) is generalized to

\[
\beta(\lambda) = -\frac{d\lambda}{dl} = -y \lambda + \pi c \lambda^2 + O(\lambda^3). \tag{8}
\]


Let us consider the situation where the theory is coupled to quantum gravity. We work in conformal gauge. At the conformal point the coupled theory is described by DDK. The metric is decomposed in a fiducial background metric and the Liouville field by

$$g_{\alpha\beta} = \hat{g}_{\alpha\beta} e^{2\rho}.$$  \hfill (9)

The Liouville part of the theory can be written as

$$S_L = \frac{1}{8\pi} \int d^2z \sqrt{\hat{g}} \left( \hat{g}^{\alpha\beta} \partial_{\alpha} \rho \partial_{\beta} \rho + Q \hat{R}_\rho + \mu \epsilon^{\gamma\rho} \right),$$ \hfill (10)

where

$$Q = \sqrt{\frac{25 - c}{3}}, \quad \gamma = \frac{1}{2} (Q - \sqrt{Q^2 - 8}) = \sqrt{\frac{25 - c}{12}} - \sqrt{\frac{1 - c}{12}}.$$ \hfill (11)

In (10) the ghost part is left out as it will play no role in the discussion to follow, and in the matter part of $S_0$ $g_{\alpha\beta}$ should be replaced by $\hat{g}_{\alpha\beta}$. Since we will be interested in the ultra-violet properties of the theory we will presently work with the cosmological term equal zero in accordance with the general folklore that this term should play no role in the ultra-violet regime \cite{8}. We have two ultra-violet cut-offs: $\hat{a}$ defined in terms of the fiducial metric $\hat{g}_{\alpha\beta}$ and the physical cut-off $a$ defined by

$$ds^2 = e^{\gamma\rho} \hat{g}_{\alpha\beta} dz^\alpha dz^\beta > a^2.$$ \hfill (12)

The theory must be independent of the cut-off $\hat{a}$ since the fiducial metric is arbitrary: the $\beta$-function must vanish. However, if the matter theory defined by (1) before coupling to gravity had a non-vanishing $\beta$-function the arguments given above can be repeated since the matter fields only couple to the fiducial metric to lowest order and we have a dependence of $\hat{a}$. This dependence can be eliminated by including new couplings between the Liouville field and the matter fields and it is a natural conjecture that the interaction between matter and gravity is uniquely fixed as a power series in the coupling constant $\lambda$ by the requirement that all cut-off dependence of $\hat{a}$ cancels. To second order in $\lambda$ it is easy to determine the coupling which leads to a vanishing $\beta$-function. A glance on (3) shows that the $\beta$-function will vanish if the operator $\int d^2z \sqrt{\hat{g}} V$ acquired a scaling dimension $y$ such that

$$y(\lambda) = \pi c \lambda.$$ \hfill (13)

While this makes no sense before coupling to gravity it is trivial to implement after coupling to gravity since the scaling dimension

$$(\Delta + \Delta) \int e^{y\rho} Q V = -\frac{y}{Q} \left( \frac{y}{Q} - Q \right) \approx y + O(y^2).$$ \hfill (14)
If we expand in powers of $\lambda$ we get:

$$\delta S = \frac{\pi}{Q} c \lambda^2 \int \rho V. \quad (15)$$

The generalization to the more realistic situation where the operator algebra is not as simple as (3) is straightforward. Let the perturbation away from the conformal point be given by:

$$S = S_0 + \lambda_i \int d^2 z V_i, \quad (16)$$

where summation over repeated indices is understood. The operator product expansion is assumed to be

$$V_i(r)V_j(0) = \frac{c_{ijk}}{r^2} V_k(0) + \cdots, \quad (17)$$

and the corresponding $\beta$-functions will be given by:

$$\beta_k(\lambda) = -\frac{d \lambda_k}{d \lambda} = \pi c_{ijk} \lambda_i \lambda_j + O(\lambda^3). \quad (18)$$

After coupling to gravity the following change in the action will ensure the vanishing of the $\beta$-function to order $O(\lambda^3)$:

$$\delta S = \frac{\pi}{Q} c_{ijk} \lambda_i \lambda_j \int \rho V_k, \quad (19)$$

a result first derived in [3]. For an earlier approach to the coupling of non-conformal matter to 2d gravity see [11].

The results can be generalized to non-linear sigma models. Let us again restrict ourselves to the simplest case where we have a non-linear sigma model on a symmetric space (this includes the $O(N)$ non-linear sigma model and the principal chiral model). Let the model before coupling to gravity be given by

$$S = \frac{1}{8\pi\lambda} \int d^2 z G_{ij}(X) \partial_\alpha X^i \partial_\alpha X^j. \quad (20)$$

The coupling to gravity just consist in replacing $d^2 z$ by $d^2 z \sqrt{g}$ and one $\partial_\alpha$ by $g^{\alpha\beta} \partial_\beta$ and in the conformal gauge any explicit dependence on the Liouville field drops out. If we expand around the Gaussian fixed point of the non-linear sigma model we will get an infinite power series of marginal perturbations, but thanks to the symmetry we know that the divergences can all be absorbed in a redefinition of the coupling constant

$$\lambda \to \lambda Z(\lambda), \quad Z(\lambda) = 1 + C \lambda \log \hat{a} + O(\lambda^2), \quad (21)$$

corresponding to a $\beta$-function

$$\beta(\lambda) = -C \lambda^2. \quad (22)$$
In (21) and (22) $C$ depends on the symmetric space. Had we been in $2 + \epsilon$ dimensions we would have picked up a linear term in (22) due to the explicit dependence on the cut-off $\hat{a}$ in the action, as follows from dimensional considerations:

$$\beta(\lambda) = \epsilon \lambda - C \lambda^2 + O(\lambda^3).$$

After coupling to gravity we can have no dependence on the cut-off $\hat{a}$ of the fiducial metric. Again, as above, we can obtain a $\beta$-function which is zero to $O(\lambda^3)$ by changing the dimension of the action via coupling to the Liouville field:

$$S = \frac{1}{8\pi} \int d^2 z \sqrt{\hat{g}} \, e^{\gamma(\lambda)\rho/Q} G_{ij}(X) \hat{g}^{\alpha\beta} \partial_\alpha X^i \partial_\beta X^j.$$ (24)

To lowest order in $\gamma$ we have

$$\beta(\lambda) = -y(\lambda) \lambda - C \lambda^2 + O(\lambda^3) = O(\lambda^3) \quad \text{for} \quad y = -C \lambda$$

and the back-reaction of gravity on the non-linear sigma model will be (to lowest order):

$$\delta S = -\frac{C}{Q8\pi} \int d^2 z \sqrt{\hat{g}} \, \rho \, G_{ij}(X) \hat{g}^{\alpha\beta} \partial_\alpha X^i \partial_\beta X^j.$$ (26)

The considerations for the non-linear sigma models are of course of a rather formal nature since the central charge $c > 1$.

### 2.2 The modified $\beta$-function

While the $\beta$-function in terms of the unphysical cut-off $\hat{a}$ by construction is zero, the change in action as a result of a change in the physical cut-off $a$ defined by (12) leads to a universal modification of the original $\beta$-function (18). The modification can be viewed as the “gravitational dressing” of the $\beta$ function. Especially when formulated in the dynamical triangulated approach [3, 4, 5] it is clear that it is the change in $a$ which governs the approach to the critical point. The renormalized theory of Liouville and matter fields was defined with respect to the fiducial cut-off $\hat{a}$. The physical cut-off $a$ appears in this formulation rather indirectly as a lower bound on the Liouville field $\rho$ through (12):

$$e^{\gamma \rho} \hat{g}_{\alpha\beta} dz^\alpha d\zeta^\beta \geq a^2 \Rightarrow \rho \geq \rho_{\min} = \frac{2}{\gamma} \log \frac{a}{\hat{r}}$$

where $\hat{r}$ is some infrared cut-off defined in the fiducial metric $\hat{g}$. Consider a change $a \to a(1 + dl)$ of the physical cut-off. According to (12) this leads to a change in the minimal allowed value of the Liouville field:

$$\rho_{\min}(a(1 + dl)) \approx \rho_{\min}(a) + \frac{2}{\gamma} dl.$$ (28)
This shift can be be viewed as a shift $\rho \to \rho + 2dl/\gamma$ while keeping $\rho_{\text{min}}$ fixed. The “running” of the coupling constants now appears by absorbing the shift $\rho \to \rho + 2dl/\gamma$ in a redefinition of the coupling constants $\lambda$ \[11, 9\]. To $O(\lambda^3)$ the only dependence on $\rho$ in the matter part of the renormalized action is found in $\delta S$ in \[19\] (or \[26\]). A shift $\rho \to \rho + 2dl/\gamma$ leads to a change of $\delta S$ by

$$dl \frac{2}{\gamma Q} c_{ijk} \lambda_i \lambda_j \int V_k,$$

and it can be absorbed to order $O(\lambda^3)$ by a change in coupling constants:

$$\lambda_k \to \lambda_k - dl \frac{2}{\gamma Q} c_{ijk} \lambda_i \lambda_j$$

The change in coupling constants needed to absorb the change in $a$ is thus the same as before coupling to gravity, except for a factor $2/\gamma Q$ and the modified $\beta$-function, $\beta_G$, defined as $-d\lambda/dl$, is related to $\beta$-function before coupling to gravity as

$$\beta_G(\lambda) = \frac{2}{\gamma Q} \beta(\lambda).$$

This result is (implicitly) in the more general discussion in \[9\] and was rediscovered in \[12\] and we see that it is also (at a formal level) valid for non-linear sigma models. We note that the sign of the $\beta$ function will not be changed by the coupling to gravity as long as $\gamma$ and $Q$ are positive, i.e as long as $c \leq 1$. In case $c > 1$ $\gamma$ becomes complex and we encounter of course in this region the well known problems of coupling of matter to 2d gravity. Note also that the correction is 1/2 for $c = 1$ and decreases as $c \to -\infty$ where $2/\gamma Q \to 1$. This is in accordance with the idea that the fluctuating geometries are suppressed for $c \to -\infty$. Indeed, the fractal dimension of space-time as defined in \[13\] is given by

$$D_F = \left(1 + \sqrt{\frac{13 - c}{25 - c}}\right) \frac{\gamma Q}{2},$$

which goes to 2 for $c \to -\infty$. Since the $\beta$-function in a fixed geometry determines the change of coupling constants with distance it is interesting to try to understand whether such an interpretation can be given after coupling to gravity. It should involve a proper treatment of the quantum average of geodesic distances as discussed above.

### 3 The effective action in target space

Consider the following action,

$$S_{n,\eta} = \frac{1}{4\pi} \int d^2 z \sqrt{g} \left\{-\eta \phi^n R + \frac{1}{2} \partial_\alpha \phi \partial^\alpha \phi + V(\phi) + \frac{1}{2} \sum_{i=1}^N \partial_\alpha f^i \partial^\alpha f^i\right\}.$$  \[33\]
The matter content of the theory is $N$ scalar fields with a Gaussian interaction and one scalar field $\phi$ which has non-trivial self-interaction via the potential $V(\phi)$. The coupling to gravity is inherent in the area element $d^2 z \sqrt{g}$ and the derivatives $\partial^a = g^{\alpha \beta} \partial_{\beta}$. The cosmological term is contained in the constant term of the potential $V(\phi)$. In addition we have introduced an explicit coupling between $\phi$ and the curvature. We will consider here only the cases $n = 0$ and $n = 1$. It would be interesting to be able to consider higher $n$’s in analogy with the non-minimal coupling of a scalar field to gravity in higher dimensions. However, in these cases we have a non-trivial interaction and we only know how to include the term as a perturbation in $\eta$, which is not what we want. The Gaussian matter fields $f^i$ will couple to gravity only by the conformal anomaly, i.e. only through their central charge $c = N$ and we have only included them in order to be able to vary the total central charge of the matter sector. The model (33) has been considered in various contexts \[14, 15\]. Let us here review the part we need for the target space formulation.

Let us work in conformal gauge by choosing a fiducial metric $\hat{g}_{\alpha \beta}$ as

$$g_{\alpha \beta} = e^{2\rho} \hat{g}_{\alpha \beta},$$

(34)

where $\rho$ represents the conformal mode. The quantum measures of $\rho, \phi$ and $f^i$ are introduced according to DDK, and the effective action can be written as

$$S_{n,\eta} = \frac{1}{4\pi} \int d^2 z \sqrt{\hat{g}} \left\{ -2n \eta \phi^{n-1} \partial_{\alpha} \phi \partial^{\alpha} \rho + \frac{1}{2} \partial_{\alpha} \phi \partial^{\alpha} \phi + \frac{\kappa}{2} \partial_{\alpha} \rho \partial^{\alpha} \rho - \hat{R}(\eta \phi^n - \frac{1}{2} \kappa \rho) + \hat{V}(\phi, \rho) \right\} + S_f + S_{\text{ghost}},$$

(35)

where

$$\kappa = \frac{24 - N}{3},$$

and $\partial^a = \hat{g}^{\alpha \beta} \partial_{\beta}$ and $\hat{R}$ is the scalar curvature for the fiducial metric $\hat{g}_{\alpha \beta}$. $S_{\text{ghost}}$ and $S_f$ play no role in the arguments to follow and we will ignore them. It is convenient to rescale $\rho \rightarrow \rho \sqrt{\kappa}$ to normalize the kinetic term. After this modification the effective action is:

$$S_{n,\eta} = \frac{1}{4\pi} \int d^2 z \sqrt{\hat{g}} \left\{ -2n \eta \phi^{n-1} \partial_{\alpha} \phi \partial^{\alpha} \rho + \frac{1}{2} \partial^2 \phi + \frac{1}{2} (\partial \rho)^2 - \hat{R}(\eta \phi^n - \frac{1}{2} \kappa \rho) + \hat{V}(\phi, \rho / \sqrt{\kappa}) \right\},$$

(36)

Since (33) is invariant under the transformation,

$$\hat{g}_{\alpha \beta} \rightarrow \hat{g}_{\alpha \beta} e^{\sigma(z)}, \quad \rho \rightarrow \rho - \frac{\sigma}{2},$$

and since the DDK measure $D\rho$ is invariant under translations, the partition function $Z(\hat{g})$, which is obtained by integrating over $\rho, \phi, f^i$ and ghosts, must be invariant
under the conformal transformation
\[ \hat{g}_{\alpha\beta} \rightarrow \hat{g}_{\alpha\beta} e^{\sigma(z)}. \] (37)

Eq. (35) (or (36)) is invariant under the above conformal transformation if \( \hat{V} \) is zero and \( n \leq 1 \) (we consider only these choices of \( n \)). For a general potential the task is to find the modification \( V(\phi) \rightarrow \hat{V}(\phi, \rho) \) consistent with the conformal invariance (37).

In the approach of DDK, \( \hat{V} \) has been determined by the requirement that the term \( \int \hat{V} \) by itself should be invariant under the conformal transformations (37) if calculated around the conformal point where \( V = 0 \), i.e. by imposing that \( \hat{V}(\phi, \rho) \) is a \((1,1)\) operator:
\[ (L_0 + \bar{L}_0)\hat{V} = 2\hat{V}. \]

If \( V \) itself has definite scaling properties a solution to this equation can be found in the form \( \hat{V}(\phi, \rho) = V(\phi)e^{\alpha \rho} \). However, writing \( V = \lambda P \), we saw in the last section that even if \( V \) was a marginal perturbation this result will only be the lowest order solution in \( \lambda \) if the \( \beta \)-function was different from zero. The same situation appears in dilaton gravity and it is an unsolved problem how to obtain the effective action which includes the gravitational effects to all orders. Here we will confine ourselves to study the problem perturbatively in the coupling constant \( \lambda \) of the potential \( V \).

Eq. (36) can be rewritten as a general non-linear sigma model:
\[ S_{\text{eff}} = \frac{1}{4\pi} \int d^2 z \sqrt{\hat{g}} \left[ \frac{1}{2} G_{\mu\nu}(X) \hat{g}^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu + \hat{R}(X) + T(X) \right], \] (38)
where \( X^\mu = (X^0, X^1) = (\rho, \phi) \). In the terminology of string theory, \( G_{\mu\nu}(X) \), \( \Phi(X) \) and \( T(X) \) represent the metric, dilaton and tachyon in the target space, respectively. They can be determined by solving the equations of zero \( \beta \) functions of the non-linear \( \sigma \) model eq. (38). As explained above the \( \beta \)-functions should be zero since the partition function, as a function of \( \hat{g} \) is conformally invariant. The solutions \( G_{\mu\nu}(X), \Phi(X) \) and \( T(X) \) obtained from the vanishing of the \( \beta \)-functions of (38) can be gotten as solutions to the classical equations of motion of the following effective action in target space [11, 17]:
\[ S_t = \frac{1}{4\pi} \int d^2 X \sqrt{G} e^{-2\Phi} \left[ R - 4(\nabla \Phi)^2 + \frac{1}{16}(\nabla T)^2 + \frac{1}{16}v(T) + \kappa \right], \] (39)
where
\[ v(T) = -2T^2 + \frac{1}{6}T^3 + \cdots \] (40)
and where \( \nabla_\mu \) denotes the covariant differentiation with respect to the metric \( G_{\mu\nu} \). The higher derivative terms are suppressed, and they will be necessary in a systematic higher order expansion. As for the powers of \( T(X) \), it may be possible to
remove the higher order terms in $v(T)$ (from $O(T^3)$) by the field redefinition in the
target space [18]. However we want to keep the relation between (35) and (39), so
we adopt $v(T)$ of eq. (40).

The equations of motion derived from the above action are the coupled equations
of $T$ and other fields, and in this way we see in a very direct way the corrections
needed for the potential $V(\phi)$. This is our main motivation for using the reformu-
lization of the problem of gravitational dressing as a set of classical equations of
motion. In addition it allows us to address the question of gravitational dressing for
perturbations which are not necessary marginal, thereby generalizing the treatment
in sect. 2.

From (39) we obtain the classical equations of motions for $T(X)$, $\Phi(X)$ and
$G_{\mu\nu}(X)$:

$$\nabla^2 T - 2\nabla \Phi \nabla T = \frac{1}{2} v'(T),$$  \hspace{1cm} (41)
$$\nabla^2 \Phi - 2(\nabla \Phi)^2 = \frac{\kappa}{2} + \frac{1}{32} v(T),$$ \hspace{1cm} (42)
$$R_{\mu\nu} - \frac{1}{2} G_{\mu\nu} R = -2\nabla_\mu \nabla_\nu \Phi + G_{\mu\nu} \nabla^2 \Phi + \frac{1}{16} \nabla_\mu T \nabla_\nu T - \frac{1}{32} G_{\mu\nu}(\nabla T)^2.$$  \hspace{1cm} (43)

The procedure for solving (41)-(43) is as follows. Consider the potential of the form

$$V(\phi) = \lambda P(\phi),$$

where $\lambda$ is a small parameter. We expand $G_{\mu\nu}$, $\Phi$ and $T$ as

$$G_{\mu\nu} = \left( \frac{1}{2} - \frac{2m}{\sqrt{\kappa}} \phi^{n-1} \right) + \lambda^2 h_{\mu\nu} + \cdots,$$ \hspace{1cm} (44)

$$\Phi = \Phi^{(0)} + \lambda^2 \Phi^{(2)} + \cdots,$$ \hspace{1cm} (45)

$$T = \lambda(T^{(0)} + \lambda T^{(1)} + \cdots),$$ \hspace{1cm} (46)

where $Q = \sqrt{\kappa}$ and $\Phi^{(0)} = \frac{1}{2} Q \rho - \eta \phi^n$. From eqs.(41)-(43) it follows that the lowest
order corrections to $G_{\mu\nu}$ and $\Phi$ are of order $O(\lambda^2)$. They are chosen as to reproduce
the original action eq.(35) with $\hat{V}(\rho, \phi) = 0$ in the limit $\lambda \to 0$.

We can now start solving eqs.(41)-(43) perturbatively with respect to $\lambda$ order by
order. The first step is to solve the lowest order equation which comes from (41).
This determines $T^{(0)}$ and is equivalent the method of DDK.

Explicitly the lowest order equation is (for $\eta = 0$):

$$\partial^2 T^{(0)} - 2 \partial \Phi^{(0)} \cdot \partial T^{(0)} = -2 T^{(0)}, \quad \Phi^{(0)} = \frac{1}{2} Q \rho.$$  \hspace{1cm} (47)
If the potential $V(\phi)$ has definite scaling properties the lowest order form of $T^{(0)}$ is the DDK ansatz $T^{(0)} = V(\phi)e^{\alpha \rho}$. This follows since the only functions of $\phi$ with definite scaling properties are exponentials, and if we use exponentials like $e^{ip\phi}$ (or better $\cos p\phi$ and $\sin p\phi$) in (47) we get the DDK result:

$$\alpha = \frac{1}{2} \left[ Q - \sqrt{Q^2 + 4p^2 - 8} \right].$$  \hfill (48)

However, since the lowest order equation (47) is linear the solution can be found for any potential by Fourier transformation. The result is

$$T^{(0)}(\phi, \rho) = e^{\frac{1}{2} Q \rho} \int d\xi P(\phi - \xi) \frac{4\pi q \rho}{\sqrt{\rho^2 + \xi^2}} K_1(q \sqrt{\rho^2 + \xi^2}), \quad q = \sqrt{1 - \frac{c}{12}}. \hfill (49)$$

For $N = 0$ we have $c = 1$, $\gamma = Q/2 = \sqrt{2}$ and the expression simplifies a little:

$$T^{(0)}(\phi, \rho) = e^{\gamma \rho} \int d\xi P(\phi - \xi) \frac{4\pi \rho}{\rho^2 + \xi^2}. \hfill (50)$$

We can then solve the next order equations by using (49) or (50) in eqs.(41)-(46). We illustrate the procedure in two simple models corresponding to $n = 0$ and $n = 1$.

### 4 n=0: The Sine-Gordon model

For $n = 0$, the first term in eq.(33) is a total divergence, so we neglect it here. As for the potential we consider the Sine-Gordon model,

$$V(\phi) = \lambda \cos(p\phi). \hfill (51)$$

The choice of this potential is dictated by our desire have a simple expression for $T^{(0)}$ as given by (49) or (50). If we take $N = 0$ we get\footnote{For a different treatment of $T^{(0)}$ see \cite{k2}.} from the above formulas:

$$T^{(0)} = \cos p\phi \, e^{\alpha \rho}, \quad \alpha = \sqrt{2} - p. \hfill (52)$$

Using this result, we obtain the next order of (41)-(43) ($O(\lambda^2)$) by expanding according to eqs.(44) and (46) (with $\eta = 0$)

$$\frac{4}{Q} (\partial_1^2 - \partial_0^2) \Phi^{(2)} + \partial_0 (h_{00} + h_{11}) - 2\partial_1 h_{10} = -\frac{1}{16Q} [(\alpha^2 + p^2) \cos(2p\phi) + \alpha^2 - p^2] e^{2\alpha \rho} \hfill (53)$$

\[\begin{align*}
- \frac{4}{Q} \partial_1 \partial_0 \Phi^{(2)} + \partial_1 h_{00} &= \frac{\alpha p}{16Q} \sin(2p\phi) e^{2\alpha \rho}, \hfill (54)
\end{align*}\]
\[
\frac{4}{Q}(\partial^2 - 2Q\partial_0)\Phi^{(2)} + \partial_0 h_{11} - 2\partial_1 h_{10} = -2Qh_{00} + \partial_0 h_{00} - \frac{2}{16Q} \cos(2p\phi) + 1) e^{2\alpha \rho} \tag{55}
\]
where \(\partial_{0,1}\) mean the partial derivative with respect to \(\rho\) and \(\phi\), respectively.

We note the following. Since we solve for the higher order terms \(h_{\mu\nu}\), \(\Phi^{(2)}\) and \(T^{(1)}\) as the response of the lowest order terms \(T^{(0)}\), \(\Phi^{(0)}\) we should only use the particular solutions which go to zero with \(T^{(0)}\). Next we will expand around the point \((p, \alpha) = (\sqrt{2}, 0)\). \(p = \sqrt{2}\) is the Kosterlitz-Thouless transition point for the Sine-Gordon equation before coupling to gravity, and exactly at this point \(V(\phi) = \lambda \cos p\phi\) is a marginal perturbation. From sect. 2 it follows that a sensible ansatz for \(h_{\mu\nu}\) is

\[
h_{\mu\nu} = \begin{pmatrix} 0 & 0 \\ 0 & h(\rho) \end{pmatrix} \tag{56}
\]
In this case, the eqs.\((53)-(55)\) can be written as,

\[
\frac{4}{Q}(\partial^2 - 2Q\partial_0)\Phi^{(2)} + \partial_0 h = -\frac{1}{16Q} [(\alpha^2 + p^2) \cos(2p\phi) + \alpha^2 - p^2] e^{2\alpha \rho}, \tag{57}
\]

\[
-\frac{4}{Q}\partial_1 \partial_0\Phi^{(2)} = \frac{\alpha p}{16Q} \sin(2p\phi) e^{2\alpha \rho}, \tag{58}
\]

\[
\frac{4}{Q}(\partial^2 - 2Q\partial_0)\Phi^{(2)} + \partial_0 h = -\frac{2}{16Q} \cos(2p\phi + 1) e^{2\alpha \rho}. \tag{59}
\]

From these equations, we get two equations for \(\Phi^{(2)}\). One is given by eq.\((58)\) and the other is obtained by subtracting \((59)\) from \((57)\). The latter is written as,

\[
(\partial_0^2 - Q\partial_0)\Phi^{(2)} = \frac{1}{128} [2\alpha^2 - Q\alpha) \cos(2p\phi) + Q\alpha - 4] e^{2\alpha \rho}. \tag{60}
\]

Both eqs.\((58)\) and \((60)\) can be solved for infinitesimal \(\alpha\) as follows,

\[
\Phi^{(2)} = \frac{1}{256} \cos(2p\phi) + \frac{1}{32Q} \rho + O(\alpha). \tag{61}
\]
By substituting this into \((57)\) (or \((59)\)), we obtain

\[
h = \frac{p^2}{16Q} \rho + O(\alpha). \tag{62}
\]

Finally, we solve for \(T^{(1)}(X)\) by taking the \(T^3\) term in \(v(T)\) into account. The equation reads:

\[
(\partial^2 - Q\partial_0 + 2)T_1 = \frac{1}{8} \cos(2p\phi + 1) e^{2\alpha \rho}, \tag{63}
\]
and its special solution near \(\alpha = 0\) is obtained as

\[
T_1 = \frac{1}{32} (1 - \cos 2p\phi) + O(\alpha), \tag{64}
\]

\[13\]
As a result, we obtain the effective action up to $O(\lambda^2)$ near $(p = \sqrt{2}, \alpha = 0)$ as follows,

$$S_{0,\text{eff}} = S_0 + \lambda^2 S_0^{(2)} + \cdots,$$

$$S_0 = \frac{1}{4\pi} \int d^2z \sqrt{g} \left\{ \frac{1}{2} (\partial \phi)^2 + \frac{1}{2} (\partial \rho)^2 + \hat{R} \Phi^{(0)} + \lambda \cos(p\phi) e^{\alpha \rho} \right\},$$

$$S_0^{(2)} = \frac{1}{4\pi} \int d^2z \sqrt{g} \left\{ \frac{1}{16Q} \rho (\partial \phi)^2 + \left[ \frac{1}{256} \cos(2p\phi) + \frac{1}{32Q} \rho \right] \hat{R} + \frac{1}{16}(1 - \cos(2p\phi)) \right\},$$

where we have neglected the terms of order $O(\lambda^2 \alpha)$ in $S_0^{(2)}$ and where $e^{\alpha \rho}$ should be understood as expanded in powers of $\alpha$, i.e. to lowest order in $\sqrt{2} - p$.

Let us now consider the renormalization group equations of the coupling constants $p$ and $\lambda$. As explained in sect. 3 the change in coupling constants should be obtained by absorbing a shift $\rho \to \rho + 2dl/\gamma = \rho + \sqrt{2}dl$. Let us write $p = \sqrt{2} + \epsilon$ and denote by $\lambda', \epsilon'$ and $\phi'$ the coupling constants and fields after the shift of $\rho$ by $dl/2$. We note that the first term in (66) is responsible for a wave function renormalization:

$$\phi^2 = (1 + \frac{\lambda^2}{8 \gamma Q} dl) \phi'^2$$

Since $p\phi = p'\phi'$ we get

$$\epsilon' = \epsilon + \frac{\sqrt{2} \lambda^2}{16} \frac{2}{\gamma Q} dl, \quad \lambda' = \lambda - \alpha \lambda \frac{2}{\gamma} dl.$$ \hspace{1cm} (68)

From (52) we have $\alpha = \sqrt{2} - p = -\epsilon$ to order $O(\lambda^2)$, and since (again to lowest order) $Q = 2\sqrt{2}$ and $\gamma = \sqrt{2}$, we can write the renormalization group equations as

$$\frac{d\lambda}{dl} = \sqrt{2} \epsilon \lambda, \quad \frac{d\epsilon}{dl} = \frac{1}{16\sqrt{2}} \lambda^2,$$ \hspace{1cm} (69)

If we compare these results with the renormalization group equations obtained without the coupling to gravity [20],

$$\frac{d\lambda}{dl} = 2\sqrt{2} \epsilon \lambda, \quad \frac{d\epsilon}{dl} = \frac{1}{8\sqrt{2}} \lambda^2,$$ \hspace{1cm} (70)

we see that they differ by the factor $1/2 = 2/\gamma Q$ in agreement with the general discussion in sect. 3. In addition we see that the term $\rho \hat{R}/32Q$ induces as change $Q \to Q + \lambda^2/32Q$. It is interesting to note that an increase in $Q$ formally is in accordance with a renormalization flow way from the ultra-violet fixed point towards a new infrared stable fixed point with a smaller $c$, as one would expect from the $c$-theorem.
5 n=1: The extended Sine-Gordon model

The model of \( n = 1 \) without the kinetic term of \( \phi \) was first proposed in [21] and examined by several people as a model of the non-critical string beyond \( c = 1 \) [22] due to the fact that the string susceptibility remained real for any value of \( c \). However, such a model constrains the value of the scalar curvature \( R \) to such an extent that it ceases to be a dynamical variable. Here we consider a model where a kinetic \( \phi \) term is added so \( \phi \) is no longer an auxiliary field and the scalar curvature a dynamical variable.

The analysis of cosmological term gives a constraint on \( N \) from the reality of the dressed factor \( e^{\gamma \rho} \). This is nothing but the reality of the string susceptibility. Consequently we first analyze the cosmological term.

5.1 The cosmological term and the string susceptibility.

We parametrize the cosmological term as,

\[
V_c = \bar{\lambda} e^{\gamma \rho}.
\]

By rescaling the variable and parameter in eq. (35) as, \( Q \rho \rightarrow \rho \) and \( \eta/Q \rightarrow \eta \), we obtain the following result from the lowest order equation for \( T \).

\[
\gamma_1 = \frac{1}{2} \left[ Q \bar{\eta} - \sqrt{Q^2 \bar{\eta}^2 - 8 \bar{\eta}} \right].
\] (71)

where \( \bar{\eta} = 1 - 4 \eta^2 \).

We demands that \( \gamma_1 \) is real and find

(i) if \( \eta^2 < \frac{1}{4} \) (\( \bar{\eta} > 0 \)) then \( N \leq 0 \), i.e. \( N = 0 \)
(ii) if \( \eta^2 > \frac{1}{4} \) (\( \bar{\eta} < 0 \)) then \( N \leq 24 \).

c could exceed one in case (ii), but the theory is not unitary since \( \det G_{\mu \nu} = 1 - 4 \eta^2 < 0 \). If we respect the unitarity of the two dimensional system, we must restrict ourself to the case (i). However, In order to see what happens in the somewhat ill-defined (non-unitary) theory, we also examine the solutions in case (ii).

The string susceptibility, which is denoted by \( \hat{\gamma} \) can be calculated by the method of DDK and we get

\[
\hat{\gamma} = 2 - \frac{\chi}{24} \left[ 24 - N + \sqrt{(24 - N)(24 - N - \frac{24}{\bar{\eta}})} \right],
\] (72)

where \( \chi = \frac{1}{4\pi} \int d^2 z \sqrt{\hat{g}} \hat{R} \). \( \hat{\gamma} \) is real if \( \gamma_1 \) is real. If we consider the case (ii), \( \hat{\gamma} \) is real even if \( c \) is larger than one, but as noted above unitarity is broken. In this sense we can not exceed the \( c = 1 \) barrier.
5.2 \( \eta^2 < \frac{1}{4} \)

Since the reflection symmetry, \( \phi \to -\phi \), is broken due to the term \( \phi R \) in this model, we extend \( V(\phi) \) as

\[
V(\phi) = \lambda [\cos(p\phi) + \delta \sin(p\phi)] \tag{73}
\]

by adding the odd term, \( \sin(p\phi) \), with a weight \( \delta \). Even if we start from the even term \( V(\phi) = \lambda \cos(p\phi) \), we will find that the odd term \( \sin(p\phi) \) is needed in order to solve the lowest order of equation (41). So it is possible to consider the odd term as a correction term due to quantum gravity. In any case, we can take the form eq. (73) with a dressed factor as the lowest order solution of eq. (41).

For the simplicity, we further change and rescale variables in \( I_{1,\eta} \) after the change, \( \sqrt{\kappa} \rho \to \rho \), as follows

\[
\rho \to \rho + 2\eta \phi, \quad \phi^2 \eta \to \phi^2.
\]

Since the Jacobian is a simple constant, there is no problem with this change of variables. The action is written as

\[
S_{1,\eta} = \frac{1}{4\pi} \int d^2z \sqrt{g} \{ \frac{1}{2} (\partial \phi)^2 + \frac{1}{2} (\partial \rho)^2 + \hat{R} \frac{Q}{2} \rho + \hat{V}(\phi, \rho + 2\eta \phi) \}, \tag{74}
\]

We have to work with the dressed operator \( e^{\alpha(\rho + \frac{2\eta}{\sqrt{1 - 4\eta^2}} \phi)} \) instead of \( e^{\alpha \rho} \). Except for this point, the analysis is parallel to the one in the previous section. In particular we note that since \( N = 0 \) we have \( c = 1 \) for \( \lambda = \delta = 0 \). This means that \( Q = 2\sqrt{2} \) and \( \gamma = \sqrt{2} \) and that we have to work close to the Kosterlitz-Thouless transition point \( p = \sqrt{2} \) in order that the perturbations are almost marginal and we can use the ansatz (56).

First we solve the lowest order equation of \( T^{(0)} \) in the form,

\[
T^{(0)} = [\cos(p\phi) + \delta \sin(p\phi)] e^{\alpha \rho + \beta \phi}, \tag{75}
\]

where

\[
\beta = \frac{2\eta}{\sqrt{1 - 4\eta^2}} \alpha.
\]

Since \( T^{(0)} \) contains two independent functions, \( \sin(p\phi) \) and \( \cos(p\phi) \) we obtain two equations of the parameters from the lowest order equation of \( T^{(0)} \),

\[
\alpha^2 + \beta^2 - p^2 + 2\beta p \delta - Q\alpha + 2 = 0, \tag{76}
\]

\[
\delta(\alpha^2 + \beta^2 - p^2) - 2\beta p - \delta(Q\alpha - 2) = 0. \tag{77}
\]

From these equations, we obtain \( \alpha = 0 = \beta \) for \( p \neq 0 \). Since we consider the case of \( p^2 = 2 \) we have that \( \alpha = 0 \). Near this point, the solution of the next order equations
with ansatz eq.(56) are as follows,
\[ h = \frac{p^2}{16Q}(1 + \delta^2)\rho + O(\alpha). \] (78)
\[ \Phi^{(2)} = \frac{1}{32Q}(1 + \delta^2)\rho + O(\alpha). \] (79)
\[ T^{(1)} = -\frac{1 - \delta^2}{48}\cos(2p\phi) - \frac{\delta}{24}\sin(2p\phi) + \frac{1 + \delta^2}{16} + O(\alpha). \] (80)

We can obtain the renormalization group equations for the coupling constants \( \lambda, \varepsilon \) (= \( p - \sqrt{2} \)) and \( \eta \) similarly to the case in the last section:
\[ \dot{\lambda} = \sqrt{2}\varepsilon\lambda, \quad \dot{\varepsilon} = \frac{1 + \delta^2}{16\sqrt{2}}\lambda^2, \quad \dot{\eta} = -\frac{1 + \delta^2}{16\sqrt{2}\varepsilon}\eta\lambda^2, \] (81)

It is possible to compare with the results obtained in [23]. Again we see the difference of a factor two of the coefficients, compared to model without coupling to gravity.

In addition eq.(79) gives a shift of the charge \( Q \) by \( \frac{\lambda^2}{32Q}(1 + \delta^2) \). This is similar to the case of \( n = 0 \), and the same remarks apply here.

### 5.3 \( \eta^2 > \frac{1}{4} \), Non-unitary Model

In this case we can repeat the analysis by making the replacement \( \phi^2 \hat{\eta} \rightarrow \phi^2 \), where \( \hat{\eta} = 4\eta^2 - 1(> 0) \). Then the action can be written as,
\[ S_{1,\eta} = \frac{1}{4\pi} \int d^2z\sqrt{g} \left\{ -\frac{1}{2}(\partial\phi)^2 + \frac{1}{2}(\partial\rho)^2 + \hat{R}\frac{Q}{2}\rho + \hat{V}(\phi, \rho + \frac{2\eta}{\sqrt{\eta}}\phi) \right\}, \] (82)
and we must use the following lowest order metric in the target space,
\[ G^{(0)}_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]

By using the same form of \( T^{(0)} \) as in (73) with \( \beta = \frac{2p}{\sqrt{4\eta^2 - 1}} \alpha \), we obtain the following results from the lowest order equation (41),
\[ \alpha^2 - \beta^2 + p^2 - 2\beta p\delta - Q\alpha + 2 = 0, \] (83)
\[ \delta(\alpha^2 - \beta^2 + p^2) + 2\beta p - \delta(Q\alpha - 2) = 0. \] (84)

These equations have no solution at \( p^2 = 2 \), and we obtain,
\[ p = 0, \quad \alpha^2 - \beta^2 - Q\alpha = -2. \] (85)

The consistent solution is then
\[ h_{\mu\nu} = 0, \]
and we can write three equations coming from eqs. (42) and (43) as

\[ (\partial_1^2 + \partial_0^2)\Phi(2) = \frac{1}{32}(\alpha^2 + \beta^2)e^{2(\alpha\rho + \beta\phi)}, \]

\[ \partial_1\partial_0\Phi(2) = \frac{\alpha\beta}{32}e^{2(\alpha\rho + \beta\phi)}, \]

\[ (\partial^2 - 2Q\partial_0)\Phi(2) = -\frac{1}{16}e^{2(\alpha\rho + \beta\phi)}, \]

They admit the solution

\[ \Phi(2) = \frac{1}{128}e^{2(\alpha\rho + \beta\phi)}. \]

Further, from the next order of eq. (41), we obtain

\[ T^{(1)} = -\frac{1}{16}e^{2(\alpha\rho + \beta\phi)}. \]

In this case the critical value of \( \alpha \) depends of \( N \) which could exceed zero but was bounded by

\[ N < 12\beta^2, \]

at the price of unitarity. Further \( \alpha \) to lowest order is given by

\[ \alpha = \frac{1}{2}\left[-Q\hat{\eta} + \sqrt{Q^2\hat{\eta}^2 + 8\hat{\eta}}\right] = \gamma_1. \]

where \( \gamma_1 \) denotes the exponential factor in the cosmological term. We can formally write down the renormalization group equation for \( \lambda \) near some value of \( \alpha \) as,

\[ \dot{\lambda} = -\alpha^2\lambda. \]

The important difference between the unitary and non-unitary models is that we can not choose \( p^2 = 2 \) in the non-unitary case and the equation for \( \alpha \) can not be obtained in this case.

6 The non-linear sigma models

Let us finally discuss the target space equations for the last class of models discussed in sec. 2, the non-linear sigma models. As already remarked in sec. 2 the discussion is somewhat formal in the sense that these models in general will have \( c > 1 \). To simplify the discussion we consider here the simplest model, the \( O(N) \) non-linear sigma model. When we couple the model to gravity by DDK quantization we get (ignoring as usual the ghost terms)

\[ S(\hat{g}, \lambda) = \frac{1}{8\pi} \int d^2z \sqrt{\hat{g}} \left\{ \partial_\beta \hat{\rho} \partial^\beta \hat{\rho} + Q\hat{R}\hat{\rho} \right\} + \frac{1}{8\pi\lambda} \int d^2z \sqrt{\hat{g}} e^{\alpha\rho} \sum_{i=1}^{N} \partial_\beta \phi^i \partial^\beta \phi^i, \]
where $\sum_{i=1}^{N} \phi^i \phi^i = 1$, and where $\rho$ as usual denotes the Liouville field. We have inserted the factor $e^{\alpha \rho}$ which we in sec. 2 argued should be present when we move away from the ultra-violet fixed point corresponding to $\lambda = 0$. We further expect $\alpha = O(\lambda)$ and can a priori only expect this interaction between the Liouville field and matter fields to be correct to this order. From this point of view it is a little misleading to write the interaction in the exponential form.

If we expand around $\phi^N = \sqrt{1 - \sum_{i=1}^{N-1} \phi^i \phi^i}$, and rescale $\phi \rightarrow \sqrt{\lambda} \phi \ (94)$ can be written as

$$S(\hat{g}, \lambda) = \frac{1}{8\pi} \int d^2 z \sqrt{\hat{g}} \left\{ \partial^2 \Phi \rho \partial^2 \rho + Q \hat{R} \rho + \sum_{i=1}^{N-1} (\partial_{\beta} \phi^i \partial^2 \phi^i) + \hat{V}(\rho, \phi) \right\}, \quad (95)$$

where the potential $\hat{V}$ is given by

$$\hat{V} = \sum_{i,j}^{N-1} \left\{ \frac{\lambda \phi^i \phi^j}{1 - \lambda \sum_{k=1}^{N-1} \phi^k} + (e^{\alpha \rho} - 1) \delta_{ij} \right\} \partial_{\beta} \phi^i \partial^2 \phi^j$$

$$= \sum_{i,j}^{N-1} \left( \lambda \phi^i \phi^j + \delta_{ij} \alpha \rho \right) \partial_{\beta} \phi^i \partial^2 \phi^j + O(\lambda^2, \alpha^2, \lambda \alpha). \quad (96)$$

With the chosen expansion the indices $\mu$ and $\nu$ in target space runs from 0 to $N-1$, where 0 corresponds to the Liouville field $\rho$ and the vector $X^\mu$ defined below (38) will in this case by written as $X^\mu = (X^0, X^i) = (\rho, \phi^i)$. The expansion of the target space metric $G_{\mu\nu}(X)$, dilaton $\Phi(X)$ and tachyon $T(X)$ will in analogy with (44)-(46) be given by

$$G_{\mu\nu} = G_{\mu\nu}^{(0)} + \lambda h_{\mu\nu} + \cdots, \quad (97)$$

$$\Phi = \Phi^{(0)} + \lambda \Phi^{(1)} + \cdots, \quad (98)$$

$$T = T^{(0)} + \lambda T^{(1)} + \cdots, \quad (99)$$

where $T^{(0)} = 0$, $\Phi^{(0)} = Q \rho / 2$, and

$$G_{\mu\nu}^{(0)} = \delta_{\mu\nu}, \quad h_{\mu\nu} = \begin{pmatrix} 0 & 0 \\ 0 & \phi^i \phi^j + A \delta_{ij} \rho \end{pmatrix}. \quad (100)$$

Since we expect that $\alpha = O(\lambda)$ we have in (100) introduced the notation

$$\alpha = A \lambda + O(\lambda^2) \quad (100)$$

From (12) and (13) we get to lowest order after a some calculations:

$$\partial^2 \Phi^{(1)} - 2 Q \partial_\rho \Phi^{(1)} = -\frac{Q}{4} (N - 1) A \quad (101)$$
\[ R^{(1)} = -4Q \partial_0 \Phi^{(1)}, \]  
\[ \lambda R^{(1)} \text{ denotes the contracted Ricci tensor corresponding to } G_{\mu \nu} \text{ to order } O(\lambda). \]

\[ R^{(1)} \text{ is simply given by the curvature dictated by the } O(N) \text{ model:} \]
\[ R^{(1)} = -(N - 1)(N - 2), \]

and (102) and (101) imply:
\[ \Phi^{(1)} = \frac{1}{4Q} (N - 1)(N - 2) \rho, \quad A = \frac{2}{Q} (N - 2). \]  

If we return to (94) we see that a change in the physical cut-off \( a \rightarrow a(1 + dl) \),
or equivalently a shift \( \rho \rightarrow \rho + 2dl/\gamma \) (where \( \gamma \) is given by (11)) can be absorbed in
a coupling constant renormalization
\[ \lambda \rightarrow \lambda Z(\lambda), \quad Z(\lambda) = e^{\frac{4(N - 2)\lambda dl}{\gamma Q}}. \]  

This leads to the \( \beta \)-function:
\[ \beta_G(\lambda) = -\frac{d\lambda}{dl} = -\frac{4(N - 2)\lambda^2}{\gamma Q} = \frac{2}{\gamma Q} \beta(\lambda). \]  

In addition the expression (103) for \( \Phi^{(1)} \) leads to a renormalization of the charge \( Q \):
\[ \tilde{Q} = Q + \frac{\lambda}{2Q} (N - 1)(N - 2) = Q - \frac{\lambda}{2Q} R^{(1)}. \]  

Again this is analogous to the result found in the Sine-Gordon model.

7 Conclusion and Discussion

We have provided some general arguments, working in conformal gauge, in favour
of a universal correction to the flat space \( \beta \)-function of marginal perturbations when
the theory is coupled to 2d quantum gravity. Our conclusions agree with the recent
observations made in [9, 12]. For three simple models we have calculated in detail the
lowest order effective actions for 2d quantum gravity coupled to matter fields which
in flat space have a non-trivial \( \beta \)-function. When the \( \beta \)-function was derived from
the effective action we found agreement with the universal correction mentioned
above. In addition we observed that the value of \( Q \) should be renormalized. This
is in agreement with general features of the renormalization group flow from an
ultra-violet to an infra-red fixed point, and in agreement with the observation that
the “gravitational dressing” of the renormalization group flow seems not to be able
change an ultra-violet stable fixed point into an infra-red stable fixed point. While the “dressing of the $\beta$-function” can be viewed as the first influence of gravity on marginal operators which perturb the matter part away from a fixed point, the change in $Q$ can be viewed as the first back-reaction of the marginal operators on gravity itself when we move away from the fixed point.

In the extended Sine-Gordon model it was from a formal point of view possible to have $c > 1 \ (N > 0)$ if we were willing to give up unitarity of the world surface non-linear sigma model. However, this seems a rather dramatic modification of the theory and in fact we found no consistent solutions to the classical equations which at the same time allowed the interpretation as a perturbation caused by marginal operators.

It would be interesting to calculate the next order correction of the $\beta$-function caused by the “gravitational dressing”. However, the arguments which used the operator product expansion can not be generalized in a straightforward manner to the $O(\lambda^3)$ corrections. Alternatively one could try first to calculate the effective action from the classical equations of motion, but a consistent calculation would need higher order terms of the tachyonic potential $v(T)$ as well as higher powers of $R$. However, these terms are not universal terms in the target space action, but depend on the renormalization scheme used for the non-linear sigma model, and the status of the results derived from such a calculation is not a priori clear to us. It seems to us a major challenge to clarify these issues and eventually apply the methods to a more complicated theory like dilaton gravity.

Finally the universal gravity correction $2/\gamma Q$ to the $\beta$-function calls upon a geometrical interpretation directly in terms of the fractal structure of space-time in quantum gravity. We have not yet found this interpretation, but it is a most fascinating topic for further study.

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**References**

[1] V. Knizhnik, A. Polyakov and A. Zamolodchikov, Mod.Phys.Lett A3 (1988) 819.
[2] F. David, Mod.Phys.Lett. A3 (1988) 1651; J. Distler and H. Kawai, Nucl.Phys. B321 (1989) 509.

[3] F. David, Nucl.Phys. B257 (1985) 43; B257 (1985) 543.

[4] J. Ambjørn, B. Durhuus and J. Fröhlich, Nucl.Phys. B257 (1985) 433; B270 (1986) 457; B275 (1986) 161.

[5] V.A. Kazakov, I. Kostov and A.A. Migdal, Phys.Lett. B157 (1985) 295.

[6] F. David, Nucl.Phys. B368 (1992) 671.

[7] J.H. Cardy, Les Houches session XLIX, 1988 (North Holland, Amsterdam, 1989)

[8] J. Polchinski, Nucl.Phys. B346 (1990) 253.

[9] C. Schmidhuber, Nucl.Phys. B404 (1993) 342.

[10] A. Tseytlin, Int.J.Mod.Phys. A5 (1990) 1833.

[11] A. Cooper, L. Susskind and L. Thorlacius, Nucl.Phys. B363 (1991) 132.

[12] I.R. Klebanov, I.I. Kogan and A.M. Polyakov, preprint PUPT-1421, September 1993, hep-th/9309106.

[13] H. Kawai and M. Ninomiya, Nucl.Phys. B336 (1990) 115.

[14] R.B. Mann, Phys.Lett. B244 (1992) 310; Phys.Rev D47 (1993) 4438.

[15] R.B. Mann, A. Shiekh and L. Tarasov, Nucl.Phys. B341 (1990) 134.

[16] J. Russo and A. Tseytlin, Nucl.Phys. B382 (1992) 259; S. Giddings and A. Strominger, Phys. Rev., D47(1993)2454; S. de Alwis, Phys. Rev., D46(1992)5429.

[17] S.R. Das and B. Sathiapalan, Phys. Rev. Lett. 56 (1986) 2664; C. Itoi and Y. Watabiki, Phys. Lett. B198(1987)486; A.A. Tseytlin, Phys. Lett. B264(1991)311.

[18] T. Banks, Nucl. Phys., B361(1991)166.

[19] A. Tseytlin, Phys.Lett. B241 (1990) 233.

[20] D.J. Amit, Y.Y. Goldschmidt and G. Grinstein, Jour. Phys. A13(1980)585.

[21] R. Jackiw, “Quantum Theory of Gravity, ed. S. Christensen (Hilger, Grisd, 1984) 403; C. Teitelboim, ibid, 327.
[22] A.H.Chamseddine, Phys. Lett. B256(1991)379;B258(1991)97; Nucl. Phys. B368(1992)98.

[23] D.Boyanovsky and R. Holman, Nucl. Phys. B332(1990)641; M.T.Grisaru, A.Lerda, S.Penati and D.Zanon, Nucl. Phys. B342(1990)564