MINIMAL HYPERSURFACES IN NEARLY $G_2$ MANIFOLDS

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Abstract. We study hypersurfaces in a nearly $G_2$ manifold. We define various quantities associated to such a hypersurface using the $G_2$ structure of the ambient manifold and prove several relationships between them. In particular, we give a necessary and sufficient condition for a hypersurface with an almost complex structure induced from the $G_2$ structure of the ambient manifold, to be nearly Kähler. Then using the nearly $G_2$ structure on the round sphere $S^7$, we prove that for a compact minimal hypersurface $M^6$ of constant scalar curvature in $S^7$ with the shape operator $A$ satisfying $|A|^2 > 6$, there exists an eigenvalue $\lambda > 12$ of the Laplace operator on $M$ such that $|A|^2 = \lambda - 6$, thus giving the next discrete value of $|A|^2$ greater than 0 and 6, thus generalizing the result of [5] about nearly Kähler $S^6$.

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1. Introduction

Let $(M^7, \mathcal{F})$ be a Riemannian manifold with a vector cross product $B$. Then they induce a $G_2$ structure $\varphi$ on $\overline{M}$, i.e., $\varphi$ is a 3-form which is non degenerate in some sense (see §2 for precise definitions). Let $M^6$ be a hypersurface of $\overline{M}$ with the induced metric $g$ from $\mathcal{F}$ and denote by $N$ the unit normal vector field of $M$ in $\overline{M}$. If we define $\xi : TM \to TM$ by $\xi(X) = B(N, X)$, where $X \in \Gamma(TM)$ and $B$ is the vector cross product, then $\xi$ is a metric compatible almost complex structure on $M$ (cf. Proposition 3.1). More generally, if $(L, g, J)$ is an almost Hermitian manifold with an almost complex structure $J$, then we have the following

Definition 1.1. Let $(L, g, J)$ be an almost Hermitian manifold with an almost complex structure $J$. Then $L$ is called a nearly Kähler manifold if $\nabla J$ is a skew-symmetric tensor, i.e.,

$$\nabla_X J = 0, \quad \forall X \in \Gamma(TM)$$

(1.1)

So a natural question is to find conditions on the oriented hypersurface $M$ so that with respect to the almost complex structure $\xi$, $(M, g, \xi)$ is a nearly Kähler manifold. Our first result is a characterization of nearly Kähler hypersurfaces of

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manifolds with a nearly $G_2$ structure (see §2 for definition). In §3, we prove the following (cf. Theorem 3.7)

**Theorem 1.2.** Let $M$ be an oriented hypersurface of a nearly $G_2$ manifold $(\overline{M}, \varphi)$. Then $(M, g, \xi)$ is a nearly Kähler structure if and only if for all $X \in \Gamma(TM)$

\[ AX = \alpha X + \beta \xi(X) \]  

(1.2)

where $A$ is the shape operator of $M$ in $\overline{M}$ and $\alpha, \beta \in C^\infty(TM)$.

For proving Theorem 1.2, we define new quantities related to a manifold with a nearly $G_2$ structure which have analogs in the study of manifolds with a nearly Kähler structure and which we hope will be of further use in the study of submanifolds of manifolds with a nearly $G_2$ structure.

In a different but related direction, suppose $M^n$ is a closed minimal hypersurface of constant scalar curvature in the unit sphere $S^{n+1}$, and let $A$ be its shape operator. A famous rigidity theorem due to the combined works of Simons [17], Lawson [12] and Chern, do Carmo, and Kobayashi [4] states that if $|A|^2 \leq n$ then $|A|^2 = 0$ or $|A|^2 = n$, where $|A|^2$ is the squared length of the shape operator. If $|A|^2 = 0$, then $M$ is isometric to the totally geodesic equatorial sphere $S^n$ in $S^{n+1}$, and if $|A|^2 = n$, then $M$ is isometric to the Clifford torus $S^k(\sqrt{\frac{n}{2}}) \times S^{n-k}(\sqrt{\frac{n-k}{n}})$. Following on his study of subsequent gaps for the scalar curvature of such hypersurfaces $M$, Chern asked the following question (cf. [20, pg.693])

**Question 1.3.** [Chern] Consider the set of all compact minimal hypersurfaces in $S^{n+1}$ with constant scalar curvature. Think of the scalar curvature as a function on this set. Is the image of the scalar curvature function a discrete set of positive numbers?

Since for any minimal hypersurface $M^n$ with scalar curvature $S$ in $S^{n+1}$, $S = n(n-1) - |A|^2$ (cf. (2.34) in §2), the above question asks whether the set of $|A|^2$ for such hypersurfaces $M$ is a discrete set.

The first two values of $|A|^2$ are known to be 0 and $n$. For the third value of $|A|^2$, Peng and Terng [16] proved that if $|A|^2 > n$, then there exists a positive constant $\delta(n)$ such that $|A|^2 > n + \delta(n)$. Also, for $n = 3$ they proved that $|A|^2 \geq 6$ and they conjectured that the third value of $|A|^2$ should be equal to $2n$. Yang and Cheng in [18] improved the constant $\delta(n)$ by proving that $\delta(n) > \frac{3}{2}n - \frac{11}{4}$ and in [19] they further improved this result by proving that if $|A|^2 > n$ then $|A|^2 > \frac{5}{6}(4n + 1)$. In [5], Deshmukh used the nearly Kähler structure on $S^6$ to prove the following theorem

**Theorem 1.4.** [Deshmukh, [5]] Let $M$ be a compact minimal hypersurface of constant scalar curvature in the unit sphere $S^6$. If the shape operator $A$ of $M$ satisfies $|A|^2 > 5$, then there exists an eigenvalue $\lambda > 10$ of the Laplace operator on $M$ satisfying $|A|^2 = \lambda - 5$.

The round unit sphere $S^7$ has a nearly $G_2$ structure, so a natural question is that whether we can say anything about the third value of $|A|^2$ for compact minimal hypersurfaces with constant scalar curvature in $S^7$ by using the nearly $G_2$ structure on it. Our next result is an analog of Theorem 1.4 for minimal hypersurfaces with constant scalar curvature in $S^7$. More precisely we prove the following

**Theorem 1.5.** Let $M^6$ be a compact minimal hypersurface of constant scalar curvature in the unit sphere $S^7$. If the shape operator $A$ of $M$ satisfies $|A|^2 > 6$, then there exists an eigenvalue $\lambda > 12$ of the Laplace operator on $M$ such that $|A|^2 = \lambda - 6$. 
The paper is organized as follows. In §2 we discuss preliminaries on vector cross products on manifolds and then proceed to define manifolds with G_{2} structures. We describe the intrinsic torsion forms of a G_{2} structure and use them to define a nearly G_{2} structure. We also discuss some notions from the geometry of submanifolds. In §3 we start by defining several quantities associated to a hypersurface of a nearly G_{2} manifold and then prove various relations among them. Using that we prove Theorem 1.2. Finally in §4, we prove Theorem 1.5.

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2. Preliminaries

2.1. Manifolds with Vector Cross Product. Let \((M^{n}, g)\) be a Riemannian manifold. An \(r\)-fold vector cross product (VCP, for short) is an alternating \(r\)-linear smooth map

\[
B : TM \times TM \times \cdots \times TM \rightarrow TM
\]

satisfying the following conditions

\[
\begin{align*}
&g(B(v_{1}, \ldots, v_{r}), v_{i}) = 0, \quad 1 \leq i \leq r \\
&\|B(v_{1}, \ldots, v_{r})\|^{2} = \|v_{1} \wedge \cdots \wedge v_{r}\|^{2}
\end{align*}
\]

for any \(v_{i} \in TM\).

Such a cross product gives rise to a \((r + 1)\) differential form \(\phi\) defined as

\[
\phi(v_{1}, v_{2}, \ldots, v_{r+1}) = g(B(v_{1}, \ldots, v_{r}), v_{r+1})
\]

The VCP is called parallel/closed if and only if the corresponding differential form is parallel/closed.

Cross products on real vector spaces were classified by Brown and Gray in [1] and global cross products on manifolds are discussed in Gray [7]. The classification of VCPS on a real vector space \(V\) with a positive definite inner product \(g\) is as follows:

1. \(r = 1\). Then a 1-fold VCP \(B\) on \(V\) is equivalent to an almost complex structure on \(V\), i.e., \(B^{2} = -I\) on \(V\). The associated VCP form is the Kähler form \(\omega\).

2. \(r = n - 1\), where \(n\) is the dimension of \(V\). An \((n - 1)\)-fold VCP \(B\) on \(V\) is the Hodge star operator \(*\) given by \(g\) on \(\Lambda^{n-1}V\) and the VCP form of degree \(n\) is the volume form on \(V\). Thus \(B\) is equivalent to an orientation.

3. \(r = 2\). A 2-fold VCP \(B\) on \(\mathbb{R}^{7}\) is a cross product defined as \(B(u, v) = \text{Im}(u, v)\), for \(u, v\) in \(\mathbb{R}^{7} \cong \text{Im}\mathbb{O}\), the set of imaginary octonions. Here, \(\text{Im}\mathbb{O}\) is the octonionic multiplication. For coordinates \(\{x_{1}, \ldots, x_{7}\}\) on \(\text{Im}\mathbb{O}\), the VCP form of degree 3 can be written as follows

\[
\varphi_{0} = dx^{123} - dx^{147} - dx^{527} - dx^{563} + dx^{415} + dx^{426} + dx^{437}
\]

where \(dx^{ijk} = dx^{i} \wedge dx^{j} \wedge dx^{k}\). Bryant [2] showed that the group of linear transformations of \(\mathbb{O}\) which preserves \(\varphi_{0}\) also preserves \(g\) and \(B\) and is the exceptional lie group \(G_{2}\) which is also the automorphism group of \(\mathbb{O}\).

4. \(r = 3\). A 3-fold VCP \(B\) on \(\mathbb{R}^{8}\) is a cross product defined as \(B(u, v, w) = \frac{1}{2}(u(\bar{v}w) - w(\bar{v}u))\) for any \(u, v, w\) in \(\mathbb{R}^{8} \cong \mathbb{O}\). For coordinates \(x_{1}, \ldots, x_{8}\) on \(\mathbb{O}\), the VCP form of degree 4 can be written as
octonionic multiplication has the following properties. For all $u, v, w$ the space of forms, mainly based on $G_2$, manifolds with $\phi_b$ bilinear form and a nowhere vanishing 7-form which defines a unique metric $\text{Stiefel-Whitney classes}$. is orientable and spin, which is equivalent to the vanishing of the first and second

The VCP form $\varphi_0$ induced by the vector cross product on $\mathbb{R}^7$ is described in (2.2). The group $G_2$ preserves $\varphi_0$ and it also preserves the metric and orientation for which $\{e_1, ..., e_7\}$ is an oriented orthonormal basis. If $\ast \varphi_0$ denotes the Hodge star determined by the metric and the orientation, then $G_2$ preserves the 4-form

$$\psi_0 = \ast \varphi_0 \varphi_0 = dx^{4567} - dx^{1523} - dx^{4163} - dx^{4127} + dx^{2637} + dx^{1537} + dx^{1526} \quad (2.6)$$

where $dx^{ijkl} = dx^i \wedge dx^j \wedge dx^k \wedge dx^l$.

Let $M$ be a 7-manifold. For $x \in M$, we denote by

$$\Lambda^3_{pos}(M)_x = \{ \varphi_x \in \Lambda^3 T^*_x M \mid \exists \text{ isomorphism } \rho : T^*_x M \to \mathbb{R}^7, \rho^* \varphi_0 = \varphi_x \}$$

The bundle $\Lambda^3_{pos}(M) = \bigcup_{x \in M} \Lambda^3_{pos}(M)_x$ is an open subbundle of $\Lambda^3 T^* M$ as $\Lambda^3_{pos}(M)_x \cong \text{GL}(7, \mathbb{R})/G_2$. A section $\varphi$ of $\Lambda^3_{pos}(M)$ is called a positive 3-form on $M$ and the space of positive 3-forms on $M$ is denoted by $\Omega^3_{pos}(M)$. Such a $\varphi$ is also called a $G_2$ structure on $M$. A $G_2$ structure exists on $M$ if and only if the manifold is orientable and spin, which is equivalent to the vanishing of the first and second Stiefel-Whitney classes.

A $G_2$ structure induces a unique metric and orientation. For a 3-form $\varphi$, we define

$$S_\varphi(u, v) = -\frac{1}{6} (u, \varphi) \wedge (v, \varphi) \wedge \varphi$$

for $u$, $v$ tangent vectors on $M$, which is a $\Omega^7(M)$-valued bilinear form. The 3-form $\varphi$ is a positive 3-form if and only if $S_\varphi$ is a tensor product of a positive definite bilinear form and a nowhere vanishing 7-form which defines a unique metric $g$ with volume form $vol_g$ by

$$g(u, v) vol_g = S_\varphi(u, v).$$
The metric and orientation determines the Hodge star operator $\star_\varphi$ and we define $\psi = \star_\varphi \varphi$.

A $G_2$ structure on $M$ induces a splitting of the spaces of differential forms on $M$ into irreducible $G_2$ representations. The space of 2-forms $\Omega^2(M)$ and 3-forms $\Omega^3(M)$ decompose as

$$\Omega^2(M) = \Omega^2_1(M) \oplus \Omega^2_{14}(M)$$  \hspace{1cm} (2.7)

$$\Omega^3(M) = \Omega^3_1(M) \oplus \Omega^3_2(M) \oplus \Omega^3_{27}(M)$$  \hspace{1cm} (2.8)

where $\Omega^k_i$ has pointwise dimension $l$. More precisely, we have the following description of the space of forms:

$$\Omega^2_1(M) = \{ f \varphi \mid f \in C^\infty(M) \}$$  \hspace{1cm} (2.12)

$$\Omega^3_1(M) = \{ X \cdot \varphi \mid X \in \Gamma(TM) \}$$  \hspace{1cm} (2.13)

$$\Omega^3_{27}(M) = \{ \gamma \in \Omega^3(M) \mid \gamma \wedge \varphi = 0 = \gamma \wedge \psi \}$$  

$$= \{ h_{ij} g^{il} dx^i \wedge (\frac{\partial}{\partial x^l}) \varphi \mid h_{ij} = h_{ji}, g^{ij} h_{ij} = 0 \}$$  \hspace{1cm} (2.14)

in local coordinates $\{x^1, ..., x^7\}$ on $M$. In (2.14), $h$ is a symmetric 2 tensor. The decompositions of $\Omega^k(M) = \Omega^k_1(M) \oplus \Omega^k_2(M) \oplus \Omega^k_{14}(M)$ and $\Omega^k(M) = \Omega^k_1(M) \oplus \Omega^k_2(M) \oplus \Omega^k_{27}(M)$ are obtained by taking the Hodge star of (2.8) and (2.7) respectively. The contractions between $\varphi$ and $\psi$ in index notation (see [3] or [11] for more details) are as follows:

$$\varphi_{ijk} \varphi_{abc} g^{kc} = g_{ia} g_{jb} - g_{ib} g_{ja} - \psi_{ijk}$$  \hspace{1cm} (2.15)

$$\varphi_{ijk} \psi_{abcd} g^{ld} = g_{ia} \varphi_{bc} + g_{ib} \varphi_{jc} + g_{ic} \varphi_{ab}$$

$$- g_{ja} \varphi_{bc} - g_{jb} \varphi_{ac} - g_{jc} \varphi_{ab}$$  \hspace{1cm} (2.16)

$$\psi_{ijkl} \psi_{abcd} g^{ik} g^{jc} g^{ld} = 24 g_{ia}$$  \hspace{1cm} (2.17)

Given a $G_2$ structure $\varphi$ on $M$, we can decompose $d\varphi$ and $d\psi$ according to (2.7) and (2.8). This defines torsion forms, which are unique differential forms $\tau_0 \in \Omega^0(M)$, $\tau_1 \in \Omega^1(M)$, $\tau_2 \in \Omega^2_1(M)$ and $\tau_3 \in \Omega^3_{27}(M)$ such that (see [11])

$$d\varphi = \tau_0 \psi + 3 \tau_1 \wedge \varphi + \star_\varphi \tau_3$$  \hspace{1cm} (2.18)

$$d\psi = 4 \tau_1 \wedge \psi + \star_\varphi \tau_2$$  \hspace{1cm} (2.19)

The full torsion tensor $T$ of a $G_2$ structure is a 2-tensor satisfying

$$\nabla_i \varphi_{jkl} = T_{im} g^{np} \psi_{pjkl}$$  \hspace{1cm} (2.20)

$$T_{im} = \frac{1}{24} (\nabla_i \varphi_{abc}) \psi_{mjkl} g^{ja} g^{kb}$$  \hspace{1cm} (2.21)

$$\nabla_m \psi_{ijkl} = - T_{mi} \varphi_{jkl} + T_{mj} \varphi_{ikl} - T_{mk} \varphi_{ijl} + T_{ml} \varphi_{ijk}$$  \hspace{1cm} (2.22)

The full torsion $T$ is related to the torsion forms by (see [11])
\[ T_{lm} = \frac{\tau_0}{4} g_{lm} - (\tau_3)_{lm} + (\tau_1)_{lm} - \frac{1}{2}(\tau_2)_{lm} \]  
(2.23)

**Remark 2.1.** Since the space \( \Omega^2_7 \) is isomorphic to the space of vector fields and hence to the space of 1-forms so in (2.23), we are viewing \( \tau_1 \) as an element of \( \Omega^2_7 \) which justifies the expression \( (\tau_1)_{lm} \). See [11] for more details.

**Definition 2.2.** A G\(_2\) structure \( \varphi \) is called torsion free if \( \nabla \varphi = 0 \) or equivalently \( T = 0 \).

A manifold \((M, \varphi)\) with a G\(_2\) structure \( \varphi \) is called a G\(_2\) manifold if it is torsion-free.

We can now define nearly G\(_2\) structure.

**Definition 2.3.** A G\(_2\) structure \( \varphi \) is a nearly G\(_2\) structure if \( \tau_0 \) is the only nonvanishing component of the torsion, i.e., \( d\varphi = \tau_0 \psi \) and \( d\psi = 0 \)

In this case, we see from (2.23) that \( T_{ij} = \frac{\tau_0}{4} g_{ij} \).

**Remark 2.4.** If \( \varphi \) is a nearly G\(_2\) structure on \( M \) then since \( d\varphi = \tau_0 \psi \), we can differentiate this to get \( d\tau_0 \wedge \psi = 0 \) and hence \( d\tau_0 = 0 \), as wedge product with \( \psi \) is an isomorphism from \( \Omega^2_7(M) \) to \( \Omega^2_7(M) \). Thus \( \tau_0 \) is a constant, if \( M \) is connected.

Given a G\(_2\) structure \( \varphi \) with torsion \( T_{lm} \), we have the expressions for the Ricci curvature \( R_{ij} \) and the scalar curvature \( S \) of its associated metric \( g \) from [11] as

\[ R_{jk} = (\nabla_j T_{lm} - \nabla_l T_{jm})\varphi_{nkl} g^{mn} g^{il} - T_{jil} g^{il} T_{lk} + \text{Tr}(T)T_{jk} - T_{jb} T_{ia} g^{il} g^{ap} \psi_{pqk} g^{bp} \]  
(2.24)

\[ S = -12 g^{il} \nabla_i (\tau_1)_{j} + \frac{21}{8} \tau_0^2 - |\tau_3|^2 + 5|\tau_1|^2 - \frac{1}{4} |\tau_2|^2 \]  
(2.25)

where \( |C|^2 = C_{ij} C_{kl} g^{ik} g^{jl} \) is the matrix norm in (2.25).

In particular, for a manifold \( M \) with a nearly G\(_2\) structure \( \varphi \), we see that

\[ R_{jk} = \frac{3}{8} \tau_0^2 g_{jk} \]  
(2.26)

\[ S = \frac{21}{8} \tau_0^2 \]  
(2.27)

Finally, we remark that \( S^7 \) with the round metric and also the squashed \( S^7 \) are examples of manifolds with nearly G\(_2\) structure (see [6] for more on nearly G\(_2\) structures. The authors in [1] call such structures nearly parallel G\(_2\) structures but we will call them nearly G\(_2\) structures.) In particular, \( S^7 \) with radius 1 has scalar curvature 42, so comparing with (2.27) we get that \( \tau_0 = 4 \).

**2.3. Geometry of Submanifolds.** In this section, we briefly recall the geometry of submanifolds. More details can be found, for example in [14]. Let \((\overline{M}, \overline{g})\) be Riemannian manifold and \((M, g)\) be an immersed orientable submanifold of \( \overline{M} \) with induced metric. Then for \( X, Y \in \Gamma(TM) \), we have

\[ \nabla_{X} Y = \nabla_{X} Y + II(X, Y) \]  
(2.28)

where \( \nabla \) is the covariant derivative on \( \overline{M} \), \( \nabla \) is the covariant derivative on \( M \) and \( II : TM \times TM \to NM \) is the second fundamental form of \( M \). Here \( NM \) is the normal bundle of \( M \) in \( \overline{M} \).

If \( M \) is an oriented hypersurface of \( \overline{M} \) and we denote by \( N \) the unit normal vector field of \( M \) in \( \overline{M} \) corresponding to this orientation, then the second fundamental form
is a multiple of $N$ and is given by the shape operator, which we denote by $A$. Here $A : TM \to TM$ is a self-adjoint linear map and (2.28) becomes

$$\nabla_X Y = \nabla_X Y + g(AX, Y)N$$  \hspace{1cm} (2.29)

We also have the Weingarten equation

$$\nabla_X N = -AX$$  \hspace{1cm} (2.30)

If $\overline{Rm}$ denotes the Riemann curvature tensor on $(\overline{M}, \overline{g})$ and $Rm$ denotes the Riemann curvature tensor on $(M, g)$, then the Gauss equation for $M$ is

$$\overline{Rm}(X, Y, Z, W) = Rm(X, Y, Z, W) - g(AX, W)g(AY, Z) + g(AX, Z)g(AY, W)$$  \hspace{1cm} (2.31)

Now suppose $\overline{M}$ is the unit sphere $S^7$ with the round metric. Then $\overline{Rm}$ as a $(3,1)-$tensor is given by $\overline{Rm}(X, Y)Z = \overline{g}(Y, Z)X - \overline{g}(X, Z)Y$. In this case (2.31) becomes

$$Rm(X, Y, Z, W) = \overline{g}(Y, Z)X - \overline{g}(X, Z)Y + g(AY, Z)AX - g(AX, Z)AY$$  \hspace{1cm} (2.32)

If $M$ is also a minimal hypersurface of $S^7$ (i.e., the mean curvature vector $H = 0$) then by taking the trace of (2.32), the Ricci and the scalar curvature of $M$ are

$$Ric(X, Y) = 5g(X, Y) - g(AX, AY)$$  \hspace{1cm} (2.33)

$$S = 30 - |A|^2$$  \hspace{1cm} (2.34)

where $|A|^2$ is the square of the length of the shape operator of $M$. We also have the Codazzi equation, which in this case is

$$\nabla_X (AY) - \nabla_Y (AX) = A[[X, Y]]$$  \hspace{1cm} (2.35)

Finally, we define totally umbilic hypersurface.

**Definition 2.5.** A hypersurface $M$ of a Riemannian manifold $\overline{M}$ is called totally umbilic at $x \in M$ if the shape operator $A$ of $M$ is a multiple of the identity map of $T_x M$. Moreover $M$ is called totally umbilic if it is totally umbilic at each of its point.

**Remark 2.6.** Throughout the paper, all quantities associated to the ambient manifold $\overline{M}$ will have a bar with them, for example the metric on $\overline{M}$ is $\overline{g}$ whereas those of the hypersurface are written without any bar.

### 3. Proof of Theorem 1.2

We start this section by defining various quantities for hypersurfaces (not necessarily minimal) of a manifold with a nearly $G_2$ structure which have analogs for hypersurfaces of a manifold with a nearly Kähler structure. Being motivated from the notion of a characteristic vector field on a manifold with an almost complex structure, we define a $(1, 1)$ tensor $\xi$ on $M^6$, induced from the octonionic multiplication on a manifold with a $G_2$ structure $(\overline{M}, \varphi)$, as follows

$$\xi(X) = B(N, X)$$  \hspace{1cm} (3.1)

where $X \in \Gamma(TM)$, $B(., .)$ is the cross product and $N$ is the unit normal to $M^6$ in $\overline{M}$. We have the following

**Proposition 3.1.** The tensor $\xi$ is a metric compatible almost complex structure on $(M^6, g)$. 
Proof. For $X \in \Gamma(TM)$, we have
\[
\xi^2(X) = \xi(B(N, X)) = B(N, B(N, X)) = -|N|^2 X + \bar{\gamma}(N, X) N = -X
\]
where the equality in the second line is from (2.4) for cross product. Hence $\xi^2(X) = -X$. Also,
\[
g(\xi(X), \xi(Y)) = g(B(N, X), B(N, Y)) = g(B(B(N, X), N), Y) = -g(B(N, B(N, X)), Y) = g(X, Y)
\]
where we have used (2.3) in going from the first to the second line, the anticommutativity of $B$ in the first equality and (2.4) and the fact that $N$ is a unit vector in the second equality of the third line.

Again, from the motivation from nearly Kähler geometry, we define a $(3, 1)$ tensor field $G$ as follows
\[
G(X, Y, Z) = (\nabla_X B)(Y, Z)
\] (3.2)
for $X, Y, Z \in \Gamma(TM)$.

Now we prove some results about $G$ and relationships between $G$ and $B$ for manifolds with a nearly $G_2$ structure.

**Proposition 3.2.** Let $\psi = \ast \varphi$ denotes the 4-form on $(\bar{M}, \varphi)$ with a nearly $G_2$ structure. Then for any vector fields $X, Y, Z, W$
\[
\bar{\gamma}(G(X, Y, Z), W) = \frac{\tau_0}{4} \psi(X, Y, Z, W)
\] (3.3)
where $\tau_0$ is as defined in (2.18).

**Proof.** If $\varphi$ is a $G_2$ structure then
\[
\varphi(X, Y, Z) = \bar{\gamma}(B(X, Y), Z)
\] (3.4)

Then from (3.4) we have
\[
\bar{\gamma}(G(X, Y, Z), W) = \bar{\gamma}(\nabla_X B)(Y, Z, W)
\]
\[
= \bar{\gamma}(\nabla_X (B(Y, Z)) - B(\nabla_X Y, Z) - B(Y, \nabla_X Z), W)
\]
\[
= \nabla_X (\varphi(Y, Z, W)) - \varphi(\nabla_X Y, Z, W) - \varphi(Y, \nabla_X Z, W)
\]
\[
- \varphi(Y, Z, \nabla_X W)
\]
\[
= (\nabla_X \varphi)(Y, Z, W)
\]
\[
= \frac{\tau_0}{4} \psi(X, Y, Z, W)
\]
where we have used $\nabla_X (B(Y, Z)) = \nabla_X (\bar{\gamma}(B(Y, Z), W)) - \bar{\gamma}(B(Y, Z), \nabla_X W)$ in going from the second to the third equality, $\nabla_i \varphi_{jkl} = \Xi_i g^{mpq} \psi_{jkl}$ and the fact that for a nearly $G_2$ structure, $T_{ij} = \frac{\tau_0}{4} g_{ij}$ in the last equality.

**Remark 3.3.** From (3.3), we see that $G$ is skew-symmetric in all of its entries.

**Proposition 3.4.** For any vector fields $X, Y, Z, W$, we have
\[
G(B(W, Z), X, Y) = \frac{\tau_0}{4} [\bar{\gamma}(X, Z) B(W, Y) + \bar{\gamma}(Y, Z) B(X, W) - \bar{\gamma}(W, X) B(Z, Y)
- \bar{\gamma}(W, Y) B(X, Z) + \varphi(X, Y, W) Z - \varphi(X, Y, Z) W]
\] (3.5)
Proof. We know from Proposition 3.2 that

\[ G(X, Y, Z) = \frac{70}{4} \psi(X, Y, Z, \cdot) \]

so

\[ G(B(X, Y), Z, W) = \frac{70}{4} \psi(B(X, Y), Z, W, \cdot)^\# \]

In local coordinates \( \{x_1, x_2, \ldots, x_7\} \), we have \( g(B(\partial_k, \partial_l), \partial_n) = \varphi_{kln} \). So

\[ G(B(\partial_k, \partial_l), \partial_n, \partial_j) = \frac{70}{4} \psi(B(\partial_k, \partial_l), \partial_n, \partial_j, \cdot)^\# \]

\[ = \frac{70}{4} \psi(\varphi_{kl} \#^\#, \partial_n, \partial_j, \cdot)^\# \]

\[ = -\frac{70}{4} \psi_{ij\cdot n} g^{\alpha\beta} \varphi_{k\alpha\beta} \]

Using the identity in (2.16)

\[ g^{\alpha\beta} \psi_{ij\cdot m} \varphi_{k\alpha\beta} = g_{kl} \psi_{ij\cdot m} + g_{kj} \varphi_{ilm} + g_{km} \varphi_{ijl} - g_{kl} \varphi_{km} - g_{lm} \varphi_{ijk} \]

we get the proposition. \qed

**Proposition 3.5.** For any vector fields \( X, Y, Z, W \), we have

\[ B(G(X, Y, Z), W) = -G(B(X, Y), Z, W) \] (3.7)

Proof. In local coordinates \( \{x_1, \ldots, x_7\} \), we have \( G(\partial_k, \partial_l, \partial_m) = \frac{70}{4} \psi_{klm}^\# \) and \( B(\partial_k, \partial_l) = \varphi_{kl}^\# \), so

\[ B(G(\partial_k, \partial_l, \partial_m), \partial_n) = \frac{70}{4} \varphi(\varphi_{kl}^\#, \partial_n, \cdot)^\# = \frac{70}{4} \varphi_{kl}^\# g^{\alpha\beta} \psi_{\alpha\beta\cdot n} \]

The proposition now follows from the last line of (3.6). \qed

We will need the expression for \( \nabla_X \xi \) later, so we have the following

**Proposition 3.6.** Let \( M \) be an oriented hypersurface of \((\overline{M}, \varphi)\) and \( \xi \) be as defined in (3.1). Then for any vector field \( X \in \Gamma(TM) \), we have

\[ (\nabla_X \xi)(Y) = G(X, N, Y) - \varphi(N, Y, AX)N - B(AX, Y) \] (3.8)

Proof. We calculate

\[ (\nabla_X \xi)(Y) = \nabla_X(\xi(Y)) - \xi(\nabla_X Y) \]

\[ = \nabla_X(B(N, Y)) - g(AX, B(N, Y))N - \xi(\nabla_X Y) \]

\[ = (\nabla_X B)(N, Y) + B(\nabla_X N, Y) + B(N, \nabla_X Y) - g(AX, B(N, Y))N \]

\[ - \xi(\nabla_X Y) \]

\[ = G(X, N, Y) - B(AX, Y) + B(N, \nabla_X Y) + g(AX, Y)B(N, N) \]

\[ - g(AX, B(N, Y))N - \xi(\nabla_X Y) \]

\[ = G(X, N, Y) - \varphi(N, Y, AX)N - B(AX, Y) \] (3.9)

where we have used (2.29) in the second equality, (2.30) and (3.2) in the fourth equality and the fact that \( B(N, N) = 0 \) in the last equality. \qed

Now we will prove Theorem 1.2 mentioned in §1, namely, we will give a necessary and sufficient condition for an oriented hypersurface of a nearly G\(_2\) manifold to be nearly Kähler. We restate the theorem.
Theorem 3.7. Let $M$ be an oriented hypersurface of a nearly $G_2$ manifold $(M, \varphi)$. Then $(M, g, \xi)$ is a nearly Kähler structure if and only if for all $X \in \Gamma(TM)$

$$AX = \alpha X + \beta \xi(X)$$

(3.10)

where $A$ is the shape operator of $M$ in $\overline{M}$ and $\alpha, \beta \in C^\infty(M)$.

Proof. We know from (1.1) that if $J$ is a metric compatible almost complex structure on $M$ then $(M, J, g)$ is nearly Kähler if and only if for all $X \in \Gamma(TM)$, we have $\nabla_X J X = 0$. From Proposition 3.1, we know that $\xi$ is a metric compatible almost complex structure on $M$. Denote by $B(X, Y)$, the tangential component of $B(X, Y)$. Using (3.8) from Proposition 3.6, for $X \in \Gamma(TM)$

$$G(X, N, X) - \varphi(N, X, AX)N - B(AX, X) = 0 \iff$$

$$\varphi(N, X, AX)N + B(AX, X)^T + g(B(AX, X), N)N = 0 \iff$$

$$B(AX, X)^T + \varphi(AX, X, N)N + \varphi(N, X, AX)N = 0 \iff$$

$$B(AX, X)^T = 0$$

(3.11)

where we used the fact that $G$ is skew-symmetric in all of its entries in going from the second line to the third.

If $X = 0$ then from (3.11), the theorem is true. So we assume that $X \neq 0$. Now if $AX = \alpha X + \beta \xi(X)$ then

$$B(AX, X)^T = B(\alpha X + \beta \xi(X), X)^T$$

$$= B(\beta \xi(X), X)^T$$

$$= \beta B(N, X)^T$$

$$= \beta (-|X|^2 N)^T$$

$$= 0$$

(3.12)

where we have used (3.1) in the third line and (2.4) in the fourth line. Thus (3.11) and (3.12) proves one direction of the theorem.

Now suppose $B(AX, X)^T = 0$. Since $AX$ is tangent to $M$ so we write $AX = \alpha X + Y$ where $\alpha$ is a function which might depend on $X$ and $g(X, Y) = 0$. So $B(Y, X)^T = 0$. Suppose $B(Y, N) = aN$ for some function $a$.

Then from (2.4) we have

$$B(B(Y, X), X) = -|X|^2 Y$$

Also, $B(B(Y, X), X) = aB(N, X) = a \xi(X)$, so we get

$$Y = -\frac{a}{|X|^2} \xi(X)$$

and hence

$$AX = \alpha X + \beta \xi(X)$$

(3.13)

where $\beta = -\frac{\alpha}{|X|^2}$. Thus (3.12) and (3.13) proves the other direction. □

Remark 3.8. Note that the proof of Theorem 3.7 remains unchanged if $G = 0$. So the above theorem also holds for hypersurfaces of $G_2$ manifolds, i.e., manifolds with torsion free $G_2$ structures.
Remark 3.9. If $\beta = 0$ and $\alpha$ is independent of $X$ then $M$ is a totally umbilic hypersurface. The proof of Theorem 3.7 shows that if $M$ is totally umbilic then it must be nearly Kähler with respect to $\xi$.

Remark 3.10. It would be interesting to find examples of hypersurfaces in a manifold with a nearly $G_2$ structure which are nearly Kähler with respect to $\xi$ but are not totally umbilic.

We will need the following Lemma in §4.

Lemma 3.11. Let $M$ be an oriented hypersurface of a nearly $G_2$ manifold $(\overline{M}, \varphi)$ and let $\xi$ be as in (3.1). Then $\text{div} \xi = 0$.

Proof. Since $A$ is a self-adjoint operator, so we choose an orthonormal frame $\{e_1, ..., e_6\}$ at a point $p \in M$ which diagonalizes $A$, i.e., $Ae_i = a_ie_i$, $\forall i$. Then for $v \in T_pM$ we compute using Proposition 3.6

$$\text{(div} \xi)_p(v) = \sum_{i=1}^{6} g((\nabla_{e_i} \xi)(v), e_i)$$

$$= \sum_{i=1}^{6} g(G(e_i, N, v) - \varphi(N, v, Ae_i)N - B(Ae_i, v), e_i)$$

$$= - \sum_{i=1}^{6} g(B(Ae_i, v), e_i) = - \sum_{i=1}^{6} \varphi(Ae_i, v, e_i) = - \sum_{i=1}^{6} \varphi(a_ie_i, v, e_i)$$

$$= 0 \quad (3.14)$$

where we used (3.8) in the second equality, Remark 3.3 in the third equality and the fact that $\varphi$ is 3-from in the last equality. $\square$

4. Proof of Theorem 1.5

In this section we will prove Theorem 1.5, stated in §1. Let $(L, g)$ be a Riemannian manifold. A vector field $X$ on $L$ is said to be a conformal vector field if

$$\mathcal{L}_X g = 2fg \quad (4.1)$$

for some $f \in C^\infty(L)$, which is called the potential of $X$. Here $\mathcal{L}_X g$ denotes the Lie derivative of $g$ with respect to $X$. If $f \equiv 0$, then $X$ is a Killing vector field. There are many non-Killing conformal vector fields on the unit sphere $S^n$ with the round metric $\overline{g}$. In particular, if $Y$ is a non-zero constant vector field on $\mathbb{R}^{n+1}$, $\overline{N}$ is the unit normal of $S^n$ in $\mathbb{R}^{n+1}$ and $Y = X + f\overline{N}$, where $X$ is the tangential component of $Y$, then using (2.29) and (2.30) and the fact that for $S^n$ as a hypersurface in $\mathbb{R}^{n+1}$, $A = -I$, we see that $\nabla f = X$ and $\nabla_w X = -fW$, and hence $\mathcal{L}_X \overline{g} = -2f \overline{g}$, so $X$ is a conformal vector field with potential $-f$. In fact, all non-Killing conformal vector fields on the unit $S^n$ arise in this manner. (see [8])

Let $M$ be an oriented compact minimal hypersurface of $S^7$ satisfying the hypotheses of Theorem 1.5, i.e., $M$ is of constant scalar curvature and the shape operator $A$ of $M$ satisfies $|A|^2 > 6$. Let $V, \tilde{V}$ be two non-Killing conformal vector fields on $S^7$ with potential functions $f, \tilde{f}$ respectively, arising from two linearly independent constant vector fields on $\mathbb{R}^8$. Let $W, \tilde{W}$ be the tangential components on $M$ of $V$ and $\tilde{V}$ respectively. Then we have $V = W + sN$ and $\tilde{V} = \tilde{W} + \tilde{s}N$, where $s, \tilde{s} : M \to \mathbb{R}$.

Using (2.29) and (2.30), for $X \in \Gamma(TM)$ we get
\[ \nabla_X W = \nabla_X V - \nabla_X (sN) \]
\[ = -f X + s A X \]  
(4.2)
\[ \nabla f = W \]  
(4.3)
\[ \nabla s = -A W \]  
(4.4)

Similarly, we get
\[ \nabla_X \tilde{W} = -\tilde{f} X + \tilde{s} A X, \quad \nabla \tilde{f} = \tilde{W} \quad \text{and} \quad \nabla \tilde{s} = -A \tilde{W} \]  
(4.5)

Now we define the function \( h : M \to \mathbb{R} \) as
\[ h = g(\xi(W), \tilde{W}) \]  
(4.6)

We are interested in finding \( \Delta_M h \). So we compute
\[ \nabla_X h = \nabla_X g(\xi(W), \tilde{W}) \]
\[ = g((\nabla_X \xi)W, \tilde{W}) + g((\nabla_X \tilde{W}), W) + g(\xi(W), \tilde{W}) \]
\[ = g(G(X, N, W) - \varphi(N, W, AX)N - B(AX, W)^T, \tilde{W}) \]
\[ + g(\xi(-f X + s A X), \tilde{W}) + g(\xi(W), -f X + \tilde{s} A X) \]
\[ = -g(G(N, W, \tilde{W}), X) - g(B(W, \tilde{W})^T, AX) + g(f \tilde{W}, X) \]
\[ - g(s \xi W, AX) - g(\tilde{f} \xi W, X) + g(\tilde{s} \xi W, AX) \]
(4.7)

so we get
\[ \nabla h = -G(N, W, \tilde{W}) - AB(W, \tilde{W})^T + f \xi \tilde{W} - s \xi \tilde{W} - \tilde{f} \xi W + \tilde{s} \xi W \]  
(4.8)

We use (4.2), (4.3), (4.4) and (4.5) to calculate the divergence of each term in (4.8). For that, we choose local orthonormal frame \( \{e_1, \ldots, e_6\} \) at \( p \in M \) such that \( A e_i = a_i e_i, \forall i \).

\[ \text{div}(f \xi W) = g(\nabla f, \xi \tilde{W}) + f \sum_{i=1}^{6} [g((\nabla_i \xi) \tilde{W}, e_i) + g(\xi(\nabla_i \tilde{W}), e_i)] \]
\[ = g(W, \xi \tilde{W}) + \sum_{i=1}^{6} g(\xi(-\tilde{f} e_i + \tilde{s} A e_i), e_i) \]
\[ = g(W, \xi \tilde{W}) + \sum_{i=1}^{6} [-f g(\xi e_i, e_i) + \tilde{s} a_i g(\xi e_i, e_i)] \]
\[ = -h \]  
(4.9)

where we have used Lemma 3.11 in the second equality and the definition of \( \xi \) to eliminate the terms inside the summation in the third equality.

Similarly
\[ \text{div}(\tilde{f} \xi W) = h \]  
(4.10)
\[
\text{div}(s\, A\xi \tilde{W}) = g(\nabla s, A\xi \tilde{W}) + s \sum_{i=1}^{6} g(\nabla_{i}(A\xi \tilde{W}), e_{i}) \\
= -g(AW, A\xi \tilde{W}) + s \sum_{i=1}^{6} [\nabla_{i}g(A\xi \tilde{W}, e_{i}) - g(A\xi \tilde{W}, \nabla_{e_{i}}e_{i})] \\
= -g(AW, A\xi \tilde{W}) + s \sum_{i=1}^{6} [g(\nabla_{i}\xi \tilde{W}, Ae_{i}) + g(\xi \tilde{W}, \nabla_{e_{i}}Ae_{i})] \\
= -g(AW, A\xi \tilde{W}) + s \sum_{i=1}^{6} [g((\nabla_{e_{i}}\xi) \tilde{W}, Ae_{i}) + g(\xi(\nabla_{e_{i}} \tilde{W}), Ae_{i})] \\
= -g(AW, A\xi \tilde{W}) \tag{4.11}
\]

where in the third equality we have used that \(\sum_{i}(\nabla A)(e_{i}, e_{i}) = \sum_{i}(\nabla_{e_{i}} Ae_{i} - A\nabla_{e_{i}} e_{i}) = 0\) which follows from the Codazzi identity (2.35) and the fact that \(M\) is minimal, (3.8), (4.5) and \(Ae_{i} = a_{i} e_{i}\) to eliminate the terms inside the summation in the second last equality.

Similarly

\[
\text{div}(\tilde{s}A\xi W) = -g(A\tilde{W}, A\xi \tilde{W}) \tag{4.12}
\]

For calculating \(\text{div}(AB(W, \tilde{W})^{T})\), we repeatedly use (2.29) and (2.30) to first compute

\[
(\nabla_{Z}B)(X, Y) = \nabla_{Z}(B(X, Y)) - B(\nabla_{Z}X, Y) - B(X, \nabla_{Z}Y) \\
= \nabla_{Z}(B(X, Y)^{T}) + \nabla_{Z}(B(X, Y), N)N - B(\nabla_{Z}X, Y) \\
- g(AZ, X)B(N, Y) - B(X, \nabla_{Z}Y) - g(AZ, Y)B(X, N) \\
= \nabla_{Z}(B(X, Y)^{T}) + g(AZ, B(X, Y)^{T})N + \nabla_{Z}(B(X, Y), N)N \\
- \nabla_{Z}(B(X, Y), AZ)N - \nabla_{Z}(B(X, Y), N)AZ - B(\nabla_{Z}X, Y) \\
- g(AZ, X)B(N, Y) - B(X, \nabla_{Z}Y) - g(AZ, Y)B(X, N) \tag{4.13}
\]

where we have written \(B(X, Y)\) as a sum of its tangential and normal components in the first term in second equality and then used (2.29) in the third equality. So we get

\[
(\nabla_{Z}B)(X, Y) = \nabla_{Z}(B(X, Y)^{T}) + \nabla_{Z}(B(X, Y), N)N + \nabla_{Z}(B(\nabla_{Z}X, Y), N)N \\
+ g(AZ, X)B(N, Y) + \nabla_{Z}(B(X, \nabla_{Z}Y), N)N \\
+ g(AZ, Y)B(X, N)N - \nabla_{Z}(B(X, Y), N)AZ - B(\nabla_{Z}X, Y) \\
- g(AZ, X)B(N, Y) - B(X, \nabla_{Z}Y) - g(AZ, Y)B(X, N) \\
= \nabla_{Z}(B(X, Y)^{T}) + \nabla_{Z}(B(X, Y), N)N + \nabla_{Z}(B(\nabla_{Z}X, Y), N)N \\
+ \nabla_{Z}(B(X, \nabla_{Z}Y), N)N - \nabla_{Z}(B(X, Y), N)AZ - B(\nabla_{Z}X, Y) \\
- B(X, \nabla_{Z}Y) - g(AZ, X)B(N, Y) - g(AZ, Y)B(X, N) \tag{4.14}
\]

where we have used \(\nabla_{Z}(B(N, V), N) = \varphi(N, V, N) = 0, \forall V\) in going from fourth to fifth equality. Now using (3.2), we see that (4.14) is
\[ \nabla_{Z}(B(X,Y)^T) = G(Z,X,Y)^T - \overline{\gamma}(B(\nabla_{Z}X,Y),N)N - \overline{\gamma}(B(X,\nabla_{Z}Y),N)N \]
\[ + \overline{\gamma}(B(X,Y),N)AZ + B(\nabla_{Z}X,Y) + B(X,\nabla_{Z}Y) + g(AZ,X)B(N,Y) + g(AZ,Y)B(X,N) \] (4.15)

Using (4.15) we calculate
\[ \text{div}(AB(W,\tilde{W})^T) = \sum_{i=1}^{6} g[(\nabla_i A)(B(W,\tilde{W})^T),e_i] + g(\nabla_{e_i}(B(W,\tilde{W})^T),Ae_i) \]
\[ = \sum_{i=1}^{6} g((G(e_i, W, \tilde{W})^T - \overline{\gamma}(B(\nabla_{e_i}W,\tilde{W}^T),N)N \]
\[ - \overline{\gamma}(B(W,\nabla_{e_i}\tilde{W}),N)N + \overline{\gamma}(B(W,\tilde{W}),N)Ae_i + B(\nabla_{e_i}W,\tilde{W}) + B(W,\nabla_{e_i}\tilde{W}) + g(\nabla_{e_i},W)B(N,\tilde{W}) + g(\nabla_{e_i},\tilde{W})B(W,N),Ae_i] \]
\[ = \sum_{i=1}^{6} \overline{\gamma}(B(N,W,\tilde{W})g(Ae_i,Ae_i) + g(-f_{e_i} + sAe_i,\tilde{W}),Ae_i) \]
\[ + g(B(W,-\tilde{f}_{e_i} + sAe_i),Ae_i) + g(e_i,AW)g(B(N,\tilde{W}),Ae_i) + g(e_i,A\tilde{W})g(B(W,N),Ae_i) \]
\[ = |A|^2h + g(AW,A\tilde{W}) - g(A\tilde{W},A\tilde{W}) \] (4.16)

where we have used Remark 3.3 to eliminate the first term inside the summation in the second equality, Proposition 3.2 in the third equality and the facts that \( \overline{\gamma}(B(a,b),c) = \varphi(a,b,c) \) and \( Ae_i = a_e e_i \) in going from the third to last equality.

For calculating \( \text{div}(G(N,W,\tilde{W})) \), we first of all note that due to Proposition 3.2, \( G(N,X,Y) \) is tangent to \( M \) for any \( X,Y \in \Gamma(TM) \). We calculate
\[ \text{div}(G(N,W,\tilde{W})) = \sum_{i=1}^{6} g(\nabla_i(G(N,W,\tilde{W})),e_i) \]
\[ = \sum_{i=1}^{6} [\nabla_i (g(G(N,W,\tilde{W})),e_i) - g(G(N,W,\tilde{W}),\nabla_i e_i)] \]
\[ = \frac{\tau_0}{4} \sum_{i=1}^{6} [(\nabla_i \psi)(N,W,\tilde{W},e_i) + \psi(\nabla_i N,W,\tilde{W},e_i) + \psi(N,\nabla_i W,\tilde{W},e_i) \]
\[ + \psi(N,W,\nabla_i \tilde{W},e_i) + \psi(N,W,\tilde{W},\nabla_i e_i) - \psi(N,W,\tilde{W},\nabla_i e_i)] \]
\[ = \frac{\tau_0}{4} \sum_{i=1}^{6} [(\nabla_i \psi)(N,W,\tilde{W},e_i) - \psi(Ae_i,W,\tilde{W},e_i)] \]
\[ = \frac{\tau_0}{4} \sum_{i=1}^{6} \frac{\tau_0}{4} [g(\xi \tilde{W},W) - g(\xi W,\tilde{W}) + \sum_{i=1}^{6} g(e_i,e_i)g(\xi W,\tilde{W})] \]
\[ = \frac{\tau_0^2}{4} h \] (4.17)
where we have used Proposition 3.2 in the third equality, (4.2), (4.5), \( Ae_i = a_i e_i \) and the fact that \( \psi \) is a 4-form to eliminate the \( \psi(N, \nabla W, \tilde{W}, e_i) \) and \( \psi(N, W, \nabla_i \tilde{W}, e_i) \) in the third equality, (2.22) (expression for \( \nabla_i \psi_{jklm} \)) and the fact that for nearly \( G_2 \) structures \( T_{ij} = \tilde{\tau}_{ij} \) in the fifth equality.

Using the fact that for unit \( S^7 \), \( \tau_0 = 4 \), (4.8), (4.9), (4.10), (4.11), (4.12), (4.16) and (4.17) we see that

\[
\Delta_M h = -4h - |A|^2 h - g(AW, A\xi \tilde{W}) + g(A\tilde{W}, A\xi W) - h + g(AW, A\xi \tilde{W})
\]

\[
- h - g(A\tilde{W}, A\xi W) \quad (4.18)
\]

so

\[
\Delta_M h = -(|A|^2 + 6)h \quad (4.19)
\]

Now if \( h \) is a constant function then (4.19) implies that \( h = 0 \), i.e., \( g(\xi(W), \tilde{W}) = 0 \). Recall that \( \tilde{W} \) is the tangential component of a non-Killing conformal vector field \( \tilde{V} \) on \( S^7 \) where \( \tilde{V} \) is the tangential component of any constant vector field on \( \mathbb{R}^8 \). The vector field \( W \) was obtained in a similar manner by taking the tangential component of a non-Killing conformal vector field \( V \) on \( S^7 \) which was obtained as the tangential component of a constant vector field on \( \mathbb{R}^8 \) which was linearly independent from the constant vector field which gives \( \tilde{V} \). So if \( g(\xi(W), \tilde{W}) = 0 \) for all \( \tilde{W} \), we get \( \xi(W) = 0 \), i.e., \( B(N, W) = 0 \). This is a contradiction because \( \xi \) is invertible and \( N \) is a unit vector. Hence \( \exists \tilde{W}, \tilde{W} \) such that \( h \) is not constant and (4.19) implies that \( h \) is an eigenfunction of \( \Delta_M \) corresponding to the eigenvalue \( \lambda = |A|^2 + 6 \). So if \( |A|^2 > 6 \) then \( \lambda > 12 \) with \( |A|^2 = \lambda - 6 \). The proof of Theorem 1.5 is now complete.

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