ON THE COMPLEMENTED SUBSPACES OF $X_p$.

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ABSTRACT. In this paper we prove some results related to the problem of isomorphically classifying the complemented subspaces of $X_p$. We characterize the complemented subspaces of $X_p$ which are isomorphic to $X_p$ by showing that such a space must contain a canonical complemented subspace isomorphic to $X_p$. We also give some characterizations of complemented subspaces of $X_p$ isomorphic to $\ell_p \oplus \ell_2$.

0. Introduction

Rosenthal [R] introduced the $L_p$ space $X_p$ in 1971. Among its interesting properties are that it contains and is contained in isomorphs of $\ell_p \oplus \ell_2$, but is not isomorphic to a complemented subspace of $\ell_p \oplus \ell_2$. These properties have made $X_p$ rather resistant to standard approaches to classifying its complemented subspaces. For example it was first proved that $X_p$ was primary in [JO2] where the device of simultaneously $L_p$, $L_2$ bounded operators was employed to prove a version of the decomposition method for $X_p$. The problem really is that the $\ell_p$ and $\ell_2$ structures in $X_p$ are mixed in a much different way than they are in $\ell_p \oplus \ell_2$ or $(\sum \ell_2)$. Let us also recall that $X_p$ or really the technique for building $X_p$ is the central device used to construct an uncountable number of separable $L_p$ spaces, [BSR]. Thus a better understanding of $X_p$ is critical for the study of the complemented subspaces of $L_p$.

In order to state precisely our results we need to introduce some special notation. Throughout this paper $w = (w_n)$ will be a sequence of positive real numbers and $2 < p < \infty$. As usual $X_{p,w}$ is the completion of $\{(a_i) : i \in \mathbb{N}, a_i \neq 0 \text{ for finitely many } i\}$ with the norm $||\{a_i\}|| = \max\{|(a_i)|_p, |(a_i)|_{2,w}\}$ where $|(a_i)|_p = [\sum_i |a_i|^p]^{1/p}$ and $|(a_i)|_{2,w} = [\sum_i |a_i|^2w_i^2]^{1/2}$. The Rosenthal space $X_p$ is $X_{p,w}$ where $w = (w_i)$ is such that for every $\epsilon > 0, \sum_{w_i < \epsilon} w_i^{2p/(p-2)} = \infty$. Throughout this paper we will always consider $X_p$ to be the subspace of $(\ell_p \oplus \ell_2)_\infty$ spanned by $((\delta_n + w_n\gamma_n))$ where $(\delta_n)$ and $(\gamma_n)$ are the usual unit vector bases of $\ell_p$ and $\ell_2$, respectively. If $E \subset \mathbb{N}$, the symbol $\omega(E)$ will be used to denote $\sum_{n \in E} w_n^{2p/(p-2)}$ which occurs frequently in computations in $X_p$. We will also need the ratio of 2-norm and $p$-norm, $|x|_2/|x|_p$, which we will denote by $r(x)$. For a subspace $Y$ of $X_p$, define $r(Y) = \sup\{r(y) : y \in Y\}$ and $h(Y) = \inf\{r(y) : y \in Y\}$. In defining functionals on $X_p$ it is convenient to use the inner product induced by the norm $| \cdot |_2$. Thus $\langle (x_n), (y_n) \rangle = \sum x_ny_nw_n^2$.

We will use standard Banach space notation and terminology as may be found in [LT]. Here subspace will mean infinite dimensional closed subspace unless otherwise
noted. The properties of $X_p$ can be found in [LT,4d] or in the original paper of Rosenthal [R].
1. Complemented subspaces of $X_p$ which contain $X_p$ complemented

In this section we will show that a complemented subspace of $X_p$ which contains a complemented subspace isomorphic to $X_p$ contains a canonical complemented copy of $X_p$. In [R] Rosenthal showed that there were nice block bases of the usual basis of $X_p$ with complemented closed linear span. The point of his construction was to make sure that the coordinate functionals of the block basis could be chosen to be bounded in both the $p$ and 2 norms. Explicitly, if for each $j \in \mathbb{N},$

$$y_j = \sum_{n=k_j+1}^{k_{j+1}} w_n^{2/(p-2)} e_n$$

where $(e_n)$ is the natural basis for $X_p$ and $(e_n^*)$ is the corresponding sequence of biorthogonal functionals, then

$$|y_j|_2 = \left[ \sum_{n=k_j+1}^{k_{j+1}} w_n^{2p/(p-2)} \right]^{1/2} \quad \text{and} \quad |y_j|_p = \left[ \sum_{n=k_j+1}^{k_{j+1}} w_n^{2p/(p-2)} \right]^{1/p}.$$  

Let

$$y_j^*(x) = |y_j|_2^{-2} \sum_{n=k_j+1}^{k_{j+1}} w_n^{2/(p-2)} w_n^2 e_n^*(x) = |y_j|_2^{-2} < y_j, x >.$$  

By applying H"older’s inequality first in $\ell_2$ and then in $\ell_{p/2}$ we see that

$$|y_j^*(x)| = |y_j|_2^{-2} \left[ \sum_{n=k_j+1}^{k_{j+1}} w_n^{2/(p-2)} w_n^2 e_n^*(x) \right]^{1/2} \leq |y_j|_2^{-2} \left[ \sum_{n=k_j+1}^{k_{j+1}} w_n^{4/(p-2)} w_n^2 \right]^{1/2} \left[ \sum_{n=k_j+1}^{k_{j+1}} w_n^2 |e_n^*(x)|^2 \right]^{1/2}.$$

$$\leq |y_j|_2^{-2} \left[ \sum_{n=k_j+1}^{k_{j+1}} w_n^{2p/(p-2)} \right]^{1/2} \min \left\{ \left[ \sum_{n=k_j+1}^{k_{j+1}} w_n^2 |e_n^*(x)|^2 \right]^{1/2}, \left[ \sum_{n=k_j+1}^{k_{j+1}} w_n^2 e_n^*(x) \right]^{(p-2)/2p} \right\} \leq \min \left\{ |x|_{[k_j+1,k_{j+1}]}^{-1/2}, |x|_{[k_j+1,k_{j+1}]}^{-1/2} \right\} \cdot |y_j|_2^{-1}.$$

Thus

$$|y_j^*(x) y_j|_2 \leq |x|_{[k_j+1,k_{j+1}]}^{-1} \quad \text{and} \quad |y_j^*(x) y_j|_p \leq |x|_{[k_j+1,k_{j+1}]}^{-1}.$$  

The important point is that this computation works because

$$|y_j|_2 = \left[ \sum_{n=k_j+1}^{k_{j+1}} w_n^{2p/(p-2)} \right]^{(p-2)/2p} \left[ \sum_{n=k_j+1}^{k_{j+1}} w_n^{2p/(p-2)} \right]^{1/p}$$

$$= \left[ \sum_{n=k_j+1}^{k_{j+1}} w_n^{2p/(p-2)} \right]^{(p-2)/2p} |y_j|_p.$$
Remark 1.1: The choice of coefficients for $y_j$ has the following geometric motivation. These coefficients give the maximum for the ratio $|\cdot|_2/|\cdot|_p$ for elements supported on $[k_j + 1, k_j + 1]$. Thus if $x$ is supported on $[k_j + 1, k_j + 1]$ and $x \in B_{\ell_p}$ (the unit ball of $\ell_p$) then $(|y_j|_p/|y_j|_2)x \in B_{|\cdot|_2}$ and there is exactly one $x$ so that the multiple has 2-norm one.

If we replace $y_j$ by any $z_j$ with the same support which satisfies

$$c|z_j|_2 \geq \left[ \sum_{n=k_j+1}^{k_j+1} w_n^{2p/(p-2)} (p-2)/2p \right] z_j|_p = \omega([k_j + 1, k_j + 1])^{p-2)/2p} z_j|_p$$

and define

$$z_j^*(x) = |z_j|_2^{-2} < z_j, x > = |z_j|_2^{-2} \sum_{n=k_j+1}^{k_j+1} e_n^*(z_j)e_n^*(x)w_n^2.$$ 

Applying Hölder’s inequality as above shows that

$$|z_j^*(x)z_j|_2 \leq |x|_{[k_j+1,k_j+1]}_2 \text{ and } |z_j^*(x)z_j|_p \leq c|x|_{[k_j+1,k_j+1]}|_p.$$ 

Combining this observation with the isomorphic classification of subspaces spanned by block basic sequences, we arrive at a prototype for a complemented subspace of $X_p$ isomorphic to $X_{p,w'}$.

Proposition 1.2. If $(z_j)$ is a normalized block basis of the natural basis of $X_p$, $(E_j)$ is a sequence of disjoint subsets of $\mathbb{N}$ and there are positive constants $c$ and $\delta$ such that for all $j$

a) $|z_j|_{E_j} \geq \delta |z_j|_2$

b) $c|z_j|_{E_j} \geq \omega(E_j)^{(p-2)/2p}$

Then $[z_j]$ is a max$\{\delta^{-1}, c\}$ complemented subspace of $X_p$ isomorphic to $X_{p,w'}$, where $w_j' = \omega(E_j)^{(p-2)/2p}$.

Proof. A block basis of the natural basis of $X_p$ spans a subspace isomorphic to $X_{p,w''}$ where $w_j'' = |z_j|_2/|z_j|_p = r(z_j)$. By Hölder’s inequality and b),

$$|z_j|_p \omega(E_j)^{(p-2)/2p} \geq |z_j|_{E_j}|_p \omega(E_j)^{(p-2)/2p} \geq |z_j|_{E_j}|_2 \geq \delta |z_j|_2.$$ 

Hence

$$\delta^{-1} \omega(E_j)^{(p-2)/2p} \geq r(z_j) \geq c^{-1} \omega(E_j)^{(p-2)/2p}$$

and thus $X_{p,w'}$ is isomorphic to $X_{p,w''}$. Define a projection onto $[z_j : j \in \mathbb{N}]$ by

$$Px = \sum_{j=1}^{\infty} z_j^*(x)z_j = \sum_{j=1}^{\infty} |z_j|_{E_j}^{-2} < z_j, x > z_j$$

$$= \sum_{j=1}^{\infty} |z_j|_{E_j}^{-2} \left[ \sum e_n^*(z_j)e_n^*(x)w_n^2 \right] z_j.$$
Clearly $P$ is the required operator if it is bounded. The computations above using b) in the form

$$(c|z_j|E_j |p|z_j|E_j |2 \geq \omega(E_j)^{(p-2)/2p}|z_j|E_j |p)$$

show that

$$||Px|| \leq \max\{\sum_{j=1}^{\infty} |z_j^*(x)z_j|_2^2, \sum_{j=1}^{\infty} |z_j^*(x)z_j|_p^p\}^{1/2}$$

$$= \max\left\{\sum_{j=1}^{\infty} |z_j^*(x)z_j|_2^2 |z_j|_E^2 |z_j|_E^2\right\}^{1/2}, \sum_{j=1}^{\infty} |z_j^*(x)z_j|_p^p |z_j|_p^p |z_j|_E |p|^{1/2}\right\}^{1/p}\right\}^{1/p}$$

$$\leq \max\left\{\sum_{j=1}^{\infty} |x|E_j |2^2 \delta^{-2}, \sum_{j=1}^{\infty} |x|E_j |p|^{1/p}|c^p\right\}^{1/p} \leq \max\{|\delta^{-1}, c| ||x||| \right\} \square$$

Next we will prove our characterization of the complemented subspaces of $X_p$ which contain $X_p$ complemented and thus are isomorphic to $X_p$, [JO2].

**Theorem 1.3.** Suppose that $X$ is a complemented subspace of $X_p$. Then the following are equivalent.

1) $X$ contains a complemented subspace isomorphic to $X_p$

2) There exist positive constants $c$ and $\delta$ such that for every $\epsilon > 0$ there is an $\epsilon', 0 < \epsilon' < \epsilon$, such that for every $N \in \mathbb{N}$ there is an $x \in X$, $||x|| = 1$ and a finite set $E \subseteq \{N, N + 1, \ldots\}$ such that

a) $||x||_{1,N} < N^{-1}$

b) $|x|E_2 \geq \delta |x|_2$

c) $\epsilon \geq c|x|_E |2 \delta^{-2} \geq \omega(E)^{(p-2)/2p} \geq \epsilon'$

**Proof.** Suppose that 2) is satisfied. Let $\epsilon_k = k^{-1}$. By induction we may choose for each $k$ a sequence $(x_{k,j})$ of norm one elements of $X$ which are a perturbation of a block basis of the basis of $X_p$ satisfying b) and c) (for $\epsilon'_k$ and $E_{k,j}$). Clearly we may assume that $E_{k,j} \cap E_{k,m} = \phi$ for $j \neq m$. By a simple diagonalization argument we can find sets $\mathcal{F}_k \subseteq \mathbb{N}$ such that $(x_{k,j})_{k=1, j \in \mathcal{F}_k}$ is equivalent to a block basis of $X_p$, the sets $E_{k,j}$, $k \in \mathbb{N}$, $j \in \mathcal{F}_k$ are disjoint, and $(\epsilon'_k)^{2p/(p-2)} |c|$ $\mathcal{F}_k \geq 1$ for each $k$. It now follows from Proposition 1.2 and standard perturbation arguments that $Y = \{x_{k,j} : k \in \mathbb{N}, j \in \mathcal{F}_k\}$ is isomorphic to $X_p$ and that $Y$ is complemented in $X_p$.

For the converse we will actually show that if we take as our isomorph of $X_p$ the special representation $X_{p,w'}$, where $w'$ is actually a doubly indexed sequence $w' = (w_{k,j})$, where $w_{k,j} = w_k$ for all $j$, $\lim_n w_k = 0$, and $\sum_{k=1}^{\infty} w_k^{2p/(p-2)} = \infty$, then the images of a subsequence of the basis satisfy the properties in 2). Thus we suppose that $Y$ is a complemented subspace of $X$ and that $T$ is an isomorphism of $X_{p,w'}$ onto $Y$. By passing to a subsequence of the basis of $X_{p,w'}$ and using a standard perturbation argument, we may assume that $Y$ is the span of a block of the basis of the containing $X_p$. Let $(y_i)$ be the normalized basis of $Y$ and let $F_i$ be the support of $y_i$, relative to the basis of $X_p$. Let $y_i^*$ denote the biorthogonal functional to $y_i$. Because $Y$ is complemented in $X_p$, we may assume that each $y_i^*$ is defined on $X_p$ and $\sup \{||y_i^*|| \leq ||T^{-1}|| ||Q|| \}$ where $Q$ is the projection onto $Y$. Because $Y$ is reflexive, $y_i^*(e_j) = 0 \iff j \neq i$ for each $i$ where $e_j$ denotes the $j$th basis vector.
of $X_p$. Thus we may assume by passing to a subsequence, a perturbation argument and perhaps enlarging the sets $F_i$ slightly that $y^*_i(x) \neq 0$ only if $x_{|F_i} \neq 0$. In other words $y^*_i(x) = y^*_i(x_{|F_i})$. Also it follows from this that the projection $Q$ onto $Y$ is given by $Q x = \sum_{i=1}^\infty y^*_i(x) y_i$.

Fix $i$ and let $E_i = \{ j \in F_i : |y_i(j)| \geq \rho \|w_j\|^{(p-2)/2} |y_i|_2^{-1/(p-2)} \}$ and assume that b) is satisfied for $y_i$ and $E_i$.

$$\omega(E_i) = \sum_{j \in E_i} w_j^{2(p-2)/p} \leq \rho^{-2} \sum_{j \in E_i} |y_i(j)|^2 |y_i|_2^{4/(p-2)} w_j^2$$

$$= \rho^{-2} |y_i|_E^2 |y_i|_2^{4/(p-2)}$$

$$\leq \rho^{-2} \delta^{-1/(p-2)} |y_i|_E^{2(p-2)/(p-2)}.$$ 

Thus if $\rho$ is independent of $i$, condition b) will imply the middle inequality in condition c) (with $c = \rho^{-2(p-2)/p} \delta^{-2/p}$). Also observe that because

$$|y_i|_p \omega(E_i)^{(p-2)/2p} \geq |y_i|_{E_i^*} \omega(E_i)^{(p-2)/2p} \geq |y_i|_{E_i} |y_i|_2 \geq \delta |y_i|_2$$

the third inequality in c) will be satisfied if $|y_i|_2$ is bounded away from zero. Hence it is sufficient to show that for some $\rho > 0$ there is a $\delta$, $0 < \delta < 1$, such that if $E_\delta = \{ i : |y_i|_{E_i} \geq \delta |y_i|_2 \}$ then for every $\epsilon_1 > 0$ there is an $\epsilon_2 > 0$ such that $\epsilon_1 \geq |y_i|_2 \geq \epsilon_2$ for infinitely many $i$ in $E_\delta$. Then c) will be satisfied with $c \epsilon_1 = \epsilon$ and $\epsilon' = \delta \epsilon_2$.

Note that

$$\sum_{j \notin E_i} |y_i(j)|^p \leq \sum_{j \notin E_i} |y_i(j)|^2 \rho^{p-2} w_j^2 |y_i|_2^{-2} \leq \rho^{-2}.$$ 

Hence $||y_i||_{E_i} \geq [1 - \rho^{p-2}]^{1/p}$ and $|y_i|_{F_i \setminus E_i} \leq \rho^{1-2/p}$. Thus if $\rho$ is small, $[y_i]_{E_i^*} : i \notin E_\delta$ is not better than $\rho^{-1+2/p}$ equivalent to $y_i : i \notin E_\delta$.

For each $K$ define $M_K = \{ i : |y_i^*|_{(y_i|_{F_i})} \leq K |y_i|_{E_i} |y_i|_2 / |y_i|_2 \}$. Observe that because $||y_i^*|| \leq ||T^{-1}|| |Q|$, $\max\{ |y_i|_{F_i \setminus E_i} |y_i|_2, |y_i|_{E_i} |y_i|_p \}$

$$\geq ||T^{-1}|| |Q| \max\{ |y_i|_2, \rho^{1-2/p} \}.$$

Thus if we consider only those $y_i$ with $|y_i|_2 \leq \rho^{1-2/p}$ we have that

$$|y_i^*|_{(y_i|_{E_i})} \geq 1 - ||T^{-1}|| |Q| \rho^{1-2/p}.$$ 

Under our assumption on the sequence $(y_i)$, the span of such $y_i$ is still isomorphic to $X_{p,w}$. From now on we will assume that $\rho^{1-2/p} \leq (||T^{-1}|| |Q| 2)^{1-2/p}$ and thus that $|y_i^*|_{(y_i|_{E_i})} \geq 1/2$ for all $i$. In this way we can work with $M_K$ instead of $E_\delta$ since for such $i$ if $i \in M_K$ then $i \in E_{1/2 K}$.

Let us now see how the projection onto $[y_i]$ acts on the span of $[y_i|_{E_i}]$. Our assumptions on the $y_i$’s imply that $Q \sum_{i=1}^\infty a_i y_i|_{E_i} = \sum_{i=1}^\infty a_i y_i^*|_{(y_i|_{E_i})} y_i$. Hence

$$|Q \sum_{i=1}^\infty a_i y_i|_{E_i} |y_i|_2 = |\sum_{i=1}^\infty a_i y_i^*|_{(y_i|_{E_i})} y_i |y_i|_2$$

$$\geq K \left( \sum |a_i|^2 |y_i|_{E_i} |y_i|_2^{-2} |y_i|_2 \right)^{1/2} = K \left( \sum |a_i|^2 |y_i|_{E_i} |y_i|_2 \right)^{1/2}.$$
If \( a_i = |y_i|_{E_i}^{2/(p-2)} \), for \( i \notin \mathcal{M}_K \) and \( i \leq N \), and 0 else, then

\[
\left| \sum_{i=1}^{N} a_i y_i |E_i| \right| = \max \left\{ \left[ \sum_{i \notin \mathcal{M}_K} |a_i|^p |y_i|_{E_i}^p \right]^{1/p}, \left[ \sum_{i \notin \mathcal{M}_K} |a_i|^2 |y_i|_{E_i}^2 \right]^{1/2} \right\}
\]

\[
\leq \max \left\{ \left[ \sum_{i \notin \mathcal{M}_K} |y_i|_{E_i}^{2p/(p-2)} |y_i|^p \right]^{1/p}, \left[ \sum_{i \notin \mathcal{M}_K} |y_i|_{E_i}^{2p/(p-2)} \right]^{1/2} \right\}
\]

\[
\leq \sum_{i \notin \mathcal{M}_K} \left[ |y_i|_{E_i}^{2p/(p-2)} \right]^{1/2}, \text{ if } \sum_{i \notin \mathcal{M}_K} |y_i|_{E_i}^{2p/(p-2)} \geq 1.
\]

This implies that

\[
||Q|| \left[ \sum_{i \notin \mathcal{M}_K} |y_i|_{E_i}^{2p/(p-2)} \right]^{1/2} \geq K \left[ \sum_{i \notin \mathcal{M}_K} |y_i|_{E_i}^{2p/(p-2)} \right]^{1/2},
\]

\[
\text{if } \sum_{i \notin \mathcal{M}_K} |y_i|_{E_i}^{2p/(p-2)} \geq 1. \text{ Therefore, if } K \geq ||Q||,
\]

\[
\sum_{i \notin \mathcal{M}_K} |y_i|_{E_i}^{2p/(p-2)} \leq 1.
\]

Because \( |y_i|_{E_i}|_p \geq \left[ 1 - \rho^{p-2} \right]^{1/p} \), (We are assuming that \( \rho < 1 \)) this implies that

\[
[y_i]_{E_i}|_p \geq [y_i]_{E_i}^{-1} : i \notin \mathcal{M}_K \]

is equivalent to the basis of \( \ell_p \). Therefore for any \( \epsilon' \) small enough only finitely many of the \( y_i \)'s with \( |y_i|_2 \geq \epsilon \) have index not in \( \mathcal{M}_K \). Indeed, if this were not the case, then there would be a subsequence of \( (y_i) \), say \( (y_i)_{i \in M} \), such that \( M \subset \mathcal{M}_K \) and \( \epsilon' > 0 \), \( |y_i|_2 \geq \epsilon \) for all \( i \in M \). However this would imply that \( Q \) is an isomorphism from \( [y_i]_{E_i} : i \in M \), which is isomorphic to \( \ell_p \), onto \( [y_i : i \in M] \), which is isomorphic to \( \ell_2 \).

It follows that \( (y_i)_{i \in \mathcal{M}_K} \) is equivalent to the basis of \( X_{\rho, w'} \), where \( w' = (w'_i) \) and for each \( \epsilon > 0 \) there is an \( \epsilon' > 0 \) such that \( \epsilon > w'_i \geq \epsilon' \), for infinitely many \( i \). Note that because \( |y_i|_{E_i}^s (y_i |E_i|) \geq 1/2 \), for any \( i \in \mathcal{M}_K \), \( |y_i|_{E_i}^s \geq \delta |y_i|_2 \), where \( \delta = 1/2K \), i.e., \( i \in \mathcal{E}_\delta \).

**Remark 1.4**: If \( X \) is not isomorphic to \( X_p \) we can use part of the proof that 1) implies 2) to get a natural way of splitting vectors in \( X \) into a piece with large ratio and a piece with small ratio. Indeed suppose \( X \) is a complemented subspace of \( X_p \) and that \( P \) is the projection onto \( X \). By \( [JJ] \) or \( [JO2] \) we may assume that \( P \) is bounded in the norm \( | \cdot |_2 \) as well. Suppose that 2) fails for \( \delta < |P|_2^{-1}, c \) and \( \epsilon \). Choose positive constants \( \epsilon' \), \( \rho \), \( \alpha \), and \( \beta \) such that

\[
\epsilon' < \min \{ \epsilon, \delta \alpha \},
\]

\[
\beta < \min \{ (1 - \delta |P|_2)/(||P||), \epsilon/c \},
\]

\[
\rho \leq \min \{ c^{-p/(p-2)} \delta^{p/(p-2)}, \beta^{p/(p-2)} \},
\]

\[
\beta > c > \max \{ \beta | \cdot | P, (1 - \beta | \cdot | P), \beta^2 | \cdot | P/(1 - \delta |P|_2) \}.
\]
Let \( N \) be an integer so that a), b), and c) of 2) fail for \( \epsilon' \) and \( N \) and suppose that \( x \in X, \ x|[1,N] = 0, \alpha < r(x) < \beta \) and \( \|x\| = 1 \). Let \( E_x = \{ j : |x(j)| \geq \rho w_j^{2/(p-2)}|x|_2^{-2/(p-2)} \} \). As in the proof above the choice of \( \rho \) guarantees that the middle inequality in 2) c) is satisfied by \( x|_{E_x} \). Because \( r(x) < \beta \leq \epsilon/c, \) the first inequality in 2) c) is also satisfied. Finally if \( |x|_{E_x} \geq \delta |x|_2, \ \omega(E_x)^{(p-2)/2p} = |x|_{E_x} \geq \delta |x|_2 \geq \delta \alpha \geq \epsilon' \) and thus all of the inequalities in c) are satisfied. The failure of 2) then implies that \( |x|_{E_x} < \delta |x|_2 \).

Let \( y = P(x|_{E_x}) \) and \( z = x - y = P(x|_{cE_x}) \). We claim that \( r(y) \leq \alpha \) and \( r(z) \geq \beta \). Indeed,

\[
|y|_2 \leq |P|_2|x|_{E_x} < |P|_2 \delta |x|_2 \leq |P|_2 \delta \beta \leq \alpha (1 - \beta \|P\|) \leq \alpha (|x|_p - \|P\| \|x|_{E_x} \|) \leq \alpha |y|_p
\]

since \( |x|_{E_x} \|_p \leq \rho^{(p-2)/p} \leq \beta \) and \( |x|_2 \leq \beta \), and

\[
|z|_2 \geq (1 - \delta |P|_2) \alpha \geq (1 - \delta |P|_2) \beta^2 \|P\|/(1 - \delta |P|_2) \geq \beta \|P\| \|x|_{E_x} \| \geq \beta |z|_p.
\]

Thus any \( x \in X \) with support in \( \{N + 1, N + 2, \ldots \} \) can be split into an element with ratio greater than \( \beta \) and one with ratio smaller than \( \alpha \). If this could be accomplished in a linear fashion it would follow that \( X \) is then isomorphic to a complemented subspace of \( \ell_p \oplus \ell_2 \).
2. Complemented subspaces of $X_p$ which are isomorphic to $\ell_p \oplus \ell_2$

In this section we look at some ways of discriminating between complemented subspaces of $X_p$ which are isomorphic to complemented subspaces of $\ell_p \oplus \ell_2$ and those isomorphic to $X_p$. First we will examine how the conditions in Theorem 1.3 fail if $X$ is isomorphic to $\ell_p \oplus \ell_2$. Below $P_n$ denotes the basis projection onto the span of the first $n$ elements of the basis of $X_p$.

**Proposition 2.1.** Suppose that $Z$, $X$, $U$, and $W$ are subspaces of $X_p$ such that $Z \subset X = U \oplus W$, $U$ is isomorphic to $\ell_2$, and $W$ is isomorphic to $\ell_p$. Suppose that $Z$ has a normalized $K$ unconditional basis $(z_n)$. Let $eta = \lim_{n \to \infty} r(z_n)$ and $eta' = \lim_{n \to \infty} \inf\{b : \text{for every } \epsilon > 0 \text{ there exists } u \in U \text{ such that } ||P_n u|| < \epsilon \text{ and } r(u) \leq b\}$. If $\beta > 0$, $\beta' \leq 1$, and $P$ is a projection from $X$ onto $Z$ then $||P|| \geq \beta'/K\beta$.

**Proof.** Let $z_n = u_n + w_n$ where $u_n \in U$ and $w_n \in W$. By passing to subsequences and a standard perturbation argument we may assume that $(u_n)$ and $(w_n)$ are block bases of the basis of $X_p$. (It could happen that $||w_n|| \to 0$, but then $\beta' \leq \beta$.) Moreover we may assume that the projection $P$ composed with the corresponding basis projection $Q$ acts disjointly with respect to the subsequence $(z_n)_{n \in M}$, i.e., $QP$ is a projection onto $[z_n : n \in M]$ and $QP u_n = \tau_n z_n$ and $QP w_n = (1 - \tau_n) z_n$. Because $(w_n)$ is equivalent to the usual unit vector basis of $\ell_p$, $p > 2$, $\beta \geq 0$, and $(z_n)$ is equivalent to the usual unit vector basis of $\ell_2$, it follows that $\tau_n \to 1$.

Because $W$ is isomorphic to $\ell_p$, $||w_n|| \to 0$ and thus $||u_n|| - ||z_n|| \to 0$. Therefore

$$\lim \sup ||u_n|| = \lim \sup \max\{||u_n||, ||u_n||_p\} \leq \lim \sup \max\{\beta, ||u_n||/r(u_n)\} \leq \max\{\beta, \beta/\beta'\} = \beta/\beta'. $$

Consequently $K||P||\beta/\beta' \geq \lim \sup ||Q||/||P|| \geq 1$. □

**Corollary 2.2.** Suppose that $X$, $U$, and $W$ are subspaces of $X_p$ which satisfy the hypotheses of Proposition 2.1 and $X$ is complemented in $X_p$ with projection $P$. Then for any $c$ and $\delta$ and $\epsilon < \beta' c \delta / \max\{c, \delta^{-1}\}$, there is no $\epsilon'$, $0 < \epsilon' < \epsilon$, such that for every $N \in \mathbb{N}$ there is an $x \in X$, $||x|| = 1$ and a finite set $E \subseteq \{N, N + 1, \ldots\}$ such that

a) $||x||_{[1, N]} < N^{-1}$
b) $||x||_E \geq \delta ||x||$
c) $\epsilon \geq c ||x||_E \geq \omega(E)^{2/2p} \geq \epsilon'$.

**Proof.** Suppose $\epsilon'$ exists for some $c$, $\delta$, and $\epsilon$. Then there is a sequence $(z_n)$ of norm one vectors in $X$ which is a perturbation of a block basis of the basis of $X_p$ and disjoint sets $(E_n)$ such that for all $n$

b) $||z_n|| \geq \delta ||z_n||$
c) $\epsilon \geq c ||z_n|| \geq \omega(E)^{2/2p} \geq \epsilon'$.

Then $[z_n : n \in \mathbb{N}]$ is complemented in $X_p$ by a projection of norm at most $\max\{c, \delta^{-1}\}$ and $r(z_n) \leq \epsilon/c\delta$. Thus by the previous proposition $||P|| \geq \beta' c \delta / \epsilon$ and hence

$$\epsilon \geq \beta' c \delta / \max\{c, \delta^{-1}\}.$$

We now turn our attention to the classification of the complemented subspaces of $X_p$. It was shown in [JO2] that if a complemented subspace of $X_p$ has an unconditional basis then it is isomorphic to $\ell_p \oplus \ell_2 \oplus \ell_\infty$ or $X_p$. In [AC] the same
conclusion was established if $X$ has a “$p$, 2 F.D.D.” Thus it seems likely that same result holds without the additional assumptions. We will next look at some well known results but recast in terms of the ratio of the 2-norm and $p$-norm.

To begin let us recall that results of Kadec and Pelczynski [KP] give a natural criterion for isomorphs of $\ell_2$ contained in $X_p$, $p > 2$, namely, a subspace $X$ of $X_p$ is isomorphic to $\ell_2$ if and only if there is a constant $C > 0$ such that $r(x) = |x|_2/|x|_p \geq C$ for all $x \in X$, i.e., $h(X) \geq C$. It follows from [JO1] that if a complemented subspace of $X_p$ does not contain $\ell_2$ then it is isomorphic to $\ell_p$. A standard gliding hump argument yields the following criterion. (Below $Q_N$ denotes the projection onto the span of the basis vectors of $X_p$ with index greater than $N$.)

**Proposition 2.3.** A complemented subspace $X$ of $X_p$ is isomorphic to $\ell_p$ if and only if for every $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that if $x \in Q_N X$ then $r(x) < \epsilon$.

Theorem 1.3 gives a criterion for identifying complemented subspaces isomorphic to $X_p$, however it seems to be rather difficult to formulate useful conditions which identify complemented subspaces isomorphic to $\ell_2 \oplus \ell_p$ or a complemented subspace of it. Here are some attempts at such criteria.

**Proposition 2.4.** Let $X$ be a complemented subspace of $X_p$ and suppose that $Z$ is a subspace of $X$ and $\epsilon$, $\beta$, and $\beta'$ are positive constants with $\epsilon \leq 1$ such that

a) for all $z \in Z$, $r(z) \geq \beta'$

b) if $x \in X$ and $r(x) > \beta$ then there exists $z \in Z$ such that $|x - z|_2 < \epsilon|x|_2$.

Then $X$ is isomorphic to a complemented subspace of $\ell_p \oplus \ell_2$ and conversely.

**Proof.** Condition a) implies that $Z$ is isomorphic to $\ell_2$. Let $Y$ be the kernel of the orthogonal projection from $X$ onto $Z$. If $y \in Y$ and $r(y) > \beta$ then by b) there exists a $z \in Z$ such that $|y - z|_2 < \epsilon|y|_2$. But $y$ is orthogonal to $z$ so we have that $|y|_2 \leq ||y|_2^2 + |z|_2^2|^{1/2} = |y - z|_2 < \epsilon|y|_2$, an impossibility. Therefore $r(y) \leq \beta$ for all $y \in Y$ and $Y$ is isomorphic to $\ell_p$.

If $X$ is isomorphic to $\ell_2$ or $\ell_p$ the converse follows easily from our earlier observations. Thus by the results of Edelstein and Wojtasczyk [EW] we may assume that $X$ is isomorphic to $\ell_p \oplus \ell_2$ and let $U$ and $W$ be the complementary subspaces with $U$ isomorphic to $\ell_2$ and $W$ isomorphic to $\ell_p$. Because $U$ is isomorphic to $\ell_2$ there is a constant $1 \geq \beta' > 0$ such that for all $u \in U$, $r(u) \geq \beta'$. We may also assume that $W$ is the kernel of the orthogonal projection $Q$ onto $U$.

Then for any $x \in X$,

$$|x - Qx|_2 \leq r(W)||x - Qx|| \leq r(W)(1 + ||Q||)||x|| \leq r(W)(1 + 1/\beta')|x|_2 \max\{1, 1/r(x)\}.$$

Because $W$ is isomorphic to $\ell_p$, it has a basis, let $R_n$ denote $I - Q$ composed with the basis projection onto the span of the first $n$ elements of the basis of $W$. Let $K = \sup||R_n||$. Choose $n$ so large that if $Y = (I - R_n)W, r(Y) < (2(1 + 1/\beta')(1 + K))^{-1}$. Then if $x \in Y + Z$ the above computation shows that b) is satisfied with $\epsilon = 1/2$. Because $R_n$ is finite rank there exists a $\beta > 1$ such that if $r(x) > \beta$ then...
Now if \( r(x) > \beta \),
\[
|x - Qx|_2 \leq (1 + ||Q||) ||Rx|| + ((I - R_n)x - Q(I - R_n)x|_2
\]
\[
\leq (1 + ||Q||) ||x||/4(1 + 1/\beta') + r(Y)||((I - R_n)x - Q(I - R_n)x||
\]
\[
\leq |x|_2 \max\{1, 1/r(x)\}/4 + r(Y)(1 + ||Q||)||I - R_n|| ||x||
\]
\[
\leq |x|_2 \max\{1, 1/r(x)\}/4 + r(Y)(1 + 1/\beta')(1 + K)|x|_2 \max\{1, 1/r(x)\}.
\]
\[
\leq (3/4)|x|_2. \quad \square
\]

**Remark 2.5:** Condition b) may be replaced by

b') if \( x \in X \) and \( r(x) > \beta'' \) then there exists \( z \in Z \) such that \( |x - z|_2 < \epsilon||x|| \).

To see this note that if b') holds then b) holds with \( \beta = \max\{\beta'', \epsilon\} \) and \( \epsilon = 1 \).

To get a similar theorem but with the hypothesis on the \( \ell_p \) part we seem to need to assume the existence of a projection.

**Proposition 2.6.** Let \( X \) be a complemented subspace of \( X_p \) and suppose that \( Y \) is the range of a projection \( P \) on \( X \) and \( \epsilon, \alpha, \text{and} \alpha' \) are positive constants with \( \epsilon \leq ||I - P||^{-1} \) such that

a) for all \( y \in Y \), \( r(y) \leq \alpha' \)

b) if \( x \in X \) and \( r(x) < \alpha \) then there exists \( y \in Y \) such that \( ||x - y|| < \epsilon||x|| \).

Then \( X \) is isomorphic to a complemented subspace of \( \ell_p \oplus \ell_2 \) and conversely.

**Proof.** Condition a) implies that \( Y \) is isomorphic to \( \ell_p \). Let \( Z \) be the kernel of the projection \( P \) from \( X \) onto \( Y \). If \( z \in Z \) and \( r(z) < \alpha \) then by b) there exists a \( y \in Y \) such that \( ||z - y|| < \epsilon||z|| \). But \( Pz = 0 \) and \( Py = y \) so we have that \( ||z|| = ||(I - P)(z - y)|| < ||I - P||\epsilon||z|| \leq ||z|| \), an impossibility. Therefore \( r(z) \geq \alpha \) for all \( z \in Z \) and \( Z \) is isomorphic to \( \ell_2 \).

As in the proof of Proposition 2.6 the converse easily reduces to the case that \( X \) is isomorphic to \( \ell_2 \oplus \ell_p \). So we again let \( U \) and \( W \) be the complementary subspaces with \( U \) isomorphic to \( \ell_2 \) and \( W \) isomorphic to \( \ell_p \) and let \( \beta' \) be a constant such that \( 1 \geq \beta' > 0 \) and for all \( u \in U, r(u) \geq \beta' \). As before we will assume that \( W \) is the kernel of the orthogonal projection \( Q \) onto \( U \).

Then for any \( x \in X \),
\[
||Qx|| = \max\{|Qx|_2, |Qx|_p\} \leq \max\{|Qx|_2, |Qx|_2/\beta'\} \leq Qx, x > 1/2 /\beta'
\]
\[
\leq ||Qx||_{2/\beta'} \leq ||Q||_{1/2} r(x)^{1/2} ||x|| /\beta'.
\]

Thus if \( r(x) < \alpha = \beta^2 ||Q||^{3}, ||x - (I-Q)x|| = ||Qx|| < ||x|| /||I - (I-Q)|| \). Because \( Y \) is isomorphic to \( \ell_p \) there is some \( \alpha' \) such that \( r(y) \leq \alpha' \) for all \( y \in Y \). \( \square \)

**Remark 2.7:** Propositions 2.3, 2.4, and 2.6 do not really use the structure of \( X_p \) and thus can be restated for complemented subspaces of \( L_p \).

Proposition 2.6 should be compared to the following result for \( X_p \) itself.

**Proposition 2.8.** There does not exist a subspace \( Y \) of \( X_p \) and positive constants \( \epsilon \) and \( \alpha, \epsilon < 1 \) such that

a) \( r(Y) < \infty \)

b) if \( x \in X \) and \( r(x) < \alpha \) then there exists a \( y \in Y \) with \( ||x - y|| \leq \epsilon||x|| \).
Proof. Suppose such a subspace exists. Then there is a normalized block basic sequence \((x_n)\) of the \(X_p\) basis such that \(\alpha > r(x_n) > \alpha/2\) for all \(n\) and such that \([x : n \in \mathbb{N}]\) is norm one complemented in \(X_p\) with projection \(P\). By b) for each \(n\) there is an element \(y_n\) of \(Y\) such that \(||x_n - y_n|| < \epsilon \cdot ||x_n||\). Because \(P\) is norm 1, \(||Py_n - x_n|| < \epsilon < 1\). Hence \(||Py_n|| > 1 - \epsilon\). By passing to a subsequence we may assume that \((y_n)\) is equivalent to the usual unit vector basis of \(\ell_p\) and that \((Py_n)\) is equivalent to a block basic sequence in \([x_n : n \in \mathbb{N}]\). But \((x_n)\) is equivalent to the unit vector basis of \(\ell_2\) and hence so is \((Py_n)\). Because \(p > 2\) this is a contradiction. \(\square\)

Remark 2.9: The above proposition fails if \(\epsilon = 1\). In this case the span of a perturbation of a natural basic sequence equivalent to the basis for \(\ell_p\) may be used for \(Y\).
ON THE COMPLEMENTED SUBSPACES OF $X_p$

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