On zeros of some composite functions

Jovita Rašytė

Department of Mathematics and Informatics, Vilnius University
Naugarduko 24, LT-03225 Vilnius
E-mail: jovita.ras@gmail.com

Abstract. We obtain an estimate of the number of zeros for the function $F(\zeta(s + imh))$, where $\zeta(s)$ is the Riemann zeta-function, and $F : H(D) \to H(D)$ is a continuous function, $D = \{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\}$.

Keywords: Riemann zeta-function, universality.

The distribution of zeros of zeta and $L$-functions is the central problem of analytic number theory, and the results in the field allow to solve many other important problems. For example, the location of non-trivial zeros of the Riemann zeta-function $\zeta(s)$, $s = \sigma + it$, has a direct relation to the distribution of prime numbers. The best result in this direction asserts that $\zeta(s) \neq 0$ in the region

$$\sigma > 1 - \frac{c}{(\log |t|)\frac{2}{3}(\log \log |t|)\frac{1}{3}}, \quad |t| \geq t_0 > 0,$$

where $c > 0$ is an absolute constant. We remind that the Riemann hypothesis says that all non-trivial zeros of $\zeta(s)$ lie on the critical line $\sigma = \frac{1}{2}$, thus by this hypothesis, $\zeta(s) \neq 0$ in the half-plane $\sigma > \frac{1}{2}$.

There are the zeta-functions for which the Riemann hypothesis is not true. For example, this holds for the Hurwitz-function $\zeta(s, \alpha)$, $0 < \alpha \leq 1$, defined, for $\sigma > 1$, by

$$\zeta(s, \alpha) = \sum_{m=0}^{\infty} \frac{1}{(m+\alpha)^s},$$

and by analytic continuation elsewhere. If $\alpha$ is a transcendental number, then $[2] \zeta(s, \alpha)$ has zeros in the strip $\frac{1}{2} < \sigma < 1$. Also, the derivative $\zeta'(s)$ has zeros in the strip $0 < \sigma < 1$.

For the investigation of zero-distribution of zeta-functions, universality theorems can be applied. The first universality theorem for the Riemann zeta-function has been proved by S.M. Voronin in [5]. The last version of this theorem is the following:

**Theorem 1.** Suppose that $K$ is a compact subset of the strip $D = \{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\}$ with connected complement, and $f(s)$ is a continuous non-vanishing function on $K$ which is analytic in the interior of $K$. Then, for every $\varepsilon > 0$,

$$\liminf_{T \to \infty} \frac{1}{T} \operatorname{meas} \left\{ \tau \in [0, T] : \sup_{s \in K} |\zeta(s + i\tau) - f(s)| < \varepsilon \right\} > 0.$$
Here $\text{meas}\{A\}$ denotes the Lebesgue measure of a measurable set $A \subset \mathbb{R}$. The proof of Theorem 1 is given, for example, in [1].

Also, a discrete version of Theorem 1 is known. Let $h > 0$ be a fixed number.

**Theorem 2.** Suppose that $K$ and $f(s)$ satisfy the hypotheses of Theorem 1. Then, for every $\varepsilon > 0$,

$$\lim\inf_{N \to \infty} \frac{1}{N+1} \sharp \left\{ 0 \leq m \leq N : \sup_{s \in K} \left| \zeta(s + imh) - f(s) \right| < \varepsilon \right\} > 0.$$ 

In [3], certain discrete universality theorems were obtained for the composite function $F(\zeta(s))$.

We recall some of them. Denote by $H(D)$ the space of analytic functions on $D$ equipped with the topology of uniform convergence on compacta, and set

$$S = \{ g \in H(D) : g^{-1}(s) \in H(D) \text{ or } g(s) \equiv 0 \}.$$ 

**Theorem 3.** Suppose that the number $\exp \left\{ \frac{2\pi k}{h} \right\}$ is irrational for all $k \in \mathbb{Z} \setminus \{0\}$, and that $F : H(D) \to H(D)$ is a continuous function such that, for every open set $G \subset H(D)$, the set $(F^{-1}G \cap S)$ is non-empty. Let $K \subset D$ be a compact subset with connected complement, and let $f(s)$ be a continuous function on $K$ which is analytic in the interior of $K$. Then, for every $\varepsilon > 0$,

$$\lim\inf_{N \to \infty} \frac{1}{N+1} \sharp \left\{ 0 \leq m \leq N : \sup_{s \in K} \left| F(\zeta(s + imh)) - f(s) \right| < \varepsilon \right\} > 0.$$ 

The next theorem is a simplification of Theorem 3.

**Theorem 4.** Suppose that the number $h$, the set $K$ and the function $f(s)$ satisfy the hypotheses of Theorem 3, and that $V > 0$ is such that $K \subset D_V$. Let $F : H(D_V) \to H(D_V)$ be a continuous function such that, for every polynomial $p = p(s)$, the set $(F^{-1}\{p\}) \cap S_V$ is non-empty. Then the assertion of Theorem 3 is true.

Now let $V$ be an arbitrary positive number. Define

$$D_V = \left\{ s \in \mathbb{C} : \frac{1}{2} < \sigma < 1, \, |t| < V \right\}$$

and

$$S_V = \{ g \in H(D_V) : g^{-1}(s) \in H(D_V) \text{ or } g(s) \equiv 0 \}.$$ 

**Theorem 5.** Suppose that the number $h$, the set $K$ and the function $f(s)$ satisfy the hypotheses of Theorem 3, and that $V > 0$ is such that $K \subset D_V$. Let $F : H(D_V) \to H(D_V)$ be a continuous function such that, for every polynomial $p = p(s)$, the set $(F^{-1}\{p\}) \cap S_V$ is non-empty. Then the assertion of Theorem 3 is true.

We note that, differently from Theorem 2, the approximated function in Theorems 3–5 is not necessarily non-vanishing.

The aim of his note is to prove the following statement.
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Theorem 6. Suppose that the number \( \exp\left(\frac{2\pi k}{h}\right) \) is irrational for all \( k \in \mathbb{Z} \setminus \{0\} \), and that the function \( F \) is as in one of Theorems 3–5. Then, for arbitrary \( \sigma_1 \) and \( \sigma_2 \), \( \frac{1}{2} < \sigma_1 < \sigma_2 < 1 \), there exists a constant \( c = c(\sigma_1, \sigma_2) > 0 \) such that the function \( F(\zeta(s + i\text{m}h)) \) has a zero in the disc

\[
|s - \hat{\sigma}| \leq \frac{\sigma_2 - \sigma_1}{2}, \quad \hat{\sigma} = \frac{\sigma_1 + \sigma_2}{2},
\]

for more than \( cN \) numbers \( m, 0 \leq m \leq N \).

First we will remind the Rouché theorem.

Lemma 1. Suppose that \( G \) is a region on the complex plane bounded by a closed continuous contour \( L \). Let \( f_1(s) \) and \( f_2(s) \) be two analytic functions on \( G \), and \( f_1(s) \neq 0 \) and \( |f_2(s)| < |f_1(s)| \) on \( L \). Then the functions \( f_1(s) \) and \( f_1(s) + f_2(s) \) have the same number of zeros on \( G \).

Proof of the lemma can be found, for example, in [4].

Proof of Theorem 6. Let

\[
\sigma_0 = \max\left(\left|\sigma_1 - \frac{3}{4}\right|, \left|\sigma_2 - \frac{3}{4}\right|\right),
\]

\( f(s) = s - \hat{\sigma} \) and \( 0 < \varepsilon < \frac{\sigma_2 - \sigma_1}{20} \). Then, in virtue of Theorems 3–5, there exists a constant \( c = c(\sigma_1, \sigma_2) > 0 \) such that, for sufficiently large \( N \),

\[
\frac{1}{N+1}\left\{0 \leq m \leq N : \sup_{|s - \frac{3}{4}| \leq \sigma_0} |F(\zeta(s + i\text{m}h)) - f(s)| < \varepsilon\right\} > c. \tag{1}
\]

The circle \( |s - \hat{\sigma}| = \frac{\sigma_2 - \sigma_1}{2} \) lies in the disc

\[
|s - \frac{3}{4}| \leq \sigma_0.
\]

Therefore, for \( m \) satisfying (1), we have that

\[
\max_{|s - \hat{\sigma}| = \frac{\sigma_2 - \sigma_1}{2}} |F(\zeta(s + i\text{m}h)) - (s - \hat{\sigma})| < \frac{\sigma_2 - \sigma_1}{20}.
\]

This shows that the functions \( (s - \hat{\sigma}) \) and

\[
F(\zeta(s + i\text{m}h)) - (s - \hat{\sigma})
\]

satisfy the hypotheses of the Rouché theorem in the disc \( |s - \hat{\sigma}| \leq \frac{\sigma_2 - \sigma_1}{2} \). However, the function \( s - \hat{\sigma} \) has precisely one zero \( s = \hat{\sigma} \) in that disc. Therefore, by the Rouché theorem, the function \( F(\zeta(s + i\text{m}h)) \) also has one zero in the disc \( |s - \hat{\sigma}| \leq \frac{\sigma_2 - \sigma_1}{2} \). Since, in view of (1) the number of such \( m, 0 \leq m \leq N \), is larger that \( cN \), this proves the theorem. □

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REZIUMĖ

**Apie kai kurių sudėtinų funkcijų nulius**

Jovita Rašytė

Tarkime, kad $\zeta(s), s = \sigma + it$, yra Rymano dzeta funkcija, $H(D), D = \{s \in \mathbb{C}: \frac{1}{2} < \sigma < 1\}$, yra analizinų funkcijų srityje $D$ erdvė, o $F: H(D) \to (D)$ yra tolydi funkcija. Straipsnyje gautas funkcijos $F(\zeta(s + imh))$ nulių skaičiaus iverčis.

**Raktiniai žodžiai:** Rymano dzeta funkcija, universalumas.