The maximum of the Gaussian $1/f^\alpha$-noise in the case $\alpha < 1$

Zakhar Kabluchko

October 12, 2010

Abstract

We prove that the appropriately normalized maximum of the Gaussian $1/f^\alpha$-noise with $\alpha < 1$ converges in distribution to the Gumbel double-exponential law.

Keywords: $1/f^\alpha$-noise, Extremes, Gaussian processes, Gumbel distribution

AMS 2000 Subject Classification: Primary, 60G15; Secondary, 60G70, 60F05

1 Introduction and statement of the result

$1/f^\alpha$-noise is usually described as a stochastic process whose spectral density is inverse proportional to some power of the frequency. $1/f^\alpha$-noises have been observed experimentally in a huge variety of physical, biological, economic systems and are believed to be ubiquitous in nature. We refer to [7] for a list of references. The aim of the present paper is to find the limiting distribution of the maximum of the $1/f^\alpha$-noise in the case $\alpha < 1$.

Let us be more precise. We define $1/f^\alpha$-noise to be a Gaussian process $\{X_n(t), t \in [-\pi, \pi]\}$ given by a finite random Fourier series

$$X_n(t) = \sum_{k=1}^{n} \sqrt{R(k)}(U_k \sin(kt) + V_k \cos(kt)),$$

where $U_k, V_k$ are independent real-valued standard Gaussian random variables and $R$ is some function regularly varying at $+\infty$ with index $-\alpha$. We will recall necessary facts about regularly varying functions in Section 3 the main example to keep in mind being $R(t) = ct^{-\alpha}$, where $c > 0$. Here we will be interested in the case $\alpha < 1$. In this case, for every $t \in [-\pi, \pi]$ the series on the right-hand side of (1) diverges as $n \to \infty$ with probability 1. The next theorem is our main result.

Theorem 1.1. Let $X_n$ be the $1/f^\alpha$-noise defined by (1), where $R : (0, \infty) \to [0, \infty)$ is an eventually monotone, regularly varying function with index $-\alpha$,.
where \(-\infty < \alpha < 1\). Let \(\sigma_n^2 = \text{Var} X_n(0)\) and \(c = 2\pi^{\frac{1}{2}} \frac{\alpha c}{1 - \alpha}\). Then, for every \(z \in \mathbb{R}\),

\[
\lim_{n \to \infty} \mathbb{P} \left[ \frac{1}{\sigma_n} \sup_{t \in [-\pi, \pi]} X_n(t) \leq \sqrt{2 \log n} + \frac{1}{\sqrt{2 \log n}} \left( \log \frac{\sqrt{c}}{\sqrt{2\pi}} + z \right) \right] = e^{-e^{-z}}.
\]

**Example 1.1.** Taking \(R(t) = 1\), we obtain a limit theorem for the maximum of a random trigonometric polynomial \(X_n(t) = \sum_{k=1}^{n} (U_k \sin(kt) + V_k \cos(kt))\).

**Remark 1.1.** The assumption \(\alpha < 1\) is crucial for the validity of Theorem 1.1. The case \(\alpha > 1\) is not interesting since in this case the series on the right-hand side of (1) converges uniformly with probability 1; see [5, Ch. VII, §1.2]. This immediately implies that the maximum of \(X_n\) converges weakly (without any normalization) to the maximum of the corresponding infinite series. Much more interesting is the case \(\alpha = 1\). A non-Gumbel limiting distribution for the maximum of the \(1/f\)-noise has been derived by non-rigorous methods in the physical literature [3]. The maximum of the \(1/f\)-noise is believed to behave similarly to the maxima of other “logarithmically correlated” fields including the two-dimensional discrete Gaussian Free Field and the Branching Brownian Motion. It has been shown recently that the maximum of the two-dimensional Gaussian Free Field recentered by its mean is tight [2]. It seems that the methods of [2] can be applied to the \(1/f\)-noise, but we will not do this here.

The rest of the paper is devoted to the proof of Theorem 1.1. Throughout, \(C\) is a large positive constant whose value may change from line to line.

## 2 Method of the proof

The idea of our proof of Theorem 1.1 is to rescale the \(1/f\alpha\)-noise in time in such a way that it becomes close to a stationary Gaussian process with differentiable sample paths. The limiting distribution for the maximum of such processes is recalled in the next theorem, see [6, Thm. 8.2.7].

**Theorem 2.1** ([6]). Let \(\{\xi(t), t \in \mathbb{R}\}\) be a stationary zero-mean, unit-variance Gaussian process with a.s. continuous paths. Suppose that the covariance function \(\rho(t) = \text{E}[\xi(0)\xi(t)]\) satisfies the following three conditions:

1. For some \(c > 0\), \(\rho(t) = 1 - ct^2 + o(t^2)\) as \(t \to 0\).
2. \(\lim_{t \to \infty} \rho(t) \log t = 0\).
3. \(\rho(t) < 1\) for \(t \neq 0\).

Then, for every \(z \in \mathbb{R}\),

\[
\lim_{n \to \infty} \mathbb{P} \left[ \sup_{t \in [0, n]} \xi(t) \leq \sqrt{2 \log n} + \frac{1}{\sqrt{2 \log n}} \left( \log \frac{\sqrt{c}}{\sqrt{2\pi}} + z \right) \right] = e^{-e^{-z}}.
\]
The following generalization of the above result to sequences of stationary Gaussian processes is due to Seleznjev [8].

**Theorem 2.2** ([8]). For every \( n \in \mathbb{N} \) let \( \{ \xi_n(t), t \in [-\frac{n}{2}, \frac{n}{2}] \} \) be a stationary zero-mean, unit-variance Gaussian process with a.s. continuous paths and covariance function \( \rho_n(t) = \mathbb{E}[\xi_n(0)\xi_n(t)] \). Suppose that

1. \( \rho_n(t) = 1 - c_n t^2 + \varepsilon_n(t) \), where \( c_n \) is a sequence satisfying \( \lim_{n \to \infty} c_n = c > 0 \) and \( \varepsilon_n(t) \) is a sequence of functions satisfying \( \lim_{t \to 0} \varepsilon_n(t)/t^2 = 0 \) uniformly in \( n \in \mathbb{N} \).
2. For every \( \varepsilon > 0 \) there is \( T = T(\varepsilon) \) such that \( \rho_n(t) \log t < \varepsilon \) for every \( n \in \mathbb{N} \), \( t \in [T(\varepsilon), \frac{n}{2}] \).
3. For some \( n_0 \in \mathbb{N} \) and every \( \varepsilon > 0 \) we have \( \sup_{n>n_0, t \in [\varepsilon, \frac{n}{2}]} \rho_n(t) < 1 \).

Then, for every \( z \in \mathbb{R} \),

\[
\lim_{n \to \infty} \mathbb{P} \left[ \sup_{t \in [-\frac{n}{2}, \frac{n}{2}]} \xi_n(t) \leq \sqrt{2 \log n + \frac{1}{\sqrt{2 \log n}} \left( \log \frac{\sqrt{c}}{2\sqrt{\pi}} + z \right)} \right] = e^{-e^{-z}}.
\]

Note that the conditions of Theorem 2.2 are just uniform versions of the conditions of Theorem 2.1. An application of Theorem 2.2 can be found in [4].

### 3 Facts about regularly varying functions

We need to recall some facts from the theory of regular variation; see [1]. A positive measurable function \( f \) defined on the positive half-axis is called *regularly varying at* \(+\infty\) with index \( \alpha \in \mathbb{R} \) (notation: \( f \in \text{RV}_\alpha \)) if for every \( \lambda > 0 \),

\[
\lim_{x \to +\infty} \frac{f(\lambda x)}{f(x)} = \lambda^\alpha.
\]  \( (2) \)

For example, the function \( f(t) = ct^\alpha \), where \( c > 0 \), is regularly varying with index \( \alpha \). A regularly varying function with index \( \alpha = 0 \) is called *slowly varying*. Any function \( f \in \text{RV}_\alpha \) can be written in the form \( f(t) = L(t)t^\alpha \), where \( L \) is slowly varying.

We will several times need the following result of Karamata [1, Prop. 1.5.8]: if \( f \in \text{RV}_\alpha \) with \( \alpha > -1 \), then

\[
\sum_{k=1}^n f(k) \sim \frac{n f(n)}{(1+\alpha)}, \quad n \to \infty.
\]  \( (3) \)

(Note that Karamata’s theorem is usually stated for the integral \( \int_1^n f(t)dt \), but the discrete version given above is also true). Also, we will need an estimate called Potter bound [1, Thm. 1.5.6]: if \( L \) is slowly varying and bounded away from 0 and \( \infty \) on every compact subset, then for every \( \delta > 0 \) there is a \( C > 0 \) such that

\[
\frac{L(x)}{L(y)} \leq C \max \left( \left( \frac{x}{y} \right)^\delta, \left( \frac{y}{x} \right)^\delta \right), \quad x, y > 0.
\]  \( (4) \)
4 Proof of the main result

Let $X_n$ be a $1/f^\alpha$-noise as in Theorem 1.1. We represent the regularly varying function $R$ in the form $R(t) = L(t)t^{-\alpha}$, where $L$ is slowly varying. The covariance function $r_n(t, s) = \mathbb{E}[X_n(t)X_n(s)]$ of the process $X_n$ is given by

$$r_n(t, s) = \sum_{k=1}^{n} R(k) \cos(k(t-s)), \quad t, s \in [-\pi, \pi].$$  \hfill (5)

In particular, for the variance $\sigma_n^2 = \text{Var} X_n(0)$ we have

$$\sigma_n^2 = \sum_{k=1}^{n} R(k) \sim \frac{nR(n)}{(1-\alpha)}, \quad n \to \infty,$$  \hfill (6)

where the last step is a consequence of (3) and the assumption $R \in \text{RV}_{-\alpha}$ with $\alpha < 1$. For every $n \in \mathbb{N}$ consider a rescaled process $\xi_n$ defined by

$$\xi_n(t) = \frac{1}{\sigma_n} X_n\left(\frac{2\pi t}{n}\right), \quad t \in \left[-\frac{n}{2}, \frac{n}{2}\right].$$  \hfill (7)

Note that $\xi_n$ is a stationary Gaussian process with zero-mean, unit-variance margins. Its covariance function $\rho_n(t) = \mathbb{E}[\xi_n(t)\xi_n(t)]$ is given by

$$\rho_n(t) = \frac{1}{\sigma_n^2} \sum_{k=1}^{n} R(k) \cos \frac{2\pi kt}{n}, \quad t \in \left[-\frac{n}{2}, \frac{n}{2}\right].$$  \hfill (8)

We claim that the sequence $\xi_n$, $n \in \mathbb{N}$, satisfies the assumptions of Theorem 2.2. We start by verifying condition 1. Write

$$\delta_{k,n}(t) = \cos\left(\frac{2\pi kt}{n}\right) - \left(1 - \frac{2\pi^2 k^2 t^2}{n^2}\right).$$  \hfill (9)

Then,

$$\rho_n(t) = 1 - \frac{2\pi^2 t^2}{n^2 \sigma_n^2} \sum_{k=1}^{n} k^2 R(k) \rho_n(t) + \frac{1}{\sigma_n^2} \sum_{k=1}^{n} R(k) \delta_{k,n}(t).$$  \hfill (10)

Note that the function $t^2 R(t)$ is regularly varying with index $2-\alpha > -1$. By (3) and (6) we have

$$c_n := \frac{2\pi^2}{n^2 \sigma_n^2} \sum_{k=1}^{n} k^2 R(k) \sim \frac{2\pi^2}{3-\alpha}, \quad n \to \infty.$$  \hfill (11)

Let us estimate the third term on the right-hand side of (10). By Taylor’s expansion, for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|\delta_{k,n}(t)| < \varepsilon t^2$ for every $n \in \mathbb{N}$, $1 \leq k \leq n$ and $|t| < \delta$. It follows that

$$|\varepsilon_n(t)| := \left|\frac{1}{\sigma_n^2} \sum_{k=1}^{n} R(k) \delta_{k,n}(t)\right| \leq \frac{\varepsilon t^2}{\sigma_n^2} \sum_{k=1}^{n} R(k) = \varepsilon t^2$$  \hfill (12)
uniformly over \( n \in \mathbb{N}, |t| < \delta \). Together with (10) and (11), this proves that condition 1 of Theorem 2.2 holds with \( c = 2\pi^2 \frac{1 - \alpha}{3 - \alpha} \).

Let us now show that condition 2 of Theorem 2.2 is satisfied. To estimate \( \rho_n(t) \) for large \( t \) we need to take into account the oscillating character of the terms on the right-hand side of (8), which suggests performing Abel’s summation. However, it can be shown that a direct application of Abel’s summation leads to a satisfactory estimate for \( \alpha < 0 \) only. So, we need a somewhat more accurate argument.

First of all, we may redefine the function \( R \) on an interval of the form \((0, A)\) to make it monotone on the whole positive half-line and bounded away from 0 on any compact set. Indeed, such a modification changes \( \rho_n(t) \) by at most \( C/\sigma_n^2 \) which is smaller than \( \varepsilon/(2 \log t) \) uniformly in \( t \). So, the modification has no influence on the validity of condition 2 of Theorem 2.2.

Let \( 1 \leq t \leq n^2 \). We will split the sum defining \( \rho_n(t) \) as follows:

\[
\rho_n(t) = \frac{1}{\sigma_n^2} \sum_{k=1}^{[n/t]-1} R(k) \cos \frac{2\pi kt}{n} + \frac{1}{\sigma_n^2} \sum_{k=\lceil n/t \rceil}^{n} R(k) \cos \frac{2\pi kt}{n} =: S_1 + S_2.
\]

(13)

The sum \( S_1 \) can be estimated in a trivial way: using the inequality \( |\cos x| \leq 1 \) and (6), we obtain

\[
|S_1| \leq \frac{1}{\sigma_n^2} \sum_{k=1}^{[n/t]} R(k) = \frac{\sigma_n^2 [n/t]}{\sigma_n^2} \leq C \frac{(n/t) R(n/t)}{R(n)} = C t^{\alpha - 1} \frac{L(n/t)}{L(n)}.
\]

(14)

The sum \( S_2 \) will be estimated by Abel’s summation. We need the Dirichlet kernel

\[
D_k(t) = \sum_{j=1}^{k} \cos(2\pi j t) = \frac{1}{2} \left( \frac{\sin((2k + 1)\pi t)}{\sin(\pi t)} - 1 \right).
\]

(15)

Since \( |t| \leq \frac{\delta}{2} \), we have \( |\sin \frac{2k \pi}{n}| \geq \kappa \frac{n}{t} \) for some \( \kappa > 0 \). It follows that for every \( k \in \mathbb{N} \),

\[
\left| D_k \left( \frac{t}{n} \right) \right| \leq \frac{1}{2} + \frac{1}{\kappa} \frac{n}{t} \leq C \frac{n}{t}.
\]

(16)

Applying Abel’s summation formula to the sum \( S_2 \) we obtain

\[
S_2 = \frac{1}{\sigma_n^2} \sum_{k=\lceil n/t \rceil}^{n-1} D_k \left( \frac{t}{n} \right) (R(k) - R(k + 1)) + \frac{1}{\sigma_n^2} D_{\lceil n/t \rceil - 1} \left( \frac{t}{n} \right) R([n/t]).
\]

(17)
Utilizing (16) and (6) we get that $|S_2|$ can be estimated from above by

$$|S_2| \leq \frac{1}{\sigma_n^2} \sum_{k=[n/t]}^{n-1} \left| D_k \left( \frac{t}{n} \right) \right| |R(k) - R(k + 1)|$$

$$+ \frac{1}{\sigma_n^2} \left| D_n \left( \frac{t}{n} \right) \right| R(n) + \frac{1}{\sigma_n^2} \left| D_{[n/t]-1} \left( \frac{t}{n} \right) \right| R([n/t])$$

$$\leq \frac{C}{tR(n)} \left( \sum_{k=[n/t]}^{n-1} |R(k) - R(k + 1)| + R(n) + R(n/t) \right).$$  \hspace{1cm} (18)

Recall that $R$ is assumed to be monotone. Depending on whether $R$ is decreasing or increasing, the expression in the brackets in (18) can be estimated from above by $2R(n/t)$ or $2R(n)$. Thus,

$$|S_2| \leq C \max \left( \frac{R(n/t)}{tR(n)} \frac{1}{t} \right) = C \max \left( t^{\alpha - 1}L(n/t) \frac{1}{L(n)} \frac{1}{t} \right).$$  \hspace{1cm} (19)

Bringing (13), (14), (19) together and employing Potter’s bound (4) we obtain that for every $\delta > 0$ there is $C > 0$ such that for all $n \in \mathbb{N}$ and $t \in [1, \frac{4}{T}]$,

$$|\rho_n(t)| \leq C \max \left( t^{\alpha - 1}L(n/t) \frac{1}{L(n)} \frac{1}{t} \right) \leq C \max \left( t^{\alpha - 1 + \delta} \frac{1}{t} \right).$$  \hspace{1cm} (20)

Recall that we assume that $\alpha < 1$. Choose $\delta > 0$ so small that $\alpha - 1 + \delta < 0$. The verification of condition 2 is completed.

Let us finally verify condition 3 of Theorem 2.2. Fix $\varepsilon > 0$. By condition 2 there is $T > 0$ such that for all $n \in \mathbb{N}$, $t \in [T, \frac{4}{T}]$ we have $\rho_n(t) < 1/2$. Thus, we have to show that for some $n_0 \in \mathbb{N}$,

$$\sup_{n > n_0, t \in [\varepsilon, T]} \rho_n(t) < 1.$$  \hspace{1cm} (21)

We can find sufficiently small $a > 0$ and $\eta > 0$ such that $|\cos \frac{2\pi kt}{n}| < 1 - \eta$ for all $n \in \mathbb{N}$, $k \in [an, 2an]$, $t \in [\varepsilon, T]$. Recalling (8) we have

$$\rho_n(t) = \frac{1}{\sigma_n^2} \sum_{k=[an, 2an]} R(k) \cos \frac{2\pi kt}{n} + \frac{1}{\sigma_n^2} \sum_{1 \leq k \leq n \atop k \not\in [an, 2an]} R(k) \cos \frac{2\pi kt}{n}.$$  \hspace{1cm} (22)

It follows that for all $n \in \mathbb{N}$ and $t \in [\varepsilon, T]$,

$$|\rho_n(t)| \leq \frac{1}{\sigma_n^2} \sum_{k=[an, 2an]} (1 - \eta)R(k) + \frac{1}{\sigma_n^2} \sum_{1 \leq k \leq n \atop k \not\in [an, 2an]} R(k) = 1 - \frac{\eta}{\sigma_n^2} \sum_{k \in [an, 2an]} R(k).$$  \hspace{1cm} (23)

Applying (8) and (11), we obtain that uniformly in $t \in [\varepsilon, T]$,

$$\limsup_{n \to \infty} |\rho_n(t)| \leq 1 - \eta \lim_{n \to \infty} \frac{2anR(2an) - anR(an)}{nR(n)} = 1 - (2^{1-\alpha} - 1)a^{1-\alpha}\eta < 1.$$  \hspace{1cm} (24)
Hence, there is \( n_0 \in \mathbb{N} \) such that (21) holds. This verifies condition 3 of Theorem 2.2.

To complete the proof of Theorem 1.1, note that \( \sigma_n^{-1} \sup_{t \in [-\pi,\pi]} X_n(t) \) has the same law as \( \sup_{t \in [-\frac{n}{2}, \frac{n}{2}]} \xi_n(t) \) and apply Theorem 2.2.

References

[1] N. H. Bingham, C. M. Goldie, and J. L. Teugels. *Regular variation*, volume 27 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, 1987.

[2] M. Bramson and O. Zeitouni. Tightness of the recentered maximum of the two-dimensional discrete Gaussian Free Field. *Preprint, available at http://arxiv.org/abs/1009.3443*, 2010.

[3] Y. V. Fyodorov and J.-P. Bouchaud. Freezing and extreme-value statistics in a random energy model with logarithmically correlated potential. *J. Phys. A: Math. Theor.*, 41:372001, 2008.

[4] Z. Kabluchko. Limiting distribution of the continuity modulus for Gaussian processes with stationary increments. *Statist. Probab. Lett.*, 79(7):953–956, 2009.

[5] J.-P. Kahane. *Some random series of functions*. D. C. Heath and Co., Lexington, Massachusetts, 1968.

[6] M. R. Leadbetter, G. Lindgren, and H. Rootzén. *Extremes and related properties of random sequences and processes*. Springer Series in Statistics. Springer-Verlag, New York, 1983.

[7] W. Li. A bibliography on 1/f noise. Available online at *http://www.nslilj-genetics.org/wli/1fnoise/*, 1996–present.

[8] O. V. Seleznjev. Limit theorems for maxima and crossings of a sequence of Gaussian processes and approximation of random processes. *J. Appl. Probab.*, 28(1):17–32, 1991.