BIHARMONIC MAPS FROM A COMPLETE RIEMANNIAN MANIFOLD INTO A NON-POSITIVELY CURVED MANIFOLD

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Abstract. We consider a biharmonic map \( \phi : (M, g) \to (N, h) \) from a complete Riemannian manifold into a Riemannian manifold with non-positive sectional curvature. For some \( \varepsilon > 0 \), assuming that \( \alpha \) satisfies \( 1 + \varepsilon < \alpha < \infty \). If such an \( \alpha \), \( \int_M |\tau(\phi)|^\alpha dv_g < \infty \) and \( \int_M |d\phi|^2 dv_g < \infty \), where \( \tau(\phi) \) is the tension field of \( \phi \), then we show that \( \phi \) is harmonic. For a biharmonic submanifold, we obtain that the above assumption \( \int_M |d\phi|^2 dv_g < \infty \) is not necessary. We also consider a complete biharmonic hypersurface \( M \) in a Riemannian manifold \( N \) with non-positive Ricci curvature. For some \( \varepsilon > 0 \), assuming that \( \alpha \) satisfies \( 1 + \varepsilon \leq \alpha < \infty \). If such an \( \alpha \), \( \int_M |H|^\alpha dv_g < \infty \), where \( H \) is the mean curvature of \( M \), then we show that \( M \) is minimal. These results give affirmative partial answers to the global version of generalized Chen’s conjecture.

1. Introduction

Theory of harmonic maps has been applied into various fields in differential geometry. Harmonic maps between two Riemannian manifolds are critical points of the energy functional \( E(\phi) = \frac{1}{2} \int_M |d\phi|^2 dv_g \), for smooth maps \( \phi : (M^m, g) \to (N^n, h) \) from an \( m \)-dimensional Riemannian manifold into an \( n \)-dimensional Riemannian manifold, where \( dv_g \) denotes the volume element of \( g \). The Euler-Lagrange equation of \( E \) is \( \tau(\phi) = \text{Trace}\nabla d\phi = 0 \), where \( \tau(\phi) \) is called the tension field of \( \phi \). \( \phi : (M, g) \to (N, h) \) is called a harmonic map if \( \tau(\phi) = 0 \).

In 1983, J. Eells and L. Lemaire [9] proposed the problem to consider biharmonic maps which are critical points of the bi-energy functional \( E_2(\phi) = \frac{1}{2} \int_M |\tau(\phi)|^2 dv_g \), on the space of smooth maps between two Riemannian manifolds. Biharmonic maps are generalized notion of harmonic maps by definition. In 1986, G. Y. Jiang [14] derived first and second variational formulas of the bi-energy and studied biharmonic maps. The Euler-Lagrange equation of \( E_2 \) is

\[
\tau_2(\phi) = -\Delta^G \tau(\phi) - \sum_{i=1}^m R^N(\tau(\phi), d\phi(e_i))d\phi(e_i) = 0,
\]

where \( \Delta^G := \sum_{i=1}^m (\nabla_{e_i} \nabla_{e_i} - \nabla_{\nabla e_i e_i}) \), and \( \nabla \) is the induced connection on \( \phi^{-1}TN \).

\( \phi : (M, g) \to (N, h) \) is a biharmonic map if \( \tau_2(\phi) = 0 \).

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One of the most interesting problem in the biharmonic theory is Chen’s conjecture. In 1988, B. Y. Chen raised the following problem:

**Conjecture 1** ([6]). *Any biharmonic submanifold in $\mathbb{E}^n$ is minimal.*

Here, we say for a submanifold $M$ in a Riemannian manifold $N$, to be biharmonic if an isometric immersion $\phi : (M, g) \to (N, h)$ is biharmonic. There are many affirmative partial answers to Chen’s conjecture. Chen’s conjecture is solved completely if $M$ is one of the following:

(a) a curve (cf. [8]),
(b) a surface in $\mathbb{E}^3$ (cf. [6]),
(c) a hypersurface in $\mathbb{E}^4$ (cf. [7], [12]).

However, we cannot apply the methods in [6], [7] and [12] to a higher dimensional manifold. In this paper, we try to think about Chen’s conjecture from a different point of view. Here we notice that since there is no assumption of completeness for submanifolds in Chen’s conjecture, in a sense it is a problem in local differential geometry. With these understandings, we reformulate Chen’s conjecture into a problem in global differential geometry (cf. [1], [17]):

**Conjecture 2.** *Any complete biharmonic submanifold in $\mathbb{E}^n$ is minimal.*

On the other hand, Chen’s conjecture was generalized as follows (cf. [5]): ”Any biharmonic submanifold in a Riemannian manifold with non-positive sectional curvature is minimal.” There are also many affirmative partial answers to this conjecture.

(a) Any biharmonic submanifold in $\mathbb{H}^3(-1)$ is minimal (cf. [4]).
(b) Any biharmonic hypersurface in $\mathbb{H}^4(-1)$ is minimal (cf. [5]).
(c) Any compact biharmonic submanifold in a Riemannian manifold with non-positive sectional curvature is minimal (cf. [14], [16]).

However, Y.-L. Ou and L. Tang gave a counterexample of this conjecture (cf. [20]). Note that there are non-minimal, biharmonic submanifolds in a sphere (cf. [14]). With these understandings, it is natural to consider the following problem.

**Conjecture 3.** *Any complete biharmonic submanifold in a Riemannian manifold with non-positive sectional curvature is minimal.*

K. Akutagawa and the author gave an affirmative partial answer to Conjecture 1 (Conjecture 2 particularly) as follows (cf. [1], [15]):

**Theorem 1.1** ([1]). *Any biharmonic properly immersed submanifold in $\mathbb{E}^n$ is minimal.*

Here, an immersed submanifold $M$ in a Riemannian manifold $N$ is said to be *properly immersed* if the immersion is a proper map. Note that the properness of the immersion implies the completeness of $(M, g)$. The author also gave an affirmative partial answer to Conjecture 3 (cf. [16]). N. Nakauchi and H. Urakawa also gave affirmative partial answers to Conjecture 3 (cf. [17], [18]).

There are not such an example that biharmonic maps from complete Riemannian manifolds into Riemannian manifolds with non-positive sectional curvature either.
N. Nakauchi, H. Urakawa and S. Gudmundsson showed the following important result (cf. [19]).

**Theorem 1.2 ([19]).** Biharmonic maps from a complete Riemannian manifold into a non-positively curved manifold with finite bi-energy and energy is harmonic.

One of the main theorems of this paper is the generalized result of Theorem 1.2 as follows (cf. Theorem 3.1).

**Theorem 1.3 (cf. Theorem 3.1).** Let \( \phi : (M, g) \rightarrow (N, h) \) be a biharmonic map from a complete Riemannian manifold \((M, g)\) into a Riemannian manifold \((N, h)\) with non-positive sectional curvature.

(i) For some \( \varepsilon > 0 \), assuming that \( \alpha \) satisfies \( 1 + \varepsilon < \alpha < \infty \). If such an \( \alpha \),

\[
\int_M |\tau(\phi)|^\alpha \, dv_g < \infty,
\]

and

\[
\int_M |d\phi|^2 \, dv_g < \infty,
\]

then \( \phi \) is harmonic.

(ii) In the case \( \text{Vol}(M, g) = \infty \). For some \( \varepsilon > 0 \), assuming that \( \alpha \) satisfies \( 1 + \varepsilon < \alpha < \infty \). If such an \( \alpha \),

\[
\int_M |\tau(\phi)|^\alpha \, dv_g < \infty,
\]

then \( \phi \) is harmonic.

If a biharmonic map \( \phi : (M, g) \rightarrow (N, h) \) is an isometric immersion, that is, \( M \) is a biharmonic submanifold in \( N \), then we have \( \tau(\phi) = mH \), where \( H \) is the mean curvature vector field of \( M \). For biharmonic submanifolds, we obtain that the assumption \( \int_M |d\phi|^2 \, dv_g < \infty \) in the above theorem is not necessary.

**Theorem 1.4 (cf. Theorem 3.4).** Let \( \phi : (M, g) \rightarrow (N, h) \) be a biharmonic isometric immersion from a complete Riemannian manifold \((M, g)\) into a Riemannian manifold \((N, h)\) with non-positive sectional curvature.

For some \( \varepsilon > 0 \), assuming that \( \alpha \) satisfies \( 1 + \varepsilon < \alpha < \infty \). If such an \( \alpha \),

\[
\int_M |H|^\alpha \, dv_g < \infty,
\]

then \( \phi \) is harmonic.

If \( \dim M = \dim N - 1 \), the mean curvature vector field \( H \) is given by \( H = H\xi \), where \( H \) is the mean curvature and \( \xi \) is the unit normal vector field. Another main theorem of this paper is the following (cf. Theorem 5.2).

**Theorem 1.5 (cf. Theorem 5.2).** Let \( \phi : (M^m, g) \rightarrow (N^{m+1}, h) \) be a biharmonic isometric immersion from an \( m \)-dimensional complete Riemannian manifold \((M^m, g)\) into an \( m + 1 \)-dimensional Riemannian manifold \((N^{m+1}, h)\) with non-positive Ricci curvature.
For some $\varepsilon > 0$, assuming that $1 + \varepsilon \leq \alpha < \infty$. If such an $\alpha$,
\[
\int_M |H|^\alpha dv_g < \infty,
\]
then $\phi$ is harmonic.

These results give affirmative partial answers to global version of generalized Chen’s conjecture (Conjecture 3).

The remaining sections are organized as follows. Section 2 contains some necessary definitions and preliminary geometric results. In section 3 we consider a biharmonic map and isometric immersion from a complete Riemannian manifold into a non-positively curved manifold. In section 4 we apply the result of Section 3 to a biharmonic submersion. In section 5 we consider a biharmonic hypersurface in a Riemannian manifold with non-positive Ricci curvature and give an affirmative partial answer to global version of generalized Chen’s conjecture.

2. Preliminaries

In this section, we shall give the definitions of harmonic maps and biharmonic maps. We also recall biharmonic submanifolds.

Let $(M, g)$ be an $m$-dimensional Riemannian manifold and $(N, h)$, an $n$-dimensional Riemannian manifold, respectively. We denote by $\nabla$ and $\nabla^N$, the Levi-Civita connections on $(M, g)$ and $(N, h)$, respectively and $\nabla$ is the induced connection on $\phi^{-1}TN$.

Let us recall the definition of a harmonic map $\phi : (M, g) \to (N, h)$. For a smooth map $\phi : (M, g) \to (N, h)$, the energy of $\phi$ is defined by
\[
E(\phi) = \frac{1}{2} \int_M |d\phi|^2 dv_g.
\]
The Euler-Lagrange equation of $E$ is
\[
\tau(\phi) = \sum_{i=1}^m \left\{ \nabla_{e_i} d\phi(e_i) - d\phi(\nabla_{e_i} e_i) \right\} = 0,
\]
where $\tau(\phi)$ is called the tension field of $\phi$. $\phi : (M, g) \to (N, h)$ is called a harmonic map if $\tau(\phi) = 0$.

In 1983, J. Eells and L. Lemaire [9] proposed the problem to consider biharmonic maps which are critical points of the bi-energy functional on the space of smooth maps between two Riemannian manifolds. In 1986, G. Y. Jiang [14] derived first and second variational formulas of bi-energy and studied biharmonic maps. For a smooth map $\phi : (M, g) \to (N, h)$, the bi-energy of $\phi$ is defined by
\[
E_2(\phi) = \frac{1}{2} \int_M |\tau(\phi)|^2 dv_g.
\]
The Euler-Lagrange equation of $E_2$ is
\[(1) \quad \tau_2(\phi) = -\Delta^\phi \tau(\phi) - \sum_{i=1}^m R^N(\tau(\phi), d\phi(e_i))d\phi(e_i) = 0,\]
where $\tau_2(\phi)$ is called the bi-tension field of $\phi$. $\phi : (M, g) \to (N, h)$ is called a biharmonic map if $\tau_2(\phi) = 0$.

We also recall biharmonic submanifolds.
Let $\phi : (M^m, g) \to (N^n, h = \langle \cdot, \cdot \rangle)$ be an isometric immersion from an $m$-dimensional Riemannian manifold into an $n$-dimensional Riemannian manifold. In this case, we identify $d\phi(X)$ with $X \in \mathfrak{X}(M)$ for each $x \in M$. We also denote by $\langle \cdot, \cdot \rangle$ the induced metric $\phi^{-1}h$. The Gauss formula is given by
\[(2) \quad \nabla^N_X Y = \nabla_X Y + B(X, Y), \quad X, Y \in \mathfrak{X}(M),\]
where $B$ is the second fundamental form of $M$ in $N$. The Weingarten formula is given by
\[(3) \quad \nabla^N_X \xi = -A_\xi X + \nabla^\perp_X \xi, \quad X \in \mathfrak{X}(M), \quad \xi \in \mathfrak{X}(M)^\perp,\]
where $A_\xi$ is the shape operator for a normal vector field $\xi$ on $M$, and $\nabla^\perp$ denotes the normal connection of the normal bundle on $M$ in $N$. It is well known that $B$ and $A$ are related by
\[(4) \quad \langle B(X, Y), \xi \rangle = \langle A_\xi X, Y \rangle.\]

For any $x \in M$, let $\{e_1, \cdots, e_m, e_{m+1}, \cdots, e_n\}$ be an orthonormal basis of $N$ at $x$ such that $\{e_1, \cdots, e_m\}$ is an orthonormal basis of $T_x M$. Then, $B$ is decomposed as
\[B(X, Y) = \sum_{\alpha=m+1}^n B_\alpha(X, Y)e_\alpha, \text{ at } x.\]

The mean curvature vector field $\mathbf{H}$ of $M$ at $x$ is also given by
\[\mathbf{H}(x) = \frac{1}{m} \sum_{i=1}^m B(e_i, e_i) = \sum_{\alpha=m+1}^n H_\alpha(x)e_\alpha, \quad H_\alpha(x) := \frac{1}{m} \sum_{i=1}^m B_\alpha(e_i, e_i).\]

If an isometric immersion $\phi : (M, g) \to (N, h)$ is biharmonic, then $M$ is called a biharmonic submanifold in $N$. In this case, we remark that the tension field $\tau(\phi)$ of $\phi$ is written as $\tau(\phi) = m\mathbf{H}$, where $\mathbf{H}$ is the mean curvature vector field of $M$. The necessary and sufficient condition for $M$ in $N$ to be biharmonic is the following:
\[(5) \quad \Delta^\phi \mathbf{H} + \sum_{i=1}^m R^N(\mathbf{H}, d\phi(e_i))d\phi(e_i) = 0.\]

3. Biharmonic maps into non-positively curved manifolds
In this section, we shall show the following theorem.
Theorem 3.1. Let \( \phi : (M, g) \to (N, h) \) be a biharmonic map from a complete Riemannian manifold \((M, g)\) into a Riemannian manifold \((N, h)\) with non-positive sectional curvature.

(i) For some \( \varepsilon > 0 \), assuming that \( \alpha \) satisfies \( 1 + \varepsilon < \alpha < \infty \). If such an \( \alpha \),

\[
\int_M |\tau(\phi)|^\alpha dv_g < \infty,
\]

and

\[
\int_M |d\phi|^2 dv_g < \infty,
\]

then \( \phi \) is harmonic.

(ii) In the case \( \text{Vol}(M, g) = \infty \). For some \( \varepsilon > 0 \), assuming that \( 1 + \varepsilon < \alpha < \infty \). If such an \( \alpha \),

\[
\int_M |\tau(\phi)|^\alpha dv_g < \infty,
\]

then \( \phi \) is harmonic.

Before to prove Theorem 3.1 we shall show the following lemma.

Lemma 3.2. Let \( \phi : (M, g) \to (N, h) \) be a biharmonic map from a complete Riemannian manifold \((M, g)\) into a Riemannian manifold \((N, h)\) with non-positive sectional curvature.

For some \( \varepsilon > 0 \), assuming that \( \alpha \) satisfies \( 1 + \varepsilon < \alpha < \infty \). If such an \( \alpha \),

\[
\int_M |\tau(\phi)|^\alpha dv_g < \infty,
\]

then \( \nabla_X \tau(\phi) = 0 \) for any vector field \( X \) on \( M \). In particular, \( |\tau(\phi)| \) is constant.

Proof. For a fixed point \( x_0 \in M \), and for every \( 0 < r < \infty \), we first take a cut off function \( \lambda \) on \( M \) satisfying that

\[
\begin{cases}
0 \leq \lambda(x) \leq 1 & (x \in M), \\
\lambda(x) = 1 & (x \in B_r(x_0)), \\
\lambda(x) = 0 & (x \notin B_{2r}(x_0)), \\
|\nabla \lambda| \leq \frac{2}{r} & (x \in M).
\end{cases}
\]

From (6), we have

\[
\int_M \langle -\Delta \tau(\phi), \lambda^2 |\tau(\phi)|^{\alpha - 2} \tau(\phi) \rangle dv_g
\]

(7)

\[
= \int_M \lambda^2 |\tau(\phi)|^{\alpha - 2} \sum_{i=1}^{m} \langle R^N(\tau(\phi), d\phi(e_i)) d\phi(e_i), \tau(\phi) \rangle dv_g \leq 0,
\]
where the inequality follows from the sectional curvature of \((N, h)\) is non-positive. By (7), we have

\[
0 \geq \int_M (-\Delta \phi \rho, \lambda^2 |\tau(\phi)|^{\alpha-2} \tau(\phi)) dv_g
\]

\[
= \int_M (\nabla \tau(\phi), \nabla (\lambda^2 |\tau(\phi)|^{\alpha-2} \tau(\phi))) dv_g
\]

\[
= \int_M \sum_{i=1}^m (\nabla_{e_i} \tau(\phi), (e_i, \lambda^2)|\tau(\phi)|^{\alpha-2} \tau(\phi) + \lambda^2 e_i \{(|\tau(\phi)|^2)^{\frac{\alpha-2}{2}}\} \tau(\phi)
\]

\[
+ \lambda^2 |\tau(\phi)|^{\alpha-2} \nabla_{e_i} \tau(\phi)) dv_g
\]

(8)

\[
= \int_M \sum_{i=1}^m (\nabla_{e_i} \tau(\phi), 2\lambda (e_i, \lambda)|\tau(\phi)|^{\alpha-2} \tau(\phi)) dv_g
\]

\[
+ \int_M \sum_{i=1}^m (\nabla_{e_i} \tau(\phi), \lambda^2 (\alpha - 2)|\tau(\phi)|^{\alpha-4} \langle \nabla_{e_i} \tau(\phi), \tau(\phi) \rangle dv_g
\]

\[
+ \int_M \sum_{i=1}^m (\nabla_{e_i} \tau(\phi), \lambda^2 |\tau(\phi)|^{\alpha-2} \nabla_{e_i} \tau(\phi)) dv_g.
\]

(i) The case of \(\alpha < 2\). We have

\[
\int_M \sum_{i=1}^m \lambda^2 |\tau(\phi)|^{\alpha-2} |\nabla_{e_i} \tau(\phi)|^2 dv_g
\]

(9)

\[
\leq -2 \int_M \sum_{i=1}^m (\nabla_{e_i} \tau(\phi), \lambda (e_i, \lambda)|\tau(\phi)|^{\alpha-2} \tau(\phi)) dv_g
\]

\[
- (\alpha - 2) \int_M \sum_{i=1}^m (\nabla_{e_i} \tau(\phi), \lambda^2 |\tau(\phi)|^{\alpha-4} \langle \nabla_{e_i} \tau(\phi), \tau(\phi) \rangle dv_g.
\]

We shall consider the first term of the right hand side of (9).

\[
-2 \int_M \sum_{i=1}^m (\nabla_{e_i} \tau(\phi), \lambda (e_i, \lambda)|\tau(\phi)|^{\alpha-2} \tau(\phi)) dv_g
\]

\[
= -2 \int_M \sum_{i=1}^m (|\tau(\phi)|^{\frac{\alpha}{2}} \nabla_{e_i} \tau(\phi), (e_i, \lambda)|\tau(\phi)|^{\frac{\alpha}{2}-1} \tau(\phi)) dv_g
\]

(10)

\[
\leq \varepsilon \int_M \sum_{i=1}^m \lambda^2 |\tau(\phi)|^{\alpha-2} |\nabla_{e_i} \tau(\phi)|^2 dv_g
\]

\[
+ \frac{1}{\varepsilon} \int_M (\nabla \lambda)^2 |\tau(\phi)|^\alpha dv_g,
\]

where the inequality of (10) follows from Young’s inequality, that is,

\[
\pm 2 \langle V, W \rangle \leq \varepsilon |V|^2 + \frac{1}{\varepsilon} |W|^2,
\]

for all positive \(\varepsilon > 0\), where we take \(V = \lambda |\tau(\phi)|^{\frac{\alpha}{2}-1} \nabla_{e_i} \tau(\phi)\) and \(W = (e_i, \lambda)|\tau(\phi)|^{\frac{\alpha}{2}-1} \tau(\phi)\).

We shall consider the second term of the right hand side of (9).
\[-(\alpha - 2) \int_M \sum_{i=1}^{m} (\nabla_{e_i} \tau(\phi), \lambda^2 |\tau(\phi)|^{\alpha - 4} (\nabla_{e_i} \tau(\phi), \tau(\phi)) \tau(\phi)) dv_g \]

(11)

\[ \leq (2 - \alpha) \int_M \sum_{i=1}^{m} |\nabla_{e_i} \tau(\phi)|^2 \lambda^2 |\tau(\phi)|^{\alpha - 2} dv_g, \]

where the inequality of (11) follows from Cauchy-Schwartz inequality. Substituting (10) and (11) into (9), we have

(12)

\[ \int_M \sum_{i=1}^{m} \lambda^2 |\tau(\phi)|^{\alpha - 2} |\nabla_{e_i} \tau(\phi)|^2 dv_g \]

\[ \leq \varepsilon \int_M \sum_{i=1}^{m} \lambda^2 |\tau(\phi)|^{\alpha - 2} |\nabla_{e_i} \tau(\phi)|^2 dv_g \]

\[ + \frac{1}{\varepsilon} \int_M (\nabla \lambda)^2 |\tau(\phi)|^\alpha dv_g \]

\[ + (2 - \alpha) \int_M \sum_{i=1}^{m} |\nabla_{e_i} \tau(\phi)|^2 \lambda^2 |\tau(\phi)|^{\alpha - 2} dv_g. \]

Thus we have

(13)

\[ (-1 - \varepsilon + \alpha) \int_M \sum_{i=1}^{m} \lambda^2 |\tau(\phi)|^{\alpha - 2} |\nabla_{e_i} \tau(\phi)|^2 dv_g \]

\[ \leq \frac{1}{\varepsilon} \int_M (\nabla \lambda)^2 |\tau(\phi)|^\alpha dv_g \]

\[ \leq \frac{1}{\varepsilon r^2} \int_M |\tau(\phi)|^\alpha dv_g. \]

Since \((M, g)\) is complete, we tend \(r\) to infinity. By the assumption \(\int_M |\tau(\phi)|^\alpha dv_g < \infty\), the right hand side of (13) goes to zero and the left hand side of (13) goes to

\[ (-1 - \varepsilon + \alpha) \int_M \sum_{i=1}^{m} |\tau(\phi)|^{\alpha - 2} |\nabla_{e_i} \tau(\phi)|^2 dv_g, \]

since \(\lambda = 1\) on \(B_r(x_0)\). By the assumption \(1 + \varepsilon < \alpha\), we have

\[ (-1 - \varepsilon + \alpha) \int_M \sum_{i=1}^{m} |\tau(\phi)|^{\alpha - 2} |\nabla_{e_i} \tau(\phi)|^2 dv_g = 0. \]

From this, we obtain for any vector field \(X\) on \(M\),

(14)

\[ \nabla_X \tau(\phi) = 0. \]

By (14),

\[ X|\tau(\phi)|^2 = 2 \langle \nabla_X \tau(\phi), \tau(\phi) \rangle = 0. \]

Therefore we obtain \(|\tau(\phi)|\) is constant.
(ii) The case $\alpha \geq 2$. From (8), we have

\[
0 \geq \int_M \sum_{i=1}^{m} (\nabla e_i \tau(\phi), 2\lambda(e_i \lambda)|\tau(\phi)|^{\alpha-2}\tau(\phi)) dv_g
\]

(15)

\[
+ \int_M \sum_{i=1}^{m} (\nabla e_i \tau(\phi), \lambda^2|\tau(\phi)|^{\alpha-2}\nabla e_i \tau(\phi)) dv_g.
\]

By using Young’s inequality, that is,

\[
\pm 2\langle V, W \rangle \leq \varepsilon |V|^2 + \frac{1}{\varepsilon} |W|^2,
\]

for all positive $\varepsilon$, we have

\[
-2 \int_M \sum_{i=1}^{m} (\nabla e_i \tau(\phi), \lambda(e_i \lambda)|\tau(\phi)|^{\alpha-2}\tau(\phi)) dv_g
\]

\[
= -2 \int_M \sum_{i=1}^{m} ((e_i \lambda)|\tau(\phi)|^{\frac{\alpha}{2}-1}\tau(\phi), \lambda|\tau(\phi)|^{\frac{\alpha}{2}-1}\nabla e_i \tau(\phi)) dv_g
\]

\[
\leq 2 \int_M |\nabla \lambda|^2|\tau(\phi)|^\alpha dv_g
\]

\[
+ \frac{1}{2} \int_M \lambda^2|\tau(\phi)|^{\alpha-2} |\nabla e_i \tau(\phi)|^2 dv_g.
\]

Substituting this into (15), we have

\[
\int_M \lambda^2|\tau(\phi)|^{\alpha-2} |\nabla e_i \tau(\phi)|^2 dv_g \leq 4 \int_M |\nabla \lambda|^2|\tau(\phi)|^\alpha dv_g
\]

(16)

\[
\leq \int_M \frac{16}{r^2} |\tau(\phi)|^\alpha dv_g.
\]

By the assumption $\int_M |\tau(\phi)|^\alpha dv_g < \infty$, the right hand side of (16) goes to zero and the left hand side of (16) goes to

\[
\int_M |\tau(\phi)|^{\alpha-2} |\nabla e_i \tau(\phi)|^2 dv_g,
\]

since $\lambda = 1$ on $B_r(x_0)$. Thus, we have

\[
\int_M |\tau(\phi)|^{\alpha-2} |\nabla e_i \tau(\phi)|^2 dv_g = 0.
\]

From this, we obtain for any vector field $X$ on $M$,

(17) \quad \nabla_X \tau(\phi) = 0.

By (17),

\[
X|\tau(\phi)|^2 = 2\langle \nabla_X \tau(\phi), \tau(\phi) \rangle = 0.
\]

Therefore we obtain $|\tau(\phi)|$ is constant.

\[\square\]

To prove Theorem 3.1, we recall Gaffney’s theorem (cf. [11]).
Proof of Theorem 3.3. Let \((M, g)\) be a complete Riemannian manifold. If a \(C^1\) \(1\)-form \(\omega\) satisfies that \(\int_M |\omega| dv_g < \infty\) and \(\int_M (\delta \omega) dv_g < \infty\), or equivalently, a \(C^1\) vector field \(X\) defined by \(\omega(Y) = \langle X, Y \rangle, \ (\forall Y \in \mathfrak{X}(M))\) satisfies that \(\int_M |X| dv_g < \infty\) and \(\int_M \text{div}(X) dv_g < \infty\), then

\[
\int_M (\delta \omega) dv_g = \int_M \text{div}(X) dv_g = 0.
\]

By using Lemma 3.2 and Theorem 3.3, we shall show Theorem 3.1.

Proof of Theorem 3.1. By Lemma 3.2, we have \(\nabla_X \tau(\phi) = 0\) for any vector field \(X\) on \(M\), and \(|\tau(\phi)| \neq 0\), then

\[
\int_M |\tau(\phi)|^\alpha dv_g = |\tau(\phi)|^\alpha \text{Vol}(M, g) = \infty,
\]

which yields the contradiction.

We shall show the case (i). Assume that a 1-form \(\omega\) on \(M\) defined by

\[
\omega(X) := |\tau(\phi)|^{\frac{2}{2}} (d\phi(X), \tau(\phi)), \ (X \in \mathfrak{X}(M)).
\]

By the assumption \(\int_M |d\phi|^2 dv_g < \infty\) and \(\int_M |\tau(\phi)|^\alpha dv_g < \infty\), we have

\[
\int_M |\omega| dv_g = \int_M \left( \sum_{i=1}^m |\omega(e_i)|^2 \right)^{\frac{1}{2}} dv_g
\]

\[
\leq \int_M |\tau(\phi)|^{\frac{2}{2}} |d\phi| dv_g
\]

\[
\leq \left( \int_M |d\phi|^2 dv_g \right)^{\frac{1}{2}} \left( \int_M |\tau(\phi)|^\alpha dv_g \right)^{\frac{1}{2}} < \infty.
\]

We consider \(-\delta \omega = \sum_{i=1}^m (\nabla_{e_i} \omega)(e_i)\).

\[
-\delta \omega \sum_{i=1}^m \nabla_{e_i} (\omega(e_i)) = -\omega(\nabla_{e_i} e_i)
\]

\[
= \sum_{i=1}^m \left\{ \nabla_{e_i} \left( |\tau(\phi)|^{\frac{2}{2}} (d\phi(e_i), \tau(\phi)) \right) - |\tau(\phi)|^{\frac{2}{2}} (d\phi(\nabla_{e_i} e_i), \tau(\phi)) \right\}
\]

\[
= \sum_{i=1}^m \left\{ |\tau(\phi)|^{\frac{2}{2}} (\nabla_{e_i} d\phi(e_i), \tau(\phi)) - |\tau(\phi)|^{\frac{2}{2}} (d\phi(\nabla_{e_i} e_i), \tau(\phi)) \right\}
\]

\[
= |\tau(\phi)|^{\frac{2}{2} + 1},
\]

\[
\text{Biharmonic maps into a non-positively curved manifold}
\]
where the third equality follows from $|\tau(\phi)|$ is constant and $\nabla_X \tau(\phi) = 0$, $(X \in \mathfrak{X}(M))$. Since $|\tau(\phi)|$ is constant and $\int_M |\tau(\phi)|^\alpha dv_g < \infty$, the function $-\delta \omega$ is also integrable over $M$. From this and [15], we can apply Gaffney’s theorem for the 1-form $\omega$. Therefore we have

$$0 = \int_M (-\delta \omega) dv_g = \int_M |\tau(\phi)|^{\beta+1} dv_g,$$

which implies that $\tau(\phi) = 0$.

If a biharmonic map $\phi : (M, g) \to (N, h)$ is an isometric immersion, that is, $M$ is a biharmonic submanifold in $N$, we obtain the following result.

**Theorem 3.4.** Let $\phi : (M, g) \to (N, h)$ be a biharmonic isometric immersion from a complete Riemannian manifold $(M, g)$ into a Riemannian manifold $(N, h)$ with non-positive sectional curvature.

For some $\varepsilon > 0$, assuming that $\alpha$ satisfies $1 + \varepsilon < \alpha < \infty$. If such an $\alpha$, $\int_M |H|^\alpha dv_g < \infty$, then $\phi$ is harmonic.

**Proof.** By Lemma [xx], we have $\nabla_X H = 0$ for any vector field $X$ on $M$ and $|H|$ is constant. Since $H$ belongs to the normal component of $T_{\phi(x)}N$ $(x \in M)$, $\langle d\phi(X), H \rangle = 0$, for any vector field $X$ on $M$. From these, we obtain

$$0 = \sum_{i=1}^m \left\{ e_i \langle d\phi(e_i), H \rangle - \langle d\phi(\nabla e_i e_i), H \rangle \right\}$$

$$= \sum_{i=1}^m \left\{ \langle \nabla e_i d\phi(e_i), H \rangle + \langle d\phi(e_i), \nabla e_i H \rangle - \langle d\phi(\nabla e_i e_i), H \rangle \right\}$$

$$= \sum_{i=1}^m \langle \nabla e_i d\phi(e_i) - d\phi(\nabla e_i e_i), H \rangle$$

$$= m \langle H, H \rangle.$$ 

Therefore $M$ is minimal, that is, $\phi$ is harmonic.

4. Biharmonic submersions into non-positively curved manifolds

In this section, we apply Theorem [xx] to submersions.

Wang and Ou showed that a biharmonic Riemannian submersion from a complete Riemannian manifold with constant sectional curvature into a Riemannian surface $(N^2, h)$ is harmonic (cf. [21]). N. Nakauchi, H. Urakawa and S. Gudmundsson applied Theorem [xx] to submersions (cf. [19]).

We first recall harmonic morphisms (cf. [2]).

Let $\phi : (M, g) \to (N, h)$ be a smooth map between Riemannian manifolds, and let $x \in M$. Then $\phi$ is called horizontally weakly conformal at $x$ if either
(i) $d\phi_x = 0$, or 
(ii) $d\phi_x$ maps the horizontal space $\mathcal{H}_x = \{\text{Ker}(d\phi_x)\}^\perp$ conformally onto $T_{\phi(x)}N$, such that 
\[ h(d\phi_x(X), d\phi_x(Y)) = \Lambda g(X, Y), \quad (X, Y \in \mathcal{H}_x). \]

The map $\phi$ is called horizontally weakly conformal on $M$ if it is horizontally weakly conformal at every point of $M$; if further, $\phi$ has no critical points, then we call it a horizontally conformal submersion. Note that if $\phi : (M, g) \to (N, h)$ is a horizontally weakly conformal map and $\dim M < \dim N$, then $\phi$ is constant.

If for every harmonic function $f : V \to \mathbb{R}$ defined on an open subset $V$ of $N$ with $\phi^{-1}(V)$ non-empty, the composition $f \circ \phi$ is harmonic on $\phi^{-1}(V)$, then $\phi$ is called a harmonic morphism. Harmonic morphisms are characterized as follows (cf. [10], [13]).

**Theorem 4.1** ([10], [13]). A smooth map $\phi : (M, g) \to (N, h)$ between Riemannian manifolds is a harmonic morphism if and only if $\phi$ is both harmonic and horizontally weakly conformal.

Let $\phi : (M, g) \to (N, h)$ be a submersion, and each tangent space $T_xM$ is decomposed as follows.
\begin{equation}
T_xM = V_x \oplus \mathcal{H}_x, \tag{20}
\end{equation}
where $V_x = \text{Ker}(d\phi_x)$ is the vertical space and $\mathcal{H}_x$ is the horizontal space. If there exists a positive $C^\infty$ function $\lambda$ on $M$ such that, for each $x \in M$,
\[ h(d\phi_x(X), d\phi_x(Y)) = \lambda^2(x)g(X, Y), \quad (X, Y \in \mathcal{H}_x), \]
then $\lambda$ is called the dilation.

When $\phi : (M^m, g) \to (N^n, h)$ $(m \geq n \geq 2)$ is a horizontally conformal submersion, the tension field $\tau(\phi)$ is given by
\begin{equation}
\tau(\phi) = \frac{n - 2}{2} \lambda^2 d\phi \left( \text{grad}_\mathcal{H} \left( \frac{1}{\lambda^2} \right) \right) - (m - n)d\phi(\mathbf{H}), \tag{21}
\end{equation}
where $\text{grad}_\mathcal{H} \left( \frac{1}{\lambda^2} \right)$ is the $\mathcal{H}$-component of the decomposition according to (20) of $\text{grad} \left( \frac{1}{\lambda^2} \right)$, and $\mathbf{H}$ is the trace of the second fundamental form of each fiber which is given by $\mathbf{H} = \frac{1}{m - n} \sum_{i=1}^m \mathcal{H}((\nabla e_i, e_i)$, where a local orthonormal frame field $\{e_i\}_{i=1}^m$ on $M$ is taken in such a way that $\{e_i|_{x}\}_{i=1}^m$ belong to $\mathcal{H}_x$ and $\{e_j|_{x}\}_{j=n+1}^m$ belong to $V_x$, where $x$ is in a neighborhood in $M$.

Then following result follows from Theorem 4.1 immediately.

**Proposition 4.2.** Let $\phi : (M^m, g) \to (N^n, h)$ $(m > n \geq 2)$ be a biharmonic horizontally conformal submersion from a complete Riemannian manifold $(M, g)$ into a Riemannian manifold $(N, h)$ with non-positive sectional curvature.

For some $\varepsilon > 0$, assuming that $\alpha$ satisfies $1 + \varepsilon < \alpha < \infty$. If such an $\alpha$, the dilation $\lambda$ satisfies that
\begin{equation}
\int_M \lambda^2 \left( \frac{n - 2}{2} \lambda^2 \text{grad}_\mathcal{H} \left( \frac{1}{\lambda^2} \right) - (m - n)\mathbf{H} \right)_g^\alpha dv_g < \infty, \tag{22}
\end{equation}
and if either $\int_M \lambda^2 dv_g < \infty$ or $\text{Vol}(M, g) = \infty$. Then, $\phi$ is a harmonic morphism.
Proof. Since $\phi: (M, g) \to (N, h)$ is a biharmonic map from a complete Riemannian manifold $(M, g)$ into a Riemannian manifold $(N, h)$ with non-positive sectional curvature, and

$$\int_M \lambda^2 \left| \frac{n - 2}{2} \lambda^2 \text{grad}_g \left( \frac{1}{\lambda^2} \right) - (m - n)H \right|^\alpha_g \, dv_g < \infty,$$

by Theorem 3.1 $\phi$ is harmonic.

Since $\phi$ is also a horizontally conformal submersion, by Theorem 4.1 $\phi$ is a harmonic morphism. □

If dim$N = 2$, Proposition 4.2 implies the following corollary.

Corollary 4.3. Let $\phi: (M^m, g) \to (N^n, h)$ ($m > n = 2$) be a biharmonic horizontally conformal submersion from a complete Riemannian manifold $(M^m, g)$ into a Riemannian manifold $(N^n, h)$ with non-positive sectional curvature.

For some $\varepsilon > 0$, assuming that $\alpha$ satisfies $1 + \varepsilon < \alpha < \infty$. If such an $\alpha$, the dilation $\lambda$ satisfies that

$$\int_M \lambda^2 \left| \left| \frac{1}{\lambda} \text{grad}_g \left( \frac{1}{\lambda^2} \right) \right|^{\alpha} \right|_g \, dv_g < \infty,$$

and if either $\int_M \lambda^2 \, dv_g < \infty$ or Vol$(M, g) = \infty$. Then, $\phi$ is a harmonic morphism.

5. Biharmonic hypersurfaces in Riemannian manifolds with non-positive Ricci curvature

In this section, we consider biharmonic hypersurfaces in Riemannian manifolds with non-positive Ricci curvature.

For any $x \in M$, let $\{e_1, \ldots, e_m, \xi\}$ be an orthonormal basis of $N$ at $x$ such that $\{e_1, \ldots, e_m\}$ is an orthonormal basis of $T_x M$. Then, the second fundamental form $B$ is written as

$$B(X, Y) = B_\xi(X, Y)\xi, \text{ at } x.$$

The mean curvature vector field $H$ of $M$ at $x$ is also given by

$$H(x) = \frac{1}{m} \sum_{i=1}^m B(e_i, e_i) = H(x)\xi, \quad H(x) := \frac{1}{m} \sum_{i=1}^m B_\xi(e_i, e_i).$$

For global version of generalized Chen’s conjecture, N. Nakauchi and H. Urakawa showed the following (cf. [17]).

Theorem 5.1 ([17]). Let $\phi: (M^m, g) \to (N^{m+1}, h)$ be a biharmonic isometric immersion from an $m$-dimensional complete Riemannian manifold $(M^m, g)$ into an $m + 1$-dimensional Riemannian manifold $(N^{m+1}, h)$ with non-positive Ricci curvature. If

$$\int_M |H|^2 \, dv_g < \infty,$$

then $\phi$ is harmonic.
The assumption $\int_M |H|^2 dv_g < \infty$ in Theorem 5.1 is natural for $\dim M = 2$. For a higher dimensional manifold, we show as follows:

**Theorem 5.2.** Let $\phi : (M^m, g) \to (N^{m+1}, h)$ be a biharmonic isometric immersion from an $m$-dimensional complete Riemannian manifold $(M^m, g)$ into an $m+1$-dimensional Riemannian manifold $(N^{m+1}, h)$ with non-positive Ricci curvature.

For some $\varepsilon > 0$, assuming that $1 + \varepsilon \leq \alpha < \infty$. If such an $\alpha$, $\int_M |H|^{\alpha} dv_g < \infty$, then $\phi$ is harmonic.

**Proof.** For a fixed point $x_0 \in M$, and for every $0 < r < \infty$, we first take a cut off function $\lambda$ on $M$ satisfying that

\[
\begin{aligned}
0 \leq \lambda(x) &\leq 1 \quad (x \in M), \\
\lambda(x) &= 1 \quad (x \in B_r(x_0)), \\
\lambda(x) &= 0 \quad (x \not\in B_{2r}(x_0)), \\
|\nabla \lambda| &\leq \frac{2}{r} \quad (x \in M).
\end{aligned}
\]

From (5) and $|H|^2 = |H|^2$, we have

\[
\int_M \langle -\Delta \phi H, \lambda^2 |H|^{\alpha-2}H \rangle dv_g
\]

\[
= \int_M \lambda^2 |H|^{\alpha-2} \sum_{i=1}^m \langle R^N(H, d\phi(e_i)) d\phi(e_i), H \rangle dv_g
\]

\[
= \int_M \lambda^2 |H|^\alpha \text{Ric}^N(\xi, \xi) dv_g \leq 0,
\]

where the inequality follows from the Ricci curvature of $(N, h)$ is non-positive. By (25), we have

\[
0 \geq \int_M \langle -\Delta \phi H, \lambda^2 |H|^{\alpha-2}H \rangle dv_g
\]

\[
= \int_M \langle \nabla H, \nabla (\lambda^2 |H|^{\alpha-2}H) \rangle dv_g
\]

\[
= \int_M \sum_{i=1}^m \langle \nabla e_i H, (e_i \lambda^2)|H|^{\alpha-2}H + \lambda^2 e_i \{(|H|^2)^{\frac{\alpha-2}{2}}\} H + \lambda^2 |H|^{\alpha-2} \nabla e_i H \rangle dv_g
\]

\[
= \int_M \sum_{i=1}^m \langle \nabla e_i H, 2\lambda(e_i \lambda)|H|^{\alpha-2}H \rangle dv_g
\]

\[
+ \int_M \sum_{i=1}^m \langle \nabla e_i H, \lambda^2 (\alpha-2)|H|^{\alpha-4} \nabla e_i H \rangle dv_g
\]

\[
+ \int_M \sum_{i=1}^m \langle \nabla e_i H, \lambda^2 |H|^{\alpha-2} \nabla e_i H \rangle dv_g.
\]
Therefore we have
\[ \int_{M} \sum_{i=1}^{m} \lambda^2 |H|^\alpha |\nabla e_i H|^2 \, dv_g \]
(26)
\[ \leq -2 \int_{M} \sum_{i=1}^{m} \langle \nabla e_i H, \lambda (e_i \lambda) |H|^\alpha \nabla^2 \rangle \, dv_g \]
\[ - (\alpha - 2) \int_{M} \sum_{i=1}^{m} \langle \nabla e_i H, \lambda^2 |H|^\alpha (\nabla e_i H, H) \rangle \, dv_g. \]

Since \( H = H^\xi \), we have
\[ \sum_{i=1}^{m} |\nabla e_i H|^2 = \sum_{i=1}^{m} \langle \nabla e_i^\perp H, \nabla e_i^\perp H \rangle + \sum_{i=1}^{m} \langle A e_i e_i \rangle \]
(27)
\[ = \sum_{i=1}^{m} \langle (e_i H) \xi_i, (e_i H) \xi_i \rangle + |H|^2 \sum_{i=1}^{m} \langle A \xi e_i, A \xi e_i \rangle \]
\[ = (\nabla H)^2 + |H|^2 |B \xi|^2. \]

From this, we obtain the left hand side of (26) as follows.
\[ \int_{M} \sum_{i=1}^{m} \lambda^2 |H|^\alpha \nabla e_i H|^2 \, dv_g \]
(28)
\[ = \int_{M} \lambda^2 |H|^\alpha - 2 ((\nabla H)^2 + |H|^2 |B \xi|^2) \, dv_g. \]

We shall consider the first term of the right hand side of (26).
\[ -2 \int_{M} \sum_{i=1}^{m} \langle \nabla e_i H, \lambda (e_i \lambda) |H|^\alpha \nabla^2 \rangle \, dv_g \]
\[ = -2 \int_{M} \sum_{i=1}^{m} \langle |H|^\alpha - 1 \nabla e_i H, (e_i \lambda) |H|^\alpha - 1 \nabla H \rangle \, dv_g \]
(29)
\[ \leq \varepsilon \int_{M} \lambda^2 |H|^\alpha - 2 \sum_{i=1}^{m} |\nabla e_i H|^2 \, dv_g \]
\[ + \frac{1}{\varepsilon} \int_{M} |\nabla \lambda|^2 |H|^\alpha \, dv_g \]
\[ = \varepsilon \int_{M} \lambda^2 |H|^\alpha - 2 ((\nabla H)^2 + |H|^2 |B \xi|^2) \, dv_g \]
\[ + \frac{1}{\varepsilon} \int_{M} |\nabla \lambda|^2 |H|^\alpha \, dv_g, \]
where the inequality of (29) follows from Young’s inequality, that is,
\[ \pm 2 \langle V, W \rangle \leq \varepsilon |V|^2 + \frac{1}{\varepsilon} |W|^2, \]
for all positive \( \varepsilon > 0 \), where we take \( V = \lambda |H|^\alpha - 1 \nabla e_i H \) and \( W = (e_i \lambda) |H|^\alpha - 1 H \).
We shall consider the second term of the right hand side of (26). Since $H = H\xi$, we have
\[
\sum_{i=1}^{m} (\nabla_{e_i} H, H)^2 = \sum_{i=1}^{m} (\nabla_{e_i}^\perp H, H)^2 = \sum_{i=1}^{m} (e_i H, H)^2 = (\nabla H)^2 |H|^2.
\]
Substituting this into the second term of the right hand of (26), we obtain
\[
-(\alpha - 2) \int_{M} \sum_{i=1}^{m} (\nabla_{e_i} H, \lambda^2 |H|^{\alpha - 4} (\nabla_{e_i} H) H) dv_g
\]
\[
= -(\alpha - 2) \int_{M} \sum_{i=1}^{m} \lambda^2 |H|^{\alpha - 4} (\nabla_{e_i} H)^2 dv_g
\]
\[
= -(\alpha - 2) \int_{M} \lambda^2 |H|^{\alpha - 2} (\nabla H)^2 dv_g.
\]
Substituting (28), (29) and (31) into (26), we have
\[
\int_{M} \lambda^2 |H|^{\alpha - 2} \left( (\nabla H)^2 + |H|^2 |B_\xi|^2 \right) dv_g
\]
\[
\leq \epsilon \int_{M} \lambda^2 |H|^{\alpha - 2} \left( (\nabla H)^2 + |H|^2 |B_\xi|^2 \right) dv_g
\]
\[
+ \frac{1}{\epsilon} \int_{M} (\nabla \lambda)^2 |H|^2 dv_g
\]
\[
- (\alpha - 2) \int_{M} \lambda^2 |H|^{\alpha - 2} (\nabla H)^2 dv_g.
\]
Thus we have
\[
(\alpha - 1 - \epsilon) \int_{M} \lambda^2 |H|^{\alpha - 2} (\nabla H)^2 dv_g + (1 - \epsilon) \int_{M} \lambda^2 |H|^{\alpha} |B_\xi|^2 dv_g
\]
\[
\leq \frac{1}{\epsilon} \int_{M} (\nabla \lambda)^2 |H|^2 dv_g
\]
\[
\leq \frac{1}{\epsilon^2} \int_{M} |H|^2 dv_g.
\]
Since $(M, g)$ is complete, we tend $r$ to infinity. By the assumption $\int_{M} |H|^2 dv_g < \infty$, the right hand side of (33) goes to zero and the left hand side of (33) goes to
\[
(\alpha - 1 - \epsilon) \int_{M} |H|^{\alpha - 2} (\nabla H)^2 dv_g + (1 - \epsilon) \int_{M} |H|^\alpha |B_\xi|^2 dv_g,
\]
since $\lambda = 1$ on $B_r(x_0)$. By $1 + \epsilon \leq \alpha < \infty$, we have
\[
(\alpha - 1 - \epsilon) \int_{M} |H|^{\alpha - 2} (\nabla H)^2 dv_g = 0.
\]
\[(35) \quad (1 - \varepsilon) \int_M |H|^\alpha |B_\xi|^2 dv_g = 0.\]

From (35), we have \(H = 0\), that is, \(\phi\) is harmonic. \(\square\)

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