An Algorithm for Limited Visibility Graph Searching
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Abstract

We study a graph search problem in which a team of searchers attempts to find a mobile target located in a graph. Assuming that (a) the visibility field of the searchers is limited, (b) the searchers have unit speed and (c) the target has infinite speed, we formulate the Limited Visibility Graph Search (LVGS) problem and present the LVGS algorithm, which produces a search schedule guaranteed to find the target in the minimum possible number of steps. Our LVGS algorithm is a “conversion” of Guibas and Lavalle’s polygonal region search algorithm.

1 Introduction

In graph search problems a team of searchers attempts to find a mobile target located in a graph. Depending on the properties of the searchers and target, different variants of the problem are obtained.

In the current paper we study a graph searching variant in which it is assumed that: (a) the visibility field of the searchers is limited (i.e., from any given position they can see only part of the graph, as described by a visibility matrix), (b) the searchers have unit speed and (c) the target has infinite speed. We formulate, under these assumptions, the Limited Visibility Graph Search (LVGS) problem and present an LVGS algorithm which, given the adjacency and visibility matrices, produces a search schedule guaranteed to find the target (if this is possible) in the minimum possible number of steps. Our LVGS algorithm is a “conversion” of the polygonal region search algorithm presented in [15].

The seminal papers for graph search problems are [5] and [29]. The basic ideas of these two works have been elaborated and extended in a large number of papers. Excellent reviews of the relevant literature appear in [14] and [7]. A large number of graph search algorithms have been presented in the literature; see for example [1, 2, 6, 25, 32, 35]. These algorithms usually depend on path- or tree-decompositions of the graph (and related methods) resulting in rather complicated implementation; in addition, the majority of the above works concerns the zero-visibility case, i.e., when the searchers can only see the vertex in which they are located. Furthermore, the search schedules studied in the graph theoretic literature often allow a searcher to move between non-adjacent vertices (i.e., not necessarily following the edges of the graph). This movement mode is called “teleporting” and is not appropriate for most realistic applications. Relatively little has been published on non-teleporting search, i.e., where the searchers move only along the graph edges. Some related approaches appear in the robotics literature, for example in [15, 19, 20, 9].

A related, but different, approach to limited visibility graph searching can be obtained in the context of pursuit games in graph. These involve one or more pursuers (i.e., searchers) who try to locate and capture an evader (i.e., target); all agents are assumed to move along the edges of the graph (and usually, but not always, with unit speed). The “classic” version of this problem is the Cops and Robbers Game (CR) first introduced in [27], in which it is assumed that the cops (searchers) are always aware of the robber’s (target’s) location; so this is a complete visibility problem. Many CR variants have been considered, obtained by varying the abilities.

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1These terms will be defined precisely in a later section.
2It is worth emphasizing that in this formulation of the problem a worst-case analysis is adopted, by which the target is essentially “factored out” of the problem; i.e., the only decision maker is the searcher. This point will be further explained in later sections.
3In this formulation we are dealing with a game in which both the pursuers and the evader are active decision makers.
and restrictions placed on both the cops and the robber. There is a large literature on the Cops and *visible* Robber problem, mainly concentrating on its theoretical aspects. An excellent and recent review of the available work appears in the book [28]. Algorithmic aspects of the problem are studied in [4] and [17].

The variant of an *invisible* or *partially visible* robber is quite close to the previously mentioned LVGS problem. The cops (resp. the robber) play a role similar to that of the searchers (resp. target); note, however, the distinction between *seeing* and *capturing* the robber. Relatively little work has been done on this problem. The first formulation appears in [34]. Further progress was achieved in the Master’s theses [21, 33] and, recently, in [11, 12, 13] (for the zero-visibility case) and in [8] (for the general case, which includes zero- and complete-visibility as special cases). A somewhat different approach to the invisible robber appears in [23, 24] and a related computational approach is presented in [22].

The approach we follow in the current paper is rather different from all of the above. As already mentioned, our formulation and solution of the problem are based on the ideas of [15, 16], which studies search of a *continuous polygonal environment*. This approach and generalizations to various geometric environments is presented more extensively in the excellent book [26].

In contrast to the previously mentioned treatments of graph search and pursuit, our LVGS algorithm is relatively simple to implement, produces non-teleporting search schedules and can be applied not only to zero-visibility but to *arbitrary* visibility specifications. On the other hand, while the algorithm can be applied to any number $K$ of searchers, the computational burden increases quickly with $K$. In short, the main usefulness of the LVGS algorithm is in solving the graph search problem under the following conditions.

1. One searcher is attempting to locate the target in a graph.
2. Both searcher and target move only along the edges of the graph; the searcher moves with unit speed while the target moves with infinite speed.
3. The *visibility* of the target is determined by the visibility matrix $B$.

The paper is organized as follows. In Section 2 we introduce preliminary definitions and notations. In Section 3 we formulate the LVGS problem and present the LVGS algorithm. Section 4 is devoted to the evaluation of the algorithm by numerical experiments. Finally, in Section 5 we summarize our work and propose directions for further research.

## 2 Preliminaries

We use the following set theoretic notation.

1. Given a set $U$, the *cardinality* of $U$ is denoted by $|U|$ and defined to be the number of elements of $U$. So, if $U = \{u_1,\ldots,u_N\}$ then $|U| = N$.
2. Given sets $U, W$ we define the *set difference* $U\setminus W$ by
   $$U\setminus W = \{u : u \in U \text{ and } u \not\in W\}.$$
3. Given a reference set $U = \{u_1,\ldots,u_N\}$, any set $W \subseteq U$ can be described by its *indicator vector* $w = [w_1, w_2,\ldots,w_N]$ where
   $$\forall n \in \{1,\ldots,N\} : w_n = \begin{cases} 1 & \text{iff } u_n \in W; \\ 0 & \text{iff } u_n \not\in W. \end{cases}$$

We use the following graph theoretic definitions and notation.

1. A *graph* is a pair $G = (V,E)$, where the *vertex set* is $V = \{x_1, x_2,\ldots,x_N\}$ and the *edge set* is $E \subseteq \{\{x,y\} : x, y \in V, x \neq y\}$;
2. a *directed graph* is a pair $G = (V,E)$, where the *vertex set* is $V = \{1, 2,\ldots,N\}$ and the *edge set* is $E \subseteq \{(x,y) : x, y \in V, x \neq y\}$.

In both of the above cases we usually (but not always) take the node set to be $V = \{1, 2,\ldots,N\}$. 
For graph $G = (V, E)$ and vertex $x \in V$:

- the neighborhood of $x$ is $N(x) = \{y : \{x, y\} \in E\}$,
- the closed neighborhood of $x$ is $N[x] = N(x) \cup \{x\}$.

For graph $G = (V, E)$ with $|V| = N$, the adjacency matrix of $G$ is an $N \times N$ matrix $A$ defined by

$$\forall x, y \in V : A_{xy} = \begin{cases} 1 & \text{iff } \{x, y\} \in E, \\ 0 & \text{iff } \{x, y\} \notin E. \end{cases}$$

For directed graph $G = (V, E)$ with $|V| = N$, the adjacency matrix of $G$ is an $N \times N$ matrix $A$ defined by

$$\forall x, y \in V : A_{xy} = \begin{cases} 1 & \text{iff } (x, y) \in E, \\ 0 & \text{iff } (x, y) \notin E. \end{cases}$$

A graph $G = (V, E)$ is uniquely specified by its adjacency matrix; hence in what follows we will sometimes say “the graph $A$”. The same is true of a directed graph.

For graph $G = (V, E)$ with $|V| = N$, a visibility matrix for $G$ is an $N \times N$ matrix $B$ defined by

$$\forall x, y \in V : B_{xy} = \begin{cases} 1 & \text{iff } y \text{ can be seen from } x, \\ 0 & \text{iff } y \text{ cannot be seen from } x. \end{cases}$$

## 3 The LVGS Problem and Algorithm

We start with an informal description of the LVGS problem.

1. We are given: the graph $G = (V, E)$, specified by its adjacency matrix $A$, and a visibility matrix $B$ for $G$.

2. At time $t = 0$ we place each of $K$ searchers at a graph vertex (a vertex can contain any number of searchers). It is also assumed that a target is located at some unknown graph vertex.

3. At discrete time steps $t \in \{1, 2, \ldots\}$ we can move each searcher to a vertex neighboring his current position.

4. The following assumptions are made for each time step $t \in \{0, 1, 2, \ldots\}$.

   (a) The visibility matrix $B$ is similar to the adjacency matrix $A$. Just like an agent located at vertex $x$ can move to vertex $y$ iff $A_{xy} = 1$, similarly, an agent located at vertex $x$, can see vertex $y$ (and its contents) iff $B_{xy} = 1$.

   (b) The target will always move to a vertex $z$ which is not seen by any searcher, provided there is a path from his current vertex to $z$ which does not pass through a searcher-visible vertex.

   (c) The process ends when the target is in a vertex which can be observed by the searchers (the target is observed, the graph is “cleared”).

5. Our goal is to clear the graph in the shortest possible number of steps.

To provide a precise mathematical formulation of the above description, we introduce the following.

**Definition 3.1** For each $t \in \{1, 2, \ldots\}$ we define the position vector by

$$x(t) = [x_1(t), \ldots, x_K(t)],$$

where $x_k(t) \in V$ is the position of the $k$-th searcher at time $t$. For all $t \geq 1$, we have $x(t) \in X = V^K$.

**Definition 3.2** We also introduce the null position vector $\lambda$; hence we write $x(0) = \lambda$ to indicate that, at $t = 0$, no searchers have been placed on the graph.
Definition 3.3 If a vertex may contain the target, the vertex is called dirty, otherwise it is called clear.

When a vertex $x$ is seen by a searcher it is cleared; it remains clear as long as there is no "free" (i.e., invisible to the searchers) path from $x$ to a dirty vertex. If, at some time, a path from a clear vertex $x$ to a dirty vertex $y$ becomes invisible to the searchers, then $x$ becomes dirty again (is recontaminated). We will denote the set of all dirty (resp. clear) vertices at time $t$ by $D(t)$ (resp. by $C(t)$). We can describe sets by their indicator vectors; we will denote the indicator vector of $D(t)$ by

$$d(t) = [d_1(t), d_2(t), ..., d_N(t)].$$

More precisely, we have the following.

Definition 3.4 For each $t \in \{0, 1, 2, \ldots\}$ we define the contamination vector by

$$d(t) = [d_1(t), d_2(t), ..., d_N(t)]$$

where $\forall t, n : d_n(t) = \begin{cases} 1 & \text{if vertex } n \text{ is dirty at time } t, \\ 0 & \text{if vertex } n \text{ is clear at time } t. \end{cases}$

We set $d(0) = [1, \ldots, 1]$, i.e., at $t = 0$ all vertices are dirty.

Next we define the state vector, which contains all the information relevant to the graph search.

Definition 3.5 For each $t \in \{0, 1, 2, \ldots\}$ we define the state vector by

$$z(t) = [x_1(t), ..., x_K(t), d_1(t), ..., d_N(t)] = [x, d].$$

For all $t \geq 1$, we have $x(t) \in V^K \times \{0, 1\}^N$. Taking in account $t = 0$, the state space (set of all possible states) is

$$Z = \left(V^K \times \{0, 1\}^N\right) \cup \{[\lambda, 1, \ldots, 1]\}.$$

It is assumed that, at every time $t$, the “search planner” knows $z(t)$; i.e., he knows the location of all searchers and the possible locations of the target.

Definition 3.6 The clear states are the elements of $Z$ which have the form

$$z = [x, 0, ..., 0],$$

and the set of all clear states will be denoted by $Z_C$. The set of all dirty states is

$$Z_D = Z \setminus Z_C$$

and the dirty states are the elements of $Z_D$.

Definition 3.7 The control vector is

$$u(t) = [u_1(t), \ldots, u_K(t)]$$

where $u_k(t)$ denotes the position to which the $k$-th searcher will be moved at time $t + 1$ (so at $t + 1$ the $k$-th searcher will be located at $x_k(t + 1) = u_k(t)$). Of course moves must be legal, i.e.,

$$\forall t, k : \{x_k(t), u_k(t)\} \in E. \quad (1)$$

To specify the “search evolution equation” we first define star-multiplication $\ast$ as follows.

Definition 3.8 Given the $L \times M$ matrix $P$ and the $M \times N$ matrix $Q$ we define the $L \times N$ matrix $P \ast Q$ as follows

$$\forall l \in \{1, 2, \ldots, L\}, n \in \{1, 2, \ldots, N\} : (P \ast Q)_{ln} = \max_{m=1,2,\ldots,M} \min(P_{lm}, Q_{mn}).$$
When the target’s speed is equal to $s$ (i.e., at every time step he can traverse $s$ edges) and assuming that no searchers exist in the graph, we have the following equation for the evolution of the dirty set:\footnote{If the target is arbitrarily fast, then it suffices to set $s = N$, the number of vertices in the graph.}

$$ \mathbf{d}(t+1) = \mathbf{d}(t) \ast \mathbf{A} \ast \cdots \ast \mathbf{A} \ast \cdots \ast \mathbf{A} $$

Now suppose a single searcher is placed at vertex $u_1$. The searcher blocks certain paths which, if taken by the target, would result in his being observed. In particular, the target will not move into any vertex which belongs to the visibility region of the searcher; similarly, the target will not move out of any vertex in the visibility field. Similar rules hold in case $K$ searchers are located in the graph.

To model this situation, we introduce the \textit{modified adjacency matrix} $\mathbf{A}(\mathbf{u})$. It describes the vertices into which the target can move without being observed by searchers located at positions $\mathbf{u} = [u_1, ..., u_K]$. In other words, $\mathbf{A}(\mathbf{u})$ is the adjacency matrix of the graph $\mathcal{G}(\mathbf{u})$ obtained by removing from the graph $\mathcal{G}$ the vertices which lie in the visibility region of the searchers (and all their incident edges).

\textbf{Definition 3.9} The \textit{modified adjacency matrix} $\mathbf{A}(\mathbf{u})$, where $\mathbf{u} = [u_1, ..., u_K]$, is defined as follows

$$ \forall x, y : \mathbf{A}_{xy}(\mathbf{u}) = \begin{cases} A_{xy} & \text{iff: } \forall k \in \{1, 2, ..., K\} : \{x, y\} \cap N[u_k] = \emptyset, \\
0 & \text{otherwise.} \end{cases} $$

Putting all of the above pieces together we can write an equation which specifies the next state of the graph search.

\textbf{Definition 3.10} The \textit{graph search evolution equation} (or simply the \textit{evolution equation}) is

$$ \begin{align*}
\mathbf{x}(t+1) &= \mathbf{u}(t), \\
\mathbf{d}(t+1) &= \mathbf{d}(t) \ast \mathbf{A}(\mathbf{u}(t)) \ast \cdots \ast \mathbf{A}(\mathbf{u}(t)).
\end{align*} $$

Now we are ready to formally define a \textit{preliminary version} of the LVGS problem as follows.

\textbf{Problem 3.11 (Preliminary LVGS Problem)} Given $A$ and $B$ choose, subject to (2)-(3), the minimum $T$ and respective $\mathbf{u}(0), ..., \mathbf{u}(T-1)$ such that $[\mathbf{x}(T), \mathbf{d}(T)] \in \mathbb{Z}_C$.

This problem can be solved by graph theoretic methods. Consider the directed graph $\mathcal{G}' = (V', E')$ where the vertices are the previously defined states (i.e., $V' = \mathbb{Z}$) and the edges correspond to valid state (i.e., an edge $(z_1, z_2)$ denotes that we can make a transition from $z_1 = (x_1, d_1)$ to $z_2 = (x_2, d_2)$). Then we can solve the LVGS problem by finding a shortest path from the vertex $(\lambda, [1, ..., 1])$ to some vertex of the form $(\mathbf{x}, [0, ..., 0]) \in \mathbb{Z}_C$, which can be solved by, e.g., Dijkstra’s algorithm.

However, in the above formulation the size of $\mathbb{Z}$ is $|\mathbb{Z}| = |V| \cdot 2^{|V|} + 1$ which grows exponentially. For instance, when the original $\mathcal{G}$ has $|V| = 10$ vertices and we employ a single searcher ($K = 1$), then $|\mathbb{Z}| = 10 \cdot 2^{10} + 1 = 10241$; with $|V| = 20$ vertices and a single searcher we have $|\mathbb{Z}| = 20 \cdot 2^{20} + 1 = 20971521$.

Consequently, in what follows we will introduce a different approach (inspired from \textbf{15} \textbf{16}) which can achieve the same goal (reaching an all-clear state) using a much smaller directed graph $\mathcal{G}_I$, the “\textit{information state graph}”; $\mathcal{G}_I$ describes how the contamination of $\mathcal{G}$ (i.e., the possible locations of the target) changes as the searcher moves in $\mathcal{G}$.

To describe the structure of $\mathcal{G}_I$, we consider the evolution of contamination as the searchers move through $\mathcal{G}$. For the sake of simplicity we will consider the case of a single searcher (the extension to the case of many searchers is straightforward). Supposing that the searcher is located on some vertex $x$, then

1. we have a \textit{visible} component, defined to be the subgraph induced by the vertices which are visible to the searcher;
2. unless the search problem is trivial, we will also have some invisible components, defined to be the connected components of the graph obtained by removing the visible component; by this definition, the vertices of the invisible components cannot be seen by the searcher.

Hence, given $A$ and $B$, and for every $x \in V$, we have a visible component and one or more invisible components. By definition, the vertices in the searcher’s visible component are clear. On the other hand, some of the vertices of the invisible components may be contaminated. Suppose that at some time the searcher is located at vertex $x$, resulting at the partition of $V$ into $M$ invisible components (we can omit the visible component, since it will always be clear); the contamination of the components is encoded by a vector $\hat{d} = [\hat{d}_1, ..., \hat{d}_M]$, where $\hat{d}_m$ equals 0 (resp. 1) iff the $m$-th component is clear (resp. dirty). The pair $\hat{z} = (x, \hat{d})$ will be called the information state and gives us all the needed information about the current state of the search.

**Example 3.12** Let us clarify the above ideas with an example. Consider the tree of Figure 1. Assume a single searcher with a visibility range of $L = 1$ (i.e., he can see all the vertices which are at distance at most one from his current position). Consequently, the visibility matrix is

$$B = \begin{bmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 \\
\end{bmatrix}$$

Assume the searcher is located on vertex 1; then he can see vertices 1, 2 and 3. Hence, as can be seen in Figure 2, the visible component is $V_1 = \{1, 2, 3\}$ and the invisible components are $V_2 = \{4\}$, $V_3 = \{5\}$, $V_4 = \{6\}$, and $V_5 = \{7\}$.

![Figure 1: A graph to be searched.](image1.png)

![Figure 2: Visible and invisible components of the graph when the searcher is at vertex 4 and has straight line visibility (he can see vertices 1, 2 and 3).](image2.png)

Now suppose the searcher performs the following search schedule: $u(0) = 1 \rightarrow u(1) = 2 \rightarrow u(2) = 1 \rightarrow u(3) = 3$. Then we have the following evolution of clear and contaminated components (in the third column, the visible

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5. Strictly speaking, the components are subgraphs of $G$ but, since $G$ is given, they are fully specified by their vertex sets. Hence “components” will also be used to refer to the corresponding vertex sets.

6. Note that the size of $\hat{d}$ is not fixed since, for different searcher positions, we will have a different number of visibility components.
component vertices are listed in italics).

| Time | Pursuer Location | Visibility Components | Contamination Vector | Information state |
|------|------------------|-----------------------|----------------------|-------------------|
| 0    | \(\lambda\)      | \{1, 2, ..., 7\}      | (1)                 | \((\lambda, 1)\)  |
| 1    | 1                | \{1, 2, 3\} \{4\} \{5\} \{6\} \{7\} | (1, 1, 1, 1)       | \((1, (1, 1, 1, 1))\) |
| 2    | 2                | \{1, 2, 3\} \{6, 7\} | (1)                 | \((2, 1)\)        |
| 3    | 1                | \{1, 2, 3\} \{4\} \{5\} \{6\} \{7\} | (0, 0, 1, 1)       | \((1, (0, 0, 1, 1))\) |
| 4    | 3                | \{1, 3, 6, 7\} \{2, 4, 5\} | (0)                 | \((3, 0)\)        |

Table 1. Visibility components, contamination vectors and information states for a walk through the binary tree of depth 2, with visibility range \(L = 1\).

The entries in the above table can be explained as follows.

1. At \(t = 0\) the searcher is outside the graph. The single invisible component contains all graph vertices.

2. At \(t = 1\) the searcher moves to vertex 1 and sees vertices 2, 3. All of these form a single clear component; each of vertices 4, 5, 6, 7 forms a separate component which is contaminated (may contain the target); hence the contamination vector is \((1, 1, 1, 1)\).

3. At \(t = 2\) the searcher moves to vertex 2. Now he sees vertices 1, 2, 4, 5, all of which form a single clear component and the vertices 3, 6, 7 forms a single contaminated component. Note that vertices 4 and 5 have been cleared and vertex 3 has been recontaminated. Hence the contamination vector is \((1)\).

4. At \(t = 3\) the searcher moves back to vertex 1. The components are the same as at time \(t = 1\), but now they are in a different contamination state, namely, \((0, 0, 1, 1)\). This is so because vertices 4 and 5 were previously clear and, though no longer directly visible, cannot be recontaminated because the target cannot move from 6 or 7 to 4 or 5 without passing through the currently visible component \{1, 2, 3\}.

5. Finally, at \(t = 4\) the searcher moves to vertex 3. The components now are \{1, 3, 6, 7\} (which is visible) and \{2, 4, 5\} (which was clear and cannot be recontaminated). Hence the contamination vector is \((0)\) and the graph has been cleared.

Note how the number of components and the vertices they include change during the search; also note that the same components can, at different times, be clear or contaminated depending on the search history.

**Example 3.13** The previous example employed “distance-based visibility”. Now we present an example with “straight-line visibility”. Consider the graph of Figure 3 in which each vertex is supposed to be actually located in the position indicated in the figure. Assume a single searcher with straight-line visibility (i.e., he can see all the vertices which lie in a straight-line path passing from his current vertex). Consequently, the visibility matrix is

\[
B = \begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

Assume the searcher is at vertex 4; then he can see vertices 3, 4, 5, 7, 8 and 9. Hence, as can be seen in Figure ??, the visible component is \(V_1 = \{3, 4, 5, 7, 8, 9\}\) and the invisible components are \(V_2 = \{1, 2\}\), \(V_3 = \{6\}\), \(V_4 = \{10\}\), and \(V_5 = \{11, 12\}\).
Now suppose the searcher performs the following search schedule: $u(0) = 1 \rightarrow u(1) = 2 \rightarrow u(2) = 3 \rightarrow u(3) = 4$, $u(4) = 9 \rightarrow u(5) = 4 \rightarrow u(6) = 5$. Then we have the following evolution of clear and contaminated components (in the third column, the visible component vertices are listed in italics).

| Time | Pursuer Location | Visibility Components | Contamination Vector | Information state |
|------|------------------|-----------------------|----------------------|-------------------|
| 0    | $\lambda$        | $\{1,2,\ldots,12\}$ | (1)                  | $(\lambda, (1))$  |
| 1    | 1                | $\{1,2\}, \{3,4,\ldots,12\}$ | (1)                  | $(1, (1))$        |
| 2    | 2                | $\{1,2,3\}, \{4,5,\ldots,12\}$ | (1)                  | $(2, (1))$        |
| 3    | 3                | $\{1\}, \{2,3,4,9\}, \{5,6,7,8\}, \{10\}, \{11,2\}$ | (0, 1, 1, 1)         | $(3, (0, 1, 1, 1))$ |
| 4    | 4                | $\{1,2\}, \{3,4,5,7,8,9\}, \{6\}, \{10\}, \{11,12\}$ | (0, 1, 1, 1)         | $(4, (0, 1, 1, 1))$ |
| 5    | 9                | $\{1,2\}, \{3,4,5,7,8,9\}, \{10\}, \{11,12\}$ | (0, 1)               | $(9, (0, 1))$     |
| 6    | 4                | $\{1,2\}, \{3,4,5,7,8,9\}, \{6\}, \{10\}, \{11,12\}$ | (0, 1, 0, 0)         | $(4, (0, 1, 0, 0))$ |
| 7    | 5                | $\{1,2,3\}, \{4,5,6,7,8\}, \{9,10,11,12\}$ | (0, 0)               | $(5, (0, 0))$     |

Table 2. Visibility components, contamination vectors and information states for a walk through the graph with straight line visibility.

The reader can check that the search schedule clears the graph, which is also seen by the final information state $(5, (0, 0))$.

We denote by $\mathcal{Z}$ the set of all information states $(x, \hat{d})$ and by $\mathcal{Z}_C$ the set of all clear information states $(x, [0,\ldots,0])$. There exists a state transition function $F : \mathcal{Z}_C \times V \rightarrow \mathcal{Z}_C$ which maps the current information state and searcher move to the next information state:

$$\mathcal{Z}(t+1) = F(\mathcal{Z}(t), u(t)).$$

While it is difficult to give a general description for $\mathcal{Z}$ and $F$, they can be easily constructed algorithmically for a given graph $G$. The information graph $G_I = (V_I, E_I)$, has $V_I = \mathcal{Z}$ and $E_I$ consists of the state transitions which are described by the state transition function $F$.

**Example 3.14** In Figure 5a we present the information graph corresponding to the graph of Example 3.12 with visibility range $L = 1$. The information graph has $|V_I| = 36$ vertices; compare this to the previously mentioned $V'$ which has $|V'| = |V| \cdot 2^{|V|} + 1 = 7 \cdot 2^7 + 1 = 897$ vertices. In this case, since the information graph is of low complexity, it is easy to check by inspection that the shortest clearing presented in Table 1 also appears in the information graph.

In Figure 5b we see the information graph corresponding to Example 3.13. The information graph has $|V_I| = 76$ vertices; compare this to the previously mentioned $V'$ which has $|V'| = |V| \cdot 2^{|V|} = 12 \cdot 2^{12} + 1 = 49,153$ vertices.
Figure 5: (a) The information digraph for Example 3.12 with visibility range $L = 1$; (b) The information digraph for Example 3.13 with straight-line visibility.

Now we can formulate the LVGS problem as follows.

**Problem 3.15 (Main LVGS Problem)** Given $A$ and $B$ choose, subject to (4), the minimum $T$ and respective $u(0), \ldots, u(T-1)$ such that $[x(T), \hat{d}(T)] \in \hat{Z}_C$.

The problem reduces to finding, in the graph $G_I$, a shortest path from the unique all-dirty state $(\lambda, [1])$ (i.e., when the searcher has not yet been placed in the graph and there is a single invisible and dirty component) to some all-clear state $(x, [0, \ldots, 0]) \in \hat{Z}_C$. Note that we also have to construct the information graph $G_I$ and the transition function $F$. The following LVGS algorithm constructs the graph and finds a shortest path. The pseudocode is rather self-explanatory; here are a few clarifications.

1. Lines 3-7: for each vertex in $G$, a group of vertices is added to $G_I$, each corresponding to a different possible information state.
2. Lines 8-15: edges are added to $G_I$ by following the searcher’s possible transitions between adjacent $G_I$ vertices, by the routine `newInfoState`.
3. Lines 6-20: using Dijkstra’s algorithm, we find a shortest path between the all-dirty vertex and every all-clear vertex of $G_I$.
4. Lines 21-22: the algorithm returns the overall shortest path.

This concludes our formulation and proposed solution of the LVGS problem. It is worth emphasizing that, in both formulation and solution, the only decision maker is the searcher; the target has been “factored out” of the problem. This is achieved by adopting a worst case analysis, namely that every possible contamination is actually realized.

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7 Actually we have two versions of this routine: (a) the first version corresponds to sequential moves (i.e., first the searcher moves with unit speed and the the target moves with unbounded speed); (b) the second version implements concurrent moves by both searcher and target and is somewhat more complicated (details are omitted, in the interest of simplicity).

8 For very large $G_I$ we can instead use a depth-first search starting at $(\lambda, (1))$; this will find a clearing search schedule, though not the shortest one.
The LVGS Algorithm

1: function LVGS(A, B)
2: Let G = (V, E) be the graph with adjacency matrix A
3: for i ∈ V do ▷ Build vertex set V_I of G_I = (V_I, E_I)
4: for every possible c(i) do ▷ c(i): contamination state of i
5: add vertex (i, c(i)) to V_I
6: end for
7: end for
8: for i ∈ V do ▷ Build edge set E_I of G_I = (V_I, E_I)
9: for j ∈ N[i] do
10: for every possible c(i) do ▷ Using algorithm newInfoState
11: calculate c(j) = newInfoState(A, B)
12: add edge (i, c(i)) → (j, c(j)) to E_I
13: end for
14: end for
15: end for
16: Let u = (λ, [1, ..., 1]), the all dirty state of V_I
17: Let V_I^c be the set of clear states of V_I
18: for v ∈ V_I^c do
19: (BestCost(v), BestPath(v)) = Dijkstra(G_I, u, v) ▷ Using Dijkstra’s algorithm
20: end for
21: Let ̂v = arg min_{v ∈ V_I^c} BestCost(v)
22: return (BestCost(̂v), BestPath(̂v))
23: end function

4 Experiments

In this section we give examples of the application of our algorithm to various graph families. We only present examples for “distance-based visibility”, i.e., the visibility matrix determined by the visibility range parameter L. Specifically, when the cop is located at vertex x, he can see the vertices y which are within L edges of x, i.e.,

$$B_{xy} = \begin{cases} 
1 & \text{iff } d(x,y) \leq L, \\
0 & \text{otherwise.}
\end{cases}$$

4.1 Paths

We first apply the LVGS algorithm to paths. This serves as a “reality check” since, clearly, a single searcher can clear a path by sweeping from one end to the other. In all our experiments the LVGS algorithm produced a clearing schedule of minimum length (minimum number of steps) which is given by the following formula: with path length N and visibility range L, the number of required steps is

$$T_{\text{min}} = \max(N - (2L + 1), 0).$$

4.2 Complete Binary Trees

The next family of graphs we examine are complete binary trees. In Table 3 we see the results for trees of depth D ∈ {1, ..., 4} and visibility range L = {1, ..., 5}. Listed in every cell is the length of an optimal search trajectory and computation time (in seconds). Note that, for D = 4 and visibility range L = 1, the cell contains ∞, which means that the searcher cannot clear the graph. Also note that computation times increase significantly with
tree depth, even for high visibility range. E.g., for $D = 4$ and visibility range $L = 3$ the cell contains n/a; this means that the algorithm did not terminate after running for 5 hours. The reason for the high execution times is that trees of higher depth must be partitioned into many components. For example, in the complete binary tree of depth 4 with visibility range $L = 3$ the searcher from vertex 1 can see the entire tree except for the leaves. Since every leaf is a different component, we have 16 invisible components which contribute to $G_{I}^{2} = 65536$ vertices. This is a case in which the computational burden of finding a shortest clearing path in $G_{I}$ is unmanageable.

| Tree Depth | Visibility Range $L$ | Tree Depth | Visibility Range $L$ |
|------------|----------------------|------------|----------------------|
| 1          | 0 0 0 0 0            | 1          | 0.48 0.05 0.02 0.02 0.04 |
| 2          | 2 0 0 0 0            | 2          | 3.52 0.15 0.09 0.01 0.01 |
| 3          | 8 2 0 0 0            | 3          | 31.14 82.22 0.65 0.61 0.41 |
| 4          | ∞ 8 n/a 0 0          | 4          | 228.65 3063.17 > 18300.00 7.13 8.23 |

Table 3: Complete binary trees: optimal length of clearing search schedule and computation times in seconds.

4.3 Grids

The next type of graph we look into are grids. In Table 4 we see search schedule lengths and computation times for $M \times N$ grids with $M, N \in \{2, ..., 6\}$. Apparently grids are actually easier to handle than trees.

| Grid Size | Visibility Range $L$ | Grid Size | Visibility Range $L$ |
|-----------|----------------------|-----------|----------------------|
| 2x2       | ∞ 0 0 0              | 2x2       | 0.81 0.03 0.02 0.01 |
| 2x3       | ∞ 0 0 0              | 2x3       | 1.31 0.05 0.02 0.01 |
| 2x4       | ∞ 1 0 0              | 2x4       | 2.01 0.91 0.04 0.01 |
| 2x5       | ∞ 2 0 0              | 2x5       | 3.70 2.45 0.12 0.05 |
| 2x6       | ∞ 3 1 0              | 2x6       | 6.05 5.17 2.48 0.14 |
| 3x3       | ∞ 0 0 0              | 3x3       | 3.21 0.11 0.04 0.01 |
| 3x4       | ∞ 1 0 0              | 3x4       | 6.32 6.19 0.18 0.05 |
| 3x5       | ∞ 2 0 0              | 3x5       | 10.40 17.66 0.48 0.22 |
| 3x6       | ∞ 3 1 0              | 3x6       | 15.89 55.65 59.82 1.38 |
| 4x4       | ∞ ∞ 1 0              | 4x4       | 7.12 15.49 8.41 0.18 |
| 4x5       | ∞ ∞ 1 0              | 4x5       | 14.01 46.19 28.38 0.76 |
| 4x6       | ∞ ∞ 3 1              | 4x6       | 19.81 67.20 86.95 35.32 |
| 5x5       | ∞ ∞ 2 0              | 5x5       | 15.89 55.65 59.82 1.38 |
| 5x6       | ∞ ∞ 3 1              | 5x6       | 41.52 139.59 182.42 110.65 |
| 6x6       | ∞ ∞ ∞ 3              | 6x6       | 32.36 90.70 204.52 159.80 |

Table 4: $M \times N$ grids: optimal length of clearing search schedule and computation times in seconds.

4.4 Randomly Generated Trees

In this section we apply the LVGS algorithm to trees in which children are generated probabilistically. Specifically, we start with a complete binary tree of depth one (one root with two children) and every vertex (except the root) can have $n \in \{0, 1, 2\}$ children with probability $p_n$ where $[p_0, p_1, p_2] = [\frac{1}{3}, \frac{1}{3}, \frac{1}{3}]$. We continue adding children to every leaf until the tree reaches a maximum depth value $D_{max}$. Hence a family of random binary trees is fully described by the parameters $\mathbf{p} = [\frac{1}{3}, \frac{1}{3}, \frac{1}{3}]$ and $D_{max}$. In Table 5 we present, for visibility range $L \in \{1, ..., 4\}$ and maximum depth $D_{max} = 5$, some statistics averaged over 100 randomly generated trees.

However, substituting a depth first search for Dijkstra’s algorithm yields a (non-optimal) clearing path in a few seconds.
Table 5: Statistics for 100 randomly generated trees with $D_{\text{max}} = 5$.

In Table 6 we present similar results for 100 trees with maximum depth $D_{\text{max}} = 6$, and for visibility range $L \in \{1, ..., 4\}$.

Table 6: Statistics for 100 randomly generated trees with $D_{\text{max}} = 6$.

4.5 Grids with Randomly Deleted Edges

In this section we use graphs which are created by deleting edges from a full grid; for brevity, these will be referred as “deleted grids”. The specific method by which the edges are deleted is the following. We start with a full grid and, for every edge $e$ of the grid:

1. If removal of $e$ would result in a disconnected graph, leave $e$ in place.
2. Otherwise remove $e$ with probability $p \in [0, 1]$ (a parameter).

The above procedure results in a deleted grid; higher $p$ values result in sparser graphs but, by construction, the resulting graph is always connected. Results for $N \times N$ deleted grids are presented in the following Tables 7, 8 and 9.

Table 7: Statistics for 100 $3 \times 3$ deleted grids.
| p = 1/2 | Visibility Range L | 1 | 2 | 3 | 4 |
|---------|---------------------|---|---|---|---|
| Number of Cleared Graphs | 44/100 | 91/100 | 96/100 | 99/100 |
| Average Clearing Length | 10.57 | 6.79 | 3.97 | 1.86 |
| Maximum Clearing Length | 13 | 10 | 8 | 6 |
| Minimum Clearing Length | 9 | 4 | 1 | 0 |
| Average Calculation Time | 15.44 | 22.13 | 17.68 | 8.92 |
| Maximum Calculation Time | 42.02 | 62.15 | 46.21 | 21.42 |
| Minimum Calculation Time | 6.80 | 8.82 | 8.72 | 0.40 |

Table 8: Statistics for 100 4 × 4 deleted grids.

| p = 1/2 | Visibility Range L | 1 | 2 | 3 | 4 |
|---------|---------------------|---|---|---|---|
| Number of Cleared Graphs | 6/100 | 67/100 | 90/100 | 95/100 |
| Average Clearing Length | 17.67 | 12.97 | 8.81 | 5.73 |
| Maximum Clearing Length | 21 | 18 | 14 | 12 |
| Minimum Clearing Length | 16 | 8 | 4 | 1 |
| Average Calculation Time | 43.33 | 99.51 | 118.8 | 100.1 |
| Maximum Calculation Time | 118.19 | 282.3 | 438.73 | 378.94 |
| Minimum Calculation Time | 18.30 | 37.70 | 43.56 | 41.53 |

Table 9: Statistics for 100 5 × 5 deleted grids.

5 Conclusion

We have formulated a graph search problem, with searchers looking for a mobile target under the following assumptions: (a) the visibility field of the searchers is limited, (b) movement is along the graph edges, (c) the searchers have unit speed and (d) the target has infinite speed.

To solve the problem we have presented the LVGS algorithm, which is a conversion the polygonal region search algorithm of [15, 16]. This algorithm is guaranteed to always find a solution if one exists. Furthermore, the LVGS algorithm can accommodate every kind of visibility conditions (0-visibility, distance-based visibility of arbitrary range and straight line visibility) using the visibility matrix. While the problem formulation can accommodate an arbitrary number of searchers and arbitrary searcher and target speed, the LVGS algorithm is particularly suited for problems involving a single searcher, movement along the edges of the graph, unit speed for the searcher and infinite speed for the target.

We conclude this paper by listing some research directions which we intend to pursue in the future.

1. The polygonal region search algorithm of [15, 16] is complex to implement and can be computationally impractical for complicated regions. An alternative strategy to handle polygonal regions is to first approximate them by discretization and then study the search problem on a graph equivalent to the discretized region. We believe it will be interesting to compare the performance of this approach combined with our LVGS algorithm to that of the original polygonal region algorithm.

2. In the Introduction we have briefly mentioned the connection of graph search to pursuit in graphs, especially to the problem of the cops and invisible (or partially visible) robber. We believe that this connection merits further study and that the LVGS algorithm could provide solutions competitive to the ones obtained by combinatorial arguments [8, 9, 10, 11, 12].

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