CONSISTENCY OF M-THEORY ON NONORIENTABLE MANIFOLDS

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Abstract. We prove that there is no parity anomaly in M-theory in the low-energy field theory approximation. Our approach is computational. We determine generators for the 12-dimensional bordism group of pin manifolds with a \( w_1 \)-twisted integer lift of \( w_4 \); these are the manifolds on which Wick-rotated M-theory exists. The anomaly cancellation comes down to computing a specific \( \eta \)-invariant and cubic form on these manifolds. Of interest beyond this specific problem are our expositions of: computational techniques for \( \eta \)-invariants, the algebraic theory of cubic forms, Adams spectral sequence techniques, and anomalies for spinor fields and Rarita-Schwinger fields.

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1. Introduction

Time-reversal symmetry is a topic of renewed interest, in part because of its prevalence in condensed matter models. Under Wick rotation time-reversal is connected to reflections in Euclidean space, and time-reversal symmetric theories may be formulated on unoriented manifolds. The obstruction to doing so is often termed a “parity anomaly”, though ‘parity’ is not synonymous with ‘time-reversal symmetry’.\(^1\) Witten [W2] recently showed that there is no anomaly for this symmetry on an M2-brane in M-theory. He suggested that we investigate the analogous issue in the bulk on suitable 11-manifolds. We do so here and prove that there is no time-reversal anomaly in M-theory.

We work in the low-energy field theory approximation to M-theory, which is classical 11-dimensional supergravity with a gravitational correction term. The theory includes a fermionic field, and so \(X\) carries a pin structure—the appropriate choice is a pin\(^+\) structure, as opposed to a pin\(^-\) structure—on the tangent bundle. The \(C\)-field in M-theory, which is odd under time-reversal symmetry, induces an additional topological structure on \(X\): a \(w_1\)-twisted integer lift of the fourth Stiefel-Whitney class \(w_4\); see [W3, §2.3]. A pin\(^+\)-manifold with a \(w_1\)-twisted integer lift of \(w_4\) is called an \(m_c\)-manifold. There are two sources of anomalies. The first is the standard fermion anomaly, though there are subtleties: the fermion field is a Rarita-Schwinger field, rather than a spinor field, and the background is a pin manifold, rather than a spin manifold. The second anomaly is nonstandard, due to the cubic form for the \(C\)-field. In the spin case Witten [W3, §4] represents the \(C\)-field as a connection on a principal \(E_8\)-bundle, and he uses this to prove that these two anomalies cancel. In the pin case this argument is not available, so we resort to a computational approach. Each anomaly is encoded in an invertible unitary topological 12-dimensional field theory, and hence is determined by its partition function. Furthermore, the partition function is a bordism invariant, so it suffices to check that the partition functions of the two theories agree on a set of generators for the appropriate bordism group. We use the Adams spectral sequence, together with computer assistance and geometric arguments, to compute a set of generators for the relevant bordism group. We deploy a mix of topological and geometric techniques to compute the partition functions on these generators, and so prove anomaly cancellation.

To define M-theory we must not only prove that anomalies cancel, but provide data which performs the anomaly cancellation. In the spin case, ignoring time-reversal symmetry, this “setting of the quantum integrand” can be achieved using Witten’s \(E_8\)-bundle technique [FM]. We do not know a canonical setting in the pin case, and indeed isomorphism classes of settings form a torsor.

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\(^1\)‘Parity’ typically refers to a spatial reflection through a point in Minkowski spacetime, relative to a splitting into time cross space. As this is orientation-preserving in even space dimensions, more relevant is reflection in a timelike hyperplane, which is always orientation-reversing. A time-reversal is reflection in a spacelike hyperplane.
over isomorphism classes of 11-dimensional invertible field theories on the same class of manifolds. The latter group is isomorphic to \( \mathbb{Z}/2\mathbb{Z} \), at least conjecturally, as we explain in §7. Since this group is nonzero, the problem of setting the quantum integrand remains open.

Now we give a more detailed summary. We begin in the expository §2 by reviewing the Wick-rotated setting for M-theory as a theory on a certain geometric bordism category. We recall that the anomaly of an 11-dimensional theory is an invertible 12-dimensional theory \( \alpha \), and that invertible topological theories may be represented as maps of spectra in stable homotopy theory. The aim of this paper is to prove Theorem 2.2: the tensor product \( \alpha_{RS} \otimes \alpha_C \) of the anomaly theories arising from the Rarita-Schwinger and \( C \)-fields of M-theory is trivializable. In §3 we define \( \alpha_{RS} \); As with the anomaly of any fermionic field its partition function is the exponential of an Atiyah-Patodi-Singer \( \eta \)-invariant. We elucidate some aspects of the general theory in Appendix A, and in §3.2 we give a general formula for the anomaly theory of a Rarita-Schwinger field; see (3.12). In our situation the anomaly partition function is independent of the Riemannian metric, so is a topological invariant. It turns out to be \( \pm 1 \) on \( \mathfrak{m}_c \)-manifolds, though at this stage the only apparent statement is that it is a root of unity. Indeed, on a general \( \text{pin}^+ \) manifold it does not necessarily have order 2. We develop formulas to compute it, following work of Donnelly, Stolz, and Zhang. Of particular interest is a topological formula which, as far as we know, has only an analytic proof in the literature [Z].\(^2\) The partition function of the anomaly theory \( \alpha_C \) is an inhomogeneous cubic polynomial in the \( C \)-field. It too is topological and by definition is equal to \( \pm 1 \). In §4 we develop an algebraic theory of the cubic form, imitating the standard algebraic theory of quadratic forms, and then define \( \alpha_C \). We also review Witten’s proof that \( \alpha_{RS} \otimes \alpha_C \) is trivializable when restricted to spin manifolds. Section 5 is a geometric interlude to review some basic spin and \( \text{pin}^+ \) manifolds and their topological invariants. We also introduce more complicated manifolds used as representative elements of bordism groups. Our main computational result, whose proof we sketch in §8, is stated as Theorem 6.1. We define a finitely generated abelian group \( A \) which surjects onto the relevant 12-dimensional bordism group, and then specify generators of \( A \) which we represent by specific 12-dimensional \( \mathfrak{m}_c \)-manifolds. For each of these we compute that the partition function of \( \alpha_{RS} \otimes \alpha_C \) vanishes, which suffices to demonstrate the anomaly cancellation. We employ a potpourri of techniques to make the computations. The aforementioned ambiguity in the definition of M-theory is discussed in Section 7. Section 8 contains a computation of the low dimensional bordism groups of \( \mathfrak{m}_c \)-manifolds. In particular, we provide a proof of Theorem 6.1. A more detailed computer-free version of these computations will appear in [GH].

Aspects of this paper have interest beyond our proof that M-theory is time-reversal invariant. This includes the algebraic theory of cubic forms in §4; our techniques to compute \( \eta \)-invariants of pin manifolds; the Adams spectral sequence techniques in §8; the discussion of spinor field anomalies in Appendix A; and the interplay between invertible unitary topological field theories and stable homotopy theory, which is developed and plays a key role in an application to condensed matter physics in [FH].

The authors take this opportunity to express our deep sense of gratitude and indebtedness to Michael Atiyah for his mentoring, encouragement, and support. Michael’s enthusiasm for mathematics and for its interaction with physics has long been an inspiration. We appreciate his unfailing

\(^2\)We thank Jonathan Campbell for pointing us to Zhang’s paper.
sense of what constitutes an enlightening and “correct” proof, and we join him in lamenting the lack of such a proof for this anomaly cancellation.

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2. Time-reversal, anomalies, and bordism

A relativistic quantum field theory on \(n\)-dimensional Minkowski spacetime \(M^n\) has a symmetry group \(\mathcal{H}_{1, n-1}\), equipped with a homomorphism to the group of isometries of \(M^n\). (See [FH, §2] for an account of symmetry groups in quantum field theory.) Divide by translations and Wick rotate to Euclidean signature to obtain a compact Lie group \(H_n\) of vector symmetries, equipped with a homomorphism \(\rho_n: H_n \to O_n\) whose image is (i) \(SO_n\) in the absence of time-reversal symmetry, or (ii) \(O_n\) if the theory has time-reversal symmetry. Eleven-dimensional M-theory has both time-reversal symmetry and fermionic fields, and no additional global symmetries, \(^3\) so the Wick-rotated symmetry group is one of the two \(\text{Pin}\) groups. Because time-reversal squares in Minkowski spacetime to \((-1)^F\), the appropriate group is \(H_{11} = \text{Pin}^+_1\); see [FH, Appendix A].

We consider M-theory on curved compact 11-dimensional Riemannian manifolds \(X\), and so we require that \(X\) have a tangential \(\text{pin}^+\) structure. \(^4\) There is an additional topological structure, first identified in [W3, §2.3]. The \(C\)-field is an abelian gauge field, thus obeys a Dirac quantization condition. The correct condition is that the de Rham cohomology class of its field strength, a closed 4-form twisted by the orientation bundle, \(^5\) refines to a \(w_1\)-twisted integer cohomology class \(c \in H^4(X; \mathbb{Z})\) whose mod 2 reduction is the fourth Stiefel-Whitney class \(w_4(X)\). Here \(\mathbb{Z}\) is the local coefficient system induced from the orientation double cover of \(X\). This motivates the following.

Definition 2.1. Let \(M\) be a \(\text{pin}^+\) manifold. An \(m_c\) structure\(^6\) on \(M\) is a \(w_1\)-twisted integer lift of \(w_4(M)\). We say \(M\) is an \(m_c\)-manifold if \(M\) is equipped with an \(m_c\) structure.

A necessary and sufficient condition to be \(m_c\) is \(\beta w_4(M) = 0\). The Wick rotation of M-theory is defined on a geometric bordism category of \(m_c\)-manifolds.

Once an \(n\)-dimensional field theory is formulated on compact Riemannian manifolds, then there is the possibility of an anomaly: the partition function may not be well-defined as a complex number, but rather may be an element of a complex line. These complex lines depend \textit{locally} on the Riemannian manifold, which is expressed by saying that they are the quantum state spaces of a field theory \(\alpha\). The theory \(\alpha\) is called a \textit{gravitational anomaly}. In addition to the coupling to a gravitational background, if the kernel of \(\rho_n: H_n \to O_n\) is nontrivial then there is also a coupling to a background gauge field, \(^7\) in which case we have a mixed gravitational and gauge anomaly. In

\(^3\)One could regard the \(C\)-field as the background field for a higher symmetry, but as the primary objects of interest are the background fields we do not pursue this point of view.

\(^4\)Equivalently, the stable normal bundle of \(X\) has a \(\text{pin}^+\) structure.

\(^5\)On the orientation double cover of \(X\) it lifts to a closed 4-form which is odd under the deck transformation.

\(^6\)The name is taken from [W2, §4.3], where it is introduced by analogy with a \(\text{Spin}^c\) structure.

\(^7\)which may be twisted by the tangent bundle, as in a \(\text{Spin}^c\) structure, for example.
most examples the anomaly theory $\alpha$ extends to an $(n+1)$-dimensional theory which has a partition function on closed $(n+1)$-manifolds. That is so in this paper. The 11-dimensional M-theory is not rigorously defined, but nonetheless we do define the 12-dimensional anomaly theory that is our main focus. Anomalies are very special among field theories: they are invertible. Recall that field theories have a composition law of tensor product, and there is a trivial theory $1$ which is an identity for this composition law. So a field theory $\alpha$ is invertible if there exists a theory $\beta$ such that $\alpha \otimes \beta$ is isomorphic to $1$. An invertible field theory has nonzero partition functions, one-dimensional state spaces, etc. We refer to [F1] and the references therein for exposition on this point of view about anomalies.

Recall that M-theory has two bosonic fields—a metric and $C$-field—and a single fermionic field—the Rarita-Schwinger field $\psi$. To analyze anomalies we work in the effective theory after integrating out $\psi$; the metric and $C$-field are treated as background fields. One source of anomalies is the fermionic integration of $\psi$, which we review in §3. Let $\alpha_{RS}$ denote that 12-dimensional anomaly theory. The other source of anomalies is the “Chern-Simons coupling” of the $C$-field, which is an inhomogeneous cubic form we review in §4. Let $\alpha_C$ denote that 12-dimensional anomaly theory. Our main result is the cancellation of these anomalies.

**Theorem 2.2.** The total anomaly theory $\alpha_{RS} \otimes \alpha_C$ is trivializable.

That is, $\alpha_{RS} \otimes \alpha_C \cong 1$. This implies that M-theory should exist as an “absolute” quantum field theory whose partition functions are complex numbers (not merely elements of an abstract complex line), whose state spaces are vector spaces (not merely well-defined as projective spaces), etc. In other words, M-theory is anomaly-free. As explained in [W2, §1] this is a strong form of the vanishing of the “parity anomaly”.

An important feature is that both $\alpha_{RS}$ and $\alpha_C$ are topological field theories. That means they are each independent of the metric and $\alpha_C$ only depends on the $C$-field through its topology.\(^8\) Furthermore, as already stated these theories are invertible. Finally, due to their physical origins\(^9\) these theories are unitary, or equivalently in the Wick-rotated version they are reflection positive. The main theorem in [FH] asserts that, assuming reasonable ansätze, reflection positive invertible topological field theories live in the world of stable homotopy theory: they are spectrum maps from a Thom spectrum to a universal target, the shifted Pontrjagin dual to the sphere spectrum. This result uses a strong form of locality—a fully extended field theory—and also a companion strong form of reflection positivity for invertible topological theories. Thus the anomaly theories are maps

$$\alpha_{RS}, \alpha_C : Mm_c \longrightarrow \Sigma^{12} IC^\times.$$  

Here $Mm_c$ is the Thom spectrum of $m_c$-manifolds: manifolds with a stable tangential pin\(^+\) structure and a $w_4$-twisted integer lift of $w_4$. We construct $Mm_c$ in §8.1. Also, $IC^\times$ is the character dual to the sphere spectrum, closely related to the Brown-Comenetz dual [BC]. The universal property which characterizes $IC^\times$ (see [FII, §5.3]) implies that the group of homotopy classes of maps (2.3) is isomorphic to the group $\text{Hom}(\pi_{12}Mm_c, \mathbb{C}^\times)$ of characters of $\pi_{12}Mm_c$. In other words,

\(^8\)A $C$-field on $X$ represents a class in the twisted differential cohomology group $\tilde{H}^4(X; \mathbb{Z})$ (§4.5); the statement is that the anomaly theory $\alpha_C$ only depends on the representative of its image under $\tilde{H}^4(X; \mathbb{Z}) \to H^4(X; \mathbb{Z})$.

\(^9\)Invertible topological theories (not necessarily unitary) have domain Madsen-Tillmann spectra; see [FHT], [S-P].
the maps (2.3) are determined up to homotopy—and the corresponding topological field theories up to isomorphism—by abelian group homomorphisms

\[ \hat{\alpha}_{RS}, \hat{\alpha}_C : \pi_{12} M \mathfrak{m} \longrightarrow \mathbb{C}^\times. \]

These homomorphisms encode the partition functions of the respective anomaly theories. We prove Theorem 2.2 by demonstrating that the product

\[ \hat{\alpha}_{RS} \cdot \hat{\alpha}_C : \pi_{12} M \mathfrak{m} \longrightarrow \mathbb{C}^\times \]

of partition functions is identically one. Both \( \hat{\alpha}_{RS} \) and \( \hat{\alpha}_C \) take values in the group \( \mathbb{T} \subset \mathbb{C}^\times \) of unit norm complex numbers. From its definition (4.52), the homomorphism \( \hat{\alpha}_C \) takes values in \( \{ \pm 1 \} \subset \mathbb{C}^\times \), and so the field theory \( \alpha_C \) has order two: its square is isomorphic to the trivial theory. It emerges from our computations that \( \hat{\alpha}_{RS} \) also has order two.\(^{10}\)

Theorem 2.2 asserts that the total anomaly is trivializable but does not specify a trivialization. (For further discussion, see [FM] where the trivialization is called a “setting of the quantum integrand”.) Homotopy classes of trivializations form a torsor over the group of invertible 11-dimensional reflection positive topological theories. That is, given one trivialization, and so in principle one realization of M-theory, any other one differs by inserting a “topological term” in the 11-dimensional theory. In §7, based on computations to appear in [GH], we discuss the following.

**Conjecture 2.6.** The group of homotopy classes of spectrum maps \( M \mathfrak{m} \rightarrow \Sigma^{11} I \mathbb{C}^\times \) is isomorphic to \( \mathbb{Z}/2\mathbb{Z} \). The partition function of the nontrivial theory is the mod 2 index of the Dirac operator.

### 3. The Rarita-Schwinger operator and \( \eta \)-invariants

The reader may want to refer to recent expositions of fermions and anomalies in [W1, W2]. We recall the relation between fermion integrals and pfaffians in §3.1. In §3.2 we indicate the modifications engendered by a Rarita-Schwinger field, as opposed to a spinor field, and then specialize to M-theory and define the Rarita-Schwinger anomaly theory \( \alpha_{RS} \). This all relies on material in Appendix A. In §3.3 we recall and prove some properties of and formulas for \( \eta \)-invariants on \( \text{pin}^+ \) 12-manifolds that we use in our subsequent computations (§6). This exponentiated \( \eta \)-invariant is topological—indeed of the Riemannian metric—and there is a topological formula (Theorem 3.28) for its value.

#### 3.1. Brief recollection of free fermionic path integrals

Suppose \( W \) is a finite dimensional complex vector space and

\[ B : W \times W \longrightarrow \mathbb{C} \]

\(^{10}\)\( \alpha_{RS} \) is pulled back from an invertible theory defined on \( M \text{TPin}^+ \) which has order \( 2^8 \).
a skew-symmetric bilinear form. We identify \( B \) as a skew-symmetric map \( W \rightarrow W^* \), and so an element \( \omega_B \in \bigwedge^2 W^* \). The natural integral on the exterior algebra is the linear map ¹¹

\[
\int_{\text{IIW}} : \bigwedge^* W^* \rightarrow \text{Det } W^*
\]

which projects a form of mixed degree to its highest degree component in \( \text{Det } W^* = \bigwedge^\text{max} W^* \). If \( \dim W = 2m \) is even, then

\[
\int_{\text{IIW}} \omega_B = \frac{\omega_B^m}{m!} = \text{pfaff } B \in \text{Det } W^*
\]

is the \textit{pfaffian} of \( B \); if \( \dim W \) is odd, then the integral vanishes. It is natural to regard \( \text{Det } W^* \) as \( \mathbb{Z}/2\mathbb{Z} \)-graded by the parity of \( \dim W \), which is equal to the parity of the dimension of the null space \( \ker B \). There is an infinite dimensional version of the pfaffian for \( W \) a Hilbert space and \( B \) a \textit{Fredholm form}: \( B \) is Fredholm if \( \ker B \) is a closed, finite dimensional subspace. Then the \( \mathbb{Z}/2\mathbb{Z} \)-graded line \( \text{Pfaff } B \) depends on \( B \) and forms a nontrivial complex line bundle over the space ²² of Fredholm skew forms, and the pfaffian elements

\[
\text{pfaff } B \in \text{Pfaff } B
\]

form a section of the Pfaffian bundle.

\textit{Remark 3.5.} There are real and quaternionic refinements; the latter applies to M-theory on spin manifolds [FM, §1].

\textit{Remark 3.6.} Suppose \( T : U \rightarrow V \) is a linear map between complex vector spaces. Set \( W = V^* \oplus U \) and

\[
B((v_1^*, u_1), (v_2^*, u_2)) = \langle v_1^*, Tu_2 \rangle - \langle v_2^*, Tu_1 \rangle.
\]

Then the Pfaffian line of \( B \) is canonically isomorphic to the determinant line of \( T \), and under that isomorphism \( \text{pfaff } B = \text{det } T \).

A spinor field in an \( n \)-dimensional relativistic field theory on Minkowski spacetime (without time-reversal symmetry) is specified by a real spinor representation \( S \) of \( \text{Spin}_{1,n-1} \) together with a symmetric nonnegative \( \text{Spin}_{1,n-1} \)-invariant bilinear form \( \Gamma : S \times S \rightarrow \mathbb{R}^{1,n-1} \); see §A.1 for details. The complexification \( S_C \) is a representation of the compact spin group \( \text{Spin}_n \). On a closed Riemannian spin \( n \)-manifold \( X \) there is an associated complex vector bundle whose sections are spinor fields \( \psi \). Define the complex skew-symmetric form

\[
B_X(\psi_1, \psi_2) = \int_X \Gamma_C(\psi_1, \nabla \psi_2) |dx|,
\]

¹¹The odd vector space \( \text{IIW} \), the parity-reversal of \( W \), has as its ring of functions the \( \mathbb{Z}/2\mathbb{Z} \)-graded exterior algebra \( \bigwedge^* W^* \). The fermionic integration (3.2) is purely algebraic—there is no measure—and it is defined on functions rather than forms or densities.

²²That space has two components distinguished by the parity of \( \dim \ker B \), the mod 2 index. Over each component the Pfaffian line bundle represents a generator of \( H^2(\cdot; \mathbb{Z}) \). The pfaffian section vanishes if and only if \( \ker B \neq 0 \).
where \( \nabla \) is induced from the Levi-Civita covariant derivative, \( \Gamma_C \) is the complexification of \( \Gamma \), and \( |dx| \) is the Riemannian measure. On appropriate function spaces \( B_X \) is Fredholm. The Feynman path integral over \( \psi \) is the formal analog of (3.3), and we define the result to be (3.4), the pfaffian element of the Pfaffian line. In particular, the fermionic path integral is anomalous.

As explained in §2 the Pfaffian line of the Dirac form \( B_X \) is the quantum state space on \( X \) of an invertible \( (n + 1) \)-dimensional field theory \( \alpha \), called the anomaly theory. To define its partition function we must use the data \( (S, \Gamma) \) to define a Dirac operator on a Riemannian spin \( (n + 1) \)-dimensional manifold \( W \). That construction is carried out in Appendix A; the partition function \( \alpha(W) \) for \( W \) closed is the exponentiated \( \eta \)-invariant (A.10).

### 3.2. The Rarita-Schwinger anomaly

The Rarita-Schwinger field occurs in theories of supergravity; it is the super-partner to the metric. In \( n \) spacetime dimensions there is an associated anomaly theory, which is an \( (n + 1) \)-dimensional invertible field theory, just as for a spinor field. Here we explain the modifications to the discussions in the previous section and §A.1 required to specify the anomaly theory. More information may be found in [FJ], [FM, Appendix A] and the references therein.

Suppose given data \( (S, \Gamma) \) for a spinor field in \( n \)-dimensional Minkowski spacetime \( M^n \), as above. Let \( V \) be the standard \( n \)-dimensional real representation of \( \text{Spin}_{1,n-1} \). The Rarita-Schwinger field is a function \( \chi : M^n \to S \otimes V \). (More precisely, we should view \( S \otimes V \) as an odd super vector space.) There is a correspondence between free fields and particles, and under this correspondence the Rarita-Schwinger field gives rise to four particles, as recounted in [FJ, §A.2]: a single spin \( 3/2 \) particle, which is the desired gravitino, as well as three spurious spin \( 1/2 \) particles. Two of the spin \( 1/2 \) particles are associated to spinor fields with values in \( S \) and the third to a spinor field with values in \( S^* \). Wick rotation of \( \chi \) proceeds by complexification, and one obtains a skew form \( B_X' \) analogous to (3.8), now built on sections of a spinor bundle tensored with the tangent bundle. To eliminate the extra spin \( 1/2 \) particles, we divide the pfaffian of \( B_X' \) by the product of the pfaffians of the forms \( B_X \) associated to the three Wick rotated spin \( 1/2 \) fields [FJ, §A.5].

We now determine the associated anomaly theory, which is an invertible \( (n + 1) \)-dimensional field theory of Riemannian spin manifolds. In Appendix A we define the anomaly theory \( \alpha_S \) associated to spinor data \( (S, \Gamma) \). There is a variation which gives the anomaly of the pfaffian of \( B_X' \). Motivated by (A.7) define\(^\text{13}\)

\[
E' = \text{Cliff}_{+2} \otimes (S \oplus S^*) \otimes \mathbb{R}^{n+1}.
\]

Let \( \text{Spin}_{n+1} \) act as after (A.7) on \( E = \text{Cliff}_{+2} \otimes (S \oplus S^*) \) and tensor with the usual vector representation on \( \mathbb{R}^{n+1} \). There is a commuting \( \text{Cliff}_{-3} \) action, as after (A.7), and the Dirac operator (A.8) and exponentiated \( \eta \)-invariant (A.10) are defined. Denote the resulting \( (n + 1) \)-dimensional theory as \( \alpha_{S\otimes V} \). Since \( \text{Spin}_n \) acts reducibly on \( \mathbb{R}^{n+1} = \mathbb{R} \oplus \mathbb{R}^n \), the specialization to a product manifold \( \mathbb{R} \times X \) gives the Dirac operator coupled to the tangent bundle plus an extra copy of the Dirac

\(^{13}\)The real Clifford algebra \( \text{Cliff}_{p,q} \) has \( p \) generators squaring to \( +1 \) and \( q \) generators squaring to \( -1 \). Set \( \text{Cliff}_{+p} = \text{Cliff}_{p,0} \) and \( \text{Cliff}_{-q} = \text{Cliff}_{0,q} \).
operator on spinor fields. Therefore, the anomaly theory associated to the pfaffian of $B'_{X}$ is

\begin{equation}
\alpha_{S\otimes V} \otimes \alpha_{S}^{(-1)}.
\end{equation}

Put (3.10) together with the anomalies of the spurious spin $1/2$ fields to obtain the total anomaly

\begin{equation}
\alpha_{S\otimes V} \otimes \alpha_{S}^{(-3)} \otimes \alpha_{S^{*}}^{(-1)}.
\end{equation}

Proposition A.18 implies that the product $\alpha_{S} \otimes \alpha_{S^{*}}$ is trivializable. Hence (3.11) is isomorphic to

\begin{equation}
\alpha_{S\otimes V} \otimes \alpha_{S}^{(-2)}.
\end{equation}

This is a general formula for the anomaly theory of a Rarita-Schwinger field in $n$ dimensions built from the spinor data $(S, \Gamma)$.

Now we specialize to $n = 11$ and M-theory on pin$^+$ manifolds. The spinor data $(S, \Gamma)$ is specified in §A.2. Let $W$ be a closed Riemannian pin$^+$ 12-manifold. We compute the partition function $\hat{\alpha}_{RS}(W)$ of the total anomaly theory (3.12). The (s)pinor bundle on $W$ is a rank 32 quaternionic bundle, and the appropriate Dirac operator (A.8) is a self-adjoint operator on its sections. The partition function (A.10) is, in this case, a ratio of exponentiated $\eta$-invariants, which we write as

\begin{equation}
\hat{\alpha}_{RS}(W) = \exp \left( 2\pi i \frac{\eta(TW - 2)}{4} \right).
\end{equation}

Here $\eta(TW - 2)$ is the difference of the $\eta$-invariant of the Dirac operator coupled to the tangent bundle and twice the $\eta$-invariant of the pure Dirac operator.

**Proposition 3.14.** The Rarita-Schwinger partition function $\hat{\alpha}_{RS}(W)$ is (i) independent of the metric on $W$, (ii) a pin$^+$ bordism invariant, and (iii) a root of unity.

**Proof.** Similar assertions for even dimensional pin$^c$ manifolds are proved in [G, §1], so we will be brief. The space of Riemannian metrics is connected (better: contractible), so it suffices to compute the derivative with respect to the metric. That variation formula [APS2] is an integral of a $w_1$-twisted 13-form over $W$; the result is a 1-form on the space of metrics. The integrand is a local invariant of the geometry by a general theory developed by Seeley [S], and here it vanishes for parity reasons; see [G, Lemma 1.5]. This proves (i). Now suppose $W$ is the boundary of a compact pin$^+$ 13-manifold $Z$. Then [APS1, Theorem 3.10] computes (3.13) as the exponential of the integral of the same $w_1$-twisted 13-form over $Z$. But as above, this 13-form vanishes identically, and so (ii) holds. For (iii) we need only use that the relevant bordism group is finite, in fact $[KT1]$

\begin{equation}
\pi_{12}MTPin^+ \cong \mathbb{Z}/2^8\mathbb{Z} \oplus \mathbb{Z}/2^4\mathbb{Z} \oplus \mathbb{Z}/2^2\mathbb{Z}.
\end{equation}
Corollary 3.16. The Rarita-Schwinger partition function factors through a homomorphism

\[ \hat{\alpha}_{RS}: \pi_{12} MTPin^+ \to \mathbb{C}^\times. \]

As reviewed in §2, the homomorphism (3.17) determines an invertible unitary topological field theory

\[ \alpha_{RS}: MTPin^+ \to \Sigma^{12} \mathbb{I} \mathbb{C}^\times \]

up to isomorphism. We stretch notation slightly and use the notation ‘\( \alpha_{RS}(W) \)’ for the partition function of a closed pin\(^+\) 12-manifold \( W \).

3.3. Properties of the \( \eta \)-invariant

On a spin manifold \( W \) the partition function \( \alpha_{RS}(W) \) has a natural logarithm defined using an integer index.

Proposition 3.19. Let \( W \) be a closed spin 12-manifold. Then \( \alpha_{RS}(W) = (-1)^{RS(W)} \), where

\[ RS(W) = \frac{1}{2} \langle \hat{A}(W) \operatorname{ch}(TW - 2), [W] \rangle. \]

Proof. As remarked after (A.22), the pin\(^+\) spinor data restricts to spin spinor data, so on a spin manifold we apply the discussion in §A.1. The crucial point, which holds in general for \( n \) odd, is that the action of the volume form \( \omega = \delta^1 \delta^2 \gamma^1 \cdots \gamma^{10} \subset \operatorname{Cliff}_{12} \) commutes with the action of \( \operatorname{Spin}_{12} \) and so anticommutes with the Dirac operator \( D^0 \). This implies that the spectrum of \( D^0 \) is invariant under \( \lambda \mapsto -\lambda \). Choose \( a \) in (A.9) to be negative and greater than the first negative eigenvalue of \( D^0 \). Then \( \eta_a(s) = \dim \ker D^0 \) for all \( s \)—the nonzero eigenvalues in the sum cancel—and so from (A.10) we have

\[ \alpha_{RS}(W) = \exp(2\pi i \dim \ker D^0/4) = (-1)^{\dim \ker D^0}. \]

The quaternionic dimension of the kernel is congruent mod 2 to the quaternionic index of the chiral Dirac operator, which maps the \(+1\)-eigenspace of \( \omega \) to its \(-1\)-eigenspace. The Atiyah-Singer index formula completes the proof.

Remark 3.22. The expansion of (3.20) in terms of Pontrjagin numbers of \( W \) is

\[ RS(W) = \left\langle \frac{97p_1^3 - 788p_1p_2 + 3952p_3}{967680}, [W] \right\rangle. \]

\[ \text{The maps (2.3), (2.4) which correspond to (3.17), (3.18) are lifted to the bordism spectrum } M\text{ of manifolds with a pin}\(^+\) \text{ structure and a } w_1\text{-twisted integer lift of } w_4. \]
Remark 3.24. In §4.3 we encounter a shift of (3.20) by an integer, namely

\[(3.25)\]
\[RS'(W) = \frac{1}{2} \left\langle \hat{A}(W) \text{ch}(TW - 4), [W] \right\rangle.\]

It has the same mod 2 reduction as \(RS(W)\).

For a real vector bundle \(V \to W\) over a closed pin\(^+\) 12-manifold set

\[(3.26)\]
\[\alpha_W(V) = \exp \left( 2\pi i \frac{\eta_W(V)}{4} \right).\]

The \(\eta\)-invariant (A.9) depends on a parameter \(a \in \mathbb{R}\), as we defined it, but the exponential (3.26) is independent of \(a\). Comparing with (3.13) our notation is \(\alpha_{RS}(W) = \alpha_W(TW - 2)\).

Proposition 3.27. Let \(V^0, V^1 \to W\) be real vector bundles over a closed pin\(^+\) 12-manifold \(W\). Then the ratio \(\alpha_W(V^0)/\alpha_W(V^1)\) of exponentiated \(\eta\)-invariants depends only on the class of the virtual bundle \([V^0] - [V^1] \in KO^0(W)\).

In particular, it does not depend on the choices of covariant derivative.

**Proof.** The independence from the covariant derivative follows from the variation formula, as in the proof of Proposition 3.14. Then simply observe that \(\alpha_W(V)\) is multiplicative: \(\alpha_W(V \oplus V') = \alpha_W(V)\alpha_W(V')\). \(\square\)

Proposition 3.14 and Proposition 3.27 suggest that there is a \(KO\)-theory formula for \(\alpha_W(V)\). Indeed, such a formula was recently proved by Zhang [Z], based on an analytic theorem of Bismut-Zhang [BZ] telling the behavior of \(\eta\)-invariants under immersions. While the other formulas and techniques in this section suffice for most of the computations in §6.2, we were only able to compute \(\alpha_{RS}\) in §6.2.6 using this topological formula.

**Theorem 3.28** (Zhang). Let \(V \to W\) be a real vector bundle over a closed pin\(^+\) 12-manifold \(W\). Let \(L \to W\) be the orientation real line bundle, \(H \to \mathbb{RP}^{20}\) the tautological line bundle, and \(\gamma: W \to \mathbb{RP}^{20}\) a map such that \(\gamma^*H \cong L\). Then

\[(3.29)\]
\[\gamma_*([V]) = 2^{11} \frac{\eta_W(V)}{4} \left( 1 - [H] \right) \text{ in } \widetilde{KO}^0(\mathbb{RP}^{20}).\]

In this formula\(^{15}\) \([V] \in KO^0(W)\) is the \(KO\)-class of \(V \to W\); the map \(\gamma\) has a spin structure induced from the pin\(^+\) structures on \(W\) and \(\mathbb{RP}^{20}\) together with a choice of isomorphism \(\gamma^*H \cong L\); and \(\gamma_*\) is the induced pushforward on \(KO\)-theory, after multiplication by the Bott class. The group \(\widetilde{KO}^0(\mathbb{RP}^{20})\) is cyclic of order \(2^{11}\) with generator \(1 - [H]\).

A pin\(^+\) structure on a smooth manifold \(M\) has an **opposite**, obtained by tensoring with the orientation double cover.

\(^{15}\)We express (3.29) in a different, but equivalent, form than [Z], and we have used the pin\(^+\) variant of his pin\(^-\) theorem (which he remarks holds in the pin\(^+\) case).
Proposition 3.30. Let $V \to W$ be a real vector bundle over a closed pin$^+$ 12-manifold $W$, and let $L \to W$ be the real line bundle associated to the orientation double cover $\pi: \tilde{W} \to W$. Then

$$\alpha_W(V \otimes L) = \alpha_W(V)^{-1}. \tag{3.31}$$

Proof. Let $\sigma$ be the deck transformation of the double cover $\pi$. Then $V$-valued spinor fields on $W$ lift to $\sigma$-invariant $\pi^* V$-valued spinor fields on $\tilde{W}$ and $V \otimes L$-valued spinor fields on $W$ lift to $\sigma$-anti-invariant $\pi^* V$-valued spinor fields on $\tilde{W}$. Hence $\eta_W(V) + \eta_W(V \otimes L) = \eta_{\tilde{W}}(\pi^* V)$. The pullback of the pin$^+$ structure on $W$ combines with the orientation of $\tilde{W}$ to produce a spin structure on $\tilde{W}$, so $\alpha_{\tilde{W}}(\pi^* V)$ is computed using the mod 2 reduction of

$$\frac{1}{2} \langle \pi^* \left[ \hat{A}(W) \text{ch}(V) \right], [\tilde{W}] \rangle, \tag{3.32}$$

as in Proposition 3.19. Since $\sigma$ is an orientation-reversing involution, it follows that the integer (3.32) equals its negative, hence vanishes. □

Proposition 3.33. Suppose $W = W' \times W''$ is the product of a pin$^+$ 4-manifold $W'$ and a spin 8-manifold $W''$. Let $V' \to W'$ and $V'' \to W''$ be real vector bundles. Then

$$\alpha_{W'}(V' \otimes V'') = \alpha_{W'}(V') \text{ind } D_{W''}(V''), \tag{3.34}$$

where the exponent is the index of the Dirac operator coupled to $V''$.

Proof. This follows directly from the topological index formula (3.29), but there is a straightforward analytic proof which we outline here. Use the setup of Appendix A. Let $\mathcal{E}^0, \mathcal{E}^1, \mathcal{E}^2, i = 0, 1$, denote the spaces of spinor fields on $W', W''$, and $D', D''$ the Dirac operators. Let $\omega = \gamma_0 \delta^1 \delta^2$ denote the volume form of the commuting Cliff$_{-3}$. Then the space of spinor fields on $W$ is $\mathcal{E}^0 \otimes \mathcal{E}^n_0 \oplus \mathcal{E}^1 \otimes \mathcal{E}^n_1$ and the Dirac operator on $W$ is $D^0_W = \omega D' \otimes id + \omega \otimes D''$. Write spectral decompositions

$$\mathcal{E}^0 = \bigoplus_{\lambda \in \text{spec } \omega D'} \mathcal{E}_\lambda^0 \tag{3.35}$$

$$\mathcal{E}^n = \bigoplus_{\mu \in \text{spec } (D'')^2} \mathcal{E}_\mu^n$$

If $\mu^2 \neq 0$ then $D^0_W$ acts on $\mathcal{E}_\lambda^0 \otimes \mathcal{E}_\mu^n \oplus \mathcal{E}_\lambda^1 \otimes \mathcal{E}_\mu^1$ with trace 0: we compute

$$D^0_W(\psi' \otimes \psi'' \pm \mu^{-1} \omega \psi' \otimes D'' \psi'') = (\lambda \mp \mu)(\psi' \otimes \psi'' \pm \mu^{-1} \omega \psi' \otimes D'' \psi'') \tag{3.36}$$

As we eventually compute using an orthogonal decomposition into finite dimensional eigenspaces, we do not worry about the topology in these tensor products.
and let ψ′, ψ″ run over orthonormal bases of \( E^0_\lambda \), \( E^0_\mu \), respectively. Hence the only contributions\(^\text{17}\) to the \( \eta \)-invariant of \( D^0_0 \) come from \( E^0_0 \otimes (\ker D^\mu_0) \) and \( E^1_0 \otimes (\ker D^\mu_0)^1 \). If \( \psi' \in E^0_\lambda \) and \( \psi'' \in (\ker D^\mu)\) \( i \), then since \( \omega D' = -D' \omega \) we compute

\[
\begin{align*}
D^0_0 (\psi' \otimes \psi''^0) &= \lambda (\psi' \otimes \psi''^0) \\
D^0_0 (\omega \psi' \otimes \psi''^1) &= -\lambda (\omega \psi' \otimes \psi''^1)
\end{align*}
\]

and (3.34) quickly follows. \( \square \)

The next result is inspired by techniques in [APS2]. Suppose \( W \) is a closed pin\(^+\) 12-manifold and \( \pi: \tilde{W} \to W \) its orientation double cover. Let \( \sigma: \tilde{W} \to \tilde{W} \) be the canonical orientation-reversing free involution. If \( P \to W \) is the principal Pin\(^+\)-bundle of frames, then \( \sigma \) lifts canonically to an involution of \( \pi^* P \to \tilde{W} \) which reverses the spin structure on \( \tilde{W} \). Suppose that \( Z \) is a compact spin 13-manifold with boundary \( \partial Z = \tilde{W} \) and \( \sigma' \) an orientation-reversing involution of \( Z \) which extends \( \sigma \) and is equipped with a lift to a spin-reversing involution of the Pin\(_{12}\)-bundle of frames. Let \( F \subset Z \) denote the fixed point set of \( \sigma' \). At an isolated fixed point \( f \in F \) the action of \( \sigma' \) on \( T_f Z \) is by \(-1\), so its lift to the pin\(^+\) frames acts by \( \pm \omega \), where \( \omega = \gamma^1 \gamma^2 \cdots \gamma^{13} \) is the volume form. Let \( i_f = \pm 1 \) denote the sign. If \( V \to W \) is a real vector bundle, assume \( \pi^* V \to \tilde{W} \) extends over \( Z \) and the involution \( \sigma' \) lifts, extending the lift of \( \sigma \) on the boundary. Let \( \tau_f \) denote the trace of the lifted action at a fixed point \( f \in F \).

**Proposition 3.38.** If \( F \) consists of isolated points, then

\[
\alpha_W (V) = \exp \left( 2\pi i \sum_{f \in F} \frac{i_f \tau_f}{28} \right).
\]

If \( V \) is the trivial real line bundle, then this is [St, Proposition 5.3], which is based on the general equivariant index theorem [Do, Theorem 1.2] for manifolds with boundary. Donnelly’s theorem identifies the contribution at a fixed point in terms of an asymptotic expansion of a heat kernel. The general cohomological expression for that contribution appears in [AS, (3.9)] in the context of the general Lefschetz theorem, and it applies to fixed point manifolds of positive dimension as well as isolated fixed points. That this is the correct fixed point contribution in Donnelly’s theorem is proved in [DP] for the signature operator. We use it for the Dirac operator and an orientation-reversing isometry in \( \S 6.2.5 \).

4. Cubic forms and the C-field

4.1. Motivation: Spin\(^c\) manifolds

Recall that the compact Lie group Spin\(^c\)_n is a group extension

\[
\begin{align*}
1 &\to \mathbb{T} \to \text{Spin}^c_n \to SO_n \to 1
\end{align*}
\]

\(^{17}\) Choose \( a \) in (A.9) to be less than zero and greater than the first negative eigenvalue of \( D_W \).
where $\mathbb{T}$ is the circle group of complex numbers of unit norm; it is defined as the quotient $(\text{Spin}_n \times \mathbb{T})/\{\pm 1\}$. Let $M$ be an $n$-dimensional Spin$^c$ manifold. A Spin$^c$ structure on $M$ is a principal Spin$^c_n$-bundle $\mathcal{B}_{\text{Spin}^c}(M) \to M$ together with an isomorphism $\mathcal{B}_{\text{Spin}^c}(M)/\mathbb{T} \cong \mathcal{B}_{SO}(M)$ with the principal $SO_n$-bundle of oriented orthonormal frames. The $\mathbb{T}$-bundle over $M$ associated to the homomorphism Spin$^c_n \to \mathbb{T}$ is called the characteristic bundle, and its first Chern class $c \in H^2(M; \mathbb{Z})$ is an integer lift of the second Stiefel-Whitney class:

\[(4.2) \quad c \equiv w_2(M) \pmod{2}.\]

Furthermore, any other Spin$^c$ structure is obtained by “tensoring” with a circle bundle $Q \to M$; the characteristic class of the new Spin$^c$ structure is $c + 2x$, where $x = c_1(Q)$. Finally, there is an involution on Spin$^c$ structures which inverts the characteristic bundle and so changes the sign of $c$.

Suppose $n = \dim M$ is even and $M$ is compact without boundary. The ($\mathbb{Z}/2\mathbb{Z}$-graded) complex spin representation of Spin$^c_n$ gives rise to a Dirac operator $D_M$ whose index is a topological invariant. It is computed by the Atiyah-Singer formula

\[(4.3) \quad \text{index } D_M = \langle \hat{A}(M)e^{c/2}, [M] \rangle,\]

where $\hat{A}(M) = 1 - p_1(M)/24 + \ldots$ and $[M]$ is the fundamental class of $M$. As a function $\kappa(c)$ of the characteristic class $c$ it is a polynomial, which for $n = 4, 6$ may be written

\[(4.4) \quad \kappa_2(c) = \frac{c^2 - \sigma(M)}{8},\]
\[(4.5) \quad \kappa_3(c) = \frac{c^3 - p_1(M)c}{48}.\]

The subscript indicates the degree of the polynomial, $\sigma(M)$ is the signature of the 4-manifold $M$, and we omit evaluation on $[M]$ from the notation for convenience. One may continue to $n = 8, 10, \ldots$ to obtain polynomials of higher degree. These polynomials satisfy a symmetry property:

\[(4.6) \quad \kappa_2(-c) = \kappa_2(c), \quad \kappa_3(-c) = -\kappa_3(c).\]

For a fixed characteristic element $c$ define $q^c : H^2(M; \mathbb{Z}) \to \mathbb{Z}$ as

\[(4.7) \quad q^c(x) = \kappa(c + 2x) - \kappa(c), \quad x \in H^2(M; \mathbb{Z}).\]

For $n = 4, 6$ we find

\[(4.8) \quad q_2^c(x) = \frac{1}{2}(x^2 + cx),\]
\[(4.9) \quad q_3^c(x) = \frac{1}{24}(p_1(M)x + 4x^3 + 6cx^2 + 3c^2x) = \frac{1}{6}x^3 + \ldots\]

The associated line bundle is often called the determinant line bundle of the Spin$^c$ structure.
Note $q_2^c$ is a quadratic refinement of the intersection pairing on $H^2(M^4; \mathbb{Z})/\text{torsion}$, and $q_3^c$ is a cubic refinement of the symmetric trilinear form on $H^2(M^6; \mathbb{Z})/\text{torsion}$.

The general mathematical problem suggested here is: Replace $c$ by a cohomology class of arbitrary even degree and extend the topological invariants (4.4), (4.5). Of course, one may pose this as well for the higher degree polynomials of $c$ deduced from the index formula (4.3) in higher dimensions. In the quadratic case we have $n = 4k$ for some $k \in \mathbb{Z}^>0$ and $c \in H^{2k}(M; \mathbb{Z})$ lies in the middle degree. The associated topological invariant was investigated by Brown [Bro] and Browder [Brd]. In this instance $c$ is an integer lift of the middle Wu class $\nu_2$ $p \in H^2(\mathbb{Z}/2\mathbb{Z})$, which may or may not exist. Corresponding geometric invariants were constructed in [HS]. We take up the next interesting case—the cubic form for $n = 12$ and deg $c = 4$—which appears in the action of the $C$-field in M-theory.

4.2. Algebraic theory of cubic forms

We begin with a review of the algebraic theory of quadratic forms. Let $L$ be a finitely generated free abelian group and $\langle \cdot, \cdot \rangle: L \times L \to \mathbb{Z}$ a nondegenerate (i.e., unimodular) symmetric bilinear form. The nondegeneracy implies the existence of a unique element $\bar{c} \in L$ such that

\begin{equation}
\langle \bar{x}, \bar{x} \rangle = \langle \bar{c}, \bar{x} \rangle \pmod{2}, \quad \bar{x} \in L \otimes \mathbb{Z}/2\mathbb{Z},
\end{equation}

since the left hand side is linear in $\bar{x}$. An element $c \in L$ with $c \equiv \bar{c} \pmod{2}$ is called characteristic. The set $L_{\text{char}} \subset L$ of characteristic elements is a torsor for $L$: if $c \in L_{\text{char}}$ and $x \in L$ then $c + 2x \in L_{\text{char}}$. Now an easy check shows that $\langle c, c \rangle \pmod{8}$ is independent of $c \in L_{\text{char}}$, so for any integer lift $\sigma \in \mathbb{Z}$ of $\langle c, c \rangle \pmod{8},$

\begin{equation}
\kappa_2(c) = \frac{\langle c, c \rangle - \sigma}{8}
\end{equation}

is an integer. It is a standard result [Se, Chapter 5] that $\sigma$ may be chosen to be the signature of $\langle \cdot, \cdot \rangle$, defined by extending the form to the real vector space $L \otimes \mathbb{R}$. This is the algebraic theory which underlies (4.4). Note $\kappa_2(-c) = \kappa_2(c)$.

We develop a similar theory for the cubic (4.5). Consider the triple $(L, \langle \cdot, \cdot, \cdot \rangle, \bar{c})$ where $L$ is a finitely generated free abelian group, $\langle \cdot, \cdot, \cdot \rangle: L \times L \times L \to \mathbb{Z}$ is a symmetric trilinear form, and $\bar{c} \in L \otimes \mathbb{Z}/2\mathbb{Z}$ is assumed to satisfy\footnote{Equation (4.12) for trilinear forms appears in Postnikov’s study [Po] of the mod 2 cohomology ring of a closed 3-manifold, for example.}

\begin{equation}
\langle \bar{c}, \bar{x}, \bar{y} \rangle \equiv \langle \bar{x}, \bar{x} \rangle + \langle \bar{x}, \bar{y}, \bar{y} \rangle \pmod{2}, \quad \bar{x}, \bar{y} \in L \otimes \mathbb{Z}/2\mathbb{Z}.
\end{equation}

As we do not know a notion of nondegeneracy for trilinear forms which guarantees the existence of $\bar{c}$, we postulate its existence. Define the torsor $L_{\text{char}} \subset L$ of characteristic elements as above. Let $L^* = \text{Hom}(L, \mathbb{Z})$ and for convenience write the trilinear form as a simple product.
Lemma 4.13. There exists a unique \( \hat{p} \in L^* \otimes \mathbb{Z}/24\mathbb{Z} \) such that

\[
(4.14) \quad \hat{p} \cdot \hat{x} \equiv 4\hat{x}^3 + 6\hat{c}\hat{x}^2 + 3\hat{c}^2 \hat{x} \pmod{24}
\]

for all \( \hat{x} \in L \otimes \mathbb{Z}/24\mathbb{Z} \) and mod 24 reductions \( \hat{c} \) of characteristic elements \( c \in L_{\text{char}} \).

Proof. Use (4.12) to check that, as a function of \( \hat{x} \), the right hand side of (4.14) defines a homomorphism \( L \otimes \mathbb{Z}/24\mathbb{Z} \to \mathbb{Z}/24\mathbb{Z} \). \( \square \)

Lemma 4.15. Let \( p \in L^* \) satisfy \( p \equiv \hat{p} \pmod{24} \). Then

\[
(4.16) \quad \frac{c^3 - p \cdot c}{24} \pmod{2}
\]

lies in \( \mathbb{Z}/2\mathbb{Z} \) and is independent of \( c \in L_{\text{char}} \). Furthermore, there exist lifts \( p \in L^* \) of \( \hat{p} \) such that this invariant vanishes, in which case

\[
(4.17) \quad \kappa_3(c) = \frac{c^3 - p \cdot c}{48}
\]

is an integer. Also, \( \kappa_3(-c) = -\kappa_3(c) \).

Proof. To check the independence of \( c \in L_{\text{char}} \), replace \( c \) in (4.16) with \( c + 2x \) for \( x \in L \) and use the fact that \( cx^2 \) is even, which follows from (4.12). To see that the fraction in (4.16) is an integer, use (4.14) and the fact that \( c^3 \) is even, which also follows from (4.12). To find the lift \( p \), if \( \hat{c} = 0 \), then any \( p \) works since we can compute (4.16) using \( c = 0 \). If \( \hat{c} \neq 0 \), and if for a chosen lift \( p \) the invariant (4.16) is nonzero, choose \( x^* \in L^* \) such that \( x^* \cdot c \) is odd for any characteristic \( c \) and replace \( p \) with \( p + x^* \). \( \square \)

4.3. The cubic form on spin 12-manifolds

In (4.5) we gave an example of the cubic form (4.17) for a closed oriented 6-manifold \( M^6 \), where \( L = H^2(M; \mathbb{Z})/\text{torsion}, \langle x, y, z \rangle = \langle x \sim y \sim z, [M] \rangle, \hat{c} = w_2(M), \) and \( p = p_1(M) \). We now consider a closed spin 12-manifold \( W^{12} \) and set

\[
L = H^4(W; \mathbb{Z})/\text{torsion}
\]

\[
(4.18) \quad \langle x, y, z \rangle = \langle x \sim y \sim z, [W] \rangle
\]

\[
\hat{c} = w_4(W)
\]

Remark 4.19. Let \( T^4 \subset H^4(W; \mathbb{Z}) \) denote the torsion subgroup, which fits into the exact sequence

\[
(4.20) \quad 0 \to T^4 \to H^4(W; \mathbb{Z}) \to L \to 0
\]

Tensoring with \( \mathbb{Z}/2\mathbb{Z} \) defines a homomorphism \( H^4(W; \mathbb{Z}) \to H^4(W; \mathbb{Z}/2\mathbb{Z}) \), and the precise definition of \( \hat{c} \) is the image of \( w_4(W) \) under the quotient map

\[
(4.21) \quad H^4(W; \mathbb{Z}/2\mathbb{Z}) \to H^4(W; \mathbb{Z}/2\mathbb{Z})/(T^4 \otimes \mathbb{Z}/2\mathbb{Z}).
\]
In the classifying space $B_{\text{Spin}}$ there is a characteristic class $\lambda \in H^4(B_{\text{Spin}}; \mathbb{Z})$ such that (i) $2\lambda = p_1$ and (ii) the image of $\lambda$ under $H^4(B_{\text{Spin}}; \mathbb{Z}) \to H^4(B_{\text{Spin}}; \mathbb{Z}/2\mathbb{Z})$ is $w_4$. A spin manifold $W$ has a corresponding integer characteristic class $\lambda(W)$. The existence of this integer lift of $w_4(W)$ implies that the image of $w_4(W)$ under (4.21) lies in the subgroup $(H^4(W; \mathbb{Z})/T^4) \otimes \mathbb{Z}/2\mathbb{Z} = L \otimes \mathbb{Z}/2\mathbb{Z}$. The computations below are written in $H^*(W; \mathbb{Z}/2\mathbb{Z})$, but the results should be interpreted in terms of this subquotient. (To do so, use the fact that torsion integer cohomology classes evaluate trivially on the fundamental class.)

**Lemma 4.22.** The Stiefel-Whitney class $\tilde{c} = w_4(W)$ of a closed spin 12-manifold $W$ satisfies (4.12).

The proof uses the Cartan formula and Adem relations for Steenrod squares, as well as the Wu formula, which states that on a closed $n$-manifold $M$ there is a class $\nu_i(M) \in H^i(M; \mathbb{Z}/2\mathbb{Z})$ such that squaring to the top,

$$Sq^i: H^{n-i}(M; \mathbb{Z}/2\mathbb{Z}) \to H^n(M; \mathbb{Z}/2\mathbb{Z}),$$

is cup product with $\nu_i(M)$. In low degrees we have

$$\begin{align*}
\nu_1 &= w_1 \\
\nu_2 &= w_1^2 + w_2 \\
\nu_3 &= w_1 w_2 \\
\nu_4 &= w_4 + w_1 w_3 + w_2^2 + w_1^4
\end{align*}$$

in terms of the Stiefel-Whitney classes of the tangent bundle. The Bockstein $\beta$ is defined as the connecting homomorphism induced from the coefficient sequence $0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$:

$$\begin{align*}
\cdots \to H^i(M; \mathbb{Z}) \xrightarrow{r} H^i(M; \mathbb{Z}/2\mathbb{Z}) \xrightarrow{\beta} H^{i+1}(M; \mathbb{Z}) \xrightarrow{2} H^{i+1}(M; \mathbb{Z}) \to \cdots
\end{align*}$$

Also, $Sq^1 = r \circ \beta$. It follows that if $x$ is an integer cohomology class, $Sq^1$ vanishes on its mod 2 reduction $x = r(x)$.

**Proof.** Omit ‘$\sim$’ from the notation for convenience. If $x, y \in H^4(W; \mathbb{Z})$ and $\bar{x}, \bar{y} \in H^4(W; \mathbb{Z}/2\mathbb{Z})$ their mod 2 reductions, then (4.24) with $w_1 = w_2 = 0$ implies

$$w_4(W)\bar{x}\bar{y} = Sq^4(\bar{x}\bar{y})$$

(4.26)

$$= (Sq^4\bar{x})\bar{y} + Sq^2\bar{x} Sq^2\bar{y} + \bar{x} Sq^4\bar{y}$$

$$= \bar{x}\bar{y} + Sq^2\bar{x} Sq^2\bar{y} + \bar{x}\bar{y},$$

since $Sq^1$ vanishes on reductions of integer classes. Then from (4.24) again

$$0 = w_2(W)\bar{x} Sq^2\bar{y}$$

$$= Sq^2(\bar{x} Sq^2\bar{y})$$

(4.27)

$$= Sq^2\bar{x} Sq^2\bar{y} + \bar{x} Sq^2 Sq^2\bar{y}$$

$$= Sq^2\bar{x} Sq^2\bar{y} + \bar{x} Sq^3 Sq^1\bar{y}$$

$$= Sq^2\bar{x} Sq^2\bar{y}.$$

$\square$
Proposition 4.28. In $B\text{Spin}$ there is a unique characteristic class $p \in H^8(B\text{Spin}; \mathbb{Z})$ with
\begin{equation}
2p = p_2 - \lambda^2.
\end{equation}
Furthermore, $p \equiv w_8 \pmod{2}$.

For a smooth manifold $M$ we obtain a characteristic class $p(M) \in H^8(M; \mathbb{Z})$, and we use the same symbol to denote its reduction modulo torsion.

Proof. $H^8(B\text{Spin}; \mathbb{Z})$ is torsionfree and $p_2 \equiv \lambda^2 \equiv w_4^2 \pmod{2}$, which proves the existence and uniqueness of $p$. To compute its reduction mod 2 we restrict to the classifying space of a maximal torus of Spin$_N$ for $N \geq 8$. The computation is carried out in [BW, §3], where $p$ is the class called `−$q_2$'. Its reduction mod 2 equals the reduction of the class called `c$_4$', which is the Stiefel-Whitney class $w_8$.

Proposition 4.30. On a closed spin 12-manifold $W$ the mod 24 reduction of $p(W)$ satisfies (4.14).

Proof. We follow Witten’s argument in [W3, §4]. Namely, a principal $E_8$-bundle over a 12-manifold is determined up to isomorphism by an element $x \in H^4(W; \mathbb{Z})$. Let $V(x)$ denote the (real) adjoint vector bundle to the principal $E_8$-bundle with characteristic class $x$, and set $c = \lambda(W) + 2x$. The Chern character of $V(x) \to W$ is
\begin{equation}
\text{ch}(V(x)) = 248 - 60x + 6x^2 - \frac{1}{3}x^3.
\end{equation}

Then a long computation verifies the following identity:
\begin{equation}
\left\langle \frac{c^3 - pc}{48} + \frac{1}{2} \hat{A}(W) \text{ch}(V(x)) + \frac{1}{4} \hat{A}(W) \text{ch}(TW - 4), [W] \right\rangle = 0.
\end{equation}

The second term is an integer; it is the KO-theory direct image of the real bundle $V(x)$, defined using the spin structure, which by the Atiyah-Singer index theorem is the index of the Dirac operator coupled to $V(x)$. Similarly, the last term is a half-integer, hence so is the cubic expression. Replace the denominator in the cubic expression by 24 to obtain an integer, and now subtract the integers for arbitrary $x$ and $x = 0$ to establish the congruence
\begin{equation}
(p_2(W) - \lambda(W)^2)x \equiv 8x^3 + 12\lambda(W)x^2 + 6\lambda(W)^2x \pmod{24},
\end{equation}
where we omit evaluation on $[W]$ from the notation for convenience. If necessary, use the last argument in Lemma 4.15 to replace $p(W)$ by $p' = p(W) + 24a$ for $a \in H^8(W; \mathbb{Z})$/torsion so that
\begin{equation}
\frac{c^3 - p'c}{48} \in \mathbb{Z}, \quad c = \lambda(W) + 2x, \quad x \in H^4(W; \mathbb{Z}),
\end{equation}
and so deduce the desired mod 24 congruence. □

---

\textsuperscript{20}A priori the Chern character is a cubic polynomial in $x$, so we need only determine the coefficients. The restriction of the adjoint representation of $E_8$ to Spin$_{16} \subset E_8$ is the sum of a half-spin representation and the adjoint representation of Spin$_{16}$. The restriction of its complexification to Spin$_3 \subset$ Spin$_{16}$ is $78V_1 \oplus 64V_2 \oplus 14V_3$, where $V_n$ is the $n$-dimensional irreducible representation of Spin$_3 \cong SU_2$; the Chern character of this representation is easily computed. Finally, the generator of $H^4(BE_8; \mathbb{Z})$ restricts to minus twice the generator of $H^4(BSU_2; \mathbb{Z})$. (The generator of $H^4(BSO_{16}; \mathbb{Z})$ restricts to the generator of $H^4(BSU_2; \mathbb{Z})$. The former pulls back to twice the generator of $H^4(B\text{Spin}_{16}; \mathbb{Z})$, whereas the latter pulls back to minus four times the generator of $H^4(BSU_2; \mathbb{Z})$.)
Note that $p(W)$ is not necessarily a distinguished lift of $\hat{p}$ described in Lemma 4.15; rather we need to add the constant term $1/4 RS'(W)$ (see (3.25)) in (4.32) is needed to obtain integrality. Define the integer-valued cubic form

\begin{equation}
(4.35) \quad \kappa_W(c) = \frac{c^3 - p(W)c}{48} + \frac{1}{2} RS'(W)
\end{equation}

on characteristic elements; it satisfies a shifted version of the symmetry (4.6):

\begin{equation}
(4.36) \quad \kappa_W(-c) = RS'(W) - \kappa_W(c).
\end{equation}

4.4. The cubic form on $\text{pin}^+ 12$-manifolds

Any manifold $M$ has a canonical orientation double cover $\widehat{M} \to M$: the fiber at $m \in M$ is the set of orientations on $T_mM$. There results a canonical local system $\hat{\mathbb{Z}} \to M$ of coefficients; we call $H^*(M; \hat{\mathbb{Z}})$ the $w_1$-twisted cohomology. An orientation is a trivialization of $\hat{\mathbb{Z}} \to M$, and on an oriented manifold $w_1$-twisted integer cohomology reduces to untwisted integer cohomology. The fundamental class $[M]$ of a closed manifold $M$ lives in $w_1$-twisted integer homology, so we can integrate $w_1$-twisted cohomology classes.

**Lemma 4.37.** Let $M$ be a closed $n$-manifold with no orientable components and $\pi: \widehat{M} \to M$ the orientation double cover. Then the image of

\begin{equation}
(4.38) \quad \pi^*: H^n(M; \hat{\mathbb{Z}}) \to H^n(\widehat{M}; \mathbb{Z})
\end{equation}

is $2H^n(\widehat{M}; \mathbb{Z})$, and if $\bar{\omega} \in H^n(M; \hat{\mathbb{Z}})$, then

\begin{equation}
(4.39) \quad \langle \bar{\omega}, [M] \rangle = \langle \frac{1}{2} \pi^*\bar{\omega}, [\widehat{M}] \rangle.
\end{equation}

As the domain and codomain of (4.38) are torsionfree, we can prove Lemma 4.37 using de Rham theory, a task we leave to the reader.

Let $W$ be a closed $\text{pin}^+$ 12-manifold $W$. The existence of a $\text{pin}^+$ structure on $W$ is equivalent to $w_2(W) = 0$, but in general $w_1(W) \neq 0$. Also, $w_3(W) = 0$ since $w_3 = Sq^1w_2 + w_1w_2$. Note then that the Wu classes (4.24) simplify to $\nu_2(W) = w_1(W)^2$ and $\nu_4(W) = w_4(W) + w_1(W)^4$. There is a short exact sequence of coefficients $0 \to \hat{\mathbb{Z}} \xrightarrow{\mu_2} \hat{\mathbb{Z}} \to \mathbb{Z}/2\mathbb{Z} \to 0$, and the connecting homomorphism in the resulting long exact sequence—(4.25) with twisted coefficients—is the twisted Bockstein $\tilde{\beta}$. In this case $r \circ \tilde{\beta} = Sq^1 + w_1$, so that if $x$ is a $w_1$-twisted integer class then

\begin{equation}
(4.40) \quad Sq^1x = w_1(M) \sim x.
\end{equation}

For a closed $\text{pin}^+$ 12-manifold $W$ we modify (4.18) to

\begin{equation}
L = H^4(W; \hat{\mathbb{Z}})/\text{torsion}
\end{equation}

\begin{equation}
\langle x, y, z \rangle = (x \sim y \sim z)[W]
\end{equation}

\begin{equation}
\bar{c} = w_4(W)
\end{equation}
Remark 4.19 applies if we replace integer cohomology with $w_1$-twisted integer cohomology and assume $W$ is an $m_c$-manifold. The dual lattice $L^* = H^8(W; \mathbb{Z})/\text{torsion}$ is untwisted integer cohomology as in the spin case.

**Proposition 4.42.** In $B\text{Pin}^+$ there is a unique characteristic class $\bar{p} \in H^8(B\text{Pin}^+; \mathbb{Z})/\text{torsion}$ whose restriction to $B\text{Spin}$ is the class $p$ of Proposition 4.28.

**Proof.** Let $\{E^p_{r,q}\}$ denote the Leray-Serre spectral sequence for the fibration

\begin{equation}
B\text{Spin} \to B\text{Pin}^+ \to \mathbb{R}P^\infty.
\end{equation}

Then $E^{0,8}_2 \cong H^8(B\text{Spin}; \mathbb{Z})$ and $E^{0,8}_\infty \cong H^8(B\text{Pin}^+; \mathbb{Z})/\text{torsion} \cong \ker(d_2): E^{0,8}_2 \to E^{2,7}_2 \cong \mathbb{Z}/2\mathbb{Z}$, since $H^7(B\text{Spin}; \mathbb{Z})$ is cyclic of order 2, generated by the integer Bockstein of $w_6$, and $H^2(\mathbb{R}P^\infty; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$. The proposition follows from $d_2(p) = 0$, which in turn follows since $d_2(p)$ is detectable mod 2 and $p \mod 2 = w_8$ survives the differentials. \hfill \Box

**Remark 4.44.** There is a (homotopy) splitting of the map $w_1$ in (4.43), namely the classifying map $\mathbb{R}P^\infty \to BO$ of the reduced tautological bundle $H \to \mathbb{R}P^\infty$, which lifts since $H$ has a pin+ structure. Then the product map

\begin{equation}
B\text{Spin} \times \mathbb{R}P^\infty \to B\text{Pin}^+
\end{equation}

is a homotopy equivalence, since it induces an isomorphism on homotopy groups. This yields an isomorphism $H^\bullet(B\text{Pin}^+; \mathbb{Z})/\text{torsion} \to H^\bullet(B\text{Spin}; \mathbb{Z})/\text{torsion}$, which re-proves Proposition 4.42.

**Proposition 4.46.** Let $W$ be a closed $m_c$ 12-manifold. Then $\bar{c} = w_4(W)$ satisfies equation (4.12). Furthermore, the mod 24 reduction of $\bar{p}(W)$, viewed as a class in $L^* = H^8(W; \mathbb{Z})/\text{torsion}$, satisfies the condition in Lemma 4.13.

**Proof.** We modify the proof of Lemma 4.22. So (4.26) becomes

\begin{equation}
(w_4(W) + w_1^4(W))\bar{x}\bar{y} = \bar{x}\bar{x}\bar{y} + \bar{x}\bar{y}\bar{y} + w_1(W)(\bar{x}S^3\bar{y} + \bar{y}S^3\bar{x}) + S^2\bar{x}S^2\bar{y}
\end{equation}

and (4.27) becomes

\begin{equation}
w_1^2(W)\bar{x}S^2\bar{y} = S^2\bar{x}S^2\bar{y} + w_1(W)\bar{x}S^3\bar{y} + \bar{x}w_1^2(W)S^2\bar{y} + \bar{x}w_1(W)S^3\bar{y},
\end{equation}

which implies $S^2\bar{x}S^2\bar{y} = 0$. Then using $\nu_3(W) = 0$ from (4.24), we find

\begin{equation}
0 = S^3(w_1(W)\bar{x}\bar{y}) = w_4^4(W)S^2(\bar{x}\bar{y}) + w_1(W)S^3(\bar{x}\bar{y}) = w_4^4(W)\bar{x}\bar{y} + w_1(W)(\bar{x}S^3\bar{y} + \bar{y}S^3\bar{x}).
\end{equation}

Combine these equations to complete the proof that $\bar{c}$ satisfies (4.12).

For the last statement in the proposition we observe that Proposition 4.42 implies $\bar{p}(W) = p(W)$ if $W$ is spin, and also if $\pi: \tilde{W} \to W$ is the orientation double cover then $\pi^*\bar{p}(W) = p(\tilde{W})$. The last statement reduces to Proposition 4.30 on orientable components of $W$, and on nonorientable components we use Lemma 4.37 to reduce to Proposition 4.30 on the orientation double cover. \hfill \Box
Lemma 4.50. Let \( W \) be a closed \( m_c \) 12-manifold and \( \tilde{c} \in H^4(W; \mathbb{Z}) \) a \( w_1 \)-twisted integer lift of \( w_4(W) \). Then

\[
\left( \begin{array}{c} \tilde{c}^3 - \tilde{p}(W)\tilde{c} \\
48
\end{array} \right) \quad (\mod \mathbb{Z})
\]

lies in \( \frac{1}{2}\mathbb{Z}/\mathbb{Z} \), is independent of the choice of \( \tilde{c} \), and is a bordism invariant of \( m_c \)-manifolds. It is additive under disjoint union.

Proof. That the fraction in (4.51) is a half-integer follows from Lemma 4.15 in the algebraic theory of cubic forms. Any \( w_1 \)-twisted integer lift of \( w_4(M) \) has the form \( \tilde{c} + 2\tilde{x} \) for some \( \tilde{x} \in H^4(W; \mathbb{Z}) \), and an easy check from (4.14) proves that (4.51) is unchanged by the replacement. If \( W = \partial Z \) is the boundary of a compact \( m_c \) 13-manifold \( Z \), then \( Z \) has a fundamental class in relative homology and the usual adjunction (integer Stokes’ theorem) argument implies that (4.51) vanishes, even before reducing modulo \( \mathbb{Z} \). □

Define

\[
\hat{\alpha}_C(W) = \exp \left( 2\pi i \frac{\tilde{c}^3 - \tilde{p}(W)\tilde{c}}{48} \right).
\]

Recall that \( Mm_c \) is the bordism spectrum of \( \text{pin}^+ \) manifolds with an \( m_c \) structure.

Corollary 4.53. The exponential of the cubic form factors through a homomorphism

\[
\hat{\alpha}_C : \pi_{12}Mm_c \longrightarrow \mathbb{C}^\times
\]

which takes values in \( \{ \pm 1 \} \subset \mathbb{C}^\times \).

As discussed in §2, the homomorphism (3.17) determines an invertible topological field theory

\[
\alpha_C : Mm_c \longrightarrow \Sigma^{12}I\mathbb{C}^\times
\]

up to isomorphism. The square \( \alpha_C^\otimes 2 \) is isomorphic to the trivial theory.

Remark 4.56. Let \( \pi : \widehat{W} \rightarrow W \) be the orientation double cover of an \( m_c \)-manifold which has no orientable components, and suppose \( \tilde{c} \in H^4(\widehat{W}; \mathbb{Z}) \) is a \( w_1 \)-twisted integer lift of \( w_4(W) \). Set \( c = \pi^*\tilde{c} \in H^4(\widehat{W}; \mathbb{Z}) \). As in the proof of Proposition 4.46 we have \( p(\tilde{W}) = \pi^*\tilde{p}(W) \). Apply Lemma 4.37 to evaluate the integer cubic form—twice (4.51)—on the orientation double cover:

\[
\left\langle \frac{\tilde{c}^3 - \tilde{p}(W)\tilde{c}}{24}, [W] \right\rangle = \left\langle \frac{\tilde{c}^3 - p(W)c}{48}, [\tilde{W}] \right\rangle.
\]
4.5. The C-field and its anomaly; cancellation on spin manifolds

The C-field in M-theory is an example of an abelian gauge field. Classically all information is captured by its field strength Ω, which is a closed 4-form. In the quantum theory Dirac’s quantization of charge applies: the de Rham cohomology class of Ω is constrained to lie in a full lattice in the degree 4 real cohomology. There is more information, as inspired by the Aharanov-Bohm effect in the case of ordinary electromagnetism and the resulting refinement of the electromagnetic field—a closed 2-form—to a connection on a principal T-bundle. In higher degrees a suitable language for quantum abelian gauge fields is differential cohomology, \(^{21}\) which is developed in [HS] in part to model the C-field; the focus there is on the M5 brane and so on a quadratic form. Here we work in the “bulk” on a Wick-rotated spacetime which we take to be an 11-dimensional Riemannian pin\(^+\) manifold \(X\). Dirac’s quantization of charge for the C-field, which is determined in [W3], is encoded by positing the C-field as a geometric representative of a \(w_1\)-twisted differential cohomology class \(^{22}\) which lifts \(w_4\) on \(X\). Locally C-fields exist but there is a global obstruction, as explained after Definition 2.1. In that spirit, a \(C\)-field \(\tilde{\Omega}\) is a differential \(mc\) structure on \(X\); a precise model is established in [HS], where it is termed a differential integral Wu structure. Its field strength \(\Omega\) is a closed \(w_1\)-twisted 4-form \(^{23}\) whose de Rham cohomology class in \(H^4(X; \mathbb{R})\) is the real image of a \(w_1\)-twisted integer lift \(\tilde{c}\) in \(H^4(X; \mathbb{Z})\) of \(w_4\) on \(X\).

The effective action of M-theory has a cubic term of the form

\[
(4.58) \quad \exp \left( \frac{2\pi i}{48} \tilde{\Omega}^3 - \tilde{p}(X)\tilde{\Omega} \right),
\]

where \(\tilde{p}(X)\) is a lift to differential cohomology of the class \(\tilde{p}(X) \in H^8(X; \mathbb{Z})/\text{torsion}\). This differential cohomology version of the cubic form is analogous to a Chern-Simons invariant. We do not need its precise definition, so will not elaborate further.

**Remark 4.59.** The \(\tilde{\Omega}^3\) term in (4.58) is part of the classical 11-dimensional supergravity action [CJS]. The \(\tilde{p}(X)\tilde{\Omega}\) term is a quantum correction, introduced in [DLM, (3.14)] in the spin case, in part inspired by [VW, §3] who introduce an analogous correction in the Type IIA superstring. We do not know of any literature about this quantum correction in the pin\(^+\) case.

**Remark 4.60.** We have only defined the class \(\tilde{p}\) in (4.58) up to an element of \(H^7(B\text{Pin}^+; \mathbb{R}/\mathbb{Z})\), but we now argue that (4.58) is independent of the lift. First, \(H^7(B\text{Pin}^+; \mathbb{R}/\mathbb{Z}) \cong H^8(B\text{Pin}^+; \mathbb{Z})_{\text{tor}}\), since \(H^7(B\text{Pin}^+; \mathbb{R}) = 0\). (\(A_{\text{tor}}\) is the torsion subgroup of the abelian group \(A\).) Recall from (4.45) that \(B\text{Pin}^+ \simeq B\text{Spin} \times \mathbb{R}P^\infty\). Then the main theorem in [Ko] implies that \(2H^8(B\text{Pin}^+; \mathbb{Z})_{\text{tor}} = 0\). Use the short exact sequence

\[
(4.61) \quad 0 \longrightarrow \frac{1}{2} \mathbb{Z}/\mathbb{Z} \longrightarrow \mathbb{R}/\mathbb{Z} \longrightarrow \mathbb{R}/\mathbb{Z} \longrightarrow 0
\]
to deduce that $H^7(B\text{Pin}^+; \frac{1}{2}\mathbb{Z}/\mathbb{Z}) \to H^7(B\text{Pin}^+; \mathbb{R}/\mathbb{Z})$ is surjective. It follows that the ambiguity in (4.58) is expressed as a characteristic number of mod 2 cohomology. Our Adams spectral sequence computation (see Figure 1 in §8) shows that there is no element of $\pi_{11}M\mathbb{m}_c$ in Adams filtration 0, and so every mod 2 characteristic number vanishes on $m_c$-manifolds.

Note that there is an ambiguity in the M-theory action from a topological term which is not mod 2 characteristic numbers but rather a mod 2 $KO$-characteristic number, a mod 2 index of a Dirac operator; see §7.

Our focus is on anomalies, and here the crucial point is that only the square of (4.58) is unambiguously defined as an element of $\mathbb{C}$. This is equivalent to the assertion that on a closed $m_c$ 12-manifold $W$ the cubic form

\[(4.62) \quad \tilde{c}^3 - \tilde{p}(W)\tilde{c} \quad \text{is integral, but is not necessarily even. Hence the square root (4.58) is an element of a complex line $\alpha_C(X)$ whose square $\alpha_C(X)^{\otimes 2}$ is trivialized. As the notation suggests, this line is the state space of the invertible 12-dimensional field theory $\alpha_C$. The field theory $\alpha_C$ is topological: it does not depend on the Riemannian metric or differential $m_c$ structure, only on the underlying topological $m_c$ structure.}

Witten’s argument [W3, §4], reproduced in the proof of Proposition 4.30, proves the Anomaly Cancellation Theorem 2.2 on spin manifolds. Let $MSpin(\beta w_4)$ denote the bordism spectrum of spin manifolds with an integer lift of $w_4$. There is a map $MSpin(\beta w_4) \to Mm_c$, where $Mm_c$ is the bordism spectrum of $m_c$-manifolds.

**Theorem 4.63** (Witten). The lift $\alpha_{RS} \otimes \alpha_C: MSpin(\beta w_4) \to \Sigma^{12}IC^x$ is trivializable.

The $E_8$-model for the $C$-field leads to a distinguished trivialization [FM].

**Proof.** Because an invertible topological field theory is determined up to isomorphism by its partition functions, to prove Theorem 4.63 we show that for any closed spin 12-manifold $W$ with an $m_c$ structure we have

\[(4.64) \quad \hat{\alpha}_{RS}(W)\hat{\alpha}_C(W) = 1.

This follows immediately from the integrality of (4.35); see Proposition 3.19 and Remark 3.24. □

## 5. Some spin and pin manifolds

This section is a geometric interlude to review and introduce some special manifolds and their topological invariants. We use these manifolds as building blocks for the closed $\text{pin}^+$ 12-manifolds we need in §6, where we also specify $m_c$ structures.
If $M$ is a smooth manifold, then we use the notations

$$w(M) = 1 + w_1(M) + w_2(M) + \cdots$$

$$p(M) = 1 + p_1(M) + p_2(M) + \cdots$$

for the total Stiefel-Whitney class and total Pontrjagin class, respectively. The former satisfies the Whitney sum formula $w(M_1 \times M_2) = w(M_1)w(M_2)$ for Cartesian products; the analogous equation for the total Pontrjagin class is true modulo torsion. Also, these characteristic classes are defined for arbitrary real vector bundles, not just the tangent bundle, and are \textit{stable} in the sense that they are unchanged by adding a trivial bundle. Recall also the characteristic class $\lambda$ of a spin manifold, or of a real vector bundle with a spin structure, characterized after (4.21); it satisfies $2\lambda = p_1$.

\section{5.1. $K3$ surface}

There is a moduli space of inequivalent complex K3 surfaces whose underlying real 4-manifolds are all diffeomorphic. For definiteness, then, we define $K \subset \mathbb{CP}^3$ as the zero locus of the quartic

$$z^0 + (z^1)^4 + (z^2)^4 + (z^3)^4 = 0,$$

where $z^0, z^1, z^2, z^3$ are the standard homogeneous coordinates on $\mathbb{CP}^3$. It is a smooth closed real 4-manifold which is simply connected, and the complex structure induces an orientation. The Chern classes can be computed from those of $\mathbb{CP}^3$ and that of the normal bundle, which is the restriction of $O(4) \to \mathbb{CP}^3$ to $K$, and from there we derive the Stiefel-Whitney and Pontrjagin classes:

$$w(K) = 1$$

$$p(K) = 1 - 48k,$$

where $k \in H^4(K; \mathbb{Z}) \cong \mathbb{Z}$ is the positive generator. In particular, $w_2(K) = 0$ and so $K$ admits a spin structure compatible with the orientation, which is unique up to isomorphism since $K$ is simply connected. Also,

$$\lambda(K) = -24k.$$

\section{5.2. Quaternionic projective plane}

Let $\mathbb{HP}^2$ denote the space of one dimensional quaternionic subspaces of the quaternionic vector space $\mathbb{H}^3$. For definiteness we let the division algebra $\mathbb{H}$ act on the right of $\mathbb{H}^3$. In coordinates write

$$\mathbb{HP}^2 = \left\{ [q^0, q^1, q^2] : q^i \in \mathbb{H} \right\} / \sim, \quad [q^0, q^1, q^2] \sim [q^0 h, q^1 h, q^2 h], \quad h \in \mathbb{H}^+.$$

$\mathbb{HP}^2$ is a simply connected 8-manifold. In fact, the filtration $\ast \subset \mathbb{HP}^1 \subset \mathbb{HP}^2$ provides a CW structure with a single 0-cell, 4-cell, and 8-cell. The simple connectivity implies that up to isomorphism $\mathbb{HP}^2$ has a unique spin structure compatible with a given orientation.
Let \( L \to \mathbb{H}P^2 \) be the tautological quaternionic line bundle; its fiber at a point \( \ell \in \mathbb{H}P^2 \) is the quaternionic line \( \ell \). There is a short exact sequence

\[
0 \to L \to \mathbb{H}^3 \to Q \to 1
\]

of (right) quaternionic vector bundles; in the middle is the trivial bundle with fiber \( \mathbb{H}^3 \) and the sequence defines the rank two quotient bundle \( Q \to \mathbb{H}P^2 \). Note that the dual \( L^\ast \cong \text{Hom}_H(L, \mathbb{H}) \) is canonically a left \( \mathbb{H} \)-module. The tangent bundle is identified as the real vector bundle \( \text{Hom}_H(L, Q) \to Q \times_H L^\ast \), and it is the quotient in the short exact sequence of real vector bundles

\[
0 \to L \otimes_H L^* \to \mathbb{H}^3 \otimes_H L^* \to Q \otimes_H L^* \to 0,
\]

so its total Pontrjagin class is the quotient

\[
p(\mathbb{H}P^2) = \frac{p(\mathbb{H}^3 \otimes_H L^*)}{p(L \otimes_H L^*)},
\]

since \( H^\ast(\mathbb{H}P^2; \mathbb{Z}) \) is torsionfree. The quaternionic line bundle \( L^* \to \mathbb{H}P^2 \) is, by restriction of scalars to \( \mathbb{C} \subset \mathbb{H} \), a rank 2 complex vector bundle isomorphic to its complex conjugate, so its total Chern class has the form \( 1 - x \), where \( x \in H^4(\mathbb{H}P^2; \mathbb{Z}) \); we call \( x \) the \textit{quaternionic first Pontrjagin class}. Restrict to \( \mathbb{H}P^1 \subset \mathbb{H}P^2 \) and fix a nonzero quaternionic functional \( \mathbb{H}^2 \to \mathbb{H} \) to define a section of \( L^* \to \mathbb{H}P^1 \) which vanishes transversely at a single point. It follows that \( x \) generates \( H^4(\mathbb{H}P^2; \mathbb{Z}) \).

Now \( L \otimes_H L^* \) splits off a trivial real line bundle, and the orthogonal rank 3 bundle is the adjoint bundle of the complex 2-plane bundle underlying \( L^* \to \mathbb{H}P^2 \); the first Pontrjagin class multiplies by 4 under the adjoint. Therefore, from (5.8)

\[
(5.9) \quad p(\mathbb{H}P^2) = \frac{(1 + x)^6}{(1 + 4x)} = 1 + 2x + 7x^2.
\]

(See [BH, §15.5] for an alternative derivation.) It follows that

\[
(5.10) \quad \lambda(\mathbb{H}P^2) = x, \quad w_4(\mathbb{H}P^2) = \bar{x},
\]

where \( \bar{x} \in H^4(\mathbb{H}P^2; \mathbb{Z}/2\mathbb{Z}) \) is the mod 2 reduction of \( x \).

\textbf{Remark 5.11.} As mentioned above, a quaternionic line bundle \( L \to X \) has a quaternionic first Pontrjagin class \( p_1^H(L) \in H^4(X; \mathbb{Z}) \) which equals minus the second Chern class after restricting scalars to \( \mathbb{C} \subset \mathbb{H} \). We can also restrict scalars to \( \mathbb{R} \subset \mathbb{H} \) to obtain a rank 4 real vector bundle \( L_\mathbb{R} \to X \), whose first Pontrjagin class satisfies \( p_1(L_\mathbb{R}) = 2p_1^H(L) \). The following general formula is useful, and can be used in the derivation of (5.9). Suppose \( R, L \to X \) are right and left quaternionic line bundles with quaternionic first Pontrjagin classes \( r, \ell \in H^4(X; \mathbb{Z}) \). Then \( R \otimes_H L \to X \) is a real 4-plane bundle with total Pontrjagin class \( 1 + 2(r + \ell) + (r - \ell)^2 \).

Use \( x^2 \in H^8(\mathbb{H}P^2; \mathbb{Z}) \cong \mathbb{Z} \) to orient \( \mathbb{H}P^2 \): choose the fundamental class such that \( \langle x^2, [\mathbb{H}P^2] \rangle = 1 \).
5.3. Bott manifold

The bordism group $\pi_8\text{Spin}$ is free abelian of rank two: there is an isomorphism

$$\pi_8\text{Spin} \rightarrow \mathbb{Z} \oplus \mathbb{Z}$$

which maps a closed spin 8-manifold to its Ahat genus and its signature. The quaternionic projective plane has $\hat{A}(\mathbb{H}\mathbb{P}^2) = 0$, $\sigma(\mathbb{H}\mathbb{P}^2) = 1$. A closed spin manifold $B$ with $\hat{A}(B) = 1$ is called a Bott manifold. We needn’t insist on vanishing signature, as that can always be achieved by connected sum with copies of $\mathbb{H}\mathbb{P}^2$ or its orientation-reversal, and indeed the Bott manifold we use has signature $-224$.

We do not know of an elementary construction of a Bott manifold. One possibility is a Riemannian manifold $B$ of special holonomy $\text{Spin}_7 \subset \text{Spin}_8$, which necessarily satisfies $\hat{A}(B) = 1$ and is simply connected; see [J, §10.6]. Closed 8-manifolds with Spin$_7$ holonomy were first produced by Joyce. A more topological approach leans on the work of Kervaire and Milnor [MK], [KM]. The Bott manifold $B$ so constructed is also simply connected, so admits a unique spin structure. Briefly, plumb together 8 copies of the disk bundle of the tangent bundle to $S^4$ according to the $E_8$ Dynkin diagram. The resulting compact 8-manifold $N$ has a boundary which is an exotic 7-sphere. The Kervaire-Milnor results imply that a connect sum of 28 copies of the exotic sphere bounds a ball, hence we define $B$ as the boundary connect sum of 28 copies of $N$ and cap off with a standard ball; see [HBJ, §6.5] for details. The manifold $B$ is almost parallelizable, i.e., admits a trivialization of the tangent bundle away from a point. This implies that $p_1(B) = 0$, and from a computation with the signature we deduce the total Pontrjagin class

$$p(B) = 1 - 1440b,$$

where $b \in H^8(B; \mathbb{Z}) \cong \mathbb{Z}$ is the positive generator. Note $\lambda(B) = 0$ and $w_4(B) = 0$. Then

$$\hat{A}(B) = \left\langle \frac{7p_1^2 - 4p_2}{5760}, [B] \right\rangle$$

implies $\hat{A}(B) = 1$. We use this Bott manifold in the sequel.

5.4. Real projective spaces

Let $L \rightarrow \mathbb{R}\mathbb{P}^n$ be the tautological real line bundle. Arguing as in the second paragraph of §5.2 we deduce that the tangent bundle to $\mathbb{R}\mathbb{P}^n$ is stably equivalent to

$$\langle n + 1 \rangle L - 1.$$

Then if $\alpha \in H^1(\mathbb{R}\mathbb{P}^n; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ is the generator, we conclude

$$w(\mathbb{R}\mathbb{P}^n) = (1 + \alpha)^{n+1}.$$
Projective 4-space $\mathbb{RP}^4$ has $w_2 = 0$, so admits a pin$^+$ structure, in fact two distinct ones which are opposite in the sense of Proposition 3.30. Of course, $w_1(\mathbb{RP}^4) = \alpha$ so that $\mathbb{RP}^4$ is not orientable, so not spin either. Also, $w_4(\mathbb{RP}^4) = \alpha^4$ is nonzero, and we fix a $w_1$-twisted lift $\tilde{c}_{\mathbb{RP}^4} \in H^4(\mathbb{RP}^4; \hat{\mathbb{Z}}) \cong \mathbb{Z}$ which is a generator.

For $n = 12$ we compute from (5.16) that $\mathbb{RP}^{12}$ is not orientable; is pin$^+$ with two opposite pin$^+$ structures; that $w_4(\mathbb{RP}^{12}) = \alpha^4$; and since $H^4(\mathbb{RP}^{12}; \hat{\mathbb{Z}}) = 0$ it does not admit an $m_c$ structure.

The $\eta$-invariants of $\mathbb{RP}^4$ and $\mathbb{RP}^{12}$ are computed in [St, Corollary 5.4]. The results are reciprocal for the two opposite pin$^+$ structures (Proposition 3.30), and we use the $\eta$-invariant to pin down a choice. Stolz’s result follows from Proposition 3.38 (see (3.26) for notation):

$$\alpha_{\mathbb{RP}^4} = \exp\left(\frac{2\pi i}{2^4}\right)$$

$$\alpha_{\mathbb{RP}^{12}} = \exp\left(\frac{2\pi i}{2^8}\right)$$

For later use we quote from [KT1] the position of these real projective spaces in pin$^+$ bordism. In dimension 4 we have

$$\pi_4M\text{Pin}^+ \cong \mathbb{Z}/2^4\mathbb{Z}$$

and $\mathbb{RP}^4$ represents a generator. In dimension 12 we have, as already quoted in (3.15),

$$\pi_{12}M\text{Pin}^+ \cong \mathbb{Z}/2^8\mathbb{Z} \oplus \mathbb{Z}/2^4\mathbb{Z} \oplus \mathbb{Z}/2^2\mathbb{Z}$$

and $\mathbb{RP}^{12}$ represents a generator of the first factor. Proposition 8.15 below proves that

$$\pi_4Mm_c \cong \mathbb{Z} \oplus \mathbb{Z}/2^5\mathbb{Z}.$$
5.5. Three special manifolds

We define three 12-dimensional manifolds $W'_0, W''_0, W_1$ which appear in Theorem 6.1 below. Each of $W'_0, W''_0$ is presented as the quotient of its orientation double cover by a free involution.

5.5.1. $W'_0$. Set

\begin{equation}
\hat{W}'_0 = S^4 \times (\mathbb{HP}^2 \# \mathbb{HP}^2),
\end{equation}

the Cartesian product of the 4-sphere and the connected sum of two quaternionic projective planes.

As an explicit model of the connected sum, fix a line through the origin in real affine space $\mathbb{A}^9$, remove two small antipodal balls from $S^8 \subset \mathbb{A}^9$ which are exchanged by the half-turn about that line, and glue in two identical copies of $\mathbb{HP}^2 \# B^8$. Then (5.23) has a free orientation-reversing involution which is the Cartesian product of the antipodal involution of $S^4$ and the half-turn of $\mathbb{HP}^2 \# \mathbb{HP}^2$ with its two fixed points. The quotient is the manifold $W'_0$. Since $\hat{W}'_0$ is simply connected, we have $\pi_1 W'_0 \cong \mathbb{Z}/2\mathbb{Z}$ and hence $H^1(W'_0; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$. Since the involution is free on $S^4$, the manifold $W'_0$ fits into a fiber bundle

\begin{equation}
\mathbb{HP}^2 \# \mathbb{HP}^2 \longrightarrow W'_0 \longrightarrow \mathbb{RP}^4.
\end{equation}

The simply connected manifold $\mathbb{HP}^2 \# \mathbb{HP}^2$ has a unique spin structure, and so the half-turn lifts to a spin automorphism. Its square is either the identity or the spin flip; we show it is the identity by computing at a fixed point on $S^8$. The differential of the half-turn is the linear map $-1$ on the 8-dimensional tangent space. The linear map $-1$ lifts to the volume form in $\text{Spin}_{8}$, which squares to $1$. Therefore, the vertical tangent bundle of (5.24) is spin, and so $w_i(W'_0), i = 1, 2,$ are pulled back from $\mathbb{RP}^4$. Using (5.16) we see that $W'_0$ is pin$^+$: it admits two opposite pin$^+$ structures.

The cohomology ring of the connected sum is

\begin{equation}
H^*(\mathbb{HP}^2 \# \mathbb{HP}^2; \mathbb{Z}) \cong \mathbb{Z}[x_1, x_2]/(x_1 x_2, x_1^2 - x_2^2), \quad \deg x_1 = \deg x_2 = 4.
\end{equation}

Let $t \in H^4(S^4; \mathbb{Z})$ denote a positive generator. Then under the antipodal involution the class $2t$ descends to the generator $\tilde{c} \in H^4(\mathbb{RP}^4; \tilde{\mathbb{Z}})$; see Lemma 4.37. Recalling (5.10) and the fact that $w_4(\mathbb{RP}^4) = \alpha^4$, as mentioned following (5.16), we deduce that $w_4(W'_0) = \alpha^4 + \tilde{x}_1 + \tilde{x}_2$, where $\tilde{x}_1$ is the mod 2 reduction of $x_1$. The class $\tilde{x}_1 + \tilde{x}_2$ is invariant under the involution of $\mathbb{HP}^2 \# \mathbb{HP}^2$ and descends to $w_4$ of the vertical tangent bundle in (5.24). Define the $w_1$-twisted integer lift $\tilde{c}'_0 \in H^4(W'_0; \tilde{\mathbb{Z}})$ of $w_4(W'_0)$ by

\begin{equation}
\pi^{\ast} \tilde{c}'_0 = 2t + x_1 - x_2,
\end{equation}

where $\pi: \hat{W}'_0 \to W'_0$ is the orientation double cover.
5.5.2. $W''_0$. Let $K_R \to S^4 = \mathbb{H}P^1$ be the underlying real 4-plane bundle of the quaternionic line bundle $K \to S^4$ with $p_1^H(K) \in H^4(S^4; \mathbb{Z})$ a positive generator. Define $W''_0 = \mathbb{P}(K_R^{\mathbb{Q}^2} \oplus \mathbb{R})$ as the total space of the real projective bundle

\[(5.27) \quad \mathbb{R}P^8 \to \mathbb{P}(K_R^{\mathbb{Q}^2} \oplus \mathbb{R}) \to \rho \to S^4.\]

Let $L \to W''_0$ be the tautological real line bundle. Since the stable tangent bundle to $S^4$ is trivial, the stable tangent bundle to $W''_0$ is the stable tangent bundle along the fibers, which is

\[(5.28) \quad (L - \mathbb{R}) + (\rho^*K_R^{\mathbb{Q}^2} \otimes L).\]

This comes from the short exact sequence $0 \to L \to \rho^*(K_R^{\mathbb{Q}^2} \oplus \mathbb{R}) \to Q \to 0$ of real vector bundles over $W''_0$ (compare (5.6)). Using (5.28) we compute $w_1(W''_0) = w_1(L)$ and $w_2(W''_0) = 0$: it suffices to restrict to a fiber of (5.27) since that restriction induces an isomorphism on $H^i(-; \mathbb{Z}/2\mathbb{Z})$, $i = 1, 2$. The orientation double cover (see (5.25)) is an $S^8$-bundle over $S^4$, which is simply connected. Hence $W''_0$ admits two opposite pin$^+$ structures.

The bundle $\rho^*K_R \otimes L$ has total Stiefel-Whitney class of the form $1 + w_4$, and it follows easily from the Whitney formula applied to (5.28) that $w_4(W''_0) = 0$.

5.5.3. $W_1$. The projective group $\mathbb{P}S_p1 \cong SO_3$ acts on $\mathbb{H}P^2$ via (see (5.5) for notation)

\[(5.29) \quad \lambda \cdot [q^0, q^1, q^2] = [\lambda q^0, \lambda q^1, \lambda q^2], \quad \lambda \in S_p1.\]

So a principal $SO_3$-bundle has an associated fiber bundle with fiber $\mathbb{H}P^2$. The action (5.29) lifts to the spin bundle of frames of $\mathbb{H}P^2$. To see this, choose a basepoint $[1, 0, 0]$ and write $\mathbb{H}P^2 = Sp_3/Sp_1 \times Sp_2$. The principal Spin$^-$-bundle of frames is associated to the principal $(Sp_1 \times Sp_2)$-bundle $Sp_3 \to \mathbb{H}P^2$ via the representation

\[(5.30) \quad Sp_1 \times Sp_2 \cong Spin_3 \times Spin_5 \to Spin_8.\]

The $\mathbb{P}Sp_1$ action fixes the basepoint, and the “diagonal” map $Sp_1 \to Sp_1 \times Sp_2 \to Spin_8$ descends to $\mathbb{P}Sp_1$. The induced map $\mathbb{P}Sp_1 \to Spin_8$ gives the desired lift. Define $W_1$ as the fiber bundle

\[(5.31) \quad \mathbb{H}P^2 \to W_1 \to \mathbb{C}P^1 \times \mathbb{C}P^1\]

obtained from the principal $SO_3$-bundle of oriented orthonormal frames of the real 3-plane bundle $O(1, 1)_R \oplus \mathbb{R} \to \mathbb{C}P^1 \times \mathbb{C}P^1$, where $O(1, 1) \to \mathbb{C}P^1 \times \mathbb{C}P^1$ is the tensor product of the hyperplane line bundles on the factors. The manifold $W_1$ is simply connected, hence orientable. The stable tangent bundle to $\mathbb{C}P^1 \times \mathbb{C}P^1$ is trivial, and the vertical tangent bundle is spin, hence $W_1$ is spin with a unique spin structure refining each orientation.

To compute the Pontrjagin classes of the vertical tangent bundle of (5.31), we use the $\mathbb{P}Sp_1$ action to construct a fiber bundle

\[(5.32) \quad \mathbb{H}P^2 \to E \to \mathbb{C}P^\infty\]
from the rank three real vector bundle $O(1)_R \oplus \mathbb{R} \to \mathbb{C}P^\infty$. The squaring map $T \to T$ induces a degree two map $f$ on $B\mathbb{T} = \mathbb{C}P^\infty$; the pullback $f^*(O(1)_R \oplus \mathbb{R}) \cong O(2)_R \oplus \mathbb{R}$ is the adjoint bundle of a principal $Sp_1 \cong SU_2$-bundle we write as the quaternionic line bundle $K \to \mathbb{C}P^8$. Then the pullback of \eqref{5.32} under $f$ is the projectivization of the rank three quaternionic vector bundle $K^{\otimes 3} \to \mathbb{C}P^8$. Let $a \in H^2(\mathbb{C}P^\infty; \mathbb{Z})$ be a generator; then the quaternionic Pontrjagin class of $K \to \mathbb{C}P^8$ is $p^H_1(K) = a^2$. Let $L \to \mathbb{P}(K^{\otimes 3})$ be the tautological quaternionic line bundle. Using the technique in §5.2, including Remark 5.11, we compute the total Pontrjagin class of the vertical tangent bundle to $E' = \mathbb{P}(K^{\otimes 3}) \to \mathbb{C}P^\infty$ as

\begin{equation}
 p(E'/\mathbb{C}P^\infty) = \frac{1 + 2(x + a^2) + (x - a^2)^2}{1 + 4x} \nonumber
\end{equation}

\begin{equation}
= 1 + (2x + 6a^2) + (7x^2 - 6a^2x) + (12a^2x^2) + \cdots
\end{equation}

where $x = p^H_1(L) \in H^4(E'; \mathbb{Z})$. Grothendieck’s formula for projective bundles\textsuperscript{27} implies

\begin{equation}
x^3 = 3a^2x^2 - 3a^4x + a^6.
\end{equation}

Use the pullback diagram

\begin{equation}
\begin{array}{ccc}
E' & \xrightarrow{\tilde{j}} & E \\
\pi' \downarrow & & \downarrow \pi \\
\mathbb{C}P^\infty & \xrightarrow{f} & \mathbb{C}P^\infty
\end{array}
\end{equation}

of fiber bundles to compute $\pi_*$ of the degree 12 Pontrjagin classes $p_3, p_1p_2, p_1^3$ of the vertical tangent bundle to $E'/\mathbb{C}P^\infty$; they pull back under $\tilde{j}$ to the corresponding Pontrjagin classes of $E'/\mathbb{C}P^\infty$. Note $f^*a = 2a$. Thus

\begin{equation}
 f^*\pi_*p_3 = \pi'_*\tilde{j}^*p_3 = \pi'_*(12a^2x^2) = 12a^2
\end{equation}

from which

\begin{equation}
\pi_*p_3 = 3a^2.
\end{equation}

Similarly,

\begin{equation}
\pi_*p_1p_2 = 18a^2,
\end{equation}

\begin{equation}
\pi_*p_1^3 = 24a^2
\end{equation}

\textsuperscript{27}Let $V \to X$ be a vector bundle of rank $r > 0$ (over $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$), $\mathbb{P}(V) \to X$ its projectivization, and $L \to \mathbb{P}(V)$ the tautological line bundle. Over $\mathbb{P}(V)$ there is a short exact sequence $0 \to L \to p^*V \to Q \to 0$ of vector bundles, where $Q \to \mathbb{P}(V)$ has rank $r - 1$. Grothendieck’s formula expresses the vanishing of its $r^{th}$ Chern or Pontrjagin class.
and hence

\[ \pi_* \lambda^3 = 3a^2. \]

Finally, we pull back by the degree \((1, 1)\) map \(\mathbb{CP}^1 \times \mathbb{CP}^1 \to \mathbb{CP}^\infty\) to compute the corresponding quantities on \(W_1\). After evaluating on the fundamental class \([\mathbb{CP}^1 \times \mathbb{CP}^1]\) we find

\[
\begin{align*}
\langle p_3(W_1), [W_1] \rangle &= 6 \\
\langle p_1p_2(W_1), [W_1] \rangle &= 36 \\
\langle p_1^3(W_1), [W_1] \rangle &= 48 \\
\langle \lambda^3(W_1), [W_1] \rangle &= 6
\end{align*}
\]

6. The anomaly cancellation

In this section we state the main computational result, Theorem 6.1, which provides generators for the dimension 12 bordism of manifolds which occur in M-theory. We give the proof in §8. Here, in six subsections, we use this bordism computation to prove the Anomaly Cancellation Theorem 2.2 by computing the invariants \(\alpha_{RS}, \alpha_C\) for a generator of each factor. We organize the presentation by Adams filtration (see §8.4). Two of the six generators are represented by spin manifolds, so in these cases the anomaly cancellation is already proved in Theorem 4.63. Nonetheless, we check directly by computing the invariants.

Since our invariants take values in a finite abelian 2-group (see Corollary 3.16 and Corollary 4.53), it suffices to compute after completing at the prime 2. (The structure of the bordism group at odd primes is much simpler, but we do not treat it here.) Let \(\mathbb{Z}_2\) denote the 2-adic numbers.

6.1. The bordism group

Recall that \(M_{m_c}\) denotes the bordism spectrum of \(m_c\)-manifolds. We use the manifolds and cohomology classes defined in §5. The cohomology class \(\lambda\) of a spin manifold is the canonical integer lift of \(w_4\).

**Theorem 6.1.** The following six \(m_c\)-manifolds generate the group \(\pi_{12} M_{m_c} \otimes \mathbb{Z}_2\):

\[
(W_0^\sigma, c_0^\sigma), \quad (W_0^\sigma, 0), \quad (W_1, \lambda) \\
(K \times \mathbb{HP}^2, \lambda), \quad (\mathbb{RP}^4, c_{\mathbb{RP}^4}) \times B, \quad (\mathbb{RP}^4 \# \mathbb{RP}^4, 0) \times B.
\]

Note that \(W_1\) and \(K \times \mathbb{HP}^2\) are spin manifolds.
6.2. Computations

As explained in (2.5), the Anomaly Cancellation Theorem 2.2 is a consequence of Theorem 6.1 and the following.

Theorem 6.3. For each of the pairs \((W, \tilde{c})\) listed in (6.2) the anomaly cancellation condition \(\alpha_{RS}(W)\alpha_C(W) = 1\) holds.

The proof of Theorem 6.3 is divided into six parts, one for each generator. It occupies the remainder of §6. Recall that \(\alpha_{RS}\) is defined in (3.13) and \(\alpha_C\) in (4.52). Strictly speaking, we do not rely on the particular \(w_1\)-twisted integer lift \(\tilde{c}\) of \(w_4\) since the mod 2 cubic invariant \(\alpha_C\) is independent of the choice (Lemma 4.50).

6.2.1. Adams filtration 4. To compute the Rarita-Schwinger anomaly partition function of \(\mathbb{RP}^4 \hat{B}\), we apply Proposition 3.27 and Proposition 3.33:

\[
\alpha_{RS}(\mathbb{RP}^4 \times B) = \frac{\alpha_{\mathbb{RP}^4}(TRP^4 + TB - 2)}{\alpha_{\mathbb{RP}^4}^{2\text{ind}D_B}}.
\]

From (5.15) the stable tangent bundle to \(\mathbb{RP}^4\) is \(5L - 1\). Proposition 3.30 implies that as far as \(\eta\)-invariants are concerned, tensoring with \(L\) induces a change of sign. Hence using Proposition 3.27 and (5.18)

\[
\alpha_{\mathbb{RP}^4}(TRP^4) = \alpha_{\mathbb{RP}^4}(5L - 1) = \alpha_{\mathbb{RP}^4}(-6) = \exp\left(-\frac{3\pi i}{4}\right).
\]

An alternative computation uses the 4-dimensional analog of Proposition 3.38 in which the denominator of \(2^8\) in (3.39) is replaced by \(2^4\). Bound the orientation double cover \(S^4\) by the closed 5-ball \(D^5\) with its antipodal involution. The pullback of \(T\mathbb{RP}^4\) extends over \(D^5\) as \(TD^5 - 1\). The trace \(\gamma_f\) at the unique fixed point is \(-5 - 1 = -6\), and we recover (6.5) from (3.39). Still another computation uses a variant of Theorem 3.28 for pin\(^+\) 4-manifolds: replace \(\mathbb{RP}^{20}\) with \(\mathbb{RP}^{12}\) and \(2^{11}\) in (3.29) with \(2^7\). Following the notation in Remark 5.22 with these replacements we compute

\[
\gamma_*(T\mathbb{RP}^4) = \gamma_*(5[H] - 1) = (5[H] - 1) \gamma_*(1) = (5[H] - 1) 8(1 - [H]) = -48(1 - [H]),
\]

and now (6.5) follows from the adapted (3.29). For the index computation on the Bott manifold we use the Atiyah-Singer index theorem and (5.13):

\[
\operatorname{ind} D_B(TB) = \hat{A}(B) \text{ch}(TB)[B] = (1 - \frac{p_2}{1440})(8 - \frac{p_2}{6})[B] = 248.
\]
Combining (5.17), (6.4), (6.5), (6.7) and \( \text{ind } D_B = 1 \) (see (5.14)) we conclude

\[
\alpha_{RS}(\mathbb{RP}^4 \times B) = \frac{e^{-3\pi i/4} e^{\pi i}}{e^{\pi i/4}} = 1.
\]

The \( C \)-field anomaly partition function is the exponential of the mod 2 reduction of the cubic form (4.62). As in §6.2 the class \( \tilde{p}(\mathbb{RP}^4 \times B) = \tilde{p}(B) = -720b \). Evaluating on the generator \( \tilde{c}_{\mathbb{RP}^4} \in H^4(\mathbb{RP}^4, \mathbb{Z}) \) we find

\[
\left( \frac{\tilde{c}_{\mathbb{RP}^4}^3 - \tilde{p}_{\mathbb{RP}^4}}{24} \right) [\mathbb{RP}^4 \times B] = 30.
\]

Since this is even, \( \alpha_C(\mathbb{RP}^4 \times B) = 1 \).

**6.2.2. Adams filtration 5.** The Rarita-Schwinger partition function only depends on the image of \( \mathbb{RP}^4 \times B \) in \( \text{pin}^+ \) bordism. The connected sum \( \mathbb{RP}^4 \# \mathbb{RP}^4 \) represents twice \( \mathbb{RP}^4 \) in \( \text{pin}^+ \) bordism, and the same is true after crossing with the Bott manifold. (See §5.4.) Thus we deduce from (6.8) that \( \alpha_{RS}((\mathbb{RP}^4 \# \mathbb{RP}^4) \times B) = \alpha_{RS}(\mathbb{RP}^4 \times B)^2 = 1 \). Since \( w_4((\mathbb{RP}^4 \# \mathbb{RP}^4) \times B) = 0 \) we can take the \( w_1 \)-twisted integer lift to vanish, and hence \( \alpha_C((\mathbb{RP}^4 \# \mathbb{RP}^4) \times B) = 1 \).

**6.2.3. Adams filtration 3.** The manifold \( W_3 = K \times \mathbb{HP}^2 \) is spin, so by Proposition 3.19 the Rarita-Schwinger anomaly partition function is the mod 2 reduction of an integer \( RS(W_3) \) defined in (3.20). Using (5.3) and (5.9) we compute

\[
\frac{1}{2} \hat{A}(W_3) \text{ch}(TW_3 - 2) = \frac{1}{2} \hat{A}(K) \hat{A}(\mathbb{HP}^2)(\text{ch}(TK) + \text{ch}(T\mathbb{HP}^2) - 2)
\]

\[
= \frac{1}{2} \left( 1 + 2k \right) \left( 1 - \frac{x}{12} \right) \left( 4 - 48k \right) + \left( 8 + 2x - \frac{5}{6}x^2 \right) - 2 \right)
\]

\[
= -kx^2 + \ldots
\]

Therefore, \( RS(W_3) = -1 \) and \( \alpha_{RS}(W_3) = -1 \).

The \( C \)-field anomaly partition function is computed from the cubic form

\[
\frac{\lambda^3 - p\lambda}{24} = \frac{(x - 24k)^3 - (3x^2 - 24xk)(x - 24k)}{24} = kx^2 + \ldots
\]

Evaluating on the fundamental class and exponentiating we deduce \( \alpha_C(W_3) = -1 \).

**6.2.4. Adams filtration 1.** The spin manifold \( W_1 \) is defined in §5.5.3; it is an \( \mathbb{HP}^2 \)-bundle over \( \mathbb{CP}^1 \times \mathbb{CP}^1 \). We plug (5.41) into (3.23) to compute \( \alpha_{RS}(W_1) = 1 \) and into (4.62) with \( \tilde{c} = \lambda \) to compute \( \alpha_C(W_1) = 1 \).
6.2.5. **Adams filtration 0, part 1.** The manifold $W'_0$ is an $(\mathbb{H}P^2 \# \mathbb{H}P^2)$-bundle over $\mathbb{R}P^4$; see (5.24). We claim that its Rarita-Schwinger anomaly partition function is trivial: $\alpha_{RS}(W'_0) = 1$. To prove this we apply Proposition 3.28. The total space (5.23) of the orientation double cover bounds $D^5 \times (\mathbb{H}P^2 \# \mathbb{H}P^2)$ with the antipodal involution on $D^5$ times the half-turn about an axis through $S^8 \subset \mathbb{H}^9$ acting on the connected sum. There are two fixed points: the center of $D^5$ times antipodal points $p, p' \in S^8$. In (3.39) the traces $\tau_p = \tau_{p'}$, and we claim $i_{p'} = -i_p$. To prove this choose the center of $S^8$ as the origin of $\mathbb{H}^9$, so identify the affine space $\mathbb{H}^9$ with the vector space $\mathbb{R}^9$. The half-turn is implemented on spinors by the element $\omega = \gamma_1 \gamma_2 \ldots \gamma_8$ in $\text{Cliff}_{+5,1}$, where we choose the axis to be the last coordinate. But the standard basis vectors $e_1, e_2, \ldots, e_8$ form an oriented basis of exactly one of $T_p S^8, T_{p'} S^8$; it is negatively oriented at the other point. So the action of frames at the other point is by the conjugate $e_1 \omega e_1^{-1} = -\omega$. Multiply by the volume element of $\text{Cliff}_{+5,1}$, which gives the action of the involution on pin$^+$ frames at the center of $D^5$.

For the $C$-field anomaly partition function we compute the cubic form on the orientation double cover using (4.57). Use the $w_1$-twisted integer lift $\tilde{c}$ specified in (5.26) via its lift $c$ to $\tilde{W}_0'$. Use (5.9) to compute that $p(\tilde{W}_0') = 3(x_1^2 + x_2^2)$. Thus

\begin{equation}
\frac{c^3 - pc}{48} = \frac{12tx^2 - 12tx^2}{48} = 0,
\end{equation}

since $x^2 = x_1^2 = x_2^2$ in $H^8(\mathbb{H}P^2 \# \mathbb{H}P^2)$; see (5.25). Therefore, $\alpha_C(W'_0) = 1$.

6.2.6. **Adams filtration 0, part 2.** The manifold $W''_0 = \mathbb{P}(2K_R \oplus \mathbb{R})$, defined in (5.27) as an $\mathbb{R}P^8$-bundle over $S^4$. Recall that $K \to S^4$ is the quaternionic line bundle with $p_H^R(K) \in H^4(S^4, \mathbb{Z})$ the positive generator. We use Theorem 3.28 to compute $\alpha_{RS}(W''_0) = 1$.

As a preliminary define $\mu \in \tilde{KO}^4(S^4)$ as $e_+(1)$ for $p: S^4 \to S^4$, and $\lambda \in \tilde{KO}^0(S^4)$ as the KO-class $[K_R] - 4$, where $K_R \to S^4$ is the real 4-plane bundle underlying $K \to S^4$. Identify $\tilde{KO}^4(S^4)$ as the Grothendieck group of quaternionic vector bundles over $S^4$ of virtual rank zero; then $\mu = [K] - [\mathbb{H}]$. Let $\pi: S^4 \to pt$ be the unique map. Then we claim

\begin{align}
\lambda \mu &= 0 \quad \tag{6.13} \\
\lambda [\mathbb{H}] &= 4\mu \quad \tag{6.14} \\
\pi_* \mu &= 1 \quad \tag{6.15} \\
\pi_* [\mathbb{H}] &= 0 \quad \tag{6.16}
\end{align}

Since we can trivialize $K_R \to S^4$ away from a point, we can arrange representatives of $\lambda, \mu$ with disjoint support, from which (6.13) follows. For (6.14) we observe that if $M$ is any quaternionic line, then there is a natural isomorphism

\begin{equation}
M^\oplus 4 \to M_R \otimes_R \mathbb{H}
\end{equation}

\begin{equation}
(\xi_1, \xi_2, \xi_3, \xi_4) \mapsto \sum_{s=1}^4 -\xi_s \otimes 1 + \xi_s i \otimes i + \xi_s j \otimes j + \xi_s k \otimes k
\end{equation}
of quaternionic vector spaces. Equation (6.15) is immediate: \( \pi_*\mu = (\pi \circ \iota)_*(1) = 1 \). Finally, \( \mathbb{H} \to S^4 \) is pulled back from \( \mathbb{H} \to \text{pt} \), so \( \pi_*(\mathbb{H}) = [\mathbb{H}]\pi_*(1) = 0 \) since \( 1 \in KO^0(S^4) \) extends over the 5-ball.

Another preliminary: If \( M_\mathbb{R} \to Y \) is the real 4-plane bundle underlying a quaternionic line bundle \( M \to Y \), then its \( KO \)-theory Euler class is

\[
(6.18) \quad [\mathbb{H}] - [M] \in KO(Y).
\]

Proof: \( M \to Y \) is associated to a principal \( Sp_1 \)-bundle via (i) the embedding \( Sp_1 \to \text{Spin}_4 \cong Sp_1 \times Sp_1 \) onto the second factor and (ii) the action of \( Sp_1 \times Sp_1 \) on \( \mathbb{H} \) in which the first factor acts trivially and the second by right multiplication. Then the \( KO \)-Euler class is pulled back from the vector bundle associated to the difference of the quaternionic half-spin representations.

Let \( J \to S^4 \) be the quaternionic line bundle with \( p_1^H(J) = -2p_1^E(K) \). Then \( K_{\mathbb{R}}^{\mathbb{R}^2} \oplus J_\mathbb{R} \to S^4 \) is trivializable. Define

\[
(6.19) \quad \gamma: W_0'' = \mathbb{P}(K_{\mathbb{R}}^{\mathbb{R}^2} \oplus \mathbb{R}) \overset{i}{\to} \mathbb{P}(K_{\mathbb{R}}^{\mathbb{R}^2} \oplus \mathbb{R} \oplus J_\mathbb{R}) \cong S^4 \times \mathbb{RP}^{12} \overset{\pi}{\to} \mathbb{RP}^{12} \overset{j}{\to} \mathbb{RP}^{20},
\]

where \( \pi \) is projection and \( j \) is a linear embedding as in Remark 5.22. Let \( L \to \mathbb{P}(K_{\mathbb{R}}^{\mathbb{R}^2} \oplus \mathbb{R} \oplus J_\mathbb{R}) \) be the tautological real line bundle. Then \( L \cong \pi^*H = \pi^*j^*H \) for \( H \to \mathbb{RP}^N \) the tautological line bundle, and \( i^*L \) is also isomorphic to the tautological line bundle. We identify \( L^* \cong L \). The normal bundle to \( i \) is the quotient of tangent bundles (see (5.28)):

\[
(6.20) \quad \left[ (i^*L - \mathbb{R}) \oplus K_{\mathbb{R}}^{\mathbb{R}^2} \oplus i^*L \oplus J_\mathbb{R} \otimes i^*L \right] / \left[ (i^*L - \mathbb{R}) \oplus K_{\mathbb{R}}^{\mathbb{R}^2} \otimes i^*L \right] \cong J_\mathbb{R} \otimes i^*L \\
\cong i^*(J_\mathbb{R} \otimes L).
\]

There is a canonical section of

\[
(6.21) \quad J_\mathbb{R} \otimes L \cong \text{Hom}(L, J_\mathbb{R}) \to \mathbb{P}(K_{\mathbb{R}}^{\mathbb{R}^2} \oplus \mathbb{R} \oplus J_\mathbb{R})
\]

given by projection \( K_{\mathbb{R}}^{\mathbb{R}^2} \oplus \mathbb{R} \oplus J_\mathbb{R} \to J_\mathbb{R} \), and its zero set is the image of \( i \). It follows that \( i_*(1) \) is the \( KO \)-Euler class of (6.21), which we compute using (6.18):

\[
(6.22) \quad i_*(1) = [\mathbb{H}] - [J \otimes L] = (1 - [L])[\mathbb{H}] + 2[L] \mu \\
= \pi^*\{ (1 - [H])[\mathbb{H}] + 2[H] \mu \}.
\]

Using (5.28) we find

\[
(6.23) \quad [TW_0'' - 2] = i^*\{ 2[H] \lambda + 9[H] + 1 \}.
\]

Combining (6.22) and (6.23) with (6.13)–(6.16) we calculate

\[
(6.24) \quad \pi_*i_*([TW_0''] - 2) = \pi_*\{ (2[H] \lambda + 9[H] + 1)((1 - [H])[\mathbb{H}] + 2[H] \mu) \} \\
= 10[H] + 10.
\]
Now \( j_*(1) = 8(1 - [H]) \) is computed in Remark 5.22, and so
\[
\begin{equation}
\tag{6.25}
j_*([H]) = [H]j_*(1) = 8[H](1 - [H]) = -j_*(1),
\end{equation}
\]
from which
\[
\begin{equation}
\tag{6.26}
\gamma_\ast([TW^n] - 2) = j_\ast \pi_\ast i_\ast ([TW^n] - 2) = 0.
\end{equation}
\]
Then \( \alpha_{RS}(W^n_0) = 1 \) follows immediately from (3.29).

The C-field anomaly is also trivial—\( \alpha_C(W^n_0) = 1 \)—since \( w_4(W^n_0) = 0 \) and we can choose the \( w_1 \)-twisted integer lift \( \tilde{c} \) in (4.52) to be zero.

7. Ambiguities in the M-theory action

As mentioned in the introduction, to define an M-theory action it is not sufficient to demonstrate the cancellation of anomalies; we must also give a trivialization of the product \( \alpha_{RS} \otimes \alpha_C \), a so-called setting of the quantum integrand. The ratio \( \beta \) of two trivializations is an invertible 11-dimensional field theory. Unitarity of M-theory requires that \( \beta \) be reflection positive. If \( \beta \) were to depend on the metric or the field strength of the C-field, then it would be detected locally. Since the local physics is fixed by considerations other than anomaly cancellation, we restrict \( \beta \) to be a topological field theory. As explained in §2 a reflection positive invertible 11-dimensional topological field theory of \( M_\mathfrak{m}_c \)-manifolds is determined by a homomorphism \( \pi_{11} M_\mathfrak{m}_c \to \mathbb{C}/\mathbb{Z} \). (Reflection positivity imposes a restriction, which is satisfied here by all such homomorphisms, since they take values in \( \{ \pm 1 \} \subset \mathbb{C}/\mathbb{Z} \).) The following conjecture describes the group of these theories.

Let \( \Sigma \) be the Klein bottle. It has four \( \text{pin}^+ \) structures of which two are nonbounding [KT2, Proposition 3.9]; fix one of those. Also, let \( S^1 \) denote the circle with its nonbounding string structure; see Remark 8.2. Define the 11-manifold
\[
\begin{equation}
\tag{7.1}
N = S^1 \times \Sigma \times B,
\end{equation}
\]
where \( B \) is the Bott manifold (§5.3). The following is based on computations to appear in [GH], and out of an abundance of caution we state it here as a conjecture, a more precise version of Conjecture 2.6.

**Conjecture 7.2.** The group \( \pi_{11} M_\mathfrak{m}_c \) is cyclic of order 2. The bordism class of the pair \( (N, 0) \) represents the generator. The mod 2 index of the \( \text{pin}^+ \) Dirac operator is an isomorphism \( \pi_{11} M_\mathfrak{m}_c \to \mathbb{Z}/2\mathbb{Z} \).

See §8.5.4 for a justification of Conjecture 7.2 using the Adams spectral sequence.

**Remark 7.3.** Index invariants of \( \text{pin}^+ \) \( n \)-manifolds correspond to index invariants of spin \( (n - 1) \)-manifolds; see [FH, §9.2.3]. Hence the mod 2 indices of spin manifolds in dimensions 9, 10 correspond to mod 2 indices of \( \text{pin}^+ \) manifolds in dimensions 10, 11. Let \( P \) be a \( \text{pin}^+ \) 10-manifold. Then the mod 2 index of the product \( S^1 \times P^{10} \) equals the mod 2 index of \( P \). We use product formulas analogous to Proposition 3.33 to compute the mod 2 index of \( \Sigma \times B \).
8. The bordism computation

In this section we present the computations which prove Theorem 6.1 and justify Conjecture 7.2. We begin in §8.1 by constructing the Thom spectrum $\text{Thom}_m$. In §8.2 we discuss some characteristic classes of $\mathfrak{m}_c$-manifolds and their behavior under transfer maps from the orientation double cover. We compute the values of some $\mathfrak{m}_c$-characteristic classes on two special manifolds in §8.3. The Adams spectral sequence is introduced in §8.4. The main work in this section occurs in §8.5. We present arguments to determine the facts we need about $\mathfrak{m}_c$-bordism groups in dimensions 11 and 12, and along the way compute low dimensional $\mathfrak{m}_c$-bordism groups.

8.1. The Thom spectrum

Our aim in this section is to justify the claim that the manifolds listed in Theorem 6.1 generate the bordism group of $\mathfrak{m}_c$-manifolds (Definition 2.1). We begin by identifying the relevant Thom spectrum.

Suppose that $(X, \zeta)$ is a space $X$ equipped with a stable vector bundle $\zeta$ of virtual dimension 0, which one may think of as a map $\zeta : X \to BO$ from $X$ to the classifying space of the infinite orthogonal group. Write $\text{Thom}(X, \zeta)$ for the Thom spectrum of $\zeta$. The homotopy groups $\pi_m \text{Thom}(X, \zeta)$ is the bordism group of triples $(M, f, \phi)$ consisting of an $m$-manifold $M$ equipped with a map $f : M \to X$, and an isomorphism

$$\pi : f^* \zeta \cong \mathbb{R}^m - TM$$

of virtual vector bundles. Put more colloquially it is the bordism group of manifolds whose stable normal bundle has a $\zeta$-structure. The bordism group of manifolds whose stable tangent bundle has a $\zeta$-structure is the homotopy group $\pi_n \text{Thom}(X, -\zeta)$.

We are interested in manifolds $M$ whose whose stable tangent bundle has a pin$^+$ structure and which are equipped with a $w_1$-twisted integer lift of $w_4$. We therefore consider the space $B\mathfrak{m}_c$ defined by the homotopy pullback square

$$(8.1) \quad \begin{array}{ccc}
B\mathfrak{m}_c & \longrightarrow & EZ/2 \times K(\mathbb{Z}, 4) \\
\downarrow & & \downarrow \\
B\text{Pin}^+ & \longrightarrow & B\mathbb{Z}/2 \times K(\mathbb{Z}/2, 4),
\end{array}$$

let $\zeta$ be the virtual vector bundle classified by the pullback

$$B\mathfrak{m}_c \to B\text{Pin}^+ \to BO,$$

and write $M\mathfrak{m}_c = \text{Thom}(B\mathfrak{m}_c, -\zeta)$. The homotopy groups $\pi_m M\mathfrak{m}_c$ are then the bordism groups of manifolds equipped with a tangential pin$^+$-structure and a $w_1$-twisted integer lift of $w_4$.

An $\mathfrak{m}_c$-manifold is a pair $(M, c)$ in which $M$ is a pin$^+$-manifold and $c$ is a $w_1$-twisted integer lift of tangential $w_4$. A Spin-manifold $M$ gives rise to an $\mathfrak{m}_c$-manifold by taking $c$ to be the tangential characteristic class $\lambda$.  

Remark 8.2. Given two \( m_c \)-manifolds \((M_1, c_1)\) and \((M_2, c_2)\) it is tempting to imagine that the product \( (M_1 \times M_2, c_1 + c_1) \) is an \( m_c \)-manifold. While it is true that \( w_4(M_1 \times M_2) = w_4(M_1) + w_4(M_2) \), the sum \( c_1 + c_2 \) doesn’t really make sense as the two summands lie in different twisted cohomology groups. The expression does make sense if \( w_1(M_2) = 0 \) and \( c_2 = 0 \), and in particular, if \( M_2 \) is a Spin-manifold equipped with a trivialization of \( \lambda \) (a String-manifold). The “Bott manifold” \( B \) of §5.3 is a String-manifold so if \((N, c)\) is an \( m_c \)-manifold, then \((N \times B, \pi^*_1 c)\) is also an \( m_c \)-manifold.

8.2. Characteristic classes

To describe bordism invariants of \( m_c \)-manifolds we will require some cohomology classes in \( Bm_c \). First note that under the equivalence

\[
BO_1 \times BSO \overset{\cong}{\to} BO
\]

the characteristic class \( w_2 \) pulls back to \((0, w_2)\). Passing to the homotopy fiber of the classifying map to \( K(\mathbb{Z}/2, 2) \) gives an equivalence

\[
BO_1 \times BSpin \to BPin^+.
\]

From this one sees that a pin\(^+\)-structure on a vector bundle \( T \) may be identified with an equivalence \( T \cong L \oplus V \) of stable vector bundles, in which \( L \) is a line bundle and \( V \) is a Spin-bundle.

Suppose \( M \) is a pin\(^+\) manifold and, using the above, regard the pin\(^+\)-structure as giving a stable isomorphism \( TM \cong L \oplus V \) with \( L \) a real line bundle and \( V \) a Spin-bundle. Set

\[
\alpha = w_1(TM) = w_1(L),
\]

\[
w_i = w_i(TM) = w_i(V), \quad 1 < 1 \leq 4,
\]

and, as in §4.3, write

\[
\lambda = \lambda(V)
\]

for the characteristic class of Spin-bundles, twice which is \( p_1 \). The mod 2 reduction of \( \lambda \) is \( w_4 \), so every pin\(^+\)-manifold has an untwisted integer lift of \( w_4 \).

Now suppose that \((M, c)\) is an \( m_c \)-manifold. The total space of the orientation double cover \( \pi : \hat{M} \to M \) is a Spin-manifold, and in fact \( T\hat{M} \) is equipped with a stable isomorphism \( T\hat{M} \cong \pi^*V \). The class \( c' = \pi^*c \) is an untwisted integer lift of \( w_4(TM) \). This specifies a class \( \iota \in H^4(\hat{M}; \mathbb{Z}) \)

satisfying

\[
2\iota = \lambda - c'
\]

(8.3)
Remark 8.4. The fact that \( \iota \) is specified uniquely and not just up to elements of order two follows from the fact that both \( \lambda \) and \( c' \) are integer lifts of \( w_4 \). The integer lifts of a fixed mod 2 cohomology class of dimension \( k \) form a torsor for integer cohomology in dimension \( k \), under the action in which an integer cohomology class \( \iota \) changes an integer lift \( c \) to \( c + 2\iota \). See [HS, §2.5] for a more systematic discussion of this from the point of view of cocycles. It is also not difficult to show (for example using (8.1)) that the classifying space \( B\text{Spin}(\beta w_4) \) for Spin bundles with an integer lift of \( w_4 \) is homotopy equivalent to \( B\text{Spin} \times K(\mathbb{Z}, 4) \) and in particular has torsion free \( H^4 \). So in fact the equation (8.3) specifies \( \iota \) uniquely as a cohomology class in \( B\text{Spin} \times \beta w_4 \).

8.2.1. Transfer. We will make use of the additive and multiplicative transfers

\[
\begin{align*}
\text{tr} : H^k(\widetilde{M}; \mathbb{Z}/2) &\to H^k(M; \mathbb{Z}/2) \\
P : H^k(\widetilde{M}; \mathbb{Z}/2) &\to H^{2k}(M; \mathbb{Z}/2).
\end{align*}
\]

Most computations of the additive transfer can be made in terms of \( \widetilde{M} \): for \( y \in H^*M \) one has

\[
\int_M \text{tr}(x) y = \int_{\widetilde{M}} (x \pi^* y).
\]

Computing the map \( P \) can be a little tricky, however there are some useful methods in special cases. For one thing, the map \( P \) is quadratic:

\[
P(x + y) = P(x) + P(y) + \text{tr}(x \tau(y)),
\]

where \( \tau \) is the cohomology homomorphism induced by the involution. One can also compute \( P(x) \) in terms of characteristic classes, when \( x \) itself is a characteristic class. In our case the following will suffice

Lemma 8.5. Suppose that \( p : \tilde{X} \to X \) is a double cover and \( W \) is a Spin-vector bundle on \( \tilde{X} \) of dimension \( d \). If \( x = w_4(W) \in H^4(\tilde{X}; \mathbb{Z}/2) \) then

\[
P(x) = w_8(p^*W - p^*\mathbb{R}^d) \in H^8(X)
\]

where, for a vector bundle \( V \), \( p^*V = \tilde{X} \times_{\mathbb{Z}/2} (V \oplus V) \). \( \Box \)

Remark 8.6. In the situation of Lemma 8.5 if \( W = p^*U \) for some vector bundle \( U \) on \( X \) then \( p^*W = U \oplus (U \otimes L) \) where \( L = \tilde{X} \times_{\mathbb{Z}/2} \mathbb{R} \), with \( \mathbb{Z}/2 \) acting on \( \mathbb{R} \) by the sign representation. In that case \( P(x) = w_8(U + U \otimes L - L^{\otimes d}) \).

8.3. Two examples

Two characteristic classes play an important role in our computation of \( \pi_{12} M \mathfrak{m}_c \). They are \( \text{tr}(\iota^3 + \iota^2 w_4) \) and \( \alpha^4 P(\iota) \).
Example 8.7. Recall from §5.5.1 the pair \((W'_0, \tilde{c}_0')\) in which

\[ W'_0 = S^4 \times \mathbb{HP}^2 \# \mathbb{HP}^2. \]

The orientation double cover is

\[ \tilde{W}'_0 = S^4 \times \mathbb{HP}^2 \# \mathbb{HP}^2 \xrightarrow{\pi} S^4 \times \mathbb{HP}^2 \# \mathbb{HP}^2 \]

and the involution of \(\mathbb{HP}^2 \# \mathbb{HP}^2\) exchanges the two summands, is orientation preserving, and has two fixed points. The \(w_1\)-twisted cohomology class \(\tilde{c}_0'\) satisfies

\[ \pi^* \tilde{c}_0' = 2t + x_1 - x_2. \]

Since \(\lambda = x_1 + x_2\) (see (5.10)) we have

\[ \iota = \frac{1}{2}(\lambda - \tilde{c}_0') \]

\[ = x_2 - t \]

\[ \iota^3 + \iota^2 w_4 = x_2^2 t \]

and so

\[ \int_{\tilde{W}'_0} \text{tr}(\iota^3 + \iota^2 w_4) = \int_{\tilde{W}'_0} (\iota^3 + \iota^2 w_4) = \int_{\tilde{W}'_0} (x_2^2 t) = 1. \]

Remark 8.8. One can check that \(\int_{W_0} \alpha^4 P(\iota) = 0\), though we will not make use of this fact.

Example 8.9. Consider the manifold \(W''_0 = P(K_{\mathbb{R}}^\otimes \otimes \mathbb{R})\) described in §5.5.2. Since the Stiefel-Whitney classes of \(K_{\mathbb{R}}^\otimes\) vanish on \(S^4\), the projective bundle formula gives

\[ H^*(W''_0 \times \mathbb{Z}/2) = \mathbb{Z}/2[t, \alpha]/(t^2, \alpha^9) \]

where \(t\) is the generator of \(H^4(S^4)\). The orientation double cover is \(S(K_{\mathbb{R}}^\otimes \otimes \mathbb{R})\) and the mod 2 reduction of

\[ \iota = \frac{1}{2}(\lambda - c) \]

is \(t = w_4(K_{\mathbb{R}})\). Since the map

\[ H^4(S^4) \to H^4(S(K_{\mathbb{R}}^\otimes \otimes \mathbb{R})) \]

is an isomorphism, we have

\[ \iota^3 + \iota^2 w_4 = 0, \]

and so

\[ \int_{W''_0} \text{tr}(\iota^3 + \iota^2 w_4) = 0. \]
For the characteristic number $\alpha^4 P(\iota)$ we first appeal to Lemma 8.5 and compute

$$P(\iota) = w_8(p_*K_\mathbb{R} - p_*\mathbb{R}^4).$$

Since $K_\mathbb{R}$ is pulled back from $W_0^n$ we are in the situation of Remark 8.6. Writing $L_\alpha$ for the real line bundle with $w_1(L_\alpha) = \alpha$ we are led to the total Stiefel-Whitney class

$$w(K_\mathbb{R} + K_\mathbb{R} \otimes L_\alpha - 4L_\alpha) = (1 + t)(1 + t + \alpha^4)(1 + \alpha)^{-4} = 1 + t^2 + \alpha^4 t + O[9] = 1 + \alpha^4 t + O[9].$$

Thus

$$P(\iota) = \alpha^4 t$$

and

$$\int_{W_0^n} \alpha^8 P(\iota) = 1.$$

### 8.4. The Adams spectral sequence

Our aim is to identify generators for $\pi_{12} M\mathfrak{m}_c$. Since our main concern is the comparison of two different homomorphisms from $\pi_{12} M\mathfrak{m}_c$ to a finite abelian 2-group, it suffices to do so after completing at 2. For this we can appeal to the Adams spectral sequence, and this can be done by computer calculation. For the purposes of this paper the authors used Mathematica to determine the mod 2 cohomology of $B\mathfrak{m}_c$ as a module over the mod 2 Steenrod algebra, and Rob Bruner’s program [Br] for computing the $E_2$-term of the Adams spectral sequence, as well as the map of Adams spectral sequences induced by the map from $M\text{Spin}$ to $M\mathfrak{m}_c$. The results are displayed in Figures 1 and 2. In [GH] a more detailed version of this computation is described, as well as means of doing it by hand.

The Adams spectral sequence begins with

$$E_2^{s,t} = \text{Ext}_A^{s,t}(H^s M\mathfrak{m}_c, \mathbb{Z}/2) = \text{Ext}_A^{s,t}(H^{s+t} M\mathfrak{m}_c, \mathbb{Z}/2),$$

with $A$ the mod 2 Steenrod algebra, and converges to the homotopy groups of the 2-adic completion $\widehat{M\mathfrak{m}_c}$ of $M\mathfrak{m}_c$. Since the homology groups of $B\mathfrak{m}_c$ are each finitely generated, the homotopy groups of $M\mathfrak{m}_c$ are each finitely generated and so $\pi_t M\mathfrak{m}_c$ is just the 2-adic completion of $\pi_{t-s} M\mathfrak{m}_c$.

Remark 8.10. For the remainder of this section all bordism groups will be 2-adically completed. Except for the appearance of the symbol $\mathbb{Z}_2$ for the 2-adic numbers, this will not be indicated in the notation.

The differential $d_r$ of the Adams spectral sequence goes from bidegree $(s,t)$ to bidegree $(s + r, t + r - 1)$. It is customary to display the Adams spectral sequence with the horizontal axis numbered by $(t - s)$ and the vertical axis $s$. With this convention the differential $d_r$ goes one square to the left and $r$-squares upward. The groups contributing to a given homotopy group lie in a column.

The “$s$” in the Adams spectral sequence direction corresponds to a decreasing filtration of stable homotopy groups known as the Adams filtration.
**Definition 8.11.** A map \( f : X \to Y \) of spectra has (mod 2) *Adams filtration greater than or equal to* \( n \) if there is a factorization

\[
X = X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \cdots \xrightarrow{f_n} X_n = Y
\]

in which each \( f_i \) induces the zero map in mod 2 cohomology.

The Adams filtration is natural in both variables, in the sense that composition with a map \( X' \to X \) or \( Y \to Y' \) sends maps of Adams filtration greater than or equal to \( n \) to maps of Adams filtration greater than or equal to \( n \).

**Definition 8.12.** A map has *Adams filtration* \( n \) if it has Adams filtration greater than or equal to \( n \) but not greater than or equal to \( n + 1 \).

The maps in Adams filtration greater than or \( n \) appear in the Adams charts with \( s \)-coordinate greater than or equal to \( n \).

The Adams spectral sequence for \( \pi_\ast X \) is a module over the Adams spectral sequence for the stable homotopy groups \( \pi_\ast S^0 \) of spheres. The element \( 2 \in \pi_0 S^0 \) has Adams filtration 1 and is represented by a class traditionally denoted as

\[
h_0 \in \text{Ext}^{1,1}_A(\mathbb{Z}/2, \mathbb{Z}/2).
\]

Multiplication by \( h_0 \) in any Adams spectral sequence is indicated in the chart by a vertical line. Similarly, the Hopf map \( \eta \in \pi_1 S^0 \) is represented by the element

\[
h_1 \in \text{Ext}^{2,1}_A(\mathbb{Z}/2, \mathbb{Z}/2)
\]

and multiplication by \( h_1 \) is indicated by a \((1, 1)\) diagonal line. A little care must be used in drawing conclusions from these notations. For example, Figure 1 shows the Adams spectral sequence for \( \pi_\ast Mm_c \). From the chart it appears that \( \pi_4 Mm_c = \mathbb{Z}/8 \oplus \mathbb{Z}_2 \), with the generator of the \( \mathbb{Z}/8 \) appearing in Adams filtration 1. However all that the chart implies is that 8 times the apparent generator in filtration 1 has Adams filtration greater than 4. Some additional argument is needed to conclude that there is an element of order 8 in filtration 1. One can conclude from the chart that \( \pi_4 \) is generated by 2 elements and has rank 1. The computation of \( \pi_4 Mm_c \) will be given in detail in §8.5.2.

The \( s = 0 \) line of the Adams spectral sequence consists of the groups

\[
\text{hom}_A(H^i X, H^0 S^0) \subset H_i X.
\]

The kernel of the higher differentials pick out the image of the Hurewicz homomorphism in \( H_i X \). When \( X = \text{Thom}(B, V) \) is a Thom spectrum it is often useful to label an element \( x \in E_2^{0,t} = \text{hom}_A(H^i X, H^0 S^0) \) with a cohomology class \( \beta \in H^i(B) \) whose image under

\[
H^i(B) \xrightarrow{\text{Thom iso}} H^i(X) \xrightarrow{x^*} H^i(S^t) = \mathbb{Z}/2
\]

(8.13)
Figure 1. The Adams spectral sequence for $\pi_* M_{\mathbb{C}}$

is non-zero. This can be a little perilous as there can be many cohomology classes having a non-zero image under given class, and some care must be taken to ensure that the labeled cohomology classes are linearly independent on the image of the Hurewicz homomorphism. In the end it provides useful information. If $x$ survives the Adams spectral sequence and is represented by a manifold $M$, the image of $\beta$ under (8.13) is

$$\int_M \beta.$$ 

Such labels therefore provide a means of identifying specific manifolds as representing a basis of the image of the Hurewicz homomorphism. The class $\beta$ is a characteristic class of some kind.

8.5. Computations

Armed with these spectral sequences, we first turn to the computation of $\pi_* M_{\mathbb{C}}$. We remind the reader of Remark 8.10, that all homotopy groups have been 2-adically completed.

8.5.1. Dimension less than 4. The homotopy fiber of the map $Bm_{\mathbb{C}} \to B\text{Pin}^+$ is the Eilenberg-MacLane space $K(\mathbb{Z}, 4)$. It follows easily from this that the map

$$\pi_* M_{\mathbb{C}} \to \pi_* M\text{TPin}^+$$

is an isomorphism for $* < 4$ and an epimorphism when $* = 4$. From [KT1] one concludes that

$$\pi_0 M_{\mathbb{C}} = \mathbb{Z}/2$$
generated by a point,
\[ \pi_0 \mathcal{M} \mathcal{M}_c = \mathbb{Z}/2 \] generated by a point
\[ \pi_1 \mathcal{M} \mathcal{M}_c = 0 \]
\[ \pi_2 \mathcal{M} \mathcal{M}_c = \mathbb{Z}/2 \] generated by \((\Sigma, 0)\)

where \(\Sigma\) is a Klein bottle in a nonbounding \(\text{pin}^+\)-structure, and
\[ \pi_3 \mathcal{M} \mathcal{M}_c = \mathbb{Z}/2 \] generated by \(S^1 \times \Sigma\),

where \(S^1\) is given the non-bounding String-structure (on \(S^1\) a String structure is equivalent to a Spin-structure).

8.5.2. **Dimension 4.** We define a homomorphism \(e : \pi_4 \mathcal{M} \mathcal{M}_c \to \mathbb{Z}_2\) by
\[ e(M, c) = \int_M c. \]

Forgetting the twisted lift of \(w_4\) gives a map
\[ u : \pi_* \mathcal{M} \mathcal{M}_c \to \pi_* \text{MTPin}^+. \]

By [KT1], the group \(\pi_4 \text{MTPin}^+\) is cyclic of order 16, with generator \(\mathbb{R}P^4\). Combined, these two homomorphisms give a map
\[ (8.14) \quad \pi_4 \mathcal{M} \mathcal{M}_c \xrightarrow{(e,u)} \mathbb{Z}_2 \oplus \mathbb{Z}/16. \]

**Proposition 8.15.** The map above gives an isomorphism of \(\pi_4 \mathcal{M} \mathcal{M}_c\) with the set of elements \((a, b) \in \mathbb{Z}_2 \oplus \mathbb{Z}/16\) with \(a \equiv b \mod 2\). The group \(\pi_4 \mathcal{M} \mathcal{M}_c\) is generated by \((\mathbb{R}P^4, \tilde{c}_{\mathbb{R}P^4})\) and \((\mathbb{R}P^4 \# \mathbb{R}P^4, 0)\).

**Proof:** By definition, the map
\[ \pi_4 \mathcal{M} \mathcal{M}_c \xrightarrow{e} \mathbb{Z}_2 \to \mathbb{Z}/2 \]

is given by \(\int_m w_4\), and the map
\[ \pi_4 \mathcal{M} \mathcal{M}_c \to \pi_4 \text{MTPin}^+ \to \pi_4 \text{MTPin}^+ \otimes \mathbb{Z}/2 = \mathbb{Z}/2 \]

is given by \(\int_M w_4^1\). From the Wu relations one knows that \(\int_M w_4(\nu) = 0\), where \(\nu\) is the stable normal bundle. Since \(w_4(\nu) = w_4^1 + (w_1^2 + w_2)w_2 + w_4\) and \(w_2 = 0\) (since \(M\) is \(\text{pin}^+\)), this implies
\[ \int_M w_4^1 = \int_M w_4, \]
so the image of (8.14) is contained in the subgroup of elements \((a, b)\) with \(a \equiv b \mod 2\). On the other hand the Adams spectral sequence shows that the kernel of \(e\) has order at most 8. The \(Mmc\)-manifold \((\mathbb{R}P^4 \# \mathbb{R}P^4, 0)\) is in the kernel of \(e\). Its image in \(\pi_4 \text{MTPin}^+ = \mathbb{Z}/16\) is \(2[\mathbb{R}P^4]\). Since \([\mathbb{R}P^4]\) generates \(\pi_4 \text{MTPin}^+\), the image of \((\mathbb{R}P^4 \# \mathbb{R}P^4, 0)\) actually has order 8. The assertion about generators follows from the computation

\[
\begin{align*}
(\mathbb{R}P^4, \tilde{c}_1) &\mapsto (1, 1) \\
(\mathbb{R}P^4 \# \mathbb{R}P^4, 0) &\mapsto (0, 2).
\end{align*}
\]

This completes the proof. □

**Example 8.16.** If \(M\) is a Spin-manifold of dimension 4 then under (8.14) one has

\[
(M, \lambda) \mapsto (\lambda(M), \lambda(M)).
\]

It follows that \((M, \lambda) \equiv \lambda(M)(\mathbb{R}P^4, \tilde{c}_1)\). In particular

\[
[K, \lambda] = -24[\mathbb{R}P^4, \tilde{c}_1]
\]

when \(K\) is a Kummer surface.

**8.5.3. Dimension 12.** Our main result in dimension 12 is the following restatement of Theorem 6.1.

**Proposition 8.17.** The group \(\pi_{12} Mmc\) is generated (over \(\mathbb{Z}_2\)) by the six manifolds

\[
(W'_0, c'_0), \quad (W'_0, 0), \quad (W_1, \lambda)
\]

\[
(K \times \mathbb{H}P^2, \lambda), \quad (\mathbb{R}P^4, \tilde{c}'_{\mathbb{R}P^4}) \times B, \quad (\mathbb{R}P^4 \# \mathbb{R}P^4, 0) \times B.
\]

The proof makes use of the following fact about Spin-bordism.

**Proposition 8.18.** The group \(\pi_{12} M\text{Spin}\) is free of rank 3, and generated by \(K \times B, K \times \mathbb{H}P^2\), and the manifold \(W_1\) described in §5.5.3, sitting in the fibration sequence

\[
\mathbb{H}P^2 \rightarrow W_1 \rightarrow \mathbb{C}P^1 \times \mathbb{C}P^1
\]

□

**Proof of Proposition 8.17:** We begin by extracting some facts from the Adams spectral sequence. First of all, the map

\[
(8.19) \quad \pi_{12} Mmc \rightarrow \mathbb{Z}/2 \oplus \mathbb{Z}/2
\]
with components
\[ \int_M \alpha^4 P(\iota) \quad \text{and} \quad \int_M \text{tr}(\iota^3 + \iota^2 w_4) \]
gives an isomorphism of the quotient of \( \pi_{12} M_{\mathfrak{m}_c} \) by the elements of positive Adams filtration with a subgroup of \( \mathbb{Z}/2 \oplus \mathbb{Z}/2 \). The computations of examples 8.7 and 8.9 show that this subgroup is in fact all of \( \mathbb{Z}/2 \oplus \mathbb{Z}/2 \). The kernel of this map contains the image of \( \pi_{12} M_{\text{Spin}} \) and the image of multiplication by \( B \). This follows from a consideration of Adams filtrations, but it is easily checked directly. Indeed if \( M \) is a Spin-manifold, then \( M \) is oriented, \( \alpha = 0 \), and \( \int_M \alpha^4 P(\iota) = 0 \). Also, since the orientation double cover of \( M \) is \( M \langle M \rangle \), one has \( \int_M \text{tr}(\iota^3 + \iota^2 w_4) = 2 \int_M (\iota^3 + \iota^2 w_4) = 0 \).

In the case \( (M, c) = (N_4, \lambda) \times B \) all of the characteristic classes to be integrated are pulled back from \( H^8(N_4) = 0 \).

Let \( J' \subset \pi_{12} M_{\mathfrak{m}_c} \) be the subgroup generated by the image of \( \pi_{12} M_{\text{Spin}} \) and the image of multiplication by \( B \), and let
\[ C = \pi_{12} M_{\mathfrak{m}_c}/J'. \]

A portion of the Adams spectral sequence for computing the map \( \pi_{12} M_{\text{Spin}} \to \pi_{12} M_{\mathfrak{m}_c} \) is shown in Figure 2. The map, which is part of the (machine) computation of Ext, can also be determined by composing with the map \( \pi_{12} M_{\mathfrak{m}_c} \to \pi_{12} M_{TP\mathbb{P}^4} \). From it one can read off that the map (8.19) gives an isomorphism
\[ C \otimes \mathbb{Z}/2 \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2. \]

Since \( C \) is finitely generated, Nakayama’s Lemma and the computations of Examples 8.7 and 8.9 show that \( C \) is generated by \( (W'_1, \mathbf{c}_0') \) and \( (W''_0, 0) \).

Let \( J \subset \pi_{12} M_{\mathfrak{m}_c} \) be the subgroup generated by
\[ (W_1, \lambda), \quad (K \times \mathbb{H} \mathbb{P}^2, \lambda), \quad (\mathbb{R} \mathbb{P}^4, \mathbf{c}'_{\mathbb{R} \mathbb{P}^4}) \times B, \quad \text{and} \quad (\mathbb{R} \mathbb{P}^4 \# \mathbb{R} \mathbb{P}^4, 0) \times B. \]

We are done if we can show that the manifolds \( (W'_1, \mathbf{c}_0') \) and \( (W''_0, 0) \) generate \( \pi_{12} M_{\mathfrak{m}_c}/J \). Note that by Proposition 8.15, the subgroup \( J \) contains \( J' \); it contains image of multiplication by \( B \) and by Proposition 8.18 it contains the image of \( \pi_{12} M_{\text{Spin}} \). By the above discussion, the manifolds \( (W_0', \mathbf{c}_0) \) and \( (W''_0, 0) \) generate \( \pi_{12} M_{\mathfrak{m}_c}/J' \), so they certainly generate \( \pi_{12} M_{\mathfrak{m}_c}/J \). This completes the proof.

8.5.4. Dimension 11. The ambiguity of the \( M \) theory action has to do with the group \( \pi_{11} M_{\mathfrak{m}_c} \). In this section we offer a tentative evaluation of this group. Since the ambiguity involves the entire group, we drop the convention that groups have been completed at 2.

Proposition 8.20. The group \( \pi_{11} M_{\mathfrak{m}_c} \) is a finite abelian 2-group.
Proof: From the theory of Serre classes, one knows that $\pi_{11} Mmc$ is finitely generated. It therefore suffices to show that

$$\pi_{11} Mmc \otimes \mathbb{Z}[1/2] = 0.$$ 

The homotopy fiber of the map $Bm_c \xrightarrow{w_1} K(\mathbb{Z}/2, 1)$ is $BSpin \times K(\mathbb{Z}, 4)$, representing $Bm_c$ as the homotopy quotient of an action of $\mathbb{Z}/2$ on $BSpin \times K(\mathbb{Z}, 4)$. A map $T \rightarrow BSpin \times K(\mathbb{Z}, 4)$ classifies a pair consisting $(V, \iota)$ consisting of a Spin bundle $V \rightarrow T$ and an element $\iota \in H^4(T; \mathbb{Z})$. The pullback of the universal $w_1$-twisted integer lift $c$ of $w_4$ is

$$c = \lambda(V) - 2\iota$$

(see (8.3)), and the generator of the $\mathbb{Z}/2$-action sends $(V, \iota)$ to

$$(V, \lambda(V) - \iota).$$

By passing to Thom spectra this depicts $Mmc$ as the homotopy quotient of the $\mathbb{Z}/2$ spectrum $MSpin \wedge K(\mathbb{Z}, 4)_+ \wedge S^{1-\sigma}$, where $\sigma$ is the sign representation of $\mathbb{Z}/2$. This means that after inverting 2, the map

$$\pi_\ast MSpin \wedge K(\mathbb{Z}, 4)_+ \rightarrow \pi_\ast Mmc$$

is projection to a summand. The claim now follows from Stong’s Theorem [Sto], which states that $\pi_{11} MSpin \wedge K(\mathbb{Z}, 4)_+ = 0$. 

□
From the discussion in the above proof it is an easy matter to compute

\[ \pi_* \text{Mm}_c \otimes \mathbb{Q}. \]

Let \( J = (j_1, \ldots, ) \) run through the sequences of non-negative integers, almost all of which are 0, and write

\[ |J| = j_1 + 2j_2 + \cdots + nj_n + \cdots p^i = p_1^{j_1} p_2^{j_2} \cdots. \]

**Proposition 8.21.** The group \( \pi_m \text{Mm}_c \otimes \mathbb{Q} \) is zero if \( m \) is not divisible by 4. The map

\[ \pi_{4n} \text{Mm}_c \otimes \mathbb{Q} \rightarrow \prod_{(k,J)<n} \mathbb{Q}. \]

\[(M,c) \rightarrow \int_M e^{2k+1}p^J\]

gives an isomorphism.

For example this implies that the group \( \pi_{12} \text{Mm}_c \) has rank 3, corresponding to the indices \( (1, (2, 0)), (1, (0, 1)), (3, (0)). \) This implies that there must be non-trivial differentials in the chart Figure 1 from dimension 13 to dimension 12.

**Remark 8.22.** In [Sto] (note after Item 6), Stong also shows that for \( * < 12 \) the groups

\[ \pi_* \text{MSpin} \wedge K(\mathbb{Z}, 4) \otimes \mathbb{Z}[1/2] \]

are torsion free. In fact his argument for dimension 8 can also be adapted to dimension 12 to establish the same conclusion for \( * = 12 \). So the above also provides an evaluation of the groups

\[ \pi_* \text{Mm}_c \otimes \mathbb{Z}[1/2], \quad * \leq 12. \]

Because of Proposition 8.20, the group \( \pi_{11} \text{Mm}_c \) can be determined from the Adams spectral sequence, which is displayed in Figure 1. The \( E_2 \)-term provides an upper bound and shows that the group has order at most 8. In the table, there are two \( d_3 \)-differentials indicated, originating in Adams filtrations 1 and 2. These should be regarded as tentative at the moment, and will appear in [GH]. Assuming them, the Adams spectral sequence shows that after 2-completion the group \( \pi_{11} \text{Mm}_c \) is cyclic of order 2, and that an isomorphism is given by the mod 2 index of the pin\(^+\)-Dirac operator. We state the outcome of this argument as a restatement of Conjecture 7.2.

**Conjecture 8.23.** The map

\[ \pi_{11} \text{Mm}_c \rightarrow \mathbb{Z}/2 \]

given by the mod 2 index of the pin\(^+\) Dirac operator is an isomorphism.

**Remark 8.24.** Let \( M \) be the product of the Bott manifold, \( S^1 \) in its non-boundary String-structure, and \((\Sigma, 0)\) where \( \Sigma \) is the Klein bottle in a nonbounding pin\(^+\)-structure (see §8.5.1). The mod 2 index of the pin\(^+\) Dirac operator on \( M \) is 1 and so the above conjecture implies that \( \pi_{11} \text{Mm}_c \) is generated by \( M \).
Appendix A. On the anomaly theory of a spinor field

In an $n$-dimensional field theory $F$ the partition function of a spinor field on a closed $n$-dimensional Riemannian manifold is the pfaffian of a Dirac operator, which is an element of a Pfaffian line, as reviewed in §3. The Pfaffian line is the quantum state space of the associated anomaly theory, which is an invertible $(n+1)$-dimensional theory $\alpha$, but initially truncated to manifolds of dimension $\leq n$, since such manifolds form the domain of $F$. To extend $\alpha$ to an $(n+1)$-dimensional theory we must define the partition function of a closed $(n+1)$-manifold, as well as an element in the state space of the boundary of a compact $(n+1)$-manifold with boundary, and these elements must satisfy a gluing law. The results in [DF] imply that an exponentiated $\eta$-invariant works as the partition function: on a compact $(n+1)$-manifold with boundary it takes values in the Pfaffian line. But to define it we must construct a Riemannian Dirac operator in $(n+1)$-dimensions from the $n$-dimensional Lorentzian data which define the spinor field. The construction was given in [FH, §9.2.5], but only in passing; in this appendix we give more detail. We discuss the base case of spin manifolds (no time-reversal symmetry) in §A.1. In §A.2 we specialize to 11 dimensions and the pin module relevant to M-theory.

A.1. The spin case in general dimensions

A spinor field in a relativistic quantum field theory is specified by [De, §6] a real spin representation $S$ of the Lorentz spin group $\text{Spin}_{1,n-1}$ together with a symmetric positive$^{28}$ $\text{Spin}_{1,n-1}$-invariant map

\[(A.1) \quad \Gamma: S \otimes S \rightarrow \mathbb{R}^{1,n-1}.\]

Thus $S$ is an ungraded module over the even subalgebra $\text{Cliff}_{n-1,1}^0$ of the Clifford algebra with $n-1$ generators squaring to $+1$ and a single generator squaring to $-1$. The pair $(S, \Gamma)$ Wick rotates to define a Dirac operator on a Riemannian spin $n$-manifold $X$ as follows. First, the complexification $S_\mathbb{C}$ is a module over the complex algebra $\text{Cliff}_{n}^0(\mathbb{C})$, so restricts to a representation of the compact spin group $\text{Spin}_{n}$. Also, $\Gamma$ complexifies to a $\text{Spin}_{n}$-equivariant morphism

\[(A.2) \quad \Gamma_\mathbb{C}: (\mathbb{C}^n)^* \otimes S_\mathbb{C} \rightarrow S_\mathbb{C}^*.\]

Let $S \rightarrow X$ be the complex vector bundle on $X$ associated to the $\text{Spin}_{n}$ representation $S_\mathbb{C}$. Then as usual the Dirac operator on $X$ is the composition

\[(A.3) \quad D_X = \Gamma_\mathbb{C} \circ \nabla: C^\infty(X; S) \rightarrow C^\infty(X; S^*),\]

where $\nabla$ is the covariant derivative on sections of $S \rightarrow X$. The operator $D_X$ is complex skew-adjoint. (The metric on $S \rightarrow X$ is constructed in the next paragraph.) For $X$ closed this operator appears in the Dirac form (3.8), and its pfaffian is the fermionic path integral (3.4).

$^{28}$The positivity condition is that $\Gamma(s, s)$ lie in the closure of the forward timelike vectors in $\mathbb{R}^{1,n-1}$. 
The construction of a Dirac operator on a Riemannian spin \( (n + 1) \)-manifold \( W \) from the data \((S, \Gamma)\) uses the Clifford linear Dirac operator [LM, §II.7] and Morita equivalence of Clifford algebras. By [De, Corollary 6.2] the data \((S, \Gamma)\) determine a unique \( \mathbb{Z}/2\mathbb{Z} \)-graded \( \text{Cliff}_{n-1,1} \)-module structure on \( S \oplus S^* \). Let \( \gamma^0 \) denote the Clifford generator with \((\gamma^0)^2 = -1\), acting as an odd endomorphism of \( S \oplus S^* \), and \( \gamma^1, \ldots, \gamma^{n-1} \) the Clifford generators with \((\gamma^i)^2 = +1\). Fix an inner product on \( S \oplus S^* \) such that the finite group consisting of products of the \( \gamma^\mu, \mu = 0, 1, \ldots, n-1 \) acts orthogonally. (It follows that \( \text{Spin}_n \) acts unitarily on \( S_C \), which induces the hermitian metric on \( S \to X \) used in (A.3).) Now

\[
\text{Cliff}_{+(n+1)} \otimes (S \oplus S^*)
\]

is a real super vector space which carries a left action of \( \text{Spin}_{n+1} \)—by left multiplication on \( \text{Cliff}_{+(n+1)} \) tensored the identity on the second factor—and a commuting left action of the real super algebra

\[
A = \text{Cliff}_{-(n+1)} \otimes \text{Cliff}_{n-1,1}.
\]

Elements of \( \text{Cliff}_{+(n+1)} \) act by right multiplication on the first factor of (A.4), tensored with the identity on the second factor, which is equivalent to a left action of \( \text{Cliff}_{-(n+1)} \), the opposite super algebra. Elements of \( \text{Cliff}_{n-1,1} \) act by the identity on the first factor of (A.4) tensored with the action above on the second factor. Now the left \( \text{Spin}_{n+1} \) action on (A.4) defines a bundle of real \( A \)-modules over \( W \) as well as a Dirac operator on its sections which commutes with the action of \( A \). We claim this solves the problem of defining an \( (n + 1) \)-dimensional Riemannian Dirac operator from \((S, \Gamma)\) which can be used in the anomaly theory \( \alpha \). To verify that claim we must: (i) define the exponentiated \( n \)-invariant of this operator, and (ii) identify the induced operator in \( n \) dimensions with (A.3).

For (i) we use a (super) Morita equivalence of \( A \) with \( \text{Cliff}_{-3} \), “canceling” the last \( n - 1 \) generators of \( \text{Cliff}_{-(n+1)} \) with the \( n - 1 \) positive generators of \( \text{Cliff}_{n-1,1} \). (The cancellation identifies \( \text{Cliff}_{-(n-1)} \otimes \text{Cliff}_{+(n-1)} \) with the super algebra of endomorphisms of the vector space \( \text{Cliff}_{+(n-1)} \), which is Morita trivial.) Under the Morita isomorphism left \( A \)-modules are identified with left \( \text{Cliff}_{-3} \)-modules, and so the \( A \)-module (A.4), rewritten as

\[
\text{Cliff}_{+(n+1)} \otimes (S \oplus S^*) \cong \text{Cliff}_{+(n-1)} \otimes \text{Cliff}_{+2} \otimes (S \oplus S^*),
\]

is identified with

\[
E = \text{Hom}_{\text{Cliff}_{+(n-1)} \otimes \text{Cliff}_{-(n-1)}} (\text{Cliff}_{+(n+1)}, \text{Cliff}_{+(n-1)} \otimes \text{Cliff}_{+2} \otimes (S \oplus S^*))
\cong \text{Cliff}_{+2} \otimes (S \oplus S^*).
\]

Let \( \delta^1, \delta^2 \) be the generators of \( \text{Cliff}_{+2} \), and \( \gamma^0 \) as above the (negative) generator of the \( \text{Cliff}_{n-1,1} \)-action on \( S \oplus S^* \). Then the \( \text{Cliff}_{-3} \) which acts on \( E \) is generated by \( \gamma^0, \delta^1, \delta^2 \), with \( \delta^1, \delta^2 \) acting by right multiplication on \( \text{Cliff}_{+2} \), tensored with the identity on \( S \oplus S^* \). Furthermore, \( \text{Spin}_{n+1} \subset \)}
Cliff\(_{(n+1)}\) acts on \(\mathbb{E}\) using left multiplication by \(\delta^1, \delta^2\) on Cliff\(_+\), tensored with the identity on \(\mathbb{S} \oplus \mathbb{S}^*\), and by \(\gamma^1, \ldots, \gamma^{n-1}\) acting on \(\mathbb{S} \oplus \mathbb{S}^*\), tensored with the identity on Cliff\(_+\). The latter actions determine an odd skew-adjoint Dirac operator \(D\) on sections of the real vector bundle \(E = E^0 \oplus E^1 \to W\) associated to the representation \(E\) of Spin\(_{n+1}\). The operator \(D\) commutes with the left Cliff\(_-\)-action on \(E\). Now \(\delta^1 \delta^2 \in \text{Cliff\(_-\)}\) acts as a complex structure on \(E\) and \(\gamma^0 \delta^2 \in \text{Cliff\(_-\)}\) acts as a complex antilinear operator which squares to \(-\text{id}_E\). Thus \(E\) has a quaternionic structure. (More simply, the ungraded algebra Cliff\(_0\) is isomorphic to the quaternion algebra.) The even self-adjoint operator

\[
D^0 := \gamma^0 \delta^1 \delta^2 D : C^\infty(W; E^0) \to C^\infty(W; E^0)
\]

commutes with this quaternionic structure. Assume \(W\) is compact without boundary. Then \(D^0\) is elliptic, so has a discrete spectrum and the eigenspaces \(E^0_\lambda\) are finite dimensional quaternionic vector spaces. Let \(a \in \mathbb{R}\) be in the complement of the spectrum. Define

\[
\eta_a(s) = \sum_{\lambda \neq 0} \text{sign}(\lambda - a) \ (\dim \mathbb{C} E^0_\lambda) \ |\lambda|^{-s} - \text{sign}(a) \dim \ker D^0, \quad \text{Re}(s) \gg 0,
\]

where the sum is over the nonzero eigenvalues of \(D^0\) and \(s\) is a complex number. According to the results of [APS1, APS2, APS3] the sum converges to a holomorphic function of \(s\) if the real part of \(s\) is sufficiently large, it has a meromorphic continuation to \(\mathbb{C}\), and \(s = 0\) is a regular point. Then

\[
\alpha(W) = \exp(2\pi i \eta_a(0)/4)
\]

is independent of \(a\). It is the partition function of the anomaly theory.

We now verify (ii) above, namely that the Dirac operator in \(n\) dimensions induced from (A.8) (see [APS1, (3.1)]) can be identified with (A.3). Let \(X\) be an \(n\)-dimensional Riemannian spin manifold and consider \(W = \mathbb{R} \times X\) with the product Riemannian metric and spin structure. Then \(W\) has a Dirac operator (A.8), which we view as complex since it commutes with the complex structure \(\delta^1 \delta^2\). Choose a local orthonormal framing on \(X\), with basis numbered \(0, 1, \ldots, n-1\), and a global coordinate \(t\) on \(\mathbb{R}\). Order the Clifford generators of Cliff\(_{(n+1)}\) as \(\delta^1, \delta^2, \gamma^1, \ldots, \gamma^{n-1}\), as in the previous paragraph. Then the Dirac operator \(D^0\) on \(W\) can be written locally as

\[
D^0 = \gamma^0 \delta^1 \delta^2 (\delta^1 \frac{\partial}{\partial t} + \delta^2 \nabla_0 + \gamma^i \nabla_i),
\]

where the sum over \(i\) runs from 1 to \(n-1\). The symbol of \(D\), evaluated on \(dt\), is induced from the algebraic operator

\[
J(x \otimes \chi) = -(-1)^{|x|} \delta^1 x \delta^1 \delta^2 \otimes \gamma^0 \chi, \quad x \in \text{Cliff\(_+\)}, \quad \chi \in \mathbb{S} \oplus \mathbb{S}^*,
\]

where \(|x| \in \mathbb{Z}/2\mathbb{Z}\) is the parity of the homogeneous element \(x\). Then \(J\) commutes with the complex structure \(\delta^1 \delta^2\), anticommutes with \(D^0\), and \(J^2 = -\text{id}\). According to [DF, §1] the induced Dirac
operator $D_X^0$ on $X$ is the operator (A.11) restricted to functions on $\mathbb{R} \times X$ which are invariant under translation in $t$, mapping the $+\sqrt{-1}$-eigenspace of $J$ to the $-\sqrt{-1}$-eigenspace of $J$. Thus, if now $x \otimes \chi$ is a section of $E^0 \to X$, we compute

\begin{equation}
D_X^0 (x \otimes \chi) = -(-1)^{|x|} (\delta^2 \nabla_0 + \gamma^i \nabla_i) (x \delta^1 \delta^2 \otimes \gamma^0 \chi).
\end{equation}

Write the $\pm \sqrt{-1}$ eigenbundles of $J$ as $E^0_\pm \to X$, and recall the complex vector bundle $S \to X$ associated to $S\mathbb{C}$. Then there are isomorphisms

\begin{equation}
S \to E^0_+
\end{equation}

$$
\psi + \sqrt{-1} \psi' \mapsto 1 \otimes \psi - \delta^1 \otimes \gamma^0 \psi - \delta^2 \otimes \gamma^0 \psi + \delta^1 \delta^2 \otimes \psi'
$$

and

\begin{equation}
S^* \to E^0_-
\end{equation}

\begin{align*}
\lambda + \sqrt{-1} \lambda' & \mapsto -1 \otimes \gamma^0 \lambda' + \delta^1 \otimes \lambda' - \delta^2 \otimes \lambda + \delta^1 \delta^2 \otimes \gamma^0 \lambda
\end{align*}

A straightforward computation demonstrates that these isomorphisms intertwine the operators $D_X^0$ in (A.13) and $D_X$ in (A.3), where the latter is

\begin{equation}
D_X = \sqrt{-1} \gamma^0 \nabla_0 + \gamma^i \nabla_i
\end{equation}

in the local moving frame on $X$. The factor $\sqrt{-1}$ comes from Wick rotation when passing from (A.1) to (A.2).

As a companion to (A.8) we have the operator

\begin{equation}
D^1 := \gamma^0 \delta^1 \delta^2 D : C^\infty (W; E^1) \to C^\infty (W; E^1)
\end{equation}

of the odd subspace of $E = D^0 \oplus E^1$. Note that swapping $S$ and $S^*$ swaps the even and odd parts of $E$; see (A.7).

**Proposition A.18.** The exponentiated $\eta$-invariant formed with $D^1$ is the reciprocal of the exponentiated $\eta$-invariant (A.10) formed with $D^0$.

**Proof.** From the definition of $D$ following (A.7), since $\text{Cliff}_{-3}$ graded commutes with $\text{Cliff}_{+(n+1)}$ it follows that $\omega = \gamma^0 \delta^1 \delta^2$ satisfies $\omega D = -D \omega$, and then since $D^i = \omega D$, $i = 0, 1$, we deduce $\omega D^0 = -D^1 \omega$. Therefore, the spectrum of $D^1$ is the negative of the spectrum of $D^0$. Then, distinguishing the $\eta$-functions (A.9) for $D^0, D^1$, we have $\eta^0_0(s) = -\eta^1_0(s)$ for all $s$. The desired conclusion follows by analytic continuation. □

The exponentiated $\eta$-invariants are the partition functions of invertible field theories $\alpha^0, \alpha^1$, and the stronger version of Proposition A.18 is that $\alpha^0$ and $\alpha^1$ are inverse theories. If both are topological, which is the case for the application to M-theory, then the stronger assertion follows from Proposition A.18 since the partition function determines the isomorphism class of the theory. Here we will not attempt to justify the stronger assertion in the non-topological case, nor the conjecture that $\alpha^0 \otimes \alpha^1$ admits a *canonical* trivialization.
A.2. The pin case in dimension 11

We describe the relevant pin representation and check that (A.11) produces the Dirac operator in 12 dimensions which appears in [St].

We follow [FH, §9.2] in which the Pin case is described by a parameter \( s = -1 \). The point is to use the embeddings [FH, Lemma 9.25] and [FH, (9.44)], which specialize to

\[
\text{Pin}_{12}^+ \rightarrow \text{Cliff}^0_{12,1}, \\
\gamma^i \mapsto \gamma^i \otimes \gamma^- 
\]

and

\[
\text{Pin}_{10,1} \rightarrow \text{Cliff}^0_{10,2}, \\
\gamma^i \mapsto \gamma^i \otimes \gamma^-, 
\]

where \( (\gamma^-)^2 = -1 \) and \( i = 1, 2, \ldots, 12 \). These give embeddings of groups \( \text{Pin}_{12}^+ \hookrightarrow \text{Spin}_{12,1} \) and \( \text{Pin}_{10,1} \hookrightarrow \text{Spin}_{10,2} \). The starting data is a real representation of \( \text{Pin}_{10,1} \) obtained by restriction from an ungraded real \( \text{Cliff}^0_{10,2} \)-module. There are isomorphisms

\[
\text{Cliff}_{10,2} \cong \text{Cliff}^+_{+8} \otimes \text{Cliff}^+_{2,2} \cong \text{End}(M^0 \oplus M^1) \otimes \text{End}(\text{Cliff}^+_{+2}) 
\]

where \( M^i \) is a real vector space of dimension 8. A minimal real \( \text{Cliff}^0_{10,2} \)-module is the even subspace

\[
S := M^0 \otimes \text{Cliff}^+_{+2} \oplus M^1 \otimes \text{Cliff}^+_{+2} 
\]

of \( (M^0 \oplus M^1) \otimes \text{Cliff}^+_{+2} \), which has real dimension 32. (We could as well take the odd subspace; see Proposition A.18.) The restriction of \( S \) to \( \text{Cliff}^0_{10,1} \subset \text{Cliff}^0_{10,2} \), or equivalently to \( \text{Spin}_{10,1} \subset \text{Spin}_{10,2} \), is irreducible. (The \( \text{Cliff}^+_{+8} \) in (A.21) splits off and one simply checks for \( \text{Cliff}^+_{2,1} \subset \text{Cliff}^+_{2,2} \).) By [De, Theorem 6.1] there is a \( \text{Spin}_{10,1} \)-invariant pairing (A.1), unique up to a positive scalar, and it is then automatically \( \text{Pin}_{10,1} \)-invariant. This defines the starting data \( (S, \Gamma) \).

The Wick rotation on 12-manifolds, carried out in the second paragraph of §A.1, is modified in the first instance by tensoring (A.4) with \( \text{Cliff}^-_{-3} \) and using the embedding (A.19), of course setting \( n = 11 \). Then the commuting super algebra (A.5) is \( \text{Cliff}_{12,1} \otimes \text{Cliff}_{10,2} \), which as before is Morita equivalent to \( \text{Cliff}_{-3} \). Then \( E = \text{Cliff}^+_{+2} \otimes (S \oplus S^*) \) is as in (A.7), but is a left \( \text{Cliff}^+_{12,1} \otimes \text{Cliff}^-_{-3} \)-module: the last Clifford generator in \( \text{Cliff}_{12,1} \) acts via the action of the last Clifford generator on the \( \text{Cliff}_{10,1} \)-module \( S \oplus S^* \). The even subspace \( E^0 \subset E \) has real dimension 128 and carries a quaternionic structure, so is a 32-dimensional quaternionic vector space. The resulting representation of \( \text{Pin}_{12}^+ \) agrees with the one described at the end of [St, §3]. (Stolz distinguishes between two representations of \( \text{Pin}_{13}^+ \), but they are isomorphic when restricted to \( \text{Pin}_{12}^+ \).)
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