A practical method for the consistent identification of a module in a dynamical network

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Abstract: We present a new and simple method for the identification of a single transfer function that is embedded in a dynamical network. In existing methods the consistent identification of the desired transfer function relies on the positive definiteness of the spectral density matrix of the vector of all node signals, and it typically requires knowledge of the topology of the whole network. The positivity condition is on the internal signals and therefore can not be guaranteed a priori; in addition it is far from necessary. The new method of this paper provides simple conditions on which nodes to excite and which nodes to measure in order to produce a consistent estimate of the desired transfer function. Just as importantly, it requires knowledge of the local topology only.

1. INTRODUCTION

The present paper is devoted to the identification of a particular transfer function (also called module or edge) within a network. This problem is far from trivial because the interconnection structure creates feedback loops, which may (and quite often do) lead to the module of interest becoming unidentifiable from available signals.

A number of recent papers have addressed the problem of identification of a specific module embedded within a dynamical network Dankers et al. (2016); Everitt et al. (2017); Gevers and Bazanella (2015); Van den Hof et al. (2013). In Dankers et al. (2016); Van den Hof et al. (2013) the focus has been on the identifiability of the desired transfer function, and on the question of which subset of node measurements will yield such identifiability. The assumption is made in these papers that the vector consisting of all the node signals in the network is informative, and this assumption is crucial for the consistent estimation of the desired transfer function. This is a strong persistence of excitation condition, which is sufficient to identify the whole network, but which is far from necessary for the identification of a single module. Moreover, it is an assumption on the internal signals rather than on the external excitation, which makes it difficult to enforce.

The question as to how to guarantee that the vector of node signals is informative by appropriately choosing the externally applied signals is an experiment design problem. This problem was first approached in Gevers and Bazanella (2015), where it was illustrated on a simple 3-node network. It was shown in that paper that the choice of informative external signals that would lead to consistent identifiability of the desired transfer function depends both on the network topology and on the chosen identification method. A framework, based on Gevers et al. (2009), was proposed to determine which external signals need to be applied in order to make the vector of node signals informative.

In Everitt et al. (2017) a new approach has been proposed for the identification of a single module. Unlike the methods presented in Dankers et al. (2016) this new approach uses all external excitation signals that enter the network. Denoting $G_{ji}$ the desired transfer function to be identified, then Everitt et al. (2017) propose a two-step procedure, one of which consists of identifying all transfers from all external signals to all the nodes that have a direct link to node $j$.

All results cited above, that deal with the identification of a single module embedded within a dynamical network, require knowledge of the topology of the whole network. In this paper we present a completely different and very simple method which requires only local information about the topology. More precisely, if the module to be identified is $G_{ji}$, then the only topological information that is required for our new method is to know either to which nodes node $i$ connects (i.e. which nodes $k$ are such that $G_{ki} \neq 0$), or which nodes connect to $j$ (i.e. which nodes $k$ are such that $G_{jk} \neq 0$).

Let us summarize this introduction as follows. The consistent identification of a specific module $G_{ji}(q)$ contains both an experiment design aspect and a computational
aspect. These can be summarized in the following three problems:

- which nodes should be excited?
- which nodes should be measured?
- how to estimate $\hat{G}_{ji}(q)$ from these signals?

The new method proposed in this paper solves all three problems at once. In addition, the method requires only local knowledge of the topology of the network, namely either the existence of the edges leaving node $i$, or the existence of the edges entering node $j$.

The paper is organized as follows. The problem is stated in section 2. In section 3 we present the standard direct Prediction Error Method (PEM) that rewrites the network as a closed-loop system and then estimates the desired $G_{ji}$ by identifying all nonzero $G_{jk}$ that appear in the same row as $G_{ji}$. In section 4 we apply this direct method to a 20-node network in order to illustrate the difficulty in arriving at excitation scenarios that yield a consistent estimate. We present the new method in section 5, and we return to the case study in section 6 to illustrate its simplicity and effectiveness.

2. PROBLEM STATEMENT

We adopt the network structure of Gevers et al. (2017), in which the outputs of the nodes are denoted \{w$_1(t), \ldots, w_L(t)\} and are related to each other and to the external excitation signals r$_i(t), i = 1, \ldots, L$ and the noise signals v$_j(t), j = 1, \ldots, L$ by the following network equations:

$$
\begin{bmatrix}
w_1(t) \\
w_2(t) \\
\vdots \\
w_L(t)
\end{bmatrix} =
\begin{bmatrix}
0 & G^0_{12}(q) & \ldots & G^0_{1L}(q) \\
G^0_{21}(q) & 0 & \ldots & G^0_{2L}(q) \\
\vdots & \vdots & \ddots & \vdots \\
G^0_{L1}(q) & G^0_{L2}(q) & \ldots & 0
\end{bmatrix}
\begin{bmatrix}
w_1(t) \\
w_2(t) \\
\vdots \\
w_L(t)
\end{bmatrix}
+ \begin{bmatrix}
r_1(t) \\
r_2(t) \\
\vdots \\
r_L(t)
\end{bmatrix} + \begin{bmatrix}
v_1(t) \\
v_2(t) \\
\vdots \\
v_L(t)
\end{bmatrix}
= G^0(q)w(t) + r(t) + v(t). \quad (1)
$$

where $q^{-1}$ is the delay operator and the superscript $0$ denotes the real value of a quantity. The matrix $G^0(q)$ will be called the network matrix and equation (1) the network model, and we will often omit the dependence on $t$ and on $q$ whenever it does not create ambiguity. We assume that the network model has the properties specified in Assumption 1 below.

Assumption 1. The network model (1) has the following properties:

1. all the transfer functions $G^0_{ij}(q)$ are proper
2. there is a delay in every loop going from any $w_j(t)$ to itself
3. the stochastic processes $v_j(t)$ are stationary, zero mean and mutually independent: $E[v_j(t)v_k(s)] = 0 \forall t, s \in \mathbb{R}$ for all $j \neq k$.
4. the external excitation signals $r_i(t)$ are quasi-stationary and uncorrelated with all noise signals $v_j(t)$
5. the network is internally stable.

In this paper we consider the problem of identifying a particular transfer function, say $G^0_{ij}(q)$, from measured node signals $w_k(t)$ and measured excitation signals $r_l(t)$.

3. BACKGROUND - THE DIRECT METHOD

Perhaps the most natural approach towards the identification of a single transfer function $G_{ij}$ in a network is to estimate it from the scalar equation of (1) in which it appears, based on the signals that appear in that equation. This corresponds to a closed-loop identification problem for a Multiple Input Single Output (MISO) feedback system with $L - 1$ inputs and one output, namely $w_j$. Let us take, without loss of generality and to ease notation, $j = 1$ and $i = 2$, so that $G_{12}$ is the desired transfer function.

We split up the vector $w$ into

$$
\begin{bmatrix}
w_1(t) \\
w_2(t) \\
\vdots \\
w_L(t)
\end{bmatrix} = \begin{bmatrix}
w_1 \\
w_2 \\
\vdots \\
w_L
\end{bmatrix} = w = \begin{bmatrix}
w_1 \\
w_2
\end{bmatrix}
\quad (2)
$$

where $\tilde{w}_2 \triangleq [w_2, \ldots, w_L]^T$. Correspondingly, we split up the matrix $G^0(q)$ into the 4-block matrix

$$
G^0 = \begin{bmatrix}
0 & (G^0_{12} \ldots G^0_{1L}) \\
(G^0_{21}) & \cdots \\
(G^0_{L1}) & \cdots \\
(G^0_{L2} \ldots 0)
\end{bmatrix}
\quad (3)
$$

which we denote as

$$
G^0 = \begin{bmatrix}
0 & \tilde{G}^1 \\
G^0_2 & \tilde{G}^0
\end{bmatrix}
\quad (4)
$$

We can now rewrite the initial network model (1) as a MISO feedback system as follows. First we rewrite (1) as

$$
\begin{bmatrix}
w_1(t) \\
\tilde{w}_2(t)
\end{bmatrix} = \begin{bmatrix}
0 & \tilde{G}^1 \\
G^0_2 & \tilde{G}^0
\end{bmatrix} \begin{bmatrix}
w_1(t) \\
\tilde{w}_2(t)
\end{bmatrix} + \begin{bmatrix}
r_1(t) \\
\tilde{r}_2(t) + \tilde{v}_2(t)
\end{bmatrix} 
\quad (5)
$$

Next we rewrite (5) in the traditional form of a MISO feedback system:

$$
w_1(t) = \tilde{G}^1 \tilde{w}_2 + r_1 + v_1 \\
= G^0_{12}w_2 + \sum_{k=3}^{L} G^0_{1k}w_k + r_1 + v_1 
\quad (6)
$$

$$
\tilde{w}_2(t) = [I - \tilde{G}^{0}_{12}]^{-1} \{G^0_{21}w_1 + \tilde{r}_2 + \tilde{v}_2\} 
\quad (7)
$$

We observe that, if the objective is to identify $G^0_{12}(q)$, then it is natural to do so by identifying $\tilde{G}^0_{12}(q)$ (of which $G^0_{12}(q)$ is the first element) directly from the scalar equation (6), by using the prediction error method. This problem setting is not the most realistic in large scale networks. Large scale networks tend to be highly sparse, meaning that most elements of $G^0_{12}(q)$ are known to be zero. With this knowledge, then the identification can proceed as described before but identifying only those elements that are known to be nonzero.

Let us define some notation in order to take advantage of the sparsity of the network. If (and only if) $G_{ik}$ is

1 From now on we omit the dependence on $q$ and $t$ whenever it creates no ambiguity.
nonzero then node $i$ is said to be an *out-neighbour* of node $k$; similarly, node $k$ is said to be an *in-neighbour* of node $i$. Observe that the in-neighbours of a node $i$ correspond to the nonzero elements of the $i$-th line of the network matrix $G^0$; similarly, the out-neighbours of node $k$ correspond to the nonzero elements of the $k$-th column of $G^0$. The set of in-neighbours of node $i$ is denoted $N^+_i$ and the set of out-neighbours of node $k$ is denoted $N^-_k$, their cardinalities being $d^+_i$ and $d^-_k$, respectively. With these definitions one can then write, in lieu of (7):

$$w_1 = \sum_{k \in N^-_i} G_{1k}(q)w_k + r_1 + v_1 \quad (9)$$

Define further the vector containing the node signals of the in-neighbours of node 1, with dimension $d^-_1$:

$$w^-_1 = [w_{11} \ldots w_{1d^-_1}]^T$$

where $l1, \ldots, l_{d^-_1}$ are the indices corresponding to the in-neighbours of node 1, and the corresponding partition of the vector $G_1$ in (6):

$$G^-_1(q) = \begin{bmatrix} G_{1,11}(q) & \cdots & G_{1,1d^-_1}(q) \end{bmatrix}.$$  

This is the vector of the nonzero transfer functions in (7). Equation (6) can then be rewritten as

$$w_1 = G^-_1(q)w^-_1 + r_1 + v_1. \quad (10)$$

Next, let us define parametrized model structures $G_{1k}(q, \theta)$ for each one of the transfer functions in the vector $G^-_1(q)$, corresponding to a parametrized $G^-_1(q, \theta)$, and a model structure $H_1(q, \theta)$ for the noise $v_1(t) = H_1^0(q)e_1(t)$ where $e_1(t)$ is a stationary zero-mean white noise process. Prediction error identification then proceeds by defining the predictor:

$$\hat{w}_1(t, \theta) = H^{-1}_1(q, \theta)[G^-_1(q, \theta)w^-_1(t) + r_1(t)]$$

and minimizing the energy of the prediction error:

$$\varepsilon(t, \theta) \triangleq w_1(t) - \hat{w}_1(t, \theta). \quad (12)$$

This direct application of prediction error identification to the network model has become known as the *direct method* in recent literature. The following theorem from Van den Hof et al. (2013) gives sufficient conditions for the direct method to succeed in providing consistent estimates.

**Theorem 3.1.** Consider a dynamic network (6) satisfying Assumption 1 and the identification of the first row of the network matrix by the direct method described above. The estimate obtained for $G^-_1(q)$ is consistent if the following two conditions are satisfied:

1. (1) there exists $\theta^0$ such that $G^-_1(q, \theta) = (G^-_1)^0(q)$ and $H_1(q, \theta) = H^0_1(q)$

2. the spectral density of $w^-_1(t)$ is positive definite for almost all $\omega$.

The problem in the direct method, and also in other known methods as described in Van den Hof et al. (2013) is condition 2 of the above theorem: it is a condition on internal signals $w_1(t)$. What is required for a proper experiment design are conditions on the external signals - $r_1(t)$ and $v_1(t)$ - that will enforce condition 2; this issue is, however, not solved in the direct method.

This problem of transfer of excitation from the external signals to the regressor used in the identification has been solved for SISO systems in Gevers et al. (2009), where necessary and sufficient conditions on the richness of the external excitation signals for closed-loop systems have been given. Although those tools can also be used in the analysis of MISO problems, as illustrated in Gevers and Bazanella (2015), no general results exist and, much more importantly, the analysis requires knowledge of the whole network.

A case study in the next section illustrates the difficulties in this experiment design problem.

4. A MOTIVATING CASE STUDY

Consider a network with $L = 20$ nodes, of which we want to identify the transfer function $G_{34}(q)$. The real network has the graph presented in Figure 1 and the real transfer function is $G_{34}^0(q) = -0.3q^{-1} + 0.8q^{-2}$. All noises $v_i$ are white. The nonzero transfer functions are all of first or second order; the full $20 \times 20$ matrix $G^0(q)$ is given in Appendix.

![Figure 1. The directed graph corresponding to the case study; the transfer function of interest is the red edge](image)

In order to apply the direct method, we need to know what are the in-neighbours of node 3; these are nodes 2, 4, 5 and 9. With these we form the regressor

$$w^-_3(t) = [w_2(t) \ w_4(t) \ w_5(t) \ w_9(t)]^T$$

We take an Output-Error-like model structure, with $H_3(q, \theta) = H_3(q) = 1$ and full-order models for the transfer functions to be identified - the desired $G_{34}$ plus $G_{32}, G_{35}$ and $G_{39}$. Then we apply the direct method with data collected from the real system under 18 different excitation scenarios: see Table 1. Each scenario consists of the excitation of a selected number of $r_i$ signals with a persistently exciting signal - stationary zero-mean white noise with unit variance. In each scenario, one thousand Monte-Carlo runs are performed, with ten thousand data in each run. The average values obtained for the parameters in eighteen different excitation scenario are given in Table 2. Recall that the real parameter values are $a_{32} = -0.3$ and $a_{35} = 0.8$. 

\[^2\] Actually it is mentioned as an open question in the last sentence of Van den Hof et al. (2013).
In the first scenario, all \( r_i \)’s are excited, which guarantees that the regressor vector \( w_i \) will have full-rank spectrum. The result of the MC runs in the space of \( G_{34} \) parameters is given in Figure 2. It is important to note that the scales of the top and bottom plots are widely different. It is seen from this Figure, as well as from Table 2, that consistent estimation is obtained, as expected. But exciting all inputs in order to identify a single transfer function is obviously far from necessary and, most importantly, far from feasible in a large network.

In fact, exciting a single input may be sufficient. We first try exciting only the node which is the input to the desired transfer function, i.e. node 4, but it does not result in an informative experiment as can be seen both by the average values in Table 2 and in Figure 3: consistency is not achieved and the variance tends to infinity. Exciting only \( r_5 \) (scenario 5 in Table 2) proves to be enough, as can be seen in Figure 3, but other “closeby” excitation scenarios, i.e. exciting neighbors of the nodes 3 and 4 involved in the desired transfer function \( G_{34} \) do not work either. On the other hand, exciting “far away” nodes, like in scenario 17, does provide consistent estimation.

### Table 1. Excitation Scenarios

| Scenario | Inputs       | Scenario | Inputs       |
|----------|--------------|----------|--------------|
| 1        | All inputs \( (r_1 - r_{20}) \) | 10       | \( r_2, r_6, r_7, r_8 \) |
| 2        | \( r_3, r_4, r_5 \) | 11       | \( r_6, r_7, r_8, r_9 \) |
| 3        | \( r_3 \) | 12       | \( r_7, r_8, r_{10} \) |
| 4        | \( r_4 \) | 13       | \( r_8, r_9, r_{10}, r_{11} \) |
| 5        | \( r_5 \) | 14       | \( r_9, r_{10}, r_{11}, r_{12} \) |
| 6        | \( r_3, r_4 \) | 15       | \( r_{10}, r_{11}, r_{12}, r_{13} \) |
| 7        | \( r_3, r_5 \) | 16       | \( r_{11}, r_{12}, r_{13}, r_{14} \) |
| 8        | \( r_4, r_5 \) | 17       | \( r_1, r_7 \) |
| 9        | \( r_1, r_2, r_6, r_7 \) | 18       | \( r_1, r_{16} \) |

In Gevers and Bazanella (2015) we have shown that, to obtain an excitation scenario for the direct method that yields informative data, may require knowledge of the whole network even though only a single transfer function is to be identified.

In this Section we present a simple method which solves both the problem of transfer of excitation and that of measurement selection. In other words, with this method we know a priori which inputs must be excited in order to obtain an informative experiment and which nodes need be measured, using only local information about the network’s topology.

The method is based on the input-output description of the system, which is obtained by rewriting it in a form that relates directly the external inputs \( r \), the disturbances \( v \) and the outputs \( w \):

\[
w(t) = (I - G^0(q))^{-1}[r(t) + v(t)]
\]

\[
= T^0(q)r(t) + \tilde{v}(t)
\]

where

\[
T^0(q) \triangleq (I - G^0(q))^{-1}, \quad \tilde{v}(t) \triangleq (I - G^0(q))^{-1}v(t).
\]

The description (14) will be called the input-output (I/O) model of the network. It is well known that one can obtain a consistent estimate \( \hat{T}(q) \) of \( T^0(q) \) from \( \{w, r\} \) data; this is an open loop MIMO identification problem.

Suppose we have an estimate \( \hat{T}(q) \) of \( T^0 \); then an estimate \( \hat{G} \) of \( G^0 \) can be obtained by solving either one of the following two equations for \( \hat{G}(q) \), each one being a set of \( L^2 \) linear equations:
\[ \hat{T}(q)(I - \hat{G}(q)) = I \quad (16) \]
\[ (I - \hat{G}(q))\hat{T}(q) = I. \quad (17) \]

If only one particular transfer function in \( G^0 \) is desired, then it can be obtained by solving a subset of these equations; this rationale is the basis of our method. To describe the method, we introduce some notations that are specific to the identification of the transfer function \( G_i \). For simplicity, we also refer to the identification of an edge of the network to refer to the identification of its transfer function.

**Notations:**
- \( G_{N^+}^+,(q) \): column vector of the out-going edges of \( i \)
- \( T_{ji}(q) \): the \( j, i \) element of matrix \( T(q) \)
- \( T_{N^+}^+,(q) \): submatrix of \( T(q) \) made up of its rows in \( N^+_i \)
- \( T_{N^+_i}^+,(q) \): \( i \)-th column of \( T_{N^+_i}^+,(q) \)
- \( T_{N^+}^+,(N^+_i) \): submatrix of \( T(q) \) made up of its rows and columns in \( N^+_i \)
- \( E_i \): \( i \)-th column of the identity matrix.

Consider the network (1) and assume that it is desired to only identify a specific transfer function \( G_{ji} \).

**Theorem 5.1.** Perform an experiment under the following conditions:
- excite node \( i \) and all its \( N^+_i \) out-neighbors with sufficiently rich signals
- measure the node signals at the \( N^+_i \) out-neighbors of node \( i \).

Under these experimental conditions and using full-order models for the elements of the matrix \( T^0 \), consistent estimates \( \hat{T}_{N^+_i}^+,(q) \) and \( \hat{T}_{N^+_i}^+,(N^+_i) \) of \( T^0_{N^+_i}^+,(N^+_i) \) and \( T^0_{N^+_i}^+,(N^+_i) \) can be obtained by standard open-loop MIMO identification. From these, a consistent estimate of \( G_{N^+_i}^+,(N^+_i) \) is obtained by

\[ \hat{G}_{N^+_i}^+,(N^+_i) = [\hat{T}_{N^+_i}^+,(N^+_i)]^{-1}\hat{T}_{N^+_i}^+,(N^+_i) \quad (18) \]

**Proof:** First note that identification of \( T^0_{N^+_i}^+,(N^+_i) \) and \( T^0_{N^+_i}^+,(N^+_i) \) is an open-loop identification problem which, under the specified experimental conditions and with full order models for the elements of these matrices, provides consistent estimates. Now, consider the system of equations

\[ T(q)(I - G(q)) = I \quad (19) \]

where it is desired to compute \( G_{ji}(q) \) from \( T(q) \). The desired \( G_{ji} \) appears only in a subset of these equations, its computation resting entirely on the solution of the \( i \)-th column of (19). In the \( i \)-th column of \( I - G \), the only nonzero elements are: the desired \( G_{ji} \), the 1 at position \((i,i)\), and the \( G_{kl} \) corresponding to the remaining \( d^+_i \) out-neighbors of \( i \). As a result, the columns of \( T \) corresponding to the zero elements of the \( i \)-th column of \( G \) do not contribute to the computation of \( G_{ji} \) using (19). It follows that these columns need not be identified and, therefore, the corresponding \( r_k \) are not required. Stated otherwise, for the identification of \( G_{ji} \), it is only required to excite node \( i \) and its \( d^+_i \) out-neighbors.

To compute \( G_{ji} \), we compute the \( i \)-th column of \((I - G), \) of which \( G_{ji} \) is an element. From (19), we thus get

\[ T(I - G)_{ji} = E_i \quad (20) \]

Now let \( C \) be a selector matrix that selects the rows of \( T \) that are in the set \( N^+_j \) of out-neighbors of \( j \). Premultiplying (20) by this \( C \) yields

\[ CT(I - G)_{ji} = C_{ji} = 0 \iff T_{N^+_j}^+,(L)(I - G)_{ji} = 0 \quad (21) \]

because the \( i \)-th column of \( C \) is zero. Since \( (I - G)_{ji} \) contains a 1 in position \( i \) and nonzero elements only in the positions corresponding to \( N^+_i \), the last equation is equivalent with

\[ T_{N^+_j}^+,(i) - T_{N^+_j}^+,(N^+_i)G_{N^+_j}^+,(i) = 0, \quad (22) \]

from which the result (18) follows. That \( T_{N^+_j}^+ \) is nonsingular follows from Proposition 5.1 in Hendrickx et al. (2017) \(^3 \).

The desired transfer function \( G_{ji} \) is an element of \( G_{N^+_j}^+,(N^+_i) \); hence Theorem 5.1 provides a new method for the identification of a single embedded module. The method rests on the identification of a submatrix of the transfer function matrix \( T^0 \); the number of elements \( T^0_{ki} \) that need to be identified is \( d^+_j \times (1 + d^+_i) \), where \( d^+_j \) is the number of out-neighbors of the input node \( i \) of the desired \( G_{ji} \). What is most important is that the theorem completely solves both the informativity and the identifiability questions for the identification of a single embedded module, namely which nodes need to be excited and which nodes need to be measured, in addition to providing a computational method for the estimation of \( G_{ji} \). Moreover, this solution requires only local information about the network’s topology, namely what nodes are the out-neighbors of node \( i \).

In the same spirit, we can derive a dual method by manipulating equation (17) instead of (16). In this case only the \( j \)-th equation is relevant and we get:

\[ (I - G)_j T = E_j^T \quad (23) \]

Taking a selector matrix \( C \) that selects the columns of \( T \) that correspond to the in-neighbors of \( j \) yields:

\[ (I - G)_j T C = C_j = 0 \iff (I - G)_j T_{L,N^-} = 0 \quad (24) \]

because the \( j \)-th line of \( C \) is zero. Since \((I - G)_j \) contains a 1 in position \( j \) and nonzero elements only in the positions corresponding to \( N^-_j \), the last equation is equivalent with

\[ T_{j,N^-} - G_{j,N^-}T_{N^-} = 0, \quad (25) \]

and we have proven the following result.

**Theorem 5.2.** Perform an experiment under the following conditions:
- excite the node signals at all \( N^-_j \) in-neighbors of node \( j \) with sufficiently rich signals
- measure node \( j \) and the node signals of its \( N^-_j \) in-neighbors.

Under these experimental conditions and using full-order models for the elements of the matrix \( T^0 \), consistent estimates \( \hat{T}_{N^-}^+,(N^-_j) \) and \( \hat{T}_{N^-}^+,(N^-_j) \) can be obtained by standard open-loop MIMO identification. From these estimates, a consistent estimate of \( G_{j,N^-} \) is given by

\[ \hat{G}_{j,N^-} = \hat{T}_{j,N^-}^+,(N^-_j)\hat{T}_{N^-}^+,(N^-_j) \quad (26) \]

\(^3\) Just apply it to \( A = C = N^+_i \).
The desired transfer function $G_{ji}$ is an element of $G_{j,N_j^-}$.

**Proof:** The proof is the exact dual of the proof of Theorem 5.1, and is therefore omitted.

The desired transfer function $G_{ji}$ is an element of $G_{j,N_j^-}$. The identification of $G_{ji}^0_j$ rests on the identification of a submatrix of $T^0$ that contains $(d_j^- + 1) \times d_j^-$ elements, where $d_j^-$ is the number of in-neighbors of the output node $j$ of the desired $G_{ji}$. Again, the identification of $G_{ji}$ using the dual method of Theorem 5.2 requires only local information about the network’s topology, namely what nodes are the in-neighbors of node $j$.

**Comment.** If local information is available about both the out-neighbors of node $i$ and the in-neighbors of node $j$, then the user has a choice, for the identification of $G_{ji}$, of applying either Theorem 5.1 or Theorem 5.2. The decision may depend on the practically available excitation and measurement scenarios. If both scenarios are possible, he/she will then obviously chose to apply the method that requires the smallest number of transfer functions $T^0_{kl}$ to be identified, by applying Theorem 5.1 if $d_i^+ \leq d_j^-$ and Theorem 5.2 otherwise.

6. THE CASE STUDY REVISITED

Let us now apply our method to the identification of $G_{34}(q)$ in our 20-node case study of section 4. It is seen in the graph that node 4 has 3 out-neighbors, i.e. $d_i^+ = 3$, while node 3 has 4 in-neighbors, i.e. $d_j^- = 4$. Thus we identify $G_{34}(q)$ using the method of Theorem 5.1, noting that the set of outneighbors of node 4 is $N_4^+ = \{3, 5, 6\}$. So we need an experiment in which we measure these three nodes and excite them plus node 4. We have excited each input $r_i$, $i = 3, 4, 5, 6$ with independent white noises with unit variance, and also applied unmeasured noise signals $v_i$, $i = 3, 4, 5, 6$ with variance $10^{-6}$.

From this experiment we have identified the nine transfer functions in the matrix $T_{N_i^+, N_j^-}$ and the three transfer functions in the vector $T_{N_i^+, N_j^-}$. We have performed black-box identification of order six models for these twelve transfer functions by the instrumental variable method in MatLab’s identification toolbox. The actual transfer functions $T^0_{kl}$ are of very large order, but models of order six were enough to get a fit above 99%. The identified transfer functions are in the form

$$T_{ij} = b_0 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3} + b_4 z^{-4} + b_5 z^{-5} + b_6 z^{-6} + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3} + a_4 z^{-4} + a_5 z^{-5} + a_6 z^{-6}$$

and their identified parameters are given below, with eight significant digits.

Some of the estimated transfer functions had unstable states that were removed. This was accomplished by determining the balanced state space realization of the unstable transfer function with the `dbalreal` command from MATLAB, which also returns the vector containing the diagonal of the observability/controlability matrix. States that the corresponded to Inf values on this vector were removed with the command `dmodred`.

| $T_{34}$ | $T_{35}$ |
|---------|---------|
| $a_1 = 3.8481674 \times 10^{-2}$ | $a_1 = 2.4570327 \times 10^{-1}$ |
| $a_2 = 3.0797313 \times 10^{-1}$ | $a_2 = 4.5393517 \times 10^{-1}$ |
| $a_3 = -6.6201436 \times 10^{-1}$ | $a_3 = -4.7766990 \times 10^{-1}$ |
| $a_4 = 1.7938664 \times 10^{-2}$ | $a_4 = -5.0273260 \times 10^{-2}$ |
| $a_5 = 3.0075344 \times 10^{-2}$ | $a_5 = -5.3388808 \times 10^{-3}$ |
| $a_6 = 0$ | $a_6 = 0$ |
| $b_0 = 1.0005030$ | $b_0 = -7.8937977 \times 10^{-2}$ |
| $b_1 = 3.1534410 \times 10^{-2}$ | $b_1 = -2.8932660 \times 10^{-1}$ |
| $b_2 = -1.7410433 \times 10^{-2}$ | $b_2 = -2.2567717 \times 10^{-1}$ |
| $b_3 = -2.2352584 \times 10^{-2}$ | $b_3 = -1.1488447 \times 10^{-1}$ |
| $b_4 = -2.0168467 \times 10^{-2}$ | $b_4 = -4.0703106 \times 10^{-2}$ |
| $b_5 = 8.5201918 \times 10^{-7}$ | $b_5 = 5.8362039 \times 10^{-3}$ |
| $b_6 = 0$ | $b_6 = -1.7599509 \times 10^{-17}$ |
\[ T_{36} \]
\[ a_1 = 3.2094476 \times 10^{-1} \]
\[ a_2 = 3.8846671 \times 10^{-1} \]
\[ a_3 = -5.6921619 \times 10^{-1} \]
\[ a_4 = -1.3026078 \times 10^{-1} \]
\[ a_5 = 2.9584499 \times 10^{-3} \]
\[ a_6 = 2.2717961 \times 10^{-2} \]
\[ b_0 = 1.4074532 \times 10^{-1} \]
\[ b_1 = -2.1822255 \times 10^{-1} \]
\[ b_2 = 1.0131479 \times 10^{-1} \]
\[ b_3 = -1.0562591 \times 10^{-1} \]
\[ b_4 = 1.7462396 \times 10^{-2} \]
\[ b_5 = 9.7104478 \times 10^{-3} \]
\[ b_6 = 1.7215627 \times 10^{-18} \]

\[ T_{53} \]
\[ a_1 = -2.1525075 \times 10^{-2} \]
\[ a_2 = 4.1218212 \times 10^{-1} \]
\[ a_3 = -6.7085915 \times 10^{-1} \]
\[ a_4 = 9.8264186 \times 10^{-2} \]
\[ a_5 = -3.8151595 \times 10^{-2} \]
\[ a_6 = 0 \]
\[ b_0 = -2.4299859 \times 10^{-4} \]
\[ b_1 = 4.2864147 \times 10^{-4} \]
\[ b_2 = 4.9627613 \times 10^{-1} \]
\[ b_3 = -2.0867406 \times 10^{-2} \]
\[ b_4 = 2.1384076 \times 10^{-2} \]
\[ b_5 = -8.0865862 \times 10^{-6} \]
\[ b_6 = 0 \]

\[ T_{54} \]
\[ a_1 = -1.9833111 \times 10^{-1} \]
\[ a_2 = 1.8199887 \times 10^{-1} \]
\[ a_3 = -7.5568457 \times 10^{-1} \]
\[ a_4 = 1.3184591 \times 10^{-1} \]
\[ a_5 = 9.6356440 \times 10^{-2} \]
\[ a_6 = 0 \]
\[ b_0 = -5.5988128 \times 10^{-4} \]
\[ b_1 = 4.9701271 \times 10^{-1} \]
\[ b_2 = -1.1035300 \times 10^{-1} \]
\[ b_3 = -9.1997426 \times 10^{-2} \]
\[ b_4 = -7.2354865 \times 10^{-3} \]
\[ b_5 = -3.5608761 \times 10^{-5} \]
\[ b_6 = 0 \]

\[ T_{55} \]
\[ a_1 = -2.4661243 \times 10^{-1} \]
\[ a_2 = 7.7627922 \times 10^{-1} \]
\[ a_3 = -7.3814227 \times 10^{-1} \]
\[ a_4 = 3.9799183 \times 10^{-1} \]
\[ a_5 = -2.7861447 \times 10^{-1} \]
\[ a_6 = 2.4048797 \times 10^{-2} \]
\[ b_0 = 1.0017417 \]
\[ b_1 = -2.7825384 \times 10^{-1} \]
\[ b_2 = 7.0393709 \times 10^{-1} \]
\[ b_3 = -8.6350269 \times 10^{-1} \]
\[ b_4 = 3.4837217 \times 10^{-1} \]
\[ b_5 = -3.0949202 \times 10^{-1} \]
\[ b_6 = -1.1725324 \times 10^{-17} \]

\[ T_{56} \]
\[ a_1 = 6.2495573 \times 10^{-2} \]
\[ a_2 = 4.2084437 \times 10^{-1} \]
\[ a_3 = -6.0218364 \times 10^{-1} \]
\[ a_4 = 7.4963364 \times 10^{-2} \]
\[ a_5 = 0 \]
\[ a_6 = 0 \]
\[ b_0 = 1.8924638 \times 10^{-2} \]
\[ b_1 = 8.5600937 \times 10^{-2} \]
\[ b_2 = -2.2788412 \times 10^{-2} \]
\[ b_3 = -5.6722905 \times 10^{-2} \]
\[ b_4 = -3.8189237 \times 10^{-6} \]
\[ b_5 = 0 \]
\[ b_6 = 0 \]

\[ T_{63} \]
\[ a_1 = 4.8106647 \times 10^{-1} \]
\[ a_2 = 7.2321028 \times 10^{-2} \]
\[ a_3 = -5.1645253 \times 10^{-1} \]
\[ a_4 = -4.1069797 \times 10^{-1} \]
\[ a_5 = 3.1824379 \times 10^{-1} \]
\[ a_6 = -9.2033189 \times 10^{-3} \]
\[ b_0 = -8.8116822 \times 10^{-4} \]
\[ b_1 = -3.8823253 \times 10^{-2} \]
\[ b_2 = -1.8354666 \times 10^{-2} \]
\[ b_3 = 2.6939801 \times 10^{-2} \]
\[ b_4 = 4.8460620 \times 10^{-3} \]
\[ b_5 = -5.1528647 \times 10^{-3} \]
\[ b_6 = -1.0676972 \times 10^{-18} \]
The direct method for the identification of a module in a network is well known and easy to apply if informative data are available. However, as we have illustrated through a 20-node example, there is no practical way to design an informative experiment for this method. Though the excitation requirements vary for different methods and for different input selections, they all suffer from this limitation.

We have presented an identification method for which the design of informative experiments is obviated. It consists in performing the identification of part of the network’s input-output model and then recovering the desired module from these identified transfer functions. Because identification of the I-O model is an open loop identification problem, it is clear which are the inputs that must be excited and the critical issue becomes to determine what is the smallest set of I-O transfer functions that must be estimated in order to be able to uniquely recover the desired module. We have shown that this smallest set depends strictly on the network’s local topology - that is, on what are the neighbours of nodes i and j - and we have provided two choices for it, one involving the in-neighbors of the end node of the desired module, and another one the out-neighbors of its source node.

We have illustrated our method by a successful application to the 20-node example.

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Appendix A. THE 20X20 MATRIX

The matrix $G^0(q)$:

$$G^0(q) = \begin{bmatrix} G_1 & G_2 \\ G_3 & G_4 \end{bmatrix}$$

where

$$G_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & G_{2,1} & 0 & 0 & 0 & 0 & G_{2,6} & G_{2,8} & 0 & 0 & 0 \\ 0 & G_{3,1} & 0 & 0 & 0 & 0 & G_{3,5} & G_{3,9} & 0 & 0 & 0 \\ 0 & G_{4,1} & 0 & 0 & 0 & 0 & G_{4,6} & G_{4,8} & 0 & 0 & 0 \\ 0 & G_{5,1} & 0 & 0 & 0 & 0 & G_{5,4} & G_{5,6} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & G_{6,4} & G_{6,5} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & G_{8,1} & 0 & 0 & 0 & 0 & G_{9,6} & G_{9,8} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$G_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & G_{7,12} & 0 & G_{7,14} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & G_{8,12} & G_{8,13} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & G_{9,12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & G_{10,12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$G_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The transfer functions are described below:
\( G_{2,1} = -1.1576491 \times 10^{-01} + 4.2048459 \times 10^{-02} \, q^{-1} \)
\( G_{2,6} = -4.9391907 \times 10^{-01} + 2.0394301 \times 10^{-01} \, q^{-1} \)
\( G_{2,8} = -3.8295603 \times 10^{-01} + 3.7364537 \times 10^{-01} \, q^{-1} \)
\( G_{3,2} = -2.3501597 \times 10^{-01} + 2.2411979 \times 10^{-01} \, q^{-1} \)
\( G_{3,4} = -0.3q^{-1} + 0.8q^{-2} \)
\( G_{3,5} = -0.5q^{-1} \)
\( G_{3,9} = -1.5484356 \times 10^{-01} + 3.5947903 \times 10^{-01} \, q^{-1} \)
\( G_{4,2} = -3.4361929 \times 10^{-01} + 2.7664996 \times 10^{-01} \, q^{-1} \)
\( G_{4,3} = q^{-1} \)
\( G_{4,6} = -4.4565148 \times 10^{-02} + 3.1267256 \times 10^{-02} \, q^{-1} \)
\( G_{4,8} = -3.0217221 \times 10^{-02} + 4.9084253 \times 10^{-01} \, q^{-1} \)
\( G_{5,1} = -4.4755747 \times 10^{-01} + 1.5153359 \times 10^{-01} \, q^{-1} \)
\( G_{5,4} = 0.5q^{-1} \)
\( G_{5,6} = -1.8258082 \times 10^{-02} + 2.6565941 \times 10^{-02} \, q^{-1} \)
\( G_{6,4} = -4.0083967 \times 10^{-02} + 2.3831631 \times 10^{-02} \, q^{-1} \)
\( G_{6,5} = -4.9526830 \times 10^{-02} + 1.8655891 \times 10^{-02} \, q^{-1} \)
\( G_{7,8} = -4.2353188 \times 10^{-02} + 1.7016841 \times 10^{-03} \, q^{-1} \)
\( G_{7,12} = -3.8831215 \times 10^{-01} + 1.6625282 \times 10^{-01} \, q^{-1} \)
\( G_{7,14} = -1.3013545 \times 10^{-01} + 3.2468616 \times 10^{-01} \, q^{-1} \)
\( G_{8,5} = -4.8312501 \times 10^{-01} + 2.9208833 \times 10^{-01} \, q^{-1} \)
\( G_{8,7} = -4.3341455 \times 10^{-02} + 2.6095021 \times 10^{-02} \, q^{-1} \)
\( G_{8,12} = -2.0610019 \times 10^{-01} + 2.6998910 \times 10^{-01} \, q^{-1} \)
\( G_{8,13} = -1.4342078 \times 10^{-02} + 3.4009137 \times 10^{-02} \, q^{-1} \)
\( G_{9,2} = -2.0348115 \times 10^{-01} + 6.7364494 \times 10^{-02} \, q^{-1} \)
\( G_{9,6} = -2.9096829 \times 10^{-01} + 6.1878182 \times 10^{-02} \, q^{-1} \)
\( G_{9,8} = -4.7096177 \times 10^{-01} + 1.6149849 \times 10^{-01} \, q^{-1} \)
\( G_{9,10} = -3.6050395 \times 10^{-02} + 4.1517977 \times 10^{-02} \, q^{-1} \)
\( G_{9,12} = -3.1296376 \times 10^{-02} + 1.1860562 \times 10^{-01} \, q^{-1} \)
\( G_{10,8} = -3.0338765 \times 10^{-01} + 4.1407173 \times 10^{-01} \, q^{-1} \)
\( G_{10,9} = -3.4408597 \times 10^{-02} + 2.3732489 \times 10^{-03} \, q^{-1} \)
\( G_{10,12} = -3.4207005 \times 10^{-02} + 4.4904179 \times 10^{-02} \, q^{-1} \)
\( G_{11,10} = 2.4710993 \times 10^{-01} q^{-1} \)
\( G_{11,12} = 2.4512609 \times 10^{-02} q^{-1} \)
\( G_{11,16} = 2.3010071 \times 10^{-01} q^{-1} \)
\( G_{12,10} = 2.0528463 \times 10^{-02} q^{-1} \)
\( G_{12,11} = 2.1646986 \times 10^{-02} q^{-1} \)
\( G_{12,14} = 2.0877819 \times 10^{-01} q^{-1} \)
\( G_{12,18} = 2.0657294 \times 10^{-01} q^{-1} \)
\( G_{13,8} = 2.1848002 \times 10^{-02} q^{-1} \)
\( G_{13,11} = 2.2137643 \times 10^{-01} q^{-1} \)
\( G_{13,14} = 2.2709971 \times 10^{-02} q^{-1} \)
\( G_{14,13} = 2.2787928 \times 10^{-02} q^{-1} \)
\( G_{14,15} = 2.4854075 \times 10^{-01} q^{-1} \)
\( G_{15,16} = 2.0964010 \times 10^{-01} q^{-1} \)
\( G_{15,18} = 2.1627442 \times 10^{-01} q^{-1} \)
\( G_{16,13} = 2.1592163 \times 10^{-01} q^{-1} \)
\( G_{17,16} = 2.4035419 \times 10^{-02} q^{-1} \)
\( G_{17,18} = 2.3030840 \times 10^{-01} q^{-1} \)
\( G_{17,19} = 2.01838053 \times 10^{-01} q^{-1} \)
\( G_{18,10} = 2.1838053 \times 10^{-01} q^{-1} \)
\( G_{18,17} = 2.2869253 \times 10^{-02} q^{-1} \)
\( G_{19,12} = 2.4800810 \times 10^{-01} q^{-1} \)
\( G_{19,14} = 1.5480058 \times 10^{-01} q^{-1} \)
\( G_{19,18} = 2.382918 \times 10^{-01} q^{-1} \)
\( G_{20,12} = 2.1965134 \times 10^{-01} q^{-1} \)
\( G_{20,13} = 2.3859570 \times 10^{-01} q^{-1} \)
\( G_{20,13} = 2.4280860 \times 10^{-01} q^{-1} \)