INVESTIGATION OF RENORMALIZATION GROUP METHODS 
FOR THE NUMERICAL SIMULATION OF ISOTROPIC 
TURBULENCE

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INTRODUCTION
Over the years, our research into turbulence at Edin-
burgh has concentrated on the application of renormaliza-
tion methods to the prediction of the energy spectrum of 
isotropic turbulence. General discussions of this work will 
be found elsewhere (McComb 1990, 1995), while accounts 
of specific progress have been given previously in this con-
ference series (McComb & Shanmugasundaram 1983, Mc-
Comb, Filipiak, Roberts & Watt, 1991).

From a practical point of view, the most promising de-
velopment in this area is undoubtedly Renormalization 
Group or RG. If we work in the Fourier representation, 
in principle, this involves the progressive averaging out of 
high-wavenumber modes in bands, with rescaling at each 
step, until a fixed point is reached. The result is, in effect, 
a ‘subgrid model’ for large-eddy simulation.

RG has enjoyed its successes in other areas of statistical 
physics. However, its application to turbulence faces sev-
eral technical difficulties, which have to be circumvented by 
working in Fourier-wavevector \( \mathbf{k} \) space and restricting 
oscillating parameter \( \eta \) to the condition \( 0 < \eta < 1 \). Work-
ing with the standard form of the solenoidal Navier-Stokes 
equation in \( k \)-space, we may write the evolution of the low-
k velocity field for \( 0 < k < k_1 \) as

\[
\frac{\partial}{\partial t} + \nu_0 k^2 \right] u^-(\mathbf{k}, t) = \left[ \frac{\partial}{\partial t} + \nu_0 k^2 \right] u^+ \right] u^+ \right) + 2 u^- \left( \mathbf{j}, t \right) u^- \left( \mathbf{k} - \mathbf{j}, t \right) + u^+ \left( \mathbf{j}, t \right) u^+ \left( \mathbf{k} - \mathbf{j}, t \right),
\]

and the evolution of the high-\( k \) velocity field for the first 
shell, \( k_1 < k < k_0 \), as

\[
\frac{\partial}{\partial t} + \nu_0 k^2 \right] u^+ \right) = \left[ \frac{\partial}{\partial t} + \nu_0 k^2 \right] u^+ \right) + 2 u^- \left( \mathbf{j}, t \right) u^- \left( \mathbf{k} - \mathbf{j}, t \right) + u^+ \left( \mathbf{j}, t \right) u^+ \left( \mathbf{k} - \mathbf{j}, t \right)
\]
really an uncontrolled approximation (Eyink, 1994).

In principle, theRG approach involves two stages: (i) Eliminate the high-k modes, \( \mathbf{u}^+ \), which appear in equation (\ref{eq:2}) for \( 0 < k < k_1 \), by solving for the mean effect of the high-k field. This results in an increment to the viscosity, i.e. \( \nu_0 \to \nu_1 = \nu_0 + \delta \nu_0 \). (ii) Rescale the basic variables, so that the Navier-Stokes equation for \( 0 < k < k_1 \) looks like the original Navier-Stokes equation for \( 0 < k < k_0 \).

Although this procedure is appealingly simple and has a clear physical interpretation, it has not proved easy to put into practice in the turbulence problem. A typical approach is to eliminate all the high-k effects in equation (\ref{eq:2}), by substituting the solution of equation (\ref{eq:4}) directly into the \( \mathbf{u}^+ \) modes in the \( \mathbf{u}^- \) equation. However, problems are then encountered because of the mode coupling between \( \mathbf{u}^- \) and \( \mathbf{u}^+ \). Even if one succeeds in carrying out the first part, the further problem of averaging out the high-k modes arises immediately, because \( \mathbf{u}^- \) and \( \mathbf{u}^+ \) are not statistically independent. This problem was avoided by Foster, Nelson and Stephen (1977; hereafter referred to as FNS) in their pioneering study of stirred fluid motion, as they restricted their attention to stirring forces which were multivariate normal and excluded the effects of the turbulence cascade. However, it has been shown that the use of a ‘filtered’ average by FNS to eliminate the \( \mathbf{u}^- \) equation is really an uncontrolled approximation (Eyink, 1994).

**Iterative-Averaging RG with Results**

Here, we follow the method of iterative averaging, which is based upon the derivation of a recurrence relation and, in principle, eliminating finite blocks of modes (i.e. high-k modes) while maintaining the form invariance of the dynamical equation. Apart from the work of FNS, elimination procedures can be performed by ‘conditional’ averaging, first introduced by McComb (1982). Further details about the conditional average have been given elsewhere (McComb, Robert and Watt, 1992). The basic ansatz of a conditional average is that a small uncertainty (\( \Phi^- \), say) at the cutoff wavenumber will generate chaotic behaviour for the high-k modes. Although the introduction of \( \Phi^- \) has been accepted, mainly due to the chaotic nature of the Navier-Stokes equations, it might be interesting to see how ‘rapidly’ chaotic behaviour develops from the given small

\[
\begin{align*}
\frac{\partial}{\partial t} + 2u^-_\beta(j,t)u^+_{\beta}(k-j,t) + u^+_\beta(j,t)u^-_{\beta}(k-j,t) \overline{\mathbf{u}^+} = & + (4)
\end{align*}
\]

where the superscripts + and − on \( M_{\alpha,\beta}(k) \) have the same significance as for \( u_\alpha(k,t) \), and the symmetrized inertial transfer operator \( M_{\alpha,\beta}(k) = (2i)^{-1} [i\bar{\gamma}D_{\alpha\beta}(k) + k_1D_{\alpha\beta}(k)] \).

\[
\text{In principle, the RG approach involves two stages: (i) Eliminate the high-k modes, } \mathbf{u}^+, \text{ which appear in equation (4) for } 0 < k < k_1, \text{ by solving for the mean effect of the high-k field. This results in an increment to the viscosity, i.e. } \nu_0 \to \nu_1 = \nu_0 + \delta \nu_0. \text{ (ii) Rescale the basic variables, so that the Navier-Stokes equation for } 0 < k < k_1 \text{ looks like the original Navier-Stokes equation for } 0 < k < k_0.\]

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\[
\begin{align*}
\Phi^- \text{ by numerical simulation. This aspect is one of our current tasks and the results will be reported in due course.}
\end{align*}
\]

The current result of the iterative-averaging calculation for the Navier-Stokes equations after first eliminating the high-k effects is

\[
\begin{align*}
\left[ \frac{\partial}{\partial t} + \nu_1 k^2 \right] u^-_\alpha(k,t) &= M^-_{\alpha,\beta}(k) \int d^3j \ u^+_\beta(j,t)u^-_{\beta}(k-j,t), \quad (5)
\end{align*}
\]

where \( \nu_1 = \nu_0 + \delta \nu_0(k) \) and

\[
\begin{align*}
\delta \nu_0(k) &= -\frac{1}{k^2} \int d^3j \ Q^+_{\beta}(j) \times \frac{\Delta}{\omega \times \Gamma} \text{Tr} \left[ M^-_{\alpha,\beta}(k) M^+_{\gamma,\sigma}(k-j) D_{\alpha\sigma}(j) \right] \frac{\nu_0^2 + \nu_0 |k-j|^2}{\nu_0^2 + \nu_0 |k-j|^2}. \quad (6)
\end{align*}
\]

Here, we consider space dimension \( d = 3 \). This result can be extended to further shells, and we have shown elsewhere (McComb and Watt, 1992) that a fixed point is reached under numerical iteration of the recursion relations (see also Figure 4). In Figure 5, we show a calculation of the Kolmogorov constant \( \alpha = 1.60 \pm 0.01 \) independent of the bandwidth of modes being eliminated for bandwidths in the range \( 0.25 \leq \eta \leq 0.45 \), in agreement with experiment.

**NUMERICAL SIMULATIONS**

Two programmes of numerical simulation are being carried out — one at the University of Edinburgh in the United Kingdom, the other at the Swiss Federal Institute of Technology, Lausanne. A large number of runs have already been carried out at Lausanne, and this paper presents some of the results obtained so far.

The simulations themselves are very similar, while the computer systems on which they are run differ greatly. At Edinburgh, work is carried out on a parallel machine, the Cray T3D, while in Lausanne a parallel-vector machine, the NEC SX-4, is used.

The simulations discussed in this paper were carried out at a resolution of 256\(^4\), requiring approximately 14 seconds of SX-4 time per time-step on a single processor.

The general method of such simulations has been well established. We follow the work of Orszag for the construction of initial velocity fields (1969) and in the use of a pseudospectral method (1971). The time integration scheme is a second-order Runge-Kutta method and partial dealiasing is achieved by way of a random-shifting method (see, for example, Rogallo, 1981).
| \( \Delta t \) | \( T \) | \( \nu_0 \) | \( k_f \) | \( k_0 \) |
|-------|-------|-------|-------|-------|
| \( 10^{-3} \) | 113.5 | \( 10^{-2} \) | 1.5 | 120 |

| \( \varepsilon \) | \( R_s \) | \( L \) | \( \lambda \) | \( \tau_E \) | \( s_3 \) | \( s_4 \) |
|-------|-------|-------|-------|-------|-------|-------|
| \( .15 \) | 190.066 | 1.431 | .246 | 1.853 | \( - .51 \) | 6.053 |

Table 1: Characteristics of the simulation

**Initial Conditions**

The simulations are started with an initial energy spectrum of the form

\[
E(k,0) = 16(2/\pi)^2 u_0^2 k_p^{-5} k^4 \exp[-2(k/k_p)^2]
\]  

(7)

where \( k_p \) is the location of the spectrum's maximum and \( u_0 \) is the required initial r.m.s. velocity.

**Forcing**

Stationary turbulence is obtained by use of a deterministic forcing term

\[
f_\varepsilon(k,t) = \begin{cases} \varepsilon u_\alpha(k,t)/(2E_f(t)) & \text{if } 0 < k < k_f, \\ 0 & \text{otherwise,} \end{cases}
\]  

(8)

where \( \varepsilon \) is the mean dissipation rate, and

\[
E_f(t) = \int_0^{k_f} E(k,t)dk.
\]  

(9)

There is no preferred direction in this forcing and the turbulence rapidly reaches a statistically isotropic and steady state.

**Statistics**

While our simulations are entirely conventional, we do not rely solely on the usual practice (as justified by isotropy) of averaging over shells in wavenumber space in order to obtain statistical quantities, but also generate many realizations in order to increase our sample size.

The main characteristics of the simulation are reported in Table 1 where \( \Delta t \) is the time step, \( T \) is the integration time, \( \nu_0 \) is the molecular viscosity, \( k_f \) is defined in (1), \( k_0 \) is the ultraviolet cut-off, \( \varepsilon \) is the mean dissipation rate, \( R_s \) is the Reynolds number based on the Taylor microscale, \( L \) is the integral scale, \( \lambda \) is the Taylor microscale, \( \tau_E \) is the turnover time and \( s_3 \) and \( s_4 \) are respectively the skewness and flatness of the velocity derivative.

The equations have been integrated for more than 60 turnover times and about 200 box-realizations of each component of the velocity field have been stored in a database. Since these box-realizations are separated by \( \approx \tau_E/4 \) they can be considered statistically independent for the middle-range-scales and the small-scales.

**RESULTS**

We wish to assess the freedom to carry out conditional averages of the type required by RG. In principle we may do this by extracting, from an ensemble of realizations of the velocity field

\[
\mathcal{X} = \{X^{(n)}_\alpha(k,t) | \alpha = 1, 2, 3; t \in [0,T]; \ 0 \leq |k| \leq k_0; n = 1, ..., N\}, \ 
\]  

(10)

two disjoint subensembles \( \mathcal{Y} \) and \( \mathcal{Z} \) chosen such that, for a prescribed \( \zeta > 0 \),

\[
\frac{|Y^{(m)}(k,t) - Z^{(m)}(k,t)|^2}{2|Y^{(m)}(k,t)|^2} \leq \zeta
\]  

for all \( 0 \leq |k| \leq k_c ; m = 1, ..., M ; t \in [0,T] \),

(11)

for each realization \( Y^{(m)} \in \mathcal{Y} \) and \( Z^{(m)} \in \mathcal{Z} \). We may then define the relative energy of the error

\[
r(|k|) = \frac{(|u(k,t) - w(k,t)|^2)}{2(u(k,t)^2)}
\]  

(12)

for all \( k \geq k_0 \).

In order to this, we have performed the following partial Fourier transform of one component of the velocity field

\[
u_\alpha(x,y,k) = \frac{1}{2\pi} \int u_\alpha(x,y,z)e^{ikz} \, dk,
\]  

(13)

then we have selected, for each box-realization, a set of realizations, say \( u_\alpha(x_i,y,k) \), where the spacing \( \delta x = |x_{i+1} - x_i| = |y_{i+1} - y_i| \) is chosen such that the realizations are (approximately) independent for the range of \( k \) we consider (if we consider only the scales such that \( k \geq k_0 \), then \( \delta x = 2\pi/k_b \)). The union of all these realizations obtained for each of the box-realizations will constitute our whole ensemble \( \mathcal{X} \). The subensemble \( \mathcal{Y} \) is formed by choosing an arbitrary subensemble of \( \mathcal{X} \). To select the subensemble \( \mathcal{Z} \), we impose the condition

\[
\frac{|Y^{(m)}(k,t) - Z^{(m)}(k,t)|^2}{2|Y^{(m)}(k,t)|^2} \leq \zeta
\]  

for all \( k_b \leq k \leq k_c ; m = 1, ..., M \).

(14)
Note that the time dependence does not appear in the equations since all the box-realizations used to form the ensemble $X$ are taken in the statistically steady regime. Figure 3(a) shows the relative energy error

$$r(k) = \frac{\langle (u(k) - w(k))^2 \rangle}{2\sigma(u(k))^2},$$

(15)

where $u \in \mathcal{Y}$ and $w \in \mathcal{Z}$ for $k_b = 10$, $k_c = 15$, $\zeta = 0.5$ and $\alpha = 1$. The number of realizations $M$ is 2533. Though the number of realizations is not large enough to have a smooth converged solution, one can see that the relaxation to a chaotic regime is indeed very fast. Figure 3(b) shows a selected set of realizations for which one can observe that the constraint imposed for $10 \leq k \leq 15$ does not prevent strong fluctuations for $k \geq 15$. The convergence of $r(k)$ is difficult to improve, due to the restriction on the number of realizations available for a given constraint.

Another natural way in which the small-scale properties of a conditional subensemble may be investigated is by studying the probability density functions (pdfs) of velocity increments. In physical-space, we can use homogeneity in the three dimensions and have sufficiently large subensembles to compute high-order statistics and pdfs. The velocity increments are defined by the following relation

$$\delta u(x, h) = u(x + h) - u(x),$$

(16)

where $h$ is a displacement vector and $x$ the position. Since the fields are statistically isotropic, we can restrict ourselves to the study of the longitudinal velocity increment $\delta v_L(h)$ which is the projection of $\delta u(h)$ on the direction of the vector $h$ and the lateral velocity increment $\delta v_h(h)$ which is the projection of $\delta u(h)$ on a direction perpendicular to $h$. For the purpose of this paper, we have only studied the longitudinal velocity increment $\delta v_L(h)$. We have selected two scales, $b_1 = \lambda/1.50$ and $b_2 = \lambda/1.26$ ($\lambda$ is the Taylor micro-scale, therefore $b_1$ is a typical scale in the inertial subrange and $b_2$ is in the dissipation subrange). The selection of the subensembles is performed using conditions of the type $a < \delta v_L(h_1) < b$. The pdfs of $\delta v_L(h_2)$ for the unconditional ensemble and for the subensembles are then compared. Figure 4 gives the normalized pdf ($\sigma$ is the standard deviation of $\delta v_L(h)$) of the unconditional ensemble for $b = b_1$ and $b = b_2$. We observe the classical result that the tails of the pdfs are growing as the scale is decreased which is the signature of growing intermittency. The pdf also shows a negative skewness which is a direct consequence of the nonlinear dynamics of the Navier-Stokes equations. Figure 5 shows the pdfs of the unconditional ensemble and of a subensemble defined by the constraint $-1 < \delta v_L(h_1) < 0$. The pdfs are almost superimposed, showing that the flow at scale $b_2$ is unaffected by the condition imposed at scale $h_1$. Figure 6 is a case for which the subensemble is much smaller due to a more restrictive condition, $1 < \delta v_L(h_1) < 4$. However, the general behavior of the pdf supports the view that the chaotic dynamics of the Navier-Stokes equations tends to restore the original distribution. Note that the skewness is incorrectly predicted and seems to be correlated with the sign of $\delta v_L(h_1)$. Figure 6 presents a case with a very strong condition, $-7 < \delta v_L(h_1) < -2$. Though the number of realizations is small, we observe that the top of the pdf is quite accurately reproduced.

**CONCLUSION**

These results, although preliminary in nature, offer crucial support to the hypothesis that a conditional average may be used to reduce the number of degrees of freedom required for the numerical simulation of turbulence. Work is continuing to make a more stringent assessment of the validity of such averages for turbulence and this includes...
carrying out simulations at higher numerical resolution. At present we are working on a $512^3$ simulation and hope to present results from this at the conference.

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