Generalising Aumann’s Agreement Theorem

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Aumann’s celebrated theorem says that a group of agents who once shared a common prior probability distribution cannot assign different posteriors to a given proposition, should these agents have common knowledge about their posteriors. In other words, rational agents cannot agree to disagree. Aumann’s agreement theorem was one of the first attempts to formalise and explore the role played by common knowledge in decision theory. Recently, we have seen a resurfacing of the debate around possible (quantum) extensions of Aumann’s results. This paper contributes to this discussion. First, as expected, we argue that agreeing to disagree is impossible in quantum theory. Secondly, and based on the quantum argument, we show that agreeing to disagree is also forbidden in any generalised probability theory.

I. INTRODUCTION

Agents cannot agree to disagree. At least that is the condensed version of J. R. Aumann’s intriguing and seminal result [1]. According to Aumann’s theorem, whenever a family of agents reach common knowledge about an event, having started from the same prior, there is no escape, they all have to agree with each other about the description of that event. We unpack the assumptions and hypotheses underlying the theorem in the sections to come. For now, it suffices to keep the simpler and condensed version in mind: rational agents cannot agree to disagree.

Aumann’s impossibility theorem is provocative for various reasons. Firstly, it explores the close connection between two intricate, imbricated and paradigmatic concepts in epistemology, that of knowledge [2] and relativism [3–5]. Basically, the theorem says that a strong notion of shared knowledge in a group implies that agents in that group must paint their world with the same colour. Some strands of relativism get immediately ruled out whenever common knowledge holds true. Secondly, the impossibility theorem opens room for distinguishing rational agents from non-rational agents. After many rounds of truthful exchanging of information, if the involved agents are still in disagreement, it may be the case that at least one of the persons involved in the discussion is not acting rationally [6]. In this sense, rather than using a betting system [7], one could, in theory, use a collective property to define rationality normatively. Finally, one may suspect Aumann’s theorem is only valid because classical probability is a coarse-grained description of nature - and that generalised probability theories may unlock other degrees of freedom that allow for escaping the theorem.

The agents were permitted to reason using more powerful resources, it might be the case that they could finally agree to disagree. It suffices to recall the old legend that goes by saying that entanglement is a trick quantum magicians use to produce phenomena that classical magicians cannot imitate.

In this work we explore extensions to the impossibility of agreeing to disagree in hybrid, post-classical scenarios. We want to show that Aumann’s intriguing theorem is not an exclusivity of classical probability theory. We will prove that the theorem carries over to any generalised probabilistic description of the nature, provided that a notion of joint state as well as a notion of conditioning are well-defined. Our hybrid scenarios mix post-classical descriptions with a classical definition for knowledge - for what is required for an agent to know that something. As will become clear later, in such scenarios, it is the set-theoretical notion of common knowledge that does all the heavy-lifting in proving the impossibility theorem.

Our manuscript is structured as follows. To facilitate the reading, in Sec. II, we review Aumann’s original argument for the impossibility of agreeing to disagree - although there are more modern approaches to Aumann’s impossibility result, we decided to stick to the original argument as it demands less in a first reading. Sec. III contains our first main result. There we prove that a hybrid quantum version of the agreement theorem also holds in a quantum-like scenario. We also introduce all the elements we need for the proof in the said section. Inspired by the quantum reformulation, we give a similar argument for GPTs. In other words, in Sec. IV, we show that Aumann’s theorem can be extended to generalised probability theories. Sec. V clarifies the main limitations and hypotheses we have used throughout this contribution. We conclude our work in Sec. VI. There we compare our findings with other similar results in the literature, and hint at possible future works.

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II. AUMANN’S ORIGINAL ARGUMENT

This section reviews Aumann’s original argument for the impossibility of agreeing to disagree. Our main aim consists in introducing the theorem and highlighting the mathematical formalism involved in the agreement result, as the latter will be preponderant in the subsequent sections. We do not intend to provide a comprehensive account of the agreement theorem, nor do we want to present an in-depth discussion of all the subsequent works [8, 9], modifications [10–12] and debates [13, 14] that have followed-up Aumann’s seminal work. We refer to [15] for an in-depth overview of the topic.

Aumann is responsible for one of the first attempts to rigorously explore the notion of common knowledge in decision theory [16]. In plain words, Aumann’s agreement theorem [1] says that whenever a group of agents had agreed in the past, if their current description of a proposition is common knowledge among them, then their descriptions must match each other’s. Slightly more precisely, if two agents and have the same priors, it is impossible that it is common knowledge among the agents that assigns to some event a probability and assigns to the same event a probability with . Statically, one can write down the main logical character of Aumann’s impossibility theorem as follows:

\begin{align}
\text{[equalpriors} & \land C(\text{posteriors})\text{]} \rightarrow \text{equalposteriors,} \quad (1)
\end{align}

where plays a role of common knowledge operator.

To fully appreciate the agreement theorem, its implications and the generalisations we want to carry out here, we need to go through a few definitions first. We need a formal mathematical background where one can talk about knowledge and common knowledge - that is another influential by-product from [1]. It will become clear that Aumann’s theorem follows directly from this mathematical model for knowledge we pass to describe.

**Definition II.1 (Knowledge Model).** A knowledge model for agents is a structure consisting of

(i) a non-empty set of states of the world,

(ii) a partition \( Q_i \) of \( \Omega \) for each \( i \in [N] \), and

(iii) a \( \sigma \)-algebra \( \Sigma \) over \( \Omega \) that includes all the elements of \( Q_1, Q_2, ..., Q_N \).

We usually denote \( Q_i(\omega) \), or \( Q_i \omega \), as that unique element of the partition \( Q_i \) containing \( \omega \in \Omega \).

**Remark:** A knowledge model \( (\Omega, Q_1, Q_2, ..., Q_N, \Sigma) \) mimics a situation where individuals are about to learn the answer to various questions - possibly through observations. The answers for agent i’s questions is codified in the partition \( Q_i = \{ Q^1_i, Q^2_i, ..., Q^{M(i)}_i \} \) representing mutually exclusive, collectively exhaustive propositions. See fig. 1.

**Definition II.2 (Knowledge).** Let \( (\Omega, Q_1, Q_2, ..., Q_N, \Sigma) \) be a knowledge model. We say that an agent knows \( E \in \Sigma \) at \( \omega \) whenever \( Q_i(\omega) \subseteq E \). The \( i \)-th knowledge operator for this model is defined as

\begin{align}
K_i : \Sigma \rightarrow \sigma \\
E \mapsto K_i(E) = \{ \omega \in \Omega | Q_i(\omega) \subseteq E \}. \quad (2)
\end{align}

Considering the reasoning of only one agent, in a knowledge model, their knowledge about an event is the set of all possible states of the world where that agent knows \( E \). The agreement theorem verses about a more refined notion that involves multiple agents - that of common knowledge. To get there, first we need to define an infinite hierarchy of group or mutual knowledge [17]. Common knowledge will be the limiting case of such nested definition.

**Definition II.3 (Mutual and Common Knowledge).** Let \( (\Omega, Q_1, Q_2, ..., Q_N, \Sigma) \) be a knowledge model and \( E \) an arbitrary event in \( \Sigma \). The mutual knowledge of \( (m+1) \)-th degree on \( E \) is recursively defined as:

\begin{align}
0-\text{th} : M_0(E) := E \\
1-\text{th} : M_1(E) := K_1(M_0(E)) \cap K_2(M_0(E)) \cap ... \cap K_N(M_0(E)) \\
2-\text{th} : M_2(E) := K_1(M_1(E)) \cap K_2(M_1(E)) \cap ... \cap K_N(M_1(E)) \\
\vdots
\end{align}

\begin{align}
(m+1)-\text{th} : M_{m+1}(E) := \bigcap_{i=1}^{N} K_i(M_m(E)). \quad (3)
\end{align}

Similarly, common knowledge is defined as

\begin{align}
C(E) := \bigcap_{m \in \mathbb{N}} M_m(E). \quad (4)
\end{align}

**Remark:** For simplicity, consider the situation involving only two agents. In this case, mutual knowledge of
second order two things. First, A knows that she knows E and that B also knows E. Second, that B knows that he knows about E and that A also knows E. What is crucial here is how mutual knowledge (of a finite degree) differs from common knowledge. Common knowledge between two agents means that a an infinite list of A knows that B knows that A knows that and so on holds true.

As thoroughly discussed in [14], common knowledge is the key concept in Aumann’s theorem. In the knowledge model we are dealing with here, the possibility of existing a set upon which the agents commonly know of puts strong constraints on the possible partitions allowed by the model. In opposition to the more dynamic of puts strong constraints on the possible partitions existing a set upon which the agents commonly know edge model we are dealing with here, the possibility of making this affirmation more precise.

Let \( D \) be a knowledge model and \( E \) an event. If \( C(E) \neq \emptyset \), then for each \( i \in [N] \) there exists a finite family \( \{D_i^1, \ldots, D_i^K\} \) such that \( C(E) = \bigcup_{i \in [K]} D_i^j \).

\[ \implies \omega \in M_m(E), \forall m \in \mathbb{N} \]
\[ \implies \omega \in E, Q_i(\omega) \subseteq E, Q_i(\omega) \subseteq K_i(E) \cap \ldots \cap K_N(E), \ldots \]
\[ \implies Q_i(\omega) \subseteq M_m(E), \forall m \implies Q_i(\omega) \subseteq C(E). \quad (5) \]

As we are considering only a finite set of states of the world, \( C(E) \) is also finite. In this case, with no loss of generality we can assume that \( C(E) = \{e_1, \ldots, e_{|E|}\} \). Now, note that \( \{Q_i(e_1), \ldots, Q_i(e_{C(E)})\} \) has at most \(|C(E)| \) disjoint elements, and that it is exactly its non-repeating elements we will use to form the set \( \{D_1^j, \ldots, D_K^j\} \).

**Remark:** As we anticipated in the paragraph above the lemma 1, the existence of a common knowledge in the knowledge model puts strong restrictions on the structure of the outcome set \( Q_1, \ldots, Q_N \) - or on the way around, only a reduced (if any) events can give rise to common knowledge between agents. Also, recalling def. II.2, note that the equality demanded by lemma 1, imposes that the common knowledge set is also known by all of the agents in the model.

Now that we have defined what common knowledge is, and how it can be characterised in terms of the partitions \( Q_1, \ldots, Q_N \) in a knowledge model, we have all ingredients to address Aumann’s agreement theorem.

**Theorem 1** (Aumann’s Thm). Let \( (\Omega, Q_1, Q_2, \ldots, Q_N, \Sigma) \) be a knowledge model and \( P : \Sigma \rightarrow [0,1] \) be a probability function over \( \Omega \). Define

\[ E := \bigcap_{i \in [N]} \{ \omega \in \Omega | P(H|Q_i(\omega)) = q_i \}, \quad (6) \]

with \( H \in \Sigma \) and \( q_1, q_2, \ldots, q_N \in [0,1] \). If \( P(C(E)) \neq \emptyset \), then

\[ q_1 = q_2 = \ldots = q_N = P(H|C(E)). \quad (7) \]

**Proof.**

\[ \frac{P(H|C(E)) = \frac{P(H \cap C(E))}{P(C(E))}}{= \frac{P(H \cap \bigcup_{j \in [K]} D_j^j)}{P(D_j^j)}} = \frac{\sum_{j \in [K]} P(H|D_j^j)P(D_j^j)}{\sum_{j \in [K]} P(D_j^j)} = q_i. \]

As the argument is valid for each agent \( i \), it follows that

\[ q_1 = q_2 = \ldots = q_N = P(H|C(E)). \quad (8) \]

**Remark:** In plain English, Aumann’s theorem says that for a given proposition \( H \), to which all the agents want to talk about, if the individual probabilities each agent assign to \( H \) is common knowledge, then these individual probabilities must be the same - even though they have been obtained by different Bayesian updates and due to different observations.

The event \( E \) defined in eq. (6) depends on the particular choice of \( H \in \Sigma \), and so \( q_1, q_2, \ldots, q_N \) is a slight abuse of notation. It would have been more precise if we have written \( E(H) \) for the event in (6), and \( q_1(H), \ldots, q_N(H) \) for the list of agent’s probabilities. In any case, the agreement theorem also holds true if we make this functional dependence more explicit - it follows a direct consequence of Aumann’s original setup.

**Theorem 2** (Aumann’s Thm - Second Version). Let \( (\Omega, Q_1, Q_2, \ldots, Q_N, \Sigma) \) be knowledge model and \( P : \Sigma \rightarrow [0,1] \) be a probability function over \( \Omega \). For each \( H \in \Sigma \), define:

\[ E_H := \bigcap_{i \in [N]} \{ \omega \in \Omega | P(H|Q_i(\omega)) = q_i(H) \}, \quad (9) \]

with \( q_1, q_2, \ldots, q_N : \Sigma \rightarrow [0,1] \). If for each \( H \) in \( E_H \), then

\[ q_1 = q_2 = \ldots = q_N = P(H|C(E_H)). \quad (10) \]
Thm. 2 is a stronger formulation of Aumann’s original theorem. Nonetheless, it still says that agents with the same priors cannot agree to disagree upon common knowledge of their posteriors. We have only made the functional dependence on \( H \) more explicit - as if Aumann’s theorem were valid for every \( H \) event [19]. We have only done so because this is exactly the format that creates the best parallel with the generalisations of the agreement theorem that we pass now to discuss.

### III. QUANTUM VERSION OF THE AGREEMENT THEOREM

This section presents a hybrid quantum version for the agreement theorem. We prove that even if the agents were allowed to use quantum systems as their probabilistic descriptions, it is also true that common knowledge of their quantum posteriors will always lead to an impossibility of agreeing to disagree. Additionally, the present section also turns explicit what is the structure in Aumann’s argument that is central not only for the original result but also for any potential generalisation of it.

That our version is a hybrid quantum version comes from the fact that we are still keeping intact the classical knowledge model of def. II.1. To a certain extent, we are dotting the agents the possibility of expanding their reasoning abilities, as they are allowed to go beyond standard probability theory and use density operators valued measures (DOVMs) as their toolbox of analysis - even though they are using these tools to deal classical data/classical set-theoretical models. We start defining what a DOVM is.

**Definition III.1 (DOVM).** Let \( \mathcal{M} = (\Omega, \Sigma) \) be a measurable space and \( \mathcal{H} \) a finite dimensional Hilbert space. A density operator valued measure (DOVM) over \( \mathcal{M} \) is a map \( \rho : \Sigma \rightarrow \mathcal{L}(\mathcal{H}) \) such that:

(i) \( \rho(\Omega) \) is a density operator;

(ii) \( \rho(\Lambda) \geq 0 \), for all \( \Lambda \in \Sigma \);

(iii) \( \rho \left( \bigcup_{j \in J} \Lambda_j \right) = \sum_{j \in J} \rho(\Lambda_j) \), for any countable family \( \{ \Lambda_j \}_{j \in J} \) of disjoint subsets in \( \Sigma \).

**Definition III.2 (Conditional State).** Let \( \rho : \Sigma \rightarrow \mathcal{L}(\mathcal{H}) \) be a DOVM. For each \( \Lambda \in \Sigma \), we define the object

\[
\rho|\Lambda := \frac{\rho(\Lambda)}{\text{Tr}[\rho(\Lambda)]}
\]

as the conditional state of the DOVM \( \rho : \Sigma \rightarrow \mathcal{L}(\mathcal{H}) \) with respect to the event \( \Lambda \).

The following two propositions help recast the notion of a DOVM in a more well-established manner.

**Proposition III.3.** Given a DOVM \( \rho : \Sigma \rightarrow \mathcal{L}(\mathcal{H}) \), the following map is a POVM on \( \text{supp}(\rho(\Omega)) \):

\[
E : \Lambda \mapsto \rho(\Omega)^{-1/2} \rho(\Lambda) \rho(\Omega)^{-1/2}
\]

**Proof.** Without lost of generality, we will assume that \( \rho(\Omega) \) has full rank. The fact that \( E \) is positive and that \( E(\Omega) = 1 \) follows directly from the definition:

\[
\langle \psi, E(\Lambda) \psi \rangle = \langle \psi, \rho(\Omega)^{-1/2} \rho(\Lambda) \rho(\Omega)^{-1/2} \psi \rangle = \langle \rho(\Omega)^{-1/2} \psi, \rho(\Lambda) \rho(\Omega)^{-1/2} \psi \rangle \geq 0, \forall \psi \in \mathcal{H}.
\]

\[
E(\Omega) = \rho(\Omega)^{-1/2} \rho(\Lambda) \rho(\Omega)^{-1/2} = 1
\]

**Proposition III.4.** Given a POVM \( E : \Sigma \rightarrow \mathcal{L}(\mathcal{H}) \) and a state \( \sigma \), the mapping

\[
\rho(\Lambda) = \sigma^{1/2} E(\Lambda) \sigma^{1/2}
\]

is a DOVM.

**Proof.** The proof of this proposition is analogous to the previous one - it is a direct consequence of the definitions. As a matter of fact,

\[
\langle \psi, \rho(\Lambda) \psi \rangle = \langle \psi, \sigma^{1/2} E(\Lambda) \sigma^{1/2} \psi \rangle = \langle \sigma^{1/2} \psi, E(\Lambda) \sigma^{1/2} \psi \rangle \geq 0.
\]

\[
\text{Tr}[\rho(\Omega)] = \text{Tr}[\sigma^{1/2} E(\Omega) \sigma^{1/2}] = \text{Tr}[\sigma^{1/2} \mathbb{1} \sigma^{1/2}] = \text{Tr}[\sigma] = 1.
\]

\[
\rho \left( \bigcup_{j \in J} \Lambda_j \right) = \sigma^{1/2} E \left( \bigcup_{j \in J} \Lambda_j \right) \sigma^{1/2} = \sigma^{1/2} \sum \sigma(\Lambda_j) \sigma^{1/2} = \sum \rho(\Lambda_j)
\]

The next theorem comprises our hybrid quantum generalisation of the agreement theorem. Its proof is entirely based on Aumann’s original argument, specially on the lemma 1 - we will comment more on this aspect later on.

**Theorem 3 (Quantum Agreement Theorem).** Let \( (\Omega, Q_1, Q_2, ..., Q_N, \Sigma) \) be a knowledge model and \( \rho : \Sigma \rightarrow \mathcal{L}(\mathcal{H}) \) be a DOVM over \( \Omega \). Define

\[
E := \bigcap_{i \in [N]} \{ \omega \in \Omega | \rho_{C(\omega)} = \sigma_i \},
\]

with \( \sigma_i \) a density operator acting on \( \mathcal{H} \). If \( \text{Tr}[\rho(C(E))] \neq 0 \), then

\[
\sigma_1 = \sigma_2 = ... = \sigma_N = \rho_{C(E)}.
\]
Proof.

\[ \rho_{|C(E)} = \frac{\rho(C(E))}{\text{Tr}[\rho(C(E))]} = \frac{\rho\left(\bigcup_{j \in [K]} D_j^i\right)}{\text{Tr}\left[\rho\left(\bigcup_{j \in [K]} D_j^i\right)\right]} \]

\[ = \frac{\sum_{j \in [K]} \rho(D_j^i)}{\text{Tr}\left[\sum_{j \in [K]} \rho(D_j^i)\right]} = \rho \cdot \left(\frac{\sum_{j \in [K]} \rho(D_j^i)}{\text{Tr}\left[\rho(D_j^i)\right]}\right) \]

\[ = \frac{\sum_{j \in [K]} \text{Tr}\left[\rho(D_j^i)\right]}{\sum_{j \in [K]} \text{Tr}\left[\rho(D_j^i)\right]} = \sigma_i. \]

(20)

As the choice of index \( i \in [N] \) has been entirely arbitrary, we can conclude that

\[ \sigma_1 = \sigma_2 = ... = \sigma_N = \rho_{|C(E)}. \]

\[ \square \]

Again, we insist on the hybrid character of our quantum version. In our theorem, instead of assigning classical probability distributions to events (or propositions), the agents are allowed to do presumably better and use a richer mathematical object - a DOVM. Still, we are using the very same classical knowledge model, based on set-theoretical assumptions, of def. II.1 as a proxy for each agent’s enquiring/learning model. The latter is exactly the reason why Aumann’s argument carries over to the quantum case.

In sum, although we have granted the agents an allegedly more resourceful object to reason about events (or propositions), they are still constrained by the impossibility of agreeing to disagree - in the presence of common knowledge. The next section shows that differently from what has been recently argued in the literature [20], agents with access to beyond quantum resources are also submitted to Aumann’s impossibility theorem - in this sense, the agreement theorem cannot be seen as a physical principle separating out certain classes of generalised probability theories.

### IV. AUumann’S THEOREM IN GTPS

The next step in our argument consists of showing that Aumann’s agreement theorem is also valid almost regardless of the underlying probabilistic theory one is working with. Our take-home message is that whenever such a theory is well-behaved concerning exclusive events and comes equipped with a well-defined notion of conditional state, then the agreement theorem must hold. To show the extension of our argument, we will base our discussion on what goes by the name of generalised probabilistic theories (GPT) - a powerful mathematical tool to study foundations of (quantum) physics [21]. In sum, this section shows how to formulate Aumann’s theorem in GPTs.

To facilitate the reading we start our discussion with a brief review of what we mean by a generalised probability theory. Then we proceed with two more definitions: state valued measure (SVM) and a conditional state. We conclude this section with our version of the agreement theorem.

**Definition IV.1 (Generalised Probability Theory).** A finite-dimensional generalised probability theory (GPT) consists of a pointed convex cone \( V^+ \) in a finite-dimensional vector space \( V \); a unit functional \( u : V^+ \to [0,1] \); a state space \( S \subseteq V^+ \) defined via

\[ S := \{ \mu \in V^+ | u(\mu) = 1 \}; \]

(22)

a set of effects \( \mathcal{E}(S) \subseteq (V^+)^* \) defined via:

\[ \mathcal{E}(S) := \{ \varphi : V^+ \to \mathbb{R} | \varphi \text{ is linear and } 0 < \varphi < u \}; \]

(23)

and finally a list of observables from \( \mathcal{E}(S) \).

**Definition IV.2 (SVM).** Let \( M = (\Omega, \Sigma) \) be a measurable space and \( (V, V^+, u) \) be an arbitrary GPT. A state valued measure (SVM) over \( M \) is a measurable function \( \mu : \Sigma \to V^+ \) satisfying:

(i) \( \mu(\Omega) \in S \);

(ii) \( \mu(\Lambda) \in V^+ \), for all \( \Lambda \in \Sigma \);

(iii) \( \mu(\bigcup_{j \in J} \Lambda_j) = \sum_{j \in J} \mu(\Lambda_j) \) for any countable family \( \{\Lambda_j\}_{j \in J} \) of disjoint subsets in \( \Sigma \).

**Definition IV.3.** Let \( \mu : \Sigma \to V^+ \) be an SVM over a measurable space \( M = (\Omega, \Sigma) \). For each \( \Lambda \in \Sigma \), we define the object

\[ \mu|_{\Lambda} := \frac{\mu(\Lambda)}{u[\mu(\Lambda)]} \]

(24)

as the conditional state of the SVM \( \mu : \Sigma \to V^+ \) with respect to the event \( \Lambda \).

At this stage, it should be clear where we are heading to. Recall that two main things were central to the machinery involved in proving the agreement theorem: for one, we needed a notion of (quantum) probability defined over the measure space that is part of the basal knowledge model - and more importantly, that this notion of probability acted as if it were an affine function on exclusive events. Secondly, we needed a meaningful definition of a conditional state so that the conditioning process is well-defined and that the law of total probability can also be worked out. Naturally, Aumann’s original framework possesses these two aspects inherently. On the other hand, it is defs. III.1 and III.2 that grant quantum theory with the said central features. Similarly, as we see below, the def. IV.2 and IV.3 play the same fundamental role in our version of the agreement theorem for GPTs.
Theorem 4 (Agreement Theorem in GPTs). Let $(\Omega, Q_1, Q_2, ..., Q_N, \Sigma)$ be a knowledge model and $\mu : \Sigma \to V^+$ be an SVM over $\Omega$. Define

$$E := \bigcap_{i \in [N]} \{ \omega \in \Omega | \mu_{|Q_i}(\omega) = \mu_i \},$$

where each $\mu_i$ is a state in $\mathcal{S}$. If $u[\mu(C(E))] \neq 0$, then

$$\mu_1 = \mu_2 = ... = \mu_N = \mu_{|C(E)}.$$  

Proof.

$$\mu_{|C(E)} = \frac{\mu(C(E))}{u[\mu(C(E))]} = \frac{\mu \left( \bigcup_{j \in [K]} D^j \right)}{u \left[ \mu \left( \bigcup_{j \in [K]} D^j \right) \right]}$$

$$= \frac{\sum_{j \in [K]} \mu(D^j)}{u \sum_{j \in [K]} \mu(D^j)} = \frac{\sum_{j \in [K]} \mu_{|D^j} \left[ \mu(D^j) \right]}{\sum_{j \in [K]} \mu(D^j)}$$

$$= \frac{\mu_i}{\sum_{j \in [K]} \mu(D^j)}$$

$$= \mu_i.$$  

As the choice of index $i \in [N]$ has been entirely arbitrary, we can conclude that

$$\mu_1 = \mu_2 = ... = \mu_N = \mu_{|C(E)}.$$  

Remark: The proof of the theorem 4 above works exactly like the previous demonstrations. That should already suffice to show that the agreement theorem is more like a mathematical statement versing about probability-like measures over a specific construction in set theory (knowledge models) than a signature of any physical restriction.

To conclude, we can say that in our hybrid GPT version of Aumann’s agreement theorem, it is also true that agents cannot agree to disagree - provided they have common knowledge. A remarkable result showing that (i) although highly counter-intuitive, the impossibility of agreeing to disagree in the presence of common knowledge is also valid in any reasonable probability theory, regardless of how abstract or general they might be; and (ii) that the agreement theorem cannot be used as a criterion to separate physical theories.

V. LIMITATIONS

This part of our manuscript is dedicated to clarifying the main assumptions and hypotheses we have used in the previous sections. In doing so, we also critically spell out the potential limitations of our results.

To begin with, we should say that our argument is grounded in a standard and set-theoretical model for knowledge - basically, the same model used by Aumann in his seminal paper [1]. Although that model does capture many elements of what we want to mean by "agent a knows $E$", it falls short in capturing other essential aspects of (multi-)agent centred knowledge. For example, there is no mention of the role of communication between the involved agents. This fact is so remarkable that within our model, it might well be the case that agents reach common knowledge without exchanging any information - not even a bit. Additionally, the very notion of questions or measurements is, to a certain extent, entirely artificial in this model. Even though one can trace a parallel with sharp measurements, knowing the outcome of a particular question does not change the underlying model. In sum, the model we are dealing with here is static and does not account for any form of communication across the parties. Models falling into the umbrella of dynamic epistemic logic try and deal with those gaps. We refer to ref. [22] and refs. [15, 18] for a critical introduction to those models. It should be explored elsewhere how to adapt these dynamic epistemic models to a hybrid scenario and prove an extended Aumann’s impossibility theorem within that framework.

Secondly, this fact might be hidden in a first reading, but the crucial ingredients we have used in our work derive from the conditional quantum state formalism put forward by the authors of [23, 24]. In so doing, we are bound to the same restrictions that also affect that formalism. Critically, in ref. [23, 24], the authors study situations in which it is not possible to assign a joint quantum state for certain causally connected regions. Just to cite a few, it is impossible to assign a well-defined causal
joint state to (i) mixed causal scenarios; (ii) to multiple time steps that are not Markovian; (iii) to pre-and-post selection scenarios; and (iv) to scenarios where one tries to learn simultaneously the outcome of a local measurement and the outcome of a remote, acausally connected measurement - see fig. 2. In our hybrid model that constraints affect the possible measurable spaces \((\Omega, \Sigma)\) and partitions \(\{A_i\}_{i \in I}\) we can consider - recall that in our definition, a DOVM and an SVM are defined as joint states over \(\Sigma\).

VI. CONCLUSIONS

This contribution has proven that Aumann’s theorem about the impossibility of agreeing to disagree is also true in a (hybrid) quantum model. In fact, paying the cost of keeping the knowledge model formulation unaltered, we have also shown that we could go beyond quantum models and that the theorem is also valid in any GPT (hybrid).

Basically, the reason why Aumann’s original argument carries over so straightforwardly to more general and abstract theories is that fact that argument is true whenever (i) a countably additive probability-like measure is defined over the same set of states of the world \(\Omega\) and the same sigma-algebra \(\Sigma\) which belongs to the knowledge model \((\Omega, Q_1, \ldots, Q_N, \Sigma)\); and (ii) that it is possible to come up with a good definition of conditioning. In this sense, the agreement theorem should be seen more like a statement about probability theories defined over knowledge models, than a criterion to delineate different physical theories.

The fact that Aumann’s theorem has appeared elsewhere in the specialised literature of quantum foundations might be not very well known. Paradoxically enough, the myriad of results indicates there is no consensus - or common knowledge. For instance, in refs. \([25, 26]\) the authors argue that in general, quantum agents may evade the agreement theorem, which would show, therefore, a clear cut between quantum and classical strategies of reasoning - although they do investigate necessary conditions in which Aumann’s result holds in a quantum-like framework. Their setup differs from ours as they base their knowledge model on operator algebras - much in the spirit of Pitowski’s works \([27]\). We also pinpoint two other aspects of refs. \([25, 26]\) that distinguish their analysis from ours. First, although they make a great effort in dealing with operator algebras, their main results are stated in terms of usual probability distributions - they use the Born rule to translate back from operators to real numbers. Second, their conditional probability is defined via an update rule that differs from our def. III.2.

Another remarkable analysis of Aumann’s agreement theorem, in the light of generalised physical theories, is the work of Contreras-Tejada et.al. \([20]\). Building their results upon the usual multipartite Bell’s scenario \([28]\), in that paper, the authors prove that the impossibility of agreeing to disagree constrains classical and quantum theories. Surprisingly, however, they exhibited examples of non-signalling boxes that avoid the agreement theorem - the paradigmatic PR-box being the extreme case of that. To do so, they had to adapt Aumann’s original framework and definitions to make them fit into the Bell’s scenario. Consequently, as in \([25, 26]\) they were bounded to use the Born rule (in the quantum case), and assume that agents describe the outcomes of their experiments with standard probability theory. Recall that even though we kept Aumann’s original knowledge model fixed, with DOVMs and SVMs we have been able to go beyond the usual probability theory.

Finally, we want to point out a potential connection of common knowledge/agreement with the emergence of an objective reality. We have mentioned en passant this possibility in the introduction, and we want to conclude our contribution by emphasising this point again. The lesson learned from Wigner’s paradox, mainly as discussed in refs. \([29, 30]\), is that in general, there is an asymmetry between Wigner’s external perspective and the friend’s internal perspective. That lack of symmetry results in Wigner’s impossibility of assigning a definite outcome for events inside the friend’s laboratory - which would lead us to abandon either Local Action or Absoluteness of Events. In ref. \([29]\) the author hints about a “law of thought” that might reconcile the two perspectives. We suggest that the common knowledge could be that law of thought the author was looking for - and that common knowledge would be the mechanism responsible for the emergence of a collective notion of objective reality, a reality where each and every agent would agree on the descriptions of events. Additionally, we feel that it should be investigated elsewhere whether a more straightforward and complete device-independent formulation of the agreement theorem may be connected with Quantum Darwinism or put constraints on possible probabilistic physical theories.

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