Smooth Solutions of the Three Dimensional
Navier-Stokes Problem

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Abstract
The aim of this paper is to solve the three dimensional Navier-Stokes problem with conservative source term when the initial conditions are divergence and curl free.

We use convolution methods with basic vector calculus to construct “well behaved” smooth solutions of the initial boundary value problem for the system of Navier-Stokes.

1 Introduction
The Navier-Stokes equations are the equations that describe the motion of usual fluids like water, air, oil. They appear in the study of many important phenomena, while the physical model leading to the Navier-Stokes equations is simple, the situation is quite different from the mathematical point of view. In particular, because of their nonlinearity, the mathematical study of these equations appeared difficult and has been open for many years.

In this paper, we study convolution techniques for solving the 3-dimensional Navier-Stokes equation when the source term is given by a conservative field.

2 Constructing Smooth Solutions
Consider the initial boundary value problem for the system of Navier-Stokes
\[
\begin{align*}
\frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla) u + \nabla p &= f,
\quad \text{div} u = 0,
\end{align*}
\]
with initial conditions
\[
u(x, 0) = u^0(x), \quad x \in \mathbb{R}^3.\]

Here, \(u^0(x)\) is a given, \(C^\infty\) divergence-free vector field on \(\mathbb{R}^3\).
This system describes the velocity \( \mathbf{u} \) and pressure \( p \) of a viscous incompressible fluid under the influence of an external force \( \mathbf{f} \). The viscosity \( \nu \) is assumed to be constant.

The system of equations (1.1) is to be solved for an unknown velocity vector
\[
\mathbf{u}(x, t) = (u^i(x, t)) \in \mathbb{R}^3
\]
and pressure \( p(x, t) \in \mathbb{R} \), defined for position \( x \in \mathbb{R}^3 \) and time \( t \geq 0 \).

**Proposition 1** Let \( \mathbf{u}^0(x) \) be a smooth vector field in \( \mathbb{R}^3 \) satisfying
\[
\text{div}\mathbf{u}^0(x) = \text{curl}\mathbf{u}^0(x) = 0
\]
If \( \mathbf{f} \) is a conservative force, then there exist \( \mathbf{u}(x, t), p(x, t) \in C^\infty(\mathbb{R}^3 \times \mathbb{R}) \) with \( \mathbf{u}(x, 0) = \mathbf{u}^0(x) \) solutions of the Navier-Stokes equation
\[
\begin{cases}
\frac{\partial \mathbf{u}}{\partial t} - \nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p = \mathbf{f} \\
\text{div}\mathbf{u} = 0.
\end{cases}
\]

**Proof** Without loss of generality, we may assume that \( \mathbf{f} = 0 \). First observe that
\[
(\mathbf{u} \cdot \nabla) \mathbf{u} = \nabla \left( \frac{1}{2} |\mathbf{u}|^2 \right) - \mathbf{u} \times \text{curl}\mathbf{u}.
\]
Equation (1.1) becomes
\[
\begin{cases}
\frac{\partial \mathbf{u}}{\partial t} - \nu \Delta \mathbf{u} + \nabla \left( \frac{1}{2} |\mathbf{u}|^2 \right) - \mathbf{u} \times \text{curl}\mathbf{u} + \nabla p = 0 \\
\text{div}\mathbf{u} = 0.
\end{cases}
\]
Let \( h \) be a smooth function with compact support satisfying
\[
\int_{\mathbb{R}^3} h(x)dx = 1
\]
Let
\[
\mathbf{u}(x, t) = \int_{\mathbb{R}^3} \mathbf{u}^0(x - ty)h(y)dy.
\]
\( \mathbf{u}(x, t) \) is \( C^\infty \) in \( \{(x, t) \in \mathbb{R}^{3+1} : t \neq 0 \} \), satisfies \( \mathbf{u}(x, 0) = \mathbf{u}^0(x) \). Moreover, we have \( \text{div}\mathbf{u} = 0 \) and \( \text{curl}\mathbf{u} = 0 \). This implies that
\[
\text{curl} \left[ \frac{\partial \mathbf{u}}{\partial t} - \nu \Delta \mathbf{u} + \nabla \left( \frac{1}{2} |\mathbf{u}|^2 \right) - \mathbf{u} \times \text{curl}\mathbf{u} \right] = 0.
\]
Hence, the vector field
\[
\frac{\partial \mathbf{u}}{\partial t} - \nu \Delta \mathbf{u} + \nabla \left( \frac{1}{2} |\mathbf{u}|^2 \right) - \mathbf{u} \times \text{curl}\mathbf{u}
\]
is conservative for all \( t \). Therefore equation (1.1) is satisfied for some \( p(x, t) \in C^\infty(\mathbb{R}^{3+1}) \). \( \blacksquare \)
In [3] a class of radial measures \( \mu \) on \( \mathbb{R}^n \) is defined so that integrable harmonic functions \( g \) on \( \mathbb{R}^n \) may be characterized as solutions of convolution equations in \( \mathbb{R}^n \). In particular, Natan Y. B. and Weit Y showed that if
\[
\varphi(x) = c_ne^{-2\pi|x|}
\]
where \( c_n = \frac{\pi^{(n+1)/2}}{\Gamma((n+1)/2)} \), then every solution of the equation \( g \ast h = g \) in \( L^1(\mathbb{R}^n, e^{-2\pi|x|} dx) \) is harmonic if and only if \( n < 9 \).

**Corollary 1**

Let
\[
u^0(x) = \nabla g(x) \quad x \in \mathbb{R}^3
\]
where \( g \) is a solution of the equation \( g \ast h = g \) in \( L^1(\mathbb{R}^3, e^{-2\pi|x|} dx) \). If \( f \) is a conservative force, then there exist \( u(x,t), p(x,t) \in C^\infty(\mathbb{R}^3 \times \mathbb{R}) \) with \( u(x,0) = u^0(x) \) solutions of the Navier-Stokes equation
\[
\begin{aligned}
&\frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla) u + \nabla p = f \\
&\text{div} u = 0.
\end{aligned}
\]

### 3 Towards a Well-behaved Solution

In this section, we propose to construct smooth solutions \( u(x,y,z,t) \) in \( \mathbb{R}^{3+1} \) with a suitable growth condition.

**Proposition 2**

There exist smooth vector fields \( u(x,y,z,t) \in C^\infty(\mathbb{R}^{3+1}, \mathbb{R}^3) \), and a smooth functions \( p(x,t) \in C^\infty(\mathbb{R}^{3+1}) \) solutions of equation (1.1) with \( u(x,y,z,t) \) periodic in \( x \) and \( y \), and whose behavior in \( z \) may be controlled by a smooth function \( \Psi : \mathbb{R} \rightarrow \mathbb{R} \) satisfying \( \Psi'' = \Psi \).

**Proof.** Let \( \alpha, \beta, \) and \( \zeta \) be nonzero real numbers such that
\[
\zeta^2 = \alpha^2 + \beta^2.
\]
Let \( \varphi \) be an even smooth function with compact support in \( \mathbb{R} \). Let \( g \) be the function defined by
\[
g(x,y,z) = \left( \int_{\mathbb{R}} \varphi(z-r)\Psi(\zeta r) dr \right) \cos(\alpha x + \beta y).
\]
\( g \) is a harmonic function in \( \mathbb{R}^3 \) that is periodic in \( x \), and \( y \).

Now define
\[
u^0(x) = \nabla g(x)
\]
where \( x = (x,y,z) \in \mathbb{R}^3 \), and
\[
u(x,t) = \int_{\mathbb{R}^3} \nu^0(x-ty)h(y) dy. \quad t \neq 0.
\]
h is as in proposition 1 a smooth function with compact support satisfying \( \int_{\mathbb{R}^3} h(x) dx = 1 \).
The proof of proposition 1 shows that $u(x,t)$ and $p(x,t)$ will solve equation (1.1) for some smooth function $p(x,t)$ in $\mathbb{R}^{3+1}$.

Moreover $u(x,t)$ satisfies the desired requirement. That is, $u(x,y,z,t)$ is periodic in $x$ and $y$, and whose behavior in $z$ depends on the growth of the function $\Psi$.

**Conclusion.** The results above guarantee the existence of smooth solutions to the three dimensional Navier-Stoke problem. A fundamental problem in analysis is to decide whether a smooth, physically reasonable solutions in the sense of [1] exist for the Navier–Stokes equations. We believed that we have given a partial answer to the question by constructing solutions that behave well in the two variables with initial conditions that are both divergence and curl free.

**References**

[1] Charles L. Fefferman: *Official Problem Description – Charles Fefferman* *Clay Mathematics Institute*

[2] JKeith J. Devlin, *The Millennium Problems: The Seven Greatest Unsolved Mathematical Puzzles of Our Time*, Basic Books (October, 2002), ISBN 0-465-01729-0.

[3] Peter D. Lax, *Functional Analysis*, Wiley-interscience (2002)

[4] Yaakov B. Natan, and Weit Y. *Integrable Harmonic functions on $R^n$* *Jour.of.Funct.Analysis 150, 471-477 (1997)*