EIGENVALUES UNDER THE BACKWARD RICCI FLOW ON
LOCALLY HOMOGENEOUS CLOSED 3-MANIFOLDS

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ABSTRACT. In this paper, we study the evolving behaviors of the first eigenvalue
of Laplace-Beltrami operator under the normalized backward Ricci flow, construct
various quantities which are monotonic under the backward Ricci flow and get
upper and lower bounds. We prove that in cases where the backward Ricci flow
converges to a sub-Riemannian geometry after a proper rescaling, the eigenvalue
evolves toward zero.

1. Introduction

The Ricci flow on a closed manifold is a flow of Riemannian metric $g(t)$ evolved
by the equation
\[ \frac{\partial g}{\partial t} = -2Rc, \quad g(0) = g_0, \]
where $Rc$ is the Ricci curvature tensor. The customary normalization is setting
$\tilde{g}(\tilde{t}) = \psi(t)g(t), \quad \tilde{t} = \int_0^t \psi(s)ds$ with $\frac{1}{\psi} \frac{\partial \psi}{\partial t} = \frac{2r}{3}$, where $r$ is the average of the scalar
curvature $R$. Setting $\tilde{\psi}(\tilde{t}) = \psi(t)$, then we have $\frac{1}{\tilde{\psi}} \frac{\partial \tilde{\psi}}{\partial \tilde{t}} = \frac{2r}{3}$ and
\[ \frac{\partial \tilde{g}}{\partial \tilde{t}} = -2\tilde{R}c + \left( \frac{\partial \ln \tilde{\psi}}{\partial \tilde{t}} \right) \tilde{g} = -2\tilde{R}c + \frac{2r}{3} \tilde{g}, \quad \tilde{g}(0) = g_0. \]

We often write the following flow
\[ (1.1) \quad \frac{\partial g}{\partial t} = -2Rc + \frac{2r}{3} g, \quad g(0) = g_0, \]
which is called the normalized Ricci flow keeping the volume constant. In dimension
3, Hamilton [7] proves that the solution to the Ricci flow converges to the constant
curvature metric on a 3-sphere if the Ricci curvature of initial metric is positive.

The eigenvalues of geometric operators under the Ricci flow are important to
understand geometry and topology of manifolds. In [18], Perelman proves the the
first eigenvalue of the Laplace-Beltrami operator with potential $R$, i.e., $-\Delta + 4R$ is
nondecreasing under the Ricci flow, where $R$ denotes the scalar curvature of metric
$g$. He also applies this to show that there are no nontrivial steady or expanding
breathers on closed manifolds. Later, Cao [1] shows that the eigenvalues of $-\Delta + \frac{R}{2}$ are nondecreasing under the Ricci flow on manifolds with nonnegative curvature operator. Using the same techniques, Li [15] extends Cao’s result to manifolds without nonnegative curvature operator. Similar results hold for the first eigenvalue of $-\Delta + aR(a \geq 1)$ along the Ricci flow [2,15]. In fact, the eigenvalues $\lambda$ are no longer differentiable in time $t$. If we denote $u$ the eigenfunction of the eigenvalue then $\lambda(u,t) = \lambda(t)$. By the eigenvalue perturbation theory, there is a $C^1$-family of smooth eigenvalues and eigenfunctions [11]. One can assume that the first eigenvalue $\lambda(t)$ and the corresponding eigenfunction $u(x,t)$ are smooth along the Ricci flow.

Cao, the author and Ling [4] derive a monotonicity formula for the first eigenvalue of $-\Delta + aR(0 < a \leq \frac{1}{2})$ on a closed surface under the Ricci flow and obtain various monotonicity formulae and estimates along the normalized Ricci flow. Our results indicate that although it is difficult to get better estimates for the eigenvalue under the Ricci flow, one can get interesting results if the problem can be dealt with by ODE techniques.

For the Laplace-Beltrami operator, Ma [16] prove the first eigenvalue on domains with Dirichlet boundary condition is nondecreasing along the Ricci flow. Ling [14] gets a sharp bound of the first eigenvalue under the normalized Ricci flow, and proves that an appropriate multiple is monotonic. The author [8] considers the eigenvalue of Laplace-Beltrami operator under the normalized Ricci flow on locally homogeneous closed 3-manifolds, constructs various monotonic quantities and gets estimates for upper and lower bounds.

For the p-Laplace operator, Wu, Wang and Zheng [20] proves the first p-eigenvalue is strictly increasing along the Ricci flow under some curvature assumption, and construct various monotonic quantities on closed Riemannian surface. Wu [19] proves the first eigenvalue monotonicity of the p-Laplace operator along the Ricci flow on closed Riemannian manifolds under some different curvature assumptions.

A Riemannian manifold $(M,g)$ is called to be locally homogeneous if for every two points $x, y \in M$, there are neighborhoods $U$ of $x$ and $V$ of $y$, and an isometry $\phi$ from $(U, g|_U)$ to $(V, g|_V)$ with $\phi(x) = y$. Furthermore, $(M,g)$ is called to be homogeneous if the isometry group is transitive i.e., $U = V = M$ for all $x$ and $y$. A result of Singer [17] says that the universal cover of a locally homogeneous manifold is homogeneous. We may study the Ricci flow of homogeneous models instead of locally homogeneous manifolds since Ricci flow commutes with the cover map.
The locally homogeneous 3-manifolds contains nine classes which can be divided into two sets. The first set consists of classes $H(3)$, $H(2) \times \mathbb{R}$ and $SO(3) \times \mathbb{R}$, where $H(n)$ denotes the group of isometries of hyperbolic $n$-space. The second set includes $\mathbb{R}$, $SU(2)$, $SL(2, \mathbb{R})$, Heisenberg, $E(1, 1)$ (the group of isometries of the plane with flat Lorentz metric) and $E(2)$ (the group of isometries of the Euclidean plane), and these are called Bianchi classes in [9].

In Bianchi classes, given an initial metric $g_0$, there is a Milnor frame $(f_1, f_2, f_3)$ such that the metric and Ricci tensor are diagonalized. Since this property is preserved by the Ricci flow, we often write

$$g = Af^1 \otimes f^1 + Bf^2 \otimes f^2 + Cf^3 \otimes f^3,$$

where $(f^1, f^2, f^3)$ is a dual frame of Milnor frame. Then the Ricci flow reduces to an ODE system involving $A$, $B$ and $C$. Since the homogeneous 3-manifolds are models of geometrization conjecture, Isenberg and Jackson [9] study the Ricci flow on such manifolds and describe their characteristic behaviors by analyzing the corresponding system. There are three types of behaviors depending on the geometry type.

Later Knopf and McLeod [12] study quasi-convergence equivalence of model geometries under the Ricci flow.

If we assume $g(t), \ t \in (-T_b, T_f)$, is the maximal solution of the forward Ricci flow (1.1), it is interesting to consider the behaviors of $g(t)$ as $t$ goes to $-T_b$. For convenience, we reverse time and consider the following backward Ricci flow equation

$$\frac{\partial g}{\partial t} = 2Rc - \frac{2r}{3}g, \ g(0) = g_0.$$

The behaviors of backward Ricci flow are described in [5] and [3], and the interesting phenomena is that the backward Ricci flow converges to a sub-Riemannian geometry after a proper rescaling. We use the simple example to explain the term of sub-Riemannian geometry. In section three, the evolving metric is

$$g = Af^1 \otimes f^1 + Bf^2 \otimes f^2 + Cf^3 \otimes f^3.$$

As $t$ goes to $-3/(16R_0)$, we get

$$A(t) \to +\infty, \ B(t) \to 0, \ C(t) \to 0.$$

Then the rescaled metric $\tilde{g}(t) = (C_0/C(t))g(t)$ converges to

$$\infty f^1 \otimes f^1 + B_0f^2 \otimes f^2 + C_0f^3 \otimes f^3.$$

Look at the dual tensor defined on the co-tangent bundle

$$Q = A^{-1}f_1 \otimes f_1 + B^{-1}_2f_2 \otimes f_2 + C^{-1}_3f_3 \otimes f_3.$$
Then the tensor $Q$ tends to

$$Q_* = B_0^{-1} f_2 \otimes f_2 + C_0^{-1} f_3 \otimes f_3.$$  

It turns out that $[f_2, f_3] = 2f_1$. Then the tensor $Q_*$ induces a natural distance function $d_*$ defined on $M$. We take the infimum of the length of all curves staying tangent to the linear span of $f_2, f_3$, which are called horizontal curves, to computed the distance. The associated geometry is called a sub-Riemannian geometry.

In this paper, we study the first eigenvalue of the Laplace-Beltrami operator under the backward Ricci flow on locally homogeneous 3-manifolds in Bianchi classes.

**Theorem 1.1.** Let $(M, g(t), t \in [0, T_+))$ be a solution to the backward Ricci flow in Bianchi classes, where $T_+ \in [0, +\infty)$ is the maximal existence time. Assume that $\lambda(t)$ is the first eigenvalue of $-\Delta$ with respect to $g(t)$. Then in cases where $g(t)$ converges to a sub-Riemannian geometry after a proper re-scaling, $\lambda(t)$ goes to zero as $t$ approaches $T_+$.

We construct various monotonic quantities and get upper and lower bounds for the eigenvalue. The behaviors of the eigenvalue are not very diverse. In many cases, the eigenvalue is decreasing after a time and goes to zero. The next of this paper is arranged as follows. In section two, we derive an evolution equation of the eigenvalue which is important to estimates. From section three to section seven, we analyze the behaviors of the eigenvalue and get estimates case by case.

2. Evolution equation of the eigenvalue

In this section, we get the following theorem.

**Theorem 2.1.** Let $(M, g(t), t \in [0, T_+))$ be a solution to the backward Ricci flow on a locally homogeneous 3-manifold. Assume that $\lambda(t)$ is the first eigenvalue of $-\Delta$ and $u(x, t) > 0$ satisfies

$$(2.1) \quad -\Delta u = \lambda u,$$

with $\int u^2(x, t) d\mu = 1$. Then along the the backward Ricci flow, we obtain

$$(2.2) \quad \frac{d}{dt} \lambda = \frac{2}{3} R \lambda - \int_{M} (2 R_{ij} \nabla_i u \nabla_j u) d\mu.$$  

The proof of this theorem is similar to Lemma 3.1 in [1].

**Proof.** By a direct calculation as in [1], we have

$$\frac{d}{dt} \lambda = \int \left( 2u R_{ij} \nabla_i \nabla_j u - \frac{2r}{3} u \Delta u \right) d\mu$$

where

$$r = \frac{\int_M R d\mu}{\frac{1}{4} M d\mu}$$
is the average of the scalar curvature, which equals $R$ on locally homogeneous manifolds. Integrating by parts and using contracted Bianchi identity, we have

$$\int 2u R_{ij} \nabla_i \nabla_j u d\mu = - \int [(2u \nabla_i R_{ij}) \nabla_j u] d\mu - \int (2R_{ij} \nabla_i u \nabla_j u) d\mu,$$

and

$$\int (2\nabla_i R_{ij}) u \nabla_j u d\mu = - \int u(\nabla_j R) \nabla_j u d\mu = \int Ru \Delta u d\mu + \int R|\nabla u|^2 d\mu = -\lambda R + \lambda R = 0.$$

We arrive at

$$\frac{d}{dt} \lambda = \frac{2}{3} R \lambda - \int (2R_{ij} \nabla_i u \nabla_j u) d\mu.$$

\[\square\]

3. Heisenberg

In this class, given a metric $g_0$, there is fixed Milnor frame such that

$$[f_2, f_3] = 2f_1, \quad [f_3, f_1] = 0, \quad [f_1, f_2] = 0.$$

Under the normalization $A_0 B_0 C_0 = 4$, the curvature components for metrics are (see examples on page 171 in [6])

$$\begin{cases} R_{11} = \frac{1}{2} A^3, \\ R_{22} = -\frac{1}{2} A^2 B \\ R_{33} = -\frac{1}{2} A^2 C, \\ R = -\frac{1}{2} A^2. \end{cases}$$

The backward Ricci flow equations are then

$$\begin{cases} \frac{d}{dt} A = \frac{4}{3} A^3, \\ \frac{d}{dt} B = -\frac{2}{3} A^2 B, \\ \frac{d}{dt} C = -\frac{2}{3} A^2 C, \end{cases}$$

and the solution is

$$\begin{cases} A = A_0 (1 + \frac{16}{3} R_0 t)^{-1/2}, \\ B = B_0 (1 + \frac{16}{3} R_0 t)^{1/4}, \\ C = C_0 (1 + \frac{16}{3} R_0 t)^{1/4}. \end{cases}$$
where $R_0 = -\frac{1}{2}A_0^2$. The metric $\tilde{g}(t) = (C_0/C(t))g(t)$ converges to a sub-Riemannian geometry.

**Theorem 3.1.** Let $\lambda(t)$ be the first eigenvalue of $-\Delta$. Assume that $B_0 \geq C_0$. Then $
abla(\lambda(t))e^{\int_0^t (\lambda(t) - \frac{R}{2} + 2R_{11})dt}$ is nondecreasing along the backward Ricci flow, and $\lambda(t)e^{\int_0^t (\lambda(t) - \frac{R}{2} + 2R_{22})dt}$ is nonincreasing. Moreover, we have

$$e^{\frac{2}{3} [1 - (1 + \frac{16R_0}{3})^{-1/2}]} (1 + \frac{16R_0}{3} )^{1/8} \lambda(0) \leq \lambda(t) \leq \lambda(0)e^{\frac{2}{3} [1 - (1 + \frac{16R_0}{3})^{1/4}]} (1 + \frac{16R_0}{3} )^{1/8}$$

for $t \in [0, -3/(16R_0)]$. As $t$ goes to $-3/(16R_0)$, $\lambda(t)$ goes to zero.

**Proof.** Assume that $B_0 \geq C_0$. By (2.2) and (3.1), have

$$\frac{2}{3} R\lambda - 2R_{11} \lambda \leq \frac{d}{dt} \lambda \leq \frac{2}{3} R\lambda - 2R_{22} \lambda.$$

Then $\lambda(t)e^{\int_0^t (\lambda(t) - \frac{R}{2} + 2R_{11})dt}$ is nondecreasing along the backward Ricci flow, and $\lambda(t)e^{\int_0^t (\lambda(t) - \frac{R}{2} + 2R_{22})dt}$ is nonincreasing.

Integrating from 0 to $t$ yields

$$e^{\frac{2}{3} [1 - (1 + \frac{16R_0}{3})^{-1/2}]} (1 + \frac{16R_0}{3} )^{1/8} \lambda(0) \leq \lambda(t) \leq \lambda(0)e^{\frac{2}{3} [1 - (1 + \frac{16R_0}{3})^{1/4}]} (1 + \frac{16R_0}{3} )^{1/8}. \quad \Box$$

4. **SU(2)**

Given a metric $g_0$, we choose a Milnor frame such that

$$[f_2, f_3] = 2f_1, \quad [f_3, f_1] = 2f_2, \quad [f_1, f_2] = 2f_3.$$

Under the normalization $A_0B_0C_0 = 4$, then the nonzero components of Ricci tensor are (see examples on page 171 in [6])

\begin{equation}
\begin{aligned}
R_{11} &= \frac{1}{2} A[A^2 - (B-C)^2], \\
R_{22} &= \frac{1}{2} B[B^2 - (A-C)^2], \\
R_{33} &= \frac{1}{2} C[C^2 - (A-B)^2],
\end{aligned}
\end{equation}

and the scalar curvature is

\begin{equation}
R = \frac{1}{2} [A^2 - (B-C)^2] + \frac{1}{2} [B^2 - (A-C)^2] + \frac{1}{2} [C^2 - (A-B)^2].
\end{equation}

The backward Ricci flow equations are

\begin{equation}
\begin{aligned}
\frac{dA}{dt} &= -\frac{2}{3} A \left[-A(2A-B-C) + (B-C)^2\right], \\
\frac{dB}{dt} &= -\frac{2}{3} B \left[-B(2B-A-C) + (A-C)^2\right], \\
\frac{dC}{dt} &= -\frac{2}{3} C \left[-C(2C-A-B) + (A-B)^2\right].
\end{aligned}
\end{equation}
Assume that $A_0 \geq B_0 \geq C_0$. Cao [5] proves the following theorem

**Theorem 4.1.**

1. If $A_0 = B_0 = C_0$, then $T_+ = \infty$ and $g(t) = g_0$.
2. If $A_0 = B_0 > C_0$, then $T_+ = \infty$, $A = B > C$ and, as $t$ goes to infinity, $A \sim \frac{8}{3}t$, $C \sim \frac{9}{16}t^{-2}$.
3. If $A_0 > B_0 \geq C_0$, then $T_+ < \infty$, $A > B \geq C$ and there are constants $\eta_1, \eta_2 \in (0, \infty)$ such that

$$A \sim \frac{\sqrt{6}}{4}(T_+ - t)^{-1/2}, \quad B \sim \eta_1(T_+ - t)^{1/4}, \quad C \sim \eta_2(T_+ - t)^{1/4}.$$

In case (3), $\bar{g}(t) = (B_0/B(t))g(t)$ converges to a sub-Riemannian geometry.

In the following, $\tau$, $c_1$ and $c_2$ denote constants which may vary from line to line and from section to section.

**Theorem 4.2.** Let $\lambda(t)$ be the first eigenvalue of $-\Delta$. Then we get

1. If $A_0 = B_0 = C_0$, then $\lambda(t) = \lambda(0)$.
2. If $A_0 = B_0 > C_0$, there is a time $\tau$ such that $\lambda(t)e^{\int_t^\tau(-\frac{2}{3}R + 2R_{11})dt}$ is nondecreasing along the backward Ricci flow, and $\lambda(t)e^{\int_t^\tau(-\frac{2}{3}R + 2R_{33})dt}$ is nonincreasing. We also get the further estimate

$$\lambda(\tau)e^{8(\tau-t)} \leq \lambda(t) \leq \lambda(\tau)\left(\frac{t}{\tau}\right)^{c_1}.$$

3. If $A_0 > B_0 \geq C_0$, then there is a time $\tau$ such that $\lambda(t)e^{\int_t^\tau(-\frac{2}{3}R + 2R_{11})dt}$ is nondecreasing along the backward Ricci flow, and $\lambda(t)e^{\int_t^\tau(-\frac{2}{3}R + 2R_{33})dt}$ is nonincreasing. We get the following estimate

$$\lambda(\tau)e^{2c_2[(T_+ - \tau)^{-1/2} - (T_+ - t)^{-1/2}]} \leq \lambda(t) \leq \lambda(\tau)\left(\frac{T_+ - t}{T_+ - \tau}\right)^{c_1}$$

As $t$ goes to $T_+$, $\lambda(t)$ approaches 0.

**Proof.**

1. If $A_0 = B_0 = C_0$, then $\lambda(t) = \lambda(0)$.
2. If $A_0 = B_0 > C_0$, then by (4.1) and (2) in Theorem 4.1 we have $R_{11} = R_{22} > 0$, $R_{33} > 0$ after a time $\tau$. It also follows that

$$R_{22} - R_{33} = \frac{1}{2}[B^3 - B(A - C)^2 - C^3 + C(A - B)^2]$$

$$= \frac{1}{2}(B - C)^2(B^2 + 2BC + C^2 - A^2)$$

$$> 0$$

for $t \geq \tau$. Thus we derive

$$\frac{2}{3}R\lambda - 2R_{11}\lambda \leq \frac{d}{dt}\lambda \leq \frac{2}{3}R\lambda - 2R_{33}\lambda$$
with \( t \geq \tau \).

Then \( \lambda(t) e^{\int_t^\tau (-\frac{2}{3}R + 2R_{11})dt} \) is nondecreasing along the backward Ricci flow, and 
\( \lambda(t) e^{\int_t^\tau (-\frac{2}{3}R + 2R_{33})dt} \) is nonincreasing.

Moreover, by (2) in Theorem 4.1 we estimate

\[
\frac{2}{3} R - 2R_{33} = \frac{1}{3} (2AB + 2BC + 2CA - A^2 - B^2 - C^2) - \left[ C^3 - C(A - B)^2 \right] \\
= \frac{1}{3} (4AC - C^2) - C^3 \\
\leq c_1 t^{-1}
\]

and

\[
\frac{2}{3} R - 2R_{11} = \frac{1}{3} (2AB + 2BC + 2CA - A^2 - B^2 - C^2) - \left[ A^3 - A(B - C)^2 \right] \\
= \frac{1}{3} (4AC - C^2) - A^3 + A(B - C)^2 \\
\geq -8
\]

for \( t \geq \tau \).

Thus we arrive at

\[-8 \leq \frac{1}{\lambda} \frac{d}{dt} \lambda \leq c_1 t^{-1}.
\]

Integrating from \( \tau \) to \( t \) gives

\[\lambda(\tau) e^{8(\tau-t)} \leq \lambda(t) \leq \lambda(\tau) \left( \frac{t}{\tau} \right)^{c_1}.
\]

(3) If \( A_0 > B_0 \geq C_0 \), then by (3) in Theorem 4.1 we have

\[R_{11} > 0, \ R_{22} < 0, \ R_{33} < 0,
\]

after a time \( \tau \). It is easy to see that

\[R_{22} - R_{33} = \frac{1}{2} [B^3 - B(A - C)^2 - C^3 + C(A - B)^2] \\
= \frac{1}{2} (B - C)(B^2 + 2BC + C^2 - A^2) \\
\leq 0
\]

if \( t \geq \tau \).

Thus we have

\[R_{11} > R_{33} \geq R_{22}
\]

and

\[\frac{2}{3} R \lambda - 2R_{11} \lambda \leq \frac{d}{dt} \lambda \leq \frac{2}{3} R \lambda - 2R_{22} \lambda
\]

with \( t \geq \tau \).
Then $\lambda(t)e^{\int_t^\tau (-\frac{2}{3}R + 2R_{11})dt}$ is nondecreasing along the backward Ricci flow, and $\lambda(t)e^{\int_t^\tau (-\frac{2}{3}R + 2R_{22})dt}$ is nonincreasing.

Furthermore, we conclude from the behavior of metric that

$$
\frac{2}{3}R - 2R_{22} = \frac{1}{3}(2AB + 2BC + 2CA - A^2 - B^2 - C^2) - [B^3 - B(A - C)^2] \\
\leq - c_1(T_+ - t)^{-1}
$$

and

$$
\frac{2}{3}R - 2R_{11} = \frac{1}{3}(2AB + 2BC + 2CA - A^2 - B^2 - C^2) - [A^3 - A(B - C)^2] \\
\geq - c_2(T_+ - t)^{-3/2}
$$

for $t \geq \tau$.

Thus we obtain

$$
-c_2(T_+ - t)^{-3/2} \leq \frac{1}{\lambda} \frac{d}{dt} \lambda \leq - c_1(T_+ - t)^{-1}.
$$

Integrating from $\tau$ to $t$ gives

$$
\lambda(\tau)e^{2c_2[(T_+ - \tau)^{-1/2} - (T_+ - t)^{-1/2}]} \leq \lambda(t) \leq \lambda(\tau) \left( \frac{T_+ - t}{T_+ - \tau} \right)^{c_1}
$$

As $t$ goes to $T_+$, $\lambda(t)$ approaches 0. \hfill \Box

5. E(1,1)

Given a metric $g_0$, we choose a fixed Milnor frame such that

$$[f_2, f_3] = 2f_1, \quad [f_3, f_1] = 0 \quad [f_1, f_2] = -2f_3.$$

Under the normalization $A_0B_0C_0 = 4$, the nonzero curvature components of the metric are

$$
\begin{align*}
R_{11} &= \frac{1}{2}A(A^2 - C^2), \\
R_{22} &= -\frac{1}{2}B(A + C)^2, \\
R_{33} &= \frac{1}{2}C(C^2 - A^2), \\
R &= -\frac{1}{2}(A + C)^2.
\end{align*}
$$

(5.1)
The backward Ricci flow equations are
\[
\begin{align*}
\frac{dA}{dt} &= \frac{2}{3} A(2A^2 + AC - C^2), \\
\frac{dB}{dt} &= -\frac{2}{3} B(A + C)^2, \\
\frac{dC}{dt} &= \frac{2}{3} C(2C^2 + AC - A^2).
\end{align*}
\tag{5.2}
\]

Assume that \(A_0 \geq C_0\). Cao [5] proved the following theorem.

**Theorem 5.1.**
(1) If \(A_0 = C_0\), then \(T_+ + \frac{3}{32} B_0 \) and
\[
A(t) = C(t) = \sqrt{\frac{6}{4}} (T_+ - t)^{-1/2}, \quad B(t) = \frac{32}{3} (T_+ - t), \quad t \in [0, T_+).
\]
(2) If \(A_0 > C_0\), then \(T_+ < \infty\), and there exist constants \(\eta_1, \eta_2 \in (0, \infty)\) such that
\[
A \sim \sqrt{\frac{6}{4}} (T_+ - t)^{-1/2}, \quad B(t) \sim \eta_1 (T_+ - t)^{1/4}, \quad C(t) \sim \eta_2 (T_+ - t)^{1/4},
\]
as \(t\) goes to \(T_+\).

In case (2), \(\bar{g}(t) = (B_0/B(t))g(t)\) converges to a sub-Riemannian geometry.

**Theorem 5.2.** Let \(\lambda(t)\) be the first eigenvalue of \(-\Delta\). Then we get

(1) If \(A_0 = C_0\), then \(\lambda(t) e^{\int_0^t (-\frac{2}{3} R) \, dt}\) is nondecreasing along the backward Ricci flow, and \(\lambda(t) e^{\int_0^t (-\frac{2}{3} R + 2 R_{11}) \, dt}\) is nonincreasing. Moreover, we get
\[
\lambda(0) \left( \frac{T_+ - t}{T_+} \right)^{1/2} \leq \lambda(t) \leq \lambda(0) \left( \frac{T_+ - t}{T_+} \right)^{1/2} e^{16 t}.
\]

(2) If \(A_0 > C_0 \geq B_0\), then there is time \(\tau\) such that \(\lambda(t) e^{\int_0^t (-\frac{2}{3} R + 2 R_{11}) \, dt}\) is nondecreasing along the backward Ricci flow, and \(\lambda(t) e^{\int_0^t (-\frac{2}{3} R + 2 R_{33}) \, dt}\) is nonincreasing. Moreover, we have
\[
\lambda(\tau) e^{2c_2 \left[(T_+ - \tau)^{-1/2} - (T_+ - t)^{-1/2}\right]} \leq \lambda(t) \leq \lambda(\tau) \left( \frac{T_+ - t}{T_+ - \tau} \right)^{c_1}.
\]
As \(t\) goes to \(T_+\), \(\lambda(t)\) approaches 0.

**Remark.** In case (2) of the above theorem, if \(B_0 > C_0\), we can get the similar estimate.

**Proof.** (1) If \(A_0 = C_0\), then by (5.1) we have
\[
R_{11} = 0, \quad R_{22} < 0, \quad R_{33} = 0.
\]

Thus we have
\[
R_{11} = R_{33} > R_{22}
\]
and
\[
\frac{2}{3} R \lambda \leq \frac{d}{dt} \lambda \leq \frac{2}{3} R \lambda - 2 R_{22} \lambda.
\]
Then \(\lambda(t) e^{\int_0^t (-\frac{2}{3} R + 2R_{11}) \, dt}\) is nondecreasing along the backward Ricci flow, and \(\lambda(t) e^{\int_0^t (-\frac{2}{3} R + 2R_{33}) \, dt}\) is nonincreasing.

Next, we do further estimates

\[
\frac{2}{3} R - 2R_{22} = -\frac{1}{3} A^2 - \frac{1}{3} C^2 - \frac{2}{3} AC + BA^2 + BC^2 + 2ABC
\]

\[
= -\frac{1}{2} (T_+ - t)^{-1} + 16
\]

and

\[
\frac{2}{3} R = -\frac{1}{2} (T_+ - t)^{-1}.
\]

Thus we arrive at

\[-\frac{1}{2} (T_+ - t)^{-1} \leq \frac{1}{\lambda} \frac{d\lambda}{dt} \leq -\frac{1}{2} (T_+ - t)^{-1} + 16.
\]

Integration from 0 to \(t\) gives

\[
\lambda(0) \left( \frac{T_+ - t}{T_+} \right)^{1/2} \leq \lambda(t) \leq \lambda(0) \left( \frac{T_+ - t}{T_+} \right)^{1/2} e^{16t}.
\]

(2) If \(A_0 > C_0\), then by (5.1) and (2) in Theorem 5.1 we have

\[R_{11} > 0, \ R_{22} < 0, \ R_{33} < 0\]

after a time \(\tau\).

Assume that \(C_0 \geq B_0\). By (5.2) we get

\[
\frac{d}{dt} \ln \frac{C}{B} = 2(AC + C^2),
\]

which implies \(C(t) > B(t)\) for all \(t > 0\). It is easy to see that

\[
R_{22} - R_{33} = \frac{1}{2} (CA^2 - BA^2 - C^3 - BC^2 - 2ABC)
\]

\[> 0\]

after a time \(\tau\).

So we arrive at

\[R_{11} > R_{22} > R_{33}\]

and

\[
\frac{2}{3} R \lambda - 2R_{11} \lambda \leq \frac{d}{dt} \lambda \leq \frac{2}{3} R \lambda - 2R_{33} \lambda
\]

with \(t \geq \tau\).

Then \(\lambda(t) e^{\int_0^t (-\frac{2}{3} R + 2R_{11}) \, dt}\) is nondecreasing along the backward Ricci flow, and \(\lambda(t) e^{\int_0^t (-\frac{2}{3} R + 2R_{33}) \, dt}\) is nonincreasing.
Moreover, we obtain

\[ \frac{2}{3} R - 2R_{33} \]
\[ = -\frac{1}{3} A^2 - \frac{1}{3} C^2 - \frac{2}{3} AC - C^3 + CA^2 \]
\[ \leq -c_1(T_+ - t)^{-1} \]

and

\[ \frac{2}{3} R - 2R_{11} \]
\[ = -\frac{1}{3} A^2 - \frac{1}{3} C^2 - \frac{2}{3} AC - A^3 + AC^2 \]
\[ \geq -c_2(T_+ - t)^{-3/2} \]

with \( t \geq \tau \).

Thus we get

\[-c_2(T_+ - t)^{-3/2} \leq \frac{1}{\lambda \, dt} \lambda \leq -c_1(T_+ - t)^{-1}.\]

Integration from \( \tau \) to \( t \) yields

\[ \lambda(\tau)e^{2c_2[(T_+ - \tau)^{-1/2} - (T_+ - t)^{-1/2}]} \leq \lambda(t) \leq \lambda(\tau) \left( \frac{T_+ - t}{T_+ - \tau} \right)^{c_1}. \]

As \( t \) goes to \( T_+ \), \( \lambda(t) \) approaches 0. \( \square \)

6. E(2)

Given a metric \( g_0 \), we choose a Milnor frame such that

\[ [f_2, f_3] = 2f_1, \quad [f_3, f_1] = 2f_2, \quad [f_1, f_2] = 0. \]

Under the normalization \( A_0B_0C_0 = 4 \), the nonzero curvature components are

\[
\begin{align*}
R_{11} &= \frac{1}{2} A(A^2 - B^2), \\
R_{22} &= \frac{1}{2} B(B^2 - A^2), \\
R_{33} &= -\frac{1}{2} C(A - B)^2, \\
R &= -\frac{1}{2}(A - B)^2.
\end{align*}
\]
Then the backward Ricci flow equations are

\[
\begin{align*}
\frac{dA}{dt} &= \frac{2}{3}A(2A + B)(A - B), \\
\frac{dB}{dt} &= -\frac{2}{3}B(2B + A)(A - B), \\
\frac{dC}{dt} &= -\frac{2}{3}C(A - B)^2.
\end{align*}
\]

(6.2)

Assume that \(A_0 \geq B_0\), Cao [5] proved the following theorem.

**Theorem 6.1.**

1. If \(A_0 = B_0\), then \(T_+ = \infty\), and \(g(t) = g_0\) for \(t \in [0, +\infty)\).
2. If \(A_0 > B_0\), then \(T_+ < \infty\), there exist two positive constants \(\eta_1, \eta_2\) such that

\[
A \sim \frac{\sqrt{6}}{4}(T_+ - t)^{-1/2}, \quad B(t) \sim \eta_1(T_+ - t)^{1/4}, \quad C(t) \sim \eta_2(T_+ - t)^{1/4},
\]

as \(t\) goes to \(T_+\).

In case (2), \(\bar{g}(t) = (B_0/B(t))g(t)\) converges to a sub-Riemannian geometry.

We will prove the following theorem.

**Theorem 6.2.** Let \(\lambda(t)\) be the first eigenvalue of \(-\Delta\). Then we get

1. If \(A_0 = B_0\), then \(g(t) = g_0\), and \(\lambda(t)\) is a constant.
2. If \(A_0 > B_0\) and \(C_0 \geq B_0\), then there is a time \(\tau\) such that \(\lambda(t)e^{\int_0^t (-\frac{2}{3}R + 2R_{11})dt}\) is nondecreasing along the backward Ricci flow, and \(\lambda(t)e^{\int_0^t (-\frac{2}{3}R + 2R_{33})dt}\) is nonincreasing. Moreover, we have

\[
\lambda(\tau)e^{2c_2[(T_+ - \tau)^{-1/2} - (T_+ - t)^{-1/2}]} \leq \lambda(t) \leq \lambda(\tau)\left(\frac{T_+ - t}{T_+ - \tau}\right)^{c_1}.
\]

As \(t\) goes to \(T_+\), \(\lambda(t)\) approaches 0.

**Remark.** In case (2) of the above theorem, if \(C_0 < B_0\), we can get the similar estimate.

**Proof.**

1. If \(A_0 = B_0\), then \(g(t) = g_0\), and \(\lambda(t)\) is independent of \(t\).
2. If \(A_0 > B_0\), then by (6.2) we have

\[
\frac{d}{dt}(A - B) = \frac{4}{3}(A - B)(A^2 + AB + B^2).
\]

So \(A - B\) is increasing and \(A(t) > B(t)\). This and (6.1) yields

\[
R_{11} > 0, \quad R_{22} < 0, \quad R_{33} < 0.
\]

Assume that \(C_0 \geq B_0\). From the equations for \(B, C\) in (6.2), we get

\[
\frac{d}{dt}\ln \frac{C}{B} = \frac{2}{3}[(2B + A)(A - B) - (A - B)^2] = 2B(A - B)
\]

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and conclude that \( \frac{C}{\pi} \) is increasing and \( C(t) > B(t) \) for \( t > 0 \).

By (6.1) and (2) in Theorem 6.1, we get
\[
R_{22} - R_{33} = \frac{1}{2}(A - B)(AC - BC - AB - B^2)
\]
\[
= \frac{1}{2}(A - B)[A(C - B) - BC - B^2]
\]
\[
> 0
\]
if \( t \geq \tau \). So we obtain \( R_{11} > R_{22} > R_{33} \) and
\[
\frac{2}{3}R\lambda - 2R_{11}\lambda \leq \frac{d}{dt}\lambda \leq \frac{2}{3}R\lambda - 2R_{33}\lambda
\]
with \( t \geq \tau \).

Then \( \lambda(t)e^{\int_{\tau}^{t}(-\frac{2}{3}R+2R_{11})dt} \) is nondecreasing along the backward Ricci flow, and \( \lambda(t)e^{\int_{\tau}^{t}(-\frac{2}{3}R+2R_{33})dt} \) is nonincreasing.

Further computations show that
\[
\frac{2}{3}R - 2R_{33}
\]
\[
= -\frac{1}{3}(A - B)^2 + C(A - B)^2
\]
\[
= -\frac{1}{3}A^2 - \frac{1}{3}B^2 + \frac{2}{3}AB + CA^2 + CB^2 - 2ABC
\]
\[
\leq -c_1(T_+ - t)^{-1}
\]
and
\[
\frac{2}{3}R - 2R_{11}
\]
\[
= -\frac{1}{3}(A - B)^2 - A(A^2 - B^2)
\]
\[
= -\frac{1}{3}A^2 - \frac{1}{3}B^2 + \frac{2}{3}AB - A^3 + AB^2
\]
\[
\geq -c_2(T_+ - t)^{-3/2}
\]
after a time \( \tau \). Thus we arrive at
\[
-c_2(T_+ - t)^{-3/2} \leq \frac{1}{\lambda} \frac{d}{dt} \lambda \leq -c_1(T_+ - t)^{-1}.
\]
Integration from \( \tau \) to \( t \) gives
\[
\lambda(\tau)e^{2c_2\left[(T_+ - \tau)^{-1/2} - (T_+ - t)^{-1/2}\right]} \leq \lambda(t) \leq \lambda(\tau) \left(\frac{T_+ - t}{T_+ - \tau}\right)^{c_1}.
\]
As \( t \) goes to \( T_+ \), \( \lambda(t) \) approaches 0. 
\( \square \)
7. $\text{SL}(2, \mathbb{R})$

Given a metric $g_0$, we choose a Milnor frame such that

$$[f_2, f_3] = -2f_1, \quad [f_3, f_1] = 2f_2, \quad [f_1, f_2] = 2f_3.$$ 

Under the normalization $A_0B_0C_0 = 4$, the nonzero curvature components are

$$R_{11} = \frac{1}{2}A[A^2 - (B - C)^2],$$
$$R_{22} = \frac{1}{2}B[B^2 - (A + C)^2],$$
$$R_{33} = \frac{1}{2}C[C^2 - (A + B)^2],$$
$$R = \frac{1}{2}[A^2 - (B - C)^2] + \frac{1}{2}[B^2 - (A + C)^2] + \frac{1}{2}[C^2 - (A + B)^2].$$

Then the backward Ricci flow equations are

$$\begin{align*}
\frac{dA}{dt} &= -\frac{2}{3}[-A^2(2A + B + C) + A(B - C)^2], \\
\frac{dB}{dt} &= -\frac{2}{3}[-B^2(2B + A - C) + B(A + C)^2], \\
\frac{dC}{dt} &= -\frac{2}{3}[-C^2(2C + A - B) + C(A + B)^2]
\end{align*}$$

Under the assumption $B_0 \geq C_0$, Cao [3, 5] proved the following theorem.

**Theorem 7.1.** The maximal existence time $T_+$ is finite. Moreover,

1. If there exists a time $t_0$ such that $A(t_0) \geq B(t_0)$, then
   $$A \sim \sqrt{\frac{6}{4}}(T_+ - t)^{-1/2}, \quad B(t) \sim \eta_1(T_+ - t)^{1/4}, \quad C(t) \sim \eta_2(T_+ - t)^{1/4}$$
   with positive constants $\eta_i, \; i = 1, 2$.

2. If there exists a time $t_0$ such that $A(t_0) \leq B(t_0) - C(t_0)$, then
   $$A \sim \eta_1(T_+ - t)^{1/4}, \quad B(t) \sim \sqrt{\frac{6}{4}}(T_+ - t)^{-1/2}, \quad C(t) \sim \eta_2(T_+ - t)^{1/4}$$
   with positive constants $\eta_i, \; i = 1, 2$.

3. If $A < B < A + C$ for all time $t \in [0, T_+]$, we arrive at
   $$A \sim \sqrt{\frac{6}{4}}(T_+ - t)^{-1/2}, \quad B(t) \sim \sqrt{\frac{6}{4}}(T_+ - t)^{-1/2}, \quad C(t) \sim \frac{32}{3}(T_+ - t).$$

In all cases, the metric $g(t)$ converges to a sub-Riemannian geometry after a proper rescaling.

We have the following theorem.
Theorem 7.2. Let $\lambda(t)$ be the first eigenvalue of $-\Delta$. Then we get

(1) If there is a time $t_0$ such that $A(t_0) \geq B(t_0)$, then there exist a time $\tau$ such that $\lambda(t)e^{\int_0^t (-\frac{4}{3}R+2R_{11})dt}$ is nondecreasing along the backward Ricci flow, and $\lambda(t)e^{\int_t^{T_+} (-\frac{4}{3}R+2R_{22})dt}$ is nonincreasing. Moreover, we have

$$\lambda(\tau)e^{2c_2[(T_+-\tau)^{-1/2}-(T_+\cdot t)^{-1/2}]} \leq \lambda(t) \leq \lambda(\tau) \left(\frac{T_+-t}{T_+-\tau}\right)^{c_1}.$$ 

As $t$ goes to $T_+$, $\lambda(t)$ approaches 0.

(2) If there exists a time $t_0$ such that $A(t_0) \leq B(t_0) - C(t_0)$, and a $t_1$ such that $A(t_1) > C(t_1)$, then there is a time $\tau$ such that $\lambda(t)e^{\int_0^t (-\frac{4}{3}R+2R_{11})dt}$ is nondecreasing along the backward Ricci flow, and $\lambda(t)e^{\int_t^{T_+} (-\frac{4}{3}R+2R_{22})dt}$ is nonincreasing. Moreover, we have

$$\lambda(\tau)e^{2c_2[(T_+-\tau)^{-1/2}-(T_+\cdot t)^{-1/2}]} \leq \lambda(t) \leq \lambda(\tau) \left(\frac{T_+-t}{T_+-\tau}\right)^{c_1},$$

and $\lambda(t)$ approaches 0 as $t$ goes to $T_+$.

(3) If $A < B < A+C$ for all time $t \in [0, T_+)$, then there is time $\tau$ such that $\lambda(t)e^{\int_0^t (-\frac{4}{3}R+2R_{11})dt}$ is nondecreasing along the backward Ricci flow, and $\lambda(t)e^{\int_t^{T_+} (-\frac{4}{3}R+2R_{22})dt}$ is nonincreasing. Moreover, we have

$$\lambda(\tau) \left(\frac{T_+-t}{T_+-\tau}\right)^{c_2} \leq \lambda(t) \leq \lambda(\tau) \left(\frac{T_+-t}{T_+-\tau}\right)^{c_1}.$$ 

Obviously, $\lambda(t)$ approaches 0, as $t$ goes to $T_+$.

Remark. If $A(t) \leq C(t)$ for all $t$ in case (2) of the above theorem, we can get the similar estimate.

Proof. (1) If there is a time $t_0$ such that $A(t_0) > B_0$, then by (7.1) and (1) in Theorem 7.1, we have

$$R_{11} > 0, \quad R_{22} < 0, \quad R_{33} < 0$$

after a time $\tau$.

Next we compare $R_{22}$ with $R_{33}$:

$$R_{22} - R_{33} = \frac{1}{2}B[B^2 - (A+C)^2] - \frac{1}{2}C[C^2 - (A+B)^2] = \frac{1}{2}(B-C)[(B+C)^2 - A^2].$$

The evolution equation of $B-C$ is

$$\frac{d}{dt}(B-C) = \frac{2}{3}[2(B^3 - C^3) + A(B^2 - C^2) - A^2(B - C)],$$

from which it follows that $B \geq C$ for all $t$.

Thus we obtain

$$R_{11} > R_{33} \geq R_{22}.$$
and
\[ \frac{2}{3}R\lambda - 2R_{11}\lambda \leq \frac{d}{dt}\lambda \leq \frac{2}{3}R\lambda - 2R_{22}\lambda \]
with \( t \geq \tau \).

Then \( \lambda(t)e^{\int_{\tau}^{t}(-\frac{2}{3}R+2R_{11})dt} \) is nondecreasing along the backward Ricci flow, and
\( \lambda(t)e^{\int_{\tau}^{t}(-\frac{2}{3}R+2R_{22})dt} \) is nonincreasing.

Moreover, we get
\[
\frac{2}{3}R - 2R_{22} = \frac{1}{3}(2BC - 2AB - 2AC - A^2 - B^2 - C^2) - B[B^2 - (A + C)^2] \\
\leq - c_1(T_+ - t)^{-1}
\]
and
\[
\frac{2}{3}R - 2R_{11} = \frac{1}{3}(2BC - 2AB - 2AC - A^2 - B^2 - C^2) - A[A^2 - (B - C)^2] \\
\geq - c_2(T_+ - t)^{-3/2}
\]
after a time \( \tau \).

Integrating from \( \tau \) to \( t \) gives
\[
\lambda(t)e^{2c_2[(T_+ - \tau)^{-1/2} - (T_+ - t)^{-1/2}]} \leq \lambda(t) \leq \lambda(\tau) \left( \frac{T_+ - t}{T_+ - \tau} \right)^{c_1}.
\]

As \( t \) goes to \( T_+ \), \( \lambda(t) \) approaches 0.

(2) If there exists a time \( t_0 \) such that \( A(t_0) \leq B(t_0) - C(t_0) \), the second case in Theorem 7.1 implies
\[
R_{11} < 0, \ R_{22} > 0, \ R_{33} < 0
\]
after a time \( \tau \).

It follows from equations for both \( A \) and \( C \) that
\[
\frac{d}{dt}\ln(A/C) = 2(A + B - C)(A + C),
\]
which implies that \( \frac{d}{dt}\ln(A/C) \) is increasing since \( B \geq C \) is preserved.

Assume that there is time \( t_1 \) such that \( A(t_1) > C(t_1) \). Then \( A(t) > C(t) \) after \( t_1 \).

The behaviors of \( A, \ B \) and \( C \) yield that
\[
R_{11} - R_{33} = \frac{1}{2}A(A^2 - (B - C)^2) - \frac{1}{2}C[C^2 - (A + B)^2] \\
= \frac{1}{2}(C - A)B^2 + \frac{1}{2}(A^3 - C^3 + CA^2 - AC^2 + 4ABC) \\
< 0
\]
after a time \( \tau \).

Thus we arrive at
\[
R_{22} > R_{33} > R_{11}
\]
and
\[ \frac{2}{3} R\lambda - 2R_{22}\lambda < \frac{d}{dt}\lambda < \frac{2}{3} R\lambda - 2R_{11}\lambda \]
with \( t \geq \tau \).

Then \( \lambda(t)e^{\int_{t}^{\tau}(-\frac{2}{3} R_{11} + 2R_{22})dt} \) is nondecreasing along the backward Ricci flow, and \( \lambda(t)e^{\int_{t}^{\tau}(-\frac{2}{3} R_{11} + 2R_{33})dt} \) is nonincreasing.

Now we estimate
\[
\frac{2}{3} R - 2R_{22} = \frac{1}{3} (2BC - 2AB - 2AC - A^2 - B^2 - C^2 - B[B^2 - (A + C)^2]) \\
\geq - c_2 (T_+ - t)^{-3/2}
\]
and
\[
\frac{2}{3} R - 2R_{11} = \frac{1}{3} (2BC - 2AB - 2AC - A^2 - B^2 - C^2 - A[A^2 - (B - C)^2]) \\
\leq - c_1 (T_+ - t)^{-1}
\]
if \( t \geq \tau \).

The above estimates imply
\[
\lambda(\tau)e^{2c_2[(T_+ - \tau)^{-1/2} - (T_+ - t)^{-1/2}]} \leq \lambda(t) \leq \lambda(\tau) \left( \frac{T_+ - t}{T_+ - \tau} \right)^{c_1},
\]
and \( \lambda(t) \) approaches 0, as \( t \) goes to \( T_+ \).

(3) If \( A < B < A + C \) for all time \( t \in [0, T_+] \), it follows from (7.1) and Theorem 7.1 that
\[
R_{11} > 0, \ R_{22} < 0, \ R_{33} < 0
\]
after a time \( \tau \).

Since \( A < B < A + C \), we obtain
\[
R_{22} - R_{33} = \frac{1}{2}(B - C)[(B + C)^2 - A^2] \geq 0.
\]

Thus we have
\[
R_{11} > R_{22} \geq R_{33}
\]
and
\[
\frac{2}{3} R\lambda - 2R_{11}\lambda < \frac{d}{dt}\lambda \leq \frac{2}{3} R\lambda - 2R_{33}\lambda
\]
with \( t \geq \tau \).

Thus \( \lambda(t)e^{\int_{t}^{\tau}(-\frac{2}{3} R_{11} + 2R_{22})dt} \) is nondecreasing along the backward Ricci flow, and \( \lambda(t)e^{\int_{t}^{\tau}(-\frac{2}{3} R_{11} + 2R_{33})dt} \) is nonincreasing.
Moreover, we compute
\[
\frac{2}{3} R - 2R_{33} \\
= \frac{1}{3} (2BC - 2AB - 2AC - A^2 - B^2 - C^2) - C[C^2 - (A + B)^2] \\
\leq - c_1 (T_+ - t)^{-1}
\]
and
\[
\frac{2}{3} R - 2R_{11} \\
= \frac{1}{3} (2BC - 2AB - 2AC - A^2 - B^2 - C^2) - A[A^2 - (B - C)^2] \\
= \frac{1}{3} (2BC - 2AB - 2AC - A^2 - B^2 - C^2) + A[A + (B - C)][B - A - C] \\
\geq - c_2 (T_+ - t)^{-1}
\]
after a time \( \tau \) since in this case the fact \( \lim_{T_+} C = 0 \) yields \( \lim_{T_+} (B - A) = 0 \).

After integrating from \( \tau \) to \( T_+ \), we get
\[
\lambda(\tau) \left( \frac{T_+ - t}{T_+ - \tau} \right)^{c_2} \leq \lambda(t) \leq \lambda(\tau) \left( \frac{T_+ - t}{T_+ - \tau} \right)^{c_1}.
\]
Obviously, \( \lambda(t) \) approaches 0, as \( t \) goes to \( T_+ \). \qed

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