GLOBAL EXISTENCE, BOUNDEDNESS AND STABILIZATION
IN A HIGH-DIMENSIONAL CHEMOTAXIS SYSTEM
WITH CONSUMPTION

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(Communicated by Hirokazu Ninomiya)

ABSTRACT. This paper deals with the homogeneous Neumann boundary-value problem for the chemotaxis-consumption system
\[
\begin{align*}
    u_t &= \Delta u - \chi \nabla \cdot (u \nabla v) + \kappa u - \mu u^2, \quad x \in \Omega, \ t > 0, \\
    v_t &= \Delta v - uv, \quad x \in \Omega, \ t > 0,
\end{align*}
\]
in N-dimensional bounded smooth domains for suitably regular positive initial data.
We shall establish the existence of a global bounded classical solution for suitably large \( \mu \) and prove that for any \( \mu > 0 \) there exists a weak solution.
Moreover, in the case of \( \kappa > 0 \) convergence to the constant equilibrium \( \left( \frac{\kappa}{\mu}, 0 \right) \) is shown.

1. Introduction. Chemotaxis is the adaptation of the direction of movement to an external chemical signal. This signal can be a substance produced by the biological agents (cells, bacteria) themselves, as is the case in the celebrated Keller-Segel model ([8], [6]) or – in the case of even simpler organisms – by a nutrient that is consumed. A prototypical model taking into account random and chemotactically directed movement of bacteria alongside death effects at points with high population densities and population growth together with diffusion and consumption of the nutrient is given by
\[
\begin{align*}
    u_t &= \Delta u - \chi \nabla \cdot (u \nabla v) + \kappa u - \mu u^2 \\
    v_t &= \Delta v - uv,
\end{align*}
\]

2010 Mathematics Subject Classification. 35Q92, 35K55, 35A01, 35B40, 35D30, 92C17.
Key words and phrases. Chemotaxis, logistic source, global existence, boundedness, asymptotic stability, weak solution.
J. Lankeit acknowledges support of the Deutsche Forschungsgemeinschaft within the project Analysis of chemotactic cross-diffusion in complex frameworks. Y. Wang was supported by the NNSF of China (no. 11501457).
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considered in a smooth, bounded domain \( \Omega \subset \mathbb{R}^N \) together with homogeneous Neumann boundary conditions and suitable initial data. Herein, \( \chi > 0, \kappa \in \mathbb{R}, \mu > 0 \) denote chemotactic sensitivity, growth rate (or death rate, if negative) and strength of the logistic saturation, respectively. The system \( (1) \), in a basic form often with \( \kappa = \mu = 0 \), appears as part of chemotaxis-fluid models intensively studied over the past few years (see e.g. the survey [1, sec. 4.1] or [2] for a recent contribution with an extensive bibliography).

Compared with the signal-production Keller-Segel model

\[
\begin{align*}
    u_t &= \Delta u - \chi \nabla \cdot (u \nabla v) + \kappa u - \mu u^2 \\
    v_t &= \Delta v - v + u,
\end{align*}
\]

which we have given in the form with logistic source terms paralleling that in \( (1) \), at first glance, \( (1) \) seems much more amenable to the global existence (and boundedness) of solutions – after all, the second equation by comparison arguments immediately provides an \( L^\infty \)-bound for \( v \).

However, such a bound is not sufficient for dealing with the chemotaxis term, and accordingly global existence and boundedness of solutions to \( (1) \) with \( \kappa = \mu = 0 \) is only known under the smallness condition

\[
\chi \|v(\cdot, 0)\|_{L^\infty(\Omega)} \leq \frac{1}{6(N + 1)}
\]

on the initial data ([20]) or in a two-dimensional setting ([27], [30] and also [12]). Their rate of convergence has been treated in [31]. In three-dimensional domains, weak solutions have been constructed that eventually become smooth [21].

Upon the choice of \( \kappa = \mu = 0 \), \( (2) \) turns into the classical Keller–Segel system, whose probably most striking feature is the possible occurrence of blow-up (c.f. [4], [28], [15]). Even more delicate results pertaining to the precise structure of the singularity formation ([5], [16]), which occurs in the form of Dirac-masses, or the type of blow-up ([14]) have been achieved. Moreover, it is known that blow-up in finite time is a generic feature for solutions emanating from radially symmetric initial data ([28], [15]).

The presence of logarithmic terms in \( (2) \) has been shown to exclude such finite-time blow-up phenomena – at least as long as \( \mu \) is sufficiently large if compared to the strength of the chemotactic effects ([26]) or if the dimension is 2 ([17]). If the quotient \( \frac{\mu}{\chi} \) is sufficiently large, solutions to \( (2) \) uniformly converge to the constant equilibrium ([29]); convergence rates have been considered in [3]. Explicit largeness conditions on \( \frac{\mu}{\chi} \) that ensure convergence, also for slightly more general source terms, can be found in [13], see also [24]. For small \( \mu > 0 \), at least global weak solutions are known to exist ([10]), and in 3-dimensional domains and for small \( \kappa \), their large-time behaviour has been investigated ([10]).

Also the chemotaxis-consumption model \( (1) \) has already been considered with nontrivial source terms in [23]. There it was proved that classical solutions exist globally and are bounded as long as \( (3) \) holds – which is the same condition as for \( \kappa = \mu = 0 \), thus shedding no light on any possible interplay between chemotaxis and the population kinetics.

In a three-dimensional setting and in the presence of a Navier-Stokes fluid, in [11] it was recently possible to construct global weak solutions for any positive \( \mu \), which moreover eventually become classical and uniformly converge to the constant equilibrium in the large-time limit.
It is the aim of the present article to prove the existence of global classical solutions if only \( \mu \) is suitably large and to show their large-time behaviour. For the case of small \( \mu > 0 \), we will prove the existence of global weak solutions (in the sense of Definition 6.1).

**What largeness condition on \( \mu \) might be sufficient for boundedness?** For the Keller-Segel type model (2) the typical condition reads: ‘If \( \mu \) is large compared to \( \chi \), then the solution is global and bounded, independent of initial data.’ In order to see why this condition would be far less natural for (1), let us suppose we are given suitably regular initial data \( u_0, v_0 \) and a corresponding solution \((u,v)\) of

\[
\begin{align*}
\begin{cases}
    u_t &= \Delta u - \chi \nabla \cdot (u \nabla v) + \kappa u - \mu u^2 \\
    v_t &= \Delta v - uv \\
    \partial_n u |_{\partial \Omega} &= \partial_n v |_{\partial \Omega} = 0 \\
    u(\cdot, 0) &= u_0, v(\cdot, 0) = v_0,
\end{cases}
\end{align*}
\]

and let us define

\[ w := \chi v. \]

Then \((u,w)\) solves

\[
\begin{align*}
\begin{cases}
    u_t &= \Delta u - \nabla \cdot (u \nabla w) + \kappa u - \mu u^2 \\
    w_t &= \Delta w - uw \\
    \partial_n u |_{\partial \Omega} &= \partial_n v |_{\partial \Omega} = 0 \\
    u(\cdot, 0) &= u_0, w(\cdot, 0) = \chi v_0,
\end{cases}
\end{align*}
\]

which is the same system, only with different chemotaxis coefficient and rescaled initial data for the second component. Consequently, in (1), large initial data equal high chemotactic strength. Hence, there cannot be any condition for global existence which includes \( \mu \) and \( \chi \parallel v_0 \parallel _{L^\infty(\Omega)} \), but not \( \parallel v_0 \parallel _{L^\infty(\Omega)} \). In light of this discussion, the requirement in Theorem 1.1 that \( \mu \) be large with respect to \( \chi \parallel v_0 \parallel _{L^\infty(\Omega)} \) seems natural. On the other hand, this observation does not preclude conditions that involve neither \( \chi \) nor \( \parallel v_0 \parallel _{L^\infty(\Omega)} \), and indeed \( \mu > 0 \) is sufficient for the global existence of weak solutions.

The first main result of the present article is global existence of classical solutions, provided that \( \mu \) is sufficiently large as compared to \( \parallel \chi v_0 \parallel _{L^\infty(\Omega)} \):

**Theorem 1.1.** Let \( N \in \mathbb{N} \) and let \( \Omega \subset \mathbb{R}^N \) be a smooth, bounded domain. There are constants \( k_1 = k_1(N) \) and \( k_2 = k_2(N) \) such that the following holds: Whenever \( \kappa \in \mathbb{R}, \chi > 0, \) and \( \mu > 0 \) and initial data

\[
\begin{align*}
\begin{cases}
    u_0 \in C^0(\Omega), & u_0 > 0 \text{ in } \Omega, \\
    v_0 \in C^1(\Omega), & v_0 > 0 \text{ in } \Omega
\end{cases}
\end{align*}
\]

are such that

\[ \mu > k_1(N)\parallel \chi v_0 \parallel _{L^\infty(\Omega)}^{\frac{1}{N}} + k_2(N)\parallel \chi v_0 \parallel _{L^\infty(\Omega)}^{2N}, \]

then the system

\[
\begin{align*}
\begin{cases}
    u_t &= \Delta u - \chi \nabla \cdot (u \nabla v) + \kappa u - \mu u^2, & x \in \Omega, t > 0, \\
    v_t &= \Delta v - uv, & x \in \Omega, t > 0, \\
    \partial_n u &= \partial_n v = 0, & x \in \partial \Omega, t > 0, \\
    u(x, 0) &= u_0(x), & v(x, 0) = v_0(x), & x \in \Omega,
\end{cases}
\end{align*}
\]
has a unique global classical solution \((u, v)\) which is uniformly bounded in the sense that there is some constant \(C > 0\) such that
\[
\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{W^{1,\infty}(\Omega)} \leq C \quad \text{for all} \quad t \in (0, \infty).
\] (6)

**Remark 1.** Here we have to leave open the question, whether, for small values of \(\mu > 0\) and large \(\chi v_0\), blow-up of solutions is possible at all. Consequently, the range of \(\mu\) in this result is not necessarily an optimal one. Nevertheless, the present condition can easily be made explicit (see Lemma 1.5 and (26) for the values of \(k_1\) and \(k_2\)). It seems worth pointing out that, in contrast to the condition (3), Theorem 1.1 admits large values of \(\chi v_0\), if only \(\mu\) is appropriately large.

The second outcome of our analysis is concerned with the large time behaviour of global solutions and reads as follows:

**Theorem 1.2.** Let \(N \in \mathbb{N}\) and let \(\Omega \subset \mathbb{R}^N\) be a bounded smooth domain. Suppose that \(\chi > 0\), \(\kappa > 0\) and \(\mu > 0\). Let \((u, v) \in C^{2,1}(\Omega \times (0, \infty)) \cap C^0(\overline{\Omega} \times [0, \infty))\) be any global bounded solution to (5) (in the sense that (6) is fulfilled) which obeys (4). Then
\[
\left\| u(\cdot, t) - \frac{\kappa}{\mu} \right\|_{L^\infty(\Omega)} \to 0
\]
and
\[
\|v(\cdot, t)\|_{L^\infty(\Omega)} \to 0
\]
as \(t \to \infty\).

**Remark 2.** This theorem in particular applies to the solutions considered in Theorem 1.1.

**Remark 3.** Boundedness is not necessary in the sense of (6); in light of Lemma 4.4, the existence of \(C > 0\) and \(p > N\) such that
\[
\|u(\cdot, t)\|_{L^p(\Omega)} \leq C \quad \text{for all} \quad t > 0
\]
would be sufficient.

**Unconditional global weak solvability.** As in the context of the signal-production Keller–Segel model (2) ([10]), global weak solutions to (5) can be shown to exist regardless of the size of initial data and for any positive \(\mu\):

**Theorem 1.3.** Let \(N \in \mathbb{N}\) and let \(\Omega \subset \mathbb{R}^N\) be a bounded smooth domain. Let \(\chi > 0\), \(\kappa \in \mathbb{R}\), \(\mu > 0\) and assume that \(u_0, v_0\) satisfy (4). Then system (5) has a global weak solution (in the sense of Definition 6.1 below).

These solutions, too, stabilize toward \((\kappa \mu, 0)\) as \(t \to \infty\), even though in a weaker sense than guaranteed by Theorem 1.2 for classical solutions:

**Theorem 1.4.** Let \(N \in \mathbb{N}\) and let \(\Omega \subset \mathbb{R}^N\) be a bounded smooth domain. Let \(\chi > 0\), \(\kappa > 0\), \(\mu > 0\) and assume that \(u_0, v_0\) satisfy (4). Then for any \(p \in [1, \infty)\) the weak solution \((u, v)\) to (5) obtained in the proof of Theorem 1.3 satisfies
\[
\|v(\cdot, t)\|_{L^p(\Omega)} \to 0 \quad \text{and} \quad \int_t^{t+1} \|u(\cdot, s) - \frac{\kappa}{\mu} \|_{L^2(\Omega)} ds \to 0
\]
as \(t \to \infty\).
Remark 4. Under the restriction \( N = 3 \), the existence of global weak solutions that eventually become smooth and uniformly converge to \((\frac{\kappa}{\mu}, 0)\) has been proven in \([11]\), where a coupled chemotaxis-fluid model is treated.

Plan of the paper. In Section 2 we will prepare some general calculus inequalities. In the following for some \( a > 0 \) we will then consider

\[
\begin{aligned}
\begin{cases}
u_{\varepsilon t} = \Delta \nu_{\varepsilon} - \chi \nabla \cdot (u_{\varepsilon} \nabla v_{\varepsilon}) + \kappa \nu_{\varepsilon} - \mu u_{\varepsilon}^2 - \varepsilon u_{\varepsilon}^2 \ln a u_{\varepsilon} \\
u_{\varepsilon t} = \Delta v_{\varepsilon} - u_{\varepsilon} v_{\varepsilon} \\
_{\varepsilon} \nu_{\varepsilon} |_{\partial \Omega} = \partial_{\nu} v_{\varepsilon} |_{\partial \Omega} = 0 \\
u_{\varepsilon}(\cdot, 0) = u_0, v_{\varepsilon}(\cdot, 0) = v_0.
\end{cases}
\end{aligned}
\]

For \( \varepsilon = 0 \), this system reduces to \((5)\); for \( \varepsilon \in (0, 1) \) we will be able to derive global existence of solutions without any concern for the size of initial data and hence obtain a suitable stepping stone for the construction of weak solutions. Beginning the study of solutions to this system in Section 3 with a local existence result and elementary properties of the solutions, we will in Section 4 consider a functional of the type \( \int_{\Omega} u^p + \int_{\Omega} |\nabla v|^2 \) and finally, aided by estimates for the heat semigroup, obtain globally bounded solutions, thus proving Theorem 1.1. In Section 5, where \( \kappa \) is assumed to be positive, we will let \( a := \frac{\mu}{\kappa} \) and employ the functional

\[
F_{\varepsilon}(t) = \int_{\Omega} u_{\varepsilon}(\cdot, t) - \frac{\kappa}{\mu} \int_{\Omega} \ln u_{\varepsilon}(\cdot, t) + \frac{\kappa}{2\mu} \int_{\Omega} v_{\varepsilon}^2(\cdot, t)
\]

in order to derive the stabilization result in Theorem 1.2 and already prepare Theorem 1.4. Section 6 finally, will be devoted to the construction of weak solutions to \((5)\), and to the proofs of Theorem 1.3 and Theorem 1.4.

Remark 5. In \([5]\), the additional term \(-\varepsilon u_{\varepsilon}^2 \ln a u_{\varepsilon}\) could be replaced by \(-\varepsilon \Phi(u_{\varepsilon})\) with some other continuous function \( \Phi \) which satisfies: \( \Phi(s) \to 0 \) as \( s \to 0 \), \( \frac{\Phi(s)}{s^2} \to \infty \) as \( s \to \infty \) and, for the stabilization results in Section 5, \( \Phi < 0 \) on \((0, \frac{\mu}{\kappa})\) as well as \( \Phi > 0 \) on \((\frac{\mu}{\kappa}, \infty)\).

We will always let

\[
a := \begin{cases}
\frac{\mu}{\kappa} & \text{if } \kappa > 0 \\
\mu & \text{if } \kappa \leq 0
\end{cases}
\]

and note that the choice for the case \( \kappa \leq 0 \) was arbitrary and that in Sections 4 and 6 the precise value of \( a \) plays no important role.

Notation. For solutions of PDEs we will use \( T_{\max} \) to denote their maximal time of existence (cf. also Lemma 3.1). Throughout the article we fix \( N \in \mathbb{N} \) and a bounded, smooth domain \( \Omega \subset \mathbb{R}^N \).

2. General preliminaries. In this section we provide some estimates that are valid for all suitably regular functions and not only for solutions of the PDE under consideration.

Lemma 2.1. a) For any \( c \in C^2(\Omega) \):

\[
|\Delta c|^2 \leq N|D^2 c|^2 \text{ throughout } \Omega.
\]
b) There are $C > 0$ and $k > 0$ such that every positive $c \in C^2(\Omega)$ fulfilling $\partial_c c = 0$ on $\partial \Omega$ satisfies
\[
-2 \int_\Omega \frac{|\Delta c|^2}{c} + \int_\Omega \frac{|
abla c|^2 \Delta c}{c^3} \leq -k \int_\Omega c |D^2 \ln c|^2 - k \int_\Omega \frac{|
abla c|^4}{c^3} + C \int_\Omega c. \tag{12}
\]

c) Let $p \in [1, \infty)$. For every $\delta > 0$ there is $C_\delta > 0$ such that every function $w \in C^2(\Omega)$ with $\partial_w w = 0$ on $\partial \Omega$ satisfies
\[
\int_{\partial \Omega} |\nabla w|^{2p-2} \partial_w |\nabla w|^2 \leq \delta \int_\Omega |\nabla w|^{2p-4} |\nabla \nabla w|^2 + C_\delta \left( \int_\Omega |\nabla w|^2 \right)^p.
\]

Proof. a) Straightforward calculations yield
\[
\left( \sum_{i=1}^N c_{x,i}^2 \right)^2 = \sum_{i,j=1}^N c_{x,i} c_{x,j} \leq \sum_{i,j=1}^N \left( \frac{1}{2} c_{x,i}^2 + \frac{1}{2} c_{x,j}^2 \right) = N \sum_{i=1}^N c_{x,i}^2 \leq N \sum_{i,j=1}^N c_{x,i} c_{x,j}.
\]

b) This is [11, Lemma 2.7 vii].

c) Combining estimates for $\partial_w |\nabla w|^2$ on the boundary with embeddings of the type $W^{r+\frac{1}{2},2} \hookrightarrow L^2(\partial \Omega)$, $r \in (0, \frac{1}{2})$ and a Gagliardo-Nirenberg inequality for fractional Sobolev spaces, this can be obtained in the same way as in [7, Prop. 3.2].

Let us now derive the following interpolation inequality on which we will rely in obtaining an estimate for $\int_\Omega u^p + \int_\Omega |\nabla v|^{2p}$ in Section 4.

**Lemma 2.2.** Let $q \in [1, \infty)$. Then for any $c \in C^2(\Omega)$ satisfying $\frac{\partial c}{\partial \nu} = 0$ on $\partial \Omega$, the inequality
\[
\|\nabla c\|_{L^{2q+2}(\Omega)}^{2q+2} \leq 2(4q^2 + N)\|c\|_{L^\infty}^2 \|\nabla c\|_{L^2(\Omega)}^{2q-1} \|D^2 c\|_{L^2(\Omega)}^2 \tag{13}
\]
holds, where $D^2 c$ denotes the Hessian of $c$.

Proof. Since $\frac{\partial c}{\partial \nu} = 0$ on $\partial \Omega$, an integration by parts yields
\[
\|\nabla c\|_{L^{2q+2}(\Omega)}^{2q+2} = -\int_\Omega c |\nabla c|^{2q} \Delta c - 2q \int_\Omega c |\nabla c|^{2q-2} \nabla c \cdot (D^2 c \cdot \nabla c).
\]

Using Young’s inequality and (11) we can estimate
\[
\left| -\int_\Omega c |\nabla c|^{2q} \Delta c \right| \leq \frac{1}{4} \int_\Omega |\nabla c|^{2q+2} + \int_\Omega c^2 |\nabla c|^{2q-2} |\Delta c|^2 \leq \frac{1}{4} \int_\Omega |\nabla c|^{2q+2} + N\|c\|_{L^\infty(\Omega)} \int_\Omega |\nabla c|^{2q-2} |D^2 c|^2. \tag{14}
\]

Likewise, we see that
\[
\left| -2q \int_\Omega c |\nabla c|^{2q-2} \nabla c \cdot (D^2 c \cdot \nabla c) \right| \leq \frac{1}{4} \int_\Omega |\nabla c|^{2q+2} + 4q^2 \|c\|_{L^\infty(\Omega)}^2 \int_\Omega |\nabla c|^{2q-2} |D^2 c|^2. \tag{15}
\]

In consequence, (14) and (15) prove (13).
3. Local existence and basic properties of solutions. We first recall a result on local solvability of \(\ref{9}\):

**Lemma 3.1.** Let \(u_0, v_0\) satisfy \(\ref{4}\), let \(\kappa \in \mathbb{R}, \mu > 0, \chi > 0\) and \(q > N\). Then for any \(\varepsilon \in [0, 1]\) there exist \(T_{\max} \in (0, \infty)\) and unique classical solution \((u_\varepsilon, v_\varepsilon)\) of system \(\ref{9}\) with \(a\) as in \(\ref{10}\) in \(\Omega \times (0, T_{\max})\) such that
\[
\begin{align*}
  u_\varepsilon &\in C^0(\overline{\Omega} \times [0, T_{\max}]) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\max})), \\
  v_\varepsilon &\in C^0(\overline{\Omega} \times [0, T_{\max}]) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\max})).
\end{align*}
\]
Moreover, we have \(u_\varepsilon > 0\) and \(v_\varepsilon > 0\) in \(\overline{\Omega} \times (0, T_{\max})\), and
\[
\text{if } T_{\max} < \infty, \text{ then } \limsup_{t \uparrow T_{\max}} (\|u_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} + \|v_\varepsilon(\cdot, t)\|_{W^{1,q}(\Omega)}) = \infty. \tag{16}
\]

*Proof.* Apart from minor adaption necessary if \(\varepsilon > 0\) (see also \(\ref{24}\) Lemma 3.1), this lemma is contained in \(\ref{27}\) Lemma 2.1. \(\square\)

Even though the total mass is not conserved, an upper bound for it can be obtained easily:

**Lemma 3.2.** Let \(u_0, v_0\) satisfy \(\ref{4}\), let \(\kappa \in \mathbb{R}, \mu > 0, \chi > 0\). Then for any \(\varepsilon \in \mathbb{R}\) the solution of \(\ref{9}\) with \(a\) as in \(\ref{10}\) satisfies
\[
\int_{\Omega} u_\varepsilon(x, t)dx \leq \max \left\{ \frac{\kappa|\Omega|}{2\mu} + \sqrt{\left( \frac{\kappa|\Omega|}{2\mu} \right)^2 + \varepsilon \frac{|\Omega|^2}{2\mu^2 e\mu} \int_{\Omega} u_0} \right\} : = m_\varepsilon \tag{17}
\]
for all \(t \in (0, T_{\max})\).

*Proof.* Because \(s^2 \ln(as) \geq -\frac{1}{2a^2 e}\) for all \(s > 0\), integrating the first equation in \(\ref{9}\) over \(\Omega\) and applying Hölder’s inequality shows that
\[
\frac{d}{dt} \int_{\Omega} u_\varepsilon \leq \kappa \int_{\Omega} u_\varepsilon - \frac{\mu}{|\Omega|} \left( \int_{\Omega} u_\varepsilon \right)^2 + \frac{\varepsilon |\Omega|^2}{2a^2 e} \text{ on } (0, T_{\max})
\]
and the claim results from an ODE-comparison argument. \(\square\)

For the second component, even uniform boundedness can be deduced instantly:

**Lemma 3.3.** Let \(u_0, v_0\) satisfy \(\ref{4}\), let \(\kappa \in \mathbb{R}, \mu > 0, \chi > 0\). Then for any \(\varepsilon \in [0, 1]\) the solution of \(\ref{9}\) with \(a\) as in \(\ref{10}\) satisfies
\[
\|v_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} \leq \|v_0\|_{L^\infty(\Omega)} \quad \text{for all } t \in (0, T_{\max}) \tag{18}
\]
and
\[
(0, T_{\max}) \ni t \mapsto \|v_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)}
\]
is monotone decreasing.

*Proof.* This is a consequence of the maximum principle and the nonnegativity of the solution. \(\square\)

Also the gradient of \(v\) can be controlled in an \(L^2(\Omega)\)-sense:

**Lemma 3.4.** Let \(u_0, v_0\) satisfy \(\ref{4}\), let \(\kappa \in \mathbb{R}, \mu > 0, \chi > 0\). There exists a positive constant \(M\) such that for all \(\varepsilon \in [0, 1]\) the solution of \(\ref{9}\) with \(a\) as in \(\ref{10}\) satisfies
\[
\int_{\Omega} |\nabla v_\varepsilon(\cdot, t)|^2 \leq M \quad \text{for all } t \in (0, T_{\max}). \tag{19}
\]
Proof. Integration by parts and the Young inequality result in
\[
\frac{d}{dt} \int_\Omega |\nabla v_\varepsilon|^2 = 2 \int_\Omega \nabla v_\varepsilon \cdot \nabla (\Delta v_\varepsilon - u_\varepsilon v_\varepsilon) \\
\leq -2 \int_\Omega |\Delta v_\varepsilon|^2 - 2 \int_\Omega |\nabla v_\varepsilon|^2 + 2 \int_\Omega v_\varepsilon (u_\varepsilon - 1) \Delta v_\varepsilon \\
\leq - \int_\Omega |\Delta v_\varepsilon|^2 - 2 \int_\Omega |\nabla v_\varepsilon|^2 + \int_\Omega v_\varepsilon^2 (u_\varepsilon - 1)^2 \\
\leq - \int_\Omega |\Delta v_\varepsilon|^2 - 2 \int_\Omega |\nabla v_\varepsilon|^2 + \|v_0\|_{L^\infty(\Omega)}^2 \int_\Omega v_\varepsilon^2 \\
+ 2 \|v_0\|_{L^\infty(\Omega)}^2 \int_\Omega u_\varepsilon + \|v_0\|_{L^\infty(\Omega)}^2 \tag{20}
\]
on (0, T_{\text{max}}). Furthermore,
\[
\frac{\|v_0\|_{L^\infty(\Omega)}^2}{\mu} \frac{d}{dt} \int_\Omega u_\varepsilon \\
\leq \frac{\kappa_+ \|v_0\|_{L^\infty(\Omega)}^2}{\mu} \int_\Omega u_\varepsilon - \|v_0\|_{L^\infty(\Omega)}^2 \int_\Omega u_\varepsilon^2 - \frac{\|v_0\|_{L^\infty(\Omega)}^2}{\mu} \int_\Omega u_\varepsilon^2 \ln(a u_\varepsilon). 
\tag{21}
\]
Adding (20) to (21) and taking into account that
\[
- \frac{\|v_0\|_{L^\infty(\Omega)}^2}{\mu} s^2 \ln s \leq \frac{\|v_0\|_{L^\infty(\Omega)}^2}{2ae^2 \mu}
\]
for any \(\varepsilon \in [0, 1]\) and \(s \geq 0\), we obtain that
\[
\frac{d}{dt} \left\{ \int_\Omega |\nabla v_\varepsilon|^2 + \frac{\|v_0\|_{L^\infty(\Omega)}^2}{\mu} \int_\Omega u_\varepsilon \right\} \\
\leq - \left( \int_\Omega |\nabla v_\varepsilon|^2 + \frac{\|v_0\|_{L^\infty(\Omega)}^2}{\mu} \int_\Omega u_\varepsilon \right) \\
+ \left( 2 \|v_0\|_{L^\infty(\Omega)}^2 + \frac{\kappa_+ + 1}{\mu} \|v_0\|_{L^\infty(\Omega)}^2 \right) \int_\Omega u_\varepsilon + \|v_0\|_{L^\infty(\Omega)}^2 + \frac{\|\Omega\| \|v_0\|_{L^\infty(\Omega)}^2}{2ae^2 \mu}. 
\]
Since Lemma 3.2 shows that \(\int_\Omega u_\varepsilon(x, t) dx \leq m_1\) for any \(\varepsilon \in [0, 1]\) and \(t \in (0, T_{\text{max}})\), a comparison argument leads to
\[
\int_\Omega |\nabla v_\varepsilon|^2 + \frac{\|v_0\|_{L^\infty(\Omega)}^2}{\mu} \int_\Omega u_\varepsilon \\
\leq \max \left\{ |\nabla v_0|^2 + \frac{\|v_0\|_{L^\infty(\Omega)}^2}{\mu} \int_\Omega u_0, \right. \\
\left. \left( 1 + \frac{\|\Omega\|}{2ae^2 \mu} \right) \|v_0\|_{L^\infty(\Omega)}^2 + \left( 2 \|v_0\|_{L^\infty(\Omega)}^2 + \frac{\kappa_+ + 1}{\mu} \|v_0\|_{L^\infty(\Omega)}^2 \right) m_1 \right\}, 
\]
holding true on (0, T_{\text{max}}), which in particular implies \([19]\).

4. Existence of a bounded classical solution. We now turn to the analysis of the coupled functional of \(\int_\Omega u^p\) and \(\int_\Omega |\nabla v|^{2p}\). We first apply standard testing procedures to gain the time evolution of each quantity.
Lemma 4.1. Let \( u_0, v_0 \) satisfy (4), let \( \kappa \in \mathbb{R}, \mu > 0, \chi > 0 \). For any \( p \in [1, \infty) \), any \( \varepsilon \in (0, 1) \), we have that the solution of (9) with \( a \) as in (10) satisfies
\[
\frac{d}{dt} \int_{\Omega} u_\varepsilon^p + \frac{2(p-1)}{p} \int_{\Omega} |\nabla u_\varepsilon|^{2}
\leq \frac{p(p-1)}{2} \chi^2 \int_{\Omega} u_\varepsilon^p |\nabla v_\varepsilon|^2 + p\kappa \int_{\Omega} u_\varepsilon^p - p\mu \int_{\Omega} u_\varepsilon^{p+1} - \varepsilon p \int_{\Omega} u_\varepsilon^{p+1} \ln a \varepsilon
\tag{22}
\]
on \( (0, T_{\text{max}}) \).

Proof. Testing the first equation in (9) against \( u_\varepsilon^{p-1} \) and using Young’s inequality, we can obtain
\[
\frac{1}{p} \frac{d}{dt} \int_{\Omega} u_\varepsilon^p = -(p-1) \int_{\Omega} u_\varepsilon^{-2} |\nabla u_\varepsilon|^2 + (p-1) \chi \int_{\Omega} u_\varepsilon^{-1} \nabla u_\varepsilon \cdot \nabla v_\varepsilon + \kappa \int_{\Omega} u_\varepsilon^p
\quad - \mu \int_{\Omega} u_\varepsilon^{p+1} - \varepsilon \int_{\Omega} u_\varepsilon^{p+1} \ln (a \varepsilon)
\leq -(p-1) \int_{\Omega} u_\varepsilon^{-2} |\nabla u_\varepsilon|^2 + \frac{p-1}{2} \int_{\Omega} u_\varepsilon^{-2} |\nabla u_\varepsilon|^2 + \frac{p-1}{2} \chi^2 \int_{\Omega} u_\varepsilon^p |\nabla v_\varepsilon|^2
\quad + \kappa \int_{\Omega} u_\varepsilon^p - \mu \int_{\Omega} u_\varepsilon^{p+1} - \varepsilon \int_{\Omega} u_\varepsilon^{p+1} \ln (a \varepsilon)
\tag{23}
\]
on \( (0, T_{\text{max}}) \), which by using the fact that
\[
\int_{\Omega} u_\varepsilon^{p-2} |\nabla u_\varepsilon|^2 = \frac{4}{p^2} \int_{\Omega} |\nabla u_\varepsilon|^2 \quad \text{on } (0, T_{\text{max}})
\]
directly results in (22).

Lemma 4.2. Let \( u_0, v_0 \) satisfy (4), let \( \kappa \in \mathbb{R}, \mu > 0, \chi > 0 \). For any \( p \in [1, \infty) \) and \( \eta > 0 \) we can find \( C_\eta > 0 \) such that for any \( \varepsilon \in (0, 1) \) the solution of (9) with \( a \) as in (10) satisfies
\[
\frac{d}{dt} \int_{\Omega} |\nabla v_\varepsilon|^{2p} + p \int_{\Omega} |\nabla v_\varepsilon|^{2p-2} |D^2 v_\varepsilon|^2
\leq p(p + N - 1 + \eta) \|v_0\|_{L^\infty(\Omega)} \int_{\Omega} u_\varepsilon^2 |\nabla v_\varepsilon|^{2p-2} + C_\eta \quad \text{on } (0, T_{\text{max}}).
\tag{24}
\]

Proof. We differentiate the second equation in (9) to compute
\[
(\|\nabla v_\varepsilon\|^2)_t = 2 \nabla v_\varepsilon \cdot \nabla \Delta v_\varepsilon - 2 \nabla v_\varepsilon \cdot \nabla (u_\varepsilon v_\varepsilon) = \Delta |\nabla v_\varepsilon|^2 - 2 |D^2 v_\varepsilon|^2 - 2 \nabla v_\varepsilon \cdot \nabla (u_\varepsilon v_\varepsilon)
\]
in \( \Omega \times (0, T_{\text{max}}) \). Upon multiplication by \( (|\nabla v_\varepsilon|^2)^{p-1} \) and integration, this leads to
\[
\frac{1}{p} \frac{d}{dt} \int_{\Omega} |\nabla v_\varepsilon|^{2p} + (p-1) \int_{\Omega} |\nabla v_\varepsilon|^{2p-4} |\nabla |\nabla v_\varepsilon|^2|^2 + 2 \int_{\Omega} |\nabla v_\varepsilon|^{2p-2} |D^2 v_\varepsilon|^2
\leq -2 \int_{\Omega} |\nabla v_\varepsilon|^{2p-2} \nabla v_\varepsilon \cdot \nabla (u_\varepsilon v_\varepsilon) + \int_{\partial\Omega} (|\nabla v_\varepsilon|^2)^{p-1} \frac{\partial}{\partial n} |\nabla v_\varepsilon|^2
\quad \tag{25}
\]
on \( (0, T_{\text{max}}) \). We choose \( \delta > 0 \) such that \( \frac{\delta(p-1)}{p-1-\delta} < \eta \) and apply Lemma 2.1(c) and Lemma 19 to find \( c_\delta > 0 \) such that
\[
\int_{\partial\Omega} (|\nabla v_\varepsilon|^2)^{p-1} \frac{\partial}{\partial n} |\nabla v_\varepsilon|^2 \leq \delta \int_{\Omega} |\nabla v_\varepsilon|^{2p-4} |\nabla |\nabla v_\varepsilon|^2|^2 + c_\delta.
\]
Lemma 4.3. Let \( \kappa \) then for every \( \mu > 0 \) the following holds: If \( \mu > \kappa \) we have that

\[
\int_\Omega |\nabla v_\varepsilon|^{2p-2} \nabla v_\varepsilon \cdot \nabla (u_\varepsilon v_\varepsilon)
\]

\[
= 2 \int_\Omega u_\varepsilon v_\varepsilon |\nabla v_\varepsilon|^{2p-2} \Delta v_\varepsilon + 2(p - 1) \int_\Omega u_\varepsilon v_\varepsilon |\nabla v_\varepsilon|^{2p-4} \nabla v_\varepsilon \cdot \nabla |\nabla v_\varepsilon|^2
\]

\[
\leq 2||v_0||_{L^\infty(\Omega)} \int_\Omega u_\varepsilon |\nabla v_\varepsilon|^{2p-2} |\Delta v_\varepsilon|
\]

\[
+ 2(p - 1)||v_0||_{L^\infty(\Omega)} \int_\Omega u_\varepsilon |\nabla v_\varepsilon|^{2p-3} \cdot |\nabla |\nabla v_\varepsilon|^2|
\]

throughout \((0, T_{\text{max}})\), were we have used Lemma 3.3. Next by Young’s inequality and Lemma 2.1 a) we have that

\[
2||v_0||_{L^\infty(\Omega)} \int_\Omega u_\varepsilon |\nabla v_\varepsilon|^{2p-2} |\Delta v_\varepsilon|
\]

\[
\leq \int_\Omega |\nabla v_\varepsilon|^{2p-2} |D^2v_\varepsilon|^2 + N||v_0||_{L^\infty(\Omega)}^2 \int_\Omega u_\varepsilon^2 |\nabla v_\varepsilon|^{2p-2},
\]

and

\[
2(p - 1)||v_0||_{L^\infty(\Omega)} \int_\Omega u_\varepsilon |\nabla v_\varepsilon|^{2p-3} \cdot |\nabla |\nabla v_\varepsilon|^2|
\]

\[
\leq (p - 1 - \delta) \int_\Omega |\nabla v_\varepsilon|^{2p-4} |\nabla |\nabla v_\varepsilon|^2|^2 + \frac{(p - 1)^2}{p - 1 - \delta} ||v_0||_{L^\infty(\Omega)}^2 \int_\Omega u_\varepsilon^2 |\nabla v_\varepsilon|^{2p-2}.
\]

Thereupon, (25) implies that with \( C_\eta := pc_\delta > 0 \)

\[
\frac{d}{dt} \int_\Omega |\nabla v_\varepsilon|^{2p} + p \int_\Omega |\nabla v_\varepsilon|^{2p-2} |D^2v_\varepsilon|^2
\]

\[
\leq p(p + N - 1 + \eta)||v_0||_{L^\infty(\Omega)}^2 \int_\Omega u_\varepsilon^2 |\nabla v_\varepsilon|^{2p-2} + C_\eta \quad \text{on } (0, T_{\text{max}}),
\]

because \( \frac{(p-1)^2}{p-1-\delta} = p - 1 + \frac{\delta(p-1)}{p-1-\delta} < p - 1 + \eta. \)

Next we will show that if \( \mu \) is suitably large, then all integrals on the right side in (22) and (24) can adequately be estimated in terms of the respective dissipated quantities on the left, in consequence implying the \( L^p \) estimate of \( \nabla u \) and the boundedness estimate for \( |\nabla v| \).

Lemma 4.3. Let \( p > 1 \). With

\[
k_1(p, N) := \frac{p(p - 1)}{p + 1} \left( \frac{4(p - 1)(4p^2 + N)}{p + 1} \right)^{\frac{1}{p}}
\]

\[
k_2(p, N) := \frac{4(p + N - 1)}{p + 1} \left( \frac{8(p - 1)(p + N - 1)(4p^2 + N)}{p + 1} \right)^{\frac{p-1}{2}}
\]

the following holds: If \( \mu > 0, \chi > 0 \) and the positive function \( v_0 \in C^1(\overline{\Omega}) \) fulfill

\[
\mu > k_1(p, N)||\chi v_0||_{L^\infty(\Omega)}^{\frac{p}{2}} + k_2(p, N)||\chi v_0||_{L^\infty(\Omega)}^{2p}
\]

then for every \( \kappa \in \mathbb{R}, 0 < u_0 \in C^0(\overline{\Omega}) \) there is \( C > 0 \) such that for every \( \varepsilon \in [0, 1) \) the solution \((u_\varepsilon, v_\varepsilon)\) of (9) with \( a \) as in (10) satisfies

\[
\int_\Omega u_\varepsilon^p(\cdot, t) + \int_\Omega |\nabla v_\varepsilon(\cdot, t)|^{2p} \leq C \quad \text{on } (0, T_{\text{max}}).
\]
If, however, \( \mu > 0, \chi > 0 \) and \( 0 < v_0 \in C^1(\bar{\Omega}) \) do not satisfy \((27)\), then for every \( \varepsilon \in (0, 1) \), \( \kappa \in \mathbb{R}, 0 < u_0 \in C^0(\bar{\Omega}) \) there is \( c_\varepsilon > 0 \) such that the solution \((u_\varepsilon, v_\varepsilon)\) of \((9)\) with \( a \) as in \((10)\) satisfies

\[
\int_\Omega u_\varepsilon^\mu (\cdot, t) + \int_\Omega |\nabla v_\varepsilon(\cdot, t)|^{2p} \leq c_\varepsilon \quad \text{on } (0, T_{\max}).
\]

**Proof.** Firstly, if \((27)\) holds, then we choose \( \eta > 0 \) such that

\[
\mu \geq k_1(p, N)\|\chi v_0\|_{L^\infty(\Omega)}^2 + k_2,\eta(p, N)\|\chi v_0\|_{L^\infty(\Omega)}^{2p},
\]

where

\[
k_{2,\eta} := \frac{4(p + N - 1 + \eta)}{p + 1} \left( \frac{8(p - 1)(p + N - 1 + \eta)(4p^2 + N)}{p + 1} \right)^{\frac{p - 1}{2}}.
\]

In the case that \((27)\) is not satisfied, \( \eta > 0 \) is arbitrary. Then Lemmata \(4.1\) and \(4.2\) show that with \( C_\eta > 0 \) as in Lemma \(4.2\)

\[
\frac{d}{dt} \left( \int_\Omega u_\varepsilon^\mu + \chi^{2p} \int_\Omega |\nabla v_\varepsilon|^{2p} \right) + \frac{2(p - 1)}{p} \int_\Omega |\nabla u_\varepsilon|^2 + p\chi^{2p} \int_\Omega |\nabla v_\varepsilon|^{2p - 2} |D^2 v_\varepsilon|^2
\leq \frac{p(p - 1)}{2} \chi^2 \int_\Omega u_\varepsilon^p |\nabla v_\varepsilon|^2 + p(p + N - 1 + \eta)\|v_0\|_{L^\infty(\Omega)}^2 \chi^{2p} \int_\Omega u_\varepsilon^p |\nabla v_\varepsilon|^{2p - 2} + pk \int_\Omega u_\varepsilon^p - p\mu \int_\Omega u_\varepsilon^{p+1} - \varepsilon \int_\Omega u_\varepsilon^{p+1} \ln a u_\varepsilon + C_\eta
\]

throughout \((0, T_{\max})\). Using Young’s inequality, we can assert that for any \( \delta_1 > 0 \),

\[
\frac{p(p - 1)}{2} \chi^2 \int_\Omega u_\varepsilon^p |\nabla v_\varepsilon|^2 \leq \frac{p(p - 1)\delta_1^{p+1}}{2(p + 1)} \chi^{2p} \int_\Omega |\nabla v_\varepsilon|^{2(p + 1)} + p^2(p - 1) \left( \frac{1}{\delta_1} \right)^{\frac{p + 1}{2}} \chi^2 \int_\Omega u_\varepsilon^{p+1}
\]

on \((0, T_{\max})\). We then apply Lemma \(2.2\) and \(\|v_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} \leq \|v_0\|_{L^\infty(\Omega)}\) to obtain

\[
\frac{p(p - 1)\delta_1^{p+1}}{2(p + 1)} \chi^{2p} \int_\Omega |\nabla v_\varepsilon|^{2(p + 1)} \leq \frac{p(p - 1)(4p^2 + N)\delta_1^{p+1}}{(p + 1)} \chi^{2p} \int_\Omega |\nabla v_\varepsilon|^{2p - 2} |D^2 v_\varepsilon|^2
\]

on \((0, T_{\max})\). If we let \( \delta_1 = \left( \frac{4(p - 1)(4p^2 + N)}{\|v_0\|_{L^\infty(\Omega)}^2} \right)^{\frac{1}{p+1}} \), \((29)\) shows that

\[
\frac{p(p - 1)}{2} \chi^2 \int_\Omega u_\varepsilon^p |\nabla v_\varepsilon|^2 \leq \frac{p}{4} \chi^{2p} \int_\Omega |\nabla v_\varepsilon|^{2p - 2} |D^2 v_\varepsilon|^2 + \frac{p^2(p - 1)}{2(p + 1)} \left( \frac{4(p - 1)(4p^2 + N)}{p + 1} \right)^{\frac{p}{2}} \|v_0\|_{L^\infty(\Omega)}^\frac{p}{2} \int_\Omega u_\varepsilon^{p+1}
\]
on \((0, T_{\text{max}})\). Similarly, for any \(\delta_2 > 0\) we have
\[
 p(p + N - 1 + \eta)\|v_0\|_{L^\infty(\Omega)}^2 \chi^{2p} \int_\Omega u_\varepsilon^2 |\nabla v_\varepsilon|^{2p - 2}
\leq \frac{p(p - 1)(p + N - 1 + \eta)\delta_2^{p+1} p \|v_0\|_{L^\infty(\Omega)}^{2p}}{p + 1} \chi^{2p} \int_\Omega |\nabla v_\varepsilon|^{2(p + 1)}
+ \frac{2p(p + N - 1 + \eta)\|v_0\|_{L^\infty(\Omega)}^{2}}{p + 1} \left(\frac{1}{\delta_2}\right)^{\frac{p+1}{2}} \chi^{2p} \int_\Omega u_\varepsilon^{p+1}
\]
(31)
on \((0, T_{\text{max}})\). Using Lemma 2.2 once more and taking
\[
\delta_2 = \left(\frac{p+1}{8(p-1)(p+N-1+\eta)(4p^2+N)}\right)^{\frac{p+1}{2}},
\]
we can obtain from (31) that
\[
p(p + N - 1 + \eta)\|v_0\|_{L^\infty(\Omega)}^2 \chi^{2p} \int_\Omega u_\varepsilon^2 |\nabla v_\varepsilon|^{2p - 2}
\leq \frac{p}{4} \chi^{2p} \int_\Omega |\nabla v_\varepsilon|^{2p - 2} |D^2 v_\varepsilon|^2
\]
(32)on \((0, T_{\text{max}})\). Combining inequalities (28), (29) and (32), we arrive at
\[
\frac{d}{dt} \left(\int_\Omega u_\varepsilon^p + \chi^{2p} \int_\Omega |\nabla v_\varepsilon|^{2p}\right) + \frac{2(p-1)}{p} \int_\Omega |\nabla u_\varepsilon^p|^2 + \frac{p}{2} \chi^{2p} \int_\Omega |\nabla v_\varepsilon|^{2p - 2} |D^2 v_\varepsilon|^2
\leq \frac{p}{2} \left(k_1(p, N)\|v_0\|_{L^\infty(\Omega)}^{\frac{p}{2}} + k_2(p, \eta, N)\|v_0\|_{L^\infty(\Omega)}^{2p} - \mu\right) \int_\Omega u_\varepsilon^{p+1}
- \varepsilon \int_\Omega u_\varepsilon^{p+1} \ln u_\varepsilon + \rho \int_\Omega \Omega u_\varepsilon^p - \frac{\rho p}{2} \int_\Omega u_\varepsilon^{p+1} + C_\eta
\]
(33)on \((0, T_{\text{max}})\).

We can moreover invoke the Poincaré inequality along with Lemma 3.2 to estimate
\[
\int_\Omega u_\varepsilon^p = \|u_\varepsilon^p\|_{L^2(\Omega)}^2 \leq c_1 \left(\|\nabla u_\varepsilon^p\|_{L^2(\Omega)}^{2} + \|u_\varepsilon^p\|_{L^2(\Omega)}^{2}\right) \leq c_2 \left(\int_\Omega \|\nabla u_\varepsilon^p\|_{L^2(\Omega)}^{2} + 1\right)
\]
on \((0, T_{\text{max}})\) with some \(c_1 > 0\) and \(c_2 > 0\). In a quite similar way, using Lemma 3.4 we obtain constants \(c_3 > 0\) and \(c_4 > 0\) such that
\[
\int_\Omega |\nabla v_\varepsilon|^{2p} = \|\nabla v_\varepsilon|^{2p}\|_{L^2(\Omega)}^2
\leq c_3 \left(\|\nabla \nabla v_\varepsilon|^{2p}\|_{L^2(\Omega)}^{2} + \|\nabla v_\varepsilon|^{2p}\|_{L^2(\Omega)}^{2}\right)
\leq c_4 \left(\int_\Omega \|\nabla v_\varepsilon|^{2p}\|_{L^2(\Omega)}^{2} + 1\right)
\leq c_4 \left(\int_\Omega \|\nabla v_\varepsilon|^{2p}\|_{L^2(\Omega)}^{2} |D^2 v_\varepsilon v_\varepsilon|^2 + 1\right) \text{ on } (0, T_{\text{max}}).
Introducing $c_5 := \min \left\{ \frac{2(p-1)}{p} \frac{p}{2}, \frac{p}{2} \right\}$ and abbreviating $y_\varepsilon(t) := \int_\Omega u_\varepsilon^p + \int_\Omega |\nabla u_\varepsilon|^{2p}$, we thus obtain from (33) that
\[ y_\varepsilon(t) + c_\varepsilon y_\varepsilon(t) \leq K \quad \text{for all } t \in (0, T_{\max}), \]
where
\[ K := p |\Omega| \left( \sup_{s > 0} (\kappa s^p - \frac{\mu}{2} s^{p+1}) + \inf_{s > 0} s^{p+1} \ln s \right) + C_\eta \quad \text{if } 27 \]
and
\[ K := C_\eta + \]
\[ p |\Omega| \sup_{s > 0} \left( \frac{1}{2} k_1(p, N) \|v_0\|_{L^p(\Omega)}^2 + \frac{1}{2} k_2(p, N) \|v_0\|_{L^p(\Omega)}^{2p} - \mu \right) s^{p+1} + \kappa s^p - \varepsilon s^{p+1} \ln a s \]
otherwise.

In consequence,
\[ y_\varepsilon(t) \leq \max \left\{ y_\varepsilon(0); \frac{K}{c_\varepsilon} \right\} \]
for all $t \in (0, T_{\max})$. We note that $K$ depends on $\varepsilon$ if and only if 27 is not satisfied.

The previous lemma ensures boundedness of $u$ in some $L^p$-space for finite $p$ only. Fortunately, this is already sufficient for the solution to be bounded – and global.

**Lemma 4.4.** Let $T \in (0, \infty]$, $p > N$, $M > 0$, $a > 0$, $\kappa \in \mathbb{R}$, $\mu > 0$. Then there is $C > 0$ with the following property: If for some $\varepsilon \in (0, 1)$, the function $(u_\varepsilon, v_\varepsilon) \in (C^0(\bar{\Omega} \times [0, T])) \cap C_{2.1}([0, T])^2$ is a solution to [3] with $a$ as in [10] such that
\[ 0 \leq u_\varepsilon, 0 \leq v_\varepsilon \quad \text{in } \Omega \times (0, T) \quad \text{and} \quad \int_\Omega u_\varepsilon^p(\cdot, t) \leq M \quad \text{for all } t \in (0, T), \]
then
\[ \|u_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} + \|\nabla v_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} \leq C \quad \text{for all } t \in (0, T). \]

**Proof.** We use the standard estimate for the Neumann semigroup ([25, Lemma 1.3]) to conclude that with some $c_1 > 0$
\[ \|\nabla v_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} \leq \|\nabla e^{t\Delta} v_\varepsilon(\cdot, 0)\|_{L^\infty(\Omega)} + \int_0^t \|\nabla e^{(t-s)\Delta} u_\varepsilon(\cdot, s)v_\varepsilon(\cdot, s)\|_{L^\infty(\Omega)} \]
\[ \leq \|\nabla v_0\|_{L^\infty(\Omega)} + c_1 \int_0^t c_1 \left( 1 + (t-s)^{\frac{1}{2}} \right) e^{\lambda_1(t-s)} \|u_\varepsilon(\cdot, s)v_\varepsilon(\cdot, s)\|_{L^p(\Omega)} \]
for all $t \in (0, T)$, where $\lambda_1$ denotes the first nonzero eigenvalue of $-\Delta$ in $\Omega$ under the homogeneous Neumann boundary conditions. Due to Lemma 3.3 and the condition on $u_\varepsilon$, we obtain $c_2 > 0$ such that
\[ \|\nabla v_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} \leq c_2 \quad \text{for all } t \in (0, T). \]

In order to obtain a bound for $u_\varepsilon$, we use the variation-of-constants formula to represent $u_\varepsilon(\cdot, t)$ as
\[ u_\varepsilon(\cdot, t) = e^{(t-t_0)\Delta} u_\varepsilon(\cdot, t_0) - \int_{t_0}^t e^{(t-s)\Delta}(\kappa u_\varepsilon(\cdot, s) - \mu u_\varepsilon^2(\cdot, s) - \varepsilon u_\varepsilon^2(\cdot, s) \ln au_\varepsilon(\cdot, s))ds, \]
\[ + \int_{t_0}^t e^{(t-s)\Delta}(\kappa u_\varepsilon(\cdot, s) - \mu u_\varepsilon^2(\cdot, s) - \varepsilon u_\varepsilon^2(\cdot, s))ds, \quad (35) \]
for each $t \in (0, T)$, where $t_0 = (t - 1) +$.

Due to the estimate

$$\kappa s - \mu s^2 - \varepsilon s^2 \ln as \leq \frac{1}{2a^2} + \sup_{\xi > 0}(\kappa \xi - \mu \xi^2) =: c_3,$$

positivity of the heat semigroup ensures that

$$\int_0^t e^{(t-s)\Delta} (\kappa u_e(\cdot, s) - \mu u_e^2(\cdot, s) - \varepsilon u_e^2(\cdot, s) \ln a u_e(\cdot, s)) \leq c_3(t - t_0) \leq c_3. \quad (36)$$

Moreover, from the maximum principle we can easily infer that

$$\|e^{(t-t_0)\Delta} u_e(\cdot, t_0)\|_{L^\infty(\Omega)} = \|e^{t\Delta} u_0\|_{L^\infty(\Omega)} \leq \|u_0\|_{L^\infty(\Omega)} \text{ if } t \in [0, 2] \quad (37)$$

and that with $c_4 > 0$ taken from [25, Lem. 1.3]

$$\|e^{(t-t_0)\Delta} u_e(\cdot, t_0)\|_{L^\infty(\Omega)} \leq \|u_e(\cdot, t_0)\|_{L^\infty(\Omega)}$$

$$\leq \|e^{t\Delta} u_e(\cdot, t_0 - 1)\|_{L^\infty(\Omega)}$$

$$\leq c_4(1 + 1 - \frac{N}{2}) \|u_e(\cdot, t_0 - 1)\|_{L^1(\Omega)} \leq c_4 m_1, \quad (38)$$

whenever $t > 2$ and with $m_1$ as in [17].

Finally, we estimate the second integral on the right hand of (35). [25, Lemma 1.3] provides $c_5 > 0$ fulfilling

$$\left\| \int_0^t e^{(t-s)\Delta} \nabla \cdot (u_e(\cdot, s) \nabla v_e(\cdot, s)) \, ds \right\|_{L^\infty(\Omega)}$$

$$\leq c_5 \int_0^t (t - s)^{-\frac{1}{2} - \frac{N}{2}} \| u_e(\cdot, s) \nabla v_e(\cdot, s) \|_{L^p(\Omega)} \, ds$$

$$\leq c_5 M c_2 \int_0^1 \sigma^{-\frac{1}{2} - \frac{N}{p}} \, d\sigma =: c_6$$

(39)

for $t \in (0, T)$. In view of (35), (37), (38), (39), we have obtained that

$$0 \leq u_e(\cdot, t) \leq \max \{\|u_0\|_{L^\infty(\Omega)}, c_4 m_1\} + c_6 + c_3$$

holds for any $t \in (0, T)$, which combined with (34) is the desired conclusion. \(\blacksquare\)

In fact, the assumption of Lemma 4.4 suffices for even higher regularity, as we will see in Lemma 5.1. For the moment we return to the proof of global existence of solutions.

**Lemma 4.5.** Let $\varepsilon \in (0, 1)$ and let $a$ be as in (10) or let $\varepsilon = 0$ and

$$\mu > k_1(N, N) \|\chi v_0\|_{L^\infty(\Omega)}^{\frac{2}{N}} + k_2(N, N) \|\chi v_0\|_{L^\infty(\Omega)}^{2N},$$

where $k_1$, $k_2$ are as in Lemma 3.3. Then the classical solution to (9) given by Lemma 3.1 is global and bounded.

**Proof.** By continuity, there is $p > N$ such that

$$\mu > k_1(p, N) \|\chi v_0\|_{L^\infty(\Omega)}^{\frac{2}{N}} + k_2(p, N) \|\chi v_0\|_{L^\infty(\Omega)}^{2p},$$

and Lemma 4.3 shows that $\int t_0 u_p^p$ is bounded on $(0, T_{\max})$. Lemma 4.4 together with Lemma 3.3 turns this into a uniform bound on $\|u_e(\cdot, t)\|_{L^\infty(\Omega)} + \|v_e(\cdot, t)\|_{W^{1, \infty}(\Omega)}$ on $(0, T_{\max})$, so that the extensibility criterion (16) shows that $T_{\max} = \infty$. \(\blacksquare\)

**Proof of Theorem 1.1** Theorem 1.1 is the case $\varepsilon = 0$ in Lemma 4.5. \(\blacksquare\)
5. Stabilization. In this section, we shall consider the large time asymptotic stabilization of any global classical bounded solution.

In a first step we derive uniform Hölder bounds that will facilitate convergence. After that, we have to ensure that solutions actually converge, and in particular must identify their limit. In the spirit of the persistence-of-mass result in [22], showing that $v \to 0$ as $t \to \infty$ would be possible by relying on a uniform lower bound for $\int_{\Omega} u$ and finiteness of $\int_0^{\infty} \int_{\Omega} uv$ (see also [11] Lemmata 3.2 and 3.3). We will instead focus on other information that can be obtained from the following showing that $v$ must identify their limit. In the spirit of the persistence-of-mass result in [22], bilization of any global classical bounded solution.

It is therefore provided

$$ u_t - \nabla \cdot (\nabla u - \epsilon t u \nabla v) = \kappa u_t - \mu u_2 - \epsilon u^2 \ln u \in \Omega \times [t, t + 2], \quad \partial_\nu u \mid_{\partial \Omega} = 0 $$

which is bounded in the sense that there exists $M > 0$ such that

$$ \|u_t(\cdot, t)\|_{L^\infty(\Omega)} + \|v_t(\cdot, t)\|_{W^{1, \infty}(\Omega)} \leq M \quad \text{for all } t \in (0, \infty). \tag{41} $$

Then there are $\alpha \in (0, 1)$ and $C > 0$ such that

$$ \|u_t\|_{C^{\alpha}([0, \infty) \times [t, t + 1 + \frac{2}{\alpha}])} + \|v_t\|_{C^{2 + \alpha}([0, \infty) \times [t, t + 1 + \frac{2}{\alpha}])} \leq C \quad \text{for all } t \in (2, \infty). $$

Proof. Due to the time-uniform ($L^\infty(\Omega)$-)bound on $v$ and on the right hand side of

$$ v_{xt} - \Delta v = u_t v_t \quad \text{in } \Omega \times [t, t + 2], \quad \partial_\nu v \mid_{\partial \Omega} = 0 $$

of which $v$ is a weak solution, [15] Thm. 1.3 immediately yields $\alpha_1 \in (0, 1)$ and $c_1 > 0$ such that

$$ \|v_t\|_{C^{\alpha_1, \frac{\alpha_1}{2}}([0, \infty) \times [t, t + 1 + \frac{2}{\alpha_1}])} \leq c_1 $$

for any $t > 0$.

Similarly, [11] provides $t$-independent bounds on the functions $\psi_0 := \frac{1}{2} \chi^2 u_t^2, \psi_1 := \chi u_t \nabla u, \psi_2 := |\chi u_t - \mu u_2 - \epsilon u^2 \ln u| \in (\Omega \times [t, t + 2], \partial_\nu u \mid_{\partial \Omega} = 0$.

An application of [13] Thm. 1.3] to solutions of

$$ u_t - \nabla \cdot (\nabla u_t - \epsilon u_t \nabla v_t) = \kappa u_t - \mu u_2 - \epsilon u^2 \ln u \in \Omega \times [t, t + 2], \partial_\nu u \mid_{\partial \Omega} = 0 $$

therefore provides $\alpha_2 \in (0, 1), \alpha_3 > 0$ such that

$$ \|u_t\|_{C^{\alpha_2, \frac{\alpha_2}{2}}([0, \infty) \times [t, t + 1 + \frac{2}{\alpha_2}])} \leq c_2 $$

for any $t > 0$.

We pick a monotone increasing function $\zeta \in C^\infty(\mathbb{R})$ such that $\zeta_{|_{(-\infty, \frac{1}{\epsilon})}} \equiv 0$, $\zeta_{|_{[1, \infty)}} \equiv 1$ and note that, for any $t_0 > 1$, the function $(x, t) \mapsto \zeta(t - t_0) v_t(x, t)$ belongs to $C^{2,1}([0, \infty) \times [t_0, t_0 + 2])$ and satisfies

$$ (\zeta v_t)_t \Delta (\zeta v_t) - u \zeta v_t + \zeta' v_t, \quad (\zeta v_t)(\cdot, t_0) = 0, \quad \partial_\nu (\zeta v_t) \mid_{\partial \Omega} = 0. $$

Due to the uniform bound for $u_\nu \zeta v_t + \zeta' v_t$ in some Hölder space, an application of [9] Thm. IV.5.3 (together with [9] Thm. III.5.1] ensures the existence of $\alpha_3 \in (0, 1)$
and \( c_3 > 0 \) such that
\[
\|v_\varepsilon\|_{C^{2+\alpha_3,1}((\Omega \times [t_0+1,t_0+2]))} = \|\zeta v_\varepsilon\|_{C^{2+\alpha_3,1}((\Omega \times [t_0+1,t_0+2]))} \\
\leq \|\zeta v_\varepsilon\|_{C^{2+\alpha_3,1}((\Omega \times [t_0,t_0+2]))} \leq c_3
\]
for any \( t_0 > 1 \).

\( \square \)

**Lemma 5.2.** Let \( u_0, v_0 \) satisfy (4) and assume that \( \mu > 0, \kappa > 0, \chi > 0, a = \frac{\mu}{\kappa} \) (as in (10)). Then for any \( \varepsilon \in [0,1) \) any solution \((u_\varepsilon, v_\varepsilon) \in C^{2,1}(\Omega \times (0,\infty)) \cap C^0(\Omega \times [0,\infty)) \) of (9) satisfies
\[
F'_{\varepsilon}(t) + \mu \int_\Omega \left( u_\varepsilon - \frac{\kappa}{\mu} \right)^2 \leq 0 \quad \text{for all } t \in (0,\infty)
\]
and, consequently, there is \( C > 0 \) such that for any \( \varepsilon \in [0,1) \)
\[
\int_0^\infty \int_\Omega \left( u_\varepsilon - \frac{\kappa}{\mu} \right)^2 \leq C.
\]

**Proof.** In fact, on \((0,\infty)\)
\[
F'_{\varepsilon} = \int_\Omega \left( u_\varepsilon - \frac{\kappa}{\mu} \right) \frac{\partial \varepsilon}{\partial t} + \kappa \int_\Omega \left( u_\varepsilon - \frac{\kappa}{\mu} \right) \left( u_\varepsilon - \frac{\kappa}{\mu} \right)
\]
\[
+ \kappa \left( \int_\Omega u_\varepsilon - \frac{\kappa}{\mu} \right) + \mu \int_\Omega v_\varepsilon \partial \varepsilon - \kappa \int_\Omega \Delta u_\varepsilon - \kappa \int_\Omega \nabla u_\varepsilon \cdot \nabla v_\varepsilon - \frac{\kappa}{\mu} \int_\Omega u_\varepsilon - \frac{\kappa^2}{\mu} \int_\Omega 1
\]
\[
\Rightarrow F'_{\varepsilon} \leq -\mu \int_\Omega \left( u_\varepsilon - \frac{\kappa}{\mu} \right)^2 - \kappa \int_\Omega \left( \nabla u_\varepsilon \right)^2 - \frac{\kappa}{2\mu} \int_\Omega \left( \nabla u_\varepsilon \right)^2 + \kappa \int_\Omega \nabla v_\varepsilon \cdot \nabla v_\varepsilon - \frac{\kappa}{\mu} \int_\Omega u_\varepsilon - \frac{\kappa^2}{\mu} \int_\Omega 1
\]
on \((0,\infty)\), which implies (42).

Building upon (43) and the second equation of (9), we can now acquire decay information about \( v_\varepsilon \):

**Lemma 5.3.** Let \( \chi > 0, \kappa > 0, \mu > 0, \) let \( u_0 \) and \( v_0 \) satisfy (4) and moreover set \( a := \frac{\mu}{\kappa} \). Then for every \( p \in [1,\infty) \) and every \( \eta > 0 \) there is \( T > 0 \) such that for every \( t > T \) and every \( \varepsilon \in [0,1) \) every global classical solution \((u_\varepsilon, v_\varepsilon) \) of (9) satisfies
\[
\|v_\varepsilon(\cdot,t)\|_{L^p(\Omega)} < \eta.
\]

**Proof.** By Lemma 5.2 we find \( c_1 > 0 \) such that for any \( \varepsilon \in [0,1) \) we may estimate
\[
\int_0^\infty \int_\Omega \left( \frac{\kappa}{\mu} - u_\varepsilon \right)^2 < c_1.
\]
Integrating the second equation of (9) shows that
\[
(0,\infty) \ni t \mapsto \int_\Omega v_\varepsilon(\cdot,t)
\]
is decreasing, and that, moreover,
\[
\frac{\kappa}{\mu} \int_0^t \int_\Omega v_\varepsilon + \int_0^t \int_\Omega (u_\varepsilon - \frac{\kappa}{\mu}) v_\varepsilon = \int_0^t \int_\Omega u_\varepsilon v_\varepsilon \leq \int_\Omega v_0
\]
for any \( t > 0 \). We conclude that for any \( t > 0 \) and any \( \varepsilon \in (0, 1) \)
\[
\int_\Omega v_\varepsilon(\cdot, t) \leq \frac{1}{t} \int_0^t \int_\Omega v_\varepsilon \leq \frac{\mu}{\kappa t} \int_\Omega v_0 + \frac{\mu}{\kappa t} \int_0^t \int_\Omega \left( \frac{\kappa}{\mu} - u_\varepsilon \right) v_\varepsilon
\]
\[
\leq \frac{\mu}{\kappa t} \int_\Omega v_0 + \frac{\mu}{\kappa t} \sqrt{\frac{\int_0^t \int_\Omega v_\varepsilon^2}{\int_0^t \int_\Omega (\frac{\kappa}{\mu} - u_\varepsilon)^2}}
\]
\[
\leq \frac{\mu}{\kappa t} \int_\Omega v_0 + \frac{\mu}{\kappa t} \sqrt{\frac{\int_\Omega v_0^2 \|v_\varepsilon\|^2_{L^\infty(\Omega)} \|v_\varepsilon\|^2_{L^1(\Omega)}}{\int_\Omega (\frac{\kappa}{\mu} - u_\varepsilon)^2}}
\]
and hence already have that \( \int_\Omega v_\varepsilon(\cdot, t) \) converges to 0 as \( t \to \infty \), uniformly with respect to \( \varepsilon \). In order to obtain \( (44) \), we invoke the additional interpolation
\[
\|v_\varepsilon(\cdot, t)\|_{L^p(\Omega)} \leq \|v_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} \|v_\varepsilon(\cdot, t)\|_{L^1(\Omega)} \leq \|v_0\|_{L^\infty(\Omega)} \|v_\varepsilon(\cdot, t)\|_{L^1(\Omega)},
\]
valid for any \( t > 0 \).

A combination of the previous lemmata in this section reveals the large time behaviour of bounded classical solutions:

**Lemma 5.4.** Let \( \kappa > 0, \mu > 0, \chi > 0 \) and let \( u_0, v_0 \) satisfy \( (1) \). For any solution \( (u, v) \in C^2((\Omega \times (0, \infty)) \cap C^0((\Omega \times [0, \infty))) \) of \( (5) \) that satisfies the boundedness condition \( (41) \), we have
\[
u(\cdot, t) \to \frac{\kappa}{\mu} \quad \text{in} \quad C^0(\Omega), \quad v(\cdot, t) \to 0 \quad \text{in} \quad C^2(\Omega).
\]

as \( t \to \infty \).

**Proof.** For \( j \in \mathbb{N} \) we define
\[
u_j(x, \tau) := u(x, j + \tau), \quad \nu_j(x, \tau) := v(x, j + \tau), \quad x \in \Omega, \tau \in [0, 1].
\]
We let \( (j_k)_{k \in \mathbb{N}} \subset \mathbb{N} \) be a sequence satisfying \( j_k \to \infty \) as \( k \to \infty \). By Lemma 5.1 there are \( \alpha \in (0, 1), C > 0 \) such that
\[
\|u_{j_k}\|_{C^{\alpha \cdot \frac{\kappa}{\mu}}(\Omega \times [0, 1])} \leq C, \quad \|v_{j_k}\|_{C^{\alpha \cdot \frac{\kappa}{\mu} + \frac{\chi}{\mu}}(\Omega \times [0, 1])} \leq C
\]
for all \( k \in \mathbb{N} \) and hence there are \( u, v \in C^{\alpha \cdot \frac{\kappa}{\mu}}(\Omega \times [0, 1]) \) such that \( u_{j_k} \to u \) in \( C^0(\Omega \times [0, 1]) \) and \( v_{j_k} \to v \) in \( C^2(\Omega \times [0, 1]) \) as \( l \to \infty \) along a suitable subsequence. According to \( (43) \) and Lemma 5.3 \( u \equiv \frac{\kappa}{\mu}, v \equiv 0 \). Because every subsequence of \( (u_j, v_j))_{j \in \mathbb{N}} \) contains a subsequence converging to \( (\frac{\kappa}{\mu}, 0) \), we conclude that \( (u_j, v_j) \to (\frac{\kappa}{\mu}, 0) \) in \( C^0(\Omega \times [0, 1]) \times C^2(\Omega \times [0, 1]) \) and hence, a fortiori, \( (45) \).

**Proof of Theorem 1.2.** The statement of Lemma 5.4 is even slightly stronger than that of Theorem 1.2. \( \square \)
6. Weak solutions. Purpose of this section is the construction of weak solutions to (5), in those cases, where Theorem 1.1 is not applicable. To this end let us first state what a weak solution is supposed to be:

**Definition 6.1.** A weak solution to (5) for initial data \((u_0, v_0)\) as in (4) is a pair \((u, v)\) of functions

\[
\begin{align*}
&u \in L^2_{\text{loc}}(\Omega \times [0, \infty)) \quad \text{with} \quad \nabla u \in L^1_{\text{loc}}(\Omega \times [0, \infty)), \\
v \in L^\infty(\Omega \times (0, \infty)) \quad \text{with} \quad \nabla v \in L^2(\Omega \times (0, \infty))
\end{align*}
\]

such that, for every \(\varphi \in C^\infty_0(\Omega \times [0, \infty))\),

\[
-\int_0^\infty \int_\Omega u \varphi_t - \int_\Omega u_0 \varphi(, 0) = -\int_0^\infty \int_\Omega \nabla u \cdot \nabla \varphi + \chi \int_0^\infty \int_\Omega u \nabla v \cdot \nabla \varphi \\
+ \kappa \int_0^\infty \int_\Omega \varphi_t - \mu \int_0^\infty \int_\Omega u^2 \varphi \\
-\int_0^\infty \int_\Omega v \varphi_t - \int_\Omega v_0 \varphi(, 0) = -\int_0^\infty \int_\Omega \nabla v \cdot \nabla \varphi - \int_0^\infty \int_\Omega u v \varphi
\]

hold true.

Some of the estimates needed for the compactness arguments in the construction of these weak solutions will spring from the following quasi-energy inequality:

**Lemma 6.2.** Let \(\mu, \chi \in (0, \infty), \kappa \in \mathbb{R}\) and let \((u_0, v_0)\) satisfy (4). There are constants \(k_1 > 0, k_2 > 0\) such that for any \(\varepsilon \in (0, 1)\) the solution of (49) with \(a\) as in (10) satisfies

\[
\frac{d}{dt} \left( \int_\Omega u \ln u + \frac{\chi}{2} \int_\Omega \frac{|\nabla v|}{v} \right) \\
+ \int_\Omega \frac{|\nabla u|^2}{v} + k_1 \int_\Omega \frac{|\nabla v|^4}{v^3} + k_1 \int_\Omega D^2 \ln v + \mu \int_\Omega u^2 + \varepsilon \int_\Omega u^2 \ln u + \varepsilon \int_\Omega u^2 \ln u \\
\leq k_2 \int_\Omega v + k_3
\]

on \((0, \infty)\).

**Proof.** According to Lemma 4.5, for any \(\varepsilon \in (0, 1)\), the solution to (49) is global, and from the second equation of (49) we obtain that

\[
\frac{d}{dt} \int_\Omega \frac{|\nabla v|^2}{v} = 2 \int_\Omega \frac{\Delta v \nabla v}{v^2} + \int_\Omega \frac{|\nabla v|^2}{v^2} \nabla v \\
= -2 \int_\Omega \frac{\Delta v v}{v^2} + 2 \int_\Omega \frac{|\nabla v|^2}{v^2} \nabla v - \int_\Omega \frac{|\nabla v|^2}{v^2} v_t \\
= -2 \int_\Omega \frac{|\nabla v|^2}{v^2} + 2 \int_\Omega u \Delta v + \int_\Omega \frac{|\nabla v|^2}{v^2} \Delta v - \int_\Omega \frac{|\nabla v|^2}{v^2} u \\
\leq -2 \int_\Omega \frac{|\nabla v|^2}{v^2} - 2 \int_\Omega \nabla u \cdot \nabla v + \int_\Omega \frac{|\nabla v|^2}{v^2} \Delta v \quad \text{on} \quad (0, \infty).
\]

Here we may rely on Lemma 2.1 b) to obtain \(k_1 > 0, k_2 > 0\) such that

\[
\frac{d}{dt} \int_\Omega \frac{|\nabla v|^2}{v^2} \leq -2 \int_\Omega \nabla u \cdot \nabla v - \frac{2k_1}{\chi} \int_\Omega v \ln v + \frac{2k_1}{\chi} \int_\Omega v^2 + \frac{2k_2}{\chi} \int_\Omega v
\]
on \((0, \infty)\). Concerning the entropy term, we compute
\[
\frac{d}{dt} \int u_{\varepsilon} \ln u_{\varepsilon} = \int u_{\varepsilon} \ln u_{\varepsilon} + \kappa \int u_{\varepsilon} - \mu \int u_{\varepsilon}^2 - \varepsilon \int u_{\varepsilon}^2 \ln au_{\varepsilon}
\]
\[
= - \int \frac{\nabla u_{\varepsilon}^2}{u_{\varepsilon}} + \chi \int \nabla u_{\varepsilon} \cdot \nabla v_{\varepsilon}
\]
\[
+ \kappa \int u_{\varepsilon} \ln u_{\varepsilon} - \mu \int u_{\varepsilon}^2 \ln u_{\varepsilon} - \varepsilon \int u_{\varepsilon}^2 \ln au_{\varepsilon}
\]
\[
+ \kappa \int u_{\varepsilon} - \mu \int u_{\varepsilon}^2 - \varepsilon \int u_{\varepsilon}^2 \ln au_{\varepsilon} \quad \text{on} \quad (0, \infty).
\]

Additionally, \(s^2 \ln as > -\frac{1}{2\sigma x^e}\) for all \(s \in (0, \infty)\), so that for all \(\varepsilon \in (0, 1)\) we have
\(-\varepsilon(s^2 \ln as) < \frac{1}{2\sigma x^e}\). Since moreover \(\lim_{\varepsilon \to \infty} (\kappa s - \mu s^2 + \kappa s \ln s - \frac{\mu}{2} s^2 \ln s) = -\infty\),
we can find \(k_3 > 0\) such that
\[
\kappa s \ln s - \frac{\mu}{2} s^2 \ln s + \kappa s - \mu s^2 - \varepsilon s^2 \ln as \leq \frac{k_3}{1}
\]
for any \(s \geq 0\) and \(\varepsilon \in (0, 1)\). Inserting this into the sum of \((48)\) and a multiple of \((47)\), we obtain \((49)\).

The following lemma serves as collection of the bounds we have prepared:

**Lemma 6.3.** Let \(\mu > 0, \chi > 0, \kappa \in \mathbb{R}\) and suppose that \(u_0, v_0\) satisfy \((4)\). Then there is \(C > 0\) and for any \(T > 0\) and \(q > N\) there is \(C(T) > 0\) such that for any \(\varepsilon \in (0, 1)\) the solution \((u_{\varepsilon}, v_{\varepsilon})\) of \((9)\) with \(a\) as in \((10)\) satisfies
\[
\int_0^T \int_\Omega u_{\varepsilon}^2 \leq C(T)
\]
\[
\int_0^T \int_\Omega \frac{\nabla u_{\varepsilon}^2}{u_{\varepsilon}} \leq C(T)
\]
\[
\int_0^T \int_\Omega |\nabla u_{\varepsilon}|^2 \leq C(T)
\]
\[
\int_0^T \int_\Omega u_{\varepsilon}^2 \ln au_{\varepsilon} \leq C(T)
\]
\[
\int_0^T \int_\Omega \varepsilon u_{\varepsilon}^2 (\ln u_{\varepsilon}) \ln au_{\varepsilon} \leq C(T)
\]
\[
\int_0^T \int_\Omega |\nabla v_{\varepsilon}|^4 \leq C
\]
\[
\int_\Omega |\nabla v_{\varepsilon}|^2 \leq C
\]
\[
\int_\infty |\nabla v_{\varepsilon}|^2 \leq C
\]
\[
\|v_{\varepsilon}\|_{L^\infty(\Omega \times (0, \infty))} \leq C
\]
\[
\|v_{\varepsilon}\|_{L^2((0,T);(W_0^{1,2}(\Omega))')} \leq C(T)
\]
\[
\|u_{\varepsilon}\|_{L^1((0,T);(W_0^{1,2}(\Omega))')} \leq C(T)
\]

If, moreover \(\kappa > 0\), then there is \(C > 0\) such that for any \(\varepsilon \in (0, 1)\) the solution \((u_{\varepsilon}, v_{\varepsilon})\) of \((9)\) with \(a = \frac{\mu}{\kappa}\) as in \((10)\) satisfies
\[
\int_0^\infty \int_\Omega \left(u_{\varepsilon} - \frac{\kappa}{\mu}\right)^2 \leq C.
\]
Proof. Boundedness of \( v_\varepsilon \) as in \([57]\) has been shown in Lemma \([3.3, 50, 51, 53, 54]\) result from Lemma \([6.2]\) by straightforward integration, as well as \([53]\) if Lemma \([3.3]\) is taken into account. Testing the second equation in \([9]\) by \( v_\varepsilon \), \([50]\) is readily obtained. By an application of Hölder’s inequality, \([52]\) immediately follows from \([50]\) and \([51]\). Moreover, \([60]\) is a consequence of \([42]\). For any \( \varphi \in C_0^\infty(\Omega \times [0, T)) \) we have

\[
\int_0^T \int_\Omega v_\varepsilon t \varphi = - \int_0^T \int_\Omega \nabla \varphi \cdot \nabla v_\varepsilon - \int_0^T \int_\Omega \varphi u_\varepsilon v_\varepsilon \\
\leq \| \nabla \varphi \|_{L^2(\Omega \times (0, T))} \| \nabla v_\varepsilon \|_{L^2(\Omega \times (0, T))} \\
+ \| v_\varepsilon \|_{L^\infty(\Omega \times (0, T))} \| u_\varepsilon \|_{L^2(\Omega \times (0, T))} \| \varphi \|_{L^2(\Omega \times (0, T))}
\]

and – by \([50], [56], [57]\) – hence \([58]\). In order to obtain \([59]\), we let \( \varphi \in (L^1((0, T); W_0^{2,q}(\Omega))^*) = L^\infty((0, T); W_0^{2,q}(\Omega)) \) with \( \| \varphi \|_{L^\infty((0, T); W_0^{2,q}(\Omega))} \leq 1 \) and have

\[
\int_0^T \int_\Omega u_\varepsilon t \varphi = \int_0^T \int_\Omega u_\varepsilon \Delta \varphi + \chi \int_0^T \int_\Omega u_\varepsilon \nabla \varphi \cdot \nabla v_\varepsilon + \kappa \int_0^T \int_\Omega u_\varepsilon \varphi - \mu \int_0^T \int_\Omega u_\varepsilon^2 \varphi \\
+ \varepsilon \int_0^T \int_\Omega \varphi u_\varepsilon^2 \ln |u_\varepsilon| \\
\leq \| u_\varepsilon \|_{L^2(\Omega \times (0, T))} \| \Delta \varphi \|_{L^2(\Omega \times (0, T))} \\
+ \chi \| u_\varepsilon \|_{L^2(\Omega \times (0, T))} \| \nabla v_\varepsilon \|_{L^2(\Omega \times (0, T))} \| \nabla \varphi \|_{L^\infty(\Omega \times (0, T))} \\
+ \| \varphi \|_{L^\infty(\Omega \times (0, T))} \varepsilon \int_0^T \int_\Omega u_\varepsilon^2 \ln |u_\varepsilon|,
\]

which, due to \([50], [56], [53]\), proves \([59]\). □

By means of compactness arguments, these estimates allow for the construction of weak solutions. This is to be our next undertaking:

Lemma 6.4. Let \( \mu > 0, \chi > 0, \kappa \in \mathbb{R} \) and assume that \( u_0, v_0 \) satisfy \([4]\). There are a sequence \( (\varepsilon_j)_{j \in \mathbb{N}}, \varepsilon_j \searrow 0 \) and functions

\[
u \in L^2_{\text{loc}}(\Omega \times [0, \infty)), \quad \nabla v \in L^4(\Omega \times [0, \infty)),
\]

\[
v \in L^\infty(\Omega \times (0, \infty)), \quad \nabla v \in L^2(\Omega \times (0, \infty))
\]

such that the solutions \( (u_\varepsilon, v_\varepsilon) \) of \([9]\) with \( a \) as in \([10]\) satisfy

\[
u_\varepsilon \rightarrow \nu \quad \text{in } L^4_{\text{loc}}([0, \infty)), \quad \nabla \nu \rightarrow \nabla \nu \quad \text{in } L^4(\Omega),
\]

\[
u_\varepsilon^2 \rightarrow \nu^2 \quad \text{in } L^4_{\text{loc}}([0, \infty)), \quad \nabla \nu_\varepsilon \rightarrow \nabla \nu \quad \text{in } L^4(\Omega),
\]

\[
u_\varepsilon \rightarrow \nu \quad \text{a.e. in } \Omega \times (0, \infty)
\]

\[
u_\varepsilon \searrow \nu \quad \text{in } L^\infty([0, \infty), L^p(\Omega)) \quad \text{for any } p \in [1, \infty]
\]

\[
\nabla v_\varepsilon \rightarrow \nabla v \quad \text{in } L^4_{\text{loc}}([0, \infty)), \quad \nabla v \rightarrow \nabla v \quad \text{in } L^4(\Omega)
\]

\[
\nabla v_\varepsilon \rightarrow \nabla v \quad \text{in } L^2((0, \infty)), \quad \nabla v \rightarrow \nabla v
\]

as \( \varepsilon = \varepsilon_j \searrow 0 \) and such that \( (\nu, v) \) is a weak solution to \([5]\).
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If additionally \( \kappa > 0 \) and \( a = \frac{\mu \kappa}{\mu} \) as in (10), then \( \varepsilon_j \) can be chosen such that additionally

\[
\varepsilon_j \rightarrow 0
\]

as \( \varepsilon = \varepsilon_j \rightarrow 0 \), and

\[
\left( u - \frac{\kappa}{\mu} \right) \in L^2(\Omega \times (0, \infty))
\]

Proof. [19, Cor. 8.4] transforms (50), (52) and (59) into (61) along a suitable sequence \( (\varepsilon_j) \); the bound in (52) enables us to find a further subsequence such that (62) holds. Similarly, (56) facilitates the extraction of a subsequence satisfying (68), and an analogous application of [19, Cor. 8.4] as before from (56), (57) and (58) provides a (non-relabeled) subsequence such that

\[
\varepsilon u_j \rightarrow v \text{ in } L^2(\Omega \times (0, \infty))
\]

along another subsequence thereof establishes (65). Also (66) is immediately obtained from (57), as is (67) from (55); (69) results from (60). For the

\[
L^1 \text{-} convergence statements in (63) and (64), mere boundedness, like obtainable from (51) and (53), even if combined with the a.e. convergence provided by (61), is insufficient for the existence of a convergent subsequence; we must, in addition, check for equi-integrability on \( \Omega \times (0, T) \) for any finite \( T > 0 \). To this purpose we note that with

\[
C(T) \text{ from (54)}
\]

If \( b \geq 0, \varepsilon \in (0, 1) \), and \( \varepsilon u^2 \ln u \geq b \),

\[
\inf_{b \geq 0} \sup_{\varepsilon \in (0, 1)} \int_0^T \int_{\{u^2 \geq b \}} \varepsilon u^2 \ln u \leq \frac{3}{\ln b} = 0
\]

and, due to (53),

\[
\inf_{b \geq 0} \sup_{\varepsilon \in (0, 1)} \int_0^T \int_{\{u^2 > b \}} u^2 \leq \inf_{b \geq 1} \sup_{\varepsilon \in (0, 1)} \int_0^T \int_{\{u^2 > b \}} u^2 \ln u \leq \frac{1}{\ln b} \leq \frac{C(T)}{\ln b} = 0.
\]

Accordingly, \( \{ \varepsilon u^2 \ln u ; \varepsilon \in (0, 1) \} \) and \( \{ u^2 ; \varepsilon \in (0, 1) \} \) are uniformly integrable, hence by (61) and the Vitali convergence theorem we can extract subsequences such that (64) and (66) hold; (63) also proves that \( u \in L^2_{loc}(\Omega \times [0, \infty)) \). Passing to the limit in each of the integrals making up a weak formulation of (9) with \( \varepsilon > 0 \), which is possible due to (61), (62), (67), (63), (64) and (66), shows that \((u, v)\) is a weak solution to (9) with \( \varepsilon = 0 \).

Proof of Theorem 1.3. The assertion of Theorem 1.3 is part of Lemma 6.4.

We will finally prove that one can expect at least some stabilization of weak solutions also. Here, the preparation in Lemma 5.3 obtained from the energy inequality for \( F \) will be crucial.
Lemma 6.5. Let $\mu > 0$, $\chi > 0$, $\kappa > 0$ and assume that $u_{0}$, $v_{0}$ satisfy (4). The weak solution $(u, v)$ obtained in Lemma 6.4 satisfies

$$\|v(\cdot, t)\|_{L^p(\Omega)} \to 0$$

(71)

for any $p \in [1, \infty)$ and

$$\int_{t}^{t+1} \|u - \frac{\kappa}{\mu} \|_{L^2(\Omega)} \to 0$$

(72)

as $t \to \infty$.

Proof. Using characteristic functions of sets $\Omega \times (t, t + 1)$ for sufficiently large $t$ as test functions in the weak-$*$-convergence statement (66), from Lemma 5.3 we obtain that for every $\eta > 0$ there is $T > 0$ such that $\|v\|_{L^\infty((T, \infty); L^p(\Omega))} < \eta$, whereas (72) is implied by (70).

Remark 6. If $N \leq 3$, the uniform bound on $\int_{t}^{t+1} \int_{\Omega} |\nabla v|^4$ contained in Lemma 6.2 proves to be sufficient for (71) even to hold for $p = \infty$, which can be used as starting point for derivation of eventual smoothness of solutions via a quasi-energy-inequality for $\int_{\Omega} \frac{u^p}{(\eta - v)^\theta}$ with suitable numbers $\theta$ and $\eta$. This result is already contained in [11].

Proof of Theorem 1.4. Lemma 6.5 is identical with Theorem 1.4. □

Acknowledgments. J. Lankeit acknowledges support of the Deutsche Forschungsgemeinschaft within the project Analysis of chemotactic cross-diffusion in complex frameworks. Y. Wang was supported by the NNSF of China (no. 11501457).

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Received August 2016; revised July 2017.

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