Comparative work for the source identification in parabolic inverse problem based on Taylor and Chebyshev wavelet methods

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In this article, we study wavelet collocation methods based on Taylor and Chebyshev wavelets for source identification in the parabolic inverse problem. In the proposed method, the highest order derivative is written in terms of the Taylor and Chebyshev wavelet series, and required unknown terms are obtained using successive integration. Taylor series approximation has been utilized to obtain the source control parameter. Convergence analysis is carried out in order to guarantee the accuracy of the method. Numerical results have been obtained based on the proposed methods, and it is shown that the Taylor wavelet method provides us with a better result than the Chebyshev wavelet method. CPU time has also been shown to ensure the efficiency of the method.

KEYWORDS
Chebyshev wavelet, control parameter, CPU time, inverse PDEs, Taylor wavelet

MSC CLASSIFICATION
65T60, 65C30, 31A30

1 | INTRODUCTION

In recent years, there has been increasing interest in the development of efficient and accurate numerical methods for the parabolic inverse problem with source control parameters. These problems have many applications in thermoelasticity, heat conduction processes, control theory, biochemistry, and population dynamics.

In this study, we propose collocation methods based on Taylor and Chebyshev wavelets to identify the source control parameter appearing in the parabolic inverse problem given by

$$\frac{\partial y}{\partial t} (x, t) = A \frac{\partial^2 y}{\partial x^2} (x, t) + B \frac{\partial y}{\partial x} (x, t) + X (t) y(x, t) + \psi(x, t), \ x \in (0, 1), \ 0 \leq t \leq T,$$

with initial condition

$$y(x, 0) = y_0(x), \ x \in (0, 1),$$

and Dirichlet boundary conditions

$$y(0, t) = f_0(t), \ y(1, t) = f_1(t), \ 0 \leq t \leq T,$$
subject to the overspecified condition

\[ y(x, t) = Q(t), \quad 0 \leq t \leq T, \]  

(1.4)

where \( A, B \in \mathbb{Z}^+, x_{in} \) is a fixed point such that \( 0 < x_{in} < 1 \). \( X \in C[0, T] \) and \( f_0, f_1, Q \in C^1(Z_T) \), where \( Z_T = \{(x, t) : 0 < x < 1, 0 < t \leq 1, \psi, y_0, f_0, f_1, \) and \( Q \) are known functions. It is also assumed that \( Q(t) \neq 0 \). We have to determine unknown functions \( y(x, t) \) and \( X(t) \) simultaneously, where \( y(x, t) \) stands for the solution of the given problem and \( X(t) \) is the source control parameter.

The existence, uniqueness, and regularity results for the parabolic inverse problem is discussed by Cannon et al. [1, 2] and Prelipliko et al. [3, 4]. Unfortunately, these type of problems turn out to be ill-posed in the sense of Hadamard because the problem is not stable. In other words, the solution to the problem does not depend continuously on the given data. Various methods, for example, the projective method [5], dual reciprocity boundary element methods [6], and potential logarithmic method [7], have been developed for the source identification in the parabolic problem.

Numerical methods based on finite difference have been studied by Dehghan [8]. They developed various other numerical methods which can be seen in Dehghan et al. [9, 10] and references therein. Liao et al. [11] proposed a high-order compact finite difference method to identify the parameter for the parabolic inverse problem. The proposed numerical scheme is fourth-order accurate in both time and space. Ritz’s least square method has been developed by Khorshidi and Yousefi [12] to identify the control parameter in the parabolic inverse problem.

In the last few years, there is an impressive increase in the use of mesh-free methods in many problems that appear in several areas of science and engineering. Ignoring mesh production and/or mesh refinement, the meshless methods lead to a significant reduction in computational cost. One of the most popular techniques of mesh-free methods is the wavelet method. Wavelet methods have been applied in various areas of science and engineering [13]. The wavelet method based on Legendre multiscaling function has been proposed by Youssri et al. [14] to solve the parabolic inverse problem.

The wavelet collocation method based on Haar wavelets has been investigated [15] to solve parabolic inverse problems. However, Haar wavelets are not even continuous. Hence, this work is an attempt to solve the parabolic inverse problem with more regular wavelets. Both Taylor and Chebyshev wavelets offer higher order approximation, fast computation, compact support, and flexibility which make them attractive options for researchers. The proposed method is an efficient and accurate numerical method in which boundary conditions are incorporated automatically. The Taylor and Chebyshev wavelet matrices are computed only once and used for different time iterations which ultimately leads to a huge reduction in the computational cost. Both these wavelets can be easily computed using recursive formulas, which makes them efficient for numerical calculations. Unfortunately, this method is very difficult to apply on a general domain, for example, circle, L-shaped domain. Besides it, these wavelets can be unstable for shock wave equations.

The wavelet collocation method based on the Chebyshev wavelet for the Burger–Huxley equation has been developed by Çelik [16]. Farooq et al. [17] developed the Chebyshev wavelet method (CWM) for fractional delay differential equations. Baghni [18] developed the second Chebyshev wavelets method for solving finite-time fractional linear quadratic optimal control problems. Esra Köse et al. [19] used the CWM to solve nonlinear time-fractional Schrödinger equation. Based on modified shifted Chebyshev polynomials, Youssri et al. [20] developed a spectral collocation method to solve initial value problems. Chebyshev collocation method for Volterra–Fredholm integral equation is proposed by Youssri et al. [21]. Atta and Youssri [22] used shifted first-kind Chebyshev polynomials to solve nonlinear time-fractional partial integro-differential equations with a weakly singular kernel. Based on the explicit Chebyshev tau method, Abd-Elhameed et al. [23] obtained spectral solution of a certain type of fractional delay differential equations. More recently, the fourth kind of CWM has been utilized in Turan Dincel and Tural Polat [24] for obtaining a numerical solution of multi-term variable-order fractional differential equations. Taylor wavelet method (TWM) has been proposed in Keshavarz et al. [25] to solve Bratu-type equations. The method has been applied to the Lane–Emden equation in the both linear and nonlinear sense in Gümgüm [26] and to the fractional pantograph equation by Vichitkunakorn et al. [27]. A numerical method based on the Taylor wavelet has been proposed in Toan et al. [28] to solve fractional delay differential equations. Rostami et al. [29] developed a numerical method based on the operational matrices of the Taylor wavelet along with the Newton method for solving a class of nonlinear partial integro-differential equations with weakly singular kernels. A numerical method based on the Taylor wavelet has been proposed by Korkut and İmamoğlu Karabaş [30] to solve the general type of KdV-Burgers’ equation.

In addition to the above-mentioned valuable studies, the main contribution of this study is to present a comparative work based on CWM and TWM for source identification in parabolic inverse problems. Rigorous functional error estimations of these wavelet methods are studied using important characteristics such as the orthogonality of the
Chebyshev wavelet and the normality of the Taylor wavelet. In what follows, the convergence analysis is carried out in order to ensure the accuracy of the proposed method.

In this paper, the topics are treated under four headings: any theoretical background of the method including the definitions and implementation of the method has been presented in Section 2. Convergence analysis is carried out in Section 3. Numerical results and discussions are presented in Section 4. The study has been finalized with a brief conclusion in Section 5.

2 | METHOD STATEMENTS

The aim of this section is to present the whole methodology. To do so, we first describe the Chebyshev and Taylor wavelets and the required first and second integral forms of both wavelets. Secondly, for any interested reader, the implementation of the method has been described explicitly.

2.1 | Basic definitions

Wavelets emerge as a strong and attractive tool which provides the use of quantities expressed by various measures of length in a natural way. Wavelets are comprised of dilation and translation of the mother wavelet by some factors. That’s why basically wavelets can vary with respect to the special choices of the mother wavelet. Due to the objective of the present study, two types of mother wavelets have been taken into account throughout the study: the Taylor wavelet and the Chebyshev wavelet. We point the reader to our fundamental references for these wavelets [16, 25, 27, 31].

Before describing the methods, it is vital to note that to alleviate the complexity of the exposition the notations have been classified for their intended use. That is, $I_{nm}(x)$ represents the wavelet methods for either the Taylor wavelets or Chebyshev wavelets. In addition, $R_{nm}(x)$ and $S_{nm}(x)$ are used to denote the first and second integral forms, respectively, for both methods.

Let $n = 1, 2, \ldots, 2^{k-1}$ for $k \in \mathbb{Z}^+$ and $m = 0, 1, \ldots, M-1$ where $M$ stands for the degree of the polynomial, the Taylor wavelets defined by Keshavarz et al. [25] can be described as follows:

$$I_{nm}(x) = \begin{cases} 2^{-k} \hat{T}_m \left(2^{k-1}x - n + 1\right), & \text{if } \frac{n-1}{2^{k-1}} \leq x < \frac{n}{2^{k-1}}; \\ 0, & \text{otherwise}, \end{cases}$$

(2.1)

where $\hat{T}_m(x) = \sqrt{2m+1}x^m$ for $m = 0, 1, 2, \ldots, M-1$. Here, $x^m$ represents the polynomial of degree $m$ which forms an orthonormal basis over the interval $[0, 1]$.

Likewise, the Chebyshev wavelets can be identified as follows:

$$I_{nm}(x) = \begin{cases} \gamma_m 2^{k-1} C_m \left(2^{k-1}x - 2n + 1\right), & \text{if } \frac{n-1}{2^{k-1}} \leq x < \frac{n}{2^{k-1}}; \\ 0, & \text{otherwise}, \end{cases}$$

(2.2)

where

$$\gamma_m = \begin{cases} \frac{\sqrt{2}}{\sqrt{\pi}}, & \text{if } m = 0; \\ \frac{2^{k-1}}{\sqrt{\pi}}, & \text{if } m \neq 0, \end{cases}$$

(2.3)

for $n = 1, 2, \ldots, 2^{k-1}$, $k \in \mathbb{Z}^+$ and $m = 0, 1, \ldots, M-1$. Notice that $C_m(x)$ stands for the Chebyshev polynomials of the first kind of degree $m$. It is worth emphasizing that Chebyshev wavelets are orthogonal weighted by $\omega \left(2^{k-1}x - 2n + 1\right) = \frac{1}{\sqrt{1-(2^{2k-2}-2n+1)^{2}}}$.

In the light of the above-mentioned definitions, any function $f(x) \in L_m^2[0, 1]$ can be expanded in terms of both the Chebyshev wavelet and the Taylor wavelet as follows:

$$f(x) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} d_{nm} I_{nm}(x),$$

(2.4)
where $d_{nm}$ are the wavelet coefficients that can vary according to the choice of wavelet methods, and they are constituted by

$$d_{nm} = \langle f(x), I_{mn}(x) \rangle = \begin{cases} \int_0^1 f(x)I_{mn}(x)dx, & \text{for the Taylor wavelet;} \\ \int_0^1 f(x)I_{mn}(x)\omega(x)dx, & \text{for the Chebyshev wavelet,} \end{cases} \tag{2.5}$$

Moreover, the approximation of $f(x)$ is given by

$$f(x) \approx \sum_{n=1}^{2^{k-1}M-1} \sum_{m=0}^{D_n} d_{nm}I_{mn}(x) = DI(x), \tag{2.6}$$

where $D$ is a $1 \times 2^{k-1}M$ vector given by

$$D = [d_{10}, d_{11}, \ldots, d_{nM-1}, d_{20}, d_{21}, \ldots, d_{2(M-1)}, \ldots, d_{2^{k-1}-1}, \ldots, d_{2^{k-1}(M-1)}], \tag{2.7}$$

and

$$I(x) = [I_{10}(x) \ldots I_{1(M-1)}(x), I_{20}(x) \ldots I_{2(M-1)}(x), \ldots, I_{2^{k-1}0}(x) \ldots I_{2^{k-1}(M-1)}(x)]^T. \tag{2.8}$$

Note that by replacing the suitable collocation points into Equation (2.8), $I(x)$ returns $2^{k-1}M \times 2^{k-1}M$ matrix.

We conclude this introductory section with the required integral forms of both wavelets. The first and the second integral forms of the Taylor wavelet are

$$R_{nm}(x) = \begin{cases} 0, & \text{if } 0 \leq x < \frac{n-1}{2^{k-1}}; \\ \frac{2^{(n-1)x/m!} \sqrt{2m+1}}{2^{(n-1)x/m!} \sqrt{2m+1}} \left( x - \frac{n-1}{2^{k-1}} \right)^{m+1}, & \text{if } \frac{n-1}{2^{k-1}} \leq x < \frac{n}{2^{k-1}}; \\ \frac{2^{(n-1)x/m!} \sqrt{2m+1}}{2^{(n-1)x/m!} \sqrt{2m+1}} \left( x - \frac{n-1}{2^{k-1}} \right)^{m+1} - P_1(x), & \text{if } \frac{n}{2^{k-1}} \leq x \leq 1, \end{cases} \tag{2.9}$$

and

$$S_{nm}(x) = \begin{cases} 0, & \text{if } 0 \leq x < \frac{n-1}{2^{k-1}}; \\ \frac{2^{(n-1)x/m!} \sqrt{2m+1}}{2^{(n-1)x/m!} \sqrt{2m+1}} \left( x - \frac{n-1}{2^{k-1}} \right)^{m+2}, & \text{if } \frac{n-1}{2^{k-1}} \leq x < \frac{n}{2^{k-1}}; \\ \frac{2^{(n-1)x/m!} \sqrt{2m+1}}{2^{(n-1)x/m!} \sqrt{2m+1}} \left( x - \frac{n-1}{2^{k-1}} \right)^{m+2} - P_2(x), & \text{if } \frac{n}{2^{k-1}} \leq x \leq 1, \end{cases} \tag{2.10}$$

where

$$P_i(x) = \sum_{j=0}^{m} \binom{m}{j} \frac{2^{(j+1)x/k!} \sqrt{2m+1}}{(j+i)!} \left( x - \frac{n}{2^{k-1}} \right)^{j+i}, \quad i = 1, 2. \tag{2.11}$$

Furthermore, the first and the second integral forms of the Chebyshev wavelet [16] are

$$R_{00}(x) = \begin{cases} 0, & \text{if } 0 \leq x < \frac{n-1}{2^{k-1}}; \\ \gamma_0 2^{-\frac{x-1}{2^{k-1}}} \left[ C_1 (\theta) + C_0 (\theta) \right], & \text{if } \frac{n-1}{2^{k-1}} \leq x < \frac{n}{2^{k-1}}; \\ \gamma_0 2^{-\frac{x-1}{2^{k-1}}} C_0 (\theta), & \text{if } \frac{n}{2^{k-1}} \leq x \leq 1, \end{cases} \tag{2.12}$$

$$R_{01}(x) = \begin{cases} 0, & \text{if } 0 \leq x < \frac{n-1}{2^{k-1}}; \\ \gamma_1 2^{-\frac{x-1}{2^{k-1}}} \left[ C_2 (\theta) - C_0 (\theta) \right], & \text{if } \frac{n-1}{2^{k-1}} \leq x < \frac{n}{2^{k-1}}; \\ 0, & \text{if } \frac{n}{2^{k-1}} \leq x \leq 1, \end{cases} \tag{2.13}$$
where $\theta = 2^k x - 2n + 1$, $\rho_m = 1 - \frac{(-1)^{m+1}}{m-1}$, $\mu_m = \frac{(-1)^{m-1}}{m-1} - \frac{(-1)^{m+1}}{m+1}$. Moreover, it is worth noting that the values of $\gamma$ are already given in Equation (2.3). Furthermore, $C_n(x)$ stands for the Chebyshev polynomials of the first kind of degree $m$. For further discussion on that topic, we refer the reader to Çelik [16] and the references therein. For the sake of simplicity and understandability, the use of unnecessary notations has been avoided. Thus, for any continuous $f(x)$ throughout this section, we have the following relations:

$$f(x) \approx \sum_{n=1}^{2^k-1} \sum_{m=0}^{M-1} d_{nm} I_{nm}(x) = DI(x),$$

$$\int_{0}^{x} f(x) dx \approx \sum_{n=1}^{2^k-1} \sum_{m=0}^{M-1} d_{nm} R_{nm}(x) = DR(x),$$

$$\int_{0}^{x} \int_{0}^{x} f(x) dx' dx'' \approx \sum_{n=1}^{2^k-1} \sum_{m=0}^{M-1} d_{nm} S_{nm}(x) = DS(x).$$
Before ending the preliminaries, a further description, which is vital for constructing the numerical scheme, is needed to give. For that purpose, this section is finalized by defining the Kronecker product which is denoted by $\otimes$ notation. The Kronecker product of two vectors mainly returns a matrix composed of the production of components of these two vectors. More precisely, for any $x = [x_1, x_2, \ldots, x_{2^{i-1}(M-1)}]$, we have

$$x \otimes S(1) = \begin{bmatrix} x_1S_{10}(1) & \cdots & x_1S_{20}(1) & \cdots & x_1S_{2^{i-1}(M-1)}(1) \\ x_2S_{10}(1) & \cdots & x_2S_{20}(1) & \cdots & x_2S_{2^{i-1}(M-1)}(1) \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ x_{2^{i-1}(M-1)-1}S_{10}(1) & \cdots & x_{2^{i-1}(M-1)-1}S_{20}(1) & \cdots & x_{2^{i-1}(M-1)-1}S_{2^{i-1}(M-1)}(1) \\ x_{2^{i-1}(M-1)-1}S_{10}(1) & \cdots & x_{2^{i-1}(M-1)-1}S_{20}(1) & \cdots & x_{2^{i-1}(M-1)-1}S_{2^{i-1}(M-1)}(1) \end{bmatrix},$$

where

$$S(1) = [S_{10}(1), \ldots, S_{1(M-1)}(1), S_{20}(1), \ldots, S_{2(M-1)}(1), \ldots, S_{2^{i-1}0}(1), \ldots, S_{2^{i-1}(M-1)}(1)].$$

### 2.2 Implementation of the method

To gain better insight into the implementation of the method, let us remind the conditions to which the parabolic inverse problem given Equation (1.1) subjects and the relations which can be obtained by using those conditions such that

$$y(x, 0) = y_0(x), \quad y_x(x, 0) = y'_0(x), \quad y_{xx}(x, 0) = y''_0(x),$$

$$y(0, t) = f_0(t), \quad y_t(0, t) = f'_0(t),$$

$$y(1, t) = f_1(t), \quad y_t(1, t) = f'_1(t),$$

$$y(x_{in}, t) = Q(t), \quad y_t(x_{in}, t) = Q'(t).$$

Let $N_t$ denote the number of divisions of the time interval such that $\Delta t = \frac{T}{N_t}$. On $t \in [t_r, t_{r+1})$, the approximate solution of Equation (1.1) can be identified as follows:

$$\frac{\partial^3 y}{\partial t \partial x^2}(x, t) \simeq \frac{\partial^3 Y}{\partial t \partial x^2}(x, t) = \sum_{n=1}^{2^{i-1}} \sum_{m=0}^{M-1} d_{nm} I_{nn}(x) = DI(x), \quad t \in [t_r, t_{r+1}).$$

Notice that $y(x, t)$ and $Y(x, t)$ represent the exact solution and numerical solution, respectively. Integrating Equation (2.25) with respect to $t$ leads to

$$\frac{\partial^2 Y}{\partial x^2}(x, t) = (t-t_r) DI(x) + \frac{\partial^2 Y}{\partial x^2}(x, t_r), \quad t \in [t_r, t_{r+1}).$$

On the other hand, integrating Equation (2.25) twice with respect to $x$ yields

$$\frac{\partial^3 Y}{\partial t \partial x}(x, t) = DR(x) + \frac{\partial^2 Y}{\partial t \partial x}(0, t),$$

$$\frac{\partial Y}{\partial t}(x, t) = DS(x) + x \frac{\partial^2 Y}{\partial t \partial x}(0, t) + \frac{\partial Y}{\partial t}(0, t).$$

Inserting 1 instead of $x$ in Equation (2.28) and applying boundary conditions (1.3) in Equation (2.28) implies

$$\frac{\partial^2 Y}{\partial t \partial x}(0, t) = f'_1(t) - f'_0(t) - DS(1),$$

$$\frac{\partial^2 Y}{\partial t \partial x}(1, t) = f'_1(t) - f'_0(t) - DS(1).$$
where \( t \) stands for the derivative notation. By inserting the relation obtained in Equation (2.29) into Equation (2.28), we have

\[
\frac{\partial Y}{\partial t}(x, t) = D(S(x) - x \otimes S(1)) + x \left( f_1'(t) - f_0'(t) \right) + f_0'(t). \tag{2.30}
\]

Notice that \( \otimes \) denotes the Kronecker product defined in Equation (2.19). Additionally, integrating Equation (2.26) twice with respect to \( x \), we get

\[
\frac{\partial Y}{\partial x}(x, t) = (t - t_r) D(R(x) - 1 \otimes S(1)) + \frac{\partial Y}{\partial x}(x, t_r) + (f_1(t) - f_0(t) - f_0(t) + f_0(t_r) - (t - t_r) DS(1). \tag{2.31}
\]

The required information about \( \frac{\partial Y}{\partial x}(0, t_r) - \frac{\partial Y}{\partial x}(0, t_r) \) can be obtained incorporating boundary conditions when \( x = 1 \). Equation (2.32) implies

\[
\left[ \frac{\partial Y}{\partial x}(0, t_r) - \frac{\partial Y}{\partial x}(0, t_r) \right] = f_1(t) - f_0(t) - f_1(t_r) + f_0(t_r) - D(1). \tag{2.33}
\]

Inserting Equation (2.33) into both Equation (2.31) and Equation (2.32) leads to

\[
\frac{\partial Y}{\partial x}(x, t) = (t - t_r) D(R(x) - S(1)) + \frac{\partial Y}{\partial x}(x, t_r) + (f_1(t) - f_1(t_r) - f_0(t) + f_0(t_r)). \tag{2.34}
\]

The required coefficient to generate the approximate solution at \( t = t_{r+1} \) can be obtained by substituting the above-mentioned equations into Equation (1.1) such that

\[
D(S(x) - x \otimes S(1)) + x \left( f_1'(t_{r+1}) - f_0'(t_{r+1}) \right) + f_0'(t_{r+1}) - A \left[ (t_{r+1} - t_r) DI(x) + \frac{\partial^2 Y}{\partial x^2}(x, t_r) \right] - B \left[ (t - t_r) D(R(x) - S(1)) + \frac{\partial Y}{\partial x}(x, t_r) + (f_1(t_{r+1}) - f_1(t_r) - f_0(t_{r+1}) + f_0(t_r)) \right] - X(t_{r+1}) [(t - t_r) D(S(x) - x \otimes S(1)) + Y(x, t_r) + f_0(t_{r+1}) - f_0(t_r)] + x (f_1(t_{r+1}) - f_1(t_r) - f_0(t_{r+1}) + f_0(t_r)) = \psi(x, t_{r+1}), t \in [t_r, t_{r+1}). \tag{2.36}
\]

It is worth reminding that by means of the conditions to which the equation subjects, we have the following relations at \( t = t_0 = 0 \).

\[
Y(x, 0) = y_0(x) \quad Y_x(x, 0) = y_0'(x) \quad Y_{xx}(x, 0) = y_0''(x). \tag{2.37}
\]

The source control parameter function needs further investigation in Equation (2.36) due to the non-availability of the solution. To do so, the overspecified condition is used. Equation (1.1) can be rewritten at a specific point \( x = x_{in} \) as follows:

\[
\frac{\partial y}{\partial t}(x_{in}, t) - A \frac{\partial^2 y}{\partial x^2}(x_{in}, t) - B \frac{\partial y}{\partial x}(x_{in}, t) - X(t) y(x_{in}, t) = \psi(x_{in}, t), 0 \leq t \leq T. \tag{2.38}
\]
Using overspecification condition (1.4) in Equation (2.38), we obtain

\[
\frac{\partial Q}{\partial t}(t) - A \frac{\partial^2 y}{\partial x^2}(x_{in}, t) - B \frac{\partial y}{\partial x}(x_{in}, t) - X(t) y(x_{in}, t) = \psi(x_{in}, t), \quad 0 \leq t \leq T. \tag{2.39}
\]

In other words,

\[
X(t) = \frac{Q'(t) - A \frac{\partial^2 y}{\partial x^2}(x_{in}, t) - B \frac{\partial y}{\partial x}(x_{in}, t) - \psi(x_{in}, t)}{Q(t)}, \quad 0 \leq t \leq T. \tag{2.40}
\]

By means of the Taylor expansion, at \( t = t_{r+1} \), the numerical solution for the source control parameter can be expressed as follows:

\[
X(t_{r+1}) = \frac{Q'(t) - A \left[ \frac{\partial^2 y}{\partial x^2}(x_{in}, t_r) + \Delta t \frac{\partial^2 y}{\partial x^2}(x_{in}, t_r) \right] - B \left[ \frac{\partial y}{\partial x}(x_{in}, t_r) + \Delta t \frac{\partial y}{\partial x}(x_{in}, t_r) \right] + \psi(x_{in}, t_r)}{Q(t)} + O(\Delta t^2). \tag{2.41}
\]

Therefore, Equation (2.36) can be reduced to an algebraic equation such that

\[
MD = b, \tag{2.42}
\]

where

\[
M = (S(x) - x \otimes S(1)) - A (t_{r+1} - t_r) I - B (t_{r+1} - t_r) (R(x) - 1 \otimes S(1)) - X(t_{r+1})(t_{r+1} - t_r) (S(x) - x \otimes S(1)), \tag{2.43}
\]

and

\[
b = \psi(x, t_{r+1}) - x \left[ f_1'(t_{r+1}) - f_0'(t_{r+1}) \right] - f_0'(t_{r+1}) + A \left( \frac{\partial^2 y}{\partial x^2}(x, t_r) \right) + B \left( \frac{\partial y}{\partial x}(x, t_r) + f_1(t_{r+1}) - f_1(t_r) - f_0(t_{r+1}) + f_0(t_r) \right) + X(t_{r+1}) (f_1(t_{r+1}) - f_1(t_r) - f_0(t_{r+1}) + f_0(t_r)). \tag{2.44}
\]

After inserting suitable collocation points and using Equation (2.41), the attained matrix equation Equation (2.42) is solved at \( t = t_{r+1} \) and wavelet coefficients \( D \) are obtained for both the Taylor wavelet and the Chebyshev wavelet via the \texttt{gmres} package in MATLAB.

### 3 CONVERGENCE ANALYSES

The analyses of the convergence of numerical methods provide the reliability of the method, which enables us to do computational work. Therefore, before implementing any numerical method, all theoretical analyses of the method should be given in an appropriate way [32–40].

The convergence and error analyses of the wavelets basis have been previously discussed by Abd-Elhameed et al. (see, for reference, [41–43]). There are various other higher order methods with polynomial approximations for which the convergence analysis has been derived by Das et al. (see [44–54]). In this study, the proposed numerical method is based on the use of both the TWM and the CWM. The convergence analysis of the proposed numerical method, of course, will rely on the functional error estimations of these wavelets. Thereby, this section is designed to state primarily the error estimations of the mentioned wavelet methods. It is followed by the convergence result of the proposed method by using the concepts of consistency and stability.
The functional accuracy estimations of the Chebyshev wavelet and the Taylor wavelet

3.1.1 The error estimations for the CWM

**Theorem 3.1.1** (Theorem 1 [55]). Let \( f(x) \in C^2[0, 1] \) with bounded second-order derivative, that is, \( |f''(x)| \leq L \). \( f(x) \) can be expanded as an infinite series of the Chebyshev wavelet, and the series converges uniformly to the function \( f(x) \), that is,

\[
f(x) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} d_{nm} I_{nm}(x).
\]

Notice that \( d_{nm} = \langle f(x), I_{nm}(x) \rangle \) where \( \langle \cdot, \cdot \rangle \) stands for the inner product in \( L^2_\delta[0, 1] \) given in Equation (2.5) and \( I_{nm}(x) \) are the Chebyshev wavelet. Moreover,

\[
|d_{nm}| \leq \frac{\gamma_m \pi L}{32n^{5/2}(m - 1)^2}, \quad m > 1,
\]

where \( n \) denotes the resolutions of the interval for which \( n = 1, 2, \ldots, 2^k - 1 \) for \( k \in \mathbb{Z}^+ \) and \( m \) stands for the degree of Chebyshev polynomials of the first kind.

**Theorem 3.1.2**. Let \( f(x) \in C^2[0, 1] \) with bounded second-order derivative, that is, \( |f''(x)| \leq L \). Then, we have the following error estimations for the CWM and its integral forms:

\[
\sigma_{nm} \leq \frac{\sqrt{\pi} L}{8} \left( \sum_{n=2^{k-1}+1}^{\infty} \sum_{m=M}^{\infty} \frac{1}{n^5 (m - 1)^4} \right)^{1/2},
\]

where \( \sigma_{nm} \) denotes the accuracy of the functional approximation of the CWM and its integral forms. Additionally, \( n \) represents the resolution of the interval for which \( n = 1, 2, \ldots, 2^k - 1 \) for \( k \in \mathbb{Z}^+ \) and \( m \) stands for the degree of Chebyshev polynomials of the first kind.

**Proof.** The functional accuracy of CWM, that is, \( I_{nm} \), has already given in the study of Sohrabi [55, Theorem 2] as follows:

\[
\sigma_{nm} \leq \frac{\sqrt{\pi} L}{8} \left( \sum_{n=2^{k-1}+1}^{\infty} \sum_{m=M}^{\infty} \frac{1}{n^5 (m - 1)^4} \right)^{1/2}.
\]

For this study, a further error analysis is required to investigate since the proposed method includes both \( R_{nm} \) and \( S_{nm} \) which stands for the first integral and the second integral approaches of the CWM, respectively. Thus, let us start with the first integral form such that

\[
\left\| \int_0^x f(\tau)d\tau - \sum_{n=1}^{2^{k-1}-1} \sum_{m=0}^{M-1} d_{nm} R_{nm}(x) \right\|^2 = \int_0^1 \left| \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} d_{nm} R_{nm}(x) - \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} d_{nm} R_{nm}(x) \right|^2 \omega(x)dx.
\]

This implies that

\[
\sigma_{nm}^2 \leq \int_0^1 \left| \sum_{n=2^{k-1}+1}^{\infty} \sum_{m=M}^{\infty} |d_{nm}|^2 \right| \int_0^x \left( \omega(\tau) \right) d\tau d\omega(x)dx
\]

\[
\leq \sum_{n=2^{k-1}+1}^{\infty} \sum_{m=M}^{\infty} |d_{nm}|^2 \int_0^1 \int_{\frac{\tau}{2^{k-1}}}^{\frac{x}{2^{k-1}}} \gamma_m^2 2^{k-1} \frac{C_m^2 (2^k \tau - 2n + 1)}{\sqrt{1 - (2^k \tau - 2n + 1)^2}} d\tau d\omega(x)dx
\]

\[
\leq \sum_{n=2^{k-1}+1}^{\infty} \sum_{m=M}^{\infty} |d_{nm}|^2 \int_0^1 \int_{-1}^{\frac{1}{2^{k-1}}} \gamma_m^2 \frac{C_m^2 (t)}{2\sqrt{1 - t^2}} dt d\omega(x)dx,
\]

where

\[
C_m^2 = \frac{\gamma_m^2}{2\sqrt{1 - t^2}}.
\]
where \( t = (2^k \tau - 2n + 1) \). Due to the orthogonality property of the Chebyshev polynomials, one can obtain

\[
\left\| \int_0^x f(\tau)d\tau - \sum_{n=1}^{2^k-1} \sum_{m=0}^{M-1} d_{nm}R_{nm}(x) \right\|^2 \leq \sum_{n=2^{k-1}+1}^{\infty} \sum_{m=M}^{\infty} |d_{nm}|^2 \gamma_m^2 \frac{\pi}{4},
\]

By virtue of \(|\gamma_m| \leq \frac{2}{\sqrt{\pi}}\), \( m = 0, 1, 2, \ldots \) and Equation (3.1), we have

\[
\sigma_{nm}^2 \leq \frac{\sqrt{\pi}L}{8} \left( \sum_{n=2^{k-1}+1}^{\infty} \sum_{m=M}^{\infty} \frac{1}{n^2(m-1)^2} \right)^{1/2}. \tag{3.3}
\]

The second integral form of the CWM follows the line of the first integral form. Besides the use of the bound of \( \gamma_m \) and Equation (3.1), some integration properties have been utilized. This yields

\[
\left\| \int_0^x \int_0^u f(\tau)d\tau d\nu - \sum_{n=1}^{2^k-1} \sum_{m=0}^{M-1} d_{nm}S_{nm}(x) \right\|^2 \leq \int_0^1 \int_0^u \int_0^1 \int_0^1 I_{nm}^2(\tau) \omega(\tau)d\tau d\nu d\omega(\nu)dx,
\]

\[
\sigma_{nm}^2 \leq \int_0^1 \int_0^u \int_0^1 \int_0^1 I_{nm}^2(\tau) \omega(\tau)d\tau d\nu d\omega(\nu)dx \tag{3.4}
\]

\[
\leq \frac{\sqrt{\pi}L}{8} \left( \sum_{n=2^{k-1}+1}^{\infty} \sum_{m=M}^{\infty} \frac{1}{n^2(m-1)^2} \right)^{1/2}.
\]

3.1.2 The error estimations for the TWM

To the best of our knowledge, there is no such analysis for the TWM.

**Theorem 3.1.3.** Let \( f(x) \in C^2[0,1] \) with bounded second-order derivative, that is, \(|f''(x)| \leq L\). \( f(x) \) can be expanded as an infinite sum of Taylor wavelets. Moreover, the series converges uniformly to \( f(x) \). That is,

\[
f(x) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} d_{nm}I_{nm}(x).
\]

Notice that \( d_{nm} = \langle f(x), I_{nm}(x) \rangle \) where \( \langle \cdot, \cdot \rangle \) stands for the inner product in \( L^2[0,1] \) given in Equation (2.5) and \( I_{nm}(x) \) are the Taylor wavelets. Moreover,

\[
|d_{nm}| \leq \frac{L \sqrt{2m+1}}{n^2(m+1)(m+2)(m+3)}, \quad m > 1.
\]

Notice that \( n \) denotes the resolutions of the interval for which \( n = 1, 2, \ldots, 2^{k-1} \) for \( k \in \mathbb{Z}^+ \) and \( m \) stands for the degree of polynomials.

**Proof.** Using the definition of \( d_{nm} \), we have

\[
d_{nm} = \int_0^1 f(x)I_{nm}(x)dx,
\]

\[
= \int_{\frac{\pi}{2}}^{\frac{\pi}{2}} f(x)2^{k-1} \tilde{T}_m(2^{k-1}x - n + 1)dx. \tag{3.5}
\]
Changing variables $\tau = 2^k x - n + 1$ leads to

$$d_{nm} = \int_0^1 f \left( \frac{\tau + n - 1}{2^{k-1}} \right) 2^{-\frac{k-1}{2}} \sqrt{2m + 1} \, d\tau.$$ 

With the help of the technique of twice integration by parts, we get

$$|d_{nm}| \leq \frac{\sqrt{2m + 1}}{2^k \sqrt{m} (m + 1)(m + 2)} \left| \int_0^1 f'' \left( \frac{\tau + n - 1}{2^{k-1}} \right) (\tau)^{m+2} d\tau \right|.$$ 

One can write by using the Cauchy–Schwartz inequality

$$|d_{nm}| \leq \frac{\sqrt{2m + 1}}{2^k \sqrt{m} (m + 1)(m + 2)} \left| \int_0^1 f'' \left( \frac{\tau + n - 1}{2^{k-1}} \right) (\tau)^{m+2} d\tau \right| \left| \int_0^1 (\tau)^m d\tau \right|.$$ 

As a result of Equation (3.6), $\sum_{n=1}^{2^k-1} \sum_{m=0}^{M-1} d_{nm}$ is absolutely convergent as $k, M \to \infty$. This guarantees the uniform convergence of $\sum_{n=1}^{2^k-1} \sum_{m=0}^{M-1} d_{nm} I_{nm}(x)$ to the function $f(x)$. That is, $\lim_{k,M \to \infty} \left| f(x) - \sum_{n=1}^{2^k-1} \sum_{m=0}^{M-1} d_{nm} I_{nm}(x) \right| \to 0$. The upcoming proofs also present further insights into the confirmation of this convergence result.

In addition to Theorem 3.1.3, Theorem 3.1.4 describes the accuracy of both the TWM and its integral forms. Let $\sigma_{nm}$ stand for the accuracy of the TWM.

**Theorem 3.1.4.** Let $f(x) \in C^2[0, 1]$ with bounded second-order derivative, that is, $|f''(x)| \leq L$. Then, we have the following error estimation for both the TWM and its integral forms:

$$\sigma_{nm} \leq L \left( \sum_{n=2^{k-1}+1}^{2^k-1} \sum_{m=M}^{M-1} \frac{2m + 1}{n^2(m + 1)^2(m + 2)^2(m + 3)^2} \right)^{1/2},$$

where $n$ denotes the resolutions of the interval for which $n = 1, 2, \ldots, 2^{k-1}$ for $k \in \mathbb{Z}^+$ and $m$ stands for the degree of polynomials.

**Proof.** To estimate the accuracy of the TWM and its integral forms, we use the standard technique as follows:

$$\left\| f(x) - \sum_{n=1}^{2^{k-1}M-1} \sum_{m=0}^{M-1} d_{nm} I_{nm}(x) \right\|^2 = \int_0^1 \left\| \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} d_{nm} I_{nm}(x) - \sum_{n=1}^{2^{k-1}M-1} \sum_{m=0}^{M-1} d_{nm} I_{nm}(x) \right\|^2 \, dx$$

$$\sigma_{nm}^2 \leq \int_0^1 \sum_{n=2^{k-1}+1}^{\infty} \sum_{m=M}^{\infty} |d_{nm}| I_{nm}^2(x) \, dx.$$
Using the triangle inequality implies that

\[
\sigma_{nm}^2 \leq \sum_{n=2^{k-1}+1}^{\infty} \sum_{m=M}^{\infty} |d_{nm}|^2 \int_0^1 I_{nn}^2(x) dx,
\]

\[
\leq \sum_{n=2^{k-1}+1}^{\infty} \sum_{m=M}^{\infty} |d_{nm}|^2 \int_0^{\frac{n}{2^{k-1}}} \left( 2^{\frac{k+1}{2}} \sqrt{2m+1} (2^{k-1}x - n + 1)^n \right)^2 dx,
\]

\[
\leq \sum_{n=2^{k-1}+1}^{\infty} \sum_{m=M}^{\infty} |d_{nm}|^2.
\]

Due to Equation (3.6), the accuracy estimation of the TWM can be defined as follows:

\[
\sigma_{nm} \leq L \left( \sum_{n=2^{k-1}+1}^{\infty} \sum_{m=M}^{\infty} \frac{2m+1}{n^2(m+1)^2(m+2)^2(m+3)^2} \right)^{1/2}
\]

On the other hand, with the help of the orthogonality property, the accuracy of the first and second integrals of the TWM can be obtained as follows:

\[
\left\| \int_0^x f(\tau) d\tau - \sum_{n=1}^{2^{k-1}-1} \sum_{m=0}^{M-1} d_{nm} R_{nm}(x) \right\|^2 = \int_0^1 \left\| \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} d_{nm} R_{nm}(x) - \sum_{n=1}^{2^{k-1}-1} \sum_{m=0}^{M-1} d_{nm} R_{nm}(x) \right\|^2 dx.
\]

This implies that

\[
\sigma_{nm}^2 \leq \sum_{n=2^{k-1}+1}^{\infty} \sum_{m=M}^{\infty} |d_{nm}|^2 \int_0^1 \int_0^x I_{nn}^2(\tau) d\tau dx
\]

\[
\sigma_{nm} \leq L \left( \sum_{n=2^{k-1}+1}^{\infty} \sum_{m=M}^{\infty} \frac{2m+1}{n^2(m+1)^2(m+2)^2(m+3)^2} \right)^{1/2},
\]

where \( t = (2^{k-1} - n + 1) \). Likewise, the second integral form of the TWM is bounded by

\[
\left\| \int_0^x \int_0^t f(u) du d\tau - \sum_{n=1}^{2^{k-1}-1} \sum_{m=0}^{M-1} d_{nm} S_{nm}(x) \right\|^2 = \int_0^1 \left\| \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} d_{nm} S_{nm}(x) - \sum_{n=1}^{2^{k-1}-1} \sum_{m=0}^{M-1} d_{nm} S_{nm}(x) \right\|^2 dx
\]

\[
\sigma_{nm}^2 \leq \int_0^1 \sum_{n=2^{k-1}+1}^{\infty} \sum_{m=M}^{\infty} |d_{nm}|^2 \int_0^x \int_0^t I_{nn}^2(u) du d\tau dx
\]

\[
\sigma_{nm} \leq L \left( \sum_{n=2^{k-1}+1}^{\infty} \sum_{m=M}^{\infty} \frac{2m+1}{n^2(m+1)^2(m+2)^2(m+3)^2} \right)^{1/2}.
\]

### 3.2 | The final convergence result of the proposed method

In light of the above-mentioned error estimations, Section 3.1 is ending up with Theorem 3.2.1 which declares the convergence result of the proposed numerical scheme with the concepts of consistency and stability.

**Theorem 3.2.1.** Let \( y \in L^2[0, 1] \cap C[0, T] \) be the exact solution of initial-boundary problem stated in Equations (1.1)–(1.4). Let \( \Delta t = \frac{T}{N_t} \) where \( N_t \) stands for the number of discretizations of the time interval. Suppose that \( Y(x, t) \) is the numerical solution obtained by the proposed method. The proposed method is convergent in the sense that...
\[ \| y(x, t_r) - Y(x, t_r) \| \leq \| y(x, 0) - Y(x, 0) \| + r \kappa \Delta t. \]

Notice that \( \kappa = \lambda \sigma_{nm}, \lambda \in \mathbb{R} \) where \( \sigma_{nm} \) represents the error estimations of the considered wavelet methods. Moreover, \( r \) represents the time step for \( r = 1, 2, \ldots, N_t \).

**Proof.** Prior to undertaking the analysis, recall the numerical solution and the exact solution at \( t = t_{r+1} \), respectively,

\[ Y(x, t_{r+1}) = (t_{r+1} - t_r) \sum_{n=1}^{2^{k-1}M-1} d_{nm}(S_{nm}(x) - x \otimes S_{nm}(1)) + Y(x, t_r) + bc(t_{r+1}^r, x), \tag{3.13} \]

and

\[ y(x, t_{r+1}) = (t_{r+1} - t_r) \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} d_{nm}(S_{nm}(x) - x \otimes S_{nm}(1)) + Y(x, t_r) + bc(t_{r+1}^r, x), \tag{3.14} \]

where \( bc(t_{r+1}^r, x) = f_0(t_{r+1}) - f_0(t_r) + x (f_1(t_{r+1}) - f_1(t_r) - f_0(t_{r+1}) + f_0(t_r)) \).

Subtracting Equation (3.13) from Equation (3.14), the local error can be defined as follows:

\[ | Y(x, t_{r+1}) - y(x, t_{r+1}) | \leq \Delta t \sum_{n=2^{k-1}+1}^{\infty} \sum_{m=M}^{\infty} | d_{nm}(S_{nm}(x) - x \otimes S_{nm}(1)) | + | Y(x, t_r) - y(x, t_r) |, \tag{3.15} \]

where \( \Delta t = (t_{r+1} - t_r) \). Notice that the terms of \( bc(t_{r+1}^r, x) \) are extracted from the exact solution; therefore, there is no contribution of \( bc(t_{r+1}^r, x) \) on the error estimation. Moreover, Equation (3.15) is hinted at a deep connection between the convergence result of the proposed method and error estimations of wavelet methods. By defining \( e_j = \left| Y(x, t_j) - y(x, t_{j-1}) \right|, j = 1, 2, \ldots, r + 1 \), we have

\[ e_r \leq e_{r-1} + \kappa \Delta t, \tag{3.16} \]

where

\[ \kappa = \begin{cases} 
\lambda \left( \sum_{n=2^{k-1}+1}^{\infty} \sum_{m=M}^{\infty} \frac{2m + 1}{n^2(m + 1)^2(m + 2)^2(m + 3)^2} \right)^{1/2}, & \text{for TWM;} \\
\lambda \left( \sum_{n=2^{k-1}+1}^{\infty} \sum_{m=M}^{\infty} \frac{1}{n^2(m - 1)^4} \right)^{1/2}, & \text{for CWM.} 
\end{cases} \tag{3.17} \]

Here, \( \lambda \) is a constant depends on \( L \) and \( \max_{1 \leq i \leq 2^{k-1}M} \). Equation (3.16) together with Equation (3.17) provides that the proposed method is consistent. Besides consistency, the controllability of the error propagation can easily be seen by induction as follows:

\[ e_1 \leq e_0 + \kappa \Delta t, \tag{3.18} \]
\[ e_2 \leq e_1 + \kappa \Delta t \leq e_0 + 2\kappa \Delta t, \tag{3.19} \]
\[ \vdots \]
\[ e_N_t \leq e_0 + N_t \kappa \Delta t. \tag{3.21} \]

Moreover, it is worth noting that at \( t = 0 \), the numerical solution is obtained by the initial condition of the equation, that is, \( e_0 = 0 \). Furthermore, based on the definition of \( \Delta t \), one can say that \( N_t \Delta t = T \). That is, the proposed method is stable. It is important to remind that \( \kappa = \lambda \sigma_{nm} \rightarrow 0 \) as \( k \) and \( M \) increase. Taking into account consistency and stability, one can be concluded that \( e_{N_t} \rightarrow 0 \) guarantees the convergence of the proposed method.
4 | NUMERICAL RESULTS AND DISCUSSION

In this section, we present numerical results based on the TWM and the CWM applied to the one-dimensional parabolic inverse problem. Numerical results are in good agreement with the exact results. To show the accuracy of the method, we have provided $L_\infty$ and $L_2$ error of the solution. The $L_\infty$ and $L_2$ error norm are defined as follows:

$$\|y(\cdot,t_r) - Y(\cdot,t_r)\|_{L_\infty} = \max_{1 \leq i \leq 2^k} |y(x_i, t_r) - Y(x_i, t_r)|,$$

and

$$\|y(\cdot,t_r) - Y(\cdot,t_r)\|_{L_2} = \frac{1}{2} \left( \sum_{i=1}^{2^k} |y(x_i, t_r) - Y(x_i, t_r)|^2 \right)^{1/2}.$$

In the numerical results, CWM denotes the Chebyshev wavelet method whereas TWM denotes the Taylor wavelet method.

Example 1.

$$\frac{\partial y}{\partial t}(x, t) = \frac{\partial^2 y}{\partial x^2}(x, t) + 2 \frac{\partial y}{\partial x}(x, t) + X(t)y(x, t) - (2 + xt^2)e^t, \quad 0 \leq x \leq 1, \quad 0 < t \leq T,$$

with initial condition

$$y(x, 0) = x, \quad 0 \leq x \leq 1,$$

and Dirichlet boundary conditions

$$y(0, t) = 0, \quad 0 < t \leq T,$$

$$y(1, t) = e^t, \quad 0 < t \leq T,$$

subject to the overspecified condition

$$y(0.5, t) = \frac{e^t}{2}, \quad 0 < t \leq T.$$

The exact solution is

$$y(x, t) = xe^t,$$

and

$$X(t) = 1 + t^2.$$

In Table 1, we have shown pointwise absolute error in the numerical solutions obtained by CWM and TWM at different $\Delta t$. As expected, when we are reducing the time step size $\Delta t$, absolute errors are becoming smaller and smaller. At very small $k$ and $M$, we have obtained accuracy of order $10^{-7}$. In other words, very small $k$ and $M$ lead to a significant reduction in the computational time. That is why the CPU time taken by the TWM is only 0.15 and 0.60 s when $\Delta t = 10^{-2}$ and $10^{-3}$.

| $x$ | $CWM \quad \Delta t = 10^{-2}$ | $TWM \quad \Delta t = 10^{-2}$ | $CWM \quad \Delta t = 10^{-3}$ | $TWM \quad \Delta t = 10^{-3}$ |
|-----|-----------------|-----------------|-----------------|-----------------|
| 0.125 | 9.812 x 10^{-3} | 5.982 x 10^{-5} | 3.702 x 10^{-4} | 4.062 x 10^{-7} |
| 0.250 | 1.738 x 10^{-2} | 1.012 x 10^{-4} | 6.106 x 10^{-4} | 6.784 x 10^{-7} |
| 0.375 | 2.245 x 10^{-2} | 1.252 x 10^{-4} | 8.021 x 10^{-4} | 8.303 x 10^{-7} |
| 0.500 | 2.452 x 10^{-2} | 1.329 x 10^{-4} | 8.834 x 10^{-4} | 8.721 x 10^{-7} |
| 0.625 | 2.318 x 10^{-2} | 1.220 x 10^{-4} | 9.265 x 10^{-4} | 8.115 x 10^{-7} |
| 0.750 | 1.854 x 10^{-2} | 9.709 x 10^{-5} | 7.580 x 10^{-4} | 6.657 x 10^{-7} |
| 0.875 | 1.073 x 10^{-2} | 5.632 x 10^{-5} | 4.562 x 10^{-4} | 4.356 x 10^{-7} |
TABLE 2  Comparison results for the control parameter at $\Delta t = 10^{-3}$, $k = 4$, and $M = 4$.

| $t$ | Exact $X$ | Pointwise absolute error |
|-----|-----------|--------------------------|
|     |           | CWM          | TWM          |
| 0.1 | 1.01      | $3.545 \times 10^{-5}$ | $3.773 \times 10^{-6}$ |
| 0.2 | 1.04      | $1.547 \times 10^{-5}$ | $3.434 \times 10^{-6}$ |
| 0.3 | 1.09      | $3.717 \times 10^{-4}$ | $3.227 \times 10^{-6}$ |
| 0.4 | 1.16      | $1.125 \times 10^{-3}$ | $3.122 \times 10^{-6}$ |
| 0.5 | 1.25      | $2.632 \times 10^{-3}$ | $3.085 \times 10^{-6}$ |
| 0.6 | 1.36      | $4.949 \times 10^{-3}$ | $3.096 \times 10^{-6}$ |
| 0.7 | 1.49      | $6.549 \times 10^{-3}$ | $3.144 \times 10^{-6}$ |
| 0.8 | 1.64      | $4.703 \times 10^{-3}$ | $3.229 \times 10^{-6}$ |
| 0.9 | 1.81      | $3.523 \times 10^{-3}$ | $3.361 \times 10^{-6}$ |
| 1.0 | 2.00      | $3.043 \times 10^{-3}$ | $3.557 \times 10^{-6}$ |

whereas the CPU time taken by the CWM is only 0.18 and 0.80 s, respectively, when $\Delta t = 10^{-2}$ and $10^{-3}$. It is evident from the comparison results in Table 1 that the TWM performs better than the CWM.

In Table 2, we have presented pointwise absolute error in the control parameter at $\Delta t = 10^{-3}$. As expected, pointwise absolute errors in the control parameter are becoming smaller and smaller, when we decrease the size of $\Delta t$. It is to be noted that a good approximation of control parameter $X(t)$ would provide a good approximation of the solution.

In Figure 1, we have shown the plot of $L_\infty$ and $L_2$ error of the solution obtained using CWM at $\Delta t = 10^{-3}$. We are able to achieve the accuracy of order $10^{-3}$ and $10^{-4}$ at very small value of $k$ and $M$. In Figure 2, the plot of $L_\infty$ and $L_2$ error of the Taylor wavelet solution has been shown at $\Delta t = 10^{-3}$. We are able to achieve the accuracy of order $10^{-7}$ at very small values of $k$ and $M$. It is evident from Figures 1 and 2 that the Taylor wavelet solution is more accurate than the Chebyshev wavelet solution at the same values of parameters.

Example 2.

$$\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2} + X(t)y + x \cos(t) - tx \sin(t), \quad 0 \leq x \leq 1, 0 < t \leq T, \quad (4.8)$$

with initial condition

$$y(x, 0) = 0, \quad 0 \leq x \leq 1, \quad (4.9)$$

and Dirichlet boundary conditions

$$y(0, t) = 0, \quad 0 < t \leq T, \quad (4.10)$$

$$y(1, t) = \sin(t), \quad 0 < t \leq T, \quad (4.11)$$
subject to the overspecified condition

\[ y(0.5, t) = 0.5 \sin t, \quad 0 < t \leq T. \]  

(4.12)

The exact solution is

\[ y(x, t) = x \sin t, \]

and

\[ X(t) = t. \]
In Table 3, we have shown pointwise absolute error in the numerical solutions obtained by CWM and TWM at $\Delta t = 10^{-2}$ and $10^{-3}$. As expected, reducing $\Delta t$ led to very small absolute errors in the numerical solutions. At $k = M = 4$, we have achieved accuracy up to order $10^{-8}$. The CPU time taken by TWM is only 0.14 and 0.64 s when $\Delta t = 10^{-2}$ and $10^{-3}$ whereas CPU time taken by CWM is only 0.15 and 0.70 s, respectively, when $\Delta t = 10^{-2}$ and $10^{-3}$. It is evident from Table 3 and CPU time that the TWM is better than the CWM in terms of numerical errors and CPU time.

In Table 4, we have shown pointwise absolute error in the $X$ at $\Delta t = 10^{-3}$. We are able to achieve the accuracy of order $10^{-4}$ in the case of the CWM whereas the accuracy of order $10^{-5}$ is achieved in the case of the TWM. TWM provides a better approximation of control parameter than the CWM.

In Figure 3, we have shown the plot of $L_{\infty}$ and $L_2$ error of the solution obtained using the CWM at $\Delta t = 10^{-3}$. We are able to achieve the accuracy of order $10^{-4}$ and $10^{-5}$ at $k = M = 4$. In Figure 4, the plot of $L_{\infty}$ and $L_2$ error of the Taylor wavelet solution has been shown at $\Delta t = 10^{-3}$. We are able to achieve the accuracy of order $10^{-7}$ at $k = M = 4$. It is clear from Figures 3 and 4 that the Taylor wavelet solution is more accurate than the Chebyshev wavelet solution.

5 | CONCLUSION

We have developed efficient and accurate numerical methods based on Taylor and Chebyshev wavelets for parameter identification in the parabolic inverse problem. Using the uniform convergence property possessed by Taylor and
Chebyshev wavelet, we have derived a rigorous convergence analysis. We have compared both the proposed methods on two parabolic inverse problems, and it is evident from the numerical result that the Taylor wavelet-based method provides better results than the CWM. Considering very few collocation points in the domain, we are able to achieve very good accuracy. Due to the very less computational cost, the CPU time taken to solve the problem is very less. The future extension in the direction of our work is to propose and analyze Taylor and Chebyshev wavelet-based collocation method for 2D and 3D parabolic inverse problems. We are also extending our method for hyperbolic inverse problems in higher dimensions.

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CONFLICT OF INTEREST STATEMENT
We declare that this work does not have any conflicts of interest.

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