Unfolding versus BRST and currents in
$Sp(2M)$ invariant higher-spin theory

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Abstract

The correspondence between BRST and unfolded formulations of field equations on group manifolds and homogeneous spaces is described. The previously introduced nonstandard BRST operator, that underlies $Sp(2M)$ invariant higher-spin field equations, is shown to admit a natural oscillator-like realization. The coordinate independent form of conserved currents in the $Sp(2M)$ invariant higher-spin theory is derived from the BRST formulation on $Sp(2M)$ extended by the Heisenberg group.
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1 Introduction

In [1] it was shown that the unfolded formulation of [2] of $Sp(2M)$ invariant higher-spin (HS) theories [3, 4] (see also [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16]) can be equivalently formulated as a BRST closure condition for a certain nonstandard BRST operator $Q$ on the semidirect product $SpH(2M) = Sp(2M) \otimes H_M$ where $H_M$ is the Heisenberg group. The aim of this paper is to clarify the invariant origin of this relation established in [1] in a particular coordinate system.

We start with the discussion of the general relation between the BRST and unfolded formulations, explaining in a coordinate-independent way that the two approaches are essentially equivalent once the BRST formulation is given in terms of Lie vector fields on a group manifold. The relation via identification of the ghost fields of the BRST formulation with the differential forms of the unfolded formulation requires however a nontrivial relation between space derivatives in the two formulations.

The proposed approach is applicable to dynamical systems formulated in nontrivial geometries of group manifolds and homogeneous spaces as well to their deformations. In particular, it is well suited for the extension of the BRST approaches to HS systems studied in [17, 18] in Cartesian coordinates in flat space or via embedding $AdS$ geometry into a higher dimensional flat space as in [19] to any coordinates and/or more complicated geometries. The same time, the equivalence between the unfolded formulation and appropriately interpreted BRST formulation in the proposed setup is by construction. Among other things, this explains that the obvious parallels between the formalisms and conclusions obtained within the BRST approach (see e.g. [18] for a recent work) with those well established in the unfolded dynamics [20, 21] (and references therein) are in no way accidental and/or surprising.

In particular, the equivalence of the two approaches explains that BRST cohomology may describe both nontrivial deformations of field equations and nontrivial closed differential forms in space-time, that can be constructed from the dynamical fields of the unfolded formulation, i.e., conserved currents. The latter relation is applied in this paper to the further study of the BRST formulation of $Sp(2M)$ invariant HS gauge theory of [1]. Firstly, we show that the complicated nonstandard BRST operator found in [1] has simple origin in the oscillator realization of the symplectic algebra. Secondly, we show that the conserved charges proposed in [16] correspond to certain BRST cohomology, which observation provides their coordinate-independent realization.

The rest of the paper is organized as follows. In Section 2 we explain the general relation between the unfolded and BRST formulations. To make the paper selfcontained, we recollect in Section 3 basic formulae of [1] on the $Sp(2M)$ invariant HS theories. The new oscillator-like realization of the nonstandard BRST operator of [1] is introduced in Section 4. The construction of closed forms from solutions of dynamical equations is presented in Section 5 first for the general case in Subsections 5.1, 5.2 and then for the $SP(2M)$ geometry in Subsections 5.3, 5.4. The latter results are applied in Section 6 to the coordinate-independent construction of conserved currents bilinear in HS fields, that reproduces the previously known results in particular coordinates. Possible applications to unfolded equations are discussed in Section 7.
2 BRST operators and unfolded equations

Consider a Lie group \( G \) and its Lie algebra \( \mathfrak{g} \). Let \( R_\alpha (\alpha = 1, \ldots \text{dim } G) \) be right Lie vector fields on \( G \) that satisfy
\[
[R_\alpha, R_\beta] = f_{\alpha\beta}^\gamma R_\gamma , \tag{2.1}
\]
where \( f_{\alpha\beta}^\gamma \) are structure constants of \( \mathfrak{g} \). Let \( T_\alpha \) form a basis of some representation \( T \) of \( \mathfrak{g} \)
\[
[T_\alpha, T_\beta] = f_{\alpha\beta}^\gamma T_\gamma , \quad [R_\beta, T_\alpha] = 0 .
\]
Provided that the ghosts \( c^a \) and \( b_a \) obey the relations
\[
[c^a, R_\beta] = 0 , \quad [b_a, R_\beta] = 0 , \quad [c^a, T_\beta] = 0 , \quad [b_a, T_\beta] = 0
\]
\[
\{c^a, b_b\} = \delta^a_b , \quad \{c^a, c^b\} = 0 , \quad \{b_a, b_b\} = 0 , \tag{2.2}
\]
the BRST operator
\[
Q = c^a(R_\alpha + T_\alpha) - \frac{1}{2} c^a c^b f_{\alpha\beta}^\gamma \tag{2.3}
\]
is nilpotent
\[
Q^2 = 0 .
\]
That the equation
\[
\{Q, c^\gamma\} = -\frac{1}{2} c^\alpha c^\beta f_{\alpha\beta}^\gamma , \tag{2.4}
\]
has the Maurer-Cartan form upon identification of \( Q \) with \( d \) suggests that, being dual to the Lie vector fields \( R_\alpha \) on \( G \), the ghosts \( c^a \) should be identified with the Cartan forms on \( G \). However, the naive identification fails because it is assumed that \( [R_\alpha, c^\beta] = 0 \) while the Lie vector fields \( R_\alpha = R_\alpha^a(x) \frac{\partial}{\partial x^a} \) do not commute to the \( x \)-dependent Cartan forms (\( x^a \) are coordinates on \( G \)).

To proceed, it is necessary to redefine the notion of the vector fields appropriately. Let
\[
R_\alpha = R_\alpha^a(x)p_a , \tag{2.5}
\]
\[
[p_a, f(x)] = \frac{\partial}{\partial x^a} f(x) , \quad [p_a, p_b] = 0 \tag{2.6}
\]
The equation (2.1) amounts to the standard Lie conditions
\[
R_\alpha^b(x) \frac{\partial}{\partial x^b} R_\beta^a(x) - R_\beta^b(x) \frac{\partial}{\partial x^b} R_\alpha^a(x) = f_{\alpha\beta}^\gamma R_\gamma^a(x) . \tag{2.7}
\]
The Cartan forms
\[
\omega^a = R^{-1}_\alpha R_\alpha^a(x) dx^a , \tag{2.8}
\]
where the differentials \( dx^a \) satisfy \( \{dx^a, dx^b\} = 0, [dx^a, f(x)] = 0 \), obey the Maurer-Cartan equation
\[
d\omega^a = -\frac{1}{2} \omega^\beta \omega^\gamma f_{\beta\gamma}^a . \tag{2.9}
\]
(We systematically skip the wedge symbol throughout this paper.) The key point is to set

\[ p_n = \frac{\partial}{\partial x^m} c^\alpha \beta R^m_\alpha \frac{\partial}{\partial x^n} (R^{-1}_m \beta). \]  

(2.10)

An elementary computation shows that \( p_n \) satisfies the relations (2.6). In addition, we can identify \( \omega^\alpha \) and \( c^\alpha \)

\[ \omega^\alpha = c^\alpha \]

since

\[ [p_n , \omega^\alpha] = \frac{\partial}{\partial x^n} (R^{-1}_m \alpha (x)) dx^m - c^\beta R^m_\beta \frac{\partial}{\partial x^n} (R^{-1}_m \alpha (x)) \bigg|_{c^\gamma = \omega^\gamma} = 0. \]

With this identification we obtain from (2.3) along with (2.10) and (2.8)

\[ Q = \omega^\alpha (R_\alpha + T_\alpha) - \frac{1}{2} \omega^\alpha \omega^\beta b_\beta \gamma f^{\alpha \beta} = \omega^\alpha (R^m_\alpha \frac{\partial}{\partial x^n} + T_\alpha) = D, \]  

(2.11)

where \( D \) is the covariant derivative in the representation \( T \) of \( G \)

\[ D = d + \omega^\alpha T_\alpha, \quad d = dx^n \frac{\partial}{\partial x^n}. \]

In these terms, the condition \( Q^2 = 0 \) is equivalent to

\[ D^2 = 0, \]

which, in turn, is the consequence of the Maurer-Cartan equation (2.9).

Thus, the BRST closure condition

\[ Q\phi = 0 \]  

(2.12)

is equivalent to the unfolded equation

\[ D\phi = 0. \]  

(2.13)

Hence, \( Q \) cohomology is equivalent to the \( D \) cohomology. In particular, in the case where the representation \( T \) is trivial, \( Q \) cohomology is equivalent to De Rham cohomology

\[ Q F(x, c) = G(x, c) \iff dF(x, \omega) = G(x, \omega). \]  

(2.14)

This simple property will be used in this paper to derive conserved HS currents from the appropriate \( Q \)-cohomology.

The equivalence between the BRST approach and unfolded approach shown in this paper contains two essential elements.

One is that the BRST operator should contain Lie vector fields of a chosen group \( G \) which form a frame of the tangent space of \( G \). As a result, the full set of derivatives on \( G \) reappears in the unfolded formulation via the exterior differential \( d \).

Another one is the relation (2.10) that tells us that, for the identification of the BRST operator \( Q \) with the exterior differential, the operator \( p \) in \( Q \) should be interpreted as a
covariant derivative (2.10) that acts on the space of ghosts (forms). Note that, in accordance with the second relation in (2.6), \( p_n \) (2.10) is flat. Indeed, the \( GL_{\dim G} \) connection in (2.10) has the standard pure gauge form with the Lie matrix \( R_n^\alpha(x) \) as the gauge function.

To arrive along these lines at interesting field equations one has to consider appropriate representations \( T \) and/or further nonstandard modifications of the BRST operator \( Q \). The examples of this construction will be considered in Section 4, where it will be shown in particular how the \( Sp(8) \) unfolded equations of [2] result from the nonstandard BRST operator.

Another interesting application is to the BRST reformulation of nonlinear HS theories that may involve higher differential forms as dynamical variables. To this end the space of ghosts \( c^\alpha \) should be extended to a larger set \( C^A \) that includes objects of different non-negative degrees \( p^A = 0, 1, \ldots \). Correspondingly, the space of ghosts \( b^\alpha \) extends to \( B^A \) of degrees \( -p^A \). The graded commutation relations are

\[
[B_A, C^B]_\pm = \delta^B_A, \quad [B_A, B_B]_\pm = 0, \quad [C^A, C^B]_\pm = 0, \quad (2.15)
\]

where

\[
[a, b]_\pm = ab - (-1)^{p(a)p(b)}ba. \quad (2.16)
\]

The idea is to extend the BRST operator as follows

\[
Q^{\mathfrak{g}} \to Q'^{\mathfrak{g}} = Q^{\mathfrak{g}} + Q \quad (2.17)
\]

where \( Q^{\mathfrak{g}} \) is the canonical BRST operator built from the vector fields of some group Lie \( G \) while \( Q \) is some other BRST operator built from the ghosts \( C \) and \( B \), that is also nilpotent \( Q^2 = 0 \). The field equations require the equivalence of the action of \( Q^{\mathfrak{g}} \) and \( Q \) on every dynamical variable \( W = W(C, B) \)

\[
Q^{\mathfrak{g}}(W) + Q(W) = 0. \quad (2.18)
\]

Upon the field redefinition explained above, this amounts to the unfolded equations

\[
dW + Q(W) = 0, \quad (2.19)
\]

in which form unfolded equations, introduced originally in [20] in the study of HS gauge theory, were discussed more recently in [21] (see also [22, 15] for more detail and references). Let us note that in this setup \( Q \) cohomology describes nontrivial deformation of the nonlinear equations (2.19).

A remarkable feature of unfolded dynamics is that it is insensitive to the dimension of space-time where the fields are defined. This property is nicely illustrated within its BRST version discussed in this paper applied to the coset space construction.

Let \( H \) be a subgroup of \( G \). The space of functions on \( G/H \) identifies with the space of solutions of the equations

\[
R_a F(G) = 0, \quad (2.20)
\]

where \( R_a \) is a subset of right vectors field of \( H \). The algebra Lie \( \mathfrak{g} \) acts on solutions of (2.20) by the left vector fields \( L_a \). The equation (2.20) results from the restriction to the sector of \( c \)-independent \( F(G) \) of the condition

\[
Q_\mathfrak{g} F(G, c) = 0, \quad (2.21)
\]
where $Q_h$ is the canonical BRST operator of $H$ and $b_a$ is realized as $\frac{\partial}{\partial c^a}$.

The extension of (2.20) to the induced module construction is

$$ (R_a + T_a) F(G) = 0, \quad (2.22) $$

where $F(G)$ is valued in some $H$-module $V$ and $T_a$ provide a representation of the Lie algebra $\mathfrak{h}$ of $H$ on $V$. In what follows we will be interested in the particular case with a one dimensional $H$–module $V$. In this case, $F(G)$ is still valued in $\mathbb{R}$ or $\mathbb{C}$ and $T_a$ is given by some constants associated to central elements of the grade zero part of $\mathfrak{h}$. The equation (2.22) results from the restriction to the sector of $c$–independent $F(G)$ of the condition (2.21). The BRST extension of the equation (2.22) to $F(G, c)$ is conveniently interpreted in terms of the Fock module generated from the vacuum that satisfies $b_a|0\rangle = 0$.

In the unfolded dynamics approach, the phenomenon illustrated by the example of a coset manifold, that a theory in a smaller space $G/H$ can be described as that in a larger space $G$, extends to less symmetric situations where the symmetry $G$ is broken or deformed. As a result, the concept of space-time dimension turns out to be dynamical in unfolded dynamics.

To apply this approach to the analysis of the $Sp(2M)$ invariant HS field equations let us first recall main ingredients of the formalism of [1].

# 3 Preliminaries

## 3.1 Heisenberg extension of symplectic group

The group $Sp(2M|\mathbb{R})$ is constituted by real matrices

$$ G = \begin{pmatrix} a^A_B & b^{AM} \\
                      c_C^B & d^M_C \end{pmatrix} \quad (3.1) $$

with $M \times M$ blocks $a^A_B$, $b^{AB}$, $c_{AB}$, $d^B_A$ that satisfy the relations

$$ a^A_C b^{DC} - a^D_C b^{AC} = 0, \quad a^A_C d^B_C - b^{AC} c_{BC} = \delta^A_B, \quad c_{BC} d^A_C - c_{AC} d^B_C = 0 \quad (3.2) $$

equivalent to the invariance condition $AR\!\!\!^\dagger = R$ for the symplectic form $R = \begin{pmatrix} 0 & I^A_B \\
                      -I^D_C & 0 \end{pmatrix}$, where $I$ is the unit $M \times M$ matrix and $A^t$ is the transposed matrix.

Any $g \in Sp(2M|\mathbb{R})$ with nondegenerate $d$ (3.1) can be represented in the form

$$ \begin{pmatrix} a^A_B & b^{AC} \\
                      c_{DB} & d^D_C \end{pmatrix} = \begin{pmatrix} \delta^A_E & X^{AF} \\
                      0 & \delta^D_F \end{pmatrix} \begin{pmatrix} A^E_G & 0 \\
                      0 & D^H_F \end{pmatrix} \begin{pmatrix} \delta^G_B & 0 \\
                      C^A_H & \delta^C_H \end{pmatrix}. \quad (3.3) $$

This gives

$$ \begin{pmatrix} a^A_B & b^{AC} \\
                      c_{DB} & d^D_C \end{pmatrix} = \begin{pmatrix} A^A_B + X^{AF} D^G_F C^C_GB \\
                      D^G_F C^C_GB & D^D_C \end{pmatrix}, \quad (3.4) $$

1Strictly speaking this is true for the so-called universal unfolded systems [22] which case is however general enough to cover all known examples of unfolded equations.
where
\[
\mathcal{A}^A_B = (d^{-1})_B^A, \quad X^{AM} = b^{AC} \mathcal{A}^M_C, \quad \mathcal{C}_{BA} = c_{CB} \mathcal{A}^C_A
\] (3.4)
can be chosen as local coordinates on \( Sp(2M|\mathbb{R}) \). Note that \( X^{BA} = X^{AB} \) and \( \mathcal{C}_{BA} = \mathcal{C}_{AB} \) by virtue of the identities
\[
-c_{BA}d^{-1}_C B + c_{BC}d^{-1}_A B = 0, \quad -b^{AB}d^{-1}_B C + b^{CB}d^{-1}_B A = 0,
\]
which follow from (3.2). Note that \( \mathcal{D}_A^B = d_A^B \).

\( Sp(2M|\mathbb{R}) \) contains the following important subgroups. The Abelian subgroup of translations \( \mathbf{T} \) consists of the elements
\[
t(X) = \begin{pmatrix} I & X \\ 0 & I \end{pmatrix}
\] (3.5)
with various \( X^{AB} = X^{BA} \). The product law in \( \mathbf{T} \) is \( t(X)t(Y) = t(X+Y) \).

Analogously, the Abelian subgroup \( S \) of special conformal transformations is constituted by the matrices (3.1) with \( a = d = I, b = 0 \). The subgroup \( GL(M) \) of generalized Lorentz transformations \( SL(M) \) and dilatations consists of the matrices (3.1) with \( b = c = 0 \) and \( a^B c_d^C = \delta^B_A \).

The lower \( P_l(\mathbb{R}) \) and upper \( P_u(\mathbb{R}) \) maximal parabolic subgroups of \( Sp(2M|\mathbb{R}) \) are
\[
P_l \ni p = \begin{pmatrix} a & 0 \\ c & d \end{pmatrix}, \quad P_u \ni p = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}.
\] (3.6)

Let \( H_M = \mathbb{R}^M \times \mathbb{R}^M \times \mathbb{R}^1 \) be the \((2M+1)\)-dimensional Heisenberg group constituted by
\[
\mathcal{F} = \{ \mathbf{f}, u \}, \quad \mathbf{f} = y^A, w_B, \quad A, B = 1, \ldots, M
\] (3.7)
with the product law
\[
\mathcal{F}_1 \circ \mathcal{F}_2 = \{ \mathbf{f}_1 + \mathbf{f}_2, u_1 + u_2 - (\mathbf{f}_1, \mathbf{f}_2) \},
\]
where \( (, ,) \) is the symplectic form
\[
(\mathbf{f}_1, \mathbf{f}_2) = y_1^A w_{2A} - y_2^A w_{1A} = -(\mathbf{f}_2, \mathbf{f}_1), \quad A = 1, \ldots, M.
\] (3.8)

\( Sp(2M|\mathbb{R}) \left( Sp(2M|\mathbb{C}) \right) \) acts canonically on \( H_M \left( {^C H}_M \right) \) which is the manifestation of the standard fact that \( Sp(2M) \) possesses the oscillator realization (see e.g. [23]). This makes it possible to introduce the group \( SpH(2M) = Sp(2M) \otimes H_M \)
\[
SpH(2M) : \mathcal{G} = \{ G, \mathcal{F} \}, \quad G \in Sp(2M), \quad \mathcal{F} = \{ \mathbf{f}, u \} \in H_M
\] (3.9)
with the product law
\[
\mathcal{G}_1 \circ \mathcal{G}_2 = \{ G_1 G_2, \mathbf{f}_1 + G_1 \mathbf{f}_2, u_1 + u_2 - (\mathbf{f}_1, G_1 \mathbf{f}_2) \},
\]
where \( (, ,) \) is the symplectic form (3.8) and
\[
G \mathbf{f} = \begin{pmatrix} a^A_B & b^{AM} \\ c_{CB} & d^C_M \end{pmatrix} \begin{pmatrix} y^B \\ w_M \end{pmatrix} = \begin{pmatrix} a^A_B y^B + b^{AM} w_M \\ c_{CB} y^B + d^C_M w_M \end{pmatrix}, \quad G \in Sp(2M), \quad \mathbf{f} = (y^A, w_B) \in H_M.
\]
Analogously, a rank $r$ Heisenberg extension $SpH_r(2M|\mathbb{A})$ is introduced for any field $\mathbb{A}$ and $r \in \mathbb{N}$ as

$$SpH_r(2M|\mathbb{A}) = Sp(2M|\mathbb{A}) \otimes H_M(\mathbb{A}) \times \cdots \times H_M(\mathbb{A})$$ (3.10)

with coordinates

$$y_j^A, \quad w_{jB}, \quad u_j, \quad j = 1, \ldots, r.$$ (3.11)

When it does not lead to misunderstandings, we will use shorthand notation like $SpH$ instead of $SpH(2M|\mathbb{R})$ etc. Note, that $SpH_1(2M) \equiv SpH(2M)$ and $SpH_0(2M) \equiv Sp(2M).

The lower quasiparabolic subgroup $PH_l(2M|\mathbb{R}) = P_l \subset H^- \subset SpH(2M|\mathbb{R})$ consists of the elements

$$PH_l(2M|\mathbb{R}) = \left\{ \begin{pmatrix} p^A_B & 0 \\ p_{CB} & p_{CD} \end{pmatrix}, 0, p_A, p \right\}.

Possible local coordinates on $SpH/PH_l$ are

$$X^{AB}, \quad Y^A = y^A - w_B X^{AB}.$$ (3.12)

Analogously we define $SpH_r/PH_{lr}$ with local coordinates $X^{AB}$ and $Y^A_1, \ldots, Y^A_r$. In the case of $r = 0$, this gives the Lagrangian Grassmannian with $X^{AB}$ being local coordinates of its big cell $M_M$. Indeed, from Eq. (3.3) it follows that any element $^{\mathrm{3.1}}$ of $Sp(2M|\mathbb{R})$ with $\det |d^{AB}| \neq 0$, which condition singles out the big cell of the Lagrangian Grassmannian, belongs to some equivalence class associated to a point of $M_M$.

### 3.2 Vector fields

Any Lie group $G$ possesses two mutually commuting sets of left and right Lie vector fields $L_\alpha$ and $R_\alpha$ ($\alpha, \beta = 1, 2, \ldots, \dim G$), where indices $\alpha, \beta, \ldots$ enumerate a basis of $G$, each forming the Lie algebra $g$ of $G$.

$$[R_\alpha, R_\beta] = f_{\alpha\beta}^\gamma R_\gamma, \quad [L_\alpha, L_\beta] = f_{\alpha\beta}^\gamma L_\gamma, \quad [R_\alpha, L_\beta] = 0.$$ (3.13)

The straightforward calculation of $Sp(2M)$ right vector fields in the coordinates (3.4), (3.11) gives

$$R_{AB} = -2A^E_A A^D_B \frac{\partial}{\partial X^{DE}} + 2A^E_{(B}C_{A)D} \frac{\partial}{\partial A^E_D} + 2C_{AD}C_{BE} \frac{\partial}{\partial C^{DE}},$$

$$R^{AB} = 2\frac{\partial}{\partial C_{AB}},$$

$$R^{A}_B = -2C_{AC} \frac{\partial}{\partial C^{BC}} - A^C_A \frac{\partial}{\partial C^{BC}}.$$

$SpH_r(2M)$ right vector fields contain in addition the Heisenberg vector fields

$$R^C_j = D^C_M \left( X^{MA} \frac{\partial}{\partial y_j^A} + \frac{\partial}{\partial w_j^M} + \left( -y_j^M + w_j^M X^{MA} \right) \frac{\partial}{\partial u_j} \right),$$ (3.15)

$$R^A_j = -A^M_A \left( \frac{\partial}{\partial y_j^M} + w_j^M \frac{\partial}{\partial u_j} \right) - C_{AN} R^N_j,$$

$$R^j = 2 \frac{\partial}{\partial u_j}.$$
where \( j = 1, \ldots, r \). Note, that for \( r = 1 \) the index \( j \) will be omitted.

In the same local coordinates, the left Heisenberg vector fields are

\[
L_j^A = \frac{\partial}{\partial y_j^A} - w_j^A \frac{\partial}{\partial u_j}, \quad L_j^A = \frac{\partial}{\partial w_j^A} + y_j^A \frac{\partial}{\partial u_j}, \quad L_j = 2 \frac{\partial}{\partial u_j},
\]

(3.16)

Nonzero commutation relations of (right) vector fields of \( \text{Sp}H_r \) are

\[
\begin{align*}
[R^A_B, R^C_E] &= \delta^C_B R^A_E - \delta^A_E R^C_B, \\
[R^A_B, R^{CE}] &= \delta^C_B R^{AE} + \delta^E_B R^{AC}, \\
[R^A_B, R_{CE}] &= -\delta^A_C R_{BE} - \delta^A_E R_{BC}, \\
[R_{AB}, R^{CE}] &= \delta^C_A R^E_B + \delta^C_B R^E_A + \delta^E_A R^C_B + \delta^E_B R^C_A
\end{align*}
\]

(3.17)

and

\[
\begin{align*}
[R^A_B, R_j^C] &= \delta^C_B R_j^A, \\
[R_{AB}, R_j^C] &= \delta^C_B R_j^A + \delta^A_j R_j^B, \\
[R_{kA}, R_j^B] &= \delta_{jk} \delta^B_A R_j.
\end{align*}
\]

(3.18)

In the case of \( r = 2 \) it is convenient to introduce

\[
y^A_{\pm} = y^1_{\pm} \pm y^2_{\pm}, \quad w^A_{\pm} = (w^1_{\pm} \pm w^2_{\pm}), \quad u_{\pm} = (u^1_{\pm} \pm u^2_{\pm}),
\]

\[
R^A_{\pm} = \frac{1}{2} (R^A_1 \pm R^A_2), \quad R^A_{\pm A} = \frac{1}{2} (R^A_{1A} \pm R^A_{2A}), \quad R_\pm = \frac{1}{2} (R_1 \pm R_2).
\]

(3.19)

Eqs. (3.15) acquire the form

\[
\begin{align*}
R_{-}^C &= D_D^C \left( \frac{\partial}{\partial w_-^D} + X^{DA} \frac{\partial}{\partial y_-^A} - \frac{1}{2} Y_+^D \frac{\partial}{\partial u_+} - \frac{1}{2} Y_-^D \frac{\partial}{\partial u_-} \right), \\
R_{- A} &= -A_D^A \left( \frac{\partial}{\partial y_-^D} + \frac{1}{2} w_-^D \frac{\partial}{\partial u_-} + \frac{1}{2} y_+^D \frac{\partial}{\partial u_+} \right) - C_{AD} R_-^D, \\
R_- &= 2 \frac{\partial}{\partial u_-},
\end{align*}
\]

(3.20)

\[
\begin{align*}
R_{+}^C &= D_D^C \left( \frac{\partial}{\partial w_+^D} + X^{DA} \frac{\partial}{\partial y_+^A} - \frac{1}{2} Y_+^D \frac{\partial}{\partial u_+} - \frac{1}{2} Y_-^D \frac{\partial}{\partial u_-} \right), \\
R_{+ A} &= -A_D^A \left( \frac{\partial}{\partial y_+^D} + \frac{1}{2} w_+^D \frac{\partial}{\partial u_+} + \frac{1}{2} y_-^D \frac{\partial}{\partial u_-} \right) - C_{AD} R_+^D, \\
R_+ &= 2 \frac{\partial}{\partial u_+},
\end{align*}
\]

(3.21)

where \( Y_\pm^D = y_\pm^D - w_{\pm A} X^{DA} \). Note that

\[
[R_{\pm A}, R_\pm^B] = \frac{1}{2} \delta^B_A R_\pm, \quad [R_{\pm A}, R_\mp^B] = \frac{1}{2} \delta^B_A R_-.
\]

The nonzero anticommutation relations of the ghosts of \( \text{sp}_2 \) are

\[
\begin{align*}
\{c^{AB}, b_{CE}\} &= \frac{1}{2} (\delta^A_E \delta^B_C + \delta^B_E \delta^A_C), \quad \{c_{AB}, b^{CE}\} = \frac{1}{2} (\delta^E_A \delta^B_C + \delta^E_B \delta^A_C), \\
\{c_A^B, b_C^E\} &= \delta^{AC}_E b^B_B, \quad \{c^A, b_C\} = \delta^A_B, \quad \{c_i^A, b_j^C\} = \delta^i_j \delta^A_B, \quad \{c, b\} = 1.
\end{align*}
\]

(3.22)
Using conventions (3.19) along with
\begin{align*}
c_{±A} &= c_{1A} ± c_{2A}, \quad c_±^A = c_1^A ± c_2^A, \quad c_± = c_1 ± c_2, \\
b_±^A &= \frac{1}{2}(b_1^A ± b_2^A), \quad b_± = \frac{1}{2}(b_1 ± b_2),
\end{align*}

one can see that the canonical BRST operator of \( SpH_2 \) is
\begin{equation}
Q_{SpH_2} = c^A R_{AB} + c^A B R_B^A + c_{AB} R^{AB} \tag{3.24}
+ c^A_{BC} b_C^A - 4 c^A_{BC} b^C + 2 c^A_{BC} b_{AC} - 2 c^A_{BC} b^{BC}
+ c_+ R_+ + c_+ R_+ A + c_+ R_+ A + c_- R_- A + c_+ R_+ A
- c_+ B c_+ b_+ A + 2 c_{ABC} b_+ A + c_+ B c_+ b_+ A - 2 c^{AB} c_+ b_+ + \frac{1}{2} c_+ c_+ A b_+
- c_+ B c_+ b_+ A + 2 c_{ABC} b_+ A + c_+ B c_+ b_+ A - 2 c^{AB} c_- b_+ + \frac{1}{2} c_- c_- A b_+
+ \frac{1}{2} c_+ c_+ A b_+ + \frac{1}{2} c_+ c_- A b_+ .
\end{equation}

### 3.3 Nonstandard BRST operator

Let some operators \( P_\alpha \) form a “closed algebra”
\[ [P_\alpha, P_\beta] = \phi_{\alpha\beta}(R) P_\gamma \] \tag{3.25}
where both \( P_\alpha(R) \) and “structure functions” \( \phi_{\alpha\beta}(R) \) belong to \( U(R) \). In general, that \( P_\gamma \) satisfy (3.25) allows one to look for a nilpotent BRST operator of the form
\begin{equation}
Q = c^\alpha P_\alpha - \frac{1}{2} \sum_{n>0} \phi_{\alpha_1...\alpha_n\alpha_{n+1}}(R) c^{\alpha_1} \ldots c^{\alpha_n} c^{\alpha_{n+1}} b_{\beta_1} \ldots b_{\beta_n}, \tag{3.26}
\end{equation}

where ghosts \( c^\alpha \) and \( b_\alpha \) obey (2.2), \( \phi_{\alpha\beta}^\gamma \) are the “structure functions” of (3.25) and \( \phi_{\alpha_1...\alpha_n\alpha_{n+1}}(R) \) for \( n > 1 \) are higher structure functions.

The nonstandard BRST operator \( Q_r \), i.e. the nilpotent operator
\begin{equation}
Q_r^2 = 0 \tag{3.27}
\end{equation}
of the form (3.26) with some nonzero higher structure functions, constructed in [1] from \( \mathfrak{sp}_r \) generators \( R_\alpha \), is
\begin{align*}
Q_r &= c^A B P^B + c_{AM} P^{AM} + c^{AB} P_{AB} + \sum_{j=1}^r (c_j P_j + c_{jA} P_j A) \\
&\quad + c^A_{BC} b_C^A - 2 c^A_{BC} b^C + 2 c^A_{BC} b_{AC} - 4 c^{AB} c_{BC} b^C A \\
&\quad - \sum_{j=1}^r c_A B c_j b_j A + \sum_{j=1}^r \nu^{-1} j \left( 2 c^{AB} c_{AB} b_j + 2 c^{AB} c_{jB} b_j R_{jA} + 4 c^{AB} c_{AC} b_j C R_{jB} \\
&\quad - 4 c^{AB} c_{BC} c_{AB} b_j b_j C - 4 c^{AB} c_{AC} c_{BE} b_j C b_j E \right)
\end{align*}
form a “closed algebra” \( \mathfrak{P}_r \) with nonzero commutation relations that follow from (3.17) and (3.18)

\[
\begin{align*}
\lbrack P_A B, P_C E \rbrack &= \delta_C B P_A E - \delta_A E P_C B, \\
\lbrack P_A B, P_{CE} \rbrack &= \delta_C P_A E + \delta_E P_A C, \\
\lbrack P_{AB}, P_{MN} \rbrack &= \delta_M P_A N + \delta_M P_B N + \delta_N P_A M + \delta_N P_B M + \sum_j \nu^{-1} j P_j (\delta_A \delta_B^N + \delta_B \delta_A^N) \\
&\quad - \sum_j \nu^{-1} j (\delta_A R_j B P_j N + \delta_B R_j A P_j N + \delta_A R_j B P_j M + \delta_B R_j A P_j M), \\
\lbrack P_{AB}, P_D C \rbrack &= \delta_D P_B C + \delta_A D P_A C, \\
\lbrack P_{AB}, P_j C \rbrack &= (\delta_C R_j B + \delta_B R_j A) P_j, \\
\lbrack P_A, P_j D \rbrack &= \delta_B P_j A.
\end{align*}
\]

Specifically, in the case of \( r = 2 \) with \( \nu_1 = -\nu_2 = \nu \) the generators \( P_\alpha \) of \( \mathfrak{P}_2 \) (3.28) are

\[
\begin{align*}
P_\alpha : \quad P_A^B &= R_A^B + \delta_A^B, \quad P^{MB} = R^{MB}, \quad P_A^i = R_i^A, \quad P_i = R_i - \nu_i, \\
P_{AB} &= R_{AB} - \sum_{i=1}^r \nu_i^{-1} R_i A R_i B.
\end{align*}
\] (3.29)

## 4 Standard oscillator realization of the nonstandard BRST operator

The appearance of the nonstandard BRST operator in [1] was a kind of mysterious and looked nontrivial. Here we show that it admits a very simple, although nonpolynomial, equivalent form resulting from the oscillator realization of \( sp(2M) \).

Let

\[
\mathcal{R}_{AB} = R_A R_B, \quad \mathcal{R}_P^S \equiv \mathcal{R}_P^S = \frac{1}{2} R_P R^S + \frac{1}{2} R^S R_P, \quad \mathcal{R}^{MN} = R^M R^N, \quad \mathcal{R} = R,
\] (4.1)

where \( R_A \) and \( R \) are vector fields of the Heisenberg algebra \( \mathfrak{h} \subset \mathfrak{sp} \), that satisfy

\[
\lbrack R_B, R_A \rbrack = \delta_B^A R.
\] (4.2)

Denoting the indices of \( \mathfrak{sp} \) by single calligraphic letters \( A, B, \ldots \), we have

\[
\lbrack \mathcal{R}_A, \mathcal{R}_C \rbrack = f_{A C}^D \mathcal{R}^D, \quad \lbrack \mathcal{R}_C, \mathcal{R}_D \rbrack = f_{C D}^A \mathcal{R}^A, \quad \lbrack \mathcal{R}_A, \mathcal{R}_D \rbrack = f_{A D}^C \mathcal{R}^C,
\] (4.3)

with the structure coefficients \( f_{A C}^D \) to be read off Eqs. (3.17), (3.18). This is nothing but the standard oscillator realization of \( sp(2M) \) [23] provided that the central element \( R \) takes some fixed nonzero value, which is indeed true for the \( Q_r \) closed elements as follows from Eq. (3.28) at \( \nu_i \neq 0 \).
Since \( R \) is nonzero, it is eligible to introduce operators \( T_A = R^{-1} R_A \):

\[
T_{AB} = R^{-1} R_A R_B, \quad T_P^S \equiv T^S_P = R^{-1} \frac{1}{2} \left( R_P R^S + R^S R_P \right), \quad T^{MN} = R^{-1} R^M R^N, \tag{4.4}
\]

that have the following commutation relations among themselves and with the vector fields of \( Sp(2M) \)

\[
[T_A, T_C] = f_{AC} \hat{D} T_D, \tag{4.5}
\]

\[
[R_A, T_C] = f_{AC} \hat{D} T_D. \tag{4.6}
\]

As a result, the operators

\[
K_A = R_A - T_A \tag{4.7}
\]

also fulfill the commutation relations of \( sp(2M) \)

\[
[K_A, K_C] = f_{AC} \hat{D} K_D \tag{4.8}
\]

and commute to all vector fields of the Heisenberg group

\[
[K_A, R_B] = 0, \quad [K_A, R^B] = 0, \quad [K_A, R] = 0.
\]

From here it follows nilpotency of any \( Q \) of the form

\[
Q = Q_K + Q_H, \quad Q^2 = 0, \tag{4.9}
\]

where \( Q_K \) has the \( sp(2M) \) canonical BRST form for the operators \( K_A \)

\[
Q_K = c^{AB} K_{AB} + c^A_B K^B_A + c_{AB} K^{AB} + c^A_{BC} c^B_C b_A^C - 4 c^{AB} c_{BC} b_A^C + 2 c^A_{BC} b_A^C - 2 c^A_{ABC} b^{BC}
\] \( \tag{4.10} \)

and \( Q_H \) is any BRST operator built from the Heisenberg vector fields. In the case of interest we set

\[
Q_H = c_A R^A + c(R - \nu). \tag{4.11}
\]

It turns out that the nonstandard BRST operator of [1] is related to \( Q \) via a canonical change of variables that preserves the commutation relations between the ghost variables and vector fields. The new realization of the nonstandard BRST operator via the oscillator realization of \( sp(2M) \) not only fully explains its origin, but also simplifies the relation of the BRST form of the dynamical equations with its unfolded formulation. Indeed, the unfolded equations, that result from the construction of Section 2, applied to the BRST operator \( Q \), are

\[
D f = \left( d - R^{-1} \omega^{AB} R_A R_B - R^{-1} \frac{1}{2} \omega^A_B \left( R^B R_A + R^A R_B \right) - R^{-1} \omega_{AB} R^A R^B \right) f = 0. \tag{4.12}
\]

At \( R \neq 0 \), these are just the \( Sp(2M) \) invariant unfolded equations proposed in [2] where the \( \omega \) dependent terms were interpreted as the \( Sp(2M) \) connection in the Fock module.

A somewhat unusual feature of the new operator \( Q \) is its nonpolynomiality in \( R \), that was not allowed in the analysis of [1]. Hence the explicit form of the relation between the two BRST operators is rather involved and also nonpolynomial.
5 Closed forms from BRST cohomology

5.1 General case

Here we consider a coordinate independent realization of the closed forms that underly the construction of HS currents of [6] and [16], using the correspondence between unfolded and BRST formulations discussed in Section 2.

Let $Q_B$ and $Q$ be two nilpotent operators

$$Q_B^2 = 0, \quad Q^2 = 0,$$

(5.1)

where $Q_B$ is a canonical BRST operator associated to some group $B \subset G$ while $Q$ is not necessarily canonical.

Consider an element

$$F = \Omega f,$$

(5.2)

where $\Omega$ belongs to the algebra $A$ generated by $R, c$ and $b$, to which $Q_B$ and $Q$ belong, and has a nonnegative ghost number $p$, while $f$ belongs to a left $A$-module and satisfies the conditions

$$Qf = 0$$

(5.3)

and

$$b_\alpha f = 0.$$

(5.4)

Eq. (5.3) is the dynamical equation obeyed by $f$ while Eq. (5.4) implies that $f$ is $c$-independent and hence should be interpreted as a zero-form in the unfolded formulation. From the equations (5.3) and (5.4) it follows that

$$P_\alpha f = 0, \quad P_\alpha = \{Q, b_\alpha\}.$$  

(5.5)

These are the independent equations encoded by the equation (5.3).

We will refer to $\Omega$ and $f$ as a $p$-form and 0-form, respectively, since they become those in the unfolded interpretation of the model. In addition it is required that, by virtue of Eqs. (5.3) and (5.4),

$$Q_B F|_N = 0,$$

(5.6)

for some submanifold $N$ of $G$. In accordance with the general analysis of Section 2 this implies that, for any orbit $O_B$ of $B$ in $G$, the pullback of the $p$-form $F = \Omega f$ to $N \cap O_B$ is closed provided that the zero-form $f$ satisfies its field equations. This acquires the interpretation of the current conservation once $f$ is expressed via bilinears of some other fields $C$ as in the examples of Section 6.

Now, let us discuss the freedom in the definition of $F$. First of all, to describe a nontrivial charge conservation, $F$ should belong to $Q_B$ cohomology on $N$. Indeed, from the analysis of Section 2 it follows that $Q_B$-exact $F$ leads to an exact form on $N \cap O_B$, hence not contributing to the integrated charge.

Another ambiguity originates from

$$F_l(\eta_l) = \eta_l \Omega f,$$

(5.7)

where $\eta_l$ is a $Q_B$ closed element of ghost number zero

$$[Q_B, \eta_l] = 0.$$  

(5.8)
Clearly, $F_l(\eta_l)$ satisfies all conditions on $F$. Note, however, that not every $Q_B$ closed $\eta_l$ leads to a nontrivial result because some contributions may vanish by virtue of the equations (5.3) and (5.4).

Alternatively, a $Q$ closed $\eta_r$ of ghost number zero makes it possible to define

$$F_r(\eta_r) = \Omega \eta_r f,$$

where

$$[Q, \eta_r] = 0.$$  

The meaning of $\eta_r$ is simple. Various $\eta_r$ describe genuine symmetries of the equation $Q f = 0$ just mapping one solution to another. Their interpretation is less trivial in terms of the rank one fields $C_1$ and $C_2$ used to compose a rank two field $f \sim C_1 C_2$ that leads to nontrivial charge conservation in the rank one model. In this case the insertion of $\eta$ affects essentially the form of the bilinear current, leading to different conserved charges. From this point of view, $\eta$ describe symmetries of the rank one field equations induced by the conserved currents upon quantization. More precisely, symmetries of rank one fields are described by the $Q_B$ cohomology of the space of $F_r(\eta_r) \cup F_l(\eta_l)$. As shown in the next section, in the cases of interest the ambiguities due to $\eta_l$ and $\eta_r$ are equivalent, i.e., $F_r(\eta_r) = F_l(\eta_l)$.

Another benefit of introducing parameters $\eta$ into the definition of conserved charges is that they allow us to extend them to a larger space. Suppose for simplicity that (5.6) is true for any $\mathcal{N}$, i.e., $\mathcal{N} = G$. The equation (5.6) then implies that $d_B F = 0$ on $G$. This allows us to integrate $F$ over submanifolds of any orbit of $B$ in $G$ (e.g., of $B$ itself).

Let us introduce an operator $\Pi_B$ that solves the equation

$$Q_G \Pi_B = \Pi_B Q_B.$$  

For any $\eta_l$, the form

$$\Phi = \eta_G \Omega f, \quad \eta_G = \Pi_B \eta_l$$

is $Q_G$ closed since

$$Q_G \eta_G = \eta_G Q_B, \quad Q_G \eta_G \Omega f = 0.$$  

It should be stressed that it is not a priori guaranteed that the equation (5.11) admits a global solution on $G$. This construction is useful to relate conserved charges that may result from integration over close surfaces in $G$ which usually give equivalent results modulo redefinition of the symmetry parameters $\eta$. In fact, the relation between different parameters is just governed by the equation (5.11) that leads to different restrictions of $\eta_G$ to different surfaces in $G$. Also let us stress that, contrary to the equation (5.8), the first of the equations (5.13) is not solved by $\eta_G = \text{const}$, i.e., the equation (5.13) reconstructs appropriate dependence of $\eta_G$ along the directions transversal to orbits of $B$. Also note that, for different subgroups $B$, this procedure may lead to different results related by a redefinition of $\eta$.

### 5.2 Nontrivial symmetries

The conserved currents of [6, 16] bilinear in HS fields in the generalized matrix space-time $\mathcal{M}_M$ depend on constant parameters $\eta^{B_1 \ldots B_n}_{A_1 \ldots A_m}$ associated to different HS symmetry
parameters. Let us show how these parameters result from the general construction of
the previous subsection.

First of all we observe that in our construction $Q_B$, $Q$ and $\Omega$ are built from the right
$\text{SpH}_2$ vector fields $R_\alpha$. Hence, any parameter $\eta \in U_2(L_\mu)$ composed of the left $\text{SpH}_2$
vector fields $L_\mu$ obey the properties of both $\eta_l$ and $\eta_r$ which, in turn, should be identified
within this class because $L_\mu$ commute to $\Omega$.

However, the space of effective symmetry parameters is smaller than $U_2(L_\mu)$ because
some of $\eta \in U_2(L_\mu)$ act trivially on $f$ that satisfy the equations (5.5). In other words,
some elements of $V \in U_2(L_\mu)$ can be represented in the form

$$V = \sum \mu a_\mu(x, \partial) P_\mu$$

where $P_\mu \in \mathfrak{g}_2$ and $a(x, \partial)$ are some differential operators on $G$. Since $\Phi_i(P)$
commute to $L_\mu$, the space $I$ of $V$ forms a two-sided ideal of $U_2(L_\mu)$. The quotient algebra
$S = U_2(L_\mu)/I$ describes true symmetries of the space of solutions of the equations (5.5).

Let $S_{\mu}^\nu(x)$ relate the right vector fields $R_\nu$ of a Lie algebra $\mathfrak{g}$ of some Lie group $G$ to
the left ones $L_\mu$,

$$L_\mu = S_{\mu}^\nu(x) R_\nu.$$ 

Clearly, in any coordinates $x^\kappa$ on $G$,

$$S_\alpha^\beta(x) = L_\alpha^\kappa(x) R^{-1}_\kappa^\beta(x), \quad \text{where} \quad L_\beta = L_\beta^\kappa \frac{\partial}{\partial x^\kappa}, \quad R_\beta = R_\beta^\kappa \frac{\partial}{\partial x^\kappa}.$$ 

From the Lie algebra commutation relations and mutual commutativity of left a right
vector fields it follows that

$$[R_\mu, S_\alpha^\beta] = -f_\mu^\lambda S_\alpha^\lambda, \quad [L_\gamma, S_\beta^\mu] = f_\gamma^\beta S_\nu^\mu, \quad -f_\beta^\mu S_\alpha^\beta S_\gamma^\mu = f_\alpha^\gamma S_\beta^\nu, \quad \text{etc.} (5.15)$$

It is convenient to use the short-hand notation

$$R_a = (R_A, R^A), \quad R_{ab} = (R_{AB}, R_A^B, R^{AB}), \quad L_a = (L_A, L^A), \quad \text{etc.}$$

Let us now consider the case of $\text{SpH}$ starting with the relations between the vector fields
$R_a$, $R(3.15)$ and $L_a$, $L(3.16)$ listed in Section 3.2 in the particular coordinates (3.4),
(3.11) for the case of $r = 1$.

One can see that

$$L_a = S_a^b R_b + S_a R, \quad L = R,$$

where

$$S_a^b = \begin{pmatrix} -D & -CD \\ XD & (A + CDX) \end{pmatrix}, \quad S_a = \begin{pmatrix} -w \\ y \end{pmatrix}.$$ (5.17)

It is convenient to use Eqs.(5.15) along with (3.18) to obtain

$$[R_{ab}, S_c^d] = -S_c^e f_{ab} e^d, \quad [R_{ab}, S_c] = 0, \quad [R_m, S_a^b] = 0,$$

$$[R_m, S_a] = -S_a^m f_m n, \quad -f_{b m} S_a^b S_c^m = f_{ac} S,$$ (5.18)
where \( f_{ac} \) is defined via \([R_a, R_b] = f_{ac} R\). Taking into account (5.18) and antisymmetry of \( f_{ac} \) in \( a \) and \( c \), from (5.16) it is elementary to obtain
\[
L(aL_c) = S_a^b S_c^d R(b R_d) + S(c^d S_a) R_d R + S_a S_c R R .
\]  
(5.19)

Denoting
\[
S_{ac}^{bd} = S_{(a}^b S_{c)}^d , \quad S_{ac}^d = S_{(c}^d S_a) , \quad S_{ac} = S_a S_c ,
T^{(l)}_{ab} = L^{-1} L(a L_b), \quad T^{(l)}_a = L_a , \quad T^{(l)} = L ,
T^{(r)}_{ab} = R^{-1} R(a R_b), \quad T^{(r)}_a = R_a , \quad T^{(r)} = R
\]
and using that \( L = R \) by virtue of (5.16), we obtain that for all sph indices \( \alpha, \beta \)
\[
T^{(l)}_\alpha = S_{\alpha \beta} T^{(r)}_{\beta} .
\]  
(5.20)

Note that, as mentioned in Section 4, \( T^{(r)}_\alpha \equiv T _\alpha \) form sph, as well as \( T^{(l)}_\alpha \). Moreover, analogously to (4.5), (4.6)
\[
[T^{(r)}_\alpha , T^{(r)}_{\beta}] = [R_\alpha , T^{(r)}_{\beta}] = f_{\alpha \beta}^{\ \ \gamma} T^{(r)}_{\gamma} ,
[T^{(l)}_\alpha , T^{(l)}_{\beta}] = [L_\alpha , T^{(l)}_{\beta}] = f_{\alpha \beta}^{\ \ \gamma} T^{(l)}_{\gamma} ,
[T^{(l)}_\alpha , T^{(r)}_{\beta}] = [L_\alpha , T^{(r)}_{\beta}] = [T^{(l)}_\alpha , R_\beta] = 0 .
\]  
(5.21)

As a result, it follows that \( S_{\alpha \beta} \), (5.16), (5.20) satisfy (5.15) where \( f_{\alpha \gamma}^{\ \ \beta} \) are sph structure constants. Indeed, from (5.20) along with (5.21) it follows for example that
\[
[R_\gamma , T^{(l)}_\alpha] = ([R_\gamma , S_\alpha]\mu] + S_{\alpha \beta} f_{\beta \gamma}^{\ \ \mu} T^{(r)}_\mu = 0 ,
[L_\gamma , T^{(l)}]_{\beta} = f_{\gamma \beta}^{\ \ \nu} S_{\nu} T^{(r)}_\mu = [L_\gamma , S_{\beta}] T^{(r)}_\mu .
\]

Therefore the vector fields
\[
\tilde{L}_\alpha = S_{\alpha \beta} R_\beta
\]
satisfy sph commutation relations and commute to \( R_\mu \). Hence, \( \tilde{L}_\alpha \) form left vector fields on SpH. Recall that, by construction, \( \tilde{L}_a = L_a \) and \( \tilde{L} = L \).

As a result, from (4.4), (5.20) and (5.19) it follows that
\[
K^{(l)}_{ab} = S_{ab}^{\ c d} K_{cd} , \quad \text{where} \quad K^{(l)}_{ab} = \tilde{L}_{ab} - T^{(l)}_{ab} , \quad K_{ab} = R_{ab} - T^{(r)}_{ab} .
\]  
(5.22)

Since \( K_{ab} \) annihilates solutions of (5.3), \( K^{(l)}_{ab} \in I \). Hence the symmetry algebra \( S \) is generated by \( L_a \). This result extends to any rank \( r \).

5.3 \( M- \) forms

Consider the Lie algebra sph\(_2\) and associated ghosts (3.22). Let
\[
\Omega = \frac{1}{M!} \varepsilon_{A_1...A_M} (c_{B_1} A_1 R_{-B_1} + \frac{1}{4} c_{A_2} A_1 R_{-}) \ldots (c_{B_M} A_M R_{-B_M} + \frac{1}{4} c_{A_M} A_M R_{-}) \equiv \ldots \equiv \sum_{k=0}^{M} 4^{k-M} k! (M-k)! \varepsilon_{A_1...A_M} c_{B_1 A_1} \ldots c_{B_k A_k} c_{A_{k+1}} \ldots c_{A_M} A_{B_1} \ldots R_{-B_k} (R_{-})^{M-k} ,
\]  
(5.23)
where $\varepsilon_{B_1...B_M}$ is the totally antisymmetric Levi-Civita symbol and "+" variables of $\text{sp}(2M)/2$ are defined in (3.19) and (3.23).

The minimal subgroup $B$ of $SpH_2$ that allows to consider the form (5.23) is $T \otimes H(y_+)$, where $T$ is the Abelian subgroup of translations (3.5) while $H(y_+)$ is the subgroup of $H_M \times H_M$ with the coordinates $y_+^A$, introduced in (3.19). The respective BRST operator is

$$Q_B = c^{AB} R_{AB} + c^+_A R_{+A} \equiv$$

$$c^{AB} P_{AB} - \nu^{-1} c^+_A R_{+A} P_+ + 4 \nu^{-1} (c^{AB} R_{-B} + \frac{1}{4} c^+_A R_-) R_{+A},$$

where $P_\alpha$ are given in (3.29).

Let from now on $f$ be a ghost independent and $Q_2$ closed function

$$Q_2 f = 0,$$

where $Q_2$ is the nonstandard BRST operator (3.28). Since $[Q_B, \Omega] = 0$, $\Omega f$ turns out to be $Q_B$ closed as a consequence of (5.5) and the fact that $(c^{AB} R_{-B} + \frac{1}{4} c^+_A R_-)^{M+1} = 0$.

Let any $B \subset SpH_2$ such that

$$Q_B \Omega f = 0$$

be called closure subgroup. The maximal closure subgroup turns out to be $SpH_+ = SpH \otimes H(y_+, u_+, w_+)$, where $H$ has coordinates $(y_+, u_+, w_+)$ (3.19). The corresponding BRST operator is

$$Q_{SpH_+} = c^{AB} R_{AB} + c^+_A R_{+A} + c_{AB} R_{AB} + c^+_A R_{+A} + c^+_A R_{+A} \quad (5.27)$$

$$+ c^{AB} c^B c^C - 4 c^{AB} b_{AC} b^{BC} + 2 c^{A B} c^{BC} b_{AC} - 2 c^{AB} c_{AC} b^{BC}$$

$$- c^{AB} c^+_A b^+_A + 2 c_{AB} c^+_B b^+_B + c^+_C c^+_A b_{+B} - 2 c^{AB} c^+_A b_{+B} + \frac{1}{2} c^+_C c^+_A b^+_A.$$ 

Indeed, using the relations and (3.29) we obtain

$$\left\{Q_{SpH_+}, c^{DE} R_{-D} + \frac{1}{4} c^+_E R_-\right\} = -2 c^{ABC} D^{DE} R_{-D} + c^+_B E (c^{BD} R_{-D} + \frac{1}{4} c^+_C R_-) \quad (5.28)$$

and

$$Q_{SpH_+} f = 4 \nu^{-1} (c^{AB} R_{-B} + \frac{1}{4} c^+_A R_-) R_{+A} f - c^+_A f. \quad (5.29)$$

From here it follows that $Q_{SpH_+} \Omega f = 0$. Clearly, any subgroup of $SpH_+$ that contains the minimal closure subgroup $T \otimes H(y_+)$, like e.g. $P_u \otimes H(y_+)$ and $P_u \otimes H(y_+, w_+, u_+)$, is also a closure subgroup of $\Omega f$. (Recall that $P_u$ is the upper parabolic subgroup of $Sp(2M/\mathbb{R})$ (3.17).)

Using Eqs. (2.14), we obtain from (5.24) that

$$Q_B \Omega f = 0 \Rightarrow d|_B \tilde{\Omega} f = 0, \quad (5.30)$$

where $d|_B$ is the exterior differential on $B$,

$$\tilde{\Omega} = \sum_{k=0}^{M} \frac{4^{-(M-k)}}{k!(M-k)!} \varepsilon_{A_1...A_M}$$

$$\omega^{B_1 A_1} \wedge \ldots \wedge \omega^{B_k A_k} \wedge \omega^{+ A_{k+1}} \wedge \ldots \wedge \omega^{+ A_M} R_{-B_1} \ldots R_{-B_k} R_-^{M-k}$$

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and \( \omega \) are Cartan forms on \( B \). In Section 6.1, we show that the pullback of the form \( \tilde{\Omega} \) to \( T \subset H(\tilde{y}_+ \subset SpH_2) \) reproduces the conserved current of [16] provided that \( f \) is bilinear in solutions of rank one field equations. The form \( \tilde{\Omega}f \) provides the coordinate independent generalization of the conserved HS currents of [6] [16].

### 5.4 3M—forms

The correspondence between conserved HS currents in four dimensional Minkowski space \( \mathcal{M}_4 \) and those in the ten dimensional matrix space \( \mathcal{M}_4 \) was established in [16]. Since the charge in \( \mathcal{M}_4 \) contains four integrations versus three in Minkowski space, the naive reduction with fourth integrations over a cyclic spin variable in \( \mathcal{M}_4 \) gives zero. To make the cycle noncontractible, a singularity (flux) should be introduced in the spinning space. It was suggested in [16] to use for this aim a generalized 2\( M \)-form current. This was achieved by introducing additional spinor variables \( W \). The corresponding currents were of the form

\[
(dW)^M (W dX + dY_+)^M \eta(W, WX + Y_+) g(W, Y_+ | X),
\]

where the parameters \( \eta \) were arbitrary functions of \( WX + Y_+ \) and \( W \) while \( g(W, Y_+ | X) \) was related to the stress tensor via the half Fourier transform that replaced \( \frac{\partial}{\partial Y_-} \) by \( W \).

This generalization allowed us to consider singular parameters \( \eta \) necessary to reproduce the standard 4\( d \) currents in Minkowski space, that was hard to achieve in the original \( M \)-form current [6], where parameters were polynomials of \( \frac{\partial}{\partial Y_-} \) and \( (X \frac{\partial}{\partial Y_-} - Y_+) \).

However the geometric meaning of the construction, and, in particular, of the half-Fourier transform was not clear in the setup of [16]. Here we introduce a geometric 3\( M \)-form current construction that reproduces that of [16]. In the new setup, the half-Fourier transform results from the integration over additional \( M \) coordinates.

Consider

\[
\Lambda = \left( c_0^{-A} c_0^{+A} \right)^M \Omega,
\]

where \( \Omega \) is of the form (5.23).

The minimal subgroup of \( SpH_2 \) that supports the form (5.32) is

\[
B = T \ltimes \mathbf{H}(y_+, w_+, u_+, y_-, u_-),
\]

where \( T \) is the Abelian subgroup of translations (3.5) while \( \mathbf{H}(y_+, w_+, u_+, y_-, u_-) \subset H_M \times H_M \) has coordinates (3.19).

The respective BRST operator \( Q_B \) can be written in the form

\[
Q_B = c^{AB} P_{AB} - \nu^{-1} c_0^{+A} R_{+A} P_- + c_B P^B + c_+ P_+ + c_- R_- + c_0^{-A} R_{-A} + 4 \nu^{-1} \left( c^{AB} R_{-B} + \frac{1}{4} c_+^{+A} R_- \right) R_{+A} - 2 c^{AB} c_+ a_{+B} + \frac{1}{2} c_+ a^- c_+ a^+ a_+ a_- + \frac{1}{2} c_+ a^- c_+ a^+ a_-,
\]

where \( P_\mu \) are given in (3.29).

One can easily see that \( \left[ Q_B, (c_0^{-A} c_0^{+A})^M \right] = 0 \) and

\[
Q_B f = \nu c_0^{-A} \Lambda f
\]
provided that ghost $a$ independent $f$ satisfies (5.5).

Note that the property (5.35) holds for a larger group $B_u = P_u \otimes \mathcal{H}(y_+, w_+, u_+, y_-, u_-)$, where $P_u$ is the upper parabolic subgroup (3.6). The group $B_u$ is maximal in the sense that further extensions lead to additional terms on the r.h.s of (5.35).

Again, using Eqs. (2.14) along with (5.24), we obtain that

$$Q_B A f = \nu c_\Lambda f \Rightarrow d|_B \tilde{A} f = \nu c_\Lambda f,$$

where $d|_B$ is the exterior differential on $B$,

$$\tilde{A} f = \left( \omega^{-A} \omega^{+A} \right)^M \tilde{\Omega} f,$$

$\tilde{\Omega}$ is of the form (5.31) and $\omega$ are Cartan forms on $B$.

Although the property (5.36) does not imply the closure of the $3M$-form $\Lambda f$ on $B$, it is closed on any submanifold $\mathcal{N}$ of $B$ such that $\omega_- \left( \omega^{-A} \omega^{+A} \right)^M \big|_{\mathcal{N}} = 0$, i.e., $\mathcal{N}$ is a kind of Pfaffian surface. This property will be used in Section 6.2 to construct conserved currents.

In fact, the formulas (5.36), (5.37) have the following interpretation. Consider the construction of Section 5.3 with the $2M$-form parameters $\phi$ that satisfy the conditions

$$c_-^A \phi = 0, \quad \phi c_+^A = 0, \quad \phi c_+^A = 0, \quad \phi c_+^A \phi = 0$$

and

$$[Q_B, \phi] = -\nu c_\phi.$$

Then, the $3M$ form

$$\Psi = \phi \Omega f$$

turns out to be $Q_B$ closed

$$Q_B \Psi = 0.$$

Here the equations (5.38) imply that $\Psi$ contains a factor of $\left( c_-^A c_+^A \right)^M$ while the equation (5.40) determines the dependence of $\phi$ on the central charge coordinate $u_-$ in such a way that the form $\Psi$ becomes closed. Note that this construction is to some extent analogous to that described in the end of Subsection 5.1.

### 6 Bilinear currents

To show that the differential forms $\tilde{\Omega}(\eta, f)$ and $\tilde{A}(\eta, f)$ introduced in Section 5 lead to bilinear conserved currents of [6, 16] we need manifest expressions for the Cartan forms on $SpH_2$. The straightforward computation gives

$$\omega^{AB} = -\frac{1}{2} D_C^A D_E^B dX^{EC},$$

$$\omega_{F^B} = -C_{FA} D_C^A D_E^B dX^{EC} - D_F^A dA^B,$$

$$\omega_{CD} = -\frac{1}{2} C_{CB} D_D^A dA^B - C_{CB} D_D^A dA^B + \frac{1}{2} dC_{CD},$$

$$\omega_+^A = -D_B^A dy_+^B + D_B^A X^{BC} dw_{+C},$$

$$\omega_+^B = -C_B dA^D + (A^C_B + C_B A^D X^{DC}) dw_{+C},$$

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\[
\omega_+ = -\frac{1}{4} w_+ B d y_+ - \frac{1}{4} y_+ B d w_+ - \frac{1}{4} w_- B d y_- + \frac{1}{4} y_- B d w_- - \frac{1}{2} d u_+ ,
\]
\[
\omega_-^A = -D_B^A d y_- + D_B^A X^{BC} d w_C ,
\]
\[
\omega_- = -\frac{1}{4} w_+ B d y_- + \frac{1}{4} y_+ B d w_- - \frac{1}{4} w_- M d y_+ + \frac{1}{4} y_- M d w_+ - \frac{1}{2} d u_- .
\]

Let us consider the cases of \( M \)-forms and \( 3M \) forms separately.

### 6.1 \( M \)-forms

To obtain the manifest formula for \( \tilde{\Omega}(\eta, f) \), note that the general solution of rank two equations (5.5) is
\[
f = \text{det}(A) \exp \left( \frac{1}{4} \nu \left( 2u_+ + w_- B Y_+ + w_+ B Y_- \right) \right) F(Y_+, Y_- | X) ,
\]
where
\[
Y_+^B = y_+^B - w_+ A X^{BA}
\]
and \( F(Y_+, Y_- | X) \) is any solution of the equation
\[
\left( \frac{\partial}{\partial X^{AB}} + 2 \nu^{-1} \frac{\partial}{\partial Y_- (B)} \frac{\partial}{\partial Y_+ (A)} \right) F = 0 .
\]

Using (6.1) along with (3.20) and (6.2) we obtain from (5.31)
\[
\tilde{\Omega}(\eta, f) = 2^{-M} \left( d X^{BA} \frac{\partial}{\partial y_- B} - \frac{1}{2} \nu (d Y_+^A + \frac{1}{2} w_+ B d X^{AB}) \right)^M \eta
\]
\[
\exp \left( \frac{1}{4} \nu \left( 2u_- + w_- B Y_+ + w_+ B Y_- \right) \right) F(Y_+, Y_- | X) ,
\]
where \( \eta \) is a free parameter of HS symmetries. That the expression (5.31) is independent of the coordinates \( A^A B \) and \( C_{AB} \) is not accidental, being a consequence of its \( Q_{SpH_+} \) closure.

To make contact with the bilinear currents of [6], consider \( B = T \otimes H(y_+) \subset SpH_2 \). Then
\[
\tilde{\Omega}(\eta, f) \big|_B = 2^{-M} \left( d X^{BA} \frac{\partial}{\partial y_- B} - \frac{1}{2} \nu d y_+^A \right)^M \eta F(y_+, y_- | X) \big|_{y_- = 0} .
\]

As mentioned in Section 5.2, \( \eta \) is a polynomial in the operators
\[
L_{+ A} = -D_A^B R_+^B - C_{BC} D_A^C R_+^B - \frac{1}{2} w_+ A R_+ - \frac{1}{2} R_- w_-, \]
\[
L_{+} = -X^{BC} D_C^A R_+^B + A^E R_+^E + D_A^C C_{EB} X^{BC} R_+^E + \frac{1}{2} y_+ A R_+ + \frac{1}{2} R_- y_-, \]
\[
L_{- A} = -D_A^B R_-^B - C_{BC} D_A^C R_-^B - \frac{1}{2} w_- A R_- - \frac{1}{2} R_- w_+, \]
\[
L_- = -X^{BC} D_C^A R_-^B + A^E R_-^E + D_A^C C_{EB} X^{BC} R_-^E + \frac{1}{2} y_- A R_- + \frac{1}{2} R_+ y_-, \]

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that can be obtained using (5.17). Modulo \( P_\mu (3.29) \), the pullback to \( B \) gives

\[
L_+^A|_B = -R_+^A, \quad L_-^A|_B = X^{BA} R_+^B + \frac{1}{2} \nu y_+^A.
\] (6.7)

It is easy to see that \( L_+^A \) and \( L_-^A \) (6.7) lead to exact forms on \( B \). Indeed, since \( \{Q_B, b_{+A}\} = R_+^A \), the \( R_+ \) dependent part of \( \eta \) is exact. Analogously, setting \( \xi^B = X^{AB} b_{+A} \), we have \( \{Q_B, \xi^B\}|_B = X^{AB} R_+^A + c^{AB} b_{+A} \) and hence

\[
(X^{AB} R_+^A + c^{AB} b_{+A}) \Omega f|_B = Q_B \Upsilon^B
\] (6.8)

for some \( \Upsilon^B \). As a result, effective parameters depend only on \( L_- \). Using (6.6) and neglecting the terms, that belong to the annihilator \( I \) generated by \( P_\mu (3.29) \), we have

\[
L_-^A|_B \simeq -R_-^A - \frac{1}{2} \nu w_+^A, \quad L_-^A|_B \simeq X^{BA} R_-^B + \frac{1}{2} \nu y_+^A.
\] (6.9)

Finally, using (5.20) we conclude that nontrivial parameters are arbitrary functions of

\[
\frac{\partial}{\partial y_-^B}, \quad X^{BA} \frac{\partial}{\partial y_-^B} - \frac{1}{2} \nu y_+^A.
\] (6.10)

Setting \( F = C^+ \left( \frac{y_+ - y_-}{2} \right) |X| C^- \left( \frac{y_- - y_+}{2} \right) |X| \), where \( C^\pm (y|X) \) satisfy the rank 1 unfolded equations

\[
\left( \frac{\partial}{\partial X_{AB}} \pm \frac{1}{2} \nu^{-1} \frac{\partial^2}{\partial y^A \partial y^B} \right) C^\pm (y|X) = 0,
\] (6.11)

we obtain from (6.4)

\[
\tilde{\Omega}(\eta, f) = 2^{-M} \left( dX^{BA} \frac{\partial}{\partial y_-^B} - \frac{1}{2} \nu dY_+^A \right)^M \eta \left( dX^{BA} \frac{\partial}{\partial y_-^B} - \frac{1}{2} \nu y_+^A \right) F(y_-|X). 
\] (6.12)

which, up to a constant, is the bilinear current of [16].

### 6.2 \( 3M \) forms

In [16] we introduced the generalized 2\( M \)-closed forms which allowed us to reproduce the usual HS charges of Minkowski space. However, neither geometric meaning of the \( M \) additional coordinates nor the origin of the half-Fourier transform applied in [16] was not clear in that paper. Here we show that the 3\( M \) form \( (5.37) \) of Subsection 5.4 naturally reproduces the results of [16].

Let \( \mathcal{N} = T \ll H(y_+, w_+, u_+, y_-, u_-)_{|u_- = \text{const}} \). Using (6.1) along with (3.20) and (6.2), we obtain from (5.37)

\[
\tilde{\Lambda}(\eta, f)|_{\mathcal{N}} = 2^{-M} \left( dX^{BA} \frac{\partial}{\partial y_-^B} - \frac{1}{2} \nu (dY_+^A + \frac{1}{2} w_+^B dX^{AB}) \right)^M \eta \exp \left( \frac{\nu}{4} w_+^B y_-^B \right) F(Y_+, Y_-|X).
\] (6.13)

Here \( F(Y_+, Y_-|X) \) is any solution of (6.3) and \( \eta \) is a function of \( L_\pm \) (6.6) up to the terms that belong to the annihilator \( I \) generated by \( P_\mu (3.29) \). Using that

\[
\omega_-(d_y^A dw_+^A)^M|_{\mathcal{N}} = 0
\] (6.14)
by virtue of (6.1) we obtain that
\[ d\tilde{\Lambda} f|_{\mathcal{N}} = 0 \quad \text{for any } Q_2 \text{ closed } f. \] (6.15)

Analogously to the case of \( M \)-forms of Subsection 6.1, one can see that the dependence of the parameters on \( L_+^A \) and \( L_-^A \) leads to exact forms.

Consider the family of surfaces \( \mathcal{N} \) of the form
\[ \mathcal{N} = \mathbb{R}^M(y_-) \times \mathbb{R}^M(w_+) \times \sigma(M), \] (6.16)
where \( \sigma(M) \) is any \( M \)-dimensional surface. In other words, we consider only such surfaces that their volume forms contain \( (dy_-^A dw_+^A)^M \).

It can be shown that all terms of the form \( \tilde{\Lambda}(\eta, f)|_{\mathcal{N}} \), that contain \( \frac{\partial}{\partial y_-^A} \), are exact on \( \mathcal{N} \). Indeed, it is evident that
\[ \varepsilon_{A_1...A_M} dy_1^{A_1} \cdots dy_M^{A_M} \frac{\partial}{\partial y_2^B} \Phi = M \left( \varepsilon_{B_1...A_M} dy_1^{A_2} \cdots dy_M^{A_M} \Phi \right) \]
provided \( \Phi \) is any form on \( \mathbb{R}^M(y) \times A(x) \). Hence, from (6.13) we obtain that
\[ \tilde{\Lambda}(\eta, f)|_{\mathcal{N}} \sim \left( -\nu \right)^M \left( dy_-^A dw_+^A \right)^M \left( dY_+^A + \frac{1}{2} w_+^B dX^{AB} \right)^M \times \]
\[ \eta \exp \left( \frac{1}{2} \nu (u_- + w_+^B y_-^B) \right) F(Y_+, y_- | X), \]
where \( \eta \) is a function of the operators \( L_-^B \) modulo (3.29) and modulo terms that contain \( \frac{\partial}{\partial y_-^B} \). From (6.9), (3.21) and (3.20) it follows that the 'effective' parameters are functions of
\[ L_-^A \simeq -\frac{1}{4} \nu w_+^A, \quad L_-^E \simeq X^{AE} \frac{1}{4} \nu w_+^A + \frac{1}{2} \nu Y_+^A. \]

This leads to the expression for the current used in [16] upon complexification to the Siegel space. The half-Fourier transform results from the integration over \( y_i \).

Indeed, setting
\[ u_- = 0, \quad -i\hbar = \frac{1}{4} \nu, \quad F(Y_+, y_- | X) = C^+(\frac{Y_+ + Y_-}{2} | X) C^-\left(\frac{Y_+ - Y_-}{2} | X\right), \]
where \( C^\pm(y | X) \) satisfy the rank one unfolded equations (6.11), the integration of (6.17) over \( y_- \) gives
\[ \varpi_{2M} \sim \left( dw_+^A \right)^M \left( dY_+^A + \frac{1}{2} w_+^B dX^{AB} \right)^M \eta \tilde{F}(Y_+, w_+ | X), \] (6.18)
where \( \tilde{F}(Y_+, w_+ | X) \) is the half-Fourier transform of \( F(Y_+, y_- | X) \)
\[ \tilde{F}(Y_+, w_+ | X) = (2\pi)^{-M/2} \int_{\mathbb{R}^M} (dy_-)^M \exp \left( -i \hbar w_+^B y_-^B \right) F(Y_+, y_- | X) \]
and \( \eta = \eta(w_+^A, \frac{1}{2} X^{AE} w_+^A + Y_+^A) \). Up to a constant, the equation (6.18) reproduces the bilinear \( 2M \)-form current of [16].
7 Conclusion

The construction of currents presented in this paper not only explains their coordinate-independent origin but also clarifies their role in the nonlinear unfolded theory. Indeed, the $M$– and $3M$–closed forms of Section 5 are analogous to the terms that glue zero-form Weyl modules to the gauge-field modules via unfolded equations of the form

$$dW + \tilde{\Omega} f = 0 \quad \text{or} \quad dV + \tilde{\Lambda} f = 0,$$

(7.1)

where $f$ belongs to the Weyl module while $W$ and $V$ should be interpreted, respectively, as a $(M - 1)$– and $(3M - 1)$–form on the respective group $B$. This structure, known since [20], is typical for the unfolded field equations.

In the context of $Sp(8)$ invariant HS theory analogous problem was recently considered in [15] where terms of this type were obtained in the sector of one-forms $W$. However, the deformation of HS field equations obtained in [15] breaks the symmetry between dotted and undotted spinors, i.e., $GL(4)$ symmetry, and hence essentially differs from the equations (7.1) which provide a new interesting framework for the deformation of HS field equations. In the case where the rank two field $f$ is represented by bilinears of the rank one fields $C$ as in Section 6 the equations (7.1) become nonlinear and should describe the HS current interactions. Given that the realization of HS currents in 4d Minkowski space of [16] was rather nontrivial, it is tempting to see how the standard HS current interaction reappears in terms of the twelve-form $\Lambda$.

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