ON THE TWISTED $q$-ZETA FUNCTIONS 
AND $q$-BERNOULLI POLYNOMIALS

Taekyun Kim, Lee Chae Jang, Seog-Hoon Rim and Hong-Kyung Pak

To the 62nd birthday of Og-Yeon Choi

Abstract. One purpose of this paper is to define the twisted $q$-Bernoulli numbers by using $p$-adic invariant integrals on $\mathbb{Z}_p$. Finally, we construct the twisted $q$-zeta function and $q$-$L$-series which interpolate the twisted $q$-Bernoulli numbers.

1. Introduction

Throughout this paper $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\mathbb{C}$ and $\mathbb{C}_p$ are respectively denoted as the ring of $p$-adic rational integers, the field of $p$-adic rational numbers, the complex number field and the completion of algebraic closure of $\mathbb{Q}_p$. The $p$-adic absolute value in $\mathbb{C}_p$ is normalized so that $|p|_p = \frac{1}{p}$. When one talks of $q$-extension, $q$ is considered in many ways such as an indeterminate, a complex number $q \in \mathbb{C}$, or a $p$-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$ one normally assumes that $|q| < 1$. If $q \in \mathbb{C}_p$, we normally assume that $|q - 1|_p < p^{-1/(p-1)}$ so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$. We use the notation as

$$[x] = [x : q] = 1 + q + \cdots + q^{x-1},$$

for $x \in \mathbb{Z}_p$.

Let $UD(\mathbb{Z}_p)$ be the set of uniformly differentiable functions on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$ the $p$-adic $q$-integral was defined as

$$I_q(f) = \int_{\mathbb{Z}_p} f(x)d\mu_q(x) = \lim_{N \to \infty} \frac{1}{[p^N]} \sum_{0 \leq x < p^N} f(x)q^x, \text{ cf. } [1, 2, 3].$$

Note that

$$I_1(f) = \lim_{q \to 1} I_q(f) = \int_{\mathbb{Z}_p} f(x)d\mu_1(x) = \lim_{N \to \infty} \frac{1}{p^N} \sum_{0 \leq x < p^N} f(x).$$

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For a fixed positive integer \(d\) with \((p, d) = 1\), let
\[
X = X_d = \lim_{N \to \infty} \mathbb{Z}/dp^N \mathbb{Z}, \quad X_1 = \mathbb{Z}/p,
\]
\[
X^* = \bigcup_{0 < a < dp} (a + dp\mathbb{Z}_p),
\]
\[
a + dp^N \mathbb{Z}_p = \{x \in \mathbb{X}|x \equiv a \pmod{dp^N}\},
\]
where \(a \in \mathbb{Z}\) lies in \(0 \leq a < dp\mathbb{N}\). Let \(\mathbb{Z}\) be the set of integers. For \(h \in \mathbb{Z}, k \in \mathbb{N}\), the \(q\)-Bernoulli polynomials were defined as

\[
(1) \quad \beta_n^{(h,k)}(x, q) = \int_{\mathbb{Z}_p^k} [x + x_1 + \cdots + x_k]^n q^{\sum_{i=1}^k x_i (h-i)} \prod_{i=1}^k d\mu_q(x_i), \quad \text{cf. [1, 2].}
\]

The \(q\)-Bernoulli polynomials at \(x = 0\) are called \(q\)-Bernoulli numbers. In [1] it was shown that the \(q\)-Bernoulli numbers were written as

\[
\beta_n^{(h,k)}(0, q) = \beta_n^{(h,k)}(0). \quad \text{Indeed, } \lim_{q \to 1} \beta_n^{(h,k)}(q) = B_n^{(k)}, \quad \text{where } B_n^{(k)} \text{ are Bernoulli numbers of order } k, \text{ see [1, 2, 3].}
\]

Let \(\chi\) be a Dirichlet character with conductor \(f \in \mathbb{N}\). Then the Dirichlet \(L\)-series attached to \(\chi\) is defined as

\[
L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}, \quad \text{for } s \in \mathbb{C}, \quad \text{cf. [7, 8].}
\]

When \(\chi = 1\), this is the Riemann zeta function. In [1], \(q\)-analogue of \(\zeta\)-function was defined as follows: For \(h \in \mathbb{Z}, s \in \mathbb{C}\),

\[
(2) \quad \zeta_q^{(h)}(s, x) = \frac{1 - s + h}{1 - s} (q - 1) \sum_{n=0}^{\infty} \frac{q^{(n+x)h}}{[n + x]^{s-1}} + \sum_{n=0}^{\infty} \frac{q^{(n+x)h}}{[n + x]^s}.
\]

Note that \(\zeta_q^{(h)}(s, x)\) is an analytic continuation on \(\mathbb{C}\) except for \(s = 1\) with

\[
(3) \quad \zeta_q^{(h)}(1 - m, x) = -\frac{\beta_{m,1}^{(h,1)}(x, q)}{m}, \quad \text{for } m \in \mathbb{N}.
\]

In [1], we easily see that

\[
(4) \quad \beta_m^{(h,1)}(x, q) = -m \sum_{n=0}^{\infty} \frac{q^{(n+x)h}[n + x]^{m-1}}{[n + x]^s} - (q - 1)(m + h) \sum_{n=0}^{\infty} \frac{q^{(n+x)h}[n + x]^m}{[n + x]^{s}}.
\]

It follows from (2) that

\[
\lim_{q \to 1} \zeta_q^{(h)}(s, x) = \zeta(s, x) = \sum_{n=1}^{\infty} \frac{1}{(n + x)^s}.
\]
By the meaning of the $q$-analogue of Dirichlet $L$-series, we consider the following $L$-series:

$$L^h_q(s, \chi) = \frac{1 - s + h}{1 - s} (q - 1) \sum_{n=1}^{\infty} \frac{q^n \chi(n)}{[n]_q^{s-1}} + \sum_{n=1}^{\infty} \frac{q^n \chi(n)}{[n]_q^s},$$

for $h \in \mathbb{Z}, s \in \mathbb{C}$. It is easy to see that $L^h_q(s, \chi)$ is an analytic continuation on $\mathbb{C}$ except for $s = 1$. For $m \geq 0$, the generalized extended $q$-Bernoulli numbers with $\chi$ are defined as

$$\beta^{(h,k)}_{m,\chi}(q) = [f]^{m-k} \sum_{i_1, \ldots, i_k=0}^{l-1} \sum_{i=1}^{l} (h-l+1)i_i \beta^{(h,k)}_{m}(\frac{l-1}{f}, q^f) \left( \prod_{j=1}^{k} \chi(i_j) \right).$$

By (5) and (6), we easily see that

$$L^h_q(1 - m, \chi) = -\frac{\beta^{(h,1)}_{m,\chi}(q)}{m},$$

for $m \in \mathbb{N}$, cf. [1].

In the present paper we give twisted $q$-Bernoulli numbers by using $p$-adic invariant integrals on $\mathbb{Z}_p$. Moreover, we construct the analogs of $q$-zeta function and $q$-$L$-series which interpolate the twisted $q$-Bernoulli numbers at negative integers.

### 2. $q$-extension of Bernoulli numbers

In this section we assume that $q \in \mathbb{C}_p$ with $|1 - q|_p < 1$. By the definition of $p$-adic invariant integrals, we see that

$$I_1(f_1) = I_1(f) + f'(x),$$

where $f_1(x) = f(x + 1)$. Let

$$T_p = \cup_{n \geq 1} C_p^n = \lim_{n \to \infty} C_p^n,$$

where $C_p^n = \{ w \mid w^{p^n} = 1 \}$ is the cyclic group of order $p^n$. For $w \in T_p$, we denote by $\phi_w : \mathbb{Z}_p \to \mathbb{C}_p$ the locally constant function $x \mapsto w^x$. If we take $f(x) = \phi_w(x)e^{tx}$, then we easily see that

$$\int_{\mathbb{Z}_p} e^{tx} \phi_w(x) d\mu_1(x) = \frac{t}{we^t - 1},$$

cf. [5].

It is obvious from (7) that

$$\int e^{tx} \chi(x) \phi_w(x) d\mu_1(x) = \sum_{i=1}^{f} \chi(i) \phi_w(i)e^{it},$$

cf. [5].
Now we define the analogue of Bernoulli numbers as follows:

\[
e^{xt} \frac{t}{we^{t} - 1} = \sum_{n=0}^{\infty} B_{n,w}(x) \frac{t^{n}}{n!},
\]

\[
\frac{\sum_{i=1}^{f} \chi(i) \phi_{w}(i)e^{it}}{w^{f}e^{f} - 1} = \sum_{n=0}^{\infty} B_{n,w,\chi} \frac{t^{n}}{n!}, \quad \text{cf. [5].}
\]

By (8), (9) and (10), it is not difficult to see that

\[
\int_{\mathbb{Z}_{p}} x^{n} \phi_{w}(x) d\mu_{1}(x) = B_{n,w} \quad \text{and} \quad \int_{X} \chi(x) x^{n} \phi_{w}(x) d\mu_{1}(x) = B_{n,w,\chi},
\]

where \( B_{n,w} = B_{n,w}(0) \).

From (11) we consider twisted \( q \)-Bernoulli numbers using \( p \)-adic \( q \)-integral on \( \mathbb{Z}_{p} \). For \( w \in T_{p} \) and \( h \in \mathbb{Z} \), we define the twisted \( q \)-Bernoulli polynomials as

\[
\beta_{m,w}^{(h)}(x, q) = \int_{\mathbb{Z}_{p}} q^{(h-1)y} w^{y}[x + y]^{m} d\mu_{q}(y).
\]

Observe that

\[
\lim_{q \to 1} \beta_{m,w}^{(h)}(x, q) = B_{m,w}(x).
\]

When \( x = 0 \), we write \( \beta_{m,w}^{(h)}(0, q) = \beta_{m,w}^{(h)}(q) \), which are called twisted \( q \)-Bernoulli numbers. It follows from (12) that

\[
\beta_{m,w}^{(h)}(x, q) = \frac{1}{(1 - q)^{m-1}} \sum_{k=0}^{m} \binom{m}{k} q^{xk} (-1)^{k} \frac{k + h}{1 - q^{h + k}w}.
\]

The Eq.(13) is equivalent to

\[
\beta_{m,w}^{(h)}(q) = -m \sum_{n=0}^{\infty} \left[ n \right]^{m-1} q^{hn} w^{n} - (q - 1)(m + h) \sum_{n=0}^{\infty} \left[ n \right]^{m} q^{hn} w^{n}.
\]

From (13), we obtain the below distribution relation for the twisted \( q \)-Bernoulli polynomials as follows: For \( n \geq 0 \),

\[
\beta_{n,w}^{(h)}(x, q) = [f]^{n-1} \sum_{a=0}^{f-1} w^{a} q^{ha} \beta_{n,w}^{(h)}(\frac{a + x}{f}, q^{f}).
\]

Let \( \chi \) be the Dirichlet character with conductor \( f \in \mathbb{N} \). Then we define the generalized twisted \( q \)-Bernoulli numbers as follows: For \( n \geq 0 \),

\[
\beta_{m,w,\chi}^{(h)}(q) = \int \chi(x) q^{(h-1)x} w^{x} [x]^{m} d\mu_{q}(x).
\]
By (15), it is easy to see that

\begin{equation}
(16) \quad \beta_{m,w,\chi}^{(h)}(q) = [f]^{m-1} \sum_{a=0}^{f-1} \chi(a)w^aq^{ha}\beta_{m,w,f}^{(h)}(\frac{q}{f}, q^f).
\end{equation}

Remark. We note that \( \lim_{q \to 1} \beta_{m,w,\chi}^{(h)}(q) = B_{m,w,\chi}, \) (see Eq. (10)).

3. \( q \)-zeta functions

In this section we assume that \( q \in \mathbb{C} \) with \( |q| < 1 \). Here we construct the twisted \( q \)-zeta function and the twisted \( q \)-\( L \)-series (see Eq.(2) and Eq.(3)). Let \( \mathbb{R} \) be the field of real numbers and let \( w \) be the \( p' \)-th root of unity. For \( q \in \mathbb{R} \) with \( 0 < q < 1 \), \( s \in \mathbb{C} \) and \( h \in \mathbb{Z} \), we define the twisted \( q \)-zeta function as

\begin{equation}
(17) \quad \zeta_{q,w}^{(h)}(s) = \frac{1-s+hw}{1-s}q^{-1}\sum_{n=1}^{\infty} \frac{q^{nh}w^n}{n^{s-1}} + \sum_{n=1}^{\infty} \frac{q^{nh}w^n}{n^{s}}.
\end{equation}

Note that \( \zeta_{q,w}^{(h)}(s) \) is an analytic continuation on \( \mathbb{C} \) except for \( s = 1 \) and \( \lim_{q \to 1} \zeta_{q,w}^{(h)}(s) = \zeta(s,w) = \sum_{n=1}^{\infty} \frac{w^n}{n^s} \), cf. [4]. We see, by (17), that

\begin{equation}
(18) \quad \zeta_{q,w}^{(s-1)}(s) = \sum_{n=1}^{\infty} \frac{q^{n(s-1)}w^n}{n^s}.
\end{equation}

In what follows, the notation \( \zeta_{q,w}^{(s-1)}(s) \) will be replaced by \( \zeta_{q,w}(s) \), that is,

\begin{equation}
\zeta_{q,w}(s) = \sum_{n=1}^{\infty} \frac{q^{n(s-1)}w^n}{n^s}.
\end{equation}

We note that Eq.(18) is the \( q \)-extension of Riemann zeta function. By (14) and (17) we give the values of \( \zeta_{q,w}^{(h)}(s) \) at negative integers as follows: For \( m \in \mathbb{N} \), we have

\begin{equation}
(19) \quad \zeta_{q,w}^{(h)}(1-m) = -\frac{\beta_{m,w}^{(h)}(q)}{m}.
\end{equation}

By (17), we also see that

\begin{equation}
(20) \quad \zeta_{q,w}(1-m) = \sum_{n=1}^{\infty} \frac{[n]^{m-1}q^{-mn}w^n}{[n]^s}.
\end{equation}

The Eq.(20) seems to be the \( q \)-analogue of Euler divergence theorem for Riemann zeta function. Now we also consider the twisted \( q \)-analogue of Hurwitz zeta function as follows: For \( s \in \mathbb{C} \), define

\begin{equation}
(21) \quad \zeta_{q,w}^{(h)}(s,x) = \frac{1-s+hw}{1-s}(q-1)\sum_{n=1}^{\infty} \frac{q^{(n+x)h}w^n}{[n+x]^{s-1}} + \sum_{n=1}^{\infty} \frac{q^{(n+x)h}w^n}{[n+x]^s}.
\end{equation}
Note that \( \zeta^{(h)}_{q,w}(s, x) \) has an analytic continuation on \( \mathbb{C} \) with only one simple poles at \( s = 1 \). By Eq.(13), Eq.(14) and Eq.(21), we obtain the following:

\[
\zeta^{(h)}_{q,w}(1 - m, x) = -\frac{\beta_{m,w}(x, q)}{m}, \quad \text{for } m > 0.
\]

Now we consider the twisted \( q \)-\( L \)-series which interpolate twisted generalized \( q \)-Bernoulli numbers as follows: For \( s \in \mathbb{C} \), define

\[
L^{(h)}_{q,w}(s, \chi) = \frac{1 - s + h}{1 - s} (q - 1) \sum_{n=1}^{\infty} \frac{q^{nh}w^{n}\chi(n)}{[n]^{s-1}} + \sum_{n=1}^{\infty} \frac{q^{nh}w^{n}\chi(n)}{[n]^{s}},
\]

where \( w \) is the \( p^r \)-th root of unity.

For any positive integer \( m \) we have

\[
L^{(h)}_{q,w}(1 - m, \chi) = -\frac{\beta_{m,w,\chi}(q)}{m}.
\]

The Eq.(22) implies that

\[
L^{(s-1)}_{q,w}(s, \chi) = \sum_{n=1}^{\infty} \frac{q^{n(s-1)}w^{n}\chi(n)}{[n]^{s}}
= [f]^{-s} \sum_{a=1}^{f} \chi(a)w^{a}q^{(s-1)a}\zeta^{f,w}_{q,f}(s, \frac{a}{f}).
\]

**Question.** Find a \( q \)-analogue of the \( p \)-adic twisted \( L \)-function which interpolates \( q \)-Bernoulli numbers \( \beta_{m,w,\chi}(q) \), cf. [3].

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Institute of Science Education, Kongju National University, Kongju 314-701, Korea, tkim@kongju.ac.kr
Department of Mathematics and Computer Science, KonKuk University, Choongju, Chungbuk 380-701, Korea, leechae.jang@kku.ac.kr
Department of Mathematics Education, Kyungpook University, Taegu, 702-701, Korea, shrim@kyungpook.ac.kr
Faculty of Informatoin and Science, Daegu Haany University, Kyungsan, 712-240, Korea, hkpak@ik.ac.kr