1. Introduction

Borcherds forms are meromorphic modular forms for arithmetic subgroups $\Gamma$ of the orthogonal group $O(n,2)$ which arise from a regularized theta lift of (vector valued) modular forms of weight $1 - \frac{n}{2}$ for $SL_2(\mathbb{Z})$ with poles at the cusp. They have interesting product expansions and explicitly known divisors (cf. [2]). In some cases they can be realized as classical modular forms, such as the difference of two modular $j$-functions or as the discriminant function $\Delta$ (see [3]). In this paper, we give a factorization of values of Borcherds forms at CM points. The main result can be viewed as a generalization of the singular moduli result (Theorem 1.3 of [9]) of Gross and Zagier. In fact, our method gives a new proof of their result, which will be discussed in a sequel to this paper.

Let $V$ be a vector space with quadratic form $Q$ of signature $(n,2)$ and let $D$ be the space of oriented negative-definite two-planes in $V(\mathbb{R})$. $D$ is the symmetric space for $O(n,2)$ and has a Hermitian structure. For example, when $n = 1$, $D \cong \mathbb{H}^+ \sqcup \mathbb{H}^-$ is the union of the upper and lower half-planes $\mathbb{H}^+$ and $\mathbb{H}^-$, respectively. Let $H = GSpin(V)$ and let $K \subset H(\mathbb{A}_f)$ be a compact open subgroup, where $\mathbb{A}_f$ is the finite adeles. We consider the quasi-projective variety

$$X_K = H(\mathbb{Q}) \setminus \left( D \times H(\mathbb{A}_f)/K \right) \cong \coprod_j \Gamma_j \setminus D^+,$$

for a finite number of arithmetic subgroups $\Gamma_j \subset H(\mathbb{Q})$, and where $D^+ \subset D$ is the subset of positively oriented two-planes.

Recall the theory of Borcherds forms on $X_K$. For a lattice $L \subset V$ with dual $L^\vee = \{ x \in V \mid (x,L) \subseteq \mathbb{Z} \}$ such that $L^\vee \supset L$, there exists a finite dimensional subspace $S_L \subset S(V(\mathbb{A}_f))$ of the Schwartz space of $V(\mathbb{A}_f)$ defined as follows. Let $\tilde{\mathbb{L}} = L \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}$. Then $S_L$ is the space of functions with support in $\tilde{L}^\vee$ which are constant on cosets of $\tilde{L}$. A natural basis of $S_L$ is

$$\{ \varphi_\eta = \text{char}(\eta + L) \mid \eta \in L^\vee/L \}$$

and $\dim S_L = |L^\vee/L|$. There exists a representation $\omega$ of (the metaplectic extension $\Gamma'$ of) $\Gamma = SL_2(\mathbb{Z})$ on $S(V(\mathbb{A}_f))$ preserving $S_L$; see section 4 of [2] for details.

A modular form $F : \tilde{\mathbb{H}} \to S_L$ of weight $1 - \frac{n}{2}$ and type $\omega$ for $\Gamma$ satisfies

$$F(\gamma \tau) = (c \tau + d)^{-1 - \frac{n}{2}} \omega(\gamma)(F(\tau))$$

for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. The function $F$ has Fourier expansion

$$F(\tau) = \sum_\eta F_\eta(\tau) \varphi_\eta = \sum_\eta \sum m c_\eta(m) q^m \varphi_\eta,$$
where \( q = e^{2\pi i \tau} \). We say that \( F \) is weakly holomorphic if only a finite number of the \( c_\eta(m) \)’s with \( m < 0 \) are non-zero. Furthermore, we call such a modular form integral if the non-positive Fourier coefficients lie in \( \mathbb{Z} \).

To a weakly holomorphic integral modular form \( F \) of weight \( 1 - \frac{n}{2} \), Borcherds attaches a function \( \Psi(F) \) (called a Borcherds form), which is a meromorphic modular form on the space \( D \times H(\mathbb{A}_f) \) with respect to \( H(\mathbb{Q}) \). The weight of \( \Psi(F) \) is \( \frac{1}{2} c_0(0) \) and the divisor of \( \Psi(F)^2 \) is given explicitly in terms of the negative Fourier coefficients of \( F \),

\[
\text{div}(\Psi(F)^2) = \sum_\eta \sum_{m>0} c_\eta(-m) Z(m, \eta, K),
\]

for divisors \( Z(m, \eta, K) \) on \( X_K \). The concrete connection between \( F \) and \( \Psi(F) \) is given by a regularized theta lift

\[
\Phi(z, h; F) := \int_{\Gamma \backslash \mathfrak{H}} (( F(\tau), \theta(\tau, z, h) )) v^{-2} du dv,
\]

where \( z \in D, h \in H(\mathbb{A}_f) \) and \( \tau = u + iv \in \mathfrak{H} \), and where \( (( F(\tau), \theta(\tau, z, h) )) \) is a theta function constructed from the Fourier expansion of \( F \); see section 2.1 for details. Since \( F \) has a pole at the cusp, this integral diverges and so it must be regularized. See [2] or section 2.1 for the exact definition of the regularized integral.

When \( z \) is not in the divisor of \( \Psi(F) \) we have

\[
(2) \quad \Phi(z, h; F) = -2 \log ||\Psi(z, h; F)||^2,
\]

where \( || \cdot || \) is the Petersson norm, suitably normalized. Our goal is to evaluate the averages of \( \Phi(F) \) over certain sets of CM points.

To define CM points, we consider a rational splitting \( V = V_+ \oplus U \), where \( V_+ \) has signature \( (n, 0) \) and \( U \) has signature \( (0, 2) \). This determines a two-point subset \( D_0 \subset D \) consisting of \( U(\mathbb{R}) \) with its two possible orientations. Let \( T = \text{GSpin}(U) \) and \( K_T = K \cap T(\mathbb{A}_f) \). We obtain a zero cycle

\[
Z(U)_K = T(\mathbb{Q}) \backslash \left( D_0 \times T(\mathbb{A}_f)/K_T \right) \hookrightarrow X_K,
\]

which we regard as a set of CM points inside of \( X_K \). The main theorem is

**Theorem 1.1.** (i) \( \Phi(F) \) is finite at all CM points.

(ii) There exist explicit constants \( \kappa_\eta(m) \) such that

\[
(3) \quad \sum_{z \in Z(U)_K} \Phi(z; F) = \frac{4}{\text{vol}(K_T)} \sum_\eta \sum_{m \geq 0} c_\eta(-m) \kappa_\eta(m),
\]

where the \( c_\eta(-m) \)’s are the negative Fourier coefficients of \( F \).

Now using relation (2) we obtain

**Corollary 1.2.** When \( Z(U)_K \) does not meet the divisor of \( \Psi(F) \), we have

\[
(4) \quad \sum_{z \in Z(U)_K} \log ||\Psi(z; F)||^2 = \frac{-2}{\text{vol}(K_T)} \sum_\eta \sum_{m \geq 0} c_\eta(-m) \kappa_\eta(m).
\]
When $Z(U)_K$ meets the divisor of $\Psi(F)$, it remains to give an interpretation of Theorem\footnote{1} in terms of the function $\Psi(F)$.

The constants $\kappa_n(m)$ come from Eisenstein series on $SL_2$. The quantity in the left hand side of \cite{3} can be written as an integral

$$
\int_{\mathcal{S}(U)} \Phi(z_0, h; F) dh = \int_{\mathcal{S}(U)} \int_{\Gamma \setminus \mathcal{S}} ((F(\tau), \theta(\tau, z_0, h)) v^{-2} dudvh,
$$

where $\mathcal{S}(U) = SO(U) \backslash SO(U)(\mathbb{A}_F)$ and $z_0 \in D_0$. Here we write the theta function as a tensor product

$$
\theta(\tau, z_0, h) = \theta(+)(\tau, z_0) \otimes \theta(-)(\tau, z_0, h)
$$

of the theta functions for $V_+$ and $U$, respectively. Then we use the contraction map $\langle \cdot, \theta_{+} \rangle$ (see section \cite{3} for details) and write

$$
\langle F(\tau), \theta(\tau, z_0, h) \rangle = \langle F(\tau), \theta_{+}(\tau, z_0) \rangle \langle \theta(-)(\tau, z_0, h) \rangle,
$$

where $\langle F, \theta_{+} \rangle \in S(U(\mathbb{A}_f))$. After some justification, the order of integration (where the inside integral is regularized) can be switched giving

$$
\int_{\Gamma \setminus \mathcal{S}} \langle F, \theta_{+} \rangle(\tau), \theta_{-}(\tau, z_0, h) dh ) v^{-2} dudv.
$$

Then by the Siegel-Weil formula, the integral of $\theta_{-}(\tau, z_0, h)$ on $\mathcal{S}(U)$ gives rise to a coherent Eisenstein series, $E(\tau, s; -1)$, of weight $-1$. For the definition of the term coherent, see \cite{3}. With this Eisenstein series, we can write \cite{3} as

$$
\int_{\Gamma \setminus \mathcal{S}} \langle F, \theta_{+} \rangle(\tau), E(\tau, 0; -1) ) v^{-2} dudv.
$$

Using Maass operators we relate $E(\tau, s; -1)$ to another Eisenstein series, $E(\tau, s; +1)$, of weight $+1$ via

$$
E(\tau, s; -1) v^{-2} = \frac{-4i}{s} \frac{\partial}{\partial \bar{s}} \{ E(\tau, s; +1) \}.
$$

One phenomenon that is very specific to the case of signature $(0, 2)$ is that the resulting Eisenstein series $E(\tau, s; +1)$ is incoherent. Hence, $E(\tau, s; +1)$ satisfies an odd functional equation with respect to $s \mapsto -s$, and, therefore, vanishes at $s = 0$. The integral \cite{3} can be evaluated using a Stokes’ Theorem argument and some convergence estimates about the Fourier coefficients of $E(\tau, s; +1)$. This leads to the constants $\kappa_n(m)$ as follows.

For $V = V_+ \oplus U$ and $L \subset V$, let $L_+ = V_+ \cap L$ and $L_- = U \cap L$. If $\mu \in L^\vee / L_-$ and $\varphi_\mu = \text{char}(\mu + L_-)$ we write

$$
E(\tau, s; \varphi_\mu, +1) = \sum_m A_\mu(s, m, v) q^m,
$$

where the Fourier coefficients have Laurent expansions

$$
A_\mu(s, m, v) = b_\mu(m, v)s + O(s^2).
$$

In order to define $\kappa_n(m)$, we first define

$$
\kappa_n^\mu(m) = \begin{cases} \lim_{v \to \infty} b_\mu(m, v) & \text{if } m > 0, \\ k_0(0) \varphi_\mu(0) & \text{if } m = 0, \\ 0 & \text{if } m < 0, \end{cases}
$$

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$$

Borcherds forms and generalizations of singular moduli.
where \( k_0(0) \) is a constant which depends on the space \( U \) (see Definition 2.17). Let
\[
L^\vee = \bigcup_{\eta}(\eta + L), \quad L = \bigcup_{\lambda}(\lambda + L_+ + L_-)
\]
and write \( \eta = \eta_+ + \eta_- \), \( \lambda = \lambda_+ + \lambda_- \). Then we define
\[
\kappa_{\eta}(m) = \sum_{\lambda} \sum_{x \in \eta_+ + \lambda_+ + L_+} \kappa^U_{\eta_+ + \lambda_+}(m - Q(x)).
\]

The space \( U \) is a rational quadratic space of signature \((0, 2)\), so \( U \cong k \) for an imaginary quadratic field \( k \) and the quadratic form is just a negative multiple of the norm-form. When \( k \) has odd discriminant and \( m \neq 0 \), then
\[
\frac{-2}{\text{vol}(K^U_K)} \kappa_{\eta}(m)
\]
is the logarithm of an integer. Thus, if \( F \) has \( c_0(0) = 0 \), so that \( \Psi(F) \) is a meromorphic function, then Corollary 1.2 shows that
\[
(7)
\prod_{z \in Z(U)_K} ||\Psi(z; F)||^2
\]
is a rational number. Moreover, if all of the negative Fourier coefficients of \( F \) are non-negative, then (7) is actually an integer. In the case of signature \((2, 2)\), similar results were obtained in [5] for certain rational functions and CM points on a Hilbert modular surface. If \( c_0(0) \neq 0 \), then there is a transcendental factor
\[
(4\pi d)^{-1} e^{2 L^U(\infty, 0)/L^U(0, 0)}
\]
appearing in (7), which is related to Shimura’s period invariant, [15], [5], [18], for the CM points in the 0-cycle \( Z(U)_K \). This factor arises from the trivialization over the CM cycle of the line bundle of which \( \Psi(F) \) defines a section.

We can say a little bit more about the rational number appearing in (7). The formulas we obtain for \( \kappa_{\eta}(m) \) tell us the explicit factorization of the rational part of (7). Then, as a consequence of Corollary 1.2, we are able to state a Gross-Zagier type of theorem about which primes can occur in the factorization. For \( F \) as in (1), define
\[
m_{\text{max}} = \max\{ m > 0 \mid c_0(-m) \neq 0 \text{ for some } \eta \}.
\]

**Theorem 1.3.** Let \(-d\) be an odd fundamental discriminant and assume \( U \cong k = \mathbb{Q}(\sqrt{-d}) \). Also assume that \( L_- \cong \mathfrak{a} \) for an \( O_k \)-ideal \( \mathfrak{a} \). Then the only primes which occur in the factorization of the rational part of
\[
\prod_{z \in Z(U)_K} ||\Psi(z; F)||^2
\]
are

(i) \( q \) such that \( q \mid d \),

(ii) \( p \) inert in \( k \) with \( p \leq dm_{\text{max}} \).

As mentioned in Theorem 1.1, one striking phenomenon that occurs in this paper is that the regularized theta lift \( \Phi(F) \) is always finite! This is interesting since the Borcherds form \( \Psi(F) \) can have zeroes or poles, and (2) only holds when the right
hand side is finite. Considering this, one might say that the theta lift is over-
regularized, and it would be interesting to find the analog of Corollary \[12\]
when \(Z(U)_K\) meets the divisor of \(\Psi(F)\).

There exists lots of recent work on singular moduli, particularly traces of singular
moduli (e.g. \[1\], \[3\] and \[19\]). By considering the case of signature \((1, 2)\), Theorem
1.3 of \[9\] can be recovered from Theorem \[17\]. The appropriate quadratic space is
\[V = \{x \in M_2(\mathbb{Z}) \mid \text{tr}(x) = 0\}\]
with \(Q(x) = \det(x)\). For a particular choice of \(F\),
\[
\prod_{z \in \mathbb{Z}(U)_K} \Psi(z; F) = \prod_{[\tau_1], [\tau_2]} \left( j(\tau_1) - j(\tau_2) \right),
\]
where \(\tau_1\) and \(\tau_2\) are CM points with relatively prime fundamental discriminants and
\([\tau_i]\) denotes an equivalence class modulo \(SL_2(\mathbb{Z})\). The right hand side of \[11\] then
gives the same factorization as in \[9\]. We will discuss this new proof of Gross-Zagier
in a subsequent paper.

2. Main theorem in the case of signature \((0, 2)\)

2.1. Basic Setup. We begin by introducing some notation and relevant back-
ground material, and we refer the reader to section 1 of \[12\] for more details. Let
\(V\) be a vector space over \(\mathbb{Q}\) of dimension \(n + 2\) with quadratic form \(Q\), of signature
\((n, 2)\), on \(V\). Let \(D\) be the space of oriented negative-definite 2-planes in \(V(\mathbb{R})\).
For \(z \in D\), let \(\text{pr}_z : V(\mathbb{R}) \rightarrow z\) be the projection map and, for \(x \in V(\mathbb{R})\), let
\(R(x, z) = -\langle \text{pr}_z(x), \text{pr}_z(x) \rangle\). Then we define
\[(x, x)_z = (x, x) + 2R(x, z),\]
and our Gaussian for \(V\) is the function
\[\varphi_\infty(x, z) = e^{-\pi(x, x)_z}.\]
For \(\tau \in \mathfrak{H}, \tau = u + iv\), let
\[g_\tau = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v^{\frac{1}{2}} \\ v^{-\frac{1}{2}} \end{pmatrix},\]
and \(g'_\tau = (g_\tau, 1) \in M_{P_2}(\mathbb{R})\). Let \(l = \frac{v}{2} - 1\), \(G = SL_2\) and \(\omega\) be the Weil represen-
tation of the metaplectic group \(G'_{\mathbb{H}}\) on \(S(V(\mathbb{A}))\), the Schwartz space of \(V(\mathbb{A})\).
If \(H = \text{GSpin}(V)\), then for the linear action of \(H(\mathbb{A}_f)\) we write \(\omega(h)\varphi(x) = \varphi(h^{-1}x)\)
for \(\varphi \in S(V(\mathbb{A}_f))\). If \(z \in D\) and \(h \in H(\mathbb{A}_f)\), we have the linear functional on
\(S(V(\mathbb{A}_f))\) given by
\[
\varphi \mapsto \theta(\tau, z; h; \varphi) = v^{-\frac{1}{2}} \sum_{x \in V(\mathbb{Q})} \omega(g'_\tau)(\varphi_\infty(\cdot, z) \otimes \omega(h)\varphi)(x).
\]
Let \(L \subset V\) be a lattice with dual
\[L^\vee = \{x \in V \mid \langle x, L \rangle \subseteq \mathbb{Z}\}\]
and let \(S_L \subset S(V(\mathbb{A}_f))\) be the space of functions with support in \(L^\vee\) and constant
on cosets of \(\hat{L}\), where \(\hat{L} = L \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}\). We remark that \(S_L\) is finite dimensional and
has a natural basis given by
\[
\{\varphi_\eta = \text{char}(\eta + L) \mid \eta \in L^\vee/L\}.
\]
We also have
\[ S(V(A_f)) = \lim_{L \to L'} S_L. \]

Let \( \Gamma' = \text{Mp}_2(\mathbb{Z}) \) be the full inverse image of \( \Gamma = \text{SL}_2(\mathbb{Z}) \subset G(\mathbb{R}) \) in \( G'_R \). For \( F : \mathfrak{H} \to S_L \), the Fourier expansion of \( F \) can be written
\[ F(\tau) = \sum_{\eta} F_{\eta}(\tau) \varphi_{\eta} = \sum_{\eta} \sum_{m} c_{\eta}(m) q^{m} \varphi_{\eta}. \]

**Definition 2.1.** We say \( F : \mathfrak{H} \to S_L \) is a weakly holomorphic modular form of weight \( 1 - \frac{n}{2} \) and type \( \omega \) for \( \Gamma' \) if
(i) \( F(\gamma' \tau) = (c \tau + d)^{1 - \frac{n}{2}} \omega(\gamma')(F(\tau)) \) for all \( \gamma' \in \Gamma' \), where \( \gamma' \mapsto \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \),
(ii) \( F \) is meromorphic at the cusp, i.e., only a finite number of the \( c_{\eta}(m) \)'s with \( m < 0 \) are non-zero.

Note that when \( n \) is even, \( \omega \) is a representation of \( G_A \) and we can just work with \( \Gamma \). The Fourier expansion in \( \text{(9)} \) is essentially the Fourier expansion given in \( \text{[2]} \), where in that paper he works with group ring elements \( e_{\eta} \in \mathbb{C}[L'/L] \) instead of the Schwartz functions \( \varphi_{\eta} \). Since the theta function \( \theta(\tau, z, h) \) is a linear functional and \( F(\tau) \in S(V(A_f)) \), we can define the \( \mathbb{C} \)-bilinear pairing
\[ (F(\tau), \theta(\tau, z, h)) = \theta(\tau, z, h; F(\tau)). \]

In terms of the Fourier expansion of \( F \), this is
\[ (F(\tau), \theta(\tau, z, h)) = \sum_{\eta} F_{\eta}(\tau) \theta(\tau, z, h; \varphi_{\eta}). \]

Note that as a function of \( \tau \), the above pairing is \( \Gamma \)-invariant (with a pole at the cusp) since the weights of \( \theta \) and \( F \) cancel and their types are dual. Using this pairing we define
\[ \Phi(z, h; F) := \int_{\mathfrak{H}} \sum_{\eta} (F(\tau), \theta(\tau, z, h)) d\mu(\tau), \]
where \( d\mu(\tau) = v^{-2} du dv \) and the integral is regularized as in \( \text{[2]} \). The regularization is defined by
\[ \int_{\mathfrak{H} \setminus \mathfrak{B}} \phi(\tau) d\mu(\tau) = \text{CT} \lim_{\sigma \to 0} \int_{\mathfrak{F}_t} \phi(\tau) v^{-\sigma} d\mu(\tau), \]
where we take the constant term in the Laurent expansion at \( \sigma = 0 \) of
\[ \lim_{t \to \infty} \int_{\mathfrak{F}_t} \phi(\tau) v^{-\sigma} d\mu(\tau), \]
defined initially for \( \text{Re}(\sigma) \) sufficiently large. Here \( \mathfrak{F} \) is the usual fundamental domain for the action of \( \Gamma \) on \( \mathfrak{H} \) and
\[ \mathfrak{F}_t = \{ \tau \in \mathfrak{F} \mid \text{Im}(\tau) \leq t \} \]
is the truncated fundamental domain.
2.2. Borcherds Forms. The space $D$ is a bounded symmetric domain. It can be viewed as an open subset $\mathbb{Q}^-$ of a quadric in $\mathbb{P}(V(\mathbb{C}))$. Explicitly,

$$D \cong \mathbb{Q}^- = \{ w \in V(\mathbb{C}) \mid (w, w) = 0, (w, \bar{w}) < 0 \}/\mathbb{C}^\times,$$

where the explicit isomorphism is $[z_1, z_2] \mapsto w = z_1 + iz_2$ for a properly oriented basis $[z_1, z_2]$. Assume $K$ is a compact open subgroup of $H(\mathbb{A}_f)$ such that $H(\mathbb{A}) = H(\mathbb{Q})H(\mathbb{R})^+K$, where $H(\mathbb{R})^+$ is the identity component of $H(\mathbb{R})$. Define

$$X_K := H(\mathbb{Q}) \backslash (D \times H(\mathbb{A}_f)/K).$$

This is the set of complex points of a quasi-projective variety rational over $\mathbb{Q}$, and if $\Gamma_K = H(\mathbb{Q}) \cap H(\mathbb{R})^+K$, then $X_K \cong \Gamma_K \backslash D^+$, where $D^+ \subset D$ is the subset of positively oriented 2-planes.

Let $L_D$ be the restriction to $D \cong \mathbb{Q}^-$ of the tautological line bundle on $\mathbb{P}(V(\mathbb{C}))$. From this we get a holomorphic line bundle $L$ on $X_K$ equipped with a natural norm, $|| \cdot ||_{\text{nat}}$, called the Petersson norm. Assume we have

$$V(\mathbb{R}) = V_0 + \mathbb{R}e + \mathbb{R}f,$$

where $e$ and $f$ are such that $(e, f) = 1, (e, e) = 0 = (f, f)$. Then $\text{sig}(V_0) = (n - 1, 1)$ and for the negative cone

$$C = \{ y \in V_0 \mid (y, y) < 0 \},$$

we have

$$D \cong \mathbb{D} := \{ z \in V(\mathbb{C}) \mid y = \text{Im}(z) \in C \}.$$ 

The explicit isomorphism is

$$\mathbb{D} \to V(\mathbb{C}), \quad z \mapsto w(z) := z + e - Q(z)f$$

composed with projection to $\mathbb{Q}^-$. The map $z \mapsto w(z)$ can be viewed as a holomorphic section of $L_D$.

We now define the notion of a modular form on $D \times H(\mathbb{A}_f)$.

**Definition 2.2.** A modular form on $D \times H(\mathbb{A}_f)$ of weight $m \in \frac{1}{2}\mathbb{Z}$ is a function $\Psi : D \times H(\mathbb{A}_f) \to \mathbb{C}$ such that

1. $\Psi(z, hk) = \Psi(z, h)$ for all $k \in K$,
2. $\Psi(\gamma z, \gamma h) = j(\gamma, z)^m \Psi(z, h)$ for all $\gamma \in H(\mathbb{Q})$, where $j(\gamma, z)$ is an automorphy factor.

Meromorphic modular forms on $D \times H(\mathbb{A}_f)$ of weight $m \in \mathbb{Z}$ can be identified with meromorphic sections of $L^{\otimes m}$. If $\Psi$ is such a meromorphic modular form, then the Petersson norm of the section $(z, h) \mapsto \Psi(z, h)w(z)^{\otimes m}$ associated to $\Psi$ is

$$||\Psi(z, h)||_{\text{nat}}^2 = ||\Psi(z, h)||^2|y|^{2m}.$$

For reasons we will see below, we renormalize $|| \cdot ||_{\text{nat}}$ and instead work with the following norm

$$||\Psi(z, h)||^2 := ||\Psi(z, h)||_{\text{nat}}^2 \left(2\pi e^{r(1)}\right)^m.$$

The “extra” constant in the metric here is related to that occurring in [13]. Borcherds proved that the regularized integral $\Phi(z, h; F)$ satisfies the equation

$$\Phi(z, h; F) = -2\log ||\Psi(z, h; F)||_{\text{nat}}^2 - c_0(0)(\log(2\pi) + \Gamma(1))$$

$$= -2\log ||\Psi(z, h; F)||^2$$
for a meromorphic modular form \( \Psi(F) \) on \( D \times H(A_f) \) of weight \( m = \frac{1}{2}c_0(0) \) when \( z \) does not lie in the divisor of \( \Psi(F) \).

**Remark 2.3.** In fact, \( \Phi(F) \) may still be finite for \( z \in D \) even if \( z \) lies in the divisor of \( \Psi \). This value of \( \Phi(F) \) must have another meaning there.

**Definition 2.4.** A Borcherds form \( \Psi(F) \) is a meromorphic modular form on \( D \times H(A_f) \) which arises (via \( \mathcal{H} \)) from the regularized theta lift of a modular form \( F \).

### 2.3. CM Points

Assume that we have a rational splitting

\[ V = V_+ \oplus U, \]

where \( V_+ \) has signature \((n, 0)\) and \( U \) has signature \((0, 2)\). This determines a two-point subset \( \{z_0^\pm\} = D_0 \subset D \) given by \( U(\mathbb{R}) \) with its two orientations. For \( z_0 \in D_0 \), we are interested in computing the integral

\[ \int_{SO(U)(\mathbb{Q}) \backslash SO(U)(A_f)} \Phi(z_0, h; F) dh. \]

Let \( T = \text{GSpin}(U) \) and note there is a natural homomorphism \( T \to H \). Let \( K \) be as in section 2.2 and define \( K_T = K \cap T(A_f) \). Consider the set of CM points

\[ Z(U)_K := T(\mathbb{Q}) \backslash \left( D_0 \times T(A_f)/K_T \right) \to X_K. \]

We want to compute

\[ \text{vol}(K_T) \sum_{z \in Z(U)_K} \Phi(z; F) = -2 \int_{SO(U)(\mathbb{Q}) \backslash SO(U)(A_f)} \Phi(z_0, h; F) dh. \]

Note that after normalizing by the volume of \( K_T \), this expression is independent of the choice of \( K \).

### 2.4. Convergence Questions and Regularization

First we consider the case when \( n = 0 \) and our space \( V = U \) is negative-definite. In this case, \( D = D_0 \), the Gaussian is \( \varphi_\infty(x) = e^{\pi(x, x)} \) and the theta function is

\[ \theta(\tau, z_0, h; \varphi) = v^\frac{1}{2} \sum_{x \in U(\mathbb{Q})} \omega(g_x) e^{\pi(x, x)} \varphi(h^{-1} x), \]

for any \( \varphi \in S(U(A_f)) \). When \( n = 0 \) and we have a lattice \( L \subset U \) we write \( \mu \in L^\vee/L \) and \( \varphi_\mu = \text{char}(\mu + L) \). Let \( F(\tau) \) be a weakly holomorphic modular form of weight \( 1 \) valued in \( S_L \), and let

\[ F(\tau) = \sum_\mu F_\mu(\tau) \varphi_\mu = \sum_\mu \sum_{m \in \mathbb{Q}} c_\mu(m) q^m \varphi_\mu, \]

where \( \mu \) runs over \( L^\vee/L \). We assume \( c_\mu(m) \in \mathbb{Z} \) for \( m \leq 0 \). The functions \( F_\mu \) are meromorphic modular forms with some real multiplier for a congruence subgroup of \( SL_2(\mathbb{Z}) \), and it will be very useful to know how large their Fourier coefficients can be.

**Lemma 2.5.** Assume \( m_\mu \in \mathbb{Z} \) is such that \( c_\mu(m_\mu) \neq 0 \) and \( c_\mu(m) = 0 \) for all \( m < m_\mu \). Then there are constants \( C \) and \( C' \) such that, for \( m > 0 \),

\[ |c_\mu(m)| \leq C' \left( (-m_\mu + 2)(m - m_\mu)^6 + m^6 e^{C' \sqrt{m}} \right), \]

where \( C \) depends on \( m_\mu \) and on the multiplier and \( C' \) depends on the polar part of \( F_\mu \).
Proof. The cusp form of weight 12, \((2\pi)^{-12}\Delta(\tau) = q \prod_{n=1}^{\infty} (1 - q^n)^{24}\), has Fourier expansion
\[
(2\pi)^{-12}\Delta(\tau) = \sum_{N=1}^{\infty} \tau(N) q^N,
\]
where \(|\tau(N)| \leq C_1 N^6\) for some constant \(C_1\). Let \(\tilde{\Delta}(\tau) = (2\pi)^{-12}\Delta(\tau)\). We can look at \(F_{\mu}/\tilde{\Delta}\), which has weight \(-11 = 1 - \frac{24}{2}\). If
\[
F_{\mu}/\tilde{\Delta} = \sum_{m=m_{\mu}-1}^{\infty} a_{\mu}(m) q^m,
\]
then for \(m > 0\), (3.38) of \([12]\) tells us there are constants \(C_2, C_3\) such that
\[
|a_{\mu}(m)| \leq C_2 m^{-\frac{24}{2}} e^{C\sqrt{m}},
\]
where \(C\) depends on \(m_\mu\) and on the multiplier. We have
\[
F_{\mu}(\tau) = \left( \sum_{N=1}^{\infty} \tau(N) q^N \right) \left( \sum_{m=m_{\mu}-1}^{\infty} a_{\mu}(m) q^m \right)
= \sum_{N=1}^{\infty} \sum_{m=m_{\mu}-1}^{\infty} \tau(N) a_{\mu}(m) q^{N+m}
= \sum_{m=m_{\mu}}^{\infty} \left[ \sum_{N=1}^{m-m_{\mu}+1} \tau(N) a_{\mu}(m-N) \right] q^m.
\]
Then
\[
|c_{\mu}(m)| = \left| \sum_{N=1}^{m-m_{\mu}+1} \tau(N) a_{\mu}(m-N) \right|
= \left| \sum_{N \geq m} \tau(N) a_{\mu}(m-N) + \sum_{0 < N < m} \tau(N) a_{\mu}(m-N) \right|
\leq C_1 \sum_{N=m}^{m-m_{\mu}+1} N^6 |a_{\mu}(m-N)| + C_1 C_2 \sum_{0 < N < m} N^6 (m-N)^{-\frac{24}{2}} e^{C\sqrt{m-N}}.
\]
We know there is a constant \(C_3\) such that \(|a_{\mu}(m)| \leq C_3\) for \(m \in \{m_\mu, \ldots, 0\}\), and thus
\[
|c_{\mu}(m)| \leq C_1 C_3 (-m_\mu + 2)(m-m_\mu)^6 + C_1 C_2 m^6 e^{C\sqrt{m}}
\leq C' \left( (-m_\mu + 2)(m-m_\mu)^6 + m^6 e^{C\sqrt{m}} \right),
\]
for some constant \(C'\).

In the \(n = 0\) case, the following over-regularization phenomenon occurs:

**Proposition 2.6.** For \(h \in H(A_f)\),
\[
\Phi(z_0, h; F) = \int_{\Gamma \setminus \delta} ((F(\tau), \theta(\tau, z_0, h)) d\mu(\tau)
\]
is always finite.
Proof. This case corresponds to signature \((2,0)\) in \([1]\). In Theorem 6.2 of \([1]\), Borcherds points out that \(\Phi\) is nonsingular except along a locally finite set of codimension 2 sub-Grassmannians \(\lambda^{\perp}\), for some negative norm vectors \(\lambda \in L\). No such vectors exist in signature \((2,0)\). For ease of the reader, we give the proof in our notation. We have

\[
\int_{\Gamma \backslash S} \left( F(\tau), \theta(\tau, z_0, h) \right) d\mu(\tau) = \text{CT}_{\sigma=0} \left\{ \lim_{t \to \infty} \int_{F_t} \theta(\tau, z_0, h; F) v^{-\sigma} d\mu(\tau) \right\},
\]

and we can write the integral on the right hand side of (15) as

\[
\int_{1}^{\frac{1}{2}} \int_{\frac{1}{2}}^{1} \theta(\tau, z_0, h; F) v^{-\sigma} d\mu(\tau) + \int_{F_1} \theta(\tau, z_0, h; F) v^{-\sigma} d\mu(\tau).
\]

The integral over the compact set \(F_1\) is finite and independent of \(t\), so we just look at the first part. By \([16]\), we have

\[
\omega(g' \tau) e^{\pi(x,x)} = v^\frac{1}{2} e(uQ(x)) e^{2\pi v Q(x)},
\]

where \(e(y) = e^{2\pi iy}\). Then (13) is

\[
\theta(\tau, z_0, h; \varphi) = \sum_{x \in U(Q)} e(uQ(x)) e^{2\pi v Q(x)} \varphi(h^{-1} x),
\]

and so the integral over \(F_t - F_1\) is

\[
\sum_{\mu} \sum_{m \in Q} \sum_{x \in U(Q)} c_{\mu}(m) \varphi_{\mu}(h^{-1} x) \int_{1}^{\frac{1}{2}} \int_{\frac{1}{2}}^{1} e(um) e(uQ(x)) e^{-2\pi v m} e^{2\pi v Q(x)} v^{-\sigma} dudv.
\]

Lemma 2.7. If \(m + Q(x) \notin \mathbb{Z}\), then \(c_{\mu}(m) = 0\).

Proof. When we consider the transformation law for \(F\), we have \(F(\tau + 1) = \omega(T)(F(\tau))\). That is, for any \(x \in U(A_f)\),

\[
\sum_{\mu} \sum_{m} c_{\mu}(m) q^m e(m) \varphi_{\mu}(x) = \omega(T) \left( \sum_{\mu} \sum_{m} c_{\mu}(m) q^m \varphi_{\mu}(x) \right)
\]

\[
= \sum_{\mu} \sum_{m} c_{\mu}(m) q^m \omega(T)(\varphi_{\mu}(x))
\]

\[
= \sum_{\mu} \sum_{m} c_{\mu}(m) q^m e(-Q(x)) \varphi_{\mu}(x).
\]

We see \(m + Q(x) \notin \mathbb{Z}\) implies \(c_{\mu}(m) = 0\). \(\square\)

For \(m + Q(x) \in \mathbb{Z}\),

\[
\int_{\frac{1}{2}}^{\frac{1}{2}} e(um) e(uQ(x)) du = \begin{cases} 1 & \text{if } m + Q(x) = 0, \\ 0 & \text{otherwise}. \end{cases}
\]
Integrating with respect to $u$ in (16) and letting $t \to \infty$ gives

\begin{equation}
\sum_{\mu} \sum_{m \in \mathbb{Q}} \sum_{m \geq 0} c_{\mu}(m) \varphi_{\mu}(h^{-1}x) \int_{1}^{\infty} e^{-4\pi m v^{-\sigma} - 1} dv.
\end{equation}

We have $m \geq 0$ since $Q(x) \leq 0$. When $m = 0$, we get

\[ \sum_{\mu} c_{\mu}(0) \varphi_{\mu}(0) \int_{1}^{t} v^{-\sigma - 1} dv = c_{0}(0) \frac{1}{\sigma}(1 - t^{-\sigma}), \]

which equals zero when we take the limit as $t \to \infty$ followed by the constant term at $\sigma = 0$. For $m > 0$, (3.35) of [12] says

\[ \int_{0}^{\infty} e^{-4\pi m v^{-\sigma} - 1} dv \leq C(\epsilon, \sigma) e^{-4\pi m} \]

for any $\epsilon$ with $0 < \epsilon < 4\pi m$, where the constant $C(\epsilon, \sigma)$ is uniform in any $\sigma$-halfplane and independent of $m$. Using this in (17), we have

\[ C(\epsilon, \sigma) \sum_{\mu} \sum_{m > 0} c_{\mu}(m) e^{-4\pi m} \sum_{x \in \mathbb{U}(Q(x)+m=0)} \varphi_{\mu}(h^{-1}x), \]

which is finite by Lemma 2.5.

2.5. Eisenstein Series. Here we give the basic definition of an Eisenstein series and some related theory when $V$ has signature $(n, 2)$ for $n$ even. What follows is a summary of the explanations given in [12] for $n$ even, and we refer the reader to that paper for the more general theory. Inside of $G_A$, we have the subgroups

\[ N_A = \{ n(b) \mid b \in A \}, \quad n(b) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \]

and

\[ M_A = \{ m(a) \mid a \in \mathbb{A}^\times \}, \quad m(a) = \begin{pmatrix} a & 0 \\ a^{-1} & 1 \end{pmatrix}. \]

Define the quadratic character $\chi = \chi_V$ of $\mathbb{A}^\times / \mathbb{Q}^\times$ by

\[ \chi(x) = (x, -\det(V)), \]

where $\det(V) \in \mathbb{Q}^\times / (\mathbb{Q}^\times)^2$ is the determinant of the matrix for the quadratic form $Q$ on $V$. For $s \in \mathbb{C}$, let $I(s, \chi)$ be the principal series representation of $G_A$. This space consists of smooth functions $\Phi(s)$ on $G_A$ such that

\[ \Phi(n(b)m(a)g, s) = \chi(a)|a|^{s+1} \Phi(g, s). \]

We have a $G_A$-intertwining map

\begin{equation}
\lambda = \lambda_V : S(V(\mathbb{A})) \to I \left( \frac{n}{2}, \chi \right),
\end{equation}

where $\lambda(\varphi)(g) = (\omega(g)\varphi)(0)$. If $K_{\infty} = SO(2)$ and $K_f = SL_2(\hat{\mathbb{Z}})$, then a section $\Phi(s) \in I(s, \chi)$ is called standard if its restriction to $K_{\infty}K_f$ is independent of $s$. The function $\lambda(\varphi)$ has a unique extension to a standard section $\Phi(s) \in I(s, \chi)$ such
that $\Phi \left( \frac{q}{r} \right) = \lambda(\varphi)$. We let $P = MN$ and define the Eisenstein series associated to $\Phi(s)$ by

$$E(g, s; \Phi) = \sum_{\gamma \in P_0 \backslash G_Q} \Phi(\gamma g, s),$$

where $G_Q$ is identified with its image in $G_\Lambda$. This series converges for $\text{Re}(s) > 1$ and has a meromorphic analytic continuation to the whole $s$-plane.

One step in proving the $(0, 2)$-Theorem is to apply Maass operators to obtain a relation between two Eisenstein series. Let

$$X_\pm = \frac{1}{2} \left( \frac{1}{2} \pm i \right) \in \mathfrak{s}\mathfrak{l}_2(\mathbb{C}).$$

For $r \in \mathbb{Z}$, let $\chi_r$ be the character of $K_\infty$ defined by

$$\chi_r(k_\theta) = e^{ir\theta}, \quad k_\theta = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \in K_\infty.$$

Let $\phi : G_{\mathbb{R}} \to \mathbb{C}$ be a smooth function of weight $l$, meaning $\phi(gk_\theta) = \chi_l(k_\theta)\phi(g)$, and let $\xi(\tau) = v^{\frac{-l}{2}}\phi(g_r)$ be the corresponding function on $\mathfrak{H}$. Then $X_\pm \phi$ has weight $l \pm 2$, and the corresponding function on $\mathfrak{H}$ is

$$v^{-\frac{l \pm 2}{2}}X_\pm \phi(g_r) = \begin{cases} \left( \frac{2iv^2}{\partial \tau} \mp \frac{1}{2} \xi \right)(\tau) & \text{for } +, \\ -2iv^2\frac{\partial \xi}{\partial \tau}(\tau) & \text{for } -. \end{cases}$$

**Lemma 2.8** (Lemma 2.7 of [12]). Let $\Phi^r_\infty(s) \in I_\infty(s, \chi)$ be the normalized eigenvector of weight $r$ for the action of $K_\infty$. Then

$$X_\pm \Phi^r_\infty(s) = \frac{1}{2}(s + 1 \pm r)\Phi^{r \pm 2}_\infty(s).$$

For $\varphi \in S(V(A_f))$, let $E(g, s; \Phi^r_\infty \otimes \lambda(\varphi))$ be the Eisenstein series of weight $r$ on $G_\Lambda$ associated to $\varphi$. For the Gaussian, $\varphi_\infty(x, z)$, we have $\lambda(\varphi_\infty) = \Phi^l_\infty \left( \frac{2}{l} \right)$, where $l = \frac{n}{2} - 1$. This means that

$$X_\pm E(g, s; \Phi^{l+2}_\infty \otimes \lambda(\varphi)) = \frac{1}{2}(s - l - 1)E(g, s; \Phi^l_\infty \otimes \lambda(\varphi)).$$

On $\mathfrak{H}$, this translates to

$$-2iv^2\frac{\partial}{\partial \tau} \left( E(\tau, s; \varphi, l + 2) \right) = \frac{1}{2} \left( s - \frac{n}{2} \right) E(\tau, s; \varphi, l),$$

where we write $E(\tau, s; \varphi, l) = v^{-\frac{l}{2}}E(g_r, s; \Phi^l_\infty \otimes \lambda(\varphi))$. One main result we need is the Siegel-Weil formula.

**Theorem 2.9** (Siegel-Weil formula). Let $V$ be a vector space of signature $(n, 2)$. Assume $V$ is anisotropic or that $\dim(V) - r_0 > 2$, where $r_0$ is the Witt index of $V$. Then $E(g, s; \varphi)$ is holomorphic at $s = \frac{n}{2}$ and

$$E \left( g, \frac{n}{2}; \varphi \right) = \frac{\alpha}{2} \int_{SO(V)(\mathbb{Q}) \backslash SO(V)(\mathbb{A})} \theta(g, h; \varphi) dh,$$

where $dh$ is Tamagawa measure on $SO(V(A))$, and $\alpha$ is $2$ if $n = 0$ and is $1$ otherwise.
Here $\theta(g,h;\varphi)$ is defined as in (4) without $v^{-\frac{1}{2}}$ and with $g$ replacing $g_i'$. The integration for $SO(U)(\mathbb{R})$ is with respect to the action $h_{\infty}^{-1}x$ in the argument of $\varphi_{\infty}$. The cases which are omitted in the Siegel-Weil formula are when $n = 1 = r_0$ ($V$ is isotropic) and $n = 2 = r_0$ ($V$ is split).

Let us now consider the situation $V = U$, $\text{sig}(U) = (0,2)$. The representation we are interested in is $I(0,\chi)$. This global principal series is a restricted tensor product of local ones,

$$I(0,\chi) = \otimes_v I_v(0,\chi_v).$$

For the local space $U_v = U(\mathbb{Q}_v)$, define the quadratic character $\chi_v$ of $\mathbb{Q}_v^\times$ by $\chi_v(x) = (x,-\det(U_v))_v$.

Let $R_v(U)$ be the maximal quotient of $S(U_v)$ on which $O(U_v)$ acts trivially. The following proposition is a special case of Proposition 1.1 of [11].

**Proposition 2.10.** (i) If $v \neq \infty$, then

$$I_v(0,\chi_v) = R_v(U_v^+) \oplus R_v(U_v^-),$$

where $U_v^\pm$ has Hasse invariant $\epsilon_v(U_v^\pm) = \pm 1$.

(ii) If $v = \infty$, then

$$I_{\infty}(0,\chi_{\infty}) = R_{\infty}(U(0,2)) \oplus R_{\infty}(U(2,0)),$$

and the spaces $U(0,2)$ and $U(2,0)$ have opposite Hasse invariants.

Recall the notion of an incoherent collection.

**Definition 2.11.** An incoherent collection $C = \{C_v\}$ of quadratic spaces is a set of quadratic spaces $C_v$ such that

1. For all $v$, $\dim_{\mathbb{Q}_v}(C_v) = 2$, and $\chi_{C_v} = \chi$.
2. For almost all $v$, $C_v \simeq U_v$.
3. (Incoherence condition) The product formula fails for the Hasse invariants:

$$\prod_v \epsilon_v(C_v) = -1.$$

Then we have, cf. (2.10) in [11],

$$I(0,\chi) \simeq \left( \bigoplus_{U'} \Pi(U') \right) \oplus \left( \bigoplus_{C} \Pi(C) \right)$$

as a sum of two irreducible pieces defined as follows. $U'$ runs over all global quadratic spaces of dimension 2 with $\chi_{U'} = \chi$, while $C$ runs over all incoherent collections of dimension 2 and character $\chi$, and

$$\Pi(U') = \otimes_v R_v(U'),$$

$$\Pi(C) = \otimes_v R_v(C).$$

For $\lambda = \lambda_U$ as in [18], we have $\lambda(\varphi_{\infty}) = \Phi_{\infty}^{-1}(0)$, where $\Phi_{\infty}^{-1}$ is the normalized eigenvector of weight $-1$ for the action of $K_{\infty}$. From the theory of principal series representations, we have $\Phi_{\infty}^{-1}(0) \in R_{\infty}(U(0,2))$ and $\Phi_{1,\infty}(0) \in R_{\infty}(U(2,0))$. Then Lemma 2.8 implies

$$X \Phi_{\infty}^{-1}(s) = \frac{1}{2} s \Phi_{1,\infty}(s),$$

where $X$ is the operator defined by

$$X f(s) = f(s+1).$$
so we see that the Maass operator $X_+$ shifts the coherent Eisenstein series $E(g, s; \Phi^{-1}_\infty \otimes \lambda(\varphi))$ to the incoherent Eisenstein series $E(g, s; \Phi^{1}_\infty \otimes \lambda(\varphi))$. Theorem 2.2 of [11] then tells us that

$$E(g, 0; \Phi^{1}_\infty \otimes \lambda(\varphi)) = 0.$$  

2.6. The $(0, 2)$-Theorem. The integral we want to compute is

$$\int_{SO(U) \setminus SO(U)(\Lambda_f)} \Phi(z_0, h; F) dh,$$

which is equal to

$$\int_{SO(U) \setminus SO(U)(\Lambda_f)} \int_{1 \setminus \mathcal{S}} ((F(\tau), \theta(\tau, z_0, h))) d\mu(\tau) dh.$$  

As in [12], we would like to be able to switch the order of integration, where the inside integral is regularized. That is, we want (22) to equal

$$\int_{\Gamma \setminus \mathcal{S}} ((F(\tau), \int_{SO(U) \setminus SO(U)(\Lambda_f)} \theta(\tau, z_0, h) dh )) d\mu(\tau).$$  

Note that $F : \mathcal{H} \to S_L$ implies $F(\tau) \in S(U(\mathbb{A}_f))^K$, where

$$K = \{ h \in H(\mathbb{A}_f) \mid h(\lambda + L) = \lambda + L, \forall \lambda \in L / L \}$$

is an open subset of $H(\mathbb{A}_f)$.

Before we justify the interchange of integrals, we need to make some remarks about our specific case. For a reference on Clifford algebras, see [9] or [10]. The Clifford algebra $C(U)$ can be written as $C(U) = C^0(U) \oplus C^1(U)$, where $C^0(U)$ and $C^1(U)$ are the even and odd parts, respectively. $C^0(U)^\times$ acts on $C^1(U)$ by conjugation. Assume $U$ has basis $\{u, v\}$ with $Q(u) = a, Q(v) = b$ and $(u, v) = 0$. Then $C(U)$ is spanned by $\{1, u, v, uv\}$ with $C^0(U) = \text{span}\{1, uv\}$ and $C^1(U) = \text{span}\{u, v\}$. By definition,

$$H = \{ g \in C^0(U)^{\times} \mid gUg^{-1} = U \}.$$  

Since $C^1(U) = U$, $H = C^0(U)^{\times}$. In $C^0(U)$ we have $(uv)^2 = -ab$, so if $k = Q(\sqrt{-ab})$, then $H \simeq k^{\times}$. This means $SO(U) \simeq k^1$ and $k^{\times} \to k^1$ is the map

$$x \mapsto \frac{x}{x^2}$$

by Hilbert's Theorem 90. We have the exact sequence

$$1 \to Z \to H \to SO(U) \to 1,$$

where $H(\mathbb{A}_f) \simeq k^{\times}_{\mathbb{A}_f}, H(\mathbb{Q}) \simeq k^{\times}, Z(\mathbb{A}_f) \simeq \mathbb{Q}^{\times}_{\mathbb{A}_f}$ and $Z(\mathbb{Q}) \simeq \mathbb{Q}^{\times}$.

**Lemma 2.12.** For any negative-definite space $U$ with quadratic form $Q$ of signature $(0, 2)$, we realize $U \simeq k$ for an imaginary quadratic field $k$ and $Q$ is given by a negative multiple of the norm-form.

If $B(h)$ is a function on $H(\mathbb{A}_f)$ which only depends on the image of $h$ in $SO(U)(\mathbb{A}_f)$, then we can view $B$ as a function on $SO(U)(\mathbb{A}_f)$ as well.
Lemma 2.13. Let \( B(h) \) be a function on \( H(\mathbb{A}_f) \) depending only on the image of \( h \) in \( SO(U)(\mathbb{A}_f) \). Assume \( B \) is invariant under \( K \) and \( H(\mathbb{Q}) \). Then
\[
\int_{SO(U)(\mathbb{Q}) \backslash SO(U)(\mathbb{A}_f)} B(h) \, dh = \text{vol}(K) \sum_{h \in H(\mathbb{Q}) \backslash H(\mathbb{A}_f)/K} B(h),
\]
and the sum is finite.

Proof. We have the exact sequence
\[
1 \to k^0_A \to k^\times_A \to \mathbb{R}^+_\times \to 1,
\]
where the map to \( \mathbb{R}^+_\times \) is the absolute value map. By the product formula, \( k^\times \subset k^0_A \) and we know \( k^\times/\mathbb{R}^+_\times \) is compact.

Lemma 2.14. \( k^\times = Q^\times_A \cdot k^0_A \).

Proof. \( Q^\times_A \) injects into \( k^\times_A \) and also maps onto \( \mathbb{R}^+_\times \). So if \( (a) \in k^\times_A \) then \( \exists (b) \in Q^\times_A \) with \( |b| = |a| \). Then \( (b) \in Q^\times_A \subset k^\times_A \) implies \( k^0_A(b) = k^0_A(a) \), so \( (a) \in Q^\times_A k^0_A \). □

Lemma 2.14 implies
\[
k^\times/\mathbb{R}^+_\times \to \mathbb{R}^+_\times Q^\times_A/\mathbb{R}^+_\times,
\]
and so \( k^\times/\mathbb{R}^+_\times \) is also compact. The set we integrate over is
\[
SO(U)(\mathbb{Q}) \backslash SO(U)(\mathbb{A}_f) = H(\mathbb{Q}) \backslash H(\mathbb{A}_f)/Z(\mathbb{A}_f) \simeq k^\times Q^\times_A \backslash k^\times_{\mathbb{A}_f}.
\]
This is compact since \( k^\times Q^\times_A \backslash k^\times_{\mathbb{A}_f} \) maps onto it. Then \( K \) is open and \( K \supset Z(\mathbb{A}_f) \) so \( H(\mathbb{Q}) \backslash H(\mathbb{A}_f)/K \) is finite. The volume term appears since \( B \) is \( K \)-invariant. □

Proposition 2.15.
\[
\int_{SO(U)(\mathbb{Q}) \backslash SO(U)(\mathbb{A}_f)} \int_{\Gamma \backslash \mathfrak{g}} ((F(\tau), \theta(\tau, z_0, h)) \, d\mu(\tau)) \, dh
= \int_{\Gamma \backslash \mathfrak{g}} ((F(\tau), \int_{SO(U)(\mathbb{Q}) \backslash SO(U)(\mathbb{A}_f)} \theta(\tau, z_0, h) \, d\mu(\tau)) \, d\mu(\tau).
\]

Proof. The main point is that since \( F(\tau) \in S(U(\mathbb{A}_f))^K \), we know
\[
\int_{\Gamma \backslash \mathfrak{g}} ((F(\tau), \theta(\tau, z_0, h)) \, d\mu(\tau)
\]
is \( K \)-invariant. So if we let
\[
B(h) = \int_{\Gamma \backslash \mathfrak{g}} ((F(\tau), \theta(\tau, z_0, h)) \, d\mu(\tau),
\]
then Lemma 2.13 says
\[
\int_{SO(U)(\mathbb{Q}) \backslash SO(U)(\mathbb{A}_f)} B(h) \, dh = \text{vol}(K) \sum_{h \in H(\mathbb{Q}) \backslash H(\mathbb{A}_f)/K} B(h)
\]
(23)
\[
= \int_{\Gamma \backslash \mathfrak{g}} \text{vol}(K) \sum_{h \in H(\mathbb{Q}) \backslash H(\mathbb{A}_f)/K} \theta(\tau, z_0, h; F(\tau)) \, d\mu(\tau),
\]
since the sum is finite. Now apply Lemma 2.13 again to \( \theta(\tau, z_0, h; F(\tau)) \) and (23) is
\[
= \int_{\Gamma \backslash \mathfrak{g}} ((F(\tau), \int_{SO(U)(\mathbb{Q}) \backslash SO(U)(\mathbb{A}_f)} \theta(\tau, z_0, h) \, d\mu(\tau)) \, d\mu(\tau).
\]
The quadratic space $U$ is anisotropic, so we can apply Theorem 2.9. This tells us that for any $\phi \in S(U(\mathbb{A}))$,

$$\int_{SO(U)(\mathbb{Q}) \backslash SO(U)(\mathbb{A})} \theta(\tau, z_0, h; \phi) dh = v^{\frac{d}{2}} E(g_\tau, 0; \phi, -1),$$

where $E(g_\tau, s; \phi, -1)$ is a coherent Eisenstein series of weight $-1$. Since $\theta(\tau, z_0, h)$ is $SO(U)(\mathbb{R})$-invariant, it suffices to integrate over $SO(U)(\mathbb{Q}) \backslash SO(U)(\mathbb{A})$. We choose a factorization for the measure $dh = dh_\infty \times dh_f$ such that $\text{vol}(SO(U)(\mathbb{R})) = 1$.

Lemma 2.16.

(i) $\int_{SO(U)(\mathbb{Q}) \backslash SO(U)(\mathbb{A})} \theta(\tau, z_0, h; \phi) dh = v^{\frac{d}{2}} E(g_\tau, 0; \phi, -1)$.

(ii) $\text{vol}(K)^{-1} = \frac{1}{2}(\#(H(\mathbb{Q}) \backslash H(\mathbb{A}))/K)$.

We let $E(\tau, s; -1) := v^{\frac{d}{2}} E(g_\tau, s; -1)$.

Then for (24) we have

$$\int_{SO(U)(\mathbb{Q}) \backslash SO(U)(\mathbb{A})} \Phi(z_0, h; F) dh = \int_{\Gamma \backslash \mathbb{H}} (\langle F(\tau), E(\tau, 0; -1) \rangle) d\mu(\tau).$$

For $F$ as in (14), the right hand side of (25) is

$$\int_{\Gamma \backslash \mathbb{H}} (\langle F(\tau), E(\tau, 0; -1) \rangle) d\mu(\tau) = \int_{\Gamma \backslash \mathbb{H}} \sum_{\mu} F_\mu(\tau) E(\tau, 0; \varphi_\mu, -1)v^{-2}dudv.$$

Let

$$I(s, t) := \int_{\mathbb{R}^+} \sum_{\mu} F_\mu(\tau) E(\tau, s; \varphi_\mu, -1)v^{-2}dudv.$$

In order to state the main theorem of this chapter, we view $U \simeq k = \mathbb{Q}(\sqrt{-d})$, where $-d$ is the discriminant of $k$, and let $\chi_d$ be the character of $\mathbb{Q}_\mathbb{A}^\times$ defined by $\chi_d(x) = (x, -d)_A$. We define the normalized $L$-series

$$\Lambda(s, \chi_d) = \pi^{-\frac{d+1}{2}} \Gamma\left(\frac{s + 1}{2}\right) L(s, \chi_d).$$

Definition 2.17. For $\varphi \in S(U(\mathbb{A}))$, let

$$E(\tau, s; \varphi, +1) = \sum_m A_\varphi(s, m, v)q^m,$$

where the Fourier coefficients have Laurent expansions

$$A_\varphi(s, m, v) = b_\varphi(m, v)s + O(s^2)$$

at $s = 0$. For any $\varphi \in S(U(\mathbb{A}))$, define

$$\kappa_\varphi(m) := \begin{cases} \lim_{v \to \infty} b_\varphi(m, v) & \text{if } m > 0, \\ k_0(0)\varphi(0) & \text{if } m = 0, \\ 0 & \text{if } m < 0, \end{cases}$$
where
\begin{equation}
(27) \quad k_0(0) = \log(d) + 2 \frac{N(1, \chi_d)}{\Lambda(1, \chi_d)}.
\end{equation}

For $\varphi = \varphi_\mu = \text{char}(\mu + L)$ we write
\[ A_\mu(s, m, v) = A_{\varphi_\mu}(s, m, v), \quad b_\mu(m, v) = b_{\varphi_\mu}(m, v), \quad \kappa_\mu(m) = \kappa_{\varphi_\mu}(m). \]

**Theorem 2.18** (The $(0, 2)$-Theorem). Let $F : \mathfrak{h} \to SL_2(\mathbb{Z}) \subset S(U(\mathfrak{h}))$ be a weakly holomorphic modular form for $SL_2(\mathbb{Z})$ of weight 1, with Fourier expansion
\[ F(\tau) = \sum_\mu F_\mu(\tau) \varphi_\mu = \sum_\mu \sum_m c_\mu(m) q^m \varphi_\mu, \]
where $\mu$ runs over $L^\vee / L$ for some lattice $L$. Also, assume $c_\mu(m) \in \mathbb{Z}$ for $m \leq 0$.

Let
\[ \Phi(z_0, h; F) = \int_{\Gamma \backslash \mathfrak{h}} \left( (F(\tau), \theta(\tau, z_0, h)) \right) d\mu(\tau). \]

Then
\[ \int_{SO(U(\mathfrak{h})) \backslash SO(U(\mathfrak{h}))} \Phi(z_0, h; F) dh = 2 \sum_\mu \sum_{m \geq 0} c_\mu(-m) \kappa_\mu(m). \]

**Proof.** Our proof is similar to that in [12]. The integral we want to compute is given by (26). Letting $l = -1$ in (19), we have
\[ E(\tau, s; \varphi_\mu, -1) = -\frac{4i}{s} \frac{\partial}{\partial \tau} \left\{ E(\tau, s; \varphi_\mu, +1) \right\}. \]

This means we can write
\[ I(s, t) = \frac{1}{2i} \int_{\mathcal{F}_t} d\left( \sum_\mu F_\mu(\tau) -\frac{4i}{s} E(\tau, s; \varphi_\mu, +1) d\tau \right). \]

By Stokes’ Theorem, this is
\[ = -\frac{2}{s} \int_{\partial \mathcal{F}_t} \sum_\mu F_\mu(\tau) E(\tau, s; \varphi_\mu, +1) d\tau \]
\[ = \frac{2}{s} \int_{\frac{1}{2} + it} \sum_\mu F_\mu(\tau) \left. E(\tau, s; \varphi_\mu, +1) \right|_{v = t} du \]
\[ = \frac{2}{s} \cdot \text{const. term of } \left( \sum_\mu F_\mu(\tau) E(\tau, s; \varphi_\mu, +1) \right) \left|_{v = t} \right. . \]

The definition of the regularized integral implies
\[ \int_{\Gamma \backslash \mathfrak{h}} \left( (F(\tau), E(\tau, 0)) \right) d\mu(\tau) = \]
\[ \text{CT} \left\{ \lim_{\sigma \to 0} \int_{\mathcal{F}_t} \sum_\mu F_\mu(\tau) E(\tau, 0; \varphi_\mu, -1) \nu^{-\sigma - 2} d\nu d\tau \right\}. \]

We need Proposition 2.5 of [12] to hold for $n = 0$. If we use Proposition 2.6 of [12] and the fact that a factor of 2 appears in the Siegel-Weil formula here, then in our notation the analogue of Proposition 2.5 of [12] is
Proposition 2.19.

\[
\begin{align*}
\text{CT}_{\sigma=0} & \left\{ \lim_{t \to \infty} \int_{\mathcal{F}_t} \sum_{\mu} F_{\mu}(\tau) E(\tau, 0; \varphi_{\mu}, -1) v^{-\sigma - 2} dudv \right\} \\
& = \lim_{t \to \infty} \left[ \int_{\mathcal{F}_t} \sum_{\mu} F_{\mu}(\tau) E(\tau, 0; \varphi_{\mu}, -1) v^{-2} dudv - 2c_0(0) \log(t) \right].
\end{align*}
\]

Proof. From Lemma 2.13, the left hand side of the desired identity is

\[
\text{vol}(K) \sum_{h} \text{CT}_{\sigma=0} \left\{ \lim_{t \to \infty} \int_{\mathcal{F}_t} ((F(\tau), \theta(\tau, z_0, h))) v^{-\sigma - 2} dudv \right\},
\]

where \(\text{vol}(K) = \frac{2}{#(H(Q) \setminus H(A_f)/K)}\). Fixing \(h\), we have

\[C \left( v, h \right) = v^{-1} \int \left( (F(\tau), \theta(\tau, z_0, h)) \right) du\]

\[= \text{const. term of } v^{-1} ((F(\tau), \theta(\tau, z_0, h)))\]

\[= \sum_{\mu} \sum_{m \in \mathbb{Q}} c_\mu(m) \sum_{x \in \mathbb{Q}} \varphi_\mu(h^{-1}x) e^{4\pi v Q(x)}.
\]

Then we write (31) as

\[\text{CT}_{\sigma=0} \left\{ \lim_{t \to \infty} \int_{1}^{t} C(v, h) v^{-\sigma - 1} dv \right\}.
\]

As in [12],

\[\int_{1}^{\infty} [C(v, h) - c_0(0)] v^{-\sigma - 1} dv
\]

is a holomorphic function of \(\sigma\). Note, this fact follows, in part, from Lemma 2.5.

For the other piece of (31) we have

\[\int_{1}^{t} c_0(0) v^{-\sigma - 1} dv = c_0(0) \frac{1}{\sigma}(1 - t^{-\sigma}).\]
This term makes no contribution when we take the limit as $t \to \infty$ followed by the constant term at $\sigma = 0$. We are left with

$$
\lim_{t \to \infty} \left[ \int_1^t C(v, h)v^{-1}dv - \int_1^t c_0(0)v^{-1}dv \right] = \lim_{t \to \infty} \left[ \int_1^t C(v, h)v^{-1}dv - c_0(0) \log(t) \right].
$$

We have the volume term in front and we sum over $h \in H(\mathbb{Q}) \backslash H(\mathbb{A}_f)/K$, so this adds on a factor of 2. □

We point out that the value $c_0(0)$ appearing in (14) and in Proposition 2.19 is independent of the choice of $L$. If we view $F(\tau) \in S(U(\mathbb{A}_f))$ as $F(\tau, x)$ for $x \in U(\mathbb{A}_f)$, then $c_0(0)$ is the zeroth Fourier coefficient of $F(\tau, 0)$. Proposition 2.19 tells us that

$$
CT_{\sigma=0} \left\{ \lim_{t \to \infty} \int_{\mathcal{F}_t} \sum_{\mu} F_\mu(\tau)E(\tau, 0; \varphi_\mu, -1)v^{-\sigma-2}dudv \right\}
$$

$$
= \lim_{t \to \infty} \left[ \int_{\mathcal{F}_t} \sum_{\mu} F_\mu(\tau)E(\tau, 0; \varphi_\mu, -1)v^{-2}dudv - 2c_0(0) \log(t) \right]
$$

$$
= \lim_{t \to \infty} \left[ I(0, t) - 2c_0(0) \log(t) \right].
$$

We need to compute $I(0, t)$. We have

(32) $A_\mu(s, m, v) = b_\mu(m, v)s + O(s^2),$

where there is no constant term in $A_\mu(s, m, v)$ since $E(\tau, s; \varphi_\mu, +1)$ vanishes at $s = 0$. Then (28) implies

$$
I(s, t) = \frac{2}{s} \sum_{\mu} \sum_{m} c_\mu(-m)A_\mu(s, m, t),
$$

so using (32) we have

(33) $I(0, t) = 2 \sum_{\mu} \sum_{m} c_\mu(-m)b_\mu(m, t).$

Now we show that parts (i) and (ii) of Proposition 2.11 of [12] hold for $n = 0$.

**Proposition 2.20.** (i) For $m < 0$, $b_\mu(m, t)$ decays exponentially as $t \to \infty$.

(ii) $\lim_{t \to \infty} \left( 2 \sum_{\mu} \sum_{m < 0} c_\mu(-m)b_\mu(m, t) \right) = 0.$

**Proof.** If $\varphi_\mu = \otimes_p \varphi_\mu, p \in S(U(\mathbb{A}_f))$ and

$$
E(\tau, s; \varphi_\mu, +1) = \sum_m E_m(\tau, s; \varphi_\mu, +1),
$$

then for $m \neq 0$ we have the product formula

$$
E_m(\tau, s; \varphi_\mu, +1) = A_\mu(s, m, v)q^m = W_{m, \infty}(\tau, s; +1) \prod_p W_{m, p}(s, \varphi_\mu, p),
$$

where $W_{m, \infty}(\tau, s; +1)$ and $W_{m, p}(s, \varphi_\mu, p)$ are the local Whittaker factors at $\infty$ and $p$, respectively. Proposition 2.6 (iii) of [12] tells us that for $m < 0,$

$$
W_{m, \infty}(\tau, 0; +1) = 0,
$$
and

\[ W'_{m,\infty}(\tau, s; +1) = \pi i q^m \int_1^\infty r^{-1} e^{-4\pi|m|v r} dr. \]

For the finite primes we have

\[ C(m) := \left( \prod_p W_{m,p}(s, \varphi_{\mu,p}) \right) \bigg|_{s=0} = O(1). \]

Then

\[ b_\mu(m, t) = C(m) W'_{m,\infty}(\tau, s; +1) \]

\[ = C(m) \pi i q^m \int_1^\infty r^{-1} e^{-4\pi|m|v r} dr, \]

and we have

\[ |b_\mu(m, t)| = O(v^{-1}|m|^{-1} e^{-4\pi|m|v}). \]

This proves (i). Part (ii) then follows from Lemma 2.5. \( \square \)

Part (ii) of Proposition 2.20 tells us that we may ignore the sum on \( m < 0 \) in (33). This means our formula for the integral is

\[ \int_{SO(U)}(U) \Phi(z_0, h; F) dh = \]

\[ \lim_{t \to \infty} \left[ 2 \sum_\mu \sum_{m \geq 0} c_\mu(-m) b_\mu(m, t) - 2c_0(0) \log(t) \right]. \]

We can improve this by looking at the \( m = 0 \) part. The analogue of Proposition 2.11 (iii) of [12] is

**Lemma 2.21.** For \( m = 0 \),

\[ b_0(0, t) - \log(t) = \log(d) + 2 \frac{\Lambda'(1, \chi_d)}{\Lambda(1, \chi_d)}, \]

and for \( \mu \neq 0, b_\mu(0, t) = 0. \)

**Proof.** By Theorem 3.1 of [17], we have

\[ E_0(\tau, s; \varphi_\mu, +1) = v^s \varphi_\mu(0) + W_{0,\infty}(\tau, s; +1) \prod_p W_{0,p}(s, \varphi_{\mu,p}) \]

\[ = v^s \varphi_\mu(0) - 2\pi i \frac{2^{-s} \Gamma(s) v^{-\frac{s+i}{2}}}{\Gamma(\frac{s}{2} + 1) \Gamma(\frac{s}{2})} \prod_p W_{0,p}(s, \varphi_{\mu,p}), \]

which by the duplication formula is

\[ = v^s \varphi_\mu(0) - \sqrt{\pi i v^{-\frac{s+i}{2}}} \frac{\Gamma(\frac{s+1}{2})}{\Gamma(\frac{s+1}{2} + 1)} \prod_p W_{0,p}(s, \varphi_{\mu,p}). \]
Theorem 5.2 of [17] implies $W_{0,p}(s, \varphi_{\mu,p}) = 0$ if $\varphi_{\mu,p}$ is not the characteristic function of the local lattice. So $b_{\mu}(0, t) = 0$ for $\mu \neq 0$. Now let $\mu = 0$. Propositions 2.1 and 6.3 of [17] imply

$$E_0(\tau, s, \varphi_0, +1) = v^{\frac{s}{2}} - \sqrt{\pi}v^{-\frac{s}{2}} \frac{\Gamma \left( \frac{s+1}{2} \right) L(s, \chi_d)}{\Gamma \left( \frac{s}{2} + 1 \right) L(s + 1, \chi_d)} C_0,$$

where

$$C_0 = 2^{\beta_2} \prod_{q \mid d} q^{-\frac{s}{2}}$$

and

$$\beta_2 = \begin{cases} 0 & \text{if } 2 \text{ is unramified}, \\ -1 & \text{if } 4 \mid d \text{ and } 8 \nmid d, \\ -\frac{1}{2} & \text{if } 8 \mid d. \end{cases}$$

Then $C_0 = d^{-\frac{3}{4}}$. We have

$$E_0(\tau, s, \varphi_0, +1) = v^{\frac{s}{2}} - v^{-\frac{s}{2}} \frac{\pi^{-\frac{s+1}{2}} \Gamma \left( \frac{s+1}{2} \right) L(s, \chi_d)}{\pi^{-\frac{s}{2}-1} \Gamma \left( \frac{s}{2} + 1 \right) L(s + 1, \chi_d)} d^{-\frac{3}{4}}$$

$$= v^{\frac{s}{2}} - v^{-\frac{s}{2}} \frac{\Lambda(s, \chi_d)}{\Lambda(s + 1, \chi_d)} d^{-\frac{3}{4}}.$$

The functional equation for $\Lambda(s, \chi_d)$ (cf. [17]) is

$$\Lambda(s, \chi_d) = d^{\frac{1}{2} - s} \Lambda(1 - s, \chi_d).$$

We normalize $E_0(\tau, s, \varphi_0, +1)$ by $d^{\frac{s+1}{2}} \Lambda(s + 1, \chi_d)$ giving

$$E_0^*(\tau, s, \varphi_0, +1) = d^{\frac{s+1}{2}} v^{\frac{s}{2}} \Lambda(1 + s, \chi_d) - d^{\frac{s+1}{2}} v^{-\frac{s}{2}} d^{-s} \Lambda(1 - s, \chi_d)$$

$$= d^{\frac{s+1}{2}} v^{\frac{s}{2}} \Lambda(1 + s, \chi_d) - d^{1-s} v^{-\frac{s}{2}} \Lambda(1 - s, \chi_d).$$

Hence,

$$E_0^{*'}(\tau, 0; \varphi_0, +1) = 2 \frac{\partial}{\partial s} \left\{ d^{\frac{s+1}{2}} v^{\frac{s}{2}} \Lambda(1 + s, \chi_d) \right\} \bigg|_{s=0}$$

$$= d^{\frac{s}{2}} \Lambda(1, \chi_d) \left\{ \log(d) + \log(v) + 2 \frac{\Lambda'(1, \chi_d)}{\Lambda(1, \chi_d)} \right\}$$

$$= h_k \left\{ \log(d) + \log(v) + 2 \frac{\Lambda'(1, \chi_d)}{\Lambda(1, \chi_d)} \right\},$$

by the residue formula. Then since $E_0^{*'}(\tau, 0; \varphi_0, +1) = h_k E'(\tau, 0; \varphi_0, +1)$, we have

$$b_0(0, t) - \log(t) = \log(d) + 2 \frac{\Lambda'(1, \chi_d)}{\Lambda(1, \chi_d)}.$$

Now the $m = 0$ part is

$$2 \sum_{\mu} c_\mu(0)b_\mu(0, t) - 2c_0(0) \log(t) = 2 \sum_{\mu \neq 0} c_\mu(0)b_\mu(0, t) + 2c_0(0)(b_0(0, t) - \log(t)),$$

and Lemma 2.21 tells us that this expression is $2c_0(0)k_0(0)$. This finishes the proof of Theorem 2.15.
3. Main theorem in the case of signature \((n, 2)\)

3.1. The Rational Splitting \(V = V_+ \oplus U\). Now we consider the general case. Assume that we have a decomposition \(V = V_+ \oplus U\) where \(V_+\) has signature \((n, 0)\) and \(U\) has signature \((0, 2)\). For \(x \in V\), write \(x = x_1 + x_2\), \(x_1 \in V_+, x_2 \in U\). Let \(z_0 \in D_0\). Then \(R(x, z_0) = -(x_2, x_2)\) so we see

\[
\varphi_\infty(x, z_0) = e^{-\pi(x, x)z_0} = e^{-\pi[(x_1, x_1)-(x_2, x_2)]} = e^{-\pi(x_1, x_1)}e^{\pi(x_2, x_2)},
\]

which is equal to \(\varphi_\infty,+(x_1)\varphi_\infty,-(x_2)\) for the Gaussians on \(V_+\) and \(U\), respectively. We also have \(\omega(g'_+^r)\varphi_\infty = \omega_+(g'_+^r)\varphi_\infty, + \otimes \omega_-(g'_+^r)\varphi_\infty, -\) for the corresponding Weil representations. For this decomposition of \(V\), we can write the theta function on \(S(V(\mathbb{A}_f))\) as a tensor product of two distributions, one on \(S(V_+(\mathbb{A}_f))\) and one on \(S(U(\mathbb{A}_f))\). To see this, let \(\varphi \in S(V(\mathbb{A}_f))\). The theta functions are linear, so it suffices to look at a factorizable Schwartz function \(\varphi = \varphi_+ \otimes \varphi_-\). This gives

\[
\theta(\tau, z_0, h; \varphi) = v^{-\frac{n}{2}} \sum_{x \in V(Q)} \omega(g'_+^r)(\varphi_\infty,+(x, z_0) \otimes \omega(h)\varphi)(x) \\
= v^{-\frac{n}{2}} \sum_{x_1, x_2} (\omega_+(g'_+^r)\varphi_\infty,+(x_1)\varphi_+(h_+^{-1}x_1))(\omega_-(g'_+^r)\varphi_\infty,-(x_2)\varphi_-(h_+^{-1}x_2)) \\
= v^{-\frac{n}{2}} \left( \sum_{x_1} \omega_+(g'_+^r)\varphi_\infty,+(x_1)\varphi_+(h_+^{-1}x_1) \right) \\
\times \left( \sum_{x_2} \omega_-(g'_+^r)\varphi_\infty,-(x_2)\varphi_-(h_+^{-1}x_2) \right) \\
= \theta_+(\tau, z_0, h_+; \varphi_+) \theta_-(\tau, z_0, h_-; \varphi_-).
\]

Hence,

\[
\theta(\tau, z_0, h) = \theta_+(\tau, z_0, h_+) \otimes \theta_-(\tau, z_0, h_-),
\]

where their respective weights are \(\frac{n}{2}\) and \(-1\). Since \(z_0\) is fixed, we write

\[
\theta_\pm(\tau, h_\pm) = \theta_\pm(\tau, z_0, h_\pm).
\]

3.2. The Contraction Map. Now we describe the main way in which we use the above factorization of the theta function. Let \(\varphi \in S(V(\mathbb{A}_f))\). Then we can write \(\varphi = \sum_j \varphi_+^j \otimes \varphi_-^j\), where \(\varphi_+^j \in S(V_+(\mathbb{A}_f)), \varphi_-^j \in S(U(\mathbb{A}_f))\) and the sum is finite. We define the contraction map

\[
(\cdot, \theta_+(\tau, h_+)) : S(V(\mathbb{A}_f)) \to S(U(\mathbb{A}_f))
\]

by

\[
\langle \varphi, \theta_+(\tau, h_+) \rangle := \sum_j \theta_+(\tau, h_+; \varphi_+^j) \varphi_-^j.
\]

It is clear that

\[
(\langle \varphi, \theta(\tau, z_0, h) \rangle) = (\langle \varphi, \theta_+(\tau, h_+) \rangle, \theta_-(\tau, h_-)).
\]

The expression on the right hand side is nice because it is the pairing of a function in \(S(U(\mathbb{A}_f))\) and the theta function for \(U\). This is just as in the \(n = 0\) case. The value of the contraction map that we are interested in is \(\langle F(\tau), \theta_+(\tau, 1) \rangle\).
Proposition 3.1. If $F : \mathfrak{H} \to S_L$ is a weakly holomorphic modular form of weight $1 - \frac{4}{\nu}$ and type $\omega$ for $\Gamma'$ whose non-positive Fourier coefficients lie in $\mathbb{Z}$, then

(i) $\langle F(\tau), \theta_+(\tau, 1) \rangle$ is a weakly holomorphic modular form of weight 1 and type $\omega_-$ for $\Gamma'$ (cf. Definition 2.1).

(ii) $\langle F(\tau), \theta_+(\tau, 1) \rangle \in S_{L_-}$ for $L_- = U \cap L$.

(iii) The non-positive Fourier coefficients of $\langle F(\tau), \theta_+(\tau, 1) \rangle$ lie in $\mathbb{Z}$.

Proof. By definition,

$$\langle F(\gamma', \tau), \theta_+(\gamma', \tau, h_+) \rangle = (c\tau + d) \left\{ \omega(\gamma')(\langle F(\tau), \omega_+(\gamma') \theta_+(\tau, h_+) \rangle) \right\}_U.$$  

Assume that $F(\tau) = \sum_j \varphi_+^j \otimes \varphi_-^j$. We have

$$\omega_+^j(\gamma')(\theta_+(\tau, h_+)) = \theta_+(\tau, h_+; \omega_+(\gamma')^{-1} \circ \cdot),$$

so (35) is

$$= (c\tau + d) \sum_j \omega_+^j(\gamma')(\varphi_+^j) \otimes \omega_-(\gamma')(\varphi_-^j), \theta_+(\tau, h_+; \omega_+(\gamma')^{-1} \circ \cdot)$$

$$= (c\tau + d) \sum_j \theta_+(\tau, h_+; \omega_+(\gamma')^{-1} \omega_+(\gamma')(\varphi_+^j)) \omega_-(\gamma')(\varphi_-^j)$$

$$= (c\tau + d) \sum_j \theta_+(\tau, h_+; \varphi_+^j) \omega_-(\gamma')(\varphi_-^j)$$

$$= (c\tau + d) \omega_-(\gamma')(\langle F(\tau), \theta_+(\tau, h_+) \rangle).$$

This proves (i).

In order to compute the Fourier expansion of $\langle F(\tau), \theta_+(\tau, h_+) \rangle$, we need the expansion of $\theta_+(\tau, h_+; \varphi_+)$ for $\varphi_+ \in S(V_+(k_f))$. We take $h_+ = 1$ since the integral we are interested in is

$$\int_{SO(U)(\mathbb{Q}) \setminus SO(U)(\mathbb{A})} \Phi(z_0, h; F) dh.$$  

The explicit $q$-expansion of $\theta_+(\tau, 1; \varphi_+)$ is obtained via the action of the Weil representation on $S(V_+(\mathbb{R}))$. In our particular case,

$$\theta_+(\tau, 1; \varphi_+) = v^{-\frac{\alpha}{4}} \sum_{x_1 \in V_+(\mathbb{Q})} \omega_+(g_\nu^\ell) \varphi_+(x_1) \varphi_+(x_1)$$

$$= v^{-\frac{\alpha}{4}} \sum_{x_1} \omega_+(g_\nu^\ell) e^{-\pi x_1, x_1} \varphi_+(x_1),$$

which by (16) is

$$= v^{-\frac{\alpha}{4}} \sum_{x_1} v^{\frac{\alpha}{2}} e^{2\pi i u_1 x_1} e^{-\pi v(x_1, x_1)} \varphi_+(x_1)$$

$$= \sum_{x_1} e^{2\pi i r_1 Q(x_1)} \varphi_+(x_1)$$

$$= \sum_{m \in \mathbb{Q}} \left( \sum_{x_1} \varphi_+(x_1) \right) q^m.$$  

(36)

Define

$$d_{\varphi_+}(m) := \sum_{x_1} \varphi_+(x_1).$$
Let $L_+ \subset V_+$ be a lattice. Note that if $\varphi_+$ is the characteristic function of a coset $\lambda_++L_+$, then $d_{\varphi_+}(m)$ is an integer which counts the number of vectors $x_1 \in \lambda_++L_+$ such that $Q(x_1) = m$. Also, $V_+ (\mathbb{Q})$ is positive definite so $m \geq 0$. \[\square\]

Now we compute the Fourier expansion of $(F(\tau), \theta_+(\tau, 1))$. We know $F(\tau) \in S_L$ for some lattice $L \subset V$. If we let $L_+ = V_+ \cap L$ and $L_- = U \cap L$, then generally the lattice $L$ does not split, i.e., $L \not
subseteq L_+ + L_-$. We have

$$L_+ + L_- \subset L \subset L^\vee \subset L_+^\vee + L_-^\vee.$$ 

Let

$$L^\vee = \bigcup_{\eta}(\eta + L), \quad L = \bigcup_{\lambda}(\lambda + L_+ + L_-),$$

where $\eta$ and $\lambda$ range over $L^\vee / L$ and $L/(L_+ + L_-)$, respectively. If we write $\eta = \eta_+ + \eta_-$ and $\lambda = \lambda_+ + \lambda_-$, then

$$L^\vee = \bigcup_{\eta, \lambda} (\eta_+ + \lambda_+ + L_+) + (\eta_- + \lambda_- + L_-).$$

Let $F(\tau) = \sum_\eta F_\eta(\tau) \varphi_{\eta+L}$ for $\varphi_{\eta+L} = \text{char}(\eta + L)$. Then

$$\varphi_{\eta+L} = \sum_\lambda \varphi_{\eta_+ + \lambda_+ + L_+} \otimes \varphi_{\eta_- + \lambda_- + L_-},$$

and we have

$$F(\tau) = \sum_\eta F_\eta(\tau) \sum_\lambda (\varphi_{\eta_+ + \lambda_+ + L_+} \otimes \varphi_{\eta_- + \lambda_- + L_-}).$$

By definition of the contraction map, this gives

$$\langle F(\tau), \theta_+(\tau, 1) \rangle = \sum_\eta \sum_\lambda F_\eta(\tau) \theta_+(\tau, 1; \varphi_{\eta_+ + \lambda_+ + L_+}) \varphi_{\eta_- + \lambda_- + L_-}.$$ 

From \[\square\], we see that

$$\langle F(\tau), \theta_+(\tau, 1) \rangle \in S_{L_-},$$

but we point out that the cosets $\eta_--\lambda_- + L_-$ need not be incongruent mod $L_-$. Let $c_\eta(m) = c_{\varphi_{\eta_+ + \lambda_+}}(m)$ and $d_{\eta_+ + \lambda_+}(m) = d_{\varphi_{\eta_+ + \lambda_+}}(m)$. Then the Fourier expansion of $\langle F(\tau), \theta_+(\tau, 1) \rangle$ is

$$\langle F(\tau), \theta_+(\tau, 1) \rangle = \sum_\eta \sum_\lambda \left( \sum_m c_\eta(m)q^m \right) \left( \sum_m d_{\eta_+ + \lambda_+}(m)q^m \right) \varphi_{\eta_- + \lambda_- + L_-}$$

$$= \sum_\eta \sum_\lambda \sum_m \left( \sum_{m_1 + m_2 = m} c_\eta(m_1)d_{\eta_+ + \lambda_+}(m_2)q^m \varphi_{\eta_- + \lambda_- + L_-} \right)$$

$$= \sum_\eta \sum_\lambda \sum_m C_{\eta, \lambda_+}(m)q^m \varphi_{\eta_- + \lambda_- + L_-},$$

where we define

$$C_{\eta, \lambda_+}(m) := \sum_{m_1 + m_2 = m} c_\eta(m_1)d_{\eta_+ + \lambda_+}(m_2).$$

The coefficients $d_{\eta_+ + \lambda_+}(m) \in \mathbb{Z}_{\geq 0}$ for $m \geq 0$ and $d_{\eta_+ + \lambda_+}(m) = 0$ if $m < 0$. So assuming $c_\eta(m) \in \mathbb{Z}$ for $m \leq 0$ implies $C_{\eta, \lambda_+}(m) \in \mathbb{Z}$ for $m \leq 0$, and this finishes the proof of Proposition \[\square\]}.
We have seen that the zeroth Fourier coefficient of the modular form $F$ is very important. For example, it gives the weight of $\Psi(F)^2$. When doing the general case, we use the contraction map to go from a modular form $F \in S(V(\Lambda_f))$ to $\langle F, \theta_+ \rangle \in S(U(\Lambda_f))$. Hence, we will want to know the zeroth coefficient of $\langle F, \theta_+ \rangle$.

For any modular form $\tilde{F} \in S(U(\Lambda_f))$, define
\[ c_0(0)(\tilde{F}) \]
to be the zeroth Fourier coefficient of $\tilde{F}$.

**Corollary 3.2.** The Fourier expansion of $(F(\tau), \theta_+ (\tau, 1))$ is
\[ \langle F(\tau), \theta_+ (\tau, 1) \rangle = \sum_{\eta} \sum_{\lambda} \sum_{m} C_{\eta, \lambda +} (m) q^m \varphi_{\eta + \lambda +}, \]
where
\[ C_{\eta, \lambda +} (m) = \sum_{m_1 + m_2 = m} c_\eta (m_1) d_{\eta + \lambda +} (m_2), \]
and
\[ c_0(0)(\langle F, \theta_+ \rangle) := c_0(0)((\langle F(\tau), \theta_+ (\tau, 1) \rangle) = \sum_{\eta} \sum_{\lambda +} C_{\eta, \lambda +} (0). \]

**3.3. The $(n, 2)$-Theorem.** Recall that the lattice $L$ may not split. For $\eta \in L^\vee / L$ and $\lambda \in L / (L_+ + L_-)$ we write $\eta = \eta_+ + \eta_-$ and $\lambda = \lambda_+ + \lambda_-$. Define
\[ \kappa_\eta (m) := \sum_{x} \sum_{x \in \eta_+ + \lambda_+ + L_+} \kappa_{\eta_+ + \lambda_-} (m - Q(x)). \]
Note that Definition 2.17 implies the sum over $x \in \eta_+ + \lambda_+ + L_+$ is finite.

**Theorem 3.4 (The $(n, 2)$-Theorem).** Let $F : \mathcal{H} \to S_L \subset S(V(\Lambda_f))$ be a weakly holomorphic modular form for $\Gamma'$ of weight $1 - \frac{2}{n}$, with Fourier expansion
\[ F(\tau) = \sum_{\eta} F_\eta (\tau) \varphi_\eta = \sum_{\eta} \sum_{m} c_\eta (m) q^m \varphi_\eta, \]
where $\varphi_\eta = \text{char}(\eta + L)$ and $\eta$ runs over $L^\vee / L$. Also, assume $c_\eta (m) \in \mathbb{Z}$ for $m \leq 0$.

Define
\[ \Phi(z, h; F) := \int_{\Gamma \setminus \mathcal{H}} ((F(\tau), \theta(\tau, z, h))) d\mu(\tau). \]
For $z_0 \in D_0$ we have
\begin{enumerate}
\item $\Phi(z_0, h; F)$ is always finite,
\item $\int_{SO(U)} (SO(V) \setminus SO(U)(\Lambda_f)) \Phi(z_0, h; F) dh = 2 \sum_{\eta} \sum_{m \geq 0} c_\eta (-m) \kappa_\eta (m).$
\end{enumerate}

**Proof.** The regularized integral is given by
\[ \Phi(z_0, h; F) = \int_{\Gamma \setminus \mathcal{H}} \phi(\tau) d\mu(\tau), \]
where the integrand is
\[ \phi(\tau) = ((F(\tau), \theta(\tau, z_0, h))) = ((\langle F(\tau), \theta_+ (\tau, 1) \rangle, \theta_- (\tau, h_-))). \]
as in \text{(34)}. Hence,
\begin{equation}
\Phi(z_0, h; F) = \Phi(z_0, h; (F(\tau), \theta_+(\tau, 1))),
\end{equation}
and Proposition 2.6 implies \text{(40)} is always finite. We remark that the regularization process does not depend on the integrand \( \phi(\tau) \).

For (ii), using \text{(40)} the desired integral can be written
\begin{equation}
\int_{SO(U) \backslash SO(U \backslash \mathbb{A}_f)} \Phi(z_0, h; F)dh
= \int_{SO(U) \backslash SO(U \backslash \mathbb{A}_f)} \Phi(z_0, h; (F(\tau), \theta_+(\tau, 1)))dh_-. \tag{41}
\end{equation}

Proposition \textbf{3.31} tells us we may apply the \((0, 2)\)-Theorem to \textbf{(41)}. Doing this we see
\begin{align}
\textbf{(41)} &= 2 \sum_{\eta} \sum_{\lambda} \sum_{m \geq 0} C_{\eta, \lambda_+}(-m) \kappa_{\eta_-, \lambda_-}(m) \\
&= 2 \sum_{\eta} \sum_{\lambda} \sum_{m \geq 0} \left( \sum_{m_1 + m_2 = -m} c_{\eta}(m_1) d_{\eta_+, \lambda_+}(m_2) \right) \kappa_{\eta_-, \lambda_-}(m) \\
&= 2 \sum_{\eta} \sum_{\lambda} \sum_{m \geq 0} \left( \sum_{m_1 \leq 0} c_{\eta}(m_1) d_{\eta_+, \lambda_+}(-m - m_1) \right) \kappa_{\eta_-, \lambda_-}(m) \\
&= 2 \sum_{\eta} \sum_{\lambda} \sum_{m \geq 0} \left( \sum_{m_1 \leq 0} c_{\eta}(-m_1) d_{\eta_+, \lambda_+}(m_1 - m) \right) \kappa_{\eta_-, \lambda_-}(m). \tag{42}
\end{align}

If \( m > m_1 \), then \( d_{\eta_+, \lambda_+}(m_1 - m) = 0 \), so
\begin{equation}
\textbf{(42)} = 2 \sum_{\eta} \sum_{\lambda} \sum_{m_1 \geq 0} c_{\eta}(-m_1) \left( \sum_{0 \leq m \leq m_1} d_{\eta_+, \lambda_+}(m_1 - m) \right) \kappa_{\eta_-, \lambda_-}(m) \tag{43}
\end{equation}

Then
\begin{align}
\sum_{0 \leq m \leq m_1} d_{\eta_+, \lambda_+}(m_1 - m) &\kappa_{\eta_-, \lambda_-}(m) \\
&= \sum_{0 \leq m \leq m_1} \left( \# \{ x \in \eta_+ + \lambda_+ + L_+ \mid Q(x) = m_1 - m \} \right) \kappa_{\eta_-, \lambda_-}(m) \\
&= \sum_{x \in \eta_+ + \lambda_+ + L_+} \kappa_{\eta_-, \lambda_-}(m_1 - Q(x)) \\
&= \sum_{x \in \eta_+ + \lambda_+ + L_+} \kappa_{\eta_-, \lambda_-}(m_1 - Q(x)),
\end{align}

since \( Q(x) \geq 0 \) and \( \kappa_{\eta_-, \lambda_-}(m) = 0 \) for \( m < 0 \). So
\begin{equation}
\textbf{(43)} = 2 \sum_{\eta} \sum_{m \geq 0} c_{\eta}(-m) \kappa_{\eta}(m). \tag{44}
\end{equation}

We now state an important corollary of Theorem \textbf{3.31} which gives the average value of the logarithm of a Borcherds form over CM points. As in section \textbf{2.36} let \( T = \text{GSpin}(U) \) and let \( K \subset H(\mathbb{A}_f) \) be a compact open subgroup such that
Write $K_T = K \cap T(\mathfrak{A}_f)$ and recall that we consider the set of CM points

$$Z(U)_K = T(\mathbb{Q}) \setminus \left( D_0 \times T(\mathfrak{A}_f)/K_T \right) \rightarrow X_K.$$  

**Corollary 3.5.** (i) When $z_0$ is not in the divisor of the Borcherds form $\Psi(F)$ (i.e., when $\mathcal{Z}_z$ holds), the result of Theorem 3.4 can be stated as

$$\sum_{z \in Z(U)_K} \log ||\Psi(z; F)||^2 = -\frac{2}{\text{vol}(K_T)} \left( \sum_{\eta} \sum_{m \geq 0} c_\eta(-m)\kappa_\eta(m) \right).$$

(ii) If $U \simeq \mathbb{Q}(\sqrt{-d})$ where $-d$ is an odd fundamental discriminant, then we have the factorization

$$\prod_{z \in Z(U)_K} ||\Psi(z; F)||^2 = \text{rat} \cdot \left( 4d\pi \right)^{-h_k/2} \left( \frac{e^{2\sum_{(0)(x)}^L}}{\pi(0)(m/2)} \right)^{k_0(0)((F, \theta_+))},$$

where $\text{rat} \in \mathbb{Q}$ and $k_0(0)((F, \theta_+))$ is the zeroth Fourier coefficient of $(F, \theta_+)$, as defined in [18]. Note that the degree of $Z(U)_K$ is $2h_k$, where $h_k$ is the class number of $k$. This factorization can also be written as

$$\prod_{z \in Z(U)_K} ||\Psi(z; F)||^2 = \text{rat} \cdot \left( 4d\pi \right)^{-h_k/2} \left( \prod_{a=1}^{d-1} \frac{\Gamma \left( \frac{a}{d} \right)^2}{\varpi_k(\chi_d(a))} \right)^{k_0(0)((F, \theta_+))},$$

where $\varpi_k$ is the number of roots of unity in $k$. The transcendental factor appearing in this factorization is related to Shimura’s period invariants [12].

(iii)

$$\log(\text{rat}) = -h_k \sum_{\eta} \sum_{m > 0} c_\eta(-m) \left( \sum_{\lambda} \sum_{\substack{x \in \eta_+ + \lambda_+ + L_+ \cr Q(x) < m}} \kappa_{\eta_+ + \lambda_+}(m - Q(x)) \right).$$

**Proof.** (i) follows from [11]. For (ii) and (iii) we have $\text{vol}(K_T) = \frac{2}{h_k}$, and we will see from Theorem 4.1 of the next section that

$$-h_k \sum_{\eta} \sum_{m > 0} c_\eta(-m) \left( \sum_{\lambda} \sum_{\substack{x \in \eta_+ + \lambda_+ + L_+ \cr Q(x) < m}} \kappa_{\eta_+ + \lambda_+}(m - Q(x)) \right)$$

is the logarithm of a rational number $\text{rat}$. From $\Lambda(s, \chi_d) = \pi^{-\frac{d+1}{2}} \Gamma \left( \frac{d+1}{2} \right) L(s, \chi_d)$, we see

$$\frac{\Lambda'(1, \chi_d)}{\Lambda(1, \chi_d)} = -\frac{1}{2} \log(\pi) + \Gamma'(1) + \frac{L'(1, \chi_d)}{L(1, \chi_d)}.$$  

So for the corresponding part of (43) that involves $k_0(0)$, we have

$$-h_k c_0(0)((F, \theta_+)) \left( \log(d) - \log(\pi) + 2\Gamma'(1) + 2 \frac{L'(1, \chi_d)}{L(1, \chi_d)} \right),$$

which equals

$$h_k c_0(0)((F, \theta_+)) \left( \frac{2L'(0, \chi_d)}{L(0, \chi_d)} - \log(4d\pi) \right).$$

The second identity in (ii) follows from the Chowla-Selberg formula (cf. Proposition 10.10 of [14]), which says

$$\frac{L'(0, \chi_d)}{L(0, \chi_d)} = \frac{\varpi_k}{2h_k} \sum_{a=1}^{d-1} \chi_d(a) \log \Gamma \left( \frac{a}{d} \right).$$
As an immediate consequence of Corollary 3.5 and Theorem 4.1 of the next section, we obtain a Gross-Zagier phenomenon about which primes can occur in the factorization of the rational part of
\[ \prod_{z \in \mathcal{Z}(U)_K} \| \Psi(z; F) \|^2. \]
For \( F \) as in (39), define
\[ m_{\text{max}} = \max \{ m > 0 \mid c_\eta(-m) \neq 0 \text{ for some } \eta \}. \]

**Theorem 3.6.** Let \( -d \) be an odd fundamental discriminant and assume \( U \simeq k = \mathbb{Q}(\sqrt{-d}) \). Then the only primes which occur in the factorization of the rational part of
\[ \prod_{z \in \mathcal{Z}(U)_K} \| \Psi(z; F) \|^2 \]
are

(i) \( q \) such that \( q \mid d \),

(ii) \( p \) inert in \( k \) with \( p \leq dm_{\text{max}} \).

Note that this fact holds for all Borcherds forms and all CM points. In addition, we point out that the modular form \( F \) is not needed in order to obtain \( m_{\text{max}} \). It can be recovered from the divisor of \( \Psi(F)^2 \) (cf. Theorem 1.3 of [12]).

4. **Explicit computation of \( \kappa_\mu(t) \) for \( t \in \mathbb{Q}_{>0} \)**

In order to compute examples of our main theorem, we need to derive explicit formulas for \( \kappa_\mu(t) \) for \( t \in \mathbb{Q}_{>0} \). Our previous discussion of the Clifford algebra of \( U \) and Lemma 2.12 imply that, without loss of generality, we may assume \( U = k \) is an imaginary quadratic field with quadratic form \( Q \) given by a negative multiple of the norm-form. In this section we assume that \( L = \mathfrak{a} \subset \mathcal{O}_k \) is an integral ideal and that \( Q(x) = -N \mathfrak{a} \mathfrak{x} \), so that \( L^\vee = \mathcal{D}^{-1} \mathfrak{a} \), where \( \mathcal{D} \) is the different of \( k \). This is certainly not the most general possible lattice. Write \( \kappa_\mu(t) \) as \( \kappa(t, \mu, \mathfrak{a}) \) for \( \mu \in \mathcal{D}^{-1} \mathfrak{a}/\mathfrak{a} \). For simplicity, we assume that \( k = \mathbb{Q}(\sqrt{-d}) \), where \( d > 3, d \equiv 3 \pmod{4} \) and is square-free, so that the prime 2 is not ramified. Let \( \chi \) be the character of \( \mathbb{Q}_k^\times \) associated to \( k \), which is defined via the global quadratic Hilbert symbol so that \( \chi(t) = (t, -d)_k \).

Then for a prime \( p \leq \infty \), the local character is \( \chi_p(t) = (t, -d)_p \) where \( (\cdot, \cdot)_p \) is the local quadratic Hilbert symbol.

Throughout this section we let \( p \) denote an unramified prime and \( q \) denote a ramified prime. Let \( \mu \) be a coset in \( \mathcal{D}^{-1} \mathfrak{a}/\mathfrak{a} \). Write \( \mu_q \) for the image of \( \mu \) under the map
\[ \mathcal{D}^{-1} \mathfrak{a}/\mathfrak{a} \rightarrow \mathcal{D}^{-1} \mathfrak{a}_q/\mathfrak{a}_q, \]
where \( \mathfrak{a}_q = \mathfrak{a} \otimes \mathbb{Z} q \). For \( t \in \mathbb{Q}_{>0} \), we introduce the function
\[ \rho(t) = \# \{ \mathfrak{a} \subseteq \mathcal{O}_k \mid N \mathfrak{a} = t \}. \]
This function factors as
\[ \rho(t) = \prod_p \rho_p(t), \]
where \( \rho_p(t) \) counts the number of ideals of norm \( t \) which are prime to \( p \).
where $\rho_p(t) = \rho(p \ord_p(t))$. The explicit formula for $\kappa(t, \mu, a)$ is given by the following theorem.

**Theorem 4.1.** For $\mu \in D^{-1}a/a$ and $t \in \mathbb{Q}_{>0}$,

$$
\kappa(t, \mu, a) = -\frac{1}{h_k} \prod_{q|d} \text{char}(Q(\mu_q) + \mathbb{Z}_q)(t) \times

\left[ \rho(dt) \sum_{q|d} \eta_q(t, \mu)(\ord_q(t) + 1) \log(q) + \eta_0(t, \mu) \sum_{p \text{ inert}} (\ord_p(t) + 1)\rho \left( \frac{dt}{p} \right) \log(p) \right],
$$

where

$$
\eta_q(t, \mu) = (1 - \chi_q(-t)) \prod_{q|d, q \neq q \text{ if } \ord_q(-t)} (1 + \chi_q(-t))
$$

and

$$
\eta_0(t, \mu) = \prod_{\mu_q = 0} (1 + \chi_q(-t)).
$$

We take $\eta_0(t, \mu) = 1$ if $\mu_q \neq 0$ for all $q | d$. For $t = 0$,

$$
\kappa(0, 0, a) = \log(d) + 2 \frac{\Lambda'(1, \chi_d)}{\Lambda(1, \chi_d)}.
$$

Note that when $\mu_q \neq 0$ for all $q | d$ we have $\eta_q(t, \mu) = 0$ for all $q$ and $\eta_0(t, \mu) = 1$, and so we get a much simpler formula in this “generic” case.

**Proof.** The value for $t = 0$ is defined in Definition 2.17. For $t > 0$, $\kappa(t, \mu, a)$ is given by the second term in the Laurent expansion of a certain Eisenstein series. These Eisenstein series have factorizations in terms of local Whittaker functions, and we use these factorizations to derive the above formula for $\kappa(t, \mu, a)$. Let $\varphi_{\mu_q}$ be the characteristic function of the coset $\mu_q$, $X = p^{-s}$, and $\tau = u + iv \in \mathfrak{H}$. Using 17 and 13, we have the following formulas for the normalized local Whittaker functions.

For $\mu = 0$, Lemma 2.3 of [13] tells us we only need to consider $t \in \mathbb{Z}$, and for $t > 0$ we have,

\begin{align}
W_{l, \infty}(\tau, s) & = \gamma_{\infty} \frac{e^{2\pi i t \tau}}{\Gamma(\frac{s}{2})} \int_{u > 2tv} e^{-2\pi u} (u - 2tv)^{-s + 1} du, \\
W_{l, p}(s, \varphi_0) & = \sum_{r=0}^{\ord_p(t)} (\chi_p(p) X)^r, \\
W_{l, q}(s, \varphi_0) & = \gamma_q \frac{1}{q} \begin{cases} 
1 + (q, -t) q X^{\ord_q(t) + 1} & \text{if } \ord_q(t) \text{ is even}, \\
1 + (q, -dt) q X^{\ord_q(t) + 1} & \text{if } \ord_q(t) \text{ is odd}.
\end{cases}
\end{align}

Here $\gamma_{\infty}$ and $\gamma_q$ are local factors which do not affect our global computations since $\gamma_{\infty} \prod_q \gamma_q = 1$, where the product is over all ramified primes. For an unramified prime $p$, the local lattice $a_p = a \otimes \mathbb{Z}_p$ is unimodular. Note that here is where we have lost generality by assuming $L = a$ is an integral ideal. Since $a_p$ is unimodular,
we only need to consider the Whittaker functions for nonzero cosets at ramified primes. For \( \mu_q \neq 0 \) we have

\[
W_{t,q}^*(s, \varphi_{\mu_q}) = \gamma_q q^{\frac{1}{2}} \chi_q(Q(\mu_q) + Z_q)(t).
\]

Note that in (48), \( W_{t,q}^*(s, \varphi_{\mu_q}) \) is either a nonzero constant or is identically zero. Following [13], the normalized Eisenstein series has Fourier coefficients given by

\[
E_t^*(\tau, s, \Phi^{1, \mu}) = v^{-\frac{1}{2}} d^{\frac{S+1}{2}} W_{t, \infty}^*(\tau, s) \prod_{q \mid d} W_{t,q}^*(s, \varphi) \prod_{p \mid d} W_{t,p}^*(s, \varphi_0).
\]

Write \( t = q^{\alpha_q} u \) where \( \alpha_q = \text{ord}_q(t) \). We now show that (47) can be combined into one nice formula.

**Lemma 4.2.** \( W_{t,q}^*(s, \varphi_0) = \gamma_q q^{-\frac{s}{2}} (1 + \chi_q(-t)X^{\alpha_q+1}) \).

**Proof.** For \( \alpha_q \) even, we have

\[
(q, -t)_q = (-t, q)_q = (-t, -1)_q(-t, -q) = (-t, -1)_q(-t, dq)_q\chi_q(-t),
\]

and

\[
(-t, -1)_q(-t, dq)_q = (-t, -dq^{-1})_q = \left(\frac{-dq^{-1}}{q}\right)^{\alpha_q} = 1.
\]

For \( \alpha_q \) odd,

\[
(q, -dt)_q = (-1)\frac{dq}{q}(q, d)_q(q, t)_q
\]

\[
= (-1, q)_q(q, d)_q(-t, -q)_q(-1, q)_q(-t, -1)_q
\]

\[
= (q, d)_q(-t, -1)_q(-t, dq)_q\chi_q(-t),
\]

and

\[
(q, d)_q(-t, -1)_q(-t, dq)_q = (q, d)_q(-t, -dq^{-1})_q
\]

\[
= (-1)^{\frac{dq-1}{2}} \left(\frac{dq^{-1}}{q}\right)^{\alpha_q} \left(\frac{-dq^{-1}}{q}\right)^{\alpha_q}
\]

\[
= 1.
\]

So (47) can be rewritten as

\[
W_{t,q}^*(s, \varphi_0) = \gamma_q q^{-\frac{s}{2}} (1 + \chi_q(-t)X^{\alpha_q+1})
\]

\[
\square
\]

Let us first compute \( \kappa(t, \mu, a) \) for \( \mu = 0 \) and \( t \in \mathbb{N} \). To do this, we need the following special values for the local Whittaker functions, cf. Lemma 2.5 and Propositions 2.6 and 2.7 of [13].

**Lemma 4.3.** At \( s = 0 \) we have

(i) \( W_{t,\infty}^*(\tau, 0) = -\gamma_{\infty} 2^{\frac{S}{2}} e(t \tau) \).

(ii) \( W_{t,p}^*(0, \varphi_0) = \rho_p(t) \), and if \( \rho_p(t) = 0 \) then

\[
W_{t,p}^*(0, \varphi_0) = \frac{1}{2} (\text{ord}_p(t) + 1) \rho_p(t) \log(p).
\]

(iii) \( W_{t,q}^*(0, \varphi_0) = \gamma_q q^{-\frac{1}{2}} (1 + \chi_q(-t)) \), and if \( \chi_q(-t) = -1 \) then

\[
W_{t,q}^*(0, \varphi_0) = \gamma_q q^{-\frac{1}{2}} (\text{ord}_q(t) + 1) \rho_q(t) \log(q).
\]
Given \( \Phi_{t,p} \), we consider different cases based on when one and only one local Whittaker function vanishes at \( s = 0 \). Since \( W_{t,∞}^*(τ, 0) \neq 0 \) for \( t ∈ \mathbb{N} \), there are two cases.

**Case 1:** \( W_{t,0}^*(0, Φ_0) = 0 \) for \( p \) unramified, \( W_{t,0}^*(0, Φ_0) ≠ 0 \) for \( q \) ramified.

\( W_{t,0}^*(0, Φ_0) = 0 \) implies that \( p \) is inert and \( ord_p(t) = odd \). Since \( W_{t,q}^*(0, Φ_0) ≠ 0 \) for \( q \) ramified, we have \( χ_q(−t) = 1 \) and \( W_{t,q}^*(0, Φ_0) = γ_q 2q^{−1} \). Computing the derivative of the Fourier coefficient we get

\[
E_{t,q}^*(τ, 0, Φ_{1,0}) = W_{t,0}^*(0, Φ_0) \left[ v^{−1/2} d^2 W_{t,∞}^*(τ, 0) \prod_{q | d} W_{t,q}^*(0, Φ_0) \prod_{p | d} W_{t,0}^*(0, Φ_0) \right]
\]

\[
= log(p) \left( \frac{t}{p} \right) \prod_{q | d} \gamma_q 2q^{−1} e(tτ) 2^{v(q)} \prod_{p | d} \rho_p(t)
\]

\[
= − log(p)(ord_p(t) + 1) \rho_q(\frac{t}{p}) e(tτ) 2^{v(q)} \prod_{p | d} \rho_p(t)
\]

since \( ρ_q(\frac{t}{p}) = 1 \) and \( ρ_p(t) = \frac{t}{p} \), and where \( v(d) \) is the number of primes dividing \( d \). So we see

(51) \( E_{t,q}^*(τ, 0, Φ_{1,0}) = − log(p)(ord_p(t) + 1) 2^{v(q)} \rho_c(\frac{t}{p}) e(tτ). \)

**Case 2:** \( W_{t,q}^*(0, Φ_0) = 0 \) for \( q \) ramified, \( W_{t,0}^*(0, Φ_0) ≠ 0 \) for \( p = q \).

\( W_{t,q}^*(0, Φ_0) = 0 \) implies \( χ_q(−t) = −1 \) while for any ramified prime \( q ≠ p \) we have \( χ_q(−t) = 1 \) and \( W_{t,q}^*(0, Φ_0) = γ_q 2q^{−1} \). In this case, we see

\[
E_{t,q}^*(τ, 0, Φ_{1,0}) = W_{t,q}^*(0, Φ_0) \left[ v^{−1/2} d^2 W_{t,∞}^*(τ, 0) \prod_{q' | d} W_{t,q'}^*(0, Φ_0) \prod_{p | d} W_{t,0}^*(0, Φ_0) \right]
\]

\[
= γ_q 2q^{−1} log(q)(ord_q(t) + 1) \rho_q(t) \times
\]

\[
= − v^{−1/2} d^2 γ_q 2q^{−1} e(tτ) 2^{v(q)} \prod_{q' | d} \gamma_{q'} 2q'^{−1} \prod_{p | d} \rho_p(t)
\]

(52) \( = − log(q)(ord_q(t) + 1) 2^{v(q)} \rho(t) e(tτ). \)

Recall that the definition of \( \kappa(t, m, a) \) involves the non-normalized Eisenstein series, and at \( s = 0 \) we have \( E_{t,q}^*(τ, 0, Φ_{1,μ}) = h_k E_{t,q}(τ, 0, Φ_{1,μ}) \). This fact and the above analysis, particularly (51) and (52), give

\[
\kappa(t, a) = \frac{2^{v(q)}}{h_k} \left( Σ_{q | d} ϵ_q(t)(ord_q(t) + 1) \rho(q) + Σ_{ρ inert} ϵ_0(t)(ord_p(t) + 1) \rho(\frac{t}{p}) log(p) \right)
\]
where
\[
\xi_q(t) = \begin{cases} 
0 & \text{if } \chi_q(t) = 0 \text{ or } \chi_q(t) = -1 = \chi_{q'}(t), \text{ for some ramified prime } q', \\
1 & \text{if } \chi_q(t) = -1 \text{ and } \chi_{q'}(t) = 1 \text{ for all ramified primes } q' \neq q,
\end{cases}
\]
and
\[
\xi_0(t) = \begin{cases} 
0 & \text{if } \chi_0(t) = -1 \text{ for some ramified prime } q, \\
1 & \text{otherwise}.
\end{cases}
\]

Now we compute \(\kappa(t, \mu, a)\) for \(\mu \neq 0\). One main thing to keep in mind is that there is at least one ramified prime \(q\) such that \(\mu_q \neq 0\), but the coset can be zero locally at other ramified primes. Write \(\mu = (\mu_p)\), where if \(p\) is unramified then \(\mu_p = 0\) and let \(\alpha(\mu) = \#\{q \text{ ramified } | \mu_q = 0\}\). Again, we consider two cases.

**Case 1:** \(W_{t,p}^*(0, \varphi_0) = 0\) for \(p\) unramified, \(W_{t,p'}^*(0, \varphi_{\mu_{p'}}) \neq 0 \forall p' \neq p\).

The formula for the derivative of the Fourier coefficient is
\[
E_t^{*'}(\tau, 0, \Phi^{1*}) = W_{t,p}^*(0, \varphi_0) \left[ e^{-\frac{\sqrt{2}}{2} d^2 W_{t,\infty}^*(\tau, 0) \prod_{p|d} W_{t,q}^*(0, \varphi_{\mu_q}) \prod_{p \neq q} W_{t,p}^*(0, \varphi_0)} \right].
\]

Then after cancelling some terms and using Lemma 4.3 and (48), we get
\[
= \log(p) \left[ \frac{1}{2} (\text{ord}_p(t) + 1) \rho_p \left( \frac{t}{p} \right) - 2e(t\tau)2^{\alpha(\mu)} \prod_{q \neq 0} \text{char}(Q(\mu_q) + Z_q)(t) \prod_{p' \neq p} \rho_{p'}(t) \right].
\]

If \(q\) is a ramified prime with \(\mu_q \neq 0\), then \(W_{t,q}^*(0, \varphi_{\mu_q}) \neq 0\) implies \(\text{ord}_q(t) = -1\). This means \(\rho_q(qt) = 1\) and this also equals \(\rho_q(dt)\). If \(\mu_q = 0\), then \(\rho_q(t) = 1 = \rho_q(dt)\). Similarly, \(\rho_p \left( \frac{t}{p} \right) = \rho_p \left( \frac{d}{p} \right)\) and \(\rho_{p'}(t) = \rho_{p'}(dt) = \rho_{p'} \left( \frac{d}{p} \right)\). We also see that if \(\mu_q = 0\), then \(\text{char}(Q(\mu_q) + Z_q)(t) = \text{char}(Z_q)(t) = 1\). So the above formula is
\[
= -2^{\alpha(\mu)} \log(p) \left( \text{ord}_q(t) + 1 \right) \rho \left( \frac{dt}{p} \right) e(t\tau) \prod_{q|d} \text{char}(Q(\mu_q) + Z_q)(t).
\]

**Case 2:** \(W_{t,q}^*(0, \varphi_0) = 0\) for \(q\) ramified, \(W_{t,p}^*(0, \varphi_{\mu_p}) \neq 0 \forall p \neq q\).

Here the derivative is given by
\[
E_t^{*'}(\tau, 0, \Phi^{1*}) = W_{t,q}^*(0, \varphi_0) \left[ e^{-\frac{\sqrt{2}}{2} d^2 W_{t,\infty}^*(\tau, 0) \prod_{q \neq q'} W_{t,q'}^*(0, \varphi_{\mu_{q'}}) \prod_{p|d} W_{t,p}^*(0, \varphi_0)} \right]
\]
\[
= \log(q) (\text{ord}_q(t) + 1) \rho_q(t) \left[ -2e(t\tau)2^{\alpha(\mu) - 1} \prod_{q \neq 0} \text{char}(Q(\mu_q) + Z_q)(t) \prod_{p|d} \rho_p(t) \right]
\]
\[
= -2^{\alpha(\mu)} \log(q) (\text{ord}_q(t) + 1) \rho(dt) e(t\tau) \prod_{q|d} \text{char}(Q(\mu_q) + Z_q)(t).
\]
Note that we do not consider the case where \( W_{t,q}^*(0, \varphi_{\mu_q}) = 0 \) for \( \mu_q \neq 0 \), since then the Whittaker function is identically zero and there is no contribution to the derivative. Formulas (53) and (54) imply that for \( \mu \neq 0 \),

\[
\kappa(t, \mu, a) = -\frac{2^{\alpha(\mu)}}{h_k} \prod_{q \mid d} \text{char}(Q(\mu_q) + \mathbb{Z}_q)(t) \times (55)
\]

\[
\left( \sum_{q \mid d} \xi_q(t, \mu)(\text{ord}_q(t) + 1)\rho(dt) \log(q) + \sum_{p \text{ inert}} \xi_0(t, \mu)(\text{ord}_p(t) + 1)\rho\left( \frac{dt}{p} \right) \log(p) \right),
\]

where

\[
\xi_q(t, \mu) = \begin{cases} 
0 & \text{if } \mu_q \neq 0, \text{ or } \mu_q = 0 \text{ and } \chi_q(-t) = 1, \text{ or } \chi_q(-t) = -1 = \chi_{q'}(-t) \\
1 & \text{if } \mu_q = 0, \chi_q(-t) = -1, \text{ and } \chi_{q'}(-t) = 1 \text{ for all ramified primes } q' \neq q \text{ with } \mu_{q'} = 0,
\end{cases}
\]

and

\[
\xi_0(t, \mu) = \begin{cases} 
0 & \text{if } \chi_q(-t) = -1 \text{ and } \mu_q = 0 \text{ for some ramified prime } q, \\
1 & \text{otherwise}.
\end{cases}
\]

If we take \( \mu = 0 \) in the above equations, we see that \( \xi_q(t, 0) = \xi_q(t), \xi_0(t, 0) = \xi_0(t) \) and \( \nu(d) = \alpha(0) \). Also, when \( \mu = 0 \) then \( t \in \mathbb{N} \) so \( \rho(dt) = \rho(t), \rho\left( \frac{dt}{p} \right) = \rho\left( \frac{1}{p} \right) \) and the characteristic functions can be ignored. This means (55) holds when \( \mu = 0 \) as well. We then note that once we sum over \( q \mid d \) with \( \mu_q = 0 \) we can replace \( 2^{\alpha(\mu)}\xi_q(t, \mu) \) with \( \eta_q(t, \mu) \) and we have

\[
\eta_0(t, \mu) = 2^{\alpha(\mu)}\xi_0(t, \mu).
\]

This finishes the proof of Theorem 4.1. \( \square \)

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