Gradient estimates for the insulated conductivity problem with inclusions of the general $m$-convex shapes

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In this paper, the insulated conductivity model with two touching or close-to-touching inclusions is considered in $\mathbb{R}^d$ with $d \geq 3$. We establish the pointwise upper bounds on the gradient of the solution for the generalized $m$-convex inclusions under these two cases with $m \geq 2$, which show that the singular behavior of the gradient in the thin gap between two inclusions is described by the first non-zero eigenvalue of an elliptic operator of divergence form on $\mathbb{S}^{d-2}$. Finally, the sharpness of the estimates is also proved for two touching axisymmetric insulators, especially including curvilinear cubes.

1 | INTRODUCTION

Assume that $D \subseteq \mathbb{R}^d$ ($d \geq 3$) is a bounded open set with $C^2$ boundary, whose interior embodies two adjacent $C^{2,\gamma}$-subdomains $D_1$ and $D_2$ with $0 < \gamma < 1$. Denote $\varepsilon := \text{dist}(D_1, D_2)$, where $\varepsilon \geq 0$. Especially when $\varepsilon = 0$, it means that $D_1$ and $D_2$ touch only at one point. Suppose also that $D_i, \ i = 1, 2,$ stay far away from the external boundary $\partial D$. Write $\Omega := D \setminus D_1 \cup D_2$. In this paper, for given boundary data $\varphi \in C^2(\partial D)$, we aim to study the singular behavior of the gradient of a solution to the insulated conductivity problem with $C^\gamma$ coefficients as follows:

$$
\begin{cases}
-\partial_i(A_{ij}(x)\partial_j u) = 0, \quad &\text{in } \Omega, \\
A_{ij}(x)\partial_j u(x)\nu_i = 0, \quad &\text{on } \partial D_i, \ i = 1, 2, \\
u = \varphi, \quad &\text{on } \partial D,
\end{cases}
$$

where the coefficient matrix $(A_{ij}(x)) \in C^\gamma$ is symmetric and verifies $\zeta I \leq A(x) \leq \frac{1}{\zeta} I$ for some positive constant $\zeta$, $\nu$ is the unit outer normal to the subdomains $D_1$ and $D_2$.

To state our main results in a precise manner, we first formulate the domain. By suitable translation and rotation of the coordinates, we have

$$D_1 := D_1^* + (0', \varepsilon/2), \quad \text{and } D_2 := D_2^* + (0', -\varepsilon/2),$$

where $D_1^*$ and $D_2^*$ are touching only at the origin and satisfy

$$D_i^* \subset \{(x', x_d) \in \mathbb{R}^d \mid (-1)^{i+1}x_d > 0\}, \quad i = 1, 2.$$

Here and throughout the paper, we denote $(d-1)$-dimensional variables and domains by adding superscript prime, for instance, $x'$ and $B'$. 

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Assume further that there exists a small \( \varepsilon \)-independent constant \( R_0 > 0 \) such that the portions of \( \partial D_1 \) and \( \partial D_2 \) around the origin are, respectively, the graphs of two \( C^{2,\gamma} \) functions \( \varepsilon / 2 + h_1(x') \) and \( -\varepsilon / 2 + h_2(x') \), where \( h_j, j = 1, 2 \) satisfy the following \( m \)-convex conditions: for \( m \geq 2 \) and \( \gamma > 0 \), \( x' \in B'_{2R_0} \),

\[
\begin{align*}
(H1) \quad & h_1(x') - h_2(x') = \kappa_0 \left( \sum_{i \in A} \kappa_i |x_i|^2 \right)^{m/2} + \sum_{j \in B} \kappa_j |x_j|^m + O(|x'|^{m+\gamma}), \\
(H2) \quad & |\nabla x' h_j(x')| \leq \tau_1 |x'|^{m-1}, j = 1, 2, \\
(H3) \quad & \|h_1\|_{C^2(B'_{2R_0})} + \|h_2\|_{C^2(B'_{2R_0})} \leq \tau_2,
\end{align*}
\]

where \( A \) and \( B \) are two sets such that \( A \cup B = \{1, \ldots, d - 1\} \) and \( A \cap B = \emptyset \), \( \kappa_i, i = 0, 1, \ldots, d - 1 \), \( \tau_1 \) and \( \tau_2 \) are all positive constants independent of \( \varepsilon \). Here and below, the notation \( O(A) \) represents that there exists a \( \varepsilon \)-independent positive constant \( C \) such that \( |O(A)| \leq CA \). We should point out that the case of \( m = 2 \) in condition (H1) corresponds to the strictly convex inclusions, that is, the principal curvatures of interfacial boundaries of inclusions are greater than zero, which has been studied in Dong et al. [1]. The results in Dong et al. [1] revealed that the gradient blow-up rate depends on the principal curvatures of the surfaces of insulators, which is different from the blow-up phenomenon occurring in the perfect conductivity problem. However, when \( m > 2 \), the principal curvatures degenerate to be zero and the surfaces of inclusions become flatter. In this case, the shapes of inclusions considered in condition (H1) are formed by the coupling of two different types of \( m \)-convex curved surfaces as follows:

\[
h_1(x') - h_2(x') = \sum_{i \in A} |x_i|^m + \sum_{j \in B} |x_j|^m + O(|x'|^{2m}), \quad \text{in } \Omega_{r_0},
\]

where \( \kappa = m^{-1}(r_1^{-m} + r_2^{-m}) \) and \( 0 < r_0 < \min\{r_1, r_2\} \). So curvilinear cube belongs to the case when \( A = \emptyset \) and \( \kappa_i = \kappa \), \( i \in B \) in condition (H1). Moreover, this type of axisymmetric inclusions has been widely used in the manufacture of composite materials due to its regular shape and fine properties.

For \( y' \in B'_{R_0}, 0 < s \leq 2R_0 \), denote a thin gap by

\[
\Omega_s(y') := \{x \in \mathbb{R}^d \mid -\varepsilon / 2 + h_2(x') < x_d < \varepsilon / 2 + h_1(x'), |x' - y'| < s\},
\]

whose upper and lower boundaries are, respectively, written by

\[
\Gamma^+_s := \{x \in \mathbb{R}^d \mid x_d = \varepsilon / 2 + h_1(x'), |x'| < s\},
\]

and

\[
\Gamma^-_s := \{x \in \mathbb{R}^d \mid x_d = -\varepsilon / 2 + h_2(x'), |x'| < s\}.
\]

For simplicity, write \( \Omega_0' \) as \( \Omega_0 \) in the case of \( y' = 0' \). Observe that from the standard elliptic estimates, we have \( \|u\|_{C^1(\Omega_0 \setminus \Omega_R)} \leq C \). Therefore, it suffices to study the insulated conductivity problem in a narrow region as follows:
\[
\begin{cases}
-\partial_i (A_{ij}(x)\partial_j u) = 0, & \text{in } \Omega_{R_0}, \\
A_{ij}(x)\partial_j u(x)\nu_i = 0, & \text{on } \Gamma^\pm_{R_0}, \\
\|u\|_{L^\infty(\Omega_{R_0})} \leq 1.
\end{cases}
\]  
(1.3)

Consider the following eigenvalue problem:

\[-\text{div}_{\mathbb{S}^{d-2}}(\kappa(\xi)^{1/2}\nabla_{\mathbb{S}^{d-2}}u(\xi)) = \lambda \kappa(\xi)u(\xi), \quad \xi \in \mathbb{S}^{d-2},\]

(1.4)

where \(\kappa(\xi) = \kappa_0 \left( \sum_{i \in A} \kappa_i |\xi_i|^2 \right)^m + \sum_{j \in B} \kappa_j |\xi_j|^m\) satisfies that \(\|\ln \kappa\|_{L^\infty(\mathbb{S}^{d-2})} < \infty\). Define the inner product as follows:

\[
\langle u, v \rangle_{\mathbb{S}^{d-2}} = \int_{\mathbb{S}^{d-2}} \kappa(\xi) uv.
\]

(1.5)

By the classical eigenvalue theory of elliptic operator of divergence form on \(\mathbb{S}^{d-2}\), we know that each eigenvalue of problem (1.4) is real and the corresponding normalized eigenfunctions form an orthonormal basis of \(L^2(\mathbb{S}^{d-2})\) under the inner product (1.5). Moreover, the first nonzero eigenvalue \(\lambda_1\) of problem (1.4) can be determined by the Rayleigh quotient:

\[
\lambda_1 = \inf_{u \neq 0, \langle u, v \rangle_{\mathbb{S}^{d-2}} = 0} \frac{\int_{\mathbb{S}^{d-2}} \kappa(\xi)|\nabla_{\mathbb{S}^{d-2}} u|^2}{\int_{\mathbb{S}^{d-2}} \kappa(\xi)|u|^2}.
\]

Denote

\[
\alpha(\lambda_1) := \frac{-(d + m - 3) + \sqrt{(d + m - 3)^2 + 4\lambda_1^2}}{2}.
\]

(1.6)

Unless otherwise stated, in the following, the constant \(C\) may change from line to line, which depends only on \(d, m, \gamma, R_0, \tau_1, \tau_2, \kappa_i, i = 0, 1, \ldots, d - 1\), and \(\|A\|_{C^7}\), but not on \(\varepsilon\). First, we establish the pointwise upper bounds on the gradient as follows.

**Theorem 1.1.** Suppose that \(D_1, D_2 \subseteq D \subseteq \mathbb{R}^d\) \((d \geq 3)\) are defined as above, conditions (H1)–(H3) hold. Let \(u \in H^1(\Omega_{R_0})\) be the solution of Equation (1.3) with \(A_{ij}(0) = \delta_{ij}\). Then

(i) if \(\varepsilon = 0\) and \(x \in \Omega_{R_0/2} \setminus \{x' = 0'\}\),

\[
|\nabla u(x)| \leq C \|u\|_{L^\infty(\Omega_{R_0})} |x'|^{\alpha(\lambda_1) - 1};
\]

(1.7)

(ii) if \(\varepsilon > 0\) is sufficiently small and \(x \in \Omega_{R_0/4}\),

\[
|\nabla u(x)| \leq C \|u\|_{L^\infty(\Omega_{R_0})} (\varepsilon + |x'|^m)^{\alpha(\lambda_1) - 1}.
\]

(1.8)

where \(\alpha(\lambda_1)\) is defined by Equation (1.6).

**Remark 1.2.** Using Lemma 5.1 in Dong et al. [1], we know that \(\lambda_1 \leq d - 2\) and the equality holds if and only if \(\kappa\) is constant. Then in the case of \(m = 2\), if condition (H1) holds with \(B = \emptyset\) and \(\kappa_i = \kappa_j, i, j \in A\), or \(B = \{1, \ldots, d - 1\}\) and \(\kappa_i = \kappa_j, i, j \in B\), or \(B \neq \emptyset, B \neq \{1, \ldots, d - 1\}\) and \(\kappa_0 \kappa_i = \kappa_j, i \in A, j \in B\), then \(\lambda_1 = d - 2\), see [2]. For \(m > 2\), if (H1) holds with \(B \neq \emptyset\) or \(B = \emptyset, \kappa_i = \kappa_j\) for some \(i_1, i_2 \in A, i_1 \neq i_2\), then \(\lambda_1 = d - 2\), see Zhao [3]. Otherwise, we have \(\lambda_1 < d - 2\).

For the purpose of establishing the optimal lower bound on the gradient, we assume that \(D = B_3\) and \(D_1, D_2 \subseteq B_4\) are two smooth \(m\)-convex inclusions such that \(\overline{D_1} \cap \overline{D_2} = \emptyset\) and the domain \(\Omega = D \setminus \overline{D_1} \cup \overline{D_2}\) are symmetric with respect to every \(x_i, i = 1, 2, \ldots, d\). The optimal lower bound on the gradient is stated as follows.
Theorem 1.4. For $d \geq 3$, let $D$, $D_1$, and $D_2$ be described as above. Conditions (H1)-(H3) hold with $A = \emptyset$ and $\gamma > 1 - \alpha(\lambda_j)$. Suppose that the eigenspace corresponding to the first nonzero eigenvalue $\lambda_1$ of Equation (1.4) contains a function, which is odd with respect to some $x_{j_0}$, $j_0 \in \{1, \ldots, d - 1\}$. Let $u \in H^1(\Omega)$ be the solution of Equation (1.1) with $\varepsilon = 0$, $\varphi = x_{j_0}$, and $A_{ij}(x) \equiv \delta_{ij}$. Then,

$$\limsup_{x \in \Omega, |x| \to 0} |x'\alpha(\lambda_j)| \|\nabla u(x)\| > \frac{1}{C},$$

where $\alpha(\lambda_j)$ is given by Equation (1.6) and the positive constant $C$ depends only on $d, m, \kappa_j, i = 0, 1, \ldots, d - 1, \text{ and upper bounds of } \|\partial D_j\|_{C^4}, j = 1, 2.$

Remark 1.5. It is worth mentioning that the validity of the assumed condition “the eigenspace corresponding to the first nonzero eigenvalue $\lambda_1$ of Equation (1.4) contains a function, which is odd with respect to some $x_{j_0}$, $j_0 \in \{1, \ldots, d - 1\}” will be demonstrated in Section 3 below. In addition, the shape of inclusions considered in Theorem 1.4 also contains curvilinear cubes with the same radii in Equation (1.2).

The paper is organized as follows. The proofs of Theorems 1.1 and 1.4 are, respectively, given in Sections 2 and 3. In the rest of the introduction, we review some earlier relevant results.

The mathematical model for the conductivity problem can be described by the following elliptic equations of divergence form:

$$\begin{align*}
\text{div}(a_k(x)\nabla u_k) &= 0, &\text{in } D, \\
u_k &= \varphi, &\text{on } \partial D, \\
a_k(x) &= \begin{cases} k \in (0, \infty), &\text{in } D_1 \cup D_2, \\
1, &\text{in } \Omega, \end{cases}
\end{align*}$$

(1.9)

where $\varphi \in C^2(\partial D)$ is a given boundary data and $k$ is called the conductivity. When $k$ tends to zero, problem (1.9) becomes the insulated conductivity problem, while it turns into the perfect conductivity problem if $k$ goes to infinity. It is well known that there always appears blow-up of the gradient $|\nabla u|$ for the insulated or perfect conductivity problem, as the distance between two adjacent inclusions approaches to zero. Babuška et al. [4] were the first to propose the problem of estimating $|\nabla u|$ in the close touching regime. In Babuška et al. [4], they analyzed computationally the damage and fracture of composites modeled by the Lamé system with finite coefficients and found that the gradient of solutions stays bounded independent of the distance between inclusions. For two touching disks, Bonnetier and Vogelius [5] proved the boundness for the conductivity problem (1.9). Li and Vogelius [6] then extended their results to general second-order elliptic equations of divergence form with piecewise Hölder coefficients. In particular, their results hold for arbitrarily smooth shape of inclusions in all dimensions. The subsequent work [7] completed by Li and Nirenberg further extended to general second-order elliptic systems of divergence form, especially covering the Lamé system. This especially demonstrates the numerical observation in Babuška et al. [4]. For more related investigations on the elliptic equation with piecewise constant coefficients, see Refs. [8–12].

Ammari et al. [13, 14] firstly made use of the layer potential techniques to establish the optimal gradient estimates for the insulated conductivity problem with two close-to-touching disks, which showed that the blow-up rate of the gradient is $\varepsilon^{-1/2}$ in two dimensions. For two adjacent spherical insulators in dimension three, Yun [15] obtained the optimal gradient estimates in the shortest segment between two inclusions and captured the blow-up rate of order $\varepsilon^\sqrt{\frac{2}{2}}$. For the general strictly convex inclusions, Bao et al. [16] developed a “flipping” technique to establish the pointwise upper bound estimate of the gradient in all dimensions as follows:

$$|\nabla u| \leq C(\varepsilon + |x'|^2)^{-1/2}, \text{ in } \Omega_{R_0},$$

(1.10)

Li and Yang [17, 18] further considered the general $m$-convex inclusions with $m \geq 2$ and obtained

$$|\nabla u(x)| \leq C(\varepsilon + |x'|^m)^{-1/m+\beta}, \text{ in } \Omega_{R_0},$$

(1.11)

for some inexplicit $\beta > 0$. This improves the result in Equation (1.10). Moreover, the upper bound (1.11) indicates that the blow-up rate $\varepsilon^{-1/m+\beta}$ will decrease as the convexity index $m$ increase. This also implies that curvilinear cubes are
superior to spheres from the view of shape design of insulated materials. For the purpose of making clear the value of $\beta$ in Equation (1.11), Weinkove [19] constructed an appropriate auxiliary function and used the maximum principle to solve an explicit constant $\beta(d)$ for two nearly touching balls in dimension greater than three. When condition (H1) becomes $(h_1 - h_2)(x') = \kappa_0|x'|^m + O(|x'|^{m+\gamma})$ in $\Omega_{R_0}$, Dong et al. [2] established the optimal upper and lower bounds on the gradient in the case of $m = 2$ and $d \geq 3$ and found that the optimal value of $\beta$ is $\beta(d, m) = -(d-1)^2 + \sqrt{(d-1)^2 + 4(d-2)} / 4$. Zhao [3] further extended the results to the case when $m > 2$ and $d \geq 3$ and revealed that the optimal gradient blow-up rate is $\varepsilon^{-1/m + \beta(d, m)}$ with $\beta(d, m) = -(d + m - 3)^2 + \sqrt{(d + m - 3)^2 + 4(d - 2)} / (2m)$. Ma [20] recently improved the auxiliary function constructed in Weinkove [19] and obtained the same $\beta$ as in Dong et al. [2].

With regard to the perfect conductivity problem, there has been a long list of literature making clear the blow-up phenomenon, for example, see Refs. [13, 14, 16, 21–27] and the references therein. For nonlinear equation, we refer to Refs. [28–30] for an interested reader.

2 THE PROOF OF THEOREM 1.1

For $\varepsilon \geq 0$, define

$$
\delta := \delta(x') = : \varepsilon + \kappa \left( \frac{x'}{|x'|} \right) |x'|^m, \quad |x'| \leq R_0,
$$

where

$$
\kappa \left( \frac{x'}{|x'|} \right) = \kappa_0 \left( \sum_{i \in A} \kappa_i |x_i|^2 |x'|^{-2} \right)^{m/2} + \sum_{j \in B} \kappa_j |x_j|^m |x'|^{-m}.
$$

For $\sigma, \tau \in \mathbb{R}$, define a norm as follows:

$$
\|F\|_{\mathcal{L}(\mathbb{R}^{d}, |x'|^m dx')} := \sup_{x' \in B'_R} |x'|^{-\sigma} (\varepsilon + |x'|^m)^{\tau-1} |F(x')|, \quad \text{with } 0 < R \leq R_0.
$$

Define the weighted space $H^1(B'_R, |x'|^m dx')$ under a weighted norm as follows:

$$
\|f\|_{H^1(B'_R, |x'|^m dx')} := \left( \int_{B'_R} |f|^2 |x'|^m dx' \right)^{1/2} + \left( \int_{B'_R} |\nabla f|^2 |x'|^m dx' \right)^{1/2}.
$$

For $0 < \rho < R_0$, write

$$
(f)_{\partial \Omega_R}^\rho := \left( \int_{\partial B'_\rho} \kappa \left( \frac{x'}{|x'|} \right) dS \right)^{-1} \int_{\partial B'_\rho} \kappa \left( \frac{x'}{|x'|} \right) f(x') dS,
$$

$$
(f)_{B'_\rho}^\rho := \left( \int_{B'_\rho} \kappa \left( \frac{x'}{|x'|} \right) dx' \right)^{-1} \int_{B'_\rho} \kappa \left( \frac{x'}{|x'|} \right) f(x') dx'.
$$

In order to prove Theorem 1.1, we need the following two propositions.

**Proposition 2.1.** For $d \geq 3, 1 + \sigma > 0, 1 + \sigma \neq \alpha(\lambda_1)$, let $\tilde{v} \in H^1(B'_R, |x'|^m dx')$ be a solution of

$$
\text{div}(\delta \nabla \tilde{v}) = \text{div}F, \quad \text{in } B'_R, \tag{2.1}
$$
with $\varepsilon = 0$ and $\|F\|_{\varepsilon, \sigma, 0, B'_{R_0}} < \infty$. Then for any $\rho \in (0, R_0)$,

$$\left( \int_{\partial B'_{\rho}} |\bar{v} - \bar{v}(0')|^2 \right)^{1/2} \leq C \|F\|_{\varepsilon, \sigma, 0, B'_{R_0}}^{\bar{\alpha}(\lambda_1)},$$

where

$$\bar{\alpha}(\lambda_1) = \min\{\alpha(\lambda_1), 1 + \sigma\}, \quad (2.2)$$

with $\alpha(\lambda_1)$ given by Equation (1.6).

Introduce some constants as follows:

$$\theta_1 = \left[ \frac{\chi_0^2}{\sum_{i \in A} \chi_i^2} + \sum_{j \in B} \chi_j^2 \chi_i^2 \right]^{1/2}, \quad (2.3)$$

$$\theta_2 = \left[ \frac{\chi_0^4}{\sum_{i \in A} \chi_i^4} \left( \sum_{i \in A} \chi_i^2 \chi_i^2 \right)^{(m-2)/2} + \sum_{j \in B} \chi_j^2 \chi_i^2 \right]^{1/4}, \quad (2.4)$$

$$\theta_3 = \left( 2 - \frac{m-2}{2} \min \left\{ 2 - \frac{m-2(b-1)}{2}, 1 \right\} \min \left\{ \chi_0 \min_{i \in A} \chi_i^{m/2}, \min_{j \in B} \chi_j \right\} \right)^{1/(m-1)}, \quad (2.5)$$

where $b := \text{card}(B)$ denotes the number of elements in set $B$. Define

$$c_0 := \min \left\{ 1, \frac{1}{4(1 + \theta_1)^{1/m}}, \frac{1}{2^m m \max\{\theta_2, 1\} \max\{\theta_3, 1\}} \right\}. \quad (2.6)$$

For $0 < \rho < R_0$, define

$$B'_{(1 \pm \varepsilon_0)\rho} := B'_{(1 + \varepsilon_0)\rho} \setminus B'_{(1 - \varepsilon_0)\rho}, \quad \text{with} \ \varepsilon_0 := 2c_0(1 + \theta_1)^{1/m}. \quad (2.7)$$

Remark that these parameters $\theta_i, i = 1, 2, 3, c_0$ and $\varepsilon_0$ are introduced to achieve the following two goals. The first goal is to give a precise characterization in terms of the equivalence of the height of small thin gap in Equation (2.35). The second one is to choose a suitable small neighborhood centered at every point of the considered thin gap for the purpose of using the “flipping argument” developed in Bao et al. [16] to derive the pointwise upper bounds on the gradient in the following. See pages 15–21 below for further explanations and more details.

**Proposition 2.2.** For $d \geq 3, 1 + \sigma > 0, 1 + \sigma \neq \alpha(\lambda_1)$, let $\bar{v} \in H^1(B'_{R_0})$ be a solution of

$$\text{div}(\delta \nabla \bar{v}) = \text{div}F, \quad \text{in} \ B'_{R_0},$$

with $\varepsilon > 0$, $\|F\|_{\varepsilon, \sigma, 0, B'_{R_0}} < \infty$ and $\|\nabla \bar{v}\|_{C, \tau, 1, B'_{R_0}} < \infty$ for $\tau \leq 1$. Then for $0 < \rho < (1 - \varepsilon_0)2R \leq (1 - \varepsilon_0)^2 R_0$,

$$\left( \int_{B'_{(1 \pm \varepsilon_0)\rho}} |\bar{v}(x') - (\bar{v})_{B'_{(1 \pm \varepsilon_0)\rho}}|^2 \right)^{1/2} \leq C \left( \frac{\rho}{R} \right)^{\alpha(\lambda_1)} \left( \int_{B'_{(1 \pm \varepsilon_0)\rho}} \left| \bar{v}(x') - (\bar{v})_{B'_{(1 \pm \varepsilon_0)\rho}} \right|^2 \right)^{1/2}$$

$$+ C \left( \frac{R}{\rho} \right)^{d + m - 2} \left[ R^{1+\sigma} \left( \frac{\sqrt{\varepsilon}}{R^{m/2}} + 1 \right) \|F\|_{\varepsilon, \sigma, 0, B'_{R_0}} + \left( \frac{\varepsilon}{R^{m}} \right)^{\bar{\beta}(\lambda_1)} R^{1-\tau} \|\nabla \bar{v}\|_{C, \tau, 1, B'_{R_0}} \right],$$
where \( \alpha(\lambda_1) \) is defined by Equation (1.6), and

\[
\beta(\lambda_1) = \begin{cases} \frac{2\alpha(\lambda_1) + d + m - 3}{2m}, & m > d + 2\alpha(\lambda_1) - 3, \\ 0, & \text{any } \alpha < 1, \\ \frac{2\alpha(\lambda_1) - 3}{m}, & m < d + 2\alpha(\lambda_1) - 3. \end{cases}
\] (2.8)

To prove Propositions 2.1 and 2.2, we start by decomposing the solution \( \overline{v} \) of Equation (2.1) as follows:

\[
\overline{v} = \overline{v}_1 + \overline{v}_2, \quad \text{in } B'_R, \quad 0 < R \leq R_0,
\] (2.9)

where \( \overline{v}_i, i = 1, 2 \), respectively, solve

\[
\begin{cases}
\text{div}(\delta \nabla \overline{v}_1) = 0, & \text{in } B'_R, \\
\overline{v}_1 = \bar{v}, & \text{on } \partial B'_R,
\end{cases}
\] (2.10)

and

\[
\begin{cases}
\text{div}(\delta \nabla \overline{v}_2) = \text{div} F, & \text{in } B'_R, \\
\overline{v}_2 = 0, & \text{on } \partial B'_R.
\end{cases}
\] (2.11)

With regard to \( \overline{v}_1 \), we obtain

**Lemma 2.3.** For \( d \geq 3 \), let \( \overline{v}_1 \in H^1(B'_R, |x'|^m \, dx') \) be a solution of Equation (2.10) with \( \varepsilon = 0 \). Then, \( \overline{v}_1 \in C^\beta(B'_R) \) for some \( \beta = \beta(d, m, \| \ln \kappa \|_{L^\infty(S^{d-2})}) \). Furthermore, for any \( 0 < \rho < R \), we have

\[
(\int_{\partial B'_R} \kappa(\frac{\rho}{\rho'}) \left| \overline{v}_1 - \overline{v}_1(0') \right|^2)^{\frac{1}{2}} \leq \left( \frac{\rho}{R} \right)^{\alpha(\lambda_1)} \left( \int_{\partial B'_R} \kappa(\frac{\rho}{\rho'}) \left| \overline{v}_1 - \overline{v}_1(0') \right|^2 \right)^{\frac{1}{2}},
\]

where \( \alpha(\lambda_1) \) is given in Equation (1.6).

**Proof.** To begin with, in view of Theorem 2.3.12 and Section 3 (see pp. 106) in Fabes et al. [31], we derive that \( \overline{v}_1 \in C^\beta(B'_R) \) for some \( \beta = \beta(d, m, \| \ln \kappa \|_{L^\infty(S^{d-2})}) \). Without loss of generality, set \( R = 1 \). Let \( x' = (r, \xi) \in (0, 1) \times S^{d-2} \). Picking a test function \( \phi(r)\psi(\xi) \) with \( \phi(r) \in C^\infty((0, 1)) \) and \( \psi(\xi) \in C^\infty(S^{d-2}) \), it follows from integration by parts that

\[
\int_{B'_1} \kappa(\xi)r^m \nabla \overline{v}_1 \nabla (\phi \psi) = \int_0^1 \int_{S^{d-2}} \kappa r^{m+d-2} \partial_r \overline{v}_1 \phi' \psi + \kappa r^{m+d-4} \nabla_{S^{d-2}} \overline{v}_1 \nabla_{S^{d-2}} \psi \phi \, d\xi \, dr
\]

\[
= - \int_{B'_1} [\kappa r^m \partial_r \overline{v}_1 + \kappa (m + d - 2)r^{m-1} \partial_r \overline{v}_1 + r^{m-2} \text{div}_{S^{d-2}}(\kappa \nabla_{S^{d-2}} \overline{v}_1)] \phi \psi.
\]

Hence \( \overline{v}_1 \) satisfies

\[
\partial_r \overline{v}_1 + \frac{m + d - 2}{r} \partial_r \overline{v}_1 + \frac{1}{\kappa(\xi)r^2} \text{div}_{S^{d-2}}(\kappa(\xi) \nabla_{S^{d-2}} \overline{v}_1) = 0, \quad \text{in } B'_1 \setminus \{0'\}.
\] (2.12)

Set \( \lambda_0 = 0 \) and \( \{\lambda_i\}_{i=1}^\infty \) denotes all the positive eigenvalues of (1.4), satisfying that \( \lambda_i < \lambda_{i+1}, i \in \{0\} \cup \mathbb{N} \). Pick a positive constant \( Y_0 \) such that \( \langle Y_0, Y_0 \rangle_{S^{d-2}} = 1 \). Denote by \( Y_{k,i} \) the corresponding eigenfunction of \( \lambda_k \), satisfying

\[
-\text{div}_{S^{d-2}}(\kappa(\xi) \nabla_{S^{d-2}} Y_{k,i}) = \lambda_k \kappa(\xi) Y_{k,i}, \quad \xi \in S^{d-2}.
\]

Moreover, \( \{Y_0\} \cup \{Y_{k,i}\}_{k,l} \) comprises an orthonormal basis of \( L^2(S^{d-2}) \) under the inner product (1.5).
Decompose $\tilde{v}_1$ as follows:

$$
\tilde{v}_1(r, \xi) = V_0(r)Y_0 + \sum_{k=1}^{\infty} \sum_{l=1}^{N(k)} V_{k,l}(r)Y_{k,l}(\xi), \quad (r, \xi) \in (0,1) \times S^{d-2},
$$

(2.13)

where $V_0(r), V_{k,l}(r) \in C^2(0,1)$ are, respectively, determined by

$$
V_0(r) = \int_{S^{d-2}} \kappa(\xi)\tilde{v}_1(r, \xi)Y_0d\xi, \quad V_{k,l}(r) = \int_{S^{d-2}} \kappa(\xi)\tilde{v}_1(r, \xi)Y_{k,l}(\xi)d\xi.
$$

By using the test functions $\kappa(\xi)Y_0$ and $\kappa(\xi)Y_{k,l}(\xi)$ for Equation (2.12) on $S^{d-2}$, we deduce that

$$
V''_0 + \frac{m+d-2}{r}V'_0 = 0, \quad V''_{k,l} + \frac{m+d-2}{r}V'_{k,l} - \frac{\lambda_k}{r^2}V_{k,l} = 0, \quad 0 < r < 1.
$$

A direct calculation gives that $V_0 = c_1 + c_2r^{3-m-d}$ and $V_{k,l} = c_3r^{\alpha(\lambda_k)+} + c_4r^{\alpha(\lambda_k)+}$ for some constants $c_i$, $i = 1, 2, 3, 4$, where

$$
\alpha(\lambda_k)_\pm := -(m + d - 3) \pm \sqrt{(m + d - 3)^2 + 4\lambda_k}.
$$

Claim that $c_2 = 0$. Otherwise, if $c_2 \neq 0$, then for any $\varphi > 0$,

$$
\int_{B'_R \setminus B'_\varphi} \kappa(\xi)\tilde{v}_1^2r^m dx' \geq \frac{1}{C} \int_{B'_R \setminus B'_\varphi} \kappa(\xi)V_0(r)^2r^m dx'
\geq \frac{1}{C} \int_{\varphi}^{1} |c_1 + c_2r^{3-m-d}|^2r^{m+d-2} dr
\geq \frac{1}{C} \left\{ \begin{array}{ll}
|\ln \varphi|, & m = 2, d = 3, \\
\varphi^{5-m-d}, & \text{otherwise},
\end{array} \right. \to \infty, \quad \text{as} \quad \varphi \to 0.
$$

This contradicts the assumed condition that $\tilde{v}_1 \in H^1(B'_R, |x'|^m dx')$. Therefore, $c_2 = 0$. By the same argument, we also obtain that $c_3 = 0$. Consequently, we have

$$
V_0(\rho) \equiv V_0(1), \quad V_{k,l}(\rho) = \rho^{\alpha(\lambda_k)+}V_{k,l}(1), \quad \rho \in (0,1).
$$

This, together with Equation (2.13), shows that

$$
\int_{\partial B'_\rho} \kappa \left( \frac{x'}{|x'|} \right) |\tilde{v}_1 - \tilde{v}_1(0')|^2 \leq \rho^{2\alpha(\lambda_1)+} \sum_{k=1}^{\infty} \sum_{l=1}^{N(k)} |V_{k,l}(1)|^2
\leq \rho^{2\alpha(\lambda_1)+} \int_{\partial B'_1} \kappa \left( \frac{x'}{|x'|} \right) |\tilde{v}_1 - \tilde{v}_1(0')|^2,
$$

where we used the fact that $\tilde{v}(0') = V_0(0)Y_0$. Moreover, we have from Equation (2.14) that $\tilde{v}(0') = V_0(\rho)Y_0 = (\tilde{v}_1)^{N(k)}_{\partial B'_\rho}$ for any $0 < \rho < 1$. The proof is complete.

For $\tilde{v}_2$, we establish its $L^\infty$ estimate by applying the Moser iteration in the following.
Lemma 2.4. For \( d \geq 3, 1 + \sigma > 0 \), let \( \bar{v}_2 \in H^1(B'_R, |x'|^m dx') \) be a solution of Equation (2.11) with \( \varepsilon = 0 \) and \( R = 1 \). Assume that \( F \in L^\infty(B'_1) \) and \( \|F\|_{\varepsilon, \sigma, 0, B'_1} < \infty \). Then,

\[
\|\bar{v}_2\|_{L^\infty(B'_1)} \leq C\|F\|_{\varepsilon, \sigma, 0, B'_1},
\]

where \( C \) is a positive constant depending only on \( d, m, \sigma \), and \( \chi_i, i = 0, 1, ..., d - 1 \), but not on \( \varepsilon \).

Proof. Without loss of generality, suppose that \( \|F\|_{\varepsilon, \sigma, 0, B'_1} = 1 \). Multiplying Equation (2.11) by \(-|\bar{v}_2|^{p-2}\bar{v}_2\) with \( p \geq 2 \) and integrating by parts, we derive

\[
(p - 1) \int_{B'_1} \kappa \left( \frac{x'}{|x'|^m} \right) |x'|^m |\nabla \bar{v}_2|^2 |\bar{v}_2|^{p-2} = (p - 1) \int_{B'_1} F \cdot \nabla \bar{v}_2 |\bar{v}_2|^{p-2}.
\]

Recall the following discrete version of Hölder’s inequality

\[
\left( \sum_{i=1}^N a_i b_i \right)^2 \leq \left( \sum_{i=1}^N a_i^2 \right) \left( \sum_{i=1}^N b_i^2 \right), \quad a_i, b_i \in \mathbb{R}, \ i = 1, ..., N, \ N \geq 1,
\] (2.15)

and the elemental inequality

\[
a^p + b^p \leq (a + b)^p \leq 2^{p-1} (a^p + b^p), \quad a, b > 0, \ p \geq 1.
\] (2.16)

For simplicity, denote \( \xi = \frac{x'}{|x'|} \in \mathbb{S}^{d-2} \). A combination of Equations (2.15)–(2.16) shows that

\[
\kappa(\xi) \leq \kappa_0 \left( \sum_{i \in A} \chi_i^2 \right)^{m \over 2} \left( \sum_{i \in A} |\xi_i|^{m \over 2} \right) + \left( \sum_{j \in B} \chi_j^2 \right)^{1 \over 2} \left( \sum_{j \in B} |\xi_j|^{m \over 2} \right)^{m \over 2}.
\]

\[
\leq \kappa_0 \left( \sum_{i \in A} \chi_i^2 \right)^{m \over 2} \left( \sum_{i \in A} |\xi_i| \right) + \left( \sum_{j \in B} \chi_j^2 \right)^{1 \over 2} \left( \sum_{j \in B} |\xi_j| \right)^{m \over 2}.
\]

\[
\leq \left[ \kappa_0 \left( \sum_{i \in A} \chi_i^2 \right)^{m \over 2} + \sum_{j \in B} \chi_j^2 \right] \left[ \left( \sum_{i \in A} |\xi_i| \right) + \left( \sum_{j \in B} |\xi_j| \right)^{m \over 2} \right],
\] (2.17)

and

\[
\kappa(\xi) \geq \kappa_0 \min_{i \in A} \kappa_i^2 \left( \sum_{i \in A} |\xi_i| \right)^{m \over 2} + \min_{j \in B} \chi_j \sum_{j \in B} |\xi_j|^{m \over 2}.
\]

\[
\geq \min \left\{ \kappa_0 \min_{i \in A} \chi_i^2, \min_{j \in B} \chi_j \right\} \left[ \left( \sum_{i \in A} |\xi_i| \right)^{m \over 2} + \sum_{j \in B} |\xi_j|^{m \over 2} \right],
\]

\[
\geq 2^{m-2 \over 2} \min \left\{ 2^{- \frac{(m-2)(m-1)}{2}}, 1 \right\} \min \left\{ \kappa_0 \min_{i \in A} \chi_i^2, \min_{j \in B} \chi_j \right\},
\] (2.18)
where in the last line, we used the fact that
\[
\left( \sum_{i \in A} |\xi_i|^2 \right)^{\frac{m}{2}} + \sum_{j \in B} |\xi_j|^m \geq \min \left\{ 2^{-\frac{(m-2)(b-1)}{2}}, 1 \right\} \left[ \left( \sum_{i \in A} |\xi_i|^2 \right)^{\frac{m}{2}} + \left( \sum_{j \in B} |\xi_j|^2 \right)^{\frac{m}{2}} \right] 
\geq 2^{-\frac{m-2}{2}} \min \left\{ 2^{-\frac{(m-2)(b-1)}{2}}, 1 \right\},
\]
with \( b := \text{card}(B) \) representing the number of elements in set \( B \). Combining Equations (2.17) and (2.18), we obtain
\[
\varphi : = \theta_3^{-m/(m-1)} \leq \kappa \left( \frac{x'}{|x'|} \right) \leq \theta_1,
\]
where \( \theta_1 \) and \( \theta_3 \) are defined by Equations (2.3) and (2.5), respectively. Note that \( |F| \leq |x'|^{\sigma + m} \) in \( B'_1 \). In light of \( 1 + \sigma > 0 \), we then deduce from Young’s inequality and Hölder’s inequality that
\[
\left\| \int_{B'_1} F \cdot \nabla \tilde{v}_2 |\tilde{v}_2|^{p-2} \right\| \leq \frac{\varphi}{2} \int_{B'_1} |x'|^m |\nabla \tilde{v}_2|^2 |\tilde{v}_2|^2 + C \int_{B'_1} |x'|^{2\sigma + m} |\tilde{v}_2|^{p-2}
\leq \frac{\varphi}{2} \int_{B'_1} |x'|^m |\nabla \tilde{v}_2|^2 |\tilde{v}_2|^2 + C \left\| \tilde{v}_2 \right\|^{p-2} \left\| \frac{d}{d+m-1+2\mu} \left( B'_1, |x'|^m dx' \right) \right\|^2
\leq \frac{\varphi}{2} \int_{B'_1} |x'|^m |\nabla \tilde{v}_2|^2 |\tilde{v}_2|^2 + C \left\| \tilde{v}_2 \right\|^{p-2} \left\| \frac{d}{d+m-1+2\mu} \left( B'_1, |x'|^m dx' \right) \right\|^2,
\]
where \( \mu \) is chosen sufficiently small to ensure that
\[
\left( \int_{B'_1} |x'|^{\sigma(d+m-1+2\mu)+m} \right)^{\frac{2}{d+m-1+2\mu}} < \infty.
\]
Consequently, combining these above facts, we deduce
\[
\frac{4}{P^2} \int_{B'_1} |x'|^m |\nabla |\tilde{v}_2|^2 |^2 = \int_{B'_1} |x'|^m |\nabla \tilde{v}_2|^2 |\tilde{v}_2|^2 \geq \left\| \tilde{v}_2 \right\|^{p-2} \left\| \frac{d}{d+m-1+2\mu} \left( B'_1, |x'|^m dx' \right) \right\|^2.
\]
We will utilize the Caffarelli–Kohn–Nirenberg inequality in Caffarelli et al. [32] having the following form:
\[
\|u\|_{L^\infty \left( B'_1, |x'|^m dx' \right)} \leq C \|\nabla u\|_{L^3 \left( B'_1, |x'|^m dx' \right)}, \quad \forall u \in H^1_0 \left( B'_1, |x'|^m dx' \right).
\]
Picking \( p = 2 \) in Equation (2.20) and applying Equation (2.21) with \( u = |\tilde{v}_2| \), we have from Hölder’s inequality that
\[
\left\| |\tilde{v}_2| \right\|_{L^2 \left( B'_1, |x'|^m dx' \right)} \leq C.
\]
For $p \geq 2$, applying Equation (2.21) with $u = |\bar{v}_2|^p$ and using Hölder’s inequality again, we obtain from Equation (2.20) that
\[
\|\bar{v}_2\|_{L^p(B'_1, |x'|^{m}dx')} \leq C\|\nabla |\bar{v}_2|^p\|_{L^2(B'_1, |x'|^{m}dx')} \leq C P^2 \|\bar{v}_2\|_{L^{(d+m-3+2\mu)p}(B'_1, |x'|^{m}dx')} \leq C P^2 \|\bar{v}_2\|_{L^{(d+m-3+2\mu)p}(B'_1, |x'|^{m}dx')},
\]
which, in combination with Young’s inequality, reads that
\[
\|\bar{v}_2\|_{L^p(B'_1, |x'|^{m}dx')} \leq \left( C P^2 \right)^{1/p} \left( \|\bar{v}_2\|_{L^{(d+m-1+2\mu)p}(B'_1, |x'|^{m}dx')} + \frac{2}{p} \right).
\]
Denote
\[
P_k = \frac{2(d + m - 1 + 2\mu)}{d + m - 3 + 2\mu} \left( \frac{d + m - 1}{d + m - 3} \cdot \frac{d + m - 3 + 2\mu}{d + m - 1 + 2\mu} \right)^k, \quad k \geq 0.
\]
After $k$ iterations, it follows from Equation (2.22) that
\[
\|\bar{v}_2\|_{L^{p_k}(B'_1, |x'|^{m}dx')} \leq \prod_{i=0}^{k-1} (C P_i^2)^{1/p_i} \|\bar{v}_2\|_{L^{p_i}(B'_1, |x'|^{m}dx')} + \sum_{i=0}^{k-1} \prod_{j=0}^{k-1-i} (C P_{k-1-j}^2)^{1/p_{k-1-j}} \frac{2}{P_i} \leq C \|\bar{v}_2\|_{L^{(d+m-1+2\mu)p}(B'_1, |x'|^{m}dx')} + C \sum_{i=0}^{k-1} \frac{1}{P_i} \leq C,
\]
where $C = C(d, m, \sigma)$. Sending $k \to \infty$, we complete the proof of Lemma 2.4.

In order to prove Proposition 2.2, we also need the following lemma.

**Lemma 2.5.** Let $\bar{w} \in H^1_0(B'_1, |x'|^{m+\beta}dx')$ with $d \geq 3$, $m \geq 2$, and $\beta < 1$. Then we have
\[
\sup_{0 < r < 1} r^{d+m-2-d-\beta} \int_{\partial B'_r} |\bar{w}|^2 \leq C \int_{B'_1} |x'|^{m+\beta} |\nabla \bar{w}|^2,
\]
where $C = C(d, m, \beta)$.

**Proof.** By denseness, it suffices to consider $\bar{w} \in C^1(B'_1)$. Utilizing Hölder’s inequality and the Fubini theorem, we obtain
\[
\int_{\partial B'_r} |\bar{w}|^2 \leq \int_{S^{d-2}} r^{d+m-2} |\bar{w}(r, \xi)|^2 d\xi \leq \int_{S^{d-2}} r^{d+m-2} \left( \int_r^{1} |\partial_s \bar{w}(s, \xi)|^2 d\xi \right)^2 d\xi \leq \int_{S^{d-2}} r^{d+m-2} \left( \int_r^{1} |\partial_s \bar{w}(s, \xi)|^2 s^\beta d\xi \right) \left( \int_r^{1} s^{-\beta} d\xi \right) d\xi \leq C \int_0^{1} \int_{S^{d-2}} s^{d+m-2+\beta} |\partial_s \bar{w}(s, \xi)|^2 ds d\xi \leq C \int_{B'_1} |x'|^{m+\beta} |\nabla \bar{w}|^2 dx'.
\]
The proof is complete. \[\square\]
We are now ready to use Lemmas 2.3–2.5 to give the proofs of Propositions 2.1 and 2.2.

**Proof of Propositions 2.1 and 2.2.** We divide into two parts to complete the proofs.

**Part 1.** Consider the case when $\varepsilon = 0$. Let $\tilde{v}(0') = 0$ and $\|F\|_{0,\sigma,0,B'_0} = 1$ without loss of generality. For $0 < \rho \leq R \leq R_0$, write

$$\omega(\rho) := \left( \int_{\partial B'_\rho} \kappa \left( \frac{x'}{|x'|} \right) |\tilde{v} - \tilde{v}_1(0')|^2 \right)^{\frac{1}{2}}.$$

Denote $\tilde{v}_2(x') := \tilde{v}_2(Rx')$ and $F(y') := R^{-(m-1)}F(Ry')$. From Equation (2.11), we obtain that $\tilde{v}_2$ solves

$$\text{div} \left[ \kappa \left( \frac{x'}{|x'|} \right) |x'|^m \nabla \tilde{v}_2 \right] = \text{div} F, \quad \text{in } B'_1,$$

where $\|\tilde{F}\|_{0,\sigma,0,B'_1} = R^{1+\sigma} \|F\|_{0,\sigma,\gamma,B'_k}$. Applying Lemma 2.4 to $\tilde{v}_2$, we have

$$\|\tilde{v}_2\|_{L^\infty(B'_R)} \leq \omega(R).$$

This, together with Equation (2.9) and Lemma 2.3, shows that

$$\omega(\rho) \leq \left( \int_{\partial B'_\rho} \kappa \left( \frac{x'}{|x'|} \right) |\tilde{v}_1 - \tilde{v}_1(0')|^2 \right)^{\frac{1}{2}} + \left( \int_{\partial B'_\rho} \kappa \left( \frac{x'}{|x'|} \right) |\tilde{v}_2 - \tilde{v}_2(0')|^2 \right)^{\frac{1}{2}}$$

$$\leq \left( \frac{\rho}{R} \right)^{\alpha(\lambda_1)} \left( \int_{\partial B'_\rho} \kappa \left( \frac{x'}{|x'|} \right) |\tilde{v}_1|^2 \right)^{\frac{1}{2}} + \left( \frac{\rho}{R} \right)^{\alpha} |\tilde{v}_1(0')| + 2 \|\tilde{v}_2\|_{L^\infty(B'_R)}$$

$$\leq \left( \frac{\rho}{R} \right)^{\alpha(\lambda_1)} \omega(R) + CR^{1+\sigma}, \quad (2.23)$$

where the facts that $\tilde{v} = \tilde{v}_1$ on $\partial B'_\rho$ and $|\tilde{v}_1(0')| = |\tilde{v}_2(0')|$ were utilized. Pick $\rho = 2^{-i-1}R_0$ and $R = 2^{-i}R_0$ in Equation (2.23) with $i = 0, \ldots, k-1, k$ is a positive integer. After $k$ iterations and in light of $1 + \sigma \neq \alpha(\lambda_1)$, we obtain

$$\omega(2^{-k}R_0) \leq 2^{-k\alpha(\lambda_1)} \omega(R_0) + C \sum_{i=1}^{k} 2^{-(k-i)\alpha(\lambda_1)} (2^{1-i}R_0)^{1+\sigma}$$

$$\leq 2^{-k\alpha(\lambda_1)} \omega(R_0) + C2^{-k\alpha(\lambda_1)}R_0^{1+\sigma} \frac{1 - 2k(\sigma(\lambda_1)-1-\sigma)}{1 - 2^\alpha(\lambda_1)-1-\sigma}$$

$$\leq 2^{-k\alpha(\lambda_1)} \left( \omega(R_0) + CR^{1+\sigma} \right),$$

where $\tilde{\alpha}(\lambda_1)$ is given by Equation (2.2). Observe that for any $\rho \in (0,R_0)$, there exists some integer $k$ such that $\rho \in (2^{-k-1}R_0, 2^{-k}R_0]$. Then we derive

$$\omega(\rho) \leq C\rho^{\tilde{\alpha}(\lambda_1)}, \quad \text{for any } \rho \in (0,R_0).$$

This, together with Equation (2.19), yields that Proposition 2.1 holds.

**Part 2.** Consider the case when $\varepsilon > 0$. Without loss of generality, assume that $\tilde{v}(0') = 0$. For simplicity, denote $\alpha := \alpha(\lambda_1)$ and $\beta := \beta(\lambda_1)$, which are, respectively, given by Equations (1.6) and (2.8). To begin with, from the mean value formula, we know

$$|\tilde{v}(x')| \leq |x'|^{1-\tau} \|\nabla \tilde{v}\|_{L^\infty(\varepsilon,\varepsilon,1,B'_R)} \quad \text{in } B'_R,$$

$$\text{in } B'_R.$$
For any $0 < R < R_0$, denote $\bar{\vartheta}_2 := \bar{\vartheta} - \bar{\vartheta}_1$, where $\bar{\vartheta} \in H^1(B'_R)$ solves Equation (2.1) with $\varepsilon > 0$ and $\bar{\vartheta}_1 \in H^1(B'_R)$, $|x'|^m \, dx'$ satisfies Equation (2.10) with $\varepsilon = 0$. From Lemma 2.3 and Equation (2.24), we obtain

$$\left( \int_{\partial B'_\rho} |\bar{\vartheta}_1 - \bar{\vartheta}_1(0')|^2 \right)^{\frac{1}{2}} \leq C \left( \frac{\rho}{R} \right)^{\alpha} R^{1-\tau} \, \|\nabla \bar{\vartheta}\|_{\varepsilon, -\tau, 1, B'_{R0}}, \quad 0 < \rho < R.$$  

For $0 < \rho < R$, applying the interior estimate in $B'_R \setminus B'_{R/2}$, we derive

$$|\nabla \bar{\vartheta}_1(x')| \leq C|x'|^{\alpha-1} R^{1-\tau} \, \|\nabla \bar{\vartheta}\|_{\varepsilon, -\tau, 1, B'_{R0}}.$$ (2.25)

Combining Lemma 2.3, Equation (2.24), and the maximum principle, we have

$$\|\bar{\vartheta}_1\|_{L^\infty(B'_R)} = \sup_{x' \in \partial B'_R \cup \{0'\}} |\bar{\vartheta}_1(x')| \leq CR^{-\tau} \, \|\nabla \bar{\vartheta}\|_{\varepsilon, -\tau, 1, B'_{R0}}.$$ (2.26)

It then follows from the boundary estimate that

$$|\nabla \bar{\vartheta}_1(x')| \leq CR^{1-\tau} \|\nabla \bar{\vartheta}\|_{\varepsilon, -\tau, 1, B'_{R0}}.$$ (2.27)

From Equations (2.25)–(2.27), we see that $\bar{\vartheta}_1 \in H^1(B'_R)$. Hence $\bar{\vartheta}_2 \in H^1(B'_R)$ verifies

$$\text{div} \left[ \varepsilon + \kappa \left( \frac{x'}{|x'|} \right) |x'|^m \right] \nabla \bar{\vartheta}_2 = \text{div} F - \varepsilon \Delta \bar{\vartheta}_1, \quad \text{for } x' \in B'_R.$$ (2.29)

Denote $\bar{\vartheta}_i(y') = \bar{\vartheta}_i(Ry')$, $i = 1, 2$, $F(y') = R^{-(m-1)}F(Ry')$, and $\bar{\varepsilon} = \varepsilon R^{-m}$. Then we have

$$\|F\|_{\varepsilon, \sigma, 0, B'_{R}} = R^{1+\sigma} \|F\|_{\varepsilon, \sigma, 0, B'_{R}}, \quad \|\nabla \bar{\vartheta}_1\|_{\varepsilon, \alpha, 1, 1, B'_{R}} = R^2 \|\nabla \bar{\vartheta}_1\|_{\varepsilon, \alpha, 1, 1, B'_{R}},$$ (2.28)

and

$$\text{div} \left[ \bar{\varepsilon} + \kappa \left( \frac{x'}{|x'|} \right) |x'|^m \right] \nabla \bar{\vartheta}_2 = \text{div} F - \bar{\varepsilon} \Delta \bar{\vartheta}_1, \quad \text{for } x' \in B'_1.$$ (2.29)

From Equations (2.25) and (2.27)–(2.28), we obtain

$$\|\nabla \bar{\vartheta}_1\|_{\varepsilon, \alpha, 1, 1, B'_{R}} \leq CR^{1-\tau} \|\nabla \bar{\vartheta}\|_{\varepsilon, -\tau, 1, B'_{R0}}.$$ (2.30)

Set $x' = (r, \xi) \in (0, 1) \times S^{d-2}$. Then multiplying Equation (2.29) by $\bar{\vartheta}_2$, we have from integration by parts that

$$\int_{B'_1} (\bar{\varepsilon} + \kappa(\xi) \rho^m) |\nabla \bar{\vartheta}_2|^2 = \int_{B'_1} F \cdot \nabla \bar{\vartheta}_2 - \bar{\varepsilon} \int_{B'_1} \nabla \bar{\vartheta}_1 \cdot \nabla \bar{\vartheta}_2.$$  

For simplicity, denote $\|F\| = \|F\|_{\varepsilon, \sigma, 0, B'_{R}}$ and $\|\nabla \bar{\vartheta}_1\| = \|\nabla \bar{\vartheta}_1\|_{\varepsilon, \alpha, 1, 1, B'_{R}}$. In view of Equation (2.19) and $2\sigma + n - 2 > -1$, it follows from Young’s inequality and Lemma 2.5 that

$$\sup_{0 < r < 1} r^{d+m-2} \int_{B'_{R_0}} |\bar{\vartheta}_2|^2 \leq \int_{B'_1} (\bar{\varepsilon} + \rho^m |\nabla \bar{\vartheta}_2|^2 \leq C \|\bar{\varepsilon}\|^2 \int_{B'_1} r^{2\sigma}(\bar{\varepsilon} + \rho^m) + C \|\nabla \bar{\vartheta}_1\|^2 \int_{B'_1} \bar{\varepsilon}^{2\alpha-2} + r^m \rho^m \leq C \|\bar{\varepsilon}\|^2 (\bar{\varepsilon} + 1) + C \|\nabla \bar{\vartheta}_1\|^2 \bar{\varepsilon}^{2\beta}.$$  

This, together with Equations (2.28) and (2.30), gives that for $0 < \rho < R$,}
\[
\int_{\partial B'_\rho} |\bar{\nu}_1|^2 \leq C \left( \frac{R}{\rho} \right)^{d+2m-2} R^{2+2\sigma} \left( \frac{\varepsilon}{R^m} + 1 \right) \|F\|_{\varepsilon, \sigma, 0, B'_{R_0}}^2 + C \left( \frac{R}{\rho} \right)^{d+2m-2} \left( \frac{\varepsilon}{R^m} \right)^{2\beta} R^{2-2\tau} \|\bar{\nu}\|_{\varepsilon, r, 1, B'_{R_0}}^2.
\]

(2.31)

Then combining Lemma 2.3 and Equation (2.31), it follows that for any \(0 < \rho < (1 - \bar{c}_0)^2R\),

\[
\int_{\partial B'_\rho} \kappa \left( \frac{x'}{|x'|} \right) |\bar{\nu}(x') - \bar{\nu}_1(0')|^2 
\leq 2 \int_{\partial B'_\rho} \kappa \left( \frac{x'}{|x'|} \right) |\bar{\nu}_1(x') - \bar{\nu}_1(0')|^2 + 2 \int_{\partial B'_\rho} \kappa \left( \frac{x'}{|x'|} \right) |\bar{\nu}_2(x')|^2 
\leq C \left( \frac{R}{\rho} \right)^{2\alpha} \int_{\partial B'_R} \kappa \left( \frac{x'}{|x'|} \right) |\bar{\nu}(x') - (\bar{\nu})_{B'_R}|^2 
+ C \left( \frac{R}{\rho} \right)^{d+2m-2} \left[ R^{2+2\sigma} \left( \frac{\varepsilon}{R^m} + 1 \right) \|F\|_{\varepsilon, \sigma, 0, B'_{R_0}}^2 + \left( \frac{\varepsilon}{R^m} \right)^{2\beta} R^{2-2\tau} \|\bar{\nu}\|_{\varepsilon, r, 1, B'_{R_0}}^2 \right].
\]

Multiplying the above by \(\rho^{d-2}\) and integrating from \((1 - \bar{c}_0)R\) to \((1 + \bar{c}_0)R\), we deduce that for any \(0 < \rho < (1 - \bar{c}_0)r\) \(\leq (1 - \bar{c}_0^2)R\),

\[
\int_{B'(1 \pm \bar{c}_0)\rho} \kappa \left( \frac{x'}{|x'|} \right) \left| \bar{\nu}(x') - \bar{\nu}(B'_{1 \pm \bar{c}_0}) \right|^2 
\leq C \left( \frac{R}{\rho} \right)^{2\alpha} \int_{B'(1 \pm \bar{c}_0)R} \kappa \left( \frac{x'}{|x'|} \right) |\bar{\nu}(x') - \bar{\nu}_1(0')|^2 
+ C \left( \frac{R}{\rho} \right)^{d+2m-2} \left[ R^{2+2\sigma} \left( \frac{\varepsilon}{R^m} + 1 \right) \|F\|_{\varepsilon, \sigma, 0, B'_{R_0}}^2 + \left( \frac{\varepsilon}{R^m} \right)^{2\beta} R^{2-2\tau} \|\bar{\nu}\|_{\varepsilon, r, 1, B'_{R_0}}^2 \right].
\]

where \(B'_{(1 \pm \bar{c}_0)\rho}\) is defined by Equation (2.7). Therefore, multiplying it by \(r^{d-2}\) and integrating from \((1 - \bar{c}_0)R\) to \((1 + \bar{c}_0)R\), we complete the proof of Proposition 2.2. \(\square\)

For any given \(x_0' = (x_1', \ldots, x_d') \in B'_{R_0/2}\), denote

\[
\delta(x_0') := \varepsilon + \kappa_0 \left( \sum_{i \in A} \kappa_i |x_i'|^2 \right)^{\frac{m}{2}} + \sum_{j \in B} \kappa_j |x_j'|^m.
\]

For \(s, t > 0\) and \(x' \in B'_{R_0}\), denote by \(Q_{s,t}(x')\) the cylinder as follows:

\[
Q_{s,t}(x') := \{y = (y', y_d) \in \mathbb{R}^d \mid |y' - x'| < s, |y_d| < t\}.
\]

For simplicity, let \(Q_{s,t} := Q_{s,t}(0')\) if \(x' = 0'\). For the convenience of presentation, in the following the domain notations such as \(Q_{s,t}(x'), \Omega_{s}(x'),\) and \(B'_t\) are used in the sense that \(Q_{s,t}(x') \setminus \{y' = 0'\}, \Omega_{s}(x') \setminus \{y' = 0'\}\) and \(B'_t \setminus \{0'\}\) if \(\varepsilon = 0\) and \(x' \neq 0'\).
Similar to Equations (2.15)–(2.19), a direct computation gives that if $\varepsilon = 0$ and $x'_0 \neq 0'$, or if $\varepsilon > 0$ and $|x'_0| \geq \varepsilon^{1/m}$,

$$\theta^{-m/(m-1)}|x'_0|^m \leq \delta \leq (1 + \theta_1)|x'_0|^m,$$

(2.32)

where $\theta_1$ and $\theta_3$ are, respectively, given by Equations (2.3) and (2.5). In view of the value of $c_0$ given in Equation (2.6), we know that $c_0 \leq \frac{1}{4}(1 + \theta_1)^{-1/m}$. This, together with Equation (2.32), shows that

$$|x'_0| - 2c_0\delta^{1/m} \geq \frac{1}{2}|x'_0|, \text{ if } \varepsilon = 0 \text{ and } x'_0 \neq 0', \text{ or if } \varepsilon > 0 \text{ and } |x'_0| \geq \varepsilon^{1/m}. \quad (2.33)$$

Observe that

$$\sum_{i=1}^{N} a_i^p \leq \left(\sum_{i=1}^{N} a_i\right)^p, \quad a_i \geq 0, \ i = 1, \ldots, N, \ N \geq 1, \ p \geq 1,$$

which, together with Equation (2.15), reads that

$$|\nabla x' \delta(x')|^2 = m^2 \kappa_0^2 \sum_{i \in A} x_i^2 |x_i|^2 \left(\sum_{i \in A} x_i^2\right)^{m-2} + m^2 \sum_{j \in B} x_j^2 |x_j|^{2(m-1)}$$

$$\leq m^2 \kappa_0^2 \left[\left(\sum_{i \in A} x_i^4\right)^{1/2} \left(\sum_{i \in A} x_i^2\right)^{m-2} \left(\sum_{i \in A} x_i^4\right)^{1/2}ight]$$

$$+ m^2 \left(\sum_{j \in B} x_j^2\right)^{1/2} \left(\sum_{j \in B} |x_j|^{4(m-1)}\right)^{1/2}$$

$$\leq m^2 \theta_2^2 \left[\left(\sum_{i \in A} x_i^4\right)^{m-1} + \left(\sum_{j \in B} x_j^2\right)^{2(m-1)}\right]^1\frac{1}{2}$$

$$\leq m^2 \theta_2^2 |x'|^{2(m-1)}, \quad (2.34)$$

where $\theta_2$ is given by Equation (2.4). Recalling the value of $c_0$ given by Equation (2.6), it follows from Equations (2.32) and (2.34) that for $0 < s \leq 2c_0\delta^{1/m}$, $x = (x', x_d) \in \Omega_s(x'_0)$,

$$|\delta(x') - \delta(x'_0)| \leq m\theta_2 |x'_0|^{m-1}|x' - x'_0|$$

$$\leq m^2 |x'_0|^{m-1} \theta_2 s \left(\sum_{j \in B} x_j^2\right)^{2(m-1)}$$

$$\leq m^2 |x'_0|^{m-1} \theta_2 ((2c_0)^{m-1} + \theta_3) c_0 \leq \frac{\delta(x'_0)}{2},$$

where $x'_0$ is some point between $x'$ and $x'_0$. This leads to that

$$\frac{1}{2} \delta(x'_0) \leq \delta(x') \leq \frac{3}{2} \delta(x'_0), \text{ in } \Omega_s(x'_0). \quad (2.35)$$

From Equation (2.35), we give a precise description for the equivalence of the height for small narrow region $\Omega_s(x'_0)$. 


For $\varepsilon \geq 0$, let
\[
\begin{aligned}
y' &= x', \\
y_d &= 2\delta_0 \left( x_d - h_2(x') + \varepsilon/2 \overline{e} + h_1(x') - h_2(x') - 1 \overline{e}/2 \right).
\end{aligned}
\tag{2.36}
\]

Under the change of variables (2.36), $\Omega_{R_0}$ becomes $Q_{R_0,\delta_0}$. It is worth pointing out that by making use of the rescaling argument for every height line segment $\delta_0$ of the thin gap $\Omega_{R_0}/2$ in Equation (2.36), we will present the proof for Theorem 1.1 in a different style (by contrast with the proofs of Theorems 1.1 and 1.3 in pages 15–22 of Dong et al. [1]). Our style of proof requires the exact value of every aforementioned parameter, including $\delta_i, i = 1, 2, 3, c_0$ and $\overline{c}_0$ defined by Equations (2.3)–(2.7). In the following, these parameters make us succeed to apply the "flipping argument" created in Bao et al. [16] to a small neighborhood centered at every point of the considered narrow region (specially, if $\varepsilon = 0$, pick $x'_0 \neq 0'$, while if $\varepsilon > 0$, choose $|x'_0| \geq \varepsilon^{1/m}$). Based on the above analysis, our style may be more favorable to deepen the readers’ understanding on the idea and schemes developed in Refs. [1, 16, 17].

Set $v(y) = u(x)$. In view of Equation (1.3), we obtain that $v$ verifies
\[
\begin{aligned}
-\partial_i (b_{ij}(y) \partial_j v(y)) &= 0, \quad \text{in } Q_{R_0,\delta_0}, \\
b_{dj}(y) \partial_j v(y) &= 0, \quad \text{on } \{y_d = \pm \delta_0\},
\end{aligned}
\tag{2.37}
\]
with $\|v\|_{L^\infty(Q_{R_0,d})} \leq 1$ and
\[
(b_{ij}(y)) = \frac{2\delta_0 (\partial_{x,y}(A_{ij} - \delta_{ij})(\partial_{x,y})^t)}{\det(\partial_{x,y})}
\begin{pmatrix}
\delta & 0 & \ldots & 0 \\
0 & \delta & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & \delta \\
\end{pmatrix}
\begin{pmatrix}
b_{1d} \\
b_{2d} \\
\vdots \\
b_{d-1d} \\
\end{pmatrix}
+ \frac{2\delta_0 (\partial_{x,y}(A_{ij} - \delta_{ij})(\partial_{x,y})^t)}{\det(\partial_{x,y})}
\begin{pmatrix}
e_1 & 0 & \ldots & 0 \\
0 & e_2 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & e_d \\
\end{pmatrix}
+ \frac{2\delta_0 (\partial_{x,y}(A_{ij} - \delta_{ij})(\partial_{x,y})^t)}{\det(\partial_{x,y})}
\begin{pmatrix}
c_{11} & c_{12} & \ldots & c_{1d} \\
c_{21} & c_{22} & \ldots & c_{2d} \\
\vdots & \vdots & \ddots & \vdots \\
c_{d1} & c_{d2} & \ldots & c_{dd} \\
\end{pmatrix}
\frac{4\delta_0^2 \sum_{i=1}^{d-1} a_{id}^2}{\delta}
\]
where elements satisfy that for $i = 1, \ldots, d - 1$, using conditions (H1) and (H2),
\[
|b_{1d}| = |b_{2d}| = | - 2\delta_0 \overline{e} \partial_{y_2}(y') - (y_d + \delta_0) \partial_{y_1}(h_1 - h_2)(y')| \leq C\delta_0 |y'|^{m-1},
\tag{2.38}
\]
and
\[
|e_i| = |O(|y'|^{m+y})| \leq C|y'|^{m+y}, \quad |e_d| \leq C\delta_0^2 |y'| y' \delta^{-1},
\tag{2.39}
\]
and every $c_{ij}$ is the element of the matrix $\frac{2\delta_0 (\partial_{x,y}(A_{ij} - \delta_{ij})(\partial_{x,y})^t)}{\det(\partial_{x,y})}$, satisfying that for $i, j = 1, \ldots, d - 1$,
\[
|c_{ij}| \leq C\delta |y'|^{l} + \delta'), \quad |c_{id}| \leq C\delta_0 (|y'| y' + \delta').
\]

In light of (1.11), a straightforward computation leads to that
\[
|\nabla_{y'} v| \leq \delta^{-1/m}, \quad |\partial_d v| \leq C\delta_0^{-1} \delta^{1-1/m}, \quad \text{in } Q_{R_0,\delta_0},
\tag{2.40}
\]
Denote
\[
\overline{v}(y') := \int_{-\delta_0}^{\delta_0} v(y', y_d) dy_d.
\tag{2.41}
\]
Thus $\hat{v}$ solves
\[
\text{div}(\delta V \hat{v}) = \text{div} F, \quad \text{in } B'_{R_0},
\] (2.42)

where $F = (F_1, \ldots, F_{d-1})$, $F_i := -b_i d \hat{v} - e_i \partial_d \hat{v} - \sum_{j=1}^{d} \frac{c_{ij}}{\delta} \partial_j \hat{v}$ for $i = 1, \ldots, d - 1$, $b_i d \hat{v}$ and $c_{ij} \partial_j \hat{v}$ denote, respectively, the averages of $b_i d \hat{v}$ and $c_{ij} \partial_j \hat{v}$ with respect to $y_d$ on $(-\delta_0, \delta_0)$. Utilizing Equations (1.11) and (2.38)–(2.40), we deduce that for $i = 1, \ldots, d - 1$,
\[
|F_i| \leq C (|y'|^{|y'|} \delta^{1-1/m} + \delta^{1+|y'|^{-1/m}}), \quad \text{in } B'_{R_0}.
\] (2.43)

We are now ready to prove Theorem 1.1.

**Proof of Theorem 1.1.** Suppose that $u(0) = 0$ and $\|u\|_{L^\infty(\Omega_{R_0})} = 1$ without loss of generality. Let $v$ and $\hat{v}$ be defined by Equations (2.37) and (2.41)–(2.42), respectively. First, we see from Equations (2.40), (2.41), and (2.43) that for $\varepsilon \geq 0$,
\[
\|\nabla \hat{v}\|_{\varepsilon, -m\sigma_0, 0, B'_{R_0}} < \infty, \quad \|F\|_{\varepsilon, -m\sigma_0, 0, B'_{R_0}} < \infty, \quad \text{with } \sigma_0 = \frac{1}{m}.
\]

Case 1. Consider the case when $\varepsilon = 0$. Applying Proposition 2.1 with $\sigma = \gamma - m\sigma_0$ and $\tau = m\sigma_0$, we obtain that for $0 < \rho < R \leq R_0$,
\[
\left( \int_{\partial B'_{\rho}} |\hat{v}(x') - \hat{v}(0')|^2 \right)^{1/2} \leq C \rho^{\tilde{\alpha}},
\] (2.44)

where $\tilde{\alpha} = \min\{\alpha(\lambda_1), 1 + \gamma - m\sigma_0\}$. In light of Equation (2.33), it then follows from Equation (2.44) that for $x_0' \in B'_{R_0/2} \setminus \{0'\}$,
\[
\int_{B'_{2c_0\delta_{0}^{1/m}(x_0')}} |\hat{v} - \hat{v}(0')|^2 \leq \int_{B'_{|x_0'|+2c_0\delta_{0}^{1/m}(0')}} |\hat{v} - \hat{v}(0')|^2 \leq C \delta_0^{\frac{2\alpha + d - 1}{m}}.
\] (2.45)

From Equation (2.40), we know
\[
|v(y', y_d) - \hat{v}(y')| \leq 2\delta_0 \max_{y_d \in (-\delta_0, \delta_0)} |\partial_d v(y', y_d)| \leq C \delta^{1-1/m}, \quad \text{in } Q_{R_0, \delta_0},
\] (2.46)

A consequence of Equation (2.45) and (2.46) shows that
\[
\int_{Q_{2c_0\delta_{0}^{1/m}, \delta_0}(x_0')} |v - \hat{v}(0')|^2 dy \leq \int_{Q_{2c_0\delta_{0}^{1/m}, \delta_0}(x_0')} 2(|v - \hat{v}|^2 + |\hat{v} - \hat{v}(0')|^2) dy \leq C \delta_0^{\frac{2\alpha}{m}}.
\]

Let
\[
\hat{u}(y) = v(\delta_{0}^{1/m} y' + x_0', \delta_{0}^{1/m} y_d) - \hat{v}(0'),
\]
\[
\hat{b}_{ij}(y) = \delta_{0}^{-1} b_{ij}(\delta_{0}^{1/m} y' + x_0', \delta_{0}^{1/m} y_d).
\]

Observe that $Q_{2c_0\delta_{0}^{1/m}, \delta_0}(x_0') \subset Q_{R_0, \delta_0}$ for $x_0' \in B'_{R_0/2} \setminus \{0'\}$, then $\hat{u}$ satisfies
\[
\begin{aligned}
-\delta_l (\hat{b}_{ij}(y) \delta_j \hat{u}(y)) &= 0, \quad \text{in } Q_{2c_0\delta_{0}^{1-1/m}}, \\
\hat{b}_{ij}(y) \partial_j \hat{u}(y) &= 0, \quad \text{on } \{y_d = \pm \delta_{0}^{1-1/m}\}.
\end{aligned}
\] (2.47)
From Equation (2.35), it follows that \( \tilde{b} := (\tilde{b}_{ij}) \) verifies
\[
\frac{L}{C} \leq \tilde{b} \leq CI, \quad \text{and} \quad \|\tilde{b}\|_{C^\mu(Q_{2\tilde{c}0\delta_0^{1-1/m}})} \leq C, \quad \text{for any } \mu \in (0, 1].
\]
(2.48)
For any \( l \in \mathbb{N} \), define
\[
S_l := \{ y \in \mathbb{R}^d \mid |y'| < 2\tilde{c}_0, (2l - 1)\delta_0^{1-1/m} < y_d < (2l + 1)\delta_0^{1-1/m} \},
\]
and
\[
S := \{ y \in \mathbb{R}^d \mid |y'| < 2\tilde{c}_0, |y_d| < 2\tilde{c}_0 \}.
\]
Especially when \( l = 0 \), \( S_0 = Q_{2\tilde{c}0\delta_0^{1-1/m}} \). We first carry out even extension of \( \tilde{v} \) in terms of \( y_d = \delta_0^{1-1/m} \) and then perform the periodic extension with the period of \( 4\delta_0^{1-1/m} \). That is, we obtain
\[
\hat{v}(y) := \tilde{v}(y', (-1)^l(y_d - 2l\delta_0^{1-1/m})), \quad \text{in } S_l, \ l \in \mathbb{Z},
\]
while, for other indices,
\[
\hat{a}_{ij}(y) := \tilde{a}_{ij}(y', (-1)^l(y_d - 2l\delta_0^{1-1/m})), \quad \text{in } S_l.
\]
Hence, \( \hat{v} \) and \( \hat{a}_{ij} \) are defined in \( Q_{2,\infty} \). Moreover, \( \hat{v} \) solves
\[
\hat{a}_{ij}(\hat{v}_{ij}) = 0, \quad \text{in } S,
\]
by utilizing the conormal boundary conditions. It then follows from Proposition 4.1 of Li and Nirenberg [7] and Lemma 2.1 of Li and Yang [17] that
\[
\|\nabla \hat{v}\|_{L^\infty(S)} \leq C \|\hat{v}\|_{L^2(S)} \leq C \delta_0^\frac{\alpha}{m}.
\]
Rescaling back to \( u \), we deduce that for \( x_0 = (x_0', x_d) \in \Omega_{R_0/2} \setminus \{x_0' = 0\} \),
\[
|\nabla u(x_0)| \leq \|\nabla u\|_{L^\infty(\Omega_{\tilde{c}0\delta_0^{1/m}(x_0')}} \leq C \delta_0^{\frac{\alpha-1}{m}} \leq C |x_0'|^{\tilde{\alpha}-1},
\]
(2.49)
where \( \tilde{\alpha} = \min\{\alpha(\lambda_1), 1 + \gamma - m\sigma_0\} \).

In light of Equation (2.49), the upper bound \( |\nabla u(x)| \leq C|x'|^{-m\sigma_0} \) has been improved to be \( |\nabla u(x)| \leq C|x'|^{\tilde{\alpha}-1} \), where \( \tilde{\alpha} - 1 = \min\{\alpha(\lambda_1) - 1, \gamma - m\sigma_0\} \). If \( 1 + \gamma - m\sigma_0 > \alpha \), the proof is complete. Otherwise, if \( 1 + \gamma - m\sigma_0 < \alpha \), then choose \( \sigma_1 = \sigma_0 - \frac{\gamma}{m} \) and repeat the above-mentioned argument. For the purpose of letting \( \alpha(\lambda_1) - 1 \neq -m\sigma_0 + k\gamma \) for any \( k \geq 1 \), we may decrease \( \gamma \) if necessary. By repeating the argument above with finite times, we prove that Equation (1.7) holds.

Case 2. Consider the case when \( \varepsilon > 0 \). Denote
\[
\omega(\rho) := \left( \int_{B_{(1+\tilde{c}_0\rho)}'(1+\tilde{c}_0\rho)} |\bar{v}(x') - (\bar{v})_{B_{(1+\tilde{c}_0\rho)}'(1+\tilde{c}_0\rho)}|^2 \right)^{1/2}, \quad \text{for } 0 < \rho < (1-\tilde{c}_0)^2R_0.
\]
Picking \( \sigma = \gamma - m\sigma_0 \) and \( \tau = m\sigma_0 \) in Proposition 2.2, we have
\[
\omega(\rho) \leq C\left( \frac{\rho}{R} \right)^{\alpha(\lambda_1)} \omega(R) + C\left( \frac{R}{\rho} \right)^{\frac{d+m-2}{2}} R^{1-m\sigma_0} \left[ R^\frac{\sqrt{\varepsilon}}{R_m/2} + 1 \right] + \left( \frac{\varepsilon}{R_m} \right)^{\beta(\lambda_1)},
\]
where $\beta(\lambda_1) > \alpha(\lambda_1)$ is defined in Equation (2.8). In the case of $\varepsilon > 0$, we pick a small positive constant $\bar{\mu}$ such that $\bar{\mu}\alpha(\lambda_1) \leq \gamma$. Then for any $\varepsilon^{m+\mu} \leq \rho < (1 - \bar{\varepsilon}_0)^2 R \leq (1 - \bar{\varepsilon}_0)^2 R_0$,

$$
\omega(\rho) \leq C \left( \frac{\rho}{R} \right)^{\alpha(\lambda_1)} \omega(R) + C \left( \frac{R}{\rho} \right)^{\frac{d+m-2}{2}} R^{1-m\sigma_0 + \mu\beta(\lambda_1)}.
$$

Making use of Lemma 5.13 in Giaquinta and Martinazzi [33], we obtain that for $\varepsilon^{m+\mu} \leq \rho < (1 - \bar{\varepsilon}_0)^2 R_0$,

$$
\omega(\rho) \leq C \rho^{\hat{\alpha}}, \quad \text{with} \quad \hat{\alpha} := \min\{\alpha(\lambda_1), 1 - m\sigma_0 + \mu\beta(\lambda_1)\}.
$$

Then for $x'_0 \in B'_0 \setminus B'_{2(1-\bar{\varepsilon}_0)}$, 

$$
\int_{B'_{2(1-\bar{\varepsilon}_0)}} \left| \tilde{u} - (\tilde{v})^{\kappa}_{B'_{(1+\bar{\varepsilon}_0)/x'_0}} \right|^2 \leq \int_{B'_{(1+\bar{\varepsilon}_0)/x'_0}} \left| \tilde{u} - (\tilde{v})^{\kappa}_{B'_{(1+\bar{\varepsilon}_0)/x'_0}} \right|^2 \leq C\delta_0^{\frac{2\hat{\alpha}}{m}}.
$$

This, together with Equation (2.46), shows that

$$
\int_{Q_{2(1-\bar{\varepsilon}_0)}} \left| u - (\tilde{v})^{\kappa}_{B'_{(1+\bar{\varepsilon}_0)/x'_0}} \right|^2 \leq C\delta_0^{\frac{2\hat{\alpha}}{m}}.
$$

Similarly as before, define

$\tilde{\theta}(y) = \theta(\delta_0^{1/m}y' + x'_0, \delta_0^{1/m}y_d) - (\tilde{\theta})^{\kappa}_{B'_{(1+\bar{\varepsilon}_0)/x'_0}}$,

$\tilde{\theta}_{ij}(y) = \delta^{-1}_{0} b_{ij}(\delta_0^{1/m}y' + x'_0, \delta_0^{1/m}y_d)$.

Therefore, $\tilde{\theta}(y)$ verifies Equation (2.47) with $\tilde{b} := (\tilde{\theta}_{ij})$ satisfying Equation (2.48). Then using the same “flipping argument” as above, we obtain

$$
|\nabla u(x'_0)| \leq \frac{\hat{\alpha}}{m} C\delta_0^{\frac{2\hat{\alpha}}{m}}, \quad \text{for} \ x'_0 \in B'_0 \setminus B'_{2(1-\bar{\varepsilon}_0)^2}.
$$

From Equation (2.50), we have

$$
\text{osc}_{\Omega^{2\varepsilon}} u \leq C \rho^{\hat{\alpha}}, \quad \text{for any} \ \frac{1}{2\varepsilon^{m+\mu}} \leq \rho < R_0/2,
$$

which, together with the maximum principle, reads that for $|x'_0| \leq 2\varepsilon^{m+\mu}$,

$$
\text{osc}_{\Omega^{2\varepsilon}} u \leq \text{osc}_{\Omega^{\varepsilon}} \frac{1}{2\varepsilon^{m+\mu}} u \leq C \varepsilon^{\frac{\hat{\alpha}}{m}}.
$$

Then for any $x_0 \in \Omega \setminus \frac{1}{2\varepsilon^{m+\mu}}$, we have

$$
\|u - u(x_0)\|_{L^\infty(\Omega^{2\varepsilon})} \leq C \varepsilon^{\frac{\hat{\alpha}}{m}}.
$$

Hence, applying the changes of variables in Equation (2.36) for $u - u(x_0)$, it follows from the same “flipping argument” and Equation (2.51) that

$$
|\nabla u(x_0)| \leq C\delta_0^{-1/m} \varepsilon^{\frac{\hat{\alpha}}{m}}, \quad \text{for} \ x_0 \in \Omega \setminus \frac{1}{2\varepsilon^{m+\mu}}.
This, in combination with Equation (2.50), shows that
\[ |∇u(x)| ≤ C δ^{-\frac{1}{m+\mu}} \frac{\Delta}{m+\mu}, \quad \text{for any } x ∈ Ω(1-\bar{c}_0)^2R_0. \]

Therefore, we improve the aforementioned upper bound \(|∇u(x)| ≤ C(\varepsilon + |x'|^m)^{-σ_0} + \frac{\Delta}{m+\mu}\) with \(\tilde{α}(λ_1) = \min\{\alpha(λ_1), 1 - mσ_0 + μβ(λ_1)\}\). If \(1 - mσ_0 + μβ(λ_1) ≥ \alpha(λ_1)\), then we have \(|∇u| ≤ C δ^{-\frac{1}{m+\mu}} \frac{\alpha(λ_1)}{m+\mu} \) in \(Ω_{R_0/2}\). By letting \(μ → 0\), we complete the proof. Otherwise, if \(1 - mσ_0 + μβ(λ_1) < \alpha(λ_1)\), then set \(σ_1 = σ_0 - \frac{1 - mσ_0 + μβ(λ_1)}{m+\mu}\) and repeat the above-mentioned argument with \(\||∇\bar{v}||_ε,−mσ_1,1,Β_{R_0}^J + ||F||_ε,γ−mσ_1,0,Β_{R_0}^J < ∞\).

Denote
\[ σ_k = σ_{k-1} - \frac{1 - mσ_{k-1} + μβ(λ_1)}{m+\mu}, \quad k ≥ 1. \]

Then we have
\[ σ_k - \frac{1}{m} = \frac{2m + μ}{m+\mu} (σ_{k-1} - \frac{1}{m}) - \frac{μβ(λ_1)}{m+\mu}. \]

Note that \(σ_0 = \frac{1}{m}\), it then follows that
\[ σ_k = σ_0 - \frac{1}{m} \sum_{i=0}^{k-1} \frac{2m + μ}{m+\mu}^i \frac{μβ(λ_1)}{m+\mu}, \]

which yields that
\[ 1 - mσ_k + μβ(λ_1) = \left( \sum_{i=0}^{k-1} \frac{2m + μ}{m+\mu}^i \frac{μβ(λ_1)}{m+\mu} + 1 \right) μβ(λ_1) → +∞, \quad \text{as } k → +∞. \]

Then there exists some \(k_0 ∈ \mathbb{N}\) such that
\[ α(λ_1) ≤ \left( \sum_{i=0}^{k_0-1} \frac{2m + μ}{m+\mu}^i \frac{μβ(λ_1)}{m+\mu} + 1 \right) μβ(λ_1) = 1 - mσ_{k_0} + μβ(λ_1), \]

and
\[ α(λ_1) > \left( \sum_{i=0}^{k_0-2} \frac{2m + μ}{m+\mu}^i \frac{μβ(λ_1)}{m+\mu} + 1 \right) μβ(λ_1), \quad \text{if } k_0 ≥ 2. \]

Therefore, after \(k_0\) iterations, we get
\[ |∇u(x)| ≤ C δ^{-\frac{1}{m+\mu}} \frac{α(λ_1)}{m+\mu} \in Ω_{R_0/4}, \]

where we used the fact that \((1 - \bar{c}_0)^2 ≥ \frac{1}{4}\). Sending \(μ → 0\), we obtain that (1.8) holds. \(\square\)

3 | THE PROOF OF THEOREM 1.4

Before giving the proof of Theorem 1.4, we first demonstrate the validity of the assumed condition “the eigenspace corresponding to \(λ_1\) contains a function, which is odd with respect to some \(x_{j_0}, j_0 \in \{1, ..., d - 1\}\).”
**Definition 3.1.** If a function space on $\mathbb{S}^{d-2} \subset \mathbb{R}^{d-1}$ is spanned by functions, which are odd with respect to some $x_i$ and even in other variables, we say that it satisfies the property $O$.

For $\mu \in \mathbb{R}$ and $d \geq 3$, consider

$$L_\mu = -\text{div}_{\mathbb{S}^{d-2}}((1 + \mu b(x))V_{\mathbb{S}^{d-2}}), \quad \text{on} \ \mathbb{S}^{d-2},$$

where $b(x) \in L^\infty(\mathbb{S}^{d-2})$. Denote by $\lambda_{1,\mu}$ and $V_{1,\mu}$ the corresponding first nonzero eigenvalue and the eigenspace of the following problem:

$$L_\mu u = \lambda (1 + \mu b(x))u.$$  \hspace{1cm} (3.1)

Recall Proposition 5.6 in Dong et al. [1] as follows:

**Lemma 3.2 Proposition 5.6 of Dong et al. [1].** Consider the eigenvalue problem (3.1). Let $b(x)$ be even in all variables and $V_{1,\mu_0}$ satisfy the property $O$ for some $\mu_0 \in \mathbb{R}$. Then there exists a small positive constant $\varepsilon_0 := \varepsilon_0(d, \|b\|_{L^\infty}, \mu_0)$ such that $V_{1,\mu}$ verifies the property $O$ for every $\mu \in (\mu_0 - \varepsilon_0, \mu_0 + \varepsilon_0)$.

Then applying Lemma 3.2 with $d \geq 3$, $\|b\|_{L^\infty} \leq 2$ and $\mu_0 = 0$, we derive a small constant $\varepsilon_0 = \varepsilon_0(d)$ such that $V_{1,\mu}$ verifies the property $O$ in $(-\varepsilon_0, \varepsilon_0)$. Note that by a rotation of the coordinates if necessary, we have $\kappa_i \geq \kappa_{i+1}$ for $i = 1, \ldots, d-2$.

Choose $\mu = \frac{\varepsilon_0}{2}$ and

$$b(x) = \frac{2}{\varepsilon_0 \kappa_{d-1}} \left[ \sum_{i=1}^{d-2} \kappa_i |x_i|^m - \left(1 - \left(1 - \sum_{i=1}^{d-2} x_i^2 \right)^{m/2} \right) x_{d-1} \right],$$  \hspace{1cm} (3.2)

which implies that $1 + \mu b = \frac{1}{\kappa_{d-1}} \sum_{i=1}^{d-1} \kappa_i |x_i|^m$. Therefore, if we assume that $\frac{\kappa_1}{\kappa_{d-1}} - 1 \leq \varepsilon_0$, then $b(x)$ defined in Equation (3.2) satisfies $\|b\|_{L^\infty} \leq 2$. In fact, since

$$1 - \left(1 - \sum_{i=1}^{d-2} x_i^2 \right)^{m/2} \geq \left(\sum_{i=1}^{d-2} x_i^2 \right)^{m/2} \geq \sum_{i=1}^{d-2} |x_i|^m,$$

then

$$|b| \leq \frac{2}{\varepsilon_0 \kappa_{d-1}} \sum_{i=1}^{d-2} (\kappa_i - \kappa_{d-1}) |x_i|^m \leq \frac{2}{\varepsilon_0} \left( \frac{\kappa_1}{\kappa_{d-1}} - 1 \right) \leq 2.$$

Consequently, under the condition of $\frac{\kappa_1}{\kappa_{d-1}} - 1 \leq \varepsilon_0$, we obtain that the eigenspace corresponding to $\lambda_1$ of the eigenvalue problem (1.4) verifies the property $O$.

We now give the proof of Theorem 1.4.

**Proof of Theorem 1.4.** For simplicity, write $\alpha := \alpha(\lambda_1)$ in the following. Assume without loss of generality that $R_0 = 1$. Denote

$$\bar{u}(x') := \int_{B_1(x')} \frac{h_2(x')}{h_2(x')} \, dx, \quad \text{in} \ \Omega_1 \setminus \{x' = 0\}.$$

Then $\bar{u}$ solves

$$\text{div} \left( \sum_{i=1}^{d-1} \kappa_i |x_i|^m \right) \nabla \bar{u} = \text{div} F, \quad \text{in} \ B_1'.$$
where \( F = (F_1, \ldots, F_{d-1}) \), \( F_i = -b_i \partial_\xi u - e \partial_\xi \bar{u} \) with \( e = O(|x'|^m) \) and
\[
b_i(x) = (h_1(x') - h_2(x')) \partial_\xi h_2(x') + (x_d - h_2(x')) \partial_\xi (h_1(x') - h_2(x')).
\]

Then we have from (H1)–(H2) and Theorem 1.1 that
\[
|F(x')| \leq C|x'|^{m-1+\gamma}, \quad \text{in } B_1' \setminus \{0'\}.
\]  

**Step 1.** Denote by \( Y_{k,j} \) the normalized eigenfunction corresponding to \( \lambda_k \), which is \((k+1)\)-th eigenvalue of problem (1.4). Then \( \{Y_{k,j}\} \) is an orthonormal basis of \( L^2(\mathbb{S}^{d-2}) \) with the inner product (1.5). From the assumed condition, we denote by \( Y_{1,j_0} \), the eigenfunction, which is odd with respect to \( x_{j_0} \). Then the eigenfunction \( Y_{1,j_0} \) corresponding to \( \lambda_1 \) on the half sphere \( \mathbb{S}^{d-2} \cap \{x_{j_0} > 0\} \) satisfies the zero Dirichlet boundary condition. \( \lambda_1 \), as the first nonzero eigenvalue of problem (1.4) on the sphere, is also the same on the half sphere. Then \( \lambda_1 \) is simple and \( Y_{1,j_0} \) keeps the same sign on the half sphere. So we assume without loss of generality that \( Y_{1,j_0} > 0 \) in \( \{x_{j_0} > 0\} \) and \( Y_{1,j_0} < 0 \) in \( \{x_{j_0} < 0\} \). Based on the fact that the domain \( \Omega := \Omega \setminus \Omega_1 \cup \Omega_2 \) is symmetric with respect to each \( x_i, 1 \leq i \leq d-1 \), and \( \varphi \) is an odd function of \( x_{j_0} \), we deduce from the classical elliptic theory that \( u \) is an odd function of \( x_{j_0} \) and thus \( \bar{u} \) is also odd in \( x_{j_0} \), and \( \bar{u}(0') = 0 \). Then we utilize the basis \( \{Y_{k,j}\} \) to carry out the decomposition as follows:
\[
\bar{u}(x') = \sum_{k=1}^{\infty} \sum_{l=1}^{N(k)} U_{k,l}(r) Y_{k,j}(\xi), \quad \text{in } B_1' \setminus \{0'\},
\]  

where \( U_{k,l} \in C([0,1)) \cap C^\infty((0,1)) \) is given by
\[
U_{k,l}(r) = \frac{\int_{S^{d-2}} (\text{div} F) Y_{1,j_0}(\xi)}{\kappa(\xi) \gamma} \int_{S^{d-2}} \frac{\partial_r F}{\gamma} + \frac{1}{r} \nabla_\xi F \xi \frac{Y_{1,j_0}(\xi)}{\kappa(\xi) \gamma} d\xi.
\]  

Making use of Equation (3.3), we obtain
\[
|A(r)| \leq C(d)r^{\gamma+\alpha-1}, \quad \text{and } |B(r)| \leq C(d)r^{\gamma+\alpha-2}, \quad \text{in } (0,1).
\]  

**Step 2.** We first demonstrate that \( U_{1,j_0} \) can be decomposed as follows:
\[
U_{1,j_0}(r) = C_1 r^\alpha + O(r^{\alpha+\gamma}),
\]  

where \( \alpha = \alpha(\lambda_1) \). Observe that \( g = r^\alpha \) satisfies \( Lg = 0 \). Define
\[
w(r) := \int_{0}^{r} \frac{1}{s^{d+m+2\alpha-2}} \int_{0}^{s} t^{d+m+2\alpha-2} H(t) dt ds, \quad \text{in } (0,1).
\]

Denote \( v = gw \). Then, \( v \) verifies
\[
Lv = \frac{G}{G'} (Gw')' = H, \quad \text{with } G = g^2 r^{d+m-2}.
\]

From Equation (5.5), we have \( |w(r)| \leq Cr^\gamma \) and thus \( |v(r)| \leq Cr^{\alpha+\gamma} \). Due to the fact that \( U_{1,j_0} - v \) stays bounded and verifies \( L(U_{1,j_0} - v) = 0 \) in \((0,1)\), we obtain that \( U_{1,j_0} = C_1 g + v \) and thus Equation (3.6) holds.
Claim that the constant $C_1 > 0$. By the symmetry and convexity of the domain, we obtain that $\partial_y x_{j_0} \geq 0$ in $\{x_{j_0} \leq 0\}$ and $\partial_y x_{j_0} \leq 0$ in $\{x_{j_0} \geq 0\}$. Then combining the assumed conditions in Theorem 1.4, $x_{j_0}$ becomes a subsolution of Equation (1.1) in $\{x_{j_0} \geq 0\}$, while it is a supersolution of Equation (1.1) in $\{x_{j_0} \leq 0\}$. This implies that $|u(x)| \geq |x_{j_0}|$ in $\Omega$ and then $|\tilde{u}(x)| \geq |x_{j_0}|$ in $B'_1$. In light of the fact that $Y_{1,j_0}$ and $x_{j_0}$ possess the same sign, we obtain

$$U_{1,j_0} = \int_{S^{d-2}} \kappa(\xi)\tilde{u}(r,\xi)Y_{1,j_0}(\xi)d\xi \geq Cr,$$

which, together with Equation (3.6) and the assumed condition that $\gamma > 1 - \alpha$, reads that $C_1 > 0$.

From Equations (3.4) and (3.6), we deduce

$$\left(\int_{S^{d-2}} \kappa(\xi)|\tilde{u}(r,\xi)|^2d\xi\right)^{1/2} \geq |U_{1,j_0}| \geq \frac{C_1}{2}r^\alpha, \quad \text{in } (0, r_0),$$

for some positive constant $r_0$. Then there exists a $\xi_0(r) \in S^{d-2}$ for any $r \in (0, r_0)$ such that

$$|\tilde{u}(r, \xi_0(r))| \geq \frac{1}{C_2}r^\alpha, \quad \text{for some constant } C_2 > 0. \quad (3.7)$$

Using Equation (1.7), we have

$$|u(r, \xi_0(r)) - \tilde{u}(r, \xi_0(r))| \leq Cr^m \sup_{h_2(x') \leq x_0, h_1(x')} |\partial_d u(r, \xi_0(r), x_d)| \leq Cr^{m+\alpha-1}. \quad (3.8)$$

From Equations (3.7) and (3.8), we deduce

$$|u(r, \xi_0(r), 0)| \geq \frac{1}{2C_2}r^\alpha, \quad \text{in } (0, r_1),$$

for some positive constant $r_1$. Write $x_0 = (r, \xi_0(r), 0)$. Utilize Equation (1.7) and pick a sufficiently large $x_0$-independent constant $C_3$ such that

$$|u(x_0C_3^{-1})| \leq C|x'|^\alpha C_3^{-\alpha} \leq \frac{1}{4C_2}|x'|^\alpha.$$

Then, there exists a point $x$ in the line segment between $x_0$ and $x_0/C_3$ such that

$$|\nabla u(x)| \geq \frac{1}{C}|x'|^{\alpha-1},$$

where $C$ is some positive constant, which may depend on $d, m, x_i, i = 1, \ldots, d - 1$, and the upper bounds of $\|\delta D_j\|_{L^4}, j = 1, 2$. The proof is complete. \hfill $\square$

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**CONFLICT OF INTEREST STATEMENT**

The authors declare no conflicts of interest.

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