Improved high-temperature expansion and critical equation of state of three-dimensional Ising-like systems

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Abstract

High-temperature series are computed for a generalized 3d Ising model with arbitrary potential. Two specific “improved” potentials (suppressing leading scaling corrections) are selected by Monte Carlo computation. Critical exponents are extracted from high-temperature series specialized to improved potentials, achieving high accuracy; our best estimates are: \( \gamma = 1.2371(4) \), \( \nu = 0.6300(23) \), \( \alpha = 0.1099(7) \), \( \eta = 0.0364(4) \), \( \beta = 0.32648(18) \). By the same technique, the coefficients of the small-field expansion for the effective potential (Helmholtz free energy) are computed. These results are applied to the construction of parametric representations of the critical equation of state. A systematic approximation scheme, based on a global stationarity condition, is introduced (the lowest-order approximation reproduces the linear parametric model). This scheme is used for an accurate determination of universal ratios of amplitudes. A comparison with other theoretical and experimental determinations of universal quantities is presented.

Keywords: Critical Phenomena, Ising Model, High-Temperature Expansion, Critical Exponents, Critical Equation of State, Universal Ratios of Amplitudes, Effective Potential.

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I. INTRODUCTION

According to the universality hypothesis, critical phenomena can be described by quantities that do not depend on the microscopic details of a system, but only on global properties such as the dimensionality and the symmetry of the order parameter. Many three-dimensional systems characterized by short-range interactions and a scalar order parameter (such as density or uniaxial magnetization) belong to the Ising universality class. This implies that the critical exponents, as well as other universal quantities, are the same for all these models. Their precise determination is therefore important in order to test the universality hypothesis.

The high-temperature (HT) expansion is one of the most effective approaches to the study of critical phenomena. Much work (even recently) has been devoted to the computation of HT series, especially for $N$-vector models and in particular the Ising model. An important issue in the analysis of the HT series is related to the presence of non-analytic corrections to the leading power-law behavior. For instance, according to the renormalization group theory (see e.g. Ref. [1]), the magnetic susceptibility should behave as

$$\chi = C t^{-\gamma} \left( 1 + a_{0,1} t + a_{0,2} t^2 + \ldots + a_{1,1} t^\Delta + a_{1,2} t^{2\Delta} + \ldots + a_{2,1} t^{\Delta^2} + \ldots \right), \quad (1.1)$$

where $t \equiv (T - T_c)/T_c$ is the reduced temperature. The leading critical exponent $\gamma$, and the correction exponents $\Delta, \Delta_2, \ldots$, are universal, while the amplitudes $C$ and $a_{i,j}$ are non-universal and should depend smoothly on any subsidiary parameter that may change $T_c$, but does not affect the nature of the transition. Non-analytic correction terms to the leading power-law behavior, represented by noninteger powers of $t$, are related to the presence of irrelevant operators. For three-dimensional Ising-like systems the existence of leading corrections described by $\Delta \simeq 0.5$ is well established. In order to obtain precise estimates of the critical parameters, the approximants of the HT series should properly allow for the confluent non-analytic corrections [2–8]. The so-called integral approximants [9] can, in principle, allow for them (see e.g. Ref. [10] for a review). However, they require long series to detect non-leading effects, and in practice they need to be biased to work well. Analyses meant to effectively allow for confluent corrections are generally based on biased approximants where the value of $\beta_c$ and the first non-analytic exponent $\Delta$ is given (see e.g. Refs. [11–15]). It is indeed expected that the leading non-analytic correction is the dominant source of systematic error.

An alternative approach to this problem is the construction of a HT expansion where the dominant confluent correction is suppressed. If the leading non-analytic terms are not anymore present in the expansion, the analysis technique based on integral approximants should become much more effective, since the main source of systematic error has been eliminated. In order to obtain an improved high-temperature (IHT) expansion, we may consider improved Hamiltonians characterized by a vanishing coupling with the irrelevant operator responsible for the leading scaling corrections. This idea has been pursued by Chen, Fisher, and Nickel [5], who studied classes of two-parameter models (such as the bcc scalar double-Gaussian and Klauder models). Such models interpolate between the spin-1/2 Ising model and the Gaussian model, and they are all expected to belong to the Ising universality class. The authors of Ref. [5] showed that improved models with suppressed leading corrections...
to scaling can be obtained by tuning the parameters (see also Refs. 8[16]). This approach has been recently considered in the context of Monte Carlo simulations 17–20, using lattice $\phi^4$ models. It is also worth mentioning the recent work 21 of Belohorec and Nickel in the context of dilute polymers, where a substantial improvement in the determination of the critical exponents $\nu$ and $\omega$ was achieved by simulating the two-parameter Domb-Joyce model.

We consider the class of scalar models defined on the simple cubic lattice by the Hamiltonian

$$\mathcal{H} = -\beta \sum_{<i,j>} \phi_i \phi_j + \sum_i V(\phi_i^2),$$  \hspace{1cm} (1.2)

where $\beta \equiv 1/T$, $<i,j>$ indicates nearest-neighbor sites, $\phi_i$ are real variables, and $V(\phi^2)$ is a generic potential (satisfying appropriate stability constraints). The critical limit of these models is expected to belong to the Ising universality class (apart from special cases corresponding to multicritical points). Using the linked cluster expansion technique, we calculated the high-temperature expansion to 20th order for an arbitrary potential, generalizing the existing expansions for the standard Ising model (see e.g. Ref. 13 for a review of the existing HT calculations). In this work we will essentially consider and present results for a potential of the form

$$V(\phi^2) = \phi^2 + \lambda_4 (\phi^2 - 1)^2 + \lambda_6 (\phi^2 - 1)^3;$$  \hspace{1cm} (1.3)

such a potential will be assumed in the following, unless otherwise stated. Within this family of potentials, improved Hamiltonians can be obtained by looking for values of the parameters $\lambda_4$ and $\lambda_6$ for which leading scaling corrections are suppressed. In particular we may keep $\lambda_6$ fixed and look for the corresponding value $\lambda_4^*$ of $\lambda_4$ that produces an improved Hamiltonian. Notice that, for generic choices of the Hamiltonian, $\lambda_4^*$ may not exist. This is the case of the $O(N)$ $\phi^4$ theory with nearest-neighbor couplings on a cubic lattice in the large-$N$ limit, where it is impossible to find a positive value of $\lambda_4$ achieving the suppression of the dominant scaling corrections. Using the estimates of the leading scaling correction amplitudes reported in Ref. 15, one can argue that the same is true for finite $N > 3$. As shown numerically by Monte Carlo simulations 17–21, $\lambda_4^*$ exists in the case $N = 1$, which is the single-component $\phi^4$ model (i.e. the model presented above with $\lambda_6 = 0$). By using finite-size techniques, Hasenbusch obtained a precise estimate of $\lambda_4^*$: $\lambda_4^* = 1.10(2)$ 21. In our work we will also consider the spin-1 (or Blume-Capel) Hamiltonian

$$\mathcal{H} = -\beta \sum_{<i,j>} s_i s_j + D \sum_i s_i^2,$$  \hspace{1cm} (1.4)

where the variables $s_i$ take the values 0, ±1. In this case the value of $D$ for which the leading scaling corrections are suppressed is $D^* = 0.641(8)$ 22.

From the point of view of the HT expansion technique, the main problem is the determination of the improved Hamiltonian. Once the improved Hamiltonian is available, the analysis of its HT series leads, as we shall see, to much cleaner and therefore reliable results. A precise estimate of the parameters associated with an improved Hamiltonian is crucial in order to obtain a substantial improvement of the IHT results. As shown in Refs. 17,20,
Monte Carlo simulations using finite-size scaling techniques seem to provide the most efficient tool for this purpose. For comparison, in the case of the pure $\phi^4$ theory ($\lambda_6 = 0$), our best estimate of $\lambda_4^*$ from the HT expansion is consistent with the above-mentioned Monte Carlo result $\lambda_4^* = 1.10(2)$, but it is affected by an uncertainty of about 10% (see Sec. III).

So we decided to follow the strategy of determining the improved Hamiltonian by Monte Carlo simulations employing finite-size scaling techniques. For the $\phi^4$ model, this work has been satisfactorily done by Hasenbusch [20]. For the $\phi^6$ model with $\lambda_6 = 1$, we performed Monte Carlo simulations in order to calculate $\lambda_4^*$, obtaining $\lambda_4^* = 1.90(4)$.

The comparison of the results obtained from the three improved Hamiltonians considered strongly supports our working hypothesis of the reduction of systematic errors in the IHT estimates, and provides an estimate of the residual errors due to the subleading confluent corrections to scaling.

The analysis of our 20th order IHT series allows us to obtain very precise estimates of the critical exponents $\gamma$, $\nu$ and $\eta$. Our estimates substantially improve previous determinations by HT or other methods.

We extended our study to the small-field expansion of the effective potential, which is the Helmholtz free energy of the model. This expansion can be parametrized in terms of the zero-momentum $n$-point couplings $g_n$ in the symmetric phase. The analysis of the IHT series provides new results for the couplings $g_n$, and leads to interesting comparisons with the estimates from other approaches based on field theory and lattice techniques. Moreover we improved the knowledge of the universal critical low-momentum behavior of the two-point function of the order parameter, which is relevant for critical scattering phenomena.

By exploiting the known analytic properties of the critical equation of state, one may reconstruct the full critical equation of state from the small-field expansion of the effective potential, which is related to the behavior of the equation of state for small magnetization in the symmetric phase. This can be achieved by using parametric representations implementing in a rather simple way the known analytic properties of the equation of state. Effective parametric representations can be obtained by parametrizing the magnetization $M$ and the reduced temperature $t$ in terms of two variables $R$ and $\theta$, setting $M \propto R^\beta \theta$, $t = R(1 - \theta^2)$, and $H \propto R^\delta h(\theta)$. In this framework, following Guida and Zinn-Justin [23], one may develop an approximation scheme based on truncations of the Taylor expansion of the function $h(\theta)$ around $\theta = 0$. Knowing a given number of terms in the small-field expansion of the effective potential, one can derive the same number of terms in the small-$\theta$ expansion of $h(\theta)$, with a dependence on an arbitrary normalization parameter $\rho$. One can try to fix $\rho$ so that this small-$\theta$ expansion has the fastest possible convergence. We propose a prescription based on the global stationarity of the truncated equation of state with respect to the arbitrary parameter $\rho$. This extends the stationarity condition of the linear model (i.e. the lowest-order non-trivial approximation) discussed in Refs. [24, 27]. Using the IHT results for $\gamma$, $\nu$ and the first few coefficients of the small-field expansion of the effective potential, we constructed approximate representations of the full critical equation of state. From them we obtained accurate estimates of many ratios of universal amplitudes. Varying the truncation order of $h(\theta)$, we observed a fast convergence, supporting our arguments.

For our readers’ convenience, we collected in Table XIII a summary of all the results obtained in this paper. There they can find new estimates of most of the universal quantities (exponents and ratios of amplitudes) introduced in the literature to describe critical
phenomena in $3d$ Ising-like systems. The only important quantity for which we have not been able to give a good estimate is the exponent $\omega$, which is related to the leading scaling corrections. We mention that a precise estimate of $\omega$ has been reported recently in Ref. [20]: $\omega = 0.845(10)$. It has been obtained by a Monte Carlo study using a finite-size scaling method.

The paper is organized as follows.

In Sec. II we discuss the main features of improved Hamiltonians from the point of view of the renormalization group.

In Sec. III we describe our Monte Carlo simulations and present the estimates of $\lambda_4^*$ for the potential (1.3) with $\lambda_6 = 1$.

Sec. IV is dedicated to the determination of the critical exponents from the IHT expansion. We present estimates of all relevant critical exponents (except for $\omega$), and compare our results with other theoretical approaches and experiments.

In Sec. V we study the small-field expansion of the effective potential. We present estimates of the first few coefficients of the expansion. We discuss the relevance of the determination of the zero-momentum four-point renormalized coupling for field-theoretical approaches (Sec. VB).

Sec. VI presents a study of the low-momentum behavior of the two-point function in the critical region. Estimates of the first few coefficients of its universal low-momentum expansion are given.

In Sec. VII we study the critical equation of state, which gives a description of the whole critical region, including the low-temperature phase. Using the estimates of the critical exponents and of the first few coefficients of the small-field expansion of the effective potential, the critical equation of state is reconstructed employing approximate parametric representations (Sec. VIIA). In Sec. VIIB we present our method, based on the global stationarity of the approximate equation of state. Relevance to the $\epsilon$-expansion is discussed in Sec. VII C. In Sec. VII D we apply the results of Sec. VII B to the computation of universal ratios of amplitudes, using as inputs the results of the IHT expansion. The results are then compared with other theoretical estimates and with experimental determinations. For sake of comparison, we also present results for the two-dimensional Ising model.

Many details of our calculations are reported in the Appendices. App. A contains a detailed description of our HT calculations, i.e. the list of the quantities we have considered and the description of the method we used to generate and analyze the HT series. We report many details and intermediate results so that the reader can judge the quality of the results we will present. In App. B we present the notations for the critical amplitudes, and report the expressions of the universal ratios of amplitudes in terms of the parametric representation of the critical equation of state. In App. C we discuss in more detail the approximation scheme for the parametric representation of the equation of state based on stationarity.

II. IMPROVED HAMILTONIANS

As discussed in the introduction, we will work with “improved” Hamiltonians, i.e. with models in which the leading correction to scaling has a vanishing (in practice very small)
amplitude.

To clarify the basic idea, let us consider a model with two relevant operators (the thermal and the magnetic ones) and one irrelevant operator. If \( \tau, \kappa, \text{and} \mu \) are the associated non-linear scaling fields, the singular part of the free energy \( F_{\text{sing}} \) has the scaling form \[28\]

\[
F_{\text{sing}}(\tau, \kappa, \mu) = |\tau|^{d \nu} f_\pm \left( |\kappa| |\tau|^{-(d+2-\eta)\mu/2}, |\mu| |\tau|^\Delta \right),
\]

where the function \( f_\pm \) depends on the phase of the model. Since the operator associated with \( \mu \) is irrelevant, \( \Delta \) is positive and \( \mu |\tau|^\Delta \to 0 \) at the critical point. Therefore we can expand the free energy obtaining

\[
F_{\text{sing}}(\tau, \kappa, \mu) = |\tau|^{d \nu} \sum_{n=0}^\infty f_{n,\pm} \left( |\kappa| |\tau|^{-(d+2-\eta)\mu/2} \right) \mu^n |\tau|^n \Delta.
\]

The presence of the irrelevant operator induces non-analytic corrections proportional to \( |\tau|^n \Delta \). Now, let us suppose that the Hamiltonian of our model depends on three parameters \( r, h, \text{and} \lambda \), where \( r \) is associated to the temperature, \( h \) is the magnetic field and \( \lambda \) is an irrelevant parameter. For each value of \( \lambda \) and for \( h = 0 \), the theory has a critical point for \( r = r_c(\lambda) \). The non-linear scaling fields \( \tau, \kappa \), and \( \mu \) are analytic functions of the parameters appearing in the Hamiltonian, and therefore we can write

\[
\tau = t + t^2 g_1(\lambda) + h^2 g_2(\lambda) + O(t^3, th^2, h^4),
\]

\[
\kappa = h \left[ 1 + t g_1(\lambda) + h^2 g_2(\lambda) + O(t^2, th^2, h^4) \right],
\]

\[
\mu = g_1(\lambda) + t g_2(\lambda) + h^2 g_3(\lambda) + O(t^2, th^2, h^4),
\]

where \( t \equiv r - r_c(\lambda) \). Substituting these expressions into Eq. \[2.2\], we see that, if \( g_{1\mu}(\lambda) \neq 0 \), the free energy has corrections of order \( t^n \Delta \). For the susceptibility in zero magnetic field we obtain the explicit formula \[29\]

\[
\chi = t^{-\gamma} \sum_{m,n=0}^\infty \chi_{1, mn}(\lambda) t^{m \Delta + n} + t^{-\alpha} \sum_{m,n=0}^\infty \chi_{2, mn}(\lambda) t^{m \Delta + n} + \sum_{n=0}^\infty \chi_{3, n}(\lambda) t^n,
\]

where the contribution proportional to \( t^{1-\alpha} \) stems from the terms of order \( h^2 \) appearing in the expansion of \( \tau \) and \( \mu \), and the last term is the contribution of the regular part of the free energy. Notice that it is often assumed that the regular part of the free energy does not depend on \( h \). If this were the case, we would have \( \chi_{3,n}(\lambda) = 0 \). However, for the two-dimensional Ising model, one can prove rigorously that \( \chi_{3,0} \neq 0 \) \[30,31\], showing the incorrectness of this conjecture. For a discussion, see Ref. \[32\].

In many interesting instances, it is possible to cancel the leading correction due to the irrelevant operator by choosing \( \lambda = \lambda^* \) such that \( g_{1\mu}(\lambda^*) = 0 \). In this case \( \mu |\tau|^\Delta \sim t^{1+\Delta} \), so that no term of the form \( t^{n \Delta + n} \), with \( n < m \), will be present. In particular the leading term proportional to \( t^\Delta \) will not appear in the expansion.

In general other irrelevant operators will be present in the theory, and therefore we expect corrections proportional to \( t^\rho \) with \( \rho = n_1 + n_2 \Delta + \sum_i m_i \Delta_i \), where \( \Delta_i \) are the exponents associated to the additional irrelevant operators. For \( \lambda = \lambda^* \) the expansion will contain only terms with \( n_1 \geq n_2 \).
It is important to notice that, by working with $\lambda = \lambda^*$, we use a Hamiltonian such that the non-linear scaling field $\mu$ vanishes at the critical point. This property is independent of the observable we are considering. Therefore all quantities will be improved, in the sense that the leading correction to scaling, proportional to $t^\Delta$, will vanish. We will call the Hamiltonians with $\lambda = \lambda^*$ “improved Hamiltonians”.

### III. Determination of the Improved Parameters

The Hamiltonian defined by Eqs. (1.2) and (1.3) with $\lambda_6 = 0$ was considered in Ref. [20], where it was shown that the leading correction to scaling cancels for $\lambda_4^* = 1.10(2)$. Here we will also consider the case $\lambda_6 = 1$, and determine the corresponding $\lambda_4^*$ using a method similar to the one discussed in Ref. [17].

The idea is the following: consider a renormalization-group invariant observable $O$ on a finite lattice $L$ and let $O^*$ be its value at the critical point, i.e.

$$O^* = \lim_{L \to \infty} \lim_{\beta \to \beta_c(L)} O(\beta, \lambda_4, L). \tag{3.1}$$

The quantity $O^*$ is a universal number and therefore it will be independent of $\lambda_4$. The standard scaling arguments predict

$$O(\beta_c(\lambda_4), \lambda_4, L) \approx O^* + a_1(\lambda_4)L^{-\omega} + a_2(\lambda_4)L^{-2\omega} + \ldots + b_1(\lambda_4)L^{-\omega_2} \ldots \tag{3.2}$$

where $\omega = \Delta/\nu$, $\omega_2 = \Delta_2/\nu$, $\Delta_2$ being the next-to-leading correction-to-scaling exponent. Since for $\lambda_4 = \lambda_4^*$, $a_1(\lambda_4^*) = a_2(\lambda_4^*) = \ldots = 0$, for $\lambda_4 \approx \lambda_4^*$ we can rewrite the previous equation as

$$O(\beta_c(\lambda_4), \lambda_4, L) \approx O^* + (\lambda_4 - \lambda_4^*) \left( a_{11}L^{-\omega} + a_{21}L^{-2\omega} + \ldots \right) + b_1(\lambda_4^*)L^{-\omega_2} \ldots \tag{3.3}$$

Now, suppose we know the exact value $O^*$, and let us define $\lambda_4^{\text{eff}}(L)$ as the solution of the equation

$$O(\beta_c(\lambda_4^{\text{eff}}(L)), \lambda_4^{\text{eff}}(L), L) = O^*. \tag{3.4}$$

From Eq. (3.3) we obtain immediately

$$\lambda_4^{\text{eff}}(L) = \lambda_4^* - \frac{b_1(\lambda_4^*)L^{\omega-\omega_2} + \ldots}{a_{11}} \tag{3.5}$$

Since $\omega_2 > \omega$, $\lambda_4^{\text{eff}}(L)$ converges to $\lambda_4^*$ as $L \to \infty$. For the 3d Ising universality class, $\omega_2 \simeq 2\omega [33, 34]$ and $\omega \simeq 0.85$ [20].

In order to apply this method in practice we need two ingredients: a precise determination of $\beta_c(\lambda_4)$ and an estimate of $O^*$.

Very precise estimates of $\beta_c(\lambda_4)$ can be obtained from the analysis of the HT series of the susceptibility $\chi$, that we have calculated to $O(\beta^2)$. For $\lambda_6 = 0$ and $1.0 \leq \lambda_4 \leq 1.2$ the values of $\beta_c(\lambda_4)$ can be interpolated by the polynomial

$$\beta_c(\lambda_4) = 0.40562043 + 0.00819000 \lambda_4 - 0.04626355 \lambda_4^3 + 0.01235674 \lambda_4^5 \pm 0.0000014. \tag{3.6}$$
In particular, for $\lambda_4 = 1.10$, we have $\beta_c(1.10) = 0.3750973(14)$, to be compared with $\beta_c(1.10) = 0.3750966(4)$ of Ref. [20]. For $\lambda_6 = 1$ and $1.8 \leq \lambda_4 \leq 2.0$ — as we shall see this is the relevant interval — we have

$$\beta_c(\lambda_4) = 0.68612192 - 0.18274273 \lambda_4 + 0.02634688 \lambda_4^2 - 0.00102710 \lambda_4^3 \pm 0.0000018. \quad (3.7)$$

The second quantity we need is an observable $O$ such that $O^*$ can be computed with high precision. We have chosen the Binder parameter

$$Q = \frac{\langle m^4 \rangle}{\langle m^2 \rangle^2}, \quad (3.8)$$

where $m$ is the magnetization. A precise estimate of $Q$ was obtained in Ref. [17] by means of a large-scale simulation of the spin-1 model. They report

$$Q^* = 0.62393(13+35+5), \quad (3.9)$$

where the error is given as the sum of three contributions: the first is the statistical error; the second and the third account for corrections to scaling. We have tried to improve this estimate by performing a high-precision Monte Carlo simulation of the Hamiltonian (1.2) for $\lambda_6 = 0$ and by computing $Q$ for $\lambda_4 = 1.10$, which is the best estimate of $\lambda_4^*$. Since this Hamiltonian is improved (i.e. the leading correction to scaling vanishes), we expect to be able to obtain a reliable estimate from simulations on small lattices for which it is possible to accumulate a large statistics. The results are reported in Table 1. There are three different sources of error: we report the statistical error, the variation of the estimate of the Binder parameter when $\lambda_4$ varies within the interval $1.08-1.12$ (due to corrections to scaling of order $L^{-\omega}$, which are not completely suppressed, since the value used for $\lambda_4$ is not exactly equal to $\lambda_4^*$), and the variation of $Q$ when $\beta_c$ varies within one error bar. The values of $L$ we use are relatively small ($L \leq 12$) and one could be afraid that next-to-leading corrections still give a non-negligible systematic deviation. Our data do not show any evidence of such an effect, and the estimates for different values of $L$ are consistent. Using the estimate obtained for $L = 12$, we get the final result

$$Q^* = 0.62388(32) \quad (3.10)$$

(the uncertainty is obtained assuming independence of systematic and statistical errors), which is in agreement with the estimate (3.4) with a slightly smaller error bar.

1We used the Brower-Tamayo algorithm [35], each iteration consisting of a Swendsen-Wang update of the sign of $\phi$ and of a Metropolis sweep.

2The simulations were performed before the appearance of Ref. [20] and the generation of the HT series, when only a very approximate expression for $\lambda_4^*$ existed. Therefore the runs were not made at the correct values of $\lambda_4$ and $\beta_c$. The values reported in Table 1 have been obtained from the Monte Carlo data by means of a standard reweighting technique.
TABLE I. For several values of lattice size $L$ and for $\lambda_6 = 0$, we report: the values of the parameters used in the Monte Carlo simulation $\lambda_{4,\text{run}}$, $\beta_{\text{run}}$, the number of Monte Carlo iterations $N_{\text{iter}}$, each iteration consisting of a standard Swendsen-Wang update and of a Metropolis sweep, and the the estimate of the Binder parameter $Q$ at $\lambda = 1.10$, $\beta = 0.3750973$. The reported error on $Q$ is the sum of three terms: the statistical error, and the errors due to the uncertainty of $\lambda^*$ and $\beta_c(\lambda)$.

| $L$ | $\lambda_{4,\text{run}}$ | $\beta_{\text{run}}$ | $N_{\text{iter}}$ | $Q$ |
|-----|----------------|-----------------|-----------------|-----|
| 6   | 1.100         | 0.375           | $91 \times 10^6$ | 0.62370(15+37+2) |
| 7   | 1.100         | 0.375           | $78 \times 10^6$ | 0.62386(17+33+2) |
| 9   | 1.080         | 0.376           | $178 \times 10^6$ | 0.62389(12+26+3) |
| 12  | 1.105         | 0.375           | $305 \times 10^6$ | 0.62387(10+25+5) |

TABLE II. For several values of lattice size $L$ and for $\lambda_6 = 1$, we report: the values of the parameters used in the Monte Carlo simulation $\lambda_{4,\text{run}}$, $\beta_{\text{run}}$, the number of Monte Carlo iterations $N_{\text{iter}}$, each iteration consisting of a standard Swendsen-Wang update and of a Metropolis sweep, and the estimate of $\lambda^\text{eff}(L)$, the solution of Eq. (3.4). The error is reported as the sum of three terms: the statistical error on $Q$, the error on $Q^*$, and the uncertainty of the critical value $\beta_c(\lambda)$.

| $L$ | $\lambda_{4,\text{run}}$ | $\beta_{\text{run}}$ | $N_{\text{iter}}$ | $\lambda^\text{eff}(L)$ |
|-----|----------------|-----------------|-----------------|-----------------|
| 6   | 1.900         | 0.427           | $100 \times 10^6$ | 1.915(9+19+1)   |
| 7   | 1.900         | 0.427           | $252 \times 10^6$ | 1.894(6+21+2)   |
| 9   | 1.920         | 0.425           | $214 \times 10^6$ | 1.897(9+25+3)   |
| 12  | 1.920         | 0.425           | $294 \times 10^6$ | 1.904(9+32+6)   |

We have next determined $\lambda^*_4$ for the model with Hamiltonian (1.2) and $\lambda_6 = 1$ using the method presented above. Estimates of $\lambda^\text{eff}(L)$ are reported in Table I from which we conclude

$$\lambda^*_4 = 1.90(4).$$

Notice that the last three points show a small upward trend which, although consistent with a statistical effect, could be a systematic increase due to the corrections of order $L^{-\omega_2+\omega'\omega}$. With the present statistical errors we cannot distinguish between these two possibilities: we have considered as our final estimate the average of the results obtained for $L = 7, 9, 12$, but we cannot exclude that the correct $\lambda^*_4$ is slightly larger than our estimate. However the quoted error should be large enough to include this systematic increase.

This effect could also be due to the fact that $Q^*$ is only approximately known. To check if this is the case, we have computed $\lambda^\text{eff}_{4,\pm}(L)$, the solutions of Eq. (3.4) with the r.h.s. replaced by $Q^* \pm \sigma_Q$, $\sigma_Q$ being the error on $Q^*$. If the increase is associated with the uncertainty of $Q^*$, we should observe that $\lambda^\text{eff}_{4,\pm}(L)$ have opposite trends, one increasing, the other decreasing. In the present case $\lambda^\text{eff}_{4,\pm}(L)$ are both increasing, thereby excluding that this effect is due to the uncertainty of $Q^*$.

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As explained in the introduction, the analysis of HT series for the determination of universal quantities is sensitive to non-analytic scaling corrections. As we will discuss below, one can use this fact to obtain a rough estimate of the optimal value of $\lambda$.

Consider for example the zero-momentum four-point coupling constant $g_4$ defined by

$$g_4 = -\frac{\chi_4}{\chi^2 \xi^3}, \quad (3.12)$$

where $\chi$, $\xi^2$ and $\chi_4$ are, respectively, the magnetic susceptibility, the second-moment correlation length and the zero-momentum four-point connected correlation function (definitions can be found in App. A1). We have chosen this observable because it appears to be affected by large corrections to scaling, but the method can be applied to any universal quantity. From the discussion of Sec. II, we have for $\beta \to \beta_c$

$$g_4(\beta) = g_4^* + c_\Delta (\beta_c - \beta)^\Delta + \ldots \quad (3.13)$$

where $g_4^*$ is a universal constant, and $c_\Delta$ is a non-universal amplitude depending on the Hamiltonian. For improved models, as discussed before, $c_\Delta = 0$. The traditional methods of analysis, e.g. those based on Padé (PA) and Dlog-Padé (DPA) approximants, are unable to handle an asymptotic behavior like (3.13) unless $\Delta$ is an integer number, thus leading to a systematic error. Integral approximants allow for non-analytic scaling corrections, but, as already said, with the series of moderate length available today, they need to be biased to give correct results: without any bias they give estimates that are similar to those of PA’s and DPA’s [14]. At present the only analyses that are able to effectively take into account the confluent corrections use biased approximants, fixing the value of $\beta_c$ and of the first non-analytic exponent $\Delta$ (see e.g. Refs. [10–12,14,36] for a discussion of this issue and for a presentation of the different methods used in the literature). The method we use has been proposed in Ref. [11] and generalized in Ref. [12]. The idea is to perform a Roskies transform (RT), i.e. the change of variables

$$z = 1 - (1 - \beta/\beta_c)^\Delta, \quad (3.14)$$

so that the non-analytic terms in $\beta_c - \beta$ become analytic in $1 - z$. Therefore the analysis of the resulting series by means of standard approximants should give correct results. For the models we are considering the exponent $\Delta$ is approximately 1/2 (e.g., Ref. [20] reports $\Delta = 0.532(6)$); for simplicity we have used the transformation (3.14) with $\Delta = 1/2$.

We have analyzed the HT expansion of $g_4$ for the model with Hamiltonian (1.2) and $\lambda_6 = 0$ for several values of $\lambda_4$. We computed PA’s, DPA’s, and first-order integral (IA1) approximants of the series in $\beta$ and of its Roskies transform (RT) in $z$. In Fig. 1 we plot the results as a function of $\lambda_4^{-1}$. The reported errors are related to the spread of the results obtained from the different approximants, see App. A4 for details. The estimates obtained from the RT’ed series are independent of $\lambda_4$ within error bars, giving the estimate $g_4^* \simeq 23.5$, in agreement with previous analyses of the HT expansion of the Ising model on various lattices using the RT or other types of biased approximants [14,15,36] (in Sec. V1 we will improve this estimate by analyzing the IHT expansion). The independence of the result from the value of $\lambda_4$ clearly indicates that the RT is effectively able to take into account the non-analytic behavior (3.13). On the other hand the analysis of the series in $\beta$ gives results
FIG. 1. Comparison of the determination of $g_4$ (plotted vs. $1/\lambda_4$) from HT series without (direct) and with the Roskies transform (RT) for the pure $\phi^4$ lattice model. The dashed line marks the more precise estimate (with its error) we derived from the analysis of the IHT expansion.

which vary with $\lambda_4$ more than the spread of the approximants: for instance, the analysis of the series of the standard Ising model, corresponding to $\lambda_4 = \infty$, gives results that differ by more than 5% from the estimate quoted above, while the spread of the approximants is much smaller. Clearly there is a large systematic error. It is important to notice that the direct analysis and the RT one coincide when $1.0 \lesssim \lambda_4 \lesssim 1.2$, i.e. in the region in which the leading non-analytic corrections are small. This fact confirms our claim that the observed discrepancies are an effect of the confluent corrections.

The results presented above can be used to obtain an estimate of $\lambda_4^*$ from the HT series alone: $\lambda_4^*$ should fall in the interval in which the direct analysis gives results compatible with those obtained from the RT’ed series. As we already mentioned, for $\lambda_6 = 0$ we obtain $\lambda_4^* = 1.1(1)$, while for $\lambda_6 = 1$ we get $\lambda_4^* = 1.9(1)$. The latter estimate was indeed the starting point of our Monte Carlo simulation. We have also tried to estimate $\lambda_4^*$ in more direct ways, but all methods we tried were even less precise.

A similar method for the determination of the improved Hamiltonian from the HT series was presented in Ref. [8]. The optimal value of the parameter (called $y$ in Ref. [8]) was determined comparing the results for the critical point $\beta_c(y)$ obtained using IA1’s and DPA’s:
$y^*$ is estimated from the value at which DPA and IA1 estimates of $\beta_c(y)$ agree between each other. It should be noticed that, for the double-Gaussian model, partial differential approximants and a later analysis of Nickel and Rehr [16] using a different method gave significantly different estimates of $y^*$.

IV. CRITICAL EXONENTS

The analysis of Sec. III is encouraging and supports our basic assumption that the systematic error due to confluent singularities is largely reduced when analyzing IHT expansions. To further check this hypothesis we will compare results obtained from different improved Hamiltonians. This will provide an estimate of the remaining systematic error which is not covered by the spread of the results from different approximants.

The definition of the quantities we have considered and a detailed description of the method we used to generate and analyze the HT series is presented in App. A.

We computed $\beta_c$ and $\gamma$ from the analysis of the HT expansion of the magnetic susceptibility. We considered integral approximants of first, second and third order. After a careful analysis we preferred the second-order integral approximants (IA2’s), which turned out to be the most stable: most of the results we present in this section and in the related App. A3 have been obtained by using IA2’s. As a further check of the effectiveness of the approximants employed, we made use of the fact that $\chi$ (and $\xi^2$) must present an antiferromagnetic singularity at $\beta_{af} = -\beta_c$ of the form [37]

$$\chi = c_0 + c_1 (\beta - \beta_{af})^{1-\alpha} + ...,$$

(4.1)

where $\alpha$ is the specific heat exponent, $c_i$ are constants and the ellipses represent higher-order singular or analytic corrections. We verified the existence of a singularity at $\beta \simeq -\beta_c$ in the approximants, and calculated the associated exponent. We also considered approximants that were biased requiring the presence of two symmetric singularities at $\beta = \pm \beta_c$ [8]; the results obtained are consistent with the predicted behavior (4.1) (see App. A3 and related Tables).

The exponent $\nu$ was obtained from the series of the second-moment correlation length

$$\xi^2 = \frac{m_2}{6\chi} \sim (\beta_c - \beta)^{2\nu},$$

(4.2)

where $m_i$ are the moments of the two-point function. We followed the procedure suggested in Ref. [38], i.e. we used the estimate of $\beta_c$ obtained from $\chi$ to bias the analysis of $\xi^2$. For this purpose we used IA’s biased to $\beta_c$. We also considered approximants biased to have a pair of singularities at $\pm \beta_c$.

In Table III we report the results obtained for the Hamiltonians (1.2) with $\lambda_6 = 0$, $\lambda_4 = 1.10$ and with $\lambda_6 = 1$, $\lambda_4 = 1.90$, and for the Hamiltonian (1.4) with $D = 0.641$. The errors are given as a sum of two terms: the first one is computed from the spread of the approximants; the second one is related to the uncertainty of the value of $\lambda_4^*$ and $D^*$, and it is evaluated by changing $\lambda_4$ in the range 1.08–1.12 for $\lambda_6 = 0$ and 1.86–1.94 for $\lambda_6 = 1$, and $D$ in the range 0.633–0.649 for the spin-1 model. There is a good agreement among the
TABLE III. Our final estimates of $\gamma$, $\nu$, $\eta$ and $\sigma$. The error is reported as a sum of two terms: the first one is related to the spread of the approximants; the second one is related to the uncertainty of the value of $\lambda^*_6$.

|       | $\gamma$       | $\nu$        | $\eta$       | $\sigma$       |
|-------|-----------------|--------------|--------------|----------------|
| $\lambda_6 = 0$ | 1.23732(24+16) | 0.63015(13+12) | 0.0364(3+1) | 0.0213(13+1) |
| $\lambda_6 = 1$ | 1.23712(26+31) | 0.63003(13+23) | 0.0363(3+2) | 0.0213(14+2) |
| spin-1 | 1.23680(30+12) | 0.62990(15+8)  | 0.0366(3+2) | 0.0202(10+1) |

estimates of $\gamma$ and $\nu$ obtained from the three improved Hamiltonians considered. This is an important check of our working hypothesis, i.e. that systematic errors due to confluent corrections are largely reduced. This will be also confirmed by the results for the universal ratios of amplitudes. We determine our final estimates by combining the results of the three improved Hamiltonians: as estimate we take the weighted average of the three results, and as estimate of the uncertainty the smallest of the three errors. We obtain for $\gamma$ and $\nu$

$$\gamma = 1.2371(4), \quad (4.3)$$
$$\nu = 0.63002(23), \quad (4.4)$$

and by the hyperscaling relation $\alpha = 2 - 3\nu$

$$\alpha = 0.1099(7). \quad (4.5)$$

In App. A3 we also report some further checks using the Monte Carlo estimate of $\beta_c$ reported in Ref. [20] to bias the analysis of the series. The results are perfectly consistent. We mention that from the analysis of the antiferromagnetic singularity, cf. Eq. (4.1), we obtain the estimate $\alpha = 0.105(10)$, which is consistent with result (4.5) obtained assuming hyperscaling.

From the results for $\gamma$ and $\nu$, we can obtain $\eta$ by the scaling relation $\gamma = (2 - \eta)\nu$. This gives $\eta = 0.0364(10)$, where the error is estimated by considering the errors on $\gamma$ and $\nu$ as independent, which is of course not true. We can obtain an estimate of $\eta$ with a smaller, yet reliable, error using the so-called critical point renormalization method (CPRM) (see Ref. [9] and references therein). We obtain the results reported in Table III, with considerably smaller errors. Our final estimate is

$$\eta = 0.0364(4). \quad (4.6)$$

Moreover using the scaling relations we obtain

$$\delta = \frac{5 - \eta}{1 + \eta} = 4.7893(22), \quad (4.7)$$
$$\beta = \frac{\nu}{2} (1 + \eta) = 0.32648(18) \quad (4.8)$$

(the error on $\beta$ has been estimated by considering the errors of $\nu$ and $\eta$ as independent).

Finally we consider the universal critical exponent describing how the spatial anisotropy, which is present in physical systems with cubic symmetry (e.g. uniaxial magnets), vanishes
when approaching the rotationally-invariant fixed point $^{39}$. For this class of systems the two-point function $G(x)$ is not rotationally invariant. Therefore non-spherical moments are in general non-vanishing, but near the critical point they are depressed with respect to spherical moments carrying the same naive physical dimensions by a factor $\xi^{-\rho}$, where $\rho$ is a universal critical exponent. From a field-theoretical point of view, space anisotropy is due to non rotationally invariant irrelevant operators in the effective Hamiltonian, whose presence depends essentially on the symmetries of the physical system, or of the lattice formulation. In Table III we report the results for $\sigma \equiv 2 - \rho$ as obtained by analyses of the first non-spherical moments (cf. Eq. (A2)) using the CPRM. The exponent $\sigma$ turns out to be very small:

$$\sigma = 0.0208(12),$$

and $\rho = 1.9792(12)$.

In Table IV we compare our results with some of the most recent estimates of the critical exponents $\gamma$, $\nu$, $\eta$, $\alpha$, and $\beta$. The table should give an overview of the state of the art for the various approaches. Let us first note the good agreement of our IHT estimates with the very precise results of the recent Monte Carlo simulations (MC) of Refs. $^{17,19,20}$. The small difference with the HT estimates of Ref. $^{13}$ (obtained from the standard Ising model) may be explained by the difficulty of controlling the effects of the confluent singularities, and by a systematic error induced by the uncertainty on the external input parameters ($\beta_c$ and $\Delta$) that are used in their biased analysis. The estimates of Refs. $^{5,7,8,16}$ have been obtained from a HT analysis of two families of models, the Klauder and the double-Gaussian models on the bcc lattice. The results of these analyses are in good agreement with our IHT estimates, especially those by Nickel and Rehr $^{17}$. The HT series for the double-Gaussian model were analyzed also in Ref. $^{8}$ where a higher estimate of $\gamma$ was obtained. As pointed out in Ref. $^{10}$, the discrepancy is essentially due to the use of a higher estimate of the improvement parameter $y^*$ with respect to that used in Ref. $^{13}$ (see the discussion at the end of Sec. III). Refs. $^{8,16}$ report also estimates of $\alpha$ obtained analyzing the singularity of the susceptibility at the antiferromagnetic critical point. The result agrees with our estimate.

The agreement with the field-theoretical calculations is overall good. The slightly larger result for $\gamma$ obtained in the analyses of Refs. $^{14,17}$ (using $O(g^7)$ series $^{16,20}$) may be due to an underestimate of the systematic error due to the non-analyticity of the Callan-Symanzik $\beta$-function. Similar results have been obtained by Kleinert, who resummed the $O(g^7)$ expansion by a variational method $^{17}$, still neglecting confluent singularities at the infrared-stable fixed point. We shall return on this point later. A better agreement is found with the analysis of the $d=3$ $g$-expansion performed by Murray and Nickel, who allow for a more general non-analytic behavior of the $\beta$-function $^{17}$. In Table IV, we quote two errors on the results of Ref. $^{16}$: the first one is the resummation error, and the second one takes into account the uncertainty of $g^*$, which is estimated to be $\sim 0.01$. The results of the $\epsilon$-expansion were obtained from the $O(\epsilon^5)$ series calculated in Refs. $^{21,22}$. We report estimates obtained by performing standard analyses (denoted as “free”) and constrained analyses $^{53}$ (denoted by “bc”) that incorporate the knowledge of the exact two-dimensional values. Both are essentially consistent with our IHT estimates, but present a significantly larger uncertainty. In Table IV we also report the results obtained by approximately solving
### TABLE IV. Theoretical estimates of critical exponents. See text for explanation of symbols in the first column. For values marked with an asterisk, the error is not quoted explicitly in the reference.

|        | \(\gamma\)          | \(\nu\)          | \(\eta\) | \(\alpha\) | \(\beta\) |
|--------|----------------------|-------------------|-----------|------------|-----------|
| IHT    | 1.2371(4)            | 0.63002(23)       | 0.0364(4) | 0.1099(7)  | 0.32648(18) |
| HT (sc) [13] | 1.2388(10)           | 0.6315(8)         |           |            |           |
| HT (bcc) [13] | 1.2384(6)            | 0.6308(5)         |           |            |           |
| HT [16] | 1.237(2)             | 0.6300(15)        | 0.0359(7) | 0.11(2)    |           |
| HT [38] | 1.239(3)             | 0.632\(+0.002\)  |           |            |           |
| HT [8]  | 1.2395(4)            | 0.632(1)          |           | 0.105(7)   |           |
| HT [7]  | 1.2378(6)            | 0.63115(30)       |           |            |           |
| HT [3]  | 1.2385(15)           |                   |           |            |           |
| HT [1]  | 1.2385(25)           | 0.6305(15)        |           |            |           |
| MC [20] | 1.2367(11)           | 0.6296(7)         | 0.0358(9) |            |           |
| MC [17] | 0.6298(5)            |                   | 0.0366(8) |            |           |
| MC [19] | 0.6294(10)           |                   | 0.0374(12)|            |           |
| MC [10] | 0.6308(10)           |                   |           |            | 0.3269(6) |
| MC [11] |                     |                   |           |            |           |
| MC [12] | 0.625(1)             | 0.025(6)          |           |            |           |
| MC [13] | 1.237(2)             | 0.6301(8)         | 0.037(3)  | 0.110(2)   | 0.3267(10) |
| \(\epsilon\)-exp-free [44] | 1.2355(50) | 0.6290(25) | 0.0360(50) |            | 0.3257(25) |
| \(\epsilon\)-exp-bc [44]  | 1.2380(50)          | 0.6305(25)        | 0.0365(50)|            | 0.3265(15) |
| \(\epsilon\)-exp-bc [14]  | 1.240(5)            | 0.631(3)          |           |            |           |
| \(d=3\) g-exp. [44]       | 1.2396(13)          | 0.6304(13)        | 0.0335(25)| 0.109(4)   | 0.3258(14) |
| \(d=3\) g-exp. [45]       | 1.241*              | 0.6305*           | 0.0347(10)|            |           |
| \(d=3\) g-exp. [46]       | 1.2378(6+18)        | 0.6301(5+11)      | 0.0355(9+6)|            |           |
| \(d=3\) g-exp. [47]       | 1.2405(15)          | 0.6300(15)        | 0.032(3)  |            |           |
| ERG [44]                    |                      | 0.618(14)         | 0.054*    |            |           |
| ERG [43]                    | 1.247*              | 0.638*            | 0.045*    |            |           |

Theoretical results represent a substantial improvement of the estimates obtained by various approaches (HT and field theory) presented in Ref. [39]. Concerning the exponent \(\sigma\) related to the rotational symmetry, the IHT results represent a substantial improvement of the estimates obtained by various approaches (HT and field theory) presented in Ref. [39].

Experimental results have been obtained studying the liquid-vapor transition in simple fluids, and the different critical transitions in multicomponent fluid mixtures, uniaxial antiferromagnetic materials and micellar systems. Many recent estimates can be found in Refs. [13, 54, 55]. In Table V we report some experimental results, most of them published after 1990. It is not a complete list of the published results, but it may be useful to get an overview of the experimental state of the art. Even if the systems studied are quite different, the results substantially agree, although, looking in more detail, as already observed in Ref. [43], one can find small discrepancies. Moreover they substantially agree with the theoretical
TABLE V. Experimental estimates of critical exponents. lv denotes the liquid-vapor transition in simple fluids, bm refers to a binary fluid mixture, ms to a uniaxial magnetic system, and mi to a micellar system.

| Ref. | γ      | ν      | η      | α      | β      |
|------|--------|--------|--------|--------|--------|
| lv   | 0.1105 | 0.0250 | 0.0270 | 0.111  | 0.324  |
|      | 0.1075 (54) |        |        |        |        |
|      | 0.1084 (23) |        |        |        |        |
|      | 0.111 (1)  | 0.324 (2) |    |        |        |
|      | 0.341 (2)  |        |        |        |        |
| bm   | 0.104  | 0.104 (11) |        | 0.327  |        |
|      | 1.233 (10) |        |        |        |        |
|      | 1.093 |        |        |        |        |
|      | 1.26 (5) | 0.64 (2) |        |        |        |
|      | 1.24 (1) | 0.606 (18) | 0.077 (44) | 0.319 (14) |        |
|      | 0.105 (8) |        |        |        |        |
|      |        |        | 0.324 (5), 0.329 (2) |        |        |
|      |        |        | 0.329 (4), 0.333 (2) |        |        |
|      |        |        | 0.336 (30) |        |        |
| ms   | 1.25 (2) | 0.64 (1) | 0.115 (4) | 0.331 (6) |        |
|      |        |        | 0.11 (3) | 0.325 (2) |        |
|      |        |        | 0.11 (3) | 0.315 (15) |        |
| mi   |        |        |        |        | 0.34 (8) |
|      | 1.17 (11) | 0.65 (4) | 0.18 (3) | 0.60 (2) |        |
|      | 1.216 (13) | 0.623 (13) | 0.039 (4) |        |        |
|      | 1.237 (7) | 0.630 (12) |    |        |        |
|      | 1.25 (2) | 0.63 (1) |    |        |        |
|      |        |        | 1.17 (11) | 0.65 (4) |        |

predictions discussed above, confirming the fact that all these transitions are in the Ising universality class. It should also be noticed that the experimental results are less accurate than the theoretical estimates.

V. THE EFFECTIVE POTENTIAL
A. Small-field expansion of the effective potential in the high-temperature phase

The effective potential (Helmholtz free energy) is related to the (Gibbs) free energy of the model. Indeed, if $M \equiv \langle \phi \rangle$ is the magnetization and $H$ the magnetic field, one defines

$$F(M) = MH - \frac{1}{V} \log Z(H),$$

where $Z(H)$ is the partition function and the dependence on the temperature is always understood in the notation.

The global minimum of the effective potential determines the value of the order parameter which characterizes the phase of the model. In the high-temperature or symmetric phase the minimum is unique with $M = 0$. According to the Ginzburg-Landau theory, as the temperature decreases below the critical value, the effective potential takes a double-well shape. The order parameter does not vanish anymore and the system is in the low-temperature or broken phase. Actually in the broken phase the double-well shape is not correct because the effective potential must be convex \[85\]. In this phase it should present a flat region around the origin.

In the high-temperature phase the effective potential admits an expansion around $M = 0$:

$$\Delta F \equiv F(M) - F(0) = \sum_{j=1}^{\infty} \frac{1}{(2j)!} a_{2j} M^{2j}.$$

The coefficients $a_{2j}$ can be expressed in terms of renormalization-group invariant quantities. Introducing a renormalized magnetization

$$\phi^2 = \frac{\xi(t, H=0)^2 M(t, H)^2}{\chi(t, H=0)},$$

where $t$ is the reduced temperature, one may write

$$\Delta F = \frac{1}{2} m^2 \phi^2 + \sum_{j=2} m^{d-j(d-2)} \frac{1}{(2j)!} g_{2j} \phi^{2j}.$$

Here $m = 1/\xi$, $g_{2j}$ are functions of $t$ only, and $d$ is the space dimension. In field theory $\phi$ is the expectation value of the zero-momentum renormalized field. For $t \to 0$ the quantities $g_{2j}$ approach universal constants (which we indicate with the same symbol) that represent the zero-momentum $2j$-point renormalized coupling constants. By performing a further rescaling

$$\phi = \frac{m^{(d-2)/2}}{\sqrt{g_4}} \phi^2$$

in Eq. (5.4), the free energy can be written as

$$\Delta F = \frac{m^d}{g_4} A(z),$$

(5.6)
where

\[ A(z) = \frac{1}{2}z^2 + \frac{1}{4!}z^4 + \sum_{j=3} \frac{1}{(2j)!} r_{2j} z^{2j}, \quad (5.7) \]

and

\[ r_{2j} = \frac{g_{2j}}{g_4^j} \quad j \geq 3. \quad (5.8) \]

One can show that \( z \propto t^{-\beta} M \), and that the equation of state can be written in the form

\[ H \propto t^{\beta \delta} \frac{\partial A(z)}{\partial z}. \quad (5.9) \]

The effective potential \( F(M) \) admits a power-series expansion also near the coexistence curve, i.e. for \( t < 0 \) and \( H = 0 \). If \( M_0 = \lim_{H \to 0^+} M(H) \), for \( M > M_0 \) (i.e. for \( H \geq 0 \)) we have

\[ \delta F \equiv F(M) - F(M_0) = \sum_{j=2} \frac{1}{j!} a_j (M - M_0)^j. \quad (5.10) \]

In terms of the renormalized magnetization \( \varphi \) we can rewrite

\[ \delta F = \frac{1}{2} m^2 \varphi^2 - \phi_0^2 = \sum_{j=3} \frac{1}{j!} m^d - j(d-2)/2 \frac{1}{j!} g_j^-(\varphi - \phi_0)^j, \quad (5.11) \]

where \( m \equiv 1/\xi^- \) and \( \xi^- \) is the second-moment correlation length defined in the low-temperature phase. For \( t \to 0^- \), the quantities \( g_j^- \) approach universal constants that represent the low-temperature zero-momentum \( j \)-point renormalized coupling constants. A simpler parametrization can be obtained if we introduce \( u \equiv \frac{M}{M_0} \),

\[ u \equiv \frac{M}{M_0}, \quad (5.12) \]

so that

\[ \delta F = \frac{m^d}{w^2} B(u), \quad (5.13) \]

where

\[ w^2 \equiv \lim_{T \to T_c^-} \lim_{H \to 0} \frac{\chi}{M^2 \xi^d}. \quad (5.14) \]

The scaling function \( B(u) \) has the following expansion

\[ B(u) = \frac{1}{2} (u - 1)^2 + \sum_{j=3} \frac{1}{j!} v_j (u - 1)^j, \quad (5.15) \]

where

\[ v_j = \frac{g_j^-}{w^{j-2}}. \quad (5.16) \]
B. The four-point zero-momentum renormalized coupling

The four-point coupling $g \equiv g_4$ plays an important role in the field-theoretic perturbative expansion at fixed dimension [87], which provides an accurate description of the critical region in the symmetric phase. In this approach, any universal quantity is obtained from a series in powers of $g$ ($g$-expansion), which is then resummed and evaluated at the fixed-point value of $g$, $g^*$ (see e.g. Refs. [47,50]). The theory is renormalized at zero momentum by requiring

$$\Gamma^{(2)}(p) = Z^{-1} \left[M^2 + p^2 + O(p^4)\right], \quad (5.17)$$

$$\Gamma^{(4)}(0, 0, 0, 0) = Z^{-2} M g. \quad (5.18)$$

When $M \to 0$ the coupling $g$ is driven toward an infrared-stable zero $g^*$ of the corresponding Callan-Symanzik $\beta$-function

$$\beta(g) \equiv M \frac{\partial g}{\partial M} \bigg|_{g_0, \Lambda}. \quad (5.19)$$

In this context a rescaled coupling is usually introduced (see e.g. Ref. [1]):

$$\bar{g} = \frac{3}{16\pi} g. \quad (5.20)$$

An important issue in this field-theoretical approach concerns the analytic properties of $\beta(g)$, that are relevant for the procedure of resummation of the $g$-expansion. General renormalization-group arguments predict a non-analytic behavior of $\beta(g)$ at $g = g^*$ [87]. One expects a behavior of the form [88]

$$\beta(g) = -\omega (g^* - g) + b_1 (g^* - g)^2 + \ldots + c_1 (g^* - g)^{1+\Delta_1} + \ldots + d_1 (g^* - g)^{\Delta_2} + \ldots \quad (5.21)$$

($\Delta = \omega \nu$ and $\Delta_2$ are scaling correction exponents). In the framework of the $1/N$ expansion of O($N$) $\phi^4$ models, the analysis [14] of the next-to-leading order of the Callan-Symanzik $\beta$-function, calculated in Ref. [89], shows explicitly the presence of confluent singularities of the form (5.21).

In the fixed-dimension field-theoretical approach, a precise determination of $g^*$ is crucial, since the critical exponents are obtained by evaluating appropriate (resummed) anomalous dimensions at $g^*$. The resummation of the $g$-expansion is usually performed following the Le Guillou-Zinn-Justin (LZ) procedure [17], which assumes the analyticity of the $\beta$-function. The presence of confluent singularities may then cause a slow convergence to the correct fixed-point value, leading to an underestimate of the uncertainty derived from stability criteria.

We have computed $g^* \equiv g_4^*$ from our IHT series by calculating the critical limit of the quantity $g_4$ defined in Eq. (3.12). A description of our analysis can be found in App. A4. The results are reported in Table VI. We find good agreement among the results of the three improved Hamiltonians, that lead to our final estimate:

$$g^* = 23.49(4), \quad \bar{g}^* = 1.402(2). \quad (5.22)$$
TABLE VI. Results for $g^*_4$, $r_6$, $r_8$, $r_{10}$, $c_2$ and $c_3$ derived from the analysis of the IHT series (see App. A). The error is reported as a sum of two terms: the first one is related to the spread of the approximants; the second one is related to the uncertainty of the value of $\lambda^*_4$.

| $\lambda_6 = 0$ | $g^*_4$ | $r_6$ | $r_8$ | $r_{10}$ | $10^4c_2$ | $10^4c_3$ |
|----------------|----------|-------|-------|--------|---------|---------|
| 23.499(16+20)  | 2.051(7+2) | 2.23(5+4) | -14(4) | -3.582(7+6) | 0.085(6) |
| $\lambda_6 = 1$ | 23.491(21+40) | 2.050(5+4) | 2.23(5+6) | -13(5) | -3.574(7+20) | 0.086(4) |
| spin-1         | 23.487(18+20) | 2.046(2+3) | 2.34(5+3) | -8(25) | -3.568(11+4) | 0.090(4) |

TABLE VII. Estimates of $\bar{g}^* \equiv 3g^*/(16\pi)$. (sc) and (bcc) in the HT estimates of Ref. [15] denote simple cubic and bcc lattice respectively. For values marked with an asterisk, the error is not quoted explicitly in the reference.

| IHT             | HT            | $\epsilon$-exp. | $d=3$ $g$-exp. | MC           | $d$-exp. | ERG        |
|-----------------|---------------|-----------------|-----------------|--------------|---------|------------|
| 1.402(2)        | 1.407(6)      | 1.397(8)        | 1.411(4)        | 1.39(3)      | 1.412(14) | 1.72(3)    |
| 1.407(6)        | 1.406(9)      | 1.391*          | 1.40*           | 1.408(12)    | 1.462(12) | 1.72*      |
| 1.414(6)        | 1.415*        | 1.415*          | 1.409(3)        | 1.416(5)     | 1.462(12) | 1.72*      |
| 1.459(9)        | 1.46(2)       | 1.46(2)         | 1.47(3)         | 1.49(3)      | 1.46(2)  | 1.72*      |
| 1.42(9)         | 1.43(2)       | 1.43(2)         | 1.44(3)         | 1.45(3)      | 1.44(2)  | 1.72*      |

Table VII presents a selection of estimates of $\bar{g}^*$ obtained by different approaches. The HT estimates of Refs. [14,15,36] were obtained by using the RT or appropriate biased approximants in order to handle the leading confluent correction. The larger result of Ref. [96] could be explained by an effect of the scaling corrections. Field-theoretical estimates are reasonably consistent, especially those obtained from a constrained analysis of the $O(\epsilon^4)$ $\epsilon$-expansion [14]. In the $d=3$ $g$-expansion approach $g^*$ is determined from the zero of $\beta(g)$ after resumming its available $O(g^7)$ series. The results obtained using the LZ resummation method [44] show a slight discrepancy from our IHT estimates. This difference can explain the apparent discrepancy found in the determination of $\gamma$. Indeed, the sensitivity of $\gamma$ to $\bar{g}^*$, quantified in Ref. [44] through $d\gamma/d\bar{g}^* \simeq 0.18$, tells us that changing the value of $\bar{g}^*$ from 1.411 (which is the value obtained from the zero of $\beta(g)$) to 1.402 shifts $\gamma$ from 1.2396 to 1.2380, which is much closer to the IHT estimate $\gamma = 1.2371(4)$. Similarly for $\nu$, using $d\nu/d\bar{g}^* \simeq 0.11$ [44], $\nu$ would change from 0.6304 to 0.6294, which is quite acceptable, since a residual uncertainty due to the resummation of $\nu(g)$ is still present. The more general analysis of the $g$-expansion of Ref. [10] leads to a smaller value $\bar{g}^* = 1.40$, with an uncertainty estimated by the authors to be about 1%. In Table VII we also report estimates obtained by approximately solving the exact renormalization group equation $[14,18]$ (ERG), and from a dimensional expansion of the Green’s functions around $d = 0$ [44] ($d$-exp.). Concerning Monte Carlo (MC) results, we mention that the result of Ref. [90] has been obtained by studying the probability distribution of the average magnetization (see also Ref. [98] for a work employing a similar approach). The other estimates have been obtained from fits to data in the neighborhood of $\beta_c$. In Ref. [18] Monte Carlo simulations were performed using the Hamiltonian [12] with $\lambda_6 = 0$ and $\lambda_4 = 1$, which is close to its optimal value. A fit to the data of $g_4$, kindly made available to us by the authors, gives the estimate $g^* = 23.41(24)$ (i.e. $\bar{g}^* = 1.397(14)$), which is in agreement with our IHT estimate. In Ref. [92] a finite-size scaling technique is used to obtain data for large correlation lengths, then the estimate of
$g_4^*$ is extracted by a fit taking into account the leading scaling correction. The Monte Carlo estimates of Refs. [94,95] were larger because the effects of scaling corrections were neglected, as already observed in Ref. [14]. A more complete list of references regarding this issue can be found in Ref. [14].

C. Higher-order zero-momentum renormalized couplings

To compute the HT series of the effective-potential parameters $r_{2j}$ defined in Eq. (5.8), we rewrite them in terms of the zero-momentum connected $2j$-point Green’s functions $\chi_{2j}$ as

$$r_6 = 10 - \frac{\chi_6 \chi_2}{\chi_4^2},$$

$$r_8 = 280 - 56 \frac{\chi_6 \chi_2}{\chi_4^2} + \frac{\chi_8 \chi_2^2}{\chi_4^3},$$

$$r_{10} = 15400 - 4620 \frac{\chi_6 \chi_2}{\chi_4^2} + 126 \frac{\chi_8 \chi_2^2}{\chi_4^3} + 120 \frac{\chi_{10} \chi_2^3}{\chi_4^4} - \frac{\chi_{10} \chi_3}{\chi_4^2} \chi_4^2,$$

etc... Details of the analysis of the series are reported in App. A4. Combining the results reported in Table VI, we obtain the following estimates:

$$r_6 = 2.048(5),$$

$$r_8 = 2.28(8),$$

$$r_{10} = -13(4).$$

From the results for $r_{2j}$ we can obtain estimates of the couplings

$$g_6 = g^2 r_6 = 1130(5),$$

$$g_8 = g^3 r_8 = 2.96(11) \times 10^4,$$

$$g_{10} = g^4 r_{10} = -4.0(1.2) \times 10^6.$$

In the literature several approaches have been used for the determination of the couplings $g_{2j}$. Table VIII presents a review of the available estimates of $r_6$, $r_8$ and $r_{10}$. We also mention the estimate $r_{10} = -10(2)$ we will obtain in Sec. VII by studying the equation of state. The agreement with the field-theoretic calculations based on the $\epsilon$-expansion [44,99] and on the $d=3$ $g$-expansion [44] is good. Precise estimates of $r_{2j}$ have also been obtained in Ref. [48] (see also Ref. [49]) by ERG, although the estimate of $g_4^*$ by the same method is not as good. Additional results have been obtained from HT expansions [36,90,91] and Monte Carlo simulations [50,52] of the Ising model. The Monte Carlo results do not agree with the results of other approaches, especially those of Refs. [21,75], which are obtained using finite-size scaling techniques. But one should consider the difficulty of such calculations due to the subtractions that must be performed to compute the irreducible correlation functions. A more complete list of references regarding this issue can be found in Refs. [23,44,99].
TABLE VIII. Estimates of $r_{2j}$. When the original reference reports only estimates of $g_{2j}$ (see Refs. [90,95,97]), the errors we quote for $r_{2j}$ have been calculated by considering the estimates of $g_{2j}$ as uncorrelated. For values marked with an asterisk, the error is not quoted explicitly in the reference.

|     | IHT       | HT        | $\epsilon$-exp. | $d=3$ g-exp. | MC        | ERG       |
|-----|-----------|-----------|-----------------|--------------|-----------|-----------|
| $r_6$ | 2.048(5)  | 1.99(6)    | 2.058(11)       | 2.053(8)     | 2.72(23)  | 2.064(36) |
|      | 2.157(18) | 2.12(12)   | 2.060*          | 3.37(11)     | 1.92*     | 1.92      |
|      | 2.25(9)   |           |                 | 3.26(26)     |           |           |
| $r_8$ | 2.28(8)   | 2.74(4)    | 2.48(28)        | 2.47(25)     | 2.47(5)   | 2.18*     |
|      |           | 2.42(30)   |                 | 2.47(5)      |           |           |
| $r_{10}$ | -13(4)   | -4(2)      | -20(15)         | -25(18)      | -18(4)    |           |
|      |           |            |                 |              |           |           |

VI. THE TWO-POINT FUNCTION

The critical behavior of the two-point correlation function $G(x)$ of the order parameter is relevant to the description of critical scattering phenomena, which can be observed in many experiments, such as light and X-ray scattering in fluids, magnets... In Born approximation the cross section $\Gamma_{fi}$ for particles of incoming momentum $p_i$ and outgoing momentum $p_f$ is proportional to the component $k = p_f - p_i$ of the Fourier transform of $G(x)$:

$$\Gamma_{fi} \propto \tilde{G}(p_f - p_i).$$

(6.1)

As a consequence of the critical behavior of the two-point function $G(x)$ at $T_c$,

$$\tilde{G}(k) \sim \frac{1}{k^{2-\eta}},$$

(6.2)

the cross section for $k \to 0$ (forward scattering) diverges as $T \to T_c$. When strictly at criticality, Eq. (6.2) holds for all $k \ll \Lambda$, where $\Lambda$ is a generic cut-off related to the microscopic structure of the statistical system, e.g. the inverse lattice spacing in the case of lattice models. In the vicinity of the critical point, where the relevant correlation length $\xi$ is large but finite, the behavior (6.2) occurs for $\Lambda \gg k \gg 1/\xi$. At low momentum, $k \ll 1/\xi$, experiments show that $G(x)$ is well approximated by a Gaussian (Ornstein-Zernike) behavior,

$$\tilde{G}(0) \simeq 1 + \frac{k^2}{M^2},$$

(6.3)

where $M \sim 1/\xi$ is a mass scale defined at zero momentum (for a general discussion see e.g. Ref. [101]). Corrections to Eq. (6.3) are present, and reflect, once more, the non-Gaussian nature of the Wilson-Fisher fixed point. The above-mentioned experimental observations, confirmed by theoretical studies [99,102], show that they are small. In the following we will improve the determination of the critical two-point function at low-momentum using IHT series.
In order to study the low-momentum universal critical behavior of the two-point function \( G(x) = \langle \phi(x)\phi(0) \rangle \), we consider the scaling function
\[
g(y) = \frac{\chi}{\tilde{G}(k)}, \quad y \equiv k^2/M^2, \quad (6.4)
\]
\((M \equiv 1/\xi\) and \(\xi\) is the second-moment correlation length\) in the critical limit \(k, M \to 0\) with \(y\) fixed. The scaling function \(g(y)\) can be expanded in powers of \(y\) around \(y = 0\):
\[
g(y) = 1 + y + \sum_{i=2}^{\infty} c_i y^i. \quad (6.5)
\]
Other important quantities which characterize the low-momentum behavior of \(g(y)\) are the critical limit of the ratios
\[
S_M \equiv \frac{M_{\text{gap}}^2}{M^2}, \quad (6.6)
\]
\[
S_Z \equiv \frac{\chi M^2}{Z_{\text{gap}}}, \quad (6.7)
\]
where \(M_{\text{gap}}\) (the mass gap of the theory) and \(Z_{\text{gap}}\) determine the long-distance behavior of the two-point function:
\[
G(x) \approx \frac{Z_{\text{gap}}}{4\pi|x|} e^{-M_{\text{gap}}|x|}. \quad (6.8)
\]
The critical limits of \(S_M\) and \(S_Z\) are related to the negative zero \(y_0\) of \(g(y)\) closest to the origin by
\[
S_M = -y_0, \quad (6.9)
\]
\[
S_Z = \left. \frac{\partial g(y)}{\partial y} \right|_{y=y_0}. \quad (6.10)
\]
The coefficients \(c_i\) can be related to the critical limit of appropriate dimensionless ratios of spherical moments of \(G(x)\) (as shown explicitly in App. A1) and can be calculated by analyzing the corresponding HT series. Some details of the analysis of our HT series are reported in App. A4. In Table VI we report the results for \(\lambda_6 = 0, 1\) and the spin-1 model. We obtain the estimates
\[
c_2 = -3.576(13) \times 10^{-4}, \quad (6.11)
\]
\[
c_3 = 0.87(4) \times 10^{-5}, \quad (6.12)
\]
and the bound
\[
-10^{-6} \lesssim c_4 < 0. \quad (6.13)
\]
The constants \(c_i\) and \(S_M\) can also be calculated by field-theoretic methods. They have been computed to \(O(\epsilon^3)\) in the framework of the \(\epsilon\)-expansion \([103]\), and to \(O(g^4)\) in the framework of the \(d=3\) \(g\)-expansion \([39]\). In Table IX we report the results of constrained analyses of the \(O(\epsilon^3)\) \(\epsilon\)-expansion of \(c_i\) and \(S_M - 1\), using exact results in \(d = 2, 1\) \((S_M = 1\) and \(c_i = 0\) in
TABLE IX. $c_i$ and $S_M - 1$ obtained from $O(\epsilon^3)$ series: unconstrained analysis (unc) and analyses constrained in dimensions $d = 1, 2$.

|       | unc      | $d = 1$ | $d = 2$ | $d = 1, 2$ |
|-------|----------|---------|---------|------------|
| $10^4(S_M - 1)$ | $-4.4(1.0)$ | $-3.3(8)$ | $-3.3(5)$ | $-3.24(36)$ |
| $10^4c_2$ | $-4.3(9)$ | $-3.2(8)$ | $-3.3(4)$ | $-3.30(21)$ |
| $10^5c_3$ | $1.13(27)$ | $0.84(22)$ | $0.76(17)$ | $0.69(10)$ |
| $10^6c_4$ | $-0.50(13)$ | $-0.37(10)$ | $-0.32(8)$ | $-0.27(5)$ |

TABLE X. Estimates of $S_M$ and $c_i$. (sc) and (bcc) denote the simple cubic and the body-centered cubic lattice respectively.

|       | IHT      | HT      | $\epsilon$-exp. | $d = 3$ g-exp. |
|-------|----------|---------|-----------------|----------------|
| $c_2$ | $-3.576(13) \times 10^{-4}$ | $-3.0(2) \times 10^{-4}$ | $-3.3(2) \times 10^{-4}$ | $-4.0(5) \times 10^{-4}$ |
|       | $-5.5(1.5) \times 10^{-4}$ (sc) | $10^2$ | $10^2$ |
|       | $-7.1(1.5) \times 10^{-4}$ (bcc) | $10^2$ | $10^2$ |
| $c_3$ | $0.87(4) \times 10^{-5}$ | $1.0(1) \times 10^{-5}$ | $0.7(1) \times 10^{-5}$ | $1.3(3) \times 10^{-5}$ |
|       | $0.5(2) \times 10^{-5}$ (sc) | $10^2$ | $10^2$ |
|       | $0.9(3) \times 10^{-5}$ (bcc) | $10^2$ | $10^2$ |
| $c_4$ | $-10^{-6} \lesssim c_4 < 0$ | $-0.3(1) \times 10^{-6}$ | $-0.6(2) \times 10^{-6}$ |
| $S_M$ | $0.999634(4)$ | $0.99975(10)$ | $0.99968(4)$ | $0.99959(6)$ |

$d = 1$; two-dimensional values will be reported in Table XI and following the method of Ref. [14]. Since the constants $c_i$ are of order $O(\epsilon^2)$, we analyzed the $O(\epsilon)$ series for $c_i/\epsilon^2$. Errors are indicative since the series are short. In Table X we compare the estimates obtained by various approaches: they all agree within the quoted errors.

As already observed in Ref. [39], the coefficients show the pattern

$$c_i \ll c_{i-1} \ll \ldots \ll c_2 \ll 1 \quad \text{for} \quad i \geq 3. \quad (6.14)$$

Therefore, a few terms of the expansion of $g(y)$ in powers of $y$ should be a good approximation in a relatively large region around $y = 0$, larger than $|y| \lesssim 1$. This is in agreement with the theoretical expectation that the singularity of $g(y)$ nearest to the origin is the three-particle cut [104,103]. If this is the case, the convergence radius $r_g$ of the Taylor expansion of $g(y)$ is $r_g = 9S_M$. Since, as we shall see, $S_M \simeq 1$, at least asymptotically we should have

$$c_{i+1} \simeq \frac{1}{9} c_i. \quad (6.15)$$

This behavior can be checked explicitly in the large-$N$ limit of the $N$-vector model [39]. In two dimensions, the critical two-point function can be written in terms of the solutions of a Painlevé differential equation [107] and it can be verified explicitly that $r_g = 9S_M$. In Table XI we report the values of $S_M$ and $c_i$ for the two-dimensional Ising model.

Assuming the pattern (6.14), we may estimate $S_M$ and $S_Z$ from $c_2$, $c_3$, and $c_4$. Indeed from the equation $g(y_0) = 0$, where $y_0 = -S_M$, we obtain
TABLE XI. Values of $S_M$ and $c_i$ for the two-dimensional Ising model in the high- and low-temperature phase.

|                     | high temperature | low temperature |
|---------------------|------------------|-----------------|
| $S_M$               | 0.999196337056   | 0.399623590999  |
| $c_2$               | $-0.7936796064 \times 10^{-3}$ | $-0.42989191603$ |
| $c_3$               | $0.109599108 \times 10^{-4}$ | 0.5256121845    |
| $c_4$               | $-0.3127446 \times 10^{-6}$ | $-0.8154613925$ |
| $c_5$               | $0.126670 \times 10^{-7}$ | 1.422603449     |
| $c_6$               | $-0.62997 \times 10^{-9}$ | $-2.663354573$  |

\[
S_M = 1 + c_2 - c_3 + c_4 + 2c_2^2 + ... \\
S_Z = 1 - 2c_2 + 3c_3 - 4c_4 - 2c_2^2 + ... \quad (6.16)
\]

where the ellipses indicate contributions that are negligible with respect to $c_4$. In Ref. [39] the relation (6.16) has been confirmed by a direct analysis of the HT series of $S_M$. From Eqs. (6.16) and (6.17) we obtain $S_M = 0.999634(4)$ (from which we can derive an estimate of the ratio $Q_\xi^+ \equiv f^{+}_{gap}/f^+ = 1.000183(2)$, cf. Eqs. (5.25) and (5.59)) and $S_Z = 1.000741(7)$.

We can also use our results to improve the phenomenological model proposed by Bray [103]. If we parametrize the large-$y$ behavior of $g(y)$ as [107]

\[
g(y)^{-1} = \frac{A_1}{y^{1-\eta/2}} \left( 1 + \frac{A_2}{y^{(1-\alpha)/(2\nu)}} + \frac{A_3}{y^{1/(2\nu)}} \right), \quad (6.18)
\]

then, by using our estimates of the critical exponents and the phenomenological function of Ref. [103], we obtain the following values for the coefficients:

\[
A_1 \approx 0.918, \quad A_2 \approx 2.55, \quad A_3 \approx -3.45. \quad (6.19)
\]

Estimating reliable errors on these results is practically impossible, since it is difficult to assess the systematic error due to the many uncontrolled simplifications that are used. It is however reassuring that they are in reasonable agreement with the $\epsilon$-expansion predictions [103]

\[
A_1 \approx 0.92, \quad A_2 \approx 1.8, \quad A_3 \approx -2.7, \quad (6.20)
\]

and with the results of a recent experimental study [61]

\[
A_1 = 0.915(21), \quad A_2 = 2.05(80), \quad A_3 = -2.95(80). \quad (6.21)
\]

Bray’s phenomenological expression makes also predictions for the coefficients $c_i$. The pattern (6.15) is built in the approach. We find $c_2 = -4.2 \cdot 10^{-4}$ and $c_3 = 1.0 \cdot 10^{-5}$, in good agreement with our IHT estimates. Therefore, Bray’s expression provides a good description of $g(y)$ for small and large values of $y$. However, in the intermediate crossover region, as already observed in Ref. [103], the agreement is worse: Bray’s interpolation is lower by 20–50% than the experimental result.

In the low-temperature phase, for $y \to 0$, the two-point function also admits a regular expansion of the form (6.5). However, the deviation from the Gaussian behavior is much...
larger. The leading coefficient \( c_2 \) is larger than \( c_2 \) by about two orders of magnitude \[108\]. Moreover, by analyzing the low-temperature series published in Ref. \[109\] one gets \( S_M^- = 0.938(8) \) (and correspondingly \( Q_\xi^- \equiv f^-_{\text{gap}}/f^+ = 1.032(4) \)). Thus \( S_M^- \) shows a much larger deviation from one (the Gaussian value) than the corresponding high-temperature phase quantity \( S_M \). The two-dimensional Ising model shows even larger deviations from Eq. (6.3), as one can see from the values of \( S_M^- \) and \( c_i^- \) reported in Table [I]. Notice that in the low-temperature phase of the two-dimensional Ising model the singularity at \( k^2 = -M^2_\text{gap} \) of \( \tilde{G}(k) \) is not a simple pole, but a branch point \[105\]. As a consequence, \( g(y) \) is not analytic for \( |y| > S_M^- \), and therefore the convergence radius of the expansion around \( y = 0 \) is \( S_M^- \). For discussions of the analytic structure of \( g(y) \) in the low-temperature phase of the three-dimensional Ising model, see e.g. Refs. \[108,103,104,110\].

VII. THE CRITICAL EQUATION OF STATE

A. The parametric representation

The critical equation of state provides relations among the thermodynamical quantities in the neighborhood of the critical temperature, in both phases. From this equation one can then derive all the universal ratios of amplitudes involving quantities defined at zero-momentum (i.e. integrated in the volume), such as specific heat, magnetic susceptibility, etc...

From the analysis of IHT series we have obtained the first few non-trivial terms of the small-field expansion of the effective potential in the high-temperature phase. This provides corresponding information for the equation of state

\[
H \propto t^{3/\delta} F(z),
\]

where \( z \propto Mt^{-\beta} \) and, using Eq. (5.3),

\[
F(z) = \frac{\partial A(z)}{\partial z} = z + \frac{1}{6} z^3 + \sum_{m=2} F_{2m+1} z^{2m+1} \tag{7.2}
\]

with

\[
F_{2m-1} = \frac{1}{(2m-1)!} \Gamma_{2m}. \tag{7.3}
\]

The function \( H(M, t) \) representing the external field in the critical equation of state (7.1) satisfies Griffith’s analyticity: it is regular at \( M = 0 \) for \( t > 0 \) fixed and at \( t = 0 \) for \( M > 0 \) fixed. The first region corresponds to small \( z \) in Eq. (7.1), while the second is related to large \( z \), where \( F(z) \) can be expanded in the form

\[
F(z) = z^\delta \sum_{n=0} F_n^\infty z^{-n/\beta}. \tag{7.4}
\]

Of course \( F_n^\infty \) are universal constants.
To reach the coexistence curve, i.e. \( t < 0 \) and \( H = 0 \), one should perform an analytic continuation in the complex \( t \)-plane \[1,23\]. The spontaneous magnetization is related to the complex zero \( z_0 \) of \( F(z) \). Therefore the description of the coexistence curve is related to the behavior of \( F(z) \) in the neighbourhood of \( z_0 \). In order to obtain a representation of the critical equation of state that is valid in the whole critical region, one may use parametric representations, which implement in a simple way all scaling and analytic properties. One parametrizes \( M \) and \( t \) in terms of \( R \) and \( \theta \) \[24–26\]:

\[
M = m_0 R^\beta \theta, \\
t = R(1 - \theta^2), \\
H = h_0 R^\beta \delta h(\theta),
\]

(7.5)

where \( h_0 \) and \( m_0 \) are normalization constants. The function \( h(\theta) \) is odd and regular at \( \theta = 1 \) and at \( \theta = 0 \). The constant \( h_0 \) can be chosen so that \( h(\theta) = \theta + O(\theta^3) \). The zero of \( h(\theta) \), \( \theta_0 > 1 \), represents the coexistence curve \( H = 0, T < T_c \). The parametric representation satisfies the requirements of regularity of the equation of state. One expects at most an essential singularity on the coexistence curve \[111\].

The relation between \( h(\theta) \) and \( F(z) \) is given by

\[
z = \rho \theta \left(1 - \theta^2\right)^{-\beta}, \\
h(\theta) = \rho^{-1} \left(1 - \theta^2\right)^{\beta\delta} F(z(\theta)),
\]

(7.6)

(7.7)

\( \theta > 0 \), and hyperscaling implies that \( \beta\delta = \beta + \gamma \). Notice that this mapping is invertible only in the region \( \theta < \theta_l \), where \( \theta_l = (1 - 2\beta)^{-1/2} \) is the solution of the equation \( z'(\theta) = 0 \). Thus the values of \( \theta \) that are relevant for the critical equation of state, i.e. \( 0 \leq \theta \leq \theta_0 \), must be smaller than \( \theta_l \). This fact will not be a real limitation for us, since the range of values of \( \theta \) involved in our calculations (which will be \( 0 \leq \theta^2 \leq \theta_0^2 \lesssim 1.40 \)) will be always far from the limiting value \( \theta_l^2 \simeq 2.88 \).

As a consequence of Eqs. (7.5), (7.6), and (7.7), we easily obtain the relationships

\[
\frac{M}{\nu^\beta} = \left(\frac{m_0}{\rho}\right) z, \\
\frac{H}{\nu^{\beta \delta}} = \left(\frac{h_0}{\rho}\right) F(z).
\]

(7.8)

We can therefore treat \( \rho \) as a free parameter, and the scaling relations between physical variables will not depend on \( \rho \), provided that \( m_0 \) and \( h_0 \) are rescaled with \( \rho \). In the exact parametric equation the value of \( \rho \) may be chosen arbitrarily but, as we shall see, when adopting an approximation procedure the dependence on \( \rho \) is not eliminated, and it may become important to choose the value of this parameter properly in order to optimize the approximation.

From \( \theta_0 \) one can obtain the universal rescaled spontaneous magnetization \[23\], i.e. the complex zero \( z_0 \) of \( F(z) \),

\[
z_0 = |z_0| e^{-i\pi \beta}, \\
|z_0| = \rho \theta_0 \left(\theta_0^2 - 1\right)^{-\beta}.
\]

(7.9)

From the function \( h(\theta) \) one can calculate the universal ratios of amplitudes. In App. \[3\] we report the definitions of the universal ratios of amplitudes that have been introduced in the literature, and the corresponding expressions in terms of \( h(\theta) \).
Expanding $h(\theta)$ in (odd) powers of $\theta$,

$$h(\theta) = \theta + \sum_{n=1} h_{2n+1} \theta^{2n+1},$$  \hspace{1cm} (7.10)

and using Eq. (7.7), one can find the relations among $h_{2n+1}$ and the coefficients $F_{2m+1}$ of the expansion of $F(z)$. The procedure is explained in App. C and the general result is:

$$h_{2n+1} = \sum_{m=0}^{n} c_{n,m} \rho^{2m} F_{2m+1},$$  \hspace{1cm} (7.11)

where

$$c_{n,m} = \frac{1}{(n-m)!} \prod_{k=1}^{n-m} (2\beta m - \gamma + k - 1);$$  \hspace{1cm} (7.12)

notice that $c_{n,n} = 1$. In general $h_{2n+1}$ depends on $\gamma$, $\beta$, and on the coefficients $F_{2m+1}$ with $m \leq n$.

We shall need the explicit form of the first two coefficients:

$$h_3 = \frac{1}{6} \rho^2 - \gamma, \hspace{1cm} (7.13)$$

$$h_5 = \frac{1}{2} \gamma(\gamma - 1) + \frac{1}{6} (2\beta - \gamma) \rho^2 + F_5 \rho^4. \hspace{1cm} (7.14)$$

B. Approximation scheme based on stationarity

In Ref. [23] Guida and Zinn-Justin use the first few coefficients of the small-$z$ expansion of $F(z)$ to get polynomial approximations of $h(\theta)$ which should provide a description that is reliable in the whole critical region. The approximations considered are truncations of the small-$\theta$ expansion of $h(\rho, \theta)$, i.e.

$$h^{(t)}(\rho, \theta) = \theta + \sum_{n=1}^{t-1} h_{2n+1}(\rho) \theta^{2n+1},$$  \hspace{1cm} (7.15)

where $h_{2n+1}(\rho)$ are given by Eq. (7.11). We follow a similar strategy, with a significant difference in the procedure adopted in order to fix the value of $\rho$.

By Eqs. (7.11) and (7.12), the coefficients $h_{2n+1}(\rho)$ included in Eq. (7.15) are written in terms of the $t$ parameters $\gamma$, $\beta$, $F_5$, ..., $F_{2t-1}$. In practice only the first coefficients of the small-$\theta$ expansion of $h(\theta)$ are well determined, since we have good estimates only for the first few $F_{2m+1}$. Once the order of the truncation has been decided, one may exploit the freedom of choosing $\rho$ to optimize the approximation of $h(\theta)$. In this way one may hope to obtain a good approximation even for small values of $t$. Ref. [23] proposes to determine the optimal value of $\rho$ by minimizing the absolute value of $h_{2t-1}(\rho)$, i.e. the coefficient of the highest-order term considered. The idea underlying this procedure is to increase the importance of small powers of $\theta$. Our approach is different.

Our starting point is the independence on $\rho$ of the scaling function $F(z)$ and, as a consequence, of all universal ratios of amplitudes that can be extracted from it. Of course,
this property does not hold anymore when we start from a truncated function $h^{(t)}(\rho, \theta)$, i.e. if we compute universal quantities from a function $F^{(t)}(\rho, z)$ defined by

$$F^{(t)}(\rho, z) \equiv \tilde{F}^{(t)}(\rho, \theta(\rho, z)), \quad (7.16)$$

where

$$\tilde{F}^{(t)}(\rho, \theta) = \frac{\rho h^{(t)}(\rho, \theta)}{(1 - \theta^2)^{\beta \delta}} \quad (7.17)$$

and $\theta(\rho, z)$ is obtained by inverting Eq. (7.6).

In order to optimize $\rho$ for a given truncation $h^{(t)}(\rho, \theta)$, we propose a procedure based on the physical requirement of minimal dependence on $\rho$ of the resulting universal function $F^{(t)}(\rho, z)$. This can be obtained by assuming $\rho$ to depend on $z$, i.e. $\rho = \rho^{(t)}(z)$, and by requiring the functional stationarity condition

$$\frac{\delta F^{(t)}(\rho^{(t)}, z)}{\delta \rho^{(t)}} = 0 \quad (7.18)$$

(see Ref. [45] and references therein for a similar technique applied to the resummation of perturbative power expansions). The non-trivial fact, even surprising at first sight, is that the solution $\rho^{(t)}(z)$ of Eq. (7.18) is constant. In other words, for any $t$ there exists a solution $\rho_t$ independent of $z$ that satisfies the global stationarity condition

$$\frac{\partial F^{(t)}(\rho, z)}{\partial \rho} \bigg|_{\rho=\rho_t} = 0. \quad (7.19)$$

This is equivalent to the fact that, for any universal ratio of amplitudes $R$, its approximation $R^{(t)}(\rho)$ (obtained from $F^{(t)}(\rho, z)$) satisfies the stationarity condition

$$\frac{dR^{(t)}(\rho)}{d\rho} \bigg|_{\rho=\rho_t} = 0. \quad (7.20)$$

The proof of Eq. (7.19) is given in App. C, where we show that the global stationarity condition amounts to requiring $\rho_t$ to be a solution of the algebraic equation

$$\left[ (2\beta - 1)\rho \frac{\partial}{\partial \rho} - 2\gamma + 2t - 2 \right] h_{2t-1}(\rho) = 0. \quad (7.21)$$

The idea behind our scheme of approximation is that, for any truncation, the stationarity condition enforces the physical request that the universal ratios of amplitudes be minimally dependent on $\rho$. To check the convergence of the approximation, one can repeat the computation of universal ratios of amplitudes from the truncated function $h^{(t)}(\rho_t, \theta)$ for different values of $t$, as long as one has a reliable estimate of $F_{2t-1}$. We have no a priori argument in favor of a fast convergence in $t$ of the universal ratios of amplitudes derived by this procedure towards their exact values. However, we may appreciate that its lowest-order implementation, corresponding to $t=2$ in Eq. (7.13), reproduces the well-known formulae of
Refs. [24–26], which give an effective optimization of the linear parametric model. Indeed we obtain from Eqs. (7.21) and (7.13) the solution
\[ \rho_2 = \sqrt{\frac{6\gamma(\gamma - 1)}{\gamma - 2\beta}}. \] (7.22)

In this case the critical equation of state and all critical amplitudes turn out to be expressible simply in terms of the critical exponents \( \beta \) and \( \gamma \). In particular, we found a closed-form expression for all \( F_{2m+1}^{(2)} \) coefficients (see App. C for a derivation):
\[ F_{2m+1}^{(2)} = \frac{(-1)^m \gamma(\gamma - 1)}{m! \rho_2^{2m}} \prod_{k=1}^{m-2} (2\beta m - \gamma - k). \] (7.23)

Wallace and Zia [27] already noticed that the minimum condition of Refs. [24–26] was equivalent to a condition of global stationarity for the linear parametric model. We have shown that such a global stationarity can be extended to other parametric models, regardless the linearity constraint, and can be used to improve the approximation.

The next truncation, corresponding to \( t = 3 \), can also be treated analytically. Since it sensibly improves the linear parametric model in the 3d Ising case, we shall present here a few details. By applying the stationarity condition (7.21) to Eq. (7.14), we obtain
\[ \rho_3 = \sqrt{\frac{(\gamma - 2\beta)(1 - \gamma + 2\beta)}{12(4\beta - \gamma)F_5}} \left( 1 - \frac{72(2 - \gamma)\gamma(\gamma - 1)(4\beta - \gamma)F_5}{(\gamma - 2\beta)^2 (1 - \gamma + 2\beta)^2} \right)^{\frac{1}{2}}. \] (7.24)

Universal ratios of amplitudes may be evaluated in terms of \( \rho_3 \); they will now depend only on the parameters \( \beta \), \( \gamma \) and \( F_5 \). Notice that the predictions of the \( t = 2 \) and \( t = 3 \) models differ from each other only proportionally to the difference between the “experimental” value of \( F_5 \) and the value predicted according to Eq. (7.23)
\[ F_5^{(2)} = \frac{(\gamma - 2\beta)^2}{72 \gamma(\gamma - 1)}. \] (7.25)

If we replace \( F_5 \) with \( F_5^{(2)} \) in the \( t = 3 \) model results, all the linear parametric model results are automatically reproduced. In the 3d Ising model, the two values differ by \( \sim 6\% \), and thus we expect comparable discrepancies for all universal ratios of amplitudes. This can be verified from the numerical results that we will present in Sec. VIII (see Table XII). All universal ratios of amplitudes obtained from the \( t = 2 \) truncation (i.e. the linear parametric model), using our estimates of \( \gamma \) and \( \beta \), differ at most by a few per cent from previously available estimates. The \( t = 3 \) and higher-order approximations are consistent with the latter. The apparent convergence in \( t \) of the results provides a further important support to this scheme.

It is worth noticing that the parametric representation of the equation of state induces parametric forms for such thermodynamics functions as the free energy and the susceptibility, as discussed in detail in App. D. When we assume a truncated form of the parametric equation of state, in general only the corresponding free energy function will admit a polynomial representation. A peculiar and possibly unique feature of our scheme is the induced
truncation of the function related to the susceptibility, which turns out to be an even polynomial of degree $2t$ in the variable $\theta$. App. C contains a more extended discussion of these and other properties of the approximation scheme based on the stationarity condition.

We have introduced our parametric representation assuming independent knowledge of $F_5, \ldots, F_{2t-1}$. It should be noticed that our results for $t = 3$ can also be used as a phenomenological parametrization, fitting the value of $F_5$ on any known universal quantity. As we will show in Sec. VII D, the difference with the linear parametric model of Refs. [24–26] is not negligible. On the other hand, our numerical estimates for $t = 4$ show that the difference from $t = 3$ is too small (compared with both theoretical and experimental precision) to justify the introduction of an additional phenomenological parameter $F_7$.

C. $\epsilon$-expansion of the parametric representation

It is interesting to compare our results with the analysis of the parametric equation of state which can be performed in the context of the $\epsilon$-expansion, generalizing results presented in Refs. [27,112].

According to Ref. [27], within the $\epsilon$-expansion it is possible to choose a value $\rho_0$ such that, for all $n \geq 2$,

$$h_{2n+1}(\rho_0) = O(\epsilon^{n+1}). \quad (7.26)$$

The calculation shows that $\rho_0 = \sqrt{2}$. We proved in App. C that Eq. (7.26) keeps holding for all choices of $\rho$ that satisfy the relation $\rho = \rho_0 + O(\epsilon)$. We can now $\epsilon$-expand our globally stationary solutions for arbitrary $t$, obtaining

$$\lim_{\epsilon \to 0} \rho_t = \rho_0. \quad (7.27)$$

As a consequence, any truncation satisfying the stationarity condition is an accurate description of the $\epsilon$-expanded parametric equation of state up to $O(\epsilon^t)$ included.

As a byproduct, we may extract from the linear model relation (7.23), expanded to $O(\epsilon^2)$, the coefficients of the $\epsilon$-expansion for $F_{2m+1}$, for $m \geq 2$:

$$F_{2m+1} = \sum_{k=1}^{\infty} f_{mk} \epsilon^k. \quad (7.28)$$

We easily obtained from Eq. (7.23) the closed form results

$$f_{m1} = \frac{(-1)^m}{6m(m-1)} \frac{1}{2^m}, \quad (7.29)$$

$$f_{m2} = f_{m1} \left[ \frac{17}{27} - \frac{m}{2} - \left( \frac{m}{3} + \frac{1}{6} \right) \sum_{k=1}^{m-2} \frac{1}{k} \right], \quad (7.30)$$

reproducing known results [23]. More generally, knowing the expansion of the coefficients $F_{2m+1}$ to $O(\epsilon^t)$ for $m < t$ is enough to reconstruct all $F_{2m+1}$ for $m \geq t$ to the same accuracy.
TABLE XII. Universal ratios of amplitudes obtained by taking different approximations of the parametric function \( h(\theta) \). Numbers marked with an asterisk are inputs, not predictions. The values \( \rho_{m,t} \) are obtained as in Ref. [23], see text for details.

| \( h^{(2)}(\rho_2, \theta) \) | \( h^{(3)}(\rho_3, \theta) \) | \( h^{(4)}(\rho_4, \theta) \) | \( h^{(5)}(\rho_5, \theta) \) | \( h^{(3)}(\rho_{m,3}, \theta) \) | \( h^{(4)}(\rho_{m,4}, \theta) \) |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| \( \rho \)      | 1.7358(12)      | 1.7407(14)      | 1.72898(3)      | 1.6868(31)      | 1.6889(26)      | 1.6513(30)      |
| \( \theta \)    | 1.3606(11)      | 1.3879(29)      | 1.372(12)       | 1.3255(53)      | 1.3310(13)      | 1.2952(27)      |
| \( F_{0}^{\infty} \) | 0.03280(14) | 0.03382(18) | 0.03374(21) | 0.03366(26) | 0.03378(18) | 0.03370(23) |
| \( |z_0| \)      | 2.825(12)       | 2.797(17)       | 2.7970(33)      | 2.8012(72)      | 2.7955(15)      | 2.7992(45)      |
| \( U_0 \)      | 0.5222(16)      | 0.5316(21)      | 0.5295(29)      | 0.5261(60)      | 0.5303(19)      | 0.5276(39)      |
| \( t_2 \)      | 4.826(11)       | 4.752(15)       | 4.769(22)       | 4.797(47)       | 4.764(13)       | 4.786(30)       |
| \( t_4 \)      | −9.737(41)      | −8.918(83)      | −9.10(18)       | −9.42(48)       | −9.061(67)      | −9.31(28)       |
| \( R_t^{+} \)  | 0.05538(13)     | 0.05681(16)     | 0.05644(32)     | 0.0558(10)      | 0.05656(14)     | 0.05606(53)     |
| \( R_t^{-} \)  | 0.021976(16)    | 0.022488(30)    | 0.02235(11)     | 0.02211(36)     | 0.022387(18)    | 0.02220(19)     |
| \( R_t^{0} \)  | 7.9789(64)      | 7.804(10)       | 7.823(18)       | 7.847(40)       | 7.8146(85)      | 7.836(25)       |
| \( R_4 \)      | 92.10(23)       | 93.91(20)       | 93.25(45)       | 91.9(1.7)       | 93.25(21)       | 92.27(83)       |
| \( v_3 \)      | 6.0116(79)      | 6.0561(68)      | 6.041(11)       | 6.010(39)       | 6.0412(73)      | 6.018(20)       |
| \( v_4 \)      | 16.320(55)      | 16.12(66)       | 16.21(11)       | 16.41(24)       | 16.239(55)      | 16.38(15)       |
| \( Q_2^{\infty} \) | 1.6775(19)     | 1.6588(27)      | 1.6624(44)      | 1.668(10)       | 1.6611(25)      | 1.6656(60)      |
| \( U_2^{+} R_5^{+} \) | 38.505(82)   | 37.09(15)       | 37.31(26)       | 37.64(55)       | 37.23(13)       | 37.50(35)       |
| \( R_3^{+} R_5^{+} \) | 0.4419(13)   | 0.4434(13)      | 0.4416(18)      | 0.4377(56)      | 0.4420(13)      | 0.4392(29)      |
| \( r_6 \)      | 1.9389(48)      | *2.048(5)       | *2.048(5)       | *2.048(5)       | *2.048(5)       | *2.048(5)       |
| \( r_8 \)      | 2.507(31)       | 2.402(39)       | *2.28(8)        | *2.28(8)        | 2.365(43)       | *2.28(8)        |
| \( r_{10} \)   | −12.612(41)     | −12.146(60)     | −10.0(1.5)      | −13(4)          | −11.80(10)      | −10.98(86)      |

D. Results

As input parameters for the determination of the functions \( h^{(t)}(\rho, \theta) \) we use the results of the IHT expansion: \( \gamma = 1.2371(4) \), \( \nu = 0.63002(23) \), \( r_6 = 2.048(5) \), \( r_8 = 2.28(8) \), \( r_{10} = -13(4) \).

In Table XII we report the universal ratios of amplitude as derived from truncations corresponding to \( t = 2, 3, 4, 5 \). We use the standard notation for the ratios of amplitudes (see e.g. Ref. [14]); all definitions can be found in Table XIII. For comparison, we also report, for \( t = 3, 4 \), the results obtained using the procedure of Ref. [23], fixing \( \rho \) to the value \( \rho_{m,t} \) which minimizes the absolute value of the \( O(\rho^{2t-1}) \) coefficient.\footnote{As already noted in Ref. [23], for \( t = 4 \) the minimum of \( h_7(\rho) \) is zero, while, for \( t = 3 \), \( h_5(\rho) \) never reaches zero.} Such results are very close to those derived from the stationarity condition; this is easily explained by the fact that the values of \( \rho_{m,t} \) are close to \( \rho_t \). The errors reported in Table XII are related to the uncertainty of the corresponding input parameters (considering them as independent). The results for \( t = 2, 3, 4 \) suggest a good convergence and give a good support to our analysis. The results for \( t = 5 \), although perfectly consistent, are less useful to check convergence, due to the large uncertainty of \( F_0 \). In Table XIII we report our final estimates, obtained using \( h^{(4)}(\rho_4, \theta) \); all the approximations reported in Table XII are consistent with them, except \( t = 2 \).

We should say that the method of Guida and Zinn-Justin to determine the optimal \( \rho \) leads to equivalent results, and shows an apparent good convergence as well. However, we believe that the global stationarity represents a more physical requirement, and it is more...
TABLE XIII. Summary of the results obtained in this paper by our high-temperature calculations (IHT), by using the parametric representation of the equation of state (IHT-PR), by analyzing the low-temperature expansion (LT), and by combining the two approaches (IHT-PR+LT). Notations are explained in App. E.

|         | IHT     | IHT-PR  | LT       | IHT-PR+LT |
|---------|---------|---------|----------|-----------|
| $\gamma$ | 1.2371(4) |         |          |           |
| $\nu$   | 0.63002(23) |       |          |           |
| $\alpha$| 0.1099(7)  |         |          |           |
| $\eta$  | 0.0364(4)  |         |          |           |
| $\beta$ | 0.32648(18) |       |          |           |
| $\delta$| 4.7893(22) |        |          |           |
| $\sigma$| 0.0208(12) |         |          |           |
| $r_6$   | 2.048(5)   |         |          |           |
| $r_8$   | 2.28(8)    |         |          |           |
| $r_{10}$| -13(4)     | -10(2)  |          |           |
| $U_0 \equiv A^+/A^-$ |         | 0.530(3) |          |           |
| $U_2 \equiv C^+/C^-$ |         | 4.77(2)  |          |           |
| $U_4 \equiv C_4^+/C_4^-$ | -9.1(2) |         |          |           |
| $R_+^\varepsilon \equiv \alpha A^+ C^+/B^2$ |         | 0.0564(3) |          |           |
| $R_-^\varepsilon \equiv \alpha A^- C^-/B^2$ |         | 0.02235(11) |          |           |
| $R_4^+ \equiv -C_4^+ B^2/(C^+)^3 = |z_0|^2$ |         | 7.82(2)  |          |           |
| $R_3 \equiv v_3 \equiv -C_3^- B/(C^-)^2$ |         | 6.041(11) |          |           |
| $R_4^- \equiv C_4^- B^2/(C^-)^3$ |         | 93.3(5)  |          |           |
| $v_4 \equiv -R_4^- + 3R_3^2$ |         | 16.21(11) |          |           |
| $Q_4^{\delta} \equiv R_\chi \equiv C^+ B^{\delta-1}/(\delta C^c)^\delta$ |         | 1.662(5) |          |           |
| $F_0^\infty$ | cf. Eq. (7.4) |        |          |           |
| $g_4^+ \equiv g \equiv -C_4^+/[(C^+)^2(f^+)^3]$ | 23.49(4) |         |          |           |
| $w^2 \equiv C^-/[B^2(f^-)^3]$ |         | 4.75(4)  | [4]       |           |
| $U_\xi \equiv f^+ / f^- = (w^2 U_2 R_4^+ / g_4^+)^{1/3}$ |         | 1.961(7) |          |           |
| $Q_\xi^+ \equiv \alpha A^+ (f^+)^3 = R_4^+ R_\xi^+ / g_4^+$ |         | 0.01880(8) |          |           |
| $R_\xi^+ \equiv (Q^+)^{1/3}$ |         | 0.2659(4) |          |           |
| $Q_-^\varepsilon \equiv \alpha A^- (f^-)^3 = R_\xi^- / w^2$ |         |          | 0.00471(5) |           |
| $Q_\xi \equiv B^2(f^+)^3/C^+ = Q_\varepsilon^+ / R_\varepsilon^+ = R_4^+ / g_4^+$ |         | 0.3330(10) |          |           |
| $g_3^- \equiv w v_3$ |         |          | 13.17(6)  |           |
| $g_4^- \equiv w^2 v_4$ |         |          | 77.0(8)   |           |
| $Q_\xi^+ \equiv f_{\varepsilon\xi}^+/f^+$ |         | 1.000183(2) |          |           |
| $Q_\xi^- \equiv f_{\varepsilon\xi}^- / f^-$ |         | 1.032(4)  | [39]       |           |
| $U_{\varepsilon\delta} \equiv f_{\varepsilon\delta}^+/f_{\varepsilon\delta}^- = \xi Q_{\varepsilon\delta}^+/Q_{\varepsilon\delta}^-$ |         | 1.901(10) |          |           |
| $Q_{\varepsilon\delta} \equiv f_{\varepsilon\delta}^- / f_{\varepsilon\delta}^+$ |         | 1.024(4)  |          |           |
| $Q_2 \equiv (f^c/f^e)^2 C^+ / C^c$ |         | 1.195(10) |          |           |
amenable to a theoretical analysis of its convergence properties. Moreover, as we have shown, it has the linear parametric model of Refs. [24–27] as the lowest-order approximation.

Estimates of other universal ratios of amplitudes can be obtained by supplementing the above results with the estimates of \( w^2 \equiv C^{-1}/[B^2(f^3)] \) and \( Q_\xi \equiv f_{\text{gap}}/f \) obtained by an analysis of the corresponding low-temperature expansion. The results so obtained are denoted by IHT-PR+LT in Table XIII. The low-temperature expansion of \( w^2 \) can be calculated to \( O(u^{21}) \) on the cubic lattice using the series published in Refs. [109,114]. The results reported in Table XIII were obtained by using the Roskies transform in order to reduce the systematic effects due to confluent singularities [14].

We also consider a parametric representation of the correlation length. Following Ref. [115] we write

\[
\xi^2/\chi = R^{-\eta}a(\theta),
\]

\[
\xi^2_{\text{gap}}/\chi = R^{-\eta}a_{\text{gap}}(\theta).
\]

We consider the simplest polynomial approximation to \( a(\theta) \) and \( a_{\text{gap}}(\theta) \):

\[
a(\theta) \approx a_0 \left(1 + c\theta^2\right),
\]

\[
a_{\text{gap}}(\theta) \approx a_{\text{gap},0} \left(1 + c_{\text{gap}}\theta^2\right),
\]

where the constants \( c \) and \( c_{\text{gap}} \) can be determined by fitting the quantities \( U_\xi \) and \( U_{\xi_{\text{gap}}} \). Then, using Eqs. (7.31), (7.32), and the parametric representation of the equation of state, one can estimate the universal ratios of amplitudes \( Q_\xi \) and \( Q_2 \) defined in Table XIII. Notice that, given the equation of state, the normalization \( a_0 \) is not arbitrary, but it may be fixed using the zero-momentum four-point coupling \( g_4 \):

\[
a_0 = (h_0/\rho)^{1/3} (m_0/\rho)^{-5/3} (g_4^*)^{-2/3},
\]

where \( h_0, \ m_0 \) and \( \rho \) have been introduced in Eqs. (7.3) and (7.6). Notice that \( a_0 \) depends only on the ratios \( h_0/\rho \) and \( m_0/\rho \), as it is required of a physical quantity. Moreover one has \( a_{\text{gap},0} = (Q_\xi^*)^2 \ a_0 \). In order to check the results obtained from the approximate expressions (7.33) and (7.34), we also considered the following parametric representation [102]:

\[
\xi^{-2} = R^{2\nu}b(\theta),
\]

\[
\xi_{\text{gap}}^{-2} = R^{2\nu}b_{\text{gap}}(\theta),
\]

and the corresponding polynomial approximations truncated to second order. The results for \( Q_\xi \) and \( Q_2 \) obtained by this second representation are perfectly consistent with those from the first one. Our final estimates of \( Q_\xi \) and \( Q_2 \) derived by the above method are reported in Table XIII.

In Table XIV we compare our results with other approaches. We find a good overall agreement. Our results appear to substantially improve the estimates of most of the universal ratios considered. In Table XIV we have collected results obtained by high-temperature and low-temperature expansions (HT,LT), Monte Carlo simulations (MC), field-theoretical methods such as \( \epsilon \)-expansion and various kinds of expansions at fixed dimension \( d = 3 \),
TABLE XIV. Estimates of the quantities in Table XIII by various approaches. The experimental data are taken from Ref. [55], unless otherwise stated. ms denotes a magnetic system; bm a binary mixture; lv a liquid-vapor transition in a simple fluid. For values marked with an asterisk, the error is not quoted explicitly in the reference.

|     | HIT–PR  | HT,LT  | MC     | $\epsilon$-exp. | $d=3$ exp. | experiments |
|-----|---------|--------|--------|-----------------|------------|-------------|
| $U_0$ | 0.530(3) | 0.523(9) | 0.560(10) | 0.527(13) | 0.537(19) | ms 0.56(2) bm 0.50(3) |
|      |         | 0.51*  |        | 0.550(12) | 0.524(10) | lv 0.53(3) |
|      |         |        |        | 0.567(16) | 0.540(11) | lv 0.53(3) |
| $U_2$ | 4.77(2)  | 4.95(15) | 4.75(3) | 4.73(16) | 4.79(10) | bm 4.6(3) |
|      |         | 5.01*  |        | 4.72(11) | 4.77(30) | lv 4.9(2) |
|      |         |        |        |         | 4.72(17) | ms 5.1(6) |
| $U_4$ | -9.1(2) | -9.0(3) | -8.6(1.5) | -9.1(6) | -9.1(6) | lv -9.1(4) |
| $R_c^+$ | 0.0564(3) | 0.0581(10) | 0.0569(35) | 0.0574(20) | 0.0594(10) | bm 0.050(15) |
| $R_c^-$ | 7.8(2) | 7.91(2) | 8.24(34) | 7.84* | 7.84* | lv 7.84| |
| $R_3 \equiv v_3$ | 6.94(11) | 6.44(30) | 5.99(5) | 6.08(6) | 6.07(19) | ms 6.0(4) |
| $v_4$ | 93.5(5) | 107(13) | 15.8(1.4) | 15.8(1.4) | 15.8(1.4) | lv 15.8(1.4) |
| $Q_\sigma^+$ | 1.662(5) | 1.57(23) | 1.648(36) | 1.669(18) | 1.75(30) | bm 1.75(30) |
| $w^2$ | 4.754(4) | 4.71(5) | 4.773(3) | 4.753* | 4.753* | bm 4.753* |
| $U_\xi$ | 1.961(7) | 1.96(1) | 1.95(2) | 1.91(12) | 2.013(28) | bm 1.93(7) |
| $Q_\gamma^+$ | 0.01880(8) | 0.01880(15) | 0.0198(19) | 0.0197* | 0.01968(15) | ms 0.0194(92) |
| $Q_\gamma^-$ | 0.0047(15) | 0.0047(20) | 0.0047(17) | 0.005(1) | 0.005(1) | ms 0.005(1) |
| $Q_c$ | 0.3330(10) | 0.324(6) | 0.328(5) | 0.331(9) | 0.335(5) | lv 0.335(5) |
| $g_3$ | 13.17(6) | 13.9(4) | 13.6(5) | 13.06(12) | 13.06(12) | lv 0.35(4) |
| $g_4$ | 77.0(8) | 85(5) | 108.7(13) | 75(7) | 75(7) | lv 75(7) |
| $Q_\xi^+$ | 1.000183(2) | 1.000183(2) | 1.00016(2) | 1.00016(2) | 1.0001(3) | 1.0001(3) |
| $Q_\xi^-$ | 1.024(4) | 1.007(3) | 1.032(9) | 1.031(6) | 1.031(6) | 1.031(6) |
| $Q_2$ | 1.195(10) | 1.17(2) | 1.17(2) | 1.17(2) | 1.17(2) | 1.17(2) |

and experiments. Concerning the HT,LT estimates, we mention the recent Ref. [113] where a review of such results is presented. The agreement with the most recent Monte Carlo simulations is good, especially with the results reported in Ref. [122], which are quite precise. However we note that the estimates of $U_0$ reported in Ref. [10] are slightly larger. Moreover there is an apparent discrepancy with the estimate of $g_\gamma^-$ of Ref. [133]. It is worth mentioning that the result of Ref. [123] was obtained simulating a four-dimensional $SU(2)$ lattice gauge model at finite temperature, whose phase transition is expected to be in the 3d Ising universality class. Field-theoretical estimates are in general less precise, although perfectly consistent. We mention that the results denoted by “$d = 3$ exp.” are obtained from different kinds of expansions: $g$-expansion [23, 14, 113, 132], minimal renormalization without $\epsilon$-expansion [120, 135], expansion in the coupling $u \equiv 3w^2$ defined in the low-temperature phase [120]. In Refs. [23, 14] Guida and Zinn-Justin used the $d = 3$ $g$- and $\epsilon$-expansion to calculate the small-field expansion of the effective potential and a parametric representation.
of the critical equation of state. We also mention the results (not included in this table) of an approach based on approximate solution of exact renormalization equations (see e.g. Refs. [48, 49, 136]). Some results can be found in Ref. [136]: \( U_2 \approx 4.29, g_3^\approx 15.24, Q_1^\approx 1.61 \) and \( U_ξ \approx 1.86 \).

We report in Table XIV experimental results for three interesting physical systems exhibiting a critical point belonging to the 3d Ising universality class: binary mixtures, liquid-vapor transitions and uni-axial antiferromagnetic systems. A review of experimental data can be found in Ref. [55]. Most of the results shown in Table XIV were reported in Refs. [23, 122]. They should give an overview of the level of precision reached by experiments.

For sake of comparison, in Table XV we report the universal ratios of amplitudes for the two-dimensional Ising model. The purely thermal results are taken from Ref. [105], where the exact two-point function has been written in terms of the solution of a Painlevé equation. \( Q^+ \) and \( Q^- \) have been computed by us solving numerically the differential equations reported in Ref. [105]. The ratios involving amplitudes along the critical isotherm can be obtained using the results reported in Ref. [138]. For the quantities that are not known exactly, we report estimates derived from the high- and low-temperature expansion. Such estimates are quite accurate and should be reliable because the leading correction to scaling is analytic, since the subleading exponent \( \Delta \) is expected to be larger than one (see e.g. Ref. [139] and references therein). In particular the available exact calculations [105] for the square-lattice Ising model near criticality have shown only analytic corrections to the leading power law. Therefore the traditional methods of series analysis should work well.

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APPENDIX A: GENERATION AND ANALYSIS OF THE HIGH-TEMPERATURE EXPANSION FOR IMPROVED HAMILTONIANS

1. Definitions

Before discussing the series computation, let us define all the quantities we are interested in and fix the notation.

Starting from the two-point function \( G(x) \equiv \langle \phi(0)\phi(x) \rangle \), we define its spherical moments

\[
m_{2j} = \sum_x (x^2)^j G(x)
\]

(\( \chi \equiv m_0 \)) and the first non-spherical moments

\[
q_{4,2j} = \sum_x (x^2)^j \left[ x^4 - 3 \frac{\chi^2}{\beta}(x^2)^2 \right] G(x)
\]

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TABLE XV. Universal ratios of amplitudes for the two-dimensional Ising model. Since the specific heat diverges logarithmically in the two-dimensional Ising model, the specific heat amplitudes $A^\pm$ are defined by $C_H \approx -A^\pm \ln t$.

| Symbol | Value |
|--------|-------|
| $\gamma$ | $7/4$ |
| $\nu$ | $1$ |
| $U_0 \equiv A^+/A^-$ | |
| $U_2 \equiv C^+/C^-$ | $37.69365201$ |
| $R^+_c \equiv A^+C^+/B^2$ | $0.31856939$ |
| $R^-_c \equiv A^-C^-/B^2$ | $0.00845154$ |
| $Q^+_1 \equiv R_\chi \equiv C^+B^{\delta-1}/(\delta C C)^\delta$ | $2$ |
| $Q^-_2 \equiv C^-/[B^2(f-)^2]$ | $0.53152607$ |
| $U_\xi \equiv f^+/f^-$ | $3.16249504$ |
| $U_{\xi_{\text{gap}}} \equiv f^+_{\text{gap}}/f^-_{\text{gap}}$ | $2$ |
| $Q^+ \equiv A^+(f^+)^2$ | $0.15902704$ |
| $Q^- \equiv A^-(f^-)^2$ | $0.015900517$ |
| $Q^+_{\xi} \equiv f^+_{\text{gap}}/f^+$ | $1.00042074$ |
| $Q^-_{\xi} \equiv f^-_{\text{gap}}/f^-$ | $1.0786828$ |
| $Q^+_{2} \equiv (f^+/f^+)^2-C^+/C^c$ | $2.8355305$ |
| $g^+ \equiv g \equiv -C^+_4/[(C^+)^2(f^+)^2]$ | $14.694(2)$ |
| $r_6$ | $3.678(2)$ |
| $r_8$ | $26.0(2)$ |
| $r_{10}$ | $275.15(15)$ |
| $v_3 \equiv -C^+_3 B/(C^-)^2$ | $33.011(6)$ |
| $v_4 \equiv -C^+_4 B^2/(C^-)^3 + 3v_3^2$ | $48.6(1.2)$ |

(Where $x^n \equiv \sum_i x_i^n$).

The second-moment correlation length is defined by

$$\xi^2 \equiv M^{-2} = \frac{m_2}{6\chi}.$$  \hfill (A3)

The coefficients $c_i$ of the low-momentum expansion of the function $g(y)$ introduced in Sec. VI can be related to the critical limit of appropriate dimensionless ratios of spherical moments, or of the corresponding weighted moments $m_{2j} \equiv m_{2j}/\chi$. Introducing the quantities

$$u_{2j} = \frac{1}{(2j+1)!} \overline{m_{2j}} M^{2j},$$  \hfill (A4)

one can define combinations of $u_{2j}$ (that we will still call $c_i$ to avoid introducing new symbols) having $c_i$ as critical limit:

$$c_2 = 1 - u_4 + \frac{1}{30} M^2,$$  \hfill (A5)

$$c_3 = 1 - 2u_4 + u_6 - \frac{1}{840} M^4,$$  \hfill (A6)

$$c_4 = 1 - 3u_4 + u_4^2 + 2u_6 - u_8 + \frac{1}{4725} M^4 + \frac{1}{60480} M^6,$$  \hfill (A7)
etc... Notice that the terms proportional to powers of $M^2$ do not contribute in the critical limit $M \to 0$, but they allow us to define improved estimators \cite{39}. Indeed in the lattice Gaussian limit, defined by the two-point function

$$\tilde{G}(k) = \frac{1}{\hat{k}^2 + M^2}, \quad \hat{k}^2 = \sum_i 4 \sin^2(k_i/2),$$

(c_i = 0 independently of $M$, and not only in the critical limit $M \to 0$.

The zero-momentum connected Green’s functions are defined by

$$\chi_{2j} = \sum_{x_2, \ldots, x_{2j}} \langle \phi(0)\phi(x_2)\ldots\phi(x_{2j-1})\phi(x_{2j}) \rangle_c;$$

(A9)

in particular, $\chi_2 \equiv \chi$.

2. Linked cluster expansion

We computed the high-temperature expansion by the linked cluster expansion (LCE) technique. A general introduction to the LCE can be found in Ref. \cite{140}. We modeled the application of the LCE to $O(N)$-symmetric models after Ref. \cite{141}.

In order to perform the LCE for the most general model described by the Hamiltonian (1.2), we parametrize the potential $V(\phi^2)$ in terms of the “single-site moments” \cite{141}

$$\hat{m}_{2k} = \frac{\Gamma\left(\frac{1}{2}N\right)}{2^k \Gamma\left(\frac{1}{2}N + k\right)} \frac{J_{N-1+2k}}{J_{N-1}}, \quad J_k = \int_0^\infty dx \, x^k \exp\left(-V(x^2)\right).$$

(A10)

We compute our series for fixed $N$, leaving all $\hat{m}_{2k}$ as free parameters: each term of the series is a polynomial in $\hat{m}_{2k}$ with rational coefficients.

With the aim of computing as many terms of the series as possible, we adopted all the technical developments of Ref. \cite{142}, and we introduced more improvements of our own; in this Appendix, we will only describe these; readers not familiar with technical details of the LCE should consult Refs. \cite{141,142}.

As discussed in Ref. \cite{142}, Sec. 3, the LCE requires a unique representation of graphs; it is convenient to implement this by defining a canonical form for the incidence matrix. The reduction to canonical form of a graph with $V$ vertices requires in principle the comparison of the $V!$ incidence matrices obtained by permutation of vertices, which is clearly unmanageable for large graphs; even with the introduction of the “extended vertex ordering” of Ref. \cite{142}, this operation remains the dominant factor in the computation time; therefore, we devoted a large effort to the optimization of this aspect of the computation. On one hand, we have perfected the extended vertex ordering, and we are able to recognize inequivalent vertices much more often. On the other hand, we search for (a subgroup of) the symmetry group of the incidence matrix, which allows us to perform explicitly only one permutation for each equivalence class. Altogether, the largest sets of vertices which are explicitly permuted are of size 5 or less (except for a few hundred diagrams requiring 6, and a handful requiring permutations of 7 or 8 elements).
The next most computer-intensive operation is the computation of embedding numbers and color factors; it is optimized by “remembering” each computed value in a table, compatibly with available memory. This is crucial for color factors, which are computed recursively, and very effective for embedding numbers.

The problem of handling integer and rational quantities which do not fit into machine precision is solved by using the GNU multiprecision (gmp) library. Neither multiprecision nor polynomials in $\hat{m}_{2k}$ are necessary for the most expensive sections of the computation; therefore they have little impact on the computation time.

In order to speed up the handling of search and insertion into ordered sets of data, we make extensive use of AVL trees (height-balanced binary trees) (cfr. e.g. Ref. [43], Chapt. 6.2.3). AVL trees are used to manipulate graph sets, multivariate polynomials, and tables of embedding numbers and of color factors.

The LCE is dramatically simplified by restricting actual computations to the set of one-particle irreducible graphs. One must however establish the relationship between the usual moments and susceptibilities and their irreducible parts. To this purpose, we found it convenient to define a generating functional of irreducible moments (irreducible momentum-space 2-point functions):

$$G^{1\pi}(\vec{p}) = \sum_{\vec{x}} \exp(i\vec{p} \cdot \vec{x}) \, G^{1\pi}(\vec{x}),$$  \hspace{1cm} (A11)

where $G^{1\pi}(\vec{x})$ is the irreducible-graph contribution to the field-field correlation. One may then prove the relationship

$$[G(\vec{p})]^{-1} = [G^{1\pi}(\vec{p})]^{-1} - 2\beta \sum_{i=1}^{d} \cos p_i.$$  \hspace{1cm} (A12)

By expanding both sides of Eq. (A12) in powers of $p_i^2$, it is trivial to establish all desired relationships, both for spherical and for non-spherical moments.

We have calculated $\chi$ and $m_2$ to 20th order, and the other moments of the two-point functions to 19th order. We have calculated $\chi_4$ to 18th order, $\chi_6$ to 17th order, $\chi_8$ to 16th order, and $\chi_{10}$ to 15th order. Using Eqs. (3.12), (5.23), (5.24), and (5.25), one can obtain the HT series corresponding to zero-momentum four-point coupling $g_4$ and the quantities $r_{2j}$ we have introduced to parametrize the effective potential.

It is useful to factorize out the leading dependence on $\beta$:

$$O = \beta^r \sum_{i=0}^{n} a_i \beta^i;$$  \hspace{1cm} (A13)

the values of $r$ and $n$ are summarized in Table XVI. In the following we will analyze the series normalized to start with $O(\beta^0)$, i.e. $a_0 + \beta a_1 + ...

We have checked our series for $\chi_{2n}$ and $m_2$ against the available series of the standard Ising model (see e.g. Refs. [13,36]); in this special case our only new result is the 18th order coefficient of the expansion of $\chi_4$:

$$a_{18}(\chi_4) = \frac{-171450770247965944104542584}{32564156625}.$$  \hspace{1cm} (A14)
We have checked the (new) series for $m_4$ and $m_6$ by changing variables to $v = \tanh \beta$ and verifying that all coefficients become integer numbers.

It would be pointless to present here the full results for an arbitrary potential: the resulting expressions are only fit for further computer manipulation. For the three potentials we are interested in, we computed $m_2$ by numerical integration (to 32 digit precision or higher). The coefficients $a_i$ of the HT series for $\lambda_6 = 0$ and $\lambda_4 = 1.1$ and for $\lambda_6 = 1$ and $\lambda_4 = 1.9$ are reported in Tables XVII and XVIII respectively. The series for the spin-1 model defined by Eq. (1.4), with $D = 0.641$, are reported in Table XIX.

3. Critical exponents

In order to determine the critical exponent $\gamma$ from the $n$th order series of $\chi$ ($n = 20$ in our case) we used quasi-diagonal first, second and third order integral approximants (IA1’s, IA2’s and IA3’s respectively).

IA1’s are solutions of the first-order linear differential equation

$$P_1(x)f'(x) + P_0(x)f(x) + R(x) = 0,$$

where the functions $P_1(x)$ and $R(x)$ are polynomials that are determined by the known $n$th order small-$x$ expansion of $f(x)$. We considered $[m_1/m_0/k]$ IA1’s with

$$m_1 + m_0 + k + 2 \geq n - p,$$

$$\text{Max } [(n - p - 2)/3] - q, 2] \leq m_1, m_0, k \leq [(n - p - 2)/3] + q,$$

where $m_1, m_0, k$ are the orders of the polynomial $P_1, P_0$ and $R$ respectively. The parameter $q$ determines the degree of off-diagonality allowed. Since the best approximants are expected to be those diagonal or quasi-diagonal, we considered sets of approximants corresponding to $q = 3$. For a given integer number $p$, only approximants using $\tilde{n}$ terms with $n \geq \tilde{n} \geq n - p$ are selected by (A16). In our analysis we considered the values $p = 0, 1$.

IA2’s are solutions of the second-order linear differential equation

$$P_2(x)f''(x) + P_1(x)f'(x) + P_0(x)f(x) + R(x) = 0.$$
We considered \([m_2/m_1/m_0/k]\) IA2's with
\[
m_2 \geq m_1 \geq m_0 \geq k \geq n - p,
\]
Max \([((n - p - 4)/4) - q, 2] \leq m_2, m_1, m_0, k \leq [(n - p - 4)/4] + q\),
(A18)
where \(m_2, m_1, m_0, k\) are the orders of the polynomial \(P_2, P_1, P_0\) and \(R\) respectively. Again the parameter \(q\) determines the degree of off-diagonality allowed, and we consider sets of approximants corresponding to \(q = 2\).

IA3's are solutions of the third-order linear differential equation
\[
P_3(x)f'''(x) + P_2(x)f''(x) + P_1(x)f'(x) + P_0(x)f(x) + R(x) = 0.
\]
(A19)
We considered \([m_3/m_2/m_1/m_0/k]\) IA3's with
\[
m_3 \geq m_2 \geq m_1 \geq m_0 \geq k \geq n - p,
\]
Max \([[(n - p - 6)/5] - q, 2] \leq m_3, m_2, m_1, m_0, k \leq [(n - p - 6)/5] + q\),
(A20)
We considered sets of approximants with \( q = 2 \).

As estimate of \( \beta_c \) and \( \gamma \) from each set of IA’s, we took the average of the values corresponding to all non-defective approximants listed above. The error quoted is the standard deviation. Approximants are considered defective when they present spurious singularities close to the real axis for \( \text{Re} \, \beta \lesssim \beta_c \). More precisely we considered defective the approximants with spurious singularities in the rectangle

\[
x_{\text{min}} \leq \text{Re} \, z \leq x_{\text{max}}, \quad |\text{Im} \, z| \leq y_{\text{max}}
\]

(A21)

where \( z \equiv \beta/\beta_c \). The special values of \( x_{\text{min}} \), \( x_{\text{max}} \) and \( y_{\text{max}} \) are fixed essentially by stability criteria, and may differ in the various analysis. One should always check that, within a reasonable and rather wide range of values, the results depend very little on the values of \( x_{\text{min}} \), \( x_{\text{max}} \), and \( y_{\text{max}} \). The condition \( \text{(A21)} \) cannot be too strict, otherwise only few approximants are left. In this case the analysis would be less robust and therefore less reliable. In the analysis of the critical exponents we fixed \( x_{\text{min}} = 0.5 \), \( x_{\text{max}} = 1.5 \) and

| \( t \) | \( \chi \) | \( m_2 \) | \( m_4 \) | \( m_6 \) |
|---|---|---|---|---|
| 0 | 0.465566267146533907 | 1.39011646285418792 | 1.39011646285418792 | 1.39011646285418792 |
| 1 | 1.3005116946285418792 | 1.3005116946285418792 | 1.3005116946285418792 |
| 2 | 3.286772712738568413 | 1.39011646285418792 | 1.39011646285418792 |
| 3 | 8.2759367606743730 | 1.39011646285418792 |
| 4 | 20.3159007243773174 | 1.39011646285418792 |
| 5 | 49.79484328088242272 | 1.39011646285418792 |
| 6 | 120.62508561530384098 | 1.39011646285418792 |
| 7 | 292.01525574713342383 | 1.39011646285418792 |
| 8 | 702.164819484216675 | 1.39011646285418792 |
| 9 | 1687.6428772292658916 | 1.39011646285418792 |
| 10 | 4038.96830004057954464 | 1.39011646285418792 |
| 11 | 9662.2792966182684635 | 1.39011646285418792 |
| 12 | 23047.113364799224307 | 1.39011646285418792 |
| 13 | 54959.286082315595970 | 1.39011646285418792 |
| 14 | 130774.5371051657548464 | 1.39011646285418792 |
| 15 | 31110.31266315734484 | 1.39011646285418792 |
| 16 | 73890.79087053751923 | 1.39011646285418792 |
| 17 | 175433.3817787454756 | 1.39011646285418792 |
| 18 | 416121.5431591307916 | 1.39011646285418792 |
| 19 | 9867152.2571694621736 | 1.39011646285418792 |
| 20 | 23372660.203457142541 | 1.39011646285418792 |

TABLE XVIII. IHT series for \( \lambda_{\max} \) and \( y_c \). The special values of \( c \) and \( \gamma \). The condition (A21) cannot be too strict, otherwise only few approximants are left. In this case the analysis would be less robust and therefore less reliable. In the analysis of the critical exponents we fixed \( x_{\text{min}} = 0.5 \), \( x_{\text{max}} = 1.5 \) and
As a further check of our analysis we used the fact that 
\[ \chi = \beta_c \text{ and } \gamma = \beta_c \text{ with } \beta_c \text{ such that } \beta_c + \beta_c^4 < \epsilon \beta_c, \] 
and extracted the estimates of \( \beta_c, \gamma \) and \( \gamma^d \) from them. As in Ref. [8], we also considered IA’s where the polynomial associated with the highest derivative of \( f(x) \) is even, i.e. it is a polynomial in \( x^2 \). We will denote them by \( b_{af} \). This ensures that if \( x_c \) is a singularity of an approximant then \( -x_c \) will also be a singular point.

In Table XX we present the results for some values of the parameters \( p, q, \epsilon \) introduced above (when the value of \( \epsilon \) is not explicitly shown it means that the corresponding constraint was not implemented). We quote the “ratio of approximate” \( r_{app} = (l - s)/t \), where \( t \) is the total number of approximants in the given set, \( l \) is the number of non-defective approximants

\[ y_{max} = 0.5. \] 
 Sometimes we also eliminated seemingly good approximants whose results were very far from the average of the other approximants (more than three standard deviations).
(passing the test \((A21)\)), and \(s\) is the number of seemingly good approximants which are excluded because their results are very far from the other approximants; \(l - s\) is the number of “good” approximants used in the analysis; notice that \(s \ll l\), and \(l - s\) is never too small. We found the IA2 analysis to give the most stable results, especially with respect to the change of the number of terms of the series considered. Therefore we consider the IA2 results to be the most reliable. Moreover, IA1’s are not completely satisfactory in reproducing the antiferromagnetic singularity when its presence is not biased. In the biased analyses where \(\beta_c\) is forced, IA1’s, IA2’s and IA3’s give almost equivalent results.

The results are quite stable, and the value of \(\gamma_{af}\) is always consistent with \(1 - \alpha \approx 0.89\). From the results of Table XX, combining the results of the IA2 and IA3 analyses (selecting the results denoted by IA2\(_{q=2,p=0}\), b\(_{af}\)IA2\(_{q=2,p=0}\), IA3\(_{q=2,p=0}\) and b\(_{af}\)IA3\(_{q=2,p=0}\)) we obtain the following estimates
\[
\beta_c = 0.3750973(14), \quad \gamma = 1.23732(24) \quad \text{for} \quad \lambda_6 = 0, \quad \lambda_4 = 1.10, \tag{A22}
\]
\[
\beta_c = 0.4269780(18), \quad \gamma = 1.23712(26) \quad \text{for} \quad \lambda_6 = 1, \quad \lambda_4 = 1.90, \tag{A23}
\]
and
\[
\beta_c = 0.3856688(20), \quad \gamma = 1.23680(30) \quad \text{for} \quad \text{spin-1}, \quad D = 0.641. \tag{A24}
\]
Taking also into account the uncertainty of \(\lambda_4^*\) and \(D^*\), we arrive at the estimates of Table XII. Notice that the value of \(\beta_c\) at \(\lambda_4 = 1.10\) and \(\lambda_6 = 0\) is in agreement with the Monte Carlo estimate of Ref. [20], i.e. \(\beta_c = 0.3750966(4)\) (where according to the author the error does not include possible systematic errors). From the analysis of the antiferromagnetic singularity using the b\(_{af}\)IA’s we obtain the following estimate for \(\alpha\):
\[
\alpha = 0.105(10), \tag{A25}
\]
which is in good agreement with the much more precise estimate \((4.3)\) obtained assuming hyperscaling.

In order to determine \(\nu\) from the analysis of the HT series of \(\xi^2\), we followed the suggestion of Ref. [38], i.e. to use the estimate of \(\beta_c\) derived from the analysis of \(\chi\) in order to bias the analysis of the series of \(\xi^2\). We analyzed the 19th order series of \(\xi^2/\beta\) and employed biased integral approximants (b\(_{\beta_c}\)IA). For instance, biased IA2’s can be obtained from the solutions of the equation
\[
(1 - x/\beta_c) P_2(x) f''(x) + P_1(x) f'(x) + P_0(x) f(x) + R(x) = 0. \tag{A26}
\]
In this case we considered the approximants satisfying the conditions
\[
m_2 + m_1 + m_0 + k + 3 \geq n - p,
\]
\[
\text{Max } \left[(n - p - 3)/4 - q, 2\right] \leq m_2, m_1, m_0, k \leq \left[(n - p - 3)/4\right] + q, \tag{A27}
\]
where, as before, \(m_i\) and \(k\) are the order of the polynomials \(P_i\) and \(R\) respectively. We also tried doubly-biased IA2 (b\(_{\pm\beta_c}\)IA2) where also a singularity at \(-\beta_c\) is forced using solutions of the equation.
TABLE XX. Results of various analyses of the 20th order IHT series for $\chi$. r-app is explained in the text. In the biased analyses forcing the value of $\beta_c$ the error is reported as a sum: the first term is related to the spread of the approximants at the central value of $\beta_c$, while the second one is related to the uncertainty of the value of $\beta_c$ and it is estimated by varying $\beta_c$.

| $\lambda_6 = 0$, $\lambda_4 = 1.08$ | $\lambda_6 = 0$, $\lambda_4 = 1.10$ | $\lambda_6 = 0$, $\lambda_4 = 1.12$ | $\lambda_6 = 1$, $\lambda_4 = 1.86$ | $\lambda_6 = 1$, $\lambda_4 = 1.90$ | $\lambda_6 = 1$, $\lambda_4 = 1.94$ |
|-----------------------------------|-----------------------------------|-----------------------------------|-----------------------------------|-----------------------------------|-----------------------------------|
| $IA_2q=2,p=0$ | $IA_2q=2,p=0$ | $IA_2q=2,p=0$ | $IA_2q=2,p=0$ | $IA_2q=2,p=0$ | $IA_2q=2,p=0$ |
| $(78 - 2)/85$ | $(62 - 1)/65$ | $(63 - 1)/65$ | $(30 - 3)/27$ | $(62 - 1)/65$ | $(63 - 1)/65$ |
| $0.3760099(21)$ | $0.3760703(18)$ | $0.3750945(27)$ | $0.3750955(18)$ | $0.3750955(18)$ | $0.3750955(18)$ |
| $1.23713(37)$ | $1.23719(30)$ | $1.23684(42)$ | $1.23699(26)$ | $1.23699(26)$ | $1.23699(26)$ |
| $0.377(4)$ | $0.377(4)$ | $0.375(3)$ | $0.376(8)$ | $0.376(8)$ | $0.376(8)$ |
| $-1.0(3)$ | $-0.903(10)$ | $-0.9(3)$ | $-0.78(8)$ | $-0.78(8)$ | $-0.78(8)$ |

| $IA_1q=3,p=0$ | $IA_1q=3,p=0$ | $IA_1q=3,p=0$ | $IA_1q=3,p=0$ | $IA_1q=3,p=0$ | $IA_1q=3,p=0$ |
| $(26 - 1)/37$ | $(24 - 2)/37$ | $(30 - 1)/65$ | $(21 - 1)/65$ | $(30 - 1)/65$ | $(30 - 1)/65$ |
| $0.3750956$ | $0.3750937(46)$ | $0.3750975(18)$ | $0.3750989(12)$ | $0.3750976(18)$ | $0.3750976(18)$ |
| $1.23701$ | $1.23763(69)$ | $1.23734(29)$ | $1.23749(19)$ | $1.23734(30)$ | $1.23734(30)$ |
| $-0.3754$ | $-0.376(6)$ | $-0.376(6)$ | $-0.375(2)$ | $-0.375(2)$ | $-0.375(2)$ |
| $-0.81$ | $-1.06$ | $-0.9(2)$ | $-0.92(2)$ | $-0.90(3)$ | $-0.90(3)$ |

| $IA_2q=2,p=0$ | $IA_2q=2,p=0$ | $IA_2q=2,p=0$ | $IA_2q=2,p=0$ | $IA_2q=2,p=0$ | $IA_2q=2,p=0$ |
| $(69 - 1)/85$ | $(62 - 1)/65$ | $(63 - 1)/65$ | $(63 - 1)/65$ | $(63 - 1)/65$ | $(63 - 1)/65$ |
| $0.3750974(21)$ | $0.3750975(19)$ | $0.3750976(18)$ | $0.3750976(18)$ | $0.3750976(18)$ | $0.3750976(18)$ |
| $1.23733(35)$ | $1.23735(31)$ | $1.23734(30)$ | $1.23734(30)$ | $1.23734(30)$ | $1.23734(30)$ |
| $-0.375(2)$ | $-0.375(2)$ | $-0.375(2)$ | $-0.375(2)$ | $-0.375(2)$ | $-0.375(2)$ |
| $-0.9(2)$ | $-0.9(2)$ | $-0.9(2)$ | $-0.9(2)$ | $-0.9(2)$ | $-0.9(2)$ |

| $IA_1q=3,p=0$ | $IA_1q=3,p=0$ | $IA_1q=3,p=0$ | $IA_1q=3,p=0$ | $IA_1q=3,p=0$ | $IA_1q=3,p=0$ |
| $(22 - 1)/37$ | $(141 - 3)/115$ | $(90 - 2)/100$ | $(61 - 1)/63$ | $(77 - 2)/85$ | $(62 - 1)/65$ |
| $0.3750966(4)$ | $0.3750966(4)$ | $0.3750966(4)$ | $0.3750966(4)$ | $0.3750966(4)$ | $0.3750966(4)$ |
| $1.23718(7+4)$ | $1.23720(2+4)$ | $1.23719(3+4)$ | $1.23719(8+7)$ | $1.23748(26)$ | $1.23748(26)$ |
| $1.1(6)$ | $1.0(5)$ | $1.0(5)$ | $1.0(6)$ | $1.0(6)$ | $1.0(6)$ |
| $-0.9(3)$ | $-0.9(3)$ | $-0.9(3)$ | $-0.9(3)$ | $-0.9(3)$ | $-0.9(3)$ |

| $IA_1q=3,p=0$ | $IA_1q=3,p=0$ | $IA_1q=3,p=0$ | $IA_1q=3,p=0$ | $IA_1q=3,p=0$ | $IA_1q=3,p=0$ |
| $(30 - 1)/65$ | $(62 - 1)/65$ | $(63 - 1)/65$ | $(63 - 1)/65$ | $(63 - 1)/65$ | $(63 - 1)/65$ |
| $0.3741199(21)$ | $0.3741199(21)$ | $0.3741199(21)$ | $0.3741199(21)$ | $0.3741199(21)$ | $0.3741199(21)$ |
| $1.23742(36)$ | $1.23742(36)$ | $1.23742(36)$ | $1.23742(36)$ | $1.23742(36)$ | $1.23742(36)$ |
| $-0.375(4)$ | $-0.375(4)$ | $-0.375(4)$ | $-0.375(4)$ | $-0.375(4)$ | $-0.375(4)$ |
| $-1.0(3)$ | $-1.0(3)$ | $-1.0(3)$ | $-1.0(3)$ | $-1.0(3)$ | $-1.0(3)$ |

| $IA_2q=2,p=0$ | $IA_2q=2,p=0$ | $IA_2q=2,p=0$ | $IA_2q=2,p=0$ | $IA_2q=2,p=0$ | $IA_2q=2,p=0$ |
| $(82 - 3)/85$ | $(82 - 3)/85$ | $(82 - 3)/85$ | $(82 - 3)/85$ | $(82 - 3)/85$ | $(82 - 3)/85$ |
| $0.3743066(21)$ | $0.3743066(21)$ | $0.3743066(21)$ | $0.3743066(21)$ | $0.3743066(21)$ | $0.3743066(21)$ |
| $1.23738(30)$ | $1.23738(30)$ | $1.23738(30)$ | $1.23738(30)$ | $1.23738(30)$ | $1.23738(30)$ |
| $-0.423(6)$ | $-0.423(6)$ | $-0.423(6)$ | $-0.423(6)$ | $-0.423(6)$ | $-0.423(6)$ |
| $-1.1(9)$ | $-1.1(9)$ | $-1.1(9)$ | $-1.1(9)$ | $-1.1(9)$ | $-1.1(9)$ |
The approximants at the central value of $\beta$ in the text. The error is reported as a sum of two terms. The first term is related to the spread of the approximants at the value of $\beta_c$, while the second one is related to the uncertainty of the value of $\beta_c$ and it is estimated by varying $\beta_c$. 

\begin{table}[h]
\centering
\begin{tabular}{ccc}
\hline
approx & $r$-app & $\nu$ & $\gamma_{\alpha}$
\
\hline
$\lambda_6 = 0$, $\lambda_4 = 1.08$ & $b_{4}I_{A2q=2,p=0}$ & $(54 - 2)/70$ & 0.63004(2+11)

$\lambda_6 = 0$, $\lambda_4 = 1.10$ & $b_{3}I_{A1q=3,p=0}$ & 35/37 & 0.63012(2+10)

$\lambda_6 = 0$, $\lambda_4 = 1.12$ & $b_{4}I_{A2q=2,p=0}$ & 51/70 & 0.63016(3+9)

& $b_{4}I_{A1q=3,p=0}$ & $(53 - 3)/55$ & 0.63015(5+10)

& $b_{4}I_{A2q=2,p=0}$ & 28/35 & 0.63009(14+10)

& $b_{4}I_{A1q=3,p=0}$ & $(64 - 3)/132$ & 0.63016(3+9)

& $b_{4}I_{A2q=2,p=0}$ & $(47 - 2)/55$ & 0.63012(3+3)

& $b_{4}I_{A2q=2,p=0}$ & (49 - 2)/70 & 0.63027(3+9)

& $b_{3}I_{A2q=1,p=0}$ & $(60 - 1)/70$ & 0.62978(1+11)

& $b_{4}I_{A1q=3,p=0}$ & 34/37 & 0.63000(2+12)

& $b_{4}I_{A2q=2,p=0}$ & $(64 - 2)/70$ & 0.63003(2+11)

& $b_{4}I_{A2q=2,p=0}$ & $(55 - 2)/55$ & 0.63003(7+11)

& $b_{4}I_{A2q=2,p=0}$ & $(113 - 4)/132$ & 0.63003(3+10)

& $b_{4}I_{A3q=2,p=0}$ & $(27 - 1)/34$ & 0.62988(17+14)

& $b_{4}I_{A2q=2,p=0}$ & $(55 - 2)/70$ & 0.63023(2+11)

& $b_{3}I_{A2q=1,p=0}$ & $(67 - 1)/70$ & 0.62999(2+13)

& $b_{4}I_{A1q=3,p=0}$ & 33/37 & 0.62988(5+15)

& $b_{4}I_{A2q=2,p=0}$ & $(66 - 1)/70$ & 0.62990(2+13)

& $b_{4}I_{A3q=2,p=0}$ & 24/35 & 0.62988(12+19)

& $b_{3}I_{A2q=1,p=0}$ & $(66 - 1)/70$ & 0.62989(2+13)

\hline
\end{tabular}
\end{table}

\[ (1 - x^2/\beta_c^2) P_2(x) f''(x) + P_1(x) f'(x) + P_0(x) f(x) + R(x) = 0. \]  

(A28)

In Table [XXI] we report the results of some of the analyses we performed. In the case of the $b_{\pm \beta_0}IA2$ analysis we also report the exponent at the antiferromagnetic singularity which turned out to be always consistent with $1 - \alpha$. The error of $\nu$ is given as a sum of two terms: the first one is computed from the spread of the approximants at $\beta_c$, the second one is related to the uncertainty of $\beta_c$. We quote as our final estimates:

\begin{align*}
\nu &= 0.63015(12) \quad \text{for} \quad \lambda_6 = 0, \quad \lambda_4 = 1.10, \\
\nu &= 0.63003(13) \quad \text{for} \quad \lambda_6 = 1, \quad \lambda_4 = 1.90.
\end{align*}

(A29) (A30)

and

\[ \nu = 0.62990(15) \quad \text{for} \quad \text{spin-1}, \quad D = 0.641. \]  

(A31)

Taking also into account the uncertainty of $\lambda_4^*$ we arrive at the estimates of Table [III]. We mention that unbiased IA analyses of the 19th series of $\xi^2/\beta$ give consistent but less precise estimates of $\beta_c$ (cf. Eqs. [A22] and [A23]) and $\nu$.

As a check of our results, we performed a biased analysis of $\chi$ and $\xi^2$ at $\lambda_6 = 0$ and $\lambda_4 = 1.10$, using the value $\beta_c = 0.3750966(4)$ obtained in Ref. [20] by Monte Carlo simulations based on finite-size scaling techniques. Although the author of Ref. [20] says that the error on $\beta_c$ does not include systematic errors, we used it as a check and found (see Tables [XX] and [XXI]) $\gamma = 1.23720(2+7)$ and $\nu = 0.63012(3+3)$ (the first error is related to the spread of the approximants at $\beta_c = 0.3750966$ and the second one to the error on $\beta_c$), which are perfectly consistent with our final estimates reported in Table [III].
TABLE XXII. Results for $\eta$ obtained using the CPRM: (a) applied to $\xi^2$ and $\chi$ (20 orders); (b) applied to $\xi^2/\beta$ and $\chi$ (19 orders). r-app is explained in the text.

| $\lambda_6 = 0$, $\lambda_4 = 1.08$ | approx | r-app | $\eta \nu$ |
|---------------------------------|--------|-------|----------|
| (a) blIA2q=2,p=0               | (95 − 1)/115 | 0.0227(4)(13) |
| (b) blIA2q=2,p=0               | 38/70 | 0.02294(8) |
| $\lambda_6 = 0$, $\lambda_4 = 1.10$ | (a) blIA2q=2,p=0 | 95/115 | 0.02280(14) |
| (b) blIA2q=2,p=0               | (150 − 1)/185 | 0.02280(16) |
| (a) blIA2q=2,p=1               | (47 − 1)/61 | 0.02280(37) |
| (b) blIA2q=2,p=0               | 28/37 | 0.02300(8) |
| (b) blIA2q=2,p=0               | 36/70 | 0.02301(8) |
| (b) blIA2q=2,p=0               | 86/132 | 0.02309(12) |
| (b) blIA2q=2,p=0               | 31/34 | 0.02311(22) |
| $\lambda_6 = 0$, $\lambda_4 = 1.12$ | (a) blIA2q=2,p=0 | 97/115 | 0.02285(15) |
| (b) blIA2q=2,p=0               | 35/70 | 0.02308(9) |
| $\lambda_6 = 1$, $\lambda_4 = 1.86$ | (a) blIA2q=2,p=0 | (94 − 3)/115 | 0.02267(12) |
| (b) blIA2q=2,p=0               | 30/70 | 0.02285(12) |
| $\lambda_6 = 1$, $\lambda_4 = 1.90$ | (a) blIA2q=2,p=0 | (90 − 2)/115 | 0.02278(12) |
| (b) blIA2q=2,p=0               | 37/70 | 0.02298(11) |
| $\lambda_6 = 1$, $\lambda_4 = 1.94$ | (a) blIA2q=2,p=0 | (92 − 2)/115 | 0.02288(13) |
| (b) blIA2q=2,p=0               | 32/70 | 0.02312(10) |
| spin-1, $D = 0.633$            | (a) blIA2q=2,p=0 | (84 − 1)/115 | 0.02292(40) |
| (b) blIA2q=2,p=0               | 36/70 | 0.02316(22) |
| spin-1, $D = 0.641$            | (a) blIA2q=2,p=0 | (85 − 2)/115 | 0.02288(40) |
| (b) blIA2q=2,p=0               | 37/70 | 0.02312(22) |
| spin-1, $D = 0.649$            | (a) blIA2q=2,p=0 | (84 − 1)/115 | 0.02285(43) |
| (b) blIA2q=2,p=0               | 37/70 | 0.02307(22) |

In order to obtain an estimate of $\eta$ without using the scaling relation $\gamma = (2 - \eta) \nu$, we employed the so-called critical point renormalization method (CPRM). The idea of the CPRM is that from two series $D(x)$ and $E(x)$ which are singular at the same point $x_0$, $D(x) = \sum_i d_i x^i \sim (x_0 - x)^{-\delta}$ and $E(x) = \sum_i c_i x^i \sim (x_0 - x)^{-\epsilon}$, one constructs a new series $F(x) = \sum_i (d_i/c_i) x^i$. The function $F(x)$ is singular at $x = 1$ and for $x \to 1$ behaves as $F(x) \sim (1 - x)^{-\phi}$, where $\phi = 1 + \delta - \epsilon$. Therefore the analysis of $F(x)$ provides an unbiased estimate of the difference between the critical exponents of the two functions $D(x)$ and $E(x)$. The series $F(x)$ may be analyzed by employing biased approximants with a singularity at $x_0 = 1$. In order to check for possible systematic errors, we applied the CPRM to the series of $\xi^2/\beta$ and $\chi$ (analyzing the corresponding 19th order series) and to the series of $\xi^2$ and $\chi$ (analyzing the corresponding 20th order series). We used IA’s biased at $x_0 = 1$. In Table XXII we present the results of the analysis for some values of the parameters $q, p$. We obtain $\eta \nu = 0.02294(20)$ at $\lambda_6 = 0$ and $\lambda_4 = 1.1$, $\eta \nu = 0.02287(20)$ at $\lambda_6 = 1$ and $\lambda_4 = 1.9$, and $\eta \nu = 0.02305(20)$ for spin-1 and $D = 0.641$. Taking again into account the uncertainty of $\lambda^*_4$ and $D^*$ we then obtain the estimate reported in Table III.

The CPRM was also employed in order to estimate the exponent $\sigma$. It was applied to the 18th order series of $\chi \xi^2/\beta$ and $q_{4,0}/\beta$. The results are displayed in Table XXIII. We find $\sigma \nu = 0.0134(8)$ for $\lambda_6 = 0$ and $\lambda_4 = 1.1$, $\sigma \nu = 0.0134(9)$ for $\lambda_6 = 1$ and $\lambda_4 = 1.9$, and $\sigma \nu = 0.0127(6)$ for spin-1 at $D = 0.641$.

4. Ratios of amplitudes

In the following we describe the analysis we employed in order to evaluate universal ratios of amplitudes, such as $g_4, r_{2j}$ and $c_i$, from the corresponding HT series. In the case of $g$, $c_2$, 

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c_3 and c_4 we analyzed the series \( \beta^{3/2}g_4 = \sum_{i=0}^{17} a_i \beta^i \), \( \beta^{-2}c_4 = \sum_{i=0}^{13} a_i \beta^i \), and \( \beta^{-2}c_4 = \sum_{i=0}^{13} a_i \beta^i \). In order to obtain estimates of the universal critical limit of \( g_4 \), \( r_{2j} \), and \( c_i \), we evaluated the approximants of the corresponding HT series at \( \beta_c \) (as determined from the analysis of the magnetic susceptibility), and multiplied by the appropriate power of \( \beta_c \).

For an \( n \)th order series we considered three sets of approximants: Padé approximants (PA’s), Dlog-Padé approximants (DPA’s) and first-order integral approximants (IA1’s).

(I) \([l/m]\) PA’s with

\[
\begin{align*}
\lambda_6 = 0, \hspace{0.5cm} \lambda_4 = 1.08 & \\
\lambda_6 = 0, \hspace{0.5cm} \lambda_4 = 1.10 & \\
\lambda_6 = 0, \hspace{0.5cm} \lambda_4 = 1.12 & \\
\lambda_6 = 1, \hspace{0.5cm} \lambda_4 = 1.86 & \\
\lambda_6 = 1, \hspace{0.5cm} \lambda_4 = 1.90 & \\
\lambda_6 = 1, \hspace{0.5cm} \lambda_4 = 1.94 & \\
\text{spin-1}, \hspace{0.5cm} D = 0.633 & \\
\text{spin-1}, \hspace{0.5cm} D = 0.641 & \\
\text{spin-1}, \hspace{0.5cm} D = 0.649 & \\
\end{align*}
\]

\[l + m \geq n - 2,
\]

\[\text{Max} \left[ n/2 - q, 4 \right] \leq l, \hspace{0.5cm} m \leq n/2 + q, \]

where \( l, m \) are the orders of the polynomials respectively in the numerator and denominator of the PA. The parameter \( q \) determines the degree of off-diagonality allowed. The best approximants should be those diagonal or quasi-diagonal. So we considered PA’s selected using \( q = 3 \). As estimate from the PA’s we take the average of the values at \( \beta_c \) of the non-defective approximants using all the available terms of the series and satisfying the condition (A33) with \( q = 2 \). The error we quote is the standard deviation around the estimate of the results from all the non-defective approximants listed above. We considered defective PA’s with spurious singularities in the rectangle defined in Eq. (A21) with \( x_{\min} = 0.9 \) (\( x_{\min} = 0 \) only in the case of \( r_{10} \) \( x_{\max} = 1.01 \) and \( y_{\max} = 0.1 \).

(II) \([l/m]\) DPA’s with

\[
\begin{align*}
\lambda_6 = 0, \hspace{0.5cm} \lambda_4 = 1.08 & \\
\lambda_6 = 0, \hspace{0.5cm} \lambda_4 = 1.10 & \\
\lambda_6 = 0, \hspace{0.5cm} \lambda_4 = 1.12 & \\
\lambda_6 = 1, \hspace{0.5cm} \lambda_4 = 1.86 & \\
\lambda_6 = 1, \hspace{0.5cm} \lambda_4 = 1.90 & \\
\lambda_6 = 1, \hspace{0.5cm} \lambda_4 = 1.94 & \\
\text{spin-1}, \hspace{0.5cm} D = 0.633 & \\
\text{spin-1}, \hspace{0.5cm} D = 0.641 & \\
\text{spin-1}, \hspace{0.5cm} D = 0.649 & \\
\end{align*}
\]

\[l + m \geq n - 2,
\]

\[\text{Max} \left[ (n - 1)/2 - q, 4 \right] \leq l, \hspace{0.5cm} m \leq (n - 1)/2 + q, \]

where \( l, m \) are the orders of the polynomials respectively in the numerator and denominator of the PA of the series of its logarithmic derivative. We again fixed \( q = 3 \). The estimate with the corresponding error is obtained as in the case of PA’s. We considered defective DPA’s with spurious singularities in the rectangle with \( x_{\min} = 0, \hspace{0.5cm} x_{\max} = 1.01 \) and \( y_{\max} = 0.1 \), cf. Eq. (A22).

(III) \([m_1/m_0/k]\) IA1’s given by Eq. (A10). The off-diagonality parameter was fixed to be \( q = 3 \), and \( p = 1 \). As estimate we take the average of the values at \( \beta_c \) of all non-defective approximants listed above. The error we quote is the standard deviation around

Table XXIII. Results for \( \sigma \) obtained using the CPRM applied to \( m_2/\beta \) and \( q_{4.0}/\beta \) (18 orders). Here we used \( x_{\min} = 0.9, \hspace{0.5cm} x_{\max} = 1.1, \hspace{0.5cm} y_{\max} = 0.1 \). r-app is explained in the text.

| approx          | r-app          | \( \sigma \nu \) |
|-----------------|----------------|---------------------|
| \( \lambda_6 = 0, \hspace{0.5cm} \lambda_4 = 1.08 \) | \( \text{BI}\; a_{q=1+p=0} \) | 29/34 | 0.0134(8) |
| \( \lambda_6 = 0, \hspace{0.5cm} \lambda_4 = 1.10 \) | \( \text{BI}\; a_{q=1+p=0} \) | 29/34 | 0.0134(8) |
| \( \lambda_6 = 0, \hspace{0.5cm} \lambda_4 = 1.12 \) | \( \text{BI}\; a_{q=2+p=0} \) | 53/62 | 0.0133(10) |
| \( \lambda_6 = 1, \hspace{0.5cm} \lambda_4 = 1.86 \) | \( \text{BI}\; a_{q=1+p=0} \) | 29/34 | 0.0133(9) |
| \( \lambda_6 = 1, \hspace{0.5cm} \lambda_4 = 1.90 \) | \( \text{BI}\; a_{q=1+p=0} \) | 29/34 | 0.0134(9) |
| \( \lambda_6 = 1, \hspace{0.5cm} \lambda_4 = 1.94 \) | \( \text{BI}\; a_{q=2+p=0} \) | 52/62 | 0.0132(12) |
| \( \text{spin-1}, \hspace{0.5cm} D = 0.633 \) | \( \text{BI}\; a_{q=1+p=0} \) | 28/34 | 0.0135(9) |
| \( \text{spin-1}, \hspace{0.5cm} D = 0.641 \) | \( \text{BI}\; a_{q=1+p=0} \) | 21/34 | 0.0127(5) |
| \( \text{spin-1}, \hspace{0.5cm} D = 0.649 \) | \( \text{BI}\; a_{q=2+p=0} \) | 37/62 | 0.0128(6) |
| \( \text{spin-1}, \hspace{0.5cm} D = 0.649 \) | \( \text{BI}\; a_{q=1+p=0} \) | 21/34 | 0.0126(5) |
TABLE XXIV. Results of PA, DPA, and IA1 analyses of the series for $g_4$, $r_2$, and $c_i$. When results are not reported, it means that for that quantity no acceptable results were obtained from that class of approximants. The fraction at subscript is the number of non-defective approximants over the total number of approximants. The last column contains the estimates obtained by combining the three classes of approximants.

| g_4 | PA          | DPA          | IA1          | combined    |
|-----|-------------|--------------|--------------|-------------|
| $\lambda_6 = 0, \lambda_4 = 1.10$ | 23.500(60)_{17/21} | 23.491(25)_{16/18} | 23.504(18)_{19/73} | 23.499(16+20) |
| $\lambda_6 = 1, \lambda_4 = 1.90$ | 23.487(45)_{17/21} | 23.474(46)_{17/18} | 23.490(24)_{17/73} | 23.491(21+40) |
| spin-1, $D = 0.641$ | 23.486(19)_{20/21} | 23.492(88)_{17/18} | 23.491(52)_{53/73} | 23.487(18+20) |
| r_6 | $\lambda_6 = 0, \lambda_4 = 1.10$ | 2.091(14)_{19/21} | 2.058(12)_{11/18} | 2.048(4)_{43/73} | 2.050(5+4) |
| $\lambda_6 = 1, \lambda_4 = 1.90$ | 2.052(12)_{16/21} | 2.063(14)_{11/18} | 2.048(4)_{33/73} | 2.050(5+4) |
| spin-1, $D = 0.641$ | 2.0493(65)_{20/21} | 2.0461(24)_{16/18} | 2.0456(16)_{23/73} | 2.046(2+3) |
| r_8 | $\lambda_6 = 0, \lambda_4 = 1.10$ | 2.23(9)_{17/18} | 2.21(13)_{17/21} | 2.23(5)_{17/69} | 2.23(5+4) |
| $\lambda_6 = 1, \lambda_4 = 1.90$ | 2.23(11)_{18/18} | 2.23(9)_{17/21} | 2.23(5)_{36/69} | 2.23(5+6) |
| spin-1, $D = 0.641$ | 2.40(8)_{16/18} | 2.31(5)_{17/21} | 2.42(13)_{26/69} | 2.34(5+3) |
| r_{10} | $\lambda_6 = 0, \lambda_4 = 1.10$ | $-14(5)_{15/21}$ | $-13.3(13)_{16/61}$ | $-14(4+0)$ |
| $\lambda_6 = 1, \lambda_4 = 1.90$ | $-14(5)_{14/21}$ | $-12(4)_{10/61}$ | $-13(5+0)$ |
| spin-1, $D = 0.641$ | $-10(21)_{13/21}$ | 4(36)_{14/61} | $-8(25+0)$ |
| 10^c_2 | $\lambda_6 = 0, \lambda_4 = 1.10$ | $-3.582(8)_{15/15}$ | $-3.580(29)_{12/12}$ | $-3.580(24)_{24/33}$ | $-3.582(7+6)$ |
| $\lambda_6 = 1, \lambda_4 = 1.90$ | $-3.574(7)_{14/15}$ | $-3.574(26)_{12/12}$ | $-3.585(38)_{24/33}$ | $-3.574(7+20)$ |
| spin-1, $D = 0.641$ | $-3.570(12)_{15/15}$ | $-3.562(30)_{11/12}$ | $-3.554(28)_{25/33}$ | $-3.568(11+4)$ |
| 10^c_3 | $\lambda_6 = 0, \lambda_4 = 1.10$ | 0.087(5)_{12/15} | 0.080(10)_{3/12} | 0.085(7)_{26/36} | 0.085(4+0) |
| $\lambda_6 = 1, \lambda_4 = 1.90$ | 0.086(9)_{11/15} | 0.078(12)_{2/12} | 0.086(5)_{26/36} | 0.086(4+0) |
| spin-1, $D = 0.641$ | 0.085(4)_{14/15} | 0.100(12)_{2/12} | 0.090(4)_{30/36} | 0.090(4+0) |

The estimate. We considered defective IA1’s with spurious singularities in the rectangle and $x_{\text{min}} = 0$, $x_{\text{max}} = 1.001$ and $y_{\text{max}} = 0.1$.

As in the case of the critical exponents, sometimes we also eliminated seemingly good approximants whose results were very far from the average of the other approximants. In order to arrive at a final estimate, the results from PA’s, DPA’s, and IA’s were then combined taking also into account the relative number of non-defective approximants (before combining the results we divided the apparent error of each set of approximants by the square root of the ratio between the number of non-defective and the total number of approximants). Of course all the above procedure to arrive at a final estimate is rather subjective. But we believe it provides reasonable estimates of the quantity at hand and its uncertainty. We report in Table XXIV the results of each set of approximants, so that the readers can judge the reliability of our final estimates. The second error in the combined estimate is related to the uncertainty of the value of $\lambda_4^*$ and $D^*$; it is estimated by varying $\lambda_4$ in the range 1.08–1.12 for $\lambda_6 = 0$, 1.86–1.94 for $\lambda_6 = 1$, and $D$ in the range 0.633–0.649 for the spin-1 model. Errors due to the uncertainty of $\beta_c$ are negligible.

We have also performed analyses of the series of $g_4$ for $\lambda_6 = 0$ and several values of $\lambda_4$ by employing the Roskies transform [11]. The idea of the Roskies transform (RT) is to perform biased analyses which take into account the leading confluent singularity. For the Ising model, where $\Delta \approx 1/2$, one replaces the variable $\beta$ in the original expansion with a new variable $z$, defined by $1 - z = (1 - \beta/\beta_c)^{1/2}$. If the original series has square-root scaling correction terms, the transformed series has analytic correction terms, which can be handled by standard PA’s or DPA’s. Note that in principle IA1’s should be able to detect the first non-analytic correction to scaling, but they probably need many more terms of the series, and practically need to be explicitly biased as in the case of PA’s and DPA’s. Indeed the
TABLE XXV. Details of the analysis of the 17th order series for $\beta^{-3/2}g_4(\beta)$ with and without the use of the RT for some values of $\lambda_4$ and $\lambda_6 = 0$. In the PA, DPA and IA analyses with RT we used $q = 2$ (other approximants turned out to be much less stable). We fixed $x_{\text{min}} = 0$, $x_{\text{max}} = 1.1$, and $y_{\text{max}} = 0.25$ for PA and DPA, and $x_{\text{min}} = 0$, $x_{\text{max}} = 1.01$ and $y_{\text{max}} = 0.25$ for IA. In order to perform a homogeneous comparison we used the same procedure for the direct analysis without RT (except that we used $x_{\text{max}} = 1.01$ and $y_{\text{max}} = 0.1$). The fraction at subscript is the number of non-defective approximants over the total number of approximants.

| $\lambda_4$ | PA | DPA | IA | combined |
|-----------|----|-----|----|----------|
| 0.5       | direct | 22.62(58)$_{10/15}$ | 22.43(16)$_{9/12}$ | 22.59(19)$_{19/37}$ | 22.48(15) |
|           | RT  | 23.75(27)$_{9/15}$  | 23.48(25)$_{8/12}$  | 23.29(50)$_{15/37}$ | 23.56(20) |
| 0.7       | direct | 23.04(28)$_{10/15}$ | 22.92(14)$_{9/12}$  | 22.95(13)$_{15/37}$ | 22.94(12) |
|           | RT  | 23.62(31)$_{3/15}$  | 23.54(20)$_{9/12}$  | 23.35(35)$_{17/37}$ | 23.54(20) |
| 1.0       | direct | 23.58(23)$_{9/15}$  | 23.40(6)$_{12/12}$  | 23.48(19)$_{15/37}$ | 23.41(17) |
|           | RT  | 23.56(27)$_{13/15}$ | 23.57(15)$_{9/12}$  | 23.45(21)$_{14/37}$ | 23.55(14) |
| 1.1       | direct | 23.50(64)$_{12/15}$ | 23.49(23)$_{11/12}$ | 23.49(16)$_{17/37}$ | 23.49(16) |
|           | RT  | 23.56(22)$_{13/15}$ | 23.59(16)$_{9/12}$  | 23.44(17)$_{14/37}$ | 23.55(13) |
| 1.2       | direct | 23.63(4)$_{10/15}$  | 23.61(9)$_{11/12}$  | 23.61(20)$_{12/37}$ | 23.61(8)  |
|           | RT  | 23.56(20)$_{13/15}$ | 23.54(34)$_{11/12}$ | 23.43(16)$_{12/37}$ | 23.52(15) |
| 1.5       | direct | 23.93(4)$_{14/15}$  | 23.92(6)$_{7/12}$   | 23.93(4)           | 23.93(4)  |
|           | RT  | 23.55(27)$_{12/15}$ | 23.53(31)$_{11/12}$ | 23.41(13)$_{11/37}$ | 23.48(16) |
| 2.0       | direct | 24.14(28)$_{15/15}$ | 24.07(17)$_{3/12}$  | 24.15(9)$_{31/37}$  | 24.15(9)  |
|           | RT  | 23.61(22)$_{12/15}$ | 23.52(18)$_{10/12}$ | 23.44(12)$_{17/37}$ | 23.50(12) |
| 3.0       | direct | 24.32(14)$_{15/15}$ | 24.32(9)$_{6/12}$   | 24.40(20)$_{15/37}$ | 24.60(12) |
|           | RT  | 23.61(23)$_{14/15}$ | 23.47(14)$_{11/12}$ | 23.34(18)$_{16/37}$ | 23.48(11) |
| $\infty$ | direct | 24.78(10)$_{15/15}$ | 24.57(19)$_{10/12}$ | 24.81(16)$_{9/37}$  | 24.75(9)  |
|           | RT  | 23.59(20)$_{13/15}$ | 23.47(11)$_{10/12}$ | 23.48(16)$_{13/37}$ | 23.50(10) |

IA1 results without the RT turn out to be substantially equivalent to those obtained using PA’s and DPA’s. In Table XXV we report the details of the analysis without and with RT for some values of $\lambda_4$ and $\lambda_6 = 0$. These results are plotted in Fig. [4].

APPENDIX B: UNIVERSAL RATIOS OF AMPLITUDES

1. Notations

Universal ratios of amplitudes characterize the behavior in the critical domain of thermodynamical quantities that do not depend on the normalizations of the external (e.g. magnetic) field, order parameter (e.g. magnetization) and temperature. Amplitude ratios of zero-momentum quantities can be derived from the critical equation of state. We consider several amplitudes derived from the singular behavior of: the specific heat

$$C_H = A^\pm |t|^{-\alpha}, \quad (B1)$$

the magnetic susceptibility

$$\chi = C^\pm |t|^{-\gamma}, \quad (B2)$$

the spontaneous magnetization on the coexistence curve

$$M = B|t|^{-\beta}, \quad (B3)$$
the zero-momentum connected $n$-point correlation functions
\[ \chi_n = C_n^\pm |t|^{-\gamma-(n-2)/\delta}. \]  \( \text{(B4)} \)

We complete our list of amplitudes by considering the second-moment correlation length
\[ \xi = f^\pm |t|^{-\nu}, \]  \( \text{(B5)} \)

and the true (on-shell) correlation length, describing the large distance behavior of the two-point function,
\[ \xi_{\text{gap}} = f_{\text{gap}}^\pm |t|^{-\nu}. \]  \( \text{(B6)} \)

One can also define amplitudes along the critical isotherm, e.g.
\[ \chi_{-1} = C_{-1}^c |H|^{-\gamma/\delta}, \]  \( \text{(B7)} \)
\[ \chi_3 = C_3^c |H|^{-2\gamma/\delta}, \]  \( \text{(B8)} \)
\[ \chi_4 = C_4^c |H|^{-3\gamma/\delta}. \]  \( \text{(B9)} \)

2. Universal ratios of amplitudes from the parametric representation

In the following we report the expressions of the universal ratios of amplitudes in terms of the parametric representation (7.5) of the critical equation of state.

The singular part of the free energy per unit volume can be written as
\[ \mathcal{F} = h_0 m_0 R^{2-\alpha} g(\theta), \]  \( \text{(B10)} \)

where $g(\theta)$ is the solution of the first-order differential equation
\[ (1 - \theta^2)g'(\theta) + 2(2 - \alpha)\theta g(\theta) = (1 - \theta^2 + 2\beta\theta^2)h(\theta), \]  \( \text{(B11)} \)

that is regular at $\theta = 1$. One may also write
\[ \chi_{-1} = \frac{h_0}{m_0} R^\gamma g_2(\theta), \quad g_2(\theta) = \frac{2\beta\delta h(\theta) + (1 - \theta^2)h'(\theta)}{(1 - \theta^2 + 2\beta\theta^2)}, \]  \( \text{(B12)} \)
\[ \chi_3 = \frac{m_0}{h_0^2} R^{-2\gamma/\delta} g_3(\theta), \quad g_3(\theta) = -\frac{(1 - \theta^2)g_2'(\theta) + 2\gamma \theta g_2(\theta)}{g_2(\theta)^3(1 - \theta^2 + 2\beta\theta^2)}, \]  \( \text{(B13)} \)
\[ \chi_4 = \frac{m_0}{h_0^3} R^{-3\gamma/\delta} g_4(\theta), \quad g_4(\theta) = \frac{(1 - \theta^2)g_3'(\theta) - 2(2\gamma + \beta)\theta g_3(\theta)}{g_2(\theta)(1 - \theta^2 + 2\beta\theta^2)}. \]  \( \text{(B14)} \)

Using the above formulae, one can then calculate the universal ratios of amplitudes:

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\[ U_0 \equiv \frac{A^+}{A^-} = (\theta_0^2 - 1)^{2-\alpha} \frac{g(0)}{g(\theta_0)}, \]  
(B15)

\[ U_2 \equiv \frac{C^+}{C^-} = (\theta_0^2 - 1)^{-\gamma} \frac{g_2(0)}{g_2(\theta_0)}, \]  
(B16)

\[ u_4 \equiv \frac{C^+_4}{C^-_4} = (\theta_0^2 - 1)^{-3\gamma - 2\beta} \frac{g_4(0)}{g_4(\theta_0)}, \]  
(B17)

\[ R_c^+ \equiv \frac{\alpha A^+ C^+}{B^2} = -\alpha (1 - \alpha)(2 - \alpha)(\theta_0^2 - 1)^{2\beta} \theta_0^2 g(0), \]  
(B18)

\[ R_c^- \equiv \frac{\alpha A^- C^-}{B^2} = \frac{R_c^+}{U_0 U_2}, \]  
(B19)

\[ v_3 \equiv v_3 \equiv -\frac{C_3^-}{(C^-)^2} = -\theta_0 g_2(\theta_0)^2 g_3(\theta_0), \]  
(B20)

\[ v_4 \equiv -\frac{C_4^-}{(C^-)^3} + 3v_3^2 = \theta_0^2 g_2(\theta_0)^3 \left[ 3 g_2(\theta_0) g_3(\theta_0)^2 - g_4(\theta_0) \right], \]  
(B21)

\[ R_4^+ \equiv -\frac{C_4^+}{(C^+)^3} = |z_0|^2, \]  
(B22)

\[ Q_1^{-\delta} \equiv R_\chi \equiv \frac{C^+ B^{\delta - 1}}{(\delta C^c)^{\delta}} = (\theta_0^2 - 1)^{-\gamma} \theta_0^{\delta - 1} h(1), \]  
(B23)

\[ F_0^\infty \equiv \lim_{z \to \infty} z^{-\delta} F(z) = \rho^{1-\delta} h(1). \]  
(B24)

Using Eq. (7.7) one can compute \( F(z) \) and obtain the small-\( z \) expansion coefficients of the effective potential \( r_{2j} \) in terms of the critical exponents and the coefficients \( h_{2l+1} \) of the expansion of \( h(\theta) \).

**APPENDIX C: APPROXIMATION SCHEME FOR THE PARAMETRIC REPRESENTATION OF THE EQUATION OF STATE BASED ON STATIONARITY**

The parametric form of the critical equation of state, described by Eqs. (7.5), (7.6), and (7.7), shows a formal dependence on the auxiliary parameter \( \rho \).

However all physically relevant amplitude ratios are independent of \( \rho \), because they may be expressed in terms of the invariant function \( F(z) \) and its derivatives, evaluated at such special values of \( z \) as \( z = 0, z = \infty \) and \( z = z_0 \), where \( F(z_0) = 0 \). Notice that, despite the apparent dependence generated by the relation \( z_0 \equiv z(\rho, \theta_0(\rho)) \), from the definition it follows that \( z_0 \) must necessarily be independent of \( \rho \).

We can exploit these facts in order to set up an approximation procedure in which the function \( h(\rho, \theta) \), entering the scaling equation of state, is truncated to some simpler (polynomial) function \( h^{(i)}(\rho, \theta) \) and the value of \( \rho \) is properly fixed to optimize the approximation.

We found that, at any given order in the truncation, it is possible and convenient to choose \( \rho \) in such a way that all the (truncated) universal amplitude ratios be simultaneously stationary against infinitesimal variations of \( \rho \) itself.

Starting from \( h^{(i)}(\rho, \theta) \) we may reconstruct the function...
\[
\bar{F}(t, \theta) = \frac{\rho h(t)(\rho, \theta)}{1 - \theta^2} \beta + \gamma. \tag{C1}
\]

In order that all truncated amplitudes be simultaneously stationary in \(\rho\), it is necessary that the function \(F(t, z) \equiv \bar{F}(t)(\rho, \theta(z), z)\) be stationary with respect to variations of \(\rho\) for any value of \(z\).

We shall prove that for any polynomial truncation \(h(t)(\rho, \theta)\) it is possible to find a value \(\rho_t\), independent of \(z\), such that

\[
\frac{\partial F(t, z)}{\partial \rho} \bigg|_{\rho = \rho_t} = 0, \tag{C2}
\]

a property which we shall term “global stationarity”.

In order to prove our statement, let us rephrase the above condition into the form

\[
\frac{\partial \bar{F}(t)(\rho, \theta)}{\partial \rho} + \frac{\partial \bar{F}(t)(\rho, \theta)}{\partial \theta} \frac{\partial \theta}{\partial \rho} = 0, \tag{C3}
\]

where the implicit function theorem allows us to write

\[
\frac{\partial \theta}{\partial \rho} = -\frac{\partial z/\partial \rho}{\partial z/\partial \theta}. \tag{C4}
\]

The definitions (7.6) and (7.7) imply

\[
\frac{\partial z}{\partial \rho} = \frac{z}{\rho}, \quad \frac{\partial z}{\partial \theta} = z \left(1 + \frac{2\beta \theta}{1 - \theta^2}\right). \tag{C5}
\]

Moreover it is trivial to show that

\[
\frac{\partial \bar{F}(t)(\rho, \theta)}{\partial \rho} = \bar{F}(t)(\rho, \theta) \left(\frac{1}{\rho} + \frac{1}{h(t)(\rho, \theta)} \frac{\partial h(t)(\rho, \theta)}{\partial \rho}\right), \tag{C6}
\]

\[
\frac{\partial \bar{F}(t)(\rho, \theta)}{\partial \theta} = \bar{F}(t)(\rho, \theta) \left(\frac{2(\beta + \gamma)\theta}{1 - \theta^2} + \frac{1}{h(t)(\rho, \theta)} \frac{\partial h(t)(\rho, \theta)}{\partial \theta}\right). \tag{C7}
\]

Substitution of these expressions into Eq. (C3) leads to the following form of the global stationarity condition:

\[
[1 - (1 + 2\gamma)\theta^2] h(t)(\rho, \theta) + [1 + (2\beta - 1)\theta^2] \rho \frac{\partial h(t)(\rho, \theta)}{\partial \rho} - [1 - \theta^2] \theta \frac{\partial h(t)(\rho, \theta)}{\partial \theta} = 0. \tag{C8}
\]

Let us now write down \(h(t)(\rho, \theta)\) as a power series in the odd powers of \(\theta\):

\[
h(t)(\rho, \theta) = \theta + \sum_{n=1}^{t-1} h_{2n+1}(\rho)\theta^{2n+1}. \tag{C9}
\]

The series-expanded stationarity condition then takes the form
\[
\sum_{n=1}^{\infty} \left\{ \left[ \frac{\partial}{\partial \rho} - 2n \right] h_{2n+1}(\rho) + \left[ (2\beta - 1)\rho \frac{\partial}{\partial \rho} - 2\gamma + 2n - 2 \right] h_{2n-1}(\rho) \right\} \theta^{2n+1} = 0, \quad (C10)
\]

with the convention \( h_1 = 1 \).

Let us now notice that, in the absence of truncations, the above equation must be identically true, since the original function \( F(z) \) is totally independent of \( \rho \). This fact implies that the coefficients of the above power-series expansion must vanish individually, and this gives us an infinite set of recursive differential equations for the functions \( h_{2n+1}(\rho) \), which must be automatically satisfied when the coefficients \( h_{2n+1}(\rho) \) are properly defined.

A truncation corresponds to arbitrarily suppressing all coefficients starting from \( h_{2t+1}(\rho) \). Hence the global stationarity condition simply amounts to requiring

\[
\left\{ \left[ (2\beta - 1)\rho \frac{\partial}{\partial \rho} - 2\gamma + 2t - 2 \right] h_{2t-1}(\rho) \right\} \theta^{2t+1} = 0, \quad (C11)
\]

because all other terms vanish. The resulting equation can be solved by choosing \( \rho_t \) such that the term in curly brackets vanishes, independent of \( \theta \). This concludes our proof.

The effectiveness of this scheme is beautifully illustrated by its lowest-order implementation, corresponding to the so-called “linear parametric model”, in the context of the three-dimensional Ising model.

Let us truncate the exact scaling function \( h(\rho, \theta) \) to its cubic approximation

\[
h^{(2)}(\rho, \theta) = \theta + h_3(\rho)\theta^3, \quad (C12)
\]

where \( h_3(\rho) \) is taken from Eq. (7.13). Substituting \( h_3 \) into the stationarity condition (C11) for \( t = 2 \) we find

\[
\frac{1}{3} (2\beta - \gamma) \rho^2 - 2\gamma(1 - \gamma) = 0, \quad (C13)
\]

which leads to

\[
\rho_2 = \sqrt{\frac{6\gamma(\gamma - 1)}{\gamma - 2\beta}}. \quad (C14)
\]

The truncated scaling function vanishes at the value \( \theta_0 \):

\[
\theta_0^2 \equiv - \frac{1}{h_3(\rho_2)} = \frac{\gamma - 2\beta}{\gamma(1 - 2\beta)}. \quad (C15)
\]

In this approximation the scaling equation of state turns out to be expressible simply in terms of the critical exponents \( \beta \) and \( \gamma \). As a consequence, all the universal ratios may then be approximated to lowest order by appropriate algebraic combinations of the critical exponents.

The above results reproduce the old formulae by Schofield, Litster, and Ho [25], who obtained expressions for critical amplitudes in terms of critical exponents from a minimum condition imposed on the predictions extracted from a parametric scaling equation of state.
In the case of the linear parametric model, the globality of the stationarity property introduced by the above authors was shown by Wallace and Zia [27], who adopted a slightly different, but essentially equivalent, formulation of the above model.

As we showed above, global stationarity can be imposed on parametric models regardless of the linearity constraint. The next truncation, corresponding to \( t = 3 \), can also be treated analytically. Our starting point will be

\[
\begin{align*}
\rho_3 &= \frac{(\gamma - 2\beta)(1 - \gamma + 2\beta)}{12(4\beta - \gamma)F_5} \left(1 - \frac{72(2 - \gamma)\gamma - 1)(4\beta - \gamma)F_5}{(\gamma - 2\beta)^2(1 - \gamma + 2\beta)^2}\right)^{1/2}.
\end{align*}
\]  

(C17)

The truncated scaling function vanishes when \( \theta \) takes the value \( \theta_0 \), which is now given by the relation

\[
\theta_0^2 = \frac{h_3(\rho_3)}{2h_5(\rho_3)} \left(1 - \frac{4h_5(\rho_3)}{h_3^2(\rho_3)}\right).
\]  

(C18)

A general feature of truncated parametric models is the possibility of making a prediction on higher-order coefficients \( F_{2n+1} \), for \( n \geq t \), in terms of lower-order coefficients. This is a natural consequence of having included by the parametrization some information on the asymptotic behavior of \( F(z) \) for large \( z \). In practice we observe that each \( F_{2n+1} \) appears first in the coefficient \( h_{2n+1} \), in the form of a free constant of integration in the solution of the recursive differential equation relating \( h_{2n+1} \) to \( h_{2n-1} \). Since a truncation corresponds to setting \( h_{2n+1} = 0 \) starting from \( n = t \), this fixes the (truncated) value of all \( F_{2n+1} \) starting from \( F_{2t+1} \).

As an important consequence of this mechanism we observe that truncated models deviate from the exact solution only proportionally to the difference between exact and predicted coefficients, and this difference may be quite small even for very low-order truncations.

In order to turn the above considerations into quantitative estimates, we need to get further insight into the properties of the functions \( h_{2n+1}(\rho) \), especially in the vicinity of the stationary point \( \rho_t \). To this purpose we introduce the expansion

\[
\begin{align*}
\rho_3 &= \frac{(\gamma - 2\beta)(1 - \gamma + 2\beta)}{12(4\beta - \gamma)F_5} \left(1 - \frac{72(2 - \gamma)\gamma - 1)(4\beta - \gamma)F_5}{(\gamma - 2\beta)^2(1 - \gamma + 2\beta)^2}\right)^{1/2}.
\end{align*}
\]  

(C19)

Substituting this expansion as an Ansatz into the recursive differential equations, we check that \( F_{2m+1} \) act as free parameters (integration constants), while the coefficients \( c_{n,m} \) must obey the following algebraic recursive equations:

\[
(n - m)c_{n,m} = [(2\beta - 1)m - \gamma + n - 1]c_{n-1,m},
\]  

(C20)

for all \( n > m \), subject to the initial conditions \( c_{m,m} = 1 \). It is possible to find a closed-form solution to Eq. (C20):
\[ c_{n,m} = \frac{1}{(n-m)!} \prod_{k=1}^{n-m} (2\beta m - \gamma + k - 1), \] \hspace{1cm} (C21)

but for our purposes the recursive equations will sometimes be more useful than their explicit solutions.

Let us define the coefficients of the \( z \)-expansion of the truncated scaling function evaluated at the stationary point:

\[ F^{(t)}(\rho_t, z) = z + \frac{1}{6}z^3 + \sum_{m=2}^{\infty} \frac{m}{m+1} F^{(t)}_{2m+1} z^{2m+1}. \] \hspace{1cm} (C22)

By definition, \( F^{(t)}_{2m+1} \) coincide with their exact value \( F_{2m+1} \) for all \( m < t \), while for \( m \geq t \) they are determined by the condition \( h_{2m+1}(\rho_t) = 0 \), which, according to Eq. (C19), implies

\[ \sum_{m=0}^{n} c_{n,m} \rho_t^{2m} F^{(t)}_{2m+1} = 0 \] \hspace{1cm} (C23)

for all \( n \geq t \).

We can now prove the following lemma:

\[ \sum_{m=1}^{n} m c_{n+1,m} \rho_t^{2m} F^{(t)}_{2m+1} = 0 \] \hspace{1cm} (C24)

holds for all \( n \geq t \).

The proof is by induction. Let us assume the lemma to hold for a given value \( n \); then, as a consequence of Eqs. (C24) and (C23), we obtain

\[ \sum_{m=0}^{n} [(2\beta - 1)m - \gamma + n] c_{n,m} \rho_t^{2m} F^{(t)}_{2m+1} = 0. \] \hspace{1cm} (C25)

Notice that the above equation holds also for the initial value \( n = t - 1 \), since in that case it coincides with the global stationarity condition.

By use of the recursion equations (C20) we now obtain

\[ \sum_{m=0}^{n} (n + 1 - m) c_{n+1,m} \rho_t^{2m} F^{(t)}_{2m+1} = 0. \] \hspace{1cm} (C26)

Because of the factor \((n + 1 - m)\) the sum can trivially be extended up to \( n + 1 \), hence:

\[ \sum_{m=0}^{n+1} c_{n+1,m} \rho_t^{2m} F^{(t)}_{2m+1} = \sum_{m=1}^{n+1} m c_{n+1,m} \rho_t^{2m} F^{(t)}_{2m+1}. \] \hspace{1cm} (C27)

The l.h.s. vanishes by definition (cf. Eq. (C23)), hence the r.h.s. vanishes and the proof is completed.

The above lemma is instrumental in evaluating the difference between the predictions originated by two subsequent truncations. By applying once more the definition of \( F^{(t)}_{2m+1} \) one can easily show that
\[
\sum_{m=0}^{n} c_{n,m} \left[ F_{t+1}^{2m+1} (F_{2m+1}^{(t+1)} - F_{2m+1}^{(t)}) + (\rho_{t+1}^{2m} - \rho_{t}^{2m}) F_{2m+1}^{(t)} \right] = 0
\]  \quad (C28)

for all \( n > t \). Let us now expand the equation to first order in the difference \( \rho_{t+1}^{2} - \rho_{t}^{2} \), and make explicit use of the lemma to obtain

\[
\sum_{m=t}^{n} c_{n,m} \rho_{t}^{2m} \left( F_{t+1}^{(t+1)} - F_{t+1}^{(t)} \right) \approx 0
\]  \quad (C29)

for all \( n > t \). It is crucial that \( F_{2m+1}^{(t+1)} - F_{2m+1}^{(t)} = 0 \) for all \( m < t \).

The equation we obtained allows us to express (within the approximation) all differences \( F_{2m+1}^{(t+1)} - F_{2m+1}^{(t)} \) in terms of the single quantity \( \delta F_{t} \equiv F_{2t+1}^{(t)} - F_{2t+1}^{(t)} \).

Knowledge of the \( c_{n,m} \) and some ingenuity lead to the explicit solution of Eq. (C29):

\[
F_{2m+1}^{(t+1)} - F_{2m+1}^{(t)} \approx \frac{\delta F_{t}}{\rho_{t}^{2(m-t)}}
\]  \quad (C30)

where, for all \( m > t \),

\[
d_{t,m} = \frac{(-1)^{m-t}}{(m-t)!} (2\beta t - \gamma) \prod_{k=1}^{m-t-1} (2\beta m - \gamma - k),
\]  \quad (C32)

and obviously \( d_{t,t} = 1 \).

As a corollary of this result, by comparing Eq. (C29) when \( t = 1 \) to Eq. (C24) when \( t = 2 \), we may write down a closed-form expression for all \( F_{2m+1}^{(2)} \) coefficients \( m \geq 1 \):

\[
F_{2m+1}^{(2)} = \frac{1}{6m} \frac{d_{1,m}}{\rho_{2}^{2(m-1)}},
\]  \quad (C33)

which completes our analysis of the linear parametric model.

One may also show that \( \delta F_{t} \) is related to the variation of \( \rho_{t} \) by the (linearized) relation

\[
\delta F_{t} \approx \frac{1 - 2\beta}{2\beta t - \gamma} \left( \sum_{m=0}^{t} m^{2} c_{t,m} \rho_{t}^{2(m-t-1)} F_{2m+1} \right) (\rho_{t+1}^{2} - \rho_{t}^{2}).
\]  \quad (C34)

Our numerical estimates, presented in Table XII, show that \( \rho_{t+1}^{2} - \rho_{t}^{2} \) is indeed small (\( \lesssim 0.01 \)).

In order to evaluate amplitude ratios, as shown in App. III, we must also reconstruct the functions \( g(\theta) \) and \( g_{2}(\theta) \), by solving Eqs. (B11) and (B12) respectively. These functions may be expanded in even powers of \( \theta \), with coefficients that are functions of \( \rho \) satisfying the same differential equations as \( h_{2n+1}^{(t)} \), Eqs. (C10). One may show that, for any given truncation \( h_{2n+1}^{(t)}(\rho, \theta) \) and arbitrary values of \( \rho \),

\[
g_{2}(\rho, \theta) = \sum_{n=0}^{\infty} \left( \sum_{m=0}^{n} c_{nm} \rho_{m}^{2m} \frac{F_{2m+1}^{(t)}}{2m + 2} \right) g^{2n+2} + A(1 - \theta^{2})^{2\beta + \gamma},
\]  \quad (C35)
where $A$ is an integration constant reflecting the arbitrariness in the zero-field value of the free energy. One may also show that for $n \geq t$,

$$
\sum_{m=0}^{n} c_{nm} \rho^{2m} \frac{F_{2m+1}^{(t)}}{2m+2} \sim \frac{(-1)^n}{(n+1)!} \prod_{k=0}^{n} (2\beta + \gamma - k)
$$

(C36)

where the terms in the r.h.s. are the coefficients of the Taylor expansion for $(1 - \theta^2)^{2\beta+\gamma}$. As a consequence the constant $A$ may always be chosen such that $g^{(t)}(\rho, \theta)$ is truncated to $O(\theta^t)$ for any arbitrary choice of $\rho$.

In turn one may also prove that, for any $h^{(t)}(\rho, \theta)$,

$$
g^{(t)}(\rho, \theta) = \frac{\left[ \sum_{n=0}^{\infty} \sum_{m=0}^{n} c_{nm} \rho^{2m}(2m+1)F_{2m+1}^{(t)} \right] \theta^{2n}}{2m+2}
$$

(C37)

Now, according to Eqs. (C23) and (C24), when we choose for $\rho$ the globally stationary value $\rho_t$, the coefficients in square brackets vanish for all $n \geq t$. As a consequence for any $t$ the value $\rho_t$ insures the truncation of $g_2^{(t)}(\rho_t, \theta)$ to $O(\theta^{t-2})$. Thus a unique feature of $\rho_t$ is the simultaneous and consistent truncation of $h(\theta)$ and $g_2(\theta)$.

Notice that we might start by imposing a global stationarity condition directly on a truncated $g_2(\rho, \theta)$, obtaining a different stationary value for $\rho$, and make use of Eq. (B12) in order to reconstruct the corresponding $h(\theta)$. However in this case, since $h(\theta)$ must be an odd function of $\theta$, there is no arbitrary integration constant (which is physically a trivial consequence of the definition of a reduced temperature) and therefore $h(\theta)$ cannot be truncated. The resulting parametric model is mathematically consistent, but in practice unappealing, because the calculation of $\theta_0$ from the equation $h(\theta_0) = 0$ and the evaluation of universal amplitude ratios becomes quite cumbersome.

The above described formalism can be usefully employed in the context of the $\epsilon$-expansion of the critical equation of state. Comparison with $\epsilon$-expansion results will also shed further light on the meaning and relevance of the results derived by the prescription of global stationarity.

Our starting point will be the result of Wallace and Zia [27], who showed that, when appropriate conditions are imposed on the zeroth order approximation, the parametric form of the critical equation of state is automatically truncated in the powers of $\theta^2$ when expanded in the parameter $\epsilon = 4 - d$. For easier comparison, notice that the parameter $b$ introduced by Schofield [24] and used by Wallace and Zia is the same as our $\theta_0$, and the variable change from $\theta_0$ to $\rho$ poses no conceptual problem.

In our reformulation one may state that, within the $\epsilon$-expansion, it is possible to choose to lowest order a value $\rho_0$ in such a way that, expanding the parametric equation of state in $\theta$ and $\epsilon$, one finds, for all $n \geq 2$,

$$
h_{2n+1}(\rho_0) = O(\epsilon^{n+1}),
$$

(C38)

and this property should survive the replacement $\rho_0 \rightarrow \rho_0 + O(\epsilon)$.

As a first application of our formalism, we can verify the consistency of the above statements, by checking that, for all $n \geq 2$, the condition
\[ \sum_{m=0}^{n} c_{n,m} \rho_0^{2m} F_{2m+1} = O(\epsilon^{n+1}) \quad (C39) \]

implies
\[ \sum_{m=1}^{n} m c_{n,m} \rho_0^{2m} F_{2m+1} = O(\epsilon^n). \quad (C40) \]

The proof is by induction. Assuming the property to hold for a given \( n \), and exploiting the fact that \( 2\beta - 1 = O(\epsilon) \), we obtain
\[ \sum_{m=0}^{n} [(2\beta - 1)m - \gamma + n] c_{n,m} \rho_0^{2m} F_{2m+1} = O(\epsilon^{n+1}). \quad (C41) \]

The initial condition, corresponding to the case \( n = 1 \), has the explicit form
\[ \gamma(\gamma - 1) + \frac{1}{6}(2\beta - \gamma) \rho_0^2 = O(\epsilon^2), \quad (C42) \]

and is a definition of \( \rho_0 \). Notice that \( \rho_0 = \lim_{\epsilon \to 0} \rho_0 \), and in the Ising model \( \rho_0^2 = 2 \).

By applying the recursion equations we then obtain
\[ \sum_{m=0}^{n} (n + 1 - m) c_{n+1,m} \rho_0^{2m} F_{2m+1} = O(\epsilon^{n+1}). \quad (C43) \]

The sum can trivially be extended to \( n + 1 \) and, recalling the hypothesis, we obtain
\[ \sum_{m=1}^{n+1} m c_{n+1,m} \rho_0^{2m} F_{2m+1} = O(\epsilon^{n+1}), \quad (C44) \]

thus completing the proof. Along the same lines it is straightforward to prove that, for all \( n \geq 2 \),
\[ \sum_{m=1}^{n} m^k c_{n,m} \rho_0^{2m} F_{2m+1} = O(\epsilon^{n-k+1}) \quad (C45) \]

for all integers \( k \leq n \). The initial condition \( (n = k) \) is trivially satisfied for all \( n \geq 2 \):
\[ \sum_{m=1}^{n} m^n c_{n,m} \rho_0^{2m} F_{2m+1} = O(\epsilon). \quad (C46) \]

As a consequence the more general statement
\[ h_{2n+1}(\rho) = O(\epsilon^{n+1}) \quad (C47) \]

holds for all \( \rho \) admitting an \( \epsilon \)-expansion and possessing the limit \( \lim_{\epsilon \to 0} \rho = \rho_0 \).

This relation implies in turn that, expanding in \( \epsilon \) the coefficients \( F_{2m+1} \) for \( m \geq 2 \) according to
\[ F_{2m+1} = \sum_{k=1}^{\infty} f_{mk} \epsilon^k, \]  

when the \( f_{mk} \) for \( m < k \) are known then all \( f_{mk} \) for \( m \geq k \) are fully determined.

As a simple application of the above we obtained the following closed form result:

\[ f_{m1} = \frac{(-1)^m}{m(m-1)} \rho_0^{-2m} \lim_{\epsilon \to 0} \frac{\gamma - 1}{\epsilon}, \]  

where \( \gamma \approx 1 + \frac{1}{6} \epsilon \) and \( \rho_0 = \sqrt{2} \).

Let us now consider the linear parametric model with global stationarity in the context of the \( \epsilon \)-expansion: \( \rho_2 \) satisfies the condition \( \rho_2 = \rho_0 + O(\epsilon) \), though it does not coincide (and is not expected to) with the \( \epsilon \)-expanded \( \rho \) value adopted by Guida and Zinn-Justin [23].

Now notice that for any higher-order truncation the stationarity condition is still solved by \( \rho_t = \rho_0 + O(\epsilon) \), as shown explicitly by the above derived Eq. (C41). As a consequence, any stationary truncation is an accurate description of the \( \epsilon \)-expanded parametric equation of state up to \( O(\epsilon^t) \) included. Actually the freedom of choosing \( \rho \) leaves such an expansion highly under-determined, and many other prescriptions might work, including that of fixing \( \rho \) (or alternatively \( \theta_0 \)) to its zeroth order value. It is however certainly pleasant to recognize that our approach based on stationarity falls naturally into the set of consistent truncations. As a side remark, notice that all the coefficients of the \( \epsilon \)-expansion of \( \rho_t \) will in general be changed order by order in \( t \), and will also in general be complex numbers. This fact will by no means affect the real character of the expanded physical amplitudes, and will not even prevent \( \rho \) and \( \theta_0 \) from taking real values in the actual three-dimensional calculations.
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