Title:
Hyperplane Arrangements with Large Average Diameter

Authors:
Antoine Deza and Feng Xie

AdvOl-Report No. 2007/05

April 2007, Hamilton, Ontario, Canada
Hyperplane Arrangements with Large Average Diameter

Antoine Deza and Feng Xie

April 2, 2007
McMaster University
Hamilton, Ontario, Canada
dez, xief@mcmaster.ca

Abstract: Let \( \Delta_A(n, d) \) denote the largest possible average diameter of a bounded cell of a simple arrangement defined by \( n \) hyperplanes in dimension \( d \). We have \( \Delta_A(n, 2) \leq 2 + \frac{2}{n-1} \) in the plane, and \( \Delta_A(n, 3) \leq 3 + \frac{4}{n-1} \) in dimension 3. In general, the average diameter of a bounded cell of a simple arrangement is conjectured to be less than the dimension; that is, \( \Delta_A(n, d) \leq d \). We propose an hyperplane arrangement with \( \binom{n-d}{d} \) cubical cells for \( n \geq 2d \). It implies that the dimension \( d \) is an asymptotic lower bound for \( \Delta_A(n, d) \) for fixed \( d \). In particular, we propose line and plane arrangements with large average diameter yielding \( \Delta_A(n, 2) \geq 2 - \frac{2\lfloor \frac{n}{2} \rfloor}{(n-1)(n-2)}(n-1)(n-2) \) and \( \Delta_A(n, 3) \geq 3 - \frac{6}{n-1} + \frac{6\lfloor \frac{n}{2} \rfloor - 2}{(n-1)(n-2)(n-3)}(n-1)(n-2)(n-3) \).

Keywords: hyperplane arrangements, bounded cell, average diameter, lower bounds

1 Introduction

Let \( A \) be a simple arrangement formed by \( n \) hyperplanes in dimension \( d \). We recall that an arrangement is called simple if \( n \geq d + 1 \) and any \( d \) hyperplanes intersect at a unique distinct point. The number of bounded cells (bounded connected component of the complement of the hyperplanes) of \( A \) is \( I = \binom{n-d}{d} \). Let \( \delta(A) \) denote the average diameter of a bounded cell \( P_i \) of \( A \); that is,

\[
\delta(A) = \frac{\sum_{i=1}^{I} \delta(P_i)}{I}
\]

where \( \delta(P_i) \) denotes the diameter of \( P_i \), i.e., the smallest number such that any two vertices of \( P_i \) can be connected by a path with at most \( \delta(P_i) \) edges. Let \( \Delta_A(n, d) \) denote the largest possible average diameter of a bounded cell of a simple arrangement defined by \( n \) inequalities in dimension \( d \). Deza, Terlaky and Zinchenko [2] conjectured that \( \Delta_A(n, d) \leq d \), and showed that if the conjecture of Hirsch holds for polytopes in dimension \( d \), then \( \Delta_A(n, d) \) would satisfy \( \Delta_A(n, d) \leq d + \frac{2d}{n-1} \). In dimension 2 and 3, they showed that \( \Delta_A(n, 2) \leq 2 + \frac{2}{n-1} \) and \( \Delta_A(n, 3) \leq 3 + \frac{4}{n-1} \). We recall that a polytope is a bounded polyhedron and that the conjecture of Hirsch, formulated in 1957 and reported in [1], states that the diameter of a polyhedron defined by
n inequalities in dimension d is not greater than n − d. The conjecture does not hold
for unbounded polyhedra. A simple line arrangement with average diameter equal to 2 − \( \frac{2}{n-1} \) was
given in [2]. We propose, in Section 2, a line arrangement with average diameter 2 − \( \frac{2}{(n-1)(n-2)} \)
and, in Section 3, a plane arrangement with average diameter 3 − \( \frac{6}{n-1} + \frac{6(\lceil \frac{d}{2} \rceil - 2)}{(n-1)(n-2)(n-3)} \), yielding
\( 2 − \frac{2}{(n-1)(n-2)} \leq \Delta_A(n, 2) \leq 2 + \frac{2}{n-1} \) and
\( 3 − \frac{6}{n-1} + \frac{6(\lceil \frac{d}{2} \rceil - 2)}{(n-1)(n-2)(n-3)} \leq \Delta_A(n, 3) \leq 3 + \frac{4}{n-1} \). In
Section 4, we propose an hyperplane arrangement with \( \binom{n-d}{d} \) cubical cells for \( n \geq 2d \). It implies
that the dimension d is an asymptotic lower bound for \( \Delta_A(n, d) \) for fixed d. For polytopes and
arrangements, we refer to the books of Grünbaum [4] and Ziegler [6] and the references therein.

2 Line Arrangements with Large Average Diameter

For \( n \geq 4 \), we consider the simple line arrangement \( A^o_{n, 2} \) made of the 2 lines \( h_1 \) and \( h_2 \) forming,
respectively, the \( x_1 \) and \( x_2 \) axis, and \( (n-2) \) lines defined by their intersections with \( h_1 \) and \( h_2 \).
We have \( h_k \cap h_1 = \{1 + (k-3)\varepsilon, 0\} \) and \( h_k \cap h_2 = \{0, 1 - (k-3)\varepsilon\} \) for \( k = 3, 4, \ldots, n-1 \), and
\( h_n \cap h_1 = \{2, 0\} \) and \( h_n \cap h_1 = \{0, 2 + \varepsilon\} \) where \( \varepsilon \) is a constant satisfying \( 0 < \varepsilon < \frac{1}{n-3} \). See
Figure 1 for an arrangement combinatorially equivalent to \( A^o_{7, 2} \).

**Proposition 1** For \( n \geq 4 \), the bounded cells of the arrangement \( A^o_{n, 2} \) consist of \( (n-2) \) triangles,
\( \frac{(n-2)(n-4)}{2} \) 4-gons, and one \( n \)-gon.

**Proof:** The first \( (n-1) \) lines of \( A^o_{n, 2} \) clearly form a simple line arrangement which bounded
cells are \( (n-3) \) triangles and \( \binom{n-3}{2} \) 4-gons. The last line \( h_n \) adds one \( n \)-gons, one triangle and
\( (n-4) \) 4-gons. \( \Box \)

**Corollary 2** We have \( \delta(A^o_{n, 2}) = 2 − \frac{2(\lceil \frac{n}{2} \rceil)}{(n-1)(n-2)} \) for \( n \geq 4 \).

**Proof:** Since the diameter of a \( k \)-gons is \( \lceil \frac{k}{2} \rceil \), we have \( \delta(A^o_{n, 2}) = 2 − 2(\frac{(n-2)}{(n-1)(n-2)}) \cdot \frac{\lceil \frac{n}{2} \rceil - 2}{(n-1)(n-2)} = 2 − \frac{2(\lceil \frac{n}{2} \rceil)}{(n-1)(n-2)} \). \( \Box \)

**Remark 3** As there is only one combinatorial type of simple line arrangement for \( n = 4 \),
we have \( \Delta_A(4, 2) = \delta(A^o_{4, 2}) = \frac{4}{3} \). For \( n = 5 \), there are 6 combinatorial types of simple line
arrangement and \( \delta(A^o_{5, 2}) \) is among the ones with maximal average diameter, i.e., \( \Delta_A(5, 2) = \delta(A^o_{5, 2}) = \frac{3}{2} \). We believe that \( \Delta_A(n, 2) = \delta(A^o_{n, 2}) = 2 − \frac{2(\lceil \frac{n}{2} \rceil)}{(n-1)(n-2)} \) for \( n \geq 4 \).

A facet of an hyperplane arrangement belongs to either zero, one or two bounded cells. We call
a facet external if it belongs to exactly one bounded cell and believe that arrangements with
large average diameter have few external facets. The first \( (n-1) \) lines of \( A^o_{n, 2} \) form the line
arrangement \( A^o_{n-1, 2} \) proposed in [2]. The arrangement \( A^o_{n, 2} \) has \( 3(n-2) \) external facets and
average diameter \( \delta(A^o_{n, 2}) = 2 − \frac{2}{n-1} \). The arrangement \( A^o_{n, 2} \) has \( 3(n-1) \) external facets. It was
hypothesized in [2] that any simple arrangement has at least \( d(n-2) \) external facets. We believe
that, in addition of maximizing the average diameter, \( A^o_{n, 2} \) minimizes the number of external
facets. Note that the envelope of the bounded cells of \( A^o_{n, 2} \) has one reflex vertex. In Section 3,
following the same approach, we generalize \( A^o_{n-1, 2} \) to dimension 3 and add one plane to reduce
the number of external facets.
3 Plane Arrangements with Large Average Diameter

For \( n \geq 5 \), we consider the simple plane arrangement \( A_{\text{o},n,3} \) made of the 3 plane \( h_1 \), \( h_2 \) and \( h_3 \) corresponding, respectively, to \( x_3 = 0 \), \( x_2 = 0 \) and \( x_1 = 0 \), and \((n - 3)\) planes defined by their intersections with the \( x_1 \), \( x_2 \) and \( x_3 \) axis. We have \( h_k \cap h_1 \cap h_2 = \{1 + 2(k - 4)\varepsilon, 0, 0\} \), \( h_k \cap h_1 \cap h_3 = \{0, 1 + (k - 4)\varepsilon, 0\} \) and \( h_k \cap h_2 \cap h_3 = \{0, 0, 1 - (k - 4)\varepsilon\} \) for \( k = 4, 5, \ldots, n - 1 \), and \( h_n \cap h_1 \cap h_2 = \{3, 0, 0\} \), \( h_n \cap h_1 \cap h_3 = \{0, 2, 0\} \) and \( h_n \cap h_2 \cap h_3 = \{0, 0, 3 + \varepsilon\} \) where \( \varepsilon \) is a constant satisfying \( 0 < \varepsilon < \frac{1}{n-3} \). See Figure 2 for an illustration of an arrangement combinatorially equivalent to \( A_{\text{o},7,3} \) where, for clarity, only the bounded cells belonging to the positive orthant are drawn.

**Proposition 4** For \( n \geq 5 \), the bounded cells of the arrangement \( A_{\text{o},n,3} \) consist of \((n - 3)\) tetrahedra, \((n - 3)(n - 4) - 1\) cells combinatorially equivalent to a prism with a triangular base, \( \binom{n-3}{3} \) cells combinatorially equivalent to a cube, and one cell combinatorially equivalent to a shell \( S_n \) with \( n \) facets and \( 2(n - 2) \) vertices. See Figure 3 for an illustration of \( S_7 \).
PROOF: For $4 \leq k \leq n - 1$, let $A^*_k, 3$ denote the arrangement formed by the first $k$ planes of $A^0_n, 3$. See Figure 4 for an arrangement combinatorially equivalent to $A^*_6, 3$. We first show by induction that the bounded cells of the arrangement $A^*_n, 3$ consist of $(n - 4)$ tetrahedra, $(n - 4)(n - 5)$ combinatorial triangular prisms and $\binom{n - 4}{3}$ combinatorial cubes. We use the following notation to describe the bounded cells of $A^*_k, 3$: $T_+$ for a tetrahedron with a facet on $h_1$ and a vertex above $h_1$; $P_\triangle$, respectively $P_\square$, for a combinatorial triangular prism with a triangular, respectively square, facet on $h_1$; and $C$, respectively $T$ and $P$, for a combinatorial cube, respectively tetrahedron and triangular prism, not touching $h_1$. The bounded cells of $A^*_{k-1, 3}$ which are to be cut by the addition of $h_k$ are marked with a bar superscript. When the plane $h_k$ is added, the cells $T_+, P_\triangle, P_\square$, and $C$ are sliced, respectively, into $T$ and $P_\triangle$, $P$ and $P_\triangle$, $P$ and $C$, and $C$ and $\bar{C}$. In addition, one $T_+$ cell and $(k - 4)$ $P_\square$ cells are created by bounding $(k - 3)$ unbounded cells of $A^*_{k-1, 3}$. Let $c(k)$ denotes the number of $C$ cells of $A^*_{k, 3}$, similarly for $C$, $T$, $T_+$, $P$, $P_\triangle$ and $P_\square$. For $A^*_4, 3$ we have $\bar{t}_+(4) = 1$ and $t(4) = p(4) = \bar{p}_\triangle(4) = \bar{p}_\square(4) = c(4) = \bar{c}(4) = 0$. The addition of $h_k$ removes and adds one $\bar{T}_+$, thus, $\bar{t}_+(k) = 1$. Similarly, all $\bar{P}_\square$ are removed.
and \((k - 4)\) are added, thus, \(\bar{p}_o(k) = (k - 4)\). Since \(t(k) = t(k - 1) + \bar{t}_+(k - 1)\) and \(\bar{p}_\Delta(k) = \bar{p}_\Delta(k - 1) + \bar{t}_+(k - 1)\), we have \(t(k) = \bar{p}_\Delta(k) = (k - 4)\). Since \(p(k) = p(k - 1) + \bar{p}_\Delta(k - 1) + \bar{p}_o(k - 1)\), we have \(p(k) = (k - 4)(k - 5)\). Since \(c(k) = \bar{c}(k - 1) + \bar{p}_o(k - 1)\), we have \(c(k) = (k - 4)\). Since \(c(k) = c(k - 1) + \bar{c}(k - 1)\), we have \(c(k) = (k - 4)\).

Therefore the bounded cells of \(A_{n-1,3}\) consist of \(t(n - 1) + \bar{t}_+(n - 1) = (n - 4)\) tetrahedra, \(p(n - 1) + \bar{p}_\Delta(n - 1) + \bar{p}_o(n - 1) = (n - 4)(n - 5)\) combinatorial triangular prisms, and \(c(n - 1) + \bar{c}(n - 1) = \binom{n - 4}{2}\) combinatorial cubes. The addition of \(h_n\) to \(A_{n-1,3}\) creates one shell \(S_n\) with 2 triangular facets belonging to \(h_2\) and \(h_3\) and one square facet belonging to \(h_1\). Besides \(S_n\), the bounded cells created by the addition of \(h_n\) are below \(h_1\) and consist of one tetrahedron, \(\binom{n - 4}{2}\) combinatorial cubes and \((2n - 9)\) combinatorial triangular prisms. □

**Corollary 5** We have \(\delta(A_{n,3}^o) = 3 - \frac{6}{n - 1} + \frac{6\lceil \frac{n}{2} \rceil - 6}{(n - 1)(n - 2)(n - 3)}\) for \(n \geq 5\).

**Proof:** Since the diameter of a tetrahedron, triangular prism, cube and n-shell is, respectively, 1, 2, 3 and \(\lceil \frac{n}{2} \rceil\), we have \(\delta(A_{n,3}^o) = 3 - \frac{6(1 + 2(n - 3) + (n - 3)(n - 4) - 1 - \frac{1}{2} - \frac{3}{2})}{(n - 1)(n - 2)(n - 3)} = 3 - \frac{6}{n - 1} + \frac{6\lceil \frac{n}{2} \rceil - 6}{(n - 1)(n - 2)(n - 3)}\). □

**Remark 6** As there is only one combinatorial type of simple plane arrangement for \(n = 5\), we have \(\Delta_{A}(5, 3) = \delta(A_{5,3}^o) = \frac{3}{2}\). For larger \(n\), the average diameter of \(A_{n,3}^o\) is not maximal as a similar but slightly more complicated arrangement gives a bit larger value.

## 4 Hyperplane Arrangements with Large Average Diameter

In Section 4.2, the arrangements \(A_{n,2}^*\) and \(A_{n,3}^*\) are generalized to an hyperplane arrangement \(A_{n,d}^*\) which contains \(\binom{d - 1}{2d}\) cubical cells for \(n \geq 2d\). It implies that the average diameter \(\delta(A_{n,d}^*)\) is arbitrarily close to \(d\) for \(n\) large enough. Thus, the dimension \(d\) is an asymptotic lower bound for \(\Delta_{A}(n, d)\) for fixed \(d\). Before presenting in Section 4.2 the arrangement \(A_{n,d}^*\), we recall in Section 4.1 the unique combinatorial structure of a simple arrangement formed by \(d + 2\) hyperplanes in dimension \(d\).

### 4.1 The average diameter of a simple arrangement with \(d + 2\) hyperplanes

Let \(A_{d+2,d}\) be a simple arrangement formed by \(d + 2\) hyperplanes in dimension \(d\). Besides simplices, the bounded cells of \(A_{d+2,d}\) are simple polytopes with \(d + 2\) facets. The \(\lceil \frac{d}{2} \rceil\) combinatorial types of simple polytopes with \(d + 2\) facets are well-known, see for example [4], but we briefly recall the combinatorial structure of \(A_{d+2,d}\) as some of the notions presented are
used in Section 4.2. As there is only one combinatorial type of simple arrangement with $d + 2$ hyperplanes, the arrangement $A_{d+2,d}$ can be obtained from the simplex $A_{d+1,d}$ by cutting off one its vertices $v$ with the hyperplane $h_{d+2}$. A prism $P$ with a simplex base is created. Let us call top base the base of $P$ which belongs to $h_{d+2}$ and assume, without loss of generality, that the hyperplane containing the bottom base of $P$ is $h_{d+1}$. Besides the simplex defined by $v$ and the vertices of the top base of $P$, the remaining $d$ bounded cells of $A_{d+2,d}$ are between $h_{d+2}$ and $h_{d+1}$. See Figure 5 illustrating the arrangement $A_{5,3}$. As the projection of $A_{d+2,d}$ on $h_{d+1}$ is combinatorially equivalent to $A_{d+1,d-1}$, the $d$ bounded cells between $h_{d+2}$ and $h_{d+1}$ can be obtained from the $d$ bounded cells of $A_{d+1,d-1}$ by the shell-lifting of $A_{d+1,d-1}$ over the ridge $h_{d+1} \cap h_{d+2}$; that is, besides the vertices belonging to $h_{d+1} \cap h_{d+2}$, all the vertices in $h_{d+1}$ (forming $A_{d+1,d-1}$) are lifted. See Figure 6 where the skeletons of the $d + 1$ bounded cells of $A_{d+2,d}$ are given for $d = 2, 3, \ldots, 6$. The shell-lifting of the bounded cells is indicated by an arrow, the vertices not belonging to $h_{d+1}$ are represented in black and the simplex cell containing $v$ is the one made of black vertices. The bounded cells of $A_{d+2,d}$ are $2$ simplices and a pair of each of the $\lfloor \frac{d}{2} \rfloor$ combinatorial types of simple polytopes with $d + 2$ facets for odd $d$. For even $d$ one of the combinatorial type is present only once. Since all the simple polytopes with $d + 2$ facets have diameter 2, we have $\delta(A_{d+2,d}) = \frac{2+2(d-1)}{d+1}$. 

Figure 4: An arrangement combinatorially equivalent to $A_{5,3}^*$.
Proposition 7 As there is only one combinatorial type of simple arrangement with \( d + 2 \) hyperplanes, we have \( \Delta_{\mathcal{A}}(d + 2, 2) = \delta(\mathcal{A}_{d+2,d}) = \frac{2d}{d+1} \).

4.2 Hyperplane Arrangements with Large Average Diameter

The arrangements \( \mathcal{A}^{*}_{n,d} \) and \( \mathcal{A}^{*}_{n,3} \) presented in Sections 2 and 3 can be generalized to the arrangement \( \mathcal{A}^{*}_{n,d} \) formed by the following \( n \) hyperplanes \( h^d_k \) for \( k = 1, 2, \ldots, n \). The hyperplanes \( h^d_k = \{ x : x_{d+1-k} = 0 \} \) for \( k = 1, 2, \ldots, d \) form the positive orthant, and the hyperplanes \( h^d_k \) for \( k = d+1, \ldots, n \) are defined by their intersections with the axes \( \bar{x}_i \) of the positive orthant. We have \( h^d_k \cap \bar{x}_i = \{ 0, \ldots, 0, 1 + (d-i)(k-d-1)\varepsilon, 0, \ldots, 0 \} \) for \( i = 1, 2, \ldots, d-1 \) and \( h^d_k \cap \bar{x}_d = \{ 0, \ldots, 0, 1 - (k-d-1)\varepsilon \} \) where \( \varepsilon \) is a constant satisfying \( 0 < \varepsilon < \frac{1}{n-d-1} \). The combinatorial structure of \( \mathcal{A}^{*}_{n,d} \) can be derived inductively. All the bounded cells of \( \mathcal{A}^{*}_{n,d} \) are on the positive side of \( h^d_k \) and \( h^d_2 \) with the bounded cells between \( h^d_3 \) and \( h^d_2 \) being obtained by the shell-lifting of a combinatorial equivalent of \( \mathcal{A}^{*}_{n-1,d-1} \) over the ridge \( h^d_2 \cap h^d_3 \), and the bounded cells on the other side of \( h^d_2 \) forming a combinatorial equivalent of \( \mathcal{A}^{*}_{n-1,1} \). The intersection \( \mathcal{A}^{*}_{n,d} \cap h^d_2 \) is combinatorially equivalent to \( \mathcal{A}^{*}_{n-1,d-1} \) for \( k = 2, 3, \ldots, d \) and removing \( h^d_2 \) from \( \mathcal{A}^{*}_{n,d} \) yields an arrangement combinatorially equivalent to \( \mathcal{A}^{*}_{n-1,d} \). See Figure 4 for an arrangement combinatorially equivalent to \( \mathcal{A}^{*}_{6,3} \).

Proposition 8 The arrangement \( \mathcal{A}^{*}_{n,d} \) contains \( \binom{n-d}{d} \) cubical cells for \( n \geq 2d \).

PROOF: The arrangements \( \mathcal{A}^{*}_{n,2} \) and \( \mathcal{A}^{*}_{n,3} \) contain, respectively, \( \binom{n-2}{d} \) and \( \binom{n-3}{d} \) cubical cells. The arrangement \( \mathcal{A}^{*}_{2d,d} \) has one cubical cell. As \( \mathcal{A}^{*}_{n,d} \) is obtained inductively from \( \mathcal{A}^{*}_{n-1,d} \) by raising \( \mathcal{A}^{*}_{n-1,d-1} \) over the ridge \( h^d_2 \cap h^d_3 \), we count separately the cubical cells between \( h^d_2 \) and \( h^d_3 \) and the ones on the other side of \( h^d_2 \). The ridge \( h^d_2 \cap h^d_3 \) is an hyperplane of the arrangements \( \mathcal{A}^{*}_{n,d} \cap h^d_2 \) and \( \mathcal{A}^{*}_{n,d} \cap h^d_3 \) which are both combinatorially equivalent to \( \mathcal{A}^{*}_{n-1,d-1} \). Removing \( h^d_2^{-1} \) from \( \mathcal{A}^{*}_{n,d} \cap h^d_2 \) yields an arrangement combinatorially equivalent to \( \mathcal{A}^{*}_{n-2,d-1} \). It implies that
Figure 6: The skeletons of the $d + 1$ bounded cells of $\mathcal{A}_{d+2,d}$ for $d = 2, 3, \ldots, 6$. 

The $(n-2)-(d-1)$ cubical cells of $\mathcal{A}_{n,d}^* \cap h_2^d$ are not incident to the ridge $h_2^d \cap h_3^d$. The shell-lifting of these $(n-d-1)$ cubical cells (of dimension $d-1$) creates $(n-d-1)$ cubical cells between $h_2^d$ and $h_3^d$. As removing $h_2^d$ from $\mathcal{A}_{n,d}^*$ yields an arrangement combinatorial equivalent to $\mathcal{A}_{n-1,d}^*$, there are $(n-1-d)$ cubical cells on the other side of $h_3^d$. Thus, $\mathcal{A}_{n,d}^*$ contains $(n-d-1) + (n-d-1) = (n-d)$ cubical cells. □

**Corollary 9** We have $\delta(\mathcal{A}_{n,d}^*) \geq \frac{d(n-d)}{n} \left( \frac{d}{n} \right)$ for $n \geq 2d$. It implies that for $d$ fixed, $\Delta_{\mathcal{A}}(n, d)$ is arbitrarily close to $d$ for $n$ large enough.

Similarly, we can inductively count $(n-d)$ simplices and $(n-d)(n-d-1)$ bounded cells of $\mathcal{A}_{n,d}^*$ combinatorially equivalent to a prism with a simplex base. We have $(n-1) - (d-1)$ simplices in $\mathcal{A}_{n,d}^* \cap h_2^d$ and, since removing $h_2^d$ from $\mathcal{A}_{n,d}^* \cap h_2^d$ yields an arrangement combinatorially
equivalent to $\mathcal{A}_{n-2,d-1}$, only one of these $(n-d)$ simplices of $\mathcal{A}_{n,d}$ is incident to the ridge $h_2^d \cap h_3^d$. Thus, between $h_2^d$ and $h_3^d$, we have one simplex incident to the ridge $h_2^d \cap h_3^d$ and $(n-d-1)$ cells combinatorially equivalent to a prism with a simplex base not incident to the ridge $h_2^d \cap h_3^d$. In addition, $(n-d-1)$ cells combinatorially equivalent to a prism with a simplex base are incident to the ridge $h_2^d \cap h_3^d$ and between $h_2^d$ and $h_3^d$. These $(n-d-1)$ cells correspond to the truncations of the simplex $\mathcal{A}_{d+1,d}$ by $h_k^d$ for $k = d + 2, d + 3, \ldots, n$. Thus, we have $2(n-d-1)$ cells combinatorially equivalent to a prism with a simplex base between $h_2^d$ and $h_3^d$. As the other side of $h_3^d$ is combinatorially equivalent to $\mathcal{A}_{n-1,d}$, it contains $(n-1-d)$ simplices and $(n-d-1)(n-d-2)$ bounded cells combinatorially equivalent to a prism with a simplex base. Thus, $\mathcal{A}_{n,d}$ has $(n-d-1)(n-d-2) + 2(n-d-1) = (n-d)(n-d-1)$ cells combinatorially equivalent to a prism with a simplex base and $(n-d)$ simplices. As a prism with a simplex base has diameter 2 and the diameter of a bounded cell is at least 1, Corollary 9 can be slightly strengthened to the following corollary.

**Corollary 10** We have $\Delta_A(n, d) \geq 1 + \frac{(d-1)^{(n-d)}+(n-d)(n-d-1)}{\binom{n-1}{d}}$ for $n \geq 2d$.

**Acknowledgments** The authors would like to thank Komei Fukuda and Christophe Weibel for their cdd [3] and minksum [5] codes which helped to investigated small simple arrangements.

**References**

[1] G. Dantzig, Linear Programming and Extensions, Princeton University Press (1963)

[2] A. Deza, T. Terlaky and Y. Zinchenko, Polytopes and arrangements : diameter and curvature, AdvOL-Report 2006/09, McMaster University (2006)

[3] K. Fukuda, cdd, http://www.ifor.math.ethz.ch/~fukuda/cdd_home/cdd.html

[4] B. Grünbaum, Convex Polytopes, Graduate Texts in Mathematics 221, Springer-Verlag (2003)

[5] C. Weibel, minksum, http://roso.epfl.ch/cw/poly/public.php

[6] G. Ziegler, Lectures on Polytopes, Graduate Texts in Mathematics 152, Springer-Verlag (1995)