Finite-dimensional approximation properties for uniform Roe algebras

Hiroki Sako

ABSTRACT

We study property A for metric spaces $X$ with bounded geometry introduced by Guoliang Yu. Property A is an amenability-type condition, which is less restrictive than amenability for groups. The property has a connection with finite-dimensional approximation properties in the theory of operator algebras. It has been already known that property A of a metric space $X$ with bounded geometry is equivalent to nuclearity of the uniform Roe algebra $C^*_u(X)$. We prove that exactness and local reflexivity of $C^*_u(X)$ also characterize property A of $X$.

1. Introduction

The notion of amenability connects functional analytic aspects of groups to geometrical aspects of them. It is characterized by the existence of Følner sets, which are finite subsets with relatively small boundaries [8]. In the theory of operator algebras, there are a number of characterizations of amenability. A typical example is Lance’s theorem. He proved that a discrete group $G$ is amenable if and only if the reduced group $C^*$-algebra $C^*_{\text{red}}(G)$ is nuclear [15, Theorem 4.2].

Property A introduced by Guoliang Yu is an amenability-type condition for discrete metric spaces. Yu proved that property A is a sufficient condition for the coarse Baum–Connes conjecture [24, Theorems 1.1 and 2.7]. An operator algebraic characterization was given by Skandalis, Tu, and Yu. Property A of a metric space $X$ with bounded geometry is equivalent to nuclearity of the uniform Roe algebra $C^*_u(X)$ [21, Theorem 5.3].

The uniform Roe algebra $C^*_u(X)$ is a natural linear representation of $X$. It is also called the translation $C^*$-algebra based on the Hilbert space $\ell^2(X)$ (Roe’s lecture note [18, Section 4.4]). In the theory of $C^*$-algebras, nuclearity, exactness, and local reflexivity are important. The purpose of this paper is to show these properties are equivalent for the algebra $C^*_u(X)$.

Theorem 1.1. For a metric space $X$ with bounded geometry, the following conditions are equivalent.

1. The space $X$ has property A.
2. The uniform Roe algebra $C^*_u(X)$ is nuclear.
3. The algebra $C^*_u(X)$ is exact.
4. The algebra $C^*_u(X)$ is locally reflexive.
5. The space $X$ has the operator norm localization property in [4].

The equivalence between property A and nuclearity of $C^*_u(X)$ was proved by Skandalis, Tu, and Yu. By the theory of $C^*$-algebra, nuclearity implies exactness, and exactness implies local...
reflexivity. It is proved in [20] that the operator norm localization property is equivalent to property A. We will deduce the operator norm localization property of \( X \) from local reflexivity of \( C^*_u(X) \) in Theorem 4.1.

The case of discrete groups has been studied by many authors. For a countable discrete group \( G \), we can define a metric function which makes \( G \) a metric space with bounded geometry. The uniform Roe algebra of \( G \) is isomorphic to \( \ell_\infty(G) \rtimes_{\text{red}} G \). The notion of an exact group was introduced by Kirchberg and Wassermann. It is characterized by exactness of the reduced group \( C^* \)-algebra \( C^*_{\text{red}}(G) \) [14, Theorem 5.2]. The class of exact groups is much larger than that of amenable groups. Ozawa proved that exactness of \( C^*_{\text{red}}(G) \) implies nuclearity of the uniform Roe algebra \( C^*_u(G) \) [16, Theorem 3]. As a corollary, exactness and nuclearity are equivalent for the uniform Roe algebra \( C^*_u(G) \). Nuclearity of \( C^*_u(G) \) is equivalent to property A of the metric space \( G \) (Higson and Roe [11, Theorem 1.1], Anantharaman-Delaroche and Renault [1]). It follows that exactness of the discrete group \( G \) is equivalent to property A of the metric space \( G \).

For a general metric space \( X \) with bounded geometry, equivalence between property A of \( X \) and exactness of \( C^*_u(X) \) had been an open problem, until the draft of this paper was announced. In [2, Corollary 30], Brodzki, Niblo, and Wright proved it in the case that the space \( X \) uniformly embeds into a discrete group.

2. Preliminaries

2.1. Notations

Let \((X, d)\) be a metric space. For \( x, y \in X \), the distance \( d(x, y) \) can be \( \infty \). For \( x \in X \) and a positive number \( R \), denote by \( N(x; R) \) the ball \( \{ y \in X | d(x, y) \leq R \} \). We assume that for every positive number \( R \), \( \sup_{x \in X} \frac{1}{2} N(x; R) < \infty \). If this condition holds, then we say that \( X \) has bounded geometry. The relation \( \{(x, y) \in X^2 | d(x, y) < \infty \} \) is an equivalence relation on \( X \). An equivalence class with respect to this relation is called a coarse connected component. Note that for a subset \( Y \subset X \), if the values of \( d|_{Y 	imes Y} \) are real numbers, then \( Y \) is contained in a coarse connected component of \( X \). A subset \( C \subset X^2 \) is said to be controlled, if \( \sup \{ d(x, y) | (x, y) \in C \} < \infty \).

2.2. Property A

Amenability for a discrete group is characterized by the existence of Følner sets, which are finite subsets with relatively small boundaries. A simple replacement of the Følner condition does not provide an appropriate notion of amenability for metric spaces. It requires a family of Følner sets.

**Definition 2.1** [24, Definition 2.1]. A discrete metric space \((X, d)\) is said to have property A if for every positive number \( \epsilon \) and every positive number \( R \), there exist a positive number \( S \) and a family of finite subsets \((A_x)_{x \in X}\) of \( X \times \mathbb{N} \) such that:

1. for every \( x \in X \), \( A_x \subset N(x; S) \times \mathbb{N} \);
2. for every \( x \in X \), \((x, 1) \in A_x \);
3. for every \( x, y \in X \), if \( d(x, y) < R \), then the symmetric difference \( A_x \triangle A_y \) satisfies \( \#(A_x \triangle A_y) < \epsilon \#(A_x \cap A_y) \).

**Example 2.2.** • Every discrete subgroup of the general linear group over a field has property A (Guentner, Higson and Weinberger [10]).
• If two groups have property A, then an amalgamated free product has property A (Tu [23, Corollary 9.5], Dykema [6], Ozawa [17, Corollary 2]).
• If a group $\Gamma$ is a hyperbolic relative to subgroups with property A, then $\Gamma$ has property A (Ozawa [17, Corollary 3]).

• Asymptotic dimension of finitely generated groups was introduced by Gromov [9, Section 1.E]. This notion can be applied for metric spaces with bounded geometry. If a metric space $X$ with bounded geometry has finite asymptotic dimension, then $X$ has property A (Higson and Roe [11, Lemma 4.3]).

2.3. Uniform Roe algebra

Every bounded linear operator $a$ on $\ell_2(X)$ is uniquely determined by the family of complex numbers

$$[a_{x,y}]_{x,y \in X} = [(a\delta_y, \delta_x)]_{x,y \in X}.$$  

We call $[a_{x,y}]_{x,y \in X}$ the matrix expression of $a$. For a positive number $R$, we say that the propagation of $a$ is at most $R$, if for every $x, y \in X$,

$$d(x, y) > R \Rightarrow a_{x,y} = 0.$$  

Denote by $E_R$ the space of all the operators on $\ell_2(X)$ whose propagation is at most $R$. We call an element in $\bigcup_R E_R$ an operator with finite propagation. The space $\bigcup_R E_R$ corresponds to the space of bounded functions on controlled sets.

**Lemma 2.3** [18, Conclusion of Lemma 4.27]. For every positive number $R$ and for every bounded function $\zeta$ on $\{(x, y) \in X^2 \mid d(x, y) \leq R\}$, there exists an element $a$ of $E_R$ such that $[a_{x,y}]_{x,y \in X} = [\zeta(x,y)]_{x,y \in X}$.

The uniform Roe algebra $C^*_u(X)$ is defined as the operator norm closure of $\bigcup_R E_R$. Since $\bigcup_R E_R$ is closed under multiplication and adjoint, the uniform Roe algebra is a $C^*$-algebra. Note that coarse connected components correspond to central projections in $C^*_u(X)$.

A coarse geometric property of $X$ sometimes implies an operator algebraic property of $C^*_u(X)$. The following is a typical example:

**Theorem 2.4** [21, Theorem 5.3]. Let $X$ be a metric space with bounded geometry. The space $X$ has property A if and only if $C^*_u(X)$ is nuclear.

2.4. Operator norm localization property

Chen, Tessera, Wang, and Yu defined the operator norm localization property in [4, Section 2]. The original definition is given for a general metric space $X$. For a metric space $X$ with bounded geometry, we use the following condition as the definition.

**Definition 2.5.** A metric space $X$ with bounded geometry is said to have the operator norm localization property, if for every $0 < c < 1$ and every positive number $R$, the following condition $(\alpha)$ holds: there exists a positive number $S$ satisfying that for every operator $a \in E_R$, there exists a unit vector $\eta \in \ell_2(X)$ such that the diameter of $\text{supp}(\eta)$ is at most $S$ and $c\|a\| \leq \|a\eta\|$.

The most important point is that $S$ is independent of the choice of $a \in E_R$. This definition is a necessary and sufficient condition of the original definition. See [20, Proposition 3.1] for the proof. We can replace the quantifier of $c$. 
Lemma 2.6 [20, Proposition 3.1]. A metric space \( X \) with bounded geometry has the operator norm localization property if there exists \( 0 < c < 1 \) such that for every positive number \( R \), condition (\( \alpha \)) holds.

The following theorem is the main result of [20]:

Theorem 2.7 [20, Theorem 4.1]. For a metric space with bounded geometry, the operator norm localization property is equivalent to property \( \alpha \).

Let us pay attention on spaces without the operator norm localization property. For a subset \( Y \) of \( X \), denote by \( P_Y \) the orthogonal projection from \( \ell_2(X) \) onto \( \ell_2(Y) \).

Lemma 2.8. If \( (X, d) \) does not have the operator norm localization property, then there exist a sequence of disjoint subsets \( V_n \) of \( X \), a sequence of positive matrices \( b_n \) acting on \( \ell_2(V_n) \) with norm \( 1 \), and a sequence \( S_n \) of positive numbers satisfying that

\[
\begin{align*}
(1) & \text{ for every } n \in \mathbb{N} \text{ and } Y \subset V_n, \text{ if } \text{diam}(Y) \leq S_n, \text{ then } \| P_Y b_n P_Y \| < 1/3; \\
(2) & \lim_n S_n = \infty; \\
(3) & \text{the propagation of } b = \sum_n b_n \text{ is finite.}
\end{align*}
\]

This lemma is essentially the same as [19, Lemma 4.2]. The author gives its proof for the reader's convenience.

Proof. Suppose that \( X \) does not have the operator norm localization property. By Lemma 2.6, there exists a positive number \( R \) such that condition (\( \alpha \)) does not hold for \( c = 1/3 \). For such \( R \), we have condition (\( \beta \)) for every positive number \( S \), there exists an operator \( a \in E_R \) such that \( \| aP_Y \| < \| a \|/3 \) whenever the diameter of \( Y \subset X \) is at most \( S \).

We prove the lemma by induction. Let \( S_1 \) be a positive number. Choose an operator \( a \) satisfying condition (\( \beta \)) for \( S = S_1 \). The operator \( a^*a \) is a member of \( E_{2R} \) and satisfies \( \| P_Y a^*a P_Y \| < \| a^*a \|/9 \) for every subset \( Y \subset X \) whose diameter is at most \( S_1 \). On the other hand, there exists a finite set \( V(1) \) of \( X \) such that \( \| a^*a \|/3 < \| P_{V(1)} a^*a P_{V(1)} \| \) and that \( V(1) \) is coarsely connected. For any subset \( Y \subset V(1) \) whose diameter is at most \( S_1 \), the inequality

\[
\| P_Y a^*a P_Y \| < \| a^*a \|/9 < \| P_{V(1)} a^*a P_{V(1)} \|/3
\]

holds. Define \( b_1 \) by \( P_{V(1)} a^*a P_{V(1)} \).

Now we assume that there exist operators \( b_1, b_2, \ldots, b_n \), finite subsets \( V(1), V(2), \ldots, V(n) \) of \( X \), and positive numbers \( S_1, \ldots, S_n \) with the following conditions.

\[
\begin{align*}
(i) & \text{ Every } V(j) \text{ is coarsely connected.} \\
(ii) & \text{ } V(1), V(2), \ldots, V(n) \text{ are disjoint.} \\
(iii) & b_1, b_2, \ldots, b_n \text{ are positive elements of } E_{2R} \text{ and satisfy } b_j = P_{V(j)} b_j P_{V(j)}. \\
(iv) & \| P_Y b_j P_Y \| < \| b_j \|/3, \text{ when the diameter of } Y \subset V(j) \text{ is at most } S_j. \\
v & S_j + 1 \leq S_{j+1}.
\end{align*}
\]

Define \( S_{n+1} \) by

\[
S_{n+1} = \max_X \text{diam} \left( \bigcup \left\{ V(j) \mid V(j) \subset \tilde{X}, 1 \leq j \leq n \right\} \right) + 1,
\]

where \( \tilde{X} \) is a coarse connected component of \( X \).
By condition \((\beta)\), there exists an operator \(a \in E_R\) such that \(\|a P Y\| < \|a\|/3\) whenever the diameter of \(Y \subset X\) is at most \(S_{n+1}\). For every coarse connected component \(\tilde{X} \subset X\), the projection \(P_{\tilde{X}}\) commutes with \(a\) and

\[
\text{diam} \left( \bigcup \left\{ V(j) \mid V(j) \subset \tilde{X}, 1 \leq j \leq n \right\} \right) < S_{n+1}.
\]

Therefore, we have \(\|a P_{\cup_{j=1}^n V(j)}\| < \|a\|/3\). There exists a finite subset \(V(n+1)\) of \(X \setminus \cup_{j=1}^n V(j)\) such that:

- \(\frac{\sqrt{2}}{2} \|a P_X \setminus \cup_{j=1}^n V(j)\| \leq \|P V(n+1)\|\);
- \(V(n+1)\) is coarsely connected.

The inequality in the first item implies

\[
\frac{1}{\sqrt{3}} \|a\| = \frac{\sqrt{3}}{2} \left( \|a\| - \frac{1}{3} \|a\| \right) < \frac{\sqrt{3}}{2} \left( \|a\| - \|a P_{\cup_{j=1}^n V(j)}\| \right)
\]

\[
\leq \frac{\sqrt{3}}{2} \|a P_X \setminus \cup_{j=1}^n V(j)\| \leq \|P V(n+1)\|.
\]

Define \(b_{n+1}\) by \(P V(n+1) a^* a P V(n+1)\). For every subset \(Y \subset V(n+1)\) whose diameter is at most \(S_{n+1}\), we have

\[
\|P_Y b_{n+1} P_Y\| = \|P_Y a^* a P_Y\| < \frac{\|a a^*\|}{9} < \frac{\|P V(n+1) a^* a P V(n+1)\|}{3} = \frac{\|b_{n+1}\|}{3}.
\]

Now we obtain operators \(b_1, \ldots, b_n, b_{n+1}\), finite subsets \(V(1), \ldots, V(n), V(n+1)\), and positive numbers \(S_1, \ldots, S_n, S_{n+1}\) with the same conditions as (i), (ii), (iii), (iv), and (v).

Iterate this procedure infinitely many times. The sequence \((S_n)\) diverges and the propagation of \(b := \sum_{j=1}^{\infty} b_j/\|b_j\|\) is at most \(2R\).

\(\square\)

### 2.5 Local reflexivity

In this paper, we consider the following properties for \(C^*\)-algebras: nuclearity, exactness, and local reflexivity. It is not hard to see that nuclearity implies exactness. Conclusions by Kirchberg [12, 13] show that exactness implies local reflexivity. They are all related to minimal tensor products between \(C^*\)-algebras. The following is the definition of local reflexivity.

**Definition 2.9** [7, Section 5]. A \(C^*\)-algebra \(B\) is said to be locally reflexive if for every finite-dimensional operator system \(V \subset B^{**}\), there exists a net of contractive completely positive maps \(\Phi_i : V \rightarrow B\) which converges to \(\text{id}_V\) in the point-weak-* topology.

We use the following proved in [7, Theorem 3.2, 5.1, and Proposition 5.3].

- A \(C^*\)-subalgebra of a locally reflexive \(C^*\)-algebra is also locally reflexive.
- If \(B\) is locally reflexive and \(K\) is an ideal of \(B\), then the exact sequence \(0 \rightarrow K \rightarrow B \rightarrow B/K \rightarrow 0\) locally splits. More precisely, for every finite-dimensional operator system \(V \subset B/K\), there exists a unital completely positive map \(\Phi : V \rightarrow B\) such that \(\Phi(a) + K = a, a \in V\).

If the algebra \(B\) is locally reflexive and \(K\) is an ideal of \(B\), then for every \(C^*\)-algebra \(C\), the naturally defined sequence \(0 \rightarrow K \otimes_{\min} C \rightarrow B \otimes_{\min} C \rightarrow B/K \otimes_{\min} C \rightarrow 0\) is exact [7, Theorem 3.2].
3. Uniform Roe algebra of a sequence of finite connected graphs

In this section, we concentrate on the case that an infinite metric space $V$ with bounded geometry is given by a sequence of finite connected graphs $(G_n = (V_n, \text{Edge}_n))_{n \in \mathbb{N}}$. We denote by $\mathbb{N}$ the set of natural numbers. We study a sequence of vector states $\omega_n = \langle \xi_n, \xi_n \rangle$ on $\mathcal{C}^*_u(V)$ given by unit vectors $\xi_n \in \ell_2(V_n)$ under the assumption that $\mathcal{C}^*_u(V)$ is locally reflexive. Our goal is Theorem 3.17. Theorem 3.17 means that the vector states can be approximated by some states which satisfy some localization property.

3.1. Action of the $C^*$-minimal tensor product

Let $(G_n = (V_n, \text{Edge}_n))_{n \in \mathbb{N}}$ be a sequence of finite connected graphs. The set $V_n$ is the vertex set of $G_n$, and $\text{Edge}_n \subset V_n^2$ is a symmetric subset of $V_n$, which is called an edge set of $G_n$. The graph metric $d_n$ on $V_n$ is defined. It is the maximal metric function on $V_n \times V_n$ satisfying that $d_n(x, y) \leq 1$ for $(x, y) \in \text{Edge}_n$.

We also introduce a metric $d$ on $V = \sqcup_n V_n$ by the following: for $x, y \in V$,

- if there exists $n \in \mathbb{N}$ such that $x, y \in V_n$, then $d(x, y) = d_n(x, y)$,
- if $x \in V_m$, $y \in V_n$, and if $m \neq n$, then $d(x, y) = \infty$.

We always assume that $V$ has bounded geometry and that $\mathcal{C}^*_u(V)$ is locally reflexive. We note that the algebra $\mathcal{C}^*_u(V)$ is a subalgebra of $\prod_{n \in \mathbb{N}} \mathcal{B}(\ell_2(V_n))$, the algebra of norm bounded sequences of matrices.

The algebra $\mathcal{C}^*_u(V)$ acts on the Hilbert space $\bigoplus_{n \in \mathbb{N}} \ell_2(V_n)$. The opposite algebra $\mathcal{C}^*_u(V)^{\text{op}} \subset \prod_{n \in \mathbb{N}} \mathcal{B}(\ell_2(V_n))^{\text{op}}$ also acts on $\bigoplus_{n \in \mathbb{N}} \ell_2(V_n)$ by its transpose:

$$(a_n)^{\text{op}}(\xi_n)_n = (a_n^T \xi_n)_n, \quad (a_n)^{\text{op}} \in \mathcal{C}^*_u(V)^{\text{op}}, \quad (\xi_n)_n \in \bigoplus_{n \in \mathbb{N}} \ell_2(V_n).$$

The $C^*$-minimal tensor product $\mathcal{C}^*_u(V) \otimes_{\text{min}} \mathcal{C}^*_u(V)^{\text{op}}$ acts on the Hilbert space $\bigoplus_{n \in \mathbb{N}} \ell_2(V_n) \otimes \bigoplus_{n \in \mathbb{N}} \ell_2(V_n)$. The closed subspace $\bigoplus_{n \in \mathbb{N}} \ell_2(V_n^2)$ of $\bigoplus_{n \in \mathbb{N}} \ell_2(V_n) \otimes \bigoplus_{n \in \mathbb{N}} \ell_2(V_n)$ is invariant under the action of the algebra $\mathcal{C}^*_u(V) \otimes_{\text{min}} \mathcal{C}^*_u(V)^{\text{op}}$.

3.2. A state $\phi_\infty$ of $\mathcal{C}^*_u(V) \otimes_{\text{min}} \mathcal{C}^*_u(V)^{\text{op}}$

Let $(\xi_n)_n$ be a sequence of unit vectors in the Hilbert space $\bigoplus_{n \in \mathbb{N}} \ell_2(V_n)$. Until Theorem 3.17, we always assume the following.

- $\xi_n$ is a positive element of $\ell_2(V_n)$ and its support is $V_n$,
- There exists a positive number $\Lambda$ such that for every $n \in \mathbb{N}$ and for every $(x, y) \in \text{Edge}_n$, $\xi_n(x)^2 \leq \Lambda \xi_n(y)^2$.

In the proof of Theorem 3.17, we prove that the general case reduces to the case that the unit vectors satisfy these conditions.

Define a sequence of unit vectors $(\Xi_n)_n \in \bigoplus_{n \in \mathbb{N}} \ell_2(V_n^2)$ by

$$\Xi_n = \sum_{x \in V_n} \xi_n(x) \delta_{(x,x)} \in \ell_2(V_n^2).$$

These vectors $\Xi_n$ define states $\phi_n(\cdot) = \langle \cdot, \Xi_n, \Xi_n \rangle$ on the $C^*$-algebra $\mathcal{C}^*_u(V) \otimes_{\text{min}} \mathcal{C}^*_u(V)^{\text{op}}$. Although our subject is the sequence of vector states $\omega_n = \langle \xi_n, \xi_n \rangle$, we concentrate on that of vector states $\phi_n = \langle \cdot, \Xi_n, \Xi_n \rangle$ for a while. Choose and fix an accumulation point $\phi_\infty$ of the sequence $(\phi_n)_n$ in the dual space $(\mathcal{C}^*_u(V) \otimes_{\text{min}} \mathcal{C}^*_u(V)^{\text{op}})^*$. Denote by $(\pi_\infty, \mathcal{H}_\infty, \Xi_\infty)$ the GNS-triple of the state $\phi_\infty$ on the algebra $\mathcal{C}^*_u(V) \otimes_{\text{min}} \mathcal{C}^*_u(V)^{\text{op}}$. For $a \in \mathcal{C}^*_u(V)$ and $a^{\text{op}} \in \mathcal{C}^*_u(V)^{\text{op}}$, we simply write
\[ \pi_\infty(a \otimes 1) = \pi_\infty(a), \]
\[ \pi_\infty(1 \otimes a^\text{op}) = \pi_\infty(a)^\text{op}. \]

The image \( \pi_\infty(C^*_n(V)) \) commutes with \( \pi_\infty(C^*_n(V))^\text{op}. \)

**Lemma 3.1.** For every \( a \in \bigcup R E_R \), there exists \( b \in \bigcup R E_R \) such that \( \pi_\infty(a)\Xi_\infty = \pi_\infty(b)^\text{op}\Xi_\infty \).

For every \( b \in \bigcup R E_R \), there exists \( a \in \bigcup R E_R \) such that \( \pi_\infty(a)\Xi_\infty = \pi_\infty(b)^\text{op}\Xi_\infty \).

Proof. We prove the first half. Let \( \langle [a_{n,x,y}]_{x,y \in V_n} \rangle_n \) be the matrix expression of a non-zero operator \( a \in C^*_n(V) \) with finite propagation. Define \( R \) by \( R = \operatorname{sup}\{d(x,y) \mid n \in \mathbb{N}, x, y \in V_n, a_{n,x,y} \neq 0\} \). Define \( b \in C^*_n(V) \) by

\[ b = \left( \begin{bmatrix} a_{n,x,y} \xi_n(y) \\ \xi_n(x) \end{bmatrix} \right)_{x,y \in V_n} \]

The vector \( (a \otimes 1)\Xi_n \) is equal to the following vector:

\[
\sum_{z \in V_n} \left( \sum_{x,y \in V_n} a_{n,x,y} \xi_n(z) \delta_z \right) \otimes \delta_z = \sum_{z \in V_n} \left( \sum_{w \in V_n} a_{n,w,z} \xi_n(z) \delta_w \right) \otimes \delta_z
\]

\[ = \sum_{w \in V_n} \delta_w \otimes \left( \sum_{z \in V_n} a_{n,w,z} \xi_n(z) \xi_n(w) \delta_z \right) \]

\[ = \sum_{w \in V_n} \delta_w \otimes \left( \begin{bmatrix} a_{n,x,y} \xi_n(y) \\ \xi_n(x) \end{bmatrix} \right)_{x,y \in V_n}^T \xi_n(w) \delta_w
\]

\[ = (1 \otimes b^\text{op})\Xi_n. \]

It follows that \( \phi_n(||(a \otimes 1) - (1 \otimes b^\text{op})||^2) = 0 \). Therefore, we have \( \phi_\infty(||(a \otimes 1) - (1 \otimes b^\text{op})||^2) = 0 \) and \( \pi_\infty(a \otimes 1)\Xi_\infty - \pi_\infty(1 \otimes b^\text{op})\Xi_\infty = 0. \)

**Lemma 3.2.** The vector \( \Xi_\infty \) is a cyclic vector of \( \pi_\infty(C^*_n(V)) \). The vector \( \Xi_\infty \) is a cyclic vector of \( \pi_\infty(C^*_n(V))^\text{op}. \)

Proof. We prove the second half. The subspace \( \operatorname{span}\{\pi_\infty(b)^\text{op}\pi_\infty(a)\Xi_\infty \mid a, b \in \bigcup R E_R\} \) of \( \mathcal{H}_\infty \) is dense. By Lemma 3.1, for every \( a \in \bigcup R E_R \), there exists \( b_1 \in \bigcup R E_R \) satisfying that \( \pi_\infty(a)\Xi_\infty = \pi_\infty(b_1)^\text{op}\Xi_\infty \). Therefore, the subspace \( \{\pi_\infty(b)^\text{op}\Xi_\infty \mid b \in \bigcup R E_R\} \) is equal to the above dense subspace. \( \square \)

### 3.3. A representation of \( \ell_\infty(\bigcup_n V_n^2) \) on \( \mathcal{H}_\infty \)

Let \( \zeta \) be a bounded function on \( \bigcup_n V_n^2 \). For an arbitrary bounded operator \( a = \langle [a_{n,x,y}]_{x,y \in V_n} \rangle_n \) on \( \ell_2(V) \) with finite propagation, since \( ||\zeta(x,y)a_{n,x,y}|| \leq ||\zeta||_\infty ||a|| \), the operator \( \langle [\zeta(x,y)a_{n,x,y}]_{x,y \in V_n} \rangle_n \) is also a bounded operator with finite propagation, by Lemma 2.3. We denote by \( M[\zeta](a) \) the operator.

**Lemma 3.3.** There exists a unique bounded operator \( \varpi_\infty(\zeta) \) on \( \mathcal{H}_\infty \) satisfying that for every \( a \in \bigcup R E_R \), \( \varpi_\infty(\zeta)\pi_\infty(a)\Xi_\infty = \pi_\infty(M[\zeta](a))\Xi_\infty. \)
Proof. For every \( a = ([a_{n,x,y}]_{x,y \in V_n})_n \in \cup_{R} E_{R} \), there exists an increasing sequence \((n(k))\) of natural numbers such that

\[
\phi_\infty(a^*a) = \lim_{k \to \infty} \phi_{n(k)}(a^*a),
\]

\[
\phi_\infty(M[\zeta](a)^*M[\zeta](a)) = \lim_{k \to \infty} \phi_{n(k)}(M[\zeta](a)^*M[\zeta](a)).
\]

For every \( k \), we have

\[
\phi_{n(k)}(a^*a) = \sum_{y \in V_n} \xi_{n(k)}(y)^2 \sum_{x \in V_n} |a_{n,x,y}|^2,
\]

\[
\phi_{n(k)}(M[\zeta](a)^*M[\zeta](a)) = \sum_{y \in V_n} \xi_{n(k)}(y)^2 \sum_{x \in V_n} |\zeta(x,y)|^2 |a_{n,x,y}|^2.
\]

It follows that \( \phi_{n(k)}(M[\zeta](a)^*M[\zeta](a)) \leq \|\zeta\|_{\infty}^2 \phi_{n(k)}(a^*a) \). Thus, we obtain the inequalities \( \phi_\infty(M[\zeta](a)^*M[\zeta](a)) \leq \|\zeta\|_{\infty}^2 \phi_\infty(a^*a) \) and

\[
\|\pi_\infty(M[\zeta](a))\Xi_\infty\| \leq \|\zeta\|_{\infty}^2 \|\pi_\infty(a)\Xi_\infty\|^2.
\]

We conclude that the mapping \( \pi_\infty(a)\Xi_\infty \mapsto \pi_\infty(M[\zeta](a))\Xi_\infty \) is well defined and bounded. By Lemma 3.2, the mapping uniquely extends to a bounded operator on the Hilbert space \( \mathcal{H}_{\infty} \).

It is also easy to show that \( \varpi_\infty : \ell_\infty(\cup_n V_n^2) \to \mathcal{B}(\mathcal{H}_{\infty}) \) is a \(*\)-homomorphism.

Note that there exists a natural unital embedding \( \iota \) of \( \ell_\infty(V) \) into \( \ell_\infty(\cup_n V_n^2) \) defined by

\[
[\iota(\zeta)](x,y) = \zeta(x), \quad \zeta \in \ell_\infty(V), (x,y) \in \cup_n V_n^2.
\]

It is straightforward to show that the representation \( \pi_\infty|\ell_\infty(V) \) of the diagonal subalgebra \( \ell_\infty(V) \) of \( C_\infty(V) \) and the representation \( \varpi_\infty \circ \iota \) are identical.

3.4. Modular operator with respect to \((\pi_\infty(C_\infty(V))^\prime\prime, \Xi_\infty)\)

Recall that \( E_{R} \) is the space of operators on \( \ell_2(V) \) whose propagations are at most \( R \). Denote by \( \mathcal{H}_{R} \) the closure of \( \pi_\infty(E_{R})\Xi_\infty \subset \mathcal{H}_{\infty} \). Note that \( \cup_{R} \mathcal{H}_{R} \) is dense in \( \mathcal{H}_{\infty} \), by Lemma 3.2.

Define a function \( H : \cup_n V_n^2 \to \mathbb{R} \) by

\[
H(x,y) = \xi_n(x)^2 / \xi_n(y)^2, \quad n \in \mathbb{N}, x, y \in V_n.
\]

Note that the function \( H \) is bounded on every controlled subset of \( \cup_n V_n^2 \). Therefore, the operator \( M[H] \) defines a bounded self-adjoint operator on \( \mathcal{H}_{R} \) for every \( R \). Such an operator on \( \cup_{R} \mathcal{H}_{R} \) uniquely extends to a self-adjoint operator on \( \mathcal{H}_{\infty} \), which is not necessarily bounded.

Denote by \( \Delta \) the extension.

We also define an anti-linear isometry \( J \) on \( \mathcal{H}_{\infty} \) by the following.

**Lemma 3.4.** There exists a unique anti-linear isometry \( J \) such that for every \( a = ([a_{n,x,y}]_{x,y \in V_n})_n \in \cup_{R} E_{R} \),

\[
J \pi_\infty(([a_{n,x,y}]_{x,y \in V_n})_n) \Xi_\infty = \pi_\infty\left(\left(\left[H(x,y)^{1/2} \overline{\pi_{n,x,y}}\right]_{x,y \in V_n}\right)_n\right) \Xi_\infty.
\]

**Proof.** Because the sequence of matrices \( ([H(x,y)^{1/2} \overline{\pi_{n,x,y}}]_{x,y \in V_n})_n \) has finite propagation and the set of entries is bounded, it gives a bounded operator by Lemma 2.3. It is straightforward to show that for every \( n \),
\[ \| [a_{n,x,y}]_{x,y \in V_n} \Xi_n \|^2 = \| \left( H(x,y)^{1/2} \begin{array}{c} a_{n,y,x} \\ \hline \end{array} \right)_{x,y \in V_n} \Xi_n \|^2, \]

\[ \phi_n (\left( [a_{n,x,y}]_{x,y \in V_n} \right)_n \left( [a_{n,x,y}]_{x,y \in V_n} \right)_n) = \phi_n \left( \left( H(x,y)^{1/2} \begin{array}{c} a_{n,y,x} \\ \hline \end{array} \right)_{x,y \in V_n} \right)_n \left( \left( H(x,y)^{1/2} \begin{array}{c} a_{n,y,x} \\ \hline \end{array} \right)_{x,y \in V_n} \right)_n). \]

The latter equality implies
\[ \phi_\infty (\left( [a_{n,x,y}]_{x,y \in V_n} \right)_n \left( [a_{n,x,y}]_{x,y \in V_n} \right)_n) = \phi_\infty \left( \left( H(x,y)^{1/2} \begin{array}{c} a_{n,y,x} \\ \hline \end{array} \right)_{x,y \in V_n} \right)_n \left( \left( H(x,y)^{1/2} \begin{array}{c} a_{n,y,x} \\ \hline \end{array} \right)_{x,y \in V_n} \right)_n). \]

\[ \| \pi_\infty (\left( [a_{n,x,y}]_{x,y \in V_n} \right)_n \Xi_n \Xi_\infty \|^2 = \| \pi_\infty (\left( \left( H(x,y)^{1/2} \begin{array}{c} a_{n,y,x} \\ \hline \end{array} \right)_{x,y \in V_n} \right)_n) \Xi_\infty \|^2. \]

Because \( \pi_\infty (\cup RE_R) \Xi_\infty \) is dense in \( H_\infty \) (Lemma 3.2), there exists a unique isometry \( J \) on \( H_\infty \) which extends the mapping
\[ \pi_\infty (\left( [a_{n,x,y}]_{x,y \in V_n} \right)_n) \Xi_\infty \rightarrow \pi_\infty (\left( \left( H(x,y)^{1/2} \begin{array}{c} a_{n,y,x} \\ \hline \end{array} \right)_{x,y \in V_n} \right)_n) \Xi_\infty. \]

It is also clear that \( J \) is anti-linear. \( \square \)

**Lemma 3.5.** \( J^2 = 1, J \Delta^{1/2} = \Delta^{-1/2} J. \)

**Proof.** For every \( (a_{n,x,y}]_{x,y \in V_n})_n \in \cup RE_R \), we obtain
\[ J J \pi_\infty (\left( [a_{n,x,y}]_{x,y \in V_n} \right)_n) \Xi_\infty \]
\[ = J \pi_\infty (\left( \left( H(x,y)^{1/2} \begin{array}{c} a_{n,y,x} \\ \hline \end{array} \right)_{x,y \in V_n} \right)_n) \Xi_\infty \]
\[ = \pi_\infty (\left( \left( H(x,y)^{1/2} \begin{array}{c} a_{n,y,x} \\ \hline \end{array} \right)_{x,y \in V_n} \right)_n) \Xi_\infty \]
\[ = \pi_\infty (\left( [a_{n,x,y}]_{x,y \in V_n} \right)_n) \Xi_\infty. \]

Therefore, we have \( J J = 1 \). We also obtain
\[ J \Delta^{1/2} \pi_\infty (\left( [a_{n,x,y}]_{x,y \in V_n} \right)_n) \Xi_\infty \]
\[ = \pi_\infty (\left( \left( H(x,y)^{1/2} \begin{array}{c} a_{n,y,x} \\ \hline \end{array} \right)_{x,y \in V_n} \right)_n) \Xi_\infty \]
\[ = \pi_\infty (\left( \left( H(x,y)^{1/2} \begin{array}{c} a_{n,y,x} \\ \hline \end{array} \right)_{x,y \in V_n} \right)_n) \Xi_\infty \]
\[ = \pi_\infty (\left( [\begin{array}{c} \{a_{n,y,x} \}_{x,y \in V_n} \right)_n) \Xi_\infty. \]
Therefore, we have $J \Delta^{1/2} = \Delta^{-1/2} J$ on every $\mathcal{H}_R$. It implies the lemma.

**Lemma 3.6.** For every $b \in \cup R E_R$, $\pi_\infty(b)^{\text{op}} \Xi_\infty = \Delta^{1/2} \pi_\infty(b) \Xi_\infty$.

**Proof.** This has been already shown in the proof of Lemma 3.1.

**Lemma 3.7.** For every $a, b \in \cup R E_R$, the following equations hold:

$$
\pi_\infty(a^* \Xi_\infty = J \Delta^{1/2} \pi_\infty(a) \Xi_\infty, \quad \pi_\infty(b^*)^{\text{op}} \Xi_\infty = J \Delta^{-1/2} \pi_\infty(b)^{\text{op}} \Xi_\infty.
$$

**Proof.** For $a = (\{a_{n,x,y}\}_{x,y \in V_n})_n$, we have already obtained the equation

$$
J \Delta^{1/2} \pi_\infty(\{a_{n,x,y}\}_{x,y \in V_n})_n \Xi_\infty = \pi_\infty(\{[\overline{a_{n,x,y}}]_{x,y \in V_n}\}_n) \Xi_\infty,
$$

in the proof of Lemma 3.5. This is the first claim. By Lemma 3.6, and by the first claim, we obtain

$$
\pi_\infty(b^*)^{\text{op}} \Xi_\infty = \Delta^{1/2} \pi_\infty(b^*) \Xi_\infty = \Delta^{1/2} J \Delta^{1/2} \pi_\infty(b) \Xi_\infty.
$$

By Lemmas 3.6 and 3.5, we obtain

$$
\Delta^{1/2} J \Delta^{1/2} \pi_\infty(b) \Xi_\infty = \Delta^{1/2} J \pi_\infty(b)^{\text{op}} \Xi_\infty = J \Delta^{-1/2} \pi_\infty(b)^{\text{op}} \Xi_\infty.
$$

Thus, we obtain the second claim.

**Lemma 3.8.** For every $a \in \cup R E_R$, $J \pi_\infty(a) J = \pi_\infty(a^*)^{\text{op}}$.

**Proof.** For every $a, b \in \cup R E_R$, by Lemma 3.6, we have

$$
J \pi_\infty(a) J \pi_\infty(b)^{\text{op}} \Xi_\infty = J \pi_\infty(a) J \Delta^{1/2} \pi_\infty(b) \Xi_\infty.
$$

By Lemma 3.7, we have

$$
J \pi_\infty(a) J \Delta^{1/2} \pi_\infty(b) \Xi_\infty = J \pi_\infty(a) \pi_\infty(b^*) \Xi_\infty = J \pi_\infty(ab^*) \Xi_\infty.
$$

Again by Lemma 3.6, we have

$$
J \pi_\infty(ab^*) \Xi_\infty = J \Delta^{-1/2} \pi_\infty(ab^*)^{\text{op}} \Xi_\infty.
$$

By Lemma 3.7, we have

$$
J \Delta^{-1/2} \pi_\infty(ab^*)^{\text{op}} \Xi_\infty = \pi_\infty(ba^*)^{\text{op}} \Xi_\infty = \pi_\infty(a^*)^{\text{op}} \pi_\infty(b)^{\text{op}} \Xi_\infty.
$$

Since $\pi_\infty(\cup R E_R)^{\text{op}} \Xi_\infty$ is dense in $\mathcal{H}_\infty$ (Lemma 3.2), we obtain $J \pi_\infty(a) J = \pi_\infty(a^*)^{\text{op}}$.

Denote by $S$ the closure of the mapping

$$
\pi_\infty(C_u(V))^{\gamma} \Xi_\infty \rightarrow \pi_\infty(C_u(V))^{\gamma} \Xi_\infty
$$

$$
a \Xi_\infty \mapsto a^* \Xi_\infty.
$$
Denote by $F$ the closure of the mapping

$$\pi_\infty(C_u^*(V))'\Xi_\infty \to \pi_\infty(C_u^*(V))'\Xi_\infty$$

$$b\Xi_\infty \mapsto b^*\Xi_\infty.$$  

It is straightforward to show that $F \subset S^*$ and $S \subset F^*$.

**Lemma 3.9.** $S = J\Delta^{1/2}, F = J\Delta^{-1/2}$.

**Proof.** By Lemma 3.7, the operators $S$ and $J\Delta^{1/2}$ are identical on the subspace $\cup R\pi_\infty(E_R)\Xi_\infty$. The subspace $\cup R\pi_\infty(E_R)\Xi_\infty$ is a core of $\Delta^{1/2}$. It follows that $J\Delta^{1/2} \subset S$. Recall that $\pi_\infty(\cup R E_R)'op \subset \pi_\infty(C_u^*(V))'$. By Lemma 3.7, the operators $F$ and $J\Delta^{-1/2}$ are identical on the subspace $\cup R\pi_\infty(E_R)'op\Xi_\infty$. Since $\pi_\infty(\cup R E_R)'op\Xi_\infty = \pi_\infty(\cup R E_R)\Xi_\infty$ (Lemma 3.1), $\pi_\infty(\cup R E_R)'op\Xi_\infty$ is a core of $\Delta^{-1/2}$. It follows that $J\Delta^{-1/2} \subset F$. Taking the adjoints of $J\Delta^{1/2} \subset S$, we obtain $S^* \subset (J\Delta^{1/2})^*$. Combining with Lemma 3.5, we have

$$J\Delta^{-1/2} \subset F \subset S^* \subset (J\Delta^{1/2})^* = \Delta^{1/2}J = J\Delta^{-1/2}.$$  

Thus, we obtain $F = J\Delta^{-1/2}$.

Taking the adjoints of $J\Delta^{-1/2} \subset F$, we obtain $F^* \subset (J\Delta^{-1/2})^*$. Combining with Lemma 3.5, we have

$$J\Delta^{1/2} \subset S \subset F^* \subset (J\Delta^{-1/2})^* = \Delta^{-1/2}J = J\Delta^{1/2}.$$  

Thus, we obtain $S = J\Delta^{1/2}$.  

**Proposition 3.10.** $\pi_\infty(C_u^*(V))'' = \pi_\infty(C_u^*(V))'$.  

**Proof.** By Lemma 3.9, $J$ is the modular conjugation of $\pi_\infty(C_u^*(V))''$ with respect to the state given by $\Xi_\infty$. By the fundamental theorem by Tomita [22, Theorem 4.1] for the left Hilbert algebra $\pi(C_u^*(V))''\Xi_\infty$, the commutant $\pi_\infty(C_u^*(V))'$ is identical to $J\pi_\infty(C_u^*(V))''J$. It follows that $\pi_\infty(C_u^*(V))'$ is generated by $J\pi_\infty(\cup R E_R)J$. By Lemma 3.8, $J\pi_\infty(\cup R E_R)J = \pi_\infty((\cup R E_R)'op)$. We conclude that $\pi_\infty(C_u^*(V))'op$ generates the commutant $\pi_\infty(C_u^*(V))'$.  

3.5. Existence of an equivariant state $\psi$ on $\mathcal{B}(\mathcal{H}_\infty)$

The strategy used in the first half of this subsection is called ‘The Trick’ in the book [3, Section 3.6]. Denote by $K$ the kernel of the representation $\pi_\infty: C_u^*(V) \to \mathcal{B}(\mathcal{H}_\infty)$. Suppose that $C_u^*(V)$ is locally reflexive. Then the sequence

$$0 \to K \otimes_{\min} C_u^*(V)'op \to C_u^*(V) \otimes_{\min} C_u^*(V)'op \to \pi_\infty(C_u^*(V)) \otimes_{\min} C_u^*(V)'op \to 0$$

is exact. Since the representation

$$\pi_\infty: C_u^*(V) \otimes_{\min} C_u^*(V)'op \to \mathcal{B}(\mathcal{H}_\infty)$$

is zero on $K \otimes_{\min} C_u^*(V)'op$, it naturally induces a representation

$$\pi_\infty(C_u^*(V)) \otimes_{\min} C_u^*(V)'op \to \mathcal{B}(\mathcal{H}_\infty).$$

By the extension theorem by Arveson, the representation

$$\pi_\infty(C_u^*(V)) \otimes_{\min} C_u^*(V)'op \to \mathcal{B}(\mathcal{H}_\infty)$$

can be extended to a unital completely positive map

$$\Phi: \mathcal{B}(\mathcal{H}_\infty) \otimes_{\min} C_u^*(V)'op \to \mathcal{B}(\mathcal{H}_\infty).$$

Since the subalgebra $\mathbb{C} \otimes_{\min} C_u^*(V)'op$ is in the multiplicative domain of $\Phi$, $\Phi$ is a $(\mathbb{C} \otimes_{\min} C_u^*(V)'op)$-module map. It follows that the image $\Phi(\mathcal{B}(\mathcal{H}_\infty))$ of the first factor is included in
the von Neumann algebra $\pi_\infty(C_\sigma^n(V))^{\text{op}}$. By Proposition 3.10, the von Neumann algebra is identical to $\pi_\infty(C_\sigma^n(V))''$. Recalling the notation, we also denote by $\Phi$ the restriction

$$\Phi : \mathcal{B}(\mathcal{H}_\infty) \to \pi_\infty(C_\sigma^n(V))''.$$  

Note that $\Phi$ is the identity map on $\pi_\infty(C_\sigma^n(V))$ and therefore $\Phi$ is a $\pi_\infty(C_\sigma^n(V))$-module map. Define a state $\psi$ on $\mathcal{B}(\mathcal{H}_\infty)$ by $\langle \Phi(\cdot) \Xi, \Xi \rangle$. We will show that the state $\psi \circ \varpi_\infty$ on $\ell_2(\sqcup_n V_n)$ is something like an invariant mean on an amenable group in Lemma 3.12. To state the claim, we need to define several notations. Take an arbitrary sequence $(\sigma_n)_n$ of partially defined injections

$$\sigma_n : \text{dom}(\sigma_n) \to \text{image}(\sigma_n),$$

from a subset of $V_n$ to a subset of $V_n$. By taking the union, we obtain a partially defined injection

$$\sigma : \sqcup_n \text{dom}(\sigma_n) \to \sqcup_n \text{image}(\sigma_n),$$

defined on the subset $\sqcup_n \text{dom}(\sigma_n)$ of $V = \sqcup_n V_n$. We always consider the case that there exists a positive constant $R$ independent of $n$ such that for arbitrary $x, y \in V_n$, if $\sigma_n(y) = x$, then $d_n(x, y) \leq R$. We call $\sigma$ a controlled partially defined injection on $V$. The map $\sigma$ induces a partial isometry $v_\sigma$ acting on $\ell_2(V)$ defined by $v_\sigma(\delta_x) = \delta_{\sigma(x)}$, $x \in \text{dom}(\sigma)$. Since $\sigma$ is controlled, $v_\sigma$ has finite propagation, and therefore is an element of $C_\sigma^n(V)$. Denote by $\sigma \times \text{id}$ the partially defined injection

$$\sigma \times \text{id} : \sqcup_n (\text{dom}(\sigma_n) \times V_n) \to \sqcup_n (\text{image}(\sigma_n) \times V_n),$$

$$(x, y) \mapsto (\sigma(x), y),$$
on $\sqcup_n V_n^2$. Note that the action of $v_\sigma$ on the Hilbert space $\ell_2(\sqcup_n V_n^2)$ is given by the partially defined injection $\sigma \times \text{id}$. The partially defined injection $\sigma \times \text{id}$ induces the $*$-isomorphisms

$$(\sigma \times \text{id})_* : \ell_\infty(\sqcup_n (\text{dom}(\sigma_n) \times V_n)) \to \ell_\infty(\sqcup_n (\text{image}(\sigma_n) \times V_n)),$$

$$(\sigma \times \text{id})^* : \ell_\infty(\sqcup_n (\text{image}(\sigma_n) \times V_n)) \to \ell_\infty(\sqcup_n (\text{dom}(\sigma_n) \times V_n)).$$

Lemma 3.11. For $\zeta \in \ell_\infty(\sqcup_n (\text{dom}(\sigma_n) \times V_n))$, we have

$$\varpi_\infty((\sigma \times \text{id})_* (\zeta)) = \pi_\infty(v_\sigma) \varpi_\infty(\zeta) \pi_\infty(v_\sigma^*).$$

Proof. For every $a = ([a_{n, x, y}]_{x, y \in V_n})_n \in \cup_R E_R$, the matrix coefficients of the operator $v_\sigma^*([a_{n, x, y}]_{x, y \in V_n})_n$ at $(x, y) \in V_n^2$ is given by

$$\begin{cases}
\begin{aligned}
a_{n, \sigma(x), y}, & \quad x \in \text{dom}(\sigma_n), \\
0, & \quad x \notin \text{dom}(\sigma_n).
\end{aligned}
\end{cases}$$

For simplicity of the notation, in the case that $x \notin \text{dom}(\sigma_n)$, we define $a_{n, \sigma(x), y}$ by $0$. For $\zeta \in \ell_\infty(\sqcup_n (\text{dom}(\sigma_n) \times V_n))$, we have

$$\begin{aligned}
\pi_\infty(v_\sigma) \varpi_\infty(\zeta) \pi_\infty(v_\sigma^*) \pi_\infty(a) \Xi_\infty \\
= \pi_\infty(v_\sigma) \varpi_\infty(\zeta) \pi_\infty(([a_{n, \sigma(x), y}]_{x, y \in V_n})_n) \Xi_\infty \\
= \pi_\infty(v_\sigma) \pi_\infty((\zeta(x, y) a_{n, \sigma(x), y})_{x, y \in V_n})_n) \Xi_\infty \\
= \pi_\infty((\zeta(\sigma^{-1}(x), y) a_{n, x, y})_{x, y \in V_n})_n) \Xi_\infty \\
= \varpi_\infty((\sigma \times \text{id})_* (\zeta)) \pi_\infty(a) \Xi_\infty.
\end{aligned}$$

$\square$
Define a positive function \( h_\sigma \in \ell_\infty(V) \) on \( \bigcup_n \text{dom}(\sigma_n) \subset V \) by
\[
h_\sigma(x) = \frac{\xi_n(\sigma(x))^2}{\xi_n(x)^2} = H(\sigma(x), x), \quad x \in \text{dom}(\sigma_n).
\]
By the assumption in Subsection 3.2, the function \( h_\sigma \) is bounded.

**Lemma 3.12.** The state \( \psi \) on \( \mathcal{B}(\mathcal{H}_\infty) \) satisfies the following.

1. For every \( a \in C_0^*(V) \), \( \psi(\pi_\infty(a)) = \phi_\infty(a) \).
2. The state \( \psi \) is equivariant with respect to the Radon–Nikodym derivative \( H \). More precisely, for every controlled partially defined injection \( \sigma \) on \( V \), for every \( c \in \mathcal{B}(\mathcal{H}_\infty) \), we have
\[
\psi(\pi_\infty(v_\sigma)c\pi_\infty(v_\sigma^*)) = \psi(\pi_\infty(h_\sigma)c).
\]

It turns out that the state \( \psi = \langle \Phi(\cdot)\Xi_\infty, \Xi_\infty \rangle \) is something similar to the hypertrace in [5, Section V].

**Proof.** Since \( \Phi \) is the identity map on \( \pi_\infty(C_0^*(V)) \), for every \( a \in C_0^*(V) \), we have
\[
\psi(\pi_\infty(a)) = \langle \Phi(\pi_\infty(a))\Xi_\infty, \Xi_\infty \rangle = \langle \pi_\infty(a)\Xi_\infty, \Xi_\infty \rangle = \phi_\infty(a).
\]
Because \( \Phi \) is a \( \pi_\infty(C_0^*(V)) \)-module map, for \( c \in \mathcal{B}(\mathcal{H}_\infty) \), we have
\[
\psi(\pi_\infty(v_\sigma)c\pi_\infty(v_\sigma^*)) = \langle \Phi(\pi_\infty(v_\sigma)c\pi_\infty(v_\sigma^*))\Xi_\infty, \Xi_\infty \rangle
\]
\[
= \langle \Phi(c)\pi_\infty(v_\sigma^*)\Xi_\infty, \pi_\infty(v_\sigma^*)\Xi_\infty \rangle.
\]
By Lemma 3.7, we have
\[
\psi(\pi_\infty(v_\sigma)c\pi_\infty(v_\sigma^*)) = \langle \Phi(c)J\Delta^{1/2}\pi_\infty(v_\sigma)\Xi_\infty, J\Delta^{1/2}\pi_\infty(v_\sigma)\Xi_\infty \rangle
\]
\[
= \langle \Phi(c)J\pi_\infty(M[H^{1/2}](v_\sigma))\Xi_\infty, J\pi_\infty(M[H^{1/2}](v_\sigma))\Xi_\infty \rangle.
\]
Since \( J \) is antilinear involutive isometry, we have
\[
\psi(\pi_\infty(v_\sigma)c\pi_\infty(v_\sigma^*)) = \langle \pi_\infty(M[H^{1/2}](v_\sigma))\Xi_\infty, J\Phi(c)J\pi_\infty(M[H^{1/2}](v_\sigma))\Xi_\infty \rangle.
\]
By Proposition 3.10, \( J\Phi(c)J \in J\pi_\infty(C_0^*(V))'J \) commutes with \( \pi_\infty(M[H^{1/2}](v_\sigma)) \). It follows that
\[
\psi(\pi_\infty(v_\sigma)c\pi_\infty(v_\sigma^*)) = \langle \pi_\infty(M[H^{1/2}](v_\sigma)^* \cdot M[H^{1/2}](v_\sigma))\Xi_\infty, J\Phi(c)J\Xi_\infty \rangle.
\]
The multiplication operator \( h_\sigma \) acting on \( \ell_2(V) \) is identical to \( M[H^{1/2}](v_\sigma)^* \cdot M[H^{1/2}](v_\sigma) \). Combining with \( J\Xi_\infty = \Xi_\infty \), we have
\[
\psi(\pi_\infty(h_\sigma)\Xi_\infty) = \langle \pi_\infty(h_\sigma)\Xi_\infty, J\Phi(c)\Xi_\infty \rangle = \langle \Phi(c)\Xi_\infty, J\pi_\infty(h_\sigma)\Xi_\infty \rangle.
\]
Because the operator \( h_\sigma \) acting on \( \ell_2(V) \) is diagonal, it satisfies
\[
\pi_\infty(h_\sigma)\Xi_\infty = \Delta^{1/2}\pi_\infty(h_\sigma)\Xi_\infty.
\]
It follows that
\[
\psi(\pi_\infty(v_\sigma)c\pi_\infty(v_\sigma^*)) = \langle \Phi(c)\Xi_\infty, J\Delta^{1/2}\pi_\infty(h_\sigma)\Xi_\infty \rangle
\]
\[
= \langle \Phi(c)\Xi_\infty, \pi_\infty(h_\sigma^*)\Xi_\infty \rangle
\]
\[
= \langle \pi_\infty(h_\sigma)\Phi(c)\Xi_\infty, \Xi_\infty \rangle
\]
\[
= \langle \Phi(\pi_\infty(h_\sigma)c)\Xi_\infty, \Xi_\infty \rangle.
\]
Thus, we conclude that \( \psi(\pi_\infty(v_\sigma)c\pi_\infty(v_\sigma^*)) = \psi(\pi_\infty(h_\sigma)c) \). \( \square \)
3.6. Almost equivariant $l_1$-functions on $\bigcup_n V_n^2$

We prove in Lemma 3.15 that for a finite family $\Sigma$ of controlled partially defined injections on $\bigcup_n V_n^2$, there exist a subsequence $(V_{n(k)})_k$ and a controlled sequence of probability measures $(p_{n(k)})_k$ on $V_{n(k)}^2$ which are almost equivariant under the action of $\Sigma$. We make use of the technique developed in [5, Lemma 2.4].

Denote by $e$ the rank one projection onto $\mathbb{C}\Xi_\infty \subset H_\infty$.

**Lemma 3.13.** There exists a net $(s_\nu)$ of density operators on $H_\infty$ satisfying the following conditions.

1. Every $s_\nu$ is a finite sum of rank one operators of the form $\pi_\infty(a)e\pi_\infty(a)^*$, where $a$ is an operator on $\ell_2(V)$ with finite propagation.
2. For every $c \in B(H_\infty)$, $\lim_\nu \text{Tr}(s_\nu c) = \psi(c)$.
3. For every controlled partially defined injection $\sigma$ on $V$,
   \[ \lim_\nu \text{Tr}(|s_\nu \pi_\infty(h_\sigma) - \pi_\infty(v_\sigma^*)s_\nu \pi_\infty(v_\sigma)|) = 0. \]

**Proof.** In the weak-$*$ topology, the state $\psi$ on $B(H_\infty)$ can be approximated by some net $(t_\nu)_{\nu \in I}$ of density operators on $H_\infty$. Because $\Xi_\infty$ is a cyclic vector of the algebra $\pi_\infty(\cup_R E_R)$, every density operator on $H_\infty$ can be approximated by finite sum of rank one operator of the form $\pi_\infty(a)e\pi_\infty(a)^*$, where $a \in \cup_R E_R$. We may assume that $t_\nu$ is such a finite sum. Let $\Sigma$ be a finite family of controlled partially defined injections on $V$. For every $\sigma \in \Sigma$, the trace class operators
   \[ t_\nu \pi_\infty(h_\sigma) - \pi_\infty(v_\sigma^*)t_\nu \pi_\infty(v_\sigma) \]
converges to 0, in the weak topology. Indeed, by Lemma 3.12, we have
   \[ \lim_\nu \text{Tr}(\{t_\nu \pi_\infty(h_\sigma) - \pi_\infty(v_\sigma^*)t_\nu \pi_\infty(v_\sigma)\}c) = \lim_\nu \text{Tr}(t_\nu \pi_\infty(h_\sigma)c) - \lim_\nu \text{Tr}(t_\nu \pi_\infty(v_\sigma)c\pi_\infty(v_\sigma)^*) = \psi(\pi_\infty(h_\sigma)c) - \psi(\pi_\infty(v_\sigma)c\pi_\infty(v_\sigma)^*) = 0, \quad c \in B(H_\infty). \]

For every $\nu \in I$, the weak closure of the set of convex combinations
   \[ \text{conv}\{t_\mu \pi_\infty(h_\sigma) - \pi_\infty(v_\sigma^*)t_\mu \pi_\infty(v_\sigma) | \mu \in I, \nu \leq \mu\} \subset \bigoplus_{\sigma \in \Sigma} B(H_\infty)^* \]
contains $(0)_{\sigma \in \Sigma}$. By the Hahn–Banach theorem, the weak closure of the convex set coincides with its norm closure. It follows that for every $\nu \in I$, there exists a density operator $s_\nu \in \text{conv}\{t_\mu | \mu \in I, \nu \leq \mu\}$ such that the net $(s_\nu)_{\nu \in I}$ converges to $\psi$ in the weak-$*$ topology of $B(H_\infty)^*$ and that
   \[ s_\nu \pi_\infty(h_\sigma) - \pi_\infty(v_\sigma^*)s_\nu \pi_\infty(v_\sigma) \]
converges to 0, in the trace 1-norm for every $\sigma \in \Sigma$. \hfill $\square$

**Lemma 3.14.** Let $s$ be a finite sum of rank one operators of the form $\pi_\infty(a)e\pi_\infty(a)^*$, where $a \in \cup_R E_R$. Then there exists a positive element $\eta$ of $l_\infty(\bigcup_n V_n^2)$ satisfying the following conditions.

1. The support of $\eta$ is controlled.
For every finite family $F$ of elements of $\ell_\infty(\sqcup_n V_n^2)$, there exists an increasing sequence $n(k)$ of natural numbers such that for every $\zeta \in F$,

$$\lim_{k \to \infty} \sum_{x,y \in V_n(k)} \zeta(x, y) \eta(x, y) \xi_{n(k)}(y)^2 = \text{Tr}(s\varpi_\infty(\zeta)).$$

Proof. Suppose that $s$ is of the form $s = \sum_{l=1}^{m} \pi_\infty(a^{(l)}) \pi_\infty(a^{(l)})^\ast$. Describe $a^{(l)}$ by $a^{(l)} = ([a^{(l)}]_{x,y} \in V_n)_{n \in \mathbb{N}}$. Define $\eta \in \ell_\infty(V)$ by $\eta(x, y) = \sum_{l=1}^{m} |a^{(l)}_{x,y}|^2$, $(x, y \in V_n^2)$. The support of $\eta$ is controlled, since the propagation of $a^{(l)}$ is finite. For every $\zeta \in \ell_\infty(\sqcup_n V_n^2)$, $\text{Tr}(s\varpi_\infty(\zeta))$ is equal to

$$\sum_{l=1}^{m} \left< \varpi_\infty(\zeta) \pi_\infty(a^{(l)}) \Xi_\infty, \pi_\infty(a^{(l)}) \Xi_\infty \right> = \sum_{l=1}^{m} \phi_\infty\left( a^{(l)}^\ast M[\zeta] a^{(l)} \right).$$

Let $F$ be an arbitrary finite family of elements of $\ell_\infty(\sqcup_n V_n^2)$. By the definition of $\phi_\infty$, there exists a subsequence $(\phi_{n(k)})$ satisfying that for every $\zeta \in F$,

$$\text{Tr}(s\varpi_\infty(\zeta)) = \lim_{k \to \infty} \sum_{l=1}^{m} \phi_{n(k)}\left( a^{(l)}^\ast M[\zeta] a^{(l)} \right).$$

Recalling the definition of $\phi_{n(k)}$, we have

$$\sum_{l=1}^{m} \phi_{n(k)}\left( a^{(l)}^\ast M[\zeta] a^{(l)} \right) = \sum_{l=1}^{m} \sum_{y \in V_n(k)} \left< M[\zeta] a^{(l)} \xi_{n(k)}(y) \delta_y, a^{(l)} \xi_{n(k)}(y) \delta_y \right>$$

$$= \sum_{l=1}^{m} \sum_{y \in V_n(k)} \sum_{x \in V_n(k)} \zeta(x, y) |a^{(l)}_{n(k), x,y}|^2 \xi_{n(k)}(y)^2$$

$$= \sum_{y \in V_n(k)} \sum_{x \in V_n(k)} \zeta(x, y) \eta(x, y) \xi_{n(k)}(y)^2. \quad \square$$

**Lemma 3.15.** Let $\Sigma$ be a finite family of controlled partially defined injections on $V$. Let $F$ be a finite family of elements in $\ell_\infty(V)$. Let $\epsilon$ be a positive number. For arbitrary $\Sigma$, $F$, and $\epsilon$, there exist

- a positive number $S$,
- an increasing sequence $(n(k))_k$ of natural numbers,
- and probability measures $p_{n(k)} \in \ell_1(V_{n(k)}^2_+)$ on $V_{n(k)}^2$,

satisfying the following conditions.

1. For every $k \in \mathbb{N}$, and for every $x, y \in V_n(k)$, if $d_{n(k)}(x, y) > S$, then $p_{n(k)}(x, y) = 0$.
2. For every $f \in F$, and for every $k \in \mathbb{N}$,

$$\left| \sum_{x \in V_n(k)} \sum_{y \in V_n(k)} f(x) p_{n(k)}(x, y) - \phi_\infty(f) \right| < \epsilon.$$  

3. For every $\sigma = (\sigma_n)_n$ in $\Sigma$, and for every $k \in \mathbb{N}$,

$$\| (\sigma \times \text{id})^\ast (p_{n(k)} \cdot \chi(\text{image}(\sigma_{n(k)}) \times V_{n(k)})) - p_{n(k)} \cdot \iota(h_{\sigma}) \|_1 < \epsilon.$$  

Here, $\chi(\cdot)$ stands for the definition function, and $\| \cdot \|_1$ stands for sum of absolute values of entries.
Proof. By Lemma 3.13, there exists a density operator $s$ such that

(i) $s$ is a finite sum of $\pi_\infty(a)e\pi_\infty(a)^*$, where $a \in \bigcup Re_R$;
(ii) for every $f \in F$,
$$|\text{Tr}(s\varpi_\infty(\ell(f))) - \phi_\infty(f)| = |\text{Tr}(s\pi_\infty(f)) - \psi_\infty(f)| < \epsilon;$$
(iii) for every $\sigma \in \Sigma$, $\text{Tr}(|s\pi_\infty(h_{\sigma}) - \pi_\infty(v_{\sigma}^\star)s\pi_\infty(v_{\sigma})|) < \epsilon$.  

By Lemma 3.14, there exists a positive element $\eta$ of $\ell_\infty(V)$ satisfying the following conditions.

(a) The support of $\eta$ is controlled.
(b) For every finite family $F$ of elements of $\ell_\infty(\bigcup_n V_n^2)$, there exists an increasing sequence $n(k)$ of natural numbers such that for every $\zeta \in F$,
$$\lim_{k \to \infty} \sum_{x,y \in V_n(k)} \zeta(x,y)\eta(x,y)\xi_{n(k)}(y)^2 = \text{Tr}(s\varpi_\infty(\zeta)).$$

We define a sequence of positive functions $p_n \in \ell_1(V_n^2)$ by $p_n(x,y) = \eta(x,y)\xi_{n(k)}(y)^2$. Since the support of $\eta$ is controlled, the union of the supports of $(p_n)_n$ is also controlled.

We define $F$ by the following collections.

- The constant function 1 on $\bigcup_n V_n^2$.
- Functions of the form $\ell(f)$, $f \in F$.
- For $\sigma = (\sigma_n) \in \Sigma$, choose a function $\zeta_\sigma \in \ell_\infty(\bigcup_n \text{dom}(\sigma_n) \times V_n)$ such that $\|\zeta_\sigma\|_\infty = 1$ and that for every $n \in \mathbb{N}$,
$$\|(\sigma_n \times \text{id})^*(p_n \cdot \chi(\text{image}(\sigma_n) \times V_n)) - p_n \cdot \ell(h_{\sigma})\|_1 = \sum_{x \in \text{dom}(\sigma_n)} \sum_{y \in V_n} (p_n(\sigma_n(x), y) - p_n(x,y)h_{\sigma}(x))\zeta_\sigma(x,y).$$

Add $(\sigma \times \text{id})_*(\zeta_\sigma)$ and $\zeta_\sigma \cdot \ell(h_{\sigma})$ to $F$.

For this collection $F$, choose an increasing sequence $n(k)$ of natural numbers satisfying the equation in condition (b). By $1 \in F$, and by condition (b), we have
$$\lim_{k \to \infty} \|p_{n(k)}\|_1 = \lim_{k \to \infty} \sum_{x,y \in V_n(k)} \eta(x,y)\xi_{n(k)}(y)^2 = \text{Tr}(s\varpi_\infty(1)) = 1.$$

For $f \in F$, we have
$$\lim_{k \to \infty} \left| \sum_{x \in V_n(k)} \sum_{y \in V_n(k)} f(x)p_{n(k)}(x,y) - \phi_\infty(f) \right|$$
$$= \lim_{k \to \infty} \left| \sum_{x \in V_n(k)} \sum_{y \in V_n(k)} f(x)\eta(x,y)\xi_{n(k)}(y)^2 - \phi_\infty(f) \right|$$
$$= |\text{Tr}(s\varpi_\infty(\ell(f))) - \phi_\infty(f)| < \epsilon.$$

At the inequality in the last line, we used condition (ii). For every $\sigma \in \Sigma$, we have
$$\lim_{k \to \infty} \|(\sigma_n \times \text{id})^*(p_{n(k)} \cdot \chi(\text{image}(\sigma_n(k) \times V_n(k))) - p_{n(k)} \cdot \ell(h_{\sigma}))\|_1$$
$$= \lim_{k \to \infty} \sum_{x \in \text{dom}(\sigma_n(k))} \sum_{y \in V_n(k)} (p_{n(k)}(\sigma_n(k)(x), y) - p_{n(k)}(x,y)h_{\sigma}(x))\zeta_{\sigma}(x,y)$$
\[ \lim_{k \to \infty} \sum_{x \in \operatorname{image}(\sigma_n(k))} \sum_{y \in V_n(k)} p_n(k)(x, y) \zeta_\sigma \left( \sigma_n^{-1}(k)(x), y \right) \]

\[ - \lim_{k \to \infty} \sum_{x \in \operatorname{dom}(\sigma_n(k))} \sum_{y \in V_n(k)} p_n(k)(x, y) h_\sigma(x) \zeta_\sigma(x, y) \]

\[ = \operatorname{Tr}(s \varpi_\infty((\sigma \times \text{id})_*(\zeta_\sigma))) - \operatorname{Tr}(s \varpi_\infty((\iota_\sigma) \cdot \zeta_\sigma))). \]

At the equality in the last line, we used condition (b). By Lemma 3.11, we obtain

\[ \lim_{k \to \infty} \|((\sigma_n(k) \times \text{id})^* (p_n(k) \cdot \chi(\operatorname{image}(\sigma_n(k)) \times V_n(k))) - p_n(k) \cdot \iota(h_\sigma)\|_1 \]

\[ = \operatorname{Tr}(s \pi_\infty(v_\sigma) \varpi_\infty(\zeta_\sigma) \pi_\infty(v_\sigma)^*) - \operatorname{Tr}(s \pi_\infty(h_\sigma) \varpi_\infty(\zeta_\sigma)) \]

\[ \leq \operatorname{Tr}(\|\pi_\infty(v_\sigma)^* s \pi_\infty(v_\sigma) - s \pi_\infty(h_\sigma)\|) \cdot \|\varpi_\infty(\zeta_\sigma)\| < \epsilon. \]

At the last inequality, we used condition (iii). Therefore, if \( k \) is large, then the sequence of probability measures \( p_n(k)/\|p_n(k)\|_1 \) satisfies the conditions in the lemma. \( \square \)

### 3.7. Another state \( \omega_\infty \) of \( C^*_u(V) \)

Let \( \xi_n \in \ell_2(V_n) \) be the sequence of positive unit vectors satisfying the conditions in Subsection 3.2. We have already studied a sequence of states \( (\phi_n)_n \) of \( C^*_u(V) \otimes_{\min} C^*_u(V)^{\text{op}} \) given by the vectors

\[ \Xi_n = \sum_{x \in V_n} \xi_n(x) \delta_{(x,x)} \in \ell_2(V_n^2). \]

Here, we consider another sequence of states \( \omega_n = (\cdot, \xi_n, \xi_n) \) on \( C^*_u(V) \) given by the vectors \( \xi_n \in \ell_2(V_n) \). The restriction of \( \phi_n \) on the first tensor factor \( C^*_u(V) \) and \( \omega_n \) are not identical, but these restrictions on the diagonal subalgebra \( \ell_\infty(V) \) are identical. We can choose an accumulation point \( \phi_\infty \) of \( (\phi_n)_n \subseteq (C^*_u(V) \otimes_{\min} C^*_u(V)^{\text{op}})^* \) and an accumulation point \( \omega_\infty \) of \( (\omega_n)_n \subseteq C^*_u(V)^* \) such that these restrictions \( \phi_\infty|_{\ell_\infty(V) \otimes \mathbb{C}} \) and \( \omega_\infty|_{\ell_\infty(V)} \) on \( \ell_\infty(V) \) are identical. The following lemma implies that \( \omega_\infty|_{\ell_\infty(V)} \) determines \( \omega_\infty \).

**Lemma 3.16.** For every controlled partially defined injection \( \sigma = (\sigma_n)_n \) on \( V = \sqcup_n V_n \), and for every \( f \in \ell_\infty(V) \), we have \( \omega_\infty(v_\sigma f) = \omega_\infty(\sqrt{h_\sigma} f) \).

**Proof.** For every \( n \in \mathbb{N} \), we have

\[ \omega_n(v_\sigma f) = \sum_{x \in \operatorname{dom}(\sigma_n)} \xi_{\sigma_n(x)}(x) \xi_{\sigma_n(x)} f(x) = \sum_{x \in \operatorname{dom}(\sigma_n)} \xi_n^2 \sqrt{h_\sigma(x)} f(x) = \omega_n \left( \sqrt{h_\sigma} f \right). \]

Therefore, we have \( \omega_\infty(v_\sigma f) = \omega_\infty(\sqrt{h_\sigma} f) \). \( \square \)

### 3.8. Localization property of the sequence of states \( (\omega_n)_n \) on \( C^*_u(V) \)

For every \( n \in \mathbb{N} \), and positive number \( S \), we denote by \( \Theta_{n,S} \) the unital completely positive map

\[ \Theta_{n,S} : \mathcal{B}(\ell_2(V_n)) \to \prod_{y \in V_n} \mathcal{B}(\ell_2(N(y; S))) \]

given by compression \( \Theta_{n,S}(a) = (P_{N(y; S)} a P_{N(y; S)})_{y \in V_n} \).

**Theorem 3.17.** Let \( (G_n = (V_n, \text{Edge}_n))_n \) be a sequence of finite connected graphs. Suppose that \( V = \sqcup_n V_n \) has bounded geometry. Let \( (\xi_n)_n \) be a sequence of unit vectors in \( \ell_2(V_n) \). Let \( G \)
be a finite family of elements in the uniform Roe algebra $C^*(V)$. Let $\epsilon$ be a positive number. If $C^*_\alpha(V)$ is locally reflexive, then for arbitrary $(\xi_n)_n$, $G$, and $\epsilon$, there exists a positive number $S$, an increasing sequence of natural numbers $(n(k))_k$, and states $\hat{\omega}_{n(k)}$ on $\prod_{y \in V_{n(k)}} B(\ell_2(N(y); S))$ such that

$$|\hat{\omega}_{n(k)}(\Theta_{n(k),S}(a)) - \langle a\xi_{n(k)}, \xi_{n(k)} \rangle| < \epsilon, \quad a \in G.$$ 

In this theorem, we do not need any assumption on the unit vectors $\xi_n \in \ell_2(V_n)$.

Proof. We first reduce our task as follows.

1. We may assume that all the elements of $G$ have finite propagation.
2. Adding edges to $G_n$, we may further assume that the propagation of elements of $G$ is at most 1.
3. There exist a finite family $G$ of $\ell_\infty(V)$ and a finite family $\Sigma$ of partially defined injections on $V$ such that for every $\sigma \in \Sigma$, the propagation of $v_\sigma$ is at most 1 and that the linear span of $\{v_\sigma f \mid \sigma \in \Sigma, f \in G\}$ contains $G$. We replace $G$ with $\{v_\sigma f \mid \sigma \in \Sigma, f \in G\}$.
4. We may assume that there exists a positive number $\Lambda$ such that for every $n \in \mathbb{N}$ and for every $(x, y) \in \text{Edge}_n$,

$$|\xi_n(y)|^2 \leq \Lambda|\xi_n(x)|^2.$$ 

5. We may assume that for every $n$, the entries of $\xi_n$ are positive.
6. We may assume that for $v_\sigma f \in G$, the sequence $(\langle v_\sigma f \xi_n, \xi_n \rangle)_n$ converges.

We prove that the above reduction processes are possible as follows.

1. All the elements of $G$ can be approximated by some operators with finite propagation.
2. For every propagation $1 \leq R$, define new edge sets $\text{Edge}_n^{(R)}$ by

$$\text{Edge}_n^{(R)} = \{(x, y) \in V_n \mid d_n(x, y) \leq R\}.$$ 

The sequence of graphs $((V_n, \text{Edge}_n^{(R)}))_n$ also has bounded geometry. The uniform Roe algebras of these sequences of vertex sets are identical.

3. Every controlled subset of $\bigcup_n V_n$ is a finite union of graphs of controlled partially defined injections, by [18, Lemma 4.10].

4. It suffices to show that there exists a subset $W_n$ of $\text{supp}(\xi_n)$ such that $\xi_n|_{W_n}$ is uniformly close to $\xi_n$ in $\ell_2$-norm and that the derivatives $|\xi_n(y)|/|\xi_n(x)|$ defined on $(x, y) \in \text{Edge}_n^{(R)}$ are uniformly bounded.

Let $(V, \text{Edge})$ be a finite connected graph. Let $\xi$ be a unit vector in $\ell_2(V)$ whose support is $V$. Let $\epsilon$ be an arbitrary positive number. Define $M$ by

$$\sup_{x \in V} \|\xi\rangle \langle \xi\rangle \mid (x, y) \in \text{Edge}\}.$$ 

Let $\Lambda$ be a positive constant larger than $M$ and 1. The constant $\Lambda$ will be determined by $\epsilon$ and $M$ later. Define a relation $\ll$ on $V$ as follows:

$$x \ll y, \text{ if } (x, y) \in \text{Edge and } \Lambda|\xi(x)|^2 \leq |\xi(y)|^2.$$ 

Define a sequence of disjoint subsets $(U(l))_l$ of $V$ as follows:

$$U(0) = \{y \in V \mid \exists x \in V, x \ll y, \neg\exists z \in V, y \ll z\},$$
$$U(1) = \{y \in V \mid \exists z \in U(0), y \ll z\},$$
$$U(l + 1) = \{y \in V \mid \exists z \in U(l), y \ll z\} \setminus \bigcup_{k=1}^l U(k).$$
By the definitions of $\ll$ and $M$, we have $\|\xi_{U(l+1)}\|^2 \leq \frac{M'}{M} \|\xi_{U(l)}\|^2$. It follows that
\[
\sum_{l=1}^{\infty} \|\xi_{U(l)}\|^2 \leq \sum_{l=1}^{\infty} \frac{M'}{M} \|\xi_{U(l)}\|^2 \leq \frac{M'}{M} \sum_{l=1}^{\infty} \|\xi_{U(l)}\|^2.
\]
Define $\Lambda$ by $(1 + 1/\epsilon)M$. Define $W$ by $V \setminus (\cup_{l=1}^{\infty} U(l))$. Thus, we have $\|\xi - \xi_{W}\|^2 \leq \sum_{l=1}^{\infty} \|\xi_{U(l)}\|^2 \leq \epsilon$. If $x, y \in W$, then $x \ll y$ does not hold. This means that $\|\xi(y)\|^2 \leq \Lambda \|\xi(x)\|^2$.

Apply the above procedure to $\text{supp}(\xi_n)$. Replace $\text{supp}(\xi_n)$ with $W_n$. We can use common $\Lambda$ for every $n$. Replace $\xi$ with $\xi_{W_n}$. Replace $G = \{v_\sigma f \mid \sigma \in \Sigma, f \in G\}$ with their restrictions on the new Hilbert space $\ell_2(U_n W_n)$. The propagation of elements of the new finite family $\mathcal{G}$ is 1.

(5) Multiplying a unitary operator $w$ in $\ell_\infty V$, we obtain a sequence of positive vectors $\xi_n \in \ell_2(V_n)$. We also need to replace $G = \{v_\sigma f \mid \sigma \in \Sigma, f \in G\}$ with their conjugations by $wv_\sigma f w^* = v_\sigma \overline{f}$.

(6) Replace $(V_n)_n$ with its subsequence such that $\lim_n \langle v_\sigma f, \xi_n \rangle$ exists for every $v_\sigma f \in \mathcal{G}$.

For the rest of the proof, we assume the above six items. We make use of states $\phi_\infty$ and $\omega_\infty$ in the previous subsections. Define $F \in \ell_\infty(V)$ by $\{\sqrt{h_\sigma} f \mid \sigma \in \Sigma, f \in G\}$. For $F$, $\Sigma$, and arbitrary $\epsilon > 0$, there exist

- a positive number $S$,
- an increasing sequence $(n(k))_k$ of natural numbers,
- and probability measures $p_{n(k)} \in \ell_1(V_n^2)$ on $V_n^2$,

satisfying the conditions in Lemma 3.15.

We first claim that for every $k$, and for every $\sigma = (\sigma_n)_n \in \Sigma$,
\[
\sum_{x \in \text{dom}(\sigma_n)} \sum_{y \in V_n} \left| \sqrt{h_\sigma(x)} p_{n(k)}(x, y) - \sqrt{p_{n(k)}(x, y)} \right| p_{n(k)}(\sigma_n(x), y)
\]
is small. By condition (4), $h_\sigma$ is at least $\Lambda^{-1}$. The above quantity is equal to
\[
\sum_{x, y} \left| \sqrt{h_\sigma(x)} p_{n(k)}(x, y) - \sqrt{p_{n(k)}(\sigma_n(x), y)} \right| p_{n(k)}(x, y)
\]
and bounded by
\[
\sqrt{\Lambda} \sum_{x, y} \left| \sqrt{h_\sigma(x)} p_{n(k)}(x, y) - \sqrt{p_{n(k)}(\sigma_n(x), y)} \right| p_{n(k)}(x, y)
\leq \sqrt{\Lambda} \sum_{x, y} \left| \sqrt{h_\sigma(x)} p_{n(k)}(x, y) - \sqrt{p_{n(k)}(\sigma_n(x), y)} \right| p_{n(k)}(x, y)
\]
\[
\times \left( \sqrt{h_\sigma(x)} p_{n(k)}(x, y) + \sqrt{p_{n(k)}(\sigma_n(x), y)} \right)
\]
\[
= \sqrt{\Lambda} \sum_{x, y} \left| h_\sigma(x) p_{n(k)}(x, y) - p_{n(k)}(\sigma_n(x), y) \right| < \sqrt{\Lambda} \epsilon.
\]
The equality in the last line is due to the last condition in Lemma 3.15.

The limit $\lim_k \langle v_\sigma f, \xi_{n(k)} \rangle$ satisfies the following:
\[
\lim_k \langle v_\sigma f, \xi_{n(k)} \rangle = \omega_\infty (v_\sigma f) = \omega_\infty \left( \sqrt{h_\sigma f} \right) = \phi_\infty \left( \sqrt{h_\sigma f} \right)
\]
\[
\sim_{\epsilon} \sum_{x \in \text{dom}(\sigma_n)} \sum_{y \in V_n} \sqrt{h_\sigma(x)} f(x) p_{n(k)}(x, y)
\]
\[
\sim \sqrt{\Lambda} \epsilon \|f\|_{\infty} \sum_{x \in \text{dom}(\sigma_n)} \sum_{y \in V_n} f(x) p_{n(k)}(x, y) p_{n(k)}(\sigma_n(x), y).
\]
Here $\sim$ means that the difference between the left-hand side and the right-hand side is at most $\square$. Define a unit vector $(q_y)_y$ in $\bigoplus_{y \in V_n(k)} \ell_2(N(y,S))$ by $q_y = \sum_x \sqrt{p_{n(k)}(x,y)} \delta_x$. For every $x,y$, we have
\[ f(x) \sqrt{p_{n(k)}(x,y)} = \left\langle v_\sigma f \sqrt{p_{n(k)}(x,y)} \delta_x, q_y \right\rangle. \]
For every $y$, we have
\[ \sum_x f(x) \sqrt{p_{n(k)}(x,y)} = \left\langle v_\sigma f q_y, q_y \right\rangle. \]
Therefore, we have
\[ \lim_k \left\langle v_\sigma f \xi_{n(k)}, \xi_{n(k)} \right\rangle \sim \sqrt{\sum_{\delta} \|f\|_{\infty} + \epsilon} \sum_{y \in V_n(k)} \left\langle v_\sigma f q_y, q_y \right\rangle. \]
Let $\hat{\omega}_{n(k)}$ be the vector state on $\prod_{y \in V_n(k)} \mathcal{B}(\ell_2(N(y,S)))$ given by the unit vector $(q_y)_y \in \bigoplus_{y \in V_n(k)} \ell_2(N(y,S))$. The right-hand side $\sum_{y \in V_n(k)} \left\langle v_\sigma f q_y, q_y \right\rangle$ is equal to
\[ \hat{\omega}_{n(k),S}(\Theta_{n(k)}(v_\sigma f)). \]

4. Main theorem

**Theorem 4.1.** For a metric space $X$ with bounded geometry, if the uniform Roe algebra $\text{C}_u^*(X)$ is locally reflexive, then $X$ has the operator norm localization property.

As a consequence, we obtain the main theorem in Section 1.

**Proof.** Suppose that $X$ does not have the operator norm localization property and that $\text{C}_u^*(X)$ is locally reflexive. By Lemma 2.8, there exist:

- a sequence of disjoint subsets $V_n$ of $X$;
- a sequence of positive matrices $b_n$ acting on $\ell_2(V_n)$ with norm 1;
- a sequence $S_n$ of positive numbers

satisfying that:

1. for every $n \in \mathbb{N}$ and $Y \subset V_n$, if $\text{diam}(Y) \leq S_n$, then $\|b_n|_{\ell_2(Y)}\| < 1/3$;
2. $\lim_n S_n = \infty$;
3. the propagation of $b := \sum_n b_n$ is finite.

Denote by $R$ the propagation of $b$.

Define a sequence of graphs $(G_n^{(R)} = (V_n, \text{Edge}_n^{(R)}))$ by
\[ \text{Edge}_n^{(R)} = \{(x,y) \in V_n \mid d(x,y) \leq R\}. \]
In the case that $G_n^{(R)}$ is not connected, choose a connected component $W_n \subset V_n$, such that the norm of $b_n P_{W_n}$ is 1. Replace $V_n$ with $W_n$. Replace $b$ with $b_n P_{W_n}$. Denote by $d_n$ the graph metric on $V_n$ defined by $\text{Edge}_n^{(R)}$. Since $d|_{V_n \times V_n} \leq Rd_n$, the uniform Roe algebra $\text{C}_u^*(\bigcup_n (V_n, d_n))$ is a non-unital subalgebra of $\text{C}_u^*(X)$. Therefore, $\text{C}_u^*(\bigcup_n (V_n, d_n))$ is also locally reflexive. Note that $b$ is an element of the subalgebra.

Since $b_n$ are positive semidefinite matrices with norm 1, there exists a unit vector $\xi_n \in \ell_2(V_n)$ satisfying that $b_n \xi_n = \xi_n$. By Theorem 3.17, there exist a positive constant $S$, infinitely many natural numbers $n$, and states $\hat{\omega}_n$ on $\prod_{y \in V_n} \mathcal{B}(\ell_2(N(y,S)))$ such that
\[ |\hat{\omega}_n(\Theta_n,S(b)) - \langle b \xi_n, \xi_n \rangle| < 1/3. \]
Choosing large $n$, we also have $S < S_n$. Since $\langle b \xi_n, \xi_n \rangle = 1$, we have
\[ \| \Theta_{n,S}(b) \| \geq |\hat{w}_n(\Theta_{n,S}(b))| > 2/3. \]
This inequality contradicts condition (1). □

References
1. C. Anantharaman-Delaroche and J. Renault, *Amenable groupoids*, Monographs of L’Enseignement Mathématique 36 (L’Enseignement Mathématique, Geneva, 2000). (With a foreword by G. Skandalis and Appendix B by E. Germain.)
2. J. Brodzki, G. A. Niblo and N. J. Wright, ‘Property A, partial translation structures, and uniform embeddings in groups’, *J. Lond. Math. Soc.* (2) 76 (2007) 479–497.
3. N. P. Brown and N. Ozawa, *$C^*$-algebras and finite-dimensional approximations*, Graduate Studies in Mathematics 88 (American Mathematical Society, Providence, RI, 2008).
4. X. Chen, R. Tessera, X. Wang and G. Yu, ‘Metric sparsification and operator norm localization’, *Adv. Math.* 218 (2008) 1496–1511.
5. E. G. Effros and U. Haagerup, ‘Lifting problems and local reflexivity for $C^*$-algebras’, *Duke Math. J.* 52 (1985) 103–128.
6. K. J. Dykema, ‘Exactness of reduced amalgamated free product $C^*$-algebras’, *Forum Math.* 16 (2004) 161–180.
7. E. Guentner, N. Higson and S. Weinberger, ‘The Novikov conjecture for linear groups’, *Publ. Math. Inst. Hautes Études Sci.* 101 (2005) 243–268.
8. E. Kirchberg, ‘On nuclear $C^*$-algebras’, *J. Funct. Anal.* 129 (1995) 35–63.
9. E. Kirchberg and S. Wassermann, ‘Exact groups and continuous bundles of $C^*$-algebras’, *Math. Ann.* 315 (1999) 169–203.
10. C. Lance, ‘On nuclear $C^*$-algebras’, *J. Funct. Anal.* 12 (1973) 157–176.
11. N. Ozawa, ‘Amenable actions and exactness for discrete groups’, *C. R. Acad. Sci. Paris Sér. I Math.* 330 (2000) 691–695.
12. N. Ozawa, ‘Boundary amenability of relatively hyperbolic groups’, *Topology Appl.* 153 (2006) 2624–2630.
13. J. Roe, *Lectures on coarse geometry*, University Lecture Series 31 (American Mathematical Society, Providence, RI, 2003).
14. J. Roe and R. Willett, ‘Ghostbusting and property A’, *J. Funct. Anal.* 266 (2014) 1674–1684.
15. G. Skandalis, J.-L. Tu and G. Yu, ‘The coarse Baum-Connes conjecture and groupoids’, *Topology* 41 (2002) 807–834.
16. M. Takesaki, *Tomita’s theory of modular Hilbert algebras and its applications*, Lecture Notes in Mathematics 128 (Springer, Berlin, 1970).
17. J.-L. Tu, ‘Remarks on Yu’s “property A” for discrete metric spaces and groups’, *Bull. Soc. Math. France* 129 (2001) 115–139.
18. G. Yu, ‘The coarse Baum-Connes conjecture for spaces which admit a uniform embedding into Hilbert space’, *Invent. Math.* 139 (2000) 201–240.
Hiroki Sako
Faculty of Engineering
Niigata University
Niigata 9502181
Japan
sako@eng.niigata-u.ac.jp