Critical formation of trapped surfaces in collisions of non-expanding gravitational shock waves in de Sitter space-time

I. Ya. Aref’eva,¹ A. A. Bagrov,¹ and E. A. Guseva²

¹Steklov Mathematical Institute, Gubkin str. 8, 119991, Moscow, Russia.
²Department of Physics, Moscow State University, 119991, Moscow, Russia.
E-mail: arefeva@mi.ras.ru, bagrov.andrey@gmail.com, g14238f@gmail.com

ABSTRACT: We study the formation of marginally trapped surfaces in the head-on collision of two shock waves in de Sitter space-time as a function of the cosmological constant and the shock wave energy. We search for a marginally trapped surface on the past light cone of the collision plane. For space-time dimensions $D \geq 3$, there exists a critical value of the shock wave energy above which there is no trapped surface of this type. For $D > 3$, the critical value of the shock wave energy depends on the de Sitter radius, and there is no this type trapped surface formation for a large cosmological constant. For $D = 3$, the critical value of the shock wave energy is independent of the cosmological constant. At the critical point, the trapped surface is finite. Below the critical energy value, the area of the trapped surface depends on the cosmological constant and the shock wave energy.

KEYWORDS: de Sitter, Trapped Surfaces, Collision of Shock Waves, Black Holes.
## Contents

1. Introduction 2
2. Setup 3
3. Marginally trapped surface in dS
   - 3.1 Solution of the marginally trapped surface equation in dS\(_4\) 6
   - 3.2 Higher-dimension cases 8
   - 3.3 Area of the marginally trapped surface below the critical point 10
4. Concluding Remarks 11
5. Appendices 12
   A. Geometric view of shock waves in plane coordinates 12
   B. Solution of the geodesics equation 13
      - B.1 The \(\sigma\)-model (\(n\)-field) approach 13
      - B.2 Focusing of geodesics 15
   C. The shock wave in the independent coordinates 16
      - C.1 From plane coordinates to independent coordinates 16
      - C.2 An analogue of the D'Eath and Payne coordinates (from the coordinates \(w, \sigma, \zeta\) to the coordinates \(W, \Sigma, \Upsilon\)) 16
      - C.3 Geodesics in terms of the independent complex conformal flat coordinates 17
      - C.4 Geodesics in terms of the independent smooth coordinates 18
   D. Trapped surface equation 18
1. Introduction

It is a well-established experimental fact that our current universe is expanding with constant acceleration, which is well described by the extremely small cosmological constant $\Lambda = 10^{-47}\text{GeV}^4$, $\Omega_\Lambda = 0.726 \pm 0.015$ [1]. There is a common opinion that two ultrarelativistic point particles in the Minkowski space-time can produce black holes [2, 3, 4]. This longstanding question has a purely theoretical meaning as well as an astrophysical one [5]. In the framework of TeV-gravity [6], black hole production in collisions of particles with the center-of-mass energy of a few TeV and their experimental signatures [7] became the subject of numerous investigations [8, 9, 10, 11, 12]. We also note a discussion of the possible production of wormholes and other more exotic objects at the LHC [13, 14, 15] (see [16] for a consideration of wormholes in astrophysics).

Our main aim in this paper is to determine if the presence of a positive cosmological constant can influence the processes of black hole formation in a collision of two ultrarelativistic point particles. Intuitively, it is almost clear that a small cosmological constant cannot have a detectible influence. Meanwhile, a positive cosmological constant generates a repulsion of matter, and a critical value of the cosmological constant $\Lambda$ should exist above which black hole formation is suppressed. Finding this critical value is worthwhile. We note that under the assumption of an asymptotically flat space-time, the presence of a trapped surface usually guarantees the existence of the event horizon [17, 18, 19, 20].

We study the formation of marginally trapped surfaces in a head-on collision of two shock waves in the de Sitter space-time as a function of the cosmological constant and the square of the shock wave energy. For $D = 4$, we show explicitly that there exists a critical value for the ratio of the shock wave energy $\bar{p}$ and the cosmological radius

$$\frac{\bar{p}}{a} = \frac{1}{4G}$$

above which there is no solution of the trapped surface equation. Here $G$ is the gravitational constant and the cosmological radius $a$ is related to the cosmological constant by $a^2 = 3/\Lambda$. Similar results were found for $D > 4$. There is a critical value for the shock wave energy above which there are no solutions of the trapped surface equation. This value depends on the cosmological radius. At the critical point, the trapped surface is finite.

This effect is similar to the emergence of critical behavior with respect to the wave width in the transverse space when a marginally trapped surface is formed in the head-on collision of two shock waves in the Minkowski space-time [21, 22]. This effect depends on the number of dimensions and is a Choptuik-like critical effect, i.e., it is similar to the critical Choptuik behavior in gravitational collapse [23] (see [24] and the references therein).
The formation of a marginally trapped surface in the collision of gravitational shock waves in $\text{AdS}_D$ was recently studied in [24, 25]. These studies are aimed at better understanding the entropy production in relativistic heavy ion collisions due to black hole production in a dual description. Despite the absence of a holographic dual description of QCD, describing the colliding heavy ions in terms of colliding gravitational shock waves in the anti-de Sitter space-time was suggested [27, 28]. Black hole formation in collisions of the dual of the nuclei in the bulk is interpreted as formation of a quark-gluon plasma [29, 30, 31]. In AdS, a dimension-dependent critical behavior with respect to the wave width in the transverse space in the formation of a marginally trapped surface in the head-on collision of two shock waves was recently found [21]. For $D = 4$ and $D = 5$, there exists a critical value of this width above which the trapped surface never forms. We note that in the AdS space-time, the obtained results are qualitatively the same as those obtained in the flat space-time.

This paper is organized as follows. We start with the setup and recall some basic facts about the generalization of the Aichelburg–Sexl shock wave geometry [32, 33, 34] to nonexpanding shock waves propagating in $D$-dimensional space-times with the cosmological constant [35, 36, 37, 38, 39, 40, 43, 42]. In Sect. 3, we calculate the critical value of cosmological constant depending on the shock wave energy below which the trapped surface occurs and give the area of the trapped surface.

2. Setup

Our main aim in this section is to present the setup for studying the formation of closed trapped surfaces in the head-on collision of two shock waves in dS space (this may be compared with the setup used to study the case without a cosmological constant in [11] and the case with the negative cosmological constant in [43, 25, 21]).

We briefly recall the results in [35, 38, 12] for the geometry of a shock wave propagating in the $D$-dimensional dS space-time (see Appendix A). In terms of the dependent plane coordinates, $(u, v, \bar{x})$, $\bar{x} = (x^2, \ldots, x^D)$, satisfying $-2uv + \bar{x}^2 = a^2$ ($a$ is the cosmological radius), the line element of the shock wave space-time is

$$ds^2 = -2du dv + d\bar{x}^2 + F(\bar{x})\delta(u)du^2. \tag{2.1}$$

The shock wave shape function $F$ is a fundamental solution of the equation

$$\left(\Delta_{S^{D-2}} + \frac{D-2}{a^2}\right) F = -16\sqrt{2}\pi G_D \bar{p} \delta(\bar{n}, \bar{n}_0), \tag{2.2}$$

where $\Delta_{S^{D-2}}$ is the Laplace–Beltrami operator on a $(D-2)$-dimensional sphere $S^{D-2}$, $\bar{n} = \bar{x}/|\bar{x}|$, $\bar{n}_0$ is the location of the particle on the sphere, $\bar{p}$ is the energy of the shock wave, and $G_D$ is the $D$-dimensional gravitational constant. The $\sqrt{2}$ in the right-hand side results from our choice of the coordinate system (if we impose $-du dv$
instead $-2du\,dv$ in the expression for the linear element, then it disappears). This metric is a solution of the Einstein equations for an energy-momentum tensor with the single nonvanishing component $T_{uu} \sim \bar{p}\delta(u)$. In the standard parameterization of the $(D-2)$-dimensional sphere by spherical angles $\vartheta_1, \ldots, \vartheta_{D-2}$, the shock wave shape function corresponding to an ultrarelativistic point particle depends on only one spherical angle $\vartheta_{D-2}$. The operator $\triangle_{\mathbb{S}^{D-2}}$ acts on $F = F(\vartheta_{D-2})$ as

$$\triangle_{\mathbb{S}^{D-2}}F = \frac{1}{a^2} \sin^{3-D}(\vartheta_{D-2}) \left( \frac{d}{d\vartheta_{D-2}} \sin^{D-3}(\vartheta_{D-2}) \frac{d}{d\vartheta_{D-2}} \right) F(\vartheta_{D-2}) \quad (2.3)$$

(see [37] regarding shock waves in dS/AdS with multipole structures).

For $D = 4$, we deal with

$$F(\xi) = 4\sqrt{2}p \left( -2 + \xi \ln \left( \frac{1 + \xi}{1 - \xi} \right) \right), \quad (2.4)$$

where $\xi = x^4/a = \cos \vartheta_2$ and $p = \bar{p}G_4$ is the rescaled energy.

We now consider a collision of two waves of the type described above. We suppose that in the region $\{u < 0\} \cup \{v < 0\}$, i.e., the part of the space-time before the collision, the metric is given by

$$ds^2 = -2du\,dv + d\vec{x}^2 + F(\xi, \xi_1)\delta(u)du^2 + F(\xi, \xi_2)\delta(v)dv^2, \quad (2.5)$$

where $\xi_1$ and $\xi_2$ are the locations of the two colliding particles (see [35] for the explicit formula for $F(\xi, \xi_i)$). In independent coordinates (see Appendix C.1), the metric is

$$ds^2 = \frac{-2dw\,d\sigma + 2d\zeta\,d\bar{\zeta} + 2H_1(\zeta, \bar{\zeta})\delta(w)dw^2 + 2H_2(\zeta, \bar{\zeta})\delta(\sigma)d\sigma^2}{[1 - (w\sigma - \zeta\bar{\zeta})/2a^2]^2}, \quad (2.6)$$

where $H_i(\zeta, \bar{\zeta}) = H(\zeta, \bar{\zeta}, \zeta_i, \bar{\zeta}_i)$, $\zeta_i = \zeta(\xi_i, \bar{\xi}_i)$, and

$$H(\zeta, \bar{\zeta}, 0, 0) = H(\zeta, \bar{\zeta}) = \frac{1}{2} \left( \frac{1}{1 + \frac{1}{2a^2}\zeta\bar{\zeta}} \right) F(\frac{1 - \zeta\bar{\zeta}/2a^2}{1 + \zeta\bar{\zeta}/2a^2}). \quad (2.7)$$

A rigorous analysis of the formation of black holes in collisions would require solving the Einstein equations in the interaction region $\{w > 0, \sigma > 0\}$ (see, e.g., [44, 41] and the references therein).

A sufficient condition for a black hole to form in the asymptotically flat case is the existence of a marginally closed trapped surface at the hypersurface $\{w \leq 0, \sigma = 0\} \cup \{w = 0, \sigma \leq 0\}$ [44, 43, 11, 13, 25]. We note that in non-asymptotically flat cases, there are no general theorems, but there is a common opinion that the existence of a marginally trapped surface can be used as an indication of black hole formation.

In the coordinates used in line element (2.6) null geodesics are discontinuous across the wave fronts, $w = 0$ and $\sigma = 0$ (see Appendix C). So using such coordinates to find this trapped surface equation is inconvenient and can be avoided by
switching to a new coordinate system \((W, \Sigma, \Upsilon, \bar{\Upsilon})\) in which the delta function terms are eliminated in the metric and the geodesics are continuous. Similar to the D’Eath and Payne coordinates \([43]\), which are closely related to the explicit form of geodesics in the Minkowski space-time with the shock wave \([34, 33]\), these coordinates are also closely related to the geodesics. This is a reason for us to study geodesics in the dS space-time with the shock wave (see Appendix B for details and references).

In these coordinates the trapped surface that we seek has two parts, which are denoted here by \(S_1\) and \(S_2\) in the respective regions \(\Sigma < 0\) and \(W < 0\). They are defined in terms of the two functions \(\Psi_1(\Upsilon, \bar{\Upsilon})\) and \(\Psi_2(\Upsilon, \bar{\Upsilon})\) by

\[
S_1 : \begin{cases}
W = 0, \\
\Sigma = -\Psi_1(\Upsilon, \bar{\Upsilon}),
\end{cases}
S_2 : \begin{cases}
\Sigma = 0, \\
W = -\Psi_2(\Upsilon, \bar{\Upsilon}),
\end{cases}
\tag{2.8}
\]

with the additional boundary conditions at the shock wave intersection \(C \subset \{W = \Sigma = 0\}\)

\[
\left.\Psi_1(\Upsilon, \bar{\Upsilon})\right|_C = 0, \quad \left.\Psi_2(\Upsilon, \bar{\Upsilon})\right|_C = 0,
\tag{2.9}
\]

and

\[
\partial_\Upsilon \Psi_1 \partial_{\bar{\Upsilon}} \Psi_2 \bigg|_C = 1. \tag{2.10}
\]

Because \(S_1\) and \(S_2\) are in the respective regions \(W < 0\) and \(\Sigma < 0\), we also have \(\Psi_1(\Upsilon, \bar{\Upsilon}) > 0\) and \(\Psi_2(\Upsilon, \bar{\Upsilon}) > 0\).

The two functions \(\Psi_1(\Upsilon, \bar{\Upsilon})\) and \(\Psi_2(\Upsilon, \bar{\Upsilon})\) must be determined by imposing the condition that the surface they define is marginally trapped \([18, 48, 46]\), i.e., that the congruence of outgoing null geodesics orthogonal to the surface has zero expansion.

The zero convergence equation for the \(D=4\) case reduces to the equation (see Appendix D)

\[
\left(\Delta_{S^2} + \frac{2}{a^2}\right) \phi_{1,2}(\Upsilon, \bar{\Upsilon}) = 0, \tag{2.11}
\]

where

\[
\Delta_{S^2} = 2 \left(1 + \frac{\Upsilon \bar{\Upsilon}}{2a^2}\right) \partial_\Upsilon \partial_{\bar{\Upsilon}} \tag{2.12}
\]

is the Laplace–Beltrami operator in complex coordinates (see Appendix D) and the functions \(\phi_{1,2}\) are related to the functions \(\Psi_{1,2}\) defining the two halves of the trapped surface and the shock wave shape functions \(H_{1,2}\) (cf. similar equations in the AdS case \([23, 21]\))

\[
\phi_{1,2} = \frac{2\Psi_{1,2} - H_{1,2}}{1 + \Upsilon \bar{\Upsilon}/2a^2}. \tag{2.13}
\]

The next question to be discussed is whether a black hole forms as a result of the head-on collision of two waves of the type described above. Head-on collisions preserve rotational symmetry around the axis of motion of massless particles, the
Because a head-on collision is $O(2)$-symmetric, the functions $\Psi_1(\Upsilon, \bar{\Upsilon})$ and $\Psi_2(\Upsilon, \bar{\Upsilon})$ describing the trapped surface are identical and depend on only the parameter $\rho^2 = \Upsilon \bar{\Upsilon}$: $\Psi_1(\Upsilon, \bar{\Upsilon}) = \Psi_2(\Upsilon, \bar{\Upsilon}) = \Psi(\rho^2)$.

3. Marginally trapped surface in dS

3.1 Solution of the marginally trapped surface equation in dS

In the case $\phi_1(\Upsilon, \bar{\Upsilon}) = \phi_2(\Upsilon, \bar{\Upsilon}) = \phi(\rho^2)$, equation (2.11) is transformed into the ordinary differential equation

$$
\left(1 + \frac{\rho^2}{2a^2}\right)^2 \left(\frac{\partial^2 \phi}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial \phi}{\partial \rho}\right) + \frac{4\phi}{a^2} = 0.
$$

(3.1)

The solutions of this equation are

$$\phi = \frac{A(\rho^2 - 2a^2) + B((\rho^2 - 2a^2) \ln \rho + 4a^2)}{\rho^2 + 2a^2},$$

(3.2)

where $A$ and $B$ are constants. Because we are interested in a solution of a homogenous equation without singularities, we take $B = 0$ and obtain the expression for the trapped surface function in terms of the shape function $H$,

$$\Psi = \frac{1}{2} H(\rho) + \left(1 + \frac{\rho^2}{2a^2}\right) \frac{A(\rho^2 - 2a^2)}{\rho^2 + 2a^2},$$

(3.3)

and also in terms of the shape function $F$,

$$\Psi = \frac{1}{4} \left(1 + \frac{\rho}{2a^2}\right) F(\rho) + \frac{1}{2a^2} A(\rho^2 - 2a^2)$$

(3.4)

or, more explicitly,

$$\Psi = \sqrt{2p} \left(1 + \frac{\rho^2}{2a^2}\right) \left(-2 + \frac{2a^2 - \rho^2}{2a^2 + \rho^2} \ln \left(\frac{2a^2}{\rho^2}\right)\right) + \frac{1}{2a^2} A(\rho^2 - 2a^2).$$

(3.5)

As mentioned above, the function $\Psi$ must satisfy the following boundary conditions in the head-on case:

$$\Psi \bigg|_C = 0,$$

(3.6)

$$\partial_\Upsilon \Psi \partial_{\bar{\Upsilon}} \Psi \bigg|_C = 1.$$

(3.7)

It is obvious that the bound $C$ is a circle $\rho = \rho_0 = \text{const}$. We hence have a system of two equations for the two constants $A$ and $\rho_0$:

$$\sqrt{2p} \left(1 + \frac{\rho_0^2}{2a^2}\right) \left(-2 + \frac{2a^2 - \rho_0^2}{2a^2 + \rho_0^2} \ln \left(\frac{2a^2}{\rho_0^2}\right)\right) + \frac{1}{2a^2} A \cdot (\rho_0^2 - 2a^2) = 0,$$

(3.8)

$$\frac{1}{4a^4 \rho_0^2} \left(2\sqrt{2pa^2} + \rho_0^2 \left(\sqrt{2p} - A + \sqrt{2p} \rho_0 \ln \left(\frac{2a^2}{\rho_0^2}\right)\right)^2\right) = 1.$$

(3.9)
Substituting $A$ from (3.8) in expression (3.3), we obtain

$$
\Psi(\rho) = \sqrt{2\rho} \left(4 \frac{\rho^2 - \rho_0^2}{\rho_0 - 2a^2} + \left(1 - \frac{\rho^2}{2a^2}\right) \ln \left(\frac{\rho_0^2}{\rho_0^2}\right)\right). 
$$

(3.10)

Equation (3.9) defining $\rho$ can be also written in terms of the initial function $F$ defining the shock wave in the plane coordinates,

$$
\frac{1}{4} \left(1 + \frac{\rho_0^2}{2a^2}\right) F'(\rho_0) + \frac{\rho_0}{2a^2 - \rho_0^2} F(\rho_0) + 2 = 0.
$$

(3.11)

Indeed, normalization condition (2.10) can be written in the form

$$
\left(\frac{d\Psi}{d\rho}\right)^2 = 4 \quad \text{or} \quad \frac{d\Psi}{d\rho} \equiv \Psi'(\rho) = \pm 2.
$$

(3.12)

We choose the minus sign. Because of relation (3.4) on $C$, we have

$$
\Psi'(\rho) \big|_C = \frac{1}{4} \left(1 + \frac{\rho_0^2}{2a^2}\right) F'(\rho_0) + \frac{\rho_0}{2a^2 - \rho_0^2} F(\rho_0),
$$

(3.13)

and we obtain (2.10) in the form

$$
\frac{1}{4} \left(1 + \frac{\rho_0^2}{2a^2}\right) F'(\rho_0) + \frac{\rho_0}{2a^2 - \rho_0^2} F(\rho_0) + 2 = 0.
$$

(3.14)

This equation is universal for an arbitrary dimension $D$. To connect with the AdS case, this equation can be rewritten in terms of the chordal coordinate related to $\rho$ by

$$
\rho = a \sqrt{\frac{2}{1 - q}}.
$$

(3.15)

After this change of variable, equation (3.11) becomes

$$
F'(q_0) + \frac{2}{1 - 2q_0} F(q_0) + \frac{8a}{\sqrt{2q_0(1 - q_0)}} = 0.
$$

(3.16)

We introduce the dimensionless parameter $x_0 = \rho_0/a$. Equation (3.4) becomes

$$
f(x_0) = \sqrt{\frac{2q_0}{p}},
$$

(3.17)

where

$$
f(x) \equiv \frac{1}{x} \frac{(2 + x^2)^2}{2 - x^2}.
$$

(3.18)

We note that in the region $0 < x < \sqrt{2}$, the function $f(x)$ has the positive minimum

$$
f'(x) \big|_{x=x_{\text{min}}} = 0, \quad x_{\text{min}} = 2 - \sqrt{2}.
$$

(3.19)
Hence, in the case
\[ \eta \equiv \frac{a}{p} < \frac{1}{\sqrt{2}} \cdot f(x_{\text{min}}) = 4, \] (3.20)
there are no solutions of (3.17) (see Fig. 1). Therefore, no solution of the trapped surface equation can be found below the critical value \( \eta_c = 4 \) of the parameter \( \eta \). In terms of the rescaling energy \( \bar{\rho} = \rho/G \), where \( G \) is the gravitational constant, this relation gives (1.1).

It is reasonable to consider two limit cases: where \( x_0 \ll 1 \) and \( p < a \) (when the particle energy is low and/or the spacetime is weakly curved; it is natural to call this the low-energy limit) and where \( x_0 \) and \( \eta \) are equal to the critical values (we call this the high-energy critical limit).

From equation (3.17) in the region \( x_0 \ll 1 \), we obtain
\[ x_0 \approx \frac{\sqrt{2}p}{a} \quad \text{and} \quad \rho_0 \approx \sqrt{2}p. \] (3.21)
And in the critical limit, we obtain
\[ p = \frac{a}{4}, \quad \rho_0 = (2 - \sqrt{2})a = (8 - 4\sqrt{2})p. \] (3.22)

Figure 1: The critical value of the parameter \( \eta = a/p \) corresponds to the minimum value of the function \( f \) (red line).

3.2 Higher-dimension cases

Similar calculations can be performed in higher dimensions. Equation (3.14) has a universal form and is independent of the dimension \( D \). The variable \( \rho \) for an arbitrary
dimension is related to \( \xi \equiv Z_D/a \) as

\[
\xi = \frac{1 - \rho^2/2a^2}{1 + \rho^2/2a^2}.
\] (3.23)

As examples, we consider the cases \( D = 5 \) and \( D = 6 \). The shape functions have the forms

\[
F_5(\xi) = \frac{3\sqrt{2\pi}p_5}{a} \frac{2\xi^2 - 1}{\sqrt{1 - \xi^2}},
\] (3.24)

\[
F_6(\xi) = 8\sqrt{2}p_6 \left( \xi \ln \left( \frac{1 + \xi}{1 - \xi} \right) + \frac{2(3\xi^2 - 2)}{3(1 - \xi^2)} \right).
\] (3.25)

where \( \xi = Z_D/a \) and \( p_D = \bar{p}G_D \). For these cases, equation (3.14) reduces to

\[
f_D(x_0) = C_D \frac{a^{D-3}}{p_D},
\] (3.26)

where \( C_D \) is a \( D \)-dependent constant (in particular, \( C_3 = 2/\pi \), \( C_5 = 32/3\pi \), and \( C_6 = 6\sqrt{2} \)) and the functions \( f_D(x) \) are given by

\[
f_D(x) = \frac{(2 + x^2)^{D-2}}{x^{D-3}(2 - x^2)}.
\] (3.27)

These functions have positive minima at the points

\[
x_{\text{min},D} = \sqrt{2(-1 + D - 2\sqrt{D - 2}) \over \sqrt{D - 3}}, \quad D > 3,
\] (3.28)

\[
x_{\text{min},3} = 0, \quad D = 3,
\] (3.29)

and the value of the functions \( f_D(x) \) at these points, \( f_{0,D} \equiv f_D(x_{\text{min},D}) \), give the critical values of the parameter \( \bar{p} \) below which trapped surfaces can be formed,

\[
\bar{p} < \bar{p}_{\text{cr},D},
\] (3.30)

\[
\bar{p}_{\text{cr},D} = a^{D-3} \frac{C_D}{G_D f_{0,D}}.
\] (3.31)

We note that this critical effect does not necessarily exclude the trapped surface in the region of interacting shock waves \( W > 0, \Sigma > 0 \).

We can also interpret formula (3.31) to mean that for a fixed value of \( \bar{p} \), there is a critical value of the de Sitter radius \( a_{cr} \),

\[
a_{cr} \equiv \left( \bar{p}G_D f_{0,D} \right)^{1/D-3}, \quad D > 3,
\] (3.32)
only above which can trapped surfaces be formed. For \( D = 3 \), the critical energy is independent of \( a \).

In particular, for \( D = 5 \) and \( D = 6 \), we have

\[
x_{\text{min},5} = -1 + \sqrt{3},
\]

\[
x_{\text{min},6} = \frac{\sqrt{6}}{3}
\]

and

\[
f_5(x_{5,\text{min}}) = 12\sqrt{3},
\]

\[
f_6(x_{6,\text{min}}) = \sqrt{6}\frac{28}{3\pi}.
\]

Hence, the boundary problem can be solved in \( D = 5 \) and \( D = 6 \) only if the conditions

\[
a^2 p_5 > \frac{9}{8}\pi \sqrt{3},
\]

\[
a^3 p_6 > \frac{128}{27}\sqrt{3}
\]

are satisfied. We thus have the same critical effect as in the four-dimensional dS space-time. The trapped surface can be formed only if the energy of colliding particles is not very high. Assuming that \( x_0 \ll 1 \), we obtain

\[
\rho_0 \approx 2 \left( \frac{\bar{\rho} G_D}{C_D} \right)^{1/(D-3)}
\]

from (3.26).

### 3.3 Area of the marginally trapped surface below the critical point

Knowing \( \rho_0 \), we can calculate the area of the trapped surface:

\[
\mathcal{A}_{\text{trap}} = 2 \int_{\rho < \rho_0} \sqrt{\det g_{\alpha\beta}} dV,
\]

where \( g_{\alpha\beta} \) is the induced metric on the trapped surface and \( dV \) is an elementary volume in the \((D-2)\)-dimensional flat space. Because the corresponding induced metric in the four- and five-dimensional space-times has the form (the form is similar in higher dimensions)

\[
g_{\alpha\beta} = \frac{1}{\mathcal{N}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad g_{\alpha\beta} = \frac{1}{\mathcal{N}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},
\]

(3.41)
the explicit expression for the area of the trapped surface is
\[ A_{\text{trap}} = 2 \cdot \text{Vol} S^{D-3} \int_{0}^{\rho_0} \frac{\frac{D-2}{2} \rho^{D-3}}{N(D-2)/2} d\rho = 2 \cdot \text{Vol} S^{D-3} \int_{0}^{\rho_0} \frac{\frac{D-2}{2} \rho^{D-3}}{(1 + \rho^2/2a^2)^{D-2}} d\rho. \] (3.42)

In particular,
\[ A^4 = 8\pi \frac{a^2 \rho_0^2}{2a^2 + \rho_0^2}, \] (3.43)
\[ A^5 = 4\sqrt{2}\pi a^3 \left( \sqrt{2} \arctan \left( \frac{\sqrt{2} \rho_0}{2a} \right) + \frac{2a\rho_0(\rho_0^2 - 2a^2)}{(2a^2 + \rho_0^2)^2} \right). \] (3.44)

We have
\[ A_{\text{LE}}^4 \approx 8\pi \frac{a^2 p^2}{a^2 + p^2} \approx 8\pi p^2 \left( 1 - \frac{p^2}{a^2} \right), \] (3.45)
\[ A_{\text{LE}}^5 \approx \frac{16\sqrt{2}\pi}{3} \rho_0^3 \approx \frac{3}{2\pi} p_5^3 \] (3.46)
in the low-energy limit and
\[ A_{\text{Cr}}^4 = (4 - 2\sqrt{2})\pi a^2, \] (3.47)
\[ A_{\text{Cr}}^5 = \frac{8\sqrt{2}\pi}{(3 - \sqrt{3})^2} (\sqrt{3} - 2) \left( 1 - 3\sqrt{2} \arctan \frac{\sqrt{2}(\sqrt{3} - 1)}{2} \right) a^3 \] (3.48)
at the critical point.

4. Concluding Remarks

We have studied the formation of marginally trapped surfaces in head-on collisions of two shock waves in the dS space-time for \( D \leq 3 \). For \( D \leq 4 \), we found the critical value of the shock wave energy dependent on the dS radius, above which the trapped surface equation has no solution. This critical behavior is similar to that found in \([21]\) and is also reminiscent of the behavior encountered in numerical simulations of gravitational collapse \([23, 24]\). For \( D = 3 \), the critical energy above which there is no trapped surface is independent of the cosmological constant.

Acknowledgments

I. A. is grateful to I. Volovich for the fruitful discussions. We are supported in part by the RFBR grant 08-01-00798 and 09-01-12179 and by the Federal Agency of Science and Innovations (contract 02.740.11.5057).
5. Appendices

A. Geometric view of shock waves in plane coordinates

A single shock wave in the $D$-dimensional dS space is shown in Fig. 2A. The dS space is represented as a hyperboloid embedded into the $(D+1)$-dimensional Minkowski space-time. The presented shock wave is located on the intersection of the hyperboloid and the plane $x^0 - x^1 = 0$,

$$u = \frac{x^0 + x^1}{\sqrt{2}}, \quad v = \frac{x^0 - x^1}{\sqrt{2}}.$$ 

The coordinates $x_2$ and $x_3$ are suppressed in this figure.

Two shock waves colliding at $u = v = 0$ are shown in Fig. 2B.

![Figure 2: A single shock wave in the dS space can be represented as the intersection of the hyperboloid and the plane $x^0 - x^1 = 0$ ($x^2$ and $x^3$ are suppressed). B. Two shock waves in the dS space. A collision of two shock waves occurs at $x^0 = 0$ and corresponds to the collision of red and yellow balls.](image)

We can also make an animation and draw the position of the single shock wave at discrete instants. In Fig. 3, we see that the shock wave is a nonexpanding one.

![Figure 3: A shock wave in the dS space at different instants of “$x^0$ time.”](image)
B. Solution of the geodesics equation

B.1 The $\sigma$-model ($n$-field) approach

In this section, we derive and solve the null-geodesic equations in the dS spacetime with a nonexpanding shock-wave. We note that applying the embedding theorems [40] requires a delicate analysis because of the nonsmoothness of the metric with the shock wave.

We use an analytical approach similar to the $\sigma$-model ($n$-field) approach and start from the Lagrangian

$$\mathcal{L} = \int d\tau \left[ \frac{dx^M(\tau)}{d\tau} G_{MN}(x(\tau)) \frac{dx^N(\tau)}{d\tau} - \lambda (x^M(\tau) g_{MN}x^N(\tau) - a^2) \right], \quad (B.1)$$

where $M, N = 0, \ldots, D$, $g_{MN}$ is the metric defined the hyperboloid, and $G_{MN}(x(\tau))$ is the metric deformed by the shock wave,

$$G_{MN} = g_{MN} + F \delta(\tau) du^2. \quad (B.2)$$

The Euler equations for this Lagrangian are

$$G_{MN} \frac{d^2 x^N(\tau)}{d\tau^2} + G_{MN} \Gamma^N_{KL} \frac{dx^K(\tau)}{d\tau} \frac{dx^L(\tau)}{d\tau} + \lambda g_{MN}x^N(\tau) = 0, \quad \lambda = -\frac{1}{a^2} (a, \dot{x} + \Gamma_{KL} \dot{x}^K \dot{x}^L)_G. \quad (B.3)$$

Using $u \delta(u) = 0$, we obtain

$$G^{MN} g_{NK} x^K = x^M, \quad (B.5)$$

and equations (B.3) and (B.4) reduce to

$$\ddot{x}^N + \Gamma^N_{KL} \dot{x}^K \dot{x}^L + \lambda x^N = 0, \quad \lambda = -\frac{1}{a^2} (a, \dot{x} + \Gamma_{KL} \dot{x}^K \dot{x}^L)_G. \quad (B.6)$$

Here and hereafter, $(x, y)_G = G_{MN} x^M y^N$. The nonvanishing components of the connection $\Gamma^M_{NK}$ are

$$\Gamma^v_{uu} = -\frac{1}{2} F \delta'(u), \quad \Gamma^v_{ui} = -\frac{1}{2} F, i \delta(u), \quad \Gamma^i_{uu} = -\frac{1}{2} F, i \delta(u). \quad (B.7)$$

Taking $(a, x)_G = a^2$ and $G_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = 0$ into account, we obtain

$$\lambda = \frac{1}{2a^2} (-F + x^i F, i) \delta(u) \dot{u}^2. \quad (B.8)$$

Substituting this expression in (B.6) and taking (B.7) into account, we obtain

$$\ddot{u} = -\frac{1}{2a^2} (-F + x^i F, i) \delta(u) \dot{u}^2 u, \quad (B.9)$$

$$\ddot{v} - \frac{1}{F} \delta'(u) \dot{u}^2 - F, i \delta(u) \dot{u} \dot{x}^i = -\frac{1}{2a^2} (-F + x^i F, i) \delta(u) \dot{u}^2 v, \quad (B.10)$$

$$\ddot{x}^i - \frac{1}{2} F, i \delta(u) \dot{u}^2 = -\frac{1}{2a^2} (-F + x^i F, i) \delta(u) \dot{u}^2 x^i. \quad (B.11)$$

- 13 -
Noting that the right-hand side of (B.9) vanishes, we obtain \( \ddot{u} = 0 \). Taking \( \tau = u \), we obtain

\[
\dddot{v} - \frac{1}{2} F' \delta(u) - F, \delta(u) \dot{x}^i = -\frac{1}{2a^2} (-F + x^i F_j) v \delta(u),
\]

(B.12)

\[
\dddot{x}^i - \frac{1}{2} F, \delta(u) = -\frac{1}{2a^2} (-F + x^j F_i) x^i \delta(u).
\]

(B.13)

We now use the ansatz (by analogy with the case of the shock wave in the Minkowski space-time [33, 34])

\[
v = v_0 + v_1 u + Q(x^i_0) \theta(u) + R(x^i_0) \theta(u) u,
\]

(B.14)

\[
x^i = x^i_0 + x^i_1 u + x^i_2 \theta(u) + S_i(x^j_0) \theta(U) U,
\]

(B.15)

\[
\dot{v} = v_1 + Q(x^i_0) \delta(u) + R(x^i_0) \theta(u),
\]

(B.16)

\[
\dot{x}^i = x^i_1 + D_i(x^j_0) \delta(u) + S_i(x^j_0) \theta(u),
\]

(B.17)

\[
\ddot{v} = Q(x^i_0) \delta'(u) + R(x^i_0) \delta(u),
\]

(B.18)

\[
\dddot{x}^i = D_i(x^j_0) \delta'(u) + S_i(x^j_0) \delta(u).
\]

(B.19)

Along the geodesics, we have the identity

\[
F(x^i) \delta'(u) = F(x^i_0) \delta'(u) - F, \delta(x^i_0) \dot{x}^i \delta(u)
\]

(B.20)

(a similar identity for the flat space-time case was used in [34]). Transforming our ansatz in order to cancel \( \delta^2(u) \), we now obtain

\[
v = v_0 + v_1 u + Q(x^i_0) \theta(u) + R(x^j_0) \theta(u) u,
\]

(B.21)

\[
x^i = x^i_0 + x^i_1 u + S_i \theta(u) u
\]

with the bounds

\[
x^2_0 = a^2,
\]

(B.22)

\[
v_0 = x^i_0 x^i_1,
\]

(B.23)

\[
v_1 = \frac{1}{2} x^2_1.
\]

(B.24)

And hence

\[
Q = \frac{1}{2} F,
\]

(B.25)

\[
R = \frac{1}{2} F, x^i_1 + \frac{1}{2a^2}(F - x^i_0 F_j)v_0 + \frac{1}{8} F^2 + \frac{1}{8a^2} (F^2 - (x^i_0 F_j)^2),
\]

(B.26)

\[
S_i = \frac{1}{2} F, x^i_1 + \frac{1}{2a^2}(F - x^j_0 F_j) x^i_0.
\]

(B.27)
We can take $x_i^1 = 0$ for simplicity. This gives $v_0 = v_1 = 0$, and we have

\[ Q = \frac{1}{2} F \]

\[ R = \frac{1}{8} F^2 + \frac{1}{8a^2} (F^2 - (x_i^0 F_i)^2), \]  

(B.28)

\[ S_i = \frac{1}{2} F_i + \frac{1}{2a^2} (F - x_i^0 F_0) x_i^0. \]

B.2 Focusing of geodesics

Before deriving the explicit expression for the trapped surface, we study the structure of geodesic beams in terms of the dependent plane coordinate $s$. We consider points that are initially on the surface $x_{20}^2 + x_{30}^2 + x_{40}^2 = a^2$ and change to the angular coordinates:

\[ x_{20} = a \cdot \sin \phi \cdot \sin \theta, \quad x_{30} = a \cdot \sin \phi \cdot \cos \theta, \quad x_{40} = a \cdot \cos \phi, \]

\[ F = 4\sqrt{2}p \left( -2 + \frac{x_{40}}{a} \ln \left( \frac{a + x_{40}}{a - x_{40}} \right) \right) = 4\sqrt{2}p \left( -2 + \cos \phi \ln \left( \frac{1 + \cos \phi}{1 - \cos \phi} \right) \right). \]

It is obvious that the term $S_i(x_j^0) \theta(u)u$ in formula (B.17) leads to the refraction of the geodesics. The refraction coefficients are $S_i(x_j^0)$. For the refraction coefficients in the angular parameterization, we have

\[ S_2 = -\sqrt{2} \frac{p \sin \theta}{a \sin \phi}, \quad S_3 = -\sqrt{2} \frac{p \cos \theta}{a \sin \phi}, \quad S_4 = \frac{\sqrt{2} p}{2} a \frac{\ln \left( \frac{1 + \cos \phi}{1 - \cos \phi} \right)}{2} \]

(B.29)

The coordinates $x_2$ and $x_3$ correspond to the physical coordinates related to the size of the beam. Hence, we can easily find the value of the affine parameter $u$, which determines the focal point of the beam (the point $x_2 = x_3 = 0$):

\[ u = \frac{a^2}{\sqrt{2}p} \cdot \sin^2 \phi. \]

The “focal length” is different for each value of the parameter $\phi$, i.e., for each circular ring of the beam cross section. This statement is illustrated in Fig. [4]. Of course, this only roughly explains why the trapped surface can exist. To find a physical meaning for this consideration, we must change coordinates to independent ones.
Figure 4: Focusing of geodesics: in the dependent coordinates, the focal length changes along with the initial conditions.

C. The shock wave in the independent coordinates

C.1 From plane coordinates to independent coordinates

To study the structure of the space-time in terms of the independent four-dimensional coordinates, it is convenient to use the complex conformal flat coordinates

\[ w = \frac{2au}{x^4 + a}, \quad \sigma = \frac{2av}{x^4 + a}, \quad \zeta = \frac{\sqrt{2a}}{x^4 + a} (x^2 + ix^3). \quad (C.1) \]

In these coordinates, the shock wave metric is

\[ ds^2 = -\frac{2dw d\sigma + 2d\zeta d\bar{\zeta} + 2H(\zeta, \bar{\zeta})\delta(w) dw^2}{[1 - (w\sigma - \zeta\bar{\zeta})/2a^2]^2}, \quad (C.2) \]

where

\[ H(\zeta, \bar{\zeta}) = \frac{1}{2} \left( 1 + \frac{1}{2a^2} \zeta \bar{\zeta} \right) F \left( \frac{1 - \zeta \bar{\zeta} / 2a^2}{1 + \zeta \bar{\zeta} / 2a^2} \right). \quad (C.3) \]

and \( F \) is given by (2.4). Hence,

\[ H(\zeta, \bar{\zeta}) = 2\sqrt{2p} \left( 1 + \frac{1}{2a^2} \zeta \bar{\zeta} \right) \left( -2 + \frac{1 - \zeta \bar{\zeta} / 2a^2}{1 + \zeta \bar{\zeta} / 2a^2} \ln \left( \frac{2a^2}{\zeta \bar{\zeta}} \right) \right). \quad (C.4) \]

C.2 An analogue of the D’Eath and Payne coordinates

(from the coordinates \( w, \sigma, \zeta \) to the coordinates \( W, \Sigma, \Upsilon \))

To eliminate \( \delta(u) \) from the metric, we use a coordinate change analogous to the change introduced in [45],

\[ w = W, \]

\[ \sigma = \Sigma + H(\Upsilon, \bar{\Upsilon})\theta(W) + W\theta(W)H_T H_{\bar{T}}, \quad (C.5) \]

\[ \zeta = \Upsilon + W\theta(W)H_{\bar{T}}, \]
where $H_Y = \partial_Y H(Y, \bar{Y})$. In these coordinates, we obtain the metric
\[
    ds^2 = \frac{-2dW \, d\Sigma + 2|dY + W\theta(W)(H_{Y\bar{Y}}dY + H_{Y\bar{Y}}d\bar{Y})|^2}{[1 - (W\Sigma - \bar{Y}\bar{Y} + W\theta(W)G)/2a^2]^2},
\]
where $G = H - YH_Y - \bar{Y}H_{\bar{Y}}$ and $H(Y, \bar{Y})$ depends on $Y, \bar{Y}$ as $H(\zeta, \bar{\zeta})$ given by (C.3) depends on $\zeta, \bar{\zeta}$.

### C.3 Geodesics in terms of the independent complex conformal flat coordinates

We have the expressions for geodesics in terms of the dependent coordinates. Using coordinate change (C.1), we obtain an expression for geodesics in terms of the independent coordinates. In the first order of the parameter $u$, we obtain
\[
    w(u) = w_1 u + \ldots, \\
    \sigma(u) = \sigma_0 + \sigma_1 u + \cdots \equiv \sigma_{0c} + \sigma_{0\theta}\theta(u) + (\sigma_{1c} + \sigma_{1\theta}\theta(u))u + \ldots, \\
    \zeta(u) = \zeta_0 + \zeta_1 u + \cdots \equiv \zeta_{0c} + \zeta_{0\theta}\theta(u) + (\zeta_{1c} + \zeta_{1\theta}\theta(u))u + \ldots,
\]
where
\[
    w_1 = \frac{2}{1 + x_0^4/a}, \quad \sigma_0 = \frac{2v_0}{1 + x_0^4/a}, \quad \sigma_{0\theta} = \frac{2Q(x_0^i)}{1 + x_0^4/a}, \\
    \sigma_{1c} = 2 \left( \frac{v_1}{1 + x_0^4/a} - \frac{x_1^4}{a(1 + x_0^4/a)^2} \right), \quad \sigma_{1\theta} = 2 \left( \frac{R(x_0^i)}{1 + x_0^4/a} - \frac{Q(x_0^i)x_1^4}{a(1 + x_0^4/a)^2} - \frac{QS^4(x_0^i)}{a(1 + x_0^4/a)^2} - \frac{S^4(x_0^i)v_0}{a(1 + x_0^4/a)^2} \right), \\
    \zeta_{0c} = \frac{\sqrt{2}z_0}{1 + x_0^4/a}, \quad \zeta_{0\theta} = 0, \quad \zeta_{1c} = \frac{\sqrt{2}z_1}{1 + x_0^4/a} - \frac{x_1^4}{a(1 + x_0^4/a)^2}, \quad \zeta_{1\theta} = \frac{\sqrt{2}S}{1 + x_0^4/a} - \frac{S^4}{a(1 + x_0^4/a)^2}.
\]
where the complex variables $S$, $z_0$, $z_1$ are related to $S^i$, $x_0^i$, $x_1^i$ by
\begin{align}
S &= S^2 + iS^3, \quad (C.17) \\
z_0 &= x_0^2 + ix_0^3, \quad z_1 = x_1^2 + ix_1^3. \quad (C.18)
\end{align}

We see that there is a discontinuity only for the $\sigma$ variable.

C.4 Geodesics in terms of the independent smooth coordinates

Using relations (C.5) between the initial independent coordinates and the smooth independent coordinates, we obtain the expression for geodesics in terms of the smooth independent coordinates up to the second order:
\begin{align}
\Sigma(w) &= (\Sigma_0c + \Sigma_0\theta(w)) + (\Sigma_1c + \Sigma_1\theta(w))w + \Sigma_2\frac{w^2}{2} + \ldots, \quad (C.19) \\
\Upsilon(w) &= \Upsilon_0 + (\Upsilon_1c + \Upsilon_1\theta(w))w + \Upsilon_2\frac{w^2}{2} + \ldots, \quad (C.20) \\
\bar{\Upsilon}(w) &= \bar{\Upsilon}_0 + (\bar{\Upsilon}_1c + \bar{\Upsilon}_1\theta(w))w + \bar{\Upsilon}_2\frac{w^2}{2} + \ldots, \quad (C.21)
\end{align}

where
\begin{align}
\Sigma_0c &= \sigma_0, \quad (C.22) \\
\Sigma_0\theta &= 0, \quad (C.23) \\
\Sigma_1c &= \sigma_1c, \quad (C.24) \\
\Sigma_1\theta &= \sigma_1\theta - w_1H(\Upsilon, \bar{\Upsilon})d\Upsilon + \bar{\Upsilon}_1H(\Upsilon, \bar{\Upsilon})d\bar{\Upsilon}, \quad (C.25) \\
\Upsilon_0 &= \zeta_0c, \quad (C.26) \\
\Upsilon_1c &= \zeta_1c, \quad (C.27) \\
\Upsilon_1\theta &= \zeta_1\theta - w_1H(\Upsilon, \bar{\Upsilon}). \quad (C.28)
\end{align}

D. Trapped surface equation

In the case of two shock waves, we deal with
\begin{align}
ds^2 &= 2g_{\Sigma\Sigma}d\Sigma d\Sigma + g_{\Upsilon\Upsilon}d\Upsilon d\Upsilon + 2g_{\Upsilon\bar{\Upsilon}}d\Upsilon d\bar{\Upsilon} + g_{\bar{\Upsilon}\bar{\Upsilon}}d\bar{\Upsilon} d\bar{\Upsilon} = \\
&= \frac{-2d\Sigma d\Upsilon + (\Sigma\theta(\Sigma) + W\theta(W))(H_{\Upsilon\Upsilon}d\Upsilon + H_{\bar{\Upsilon}\bar{\Upsilon}}d\bar{\Upsilon})^2}{[1 - (W\Sigma - \Sigma\theta(\Sigma) + W\theta(W))^2]^{2}}. \quad (D.1)
\end{align}

Because of the notation symmetry ($W \leftrightarrow \Sigma$), we may analyze only one part of the trapped surface.
The null geodesics passing through the surface
\[ \Sigma = \Psi(\Upsilon, \bar{\Upsilon}), \quad W = 0 \] (D.2)
can be specified by the tangent vectors (see Subsections C.3 and C.4):
\[ \xi^W = w_1, \quad \xi^\Sigma = \sigma_1, \quad \xi^\Upsilon = \zeta_1, \quad \xi^{\bar{\Upsilon}} = \bar{\zeta}_1, \] (D.3)
where
\[ \sigma_1 = -\frac{1}{2g_{W\Sigma}w_1}(g_{\Upsilon\Upsilon}\zeta_1^2 + 2g_{T\bar{T}\Sigma}\bar{\zeta}_1\zeta_1 + g_{\bar{T}\bar{T}\Sigma}\bar{\zeta}_1^2). \] (D.4)
We also suppose that
\[ (\xi, K_a) = 0, \quad a = \zeta, \bar{\zeta}, \] (D.5)
for
\[ K_a^M = (0, -\partial_a \Psi, \delta^b_a) \] (D.6)
(our calculations are very close to those in the review section in [28]). This gives
\[ -g_{W\Sigma}\partial_a \Psi w_1 + g_{ab}\zeta^b = 0. \] (D.7)
From (D.7), we have
\[ \zeta^a = w_1 g^{ab}g_{W\Sigma}\partial_b \Psi, \] (D.8)
where \( g^{ab} \) is the inverse metric for \( g_{ab} \). Therefore, we have
\[ \xi^W = w_1, \quad \xi^\Sigma = \sigma_1 = -\frac{w_1 g_{W\Sigma}}{2 \det(g^{ab})}(g_{T\bar{T}}\partial_T \Psi - 2g_{T\bar{T}\Sigma}\partial_T \Psi + g_{\bar{T}\bar{T}\Sigma}\partial_T \Psi), \] (D.10)
\[ \xi^\Upsilon = \zeta_1 = \frac{w_1 g_{W\Sigma}}{\det(g^{ab})}(g_{\Upsilon\bar{\Upsilon}}\partial_{\bar{\Upsilon}} \Psi - g_{\Upsilon\bar{\Upsilon}\Sigma}\partial_{\bar{\Upsilon}} \Psi), \] (D.11)
\[ \xi^{\bar{\Upsilon}} = \bar{\zeta}_1 = \frac{w_1 g_{W\Sigma}}{\det(g^{ab})}(g_{\bar{T}\bar{T}}\partial_{\bar{T}} \Psi - g_{\bar{T}\bar{T}\Sigma}\partial_{\bar{T}} \Psi). \] (D.12)
For \( \xi_M = g_{MN}\xi^N \) at the point \( W = 0 \), we have
\[ \xi_M = \left( -\frac{\partial_T \Psi}{1 + \bar{\Upsilon}^2/2a^2}, \frac{1}{1 + \bar{\Upsilon}^2/2a^2}, -\frac{\partial_T \Psi}{1 + \bar{\Upsilon}^2/2a^2}, \frac{\partial_T \Psi}{1 + \bar{\Upsilon}^2/2a^2} \right). \] (D.13)
For the convergence at the surface \( W = 0, \Sigma = -\Psi(\zeta, \bar{\zeta}) \), we obtain
\[ \theta = h^{MN}\nabla_N \xi_M, \] (D.14)
where
\[ h^{MN} = K_a^M g^{a\beta} K_\beta^N \] (D.15)
and $g^{\alpha \beta}$ is the inverse of the metric induced on the trapped surface,

$$
g_{\alpha \beta} = K^M_{\alpha} g_{M N} K^N_{\beta} = \begin{pmatrix} g_{\Upsilon \Upsilon} & g_{\Upsilon \bar{\Upsilon}} \\ g_{\Upsilon \bar{\Upsilon}} & g_{\bar{\Upsilon} \bar{\Upsilon}} \end{pmatrix}. \tag{D.16}$$

The components of the connection for metric (D.1) in the coordinates $X^M = (W, \Sigma, \Upsilon, \bar{\Upsilon})$, $M, N = 0, 1, 2, 3$, are

$$
\begin{align*}
\Gamma^1_{11} &= -\frac{\partial_{\Sigma} N}{N}, & \Gamma^1_{12} &= -\frac{\partial_{\Upsilon} N}{2N}, & \Gamma^2_{12} &= -\frac{\partial_{\Sigma} N}{2N}, \\
\Gamma^1_{22} &= \frac{H_{\Upsilon \bar{\Upsilon}}}{2}, & \Gamma^2_{22} &= -\frac{\partial_{\Upsilon} N}{N}, & \Gamma^1_{13} &= -\frac{\partial_{\Sigma} N}{2N}, \\
\Gamma^1_{23} &= -\frac{\partial_{\Sigma} N}{2N}, & \Gamma^1_{23} &= -\frac{1}{2N} (\partial_W N - H_{\Upsilon \bar{\Upsilon}} N), & \Gamma^0_{23} &= -\frac{\partial_{\Sigma} N}{2N}, \\
\Gamma^3_{33} &= \frac{H_{\Upsilon \bar{\Upsilon}}}{2}, & \Gamma^3_{33} &= -\frac{\partial_{\Upsilon} N}{N},
\end{align*} \tag{D.17}$$

where

$$\mathcal{N} = \left[1 - (W \Sigma - \Upsilon \bar{\Upsilon} + (\Sigma \theta(\Sigma) + W \theta(W)) G^1/2a^2\right]^2. \tag{D.21}$$

At the point $W = 0$, the tensor $h^{MN}$ can be represented as

$$
h^{MN} = \frac{1}{\det(g_{\alpha \beta})} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -2\partial_{\Upsilon} \Psi \partial_{\Upsilon} \Psi & -\partial_{\Upsilon} \Psi & \partial_{\Upsilon} \Psi \\ 0 & -\partial_{\Upsilon} \Psi & 0 & -1 \\ 0 & -\partial_{\Upsilon} \Psi & -1 & 0 \end{pmatrix}. \tag{D.22}$$

Finally, the equation for the trapped surface can be found by direct calculations:

$$
\left(\left(1 + \frac{\Upsilon \bar{\Upsilon}}{2a^2}\right)^2 \partial_{\Upsilon \bar{\Upsilon}} + \frac{1}{a^2}\right) \frac{2\Psi - H}{1 + \Upsilon \bar{\Upsilon}/2a^2} = 0. \tag{D.23}
$$
References

[1] E. Komatsu et al. [WMAP Collaboration], Five-year Wilkinson microwave anisotropy probe (WMAP) observations: Cosmological interpretation, Astrophys. J. Suppl. 180 (2009) 330 [0803.0547]

[2] G. ’t Hooft, Gravitational collapse and particle physics, Proceedings: Proton-antiproton collider physics, Aachen, 1986, pp. 669–688;
G. ’t Hooft, Graviton dominance in ultrahigh-energy scattering, Phys. Lett. B198 (1987) 61;
G. ’t Hooft, On the factorization of universal poles in a theory of gravitating point particles, Nucl. Phys. B304 (1988) 867

[3] D. Amati, M. Ciafaloni and G. Veneziano, Superstring collisions at Planckian energies, Phys. Lett. B197 (1987) 81;
D. Amati, M. Ciafaloni and G. Veneziano, Classical and quantum gravity effects from Planckian energy superstring collisions, Int. J. Mod. Phys. A3 (1988) 1615;
D. Amati, M. Ciafaloni and G. Veneziano, Can space-time be probed below the string size? Phys. Lett. B216 (1989) 41;
D. Amati, M. Ciafaloni and G. Veneziano, Higher order gravitational deflection and soft bremsstrahlung in Planckian energy superstring collisions, Nucl. Phys. B347 (1990) 550;
D. Amati, M. Ciafaloni and G. Veneziano, Planckian scattering beyond the semiclassical approximation, Phys. Lett. B289 (1992) 87

[4] I. Ya. Aref’eva, K. Viswanathan and I. V. Volovich, Planckian-energy scattering, colliding plane gravitational waves and black hole creation, Nucl. Phys. B452 (1995) 346; Erratum, ibid. B462 (1996) 613 [hep-th/9412157];
I. Ya. Aref’eva, K. S. Viswanathan and I. V. Volovich, On black hole creation in Planckian energy scattering, Int. J. Mod. Phys. D5 (1996) 707 [hep-th/9512170]

[5] R. Narayan, Black holes in Astrophysics, New J. Phys. 7 (2005) 199 [gr-qc/0506078]

[6] N. Arkani-Hamed, S. Dimopoulos and G. R. Dvali, The hierarchy problem and new dimensions at a millimeter, Phys. Lett. B429 (1998) 263 [hep-ph/9803315];
I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos and G. R. Dvali, New dimensions at a millimeter to a Fermi and superstrings at a TeV, Phys. Lett. B436 (1998) 257 [hep-ph/9804398]

[7] G. F. Giudice, R. Rattazzi and J. D. Wells, Quantum gravity and extra dimensions at high-energy colliders, Nucl. Phys. B544 (1999) 3 [hep-ph/9811291];
G. F. Giuduce, R. Rattazzi and J. D. Wells, Transplanckian collisions at the LHC and beyond, Nucl. Phys. B630 (2002) 293 [hep-ph/0112161]

[8] T. Banks and W. Fischler, A model for high energy scattering in quantum gravity [hep-th/9906038]
[9] I. Ya. Aref’eva, *High-energy scattering in the brane world and black hole production*, *Part. Nucl.* **31** (2000) 169 [hep-th/9910269]

[10] S. Dimopoulos and G. Landsberg, *Black holes at the LHC*, *Phys. Rev. Lett.* **87** (2001) 161602 [hep-ph/0106295]

[11] S. B. Giddings and S. Thomas, *High energy colliders as black hole factories: The end of short distance physics*, *Phys. Rev.* **D65** (2002) 056010 [hep-ph/0106219];
D. M. Eardley and S. B. Giddings, *Classical black hole production in high-energy collisions*, *Phys. Rev.* **D66** (2002) 044011 [gr-qc/0201034]

[12] A. Ringwald and H. Tu, *Collider versus cosmic ray sensitivity to black hole production*, *Phys. Lett.* **B525** (2002) 135 [hep-ph/0111042];
E. J. Ahn, M. Cavaglià and A. V. Olinto, *Brane factories*, *Phys. Lett.* **B551** (2003) 1 [hep-th/0201042];
S. N. Solodukhin, *Classical and quantum cross-section for black hole production in particle collisions*, *Phys. Lett.* **B533** (2002) 153 [hep-ph/0201248];
E. Kohlprath and G. Veneziano, *Black holes from high-energy beam-beam collisions*, *J. High Energy Phys.* **06** (2002) 057 [arXiv:gr-qc/0203093];
H. Yoshino and Ya. Nambu, *Black hole formation in the grazing collision of high-energy particles*, *Phys. Rev.* **D67** (2003) 024009 [gr-qc/0209003];
O. I. Vasilenko, *Trapped surface formation in high-energy black holes collision* [hep-th/0305067];
H. Yoshino and V. S. Rychkov, *Improved analysis of black hole formation in high-energy particle collisions*, *Phys. Rev.* **D71** (2005) 104028 [hep-th/0503171];
M. Cavaglià, *Black hole and brane production in TEV gravity: A review*, *Int. J. Mod. Phys.* **A18** (2003) 1843 [hep-ph/0210296];
P. Kanti, *Black holes in theories with large extra dimensions: A review*, *Int. J. Mod. Phys.* **A19** (2004) 4899 [hep-ph/0402168];
S. B. Giddings and V. S. Rychkov, *Black holes from colliding wavepackets*, *Phys. Rev.* **D70** (2004) 104026 [hep-th/0409131];
V. Cardoso, E. Berti and M. Cavaglià, *What we (don’t) know about black hole formation in high-energy collisions*, *Class. Quant. Grav.* **22** (2005) L61-R84 [hep-ph/0505125];
G. L. Landsberg, *Black holes at future colliders and beyond*, *J. Phys.* **G32** (2006) R337 [hep-ph/0607297];
D. M. Gingrich, *Black hole cross section at the LHC*, *Int. J. Mod. Phys.* **A21** (2006) 6653 [hep-ph/0609055];
H. Stoecker, *Mini black holes in the first year of the LHC: Discovery through di-jet suppression, multiple mono-jet emission and ionizing tracks in ALICE*, *J. Phys.* **G32** (2006) S429;
B. Koch, M. Bleicher and H. Stoecker, *Black holes at LHC?* *J. Phys.* **G34** (2007) S535 [hep-ph/0702187];
N. Kaloper and J. Terning, *How black holes form in high energy collisions* [0705.0408];
M. Cavaglia et al., Signatures of black holes at the LHC [0707.0317];
P. Mende and L. Randall, Black holes and quantum gravity at the LHC [0708.3017];
S. B. Giddings, High-energy black hole production [0709.1107]

[13] I. Ya. Aref’eva and I. V. Volovich, Time machine at the LHC, Int. J. Geom. Meth. Mod. Phys. 05 (2008) 641 [0710.2696]

[14] A. Mironov, A. Morozov and T. N. Tomaras, If LHC is a mini-time-machines factory, can we notice? [0710.3395]

[15] P. Nicolini and E. Spallucci, Noncommutative geometry inspired wormholes and dirty black holes [0902.4654]

[16] I. D. Novikov, N. S. Kardashev and A. A. Shatskiy, The multicomponent universe and the astrophysics of wormholes, Phys. Usp. 50 (2007) 965 [Usp. Fiz. Nauk 177 (2007) 1017]

[17] S. W. Hawking and R. Penrose, The singularities of gravitational collapse and cosmology, Proc. Roy. Soc. Lond. A314 (1970) 529

[18] S. W. Hawking and G. R. F. Ellis, The large scale structure of space-time Cambridge Univ. Press (1973)

[19] G. W. Gibbons, Some comments on gravitational entropy and the inverse mean curvature flow, Class. Quant. Grav. 16 (1999) 1677 [hep-th/9809167]

[20] R. Penrose, The question of cosmic censorship, J. Astrophys. Astron. 20 (1999) 233

[21] L. Alvarez-Gaume, C. Gomez, A. S. Vera, A. Tavanfar and M. A. Vazquez-Mozo, Critical formation of trapped surfaces in the collision of gravitational shock waves [0811.3969]

[22] D. Amati, M. Ciafaloni and G. Veneziano, Towards an S-matrix description of gravitational collapse, J. High Energy Phys. 02 (2008) 049 [0712.1209];
G. Veneziano and J. Wosiek, Exploring an S-matrix for gravitational collapse, J. High Energy Phys. 09 (2008) 023 [0804.3321];
G. Veneziano and J. Wosiek, Exploring an S-matrix for gravitational collapse II: A momentum space analysis, J. High Energy Phys. 09 (2008) 024 [0805.2973]

[23] M. W. Choptuik, Universality and scaling in gravitational collapse of a massless scalar field, Phys. Rev. Lett. 70 (1993) 9

[24] C. Gundlach and J. M. Martín-García, Critical phenomena in gravitational collapse, Living Rev. Rel. 10 (2007) 5 [0711.4620]

[25] S. S. Gubser, S. S. Pufu and A. Yarom, Entropy production in collisions of gravitational shock waves and of heavy ions, Phys. Rev. D78 (2008) 066014 [0805.1551]
[26] S. S. Gubser, S. S. Pufu and A. Yarom, *Off-center collisions in AdS with applications to multiplicity estimates in heavy-ion collisions* [0902.4062]

[27] H. Nastase, *On high energy scattering inside gravitational backgrounds* [hep-th/0410124]

[28] H. Nastase, *The RHIC fireball as a dual black hole* [hep-th/0501068]

[29] E. Shuryak, S.-J. Sin and I. Zahed, *A gravity dual of RHIC collisions*, *J. Korean Phys. Soc.* **50** (2007) 384 [hep-th/0511199]

[30] A. J. Amsel, D. Marolf and A. Virmani, *Collisions with black holes and deconfined plasmas*, *JHEP* **04** (2008) 025 [0712.2221]

[31] D. Grumiller and P. Romatschke, *On the collision of two shock waves in AdS5* [0803.3226]

[32] P. Aichelburg and R. Sexl, *On the gravitational field of a massless particle*, *Gen. Rel. and Grav.* **2** (1971) 303

[33] T. Dray and G. ’t Hooft, *The gravitational shock wave of a massless particle*, *Nucl. Phys.* **B253** (1985) 173

[34] V. Ferrari, P. Pendenza and G. Veneziano, *Beam-like gravitational waves and their geodesics*, *Gen. Rel. and Grav.* **20** (1988) 1185

[35] M. Hotta and M. Tanaka, *Shock wave geometry with non-vanishing cosmological constant*, *Class. Quant. Grav.* **10** (1993) 307

[36] K. Sfetsos, *On gravitational shock waves in curved space-times*, *Nucl. Phys.* **B436** (1995) 721 [hep-th/9408169]

[37] J. Podolsky and J. B. Griffiths, *Impulsive waves in de Sitter and anti-de Sitter space-times generated by null particles with an arbitrary multipole structure*, *Class. Quant. Grav.* **15** (1998) 453 [gr-qc/9710049]

[38] G. T. Horowitz and N. Itzhaki, *Black holes, shock waves, and causality in the AdS/CFT correspondence*, *JHEP* **02** (1999) 010 [hep-th/9901012]

[39] J. Podolsky and J. B. Griffiths, *Nonexpanding impulsive gravitational waves with an arbitrary cosmological constant*, *Phys. Lett.* **A261** (1999) 1 [gr-qc/9908008]; J. Podolsky, *Exact impulsive gravitational waves in spacetimes of constant curvature* [gr-qc/0201029]

[40] J. Podolsky and M. Ortaggio, *Symmetries and geodesics in (anti-)de Sitter space-times with nonexpanding impulsive waves*, *Class. Quant. Grav.* **18** (2001) 2689 [gr-qc/0105065]

[41] R. Emparan, *Exact gravitational shockwaves and Planckian scattering on branes*, *Phys. Rev.* **D64** (2001) 024025 [hep-th/0104009]
[42] G. Esposito, R. Pettorino and P. Scudellaro, *On boosted space-times with cosmological constant and their ultrarelativistic limit* [gr-qc/0606126]

[43] K. Kang and H. Nastase, *High energy QCD from Planckian scattering in AdS and the Froissart bound*, Phys. Rev. D72 (2005) 106003 [hep-th/0410173];
K. Kang and H. Nastase, *Planckian scattering effects and black hole production in low M(Pl) scenarios*, Phys. Rev. D71 (2005) 124035 [hep-th/0409099]

[44] R. Penrose, *Results presented at the Cambridge University Seminar* (1974) [unpublished]

[45] P. D. D’Eath and P. N. Payne, *Gravitational radiation in high speed black hole collisions. 1. Perturbation treatment of the axisymmetric speed of light collision*, Phys. Rev. D46 (1992) 658;
P. D. D’Eath and P. N. Payne, *Gravitational radiation in high speed black hole collisions. 2. Reduction to two independent variables and calculation of the second order news function*, Phys. Rev. D46 (1992) 675;
P. D. D’Eath and P. N. Payne, *Gravitational radiation in high speed black hole collisions. 3. Results and conclusions*, Phys. Rev. D46 (1992) 694

[46] R. M. Wald, *General Relativity*, Univ. Chicago Press (1984)

[47] J. B. Griffiths, *Colliding plane waves in general relativity*, Clarendon (1991)

[48] E. Poisson, *A relativist’s toolkit: The mathematics of black-hole mechanics*, Cambridge (2004)