On the Proximity Factors of Lattice Reduction-Aided Decoding

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Abstract—Lattice reduction-aided decoding features reduced decoding complexity and near-optimum performance in multi-input multi-output communications. In this paper, a quantitative analysis of lattice reduction-aided decoding is presented. To this aim, the proximity factors are defined to measure the worst-case losses in distances relative to closest point search (in an infinite lattice). Upper bounds on the proximity factors are derived, which are functions of the dimension \( n \) of the lattice alone. The study is then extended to the dual-basis reduction. It is found that the bounds for dual basis reduction may be smaller. Reasonably good bounds are derived in many cases. The constant bounds on proximity factors not only imply the same diversity order in fading channels, but also relate the error probabilities of (infinite) lattice decoding and lattice reduction-aided decoding.

Index Terms—coding bounds, dual basis, lattice decoding, lattice reduction, minimum distance, multi-input multi-output communications.

EDICS: MSP-CODR or SPC-PERF

I. INTRODUCTION

The linear multi-input multi-output (MIMO) model covers a range of problems in communications, such as code-division multiple access (CDMA), inter-symbol interference (ISI) channels, linearly precoded orthogonal frequency-division multiplexing (OFDM), multi-antenna fading channels with or without linear encoding, multi-antenna broadcast, and cooperative diversity [1]–[4]. For moderate to large problem sizes, (near-)optimum decoding for MIMO systems represents a challenging problem in communication theory. The naive exhaustive search for maximum-likelihood (ML) decoding suffers from exponential complexity. Suboptimum strategies such as zero-forcing (ZF) and successive interference cancelation (SIC) sometimes incurs heavy performance loss. During the past decades, several improved strategies have been developed, especially in the context of multiuser detection and equalization. Sequential decoding [5], branch-and-bound search [6], and semidefinite relaxation [7] represent the state-of-the-art.

The theory of lattices is a powerful, distinctive approach to fast MIMO decoding [8]–[10]. In digital communications, the signal constellation or codebook is often drawn from a lattice, especially as the spectrally efficient high-order quadrature amplitude modulation (QAM) is increasingly used. Lattice decoding exploits the structure of a lattice to significantly reduce the decoding complexity. Decoding corresponds to the closest vector problem (CVP) for a lattice. In general, solving the CVP may consist of two stages: lattice reduction and a local search usually implemented by sphere decoding. While sphere decoding dramatically lowers the decoding complexity at high signal-to-noise ratio (SNR) [11], it could be computationally intensive at low to moderate SNR’s. Furthermore, it is known that its complexity grows exponentially with the system size \( n \) for any fixed SNR [12]. Thus, to further widen the decoding speed bottleneck, some performance has to be traded for complexity, i.e., the CVP needs to be solved approximately. The problem of solving CVP approximately was first addressed by Babai in [13], which in essence applies ZF or SIC on a reduced lattice. This technique of approximate lattice decoding is referred to as lattice-reduction-aided decoding in communications literature [14], [15], where the expense associated with lattice reduction can be shared if the channel matrix keeps constant during a frame of data.

Contributions of this paper: This paper presents a quantitative understanding of approximate lattice decoding. More precisely, we shall develop a systematic approach to characterize its performance in terms of proximity factors, i.e., the worst-case loss in distances relative to (infinite) lattice decoding, which relates the probabilities of (infinite) lattice decoding and approximate lattice decoding. This approach is justified by the fact that lattice decoding often ignores the boundary of finite signals to reduce decoding complexity [16], [17]. In this paper, the proximity factors are found to be bounded above by a function of the dimension of the lattice alone. In other words, the output of approximate lattice decoding is in proximity to that of lattice decoding. As an alternative to reducing the primal basis, one can reduce the dual basis. This technique has received less attention thus far. In this paper, we find that in some cases dual basis reduction results in a smaller upper bound on the proximity factor. The derived bounds serve two purposes: one is to bound the performance in its own right, the other is to give insights into approximate lattice decoding. The bounds apply to both fixed and random channels, and hold irrespective of fading statistics.

Relations to prior works: A quantitative analysis was first given by Yao and Wornell [14], who showed that the loss is not greater than 3 dB for 2-dimensional complex Gauss reduction. Our idea of quantifying the performance losses using constant bounds in the general case started in [18] which used the LLL reduction in differential lattice decoding. This technique has been subsequently developed and improved in a series of conference papers [19]–[21], eventually leading to the current paper.

Independently, the full receive diversity of Lenstra, Lenstra and Lovász (LLL) reduction in uncoded MIMO fading channels was shown in [22], [23] by using different approaches. It was further shown in [17] that, remarkably, minimum mean-square error (MMSE) based lattice-reduction aided decoding achieves the optimal diversity and multiplexing tradeoff. We would like to point out that the diversity order derived in those

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papers does not fully characterize the error performance, as the SNR gap could be arbitrary for the same diversity order. Therefore, proximity factors offer an approach to performance analysis complementary to [17], [22], [23] and could deepen our understanding about approximate lattice decoding. The derivations in those papers made use of the orthogonality defect of a lattice, which could also be used to derive upper bounds on proximity factors. However, the bounds derived in this paper are better.

Our analysis cannot capture the boundary effect associated with lattice decoding. The limitation of lattice decoding ignoring the boundary was pointed out in [24]. Yet it was shown in [16], [17] that the limitation is due to the naive implementation of lattice decoding. With MMSE preprocessing, the performance loss relative to ML decoding is insignificant, and in fact lattice decoding achieves the optimum diversity and multiplexing tradeoff [16], [17].

Organization: The rest of the paper is organized as follows. Section II describes the system model. Section III presents the basics of lattices and lattice reduction that are essential to the subsequent development of the theory in this paper. In Section IV, we define the proximity factors of lattice-reduction-aided decoding and show the main results. In Sections V and VI, we derive the proximity factors for reducing the primal and dual basis, respectively. Section VII presents numerical results and then ends with a discussion.

Notation: Matrices and column vectors are denoted by upper and lowercase boldface letters (unless otherwise stated), and the transpose, inverse, pseudoinverse of a matrix B by $B^T$, $B^{-1}$, and $B^+$, respectively. For convenience, write $B^{-T} = (B^{-1})^T$ and $B^{T^T} = (B^+)^T$. The inner product in the Euclidean space between vectors $u$ and $v$ is defined as $\langle u, v \rangle = u^T v$, and the Euclidean length $\|u\| = \sqrt{u^T u}$.

II. MIMO DECODING

We consider a real-valued system model. For complex-valued system, an equivalent real system model with a doubled dimension is used. Reduction of complex lattices without using the equivalent real model is a topic that has been investigated elsewhere [25]. Let $x = (x_1, ..., x_n)^T$ be the $n \times 1$ data vector, where each symbol $x_i$ is chosen from a finite subset of the integer set $\mathbb{Z}$. With proper scaling and shifting [15], one has the generic $m \times n$ MIMO system model

$$y = Bx + n$$

where $m \geq n$, $y, n \in \mathbb{R}^m$ denote the channel output and noise vectors, respectively, and $B \in \mathbb{R}^{m \times n}$ is the $m \times n$ full column-rank matrix. The entries of $n$ are i.i.d. normal with zero mean and variance $\sigma^2$. The expression of matrix $B$ depends on the problem at hand. For example, $B$ is the channel matrix for uncoded MIMO systems; $B$ is composed of the channel matrix and the generating matrix of the encoding lattice for spacetime block codes [2]; $B$ involves the product-channel matrix for cooperative communications [4].

For such a system model, the ML decoder is given by

$$\hat{x} = \arg \min_{x \in C} \|y - Bx\|^2$$

where $C$ stands for the transmitter finite set. Note that the complexity of the standard ML decoding that uses exhaustive search is exponential in $n$, and also increases with the alphabet size.

There are linear and nonlinear decoders with cubic complexity. In the linear ZF strategy, $y$ is multiplied on the left by the pseudoinverse of $B$, to yield the detection rule

$$\hat{x} = Q \{B^+ y\}$$

where $Q(\cdot)$ denotes the quantization to the nearest integer within the signal boundary. A well-known drawback of ZF is the effect of noise amplification when the channel matrix $B$ is ill-conditioned. By introducing decision feedback in the detection process, the nonlinear SIC strategy has better performance. One way to do SIC is to perform the QR decomposition $B = QR$, where $Q$ has orthogonal columns and $R$ is an upper triangular matrix [26]. Multiplying (1) on the left with $Q^T$ we have

$$y' = Q^T y = Rx + n'.$$

In SIC, the last symbol $x_n$ is estimated first as $\hat{x}_n = Q(y_n' / r_{n,n})$. Then the estimate is substituted to remove the interference term in $y_{n-1}'$ when $x_{n-1}$ is being estimated. The procedure is continued until the first symbol is detected. That is, we have the following recursion:

$$\hat{x}_i = Q \left\{ \frac{y_i' - \sum_{j=i+1}^n r_{i,j} \hat{x}_j}{r_{i,i}} \right\}$$

for $i = n, n-1, ..., 1$.

ZF and SIC may incur heavy performance loss. For example, in single-user uncoded MIMO fading channels, ZF and SIC are only able to achieve the first-order diversity in an $n \times n$ system [27], i.e., that of a single antenna, which is far below the order $n$ achieved by ML decoding. In fact, the diversity order of SIC cannot be increased even with ordering [28].

The basic idea behind approximate lattice decoding is to use lattice reduction in conjunction with traditional low-complexity decoders. With lattice reduction, the basis $B$ is transformed into a new basis consisting of roughly orthogonal vectors (this is always possible in a sense defined later)

$$B' = BU$$

where $U$ is a unimodular matrix, i.e., $U$ contains only integer entries and the determinant $\det U = \pm 1$. Indeed, we have the equivalent channel model

$$y = B'U^{-1}x + n = B'x' + n, \quad x' = U^{-1}x.$$ Then conventional decoders (ZF or SIC) are applied on the reduced basis. For example, ZF obtains the estimation

$$\hat{x}' = Q \left\{ \{B'\}^y \right\}.$$ This estimate is then transformed back into $\hat{x} = U\hat{x}'$. Since the equivalent channel is much more likely to be well-conditioned, the effect of noise enhancement will be moderated. Note that due to the linear transform $U^{-1}$, it is no longer easy to control the boundary; thus it is typical to quantize to the nearest integer ignoring the signal boundary in [2].

Both linear and nonlinear detectors can also be designed with respect to the MMSE criterion. The extension to the
MMSE criterion is straightforward by dealing with the augmented channel matrix \cite{29}:

\[
B_a = \begin{bmatrix} B \\
\sigma I \end{bmatrix}.
\] (8)

When applied to the augmented channel matrix, ZF and SIC are equivalent to the standard MMSE and MMSE-SIC, respectively. In fact, MMSE is essential for infinite lattice decoding to achieve the optimum diversity and multiplexing tradeoff for finite constellations \cite{16}, \cite{17}. The proximity factors derived in this paper apply to the MMSE criterion as well, with the understanding that ZF is replaced by MMSE, and SIC replaced by MMSE-SIC.

III. LATTICES AND LATTICE REDUCTION

A lattice in the \(m\)-dimensional embedding Euclidean space \(\mathbb{R}^m\) is generated as the integer linear combination of some set of linearly independent vectors \cite{30}:

\[
L \triangleq L(B) = \left\{ \sum_{i=1}^{n} x_i b_i \mid x_i \in \mathbb{Z}, i = 1, \ldots, n \right\}
\] (9)

where \(B = [b_1, \ldots, b_n]\) is called the basis of \(L\).

A lattice \(L\) can be generated by infinitely many bases, of which one would like to select one that is in some sense nice or reduced. In many applications, it is advantageous to have the basis vectors as short as possible. Therefore, lattice reduction, the shortest vector problem (SVP) and CVP are closely related problems. All bases of the lattice arise by the transformation \(B' = BU\), where \(U\) is a unimodular matrix.

The dual lattice \(L^*\) of a lattice \(L\) is defined as those vectors \(u\), such that the inner product \(\langle u, v \rangle \in \mathbb{Z}\), for all \(v \in L\) \cite{30}. One might define the dual basis as \(B^T\). In this paper, we follow the definition of the dual basis \(B^* \triangleq B^{T*}J\) in \cite{31},

\[
J \triangleq \begin{bmatrix} 0 & \cdots & 0 & 1 \\
0 & \cdots & 1 & 0 \\
\vdots & & \ddots & \\
1 & \cdots & 0 & 0 \end{bmatrix}
\]

is the flipping matrix, i.e., it reverses the columns of \(B^{T*}\) in the left-right direction. Under this definition, the dual basis satisfy

\[
\langle b_i, b_k^* \rangle = \delta_{i,n-k+1}.
\] (10)

where \(b_k^*\) is the \(k\)-th column of \(B^*\) and \(\delta_{i,k}\) is the Kronecker delta. We have \(\det L^* = 1/\det L\) and \(L^{**} = L\).

For the sake of convenience, reduction of the dual basis will be referred to as dual reduction. Dual reduction still results in a reduced basis of the primary lattice. To see this, suppose \(U^*\) is the unimodular matrix arising from reducing the dual basis \(B^*\), then the corresponding reduced primary basis is given by

\[
B' = (B^{T*}JU^*)^{T*}J = BJ(U^*)^{T*}J
\]

Since \(J(U^*)^{T*}J\) is also a unimodular matrix, \(B'\) must be a basis of the primary lattice. In approximate lattice decoding, we can use \(J(U^*)^{T*}J\) as the transformation matrix in (6), whereas anything else remains the same.

\footnote{Interestingly, this formulation conveniently solves the problem of lattice reduction in the under-determined case \(m < n\).}

Let \(\lambda(L)\) be the length of the shortest vector in \(L\). The Hermite constant is defined as \cite{30}

\[
\gamma_n \triangleq \sup_L \frac{\lambda^2(L)}{\det^{2/n}(L)}.
\] (11)

where the suprema is taken over all lattices \(L\) of dimension \(n\). Hermite showed that \(\gamma_n\) is bounded \cite{30}. Mordell showed the inequality \(\gamma_n^{-2} \leq \gamma_{n-1}^{-2}\) \cite{32}. The upper bound was improved by Blichfeldt \cite{33}

\[
\gamma_n \leq \frac{2}{\pi} \left( 2 + \frac{n-1}{2} \right),
\] (12)

which has an asymptotic value \(n/\pi e\). Even better estimates are known as \(n\) goes to infinity \cite{31}:

\[
\frac{n}{2\pi e} + o(n) \leq \gamma_n \leq \frac{0.872n}{\pi e} + o(n), \quad n \to \infty.
\] (13)

Moreover, the Hermite constant is known exactly for \(n \leq 8\):

\[
\gamma_1 = 1, \quad \gamma_2 = 4/3, \quad \gamma_3 = 2, \quad \gamma_4 = 4, \quad \gamma_5 = 8, \quad \gamma_6 = 64/3, \quad \gamma_7 = 64, \quad \gamma_8 = 98/3.
\] (14)

The theory of lattice reduction \cite{30} started with Gauss who introduced the concept of a lattice and found short vectors in 2-dimensional lattices. Reduction for general \(n\) was proposed by Hermite. More manageable versions were given by Minkowski \cite{34}, and Korkin and Zolotarev (KZ) \cite{35}, which try to find short basis vectors recursively. For a Minkowski-reduced basis \(B, b_1\) is a shortest nonzero vector in \(L\), and \(b_i\) for \(i = 2, \ldots, n\), is the shortest vector in \(L\) such that \([b_1, \ldots, b_i]\) may be extended to a basis of \(L\). In KZ reduction, this is performed in the linear space orthogonal to the vectors already found. In 1982, Lenstra, Lenstra and Lovász proposed the first polynomial time algorithm which finds a vector not longer than \(2^{(n-1)/2}\) times the shortest nonzero vector.

Both KZ and LLL reduction can be seen as possible generalizations of the Gauss reduction, and they can be defined in terms of the Gram-Schmidt orthogonalization. One can compute the Gram-Schmidt orthogonalization by the recursion \cite{36}

\[
\hat{b}_i = b_i - \sum_{j=1}^{i-1} \mu_{i,j} \hat{b}_j, \quad \text{for } i = 1, \ldots, n
\] (15)

where \(\mu_{i,j} = \langle b_i, \hat{b}_j \rangle/||\hat{b}_j||^2\). Note that the Gram-Schmidt orthogonalization depends on the order of the basis vectors. In matrix notation, Gram-Schmidt orthogonalization can be written as \(B = \hat{B}\mu^T\), where \(\hat{B} = [\hat{b}_1, \ldots, \hat{b}_n]\), and \(\mu\) is a lower-triangular matrix with unit diagonal elements. In relation to the QR decomposition, one has \(\hat{b}_{i,j} = r_{i,j}/r_{i,i}\) and \(\hat{b}_i = r_{i,i} \cdot q_i\) where \(q_i\) is the \(i\)-th column of \(Q\).

A. KZ Reduction

A basis \(B\) is said to be KZ-reduced if it satisfies the following conditions \cite{31}:

\begin{itemize}
  \item \(b_1\) is the shortest nonzero vector of \(L\);
  \item \(|\mu_{i,j}| \leq 1/2\), for \(1 \leq j < i \leq n\);
  \item If \(L^{(n-1)}\) denotes that orthogonal projection of \(L\) on the orthogonal complement of \(\mathbb{R}b_1\), then the projections \(b_{n-1} - \mu_{i,1} b_i\) of \(b_2, \ldots, b_n\) yield a KZ basis of \(L^{(n-1)}\).
\end{itemize}

Equivalently, \(\hat{b}_1\) is a shortest vector of \(L^{(n-1)}\).
Finding a KZ basis is polynomial-time equivalent to finding the shortest vector. Hence, the KZ basis is in general the same as the Gaussian reduced basis.

**B. LLL Reduction**

A basis is LLL reduced if [37]
- $|\mu_{i,j}| \leq 1/2$, for $1 \leq j < i \leq n$;
- $\|b_i\|^2 + \mu_{i-1,j}^2 \|b_{i-1}\|^2 \geq \delta \|b_{i-1}\|^2$,

The first clause is called size reduction, and the second is known as the Lovász condition. The parameter $\delta = 3/4$ for a tradeoff between quality and complexity. It is easy to see from Fig. 1 that $d_{i,LD} = \|v_i\|/2$. The decision regions of both ZF and SIC are polyhedra with $2n$ facets and are symmetric with respect to the origin. Specifically, the ZF decision region is just the fundamental parallelepiped $P(B) = \{Ba_i \mid |a_i| \leq 1/2\}$ centered at $x = 0$. Meanwhile, the SIC decision region is a rectangle specified by the Gram-Schmidt vectors $\{Ba_i \mid |a_i| \leq 1/2\}$ [14], [18], [40].

Let $d_{i,LD}$ be the Euclidean distance from point $0$ to the $i$-th facet of $R_{LD}$. It is easy to see from Fig. 1 that $d_{i,LD} = \|v_i\|/2$. The decision regions of both ZF and SIC are polyhedra with $2n$ facets and are symmetric with respect to the origin. Specifically, the ZF decision region is just the fundamental parallelepiped $P(B) = \{Ba_i \mid |a_i| \leq 1/2\}$ centered at $x = 0$. Meanwhile, the SIC decision region is a rectangle specified by the Gram-Schmidt vectors $\{Ba_i \mid |a_i| \leq 1/2\}$ [14], [18], [40].

Let $d_{i,ZF}$ and $d_{i,SIC}$ be the Euclidean distance from point $0$ to the $i$-th facet of the decision regions of ZF and SIC, respectively. We are concerned with a spectrum of $n$ distances due to symmetry. It can be seen from Fig. 1 that the distance spectrum of ZF is given by

$$d_{i,ZF} = \frac{1}{2}\|b_i\| \sin \theta_i, \quad i = 1, ..., n$$

where $\theta_i$ denotes the angle between $b_i$ and the linear space spanned by the other $n-1$ basis vectors. In Appendix I we derive a formula to calculate $\theta_i$ from $B$ and its Gram-Schmidt orthogonalization, which will be used in later sections to bound $d_{i,ZF}$.

For SIC, the distance spectrum is given by

$$d_{i,SIC} = \frac{1}{2}\|b_i\| \sin \phi_i, \quad i = 1, ..., n$$

where $\phi_i$ is the angle between $b_i$ and the hyperplane spanned by $b_1, ..., b_{i-1}$. Note that $\|b_i\| \sin \theta_i$ is the distance between the vector point $b_i$ and the linear space spanned by the other $n-1$ basis vectors. Since $b_1$ only needs to be orthogonal to $b_1, ..., b_{i-1}$, we must have $\theta_i \leq \phi_i$ and hence $d_{i,ZF} \leq d_{i,SIC}$.

The minimum distance $d_{i,LD} = \inf \{d_{i,LD}\} = \lambda(L)/2$ plays an important role. We are motivated to define the proximity factors measuring the proximity between the performance of (infinite) lattice decoding and approximate lattice decoding as follows:

$$\rho_{i,ZF} = \sup_{B \in B_{\text{Recode}}} \frac{d_{i,ZF}^2}{d_{i,ZF}^2} = \sup_{B \in B_{\text{Recode}}} \frac{\lambda^2(L)}{\|b_i\|^2 \sin^2 \theta_i}$$

$$\rho_{i,SIC} = \sup_{B \in B_{\text{Recode}}} \frac{d_{i,SIC}^2}{d_{i,SIC}^2} = \sup_{B \in B_{\text{Recode}}} \frac{\lambda^2(L)}{\|b_i\|^2}$$

(16)

**Fig. 1.** Decision regions of ZF (dotted), SIC (dash-dotted) and (infinite) lattice decoding (solid) and the corresponding distances in a 2-dimensional lattice.
where the supremum is taken over the set $\mathcal{B}_{\text{Reduced}}$ of basis matrices satisfying a certain reduction criterion for any $n$-dimensional lattices $L$. Furthermore, we define $\rho_{\text{ZF}} \triangleq \max_{1 \leq i \leq n} \rho_i, \text{ZF}$ and $\rho_{\text{SIC}} \triangleq \max_{1 \leq i \leq n} \rho_i, \text{SIC}$, which quantify the worst-case loss of approximate lattice decoding in the minimum squared Euclidean distance. It is noteworthy that the proximity factors are not a function of any particular lattice $L$, and that without reduction they are unbounded. However, we shall show in the latter Sections that with lattice reduction the proximity factor is bounded by a constant that is a function of $n$ only.

B. Distance Properties of Dual Reduction

By \((10)\), $b^*_{n-i+1}$ of the dual basis is perpendicular to the \((n-1)\)-dimensional hyperplane spanned by vectors $b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_n$. Furthermore, since $(\pi/2 - \theta_i)$ is the angle between $b_i$ and $b^*_{n-i+1}$, we have

$$\langle b_i, b^*_{n-i+1} \rangle = \|b^*_{n-i+1}\| \cdot \|b_i\| \cos(\pi/2 - \theta_i) = \|b^*_{n-i+1}\| \cdot \|b_i\|.$$ 

Therefore, the following relation holds for dual reduction:

$$d_{i, \text{ZF}} = \frac{1}{2} \|b_i\| \sin \theta_i = \frac{1}{2} \|b^*_{n-i+1}\|^{-1}.$$ \hspace{1cm} (17)

It indicates that we can have large distances in ZF if the vectors of the dual basis are short.

**Proposition 1:** Let $B = \hat{B} \mu^T$ and $B^* = \hat{B}^*(\mu^*)^T$ be the Gram-Schmidt orthogonalization of the primal basis $B$ and dual basis $B^*$, respectively. Then

$$\hat{B}^* = \hat{B}^T J,$$

$$\mu^* = J \mu^{-T} J.$$ \hspace{1cm} (18)

**Proof:** By definition, we have that for matrix $B$ with full column rank

$$B^T = [(B^T B)^{-1} B^T]^T = B (B^T B)^{-T}.$$ Substituting $B = \hat{B} \mu^T$, we arrive at

$$B^T = \hat{B} \mu^T (\mu B^T \mu B^T)^{-T} = \hat{B} \hat{B}^T \hat{B}^{-T} \mu^{-1} = \hat{B}^T \mu^{-1}.$$ Hence the reversed dual basis can be written as

$$B^* = B^T J = \hat{B}^T \mu^{-1} J.$$ As $JJ = I$, it can be rewritten into

$$B^* = \hat{B}^T J \cdot \mu^{-1} J.$$ Since $J \mu^{-1} J$ is upper triangular with unit diagonal elements, and since the columns of $\hat{B}^T J$ are orthogonal to each other, this is exactly the Gram-Schmidt orthogonalization of $B^*$. Therefore, we have $\hat{B}^* = \hat{B}^T J$ and $(\mu^*)^T = J \mu^{-1} J$. As $J = J^T$, $\mu^* = J \mu^{-T} J$ holds. \hfill \Box

Meanwhile, by \((19)\),

$$d_{i, \text{SIC}} = \frac{1}{2} \|b_i\| = \frac{1}{2} \|b^*_{n-i+1}\|^{-1}.$$ \hspace{1cm} (20)

It shows that SIC can have large distances if the reversed dual basis has short Gram-Schmidt vectors.

C. Performance Bounds for Approximate Lattice Decoding

For each decoder, an error occurs when the noise falls outside of $\mathcal{R}$. Accordingly, given the basis $B$, the error probability for vector $x$ is given by

$$P_e(B) = P(\hat{x} \neq 0 | x = 0) = P(n \in \mathcal{R}).$$ \hspace{1cm} (21)

To keep the results general, we write SNR $= c/\sigma^2$, where $c$ is a constant depending on the problem. By the symmetry of the Voronoi cell (i.e., there are at least two vectors with the shortest length,) we have the lower bound on the conditional decoding error probability of (infinite) lattice decoding

$$P_{e,\text{LD}}(\text{SNR}, B) \geq 2Q \left( \frac{d_{\text{LD}}}{\sigma} \right) = 2Q \left( \sqrt{\frac{d_{\text{LD}}^2 \cdot \text{SNR}}{c}} \right).$$ \hspace{1cm} (22)

Meanwhile, the union bound on the conditional error probability of ZF reads

$$P_{e,\text{ZF}}(\text{SNR}, B) \leq 2 \sum_{i=1}^{n} Q \left( \frac{d_{i, \text{ZF}}}{\sigma} \right)$$ \hspace{1cm} (23)

where the factor 2 is due to symmetry. The union bound for SIC admits a form similar to \((23)\). Given the same basis matrix $B$, the conditional error probability of lattice reduction-aided ZF can be bounded above as

$$P_{e,\text{ZF}}(\text{SNR}, B) \leq 2 \sum_{i=1}^{n} Q \left( \frac{d_{i, \text{ZF}}}{\sigma} \right) = 2 \sum_{i=1}^{n} Q \left( \sqrt{\frac{d_{i, \text{LD}}^2 \cdot \text{SNR}}{c \cdot \rho_{i, \text{ZF}}}} \right).$$ \hspace{1cm} (24)

since $d_{i, \text{ZF}} \geq \rho_{i, \text{ZF}} \cdot d_{i, \text{LD}}^2$ by definition \((16)\) and since $Q(\cdot)$ is a decreasing function. It is worth pointing out that while the distance $d_{i, \text{LD}}$ is a function of $B$, $\rho_{i, \text{ZF}}$ is not.

Now, combining \((22)\) and \((24)\), we have

$$P_{e,\text{ZF}}(\text{SNR}, B) \leq \sum_{i=1}^{n} P_{e,\text{LD}} \left( \text{SNR} \left/ \rho_{i, \text{ZF}} \right. \right).$$ \hspace{1cm} (25)

Since \((25)\) holds for any $B$, averaging out $B$ we obtain

$$P_{e,\text{ZF}}(\text{SNR}) \leq \sum_{i=1}^{n} P_{e,\text{LD}} \left( \text{SNR} \left/ \rho_{i, \text{ZF}} \right. \right)$$ \hspace{1cm} (26)

for arbitrary SNR. In particular,

$$P_{e,\text{ZF}}(\text{SNR}) \leq n P_{e,\text{LD}} \left( \text{SNR} \left/ \rho_{\text{ZF}} \right. \right).$$ \hspace{1cm} (27)

The relations \((26)\) and \((27)\) hold irrespective of fading statistics, and similar relations exist for SIC. They reveal, in a quantitative manner, that approximate lattice decoding performs within a constant bound from lattice decoding. The mere effect on the error rate curve is a shift from that of lattice decoding, up to a multiplicative factor $n$, which obviously does not change the diversity order. In other words, the diversity order is the same as that of lattice decoding. Therefore, existing results on the diversity order
of lattice decoding can be extended to approximate lattice decoding. In particular, since lattice decoding with MMSE pre-processing achieves the optimal diversity-multiplexing tradeoff for approximately universal codes [17], approximate lattice decoding with MMSE pre-processing also does. Moreover, since lattice decoding achieves full receive diversity in the uncoded V-BLAST system [22], approximate lattice decoding also achieves full diversity. This provides an alternative way of showing the diversity order of lattice-reduction aided decoding given in [17], [22].

D. Main Results

Table I shows the main results of this paper, i.e., the bounds on proximity factors for reducing the primary basis and for reducing the dual basis. The derivations will be given in the following two sections. Notice that the value 4/3 is exact for Gaussian reduction, and could be obtained by specializing the bounds for LLL or KZ reduction. In the meantime, the bounds for LLL and KZ reduction in Table I are less explicit, i.e., they are expressed in the parameter \( \beta \) for LLL reduction and in the Hermite and KZ constants for KZ reduction. To give the readers a feeling about the dependence of the bounds on \( n \), Table II shows more explicit bounds for LLL and KZ reduction, where the bounds are however less tight. For LLL reduction, the bounds on proximity factors are exponential; if expressed in dB, they are linear with \( n \). It is seen that reducing the dual basis leads to smaller bounds in many cases. Namely, when the dual basis is reduced, LLL-ZF has a smaller base in its exponential bound, while KZ reduction results in polynomial bounds on proximity factors for both ZF and SIC. It is also interesting that, unlike primal basis reduction, LLL-ZF and LLL-SIC have the same bounds on proximity factors when the dual basis is reduced.

V. PRIMAL BASIS REDUCTION

With lattice reduction, the supremum in (16) is taken over the set of reduced bases of the lattice \( L \). Consequently, the proximity factors will be bounded. In this Section, we shall derive explicit upper bounds for LLL and KZ reductions. The bounds are expressed in closed form and turn out to be functions of \( n \) only.

A. LLL Reduction

The derivation for LLL reduction will be done by adapting the techniques of the original LLL paper [37] and Babai [13]. While the exponential bounds in [13], [37] have the best base, in this subsection we will improve the bounds by a constant factor.

Substituting this into the representation of \( b_i \) in Gram-Schmidt vectors

we obtain

\[
\|b_i\|^2 \leq \left( 1 + \sum_{j=1}^{i-1} \beta^{-j} / 4 \right) \|\hat{b}_i\|^2.
\]

Note that (31) was also given in [37] for the special case \( \delta = 3/4 \).

Let \( B_{\text{LLL}} \) be the set of LLL-reduced basis matrices. Since \( \lambda(L) \leq \|b_i\| \), the loss in the \( i \)-th squared Euclidean distance is bounded by

\[
\rho_{i, \text{SIC}} = \sup_{B \in B_{\text{LLL}}} \frac{\lambda^2(L)}{\|b_i\|^2}
\leq \left( 1 + \frac{\beta^i - \beta}{4(\beta - 1)} \right) \sup_{B \in B_{\text{LLL}}} \frac{\lambda^2(L)}{\|b_i\|^2}
\leq 1 + \frac{\beta^i - \beta}{4(\beta - 1)}.
\]

It is easy to verify that when \( \beta \geq 4/3 \) (which is the case for LLL reduction) (32) is smaller than the bound

\[
\rho_{i, \text{SIC}} \leq \beta^{i-1}
\]

directly obtainable from [37]. Obviously,

\[
\rho_{\text{SIC}} \leq 1 + \frac{\beta^n - \beta}{4(\beta - 1)} \leq \beta^{n-1}.
\]

2) ZF: The derivation for LLL-ZF needs the following lemma. It improves Babai’s lower bound on \( \theta_i \) [13] and extends it to arbitrarily values of \( \delta \in (1/4, 1) \). As recognized by Babai [13], the lower bound on \( \theta_i \) describes a geometric feature of an LLL-reduced basis, which is of independent interest. The proof is given in Appendix B.

Lemma 1: If \( B \) is an LLL-reduced basis of lattice \( L(B) \), then

\[
\sin^2 \theta_i \geq \left[ \frac{\beta}{9\beta - 4} \left( \frac{9\beta}{4} \right)^{-n-i} + \frac{8\beta - 4}{9\beta - 4} \right]^{-1} \times \left[ 1 + \frac{\beta^i - \beta}{4(\beta - 1)} \right]^{-1}
\]

for \( 1 \leq i \leq n \).

It is worth mentioning that the lower bound of Lemma 1 is tight. The equality is achieved when \( \|\hat{b}_j\|^2 = \beta^{i-j} \|\hat{b}_i\|^2 \) and \( \mu_{i,j} = -1/2 \) for all \( 1 \leq j < i \leq n \). This can be seen from the proof.

| Primal Reduction | LLL-ZF | LLL-SIC | KZ-ZF | KZ-SIC |
|------------------|--------|--------|------|------|
|                  | \( (2^{\frac{n}{2}})^n \) | \( 2^n \) | \( (2^{\frac{n}{2}})^n \) | \( n^{1+\ln n} \) |
|                  | \( 2^n \) | \( 2^n \) | \( n^{1+\ln n} \) | \( n^2 \) |
TABLE I

Upper Bounds on Proximity Factors (n: dimension; \( \beta = 1/(\delta - 1/4) \), \( 1/4 < \delta \leq 1 \); \( \gamma_j \): Hermite constant; \( \xi \): KZ constant)

| Reduction       | Gauss-ZF | Gauss-SIC | LLL-ZF | LLL-SIC | KZ-ZF | KZ-SIC |
|-----------------|----------|-----------|--------|---------|-------|--------|
| Primal          | \( \frac{4}{3} \) | \( \frac{4}{3} \) | \( 9\beta/4 \) \( \beta^{n-1} \) | \( 1 + \frac{4n}{(\beta - 1)} \) | \( \xi_n \) | \( \xi_n \) |
| Dual            | \( \frac{4}{3} \) | \( \frac{4}{3} \) | \( \beta^{n-1} \) | \( \beta^{n-1} \) | \( 1 + \frac{1}{4} \sum_{j=2}^{n} \gamma_j^2 \) | \( \max_{1 \leq j \leq n} \gamma_j^2 \) |

Applying Lemma 1 and the trivial inequality \( \|\hat{b}_i\| \geq \lambda(L) \) to

\[
\rho_{i,ZF} \leq \sup_{B \in B_{L^2}} \frac{\lambda^2(L)}{\|\hat{b}_i\|^2 \sin^2 \theta_i},
\]

we obtain the following theorem:

**Theorem 2:** For LLL-ZF, the proximity factors are bounded by

\[
\rho_{i,ZF} \leq \left[ \frac{\beta}{9\beta - 4} \left( \frac{9\beta}{4} \right)^{n-i} + \frac{8\beta - 4}{9\beta - 4} \right] \left[ 1 + \frac{1}{4} \beta^i - \beta \right].
\]

Since the upper bound is maximized when \( i = 1 \), we have

\[
\rho_{ZF} \leq \frac{\beta}{9\beta - 4} \left( \frac{9\beta}{4} \right)^{n-1} + \frac{8\beta - 4}{9\beta - 4}.
\]

This is better than the bound \( \rho_{ZF} \leq \left( \frac{\beta}{4} \right)^{n-1} \) obtained previously in [19].

**B. KZ Reduction**

1) SIC: To derive the proximity factors for SIC with KZ reduction, we make use of the KZ constant defined in [42] as

\[
\xi_n \triangleq \sup_{B \in B_{KZ}} \frac{\|b_n\|^2}{\|\hat{b}_n\|^2}
\]

where \( B_{KZ} \) is the set of all KZ-reduced bases of dimension \( n \).

The KZ constant is bounded in terms of the Hermite constant by [42]

\[
\xi_n \leq \gamma_n \gamma_n^{1/(n-1)} \gamma_n^{1/(n-2)} \ldots \gamma_2
\]

for \( n \geq 2 \). Since \( \|b_n\| = \lambda(L) \) in a KZ-reduced lattice, we have

\[
\rho_{i,SIC} = \xi_i.
\]

To obtain \( \rho_{i,SIC} \) for \( i < n \), note that if \( B \) is a KZ-reduced basis of lattice \( L(B) \), then \( B_i = [b_1, \ldots, b_i] \) is a KZ reduced basis of the sublattice \( L_i = L(B_i) \). This follows from the definition of KZ reduction and the fact that \( L_i \subseteq L \). Accordingly, we have

**Theorem 3:** For KZ-SIC, the proximity factors are bounded by

\[
\rho_{i,SIC} = \xi_i.
\]

Since the KZ constant \( \xi_n \) is a nondecreasing function of \( n \) [42], we have \( \rho_{SIC} = \xi_n \), i.e., the KZ constant exactly quantifies the worst-case performance loss of SIC. The KZ constant is exactly known for \( n = 2 \) and \( n = 3 \): \( \xi_2 = 4/3 \) and \( \xi_3 = 3/2 \) [42]. It is also known that [31], [42]

\[
\xi_n \leq n^{1+ln n},
\]

which is sub-exponential.

2) ZF:

**Theorem 4:** For KZ-ZF, the proximity factors are bounded by

\[
\rho_{i,ZF} \leq \xi_i + \sum_{j=1}^{n-i} \frac{1}{3} \left( \frac{3}{2} \right)^{2j} \xi_{i+j}.
\]

**Proof:** By (60), we have

\[
\rho_{i,ZF} = \sup_{B \in B_{KZ}} \frac{\lambda^2(L)}{\|\hat{b}_i\|^2 \sin^2 \theta_i} \leq \sup_{B \in B_{KZ}} \frac{\lambda^2(L)}{\sum_{j=i}^{n} \gamma_j^2 \|\hat{b}_j\|^2}.
\]

Applying (65) yields

\[
\rho_{i,ZF} \leq \sup_{B \in B_{KZ}} \left[ \frac{\lambda^2(L)}{\|\hat{b}_i\|^2} + \sum_{j=1}^{n-i} \left( \frac{1}{3} \right) \left( \frac{3}{2} \right)^{2j} \xi_{i+j} \right].
\]

Using the facts that \( \lambda^2(L) = \|\hat{b}_1\|^2 \) and \( \|\hat{b}_1\|^2 \leq \xi_j \|\hat{b}_j\|^2 \) for a KZ-reduced basis, we obtain (42).

The maximum is attained when \( i = 1 \), and

\[
\rho_{ZF} \leq 1 + \sum_{j=1}^{n-i} \left( \frac{1}{3} \right) \left( \frac{3}{2} \right)^{2j} \xi_{j+1}.
\]

Using the upper bound (41), we obtain a more explicit form

\[
\rho_{ZF} \leq \left( \frac{9}{4} \right)^{n-1} \cdot n^{1+ln n}.
\]

**VI. Dual Basis Reduction**

In this Section, we shall derive upper bounds on the proximity factors when the dual rather than the primal basis is reduced.

**A. Dual LLL Reduction**

1) SIC:

**Theorem 5:** For SIC with dual LLL reduction, the proximity factors are bounded by

\[
\rho_{i,SIC} \leq \beta^{i-1}.
\]

**Proof:** A scrutiny into the derivation of \( \rho_{SIC} \) with primal-basis reduction reveals that the following condition is crucial:

\[
|\mu_{i,i-1}| \leq 1/2, \quad 1 < i \leq n.
\]

A basis satisfies the Lovász condition and (47) is called an effectively LLL-reduced basis [41]. Effective LLL reduction produces a basis with the same set of Gram-Schmidt vectors. Accordingly, with SIC, it yields the same output as the standard LLL reduction. Moreover, if a basis is effectively...
LLL-reduced, so is its dual basis [41]. This is because the inverse of the coefficient matrix \( \mu \) admits the form [43]
\[
\mu^{-1} = \begin{bmatrix}
1 & 0 & \cdots & 0 & 0 \\
-\mu_{2,1} & 1 & \cdots & 0 & 0 \\
\times & -\mu_{3,2} & 1 & \cdots & 0 \\
\times & \cdots & \times & -\mu_{n,n-1} & 1
\end{bmatrix}
\]
and the Lovász condition can be shown to be satisfied for the dual basis.

Although other off-diagonal elements may have absolute values larger than 1/2, it does not really cause a problem, since we still have
\[
\rho_{i, \text{SIC}} = \sup_{B^* \in \mathcal{B}_{\text{LLL}}} \frac{\lambda^2(L)}{\| \hat{b}_i \|^2} \leq \beta^{i-1} \sup_{B^* \in \mathcal{B}_{\text{LLL}}} \frac{\lambda^2(L)}{\sum_{j=1}^{n} \| b_j \|^2} \leq \beta^{i-1}.
\]

Note that this is the same as (33) for primal LLL reduction. Accordingly, we expect similar performance for primal and dual LLL reductions.

2) ZF:

**Theorem 6**: For ZF with dual LLL reduction, the proximity factors are bounded by
\[
\rho_{i, \text{ZF}} \leq \left( 1 + \frac{\beta}{4}, \frac{\beta^n - i - 1}{\beta - 1} \right)^{\beta^{i-1}}.
\]

**Proof**: The derivation of the bound on \( \rho_{i, \text{ZF}} \) is analogous to that for primal LLL reduction. We start by
\[
\rho_{i, \text{ZF}} \leq \sup_{B^* \in \mathcal{B}_{\text{LLL}}} \frac{\lambda^2(L)}{\sum_{j=1}^{n} \| b_{i+j} \|^2},
\]
which holds for any bases. Incorporating the lower bound [69] in Appendix III we deduce
\[
\rho_{i, \text{ZF}} \leq \sup_{B^* \in \mathcal{B}_{\text{LLL}}} \lambda^2(L) \left( \| \hat{b}_i \|^2 + \frac{1}{4} \sum_{j=1}^{n-i} \| b_{i+j} \|^2 \right).
\]

Besides, the primal basis is effectively LLL-reduced when the dual basis is LLL-reduced. Thus, \( \| \hat{b}_i \|^2 \geq \beta^{-(i-1)} \| b_1 \|^2 \). We obtain
\[
\rho_{i, \text{ZF}} \leq \sup_{B^* \in \mathcal{B}_{\text{LLL}}} \frac{\lambda^2(L)}{\| b_1 \|^2} \left( 1 + \frac{1}{4} \sum_{j=1}^{n-i} \beta^j \right)^{\beta^{i-1}}
\]
\[
= \left( 1 + \frac{1}{4} \sum_{j=1}^{n-i} \beta^j \right)^{\beta^{i-1}},
\]
which leads to (49).

Since the right-hand side of (49) is an increasing function of \( i \) for \( \beta \geq 4/3 \), we have
\[
\rho_{\text{ZF}} \leq \beta^{i-1}.
\]

We emphasize that the worst case bound (52) is the same as dual LLL-SIC and is close to that for SIC with primal LLL reduction.

### B. Dual KZ Reduction

1) SIC: It is shown in [31] that if a reversed dual basis is KZ reduced, then \( \gamma_i \| \hat{b}_i \| \geq \lambda(L) \). Accordingly, we have

**Theorem 7**: For SIC with dual KZ reduction, the proximity factors are bounded by
\[
\rho_{i, \text{SIC}} \leq \gamma_i^2.
\]

Although the Hermite constant is very likely to be an increasing function, it has never been proved [31]. Hence, we bound the overall proximity factor for SIC by
\[
\rho_{\text{SIC}} \leq \max_{1 \leq i \leq n} \gamma_i^2.
\]

Since \( \gamma_n \leq 2n/3 \), we have \( \rho_{\text{SIC}} \leq (2n/3)^2 < n^2 \). A better bound can be obtained by employing Blichfeldt’s bound on the Hermite constant:
\[
\rho_{\text{SIC}} \leq \left( \frac{2}{\pi} \right)^{2(n+n)} \left( 2 + \frac{n}{2} \right).
\]

We claim that \( \gamma_i^2 \) is never greater than the bound on the KZ constant (40). We prove this by induction. When \( i = 1 \), \( \xi_1 = \gamma_1^2 = 1 \), and the claim is true. Suppose it is true when \( k = i - 1 \), from which we deduce \( \gamma_{i-1} \leq \lambda_1/(i-2) \cdots \gamma_2 \). When \( k = i \) we have
\[
\gamma_i \gamma_{i-1}/(i-1) \gamma_{i-2}/(i-2) \cdots \gamma_2 \leq \gamma_i \gamma_{i-1}/(i-1) \gamma_{i-1}/(i-1) = \gamma_i \gamma_1/(i-1)^2.
\]

By Mordell’s inequality \( \gamma_i \gamma_{i-2} \leq \gamma_{i-1}^2 \) [32], this is not greater than one. Thus the claim is true for \( k = i \) as well.

Therefore, reducing the reversed dual basis again has a smaller bound on \( \rho_{i, \text{SIC}} \) than reducing the primal basis.

2) ZF: Substituting \( \gamma_j \| \hat{b}_j \| \geq \lambda(L) \) for dual KZ reduction [31] into (50), we arrive at

**Theorem 8**: For ZF with dual KZ reduction, the proximity factors are bounded by
\[
\rho_{i, \text{ZF}} \leq \gamma_i^2 + \frac{1}{4} \sum_{j=1}^{n} \gamma_j^2.
\]

We claim that the right-hand side of (56) decreases with \( i \). To see this, it is sufficient to show \( \gamma_i^2 \geq \frac{1}{4} \gamma_{i+1}^2 \). We apply Mordell’s inequality together with the inequality \( \gamma_n \leq 2/\sqrt{3} \) \( n^{-1} \), yielding
\[
\gamma_{i+1} \leq \gamma_1/(i-1) = \gamma_{i+1}^{1/(i-1)} \leq \gamma_i : 2/\sqrt{3}.
\]

Thus the claim is proven.

Accordingly, the right-hand side of (56) attains the maximum when \( i = 1 \), and
\[
\rho_{\text{ZF}} \leq 1 + \frac{1}{4} \sum_{j=2}^{n} \gamma_j^2.
\]

Using the bound \( \gamma_j \leq 2j/3 \) for \( j \geq 2 \), we can obtain
\[
\rho_{\text{ZF}} \leq \frac{8}{9} + \frac{1}{4} \frac{n(n+1)(2n+1)}{6}.
\]

The right side of (58) is never larger than \( n^3/27 \) as \( n \) goes large.
as far as the Gram-Schmidt vectors are concerned, gaining due to dual KZ reduction is not significant. The reason is much smaller. Meanwhile, we can see that, for SIC, the bound for reducing the dual basis is smaller for the purpose of ZF.

Obviously, for ZF, the bound for reducing the dual basis is smaller for the purpose of ZF.

LLL reduction. Obviously, the bound for reducing the dual basis is smaller than those for LLL reduction. This is expected since KZ reduction is a stronger notion of lattice reduction.

To visualize the behaviors of proximity factors, we plot the best upper bounds in Table 1. In Fig. 2, we show the upper bounds on the proximity factors for KZ reduction and for LLL reduction with best performance (i.e., δ = 1). For KZ reduction, we employ (44) and (40) in conjunction with the bound (12) on the Hermite constant and the exact value (14) for n ≤ 8. It is clear that the bounds on proximity factors for KZ reduction are smaller than those for LLL reduction. This is expected since KZ reduction is a stronger notion of lattice reduction.

Fig. 3 shows the upper bounds on ρZF for primal and dual LLL reduction. Obviously, the bound for reducing the dual basis is smaller for the purpose of ZF.

In Fig. 4 we show the upper bounds on the proximity factors for KZ reduction of the primal as well as dual bases. Obviously, for ZF, the bound for reducing the dual basis is much smaller. Meanwhile, we can see that, for SIC, the gain due to dual KZ reduction is not significant. The reason is that as far as the Gram-Schmidt vectors are concerned, the primal and dual KZ reduction are in fact not too much different, as also recognized in [10]. Because the product \( \prod_{i=1}^n \|b_i^\dagger\| = \det L \) is invariant under lattice reduction, choosing short Gram-Schmidt vectors in the beginning will force long Gram-Schmidt vectors in the end, and vice versa.

Compared with primal basis reduction, the dual-basis reformulation is more natural in the following senses.

For ZF, we have \( B^\dagger y = x + B^\dagger n \). Then, the noise variance associated with the \( i \)-th output is proportional to \( \|b_i^\dagger\|^2 \). Obviously, it makes sense to minimize \( \|b_i^\dagger\|^2 \).

Most existing reduction notions aim to find short vectors one after another; hence when applied to the dual basis, they can be viewed as various greedy heuristics of finding the optimal bases for decoding purposes. In particular, the dual Minkowski reduction can be viewed as a greedy algorithm to approximate the ZF criterion, while the dual KZ reduction as a greedy algorithm to approximate the SIC criterion, since they will successively find the shortest vector and the shortest Gram-Schmidt vector for the dual basis, respectively.

To verify the benefit of dual reduction for LLL-ZF, we simulate the BER performance of an 8 × 8 MIMO system with i.i.d. complex standard normal channel coefficients. The complex basis matrix \( H \) is converted into its real equivalent by

\[
B = \begin{bmatrix}
\Re(H) & -\Im(H) \\
\Im(H) & \Re(H)
\end{bmatrix},
\]

and then \( B \) is reduced. The signal constellation at each antenna is 64QAM with Gray mapping. With such a signal model, the SNR at each receive antenna is defined as

\[
\text{SNR} = n_T E_x x^2 / \sigma^2,
\]

where \( n_T \) is the number of transmit antennas. Fig. 5 shows the simulated BER of ZF with primal LLL reduction and dual LLL reduction. When the decoded lattice point falls outside of the signal boundary, we simply round it componentwise back to the 64QAM alphabet. It is seen that the latter exhibits about 3 dB gain at the BER of \( 10^{-5} \). Lattice decoding ignoring the boundary is also simulated, whose performance is about 1.5 dB worse than ML decoding at the BER of \( 10^{-5} \).

A similar amount of gain is observed for ZF with primal and dual KZ reduction. In the simulation of SIC, though, only
a small difference in the BER is observed between primal and dual reduction. This is not surprising given their close proximity factors (cf. Table I).

We assess the tightness of the bounds by means of numerical experimentation. Each time, a basis matrix \( \mathbf{B} \) is randomly generated and reduced, and then the minimum distance of the decision region (for ZF and SIC, respectively) as well as the shortest vector is found. For each value of \( n \), an empirical proximity factor is obtained as the maximum over a ensemble of 10000 i.i.d. Gaussian matrices \( \mathbf{B} \). While there is no guarantee that this maximum reaches the worst-case bound, it should be a reasonable indicator of the theoretical proximity factor. In particular, it can serve as a experimental lower bound on the theoretical proximity factor. Fig. 5 shows the results for (primal-basis) LLL reduction with \( \delta = 3/4 \). The numerical results (in dB) still grow linearly, but at a lower slope. Fig 6 also shows the real SNR gap to ML decoding observed in computer simulation for \( n = 8, 12, 16 \). To estimate the real SNR gap, the BER performance is simulated, and curve like those in Fig. 5 are obtained. It can be seen that the real SNR gap is even smaller. This behavior is consistent with the common belief that in practice LLL reduction performs better than the theoretical upper bounds. Fig. 7 demonstrates the numerical results for KZ reduction. The similar trend can be seen.

In summary, we have shown that lattice reduction-aided decoding is in proximity to (infinite) lattice decoding. We have derived analytic bounds on proximity factors that quantify the worst-case gap. The same diversity order as that of lattice decoding in MIMO fading channels comes as a direct consequence. According to the analysis, dual lattice reduction could give smaller upper bounds, especially in ZF where performance gain has been observed in computer simulation. Consequently. According to the analysis, dual lattice reduction could give smaller upper bounds, especially in ZF where performance gain has been observed in computer simulation. Furthermore, we point out several open issues of lattice reduction-aided decoding.

Firstly, there is room to tighten the bounds derived in this paper. In Table I the bounds are tight for Gauss reduction and for KZ-SIC with primal reduction. The bound is very likely to be tight for KZ-SIC with dual reduction; it will be tight as long as the Hermite constant is an increasing function which is very likely to be true. The bound is quite good for KZ-ZF with dual reduction (cf. Fig. 4). However, the bounds are not tight for LLL reduction and KZ-ZF with primal reduction. For LLL reduction, we have applied the trivial bound \( \lambda_2(L)/\|\mathbf{b}_1\|^2 \leq 1 \) or \( \lambda_2(L)/\|\mathbf{b}_1\|^2 \leq 1 \). But this is likely to loosen the bound, since \( \beta^{-n-1} \leq \lambda_2(L)/\|\mathbf{b}_1\|^2 \) and \( \beta^{-n-1} \leq \lambda_2(L)/\|\mathbf{b}_1\|^2 \) for LLL reduction [37].

Secondly, we emphasize that the worst-case bounds should be carefully interpreted. The average-case performance may be a more meaningful measure if the basis \( \mathbf{B} \) is random. A smaller proximity factor does not necessarily guarantee better average performance. For example, KZ reduction must have smaller approximation factors than LLL reduction, but their BER performance appears to be very close in MIMO fading channels [15]. Hence an average-case analysis will complement the worst-case analysis. In simulations of LLL
reduction, we observed that the SNR gap (in dB) indeed widens linearly with the dimension, but at a lower slope. This behavior is consistent with the common belief that in practice LLL reduction performs better than the theoretic upper bounds. Recently, Nguyen and Stehlé showed that the average behavior of LLL reduction is still exponential, but at a slower rate [44].

Thirdly, the analysis of dual reduction naturally raises the issue of simultaneously reducing the primal and dual bases. Methods of simultaneous reduction exist, notably those of Seyer [45] and Howgrave-Graham [46].

APPENDIX I
The Angle $\theta_i$ of a Basis

We derive a formula for angle $\theta_i$ of a basis $B$, which is used to compute the distance of $ZF$. Following Babai [13], let $a = \sum_{j \neq i} a_j b_j$ $(a_j \in R$ for $j \neq i$) be an arbitrary vector on the hyperplane spanned by vectors $b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_n$. The vector $z = a - b_i$ will attain its minimum length if only if it is perpendicular to the hyperplane. We express $z$ in terms of the Gram-Schmidt vectors as $z = \sum_{j=1}^n r_j \hat{b}_j$. Set $\alpha_i = -1$. Then we have the expression [37]

$$r_j = \alpha_j + \sum_{t=j+1}^n \alpha_t u_{t,j}.$$ 

We can see

$$\|b_i\| \sin^2 \theta_i = \min \|z\|^2 = \min \sum_{j=1}^n r_j^2 \|\hat{b}_j\|^2$$

(59)

where the minimum is taken over the coefficients $\alpha_j$. It is not difficult to see that $r_j = 0$ for $j < i$ if $z$ is orthogonal to the hyperplane. Accordingly, we may exclude the $j < i$ terms, to yield

$$\|b_i\| \sin^2 \theta_i = \sum_{j=i}^n r_j^2 \|\hat{b}_j\|^2.$$ 

(60)

This was also done by Babai [13]. Here we have explicitly shown that excluding the $j < i$ terms does not weaken the bound.

For notational convenience, define $\alpha = [\alpha_i, \ldots, \alpha_n]^T$, $r = [r_1, \ldots, r_n]^T$, $M$ be $(n - i + 1) \times (n - i + 1)$ bottom-right submatrix of $\mu$, $A$ be the diagonal matrix diag($\|\hat{b}_1\|^2, \|\hat{b}_{i+1}\|^2, \ldots, \|\hat{b}_n\|^2$), and $M = MAM^T$. We have the expressions $r = M^T \alpha$ and

$$\sum_{j=1}^n r_j^2 \|\hat{b}_j\|^2 = r^T A r = \alpha^T A \alpha.$$ 

We want to find the minimum of $\alpha^T A \alpha$ subject to the constraint $\alpha_i = -1$. This can be achieved by using the Lagrangian multiplier method. Define the objective function

$$f = \alpha^T A \alpha + s(\alpha_i + 1).$$

Nulling the partial derivative of $f$ with respect to $\alpha$, we have

$$2A \alpha + se_1 = 0.$$ 

where the unit vector $e_1$ has all zero elements except that the first is one. From this we obtain $\alpha = -\frac{s}{2} A^{-1} e_1$. Furthermore, since we know $\alpha_i = -\frac{s}{2} (A^{-1})_{1,1} = -1$, we have $s = 2/(A^{-1})_{1,1}$. Therefore,

$$f_{min} = \frac{s^2}{4} e_i^T A^{-1} e_i = \frac{s^2}{4} (A^{-1})_{1,1} = \frac{1}{(A^{-1})_{1,1}},$$

(61)

and from (59) we deduce

$$\sin^2 \theta_i = \frac{1}{\|b_i\|^2 (A^{-1})_{1,1}}.$$ 

(62)

We have expressed $\theta_i$ in terms of $b_i$ and the Gram-Schmidt orthogonalization of $B$. Given $B$, the angle $\theta_i$ can easily be calculated.

APPENDIX II
Proof of Lemma [1]

When deriving a lower bound on $\theta_i$, Babai loosened the bound in several steps. It is natural to ask if his bound can be improved. In Appendix II we have seen that excluding the $j < i$ terms from (59) does not weaken the bound. Babai showed [13]

$$\sum_{j=i}^n r_j^2 \|\hat{b}_j\|^2 \geq (2/9)^{n-i} \|\hat{b}_i\|^2.$$ 

(63)

Here we derive a better lower bound by using a different approach. Basically, we examine the maximum of the first element of matrix $A^{-1} = M^{-T} A^{-1} M^{-1}$ for a size-reduced (lower-triangular) matrix $M$. To do so, we need the following lemma:

**Lemma 2:** The absolute value of the $(i, j)$-th entry of the inverse $M^{-1}$ of an $l \times l$ size-reduced matrix $M$ is not greater than

$$\frac{1}{3} \left( \frac{3}{2} \right)^{i-j}, \quad 1 \leq j < i \leq l,$$

(64)

and this is achieved when all off-diagonal elements of $M$ are equal to $-1/2$.

**Proof:** We prove it by induction on the dimension $l$. Note that $M^{-1}$ is lower triangular, with unit diagonal elements.

Lemma [2] is obviously true when $l = 2$, since the off-diagonal element of $M^{-1}$ is $-m_{2,1}$, and the maximum 1/2 is achieved when $m_{2,1} = -1/2$.

Suppose Lemma [2] is true for $l = k - 1$. When $l = k$, we partition the matrix $M$ into the form

$$M = \begin{bmatrix} M_{k-1} & 0 \\ h^T & 1 \end{bmatrix}.$$ 

Using the formula for the inverse of a partitioned matrix [47], $M^{-1}$ can be expressed as

$$M^{-1} = \begin{bmatrix} M_{k-1}^{-1} & 0 \\ -h^T M_{k-1}^{-1} & 1 \end{bmatrix}.$$ 

Here, the $(i,j)$-th element of $M_{k-1}^{-1}$ is given by (64) for $1 \leq j < i \leq k - 1$. Since they are all positive, each element of $-h^T M_{k-1}^{-1}$ is maximized when $h$ is an all $-1/2$ vector.

Consider the first element of $-h^T M_{k-1}^{-1}$, given by

$$\frac{1}{2} \left[ 1 + \sum_{j=1}^{k-2} \frac{1}{3} \left( \frac{3}{2} \right)^j \right] = \frac{1}{3} \left( \frac{3}{2} \right)^{k-1},$$
which verifies (63). The other elements of $-h^T M^{-1}_{k-1}$ can be verified in the same way.

Denote by $p$ the first column of $M^{-1}$. Then we have

$$\begin{align*}
(A^{-1})_{1,1} &= p^T A^{-1} p \\
&\leq \|\hat{b}_i\|^{-2} + \sum_{j=1}^{n-i} \left(\frac{1}{3}\right)^2 \left(\frac{3}{2}\right)^j \|\hat{b}_{i+j}\|^{-2}. \quad (65)
\end{align*}$$

Note that the only condition for (65) is that the basis is size-reduced, i.e., $|\mu_{i,j}| \leq 1/2$ for $i < j$. The choice of the parameter $\delta$ would have no effect on this condition.

For an LLL-reduced basis, $\|\hat{b}_{i+j}\|^2 \geq \beta^{-j} \|\hat{b}_i\|^2$. Hence,

$$\begin{align*}
(A^{-1})_{1,1} &\leq \left[1 + \sum_{j=1}^{n-i} \left(\frac{1}{3}\right)^2 \left(\frac{3}{2}\right)^j \beta^j\right] \|\hat{b}_i\|^{-2} \\
&= \left[\frac{\beta}{9\beta - 4} \left(\frac{9\beta}{4}\right)^{n-i} + \frac{8\beta - 4}{9\beta - 4}\right] \|\hat{b}_i\|^{-2}, \quad (66)
\end{align*}$$

and correspondingly,

$$\sum_{j=i}^{n} r_j^2 \|\hat{b}_j\|^2 \geq \left[\frac{\beta}{9\beta - 4} \left(\frac{9\beta}{4}\right)^{n-i} + \frac{8\beta - 4}{9\beta - 4}\right]^{-1} \|\hat{b}_i\|^2. \quad (67)$$

When $\beta = 2$, the new bound is asymptotically tighter by a factor of 7 than Babai’s lower bound (67).

Substituting (31) and (66) into (62), we obtain Lemma 1.

Note that we have also incorporated the better estimate (31) of the length of $b_i$.

### APPENDIX III

#### LOWER BOUND FOR DUAL SIZE REDUCTION

We derive the counterparts of (65) and (67) when the reversed dual basis is size-reduced. The notation in Appendix II is followed. The difference is that, by the following lemma, all off-diagonal elements of $M^{-1}$ lie in interval $[-1/2, 1/2]$.

**Lemma 3:** If the dual basis is size-reduced, then the off-diagonal elements of $\mu^{-1}$ lie in the interval $[-1/2, 1/2]$.

**Proof:** By definition, the off-diagonal elements of $\mu$ lie in $[-1/2, 1/2]$ if the reversed dual basis $B^*$ is size-reduced. By Proposition 1 $\mu^{-1} = J(\mu^*)^T J$. This proves Lemma 3. ■

Consequently,

$$\begin{align*}
(A^{-1})_{1,1} &= p^T A^{-1} p \\
&\leq \|\hat{b}_i\|^{-2} + \frac{1}{4} \sum_{j=1}^{n-i} \|\hat{b}_{i+j}\|^{-2}, \quad (68)
\end{align*}$$

and

$$\begin{align*}
\sum_{j=i}^{n} r_j^2 \|\hat{b}_j\|^2 \geq \left(\|\hat{b}_i\|^{-2} + \frac{1}{4} \sum_{j=1}^{n-i} \|\hat{b}_{i+j}\|^{-2}\right)^{-1} \|\hat{b}_i\|^2. \quad (69)
\end{align*}$$

In contrast, when the primal basis is size-reduced, the off-diagonal elements of $M^{-1}$ can only be bounded by (63), which can be much larger than $1/2$. This analysis clearly shows that it is the dual basis, rather than the primal basis, that should be reduced for the purpose of ZF.

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