The Dynamical Correlation Function of the XXZ Model

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Abstract

We perform a spectral decomposition of the dynamical correlation function of the spin 1/2 XXZ model into an infinite sum of products of form factors. Beneath the four-particle threshold in momentum space the only non-zero contributions to this sum are the two-particle term and the trivial vacuum term. We calculate the two-particle term by making use of the integral expressions for form factors provided recently by the Kyoto school. We evaluate the necessary integrals by expanding to twelfth order in $q$. We show plots of $S(w, k)$, for $k = 0$ and $\pi$ at various values of the anisotropy parameter, and for fixed anisotropy at various $k$ around 0 and $\pi$.

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1 Introduction

In the last two years, a remarkable series of papers have appeared in which the consequences of the quantum affine symmetry of quantum spin chains is explored [1, 2, 3, 4, 5]. In the simplest case, the spin chain is the antiferromagnetic spin-1/2 XXZ model, and the associated non-abelian symmetry that of $U_q(\hat{sl}(2))$ [1, 2]. The $U_q(\hat{sl}(2))$ symmetry manifests itself in different ways. Firstly, the Hamiltonian of this model commutes with the algebra directly. Secondly, the model has a vertex operator algebra as a dynamical symmetry. These vertex operators are in turn intertwiners of certain $U_q(\hat{sl}(2))$ modules. However, we do not wish to review the work of the Kyoto School here. An excellent and accessible reference is the recent book by Jimbo and Miwa [6]. It is sufficient for our purposes to note that this approach leads to exact expressions for both correlations functions and form factors of local operators of the XXZ model. These objects are given as the trace of vertex operators over irreducible highest weight representations of $U_q(\hat{sl}(2))$. Using the free field representation of the algebra these traces may be evaluated to yield integral expressions.

Perhaps somewhat surprisingly, one-dimensional quantum spin chains are relevant to certain physical systems (crystals that have a much stronger interaction in one direction, making them effectively one-dimensional - for reviews of theoretical approaches and experimental data, see [7, 8] and references therein). A quantity of key interest in this context is the dynamical correlation function (defined in the next section). This quantity is related to neutron scattering data [9]. In this paper we calculate the dynamical correlation of the spin-1/2 XXZ model by performing a spectral decomposition, i.e., we re-express the correlation function as an infinite sum over products of form factors. Since we are calculating the dynamical correlation function in momentum space, all terms other than the vacuum and two-particles one vanish if we restrict ourselves to beneath the four-particle threshold (odd particle terms are always zero). This may seem an artificial restriction, but we are lucky in that experimentalist are, for the most part, only interested in the region around the two-particle threshold. The vacuum term is trivial to evaluate. It gives a delta function peak at $w = 0$ and $k = \pi$.

We evaluate the two-particle form factor by expanding the integral expression of the
Kyoto School in $q$ (to twelfth order). Up to some subtleties about the integration contour (which we explain), the coefficients are then relatively easy to work out.

In Section 2, we define the XXZ model, and describe the spectral decomposition of the dynamical correlation function. In Section 3, we write down the form factor expressions derived by the Kyoto School. We describe the general method of $q$-expanding such integrals, and evaluate the specific integrals required to twelfth order in $q$. In Section 4, we present the results of some consistency checks on our form factors. We calculate all the contributions to the correlation function, and show graphically how the correlation function behaves in various regions of parameter space defined by energy, momentum and $q$. Finally, in Section 5, we draw some conclusions.

2 The Dynamical Correlation Function of the XXZ Model

2.1 The XXZ Hamiltonian

The Hamiltonian of the spin 1/2 XXZ Heisenberg quantum spin chain is \[ \mathcal{H} = -\frac{1}{2} \sum_{i=-\infty}^{\infty} (\sigma_i^1 \sigma_{i+1}^1 + \sigma_i^2 \sigma_{i+1}^2 + \Delta \sigma_i^3 \sigma_{i+1}^3), \] (2.1)
where $\Delta = (q + q^{-1})/2$ is an isotropy parameter. Here $\sigma_i^j$ are the Pauli matrices acting at the $i$th site, with the Hamiltonian acting formally on the infinite tensor product \[ W = \ldots V \otimes V \otimes V \otimes V \ldots, \] (2.2)
where $V \simeq \mathbb{C}^2$. We consider the model in the massive antiferromagnetic phase $\Delta < -1$, which we parametrise by $-1 < q < 0$.

2.2 The dynamical correlation function

The dynamical correlation function we consider is

\[
S^i(w, k) = \int_{-\infty}^{\infty} dt \sum_{p \in \mathbb{Z}} e^{i(wt-kp)} S^i(t, p),
\]
\[
S^i(t, p) = \langle i \langle vac | \mathbf{g}(t, p) \cdot \mathbf{g}(0, 0) | vac >_i, \]

2
where $\sigma^j(t,p)$ refers to $\sigma^j$ acting on the site at position $p$ at time $t$ (in Minkowski space), and $i = 0, 1$ refers to a choice of one of the two antiferromagnetic boundary conditions. We may define these boundary conditions by arbitrarily choosing the eigenvalue of $\sigma^3$ at the $j$’th site, $s^j = (-1)^{i+j}$, for $|j| >> 1$. The dynamical correlation function is of physical interest because of its connection with neutron scattering data [8, 9]. For a physical system described by the Hamiltonian [2,1], the cross section for scattering of neutrons that give up energy $w$ and momentum $k$ is proportional to $S^i(w,k)$. In this paper we describe a method of calculating this function as an expansion in the parameter $q$.

2.3 Decomposition of the identity

We evaluate 2.3 by performing a spectral decomposition (a similar method for calculating the dynamical correlation function of the $s = 1$ XXX model, as approximated by an O(3) non-linear sigma model, was used in reference [13]). That is, we insert a complete set of energy eigenstates between the Pauli matrices. The Kyoto School has shown how to construct these states within the context of the representation theory of the quantum affine algebra $U_q(\widehat{sl}(2))$. We don’t wish to go into excessive detail about this rather long story, but rather pull results from the literature as necessary. A complete discussion of this subject is given in reference [6].

Jimbo and Miwa have conjectured (and given strong plausibility arguments for) the completeness of the following decomposition of the identity in terms of the eigenstates of the Hamiltonian [6]:

$$ I = \sum_{i=0,1} \sum_{n \geq 0} \sum_{\epsilon_n, \cdots, \epsilon_1} \frac{1}{n!} \int \bar{d}\xi_1 \cdots \bar{d}\xi_n |\xi_n, \cdots, \xi_1 >_{\epsilon_n, \cdots, \epsilon_1} ; i_{\epsilon_n, \cdots, \epsilon_1} < \xi_1, \cdots, \xi_n|, $$

(2.4)

where, $\bar{d}\xi = d\xi/(2\pi iz)$, the contours are around the unit circle, $|\xi_i| = 1$, and the pseudoparticle ‘charges’ are $\epsilon_i = \pm 1$.

The action of the spatial shift operator and Hamiltonian on multiparticle eigenstates is given by

$$ T|\xi_1, \cdots, \xi_n >_i = \prod_{i=1}^{n} \tau(\xi_i)^{-1}|\xi_1, \cdots, \xi_n >_{1-i}, $$

$$ H_{XXZ}|\xi_1, \cdots, \xi_n >_i = \sum_{i=1}^{n} E(\xi_i)|\xi_1, \cdots, \xi_n >_i, $$

(2.5)
where,
\[
\tau(\xi) = \exp(-1/2 \xi^2), \\
E(\xi) = \frac{1}{2} \frac{\xi^2}{\sqrt{\pi}} \frac{d}{d\xi} \log \tau(\xi),
\]
and \(\theta_a(b) = (a; a)_{\infty}(b; a)_{\infty}(ab^{-1}; a)_{\infty}; (a; b)_\infty = \prod_{n=0}^{\infty} (1 - ab^n)\). For later purposes, we also define \((a; b, c) = \prod_{n,m=0}^{\infty} (1 - ab^n c^m)\). As before, \(i = 0, 1\) corresponds to a choice of one of the boundary conditions. The states \(|\xi_n\rangle\) correspond to the hole states in the Bethe Ansatz spectrum \([14]\).

Inserting this expression for the identity into \([2.3]\), and making use of the properties,
\[
\sigma^j(t, p) = e^{iHt}T^{-p}\sigma^j(0, 0)T^pe^{-iHt}, \\
T|\text{vac} > _i = |\text{vac} > _{1-i},
\]
we obtain the following expression for the correlation function:
\[
S^i(w, k) = \sum_{n \geq 0} S^i_n(w, k), \quad \text{where}
\]
\[
S^i_n(w, k) = \sum_{\nu} \sum_{\epsilon_n, \cdots, \epsilon_1} \frac{1}{n!} \oint d\xi_1 \cdots d\xi_n \left(\frac{e^{i\epsilon_k}}{\tau(\xi_1) \cdots \tau(\xi_n)}\right)^p \delta(w - E(\xi_1) \cdots - E(\xi_n))
\]
\[
\times (i < \text{vac} |\psi(0, 0)|\xi_n, \cdots, \xi_1 >_{\epsilon_n, \cdots, \epsilon_1}; i+ p _{\epsilon_n, \cdots, \epsilon_1} < \xi_1, \cdots, \xi_n |\psi(0, 0)|\text{vac} > _i,
\]
where the boundary conditions are understood as modulo 2. Using the delta function \(\Delta(z) \equiv \sum_p z^p\), we may rewrite this expression as
\[
S^i_n(w, k) = \sum_{\nu} \sum_{\epsilon_n, \cdots, \epsilon_1} \frac{1}{n!} \oint d\xi_1 \cdots d\xi_n \Delta \left(\frac{e^{i\epsilon_k}}{\tau(\xi_1) \cdots \tau(\xi_n)}\right) \delta(w - E(\xi_1) \cdots - E(\xi_n))
\]
\[
\times (i < \text{vac} |\psi(0, 0)|\xi_n, \cdots, \xi_1 >_{\epsilon_n, \cdots, \epsilon_1}; i+ p _{\epsilon_n, \cdots, \epsilon_1} < \xi_1, \cdots, \xi_n |\psi(0, 0)|\text{vac} > _i
\]
\[
+ \left(\frac{e^{i\epsilon_k}}{\tau(\xi_1) \cdots \tau(\xi_n)}\right)^{1-i} \times \text{vac} |\psi(0, 0)|\xi_n, \cdots, \xi_1 >_{\epsilon_n, \cdots, \epsilon_1; 1-i} \epsilon_1, \cdots, \epsilon_n < \xi_1, \cdots, \xi_n |\psi(0, 0)|\text{vac} > _i.
\]

Two of the integrals may be carried out by making use of the identities,
\[
\oint \frac{d\xi}{2\pi i} \delta(f(\xi))g(\xi) = \sum_{\xi_0} \frac{g(\xi_0)}{|\xi_0| |f(\xi_0)|}, \quad \{\xi_0 | f(\xi_0) = 0, |\xi_0| = 1\},
\]
\[
\oint \frac{d\xi}{2\pi i} \Delta(f(\xi))g(\xi) = \sum_{\xi_0} \frac{g(\xi_0)}{|\xi_0| |\log f(\xi_0)|}, \quad \{\xi_0 | f(\xi_0) = 1, |\xi_0| = 1\}.
\]

The first observation we make about \([2.9]\) is that, for a given choice of \(w\) and \(k\), the integral is only non-zero if their exist \(\xi_1, \cdots, \xi_n\) such that both, \(\tau(\xi_1)^2 \cdots \tau(\xi_n)^2 = e^{2ik}\), and, \(E(\xi_1) + \cdots + E(\xi_n) = w\) hold (that is, there is momentum and energy conservation). In particle physics language, \(w\) needs to be greater than the n-particle threshold. (If we were
dealing with free relativistic bosons, the analogous argument would imply that the energy and momentum conservation conditions could be met only with \( w^2 > k^2 + n^2 m^2 \). Putting the argument another way, for a given choice of \( w \) and \( k \), the series \( 2.3 \) terminates.

Charge conservation in this model implies that \( S^i_{n \text{ odd}}(w, k) \) vanishes. (This point is clear within the context of the Kyoto School approach when form factors are expressed as traces over vertex operators. Ultimately, the charge conservation is related to the charge conservation in the Boltzmann weights of the corresponding six vertex model \([15]\).) So up to the four-particle threshold, there is an exact equality between \( S^i(w, k) \) and \( S^i_0(w, k) + S^i_2(w, k) \).

The existence and location of the two and four particle thresholds are shown in Figs. 1 and 2. In Fig. 1, we simply choose a large number of \((\xi_1, \xi_2)\) pairs randomly distributed over \(|\xi_i| = 1\), and plot \( w = E(\xi_1) + E(\xi_2) \) versus \( k = -i \log(\tau(\xi_1) + \tau(\xi_2)) \) (we evaluate \( E(\xi_i) \) and \( \tau(\xi_i) \) by the \( q \) expansions given in Section 4). In Fig. 2, we do the same but with \((\xi_1, \cdots, \xi_4)\).

The thresholds are the lower edges of the two scatter plots.

The vacuum, or zero-particle, term, is given by

\[
S^i_0(w, k) = \Delta(e^{2ik})\delta(w) \left( i < \text{vac}|\sigma(0, 0)|\text{vac} >_i + e^{ik} 1_{-i} < \text{vac}|\sigma(0, 0)|\text{vac} >_{1-i} \right)
\]

(2.11)

Again, charge conservation implies that \( i < \text{vac}|\sigma^\pm|\text{vac} >_i = 0 \). Using a \( \mathbb{Z}_2 \) symmetry given in \([3]\):

\[
i < \text{vac}|\sigma^z(0, 0)|\text{vac} >_i = -1 - i < \text{vac}|\sigma^z(0, 0)|\text{vac} >_{1-i},
\]

(2.12)

we can simplify this expression to give

\[
S^i_0(w, k) = \Delta(e^{2ik})\delta(w) \left( i < \text{vac}|\sigma^z(0, 0)|\text{vac} >_i \right)^2 (1 - e^{ik}).
\]

(2.13)

So the vacuum contribution to \( S^i(w, k) \) is a delta function peak at \( k = \pi \). The coefficient \( i < \text{vac}|\sigma^z(0, 0)|\text{vac} >_i \) is non-zero, and is in fact the staggered polarization = \( (q^2; q^2)_\infty^2 / (-q^2; q^2)_\infty^2 \) derived by Baxter \([13]\), and reproduced by the Kyoto School (by explicitly integrating the relevant integral formula \([2]\)).

In order to calculate the two particle contribution, we use the identities \( 2.10 \) to carry out the two integrals. This gives,

\[
S^i_2(w, k) = \sum_{\varepsilon_2, \varepsilon_1} \sum_{\xi_1, \xi_2} \frac{1}{\tau(\xi_1, \xi_2)} \left( i < \text{vac}|\sigma|\xi_2, \xi_1 >_{\varepsilon_2, \varepsilon_1; i} i_{\varepsilon_1, \varepsilon_2} < \xi_1, \xi_2|\sigma|\text{vac} >_i + \left( \frac{e^{ik}}{\tau(\xi_1)\tau(\xi_n)} \right) 1_{-i} < \text{vac}|\sigma|\xi_2, \xi_1 >_{\varepsilon_2, \varepsilon_1; 1-i} i_{\varepsilon_1, \varepsilon_2} < \xi_1, \xi_2|\sigma|\text{vac} >_i \right).
\]

(2.14)
Here, the sum is over all \((\xi_1, \xi_2)\) which are solutions of \(\tau(\xi_1)^2\tau(\xi_2)^2 = e^{2ik}\) and \(E(\xi_1) + E(\xi_2) = w\). The Jacobian factor \(c(\xi_1, \xi_2)\) is given by

\[
c(\xi_1, \xi_2) = \frac{1}{q-q^{-1}}|\xi_1 E'(\xi_1)E(\xi_2) - \xi_2 E'(\xi_2)E(\xi_1)|.
\] (2.15)

### 3 Evaluation of Form Factors

In this section we shall give expressions for the form factors appearing in equation 2.14.

#### 3.1 The results of the Kyoto School

The Kyoto School have shown how to calculate form factors of the XXZ model as traces of vertex operators over infinite-dimensional irreducible highest weight modules of \(U_q(sl(2))\) - see [6] and references therein. By bosonizing the vertex operators, and representing the highest weight modules in terms of a bosonic Fock space, they are able to write down explicit expressions (generally given in terms of multiple integrals of elliptic functions) for both form factors and correlation functions. The following expressions for the form factors of interest are given in reference [6]:

\[ i<\text{vac}|\sigma^+|\xi_2, \xi_1 >_{+,+,\cdots} = (-q)^{1-i} \xi_1^{-i} \xi_2^{2-i} (q^2)_{4}(q^4)_{4}^{3} \rho(q)^{2} \gamma(u_2/u_1) \theta_{q}^{\frac{u_1}{u_2}} \frac{\theta_{q}(u_1)}{\theta_{q}(u_1)} \frac{\theta_{q}(u_2)}{\theta_{q}(u_2)}, \] (3.16)

\[ i<\text{vac}|\sigma^z|\xi_2, \xi_1 >_{+,+,\cdots} = (-q)^{-i}(1-q^{-2}) (q^2)^{3} (q^4)^{3} \rho(q)^{2} \gamma(u_2/u_1) \] \[ \times \frac{\theta_{q}(u_1)}{\theta_{q}(u_1)} \frac{\theta_{q}(u_2)}{\theta_{q}(u_2)} \prod_{k=1,2} \frac{\xi_{k}^{1-i}(-qu_{k}/w)_{4}(-q^{2}w/w_{k})_{4}}{(-qu_{k}/w)_{4}(-q^{2}w/w_{k})_{4}}, \] (3.17)

where \(u_i = -\xi_i^2\). We introduce the notation \((a)_n = (a; q^n)_{\infty}\) and \((a)_{n,m} = (a; q^n, q^m)_{\infty}\). \(\gamma\) and \(\rho\) are defined by

\[
\gamma(a) = \frac{(q^4u)_{4}(u^{-1})_{4}}{(q^4u)_{4}(q^4u^{-1})_{4}},
\] (3.18)

\[
\rho(q) = \frac{(q^4a)_{4}}{(q^4a)_{4}}.
\]
The contours $C, C_{\pm}$ are defined by

\begin{align*}
C : & \quad u_i q^{4n} < v < u_i q^{2-4n} \quad \text{for } n \geq 0, \\
C_{\pm} : & \quad q^{(2n+1)\pm1} < w < q^{-\left(\frac{2n+1}{2}\right)\pm1} \quad \text{for } n \geq 0, \quad \text{and}, \\
& \quad -v q^{2n-1} < w < -v q^{-2n} \quad \text{for } n \geq 0, \quad \text{or equiv.,} \\
& \quad -w q^{2n+3} < v < -w q^{-2n+1} \quad \text{for } n \geq 0.
\end{align*}

(3.19)

We use the $>$, $<$ signs to imply that the contour on the left-hand side runs outside or inside a series of poles. The contour $C$ is not a circle, because $|u_i q^2| < |u_i|$, but it is nevertheless well defined - it simply loops inside the pole at $u_i q^2$.

The remaining non-zero form factors are defined through the $\mathbb{Z}_2$ symmetry of the theory\cite{6} by

\begin{align*}
& \quad i \langle \text{vac}|\sigma^-|\xi_2, \xi_1 >_{++,i} = 1_{-i} < \text{vac}|\sigma^+|\xi_2, \xi_1 >_{--1-1-i}, \\
& \quad i \langle \text{vac}|\sigma^z|\xi_2, \xi_1 >_{+-i} = -1_{-i} < \text{vac}|\sigma^z|\xi_2, \xi_1 >_{+-1-1-i}.
\end{align*}

(3.20)

All non-zero dual form factors are also defined in terms of (3.16) and (3.17) via the symmetry\cite{3}

\begin{align*}
& \quad i_{\epsilon_1, \epsilon_2} < \xi_1, \xi_2 | O | \text{vac} >_{i} = i < \text{vac}|O|-q \xi_1, -q \xi_2 >_{-\epsilon_1, \epsilon_2; i},
\end{align*}

(3.21)

where $O$ represent any local operator in the theory.

### 3.2 Evaluation of the integral

To obtain a useful expression for the dynamical correlation function $2.14$, we need to evaluate the integral $3.17$. It is a daunting task to attempt this directly, and we instead employ the technique of $q$ expanding the integral. In order to illustrate some of the subtleties of this technique, we apply it first to a simpler integral $I(k, l)$, defined by

\begin{equation}
I(k, l) = \oint_{C} \frac{1}{(q^k \xi)^4(q^l / \xi)^4},
\end{equation}

(3.22)

where the $k$ and $l$ are integers, and the contour is chosen as $C : q^{i+4n} < \xi < q^{-k-4n}$ for $n \geq 0$. We consider three generic cases.
In this case we can simply expand each term in the products in the denominator as a series, i.e.,

\[ I(k, l) = \oint \bar{d}\xi \prod_{n \geq 0} \sum_{m_1 \geq 0} (\xi q^{k+4n})^{m_1} \sum_{m_2 \geq 0} (\xi^{-1} q^{l+4n})^{m_2}. \]  

(3.23)

We may rewrite this as,

\[ I(k, l) = \oint \bar{d}\xi \sum_{n \geq 0} a_n(\xi) q^n, \]  

(3.24)

where for any \( n \), \( a_n(\xi) \) is a finite polynomial in \( \xi \). We calculate the residue at \( \xi = 0 \) by extracting the constant term in \( a_n(\xi) \). As an example we find,

\[ I(2, 2) = 1 + q^4 + 3q^{12} + 12q^{16} + 21q^{20} + 38q^{24} + 63q^{28} + 106q^{32} \cdots. \]  

(3.25)

Consider the example \( I(-1, 3) \), where \( C : q^3 < \xi < q \). Now, we cannot expand the denominator in positive powers of \( q \) as \( |\xi/q| < 0 \) in the \((1 - \xi/q)\) term. To be able to do so we must shift the contour outside of the \( \xi = q \) pole, i.e.,

\[ I(-1, 3) = -\oint \bar{d}\xi \frac{q}{\xi(1 - q/\xi)(q^3/\xi)_4(q^9/\xi)_4} + \frac{1}{(q^4)_4(q^2)_4}, \]  

(3.26)

where \( C' : q < \xi < q^{-3} \), and the second term comes from the residue at \( \xi = q \). We can then calculate the integral over the contour \( C' \) by the method used in the previous case.

In each of these cases we can \( q \) expand the integrand without problem, but we also obtain a pole at \( \xi = 1 \) which we must exclude or include for the two cases respectively.

We now show how we have applied this \( q \) expansion technique to the integral \( 3.17 \). The location of poles in the integrand is determined by the following product of terms in the denominator:

\[(w)_2(q^2/w)_2(-q^{-1}v/w)_2(-q^3w/v)_2(u_1/v)_4(u_2/v)_4(q^2v/u_1)_4(q^{-2}v/u_2)_4.\]  

(3.27)

We fix \( w \) at a point in the \( C_+ \) or \( C_- \) band (\( C_\pm \) are genuinely bands, unlike \( C \)) and carry out the \( v \) integration first. In order to \( q \) expand we must shift the \( v \) contour outside of the
\( v = q^2u_2 \) and \( v = -qw \) poles, in order that it lies within the bands \(|u_i| < |v| < |q^{-2}u_i|\) and \(|qw| < |v| < |q^{-1}w|\) (we call this contour \( C' \)). The entire integrand may then be expanded. In the \( v \) integration of this expanded integrand we are left (at any order in \( q \)) with poles at \( v = u_1, u_2, 0 \). In the subsequent \( w \) integration we get poles at \( w = 0 \), and also at \( w = 1 \) for the \( C_\rightarrow \) contour. It remains to calculate the \( w \) integrals associated with the residues at the \( v = q^2u_2 \) and \( v = -qw \) poles. At \( v = q^2u_2 \) we get the following \( w \) poles in the denominator,

\[
(w)_2(q^2/w)_2(-qu_2/w)_2(-qw/u_2)_2.
\] (3.28)

However the third factor is partially cancelled by a term in the numerator leaving us with,

\[
(w)_2(q^2/w)_2(-q^3u_2/w)_4(-qw/u_2)_2.
\] (3.29)

In determining the location of the \( w \) contour with respect to the \( u_2 \) dependent poles we encounter a new subtlety. The origin of the rather curious original \( C \) contour was that it arose from the sum of two expressions with contours \( C_1 : |q^4u_i| < |v| < |q^2u_i| \) and \( C_2 : |u_i| < |v| < |q^{-2}u_i| \) \[6,\]. (We don’t wish to go into the origin or form of these two expressions - yet again we refer the reader to reference [6].) Compatibility of these contours with \(|wq^3| < |v| < |wq|\), requires \(|q^4u_i| < |wq| < |u_i|\) and \(|u_i| < |wq| < |q^{-4}u_i|\) respectively - which are of course incompatible with each other. So in order to work out whether we should choose \(|w| < |u_2/q|\) or \(|w| > |u_2/q|\) in the integral associated with \( 3.29 \), we must determine whether the \( v = q^2u_2 \) arises from the part of the original integral associated with the contour \( C_1 \) or the part with contour \( C_2 \). In fact it arises from the \( C_1 \) part and so we must choose \(|wq| < |u_2|\). This is convenient, because it means that we can carry out the \( q \) expansion without shifting the contour any further.

Now consider the \( w \) integral associated with the residue at the \( v = -qw \) pole. In this case we are left with the following \( w \) dependent terms in the denominator,

\[
(-u_1/(qw))_4 (-u_2/(qw))_4.
\] (3.30)

Again, other products cancel with those in the denominator. We find that these poles arise only from terms associated with the \( C_2 \) contribution to the original integral. This means that we must choose \(|u_i| < |wq|\). This causes a problem because we cannot then expand the
(1 + u_2/(qw)) and (1 + u_2/(qw)) factors in positive powers of q. Hence we must shift our contour to |qw| < |ui|, which requires that we evaluate the residues at two extra poles.

A complete list of the combinations of poles which we must evaluate in order to obtain a complete q expansion for the form factor 3.17 is given in Table 1. The sign (±) in the table indicates the sign of the contribution of a particular residue to the overall integral.

**Table 1 - Location of residues evaluated in association with the form factor 3.17**

| v   | w         |
|-----|-----------|
| 0   | (+)0 (+)1 |
| 1   | (+)0 (+)1 |
| u_1 | (+)0 (+)1 |
| u_2 | (+)0 (+)1 |
| q^2 u_2 | (-)0 (-)1 |
| -qw | (-)0 (-) - u_1/q (-) - u_2/q |

The technique for calculating \( i < \text{vac} | \sigma^z | -q \xi, -q \xi >_{+;i} \) is completely analogous (although not identical as different poles contribute).

## 4 Results

In practice, as a non-trivial check of our technique, we calculated each of the following eight integral form factors separately: \( i < \text{vac} | \sigma^z | \xi_1, \xi_2 >_{+;i}, \) \( i < \text{vac} | \sigma^z | \xi_1, \xi_2 >_{-;i}, \) \( i < \text{vac} | \sigma^z | -q \xi_1, -q \xi_2 >_{+;i} \) and \( i < \text{vac} | \sigma^z | -q \xi_1, -q \xi_2 >_{-;i} \) (each for \( i = 0, 1 \)). We did this to 6th order in \( q \), and found that the \( \mathbb{Z}_2 \) symmetry relations 3.20 held. In addition we found that the dual form factors were precisely the complex conjugates of the form factors, i.e.,

\[
  i < \text{vac} | \sigma^z | -q \xi_1, -q \xi_2 >_{-;i} = i < \text{vac} | \sigma^z | \xi_2^{-1}, \xi_1^{-1} >_{\epsilon_2, \epsilon_1; i} .
\]  

This equality is also true for the exact expression 3.16, i.e.,

\[
  i < \text{vac} | \sigma^+ | -q \xi_1, -q \xi_2 >_{-;i} = i < \text{vac} | \sigma^- | \xi_2^{-1}, \xi_1^{-1} >_{\epsilon_2, \epsilon_1; i} .
\]
Having established these identities (to sixth order in $q$ at least), there are only two independent sets of form factors (the members of each set being related by $\mathbb{Z}_2$ symmetry or conjugation). We then proceeded to calculate one member from each set, in practice $0 < \langle \text{vac}|\sigma^z| - q\xi_1, -q\xi_2 >_{+;0}$ and $1 < \langle \text{vac}|\sigma^z| - q\xi_1, -q\xi_2 >_{+;1}$, to twelfth order in $q$. The results are given in Tables 2 and 3 at the end of the paper.

Using these results, we calculated the two quantities required in the $\sigma^z$ contribution to formula 2.14 (to order $q^{12}$),

$$f^z_I(\xi_1, \xi_2) = \langle \text{vac}|\sigma^z|\xi_2, \xi_1 >_{+;i} \langle \text{vac}|\sigma^z| - q\xi_1, -q\xi_2 >_{+;i},$$

$$f^{zII}_I(\xi_1, \xi_2) = \langle \text{vac}|\sigma^z|\xi_2, \xi_1 >_{+;1-i} \langle \text{vac}|\sigma^z| - q\xi_1, -q\xi_2 >_{+;i}.$$  \hfill (4.33)

$f^z_I$ and $f^{zII}_I$ are both actually independent of $i$ because of the $\mathbb{Z}_2$ symmetry. This is also true for the exact $\sigma^+$ and $\sigma^-$ terms,

$$f^\pm_I(\xi_1, \xi_2) = \langle \text{vac}|\sigma^\pm|\xi_2, \xi_1 >_{+;i} \langle \text{vac}|\sigma^\mp| - q\xi_1, -q\xi_2 >_{+;i},$$

$$f^{\pm II}_I(\xi_1, \xi_2) = \langle \text{vac}|\sigma^\pm|\xi_2, \xi_1 >_{+;1-i} \langle \text{vac}|\sigma^\mp| - q\xi_1, -q\xi_2 >_{+;i}. \hfill (4.34)$$

Thus $S^I_2(w, k)$ is also independent of $i$. Furthermore we find, from our explicit results that,

$$f^z_I(\xi_1, \xi_2) = f^z_I(\pm\xi_1, \mp\xi_2) = f^z_I(\xi_2, \xi_1),$$

$$f^{zII}_I(\xi_1, \xi_2) = -f^{zII}_I(\pm\xi_1, \mp\xi_2) = f^{zII}_I(\xi_2, \xi_1). \hfill (4.35)$$

Again, the same properties hold for the corresponding $\sigma^\pm$ form factors.

This later fact is important when we look more closely at formula 2.14, which written in terms of the above functions, becomes,

$$S^I_2(w, k) = \sum_{\xi_1, \xi_2} \frac{1}{4c(\xi_1, \xi_2)} \left( 2 \left( f^z_I(\xi_1, \xi_2) + f^{-z}_I(\xi_1, \xi_2) \right) + f^{zII}_I(\xi_1, \xi_2) \right)$$

$$+ \left( \frac{\epsilon^{ik}}{\tau(\xi_1)\tau(\xi_2)} \right) \left( 2 \left( f^{zII}_I(\xi_1, \xi_2) + f^{-zII}_I(\xi_1, \xi_2) \right) + f^{zII}_I(\xi_1, \xi_2) \right). \hfill (4.36)$$

It would be nice to able obtain $S^I_2(w, k)$ without performing the rather tedious task of solving the energy and momentum constraints

$$\tau(\xi_1)^2\tau(\xi_2)^2 = e^{2ik},$$

$$E(\xi_1) + E(\xi_2) = w, \hfill (4.37)$$
to get all the $\xi_1$ and $\xi_2$ roots for a given $w$ and $k$. This can be done, if given one pair of $\xi_1$ and $\xi_2$, that give a particular $(w,k)$ (via the energy relation and $\tau(\xi_1)\tau(\xi_1) = e^{-ik}$), one can get all the other solutions of the more general constraint $4.37$. From the explicit form of $\tau(\xi)$ and $E(\xi)$ (either 2.6 or 4.39), we find that the set of such solutions is, $(\xi_1,\xi_2), (\pm \xi_1, \mp \xi_2), (-\xi_2, -\xi_1)$, and another four with $\xi_1 \leftrightarrow \xi_2$. Given, the properties 4.39, and the fact that $c(\xi_1,\xi_2) = c(\pm \xi_1, \mp \xi_2) = c(\xi_2,\xi_1)$, we can label $S^i_2(w,k)$ in terms of the original $(\xi_1,\xi_2)$ pair, and write,

$$
S^i_2(\xi_1,\xi_2) = \frac{2}{c(\xi_1,\xi_2)} \left( 2 \left( f^{+I}(\xi_1,\xi_2) + f^{-I}(\xi_1,\xi_2) \right) + f^{zI}(\xi_1,\xi_2) \right)
+ 2 \left( f^{+II}(\xi_1,\xi_2) + f^{-II}(\xi_1,\xi_2) \right) + f^{zII}(\xi_1,\xi_2) \right). (4.38)
$$

For possible applications to neutron scattering data, it is useful, to work out $S^i_2(w,k)$ as a function of $w$ for fixed $k$ values. We do this by numerically solving the momentum constraint $\tau(\xi_1)\tau(\xi_2) = e^{ik}$ for a range of $(\xi_1,\xi_2)$ pairs, and calculating the associated $w$ value and $S^i_2(w,k)$ for each pair. In order to do this we identify $\tau(\xi) = e^{-np(\theta)}$, $\xi_i = e^{\theta_i}$, and use suitably truncated versions of the following $q$ expansions of the functions 2.6 (we calculate $c(\xi_1,\xi_2)$ in the same fashion):

$$
p(\theta) = \theta + 2 \sum_{m>0} \frac{\sin(2m\theta)q^m}{m(1+q^m)}
E(\theta) = \frac{(q-q^{-1})}{2} \left\{ \sum_{m>0} \frac{4\cos(2m\theta)q^m}{m(1+q^m)} \right\}. (4.39)
$$

In Figs 3 and 4, we plot $S_2(w,k=0)$ and $S_2(w,k=\pi)$ for a range of $q$ values. We find, in each case, that there is a rounded peak above the two-particle threshold. The location of the threshold moves to smaller $w$ at $q$ decreases towards $-1$. This is as expected since, from 4.39, there is a mass gap in the theory,

$$
E(p = 0) = \frac{(q-q^{-1})}{2} \sum_{m>0} \frac{4q^m}{m(1+q^m)}, (4.40)
$$

which vanishes as $q \to -1$. The results at $k = 0$ and $k = \pi$ are similar in form, although different in scale. The results at $k = 0$ and $k = \pi$ are different since $S^i_2(\pm \xi_1, \mp \xi_2) \neq S^i_2(\xi_1,\xi_2)$ - as follows from equation 4.38. Our expressions for $S_2(w,k=0)$ and $S_2(w,k=\pi)$ are easily convergent at all three $q$ values. As $|q|$ increases, the peaks become infinite spikes at, or very close to, the threshold (at least on the scale of the plots shown).
In Figs 5 and 6, we plot $S_2(w, k)$ for a range of values at around $k = 0$ and $k = \pi$, and at fixed $q = -0.2$. The location of the peaks move, as the location of the two-particle threshold moves as in Fig. 1. A rather curious fact is that the peak increases in height as $k$ is increased from zero, and decreases in height at $k$ is increased from $\pi$.

5 Conclusions

One of the criticisms sometimes made of the recent work on quantum spin chains, is that, despite all of the mathematical insights gained along the way, in the end one is left with formulae for correlation functions and form-factors which are intractable. We hope we have shown that, at the very least, these integral formulae can be regarded as generating expressions for $q$ expansions of physically interesting quantities. We also hope that we have shown how, by making use of the expressions for form factors, one can use this approach to analyse dynamical quantities - rather that the short distance equal-time correlation functions to which it has formerly been applied. Of course, we have also calculated a quantity which, closer to $q = -1$ may be relevant to the anisotropic spin-chains found in nature. We hope to comment on such applications in the future.

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\[
0 < v a c | \sigma^* |- q \xi_1 < q \xi_2 > - +.0 = -1 < v a c | \sigma^* |- q \xi_1 < q \xi_2 > - +.1 \\
= 0 < v a c | \sigma^* | \xi_2^1, \xi_1^1 > + - .0 = -1 < \xi | \sigma^* | \xi_2^1, \xi_1^1 > - +.1
\]

| \( q \) | coefficient |
|---|---|
| 1 | \(- \frac{2 \xi_1}{\xi_2} + \frac{2 \xi_2}{\xi_1} \) |
| 2 | \(- \frac{2 \xi_1}{\xi_2} + \frac{2 \xi_2}{\xi_1} \) |
| 3 | \(- \frac{2 \xi_1}{\xi_2} - \frac{2 \xi_1^3}{\xi_2^3} + \frac{2 \xi_1}{\xi_2} - \frac{4 \xi_1^5}{\xi_2^5} + \frac{2 \xi_2}{\xi_1} \) |
| 4 | \(- \frac{2 \xi_1}{\xi_2^3} - \frac{2 \xi_1^3}{\xi_2^3} + \frac{4 \xi_1}{\xi_2^3} + \frac{2 \xi_2}{\xi_1^5} - \frac{4 \xi_1^7}{\xi_2^7} - 2 \xi_1 \xi_2 + \frac{2 \xi_2^3}{\xi_1^3} + \frac{2 \xi_2^3}{\xi_1^3} \) |
| 5 | \(- \frac{2 \xi_1}{\xi_2^3} - \frac{2 \xi_1^3}{\xi_2^3} - \frac{2 \xi_2}{\xi_1^5} + \frac{2 \xi_1}{\xi_2^3} + \frac{4 \xi_1^5}{\xi_2^5} - \frac{2 \xi_1}{\xi_2^3} + \frac{2 \xi_2}{\xi_1^5} - \frac{2 \xi_2}{\xi_1^5} + \frac{2 \xi_2^3}{\xi_1^3} + \frac{6 \xi_2^5}{\xi_1^5} + \frac{2 \xi_2^3}{\xi_1^3} \) |
| 6 | \(+ 4 \xi_1 \xi_2 + \frac{2 \xi_2^3}{\xi_1^3} + \frac{6 \xi_2^5}{\xi_1^5} - \frac{4 \xi_1^3}{\xi_2^3} + \frac{2 \xi_2^3}{\xi_1^3} + \frac{2 \xi_2^3}{\xi_1^3} \) |
| 7 | \(- \frac{2 \xi_1}{\xi_2^3} - \frac{2 \xi_1^3}{\xi_2^3} - \frac{2 \xi_1^3}{\xi_2^3} - \frac{2 \xi_1^3}{\xi_2^3} + \frac{4 \xi_1^5}{\xi_2^5} + \frac{2 \xi_1}{\xi_2^3} + \frac{4 \xi_1^5}{\xi_2^5} + \frac{2 \xi_1^3}{\xi_2^3} + \frac{2 \xi_1^3}{\xi_2^3} - \frac{2 \xi_1^3}{\xi_2^3} + \frac{2 \xi_1^3}{\xi_2^3} + \frac{2 \xi_1^3}{\xi_2^3} + \frac{2 \xi_1^3}{\xi_2^3} \) |
| 8 | \(+ 6 \xi_2^3 + \frac{2 \xi_1^3}{\xi_2^3} + \frac{4 \xi_1^5}{\xi_2^5} + \frac{2 \xi_1^3}{\xi_2^3} + \frac{4 \xi_1^5}{\xi_2^5} + \frac{2 \xi_1^3}{\xi_2^3} + \frac{4 \xi_1^5}{\xi_2^5} + \frac{2 \xi_1^3}{\xi_2^3} + \frac{2 \xi_1^3}{\xi_2^3} + \frac{2 \xi_1^3}{\xi_2^3} + \frac{2 \xi_1^3}{\xi_2^3} \) |
| 9 | \(- \frac{2 \xi_1}{\xi_2^3} + \frac{2 \xi_1^3}{\xi_2^3} + \frac{2 \xi_1^3}{\xi_2^3} + \frac{2 \xi_1^3}{\xi_2^3} + \frac{4 \xi_1^5}{\xi_2^5} + \frac{2 \xi_1^3}{\xi_2^3} + \frac{4 \xi_1^5}{\xi_2^5} + \frac{2 \xi_1^3}{\xi_2^3} + \frac{4 \xi_1^5}{\xi_2^5} + \frac{2 \xi_1^3}{\xi_2^3} + \frac{4 \xi_1^5}{\xi_2^5} + \frac{2 \xi_1^3}{\xi_2^3} + \frac{4 \xi_1^5}{\xi_2^5} \) |
| 10 | \(- \frac{2 \xi_1}{\xi_2^3} + \frac{2 \xi_1^3}{\xi_2^3} + \frac{2 \xi_1^3}{\xi_2^3} + \frac{2 \xi_1^3}{\xi_2^3} + \frac{2 \xi_1^3}{\xi_2^3} + \frac{4 \xi_1^5}{\xi_2^5} + \frac{2 \xi_1^3}{\xi_2^3} + \frac{4 \xi_1^5}{\xi_2^5} + \frac{2 \xi_1^3}{\xi_2^3} + \frac{4 \xi_1^5}{\xi_2^5} + \frac{2 \xi_1^3}{\xi_2^3} + \frac{4 \xi_1^5}{\xi_2^5} \) |
Table 2 continued.

| $q^\#$ | coefficient |
|-------|-------------|
| 11    | \[
-\frac{2}{\xi_1 \xi_2} - \frac{2 \xi_1}{\xi_2^3} + \frac{2}{\xi_1^3 \xi_2^2} - \frac{2}{\xi_1^3 \xi_2} + \frac{2 \xi_1}{\xi_2^2} + \frac{2}{\xi_1 \xi_2^2} - \frac{2 \xi_1}{\xi_2^3} + \frac{4}{\xi_1^2 \xi_2} + \frac{2 \xi_1^2}{\xi_2} + \frac{8 \xi_1^9}{\xi_2} - \frac{2}{\xi_1^7 \xi_2} - \frac{2 \xi_1^3}{\xi_2^2} - \frac{8 \xi_1^7}{\xi_2^2} - \frac{4}{\xi_1^3 \xi_2^2} - \frac{2 \xi_1^3}{\xi_2^3} + \frac{4}{\xi_1^3 \xi_2} + \frac{2 \xi_1^3}{\xi_2^2} - \frac{2}{\xi_1^5 \xi_2} - \frac{2 \xi_1}{\xi_1^3 \xi_2} - \frac{2 \xi_1}{\xi_1^5 \xi_2} + \frac{2 \xi_1}{\xi_1^3 \xi_2^2} - \frac{2 \xi_1}{\xi_1^5 \xi_2^2} + \frac{2 \xi_1}{\xi_1^3 \xi_2^3} - \frac{2 \xi_1}{\xi_1^5 \xi_2^3} + \frac{2 \xi_1}{\xi_1^3 \xi_2^4} - \frac{2 \xi_1}{\xi_1^5 \xi_2^4} - \frac{2 \xi_1}{\xi_1^3 \xi_2^5} + \frac{2 \xi_1}{\xi_1^5 \xi_2^5} - \frac{2 \xi_1}{\xi_1^3 \xi_2^6} + \frac{2 \xi_1}{\xi_1^5 \xi_2^6} - \frac{2 \xi_1}{\xi_1^3 \xi_2^7} + \frac{2 \xi_1}{\xi_1^5 \xi_2^7} - \frac{2 \xi_1}{\xi_1^3 \xi_2^8} + \frac{2 \xi_1}{\xi_1^5 \xi_2^8} - \frac{2 \xi_1}{\xi_1^3 \xi_2^9} + \frac{2 \xi_1}{\xi_1^5 \xi_2^9} - \frac{2 \xi_1}{\xi_1^3 \xi_2^{10}} + \frac{2 \xi_1}{\xi_1^5 \xi_2^{10}} - \frac{2 \xi_1}{\xi_1^3 \xi_2^{11}} + \frac{2 \xi_1}{\xi_1^5 \xi_2^{11}}|
\] |
| 12    | \[
-\frac{2}{\xi_1 \xi_2} + \frac{2}{\xi_1^3 \xi_2} - \frac{2}{\xi_1^3 \xi_2^2} - \frac{2 \xi_1^5}{\xi_2} + \frac{2}{\xi_1 \xi_2^2} + \frac{2}{\xi_1^3 \xi_2} + \frac{4}{\xi_1 \xi_2^2} + \frac{2 \xi_1}{\xi_2^2} - \frac{2 \xi_1}{\xi_1^5 \xi_2} + \frac{2 \xi_1}{\xi_2^2} + \frac{2 \xi_1}{\xi_1^5 \xi_2^2} - \frac{2 \xi_1}{\xi_1^3 \xi_2^3} + \frac{2 \xi_1}{\xi_2^2} + \frac{2 \xi_1}{\xi_1^3 \xi_2^2} - \frac{2 \xi_1}{\xi_1^5 \xi_2^2} + \frac{2 \xi_1}{\xi_1^3 \xi_2^3} - \frac{2 \xi_1}{\xi_1^5 \xi_2^3} + \frac{2 \xi_1}{\xi_1^3 \xi_2^4} - \frac{2 \xi_1}{\xi_1^5 \xi_2^4} - \frac{2 \xi_1}{\xi_1^3 \xi_2^5} + \frac{2 \xi_1}{\xi_1^5 \xi_2^5} - \frac{2 \xi_1}{\xi_1^3 \xi_2^6} + \frac{2 \xi_1}{\xi_1^5 \xi_2^6} - \frac{2 \xi_1}{\xi_1^3 \xi_2^7} + \frac{2 \xi_1}{\xi_1^5 \xi_2^7} - \frac{2 \xi_1}{\xi_1^3 \xi_2^8} + \frac{2 \xi_1}{\xi_1^5 \xi_2^8} - \frac{2 \xi_1}{\xi_1^3 \xi_2^9} + \frac{2 \xi_1}{\xi_1^5 \xi_2^9} - \frac{2 \xi_1}{\xi_1^3 \xi_2^{10}} + \frac{2 \xi_1}{\xi_1^5 \xi_2^{10}} - \frac{2 \xi_1}{\xi_1^3 \xi_2^{11}} + \frac{2 \xi_1}{\xi_1^5 \xi_2^{11}}|
\] |
\[ 1 < \nu c | \sigma^* | - q \xi_1, - q \xi_2 >_{+,-,1} = -0 < \nu c | \sigma^* | - q \xi_1, - q \xi_2 >_{+,-,0} = 1 < \nu c | \sigma^* | \xi_2^{-1}, \xi_1^{-1} >_{+,-,1} = -0 < \xi | \sigma^* | \xi_2^{-1}, \xi_1^{-1} >_{+,-,0} \]

| q# | coefficient |
|----|-------------|
| 1  | \(-\frac{2}{\xi_1} + \frac{2}{\xi_2}\) |
| 2  | \(-\frac{2}{\xi_1} + \frac{2}{\xi_2} - \frac{2\xi_1^2}{\xi_2}\) |
| 3  | \(-\frac{2}{\xi_1^2} + \frac{6}{\xi_1^2} + 2\xi_1^2 + \frac{2}{\xi_2^2} - \frac{4}{\xi_2^2} - 2\xi_2^2 - \frac{2\xi_2^2}{\xi_1}\) |
| 4  | \(-\frac{2}{\xi_1^2} + \frac{4}{\xi_1^2} + \frac{2}{\xi_2^2} - \frac{4}{\xi_2^2} + \frac{2\xi_1^2}{\xi_2^2} + \frac{2\xi_1^2}{\xi_1\xi_2^2} + \frac{8\xi_1^2}{\xi_2^2} - 2\xi_2^2 - \frac{2\xi_2^2}{\xi_1}\) |
| 5  | \(-\frac{2}{\xi_1^2} + \frac{4}{\xi_1^2} + \frac{6}{\xi_1^2} - \frac{4\xi_1^2}{\xi_2^2} + \frac{2}{\xi_1^2} + \frac{2}{\xi_2^2} + \frac{2}{\xi_1^2} + 6\xi_2^2 - \frac{2\xi_2^2}{\xi_1} + \frac{6\xi_2^2}{\xi_1} - \frac{2\xi_2^2}{\xi_1^2} - \frac{2\xi_2^2}{\xi_1} \) |
| 6  | \(-\frac{8}{\xi_1^2} + \frac{4}{\xi_1^2} + \frac{2}{\xi_1} + 2\xi_1^4 + \frac{2}{\xi_1^2} - \frac{4}{\xi_1^2} + \frac{2}{\xi_1} + \frac{2}{\xi_1^2} + \frac{2}{\xi_1} - \frac{2}{\xi_1^2} + \frac{\xi_1^2}{\xi_1^2}\) |
| 7  | \(-\frac{2}{\xi_1^2} + \frac{4}{\xi_1^2} + \frac{2}{\xi_1} + \frac{2}{\xi_1^2} + \frac{2}{\xi_1} - \frac{4}{\xi_1^2} + \frac{2}{\xi_1} + \frac{2}{\xi_1} + \frac{2}{\xi_1^2} + \frac{2}{\xi_1} + \frac{2}{\xi_1^2} + \frac{\xi_1^2}{\xi_1^2}\) |
| 8  | \(-\frac{2}{\xi_1^2} + \frac{4}{\xi_1^2} + \frac{2}{\xi_1} + \frac{2}{\xi_1^2} + \frac{2}{\xi_1} - \frac{4}{\xi_1^2} + \frac{2}{\xi_1} + \frac{2}{\xi_1} + \frac{2}{\xi_1^2} + \frac{2}{\xi_1} + \frac{\xi_1^2}{\xi_1^2}\) |
| 9  | \(-\frac{2}{\xi_1^2} + \frac{4}{\xi_1^2} + \frac{2}{\xi_1} + \frac{2}{\xi_1^2} + \frac{2}{\xi_1} - \frac{4}{\xi_1^2} + \frac{2}{\xi_1} + \frac{2}{\xi_1} + \frac{2}{\xi_1^2} + \frac{2}{\xi_1} + \frac{\xi_1^2}{\xi_1^2}\) |
| 10 | \(-\frac{2}{\xi_1^2} + \frac{4}{\xi_1^2} + \frac{2}{\xi_1} + \frac{2}{\xi_1^2} + \frac{2}{\xi_1} - \frac{4}{\xi_1^2} + \frac{2}{\xi_1} + \frac{2}{\xi_1} + \frac{2}{\xi_1^2} + \frac{2}{\xi_1} + \frac{\xi_1^2}{\xi_1^2}\) |
| 11 | \(-\frac{2}{\xi_1^2} + \frac{4}{\xi_1^2} + \frac{2}{\xi_1} + \frac{2}{\xi_1^2} + \frac{2}{\xi_1} - \frac{4}{\xi_1^2} + \frac{2}{\xi_1} + \frac{2}{\xi_1} + \frac{2}{\xi_1^2} + \frac{2}{\xi_1} + \frac{\xi_1^2}{\xi_1^2}\) |
| 12 | \(-\frac{2}{\xi_1^2} + \frac{4}{\xi_1^2} + \frac{2}{\xi_1} + \frac{2}{\xi_1^2} + \frac{2}{\xi_1} - \frac{4}{\xi_1^2} + \frac{2}{\xi_1} + \frac{2}{\xi_1} + \frac{2}{\xi_1^2} + \frac{2}{\xi_1} + \frac{\xi_1^2}{\xi_1^2}\) |
Figure Captions

Fig. 1 A scatter plot of $w = E(\xi_1) + E(\xi_2)$ vs $k = -i \log(\tau(\xi_1)\tau(\xi_2))$ for $(\xi_1, \xi_2)$ randomly distributed over $0 < \theta_i \leq 2\pi$ (where $\xi_i = e^{i\theta_i}$). The lower limit of the points indicates the location of the two-particle threshold. Only the $0 < k < 2\pi$ portion of the plot is shown.

Fig. 2 A scatter plot of $w = E(\xi_1) + E(\xi_2) + E(\xi_3) + E(\xi_4)$ vs $k = -i \log(\tau(\xi_1)\tau(\xi_2)\tau(\xi_3)\tau(\xi_4))$ for $(\xi_1, \xi_2, \xi_3, \xi_4)$ randomly distributed over $0 < \theta_i \leq 2\pi$ (where $\xi_i = e^{i\theta_i}$). The lower limit of the points indicates the location of the four-particle threshold. Only the $0 < k < 2\pi$ portion of the plot is shown.

Fig. 3 $S_2(w, k = 0)$ vs $w$ for a range of $q$ values.

Fig. 4 $S_2(w, k = \pi)$ vs $w$ for a range of $q$ values.

Fig. 5 $S_2(w, k)$ vs $w$ for a range of $k$ values close to $k = 0$, and for fixed $q = -0.2$.

Fig. 6 $S_2(w, k)$ vs $w$ for a range of $k$ values close to $k = \pi$, and for fixed $q = -0.2$. 
Fig. 1. Two Particle Threshold at $q=-0.2$

Fig. 2. Four Particle Threshold at $q=-0.2$
Fig 3. $k=0$

Fig 4. $k=3.1416$

$S(w, k=0)$

$S(w, k=\pi)$
Fig 5. \( q = -0.2 \)

\[ S(w, k) \]

- \( k = 0 \)
- \( k = 0.392 \)
- \( k = 0.577 \)

Fig 6. \( q = -0.2 \)

\[ S(w, k) \]

- \( k = \pi \)
- \( k = 3.288 \)
- \( k = 3.632 \)
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