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EXPONENTS FOR THREE-DIMENSIONAL SIMULTANEOUS DIOPHANTINE APPROXIMATIONS

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Abstract. Let $\Theta = (\theta_1, \theta_2, \theta_3) \in \mathbb{R}^3$. Suppose that $1, \theta_1, \theta_2, \theta_3$ are linearly independent over $\mathbb{Z}$. For Diophantine exponents

$$\alpha(\Theta) = \sup\{\gamma > 0 : \limsup_{t \to +\infty} t^\gamma \psi_\Theta(t) < +\infty\},$$

$$\beta(\Theta) = \sup\{\gamma > 0 : \liminf_{t \to +\infty} t^\gamma \psi_\Theta(t) < +\infty\}$$

we prove

$$\beta(\Theta) \geq \frac{1}{2} \left( \frac{\alpha(\Theta)}{1 - \alpha(\Theta)} + \sqrt{\left( \frac{\alpha(\Theta)}{1 - \alpha(\Theta)} \right)^2 + \frac{4\alpha(\Theta)}{1 - \alpha(\Theta)}} \right) \alpha(\Theta).$$

Keywords: Diophantine approximations, Diophantine exponents, Jarník’s transference principle

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1. DIOPHANTINE EXPONENTS

Let $\Theta = (\theta_1, \ldots, \theta_n)$ be a real vector. We deal with the function

$$\psi_\Theta(t) = \min_{x \leq t} \max_{1 \leq i \leq n} \|\theta_i x\|.$$

Here the minimum is taken over positive integers $x$ and $\| \cdot \|$ stands for the distance to the nearest integer.

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Suppose that at least one of the numbers $\theta_1, \ldots, \theta_n$ is irrational. Then $\psi_\Theta(t) > 0$ for all $t \geq 1$. The uniform Diophantine exponent $\alpha(\Theta)$ is defined as the supremum of the set
\[ \{ \gamma > 0 : \lim_{t \to +\infty} t^{\gamma} \psi_\Theta(t) < +\infty \}. \]

It is a well-known fact that for all $\Theta$ one has
\[ \frac{1}{n} \leq \alpha(\Theta) \leq 1. \]

The ordinary Diophantine exponent $\beta(\Theta)$ is defined as the supremum of the set
\[ \{ \gamma > 0 : \liminf_{t \to +\infty} t^{\gamma} \psi_\Theta(t) < +\infty \}. \]

Obviously
\begin{equation}
(1) \quad \beta(\Theta) \geq \alpha(\Theta).
\end{equation}

2. Functions

For each $\alpha \in \left[\frac{1}{3}, 1\right)$, define
\[ g_1(\alpha) = \frac{\alpha}{1 - \alpha} \]
and
\[ g_2(\alpha) = \frac{\alpha(1 - \alpha) + \sqrt{\alpha(\alpha^3 + 6\alpha^2 - 7\alpha + 4)}}{2(2\alpha^2 - 2\alpha + 1)}. \]

The value $g_2(\alpha)$ is the largest root of the equation
\[ (2\alpha^2 - 2\alpha + 1)x^2 + \alpha(\alpha - 1)x - \alpha = 0. \]

Note that
\[ g_2(1/3) = g_2(1) = 1, \]
and for $1/3 < \alpha < 1$ one has $g_2(\alpha) > 1$. Let $\alpha_0$ be the unique real root of the equation
\[ x^3 - x^2 + 2x - 1 = 0. \]

In the interval $1/3 < \alpha < \alpha_0$ one has
\begin{equation}
(2) \quad g_2(\alpha) > \max (1, g_1(\alpha)).
\end{equation}
In the interval $\alpha_0 \leq \alpha < 1$ we see that

$$g_2(\alpha) \leq g_1(\alpha).$$

We define one more function. Put

$$g_3(\alpha) = \frac{1}{2} \left( \frac{\alpha}{1 - \alpha} + \sqrt{\left( \frac{\alpha}{1 - \alpha} \right)^2 + \frac{4\alpha}{1 - \alpha}} \right).$$

Simple calculation shows that

$$g_3(\alpha) > \max(g_1(\alpha), g_2(\alpha)) \quad \forall \alpha \in \left(\frac{1}{3}, 1\right).$$

3. Jarník’s result

In a fundamental paper [1] V. Jarník proved the following theorem.

**Theorem 1.** Let $\psi(t)$ be a continuous function in $t$, decreasing to zero as $t \to +\infty$. Suppose that the function $t\psi(t)$ increases to infinity as $t \to +\infty$. Let $\varrho(t)$ be the inverse function to the function $t\psi(t)$. Put

$$\varphi^{[\psi]}(t) = \psi\left( \varrho\left( \frac{1}{6\psi(t)} \right) \right).$$

Suppose that $n \geq 2$ and among numbers $\theta_1, \ldots, \theta_n$ there exist at least two numbers which, together with 1, are linearly independent over $\mathbb{Z}$. Suppose that

$$\psi_{\Theta}(t) \leq \psi(t)$$

for all $t$ large enough. Then there exist infinitely many integers $x$ such that

$$\max_{1 \leq j \leq n} \|x\theta_j\| \leq \varphi^{[\psi]}(x).$$

The next Jarník’s result on Diophantine exponents is an obvious corollary of Theorem 1.
Theorem 2. Suppose that $n \geq 2$ and among numbers $\theta_1, \ldots, \theta_n$ there exist at least two numbers which, together with 1, are linearly independent over $\mathbb{Z}$. Then

$$\beta(\Theta) \geq \alpha(\Theta)g_1(\alpha(\Theta)).$$

To obtain Theorem 2 from Theorem 1 one takes $\psi(t) = t^{-\alpha}$ with $\alpha < \alpha(\Theta)$.

On the other hand, V. Jarník [1] proved that there exists a collection of numbers $\Theta = (\theta_1, \ldots, \theta_n)$ such that $1, \theta_1, \ldots, \theta_n$ are linearly independent over $\mathbb{Z}$ and

$$\beta(\Theta) < \frac{\alpha(\Theta)}{1 - \alpha(\Theta)}.$$

In the case $n = 2$ the lower bound in Jarník’s Theorem 2 is optimal. The following result was proved by M. Laurent [2].

Theorem 3. For any $\alpha, \beta > 0$ satisfying

$$\frac{1}{2} \leq \alpha \leq 1, \quad \beta \geq \alpha g_1(\alpha)$$

there exists a vector $\Theta = (\theta_1, \theta_2) \in \mathbb{R}^2$ such that

$$\alpha(\Theta) = \alpha, \quad \beta(\Theta) = \beta.$$

This result is a corollary of a general theorem concerning four two-dimensional Diophantine exponents.

Note that in the case $n = 3$ the bound in Theorem 2 in the range $1/n \leq \alpha < \frac{1}{2}$ is weaker than the trivial bound (1).

N. Moshchevitin [3] (see also [4], Section 5.2) improved Jarník’s result in the case $n = 3$ and for $\alpha \in (\frac{1}{3}, \alpha_0)$. He obtained

Theorem 4. Suppose that $m = 1, n = 3$ and the collection $\Theta = (\theta_1, \theta_2, \theta_3)$ consists of numbers which, together with 1, are linearly independent over $\mathbb{Z}$. Then

$$\beta(\Theta) \geq \alpha(\Theta)g_2(\alpha(\Theta)).$$

In the case $n = 3$, Theorems 2 and 4 together give an estimate which is better than the trivial estimate (1) for all admissible values of $\alpha(\Theta)$.

4. NEW RESULT

In this paper we give a new lower bound for $\beta(\Theta)$ in terms of $\alpha(\Theta)$. From (4) it follows that this bound is better than all the previous bounds (Theorems 2 and 4) for all admissible values of $\alpha(\Theta)$. 

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**Theorem 5.** Suppose that \( m = 1, n = 3 \) and the vector \( \Theta = (\theta_1, \theta_2, \theta_3) \) consists of numbers linearly independent, together with 1, over \( \mathbb{Z} \). Then

\[
\beta(\Theta) \geq \alpha(\Theta) g_3(\alpha(\Theta)).
\]

Sections 5, 6, 7 below contain auxiliary results. Theorem 5 is proved in Section 8.

5. **Best approximations**

For each integer \( x \), put

\[
\zeta(x) = \max_{1 \leq j \leq n} \| \theta_j x \|.
\]

A positive integer \( x \) is said to be a *best approximation* if

\[
\zeta(x) = \min_{x'} \zeta(x'),
\]

where the minimum is taken over all \( x' \in \mathbb{Z} \) such that

\[
0 < x' \leq x.
\]

Consider the case when all numbers 1 and \( \theta_j, 1 \leq j \leq n \) are linearly independent over \( \mathbb{Z} \). Then all best approximations lead to sequences

\[
x_1 < x_2 < \ldots < x_\nu < x_{\nu+1} < \ldots,
\]

\[
\zeta(x_1) > \zeta(x_2) > \ldots > \zeta(x_\nu) > \zeta(x_{\nu+1}) > \ldots.
\]

We use the notation

\[
\zeta_\nu = \zeta(x_\nu).
\]

Choose \( y_{1,\nu}, \ldots, y_{n,\nu} \in \mathbb{Z} \) such that

\[
\| \theta_j x_\nu \| = | \theta_j x_\nu - y_{j,\nu} |.
\]

We define

\[
z_\nu = (x_\nu, y_{1,\nu}, \ldots, y_{n,\nu}) \in \mathbb{Z}^{n+1}.
\]

If \( \psi(t) \) is a continuous function decreasing to 0 as \( t \to \infty \), with

\[
\psi(\Theta)(t) \leq \psi(t),
\]

then one easily sees that

\[
(5) \quad \zeta_\nu \leq \psi(x_{\nu+1}).
\]

Some useful fact about best approximations can be found in [4].
Lemma 1. Suppose that all vectors of the best approximations $z_l$, $\nu \leq l \leq k$ lie in a certain two-dimensional linear subspace $\pi \subset \mathbb{R}^4$. Consider the two-dimensional lattice $\Lambda = \pi \cap \mathbb{Z}^4$ with the two-dimensional fundamental volume $\det \Lambda$. Then for all $l$ from the interval $\nu \leq l \leq k-1$ one has

$$C_1 \det \Lambda \leq \zeta_l x_{l+1} \leq 2 \det \Lambda$$

where $C_1 = \left(2 \sqrt{3(1 + (|\theta_1| + \frac{1}{2})^2 + (|\theta_2| + \frac{1}{2})^2 + (|\theta_3| + \frac{1}{2})^2)}\right)^{-1}$. In particular,

$$\det \Lambda \geq \frac{\min(\zeta_\nu x_\nu + 1, \zeta_{k-1} x_k)}{2}.$$

Proof. The parallelepiped

$$\Omega_l = \left\{ z = (x, y_1, y_2, y_3) : |x| < x_{l+1}, \max_{1 \leq j \leq 3} |\theta_j x - y_j| < \zeta_l \right\}$$

has no non-zero integer points inside for every $l$. Consider the two-dimensional 0-symmetric convex body

$$\Xi_l = \Omega_l \cap \pi.$$

One can see that the two-dimensional Lebesgue measure $\mu(\Xi_l)$ of $\Xi_l$ admits the following lower and upper bounds:

$$2 \zeta_l x_{l+1} \leq \mu(\Xi_l) \leq 4 \sqrt{3(1 + (|\theta_1| + \frac{1}{2})^2 + (|\theta_2| + \frac{1}{2})^2 + (|\theta_3| + \frac{1}{2})^2)} \zeta_l x_{l+1}.$$

We see that there is no non-zero point of $\Lambda$ inside $\Xi_l$ and that there are two linearly independent points $z_l, z_{l+1} \in \Lambda$ on the boundary of $\Xi_l$. So obviously

$$2 \det \Lambda \leq \mu(\Xi_l).$$

From the Minkowski convex body theorem it follows that

$$\mu(\Xi_l) \leq 4 \det \Lambda.$$

Now (6) follows from (8, 9, 10). Lemma is proved. □
7. Three-dimensional subspaces

Consider three consecutive best approximation vectors \( z_{l-1}, z_l, z_{l+1} \). Suppose that these vectors are linearly independent. Consider the three-dimensional linear subspace

\[
\Pi_l = \text{span}(z_{l-1}, z_l, z_{l+1}).
\]

Consider the lattice

\[
\Gamma_l = \Pi_l \cap \mathbb{Z}^4
\]

with the fundamental volume \( \det \Gamma_l \). Let \( \Delta \) be the three-dimensional volume of the three-dimensional simplex \( S \) with vertices \( 0, z_{l-1}, z_l, z_{l+1} \). We see that

\[
(11) \quad \Delta \geq \frac{\det \Gamma_l}{6}.
\]

Consider determinants

\[
(12) \quad \Delta_1 = -\begin{vmatrix}
{x_l-1} & {y_{2,l-1}} & {y_{3,l-1}} \\
{x_l} & {y_{2,l}} & {y_{3,l}} \\
{x_{l+1}} & {y_{2,l+1}} & {y_{3,l+1}}
\end{vmatrix}, \quad \Delta_2 = \begin{vmatrix}
{x_l-1} & {y_{1,l-1}} & {y_{3,l-1}} \\
{x_l} & {y_{1,l}} & {y_{3,l}} \\
{x_{l+1}} & {y_{1,l+1}} & {y_{3,l+1}}
\end{vmatrix},
\]

\[
\Delta_3 = -\begin{vmatrix}
{x_l-1} & {y_{1,l-1}} & {y_{2,l-1}} \\
{x_l} & {y_{1,l}} & {y_{2,l}} \\
{x_{l+1}} & {y_{1,l+1}} & {y_{2,l+1}}
\end{vmatrix}.
\]

The absolute values of these determinants are equal to the three-dimensional volumes of the projections of the simplex \( S \) onto the three-dimensional coordinate subspaces (\( \{y_1 = 0\}, \{y_2 = 0\} \) and \( \{y_3 = 0\} \) respectively) multiplied by 6.

Note that for \( j = 1, 2, 3 \) one has

\[
(13) \quad |\Delta_j| \leq 6 \zeta_{l-1} \zeta_l x_{l+1}.
\]

**Lemma 2.** Among determinants (12) there exists a determinant with absolute value \( \geq C_2 \Delta \), where \( C_2 = 2/(2 + \max_{1 \leq i \leq 3} |\theta_i|) \).

**Proof.** Consider the determinant

\[
\Delta_0 = \begin{vmatrix}
{y_{1,l-1}} & {y_{2,l-1}} & {y_{3,l-1}} \\
{y_{1,l}} & {y_{2,l}} & {y_{3,l}} \\
{y_{1,l+1}} & {y_{2,l+1}} & {y_{3,l+1}}
\end{vmatrix}
\]

and the vector

\[
w = (\Delta_0, \Delta_1, \Delta_2, \Delta_3) \in \mathbb{Z}^4.
\]
We see that \( w \) is orthogonal to the subspace \( \Pi_i \), that is
\[
\Delta_0 x_j + \Delta_1 y_{1,j} + \Delta_2 y_{2,j} + \Delta_3 y_{3,j} = 0, \quad j = l - 1, l, l + 1.
\]
So
\[
\Delta_0 = -\sum_{i=1}^{3} \Delta_i \frac{y_{i,l}}{x_l} = -\sum_{i=1}^{3} \Delta_i \left( \frac{y_{i,l}}{x_l} - \theta_i \right) - \sum_{i=1}^{3} \Delta_i \theta_i.
\]
As \( |y_{i,l}/x_l - \theta_i| \leq 1 \) we see that
\[
|\Delta_0| \leq (1 + \max_{1 \leq i \leq 3} |\theta_i|)(|\Delta_1| + |\Delta_2| + |\Delta_3|).
\]
However,
\[
36\Delta^2 = \Delta_0^2 + \Delta_1^2 + \Delta_2^2 + \Delta_3^2.
\]
From (14), (15) we deduce the inequality
\[
\Delta \leq \frac{1}{6} \left( 2 + \max_{1 \leq i \leq 3} |\theta_i| \right) (|\Delta_1| + |\Delta_2| + |\Delta_3|),
\]
and the lemma follows.

8. Proof of Theorem 5

Take \( \alpha < \alpha(\Theta) \). Then
\[
\zeta_l \leq x_{l+1}^{-\alpha}
\]
for all \( l \) large enough.

Consider best approximation vectors \( z_\nu = (x_\nu, y_{1,\nu}, y_{2,\nu}, y_{3,\nu}) \). From the condition that the numbers \( 1, \theta_1, \theta_2, \theta_3 \) are linearly independent over \( \mathbb{Z} \) we see that there exist infinitely many pairs of indices \( \nu < k, \nu \to +\infty \) such that
- both the triples \( z_{\nu-1}, z_\nu, z_{\nu+1}; \quad z_{k-1}, z_k, z_{k+1} \)
- consist of linearly independent vectors;
- there exists a two-dimensional linear subspace \( \pi \) such that
\[
z_l \in \pi, \quad \nu \leq l \leq k; \quad z_{\nu-1} \notin \pi, \quad z_{k+1} \notin \pi;
\]
the vectors $z_{\nu-1}, z_{\nu}, z_{k}, z_{k+1}$ are linearly independent. 
Consider the two-dimensional lattice $$\Lambda = \pi \cap \mathbb{Z}^4.$$ 
By Lemma 1, its two-dimensional fundamental volume $\det \Lambda$ satisfies

(17) $\det \Lambda \asymp \zeta_{\nu} x_{\nu+1} \asymp \zeta_{k-1} x_k.$

Consider the two dimensional orthogonal complement $\pi^\perp$ to $\pi$ and the lattice $$\Lambda^\perp = \pi^\perp \cap \mathbb{Z}^4.$$ It is well-known that

(18) $\det \Lambda^\perp = \det \Lambda.$

Consider the lattices $$\Gamma_{\nu} = (\text{span}(z_{\nu-1}, z_{\nu}, z_{\nu+1})) \cap \mathbb{Z}^4, \quad \Gamma_{k} = (\text{span}(z_{k-1}, z_{k}, z_{k+1})) \cap \mathbb{Z}^4$$ and primitive integer vectors $w_{\nu}, w_{k} \in \mathbb{Z}^4$ which are orthogonal to $\Pi_{\nu} = \text{span}(z_{\nu-1}, z_{\nu}, z_{\nu+1}), \Pi_{k} = \text{span}(z_{k-1}, z_{k}, z_{k+1})$ respectively. Obviously

$$w_{\nu}, w_{k} \in \Lambda^\perp.$$ 

Put $$b = \frac{1}{2} \left( -\frac{\alpha}{1-\alpha} + \sqrt{\left( \frac{\alpha}{1-\alpha} \right)^2 + \frac{4\alpha}{1-\alpha}} \right) \in (0,1), \quad a = 1 - b,$$ so $$\frac{\alpha}{1-\alpha} + b = g_3(\alpha).$$ 
Then $\det \Lambda^\perp \leq |w_{\nu}| \cdot |w_{k}|,$
where $| \cdot |$ stands for the Euclidean norm, and so we obtain that either

(19) $\det \Gamma_{\nu} = |w_{\nu}| \geq (\det \Lambda^\perp)^a = (\det \Lambda)^a$
or
\begin{equation}
\det \Gamma_k = |w_k| \geq (\det \Lambda^\perp)^b = (\det \Lambda)^b
\end{equation}
(using (18)).

If (19) holds then by Lemma 2, (13), (11) and (17) we see that
\[
\zeta_{\nu-1} \zeta_\nu x_{\nu+1} \gg |\Delta_j| \gg \det \Gamma_\nu \gg (\det \Lambda)^a \gg (\zeta_\nu x_{\nu+1})^a
\]
(here $\Delta_j$ is the determinant from Lemma 2 applied to the lattice $\Gamma = \Gamma_\nu$). From the definition of $a$ and (16) we see that
\[
x_{\nu+1} \gg \Theta x_{\nu}^{\alpha g_3(\alpha)}.
\]
We apply (16) again to obtain
\[
\zeta_\nu \ll \Theta x_{\nu}^{-\alpha g_3(\alpha)}.
\]

If (20) holds then by Lemma 2, (13), (11) and (17) we see that
\[
\zeta_{k-1} \zeta_k x_{k+1} \gg |\Delta_{j'}| \gg \det \Gamma_k \gg (\det \Lambda)^b \gg (\zeta_{k-1} x_k)^b
\]
(here $\Delta_{j'}$ is the determinant from Lemma 2 applied to the lattice $\Gamma = \Gamma_k$). From the definition of $b$ and (16) we see that
\[
x_{k+1} \gg \Theta x_{k}^{\alpha g_3(\alpha)}.
\]
We apply (16) again to obtain
\[
\zeta_k \ll \Theta x_{k}^{-\alpha g_3(\alpha)}.
\]

Theorem 5 is proved. \hfill \Box

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\[
H(U \cap V)H(U + V) \ll_n H(U)H(V).
\]
To prove our Theorem 5 one can use this inequality for
\[
U = \text{span}(z_{\nu-1}, z_\nu, z_{\nu+1}), \quad V = \text{span}(z_{k-1}, z_k, z_{k+1}).
\]
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