On the Degree Growth of Birational Mappings in Higher Dimension

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§0. Introduction

Let \( f : \mathbb{C}^d \to \mathbb{C}^d \) be a birational map. The problem of determining the behavior of the iterates \( f^n = f \circ \cdots \circ f \) is very interesting but not well understood. A basic property of a rational map is its degree (see §1 for the definition). Another quantity is the dynamical degree \( \delta(f) = \lim_{n \to \infty} (\deg(f^n))^{1/n} \), which is invariant under birational self-maps of \( \mathbb{C}^d \) (see [BV]). It has been called “complexity” by some physicists (see [BM] and [AABM]), and its logarithm has been called “algebraic entropy” in [BV].

One aspect of a birational map is that its birational conjugacy class does not have a well-defined “domain.” Namely, if \( f : X \to X \) is a (birational) dynamical system, then any birational equivalence \( h : X \to \tilde{X} \) will convert \( f \) to a (birational) dynamical system on \( \tilde{f} = h \circ f \circ h^{-1} : \tilde{X} \to \tilde{X} \). This may serve as a significant change of the presentation of \( f \), since the cohomology groups of \( X \) and \( \tilde{X} \) may have different dimensions. There is a well-defined pull-back map on cohomology \( f^* : H^{1,1}(X) \to H^{1,1}(X) \). Thus we can define \( \delta(f) \) more generally as \( \lim_{n \to \infty} ||(f^n)^*||^{1/n} \), which is the exponential rate of growth of the action of \( f^n \) on \( H^{1,1} \). Dinh and Sibony [DS] showed that this more general \( \delta(f) \) is birationally invariant.

The passage to cohomology may or may not be compatible with the dynamical system, depending on whether \( (f^*)^n = (f^n)^* \) holds on \( H^{1,1} \). In case this holds, we say that \( f \) is \( H^{1,1} \)-regular, or simply 1-regular. And if \( f \) is 1-regular, then it follows that \( \delta \) is the spectral radius of \( f^* \). We will pursue the study of \( \delta \) using what might be called a method of regularization. The first step of this method is to replace the pair \( (f, X) \) by a regular pair \( (\tilde{f}, \tilde{X}) \). This is done by finding a new complex manifold \( \tilde{X} \) which is birationally equivalent to \( X \). The second step, then, is to determine \( \tilde{f}^* \) and its spectral radius.

In this paper, we show how the method of regularization may be carried out on certain sub-classes of the family of maps which have the form \( f = L \circ J \), where \( L \) is an invertible linear map, and \( J(x_1, \ldots, x_d) = (x_1^{-1}, \ldots, x_d^{-1}) \). Generally speaking, a birational map has subvarieties that are mapped to lower dimensional sets, and it has lower dimensional subvarieties that are blown up to sets of higher dimension. In the case of \( f = L \circ J \), the coordinate hypersurfaces \( \{x_j = 0\} \) are blown down to points, and the points \( e_\ell = [0 : \ldots : 1 : \ldots : 0] \) are blown back up to hypersurfaces. This interplay between the blowing down and the blowing up serves as the key for degree growth. The essential objects are those orbits of the following form: \( \{x_j = 0\} \to \ast \to \cdots \to \ast \to e_\ell \). That is, they start at a hypersurface which is blown down to a point, and the orbit of this point lands on a point of indeterminacy which is blown back up to a hypersurface. We say that such orbits are singular. In §4 we define the class of elementary maps, which are characterized by the property that they are locally biholomorphic at the intermediate points of singular orbits. The 1-regularization of an elementary map is obtained by blowing up the points.
of the singular orbits. The singular orbits of an elementary map can be organized into an orbit list structure \( \mathcal{L}^c \), \( \mathcal{L}^o \), which consists of two sets of lists of positive integers. It is then shown how \( \delta = \delta(\mathcal{L}^c, \mathcal{L}^o) \) is determined by this list structure: an expression for \( \hat{f}^* \) is given in \( \S 4 \), and the characteristic polynomial is given in the Appendix. In \( \S 7 \) we illustrate this method by carrying out the procedure of finding orbit lists for some maps that have appeared in the mathematical physics literature.

In general we have \( \delta \leq \deg(f) \), and the existence of a singular orbit causes the inequality to be strict (Theorem 4.2). It is often desirable to have some way of estimating \( \delta \) without actually computing it. To this end, we give a way of comparing \( \delta \) for two maps \( f \) and \( \hat{f} \) with list structures \( \mathcal{L}^c \), \( \mathcal{L}^o \) and \( \hat{\mathcal{L}}^c \), \( \hat{\mathcal{L}}^o \), respectively. In \( \S 5 \) we give a number of comparison results; here are two examples. Theorem 5.1 shows that if \( f \) and \( \hat{f} \) have the same list structures, except that the orbits of \( \hat{f} \) are longer, then \( \delta(f) \leq \delta(\hat{f}) \). In Theorem 5.3, we show that adding a complete orbit list decreases \( \delta \). If we simply add a new singular orbit of length \( M \) to one of the orbit lists, then whether \( \delta \) is increased or decreased depends on the size of \( M \).

In \( \S 6 \) we introduce linear maps \( L_p \), which are determined by a permutation \( p \). The rest of \( \S 6 \) will be devoted to a consideration of the case where \( p = I \) is the identity permutation. The maps \( f = L_I \circ J \) are the Noetherian maps which were defined in the work [BHM]. Our primary motivation in this section is to show how the work of \( \S 4 \) applies to give \( \deg(f^n) \), \( n \geq 0 \) for these maps. These same numbers were conjectured in [BHM].

For more general permutations \( p \), the map \( f_p = L_p \circ J \) can lead to some complicated examples of orbit collision. The expression orbit collision refers to the fact that a singular orbit \( \{ x_j = 0 \} \to * \to \ldots \to \sigma_k \to \ldots \to e_\ell \) can contain a point \( \sigma_k \in \{ x_k = 0 \} - \mathcal{I} \) where \( f \) is smooth but not locally invertible. Thus the singular orbit starting at \( \{ x_j = 0 \} \) contains the singular orbit starting at \( \{ x_k = 0 \} \) (and possibly others). In \( \S 8 \) we define singular chains. The singular chain structure is used both to construct the 1-regularization \( (f^*_X, X) \) of \( (f_p, \mathcal{P}) \) and to write down \( f^*_X \). As before, \( \delta(f_p) \) is given by the spectral radius of \( f^*_X \).

In this paper we follow up on ideas of Diller and Favre [DF], Guedj [G] and Boukraa, Hassani and Maillard [BHM]. The paper [DF] shows that 1-regularization is possible for all birational maps in dimension two, and this forms the basis for their penetrating analysis. The possibility of extending this approach to the case of higher dimension was discussed in [BHM]. Birational maps are more complicated in higher dimension, however, and [BHM] proposed a family of birational maps as a model family of mappings to analyze. In their study of these mappings, they identify the integrable cases and give a numerical description of \( \delta(f) \) in many cases. This family was chosen in part because it resembles maps that arise in mathematical physics (see [AABHM], [BTR], and [RGMR]). Our analysis of elementary maps evolved from an effort to understand questions posed in [BHM].

The contents of this paper are as follows. In \( \S 1 \) we assemble some basic concepts concerning rational maps, and we formulate the property \( (1.1) \) which we use in constructing 1-regularizations. In \( \S 2 \) we recall the basic relationship between \( (f^*)^n \) on cohomology and the degree of \( f^n \). In \( \S 3 \) we show how to 1-regularize the basic mapping \( J \). Then in \( \S 4 \) we apply this to elementary maps and show how to compute the induced mapping \( f^* \) on \( H^{1,1} \). At this stage, we may compute the characteristic polynomial \( \chi_f \) of \( f^* \). The actual computation is deferred to Appendix A so that our discussion of \( \delta \) is not interrupted.
§5 shows how to use the formula for $\chi_f$ to obtain comparison theorems for $\delta$. In §6 we introduce the family of linear transformations $L_p$, which depend on a permutation $p$, and we give the degree growth for Noetherian maps. In §7 we analyze some mappings that have appeared in the mathematical physics literature. In §8 we give the general method for 1-regularization of permutation mappings.

§1. Polynomial and Rational Maps

We will review some of the basic properties of the dynamical degree of birational maps. We refer the reader to [RS] and [S] for further details for birational maps of $\mathbb{P}^d$ and to [DS] for the case of more general manifolds.

A polynomial is a finite sum $p(x) = \sum c_{i_1...i_d}x_1^{i_1}\cdots x_d^{i_d}$. We define the degree of a monomial as $\deg(x_1^{i_1}\cdots x_d^{i_d}) = i_1 + \cdots + i_d$, and we define $\deg(p)$ to be the maximum of $i_1 + \cdots + i_d$ for all nonzero coefficients $c_{i_1...i_d}$. A mapping $p = (p_1, \ldots, p_d)$ is said to be a polynomial mapping if each coordinate function is polynomial. Similarly, a mapping $f = (f_1, \ldots, f_d)$ is said to be rational if each coordinate function is rational, i.e., each $f_j = p_j/q_j$ is a quotient of polynomials.

We will use complex projective space $\mathbb{P}^d$ as a compactification of $\mathbb{C}^d$. Recall that

$$\mathbb{P}^d = \{[x_0 : \cdots : x_d] : x_j \in \mathbb{C}, \text{ and the } x_j \text{ are not all 0}\},$$

where the notation $[x_0 : \cdots : x_d]$ denotes homogeneous coordinates, i.e., $[x_0 : \cdots : x_d] = [\zeta x_0 : \cdots : \zeta x_d]$ for all nonzero $\zeta \in \mathbb{C}$.

A rational map $f = (p_1/q_1, \ldots, p_d/q_d)$ of $\mathbb{C}^d$ induces a (partially defined) holomorphic map $\hat{f} = [\hat{f}_0 : \cdots : \hat{f}_d]$ of projective space. Let us describe how to obtain $\hat{f}$ from $f$. We add the variable $x_0$ and convert $f$ to a homogeneous function

$$\hat{f} : [x_0 : \cdots : x_d] \mapsto \left[1 : \frac{p_1(x_1/x_0, \ldots, x_d/x_0)}{q_1(x_1/x_0, \ldots, x_d/x_0)} : \cdots : \frac{p_d(x_1/x_0, \ldots, x_d/x_0)}{q_d(x_1/x_0, \ldots, x_d/x_0)}\right]$$

$$= \left[1 : \frac{\tilde{p}_1(x_0, \ldots, x_d)}{\tilde{q}_1(x_0, \ldots, x_d)} : \cdots : \frac{\tilde{p}_d(x_0, \ldots, x_d)}{\tilde{q}_d(x_0, \ldots, x_d)}\right].$$

Note that the first line is homogeneous of degree zero, and thus $\tilde{p}_j$ and $\tilde{q}_j$ are homogeneous polynomials with $\deg(\tilde{p}_j) = \deg(\tilde{q}_j)$. We passed from the first equation to the second by multiplying the numerators and denominators by powers of $x_0$. Let $Q = \tilde{q}_1 \cdots \tilde{q}_d$. On the dense set where $Q \neq 0$, we do not change $\hat{f}$ if we multiply by $Q$. Thus $\hat{f}$ is also given by the map $x \mapsto [Q(x) : Q(x)\tilde{p}_1(x)/\tilde{q}_1(x) : \cdots : \tilde{p}_d(x)/\tilde{q}_d(x)]$. Thus we have represented $\hat{f}$ by a polynomial mapping to projective space. To obtain $\hat{f} = [\hat{f}_0 : \cdots : \hat{f}_d]$, we divide out the greatest (polynomial) factor. After this is done, there is no polynomial that divides all the $\hat{f}_j$, and we define $\deg(f) := \deg(\hat{f}_0) = \cdots = \deg(\hat{f}_d)$. We may also take the $n$-fold composition $f^n = f \circ \cdots \circ f$, and perform the same passage to a map $\hat{f}^n$ on projective space. Since we may divide out a (possibly larger) common factor at the end, it is evident that

$$\deg(f^n) \leq (\deg(f))^n.$$
The indeterminacy locus is the set
\[ \mathcal{I}(f) = \{ x \in \mathbb{P}^d : \hat{f}_0(x) = \cdots = \hat{f}_d(x) = 0 \}. \]

It is evident that \( \hat{f} \) defines a holomorphic mapping of \( \mathbb{P}^d - \mathcal{I}(f) \) to \( \mathbb{P}^d \). We denote this simply by \( f \); we reserve the notation \( \hat{f} \) for the multiple-valued mapping which will be defined below. In fact, \( \mathbb{P}^d - \mathcal{I} \) is the maximal domain on which \( f \) can be extended to be analytic. For if \( \hat{a} \in \mathcal{I}(\hat{f}) \), there is no neighborhood \( \omega \) of \( \hat{a} \) such that \( f(\omega - \mathcal{I}) \) is relatively compact in any of the affine coordinate charts \( U_j = \{ x_j \neq 0 \} \). For in this case, at least one of the coordinate functions, say \( \hat{f}_1 \), has no common factor with \( \hat{f}_j \). Thus \( (\omega - \{ \hat{f}_j = 0 \}) \ni x \mapsto \hat{f}_1(x)/\hat{f}_j(x) \) will take on all complex values. For \( a \in \mathbb{P}^d \), let us define the cluster set \( Cl_f(a) \) to be the set of all limits of \( f(a') \) for \( a' \in \mathbb{P}^d - \mathcal{I}, a' \to a \). The cluster set is connected and compact. By the arguments above, it follows that \( Cl_f(a) \) contains more than one point exactly when \( a \in \mathcal{I} \).

Let us consider the graph of the restriction of \( f \) to \( \mathbb{P}^d - \mathcal{I} \):
\[ \Gamma_f = \{ (x, y) : x \in \mathbb{P}^d - \mathcal{I}, y = f(x) \}. \]

Thus \( \Gamma_f \) is a subvariety of \( (\mathbb{P}^d - \mathcal{I}) \times \mathbb{P}^d \), and by \( \hat{\Gamma}_f \) we denote the closure of \( \Gamma_f \) inside \( \mathbb{P}^d \times \mathbb{P}^d \). Thus \( \hat{\Gamma}_f \) is an algebraic variety. Let \( \pi_1 \) (resp. \( \pi_2 \)) denote the projections of \( \hat{\Gamma}_f \) to the first (resp. second) coordinate. Now we may define the multiple-valued mapping \( \hat{f}(x) := \pi_2 \circ \pi_1^{-1}(x) \), and thus \( \hat{f}(x) \) is a subvariety of \( X \) for each \( x \). We have \( \hat{f}(x) = Cl_f(x) \), and the dimension of \( \hat{f}(x) \) is greater than zero exactly when \( x \in \mathcal{I} \).

A projective manifold \( X \) is said to be rational if it is birationally equivalent to \( \mathbb{P}^d \). The discussion above applies to rational manifolds. Let \( \mathcal{T}(X) \) denote the set of positive, closed currents on \( X \) of bidegree \((1,1)\). One of the well-known properties of a positive, closed \((1,1)\)-current \( T \) is that it has a local potential \( p \) and can be written locally as \( T = dd^c p \). Following Guedj [G] we use local potentials to define the induced pull-back map \( \Phi_f : \mathcal{T}(X) \to \mathcal{T}(X) \). Namely, if \( T \in \mathcal{T}(X) \), and if \( x_0 \in X - \mathcal{I} \), then \( T \) has a local potential \( p \) in a neighborhood for \( f(x_0) \), and we define the pullback \( f^*T := dd^c(p \circ f) \) in a neighborhood of \( x_0 \). This yields a well-defined, positive, closed \((1,1)\)-current on the set \( X - \mathcal{I} \). Now by [HP], the set \( \mathcal{I} \), being a subvariety of codimension at least 2, is a “removable singularity” for a positive, closed \((1,1)\) current. This means two things. First, the current \( f^*T \) has finite total mass, so it may be considered to be a \((1,1)\)-form whose coefficients are (complex, signed) measures with finite total mass. This allows us to define \( \Phi_f(T) := \hat{f}^*T \) as the current obtained by extending these measures “by zero” to \( X \), i.e. by assigning zero mass to the set \( \mathcal{I} \). Second, the current \( \Phi_f(T) \) is closed. Thus \( \Phi_f(T) \in \mathcal{T}(X) \).

The currents we will use are currents of integration. Specifically, if \( V \) is a subvariety of pure codimension 1, then we define the current of integration \([V]\) as an element of the dual space to the space of smooth \((d-1,d-1)\)-forms: \( \varphi \mapsto \langle [V], \varphi \rangle := \int_V \varphi \). By a classic theorem of Lelong, \([V]\) is well-defined and is a positive, closed \((1,1)\)-current. If \( V \) is defined locally as \( \{ h = 0 \} \) for some holomorphic function \( h \), then \( \frac{1}{2 \pi} \log |h| \) is a local potential for \([V]\), which means that locally, \([V] = \frac{1}{2 \pi} dd^c \log |h| \). The pull-back of the current in this case is simply the preimage: \( \Phi_f([V]) = [(f|_{X-\mathcal{I}})^{-1}V] \).
An irreducible subvariety \( V \) will be said to be exceptional if \( V - \mathcal{I}(f) \neq \emptyset \), and if \( \dim(f(V - \mathcal{I}(f))) < \dim(V) \). The exceptional locus for \( f \), written \( \mathcal{E}(f) \) is the union of all irreducible exceptional varieties.

We will use the following condition:

For every exceptional hypersurface \( V \) and every \( n > 1 \),

\[
\text{the image } \hat{f}^n(V - \mathcal{I}) \text{ has codimension strictly greater than one.} (1.1)
\]

Note that if \( V \) is exceptional, then \( f(V - \mathcal{I}) = \hat{f}(V - \mathcal{I}) \) will be contained in a subvariety of codimension at least 2. The only possibility that the codimension could jump to 1 for \( n = 2 \) would come from the action of \( \hat{f} \) on \( \mathcal{I} \cap (f(V - \mathcal{I})) \). Thus (1.1) depends on the behavior of \( \hat{f} \) at certain points of \( \mathcal{I} \). A related condition, called algebraic stability, was introduced in [FS] for maps of \( \mathbb{P}^d \) and is equivalent to \( \deg(f^n) = (\deg(f))^n \) for all \( n \geq 0 \).

**Proposition 1.1.** If (1.1) holds, then \( (\Phi f)^n = \Phi f^n \).

**Proof.** Let us fix \( T \in \mathcal{T}(X) \). Since \( f \) and \( f^2 \) are both holomorphic on \( X - \mathcal{I}(f) \cup f^{-1}(\mathcal{I}(f)) \), we see that \( (f^2)^*T = (f^*)^2T \) on this set. Now \( \mathcal{I}(f^2) \subset \mathcal{I}(f) \cup f^{-1}(\mathcal{I}(f)) \), and let us write \( V = (\mathcal{I}(f) \cup f^{-1}(\mathcal{I}(f))) - (\mathcal{I}(f^2)) \). It suffices to show that \( (f^2)^*T = 0 \) on \( V \). Since \( T \) has codimension 1, it puts zero mass on any subvariety of codimension two. Thus we may suppose that \( W \) is an irreducible component of \( V \) of codimension 1. Now by the construction of \( V \), we have \( f(W - \mathcal{I}(f)) \subset \mathcal{I}(f) \). Thus \( W \) is an exceptional hypersurface. Thus \( \hat{f}(f(W - \mathcal{I}(f))) \) is a subvariety of codimension at least 2. It follows that \( T \) puts no mass on this subvariety, and thus \( (f^2)^*T \) puts no mass on \( W \). We conclude, then that \( (f^2)^*T \) puts no mass on \( V \). Since \( \Phi \) is obtained by extending by zero, we conclude that \( \Phi f^n T = \Phi^2 f^n T \). The proof for \( n > 2 \) is similar. QED

Let us recall the cohomology group \( H^{1,1}(X) \) which is given as the set of smooth, \( d \)-closed \((1,1)\)-forms modulo \( d \)-exact \(1\)-forms. If \( \omega \) is a smooth \((1,1)\)-form, then it acts on a \((d - 1, d - 1)\)-form \( \xi \) as \( \xi \mapsto \langle \omega, \xi \rangle = \int_X \omega \wedge \xi \). Thus \( \omega \) defines a \((1,1)\)-current. For \( T \in \mathcal{T}(X) \) there is a smooth \((1,1)\)-form \( \omega_T \) such that the current \( T - \omega_T \) is \( d \)-exact. The cohomology class \( \omega_T \in H^{1,1}(X) \) is uniquely defined, so we have a map \( \mathcal{T}(X) \to H^{1,1}(X) \).

If \( V \) is a codimension 1 subvariety of \( X \), we let \( \{V\} \in H^{1,1}(X) \) denote the cohomology class \( \omega_{[V]} \) corresponding to the current of integration \( [V] \). The map \( \Phi f \) is consistent with this passage to cohomology: \( \{\Phi f T\} = f^*\{T\} \). We will say that \( f \) is 1-regular if \( (f^*)^n = (f^n)^* \) holds on \( H^{1,1}(X) \). The following is a consequence of Proposition 1.1:

**Proposition 1.2.** If (1.1) holds, then \( f \) is 1-regular.

§2. Degrees of the Iterates

A (complex) hyperplane \( \mathcal{H} \subset \mathbb{P}^d \), defined by a linear equation \( \mathcal{H} = \{\ell(x) = 0\} \), gives a generator \( H := \{\mathcal{H}\} \) of \( H^{1,1}(\mathbb{P}^d) \). If \( V = \{h(x) = 0\} \subset \mathbb{P}^d \) is a hypersurface of degree \( m \), then \( h/\ell^m \) is well-defined as a function on \( \mathbb{P}^d \), so we have \( [V] - [\mathcal{H}] = \frac{1}{2\pi} d\mu \log |h/\ell^m| \). Thus \( \{V\} = mH \), so the cohomology class of \( \{V\} \) corresponds to the degree of \( V \). By definition, \( \deg(f) \) is the degree of the homogeneous polynomials defining \( f \), and we have
$f^*H = \deg(f)H$. We will make use of the action on cohomology as a way of computing $\deg(f)$.

The manifolds we will work with are obtained from $\mathbb{P}^d$ by blowing up points (the “blowing up” construction will be given in §3). This means that there is a sequence of spaces $X_1, X_2, \ldots, X_m$ such that $X_1 = \mathbb{P}^d$ and $X_m = X$, and for each $j$ we have the following: there is a projection $\pi_j : X_j \to X_{j-1}$ and a finite set $S_{j-1} \subset X_{j-1}$ such that $\pi_j : X_j - \pi_j^{-1}S_{j-1} \to X_{j-1} - S_{j-1}$ is biholomorphic, and for each $s \in S$, the exceptional fiber is $\pi_j^{-1}s \simeq \mathbb{P}^{d-1}$.

For such $X$, we may describe $H^{1,1}(X)$ by the following inductive procedure. We start with $H$ as a basis of the $(1,1)$-cohomology of $X_1 = \mathbb{P}^d$. Now suppose we have a basis $\mathcal{B}_{j-1}$ for $H^{1,1}(X_{j-1})$. We define a basis $\mathcal{B}_j$ for $H^{1,1}(X_j)$ by taking the elements $\pi_j^*b$ for $b \in \mathcal{B}_{j-1}$, together with the classes of the exceptional fibers: $\{\pi_j^{-1}s\}$ for all $s \in S_{j-1}$.

In particular, let us take $H_X := \pi^*\{h\}$ as the first element of our basis $\mathcal{B}$ of $H^{1,1}(X)$. The rest of the basis elements can be taken to be exceptional fibers. Let us suppose that the degree of $f$ is $m$. Thus $f_X^*H_X = mH_X + E$, where $E$ denotes a sum over multiples of other basis elements from $\mathcal{B}$. The reason for the $m$ on the right hand side is as follows. A generic line $L \subset \mathbb{P}^d$ does not intersect any of the centers of blow-up, so $\pi^{-1}$ is well defined in a neighborhood of $L$. Thus $\pi^{-1}L$ intersects $f_X^*H_X$ with multiplicity $m$, since that is the multiplicity of intersection between $L$ and $f^*H$.

Now let $M$ be the matrix which represents $f_X^*$ with respect to the basis $\mathcal{B} = \{H_X, \ldots\}$. If $f_X$ is 1-regular, then $M^\alpha$ is the matrix representation for $(f_X^*)^\alpha$ with respect to $\mathcal{B}$. Since $H_X$ is the first element of $\mathcal{B}$, we have $d_n = (M^\alpha)_{1,1}$.

There are various ways of representing $d_n$. Let $\lambda_1, \ldots, \lambda_N$ denote the eigenvalues of the matrix $M$, and let

$$\chi(x) = \prod_{j=1}^N (x - \lambda_j) = x^N + \lambda_{N-1}x^{N-1} + \cdots + \lambda_0$$

be the characteristic polynomial of $M$. Let us first suppose that the $\lambda_j$ are nonzero and have multiplicity one. If we diagonalize $M$, we find constants $c_1, \ldots, c_N$ such that

$$d_n = (M^\alpha)_{1,1} = c_1\lambda_1^n + \cdots + c_N\lambda_N^n. \quad (2.1)$$

Thus $\{d_n\}$ satisfies the recursion formula

$$d_{n+N} = \alpha_{N-1}d_{n+N-1} + \cdots + \alpha_0d_n \quad (2.2)$$

where the coefficients $\alpha_j$ are determined by the characteristic polynomial since we have $\alpha_j = -\chi_j$ for $0 \leq j \leq N - 1$.

Another way of producing the sequence $\{d_n\}$ is to find polynomials $p(x)$ and $q(x)$ such that

$$\frac{p(x)}{q(x)} = \sum_{n=0}^\infty d_nx^n.$$
If we write \( q(x) = \prod (x - r_j) \) and if \( \deg(p) < \deg(q) \), then we may expand \( p(x)/q(x) \) into partial fractions we obtain

\[
\frac{p(x)}{q(x)} = \sum_j \frac{a_j}{r_j - x} = \sum_{n=0}^{\infty} \left( \sum_j \frac{a_j}{r_j^n} \right) x^n.
\]  

Comparing (2.1) and (2.3), we see that after renumbering the \( r_j \) if necessary, we must have \( r_j^{-1} = \lambda_j \). Thus \( \chi \) and \( q \) essentially determine each other:

\[
q(x)\chi(0) = x^N\chi(1/x).
\]  

Now let us suppose that the eigenvalues are nonzero (and not necessarily simple). Then we may approximate \( M \) by a matrix \( M' \) with simple eigenvalues. Equations (2.2) and (2.4) will hold for every such \( M' \). Since the characteristic polynomial and the coefficients \( ((M')^n)_{1,1} \) depend continuously on \( M' \), equations (2.2) and (2.4) will continue to hold for \( M \). It is not hard to adapt (2.4) to the case of zero eigenvalues.

**Theorem 2.1.** If \( \delta > 0 \), then \( \delta \) is the largest real zero of the characteristic polynomial \( \chi(x) \).

**Proof.** Let \( S \) denote the cyclic subspace spanned by \( \{M^n e_1 : n \geq 0\} \), and let \( M_S \) be the restriction of \( M \) to \( S \). For convenience, let us assume first that \( M_S \) is diagonalizable, and let \( \lambda_1, \ldots, \lambda_N \) be the eigenvalues with corresponding eigenvectors \( v_1, \ldots, v_N \). We may write \( e_1 = \sum c_j v_j \) and \( M^n e_1 = \sum c_j \lambda_j^n v_j \). Since \( e_1 \) is cyclic on \( S \), there are nonzero numbers \( a_j \) such that

\[
d_n = e_1 \cdot M^n e_1 = \sum_{j=1}^{N} a_j \lambda_j^n.
\]  

Now at least one of the \( \lambda_j \) must have modulus \( \delta \) (the spectral radius of \( M_S \)). We claim that since \( d_n \geq 0 \), it follows that \( \delta \) itself must be an eigenvalue.

Let us suppose, by way of contradiction, that \( \delta \neq \lambda_j \) is not an eigenvalue. It is easy to reduce to the case where all the eigenvalues in (2.5) have modulus \( \delta \). Let us write \( \lambda_j = \delta e^{2\pi i \theta_j} \) with \( 0 < \theta_j < 1 \). We consider two cases separately. The first case is that all \( \theta_j \) are rational. Thus all the \( \lambda_j \) are \( M \)th roots of unity for some \( M \). It follows that \( \sum_{n=1}^{M} \lambda_j^n = 0 \), and \( d_n \geq 0 \), so we must have \( d_n = 0 \) for all \( n \). However, since the vectors \( \{ (\lambda_1^n, \ldots, \lambda_N^n) \}, 1 \leq n \leq N \) are a basis for \( \mathbb{C}^N \), the condition \( d_n = 0 \) for \( 1 \leq n \leq M \) implies that the \( a_j \) all vanish, which is a contradiction.

The other case is where the \( \theta_j \) are not all rational. We consider the closed (Lie) subgroup \( G \) of \( \mathbb{R}^N/\mathbb{Z}^N \) generated by \( \{ n(\theta_1, \ldots, \theta_N) : n \in \mathbb{Z} \} \). Let us define \( h(\xi_1, \ldots, \xi_N) = \sum a_j e^{2\pi i \xi_j} \) for \( (\xi_1, \ldots, \xi_N) \in \mathbb{R}^N/\mathbb{Z}^N \). By continuity, we have \( h(\xi) \geq 0 \) for \( \xi \in G \). Since \( G \) is a closed subgroup, it contains a point \( (\phi_1, \ldots, \phi_N) \) with rational coordinates. Thus \( h(n\xi) \geq 0 \) for all \( n \geq 0 \). Arguing as before, we reach the same contradiction that the \( a_j \) all vanish.

Finally, if \( M_S \) is not diagonalizable, we may decompose it into cyclic subspaces corresponding to the different eigenvalues, and the \( a_j \) in the formula above become polynomials.
of \( n \). Now we may replace the polynomials by the highest degree coefficients and proceed as before.

§3. 1-Regularization of \( J \)

A basic map we work with is defined by

\[
J[x_0 : x_1 : \cdots : x_d] = [x_0^{-1} : x_1^{-1} : \cdots : x_d^{-1}] = [x_0 : x_1 : \cdots : x_d]
\]

where we write \( x_j = \prod_{i \neq j} x_j \). This is an involution because

\[
J^2[x_0 : \cdots : x_d] = [(x_0 \cdots x_d)^{d-1}x_0 : \cdots : (x_0 \cdots x_d)^{d-1}x_d] = [x_0 : \cdots : x_d]
\]

for \( x_0 \cdots x_d \neq 0 \). In order to discuss the behavior of \( J \), we introduce some notation. For a subset \( I \subset \{0, 1, \ldots, d\} \), we write the complement as \( \hat{I} = \{0, \ldots, d\} - I \). Let us set

\[
\Sigma_I = \{x \in \mathbb{P}^d : x_i = 0, \forall i \in I\}, \quad \Sigma_I^* = \{x \in \Sigma_I : x_j \neq 0, \forall j \in \hat{I}\}.
\]

Thus \( \Sigma_I \) is closed, and \( \Sigma_I^* \) is a dense, open subset of \( \Sigma_I \). We see that the indeterminacy locus is given by \( \mathcal{I}(J) = \bigcup_{|I| \geq 2} \Sigma_I \). In fact, we have a stratification given by

\[
\mathcal{I}(J) = \bigcup_{j=0}^d (\Sigma_j - \Sigma_j^*) = \bigcup_{|I| \geq 2} \Sigma_I^*.
\]

The action of \( J \) corresponds to the involution \( I \leftrightarrow \hat{I} \) of the set of subsets of \( \{0, \ldots, d\} \): for \( |I| \geq 2 \), \( \hat{J} \) acts as

\[
\hat{J} : \Sigma_I^* \ni p \mapsto \hat{J}(p) = \Sigma_j.
\]

The points of \( \mathbb{P}^d - \mathcal{I}(J) \) where \( J \) is not a local diffeomorphism is \( \bigcup_{j=0}^d \Sigma_j^* \). We use the notation

\[
eq_j := \Sigma_j = \{[0 : \cdots : 1 : 0 : \cdots : 0]\},
\]

so \( J(\Sigma_j^*) = e_j \), and the exceptional locus is \( \mathcal{E} = \Sigma_0 \cup \ldots \cup \Sigma_d \).

\( J \) is not 1-regular, since we have \( \hat{J} : \Sigma_j^* \to e_j \to \Sigma_j \). We show here how a 1-regularization of \( J \) may be obtained by blowing up. Let \( \pi : X \to \mathbb{C}^d \) denote the space \( \mathbb{C}^d \) blown up at the origin. We represent the blow-up as

\[
X = \{(z_1, \ldots, z_d), [\xi_1 : \cdots : \xi_d] \in \mathbb{C}^d \times \mathbb{P}^{d-1} : z_i \xi_j = z_j \xi_i, 1 \leq i, j \leq d\},
\]

and \( \pi(z, \xi) = z \). Thus \( X \) is a smooth \( d \)-dimensional submanifold of \( \mathbb{C}^d \times \mathbb{P}^{d-1} \). Let \( E := \pi^{-1}(0) \) denote the fiber over the origin. Thus \( E \cong \mathbb{P}^{d-1} \), and \( \pi : X - E \to \mathbb{C}^d - \{0\} \) is biholomorphic; the inverse map is given by \((z_1, \ldots, z_d) \mapsto ((z_1, \ldots, z_d), [\hat{z}_1 : \cdots : \hat{z}_d])\), for \( z \neq (0, \ldots, 0) \). If \( V \) is a complex subvariety of \( \mathbb{C}^d \), we identify it as a subvariety \( V_X \subset X \) as follows: by \( V_X \), we mean the closure of \( \pi^{-1}(V - \{0\}) \) inside \( X \). Thus if \( 0 \in V \), this is a
proper subset of $\pi^{-1} V = V_X \cup E$. When there is no danger of confusion, we will write $V$ for $V_X$.

Let us see how the operation of blow-up modifies the map $J$. We may identify a neighborhood of $e_0 = [1 : 0 : \cdots : 0] = J(\Sigma^*_0)$ inside $\mathbf{P}^d$ with $C^d$ via the map $[1 : z_1 : \cdots : z_d] \mapsto (z_1, \ldots, z_d)$. Performing the blow-up $\pi : X \to C^d$ at $e_0$ in the range of $J$ induces a (partially defined) map $J_X : C^d \to X$, given by

$$J_X := \pi^{-1} \circ J : \{ x \in \mathbf{P}^d : x_1, \ldots, x_d \neq 0 \} \to X.$$ 

For $x_0 \neq 0$, we have

$$J_X : [x_0 : x_1 : \cdots : x_d] \mapsto [x_0^{-1} : x_1^{-1} : \cdots : x_d^{-1}] = [1 : \frac{x_0}{x_1} : \cdots : \frac{x_0}{x_d}]$$

$$\leftrightarrow \left( z_1 = \frac{x_0}{x_1}, \ldots, z_d = \frac{x_0}{x_d} \right) \mapsto \pi^{-1}(z_1, \ldots, z_d) \in X.$$ 

Thus, letting $x_0 \to 0$, we obtain the map $J_X : \Sigma^*_0 \to E \cong \mathbf{P}^{d-1}$ given by

$$J_X(0 : x_1 : \cdots : x_d) = [\xi_1 = x_1^{-1} : \cdots : \xi_d = x_d^{-1}].$$

We see that $J_X$ is a local diffeomorphism at points of $\Sigma^*_0$.

The process of blowing up a point is in fact local and can be performed at any point of a complex manifold. Let $\pi : X \to \mathbf{P}^d$ denote the complex manifold obtained by blowing up at the centers $\{ e_0, \ldots, e_d \}$, and let $E_j = \pi^{-1} e_j$ denote the exceptional fiber over $e_j$.

Let us describe the induced birational map $J_X : X \to X$. Since

$$\pi : X - \bigcup_{j=0}^d E_j \to \mathbf{P}^d - \{ e_0, \ldots, e_d \}$$

is a biholomorphism, it follows that $\mathcal{I}(J_X) \cap (X - \bigcup E_j) = \bigcup_{|I| \geq 2} \Sigma_I \cap (X - \bigcup E_j)$. Further, the calculation above showed that $J_X|\Sigma^*_I$ is essentially $J$, and thus $J_X|E_j$ is essentially $J$ on $\mathbf{P}^{d-1}$. Thus we conclude that $\mathcal{I}(J_X) = \bigcup_{|I| \geq 2} \Sigma_I$, where $\Sigma_I \subset X$ is interpreted as above. In particular, $\mathcal{I} \cap E_j$ has codimension $2$ in $E_j$. Now the restriction of $J_X$ to $X - \bigcup \Sigma_I$ may be identified with the restriction of $J$ to $\mathbf{P}^d - \bigcup \Sigma_j$, which is a diffeomorphism. We have also seen that $J_X$ is a local diffeomorphism on $\bigcup \Sigma^*_I$. Thus $J_X$ is a local diffeomorphism at all points of $X - \mathcal{I}(J_X)$. This means that the exceptional locus is empty, and thus $J_X$ is 1-regular.

If $d = 2$, then $J_X$ is in fact holomorphic, i.e., $\mathcal{I}(J_X) = \emptyset$. If $2 \leq |I| < d$, then $\Sigma_I \cap (\mathbf{P}^d - \{ e_0, \ldots, e_d \}) \neq \emptyset$, so $\Sigma_I \cap \Sigma_j$ intersects the exceptional fibers $E_j$ for all $j$ such that $j \notin I$.

Let us remark that a linear map $L$ also induces a birational map $L_X = \pi^{-1} \circ L \circ \pi : X \to X$. It is evident that $\mathcal{I}(L_X) \subset \{ e_0, \ldots, e_d \}$ and $\mathcal{E}(L_X) \subset E_0 \cup \ldots \cup E_d$. If $L^{-1} e_i = e_i$ for some $i$ and $j$, then $L_X$ is biholomorphic in a neighborhood of $E_i$ and maps $E_i$ to $E_j$. If $L^{-1} e_j$ is not one of these points $e_i$, then $L^{-1} e_j \in \mathcal{I}(L_X)$, and $L(L^{-1} e_j) = E_j$. And
if $Le_j$ is not one of the $e_i$, then $L_X E_j$ is a point, so $E_j \in \mathcal{E}(L_X)$. We see that $L_X$ fails to be 1-regular exactly when there is a point $e_i$ such that $Le_i$ is not one of the $e_j$’s, but $L^n e_i = e_j$ for some $n \geq 2$ and some $j$.

From the discussion of the previous paragraph, we can deduce the action of $L_X^*$ on $H^{1,1}(X)$. Namely, let $H_X$ denote the cohomology class of a hyperplane. Since neither $H$ and $L H$ will contain any of the $e_i$’s for generic $H$, we see that $L_X^* H_X = H_X$. Further, we have $L_X^* E_i = E_j$ for the pairs $(i,j)$ such that $Le_j = e_i$. Let $M$ be the $(d+1) \times (d+1)$ matrix such that $m_{i,j} = 1$ if $Le_j = e_i$ and 0 otherwise. Then, with respect to the basis $B = \{H_X, E_0, \ldots, E_d\}$ of $H^{1,1}(X)$, we have

$$L_X^* = \begin{pmatrix} 1 & 0 \\ 0 & M \end{pmatrix}. $$

Next we discuss the action induced by $J_X$ on $H^{1,1}(X)$. Let $H \in H^{1,1}(\mathbb{P}^d)$ denote the class of a hyperplane, and let $H_X = \pi^* H$ denote the induced class in $H^{1,1}(X)$. When there is no danger of confusion, we will also denote $H_X$ simply by $H$. Let $E_j \in H^{1,1}(X)$ denote the cohomology class induced by $E_j$. Thus $\{H_X, E_0, \ldots, E_d\}$ is a basis for $H^{1,1}(X)$, and we will represent $J_X^*$ as a matrix with respect to this basis.

Let $\{\Sigma_0\} \in H^{1,1}(X)$ denote the class induced by $\Sigma_0$. We wish to represent $\{\Sigma_0\}$ in terms of our basis. Let us start by observing that $\Sigma_0$ is a hyperplane in $\mathbb{P}^d$, and so $\Sigma_0 = H \in H^{1,1}(\mathbb{P}^d)$. Thus we have $H_X = \pi^* H = \pi^* \Sigma_0$. By our formula for the pullback of a current, we have that $\pi^* \Sigma$ will correspond to the current of integration over $\pi^{-1} \Sigma_0 = \Sigma_0 \cup E_1 \cup \ldots \cup E_d$. (Since $e_0 \notin \Sigma_0$, the divisor $E_0$ will not be involved.)

It remains to determine the multiplicities of the different components. Now on the set $x_i \neq 0$, the current $\{\Sigma_0\}$ is represented by the potential $\log |x_0/x_i|$. Let us choose $i = d$ for convenience, and use affine coordinates $z_0 = x_0/x_d, \ldots, z_{d-1} = x_{d-1}/x_d$. In these coordinates, the potential for $\{\Sigma_0\}$ is given by $h := \log |z_0|$. Let us work in a neighborhood of the exceptional fiber $E_k$ for some $1 \leq k \leq d - 1$. On the dense open subset $\xi_0 \neq 0$, we may write a point of the fiber as $[1 : \xi_1 : \ldots : \xi_{d-1}]$. With this, we define an affine coordinate system

$$(z_0, \xi_1, \ldots, \xi_{d-1}) \mapsto ((z_0, \xi_1 z_0, \ldots, 1 + \xi_{k+1} z_0, \xi_{k+2} z_0, \ldots, \xi_{d-1} z_0), [1 : \xi_1 : \ldots : \xi_{d-1}]).$$

In this coordinate system, we see that $E_k$ is given by $z_0 = 0$. The potential for the current $\pi^* \{\Sigma_0\}$ is then given by $\pi^* h = h \circ \pi$. It follows that the multiplicities are one, so

$$\pi^* \Sigma_0 = \{\Sigma_0\} + \sum_{i \neq 0} E_i. $$

Combining this with the previous equation, we obtain

$$\{\Sigma_j\} = H_X - \sum_{i \neq j} E_i. $$

We have seen that $J_X$ is a diffeomorphism from $\Sigma_j^*$ to its image in $E_j$. Since $J_X$ induces a diffeomorphism outside a subvariety of codimension 2, and we are pulling back cohomology classes of codimension one, it follows that

$$J_X^* E_j = \{\Sigma_j\} = H_X - \sum_{i \neq j} E_i.$$
Next we need to determine $J_X^* H_X$. A generic hyperplane $H$ in $\mathbb{P}^d$ does not meet any of the $e_j$ and may be considered to be a subset of $X$. Thus it generates $H_X$. Thus we consider the restriction $J|_{X - \mathcal{I}}$ and determine the class $\{ (J^{-1} H) - \mathcal{I} \} = J_X^* H_X \in H^{1,1}(X)$. Let us start with the observation which connects $H$, $H_X$, and the preimage of $H$:

$$d \cdot H_X = \pi^*(d \cdot H) = \pi^*(J^* H) = \pi^*\{ J^{-1} H \}.$$ 

A hyperplane has the form $H = \{ h = 0 \}$ for some $h = \sum a_j x_j$. Thus $J^{-1} H = \{ \sum a_j x_j = 0 \}$, and $\log h \circ J \circ \pi$ will be a potential for $\pi^*\{ J^{-1} H \}$. The element $\pi^*\{ J^{-1} H \} \in H^{1,1}(X)$ will be $\{ J^{-1} H \}$ and a linear combination of the $E_j$. We need only determine the multiplicities of the $E_j$. Let us consider $E_d$. Since $x_d \neq 0$, we work in an affine coordinate system $(z_0, \ldots, z_{d-1})$. We write points in the fiber as $[\xi_0 : \ldots : \xi_{d-1}]$. On a dense open subset of the fiber we have $\xi_0 \neq 0$, and so with the same coordinate system as above we have

$$h \circ J \circ \pi = A(\xi) z_0^{d-1} + B(\xi) z_0^d$$

For generic $\xi$, $A(\xi) \neq 0$, so this vanishes to order $d - 1$ in $z_0$, and thus the multiplicity of $E_d$ (and all $E_j$) is $d - 1$. This gives

$$\pi^*\{ J^{-1} H \} = \{ J^{-1} H \} + \sum_j (d - 1) E_j.$$ 

Finally, we use the fact that $J_X^* H_X = \{ J^* H - \mathcal{I} \}$ to conclude that

$$J_X^* H_X = d \cdot H_X + \sum_j (1 - d) E_j.$$ 

Thus we may write the action on $H^{1,1}$ with respect to our basis in matrix form:

$$
\begin{pmatrix}
  d & 1 & 1 & \ldots & 1 \\
 1 - d & 0 & -1 & \ldots & -1 \\
 1 - d & -1 & 0 & \ldots & -1 \\
 \vdots & \vdots & \vdots & \ddots & \vdots \\
 1 - d & -1 & -1 & \ldots & 0 \\
\end{pmatrix}
\quad \text{(3.1)}
$$

The fact that this matrix is an involution corresponds to the fact that $J_X$ is 1-regular.

§4. Elementary Mappings

This Section is devoted to a discussion of mappings of the form $f = L \circ J$, where $L$ is a linear map of $\mathbb{P}^d$, and $J$ is as in the previous section. For $p \in X$, we define the orbit $\mathcal{O}(p)$ as follows. If $p \in \mathcal{E} \cup \mathcal{I}$, then $\mathcal{O}(p) = \{ p \}$. If there is an $N \geq 1$ such that $f^j p \notin \mathcal{E} \cup \mathcal{I}$ for $0 \leq j \leq N - 1$ and $f^N p \in \mathcal{E} \cup \mathcal{I}$, then $\mathcal{O}(p) = \{ p, f p, \ldots, f^N p \}$. Otherwise, we have $f^j p \notin \mathcal{E} \cup \mathcal{I}$ for all $j \geq 0$, and we set $\mathcal{O}(p) = \{ p, f p, f^2 p, \ldots \}$. In the first two cases (when the orbit is finite), we say that the orbit is singular. Otherwise, we say that the orbit is nonsingular.
The orbits starting at the image points of the exceptional hypersurfaces $\Sigma_j$, for $0 \leq j \leq d$ have special importance. We will use the notation $\alpha_j = f\Sigma_j^* = L\alpha_j$ for the image of the $j$th exceptional hypersurface, which is identified with $j$th column of the matrix $L$; and we let $O_j := O(f\Sigma_j^*) = O(\alpha_j)$ denote its orbit. We say that the mapping $f$ is elementary if for each $0 \leq j \leq d$, the orbit $O_j$ is either nonsingular or, if it is singular, it ends at one of the points $\{e_0, \ldots, e_d\}$.

Now suppose that $f$ is elementary. We define an orbit list to be a list of singular orbits of exceptional components $O_i = O(\alpha_i)$ with sequential indices:

$$\mathcal{L} = \{O_a, O_{a+1}, O_{a+2}, \ldots, O_{a+\mu}\}$$

such that if $0 \leq j < \mu$, then the endpoint of $O_{a+j}$ is $e_{a+j+1}$. In other words, if $j < \mu$, then the orbit $O_{a+j}$ must be singular, and the ending index $k$ of the endpoint $e_k$ of this orbit is the beginning index of the next orbit in the list. Let us suppose that the last orbit in the list, $O_{a+\mu}$, ends at the point $e_k$. We say that the list $\mathcal{L}$ is open if the orbit $O_k = O(\alpha_k)$ is nonsingular. We say that $\mathcal{L}$ is closed if $k = a$.

Renumbering the variables, if necessary, we may group the orbits into maximal orbit lists $\mathcal{L}_1, \ldots, \mathcal{L}_\nu$. It follows from the maximality that each $\mathcal{L}_j$ is either open or closed. Let us define $\mathcal{A}$ to be the set of indices $i$ such that $O_i$ is a singular orbit and is the first orbit in an open orbit list. Let $\Omega$ consist of the indices $j$ such that $e_j$ is the endpoint of a singular orbit.

Now we construct the 1-regularization of $f$. Let $S = \{i : O_i$ is singular$\}$, and let $O_S := \bigcup_{i \in S} O_i$. Let $\pi : X \to \mathbf{P}^d$ be the space obtained by blowing up each of the points of $O_S$. For $p \in O_S$, we let $F_p$ denote the exceptional fiber $\pi^{-1}p$ in $X$ over $p$, and we also let $F_p$ denote the induced cohomology class in $H^{1,1}(X)$. Repeating the reasoning of the previous section, we see that the hypersurfaces $\Sigma_j \subset X$, $j \in S$, are not exceptional for the induced birational map $f_X : X \to X$. Thus $f_X$ is 1-regular.

Let us determine the induced mapping $f_X^*$ on $H^{1,1}(X)$. The class $H_X$, together with the classes $F(p)$ for $p \in O_S$, form a basis for $H^{1,1}(X)$. For $i \in S$, we have

$$\Sigma_i \to f\Sigma_i^* = \alpha_j \to \cdots \to f^{n_j-1}\alpha_j = f^n\Sigma_i^* = e_{\beta_i}.$$

At each of the points $f^j\alpha_i$, $0 \leq j \leq n_i - 2$, $f$ is locally biholomorphic, so $f_X$ induces a biholomorphic map of a neighborhood of the fiber $f_X : F_{f^j\alpha_i} \to F_{f^{j+1}\alpha_i}$. We conclude that

$$f_X^*F_{f^j\alpha_i} = F_{f^{j+1}\alpha_i} \quad \text{for } 0 \leq j \leq n_i - 1 \quad (4.1)$$

and

$$f_X^*F_{\alpha_i} = \{\Sigma_i\}$$

where $\{\Sigma_i\}$ denotes the class induced by $\Sigma_i$ in $H^{1,1}(X)$. As in the previous section, we have

$$\{\Sigma_i\} = H_X - \sum_p F_p,$$

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where the sum is taken over all blow-up centers which belong to \( \Sigma_i \cap I \). The set of blow-up centers which belong to \( I \) is \( \Omega \), and the only question is whether \( i \in \Omega \). In fact, we have \( i \in \Omega \) if \( i \notin A \). Thus we have
\[
\begin{align*}
    f_X^* F_{a_i} &= H_X - F_{e_i} + F_{e_i}, \\
    f_X^* F_{a_i} &= H_X - F_{e_i} + F_{e_i}.
\end{align*}
\] (4.2)
where we have adopted the notation \( F_{\Omega} := \sum_{i \in \Omega} F_{e_i} \). Finally, to pull back the class of a hyperplane, we use the fact that \( L \) is biholomorphic, and for a generic hyperplane \( H \), the preimage \( L^{-1} H \) is again a generic hyperplane. Thus we may use \( f_X^{-1} = J_X^{-1} \circ L^{-1} \) and argue as in the previous section to find:
\[
f_X^* H_X = \{ f_X^{-1} H \} = \{ J_X^{-1} H \} = d H_X + (1 - d) F_{\Omega}.
\] (4.3)

Let \( L^c = \{ \{ O_{a_1}, \ldots, O_{a_1 + \mu_1} \}, \{ O_{a_2}, \ldots, O_{a_2 + \mu_2} \}, \ldots, \{ O_{a_m}, \ldots, O_{a_m + \mu_m} \} \} \) denote a listing of the set of closed orbit lists, and let \( L^o \) be a listing of the set of those open orbit lists which contain singular orbits. For an orbit \( O \), we let \( |O| \) denote its length, and by \# \( L^c \) we denote the set of lists of lists of orbit lengths
\[
\# L^c = \{ \{ |O_{a_1}|, \ldots, |O_{a_1 + \mu_1}| \}, \ldots, \{ |O_{a_m}|, \ldots, |O_{a_m + \mu_m}| \} \}.
\]

We see that the mapping \( f_X^* \) is determined by \# \( L^c \) and \# \( L^o \). Thus we have the following:

**Theorem 4.1.** If \( f = L \circ J \) is elementary, then the dynamic degree \( \delta(f) \) is determined by \# \( L^c \) and \# \( L^o \).

Henceforth, we will abuse notation and simply write \( L^c \) and \( L^o \) for the orbit list structure \# \( L^c \), \# \( L^o \), since only the lengths of the orbits (and not the specific points) are used in computing \( f^* \) and \( \delta \). Thus we may consider \( \delta(L^c, L^o) \) to be a number which is determined by two sets of lists of positive integers. In addition, the characteristic polynomial may be explicitly computed in terms of \( L^c \) and \( L^o \). This is done in the Appendix. By Theorem 2.1, \( \delta \) is the largest real zero of \( \chi(x) \), and by Theorem A.1 \( \chi(d) > 0 \), so we have the following:

**Theorem 4.2.** If \( f \) is an elementary mapping of \( \mathbf{P}^d \) with at least one singular orbit, then \( \delta < d \).

\section*{§5. Comparison Results}

We saw in the previous section that for an elementary mapping, \( \delta \) is determined by the orbit list structure \( L^c, L^o \). Here we develop some results which may be interpreted as giving monotonicity properties of this dependence, or equally well, as giving a method of comparing \( \delta \) whenever the orbit lists may be compared. Let us describe how to compare orbit lists and lists of lists. Our first comparison theorem involves lists with the same structure pattern but different orbit lengths. Let \( L = \{ N_1, \ldots, N_\ell \} \) and \( \hat{L} = \{ \hat{N}_1, \ldots, \hat{N}_\ell \} \) be the structures of two orbit lists. We say that \( \hat{L} \) has longer orbits than \( L \) if \( \ell = \hat{\ell} \) and \( |\hat{N}_i| \geq |N_i| \) for all \( 1 \leq i \leq \ell \). If these are closed orbit lists, we may also allow circular permutations of the orbits in our comparison. Now if \( L = \{ L_1, \ldots, L_\mu \} \) and \( \hat{L} = \{ \hat{L}_1, \ldots, \hat{L}_\mu \} \) are lists of lists, we say that \( \hat{L} \) has longer orbits than \( L \) if they are both of the same type (either open or closed) and if \( \hat{\mu} = \mu \), and after a possible permutation of the index set \( \{ 1, \ldots, \mu \} \), the list \( \hat{L}_j \) has longer orbits than \( L_j \) for each \( 1 \leq j \leq \mu \).
Theorem 5.1. Let \( f \) and \( \hat{f} \) be two elementary maps of \( \mathbb{P}^d \). Let \( \mathcal{L}^o, \mathcal{L}^c \) (respectively, \( \hat{\mathcal{L}}^c, \hat{\mathcal{L}}^o \)) be the orbit list structure of \( f \) (respectively, \( \hat{f} \)). If \( \hat{\mathcal{L}}^c, \hat{\mathcal{L}}^o \) has longer orbits than \( \mathcal{L}^c, \mathcal{L}^o \), then \( \delta(\hat{f}) \geq \delta(f) \). If \( \delta(f) > 1 \), then the inequality is strict.

Proof. Proceeding by induction, we may assume that all the orbit lengths except one are the same. We will suppose that the orbit length changes inside one of the closed orbit lists. (The proof of the case if the orbit list is open is similar.) Without loss of generality, we may suppose that the orbit which is changed is the first orbit inside \( \mathcal{L}^c \), and its length is \( N_{1,1} \), and that the orbit length is \( \hat{N}_{1,1} = N_{1,1} + 1 \) inside \( \hat{\mathcal{L}}^c \). Let \( \chi \) (respectively \( \hat{\chi} \)) denote the characteristic polynomial corresponding to \( \mathcal{L}^c, \mathcal{L}^o \) (respectively \( \hat{\mathcal{L}}^c, \hat{\mathcal{L}}^o \)). Now recall from (A.3) that the characteristic polynomial has the general form

\[
\chi(x) = (x - d)T_1\prod T + (x - 1)S_1\prod T + (x - 1)T_1\sum S(\prod T).
\]  

(5.1)

Since the orbit lists agree except at the first orbit of the first list, we have \( T_i = \hat{T}_i \) and \( S_i = \hat{S}_i \) except for \( i = 1 \). For \( x > 1 \), we have

\[
\gamma := \frac{\hat{T}_c^1(x)}{T_1^c(x)} = \frac{x\prod x^{N_{i,j} - 1} - 1}{\prod x^{N_{i,j} - 1}} = x + \frac{x - 1}{\prod x^{N_{i,j} - 1}} > x.
\]

Similarly, we find that for \( x > 1 \), we have

\[
\rho := \frac{\hat{S}_c^1(x)}{S_1^c(x)} = x - \frac{(x - 1) \cdot \text{(positive terms)}}{S_1^c(x)} < x.
\]

It follows that \( \rho < \gamma \). Substituting \( \rho \) and \( \gamma \) into (5.1), we find that if \( 1 < x < d \), then

\[
\hat{\chi}(x) = \gamma(x - 1)T_1\prod T + \rho(x - 1)S_1\prod T + \gamma(x - 1)T_1\sum S\prod T
\]

and thus \( \hat{\chi}(x) < \chi(x) \), since \( T_i, S_i > 0 \), and \( \rho < \gamma \).

Finally, if we set \( x = \delta \), then by Theorem 2.1, we have \( \chi(\delta) = 0 \), which gives \( \hat{\chi}(\delta) < 0 \). Thus the largest root of \( \hat{\chi} \) will be greater than \( \delta \). This gives us the desired result. QED

Next we discuss the limiting behavior as the length of one (or several) of the orbits becomes unbounded.

Theorem 5.2. Let \( f \) be an elementary map of \( \mathbb{P}^d \) with orbit structure \( \mathcal{L}^c, \mathcal{L}^o \) and with \( \delta(f) > 1 \). Let \( \mathcal{L} = \{N_1, \ldots, N_i\} \) be one of the orbit lists in this structure, and let

\[
\delta_i := \lim_{N_i \to \infty} \delta(f).
\]

Then \( \delta_i \) is the dynamical degree corresponding to the orbit list structure \( \hat{\mathcal{L}}^c, \hat{\mathcal{L}}^o \) which is obtained as follows:

Closed case: If \( \mathcal{L} \) is a closed orbit list, then \( \hat{\mathcal{L}}^c \) is obtained by deleting \( \mathcal{L} \) from \( \mathcal{L}^c \) and adding the open list \( \{N_{i+1}, \ldots, N_i, N_1, \ldots, N_{i-1}\} \) to \( \mathcal{L}^o \).
Open case: If $\mathcal{L}$ is an open orbit list, then we set $\hat{\mathcal{L}}^c = \mathcal{L}^c$ and replace $\mathcal{L}$ in $\mathcal{L}^o$ according to the following cases:
If $i = 1$, replace $\mathcal{L}$ by the open list $\{N_2, \ldots, N_\ell\}$.
If $1 < i < \ell$, replace $\mathcal{L}$ by the pair of open lists $\{N_1, \ldots, N_{i-1}\}$ and $\{N_{i+1}, \ldots, N_\ell\}$.
If $i = \ell$, replace $\mathcal{L}$ by the open list $\{N_1, \ldots, N_{\ell-1}\}$.

Proof. There are four cases to consider. The proofs of all these cases are similar, so we consider only the first case. Since $\mathcal{L}$ is closed, we may perform a circular permutation so that we have $i = \ell$. Let $\chi(x)$ denote the characteristic polynomial as given by the formula (A.3), and let $\hat{\chi}$ denote the characteristic polynomial for the orbit structure obtained from $\mathcal{L}^c, \mathcal{L}^o$ by replacing the list $\{N_1, \ldots, N_\ell\}$ by $\{N_1, \ldots, N_{\ell-1}\}$. Inspecting the formula (A.3), we may write

$$x^{-N_\ell}\chi(x) = \hat{\chi}(x) + O(x^{-N_\ell})$$

for $x > 1$. For each value of $N_\ell$, we let $\delta_{N_\ell}$ denote the corresponding dynamical degree, which is also the largest real zero of $\chi$. By Theorem 5.1, $\delta_{N_\ell}$ is monotone increasing. Thus $\delta_{N_\ell} \to 0$. We conclude that the $O$ term in (5.2) vanishes as $N_\ell \to \infty$, and so the limiting value, $\hat{\delta}_i$ is the largest real zero of $\hat{\chi}$. QED

Theorem 5.3. Let $\mathcal{L}^c, \mathcal{L}^o$ be the orbit list structure of an elementary map of $\mathbb{P}^d$. If we let $\hat{\mathcal{L}}^c, \hat{\mathcal{L}}^o$ be the orbit list structure obtained by adding an orbit list to $\mathcal{L}^c$ or $\mathcal{L}^o$, then $\delta(\hat{\mathcal{L}}^c, \hat{\mathcal{L}}^o) \leq \delta(\mathcal{L}^c, \mathcal{L}^o)$.

Proof. For the new orbit list, let $T(x)$ and $S(x)$ denote the polynomials corresponding to the definitions in (A.1-2). Let $\chi$ denote the characteristic polynomial corresponding to the old orbit list structure, and let $\hat{\chi}$ denote the characteristic polynomial corresponding to the new one. Thus we have

$$\hat{\chi}(x) = (x - d)T(x)\prod T + (x - 1)S(x)\prod T + (x - 1)T(x)\sum S\prod T$$

$$= T(x)\chi(x) + (x - 1)S(x)\prod T,$$

where the notation $\prod'$ means we are taking the product over all of the polynomials $T_i^c$ and $T_i^o$, except the new $T(x)$. If we let $x = \hat{\delta}$ be the largest zero of $\hat{\chi}$, then we have

$$0 = T(\hat{\delta})\chi(\hat{\delta}) + (\hat{\delta} - 1)S(\hat{\delta})[\cdots].$$

Since $\hat{\delta} \geq 1$, it follows that all the terms except $\chi(\hat{\delta})$ on the right hand side of the equation are positive, so $\chi(\hat{\delta}) \leq 0$. Thus the largest zero of $\chi$ is greater than or equal to $\hat{\delta}$. QED

Theorem 5.4. Let $\mathcal{L}^c, \mathcal{L}^o$ denote the orbit list structure of an elementary map of $\mathbb{P}^d$, and let $\mathcal{L} = \{N_1, \ldots, N_\ell\}$ denote the structure of one of the lists. For $1 \leq j < \ell$ there is a number $M^* = M^*(j, \mathcal{L}, \mathcal{L}^c, \mathcal{L}^o)$ with the following property: Given $M$, we let $\mathcal{L}(j) = \{N_1, \ldots, N_j, M, N_{j+1}, \ldots, N_\ell\}$ denote the list obtained by adding an orbit of length $M$ at the $(j + 1)$st place in the list $\mathcal{L}$. Let $\hat{\mathcal{L}}^c, \hat{\mathcal{L}}^o$ denote the new orbit list structure obtained by
replacing \( L \) with \( L(j) \). Then if \( M < M^* \), we have \( \delta(\hat{L}^c, \hat{L}^o) < \delta(L^c, L^o) \); if \( M > M^* \), we have \( \delta(\hat{L}^c, \hat{L}^o) > \delta(L^c, L^o) \).

**Remark.** In the open case, if \( j = 0 \) or \( j = \ell \), then by Theorems 5.1 and 5.2 we can only reduce \( \delta \) by adding an orbit in the \( j \)th position; this means that \( M^* = \infty \).

**Proof.** Let us assume that \( L \) is an open orbit list. (The proof for the case of a closed orbit list is similar.) Without loss of generality we may suppose that the orbit list \( L \) is the orbit list inside \( L^o \). Let us define

\[
\phi(x, M) = x^M[1 + \sum_{i=1}^{j-1} \prod_{k=1}^i x^{N_k}][1 + \sum_{i=j+2}^\ell \prod_{k=i}^\ell x^{N_k}]
- [1 + \sum_{i=1}^j \prod_{k=1}^i x^{N_k}][1 + \sum_{i=j+1}^\ell \prod_{k=i}^\ell x^{N_k}].
\]

For the new orbit list structure we have

\[
\hat{S}_1(x) = x^M S_1(x) - \phi(x, M).
\]

We have

\[
\gamma = \frac{\hat{T}_1(x)}{T_1(x)} = x^M
\]

and

\[
\rho = \frac{\hat{S}_1(x)}{S_1(x)} = x^M - \frac{\phi(x, M)}{S_1(x)}.
\]

It is clear that for fixed \( x \), \( \phi(x, M) \) is strictly increasing in \( M \), and thus there is a unique \( M^* = M^*(x) \) such that \( \phi(x, M^*(x)) = 0 \). Now let \( \delta = \delta(L^c, L^o) \). If \( M > M^*(\delta) \), then \( \phi(\delta, M^*) > 0 \), which implies that \( \rho(\delta) < \gamma(\delta) \). This implies that \( \hat{\chi}(\delta) < \gamma(\delta)\chi(\delta) = 0 \). This implies that \( \hat{\delta} > \delta \).

By inspection, \( M^*(x) \) is monotone increasing in \( x \), for \( 1 \leq x \leq d \). Thus for \( x > \delta \) we have \( M^*(x) > M^*(\delta) \). This implies that \( 0 = \phi(x, M^*(x)) \geq \phi(x, M^*(\delta)) \). This means that if \( M < M^* \) and \( x > \delta \), then \( \phi(x, M) \leq 0 \), which in turn implies that \( \rho \geq \gamma \) and so \( \hat{\chi}(x) \geq \gamma(x)\chi(x) > 0 \). From this we conclude that \( \hat{\delta} \leq \delta \). QED

**Theorem 5.5.** Let \( L^c, L^o \) be an orbit list structure, and let \( \hat{L}^c, \hat{L}^o \) be the orbit list structure obtained by removing an orbit list \( L \) from \( L^o \) and adding \( L \) to \( L^c \), i.e., we move an orbit list from \( L^o \) to \( L^c \). Then \( \delta(\hat{L}^c, \hat{L}^o) \leq \delta(L^c, L^o) \), and the inequality is strict unless \( \delta(L^c, L^o) = 1 \).

**Proof.** Without loss of generality we may assume that \( L = L_1 \) is the first list in \( L^o \), and we move \( L \) to the first list of \( L^c \). Let \( T_1^o, S_1^o, S_1^c \) be defined as in (A.1). Let \( \chi \) (resp. \( \hat{\chi} \)) be the characteristic polynomial corresponding to \( L^c, L^o \) (resp. \( \hat{L}^c, \hat{L}^o \)). Then we have

\[
\chi(x) = (x - d)T_1^o \prod T + (x - 1)S_1^o \prod T + (x - 1)T_1^o \sum S \prod T.
\]
Since the only difference between $\chi$ and $\hat{\chi}$ arises from the change of $L$, we have

$$\gamma = \frac{T^c_1(x)}{T^c_0(x)} = \frac{x^{|L^c|} - 1}{x^{|L^c|}} < 1$$

and

$$\rho = \frac{S^c_1(x)}{S^c_0(x)} = \frac{S^c_0(x) + \text{positive}}{S^c_0(x)} > 1$$

for $x > 0$. It follows that $\hat{\chi}(x) > \chi(x)$ for $1 < x < d$ and thus $\hat{\delta} \leq \delta$. QED

**Theorem 5.6.** Let $L^c, L^o$ be an orbit list structure. Suppose that $L = \{1, \ldots, 1\}$ is one of the lists inside $L^c$. Let $\hat{L}^c, \hat{L}^o$ be the orbit list structure obtained by setting $\hat{L}^o = L^o$ and replacing the list $\{1, \ldots, 1\}$ in $L^c$ by $n$ closed lists $\{1\}, \ldots, \{1\}$. Then $\hat{\delta}(\hat{L}^c, \hat{L}^o) \geq \delta(L^c, L^o)$. If $n \geq 4$, then the inequality is strict.

**Proof.** To fix notation, let us write $L = L_1 = \{1, \ldots, 1\}$ be the first list of $L^c$, and we change $L$ to the first $n$ lists $\hat{L}_1 = \{1\}, \ldots, \hat{L}_n = \{1\}$ in $\hat{L}^c$. Let $\chi$ and $\hat{\chi}$ be the corresponding characteristic polynomials. Then

$$\chi(x) = (x - d)T_1 \prod T + (x - 1)S_1 \prod T + (x - 1)T_1 \sum S(\prod T)$$

and

$$\hat{\chi}(x) = (x - d)(\prod \hat{T}_i) \prod (n) T + (x - 1) \sum S_i(\prod \hat{T}_i) \prod (n) T +$$

$$+ (x - 1)(\prod \hat{T}_i) \sum S(\prod (n) T).$$

Thus we have

$$\gamma = \frac{\prod^n \hat{T}}{T_1} = \frac{(x - 1)^n}{(x^n - 1)} = \frac{(x - 1)^{n-1}}{x^{n-1} + \cdots + 1}$$

and

$$\rho = \frac{\sum S_i(\prod \hat{T})}{S_1} = \frac{n(x - 1)^{n-1}}{n + \sum_{j=1}^{n-1} x^j} = \frac{(x - 1)^{n-1}}{1 + \sum_{j=1}^{n-1} x^j}.$$

Hence if $1 \leq x \leq d$ we have $\rho \leq \gamma$ and therefore $\hat{\chi}(x) \leq \chi(x)$. Thus $\hat{\delta} \geq \delta$. QED

**Theorem 5.7.** Let $f$ be an elementary mapping of $P^d$ with $k$ singular orbits. Then $\delta(f) \geq \delta(L^c, L^o)$, where $L^c = \{(1, \ldots, 1)\}$, $|L^c| = k$, and $L^o = \emptyset$.

**Proof.** Let $L^c_f, L^o_f$ be the orbit list structure of $f$. By Theorem 5.1, $\delta$ will decrease if we make all the orbits have length equal to 1. By Theorem 5.5, $\delta$ will be also be decreased if we change all open orbit lists to closed orbit lists. Finally, by Theorem 5.6, $\delta$ will be decreased if we join all the orbit lists to one orbit list $\{1, \ldots, 1\}$. QED
§6. Permutation mappings: I

Next we define the family of permutation maps. In this section we will direct our attention to the case of the identity permutation: this is the family of mappings introduced in [BHM]. We will see that our discussion of elementary mappings applies in this case; in particular, these mappings have orbit list structure given by $\mathcal{L}^c = \{\{N_1\}, \ldots, \{N_\ell\}\}$ and $\mathcal{L}^o = \emptyset$. Using this, we will give proofs of some conjectures from [BHM].

Let us define

$$
\mathcal{D}_j = \{[x_0 : \ldots : x_d] \in \mathbb{P}^d : x_{i_1} = x_{i_2} \text{ for all } i_1, i_2 \neq j\}.
$$

By $\eta_j(c) = (x_0, \ldots, x_d) \in \mathcal{D}_j$ we denote the point such that $x_j = c - 1$, and $x_i = c$ for all indices $i \neq j$. With this notation we have $\eta_j(0) = e_j$. Let $a_0, \ldots, a_d \in \mathbb{C}$ be constants satisfying

$$a_0 + a_1 + \cdots + a_d = 2. \quad (6.1)$$

Let $\varphi$ be a permutation of the set $\{0, 1, \ldots, d\}$, and let $P = (P_{i,j})_{0 \leq i, j \leq d}$ be the associated permutation matrix, i.e., $P_{i,j} = \delta_{i,\varphi(j)}$. Let us define the $(d+1) \times (d+1)$ matrix

$$L = [\eta_{\varphi(0)}(a_0), \ldots, \eta_{\varphi(d)}(a_d)] = 
\begin{pmatrix}
a_0 & a_1 & \ldots & a_d \\
\vdots & \vdots & & \vdots \\
a_0 & a_1 & \ldots & a_d
\end{pmatrix} - P.
$$

We set $f = L \circ J$ and define

$$\alpha_j := \eta_{\varphi(j)}(a_j), \quad \beta_j := \eta_{\varphi(j)}(1 - a_j), \quad \sigma_j = \eta_j(1). \quad (6.2)$$

It follows that $f(\Sigma_i \cap \mathcal{D}_i) = \sigma_i$. If (6.1) and (6.2) hold, we will refer to $f = L \circ J$ as a permutation mapping. In this case we see by the following Lemma that $f$ permutes the diagonals $\mathcal{D}_j$ according to the permutation $\varphi$.

**Lemma 6.1.** If (6.1) and (6.2) hold, then for each $j$ and each $c \in \mathbb{C}$, we have $f(\mathcal{D}_j - e_j) \subset \mathcal{D}_{\varphi(j)}$, and in fact:

$$h(\eta_j(c)) = \eta_{\varphi(j)}(c + a_j - 1). \quad (6.3)$$

We are especially interested in the orbits $\mathcal{O}(\alpha_j)$ and $\mathcal{O}(\beta_j)$. Since $\alpha_j$ and $\beta_j$ both belong to $\mathcal{D}_j$, we see that these orbits can be singular (i.e., they can enter $\mathcal{I} \cup \mathcal{E}$) only if they end in $e_j$ or $\sigma_j$. By the Lemma, $\mathcal{O}(\alpha_j)$ is singular exactly when one of two things happens: either

$$a_j + a_{\varphi(j)} + \ldots + a_{p^{N-1}(j)} - (N - 1) = 0, \quad (6.4)$$

in which case the orbit of $\alpha_j = \eta_{\varphi(j)}(a_j)$ ends in $f^{N-1}\alpha_j = e_{p^{N}(j)}$, or

$$a_j + a_{\varphi(j)} + \ldots + a_{p^{N-1}(j)} - (N - 1) = 1, \quad (6.5)$$

in which case it ends in $\sigma_{p^{N}(j)}$. Similarly, the orbit $\mathcal{O}(\beta_j)$ is singular exactly when either

$$1 - a_j + a_{\varphi(j)} + \ldots + a_{p^{N-1}(j)} - (N - 1) = 0, \quad (6.6)$$

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or

\[ 1 - a_j + a_{p(j)} + \ldots + a_{p^{n-1}(j)} - (N - 1) = 1. \] (6.7)

For the rest of this section we suppose that \( p \) is the identity. In this case, the only way we can have a singular orbit is in case (6.4), which becomes

\[ a_j = \frac{N - 1}{N}. \] (6.8)

It follows that all singular orbit lists consists of single orbits and are closed. Thus the orbit list structure of our map is \( \mathcal{L}^c = \{\{N_0\}, \{N_1\}, \ldots, \{N_k\}\} \), \( \mathcal{L}^o = \emptyset \). By Theorem A.1, the characteristic polynomial is given as

\[ \chi(x) = (x - d) \prod_{j=0}^{k} (x^{N_j} - 1) + (x - 1) \sum_{j=0}^{k} \prod_{i \neq j} (x^{N_i} - 1). \] (6.9)

We note from the formula for \( \chi(x) \) that \( f^* \) has a zero eigenvalue exactly when \( \chi(0) = k + 1 - d = 0 \). The case \( k = d - 1 \) may be considered to be as close as possible to the integrable case and is of particular interest. It was conjectured in [BHM] that the denominator of the generating function in this case should be

\[ q(x) = 1 - (k + 1)x - \sum_{i=1}^{k+1} (-1)^i (kx + i - 1) S_i(N_1, \ldots, N_n), \]

where \( S_i(N_1, \ldots, N_{k+1}) = \sum_{i_1 < \ldots < i_k} x^{N_{i_1} + \ldots + N_{i_k}} \). This is a consequence of formula (2.4) applied to the characteristic polynomial \( \chi(x) \) as given in (6.9).

Without loss of generality, we will assume that \( N_0 \leq N_1 \leq \ldots \leq N_k \). If some of the \( N_j \) are equal to 1, we define \( 1 \leq \ell \leq k + 1 \) by the condition that \( N_0 = \ldots = N_{\ell-1} = 1 \) and \( N_\ell > 1 \). In this case we have

\[ \chi(x) = (x - 1)^\ell \left[ (x - \bar{d}) \prod_{j=\ell}^{k} (x^{N_j} - 1) + (x - 1) \sum_{j=\ell}^{k} \prod_{i=\ell, i \neq j}^{k} (x^{N_i} - 1) \right], \] (6.10)

where we set \( \bar{d} = d - \ell \). There are cases which turn out to be particularly simple:

(a) \( k = d - 2 \) and \( N_0 = \ldots = N_k = 1 \), or

(b) \( k = d - 1 \), and \( N_0 = \ldots = N_{k-1} = 1 \), or

(c) \( k = d - 1 \), \( N_0 = \ldots = N_{k-2} = 1 \), and \( N_{k-1} = N_k = 2 \). (6.11)

In connection with conditions (6.10) and (6.11), we note that the eigenspace of \( M \) corresponding to eigenvalue 1 is given by

\[ \{(\tau; \alpha_0, \ldots, \alpha_0; \ldots; \alpha_k, \ldots, \alpha_k) : (d - 1)\tau + \alpha_0 + \ldots + \alpha_k = 0\}. \]

The codimension of this space is \( 1 + \sum_{j=0}^{k} (N_j - 1) \).
**Theorem 6.2.** If (6.11) holds, then $d_n$ grows at most linearly.

**Proof.** Let $M$ be the matrix representing $f^*_X$. In case (6.11a), $M$ is a $d \times d$ version of the matrix in (3.1), and thus $d_n = (d - 1)n + 1$.

In case (6.11b) we have $\ell = k$ and $\chi = (x - 1)^{\ell}x^{N_k}$. The matrix $M$ is expanded from the previous case; the lower right hand 0 is replaced by the $N_k \times N_k$ block $\begin{bmatrix} 0 & & \\ & 1 & \\ & & 1 \end{bmatrix}$. Thus $M$ has size $(1 + \ell + N_k) \times (1 + \ell + N_k)$ and rank $\ell + N_k$. The null space of $M$ is one-dimensional, and there is a $N_k \times N_k$ block $\begin{bmatrix} 0 & & \\ & 1 & \\ & & 1 \end{bmatrix}$ in its Jordan canonical form. The matrix $M - I$ is seen to have rank $N_k$, so the rest of the Jordan canonical form consists of an identity matrix. It follows that $M^n = M^{N_k}$ for $n \geq N_k$.

In case (6.11c) we have $\ell = k - 1 = d - 2$, $\bar{d} = 2$, and $\chi(x) = x(x + 1)(x - 1)^{d+1}$. The rank of $M - I$ is seen to be 3. This means that the space of eigenvectors with eigenvalue 1 has dimension $d$. Thus the Jordan canonical form has a diagonal portion and a $2 \times 2$ block with eigenvalue 1. The diagonal portion consists of $d - 1$ ones, a zero and a minus one. Thus $d_{2n}$ and $d_{2n+1}$ are each a linear function of $n$. This completes the proof.

**Theorem 6.3.** If (6.8) holds for at least one $j$, and if (6.11) does not hold, then $1 < \delta < d$ and $d - 1 \leq \delta \leq \bar{d}$.

**Proof.** We saw at the end of §4 that if (6.8) holds for at least one $j$, then $\delta < d$. Now we show that $1 < \delta$. By Theorem 2.1, $\delta$ is the largest real zero of $\chi(x)$. Thus we will show that if (6.11) does not hold, then $\chi$ has a zero in the interval $(1, d)$. Let us expand $\chi$ in a Taylor series about the point $x = 1$. We find $\chi(x) = C(x - 1)^{k+1} + O((x - 1)^{k+2})$, with

$$C = (1 - d)N_0 \cdots N_k + \sum_{j=0}^{k} \prod_{i \neq j} N_i.$$ 

We will show that if (6.11) does not hold, then $\chi < 0$ on some interval $(1, 1 + \epsilon)$ and thus $\chi$ will have a zero in $(1 + \epsilon, d)$.

First suppose that $k \leq d - 2$. Then $1 - d \leq -(k + 1)$ and $\prod_{j \neq i} N_i \leq N_1 \cdots N_k$, so

$$C \leq -(k + 1)N_0 \cdots N_k + (k + 1)N_1 \cdots N_k.$$ 

This gives $C < 0$ unless $N_0 \cdots N_k = \prod_{i \neq j} N_i$, in which case $N_0 = N_1 = \ldots = N_k = 1$. This is case (6.11a).

Now suppose $k = d - 1$, let $\ell$ be as above, and factor $\chi = (x - 1)^{\ell}q(x)$ as in (6.10). Expanding $q(x)$ about $x = 1$, we obtain $q(x) = \tilde{C}(x - 1)^{k-\ell+1} + O((x - 1)^{k-\ell+2})$, where

$$\tilde{C} = (1 - \bar{d})N_\ell \cdots N_k + \sum_{j=\ell}^{k} \prod_{\ell \leq i \leq k} N_i,$$
where $\prod'$ means that the product is taken over $i \neq j$. Since $N_\ell \geq 2$, we have

$$\tilde{C} \leq 2(\ell + 1 - d)N_{\ell+1} \cdots N_k + (kl - \ell + 1)N_{\ell+1} \cdots N_k = (\ell + 2 - d)N_{\ell+1} \cdots N_k.$$  

Now we may assume $\ell \leq k - 1$, for otherwise we are in case (6.11b). Thus $\tilde{C} < 0$ unless $\ell = d - 2$, and thus $d - 1 = k$. We have already handled the case $\ell = k = d - 1$. Thus we have $\tilde{C} < 0$ unless $\ell = k - 1 = d - 2$. By our formula, then,

$$\tilde{C} = (d - 2 + 1 - d)N_{k-1}N_k + N_{k-1} + N_k = -N_{k-2}N_k + N_{k-1} + N_k,$$

which is strictly negative unless $N_{k-1} = n_k = 2$, which is case (6.11c). This completes the proof.

**Theorem 6.4.** If (6.11) does not hold, then $\delta$ is a simple eigenvalue for $f^*$. If, in addition, $\bar{d} \geq 3$, then $\delta$ is the unique root of $\chi$ in the interval $[2, d]$.

**Proof.** By (6.10) we may assume that $\bar{d} = d$, which is to say that $N_j \geq 2$ for all $j$. We will suppose that $x > 1$ is a zero of $\chi$, and we will show that for such a zero we have $\chi'(x) > 0$. If we divide $\chi$ by $(x^{N_j} - 1)$, the condition that $\chi(x) = 0$ is equivalent to

$$(x - d) \prod_{i \neq j}(x^{N_i} - 1) = -\frac{x - 1}{x^{N_j} - 1} \prod_{i \neq j}(x^{N_i} - 1) - (x - 1) \sum_{i \neq j} \prod''(x^{N_\ell} - 1),$$

where $\prod''$ indicates a product over all $\ell$ distinct from $i$ and $j$. In order to compute $\chi'(x)$ we first use the product rule and then we substitute the identity above to obtain

$$\chi'(x) = \left(\prod_j(x^{N_j} - 1)\right) \left[1 + \sum_j \left\{1 - (x - 1)\frac{N_j x_j^{N_j-1}}{x^{N_j} - 1}\right\} \frac{1}{x^{N_j} - 1}\right].$$

We abbreviate this as

$$\chi'(x) = \left(\prod_j(x^{N_j} - 1)\right) \left[1 + \sum_j \varphi_j(x)\right],$$

where each $\varphi_j$ has the form

$$\varphi(x) = \frac{1}{x^N - 1} + (1 - x)\frac{N x^{N-1}}{(x^N - 1)^2}.$$

Since the product in the formula for $\chi'$ is strictly positive for $x > 1$, it suffices to show that $\varphi_j(x) > 0$ for each $j$. In fact, we have $\varphi(x) > 0$ for all $x \geq 2$ and $N \geq 2$. For this, we note that $\lim_{x \to \infty} \varphi(x) = 0$, and we show that

$$\varphi'(x) = \frac{N x^{N-2} - 3(x - 1)x^N}{(x^N - 1)^3} \left[3(N - 1)x^{N+1} - (3N - 1)x^N - (N - 3)x + (N - 1)\right] < 0.$$
This is equivalent to showing that the expression in square brackets is positive for all \( N \geq 2 \) and \( x \geq 2 \). This is elementary, and so we conclude that \( \chi'(x) > 0 \) for every zero of \( \chi \) in the interval \([2, d]\). Thus there can be no more than one zero in \([2, d]\).

If \( d \geq 3 \), then as was observed above, \( 2 \leq d - 1 \leq \delta \leq d \). Thus \( \delta \) is a simple zero of \( \chi \). If \( d = 2 \), then \( 1 \leq \delta \leq 2 \), and so the arguments above do not apply directly. However, the case \( d = 2 \) may be broken into three subcases (1) \( k = 0, N_0 \geq 2 \), (2) \( k = 1, N_0 = 2 < N_1 \), and (3) \( k = 1, 3 \leq N_0 \leq N_1 \). The computations are similar to what we have done already, so we omit the details.

§7. Examples

A number of mappings of the form \( L \circ J \) have arisen in the mathematical physics literature. Let us show how the preceding discussion may be applied to yield the degree complexity of these maps. The third example will lead us to some non-elementary maps, and our treatment of them will foreshadow the technique we use in §8.

Example 7.1. We consider the (families of) matrices:

\[
A_1 = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 2 & -1 + q^2 & -1 + q^2 \\ 2 & -1 + q & -1 - q \\ 2 & -1 - q & -1 + q \end{pmatrix}
\]

\[
A_3 = \begin{pmatrix} 1 & \frac{1}{\ell(1+2\ell)} & \frac{1-3/\ell}{(1+2\ell)(1+\ell)} \\ \frac{1-2/\ell}{\ell(1+2\ell)} & 1 & \frac{3+2\ell}{1+\ell} \\ \frac{1-3/\ell}{(1+2\ell)(1+\ell)} & \frac{3+2\ell}{1+\ell} & 1 \end{pmatrix}
\]

We set \( f = A_j \circ J \) for \( j = 1, 2, 3 \). (The case of matrix \( A_1 \) arises, for instance, in [BMV], \( A_2 \) in [BMV] and [V], and \( A_3 \) is found in [R3]; and also in [R1] for the special case \( \ell = 1 \).)

In each case we have

\[
\Sigma_j \to \alpha_j \to e_j, \quad 0 \leq j \leq 2.
\]

Thus the orbits \( O_j \) are singular, and \( |O_j| = 2 \) for \( 0 \leq j \leq 2 \). In other words, the singular orbit list structure is \( \mathcal{L}^c = \{ \{2\}, \{2\}, \{2\} \} \) and \( \mathcal{L}^s = \emptyset \). If we write \( f_X^n \) according to equations (4.1) (4.2) and (4.3), we find that \( (f_X)^6 \) is the identity. Thus \( \deg(f^n) \) is bounded since it is a periodic sequence of period 6.

Example 7.2. We consider the matrices:

\[
B_1 = \frac{1}{9} \begin{pmatrix} 1 & -8 & 16 \\ -2 & 7 & 4 \\ 4 & 4 & 1 \end{pmatrix}, \quad B_2 = \frac{1}{10} \begin{pmatrix} 1 & 22 & 77 \\ 1 & -8 & 7 \\ 1 & 2 & -3 \end{pmatrix}.
\]

(We have taken \( B_1 \) from [R2] and \( B_2 \) from [V].) Let \( g_j = B_j \circ J \). It follows that for both \( g_1 \) and \( g_2 \) we have:

\[
\Sigma_0^* \to * \to e_0.
\]

In the case of \( g_1 \), the point \([1 : 1 : 1]\) is a parabolic fixed point, and the orbits \( O_1 \) and \( O_2 \) are in the attracting basin of \([1 : 1 : 1]\), so they are both nonsingular. Similarly, in}
the case of $g_2$, the orbits $O_1$ and $O_2$ are in the basin of an attracting 2-cycle and thus are both nonsingular. We conclude that the singular orbit list structure for both $g_1$ and $g_2$ is $L^c = \{\{2\}\}$, $L^o = \emptyset$, and so $\delta(g_1) = \delta(g_2) = (1 + \sqrt{5})/2$ is the largest root of $x^2 - x - 1 = 0$.

**Example 7.3.** Consider the family of matrices:

$$C(q) = \begin{pmatrix} 2 & -1 - q^2 & -1 - q^2 \\ 2 & -1 + q & -1 - q \\ 2 & -1 - q & 1 + q \end{pmatrix}$$

with $q \neq 0$, which is considered in [V]. If we set $h = C(q) \circ J$, then $\Sigma_0 \to \alpha_0 \to e_0$ for all $q$. The natures of the orbits $O_1$ and $O_2$, however, are dependent on $q$. For generic $q$, the orbits $O_1$ and $O_2$ are nonsingular, so we have $L^c = \{\{2\}\}$, $L^o = \emptyset$, so $\delta = (1 + \sqrt{5})/2$ as in Example 2.

Now let us show what happens in the singular cases. Our purpose here is to show how the methods of §4 can be used to treat the different cases that can arise. First let us handle the most singular cases:

**Case** $q = -1$. $\alpha_0 \to e_0$, $\alpha_1 = e_1$, $\alpha_2 = e_2$. In this case we have $L^c = \{\{2\}, \{1\}, \{1\}\}$, $L^o = \emptyset$, and the degrees are periodic of period 3.

**Case** $q = 1$. $\alpha_0 \to e_0$, $\alpha_1 = e_2$, $\alpha_2 = e_1$. In this case we have $L^c = \{\{2\}, \{1, 1\}\}$, $L^o = \emptyset$, and the degrees are periodic with period 6.

In every case, we have $\alpha_0 \to e_0$, so we pass to the map $h : X \to X$ obtained by blowing up the orbit $O_0 = \{\alpha_0, e_0\}$. Now observe:

$$\text{If } h[a : b : c] = [a' : b' : c'], \text{ then } h[a : c : b] = [a' : c' : b']\tag{7.1}$$

It follows from (7.1) that $h^n \alpha_1 \in I \cup E$ if and only if $h^n \alpha_2 \in I \cup E$. It follows that we may proceed by induction on $n := |O_1| = |O_2|$. Let us define

$$S' := \{x_0 - x_2 = 0\}, \quad S'' := \{(1 + q)x_1 + (1 - q)x_2 = 0\}.$$ 

We have $h : S' \leftrightarrow S''$. Since $\alpha_1 \in S''$, it follows that $h^n \alpha_1 \in S'$ when $n$ is odd and $h^n \alpha_1 \in S''$ when $n$ is even. By (7.1), we have

$$\{x_0 - x_1 = 0\} \leftrightarrow \{(1 + q)x_2 + (1 - q)x_1 = 0\},$$

so an analogous discussion applies to the orbit of $\alpha_2$.

**Case** $n = |O_1| = |O_2|$ is even. If $n$ is even, then $h^{n-1} \alpha_j \in E \cup I$. If $h^{n-1} \alpha_j \in I$, then $h^{n-1} \alpha_j \in S' \cap I$, and we have $h^{n-1} \alpha_j = e_j$ for $j = 1, 2$. Thus our orbit structure is $L^c = \{\{2\}, \{n\}, \{n\}\}$ and $L^o = \emptyset$. If $n = 2$, then $d_n$ is periodic of period 6, as in Example 7.1. If $n \geq 6$, then the degree complexity $\delta_n$ is the largest root of the polynomial $x^{n+2} - x^{n+1} - x^n + x^2 + x - 1$.

The other possibility is that $h^{n-1} \alpha_1 = [1 : 0 : 1] \in S' \cap \Sigma_j$. In this case, we have $h^n \alpha_1 = \alpha_1$. By (7.1), a similar argument applies to $\alpha_2$. Thus $\alpha_1$ and $\alpha_2$ are periodic, and
the orbits $\mathcal{O}_1$ and $\mathcal{O}_2$ are essentially nonsingular. Thus we have $\mathcal{L}^c = \{\{2\}\}$, and we have $\delta = (1 + \sqrt{5})/2$ as in Example 7.2.

**Case** $n = |\mathcal{O}_1| = |\mathcal{O}_2|$ is odd. In this case we have $h^{n-1} \alpha_1 \in S'' \cap \mathcal{E} \cup \mathcal{I}$. We cannot have $h^{n-1} \alpha_1 \in \mathcal{I} \cap S''$, since the point $e_0$ has been blown up, and we can only reach the blow-up fiber $\mathcal{F}_{e_0}$ through the fiber $\mathcal{F}_{\alpha_0}$, and we can reach $\mathcal{F}_{\alpha_0}$ only through $\Sigma_0$. Thus we must have $h^{n-1} \alpha_1 = [0 : 1 - q : 1 + q] \in S'' \cap \Sigma_0$. Let us consider this point as the endpoint of the curve $t \mapsto [t : 1 - q : 1 + q]$ as $t \to 0$. Thus $h$ maps this to the curve

\[ t \to (1, 1, 1) + t(1 + q^2, 1 - q^2, 1 + q^2) + O(t^3) \]

which lands at a point of the fiber $\mathcal{F}_{\alpha_0}$, and then to the curve

\[ t \to (1, 0, 0) + t(0, 1 - q, 1 + q) + t^2(1 + q^2, 2(1 + q)^{-1}, 2(1 - q)^{-1}) + O(t^3), \]

which lands at a point of the fiber $\mathcal{F}_{e_0}$. The next image of this curve lands at

\[ \beta_1 := h^{n+2} \alpha_1 = [1 + q^2 : 1 - q^2 : 1 + q^2] \in \mathbb{P}^2 - \{\alpha_0, e_0\}, \]

and we are back to a “normal” point of $S'$.

At this stage there are two possibilities. First, it is possible that $\mathcal{O}(\beta_1)$ is nonsingular. We conclude, then that $h : X \to X$ is 1-regular, and we have $\delta = (1 + \sqrt{5})/2$ as in Example 2. The other possibility is that $\mathcal{O}(\beta_1)$ is singular. This means that $h^{j_i} \beta_1 \in X - (\mathcal{I} \cup \mathcal{E})$ for $0 \leq j < j_1$, and $h^{j_1} \beta_1 \in \mathcal{I} \cup \mathcal{E}$. First, we see that $h^{j_1} \beta_1$ cannot be in $S'' \cap \mathcal{E}$. For in this case we must have $h^{j_1} \beta_1 = [0 : 1 - q : 1 + q]$ as before. But this is not possible since we have remained inside points where $h$ is a diffeomorphism. On the other hand, if we have $h^{j_1} \beta_1 \in S'$, then we must have $h^{j_1} \beta_1 = [1 : 0 : 1]$. We have $h^{j_1} \beta_1 \in S'$ if $j_1$ is even. Thus $h^{j_1 + 1} \beta_1 = \alpha_1$, and $\alpha_1$ is periodic. A similar argument shows that both $\alpha_1$ and $\alpha_2$ are periodic in this case. Thus $\mathcal{O}_1$ and $\mathcal{O}_2$ are both essentially nonsingular, and we are in the case of Example 2 again.

**Sub-case** $\mathcal{O}(\beta_1)$ is singular. The other possibility is that $\mathcal{O}(\beta_1)$ ends at the point $e_1 \in S'$. In this case, $\mathcal{O}(\beta_2)$ also ends at $e_2$, and $|\mathcal{O}(\beta_1)| = |\mathcal{O}(\beta_2)|$. This sub-case is not elementary, and here we must perform a second series of blow-ups. Let $\tilde{\mathcal{O}}(\alpha_1)$ denote the orbit in $X$, starting with $\alpha_1$. Figure 1 shows $\tilde{\mathcal{O}}(\alpha_1)$ and $\tilde{\mathcal{O}}(\alpha_2)$ in the space $X$. On the top row, the portion $\alpha_1 \to (\ast)_1 \to [0 : 1 - q : 1 + q]$ is the orbit $\mathcal{O}_1 = \mathcal{O}(\alpha_1)$, and $(\ast)_1$ indicates the points in the middle of the orbit. The image $h[0 : 1 - q : 1 + q]$ is indicated by the subscript $\alpha_0$ (base point) and fiber coordinate $[1 + q^2 : 1 - q^2 : 1 + q^2] \in \mathcal{F}(\alpha_0)$. We will use the notation

\[ \tau_{0, 1} = \begin{bmatrix} 0 & 1 - q \\ 1 + q \\ 1 + q \end{bmatrix}_{e_0}, \quad \tau_{0, 2} = \begin{bmatrix} 0 \\ 1 + q \\ 1 - q \end{bmatrix}_{e_0} \]

for the points of the orbit that are in the fiber $\mathcal{F}(e_0)$. The bottom row shows the orbit of $\Sigma_0$, which contains the two points of exceptional fibers $\mathcal{F}(\alpha_0)$ and $\mathcal{F}(e_0)$. 

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We see that under

\[
\Sigma_1 \to \alpha_1 \to (*)_1 \to \begin{bmatrix} 0 \\ 1 - q \\ 1 + q \end{bmatrix} \to \begin{bmatrix} 1 + q^2 \\ 1 - q^2 \\ 1 + q^2 \end{bmatrix}_\alpha \to \tau_{0,1} \to \beta_1 \to (**)_1 \to e_1
\]

\[
\Sigma_2 \to \alpha_2 \to (*)_2 \to \begin{bmatrix} 0 \\ 1 + q \\ 1 - q \end{bmatrix} \to \begin{bmatrix} 1 + q^2 \\ 1 - q^2 \\ 1 - q^2 \end{bmatrix}_\alpha \to \tau_{0,2} \to \beta_2 \to (**)_2 \to e_2
\]

\[
\Sigma_0 \to \mathcal{F}(\alpha_0) \to \mathcal{F}(e_0)
\]

Figure 1

We let \( \pi : X_2 \to X \) denote the space obtained from \( X \) by blowing up the points of \( \mathcal{O}(\alpha_1) \cup \mathcal{O}(\alpha_2) \). Let \( h_2 : X_2 \to X_2 \) denote the induced birational map. We see that in \( X_2 \) the curves \( \Sigma_1 \) and \( \Sigma_2 \) are no longer exceptional, so \( h_2 \) is 1-regular. Let us determine the mapping on cohomology. As a basis for \( H^{1,1}(X_2) \) we take \( H_{X_2} \), together with all the fibers indicated in Figure 1. For instance, we take the fibers \( \mathcal{F}(p) \) for all the points \( p \in (**)_1 \). We see that under \( h_2^* \) we have:

\[
\mathcal{F}(e_0) \to \mathcal{F}(\alpha_0) \to \{\Sigma_0\}_{X_2}
\]  

(7.2)

and

\[
\mathcal{F}(e_1) \to \mathcal{F}(p) \to \cdots \to \mathcal{F}(p) \to \mathcal{F}(\beta_1) \to \cdots \to \mathcal{F}(\alpha_1) \to \{\Sigma_1\}_{X_2},
\]  

(7.3)

where the \( p \) are points in the \( (**)_1 \) portion of the orbit, taken in inverse order. Similarly, we note that the centers of blow-up inside \( \Sigma_0 \) are \( e_1, e_2, [0 : 1 - q : 1 + q] \), and \([0 : 1 + q, 1 - q] \). Thus we have

\[
\{\Sigma_0\}_{X_2} = H_{X_2} - \mathcal{F}(e_1) - \mathcal{F}(e_2) - \mathcal{F} \begin{bmatrix} 0 \\ 1 - q \\ 1 + q \\ 1 - q \end{bmatrix} = \mathcal{F}(\tau_{0,2}) \cdot \begin{bmatrix} 0 \\ 1 - q \\ 1 + q \end{bmatrix}.
\]  

(7.4)

On the other hand, \( e_0, e_2 \) are the base points in \( \Sigma_1 \). The closure of \( \Sigma_1 \) in \( X_2 \) intersects the fiber \( \mathcal{F}(e_0) \) at the point with (fiber) coordinate \([0 : 0 : 1]\). Since \( q^2 \neq 1 \), it follows that this is distinct from the base points \( \tau_{0,1} \) and \( \tau_{0,2} \) of the second level blow up. Thus we have

\[
\{\Sigma_1\}_{X_2} = H_{X_2} - \mathcal{F}(e_0) - \mathcal{F}(e_2) - \mathcal{F}(\tau_{0,1}) - \mathcal{F}(\tau_{0,2})
\]  

(7.5)

and

\[
\{\Sigma_2\}_{X_2} = H_{X_2} - \mathcal{F}(e_0) - \mathcal{F}(e_1) - \mathcal{F}(\tau_{0,1}) - \mathcal{F}(\tau_{0,2}).
\]  

(7.6)

Finally, we must evaluate \( h_2^* H_{X_2} \). If \( H \) is a general hypersurface in \( \mathbb{P}^2 \), then \( \{e_0, e_1, e_2 \} \subset h^{-1} H = J^{-1} L^{-1} H \). We have seen that \( J_X \) is nonconstant on the fibers \( \mathcal{F}(e_j) \), so \( e_0, [1 + q^2 : 1 - q^2 : 1 + q^2] \) will not be contained in \( J_X^{-1} H \) for generic \( H \). We conclude that

\[
h_2^* H_{X_2} = 2H_{X_2} - \mathcal{F}(e_0) - \mathcal{F}(e_1) - \mathcal{F}(e_2) - \mathcal{F}(\tau_{0,1}) - \mathcal{F}(\tau_{0,2}).
\]  

(7.7)
Equations (7.2–7) serve to define the linear transformation $f^*$. Assuming that $|\mathcal{O}(\alpha_1)| = |\mathcal{O}(\beta_1)| = n$, the characteristic polynomial of this transformation turns out to be the same as the characteristic polynomial corresponding to the elementary map $L^c = \{\{2\}, \{2n + 2\}, \{2n + 2\}\}$. $L^o = \emptyset$.

**Observed cases.** The first few cases with $n$ even occur for $n = 2$ if $3 + q^2 = 0$, $n = 6$ if $5 + 10q^2 + q^4 = 0$, $n = 10$ if $7 + 35q^2 + 21q^4 + q^6 = 0$. In all of these cases the orbit $O_j$ ends with $e_j$. The first few cases with $n$ odd occur for $n = 1$ if $1 + q^2 = 0$, $n = 3$ if $1 + 3q^2 = 0$, $n = 5$ if $q^4 + 6q^2 + 1 = 0$, $n = 7$ if $1 + 10q^2 + 5q^4 = 0$, $n = 9$ if $1 + 14q^2 + q^4 = 0$, $n = 11$ if $1 + 21q^2 + 35q^4 + 7q^6 = 0$. In all of the odd cases, the orbit $O(\beta_j)$ is singular, and $|\mathcal{O}(\alpha_j)| = |\mathcal{O}(\beta_j)|$. In both the even and odd cases, $\delta_n$ is less than the generic case $(1 + \sqrt{5})/2$, and we see from the defining equations that $\delta_n \to (1 + \sqrt{5})/2$ as $n \to \infty$.

§8. Permutation mappings: Orbit collision, orbit separation.

Here we continue our discussion of permutation mappings. Like Noetherian mappings, the permutation mappings have the form $f = L \circ J$. As before, the key to understanding these mappings is understanding what happens with the orbits of the points $\alpha_j := f\Sigma_k^*$. If such an orbit is singular, it ends at a point $e_k$ or a point $\sigma_k \in \Sigma_k^*$. The first case corresponds to elementary behavior and has been treated above. The second case, corresponding to (6.4–7), is an example of non-elementary behavior; in this case the orbit $O(\alpha_j)$ “joins” the orbit $O(\alpha_k)$. We refer to this as an orbit collision. Further orbit collisions are also possible, with $O(\alpha_k)$ joining $O(\alpha_n)$, etc. Our interest in §8 is to show that the method of regularization can be applied to the case of orbit collisions. This leads us to perform multiple blow-ups over a fixed base point, which provides a new manifold in which these orbits are separated, and the induced map is 1-regular.

Let $\mathcal{S}$ denote the set of all orbits $O(\alpha_j)$, $O(\beta_j)$, $0 \leq j \leq d$, which are singular. There are four possible types of singular orbits in $\mathcal{S}$: $\alpha e$, $\alpha \sigma$, $\beta e$ and $\beta \sigma$, depending on the type of starting point and the type of ending point. Now we define admissible chains of singular orbits. The admissible chains of the first generation are the singular chains starting with an $\alpha$ and ending with an $e$. We denote the chains of the first generation by $C^1$. Now let us proceed inductively, assuming that we have defined the admissible chains $C^j$ at generation $j$. An admissible chain of generation $j + 1$ will be a finite sequence of singular orbits of $\mathcal{S}$ which has the following form: $\mathcal{S} \mathcal{C} \mathcal{S} \mathcal{C} \ldots \mathcal{C} \mathcal{S}$, which means that we start and end with orbits of $\mathcal{S}$ and in the middle, we alternate between $\mathcal{S}$ and $\mathcal{C} = C^1 \cup \ldots \cup C^j$. By convention $C^{j+1}$ is disjoint from $C^1 \cup \ldots \cup C^j$. In addition, the sequence must obey the following rules: The first orbit starts with an $\alpha$; the last orbit ends with an $e$, and the permissible transitions between $\mathcal{S}$ and $\mathcal{C}$ are $e_i \to \beta_{p(i)}$ and $\sigma_i \to \alpha_{p(i)}$. In other words, suppose that $O'$ is an orbit from $\mathcal{S}$ which is followed by a chain $O'' \ldots O''' \in \mathcal{C}$. If $O'$ ends with $e_\ell$, then $O''$ must begin with $\beta_{p(\ell)}$; and if $O'''$ ends with $\sigma_\kappa$, and if $O'''$ is followed by an orbit $O'''' \in \mathcal{S}$, then $O'''''$ must begin with $\alpha_{p(\kappa)}$. The process of constructing chains is finite, so there is a maximum generation $\kappa$ that can occur. Thus $C^1 \cup C^2 \cup \ldots \cup C^\kappa$ is the set of admissible chains.

We illustrate this with an example, which is sketched in Figure 2. This corresponds to the cyclic permutation $p = (1, 2, 3, \ldots, N)$ with $N$ greater than 14. The singular orbits $\mathcal{S}$ are $\{\alpha_1, \sigma_2\}$, $\{\alpha_3, \sigma_4\}$, $\{\alpha_5, e_6\}$, $\{\beta_7, e_8\}$, etc. These are inside the bottom row of Figure 26.
1. To conserve space, we have constructed all of these orbits to have (minimal) length 2. The chains of the \( j \)th generation may be read off from the \( C^j \) row of the matrix by joining adjacent dots, moving from left to right. Thus \( C^1 = \{ \{\alpha_5, e_6\}, \{\alpha_{11}, e_{12}\} \} \) consists of two chains, and \( C^2 = \{\alpha_3, \ldots, e_8\} \) and \( C^3 = \{\alpha_1, \ldots, e_{14}\} \) each contain one chain.

\[
\begin{array}{cccccccccccc}
C^3 & & & & & & & & & & & \\
C^2 & & & & & & & & & & & \\
C^1 & & & & & & & & & & & \\
\alpha_1 & \sigma_2 & \alpha_3 & \sigma_4 & \alpha_5 & e_6 & \beta_7 & e_8 & \beta_9 & \sigma_{10} & \alpha_{11} & e_{12} & \beta_{13} & e_{14}
\end{array}
\]

Figure 2. Singular Chains

In order to construct a 1-regularization of \( f \), we perform multiple blow-ups of points of \( P^d \), determined by the structure of the chains. We define the *height* of a point \( p \in P^d \), written \( h(p) \), to be the number of chains \( \gamma \) that contain \( p \). Note that if \( \gamma \in C^j \) is a singular chain, the height \( h(p) \) changes by at most 1 as we step forward from one point \( p \in \gamma \) to the next one. Let \( \mathcal{P} \) denote the set of points \( p \) of \( P^d \) which occur in singular chains. Thus \( \mathcal{P} = \{ p : h(p) > 0 \} \). Observe that \( \alpha_j, \beta_j \in \mathcal{D}_j \), and thus \( \mathcal{P} \subset \bigcup_{j=0}^{\infty} \mathcal{D}_j \).

We will define the space \( X \) by blowing up \( h(p) \) times over each \( p \in \mathcal{P} \). Let \( \pi^1 : X^1 \to P^d \) denote the space obtained by blowing up \( P^d \) at each \( p \in \mathcal{P} \). As we construct manifolds \( \pi^k : X^k \to X^{k-1} \), it will be convenient to let \( \mathcal{D}_j \subset X^k \) denote the strict transform of \( \mathcal{D}_j \), i.e. the closure of \( (\pi^k)^{-1} (\mathcal{D}_j - \{p\}) \) in \( X^k \). The strict transform of \( \mathcal{D}_j \) in \( X^1 \) intersects \( \mathcal{F}^1(p) \) transversally at a point \( p^1 := \mathcal{D}_j \cap \mathcal{F}^1(p) \). We now construct \( \pi^2 : X^2 \to X^1 \) by blowing up all the points \( p^1 \in \mathcal{F}^1(p) \) for which \( h(p) > 1 \). Let \( \mathcal{F}^2(p) = (\pi^2)^{-1} (p^1) \) denote the new fiber. To simplify our notation we write \( \mathcal{F}^1(p) \) for the strict transform of \( \mathcal{F}^1(p) \) inside \( X^2 \). This abuse of notation causes no problem because \( \mathcal{F}^1(p) \cap \mathcal{F}^2(p) \) has codimension 2. Since \( p^1 \in \mathcal{D}_j \), it follows that \( \mathcal{D}_j \) intersects \( \mathcal{F}^2(p) \) transversally at a point \( p^2 \). We continue the blow-up process at the points \( p^2 \) for which \( h(p) > 2 \). We continue in this way until we reach the maximum value of \( h \); thus we construct the space \( X \).

It follows that over every point \( p \in \mathcal{P} \), we have exceptional fibers \( \mathcal{F}^j(p), 1 \leq j \leq h(p) \). For simplicity of notation, we let \( \mathcal{F}^j(p) \) denote its corresponding class in \( H^{1,1} \). These cohomology classes, together with the class \( H_\mathcal{X} \) of a hyperplane, generate \( H^{1,1}(X) \). We find it convenient to use the notation \( \mathcal{F}(p) = \sum_{j=1}^{h(p)} \mathcal{F}^j(p) \).

Let us describe how the induced map \( f_X : X \to X \) maps the various exceptional fibers. Let us start with a singular chain \( \mathcal{O} = \{ \alpha_j, \ldots, e_k \} \) of the first generation. By §3, the first and last maps in the sequence

\[
\Sigma_{p-1(j)} \to \mathcal{F}^1(\alpha_j) \to \cdots \to \mathcal{F}^1(e_k) \to L \Sigma_k
\]

have (maximal) generic rank \( d \). Since \( f \) is locally biholomorphic at each point of \( \mathcal{O} - \{e_k\} \), the rest of the maps are biholomorphic in a neighborhood of the fibers. In particular, none of the hypersurfaces in (8.1) is exceptional.

Next, consider a singular chain of the 2nd generation. We may suppose that the chain has the form \( \mathcal{O}'\mathcal{O}''\mathcal{O}''' \), where \( \mathcal{O}' = \{ \alpha_i, \ldots, \sigma_{p-1(j)} \} \), \( \mathcal{O}'' = \{ \alpha_j, \ldots, e_k \} \), and \( \mathcal{O}''' = \ldots \).
\{\beta_{p(k)}, \ldots, e_\ell\}. We will now claim that \(f_X\) induces full rank mappings

\[
\Sigma_{p^{-1}(i)} \rightarrow \mathcal{F}^1(\alpha_i) \rightarrow \cdots \rightarrow \mathcal{F}^1(\sigma_{p^{-1}(j)}) \rightarrow \mathcal{F}^2(\alpha_j) \rightarrow \cdots \\
\cdots \rightarrow \mathcal{F}^2(e_k) \rightarrow \mathcal{F}^1(\beta_{p(k)}) \rightarrow \cdots \rightarrow \mathcal{F}^1(e_\ell) \rightarrow L\Sigma_\ell.
\] (8.2)

We need to discuss the arrows marked with “1,” “2,” and “3.” The arrows marked “2” come about because \(f\) maps \(D_j\) to \(D_{p(j)}\), and so \(f\) maps \(p^1\) to \((fp)^1\). Since \(f\) is locally biholomorphic at \(p\), it follows that \(f\) is biholomorphic in a neighborhood of \(\mathcal{F}^2(p)\).

To analyze maps “1” and “3,” we give a local coordinate system on \(\mathcal{F}\). Let us describe the induced mapping \(\mathcal{F}\).

Now let us write \(\Sigma\) with an affine coordinate chart about \(P\) with a point in the fiber \(F\) \(\gamma\). If \(\gamma\) \(\in F\) \(\gamma\), then the endpoints are distinct chains, then the endpoints are distinct \(s(\gamma') \neq s(\gamma'')\), and \(\omega(\gamma') \neq \omega(\gamma'')\). The mapping on cohomology is given by:

\[
\mathcal{F}^1(e_\omega(\gamma')) \rightarrow \cdots \rightarrow \mathcal{F}^1(\alpha_{p(s(\gamma))}) \rightarrow \Sigma_\sigma(\gamma).
\] (8.4)

We see that every \(\mathcal{F}^j(p)\) is contained in a unique singular chain \(\gamma\), so (8.4) tells how \(f_X^*\) acts on each \(\mathcal{F}^j(p)\).

Now let us write \(\Sigma_0\) with respect to our basis. The only centers of blow up inside \(\Sigma_0\) are \(\{\sigma_0, e_1, \ldots, e_d\} = P \cap \Sigma_0\). Now each \(D_0\) is the line connecting \(\sigma_0\) to \(1 \notin \Sigma_0\). Thus \(D_0\) intersects \(\Sigma_0\) transversally. Thus \(D_0 \cap \mathcal{F}^1(\sigma_0)\) is disjoint from \(\Sigma_0 \cap \mathcal{F}^1(\sigma_0)\). Similarly, for
$j \neq 0$, $D_j$ is the line connecting $e_0$ to $1 \notin \Sigma_0$. Thus $D_j$ intersects $\Sigma_0$ transversally. Thus $D_j \cap F^1(e_0)$ is disjoint from $\Sigma_0 \cap F^1(\sigma_0)$. We conclude that

$$\{\Sigma_0\}_X = H_X - \sum_{j \in P-e_0} \hat{F}(e_j) - \hat{F}(\sigma_0) \in H^{1,1}(X).$$  \hspace{1cm} (8.5)

By a similar argument, we have

$$f^*H_X = d \cdot H_X + (1 - d) \sum_{e_j \in P} \hat{F}(e_j).$$  \hspace{1cm} (8.6)

It follows that $f_X^*$ is given by (8.4–6).

**Appendix: Characteristic Polynomial**

Let $L = \{L_1, \ldots, L_\mu\} = \{N_1, 1, \ldots, N_1, \ell_1\}, \{N_2, 1, \ldots, N_2, \ell_2\}, \ldots, \{N_\mu, 1, \ldots, N_\mu, \ell_\mu\}$ denote the set of lists of lengths of orbits inside orbit lists. Let us fix an orbit list $L_i$ and let $M$ denote a subset of indices $\{1, \ldots, \ell_i\}$. We define

$$|L_i|_M = \sum_{j \in M} N_{i,j}.$$

If $L_i = L_i^o$ is an open orbit list, we define

$$T_i^o(x) = x^{|L_i^o|}, \quad \text{and} \quad S_i^o(x) = \sum_M x^{|L_i^o|_M + 1}$$  \hspace{1cm} (A.1)

where the summation is taken over all $M = \{1, \ldots, \ell_i\} - I$, where $I$ is a proper sub-interval of $\{1, \ldots, \ell_i\}$. That is, $I$ is nonempty and not the whole interval. Let us consider the example of an open orbit list $L_1^o = \{7, 10, 8\}$. Then $\ell_1 = 3$, and $|L_1^o| = 7 + 10 + 8$, so $T_i^o(x) = x^{25}$. The proper sub-intervals of $\{1, \ldots, \ell_i\} = \{1, 2, 3\}$ are $I = \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}$. Thus the possibilities for $M = \{1, 2, 3\} - I$ are $\{2, 3\}, \{1, 3\}, \{1, 2\}, \{3\}, \{1\}$. This gives

$$S_i^o(x) = 1 + x^{10+8} + x^{7+8} + x^{7+10} + x^8 + x^7.$$

If $L_i = L_i^c$ is a closed orbit list, then we consider $\{1, \ldots, \ell_i\}$ to be an interval with a cyclic ordering. In this case we define

$$T_i^c(x) = x^{|L_i^c|} - 1, \quad \text{and} \quad S_i^c(x) = \sum_M x^{|L_i^c|_M + \ell_i^c}$$  \hspace{1cm} (A.2)

where the summation is taken over all $M = \{1, \ldots, \ell_i\} - I$, where $I$ is a proper cyclic sub-interval of $\{1, \ldots, \ell_i\}$. That is, $I$ is nonempty and not the whole interval.
Theorem A.1. The characteristic polynomial for the matrix representation (4.1–3) is:

\[
\chi(x) = (x - d) \prod_{i=1}^{\mu_c} T_i^c(x) \prod_{j=1}^{\mu_o} T_j^o(x) + (x - 1) \sum_{i=1}^{\mu_c} S_i^c \{ \prod_{k \neq i} T_k^c(x) \prod_{j=1}^{\mu_o} T_j^o(x) \}
\]

\[+ (x - 1) \sum_{j=1}^{\mu_o} S_j^o \{ \prod_{k \neq j} T_k^o(x) \prod_{i=1}^{\mu_c} T_i^c(x) \}. \tag{A.3}\]

For each orbit \(O_{i,j}\), there is an exceptional locus \(\Sigma_{i,j}\) and its image \(\alpha_{i,j}\). We have the corresponding subset of ordered basis with \(N_{i,j}\) elements, \(B_{i,j} = (\mathcal{F}_{N_{i,j}}^{i,j}, \mathcal{F}_{N_{i,j}+1}^{i,j}, \ldots, \mathcal{F}_1^{i,j})\) where \(\mathcal{F}_{k}^{i,j}\) is the exceptional fiber \(\pi^{-1} p\) in \(X\) over \(p = f^{N_{i,j}-k}(\alpha_{i,j})\). Let us write down the ordered basis as

\[
H_X, B_{c,1}, \ldots, B_{c,\mu_c}, B_{o,1}, \ldots, B_{o,\mu_o}, L_{\mu_c, 1}, \ldots, L_{\mu_c, \mu_o}. \]

To explain the matrix representation and computation for the characteristic polynomial, let us use \(c_{i,j}^c\) for the column corresponding to the last element of \(B_{c,i,j}\) and \(c_{i,j}^o\) for the column corresponding to the last element of \(B_{o,i,j}\). We also use \(r_{i,j}^c\) for the row corresponding to the first element of \(B_{c,i,j}\) and \(r_{i,j}^o\) for the row corresponding to the first element of \(B_{o,i,j}\). Using the ordered basis the resulting matrix is

\[
M = \begin{pmatrix}
\begin{array}{ccccccc}
d & B_1^c & \cdots & B_{\mu_c}^c & B_1^o & \cdots & B_{\mu_o}^o \\
C_i^c & D_i^c & \cdots & \cdots & \cdots & \cdots & \cdots \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
C_{\mu_c}^c & D_{\mu_c}^c & \cdots & \cdots & \cdots & \cdots & \cdots \\
C_i^o & D_i^o & \cdots & \cdots & \cdots & \cdots & \cdots \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
C_{\mu_o}^o & D_{\mu_o}^o & \cdots & \cdots & \cdots & \cdots & \cdots 
\end{array}
\end{pmatrix}
\]

where \(B_i^c\) is the \(|L_i^c| \times 1\) matrix where 1 marks the end of each subset \(B_{c,i,j}\) of the basis, and thus

\[
B_i^c = [0, \ldots, 0, 1; 0, \ldots, 0, 1; 0, \ldots, 0, 1; \ldots]'
\]

and \(C_i^c\) is the \(1 \times |L_i^c|\) matrix with 1 marking the beginning of each subset \(B_{c,i,j}\) of the basis \(B_{i,\ell_i^c}, \ldots, B_{i,1}\). Thus

\[
C_i^c = [1 - d, 0, \ldots, 0; 1 - d, 0, \ldots, 0; \ldots].
\]

\(B_i^o\) and \(C_i^o\) are constructed similarly. \(D_i^c\) is the \(|L_i^c| \times |L_i^c|\) matrix with 1 in the lower off-diagonal positions and 0 for \((c_{i,j}^c, r_{i,j+1}^c)\) for \(j = 1, \ldots, \ell_i^c - 1\) and \((c_{i,\ell_i^c}, r_{i,1}^c)\) entries. \(D_i^o\) is the \(|L_i^o| \times |L_i^o|\) matrix with -1 in \((c_{i,j}^o, r_{i,j+1}^o)\) position and 0 in the lower off-diagonal positions and 0 in \((c_{i,j}^o, r_{i,j+1}^o)\) for \(j = 1, \ldots, \ell_i^o - 1\) positions. Otherwise, \((c_{i,j}^c, r_{k,t}^c), (c_{i,j}^c, r_{k,t}^o), (c_{i,j}^o, r_{k,t}^c), (c_{i,j}^o, r_{k,t}^o)\) have -1 for every remaining combination of \(i, j, k, t\). All other points are zeros.

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The characteristic polynomial is \( \chi(x) = \pm \det(M - xI) \) with the sign chosen to make it monic. Now we perform some row operations on \( M - xI \). The top row of \( M - xI \) is \([d - x, B_1^c, \ldots, B_{\mu_c}^c, \ldots]\). We add this to every row \( r_{i,j}^c \) and \( r_{i,j}^o \). This produces the matrix

\[
\tilde{M} = \begin{pmatrix}
 d - x & B_1^c & \ldots & B_{\mu_c}^c & B_1^o & \ldots & B_{\mu_o}^o \\
 \tilde{C}_1^c & A_1^c \\
 \vdots & \ddots & \ddots \\
 \tilde{C}_{\mu_c}^c & A_{\mu_c}^c \\
 \tilde{C}_1^o & A_1^o \\
 \vdots & \ddots & \ddots \\
 \tilde{C}_{\mu_o}^o & A_{\mu_o}^o
\end{pmatrix}
\]

where \( \tilde{C}_i^c, \tilde{C}_i^o \) are the same as \( C_i^c, C_i^o \) except that the entry \( 1 - d \) is now changed to \( 1 - x \). The new diagonal blocks \( A_i^c \) have \(-x\) in the diagonal, \( 1 \) in the lower off-diagonal and \( 1 \) in the right hand column of the first row, and \( A_i^o \) have \(-x\) in the diagonal and \( 1 \) in the lower off-diagonal. That is

\[
A_i^c = \begin{pmatrix}
 -x & 1 \\
 1 & -x \\
 \ddots & \ddots \\
 1 & -x
\end{pmatrix}_{|L_i^c| \times |L_i^c|},
A_i^o = \begin{pmatrix}
 -x & 1 \\
 1 & -x \\
 \ddots & \ddots \\
 1 & -x
\end{pmatrix}_{|L_i^o| \times |L_i^o|}
\]

The main thing we have accomplished is that except for the first row and column, our matrix \((\tilde{M})\) has block diagonal form with blocks \( A_i^c \)'s, and \( A_i^o \)'s. The determinant of the each block has the following form:

**Lemma A.2.**

\[
T_i^c(x) = (-1)^{|L_i^c|} \det A_i^c = x^{|L_i^c|} - 1
\]

and

\[
T_i^o(x) = (-1)^{|L_i^o|} \det A_i^o = x^{|L_i^o|}.
\]

We will evaluate the determinant of \((\tilde{M})\) by expanding in minors, going down the left hand column. 1,1-minor is already block diagonal matrix, thus we use Lemma A.2 to take the determinant:

\[
\epsilon \prod_{i=1}^{\mu_c} T_i^c(x) \prod_{j=1}^{\mu_o} T_j^o(x)
\]

where \( \epsilon = (-1)^{\sum_{i=1}^{\mu_c} |L_i^c| + \sum_{i=1}^{\mu_o} |L_i^o|} \). Now we consider the \((r_{i,j}, 1)\)-minor which is obtained by eliminating the first column and the \( r_{i,j} \) row. For us it is more convenient to move the first row to the \( r_{i,j} \) position by interchanging two rows. The resulting matrix is almost block diagonal except the \( r_{i,j} \) row and its determinant is \((-1)\) times the determinant of the corresponding minor. Let us denote \( \hat{A}_i(j) \) for the diagonal block obtained from \( A_i \) by
replacing the row \( r_{i,j} \) by \( B_i \). For a fixed \( i \), \( r_{i,j}, 1\) minor for all \( 1 \leq j \leq \ell_i \) correspond to \( \mathcal{L}_i \). The sum of determinant of \( \hat{A}_i(j) \) for \( 1 \leq j \leq \ell_i \) is following:

**Lemma A.3.**

\[
S_c^i(x) = (-1)^{|\mathcal{L}_c^i|+1} \sum_{j=1}^{\ell_i} \det \hat{A}_i^c(j)
\]

and

\[
S_o^i(x) = (-1)^{|\mathcal{L}_o^i|+1} \sum_{j=1}^{\ell_o} \det \hat{A}_i^o(j).
\]

**Proof.** Let us consider a closed orbit list \( \mathcal{L}_c = \{N_1, \ldots, N_{\ell} \} \). We will compute the determinant of \( A_c^c(j) \) by expanding in minors, going to the right the \( r_j \) row. Notice that the 1’s in \( r_j \) row, are not on the diagonal. The minor corresponding the \( k \)-th 1 on the left hand side of the diagonal, i.e. \( k < j \), is \( x^{N_k+\cdots+N_{j-1}} \). The minor corresponding the \( k \)-th 1 on the left hand side of the diagonal, i.e. \( i \geq j \), is \( x^{N_1+\cdots+N_k-1+N_{j+1}+\cdots+N_{\ell}} \). Summing all these monomials for all \( k \)’s, we have the determinant of \( A_c^c(j) \). For the open orbit list, all the minor corresponding the \( k \)-th 1 on the left hand side of the diagonal, i.e. \( k < j \), are zero since the corresponding matrices are lower triangular matrix with zeros on the diagonal.

For the \( (r_{i,j}^c, 1) \)-minor, 1’s in the off-diagonal block would produce zero block on the diagonal. For the 1’s on the diagonal block will keep other diagonal blocks as diagonal blocks. Thus by Lemma A.2 and Lemma A.3, sum of the determinant of the \( (r_{i,j}^c, 1) \)-minor for \( j = 1, \ldots, \ell_i \) is given by

\[
\epsilon S_c^i(x) \prod_{k \neq i} T_k^c(x) \prod_{j=1}^{\mu_o} T_j^o(x)
\]

where \( \epsilon = (-1)^{\sum_{i=1}^{\mu_c} |\mathcal{L}_c^i|+\sum_{i=1}^{\mu_o} |\mathcal{L}_o^i|} \).

**Proof of Theorem A.1.** Combining the previous arguments completes the proof.

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