On the Structure of the Square of a $C_0(1)$ Operator*

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Dedicated to I.B. Simonenko on his seventieth birthday

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While the model theory for contraction operators (cf. [4]) is always a useful tool, it is particularly powerful when dealing with $C_0(1)$ operators. Recall that an operator $T$ on a Hilbert space $H$ is a $C_0(N)$-operator ($N = 1, 2, \ldots$) if $\|T\| \leq 1$, $T^n \to 0$ and, $T^n \to 0$ (strongly) when $n \to \infty$ and rank$(1 - T^*T) = N$. In particular, a $C_0(1)$ operator is unitarily equivalent to the compression of the unilateral shift operator $S$ on the Hardy space $H^2$ to a subspace $H^2 \ominus mH^2$ for some inner function $m$ in $H^\infty$.

In this note we use the structure theory to determine when the lattices of invariant and hyperinvariant subspaces differ for the square $T^2$ of a $C_0(1)$ operator and the relationship of that to the reducibility of $T^2$. To accomplish this task we first determine very explicitly the characteristic operator function for $T^2$ and use the representation obtained to determine when the operator is irreducible. While every operator $T$ in $C_0(1)$ is irreducible, it does not follow that $T^2$ is necessarily irreducible, that is, has no reducing subspaces. In particular, we characterize those $T$ in $C_0(1)$ for which $T^2$ is irreducible but for which the lattices of invariant and hyperinvariant subspaces for $T^2$ are distinct.

Finally, we provide an example of an operator $X$ on a four dimensional Hilbert space for which the two lattices are distinct but $X$ is irreducible, and show that such an example is not possible on a three dimensional space.

This work was prompted by a question to the first author from Ken Dykema (Sect. 2, [2]) concerning hyperinvariant subspaces in von Neumann algebras. He asked whether the lattices of invariant and hyperinvariant subspaces for an irreducible matrix must coincide. He provides an

*2000 AMS Classification: 47A15, 47A45.

Keywords: $C_0$ operators, invariant subspace lattice.
example in [2] on a six-dimensional Hilbert space showing that this is not the case.

We assume that the reader is familiar with the concepts and notation in [1] and [4].

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Let $T \in C_0(1)$ on $H$, dim $H \geq 2$. WLOG we can assume

\[(1.1) \quad T = P_H S|_H, \text{ where } H = H^2 \oplus mH^2, (S h) z = z h(z)(z \in D, h \in H^2), m \in H^\infty, m \text{ inner.}\]

Define

\[(1.2) \quad \Theta(\lambda) = \frac{1}{2} \begin{bmatrix} b(\lambda) & \lambda d(\lambda) \\ d(\lambda) & b(\lambda) \end{bmatrix} \quad (\lambda \in D), \]

where

\[(1.3a) \quad b(\lambda) = m(\sqrt{\lambda}) + m(-\sqrt{\lambda}) \quad (\lambda \in \mathbb{D}) \quad \text{and} \quad d(\lambda) = \frac{m(\sqrt{\lambda}) - m(-\sqrt{\lambda})}{\sqrt{\lambda}} \quad (0 \neq \lambda \in \mathbb{D}) \]

\[(1.3b) \quad d(0) = 2m'(0). \]

Lemma 1. The matrix function $\Theta(\cdot)$ is inner, pure and (up to a coincidence) the characteristic operator function of $T^2$.

Proof. For $h \in H^2$ write

\[(1.4a) \quad h(\lambda) = h_0(\lambda^2) + \lambda h_1(\lambda^2) \quad (\lambda \in \mathbb{D}). \]

Clearly $h_0(\cdot), h_1(\cdot) (= h_0(\lambda), h_1(\lambda), \lambda \in \mathbb{D})$ belong to $H^2$. Define $W: H^2 \mapsto H^2 \oplus H^2 (= H^2(\mathbb{C}^2))$ by

\[(1.4b) \quad Wh = h_0 \oplus h_1, \text{ where } h \text{ is given by } (1.4a). \]

Then $W$ is unitary and

\[(1.5) \quad WS^2 = (S \oplus S)W. \]

Consequently,

\[(1.6) \quad WT^2 = WP_H S^2 = P_W H WS^2 = P_W (S \oplus S)W; \]
Moreover, since $S^2mH^2 \subset mH^2$ we also have

$$(S \oplus S)WmH^2 = WS^2mH^2 \subset WmH^2$$

and therefore

$$P_{WH}(S \oplus S) = P_{WH}(S \oplus SP_{WH}) = WP_HW^*(S \oplus S)P_{WH} =$$

$$= WP_HS^2W^*P_{WH} = WT^2P_HW^* =$$

$$= |H|T^2(W|H|^*).$$

These relationships show that $S \oplus S$ is an isometric lifting of $T_0 = P_{WH}(S \oplus S)|WH$ and that this operator is unitarily equivalent to $T^2$. Moreover, since

$$\int_0^\infty (S \oplus S)^nWH = H^2 \oplus H^2$$

is obvious, $S \oplus S$ is the minimal isometric lifting of $T = |H|T^2(W|H|^*)$.

Further,

$$WmH^2 = \{W(m_0(\lambda^2) + \lambda m_1(\lambda^2))(h_0(\lambda^2) + \lambda h_1(\lambda^2)): \ h \in H^2\}$$

$$= \{W[(m_0 h_0)(\lambda^2) + \lambda^2(m_1 h_1)(\lambda^2) + \lambda(m_0 h_1 + m_1 h_0)(\lambda^2)): \ h \in H^2\} =$$

$$= \{(m_0 h_0)(\lambda) + \lambda(m_1 h_1)(\lambda) \oplus (m_0 h_1 + m_1 h_0)(\lambda): \ h \in H^2\} =$$

$$= \left\{ \begin{bmatrix} m_0 & \lambda m_1 \\ m_1 & m_0 \end{bmatrix} (h_0 \oplus h_1): \ h \in H^2 \right\} = \begin{bmatrix} m_0 & \lambda m_1 \\ m_1 & m_0 \end{bmatrix} H^2 \oplus H^2.$$ 

Note that the above computations also prove that

$$(1.8) \quad (Wm(S)W^*)(h_0 \oplus h_1) = \begin{bmatrix} m_0 & \lambda m_1 \\ m_1 & m_0 \end{bmatrix} h_0 \oplus h_1 \quad (h_0 \oplus h_1 \in H^2 \oplus H^2).$$

Since $m(S)$ is isometric, so is $Wm(S)W^*$, that is,

$$(1.9) \quad M(\lambda) \equiv \begin{bmatrix} m_0(\lambda) & \lambda m_1(\lambda) \\ m_1(\lambda) & m_0(\lambda) \end{bmatrix} \text{ is inner.}$$

Consequently, $T_0$ is the compression of $S \oplus S$ to

$$(1.10) \quad WH = (H^2 \oplus H^2) \ominus M(H^2 \oplus H^2).$$

Moreover, it is clear that

$$m_0(\lambda) = \frac{1}{2} b(\lambda), \quad m_1(\lambda) = \frac{1}{2} d(\lambda) \quad (\lambda \in \mathbb{D})$$

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so that the matrix \( M(\cdot) \) defined by (1.9) is identical to the matrix \( \Theta(\cdot) \) defined by (1.2).

Note that
\[
\Theta(0) = \begin{bmatrix} m(0) & 0 \\ m'(0) & m(0) \end{bmatrix}
\]
and
\[
\Theta(0)^*\Theta(0) = \begin{bmatrix} |m(0)|^2 + |m'(0)|^2 & \overline{m'(0)}m(0) \\ \overline{m(0)}m'(0) & |m(0)|^2 \end{bmatrix}.
\]
If \( \Theta(0) \) were not pure, then \( \Theta(0)^*\Theta(0) \) would have the eigenvalue 1 and therefore the other eigenvalue must be \( |m(0)|^4 \). Taking traces we have
\[
2|m(0)|^2 + |m'(0)|^2 = 1 + |m(0)|^4.
\]
This implies that the modulus of the analytic function \( \tilde{m}(\lambda) \) defined by
\[
\lambda\tilde{m}(\lambda) = \frac{m(\lambda) - m(0)}{1 - \overline{m(0)}m(\lambda)} \quad (\lambda \in \mathbb{D}, \lambda \neq 0)
\]
and
\[
\tilde{m}(0) = \frac{m'(0)}{1 - |m(0)|^2}
\]
attains its maximum (= 1) at \( \lambda = 0 \). By virtue of the maximum principle, \( \tilde{m}(\lambda) = c = \text{constant, } |c| = 1 \). Thus
\[
m(\lambda) \equiv c \left( \frac{\lambda + \tilde{m}(0)}{1 + \lambda\tilde{m}(0)} \right) \quad (\lambda \in \mathbb{D})
\]
and
\[
2 \leq \dim H = \dim(H^2 \ominus mH^2) = 1,
\]
which is a contradiction.

We conclude that \( \Theta(\cdot) \) is pure and, by virtue of (1.10) (recall \( \Theta(\lambda) \equiv M(\lambda) \)), that \( \Theta(\cdot) \) is the characteristic operator function of \( T_0 \) and hence (up to a coincidence) also the characteristic operator function of \( T^2 \). This concludes the proof of the lemma. \( \square \)

Note that the preceding result also shows that \( T^2 \) is a \( C_0(2) \) operator.

Our next step is to characterize in terms of \( \Theta(\lambda) \) the reducibility of \( T^2 \).
Lemma 2. The operator $T^2$ is reducible if and only if there exist $Q_i = Q_i^* = Q_i^2$, $Q_i \in \mathcal{L}(\mathbb{C}^2)$ ($i = 1, 2$) so that

\begin{equation}
\Theta(\lambda)Q_2 = Q_1\Theta(\lambda) \quad (\lambda \in \mathbb{D})
\end{equation}

and $0 \neq Q_i \neq I_{\mathbb{C}^2}$ ($i = 1, 2$).

Proof. If $Q_1, Q_2$ as above exist, then (since rank $Q_1 = 1 = \text{rank } Q_2$) there exist unitary operators in $\mathcal{L}(\mathbb{C}^2)$ so that

\begin{equation}
W_1\Theta(\lambda)W_2 = \begin{bmatrix}
\theta_1(\lambda) & 0 \\
0 & \theta_2(\lambda)
\end{bmatrix} \quad (\lambda \in \mathbb{D})
\end{equation}

for functions $\theta_1(\cdot), \theta_2(\cdot)$.

Indeed, if $W_1$ and $W_2$ are unitary operators in $\mathcal{L}(\mathbb{C}^2)$ such that

\begin{equation}
Q_1\mathbb{C}^2 = W_1^*(\mathbb{C} \oplus \{0\}), \quad Q_2\mathbb{C}^2 = W_2(\mathbb{C} \oplus \{0\}),
\end{equation}

then

\begin{align*}
W_1\Theta(\lambda)W_2 & = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \Theta(\lambda)W_2 \\
& = W_1 \left\{ \Theta(\lambda)W_2 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \Theta(\lambda) \right\} W_2 \\
& = W_1(\Theta(\lambda)Q_2 - Q_1\Theta(\lambda)) = 0.
\end{align*}

Thus $\mathbb{C} \oplus \{0\}$ (and hence also $\{0\} \oplus \mathbb{C}$) reduces $W_1\Theta(\lambda)W_2$ and consequently this operator has the form (2.2).

Clearly the $\theta_1, \theta_2$ in (2.2) are inner (and non-constant). Let

\begin{equation}
T_i = P_{H_i}S|_{H_i}, \quad \text{where } H_i = H^2 \ominus \theta_iH^2 \quad (i = 1, 2).
\end{equation}

Then the characteristic operator function of $T_1 \oplus T_2$ is the right hand side of (2.2) which coincides with $\Theta(\lambda)$. Thus $T^2$ and $T_1 \oplus T_2$ are unitarily equivalent.

Conversely, if $T^2$ is reducible then $T^2$ is unitarily equivalent to the direct sum $T_1' \oplus T_2'$, where $T_i' = T^2|_{H_i}$ ($i = 1, 2$), $H_1, H_2$ are reducing subspaces for $T^2$, and $H = H_1 \oplus H_2$. Clearly each $T_i' \in \mathcal{C}_{00}$ and since the defect indices of the $T_i'$s sum up to 2, it follows that each $T_i' \in \mathcal{C}_{0}(1)$. Thus the characteristic operator function of $T_1' \oplus T_2'$ coincides with

\begin{equation}
\begin{bmatrix}
\theta_1(\lambda) & 0 \\
0 & \theta_2(\lambda)
\end{bmatrix},
\end{equation}
where $\theta_i$ is the characteristic function of $T_i' \ (i = 1, 2)$. Again $\Theta(\lambda)$ is connected to (2.4) by a relation of the form (2.2), that is,

$$\Theta(\lambda) \equiv W_1^* \begin{bmatrix} \theta_1(\lambda) & 0 \\ 0 & \theta_2(\lambda) \end{bmatrix} W_2^*,$$

where $W_1, W_2$ are again unitary. Then

$$Q_1 = W_1^* \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} W_1, \quad Q_2 = W_2 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} W_2^*$$

satisfy (2.1).

**Remark.** Note that in (2.1), the orthogonal projections $Q_1, Q_2$ are of rank one. Such a projection $Q$ is of the form

$$Q = f \otimes f = \begin{bmatrix} |f_1|^2 & f_1 \bar{f}_2 \\ f_2 \bar{f}_1 & |f_2|^2 \end{bmatrix},$$

where

$$f = f_1 \oplus f_2 \in \mathbb{C}^2, \quad \|f\| = 1.$$

Thus

$$Q = \begin{bmatrix} q & r \bar{\theta} \\ r \theta & 1 - q \end{bmatrix}, \text{ where } 0 \leq q \leq 1, |\theta| = 1, r = (q(1 - q))^{1/2}.$$

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In this paragraph we study the relation (2.1) using the representation (2.6) for $Q = Q_i \ (i = 1, 2)$ and the form (1.2) of $\Theta(\lambda)$. Thus we have

$$\begin{bmatrix} b(\lambda) & \lambda d(\lambda) \\ d(\lambda) & b(\lambda) \end{bmatrix} \begin{bmatrix} q_2 & r_2 \bar{\theta}_2 \\ r_2 \theta_2 & 1 - q_2 \end{bmatrix} = \begin{bmatrix} q_1 & r_1 \bar{\theta}_1 \\ r_1 \theta_1 & 1 - q_1 \end{bmatrix} \begin{bmatrix} b(\lambda) & \lambda d(\lambda) \\ d(\lambda) & b(\lambda) \end{bmatrix},$$

where

$$0 \leq q_1, q_2 \leq 1, |\theta_1| = |\theta_2| = 1, r_i = (q_i(1 - q_i))^{1/2} \ (i = 1, 2).$$

We begin by noting that

$$|b(\lambda)|^2 + |d(\lambda)|^2 \neq 0 \quad (\lambda \in \mathbb{D}),$$

since otherwise we would have $m(\lambda) \equiv 0$. In discussing (3.1) we will consider several cases:
**Case I.** If \( b(\lambda) \equiv 0 \ (\lambda \in \mathbb{D}) \), then \((3.1)\) becomes:

\[
\begin{bmatrix}
\lambda d(\lambda) r_2 \theta_2 & \lambda d(\lambda)(1 - q_2) \\
d(\lambda) q_2 & d(\lambda) r_2 \bar{\theta}_2
\end{bmatrix} = \begin{bmatrix}
r_1 \bar{\theta}_1 d(\lambda) & \lambda q_1 d(\lambda) \\
(1 - q_1) d(\lambda) & r_1 \theta_1 d(\lambda)
\end{bmatrix}
\]

which is possible if and only if \( r_1 = 0 = r_2 \) and \( q_1 = 1 - q_2 \). In this case \( T^2 \) is reducible.

**Case II.** If \( d(\lambda) \equiv 0 \ (\lambda \in \mathbb{D}) \), then

\[
Q_2 = Q_1 = \text{any } Q = Q^* = Q^2 \text{ with rank } Q = 1
\]

and again \( T^2 \) is reducible.

**Case III.** If \( b(\lambda) \neq 0, d(\lambda) \neq 0 \ (\lambda \in \mathbb{D}) \), then \((3.1)\) is equivalent to the equations

\[
\begin{align*}
b(q_2 - q_1) &= d(r_1 \bar{\theta}_1 - \lambda r_2 \theta_2), & b(r_2 \bar{\theta}_2 - r_1 \bar{\theta}_1) &= \lambda d(q_1 + q_2 - 1) \\
d(q_1 + q_2 - 1) &= b(r_1 \theta_1 - r_2 \theta_2), & b(q_2 - q_1) &= d(r_2 \bar{\theta}_2 - \lambda r_1 \theta_1),
\end{align*}
\]

which in turn are equivalent to

\[
\begin{cases}
r_1 \theta_1 = r_2 \theta_2, & q_2 + q_1 = 1 \\
(3.4) & b(\lambda)(1 - 2q_1) \equiv d(\lambda)(\bar{\theta}_1 - \lambda \theta_1)r_1 \ (\lambda \in \mathbb{D}).
\end{cases}
\]

In \((3.4)\), \( q_1 = 1/2 \), if and only if \( r_1 = 0 \), i.e. \( q_1 = 0 \) or \( 1 \), a contradiction. Thus we can divide by \( 1 - 2q_1 \) and \((3.4)\) implies (with \( \theta = \theta_1 \))

\[
\begin{cases}
\begin{align*}
b(\lambda) &\equiv d(\lambda)(\bar{\theta} - \lambda \theta)\rho \quad (\lambda \in \mathbb{D}) \\
&\text{for some } \rho \in \mathbb{R}, \rho \neq 0.
\end{align*}
\end{cases} \tag{3.5}
\]

Conversely, if \((3.5)\) holds, then setting

\[
q_1 = \frac{1}{2} \pm \frac{1}{2} \frac{1}{(4\rho^2 + 1)^{1/2}} \quad \text{(according to whether } \rho \lesssim 0),
\]

and \( q_2 = 1 - q_1, \theta_2 = \theta_1 = \theta \), we obtain \((3.4)\).

We now summarize our discussion in terms of \( m(\cdot) \) (see \((1.3a), (1.3b)\)), instead of \( b(\cdot) \) and \( d(\cdot) \), obtaining the following:

**Lemma 3.** The operator \( T^2 \) is reducible if and only if one of the following conditions holds:

\[
\begin{align*}
(3.6) & \quad m(-\lambda) \equiv -m(\lambda) \quad (\forall \lambda \in \mathbb{D}) \quad \text{(Case I above)}; \\
(3.7) & \quad m(-\lambda) \equiv m(\lambda) \quad (\forall \lambda \in \mathbb{D}) \quad \text{(Case II above)};
\end{align*}
\]

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or there exist \( \rho \in \mathbb{R}, \rho \neq 0 \) and \( \theta \in \mathbb{C}, |\theta| = 1 \), such that the function

\[
(3.8a) \quad n(\lambda) \equiv m(\lambda)(\rho \theta \lambda^2 + \lambda - \rho \bar{\theta}) \quad (\lambda \in \mathbb{D})
\]

satisfies

\[
(3.8b) \quad n(\lambda) \equiv n(-\lambda) \quad (\lambda \in \mathbb{D}) \quad (\text{Case III above}).
\]

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We shall now give a more transparent form to conditions (3.8a), (3.8b) above. To this end note that

\[
\rho \theta \lambda^2 + \lambda - \rho \bar{\theta} \equiv \rho \theta (\lambda - \delta_+ \bar{\theta})(\lambda - \delta_- \bar{\theta}),
\]

where

\[
(4.1) \quad \delta_\pm = \frac{-1 \pm \sqrt{4\rho^2 + 1}}{2\rho}.
\]

Thus (with \( \mu = \bar{\theta} \delta_\pm \)), we have

\[
(4.2) \quad \rho \theta \lambda^2 + \lambda - \rho \bar{\theta} = -\rho \delta_- (\lambda - \mu)(1 + \bar{\mu} \lambda).
\]

Using this representation in (3.8a), condition (3.8b) becomes

\[
m(\lambda)(\lambda - \mu)(1 + \bar{\mu} \lambda) \equiv m(-\lambda)(-\lambda - \mu)(1 - \bar{\mu} \lambda) \quad (\lambda \in \mathbb{D}),
\]

which can be written (since \( 0 < |\mu| < 1 \)) as

\[
(4.3) \quad m(\lambda) \frac{\lambda - \mu}{1 - \bar{\mu} \lambda} \equiv m(-\lambda) \frac{(-\lambda) - \mu}{1 - \bar{\mu}(-\lambda)} \quad (\lambda \in \mathbb{D}).
\]

Thus \( m(-\mu) = 0 \) and therefore

\[
(4.4) \quad m(\lambda) = p(\lambda) \frac{\lambda + \mu}{1 + \bar{\mu} \lambda} \quad (\lambda \in \mathbb{D}),
\]

where \( p(\cdot) \in H^\infty \) is an (other) inner function. Obviously (4.3) is equivalent to

\[
(4.5) \quad p(\lambda) \equiv p(-\lambda) \quad (\lambda \in \mathbb{D}).
\]

This discussion together with Lemma 3 readily yields the following
Theorem 1. The operator $T^2$ is reducible iff either

(4.6) $m(\lambda) = m(-\lambda)$ \hspace{1em} (\lambda \in \mathbb{D})

or there exists a $\mu \in \mathbb{D}$ such that

(4.7) $m(\lambda) \equiv p(\lambda) \frac{\lambda + \mu}{1 + \bar{\mu}\lambda}$ \hspace{1em} (\lambda \in \mathbb{D}),

where $p(\cdot) \in H^\infty$ satisfies

(4.8) $p(\lambda) \equiv p(-\lambda)$ \hspace{1em} (\lambda \in \mathbb{D}).

Remark. Case (3.6) is contained in the second alternative above when $\mu = 0.$

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In order to study the lattices $\text{Lat}\{T^2\}$ and $\text{Lat}\{T^2\}'$ we first bring together the following characterization of the $C_0(N)$ operators that are multiplicity free.

Proposition 1. Let $\tilde{T}$ be a $C_0(N)$ operator. Then the following statements are equivalent.

1. $\tilde{T}$ is multiplicity free (that is, $\tilde{T}$ has a cyclic vector).
2. $\text{Lat}\{\tilde{T}\} = \text{Lat}\{\tilde{T}\}'.
3. The minors of the characteristic matrix function of order $N - 1$ have no common inner divisor.

Proof. The equivalence of (1) and (3) is contained in the equivalence of (i) and (ii) in Theorem 2 in [3]. The implication (1) implies (2) is an easy corollary of the implication (i) implies (vi) of the same theorem and is contained in Corollary 2.14 in Chapter 3 of [1]. Finally, implication (3) implies (1) proceeds from the following lemma.

Lemma 4. Let $T$ be an $C_0$ operator on the Hilbert space $\mathcal{H}$ and $f$ a maximal vector for $T.$ Then $f$ is cyclic for $\{T\}'.$

Proof. Let $\mathcal{M}$ be the cyclic subspace for $\{T\}'$ generated by $f$ and write $T \sim (T^* X \ T')$ for the decomposition $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp.$ Since $\mathcal{M}$ is hyperinvariant for $T$, it follows from Corollary 2.15 in Chapter 4 of [1], that the minimal functions satisfy $m_T = m_{T'} \cdot m_{T''}.$ However, $f$ maximal for $T$ implies that $m_{T'} = m_T$ and hence $m_{T''} = 1.$ Therefore, $\mathcal{M}^\perp = (0)$ or $\mathcal{M} = \mathcal{H}$ which completes the proof.
Our next aim is to characterize the case when the operator $T^2$ is multiplicity free. According to Proposition 1, that happens if and only if

$$b(\lambda), d(\lambda) \text{ and } \lambda d(\lambda)$$

have no common nontrivial inner divisor. Let $q(\lambda)$ be an inner divisor of $b(\lambda)$ and $d(\lambda)$, that is,

(6.1a) \hspace{1cm} m(\sqrt{\lambda}) + m(-\sqrt{\lambda}) \equiv q(\lambda)r(\lambda) \quad (\lambda \in \mathbb{D})

(6.1b) \hspace{1cm} m(\sqrt{\lambda}) - m(-\sqrt{\lambda}) \equiv q(\lambda)\lambda s(\lambda)

for some $r, s \in H^\infty$. It follows that

(6.2) \hspace{1cm} m(\lambda) \equiv q(\lambda^2)(r(\lambda^2) - \lambda s(\lambda^2)),$$

that is, $m(\lambda)$ has an even inner divisor.

Conversely, if $m(\cdot)$ has an inner divisor (in $H^\infty$) $p(\cdot)$ satisfying

(6.3) \hspace{1cm} p(\lambda) \equiv p(-\lambda),

then $q(\lambda) = p(\sqrt{\lambda}) = p(-\sqrt{\lambda})$ is in $H^\infty$ and inner. Thus $m(\lambda)$ can be represented as in (6.2) and clearly (6.2) implies (6.1a), (6.1b). Thus we obtained the following:

**Theorem 2.** The operator $T^2$ is multiplicity free iff the characteristic function $m(\lambda)$ for $T$ has no nontrivial inner divisor $p(\lambda)$ in $H^\infty$ such that (see (6.3))

$$p(\lambda) \equiv p(-\lambda) \quad (\forall \lambda \in \mathbb{D}).$$

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Our main result is now a direct consequence of Theorems 1 and 2 and Proposition 1, namely

**Theorem 3.** Let $T \in C_0(1)$ satisfy:

(A) \hspace{1cm} m_T(\lambda) \neq m_T(-\lambda)

(B) \hspace{1cm} For $m_T(\lambda_0) = 0$, $\lambda_0 \in \mathbb{D}$, the function

$$m_{T,\lambda_0}(\lambda) = m_T(\lambda) / \frac{\lambda - \lambda_0}{1 - \lambda_0 \lambda} \quad (\lambda \in \mathbb{D})$$
is not even, that is,
\[ m_{T,\lambda_0}(\lambda) \not\equiv m_{T,\lambda_0}(-\lambda). \]

(C) There exists a nontrivial inner divisor \( p(\lambda) \) (in \( H^\infty \)) of \( m_T(\lambda) \) such that
\[ p(\lambda) \equiv p(-\lambda). \]

Then

(D) \( T^2 \) is irreducible, and

(E) \( \text{Lat} \ T^2 \not= \text{Lat}\{T^2\} \).

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Remarks

a) Let

\[ m_T(\lambda) = \frac{\lambda - \lambda_1}{1 - \lambda \lambda_2} \frac{\lambda - \lambda_2}{1 - \lambda_2 \lambda} \frac{\lambda - \lambda_3}{1 - \lambda_3 \lambda} \quad (\lambda \in \mathbb{D}), \]

where \( \lambda_1, \lambda_2, \lambda_3 \in \mathbb{D}, \lambda_2^2 \neq \lambda_1 \). Then \( m \) fulfills the solutions (A), (B), (C) in Theorem 3, \( T^2 \) satisfies (D) and (E) above and hence \( \text{dim} H = 4 \).

b) If \( \text{dim} H = 3 \) then

\[ m_T(\lambda) = \frac{\lambda - \lambda_1}{1 - \lambda \lambda_2} \frac{\lambda - \lambda_2}{1 - \lambda_2 \lambda} \frac{\lambda - \lambda_3}{1 - \lambda_3 \lambda} \quad (\lambda \in \mathbb{D}) \]

with some \( \lambda_1, \lambda_2, \lambda_3 \in \mathbb{D} \). If \( m_T \) satisfies (C) then \( (\lambda_1 + \lambda_2)(\lambda_2 + \lambda_3)(\lambda_3 + \lambda_1) = 0 \) and \( m_T(\lambda) \) has the form (upon relabelling the \( \lambda_i \)'s)

\[ m_T(\lambda) = \frac{\lambda^2 - \lambda_1^2}{1 - \lambda^2 \lambda_2^2} \frac{\lambda - \lambda_2}{1 - \lambda_2 \lambda} \quad (\lambda \in \mathbb{D}). \]

Consequently \( m_T \) does not satisfy (B). Thus for Theorem 3 to hold it is necessary that \( \text{dim} H \geq 4 \).

3) Let \( m_T \) be singular, that is,

\[ m_T(\lambda) = \exp \left[ -\frac{1}{2\pi} \int_0^\pi \frac{e^{it} + \lambda}{e^{it} - \lambda} d\mu(e^{it}) \right] \]

with \( \mu \) a singular measure on \( \partial \mathbb{D} = \{e^{it}: 0 \leq t < 2\pi\} \). Assume that there exists a Borel set \( \Omega \subset \partial \mathbb{D} \) so that

\[ \mu(\Omega) = \mu(\partial \mathbb{D}), \quad \mu(\{\lambda: \lambda \in \Omega\}) = 0. \]
(e.g. $\mu = \delta_1$, the point mass at 1). Then

\begin{equation}
\text{Lat}\{T^2\} = \text{Lat}\{T^2\}' = \text{Lat}\{T\}.
\end{equation}

Indeed, in this case (C) above does not hold.

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