ON SOME NONLINEAR OPERATORS, FIXED-POINT THEOREMS AND NONLINEAR EQUATIONS

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Abstract. In this article we discuss the solvability of some class of fully nonlinear equations, and equations with p-Laplacian in more general conditions by using a new approach given in [1] for studying the nonlinear continuous operator. Moreover we reduce certain general results for the continuous operators acting on Banach spaces, and investigate their image. Here we also consider the existence of a fixed-point of the continuous operators under various conditions.

1. Introduction

In the present paper we consider the boundary-value problem for the fully nonlinear equation of the second order

\[(1.1) \quad F(x, u, Du, \Delta u) = h(x), \quad x \in \Omega,\]

and also for the nonlinear equations with p-Laplacian that depend upon the parameters \(\lambda\) and \(\mu\)

\[(1.2) \quad - \nabla \left( |\nabla u|^{p-2} \nabla u \right) + G(x, u, Du, \lambda, \mu) = h(x), \quad x \in \Omega,\]

on the smooth bounded domain \(\Omega \subset \mathbb{R}^n (n \geq 1)\), where \(F(x, \xi, \eta, \zeta)\) and \(G(x, \xi, \eta, \lambda, \mu)\) are Caratheodory functions. We deal with the properties of the nonlinear operators generated by the posed problems and study the solvability of these problems by using the general results of such type as in [1]. It should be noted that equations of such type arise in the diffusion processes, reaction-diffusion processes etc., in the steady-state case (see, for example [2 - 13] and their references). Furthermore we discuss some of the nonlinear continuous mappings acting on Banach spaces and an equation (inclusion) with mappings of such type.

The problems of such type were studied earlier under various conditions in the semilinear ([2, 3, 7 - 11, 13] etc.) and in the fully nonlinear cases ([4, 12, 14] etc.). In the mentioned articles, the known general results having such conditions that cannot be applicable to the problems considered here, were used, whereas in this article we want to investigate these problems under more general conditions. Therefore we need to use a general result that will be applicable to the considered problems here and consequently, the conditions of the general result differ from the conditions of the known results. The results of [1] and the general results adduced here allow us to study the imposed problem in more general conditions.

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So for our goal we lead to a fixed-point theorem for nonlinear continuous mappings acting on Banach spaces and also a solvability theorem for nonlinear equations involving continuous operators. These general results allow us to study various nonlinear mappings and also nonlinear problems under more general conditions. Therefore we investigate boundary value problems (BVP) for the nonlinear differential equations by using the mentioned general results. Moreover we obtain the existence of the fixed point for the operator associated with imposed problem. We also note that Theorem 1 of present article is a result of the type of Lax-Milgram theorem.

This article is constructed in the following way. In section 2, we lead a general theorem that shows how one can locally determine the image of a subset of the domain of definition under the continuous mapping acting in Banach space (cf. [1, 8, 11 - 14] etc.). From this theorem we deduce the solvability theorem and an existence of the fixed point of the mapping associated with imposed problem, and consequently, an existence of the fixed point of the mapping associated with the problem, considered in the next section. In section 3, we prove a solvability theorem for an equation with the perturbed operator on Banach spaces, and then in sections 4 and 5 by using these results we study the solvability of the boundary value problem for various classes of nonlinear differential equations.

2. Some General Results on Solvability

Let $X,Y$ be reflexive Banach spaces and $X^*, Y^*$ be their dual spaces, moreover let $Y$ be reflexive with strictly convex norm together with $Y^*$ (this condition is not complementary condition; see, for example, [15]) and $f : D(f) \subseteq X \rightarrow Y$ be an operator.

So, we will conduct here the special case of the main result of [1]. Consider the following conditions. Let the closed ball $B^X_{r_0}(0)$ of $X$ be contained in $D(f)$, i.e. $B^X_{r_0}(0) \subseteq D(f) \subseteq X$ and on $B^X_{r_0}(0)$ are fulfilled the conditions:

(i) $f : B^X_{r_0}(0) \subseteq D(f) \subseteq X \rightarrow Y$ be a continuous operator that bounded on $B^X_{r_0}(0)$, i.e.

$$\|f(x)\|_Y \leq \mu (\|x\|_X), \quad \forall x \in B^X_{r_0}(0);$$

(ii) there is a mapping $g : D(g) \subseteq X \rightarrow Y^*$ such that $D(f) \subseteq D(g)$, and for any $S^X_r(0) \subseteq B^X_{r_0}(0), 0 < r \leq r_0$, cl $g(S^X_r(0)) = g(S^X_{r_0}(0)) \equiv S^{Y^*}_r(0), S^X_r(0) \subseteq g^{-1}(S^{Y^*}_{r_0}(0))$

$$\langle f(x), g(x) \rangle \geq \nu (\|x\|_X) \|x\|_X, \quad \text{a.e. } x \in B^X_{r_0}(0) \quad \& \quad \nu (r_0) \geq \delta_0 > 0$$

hold\footnote{In particular, the mapping $g$ can be a linear bounded operator as $g \equiv L : X \rightarrow Y^*$ that satisfies the conditions of (ii).}, where $\mu : R^1_+ \rightarrow R^1_+$ and $\nu : R^1_+ \rightarrow R^1$ are continuous functions ( $\mu, \nu \in C^0$), moreover $\nu$ is the nondecreasing function for $\tau \geq \tau_0, r_0 \geq \tau_0 \geq 0$; $\tau_0, \delta_0 > 0$ are constants;

(iii) almost each $\bar{x} \in intB^X_{r_0}(0)$ possesses a neighborhood $V_\varepsilon(\bar{x}), \varepsilon \geq \varepsilon_0 > 0$ such that the following inequality

$$\|f(x_2) - f(x_1)\|_Y \geq \Phi (\|x_2 - x_1\|_X, \bar{x}, \varepsilon),$$

\footnote{In particular, the mapping $g$ can be a linear bounded operator as $g \equiv L : X \rightarrow Y^*$ that satisfies the conditions of (ii).}
holds for any $\forall x_1, x_2 \in V_c(\bar{x}) \cap B^X_{r_0}(0)$, where $\Phi(\tau, \bar{x}, \varepsilon) \geq 0$ is a continuous function at $\tau$ and $\Phi(\tau, \bar{x}, \varepsilon) = 0 \iff \tau = 0$ (in particular, $\bar{x} = 0$, $\varepsilon = \varepsilon_0 = r_0$ and $V_c(\bar{x}) = V_{r_0}(0) \equiv B^X_{r_0}(0)$, consequently $\Phi(\tau, 0, r_0) = \Phi(\tau, \bar{x}, \varepsilon)$).

**Theorem 1.** Let $X, Y$ be Banach spaces such as above and $f : D(f) \subseteq X \rightarrow Y$ be an operator. Assume that on the closed ball $B^X_{r_0}(0) \subseteq D(f) \subseteq X$ the conditions (i) and (ii) are fulfilled then the image $f(B^X_{r_0}(0))$ contains an everywhere dense subset of $M$ that has the form

$$M \equiv \{ y \in Y \mid \langle y, g(x) \rangle \leq \langle f(x), g(x) \rangle, \forall x \in S^X_{r_0}(0) \}.$$ 

Furthermore if in addition the image $f(B^X_{r_0}(0))$ of the ball $B^X_{r_0}(0)$ is closed or the condition (iii) is fulfilled then the image $f(B^X_{r_0}(0))$ is a bodily subset of $Y$, moreover $f(B^X_{r_0}(0))$ contains the above bodily subset $M$.

The proof of this theorem is obtained from general result that was proven in [14] (see also, [1])

**Remark 1.** It is easy to see that the condition $B^X_{r_0}(0) \subseteq D(f)$ is not essential, because if $D(f)$ comprises a bounded closed subset $U(x_0) \subseteq X$ of some element $x_0 \in D(f)$ such that $U(x_0)$ is topologically equivalent to $B^X_{r_0}(0)$ and $U(x_0) \subseteq D(f) \cap D(g)$, then we can formulate the conditions and statement of this theorem analogously, i.e. for this we determine the operator $\tilde{f}(x) = f(x) - f(x_0)$ and assume that

$$\|\tilde{f}(x)\|_X \leq \mu(\|x - x_0\|_X),$$

holds for any $x \in U(x_0)$ and

$$(2.1) \quad \langle \tilde{f}(x), g(x - x_0) \rangle \geq \nu(\|x - x_0\|_X) \|x - x_0\|_X,$$

holds for almost all $x \in U(x_0)$. Moreover $\nu(\|x - x_0\|_X) \geq \delta_0 > 0$ for any $x \in \partial U(x_0)$, where $g : D(g) \subseteq X \rightarrow Y^*$ such that $D(f) \subseteq D(g)$ and $g$ satisfies a claim analogously of the condition (ii) respect to $U(x_0)$. In this case, we define subset $\tilde{M}_{x_0}$ in the form

$$\tilde{M}_{x_0} \equiv \{ y \in Y \mid 0 \leq \langle f(x) - y, g(x - x_0) \rangle, \forall x \in \partial U(x_0) \}.$$ 

2. In the formulation of Theorem 1 we use the equality $\|g(x)\|_{Y^*} \equiv \|x\|_X$ that can be determined by the known way, i.e. $g(x) \equiv \frac{\|x\|_X}{\|g(x)\|_{Y^*}} g(x)$ for any $x \in D(g) \subseteq X$.

Condition (iii) of Theorem 1 can be generalized, for example, as in the following proposition.

**Corollary 1.** Let all conditions of Theorem 1 be fulfilled except inequality (2.2), and instead of that, let the following inequality

$$(2.3) \quad \|f(x_2) - f(x_1)\|_Y \geq \Phi(\|x_2 - x_1\|_X, \bar{x}, \varepsilon) + \psi(\|x_1 - x_2\|_Z, \bar{x}, \varepsilon),$$

be fulfilled for any $x_1, x_2 \in V_c(\bar{x}) \cap B^X_{r_0}(0)$, where $\Phi(\tau, \bar{x}, \varepsilon)$ is a function such as in the condition (iii), $Z$ is a Banach space and the inclusion $X \subset Z$ is compact, and $\psi(\cdot, \bar{x}, \varepsilon) : R^1_+ \rightarrow R^1$ is a continuous function at $\tau$ and $\psi(0, \bar{x}, \varepsilon) = 0$. Then the statement of Theorem 1 is correct.

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We note that Theorem 1 is a generalization of Theorems of such type from

Solntanov, K.N.: On equations with continuous mappings in Banach spaces. Funct. Anal. Appl. 33, (1999) 1, 76-81.
Note. It should be noted that this result is a generalization of the known Lax-Milgram theorem to the nonlinear case in the class of Banach spaces when all conditions of this theorem are fulfilled on whole space. Indeed, we can formulate the Lax-Milgram theorem for the linear operator $T$ acting on the real Hilbert space $X$ in the form:

a) There exists a positive constant $\gamma$ such that

$$|(Tx, y)| \leq \gamma \|x\|_X \cdot \|y\|_X, \quad \forall (x, y) \in X \times X;$$

that is equivalent to boundedness of operator $T : X \rightarrow X$.

b) there exists a positive constant $\delta$ such that

$$(Tx, x) \geq \delta \|x\|_X^2;$$

that is equivalent to the coerciveness of $T : X \rightarrow X$, i.e. $T$ satisfies the condition (ii) for the special case $(g \equiv \text{id})$.

Then equation $Tx = y$ is solvable for any $y \in X$.

From the condition b, it follows that

$$\|T(x_1 - x_2)\|_X \geq \delta \|x_1 - x_2\|_X;$$

i.e. inequality (2.2) holds for any $x_1, x_2 \in X$.

From Theorem 1 it immediately follows that

**Theorem 2.** (Fixed-Point Theorem). Let $X$ be a reflexive separable Banach space and $f_1 : D(f_1) \subseteq X \rightarrow X$ be a bounded continuous operator. Moreover, let on a closed ball $B^X_{r_0}(x_0) \subseteq D(f_1)$, where $x_0 \in D(f_1)$, operator $f \equiv \text{Id} - f_1$ satisfy the following conditions

$$\|f_1(x) - f_1(x_0)\|_X \leq \mu (\|x - x_0\|_X), \quad \forall x \in B^X_{r_0}(x_0),$$

(2.4) \hspace{1cm} \langle f(x) - f(x_0), g(x - x_0) \rangle \geq \nu (\|x - x_0\|_X) \|x - x_0\|_X, \quad \forall x \in B^X_{r_0}(x_0),$$

and almost each $\bar{x} \in \text{int} B^X_{r_0}(x_0)$ possesses a neighborhood $V_\varepsilon(\bar{x})$, $\varepsilon \geq \varepsilon_0 > 0$ such that the following inequality

$$\|f(x_2) - f(x_1)\|_X \geq \varphi (\|x_2 - x_1\|_X, \bar{x}, \varepsilon),$$

holds for any $x_1, x_2 \in V_\varepsilon(\bar{x}) \cap B^X_{r_0}(x_0)$, where $g : D(g) \subseteq X \rightarrow X^*$ such that $B^X_{r_0}(0) \subseteq D(g)$ and $g$ satisfies condition (ii), $\mu$ and $\nu$ are such functions as in Theorem 1, function $\varphi (\tau, \bar{x}, \varepsilon)$ has such a form as the right hand side of inequality (2.3) (in particular, $g \equiv J : X \rightrightarrows X^*$, i.e. $g$ be a duality mapping). Then the operator $f_1$ possesses a fixed-point on the ball $B^X_{r_0}(x_0)$.

**Definition 1.** We call that an operator $f : D(f) \subseteq X \rightarrow Y$ possesses the P-property if any precompact subset $M$ of $Y$ from $\text{Im} f$ has a (general) subsequence $M_0 \subseteq M$ such that there exists a precompact subset $G$ of $X$ that satisfies the inclusions $f^{-1}(M_0) \subseteq G$ and $f(G \cap D(f)) \subseteq M_0$.

**Notation 1.** We can take the following condition instead of condition (iii) of Theorem 1: $f$ possesses the P-property on the ball $B^X_{r_0}(0)$. It should be noted that an operator $f : D(f) \subseteq X \rightarrow Y$ possesses of the P-property if $f^{-1}$ is a lower or upper semi-continuous mapping.
In the above results for the completeness of the image (Im $f$) of the imposed operator $f$, the condition (iii) and P-property (and also the generalizations of the conditions (iii)) are used. But there are some other types of the complementary conditions on $f$ under which Im $f$ will be a closed subset. These types of conditions are described in [1, 12, 14]. Therefore we do not conduct them here again.

3. General Result on Solvability of Perturbed Equation

Now, we lead a solvability theorem for the perturbed nonlinear equation in the Banach spaces, proved by using Theorem 1 and Corollary 1.

Let $X, Y$ be reflexive Banach spaces and $X^*, Y^*$ be their dual spaces, let $F : D(F) \subseteq X \rightarrow Y$ be a nonlinear operator that has the representation $F(x) \equiv F_0(x) + F_1(x)$ for any $x \in D(F)$, where $F_i : D(F_i) \subseteq X \rightarrow Y$, $i = 0, 1$ are some operators such that $D(F_0) \cap D(F_1) \equiv G \subseteq X$ and $G \neq \emptyset$.

Consider the following equation

\[ F(x) \equiv F_0(x) + F_1(x) = y, \quad y \in Y, \]

where $y$ is an element of $Y$.

Let $B^X_r(0) \subseteq G \subseteq X$ be a closed ball, $r > 0$ be a number. We set the following conditions

1) $F_0 : B^X_r(0) \rightarrow Y$ is the continuous operator with its inverse operator $F_0^{-1}$,
2) $F_1 : B^X_r(0) \rightarrow Y$ is a nonlinear continuous operator;
3) there are such continuous functions $\mu_1 : R^1_+ \rightarrow R^1_+$, $i = 1, 2$ and $\nu : R^1_+ \rightarrow R^1$ that the following inequalities

\[ \|F_0(x)\|_Y \leq \mu_1(\|x\|_X) \quad \& \quad \|F_1(x)\|_Y \leq \mu_2(\|x\|_X), \]
\[ \langle F_0(x) + F_1(x), g(x) \rangle \geq c \langle F_0(x), g(x) \rangle \geq \nu(\|x\|_X)\|x\|_X, \]

hold for any $x \in B^X_r(0)$, moreover $\nu(r) \geq \delta_0$ holds for some number $\delta_0 > 0$, where the mapping $g : B^X_r(0) \subseteq D(g) \subseteq X \rightarrow Y^*$ fulfills the conditions of Theorem 1, where $c > 0$ is a constant;
4) almost each $\bar{x} \in intB^X_r(0)$ possesses a neighborhood $B^X_\varepsilon(\bar{x})$, $\varepsilon \geq \varepsilon_0 > 0$, such that the following inequality

\[ \|F(x_1) - F(x_2)\|_Y \geq c_1(\|x_1, x_2\|)\|F_0(x_1) - F_0(x_2)\|_Y, \]

holds for any $x_1, x_2 \in B^X_\varepsilon(\bar{x})$ and some number $\varepsilon_0 > 0$, where $c_1(\|\bar{x}\|_X, \varepsilon) > 0$ is bounded for each $\bar{x} \in intB^X_r(0)$;

or
4') operator $F : B^X_r(0) \subset X \rightarrow Y$ possesses P-property, except for the existence of the inverse operator $F_0^{-1}$ in condition 1.

Then the following statement is true by Theorem 1.

**Theorem 3.** Let conditions 1, 2, 3, 4 (or 1, 2, 3, 4') be fulfilled then equation (3.1) has a solution in the ball $B^X_r(0)$ for any $y \in Y$ that fulfills the following inequality

\[ \langle y, g(x) \rangle \leq \nu(\|x\|_X)\|x\|_X, \quad \forall x \in S^X_r(0). \]
4. Fully Nonlinear Equations of Second Order

Now, we study some nonlinear BVP with using the general results. Let \( \Omega \subset \mathbb{R}^n \) \((n \geq 1)\) be an open bounded domain with sufficiently smooth boundary \( \partial \Omega \).

Consider the following problem

\[
(4.1) \quad f(u) \equiv -\Delta u + F(x, u, Du, \Delta u) = 0, \quad x \in \Omega, \quad u \mid \partial \Omega = 0,
\]

where \( F(x, \xi, \eta, \zeta) \) is a Carathéodory function on \( \Omega \times \mathbb{R}^2 \times \mathbb{R}^n \) as \( F : \Omega \times \mathbb{R}^2 \times \mathbb{R}^n \to \mathbb{R}^1, \quad D \equiv (D_1, D_2, \ldots, D_n), \Delta \equiv \sum_{i=1}^n D_i^2 \) (is Laplacian), \( D_i \equiv \frac{\partial}{\partial x_i} \).

Let the following conditions be fulfilled

(i) there are Carathéodory functions \( F_0, F_1, F_2 : \Omega \times \mathbb{R}^n \to \mathbb{R}^1 \) such that \( F(x, \xi, \eta, \zeta) = F_0(x, \xi) + F_1(x, \xi, \eta) + F_2(x, \xi, \eta, \zeta) \) for any \((x, \xi, \eta, \zeta) \in \Omega \times \mathbb{R}^1 \times \mathbb{R}^n \times \mathbb{R}^1\), moreover

(a) there exist a Carathéodory function \( a_1(x, \xi) \) and numbers \( m_0, \mu \geq 0, \mu, M > 0, \mu + 2 > 2\mu \) such that

\[
F_0(x, \xi) \equiv M - |\xi|^\mu + a_1(x, \xi),
\]

(4.2) \quad \quad \quad |a_1(x, \xi)| \leq m_0 \cdot |\xi|^\mu + \psi(x), \quad \psi \in L_p(\Omega), \quad p > 2,

hold for a.e. \( x \in \Omega \) and any \( \xi \in \mathbb{R}^1 \), and

(b) there exist a number \( \rho : 2 \geq \rho \geq 0 \) and a nonnegative Carathéodory function \( m_1(x, \xi, \eta) \geq 0 \) such that

\[
|F_1(x, \xi, \eta)| \leq m_1(x, \xi, \eta) |\eta|^\rho + k(x), \quad \forall (x, \xi, \eta) \in \Omega \times \mathbb{R}^1 \times \mathbb{R}^n,
\]

holds, where \( m_1(x, \xi, \eta) \leq M_1(x), \) and \( 2 \left( \hat{C} (\mu, \rho, n) \|M_1\|_\infty^2 \right)^2 \leq M, \quad k \in L^{p_1}(\Omega), \quad p_1 > 2, \)

where \( \hat{C} (\mu, \rho, n) \) is the coefficient of Gagliardo-Nirenberg-Sobolev (G-N-S) inequality (see, [16]).

c) there exist Carathéodory functions \( c(x, \xi, \eta, \zeta), \bar{F}_2(x, \xi, \eta) \) and a continuous function \( k(\zeta) \) such that the following inequalities

\[
|F_2(x, \xi, \eta, \zeta) - \bar{F}_2(x, \xi, \eta, \zeta_1)| \leq c(x, \xi, \eta, \zeta, \zeta_1) |\zeta - \zeta_1|,
\]

(4.4) \quad \quad \quad |F_2(x, \xi, \eta, \zeta) - \bar{F}_2(x, \xi, \eta, \eta_1, \zeta_1)| \leq k(\zeta) \left| \bar{F}_2(x, \xi, \eta, \eta_1) - \bar{F}_2(x, \xi, \eta, \zeta_1) \right|,

hold for a.e. \( x \in \Omega \) and any \((\xi, \eta), (\xi_1, \eta_1) \in \mathbb{R}^{n+1}, \forall \zeta, \zeta_1 \in \mathbb{R}^1, \) and \( F_2(x, \xi, \eta, 0) = 0 \)

moreover there exists a function \( \psi_1(x) \geq 0, c(x, \xi, \eta, \zeta) \leq \psi_1(x) \) hold for a.e. \( x \in \Omega \) and any \((\xi, \eta, \zeta) \in \mathbb{R}^N, \) here \( \|\psi_1\|_{L^\infty(\Omega)} \equiv \|\psi_1\|_\infty \leq 4^{-1} \).

\[
\|D^{\beta} v\|_{p_0} \leq C(p_0,p_1,p_2,l,s,n) \times \left[ \sum_{|\alpha| \leq l} \|D^{\alpha} v\|_{p_1} \right]^{\theta} \times \|v\|_{p_2}^{1-\theta},
\]

where \( \theta \in \left[ \frac{n}{p_0} - s, 1 \right] \) and satisfy the equation

\[
\frac{n}{p_0} - s = \theta \left( \frac{n}{p_1} - l \right) + (1 - \theta) \frac{n}{p_2},
\]

the constant \( C(p_0,p_1,p_2,l,s,n) \) is independent of \( v(x) \); \( p_0, p_1, p_2 \geq 1, 0 \leq s < l, |\beta| = \sum_{i=1}^n \beta_i = s, \beta_i \geq 0 \) be some numbers.
Assume the following denotations: \( \|u\|_{L^p(\Omega)} \equiv \|u\|_p \) for any \( p \in [1, \infty] \), and \( \|u\|_{W^{1,p}(\Omega)} \equiv \|u\|_{l,p} \), for \( u \in W^{1,p}(\Omega) \), \( l \geq 1 \).

So, we consider the operator \( f : W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega) \rightarrow L^2(\Omega) \), which is generated by the imposed problem.

**Theorem 4.** Let conditions (i), (a), (b), (c) be fulfilled and parameters \( \mu \) and \( \rho \) satisfy the following relations

\[
1 < \mu \leq \frac{4}{n - 4} \quad \text{if } n \geq 5 \quad \text{and} \quad 1 < \mu < \infty \quad \text{if } n = 2, 3, 4, \\
\rho \geq \frac{(\mu + 2)n}{2(n + \mu)} \quad \text{and} \quad \rho \leq 1 + \min \left\{ \frac{\mu + 2}{n + \mu}, \frac{2}{\mu - 2}, \frac{\mu}{\mu + 2} \right\}.
\]

Then problem (4.1) is solvable in \( W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega) \).

**Proof.** For the proof, it is enough to show that the operator \( f : W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega) \rightarrow L^2(\Omega) \) fulfills all conditions of Theorem 1 (or Theorem 3). For this, we will estimate the following dual form

\[
\langle f(u), g(u) \rangle = \langle -\Delta u + F(x, u, Du, \Delta u), -\Delta u \rangle,
\]

where the operator \( g \) is determined in the form \( g \equiv -\Delta : W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega) \rightarrow L^2(\Omega) \), then we have

\[
\langle -\Delta u + F(x, u, Du, \Delta u), -\Delta u \rangle = \|\Delta u\|^2_2 + \langle F(x, u, Du, \Delta u), -\Delta u \rangle,
\]

consequently we need to estimate the second term of this equation, i.e. the dual form: \( \langle F(x, u, Du, \Delta u), -\Delta u \rangle \).

Thus using conditions (i), (a), (b), (c) we obtain

\[
\langle F(x, u, Du, \Delta u), -\Delta u \rangle = -\langle F_0(x, u), \Delta u \rangle - \langle F_1(x, u, Du), \Delta u \rangle - \langle F_2(x, u, Du, \Delta u), \Delta u \rangle - \langle F_3(x, u, Du, \Delta u), \Delta u \rangle
\]

for which we will use the known inequality (G-N-S). Using that we get

\[
\langle F(x, u, Du, \Delta u), -\Delta u \rangle \geq (M(\varepsilon_1)(\mu + 1) - (\varepsilon + 2^{-1}) \|\Delta u\|^2_2 - C\left(\varepsilon, \varepsilon_1, M_1, \|\psi\|_p, \|k\|_p\right), \quad \varepsilon \in (0, 1).
\]

Hence we need to estimate the term \( \|\nabla u\|^\theta_2 \) by using \( \|u\|_{1,2}^\mu \|u\|_{1,2}^\varepsilon \) and \( \|u\|_{1,2}^2 \) for which we will use the known inequality (G-N-S). Using that we get

\[
\langle F(x, u, Du, \Delta u), -\Delta u \rangle \geq (M(\varepsilon_1)(\mu + 1) - (\varepsilon + 2^{-1}) \|\Delta u\|^2_2 - C\left(\varepsilon, \varepsilon_1, M_1, \|\psi\|_p, \|k\|_p\right), \quad \varepsilon \in (0, 1)\]
\begin{equation}
(4.7) \quad \left( \bar{C} (\mu, \rho, n) \|M_1\|_\infty^2 \right)^2 \|u\|_{L^{\mu+2}}^{\mu+2} - C \left( \varepsilon, \varepsilon_1, m_0, \|\psi\|_p, \|k\|_p \right),
\end{equation}
where \( \left( \bar{C} (\mu, \rho, n) \|M_1\|_\infty^2 \right)^2 < M \) by the condition (b).

Now if we take into account (4.6) and (4.7), we get

\[
\langle f (u), g (u) \rangle = \langle -\Delta u + F (x, u, Du, \Delta u), -\Delta u \rangle \geq \left( \frac{1}{4} - \varepsilon \right) \|\Delta u\|_2^2 + \frac{\bar{M} (\mu + 1)}{4} \|u\|_2^2 \nabla u \|_2^2 - C \left( \varepsilon, \varepsilon_1, m_0, \|\psi\|_p, \|k\|_p \right).
\]

So, condition 3 of Theorem 3 is fulfilled since the operator \( f : W^{2,2} (\Omega) \cap W_0^{1,2} (\Omega) \rightarrow L^2 (\Omega) \) is bounded, that can be seen easily from its expression and the conditions of this theorem.

Thus it follows that problem (4.1) is densely solvable in \( L^2 (\Omega) \). Consequently, it remains to show that image \( f \left( W^{2,2} (\Omega) \cap W_0^{1,2} (\Omega) \right) \) is closed in the space \( L^2 (\Omega) \).

Let \( h_0 \in L^2 (\Omega) \) then there is a sequence \( \{h_m\}_{m=1}^\infty \subset \text{Im } f \subseteq \text{ that } L^2 (\Omega) \) converges to the given \( h_0 \) in \( L^2 (\Omega) \), as \( c_1 \text{ Im } f \equiv L^2 (\Omega) \). For any \( h_m \) we have the subset \( f^{-1} (h_m) \) and \{\( \{h_m\} \) is a bounded subset in \( L^2 (\Omega) \) then there is a bounded subset \( G \) of \( W^{2,2} (\Omega) \cap W_0^{1,2} (\Omega) \) such that \( G \cap f^{-1} \left( \{h_m\}_{m=1}^\infty \right) \neq \emptyset \), in addition \( G \cap f^{-1} (h_m) \neq \emptyset \) for any \( h_m \). Then we can choose a sequence \( \{u_m\} \subset G \) such that \( f (u_m) = h_m \), which belongs to the bounded subset \( G \). From here, by using the reflexivity of the space \( W^{2,2} (\Omega) \cap W_0^{1,2} (\Omega) \), we can select a subsequence \( \{u_{m_k}\}_{k=1}^\infty \subset \{u_m\}_{m=1}^\infty \) that is a weakly convergent sequence in \( W^{2,2} (\Omega) \cap W_0^{1,2} (\Omega) \), i.e. there is an element \( u_0 \) such that \( u_{m_k} \rightharpoonup u_0 \) in \( W^{2,2} (\Omega) \cap W_0^{1,2} (\Omega) \) (may be after the choice of a subsequence of \( \{u_{m_k}\}_{k=1}^\infty \), and consequently \( u_{m_k} \rightharpoonup u_0 \) in \( W^{1,p} (\Omega), 1 \leq p < 2^* \).

Thus, we get
\[
F_0 (x, u_{m_k}) \rightarrow F_0 (x, u_0), \quad F_1 (x, u_{m_k}, Du_{m_k}) \rightarrow F_1 (x, u_0, Du_0)
\]
in \( L^2 (\Omega) \) by the conditions of Theorem 4 that \( F_i : W^{2,2} (\Omega) \cap W_0^{1,2} (\Omega) \rightarrow L^2 (\Omega) \), \( i = 0, 1 \), are continuous operators.

On the other hand for any \( \varepsilon > 0 \) there exist \( m_k, m_l \geq m_k (\varepsilon) \geq 1 \) such that the inequality
\[
\varepsilon > \|h_{m_k} - h_{m_l}\|_2 \approx \|F (u_{m_k}) - F (u_{m_l})\|_2 \geq \frac{3}{4} \|\Delta (u_{m_k} - u_{m_l})\|_2 + \bar{M} \|u_{m_k} - u_{m_l}\|_2^{\mu+1} \left( 2\left( \mu+1 \right) \right) - \|F_0 (x, u_{m_k}) - F_0 (x, u_{m_l})\|_2 - \|F_1 (x, u_{m_k}, Du_{m_k}) - F_1 (x, u_{m_l}, Du_{m_l})\|_2 - \frac{k (\|\Delta u_{m_k}\|_2 \|F_2 (x, u_{m_k}, Du_{m_k}) - F_2 (x, u_{m_l}, Du_{m_l})\|_2)}{2} \right),
\]
holds, where \( u_{m_k} \rightharpoonup u_0 \) in \( W^{1,p} (\Omega), 1 \leq p < 2^* \) and \( u_{m_k} \rightharpoonup u_0 \) in \( W^{2,2} (\Omega) \). Hence we obtain that the last terms of (4.9) converge to zero under \( m_k \nrightarrow \infty \), then we get
\[
\|\Delta (u_{m_k} - u_{m_l})\|_2 \nrightarrow 0 \quad \text{if } \quad m_k, m_l \nrightarrow \infty.
\]

Consequently \( \Delta u_{m_k} \rightharpoonup \Delta u_0 \) in \( L^2 (\Omega) \), \( F_2 (x, u_{m_k}, Du_{m_k}, \Delta u_{m_k}) \rightarrow F_2 (x, u_0, Du_0, \Delta u_0) \) in \( L^2 (\Omega) \) and from the equality
\[
\langle (-\Delta u_{m_k} + F (x, u_{m_k}, Du_{m_k}, \Delta u_{m_k}), v) \rangle = \langle -\Delta u_{m_k}, v \rangle + \langle F_0 (x, u_{m_k}), v \rangle - \langle F_1 (x, u_{m_k}, Du_{m_k}), v \rangle - \langle F_2 (x, u_{m_k}, Du_{m_k}, \Delta u_{m_k}), v \rangle = \langle h_{m_k}, v \rangle, \quad \forall v \in L^2 (\Omega) \& \forall k \geq 1,
\]
we obtain that
\[ \langle -\Delta u - F(x, u, Du, \Delta u), v \rangle = \langle h_0, v \rangle, \quad \forall v \in L^2(\Omega). \]
Hence it follows that \( h_0 \in \text{Im } f \), i.e. \( \text{Im } f \equiv f \left( W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega) \right) \equiv L^2(\Omega). \)

**Remark 2.** The result of this theorem shows that we can consider the following problem
\[ -\Delta u + M \, |u|^\mu \, u = -a_1(x, u) - F_1(x, u, Du) - F_2(x, u, Du, \Delta u) \]
let the operators \( G_0 : W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega) \to L^2(\Omega) \) and \( G_1 : W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega) \to L^2(\Omega) \) be defined by the expressions
\begin{align*}
G_0(u) &\equiv -\Delta u + M \, |u|^\mu \, u, \quad G_1(u) \equiv -a_1(x, u) - F_1(x, u, Du) - F_2(x, u, Du, \Delta u)
\end{align*}
respectively, then existence of \( G_0^{-1} : L^2(\Omega) \to W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega) \) is known and \( G_0^{-1} \) is a bounded continuous operator. Now we determine the operator \( G(u) \equiv (G_0^{-1} \circ G_1)(u) \) that acts from \( W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega) \) to \( W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega) \) and is bounded continuous operator under the conditions of Theorem 4. Hence we obtain that the operator \( G \) possesses a fixed point that allows us to investigate the problem on the existence of the eigenvalue of the operator \( G_0 \) relative to the operator \( G_1 \), i.e. study of the problem \( G_0(u) = \lambda G_1(u) \). But in this case we can only conclude that \( \lambda \) will be dependent on \( u \).

**Remark 3.** From the proof of this theorem it follows that the result of such type is true and in the case of the operator \( F(x, u, Du, \Delta u) \) in the problem (4.1) is independent of \( \Delta u \), i.e. it has the representation \( F(x, \xi, \eta, \zeta) \equiv F(x, \xi, \eta) \equiv F_0(x, \xi) + F_1(x, \xi, \eta) \) for \( (x, \xi, \eta) \in \Omega \times \mathbb{R}^{n+1} \).

5. **Nonlinear Equation with \( p \)-Laplacian**

On the open bounded domain \( \Omega \subset \mathbb{R}^n \) with sufficiently smooth boundary \( \partial \Omega \) consider the following problem
\begin{align}
(5.1) \quad f(u) &\equiv -\nabla \left( |\nabla u|^{p-2} \nabla u \right) + G(x, u, Du, \lambda, \mu) = h(x), \quad x \in \Omega \subset \mathbb{R}^n, \\
(5.2) \quad u |_{\partial \Omega} &= 0, \quad n \geq 1, \quad \Omega \in \text{Lip}, \quad h \in W^{-1,q}(\Omega). 
\end{align}
Assume that
\begin{equation}
(5.3) \quad G(x, \xi, \eta, \lambda) = \mu G_0(x, \xi) + \lambda G_1(x, \xi, \eta),
\end{equation}
holds for a.e. \( x \in \Omega \) and any \( (\xi, \eta, \lambda) \in \mathbb{R}^{n+1} \), where \( G_1(x, \xi, \eta) \) and \( G_0(x, \xi) \) are some Carathéodory functions, \( \lambda \in \mathbb{R}, \mu \geq 0 \) are some parameters.

**5.1. Dense solvability.** Let the following conditions
\begin{align}
(5.4) \quad |G_0(x, \xi)| &\geq a_0(x) \, |\xi|^{p_0} - a_1(x), \quad a_0(x) \geq A_0 > 0; \\
&|G_0(x, \xi)| \leq \bar{a}_0(x) \, |\xi|^{p_0-1} + \bar{a}_1(x), \quad \bar{a}_0(x), \bar{a}_1(x) \geq 0,
\end{align}
\begin{align}
(5.5) \quad |G_1(x, \xi, \eta)| &\leq b_0(x) \, |\eta|^{p_1} + b_1(x) \, |\xi|^{p_2} + b_2(x), \quad b_j(x) \geq 0, \quad j = 0, 1, 2,
\end{align}
hold for a.e. \( x \in \Omega \) and any \( (\xi, \eta, \lambda) \in \mathbb{R}^{n+1} \) where \( p_0, p_1, p_2 \geq 0, \, p > 1 \) are some numbers, \( a_k(x), \bar{a}_k(x) \) and \( b_j(x) \) are some functions, \( k = 0, 1 \) and \( j = 0, 1, 2 \).
Here we study the solvability of problem (5.1)-(5.2) in the generalized sense, i.e. a function \( u \in W_0^{1,p} (\Omega) \) is called a solution of the problem (5.1), (5.2) if \( u \) satisfies the equation
\[
\langle f (u), v \rangle = \langle h, v \rangle, \quad v \in W_0^{1,p} (\Omega),
\]
for any \( v \in W_0^{1,p} (\Omega) \).

**Theorem 5.** Let conditions (5.3) - (5.5) be fulfilled and \( p_0 - 1 \geq p_2 \geq 0, p > 1, p > p_1 \geq 0 \). Moreover, let \( a_k (x), \tilde{a}_k (x), b_j (x) \) be such functions that \( a_0, \tilde{a}_0 \in L^\infty (\Omega), a_1, \tilde{a}_1 \in L^p (\Omega) \) and \( b_0, b_1 \in L^q (\Omega) \). If \( p_1, p_0 \) satisfy the inequalities \( p_1 \leq p - \frac{p}{p_0}, p_0 \leq p^* \equiv \frac{np}{n-p} \), then there exist a subset \( \mathcal{M} \subseteq W^{-1,q} (\Omega) \) and some numbers \( \mu > 0, C_0 > 0 \) such that \( \overline{\mathcal{M}} \subseteq W^{-1,q} (\Omega), q = \frac{p}{p-1} = p' \), and \( \mu \geq \mu_0 > 0 \), \( \lambda : |\lambda| \leq C_0 \), problem (5.1)-(5.2) is solvable in \( W_0^{1,p} (\Omega) \) for any \( h \in \mathcal{M} \); moreover if \( \bar{p} = p_0 \) or \( p_2 + 1 = p_0 \), then \( C_0 \equiv C_0 (A_0, b_0, b_1, \mu) \) is sufficiently small number.

**Proof.** Let \( u \in W_0^{1,p} (\Omega) \cap L^{p_0} (\Omega) \) and consider the dual form
\[
\langle f (u), u \rangle = \| \nabla u \|^p_p + (G (x, u, Du), \lambda), u =
\]
\[
\| \nabla u \|^p_p + \langle \mu G_0 (x, u), u \rangle + \lambda (G_1 (x, u, Du), u),
\]
then by using conditions (5.4) and (5.5), we get
\[
\langle f (u), u \rangle \geq \| \nabla u \|_p^p + \mu \left( a_0 (x) |u|^{p_0-2} u, u \right) - \mu \| a_1 \|_1 -
\]
\[
|\lambda| \langle b_0 (x) |\nabla u|^{p_1}, |u| \rangle - |\lambda| \langle b_1 (x) |u|^{p_2}, |u| \rangle - |\lambda| \langle b_2 (x), |u| \rangle,
\]
or
\[
(5.6) \quad |\lambda| \int_\Omega b_1 (x) |u|^{p_2+1} dx - |\lambda| \int_\Omega b_2 (x) |u| dx.
\]
Since \( b_j \in L^\infty (\Omega), j = 0, 1 \), it is enough to estimate first integral of the right side of inequality (5.6). For this, we have
\[
|\lambda| \int_\Omega b_0 (x) |\nabla u|^{p_1} |u| dx \leq |\lambda| \| b_0 \|_\infty \left[ \varepsilon \| \nabla u \|_p^p + c (\varepsilon) \| u \|_p^p \right].
\]
By using the last inequality in (5.6) and taking into account the condition on \( p_1 \), we obtain
\[
(5.6) \quad (1 - \varepsilon |\lambda| \| b_0 \|_\infty) \| \nabla u \|_p^p + (\mu A_0 - \varepsilon_1) \| u \|_p^p -
\]
\[
(1 - \varepsilon |\lambda| \| b_0 \|_\infty) \| u \|_p^p - |\lambda| \int_\Omega b_1 (x) |u|^{p_2+1} dx - C_{\varepsilon_1} \left( |\lambda|, \mu, \| a_1 \|_q, \| b_1 \|_q \right).
\]
since \( p_1 \leq p \left( 1 - p_0^{-1} \right) \) by the conditions \( \bar{p} \leq p_0 \). Hence either of these cases take place: \( \bar{p} < p_0 \) and \( p_2 + 1 < p_0 \) or one of equations \( \bar{p} = p_0 \) or \( p_2 + 1 = p_0 \) holds, if \( \bar{p} < p_0 \) and \( p_2 + 1 < p_0 \) then we can estimate it by second term from right side of the previous inequality with using Young inequality, and if \( \bar{p} = p_0 \) or \( p_2 + 1 = p_0 \) then
it is enough to choose number $|\lambda|$ sufficiently small. Thus we obtain the following inequality:
\[
\langle f(u), u \rangle \geq (1 - \varepsilon |\lambda| \|b_0\|_{\infty}) \|\nabla u\|_p^p + (\mu A_0 - \varepsilon_1 - \varepsilon_2) \|u\|_{p_0}^{p_0} - C_{x_1} (|\lambda|, \mu, \varepsilon_2, \|a_1\|, \|b_1\|, \|b_2\|).
\]
Consequently, inequality (2.1) of Theorem 1 is fulfilled, $|\lambda|$ must be sufficiently small, i.e. the statement of Theorem 5 is true since the operator
\[
f : W^{1,p}_0(\Omega) \cap L^{p_0}(\Omega) \to W^{-1,q}(\Omega) + L^{q_0}(\Omega), \quad q_0 = \frac{p_0}{p_0 - 1},
\]
is bounded by virtue of the obtained estimations here. \qed

5.2. Everywhere solvability. Now we reduce the conditions under which imposed problem (5.1)-(5.2) is everywhere solvable. Let conditions (5.3) - (5.5) be fulfilled and consider the following conditions
\[
|G_0(x, \xi) - G_0(x, \xi_1)| \leq c_0 \left( x, \tilde{\xi} \right) |\xi - \xi_1|,
\]
(5.7)
\[
|G_1(x, \xi, \eta) - G_1(x, \xi_1, \eta_1)| \leq c_1 \left( x, \tilde{\xi}, \tilde{\eta} \right) |\eta - \eta_1|,
\]
hold for a.e. $x \in \Omega$, and any $(\xi, \eta), (\xi_1, \eta_1) \in \mathbb{R} \times \mathbb{R}^n$, where $c_0(x, \xi), c_1(x, \xi, \eta)$ are some Caratheodory functions such that $\tilde{\xi} = \tilde{\xi}(\xi, \xi_1), \tilde{\eta} = \tilde{\eta}(\eta, \eta_1)$ are continuous functions, and moreover $c_1(x, v, \nabla v), c_0(x, v)$ are bounded operators such that if $v(x)$ belongs to a bounded subset $D$ of $W^{1,p}_0(\Omega) \cap L^{p_0}(\Omega)$, i.e. if $\|v\|_{W^{1,p}_0(\Omega) \cap L^{p_0}(\Omega)} \leq K_0$, then
\[
\|c_1(x, v, \nabla v)\|_{L^\infty(\Omega)} \leq K_1, \quad \|c_0(x, v)\|_{L^\infty(\Omega)} \leq K_2,
\]
for some numbers $K_0, K_1, K_2 > 0$, i.e.
(5.8)
\[
c_j(x, \cdot, \cdot) : W^{1,p}_0(\Omega) \cap L^{p_0}(\Omega) \to L^\infty(\Omega), \quad j = 0, 1,
\]
are bounded operators.
Then
\[
|\langle G_1(x, u, \nabla u) - G_1(x, v, \nabla v), u - v \rangle| \leq \|c_1(x, u, \nabla u, \nabla v)\|_{\infty} \|\nabla u - \nabla v\|_p \|u - v\|_q,
\]
holds for any $u, v \in W^{1,p}_0(\Omega) \cap L^{p_0}(\Omega)$. Hence we get
\[
\langle f(u) - f(v), u - v \rangle = \langle |\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v, \nabla (u - v) \rangle + \mu \langle G_0(x, u) - G_0(x, v), u - v \rangle + \lambda \langle G_1(x, u, Du) - G_1(x, v, Dv), u - v \rangle \geq \tilde{C}_0 \|\nabla (u - v)\|_p^p + \mu \left( a(x) \left| u|^{p_0-2} - |v|^{p_0-2} \right|, u - v \right),
\]
and we obtain the following inequality by using conditions (5.4), (5.7) and (5.8)
\[
\langle f(u) - f(v), u - v \rangle \geq \tilde{C} \|\nabla (u - v)\|_p^p + \tilde{A}_0 \|u - v\|_{p_0}^{p_0} - \|c_1(x, u, \nabla u, \nabla v)\|_{\infty} \|\nabla u - \nabla v\|_p \|u - v\|_q.
\]
Consequently, we have
\[
\|f(u) - f(v)\|_{W^{-1,q}_0(\Omega)} \cdot \|\nabla (u - v)\|_p \geq \tilde{C} \|\nabla (u - v)\|_p^p + \tilde{A}_0 \|u - v\|_{p_0}^{p_0} - \|c_1(x, u, \nabla u, \nabla v)\|_{\infty} \left[ \varepsilon \|\nabla (u - v)\|_p^p + c(\varepsilon) \|u - v\|_q^q \right],
\]
or
\[
\|f(u) - f(v)\|_{W^{-1,q}_0(\Omega)} \cdot \|\nabla (u - v)\|_p \geq \tilde{C}_1 \|\nabla (u - v)\|_p^{p-1} + \tilde{A}_1 \|u - v\|_{p_0}^{p_0} - \|c_1(x, u, \nabla u, \nabla v)\|_{\infty} \left[ \varepsilon \|\nabla (u - v)\|_p^{p-1} + c(\varepsilon) \|u - v\|_q^{q-1} \right].
\]
Thus we get that the conditions of Corollary 1 are fulfilled, i.e. if we continue this proof as in the section 4 then we obtain that the following result is true.

**Theorem 6.** Let conditions (5.3)-(5.5), (5.7) and (5.8) be fulfilled and the numbers \( \lambda, \mu \) satisfy the conditions of Theorem 5, then problem (5.1)-(5.2) is solvable in \( W^{1,p}_0(\Omega) \cap L^{p_0}(\Omega) \) for any \( h \in W^{-1,q}(\Omega) \).

**Remark 4.** It should be noted that a remark similar to remark 2 takes place for the problem investigated here. Moreover, we can obtain the same conclusion for the considered general case. Therefore, for the investigation of the spectrum of the nonlinear operators by using the fixed-point theorem mentioned above, we need to consider some particular cases of these problems.

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