Modeling, analysis, and simulation of X-shape quasi-zero-stiffness-roller vibration isolators*

Xiaoye MAO¹, Mengmeng YIN¹, Hu DING¹,†, Xiaofeng GENG¹, Yongjun SHEN²,†, Liqun CHEN¹

1. Shanghai Institute of Applied Mathematics and Mechanics, Shanghai Key Laboratory of Mechanics in Energy Engineering, School of Mechanics and Engineering Science, Shanghai University, Shanghai 200444, China;
2. State Key Laboratory of Mechanical Behavior and System Safety of Traffic Engineering Structures, Shijiazhuang Tiedao University, Shijiazhuang 050043, China

(Received Dec. 26, 2021 / Revised Mar. 11, 2022)

Abstract Existing quasi-zero stiffness (QZS) isolators are reviewed. In terms of their advantages, a novel X-shape QZS isolator combined with the cam-roller-spring mechanism (CRSM) is proposed. Different from the existing X-shape isolators, oblique springs are used to enhance the negative stiffness of the system. Meanwhile, the CRSM is used to eliminate the gravity of the loading mass, while the X-shape structure leaves its static position. The existing QZS isolators are demonstrated and classified according to their nonlinearity mechanisms and classical shapes. It is shown that the oblique spring can realize negative stiffness based on the simplest mechanism. The X-shape has a strong capacity of loading mass, while the CRSM can achieve a designed restoring force at any position. The proposed isolator combines all these advantages together. Based on the harmonic balance method (HBM) and the simulation, the displacement transmissibilities of the proposed isolator, the X-shape isolators just with oblique springs, and the X-shape isolators in the traditional form are studied. The results show that the proposed isolator has the lowest beginning isolation frequency and the smallest maximum displacement transmissibility. However, it still has some disadvantages similar to the existing QZS isolators. This means that its parameters should be designed carefully so as to avoid becoming a bistable system, in which there are two potential wells in the potential energy curve and thus the isolation performance will be worsened.

Key words quasi-zero stiffness (QZS), cam-roller, X-shape isolator, nonlinear isolation

Chinese Library Classification O322
2010 Mathematics Subject Classification 74H45

* Citation: MAO, X. Y., YIN, M. M., DING, H., GENG, X. F., SHEN, Y. J., and CHEN, L. Q. Modeling, analysis, and simulation of X-shape quasi-zero-stiffness-roller vibration isolators. Applied Mathematics and Mechanics (English Edition), 43(7), 1027–1044 (2022) https://doi.org/10.1007/s10483-022-2871-6
† Corresponding authors, E-mails: dinghu3@shu.edu.cn; shenyongjun@126.com
Project supported by the National Natural Science Foundation of China (No. 12002195), the National Science Fund for Distinguished Young Scholars of China (No. 12025204), the Program of Shanghai Municipal Education Commission of China (No. 2019-01-07-00-09-E00018), and the Pujiang Project of Shanghai Science and Technology Commission of China (No. 20PJ1404000)
©The Author(s) 2022
1 Introduction

Quasi-zero stiffness (QZS) isolators isolate low and ultra-low frequency vibrations\cite{1}. By eliminating the linear stiffness of the suspension, QZS can produce an ultra-low dynamic stiffness at the vertical direction, and the dynamic stiffness could be zero theoretically. Therefore, this kind of isolation has a very low working frequency and a strong static supporting capacity. Consequently, it can also be called high-static-low-dynamic isolation. Applying this advantage, this design is successfully applied to the gravity counteraction for astronautic device tests\cite{2-3} and gravity meter\cite{4}. Now, it has been widely studied for more than 60 years\cite{5-6}. The key of QZS is to realize the negative stiffness\cite{7}. According to the existing studies, the structure can be classified into two typical kinds, i.e., geometric nonlinearity and intrinsic nonlinearity.

The geometric nonlinearity is realized via some special mechanical structure designs. T-shape, X-shape, and cam-roller-spring mechanism (CRSM) QZS isolators are typical examples. T-shape QZS isolators with oblique coil springs have the simplest QZS mechanism among current QZS isolators. Although the stiffness of an oblique spring is linear along its axis, non-linearity exists in the direction perpendicular to the axis. The stiffness of the support spring can be eliminated, while the oblique spring is pre-pressed under a certain length. Zhang et al.\cite{8} verified the advantage of this kind of isolator via an experiment. They showed that the natural frequency of the isolator could be adjusted even to 1 Hz. However, they did not study the QZS design in detail. Based on the same structure, Carrella et al.\cite{1} found that there was a unique relationship between the geometry and the stiffness of the spring, yielding a system with zero dynamic stiffness at the static equilibrium position. The force transmissibility\cite{9} and the displacement transmissibility\cite{10} of a QZS isolator are also defined. It has found that the displacement transmissibility can be unbounded unless the damping is greater than a threshold value. But the same thing did not happen to the force transmissibility. Except the harmonic excitation, the shock performance\cite{11} and the local and global bifurcations\cite{12} were also studied. It is found that when the shock amplitude is small, the nonlinearity is beneficial, and the QZS isolator has the best shock performance. Although the advantages of QZS isolators have been proven, the disadvantages have also been found. QZS isolators are sensitive to the static load. However, although the perfect performance was destroyed, QZS isolators had a dramatically better performance than the deduced linear isolators\cite{13}. Another disadvantage is that the displacement can be unbounded. Although the displacement can be avoided under a big damping\cite{10}, the transmissibility will be worsen in the working region. Fortunately, this can be overcome via a nonlinear damping\cite{14}. The nonlinear damping can restrain the resonance peak while keeps the same transmissibility in the high-frequency region. Now, the mechanism of T-shape QZS isolators with oblique coil springs can be applied to engineering maturely\cite{15}.

The T-shape QZS isolators with oblique coil springs usually have two oblique springs. But, the two oblique springs are not strong enough for engineering applications. Xu et al.\cite{16} proposed a QZS platform with four same oblique springs circumferentially installed, which could enhance the performance of the QZS isolation to 0.5 Hz. The oblique springs opposite to each other must be the same, but this does not mean that the oppositely arranged oblique springs in other angles must be the same. Zhao et al.\cite{17-18} proposed a limb-like QZS isolator with two different pairs of oblique springs. Compared with the corresponding QZS isolator with just one pair of oblique springs, the QZS isolator with two pairs of oblique springs could achieve a lower dynamic stiffness in a much wider region around the static equilibrium position. Zhao et al.\cite{19} studied the performance of a QZS isolator with three pairs of oblique springs, and achieved a nearly horizontal straight line near the equilibrium position which could further widen the QZS region. The above-mentioned QZS isolators all have only a single layer. Lu et al.\cite{20-21} studied a two-layer QZS isolator, and obtained that the nonlinearity in the upper stage had very little influence on the isolation performance while the nonlinearity in the lower stage could significantly improve the effectiveness of the isolation system. Besides, this type of QZS isolators preferred
a high damping in the upper stage and a very low damping in the lower stage. Wang et al.\cite{22} compared a single layer QZS isolator with a two-layer one, and concluded that the two-layer isolator could achieve a better isolation performance in the higher isolation frequency band. Deng et al.\cite{23} proposed a more complex QZS isolator with multi-layers of oblique springs, and utilized the multi-layer structure of the neck’s cervical spine. They found that more layers could further enhance the isolation performance, e.g., lower working frequency, lower resonance peak, and weaker vibration in the high-frequency region. However, the stability of the structure was a risk in engineering. Except the isolation in the vertical direction, QZS is also used to isolate the horizontal excitation. Xu and Sun\cite{24} discussed a multi-direction isolator with QZS and time-delayed active control. Zhu et al.\cite{25} designed a QZS prototype for the aseismic protection with two pairs of oblique springs in the orthogonal and horizontal directions.

Stressed beams can provide negative stiffness in the orthogonal direction to their axes, and thus T-shape QZS isolators with stressed beams have also been investigated. As early as 1959, Tobias\cite{26} has proposed a low frequency isolator with the SiN beam or buckled leaf spring. However, he just showed the static analysis. In 1991, Platus\cite{27} designed a 6 degrees-of-freedom (DOF) prototype for the protection of micro-electronics, and the QZS isolator had an isolation frequency below 0.2 Hz. At the same time, Woodard and Housner\cite{3} combined the cable suspension for flight mode tests with the stressed beam QZS. They found that although the system was very close to the free state, it was sensitive to imperfections. Latterly, Liu et al.\cite{28} investigated the parameters of this kind of QZS isolators in detail, and provided some useful guidelines for choosing the system parameters such as the properties of the beams and the stiffness relationship between the beams and the linear spring. Except the force transmissibility and the displacement transmissibility, the power flow was also used to assess the performance of the stressed beam QZS isolator\cite{29}.

T-shape QZS isolators with magnetic springs are also studied, owing to that magnetic force is also a kind of restoring force. Xu et al.\cite{30} proposed a nonlinear magnetic low-frequency vibration isolator with the character of QZS, and shared a tuning technique for adapting to the change of loading mass. Jiang et al.\cite{31} investigated an electromagnetic-air hybrid QZS isolator, and showed that the negative stiffness could be adjusted easily, owing to the electromagnets.

There are T-shape QZS isolators with other negative stiffness. Lan et al.\cite{32} used the planar spring instead of the coil spring to realize a QZS isolator. The planar spring was built by a string of cell structures. The advantage of the planar spring to the coil spring is that it makes the assembled isolator very compact. To further lower the beginning frequency, Wang et al.\cite{33} proposed a double QZS isolator, in which each oblique spring was replaced by a T-shape QZS mechanism. They concluded that the double QZS isolator had a better isolation performance than the single T-shape one. However, if the T-shape QZS isolator needs a heavy loading capacity, the main spring must be very strong. This brings a disadvantage that the oblique spring should be very strong accordingly to eliminate the linear stiffness of the main spring. As a result, the nonlinearity of the system is strong. It can induce strong nonlinear behaviors such as jumping and bifurcation. Besides, it is easy to lose its stability under this situation\cite{34}.

Since T-shape QZS isolators work in weak response regions, X-shape QZS isolators are designed, which are also named as scissor-like QZS isolators. X-shape QZS isolators can be classified according to the structure layer.

Among the existing studies on X-shape QZS isolators, single layer X-shape QZS isolator is not the mainstream. However, it has the simplest structure. Zhang and Zhao\cite{35} investigated a typology. It had six assembly types of linear springs, and could cover most of X-shape QZS isolators and realize QZS solely or conjunctively. A rhombic structure isolator can be treated as the variation of an X-shape QZS isolator. Cheng et al.\cite{36} found that the nonlinear damping could eliminate the unbounded response of the displacement transmissibility. Cheng et al.\cite{37} considered the load mismatch, and found that the QZS isolator still outperformed the equivalent linear isolator. Wang et al.\cite{38} connected the upper platform of the isolator with an inerter to
enhance the performance. Based on the single layer X-shape QZS isolator, multi-layer X-shape QZS isolators are studied, including multi-layer symmetrical X-shape QZS isolators and multi-layer unsymmetrical X-shape QZS isolators.

If the vertical displacement caused by a unit angle of an X arm is marked as $y$, an $n$-layer symmetrical X-shape isolator has a displacement of $ny$ for the loading platform at the top of the structure. Thus, the equivalent mass of the load will be $n^2$ times of the single layer. As the result, the isolator will provide a lower beginning isolation frequency. The same mechanism can also be found in the lever-type isolator. Based on this opinion, Liu et al. used a lever to amplify the inertia of the top platform and the loading mass on a 3-layer X-shape QZS isolator, and verified that the hybrid isolator had an ultra-low isolation frequency. Liu et al. designed a more complex isolator with a linear isolator and a 3-layer X-shape QZS isolator, both of which had a lever mechanism, and achieved very beneficial nonlinear stiffness and damping properties. Sun and Jing investigated a 3-layers X shape isolator theoretically, considering these parameters and the planar motion. Jing et al. discussed the critical design parameters of this kind of isolator in detail. Wu et al. explored this structure to a 6-DOF isolation platform. Feng and Jing investigated the protection of human body. Except the vertical protection, X-shape QZS isolators can also be used to isolate the excitation on the horizontal plane. Besides, oblique springs also can be used in the stiffness design of X-shape QZS isolators. Owing to the better performance than single layer X-shape isolator, multi-layer X-shape isolators have been used in vehicle seats. The multi-layer symmetrical X-shape can be varied by dividing the X-shape along the main vertical axis with the layer shape of < or >. It can achieve a good performance, but need a vertical guide rail. Since half of the X-shape is similar to the skeleton of animal’s leg, this kind of isolator has a name of bio-inspired structure. Dai et al. employed this inspiration on the application to on-orbit capture. Wang et al. proposed an on-orbit capture structure based on the same idea.

Except the symmetrical structure, researchers also paid their attention to the unsymmetrical structure. Wu et al. demonstrated an X-shape isolator with unequal length arms. Hu and Jing designed a 6-DOF Stewart platform with unsymmetrical X-shape legs, which could achieve high-static-low-dynamic-stiffness isolation in all the six directions. Wang et al. compared the symmetrical X-shape with the unsymmetrical one, and found that the asymmetric structure could produce lower natural frequency by properly adjusting the rod-length ratio.

Since the X-shape has many design parameters, it is now a hot topic for QZS isolators, and this provides more choices for the nonlinearity design.

The CRSM can also provide negative stiffness. Hence, it can be used to realize QZS while combining with the positive stiffness supporting spring. The conceptual design of QZS isolators with the CRSM firstly appeared in a monograph in 1989. However, the detailed analysis and experiment have not been carried out until Zhou and Wang shared their work in 2015. According to their conclusions, this type of isolator had a particular behavior compared with the traditional ones, which meant that the peak transmissibility and starting frequency of isolation never overshot those of the linear counterpart, no matter how large the excitation amplitude was. To enhance the isolation performance, Wang et al. provided a 2-layer CRSM vibration isolator. Zhou et al. proposed a 6-DOF QZS platform with 3 CRSM isolators at each edge point of the bearing surface. Instead of the coil spring used in the CRSM, they used a leaf spring to make the CRSM leg more compact. Zhou et al. used the CRSM to design a torsional isolation for rotor systems, and discussed the sensibility of the parameter errors. In view of the complex of the nonlinearity produced by the cam-roller, Sun et al. used the quadratic polynomial trajectory in the CRSM, which could keep the performance on the vibration isolation while simplify the manufacture and the calculation. If the surface of the cam-roller has a ball shape, it can achieve negative stiffness in all directions. Based on this opinion, Zhang et al. proposed a torsion-translational vibration isolator with the
CRSM. Usually, the CRSM structure is symmetric. In this way, the direction of the motion is along the symmetric axis. By eliminating one side of the CRSM, Ye et al.\cite{68} also realized a unique QZS isolator for translational and rotational vibrations. The CRSM has a unique benefit that the track of the roller can be designed arbitrarily. This means that the restoring force at any position could be designed according to the demand, unlike the restoring force of other QZS isolators governed by the trigonometric function or the interaction of the magnet. Yao et al.\cite{69–70} designed a variation of the CRSM by replacing the traditional cam by a well-designed track. It helps the CRSM isolator achieve a big motion region, and thus makes the restoring force discontinuous\cite{71}.

The intrinsic nonlinearity can also be applied to achieve negative stiffness. One typical example is the magnet.

Robertson et al.\cite{72} used the cube magnet to design a levitation system to achieve negative stiffness, and proposed four design criteria based on the magnet size and the gap between the fixed magnets. Wu et al.\cite{73} designed a novel magnetic spring with a negative stiffness (MS-NS) isolator. Both of the theoretical analysis and the experiment verified that the MS-NS could realize low-frequency vibration isolation without increasing the static deflection. Zheng et al.\cite{74} used a pair of coaxial ring permanent magnets to provide negative stiffness, and proposed the designing procedure for this kind of magnetic spring. Latterly, Dong et al.\cite{75–76} developed a negative magnetic stiffness isolator with a spatial pendulum, but the obtained dynamic coupling was so weak that the responses in different directions did not interact with each other. Zheng et al.\cite{77} proposed a torsion magnetic spring with negative stiffness for rotor systems. Zhou et al.\cite{78} and Wang et al.\cite{79} used magnetic springs to realize the QZS isolation of the neonatal transport, by which the random disturbance caused by the floor could be eliminated. Since magnets can realize magnetic levitation, magnet-based QZS isolators can achieve the isolation performance in all freedom directions of the loading mass\cite{80–81}, including the motion in the horizontal plane\cite{82}. By replacing the spring of a 6-DOF Stewart platform by the magnetic QZS, Zheng et al.\cite{83} combined the intrinsic nonlinearity and the geometrical nonlinearity, by which the resonance frequencies of the Stewart isolator were reduced and the isolation performance in all six directions was improved. However, the above works all use permanent magnets. Since electro-magnets can also achieve negative stiffness, QZS isolators based on electro-magnets can adjust the stiffness according to different loading mass. In this way, there comes the semi-active QZS isolator\cite{84–85}.

Excepte the above structures, there are some other kinds of QZS isolators. Araki et al.\cite{86} used super-elastic Cu-Al-Mn shape memory alloy (SMA) bars to achieve large loading capacity as well as large stroke length while keeping the QZS mechanism simple and compact. Disk spring can also produce negative stiffness since it has two potential energy wells\cite{87}. Zhao et al.\cite{88} combined the V-shape lever, the plate spring, and the cross-shaped structure together to achieve a special QZS isolator. Yan et al.\cite{89} developed the QZS isolator for bistable region, and provided the guidelines for the design, analysis, and optimization. The origami is also a distinct mechanism for the QZS isolator\cite{90–92}.

Now, QZS isolators have been successfully used in floating rafts of engines\cite{93}, one-dimensional structures\cite{94–96}, and bridges\cite{97}. It is still a hot topic for passive isolation. Based on the survey above, one can obtain the following design rules for QZS isolators.

(i) An elastic component with two potential energy wells provides negative stiffness around the local position of the maximum potential energy. Combining with the supporting spring with positive stiffness, QZS or high-static-low-dynamic mechanism can be achieved.

(ii) Multiple layers of QZS can enhance the transmissibility performance. The vibration energy will be rejected and reflected multiple times during the transfer path.

(iii) The increase in the inertia of the loading mass can lower the beginning isolation frequency of the isolator. It can be achieved by amplifying the displacement of the loading platform with special mechanical structures.
Strong nonlinear damping is beneficial to the reduction in the transmissibility peak while keeps the transmissibility in the high-frequency region the same.

QZS isolators are sensitive to the design error or assemble error. However, it still has a isolation performance better than the corresponding linear isolator.

During the design of QZS isolators, one should avoid the negative error which may make the system become bistable.

In this paper, by means of the simple mechanism of oblique springs, the strong loading capacity and the inertia amplification effect of the multi-layer X-shape structure, and the arbitrary design of the CRSM, a novel X-shape QZS-roller (X-QZS-R) isolator is designed. First, the governing equation is established. Second, the equation is discussed with the harmonic balance method (HBM) and verified by simulation. Finally, the parameters are discussed in detail, and the performances of the designed isolator with such parameters are compared with the traditional QZS isolators degenerated from it.

2 Structural design and parameter design

2.1 Specification of the structural design

Figure 1 presents the model diagram of the X-QZS-R isolator. The loading platform is supported to the base through the X-shape rods with a pair of arms with the length \( l_r \) and a pair of arms with the length \( 2l_r \). The two bottom rods are symmetrically connected to the linear spring \( k_1 \) and the damping \( c \) to support the whole device. The pallet can vibrate on the vertical guide rod, while the angle between the X-shape rod and the horizontal direction changes accordingly, as plotted in Fig. 1(a). A pair of oblique springs are symmetrically fixed on the vertical guide rod through a fixed bracket. A concave plate is fixed on the pallet. Two rollers are installed in the front end of the spring \( k_2 \) to contact with the concave plate, as depicted in Fig. 1(b). The shape of the concave plate can be designed to the ideal purpose, which can help the X-shape structure to eliminate the gravity of the main mass all the time at any position. A simplification claimed here is that the mass of these accessories are neglected so as to simplify the modeling process.

![Diagram of the X-QZS-R isolator model](color online)

When the mass \( m \) is put on the platform, the spring marked as \( k_1 \) is elongated \( \delta_x \) in the horizontal rail direction to maintain the static balance of the whole system. \( \alpha \) is the angle between the X-shape bottom rod and the horizontal direction.

In static balance, the spring \( k_2 \) has the original length without being squeezed. When the base is vertically excited with the amplitude \( z \), the displacement \( u \) of the mass \( m \) with the amplitude \( y \) relative to the base is expressed as \( u = y - z \).
To form QZS, the negative stiffness, yielded by two pairs of symmetrically squeezed springs \( k_3 \) and X-shape structures, is cancelled out by the positive stiffness generated by a pair of rollers and concave plates. Meanwhile, \( k_2 \) is compressed by \( w \), and the angular displacement of the X-shape rod is \( \theta \).

### 2.2 Specification of the parameter design

Assume that the system remains stationary. Then, analyze the force balance at the point where the X-shape rod is connected to the base (see Fig. 2). In Fig. 2, \( f_{y1} \) is the load in the vertical direction, \( f_x \) is the resultant force in the horizontal direction, and \( f_{axle} \) is the force along the rod direction.

![Fig. 2 Schematic diagram of the force on the end point of the bottom rod of the X-shape isolator](image)

Based on Figs. 1 and 2, the following equations are obtained:

\[
f_{y1} = f_{axle} \sin(\alpha + \theta), \quad f_x = f_{axle} \cos(\alpha + \theta).
\]

If the displacement of the mass at the top is marked as \( u \), the vertical displacement of the cross-point near the bottom is \( u/3 \). Meanwhile, the length of the rod is always \( l_r \). Based on this geometrical relationship, Eq. (1) becomes

\[
f_{y1} = \frac{f_x}{\cot(\alpha + \theta)} = \frac{l_r \sin \alpha + u/3}{\sqrt{l_r^2 - (l_r \sin \alpha + u/3)^2}} f_x.
\]

The static equilibrium compression \( \delta_x \) of the spring \( k_1 \) is

\[
\delta_x = \frac{mg \cot \alpha}{2k_1}.
\]

Correspondingly, the static equilibrium displacement \( \delta_y \) in the vertical direction is

\[
\delta_y = 3(\sqrt{l_r^2 - (l_r \cos \alpha - \delta_x)^2} - l_r \sin \alpha).
\]

When the mass is affected by the external force \( P_1 \), according to the force balance, the following equations can be obtained:

\[
f_x = k_1(-\delta_x + x) + 2k_h \left( 1 - \frac{l_{s0}}{\sqrt{x^2 + l_s^2}} \right) x, \quad 2f_{y1} = P_1 - mg,
\]

where \( l_s \) and \( l_{s0} \) are the set length and the original length of the squeezed spring \( k_h \), respectively, and \( x \) is the lateral displacement. The relationship between \( x \) and \( u \) is

\[
x = l_r \cos \alpha - \frac{1}{3} \sqrt{9l_r^2(\cos \alpha)^2 - 6l_r u \sin \alpha - u^2}.
\]

Therefore, the expression of \( P_1 \) can be obtained as follows:

\[
P_1 = mg + \frac{2(l_r \sin \alpha + 1/3u)}{\sqrt{l_r^2 - (l_r \sin \alpha + 1/3u)^2}} \left( k_1 \left( -\frac{1}{2} \frac{mg \cot \alpha}{k_1} + l_r \cos \alpha - \sqrt{l_r^2 - (l_r \sin \alpha + 1/3u)^2} \right) \right.
\]

\[
+ 2k_h(l_r \cos \alpha - \sqrt{l_r^2 - (l_r \sin \alpha + 1/3u)^2}) \cdot \left( 1 - \frac{l_{s0}}{\sqrt{(l_r \cos \alpha - \sqrt{l_r^2 - (l_r \sin \alpha + 1/3u)^2})^2 + l_s^2}} \right).
\]
Now, it is clear that the gravity of the mass is eliminated just when \( u = 0 \) based on Eq. (7), which means that an especially force should be designed to help the X-QZS to balance the gravity. This force is marked as

\[
Q = mg(1 - \cot \alpha \cdot \tan(\alpha + \theta)).
\]  

(8)

At this step, the vertical restoring force for the mass is

\[
P = P_1 + Q.
\]  

(9)

To realize the auxiliary force, a track-roller structure is attached to the X-QZS. It is the X-QZS-R isolator, in which R means the roller. In Fig. 1, it is marked as the concave plate. The trajectory of the track, which can provide the auxiliary force, is given in Fig. 3. Now, assuming that the expression of the roller trajectory can be expressed as a polynomial as follows:

\[
w(u) = \begin{cases} 
\sum_{i=1}^{n} s_i u^i, & u \geq 0, \\
-\sum_{i=1}^{n} s_i u^i, & u < 0,
\end{cases}
\]  

(10)

where \( n \) and \( s_i \) are positive integers.

Suppose that the resultant force in the horizontal direction is \( f_w \), the resultant force in the vertical direction is \( f_{y2} \), the resultant force \( f_n \) acting on the roller is along the track normal, and the angle between \( f_n \) and the horizontal direction is \( \beta \). Then, one can obtain

\[
f_{y2} = f_w \tan \beta, \quad \tan \beta = w'(u).
\]  

(11)

The roller structure is under the action of the external force \( Q \). \( f_w \) and \( f_{y2} \) are

\[
f_w = k_2 w, \quad 2f_{y2} = Q.
\]  

(12)

Then, the following equation is obtained:

\[
Q = 2k_2 w'w = 2k_2(\sum ns_n u^{n-1})(\sum s_n u^n) = 2k_2(\sum r_n u^n + o(u^{n+1})),
\]  

(13)

where \( o(u^{n+1}) \) is the high-order terms, and can be ignored. Carrying out the \( n \)th-order Taylor expansion of \( Q \) at the zero point based on Eq. (8), the following expression can be obtained:

\[
Q = \sum Q^{(n)}_{n!} u^n + R_n(u),
\]  

(14)
where $Q^{(n)}$ is the $n$th-order differential of $Q$. Therefore, the trajectory $r_n$ in Eq. (13) can be solved out, which means that the trajectory is designed.

Define the ratio of the static stiffness of the vibration isolation structure to the linear stiffness $2k_2$ as $K$ by

$$K = \frac{1}{2k_1} \frac{\partial P}{\partial u}.$$  \hfill (15)

The parameters in Table 1 are in the same order of magnitude as those in Ref. [64]. To obtain a wide QZS region, the potential energy is integrated based on Eq. (9) to $u$. The potential energy curve around $u = 0$ is flat, and the corresponding stiffness must be very weak. Hence, the set length of the squeezed spring together with its stiffness can be determined. Figure 4 shows the curvature of the potential energy when $u = 0$. The system parameters of the X-shape isolator are given in Table 2. In theory, the set length of the squeezed spring is 0.0997 m. However, it produces a non-zero static equilibrium position. As the result, the set length in the present work is 0.1 m, which has the error of 0.3% to the theoretical value.

**Table 1**  System parameters of the X-shape isolator

| Parameter                      | Notation | Value | Unit   |
|-------------------------------|----------|-------|--------|
| Mass                          | $m$      | 4     | kg     |
| Load-bearing spring stiffness  | $k_1$    | 2000  | N·m$^{-1}$ |
| Original length of the squeezed spring | $l_{s0}$ | 0.12  | m      |
| Length of the short rod       | $l$      | 0.2   | m      |
| Angle of the long rod         | $\alpha$| $\pi/4$ | rad   |

**Fig. 4**  Curvature of the potential energy around $u = 0$ (color online)

**Table 2**  Set system parameters of the X-shape isolator

| Parameter                      | Notation | Value | Unit   |
|-------------------------------|----------|-------|--------|
| Mass                          | $m$      | 4     | kg     |
| Squeezed spring stiffness     | $k_h$    | 3550  | N·m$^{-1}$ |
| Length of the squeezed spring | $l_s$    | 0.10  | m      |

At this step, the structure parameters are all determined. If there is no special description in this section, the parameters will remain the same during the whole investigation. Figure 5 shows the potential energy changing with $k_h$ based on the parameters listed in Table 2. It can be seen that with the increase in the squeezed spring stiffness, a new equilibrium position occurs. The setting equilibrium position, when $u = 0$, becomes an unstable position. This is forbidden for the X-QZS-R isolator. On contrast, the small squeezed spring stiffness brings a steep potential well. This is also negative for the X-QZS-R isolator. When $k_h = 3550$ N·m$^{-1}$, the potential well has the flattest bottom.
If the basement excitation is marked as $z$, the absolute vertical displacement $y$ equals $u + z$. According to Newton’s second law, the following governing equation can be obtained:

$$
m\left(\frac{d^2 u}{dt^2} + \frac{d^2 z}{dt^2}\right) + k_1 \sum_{i=1}^{5} s_k u^i + k_h \sum_{i=1}^{5} s_h u^i + 2k_2 \left(\sum_{i=1}^{n} s_i u^i \sum_{i=1}^{n} i s_i u^{i-1}\right) + mg \sum_{i=1}^{5} s_G u^i = 0. \quad (16)$$

Meanwhile, considering a parallel damper $c$ to $k_1$, the governing equation is

$$
m\left(\frac{d^2 u}{dt^2} + \frac{d^2 z}{dt^2}\right) + k_1 \sum_{i=1}^{5} s_k u^i + c \sum_{i=1}^{4} s_c u^i + k_h \sum_{i=1}^{5} s_h u^i + 2k_2 \left(\sum_{i=1}^{n} s_i u^i \sum_{i=1}^{n} i s_i u^{i-1}\right) + mg \sum_{i=1}^{5} s_G u^i = 0. \quad (17)$$

Based on this equation, the response can be discussed analytically.

### 3 Processing of the analytical solution and simulation

Assume that the solution $u$ to Eq. (17) satisfies

$$u(t) = a_0(t_0) + \sum_{i=1}^{n} a_i(t_0) \sin(i\omega t) + \sum_{i=1}^{n} b_i(t_0) \cos(i\omega t), \quad (18)$$

where $i$ is the harmonic order, and $t_0$ stands for a slow time. Based on this assumption, the stability of the analytical solution can be determined\(^{98-99}\). Therefore, the velocity response and the acceleration response of the mass $m$ are, respectively, deduced as follows:

$$\begin{align*}
\frac{du(t)}{dt} &= D_1 a_0 + \sum_{i=1}^{n} (D_1 a_i - i\omega b_i) \sin(i\omega t) + \sum_{i=1}^{n} (D_1 b_i + i\omega a_i) \cos(i\omega t), \\
\frac{d^2 u(t)}{dt^2} &= D_2 a_0 + \sum_{i=1}^{n} (D_2 a_i - 2i\omega D_1 b_i + \omega^2 a_i) \sin(i\omega t) \\
&\quad + \sum_{i=1}^{n} (D_2 b_i + 2i\omega D_1 a_i + \omega^2 b_i) \cos(i\omega t),
\end{align*} \quad (19)$$

Fig. 5  Potential energy changing with the squeezed spring stiffness (color online)
where $D_1$ and $D_2$ are the first-order and second-order derivations to the slow time, respectively. According to the HBM, the following equations can be obtained:

\[
\begin{cases}
D_2 a_0 = H_0(V), \\
D_2 a_i - 2i\omega D_1 b_i + \omega^2 a_i = H_{s,i}(V), \\
D_2 b_i + 2i\omega D_1 a_i + \omega^2 b_i = H_{c,i}(V),
\end{cases}
\]

\begin{equation}
V = \left( a_0, a, b, \frac{da_0}{dt}, D_1 a, D_1 b \right),
\end{equation}

\begin{equation}
a = (a_1, a_2, \cdots, a_n)^T, \quad D_1 a = (D_1 a_1, D_1 a_2, \cdots, D_1 a_n)^T,
\end{equation}

\begin{equation}
b = (b_1, b_2, \cdots, b_n)^T, \quad D_1 b = (D_1 b_1, D_1 b_2, \cdots, D_1 b_n)^T.
\end{equation}

When those derivations to the slow time are 0, the response is steady, which means that those harmonic coefficients are constants. After all those coefficients are solved out, the mass response can be superposed by the corresponding harmonics, including both of the relative displacement $u$ and the absolute displacement $y$.

Define the transmissibility $T$ of the base displacement and the peak transmissibility $T_p$ by

\begin{equation}
T = 20\log_{10} \frac{|y_{\text{max}} - y_{\text{min}}|}{2\zeta_0}.
\end{equation}

Then, the isolation efficiency can be quantized.

To verify the analytic results, the Runge-Kutta method is used to numerically solve Eq. (17).

Figure 6 shows the convergence of the HBM. The results show that the 2nd-order HBM loses its accuracy at the tops of these curves, while the 3rd-order HBM is the best accurate. Considering the calculation cost, the 3rd-order HBM is a proper choice.

![Fig. 6 Convergence of different orders of the HBM (color online)](image)
Figure 7 shows the accuracy verification of the 3rd-order HBM with the simulation. It can be seen that the response has a mixed softening and hardening characteristic. This is caused by the residual static load as the set length of the squeezed spring is not set strictly. The same phenomenon can also be found in Ref. [100]. Figure 7(a) shows the harmonics, while Fig. 7(b) shows the total responses. These figures indicate that the analytical method has a good agreement with the simulation. The stability can be determined via the processing proposed by Luo and Huang [98, 101]. In Fig. 7(b), the stable analytical response is demonstrated by the solid line, and the unstable result is given by the dashed line. The turning points of these curves, which are directed by arrows, are Hopf bifurcation points.

\[
\Omega / \sqrt{k_1} = \frac{2 \left( l_r \sin \alpha + \frac{1}{3} u \right)}{l^2 - \left( l_r \sin \alpha + \frac{1}{3} u \right)^2} \times \left( \frac{-mg \cot \alpha}{2k_1} + l_r \cos \alpha - \sqrt{l_r^2 - \left( l_r \sin \alpha + \frac{1}{3} u \right)^2} \right). \tag{23}
\]

As the result, the governing equation can be degenerated from Eq. (18), which is marked as X-LS. L means linear, and corresponds to QZS as the X-LS just contains the linear spring. Meanwhile, an X-QZS isolator without the track roller is degenerated from Eq. (18) by eliminating \( P_2 \) from it. Under the same basement excitation, the transmissibilities of these three isolators are compared in Fig. 8. The beginning isolation frequency of the X-QZS-R isolator is 0.45 Hz with the maximum displacement transmissibility of 27.21 dB. The X-QZS has the beginning frequency of 1.6 Hz and the maximum transmissibility of 53.84 dB. The X-LS has the beginning frequency of 3.8 Hz and the maximum transmissibility of 69.81 dB. Obviously, the proposed isolator has the best performance with the smallest ineffective bandwidth and the smallest resonance peak.

4 Discussion on the advantage and efficiency

In this section, the efficiency of the proposed X-QZS-R isolator will be discussed based on the HBM.

The traditional X-shape isolator just consists of the restoring force produced by \( k_1 \), which means

\[
P_1 = mg + \frac{2 \left( l_r \sin \alpha + \frac{1}{3} u \right)}{\sqrt{l_r^2 - \left( l_r \sin \alpha + \frac{1}{3} u \right)^2}} k_1 \left( \frac{-mg \cot \alpha}{2k_1} + l_r \cos \alpha - \sqrt{l_r^2 - \left( l_r \sin \alpha + \frac{1}{3} u \right)^2} \right). \tag{23}
\]
5 Conclusions

The existing QZS isolators are investigated, and their advantages and disadvantages are summarized. Based on these, a novel QZS isolator centered on the X-shape structure is designed together with oblique springs and the CRSM with the HBM. The simulation verifies the accuracy of the analytical solution. Based on the discussion above, the following conclusions are obtained.

(i) Compared with degraded isolators, the proposed isolator has a better isolation performance. It has a low beginning isolation frequency and a small transmissibility peak.

(ii) A strong damping is beneficial for the isolator performance. It reduces the peak transmissibility, and avoids strong nonlinear responses. The transmissibility in the high-frequency region is worsen because of the linear part caused by the design error, and thus a strong damping should be chosen.

(iii) An oblique spring should achieve a high density of potential energy, which will produce a strong negative stiffness and a wide QZS region by increasing the stiffness or the compression ratio of the oblique spring.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. To view a copy of this licence, visit http://creativecommons.org/licenses/by/4.0/.

References

[1] CARRELLA, A., BRENNAN, M. J., and WATERS, T. P. Static analysis of a passive vibration isolator with quasi-zero-stiffness characteristic. *Journal of Sound and Vibration*, 301, 678–689 (2007)

[2] IKEGAMI, R. E. A. Zero-G ground test simulation methods. *Proceedings of the 11th Aerospace Testing Seminar*, Institute of Environmental Science, Manhattan Beach (1988)

[3] WOODARD, S. E. and HOUSNER, J. M. The nonlinear behavior of a passive zero-spring-rate suspension system. *29th Structures, Structural Dynamics and Materials Conference*, Reston (1998)

[4] LACOSTE, L. LaCoste and Romberg straight-line gravity meter. *Geophysics*, 48, 606–610 (1983)

[5] IBRAHIM, R. A. Recent advances in nonlinear passive vibration isolators. *Journal of Sound and Vibration*, 314, 371–452 (2008)

[6] GUO, L. C., WANG, X., FAN, R. L., and BI, F. R. Review on development of high-static-low-dynamic-stiffness seat cushion mattress for vibration control of seating suspension system. *Applied Sciences*, 10, 2887 (2020)
Recent advances in quasi-zero-stiffness vibration isolation systems. *Applied Mechanics and Materials*, 397-400, 295–303 (2013)

Zhang, J. Z., Li, D., Chen, M. J., and Dong, S. An ultra-low frequency parallel connection nonlinear isolator for precision instruments. *Key Engineering Materials*, 257-258, 231–236 (2004)

Carrella, A., Brennan, M. J., Kovacic, I., and Waters, T. P. On the force transmissibility of a vibration isolator with quasi-zero-stiffness. *Journal of Sound and Vibration*, 322, 707–717 (2009)

Carrella, A., Brennan, M. J., Waters, T. P., and Lopes, V., Jr. Force and displacement transmissibility of a nonlinear isolator with high-static-low-dynamic-stiffness. *International Journal of Mechanical Sciences*, 55, 22–29 (2012)

Tang, B. and Brennan, M. J. On the shock performance of a nonlinear vibration isolator with high-static-low-dynamic-stiffness. *International Journal of Mechanical Sciences*, 81, 207–214 (2014)

Hao, Z. F., Cao, Q. J., and Wiercigroch, M. Nonlinear dynamics of the quasi-zero-stiffness SD oscillator based upon the local and global bifurcation analyses. *Nonlinear Dynamics*, 87, 987–1014 (2017)

Abolfathi, A., Brennan, M. J., Waters, T. P., and Tang, B. On the effects of mistuning a force-excited system containing a quasi-zero-stiffness vibration isolator. *Journal of Vibration and Acoustics*, 137, 044502 (2015)

Peng, Z. K., Lang, Z. Q., Zhao, L., Billings, S. A., Tomlinson, G. R., and Guo, P. The force transmissibility of MDOF structures with a non-linear viscous damping device. *International Journal of Non-Linear Mechanics*, 46, 1305–1314 (2011)

Wang, Y., Li, S. M., Cheng, C., and Su, Y. Q. Adaptive control of a vehicle-seat-human coupled model using quasi-zero-stiffness vibration isolator as seat suspension. *Journal of Mechanical Science and Technology*, 32, 2973–2985 (2018)

Xu, D. L., Zhang, Y. Y., Zhou, J. X., and Lou, J. J. On the analytical and experimental assessment of the performance of a quasi-zero-stiffness isolator. *Journal of Vibration and Control*, 20, 2314–2325 (2014)

Zhao, F., Ji, J. C., Ye, K., and Luo, Q. T. Increase of quasi-zero stiffness region using two pairs of oblique springs. *Mechanical Systems and Signal Processing*, 144, 106975 (2020)

Zhao, F., Ji, J. C., Luo, Q. T., Cao, S. Q., Chen, L. M., and Du, W. L. An improved quasi-zero stiffness isolator with two pairs of oblique springs to increase isolation frequency band. *Nonlinear Dynamics*, 104, 349–365 (2021)

Zhao, F., Ji, J. C., Ye, K., and Luo, Q. T. An innovative quasi-zero stiffness isolator with three pairs of oblique springs. *International Journal of Mechanical Sciences*, 192, 106093 (2021)

Lu, Z. Q., Brennan, M. J., Yang, T. J., Li, X. H., and Liu, Z. G. An investigation of a two-stage nonlinear vibration isolation system. *Journal of Sound and Vibration*, 332, 1456–1464 (2013)

Lu, Z. Q., Yang, T. J., Brennan, M. J., Li, X. H., and Liu, Z. G. On the performance of a two-stage vibration isolation system which has geometrically nonlinear stiffness. *Journal of Vibration Acoustics*, 136, 064501 (2014)

Wang, Y., Li, S. M., Neilid, S. A., and Jiang, J. Z. Comparison of the dynamic performance of nonlinear one and two degree-of-freedom vibration isolators with quasi-zero stiffness. *Nonlinear Dynamics*, 88, 635–654 (2017)

Deng, T. C., Weng, G. L., Ding, H., Lu, Z. Q., and Chen, L. Q. A bio-inspired isolator based on characteristics of quasi-zero stiffness and bird multi-layer neck. *Mechanical Systems Signal Processing*, 145, 106967 (2020)

Xu, J. and Sun, X. T. A multi-directional vibration isolator based on quasi-zero-stiffness structure and time-delayed active control. *International Journal of Mechanical Sciences*, 100, 126–135 (2015)
[25] ZHU, G. N., LIU, J., CAO, Q. J., CHENG, Y. F., LU, Z. C., and ZHU, Z. B. A two degree of freedom stable quasi-zero stiffness prototype and its applications in aseismic engineering. *SCIENCE CHINA Technological Sciences*, **63**, 496–505 (2020)

[26] TOBIAS, S. A. Design of small isolator units for the suppression of low frequency vibration. *Journal of Mechanical Engineering Science*, **26**, 280–292 (1959)

[27] PLATUS, D. L. Negative-stiffness-mechanism vibration isolation systems. *Optics and Metrology*, **1619**, 44–54 (1991)

[28] LIU, X. T., HUANG, X. C., and HUA, H. X. On the characteristics of a quasi-zero stiffness isolator using Euler buckled beam as negative stiffness corrector. *Journal of Sound and Vibration*, **332**, 3359–3376 (2013)

[29] YANG, J., XIONG, Y. P., and XING, J. T. Dynamics and power flow behaviour of a nonlinear vibration isolation system with a negative stiffness mechanism. *Journal of Sound and Vibration*, **332**, 167–183 (2013)

[30] XU, D. L., YU, Q. P., ZHOU, J. X., and BISHOP, S. R. Theoretical and experimental analyses of a nonlinear magnetic vibration isolator with quasi-zero-stiffness characteristic. *Journal of Sound and Vibration*, **332**, 3377–3389 (2013)

[31] JIANG, Y. L., SONG, C. S., DING, C. M., and XU, B. H. Design of magnetic-air hybrid quasi-zero stiffness vibration isolation system. *Journal of Sound and Vibration*, **477**, 115346 (2020)

[32] LAN, C. C., YANG, S. A., and WU, Y. S. Design and experiment of a compact quasi-zero-stiffness isolator capable of a wide range of loads. *Journal of Sound and Vibration*, **333**, 4843–4858 (2014)

[33] WANG, K., ZHOU, J. X., CHANG, Y. P., OUYANG, H. J., XU, D. L., and YANG, Y. A nonlinear ultra-low-frequency vibration isolator with dual quasi-zero-stiffness mechanism. *Nonlinear Dynamics*, **101**, 755–773 (2020)

[34] AHN, H. J. Performance limit of a passive vertical isolator using a negative stiffness mechanism. *Journal of Mechanical Science and Technology*, **22**, 2357–2364 (2008)

[35] ZHANG, W. and ZHAO, J. B. Analysis on nonlinear stiffness and vibration isolation performance of clipper-like structure with full types. *Nonlinear Dynamics*, **86**, 17–36 (2016)

[36] CHENG, C., LI, S. M., WANG, Y., and JIANG, X. X. Force and displacement transmissibility of a quasi-zero stiffness vibration isolator with geometric nonlinear damping. *Nonlinear Dynamics*, **87**, 2267–2279 (2017)

[37] CHENG, C., LI, S. M., WANG, Y., and JIANG, X. X. Resonance of a quasi-zero stiffness vibration system under base excitation with load mismatch. *International Journal of Structural Stability and Dynamics*, **18**, 1850002 (2018)

[38] WANG, Y., LI, H. X., CHENG, C., DING, H., and CHEN, L. Q. Dynamic performance analysis of a mixed-connected inerter-based quasi-zero stiffness vibration isolator. *Structural Control & Health Monitoring*, **27**, e2604 (2020)

[39] SUN, X. T., JING, X. J., and CHENG, L. Vibration isolation via a scissor-like structured platform. *Journal of Sound and Vibration*, **333**, 2404–2420 (2014)

[40] YAN, B., WANG, Z. H., MA, H. Y., BAO, H. H., WANG, K., and WU, C. Y. A novel lever-type vibration isolator with eddy current damping. *Journal of Sound and Vibration*, **494**, 115862 (2021)

[41] LIU, C. C., JING, X. J., and LI, F. M. Vibration isolation using a hybrid lever-type isolation system with an X-shape supporting structure. *International Journal of Mechanical Sciences*, **98**, 169–177 (2015)

[42] LIU, C. C., JING, X. J., and CHEN, Z. B. Band stop vibration suppression using a passive X-shape structured lever-type isolation system. *Mechanical Systems and Signal Processing*, **68-69**, 342–353 (2016)

[43] SUN, X. T. and JING, X. J. Analysis and design of a nonlinear stiffness and damping system with a scissor-like structure. *Mechanical Systems and Signal Processing*, **66-67**, 723–742 (2016)

[44] SUN, X. T. and JING, X. J. A nonlinear vibration isolator achieving high-static-low-dynamic stiffness and tunable anti-resonance frequency band. *Mechanical Systems and Signal Processing*, **80**, 166–188 (2016)
[45] JING, X. J., ZHANG, L. L., JIANG, G. Q., FENG, X., GUO, Y. Q., and XU, Z. D. Critical factors in designing a class of X-shaped structures for vibration isolation. *Engineering Structures*, 199, 109659 (2019)

[46] WU, Z. J., JING, X. J., SUN, B., and LI, F. M. A 6DOF passive vibration isolator using X-shape supporting structures. *Journal of Sound and Vibration*, 380, 90–111 (2016)

[47] FENG, X. and JING, X. J. Human body inspired vibration isolation: beneficial nonlinear stiffness, nonlinear damping & nonlinear inertia. *Mechanical Systems and Signal Processing*, 117, 786–812 (2019)

[48] SUN, X. T. and JING, X. J. Multi-direction vibration isolation with quasi-zero stiffness by employing geometrical nonlinearity. *Mechanical Systems and Signal Processing*, 62-63, 149–163 (2015)

[49] BIAN, J. and JING, X. J. Analysis and design of a novel and compact X-structured vibration isolation mount (X-mount) with wider quasi-zero-stiffness range. *Nonlinear Dynamics*, 101, 2195–2222 (2020)

[50] GUO, L. C., KHIU, A., FAN, R. L., and WANG, X. Analysis of a passive scissor-like structure isolator with quasi-zero stiffness for a seating system vibration-isolation application. *International Journal of Vehicle Design*, 82, 224–240 (2020)

[51] YAN, G., ZOU, H. X., WANG, S., ZHAO, L. C., GAO, Q. H., TAN, T., and ZHAN, W. M. Large stroke quasi-zero stiffness vibration isolator using three-link mechanism. *Journal of Sound and Vibration*, 478, 115344 (2020)

[52] DAI, H. H., JING, X. J., SUN, C., WANG, Y., and YUE, X. K. Accurate modeling and analysis of a bio-inspired isolation system: with application to on-orbit capture. *Mechanical Systems and Signal Processing*, 109, 111–133 (2018)

[53] DAI, H. H., JING, X. J., WANG, Y., YUE, X. K., and YUAN, J. P. Post-capture vibration suppression of spacecraft via a bio-inspired isolation system. *Mechanical Systems and Signal Processing*, 105, 214–240 (2018)

[54] WANG, X., YUE, X. K., DAI, H. H., and YUAN, J. P. Vibration suppression for post-capture spacecraft via a novel bio-inspired Stewart isolation system. *Acta Astronautica*, 168, 1–22 (2020)

[55] WANG, X., YUE, X. K., WEN, H. W., and YUAN, J. P. Hybrid passive/active vibration control of a loosely connected spacecraft system. *Computer Modeling in Engineering & Sciences*, 122, 61–87 (2020)

[56] WU, Z. J., JING, X. J., BIAN, J., LI, F. M., and ALLEN, R. Vibration isolation by exploring bio-inspired structural nonlinearity. *Bioinspiration & Biomimetics*, 10, 056015 (2015)

[57] HU, F. Z. and JING, X. J. A 6-DOF passive vibration isolator based on Stewart structure with X-shaped legs. *Nonlinear Dynamics*, 91, 157–185 (2018)

[58] WANG, Y. and JING, X. J. Nonlinear stiffness and dynamical response characteristics of an asymmetric X-shaped structure. *Mechanical Systems and Signal Processing*, 125, 142–169 (2019)

[59] WANG, Y., JING, X. J., and GUO, Y. Q. Nonlinear analysis of a bio-inspired vertically asymmetric isolation system under different structural constraints. *Nonlinear Dynamics*, 95, 445–464 (2019)

[60] ALABUZHEV, P. M. and RIVIN, E. I. *Vibration Protecting and Measuring Systems with Quasi-zero Stiffness*, Hemisphere Publishing Corporation, New York (1989)

[61] ZHOU, J. X., WANG, X. L., XU, D. L., and BISHOP, S. Nonlinear dynamic characteristics of a quasi-zero stiffness vibration isolator with cam-roller-spring mechanisms. *Journal of Sound and Vibration*, 346, 53–69 (2015)

[62] WANG, X. L., ZHOU, J. X., XU, D. L., OUYANG, H. J., and DUAN, Y. Force transmissibility of a two-stage vibration isolation system with quasi-zero stiffness. *Nonlinear Dynamics*, 87, 633–646 (2017)

[63] ZHOU, J. X., XIAO, Q. Y., XU, D. L., OUYANG, H. J., and LI, Y. L. A novel quasi-zero-stiffness strut and its applications in six-degree-of-freedom vibration isolation platform. *Journal of Sound and Vibration*, 394, 59–74 (2017)

[64] ZHOU, J. X., XU, D. L., and BISHOP, S. A torsion quasi-zero stiffness vibration isolator. *Journal of Sound and Vibration*, 338, 121–133 (2015)
[65] WANG, K., ZHOU, J. X., and XU, D. L. Sensitivity analysis of parametric errors on the performance of a torsion quasi-zero-stiffness vibration isolator. *International Journal of Mechanical Sciences*, 134, 336–346 (2017)

[66] SUN, M. N., DONG, Z. X., SONG, G. Q., SUN, X. W., and LIU, W. J. A vibration isolation system using the negative stiffness corrector formed by cam-roller mechanisms with quadratic polynomial trajectory. *Applied Sciences-Basel*, 10, 3573 (2020)

[67] ZHANG, Q. L., XIA, S. Y., XU, D. L., and PENG, Z. K. A torsion-translational vibration isolator with quasi-zero stiffness. *Nonlinear Dynamics*, 99, 1467–1488 (2020)

[68] YE, K., JI, J. C., and BROWN, T. A novel integrated quasi-zero stiffness vibration isolator for coupled translational and rotational vibrations. *Mechanical Systems and Signal Processing*, 149, 107340 (2021)

[69] YAO, Y. H., LI, H. G., LI, Y., and WANG, X. J. Analytical and experimental investigation of a high-static-low-dynamic stiffness isolator with cam-roller-spring mechanism. *International Journal of Mechanical Sciences*, 186, 134–142 (2020)

[70] YAO, Y. H., WANG, X. J., and LI, H. G. Design and analysis of a high-static-low-dynamic stiffness isolator using the cam-roller-spring mechanism. *Journal of Vibration and Acoustics*, 142, 1–24 (2020)

[71] YE, K., JI, J. C., and BROWN, T. Design of a quasi-zero stiffness isolation system for supporting different loads. *Journal of Sound and Vibration*, 471, 115198 (2020)

[72] ROBERTSON, W. S., KIDNER, M. R. F., CAZZOLATO, B. S., and ZANDER, A. C. Theoretical design parameters for a quasi-zero stiffness magnetic spring for vibration isolation. *Journal of Sound and Vibration*, 326, 88–103 (2009)

[73] WU, W. J., CHEN, X. D., and SHAN, Y. H. Analysis and experiment of a vibration isolator using a novel magnetic spring with negative stiffness. *Journal of Sound and Vibration*, 333, 2958–2970 (2014)

[74] ZHENG, Y. S., ZHANG, X. N., LUO, Y. J., YAN, B., and MA, C. C. Design and experiment of a high-static-low-dynamic stiffness isolator using a negative stiffness magnetic spring. *Journal of Sound and Vibration*, 360, 31–52 (2016)

[75] DONG, G. X., ZHANG, X. N., LIU, Y. J., ZHANG, Y. H., and XIE, S. L. Analytical study of the low frequency multi-direction isolator with high-static-low-dynamic stiffness struts and spatial pendulum. *Mechanical Systems and Signal Processing*, 110, 521–539 (2018)

[76] DONG, G. X., ZHANG, X. N., XIE, S. L., YAN, B., and LIU, Y. J. Simulated and experimental studies on a high-static-low-dynamic stiffness isolator using magnetic negative stiffness spring. *Mechanical Systems and Signal Processing*, 86, 188–203 (2017)

[77] ZHENG, Y. S., ZHANG, X. N., LIU, Y. J., ZHANG, Y. H., and XIE, S. L. Analytical study of a quasi-zero stiffness coupling using a torsion magnetic spring with negative stiffness. *Mechanical Systems and Signal Processing*, 100, 135–151 (2018)

[78] ZHOU, J. X., WANG, K., XU, D. L., OUYANG, H. J., and FU, Y. M. Vibration isolation in neonatal transport by using a quasi-zero-stiffness isolator. *Journal of Vibration and Control*, 24, 3278–3291 (2018)

[79] WANG, Q., ZHOU, J. X., XU, D. L., and OUYANG, H. J. Design and experimental investigation of ultra-low frequency vibration isolation during neonatal transport. *Mechanical Systems and Signal Processing*, 139, 106633 (2020)

[80] ZHU, T., CAZZOLATO, B., ROBERTSON, W. S. P., and ZANDER, A. Vibration isolation using six degree-of-freedom quasi-zero stiffness magnetic levitation. *Journal of Sound and Vibration*, 358, 48–73 (2015)

[81] KAMARUZAMAN, N. A., ROBERTSON, W. S. P., GHAYESH, M. H., CAZZOLATO, B. S., and ZANDER, A. C. Six degree of freedom quasi-zero stiffness magnetic spring with active control: theoretical analysis of passive versus active stiffness for vibration isolation. *Journal of Sound and Vibration*, 502, 116086 (2021)

[82] LIU, C. R., ZHAO, R., YU, K. P., and LIAO, B. P. In-plane quasi-zero-stiffness vibration isolator using magnetic interaction and cables: theoretical and experimental study. *Applied Mathematical Modelling*, 96, 497–522 (2021)
[83] ZHENG, Y. S., LI, Q. P., YAN, B., LOU, Y. J., and ZHANG, X. N. A Stewart isolator with high-static-low-dynamic stiffness struts based on negative stiffness magnetic springs. *Journal of Sound and Vibration*, **422**, 390–408 (2018)

[84] YUAN, S. J., SUN, Y., WANG, M., DING, J. H., ZHAO, J. L., HUANG, Y. N., PENG, Y., XIE, S. R., LOU, J., PU, H. Y., LIU, F. Q., BAI, L., and YANG, X. D. Tunable negative stiffness spring using maxwell normal stress. *International Journal of Mechanical Sciences*, **193**, 106127 (2021)

[85] PU, H. Y., YUAN, S. J., PENG, Y., MENG, K., ZHAO, J. L., XIE, R. Q., HUANG, Y. N., SUN, Y., YANG, Y., XIE, S. R., LOU, J., and CHEN, X. D. Multi-layer electromagnetic spring with tunable negative stiffness for semi-active vibration isolation. *Mechanical Systems and Signal Processing*, **121**, 942–960 (2019)

[86] ARAKI, Y., KIMURA, K., ASAI, T., MASUI, T., OMORI, T., and KAINUMA, R. Integrated mechanical and material design of quasi-zero-stiffness vibration isolator with superelastic Cu-Al-Mn shape memory alloy bars. *Journal of Sound and Vibration*, **358**, 74–83 (2015)

[87] MENG, L. S., SUN, J. G., and WU, W. J. Theoretical design and characteristics analysis of a quasi-zero stiffness isolator using a disk spring as negative stiffness element. *Shock and Vibration*, **2015**, 813763 (2015)

[88] ZHOU, X. H., SUN, X., ZHAO, D. X., YANG, X., and TANG, K. H. The design and analysis of a novel passive quasi-zero stiffness vibration isolator. *Journal of Vibration Engineering & Technologies*, **9**, 225–245 (2021)

[89] YAN, B., YU, N., MA, H. Y., and WU, C. Y. A theory for bistable vibration isolators. *Mechanical Systems and Signal Processing*, **167**, 108507 (2022)

[90] ISHIDA, S., SUZUKI, K., and SHIMOSAKA, H. Design and experimental analysis of origami-inspired vibration isolator with quasi-zero-stiffness characteristic. *Journal of Vibration and Acoustics*, **139**, 051004 (2017)

[91] SADEGHI, S. and LI, S. Y. Fluidic origami cellular structure with asymmetric quasi-zero stiffness for low-frequency vibration isolation. *Smart Materials and Structures*, **28**, 065006 (2019)

[92] LIU, S. W., PENG, G. L., and JIN, K. Design and characteristics of a novel QZS vibration isolation system with origami-inspired corrector. *Nonlinear Dynamics*, **106**, 255–277 (2021)

[93] LI, Y. L. and XU, D. L. Vibration attenuation of high dimensional quasi-zero stiffness floating raft system. *International Journal of Mechanical Sciences*, **126**, 186–195 (2017)

[94] DING, H. and CHEN, L. Q. Nonlinear vibration isolation for fluid-conveying pipes using quasi-zero stiffness characteristics. *Mechanical Systems and Signal Processing*, **121**, 675–688 (2019)

[95] DING, H., LU, Z., and CHEN, L. Q. Nonlinear isolation of transverse vibration of pre-pressure beams. *Journal of Sound and Vibration*, **442**, 738–751 (2019)

[96] BOUNA, H. S., NBENDJO, B. R. N., and WOAFO, P. Isolation performance of a quasi-zero stiffness isolator in vibration isolation of a multi-span continuous beam bridge under pier base vibrating excitation. *Nonlinear Dynamics*, **100**, 1125–1141 (2020)

[97] LUNA, A. C. J. and HUANG, J. Analytical dynamics of period-m flows and chaos in nonlinear systems. *International Journal of Bifurcation and Chaos*, **22**, 1250093 (2012)

[98] LUNA, A. C. J. and HUANG, J. Analytical solutions for asymmetric periodic motions to chaos in a hardening Duffing oscillator. *Nonlinear Dynamics*, **72**, 417–438 (2013)

[99] KOVACIC, I., BRENNAN, M. J., and LINETON, B. Effect of a static force on the dynamic behaviour of a harmonically excited quasi-zero stiffness system. *Journal of Sound and Vibration*, **325**, 870–883 (2009)

[100] LUNA, A. C. J. and HUANG, J. Approximate solutions of periodic motions in nonlinear systems via a generalized harmonic balance. *Journal of Vibration and Control*, **18**, 1661–1674 (2012)