Asymptotic normality and consistency of a two-stage generalized least squares estimator in the growth curve model

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Let \( Y = X\Theta Z' + E \) be the growth curve model with \( E \) distributed with mean 0 and covariance \( I_n \otimes \Sigma_1 \), where \( \Theta, \Sigma \) are unknown matrices of parameters and \( X, Z \) are known matrices. For the estimable parametric transformation of the form \( \gamma = C\Theta D' \) with given \( C \) and \( D \), the two-stage generalized least-squares estimator \( \hat{\gamma}(Y) \) defined in (7) converges in probability to \( \gamma \) as the sample size \( n \) tends to infinity and, further, \( \sqrt{n}[\hat{\gamma}(Y) - \gamma] \) converges in distribution to the multivariate normal distribution \( N(0, (CR^{-1}C') \otimes (D(\Sigma^{-1}Z)^{-1}D')) \) under the condition that \( \lim_{n \to \infty} X'X/n = R \) for some positive definite matrix \( R \). Moreover, the unbiased and invariant quadratic estimator \( \hat{\Sigma}(Y) \) defined in (6) is also proved to be consistent with the second-order parameter matrix \( \Sigma \).

Keywords: asymptotic normality; consistent estimator; estimation; generalized least-squares estimator; growth curve model

1. Introduction

The growth curve model is defined as

\[
Y = X\Theta Z' + E, \quad E \sim G(0, I_n \otimes \Sigma),
\]

where \( Y \) is an \( n \times p \) matrix of observations, \( X \) and \( Z \) are known \( n \times m (n > m) \) and \( p \times q (p > q) \) full-rank design matrices, respectively, \( \Theta \) is an unknown \( m \times q \) matrix, called the first-order parameter matrix, and \( \Sigma \) is an unknown positive definite matrix of order \( p \), called the second-order parameter matrix. \( E \) follows a general continuous distribution \( G \) with mean matrix \( 0 \) and Kronecker product structure covariance matrix \( I_n \otimes \Sigma \).

Model (1) was proposed by Potthoff and Roy [11] under the normality assumption of the error matrix \( E \). Since then, parameter estimation, hypothesis testing and prediction of future values have been investigated by numerous researchers, generating a substantial amount of literature concerning the model.

In what follows, we give a brief review of the literature on large sample properties for the growth curve model, a particular kind of multivariate regression model. Chakravorti [2] presented the asymptotic properties of the maximum likelihood estimators. Žežula [15] investigated the
asymptotic properties of the growth curve model with covariance components. Gong [4] gave
the asymptotic distribution of the likelihood ratio statistic for testing sphericity. Bischoff [1]
considered some asymptotic optimal tests for some growth curve models under non-normal error
structure. However, no work has been done on the asymptotic normality and consistency of two-
stage generalized least-squares estimators of the first-order parameter matrix for the growth curve
model (1).

In this paper, we shall investigate the consistency and asymptotic normality of a two-stage
generalized least-squares estimator \( \hat{\gamma}(\mathbf{Y}) \) for the estimable parametric transformation of the form
\( \mathbf{y} = \mathbf{C} \Theta \mathbf{D}' \) with respect to the first-order parameter matrix \( \Theta \). In addition, we shall demonstrate
the consistency of a known quadratic covariance estimator \( \hat{\Sigma}(\mathbf{Y}) \) with the second-order parameter
matrix \( \Sigma \) (see Žežula [14]).

Readers are referred to Eicker [3], Theil [13] and Nussbaum [10] for results on the large sample
properties of the least-squares estimators for ordinary univariate and multivariate regression
models.

This paper is divided into four sections. Some preliminaries are presented in Section 2. In par-
ticular, for the estimable parametric transformation of the form \( \mathbf{y} = \mathbf{C} \Theta \mathbf{D}' \), a two-stage gener-
alized least-squares estimator \( \hat{\gamma}(\mathbf{Y}) \) is defined in (7). The consistency of the estimator \( \hat{\gamma}(\mathbf{Y}) \) and
the consistency of the known quadratic estimator \( \hat{\Sigma}(\mathbf{Y}) \) defined in (6) are investigated in Section 3. Finally, in Section 4, the asymptotic normality of the two-stage generalized least-squares estimator \( \hat{\gamma}(\mathbf{Y}) \) is obtained under a certain condition.

2. Preliminaries

Throughout this paper, the following notation is used. Let \( \mathcal{M}_{n \times p} \) denote the set of all \( n \times p \)
matrices. Let \( \mathbf{A}' \) denote the transpose of the matrix \( \mathbf{A} \). Let \( \text{tr}(\mathbf{A}) \) denote the trace of the matrix
\( \mathbf{A} \). Let \( \mathbf{I}_n \) denote the identity matrix of order \( n \). For a sequence of numbers \( \{a_n\} \) and a sequence
of numbers \( \{b_n\} \), we say that \( a_n = O(b_n) \) if there is a constant \( c \) such that \( \limsup |a_n/b_n| \leq c \);
we say that \( a_n = o(b_n) \) if \( \lim a_n/b_n = 0 \). For an \( n \times p \) matrix \( \mathbf{Y} \), we write \( \mathbf{Y} = [\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n]' \),
\( \mathbf{y}_i \in \mathbb{R}^p \), where \( \mathbb{R}^p \) is the \( p \)-dimensional real space, and vec\((\mathbf{Y}')\) denotes the \( np\)-dimensional
vector \([\mathbf{y}'_1, \mathbf{y}'_2, \ldots, \mathbf{y}'_n]'\). Here, the vec operator transforms a matrix into a vector by stacking the
columns of the matrix one under another. \( \mathbf{Y} \sim \mathcal{G}(\mathbf{M}, \mathbf{I}_n \otimes \Sigma) \) means that \( \mathbf{Y} \) follows a general
continuous distribution \( \mathcal{G} \) with \( \text{E}(\mathbf{Y}) = \mathbf{M} \) and that \( \mathbf{I}_n \otimes \Sigma \) is the covariance matrix of the vector
vec\((\mathbf{Y}')\); see Muirhead [9], Section 3.1. The Kronecker product \( \mathbf{A} \otimes \mathbf{B} \) of matrices \( \mathbf{A} \) and \( \mathbf{B} \) is
defined to be \( \mathbf{A} \otimes \mathbf{B} = (a_{ij}\mathbf{B}) \). We then have vec\((\mathbf{ABC}) = (\mathbf{C}' \otimes \mathbf{A})\text{vec}(\mathbf{B}) \). Let \( \mathbf{A}^+ \) denote the
Moore–Penrose inverse of \( \mathbf{A} \) and \( \mathbf{P}_X = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}' \) be the projection onto the column space
\( \mathcal{C}(\mathbf{X}) \) of a matrix \( \mathbf{X} \) along the orthogonal complement \( \mathcal{C}(\mathbf{X})^\perp \) of \( \mathcal{C}(\mathbf{X}) \).

Given \( \mathbf{A} \in \mathcal{M}_{n \times p} \) and \( \mathbf{B} \in \mathcal{M}_{p \times s} \), a linear parametric function \( \mathbf{B}'\mathbf{\beta} \) is called estima-
tible with respect to \( \mathbf{A} \) if there exists some \( \mathbf{T} \in \mathcal{M}_{n \times s} \) such that \( \text{E}(\mathbf{T}'\mathbf{A}\mathbf{\beta}) = \mathbf{B}'\mathbf{\beta} \) for all \( \mathbf{\beta} \in \mathbb{R}^p \); see Hu and
Shi [6] for a more detailed description.

Note that the first-order parameter \( \Theta \) in model (1) is defined before a design is planned and(observation \( \mathbf{Y} \) is obtained. Thus, the rows of the design matrix \( \mathbf{X} \) in model (1) are added one after
another and the term \( \mathbf{Z} \) in model (1) does not depend on the sample size \( n \); see the example in
Potthoff and Roy [11]. So, we shall only consider the case of full-rank matrices \( \mathbf{X} \) and \( \mathbf{Z} \).
As discussed in Potthoff and Roy [11], hypotheses of the form $C\Theta D' = 0$ under model (1) are usually considered, where $C \in \mathcal{M}_{s \times m}$ and $D \in \mathcal{M}_{t \times q}$. Thus, in this paper, we shall consider the estimator of the parametric transformation $\gamma = C\Theta D'$ of $\Theta$ with given matrices $C \in \mathcal{M}_{s \times m}$ and $D \in \mathcal{M}_{t \times q}$.

We shall begin by reviewing the case of a known second-order parameter matrix $\Sigma$, say $\Sigma_0$. According to the theory of least squares (see, e.g., Rao [12], 4a.2), the normal equations of model (1) are $X'\Theta \Sigma_0^{-1}Z = X'\gamma \Sigma_0^{-1}Z$. The least-squares estimator $\hat{\Theta}_0$ of $\Theta$ is given by

$$\hat{\Theta}_0 = (X'X)^{-1}X'Y\Sigma_0^{-1}Z(Z'\Sigma_0^{-1}Z)^{-1}. \quad (2)$$

Since

$$(X'X)^{-1}X' = (X'X)^{-1}X'P_X$$

and

$$Z(Z'\Sigma_0^{-1}Z)^{-1}Z' = (P_Z \Sigma_0^{-1}P_Z)^+, \quad (3)$$

(2) can be written as

$$\hat{\Theta}_0 = (X'X)^{-1}X'P_X Y\Sigma_0^{-1}(P_Z \Sigma_0^{-1}P_Z)^+Z(Z'Z)^{-1}. \quad (4)$$

Let

$$\hat{\gamma}_0 = C\hat{\Theta}_0 D'. \quad (5)$$

The mean and covariance of $\hat{\gamma}_0$ are, respectively, $C\Theta D'$ and $(C(X'X)^{-1}C') \otimes (D(Z'\Sigma_0^{-1}Z)^{-1}D')$.

In addition, it follows from Rao [12], 4a.2, that $\gamma = C\Theta D'$, for any matrices $C \in \mathcal{M}_{s \times m}$ and $D \in \mathcal{M}_{t \times q}$, is an estimable parametric transformation if matrices $X$ and $Z$ are of full rank. So, $\hat{\gamma}_0$ defined in (5) is said to be a least-squares estimator of the estimable parametric transformation $\gamma = C\Theta D'$. It is easily derived from 4a.2 of Rao [12] that $\hat{\gamma}_0$ is the best linear unbiased estimator (BLUE) of $\gamma$.

Now, we shall focus our attention on the case of an unknown $\Sigma$.

Let

$$\hat{\Sigma}(Y) = Y' W Y, \quad W \equiv \frac{1}{n - \text{rank}(X)} (I - P_X). \quad (6)$$

It is well known that $\hat{\Sigma}(Y)^{-1}$ is positive definite with probability 1 (see the proof of Muirhead [9], Theorem 3.1.4), Žežula [14], Theorem 3.7, tells us that $\hat{\Sigma}(Y)$ is a uniformly minimum variance unbiased invariant estimator of $\Sigma$ under the assumption of normality. This estimator $\hat{\Sigma}(Y)$ is often used to find the first-stage estimator; see, for example, Žežula [16]. We shall also take the estimator as the first-stage estimator in our following discussion.

In (5), an unbiased least-squares estimator of $\gamma$ is given when $\Sigma$ is known. However, when $\Sigma$ is unknown, if we write $\hat{\Theta} \equiv (X'X)^{-1}X'Y\Sigma^{-1}Z(Z'\Sigma^{-1}Z)^{-1}$, then the statistic $\hat{\gamma} \equiv C\Theta D'$ depends on $\Sigma$. In this case, we shall use a method called two-stage estimation to find an estimator, which is denoted by $\hat{\gamma}(Y)$: first, based on data $Y$, we find a first-stage estimator $\hat{\Sigma}$ of $\Sigma$; second, replace the unknown $\Sigma$ with the first-stage estimator $\hat{\Sigma}$ and then find $\hat{\Theta}$ through the normal equations of model (1).
We take \( \hat{\Sigma}(Y) \) in (6) as the first-stage estimator \( \hat{\Sigma} \). Replacing \( \Sigma \) in (4) with \( \hat{\Sigma}(Y) \), (5) can be expressed as

\[
\hat{y}(Y) = C(X'X)^{-1}X'Y\hat{\Sigma}^{-1}(Y)Z(Z'\hat{\Sigma}^{-1}(Y)Z)^{-1}D'.
\] (7)

Let

\[
H(Y) = \hat{\Sigma}^{-1}(Y)(P_Z\hat{\Sigma}^{-1}(Y)P_Z)^+.
\] (8)

Then, by (3), (7) can be rewritten as

\[
\hat{y}(Y) = C(X'X)^{-1}X'YH(Y)Z(Z'Z)^{-1}D'.
\] (9)

The estimator \( \hat{y}(Y) \) is said to be a two-stage generalized least-squares estimator of the estimable parametric transformation \( y = C\Theta D' \).

In the special case of \( C \) and \( D \) being identity matrices, the estimable parametric transformation \( y \) is the first-order parameter matrix \( \Theta \). By (9) or (4), we have

\[
\hat{\Theta}(Y) = (X'X)^{-1}X'YH(Y)Z(Z'Z)^{-1}.
\] (10)

The following lemma concerns the unbiasedness of the estimator \( \hat{y}(Y) \) under the assumption that \( \mathcal{E} \) is symmetric about the origin.

**Lemma 2.1.** Assume that the distribution of \( \mathcal{E} \) is symmetric about the origin. Then the statistic \( \hat{y}(Y) \) defined in (9) is an unbiased estimator of the estimable parametric transformation \( y \).

**Proof.** Since \( \hat{\Sigma}(Y) = \hat{\Sigma}(\mathcal{E}) = \hat{\Sigma}(-\mathcal{E}) \), \( \hat{y}(Y) \) can be expressed as

\[
\hat{y}(Y) = C(X'X)^{-1}X'X\Theta Z'\hat{\Sigma}^{-1}(\mathcal{E})Z(Z'\hat{\Sigma}^{-1}(\mathcal{E})Z)^{-1}D' \\
+ C(X'X)^{-1}X'\Theta Z'\hat{\Sigma}^{-1}(\mathcal{E})Z(Z'\hat{\Sigma}^{-1}(\mathcal{E})Z)^{-1}D'.
\]

Let

\[
M(\mathcal{E}) = C(X'X)^{-1}X'\mathcal{E}Z'\hat{\Sigma}^{-1}(\mathcal{E})Z(Z'\hat{\Sigma}^{-1}(\mathcal{E})Z)^{-1}D'.
\]

Then \( M(-\mathcal{E}) = -M(\mathcal{E}) \) and hence \( E(M(\mathcal{E})) = 0 \). Thus, \( E(\hat{y}(Y)) = C\Theta D' \). This completes the proof. \( \square \)

**3. Consistency**

Since \( Y \) is associated with sample size \( n \), we shall use \( Y_n \) to replace \( Y \) in (9) and then investigate the consistency of the estimator \( \hat{\Sigma}(Y_n) \), as well as the consistency of the related estimator \( \hat{y}(Y) \), as the sample size \( n \) tends to infinity. Note that \( X \) and \( \mathcal{E} \) are also associated with the sample size \( n \).

Recall that an estimator of \( \Sigma \) of the form \( Y_n'W^*Y_n \) is unbiased and invariant if and only if \( \text{tr}(W^*) = 1 \) and \( W^*X = 0 \); see Žežula [14]. Hence, the statistic \( \hat{\Sigma}(Y_n) = Y_n'WY_n' \) defined in (6)
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is an unbiased and invariant estimator of $\Sigma$ without the assumption of normality. Moreover, under the assumption of normality, the estimator $\hat{\Sigma}(Y_n)$ follows a Wishart distribution; see Hu [5].

Now, we shall investigate the consistency property of the estimator $\hat{\Sigma}(Y_n)$.

**Theorem 3.1.** For model (1), the statistic $\hat{\Sigma}(Y_n)$ defined in (6) is a consistent estimator of the second-order parameter matrix $\Sigma$.

**Proof.** Since $Y_n' W Y_n = (Y_n - X\Theta Z)' W (Y_n - X\Theta Z)$, in the following discussion we can assume without loss of generality that $X\Theta Z' = 0$. So, by (6),

$$\hat{\Sigma}(Y_n) = \frac{n}{n-m} \left( \frac{1}{n} \sum_{l=1}^{n} E_l E_l' - \frac{1}{n} E' P X E \right), \tag{11}$$

where $E = (E_1, E_2, \ldots, E_n) \sim \mathcal{G}(0, I_n \otimes \Sigma)$. Note that $(E_l E_l')_{l=1}^{n}$ is a random sample from a population with mean $E(E_l E_l') = \Sigma$. According to Kolmogorov’s strong law of large numbers (see Rao [12], 2c.3 (iv)),

$$\frac{1}{n} \sum_{l=1}^{n} E_l E_l' \text{ converges almost surely to } \Sigma. \tag{12}$$

Letting $\varepsilon > 0$, by Chebyshev’s inequality and the fact that $E(Y' W Y) = \text{tr}(W) \Sigma + E(Y)' W X E(Y)$, we have

$$P\left( \left\| \frac{1}{\sqrt{n}} P X E \right\| \geq \varepsilon \right) \leq \frac{1}{n \varepsilon^2} E[\text{tr}(E' P X E)] = \frac{1}{n \varepsilon^2} \text{tr}(E[E' P X])
= \frac{1}{n \varepsilon^2} \text{tr}(I_n \text{tr}(\Sigma) P X) = \frac{1}{n \varepsilon^2} \text{tr}(P X) \text{tr}(\Sigma).$$

Since $\text{tr}(P X) = \text{rank}(X)$ is a constant, $P(\left\| \frac{1}{\sqrt{n}} P X E \right\| \geq \varepsilon)$ tends to 0 as the sample size $n$ tends to infinity. So,

$$\frac{1}{\sqrt{n}} P X E \text{ converges in probability to } 0. \tag{13}$$

Since convergence almost surely implies convergence in probability, by (12) and (13), we obtain from (11) that $\hat{\Sigma}(Y_n)$ converges to $\Sigma$ in probability. This completes the proof. $\Box$

Now, we focus our attention on the consistency of the estimator $\hat{Y}(Y_n)$. We first prove the following lemma.

**Lemma 3.2.** $H(Y_n)$ converges in probability to $H$, where $H(Y_n)$ is defined in (8) and $H = \Sigma^{-1}(P Z \Sigma^{-1} P Z)'$. 

Proof. Note that the function $A$ to $A^+$ is not continuous. Since $\hat{\Sigma}^{-1}(Y_n)$ is positive definite with probability 1, by Lehmann [7], Lemma 5.3.2, and Theorem 3.1, we have

$$\hat{\Sigma}^{-1}(Y_n) \text{ converges in probability to } \Sigma^{-1}. \quad (14)$$

Write

$$P_Z = O \Lambda O', \quad Q_n = O' \hat{\Sigma}(Y_n) O,$$

where $O$ is a $p \times p$ orthogonal matrix, $\Lambda = \text{diag}[0, I_q]$ with $q = \text{rank}(Z)$ and

$$Q_n^{-1} = O' \hat{\Sigma}^{-1}(Y_n)O = \begin{bmatrix} G_{11}(Y_n) & G_{12}(Y_n) \\ G_{21}(Y_n) & G_{22}(Y_n) \end{bmatrix} = [G_{ij}(Y_n)]_{2 \times 2}$$

with $G_{22}(Y_n)$ a $q \times q$ random matrix. By (14), for any $i, j = 1, 2, G_{ij}(Y_n)$ converges in probability to $G_{ij}$. Note that $(P_Z CP_Z)^+ = P_Z (P_Z CP_Z)P_Z$. Then

$$H(Y_n) = \hat{\Sigma}^{-1}(Y_n)(P_Z \hat{\Sigma}^{-1}(Y_n)P_Z)^+ = OQ_n^{-1} AO'(OAOQ_n^{-1}AO')^+ OAO'$$

$$= O \begin{bmatrix} 0 & G_{12}(Y_n) \\ 0 & G_{22}(Y_n) \end{bmatrix} O'(O \text{diag}[0, G_{22}(Y_n)]O')^+ OAO'$$

$$= O \begin{bmatrix} 0 & G_{12}(Y_n) \\ 0 & G_{22}(Y_n) \end{bmatrix} \text{diag}[0, G_{22}^{-1}(Y_n)]AO' = O \begin{bmatrix} 0 & G^*(Y_n) \\ 0 & I_q \end{bmatrix} O',$$

where $G^*(Y_n) = G_{12}(Y_n)G_{22}^{-1}(Y_n)$. Similarly, $H$ can be decomposed as

$$H = O \begin{bmatrix} 0 & G_{12}G_{22}^{-1} \\ 0 & I_q \end{bmatrix} O'.$$

Since $G^*(Y_n)$ converges in probability to $G_{12}G_{22}^{-1}$, we conclude that

$$\begin{bmatrix} 0 & G^*(Y_n) \\ 0 & I_q \end{bmatrix} \text{ converges in probability to } \begin{bmatrix} 0 & G_{12}G_{22}^{-1} \\ 0 & I_q \end{bmatrix},$$

namely, $H(Y_n)$ converges in probability to $H$. This completes the proof. \qed

Based on Lemma 3.2, we obtain the following consistency result for the estimator $\hat{\gamma}(Y_n)$.

Theorem 3.3. Assume that

$$\lim_{n \to \infty} \frac{1}{n}(X'X) = R, \quad (15)$$

where $R$ is a positive definite matrix. Then the statistic $\hat{\gamma}(Y_n)$ is a consistent estimator of the estimable parametric transformation $\gamma = C\Theta D'$. 

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**Proof.** To prove that \( \hat{\gamma}(Y_n) \) is a consistent estimator of \( \gamma \), by Slutsky’s theorem (see Lehmann and Romano [8], Theorem 11.2.11), it suffices to show that \( \hat{\Theta}(Y_n) \) is a consistent estimator of \( \Theta \).

Replacing \( X \) with \( X\Theta Z' + E \) in (10), we decompose \( \hat{\Theta}(Y_n) \) as \( E_n + F_n \), where

\[
E_n = (X'X)^{-1}X'X\Theta Z'\hat{H}(Y_n)Z(Z'Z)^{-1}
\]

and

\[
F_n = (X'X)^{-1}X'\hat{E}H(Y_n)Z(Z'Z)^{-1}.
\]

Note that \( A'A(A'A)^{-1}B = B \) if \( B \) is estimable with respect to \( A \). Since \( Z'P_Z = Z' \), we obtain

\[
E_n = \Theta Z'\hat{\Sigma}^{-1}(Y_n)(P_Z\hat{\Sigma}^{-1}(Y_n)P_Z)^+Z(Z'Z)^{-1}
\]

\[
= \Theta Z'P Z\hat{\Sigma}^{-1}(Y_n)P Z(P Z\hat{\Sigma}^{-1}(Y_n)P Z)^+Z(Z'Z)^{-1}
\]

and

\[
F_n = n(X'X)^{-1}\left(\frac{1}{\sqrt{n}}X'\right)\left(\frac{1}{\sqrt{n}}P X\hat{E}\right)H(Y_n)Z(Z'Z)^{-1}.
\]

By (15), \( X' / \sqrt{n} \) are bounded. In fact, the elements of \( X' / \sqrt{n} \) are at most of order \( n^{-1/2} \) (see the proof of Lemma 4.1 below). Then, by (13), (15) and Lemma 3.2, \( F_n \) converges in probability to \( 0 \). So, \( \hat{\Theta}(Y_n) \) converges in probability to \( \Theta \). This completes the proof.

**4. Asymptotic normality**

We investigated the consistency of the estimator \( \hat{\gamma}(Y_n) \) in Section 3. In this section, we shall investigate the asymptotic normality of \( \sqrt{n}[\hat{\gamma}(Y_n) - \gamma] \).

First, we shall prove the following lemma.

**Lemma 4.1.** Suppose that condition (15) holds. Let \( S = (XX)^{-1}X = (s_1, s_2, \ldots, s_n)_{m \times n} \), where \( s_i \) is the \( i \)th column of \( (XX)^{-1}X \). Then, for any \( l \in \{1, 2, \ldots, n\} \), the \( m \) elements of \( \sqrt{n}s_i \) are \( O(n^{-1/2}) \).

**Proof.** Write \( V = \frac{1}{\sqrt{n}}X' = [v_1, v_2, \ldots, v_n] \). The transpose of \( v_l \) is an \( m \)-element row vector,

\[
v_l' = \left( \frac{1}{\sqrt{n}}x_{l1}, \frac{1}{\sqrt{n}}x_{l2}, \ldots, \frac{1}{\sqrt{n}}x_{lm} \right),
\]

where \( X = [x_{ij}]_{n \times m} \). By (15), \( VV' = n^{-1}XX' \) converges to a positive definite matrix \( R \). So, the elements of \( VV' = v_1v_1' + v_2v_2' + \cdots + v_nv_n' \) are bounded. We claim that for any \( l \in \{1, 2, \ldots, n\} \), the \( m \) elements of \( v_l \) are all \( O(n^{-1/2}) \).
If this is not true, we can assume, without loss of generality, that one element of \( v_n \) is \( O(np^{-1/2}) \) with \( p > 0 \). Then one element of \( v_n'v_n' \) would be \( O(n^2p^{-1}) \). Hence, the corresponding element in matrix \( \mathbf{V}' = v_1v_1' + v_2v_2' + \cdots + v_nv_n' \) would be \( O(n^2p) \), which is not bounded. This contradicts condition (15).

Note that

\[
(\sqrt{n}s_1, \sqrt{n}s_2, \ldots, \sqrt{n}s_n) = \sqrt{n}(X'X)^{-1}X' = n(X'X)^{-1} \frac{1}{\sqrt{n}} X',
\]

\[
= n(X'X)^{-1}[v_1, v_2, \ldots, v_n],
\]

namely, for \( l = 1, 2, \ldots, n \), \( \sqrt{n}s_l = n(X'X)^{-1}v_l \). Thus, for \( l = 1, 2, \ldots, n \), the \( m \) elements of \( \sqrt{n}s_l \) are also \( O(n^{-1/2}) \). This completes the proof. \( \square \)

Now, we shall show the following important result on the asymptotic normality of \( \sqrt{n}[\hat{\gamma}(Y_n) - \gamma] \).

**Theorem 4.2.** Under the assumption of condition (15), \( \sqrt{n}[\hat{\gamma}(Y_n) - \gamma] \) converges in distribution to \( \mathcal{N}_{s \times t}(0, (CR^{-1}C') \otimes (D(Z'\Sigma^{-1}Z)^{-1}D')) \).

**Proof.** First, by (8), we rewrite \( \gamma \) and \( \hat{\gamma}(Y_n) \) as

\[
\gamma = CSX\Theta'Z'H(Y_n)KD'
\]

and

\[
\hat{\gamma}(Y_n) = CSY_nH(Y_n)KD',
\]

where \( K = Z(Z'Z)^{-1} \). So,

\[
\hat{\gamma}(Y_n) - \gamma = CSY_nH(Y_n)KD' - CSX\Theta'Z'H(Y_n)KD'
\]

\[
= CS(Y_n - X\Theta'Z')H(Y_n)KD'
\]

\[
= CS\mathcal{E}H(Y_n)KD' = CL_nH(Y_n)KD',
\]

where \( L_n \equiv S\mathcal{E} \). Further, \( L_n \) is expressed as

\[
L_n = \sum_{l=1}^{n} s_l\mathcal{E}_l',
\]

where \( s_l \) is the \( l \)th column vector of \( S \) and \( \mathcal{E}_l' \) is the \( l \)th row vector of the matrix \( \mathcal{E} \) with \( \mathcal{E} \sim \mathcal{G}(0, I_n \otimes \Sigma) \).

Next, we shall find the limiting distribution of \( \sqrt{n}[\hat{\gamma}(Y_n) - \gamma] \) through showing that

\[
\sqrt{n}L_n \text{ converges in distribution to } \mathcal{N}_{m \times p}(0, R^{-1} \otimes \Sigma).
\]
Since \( \{e_i^l\}_{i=1}^n \) are independent and identically distributed, for \( t \in \mathcal{M}_{p \times m} \), the characteristic function \( \Psi_n(t) \) of \( \sqrt{n}L_n' \) is given by

\[
\Psi_n(t) = E(\exp\{i \text{tr}(t' \sqrt{n}L_n')\}) = \prod_{l=1}^n \Phi(\sqrt{n}ts_l),
\]

where \( \Phi(\cdot) \) is the characteristic function of \( e_i^l \).

Recall that for \( u \) in the neighborhood of 0,

\[
\ln(1 - u) = -u + f(u) \quad \text{with} \quad f(u) = \frac{1}{2}u^2 + o(u^2).
\]  \hspace{1cm} (19)

If we write \( p(u) = f(u)/u \), then from (19),

\[
p(u) = o(u) \quad \text{as} \quad u \to 0. \]  \hspace{1cm} (20)

Also,

\[
\Phi(x) = 1 - \frac{1}{2}x'\Sigma x + g(x) \quad \text{for} \quad x \in \mathbb{R}^m \quad \text{and} \quad g(x) = o(\|x\|^2) \quad \text{as} \quad x \to 0. \]  \hspace{1cm} (21)

For \( \varepsilon > 0 \), there exists \( \delta(\varepsilon) > 0 \) such that

\[
|g(x)| < \varepsilon \|x\|^2 \quad \text{as} \quad 0 < \|x\| < \delta(\varepsilon). \]  \hspace{1cm} (22)

Therefore, by (19) and (21), the characteristic function of \( \sqrt{n}L_n' \) can be decomposed as

\[
\Psi_n(t) = \exp\left\{ \sum_{l=1}^n \ln\left( \Phi(\sqrt{n}ts_l) \right) \right\}
\]  
\[
= \exp\left\{ \sum_{l=1}^n \ln\left( 1 - \frac{n}{2}s_l't'\Sigma ts_l + g(\sqrt{n}ts_l) \right) \right\}
\]  
\[
= \exp\left\{ \sum_{l=1}^n \left[ -\frac{1}{2}ns_l't'\Sigma ts_l + g(\sqrt{n}ts_l) + f\left( \frac{1}{2}ns_l't'\Sigma ts_l - g(\sqrt{n}ts_l) \right) \right] \right\}
\]  
\[
= \exp\left\{ -\frac{1}{2}\alpha_n + \beta_n + \eta_n \right\},
\]

where

\[
\alpha_n = \sum_{l=1}^n ns_l't'\Sigma ts_l = \text{tr}\left( \sum_{l=1}^n ns_l't'\Sigma ts_l \right),
\]

\[
\beta_n = \sum_{l=1}^n g(\sqrt{n}ts_l),
\]

and

\[
\eta_n = \sum_{l=1}^n f\left( \frac{1}{2}ns_l't'\Sigma ts_l - g(\sqrt{n}ts_l) \right).
\]
and
\[ \eta_n = \sum_{l=1}^{n} f\left(\frac{1}{2}nts_l' \Sigma ts_l - g(\sqrt{nts_l})\right). \]

For \(\alpha_n\), we have
\[ \alpha_n = \text{tr}(\Sigma tnt's') = \text{tr}(\Sigma tnt(X'X)^{-1}t'). \tag{24} \]

By (15),
\[ \lim_{n \to \infty} \alpha_n = \text{tr}(R^{-1}t'\Sigma t) = (\text{vec}(t))'(R^{-1} \otimes \Sigma) \text{vec}(t). \tag{25} \]

For \(\beta_n\), by Lemma 4.1 and the continuity of \(ts_l\), for the \(\delta(\varepsilon) > 0\) in (22), there is an integer \(N(\varepsilon) > 0\) such that for \(n > N(\varepsilon)\),
\[ 0 < \| \sqrt{nts_l} \| < \delta(\varepsilon) \quad \text{for all} \quad l = 1, 2, \ldots, n. \tag{26} \]

If we take \(n > N(\varepsilon)\), then by (22) and (26),
\[ |g(\sqrt{nts_l})| < \| \sqrt{nts_l} \|^2 \varepsilon. \tag{27} \]

So,
\[ |\beta_n| < \sum_{l=1}^{n} \| \sqrt{nts_l} \|^2 \varepsilon = \varepsilon \text{tr}(ntSS't') = \varepsilon \text{tr}(tn(X'X)^{-1}t'). \]

So, by (27), \(\lim_{n \to \infty} |\beta_n| \leq \varepsilon \text{tr}(tR^{-1}t')\). Since \(\varepsilon > 0\) is arbitrary, we obtain
\[ \lim_{n \to \infty} \beta_n = 0. \tag{28} \]

For \(\eta_n\), let
\[ \lambda_l = \frac{1}{2}(\sqrt{nts_l})' \Sigma (\sqrt{nts_l}) - g(\sqrt{nts_l}). \]

So, by (27),
\[ |\lambda_l| < \frac{1}{2}(\sqrt{nts_l})' \Sigma (\sqrt{nts_l}) + \| \sqrt{nts_l} \|^2 \varepsilon. \tag{29} \]

Take \(n > N(\varepsilon)\). By Lemma 4.1, the continuity of \(ts_l\) and (20), increasing \(N(\varepsilon)\) if necessary, we may suppose that for all \(l\), \(|p(\lambda_l)| < \varepsilon\). Since \(f(\lambda_l) = p(\lambda_l)\lambda_l\),
\[ |\eta_n| = \sum_{l=1}^{n} |f(\lambda_l)| = \sum_{l=1}^{n} |p(\lambda_l)||\lambda_l| \leq \sum_{l=1}^{n} \varepsilon |\lambda_l|. \]

So, by (29),
\[ |\eta_n| \leq \sum_{l=1}^{n} \left[ \frac{\varepsilon}{2} \text{tr}(\sqrt{nts_l}' \Sigma t\sqrt{nts_l}) + \| \sqrt{nts_l} \|^2 \varepsilon^2 \right] \]
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or

\[ |\eta_n| \leq \sum_{l=1}^{n} \left[ \frac{n\varepsilon}{2} \text{tr}(t' \Sigma ts'_{l}) + \text{tr}(\sqrt{n}ts_i(\sqrt{n}ts_i)'\varepsilon^2) \right], \]

namely,

\[ |\eta_n| \leq \frac{\varepsilon}{2} \text{tr}(t' \Sigma tnSS') + \text{tr}(tnSS't')\varepsilon^2. \quad (30) \]

Note that \(nSS' = n(X'X)^{-1}\). Since \(\varepsilon\) is arbitrary, by \((15)\) and \((30)\),

\[ \lim_{n \to \infty} \eta_n = 0. \quad (31) \]

By \((25)\), \((28)\) and \((31)\), we obtain from \((23)\) that

\[ \lim_{n \to \infty} \frac{\psi_1n(t)}{\Psi_1n(t)} = \exp\left\{ -\frac{1}{2} (\text{vec}(t))'(R^{-1} \otimes \Sigma) \text{vec}(t) \right\}. \quad (32) \]

So, by Lévy’s continuity theorem, \(\sqrt{n}L_n\) in \((17)\) converges in distribution to the normal distribution \(N_{s \times t}(0, CR^{-1}C' \otimes (DK'H'SHKD'))\).

Finally, by Lemma 3.2, \((16)\), \((18)\) and Muirhead \[9\], Theorem 1.2.6, we obtain that

\[ \sqrt{n}\hat{\gamma}(Y_n) - \gamma \] converges in distribution to \(N_{s \times t}(0, (CR^{-1}C' \otimes (DTD'))), \]

where \(T = (Z'Z)^{-1}Z'(PZ\Sigma^{-1}PZ)^+Z(Z'Z)^{-1} = (Z'\Sigma^{-1}Z)^{-1} \) (see \((3)\)). Thus, the proof is complete. \(\square\)

Under model \((1)\), hypotheses of the form

\[ H: \gamma \equiv C\Theta D' = 0 \]

are usually considered; see Potthoff and Roy \[11\].

From Theorem 4.2 and Slutsky’s theorem, the following corollary provides the asymptotic behavior of \(\sqrt{n}\hat{\gamma}(Y_n)\) under \(H\).

**Corollary 4.3.** Under the assumption of condition \((15)\), if matrices \(C(X'X)^{-1}C'\) and \(D(Z'\hat{\Sigma}^{-1}(Y_n)Z)^{-1}D'\) are non-singular, then the statistic \((Cn(X'X)^{-1}C')^{-1/2} \sqrt{n}\hat{\gamma}(Y_n)(D(Z' \times \hat{\Sigma}^{-1}(Y_n)Z)^{-1}D')^{-1/2} under H converges in distribution to \(N_{s \times t}(0, I)\).**

**Remark 4.4.** Lemma 2.1 tells us that \(\hat{\gamma}(Y)\) is an unbiased estimator of \(\gamma\) under the assumption of \(E\) being symmetric about the origin. In general, it is very difficult to obtain the covariance matrix of \(\hat{\gamma}(Y)\), even under the assumption of normality. However, under condition \((15)\), Theorem 4.2
gives us an approximate covariance matrix \((C(X'X)^{-1}C') \otimes (D(Z' \hat{\Sigma}^{-1}(Y)Z)^{-1}D')\) of \(\hat{\gamma}(Y)\) for large sample size \(n\), without the assumption of normality.

We now conclude this paper by discussing the example in Potthoff and Roy [11]. No assumption of normality is made in our discussion.

**Example 4.5.** There are \(m\) groups of animals, with \(r\) animals in the \(j\)th group and each group being subjected to a different treatment. Animals in all groups are measured at the same \(p\) time points, \(t_1, t_2, \ldots, t_p\). The observations of different animals are independent, but the \(p\) observations on each animal are assumed to have a covariance matrix \(\Sigma_1\).

Based on the problem and our discussion, \(m\) remains constant, while \(r\) tends to infinity.

For \(i = 1, 2, \ldots, m\), the growth curve associated with the \(i\)th group is

\[
\theta_{i0} + \theta_{i1}x + \theta_{i2}x^2 + \cdots + \theta_{i(q-1)}x^{q-1}.
\]

Put

\[
X = (x_1, x_2, \ldots, x_m),
\]

where \(x_i = [\delta_{i1}e'_r, \delta_{i2}e'_r, \ldots, \delta_{it}e'_r]', e_r = (1, 1, \ldots, 1)' \in \mathbb{R}^r\), \(\delta_{ij}\) are the Kronecker symbols, \(n = rm\),

\[
\theta_i = (\theta_{i0}, \theta_{i1}, \theta_{i2}, \ldots, \theta_{i(q-1)}), \quad \Theta = (\theta'_1, \theta'_2, \ldots, \theta'_m)'.
\]

and

\[
Z = \begin{bmatrix}
1 & t_1 & t_1^2 & \cdots & t_1^{q-1} \\
1 & t_2 & t_2^2 & \cdots & t_2^{q-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & t_p & t_p^2 & \cdots & t_p^{q-1}
\end{bmatrix}.
\]

The observation data matrix \(Y_n\) can be written as

\[
Y_n = X\Theta Z' + \mathcal{E},
\]

where \(\mathcal{E} = (\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_n)'\) with \(\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_n\) being independent and identically distributed with mean \(0\) and covariance \(\Sigma\). Then, by (14),

\[
R^{-1} = \lim_{r \to \infty} n(X'X)^{-1} = mI.
\]

By (10),

\[
\hat{\Theta}(Y_n) = \frac{m}{n} X'Y_n \hat{\Sigma}^{-1}(Y_n)(P_Z(\hat{\Sigma}^{-1}(Y_n))P_Z)^+K.
\]

For the estimable parametric transformation of the form \(\gamma = C\Theta D'\) with given \(C \in \mathcal{M}_{s \times m}\) and \(D \in \mathcal{M}_{t \times q}\), the two-stage generalized least-squares estimator is given by

\[
\hat{\gamma}(Y_n) = C\hat{\Theta}(Y_n)D'.
\]
It follows from Theorem 4.2 that $\sqrt{n}[\hat{\gamma}(Y_n) - \gamma]$ converges in distribution to the normal distribution $N_{s \times t}(0, (mCC') \otimes (D(Z'\Sigma^{-1}Z)^{-1}D'))$.

Moreover, if we try to test that all $m$ growth curves are equal, except possibly for the additive constant $\theta_i$, then we take $C$ to be a matrix whose last column contains all $-1$'s and whose first $(m - 1)$ columns constitute the identity matrix, and $D$ to be a $(q - 1) \times q$ matrix whose first column contains all $0$'s and whose last $(q - 1)$ columns constitute the identity matrix, namely, taking

$$C = [1_{m-1} - 1_{m-1}]_{(m-1) \times m}, \quad D = [0 \ I_{q-1}]_{(q-1) \times q},$$

where $1_{m-1} = (1, 1, \ldots, 1)'$, and hypothesis $H_0: C\Theta D' = 0$. Obviously, matrices $C(X'X)^{-1}C'$ and $D(Z'\hat{\Sigma}^{-1}(Y_n)Z)^{-1}D'$ are non-singular. It follows from Corollary 4.3 that statistic $(Cn(X'X)^{-1}C')^{-1/2}\sqrt{n}[\hat{\gamma}(Y_n)(D(Z'\hat{\Sigma}^{-1}(Y_n)Z)^{-1}D')^{-1/2}$ under $H_0$ converges in distribution to $N_{(m-1) \times (q-1)}(0, I)$.

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