Complex Interpolation of Weighted Besov- and Lizorkin-Triebel Spaces (long version)

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Abstract

We study complex interpolation of weighted Besov and Lizorkin-Triebel spaces. The used weights \(w_0, w_1\) are local Muckenhoupt weights in the sense of Rychkov. As a first step we calculate the Calderón products of associated sequence spaces. Finally, as a corollary of these investigations, we obtain results on complex interpolation of radial subspaces of Besov and Lizorkin-Triebel spaces on \(\mathbb{R}^d\).

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1 Introduction

Nowadays interpolation theory has been established as an important tool in various branches of mathematics, in particular in analysis of PDE’s. Within the known interpolation methods the complex interpolation method of Calderón, denoted by $[\cdot, \cdot]_{\Theta}$, is of particular importance and probably the most often used one.

Let $L_p(\mathbb{R}^d, w)$ denote the weighted Lebesgue space with weight $w$. Here in this paper we study generalizations of the following formula

$$
\left[ L_{p_0}(\mathbb{R}^d, w_0), L_{p_1}(\mathbb{R}^d, w_1) \right]_{\Theta} = L_p(\mathbb{R}^d, w), \quad 1 \leq p_0, p_1 < \infty, \quad (1)
$$

where $p = \frac{1}{p} = \frac{1}{p_0} + \frac{\Theta}{p_1}$, $w := w_0^{(1-\Theta)p/p_0} w_1^{\Theta p/p_1}, \quad (2)$

see, e.g., [1, Theorem 5.5.3] or [2, Theorem 1.18.5]. We shall replace the weighted Lebesgue spaces $L_p(\mathbb{R}^d, w)$ by weighted Besov and Lizorkin-Triebel spaces. There are already some contributions dealing with this problem. Let us mention here Bownik [3] and Wojciechowska [4]. But both authors only deal with the case $w = w_0 = w_1$.

Bownik considers weights related to doubling measures and Wojciechowska is dealing with local Muckenhoupt weights (as we shall do in most of the cases). Whereas in (1) it will be enough that $w_0$ and $w_1$ are positive, in the generalizations, we have in mind, it is not clear what is the correct class of weights. It seems that necessary conditions concerning the weights are not known in this context.

To calculate

$$
\left[ F_{s_0, p_0, q_0}^{s_0}(\mathbb{R}^d, w_0), F_{p_1, q_1}^{s_1}(\mathbb{R}^d, w_1) \right]_{\Theta} \quad \text{and} \quad \left[ B_{s_0, p_0, q_0}^{s_0}(\mathbb{R}^d, w_0), B_{p_1, q_1}^{s_1}(\mathbb{R}^d, w_1) \right]_{\Theta} \quad (3)
$$

we shall apply a method which has been used by Bownik [3] and Wojciechowska [4] as well. First we shall calculate the Calderón products of associated sequence spaces. Afterwards we shall use the known coincidence of Calderón products and the complex method of interpolation (under certain extra conditions) to shift the results to the complex interpolation of these sequence spaces. Finally, these results are lifted by wavelet isomorphisms to the level of function spaces. This method has been developed by Calderón [5], Frazier, Jawerth [6], Mendez, Mitrea [7] and Kalton, Mayboroda, Mitrea [8]. The latter two references are connected with the extension of the complex method to certain quasi-Banach spaces. Also in our paper we shall work with quasi-Banach spaces. In fact, we will allow the maximal range of the parameters in (3) with the exception of $F_{s_0, q}^{s_0}(\mathbb{R}^d, w)$. Of course, it would be interesting to incorporate these spaces as well but this requires additional effort.

The paper is organized as follows. In Section 2 we recall the basic notions for the complex method and also describe the state of the art in the unweighted case. The
next section is devoted to the calculation of the Calderón products of some weighted sequence spaces. For \( w_0, w_1 \in A^{loc}_\infty \) we define

\[
w(x) := w_0(x) \left( \frac{(1-\Theta)}{p_0} \right)^{\frac{q_0}{p_1}} w_1(x)^{\frac{q_1}{p_1}}, \quad x \in \mathbb{R}^d.
\]

Then we will establish the formulas

\[
f_{s_0, p_0, q_0}^s(\mathbb{R}^d, w_0) \left( 1-\Theta \right) f_{s_1, p_1, q_1}^s(\mathbb{R}^d, w_1)^\Theta = f_{p, q}^s(\mathbb{R}^d, w)
\]

as well as

\[
b_{s_0, p_0, q_0}^s(\mathbb{R}^d, w_0) \left( 1-\Theta \right) b_{s_1, p_1, q_1}^s(\mathbb{R}^d, w_1)^\Theta = b_{p, q}^s(\mathbb{R}^d, w)
\]

under rather general conditions on the parameters. This will be the most complicated part. In Section 4 we deal with complex interpolation of weighted Besov and Lizorkin-Triebel spaces. On the one side we simply shift here the results, obtained for Calderón products, to complex interpolation formulas, on the other side we apply the result of Shestakov [9] to calculate (3) also in some of those situations where both spaces are not separable. Finally, in Section 5 we apply the results obtained before to derive some complex interpolation formulas for radial subspaces of Besov and Lizorkin-Triebel spaces. This was actually the original motivation for our work.

Definitions and some properties of the classes of weights and classes of function spaces under consideration here are collected in the Appendix at the end of this paper.

**Notation**

As usual, \( \mathbb{N} \) denotes the natural numbers, \( \mathbb{N}_0 := \mathbb{N} \cup \{0\} \), \( \mathbb{Z} \) denotes the integers and \( \mathbb{R} \) the real numbers. For the complex numbers we use the symbol \( \mathbb{C} \), for the Euclidean \( d \)-space we use \( \mathbb{R}^d \) and \( \mathbb{Z}^d \) to denote the collection of all elements in \( \mathbb{R}^d \) having integer components. At very few places we shall need the Fourier transform \( \mathcal{F} \) as well as its inverse transformation \( \mathcal{F}^{-1} \), always defined on the Schwartz space \( \mathcal{S}'(\mathbb{R}^d) \) of tempered distributions.

If \( X \) and \( Y \) are two quasi-Banach spaces, then the symbol \( X \hookrightarrow Y \) indicates that the embedding is continuous. As usual, the symbol \( c \) denotes positive constants which depend only on the fixed parameters \( s, p, q \) and probably on auxiliary functions, unless otherwise stated; its value may vary from line to line. Sometimes we will use the symbols “\( \lesssim \)” and “\( \gtrsim \)” instead of “\( \leq \)” and “\( \geq \)”, respectively. The meaning of \( A \lesssim B \) is given by: there exists a constant \( c > 0 \) such that \( A \leq c B \). Similarly \( \gtrsim \) is defined. The symbol \( A \asymp B \) will be used as an abbreviation of \( A \lesssim B \asymp A \).

Inhomogeneous weighted Besov and Lizorkin-Triebel spaces are denoted by \( B_{p,q}^s(\mathbb{R}^d, w) \) and \( F_{p,q}^s(\mathbb{R}^d, w) \), respectively. If the weight is identically 1, we shall drop \( w \) in notation.
In case there is no reason to distinguish between these two scales we will use the notation $A^p_{s,q}(\mathbb{R}^d, w)$. Definitions, properties as well as some references are given in the Appendix.

## 2 Complex interpolation of Besov and Lizorkin-Triebel spaces - the state of the art

For convenience of the reader we recall some notions from interpolation theory as well as some results in the framework of Besov and Lizorkin-Triebel spaces. For the basics of interpolation theory we refer to the monographs [1, 2, 10, 11].

### 2.1 The complex method of interpolation

The complex method in case of interpolation couples of Banach spaces is a well-studied subject, see the quoted monographs above. Here we are interested in the complex method in case of interpolation couples of certain quasi-Banach spaces. For that reason we give some details. We follow [8], see also [7] and [12].

**Definition 1** A quasi-Banach space $(X, \| \cdot |X\|)$ is called analytically convex if there is a constant $c$ such that for every polynomial $P : \mathbb{C} \to X$ we have

$$\|P(0)|X\| \leq c \max_{|z|=1} \|P(z)|X\|.$$

In the framework of analytically convex quasi-Banach spaces the Maximum Modulus Principle holds. Let

$$S_0 := \{ z \in \mathbb{C} : 0 < \Re z < 1 \} \quad \text{and} \quad S := \{ z \in \mathbb{C} : 0 \leq \Re z \leq 1 \} .$$

**Proposition 2** For a quasi-Banach space $(X, \| \cdot |X\|)$ the following conditions are equivalent:

(i) $X$ is analytically convex.

(ii) There exists a constant $c$ such that

$$\max \{ \|f(z)|X\| : z \in S_0 \} \leq c \max \{ \|f(z)|X\| : z \in S \setminus S_0 \}$$

for any function $f : S \to X$, analytic on $S_0$ and continuous and bounded on $S$.

We refer to [8, Theorem 7.4]. Based on this property the following definition makes sense.
Definition 3 Let \((X_0, X_1)\) be an interpolation couple of quasi-Banach spaces, i.e., \(X_0\) and \(X_1\) are continuously embedded into a larger topological vector space \(Y\). In addition, let \(X_0 + X_1\) be analytically convex. Let \(\mathcal{A}\) be the set of all bounded and analytic functions \(f : S_0 \to X_0 + X_1\), which extend continuously to the closure \(S\) of the strip \(st\) the traces \(t \mapsto f(j + it)\) are bounded continuous functions into \(X_j, j = 0, 1\). We endow \(\mathcal{A}\) with the quasi-norm
\[
\| f \|_{\mathcal{A}} := \max \left\{ \sup_{t \in \mathbb{R}} \| f(it) \|_{X_0}, \sup_{t \in \mathbb{R}} \| f(1 + it) \|_{X_1} \right\}.
\]

Let \(0 < \Theta < 1\). Further, we define \([X_0, X_1]_\Theta\) to be the set of all \(x \in \mathcal{A}(\Theta) := \{ f(\Theta) : f \in \mathcal{A} \}\) and
\[
\| x \|_{[X_0, X_1]_\Theta} := \inf \left\{ \| f \|_{\mathcal{A}} : f(\Theta) = x \right\}.
\]

Remark 1 Any Banach space is analytically convex. Hence, if \((X_0, X_1)\) is an interpolation couple of Banach spaces, this reduces to the standard definition of \([X_0, X_1]_\Theta\).

Lemma 4 Let \(0 < q \leq \infty\), \(s \in \mathbb{R}\) and \(w \in A^{\text{loc}}_{\infty}\).

(i) Let \(0 < p < \infty\). Then \(F^{s}_{p,q}(\mathbb{R}^d, w)\) is analytically convex.

(ii) Let \(0 < p \leq \infty\). Then \(B^{s}_{p,q}(\mathbb{R}^d, w)\) is analytically convex.

Proof. If \(X\) is an analytically convex quasi-Banach space and \(Y\) is a closed subspace of \(X\) then \(Y\) is analytically convex, see Proposition 7.5 in [8]. By means of Proposition 3.7 it will be enough to prove analytic convexity for the sequence spaces \(F^{s}_{p,q}(\mathbb{R}^d, w)\) and \(B^{s}_{p,q}(\mathbb{R}^d, w)\). In contrast to the function spaces the sequence spaces are quasi-Banach lattices. We need a further notion. A quasi-Banach lattice of functions \((X, \| \cdot \|_X)\) is called lattice \(r\)-convex if
\[
\left\| \left( \sum_{j=1}^{m} |f_j|^r \right)^{1/r} |X| \right\| \leq \left( \sum_{j=1}^{m} \| f_j \|_X^r \right)^{1/r}
\]
for any finite family \(\{f_j\}_{1 \leq j \leq m}\) of functions from \(X\).

There are simple criteria for a quasi-Banach lattice of functions to be analytically convex, see [8, Theorem 7.8]: \(X\) is analytically convex if, and only if, \(X\) is lattice \(r\)-convex for some \(r > 0\).

It remains to show that the sequence spaces \(a^{s}_{p,q}(\mathbb{R}^d, w), a \in \{b, f\}\), are lattice \(r\)-convex. This holds with \(r \leq \min(p, q, 1)\) by standard arguments (use the generalized Minkowski inequality).

Remark 2 The unweighted case was considered in Mendez and Mitrea [7], see also Kalton, Mayboroda and Mitrea [8, Proposition 7.7]. For weights related to doubling measures the statement has been settled by Bownik [3].
2.2 The state of the art

For convenience of the reader we recall what is known in the unweighted situation. Comments to the weighted case will be given within the text.

**Proposition 5** Let $0 < p_0, p_1 \leq \infty$, $0 < q_0, q_1 \leq \infty$, $s_0, s_1 \in \mathbb{R}$, and $0 < \Theta < 1$. Define

$$s := (1 - \Theta) s_0 + \Theta s_1, \quad \frac{1}{p} := \frac{1 - \Theta}{p_0} + \frac{\Theta}{p_1} \quad \text{and} \quad \frac{1}{q} := \frac{1 - \Theta}{q_0} + \frac{\Theta}{q_1}. \quad (4)$$

(i) Let $\min(p_0 + q_0, p_1 + q_1) < \infty$. Then we have

$$B^s_{p,q}(\mathbb{R}^d) = \left[ B^{s_0}_{p_0,q_0}(\mathbb{R}^d), B^{s_1}_{p_1,q_1}(\mathbb{R}^d) \right]_\Theta. \quad (5)$$

(ii) Let $\max(p_0, p_1) < \infty$ and $\min(q_0, q_1) < \infty$. Then we have

$$F^s_{p,q}(\mathbb{R}^d) = \left[ F^{s_0}_{p_0,q_0}(\mathbb{R}^d), F^{s_1}_{p_1,q_1}(\mathbb{R}^d) \right]_\Theta. \quad (6)$$

**Remark 3** (i) Proposition 5 in this generality can be found in Frazier and Jawerth [6] (the F-case) and in Kalton, Mayboroda, Mitrea [8, Theorem 9.1](F- and B-case). With the extra condition $s_0 \neq s_1$ one can find [5] also in Mendez, Mitrea [7]. However, Proposition 5 has many forerunners in case $\min(p_0, p_1, q_0, q_1) \geq 1$, e.g., Calderón, J.L. Lions, Magenes, Taibleson, Grisvard, Schechter, Peetre and Triebel. We refer to [1, 13, 2] and the references given there.

(ii) Let us mention, that formula (6) remains true in case if either $\max(p_0, q_0) < \infty$ or $\max(p_1, q_1) < \infty$, see [6] and [8]. However, in our paper we shall not deal with the spaces $F^s_{\infty,q}(\mathbb{R}^d)$ and its weighted counterparts.

(iii) The counterpart of (6) for anisotropic Lizorkin-Triebel spaces (more exactly, the generalization to) has been proved by Bownik [3].

(iv) It is of certain interest to notice that in some cases complex interpolation of pairs of Besov spaces does not result in a Besov space. More exactly, if $1 < p < \infty$, $s_0, s_1 \in \mathbb{R}$, $s_0 \neq s_1$, $s := (1 - \Theta) s_0 + \Theta s_1$, then

$$B^s_{p,\infty}(\mathbb{R}^d) = \left[ B^{s_0}_{p,\infty}(\mathbb{R}^d), B^{s_1}_{p,\infty}(\mathbb{R}^d) \right]_\Theta, \quad (7)$$

where $\hat{B}^s_{p,\infty}(\mathbb{R}^d)$ denotes the closure of the set of test functions in $B^s_{p,\infty}(\mathbb{R}^d)$, a space strictly smaller than $B^s_{p,\infty}(\mathbb{R}^d)$. We refer to [2 Theorem 2.4.1]. Later on, see Subsection 4.3 we shall supplement this formula.

(v) In [1 Theorem 6.4.5] the following formula is claimed to be true:

$$B^s_{p,q}(\mathbb{R}^d) = \left[ B^{s_0}_{p_0,q_0}(\mathbb{R}^d), B^{s_1}_{p_1,q_1}(\mathbb{R}^d) \right]_\Theta, \quad (8)$$
where $1 \leq p_0, p_1, q_0, q_1 \leq \infty$, $s_0 \neq s_1$ and $s, p, q$ as in (4). Of course, this is in contradiction with part (iv) in case $q = q_0 = q_1 = \infty$. We do not believe in this formula in cases, when both spaces $B^{s_0}_{p_0,q_0}(\mathbb{R}^d)$ and $B^{s_1}_{p_1,q_1}(\mathbb{R}^d)$ are not separable, see Subsection 4.3 for more information.

(vi) There is a number of further methods of interpolation where the outcome is known in case of pairs of either Besov or Lizorkin-Triebel spaces. Most prominent is the real method of interpolation. For corresponding results we refer to [1, Theorem 6.4.5] and [14, 2.4]. Triebel [14, 2.4] also had invented a certain modification of the complex method and has been able to prove the counterparts of (5), (6) for this modified complex method. However, it is not known whether this modified method has the interpolation property. Frazier, Jawerth [6] and Bownik [3] also studied the $\pm$-method of Gustaffson and Peetre, denoted by $\langle X_0, X_1, \theta \rangle$, and the method $\langle X_0, X_1 \rangle_\theta$, due to Gagliardo. Then Proposition 5(ii) remains true also for these methods.

3 Calderón products of sequence spaces associated to weighted Besov and Lizorkin-Triebel spaces

After having introduced the necessary definitions in Subsection 3.1 we shall first deal with the Calderón products of the sequence spaces $f_{p,q}^s(\mathbb{R}^d, w)$ (originated from the ingenious proof of Frazier and Jawerth in the unweighted situation). In the third subsection we shall investigate Calderón products of the sequence spaces $b_{p,q}^s(\mathbb{R}^d, w)$ by employing a totally different method.

3.1 Definition and basic properties of the Calderón product

Let $(\mathfrak{X}, S, \mu)$ be a $\sigma$–finite measure space and let $\mathfrak{M}$ be the class of all complex–valued, $\mu$–measurable functions on $\mathfrak{X}$. Then a quasi-Banach space $X \subset \mathfrak{M}$ is called a quasi-Banach lattice of functions if for every $f \in X$ and $g \in \mathfrak{M}$ with $|g(x)| \leq |f(x)|$ for $\mu$–a.e. $x \in \mathfrak{X}$ one has $g \in X$ and $\|g\|_X \leq \|f\|_X$.

Definition 6 Let $(\mathfrak{X}, S, \mu)$ be a $\sigma$–finite measure space and let $\mathfrak{M}$ be the class of all complex–valued, $\mu$–measurable functions on $\mathfrak{X}$. Let $X_j \subset \mathfrak{M}$, $j = 0, 1$, be quasi-Banach lattices of functions. Let $0 < \Theta < 1$. Then the Calderón product $X_0^{1-\Theta} X_1^\Theta$ of $X_0$ and $X_1$ is the collection of all functions $f \in \mathfrak{M}$ s.t. the quasi-norm

$$
\|f\|_{X_0^{1-\Theta} X_1^\Theta} := \inf \left\{ \|f_0\|_{X_0}^{1-\Theta} \|f_1\|_{X_1}^\Theta : |f| \leq |f_0|^{1-\Theta} |f_1|^\Theta \mu \text{– a.e., } f_j \in X_j, j = 0, 1 \right\}
$$
is finite.

**Remark 4**

(i) Calderón products have been introduced by Calderón [5, 13.5] (in a little bit different form which coincides with the above one). The usefulness of this method and its limitations have been perfectly described by Frazier and Jawerth [6] which we quote now: Although restricted to the case of a lattice, the Calderón product has the advantage of being defined in the quasi-Banach case, and, frequently, of being easy to compute. It has the disadvantage that the interpolation property (i.e., the property that a linear transformation \( T \) bounded on \( X_0 \) and \( X_1 \) should be bounded on the spaces in between) is not clear in general.

(ii) A further remark to the literature. Calderón products are not investigated in the most often quoted books on interpolation theory: Bergh and Lofström [1] (except a short remark on page 129), Triebel [2] and Bennett and Sharpley [11]. However, in the monographs of Kreĭn, Petunin and Semënov [10, pp. 242-246], Brudnyi and Kruglyak [15, 4.3] and in the lecture note of Maligranda [16] a few informations about Calderón products can be found, sometimes in the more general framework of Calderón-Lozanovskii constructions. All these references are concerned with Banach spaces. Since we need this concept in quasi-Banach spaces as well, we refer in addition to Nilsson [17], Frazier and Jawerth [6], Kalton and Mitrea [12], Mendez and Mitrea [7], Kalton, Mayboroda and Mitrea [8] and Yang, Yuan and Zhuo [18].

We collect a few useful properties for later use, see [18].

**Lemma 7** Let \((\mathcal{X}, \mathcal{S}, \mu)\) be a \( \sigma \)-finite measure space and let \( \mathcal{M} \) be the class of all complex-valued, \( \mu \)-measurable functions on \( \mathcal{X} \). Let \( X_j \subset \mathcal{M} \), \( j = 0, 1 \), be quasi-Banach lattices of functions. Let \( 0 < \Theta < 1 \).

(i) Then the Calderón product \( X_0^{1-\Theta}X_1^{\Theta} \) is a quasi-Banach space.

(ii) Define \( X_0^{1-\Theta}X_1^{\Theta} \) as the collection of all \( f \) s.t. there exist a positive real number \( \lambda \) and elements \( g \in X_0 \) and \( h \in X_1 \) satisfying

\[
|f| \leq \lambda |g|^{1-\Theta} |h|^{\Theta}, \quad \|g\|_{X_0} \leq 1, \quad \|h\|_{X_1} \leq 1.
\]

We put

\[
\| f \|_{X_0^{1-\Theta}X_1^{\Theta}} := \inf \left\{ \lambda > 0 : |f| \leq \lambda |g|^{1-\Theta} |h|^{\Theta}, \quad \|g\|_{X_0} \leq 1, \quad \|h\|_{X_1} \leq 1 \right\}.
\]

Then \( X_0^{1-\Theta}X_1^{\Theta} = X_0^{1-\Theta}X_1^{\Theta} \) follows with equality of quasi-norms.

Here is one well-known example of a Calderón product which can be easily calculated. Let \( w : \mathbb{R}^d \to [0, \infty) \) be measurable and positive a.e.. The weighted Lebesgue
space $L_p(\mathbb{R}^d, w)$ is the collection of all measurable functions $f : \mathbb{R}^d \to \mathbb{C}$ such that
\[
\| f |_{L_p(\mathbb{R}^d, w)} \| := \left( \int_{\mathbb{R}^d} |f(x)|^p w(x) \, dx \right)^{1/p} < \infty .
\]

In case $p = \infty$ we shall use the convention $L_{\infty}(\mathbb{R}^d, w) := L_{\infty}(\mathbb{R}^d)$, i.e., we always take $w \equiv 1$.

**Lemma 8** Let $0 < \Theta < 1$, $0 < p_0, p_1 \leq \infty$ and let $w_j : \mathbb{R}^d \to [0, \infty)$, $j = 0, 1$, be measurable and positive a.e.. We define
\[
\frac{1}{p} := \frac{1 - \Theta}{p_0} + \frac{\Theta}{p_1} \quad \text{and} \quad \omega := \omega_0 \frac{(1 - \Theta) p_0}{\omega_1 p_1} .
\]
Then
\[
L_{p_0}(\mathbb{R}^d, w_0)^{1 - \Theta} L_{p_1}(\mathbb{R}^d, w_1)^{\Theta} = L_p(\mathbb{R}^d, w)
\]
with coincidence of the quasi-norms.

**Proof.** Step 1. Let $\max(p_0, p_1) < \infty$.

**Substep 1.1** We prove $L_{p_0}(\mathbb{R}^d, w_0)^{1 - \Theta} L_{p_1}(\mathbb{R}^d, w_1)^{\Theta} \subset L_p(\mathbb{R}^d, w)$. Let $f \in L_{p_0}(\mathbb{R}^d, w_0)^{1 - \Theta} L_{p_1}(\mathbb{R}^d, w_1)^{\Theta}$. Then there exist $f_0, f_1$ s.t.
\[
|f(x)| \leq |f_0(x)|^{1 - \Theta} |f_1(x)|^\Theta \quad \text{a.e. in } \mathbb{R}^d
\]
and $f_j \in L_{p_j}(\mathbb{R}^d, w_j)$, $j = 0, 1$. We employ this inequality together with Hölder’s inequality and obtain
\[
\left( \int_{\mathbb{R}^d} |f(x)|^p w(x) \, dx \right)^{1/p} \leq \left( \int_{\mathbb{R}^d} |f_0(x)|^{(1 - \Theta)p} w_0(x)^{(1 - \Theta) p_0} |f_1(x)|^{\Theta p} w_1(x)^{\Theta p_1} \, dx \right)^{1/p}
\]
\[
\leq \left( \int_{\mathbb{R}^d} |f_0(x)|^{p_0} w_0(x) \, dx \right)^{(1 - \Theta)/p_0} \left( \int_{\mathbb{R}^d} |f_1(x)|^{p_1} w_1(x) \, dx \right)^{\Theta/p_1} .
\]
Hence, $f \in L_p(\mathbb{R}^d, w)$ and $\| f |_{L_p(\mathbb{R}^d, w)} \| \leq \| f |_{L_{p_0}(\mathbb{R}^d, w_0)^{1 - \Theta} L_{p_1}(\mathbb{R}^d, w_1)^{\Theta}} \|$.

**Substep 1.2**. We prove $L_p(\mathbb{R}^d, w) \subset L_{p_0}(\mathbb{R}^d, w_0)^{1 - \Theta} L_{p_1}(\mathbb{R}^d, w_1)^{\Theta}$. For given $f \in L_p(\mathbb{R}^d, w)$ we define
\[
f_0(x) := |f(x)|^{p/p_0} \left( \frac{w(x)}{w_0(x)} \right)^{1/p_0} \quad \text{and} \quad f_1(x) := |f(x)|^{p/p_1} \left( \frac{w(x)}{w_1(x)} \right)^{1/p_1} .
\]
Then $f_j \in L_{p_j}(\mathbb{R}^d, w_j)$, $j = 0, 1$, which implies that $f \in L_{p_0}(\mathbb{R}^d, w_0)^{1 - \Theta} L_{p_1}(\mathbb{R}^d, w_1)^{\Theta}$ and
\[
\| f |_{L_{p_0}(\mathbb{R}^d, w_0)^{1 - \Theta} L_{p_1}(\mathbb{R}^d, w_1)^{\Theta}} \| \leq \| f |_{L_p(\mathbb{R}^d, w)} \|^{(1 - \Theta)p/p_0} \| f |_{L_p(\mathbb{R}^d, w)} \|^{\Theta p/p_1}
\]
\[
= \| f |_{L_p(\mathbb{R}^d, w)} \| .
\]
Step 2. Let $\min(p_0, p_1) < \max(p_0, p_1) = \infty$. We shall concentrate on the case $0 < p_0 < p_1 = \infty$. Then, by our convention, $w_1 := 1$. The modifications, needed in Substep 1.1, are obvious. The function $f_1$, used in Substep 1.2, is now given by $f_1 = 1$. With this choice the needed arguments are the same.

Step 3. Let $p_0 = p_1 = \infty$. The proof of $L_\infty(\mathbb{R}^d)^1 - \Theta L_\infty(\mathbb{R}^d)^\Theta = L_\infty(\mathbb{R}^d)$ is obvious.

**Remark 5** In case $1 \leq p_0, p_1 \leq \infty$ this result can be found in [15, Exercise 4.3.8]. For the unweighted case we also refer to [10, formula 1.6.1 on page 2.4.6] and [16, Exercise 3 on page 179].

Weighted $L_p$-spaces are lattice $r$-convex with $r \leq \min(1, p)$, hence analytically convex, see the proof of Lemma 4 for an explanation of this notion and [8, Theorem 7.8] for a proof. Hence, complex interpolation of pairs of weighted $L_p$-spaces makes sense. There are nice connections between complex interpolation spaces and the corresponding Calderón product, see the original paper of Calderón [5] or Theorem 7.9 in [8].

**Proposition 9** Let $(X, S, \mu)$ be a complete separable metric space, let $\mu$ be a $\sigma$–finite Borel measure on $X$, and let $X_0, X_1$ be a pair of quasi-Banach lattices of functions on $(X, \mu)$. Then, if both $X_0$ and $X_1$ are analytically convex and separable, it follows that $X_0 + X_1$ is analytically convex and

$$[X_0, X_1]_\Theta = X_0^{1-\Theta} X_1^\Theta, \quad 0 < \Theta < 1. \tag{10}$$

Lemma 8 and Proposition 9 immediately imply the following extension of (1).

**Corollary 10** Let $0 < \Theta < 1$, $0 < p_0, p_1 < \infty$ and let $w_j : \mathbb{R}^d \to [0, \infty), \ j = 0, 1$, be measurable and positive a.e.. Let $p$ and $w$ be defined as in (4). Then

$$\left[ L_{p_0}(\mathbb{R}^d, w_0), L_{p_1}(\mathbb{R}^d, w_1) \right]_\Theta = L_p(\mathbb{R}^d, w)$$

in the sense of equivalence of quasi-norms.

**Remark 6** Also in Gustavsson [19] and Nilsson [17] interpolation of $L_{p_0}(\mathbb{R}^d, w_0)$ and $L_{p_1}(\mathbb{R}^d, w_1)$ is discussed for the full range of $p_0$ and $p_1$. They considered $\langle L_{p_0}(\mathbb{R}^d, w_0), L_{p_1}(\mathbb{R}^d, w_1) \rangle$, where $\langle \cdot, \cdot \rangle$ denotes an interpolation method introduced by Gagliardo and coincides with the Calderón product under certain conditions, see [17].
3.2 Calderón products of $f^s_{p,q}(\mathbb{R}^d, w)$ spaces

In case of weighted Besov or Lizorkin-Triebel spaces there exist wavelet isomorphisms which relate these spaces to weighted sequence spaces, see the Appendix for more details. We first study Calderón products of these sequence spaces.

Here we are going to use the following abbreviations. By

$$Q_{j,k} := \{ x \in \mathbb{R}^d : 2^{-j}k_\ell \leq x_\ell < 2^{-j}(k_\ell + 1), \ \ell = 1, \ldots, d \}, \ j \in \mathbb{N}_0, \ k \in \mathbb{Z}^d,$$

we denote the dyadic cubes in $\mathbb{R}^d$ (with volume $\leq 1$). The symbol $X_{j,k}$ is used for the characteristic function of the cube $Q_{j,k}$.

**Definition 11** Let $0 < q \leq \infty$, $s \in \mathbb{R}$ and let $w : \mathbb{R}^d \to [0, \infty)$ be a nonnegative measurable function. In case $0 < p < \infty$ we define

$$f^s_{p,q}(\mathbb{R}^d, w) := \left\{ \{\lambda_{j,k}\}_{j,k} : \lambda_{j,k} \in \mathbb{C}, \right\}$$

$$\| (\lambda_{j,k}) |f^s_{p,q}(\mathbb{R}^d, w)\| := \left\| \left( \sum_{j=0}^{\infty} \sum_{k \in \mathbb{Z}^d} 2^{sjq} |\lambda_{j,k}|^q X_{j,k}(\cdot) \right)^{1/q} \left\| L_p(\mathbb{R}^d, w) \right\| < \infty \right\}. \quad (11)$$

Obviously the spaces $f^s_{p,q}(\mathbb{R}^d, w)$ are quasi-Banach lattices. For us only those weights $w$ will be of interest which are locally integrable and satisfy

$$0 < w(Q_{j,k}) := \int_{Q_{j,k}} w(x) \, dx < \infty \quad \text{for all} \quad j \in \mathbb{N}_0, \ k \in \mathbb{Z}^d. \quad (12)$$

**Remark 7** (i) In case $w(x) = 1$ for all $x \in \mathbb{R}^d$ we are back in the unweighted situation. The associated sequence spaces are denoted simply by $f^s_{p,q}(\mathbb{R}^d)$.

(ii) Let $w$ satisfy (12). Let $\hat{f}^s_{p,q}(\mathbb{R}^d, w)$ denote the closure of the finite sequences in $f^s_{p,q}(\mathbb{R}^d, w)$. Then

$$\hat{f}^s_{p,q}(\mathbb{R}^d, w) = f^s_{p,q}(\mathbb{R}^d, w) \iff q < \infty.$$ $\hat{f}^s_{p,q}(\mathbb{R}^d, w)$ is a proper subspace of $f^s_{p,\infty}(\mathbb{R}^d, w)$.

(iii) Let $w$ satisfy (12). It is easily checked that $f^s_{p,q}(\mathbb{R}^d, w)$ is separable if, and only if, $q < \infty$.

(iv) Frazier and Jawerth have introduced also the spaces $f^s_{\infty,q}(\mathbb{R}^d)$. Here in this paper we shall not deal with these classes.

Now we turn to the investigation of the Calderón products of these sequence spaces. As mentioned above we are interested in the most general situation (except the use of $f^s_{\infty,q}(\mathbb{R}^d, w)$). The class of weights, we are dealing with, is $A^\text{loc}_\infty$, see the Appendix for the definition. With certain care we shall study also the limiting situations

$$\max(q_0, q_1) = \infty.$$
Theorem 12 Let $0 < \Theta < 1$. Let $0 < p_0, p_1 < \infty$, $0 < q_0, q_1 \leq \infty$ and $s_0, s_1 \in \mathbb{R}$. We put
\[
\frac{1}{p} := \frac{1 - \Theta}{p_0} + \frac{\Theta}{p_1}, \quad \frac{1}{q} := \frac{1 - \Theta}{q_0} + \frac{\Theta}{q_1} \quad \text{and} \quad s := (1 - \Theta) s_0 + \Theta s_1. \tag{13}
\]
Let $w_0, w_1 \in A_{\infty}^{loc}$ and define $w$ by the formula
\[
w := w_0^{\frac{(1 - \Theta) p_0}{p_0}} w_1^{\frac{\Theta p_1}{p_1}}. \tag{14}\]
Then
\[
f_{p_0, q_0}^s(\mathbb{R}^d, w_0)^{1 - \Theta} f_{p_1, q_1}^{s_1}(\mathbb{R}^d, w_1)^{\Theta} = f_{p, q}^s(\mathbb{R}^d, w) \tag{15}\]
holds in the sense of equivalent quasi-norms.

Proof. By Lemma 35 the weight $w$ belongs to $A_{\infty}^{loc}$, i.e., our sequence spaces $f_{p, q}^s(\mathbb{R}^d, w)$ are well-defined. Muckenhoupt weights and therefore also local Muckenhoupt weights can not vanish on a set of positive measure. Hence, (12) holds for $w_0, w_1$ and $w$.

Step 1. We shall prove
\[
f_{p_0, q_0}^s(\mathbb{R}^d, w_0)^{1 - \Theta} f_{p_1, q_1}^{s_1}(\mathbb{R}^d, w_1)^{\Theta} \hookrightarrow f_{p, q}^s(\mathbb{R}^d, w).
\]
We suppose that sequences $\lambda := (\lambda_{j, k})_{j, k}$, $\lambda^{\ell} := (\lambda_{j, k}^{\ell})_{j, k}$, $\ell = 0, 1$, are given and that
\[
|\lambda_{j, k}| \leq |\lambda_{j, k}^0|^{1 - \Theta} \cdot |\lambda_{j, k}^1|^{\Theta}
\]
holds for all $j \in \mathbb{N}_0$ and all $k \in \mathbb{Z}^d$. We have to show that there exists a constant $c$ s.t.
\[
\| \lambda f_{p, q}^s(\mathbb{R}^d, w) \| \leq c \| \lambda^0 f_{p_0, q_0}^s(\mathbb{R}^d, w_0) \|^{1 - \Theta} \cdot \| \lambda^1 f_{p_1, q_1}^{s_1}(\mathbb{R}^d, w_1) \|^{\Theta}
\]
holds for all such $\lambda, \lambda^0, \lambda^1$. But this follows directly by Hölder’s inequality (with $c = 1$).

Step 2. Now we turn to the proof of
\[
f_{p, q}^s(\mathbb{R}^d, w) \hookrightarrow f_{p_0, q_0}^s(\mathbb{R}^d, w_0)^{1 - \Theta} f_{p_1, q_1}^{s_1}(\mathbb{R}^d, w_1)^{\Theta}.
\]
We assume in addition $\max(q_0, q_1) < \infty$. Let the sequence $\lambda \in f_{p, q}^s(\mathbb{R}^d, w)$ be given. We have to find sequences $\lambda^0$ and $\lambda^1$ such that $|\lambda_{j, k}| \leq |\lambda_{j, k}^0|^{1 - \Theta} \cdot |\lambda_{j, k}^1|^{\Theta}$ for every $j \in \mathbb{N}_0$ and $k \in \mathbb{Z}^d$ and
\[
\| \lambda^0 f_{p_0, q_0}^s(\mathbb{R}^d, w_0) \|^{1 - \Theta} \cdot \| \lambda^1 f_{p_1, q_1}^{s_1}(\mathbb{R}^d, w_1) \|^{\Theta} \leq c \| \lambda f_{p, q}^s(\mathbb{R}^d, w) \| \tag{16}
\]
with some constant $c$ independent of $\lambda$. We follow ideas of the proof of Theorem 8.2 in Frazier and Jawerth [6], see also Bownik [3]. Since $w_0, w_1$ are local Muckenhoupt weights, they are positive and finite a.e., hence, also $w$ is positive and finite a.e.. Let
\[
A := \left\{ x \in \mathbb{R}^d : 0 < \frac{w(x)}{w_0(x)} < \infty \quad \text{and} \quad 0 < \frac{w(x)}{w_1(x)} < \infty \right\}.
\]
The functions \( w, w_0 \) and \( w_1 \) are locally integrable and positive a.e., therefore \( \mathbb{R}^d \setminus A \) is a set of measure zero. We put

\[
A_\ell := \left\{ x \in A : \left( \sum_{j=0}^{\infty} \sum_{k \in \mathbb{Z}^d} 2^{jsq} |\lambda_{j,k}|^q X_{j,k}(x) \right)^{1/q} \cdot \left( \frac{w(x)}{w_0(x)} \right)^{\frac{1}{p_0} - \frac{1}{q_0}} > 2^\ell \right\},
\]

\( \ell \in \mathbb{Z} \). Obviously \( A_{\ell+1} \subset A_\ell \), \( \ell \in \mathbb{Z} \). Now we introduce a (partial) decomposition of \( \mathbb{N}_0 \times \mathbb{Z}^d \) by taking

\[
C_\ell := \left\{ (j, k) : |Q_{j,k} \cap A_\ell| > \frac{|Q_{j,k}|}{2} \quad \text{and} \quad |Q_{j,k} \cap A_{\ell+1}| \leq \frac{|Q_{j,k}|}{2} \right\}, \quad \ell \in \mathbb{Z}.
\]

The sets \( C_\ell \) are pairwise disjoint, i.e., \( C_\ell \cap C_m = \emptyset \) if \( \ell \neq m \).

**Substep 2.1.** We claim that \( \lambda_{j,k} = 0 \) holds for all tuples \( (j, k) \not\in \bigcup_\ell C_\ell \). Let us consider one such tuple \( (j_0, k_0) \) and let us choose \( l_0 \in \mathbb{Z} \) arbitrarily. As \( (j_0, k_0) \not\in C_{l_0} \), then either

\[
|Q_{j_0,k_0} \cap A_{l_0}| \leq \frac{|Q_{j_0,k_0}|}{2} \quad \text{or} \quad |Q_{j_0,k_0} \cap A_{l_0+1}| > \frac{|Q_{j_0,k_0}|}{2}.
\]

Let us assume for the moment that the second condition is satisfied. By induction on \( \ell \) it follows

\[
|Q_{j_0,k_0} \cap A_{l+1}| > \frac{|Q_{j_0,k_0}|}{2} \quad \text{for all} \quad \ell \geq \ell_0.
\]

Let \( D := \bigcap_\ell Q_{j_0,k_0} \cap A_\ell \). The family \( \{Q_{j_0,k_0} \cap A_\ell\}_\ell \) is a decreasing family of sets, i.e., \( Q_{j_0,k_0} \cap A_{\ell+1} \subset Q_{j_0,k_0} \cap A_\ell \). Therefore, in view of (18), the measure of the set \( D \) is larger than or equal to \( \frac{|Q_{j_0,k_0}|}{2} \). We obtain

\[
\| \lambda \left| f_{p,q}^s(\mathbb{R}^d, w) \right| \|^p := \left\| \left( \sum_{j=0}^{\infty} \sum_{k \in \mathbb{Z}^d} 2^{sjq} |\lambda_{j,k}|^q X_{j,k}(\cdot) \right)^{1/q} \left| L_p(\mathbb{R}^d, w) \right| \right\|^p
\]

\[
> \int_{Q_{j_0,k_0} \cap A_\ell} \left( 2^\ell \left( \frac{w_0(x)}{w(x)} \right)^{\frac{1}{p_0} - \frac{1}{q_0}} \right)^p w(x) \, dx
\]

\[
> 2^{\ell p} \int_D w_0(x)^{\frac{p}{q_0} - \frac{1}{q_0}} w(x)^{\frac{1}{p_0} - \frac{1}{q_0}} \, dx.
\]

The norm \( \| \lambda \left| f_{p,q}^s(\mathbb{R}^d, w) \right| \|^p \) is finite since \( \lambda \in f_{p,q}^s(\mathbb{R}^d, w) \). In consequence the integral over \( D \) is a finite positive number. We recall that the function we integrate is positive a.e. and \( |D| \geq \frac{|Q_{j_0,k_0}|}{2} \). Letting \( \ell \) tend to infinity we get a contradiction. Hence, we have to turn in (17) to the situation where the first condition is satisfied. We claim

\[
|Q_{j_0,k_0} \cap A_\ell| \leq \frac{|Q_{j_0,k_0}|}{2} \quad \text{for all} \quad \ell \leq \ell_0.
\]

Again this follows by induction on \( \ell \) using \( (j_0, k_0) \not\in \bigcup_\ell C_\ell \). Obviously this yields

\[
|Q_{j_0,k_0} \cap A_\ell^c| \geq 2^{-j_0d-1} \quad \text{for all} \quad \ell \leq \ell_0.
\]

(19)
Let now $E := \bigcap_{\ell} Q_{j_0,k_0} \cap A \cap A_{\ell}^c$. The family $\{Q_{j_0,k_0} \cap A \cap A_{\ell}^c\}_{\ell}$ satisfies

$$(Q_{j_0,k_0} \cap A \cap A_{\ell-1}) \subset (Q_{j_0,k_0} \cap A \cap A_{\ell}).$$

Therefore, in view of (19), the measure of the set $E$ is larger than or equal to $\frac{|Q_{j_0,k_0}|}{2}$.

By selecting a point $x \in E$ we conclude that

$$2^{j_0} |\lambda_{j_0,k_0}| \leq \left( \sum_{j=0}^{\infty} \sum_{k \in \mathbb{Z}^d} 2^{jsq} |\lambda_{j,k}|^q X_{j,k}(x) \right)^{1/q} \leq 2^{\ell} \left( \frac{w_0(x)}{w(x)} \right)^{\frac{1}{p_0}} \frac{1}{p_0} \frac{1}{q_0},$$

for any $\ell \leq \ell_0$. Now, for $\ell$ tending to $-\infty$ the claim, namely $\lambda_{j_0,k_0} = 0$, follows.

**Substep 2.2.** If $(j, k) \not\in \bigcup_{\ell \in \mathbb{Z}} C_{\ell}$, then we define $\lambda_{j_0} := \lambda_{j_0,k_0} := 0$. If $(j, k) \in C_{\ell}$, we put

$$\lambda_{j_0} := 2^{\ell \gamma} 2^{j_u} |\lambda_{j,k}|^{q/q_0} \quad \text{and} \quad \lambda_{j_0} := 2^{\ell \delta} 2^{j_v} |\lambda_{j,k}|^{q/q_1},$$

where

$$\gamma := \frac{p}{p_0} - \frac{q}{q_0}, \quad \delta := \frac{p}{p_1} - \frac{q}{q_1}, \quad u := \frac{q_0}{q_1} \left( \frac{s_1}{q_1} - \frac{s_0}{q_0} \right), \quad v := q(1 - \Theta) \left[ \frac{s_0}{q_1} - \frac{s_1}{q_0} \right].$$

We observe, that

$$(\lambda_{j_0,k_0})^{1-\Theta} \cdot (\lambda_{j_0,k_0})^\Theta = 2^{\ell [\gamma(1 - \Theta) + \Theta]} \cdot 2^{j[u(1-\Theta) + v\Theta]} : |\lambda_{j,k}| = |\lambda_{j_0,k_0}|,$$

which holds now for all pairs $(j, k)$. To prove (16), it will be sufficient to establish the following two inequalities

$$\|\lambda_0\|_{f_{p_0,q_0}^s(\mathbb{R}^d, w_0)} \lesssim \|\lambda\|_{f_{p,q}^s(\mathbb{R}^d, w)}^{p/p_0} \quad (20)$$

$$\|\lambda_1\|_{f_{p_1,q_1}^s(\mathbb{R}^d, w_1)} \lesssim \|\lambda\|_{f_{p,q}^s(\mathbb{R}^d, w)}^{p/p_1}. \quad (21)$$

**Substep 2.3.** First we deal with (20) under the condition $\gamma \geq 0$. Our restrictions on $p_0, p_1$ and $q_0, q_1$ are symmetric. It follows from (13) that

$$\min \left( \frac{p_0}{q_0}, \frac{p_1}{q_1} \right) \leq \frac{p}{q} \leq \max \left( \frac{p_0}{q_0}, \frac{p_1}{q_1} \right).$$

As $\gamma \geq 0$, we get also $p/p_0 \geq q/q_0$ and $\delta \leq 0$. By employing the sets $C_{\ell}$ we derive

$$\|\lambda_0\|_{f_{p_0,q_0}^s(\mathbb{R}^d, w_0)} = \left\| \left( \sum_{j=0}^{\infty} \sum_{k \in \mathbb{Z}^d} 2^{j_0q_0} (\lambda_{j_0,k}^0)^{q_0} X_{j,k}(\cdot) \right)^{1/q_0} L_{p_0}(\mathbb{R}^d, w_0) \right\|$$

$$= \left\| \left( \sum_{j=0}^{\infty} \sum_{k \in \mathbb{Z}^d} 2^{j_0q_0} 2^{\ell \gamma q_0} 2^{j_u q_0} |\lambda_{j,k}|^{q_0} X_{j,k}(\cdot) \right)^{1/q_0} L_{p_0}(\mathbb{R}^d, w_0) \right\|$$

$$= \left\| \left( \sum_{j=0}^{\infty} \sum_{k \in \mathbb{Z}^d} f_{j,k}(\cdot)^{Q} \right)^{1/Q} L_{P}(\mathbb{R}^d, w_0) \right\|^{P/p_0},$$

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where
\[ f_{j,k}(\cdot) := \left( 2^{j+q_0} 2^{j+q_0} 2^{j+q_0} |\lambda_{j,k}|^q X_{j,k}(\cdot) \right)^{1/q_0}, \quad (j, k) \in C_\ell, \]
and \( P = \frac{wp}{p_0} \) are chosen such that \( w_0 \in A_P, 1 < P < \infty, 1 < Q \leq \infty \). Next we apply the weighted vector-valued maximal inequality \((\text{III})\) for the local Hardy-Littlewood maximal function from the Appendix together with the estimate
\[ X_{j,k}(x) \leq 2 (M_{loc} X_{j,k} \cap A_\ell)(x), \quad x \in \mathbb{R}^d, \quad (j, k) \in C_\ell. \]
Using \( u + s_0 = \frac{aq}{q_0} \), \( \gamma \geq 0 \) and
\[ \bigcup_{\ell = -\infty}^{\infty} (A_\ell \setminus A_{\ell+1}) = \bigcup_{\ell = -\infty}^{\infty} A_\ell \]
we may further proceed
\[
\| \lambda^0 \|_{f_p^{s_0, q_0}(\mathbb{R}^d, w_0)} \leq \left\| \left( \sum_{\ell = -\infty}^{\infty} \sum_{(j, k) \in C_\ell} 2^{j+q_0} 2^{j+q_0} 2^{j+q_0} |\lambda_{j,k}|^q X_{Q_{j,k} \cap A_\ell}(\cdot) \right)^{1/q_0} \bigg| L_{p_0}(\mathbb{R}^d, w_0) \right\|
\]
\[
= \left\| \left( \sum_{\ell = -\infty}^{\infty} \sum_{(j, k) \in C_\ell} 2^{j+q_0} 2^{j+q_0} |\lambda_{j,k}|^q X_{Q_{j,k} \cap A_\ell}(\cdot) \right)^{1/q_0} \bigg| L_{p_0}(\mathbb{R}^d, w_0) \right\|
\]
\[
\leq \left\| \sum_{\ell = -\infty}^{\infty} X_{A_\ell \setminus A_{\ell+1}}(\cdot) \left( \sum_{\ell = -\infty}^{\infty} \sum_{(j, k) \in C_\ell} 2^{j+q_0} 2^{j+q_0} |\lambda_{j,k}|^q X_{Q_{j,k} \cap A_\ell}(\cdot) \right)^{1/q_0} \bigg| L_{p_0}(\mathbb{R}^d, w_0) \right\|
\]
\[
\leq \left\| \sum_{\ell = -\infty}^{\infty} X_{A_\ell \setminus A_{\ell+1}}(\cdot) \left( \sum_{\ell = -\infty}^{\infty} \sum_{(j, k) \in C_\ell} 2^{j+q_0} 2^{j+q_0} |\lambda_{j,k}|^q X_{j,k}(\cdot) \right)^{1/q_0} \bigg| L_{p_0}(\mathbb{R}^d, w_0) \right\|
\]
Introducing the abbreviation \( D_L := \bigcup_{m \leq L} C_m \) and using that \( \gamma \geq 0 \), we obtain
\[
\| \lambda^0 \|_{f_p^{s_0, q_0}(\mathbb{R}^d, w_0)} \leq \left\| \sum_{\ell = -\infty}^{\infty} X_{A_\ell \setminus A_{\ell+1}}(\cdot) 2^{L\gamma} \left( \sum_{(j, k) \in D_L} 2^{j+q} |\lambda_{j,k}|^q X_{j,k}(\cdot) \right)^{1/q_0} \bigg| L_{p_0}(\mathbb{R}^d, w_0) \right\|
\]
\[
\leq \left\| \sum_{\ell = -\infty}^{\infty} X_{A_\ell \setminus A_{\ell+1}}(\cdot) 2^{L\gamma} \left( \sum_{j=0}^{\infty} \sum_{k \in \mathbb{Z}^d} 2^{j+q} |\lambda_{j,k}|^q X_{j,k}(\cdot) \right)^{1/q_0} \bigg| L_{p_0}(\mathbb{R}^d, w_0) \right\|
\]
Let
\[ f(\cdot) := \left( \sum_{j=0}^{\infty} \sum_{k \in \mathbb{Z}^d} 2^{j+q} |\lambda_{j,k}|^q X_{j,k}(\cdot) \right)^{1/q} . \]
We employ the definition of \( A_L \) and find
\[
\| \lambda^0 \|_{f_p^{s_0, q_0}(\mathbb{R}^d, w_0)} \leq \left\| \sum_{\ell = -\infty}^{\infty} X_{A_\ell \setminus A_{\ell+1}}(\cdot) f(\cdot) \left( \frac{w(\cdot)}{w_0(\cdot)} \right)^{\frac{1}{p_0}} \left( \frac{1}{q_0} \right)^{\frac{1}{q_0} - \gamma} f(\cdot)^{\frac{q}{q_0}} \bigg| L_{p_0}(\mathbb{R}^d, w_0) \right\|
\]
\[
= \left\| f^{\gamma + q/q_0} \left( \frac{w(\cdot)}{w_0(\cdot)} \right) \bigg| L_{p_0}(\mathbb{R}^d, w_0) \right\|
\]
\[
= \left\| f^{\frac{q}{q_0}} (w/w_0)^{\frac{1}{p_0}} \bigg| L_{p_0}(\mathbb{R}^d, w_0) \right\|
\]
\[
= \left\| f \left| L_p(\mathbb{R}^d, w) \right|^{p/p_0} = \| \lambda f^{p/q}(\mathbb{R}^d, w) \|^{p/p_0} \right. . \]
Substep 2.4. Now we prove (21). We only make some comments concerning necessary modifications in comparison with Substep 2.3. We first point out, that the identity
\[
\left( \frac{w(x)}{w_0(x)} \right)^{\frac{1}{p_0}} = \left( \frac{w(x)}{w_1(x)} \right)^{\frac{1}{p_1}}, \quad x \in A
\]
raised to the appropriate power gives
\[
\left( \frac{w(x)}{w_0(x)} \right)^{\frac{1}{p_0} \frac{1}{p_0} \frac{1}{q_0}} = \left( \frac{w(x)}{w_1(x)} \right)^{\frac{1}{p_1} \frac{1}{p_1} \frac{1}{q_1}}, \quad x \in A. \tag{22}
\]
This means, that the definition of the sets $A_\ell$ and $C_\ell$ does not change, if we replace $(w_0, p_0, q_0)$ by $(w_1, p_1, q_1)$. As $\delta \leq 0$ in this case, we are forced to replace the sets $Q_{j,k} \cap A_\ell$ by $Q_{j,k} \cap A_{\ell+1}^c$. Observe that $|Q_{j,k} \cap A_{\ell+1}^c| \geq \frac{|Q_{j,k}|}{2}$ and hence
\[
\mathcal{X}_{j,k}(x) \leq 2 (M^\text{loc} \mathcal{X}_{Q_{j,k} \cap A_{\ell+1}^c})(x), \quad x \in \mathbb{R}^d, \ (j,k) \in C_\ell.
\]
This, together with $v + s_1 = sq/q_1$ and the maximal inequality (21), leads to
\[
\| \lambda^1 f_{p_1,q_1}^{s_1}(\mathbb{R}^d, w_1) \| \lesssim \left( \sum_{\ell = -\infty}^{\infty} \sum_{(j,k) \in C_\ell} 2^{\delta q_1} \lambda_{j,k} |Q_{j,k} \cap A_{\ell+1}^c(\cdot)| \right)^{1/q_1} \left( \int_{\mathbb{R}^d} (w_1) \right) ||L_{p_1}(\mathbb{R}^d, w_1)||
\]
\[
\leq \left( \sum_{\ell = -\infty}^{\infty} \sum_{(j,k) \in C_\ell} 2^{\delta q_1} \lambda_{j,k} |Q_{j,k} \cap A_{\ell+1}^c(\cdot)| \right)^{1/q_1} \left( \int_{\mathbb{R}^d} (w_1) \right) ||L_{p_1}(\mathbb{R}^d, w_1)||
\]
\[
\leq \left( \sum_{L = -\infty}^{\infty} \mathcal{X}_{j,k}^{A_{\ell+1} \setminus A_{L}^c(\cdot)} \left( \sum_{\ell = -\infty}^{\infty} \sum_{(j,k) \in C_\ell} 2^{\delta q_1} \lambda_{j,k} |Q_{j,k} \cap A_{\ell+1}^c(\cdot)| \right)^{1/q_1} \left( \int_{\mathbb{R}^d} (w_1) \right) ||L_{p_1}(\mathbb{R}^d, w_1)||
\]
\[
\leq \left( \sum_{L = -\infty}^{\infty} \mathcal{X}_{j,k}^{A_{\ell+1} \setminus A_{L}^c(\cdot)} \left( \sum_{\ell = -\infty}^{\infty} \sum_{(j,k) \in C_\ell} 2^{\delta q_1} \lambda_{j,k} |Q_{j,k} \cap A_{\ell+1}^c(\cdot)| \right)^{1/q_1} \left( \int_{\mathbb{R}^d} (w_1) \right) ||L_{p_1}(\mathbb{R}^d, w_1)||
\]
Defining $E_L := \bigcup_{m \geq L-1} C_m$ and making use of $\delta \leq 0$ we obtain
\[
\| \lambda^1 f_{p_1,q_1}^{s_1}(\mathbb{R}^d, w_1) \| \lesssim \left( \sum_{L = -\infty}^{\infty} \mathcal{X}_{j,k}^{A_{\ell+1} \setminus A_{L}^c(\cdot)} \right) 2^{\delta q_1} \left( \sum_{(j,k) \in E_L} 2^{sq} \lambda_{j,k} |Q_{j,k} \cap A_{\ell+1}^c(\cdot)| \right)^{1/q_1} \left( \int_{\mathbb{R}^d} (w_1) \right) ||L_{p_1}(\mathbb{R}^d, w_1)||.
\]
As above we put
\[
f(\cdot) := \left( \sum_{j=0}^{\infty} \sum_{k \in \mathbb{Z}^d} 2^{jsq} \lambda_{j,k} |Q_{j,k}(\cdot)| \right)^{1/q}.
\]
The definition of $A_L$ yields
\[
2^L < f(x) \left( \frac{w(x)}{w_1(x)} \right)^{\frac{1}{p_1} \frac{1}{p_1} \frac{1}{q_1}} \leq 2^{L+1}, \quad x \in A_{\ell+1}^c \setminus A_{L}^c.
\]
see (22). Inserting this in the previous inequality we get

\[ \| \lambda^1 f_{p_1,q_1}^{s_1}(\mathbb{R}^d, w_1) \| \lesssim \left( \sum_{L=-\infty}^{\infty} X_{A_{L+1} \setminus A_L}(\cdot) f^{\delta+\eta}(\cdot) \left( \frac{w(\cdot)}{w_1(\cdot)} \right) \right)^{\frac{1}{p_1}} \frac{1}{|L|^{\delta}} \left| L_{p_1}(\mathbb{R}^d, w_1) \right| \]

\[ \lesssim \| f^{\frac{p}{p_1}} (w/w_1)^{\frac{1}{p}} \| L_{p_1}(\mathbb{R}^d, w_1) \|
\]

\[ = \| f \|_{L_p(\mathbb{R}^d, w)} \|^{p/p_1} = \| \lambda f^{s}(\mathbb{R}^d, w) \|^{p/p_1}. \]

This proves (21).

**Step 3.** Let max(q_0, q_1) = \infty.

**Substep 3.1.** First we consider 0 < q_1 < q_0 = \infty. We shall discuss the needed modifications only. As in Step 2, if (j, k) \notin \bigcup_{\ell \in \mathbb{Z}} C_\ell, then we define \lambda^0_{j,k} = \lambda^1_{j,k} = 0. If (j, k) \in C_\ell, we put

\[ \lambda^0_{j,k} := 2^{j_\gamma} 2^{j_\eta} \quad \text{and} \quad \lambda^1_{j,k} := 2^{\delta_\gamma} 2^{j_\eta} |\lambda_{j,k}|^{q/q_1}, \]

where

\[ \gamma := \frac{p}{p_0}, \quad \delta := \frac{p}{p_1} - \frac{q}{q_1}, \quad u := -q \Theta \frac{s_0}{q_1}, \quad v := q (1 - \Theta) \frac{s_0}{q_1}. \]

This implies (\lambda^0_{j,k})^{1-\Theta} (\lambda^1_{j,k})^\Theta = |\lambda_{j,k}|. Again we are going to establish the inequalities (20) and (21). Since there is nothing changed with respect to (21) we deal with (20). Obviously \gamma > 0. Formally it looks like that we lost the influence of \lambda. However, this is not true. By employing the same arguments as in Substep 2.3 and u + s_0 = 0 we obtain

\[ \| \lambda^0 f^{s_0}_{p_0,\infty}(\mathbb{R}^d, w_0) \| \lesssim \sup_{\ell \in \mathbb{Z}} \sup_{(j,k) \in C_\ell} 2^{j_\gamma} X_{Q_{j,k} \cap A_\ell}(\cdot) \| L_{p_0}(\mathbb{R}^d, w_0) \| \]

\[ \leq \left( \sum_{L=-\infty}^{\infty} X_{A_{L+1} \setminus A_L}(\cdot) \right) \sup_{-\infty < \ell \leq L} \sup_{(j,k) \in C_\ell} 2^{j_\gamma} X_{j,k}(\cdot) \| L_{p_0}(\mathbb{R}^d, w_0) \| \]

\[ \leq \left( \sum_{L=-\infty}^{\infty} X_{A_{L+1} \setminus A_L}(\cdot) \right) 2^{\gamma} \sup_{(j,k) \in D_\ell} X_{j,k}(\cdot) \| L_{p_0}(\mathbb{R}^d, w_0) \| \]

Next we use the definition of the set A_L. In case x \in X_{A_L \setminus A_{L+1}} we conclude

\[ 2^{\gamma} < \left( \sum_{j=0}^{\infty} \sum_{k \in \mathbb{Z}^d} 2^{j_\gamma q} |\lambda_{j,k}|^q X_{j,k}(x) \right)^{1/q} \left( \frac{w(x)}{w_0(x)} \right)^{1/p} \leq 2^{L+1}. \]
We insert this into the previous inequality and find
\[
\|\lambda^0 f_{p_0,\infty}^s(\mathbb{R}^d, w_0)\|
\leq \left\| \sum_{L=-\infty}^{\infty} \mathcal{X}_{A_L \setminus A_{L+1}}(\cdot) \left( \sum_{j=0}^{\infty} \sum_{k \in \mathbb{Z}^d} 2^{jsq} |\lambda_{j,k}|^q \mathcal{X}_{j,k}(x) \right)^{\gamma/q} \left( \frac{w(x)}{w_0(x)} \right)^{\gamma/p} \right\|_{L_{p_0}(\mathbb{R}^d, w_0)}
\]
\[
\leq \left\| \left( \sum_{j=0}^{\infty} \sum_{k \in \mathbb{Z}^d} 2^{jsq} |\lambda_{j,k}|^q \mathcal{X}_{j,k}(x) \right)^{\gamma/q} \left( \frac{w(x)}{w_0(x)} \right)^{1/p_0} \right\|_{L_{p_0}(\mathbb{R}^d, w_0)}
\]
\[
= \left\| \left( \sum_{j=0}^{\infty} \sum_{k \in \mathbb{Z}^d} 2^{jsq} |\lambda_{j,k}|^q \mathcal{X}_{j,k}(x) \right)^{1/q} \right\|_{L_p(\mathbb{R}^d, w)}^{p/p_0},
\]
as we wanted to prove.

Substep 3.2. Observe that the case \(0 < q_0 < q_1 = \infty\) follows by symmetry.

Substep 3.3. It remains to study the case \(q_0 = q_1 = \infty\). If \((j, k) \in C_\ell\), we choose \(\lambda_{j,k}^i\), \(i = 0, 1\), s.t.
\[
\lambda_{j,k}^0 := 2^{\ell \gamma} 2^{ju} |\lambda_{j,k}| \quad \text{and} \quad \lambda_{j,k}^1 := 2^{\ell \delta} 2^{iv} |\lambda_{j,k}|,
\]
where
\[
\gamma := \frac{p}{p_0} - 1, \quad \delta := \frac{p}{p_1} - 1, \quad u := \Theta (s_1 - s_0), \quad v := (1 - \Theta)(s_0 - s_1).
\]
Again this implies \((\lambda_{j,k}^0)^{1-\Theta} (\lambda_{j,k}^0)^{\Theta} = |\lambda_{j,k}|\). Without loss of generality we may assume that \(p_0 \leq p \leq p_1\), i.e., \(\gamma \geq 0\). Now, using \(u + s_0 = s\), we may proceed as in Substep 3.1 and obtain
\[
\|\lambda^0 f_{p_0,\infty}^s(\mathbb{R}^d, w_0)\| \leq \left\| \sum_{L=-\infty}^{\infty} \mathcal{X}_{A_L \setminus A_{L+1}}(\cdot) 2^{L \gamma} \sup_{(j,k) \in D_L} 2^{js} |\lambda_{j,k}| \mathcal{X}_{j,k}(x) \right\|_{L_{p_0}(\mathbb{R}^d, w_0)}.
\]
Employing the definition of the set \(A_L\) we conclude
\[
2^L < \sup_{j \in \mathbb{N}_0} \sup_{k \in \mathbb{Z}^d} 2^{js} |\lambda_{j,k}| \mathcal{X}_{j,k}(x) \left( \frac{w(x)}{w_0(x)} \right)^{\frac{1}{p_0} \frac{1}{p_0 - 1}} \leq 2^{L + 1}, \quad x \in \mathcal{X}_{A_L \setminus A_{L+1}}.
\]
We insert this into the previous inequality and find
\[
\|\lambda^0 f_{p_0,\infty}^s(\mathbb{R}^d, w_0)\|
\leq \left\| \sum_{L=-\infty}^{\infty} \mathcal{X}_{A_L \setminus A_{L+1}}(\cdot) \left( \sup_{j \in \mathbb{N}_0} \sup_{k \in \mathbb{Z}^d} 2^{js} |\lambda_{j,k}| \mathcal{X}_{j,k}(x) \right)^{\gamma + 1} \left( \frac{w(x)}{w_0(x)} \right)^{\frac{1}{p_0} \frac{1}{p_0 - 1}} \right\|_{L_{p_0}(\mathbb{R}^d, w_0)}
\]
\[
\leq \left\| \left( \sup_{j \in \mathbb{N}_0} \sup_{k \in \mathbb{Z}^d} 2^{js} |\lambda_{j,k}| \mathcal{X}_{j,k}(x) \right)^{p/p_0} \left( \frac{w(x)}{w_0(x)} \right)^{\frac{1}{p_0}} \right\|_{L_{p_0}(\mathbb{R}^d, w_0)}
\]
\[
= \left\| \sup_{j \in \mathbb{N}_0} \sup_{k \in \mathbb{Z}^d} 2^{js} |\lambda_{j,k}| \mathcal{X}_{j,k}(x) \right\|_{L_p(\mathbb{R}^d, w)}^{p/p_0},
\]
as we wanted to prove. The estimate of \(\|\lambda^1 f_{p_1,\infty}^s(\mathbb{R}^d, w_1)\|\) follows by similar arguments. 
\[\blacksquare\]
Remark 8 (i) The identity
\[ f_{p_0,q_0}(\mathbb{R}^d)^{1-\Theta} f_{p_1,q_1}(\mathbb{R}^d)^\Theta = f_{p,q}(\mathbb{R}^d) \]
i.e., formula (15) with \( w_0 = w_1 \equiv 1 \), has been proved in Frazier and Jawerth [6]. Our proof, given here, is just an adaptation to the weighted situation. However, let us mention, that we had some advantage from Bownik’s proof in [3], in particular in case \( \max(q_0, q_1) = \infty \). In fact, Bownik had considered the situation
\[ f_{p_0,q_0}(\mathbb{R}^d, \mu)^{1-\Theta} f_{p_1,q_1}(\mathbb{R}^d, \mu)^\Theta = f_{p,q}(\mathbb{R}^d, \mu), \]
where \( \mu \) is a doubling measure. In Substep 2.1 we also used some ideas from Yang, Yuan and Zhuo [18]. These three authors dealt with extensions to sequence spaces related to Lizorkin-Triebel spaces built on Morrey spaces. Further we would like to mention that Wojciechowska [4] recently proved
\[ f_{p_0,q_0}(\mathbb{R}^d, w)^{1-\Theta} f_{p_1,q_1}(\mathbb{R}^d, w)^\Theta = f_{p,q}(\mathbb{R}^d, w), \]
where \( w \in A^{\ell oc}_{\infty} \). The class \( A^{\ell oc}_{\infty} \) and the set of doubling measures are incomparable (more exactly, there exists weights in \( A^{\ell oc}_{\infty} \) which are exponentially growing and therefore do not induce a doubling measure, vice versa there are doubling measures which do not induce a weight in \( A_{\infty} \)). An example of a doubling measure such that the associated weight does not belong to \( A^{\ell oc}_{\infty} \) can be found in Wik’s paper [20]. Furthermore, we refer to [21, §1.8.8] for an example of a doubling measure, which is singular with respect to the Lebesgue measure, i.e. without any associated weight.

Theorem 12 with \( w_0 \neq w_1 \) seems to be a novelty. However, as the previously mentioned results of Bownik indicate, there is some hope to extend it (as well as Corollary 22) to larger classes of weights.

(ii) Frazier, Jawerth [6] and Bownik [3] have also treated extensions of Theorem 12 to \( \max(p_0, p_1) = \infty \).

At the end of this subsection we would like to discuss the class of admissible weights. We had concentrated on the class \( A^{\ell oc}_{\infty} \) in Theorem 12. We do not believe that this is the end of the story and expect that Theorem 12 holds also for more general weights. Let us assume for the moment that Proposition 37 extends to some weights \( w_0, w_1 \) (not necessarily belonging to \( A^{\ell oc}_{\infty} \)). In addition we need the identification of \( L_{p_j}(\mathbb{R}^d, w_j) \) and \( F_{p_j,2}(\mathbb{R}^d, w_j) \), \( 1 \leq p_j < \infty \), \( j = 0, 1 \), which is known to be true only for the class \( A^{\ell oc}_{\infty} \), see Rychkov [22]. These two properties are also needed for the associated weight \( w \). Then Lemma 8 implies that
\[ F_{p_0,2}(\mathbb{R}^d, w_0)^{1-\Theta} F_{p_1,2}(\mathbb{R}^d, w_1)^\Theta = F_{p,2}(\mathbb{R}^d, w) \]
and therefore
\[ f^{d/2}_p(R^d, w_0)^{1-\Theta} f^{d/2}_p(R^d, w_1)^{\Theta} = f^{d/2}_p(R^d, w). \]

### 3.3 Calderón products of \( b_{p,q}^s(R^d, w) \) spaces

**Definition 13** Let \( 0 < p, q \leq \infty \), \( s \in \mathbb{R} \) and let \( w : \mathbb{R}^d \to [0, \infty) \) be a nonnegative and locally integrable function. We put

\[ b_{p,q}^s(R^d, w) := \left\{ \{\lambda_{j,k}\}_{j,k} : \lambda_{j,k} \in \mathbb{C}, \right\} \]

\[ \| (\lambda_{j,k}) | b_{p,q}^s(R^d, w) \| := \left( \sum_{j=0}^{\infty} \left( \sum_{k \in \mathbb{Z}^d} 2^{sj} |\lambda_{j,k}| X_j(k) \right) \right)^{1/q} < \infty. \]  

(25)

**Remark 9**

(i) In case \( w(x) = 1 \) for all \( x \in \mathbb{R}^d \) we are back in the unweighted situation. The associated sequence spaces are denoted simply by \( b_{p,q}^s(R^d) \).

(ii) Let \( w \) satisfy (12). Let \( \check{b}_{p,q}^s(R^d, w) \) denote the closure of the finite sequences in \( b_{p,q}^s(R^d, w) \). We have

\[ \check{b}_{p,q}^s(R^d, w) = b_{p,q}^s(R^d, w) \iff \max(p, q) < \infty. \]

If \( \max(p, q) = \infty \), then \( \check{b}_{p,q}^s(R^d, w) \) is a proper subspace of \( b_{p,q}^s(R^d, w) \).

(iii) Let \( w \) satisfy (12). It is easily checked that \( b_{p,q}^s(R^d, w) \) is separable if, and only if, \( \max(p, q) < \infty \).

Before we are turning to a description of the associated Calderón products we would like to introduce a second type of sequence spaces. Observe

\[ \left\| \sum_{k \in \mathbb{Z}^d} 2^{js} |\lambda_{j,k}| X_j(k) \right\| p \left( \int_{Q_{j,k}} w(x) \, dx \right)^{1/p}. \]

Now we shall replace \( \int_{Q_{j,k}} w(x) \, dx \) by the positive real number \( y_{j,k} \), i.e., instead of a weight function we are using a sequence of positive real numbers.

**Definition 14** Let \( 0 < p, q \leq \infty \), \( s \in \mathbb{R} \) and let \( y := (y_{j,k})_{j,k} \) be a sequence of positive real numbers. We put

\[ b_{p,q}^s(R^d, s - y) := \left\{ \{\lambda_{j,k}\}_{j,k} : \lambda_{j,k} \in \mathbb{C}, \right\} \]

\[ \| (\lambda_{j,k}) | b_{p,q}^s(R^d, s - y) \| := \left( \sum_{j=0}^{\infty} \left( \sum_{k \in \mathbb{Z}^d} 2^{js} p |\lambda_{j,k}| y_{j,k} \right)^{q/p} \right)^{1/q} < \infty. \]  

(26)
Remark 10 Each space \( b^s_{p,q}(\mathbb{R}^d, w), \ w \in \mathcal{A}^\ell_{\infty} \), can be interpreted as a space \( b^s_{p,q}(\mathbb{R}^d, s - y) \) by taking

\[
y_{j,k} := \int_{Q_{j,k}} w(x) \, dx, \quad j \in \mathbb{N}_0, \quad k \in \mathbb{Z}^d.
\]

As a consequence of the formula

\[
b^{s_0}_{p_0,q_0}(\mathbb{R}^d)^{1-\Theta} b^{s_1}_{p_1,q_1}(\mathbb{R}^d)^{\Theta} = b^s_{p,q}(\mathbb{R}^d),
\]
due to Kalton, Mayboroda, Mitrea [8, Proposition 9.3], we derive the following.

Corollary 15 Let \( 0 < \Theta < 1 \). Let \( 0 < p_0, p_1 \leq \infty, \ 0 < q_0, q_1 \leq \infty \) and \( s_0, s_1 \in \mathbb{R} \). Let \( p, q \) and \( s \) be as in (13). Let \( y^0 := (y^0_{j,k})_{j,k}, y^1 := (y^1_{j,k})_{j,k} \) be sequences of positive real numbers. We put

\[
y_{j,k} := (y^0_{j,k})^{(1-\Theta)p_0}_p (y^1_{j,k})^{\Theta q_1}_q.
\]

Then

\[
b^{s_0}_{p_0,q_0}(\mathbb{R}^d, s - y^0)^{1-\Theta} b^{s_1}_{p_1,q_1}(\mathbb{R}^d, s - y^1)^{\Theta} = b^s_{p,q}(\mathbb{R}^d, s - y)
\]

holds in the sense of equivalent quasi-norms.

Proof. Of course, \( y := (y_{j,k})_{j,k} \) is a sequence of positive real numbers.

Step 1. A preparation. Let \( \varrho := (\varrho_{j,k})_{j,k} \) be a sequence of positive real numbers. We introduce the associated family of mappings

\[
T_{\varrho,p} : (\lambda_{j,k})_{j,k} \mapsto (\lambda_{j,k} \cdot \varrho^{1/p}_{j,k})_{j,k},
\]

where \( 0 < p \leq \infty \) is fixed. Obviously, \( T_{\varrho,p} \) is an isomorphism considered as a mapping of \( b^s_{p,q}(\mathbb{R}^d, s - \varrho) \) onto \( b^s_{p,q}(\mathbb{R}^d) \) for all \( s \) and all \( q \).

Step 2. Again the embedding

\[
b^{s_0}_{p_0,q_0}(\mathbb{R}^d, s - y^0)^{1-\Theta} b^{s_1}_{p_1,q_1}(\mathbb{R}^d, s - y^1)^{\Theta} \hookrightarrow b^s_{p,q}(\mathbb{R}^d, s - y)
\]

follows by repeated use of Hölder’s inequality.

Step 3. We deal with

\[
b^s_{p,q}(\mathbb{R}^d, s - y) \hookrightarrow b^{s_0}_{p_0,q_0}(\mathbb{R}^d, s - y^0)^{1-\Theta} b^{s_1}_{p_1,q_1}(\mathbb{R}^d, s - y^1)^{\Theta}.
\]

Let the sequence \( \lambda \in b^s_{p,q}(\mathbb{R}^d, s - y) \) be given. We have to find sequences \( \lambda^0 \) and \( \lambda^1 \) such that

\[
|\lambda_{j,k}| \leq |\lambda^0_{j,k}|^{1-\Theta} \cdot |\lambda^1_{j,k}|^{\Theta} \quad \text{for every} \quad j \in \mathbb{N}_0 \quad \text{and} \quad k \in \mathbb{Z}^d
\]

and

\[
\|\lambda^0 b^{s_0}_{p_0,q_0}(\mathbb{R}^d, w_0)\|^{1-\Theta} \cdot \|\lambda^1 b^{s_1}_{p_1,q_1}(\mathbb{R}^d, w_1)\|^{\Theta} \leq c \|\lambda b^s_{p,q}(\mathbb{R}^d, w)\|,
\]
with some constant \( c \) independent of \( \lambda \). Now we are going to use the unweighted case

\[
b_{p,q}^*(\mathbb{R}^d) = b_{p_0,q_0}^0(\mathbb{R}^d)^{1-\Theta} b_{p_1,q_1}^{s_1}(\mathbb{R}^d)^\Theta,
\]

see Kalton, Mayboroda, Mitrea \[8\] Proposition 9.3. Let \( \gamma_{j,k} := \lambda_{j,k} \cdot y_{j,k}^{1/p}, \ j \in \mathbb{N}_0, \ k \in \mathbb{Z}^d \). Since \( \gamma := (\gamma_{j,k})_{j,k} \in b_{p,q}^*(\mathbb{R}^d) \) this implies the existence of sequences \( \gamma^0 := (\gamma^0_{j,k})_{j,k} \in b_{p_0,q_0}^0(\mathbb{R}^d), \gamma^1 := (\gamma^1_{j,k})_{j,k} \in b_{p_1,q_1}^{s_1}(\mathbb{R}^d) \) s.t. \( |\gamma_{j,k}| \leq |\gamma^0_{j,k}|^{1-\Theta} \cdot |\gamma^1_{j,k}|^\Theta \) for all \( j \in \mathbb{N}_0 \) and all \( k \in \mathbb{Z}^d \) and

\[
|\gamma^0_{j,k}|^{1-\Theta} \cdot |\gamma^1_{j,k}|^\Theta \leq c \|b_{p,q}^*(\mathbb{R}^d)\|
\]

with some constant \( c \) independent of \( \gamma \). We define

\[
\lambda^0_{j,k} := \frac{\gamma^0_{j,k}}{(y^0_{j,k})^{1/p_0}} \quad \text{and} \quad \lambda^1_{j,k} := \frac{\gamma^1_{j,k}}{(y^1_{j,k})^{1/p_1}}.
\]

The sequences \( \lambda^0, \lambda^1 \) obviously meet the above requirements.

For nonnegative and locally integrable functions \( w_0, w_1 \), satisfying (12), we define

\[
y^i_{j,k} := \int_{Q_{j,k}} w_i(x) \, dx, \quad j \in \mathbb{N}_0, \ k \in \mathbb{Z}^d, \ i = 0, 1.
\]

Then

\[
b_{p_0,q_0}^0(\mathbb{R}^d,w_0)^{1-\Theta} b_{p_1,q_1}^{s_1}(\mathbb{R}^d,w_1)^\Theta = b_{p_0,q_0}^0(\mathbb{R}^d,s-y^0)^{1-\Theta} b_{p_1,q_1}^{s_1}(\mathbb{R}^d,s-y^1)^\Theta
\]

follows by using the natural interpretation, see (27). However, without extra conditions on the functions \( w_0 \) and \( w_1 \) we can not interpret \( b_{p,q}^*(\mathbb{R}^d,s-y) \) as the space \( b_{p,q}^*(\mathbb{R}^d,w) \), where the function \( w \) is defined as in (14). A sufficient condition consists in

\[
\int_{Q_{j,k}} w(x) \, dx \simeq \left( \int_{Q_{j,k}} w_0(x) \, dx \right)^{(1-\Theta)p/p_0} \left( \int_{Q_{j,k}} w_1(x) \, dx \right)^{\Theta p/p_1}
\]

for all \( j \) and all \( k \). This gives rise to the following definition.

**Definition 16** Let \( 0 < p_0, p_1 \leq \infty \) and \( 0 < \Theta < 1 \). Define \( p \) by \( 1/p := (1-\Theta)/p_0 + \Theta/p_1 \). Let \( w_j : \mathbb{R}^d \to [0,\infty), \ j = 0, 1, \) be nonnegative and locally integrable functions s.t. (12) is satisfied. We say that the pair \( (w_0,w_1) \) belongs to the class \( \mathcal{W}(\Theta,p_0,p_1) \) if

\[
\int_{Q_{j,k}} w_0(x)^{(1-\Theta)p/p_0} w_1(x)^{\Theta p/p_1} \, dx \simeq \left( \int_{Q_{j,k}} w_0(x) \, dx \right)^{(1-\Theta)p/p_0} \left( \int_{Q_{j,k}} w_1(x) \, dx \right)^{\Theta p/p_1}
\]

holds for all \( j \in \mathbb{N}_0 \) and all \( k \in \mathbb{Z}^d \).
Lemma 17 Suppose \( w_0, w_1 \in A_{\infty}^{\text{loc}} \). Then the pair \( (w_0, w_1) \) belongs to all classes \( \mathfrak{W}(\Theta, p_0, p_1) \).

Proof. The inequality

\[
\int_{Q_{j,k}} w(x) \, dx \leq \left( \int_{Q_{j,k}} w_0(x) \, dx \right)^{\frac{(1-\Theta)p}{p_0}} \left( \int_{Q_{j,k}} w_1(x) \, dx \right)^{\frac{\Theta p}{p_1}}
\]

is a consequence of Hölder’s inequality. To prove the opposite inequality we shall use the following characterization of \( A_{\infty}^{\text{loc}} \): a weight \( v \) belongs to \( A_{\infty}^{\text{loc}} \) if, and only if, there exists a constant \( C > 0 \) s.t.

\[
\frac{1}{|Q|} \int_Q v(x) \, dx \leq C \exp \left( \frac{1}{|Q|} \int_Q \log v(x) \, dx \right)
\]

holds for all cubes (with sides parallel to the axes) and volume \( \leq 1 \), cf. Theorem 2.15, p. 407, in [23]. Using this inequality we obtain, for all such cubes \( Q \),

\[
\left( \frac{1}{|Q|} \int_Q w_0(x) \, dx \right)^{\frac{(1-\Theta)p}{p_0}} \leq C_0 \exp \left( \frac{1-\Theta}{p_0} \int_Q \log w_0(x) \, dx \right)
\]

and

\[
\left( \frac{1}{|Q|} \int_Q w_1(x) \, dx \right)^{\frac{\Theta p}{p_1}} \leq C_1 \exp \left( \frac{\Theta p}{p_1} \int_Q \log w_1(x) \, dx \right).
\]

Multiplying these inequalities and applying Jensen’s inequality yields

\[
\left( \int_{Q_{j,k}} w_0(x) \, dx \right)^{\frac{(1-\Theta)p}{p_0}} \left( \int_{Q_{j,k}} w_1(x) \, dx \right)^{\frac{\Theta p}{p_1}} \leq C_0 C_1 |Q| \exp \left( \frac{1}{|Q|} \int_Q \log (w_0(x))^{\frac{(1-\Theta)p}{p_0}} \, dx + \frac{1}{|Q|} \int_Q \log (w_1(x))^{\frac{\Theta p}{p_1}} \, dx \right)
\]

\[
= C |Q| \exp \left( \frac{1}{|Q|} \int_Q \log \left( (w_0(x))^{\frac{(1-\Theta)p}{p_0}} (w_1(x))^{\frac{\Theta p}{p_1}} \right) \, dx \right)
\]

\[
\leq C \int_Q (w_0(x))^{\frac{(1-\Theta)p}{p_0}} (w_1(x))^{\frac{\Theta p}{p_1}} \, dx
\]

\[
= C \int_Q w(x) \, dx,
\]

which completes the proof of the equivalence.

Corollary 18 Let \( 0 < \Theta < 1 \). Let \( 0 < p_0, p_1 \leq \infty \), \( 0 < q_0, q_1 \leq \infty \) and \( s_0, s_1 \in \mathbb{R} \). Let \( p, q \) and \( s \) be as in (13). Let \( w_0, w_1 \in A_{\infty}^{\text{loc}} \) and \( w \) as in (14). Then

\[
b_{p_0,q_0}^{s_0}(\mathbb{R}^d, w_0)^{1-\Theta} b_{p_1,q_1}^{s_1}(\mathbb{R}^d, w_1)^{\Theta} = b_{p,q}^{s}(\mathbb{R}^d, w)
\]

holds in the sense of equivalent quasi-norms.
Proof. This is an immediate consequence of Lemma \ref{lem:17} and Corollary \ref{cor:15}. ■

Remark 11 (i) Any extension of Lemma \ref{lem:17} yields an extension of Corollary \ref{cor:18}. From this point of view it would be of interest to characterize the classes $\mathfrak{W}(\Theta, p_0, p_1)$.

(ii) The formula
\begin{equation}
\hat{b}_{s_0}^{s_0}(\mathbb{R}^d)^{1-\Theta} b_{s_1}^{s_1}(\mathbb{R}^d)^{\Theta} = \hat{b}_{s,q}^{s}(\mathbb{R}^d)
\end{equation}
has been proved by Mendez and Mitrea \cite{7} under the additional restriction $s_0 \neq s_1$. Later on this restriction has been removed by Kalton, Mayboroda and Mitrea \cite{8}. Also in the situation of the $b$-spaces the case of different weights seems to be new.

3.4 Calderón products of $\hat{a}_{p,q}^s(\mathbb{R}^d, w)$ spaces

It is of certain use to study Calderón products of the spaces $\hat{f}_{p,q}^s(\mathbb{R}^d, w)$ and $\hat{b}_{p,q}^s(\mathbb{R}^d, w)$ separately.

Theorem 19 Let $0 < \Theta < 1$. Let $0 < p_0, p_1 < \infty$, $0 < q_0, q_1 \leq \infty$ and $s_0, s_1 \in \mathbb{R}$. Let $p,q$ and $s$ be defined as in \eqref{eq:15}. Let $w_0, w_1 \in A_{\infty}^{loc}$ and define $w$ by the formula \eqref{eq:14}. Then
\begin{equation}
\hat{f}_{p_0,q_0}^{s_0}(\mathbb{R}^d, w_0)^{1-\Theta} f_{p_1,q_1}^{s_1}(\mathbb{R}^d, w_1)^{\Theta} = \hat{f}_{p,q}^{s}(\mathbb{R}^d, w)
\end{equation}
and
\begin{equation}
\hat{f}_{p_0,q_0}^{s_0}(\mathbb{R}^d, w_0)^{1-\Theta} f_{p_1,q_1}^{s_1}(\mathbb{R}^d, w_1)^{\Theta} = f_{p_0,q_0}^{s_0}(\mathbb{R}^d, w_0)^{1-\Theta} f_{p_1,q_1}^{s_1}(\mathbb{R}^d, w_1)^{\Theta} = \hat{f}_{p,q}^{s}(\mathbb{R}^d, w)
\end{equation}
hold in the sense of equivalent quasi-norms.

Proof. The cases $\max(q_0, q_1) < \infty$ are already covered by Theorem \ref{thm:12}. We may concentrate on $\max(q_0, q_1) = \infty$. It will be enough to make some comments to the needed modifications in the proof of Theorem \ref{thm:12}

Step 1. We shall prove
\begin{equation}
\hat{f}_{p_0,\infty}^{s_0}(\mathbb{R}^d, w_0)^{1-\Theta} f_{p_1,\infty}^{s_1}(\mathbb{R}^d, w_1)^{\Theta} \hookrightarrow \hat{f}_{p,q}^{s}(\mathbb{R}^d, w).
\end{equation}

We suppose that sequences $\lambda := (\lambda_{j,k})_{j,k}$, $\lambda^\ell := (\lambda^\ell_{j,k})_{j,k}$, $\ell = 0, 1$, are given and that
\begin{equation}
|\lambda_{j,k}| \leq |\lambda^0_{j,k}|^{1-\Theta} \cdot |\lambda^1_{j,k}|^\Theta
\end{equation}
holds for all $j \in \mathbb{N}_0$ and all $k \in \mathbb{Z}^d$. Now the essential observation is that if $\lambda^0 := (\lambda^0_{j,k})_{j,k}$ is a finite sequence (i.e. only a finite number of the $\lambda^0_{j,k}$ is not vanishing), then $\lambda$ has the same property. Employing Step 1 of the proof of Theorem \ref{thm:12} we know that
\begin{equation}
\|\lambda f_{p,q}^s(\mathbb{R}^d, w)\| \leq \|\lambda^0 f_{p_0,\infty}^{s_0}(\mathbb{R}^d, w_0)\|^{1-\Theta} \cdot \|\lambda^1 f_{p_1,\infty}^{s_1}(\mathbb{R}^d, w_1)\|^\Theta
\end{equation}
holds for all such $\lambda, \lambda^0, \lambda^1$.

For $\lambda \in f^s_{p,q}(\mathbb{R}^d, w)$ we define the $M$-cutoff sequence $\lambda^{(M)}$, $M \in \mathbb{N}$, by putting $\lambda^{(M)}_{j,k} = 0$ if $j > M$ or $\sup_i |k_i| > M$ and $\lambda^{(M)}_{j,k} = \lambda_{j,k}$ otherwise. Then $\lambda \in f^s_{p,q}(\mathbb{R}^d, w)$ if and only if $\lambda^{(M)}$ converge to $\lambda$ in $f^s_{p,q}(\mathbb{R}^d, w)$ if $M \to \infty$, cf. [18]. Now (31) implies that

$$|\lambda_{j,k} - \lambda_{j,k}^{(M)}| \leq |\lambda^0_{j,k} - \lambda_{j,k}^{0,(M)}|^{1-\Theta} \cdot |\lambda_{j,k}^1|^{\Theta},$$

thus

$$\|\lambda - \lambda^{(M)}|f^s_{p,q}(\mathbb{R}^d, w)\| \leq \|\lambda^0 - \lambda^{0,(M)}|f^{s_0}_{p_0,q_0}(\mathbb{R}^d, w_0)\|^{1-\Theta} \cdot \|\lambda^1|f^{s_1}_{p_1,q_1}(\mathbb{R}^d, w_1)\|^{\Theta}.$$ 

Hence, $\lambda \in f^s_{p,q}(\mathbb{R}^d, w)$ and from this the claim follows. The embedding

$$f^{s_0}_{p_0,q_0}(\mathbb{R}^d, w_0)^{1-\Theta} f^{s_1}_{p_1,q_1}(\mathbb{R}^d, w_1)^{\Theta} \hookrightarrow f^s_{p,q}(\mathbb{R}^d, w)$$

follows by symmetry. Finally, observe

$$f^{s_0}_{p_0,q_0}(\mathbb{R}^d, w_0)^{1-\Theta} f^{s_1}_{p_1,q_1}(\mathbb{R}^d, w_1)^{\Theta} \hookrightarrow f^s_{p,q}(\mathbb{R}^d, w).$$

**Step 2.** Now we turn to the proof of

$$f^s_{p,q}(\mathbb{R}^d, w) \hookrightarrow f^{s_0}_{p_0,q_0}(\mathbb{R}^d, w_0)^{1-\Theta} f^{s_1}_{p_1,q_1}(\mathbb{R}^d, w_1)^{\Theta}.$$ 

Let the sequence $\lambda \in f^s_{p,q}(\mathbb{R}^d, w)$ be given. We have to find sequences $\lambda^0 \in f^{s_0}_{p_0,q_0}(\mathbb{R}^d, w_0)$ and $\lambda^1 \in f^{s_1}_{p_1,q_1}(\mathbb{R}^d, w_1)$ such that $|\lambda_{j,k}| \leq |\lambda^0_{j,k}|^{1-\Theta} \cdot |\lambda^1_{j,k}|^{\Theta}$ for every $j \in \mathbb{N}_0$ and $k \in \mathbb{Z}^d$ and

$$\|\lambda^0|f^{s_0}_{p_0,q_0}(\mathbb{R}^d, w_0)\|^{1-\Theta} \cdot \|\lambda^1|f^{s_1}_{p_1,q_1}(\mathbb{R}^d, w_1)\|^{\Theta} \leq c \|\lambda|f^s_{p,q}(\mathbb{R}^d, w)\|$$

with some constant $c$ independent of $\lambda$. For the moment we suppose that $\lambda$ is a finite sequence. Since $\max(q_0, q_1) = \infty$, we have to use the formulas (23) and (24) from Step 3 of the proof of Theorem [12]. In case $q_0 = q_1 = \infty$ it is immediate that $\lambda^0$ and $\lambda^1$ are also finite sequences. For the case $0 < q_1 < q_0 = \infty$ we need a simple modification (and similarly in case $0 < q_0 < q_1 = \infty$). Obviously we may assume that $\lambda^1$ is a finite sequence. We define

$$\lambda^0_{j,k} := \begin{cases} 2^{t_j} 2^{u_k} & \text{if } \lambda^1_{j,k} \neq 0, \\ 0 & \text{otherwise}. \end{cases}$$

Then also $\lambda^0$ is a finite sequence and (31) is guaranteed by Step 3 of the proof of Theorem [12]. Now we turn to a Cauchy sequence $(\lambda^{(M)})_M$ of finite sequences, convergence in $f^s_{p,q}(\mathbb{R}^d, w)$ with limit $\lambda$. Using the formulas (23) and (24) we obtain a sequence $\lambda^{1(M)}$ of finite sequences which is convergent in $f^{s_1}_{p_1,q_1}(\mathbb{R}^d, w_1)$ with limit $\lambda^1$. Similarly, taking into account the above modification, we get a sequence $\lambda^{0(M)}$ of finite sequences which
is convergent in \( f^{s_0}_{p_0,q_0}(\mathbb{R}^d, w_0) \) with limit \( \lambda^0 \). In both cases convergence is derived by using the inequalities

\[
\|\lambda^0 f^{s_0}_{p_0,q_0}(\mathbb{R}^d, w_0)\| \lesssim \|\lambda f^s_{p,q}(\mathbb{R}^d, w)\|^{p/p_0}
\]

\[
\|\lambda f^{s_1}_{p_1,q_1}(\mathbb{R}^d, w_1)\| \lesssim \|\lambda f^s_{p,q}(\mathbb{R}^d, w)\|^{p/p_1},
\]

see again Step 3 of the proof of Theorem 12. Since

\[
\hat{f}^{s_0}_{p_0,q_0}(\mathbb{R}^d, w_0)^{1-\Theta} \hat{f}^{s_1}_{p_1,q_1}(\mathbb{R}^d, w_1)^{\Theta} \hookrightarrow \hat{f}^{s_0}_{p_0,q_0}(\mathbb{R}^d, w_0)^{1-\Theta} \hat{f}^{s_1}_{p_1,q_1}(\mathbb{R}^d, w_1)^{\Theta}
\]

as well as

\[
\hat{f}^{s_0}_{p_0,q_0}(\mathbb{R}^d, w_0)^{1-\Theta} \hat{f}^{s_1}_{p_1,q_1}(\mathbb{R}^d, w_1)^{\Theta} \hookrightarrow \hat{f}^{s_0}_{p_0,q_0}(\mathbb{R}^d, w_0)^{1-\Theta} \hat{f}^{s_1}_{p_1,q_1}(\mathbb{R}^d, w_1)^{\Theta}
\]

the proof is complete.

**Remark 12** In case \( w = w_0 = w_1 \equiv 1 \) we refer to Yang, Yuan and Zhuo [18] for a partially different proof.

To derive the counterpart for the \( b \)-spaces we will not use the method of Kalton, Mayboroda and Mitrea [8]. These authors reduced the proof of

\[
b^{s}_{p,q}(\mathbb{R}^d) = b^{s_0}_{p_0,q_0}(\mathbb{R}^d)^{1-\Theta} b^{s_1}_{p_1,q_1}(\mathbb{R}^d)^{\Theta}
\]

to the complex interpolation formula

\[
B^{s}_{p,q}(\mathbb{R}^d) = [B^{s_0}_{p_0,q_0}(\mathbb{R}^d), B^{s_1}_{p_1,q_1}(\mathbb{R}^d)]_{\Theta}.
\]

This time our aim will consists in proving complex interpolation formulas based on assertions on Calderón products. Our proof will be based on the results in the un-weighted case, i.e.,

\[
\hat{b}^{s_0}_{p_0,q_0}(\mathbb{R}^d)^{1-\Theta} \hat{b}^{s_1}_{p_1,q_1}(\mathbb{R}^d)^{\Theta} = \hat{b}^{s}_{p,q}(\mathbb{R}^d)
\]

and

\[
\hat{b}^{s_0}_{p_0,q_0}(\mathbb{R}^d)^{1-\Theta} b^{s_1}_{p_1,q_1}(\mathbb{R}^d)^{\Theta} = b^{s_0}_{p_0,q_0}(\mathbb{R}^d)^{1-\Theta} \hat{b}^{s_1}_{p_1,q_1}(\mathbb{R}^d)^{\Theta} = \hat{b}^{s}_{p,q}(\mathbb{R}^d)
\]

in the sense of equivalent quasi-norms. For this we refer to the recent paper of Yang, Yuan and Zhuo [18].

**Theorem 20** Let \( 0 < \Theta < 1 \). Let \( 0 < p_0, p_1, q_0, q_1 \leq \infty \) and \( s_0, s_1 \in \mathbb{R} \). Let \( p, q \) and \( s \) be defined as in [13]. Let \( w_0, w_1 \in \mathcal{A}^{\infty}_{1 \infty} \) and define \( w \) by the formula [13]. Then

\[
\hat{b}^{s_0}_{p_0,q_0}(\mathbb{R}^d, w_0)^{1-\Theta} \hat{b}^{s_1}_{p_1,q_1}(\mathbb{R}^d, w_1)^{\Theta} = \hat{b}^{s}_{p,q}(\mathbb{R}^d, w)
\] (32)
and
\[
\dot{b}_{p_0,q_0}^{s_0}(\mathbb{R}^d, w_0)^{1-\Theta} \dot{b}_{p_1,q_1}^{s_1}(\mathbb{R}^d, w_1)^{\Theta} = \dot{b}_{p_0,q_0}^{s_0}(\mathbb{R}^d, w_0)^{1-\Theta} \dot{b}_{p_1,q_1}^{s_1}(\mathbb{R}^d, w_1)^{\Theta} = \dot{b}_{p,q}^{s}(\mathbb{R}^d, w) \tag{33}
\]
hold in the sense of equivalent quasi-norms.

**Proof.** Step 1. Concerning the embedding
\[
\dot{b}_{p_0,q_0}^{s_0}(\mathbb{R}^d, w_0)^{1-\Theta} \dot{b}_{p_1,q_1}^{s_1}(\mathbb{R}^d, w_1)^{\Theta} \hookrightarrow \dot{b}_{p,q}^{s}(\mathbb{R}^d, w)
\]
the arguments from Step 1 of the proof of Theorem 19 carry over to the present situation. The embedding
\[
\dot{b}_{p_0,q_0}^{s_0}(\mathbb{R}^d, w_0)^{1-\Theta} \dot{b}_{p_1,q_1}^{s_1}(\mathbb{R}^d, w_1)^{\Theta} \hookrightarrow \dot{b}_{p,q}^{s}(\mathbb{R}^d, w)
\]
follows by symmetry.

**Step 2.** It remains to prove
\[
\dot{b}_{p,q}^{s}(\mathbb{R}^d, w) \hookrightarrow \dot{b}_{p_0,q_0}^{s_0}(\mathbb{R}^d, w_0)^{1-\Theta} \dot{b}_{p_1,q_1}^{s_1}(\mathbb{R}^d, w_1)^{\Theta}.
\]
Here we proceed as in Step 3 of the proof of Corollary 15 taking into account Lemma 17 and the formulas (32), (33).

\[\square\]

## 4 Complex interpolation of weighted Besov and Lizorkin-Triebel spaces

Now we transfer our results on Calderón products into results on complex interpolation.

### 4.1 Complex interpolation of the spaces \( a_{p,q}^{s}(\mathbb{R}^d, w) \)

We have to take into account the following supplement to Proposition 9.

**Lemma 21** Let \( X_0, X_1 \) be a pair of quasi-Banach sequence lattices. Then, if both \( X_0 \) and \( X_1 \) are analytically convex and at least one is separable, it follows that \( X_0 + X_1 \) is analytically convex and
\[
[X_0, X_1]_{\Theta} = X_0^{1-\Theta} X_1^{\Theta}, \quad 0 < \Theta < 1.
\]

**Proof.** This is the contents of the Remark in front of Theorem 7.10 in [8], see also [7]. \[\square\]

We only need to summarize what we did before.
Corollary 22 Let $0 < \Theta < 1$, $0 < p_0, p_1 \leq \infty$, $0 < q_0, q_1 \leq \infty$ and $s_0, s_1 \in \mathbb{R}$. Let $p, q$ and $s$ be as in (13). Let $w_0, w_1 \in A_{\infty}^{loc}$ and define $w$ by the formula (14).

(i) If $\max(p_0, p_1) < \infty$ and $\min(q_0, q_1) < \infty$, then

$$[f_{p_0, q_0}^{s_0}(\mathbb{R}^d, w_0), f_{p_1, q_1}^{s_1}(\mathbb{R}^d, w_1)]_{\Theta} = f_{p, q}^{s}(\mathbb{R}^d, w)$$

holds in the sense of equivalent quasi-norms.

(ii) If $\max(p_0, q_0) < \infty$ or $\max(p_1, q_1) < \infty$, then

$$[b_{p_0, q_0}^{s_0}(\mathbb{R}^d, w_0), b_{p_1, q_1}^{s_1}(\mathbb{R}^d, w_1)]_{\Theta} = b_{p, q}^{s}(\mathbb{R}^d, w)$$

holds in the sense of equivalent quasi-norms.

**Proof.** Step 1. Proof of (i). We have to combine Theorem 12, Lemma 4 and Lemma 21. Because of $f_{p, q}^{s}(\mathbb{R}^d, w)$ is separable if, and only if, $q < \infty$ the claim follows under the condition $\max(q_0, q_1) < \infty$. Taking into account Lemma 21 we replace $\max(q_0, q_1) < \infty$ by $\min(q_0, q_1) < \infty$.

Step 2. Proof of (ii). This time we combine Corollary 18, Lemma 4 and Lemma 21. Because of $b_{p, q}^{s}(\mathbb{R}^d, w)$ is separable if, and only if, $\max(p, q) < \infty$ the claim follows under the condition $\max(p_0, p_1, q_0, q_1) < \infty$. By means of Lemma 21 we may replace $\max(p_0, p_1, q_0, q_1) < \infty$ by $\max(p_0, q_0) < \infty$ or $\max(p_1, q_1) < \infty$.

Remark 13 In case $w_0 = w_1 \equiv 1$ this has been proved in Frazier, Jawerth [6] (f-case), Mendez, Mitrea [7] (b-case with $s_0 \neq s_1$) and Kalton, Mayboroda and Mitrea [8] (b-case). The formula

$$[f_{p_0, q_0}^{s_0}(\mathbb{R}^d, \mu), f_{p_1, q_1}^{s_1}(\mathbb{R}^d, \mu)]_{\Theta} = f_{p, q}^{s}(\mathbb{R}^d, \mu)$$

with $\mu$ being a doubling measure has been established by Bownik [3]. In case $w_0 = w_1 = w \in A_{\infty}^{loc}$

$$[f_{p_0, q_0}^{s_0}(\mathbb{R}^d, w_0), f_{p_1, q_1}^{s_1}(\mathbb{R}^d, w_1)]_{\Theta} = f_{p, q}^{s}(\mathbb{R}^d, w)$$

has been proved in Wojciechowska [4].

The same type of arguments, this time applied with Theorem 19 instead of Theorem 12 and with Theorem 20 instead of Corollary 18 yields the next interesting result. Observe, that all spaces $\hat{a}_{p, q}^{s}(\mathbb{R}^d, w)$, $a \in \{b, f\}$, are separable and analytically convex (use Lemma 4).
Corollary 23 Let $0 < \Theta < 1$, $0 < p_0, p_1 \leq \infty$, $0 < q_0, q_1 \leq \infty$ and $s_0, s_1 \in \mathbb{R}$. Let $p,q$ and $s$ be as in (13). Let $w_0, w_1 \in \mathcal{A}_\infty^{\text{loc}}$ and define $w$ by the formula (14).

(i) If $\max(p_0, p_1) < \infty$, then

$$\left[ \hat{f}^{s_0}_{p_0, q_0}(\mathbb{R}^d, w_0), \hat{f}^{s_1}_{p_1, q_1}(\mathbb{R}^d, w_1) \right]_{\Theta} = \left[ \hat{f}^{s_0}_{p_0, q_0}(\mathbb{R}^d, w_0), \hat{f}^{s_1}_{p_1, q_1}(\mathbb{R}^d, w_1) \right]_{\Theta}$$

holds in the sense of equivalent quasi-norms.

(ii) Always we have

$$\left[ \hat{b}^{s_0}_{p_0, q_0}(\mathbb{R}^d, w_0), \hat{b}^{s_1}_{p_1, q_1}(\mathbb{R}^d, w_1) \right]_{\Theta} = \left[ \hat{b}^{s_0}_{p_0, q_0}(\mathbb{R}^d, w_0), \hat{b}^{s_1}_{p_1, q_1}(\mathbb{R}^d, w_1) \right]_{\Theta} = \hat{b}^{s}_{p, q}(\mathbb{R}^d, w)$$

in the sense of equivalent quasi-norms.

Remark 14 In case $w_0 = w_1 \equiv 1$, Corollary 23 can be found in Yang, Yuan and Zhuo [18].

4.2 Complex interpolation of weighted Besov and Lizorkin-Triebel spaces

Now we turn to the complex interpolation of the distribution spaces $F^s_{p,q}(\mathbb{R}^d, w)$ and $B^s_{p,q}(\mathbb{R}^d, w)$. For a definition of these classes we refer to the Appendix. Observe that neither $F^s_{p,q}(\mathbb{R}^d, w)$ nor $B^s_{p,q}(\mathbb{R}^d, w)$ are quasi-Banach lattices in general. Here our main result is as follows.

Theorem 24 Let $0 < p_0, p_1 \leq \infty$, $0 < q_0, q_1 \leq \infty$, $s_0, s_1 \in \mathbb{R}$, $0 < \Theta < 1$ and define $p, q$ and $s$ according to (13). Let $w_0, w_1 \in \mathcal{A}_\infty^{\text{loc}}$ be local Muckenhoupt weights and $w$ defined as in (14).

(i) If $\max(p_0, p_1) < \infty$ and $\min(q_0, q_1) < \infty$, then

$$\left[ F^{s_0}_{p_0, q_0}(\mathbb{R}^d, w_0), F^{s_1}_{p_1, q_1}(\mathbb{R}^d, w_1) \right]_{\Theta} = F^{s}_{p, q}(\mathbb{R}^d, w)$$

holds in the sense of equivalent quasi-norms.

(ii) If $\max(p_0, q_0) < \infty$ or $\max(p_1, q_1) < \infty$, then

$$\left[ B^{s_0}_{p_0, q_0}(\mathbb{R}^d, w_0), B^{s_1}_{p_1, q_1}(\mathbb{R}^d, w_1) \right]_{\Theta} = B^{s}_{p, q}(\mathbb{R}^d, w)$$

holds in the sense of equivalent quasi-norms.

Proof. It is enough to combine Corollary 22 and Proposition 37 (in the Appendix). If either $p_0 = \infty$ or $p_1 = \infty$, then one has to take into account also Remark 22. \qed

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Remark 15 The above results complement the knowledge on interpolation of weighted Besov and Lizorkin-Triebel spaces with Muckenhoupt weights. Bownik [3] has proved
\[
[F^{s_0}_{p_0,q_0}(\mathbb{R}^d, \mu), F^{s_1}_{p_1,q_1}(\mathbb{R}^d, \mu)]_\Theta = F^s_{p,q}(\mathbb{R}^d, \mu),
\]
where \( \mu \) is a doubling measure. Furthermore, Wojciechowska [4] recently proved
\[
[F^{s_0}_{p_0,q_0}(\mathbb{R}^d, w), F^{s_1}_{p_1,q_1}(\mathbb{R}^d, w)]_\Theta = F^s_{p,q}(\mathbb{R}^d, w),
\]
where \( w \in \mathcal{A}^{\ell \text{oc}}_\infty \). For various interpolation formulas for the real method we refer to Bui [24] (\( w \in \mathcal{A}_\infty \)) and Rychkov [22] (\( w \in \mathcal{A}^{\ell \text{oc}}_\infty \)).

We finish this subsection by formulating a consequence of Corollary 23. Let \( \hat{A}^{s}_{p,q}(\mathbb{R}^d, w) \) denote the closure of the test functions in \( A^s_{p,q}(\mathbb{R}^d, w) \). Using the same arguments as above, but replacing Proposition 37 by Proposition 38, we obtain the following.

Theorem 25 Let \( 0 < \Theta < 1, 0 < p_0, p_1 \leq \infty, 0 < q_0, q_1 \leq \infty \) and \( s_0, s_1 \in \mathbb{R} \). Let \( p, q \) and \( s \) be as in (13). Let \( w_0, w_1 \in \mathcal{A}^{\ell \text{oc}}_\infty \) and define \( w \) by the formula (14).

i) If \( \max(p_0, p_1) < \infty \), then
\[
[F^{s_0}_{p_0,q_0}(\mathbb{R}^d, w_0), F^{s_1}_{p_1,q_1}(\mathbb{R}^d, w_1)]_\Theta = [F^{s_0}_{p_0,q_0}(\mathbb{R}^d, w_0), F^{s_1}_{p_1,q_1}(\mathbb{R}^d, w_1)]_\Theta = [F^{s_0}_{p_0,q_0}(\mathbb{R}^d, w_0), F^{s_1}_{p_1,q_1}(\mathbb{R}^d, w_1)]_\Theta = F^s_{p,q}(\mathbb{R}^d, w)
\]
holds in the sense of equivalent quasi-norms.

ii) Always we have
\[
[B^{s_0}_{p_0,q_0}(\mathbb{R}^d, w_0), B^{s_1}_{p_1,q_1}(\mathbb{R}^d, w_1)]_\Theta = [B^{s_0}_{p_0,q_0}(\mathbb{R}^d, w_0), B^{s_1}_{p_1,q_1}(\mathbb{R}^d, w_1)]_\Theta = [B^{s_0}_{p_0,q_0}(\mathbb{R}^d, w_0), B^{s_1}_{p_1,q_1}(\mathbb{R}^d, w_1)]_\Theta = B^s_{p,q}(\mathbb{R}^d, w)
\]
in the sense of equivalent quasi-norms.

Remark 16 (i) Of course, Theorem 25 is only of interest in the cases \( \max(p_0, p_1, q_0, q_1) = \infty \). All other cases are covered by Theorem 24.

(ii) Theorem 25 has been known since a long time for Besov spaces in the case \( w_0 = w_1 \equiv 1, 1 < p_0, p_1 < \infty \). We refer to Triebel [2, Remark 2.4.1/3]. Theorem 25 in case \( w_0 = w_1 \equiv 1 \) and arbitrary \( p \)'s and \( q \)'s has been proved recently in Yang, Yuan, Zhuo [18].
4.3 Shrinking some gaps

The results in Subsection 4.2 do not cover all possible interpolation couples of the form

\[ [A_{p_0, q_0}^{s_0}(\mathbb{R}^d, w_0), A_{p_1, q_1}^{s_1}(\mathbb{R}^d, w_1)]_\Theta, \quad A \in \{B, F\}. \]

Those cases, where both spaces are not separable, are not covered. This means that a description of

\[
\begin{align*}
[F_{p_0, \infty}^{s_0}(\mathbb{R}^d, w_0), F_{p_1, \infty}^{s_1}(\mathbb{R}^d, w_1)]_\Theta & \quad 0 < p_0, p_1 < \infty, \\
[B_{p_0, \infty}^{s_0}(\mathbb{R}^d, w_0), B_{p_1, \infty}^{s_1}(\mathbb{R}^d, w_1)]_\Theta & \quad 0 < p_0, p_1 \leq \infty, \\
[B_{\infty, q_0}^{s_0}(\mathbb{R}^d, w_0), B_{\infty, q_1}^{s_1}(\mathbb{R}^d, w_1)]_\Theta & \quad 0 < q_0, q_1 \leq \infty, \quad \min(q_0, q_1) < \infty,
\end{align*}
\]

is still open. We can not fill this gap. However, we can make it smaller. The method we will apply is based on a result of Shestakov [9] (see also [15, Remark 4.3.5, pp. 557]):

for Banach lattices \(X_0, X_1\) and \(0 < \Theta < 1\) we have the identity

\[ [X_0, X_1]_\Theta = X_0^1 \cap X_1^1, \quad Y_\Theta := X_0^{1-\Theta} X_1^\Theta. \] (35)

To use these results we have to switch to sequence spaces for a moment. Theorem 12 and Corollary 18 combined with (35) yield

\[
[a_{p_0, q_0}^{s_0}(\mathbb{R}^d, w_0), a_{p_1, q_1}^{s_1}(\mathbb{R}^d, w_1)]_\Theta \hookrightarrow a_{p, q}^{s}(\mathbb{R}^d, w), \quad a \in \{b, f\},
\]

since

\[
\overline{a_{p_0, q_0}^{s_0}(\mathbb{R}^d, w_0) \cap a_{p_1, q_1}^{s_1}(\mathbb{R}^d, w_1)}^{a_{p, q}^{s}(\mathbb{R}^d, w)} \hookrightarrow a_{p, q}^{s}(\mathbb{R}^d, w).
\]

An application of the trivial embedding

\[
[A_{p_0, q_0}^{s_0}(\mathbb{R}^d, w_0), A_{p_1, q_1}^{s_1}(\mathbb{R}^d, w_1)]_\Theta \hookrightarrow [A_{p_0, q_0}^{s_0}(\mathbb{R}^d, w_0), A_{p_1, q_1}^{s_1}(\mathbb{R}^d, w_1)]_\Theta, \quad A \in \{B, F\},
\]

and of Proposition 37 yield the following lemma.

Lemma 26 Let \(1 \leq p_0, p_1 \leq \infty, 1 \leq q_0, q_1 \leq \infty, s_0, s_1 \in \mathbb{R}, 0 < \Theta < 1\) and define \(p, q\) and \(s\) according to (13). Let \(w_0, w_1 \in \mathcal{A}_{\infty}^{\text{loc}}\) be local Muckenhoupt weights and \(w^0, w^1\) defined as in (13).

(i) If \(\max(p_0, p_1) < \infty\), then

\[
\mathcal{F}_{p, q}^s(\mathbb{R}^d, w) \hookrightarrow [F_{p_0, q_0}^{s_0}(\mathbb{R}^d, w_0), F_{p_1, q_1}^{s_1}(\mathbb{R}^d, w_1)]_\Theta \hookrightarrow \mathcal{F}_{p, q}^s(\mathbb{R}^d, w)
\]

holds.

(ii) Always we have

\[
\mathcal{B}_{p, q}^s(\mathbb{R}^d, w) \hookrightarrow [B_{p_0, q_0}^{s_0}(\mathbb{R}^d, w_0), B_{p_1, q_1}^{s_1}(\mathbb{R}^d, w_1)]_\Theta \hookrightarrow \mathcal{B}_{p, q}^s(\mathbb{R}^d, w).
\]
Remark 17 Let \( w \in \mathcal{A}^{\text{loc}}_\infty \). The restriction to Banach spaces in Lemma 26 is caused by the use of (35). Recently, Yang, Yuan and Zhuo [18] have proved

\[
[a_{p_0,q_0}^s(\mathbb{R}^d, w), a_{p_1,q_1}^s(\mathbb{R}^d, w)]_\Theta \hookrightarrow a_{p,q}^s(\mathbb{R}^d, w), \quad a \in \{b, f\},
\]

without restrictions on the parameters. In fact, they did it in the unweighted case, but it can be immediately lifted to the present situation. Hence, Lemma 26 remains true for values of \( p_0, p_1, q_0, q_1 < 1 \) if restricted to the case of one weight.

As mentioned above, in some situations we can go one step further.

Theorem 27 Let \( s_0, s_1 \in \mathbb{R} \) and \( 0 < \Theta < 1 \). Further, let either \( 1 \leq p_0 = p_1 < \infty \) and \( s_0 > s_1 \) or \( 1 \leq p_0 < p_1 < \infty \) and

\[
s_0 - \frac{d}{p_0} \geq s_1 - \frac{d}{p_1}.
\]

As always \( p, q \) and \( s \) are defined according to (13). Let \( w \in \mathcal{A}^{\text{loc}}_\infty \) be a local Muckenhoupt weight. Then

\[
\hat{f}^s_{p,\infty}(\mathbb{R}^d, w) = [f_{p_0,\infty}^{s_0}(\mathbb{R}^d, w), f_{p_1,\infty}^{s_1}(\mathbb{R}^d, w)]_\Theta.
\]

holds.

Proof. The conditions guarantee

\[
f_{p_0,\infty}^{s_0}(\mathbb{R}^d) \hookrightarrow f_{p_1,\infty}^{s_1}(\mathbb{R}^d) \hookrightarrow f_{p_1,\infty}^{s_1}(\mathbb{R}^d),
\]

see [14, Theorem 2.7.1]. By means of Proposition 37 this yields

\[
f_{p_0,\infty}^{s_0}(\mathbb{R}^d) \hookrightarrow f_{p_1,\infty}^{s_1}(\mathbb{R}^d) \hookrightarrow f_{p_{1,\infty}}^{s_1}(\mathbb{R}^d),
\]

which can be lifted to the weighted case

\[
f_{p_0,\infty}^{s_0}(\mathbb{R}^d, w) \hookrightarrow f_{p_1,\infty}^{s_1}(\mathbb{R}^d, w) \hookrightarrow f_{p_{1,\infty}}^{s_1}(\mathbb{R}^d, w)
\]

by using an appropriate isomorphism. This implies that

\[
f_{p_0,\infty}^{s_0}(\mathbb{R}^d, w) \cap f_{p_1,\infty}^{s_1}(\mathbb{R}^d, w) = f_{p_0,\infty}^{s_0}(\mathbb{R}^d, w).
\]

We claim

\[
\frac{f_{p_0,\infty}^{s_0}(\mathbb{R}^d, w)}{f_{p_1,\infty}^{s_1}(\mathbb{R}^d, w)} = \hat{f}^s_{p,\infty}(\mathbb{R}^d, w).
\]

To prove this, observe \( \hat{f}^s_{p,\infty}(\mathbb{R}^d, w) = f_{p,\infty}^{s_1}(\mathbb{R}^d, w) \). Hence, finite sequences are dense in the set \( f_{p_0,\infty}^{s_0}(\mathbb{R}^d, w) \) when equipped with the quasi-norm of \( f_{p_1,\infty}^{s_1}(\mathbb{R}^d, w) \). From the trivial embedding \( f_{p,\infty}^{s_1}(\mathbb{R}^d, w) \hookrightarrow f_{p,\infty}^{s_1}(\mathbb{R}^d, w) \) we derive the density of the finite sequences in
the set $f_{p_0,\infty}^{s_0}(\mathbb{R}^d, w)$ when equipped with the quasi-norm of $f_{p,\infty}^{s}(\mathbb{R}^d, w)$. Shestakov’s identity \cite{Shestakov} yields

$$\hat{f}_{p,\infty}^{s}(\mathbb{R}^d, w) = [f_{p_0,\infty}^{s_0}(\mathbb{R}^d, w), f_{p_1,\infty}^{s_1}(\mathbb{R}^d, w)]_{\Theta}.$$

By means of Proposition \ref{prop:identity} we complete the proof. □

Arguing as before we can prove the following counterpart for Besov spaces. For the embedding relations we refer to \cite[2.7.1]{Triebel}.

**Theorem 28** Let $1 \leq q_0, q_1 \leq \infty$, $s_0, s_1 \in \mathbb{R}$ and $0 < \Theta < 1$. Further, let

$$s_0 - \frac{d}{p_0} > s_1 - \frac{d}{p_1},$$

and either $1 \leq p_0 < p_1 \leq \infty$ or $1 \leq p_0 = p_1 < \infty$. As always $p, q$ and $s$ are defined according to \cite{Triebel}. Let $w \in A_{\text{loc}}^{\infty}$ be a local Muckenhoupt weight. Then

$$\hat{B}_{p,q}^{s}(\mathbb{R}^d, w) = [B_{p_0,q_0}^{s_0}(\mathbb{R}^d, w), B_{p_1,q_1}^{s_1}(\mathbb{R}^d, w)]_{\Theta}.$$

holds.

**Remark 18** (i) The case $w \equiv 1$, $1 < p_0 = p_1 < \infty$, $q_0 = q_1 = q = \infty$ has been known before, we refer to Triebel \cite[Theorem 2.4.1]{Triebel}.

(ii) Let us comment on the conditions \eqref{eq:condition} and \eqref{eq:condition2}. Let $w \equiv 1$, $1 \leq p_0 < p_1 \leq \infty$ and

$$s_0 - \frac{d}{p_0} \leq s_1 - \frac{d}{p_1},$$

Furthermore, for $j \in \mathbb{N}_0$ let $K_j$ be a subset of $\mathbb{Z}^d$ with cardinality

$$\#K_j = \left[2^{-j\{(s_1-s_0)\cdot \frac{1}{p_1-\frac{1}{p_0}-d}\}}\right],$$

where $[t]$ denotes the smallest integer larger than or equal to $t \in \mathbb{R}$. We define a sequence $\lambda := \{\lambda_{j,k}\}_{j,k}$ by

$$\lambda_{j,k} = \begin{cases} 2^j \cdot \frac{p_1-s_1-p_0}{p_0-p_1} & \text{if } k \in K_j, \\ 0 & \text{otherwise.} \end{cases}$$

A simple calculation shows that $\lambda \in b_{p,\infty}^{s}(\mathbb{R}^d) \setminus \hat{b}_{p,\infty}^{s}(\mathbb{R}^d)$ as well as $\lambda \in b_{p_0,\infty}^{s_0}(\mathbb{R}^d) \cap b_{p_1,\infty}^{s_1}(\mathbb{R}^d)$. The result of Shestakov \cite{Shestakov} then yields

$$\lambda \in b_{p_0,\infty}^{s_0}(\mathbb{R}^d) \cap b_{p_1,\infty}^{s_1}(\mathbb{R}^d) \setminus \hat{b}_{p,\infty}^{s}(\mathbb{R}^d) = [b_{p_0,\infty}^{s_0}(\mathbb{R}^d), b_{p_1,\infty}^{s_1}(\mathbb{R}^d)]_{\Theta}.$$

Hence, the embedding $\hat{b}_{p,q}^{s}(\mathbb{R}^d) \hookrightarrow [b_{p_0,q_0}^{s_0}(\mathbb{R}^d), b_{p_1,q_1}^{s_1}(\mathbb{R}^d)]_{\Theta}$ is strict in this case. With other words, \cite{Shestakov} is also necessary for the validity of the interpolation formula in Theorem \ref{thm:interpolation}.
5 Complex interpolation of radial subspaces of Besov and Lizorkin-Triebel spaces

In a series of papers the authors have studied radial subspaces of Besov and Lizorkin-Triebel spaces, see \cite{25, 26, 27}. The motivation came from the interesting interplay of decay and smoothness properties of radial functions as expressed in its simplest form in the radial Lemma of Strauss \cite{28} and with important applications for the compactness of some embeddings. We refer also to Lions \cite{29} and Cho and Ozawa \cite{30} in this connection. Let us recall the convention, that if $E$ denotes a space of functions on $\mathbb{R}^d$ then by $RE$ we mean the subset of radial functions in $E$ and we endow this subset with the same quasi-norm as the original space.

5.1 The main result for radial subspaces

In \cite{31} one of the authors has proved that in case $p, q \geq 1$ the spaces $RB^s_{p,q}(\mathbb{R}^d)$, $RF^s_{p,q}(\mathbb{R}^d)$ are complemented subspaces of $B^s_{p,q}(\mathbb{R}^d)$ and $F^s_{p,q}(\mathbb{R}^d)$, respectively. By means of the method of retraction and coretraction, see, e.g., Theorem 1.2.4 in \cite{2}, this allows to transfer the interpolation formulas for the original spaces $B_s^{p,q}(\mathbb{R}^d)$, $F_s^{p,q}(\mathbb{R}^d)$ to their radial subspaces.

Such simple arguments do not seem to be available in case of quasi-Banach spaces. In \cite{26} we announced the following result concerning complex interpolation of radial subspaces of Besov and Lizorkin-Triebel spaces.

**Theorem 29** Let $0 < p_0, p_1 < \infty$, $0 < q_0, q_1 \leq \infty$, $s_0, s_1 \in \mathbb{R}$, and $0 < \Theta < 1$. Define $s := (1 - \Theta) s_0 + \Theta s_1$,

$$\frac{1}{p} := \frac{1 - \Theta}{p_0} + \frac{\Theta}{p_1} \quad \text{and} \quad \frac{1}{q} := \frac{1 - \Theta}{q_0} + \frac{\Theta}{q_1}.$$ 

(i) Let $\min(q_0, q_1) < \infty$. Then we have

$$RB^s_{p,q}(\mathbb{R}^d) = \left[ RB^{s_0}_{p_0,q_0}(\mathbb{R}^d), RB^{s_1}_{p_1,q_1}(\mathbb{R}^d) \right]_{\Theta}.$$ 

(ii) Let $\min(q_0, q_1) < \infty$. Then we have

$$RF^s_{p,q}(\mathbb{R}^d) = \left[ RF^{s_0}_{p_0,q_0}(\mathbb{R}^d), RF^{s_1}_{p_1,q_1}(\mathbb{R}^d) \right]_{\Theta}.$$ 

5.2 The proof of Theorem 29

We are going to prove Theorem 29. Our main idea consists in a reduction of this problem to the weighted case. Therefore we need to recall some results from \cite{26}.
For a real number $s$ we denote by $[s]$ the integer part, i.e. the largest integer $m$ such that $m \leq s$. We put $w_{d-1}(t) := |t|^{d-1}$, $t \in \mathbb{R}$, $d \geq 2$. Finally, if $f : \mathbb{R}^d \to \mathbb{C}$ is a radial function, we denote by
\[
(\text{tr } f)(t) = f(t, 0, \ldots, 0), \quad t \in \mathbb{R}
\]
its trace. The corresponding extension operator is then given by
\[
(\text{ext } g)(x) = g(|x|), \quad x \in \mathbb{R}^d,
\]
where $g$ is an even function on $\mathbb{R}$.

**Proposition 30** Let $d \geq 2$, $0 < p < \infty$ and $0 < q \leq \infty$.
(i) Suppose either $s > d \left(\frac{1}{p} - \frac{1}{q} \right)$ or $s = d \left(\frac{1}{p} - \frac{1}{q} \right)$ and $q \leq 1$. Then the mapping $\text{tr}$ is a linear isomorphism of $RB_{p,q}^s(\mathbb{R}^d)$ onto $RB_{p,q}^s(\mathbb{R}, w_{d-1})$ with inverse $\text{ext}$.
(ii) Suppose either $s > d \left(\frac{1}{p} - \frac{1}{q} \right)$ or $s = d \left(\frac{1}{p} - \frac{1}{q} \right)$ and $0 < p \leq 1$. Then the mapping $\text{tr}$ is a linear isomorphism of $RF_{p,q}^s(\mathbb{R}^d)$ onto $RF_{p,q}^s(\mathbb{R}, w_{d-1})$ with inverse $\text{ext}$.

**Remark 19** Let $p > 1$. As it is well known, the weight $w(x) := |x|^\alpha$ belongs to $A_p(\mathbb{R})$ if, and only if, $-1 < \alpha < p - 1$. This implies $w_{d-1} \in A_p(\mathbb{R})$ for any $p > d$, see [21, pp. 218].

We would like to apply Theorem [24] with respect to $RB_{p,q}^s(\mathbb{R}, w_{d-1})$ and $RF_{p,q}^s(\mathbb{R}, w_{d-1})$, respectively. Therefore we consider the following mapping:
\[
Tf(x) := \frac{1}{2} (f(x) + f(-x)), \quad x \in \mathbb{R}.
\]
Of course,
\[
T \in \mathcal{L}(A_{p,q}^s(\mathbb{R}, w_{d-1}), RA_{p,q}^s(\mathbb{R}, w_{d-1})), \quad A \in \{B, F\}.
\]
Furthermore, $Tf = f$ if $f \in RA_{p,q}^s(\mathbb{R}, w_{d-1})$, i.e., $RA_{p,q}^s(\mathbb{R}, w_{d-1})$ is a retract of $A_{p,q}^s(\mathbb{R}, w_{d-1})$. By the standard method of retraction and coretraction, see [21, 1.2.4] and Lemma 7.11 in [8], we obtain the following.

**Lemma 31** Let $d \geq 2$, $0 < \Theta < 1$, $0 < p_0, p_1 \leq \infty$, $0 < q_0, q_1 \leq \infty$, $s_0, s_1 \in \mathbb{R}$ and define $p, q$ and $s$ according to (13).
(i) If $\max(p_0, p_1) < \infty$ and $\min(q_0, q_1) < \infty$, then
\[
[RF_{p_0,q_0}^{s_0}(\mathbb{R}, w_{d-1}), RF_{p_1,q_1}^{s_1}(\mathbb{R}, w_{d-1})]_{\Theta} = RF_{p,q}^{s}(\mathbb{R}, w_{d-1})
\]
holds in the sense of equivalent quasi-norms.
(ii) If $\max(p_0, q_0) < \infty$ or $\max(p_1, q_1) < \infty$, then
\[
[RB_{p_0,q_0}^{s_0}(\mathbb{R}, w_{d-1}), RB_{p_1,q_1}^{s_1}(\mathbb{R}, w_{d-1})]_{\Theta} = RB_{p,q}^{s}(\mathbb{R}, w_{d-1})
\]
holds in the sense of equivalent quasi-norms.
The next step consists in combining Lemma 31 and Proposition 30.

**Lemma 32** Let \( d \geq 2, \) \( 0 < \Theta < 1, \) \( 0 < p_0, p_1 < \infty, \) \( 0 < q_0, q_1 \leq \infty, \) \( s_0, s_1 \in \mathbb{R} \) and define \( p, q \) and \( s \) according to (13). Furthermore, we suppose

\[
s_i > d \left( \frac{1}{p_i} - \frac{1}{d} \right), \quad i = 0, 1,
\]

If \( \min(q_0, q_1) < \infty, \) then

\[
[RF_{p_0,q_0}^s(\mathbb{R}^d), RF_{p_1,q_1}^s(\mathbb{R}^d)]_\Theta = RF_{p,q}^s(\mathbb{R}^d)
\]

and

\[
[RB_{p_0,q_0}^s(\mathbb{R}^d), RB_{p_1,q_1}^s(\mathbb{R}^d)]_\Theta = RB_{p,q}^s(\mathbb{R}^d)
\]

hold in the sense of equivalent quasi-norms.

**Proof of Theorem 29** It is enough to remove the restrictions for \( s_0 \) and \( s_1 \) in Lemma 32, see (39). But this is an easy task. Let \( \sigma \in \mathbb{R}. \) We consider the family of lifting operators

\[
I_\sigma f := \mathcal{F}^{-1} \left[ (1 + |\xi|^2)^{\sigma/2} \mathcal{F} f(\xi) \right](\cdot), \quad f \in \mathcal{S}'(\mathbb{R}^d).
\]

As it is well known, see, e.g., [14, Theorem 2.3.8], \( I_\sigma \) is an isomorphism, which maps \( A_{s,p,q}^s(\mathbb{R}^d) \) onto \( A_{s-\sigma,p,q}^s(\mathbb{R}^d). \) By standard properties of the Fourier transform we deduce that \( f \in RA_{p,q}^s(\mathbb{R}^d) \) implies \( I_\sigma f \in RA_{p,q}^{s-\sigma}(\mathbb{R}^d). \) By the same argument, \( I_\sigma \) is an isomorphism, which maps \( RA_{p,q}^s(\mathbb{R}^d) \) onto \( RA_{p,q}^{s-\sigma}(\mathbb{R}^d). \) Hence,

\[
[I_\sigma(RF_{p_0,q_0}^s(\mathbb{R}^d)), I_\sigma(RF_{p_1,q_1}^s(\mathbb{R}^d))]_\Theta = I_\sigma(RF_{p,q}^s(\mathbb{R}^d)).
\]

Now, Theorem 29 follows from Lemma 32 by choosing \( \sigma \) appropriate.

### 6 Appendix – Muckenhoupt weights and function spaces

For convenience of the reader we collect some definitions and properties around Muckenhoupt weights and associated weighted function spaces.

#### 6.1 Muckenhoupt weights

A weight function (or simply a weight) is a nonnegative and measurable function on \( \mathbb{R}^d. \) We collect a few facts including the definition of Muckenhoupt and local Muckenhoupt weights. As usual, \( p' \) is related to \( p \) via the formula \( 1/p + 1/p' = 1. \)
Definition 33 Let $1 < p < \infty$. Let $w$ be a nonnegative, locally integrable function on $\mathbb{R}^d$.

(i) Then $w$ belongs to the Muckenhoupt class $A_p$, if

$$A_p(w) := \sup_B \left( \frac{1}{|B|} \int_B w(x)\,dx \right)^{1/p} \cdot \left( \frac{1}{|B|} \int_B w(x)^{-p'/p}\,dx \right)^{1/p'} < \infty,$$

where the supremum is taken with respect to all balls $B$ in $\mathbb{R}^d$.

(ii) The weight $w$ belongs to the local Muckenhoupt class $A_{\text{loc}}^p$, if we restrict the set of admissible balls in the supremum in (i) to those with volume $\leq 1$. We put

$$A_{\text{loc}}^p(w) := \sup_{|B|\leq 1} \left( \frac{1}{|B|} \int_B w(x)\,dx \right)^{1/p} \cdot \left( \frac{1}{|B|} \int_B w(x)^{-p'/p}\,dx \right)^{1/p'} < \infty.$$

The classes $A_\infty$ and $A_{\text{loc}}^\infty$ are defined as

$$A_\infty := \bigcup_{p>1} A_p \quad \text{and} \quad A_{\text{loc}}^\infty := \bigcup_{p>1} A_{\text{loc}}^p,$$

respectively.

Remark 20 (i) Good sources for Muckenhoupt weights are Stein [21], García-Cuerva and Rubio de Francia [23] and the graduate text [32] of Duoandikoetxea.

(ii) The classes of local Muckenhoupt weights $A_{\text{loc}}^p$ have been introduced by Rychkov [22].

The following lemma of Rychkov [22] will be of some use.

Lemma 34 Let $1 < p \leq \infty$, $w \in A_{\text{loc}}^p$, and $Q$ be a cube with sides parallel to the axes and volume 1. Then there exists a weight $\overline{w} \in A_p$ s.t.

$$\overline{w} = w \quad \text{on } Q \quad \text{and} \quad A_p(\overline{w}) \leq c A_{\text{loc}}^p(w).$$

Here $c$ does not depend on $w$ and $Q$.

By $Mf$ we denote the Hardy-Littlewood maximal function of $f$, given by

$$Mf(x) := \sup_B \frac{1}{|B|} \int_B |f(y)|\,dy,$$

where the supremum is taken with respect to all balls in $\mathbb{R}^d$ with center $x$. Furthermore, by $M_{\text{loc}}$ we denote the following local counterpart of $M$, namely

$$M_{\text{loc}} f(x) := \sup_Q \frac{1}{|Q|} \int_Q |f(y)|\,dy,$$
where the supremum is taken with respect to all cubes $Q$ containing $x$, with sides parallel to the axes and volume $\leq 1$. The weighted Lebesgue space $L_p(\mathbb{R}^d, w)$ is the collection of all measurable functions $f : \mathbb{R}^d \to \mathbb{C}$ such that

$$\|f|L_p(\mathbb{R}^d, w)\| := \left( \int_{\mathbb{R}^d} |f(x)|^p w(x) \, dx \right)^{1/p} < \infty.$$ 

In case $p = \infty$ we are back in the unweighted situation, i.e., $w \equiv 1$. We shall also need the following maximal inequality. Let $1 < p < \infty$, $1 < q \leq \infty$ and $w \in A_{\elloc}$. Then there exists a constant $c$ such that

$$\left( \int_{\mathbb{R}^d} \left( \sum_{j=0}^{\infty} |Mf_j(x)|^q \right)^{p/q} w(x) \, dx \right)^{1/p} \leq c \left( \int_{\mathbb{R}^d} \left( \sum_{j=0}^{\infty} |f_j(x)|^q \right)^{p/q} w(x) \, dx \right)^{1/p},$$ 

holds for all sequences $(f_j)_j \subset L_p(\mathbb{R}^d, w)$, see [33], [34] or [21, Theorem V.3.1]. Rychkov [22] proved the local version: for $1 < p < \infty$, $1 < q \leq \infty$ and $w \in A_{\elloc}$ there exists a constant $c$ s.t.

$$\left( \int_{\mathbb{R}^d} \left( \sum_{j=0}^{\infty} |M_{\elloc}f_j(x)|^q \right)^{p/q} w(x) \, dx \right)^{1/p} \leq c \left( \int_{\mathbb{R}^d} \left( \sum_{j=0}^{\infty} |f_j(x)|^q \right)^{p/q} w(x) \, dx \right)^{1/p},$$ 

holds for all sequences $(f_j)_j \subset L_p(\mathbb{R}^d, w)$.

We need one further property of Muckenhoupt weights.

**Lemma 35** Let $0 < \Theta < 1$ and $0 < p_0, p_1 \leq \infty$. We put

$$\frac{1}{p} := \frac{1 - \Theta}{p_0} + \frac{\Theta}{p_1}.$$

(i) Let $w_0, w_1 \in A_{\infty}$ and define

$$w := w_0 \left( \frac{1 - \Theta}{p_0} \right)^{\frac{p_0}{p_1}} \frac{\phi_{p_0}}{w_1^{\frac{\Theta}{p_1}}}.$$

Then $w \in A_{\infty}$.

(ii) If $w_0, w_1$ belong to $A_{\infty}^{\elloc}$, then $w \in A_{\infty}^{\elloc}$.

**Proof.** Step 1. We prove (i). If $w_0, w_1 \in A_{\infty}$ then there exist $r_0, r_1 \in (1, \infty)$ such that $w_i \in A_{r_i}$, $i = 0, 1$. First, let $\max(p_0, p_1) < \infty$. If $r_i \leq p_i$, then by the monotonicity of Muckenhoupt classes $w_i \in A_{p_i}$, and this implies that $w \in A_{p_i}$, see Stein [21, V.6.1(a), pp. 218]. If $\max \left( \frac{r_0}{p_0}, \frac{r_1}{p_1} \right) > 1$, then we choose $\alpha > \max \left( \frac{r_0}{p_0}, \frac{r_1}{p_1} \right)$. Now $w_i \in A_{\alpha p_i}$, $i = 0, 1$, follows because of

$$\frac{1}{\alpha p} = \frac{1 - \Theta}{\alpha p_0} + \frac{\Theta}{\alpha p_1} \quad \text{and} \quad w = \frac{(1-\Theta)\alpha p}{\alpha p_0} \frac{\phi_{r_0}}{w_1^{\Theta \frac{r_1}{p_1}}}.$$

So applying the same argument as before we get $w \in A_{\alpha p} \subset A_{\infty}$. For the remaining cases $\max(p_0, p_1) = \infty$ it is enough to observe that the function $w(x) = 1$, $x \in \mathbb{R}^d$,
belongs to $\mathcal{A}_\infty (p_0 = p_1 = \infty)$ and in case $0 < p_0 < \infty = p_1$ we have $(1 - \theta)p/p_0 = 1$, i.e., $w = w_0$.

Step 2. The monotonicity of the local Muckenhoupt classes $\mathcal{A}_p^{loc}$ has been proved in [22]. The above used result from [21, V.6.1(a), pp. 218] is based on Hölder’s inequality. For that reason it carries over to the local situation. Alternatively one could argue with Lemma 34.

6.2 Weighted Besov and Lizorkin-Triebel spaces

Now we introduce weighted Besov and Lizorkin-Triebel spaces. Since we work with local Muckenhoupt weights we need larger space of distributions than the spaces of tempered distributions. Recall that the class $\mathcal{A}_\infty^{loc}$ contains weights of exponential growth. We follow the ideas of Rychkov that are based on local reproducing formula, cf. [22].

Let $S_e(\mathbb{R}^d)$ denote the set of all $\psi \in C^\infty(\mathbb{R}^d)$ such that

$$q_N(\psi) = \sup_{x \in \mathbb{R}^d} e^{N|x|} \left| \sum_{|\alpha| \leq N} \partial^\alpha \psi(x) \right| < \infty \quad \text{for all } N \in \mathbb{N}_0.$$ 

Then $S_e(\mathbb{R}^d)$, equipped with the topology generated by the system of semi-norms $q_N$, is a locally convex space. Its dual space $S'_e(\mathbb{R}^d)$ can be identified with a subspace of the space of distributions $\mathcal{D}'(\mathbb{R}^d)$ in the obvious way.

Let $\varphi_0 \in C^\infty_0(\mathbb{R}^d)$ and $\varphi(x) = \varphi_0(x) - 2^{-d} \varphi_0(\frac{x}{2})$ be functions such that

$$\int_{\mathbb{R}^d} \varphi_0(x)dx \neq 0 \quad \text{and} \quad \int_{\mathbb{R}^d} x^\beta \varphi(x)dx = 0$$

for any multiindex $\beta \in \mathbb{N}_0^d$, $|\beta| \leq B$, where $B$ is a fixed natural number.

Definition 36 Let $0 < q \leq \infty$, $s \in \mathbb{R}$ and $w \in \mathcal{A}_\infty^{loc}$. Moreover let $B > |s|$.

(i) Let $0 < p < \infty$. Then the weighted Besov space $B^s_{p,q}(\mathbb{R}^d, w)$ is the collection of all $f \in S'_e(\mathbb{R}^d)$ such that

$$\| f | B^s_{p,q}(\mathbb{R}^d, w) \| := \left( \sum_{j=0}^{\infty} 2^{jsq} \| \varphi_j * f | L_p(\mathbb{R}^d, w) \|^q \right)^{1/q} < \infty.$$ 

(ii) Let $0 < p < \infty$. Then the weighted Triebel-Lizorkin space $F^s_{p,q}(\mathbb{R}^d, w)$ is the collection of all $f \in S'_e(\mathbb{R}^d)$ such that

$$\| f | F^s_{p,q}(\mathbb{R}^d, w) \| := \left( \sum_{j=0}^{\infty} 2^{jsq} \| \varphi_j * f \|^q \right)^{1/q} \| L_p(\mathbb{R}^d, w) \| < \infty.$$
Remark 21 (i) For $w \equiv 1$ we are in the unweighted case. The associated spaces are denoted by $B_{p,q}^s(\mathbb{R}^d)$ and $F_{p,q}^s(\mathbb{R}^d)$. The above definition coincides with characterization of $B_{p,q}^s(\mathbb{R}^d)$ and $F_{p,q}^s(\mathbb{R}^d)$ by so called local means, cf. [40] and [37].

(ii) Observe, that we did not define weighted spaces with $p = \infty$. However, it will be convenient for us to use the convention $B_{\infty,q}^s(\mathbb{R}^d,w) := B_{\infty,q}^s(\mathbb{R}^d)$. 

(iii) Weighted Besov and Lizorkin-Triebel spaces with $w \in A_{\infty}$ have been first studied systematically by Bui [24, 35], cf. also [36] and [37]. In addition we refer to Haroske, Piotrowska [38] and [39]. Standard references for unweighted spaces are the monograph’s [13, 14, 40, 41] as well as [6]. These classes with $A_{\text{loc}}^\infty$ weights have been treated by Rychkov [22], Izuki and Sawano [42], and Wojciechowska [43, 4]. Different types of measure have been considered by Bownik and Ho [44] and Bownik [3].

Wavelet characterizations of weighted spaces

Here we need the following result.

Proposition 37 Let $0 < p < \infty$, $0 < q \leq \infty$ and $s \in \mathbb{R}$. Let $w \in A_{\infty}^{\text{loc}}$. Then there exists a linear isomorphism $T$ which maps $B_{p,q}^s(\mathbb{R}^d,w)$ onto $b_{p,q}^{s+d/2}(\mathbb{R}^d,w)$ and $F_{p,q}^s(\mathbb{R}^d,w)$ onto $f_{p,q}^{s+d/2}(\mathbb{R}^d,w)$.

Remark 22 (i) The mapping $T$ is generated by an appropriate wavelet system. A proof of Proposition 37 can be found in Izuki and Sawano [42], cf. also Wojciechowska [43] for a different proof. In case of Besov spaces and $w \in A_{\infty}$ we also refer to [39] and Bownik, Ho [44] in this context.

(ii) Proposition 37 extends to $p = \infty$ for Besov spaces, see Remark 21. Wavelet characterizations of unweighted Besov spaces are proved at various places, we refer to Meyer [45], Kahane and Lemarie-Rieusset [46], Triebel [41, 3.1.3] and Wojtaszczyk [47].

There is a little supplement to the previous proposition dealing with the classes $\dot{A}_{p,q}^s(\mathbb{R}^d,w)$ and $\dot{a}_{p,q}^s(\mathbb{R}^d,w)$ ($A \in \{B,F\}$, $a \in \{b,f\}$).

Proposition 38 Let $0 < q \leq \infty$, $s \in \mathbb{R}$ and $w \in A_{\infty}^{\text{loc}}$.

(i) Let $0 < p < \infty$. Then there exists a linear isomorphism $T$ which maps $\dot{B}_{p,q}^s(\mathbb{R}^d,w)$ onto $\dot{b}_{p,q}^{s+d/2}(\mathbb{R}^d,w)$ and $\dot{F}_{p,q}^s(\mathbb{R}^d,w)$ onto $\dot{f}_{p,q}^{s+d/2}(\mathbb{R}^d,w)$.

(ii) There exists a linear isomorphism $T$ which maps $\dot{B}_{\infty,q}^s(\mathbb{R}^d,w)$ onto $\dot{b}_{\infty,q}^{s+d/2}(\mathbb{R}^d,w)$.

The proof of Proposition 38 follows the same pattern as the proof of Proposition 37. We leave out the details but see [18, Proposition 2.1] for the unweighted case.
References

[1] Bergh, J. and Löfström, J., Interpolation spaces. An Introduction, Springer Verlag, New York, 1976

[2] Triebel, H., Interpolation Theory, Function Spaces, Differential Operators, North Holland, Amsterdam, 1978

[3] Bownik, M., Duality and interpolation of anisotropic Triebel-Lizorkin spaces, Math. Z. 259, no. 1, 131-169 (2008)

[4] Wojciechowska, A., Multidimensional wavelet bases in Besov and Lizorkin-Triebel spaces, PhD-thesis, Adam Mickiewicz University Poznań, Poznań, 2012

[5] Calderón, A.P., Intermediate spaces and interpolation, the complex method, Studia Math. 24, 113-190 (1964)

[6] Frazier, M. and Jawerth, B., A discrete transform and decompositions of distribution spaces, J. Funct. Anal. 93, 34-170 (1990)

[7] Mendez, O. and Mitrea, M., The Banach envelopes of Besov and Triebel–Lizorkin spaces and applications to partial differential equations, J. Fourier Anal. Appl. 6, 503-531 (2000)

[8] Kalton, N., Mayboroda, S. and Mitrea, M., Interpolation of Hardy-Sobolev-Besov-Triebel-Lizorkin spaces and applications to problems in partial differential equations, Interpolation theory and applications, Contemporary Math. 445, 121-177 (2007)

[9] Shestakov, V.A., On complex interpolation of Banach spaces of measurable functions, Vestnik Leningrad Univ. Math. 19, 64-68 (1974)

[10] Kreĭn, S.G., Petunin, Yu.I. and Semēnov, E.M., Interpolation of linear operators, Nauka, Moscow, 1978, engl. translation AMS, Providence, R.I., 1982

[11] Bennett, C. and Sharpley, R., Interpolation of Operators, Academic Press, Boston, 1988

[12] Kalton, N. and Mitrea, M., Stability results on interpolation scales of quasi-Banach spaces and applications, Trans. Amer. Math. Soc. 350, 3903-3922 (1998)
[13] Peetre, J., New Thoughts on Besov Spaces, Duke Univ. Press, Durham, 1976

[14] Triebel, H., Theory of Function Spaces, Birkhäuser, Basel, 1983

[15] Brudnyi, Yu.A. and Kruglyak, N.Ya., Interpolation Functors and Interpolation Spaces, North Holland, Amsterdam, 1991

[16] Maligranda, L., Orlicz Spaces and Interpolation, Seminars in Math., Dept. of Math., Univ. Estadual de Campinas, Campinas, 1989

[17] Nilsson, P., Interpolation of Banach lattices, Studia Math. 82, 135-154 (1985)

[18] Yang, D., Yuan, W. and Zhuo, C., Complex interpolation on Besov-type and Triebel-Lizorkin-type spaces, Analysis and Appl. (to appear)

[19] Gustavsson, J., On interpolation of weighted $L_p$-spaces and Ovchinnikov’s theorem, Studia Math. 72, 237-251 (1982)

[20] Wik, I., On Muckenhoupt’s classes of weight functions, Studia Math. 94, 245-255 (1989)

[21] Stein, E.M., Harmonic Analysis, Real-Variable Methods, Orthogonality, and Oscillatory Integrals, Princeton Univ. Press, Princeton, 1993

[22] Rychkov, V.S., Littlewood-Paley theory and function spaces with $A^\text{loc}_{p}$ weights, Math. Nachr. 224, 145-180 (2001)

[23] García-Cuerva, J. and Rubio de Francia, J., Weighted Norm Inequalities and Related Topics, Graduate Studies in Math. 29, North-Holland, Amsterdam, 1985

[24] Bui, H.-Q., Weighted Besov and Triebel spaces : Interpolation by the real method, Hiroshima Math. J. 12, no. 3, 581-605 (1982)

[25] Sickel, W. and Skrzypczak, L., Radial subspaces of Besov and Lizorkin-Triebel spaces: extended Strauss lemma and compactness of embeddings, J. Fourier Anal. Appl. 6, 639-662 (2000)

[26] Sickel, W., Skrzypczak, L. and Vybíral, J., On the interplay of regularity and decay in case of radial functions I. Inhomogeneous spaces, Comm. Contemp. Math. 14, No. 1 (2012)
[27] Sickel, W. and Skrzypczak, L., On the interplay of regularity and decay in case of radial functions II. Homogeneous spaces, J. Fourier Anal. Appl. 18, 548-582 (2012)

[28] Strauss, W.A., Existence of solitary waves in higher dimensions, Comm. in Math. Physics 55, 149-162 (1977)

[29] Lions, P.L., Symmetrie et compacité dans les espaces de Sobolev, J. Funct. Anal. 49, 315-334 (1982)

[30] Cho, Y. and Ozawa, T., Sobolev inequalities with symmetry, Commun. Contemp. Math. 11, 355-365 (2009)

[31] Skrzypczak, L., Rotation invariant subspaces of Besov and Triebel-Lizorkin spaces: compactness of embeddings, smoothness and decay properties, Revista Mat. Iberoamericana 18, 267-299 (2002)

[32] Duoandikoetxea, J., Fourier Analysis, Graduate Studies in Math. 29, AMS, Providence, Rhode Island, 2001

[33] Kokilashvili, V.N., Maximal inequalities and multipliers in weighted Lizorkin-Triebel spaces, Dokl. Akad. Nauk. SSSR 239 (1), 42-45 (1978)

[34] Andersen, K.F. and John, R.T., Weighted inequalities for vector-valued maximal functions and singular integrals, Studia Math. 69, 19-31 (1980/81)

[35] Bui, H.-Q., Characterizations of weighted Besov and Triebel-Lizorkin spaces via temperatures, J. Funct. Anal. 55, no. 1, 39-62 (1984)

[36] Bui, H.-Q., Paluszyński, M. and Taibleson, M.H., A maximal function characterization of weighted Besov-Lipschitz and Triebel-Lizorkin spaces, Studia Math. 119, no. 3, 219-246 (1996)

[37] Bui, H.-Q., Paluszyński, M. and Taibleson, M.H., Characterization of the Besov-Lipschitz and Triebel-Lizorkin spaces. The case $q < 1$, J. Fourier Anal. Appl. 3, 837-846 (1997)

[38] Haroske, D.D. and Piotrowska, I., Atomic decompositions of function spaces with Muckenhoupt weights; an example from fractal geometry, Math. Nachr. 281, 1476-1494 (2008)
[39] Haroske, D.D. and Skrzypczak, L., Entropy and approximation numbers of embeddings of function spaces with Muckenhoupt weights, I, Rev. Mat. Complut. 21, 135-177 (2008)

[40] Triebel, H., Theory of Function Spaces II, Birkhäuser, Basel, 1992.

[41] Triebel, H., Theory of Function Spaces III, Birkhäuser, Basel, 2006.

[42] Izuki, M. and Sawano, Y., Wavelet bases in the weighted Besov and Triebel-Lizorkin spaces with $A^\text{loc}_p$-weights, J. Approx. Theory 161, 656-673 (2009)

[43] Wojciechowska, A., Local means and wavelets in function spaces with local Muckenhoupt weights, In: Function Spaces IX, Banach Center Public. 92, Warszawa, 399-412 (2011)

[44] Bownik, M. and Ho, K.-P., Atomic and molecular decompositions of anisotropic Triebel-Lizorkin spaces, Trans. Amer. Math. Soc. 358, no. 4, 1469-1510 (2006)

[45] Meyer, Y., Wavelets and Operators, Cambridge Univ. Press, Cambridge, 1992

[46] Kahane, J.-P. and Lemarie-Rieuseut, P.-G., Fourier Series and Wavelets, Gordon and Breach Publ., 1995

[47] Wojtaszczyk, P., A Mathematical Introduction to Wavelets, Cambridge Univ. Press, Cambridge, 1997