LEHMER PAIRS AND DERIVATIVES OF HARDY’S Z-FUNCTION

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Abstract. Occurrences of very close zeros of the Riemann zeta function are strongly connected with Lehmer pairs and with the Riemann Hypothesis. The aim of the present note is to derive a condition for a pair of consecutive simple zeros of the $\zeta$-function to be a Lehmer pair in terms of derivatives of Hardy’s $Z$-function. Furthermore, we connect Newman’s conjecture with stationary points of the $Z$-function, and present some numerical results.

1. INTRODUCTION

Hardy’s $Z$-function is defined by:

$$Z(t) := e^{i\vartheta(t)} \left( \frac{1}{2} + it \right), \quad \vartheta(t) := \frac{1}{2} \log \frac{\Gamma \left( \frac{1}{4} + \frac{it}{2} \right)}{\Gamma \left( \frac{1}{4} - \frac{it}{2} \right)} - \frac{\log \pi}{2} t.$$

In 1956, D. H. Lehmer found a pair of very close zeros of the $Z$-function. Visually this means that the graph $Z(t)$ barely crosses the $t$-axis at these zeros, and this poses a threat to the validity of the Riemann Hypothesis, see [Edw01]. The occurrence of such pairs of close zeros is known as Lehmer’s phenomenon and pairs are called Lehmer pairs. The paper [CSV94] gives a precise meaning to the notion of Lehmer pairs and proves a result which is today’s most promising method to find lower bounds $\lambda$ of the de Bruijn-Newman constant $\Lambda$. It should be noted that the Riemann Hypothesis is equivalent to $\Lambda \leq 0$ and an infinite number of Lehmer pairs implies $\Lambda \geq 0$, which is known as Newman’s conjecture.

Let $\{\gamma_1, \gamma_2\}$ be a pair of two simple zeros of the $Z$-function and define

$$g_{\{\gamma_1, \gamma_2\}} := (\gamma_1 - \gamma_2)^2 \sum_{\gamma \notin \{\gamma_1, \gamma_2\}} \frac{1}{(\gamma - \gamma_1)^2} + \frac{1}{(\gamma - \gamma_2)^2}$$

(1)

where $\gamma$ goes through all zeros of the $Z$-function. Assuming the Riemann Hypothesis, a pair $\{\gamma_-, \gamma_+\}$ of two consecutive simple zeros of the $Z$-function is said to be a Lehmer pair if $g_{\{\gamma_-, \gamma_+\}} < 4/5$. If this is the case, then

$$\lambda_{\{\gamma_-, \gamma_+\}} := \frac{(\gamma_+ - \gamma_-)^2}{2g_{\{\gamma_-, \gamma_+\}}} \left( \left( 1 - \frac{5}{4} g_{\{\gamma_-, \gamma_+\}} \right)^{\frac{2}{5}} - 1 \right) \leq \Lambda.$$

Because the bound $4/5$ is the least restrictive possible that allows the proof of the above inequality to go through, it is not surprising that Lehmer pairs are not so rare; the first thousand zeros contain 48 Lehmer pairs.

In order to produce an “analytic” definition of Lehmer pair, Stopple gives in [Sto17] Theorem 1 a more restrictive definition in terms of the first three derivatives of the Riemann xi-function $\Xi(t)$ at the pair’s zeros. In this note we extend his result to derivatives of the $Z$-function.

2010 Mathematics Subject Classification. 11M06, 11M26.
Key words and phrases. Lehmer pair, Hardy’s $Z$-function, Riemann hypothesis.
Theorem 1. Let $Z(t)$ be Hardy’s $Z$-function and define the real function $\tilde{F}$ by

$$\tilde{F}(t) := -\frac{Z''(t)}{Z'(t)} + \frac{3}{4} \left( \frac{Z''}{Z'} \right)^2 (t). \quad (2)$$

Let $\{\gamma_1, \gamma_2\}$ be a pair of two simple zeros of the $Z$-function and define

$$\hat{g}_{\{\gamma_1, \gamma_2\}} := \frac{1}{3} (\gamma_1 - \gamma_2)^2 \left( \tilde{F}(\gamma_1) + \tilde{F}(\gamma_2) \right) - 2.$$ 

Under the Riemann Hypothesis we have

$$0 < \hat{g}_{\{\gamma_1, \gamma_2\}} - \hat{g}_{\{\gamma_1, \gamma_2\}} < 3 \left( \gamma_1 - \gamma_2 \right)^2 \left( \frac{1}{\gamma_1^2} + \frac{1}{\gamma_2^2} \right) < 3\hat{g}_{\{\gamma_1, \gamma_2\}}. \quad (3)$$

This estimate is obviously very good for consecutive and relatively large zeros. The following immediate corollary gives conditions for a pair of zeros to be a Lehmer pair.

Corollary 1. With notations and assumptions as in Theorem 1, the value $\hat{g}_{\{\gamma_1, \gamma_2\}}$ is always strictly positive. If $\{\gamma_-, \gamma_+\}$ is a Lehmer pair, then $\hat{g}_{\{\gamma_-, \gamma_+\}} < 4/5$. If

$$\hat{g}_{\{\gamma_-, \gamma_+\}} < \frac{4}{5} - 3 (\gamma_+ - \gamma_-)^2 \left( \frac{1}{\gamma_+^2} + \frac{1}{\gamma_-^2} \right), \quad (4)$$

then $\{\gamma_-, \gamma_+\}$ is a Lehmer pair.

Observe that the expression on the right hand side of (4) is very close to $4/5$ for large zeros, especially when we have a good candidate for a Lehmer pair. Therefore, it is reasonable to conjecture that we can omit this small term without changing the conclusion of Corollary 1. We also conjecture that $\inf \{\hat{g}_{\{\gamma_-, \gamma_+\}}\} = 0$.

In Section 3 we briefly analyse a stronger condition for pairs of zeros to be a Lehmer pair. We take a similar approach to that of Stopple in the second part of his paper, but, in our case, working with stationary points of the $Z$-function instead of the $\zeta$-function, see Theorem 2.

The main reason for considering such a rephrasing of the definition of Lehmer pair was a naïve question by the present author, if it is possible to prove that a given pair is Lehmer by simply computing derivatives. Unfortunately, values of the $\Xi$-function are very small, even for small $t$, and therefore inappropriate for numerical calculations. In Section 4 we demonstrate that it is possible to calculate derivatives of the $Z$-function by methods of numerical integration. Therefore, it is an interesting question to see whether Corollary 1 can be practical as a means of testing for Lehmer pairs.

2. Some lemmas

The following lemma is crucial in Stopple’s approach and also in ours. Its proof is very simple, but for the convenience of the reader we include it here.

Lemma 1. Assume that $f(s)$ and $g(s)$ are holomorphic functions on some domain $\Omega \subseteq \mathbb{C}$ such that $f(s) = (s - z)g(s)$ and $g(z) \neq 0$ for some $z \in \Omega$. Then

$$\left( \left( \frac{f''}{f'} \right)' + \frac{1}{4} \left( \frac{f''}{f'} \right)^2 \right) (z) = 3 \left( \frac{g'}{g} \right)' (z).$$

Proof. We have $f'(s) = g(s) + (s - z)g'(s)$, $f''(s) = 2g'(s) + (s - z)g''(s)$ and $f'''(s) = 3g''(s) + (s - z)g'''(s)$. Therefore

$$\left( \frac{f''}{f'} \right)' (z) = \frac{f''}{f'} (z) - \left( \frac{f''}{f'} \right)^2 (z) = 3 \frac{g''}{g} (z) - 4 \left( \frac{g'}{g} \right)^2 (z)$$

and we get the desired equality. \qed
Lemma 2. Let \( \zeta(s, z) = \sum_{n=0}^{\infty} (z + n)^{-s} \) be the Hurwitz zeta-function. For \( t > 1 \) we have

\[
 \frac{-56t^2}{(4t^2 + 1)^2} < \Re \left\{ \zeta \left( 2, \frac{1}{4} + it/2 \right) \right\} < \frac{3}{2t^2}.
\]

Proof. Choose an arbitrary \( t > 1 \). It is easy to see that \( \Re \{\zeta(2, 1/4 + it/2)\} = \sum_{n=0}^{\infty} f(n) \) where

\[
f(n) := 16 \left( (1 + 4n)^2 - 4t^2 \right) \left( (1 + 4n)^2 + 4t^2 \right)^{-2}.
\]

We are interested in the behavior of this function on the interval \([0, \infty)\); at \( n = 0 \) it is negative, on the interval \([0, (2\sqrt{3}t - 1)/4]\) it increases to positive maximum \(1/2t^{-2}\), then it decreases to zero at infinity. Because of this behavior we have

\[
\sum_{n=0}^{\infty} f(n) > f(0) + \int_{0}^{\infty} f(n)dn - \frac{1}{2t^2} = -\frac{(4t^2 - 1) (28t^2 - 1)}{2t^2 (4t^2 + 1)^2}
\]

from which the left side of (5) follows. On the other hand,

\[
\sum_{n=0}^{\infty} f(n) < \int_{0}^{\infty} f(n)dn + \frac{1}{2t^2} < \frac{3}{2t^2}
\]

and the proof is complete. \( \square \)

Lemma 3. Let \( 0 < \gamma_1' \leq \gamma_2' \leq \gamma_3' \leq \ldots \) denote ordinates of zeros of the \( \zeta \)-function in the upper half-plane. Then \( \gamma_n' - \gamma_n < 7 \) for every \( n \in \mathbb{N} \).

Proof. Denote by \( N(T) \) the number of \( \gamma_n' \)'s not exceeding \( T \). One form of the Riemann-von Mangoldt formula is

\[
\left| N(T) - \frac{T}{2\pi} \log \frac{T}{2\pi e} - \frac{7}{8} \right| \leq a \log T + b \log \log T + c + \frac{1}{5T}
\]

for \( T \geq e \) and suitable positive constants \( a, b, c \). The best known values are \( a = 0.110, b = 0.290 \) and \( c = 2.290 \), see [PT15]. From (3) we obtain

\[
N(T + H) - N(T) > \left( \frac{H}{2\pi} - 2a \right) \log (T + H) - 2b \log \log (T + H)
+ \frac{T}{2\pi} \log \left( 1 + \frac{H}{T} \right) - \frac{H}{2\pi} \log (2\pi e) - 2c - \frac{2}{5T}.
\]

Therefore, for \( H > 4\pi a \approx 1.38 \) a constant \( T_0 \) exists such that \( \gamma_{n+1}' - \gamma_n' \leq H \) for every \( \gamma_n' > T_0 \). The main advantage of (7) is that we can calculate \( T_0 \) for a given \( H \).

Since \( \gamma_2' - \gamma_1' < 7 \), it is enough to demonstrate

\[
\gamma_{n+1}' - \gamma_n' < H = 6 \quad \text{for} \quad n \geq 2.
\]

With help of Mathematica we obtain from (7) the bound \( T_0 = 35370 \) (just for comparison, \( H = 1.4 \) gives \( T_0 = 4.7 \times 10^{140} \)). Furthermore, it is known (see [Tru14] p. 281) that for \( T \in [0, 6.8 \times 10^5] \) inequality (3) is true for constants \( a = b = 0 \) and \( c = 2 \). This fact lowers our bound to \( T_0 = 412 \). Up to this value we easily verified (8) by computer. \( \square \)
3. Proof of Theorem 1

Define the function $H$ by

$$H(s) := \frac{1}{2} (1 - s) s \pi^{-s/2} \Gamma \left( \frac{s}{2} \right). \quad (9)$$

Then the Riemann $\xi$-function is given by $\xi(s) := H(s) \zeta(s)$ and $\Xi(t) = \xi(1/2 + it)$. Let $\rho_0$ be some simple zero of $\xi(s)$. By the Hadamard product formula for the $\xi$-function we have

$$\xi(s) = - (s - \rho_0) \prod_{\rho \neq \rho_0} e^{\rho s + s/\rho_0 - 1} \left( 1 - \frac{s}{\rho} \right) e^{s/\rho}$$

where $B$ is some constant. Applying Lemma 1 on (10), we get

$$F(\rho_0) := \frac{1}{3} \left( \left( \frac{\xi''}{\xi'} \right)' + \frac{1}{4} \left( \frac{\xi''}{\xi'} \right)^2 \right) (\rho_0) = - \sum_{\rho \neq \rho_0} (\rho - \rho_0)^{-2}$$

$$= \frac{1}{3} \left( \frac{\xi''}{\xi'} + \frac{3}{4} \left( \frac{\xi''}{\xi'} \right)^2 \right) (\rho_0).$$

If we take two simple zeros $\rho_1$ and $\rho_2$ and define $\Delta := \rho_1 - \rho_2$, then (11) gives us

$$\Delta^2 \sum_{\rho \notin \{\rho_1, \rho_2\}} \frac{1}{(\rho - \rho_1)^2} + \frac{1}{(\rho - \rho_2)^2} = - \Delta^2 (F(\rho_1) + F(\rho_2)) - 2 \quad (12)$$

The left hand side of (12) is very similar to expression (11). Indeed, if we assume the Riemann Hypothesis, then $g_{(\gamma_1, \gamma_2)} = (\gamma_1 - \gamma_2)^2 (F(\gamma_1) + F(\gamma_2)) - 2$ where

$$F(t) = - \frac{1}{3} \left( \left( \frac{\Xi''}{\Xi'} \right)' + \frac{1}{4} \left( \frac{\Xi''}{\Xi'} \right)^2 \right) (t).$$

We would like to express function $F$ firstly in terms of the $\zeta$-function and then in terms of the $Z$-function. Taking $\zeta(\rho_0) = 0$ into account, we simply differentiate $\xi(s) = H(s) \zeta(s)$ to find that

$$\frac{\xi''}{\xi'} (\rho_0) = \frac{3}{H} (\frac{H''}{H} (\rho_0) + \frac{3}{H} (\frac{H'}{H} (\rho_0) + \frac{\xi''}{\zeta'} (\rho_0),$$

$$= \frac{3}{4} \left( \frac{\xi''}{\xi'} \right)^2 (\rho_0) = 3 \left( \frac{H'}{H} \right)^2 (\rho_0) + \frac{3}{4} \frac{H' \xi''}{H \zeta'} (\rho_0) + \frac{3}{4} \left( \frac{\xi''}{\xi'} \right)^2 (\rho_0)$$

and finally

$$F(\rho_0) = \left( \frac{H'}{H} \right)' (\rho_0) + \frac{1}{3} \widetilde{F}(\rho_0) \text{ where } \widetilde{F}(\rho_0) := \left( \frac{\xi''}{\xi'} - \frac{3}{4} \left( \frac{\xi''}{\xi'} \right)^2 \right) (\rho_0). \quad (13)$$

By (9) we obtain

$$\left( \frac{H'}{H} \right)' (\rho_0) = (\log H)'' (\rho_0) = - \frac{1}{(1 - \rho_0)^2} - \frac{1}{\rho_0^2} + \frac{1}{4} \frac{\zeta(2, \rho_0)}{2}. \quad (14)$$

Let $\rho_0 = 1/2 + i\gamma_0$. Straightforward calculations with $Z(\gamma_0) = 0$ in mind give us

$$\zeta'(\rho_0) = - i e^{-i\theta(\gamma_0)} Z'(\gamma_0),$$

$$\zeta''(\rho_0) = e^{-i\theta(\gamma_0)} (2i \theta'(\gamma_0) Z'(\gamma_0) - Z''(\gamma_0)), $$

$$\zeta'''(\rho_0) = e^{-i\theta(\gamma_0)} \left( (3 \theta''(\gamma_0) - 3i \theta'^2(\gamma_0) \right) Z'(\gamma_0) + 3 \theta'(\gamma_0) Z''(\gamma_0) + i Z'''(\gamma_0).$$
From this, (13) and (2) we obtain $\hat{F}(\rho_0) = \hat{F}(\gamma_0) + i3\theta''(\gamma_0)$ where this is indeed the decomposition into the real and the imaginary part of $F(\rho_0)$. By (14) we have

$$
\left( \frac{H'}{H} \right)'(\rho_0) = \frac{2\gamma_0^2 - 1/2}{(\gamma_0^2 + 1/4)^2} + \frac{1}{4} \zeta \left( 2, \frac{1}{4} + i\frac{\gamma_0}{2} \right).
$$

Since $\Im \{ \zeta (2,\rho_0/2) \} = -4\theta''(\gamma_0)$ by the definition of the Hurwitz zeta-function and the function $\vartheta$, $F(\rho_0) = (1/3)\hat{F}(\gamma_0) + \varepsilon(\gamma_0)$ where

$$
\varepsilon(\gamma_0) := \frac{2\gamma_0^2 - 1/2}{(\gamma_0^2 + 1/4)^2} + \frac{1}{4} \Re \left\{ \zeta \left( 2, \frac{1}{4} + i\frac{\gamma_0}{2} \right) \right\}.
$$

By Lemma 2 we have $0 < \varepsilon(\gamma_0) < 3\gamma_0^{-2}$, and by (12) we have

$$
\hat{g}(\gamma_1, \gamma_2) = \hat{g}(\gamma_1, \gamma_2) + (\gamma_1 - \gamma_2)^2 (\varepsilon(\gamma_1) + \varepsilon(\gamma_2)).
$$

The combination of both gives the first two inequalities in (3).

To obtain the third inequality we need Lemma 3. Since $\gamma_2' - \gamma_1' < 7 < \gamma_1'/2$ by concrete calculations, it follows $\gamma_n' + 1 - \gamma_n' < \gamma_n'/2$ for every $n \in \mathbb{N}$. Assuming the Riemann Hypothesis, by (11) this implies $\hat{g}(\gamma_1, \gamma_2) > 4(\gamma_1 - \gamma_2)^2 (\gamma_1^{-2} + \gamma_2^{-2})$ for every distinct zeros $\gamma_1, \gamma_2 \geq \gamma_1'$ of the Z-function. This completes the proof of Theorem 1.

4. Stationary points of the Z-function

Writing $Pf'(t) := (f''/f')'(t)$ for the pre-Schwarzian derivative of $f'$, the assertion $-(\gamma_1 - \gamma_2)^2 (P\Xi(\gamma_1) + P\Xi'(\gamma_1)) < 42/5$ implies that $\{\gamma_1, \gamma_2\}$ is a Lehmer pair. Stopple named such a pair a strong Lehmer pair and showed by concrete calculations that this is indeed a much stronger condition. Define

$$
\hat{g}(\gamma_1, \gamma_2) := \frac{1}{3} (\gamma_1 - \gamma_2)^2 (PZ'(\gamma_1) - PZ'(\gamma_2)) = 2.
$$

By Corollary 1 if

$$
\hat{g}(\gamma_1, \gamma_2) < \frac{4}{5} - 3(\gamma_1 - \gamma_2)^2 \left( \frac{1}{\gamma_1^2} + \frac{1}{\gamma_2^2} \right),
$$

then $\{\gamma_1, \gamma_2\}$ is a Lehmer pair.

Stopple obtained in [Sto17] Theorem 3 a representation of $P\Xi'(\gamma)$ in terms of nearby zeros of $\zeta'(s)$. We can derive a much simpler expression for $PZ'(\gamma)$ of a similar nature. The Z-function is on the right half-plane $\mathbb{H}_R := \{ z \in \mathbb{C} : \Re \{ z \} > 0 \}$ a holomorphic function. R. Hall proved in [Hal04] Theorem 2 that, assuming the Riemann Hypothesis, all zeros $u$ of $Z'(z)$ in $\mathbb{H}_R$ are real. Furthermore, Lehmer’s theorem [Edw01] [8.3] asserts that such a point $u$ is unique between consecutive zeros of the Z-function. We denote this nondecreasing sequence by $\{u_n\}$. For $t > e$ define the interval $I_t := [t - 1/\log \log t, t + 1/\log \log t]$. We further assume the hypothesis

$$
\frac{1}{\zeta'(1/2 + i\gamma)} = O(\gamma)
$$

for all positive zeros $\gamma$ of the Z-function. This follows from the “weak” Mertens’ conjecture

$$
\int_1^T \left( \frac{M(x)}{x} \right)^2 \, dx = O(\log T)
$$

where $M(x) = \sum_{n \leq x} \mu(n)$ and $\mu$ is the Möbius function, see Section 14.29 and proof of 14.30 in [Fit86]. Observe that (10) implies the simplicity of zeros. Assuming all
this, with \( r = 1/\log \log \gamma \) and \( s_0 = 1/2 + i\gamma \) a similar procedure as in the proof of 
\[ \text{Sto17} \] Lemma 2] gives us
\[
-PZ'(\gamma) = \sum_{u \in \gamma} \frac{1}{(\gamma - u)^2} + O(\log \gamma \log \log \gamma). 
\tag{17}
\]
A straightforward interpretation of this estimate is contained in the following theorem.

**Theorem 2.** Assume the Riemann Hypothesis and [11]. Let \( \gamma_n \) be the \( n \)-th zero and \( u_n \in (\gamma_n, \gamma_{n+1}) \) the \( n \)-th stationary point of \( Z(t) \). Define

\[
\alpha_n := \frac{u_n - \gamma_n}{\gamma_{n+1} - \gamma_n}. 
\]
If the set
\[
\mathcal{Z} := \{ n \geq 2 : \gamma_{n+1} - \gamma_n < \frac{1}{\log \gamma_n}, \{u_{n-1}, u_{n+1}\} \not\subset \mathcal{I}_{\gamma_n} \cup \mathcal{I}_{\gamma_{n+1}}, \alpha_n \in [0.44, 0.56] \}
\]
is infinite, then the de Bruijn-Newman constant \( \Lambda \) is 0.

**Proof.** From [17] it follows that
\[
g'(\gamma_n) = \frac{1}{3} \sum_{u \in \gamma_n} \left( \frac{\gamma_{n+1} - \gamma_n}{u - \gamma_n} \right)^2 
+ \frac{1}{3} \sum_{u \in \gamma_{n+1}} \left( \frac{\gamma_{n+1} - \gamma_n}{\gamma_{n+1} - u} \right)^2 - 2 
+ \left( \gamma_{n+1} - \gamma_n \right)^2 \log \gamma_{n+1} \log \gamma_n \log \gamma_{n+1} . 
\tag{18}
\]
The first assertion from the set \( \mathcal{Z} \) guarantees that the error term tends to zero at infinity. Thus, for \( \varepsilon = 10^{-3} \) there exists an infinite subset \( \mathcal{Z}' \subset \mathcal{Z} \) such that for every \( n \in \mathcal{Z}' \) the absolute values of the error terms in [15] and [15] are not greater than \( \varepsilon \). By the second assertion, the only contributing stationary point in the above sums is \( u_n \). By the third assertion, for \( n \in \mathcal{Z}' \) we have
\[
g'(\gamma_n) < \frac{1}{3} \left( \frac{1}{\alpha_n^2} + \frac{1}{(1 - \alpha_n)^2} \right) - 2 + \varepsilon < 0.79 + \varepsilon < \frac{4}{5} - \varepsilon.
\]
Therefore, there exist an infinite number of Lehmer pairs, which implies \( \Lambda = 0 \). 

Famous Montgomery’s pair correlation conjecture states that
\[
\lim \inf_{n \to \infty} (\gamma_{n+1} - \gamma_n) \log \gamma_n = 0,
\]
see [BMN10] for a brief overview of the problem. This implies that the first assertion from the set \( \mathcal{Z} \) is true for an infinite number of pairs. The author has calculated that the first two million zeros include 4637 pairs of zeros which satisfy the first assertion, while 1901 pairs actually belong to the set \( \mathcal{Z}' \).

5. Numerical results

We can use Cauchy’s integral formula to express higher derivatives since the \( Z \)-function is holomorphic in a neighborhoods of its zeros. This gives
\[
Z^{(n)}(x) = \frac{n!}{\pi^n} \int_0^1 e^{-2\pi n t} Z(x + re^{2\pi it}) dt = \frac{n!}{\pi^n} \int_0^1 G(t) dt
\tag{19}
\]
where
\[
G(t) := \Re \{ Z(x + re^{2\pi it}) \} \cos(2\pi nt) + \Im \{ Z(x + re^{2\pi it}) \} \sin(2\pi nt)
\]
is a real function. We used Mathematica’s function \texttt{RiemannSiegelZ} for the approximate calculation of the integral [19] by the composite trapezoidal rule with tolerance \( 10^{-7} \). Observe that the parameter \( n \) is involved only in the trigonometric
functions which enables the simultaneous calculation of derivatives. The results we get for three pairs are summarized in Table 1.

| $n$ | $\gamma_n$ | $Z'(\gamma_n)$ | $Z''(\gamma_n)$ | $Z'''(\gamma_n)$ |
|-----|-------------|-----------------|-----------------|-----------------|
| 34  | 111.02955354 | $-1.590846$     | 3.8401          | 1.4834          |
| 35  | 111.87465918 | 1.361151        | 2.2575          | $-4.6657$       |
| 6709| 7005.06286617| $0.414558$      | $-21.3028$      | $-57.1590$      |
| 6710| 7005.10056467| $-0.427037$     | $-23.2882$      | $-47.9737$      |
| 1048449114| 388858886| $0.008173$      | 150.552         | 123.981         |
| 1048449115| 388858886| $0.008173$      | 150.565         | $122.665$       |

Table 1. Values of derivatives of the $Z$-function for three pairs of zeros.

The first pair is actually the first Lehmer pair while the second pair is probably the most famous one because Lehmer found it. The third pair is used in [COSV93] in order to obtain a lower bound for the de Bruijn-Newman constant.

These numbers are in agreement with calculations derived by the Python’s mpmath function siegelz(z, derivative=n). This function gives for $1 \leq n \leq 4$ the $n$-th derivative of the $Z$-function at $z$.

| $n$ | $\sim (\gamma_n - \gamma_{n+1})^2 PZ'(\gamma_n)$ | $\hat{g}$ | $\hat{\hat{g}}$ |
|-----|-----------------------------------------------|----------|--------------|
| 34  | 4.83                                          | 0.57     | 1.08         |
| 35  | 4.41                                          |          |              |
| 6709| 3.95                                          | $7 \times 10^{-3}$ | 0.67       |
| 6710| 4.07                                          |          |              |
| 1048449114| 4.00                                         |          |              |
| 1048449115| 4.00                                         |          |              |

Table 2. Values of $\hat{g}$ and $\hat{\hat{g}}$ for three pairs of zeros.

We use the values in Table 1 to obtain $\hat{g}$ and $\hat{\hat{g}}$ for corresponding pairs of zeros. The results are displayed in Table 2. Observe that the value $\hat{g}$ for the second and the third pair is very close to zero while $\hat{\hat{g}}$ is approximately $2/3$. By Corollary 1 we know that $\hat{g}$ must be positive. Therefore, we have further evidence of “the near failure of the Riemann Hypothesis”. On the other hand, by Theorem 2 the value $2/3$ is “optimal” for $\hat{g}$ in the sense that the stationary point is exactly in the middle of the pair’s zeros ($\alpha_n = 1/2$) and the error term is small enough to be neglected. This is expected to be true for extremely close zeros.

Acknowledgements. We would like to thank Jeffrey Stopple for his interest in the subject, and Timothy Trudgian for his helpful remarks.

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