VISUALIZING MODULAR FORMS

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Abstract. We describe practical methods to visualize modular forms. We survey several current visualizations. We then introduce an approach that can take advantage of colormaps in python’s matplotlib library and describe an implementation.

1. Introduction

1.1. Motivation. Graphs of real-valued functions are ubiquitous and commonly serve as a source of mathematical insight. But graphs of complex functions are simultaneously less common and more challenging to make. The reason is that the graph of a complex function is naturally a surface in four dimensions, and there are not many intuitive embeddings available to capture this surface within a 2d plot.

In this article, we examine different methods for visualizing plots of modular forms on congruent subgroups of SL(2, Z). These forms are highly symmetric functions and we should expect their plots to capture many distinctive, highly symmetric features.

In addition, we wish to take advantage of the broader capabilities that exist in the python/SageMath data visualization ecosystem. There are a vast number of color choices and colormaps implemented in terms of python’s matplotlib library [Hun07]. While many of these are purely aesthetic, some offer color choices friendly to color blind viewers. Further, some are designed with knowledge of color theory and human cognition to be perceptually uniform with respect to human vision. We describe this further in §3.

1.2. Broad Overview of Complex Function Plotting. Over the last 20 years, different approaches towards representing graphs of complex functions have emerged. The most commonly used approach is one first introduced by Frank Farris in a review [Far98] of Needham’s Visual Complex Analysis.

The idea is to represent the output of a complex function though color. Frequently, the idea is to use hue to represent argument and lightness to represent magnitude. Or more generally, the idea is to associate a color to each point in the complex plane and to color the domain of a complex

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function $f$ by the color of $f(z)$. This technique is now called *domain coloring*. Using hue and lightness to represent argument and magnitude is the default complex plotting method in SageMath [Sag20], as well as other common complex plotting libraries.

Unfortunately, it is often difficult to distinguish between hue and lightness — and in particular it is challenging to determine if two points with different hues have the same lightness. Thus some variations omit a dimension from the graph and plot only the magnitude or phase. Other variations specially color the complex plane.

In [WS10], Wegert describes an approach using domain coloring, but emphasizing phase plots as a visual tool. He also produces phase plots with small brightness adjustments near certain magnitude thresholds. The effect is similar to looking at a contour map of a landscape, except that each pixel’s color carries meaning (the phase of $f$ evaluated at that point).

1.3. Paper Overview. In §2, we give a short survey of common existing complex plotting techniques applied to modular forms. We include images coming from the ideas of Farris and Wegert, built using SageMath. We also reproduce the plots associated to modular forms on the L-Function and Modular Form Database (LMFDB [LMF19]).

In §3, we describe how one can incorporate pre-established colormaps into modular form visualizations. And in §3.1, we discuss one such implementation in terms *matplotlib*. This is a new implementation.

Throughout, we revisit the same basic figures with each visualization technique. We plot two different modular forms on two different domains in each technique. Each plot reveals different behaviors of the underlying forms.

2. Survey of Visualizations

2.1. Halfplane and Disk Models. Plots of modular forms typically represent the form either on the upper halfplane or on the Poincaré disk. We first describe the relationship between these two types of plots, and then survey different visualizations of forms in each plot type.

Let $\mathcal{H} = \{x + iy : y > 0\}$ denote the complex upper halfplane. A modular form of weight $k$ and nebentypus $\chi$ on a congruence subgroup $\Gamma \subseteq \text{SL}(2, \mathbb{Z})$ is a holomorphic function $f : \mathcal{H} \to \mathbb{C}$ satisfying

$$f(\gamma z) = \chi(d)(cz + d)^kf(z) \quad \forall \gamma = \left(\begin{array}{cc} a & b \\ c & d \end{array}\right) \in \Gamma,$$

as well as certain growth conditions at the cusps of $\Gamma\backslash\mathcal{H}$. See [DS05] for complete definitions.

To the congruence subgroup $\Gamma$ is associated a positive integer $N$ called the level. The automorphy condition (1) implies strong symmetry conditions on $f$ in terms of $N$. In particular, $\left(\begin{array}{cc} 1 & N \\ 0 & 1 \end{array}\right) \in \Gamma$ implies that $f$ will be periodic with period $N$. Thus one can constrain a plot to a vertical strip of width $N$ without omitting any information.
In practice, one would cut off the vertical strip at some height $H$. If $f$ is a cuspidal form, so that $\lim_{y \to \infty} f(x + iy) = 0$, then heuristically most interesting behavior of the form will lie within a well-chosen box $[0, N] \times [0, H]$. We refer to this type of plot as a plot of $f$ in the upper halfplane.

Alternately, the upper halfplane is conformally equivalent to the Poincaré disk $\mathbb{D} = \{ x + iy : x^2 + y^2 < 1 \}$. There are infinitely many such maps taking $\mathbb{D}$ to $\mathcal{H}$, but we choose the Möbius transform $\phi(z) = \frac{1 - iz}{z - i}$.

Under $\phi$, the points $-i, 0, i$ in $\mathbb{D}$ are mapped to $0, i, \infty$, respectively, in $\mathcal{H}$. Thus the apparent vertical orientation remains fixed in both models.

Plotting a modular form $f$ on $\mathbb{D}$ is a complete picture; values of $f$ at every point of $\mathcal{H}$ will be represented through such a plot. We refer this type of plot as a plot of $f$ in the disk.

2.2. Visualizations. Let $\Delta(z)$ denote the Ramanujan Delta function, the unique holomorphic cuspidal form of weight 12 on $\text{SL}(2, \mathbb{Z})$. And let $g(z)$ denote the unique holomorphic cuspidal form of weight 4 on $\Gamma(5)$. In the plots that follow, we will show these two forms.¹ For plots on $\mathcal{H}$, we plot $\Delta(z)$ on $[-1, 1] \times [0, 2]$ and $g(z)$ on $[-2.5, 2.5] \times [0, 2]$ in all images below.

We plot two different forms with a variety of different visualizations and discuss each in turn.

**Remark 1.** We do not intend to emphasize one type of visualization as being better than all the other. As we will see, each type has flaws and only presents a single view. For exploratory analysis, using a combination of these visualizations will likely lead to a more accurate overall impression.

2.2.1. Standard domain coloring. In Figure 1, we present default plots (as created by `complex_plot` in SageMath). In these plots, the arg $z$ is represented by hue and $|z|$ is represented by brightness, where 0 is black and $\infty$ is white.

The fact that $g$ and $\Delta$ are both cuspidal forms is immediately clear as each plot is dominated by the color black (corresponding to the overall size being small). Further, a closer look will reveal that each cusp appears to be visibly dark. It is possible to adjust the mapping from magnitude to brightness, but in practice these plots look roughly the same for a wide choice of maps and it takes quite a bit of tuning to choose an appropriate map.²

2.2.2. Magnitude plot without color. We now produce plots after ignoring the phase of $f(z)$ and instead only using the magnitude $|f(z)|$. There remains a choice of how to map magnitude to brightness; it is common to

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¹To produce the plots, we approximate each modular form by its first 300 Fourier coefficients and evaluate the function on a $500 \times 500$ grid, and store it as a PNG.

²SageMath does not expose the map from magnitude to brightness. In order to adjust this map, it is necessary to copy and modify SageMath’s source.
use maps of the form \( \arctan(\log(|f(z)|^\alpha + 1)) \) for some \( 0 < \alpha < 1 \). Taking \( \alpha = 0.25 \) yields the plots in Figure 2. We note that adjusting the power \( \alpha \) is the most natural and clearest way to adjust the overall darkness/lightness scale of this style of plot: larger alpha causes points near zeros to be much darker and points near poles to be much lighter; smaller alpha decreases this effect.

Qualitatively, each of these pictures offers very similar information as those in Figure 1. The lack of phase information makes it more challenging to distinguish between the images of \( \Delta \) and \( g \) in the disk, however.

2.2.3. **Magnitude plot with periodic linear color.** In the LMFDB, magnitude is mapped periodically to color. That is, hue represents the magnitude modulo 1, where blue is zero, and increases through purple, red, orange, yellow, and so on. In Figure 3, we mimic the plots in the LMFDB attained by ignoring the phase of mapping the magnitude mod 1 to hue.

Each form has a label on the LMFDB, and it is possible to link directly to each page for reference. The \( \Delta \) function has label 1.12.a.a\(^3\) and \( g \) has label 5.4.a.a\(^4\). Each page has a medallion with a plot similar to those displayed here.

\(^3\)https://www.lmfdb.org/ModularForm/GL2/Q/holomorphic/1/12/a/a/
\(^4\)https://www.lmfdb.org/ModularForm/GL2/Q/holomorphic/5/4/a/a/
In these plots it is now clear that the magnitude of these two forms is very small most of the time. The large bluish blob corresponds to a region where the magnitude is never more than 1. The larger weight of $\Delta(z)$ corresponds to more rapid growth away from the cusps, as evidenced by the well-defined tightly packed lines.

For $g$ this is a very accurate plot. But $\Delta(z)$ changes too rapidly near the boundary, causing a static-like appearance. Increasing resolution helps, but in practice it is not feasible to produce sufficiently high resolution images to completely make sense of the boundary.

2.2.4. Magnitude plot with periodic logarithmic spacing. In Figure 4, we change yet again the map from magnitude to color. We now allow the hue to represent $\log_\alpha(|f(z)|) \mod 1$ for some base $\alpha$. Thus two consecutive bands of red will correspond to the points in one band being $\alpha$ times the size of the points in the other.

We note that adjusting the factor between consecutive bands (i.e. choosing the base of the logarithm in the map from magnitude to color) controls the number of spacing of bands, and thus it is possible to make the plots visually simpler or more complex by experimenting with this parameter. The plots in Figure 4 were produced with $\alpha = 7$. 
The effect in these plots is similar to giving a higher resolution of Figure 3. A large amount of detail and data is visible within these plots; however this is double-edged, and the plots are now busier.

2.2.5. Pure phase plots. Following the ideas of Wegert [WS10], we now consider phase plots. In Figure 5, we ignore magnitude completely and only consider the phase. There is a natural mapping from phase to hue: we allow hue to represent \( \text{arg}(z)/2\pi \), and the natural periodicity of phase works well with the natural periodicity of the color wheel.

Qualitatively, less behavior of each modular form is evident in its phase plot in comparison to its magnitude plot. This is partly due to the lack of poles and infrequent zeros away from the boundary. Poles and zeros are singularities in phase plots, and the behavior of the colors around these points describes the order and nature of these points. In our opinion, pure phase plots should not be used on their own to study or describe modular forms.

But in some sense, pairing each image in Figure 5 with its corresponding image in Figure 3 or Figure 4 completely represents the modular form on the indicated domain.
2.2.6. Phase plots with magnitude contours. Again following Wegert [WS10], we create phase plots with contours representing magnitude. We indicate the contours by adjusting the lightness: below each contour we adjust the lightness to be darker and above each contour we adjust the lightness to be lighter. The sudden change from darker to lighter indicates the contour itself.

The phase naturally maps to hue as in the pure phase plots. Contours could be chosen with spacing defined linearly as in §2.2.3 or logarithmically as in §2.2.4. In these plots, we have chosen logarithmic spacing with respect to \( \log_2 \). That is, two consecutive contour lines indicate that the points along one have twice the magnitude of the points along the other.

Remark 2. We emphasize that it is not necessary to compute the contours themselves. Instead, one defines a coloring of the complex plane that includes brightness adjustments along the desired contours. In this case, we defined brightness adjustments around points of magnitude \( 2^n \) for integers \( n \). Then simply plotting the function with respect to this domain coloring causes the contours to appear. This is a key idea why approaches based on domain coloring are simple to compute.

These plots carry all the information visible in the pure phase plots of §2.2.5 and some of the information contained within the logarithmically
Figure 5. Phase plots. Magnitude is ignored, and hue represents the phase. On the left is $\Delta(z)$. On the right is $g(z)$.

spaces magnitude plots of §2.2.4, but these two sets of information are not displayed equally. The colors grab attention far more than the brightness adjustments around contours — analogous to the lightness difficulties for the standard domain coloring plots of §2.2.1. These plots emphasize argument more than magnitude.

3. Choosing Color

We surveyed many different visualizations in §2, but one thing remained constant: the choices of color. In most complex function visualization packages, there is only one color choice available. All visualizations in §2 were created using the only colorscheme for complex.plot available in SageMath.$^5$

Unfortunately, this colorscheme has uneven perceptual contrast. That is, it has points of locally high color contrast (giving the impression of additional local variation) and points of locally low color contrast (hiding local variation). For example, in the scheme used above, there is particularly high color contrast on the border of green and yellow, and particularly low color contrast in the middle greens. Thus in Figure 5, for instance, green is

$^5$The behavior in matlab and maple appears similar, though it is possible to define a colorscheme. On the other hand Mathematica has an extensive library of complex plotting colorschemes.
Figure 6. Phase plots with magnitude indicated along contours. Away from the contours, magnitude is ignored. Hue represents the phase. On the left is $\Delta(z)$. On the right is $g(z)$.

particularly visible and the yellow-green border is particularly striking, even though there is no underlying feature that this is identifying.

In his excellent article [Kov15], Kovesi describes this problem and details how to create colormaps that are perceptually uniform — that don’t have uneven perceptual contrast. His work led to the creation of new default colorschemes for python’s matplotlib plotting library.

While studying visualizations of modular forms, we incorporated matplotlib colorschemes into our plotting routines. In particular, we implemented phase plots and phase plots with contours using these colorschemes. In this section, we describe this implementation and give examples of the resulting plots.

Remark 3. SageMath and matplotlib share many plotting routines, and newer versions of SageMath will default to these new perceptually uniform colormaps for real-valued plotting. But SageMath’s complex_plot does not directly use matplotlib.

We are working towards making these new plotting routines easy to use and publicly available, and possibly incorporating them into SageMath.

3.1. Implementation Overview. Some of the details of this implementation are specific to SageMath, but the overall idea readily generalizes.
(1) Begin with a desired complex function $f$, a region $\Omega \subset \mathbb{C}$, and a desired colormap $cm$.

(2) Evaluate $\arg(f(z))$ and $|f(z)|$ for each point in an $M \times N$ grid containing $\Omega$. (Optionally mask points $(m,n)$ in this grid outside of $\Omega$ to avoid unnecessary computation. For example, to produce the plots of $\Delta$ and $g$ in the disk, we mask points $z$ with $|z| \geq 1$.) This gives an $M \times N \times 2$ array.

(3) Map each phase to $[0, 1]$. This is most often done through the natural map $\theta \mapsto (\theta/2\pi) \mod 1$, but note that it is possible to add an offset $\Theta$ through $\theta \mapsto (\theta/2\pi + \Theta) \mod 1$. This might be done to choose which phase corresponds to the endpoints of the colormap. (See plots in Figure 8 for examples).

(4) Each matplotlib colormap is implemented as a function $cm : [0, 1] \to \text{RGB}$, where RGB $\cong [0, 1]^3$ is red-green-blue-alpha colorspace. Thus apply $cm$ to each mapped phase.

If the intention is to produce a pure phase plot, then (after discarding the magnitude) one now has an $M \times N \times \text{RGB}$ array, and one now produces a plot using this data in this grid, either directly or using a plotting library. For example, matplotlib's method `matplotlib.pyplot.imshow` will directly plot this array.

However, if the intention is to produce a phase plot with (logarithmically spaced) magnitude contours, then additional work is necessary. We would like to adjust the brightness as done in §2.2.6, but the brightness of an RGB pixel is nontrivial to work with directly.

Instead we convert each pixel to HSL colorspace (hue-saturation-lightness) and adjust lightness. As most plotting libraries use RGB (as does the underlying hardware), we convert back to RGB prior to plotting. Thus to produce a phase plot with magnitude contours we perform these additional steps.

(5) Map each RGB element to HSL colorspace, where HSL refers to “hue-saturation-lightness”. In HSL, it is possible to directly manipulate the lightness. This gives an $M \times N \times 3$ array.

(6) Map each magnitude to a lightness adjustment in the range $[-1, 1]$. For example, to have logarithmic contours, one could use a map of the form $(\log_2(|f(z)|) \mod 1)/2$ with a fixed lightness adjustment in a neighborhood of 0. This has the effect of creating logarithmically spaced contours. This gives an $M \times N \times 1$ array.

We note that we refer to $x \mod 1$ to mean the least nonnegative number of the form $x + \ell$ for $\ell \in \mathbb{Z}$, i.e. the “computer science” modulo operation.

(7) For each $(m,n)$, apply the corresponding lightness adjustment to each HSL value. (It may be necessary to renormalize lightness at this step). This gives a single $M \times N \times 3$ array.

(8) Map each HSL back to RGB to get the final $M \times N \times 3$ array.
And now one can directly plot (or use a library like `matplotlib` as noted above).

**Remark 4.** Each step can be done very quickly if one uses vectorized array operations as in NumPy [Oli06]. We also note that python’s `colorsys` library implements NumPy-vectorized conversions between RGB and HSL. Then the most time-consuming step is then the evaluation of $f$ on the $M \times N$ grid, as should be expected.

3.2. **Example visualizations with colormaps.** We now give example visualizations with `matplotlib` colormaps. We pay particular attention to the two new perceptually uniform colormaps in `matplotlib`, called `twilight` and `viridis`.

The `twilight` colormap is both perceptually uniform and cyclic. This means that the two ends of the colormap line up perfectly, and thus it is a good match to represent phase in phase plots. In Figure 7, we plot $\Delta$ and $g$ with `twilight`, including examples with contours and without contours.

In contrast to previous plots, these visualizations feel balanced. Compare in particular the left plots of Figure 7 to the left plots of Figure 5. But we note that perceptually uniform doesn’t mean perfect — there is a sensation of peaks and troughs resulting from the light and dark areas of the plots.

![Figure 7](image-url)

**Figure 7.** Phase plots with the `matplotlib` perceptually uniform colormap `twilight` applied. On the left is $\Delta(z)$ without contours. In the middle is $\Delta(z)$ with logarithmically spaced contours. On the right is $g(z)$ with logarithmically spaced contours.
The *viridis* colorscheme is not cyclic. There is a sharp difference in colors at the two ends of the colorscheme. Plotting a modular form in this colorscheme thus highlights the points where the phase crosses over the end of the colorscheme.

In many cases, this may be undesirable. For exploratory visualization, using a cyclic perceptually uniform colorscheme often seems superior.

But more generally one can imagine wanting to emphasize certain aspects of a complex plot, and using a non-cyclic colorscheme to highlight phase-changes is just one method.

In Figure 8 we have included three *viridis* plots (the top row) as well as three plots in other colorschemes. The bottom middle plot is the non-cyclic colorscheme *coolwarm*, normalized so that the sharp distinction between red and blue occurs when the argument of $\Delta$ is $3\pi/2$. The bottom right plot is the discrete colorscheme *Paired*, which has the effect of coloring different regions of the complex plane based on discrete argument sectors.

While each plot in 8 may not be appropriate for initial exploratory visualization, they demonstrate the ability to choose colorschemes to highlight certain behaviors or areas of the function.

**Remark 5.** Using specifically chosen domain colorings to highlight certain behaviors is not a new concept. For example, Frank Farris has produced several pieces of art based on domain colorings. See his website [http://math.scu.edu/~ffarris/](http://math.scu.edu/~ffarris/) for more.

But the idea of using colormaps for exploratory complex function visual analysis is young, and there remains much to be discovered.

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Figure 8. Phase plots with a variety of colorschemes from matplotlib applied. The top row is the perceptually uniform colorscheme viridis. The bottom row uses the continuous colorschemes ocean and coolwarm, followed by the discrete colorscheme Paired. In the bottom-middle plot, we have chosen a colorscheme that is sharply discontinuous when the phase is $3\pi/2$.

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