Universal Character and Large $N$ Factorization in Topological Gauge/String Theory

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Abstract

We establish a formula of the large $N$ factorization of the modular $S$-matrix for the coupled representations in $U(N)$ Chern-Simons theory. The formula was proposed by Aganagic, Neitzke and Vafa, based on computations involving the conifold transition. We present a more rigorous proof that relies on the universal character for rational representations and an expression of the modular $S$-matrix in terms of the specialization of characters.
1 Introduction

In [1] a remarkable relation

\[ Z_{BPS} \simeq |Z_{top}|^2, \]  

(1.1)

was proposed. The proposal (1.1) relates the partition function \( Z_{BPS} \) of counting BPS black holes formed by \( D \)-branes and topological string amplitudes \( Z_{top} \). While several attempts have been made at making (1.1) more precise, the conjecture has been tested for non-compact Calabi-Yau 3-folds in [2],[3],[4],[5], where BPS microstates arise from \( D4 \)-branes wrapping over a line bundle \( \mathcal{O}(-n) \to \Sigma \) on a Riemann surface \( \Sigma \). The (electric) charges of black hole are given by the numbers of \( D0 \) and \( D2 \) branes bound to \( D4 \) branes. When we have \( N \) \( D4 \) branes, the partition function \( Z_{BPS} \) is computed from \( U(N) \) topological Yang-Mills theory on the world volume of \( D4 \) branes and it has been shown that it is reduced to the \( q \)-deformed Yang-Mills theory on \( \Sigma \) [3]. The large \( N \) factorization of two dimensional Yang-Mills theory and its interpretation by a closed string theory were given in [6],[7]. The group theory of coupled representation labeled by a pair of Young diagrams was employed there. In the \( q \)-deformed case, the key to the large \( N \) factorization of the partition function is the following formula given in [1];

\[ q^{\nu_N^2 + \frac{N}{2}} S_{PQ}(q, N) = M(q) \eta(q)^N q^{\frac{1}{2}(\kappa_{Q_+} + \kappa_{Q_-})} \cdot q^{\frac{N}{2}(\sum |P_+| + |P_-| + |Q_+| + |Q_-|)} \times \sum_R (-1)^{|R|} q^{-|R|} C_{P_+ Q_+ R}(q) C_{P_- Q_- R^c}(q), \]  

(1.2)

for an extension \( S_{PQ} \) of the modular matrix of the \( U(N) \) Chern-Simons theory to the coupled representations \( P, Q \). In view of the conjecture (1.1), it is important that in (1.2) we have the topological vertex \( C_{PQR}(q) \), which is a building block of all genus topological string amplitudes on local toric Calabi-Yau 3-folds [8],[9],[10]. On the other hand, the modular \( S \)-matrix \( S_{PQ} \) appears as a building block for \( Z_{BPS} \) [9],[5]. For example, for \( D4 \) branes wrapping over a non-compact four cycle \( \mathcal{O}(-n) \to \mathbb{P}^1 \), the BPS partition function is

\[ Z(\mathbb{P}^1, p) = \sum_R (S_{R\bullet})^2 q^{\frac{2}{2} C_2(R)} e^{i \theta C_1(R)}, \]  

(1.3)
where $C_1(\mathcal{R})$ and $C_2(\mathcal{R})$ are the Casimir of irreducible $U(N)$ representation $\mathcal{R}$ and $\theta$ is the $\theta$ angle of the $q$-deformed $YM_2$. In this paper we will use $\bullet$ for the trivial representation. We note that $S_{\mathcal{R}\bullet}$ is related to the quantum dimension by $S_{\mathcal{R}\bullet} = S_{\bullet\bullet} \cdot \dim_q \mathcal{R}$. Up to the normalization associated with $S_{\bullet\bullet}$, the formula (1.2) gives

$$S_{\mathcal{R}\bullet}(q, N) = q^{\frac{N}{2}(|R| + |R|-)} \sum_Q (-1)^{|Q|} q^{-N|Q|} C_{R+,\bullet} Q(q) C_{R-,\bullet} Q^*(q).$$  

(1.4)

Since only the trivial representation $Q = \bullet$ survives in the large $N$ limit, we obtain

$$Z(\mathbb{P}^1, p) \sim Z_{\text{top}}^+ \cdot Z_{\text{top}}^-,$$

$$Z_{\text{top}}^\pm := \sum_{R^\pm} (C_{R^+\bullet}(q))^2 q^{\frac{N}{2} C_2(R^\pm)} e^{\pm i \theta C_1(R^\pm)}.$$  

(1.5)

The factor $Z_{\text{top}}^\pm$ is the topological string amplitude on the local toric Calabi-Yau manifold $O(p-2) \oplus O(-p) \to \mathbb{P}^1$ with an appropriate (complexified) Kähler parameter. Thus we find the topological string amplitudes in the factorization of $Z_{\text{BPS}}$ at large $N$.

In [4] the validity of the formula (1.2) has been argued by comparing the open/closed topological string amplitudes related by the conifold transition. The main purpose of this article is to give a more rigorous proof based on the universal character of the coupled representation labeled by a pair of partitions $(\lambda, \mu)$ or Young diagrams. The Schur function $s_\lambda(x)$ is nothing but the universal character of irreducible (polynomial) representation corresponding to a single partition $\lambda$. The coupled representation used in [4] is what is called rational representation in mathematics literature and the universal character $s_{[\lambda,\mu]}(x, y)$ of the rational representation is a generalization of the Schur function defined, for example, in [12]. This article is organized as follows; In section 2 we introduce the universal character for rational representation following [12]. We will see that $s_{[\lambda,\mu]}(x, y)$ is expanded in terms of (skew) Schur functions. In section 3 we review how the Hopf link invariants, or the normalized modular $S$-matrices of the Chern-Simons-WZW theory, are expressed in terms of specialization of Schur functions [13]. These facts are basic ingredients in our proof of (1.2), since it is known that the topological vertex has an

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1We use $C_1(\mathcal{R}) = C_1(R_+) - C_1(R_-)$ and $C_2(\mathcal{R}) = C_2(R_+) + C_2(R_-) + O(1/N)$.

2See also [11] for the relation of topological string amplitudes to the $q$-deformed 2D Yang-Mills theory.
expression in terms of specialization of skew Schur functions \[14\]. Finally we prove (1.2) in section 4 by using the formulas for the specialization of skew Schur function. It might be interesting to investigate some applications of the formula (1.2) to the large $N$ limit of the discrete matrix model and the generalized two-dimensional Yang-Mills theory \[15\].

2 Universal character for rational representation

A finite dimensional representation $X \to \rho(X)$ of $GL(N, \mathbb{C})$\[3\] is called polynomial or rational, if the matrix components of $\rho(X)$ are polynomial or rational functions of the components of $X$, respectively. For example the determinant $\rho(X) = \det X$ is a one-dimensional polynomial representation. The Cartan subalgebra of $GL(N, \mathbb{C})$ is the set of diagonal matrices $\mathfrak{h}_N := \{X; X = \text{diag.}(x_1, x_2, \ldots, x_N)\}$. The weight lattice is generated by $\epsilon_i : \mathfrak{h}_N \to \mathbb{C}$, $\epsilon_i(X) := x_i$. It is well-known that the irreducible polynomial representations of $GL(N, \mathbb{C})$ are labeled by partitions $\mu : \mu_1 \geq \mu_2 \geq \cdots \geq \mu_{\ell(\mu)} > 0$ of length $\ell(\mu) \leq N$. The highest weight of the representation $\rho_\mu$ labeled by $\mu$ is $\mu_1 \epsilon_1 + \mu_2 \epsilon_2 + \cdots + \mu_N \epsilon_N$ and the character is known as the Schur polynomial $\text{Tr}_{\rho_\mu} X = s_\mu(x_1, \cdots, x_N)$, where $(x_1, \cdots, x_N)$ are eigenvalues of $X$. For a fixed partition $\mu$, the projective limit $s_\mu(x) = \lim_{\leftarrow} s_\mu(x_1, \cdots, x_N)$ has the meaning in the ring of symmetric functions $\Lambda_x$, which is the projective limit of the algebra of symmetric polynomials in $N$ variables. In this sense the Schur function $s_\mu(x) \in \Lambda_x$ is sometimes called universal character of polynomial representation.

The irreducible rational representations of $GL(N, \mathbb{C})$ were classified by Schur. He showed any rational representation is of the form $X \to (\det X)^r \rho(X)$, where $r \in \mathbb{Z}$ and $\rho$ is a polynomial representation. That is, the denominator of rational representation is a power of the determinant. We may identify the integer $r$ with the $U(1)$ charge of the rational representation. The highest weight of the determinant representation is $\sum_{i=1}^N \epsilon_i$. Thus a complete set of inequivalent rational representations of $GL(N, \mathbb{C})$ is indexed by $N$-tuples $\lambda \in \mathbb{Z}^N : \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$ and the highest weight

\[3\] $U(N)$ and $SU(N)$ are the compact real forms of $GL(N, \mathbb{C})$ and $SL(N, \mathbb{C})$, respectively.
is given by \( \sum_{i=1}^{N} \lambda_i \epsilon_i \). The polynomial representations are characterized by the condition \( \lambda_N \geq 0 \). Separating the (strictly) positive and negative parts of \( \lambda \), we can also label them in terms of a pair of partitions \((\lambda_+, \lambda_-)\) with \( \ell(\lambda_+) + \ell(\lambda_-) \leq N \) so that \((\lambda_1, \lambda_2, \cdots, \lambda_N) = (\lambda_{+1}, \cdots, \lambda_{+\ell(\lambda_+)}, 0, \cdots, 0, -\lambda_{-\ell(\lambda_-)}, \cdots, -\lambda_{-1})\). In \( SU(N) \) gauge theory the difference of polynomial representations and rational representations is irrelevant, since the determinant representation is trivial. However, for \( U(N) \) gauge theory the difference is important. In fact Gross and Taylor employed the rational representation, which they called composite representation, to work out the large \( N \) factorization of the partition function of two-dimensional Yang-Mills theory \([6],[7]\).

![Figure 1: Rational representation as a tensor product of the determinant (the shaded rectangle) and a polynomial representation.](image)

Let \( \lambda_\pm \) be a partition of \( |\lambda_\pm| := \sum_{i=1}^{\ell(\lambda_\pm)} \lambda_{\pm, i} \). Then it is known that the rational representation \( \mathcal{R} \) labeled by \((\lambda_+, \lambda_-)\) occurs in the tensor product of \( |\lambda_+| \) copies of the defining representation \( \mathbb{C}^N \) and \( |\lambda_-| \) copies of the contragredient representation \( (\mathbb{C}^N)^* \);

\[
T_{|\lambda_+|,|\lambda_-|} := \left( |\lambda_+| \otimes \mathbb{C}^N \right) \bigotimes \left( |\lambda_-| \otimes (\mathbb{C}^N)^* \right). \tag{2.1}
\]

The universal character of the rational representation has been introduced in \([12]\). Let us denote the universal character of the rational representation \( \mathcal{R} \) by \( s_{|\lambda_+|,|\lambda_-|}(x, y) \), where the variables \( x \) and \( y \) are associated with the defining representation \( \mathbb{C}^N \) and the contragredient representation \( (\mathbb{C}^N)^* \), respectively. Consequently, \( s_{|\lambda_+|,|\lambda_-|}(x, y) \) is in the tensor product of universal character rings \( \Lambda_x \otimes \Lambda_y \) and it can be expanded in terms of the
products of Schur functions. Let us introduce the Littlewood-Richardson coefficients $c^\mu_{\eta,\nu}$ defined by the relation:

$$s_\eta(x)s_\nu(x) = \sum_\mu c^\mu_{\eta,\nu} s_\mu(x).$$

(2.2)

The coefficients $c^\mu_{\eta,\nu}$ are non negative integers. In [12] the expansion

$$s_{[\lambda_+,\lambda_-]}(x,y) = \sum_{\tau,\gamma,\nu} (-1)^{|\tau|} c^\gamma_{\eta,\tau} c^\nu_{\nu,\tau} s_\eta(x)s_\nu(y),$$

(2.3)

and its inverse relation

$$s_\eta(x)s_\nu(y) = \sum_{\tau,\lambda_+,\lambda_-} c^\eta_{\lambda_+,\tau} c^\nu_{\lambda_-,\tau} s_{[\lambda_+,\lambda_-]}(x,y),$$

(2.4)

were proved. The partition $\tau^t$ in (2.3) is defined by the transpose of the Young diagram.

Using the skew Schur function defined by

$$s_{\mu/\nu}(x) = \sum_\tau c^\mu_{\tau,\nu} s_\tau(x),$$

(2.5)

we can write the expansion (2.3) as

$$s_{[\lambda_+,\lambda_-]}(x,y) = \sum_\tau (-1)^{|\tau|} s_{\lambda_+/\tau}(x)s_{\lambda_-/\tau^t}(y).$$

(2.6)

Finally we note that there is the embedding $GL(N, \mathbb{C}) \ni g \mapsto (g, (g^t)^{-1}) \in GL(N, \mathbb{C}) \times GL(N, \mathbb{C})$, which is a group homomorphism. We can use the induced homomorphism $\Lambda_x \otimes \Lambda_y \to \Lambda_x$ obtained by putting $y = x^{-1}$ to send the above formulas to relations in the universal character ring $\Lambda_x$ of $GL(N, \mathbb{C})$.

### 3 Hopf link invariants and specialization of Schur function

Both the elementary symmetric functions $e_1(x), \cdots, e_n(x), \cdots$ and the complete symmetric functions $h_1(x), \cdots, h_n(x), \cdots$ are $\mathbb{Z}$-basis of the ring $\Lambda_x$ of the symmetric functions.
They are defined by the following generating functions;

\[ E(t, x) := 1 + \sum_k e_k(x)t^k = \prod_{i=1}^{\infty} (1 + x_it) , \quad (3.1) \]

\[ H(t, x) := 1 + \sum_k h_k(x)t^k = \prod_{i=1}^{\infty} (1 - x_it)^{-1} , \quad (3.2) \]

that satisfy \( E(t, x)H(-t, x) = 1 \). We note that the power sum functions \( p_1(x), \ldots, p_n(x), \ldots \)
are not \( \mathbb{Z} \)-basis, but \( \mathbb{Q} \)-basis of the symmetric functions. The generating function is

\[ P(t, x) := \sum_k p_k(x)t^{k-1} = \frac{d}{dt} \log H(t, x) . \quad (3.3) \]

Any symmetric function \( f \in \Lambda_x \) can be written as a polynomial \( f(e_1, \ldots, e_n, \ldots) \) in the elementary symmetric functions. For example, the Jacobi-Trudi formula gives the Schur function in terms of \( \{e_i(x)\} \);

\[ s_\mu(x) = \det \left( e_{\mu_i-i+j}(x) \right) . \quad (3.4) \]

We can define a specialization of the elementary symmetric functions by taking the generating function \( E(t) = 1 + \sum_{n=1}^{\infty} e_nt^n \), to be any formal power series with the leading coefficient 1. We denote \( f(e_1, \ldots, e_n, \ldots) \) as \( f(E(t)) \), which gives a specialization of the symmetric function \( f \). A basic example of such a specialization of symmetric functions is given in [16] (Examples I-2.5 and I-3.3). When we take the generating functions

\[ H(t) = \prod_{i=0}^{\infty} \frac{1 - bq^it}{1 - aq^it} , \quad E(t) = \prod_{i=0}^{\infty} \frac{1 + aq^it}{1 + bq^it} , \quad (3.5) \]

we have

\[ h_n(q) = \prod_{i=1}^{n} \frac{a - bq^{i-1}}{1 - q^i} , \quad e_n(q) = \prod_{i=1}^{n} \frac{aq^{i-1} - b}{1 - q^i} . \quad (3.6) \]

In this specialization the power sum function is

\[ p_n(q) = \frac{a^n - b^n}{1 - q^n} , \quad (3.7) \]

and the Schur function is given by

\[ s_\mu(q) = q^{n(\mu)} \prod_{(i,j) \notin \mu} \frac{a - bq^{(i,j)}}{1 - q^{h(i,j)}} , \quad (3.8) \]
where \( n(\mu) \) is
\[
n(\mu) = -\frac{1}{2} \sum_{(i,j) \in \mu} (1 + c(i,j) - h(i,j)) .
\] (3.9)

We have introduced the standard notation for the content \( c(i,j) := j - i \) and the hook length \( h(i,j) := \mu_i + \mu_j^t - i - j + 1 \) of the box at \((i,j)\) in the Young diagram.

Morton and Lukac used this kind of specialization of Schur functions to express the quantum dimension and Hopf link invariants \[13\]. For example, when \( a = \lambda^{-1}, b = 1, \)
\[
s_\mu(q) = q^{\frac{1}{2} \sum_{(i,j) \in \mu} (h(i,j) - c(i,j) - 1)} \prod_{(i,j) \in \mu} \frac{\lambda^{-1} - q^{c(i,j)}}{1 - q^{h(i,j)}} ,
\]
\[
= (q\lambda)^{-|\mu|} \prod_{(i,j) \in \mu} \frac{[c(i,j)]_\lambda}{[h(i,j)]} .
\] (3.10)

We find the quantum dimension \( \dim_q R = \prod_{(i,j) \in \mu} [c(i,j)]_{\lambda} \) of the representation \( R \) defined by the partition \( \mu \). In the \( SU(N) \) Chern-Simons theory at level \( k, q = \exp(\frac{2\pi i}{N+k}) \) and \( \lambda = q^N \). Thus we have
\[
W_\mu(q, \lambda) := \dim_q R = \lambda^{|\mu|} s_\mu (E_\bullet(t; q, \lambda)) ,
\] (3.11)

where
\[
E_\bullet(t) := 1 + \sum_{n=1}^{\infty} (q^{-\frac{1}{2}} t)^n \prod_{i=1}^{n} \frac{1 - \lambda^{-1} q^{i-1}}{q^{i-1} - 1} = \prod_{i=1}^{\infty} \frac{1 + \lambda^{-1} q^{-i+\frac{1}{2}} t}{1 + q^{i-\frac{1}{2}} t} .
\] (3.12)

We have absorbed the overall factor \( q^{-|\mu|} \) in (3.10) by the shift of the exponent \( q^i \to q^{i-\frac{1}{2}} \).

In this paper we will take the convention\(^4\) \( q = \exp(g_s) \) with \( g_s \) being string coupling (the parameter of genus expansion of topological string theory). Thus the relevant region of \( q \) in our convention is \(|q| > 1\). Since the expression in (3.12) is that for \(|q| < 1\), we make an analytic continuation to the region \(|q| > 1\) \[17, 18\], to obtain
\[
E_\bullet(t) = \prod_{i=1}^{\infty} \frac{1 + q^{-i+\frac{1}{2}} t}{1 + \lambda^{-1} q^{-i+\frac{1}{2}} t} .
\] (3.13)

We note that \( SU(N) \) specialization \( \lambda = q^N \) gives
\[
E_\bullet(N)(t) = \prod_{i=1}^{\infty} \frac{1 + q^{-i+\frac{1}{2}} t}{1 + q^{-N-i+\frac{1}{2}} t} = \prod_{i=1}^{N} \left(1 + q^{-i+\frac{1}{2}} t\right) ,
\] (3.14)

\(^4\)This is the same as \[9\], but different from \[3, 4\] and \[5\], where \( q = \exp(-g_s) \).
which is the same as the generating function (3.1) for the specialization $x_i = q^{-i + \frac{1}{2}}$ ($1 \leq i \leq N$), $x_j = 0$ ($N < j$). Similarly, the Hopf link invariants are also expressed in terms of an appropriate specialization of Schur function as follows;

$$W_{\mu\nu}(q, \lambda) = W_\mu(q, \lambda) \lambda^{\nu/2} s_\nu(E_\mu(t : q, \lambda)) \;,$$

where

$$E_\mu(t) := \prod_{i=1}^{\ell(\mu)} \frac{1 + q^{\mu_i - i + \frac{1}{2}} t}{1 + q^{-i + \frac{1}{2}} t} \cdot E_\mu(t) = \prod_{i=1}^{\infty} \frac{1 + q^{\mu_i - i + \frac{1}{2}} t}{1 + \lambda^{-1} q^{-i + \frac{1}{2}} t}.$$  \hspace{1cm} (3.16)

When $N \to \infty$, $\lambda^{-1} = q^{-N} \to 0$ for $|q| > 1$ and consequently

$$W_{\mu\nu}(q) \simeq \lambda^{\frac{1}{2}(|\mu| + |\nu|)} s_\mu(q^\rho) s_\nu(q^{\mu + \rho}) \;,$$

where $q^\rho$ and $q^{\mu + \rho}$ mean the specialization $x_i = q^{-i + \frac{1}{2}}$ and $x_i = q^{\mu_i - i + \frac{1}{2}}$, respectively.

### 4 Large $N$ factorization of the modular matrix of $U(N)$ CS theory

The modular S-matrix $S_{PQ}$ of the $U(N)$ Chern-Simons theory for coupled representations $P, Q$ is defined by

$$S_{PQ}(q, N) = \sum_{w \in S_N} (-1)^w q^{-w(P + \rho_N) : (Q + \rho_N)} \;,$$

where the symmetric group $S_N$ is the Weyl group of $U(N)$ and $\rho_N$ is the Weyl vector with the components $\frac{1}{2}(N - 2i + 1)$, $i = 1, \cdots, N$. Note that the same notation $P, Q$ is used for the highest weight. When $P$ and $Q$ are integrable representations $P, Q$ of $SU(N)$ affine Lie algebra at level $k$, (4.1) gives the matrix element of the $S$-transformation on the space of conformal blocks, which is the physical Hilbert space of the Chern-Simons theory on $T^2 \times \mathbb{R}$. The Hopf link invariants are obtained as the normalized $S$-matrix elements; $W_{PQ} = S_{PQ}/S_{**}$. The definition (4.1) is a formal extension of the modular
S-matrix to rational representations of $U(N)$. Recall that $q = \exp \left( \frac{2\pi i}{N+k} \right) = \exp(g_s)$. In [4] the following formula for $S_{\mathcal{PQ}}$ was claimed;

$$q^{\kappa^2 + \frac{N}{2N}} S_{\mathcal{PQ}}(q, N) = M(q) \eta(q)^N \frac{1}{q^{2\kappa_{Q_+} + \kappa_{Q_-}}} \cdot q^{\frac{N}{2}(|P_+| + |P_-| + |Q_+| + |Q_-|)}$$

$$\times \sum_R (-1)^{|R|} q^{-N|R|} C_{P_+Q_+ R}(q) C_{P_-Q_- R^t}(q) , \quad (4.2)$$

where

$$\mathcal{P} = [P_+, P_-], \quad \mathcal{Q} = [Q_+, Q_-] , \quad (4.3)$$

are two coupled representations and $C_{PQ R}(q)$ is the topological vertex\(^5\). For the representation $R$ corresponding to a partition $\mu^R$, $\kappa_R := 2 \sum_{(i,j) \in \mu^R} c(i,j)$. In this section we identify the representations, the partitions and the Young diagrams and often use the same notation for them. The MacMahon function $M(q)$ and the eta function $\eta(q)$ defined by

$$M(q) = \prod_n \frac{1}{(1-q^n)^n} , \quad \eta(q) = q^{1/24} \prod_n (1-q^n) , \quad (4.4)$$

appear as overall factors and they are related to the normalization of the Hopf link invariants (see Appendix).

Let us look at some special cases of the formula (4.2). When $P_\pm = Q_\pm = \bullet$, $S_{\bullet\bullet}$ is nothing but the partition function of $U(N)$ Chern-Simons theory on $S^3$ and (4.2) gives

$$q^{\frac{N^2}{2}} S_{\bullet\bullet} = M(q) \eta(q)^N \sum_R (-1)^{|R|} q^{-N|R|} C_{\bullet\bullet R}(q) C_{\bullet\bullet R^t}(q) ,$$

$$= M(q) \eta(q)^N \prod_{n=1}^{\infty} (1 - \lambda^{-1} q^{-n})^n$$

$$= M(q) \eta(q)^N \exp \left( - \sum_{n=1}^{\infty} \frac{e^{-tn}}{n(q^n - q^{-n})^2} \right) . \quad (4.5)$$

If we identify the 't Hooft coupling $t = N g_s$ with the Kähler parameter of the resolved conifold geometry, we recover the basic example of gauge/geometry correspondence of Gopakumar-Vafa [19]. When $P_- = Q_- = \bullet$, the left hand side is the usual modular

\(^5\)In [3] $q := \exp(-g_s)$, which is different from ours.
matrix of $\mathcal{U}(N)$ theory and (4.2) implies
\[
W_{PQ}(q, N) = \frac{S_{PQ}(q, N)}{S_{\bullet \bullet}(q, N)} = q^{1/2}e^{\frac{2}{N}(|P|+|Q|)} \sum_{n=1}^{\infty} (1 - \lambda^{-n})^{-n} \sum_{R} (-1)^{|R|} q^{-N|R|} C_{PQ}C_{P'Q'}C_{\bullet \bullet}C_{P'Q'}^\bullet, (4.6)
\]
which has been derived in [17]. When $N \to \infty$ only the trivial representation survives in the sum and
\[
W_{PQ}(q) = q^{1/2}e^{\frac{2}{N}(|P|+|Q|)} C_{PQ}^\bullet(q). (4.7)
\]
We recover the formula (3.17) for $W_{PQ}(q)$.

Topological vertex is expressed in terms of the skew Schur function [14];
\[
C_{R_1R_2R_3}(q) = q^{\frac{3}{2}r_3} s_{R_2}(q^\rho) \sum_{Q} s_{R_1/Q}(q^{\mu_{R_2}+\rho}) s_{R_1/Q}(q^{\mu_{R_2}+\rho}). (4.8)
\]
Using the Cauchy formula
\[
\sum_{R} s_{R/R_1}(x)s_{R'/R_2}(y) = \prod_{i,j \geq 1} (1 + x_i y_j) \sum_{Q} s_{R_1/Q}(x)s_{R_1/Q}(y), (4.9)
\]
and the relation
\[
s_{P/Q}(q^{R+\rho}) = (-1)^{|P|-|Q|} s_{P'/Q'}(q^{-R'-\rho}), (4.10)
\]
we can compute $^6$
\[
q^{\frac{2}{N}(\kappa_{Q_+}+\kappa_{Q_-})} \sum_{R} (-1)^{|R|} q^{-N|R|} C_{RP_+Q_+}(q)C_{RP_-Q_-}(q) = s_{P_+(q^\rho)}s_{P_-(q^\rho)} \prod_{1 \leq i,j} \left( 1 - \lambda^{-1}q^{P_{+,i}+P_{-,j}-i-j+1} \right) \sum_{V} (-\lambda)^{-|V|} s_{Q_+/V}(\lambda^{-1}q^{-P_{+,P_{+,Q_+}}}) s_{Q_-/V}(\lambda^{-1}q^{-P_{-,P_{-,Q_-}}}), (4.11)
\]
where the Schur function with two variables $s_{R}(x, y)$ is defined by
\[
s_{R}(x, y) = \sum_{P,Q} c_{PQ}^R s_{P}(x)s_{Q}(y) = \sum_{Q} s_{R/Q}(x)s_{Q}(y). (4.12)
\]

\textsuperscript{6}We use the cyclic symmetry of $C_{R_1R_2R_3}$. This type of computation appears in confirming the flop invariance of topological string amplitudes [17, [20].
The corresponding generating function of the elementary symmetric functions is

\[ E(t) = \prod_{i=1}^{\infty} (1 + x_i t) \prod_{j=1}^{\infty} (1 + y_j t) \]  

(4.13)

The cyclic symmetry of the topological vertex,

\[ \sum_{R} (-1)^{|R|} q^{-N|R|} C_{RP+*}(q) C_{R'P-•}(q) = \sum_{R} (-1)^{|R|} q^{-N|R|} C_{P+*R}(q) C_{P-•R'}(q) , \]  

(4.14)

implies

\[ s_{P+}(q^\rho) s_{P-}(q^\rho) \prod_{1 \leq i,j} \left( 1 - \lambda^{-1} q^{P_+,i+P'_-,j-i-j+1} \right) \]

\[ = \prod_{n=1}^{\infty} (1 - \lambda^{-1} q^{-n})^n \lambda^{-|P|-|V|} \sum_{V} (-1)^{|V|} s_{P+/V^t}(\lambda^{-1} q^{-\rho}, q^\rho) s_{P-/-V^t}(\lambda q^\rho, q^{-\rho}) . \]  

(4.15)

Hence we finally obtain

\[ q^{\frac{1}{2}(\kappa_{Q+}+\kappa_{Q-})} \sum_{R} (-1)^{|R|} q^{-N|R|} C_{RP+*}(q) C_{R'P-•}(q) \]

\[ = \prod_{n=1}^{\infty} (1 - \lambda^{-1} q^{-n})^n \lambda^{-|P_-|+|Q_-|} \sum_{V} (-1)^{|V|} s_{P+/V^t}(\lambda^{-1} q^{-\rho}, q^\rho) s_{P-/-V^t}(\lambda q^\rho, q^{-\rho}) \]

\[ \sum_{V} (-1)^{|V|} s_{Q+/V^t}(\lambda^{-1} q^{-P_-+\rho}, q^{P_-+\rho}) s_{Q-/-V^t}(q^{-P_-+\rho}, \lambda q^{P_-+\rho}) . \]  

(4.16)

On the other hand, the result reviewed in section 3 formally implies an expression of \( S_{PQ} \) in terms of the specialization the Schur function \( s_{\mathcal{R}}(x) \):

\[ W_{PQ}(q, N) := \frac{S_{PQ}(q, N)}{S_{••}(q, N)} = \lambda^{\frac{1}{2}(|P|+|Q|)} s_{P}(E_{•}(t; q, \lambda)) s_{Q}(E_{•}(t; q, \lambda)) . \]  

(4.17)

But, we have to make it precise what \( s_{\mathcal{R}}(x) \) and \( E_{\mathcal{R}}(t; q, \lambda) \) mean for the rational representation \( \mathcal{R} \). Firstly, the Schur function \( s_{\mathcal{R}}(x) \) should be regarded as the universal character of rational representation defined by the pair of partitions \((R_+, R_-)\). The generating function \( E_{\mathcal{R}}(t) \) that defines the specialization can be obtained by looking at the Young diagram of the rational representation \( \mathcal{R} = [R_+, R_-] \) (see Figure 2). Assuming
\[ \ell(R_+) + \ell(R_-) \leq N, \text{ we find} \]
\[
E_R(t) = \prod_{i=1}^{\ell(R_+)} \frac{1 + q^{R_+,-i+\frac{1}{2}t}}{1 + q^{-i+\frac{3}{2}t}} \prod_{j=1}^{\ell(R_-)} \frac{1 + q^{-N-R_-,j+\frac{1}{2}t}}{1 + q^{-N+j-\frac{3}{2}t}} \cdot E_\bullet(t),
\]
\[
= \prod_{i=1}^{\infty} \frac{1 + q^{R_+,-i+\frac{1}{2}t}}{1 + q^{-i+\frac{3}{2}t}} \prod_{j=1}^{\infty} \frac{1 + \lambda^{-1}q^{-R_-,j+\frac{1}{2}t}}{1 + \lambda^{-1}q^{j-\frac{3}{2}t}} \prod_{k=1}^{\infty} \frac{1 + q^{-k+\frac{1}{2}t}}{1 + \lambda^{-1}q^{-k+\frac{3}{2}t}},
\]
\[
= \prod_{i=1}^{\infty} \left(1 + q^{R_+,-i+\frac{1}{2}t}\right) \prod_{j=1}^{\infty} \left(1 + \lambda^{-1}q^{-R_-,j+\frac{1}{2}t}\right). \tag{4.18}
\]

Figure 2: Extended Young diagram for rational representation labeled by \((R_+, R_-)\).

Since the universal character of the rational representation \(R\) is given by
\[
s_R(x) = \sum_Q (-1)^{|Q|} s_{R_+/Q}(x) s_{R_-/Q'}(x^{-1}), \tag{4.19}
\]
and the specialization of \(s_R(x)\) is defined by \(E_R(t)\) of (4.18), we obtain
\[
W_{PQ}(q, N) = \lambda^{\frac{1}{2}(|P|+|Q|)} \sum_U (-1)^{|U|} s_{P_+/U}(\lambda^{-1}q^{-\rho}, q^\rho) s_{P_-/U^t}(\lambda q^\rho, q^{-\rho})
\]
\[
\quad \sum_V (-1)^{|V|} s_{Q_+/V}(\lambda^{-1}q^{-P_-\rho}, q^{P_+\rho}) s_{Q_-/V^t}(\lambda q^{P_+\rho}, q^{-P_-\rho}). \tag{4.20}
\]
Comparing (4.16) with (4.20), we see that the identification \(|P| = |P_+| - |P_-|, |Q| = |Q_+| - |Q_-|\) completes the proof of (4.2).

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Appendix : Normalization of Hopf Link Invariants

The normalization factor of the Hopf link invariants $S_{\bullet \bullet}(q, N)$ is the partition function of the Chern-Simons theory on $S^3$. In topological string theory it is the contribution of constant maps to topological string amplitudes [19]. For $SU(N)$ theory it is given by

$$S_{\bullet \bullet}(q, N) = \prod_{1 \leq i < j \leq N} \left( q^{\frac{1}{2}(j-i)} - q^{-\frac{1}{2}(j-i)} \right) ,$$

$$= \exp \left[ - \sum_{1 \leq i < j \leq N} \left( \frac{j-i}{2} \log q - \log(1 - q^{j-i}) \right) \right]. \quad (A.1)$$

Using the strange formula for $SU(N)$

$$\frac{1}{2} \sum_{1 \leq i < j \leq N} (j-i) = \frac{1}{12} N(N^2 - 1) = \rho_N^2 , \quad (A.2)$$

and

$$\sum_{1 \leq i < j \leq N} \log(1 - q^{j-i}) = - \sum_{m=1}^{\infty} \left[ \frac{Nq^m}{m(1-q^m)} - \frac{q^m - q^{m(N+1)}}{m(1-q^m)^2} \right] , \quad (A.3)$$

we find

$$q^{\rho_N^2 + \frac{N}{2}} S_{\bullet \bullet}(q, N) = M(q) \eta(q)^N N_0(q, \lambda) , \quad (A.4)$$

where $\lambda = q^N$ and

$$N_0(q, \lambda) = \exp \left( - \sum_{n=1}^{\infty} \frac{q^n}{n(1-q^n)^2} \lambda^n \right) , \quad (A.5)$$

is regarded as non-perturbative correction to the constant map contributions.

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