Quantum anholonomy with topology change

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We study a family of closed quantum graphs described by one singular vertex of order \( n = 4 \). By suitable choice of the parameters specifying the singular vertex, we can construct a closed sequence of paths in the parameter space that physically corresponds to the smooth interpolation of different topologies - a ring, separate two lines, separate two rings, two rings with a contact point. We find that the spectrum of a quantum particle on this family of graphs shows the quantum anholonomy.

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INTRODUCTION

The idea of quantum mechanics on graphs has been introduced in the last century. During the last 30 years, it has been developed into a powerful tool for the study of particles on tiny graph-like structures [5]. Since a graph is a one-dimensional variety, the analysis of a particle on a graph consists in solving a system of ordinary differential equations, which makes the graph models simple from the mathematical point of view. Therefore, quantum mechanics on graphs can serve also as a useful laboratory for the study of various complex quantum phenomena in a simple setting.

Here we examine the influence of the topology of quantum graphs on spectral properties, taking inspiration from the works [1] and [7]. We may regard that the topology change mimics a macroscopic or violent variation of the \( n \) equations connecting the boundary values and derivatives that can be written in the form [6]

\[
A\Psi + B\Psi' = 0,
\]

where \( \Psi, \Psi' \) are vectors formed from the values of the wave function in the vertex and from its derivatives (taken in the outgoing sense), respectively, and \( A \) and \( B \) are \( n \times n \) complex matrices with the property

\[
AB^\dagger = BA^\dagger \quad \text{and} \quad \text{rank}(A|B) = n
\]

with \( (A|B) \) denoting the \( n \times 2n \) matrix composed from the columns of the matrices \( A \) and \( B \). For the graph depicted in Figure 1, we have

\[
\Psi = \begin{pmatrix}
\psi_1(0) \\
\psi_1(L_1) \\
\psi_2(0) \\
\psi_2(L_2)
\end{pmatrix}, \quad \Psi' = \begin{pmatrix}
\psi'_1(0) \\
-\psi'_1(L_1) \\
\psi'_2(0) \\
-\psi'_2(L_2)
\end{pmatrix}.
\]

In our model, we focus on a specific subset of all possible...
connection conditions \([1]\), namely

\[
\begin{pmatrix}
1 & t b(\theta) & t a(\theta) & s d(\theta) \\
0 & 1 - c(\theta) & 0 & t a(\theta) \\
0 & 0 & c(\theta) & t b'(\theta) \\
0 & 0 & 0 & 0
\end{pmatrix} \Psi' = \begin{pmatrix}
0 & -t b(\theta) & c(\theta) & 0 \\
-t a(\theta) & 0 & 0 & 0 \\
-s d(\theta) & -t a(\theta) & -t b'(\theta) & 1
\end{pmatrix} \Psi \tag{4}
\]

where \(t\) and \(s\) are positive constants, and \(a(\theta), b(\theta), b'(\theta), c(\theta)\) and \(d(\theta)\) are real periodic functions of \(\theta\) with period \(2\pi\) that vary within the range between 0 and 1. In addition, it can be easily shown that the functions must satisfy

\[a(\theta) \neq 0 \Rightarrow c(\theta) = 0, \tag{5}\]
\[b(\theta), b'(\theta) \neq 0 \Rightarrow c(\theta) = 1 \tag{6}\]

for all \(\theta \in [0, 2\pi]\) in order that the matrices \(A, B\) involved in the connection condition \([4]\) obey \([2]\).

#### Long Cycle

At first we consider the boundary conditions \([4]\) with functions \(a(\theta), b(\theta), b'(\theta), c(\theta)\) and \(d(\theta)\) chosen as

\[a(\theta) = \cos \theta - \frac{1}{2} + \left| \cos \theta - \frac{1}{2} \right|, \]
\[b(\theta) = -\cos \theta - \frac{1}{2} + \left| \cos \theta + \frac{1}{2} \right|, \]
\[b'(\theta) = \frac{1}{4 - 2\sqrt{3}} \left\{ \sin \theta - \sqrt{3} \cos \theta - \sqrt{3} - \left| \sin \theta - \sqrt{3} \cos \theta - \sqrt{3} \right| \right\}, \tag{7}\]
\[c(\theta) = \frac{1}{\sqrt{3} - 1} \left\{ \frac{\sqrt{3}}{2} - \frac{1}{2} + \left| \sin \frac{\theta}{2} - \frac{1}{2} \right| - \left| \sin \frac{\theta}{2} - \sqrt{3} \frac{2}{2} \right| \right\}, \tag{8}\]
\[d(\theta) = \frac{1}{2\sqrt{3}} \left\{ |\sin \theta| - \sin \theta + \sqrt{3} - |\sin \theta| - \sin \theta - \sqrt{3} \right\}. \tag{9}\]

The graphs of these functions for \(\theta \in [0, 2\pi]\) are plotted in Fig. 2. The interval \([0, 2\pi]\) can be divided into six regions, in each of them the vector \((a(\theta), b(\theta), b'(\theta), c(\theta), d(\theta))\) has a different structure of zero and nonzero entries. As we show below, these regions correspond to mutually different topologies of the system. It extends the basic idea of parametric systems with a varying topology introduced in works of Balachandran et. al. \([1]\) and also in \([7]\). Our use of a single \(n = 4\) singular vertex makes the treatment more systematic and unified.

![Fig. 2. Graphs of functions defined by (7) for \(\theta \in [0, 2\pi]\).](image)

**Region I** For \(\theta \in [0, \frac{\pi}{3}]\), we have

\[
\begin{pmatrix}
1 & t \cdot a(\theta) \\
1 & t \cdot a(\theta)
\end{pmatrix} \Psi' = \begin{pmatrix}
- t \cdot a(\theta) & 1 \\
-t \cdot a(\theta) & 1
\end{pmatrix} \Psi, \tag{11}
\]

where \(t a(\theta)\) takes the value \(t\) at \(\theta = 0\) and it continuously decreases down to 0 at \(\theta = \frac{\pi}{3}\). With regard to the structure of \([11]\), the endpoint 1 is joined to the endpoint 2 and the endpoint 3 is joined to the endpoint 4. On the other hand, the endpoints 1 and 2 are disjoint, so are 3 and 4. Therefore, the system is topologically equivalent to a single ring, see Fig. 3.

**Region II** For \(\theta \in [\frac{\pi}{3}, \frac{2\pi}{3}]\), we have

\[
\begin{pmatrix}
1 & 1 - c(\theta) \\
1 & c(\theta)
\end{pmatrix} \Psi' = \begin{pmatrix}
c(\theta) & 1 - c(\theta) \\
c(\theta) & 1
\end{pmatrix} \Psi, \tag{12}
\]

where we start with the value \(c(\frac{\pi}{3}) = 0\) and continuously increase it up to \(c(\frac{2\pi}{3}) = 1\). Since all four inner endpoints are disjoint, the system is topologically equivalent to two separate lines.
Region III For $\theta \in [\frac{2\pi}{3}, \pi]$, we have

$$
\begin{pmatrix}
1 & t \cdot b(\theta) \\
1 & t \cdot b'(\theta)
\end{pmatrix}
\begin{pmatrix}
\Psi \\
\Psi'
\end{pmatrix}
= 
\begin{pmatrix}
-s & \theta \\
-t & \theta
\end{pmatrix}
\begin{pmatrix}
\Psi \\
\Psi'
\end{pmatrix},
$$

(13)

where $tb(\theta)$ continuously increases from $tb(\frac{2\pi}{3}) = t$ to $tb(\pi) = t$, while $tb'(\theta)$ takes the value $tb'(\theta) = 0$ at $\theta = \frac{2\pi}{3}$, continuously increases up to $t$ at $\theta = \frac{5\pi}{6}$ and comes down back to 0 at $\theta = \pi$. The connection condition (13) implies disjoints endpoints 1 and 3, as well as 2 and 4, making the system topologically equivalent to two separate rings.

Region IV For $\theta \in [\pi, \frac{4\pi}{3}]$, we have

$$
\begin{pmatrix}
1 & t \cdot b(\theta) \\
1 & s \cdot d(\theta)
\end{pmatrix}
\begin{pmatrix}
\Psi \\
\Psi'
\end{pmatrix}
= 
\begin{pmatrix}
-s & \theta \\
-t & \theta
\end{pmatrix}
\begin{pmatrix}
\Psi \\
\Psi'
\end{pmatrix},
$$

(14)

where $tb(\theta)$ continuously decreases from $tb(\frac{4\pi}{3}) = t$ to $tb(\pi) = 0$, while $sd(\theta)$ continuously increases from $sd(\frac{2\pi}{3}) = 0$ up to $sd(\frac{4\pi}{3}) = s$. The system is topologically equivalent to a ring and a line connected at a single point since three of four endpoints, namely 1, 2 and 4, are connected in the node. Note that at the juncture of regions III and IV, the system is topologically equivalent to a separated line and a ring.

Region V For $\theta \in [\frac{4\pi}{3}, \frac{5\pi}{3}]$, we have

$$
\begin{pmatrix}
1 & 1 - c(\theta) \\
1 & c(\theta)
\end{pmatrix}
\begin{pmatrix}
\Psi \\
\Psi'
\end{pmatrix}
= 
\begin{pmatrix}
c(\theta) & 1 - c(\theta) \\
-s & 1
\end{pmatrix}
\begin{pmatrix}
\Psi \\
\Psi'
\end{pmatrix},
$$

(15)

where $c$ continuously decreases from $c(\frac{4\pi}{3}) = 1$ down to $c(\frac{5\pi}{3}) = 0$. The system is topologically equivalent to a single line, since only endpoints 1 and 4 are connected, whereas 2 and 3 are disjoint.

Region VI For $\theta \in [\frac{5\pi}{3}, 2\pi]$, we have

$$
\begin{pmatrix}
1 & t \cdot a(\theta) \\
1 & t \cdot a'(\theta)
\end{pmatrix}
\begin{pmatrix}
\Psi \\
\Psi'
\end{pmatrix}
= 
\begin{pmatrix}
-t \cdot a(\theta) & 1 \\
-s \cdot d(\theta) & -t \cdot a(\theta)
\end{pmatrix}
\begin{pmatrix}
\Psi \\
\Psi'
\end{pmatrix},
$$

(16)

where $ta(\theta)$ continuously increases from $ta(\frac{4\pi}{3})$ up to $ta(2\pi) = t$, while $sd(\theta)$ continuously decreases from $sd(\frac{5\pi}{3}) = s$ down to $sd(2\pi) = 0$. In this region, the system is topologically equivalent to two rings connected at a single point since all four endpoints are connected together at the central node.

FIG. 3. Cycle of topological changes corresponding to functions (7).

Therefore, with the parameter change $\theta = 0 \rightarrow 2\pi$, seven distinct topologies are traversed in a continuous manner. The situation is illustrated in Fig. 3. The system is never identical for any two different values of $\theta$ in $[0, 2\pi]$, because the the functions $a(\theta)$ to $d(\theta)$ form district combinations of patterns in all six regions.

Short Cycle

As an additional model, we consider a simplified parametric cycle represented by the boundary condition (14) with

$$
\begin{align*}
a(\theta) &= \frac{1}{2} (\cos \theta + |\cos \theta|), \\
b(\theta) &= b'(\theta) = 0, \\
c(\theta) &= \frac{1}{2} (-\cos \theta + |\cos \theta|), \\
d(\theta) &= \frac{1}{2} (-\sin \theta + |\sin \theta|).
\end{align*}
$$

(17)

The graphs of these functions are plotted in Fig. 4. This
system traverses four distinct segments in the parameter space of $\theta$, as $\theta$ runs from 0 to $2\pi$. The segments correspond to four topologies, cf. Fig. 5. Namely, a ring, two disjoint lines, a line attached to a ring, and two rings connected at a point.

![Figure 5](image1)

**FIG. 5.** Cycle of topological changes corresponding to functions (17).

### QUANTUM ANHOLONOMY

The system wave functions can be written in the form

$$
\begin{align*}
\psi_1(x_1) &= \alpha_1 \sin kx_1 + \beta_1 \cos kx_1, \\
\psi_2(x_2) &= \alpha_2 \sin kx_2 + \beta_2 \cos kx_2.
\end{align*}
$$

In this notation, we have

$$
\Psi = U \begin{pmatrix} \alpha_1 \\ \beta_1 \\ \alpha_2 \\ \beta_2 \end{pmatrix}, \quad \Psi' = kV \begin{pmatrix} \alpha_1 \\ \beta_1 \\ \alpha_2 \\ \beta_2 \end{pmatrix}
$$

with

$$
U = \begin{pmatrix} 1 \\ \sin kL_1 \cos kL_1 \\ 1 \\ \sin kL_2 \cos kL_2 \end{pmatrix}, \quad V = \begin{pmatrix} 1 \\ -\cos kL_1 \sin kL_1 \\ 1 \\ -\cos kL_2 \sin kL_2 \end{pmatrix}.
$$

The spectra of the system can be calculated from the secular equation

$$
|AU + kBV| = 0.
$$

It turns out that certain choices of $s, t$ and ratio $L_1/L_2$ result in a quantum anholonomy. Below we present the results, separately for the long and the short cycle.

**Long Cycle**

We first consider the long cycle of parameter change represented by (7). We performed a numerical calcula-
FIG. 7. Anholonomy as a result of the long cycle topology change for $t = 0.5$, $s = 0.5$. The ratio $L_1/L_2$ is chosen as the bronze mean (top), silver mean (middle) and golden mean (bottom).

FIG. 8. Anholonomy as a result of the short cycle topology change for $t = 0.5$, $s = 0.5$. The ratio $L_1/L_2$ is chosen as the bronze mean (top), silver mean (middle) and golden mean (bottom).

We explain the numerical result shown in Figure 6. The spectra have period $2\pi$ as a function of $\theta$. This is because all the Hamiltonians are $2\pi$ periodic by construction. On the other hand, however, some wavenumber do not have the $2\pi$ periodicity. For example, let us keep track of the wavenumber (say $k_0$) of the ground state at $\theta = 0$ in the uppermost figure. Around $\pi/3 < \theta < 2\pi/3$, $k_0$ increase suddenly and form three crossings with another eigenstates. Because the graphs consist of two disjoint parts, these crossings are not avoided. The eigenfunction corresponding to $k_0$ is localized within a disjoint fragment, and the other three eigenfunctions involving the crossings are localized in the other part. Hence, as a result of a $2\pi$-cycle of $\theta$, $k_0$ becomes the wavenumber of the third excited state. This is an example of anholonomy in eigenvalue and eigenspace. Some other wavenumbers, e.g., 4-th excited state at $\theta = 0$ in the leftmost figure, are not involved by the anholonomy. By changing the ratio $L_1/L_2$, we see various patterns of the quantum anholonomy. Also, the anholonomies appear in various choice of the parameters, cf. Figure 7.

Short Cycle

Let us proceed to the simplified cycle represented by (17), in which four distinct topologies are connected in a single sequence. The spectra as functions of the angle parameter $\theta \in [0, 2\pi)$ have been calculated for the choice of parameters $s = 0.5$, $t = 0.5$ and $L_1/L_2$ attaining the
value of the golden mean, the silver mean and the bronze mean. The results are depicted in Fig. 8.

**ON THE BOUNDARY CONDITION**

FIG. 9. Finite approximation of the physical meaning of the boundary condition (4) for $\theta$ belonging to the sectors I and II. The strengths of the $\delta$ potentials in the vertices 1, 2, 3, 4 are given by the following formulas. Left: $v_1 = v_2 = \frac{(t \theta^2 - t)}{\epsilon}$, $v_3 = v_4 = \frac{1}{\epsilon}$, $v_1 = \frac{1}{\epsilon}$. Right: $v_1 = 0$, $v_2 = \frac{1}{\epsilon}$, $v_3 = \frac{1}{\epsilon}$, $v_4 = \frac{1}{\epsilon}$.

FIG. 10. Finite approximation of the physical meaning of the boundary condition (4) for $\theta$ belonging to the sectors III and IV. The strengths of the $\delta$ potentials in the vertices 1, 2, 3, 4 are given by the following formulas. Left: $v_1 = \frac{(t \theta^2 - t)}{\epsilon}$, $v_2 = \frac{1}{\epsilon}$, $v_3 = \frac{(t \theta^2 - t \theta')}{\epsilon}$, $v_4 = \frac{1}{\epsilon}$. Right: $v_1 = \frac{(t \theta^2 - t \theta')}{\epsilon}$, $v_2 = \frac{1}{\epsilon}$, $v_3 = \frac{1}{\epsilon}$, $v_4 = \frac{1}{\epsilon}$.

FIG. 11. Finite approximation of the physical meaning of the boundary condition (4) for $\theta$ belonging to the sectors V and VI. The strengths of the $\delta$ potentials in the vertices 1, 2, 3, 4 are given by the following formulas. Left: $v_1 = \frac{1}{\epsilon}$, $v_2 = \frac{1}{\epsilon}$, $v_3 = \frac{1}{\epsilon}$, $v_4 = \frac{1}{\epsilon}$. Right: $v_1 = \frac{(t \theta^2 - t \theta - t \theta - 3 s \theta)}{\epsilon}$, $v_2 = \frac{1}{\epsilon}$, $v_3 = \frac{1}{\epsilon}$, $v_4 = \frac{1}{\epsilon}$.

The boundary condition (4) represents a certain singular potential in the vertex that has in general a highly nontrivial nature. As it has been shown in [3, 8], the physical content of a generic boundary condition corresponds to a set of infinitely strong $\delta$ potentials sited in points in an infinitely small web located in the vertex. We sketch how the webs look like for the boundary conditions of our model in Figures 9, 10 and 11. In all of the finite approximations, the parameter $\epsilon$ has to be chosen small with respect to the wavelength of the particle. For $\epsilon \to 0$, the approximation effectively produces the boundary conditions (4). The result has been obtained using the formulas from [3, 8]. It illustrates in a simple way the relation between the global topology of the system and the topology of the internal infinitely small web.

**DISCUSSION**

The creation of the anholonomy in our model can be explained in the following way. In regions I, IV, V and VI, when the graph consists of a single component, the eigenvalues as functions of the system parameter $\theta$ can change their values, but they never cross each other, because generically there will be avoided crossings. However, in regions II and III, the graph is separated into two components (two lines or two rings), i.e., the system is in fact made up of two independent entities, both having its own energy levels that can independently vary with change of the system parameter $\theta$. Therefore, if the parametrization of boundary conditions is conveniently chosen, the eigenvalues as functions of $\theta$ can cross each other, which leads to the anholonomy effect.

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[1] A. P. Balachandran, G. Bimonte, G. Marmoc, and A. Simon. *Topology change and quantum physics*. Nucl. Phys. B **446**, 299–314, 1995.
[2] M. V. Berry. *Quantal Phase Factors Accompanying Adiabatic Changes*. Proc. R. Soc. London A **392**, 45–57 (1984).
[3] T. Cheon, P. Exner and O. Turek. Approximation of a general singular vertex coupling in quantum graphs. *Ann. Phys.* **325**, 548–578, 2010.
[4] T. Cheon and A. Tanaka. New Anatomy of Quantum Holonomy. *Europhys. Lett.* **85**, 20001(5p), 2009.
[5] P. Exner, J.P. Keating, P. Kuchment, T. Sunada, A. Teplyaev, eds. *Analysis on Graphs and Applications*. AMS Proc. of Symposia in Pure Math. Ser., vol. 77, Providence, R.I., 2008, and references therein.
[6] V. Kostrykin, R. Schrader. Kirchhoff’s rule for quantum wires. *J. Phys. A: Math. Gen.* **32**, 595–630, 1999.
[7] A. D. Shapere, F. Wilczek, Z. Xiong. Models of topology change. *arXiv*:1210.3545.
[8] O. Turek, T. Cheon. Potential-controlled filtering in quantum star graphs. *Ann. Phys. NY* **330**, 104–141, 2013.
[9] F. Wilczek and A. Zee. Appearance of Gauge Structure in Simple Dynamical Systems. *Phys. Rev. Lett.* **52**, 2111–2114, 1984.