CURVES ON A DOUBLE SURFACE

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Dedicated to Silvio Greco on the occasion of his sixtieth birthday

Abstract. Let $X$ be a doubling of a smooth surface $F$ in a smooth threefold and let $C \subset X$ be a locally Cohen-Macaulay curve. Then $C$ gives rise to two effective divisors on $F$, namely the curve part $P$ of $C \cap F$ and the curve $R$ residual to $C \cap F$ in $C$. We show that a general deformation of $R$ on $F$ lifts to a deformation of $C$ on $X$ when a certain cohomology group vanishes and give applications to the study of Hilbert schemes of locally Cohen-Macaulay space curves.

1. Introduction

It is usually difficult to determine when a fixed curve $C \subset \mathbb{P}^3$ is in the closure of another family of curves. Beyond semicontinuity conditions, there are few known obstructions. Hartshorne showed that curves of certain degree and genus cannot specialize to stick figures by analyzing the specific quadric and cubic surfaces on which the curves lie [H97]. The problem reduces to that of linear equivalence if all the curves lie on a fixed smooth surface $F$, since the families of divisor classes are both open and closed in the corresponding Hilbert scheme [F68]. When the curves in question are nonreduced, smooth surfaces are of little help.

In our analysis of curves of degree four in $\mathbb{P}^3$ [NS01], we used families of curves on double planes and double quadric surfaces to produce various specializations in the Hilbert schemes: these were critical in determining irreducible components and showing connectedness. If $X$ is a doubling of a smooth surface $F$ in a smooth threefold - or more generally, if $X$ is a ribbon supported on $F$ in the sense of Bayer and Eisenbud [BE95] - we will describe the Hilbert scheme of curves on $X$, using as our model the rather complete study of curves on a double plane in $\mathbb{P}^3$ [HS00].

In section 2 we describe the natural triple $T(C)$ associated to a curve $C \subset X$ defined in [HS00]: the scheme-theoretic intersection $C \cap F$ has a divisorial part $P$ and a zero-dimensional part $Z$, and when we form the residual curve $R$ to $C \cap F$ in $C$, we obtain the triple $T(C) = \{Z, R, P\}$. Here $Z$ is a generalized Gorenstein divisor on $R$ and $R \subset P$ are effective divisors on $F$. We describe the curves $C$ giving rise to a fixed triple $\{Z, R, P\}$ and give practical conditions (2.3 and 2.5) under which this space is non-empty. The existence of such curves $C$ is subtle when our conditions fail.

Using the triples above, we stratify $H_{d,g}(X)$ in section 3 to obtain locally closed $H_{z,r,p} \subset H_{d,g}(X)$ with natural projection maps $t$ to the relevant Hilbert flag schemes $D_{z,r,p}$. Relativizing results from section 2, we find (3.1) that $t$ has the local structure of an open immersion followed by an affine bundle projection over the locus $V \subset D_{z,r,p}$ of triples satisfying $H^1(\mathcal{O}_R(Z + P - F)) = 0$ (the fibres are nonempty if condition
(2) of (2.3) holds). Combining with the structure of the projection map $D_{z,r} \to H_r$ (3.5), we find (Theorem 3.10) that if $C \subset X$ is a curve whose triple $\{Z, R, P\}$ satisfies $H^1(O_R(Z + P - F)) = 0$, then a general deformation of $R$ lifts to a deformation of $C$. We close with some applications to families of space curves.

2. Curves on a ribbon

Let $F$ be a smooth surface over an algebraically closed ground field $k$. If $F$ is contained in a smooth threefold $T$ and $X$ is the effective divisor $2F$ on $T$, then

1. $\text{Supp} X = F$;
2. $\mathcal{I}_{F,X} \cong \mathcal{O}_F(-F)$ is an invertible $\mathcal{O}_F$-module.

In other words, $X$ is a ribbon over $F$ in the sense of Bayer and Eisenbud [BE95]. Since $F$ is smooth, any ribbon $X$ is locally split, hence appears locally as a doubling of $F$ in a smooth threefold. We use the notation $\mathcal{O}_F(-F) = \mathcal{I}_{F,X}$ and $\mathcal{O}_F(F) = \mathcal{H}\text{om}_{\mathcal{O}_F}(\mathcal{I}_{F,X}, \mathcal{O}_F)$.

Here we will further assume $X$ is projective, although many of our constructions work more generally.

We will study curves on a ribbon $X$ over $F$ using the triples introduced in [HS00]. We adopt the following conventions: A subscheme $C \subset X$ is a curve if all of its associated points have dimension one, thus $C$ is locally Cohen-Macaulay of pure dimension one or empty. If $Y$ is a subscheme of $X$, $\mathcal{I}_Y$ denotes the ideal sheaf of $Y$ in $X$. If $R$ is a Gorenstein scheme and $Z$ a generalized divisor on $R$ [H194], then $\mathcal{O}_R(Z) = \mathcal{H}\text{om}(\mathcal{I}_{Z,R}, \mathcal{O}_R)$ denotes the reflexive sheaf associated to the divisor $Z$. If further $R \subset F$, we write $\mathcal{O}_R(Z - F)$ for $\mathcal{O}_R(Z) \otimes \mathcal{O}_F(-F)$.

**Proposition 2.1.** To each curve $C$ in $X$ is associated a triple $T(C) = \{Z, R, P\}$ in which $R \subset P$ are effective divisors on $F$, $Z \subset R$ is Gorenstein and zero-dimensional (possibly empty), and

\[ \mathcal{I}_{P,C} \cong \mathcal{O}_R(Z - F). \]

The arithmetic genera are related by

\[ p_a(C) = p_a(P) + p_a(R) + \deg R \mathcal{O}_R(F) - \deg(Z) - 1. \]

**Proof.** We proceed as in [HS00, §2]. Extracting the possible embedded points from the one dimensional scheme-theoretic intersection $C \cap F \subset F$, we may write

\[ \mathcal{I}_{C \cap F,F} = \mathcal{I}_{Z,F}(-P) \]

where $P$ is an effective divisor and $Z$ is zero-dimensional. The inclusion $P \subset C \cap F$ yields a commutative diagram

\[
\begin{array}{cccccc}
0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & \mathcal{I}_{R,F}(-F) & \rightarrow & \mathcal{I}_{C,X} & \rightarrow & \mathcal{I}_{Z,F}(-P) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & \mathcal{O}_F(-F) & \rightarrow & \mathcal{I}_{P,X} & \rightarrow & \mathcal{O}_F(-P) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & \mathcal{O}_R(-F) & \rightarrow & \mathcal{L} & \rightarrow & \mathcal{O}_Z(-P) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0 & & 0 & & 0 \\
\end{array}
\]
which defines the residual scheme $R$ to $C \cap F$ in $C$. The inclusion $O_R(-F) \hookrightarrow O_C$ shows that the associated points of $R$ are among those of $C$, hence $R$ is a curve. By construction, $P$ is the largest curve in $F \cap C$, hence $R \subseteq P$ and $C \subset F$ if and only if $R$ is empty.

We now show that $Z$ is Gorenstein on $R$ and that $L \cong O_R(Z - F)$ is a rank one reflexive $O_R$-module. In view of the bottom row of diagram \ref{eq:1}, the submodule $I_R L \subset L$ is supported on $Z$, but $L = I_{P,C} \subset O_C$ has only associated points of dimension one (because $C$ is purely one-dimensional), hence $I_R L = 0$ and $L$ is an $O_R$-module. It follows that $O_Z(-P)$ is an $O_R$-module as well, hence $Z \subset R$.

Applying the bifunctor $\mathcal{H}om_{O_R}(-, -)$ to the sequence $0 \rightarrow I_{Z,R} \rightarrow O_R \rightarrow O_Z \rightarrow 0$ and the bottom row of diagram \ref{eq:1} we obtain

$$
\begin{array}{cccc}
0 & \to & O_R(-F) & \to & 0 \\
\downarrow & & \downarrow & \alpha \\
0 & \to & O_R(Z - F) & \to & \mathcal{E}xt^1_{O_R}(O_Z, O_R(-F)) \\
\downarrow & & \downarrow & 0 \\
0 & \to & L & \to & \mathcal{H}om_{O_R}(I_{Z,R}, L) \\
\downarrow & & \downarrow & \gamma \\
O_Z(-P) & \to & O_Z(-P) & \to & \mathcal{H}om_{O_R}(I_{Z,R}, O_Z(-P))
\end{array}
$$

The morphisms $\pi$, $\phi$ and $\alpha$ are injective because $\mathcal{H}om_{O_R}(O_Z, L) = 0$ as $L$ has no zero dimensional associated point.

Since $F$ is smooth, $R$ is Gorenstein, hence $\omega_Z \cong \mathcal{E}xt^1_{O_R}(O_Z, O_R(-F))$ and $\alpha$ can be thought as a morphism $O_Z \to \omega_Z$. Since $O_Z$ and $\omega_Z$ have the same length, $\alpha$ is an isomorphism (which explains the 0 map at the right of the diagram) and $Z$ is Gorenstein.

Thinking of $L$ and $O_R(Z - F)$ as subsheaves of $\mathcal{H}om_{O_R}(I_{Z,R}, L)$, the 0 at the bottom of the diagram yields $L \subset O_R(Z - F)$ while the 0 at the right gives $O_R(Z - F) \subset L$, hence $L = O_R(Z - F)$.

For the arithmetic genus formula, note that $p_a(C) - p_a(P) = -\chi I_{P,C} = -\chi L$, which can be read off from the bottom row of diagram \ref{eq:1}, keeping in mind that $\deg Z = \chi O_Z$ and $\deg R \mathcal{E} = \chi \mathcal{E} - \chi O_R$ for an invertible sheaf $\mathcal{E}$ on $R$.

\[\square\]

**Proposition 2.2.** Given a triple $\{Z, R, P\}$ of closed subschemes of $F$ as above, the set of curves $C \subset X$ with $T(C) = \{Z, R, P\}$ is in one-to-one correspondence with an open subset of the vector space $H^0(R, O_R(Z + P - F)) \cong \text{Hom}_R(O_R(-P), O_R(Z - F))$.

**Proof.** We study the fibres of the map $C \mapsto T(C)$. We have seen that the bottom row of diagram \ref{eq:1} is a sequence of $O_R$-modules, hence tensoring with $O_R$ we obtain a new diagram

$$
\begin{array}{cccc}
0 & \to & O_R(-F) & \to & I_P \otimes O_R \\
\downarrow & \circlearrowright & \downarrow & \circlearrowright \\
0 & \to & O_R(Z - F) & \to & O_Z(-P)
\end{array}
$$

(3)

\[\square\]
in which \( \phi \) is surjective, the top row is the conormal sequence of \( P \) in \( X \) restricted to \( R \), and the bottom row is obtained by dualizing \( 0 \to \mathcal{I}_{Z,R} \to \mathcal{O}_R \to \mathcal{O}_Z \to 0 \).

It is clear that any surjection \( \phi \) with \( \phi \circ \tau = \sigma \) yields a curve with triple \( \{Z, R, P\} \).

As in \cite{HS00}, we obtain a one-to-one correspondence between curves \( C \) in \( X \) with triple \( \{Z, R, P\} \) and surjections \( \phi \) satisfying \( \phi \circ \tau = \sigma \). Since the cokernel of \( \tau \) is isomorphic to \( \mathcal{O}_R(\mathcal{O}_R(-P), \mathcal{O}_R(Z-F)) \).

It is useful to know when a triple actually arises from a curve.

**Proposition 2.3.** Let \( \{Z, R, P\} \) be a triple of subschemes of \( F \) as in proposition 2.1. Suppose that

1. \( H^1(R, \mathcal{O}_R(Z + P - F)) = 0 \); and
2. the map \( H^0(\mathcal{O}_R(Z + P - F)) \otimes \mathcal{O}_R \to \mathcal{O}_Z \) induced by \( \gamma \) is surjective.

Then the set of curves \( C \subset X \) with \( T(C) = \{Z, R, P\} \) is parametrized by a non-empty open subset \( U \subset H^0(R, \mathcal{O}_R(Z + P - F)) \) of dimension \( \deg Z + \chi \mathcal{O}_R(P - F) \).

**Proof.** The triple \( \{Z, R, P\} \) gives rise to the two exact rows of diagram \( B \). Condition (1) gives

\[
\operatorname{Ext}^1(\mathcal{O}_R(-P), \mathcal{O}_R(Z - F)) \cong H^1(\mathcal{O}_R(Z + P - F)) = 0,
\]

hence there exists \( \phi_0 \in \operatorname{Hom}(\mathcal{I}_P \otimes \mathcal{O}_R, \mathcal{O}_R(Z - F)) \) such that \( \phi_0 \circ \tau = \sigma \). Moreover, any such morphism \( \phi \) can be written \( \phi = \phi_0 + \alpha \circ \pi \) for \( \alpha \in \operatorname{Hom}(\mathcal{O}_R(-P), \mathcal{O}_R(Z - F)) \subset \operatorname{Hom}(\mathcal{I}_P \otimes \mathcal{O}_R, \mathcal{O}_R(Z - F)) \).

Let \( \bar{\phi}_0 : \mathcal{O}_R(-P) \to \mathcal{O}_Z(-P) \) be the morphism induced by \( \phi_0 \). The snake lemma shows that the morphism \( \phi_0 + \alpha \circ \pi \) is surjective if and only if \( \bar{\phi}_0 + \gamma \circ \alpha \) is. Tensoring with \( \mathcal{O}_R(P) \), we view \( \alpha \) as a global section of \( \mathcal{O}_R(Z + P - F) \). The images of these global section under \( \gamma \) generate \( \mathcal{O}_Z \) of by condition (2). Since \( Z \) is finitely supported, it follows that for a general such section \( s \in H^0(\mathcal{O}_R(Z + P - F)) \), the global section \( \gamma(s) + \phi_0(1) \) is a unit in \( \mathcal{O}_{Z,z} \) at each point \( z \in Z \). Thus \( \alpha \) with \( \alpha(1) = s \) corresponds to a surjective morphism \( \phi \).

**Example 2.4.** The hypotheses of Proposition 2.3 are not necessary for the existence of a curve \( C \) with a given triple. Let \( F \subset \mathbb{P}^3 \) be a smooth surface of degree \( d = \deg F \) which contains a line \( L \). The effective divisor \( X = 2F \) on \( \mathbb{P}^3 \) is a ribbon over \( F \) which contains all double lines \( C \) supported on \( L \). If \( p_a(C) \neq 1 - d \), then \( C \not \subset F \) and the triple \( T(C) \) must take the form \( \{Z, L, L\} \), where \( Z \) is an effective divisor of degree \( d - 1 - p_a(C) \) on \( L \). Since \( \mathcal{O}_L(Z + L - F) \cong \omega_L(\deg Z + 4 - 2 \deg F) \) it is clear that \( H^1(\mathcal{O}_L(Z + L - F)) \neq 0 \) and \( H^0(\mathcal{O}_L(Z + L - F)) = 0 \) for \( d \gg 0 \), hence neither hypothesis of 2.3 hold, yet the existence of \( C \) shows that the there are curves with the triple \( \{Z, L, L\} \). Replacing \( L \) by any smooth curve gives similar examples.

**Remark 2.5.** The following practical conditions imply the hypotheses of Prop. 2.3:

1. \( H^1(R, \mathcal{O}_R(Z + P - F)) = 0 \) and \( \mathcal{O}_R(Z + P - F) \) is generated by global sections.
2. \( H^1(R, \mathcal{O}_R(Z + P - F - H)) = 0 \) for some very ample divisor \( H \) on \( R \).
3. \( H^1(R, \mathcal{O}_R(P - F)) = 0 \).
Indeed, the first condition is clearly stronger than the hypotheses of 2.3. The second condition implies the first by Mumford’s regularity theorem. If the third condition is satisfied, then for any effective generalized divisor \( Z \subset R \) the exact sequence

\[
0 \to \mathcal{O}_R(P - F) \to \mathcal{O}_R(Z + P - F) \to \mathcal{O}_Z \to 0
\]

shows that \( H^1(R, \mathcal{O}_R(Z + P - F)) = 0 \) and that \( \gamma \) is surjective on global sections, which implies hypothesis (2) of 2.3 since \( Z \) has finite length.

**Remark 2.6.** Perhaps the simplest situation occurs when the restriction to \( R \) of the conormal sequence associated to \( P \subset F \) (the top row of diagram 3) splits. This happens in case (3) of Remark 2.5.

1. This splitting occurs if and only if the triple \( \{\emptyset, R, P\} \) arises from a curve \( C \), since in this case \( \mathcal{O}_Z = 0 \) and \( \sigma \) is the identity map. In case \( R = P \) is a general smooth space curve, we expect that \( \{\emptyset, R, P\} \) does not arise from a curve \( C \), as this would be equivalent to splitting of the normal bundle.

2. If \( P \) is the intersection of a surface \( F \subset \mathbb{P}^n \) with a hypersurface \( H \) of degree \( d \), then restricting the natural map \( \mathcal{O}_{\mathbb{P}^n}(-d) \to \mathcal{I}_P \) to \( R \) provides such a splitting and the triple \( \{\emptyset, P, P\} \) arises from the curve \( X \cap H \). If \( F \) is a general surface of degree \( \geq 4 \) in \( \mathbb{P}^3 \), then every curve \( P \subset F \) arises in this way since \( \text{Pic} F \cong \mathbb{Z} \) with \( \mathcal{O}_S(1) \) as generator [GH85].

3. When the splitting does occur, there is no obstruction to finding maps \( \phi \) such that \( \phi \circ \tau = \sigma \) and the existence of a surjective such \( \phi \) is equivalent to condition (2) of Proposition 2.3.

**Example 2.7.** If \( X = 2H \) is a double plane in \( \mathbb{P}^3 \), then every triple \( \{Z, R, P\} \) with \( Z \) Gorenstein arises from a curve \( C \subset X \) by Remark 2.5(3) cf. [HS00].

In the next three examples, we consider behavior of triples \( \{Z, R, P\} \) for the double quadric \( X = 2Q \subset \mathbb{P}^3 \), using the standard isomorphism \( \text{Pic} Q \cong \mathbb{Z} \oplus \mathbb{Z} \) [H77, II, 6.6.1].

**Example 2.8.** If \( P \) has type \((a, b)\) with \( a, b > 0 \) and \( R < P \), then every triple \( \{Z, R, P\} \) with \( Z \) Gorenstein arises from a curve \( C \subset X \). Indeed, the exact sequence

\[
0 \to \mathcal{O}_Q(P - R - Q) \to \mathcal{O}_Q(P - Q) \to \mathcal{O}_R(P - Q) \to 0
\]

yields the vanishing of Remark 2.5(3). Since Remark 2.6 applies here, it is common for the normal bundle of a curve \( P \subset Q \) to split when restricted to a proper subcurve \( R \), while this is quite rare when \( R = P \) [Hu82].

**Example 2.9.** If \( 0 < a \leq b \) and \( R = P \), then \( H^1(\mathcal{O}_R(P - Q)) \cong k \) and condition 2.5(3) fails.

1. If \( a = b \), then \( P \) is a complete intersection and the triple \( \{\emptyset, P, P\} \) arises from the complete intersection of \( X \) and a surface of degree \( a \) containing \( P \). Not every triple \( \{Z, P, P\} \) arises from a curve, however: if \( P \) has type \((1, 1)\) and \( \deg(Z) = 1 \), the triple \( \{Z, P, P\} \) could only be associated to a curve of degree 4 and genus 2 by (2.1), but there is no such curve in \( \mathbb{P}^3 \) [H91, 3.1 and 3.3]. On the other hand, if \( \deg(Z) \geq 2 \) and \( P \) is a smooth conic, then there exists a curve \( C \subset X \) with \( T(C) = \{Z, P, P\} \) by Proposition 2.3.
2. If $1 = a < b$ or $(a, b) = (1, 0)$ and $P$ is a smooth rational curve, then the triple \{\(Z, P, P\)\} arises from a curve if and only if \(\text{deg}(Z) > 0\) (condition (2) of 2.3) fails when \(\text{deg}(Z) = 1\), but any nonzero map \(\phi\) in diagram (3) is surjective in this case. The normal bundle splits (as described in [Hu82, Theorem 1] over \(\mathbb{C}\)), but the top row of diagram (3) does not.

3. $1 < a < b$ and the pair \(Z \subset P\) is sufficiently general with \(\text{deg}(Z) > 0\), then the triple \{\(Z, P, P\)\} arises from a curve on \(X\). Since the proof uses a degeneration argument, we postpone it until the following section on families (see Example 3.12 below).

Example 2.10. Now suppose that $P$ has type \((0, b)\) for some $b > 0$.

1. Suppose that $R \subset P$ is a disjoint union of reduced lines. Then applying Example 2.3(2) above to each line $L \subset R$, we see that the triple \{\(Z, R, P\)\} arises from a curve $C \subset X$ if and only if $Z \cap L \neq \emptyset$ for each line $L \subset R$ if and only if $H^1(\mathcal{O}_R(Z + P - Q)) \cong H^1(\mathcal{O}_R(Z - Q)) = 0$.

2. Let $R \subset P$ be a double line on $Q$. In this case $Z$ need not be contained in the underlying reduced line. In fact, if $L$ is the underlying support, then the triple $Z \subset R \subset P$ satisfies the conditions of 2.3(2) if $2 \leq \text{deg}(Z \cap L) \leq \text{deg}(Z) - 2$. To see this, let $W = Z \cap L$ and let $Y$ be the residual scheme to $W$ in $Z$. Since $R^2 = 0$, the sequence relating $Y$ and $W$ to $Z$ takes the form

$$0 \to \mathcal{I}_{Y,L} \to \mathcal{I}_{Z,R} \to \mathcal{I}_{W,L} \to 0$$

and applying $\text{Hom}_{\mathcal{O}_R}(\cdot, \mathcal{O}_R)$ yields the exact sequence

$$0 \to \mathcal{O}_L(W) \to \mathcal{O}_R(Z) \to \mathcal{O}_L(Y) \to 0$$

(one checks locally that $\text{Ext}^1_{\mathcal{O}_R}(\mathcal{O}_L, \mathcal{O}_R) = 0$ and $\text{Hom}_{\mathcal{O}_R}(\mathcal{O}_L(a), \mathcal{O}_R) \cong \mathcal{O}_L(-a)$ by [H77, III,6.7]). Tensoring by $\mathcal{O}_R(-Q - H)$ and taking the long exact cohomology sequence now gives the desired vanishing. One can formulate more complicated criteria for higher order multiple lines on $Q$.

3. Families

In this section we study families of curves in $X$ and their corresponding triples. We prove that, if $V$ denotes the open subset of the flag Hilbert scheme $D$ consisting of triples satisfying the conditions of Proposition 2.3, the set of curves $C$ with $T(C) \in V$ is an open dense subset of an affine fibre bundle over $V$ (3.1). Combining this with the structure of the projection maps on Hilbert flag schemes (3.3), we find that a curve $C$ with triple \{\(Z, R, P\)\} satisfying the first condition of Proposition 2.3 is the flat limit of curves with triples \{\(Z', R', P'\)\} for which $R'$ is general (3.10).

We first extend our constructions to the relative case. Let $\text{Sch}_k$ be the category of locally Noetherian schemes over the ground field $k$. For $S \in \text{Sch}_k$, let $H(S)$ be the set of families of curves $C \subset X \times S$ such that the sheaves $\mathcal{O}_C, \mathcal{O}_{C \cap (F \times S)}$ and $\mathcal{E}_C = \text{Ext}^1_{\mathcal{O}_{F \times S}}(\mathcal{I}_{C \cap (F \times S)}, \mathcal{O}_{F \times S})$ are all flat over $S$. Then $H : \text{Sch}_k \to \text{Sets}$ defines a contravariant functor: if $\phi : T \to S$ is a morphism in $\text{Sch}_k$, we define $H(\phi) : H(S) \to H(T)$ by sending a family $C \in H(S)$ to its pull-back $C_T = C \times_S T$.

We have to check that this is well defined, i.e., that $C_T \in H(T)$. Here the point is that $\mathcal{E}_{C_T}$ is the pullback of $\mathcal{E}_C$; indeed, on the fibres we have $\text{Ext}^2(\mathcal{I}_{C \cap F}, \mathcal{O}_F) = 0$, so the theorem of base change for the $\text{Ext}$ functors [BPS81, JS90] tells us that $\mathcal{E}_C$ commutes...
Theorem 3.1. Let $t$ be a morphism of schemes satisfying $H$ section of $f$ fibre is an invertible sheaf, hence so is $H$. Let $X$ be a scheme and $Z \subseteq Y \subseteq P$ are closed subschemes of $F \times S$, flat over $S$, such that for every closed point $s \in S$ the triple $\{Z_s, Y_s, P_s\}$ is precisely the triple $T(C_s)$.

To construct $P$, we need to show that the sheaf $\mathcal{H}_C = \mathcal{H}om_{\mathcal{O}_{F \times S}}(\mathcal{I}_{C,F}, \mathcal{O}_{F \times S})$ is an invertible sheaf on $F \times S$. By definition of $H(S)$, we know that $\mathcal{E}_C$ is flat over $S$, and its formation commutes with base change. The theorem of base change for the functors $\mathcal{E}xt^i$ implies that $\mathcal{H}_C$ itself is flat over $S$ and commutes with base change.

In particular, the natural map $\mathcal{H} \otimes k(s) \to \mathcal{H}om(\mathcal{I}_{C,F}, \mathcal{O}_F)$ is an isomorphism for every closed point $s \in S$. Thus the restriction of $\mathcal{H}$ to each fibre is an invertible sheaf, hence so is $\mathcal{H}$.

By a standard argument [K96, 7.4.1], the inclusion $\mathcal{I}_{C,F} \hookrightarrow \mathcal{O}_{F \times S}$ defines a global section of $\mathcal{H}$ whose zero scheme is an effective Cartier divisor $P \subseteq F \times S$, flat over $S$.

Now define $Z(C) \subseteq F \times S$ to be the residual scheme to $P$ in $C \cap (F \times S)$, so that $\mathcal{I}_{C\cap(F\times S)} = \mathcal{I}_P \mathcal{I}_Z$. To see that $Z$ is flat over $S$ we note $\mathcal{O}_Z(-P) \cong \mathcal{I}_P, C \cap (F \times S)$ and use [K96, 7.4.1].

Finally, define $R(C) \subseteq X \times S$ to be the residual scheme to the intersection of $C$ with $F \times S$. The exact sequence

$$0 \to \mathcal{O}_R(-F \times S) \to \mathcal{O}_C \to \mathcal{O}_{C,F} \to 0$$

shows that $R$ is flat over $S$, and that for each $s \in S$ the fibre $R_s$ is the residual scheme to the intersection of $C_s$ with $F$. Since $Z_s \subseteq R_s \subseteq P_s$ for each $s \in S$, we have $Z \subseteq R \subseteq P$.

Summing up, to any $C \in H(S)$ we can associate a triple $T(C) = \{Z, R, P\}$ where $Z \subseteq Y \subseteq P$, are closed subschemes of $F \times S$, flat over $S$, and this construction is compatible with base change. Thus we have a natural transformation $T : H \to D$ where $D$ is the functor that to a scheme $S$ associates flags $Z \subset R \subset P \subset F \times S$, with $Z, R, P$ flat over $S$, $Z$ zero dimensional, and $R \subset P$ effective Cartier divisors.

Both $H$ and $D$ are represented by quasiprojective schemes. This is well known for $D$. Using Mumford’s flattening stratification, we see $H$ is representable by a subscheme of the Hilbert scheme of curves in $X$. Since giving the Hilbert polynomials of $C$, $C \cap (F \times S)$ and $\mathcal{E}_C$ is the same as giving the Hilbert polynomials of $Z$, $R$ and $P$, $H$ is represented by the disjoint union of locally closed subschemes $H_{z,r,p}$ of the Hilbert scheme of curves in $X$, where $\{z,r,p\}$ vary in the set of possible Hilbert polynomials for $Z, R$ and $P$. Furthermore, the natural transformation $T$ induces a morphism of schemes $t : H_{z,r,p} \to D_{z,r,p}$.

**Theorem 3.1.** Let $V \subseteq D_{z,r,p}$ be the open subscheme corresponding to triples $\{Z, R, P\}$ satisfying $H^1(\mathcal{O}_R(Z + P - F)) = 0$. Then the map $t^{-1}(V) \to V$ has the structure of an open immersion followed by a projection from an affine bundle over $V$. 
Proof. Given a triple \( \{Z, R, P\} \in D(S) \), we define
\[
\mathcal{O}_R(Z - F \times S) = \text{Hom}_{\mathcal{O}_R}(\mathcal{I}_{Z,R}, \mathcal{O}_R(-F \times S)).
\]

If \( s \in S \) is a closed point, we have \( \mathcal{E}xt^1_{\mathcal{O}_{Rs}}(\mathcal{I}_{Zs,Rs}, \mathcal{O}_{Rs}(-F)) = 0 \) because \( R_s \) is Gorenstein. It follows that \( \mathcal{O}_R(Z - F \times S) \) is flat over \( S \) and its formation commutes with base change \([BPS80, JS90]\), so that for every morphism \( g : T \to S \) in \( \text{Sch}_k \) the pull back of \( \mathcal{O}_R(Z - F \times S) \) is \( \mathcal{O}_{R_T}(Z_T - F \times T) \). Hence there is a functor \( A = A_{z,r,p} \) that assigns to the scheme \( S \) the set of flat families of flags \( Z \subset R \subset P \subset F \times S \) with Hilbert polynomials \( z, r, p \) along with a morphism \( \phi : \mathcal{I}_P \otimes \mathcal{O}_R \to \mathcal{O}_R(Z - F \times S) \).

The exact sequence
\[
0 \to \mathcal{O}_R(-F \times S) \xrightarrow{\tau} \mathcal{I}_P \otimes \mathcal{O}_R \xrightarrow{\pi} \mathcal{O}_R(-P) \to 0.
\]

and the sequence
\[
0 \to \mathcal{O}_R(Z - F \times S) \xrightarrow{\sigma} \mathcal{O}_R(-F \times S) \to \mathcal{E}xt^1(\mathcal{O}_{Zs}, \mathcal{O}_{Rs}(-F \times S)) \to 0
\]

obtained by dualizing
\[
0 \to \mathcal{I}_{Z,R} \to \mathcal{O}_R \to \mathcal{O}_Z \to 0
\]

are both compatible with base change, thus \( A \) has a subfunctor \( M = M_{z,r,p} \) corresponding to morphisms \( \phi \) satisfying \( \phi \circ \tau = \sigma \).

Now we claim that \( H_{z,r,p} \) is an open subfunctor of \( M \). Indeed, given \( C \in H(S) \), we may write a diagram analogous to diagram (2):

\[
\begin{array}{ccccccccc}
0 & \to & \mathcal{O}_R(-F \times S) & \xrightarrow{\tau} & \mathcal{I}_P \otimes \mathcal{O}_R & \xrightarrow{\pi} & \mathcal{O}_R(-P) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\mathcal{I}_C \otimes \mathcal{O}_R & \to & \mathcal{I}_{Z,R}(-P) & \to & 0
\end{array}
\]

As in the proof of [2.1], we obtain a morphism \( \psi : \mathcal{L} \to \mathcal{O}_R(Z - F \times S) \). These sheaves are flat over \( S \) and compatible with pull back. Since \( \psi \) induces isomorphisms \( \psi_s \) on the fibres by the proof of [2.1], \( \psi \) is an isomorphism. Thus the diagram gives us a morphism \( \phi : \mathcal{I}_P \otimes \mathcal{O}_R \to \mathcal{O}_R(Z - F) \) with \( \phi \circ \tau = \sigma \), and we obtain a natural transformation from \( H \) to \( M \) that makes \( H \) into a subfunctor of \( M \). It is open because it corresponds to the open condition that the map \( \phi \) be surjective.

It remains to show that when we take inverse images over \( V \subset D \), the induced map \( M_V \to V \) has the structure of an affine bundle. Let \( U \subset V \) be an affine open set
equipped with universal flat flag

\[ Z \subset R \subset P \subset F \times U \]

\[ \downarrow f \]

\[ U. \]

Since \( H^1(\mathcal{O}_R(Z_u + P_u - F)) = 0 \) for each \( u \in U \), we deduce [H77, III, 8.5 and 12.9] that \( R^1f_*\mathcal{O}_R(Z + P - F) = 0 \) and hence that

\[ \text{Ext}^1(\mathcal{O}_R(-P), \mathcal{O}_R(Z - F)) \cong H^1(\mathcal{O}_R(Z + P - F)) = 0. \]

In particular, there exists \( \phi_0 : \mathcal{I}_P \otimes \mathcal{O}_R \to \mathcal{O}_R(Z - F) \) such that \( \phi_0 \circ \tau = \sigma. \)

Now let \( G : \text{Sch}_U \to \text{Sets} \) be the functor that to a scheme \( T \) over \( U \) associates the set

\[ G(T) = \text{Hom}_{RT}(\mathcal{O}_{RT}(-P_T), \mathcal{O}_{RT}(Z_T - F_T)). \]

By the lemma 3.2 below, \( \mathcal{E} = f_*\text{Hom}_{\mathcal{O}_R}(\mathcal{O}_R(-P), \mathcal{O}_R(Z - F)) \) is locally free on \( U \), and \( G \) is represented by the geometric vector bundle \( B \rightarrow U \) whose sheaf of sections is \( \mathcal{E} \). Thus there is a universal map \( \alpha : \mathcal{O}_{R_B}(-P_B) \to \mathcal{O}_{R_B}(Z_B - F) \) on the pullback of the universal flag to \( B \). We now show that the pair \((B, \phi = p^*(\phi_0) + \alpha \circ \pi) \) represents \( M_U \).

To this end, let \( S \) be a scheme, \( Z_S \subset R_S \subset P_S \subset F \times S \) be a flag corresponding to a map \( h : S \to D \) that factors through \( U \), and \( \psi : \mathcal{I}_{P_S} \otimes \mathcal{O}_{R_S} \to \mathcal{O}_{R_S}(Z - F) \) be a map satisfying \( \psi \circ \tau_S = \sigma_S \). By construction the map \( \psi - h^*(\phi_0) \) is the image of a map in \( \text{Hom}(\mathcal{O}_{R_S}(-P_S), \mathcal{O}_{R_S}(Z_S - F_S)) \), hence the universal property of \( B \to S \) yields a unique lifting \( \tilde{h} : S \to B \) of \( h \). Moreover, it is clear from construction that \( \psi = \tilde{h}^*(\phi) \). This shows that \((B, \phi) \) represents \( M_U \), finishing the proof.

The following lemma, which we used in the above proof, is an immediate consequence of the theorems of base change for cohomology and for the \( \text{Ext} \) functors.

**Lemma 3.2.** Let \( f : R \to U \) be a morphism of locally Noetherian schemes over \( k \), and let \( \mathcal{F}, \mathcal{G} \) be coherent sheaf on \( R \). Let \( G = G_{\mathcal{F}, \mathcal{G}} : \text{Sch}_U \to \text{Sets} \) be the contravariant functor that to a locally Noetherian \( U \)-scheme \( T \) associates the set

\[ G(T) = \text{Hom}_{RT}(\mathcal{F}_T, \mathcal{G}_T) \]

where \( R_T, \mathcal{F}_T, \mathcal{G}_T \) are the base extensions to \( T \). Suppose that \( f \) is projective and flat, and \( \mathcal{F}, \mathcal{G} \) are flat over \( U \). Furthermore, suppose that for every point \( u \in U \):

1. \( \text{Ext}^1_{\mathcal{O}_{R_u}}(\mathcal{F}_u, \mathcal{G}_u) = 0; \)
2. \( H^1(R_u, \text{Hom}_{\mathcal{O}_{R_u}}(\mathcal{F}_u, \mathcal{G}_u)) = 0. \)

Then the sheaf \( \mathcal{E} = f_*\text{Hom}_{\mathcal{O}_R}(\mathcal{F}, \mathcal{G}) \) is locally free on \( U \), and \( G \) is represented by the geometric vector bundle over \( U \) whose sheaf of sections is \( \mathcal{E} \).

**Corollary 3.3.** Let \( Y \) be an irreducible component of \( D_{z,r,p} \) and let \( U \subset Y \) be the open subset consisting of triples \( \{Z, R, P\} \) for which \( H^1(\mathcal{O}_R(Z + P - F)) = 0 \). If \( t^{-1}(U) \) is nonempty, then \( t^{-1}(U) \) is an irreducible component of \( H_{z,r,p} \).

**Proof.** From the structure of \( t \) given in Theorem 3.1, \( t^{-1}(U) \subset H_{z,r,p} \) is an irreducible open subset of \( t^{-1}(Y) \). Let \( W \) be an irreducible component of \( H_{z,r,p} \) containing \( \overline{t^{-1}(U)} \). Then \( t(W) \) is irreducible and contains a nonempty open subset of \( U \) \((t|_{\overline{t^{-1}(U)}}) \) is an
open map by \[3.1\], hence \(Y = \overline{U} \subset t(W)\). Since \(Y\) is an irreducible component, we must have \(Y = \overline{t(W)}\), hence \(W \subset t^{-1}(Y)\). It follows that \(t^{-1}(U) = t^{-1}(U) \cap W\) is a nonempty open subset of \(W\) and \(W = t^{-1}(U)\).

\textbf{Remark 3.4.} Note that \(t^{-1}(V)\) (resp. \(t^{-1}(U)\)) may be empty in Theorem \[3.1\] (resp. Cor. \[3.3\]), as in the case of the smooth conic of type \((1, 1)\) on the quadric surface and \(\deg Z = 1\) (Example \[2.9\](1)). These sets are guaranteed to be nonempty if there exists a triple \(\{Z, R, P\}\) in \(V\) (resp. \(U\)) satisfying condition two of Proposition \[2.3\].

To use Corollary \[3.3\], we need to understand the Hilbert scheme of flags \(D_{z,r,p}(F)\). Now \(D_{z,r,p}\) breaks up as the disjoint union of closed subschemes \(D_{z,\xi,\eta}\) where \(\xi\) (resp. \(\eta\)) varies in the set of numerical equivalence classes of divisors in \(F\) with Hilbert polynomial \(r\) (resp. \(p\)). We have a decomposition

\[D_{z,\xi,\eta} \cong D_{z,\xi} \times H_{\eta-\xi}\]

where \(D_{z,\xi}\) denotes the Hilbert scheme of flags \(Z \subset R \subset F\), with \(Z\) zero dimensional of degree \(z\), and \(R\) an effective divisor of class \(\xi\), and \(H_{\eta-\xi}\) is the Hilbert scheme of effective divisors in \(F\) of class \(\eta - \xi\) - this because we can tack on the effective divisor \(P - R\) after choosing the flag \(Z \subset R\). The following lemma helps to identify the irreducible components of the Hilbert flag scheme.

\textbf{Lemma 3.5.} Let \(q : D_{z,\xi} \to H_{\xi}\) be the projection. Then \(q\) is surjective and maps generic points of \(D_{z,\xi}\) to generic points of \(H_{\xi}\).

\textbf{Proof.} The argument is due to Brun and Hirschowitz [BH87, 3.2]. Since \(q\) is proper and surjective, it is enough to show that, if \(A\) is an irreducible component of \(D_{z,\xi}\) and \(B\) is an irreducible component of \(H_{\xi}\) that contains \(q(A)\), then \(B = q(A)\). Let \(M = \text{Hilb}_z(F)\). \(D_{z,\xi}\) is constructed as the scheme of zeros of a global section of a rank \(z\) vector bundle on \(M \times H_{\xi}\) [K87, S86].

Thus the codimension of \(D_{z,\xi}\) in \(M \times H_{\xi}\) is \(\leq z\) at each point. In particular, the irreducible component \(A\) has dimension at least \(\dim B + z\).

On the other hand, let \(J \subset B\) denote the image of \(A\). The fibre over any fixed curve \(Y \in B\) has dimension \(\leq z\) by the theorem of Briançon [B77, I77] which describes the punctual Hilbert scheme. It follows that

\[\dim B + z \leq \dim A \leq \dim J + z,\]

hence these are equalities and \(J = B\).

\textbf{Remark 3.6.} If \(Z\) is Cartier on \(R\), it follows from deformation theory that the map \(q\) of lemma 3.4 is smooth at the point \((Z, R)\) of \(D\), because \(H^1(N_{Z,R}) = 0\).

\textbf{Remark 3.7.} If \(B \subset H_{\xi}\) is an irreducible component whose general member is a smooth connected curve, then \(q^{-1}(B)\) is an irreducible component of \(D_{z,\xi}\) ([HS00, 4.3]). Indeed, the irreducible components of \(D_{z,\xi}\) contained in \(q^{-1}(B)\) map dominantly to \(B\), but the general fibre of \(q\) is irreducible, so there is only one such component.
Example 3.8. (1) If \( F = \mathbb{P}^2 \), the class of a divisor is determined by its degree \( d \). If \( B = \mathbb{P}^1 H^0(\mathcal{O}_{\mathbb{P}^2}(d)) \), then \( D_{x,d} = q^{-1}(B) \) is irreducible by Remark [BF86, 3.4]. It now follows from Corollary 3.3 and Remark 2.3(3) that the schemes \( H_{z,r,p} \) are irreducible. In fact, their closures are precisely the irreducible components of \( H_{d,g}(X) \) [HS01, 5.1].

Example 3.9. If \( F = Q \subset \mathbb{P}^3 \) is the smooth quadric surface, then the numerical equivalence class of a divisor is determined by its bidegree \( (a,b) \) [H77, II,6.6.1] and \( H_{a,b} = |\mathcal{O}_Q(a,b)| \) is a projective space. If \( a \) and \( b \) are both positive, then the general element of \( H_{a,b} \) is smooth and irreducible, hence \( D_{z,(a,b)} \) is irreducible by Remark 3.4.

If \( a = 0 \) and \( b > 0 \), then the general element in \( H_{a,b} \) is a disjoint union of \( b \) lines and \( D_{z,(a,b)} \) has irreducible components corresponding to various partitions of \( z \) as a sum of \( b \) non-negative integers, depending on how the zero-dimensional scheme \( Z \) is distributed among the generic lines in the family. In particular, \( q^{-1}(H_{a,b}) \) is not irreducible unless \( z \leq 1 \) or \( b = 1 \).

We now prove that if \( T(C) = \{Z,R,P\} \) satisfies \( H^1(\mathcal{O}_R(Z + P - F)) = 0 \), then a general deformation of \( R \) lifts to a deformation of \( C \).

Theorem 3.10. Let \( C \subset X \) be a curve with triple \( T(C) = \{Z,R,P\} \) such that \( H^1(\mathcal{O}_R(Z + P - F)) = 0 \). Suppose that \( B \) is an irreducible component of \( H_r(F) \) containing \( R \). Then there is an irreducible component \( W \) of \( H_{z,r,p} \) containing \( C \) such that the natural map \( H_{z,r,p} \to H_r \) induces a dominant map \( W \to B \).

Proof. By Lemma 3.3 there is an irreducible component \( X \subset D_{z,r} \) containing \( (Z,R) \) such that \( q(X) = B \). Since \( D_{z,\xi,\eta} \cong D_{z,\xi} \times H_{\eta-\xi} \) (here \( \eta \) and \( \xi \) are the numerical equivalence classes of \( P \) and \( R \); see discussion following Cor. 3.3), we obtain an irreducible component \( Y = X \times K \) of \( D_{z,\xi,\eta} \) containing \( T(C) \) and mapping dominantly to \( B \) for a suitable irreducible component \( K \subset H_{\eta-\xi} \). Letting \( U \subset Y \) be the open set of triples for which the vanishing occurs, \( U \) is dense in \( Y \) and the generic point of \( U \) maps to the generic point of \( B \). By Lemma 3.3 \( W = \overline{t^{-1}(U)} \) is an irreducible component of \( H_{z,r,p} \), and by construction the generic point of \( t^{-1}(U) \) maps to the generic point of \( B \).

Example 3.11. The conclusion of Theorem 3.10 fails for a general thick 4-line \( C \) of genus \( g \) on the double quadric \( X = 2Q \) in \( \mathbb{P}^3 \). Recall that a thick 4-line is a curve of degree 4 supported on a line \( L \) and containing the first infinitesimal neighborhood \( L^{(2)} \) [BF86]. We claim that such a curve is not a flat limit of disjoint unions of double lines on \( X \). To see this, we first note that the family of double lines of genus \( g_1 \) with fixed support is irreducible of dimension \( 1 - 2g_1 \) by [X97, 1.6]. Since the lines on \( Q \) form a one-dimensional family, the disjoint unions of two double lines of genera \( g_1 \) and \( g_2 \) form a family of dimension \( 4 - 2g_1 - 2g_2 = 2 - 2g \).

On the other hand, the thick 4-lines on fixed support \( L \) are determined by surjections in

\[
\text{Hom}(\mathcal{I}_L, \mathcal{O}_L(-g - 1)) \cong \text{Hom}(\mathcal{O}_L(-2)^3, \mathcal{O}_L(-g - 1)) \cong H^0(\mathcal{O}_L(-g + 1)^3)
\]

by [BF86, §4], hence these form an irreducible family of dimension \( 5 - 3g \). We are interested in the subset of those which send the equation of \( X \) to zero. If \( L = \{x = y = 0\} \) and \( Q = \{xz - yw = 0\} \), then \( X = \{x^2z^2 - 2xyzw + y^2w^2 = 0\} \).
and hence the thick 4-lines with support \( L \) lying on \( X \) correspond to the triples \( \{(a, b, c) \in H^0(\mathcal{O}_L(-g + 1)^3) : az^2 - 2zw+b + cu^3 = 0\} \). These form a vector subspace of codimension \(-g + 4\) (provided char \( k \neq 2\)), hence the family has dimension \( 1 - 2g \). Varying the support line \( L \) on \( Q \), we obtain a family of dimension \( 2 - 2g \) and conclude that the general thick 4-line \( C \) cannot be the limit of a family whose general member is a disjoint union of two double lines.

**Example 3.12.** In Example 2.9(3) we claimed that if \( X = 2Q \subset \mathbb{P}^3 \) is the double quadric, then the general triple \( \{Z, P, P\} \) arises from a curve on \( X \) if \( \deg Z > 0 \) and \( P \) has type \((a, b)\) with \( 1 < a < b \). We now explain why.

Let \( C \) be a smooth rational curve of type \((1, b - a + 1)\) on \( Q \) and \( Z \subset C \) a divisor with \( \deg Z > 0 \). By Example 2.9(2), \( H^1(\mathcal{O}_C(Z + C - Q)) = 0 \) and the triple \( \{Z, P, P\} \) arises from a curve \( \tilde{C} \) on \( X \). If \( H \) is a general hypersurface of degree \( a - 1 \), then \( H \cap \tilde{C} \) consists of \((a - 1)(b - a + 2)\) double points and \( H \cap Z = \emptyset \). Letting \( \tilde{E} = H \cap X \), we have \( T(\tilde{E}) = \{0, E, E\} \) where \( E \) is a divisor on \( Q \) of type \((a - 1, a - 1)\) (see 2.3(1)). The triple for \( \tilde{C} \cup \tilde{E} \) has form \( \{\tilde{Z}, C \cup E, C \cup E\} \), but \( Z \subset \tilde{Z} \) by local considerations and the genus formula forces \( \tilde{Z} = \tilde{Z} \), hence \( T(\tilde{C} \cup \tilde{E}) = \{Z, C \cup E, C \cup E\} \).

Since \( H^1(\mathcal{O}_C(Z + C - Q)) = 0 \), when we tensor the exact sequence
\[
0 \to \mathcal{O}_C(C) \to \mathcal{O}_{CUE}(C + E) \to \mathcal{O}_E(C + E) \to 0
\]
by \( \mathcal{O}_{CUE}(Z - Q) \) we see that \( H^1(\mathcal{O}_{CUE}(Z + C + E - Q)) = H^1(\mathcal{O}_P(Z + P - Q)) = 0 \).

Note here that \( H^1(\mathcal{O}_E(Z + C + E + Q)) = 0 \) via the exact sequence
\[
0 \to \mathcal{O}_E(Z + E - Q) \to \mathcal{O}_E(Z + E + C - Q) \to \mathcal{O}_{E+C}(Z + E + C - Q) \to 0.
\]

We can now apply Theorem 3.10 and its proof to \( \tilde{C} \cup \tilde{E} \). By Remark 3.7(b), \( B = H_{(a,b)} \) is irreducible as is \( D_{z_r} \cong D_{z,x,p} \), hence the general triple \( \{Z, P, P\} \) with \( \deg Z > 0 \) and \( P \) of type \((a, b)\) arises from a curve.

**Example 3.13.** Let \( W \) be a quasi-primitive triple line of type \((0, b)\) in \( \mathbb{P}^3 \) for some \( b \geq 0 \). Then the underlying double line \( D \) necessarily lies on a smooth quadric surface \( Q \) \( \text{[N97, 1.5]} \) and hence \( W \) lies on the double quadric \( X = 2Q \). The associated triple is \( T(W) = \{Z, L, D\} \), where \( L \) is the support of \( W \) and \( Z \subset L \) is a divisor of degree \( b + 2 \) by the genus formula of Proposition 2.1 \((g(W) = -2 - b \text{ by } \text{[N97, 2.3a]})) \). If \( H \) denotes the hyperplane divisor, then
\[
H^1(\mathcal{O}_L(Z + D - Q - H)) \cong H^1(\mathcal{O}_L(b - 1)) = 0
\]
since \( b \geq 0 \) and Remark 2.3(2) applies. We deduce from Theorem 3.10 that \( W \) is the limit of a family of curves on \( 2Q \) whose general member is the disjoint union of a line and a double line. This generalizes the deformation used in the proof of \([N97, 3.3] \).

**Example 3.14.** This is the example that inspired the present paper. Let \( R = P \) be a double line \( 2L \) on the smooth quadric surface \( Q \subset \mathbb{P}^3 \). Let \( c \geq b \geq 0 \) be integers and let \( Z \subset R \) be a divisor consisting of \( c - b \) simple points and \( b + 2 \) double points which are not contained in \( L \). One can show \([NS01, 3.2] \) that the triple \( \{Z, R, P\} \) arises from a general quasiprimitive 4-line \( C \) of type \((0, b, c) \). Since \( Z \) contains \( \geq 2 \) double points not contained in \( L \), the sufficient condition of Example 2.9(b) holds - thus the conditions of Proposition 2.3 hold for the triple \( \{Z, R, P\} \) and we can apply Theorem...
to see that $C$ is in the closure of some family of disjoint unions of double lines, since the general member of $|O_Q(0,2)|$ is a disjoint union of two lines.

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