On diffusion of large matrices

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Abstract. We briefly review the method of free random variables, its relation to random matrices and possible applications in a context of the stochastic diffusion theory. In order to demonstrate the use of the approach, the formalism is applied to study an additive matrix diffusion and a matrix analogue of a multiplicative Brownian walk.

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1. Introduction

Random matrix theory (RMT) represents a powerful tool in several statistical problems, where degrees of freedom can be encoded as elements of certain ensembles of large matrices. Applications of the RMT cover practically all branches of theoretical physics (spectral properties

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of excited nuclear, atomic and molecular systems, simplicial gravity, theory of mesoscopic conductance or Euclidean quantum chromodynamics [1]—to mention only the most known) and are more and more frequent in recent interdisciplinary research (telecommunication [2, 3], biophysical systems [4]–[7] and economic analysis [8, 9]). The origin of such an omnipresence of the RMT in so many and so different branches of science is quite generic and deep. RMT can be viewed as a generalization of the classical probability calculus, where the concept of probability density distribution for a one-dimensional random variable is generalized onto an averaged spectral distribution of the ensemble of large, non-commuting random matrices. Such a structure exhibits several phenomena known in classical probability theory, including central limit theorems [10]. Actually, the celebrated Wigner semicircle law for the eigenvalue distribution of large matrices with a finite second spectral moment is nothing but the matrix-analogue for similar limiting distribution in the classical probability theory, i.e. the Gaussian distribution.

Most of the applications of random matrices correspond to the ‘static case’, when we are only interested in spectral properties of certain ensembles, and not in the properties of evolution of such ensembles as a function of some external parameter. We name this parameter generically as ‘time’ $\tau$; however its meaning is much richer: $\tau$ can also represent the length of a disordered wire, inverse of the temperature, rapidity, strength of the coupling constant, surface of the string, etc. In all these cases, matrix ensembles are becoming evolving dynamical systems. Sample examples cover real time evolution triggered by stochastic Hamiltonians [11], two-dimensional quantum chromodynamics [12], evolution of Polyakov and Wilson lines [13], DMPK evolution for disordered wires [14] and many others.

It is obvious that such processes constitute a rich class of stochastic evolution. One may think of constructing various ensembles of random matrices (symmetric, Hermitian, complex, unitary, etc) which, when subject to several independent increments, form eventually a matrix-valued stochastic process. It is surprising, however, that such a unique structure has not yet been fully explored. Some well-known examples in mathematics are usually hardly known to physicists, and physical applications are often confined to few particular problems, missing a broader context of the stochastic matrix evolution.

One of the goals of this paper is to familiarize the concept of evolving random matrices, using the analogues borrowed from Brownian walk and from the Einstein–Smoluchowski concepts of diffusion. In section 2 we introduce, after mathematicians [10, 15, 16] the concept of freeness, i.e. the matrix analogue of the independence. Then we show how remarkably free random variables (FRVs) allow us to solve the problem of convolution of random, non-commuting matrices. This law leads directly to a matrix analogue of the central limit theorem. Equipped with the formalism of independent (free) matrix variables, we can start building a series of free matrix-valued increments. In section 3, we demonstrate the results of such a process for the analogue of an additive Brownian walk. In section 4, we parallel the construction for the multiplicative Brownian walk. In the matrix case, this corresponds to an infinite, ordered product of non-commuting matrices, so the usual trick from classical probability theory of taking logarithm and reducing the problem to an additive one (log-normal distribution) fails and a more subtle construction is required. For simplicity, we restrict in this paper to hermitian random ensemble, i.e. Gaussian unitary ensemble (GUE), noting that generalizations for other ensembles, including non-Gaussian and even Lévy-like matrix analogues are possible. The resulting spectral distribution for multiplicative diffusion triggered by GUE was solved by us in [17] (to which we refer as paper I), with the help of dedicated diagrammatic techniques. Here, using the techniques of FRVs, we obtain a simple prescription for the analytic function generating all the
moments’ of the above-mentioned spectral distribution. This is the first new result of this work. Then, we observe that the obtained analytic function is identical to the so-called holomorphic solution discussed in I. In this way we have established a direct link between diagrammatics of I and the concept of FRVs. On the basis of the above observation, we formulate duality hypothesis, which we justify at the end of section 4. As far as we know, such duality has never been noticed in the literature. We conclude with final remarks in section 5, listing also some intriguing open problems.

2. FRVs

In order to make the analogy between the independence (classical probability calculus) and freeness (matrix or non-commutative probability calculus), we start from a well-known textbook example. Let us consider a Gaussian distribution in the classical theory of probability, i.e. the distribution

$$p(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad (1)$$

with the unit variance $\sigma = 1$. Now let us ask the question: what is the distribution of the sum of two identical independent Gaussian distributions, i.e. what is the distribution of

$$S_2 = x_1 + x_2, \quad (2)$$

where $P_2(S_2) = p(x_1) \otimes p(x_2)$? Since the Fourier transformation unwinds the convolution, the trick is to introduce the Fourier transforms (characteristic functions):

$$\phi(q) = \int p(x) e^{iqx} dx, \quad p(x) = \frac{1}{2\pi} \int \phi(q) e^{-iqx} dq. \quad (3)$$

Then, $\Phi_2(q) = \phi(q) \cdot \phi(q)$. The inverse Fourier transformation of the product gives

$$P_2(S_2) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2}} \exp \left[ -\frac{x^2}{2(\sqrt{2})^2} \right], \quad (4)$$

i.e. again the normal distribution but with the new variance $\sigma_2 = \sqrt{2}\sigma = \sqrt{2}$. One may say that the logarithm of the characteristic function

$$\ln \Phi_2(q) = \ln \phi(q) + \ln \phi(q) \quad (5)$$

realizes a certain additivity law for Gaussian variables (2). The characteristic function serves (by construction) as the generating function of the moments $m_n$ and respectively, cumulants $k_n$ of the probability distribution $p(x)$

$$\phi(q) \equiv \sum_{n=0}^{\infty} \frac{m_n}{n!} q^n \equiv \exp \left[ \sum_{n=1}^{\infty} \frac{k_n}{n!} q^n \right]. \quad (6)$$

Therefore, additivity law (5) simply means the additivity of the cumulants for the probability distributions. Now we would like to find an analogue of this construction for the random matrix
ensembles. Consider a Gaussian ensemble built of large \( N \times N \) hermitian matrices \( \mathbf{M} \), for which the pertinent resolvent (Green’s function) is generically given by

\[
G(z) = \int \frac{\mathcal{D}\mathbf{M}}{N} e^{-NV(M)} \frac{1}{N} \text{Tr} \left( \frac{1}{z - \mathbf{M}} \right),
\]

(7)

where the potential \( V(M) \) equals \( \frac{1}{2} \text{Tr} \mathbf{M}^2 \). For simplicity we put the scale of the Gaussian distribution equal to one. This Green’s function fulfils in the limit \( N \to \infty \) a simple algebraic equation

\[
G(z) = (z - G(z))^{-1},
\]

and the normalizable solution of this quadratic equation reads

\[
G(z) = \frac{1}{2} (z - \sqrt{z^2 - 4}).
\]

(8)

Using the fact that the averaged spectral distribution is related to the discontinuities of the Green’s function\(^1\)

\[
\rho(\lambda) = -\frac{1}{\pi} \lim_{\epsilon \to \infty} \text{Im} G(z)|_{z = \lambda + i\epsilon},
\]

(9)

we recover Wigner’s semicircle law

\[
\rho(\lambda) = \frac{1}{2\pi} \sqrt{4 - \lambda^2}.
\]

(10)

This Wigner semicircle law constitutes a matrix analogue of the Gaussian distribution. Let us now try to add two independent Hermitian matrix Gaussian ensembles, as we did in the case of commuting variables of the classical probability theory. In other words, we are asking now what is the distribution of the eigenvalues, if the corresponding Green’s function is given by

\[
G_{1+2}(z) = \int \frac{\mathcal{D}\mathbf{M}_1}{N} \frac{\mathcal{D}\mathbf{M}_2}{N} e^{-NV_1(M_1)} e^{-NV_2(M_2)} \frac{1}{N} \text{Tr} \left( \frac{1}{z - (\mathbf{M}_1 + \mathbf{M}_2)} \right).
\]

(11)

Note that the ‘convolution’ is matrix-valued, and the matrices \( \mathbf{M}_1 \) and \( \mathbf{M}_2 \) do not commute, so mathematically the problem is very hard to solve.

FRV calculus provides a surprisingly simple solution to this problem. We first state the result. To construct the analogue of the additivity law (5), we proceed as follows: firstly, let us find the functional inverse of the Green’s functions for ensembles \( \mathbf{M}_1 \) and \( \mathbf{M}_2 \), i.e. find the solution \( G[B(z)] = z \) for every \( z \) for both ensembles. Then, define \( R(z) = B(z) - 1/z \) for both ensembles. The function \( R \) is additive

\[
R_{1+2}(z) = R_1(z) + R_2(z).
\]

(12)

Proceeding now in the reverse order one can reconstruct \( G_{1+2}(z) \).

We exemplify this construction considering two identical matrix Gaussian ensembles (7). Since for both the Green’s function is given by (8), substituting in both \( z \to B_i(z) \) for \( i = 1, 2 \) and using the definition of the functional inverse we arrive at

\[
B_1(z) = B_2(z) = z + 1/z,
\]

(13)

\(^1\) It is obvious using the special functions property \( \frac{1}{x+i\epsilon} = \text{PV} \frac{1}{x} - i\pi\delta(x) \).
so \( R_1(z) = R_2(z) = z \) and the additivity law gives

\[
R_{1+2}(z) = R_1(z) + R_2(z) = z + z = 2z
\]  

(14)

or, equivalently, \( B_{1+2}(z) = 2z + 1/z \). By substitution \( z \rightarrow G_{1+2}(z) \) and by the definition of the functional inverse, we immediately get

\[
G_{1+2}(z) = \frac{1}{4}(z - \sqrt{z^2 - 8}).
\]  

(15)

The discontinuity of this function yields the spectral distribution

\[
\rho_{1+2}(\lambda) = \frac{1}{4\pi} \sqrt{8 - \lambda^2}.
\]  

(16)

We thus conclude that the matrix convolution of two Gaussian ensembles is again a Gaussian ensemble, but now rescaled by a factor \( \sqrt{2} \), in analogy to the previous exercise from classical probability calculus.

The additivity law (12) in non-commutative probability theory of random matrices forms an analogue of the additivity law for the logarithms of the characteristic functions in the case of classical probability theory.

We may ask the question: what is the origin of such a construction? The answer lies in the definition of the freeness. For non-commuting random variables \( M_1, M_2 \), freeness means that

\[
E[p_1(M_1)r_1(M_2)p_2(M_1)r_2(M_2) \cdots] = 0,
\]  

(17)

provided that \( E[p_j(M_1)] = E[r_j(M_2)] = 0 \) where \( p_j, r_j \) are any polynomials. In case of matrices, an exemplary expectation value \( E(M_1) \) is defined as \( \lim_{N \rightarrow \infty} \frac{1}{N} \langle \text{Tr} M_1 \rangle \), where \( \langle \cdots \rangle \) denotes averaging over the matrix-valued measure; and similarly for other polynomials. Definition (17) gives the rule how to calculate mixed moments in terms of the moments of \( M_1 \) and in terms of the moments of \( M_2 \). In particular, \( E[M_1M_2] = 0 \) if \( E[M_1] = E[M_2] = 0 \). If the variables are not centred (the mean does not vanish), we play the trick by redefining them as \( N_i = M_i - E[M_i] \). Then the product \( E[N_1N_2] \) vanishes by the definition, leading to the law for the mixed moment:

\[
E[M_1M_2] = E[M_1]E[M_2].
\]  

(18)

Similarly, we may calculate more complicated moments:

\[
E[M_1M_2M_1M_2] = E[M_1^2]E[M_2]E[M_2] + E[M_1]E[M_1]E[M_2^2] - E[M_1]E[M_2]E[M_1]E[M_2],
\]

\[
E[M_1M_1M_2M_2] = E[M_1^2]E[M_2^2].
\]  

(19)

Note that the above two moments are different, since now variables \( M_1 \) and \( M_2 \) are matrices and do not commute. So freeness is, in some sense, a much more restrictive property than the well-known independence. The additivity law becomes more obvious, if instead of spectral moments we use spectral cumulants \([15]\). Then the analytic function \( R(z) \) is just the generating function of free cumulants \( K_n \) (note the absence of factorial comparing to the classical case, due to the non-commutativity of the matrix-valued entries)

\[
R(z) = \sum_{n=1}^{\infty} K_n z^n,
\]  

(20)
hence the additivity law (12) means the additivity of free spectral cumulants, in full analogy to the similar additivity law (6) for the cumulants of the sum of independent classical random variables [18]. The reader can easily check that, since for the Gaussian measure (for any $M_1$ or $M_2$) explicitly,

$$\langle M_{ab}M_{cd} \rangle = \frac{1}{N} \delta_{ad} \delta_{bc}, \quad (21)$$

all cumulants in the large $N$ limit reproduce the freeness property, hence the simple addition law for the above studied matrix convolution of two GUEs.

The above example demonstrated that the techniques of FRV offer a powerful shortcut when we seek the distribution of variables coming from ‘sum’ of different ensembles. The above construction could be generalized to higher-order polynomial measures, sums of random and deterministic ensembles, free products of ensembles and also to strictly non-hermitian ensembles [19, 20]. For completeness, we mention also that the crucial question of stable distributions (Lévy–Khintchine distributions with power-law tails) can be also positively answered in the matrix probability calculus [21], leading to a new class of matrix ensembles. Despite spectral moments being divergent, one can show that there exists a Green’s function $G(z)$ satisfying an algebraic equation in large $N$ limit [22] ($\alpha \neq 1$)

$$bG^a(z) - (z - a)G(z) + 1 = 0, \quad (22)$$

in the upper half-plane, and follows by Cauchy reflection in the lower half-plane. The spectra of such ensembles exhibit a power law for large eigenvalues, $\rho(\lambda) \sim 1/\lambda^{\alpha+1}$, where again $\alpha$ belongs exclusively to the interval $(0, 2]$ in a surprising analogy to the classical probability case. The parameter $b$ is related to the Lévy index $\alpha$, asymmetry $\beta$, and range or scale $\gamma$. Further details are discussed in [22]. We would like to mention that the explicit form of the matrix measure is known only in few cases. Even in these cases, Lévy ensembles exhibit a non-trivial behaviour, and the potentials are non-analytic, contrary to the broadly studied polynomial cases.

In the next section, we will use the power of FRV and resulting additivity law to construct the simplest matrix analogue of the matrix-valued Gaussian process.

3. Additive matrix diffusion

Let us remind the diffusion process, where stochastic variable $y$ undergoes an evolution

$$dy = \mu dt + \sigma dx_t. \quad (23)$$

In the above stochastic differential equation, $dx_t$ represents the Wiener process obeying

$$\langle dx_t \rangle = 0, \quad \langle dx_t^2 \rangle = dt \quad (24)$$

and $\mu$ represents the drift. After averaging over independent identical distributions (iid) of the Gaussian variables, we recover the well-known solution for the probability density of $y(\tau)$ fulfilling the heat equation.

We would like to parallel the above construction, in the case when the independent increments in the diffusion process are replaced by free, large and non-commuting random
matrices. A natural, finite time-interval definition of the Gaussian increment is \( Y_i = \sqrt{\tau/K} X_i \), where we divided the finite time interval \([0, \tau]\) into \( K \) infinitesimal steps and \( X_i \) belong to independent free GUEs. Note that due to the properties of GUE, each matrix measure fulfills
\[
\langle Y \rangle = 0, \quad \langle Y^2 \rangle \sim d \tau,
\]
where \( d \tau = \tau/K \), i.e. behaves as an analogue of the Wiener measure. We can now study the problem, how some initial spectral distribution (e.g. Wigner semicircle) behaves under a series of identical, free increments of Gaussian origin.

The simplest tool to investigate this problem is the FRV calculus. First, we note that the Green’s function of the matrix ensemble \( \sqrt{\tau/K} X \) is related to the \( R \)-transform \( R(z) = (\tau/K) z \), i.e. it corresponds to the rescaling of the variance. The algebraic Brownian walk is then realized trivially as a sum of subsequent \( R \) transformations
\[
R(z) = \lim_{K \to \infty} R_K(z) = \lim_{K \to \infty} \sum_{i} R_i = K \frac{\tau}{K} z = \tau z,
\]
and the corresponding Green’s function reads
\[
G(z) = \frac{1}{2t}(z - \sqrt{z^2 - 4(1+\tau)}).
\]

By taking the imaginary part, we get the evolving-in-time semi-circle law. Note that the edges of the spectrum ‘diffuse’ with time, i.e. \( \lambda^2 \sim \tau \), represents diffusion in the space of the eigenvalues of the ensemble. Let as also recall, after Voiculescu, that the Green function for the variable \( Z = A + \sqrt{\tau} X \), where \( A, X \) are Gaussian FRV, fulfills the complex Burgers equation. Indeed, the \( R \)-transformation for variable \( Z \) reads \( R(z) = (1+\tau) z \). The so-called Blue’s function (functional inverse of the Green’s function) reads \( B(z) \equiv R(z) + 1/z = (1+\tau) z + 1/z \). The functional inverse defined as \( G[B(z)] = z \) reads explicitly
\[
G(z, \tau) = \frac{z - \sqrt{z^2 - 4(1+\tau)}}{2(1+\tau)}.
\]

We would like to mention, after \[10\], that solution (28) fulfills a nonlinear differential equation (the complex Burgers equation)
\[
\partial_t G(z, \tau) + G(z, \tau) \partial_z G(z, \tau) = 0,
\]
supplemented by the initial condition
\[
G(z, \tau = 0) = \frac{1}{2}(z - \sqrt{z^2 - 4}).
\]

Hence the complex Burgers equation is a FRV analogue of the heat equation, corresponding to the standard convolution of one-dimensional Gaussian variables.
4. Multiplicative matrix Brownian walk

The analogue of a one-dimensional diffusion process for a geometric random walk is given by the stochastic differential equation

$$\frac{dy}{y} = \mu \, dt + \sigma \, dx_t,$$

(31)

with the same notation as in the previous section. Again, by averaging $y(\tau)$, we recover the well-known solution for the probability density, the so-called log-normal law:

$$p(y, \tau|y_0, 0) = \frac{1}{y \sqrt{2\pi \tau}} \exp \left[ -\frac{\left(\log\left(y/y_0\right) - \mu \tau + \frac{1}{2} \sigma^2 \tau\right)^2}{2\sigma^2 \tau} \right].$$

(32)

Is it possible to find a matrix analogue of the above multiplicative diffusion process? Note that in the case of classical probability theory, the product of random variables may be easily converted into an additive process by taking the logarithm of the product of random variables. This trick fails here, since matrices are non-commuting objects. One way out is to introduce another class of random matrix ensembles, i.e. circular unitary ensembles (CUEs). Eigenvalues of such an ensemble lie on the unit circle, and the product of unitary matrices is again a unitary matrix, so the multiplicative process corresponds to the diffusion of the initial measure over the unit circle as a function of some external parameter $\tau$. This is a case considered recently by one of the authors [23], with several non-trivial links to the existing random models.

In this work we take, however, another course of action, sticking to the GUE, in the case of the multiplicative walk also. Then the pertinent object is the product of random matrix-valued variables [17, 24, 25],

$$Y_\tau = \prod_{i=1}^{K} \left( 1 + \sqrt{\frac{\tau}{K}} X_i \right),$$

(33)

where $X_i$ represent large (infinite) Hermitian Gaussian-free matrices belonging to GUE with the unit variance. Indeed, restricting to a zero drift, the above infinite product is the matrix analogue of the evolution operator for the vector $y$ undergoing multiplicative random changes.

The problem is, however, much more complex. Note that the product of Hermitian matrices is, in general, no longer a Hermitian matrix, which means that the eigenvalues start to diffuse over the whole complex plane, and do not stay on the real axis. This problem was recently solved by us using diagrammatic tools of RMT [17]. In particular, we managed to get the analytical form of the evolution of the envelope of the eigenvalues as a function of ‘time’ $\tau$, we also found the eigenvalue distribution and identified a new, topological phase transition in the spectra of the evolution operator.

Here we would like to demonstrate that several non-trivial features of the evolution operator may also be addressed using the FRVs calculus. We may apply these methods here since the notion of freeness of random matrix ensembles is purely combinatorial and so hermiticity properties are not essential. As long as one is calculating moments, these methods work perfectly since moments of a product are just some specific ‘mixed’ moments of the various free random matrix ensembles. The necessity of using more complex diagrammatic methods as in [17] arises when
we are interested in more complete information about the product such as the density of complex eigenvalues, etc.

Firstly, we derive the generating function for all the spectral moments of (33). We use here another remarkable tool of this calculus which allows us to find a generating function for all moments of the product, provided the spectral properties of the individual ensembles are known. The operational procedure is again surprisingly simple: let us assume that we know Green’s functions \( G_i(\zeta) \) for each of the Hermitian ensembles \( M_i \). We construct an auxiliary analytical function for each of them, defined by the condition

\[
\frac{1}{\chi_i} G_i \left( \frac{1}{\chi_i} \right) - 1 = \zeta, \tag{34}
\]

and then we redefine the result, getting \( S_i(\zeta) \equiv \frac{1+\zeta}{\zeta} \chi_i(\zeta) \). The resulting S-transform is multiplicative, i.e.

\[
S_{1 \times 2}(\zeta) = S_1(\zeta) \cdot S_2(\zeta), \tag{35}
\]

represents the multiplication law for the spectral cumulants. Note that by reconstructing the final Green’s function, we can reach only the information about the moments of the complex eigenvalues; in particular, we cannot by this method get the non-analytic properties like the complex \( \rho(z, \bar{z}) \) distribution of the eigenvalues over the complex plane.

Let us apply now the S-transform for the product of infinitely many random matrices (33). Firstly, we have to analyse the properties of a single matrix in the infinite product, i.e. to find the Green’s function for the variable \( 1 + \sqrt{\tau/KX} \). We encounter the problem of addition of the GUE ensemble to the deterministic entry (here unit matrix), described by \( G_1(\zeta) = (\zeta - 1)^{-1} \). The addition law gives immediately the final answer, i.e. Green’s function for the ensemble \( 1 + \sqrt{\tau/KX} \) reads

\[
G(\zeta) = \frac{1}{2\beta}(\zeta - 1 \pm \sqrt{(1 - \zeta)^2 - 4\beta}), \tag{36}
\]

where \( \beta = \tau/K \). Indeed, it represents a semicircle law shifted by a unit, as expected from the change of the variable \( z \rightarrow (z - 1) \). We construct then the auxiliary function \( \chi(z) \), defined above. A short calculation yields

\[
\chi(z) = \frac{-1 \pm \sqrt{1 + 4\beta z}}{2\beta(1 + z)}. \tag{37}
\]

Finally, we define the S-transform as \( S(\zeta) = (1 + \zeta)\chi(z)/z \). The S-transform allows multiplication of the ensembles [10], hence we have to calculate the limit

\[
S_Y(\zeta) = \lim_{k \to \infty} S_K = \lim_{k \to \infty} \prod_k S_i = \lim_{k \to \infty} \left( \frac{-1 + \sqrt{1 + 4\tau z/K}}{2\tau z/K} \right)^k. \tag{38}
\]

The surprisingly simple answer reads

\[
S_Y(\zeta) = \exp(-\tau z). \tag{39}
\]
We now repeat the procedure in reverse order, first reconstructing $\chi(z)$, then reading out the Green’s function for the ensemble $Y$,  

$$
G_Y(z, \tau) = \frac{1 + f(z, \tau)}{z},
$$

(40)

where $f$ is the solution of the transcendental equation  

$$
z f = (1 + f)e^{zf}.
$$

(41)

Note that, by definition, $G_Y(z, \tau)$ is a holomorphic function (defined for large $z$) generating spectral moments of $Y_\tau$, i.e.

$$
G_Y(z, \tau) = \sum_{n=0}^{\infty} \text{Tr} Y^n_\tau \frac{1}{z^{n+1}}.
$$

(42)

Let us comment on the relation of this simple result (40) to the much more complicated expressions in [17]. Equation (40) is the generating function for the moments and hence is the Green’s function outside the domain of eigenvalues. The non-holomorphic expressions given in [17] are in contrast defined on the domain of eigenvalues and give the complete information on the eigenvalue density. Of course, the holomorphic result (40) can also be obtained using the methods of [17].

Let us observe that $S_Y$ fulfils a simple equation  

$$
z \partial_z S - \tau \partial_{\tau} S = 0.
$$

(43)

By differentiating (41) with respect to the variables $\tau$, $z$, we get the following evolution equation  

$$
\partial_{\tau} f + zf \partial_z f = 0
$$

(44)

or equivalently  

$$
\partial_{\tau} f + f \partial_{\ln z} f = 0,
$$

(45)

supplemented by the boundary condition $f(\tau = 0, z) = (z - 1)^{-1}$. This is the Burgers equation where spatial evolution is governed by $\ln z$. Note, however, that this time the semicircle solution of the additive Burgers equation (28), with $z$ replaced by $\ln z$, does not fulfil (44). For readers familiar with our work on infinite products of random matrices based on diagrammatic techniques [17] (referred to also as I), we would like to mention that by the above construction, we have explicitly established the link between FRVs calculus and diagrammatic methods of I. We have checked that the holomorphic solution for the infinite product of hermitian matrices derived in I is identical to the solution of the complex log-Burgers equation defined here. This fact tempts us to formulate a rather intriguing duality hypothesis: in the generic case of non-hermitian ensembles (i.e. where spectra are complex) both solutions, i.e. the holomorphic one and the non-holomorphic one, carry full information on the spectral moments of the eigenvalue distributions.

In the literature, only the non-holomorphic solution was considered to be the relevant one, and the holomorphic one was usually discarded as a ‘spurious one’. However, since holomorphic and non-holomorphic solutions match on the boundary of the support of eigenvalues, integrating
the spectrum with arbitrary monomial (moment) along any contour non-crossing the support of the eigenvalues, demonstrates that the holomorphic function contains the full information about all moments of the final, complex spectral distribution. As far as we know, the above simple observation on the spectral moments’ duality of both solutions has never been made in the literature before. We have checked that all solved cases of the non-Hermitian ensembles known fulfil this duality.

We stress that in the case of multiplicative diffusion generated by the GUE ensemble, it is impossible to reconstruct the distribution of the eigenvalues from the knowledge of the moments only, since the eigenvalues in this case form two-dimensional islands on the complex plane and are described by non-analytic functions [17]. This is, however, not the case of the diffusion governed by the CUE. Since the eigenvalues in this case remain all the time on the unit circle, one can infer the spectral density from the knowledge of spectral moments in a unique way [23]. It is rather remarkable that the spectral diffusion caused by two different ensembles (free hermitian increments in the case of GUE versus free unitary increments in the case of CUE) are governed by the identical, simple S-transformation $S = \exp(-\tau z)$ (cf our formulae (39) and [23]).

5. Summary

In this paper, we attempted to introduce the reader to the problem of Brownian-like walks in the space of the spectra of large matrices with the help of powerful tools of FRVs calculus. There are several intriguing links pointing that an FRV may constitute an important basis for solving several statistical problems of complex origin.

Firstly, let us mention that despite FRV being a rather new concept (1990s), it appeared already, although often implicitly, in several problems in physics. The reason why Anderson’s model is solvable in the large $N$ limit is due to the fact that it can be mapped directly onto the FRV structures. In particular, ‘coherent potential approximation’ is just the representation of additive law for large matrices (R-transform), as proven in [26]. A similar analogy holds in two-dimensional quantum chromodynamics, where solution of the theory can be mapped on the multiplicative properties of ‘plaquettes’, fulfilling the S-transformation property [12]. There are also several other physical cases, where universal, Burgers-like evolution appears [14], hinting probably at the underlying FRV structures.

Secondly, in the era where enormous amount of information (e.g. capabilities of computers) is stored in the form of large matrices, it is tempting to develop a universal tool, replacing the classical methods of probability calculus by spectral analysis, in order to avoid redundant information and to isolate signals from noise. In particular, the assumption of freeness corresponds to extremization of the SINR (signal-to-noise ratio) in several fundamental problems of telecommunication [3]. FRVs calculus, augmented by the dynamical properties may play a unique role in contemporary analysis of other real complex systems as well.

Thirdly, FRVs form an exciting and rich mathematical structure, still not well understood. Matrix probability calculus features several non-trivial analogies to its classical counterparts. The class of stochastic differential equations governed by FRVs is very rich, including the cases of sub- and super-diffusive processes. The fundamental questions concerning FRV definition of entropy and the relation of Langevin-like to Fokker–Planck-like dynamics have only very recently been posed by the mathematicians [27].

Today, precisely 100 years after seminal works on Brownian motion, we may view as a great challenge a rigorous FRV-generalization of the basic concepts of statistical physics.
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