THE BOUNDED EULER CLASS AND THE SYMPLECTIC ROTATION NUMBER

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ABSTRACT. Ghys [4] established the relationship between the bounded Euler class in $H^2_b(\text{Homeo}_+(S^1); \mathbb{Z})$ and the Poincaré rotation number, that is, he proved that the pullback of the bounded Euler class under a homomorphism $\varphi: \mathbb{Z} \to \text{Homeo}_+(S^1)$ coincides with the Poincaré rotation number of $\varphi(1)$. In this paper, we extend the above result to the symplectic group in some sense, and clarify the relationship between the bounded Euler class in $H^2_b(\text{Sp}(2n; \mathbb{R}); \mathbb{Z})$ and the symplectic rotation number investigated in Barge-Ghys [1].

1. INTRODUCTION AND STATEMENT OF RESULT

Let $G = \text{Homeo}_+(S^1)$ be the group of orientation-preserving homeomorphisms on the circle $S^1$ and $\tilde{G}$ denote its universal cover. It is known that $\tilde{G}$ is identified with the group of self-homeomorphisms of $\mathbb{R}$ which commute with translations by integers:

$$\tilde{G} \cong \{ \tilde{g} \in \text{Homeo}_+(\mathbb{R}) \mid \forall x \in \mathbb{R}, \tilde{g}(x + 1) = \tilde{g}(x) + 1 \}.$$ 

It is also known that there exist mappings $\tau: \tilde{G} \to \mathbb{R}$ and $\rho: G \to \mathbb{R}/\mathbb{Z}$ called, respectively, the Poincaré translation number and the Poincaré rotation number, which are defined by the formulae:

$$\tau(\tilde{g}) = \lim_{n \to \infty} \frac{\tilde{g}^n(0)}{n}; \quad \rho(g) = \tau(\tilde{g}) \mod \mathbb{Z},$$

where $\tilde{g}$ is any lift of $g$ in the second formula.

Let $H^\bullet(\Gamma; \mathbb{Z})$ (resp. $H^\bullet_b(\Gamma; \mathbb{Z})$) denote the group cohomology (resp. the bounded cohomology) of a discrete group $\Gamma$ with coefficients in $\mathbb{Z}$ (see Section 2.). For $\Gamma = \mathbb{Z}$, it is known that a homomorphism $\mathbb{R} \to H^2_b(\mathbb{Z}; \mathbb{Z}); r \mapsto -[c_r]$ induces an isomorphism $\mathbb{R}/\mathbb{Z} \cong H^2_b(\mathbb{Z}; \mathbb{Z})$. Here the bounded 2-cocycle $c_r: \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ is given by

$$c_r(n, m) = \lfloor rn \rfloor + \lfloor rm \rfloor - \lfloor r(n + m) \rfloor,$$

where $\lfloor \cdot \rfloor$ denotes the floor function. On the other hand, for $\Gamma = G$, it is known that the universal cover $\tilde{G} \to G$ induces a central $\mathbb{Z}$-extension

$$0 \to \mathbb{Z} \to \tilde{G} \to G \to 1,$$
and this central extension determines the Euler class $e \in H^2(G; \mathbb{Z})$. Furthermore, the Euler class $e$ has a bounded representative (see, e.g., [4, Lemma 6.3]) and the resulting class is called the bounded Euler class $e_b \in H^2_b(G; \mathbb{Z})$.

Ghys investigated these objects, and found the following relationship between the bounded Euler class $e_b$ and the Poincaré rotation number:

**Theorem 1.1** ([4, Theorem 6.4]). Given $g \in G$, let $\varphi_g : \mathbb{Z} \to G$ be a homomorphism defined by $\varphi_g(k) = g^k$. Then, under the identification $H^2_b(\mathbb{Z}; \mathbb{Z}) \cong \mathbb{R}/\mathbb{Z}$, the pullback $\varphi_g^*(e_b)$ of the bounded Euler class coincides with the Poincaré rotation number of $g$. \hfill $\blacksquare$

For more details, see Ghys’ survey article [4].

In the paper Barge-Ghys [1], they defined the symplectic rotation number and the symplectic translation number on the symplectic group $Sp(2n; \mathbb{R})$ and its universal cover, respectively, and gave the way to compute the symplectic rotation number for a symplectic matrix in terms of its eigenvalues.

One can easily see that the definitions of these maps are similar to those of Poincaré’s. Furthermore, as in the case of $\text{Homeo}_+(S^1)$, the universal cover of $Sp(2n; \mathbb{R})$ induces a central $\mathbb{Z}$-extension, which corresponds to the bounded Euler class in $H^2_b(Sp(2n; \mathbb{R}); \mathbb{Z})$. In this paper, we clarify the relationship between the bounded Euler class and the symplectic rotation number. The following is our main result, which is comparable to Theorem 1.1 due to Ghys:

**Theorem 1.2.** Given a symplectic matrix $g \in Sp(2n; \mathbb{R})$, let $\varphi_g : \mathbb{Z} \to Sp(2n; \mathbb{R})$ be a homomorphism defined by $\varphi_g(k) = g^k$. Then, under the identification $H^2_b(\mathbb{Z}; \mathbb{Z}) \cong \mathbb{R}/\mathbb{Z}$, the pullback of the bounded Euler class by $\varphi_g$ coincides with the symplectic rotation number of $g$. \hfill $\blacksquare$

2. Preliminaries

In this section, we review the definitions and properties of some concepts which we will need to state and prove our result. Let $G$ be a group and $A$ an abelian group. For each integer $k \geq 0$, let $C^k(G, A)$ denote the set of all mappings of $G$ into $A$, and define $\delta : C^k(G, A) \to C^{k+1}(G, A)$ by

$$
\delta c(g_1, \ldots, g_{k+1}) = c(g_2, \ldots, g_{k+1}) + \sum_{i=1}^{k} (-1)^i c(g_1, \ldots, g_i g_{i+1}, \ldots, g_{k+1}) + (-1)^{k+1} c(g_1, \ldots, g_k).
$$
Then \((C^\bullet(G, A), \delta)\) constitutes a cochain complex and its cohomology \(H^\bullet(G; A)\) is called the group cohomology of \(G\) with coefficients in \(A\). It is known that the second group cohomology \(H^2(G; A)\) and the set of equivalence classes of central \(A\)-extensions of \(G\) are in 1-1 correspondence: Here a central \(A\)-extension of \(G\) is a short exact sequence

\[
0 \rightarrow A \xrightarrow{i} \Gamma \xrightarrow{p} G \rightarrow 1
\]

of groups with \(i(A) \subset Z(\Gamma)\), where \(Z(\Gamma)\) is the center of \(\Gamma\). Sometimes we simply refer \(\Gamma\) as a central \(A\)-extension of \(G\).

The correspondence is given as follows: For a central \(A\)-extension \(\Gamma\) of \(G\), choose a set-theoretic section \(s: G \rightarrow \Gamma\) of a surjection \(p: \Gamma \rightarrow G\). Since \(s(g_1 g_2)^{-1}s(g_1)s(g_2) \in \ker(p) = i(A)\) for each \(g_1, g_2 \in G\), we can define a 2-cochain \(\chi: G^2 \rightarrow A\) by the formula

\[
i(\chi(g_1, g_2)) = s(g_1 g_2)^{-1}s(g_1)s(g_2).
\]

Then it is straightforward to check that \(\chi\) is indeed a 2-cocycle and a cohomology class \([\chi] \in H^2(G; A)\) determined by \(\chi\) is independent of the choice of a section. This cohomology class is called the Euler class of the extension and denoted by \(e(\Gamma)\). Assigning \(e(\Gamma)\) to the equivalence class of \(\Gamma\) is the required correspondence.

If \(A\) has a norm (e.g., \(A = \mathbb{Z}\) or \(\mathbb{R}\) with the usual absolute value as its norm), we can consider a subcomplex \(C^\bullet_b(G, A)\) of \(C^\bullet(G, A)\) consisting of bounded cochains, and its cohomology denoted by \(H^\bullet_b(G; A)\) is called the bounded cohomology of \(G\) with coefficients in \(A\). Given a central \(A\)-extension \(\Gamma\) of \(G\), if the 2-cocycle \(\chi: G^2 \rightarrow A\) defined above is bounded, then its cohomology class \([\chi] \in H^2_b(G; A)\) is called the bounded Euler class of the extension \(\Gamma\) and denoted by \(eb(\Gamma)\).

A 1-cochain \(f \in C^1(G, \mathbb{R})\) is called a quasi-morphism if its coboundary is bounded, i.e.,

\[
\sup_{g_1, g_2 \in G} |f(g_1) + f(g_2) - f(g_1 g_2)| < +\infty
\]

holds. A quasi-morphism \(f: G \rightarrow \mathbb{R}\) is called a homogeneous quasi-morphism if \(f(g^n) = n \cdot f(g)\) for all \(g \in G\) and \(n \in \mathbb{Z}\). There exists a way to construct a homogeneous quasi-morphism out of a quasi-morphism:

**Proposition 2.1.** For a quasi-morphism \(f: G \rightarrow \mathbb{R}\), define a map \(\overline{f}: G \rightarrow \mathbb{R}\) by the formula

\[
\overline{f}(g) = \lim_{n \to \infty} \frac{f(g^n)}{n}.
\]
Then this map is a unique homogeneous quasi-morphism on $G$ which satisfies the condition $\sup_{g \in G} |f(g) - f(g')| < +\infty$. The map $\overline{f}$ is called the homogenization of $f$.

Note that if $f \colon G \to \mathbb{Z}$ is a quasi-morphism, then its coboundary $\delta f$ is a bounded 2-cocycle, and hence defines a bounded cohomology class in $H^2_b(G; \mathbb{Z})$. Using this remark, we can determine the second bounded cohomology of $\mathbb{Z}$ with coefficients in $\mathbb{Z}$:

**Proposition 2.2.** For a real number $r \in \mathbb{R}$, define a 1-cochain $\beta_r \colon \mathbb{Z} \to \mathbb{Z}$ by $\beta_r(n) = \lfloor rn \rfloor$, where $\lfloor \cdot \rfloor : \mathbb{R} \to \mathbb{Z}$ is the floor function. Then $\beta_r$ is a quasi-morphism, and a homomorphism $\mathbb{R} \to H^2_b(\mathbb{Z}; \mathbb{Z})$ defined by $r \mapsto [\delta \beta_r]$ induces an isomorphism $\mathbb{R}/\mathbb{Z} \cong H^2_b(\mathbb{Z}; \mathbb{Z})$.

Consult Frigerio [2] for proofs and further details.

Now recall that a connection cochain introduced in Moriyoshi [5]:

**Definition 2.3.** Let $0 \to A \to \Gamma \to G \to 1$ be a central $A$-extension of $G$. A **connection cochain** is a 1-cochain $\tau : \Gamma \to A$ which satisfies the condition

$$\tau(\gamma \cdot i(a)) = \tau(\gamma) + a$$

for all $\gamma \in \Gamma, a \in A$.

**Remark 2.4.** Let $\Gamma$ be a central $A$-extension of $G$. Then there exists a 1-1 correspondence between the sets of sections of $\Gamma \to G$ and the sets of connection cochains for this extension.

**Remark 2.5.** Given a central $A$-extension $\Gamma$ of $G$, let $B$ be an abelian group and $j : A \to B$ a homomorphism. Then a 1-cochain $\tau : \Gamma \to B$ which satisfies the condition

$$\tau(\gamma \cdot i(a)) = \tau(\gamma) + j(a)$$

for all $\gamma \in \Gamma, a \in A$ is called a $B$-valued connection cochain. There exists a way to construct a central $B$-extension $\Gamma_B$ of $G$, and then $\tau$ induces a connection cochain $\tau_B : \Gamma_B \to B$ for this extension.

The fundamental property of connection cochain is the following

**Proposition 2.6.** Let $0 \to A \to \Gamma \to G \to 1$ be a central $A$-extension of $G$. For a connection cochain $\tau : \Gamma \to A$, there exists a unique 2-cocycle $\sigma : G^2 \to A$ such that $\delta \tau = p^* \sigma : G^2 \to A$. Moreover the Euler class $e(\Gamma) \in H^2(G; A)$ coincides with $[-\sigma]$.
Finally, we recall that a group \( G \) is **uniformly perfect** if there exists an integer \( k \) such that any element of the group \( G \) is a product of at most \( k \) commutators. The fundamental property of central extension of a uniformly perfect group is

**Proposition 2.7** ([1, p.236f.]). Let \( 0 \to \mathbb{Z} \to \Gamma \to G \to 1 \) be a central \( \mathbb{Z} \)-extension of a uniformly perfect group \( G \). Then there exists at most one function \( \tau : \Gamma \to \mathbb{R} \) which is a homogeneous quasi-morphism and, simultaneously, an \( \mathbb{R} \)-valued connection cochain. □

The groups \( \text{Homeo}_+ (S^1) \) and \( Sp(2n; \mathbb{R}) \) are both uniformly perfect, and then this proposition is essential to define the rotation number \( \rho \).

### 3. Proof of Main Theorem

Let \( Sp(2n; \mathbb{R}) \) be the group of symplectic matrices with respect to the standard symplectic form on \( \mathbb{R}^{2n} \). Since \( \pi_1(\text{Sp}(2n; \mathbb{R})) \cong \mathbb{Z} \), its universal covering group \( p : \tilde{\text{Sp}}(2n; \mathbb{R}) \to \text{Sp}(2n; \mathbb{R}) \) induces a central \( \mathbb{Z} \)-extension

\[
0 \to \mathbb{Z} \overset{i}{\to} \tilde{\text{Sp}}(2n; \mathbb{R}) \overset{p}{\to} \text{Sp}(2n; \mathbb{R}) \to 1.
\]

There exist several quasi-morphisms on the group \( \tilde{\text{Sp}}(2n; \mathbb{R}) \) which are also \( \mathbb{R} \)-valued connection cochains ([1, C-1]), but according to Proposition 2.7, their homogenization which we will denote \( \tau : \tilde{\text{Sp}}(2n; \mathbb{R}) \to \mathbb{R} \) is uniquely determined. The map \( \tau \) is called the **symplectic translation number**.

Note that for each \( g \in \text{Sp}(2n; \mathbb{R}) \), the value \( \tau(\tilde{g}) \mod \mathbb{Z} \) is independent of the choice of a lift \( \tilde{g} \in p^{-1}(g) \). This is because \( \tau \) is an \( \mathbb{R} \)-valued connection cochain. Then define the **symplectic rotation number** \( \rho : \text{Sp}(2n; \mathbb{R}) \to \mathbb{R}/\mathbb{Z} \) by the formula

\[
\rho(g) = \tau(\tilde{g}) \mod \mathbb{Z},
\]

where \( \tilde{g} \in p^{-1}(g) \) is any lift of \( g \).

Let \( s : \text{Sp}(2n; \mathbb{R}) \to \tilde{\text{Sp}}(2n; \mathbb{R}) \) be a section of \( p \) corresponding to a connection cochain \( \lfloor \cdot \rfloor \circ \tau : \tilde{\text{Sp}}(2n; \mathbb{R}) \to \mathbb{Z} \). Then, by Proposition 2.6, there exists a 2-cocycle \( \sigma : \text{Sp}(2n; \mathbb{R}) \times \text{Sp}(2n; \mathbb{R}) \to \mathbb{Z} \) such that \( p^* \sigma = \delta(\lfloor \cdot \rfloor \circ \tau) \). Since \( \lfloor \cdot \rfloor \circ \tau \) is a quasi-morphism, the cocycle \( - \sigma \) is bounded, and hence defines the bounded Euler class \( e_b = [-\sigma] \in H_b^2(\text{Sp}(2n; \mathbb{R}); \mathbb{Z}) \). Note that the homogenization of \( \lfloor \cdot \rfloor \circ \tau \) is nothing but the symplectic translation number \( \tau \).
Theorem 3.1. Given a symplectic matrix $g \in \text{Sp}(2n;\mathbb{R})$, let $\varphi_g : \mathbb{Z} \to \text{Sp}(2n;\mathbb{R})$ be a homomorphism defined by $\varphi_g(k) = g^k$. Then, under the identification $H^2_b(\mathbb{Z};\mathbb{Z}) \cong \mathbb{R}/\mathbb{Z}$, the pullback of the bounded Euler class $e_b \in H^2_b(\text{Sp}(2n;\mathbb{R});\mathbb{Z})$ by $\varphi_g$ coincides with the symplectic rotation number of $g$, that is,

$$\varphi_g^*(e_b) = \rho(g)$$

holds.

Proof. For $g \in \text{Sp}(2n;\mathbb{R})$, define a quasi-morphism $\beta : \mathbb{Z} \to \mathbb{Z}$ by

$$\beta(k) = \lfloor \tau(s(g)^k) \rfloor.$$ 

Then we have

$$\varphi_g^*(\sigma(k, \ell)) = \sigma(g^k, g^\ell) = p^*\sigma(s(g)^k, s(g)^\ell) = \delta([\cdot] \circ \tau)(s(g)^k, s(g)^\ell) = \delta \beta(k, \ell),$$

for all $k, \ell \in \mathbb{Z}$. Let $\overline{\beta} : \mathbb{Z} \to \mathbb{R}$ be the homogenization of $\beta$, and set $r = \overline{\beta}(1) \in \mathbb{R}$:

$$r = \overline{\beta}(1) = \lim_{k \to \infty} \frac{1}{k} \beta(k) = \lim_{k \to \infty} \frac{1}{k} \lfloor \tau(s(g)^k) \rfloor = \tau(s(g)).$$

Since the symplectic translation number $\tau$ is a homogeneous quasi-morphism,

$$\beta(k) = \lfloor \tau(s(g)^k) \rfloor = \lfloor \tau(s(g)) \cdot k \rfloor = \beta_r(k),$$

for all $k \in \mathbb{Z}$, and hence $\beta = \beta_r$. Therefore, under the identification $H^2_b(\mathbb{Z};\mathbb{Z}) \cong \mathbb{R}/\mathbb{Z}$ (Proposition 2.2), we have

$$\varphi_g^*(e_b) = \varphi_g^*(-\sigma) = -[\delta \beta_r] = r \text{ mod } \mathbb{Z} = \rho(g),$$

as desired. \qed

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