Towards the classification of odd dimensional positively curved reversible homogeneous Finsler spaces

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Abstract

In this paper, we use the flag curvature formula for homogeneous Finsler spaces from our earlier work to classify odd dimensional smooth coset spaces admitting positively curved reversible homogeneous Finsler metrics. We will show the majority in L. Bérard-Bergery’s classification of odd dimensional positively curved Riemannian homogeneous spaces can be generalized to this Finslerian situation.

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1 Introduction

Finding new examples of compact manifolds which admits Riemannian metrics of positive sectional curvature is one of the central problems in Riemannian geometry. Its homogeneous version, the classification of positively curved Riemannian homogeneous spaces, has been completed by several classical works in this field, i.e.\([3,16,1,2]\). Notice that in \([3]\), M. Berger missed one in his classification of positively curved normal homogeneous spaces, which is pointed out by B. Wilking \([17]\). In the classification of odd dimensional positively curved Riemannian homogeneous space \([2]\) by L. Bérard Bergery, a gap has been found recently by the first author and J. A. Wolf, which is

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corrected by B. Wilking [24]. With some tools borrowed from [18], B. Wilking and W. Ziller provided an alternative proof of [2] in one of their most recent preprint [19].

In homogeneous Finsler geometry, we study the following relevant problem, which is also of great theoretical significance.

**Problem 1.1** Classify all smooth coset spaces $G/H$ which admit a $G$-homogeneous Finsler metric of positive flag curvature.

For simplicity, we will call a homogeneous space *positively curved* when it admits a homogeneous Finsler metric of positive flag curvature, or it has been given such a metric. Any positively curved homogeneous space must be compact by the theorem of Bonnet-Myers.

Problem [1.1] was first studied by S. Deng and Z. Hu in [13], in which they classified homogeneous Randers metrics with positive flag curvature and vanishing S-curvature. Their classification is also valid for homogeneous $(\alpha, \beta)$-spaces with positively flag curvature and vanishing S-curvature [22].

We have also made big progress on this classification with more generality. In [21], the first author and S. Deng classified positively curved normal homogeneous Finsler spaces, generalizing [3]. In [23], together with S. Deng and Z. Hu, we classified even dimensional positively curved homogeneous Finsler spaces, generalizing [16].

In [23], we have found a very useful flag curvature formula for homogeneous Finsler spaces (see Theorem 3.1). In this paper, we will continue using it on the classification of odd dimensional positive curved homogeneous Finsler spaces. The general theme for the classification is from [21]. For any positively curved homogeneous Finsler space $(G/H, F)$ with a bi-invariant orthogonal decomposition $g = h + m$ for the compact Lie group $g$, and a fundamental Cartan subalgebra $t$ of $g$ (i.e. $t \cap h$ is a Cartan subalgebra of $h$), we divide our discussion into the following three occasions:

**Occasion I.** Each root plane of $h$ is also a root plane of $g$.

**Occasion II.** There are roots $\alpha$ and $\beta$ of $g$ of $g$ from different simple factors, such that $pr_h(\alpha) = pr_h(\beta) = \alpha'$ is a root of $h$.

**Occasion III.** There are a linearly independent pair of roots $\alpha$ and $\beta$ of $g$ from the same simple factor, such that $pr_h(\alpha) = pr_h(\beta) = \alpha'$ is a root of $h$.

The classification is only up to local isometry. So we introduce the equivalence (see Subsection 2.5) to specify some typical procedures which results local isometries, like changing $G$ to its covering group, changing $H$ while fixing its identity component, cancel common product factors from $G$ and $H$, changing $H$ within $G$ by an isomorphism, and so on. This terminology may save us from unnecessary complexity in the discussion and statement of the results.

In our practical usage of the flag curvature formula in Theorem 3.1, we found that assuming $F$ is reversible can provide great convenience for our discussion. Restriction to reversible Finsler metrics would not ruin too much the generality, and it still contain the Riemannian ones as a special case. It implies the classification result in this paper also verifies the one in [2] by L. Bérard-Bergery.
The main results of this paper, Theorem 5.1, Theorem 6.2 and Theorem 6.3 can be summarized together to the following main theorem.

The main theorem of this paper is the following.

**Theorem 1.2** Let \((G/H, F)\) be an odd dimensional positively curved reversible homogeneous Finsler space. Then we have the following:

1. If it belongs to Occasion I, then up to equivalence, either \(G\) is simple, or \(G/H\) is one of the homogeneous spheres
   
   \[ S^{2n-1} = \frac{U(n)}{U(n-1)} \text{ and } S^{4n-1} = \frac{Sp(n)U(1)}{Sp(n-1)U(1)} \text{ for } n > 1 \]
   
   or the U(3)-homogeneous Aloff-Wallach’s spaces.

2. If it belongs to Occasion II, then up to equivalence, \(G/H\) is one of the homogeneous spheres
   
   \[ S^3 = \frac{SO(4)}{SO(3)} \text{ and } S^{4n-1} = \frac{Sp(n)Sp(1)}{Sp(n-1)Sp(1)} \text{ for } n > 1, \]
   
   or Wilking’s space \(SU(3) \times SO(3)/U(2)\).

3. If it belongs to Occasion III, then up to equivalence, \(G/H\) is one of the homogeneous spheres
   
   \[ S^{2n-1} = \frac{SO(2n)}{SO(2n-1)} \text{ with } n > 2, \]
   \[ S^7 = \frac{Spin(7)}{G_2}, \text{ and } S^{15} = \frac{Spin(9)}{Spin(7)}, \]
   
   or Berger’s spaces
   
   \[ SU(5)/Sp(2)U(1) \text{ and } Sp(2)/SU(2). \]

Notice \(S^{2n-1} = \frac{SO(2n)}{SO(2n-1)}\) and \(S^7 = \frac{Spin(7)}{G_2}\) only admit Riemannian metrics of positive constant curvature as their homogeneous Finsler metrics. Aloff-Wallach’s spaces admit non-Riemannian homogeneous Randers metrics or \((\alpha, \beta)\)-metrics with positive flag curvature and vanishing S-curvature [13] [22]. The others listed in Theorem 1.2 admit non-Riemannian positively curved normal homogeneous Finsler metrics [21].

The main theorem is a summarization of Theorem 5.1, Theorem 6.2 and Theorem 6.3 which will be proved separately. It successfully generalize the majority of the classification result in [2]. But it is not complete for Occasion I when \(G\) is compact simple. That is why the homogeneous spheres \(SU(n)/SU(n-1), Sp(n)/Sp(n-1),\) and the SU(3)-homogeneous Aloff-Wallach’s spaces are missing in Theorem 1.2. The method in this paper is originated from the most traditional algebraic one, i.e. to prove a homogeneous space is not be positively curved, we look for a linearly independent commuting pair \(u\) and \(v\) from \(\mathfrak{m}\) such that the sectional (or flag) curvature for the plane spanned by them (the flag pole needs to be specified for the flag curvature) vanishes. But this method is not valid for some rare cases in Occasion I [24].

In Section 2, we give a brief summary of basic notions in Finsler geometry and homogeneous Finsler geometry, and define the equivalence which will be used throughout
this paper. In Section 3, we present the general theme for the classification of odd dimensional positively curved homogeneous Finsler spaces, including the flag curvature formula, the rank equality, and some useful lemmas. In Section 4 and Section 5, we discuss the classification of odd dimensional positively curved reversible homogeneous Finsler spaces in Occasion III. In Section 6, we discuss the classification of odd dimensional positively curved reversible homogeneous Finsler spaces in Occasion II and I. Section 7 is the appendix where we summarize the presentation of root systems of compact simple Lie algebras which have been used in earlier discussions.

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2 Preliminaries

We will only consider connected smooth manifolds and Lie groups in this work. In this section, we summarize some background knowledge in Finsler geometry. See [6] and [8] for more details.

2.1 Minkowski norm and Finsler metric

A Minkowski norm on a real vector space $V$, $\dim V = n$, is a continuous real-valued function $F : V \rightarrow [0, +\infty)$ satisfying the following conditions:

1. $F$ is positive and smooth on $V \setminus \{0\}$;
2. $F(\lambda y) = \lambda F(y)$ for any $\lambda > 0$;
3. With respect to any linear coordinates $y = y^i e_i$, the Hessian matrix

$$ (g_{ij}(y)) = \left( \frac{1}{2} [F^2]_{y^i y^j} \right) $$

is positive definite at any nonzero $y$.

The Hessian matrix $(g_{ij}(y))$ and its inverse $(g^{ij}(y))$ can be used to raise up and lower down indices of relevant tensors in Finsler geometry.

At each $y \neq 0$, the Hessian matrix $(g_{ij}(y))$ defines an inner product $\langle \cdot, \cdot \rangle_y$ on $V$ by

$$ \langle u, v \rangle_y = g_{ij}(y) u^i v^j, $$

where $u = u^i e_i$ and $v = v^i e_i$. The above inner product is also denoted as $\langle \cdot, \cdot \rangle^F_y$ to specify the norm. Sometimes it is shortened as $g_y$ or $g^F_y$. This inner product can also be expressed as

$$ \langle u, v \rangle_y = \frac{1}{2} \frac{\partial^2}{\partial s \partial t} [F^2(y + su + tv)]_{s=t=0}. \quad (2.2) $$

It is easy to check that the above definition is independent of the choice of linear coordinates.
Let $M$ be a smooth manifold with dimension $n$. A Finsler metric $F$ on $M$ is a continuous function $F : TM \to [0, +\infty)$ such that it is positive and smooth on the slit tangent bundle $TM \setminus 0$, and its restriction in each tangent space is a Minkowski norm. Then $(M, F)$ is called a Finsler manifold or a Finsler space.

Here are some important examples.

Riemannian metrics are a special class of Finsler metrics such that the Hessian matrix only depend on $x \in M$. For a Riemannian manifold, the metric is often referred to the global smooth section $g_{ij} dx^i dx^j$ of $\text{Sym}^2(T^*M)$. Unless otherwise stated, we mainly deal with non-Riemannian metrics in Finsler geometry in this paper.

Randers metrics are the simplest and the most important class of non-Riemannian metrics in Finsler geometry. They are defined by $F = \alpha + \beta$, in which $\alpha$ is a Riemannian metric and $\beta$ is a 1-form. The notion of Randers metrics has been naturally generalized to $(\alpha, \beta)$-metrics. An $(\alpha, \beta)$-metric is a Finsler metric of the form $F = \alpha \phi(\beta/\alpha)$, where $\phi$ is a positive smooth real function, $\alpha$ is a Riemannian metric and $\beta$ is a 1-form. In recent years, there have been a lot of research works concerning $(\alpha, \beta)$-metrics as well as Randers metrics.

Recently, we have defined and studied $(\alpha_1, \alpha_2)$-metrics and suggested more generalized $(\alpha_1, \alpha_2, \ldots, \alpha_k)$-metrics [11][23]. They naturally appear in the study of homogeneous Finsler geometry.

A Minkowski norm or a Finsler metric is called reversible if $F(y) = F(-y)$ for any $y \in V$ or $F(x, y) = F(x, -y)$ for any $x \in M$ and $y \in T_x M$. Obviously Riemannian metrics are reversible. Non-Riemannian Randers metrics are always non-reversible. There are non-Riemannian reversible $(\alpha, \beta)$-metrics in which the function $\phi$ is an even function, and there are much more non-reversible $(\alpha, \beta)$-metrics.

2.2 Geodesic spray and geodesic

On a Finsler space $(M, F)$, a local coordinate system $\{x = (x^i) \in M; y = y^i \partial_{x^i} \in T_x M\}$ on $TM$ is called a standard local coordinates system.

The geodesic spray is a vector field $G$ globally defined on $TM \setminus 0$. In a standard local coordinate system, it can be presented as

$$G = y^i \partial_{x^i} - 2G^i \partial_{y^i},$$

in which

$$G^i = \frac{1}{4} g^{ij} (F^2)_{x^k y^i y^k} - [F^2]_{x^i}.$$  \hspace{1cm} (2.4)

A non-constant curve $c(t)$ on $M$ is called a geodesic if $(c(t), \dot{c}(t))$ is an integration curve of $G$, in which the tangent field $\dot{c}(t) = \frac{d}{dt} c(t)$ along the curve gives the speed. For any standard local coordinates, a geodesic $c(t) = (c^i(t))$ satisfies the equations

$$\ddot{c}^i(t) + 2G^i(c(t), \dot{c}(t)) = 0.$$ \hspace{1cm} (2.5)

It is well known that $F(c(t), \dot{c}(t)) \equiv \text{const}$, i.e. the geodesic we will consider are only geodesics of nonzero constant speed.
2.3 Riemannian curvature and flag curvature

On a Finsler manifold, we have a similar curvature as in the Riemannian case, which is called the Riemannian curvature. It can be defined either by the Jacobi field or the structure equation for the curvature of the Chern connection.

For a standard local coordinate system, the Riemannian curvature is a linear map
\[ R_y = R^i_k(y) \partial_{x^i} \otimes dx^k : T_x M \to T_x M, \]
defined by
\[
R^i_k(y) = 2 \partial_{x^k} G^i - y^j \partial^2 y^i y^k G^j + 2 G^i \partial y^j y^k G^j - \partial y^j G^i \partial_y G^j. \tag{2.6}
\]
When the metric needs to be specified, the Riemannian curvature is denoted as
\[ R^F_y = (R^F)_{y}^i_k(y) \partial_{x^i} \otimes dx^k. \]
The Riemannian curvature \( R_y \) is self-adjoint with respect to \( \langle \cdot, \cdot \rangle_y \).

Using the Riemannian curvature, we can generalize the notion of sectional curvature to the flag curvature in Finsler geometry. Let \( y \in T_x M \) be any nonzero tangent vector and \( P \) any tangent plane in \( T_x M \) containing \( y \), linearly spanned by \( y \) and \( v \) for example. Then the flag curvature of the pair of pole and flag \((y, P)\) is given by
\[
K(x, y, y \wedge v) = \frac{\langle R_y v, v \rangle_y}{\langle y, y \rangle_y \langle v, v \rangle_y - \langle y, v \rangle_y^2}. \tag{2.7}
\]
The flag curvature in (2.7) does not depend on the choice of \( v \) but only on \( y \) and \( P \). Sometimes we will also write the flag curvature of a Finsler metric \( F \) as \( K^F(x, y, y \wedge v) \) or \( K^F(x, y, P) \) to indicate the metric explicitly.

2.4 Totally geodesic submanifold

Any submanifold \( N \) of a Finsler space \( (M, F) \) can be naturally endowed with a submanifold Finsler metric, denoted as \( F|_N \). At each point \( x \in N \), the Minkowski norm \( F|_N(x, \cdot) \) is just the restriction of the Minkowski norm \( F(x, \cdot) \) to \( T_x N \). We say that \((N, F|_N)\) is a Finsler submanifold or a Finsler subspace.

A Finsler subspace \((N, F|_N)\) of \((M, F)\) is called totally geodesic if any geodesic of \((N, F|_N)\) is also a geodesic of \((M, F)\). For a standard local coordinate system \((x^i, y^j)\) such that \( N \) is locally defined by \( x^{k+1} = \cdots = x^n = 0 \), the totally geodesic condition can be equivalently given as
\[
G^i(x, y) = 0, \quad k < i \leq n, x \in N, y \in T_x N. \]
A direct calculation shows that in this case the Riemannian curvature \( R^F_y : T_x N \to T_x N \) of \((N, F|_N)\) is just the restriction of the Riemannian curvature \( R^y \) of \((M, F)\), where \( y \) is a nonzero tangent vector of \( N \) at \( x \in N \). Therefore we have

**Proposition 2.1** Let \((N, F|_N)\) be a totally geodesic submanifold of \((M, F)\). Then for any \( x \in N, y \in T_x N\setminus 0, \) and a tangent plane \( P \subset T_x N \) containing \( y \), we have
\[
K^{F|_N}(x, y, P) = K^F(x, y, P). \tag{2.8}
\]

As in Riemannian geometry, the set of common fixed points for some isometries of \((M, F)\) is a union of totally geodesic sub-manifolds of \((M, F)\) \([7]\).
2.5 General setup in homogeneous Finsler geometry

A connected Finsler manifold $(M, F)$ is called a **homogeneous Finsler space**, or $F$ is called a **homogeneous Finsler metric**, if the connected isometry group $I_0(M, F)$ of $(M, F)$ acts transitively on $M$. It is shown in [9] that $G = I_0(M, F)$ is a Lie transformation group. Let $H$ be the compact isotropic subgroup at a point $o \in M$. Then $M$ is diffeomorphic to the smooth coset space $G/H$, associated with a canonical smooth projection map $\pi : G \to M = G/H$ such that $\pi(e) = o$. The tangent space $T_oM$ can be naturally identified with $m = g/h$, in which $g$ and $h$ are Lie algebras of $G$ and $H$ respectively. The isotropy action of $H$ on $T_oM$ coincides with the induced $\text{Ad}(H)$-action on $m$. In the occasions we discuss later, $m$ can be realized as a complement of $h$ in $g$ which is preserved by $\text{Ad}(H)$-actions, then we have an $\text{Ad}(H)$-invariant decomposition $g = h + m$ satisfying the reductive condition $[h, m] \subseteq m$.

When $(M, F)$ is positively curved, by the theorem of Bonnet-Myers, $M$ must be compact, which implies $G = I_0(M, F)$ be also compact. Choose any bi-invariant inner product for $g$, then we can realize $m$ as the bi-invariant orthogonal complement of $h$, in this case, the decomposition $g = h + m$ is called a **bi-invariant orthogonal decomposition** for the homogeneous space $G/H$.

Notice for any closed connected subgroup $G$ of $I_0(M, F)$ which acts transitively on $M$, we have a corresponding presentation $M = G/H$. The most typical example is the nine classes of homogeneous spheres [4]. For the convenience of later discussion, we will consider a slightly more general situation than this, i.e. we only require the positively curved homogeneous Finsler space $M = G/H$ satisfies $g$ is compact (i.e. $G$ is quasi-compact). The notion of bi-invariant orthogonal decomposition is still valid in this case.

To simplify our discussion and avoid the unnecessary iterate in the classification results, we will not distinguish homogeneous Finsler spaces which are locally isometric to each other. In particular, we will call $(G_1/H_1, F_1)$ and $(G_2/H_2, F_2)$ (with corresponding bi-invariant orthogonal decompositions for the compact Lie groups $g_1$ and $g_2$ respectively) **equivalent** if one of the following is satisfied

1. $G_1$ is a covering group of $G_2$, $H_1$ has the identity component as $H_2$, and $F_1$ is naturally induced from $F_2$, up to a scalar change;

2. $G_1 = G_2 \times G'$, $H_1 = H_2 \times G'$, moreover $F_1$ and $F_2$ are induced from the same Minkowski norm, when $m_1$ and $m_2$ are naturally identified as the same;

3. there is a group isomorphism from $G_1$ to $G_2$, which maps $H_1$ to $H_2$ and induces the isometry from $F_1$ to $F_2$.

Above description in fact defines an **equivalent relation** between compact homogeneous Finsler spaces $G/H$ with $g = \text{Lie}(G)$ compact.
3 The general theme towards the classification of positively curved reversible homogeneous Finsler spaces

3.1 A flag curvature formula for homogeneous Finsler spaces

In [23], we have proven the following theorem,

**Theorem 3.1** Let \((G/H, F)\) be a connected homogeneous Finsler space, and \(g = h + m\) be an \(\text{Ad}(H)\)-invariant decomposition for \(G/H\). Then for any linearly independent commuting pair \(u\) and \(v\) in \(m\) satisfying \(\langle [u, m], u \rangle^F_u = 0\), we have

\[
K^F(o, u, u \wedge v) = \frac{\langle U(u, v), U(u, v) \rangle^F_u}{\langle u, u \rangle^F_u \langle v, v \rangle^F_u - \langle u, v \rangle^F_u \langle u, v \rangle^F_u},
\]

in which \(U(u, v) \subset m\) satisfies

\[
\langle U(u, v), w \rangle^F_u = \frac{1}{2} (\langle [w, u]_m, v \rangle^F_u + \langle [w, v]_m, u \rangle^F_u), \text{ for any } w \in m,
\]

where \([\cdot, \cdot]_m = \text{pr}_m \circ [\cdot, \cdot]_m\) and \(\text{pr}_m\) is the projection with respect to the given \(\text{Ad}(H)\)-invariant decomposition.

Through the flag curvature formula in Theorem 3.1 is not the most general one (see below for the most general one given by L. Huang), it is exactly the convenient one for our usage.

In [23], we introduced the submersion technique to prove this theorem. To make this work more self contained, we quote another shorter proof by L. Huang, which can also be found in [23].

In [14], L. Huang uses invariant frames to give explicit formulas for curvatures of homogeneous Finsler spaces.

To introduce his formula for the Riemann curvatures of homogeneous Finsler spaces, we first define the *spray vector field* \(\eta : m\{0\} \to m\) and the *connection operator* \(N : (m\{0\}) \times m \to m\). For any \(u \in m\{0\}\), \(\eta(u)\) is defined by

\[
\langle \eta(u), w \rangle^F_u = \langle u, [w, u]_m \rangle^F_u, \quad \forall v \in m,
\]

and \(N(u, \cdot)\) is a linear operator on \(m\) determined by

\[
2\langle N(u, w_1), w_2 \rangle^F_u = \langle [w_2, w_1]_m, u \rangle^F_u + \langle [w_2, u]_m, w_1 \rangle^F_u + \langle [w_1, u]_m, w_2 \rangle^F_u - 2C^F_u(w_1, w_2, \eta(u)), \quad \forall w_1, w_2 \in m.
\]

Using these two notions, L. Huang has proved the following formula for Riemannian curvature \(R_u : T_o(G/H) \to T_o(G/H)\),

\[
\langle R_u(w), w \rangle^F_u = \langle [[w, u]_b, w], u \rangle^F_u + \langle \tilde{R}(u) w, w \rangle^F_u, \quad w \in m,
\]

in which the linear operator \(\tilde{R}(u) : m \to m\) is given by

\[
\tilde{R}(u) = D_{\eta(u)} N(u, w) - N(u, N(u, w)) + N(u, [u, w]_m) - [u, N(u, w)]_m,
\]

8
where \( D_{\eta(u)}N(u, w) \) is the derivative of \( N(\cdot, w) \) at \( u \in \mathfrak{m}\setminus\{0\} \) in the direction of \( \eta(u) \), especially it is 0 when \( \eta(u) = 0 \).

Now suppose \( u \in \mathfrak{m}\setminus\{0\} \) satisfies \( \langle [u, \mathfrak{m}], u \rangle^F_u = 0 \), i.e. \( \eta(u) = 0 \). For any \( v \in \mathfrak{m} \) which commutes with \( u \), we have \( N(u, v) = U(u, v) \) in the theorem, so

\[
(R_u(v), v)^F_u = -\langle N(u, N(u, v)), v \rangle^F_u - \langle [u, N(u, v)], v \rangle^F_u = -\frac{1}{2} \langle [[v, N(u, v)]_\mathfrak{m}, u]^F_u + \langle [N(u, v), u]_\mathfrak{m}, v \rangle^F_u + \langle [N(u, v), u], v \rangle^F_u \]
\]

\[
= \frac{1}{2} \langle [[N(u, v), v], u]^F_u + \langle [N(u, v), u], v \rangle^F_u \rangle
\]

\[
= \langle U(u, v), N(u, v) \rangle^F_u = (U(u, v), U(u, v))^F_u.
\]

From this calculation, the flag curvature formula in Theorem 3.1 follows immediately.

### 3.2 The totally geodesic technique and the rank equality

Assume \((G/H, F)\) is a positively curved homogeneous Finsler space, with a bi-invariant orthogonal decomposition \( \mathfrak{g} = \mathfrak{h} + \mathfrak{m} \) for the compact Lie group \( \mathfrak{g} \).

Let \( \mathfrak{t} \) be a Cartan subalgebra of \( \mathfrak{g} \) such that \( \mathfrak{t} \cap \mathfrak{h} \) is a Cartan subalgebra of \( \mathfrak{h} \). For simplicity, we will just call this \( \mathfrak{t} \) a fundamental Cartan subalgebra. Taking any subalgebra \( \mathfrak{t}' \) of \( \mathfrak{t} \cap \mathfrak{h} \), denote this \( \mathfrak{t} \) a fundamental Cartan subalgebra. Taking any subalgebra \( \mathfrak{t}' \) of \( \mathfrak{t} \cap \mathfrak{h} \), denote this \( \mathfrak{t} \) a fundamental Cartan subalgebra. Taking any subalgebra \( \mathfrak{t}' \) of \( \mathfrak{t} \cap \mathfrak{h} \), denote this \( \mathfrak{t} \) a fundamental Cartan subalgebra. Taking any subalgebra \( \mathfrak{t}' \) of \( \mathfrak{t} \cap \mathfrak{h} \), denote this \( \mathfrak{t} \) a fundamental Cartan subalgebra. Taking any subalgebra \( \mathfrak{t}' \) of \( \mathfrak{t} \cap \mathfrak{h} \), denote this \( \mathfrak{t} \) a fundamental Cartan subalgebra. Taking any subalgebra \( \mathfrak{t}' \) of \( \mathfrak{t} \cap \mathfrak{h} \), denote this \( \mathfrak{t} \) a fundamental Cartan subalgebra. Taking any subalgebra \( \mathfrak{t}' \) of \( \mathfrak{t} \cap \mathfrak{h} \), denote this \( \mathfrak{t} \) a fundamental Cartan subalgebra. Thus the connected component of \( \mathfrak{g}^\perp \) in \( \mathfrak{g} \) is the centralizer \( C_{\mathfrak{g}}(\mathfrak{t}') \) of \( \mathfrak{t}' \) as \( \mathfrak{g}' \) and \( H' = \mathfrak{g}' \cap H \).

Then \((G'/H', F|_{G'/H'})\) is a homogeneous submanifold of \((G/H, F)\).

Then we have the following useful lemma.

**Lemma 3.2** Keep all above notations. Then \((G'/G' \cap H, F|_{G'/H'})\) is totally geodesic in \((G/H, F)\). In particular, when \(G/H\) admits positively curved homogeneous Finsler metrics and \(\dim G'/H' > 1\), then \(G'/H'\) also admits positively curved homogeneous Finsler metrics.

**Proof.** Here we present two proves. The first proof uses Corollary II.5.7 of [5], i.e. the set of all common fixed points of \( T' \) is a disconnected union of finite orbit of \( N_G(T') = \{ g \in G|g^{-1}T'g = T' \} \). Thus the connected component of \( N_G(T') \cdot o \) containing \( o = eH \), which coincides with \( G'/H' \), is a totally geodesic submanifold of \((G/H, F)\). So \((G'/H', F|_{G'/H'})\) is positively curved if its dimension is bigger than 1 and \((G/H, F)\) is positively curved.

The second proof is by direct calculation, using the geodesic spray formula for homogeneous Finsler spaces [12].

The Lie algebra \( \mathfrak{g}' \) of \( G' \) is the centralizer \( C_{\mathfrak{g}}(\mathfrak{t}') \) of \( \mathfrak{t}' \) in \( \mathfrak{g} \). Because \( \mathfrak{t}' \subset \mathfrak{h} \), we also have the decomposition \( \mathfrak{g}' = (\mathfrak{g}' \cap \mathfrak{h}) + (\mathfrak{g}' \cap \mathfrak{m}) \), in which \( \mathfrak{g}' \cap \mathfrak{h} = C_{\mathfrak{h}}(\mathfrak{t}') \) is the Lie algebra of \( H' \). For \( \mathfrak{g}'^\perp = [\mathfrak{t}', \mathfrak{g}] \) with respect to the bi-invariant metric, we also have

\[
[t', \mathfrak{g}] = [t', \mathfrak{h}] + [t', \mathfrak{m}] = (\mathfrak{g}' \cap \mathfrak{h}) + (\mathfrak{g}' \cap \mathfrak{m}).
\]

Let \( v_1, \ldots, v_m, v_{m+1}, \ldots, v_{m+n} \) be an orthogonal basis of \( \mathfrak{m} \) with respect to the chosen bi-invariant inner product, such that the first \( m \) basis vectors belong to \( \mathfrak{g}' \cap \mathfrak{m} \). For each \( v_i \in \mathfrak{m} \subset \mathfrak{g} \), we denote the Killing vector field it defines on \( M = G/H \) as \( X_i \), i.e. \( X_i \) is induced by the right invariant vector field of \( G \) which takes the value \( v_i \in \mathfrak{g} = T_eG \) at
e. Then the restrictions of $X_i$, $1 \leq i \leq m$ are Killing vector fields of $(M', F|_{M'})$. Around $o$, we have linear coordinates $y = y^i X_i$ for $y \in TM$ and corresponding local tangent vector fields $\partial_{y^i}$ on $T(G/H)\backslash 0$.

We will to look at the geodesic spray $G(o,y)$ of $(M,F)$ at $o$ when $y$ is spanned by $v_1, \ldots, v_m$.

In [20], we have proven,

$$G(o,y) = y^i \dot{X}_i + g^{il} c_{ijl} g_{khly} y^j \partial_{y^l},$$

(3.10)
in which $\dot{X}_i$ is the tangent vector field $X_i$ naturally defined on $T(TM\backslash \{0\})$, and the coefficients $c_{ij}$ are defined by $[v_i, v_j]_m = c_{ij} v_k$. Because $[c_o(t'), [t', g]] \subset [t', g]$, we have $[g' \cap m, [t', g] \cap m] \subset [t', g] \cap m$, i.e. $c_{ij}^{k} = 0$, when $i \leq m$, $j > m$ and $k \leq m$. On the other hand, because $F$ is Ad($H$)-invariant, by [9],

$$\langle [h,v], w \rangle_F + \langle v, [h,w] \rangle_F = -2 C_u([h,y], v, w), \quad \forall h \in h, v \in g' \cap m, w \in m.$$

Especially for $h \in t'$, we have $[h,v] = [h,y] = 0$ and then

$$\langle g' \cap m, [t', m] \rangle_u^F = \langle g' \cap m, [t', g] \cap m \rangle_u^F = 0,$$

i.e. $g'' = 0$ when $y^k = 0$ for $k > m$, and $1 \leq i \leq m < j \leq m + n$. So when $y^k = 0$ for $k > m$, the only nonzero terms in the right side of (3.10) are $y^i X_i$ with $i \leq m$, and $g^{il} c_{ijl} g_{khly} y^j \partial_{y^l}$ with $i,j,k, h, l \leq m$, which gives the geodesic spray of $(G'/H', F|_{G'/H'})$ at $(o,y)$. Thus we have proven our assertion.

By the homogeneity, changing $o$ to other $x = g \cdot o$ with $g \in G'$ will only result the change by an Ad($g$)-action, i.e. the above argument is still valid, which proves $(G'/H', F|_{G'/H'})$ is totally geodesic in $(G/H, F)$. □

From the first proof, we see the lemma is still valid with $T'$ changed to other subgroups in $H$, but it may not be convenient to calculate $G'$. Up to equivalence $G'$ and $H'$ contains the common factor $T'$ which can be cancelled.

One of the most immediate application of Lemma 3.2 is to assume $t' = t \cap h$, $F|_{G'/H'}$ induces a left invariant Finsler metric $F''$ on the compact Lie group $G''$ with $\text{Lie}(G'') = c_o(t \cap h) \cap m$. When dim $G'' > 1$, $F'$ is positively curved. Thus by Theorem 5.1 of [10], $G'' = U(1)$, SU(2) or SO(3), i.e. we have the following rank equality which is a special case of Theorem 5.2 in [23].

**Corollary 3.3** Let $(G/H, F)$ be an odd dimensional positively curved homogeneous Finsler space with compact $\mathfrak{g} = \text{Lie}(G)$, then $\text{rk} \mathfrak{g} = \text{rk} h + 1$.

### 3.3 Notation for the Lie algebras and root systems

Assume $(G/H, F)$ is an odd dimensional positively curved homogeneous Finsler space with a bi-invariant orthogonal decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ for the compact Lie algebra $\mathfrak{g} = \text{Lie}(G)$. The orthogonal projections to the $\mathfrak{h}$-factor and $\mathfrak{m}$-factor are denoted as $\text{pr}_h$ and $\text{pr}_m$ respectively.

We can find a fundamental Cartan subalgebra $\mathfrak{t}$ (i.e. $\mathfrak{t} \cap \mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{h}$). Our later discussions about root systems, root planes, etc, are with respect to $\mathfrak{t}$.
when they are for $g$, or with respect to $t \cap h$ when they are for $h$. It is not hard to see $t$ is a splitting Cartan subalgebra, i.e.

$$t = (t \cap h) + (t \cap m).$$

By Corollary 3.3, $\dim(t \cap m) = 1$.

The maximal torus of $G$ (and $H$ resp.) corresponding to $t$ (and $t \cap h$ resp.) will be denoted as $T$ (and $T_H$ respectively). We have the following decomposition of $g$ with respect to $\text{Ad}(T)$-actions,

$$g = t + \sum_{\alpha \in \Delta_g} g_{\pm \alpha},$$

(3.11)
in which $\Delta_g \subset t$ is the root system of $g$, and each $g_{\pm \alpha}$ (which is the same when $\alpha$ is changed to $-\alpha$) is an two dimensional irreducible representation of $\text{Ad}(T)$-actions, called a root plane. Here by the chosen bi-invariant inner product, the roots are regarded as vectors in $t$ rather than vectors in $t^*$. For the compact Lie algebra $h = \text{Lie}(H)$, we have a similar decomposition with respect to $\text{Ad}(T_H)$-actions. The root planes of $h$ are denoted as $h_{\pm \alpha'}$ in which $\alpha' \in t \cap h$ is a root of $h$ in the root system $\Delta_h \subset t \cap h$.

There is another decomposition of $g$ with respect to $\text{Ad}(T_H)$-actions, i.e.

$$g = \sum_{\alpha' \in t \cap h} \hat{g}_{\pm \alpha'},$$

(3.12)
in which

$$\hat{g}_{\pm \alpha'} = \sum_{pr_h(\alpha) = \alpha'} g_{\pm \alpha}, \quad \forall \alpha' \neq 0,$$

and $\hat{g}_0 = t + g_{\pm \alpha}$ when there is a root $\alpha$ of $g$ contained in $t \cap m$, or $\hat{g}_0 = t$ otherwise. This $\text{Ad}(T_H)$-invariant decomposition is compatible with the bi-invariant orthogonal decomposition in the sense that

$$\hat{g}_{\pm \alpha'} = (\hat{g}_{\pm \alpha'} \cap h) + (\hat{g}_{\pm \alpha'} \cap m).$$

To be more precise, we have the following easy lemma, which will be repeatedly used without mentioning it.

**Lemma 3.4** Let $\alpha'$ be any vector of $t \cap h$, then we have the following:

1. If $\alpha' \in \Delta_h$, we have $\hat{g}_{\pm \alpha'} = (\hat{g}_{\pm \alpha'} \cap h) + (\hat{g}_{\pm \alpha'} \cap m)$, in which $\hat{g}_{\pm \alpha'} \cap h = g_{\pm \alpha'}$;

2. If $\alpha' \notin \Delta_h$, we have $g_{\pm \alpha'} \subset m$. In particular, $\hat{g}_0 \subset m$, and $g_{\pm \alpha} \subset m$, if $pr_h \alpha \notin \Delta_h$.

For the bracket relation between root planes, we have the following well known formula,

$$[g_{\pm \alpha}, g_{\pm \beta}] \subset g_{\pm (\alpha + \beta)} + g_{\pm (\alpha - \beta)},$$

(3.13)

where $g_{\pm \alpha}$ and $g_{\pm \beta}$ are different root planes, i.e. $\alpha \neq \pm \beta$, and each term of the right side can be 0 when the corresponding vector is not a root of $g$. It can be included in the following lemma.
Lemma 3.5 Keep all above notations.

(1) For any root $\alpha$ of $\mathfrak{g}$, $[\mathfrak{g}_{\pm \alpha}, \mathfrak{g}_{\pm \alpha}] = \mathbb{R}\alpha$.

(2) Let $\alpha$ and $\beta$ be two linearly independent roots of $\mathfrak{g}$. If $\alpha \pm \beta$ are not roots of $\mathfrak{g}$, then $[\mathfrak{g}_{\pm \alpha}, \mathfrak{g}_{\pm \beta}] = 0$; if one of $\alpha \pm \beta$ is a root, and the other is not, then $[\mathfrak{g}_{\pm \alpha}, \mathfrak{g}_{\pm \beta}] = \mathfrak{g}_{\pm (\alpha \pm \beta)}$, the nonzero one; If both $\alpha \pm \beta$ are roots of $\mathfrak{g}$, then $[\mathfrak{g}_{\pm \alpha}, \mathfrak{g}_{\pm \beta}]$ is a cone in $\mathfrak{g}_{\pm (\alpha + \beta)} + \mathfrak{g}_{\pm (\alpha - \beta)}$.

(3) In (2), if $[\mathfrak{g}_{\pm \alpha}, \mathfrak{g}_{\pm \beta}]$ is the 2-dimensional root plane for $\alpha \pm \beta$, then for any nonzero vector $v \in \mathfrak{g}_{\pm \alpha}$, the linear map $\text{ad}(v)$ is an isomorphism from $\mathfrak{g}_{\pm \beta}$ onto $[\mathfrak{g}_{\pm \alpha}, \mathfrak{g}_{\pm \beta}] = \mathfrak{g}_{\pm (\alpha \pm \beta)}$.

The root systems of compact simple Lie algebras $\mathfrak{a}_n$, $\mathfrak{g}_2$ and the presentation of root planes for the classical cases $\mathfrak{a}_n$-$\mathfrak{o}_n$ are listed in the appendix (Section 7).

3.4 The three Occasions and the reversibility assumption

Keep all above assumptions and notations. Suggested by the general theme for our classification of positively curved normal homogeneous Finsler spaces, we only need to discuss separately for the following three occasions:

**Occasion I.** Each root plane of $\mathfrak{h}$ is also a root plane of $\mathfrak{g}$.

**Occasion II.** There are roots $\alpha$ and $\beta$ of $\mathfrak{g}$ from different simple factors, such that $\text{pr}_\mathfrak{h}(\alpha) = \text{pr}_\mathfrak{h}(\beta) = \alpha'$ is a root of $\mathfrak{h}$.

**Occasion III.** There are a linearly independent pair of roots $\alpha$ and $\beta$ of $\mathfrak{g}$ from the same simple factor, such that $\text{pr}_\mathfrak{h}(\alpha) = \text{pr}_\mathfrak{h}(\beta) = \alpha'$ is a root of $\mathfrak{h}$.

In later sections, we will restrict our discussion to reversible Finsler metrics (i.e. $F(x, y) = F(x, -y)$ for any $y \in T_x(G/H)$). The reason for adding this condition for $F$ will be seen in the next subsection.

It turns out that with the reversibility assumption for $F$, Occasion II is the easiest one to study. Occasion III contains a lot of case-by-case discussions. But we are lucky in this occasion because the root $\alpha'$ of $\mathfrak{h}$ can still provide a big help. Occasion I turns out to be the most difficult one, for which we can only get some partial result.

Adding the reversibility assumption will not ruin too much the generality of this classification work. And it provides another way to see the classification result in [2] is correct.

3.5 The key lemmas with the reversibility assumption for the metric

Since this subsection, we will assume $(G/H, F)$ is an odd dimensional positively curved reversible homogeneous Finsler space, with a bi-invariant orthogonal decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ for the compact Lie algebra $\mathfrak{g} = \text{Lie}(G)$, and a fundamental Cartan subalgebra $\mathfrak{t}$. Keep all relevant notations as before.

The following three lemmas are about the orthogonal decomposition of $\mathfrak{m}$ with respect to the inner product $\langle \cdot, \cdot \rangle^F_u$. It is one of the most crucial issue in later discussions.
Lemma 3.6 Keep above assumptions and notations.

(1) Let \( u \) be a nonzero vector in \( \hat{g}_0 \subset m \). Then with respect to \( \langle \cdot, \cdot \rangle_u^F \), \( m \) can be orthogonal decomposed as the sum of \( \hat{m}_{\pm \alpha'} = \hat{g}_{\pm \alpha'} \cap m \) for all \( \alpha' \in t \cap h \). In particular, \( \hat{m}_0 = \hat{g}_0 \).

(2) If \( \dim \hat{g}_0 = 3 \), we can choose the suitable fundamental Cartan subalgebra \( t \), such that for any nonzero vector \( u \in t \cap m \), \( \langle t \cap m, g_{\pm \alpha} \rangle_u^F = 0 \), in which \( \alpha \) is the root in \( t \cap m \).

Proof. (1) Let \( T_H \) be the torus in \( H \) with \( \text{Lie}(T_H) = t \cap h \). Because both \( F \) and \( u \in \hat{g}_0 \) is \( \text{Ad}(T_H) \)-invariant, the inner product \( \langle \cdot, \cdot \rangle_u^F \) is \( \text{Ad}(T_H) \)-invariant. The decomposition given in the lemma corresponds to different irreducible representations of \( T_H \), thus it is an orthogonal decomposition with respect to \( \langle \cdot, \cdot \rangle_u^F \).

(2) Let \( u \) be a \( F \)-unit vector in \( \hat{g}_0 \) such that \( ||u||_{bi} \) reaches the maximum among all \( F \)-unit vectors. Then \( t_0 = t \cap h + \mathbb{R} u \) is also a fundamental Cartan subalgebra of \( g \). Notice the subspace \( \hat{g}_{\pm \alpha'} \) for each \( \alpha' \in t \cap h \) is not changed when \( t \) is changed with \( t_0 \). The bi-invariant orthogonal complement \( u^\perp \cap \hat{g}_0 \) of \( u \) in \( \hat{g}_0 \) is a root plane \( g_{\pm \alpha} \) for \( t_0 \). Our assumption on \( u \) implies

\[
(\langle t_0 \cap m, g_{\pm \alpha} \rangle_u^F = \langle \mathbb{R} u, u^\perp \cap g_0 \rangle_u^F = 0,
\]

which ends the proof of the lemma.

Lemma 3.7 Keep above assumptions and notations. For any nonzero \( u \in m \) in a root plane \( g_{\pm \alpha} \) with \( \alpha' = \text{pr}_h(\alpha) \neq 0 \). Denote the bi-invariant orthogonal complement of \( \alpha \) in \( t \cap h \) as \( t' \), and the bi-invariant orthogonal projection to \( t' \) as \( \text{pr}_{t'} \). Then with respect to \( \langle \cdot, \cdot \rangle_u^F \), \( m \) can be orthogonally decomposed as the sum of

\[
\hat{m}_{\pm \gamma} = \bigoplus_{\text{pr}_{t'}(\gamma) = \gamma''} g_{\pm \gamma} \cap m = \bigoplus_{\text{pr}_{t'}(\gamma) = \gamma''} (\hat{g}_{\pm \gamma'} \cap m)
\]

in which \( \tau \) is a root of \( g \) with \( \text{pr}_{t'}(\tau) = \gamma'' \). In particular, \( \hat{m}_0 = (\sum_{\gamma \in \mathbb{R} + t \cap m} g_{\pm \gamma}) \cap m \).

Proof. Let \( T' \) be the torus in \( H \) with \( \text{Lie}(T') = t' \). Because both \( F \) and \( u \) are \( \text{Ad}(T') \)-invariant, so the inner product \( \langle \cdot, \cdot \rangle_u^F \) on \( m \) is \( \text{Ad}(T') \)-invariant. The decomposition given in the lemma corresponds to different irreducible representations of \( T' \), thus it is an orthogonal decomposition with respect to \( \langle \cdot, \cdot \rangle_u^F \).

The reversibility assumption is used in the following lemma.

Lemma 3.8 Keep above assumptions and notations. Then for any nonzero vector \( u \in \hat{m}_{\pm \alpha'} = \hat{g}_{\pm \alpha'} \cap m \) with \( \alpha' \neq 0 \), and any \( \beta' \in t \cap h \) such that \( \beta' \) is not an even multiple of \( \alpha' \), then we have

\[
\langle \hat{m}_{\pm \beta'}, \hat{g}_0 \rangle_u^F = 0.
\]

In particular, we have

\[
\langle \hat{m}_{\pm \alpha'}, \hat{g}_0 \rangle_u^F = 0.
\]
Proof. If $\hat{m}_{\pm, \beta} = 0$, there is nothing to prove. Otherwise $\dim \hat{m}_{\pm, \beta} = 2k > 0$ is even. We can find an element $g$ in the maximal torus $T_H$ of $H$, and suitable bi-invariant orthonormal basis $\{u_1, v_1, u_2, v_2, \ldots, u_k, v_k\}$ for $\hat{g}_{\pm, \beta} \cap m$ such that $\text{Ad}(g)|\hat{g}_{\pm, \alpha'} = -\text{Id}$, $\text{Ad}(g)|\hat{g}_{\pm, m} = \text{Id}$, and for each $i$, $\text{Ad}(g)|\hat{m}_{u_i, v_i}$ is the anticlockwise rotation $R(\theta)$ with the angle $\theta \in (0, 2\pi)$.

Because $F$ is $\text{Ad}(g)$-invariant, for any $w_1 \in \mathbb{R}t_i + \mathbb{R}v_i$ and $w_2 \in \hat{g}_0 \cap m$ we have

$$\langle w_1, w_2 \rangle_F^\alpha = \langle \text{Ad}(g)w_1, \text{Ad}(g)w_2 \rangle_{\text{Ad}(g)\alpha} = \langle R(\theta)w_1, w_2 \rangle_{\alpha} = \langle R(\theta)w_1, w_2 \rangle_F^\alpha.$$

Repeating this procedure, we get

$$\langle w_1, w_2 \rangle_F^\alpha = \langle R(n\theta)w_1, w_2 \rangle_{\alpha} = \langle R(n\theta)w_1, w_2 \rangle_F^\alpha$$

for each $n \in \mathbb{N}$. So

$$\langle w_1, w_2 \rangle_F^\alpha = \lim_{n \to \infty} \langle \frac{1}{n}R(\theta)w_1 + \cdots + R(n\theta)w_1, w_2 \rangle_F^\alpha = 0.$$

Let $i$ exhausts all integers from 1 to $k$, then the above argument proves the lemma. ■

We will also repeatedly use the following two key lemmas in later discussion.

Lemma 3.9 Let $F$ be a positively curved homogeneous Finsler metric on the odd dimensional coset space $G/H$. Keep all relevant notations as before. If $\alpha$ is root of $\hat{g}$ contained by $t \cap \mathfrak{h}$, and it is the only root of $\hat{g}$ contained in $\alpha + (t \cap m)$, then it is a root of $\mathfrak{h}$ and $\mathfrak{h}_{\pm, \alpha} = \hat{g}_{\pm, \alpha} = \mathfrak{g}_{\pm, \alpha}$.

Proof. We only need to prove $\alpha$ is a root of $\mathfrak{h}$. The other statement follows easily.

Assume conversely $\alpha$ is not a root of $\mathfrak{h}$, then $\mathfrak{g}_{\pm, \alpha} = \hat{g}_{\pm, \alpha}$ is contained by $\mathfrak{m}$. Using (2) of Lemma 3.6 if $\dim \hat{g}_0 = 3$, we can find a suitable fundamental Cartan subalgebra $t$ and a nonzero $u$ in $t \cap m$ such that

$$\langle u^\perp \cap \hat{g}_0, u \rangle_F^\alpha = 0,$$

in which $u^\perp \cap \hat{g}_0$ is the bi-invariant orthogonal complement of $u$ in $\hat{g}_0$. Let $v$ be any nonzero vector in $\mathfrak{g}_{\pm, \alpha}$. Because $\alpha \in t \cap \mathfrak{h}$, it is obvious to see $u$ and $v$ are a linearly independent commuting pair in $\mathfrak{m}$.

Let $\alpha' = \text{pr}_\mathfrak{h}(\alpha)$. Direct calculation shows

$$[u, m]_m \subset u^\perp \cap \hat{g}_0 + \sum_{\gamma' \neq \alpha'} \hat{g}_{\pm, \gamma'},$$

so by (3.14) and (1) of Lemma 3.6 we have

$$\langle [u, m]_m, u \rangle_F^\alpha = \langle [u, m]_m, v \rangle_F^\alpha = 0.$$

Direct calculation also shows

$$[v, m]_m \subset \sum_{\gamma' \neq 0} \hat{g}_{\pm, \gamma'},$$

so by (1) of Lemma 3.6 we have

$$\langle [v, m]_m, u \rangle_F^\alpha = 0.$$

Summarizing (3.15) and (3.16), we get $U(u, v) = 0$ and $K^F(\alpha, u, u \wedge v) = 0$ by Theorem 3.1. This is a contradiction. ■

14
Lemma 3.10 Let $F$ be a reversible positively curved homogeneous Finsler metric on the odd dimensional coset space $G/H$. Keep all relevant notations as before. Then there does not exist two linearly independent roots $\alpha$ and $\beta$ of $g$ such that

(1) $\alpha$ and $\beta$ are not roots of $h$;
(2) $\alpha \pm \beta$ are not roots of $g$;
(3) $\pm \alpha$ are the only roots of $g$ in $\mathbb{R}\alpha + t \cap m$;
(4) $\pm \beta$ are the only roots of $g$ in $\mathbb{R}\alpha \pm \beta + t \cap m$.

Proof. Assume conversely that there are roots $\alpha$ and $\beta$ of $g$ satisfying (1)-(4) of the lemma. Denote $\alpha' = \text{pr}_h(\alpha)$ and $\beta' = \text{pr}_h(\beta)$. Then $g_{\pm \alpha}$ must be contained in $m$, otherwise by (3) of the lemma, $g_\alpha = \hat{g}_\alpha$ is a root plane in $h$, and then $\mathbb{R}\alpha = [g_{\pm \alpha}, g_\alpha] \subset h$, i.e. $\alpha$ is a root of $h$, which is a contradiction to (1). Similarly, by (4) of the lemma, $g_{\pm \beta} = \hat{g}_{\pm \beta}$ is also contained in $m$.

First we consider the situation that $\alpha' \neq 0$, i.e. $\alpha$ is not contained by $t \cap m$. Let $u$ and $v$ be any nonzero vectors in $g_{\pm \alpha}$ and $g_{\pm \beta}$ respectively. By (2) of the lemma and the previous argument, they are a linearly independent commuting pair in $m$.

Let $u'$ be another nonzero vector in $g_{\pm \alpha}$ such that $\langle u, u' \rangle_{bi} = 0$. By the Ad$(T_H)$-invariance of $F|_{g_{\pm \alpha}}$, we see it coincides with the restriction of the bi-invariant inner product up to scalar changes, so we have

$$\langle u^\perp \cap g_{\pm \alpha}, u \rangle_u^F = \langle \mathbb{R}u', u \rangle_u^F = 0,$$

in which $u^\perp \cap g_{\pm \alpha} = \mathbb{R}u'$ is the bi-invariant orthogonal complement of $u$ in $g_{\pm \alpha}$.

Let $t'$ be the bi-invariant orthogonal complement of $\alpha$ in $h$, and $\text{pr}_{t'}$ be the orthogonal projection to $t'$ with respect to the bi-invariant inner product. By Lemma 3.7 with respect to $\langle \cdot, \cdot \rangle_u^F$, $m$ can be orthogonally decomposed as the sum of

$$\hat{m}_{\pm \gamma'} = \bigoplus_{\text{pr}_{t'}(\gamma') = \gamma'} g_{\gamma'} \cap m$$

for all different $\{ \pm \gamma' \} \subset t'$. In particular, (3) and (4) of the lemma indicates

$$\hat{g}_0 = t \cap m, \hat{m}_0 = t \cap m + g_{\pm \alpha}, \text{ and } \hat{m}_{\pm \beta'} = g_{\pm \beta}.$$

Using (2) of the lemma, direct calculation shows

$$[u, m] \subset t \cap m + u^\perp \cap g_{\pm \alpha} + \bigoplus_{\gamma'' \neq 0, \gamma' \neq \pm \beta'} \hat{m}_{\pm \gamma''},$$

so by Lemma 3.8, Lemma 3.7 and (3.17), we have

$$\langle [u, m]_m, u \rangle_u^F = \langle [u, m]_m, v \rangle_u^F = 0.$$

Direct calculation also shows

$$[v, m]_m \subset \hat{g}_0 + \bigoplus_{\gamma'' \neq 0} \hat{m}_{\pm \gamma''},$$

15
so by Lemma 3.8 and Lemma 3.7, we have

\[ \langle [v, m], w \rangle^F = 0 \]  \hspace{1cm} (3.20)

Summarizing (3.19) and (3.20), we get \( U(u, v) = 0 \) and \( K^F(o, u, u \wedge v) = 0 \) by Theorem 3.1. This is a contradiction.  

Notice Lemma 3.6, Lemma 3.7 and Lemma 3.9 does not require \( F \) to be reversible. For most cases in later discussions, the key lemmas are enough for us to solve the problem. But there are a few situation (Subsection 5.5 for example), that we need to go back to Theorem 3.1 for the conclusion.

4 Occasion III: the general reduction and the discussion for classical cases

4.1 General reductions for the discussion of Occasion III

Assume \((G/H, F)\) is an odd dimensional positively curved reversible homogeneous Finsler space in Occasion III, i.e. with respect to the bi-invariant decomposition \( g = h + m \) for the compact Lie algebra \( g = \text{Lie}(G) \), and a fundamental Cartan subalgebra \( t \), there are roots \( \alpha \) and \( \beta \) of \( g \) from the same simple factor, \( \alpha \neq \pm \beta \), such that \( \text{pr}_h(\alpha) = \text{pr}_h(\beta) = \alpha' \) is a root of \( h \). Then obviously \( t \cap m \) is spanned by \( \alpha - \beta \).

First we will prove the following lemma.

**Lemma 4.1** Let \((G/H, F)\) be an odd dimensional positively curved reversible homogeneous Finsler space in Occasion III. Keep all relevant notations. Then \((G/H, F)\) is equivalent to a positively curved reversible homogeneous Finsler space \((G'/H', F')\) in which \( G' \) is a compact simple Lie group.

**Proof.** Denote the direct sum decomposition of \( g \) as

\[ g = g_0 \oplus g_1 \oplus \cdots \oplus g_n, \]

in which \( g_0 \) is Abelian, and for each \( i > 0 \), \( g_i \) is a simple ideal of \( g \). Assume \( \alpha \) and \( \beta \) are roots of \( g_1 \). Then obviously the Abelian factor \( g_0 \) is contained in \( h \).

Consider any root \( \gamma \) of \( g_i \) for \( i > 1 \). Obviously \( \gamma \) is the only root contained in \( \gamma + t \cap m \), so by Lemma 3.9 \( \gamma \) is a root of \( h \) and \( g_{\pm \gamma} = h_{\pm \gamma} \) is contained in \( h \). Because each simple factor \( g_i \) for \( i > 1 \) is algebraically generated by its root planes, we see \( g_i \subset h \) for \( i > 1 \). Let \( G'/H' \) be the homogeneous space corresponding to the pair \((g_1, h_1)\), then it admits a homogeneous Finsler metric \( F' \) naturally induced by \( F \), such that \((G/H, F)\) is equivalent to \((G'/H', F')\), which ends the proof of the lemma.  

Notice because Lemma 3.9 does not require the reversibility of the metric, Lemma 4.1 is also valid for non-reversible \( F \).

Before the case by case discussion with respect to the compact simple Lie algebra \( g \), there are some common situations which can be uniformly crossed out. We summarize them as the following lemma.
Lemma 4.2 Let \( (G/H,F) \) be an odd dimensional positively curved reversible homogeneous Finsler space in Occasion III, with a compact simple \( g = \text{Lie}(G) \). Then for any two different roots \( \alpha \) and \( \beta \) such that \( \text{pr}_t(\alpha) = \text{pr}_t(\beta) = \alpha' \) is a root of \( h \), the angle between \( \alpha \) and \( \beta \) can not be \( \pi/3 \) or \( 2\pi/3 \).

Proof. Firstly, we assume \( g \neq g_2 \) and prove the angle between \( \alpha \) and \( \beta \) can not be \( \pi/3 \).

Assume conversely the angle between \( \alpha \) and \( \beta \) is \( \pi/3 \). Let \( t' = \alpha'^2 \cap t \cap h = \langle \mathbb{R}\alpha + \mathbb{R}\beta \rangle^1 \cap t \) be the bi-invariant orthogonal complement of \( \alpha' \) in \( t \cap h \), and \( T' \) be the corresponding torus in \( H \). Notice \( \text{Lie}(C_G(T')) = t' + a_2 \), such that \( \alpha \) and \( \beta \) are roots of the \( a_2 \)-factor. By Lemma 3.9, there is a positively curved homogeneous Finsler space \( (G''/H'',F'') \), in which \( g'' = \text{Lie}(G'') = \mathfrak{su}(3) \), and \( h'' = \text{Lie}(H'') = a_1 \) is linearly spanned by

\[
\begin{align*}
 w_1 &= \sqrt{-1} \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\
 w_2 &= \sqrt{-1} \begin{pmatrix} 0 & \bar{a} & \bar{b} \\ a & 0 & 0 \\ b & 0 & 0 \end{pmatrix}, \\
 w_3 &= \frac{1}{3} [w_1, w_2] = \begin{pmatrix} 0 & \bar{a} & \bar{b} \\ -a & 0 & 0 \\ -b & 0 & 0 \end{pmatrix},
\end{align*}
\]

in which \( a, b \in \mathbb{C} \) with \( (a, b) \neq (0, 0) \). But then \([w_3, w_1]\) is not contained by \( h'' \). This is a contradiction.

Secondly, we prove the angle between \( \alpha \) and \( \beta \) can not be \( 2\pi/3 \). Assume conversely it is, which implies \( \alpha' = \frac{1}{3}(\alpha + \beta) \) is a root of \( h \). But \( \gamma = 2\alpha' = \alpha + \beta \) is a root of \( g \) contained in \( t \cap h \), and it is the only root contained in \( \gamma + (t \cap m) \). So by Lemma 3.9, \( \gamma = 2\alpha' \) is also a root of \( h \). This is a contradiction.

Lastly, we assume \( g = g_2 \) and prove the angle between \( \alpha \) and \( \beta \) can not be \( \pi/3 \). If \( \alpha \) and \( \beta \) are short roots, then they can be changed to long roots with an angle \( 2\pi/3 \), which has already been proven to be impossible. If \( \alpha \) and \( \beta \) are long roots, then \( \alpha' = \frac{1}{3}(\alpha + \beta) \) is a root of \( h \). By Lemma 3.9 and similar argument as above, the short root \( \gamma = \frac{1}{3}(\alpha + \beta) = \frac{2}{3}\alpha' \) is also a root of \( h \). This is a contradiction.

Now we start the case by case discussion which will last two sections. In each case, we use the standard presentation of the root systems (see the Section 7), and divide the discussion into subcases with respect to the rank of \( G \), the long/short roots choices of \( \alpha \) and \( \beta \) and the angle between \( \alpha \) and \( \beta \). Using Weyl group actions and more outer automorphisms for \( d_n \) and \( e_6 \), the subcases can be reduced to the following.

4.2 The case \( g = a_n \)

We only need to consider the following subcases.

**Subcase 1.** Assume \( n = 3 \), \( \alpha = e_1 - e_4 \) and \( \beta = e_3 - e_2 \).

Then \( t \cap m = \mathbb{R}(e_1 + e_2 - e_3 - e_4) \) and \( \alpha' = \frac{1}{4}(e_1 - e_2 + e_3 - e_4) \) is a root of \( h \). By Lemma 3.9, \( e_1 - e_2 \) and \( e_3 - e_4 \) are roots of \( h \). Notice \( g_{\pm(e_1 - e_2)} = g_{\pm(e_1 - e_2)} \) is a root.
plane of \( h \). Let \( \beta' = \frac{1}{2}(-e_1 + e_2 + e_3 - e_4) \in t \cap h \). Then any non zero \( u \in g_{\pm(e_1-e_2)} \subset h \) provides a linear isomorphism

\[
\text{ad}(u) : \hat{g} \pm \alpha' = g_{\pm(e_1-e_4)} + g_{\pm(e_2-e_3)} \rightarrow \hat{g} \pm \beta' = g_{\pm(e_2-e_4)} + g_{\pm(e_1-e_3)}.
\]  

(4.21)

Because \( u \in h \), thus \( \text{ad}(u) \) preserve the bi-invariant orthogonal decomposition. So \( \beta' = \frac{1}{2}(-e_1 + e_2 + e_3 - e_4) \) is also a root of \( h \). Now we see \( h = b_2 \) and we have got all roots of \( h \). We will see up to conjugation, \( h \) is uniquely determined. By (4.21), it is easy to see \( h \) is uniquely determined by determining \( h_{\pm \alpha'} \). Let \( g' \) be the subalgebra of \( g \) isomorphic to \( a_1 \oplus a_1 \), defined by

\[
g' = \mathbb{R} \alpha + \mathbb{R} \beta + g_{\pm(e_1-e_4)} + g_{\pm(e_2-e_3)},
\]

\( h' \) be the subalgebra of \( g' \) defined by \( h' = \mathbb{R} \alpha' + h_{\pm \alpha'} \), and \( t' = t \cap g' = \mathbb{R} \alpha + \mathbb{R} \beta \) be a fundamental Cartan subalgebra of \( g' \). Then we also have the induced bi-invariant orthogonal decomposition \( g' = h' + m' \), such that \( m' = m \cap g' \) and \( t' \cap m' = t \cap m \). Notice \( h' \) can not have nonzero intersection with the two simple factors of \( g' \), otherwise, we can apply \( \text{Ad}(t \cap h) \)-actions to show \( h_{\pm \alpha'} \) is the root plane \( g_{\pm(e_1-e_4)} \) or \( g_{\pm(e_2-e_3)} \) of \( g \). But \( \alpha' \subset [h_{\pm \alpha'}, h_{\pm \alpha'}] \) is not a root of \( g \). This is a contradiction.

The following lemma will be useful to discussion this situation here and later.

**Lemma 4.3** Let \( g' = g_1 \oplus g_2 = a_1 \oplus a_1 \) be endowed with a bi-invariant product. Assume that \( t' \) is a Cartan subalgebra, and \( h' \) and \( h'' \) are subalgebras of \( g' \) isomorphic to \( a_1 \) satisfying the following conditions:

1. \( h' \cap t' = h'' \cap t' \) is one dimensional.
2. \( h' \cap g_i = h'' \cap g_i = 0, \ i = 1, 2. \)
3. \( h' \cap (h' \cap t')^\perp \subset t'^\perp \), and \( h'' \cap (h'' \cap t')^\perp \subset t'^\perp \), in which the orthogonal complements are taken with respect to the chosen bi-invariant inner product on \( g \).

Then there is an \( \text{Ad}(\exp t') \)-action which maps \( h' \) to \( h'' \).

**Proof.** We first give a definition. For a compact Lie algebra of type \( a_1 \) endowed with a bi-invariant inner product, we call an orthogonal basis \( \{u_1, u_2, u_3\} \) standard, if all basis vectors have the same length, and they satisfy the condition \( [u_i, u_j] = u_k \) for \( (i, j, k) = (1, 2, 3), (2, 3, 1) \) and \( (3, 1, 2) \). The length \( c \) of each \( u_i \) is a constant which only depends on the scale of the bi-invariant inner product. The bracket \( u_3' = [u_1', u_2'] \) of any two orthogonal vectors with length \( c \) is also a vector with a length \( c \), and \( \{u_1', u_2', u_3'\} \) is a standard basis as well.

Now we come back to prove the lemma. Let \( c_1 \) and \( c_2 \) be the length of the standard basis vectors for \( g_1 \) and \( g_2 \) respectively. We can choose standard bases \( \{u_1, u_2, u_3\} \) and \( \{v_1, v_2, v_3\} \) for \( g_1 \) and \( g_2 \) as follows. First, we choose vectors \( u_1 \) and \( v_1 \) from \( t' \cap g_1 \) and \( t' \cap g_2 \) with length \( c_1 \) and \( c_2 \) respectively. Then we freely choose any vectors \( u_2 \) of length \( c_1 \) from \( t'^\perp \cap g_1 \) and set \( u_3 = [u_1, u_2] \). By (2) and (3) in the lemma, we can find a vector of \( h' \) from \( u_2 + g_1 \cap t'^\perp \). Its \( g_2 \)-factor is not 0, which can be positively scaled to the vector \( v_2 \) with the length \( c_2 \). Finally, we set \( v_3 = [v_1, v_2] \).
Now suppose \( \mathfrak{h}' \) is linearly spanned by \( u_1 + av_1, u_2 + bv_2 \) and their bracket
\[
[u_1 + av_1, u_2 + bv_2] = u_3 + abv_3,
\]
where \( a \) is a fixed nonzero constant and \( b > 0 \). As a Lie algebra, it satisfies \([u_2 + bv_2, u_3 + abv_3] = u_1 + av_1\), hence \( b = 1 \).

With \( \mathfrak{h}' \) changed to \( \mathfrak{h}'' \), the same argument above can also provide standard basis \( \{u_1', u_2', u_3'\} \) and \( \{v_1', v_2', v_3'\} \) for \( \mathfrak{g}_1 \) and \( \mathfrak{g}_2 \) respectively, satisfying \( u_i' = u_i \) for each \( i \), and \( v_i' = v_i \). Then it is easy to see that there exists a real number \( t \) such that \( \text{Ad}(\exp(tv_1)) \) maps \( v_2 \) to \( v_2' \), \( v_3 \) to \( v_3' \), keeping \( v_1 \) and all vectors \( u_i \) unchanged. So it map \( \mathfrak{h}' \) isomorphically to \( \mathfrak{h}'' \).

By Lemma 1.1.3, it is easy to see \( \mathfrak{h}_{\pm \alpha'} \) is uniquely determined, up to \( \text{Ad}(\exp(t)) \)-actions which preserve all roots and root planes of \( \mathfrak{g} \). So \( \mathfrak{h} \) is conjugate to the most standard \( \mathfrak{sp}(2) \) in \( \mathfrak{su}(4) \) which makes \( G/H \) a symmetric space. Because \( a_3 = \mathfrak{d}_3 \), \( G/H \) is equivalent to a standard Riemannian sphere \( S^5 = \text{SO}(6)/\text{SO}(5) \) with constant positive curvature.

In this subcase, we can also directly prove \( G/H \) is a symmetric homogeneous space, i.e. \([m,m] \subset \mathfrak{h}\), and apply the classification of symmetric homogeneous spaces. But then the argument can be used for some later situations.

**Subcase 2.** Assume \( n = 4 \), \( \alpha = e_1 - e_4 \) and \( \beta = e_3 - e_2 \).

Then \( \mathfrak{t} \cap \mathfrak{m} = \mathbb{R}(e_1 + e_2 - e_3 - e_4) \) and \( \alpha' = \frac{1}{2}(e_1 - e_2 + e_3 - e_4) \) is a unit root of \( \mathfrak{h} \). Notice \( \text{pr}_\mathfrak{h}(e_i - e_j) \) for \( 1 \leq i \leq 4 \) cannot be a root of \( \mathfrak{h} \) because it is not orthogonal to \( \alpha' \) and its length is \( \sqrt{2} \). So all roots of \( \mathfrak{h} \) must be of the form \( \text{pr}_\mathfrak{h}(e_i - e_j) \) with \( 1 \leq i < j \leq 4 \). Similar argument as for Subcase 1 then shows the root system of \( \mathfrak{h} \) is of type \( \mathfrak{b}_2 = \mathfrak{c}_2 \), i.e. \( \mathfrak{h} = \mathbb{R}(e_1 + e_2 + e_3 + e_4 - 4e_5) \oplus \mathfrak{h}' \), in which \( \mathfrak{h}' \) is the standard \( \mathfrak{sp}(2) \) in \( \mathfrak{su}(4) \) corresponding to \( e_5 \) for \( 1 \leq i \leq 4 \), up to \( \text{Ad}(\mathfrak{su}(4)) \)-actions. So \( G/H \) is equivalent to the Berger’s space \( \text{SU}(5)/\text{Sp}(2)U(1) \), which admits positive curved normal homogeneous metrics.

**Subcase 3.** Assume \( n > 4 \), \( \alpha = e_1 - e_4 \) and \( \beta = e_3 - e_2 \).

Then \( \mathfrak{t} \cap \mathfrak{m} = \mathbb{R}(e_1 + e_2 - e_3 - e_4) \) and \( \alpha' = \frac{1}{2}(e_1 - e_2 + e_3 - e_4) \) is a unit root of \( \mathfrak{h} \). Take \( \gamma_1 = e_1 - e_5 \) and \( \gamma_2 = e_2 - e_6 \), then they satisfy all conditions (1)-(4) of Lemma 3.10 which is impossible by that lemma.

### 4.3 The case \( \mathfrak{g} = \mathfrak{b}_n \) with \( n > 1 \)

We only need to consider the following subcases.

**Subcase 1.** \( \alpha = e_1 + e_2, \beta = e_2 \).

Then \( \mathfrak{t} \cap \mathfrak{m} = \mathbb{R}e_1 \) and \( \alpha' = e_2 \) is a root of \( \mathfrak{h} \), with
\[
\mathfrak{h}_{\pm e_2} \subset \mathfrak{g}_{\pm e_2} = \mathfrak{g}_{\pm(e_2-e_1)} + \mathfrak{g}_{\pm e_2} + \mathfrak{g}_{\pm(e_2+e_1)}.
\]

Denote \( \mathfrak{g}' = \mathbb{R}e_1 + \mathbb{R}e_2 + \sum_{a,b} \mathfrak{g}_{\pm(ae_1+be_2)} \) and \( \mathfrak{g}'' = \mathbb{R}e_1 + \mathfrak{g}_{\pm e_1} \). They are Lie algebras of types \( \mathfrak{b}_2 = \mathfrak{so}(5) \) and \( \mathfrak{a}_1 \) respectively. The subalgebra \( \mathfrak{h} \cap \mathfrak{g}' \) of type \( \mathfrak{a}_1 \) is linearly
spanned by

\[
u = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \in \mathbb{R}e_2,
\]

\[
v = \begin{pmatrix} 0 & 0 & 0 & -a & -a' \\ 0 & 0 & 0 & -b & -b' \\ 0 & 0 & 0 & -c & -c' \\ a & b & c & 0 & 0 \\ a' & b' & c' & 0 & 0 \end{pmatrix} \in \mathfrak{h}_{\pm e_2},
\]

and

\[
w = [u, v] = \begin{pmatrix} 0 & 0 & 0 & a' & -a \\ 0 & 0 & 0 & b' & -b \\ 0 & 0 & 0 & c' & -c \\ -a' & -b' & -c' & 0 & 0 \\ a & b & c & 0 & 0 \end{pmatrix},
\]

in which \(a, b, c, a', b'\) and \(c'\) \(\in \mathbb{R}^6\) with \(a^2 + b^2 + c^2 + a'^2 + b'^2 + c'^2 > 0\). Because \([v, w] \in \mathfrak{h} \cap \mathfrak{g}'\), direct calculation shows \((a, b, c)\) and \((a', b', c')\) are linearly dependent vectors. Using a suitable isomorphism \(l \in \text{Ad}(\exp \mathfrak{g}')\) of \(\mathfrak{g}\), we can make \(b = c = b' = c' = 0\), i.e. up to equivalence, we can assume \(\mathfrak{h}_{\pm e_2} = \mathfrak{g}_{\pm e_2}\), and thus \(\mathfrak{g}_{\pm(e_2+e_1)} \subset \mathfrak{m}\).

By Lemma 3.9, any root \(\pm e_i \pm e_j\) of \(\mathfrak{g}\) with \(1 < i < j\) are roots of \(\mathfrak{h}\) and \(\mathfrak{g}_{\pm(e_i \pm e_j)} = \mathfrak{g}_{\pm(e_i \pm e_j)}\). By the linear isomorphism \(\text{ad}(w)\) between \(\mathfrak{g}_{\pm e_2}\) and \(\mathfrak{g}_{\pm e_i}\), for any nonzero vector \(w \in \mathfrak{g}_{\pm(e_2-e_1)}\) with \(i > 2\), we can see \(\mathfrak{g}_{\pm e_i} \subset \mathfrak{h}\) and \(\mathfrak{g}_{\pm(e_i \pm e_1)} \subset \mathfrak{m}\) for any \(i \geq 2\). To summarize, we have

\[
m = \mathbb{R}e_1 + \mathfrak{g}_{\pm e_1} + \sum_{i=2}^{n}(\mathfrak{g}_{\pm(e_i \pm e_1)} + \mathfrak{g}_{\pm(e_i \pm e_1)}).
\]

Let \(\{u, u'\}\) be a bi-invariant orthonormal basis of \(\mathfrak{g}_{\pm(e_1+e_2)}\) and choose a nonzero vector \(v\) from \(\mathfrak{g}_{\pm(e_1-e_2)}\) such that \(\langle u', v \rangle^F_u = 0\). Because the Minkowski norm \(F|_{\mathfrak{g}_{\pm(e_1+e_2)}}\) is \(\text{Ad}(\exp(\mathbb{R}e_2))\)-invariant, it coincides with the restriction of the bi-invariant inner product up to scalar changes. So we have

\[
\langle u', u \rangle^F_u = \langle [u, e_1], u \rangle^F_u = \langle [u, e_2], u \rangle^F_u = 0.
\]

Direct calculation shows

\[
[u, [u, m]]_{m} \subset \mathbb{R}[e_1, u] + \mathbb{R}e_1 \subset \mathbb{R}u' + \hat{\mathfrak{g}}_0,
\]

so by Lemma 3.8

\[
\langle v, \hat{\mathfrak{g}}_0 \rangle^F_u = \langle u, \hat{\mathfrak{g}}_0 \rangle^F_u = 0,
\]

then using our assumptions on \(u\) and \(v\), we have

\[
\langle [u, m], u \rangle^F_u = \langle \mathbb{R}u', u \rangle^F_u = 0,
\]

and

\[
\langle [u, m], v \rangle^F_u = \langle \mathbb{R}u', u \rangle^F_u + \langle \hat{\mathfrak{g}}_0, v \rangle^F_u = 0.
\]

Because \(e_2 \in \mathfrak{h}\), by Theorem 1.3 of [10], we have

\[
\langle [e_2, v], u \rangle^F_u = -\langle [e_2, u], v \rangle^F_u - 2C^F_u(u, v, [e_2, u]),
\]

20
in the right side of which, the first term vanishes by (4.23) and the second term vanishes by the basic properties of Cartan tensors. So we have

\[ \langle [e_1, v], u \rangle_u^F = \langle [e_2, v], u \rangle_u^F = 0. \quad (4.27) \]

Direct calculation shows

\[ [v, m]_m \subset \mathbb{R}[e_1, v] + \hat{g}_0, \]

so by Lemma 3.8 and (4.27), we have

\[ \langle [v, m]_m, u \rangle_u^F = \langle \mathbb{R}[e_1, v], u \rangle_u^F + \langle \hat{g}_0, u \rangle_u^F = 0. \quad (4.28) \]

Summarize (4.21), (4.25) and (4.28), we have \( U(u, v) = 0 \) and \( K^F(\alpha, u, u \wedge v) = 0 \) by Theorem 3.1 which is a contradiction.

**Subcase 2.** \( \alpha = e_1 + e_2, \beta = e_2 - e_1. \)

This subcase has been covered by Subcase 1.

**Subcase 3.** \( n = 4, \alpha = e_1 + e_2, \beta = -e_3 - e_4. \)

Then \( t \cap m = \mathbb{R}(e_1 + e_2 + e_3 + e_4) \) and \( \alpha' = \frac{1}{2}(e_1 + e_2 - e_3 - e_4) \) is a root of \( \neu {\mathfrak{h}} \) with \( \mathfrak{h} \) \( \pm \alpha' \subset \mathfrak{g} \) \( \pm \alpha' = \mathfrak{g} \) \( \pm (e_1 + e_2 + e_3 + e_4). \)

The root system of \( \neu {\mathfrak{g}} \) is similar to Subcase 3, or Subcase 1 for \( \alpha_n. \) Obviously \( \mathfrak{h} \) \( \pm \alpha' \) is not a root plane of \( \mathfrak{g}. \) By Lemma 3.9, whenever \( 1 \leq i < j \leq 4, \) the root \( e_i - e_j \) of \( \mathfrak{g} \) is also a root of \( \mathfrak{h} \) with \( \mathfrak{h} \) \( \pm (e_i - e_j) = \mathfrak{g} \) \( \pm (e_i - e_j). \)

By the action \( \text{ad}(u) \) for any non zero vector \( u \in \mathfrak{g} \) \( \pm (e_i - e_j) \subset \mathfrak{h}, \) \( (i, j) = (2, 3) \) or \( (2, 4), \)

\[ \beta' = \frac{1}{2}(e_1 + e_3 - e_2 - e_4) \]

and \( \gamma' = \frac{1}{2}(e_1 + e_4 - e_2 - e_3) \) are also roots of \( \mathfrak{h}. \) We see \( \mathfrak{h} \) is of type \( \mathfrak{b}_3, \) and it can be totally determined by the choice of \( \mathfrak{h} \) \( \pm \alpha'. \)

Using Lemma 4.3 we see, up to \( \text{Ad}(G) \) -actions, \( \mathfrak{h} \) can be chosen to be the most standard one defining homogeneous Finsler sphere \( S^{15} = \text{Spin}(9)/\text{Spin}(7) \) on which we can find positively curved homogeneous Finsler metrics.

**Subcase 4.** \( n > 4, \alpha = e_1 + e_2 \) and \( \beta = -e_3 - e_4. \)

Then \( t \cap m = \mathbb{R}(e_1 + e_2 + e_3 + e_4) \) and \( \alpha' = \frac{1}{2}(e_1 + e_2 - e_3 - e_4) \) is a unit root of \( \mathfrak{h}. \)

Take \( \gamma_1 = e_+ e_5 \) and \( \gamma_2 = e_1 - e_5, \) then they satisfy (1)-(4) of Lemma 3.10 which is impossible by that lemma.

**Subcase 5.** \( n = 3, \alpha = e_1 + e_2 \) and \( \beta = -e_3. \)

Then \( t \cap m = \mathbb{R}(e_1 + e_2 + e_3) \) and \( \alpha' = \frac{1}{2}(e_1 + e_2 - 2e_3) \) is a root of \( \mathfrak{h}. \) The argument is similar to Subcase 3, or Subcase 1 for \( \alpha_n. \) Using Lemma 3.9 and Lemma 3.11 the root system of \( \mathfrak{h} \) contains the following roots,

\[ \pm (e_i - e_j) \text{ for all } 1 \leq i < j \leq 3, \text{ and } \frac{1}{3}(e_1 + e_2 + e_3) - e_i \text{ for all } 1 \leq i \leq 3. \]

The subalgebra \( \mathfrak{h} \) is of type \( \mathfrak{g}_2. \) It can be totally determined by the choice of

\[ \mathfrak{h} \subset \hat{\mathfrak{g}} \subset \mathfrak{g}_2 = \mathfrak{g}_2 = \mathfrak{g}_2(e_1 + e_2) + \mathfrak{g}_2(e_3). \]

Using Lemma 4.3 we see up to \( \text{Ad}(G) \) -actions, there exist only one such \( \mathfrak{h} \) which defines the homogeneous sphere \( S^7 = \text{Spin}(7)/\text{G}_2. \) Notice in this situation the isotropy action is transitive, so any homogeneous Finsler metric on it must be Riemannian.
with positive constant curvature. So in this subcase \((G/H, F)\) is equivalent to the Riemannian homogeneous sphere \(S^7 = \text{Spin}(7)/G_2\) of positive constant curvature.

**Subcase 6.** \(n > 3, \alpha = e_1 + e_2, \) and \(\beta = -e_3.\)

Then \(t \cap m = \mathbb{R}(e_1 + e_2 + e_3)\) and \(\alpha' = \frac{1}{2}(e_1 + e_2 - 2e_3)\) is a root of \(\mathfrak{h}.\) Take \(\gamma_1 = e_1 + e_4\) and \(\gamma_2 = e_1 - e_4,\) then they satisfies (1)-(4) of Lemma 3.10 which is impossible by that lemma.

**Subcase 7.** \(\alpha = e_1, \beta = e_2.\)

Then \(t \cap m = \mathbb{R}(e_1 - e_2)\) and \(\alpha' = \frac{1}{2}(e_1 + e_2)\) is a root of \(\mathfrak{h}.\) By Lemma 3.9
\[2\alpha' = e_1 + e_2\] is also a root of \(\mathfrak{h}.\) This is a contradiction.

**Subcase 8.** \(n = 2, \alpha = e_1 + e_2, \) and \(\beta = -e_1.\)

Then \(t \cap m = \mathbb{R}(2e_1 + e_2)\) and \(\alpha' = \frac{1}{2}e_1 + \frac{2}{3}e_2\) is a root of \(\mathfrak{h}.\) The subalgebra \(\mathfrak{h}\) is of type \(\mathfrak{a}_1,\) which is totally determined by the choice of \(\mathfrak{h}_{\pm\alpha'}\) in \(\mathfrak{h}_{\pm\alpha'} = \mathfrak{g}_{\pm(e_1 + e_2)} + \mathfrak{g}_{\pm e_1}.\) Using Lemma 4.3 we see \(G/H\) is uniquely determined up to equivalence, i.e. the Berger’s space \(\text{Sp}(2)/\text{SU}(2)\). Positively curved normal homogeneous metrics has been found on it.

**Subcase 9.** \(n > 2, \alpha = e_1 + e_2, \) and \(\beta = -e_1.\)

Then \(t \cap m = \mathbb{R}(2e_1 + e_2)\) and \(\alpha' = -\frac{1}{2}e_1 + \frac{2}{3}e_2\) is a root of \(\mathfrak{h}.\) Take \(\gamma_1 = e_1 + e_3\) and \(\gamma_2 = e_1 - e_3.\) Then \(\gamma_1\) and \(\gamma_2\) satisfy (1)-(3) but not (4) of Lemma 3.10 i.e. \(\pm\gamma_1\) are the only roots of \(g\) in \(\mathfrak{g}_{\gamma_1} + t \cap m,\) and all roots of \(g\) in \(\mathfrak{g}_{\pm \gamma_2} + \mathbb{R}\gamma_1 + t \cap m\) are \(\pm\gamma_2 = \pm(e_1 - e_3)\) and \(\pm\gamma_3 = \pm e_2.\) Choosing \(u\) and \(v\) from \(\mathfrak{g}_{\pm\gamma_1}\) and \(\mathfrak{g}_{\pm\gamma_2}\) as in the proof for Lemma 3.10 then we can similarly get
\[\langle [u, m]_m, u \rangle^F_u = \langle [v, m]_m, u \rangle^F_u = 0. \tag{4.29}\]

Notice \(\gamma_1 = e_1 + e_3\) and \(\gamma_3 = e_2\) also satisfy (2) of Lemma 3.10 i.e. \(\gamma_1 \pm \gamma_3\) are not roots of \(g,\) by which, together with Lemma 3.7 we can get
\[\langle [u, m]_m, v \rangle^F_u = 0. \tag{4.30}\]

Summarizing (4.29) and (4.30), we get \(U(u, v) = 0\) and then \(K^F(o, u, u \wedge v) = 0\) by Theorem 3.1 which is a contradiction.

### 4.4 The case \(g = \mathfrak{c}_n\) with \(n > 2\)

We only need to consider the following subcases.

**Subcase 1.** \(\alpha = 2e_1, \beta = e_1 + e_2.\)

Then \(t \cap m = \mathbb{R}(e_1 - e_2)\) and \(\alpha' = \beta = e_1 + e_2\) is a root of \(\mathfrak{h}.\) Let \(t'\) be the subalgebra of \(t \cap \mathfrak{h}\) spanned by \(\{e_3, \ldots, e_n\},\) and \(T'\) the corresponding sub-torus in \(T \cap H.\) The Lie algebra of \(C_G(T')\) is \(t' \oplus g''\), in which \(g''\) is of type \(b_2.\) Using Lemma 3.2 we get a positively curved homogeneous Finsler space \(SO(5)/SO(3)\) considered in Subcase 1 for \(b_n,\) which has been proven impossible.

**Subcase 2.** \(\alpha = 2e_1, \beta = 2e_2.\)

This subcase has been covered by the previous one.

**Subcase 3.** \(\alpha = 2e_1, \beta = -e_2 - e_3.\)
Then \( t \cap m = \mathbb{R}(2e_1 + e_2 + e_3) \) and \( \alpha' = \frac{2}{3}e_1 - \frac{2}{3}e_2 - \frac{2}{3}e_3 \) is a root of \( \mathfrak{h} \). Take \( \gamma_1 = 2e_2 \) and \( \gamma_2 = 2e_3 \). Then they satisfy (1)-(4) of Lemma 3.10, which is impossible by that lemma.

**Subcase 4.** \( \alpha = e_1 + e_2, \beta = e_1 - e_2. \)

Then \( t \cap m = \mathbb{R}e_2 \) and \( \alpha' = e_1 \) is a root of \( \mathfrak{h} \). By Lemma 3.10, \( 2\alpha' = 2e_1 \) is also a root of \( \mathfrak{h} \). This is a contradiction.

**Subcase 5.** \( \alpha = e_1 + e_2, \beta = -e_3 - e_4. \)

Then \( t \cap m = \mathbb{R}(e_1 + e_2 + e_3 + e_4) \) and \( \alpha' = \frac{1}{2}(e_1 + e_2 - e_3 - e_4) \) is a root of \( \mathfrak{h} \). Take \( \gamma_1 = 2e_1 \) and \( \gamma_2 = 2e_2 \), then they satisfy (1)-(4) of Lemma 3.10, which is impossible by Lemma 3.10.

**Subcase 6** \( n > 2, \alpha = 2e_1 \) and \( \beta = -e_1 - e_2. \)

Then \( t \cap m = \mathbb{R}(3e_1 + e_2) \) and \( \alpha' = \frac{1}{3}e_1 - \frac{2}{3}e_2 \) is a root of \( \mathfrak{h} \). Take \( \gamma_1 = e_1 + e_3 \) and \( \gamma_2 = 2e_2 \), then they satisfy (1)-(4) of Lemma 3.10, which is impossible by that lemma.

### 4.5 The case \( \mathfrak{g} = \mathfrak{o}_n \) with \( n > 3 \)

We only need to consider the following subcases.

**Subcase 1.** \( \alpha = e_1 + e_2 \) and \( \beta = e_2 - e_1 \).

Then \( t \cap m = \mathbb{R}e_1 \) and \( \alpha' = e_2 \) is a root of \( \mathfrak{h} \). As we have discussed for Subcase 3 and 5 in \( \mathfrak{b}_n \), we can apply Lemma 3.9, Lemma 3.10 and similar argument as Subcase 1 for \( \mathfrak{a}_n \) (which in fact is a special situation of this subcase), to show \( \mathfrak{h} \) is of type \( \mathfrak{b}_{n-1} \) with all the roots given by

\[ \pm e_i \pm e_j \] for \( 1 < i < j \leq n \) and \( \pm e_i \) for \( 1 < i \leq n \).

Using Lemma 4.3 we can show \( \mathfrak{h} \) is the most standard one up to \( \text{Ad}(G) \)-actions, such that the homogeneous Finsler space \( (G/H, F) \) is equivalent to Riemannian symmetric sphere \( \text{SO}(2n)/\text{SO}(2n-1) \) of positive constant curvature.

Notice when \( n = 4, \mathfrak{o}_4 \) has more outer automorphism. So this case covers the next subcase for \( \alpha \) and \( \beta \) with \( n = 4 \).

**Subcase 2.** \( n > 4, \alpha = e_1 + e_2 \) and \( \beta = -e_3 - e_4. \)

Then \( t \cap m = \mathbb{R}(e_1 + e_2 + e_3 + e_4) \) and \( \alpha' = \frac{1}{2}(e_1 + e_2 - e_3 - e_4) \) is a root of \( \mathfrak{h} \) with the length 1. Take \( \gamma_1 = e_1 + e_5 \) and \( \gamma_2 = e_1 - e_5 \), then they satisfy (1)-(4) of Lemma 3.10, which is impossible by that lemma.

### 5 Occasion III continued: the discussion for exceptional cases and the summarization

We continue with the case by case discussion of the last section, and summarize a theorem at the end, which is one of the main result of this paper.

#### 5.1 The case \( \mathfrak{g} = \mathfrak{e}_6 \)

Notice we can always change the orthogonal pair of roots \( \alpha \) and \( \beta \) to be of the form \( \pm e_i \pm e_j \). Up to Weyl group action induced by \( \mathfrak{d}_5 \), there are two subcases: (1) \( \alpha = e_1 + e_2 \)
and $\beta = e_2 - e_1$; (2) $\alpha = e_1 + e_2$ and $\beta = -e_3 - e_4$. Using the outer automorphisms of $\mathfrak{e}_6$ as well as Weyl group actions, the second subcase can be reduced to the first one. So we can assume $\alpha = e_1 + e_2$ and $\beta = e_2 - e_1$. Then $t \cap m = \mathbb{R}e_1$, and $\alpha' = e_2$ is a root of $\mathfrak{h}$. Take

$$\gamma_1 = -\frac{1}{2} e_1 + \frac{1}{2} e_2 + \frac{1}{2} e_3 + \frac{1}{2} e_4 + \frac{1}{2} e_5 + \frac{\sqrt{3}}{2} e_6,$$

$$\gamma_2 = -\frac{1}{2} e_1 - \frac{1}{2} e_2 - \frac{1}{2} e_3 - \frac{1}{2} e_4 - \frac{1}{2} e_5 + \frac{\sqrt{3}}{2} e_6.$$  

They satisfy (1)-(4) of Lemma 3.10 which is impossible by Lemma 3.10.

### 5.2 The case $\mathfrak{g} = \mathfrak{e}_7$

For any orthogonal pair of roots $\alpha$ and $\beta$ of $\mathfrak{g}$, we can always use some Weyl group action to change $\beta$ to $\sqrt{2} e_7$. Then because $\beta$ is orthogonal to $\alpha$, at the same time, $\alpha$ is changed to the form $\pm e_i \pm e_j$ with $1 \leq i < j \leq 6$. Using Weyl group actions induced by $\delta_0$, we can change $\alpha$ to be $e_1 + e_2$ while keeping $\beta = \sqrt{2} e_7$ fixed. So essentially there is only one subcase we need to consider, for example we can take $\alpha = e_1 + e_2$ and $\beta = e_2 - e_1$. Then $t \cap m = \mathbb{R}e_1$ and $\alpha' = e_2$ is a root of $\mathfrak{h}$. Take

$$\gamma_1 = -\frac{1}{2} e_1 + \frac{1}{2} e_2 + \frac{1}{2} e_3 + \frac{1}{2} e_4 + \frac{1}{2} e_5 + \frac{\sqrt{2}}{2} e_7,$$

$$\gamma_2 = \frac{1}{2} e_1 - \frac{1}{2} e_2 - \frac{1}{2} e_3 - \frac{1}{2} e_4 + \frac{1}{2} e_5 + \frac{\sqrt{2}}{2} e_7.$$  

Then they satisfy (1)-(4) of Lemma 3.10 which is impossible by that lemma.

### 5.3 The case $\mathfrak{g} = \mathfrak{e}_8$

Notice by suitable Weyl group actions, we can always change $\alpha$ and $\beta$ to be of the form $\pm e_i \pm e_j$. Then we only need to consider the following two subcases.

**Subcase 1.** $\alpha = e_1 + e_2$ and $\beta = e_2 - e_1$.

Then $t \cap m = \mathbb{R}e_1$ and $\alpha' = e_2$ is a root of $\mathfrak{h}$. Take

$$\gamma_1 = \frac{1}{2} e_1 + \frac{1}{2} e_2 + \frac{1}{2} e_3 + \frac{1}{2} e_4 + \frac{1}{2} e_5 + \frac{1}{2} e_6 + \frac{1}{2} e_7 + \frac{1}{2} e_8,$$

$$\gamma_2 = -\frac{1}{2} e_1 - \frac{1}{2} e_2 - \frac{1}{2} e_3 - \frac{1}{2} e_4 + \frac{1}{2} e_5 + \frac{1}{2} e_6 + \frac{1}{2} e_7 + \frac{1}{2} e_8.$$  

Then they satisfy (1)-(4) of Lemma 3.10 which is impossible by that lemma.

**Subcase 2.** $\alpha = e_1 + e_2$ and $\beta = -e_3 - e_4$.

Then $t \cap m = \mathbb{R}(e_1 + e_2 + e_3 + e_4)$ and $\alpha' = \frac{1}{2} (e_1 + e_2 - e_3 - e_4)$ is a root of $\mathfrak{h}$. Take $\gamma_1 = e_1 + e_5$ and $\gamma_2 = e_2 + e_6$, then they satisfy (1)-(4) of Lemma 3.10 which is a contradiction by Lemma 3.10.
5.4 The case \( g = f_4 \)

Notice by Weyl group actions, any short root of \( f_4 \) can be changed to \( e_1 \), and then any orthogonal pair of short roots of \( f_4 \) can be changed to \( e_1 \) and \( -e_2 \). In particular, using the reflections for the roots \( \frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4) \), any orthogonal pair of long roots can be changed to the pair \( e_1 \pm e_2 \).

**Subcase 1.** \( \alpha = e_1 + e_2 \) and \( \beta = e_2 \).

Then \( t \cap m = \mathbb{R}e_1 \) and \( \alpha' = e_2 \) is a root of \( \mathfrak{h} \). Let \( t' \) be the subalgebra of \( t \cap \mathfrak{h} \) spanned by \( e_3 \) and \( e_4 \), and \( T' \) be the closed sub-torus in \( T \cap H \) with \( \text{Lie}(T') = t' \). Using Lemma 3.2 for \( T' \), we get a positively curved homogeneous Finsler space \( SO(5)/SO(3) \) considered in Subcase 1 for \( \mathfrak{b}_n \), which has been proven impossible already.

**Subcase 2.** \( \alpha = e_1 + e_2 \) and \( \beta = e_2 - e_1 \).

This subcase has been covered by the previous one.

**Subcase 3.** \( \alpha = e_1 + e_2 \) and \( \beta = -e_3 \).

Then \( t \cap m = \mathbb{R}(e_1 + e_2 + e_3) \) and \( \alpha' = \frac{1}{2}e_1 + \frac{1}{3}e_2 - \frac{2}{3}e_3 \) is a root of \( \mathfrak{h} \) with length \( \sqrt{\frac{2}{3}} \), with \( h_{\pm \alpha'} \subset \mathfrak{g}_{\pm \alpha'} = \mathfrak{g}_{\pm(e_1+e_2)} + \mathfrak{g}_{e_3} \). By Lemma 3.3, \( e_4 \) are roots of \( \mathfrak{h} \), and \( \mathfrak{h}_{e_4} = \mathfrak{g}_{e_4} = \mathfrak{g}_{e_4} \). Notice \( \mathfrak{pr}_h(e_4 - e_3) \) is not orthogonal to \( \alpha' \) and has the length \( \sqrt{\frac{2}{3}} \). So \( \mathfrak{pr}_h(e_4 - e_3) \) is not a root of \( \mathfrak{h} \), and then \( \mathfrak{g}_{\pm(e_4-e_3)} \subset m \). Therefore we have

\[
\mathfrak{g}_{\pm e_1} = [\mathfrak{g}_{e_4}, \mathfrak{g}_{(e_4-e_3)}] \subset m,
\]

and thus \( h_{\pm \alpha'} = \mathfrak{g}_{\pm(e_1+e_2)} \). Then we can get

\[
\alpha' = \frac{1}{3}e_1 + \frac{1}{3}e_2 - \frac{2}{3}e_3 \subset [h_{\pm \alpha'}, h_{\pm \alpha'}] = [\mathfrak{g}_{\pm(e_1+e_2)}, \mathfrak{g}_{\pm(e_1+e_2)}] = \mathbb{R}(e_1 + e_2),
\]

which is obviously a contradiction.

**Subcase 4.** \( \alpha = e_1 \) and \( \beta = -e_2 \).

Then \( t \cap m = \mathbb{R}(e_1 + e_2) \) and \( \alpha' = \frac{1}{3}(e_1 - e_2) \) is a root of \( \mathfrak{h} \). By Lemma 3.10, \( e_1 - e_2 = 2\alpha' \) is also a root of \( \mathfrak{h} \), which is a contradiction.

**Subcase 5.** \( \alpha = e_1 + e_2 \) and \( \beta = -e_2 \).

Then \( t \cap m = \mathbb{R}(e_1 + 2e_2) \), and \( \alpha' = \frac{2}{3}e_1 - \frac{1}{3}e_2 \) is a root of \( \mathfrak{h} \) of length \( \sqrt{\frac{1}{3}} \), with \( h_{\pm \alpha'} \subset \mathfrak{g}_{\pm \alpha'} = \mathfrak{g}_{\pm(e_1+e_2)} + \mathfrak{g}_{e_3} \). By Lemma 3.3, \( e_3 \) is a root of \( \mathfrak{h} \) and \( h_{e_3} = \mathfrak{g}_{e_3} = \mathfrak{g}_{e_3} \).

The vector \( \mathfrak{pr}_h(e_2 + e_3) \) is not a root of \( \mathfrak{h} \), since it is not orthogonal to \( \alpha' \) and its length is \( \sqrt{\frac{2}{3}} \). So \( \mathfrak{g}_{(e_2+e_3)} \subset m \). So we have

\[
\mathfrak{g}_{\pm e_2} = [\mathfrak{g}_{(e_2+e_3)}, \mathfrak{g}_{e_3}] \subset m.
\]

This implies \( h_{\pm \alpha'} = \mathfrak{g}_{\pm(e_1+e_2)} \). We can get a contradiction by a similar argument as in Subcase 3.

There is another way to get the contradiction. Let \( t' = \mathbb{R}e_4 \) and \( T' \) be the corresponding closed one-parameter subgroup in \( H \). Using Lemma 3.2, we get a positively curved homogeneous Finsler space which has been proven impossible in Subcase 9 for \( \mathfrak{b}_n \).
5.5 The case $g = g_2$

If the angle between $\alpha$ and $\beta$ is $\pi/6$ or $\pi/2$, we can find a pair of short roots $\alpha_1$ and $\beta_1$ of $g$, such that the angle between $\alpha$ and $\beta$ is $\pi/3$, and $\alpha' = \text{pr}(\alpha_1) = \text{pr}(\beta_1)$ is a root of $h$, for which We can get the contradiction by Lemma 4.2.

We only need to consider the case $\alpha$ is a long root, $\beta$ is a short, and the angle between them is $5\pi/6$. Then $\alpha' = \text{pr}_h(\alpha) = \text{pr}_h(\beta)$ is a root of $h$, and $\beta$ is of type $a_1$. Let $\gamma_1 = \alpha + 3\beta$ and $\gamma_2 = \alpha + \beta$. Take any non-zero vectors $u \in g_{+\gamma_1}$ and $v \in g_{+\gamma_2}$. It is not hard to see the long root $\gamma_1$ and the short root $\gamma_2$ are orthogonal to each other, and $\gamma_1 \pm \gamma_2$ are not roots of $g$. So $u$ and $v$ are a linearly independent commuting pair. Denote the anticlockwise rotation as $R(\theta)$, in which $\theta$ is angle. We can find a group element $g \in T \cap H$, and suitable bases for each of the subspaces of $m$ below, such that

\[
\text{Ad}(g)|_{\cap m} = \text{Id}, \\
\text{Ad}(g)|_{\cap m} = R(\pi/4), \\
\text{Ad}(g)|_{\cap m} = R(\pi/2), \\
\text{Ad}(g)|_{\cap m} = R(3\pi/4), \\
\text{Ad}(g)|_{\cap m} = R(\pi) = \text{Id}, \\
\text{Ad}(g)|_{\cap m} = R(5\pi/4).
\]

Denote each above subspace on which $\text{Ad}(g)$ acts by $R(k\pi/4)$ as $m_k$, $k = 1, \ldots, 5$, and in particular $m_0 = \cap m$, $m_2 = g_{+\gamma_2}$ and $m_4 = g_{+\gamma_1}$. By Lemma 5.8, we have

\[
\langle m_4, m_i \rangle^F_u = 0, \quad \forall i \neq 4. \tag{5.31}
\]

For any $v' \in m_2$ and $w' \in m_i$ with $i \neq 2$, we have

\[
\langle v', w' \rangle^F_u = \langle \text{Ad}(g)v', \text{Ad}(g)w' \rangle^F_{\text{Ad}(g)u} = \langle R(\pi/2)v', R(i\pi/4)w' \rangle^F_u \\
= \langle R(\pi/2)v', R(i\pi/4)w' \rangle^F_u = \langle R(\pi/2)^2v', R(i\pi/4)^2w' \rangle^F_u \\
= \langle -v', R((i - 2)\pi/2)w' \rangle^F_u = \langle v', R((i - 2)\pi/2)w' \rangle^F_u.
\]

By similar argument as in the proof for Lemma 5.8, we get

\[
\langle m_2, m_i \rangle^F_u = 0, \quad \forall i \neq 2. \tag{5.32}
\]

By the $\text{Ad}(T \cap H)$-invariance, the Minkowski norm $F|_{m_4}$ coincides with the restriction of the bi-invariant inner product up to a scalar change, so

\[
\langle [u, t], u \rangle^F_u = 0. \tag{5.33}
\]

Direct calculation shows,

\[
[u, m]_m \subset m_0 + m_1 + m_3 + [u, t] + m_5,
\]

and

\[
[v, m]_m \subset m_0 + m_1 + m_2 + m_3 + m_5.
\]

So by (5.31), (5.32) and (5.33), we have

\[
\langle [u, m]_m, u \rangle^F_u = \langle [u, m]_m, v \rangle^F_u = \langle [v, m]_m, u \rangle^F_u = 0. \tag{5.34}
\]

By Theorem 3.1 we get $U(u, v) = 0$ and thus $K^F(\alpha, u, u \wedge v) = 0$, which is a contradiction.
5.6 Summarization

We summarize all the discussions in Section 4 and Section 5 as the following theorem, which completely classifies odd dimensional positively curved reversible homogeneous Finsler spaces in Occasion III.

**Theorem 5.1** Let \((G/H, F)\) be an odd dimensional positively curved reversibly homogeneous Finsler space of Occasion III, i.e. with respect to a bi-invariant orthogonal decomposition \(g = h + m\) for the compact Lie group \(g\), and a fundamental Cartan subalgebra \(t\), there are roots \(\alpha\) and \(\beta\) of \(g\) from the same simple fact, such that \(\alpha \neq \pm \beta\) and \(pr_h(\alpha) = pr_h(\beta) = \alpha'\) is a root of \(h\). Then \((G/H, F)\) is equivalent to the homogeneous Finsler metric on one of following:

1. odd dimensional Riemannian symmetric spheres \(S^{2n-1} = SO(2n)/SO(2n-1)\) with \(n > 2\);
2. homogeneous sphere \(S^7 = \text{Spin}(7)/G_2\) and \(S^{15} = \text{Spin}(9)/\text{Spin}(7)\);
3. Berger’s spaces \(\text{SU}(5)/\text{Sp}(2)U(1)\) and \(\text{Sp}(2)/\text{SU}(2)\).

6 Discussion for Occasion II and I

In this section we will consider odd dimensional positively curved reversible homogeneous Finsler spaces in Occasion II and Occasion I.

6.1 Discussion for Occasion II

Let \((G/H, F)\) be an odd dimensional positively curved reversible homogeneous Finsler space in Occasion II, i.e. respect to the bi-invariant orthogonal decomposition \(g = h + m\) for the compact Lie algebra \(g = \text{Lie}(G)\) and the fundamental Cartan subalgebra \(t\), there exists two roots \(\alpha\) and \(\beta\) of \(g\) from different simple factors such that \(pr_h(\alpha) = pr_h(\beta) = \alpha'\) is a root of \(h\). In this situation \(\alpha'\) is a linear combination of \(\alpha\) and \(\beta\) with two nonzero coefficients, thus \(h_{\pm \alpha'} \subset \hat{g}_{\pm \alpha} + \hat{g}_{\pm \beta}\) can not be a root plane of \(g\), or equivalently \(g_{\pm \alpha}\) and \(g_{\pm \beta}\) are not contained in \(h\) or \(m\).

Firstly, we can find a direct sum decomposition

\[ g = g_1 \oplus \cdots \oplus g_n \oplus \mathbb{R}^m, \]

such that each \(g_i\) is a simple ideal of \(g\), moreover \(\alpha\) and \(\beta\) are roots of \(g_1\) and \(g_2\) respectively. Because \(t \cap m = \mathbb{R}(\alpha - \beta) \subset g_1 \oplus g_2\), the Abelian factor of \(g\) and \(t \cap g_i\) for each \(i > 2\) are contained in \(t \cap h\). It is also obvious to see for each root \(\gamma\) of \(g\) with \(\gamma \neq \pm \alpha, \gamma \neq \pm \beta\) and \(pr_h(\gamma) = \gamma'\), we have \(g_{\pm \gamma} = \hat{g}_{\pm \gamma}\) is contained in either \(h\) or \(m\).

Secondly, we prove each \(g_i\) for \(i > 2\) is contained by \(t \cap h\). Because each simple factor \(g_i\) can be algebraically generated by its root planes, We only need to prove that each root plane of \(g_i\) for \(i > 2\) is contained in \(h\). Let \(\gamma\) be any root of \(g_i\). When \(i > 2\), \(\gamma\) is contained in \(t \cap h\) and it is the only root of \(g\) in \(\gamma + (t \cap m)\). By Lemma 3.9 \(\gamma\) is a root of \(h\) and \(g_{\pm \gamma} = \hat{g}_{\pm \gamma} = h_{\pm \gamma} \subset h\).
Lastly, we consider roots of \( g_1 \) and \( g_2 \). Up to the equivalence, we can just assume \( g = g_1 \oplus g_2 \). Let \( \gamma \) be any root of \( g_1 \) such that \( \gamma \neq \pm \alpha \). Because it is the only root of \( g_1 \) contained in \( \gamma + (t \cap m) \), by Lemma 3.9 we have either \( g_{\pm \gamma} \subset m \), when \( \gamma \) is not orthogonal to \( \alpha \), or \( g_{\pm \gamma} \subset h \), if \( \gamma \subset t \cap h \cap g_1 \). For any root of \( g_2 \), we have a similar statement.

We claim there cannot exist two roots \( \gamma_1 \) and \( \gamma_2 \) of \( g_1 \) and \( g_2 \) respectively, such that their root planes are contained in \( m \). Assume conversely there exist such \( \gamma_1 \) and \( \gamma_2 \), then \( \gamma_1 \neq \pm \alpha, \gamma_2 \neq \pm \beta \) and they satisfy (1)-(4) of Lemma 3.10 which is impossible by that lemma.

Without loss of generality, we can then assume all roots of \( g_1 \) other than \( \pm \alpha \) are roots of \( h \), thus they are bi-invariant orthogonal to \( \pm \alpha \). Because \( g_1 \) is simple, \( g_1 \) is of type \( a_1 \) with the only roots \( \pm \alpha \). Next we consider \( g_2 \). We have the following lemma.

**Lemma 6.1** Following above assumptions and notations. Then there do not exist a pair of roots \( \gamma_1 \) and \( \gamma_2 \) of \( g_2 \) satisfying the following conditions:

1. \( \gamma_1 \neq \pm \gamma_2, \gamma_1 \neq \pm \beta \) and \( \gamma_2 \neq \beta \);
2. \( \gamma_1 \) and \( \gamma_2 \) are not roots of \( h \);
3. \( \gamma_1 \pm \gamma_2 \) are not roots of \( g \).

**Proof.** Assume conversely there are roots \( \gamma_1 \) and \( \gamma_2 \) of \( g_2 \) satisfying (1)-(3) of the lemma. It is easy to see \( \gamma_1 \) is the only root in \( \gamma_1 + \mathbb{R}(\alpha - \beta) \). On the other hand, if for some real numbers \( t_1 \) and \( t_2 \), \( \gamma_3 = \gamma_2 + t_1 \gamma_1 + t_2 (\alpha - \beta) \) is a root of \( g \) other than \( \gamma_2 \), then \( t_2 \in \{-1,0,1\} \). If \( t_2 = 0 \) then \( \gamma_3 = \gamma_2 + t_1 \gamma_1 \) with \( t_1 \neq 0 \) is a root of \( g_2 \). This is impossible because \( \gamma_1 \pm \gamma_2 \) are not roots of \( g_2 \). If \( t_2 = \pm 1 \) then \( \pm \beta = t_1 \gamma_1 + \gamma_2 \) is a root of \( g_2 \) other than \( \gamma_2 \). Similarly we can get the contradiction. Now we see \( \gamma_1 \) and \( \gamma_2 \) satisfy (1)-(4) of Lemma 3.10 which is impossible by that lemma. \( \blacksquare \)

Let \( \mathfrak{t} \) be the subalgebra of \( g_2 \) generated by \( g_{\pm \beta} \) and \( h \cap g_2 \). It has the same rank as \( g_2 \) and it can be decomposed as a direct sum \( \mathfrak{t} = a_1 \oplus h \oplus g_2 \), in which the \( a_1 \)-factor is generated by \( g_{\pm \beta} \). By Lemma 6.1 the pair \((g_2, \mathfrak{t})\) satisfies the condition (A) in [16], by Proposition 6.1 of [16], all possible \((g_2, \mathfrak{t})\) are the following,

\[
(g_2, \mathfrak{t}) = (a_1, a_1), (a_2, a_1 \oplus \mathbb{R}) \text{ or } (c_n, a_1 \oplus c_{n-1}).
\]

Correspondingly, all possible \((g, \mathfrak{h})\) are the following,

\[
(g, \mathfrak{h}) = (a_1 \oplus a_1, a_1), (a_1 \oplus a_2, a_1 \oplus \mathbb{R}) \text{ or } (a_1 \oplus c_n, a_1 \oplus c_{n-1}),
\]
in which the \( a_1 \)-factor in \( h \) is a diagonal one. Thus the corresponding homogeneous Finsler space is equivalent to the symmetric homogeneous sphere \( S^3 = SO(4)/SO(3) \), Wilking’s space \( SU(3) \times SO(3)/U(2) \) which coincides with \( S_{1,1} \) among Aloff-Wallach’s spaces [1][17], and the homogeneous sphere \( S^{4n-1} = Sp(n)Sp(1)/Sp(n-1)Sp(1) \).

To summarize, we have the following theorem classifying odd dimensional positively curved reversible homogeneous Finsler spaces in Occasion II.

**Theorem 6.2** Let \((G/H, F)\) be an odd dimensional positively curved reversible homogeneous Finsler space of Occasion II, i.e. with respect to a bi-invariant orthogonal
decomposition \( g = h + m \) for the compact Lie algebra \( g = \text{Lie}(G) \) and a fundamental Cartan subalgebra \( t \) of \( g \), there are roots \( \alpha \) and \( \beta \) of \( g \) from different simple factors such that \( \text{pr}_h(\alpha) = \text{pr}_h(\beta) = \alpha' \) is a root of \( h \). Then \( (G/H, F) \) is equivalent to the homogeneous Finsler metric on one of following:

1. the symmetric homogeneous sphere \( S^3 = \text{SO}(4)/\text{SO}(3) \);
2. the homogeneous spheres \( \text{Sp}(n)\text{Sp}(1)/\text{Sp}(n-1)\text{Sp}(1) \);
3. Wilking’s space \( \text{SU}(3) \times \text{SO}(3)/\text{U}(2) \).

6.2 Discussion for Occasion I

Assume \( (G/H, F) \) is an odd dimensional positively curved reversible homogeneous Finsler space in Occasion I, i.e. with respect to a bi-invariant orthogonal decomposition \( g = h + m \) for the compact Lie algebra \( g = \text{Lie}(G) \) and a fundamental Cartan subalgebra \( t \), each root plane of \( h \) is also a root plane of \( g \). Keep all relevant notations as before. The root system of \( h \) is then a subset of the root system of \( g \), i.e. \( \Delta_h \subset \Delta_g \cap h \).

For each root \( \alpha \) of \( g \), either \( g_{\alpha} \subset h \) or \( g_{\alpha} \subset m \). The argument is not affected by the choice of the bi-invariant inner product on \( g \).

Assume \( g \) has the following direct sum decomposition,

\[
g = g_0 \oplus g_1 \oplus \cdots \oplus g_n, \tag{6.35}
\]

in which \( g_0 \) is Abelian and each other \( g_i \) is a simple ideal. Let \( w = w_0 + \cdots + w_n \) be the decomposition according to (6.35) for any nonzero vector \( w \) in \( t \cap m \). It is not hard to see from Lemma 3.9, for each \( i > 0 \), \( g_i \) is contained in \( h \) iff \( w_i = 0 \).

**Case 1.** Assume \( w_0 \neq 0 \).

Then any two roots \( \alpha \) and \( \beta \) of \( g \), such that they are not roots of \( h \), and \( \alpha \geq \beta \) are not roots of \( g \), satisfies (1)-(4) of Lemma 3.10 which is impossible by that lemma. Let \( \mathfrak{t} \) be subalgebra generated by \( h \) and \( t \), then \( \mathfrak{t} = h \oplus t \cap m \). There is a closed subgroup \( K \) of \( G \) with \( \text{Lie}(K) = \mathfrak{t} \) and \( \text{rk} K = \text{rk} G \). The pair \( (g, \mathfrak{t}) \) satisfies the Condition (A) in [16], thus we can arrange the decomposition (6.35) of \( g \) such that

\[
\mathfrak{t} = g_0 \oplus \mathfrak{t}_1 \oplus g_2 \oplus \cdots \oplus g_n,
\]

in which \((g_1, \mathfrak{t}_1) = (a_n, a_{n-1} \oplus \mathbb{R}), (c_n, c_{n-1} \oplus \mathbb{R}), \) or \((a_2, \mathbb{R} \oplus \mathbb{R})\).

Notice the others in Wallach’s list can not provide the Abelian factor required for this situation. If \( g_1 = a_2 \), by Lemma 3.9 no root of \( g_1 \) can be contained in \( t \cap h \). So correspondingly \( (G/H, F) \) is equivalent to one of the following:

1. the homogeneous sphere

\[
S^{2n-1} = U(n)/U(n-1) \text{ or } S^{4n-1} = \text{Sp}(n)\text{U}(1)/\text{Sp}(n-1)\text{U}(1) \text{ for } n > 1;
\]
(2) the U(3)-homogeneous presentations of Aloff-Wallach’s spaces $S_{k,l} = U(3)/T^2$, in which $T^2$ is a two dimensional torus of diagonal matrices such that it does not contain the center of U(3) and

$$T^2 \cap SU(3) = U_{k,l} = \{ \text{diag}(z^k, z^l, z^{-k-l}) | z \in \mathbb{C}, |z| = 1 \},$$

where $k$ and $l$ are integers satisfying $kl(k + l) \neq 0$.

Notice the same SU(3)-homogeneous space $S_{k,l}$ can have infinitely many different presentation as U(2)-homogeneous spaces.

**Case 2.** Assume $w = w_1 + w_2$ with nonzero $w_1$ and $w_2$.

Up to equivalence, we can just assume $g = g_1 \oplus g_2$ for the argument in this case.

Suppose there exist a root $\alpha \notin \mathbb{R}w_1$ of $g_1$ and a root $\beta \notin \mathbb{R}w_2$ of $g_2$ such that they are not roots of $h$, then $\alpha$ and $\beta$ satisfy (1)-(4) of Lemma 3.10 which is impossible by that lemma. Without loss of generality, we can assume all roots of $g_1$ outside $\mathbb{R}w_1$ are roots of $h$, i.e. contained by $t \cap h$. By the simpleness of $g_1$, we must have $g_1 = a_1$ with the only roots $\pm \alpha$ in $t \cap g_1 = \mathbb{R}w_1$. There are two subcases we need to consider.

**Subcase 1.** There exists a root $\beta$ of $g_2$ contained in $\mathbb{R}w_2$.

Obviously $\alpha$ and $\beta$ are not roots of $h$, i.e. their root planes are contained in $m$. Let $t'$ be the bi-invariant orthogonal complement of $w_2$ in $g_2$ and $T'$ be the corresponding torus in $H$. Using Lemma 3.2 we get a positively curved reversible homogeneous Finsler space $SU(2) \times SU(2)/U(1)$ in Occasion I, in which $U(1)$ is not contained by either simple factor. In the following, we only need to consider this situation, i.e. we will assume $g_2 = a_1$ as well, with the only roots $\pm \beta$. We will choose the bi-invariant inner product on $g = g_1 \oplus g_2 = a_1 \oplus a_1$ such that its restriction on each factor has the same scale. By suitably re-ordering the two simple factors, we can assume $\alpha + \beta \in t \cap m$ with $|c| \geq 1$. Denote $\alpha' = \text{pr}_h(\alpha)$ and $\beta' = \text{pr}_h(\beta)$. Then our assumption implies $\beta'$ not be an even multiple of $\alpha'$.

Let $\{u, u'\}$ be a bi-invariant orthonormal basis of $g_{\pm \alpha}$, and $v$ any nonzero vector in $g_{\pm \beta}$ such that $\langle u', v \rangle_u^F = 0$. Obviously $u$ and $v$ are a linearly independent commuting pair.

Because of the Ad($H$)-invariance, the Minkowski norm $F|_{g_{\pm \alpha}}$ coincides with the bi-invariant inner product up to scalar changes, thus we have

$$\langle u', u \rangle_u^F = \langle [t, u], u \rangle_u^F = 0.$$

By our previous assumption and Lemma 3.8,

$$\langle t \cap m, u \rangle_u^F = \langle t \cap m, v \rangle_u^F = 0. \quad (6.36)$$

Direct calculation shows

$$[u, m]_m = t \cap m + [t, u], \quad (6.37)$$

so by (6.36), we get

$$\langle [u, m]_m, u \rangle_u^F = \langle t \cap m, u \rangle_u^F + \langle R u', u \rangle_u^F = 0, \quad (6.38)$$

and

$$\langle [u, m]_m, v \rangle_u^F = \langle t \cap m, v \rangle_u^F + \langle R u', v \rangle_u^F = 0. \quad (6.39)$$
Let \( v \) be a nonzero vector in \( \mathfrak{g}_{\pm \beta} \) such that
\[
\langle u', v \rangle^F_u = \langle [t \cap \mathfrak{h}, u], v \rangle^F_u = 0.
\]

Direct calculation also shows
\[
[v, m]_m = t \cap m + [t \cap m, v] = t \cap m + [t \cap \mathfrak{h}, v]. \tag{6.40}
\]
For any \( w' \in t \cap \mathfrak{h} \), using Theorem 3.1 of [9], we have
\[
\langle [w', v], u \rangle^F_u = -\langle v, [w', u] \rangle^F_u - 2C^F_u([w', u], v, u) = 0,
\]
by our assumption on \( v \) and the basic properties of Cartan tensor. So by Lemma 3.8 and (6.40) we have
\[
\langle [v, m]_m, u \rangle^F_u = \langle [v, t \cap \mathfrak{h}], u \rangle^F_u = 0. \tag{6.41}
\]

Summarizing (6.38), (6.39) and (6.41), we get \( U(u, v) = 0 \) and \( K^F(o, u, u \wedge v) = 0 \) by Theorem 3.1, which is a contradiction.

**Subcase 2.** Assume there does not exist any root of \( \mathfrak{g}_2 \in \mathbb{R}w_2 \).

Then by the simpleness of \( \mathfrak{g}_2 \), there is a root \( \beta \) of \( \mathfrak{g}_2 \) which is not bi-invariant orthogonal to \( w_2 \). Let \( u \) and \( v \) be any nonzero vectors in \( \mathfrak{g}_{\pm \alpha} \) and \( \mathfrak{g}_{\pm \beta} \) respectively. Obviously they are a linearly independent commuting pair. Let \( t' = t \cap \mathfrak{h} \cap \mathfrak{g}_2 \) which coincides with \( w_2 \cap t \cap \mathfrak{g}_2 \), the bi-invariant orthogonal complement of \( w_2 \) in \( t \cap \mathfrak{g}_2 \). Denote \( T' \) the corresponding torus in \( H \). The inner product \( \langle \cdot, \cdot \rangle^F \) is \( \text{Ad}(T') \)-invariant, so by Lemma 3.7 with respect to this inner product, \( m \) can be orthogonally decomposed as the sum of \( m' = \hat{m}_0 = t \cap m + \mathfrak{g}_{\pm \alpha} \) for the trivial irreducible \( T' \)-representation and \( m'' \subset \mathfrak{g}_2 \) for nontrivial irreducible \( T' \)-representations. Notice \( m'' \) is the sum of some root planes in \( \mathfrak{g}_2 \), moreover \( u \) and \( v \) are contained by \( m' \) and \( m'' \) respectively.

Direct calculation shows
\[
[u, m]_m = t \cap m + [t, u] \subset m' \text{ and } [v, m]_m \subset t \cap m + m''. \tag{6.42}
\]
The \( \text{Ad}(T' \cap H) \) invariance of \( F|_{\mathfrak{g}_{\pm \alpha}} \) indicate \( F|_{\mathfrak{g}_{\pm \alpha}} \) coincides with the restriction of a bi-invariant inner product up to scalar changes, i.e. we have
\[
\langle [t, u], u \rangle^F_u = \langle [t \cap \mathfrak{h}, u], u \rangle^F_u = 0. \tag{6.43}
\]
By Lemma 3.8
\[
\langle t \cap m, u \rangle^F_u = 0. \tag{6.44}
\]
Summarizing (6.42), (6.43) and (6.44), we have
\[
\langle [u, m]_m, u \rangle^F_u = \langle [u, m]_m, v \rangle^F_u = \langle [v, m]_m, u \rangle^F_u = 0,
\]
from which we can get \( U(u, v) = 0 \) and \( K^F(o, u, u \wedge v) = 0 \) by Theorem 3.1. This is a contradiction.

**Case 3.** Assume \( w = w_1 + \cdots + w_m \) with \( m > 2 \) and each \( w_i \) nonzero.

If there is a root \( \alpha \notin \mathbb{R}w_1 \) of \( \mathfrak{g}_1 \), and a root \( \beta \notin \mathbb{R}w_2 \) of \( \mathfrak{g}_2 \) such that they are not roots of \( \mathfrak{h} \), then they satisfy (1)-(4) of Lemma 3.10, which is impossible by that lemma. Similar to the previous case, we can assume \( \mathfrak{g}_1 = \mathfrak{g}_2 = \mathfrak{a}_1 \). Let \( \pm \alpha \) and \( \pm \beta \) be the roots.
of $g_1$ and $g_2$ respectively. Notice there is no roots contained in $\mathbb{R}(w_2 + \cdots + w_m)$, so similar argument as for Subcase 2 of the previous case can be applied here to deduce the contradiction.

The above discussion can be summarized to the following theorem.

**Theorem 6.3** Let $(G/H, F)$ be an odd dimensional positively curved reversible homogeneous Finsler space in Occasion I, i.e. with respect to a bi-invariant orthogonal decomposition $g = \mathfrak{h} + \mathfrak{m}$ for the compact Lie algebra $g = \text{Lie}(G)$ and a fundamental Cartan subalgebra $t$ of $g$, each root plane of $\mathfrak{h}$ is also a root plane of $g$. Assume $(G/H, F)$ is not equivalent to any $(G'/H', F')$ with a compact simple $G'$, then it is equivalent to one of the following:

1. the homogeneous sphere
   
   $S^{2n-1}U(n)/U(n-1)$, or $S^{4n-1}Sp(n)U(1)/Sp(n-1)U(1)$, $\forall n > 1$;

2. the $U(3)$-homogeneous Aloff-Wallach's spaces.

### 6.3 Remarks on the cases with normal homogeneity and Riemannian homogeneity

When we continue the classification for the odd dimensional positively curved reversible homogeneous Finsler spaces $G/H$ in Occasion I with $G$ simple, we meet both technical and substantial difficulties. But if we additionally assume $G/H$ is normal homogeneous or Riemannian, then the classification can be completed.

When $G/H$ is normal homogeneous, $F$ is subduced by a bi-invariant Finsler metric on $G$. Let $\mathfrak{k}$ be the subalgebra generated by $\mathfrak{h}$ and $\mathfrak{t}$, and $K$ be the closed subgroup of $G$ with $\text{Lie}(K) = \mathfrak{k}$, then $\mathfrak{k} = \mathfrak{t} \cap \mathfrak{m} \oplus \mathfrak{h}$. The same bi-invariant Finsler metric on $G$ defines another normal homogeneous Finsler metric $\bar{F}$ on $G/K$, such that the natural projection from $G/H$ to $G/K$ is a Finslerian submersion. Because $\dim G/H > 1$, $G/K$ is an even dimensional coset space admitting positively curved normal homogeneous Finsler metrics, which has been classified in [21]. From this clue, we can easily find the missing homogeneous spheres

$$S^{2n-1} = SU(n)/SU(n-1) \text{ and } S^{4n-1} = Sp(n)/Sp(n-1), \forall n > 1.$$  

When $G/H$ is Riemannian, the metric is induced by an $\text{Ad}(H)$-invariant inner product $\langle \cdot, \cdot \rangle$. The submersion technique described above still works when there do not exist two different roots $\alpha$ and $\beta$ such that $\alpha - \beta \in \mathfrak{t} \cap \mathfrak{m}$. Taking $\beta = -\alpha$, the assumption also implies there not exist any root contained in $\mathfrak{t} \cap \mathfrak{m}$. In fact if $g$ is simple, and there is a root contained in $\mathfrak{t} \cap \mathfrak{m}$, we can always find roots $\alpha$ and $\beta$, such that $\alpha \neq \beta$ and $\alpha - \beta \in \mathfrak{t} \cap \mathfrak{m}$.

Notice the lemmas in Subsection 3.5 can strengthened in Riemannian geometry. For example, Lemma 3.10 can be strengthened to the following one.
Lemma 6.4 Let \( G/H \) be an odd dimensional positively curved Riemannian homogeneous space, with a bi-invariant orthogonal decomposition \( g = h + m \) and a fundamental Cartan subalgebra \( t \). Then there do not exist two roots \( \alpha \) and \( \beta \) of \( g \), satisfying the following conditions:

1. \( \alpha \) and \( \beta \) are not roots of \( h \);
2. \( \alpha \pm \beta \) are not roots of \( g \);
3. \( \alpha \) is the only roots contained in \( \alpha + t \cap m \);
4. \( \beta \) is the only roots contained in \( \beta + t \cap m \).

Assuming there exist roots \( \alpha \) and \( \beta \) of \( g \) such that \( \alpha \neq \pm \beta \) and \( \alpha - \beta \in t \cap m \), we can use the \( \text{Ad}(H) \)-invariance of the inner product \( \langle \cdot, \cdot \rangle \) and those strengthened key lemmas, to discuss each possible case. Notice in the situation of \( \text{Sp}(2)/U(1) \) in [24], it is positively curved for all commuting pairs, so B. Wilking’s method is essential here. The case by case discussion is not hard, by too long to be presented here. See [2] for the original proof (with the correction in [24]), or the recent paper [19] for the much shorter proof.

7 Appendix: the root systems of compact simple Lie algebras

For each simple Lie algebra \( g \) we recall the Bourbaki description of the root system \( \Delta_g \), and the root planes in the classical cases.

1. The case \( g = a_n = \mathfrak{su}(n + 1) \) for \( n > 0 \). Let \( \{e_1, \ldots, e_{n+1}\} \) denote the standard orthonormal basis of \( \mathbb{R}^{n+1} \). Then \( t \) can be isometrically identified with the subspace \( (e_1 + \cdots + c_{n+1})^\perp \subset \mathbb{R}^{n+1} \). The root system \( \Delta \) is

\[ \{\pm(e_i - e_j) \mid 1 \leq i < j \leq n+1\}. \]

Let \( E_{i,j} \) be the matrix with all zeros except for a 1 in the \((i, j)\) place. Then

\[ e_i = \sqrt{-1}E_{i,i} \in \mathfrak{su}(n+1), \text{ and} \]
\[ g_{\pm(e_i-e_j)} = \mathbb{R}(E_{i,j} - E_{j,i}) + \mathbb{R}\sqrt{-1}(E_{i,j} + E_{j,i}). \]

2. The case \( g = b_n = \mathfrak{so}(2n + 1) \) for \( n > 1 \). The Cartan subalgebra \( t \) can be isometrically identified with \( \mathbb{R}^n \) with the standard orthonormal basis \( \{e_1, \ldots, e_n\} \). The root system \( \Delta \) is

\[ \{\pm e_i \mid 1 \leq i \leq n\} \cup \{\pm e_i \pm e_j \mid 1 \leq i < j \leq n\}. \]
Using matrices, we have

\[
\begin{align*}
e_i &= E_{2i,2i+1} - E_{2i+1,2i}, \\
g_{\pm e_i} &= \mathbb{R}(E_{2i,1} - E_{1,2i}) + \mathbb{R}(E_{2i+1,1} - E_{1,2i+1}), \\
g_{\pm(e_i-e_j)} &= \mathbb{R}(E_{2i,2j} + E_{2i+1,2j+1} - E_{2j,2i} - E_{2j+1,2i+1}) \\
&\quad + \mathbb{R}(E_{2i,2j+1} - E_{2i+1,2j} + E_{2j,2i+1} - E_{2j+1,2i}), \\
g_{\pm(e_i+e_j)} &= \mathbb{R}(E_{2i,2j} - E_{2i+1,2j+1} - E_{2j,2i} + E_{2j+1,2i+1}) \\
&\quad + \mathbb{R}(E_{2i,2j+1} + E_{2i+1,2j} - E_{2j,2i+1} - E_{2j+1,2i}).
\end{align*}
\]

(3) The case \( g = \mathfrak{c}_n = \mathfrak{sp}(n) \) for \( n > 2 \). As before \( t \) is isometrically identified with \( \mathbb{R}^n \) with the standard orthonormal basis \( \{e_1, \ldots, e_n\} \). The root system \( \Delta \) is

\[
\{ \pm 2e_i \mid 1 \leq i \leq n \} \cup \{ \pm e_i \pm e_j \mid 1 \leq i < j \leq n \}.
\]

Using matrices, we have

\[
\begin{align*}
e_i &= \mathbf{i}E_{i,i}, \\
g_{\pm 2e_i} &= \mathbb{R}\mathbf{j}E_{i,i} + \mathbb{R}\mathbf{k}E_{i,i}, \\
g_{\pm(e_i-e_j)} &= \mathbb{R}(E_{i,j} - E_{j,i}) + \mathbb{R}(E_{i,j} + E_{j,i}), \text{ and} \\
g_{\pm(e_i+e_j)} &= \mathbb{R}\mathbf{j}(E_{i,j} + E_{j,i}) + \mathbb{R}\mathbf{k}(E_{i,j} + E_{j,i}).
\end{align*}
\]

(4) The case \( g = \mathfrak{d}_n = \mathfrak{so}(2n) \) for \( n > 3 \). The Cartan subalgebra \( t \) is identified with \( \mathbb{R}^n \) with the standard orthonormal basis \( \{e_1, \ldots, e_n\} \). The root system \( \Delta \) is

\[
\{ \pm e_i \pm e_j \mid 1 \leq i < j \leq n \}.
\]

In matrices, we have formulas for the \( e_i \) and for the root planes for \( e_i \pm e_j \) similar to those in the case of \( \mathfrak{b}_n \), i.e.

\[
\begin{align*}
e_i &= E_{2i-1,2i} - E_{2i,2i-1}, \\
g_{\pm(e_i-e_j)} &= \mathbb{R}(E_{2i-1,2j-1} + E_{2i,2j} - E_{2j-1,2i-1} - E_{2j,2i}) \\
&\quad + \mathbb{R}(E_{2i-1,2j} - E_{2i,2j-1} + E_{2j-1,2i} - E_{2j,2i-1}), \text{ and} \\
g_{\pm(e_i+e_j)} &= \mathbb{R}(E_{2i-1,2j-1} - E_{2i,2j} - E_{2j-1,2i-1} + E_{2j,2i}) \\
&\quad + \mathbb{R}(E_{2i-1,2j} + E_{2i,2j-1} - E_{2j-1,2i} - E_{2j,2i-1}).
\end{align*}
\]

(5) The case \( g = \mathfrak{e}_6 \). The Cartan subalgebra \( t \) can be isometrically identified with \( \mathbb{R}^6 \) with the standard orthonormal basis \( \{e_1, \ldots, e_6\} \). The root system is

\[
\{ \pm e_i \pm e_j \mid 1 \leq i < j \leq 5 \} \cup \{ \pm \frac{1}{2} e_1 \pm \cdots \pm \frac{1}{2} e_5 \pm \frac{\sqrt{3}}{2} e_6 \text{ with odd number of } +'s \}.
\]

It contains a root system of type \( \mathfrak{d}_5 \).
(6) The case $g = e_7$. The Cartan subalgebra can be isometrically identified with $\mathbb{R}^7$ with the standard orthonormal basis $\{e_1, \ldots, e_7\}$. The root system is
\[
\{\pm e_i \pm e_j | 1 \leq i < j < 7\} \cup \{\pm \sqrt{2}e_7; \frac{1}{2}(\pm e_1 \pm \cdots \pm e_6 \pm \sqrt{2}e_7)\}
\]
with an odd number of plus signs among the first six coefficients.

It contains a root system of $\mathfrak{d}_6$.

(7) The case $g = e_8$. The Cartan subalgebra can be isometrically identified with $\mathbb{R}^8$ with the standard orthonormal basis $\{e_1, \ldots, e_8\}$. The root system $\Delta$ is
\[
\{\pm e_i \pm e_j | 1 \leq i < j < 8\} \cup \frac{1}{2}(\pm e_1 \pm \cdots \pm e_8)\text{ with an even number of +’s}.\]

It contains a root system of $\mathfrak{d}_8$.

(8) The case $g = f_4$. The Cartan subalgebra is isometrically identified with $\mathbb{R}^4$ with the standard orthonormal basis $\{e_1, \ldots, e_4\}$. The root system is
\[
\{\pm e_i | 1 \leq i \leq 4\} \cup \{\pm e_i \pm e_j | 1 \leq i < j \leq 4\} \cup \frac{1}{2}(\pm e_1 \pm \cdots \pm e_4).\]

It contains the root system of $\mathfrak{b}_4$.

(9) The case $g = g_2$. The Cartan subalgebra is isometrically identified with $\mathbb{R}^2$ with the standard orthonormal basis $\{e_1, e_2\}$. The root system $\Delta$ is
\[
\{(\pm \sqrt{3}, 0), (\pm \frac{\sqrt{3}}{2}, \pm \frac{3}{2}), (0, \pm 1), (\pm \frac{\sqrt{3}}{2}, \pm \frac{1}{2})\}.
\]

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