BIGERBES

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ABSTRACT. The bigerbes introduced here give a refinement of the notion of 2-gerbes, representing degree four integral cohomology classes of a space. Defined in terms of bisimplicial line bundles, bigerbes have a symmetry with respect to which they form ‘bundle 2-gerbes’ in two ways; this structure replaces higher associativity conditions. We provide natural examples, including a Brylinski-McLaughlin bigerbe associated to a principal $G$-bundle for a simply connected simple Lie group. This represents the first Pontryagin class of the bundle, and is the obstruction to the lifting problem on the associated principal bundle over the loop space to the structure group consisting of a central extension of the loop group; in particular, trivializations of this bigerbe for a spin manifold are in bijection with string structures on the original manifold. Other natural examples represent ‘decomposable’ 4-classes arising as cup products, a universal bigerbe on $K(\mathbb{Z}, 4)$ involving its based double loop space, and the representation of any 4-class on a space by a bigerbe involving its free double loop space. The generalization to ‘multigerbes’ of arbitrary degree is also described.

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Gerbes provide a (more or less) geometric representation of integral cohomology 3-classes on a space \([8, 5]\). Bundle gerbes, introduced by Murray in \([16]\), are particularly geometric and have a well-known application in the form of the ‘lifting bundle gerbe’, representing the obstruction to the extension of a principal \(G\)-bundle to a principal bundle with structure group a \(U(1)\) central extension of \(G\). Here we present a direct extension of the notion of a bundle gerbe to obtain a similar representation of integral 4-classes. These \textit{bigerbes} are special cases, in a sense more rigid, of the bundle 2-gerbes as defined by Stevenson \([22]\), which in turn are a more geometric version of 2-gerbes as defined by Breen \([3]\). In particular our bigerbes induce bundle 2-gerbes in two ways. One application of this notion is to Brylinski-McLaughlin (bi)gerbes, corresponding to the existence of an extension of the principal bundle over the loop space induced by a principal \(G\)-bundle over the original space, to a bundle with structure group a central extension of the loop group \([4]\).

A gerbe may be defined as a simplicial object \([17, 22]\). We work in the context of locally split maps, which is to say continuous maps \(\pi: Y \to X\), with local right inverses over an open cover of the topological space \(X\). Such a map determines an associated simplicial space, \(Y^{[n]}\), over \(X\), formed from the fiber products \(Y^{[k]} = Y \times_X \cdots \times_X Y:\)

\[
Y \quad \leftrightarrow \quad Y^{[2]} \quad \leftrightarrow \quad Y^{[3]} \quad \ldots \ldots.
\]

This constitutes a contravariant functor \(\Delta \to \text{Top}/X\), where \(\Delta\) denotes the simplex category with objects the sets \(\mathbb{n} = \{1, \ldots, n\}\) for \(n \in \mathbb{N}_0\) with morphisms the order preserving maps between these, and \(\text{Top}/X\) denotes the category of spaces with commuting maps to \(X\). Functions on \(Y^{[n]}\) admit a simplicial differential, denoted by \(d\), by taking the alternating sum of the pull-backs, and this operation extends to line (or circle) bundles and sections thereof by taking the alternating tensor product of the pull-backs.

A bundle gerbe on \(X\) is specified in terms of the simplicial space \((1)\) by the prescription of a complex line bundle \(L\) over \(Y^{[2]}\) such that \(dL\) over...
$Y^{[3]}$ has a section $s$ which pulls back to be the canonical section of $d^2 L$ over $Y^{[4]}$. The important special case of the lifting bundle gerbe is obtained when $\pi : E \to X$ is a principal $G$-bundle; then there is a natural map $E^{[2]} \to G$, and the line bundle is the pull-back of the line bundle over $G$ associated to a given central extension of $G$ by $\mathbb{C}^*$ or $U(1)$.

Our notion of a bigerbe is based on a *split square* of maps. This is a commutative square of locally split maps

$$
\begin{array}{ccc}
Y_2 & \leftarrow & W \\
\downarrow & & \downarrow \\
X & \leftarrow & Y_1
\end{array}
$$

(2)

with the additional property that the induced map

$$
W \to Y_1 \times_X Y_2
$$

is also locally split (and in particular surjective).

Such a split square induces a bisimplicial space $W^{[\bullet,\bullet]}$ over $X$:

$$
\begin{array}{c}
W^{[3]} \leftarrow W^{[1,3]} \leftarrow W^{[2,3]} \equiv W^{[3,3]} \equiv W^{[3,2]} \equiv W^{[3,1]} \equiv W^{[2,3]} \equiv W^{[2,2]} \equiv W^{[2,1]} \equiv W^{[3,1]} \equiv W^{[2,1]} \equiv W^{[1,1]} \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
Y_2 \leftarrow W^{[1,2]} \leftarrow W^{[2,2]} \equiv W^{[3,2]} \equiv W^{[3,1]} \equiv W^{[2,2]} \equiv W^{[2,1]} \equiv W^{[3,1]} \equiv W^{[2,1]} \equiv W^{[1,1]} \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
Y_2 \leftarrow W^{[1,1]} \leftarrow W^{[2,1]} \equiv W^{[3,1]} \equiv W^{[2,1]} \equiv W^{[1,1]} \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
X \leftarrow Y_1 \leftarrow Y_1^{[2]} \equiv Y_1^{[3]} \equiv Y_1^{[4]} \equiv Y_1^{[3]} \equiv Y_1^{[2]} \equiv Y_1^{[1]} \equiv Y_1
\end{array}
$$

(3)

where the left column and bottom row are the standard simplicial spaces as in (1) and the interior spaces are given inductively by

$$
W^{[m,n]} = W^{[m,1]} \times_{Y_1^{[n]}} \cdots \times_{Y_1^{[n]}} W^{[m,1]}. 
$$

The result is a commutative diagram in which all rows and columns are simplicial spaces. There are then two commuting simplicial differentials, $d_1$ and $d_2$, corresponding to the horizontal and vertical maps, respectively.

**Definition.** A bigerbe on the bisimplicial space (3) corresponding to a locally split square (2) is specified by a (locally trivial) complex line bundle $L$ over $W^{[2,2]}$ with $d_1 L$ and $d_2 L$, over $W^{[3,2]}$ and $W^{[2,3]}$ respectively, having trivializing sections $s_i$, for $i = 1, 2$, such that $ds_i$ is the canonical trivialization of $d_i^2 L$ and $d_2 s_1 = d_1 s_2$.

As for bundle gerbes, there are straightforward notions of products, inverses, pullbacks, and morphisms of bigerbes, for which the characteristic 4-class defined below behaves naturally.
As noted above, among the natural examples is the Brylinski-McLaughlin bigerbe. Suppose that \( E \to X \) is a principal \( G \)-bundle over a manifold with structure group a compact, connected, simply connected, simple Lie group. Then

\[
\begin{array}{c}
E & \leftarrow & PE \\
\downarrow & & \downarrow \\
X & \leftarrow & PX
\end{array}
\]

(5)

is a split square where \( PX \) and \( PE \) are the respective (based) path spaces, the vertical arrows are projections and the horizontal arrows are the endpoint maps. In the resulting bisimplicial space \( W^{[2,1]} = \Omega E \) is the based loop space of \( E \) which is a principal bundle with structure group the based loop group \( \Omega G \) of \( G \). The central extensions

\[
1 \to U(1) \to \hat{\Omega G} \to \Omega G \to 1
\]

(6)

are classified by \( H^3(G; \mathbb{Z}) = H^3_G(G; \mathbb{Z}) = \mathbb{Z} \), and the associated line bundle for such a central extension pulls back over \( W^{[2,2]} = \Omega E^{[2]} \) to a line bundle \( Q \) determining a bigerbe. Here the triviality of \( d_1 Q \) is the multiplicativity of the central extension, as for a lifting gerbe, whereas the (consistent) triviality of \( d_2 Q \) corresponds to the so-called ‘fusion’ property of the central extension with respect to certain configurations of loops [24, 29, 11], and which is equivalent to the gerbe property with respect to the path fibration \( PE^{[2]} \to E^{[2]} \). Incorporation of an additional ‘figure-of-eight’ condition as in [11, 12] — a condition related to the simplicial space of products of \( X \) discussed in §3 — promotes this to a bigerbe involving free loops and paths, representing the obstruction of the lift of \( LE \to LX \) to a ‘loop-fusion’ \( \hat{LG} \)-bundle which is discussed further below.

There are various 2-gerbe versions of this in the literature. In [4], Brylinski and McLaughlin define a 2-gerbe in the sense of Breen by pulling back the canonical gerbe on \( G \) (corresponding to the given class in \( H^3(G; \mathbb{Z}) \)) to \( E^{[2]} \) by the difference map, particularly in the universal case where \( X = BG \). In [6], and later [28], a similar construction was used to produce a bundle 2-gerbe in the sense of Stevenson. Furthermore, in [4], the authors discuss a correspondence between the 2-gerbe and the problem of extending the structure group of the free loop space \( LX \) from \( LG \) to \( \hat{LG} \). The bigerbe above demonstrates this correspondence explicitly.

Returning to the simplicial space, (1), arising from any locally split map, the simplicial differentials extend to the Čech cochain spaces over the \( Y^{[k]} \) with respect to a system of ‘admissible’ covers, as defined in §1.1 and §1.2, which can be taken to be arbitrarily fine. Then the simplicial complex

\[
0 \to \check{C}^\ell (X, A) \xrightarrow{d} \check{C}^\ell (Y, A) \xrightarrow{d} \check{C}^\ell (Y^{[2]}, A) \xrightarrow{d} \ldots
\]

(7)
is exact (see Proposition 2.3), with a homotopy inverse arising from local sections over an open cover. The simplicial differential commutes with the Čech differential resulting in a double complex.

For a bundle gerbe, the representative \( c(L) \) of the Chern class of \( L \) can be chosen to be a pure cocycle: \( \delta c(L) = dc(L) = 0 \). From the exactness of the simplicial differential this class descends:

\[
c(L) = -d\beta, \quad \delta \beta = d\alpha \quad \text{for some } \beta \in \check{C}^1(Y; \mathbb{C}^*), \alpha \in \check{C}^2(X; \mathbb{C}^*)
\]

and then \( \text{DD}(L) \in \check{H}^3(X; \mathbb{Z}) \), the image of \([\alpha]\) in \( \check{H}^2(X; \mathbb{C}^*) \) under the Bockstein isomorphism, is the Dixmier-Douady class of the gerbe. This is not the original definition of the Dixmier-Douady class of a bundle gerbe as in [16, 17]; we show that it is equivalent below in Proposition 2.8 and use the simplicial characterization to prove that a locally split map \( \pi: Y \to X \) supports a bundle gerbe with a given 3-class on \( X \) if and only if the class vanishes when pulled back to \( Y \) (Theorem 2.10), and also to classify trivializations of bundle gerbes (in Proposition 2.11).

For a general bigerbe there is a similar Čech analysis in terms of the triple complex, formed by the three commuting differentials \( \delta, d_1 \) and \( d_2 \), on the Čech spaces over the bisimplicial space (3). Now the Chern class \( c(L) \in \check{C}^1(W^{[2,1]}; \mathbb{C}^*) \) can again be chosen to be a pure cocycle: \( \delta c(L) = d_1 c(L) = d_2 c(L) = 0 \). As a consequence it descends to a cocycle on \( Y^{[2]} \):

\[
c(L) = d_2 \beta_1, \quad d_1 \beta_1 = 0, \quad \beta_1 \in \check{C}^1(W^{[2,1]}; \mathbb{C}^*)
\]

\[
\delta \beta_1 = -d_2 \lambda_1, \quad d_1 \lambda_1 = 0, \quad \lambda_1 \in \check{C}^2(Y^{[2]}; \mathbb{C}^*),
\]

essentially as for the gerbe. Thus the image of \([\lambda_1]\) in \( \check{H}^3(Y^{[2]}; \mathbb{Z}) \) (under the Bockstein isomorphism) is the Dixmier-Douady class of \( L \) as a gerbe over \( Y^{[2]} \). Significantly however, \( \lambda_1 \) is naturally a simplicial cocycle — a pure cocycle in the \( \delta, d_1 \) complex — and so \( d_1 \lambda_1 = 0 \).

In view of this, the simplicial class further descends under \( d_1 \):

\[
\lambda_1 = -d_1 \mu_1, \quad \delta \mu_1 = d_1 \gamma, \quad \delta \gamma = 0, \quad \mu_1 \in \check{C}^2(Y_1; \mathbb{C}^*), \quad \gamma \in \check{C}^3(X; \mathbb{C}^*).
\]

The Bockstein image \( G(L) \in \check{H}^4(X; \mathbb{Z}) \) of \([\gamma]\) is the characteristic 4-class associated to the bigerbe.

The symmetry of the bigerbe allows \( Y_1 \) and \( Y_2 \) to be interchanged, but this also reverses the sign of \( G(L) \).

**Theorem** (Thm 4.15, Thm. 4.17).

(i) The characteristic 4-class of a bigerbe is natural with respect to pullbacks, morphisms, products, and inverses. It vanishes if and only if the bigerbe admits a trivialization, and two bigerbes have the same 4-class if and only if they are stably isomorphic.

(ii) The bisimplicial space generated by a split square, as in (2), supports a bigerbe for a given class in \( \check{H}^4(X; \mathbb{Z}) \) if and only if this class lifts to
the $Y_i$ to be trivial, with primitives which when pulled back to $W$ have exact difference.

For the Brylinski-McLaughlin bigerbe $Q$ associated to a principal bundle $E \to X$ with structure group a compact, connected, simply connected and simple Lie group $G$, the 4-class $G(Q) = [\gamma]$ is the transgression to $X$ of the 3-class on $G$ corresponding to a central extension $\hat{L}G$ of $LG$:

$$X \quad \underset{\b}{\longleftarrow} \quad E \quad \underset{\b}{\leftrightarrow} \quad E^[2] \quad \underset{q}{\longrightarrow} \quad G,$$

$$\alpha \in H^3(G;\mathbb{Z}), \quad q^*\alpha = d\mu, \quad d\gamma = \delta\mu. \quad (11)$$

**Theorem** (Thm. 5.9, Thm. 5.11). The Brylinski-McLaughlin bigerbe $Q \to LE^[2]$ has characteristic class $G(Q) = p_1(E) \in \hat{H}^4(X;\mathbb{Z})$, the vanishing of which is equivalent to the existence of a ‘loop-fusion’ (meaning fusion and figure-of-eight, see §3) $\hat{L}G$ lift of the $LG$-bundle $LE \to LX$. Such lifts, which are equivalent to certain trivializations of $Q$, are classified by $\hat{H}^3(X;\mathbb{Z})$.

In particular this applies to the spin frame bundle of a spin manifold. There it represents the obstruction to a lift of the principal loop spin bundle over the loop space to a loop-fusion principal bundle for the basic central extension of the loop spin group. The obstruction is then the Pontryagin class, usually denoted $\frac{1}{2}p_1$ because of its relation to the Pontryagin class of the oriented orthogonal frame bundle, of the spin bundle [29, 6, 11]. Such loop-fusion lifts are, by the above theorem, in bijection with so-called ‘string structures’ on the manifold (see Corollary 5.13).

In addition to the Brylinski-McLaughlin bigerbes, we provide other natural examples of bigerbes representing ‘decomposable’ 4-classes which are the cup product of either 2-classes or a 1-class and a 3-class (see §5.1). Moreover we show that for a simply connected and locally contractible space $X$, every 4-class is represented by a bigerbe with respect to the locally split square in which the $Y_i$ are the based path spaces $PX$ and $W = PPX$ is the based mapping space of the square into $X$ (see §5.5). In particular $K(\mathbb{Z}, 4)$ supports a universal bigerbe. Likewise, for $X$ not necessarily simply connected, we show that every 4-class is represented by a bigerbe using the free path spaces; in particular $W^[2,2] = LLX$ is the double free loop space in this case.

There is a direct extension of bigerbes to ‘multigerbes’, higher versions of (bi)gerbes in which the locally split squares are replaced by $n$-cubes, the line bundles satisfy simplicial conditions with respect to $n$ commuting differentials; these represent cohomology classes of degree $n + 2$. The symmetry of the (multi)simplicial conditions replaces the ever higher and more complex ‘associativity’ conditions associated to higher gerbes. This extension is quite straightforward, and for this reason, and since we are unaware of examples apart from decomposable and path multigerbes, we only outline the theory briefly at the end of this paper.
In order to restrict attention to a simple category of topological spaces, and to avoid expanding the paper further, we do not develop the theory of connections on bigerbes here, though this will be done in a future work. We also do not discuss here the bigerbe analogue of bundle gerbe modules or the related theory of generalized morphisms due to Waldorf [26].

Section 1 below contains a discussion of covers and locally split maps, which is the context for the rest of the paper, and our notation for Čech theory is introduced in §1.2.

Section 2 is a review of Murray’s theory of bundle gerbes (without connections), as developed from the Čech-simplicial point of view, with the basic properties of bundle gerbes over split maps recalled in §2.1. The extension of the Čech cohomology complex to a bicomplex over the simplicial space of a split map in §2.2 leads to an alternative definition of the Dixmier-Douady class for a bundle gerbe in §2.3, the classification of gerbes over a given split map in §2.4, and the classification of trivializations in §2.5. Examples of bundle gerbes are recalled in §2.6.

A ‘product-simplicial’ condition, which we refer to as ‘doubling’, on bundle gerbes is defined in §3.1 with particular application to the free loop space, and the connection with results from [12] is described in §3.2.

In §4.1 the basic properties of locally split squares of maps are given, leading to the definition of bigerbes in §4.2. The characteristic 4-class of a bigerbe is obtained in §4.3 and conversely §4.4 contains a necessary and sufficient condition for representability of a 4-class over a given locally split square.

Section §5 is devoted to examples, with explicit bigerbes corresponding to decomposable classes constructed in §5.1, extending some of the results of [14]. After a brief discussion of doubling for bigerbes in §5.2, our main application of bigerbes — the Brylinski-McLaughlin lifting gerbe — is discussed in §5.3 and its relation to string structures is discussed in §5.4. Further examples of path bigerbes can be found in §5.5, and finally, we end with a brief account of multigerbes in §6.

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1. Čech Theory

1.1. Covers and locally split maps. Since we will make substantial use of Čech theory, we start with some conventions on spaces, open covers, and maps. We work throughout in the standard topologist’s category of compactly generated Hausdorff spaces and continuous maps, without additional conditions unless explicitly noted.

An open cover of a topological space $X$ is a collection of open sets, $\mathcal{U}$, for which $X = \bigcup_{U \in \mathcal{U}} U$. Such a cover defines an étale space by taking the disjoint union

$$\text{Et}(\mathcal{U}) = \bigsqcup_{U \in \mathcal{U}} U \longrightarrow X$$

with the map to $X$ consisting of the inclusion map on each $U$.

Note that since the individual sets may not be connected, it is not generally possible to recover the collection $\mathcal{U}$ from Et$(\mathcal{U})$ without specifying additional indexing information. We regard $X$ as its own minimal cover.

If $\mathcal{U}$ and $\mathcal{V}$ are covers of $X$ and $Y$ then a map of covers is a continuous map $g : \text{Et}(\mathcal{U}) \longrightarrow \text{Et}(\mathcal{V})$ where each element $U \in \mathcal{U}$ is mapped to a specific element $V \in \mathcal{V}$. In particular, there is an underlying map of index sets.

Definition 1.1. A continuous map $\pi : Y \longrightarrow X$ of topological spaces is locally split if it admits continuous local sections; thus $\pi$ is surjective and there exists a cover $\mathcal{U}$ of $X$ with respect to which the local sections constitute a continuous map of covers $s : \text{Et}(\mathcal{U}) \longrightarrow Y$ such that $\pi \circ s : \text{Et}(\mathcal{U}) \longrightarrow X$ is inclusion of the cover in $X$. In particular the inclusion map of covers $\text{Et}(\mathcal{U}) \longrightarrow X$ is itself locally split.

If $\mathcal{U}$ and $\mathcal{V}$ are covers of the same space $X$ then a map of covers such that

$$\begin{array}{ccc}
\text{Et}(\mathcal{V}) & \longrightarrow & \text{Et}(\mathcal{U}) \\
\downarrow & & \downarrow \\
X & & X
\end{array}$$

(1.1)

commutes makes $\mathcal{V}$ a refinement of $\mathcal{U}$; often in the literature the underlying map of index sets is omitted but we always retain it, even if implicitly. In this way the covers of $X$ define a category with refinements as morphisms. Observe also that if $\mathcal{V}$ is a cover of Et$(\mathcal{U})$ considered as a space, then $\mathcal{V}$ is a cover of $X$ as well, the composite map Et$(\mathcal{V}) \longrightarrow X$ is an inclusion map of covers, and $\mathcal{V}$ constitutes a refinement of $\mathcal{U}$.

If $f : Y \longrightarrow X$ is a continuous map then the pullback

$$f^{-1}\mathcal{U} = \{ f^{-1}(U) : U \in \mathcal{U} \}$$

(1.2)

is a cover of $Y$ and $f$ lifts to a well defined map of covers $f : \text{Et}(f^{-1}\mathcal{U}) \cong f^{-1}\text{Et}(\mathcal{U}) \longrightarrow \text{Et}(\mathcal{U})$. If $\mathcal{U}$ and $\mathcal{V}$ are both covers of $X$, then

$$\mathcal{U} \cap \mathcal{V} = \{ U \cap V : U \in \mathcal{U}, V \in \mathcal{V} \}$$

(1.3)
is a cover of $X$ mutually refining $\mathcal{U}$ and $\mathcal{V}$; indeed, this is the same thing as pulling back $\mathcal{V}$ to a cover of $\text{Et}(\mathcal{U})$ by the inclusion map to $X$ or vice versa, and $\text{Et}(\mathcal{U} \cap \mathcal{V}) \cong \text{Et}(\mathcal{U}) \times_X \text{Et}(\mathcal{V})$. Note that

$$\mathcal{U}^{(\ell)} = \mathcal{U} \cap \cdots \cap \mathcal{U}$$

gives the cover by $\ell$-fold intersections of sets in $\mathcal{U}$ (we refer to $\text{Et}(\mathcal{U}^{(\ell)})$ as a Čech space) and that there is a canonical identification

$$f^{-1}(\mathcal{U}^{(\ell)}) \cong (f^{-1}\mathcal{U})^{(\ell)}$$  \hspace{1cm} (1.4)

of covers over $Y$ whenever $\mathcal{U}$ is a cover of $X$ and $f : Y \to X$ is continuous.

1.2. Čech cohomology. The standard definition of Čech cohomology proceeds by fixing an open cover $\mathcal{U}$ of $X$ and taking the homology $\check{H}^\bullet(\mathcal{U}; A)$ of the cochain complex $(\check{C}^\bullet(\mathcal{U}; A), \delta) = (\Gamma(\mathcal{U}^{(\bullet+1)}; A), \delta)$, where $A$ is a sheaf of abelian groups on $X$, $\Gamma(\mathcal{U}^{(\bullet+1)}; A)$ denotes the group of local sections of $A$ on the intersection cover $\mathcal{U}^{(\bullet+1)}$ and $\delta : \Gamma(\mathcal{U}^{(\bullet)}; A) \to \Gamma(\mathcal{U}^{(\bullet+1)}; A)$ is given by the alternating sum of the pullbacks by the various inclusion maps $\mathcal{U}^{(\bullet+1)} \to \mathcal{U}^{(\bullet)}$. For our purposes, $A$ will always be a fixed topological abelian group such as $\mathbb{C}$, $\mathbb{C}^*$, $\mathbb{Z}$, or $\text{U}(1)$ and we will work on the sheaf of continuous maps to $A$, so that

$$\check{C}^\bullet(\mathcal{U}; A) = C(\text{Et}(\mathcal{U}^{(\bullet+1)}); A)$$

is a space of continuous maps to $A$ from the étale space. The full Čech cohomology is defined as the direct limit

$$\check{H}^\bullet(X; A) = \lim_{\mathcal{U}} \check{H}^\bullet(\mathcal{U}; A)$$  \hspace{1cm} (1.5)

under refinement. Here the limit is taken with respect to the directed set of covers of $X$, where $\mathcal{U} < \mathcal{V}$ if there exists a refinement $\text{Et}(\mathcal{U}) \to \text{Et}(\mathcal{V})$ inducing a pullback $\check{C}^\bullet(\mathcal{V}; A) \to \check{C}^\bullet(\mathcal{U}; A)$; while there may be many such distinct refinements, the associated pullbacks are chain homotopic and the induced maps on cohomology agree. We use the standard terminology of cochains, cocycles and coboundaries for elements of $\check{C}^\bullet(X; A)$, and also borrow the adjectives closed and exact from de Rham theory for cocycles and coboundaries, respectively.

**Proposition 1.2.** Fix a topological abelian group $A$, and suppose a continuous map $f : Y \to X$ of topological spaces lifts to a map of covers $\tilde{f} : \text{Et}(Y) \to \text{Et}(\mathcal{U})$ for covers $\mathcal{V}$ and $\mathcal{U}$ of $Y$ and $X$, respectively. Then this induces a chain map

$$\tilde{f}^* : \check{C}^\bullet(\mathcal{U}; A) \to \check{C}^\bullet(\mathcal{V}; A)$$

which descends to the pull-back functor $f^* : \check{H}^\bullet(X; A) \to \check{H}^\bullet(Y; A)$ on cohomology.
Proof. The map of covers induces a map of covers $\text{Et}(\mathcal{V}(\ell)) \rightarrow \text{Et}(\mathcal{U}(\ell))$ for each $\ell$, giving a chain map $\tilde{C}_{\mathcal{V}}^d(Y; A) \rightarrow \tilde{C}_{\mathcal{U}}^d(X; A)$, and the natural maps $\text{Et}(\mathcal{V}(\ell)) \rightarrow \text{Et}(\mathcal{U}(\ell))$ commute with the inclusions $\text{Et}(\mathcal{U}(\ell)) \rightarrow \text{Et}(\mathcal{U}(\ell-1))$ in each factor of the $\ell$-fold commute, descending in the direct limit in cohomology to the map

$$
\text{Et}(\mathcal{V}^{(\ell)}) \xrightarrow{\tilde{f}} \text{Et}(\mathcal{U}^{(\ell)})
$$

commutes, descending in the direct limit in the map $f^* : H^*(Y; A) \rightarrow H^*(X; A)$. \hfill \Box

We proceed to define a more limited form of pullback on chains with respect to the sections of a locally split map. This is essential to the exactness of the simplicial complex (7) of Čech chains on the simplicial space induced from a locally split map mentioned in the introduction and discussed in detail in §2.2.

**Definition 1.3.** For a locally split map $\pi : Y \rightarrow X$, an admissible pair of covers consists of covers $\mathcal{U}$ of $X$ and $\mathcal{V}$ of $Y$ along with maps of covers $\tilde{\pi} : \text{Et}(\mathcal{V}) \rightarrow \text{Et}(\mathcal{U})$ and $\tilde{s} : \text{Et}(\mathcal{U}) \rightarrow \text{Et}(\mathcal{V})$ such that

(i) $\tilde{\pi}$ is a lift of $\pi$,

(ii) $\tilde{s}$ is a lift of a local section $s : \text{Et}(\mathcal{U}) \rightarrow Y$,

(iii) $\tilde{s}$ is a section (right inverse) of $\tilde{\pi}$, i.e. $\tilde{\pi}\tilde{s} = \text{Id}_{\mathcal{U}} : \text{Et}(\mathcal{U}) \rightarrow \text{Et}(\mathcal{U})$, and

(iv) for each $U_1, U_2 \in \mathcal{U}$ such that $U_1 \cap U_2 \neq \emptyset$, there exists $V_2 \in \tilde{\pi}^{-1}(U_2) \subset \mathcal{V}$ such that $\tilde{s}_{U_1} : U_1 \cap U_2 \rightarrow V_2$, i.e., the section defined on $U_1$ restricts over $U_2$ to a map into some $V_2$ over $U_2$.

The key consequence of (iv) is that for each $\ell \geq 1$ there exist canonical sections $\tilde{s}_\ell : \mathcal{U}^{(\ell)} \rightarrow \mathcal{V}^{(\ell)}$ of $\tilde{\pi} : \mathcal{V}^{(\ell)} \rightarrow \mathcal{U}^{(\ell)}$ defined by

$$
\tilde{s}_\ell = \tilde{s}_{U_1} : U_1 \cap U_2 \cap \cdots \cap U_\ell \rightarrow V_1 \cap V_2 \cap \cdots \cap V_\ell
$$

(1.6)

so that $\mathcal{U}^{(\ell)}$ and $\mathcal{V}^{(\ell)}$ also constitute an admissible pair.

**Lemma 1.4.** If $\pi : Y \rightarrow X$ is a locally split map with arbitrary covers $\mathcal{U}$ of $Y$ and $\mathcal{V}$ of $X$, respectively, admitting a section $s : \text{Et}(\mathcal{U}) \rightarrow Y$, then there exists an admissible pair $\mathcal{U}'$ and $\mathcal{V}'$ refining $\mathcal{U}$ and $\mathcal{V}$ such that $\tilde{s}$ lifts $s$.

In fact, if $\pi_i : Y_i \rightarrow X, i = 1, \ldots, N$, is a finite number of locally split maps over $X$, then any covers $\mathcal{U}$ of $X$ and $\mathcal{V}_i$ of $Y_i$ can together be refined to $\mathcal{U}'$ and $\mathcal{V}'_i$, respectively, constituting an admissible pair for each $i$ simultaneously, with $\tilde{s}_i$ lifting any given sections $s_i : \text{Et}(\mathcal{U}) \rightarrow Y_i$. 

Remark. Since admissible covers may be taken to refine arbitrary covers of $X$ and $Y$, it follows that covers participating in admissible pairs are *final* in the directed set of all covers; in particular the direct limits $\tilde{H}^\ell(X; A) = \lim_\mathcal{U} \tilde{H}^\ell_\mathcal{U}(X; A)$ and $\tilde{H}^\ell(Y; A) = \lim_\mathcal{V} \tilde{H}^\ell_\mathcal{V}(Y; A)$ are equivalent to the direct limit taken over admissible covers alone [13, Thm. 1, p. 213]. Equivalently, any class in the cohomology of $X$ or $Y$ may be represented as a Čech cochain with respect to a cover which is part of an admissible pair.

Proof. We construct the refinement in two steps, first ‘pulling down’ $\mathcal{V}$ to get $\mathcal{U}' = s^{-1}\mathcal{V}$ and then ‘pulling up’ $\mathcal{U}'$ to get $\mathcal{V}' = \pi^{-1}\mathcal{U}' \cap \mathcal{V}$. More precisely, if $s_U : U \to Y$ is the section of $\pi$ over $U \in \mathcal{U}$ then the sets $s_U^{-1}V \subset U$ for $V \in \mathcal{V}$, define a cover of $U$, and hence together a refinement of $\mathcal{U}$ as a cover of $X$. This is the cover $\mathcal{U}' = s^{-1}\mathcal{V}$ defined above. Then $s$ lifts to a naturally defined map of covers $\tilde{s} : \text{Et}(\mathcal{U}') \to \text{Et}(\mathcal{V})$ given by $s_U : s_U^{-1}V \to V$. The pull-back $\pi^{-1}\mathcal{U}'$ of the new cover of $X$ defines a refinement $\mathcal{V}' = \pi^{-1}\mathcal{U}' \cap \mathcal{V}$ of $\mathcal{V}$ to which $\pi$ lifts as a map of covers $\tilde{\pi} : \text{Et}(\mathcal{V}') \to \text{Et}(\mathcal{U}')$. Moreover, as a consequence of the fact that $\pi s = \text{Id}$ on each element $U \in \mathcal{U}'$, it follows that $\tilde{s}$ lifts to a map of covers $\tilde{s} : \text{Et}(\mathcal{U}') \to \text{Et}(\mathcal{V}')$, and $\tilde{\pi} \tilde{s} = \text{Id}$.

To see that this satisfies the final condition of Definition 1.3, note that by definition each $U'_i = U_1 \cap s_{U_1}^{-1}(V)$ for some elements $U_1 \in \mathcal{U}$ and $V \in \mathcal{V}$, respectively. Then $V'_1 = V \cap \pi^{-1}(U'_1)$ contains the image of $U'_1$ under the lifted section and the other elements $V'_2 = V \cap \pi^{-1}(U'_2)$ contain the image of $U'_1 \cap U'_2$ under $s_{U_1}$.

Given a finite number of locally split maps $\pi_i : Y_i \to X$, we set $\mathcal{U}' = \bigcap_i s_i^{-1}\mathcal{V}_i$, which refines each $s_i^{-1}\mathcal{V}_i$ and hence lifts the map of covers $\tilde{s}_i$ defined above, followed by $\mathcal{V}'_i = \pi_i^{-1}\mathcal{U}' \cap \mathcal{V}$ for each $i$. 

In preparation for §2.2 we make the following observations.

Lemma 1.5. Denote by $Y^{[k]} = Y \times_X \cdots \times_X Y$ the $k$-fold fiber product of $\pi : Y \to X$ and fix an admissible pair of covers $(\mathcal{U}, \mathcal{V})$ for $(X, Y)$. Then for each $k \geq 2$,

(i) $Y^{[k]} := \mathcal{V} \times_\mathcal{U} \cdots \times_\mathcal{U} \mathcal{V}$, the $k$-fold fiber product of $\tilde{\pi} : \mathcal{V} \to \mathcal{U}$, is a cover of $Y^{[k]}$, 

(ii) The projection maps $\pi_j : Y^{[k]} \to Y^{[k-1]}$, for $0 \leq j \leq k-1$, lift canonically to maps of covers $\tilde{\pi}_j : \text{Et}(\mathcal{V}^{[k-1]}) \to \text{Et}(\mathcal{V}^{[k-1]})$ for each $\ell$,

(iii) The sections $\tilde{s}_\ell$ of Definition 1.3.(iv) determine a map of covers $\tilde{s}_\ell : \text{Et}(\mathcal{V}^{[k-1]}) \to \text{Et}(\mathcal{V}^{[k]})$ for each $\ell$ such that

\begin{equation}
\tilde{\pi}_j \tilde{s}_\ell = \begin{cases} 
\text{Id} & j = 0 \\
\tilde{s}_\ell \tilde{\pi}_j \tilde{s}_{\ell-1} & 1 \leq j \leq k-1
\end{cases}
\end{equation}

Proof. It is clear that $Y^{[k]}$ is an open cover of $Y^{[k]}$. Note that there is a canonical identification $(\mathcal{V}^{[k]})^{(\ell)} \cong (\mathcal{V}^{(\ell)})^{[k]}$, the latter fiber product taken
are both identified with pairs of points \((y, y')\) such that \(\pi(y) = \pi(y') \in U_1 \cap U_2\) with \(y \in V_1 \cap V_2\) and \(y' \in V'_1 \cap V'_2\), with similar identification holding in general.

With this observation, the lifts \(\tilde{\pi}_j\) are simply the fiber product projections \((\mathcal{V}^{(\ell)})^{[k]} \to (\mathcal{V}^{(\ell)})^{[k-1]}\), defined pointwise by \((y_0, \ldots, y_{k-1}) \mapsto (y_0, \ldots, \tilde{y}_j, \ldots, y_{k-1})\), with \(\tilde{\cdot}\) denoting omission of the corresponding variable. The section \(\tilde{s}_1 : \text{Et}((\mathcal{V}^{[k-1]})^{(\ell)}) \to \text{Et}((\mathcal{V}^{[k]})^{(\ell)})\) is defined by \(\tilde{s} \circ \pi \times 1\) (where \(\tilde{\pi} : (\mathcal{V}^{[k-1]} \to \mathcal{U}\) and \(\tilde{s} : \mathcal{U} \to \mathcal{V}\) acting pointwise by

\[ (y_0, \ldots, y_{k-2}) \mapsto (\tilde{s}(\pi(y_0)), y_0, \ldots, y_{k-2}) \]

and \(\text{Et}((\mathcal{V}^{[k-1]})^{(\ell)}) \to \text{Et}((\mathcal{V}^{[k]})^{(\ell)})\) is defined similarly by \(\tilde{s}_\ell \circ \tilde{\pi} \times 1\), using the identification \((\mathcal{V}^{[k]})^{(\ell)} \cong (\mathcal{V}^{(\ell)})^{[k]}\), from which (1.7) follows directly. \(\square\)

**Lemma 1.6.** Given an arbitrary cover \(\mathcal{O}\) of \(Y^{[k]}\), there exists an admissible pair of covers \((\mathcal{U}, \mathcal{V})\) for \(\pi : Y \to X\) such that \(\mathcal{V}^{[k]}\) refines \(\mathcal{O}\). Thus the covers of the form \(\mathcal{V}^{[k]}\) for admissible \((\mathcal{U}, \mathcal{V})\) are final in the directed set of all covers on \(Y^{[k]}\), and in particular every cohomology class in \(\tilde{H}^*(Y^{[k]}; A) = \lim_{\mathcal{O}} \tilde{H}^*_S(Y^{[k]}; A)\) has a representative in \(C^\bullet_{\mathcal{V}^{[k]}}(Y^{[k]}; A)\) for some admissible pair.

**Proof.** Since \(Y^{[k]}\) is topologized as a subspace of the product \(Y^k\), for each \(O \in \mathcal{O}\) there is some open \(\tilde{O} \subset Y^k\) in the product topology such that \(O = \tilde{O} \cap Y^{[k]}\), and \(\tilde{O}\) is a union of sets of the form \(O_1 \times \cdots \times O_k\) for open \(O_i \subset Y\). Taking the set of all such \(O_i\) over \(O \in \mathcal{O}\) gives a cover \(\mathcal{V}'\) of \(Y\) such that \(\mathcal{V}' \times_X \cdots \times_X \mathcal{V}'\) is a refinement of \(\mathcal{O}\), and then invoking Lemma 1.4 produces an admissible pair such that \(\mathcal{V}^{[k]} = \mathcal{V} \times_U \cdots \times_U \mathcal{V}\) refines \(\mathcal{V}' \times_X \cdots \times_X \mathcal{V}'\) and therefore \(\mathcal{O}\). \(\square\)

## 2. Bundle gerbes

### 2.1. Simplicial line bundles

We recall the notion of a bundle gerbe [16], which for our purposes is most efficiently defined in terms of simplicial line bundles.

Denote by \(Y^{[k]}\) the \(k\)-fold fiber product \(Y \times_\pi \cdots \times_\pi Y\), with projection maps \(\pi_j : Y^{[k]} \to Y^{[k-1]}, j = 0, 1, \ldots, k - 1\), where \(\pi_j(y_0, \ldots, y_{k-1}) = (y_0, \ldots, \tilde{y}_j, \ldots, y_{k-1})\) omits the \(j\)th factor enumerated from 0. Then

\[ X \leftarrow Y \leftrightarrow Y^{[2]} \leftrightarrow Y^{[3]} \ldots \tag{2.1} \]

is a simplicial space with face maps \(\pi_j\) and degeneracy maps the fiber diagonal maps \(Y^{[k-1]} \to Y^{[k]}\) (of which we will not make use). More precisely, \(Y^{[\bullet]}\) is a simplicial space over \(X\), meaning that all maps commute with the
projections $\pi : Y^{[k]} \to X$, and $X$ itself may be regarded as an augmentation in (2.1). For notational convenience, we set $Y^{[1]} = Y$ and $Y^{[0]} = X$, with $\pi_0 = \pi : Y^{[1]} \to Y^{[0]}$.

**Remark.** Our enumeration (which is geometrically natural here) differs unfortunately from the standard convention for simplicial spaces, under which one would typically write $Y_0 = Y$ (as the image of the 0 simplex), $Y_1 = Y^{[2]}$ (as the image of the 1-simplex), etc., augmented by $Y_{-1} = X$. For consistency we use this alternative convention throughout, and beg the pardon of readers who would prefer to use the standard one.

Given a complex line bundle $L \to Y^{[k]}$, its differential is defined to be the line bundle

$$dL := \bigotimes_{i=0}^{k} \pi_i^* L^{(-1)^i} \to Y^{[k+1]}.$$  \hspace{1cm} (2.2)

Using the commutation relations between the $\pi_j$, it follows that $d^2 L = d(dL)$ is canonically trivial over $Y^{[k+2]}$.

**Remark.** While we will mostly work with complex line bundles, we could equivalently take $L$ to be a principal $\mathbb{C}^*$ or $U(1)$ bundle. At times we will use these objects interchangeably without further elaboration.

A bundle gerbe $(L, Y, X)$ as defined by Murray is equivalent to a simplicial line bundle on the simplicial space $Y^{[*]}$ in the sense of Brylinski and McLaughlin [4]; this consists of a complex line bundle $L \to Y^{[2]}$ along with a trivialization of the bundle

$$dL = \pi_0^* L \otimes \pi_1^* L^{-1} \otimes \pi_2^* L$$

over $Y^{[3]}$, which in turn induces the canonical trivialization of $d^2 L$ when pulled back over $Y^{[4]}$. The trivialization of $dL \to Y^{[3]}$ is equivalent to the ‘gerbe (or groupoid) product’ isomorphism

$$\phi : \pi_2^* L \otimes \pi_0^* L \xrightarrow{\cong} \pi_1^* L,$$

which multiplies (composes) pairs of respective elements in the fibers $L_{y_0,y_1}$ and $L_{y_1,y_2}$ to get elements in $L_{y_0,y_2}$; the condition that the trivialization coincide with the canonical one on $d^2 L$ over $Y^{[4]}$ is equivalent to associativity of this product.

A bundle gerbe $(L, Y, X)$ is trivial if there exists a bundle $L' \to Y$ and an isomorphism $L \cong dL'$ on $Y^{[2]}$; such an isomorphism is called a trivialization of $L$.

If $(L, Y, X)$ is a bundle gerbe on $X$, then its pullback by a continuous map $f : X' \to X$ is the bundle gerbe $(f^* L, f^* Y, X')$; here we use the naturality $f^*(Y \times_X Y) \cong f^*(Y) \times_{X'} f^*(Y)$ and denote the resulting map $f^* Y^{[2]} \to Y^{[2]}$ by $\tilde{f}$. Likewise, the product of two bundle gerbes $(L_i, Y_i, X), i = 1, 2$ on $X$ is given by

$$(L_1 \otimes L_2, Y_1 \times_Y Y_2, X)$$
where \( L_1 \otimes L_2 \) is shorthand for \( \text{pr}_1^*L_1 \otimes \text{pr}_2^*L_2 \) with \( \text{pr}_i : Y_1 \times_X Y_2 \rightarrow Y_i \) denoting the projections from the fiber product. It is straightforward to verify that this is a bundle gerbe, which we denote for simplicity as \( L_1 \otimes L_2 \).

The definitions of product and pullback implicitly use the following standard result which we record for later use.

**Lemma 2.1.** Pullbacks and fiber products of locally split maps are locally split. More precisely, if \( \pi : Y \rightarrow X \) is locally split and \( f : X' \rightarrow X \) is continuous, then \( f^*Y \rightarrow X' \) is locally split, and if \( \pi' : Y' \rightarrow X \) is another locally split map, then \( \pi \times \pi' : Y \times_X Y' \rightarrow X \) is locally split.

More generally, a (strong) morphism \((L', Y', X') \rightarrow (L, Y, X)\) of bundle gerbes consists of a map \( \tilde{f} : Y' \rightarrow Y \) covering a map \( f : X' \rightarrow X \) along with an isomorphism \( L \cong L' \) over \( \tilde{f}^*[2] : Y'^*[2] \rightarrow Y^*[2] \) which intertwines the sections of \( dL \) and \( dL' \); a (strong) isomorphism is a morphism for which \( X = X' \) and \( f = \text{Id} \). In particular, a morphism \( f : (L', Y', X') \rightarrow (L, Y, X) \) is equivalent to an isomorphism \( (L', Y', X') \cong (f^*L, f^*Y, X') \).

Finally, two gerbes \((L_i, Y_i, X), i = 1, 2\) are said to be stably isomorphic if \( L_1 \otimes L_2^{-1} \) is trivial, or equivalently, there exist trivial gerbes \((T_i, Z_i, X)\) such that

\[
L_1 \otimes T_1 \cong L_2 \otimes T_2,
\]

in the sense of a strong isomorphism over a space \( Z^*[2] \) where \( Z \rightarrow X \) admits maps to \( Y_1, Y_2, Z_1, \) and \( Z_2 \). This is strictly weaker than an isomorphism as defined above, and was introduced in [17] in order to obtain a classification of bundle gerbes up to stable isomorphism by their Dixmier-Douady class.

**Remark.** There is a weaker notion of gerbe morphism due to Waldorf [26], which naturally incorporates the theory of gerbe modules, and has the property that the invertible morphisms are precisely the stable isomorphisms; moreover a trivialization becomes the same thing as an isomorphism to a canonical trivial gerbe over \( X \). Because we leave the generalization to bigerbes of Waldorf’s morphisms to a future work, we will not pursue this further here.

### 2.2. Simplicial Čech theory

A primary motivation for the definition of gerbes is that they represent a class in \( \tilde{H}^3(X; \mathbb{Z}) \cong \tilde{H}^2(X; \mathbb{C}^*) \) — the Dixmier-Douady class; for a bundle gerbe \((L, Y, X)\) this will be denoted by \( \text{DD}(L) \). This class is natural with respect to products, pullbacks, and inverses, and it determines \((L, Y, X)\) up to stable isomorphism [17]. We give an alternative (though not necessarily simpler) derivation of these facts based on a closer study of simplicial spaces; this approach is used in the generalization to bigerbes below. In doing so we identify the 3-classes on \( X \) which can be represented by a bundle gerbe with respect to a given locally split map \( Y \rightarrow X \), and recover the classification of the trivializations of a trivial bundle gerbe.
Consider the simplicial space, (2.1), consisting of the fiber products of a locally split map \( \pi : Y \to X \), and fix an admissible pair of covers \((U, V)\). Proposition 1.2 and Lemma 1.5 show that the induced maps \( \pi^*_j : \check{C}^\bullet_{Y[k]}(Y^{[k]}; A) \to \check{C}^\bullet_{Y[k+1]}(Y^{[k+1]}; A) \) are chain maps and the simplicial differential on Čech cochains is defined by

\[
d = \sum_{j=0}^{k} (-1)^j \pi^*_j : \check{C}^\bullet_{Y[k]}(Y^{[k]}; A) \to \check{C}^\bullet_{Y[k+1]}(Y^{[k+1]}; A),
\]

so \( d^2 = 0 \) and \( d\delta = \delta d \).

Thus, \((\check{C}^\bullet_{Y}(Y^{[\bullet]}; A), d, \delta)\) forms a double complex

\[
\begin{array}{ccc}
\vdots & \vdots & \vdots \\
\check{C}^0(Y^{[2]}) & \delta & \check{C}^1(Y^{[2]}) & \delta & \check{C}^2(Y^{[2]}) & \delta \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\check{C}^0(Y) & \delta & \check{C}^1(Y) & \delta & \check{C}^2(Y) & \delta \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\check{C}^0(X) & \delta & \check{C}^1(X) & \delta & \check{C}^2(X) & \delta \\
0 & 0 & 0 & & & \\
\end{array}
\]

where we have omitted the coefficient group and covers from the notation for simplicity. We take the row \( \check{C}^\bullet(Y) \) to have vertical degree 0 in (2.3) (corresponding to its true simplicial degree), so the row \( \check{C}^\bullet(X) \) has degree \(-1\). Related versions of this Čech-simplicial double complex appear more generally in algebraic geometry [7], and are also discussed in [4] in the context of simplicial gerbes.

**Convention 2.2.** Our convention for double (and higher) complexes is that the two differentials \( d \) and \( \delta \) commute, as above. This necessitates the introduction of a sign (depending on an ordering of the differentials) in the total differential, which we take to be

\[
D = \delta + (-1)^p d \quad \text{on} \quad \check{C}^p(Y^{[q]}; A).
\]

Another possible sign convention is given by changing the formal order of \( d \) and \( \delta \), namely \( D' = d + (-1)^{q+1}\delta \) (recalling that \( Y^{[q]} \) has vertical degree \( q - 1 \)). This is intertwined with \( D \) via the automorphism \((-1)^{p(q+1)}\) of the double complex.

In general, whenever we have a multicomplex \( C^{p_1,\ldots,p_k} \) with \( k \) commuting differentials \( d_1, d_2, \ldots, d_k \), the total differential will be defined inductively by

\[
D_k = D_{k-1} + (-1)^{p_1+\cdots+p_{k-1}}d_k
\]

\[
= d_1 + (-1)^{p_1}d_2 + (-1)^{p_1+p_2}d_3 + \cdots + (-1)^{p_1+\cdots+p_{k-1}}d_k.
\]

Switching the order of the indices requires composing with an automorphism given in each degree by an appropriate power of \(-1\) as above.
Proposition 2.3. For an admissible pair \((U, V)\), the simplicial chain complex

\[
0 \longrightarrow \check{C}^\ell_U(X; A) \xrightarrow{d} \check{C}^\ell_V(Y; A) \xrightarrow{d} \check{C}^\ell_{V[2]}(Y^{[2]}; A) \xrightarrow{d} \cdots \quad (2.4)
\]

is exact. In particular, a collection of local sections \(s : \text{Et}(U) \longrightarrow Y\) determines a chain homotopy contraction via the maps in Lemma 1.5.

Compare the exactness of the de Rham complex in §8 of [16]. This may be understood as a manifestation of the fact that the geometric realization of the simplicial set \(Y^{[\bullet]}\) is known to be homotopy equivalent to \(X\).

Proof. The chain contraction is defined by

\[
\tilde{s}_\ell^* : \check{C}^\ell_{V[k]}(Y^{[k]}; A) \longrightarrow \check{C}^\ell_{V[k-1]}(Y^{[k-1]}; A)
\]

for \(k \geq 2\), where \(\tilde{s}_\ell^*\) is the map of covers in Lemma 1.5.(iii), and at the bottom by \(\tilde{s}_\ell : \check{C}^\ell_V(Y; A) \longrightarrow \check{C}^\ell_U(X; A)\) where \(\tilde{s}_\ell\) is as in Definition 1.3.(iv). That \(\tilde{s}_\ell^*\) forms a chain homotopy retraction follows directly from (1.7). □

Remark. It is important that the maps \(s_\ell^*\) do not generally commute with the Čech differential; in other words, we do not obtain a chain map, and in particular we do not claim that \(s_\ell^*\) descends to cohomology. Indeed the lift, \(s_\ell\), of \(s\) in (1.6) corresponds to an (arbitrary) preference for the map corresponding to the first factor, \(U_1\), in the \(\ell\)-fold intersection.

2.3. Dixmier-Douady class of a gerbe. The setting for our analysis of the Čech cohomology of a bundle gerbe is the truncated complex

\[
\begin{array}{ccc}
\check{Z}_V^{0[2]}(Y^{[2]}; A) & \xrightarrow{\delta} & \check{Z}_V^{1[2]}(Y^{[2]}; A) \\
\uparrow d & & \uparrow d \\
\check{C}_V^0(Y; A) & \xrightarrow{\delta} & \check{C}_V^1(Y; A) \\
\uparrow d & & \uparrow d \\
0 & & 0
\end{array}
\quad (2.5)
\]

where we have omitted the bottom row of (2.3), and where

\[
\begin{align*}
\check{Z}_V^{\ell[2]}(Y^{[2]}; A) & := \text{Ker} \left\{ d : \check{C}_V^{\ell[2]}(Y^{[2]}; A) \longrightarrow \check{C}_V^{\ell[3]}(Y^{[3]}; A) \right\} \\
& = \text{Im} \left\{ d : \check{C}_V^{\ell}(Y; A) \longrightarrow \check{C}_V^{\ell[2]}(Y^{[2]}; A) \right\}.
\end{align*}
\]

Denote by

\[
\check{H}_Z^\bullet(Y^{[2]}; A) := \lim_{\mathcal{V}[2]} H^\bullet_{\mathcal{V}[2]}(\check{Z}(Y^{[2]}; A), \delta)
\]

the Čech cohomology of the simplicially trivial classes on \(Y^{[2]}\), or the horizontal cohomology of the top row in (2.5), taken in the direct limit over covers of the form \(\mathcal{V}^{[2]}\) for admissible pairs. For later use we note the following result.

Lemma 2.4. There is a natural Bockstein isomorphism \(\check{H}_Z^\bullet(Y^{[2]}; \mathbb{C}^*) \cong \check{H}_Z^{\bullet+1}(Y^{[2]}; \mathbb{Z})\).
Proof. Regarding the chain complexes \( \check{Z}^\bullet_{\mathcal{V}[2]}(Y[2]; A) \) for an abelian group \( A \) as the image under \( d \) of \( \check{C}^\bullet(Y; A) \), it follows both that the coefficient sequence
\[
0 \longrightarrow \check{Z}^\bullet_{\mathcal{V}[2]}(Y[2]; \mathbb{Z}) \longrightarrow \check{Z}^\bullet_{\mathcal{V}[2]}(Y[2]; \mathbb{C}) \xrightarrow{exp(2\pi i \cdot \cdot \cdot)} \check{Z}^\bullet_{\mathcal{V}[2]}(Y[2]; \mathbb{C}^*) \longrightarrow 0 \quad (2.6)
\]
is short exact, and that \( \check{Z}^\bullet_{\mathcal{V}[2]}(Y[2]; \mathbb{C}) \) is acyclic, from which the long exact sequence for \( (2.6) \) degenerates to the Bockstein isomorphism. \( \square \)

**Theorem 2.5.** The total cohomology of the double complex \((2.5)\) is isomorphic to \( \check{H}^\bullet_{\mathcal{U}}(X; A) \).

**Proof.** Owing to exactness of the columns, the \((d, \delta)\) spectral sequence of \((2.5)\) degenerates at the \( E_1 \) page to
\[
\begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & 0 \\
\check{C}^0_{\mathcal{U}}(X) & \delta & \check{C}^1_{\mathcal{U}}(X) & \delta & \check{C}^2_{\mathcal{U}}(X) & \delta \\
\end{array}
\]
and therefore stabilizes at \( E_2 \) to the cohomology \( \check{H}^\bullet_{\mathcal{U}}(X; A) \). \( \square \)

Next we will show that a bundle gerbe is represented by a pure cocycle in the double complex \((2.5)\) concentrated at \( \check{Z}^1(Y[2]; \mathbb{C}^*) \), so with \( \delta c(L) = 0 \) and \( dc(L) = 0 \) and this descends to the Dixmier-Douady class.

**Proposition 2.6.** A bundle gerbe \((L, Y, X)\) has Chern class represented by \( c(L) \in \check{Z}^1_{\mathcal{V}[2]}(Y[2]; \mathbb{C}^*) \) for some admissible pair of covers \((\mathcal{U}, \mathcal{V})\); in particular,
\[
c(L) \in \check{H}^1_Z(Y[2]; \mathbb{C}^*) \cong \check{H}^2_Z(Y[2]; \mathbb{Z}). \quad (2.7)
\]
Conversely, any such class determines a bundle gerbe, and \( L \) admits a trivialization if and only if \( [c(L)] \in d\check{H}^1(Y; \mathbb{C}^*) \).

**Proof.** As a complex line bundle, \( L \) gives rise to a Chern cocycle \( c(L) \in \check{C}^1_{\mathcal{O}}(Y[2], \mathbb{C}^*) \) for some cover \( \mathcal{O} \) of \( Y[2] \), which by Lemma 1.5 may be assumed to be of the form \( \mathcal{O} = \mathcal{V}[2] \) for an admissible pair, and \( dc(L) \in \check{C}^1_{\mathcal{V}[3]}(Y[3], \mathbb{C}^*) \) represents the bundle \( dL \) on \( Y[3] \). The trivialization of \( dL \) is encoded by \( \gamma \in \check{C}^0_{\mathcal{V}[3]}(Y[3], \mathbb{C}^*) \) such that \( \delta \gamma = dc(L) \), and the fact that this induces the canonical trivialization of \( d^2L \) on \( Y[4] \) means that \( d^2 = 0 \). Thus by exactness we can alter \( c(L) \) by \( \delta \) applied to a \( d \)-pre-image of \( \gamma \) to arrange that \( dc(L) = 0 \).

Altering \( c(L) \in \check{Z}^1_{\mathcal{V}[3]}(Y[3]; \mathbb{C}^*) \) by \( \delta \beta \) for \( \beta \in \check{Z}^0_{\mathcal{V}[2]}(Y[2]; \mathbb{C}^*) \) amounts to applying an automorphism to \( L \longrightarrow Y[2] \) which does not change the trivialization of \( dL \longrightarrow Y[3] \), so the Chern class in \( \check{H}^1_Z(Y[2]; \mathbb{C}^*) \) is well-defined.

Conversely, given \( \alpha \in \check{Z}^1_{\mathcal{V}[2]}(Y[2]; \mathbb{C}^*) \) representing a cocycle with respect to some fixed open cover \( \mathcal{V}[2] \) of \( Y[2] \) associated to an admissible pair, the
usual construction uses $\alpha$ on $(V^{[2]})^{(2)}$ to assemble a line bundle $L \rightarrow Y^{[2]}$ out of trivial bundles on $V \in V^{[2]}$. Then since $d\alpha = 0$ it follows that $dL$ is assembled trivially out of trivial bundles on the open cover $V^{[3]}$ of $Y^{[3]}$, and hence is globally trivial (with the trivialization agreeing with the canonical one on $d^2L$).

Finally, $L$ admits a trivialization $L \cong dQ$ for some $Q \rightarrow Y$, if and only if $c(L) = dc(Q) \in \check{H}^2(Y^{[2]}; \mathbb{C}^*)$ where $c(Q) \in H^1_Y(V; \mathbb{C}^*)$ is the Chern class of $Q$. □

**Definition 2.7.** The Dixmier-Douady class of a bundle gerbe $(L, Y, X)$ is the image $\text{DD}(L) \in \check{H}^2(X; \mathbb{C}^*) = \lim U \check{H}^2(U)(X; \mathbb{C}^*) \cong \check{H}^3(X; \mathbb{Z})$ of the hypercohomology class of $c(L) \in \check{Z}_V^{1}(Y^{[2]}; \mathbb{C}^*)$ in the double complex (2.5) and is obtained explicitly by a zig-zag construction

\[
\begin{array}{c}
0 \\
1 \\
\downarrow \\
\beta \\
\downarrow d \\
\delta \\
\downarrow \\
\downarrow d \delta \\
\downarrow \\
\text{DD}(L) \\
0
\end{array}
\]

The sign in $-d\beta = c(L)$ arises from the fact that the total differential $D$ involves the term $-d$ on that column according to Convention 2.2.

Note that if $\pi_i : Y_i \rightarrow X_i$, $i = 1, 2$ are locally split maps which are intertwined by $f : X_1 \rightarrow X_2$ and $\tilde{f} : Y_1 \rightarrow Y_2$, then Čech cochain maps determined by $\tilde{f}^{[k]} : Y_1^{[k]} \rightarrow Y_2^{[k]}$ as in Proposition 1.2 together form a morphism $\check{C}^\bullet(Y_2^{[*]}; A) \rightarrow \check{C}^\bullet(Y_1^{[*]}; A)$ of double complexes, in that the $(\tilde{f}^{[k]})^*$ commute with $\delta$ and $d$. Indeed, we may fix an admissible pair $(U_2, V_2)$ for $(Y_2, X_2)$ and then invoke Lemma 1.4 to obtain a refinement of $(f^{-1}U_2, \tilde{f}^{-1}V_2)$ by an admissible pair $(U_1, V_1)$, and then it follows that there are maps of covers $\tilde{f}^{[k]} : \text{Et}(V_1^{[k]}) \rightarrow \text{Et}(V_2^{[k]})$ the pullbacks of which intertwine the two double complexes.

**Proposition 2.8.** The Dixmier-Douady class as defined above coincides with the definition given by Murray and is natural with respect to inverse, product, and pullback; it vanishes if and only if the gerbe is trivial.

**Proof.** The (well-known) naturality properties follow directly from the preceding remark. To see the coincidence of our definition with that of Murray given in [16], we first recall the latter.

Suppose $s : U \rightarrow Y$ is a set of local sections of the locally split map, and consider the pullback $L' = (s^2)^*L$ to $U^{(2)}$ of $L$ via the map $s^2 : U^{(2)} \rightarrow Y^{(2)}$. Since $L$ is locally trivial, this cover can be refined so that $L'$ is trivial over each component, and so has a nonvanishing section $\sigma : U^{(2)} \rightarrow L'$. The
trivialization of $dL \to Y^{[3]}$ pulls back to give a trivialization of $\delta L' = (s^3)^*dL \to U^{(3)}$ which allows $g := \delta \sigma$ to be regarded as a cochain $g : U^{(3)} \to \mathbb{C}^*$ and the associativity condition over $Y^{[4]}$ implies that $g$ is closed, hence $[g] \in \check{H}^2_U(X; \mathbb{C}^*) \cong H^3(X; \mathbb{Z})$ is defined to be the Dixmier-Douady class.

To see that this is equivalent to Definition 2.7, it suffices to show that $[g]$ represents the image of $c(L)$ in the total cohomology of the double complex $(\check{C}^\bullet(U^\ell)); \mathbb{C}^*)$, where we use $Et(U) \to X$ itself as the locally split map. For convenience we suppose that $U$ is a ‘good cover’, meaning that each element of $U^{(\ell)}$ is contractible for each $\ell$. Note that by this contractibility, the Čech cohomology $\check{H}^\bullet(U^{(\ell)}); \mathbb{C}^*)$ of the space $Et(U^{(\ell)})$ is trivial except in degree 0 where

$$\check{H}^0(Et(U^{(\ell)}); \mathbb{C}^*) = \Gamma(U^{(\ell)}; \mathbb{C}^*) = \check{C}^{\ell-1}_U(X; \mathbb{C}^*).$$

Thus the $(\delta, d)$ spectral sequence of the double complex

\[
\begin{array}{cccc}
\check{C}^0(Et(U^{(3)})) & \delta & \check{C}^1(Et(U^{(3)})) & \delta & \check{C}^2(Et(U^{(3)})) \\
\check{C}^0(Et(U^{(2)})) & \delta & \check{C}^1(Et(U^{(2)})) & \delta & \check{C}^2(Et(U^{(2)})) \\
\check{C}^0(Et(U)) & \delta & \check{C}^1(Et(U)) & \delta & \check{C}^2(Et(U))
\end{array}
\]

degenerates at the $E_1$ page to

\[
\begin{array}{cccc}
\check{C}^0(Et(U^{(3)})) & \delta & \check{C}^1(Et(U^{(3)})) & \delta & \check{C}^2(Et(U^{(3)})) \\
\check{C}^0(Et(U^{(2)})) & \delta & \check{C}^1(Et(U^{(2)})) & \delta & \check{C}^2(Et(U^{(2)})) \\
\check{C}^0(Et(U)) & \delta & \check{C}^1(Et(U)) & \delta & \check{C}^2(Et(U))
\end{array}
\]

degenerates at $E_1$ page to

\[
\begin{array}{cccc}
\check{C}^0(Et(U^{(3)})) & \delta & \check{C}^1(Et(U^{(3)})) & \delta & \check{C}^2(Et(U^{(3)})) \\
\check{C}^0(Et(U^{(2)})) & \delta & \check{C}^1(Et(U^{(2)})) & \delta & \check{C}^2(Et(U^{(2)})) \\
\check{C}^0(Et(U)) & \delta & \check{C}^1(Et(U)) & \delta & \check{C}^2(Et(U))
\end{array}
\]

with the simplicial differential now identified with the Čech differential on $\check{C}^\bullet_U(X)$, and then stabilizes at $E_2$ to give $\check{H}^0(U^\ell; \mathbb{C}^*) \cong H^{\ell+1}(X; \mathbb{Z})$. The image of $[L'] \in \check{C}^1(Et(U^{(2)})); \mathbb{C}^*)$ in the total cohomology $H^3(X; \mathbb{Z})$ is therefore equivalently represented by its image in $\check{H}^0(U^{(3)}; \mathbb{C}^*) = \check{C}^2_U(X; \mathbb{C}^*)$ on the $E_1$ page above, and Murray’s construction gives an explicit zig-zag

\[
\begin{array}{c}
0 \\
\uparrow \\
\uparrow \\
\sigma \quad [L'] \quad 0
\end{array}
\]

realizing $[g]$ as $\text{DD}(L)$. \qed
2.4. Representability of 3-classes. We proceed to give a characterization of the classes in $H^3(X; \mathbb{Z})$ which are represented by bundle gerbes $(L, Y, X)$ for a given locally split map $Y \to X$. Note that the augmented double complex

$$
\begin{array}{cccc}
0 & 0 & 0 \\
\delta & \delta & \delta \\
\delta & \delta & \delta \\
\delta & \delta & \delta \\
\delta & \delta & \delta \\
0 & 0 & 0
\end{array}
$$

for any admissible pair $(U, V)$ has exact columns and therefore trivial total cohomology. Since the $(\delta, d)$ spectral sequence of this complex (beginning with the horizontal differential) must necessarily stabilize at the $E_3$ page (as there are only three rows), it follows that the $E_2$ differentials are necessarily isomorphisms, which we record in the following form after passing to the direct limit over covers in cohomology.

**Theorem 2.9.** There are isomorphisms

$$\text{Ker} \{ \pi^* : \tilde{H}^{k+1}(X; A) \to \tilde{H}^{k+1}(Y; A) \} \cong \tilde{H}^k_Z(Y; Z)/\tilde{H}^k(Y; A) \quad (2.10)$$

for each $k \in \mathbb{N}$ and coefficient group $A$; these isomorphisms are natural with respect to pullback by maps

$$
\begin{array}{ccc}
Y_1 & \xrightarrow{\tilde{f}} & Y_2 \\
\downarrow \pi_1 & & \downarrow \pi_2 \\
X_1 & \xrightarrow{f} & X_2
\end{array}
$$

of locally split spaces, and also with respect to the Bockstein isomorphisms $\tilde{H}^{k+1}_Z(Y; \mathbb{Z}) \cong \tilde{H}^k_Z(Y; \mathbb{C}^*)$ and $\tilde{H}^{k+1}(X; \mathbb{Z}) \cong \tilde{H}^k(X; \mathbb{C}^*)$.

**Remark.** It is reasonable to call the isomorphism (2.10) transgression from classes on $X$ to (equivalence classes of) classes on $Y$. The map is realized at the chain level by the zig-zag (2.8).

In particular there is a natural isomorphism

$$\text{Ker} \{ \pi^* : \tilde{H}^3(X; Z) \to \tilde{H}^3(Y; Z) \} \cong \tilde{H}^2_Z(Y; \mathbb{Z})/\tilde{H}^2(Y; \mathbb{Z})$$

$$\cong \tilde{H}^2_Z(Y; \mathbb{C}^*)/\tilde{H}^1(Y; \mathbb{C}^*) \quad (2.11)$$

under which the Chern class $c(L) \in \tilde{H}^1_Z(Y; \mathbb{C}^*)/d\tilde{H}^1(Y; \mathbb{C}^*)$ is the image of $DD(L)$ in $\tilde{H}^3(X; \mathbb{Z})$. We see again that $DD(L) = 0$ if and only if $c(L) \in d\tilde{H}^1(Y; \mathbb{C}^*)$, which by Proposition 2.6 holds precisely when $L$ is trivial. In combination with Proposition 2.6 this proves the following result.
Theorem 2.10. A class $\alpha \in H^3(X; \mathbb{Z})$ is represented by a bundle gerbe $(L, Y, X)$ for a given locally split map $\pi : Y \to X$ if and only if $\pi^* \alpha = 0 \in H^3(Y; \mathbb{Z})$.

Remark. Another direct (and more geometric) way to show Theorem 2.10 is to use $\text{BPU}(H)$ as a $K(\mathbb{Z}, 3)$, where $H$ is an infinite dimensional separable Hilbert space. Here $\text{PU}(H) = U(H)/U(1)$ denotes the projective unitary group and by Kuiper’s theorem $U(H)$ is contractible, making $\text{PU}(H)$ a $K(\mathbb{Z}, 2)$. Thus $\alpha \in H^3(X; \mathbb{Z})$ is classified by a map (up to homotopy) to $\text{BPU}(H)$ and represented by a $\text{PU}(H)$ bundle $E \to X$. If $\pi^* \alpha = 0 \in H^3(Y; \mathbb{Z})$, it follows that $\pi^* E \to Y$ admits a global section $s : Y \to \pi^* E$. Then on $Y^{[2]}$ the shift map composed with $s^{[2]}$ determines a map $\chi : Y^{[2]} \to \text{PU}(H)$, along which the universal line bundle can be pulled back to give a simplicial bundle $L = \chi^* U(H) \to Y^{[2]}$ with $\text{DD}(L) = \alpha$.

2.5. Classification of trivializations. Suppose $(L, Y, X)$ is a trivial gerbe. There is an action on the set of trivializations of $L$ by $H^2(X; \mathbb{Z})$ (in the form of equivalence classes of line bundles) as follows. Given a line bundle $P \to Y$ trivializing $L$, so $dP \cong L$, and $\alpha \in H^2(X; \mathbb{Z})$ representing a line bundle $Q \to X$, the bundle $P \otimes \pi^* Q = P \otimes dQ \to Y$ is another trivialization of $L$ in light of the fact that $d^2 Q$ is canonically trivial.

Proposition 2.11. Let $(L, Y, X)$ be a trivial gerbe. Then the set of trivializations of $L$ is a torsor for the group $\text{Im} \{ \pi^* : H^2(X; \mathbb{Z}) \to H^2(Y; \mathbb{Z}) \}$.

Proof. Clearly the action of $H^2(X; \mathbb{Z})$ factors through its image in $H^2(Y; \mathbb{Z})$; to see that this image acts transitively suppose $P \to Y$ and $P' \to Y$ are two trivializations of $L$, represented by classes $[P], [P'] \in \tilde{H}^2(Y; \mathbb{Z})$. Then $d([P] - [P']) = 0 \in H^3_Z(Y^{[2]}; \mathbb{Z})$, and from the $(\delta, d)$ spectral sequence for (2.9), the $E_2$ term associated to $\tilde{C}^2(Y; \mathbb{Z})$ of which must vanish identically, it follows that

$$\text{Ker} \left\{ \delta : \tilde{H}^2(Y; \mathbb{Z}) \to \tilde{H}^3_Z(Y^{[2]}; \mathbb{Z}) \right\} = \text{Im} \left\{ \pi^* : \tilde{H}^2(X; \mathbb{Z}) \to \tilde{H}^2(Y; \mathbb{Z}) \right\},$$

(2.12)

and hence $[P] - [P'] = \pi^*[Q]$ for some $[Q] \in \tilde{H}^2(X; \mathbb{Z})$, represented by a line bundle $Q \to X$. \hfill $\square$

2.6. Decomposable and universal gerbes. One consequence of Theorem 2.10 is the existence of the decomposable gerbes of [14]. Given a 3-class on $X$ which is the cup product $\alpha \cup \beta$ of $\alpha \in H^2(X; \mathbb{Z})$ and $\beta \in H^1(X; \mathbb{Z})$, we may take $\pi : Y \to X$ to be the circle bundle with Chern class $c(Y) = \alpha$, and then since $Y$ is canonically trivial when pulled back to itself, it follows that $\pi^*(\alpha \cup \beta) = 0 \cup \pi^* \beta = 0 \in H^3(Y; \mathbb{Z})$, so by Theorem 2.10 the following is immediate.

Proposition 2.12. For every $\alpha \in H^2(X; \mathbb{Z})$ and $\beta \in H^1(X; \mathbb{Z})$, the circle bundle $Y \to X$ with $c(Y) = \alpha$ supports a bundle gerbe $(L, Y, X)$ with $\text{DD}(L) = \alpha \cup \beta$. 
Remark. Note that [14] goes further for $X$ a smooth manifold by constructing a connection on the gerbe from a connection on $Y$ and a function $u \in C^\infty(X; U(1))$ representing $\beta$.

In fact, the image of $\alpha \in H^2(X; \mathbb{Z})$ in $H^1_Z(Y^{[2]}; \mathbb{Z})/H^1(Y; \mathbb{Z})$ with respect to the isomorphism (2.11) has a geometric interpretation that will be of use in the construction of decomposable bigerbes in §5.1.

Lemma 2.13. Let $\pi : Y \to X$ be a circle bundle with $c(Y) = \alpha \in H^2(X; \mathbb{Z})$. Then the image of $\alpha$ under the isomorphism (2.11) coincides with the pullback to $Y^{[2]}$ of the generator of $H^1(U(1); \mathbb{Z})$ by the shift map

$$\chi : Y^{[2]} \to U(1), \quad y_2 = \chi(y_1, y_2)y_1 \quad \text{for} \quad (y_1, y_2) \in Y^{[2]}.$$  

Proof. This is easiest to see with $U(1)$ coefficients. With respect to the isomorphism $H^1(U(1); \mathbb{Z}) \cong H^0(U(1); U(1))$ the generator corresponds to the identity map, so it suffices to show that the image of $c(Y) \in H^1(X; U(1))$ is represented by $\chi$, itself regarded as a class in $H^0(Y^{[2]}; U(1))$.

Let $\alpha \in \hat{C}^1_U(X; U(1))$ represent $c(Y)$; explicitly, we may take $\alpha$ to be defined with respect to a cover $i : U \to X$ with respect to which $Y$ is (locally) trivialized by $h : i^*Y \to U \times U(1)$, and we may abuse notation to write $\alpha = \delta h$, meaning $\alpha : U^{(2)} \to U(1)$ is defined so that $\delta h = 1 \times \alpha : U^{(2)} \times U(1) \to U^{(2)} \times U(1)$. Now $\pi^*Y = Y^{[2]} \to Y$ is globally trivialized by $1 \times \chi$, to which $\pi^*h$ may be compared to write $\pi^*h = \gamma \chi$ for $\gamma \in \hat{C}^0_{\pi^{-1}U}(Y; U(1))$ and then

$$d\alpha = \pi^*\alpha = \delta \pi^*h = \delta \gamma \delta \chi = \delta \gamma \in \hat{C}^1_{\pi^{-1}U}(Y; U(1)).$$

Finally, it follows by a straightforward computation that $d\gamma = \chi^{-1}$ in $\hat{C}^0_{\pi^{-1}U}(Y^{[2]}; U(1))$, and then the result follows in observance of Convention 2.2. \$ \square$

It also follows from Theorem 2.10 that for a connected, locally contractible, space $X$, a gerbe can be constructed representing any integral three class using the (based) path fibration $P^3X \to X$. Indeed, the hypotheses on $X$ imply that the end-point map $P^3X \to X$ is locally split, and since $P^3X$ is contractible, any 3-class on $X$ vanishes when lifted to $P^3X$. The fiber product $P^3X$ may be identified with the based loop space $\Omega X$, and the isomorphism (2.11) takes the form

$$H^3(X; \mathbb{Z}) \cong \tilde{H}^3_Z(\Omega X; \mathbb{C}^*),$$

from which we recover the following well-known result.

Theorem 2.14. For a connected, locally contractible, space $X$ each $\alpha \in H^3(X; \mathbb{Z})$ corresponds to a unique bundle gerbe $L \to \Omega X$ (up to simplicial isomorphisms of the line bundle) with $\text{DD}(L) = \alpha$. 


This ‘canonical gerbe’ on the loop space goes back at least to Brylinski [5]. Murray defines a bundle gerbe version in [16] under the assumption that $X$ is 2-connected; a hypothesis which is removed in [6].

In particular, since $K(\mathbb{Z}, 3)$ may be realized as a CW complex, its path space carries a universal gerbe.

The simplicial structure on $\Omega X$ coming from $P[k]X$ is related to what has been called the fusion product in the literature [24, 27, 29, 12]. A point $\gamma = (\gamma_0, \gamma_1, \gamma_2) \in P[3]X$ consists of three paths with common endpoints and so defines three loops, $\ell_i = \pi_i \gamma \in \Omega X = P[2]X$, $i = 0, 1, 2$, by the simplicial maps, and we say $\ell_1 = (\gamma_0, \gamma_2)$ is the fusion product of $\ell_2 = (\gamma_0, \gamma_1)$ and $\ell_0 = (\gamma_1, \gamma_2)$.

A fusion structure on a line bundle $L \rightarrow \Omega X$ is a collection of associative isomorphisms

$$L_{\ell_1} \cong L_{\ell_2} \otimes L_{\ell_0}$$

for all such triples, which is equivalent to a simplicial line bundle structure on $L$ with respect to $P[\bullet]X$. In this language then, Theorem 2.14 shows that fusion line bundles on $\Omega X$, which are equivalent to bundle gerbes $(L, PX, X)$, are classified by $H^3(X; \mathbb{Z})$ (see also Waldorf’s related results in [27]).

### 3. Doubling and the free loop space

#### 3.1. Simplicial bundle gerbes and figure-of-eight

Replacing the simplicial line bundle in the definition of a bundle gerbe with a bundle gerbe over $X_2$ of a simplicial space $X_{\bullet}$ leads to the notion of a simplicial bundle gerbe, which has been defined by Stevenson [22] and is the setting for his definition of bundle 2-gerbes. Here we consider a more limited ‘product-simplicial’ version, which we call simply doubled, of this theory, not yet to obtain a version of 2-gerbes as we shall do in §4 below, but rather to promote the examples of bundle gerbes involving the based loop space $\Omega X$ to those involving the free (unbased) loop space $LX$ by satisfying an additional condition with respect to the simplicial space $\{X^k : k \in \mathbb{N}\}$ of products, with face maps the projections; this space is often denoted by $EX$ in the literature.

While we specialize to this simplicial space of products below, we proceed for the moment in some generality for an arbitrary simplicial space $X_{\bullet}$, where we continue to use our unusual enumeration convention. Suppose then that $(L, Y_2, X_2)$ is a bundle gerbe over $X_2$. Using products, inverses and pullbacks, we may define the gerbe

$$\partial L := \pi_0^* L \otimes \pi_1^* L^{-1} \otimes \pi_2^* L$$

over $X_3$, where $\pi_j : X_3 \rightarrow X_2$ for $j = 0, 1, 2$ are the face maps of the simplicial space.

**Definition 3.1.** A simplicial trivialization of a bundle gerbe $L$ over $X_2$ is a trivialization of the bundle gerbe $\partial L$ over $X_3$. It follows by naturality that
for such a gerbe the Dixmier-Douady class $\DD(L) \in H^3(X_2; \mathbb{Z})$ satisfies
\[ \partial \DD(L) := \sum_{j=0}^{2} (-1)^j \pi_j^* \DD(L) = \DD(\partial L) = 0 \in H^3(X_3; \mathbb{Z}). \] (3.1)

Note that the gerbe $\partial L$ is defined a priori with respect to the locally split map
\[ \pi_0^* Y_2 \times_{X_3} \pi_1^* Y_2 \times_{X_3} \pi_2^* Y_2 \to X_3. \] (3.2)

However, using the notion of gerbe morphism, we may specialize to the setting in which there exists a locally split map $Y_3 \to X_3$ for some fixed space $Y_3$, along with lifts $\tilde{\pi}_j : Y_3 \to Y_2$ of the $\pi_j : X_3 \to X_2$ for $j = 0, 1, 2$.

Indeed, it then follows that $Y_3$ maps through the product space (3.2), and we may require that
\[ \tilde{\pi}_0^* L \otimes \tilde{\pi}_1^* L^{-1} \otimes \tilde{\pi}_2^* L \to Y_3^{[2]} \] (3.3)
is trivial as a bundle gerbe over $X_3$, where we continue to denote the extensions of $\tilde{\pi}_j$ as maps from $Y_3^{[2]}$ to $Y_2^{[2]}$ by the same notation. When such data is available, we will abuse notation by referring to (3.3) itself as $\partial L$ (as these are (strongly) isomorphic as bundle gerbes over $X_3$) and a trivialization of (3.3) is a simplicial trivialization of $L$. Explicitly, this then is the data of a line bundle $S \to Y_3$ such that $dS \cong \partial L$, as summarized in the diagram
\[ \begin{array}{ccc}
L & \to & Y_2^{[2]} \equiv Y_3^{[2]} \leftarrow \partial L \cong dS \\
\downarrow & & \downarrow \\
Y_2 & \equiv & Y_3 \leftarrow S \\
\downarrow & & \downarrow \\
X_1 & \equiv & X_2 \equiv X_3.
\end{array} \] (3.4)

By naturality of the Dixmier-Douady class, the conclusion (3.1) remains valid.

**Remark.** We do not require that the split maps $Y_\bullet \to X_\bullet$ be compatible by the $\pi_j$ in the sense of §1.2. We also do not require that $Y_\bullet$ extend to form (part of) a simplicial space over $X_\bullet$, as indeed our example of interest will not. By contrast, in the setting of the biggeres defined in §4 below, we will employ a *bisimplicial space* of compatible locally split maps.

We now specialize to the case in which $X_\bullet = X^\bullet$ consists of products of a fixed space $X$. As a special case of the fiber product construction over the unique map $\pi : X \to *$ to a 1-point space, this map is globally split, with section $s : * \to x_\ast \in X$ for any choice of $x_\ast \in X$. The Čech theory constructions of §1.1 and §2.2 give a map $s^* : \check{C}^{\bullet}(X^k; A) \to \check{C}^{\bullet}_{k-1}(X^{k-1}; A)$ which in this case does commute with the Čech differential (since $s$ is global), and hence descends to a chain homotopy contraction for each $\ell$ of the cohomology complex
\[ 0 \to \check{H}^\ell(X; A) \to \check{H}^\ell(X^2; A) \to \check{H}^\ell(X^3; A) \to \cdots \] (3.5)
which is therefore exact. (This is a reflection of the well-known fact that the geometric realization $|EX|$ of the simplicial set $EX$ is contractible.) Indeed, denoting by $s = s \times 1 \times \cdots \times 1 : X^k \to X^{k+1}$ the map $(x_0, \ldots, x_{k-1}) \mapsto (x_s, x_0, \ldots, x_{k-1})$, it follows that $s^*\partial + \partial s^* = 1$ on $H^*(X^k; A)$. Note that throughout this section and below, we denote this product simplicial differential by $\partial = \sum_{j=0}^{k-1} (-1)^j \pi_j^*$ rather than $d$ to avoid confusion whenever both appear together.

As a consequence of (3.5) and (3.1), we have the following result.

**Proposition 3.2.** For a gerbe $(L, Y_2, X^2)$ with simplicial trivialization over $X^3$, the Dixmier-Douady class of $L$ descends from $X^2$ to $X$ itself, so

$$\text{DD}(L) \in H^3(X; \mathbb{Z})$$

is well-defined.

We refer to such a gerbe as a doubled gerbe.

**Remark.** When $X_\bullet = X^{[\bullet]}$ is a more general simplicial space of fiber products of a locally split map $X = X_1 \to X_0$, Stevenson in [22] defines additional conditions for a simplicial gerbe, including higher associativity conditions over $X_4$ and $X_5$ under which the class of a bundle gerbe further descends to a degree four cohomology class on $X_0$ and such an object is defined to be a bundle 2-gerbe on $X_0$. Here we only use the simplicial condition to descend the 3-class from $X^2 = X^2$ to $X_1 = X$, and will not make use of these additional conditions.

The locally split map of present interest consists of the free (unbased) path space

$$IX = C([0, 1]; X) \to X^2, \quad \gamma \mapsto (\gamma(0), \gamma(1)),$$

mapping to $X^2$ by the evaluation map on both endpoints. Instead of the fiber products of the pullbacks of $IX$ to $X^3$ we will take $Y_3 = IX$ also, with evaluation map

$$IX \to X^3, \quad \gamma \mapsto (\gamma(0), \gamma(\frac{1}{2}), \gamma(1)),$$

mapping to the midpoint as well as the endpoints. For disambiguation, we will often distinguish these two incarnations of the free path space by writing them as $I_2X$ and $I_3X$, respectively. The three liftings $\tilde{\pi}_j$ of the projection maps $\pi_j : X^3 \to X^2$ taking $\gamma \in I_3X$ to $I_2X$ are obtained by reparameterizing to obtain the three paths

$$\tilde{\pi}_1 \gamma(t) = \gamma(t), \quad \tilde{\pi}_2 \gamma(t) = \gamma(\frac{t}{2}), \quad \text{and} \quad \tilde{\pi}_0 \gamma(t) = \gamma(\frac{1}{2} + t).$$

**Remark.** While it is possible to continue to the right, with $Y_n = I_nX$ mapping to $X^n$ by evaluating along $n$ points, the need to reparameterize paths to define the lifts $\tilde{\pi}_j$ means that these do not satisfy the simplicial relations, so we do not in fact obtain simplicial spaces $Y_\bullet^{[k]}$ over $X_\bullet$. In particular, the associated maps $\partial$ on line bundles do not form a complex, i.e., $\partial^2 L$ is not canonically trivial, except at the bottom level.
Observe that \( Y_2 = I_2 X \) may be naturally identified with the free loop space, \( LX = C(\mathbb{R} / 2\pi \mathbb{Z}; X) \), while the space \( Y_3 = I_3 X \) consists of pairs of paths which coincide at their midpoint in addition to their endpoints. The latter may be identified with those loops \( \ell \) in \( LX \) for which \( \ell(\pi / 2) = \ell(3\pi / 2) \), which we call figure-of-eight loops, and we accordingly denote the figure-of-eight loop (sub)space by
\[
L_8 X \cong I_3 X.
\]

In fact, in this case the product doubling condition for a gerbe over \( X^2 \) can be strengthened.

**Lemma 3.3.** A gerbe \((L, IX, X^2)\) or \((L, IX, X^3)\) is trivial if and only if \( L \rightarrow LX \) (resp. \( L \rightarrow L_8 X \)) is trivial as a line bundle. In particular, a gerbe \((L, IX, X^2)\) is doubled if and only if \( \partial L \rightarrow L_8 X \) is a trivial line bundle.

**Proof.** Retraction of paths onto their initial points determines a deformation retract of \( I_k X \) onto \( X \), with respect to which the two simplicial maps
\[
I_k X \longrightarrow I_k X
\]
both become identified with the evaluation map \( I_k X \rightarrow X \) at a single parameter value. Thus every line bundle \( P \rightarrow I_k X \) is isomorphic to a bundle \( Q \) pulled back from \( X \), and then \( dP \cong Q \otimes Q^{-1} \rightarrow I_k X \) is isomorphic to a trivial bundle. This result is independent of \( k \).

Alternatively, we may use the equality (2.12) proved in Proposition 2.11, which here takes the form
\[
\text{Ker} \left\{ d : H^2(I_k X; \mathbb{Z}) \cong H^2(X; \mathbb{Z}) \longrightarrow H^2(I_k X; \mathbb{Z}) \right\} = H^2(X; \mathbb{Z}),
\]

since with respect to the retraction \( IX \simeq X \) the map \( \pi : I_k X \rightarrow X^k \) is identified with the diagonal map \( \Delta : X \rightarrow X^k \). Since \( \Delta^* \) is surjective on cohomology, it follows that \( d \equiv 0 : H^2(I_k X; \mathbb{Z}) \rightarrow H^2(I_k X; \mathbb{Z}) \), so every trivial gerbe is in fact trivial as a line bundle.

**Remark.** In other words, there are no ‘nontrivial trivial gerbes’ with respect to the path spaces. This seems at first confusing in light of Proposition 2.11, since the classifying set \( \text{Im} \left\{ \pi^* : H^2(X^2; \mathbb{Z}) \rightarrow H^2(IX; \mathbb{Z}) \right\} \cong H^2(X; \mathbb{Z}) \) of gerbe trivializations may well be nontrivial, yet these facts are not inconsistent. Indeed, while the only trivial gerbe with respect to \( IX \rightarrow X^2 \) is the equivalence class of the trivial line bundle on \( LX \), the set of gerbe trivializations of this trivial gerbe may itself be nontrivial.

On the other hand, we could restrict consideration to doubled trivializations, meaning line bundles \( P \rightarrow I_2 X \) with \( dP = L \) such that \( \partial P \rightarrow I_3 X \) is a trivial bundle. The set of these doubled trivializations is indeed trivial, since under the retractions \( I_k X \simeq X \), the reparameterization maps
\[ \tilde{\pi}_j : I_3 X \to I_2 X \] become the identity, and the operator \( \partial \cong \text{Id}^* - \text{Id}^* + \text{Id}^* \) likewise becomes the identity, so triviality of \( \partial P \) implies triviality of \( P \) itself.

The extension of this notion of doubling will be important in the setting of the Brylinski-McLaughlin bigerbe in §5.3.

From the point of view of fusion line bundles on loop space, the doubling property corresponds to the ‘figure-of-eight’ condition, as defined in [12, 11]. The following definition is therefore just a repackaging of the above in a different language.

**Definition 3.4.** A loop-fusion structure on a line bundle \( L \to LX \) is a fusion structure, meaning a trivialization of \( dL \to I_3^3 X \) inducing the canonical trivialization of \( d^2 L \to I_2^4 X \), along with the figure-of-eight condition that \( \partial L \to L_8X \cong I_3^2 X \) is trivial as a line bundle. An isomorphism of loop-fusion line bundles is a line bundle isomorphism which intertwines the fusion structures.

**Theorem 3.5.** The following are naturally in bijection:

(i) The set of doubled gerbes \((L, IX, X^2)\) up to strong isomorphism.

(ii) The set of loop-fusion line bundles on \( LX \) up to isomorphism.

(iii) \( H^3(X; \mathbb{Z}) \).

**Proof.** Equivalence of the first two is a consequence of Lemma 3.3 and Definition 3.4. Doubled gerbes \((L, IX, X^2)\) are classified by their Dixmier-Douady class, which descends to \( X \), as noted above, and that every element in \( H^3(X; \mathbb{Z}) \) is represented by a doubled gerbe (equivalently, loop-fusion line bundle) \( L \to LX \) follows from Theorem 3.6 below.

**Remark.** The figure-of-eight structure is weaker than other conditions that have been considered in the literature, such as thin homotopy equivariance in [27], or reparameterization equivariance in [11], which likewise identify categories of fusion line bundles on \( LX \) with gerbes on \( X \).

### 3.2. Loop-fusion cohomology.

In fact, applying the above considerations to Čech theory in place of line bundles leads to a general result, which recovers the main theorem in our previous paper [12]. There we defined loop-fusion cohomology on \( LX \), which in the present language is equivalent to the group

\[ \tilde{H}_\ell^X(LX; A) = \text{Ker}\left\{ \partial : \tilde{H}_Z^\ell(LX; A) \to \tilde{H}_Z^\ell(L_8X; A) \right\}. \]

In particular, the set \( \tilde{H}_\ell^2(LX; \mathbb{Z}) \) classifies loop-fusion line bundles up to isomorphism.

**Theorem 3.6 ([12]).** For each \( \ell \in \mathbb{N} \) and topological abelian group \( A \), there is an isomorphism

\[ \tilde{H}^\ell(X; A) \cong \tilde{H}_\ell^{\ell-1}(LX; A). \]
It is additionally shown in [12] that the isomorphism descends via the forgetful map \( \hat{H}^\bullet(LX; A) \to \hat{H}^\bullet(LX; A) \) to the transgression homomorphism \( \hat{H}^\ell(X; A) \to \hat{H}^{\ell-1}(LX; A) \); recall that the latter is defined by composing the pullback along the evaluation map \( S^1 \times LX \to LX \) with the pushforward along the projection \( S^1 \times LX \to LX \) (given by cap product with the fundamental class of \( S^1 \)).

**Proof.** The result follows from naturality of the isomorphism (2.10), and exactness of (3.5). Applied to the three maps from \( I^k_3 X \) to \( I^k_2 X \) this yields an isomorphism (omitting the coefficient group for brevity)

\[
\Ker \partial \cap \Ker \pi^* \subset \hat{H}^{\ell}(X^2)
\]

(3.6)

However, as noted in the proof of Lemma 3.3, the deformation retraction of the free path spaces \( I^k X \) onto \( X \) implies that \( d : \hat{H}^{\ell-1}(I^k X; A) \to \hat{H}^{\ell-1}(I^k X; A) \) is trivial, so the quotients in (3.6) disappear. Moreover, by exactness of (3.5), the kernel of \( \partial \) in \( \hat{H}^{\ell}(X^2; A) \) is the image of \( \hat{H}^{\ell}(X; A) \) and this is automatically in the kernel of \( \pi^* \) under the retraction \( I^k X \simeq X \), so (3.6) simplifies to

\[
\hat{H}^{\ell}(X; A) \cong \Ker \left\{ \partial : \hat{H}^{\ell-1}(I^2_2 X; A) \to \hat{H}^{\ell-1}(I^2_3 X) \right\} = \hat{H}^{\ell-1}(LX; A)
\]

as claimed. \qed

4. **Bundle bigerbes**

4.1. **Locally split squares.** Bigerbes as introduced below are based on the following notion.

**Definition 4.1.** A commutative diagram

\[
\begin{array}{ccc}
Y_2 & \leftarrow & W \\
\pi_2 \downarrow & & \downarrow \\
X & \leftarrow & Y_1
\end{array}
\]

(4.1)

is a **locally split square** if \( Y_i \to X \), \( i = 1, 2 \) and the induced map \( W \to Y_1 \times_X Y_2 \) are locally split.

There is manifest symmetry in the definition. Note that in fact all four maps in (4.1) are locally split; indeed, if \( s_1 : \tun{U} \to Y_1 \) are local sections of \( \pi_1 \) for some cover \( \tun{U} \) of \( X \), then \( 1 \times (\tilde{s}_1 \circ \pi_2) : \tun{U}^{-1} \to Y_1 \times_X Y_2 \) form local sections of the projection \( Y_1 \times_X Y_2 \to Y_2 \); moreover, refining this cover \( \pi_2^{-1} \) of \( Y_2 \) if necessary, these may then be composed with local sections of \( W \to Y_1 \times_X Y_2 \) to obtain local sections of \( W \to Y_2 \), and similarly for \( Y_1 \).
Definition 4.2. Given a locally split square as above, an **admissible set of covers** is a set of covers \( \mathcal{U}_X, \mathcal{U}_Y, \mathcal{U}_Z, \) and \( \mathcal{U}_W \) for \( X, Y, Z, \) and \( W, \) respectively, for which each pair \((\mathcal{U}_X, \mathcal{U}_Y), (\mathcal{U}_X, \mathcal{U}_Z), (\mathcal{U}_Y, \mathcal{U}_W), \) and \((\mathcal{U}_Z, \mathcal{U}_W)\) is an admissible pair and such that the following four diagrams commute:

\[
\begin{array}{ccc}
\text{Et}(\mathcal{U}_Y) & \leftarrow & \text{Et}(\mathcal{U}_W) \\
\downarrow & & \downarrow \\
\text{Et}(\mathcal{U}_X) & \leftarrow & \text{Et}(\mathcal{U}_Y)
\end{array} \quad \begin{array}{ccc}
\text{Et}(\mathcal{U}_Y) & \rightarrow & \text{Et}(\mathcal{U}_W) \\
\uparrow & & \uparrow \\
\text{Et}(\mathcal{U}_X) & \rightarrow & \text{Et}(\mathcal{U}_Y)
\end{array}
\]

\[
\begin{array}{ccc}
\text{Et}(\mathcal{U}_Y) & \rightarrow & \text{Et}(\mathcal{U}_W) \\
\downarrow & & \downarrow \\
\text{Et}(\mathcal{U}_X) & \rightarrow & \text{Et}(\mathcal{U}_Y)
\end{array} \quad \begin{array}{ccc}
\text{Et}(\mathcal{U}_Y) & \leftarrow & \text{Et}(\mathcal{U}_W) \\
\uparrow & & \uparrow \\
\text{Et}(\mathcal{U}_X) & \leftarrow & \text{Et}(\mathcal{U}_Y)
\end{array}
\]

Lemma 4.3. Given any set of covers for the spaces in a locally split square \((X, Y_1, Y_2, W)\), there exist refinements constituting an admissible set.

In fact, given any finite set of locally split squares over \( X \) which may share some of the same maps \( \pi_i : Y_i \rightarrow X \) and any initial set of covers of these, there exists a set of covers for all spaces which forms an admissible set for each square separately, and for which the section maps lift any initially prescribed local sections of the maps \( Y_i \rightarrow X \) or \( W \rightarrow Y_1 \times Y_2 \).

Proof. Let \( \mathcal{U}'_{\bullet} : \bullet \in \{X, Y_1, Y_2, W\} \) be the initial arbitrary set of covers for a given square. Without loss of generality by the stronger statement of Lemma 1.4, we may assume that \((\mathcal{U}'_X, \mathcal{U}'_Y), i = 1, 2\) form an admissible pair lifting prescribed local sections \( s_1 \) and \( s_2 \) of \( \pi_1 \) and \( \pi_2 \). Under this condition, it follows that \( \mathcal{U}'_{\bullet} \) support sections \( \sigma_i \) of the fiber product projections \( Y_1 \times_X Y_2 \rightarrow Y_i \) defined by \( \sigma_1 := 1 \times (s_2 \circ \pi_1) \) and \( \sigma_2 := (s_1 \circ \pi_2) \times 1 \), respectively (here \( 1 \) denotes the inclusion of the cover \( : \mathcal{U}'_i \rightarrow Y_i \)), giving a diagram

\[
\begin{array}{ccc}
\text{Et}(\mathcal{U}'_X) & \xrightarrow{\sigma_2} & Y_1 \times X Y_2 \\
\downarrow \tilde{\sigma}_2 & & \downarrow \tilde{\sigma}_1 \\
\text{Et}(\mathcal{U}'_X) & \xrightarrow{\tilde{\sigma}_1} & \text{Et}(\mathcal{U}'_X)
\end{array}
\]

for which the following three squares commute:

\[
\begin{array}{ccc}
\text{Et}(\mathcal{U}'_X) & \rightarrow & Y_1 \times X Y_2 \\
\uparrow & & \uparrow \\
\text{Et}(\mathcal{U}'_X) & \rightarrow & \text{Et}(\mathcal{U}'_X)
\end{array} \quad \begin{array}{ccc}
\text{Et}(\mathcal{U}'_X) & \rightarrow & Y_1 \times X Y_2 \\
\downarrow & & \downarrow \\
\text{Et}(\mathcal{U}'_X) & \rightarrow & \text{Et}(\mathcal{U}'_X)
\end{array} \quad \begin{array}{ccc}
Y_2 & \leftarrow & Y_1 \times X Y_2 \\
\uparrow & & \uparrow \\
Y_1 & \leftarrow & \text{Et}(\mathcal{U}'_X)
\end{array}
\]

We also fix an initial cover \( \mathcal{U}'_{Y_1 \times X Y_2} \) of \( Y_1 \times X Y_2 \) which supports a prescribed section \( t : \text{Et}(\mathcal{U}'_{Y_1 \times X Y_2}) \rightarrow W \) of the locally split map \( p : W \rightarrow Y_1 \times X Y_2 \).

This is the starting point, and we set \( \mathcal{U}'_{\bullet} = \mathcal{U}'_{\bullet} \) for \( \bullet \in \{X, Y_1, Y_2, Y_1 \times X Y_2, W\} \), where the superscript denotes the step in the process. As in the proof of Lemma 1.4, we proceed to refine the covers in two steps, first “pulling down” by the sections, and then “pulling up” by the projections.
**Step 1:** Starting at the top right, we subsequently set \( U^1_\nu = U^0_\nu \), \( U^1_{Y_1 \times X Y_2} = U^0_{Y_1 \times X Y_2} \cap t^{-1}(U^0_Y) \), \( U^1_{Y_1} = U^0_{Y_1} \cap \sigma^{-1}_i(U^1_{Y_1 \times X Y_2}) \), and
\[
U^1_X = U^0_X \cap (\sigma_1 \circ \tilde{s}_1)^{-1}(U^1_{Y_1 \times X Y_2}) \cong U^0_X \cap \tilde{s}_1^{-1}(U^1_{Y_1}) \cong U^0_X \cap (\sigma_2 \circ \tilde{s}_2)^{-1}(U^1_{Y_1 \times X Y_2}) \cong U^0_X \cap \tilde{s}_2^{-1}(U^1_{Y_2}) \tag{4.2}
\]
after which we have maps of covers constituting the commutative diagram
\[
\begin{array}{c}
\text{Et}(U^1_\nu) \\
\uparrow \tilde{s}_2 \quad \uparrow \tilde{s}_1 \\
\text{Et}(U^1_{Y_1 \times X Y_2}) \\
\end{array}
\]
and which each constitute sections the appropriate map \( Y_i \to X \), \( Y_1 \times X Y_2 \to Y_i \), or \( W \to Y_1 \times X Y_2 \). The only nontrivial assertion is the horizontal identifications in (4.2), but this follows easily from viewing the refinements as pullbacks in light of naturality of pullback. Alternatively it can be checked directly: given \( U \in U^0_X \) mapping by \( \tilde{s}_1 \) into \( V_1 \in U^0_{Y_1} \), say, the latter becomes refined into sets of the form \( V_1 \cap \sigma_1^{-1}(W) \) for \( W \in U^1_{Y_1 \times X Y_2} \), and then \( U \) decomposes into sets of the form \( U \cap \tilde{s}_1^{-1}(V_1 \cap \sigma_1^{-1}(W)) = U \cap (\sigma_1 \circ \tilde{s}_1)^{-1}(W) \). For the stronger version involving multiple split squares over \( X \), we note that \( U^1_{Y_1} \) and then subsequently \( U^1_X \) may be further refined while retaining the diagram (4.3); in particular, we can replace these by the mutual refinements of the Step 1 covers for each locally split square in which the spaces are involved.

**Step 2:** Starting at the bottom left, we subsequently set \( U^2_X = U^1_X \), \( U^2_{Y_i} = U^1_{Y_i} \cap \sigma_i^{-1}(U^1_X) \), \( U^2_{Y_1 \times X Y_2} = U^1_{Y_1 \times X Y_2} \cap (U^2_X \times U^2_{Y_1} \times U^2_{Y_2}) \), and \( U^2_W = U^2_W \cap p^{-1}(U^2_{Y_1 \times X Y_2}) \), after which we have maps of covers
\[
\begin{array}{c}
\text{Et}(U^2_\nu) \\
\uparrow \tilde{s}_2 \quad \uparrow \tilde{s}_1 \\
\text{Et}(U^2_{Y_1 \times X Y_2}) \\
\end{array}
\]

such that each subsequent pair is admissible. Indeed, the only assertion to check admissibility of the pair \((U^2_X, U^2_{Y_1 \times X Y_2})\), since \( U^2_{Y_1 \times X Y_2} \) is defined in a different way here compared to Lemma 1.4.

To check this, first note that \( U^1_{Y_1 \times X Y_2} \) is in general finer than either \( U^1_{Y_1 \times X Y_2} \cap \sigma_i^{-1}(U^1_{Y_i}) \) for \( i = 1, 2 \), and in fact it factors through both of
these since $\text{pr}_1^{-1}(U^2_{Y_1}) = U^2_{X_3} \times_X Y_2$ and $\text{pr}_2^{-1}(U^2_{Y_2}) = Y_1 \times_X U^2_{Y_1}$. It follows from this that $\tilde{p}_1$ lift to $U^2_{Y_1 \times_X Y_2}$ and $\pi_1 \circ \tilde{p}_1 = \tilde{p}_2 \circ \tilde{p}_2$. To see that $\tilde{\sigma}_i$ lift, consider for example a $V \in U^2_{Y_2}$ mapping by $\tilde{\sigma}_2$ to $W \in U^1_{Y_1 \times_X Y_2}$ after step 1. In Step 2, $V$ is refined into sets of the form $V \cap \pi^{-1}_2(U)$ for $U \in U^2_X$, and we have a diagram

$$
V \cap \pi^{-1}_2(U) \xrightarrow{\tilde{\sigma}_2} W
$$

for some $V' \cap \pi^{-1}_1(U) \in U^2_{Y_1}$. But then $\tilde{\sigma}_2$ lifts canonically into a map

$$
\tilde{\sigma}_2 : V \cap \pi^{-1}_2(U) \rightarrow W \cap (V \times_U V') \in U^2_{Y_1 \times_X Y_2}.
$$

Finally, to see that the final (pairwise intersection) property for admissibility holds, take $V_1, V_2 \in U^2_{Y_2}$ with non-empty intersection. By construction the $V_i$ are associated to $U_i \in U^2_X$ for $i = 1, 2$, and by the intersection property for $\tilde{s}_1$ it follows that $(\tilde{s}_1)_U : U_1 \cap U_2 \rightarrow V'_1 \cap V'_2$ (more precisely, the restriction to $U_1 \cap U_2$ of $\tilde{s}_1$ as defined on $U_1$) maps into some $V'_2 \in U^2_{Y_1}$ over $U_2$. It follows then that the restriction to $V_1 \cap V_2$ of $\tilde{\sigma}_2$ (as defined on $V_1$), in fact maps into $(W \cap V_1 \times_U V'_1) \cap (W \cap V_2 \times_U V'_2)$. In case multiple split squares over $X$ are being considered, note that the refinements in Step 2 are mutually consistent, so there are no additional considerations to this step in this case.

The admissible set is given by $U^*_\bullet = U^2_\bullet$ for $\bullet \in \{X, Y_1, Y_2, W\}$, with maps of covers between the $U^*_Y$ and $U^*_W$ given by compositions $\tilde{p}_i \circ \tilde{p}$ and $\tilde{t} \circ \tilde{\sigma}_i$. The required commutativity of the four squares follows from the like commutativity of the squares in step 2 for the spaces $X, Y_1, Y_2$, and $Y_1 \times_X Y_2$. □

As in §2.1, let $Y^*_{i[k]}$ be the $k$-fold fiber product $Y_i \times_X \cdots \times_X Y_i$ for $i = 1, 2$. Then $Y^*_{1[1]}$ and $Y^*_{2[1]}$ each form simplicial spaces over $X$, giving the bounding column and row in (4.4) below.

Setting $W^{[1,1]} = W$ and

$$
W^{[1,k]} = W \times Y_1 \cdots \times Y_1 W, \quad \text{and} \quad W^{[k,1]} = W \times Y_2 \cdots \times Y_2 W,
$$

with projection maps $W^{[1,k]} \rightarrow W^{[1,k-1]}$ and $W^{[k,1]} \rightarrow W^{[k-1,1]}$ gives simplicial spaces over $Y_1$ and $Y_2$ extending above and to the right of $W^{[1,1]}$ in (4.4).

That the rest of the quadrant can then be filled out unambiguously by fiber products is a consequence of the following result.

**Proposition 4.4.** For each $n$ and $m$, there is a natural isomorphism

$$
W^{[m,n]} := W^{[m,1]} \times_{Y_1^{[m]}} \cdots \times_{Y_1^{[m]}} W^{[m,1]} \cong W^{[n,1]} \times_{Y_2^{[n]}} \cdots \times_{Y_2^{[n]}} W^{[n,1]}.
$$
Proof. Both sides may be identified with the set of tuples \((w_{i,j} : 1 \leq i \leq m, 1 \leq j \leq n) \in W^{mn}\) such that for each \(i\), \((w_{i,1}, \ldots, w_{i,n})\) all map to a fixed \(y_{1,i} \in Y_1\) and for each \(j\), \((w_{1,j}, \ldots, w_{m,j})\) all map to a fixed \(y_{2,j} \in Y_2\), and where every \(y_{1,i}\) and \(y_{2,j}\) sit over a fixed \(x \in X\). 

The spaces \(W^{[\bullet, \bullet]}\) in the resulting diagram

\[
\begin{array}{cccccc}
Y_2^{[3]} & \leftarrow & W^{[1,3]} & \leftrightarrow & W^{[2,3]} & \leftrightarrow & W^{[3,3]} & \ldots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
Y_2^{[2]} & \leftarrow & W^{[1,2]} & \leftrightarrow & W^{[2,2]} & \leftrightarrow & W^{[3,2]} & \ldots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
Y_2 & \leftarrow & W^{[1,1]} & \leftrightarrow & W^{[2,1]} & \leftrightarrow & W^{[3,1]} & \ldots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
X & \leftarrow & Y_1 & \leftrightarrow & Y_1^{[2]} & \leftrightarrow & Y_1^{[3]} & \ldots \\
\end{array}
\]

form a bisimplicial space over \(X\), meaning a functor \(\Delta^{op} \times \Delta^{op} \to \text{Top}/X\) where \(\Delta\) is the simplex category. In particular, \(W^{[m, \bullet]}\) and \(W^{[\bullet, n]}\) are simplicial spaces over \(Y_1^{[m]}\) and \(Y_2^{[n]}\), respectively, and the squares commute for consistent choices of maps. For notational convenience, we also set \(W^{[k,0]} = Y_1^{[k]}\), \(W^{[0,k]} = Y_2^{[k]}\), and \(W^{[0,0]} = X\).

4.2. Bigerbes. If \(L \to W^{[m,n]}\) is a line bundle over one of the spaces in (4.4) then its two simplicial differentials are

\[
d_1L = \bigotimes_{i=0}^{m} (\pi^1_i)^*L(-1)^i \to W^{[m+1,n]} \quad \text{and} \quad d_2L = \bigotimes_{i=0}^{n} (\pi^2_i)^*L(-1)^i \to W^{[m,n+1]},
\]

where \(\pi^1_j : W^{[m+1,n]} \to W^{[m,n]}\), \(0 \leq j \leq m\) and \(\pi^2_j : W^{[m,n+1]} \to W^{[m,n]}\), \(0 \leq j \leq n\) denote the fiber projection maps. The bundles \(d_1d_1L\) and \(d_2d_2L\) are canonically trivial, and there is a natural isomorphism \(d_1d_2L \cong d_2d_1L\).

Definition 4.5. A bigerbe consists of a locally split square \((W, Y_2, Y_1, X)\), a line bundle \(L \to W^{[2,2]}\), and trivializations of \(d_1L\) and \(d_2L\), which induce the same trivialization of \(d_1d_2L\) and which induce the canonical trivializations of \(d_1^2L\) and \(d_2^2L\). We denote the bigerbe by \((L, W, Y_2, Y_1, X)\) or simply \(L\).

For reasons that will become clear below, the order of the spaces \(Y_1\) and \(Y_2\), or equivalently the orientation of the square (4.1), is part of the data of the bigerbe.

Definition 4.6. If \(Q_1 \to W^{[1,2]}\) and \(Q_2 \to W^{[2,1]}\) are line bundles which are simplicial with respect to \(d_2\) and \(d_1\), respectively — so \(d_2Q_1\) over \(W^{[1,3]}\)
is equipped with a trivialization inducing the canonical trivialization of $d_2^2 Q_1$ and similarly for $Q_2$ — then $d_1 Q_1 \otimes d_2 Q_2^{-1}$ has a canonical bigerbe structure. A bigerbe $L$ is said to be trivial if

$$L \cong d_1 Q_1 \otimes d_2 Q_2^{-1},$$

for $Q_1$ and $Q_2$ as above, with the isomorphism identifying the bigerbe structure on $L$ with the canonical one on $d_1 Q_1 \otimes d_2 Q_2^{-1}$; such an isomorphism is referred to as a trivialization of $L$.

In particular, $L$ is trivial if either

(i) $L \cong d_1 P$ where $P \to W^{[1,2]}$ is a line bundle with trivialization $d_2 P \cong \mathbb{C}$ inducing the canonical trivialization of $d_2^2 P$, or

(ii) $L \cong d_2 Q$ where $Q \to W^{[2,1]}$ is a line bundle with trivialization $d_1 Q \cong \mathbb{C}$ inducing the canonical trivialization of $d_1^2 Q$,

as in either case we can take the trivial bundle on the other factor.

As for ordinary bundle gerbes, we proceed to define pullbacks and products of bigerbes.

**Lemma 4.7.** If $(W, Y_2, Y_1, X)$ and $(W', Y_2', Y_1', X)$ are locally split squares over $X$ and $f : X' \to X$ is a continuous map, then

(i) $(f^*(W), f^*(Y_2), f^*(Y_1), X')$ is a locally split square over $X'$, and

(ii) $(W \times_X W', Y_2 \times_X Y_2', Y_1 \times_X Y_1', X)$ is a locally split square over $X$.

**Proof.** By hypothesis there are covers $U_i \to X$ admitting sections $s^i : U_i \to Y_i$ of $\pi_i : Y_i \to X$ and a cover $V \to Y_1 \times_X Y_2$ admitting a section $t : V \to W$ of the universal map $p : W \to Y_1 \times_Y Y_2$.

Pullback of these by $f$ gives covers $f^{-1} U_i \to X'$ and sections $f^* s^i : f^{-1} U_i \to f^* Y_i$, with $f^* s^i = 1 \times s^i \circ f : f^{-1} U_i \to X' \times_X Y_i = f^* Y_i$, where the section is composed with the lift $f : f^{-1} U_i \to U_i$ and 1 denotes the inclusion map $f^{-1} U_i \to X'$ of covers. Similarly, if $\tilde{f} : f^*(Y_1 \times_X Y_2) \to Y_1 \times_X Y_2$ denotes the natural lift over $f$, then $\tilde{f}^* V \to f^*(Y_1 \times_X Y_2)$ supports the section $\tilde{f}^* t : \tilde{f}^{-1} V \to f^* W$ of the natural map $f^* W \to f^*(Y_1 \times_Y Y_2) \cong (f^* Y_1) \times_X (f^* Y_2)$, proving (i).

For the fiber product, the cover $U_i \cap U'_i \to X$ admits sections $s^i \times (s')^i$ of $Y_i \times_X Y_i'$, and then $(Y_1 \times_X Y_1') \times_X (Y_2 \times_X Y_2') \cong (Y_1 \times_Y Y_2) \times_X (Y_1' \times_Y Y_2')$ may be equipped with the cover $V \times_X V'$, which admits the section $t \times t'$ to $W \times_X W'$, proving (ii).

**Definition 4.8.** If $L \to W^{[2,2]}$ is a bigerbe with respect to the locally split square $(W, Y_2, Y_1, X)$, and $f : X' \to X$ is a continuous map, then the pullback of $L$ is the bigerbe $\tilde{f}^* L \to f^*(W^{[2,2]})$ with respect to the locally split square $(f^*(W), f^*(Y_2), f^*(Y_1), X')$.

If $L = (L, W, Y_2, Y_1, X)$ and $L' = (L', W', Y_2', Y_1', X)$ are bigerbes on $X$, then the product of $L$ and $L'$ is the bigerbe

$$(L \otimes L', W \times_X W', Y_2 \times_X Y_2', Y_1 \times_X Y_1', X).$$
Next we define (strong) morphisms and stable isomorphisms for bigerbes. A morphism of locally split squares \((W', Y'_2, Y'_1, X') \rightarrow (W, Y_2, Y_1, X)\) is a collection of maps from each space in the first square to the corresponding space in the second, with each of the relevant squares commuting. As for bundle gerbes we do not require compatibility in the sense of §1.2 of the locally split maps of the first square with those of the second. By naturality of fiber products, these maps extend to maps \(W'^{[m,n]} \rightarrow W^{[m,n]}\) for each \((m, n) \in \mathbb{N}_0^2\) commuting with the various fiber projections in both directions. By abuse of notation we will denote all such maps by a single letter, say \(f : W'^{[m,n]} \rightarrow W^{[m,n]}\).

**Definition 4.9.** If \((L, W, Y_2, Y_1, X)\) and \((L', W', Y'_2, Y'_1, X')\) are bigerbes over \(X\) and \(X'\) respectively, then a (strong) morphism \(f : (W', Y'_2, Y'_1, X') \rightarrow (W, Y_2, Y_1, X)\) and an isomorphism \(L' \cong f^*L\) over \(W'^{[2,2]}\) which intertwines the sections of \(d_iL\) and \(d_if^*L\) for \(i = 1, 2\). A (strong) isomorphism is a morphism for which \(X = X'\) and \(f = \text{Id} : X \rightarrow X'\).

Finally, a stable isomorphism of bigerbes \(L\) and \(L'\) over \(X\) is a (strong) isomorphism

\[ L \otimes T \cong L' \otimes T \]

where \(T\) and \(T'\) are trivial bigerbes.

### 4.3. The 4-class of a bigerbe

We proceed to define the cohomology 4-class of a bigerbe using Čech theory as in §2.2 and §2.3, starting with the following generalization of Lemma 1.5.

**Lemma 4.10.** Fix an admissible set of covers for a locally split square \((X, Y_1, Y_2, W)\). Then

(i) \[ U_W^{[m,n]} := U_W^{[m,1]} \times U_Y^{[m]} \cdots \times U_Y^{[m]} U_W^{[1,1]} \cong U_W^{[1,n]} \times U_Y^{[n]} \cdots \times U_Y^{[n]} U_W^{[1,1]} \]

is a well-defined cover of \(W^{[m,n]}\) for each \(m\) and \(n\).

(ii) The various projection maps \(\pi^1_i : W^{[m,n]} \rightarrow W^{[m-1,n]}\) and \(\pi^2_j : W^{[m,n]} \rightarrow W^{[m,n-1]}\) for \(0 \leq i \leq m - 1\) and \(0 \leq j \leq n - 1\) lift canonically to maps of covers \(\tilde{\pi}^1_i : \text{Et}((U_W^{[m,n]})(\ell)) \rightarrow \text{Et}((U_W^{[m-1,n]})(\ell))\) and \(\tilde{\pi}^2_j : \text{Et}((U_W^{[m,n-1]})(\ell)) \rightarrow \text{Et}((U_W^{[m,n-1]})(\ell))\) for each \(\ell\).

(iii) The lifted sections on the admissible set of covers determine maps of covers \(\tilde{s}^1_\ell : \text{Et}((U_W^{[m-1,n]})(\ell)) \rightarrow \text{Et}((U_W^{[m,n]})(\ell))\) and \(\tilde{s}^2_\ell : \text{Et}((U_W^{[m,n-1]})(\ell)) \rightarrow \text{Et}((U_W^{[m,n]})(\ell))\) for each \(\ell\) such that \(\tilde{s}^1_\ell\) commutes with the projections \(\tilde{\pi}^1_j\), \(\tilde{\pi}^2_j\), and satisfies

\[ \tilde{s}^1_j \tilde{\pi}^1_{j-1} = \begin{cases} 1 & j = 0 \\ \tilde{\pi}^1_j & 1 \leq j \leq m \end{cases} \]

and similarly for \(\tilde{s}^2_\ell\) with respect to \(\tilde{\pi}^1_j\) and \(\tilde{\pi}^2_j\).
Proof. The same argument as in Proposition 4.4 shows that $\mathcal{U}_W^{[m,n]}$ is well-defined, namely both sides are identified with tuples

$$(w_{i,j} \in W_{i,j} : 1 \leq i \leq m, 1 \leq j \leq n), \quad W_{i,j} \in \mathcal{U}_W$$

such that for each $i$, $(w_{i,1}, \ldots, w_{i,n})$ all map to a fixed $y_{1,i} \in V_{1,i}$ for some $V_{1,i} \in \mathcal{U}_Y$, and for each $j$, $(w_{1,j}, \ldots, w_{m,j})$ all map to a fixed $y_{2,j} \in V_{2,j}$ for some $V_{2,j} \in \mathcal{U}_Y$, with $(y_{1,1}, \ldots, y_{1,m}) \in V_{1,1} \times_U \cdots \times_U V_{1,m} \in \mathcal{U}_{V^{[m]}}$ and $(y_{2,1}, \ldots, y_{2,n}) \in V_{2,1} \times_U \cdots \times_U V_{2,n} \in \mathcal{U}_Y^{[n]}$.

Then the arguments in Lemma 1.5 apply, viewing each row or column of (4.4) as a simplicial space, with commutativity of the vertical and horizontal maps of covers deriving from their commutativity at the bottom level for the locally split square. \hfill $\square$

The abelian groups of Čech cochains $\check{C}_U^\ell(W^{[j,k]}; A) := \check{C}_U^\ell(W^{[j,k]}; A)$ are defined for each $\ell$, $j$, and $k \in \mathbb{N}_0$. Using Lemma 4.10 we may define

$$d_1 = \sum_{j=0}^m (-1)^j (\pi_j^1)^* : \check{C}_U^\ell(W^{[m,n]}; A) \longrightarrow \check{C}_U^\ell(W^{[m+1,n]}; A), \quad \text{and}$$

$$d_2 = \sum_{j=0}^n (-1)^j (\pi_j^2)^* : \check{C}_U^\ell(W^{[m,n]}; A) \longrightarrow \check{C}_U^\ell(W^{[m,n+1]}; A)$$

which are differentials commuting with one another and with the Čech differential $\delta$. Thus $\check{C}_U^\ast(W^{[\bullet,\bullet]}; A), \delta, d_1, d_2)$ forms a triple complex, and combining Lemma 4.10 and Proposition 2.3 leads to the following result.

**Proposition 4.11.** An admissible set of covers determines commuting homotopy contractions for the $d_i$ which commute with the other simplicial differential, and for each fixed $\ell$ and $k$, the subcomplex

$$\left( \text{Ker} \left\{ d_1 : \check{C}_U^\ell(W^{[j,k]}; A) \longrightarrow \check{C}_U^\ell(W^{[j+1,k]}; A) \right\} \right)$$

is exact and similarly with indices reversed.

Just as a bundle gerbe has a Dixmier Douady class in $H^3(X; \mathbb{Z})$, a biggerbe determines a characteristic class in $H^3(X; \mathbb{Z})$. To see this, consider the truncation of the triple complex $\check{C}_U^\ast(W^{[\bullet,\bullet]}; A), \delta, d_1, d_2)$ which we denote by $Z_U^\ast(W^{[\bullet,\bullet]}; A), \delta, d_1, d_2)$, where

$$Z_U^\ast(W^{[j,k]}; A) = \begin{cases} 
\check{C}_U^\ell(W^{[j,k]}; A), & [j,k] = [1,1] \\
\text{Ker } d_1 \subset \check{C}_U^\ell(W^{[2,1]}; A) & [j,k] = [2,1] \\
\text{Ker } d_2 \subset \check{C}_U^\ell(W^{[1,2]}; A) & [j,k] = [1,2] \\
\text{Ker } d_1 \cap \text{Ker } d_2 \subset \check{C}_U^\ell(W^{[2,2]}; A) & [j,k] = [2,2] \\
0 & \text{otherwise}
\end{cases}$$
Suppressing the Čech direction, we may depict the truncated complex as

\[
\begin{array}{c}
\vdots \\
0 & 0 \\
0 & d_1 \downarrow & d_2 \uparrow & d_1 \downarrow
\end{array}
\]
\[
\begin{array}{c}
Z_U^\bullet(W^{[1,2]}; A) \quad d_1 & Z_U^\bullet(W^{[2,2]}; A) & d_1 \downarrow 0 \\
\delta & \delta & \delta
\end{array}
\]
\[
\begin{array}{c}
\delta & \delta & \delta
\end{array}
\]
\[
\begin{array}{c}
C_U^\bullet(W^{[1,1]}; A) \quad d_1 & Z_U^\bullet(W^{[2,1]}; A) & d_1 \downarrow 0.
\end{array}
\]

In particular, the leftmost column and bottom row of (4.5) are taken to have \(d_i\) degree 0. Then following Convention 2.2, the total differential on (4.5) is

\[D = \delta + (-1)^\ell d_1 + (-1)^{\ell+m+1} d_2 \quad \text{on } Z_U^\bullet(W^{[m,n]}; A)\]

(4.6)
since \(C_U^\bullet(W^{[m,n]})\) occupies the \((m-1, n-1)\) coordinate in the \((d_1, d_2)\) plane.

Employing a spectral sequence argument twice immediately yields the following result.

**Proposition 4.12.** The triple complex \((Z_U^\bullet(W^{[\bullet,\bullet]}; A), \delta, d_1, d_2)\) has total cohomology isomorphic to the ordinary cohomology \(H^\bullet_U(X; A)\) of \(X\).

**Proof.** The total differential (4.6) of the \((\delta, d_1, d_2)\) triple complex can be written as \(D = D_1 + (-1)^\ell d_1 + (-1)^{\ell+m+1} d_2\) on \(Z_U^\bullet(W^{[m,n]}; A)\), where \(D_1 = \delta + (-1)^\ell d_1\) is the total differential of the \((\delta, d_1)\) double complex. By exactness of \(d_2\), the total cohomology of the \((D_1, d_2)\) double complex is isomorphic to the cohomology of the \(D_1\) (double) complex \(C_U^\bullet(Y_1^{[\bullet]}; A)\), which in turn is isomorphic to \(H^\bullet_U(X; A)\) as in Theorem 2.5.

**Lemma 4.13.** The line bundle \(L \rightarrow W^{[2,2]}\) of a bigerbe determines a pure cocycle \(c(L) \in Z_U^1(W^{[2,2]}; C^\bullet)\) in the triple complex (4.5) for some admissible set of covers, and conversely any line bundle with \(c(L) \in Z_U^1(W^{[2,2]}; C^\bullet)\) determines a bigerbe. Moreover, the pure cocycle \(c(L) \in Z_U^1(W^{[2,2]}; C^\bullet)\) is a coboundary if and only if \(L\) admits a trivialization.

**Proof.** The line bundle \(L\) is represented by its ‘transition’ Chern class on some cover, which by arguments along the lines of Lemma 1.6 can be assumed to be of the form \(U_{W^{[2,2]}}\) for some admissible set of covers, and hence by an element \(c(L) \in C_U^1(W^{[2,2]}; C^\bullet)\) such that \(\delta c(L) = 0\). The simplicial trivializations of \(d_i L, i = 1, 2\) are represented by elements \(\alpha_1 \in C_U^0(W^{[3,2]})\) and \(\alpha_2 \in C_U^0(W^{[2,3]})\) such that \(d_i \alpha_i = 0, \delta \alpha_i = d_i c(L),\) and \(\delta (d_2 \alpha_1 - d_1 \alpha_2) = 0\). In other words, the triple \((c(L), -\alpha_1, \alpha_2)\) forms a cocycle in the triple complex \((C_U^\bullet(W^{[\bullet,\bullet]}; C^\bullet), \delta, d_1, d_2)\). Now, by exactness, we may obtain \(d_i\) preimages \(\beta_i\) of the \(\alpha_i\), and then \(c(L)\) can be altered by the image under \(\delta\) of the \(\beta_i\) to obtain a pure cocycle, which we again denote by \(c(L) \in Z_U^1(W^{[2,2]}; C^\bullet)\).

A coboundary for \(c(L)\) in the triple complex consists of a triple \((\alpha, \beta, \gamma)\) where \(\alpha \in Z_U^0(W^{[2,2]}), \beta \in Z_U^1(W^{[1,2]}),\) and \(\gamma \in Z_U^1(W^{[2,1]})\) such that \(\delta \beta = 0\) and \(\delta \gamma = 0,\) and

\[
D(\alpha, \beta, \gamma) = \delta \alpha - d_1 \beta + d_2 \gamma = c(L).
\]

(4.7)
Let $(\text{coboundary} (4.7))$. □

For an explicit zig-zag construction of $G$ the order of $Y$ isomorphic (with isomorphism determined by $\alpha$) to $d_1 P \otimes d_2 Q^{-1}$, i.e., $L$ is trivial. Conversely, a trivialization of the bigerbe $L$ determines such a coboudnary (4.7).

**Definition 4.14.** Let $(L, W, Y_2, Y_1, X)$ be a bigerbe over $X$. The characteristic 4-class of $L$ is the image $G(L) \in H^4(X; \mathbb{Z}) \cong H^3(X; \mathbb{C}^*)$ of the hypercohomology class of $c(L) \in \tilde{Z}^4_{\mathcal{U}}(W^{[2,2]}; \mathbb{C}^*)$ in the triple complex (4.5).

For an explicit zig-zag construction of $G(L)$ from $c(L)$, see (4.10) and (4.11) in the proof of Theorem 4.17 below.

Because of the need to introduce signs in the $(\delta, d_1, d_2)$ total complex following Convention 2.2, the sign of the class $G(L)$ in the proof of Theorem 4.17 below.

Alternatively, from the explicit zig-zag (4.10) and (4.11) it follows that $c(L)$ is the double transgression of $G(L) = G(L')$ (in the sense of the isomorphism (2.10)) first to $H^3_Z(Y_1^{[2]}; \mathbb{Z})$ and then to $H^3_Z(Y_2^{[2]}; \mathbb{Z})$. The transgression the other way, first to $Y_2^{[2]}$ and then to $W^{[2,2]}$ has the opposite image $-c(L)$.

**Theorem 4.15.** The characteristic 4-class $G(L)$ vanishes if and only if $L$ is trivial as a bigerbe, and is natural with respect to pullback, product and inverses in that

$$G(f^* L) = f^* G(L), \quad G(L_1 \otimes L_2) = G(L_1) + G(L_2), \quad G(L^{-1}) = -G(L)$$

A morphism $f : (L', W', Y_2', Y_1', X') \rightarrow (L, W, Y_2, Y_1, X)$ of bigerbes induces an equality $f^* G(L) = G(L')$, and two bigerbes $L$ and $L'$ over $X$ satisfy $G(L) = G(L')$ if and only if they are stably isomorphic.

**Proof.** That $G(L) = 0$ if and only if $L$ admits a trivialization was proved in Lemma 4.13. The pullback of a locally split square over $X$ by a continuous map $f : X' \rightarrow X$ induces natural maps $f^* W^{[m,n]} \rightarrow W^{[m,n]}$ commuting with each $\pi_j$, and thus a map $f^* c_{\mathcal{U}}(W^{[••]}) \rightarrow \tilde{C}_{\mathcal{U}'}(f^* W^{[••]})$ of triple complexes, where $\mathcal{U}'_*$ is an admissible set of covers refining $f^{-1}(\mathcal{U}_*)$. The naturality of $G$ with respect to pullbacks and morphisms is then a consequence of the naturality of the spectral sequences which identify the total cohomology of the triple complex with the cohomology of $X$ and $X'$, respectively. Naturality with respect to products and inverses is a direct consequence of the fact
that we can take \([L^{-1}] = -c(L)\) and \([L_1 \otimes L_2] = [pr^*L_1] + [pr^*L_2]\) as representatives. Finally, if \(L\) and \(L'\) are stably isomorphic, then \(G(L) = G(L')\) by triviality and products, and conversely if \(G(L) = G(L')\), then \(L^{-1} \otimes L' = T\) is trivial, from which a stable isomorphism \(L \otimes T \cong L'\) may be constructed. \(\square\)

4.4. **Representability of 4-classes.** To characterize those 4-classes which are represented by bigerbs over a given locally split square we follow a similar argument to that in §2.4, though it is necessary in this case to go further in a spectral sequence for the triple complex. Consider the augmented triple complex

\[
\begin{array}{ccc}
0 & 0 & 0 \\
d_2 \uparrow & d_2 \uparrow & d_2 \uparrow \\
\hat{Z}^*(Y_2^{[2]}) & \hat{Z}^*(W^{[1,2]}) & \hat{Z}^*(W^{[2,2]}) \\
d_2 \uparrow & d_2 \uparrow & d_2 \uparrow \\
\hat{C}^*(Y_2) & \hat{C}^*(W^{[1,1]}) & \hat{Z}^*(W^{[2,1]}) \\
d_2 \uparrow & d_2 \uparrow & d_2 \uparrow \\
\hat{C}^*(X) & \hat{C}^*(Y_1) & \hat{Z}^*(Y_1^{[2]}) \\
d_1 \downarrow & d_1 \downarrow & d_1 \downarrow \\
\end{array}
\]  

(4.8)

with the leftmost column and bottom row considered as degree \(-1\) for \(d_1\) and \(d_2\), respectively.

**Lemma 4.16.** Fix \(\ell \geq 1\) and an abelian group \(A\). Suppose \([\alpha] \in \hat{H}^\ell(X; A)\) satisfies \(\pi^*_i[\alpha] = 0 \in \hat{H}^\ell(Y_i; A)\) for \(i = 1, 2\). Then there is a well-defined transgression class defined by

\[
\text{Tr}[\alpha] = [d_1 \beta_2 - d_2 \beta_1] \in \hat{H}^{\ell-1}(W; A)/\left(\hat{H}^{\ell-1}(Y_1; A) \oplus \hat{H}^{\ell-1}(Y_2; A)\right)
\]

(4.9)

where \(\beta_i \in \hat{C}^{\ell-1}_U(Y_i; A)\) are any elements satisfying \(\delta \beta_i = \pi^*_i \alpha \in \hat{C}^{\ell}_U(Y_i; A)\) for a representative \(\alpha \in \hat{C}^\ell_U(X; A)\) for an appropriate admissible set of covers \(U^*\).

**Remark.** This transgression can be understood as the \(W^{[1,1]}\) component of the \(E_2\) page differential of the \((\delta, D_{12})\) spectral sequence of \((\hat{C}^\ell_U(W^{[k,*]}), \delta, D_{12})\) applied to \([\alpha]\), where we have rolled up \(d_1\) and \(d_2\) into a total differential \(D_{12} = d_1 \pm d_2\).

In fact, to observe the sign convention discussed in Convention 2.2 we should properly define \(\text{Tr}[\alpha]\) as the class \([-(-1)^{\ell \ell} d_2 \beta_1 + (-1)^{\ell+1} d_1 \beta_2]\) where \(\delta \beta_1 = (-1)^{\ell} d_1 \alpha\) and \(\delta \beta_2 = (-1)^{\ell+1} d_2 \alpha\), but then cancellation of the two factors of \((-1)^{\ell+1}\) and exchanging \(\beta_1\) with \(-\beta_1\) makes this equivalent to the definition given above.

**Proof.** With \(\alpha, \beta_1, \) and \(\beta_2\) as above for a fixed admissible set of covers, it follows that \(d_1 \beta_2 - d_2 \beta_1\) is a cocycle since

\[
\delta(d_1 \beta_2 - d_2 \beta_1) = d_1 \pi^*_2 \alpha - d_2 \pi^*_1 \alpha = d_1 d_2 \alpha - d_2 d_1 \alpha = 0.
\]

Another choice of representative \(\alpha' = \alpha + \delta \gamma\) can be incorporated as a different choice \(\beta'_i = \beta_i + d_i \gamma\) of the \(\beta_i\); moreover if \(\beta'_i \in \hat{C}^{\ell-1}_U(Y_i)\) are
another choice of bounding chains for $\pi_i\alpha$, then $\delta(\beta_i - \beta'_i) = 0$ and

$$(d_1\beta_2 - d_2\beta_1) - (d_1\beta'_2 - d_2\beta'_1) = d_1(\beta_2 - \beta'_2) + d_2(\beta'_1 - \beta_1)$$

is in the image under $[d_2 \ d_1]$ of $\check{H}^{\ell-1}(Y_1) \oplus \check{H}^{\ell-1}(Y_2)$. \hfill \qedsymbol

**Theorem 4.17.** A locally split square $(W, Y_2, Y_1, X)$ supports a bigerbe with a given class $[\alpha] \in H^3(X; \mathbb{Z})$ if and only if

(i) $\pi_i^*[\alpha] = 0 \in H^4(Y_i; \mathbb{Z})$ for $i = 1, 2$ and

(ii) $\text{Tr}[\alpha] = 0 \in H^3(W; \mathbb{Z})/(H^3(Y_1; \mathbb{Z}) \oplus H^3(Y_2; \mathbb{Z}))$.

**Proof.** By naturality of the Bockstein isomorphism, it suffices to work one degree lower with $\mathbb{C}^*$ coefficients. Thus suppose $\alpha \in \check{C}^3_{ul}(X; \mathbb{C}^*)$ represents $[\alpha]$. Since by hypothesis $\text{Tr}[\alpha]$ vanishes, there exist representatives $\beta_i \in \check{C}^2_{ul}(Y_i; \mathbb{C}^*)$ such that $[d_1\beta_2 - d_2\beta_1] = 0 \in \check{H}^2_{ul}(W; \mathbb{C}^*)$; thus $d_1\beta_2 - d_2\beta_1 = \delta\gamma$ for $\gamma \in \check{C}^1_{ul}(W; \mathbb{C}^*)$. Then we claim $d_1d_2\gamma = d_2d_1\gamma \in \check{Z}^1_{ul}(W^{[2,2]}; \mathbb{C}^*)$ is a pure cocycle and that a bigerbe $L \to W^{[2,2]}$ with

$$c(L) = -d_1d_2\gamma$$

satisfies $G(L) = [\alpha]$. Indeed, it is obvious that $d_i(d_1d_2\gamma) = 0$ for $i = 1, 2$; moreover $\delta d_1d_2\gamma = d_1d_2\delta\gamma = d_1d_2(d_1\beta_2 - d_2\beta_1) = 0$ as well, so by Lemma 4.13, $d_1d_2\gamma = -c(L) \in \check{Z}^1_{ul}(W^{[2,2]}; \mathbb{C}^*)$ for a bigerbe $L \to W^{[2,2]}$.

To see that $G(L) = \alpha$, we follow the proof of Proposition 4.12, carefully observing the sign convention (4.6) and observe that

$$
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 d_2 \downarrow \\
 -d_1\gamma
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
 d_1 = \delta - d_1
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
 (d_2d_1\beta_1, 0)
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
 -d_2 \downarrow \\
 (-d_1\beta_1, 0)
\end{array}
\end{array}
\end{array}
\tag{4.10}
$$

is a zig-zag which identifies $-d_1\beta_1 \in \check{Z}^2_{ul}(Y_1^{[2]}; \mathbb{C}^*)$ as a pure cocycle representing the image of $c(L)$ in the $E_1$ page of the $(d_2, D_1 = \delta \pm d_1)$ spectral sequence of the triple complex (4.5) which collapses to the $D_1$ cohomology of $\check{Z}^\bullet_{ul}(Y_1^{[*]}; \mathbb{C}^*)$. Then, as in (2.8),

$$
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 d_1 \downarrow \\
 -\beta_1
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
 \delta
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
 -d_1\alpha
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 -d_1 \downarrow \\
 \alpha
\end{array}
\end{array}
\end{array}
\end{array}
\tag{4.11}
$$

is a further zig-zag which identifies $\alpha \in \check{C}^3_{ul}(X; \mathbb{C}^*)$ as the image of $c(L)$ in the $E_1$ page of the $(d_1, \delta)$ spectral sequence of $\check{Z}^\bullet_{ul}(Y_1^{[*]}; \mathbb{C}^*)$ representing the class $G(L) = [\alpha]$.

Conversely, to show necessity of this condition, suppose that $L \to W^{[2,2]}$ is a bigerbe. As shown in Lemma 4.13 this generates a Čech cocycle, $\lambda =
c(L) \in \check{Z}_d^2(W^{[2,2]}; \mathbb{C}^*) \text{ with respect to some admissible set of covers, with values in } \mathbb{C}^* \text{ which is a pure cocycle in the triple complex:}
\delta \lambda = d_1 \lambda = d_2 \lambda = 0. \quad (4.12)

Using the exactness of the simplicial complexes we may pull this back under the two homotopy contractions \((s_1^*)^*\) and \((s_2^*)^*\) giving
\gamma \in \check{C}_d^1(W; \mathbb{C}^*), \quad d_1d_2\gamma = d_2d_1\gamma = \lambda. \quad (4.13)

Consider the Čech differential \(\delta \gamma \in \check{C}_d^2(W; \mathbb{C}^*)\). The images of this,
\[d_1\delta \gamma = \delta d_1 \gamma \in \check{C}_d^2(W^{[2,1]}; \mathbb{C}^*) \text{ and } d_2\delta \gamma \in \check{C}_d^2(W^{[1,2]}; \mathbb{C}^*)\]
are pure cocycles in the triple complex, since \(\lambda\) is closed. Thus \(d_2\delta \gamma\) descends to a uniquely defined Čech cocycle \(\mu_2 \in \check{C}_d^2(Y_2^{[2]}; \mathbb{C}^*)\) with \(d_1\mu_2 = d_2\delta \gamma\). Note that \(d_\mu_2 = 0\) by injectivity of \(d_1\) at the bottom level. Under \((s_2^*)^*\) this in turn pulls back to \(\beta_2 \in \check{C}_d^2(Y_1^{[2]}; \mathbb{C}^*)\) with \(d_2\beta_2 = \mu_2\). Now \(d_2(\delta \gamma - d_1\beta_2) = 0\) by construction, so there is a unique \(\beta_1 \in \check{C}_d^2(Y_1; \mathbb{C}^*)\) such that
\[-d_2\beta_1 = \delta \gamma - d_1\beta_2. \quad (4.14)\]
It follows that \(\mu_1 = d_1\beta_1\) satisfies \(d_1\mu_1 = 0\) and \(d_\mu_1 = 0\) (by injectivity of \(d_2\) on \(\check{C}_d^2(Y_1^{[2]}; \mathbb{C}^*)\) and the fact that \(d_\delta \mu_1 = d_\delta d_1 \beta_1 = -\delta^2 \delta \gamma = 0\).

Thus \(\delta \beta_2\) and \(\delta \beta_1\) descend, from \(Y_2\) and \(Y_1\) respectively, to define cocycles in \(\check{C}_d^3(X; \mathbb{C}^*)\); moreover these must be the same cocycle \(\alpha \in \check{C}_d^3(X; \mathbb{C}^*)\) by injectivity of \(d_1\) and \(d_2\) and the fact that \(d_1\delta \beta_2 = d_2\delta \beta_1\), so this represents the 4-class of the biggerbe. This shows that the difference \(d_1\beta_2 - d_2\beta_1\) is exact and the criterion therefore holds. \(\square\)

There is an analogue of Proposition 2.11 classifying trivializations of bundle gerbes.

**Proposition 4.18.** The trivializations of a bundle biggerbe \((L, W, Y_2, Y_1, X)\) form a torsor for the group
\[
\text{Im} \left\{ d_2 : \check{H}_2^a(Y_1^{[2]}; \mathbb{Z}) \rightarrow \check{H}_2^a(W^{[2,1]}; \mathbb{Z}) \right\}
\oplus \text{Im} \left\{ d_1 : \check{H}_2^a(Y_2^{[2]}; \mathbb{Z}) \rightarrow \check{H}_2^a(W^{[1,2]}; \mathbb{Z}) \right\}
\quad (4.15)
\]
where \(\check{H}_2^a\) denotes the \(\delta\) cohomology of the associated space in the diagram (4.8).

In specific cases, as for the Brylinski-McLaughlin biggerbe in §5.3, this may be simplified further.

**Proof.** If \((Q_1, Q_2)\) and \((Q'_1, Q'_2)\) are two trivializations of a biggerbe \(L\), then \(P_1 = Q'_1 \otimes Q_1^{-1} \rightarrow W^{[1,2]}\) and \(P_2 = Q'_2 \otimes Q_2^{-1} \rightarrow W^{[2,1]}\) are line bundles represented by Čech cocycles \(\alpha_i = c(P_i)\) satisfying \(d_1\alpha_i = d_2\alpha_i = \delta \alpha_i = 0\) for \(i = 1, 2\). Exactness of the rows and columns of (4.5) gives the existence of \(\beta_1 \in \check{Z}^1(Y_2^{[2]}; \mathbb{C}^*)\) and \(\beta_2 \in \check{Z}^1(Y_1^{[2]}; \mathbb{C}^*)\) satisfying \(d_2\beta_1 = \alpha_1\) and \(d_1\beta_2 = \alpha_2\), \(d_i\beta_i = 0\) and (by injectivity) \(\delta \beta_i = 0\). It is straightforward
to see that \([\alpha_1] \in \text{Im} \left\{ d_1 : \check{H}^1_{\mathbb{Z}}(Y_2^{[2]}; \mathbb{C}^*) \to \check{H}^1_{\mathbb{Z}}(W^{[1,2]}; \mathbb{C}^*) \right\} \) is well-defined independent of choices, and similarly for \([\alpha_2]\). Conversely, given elements in (4.15) corresponding to line bundles on \(W^{[1,2]}\) and \(W^{[2,1]}\) coming from simplicial line bundles \(Y_2^{[2]}\) and \(Y_1^{[2]}\), respectively, a trivialization \((Q_1, Q_2)\) may be altered to give a different trivialization of the same bigerbe. □

5. Examples of Bigerbes

5.1. Decomposable Bigerbes. As for the decomposable bundle gerbes discussed in §5.1, we consider the special classes of bigerbes corresponding to decomposable classes in \(H^4(X; \mathbb{Z})\). These are either of the form \(\alpha_1 \cup \alpha_2\) with \(\alpha_i \in H^2(X; \mathbb{Z})\) or of the form \(\rho \cup \alpha\) with \(\rho \in H^1(X; \mathbb{Z})\) and \(\alpha \in H^3(X; \mathbb{Z})\). Stuart Johnson in his PhD thesis, [9], makes related constructions in the setting of 2-gerbes.

From Theorem 4.17 it follows that if, for \(i = 1, 2\), \(\alpha_i \in H^2(X; \mathbb{Z})\) and \(\pi_i : Y_i \to X\), are locally split maps such that \(\pi^*_i \alpha_i = 0 \in H^2(Y_i; \mathbb{Z})\) then the cup product \(\alpha_1 \cup \alpha_2\) is represented by a bigerbe over the locally split square \((Y_1 \times_X Y_2, Y_2, Y_1, X)\). Indeed, in Čech theory if \(\rho \in C^1(Y_i; \mathbb{Z})\) are primitives for the \(\pi^*_i \alpha_i\) then \(\rho_1 \cup \alpha_2\) are primitives for \(\alpha_1 \cup \rho_2\) on \(Y_1\) and \(Y_2\), respectively, and pulled back to \(Y_1 \times_X Y_2\) their difference, \(\alpha_1 \cup \rho_2 - \rho_1 \cup \alpha_2\) has primitive \(\rho_1 \cup \rho_2\).

If the spaces \(Y_i\) are the total spaces of circle bundles representing the 2-classes the bigerbe is given quite explicitly in terms of the classifying line bundle, for decomposed 2-forms, over the torus.

Lemma 5.1. The fundamental line bundle on \(\mathbb{T}^2\) (with Chern class generating \(H^2(\mathbb{T}^2; \mathbb{Z}) = \mathbb{Z}\)) has a ‘bimultiplicative’ representative \(\mathbb{T}^2 \to \mathbb{Z}^2\), meaning there are natural isomorphisms between fibers \(S_{x_1+x_2, y_1+y_2} \cong S_{x_1, y_1} \otimes S_{x_2, y_2}\) and \(S_{x_1+y_1, y_2+y_2} \cong S_{x_1, y_1} \otimes S_{x_2, y_2}\) such that

\[
\begin{array}{ccc}
S_{x_1+x_2, y_1+y_2} & \to & S_{x_1, y_1+y_2} \otimes S_{x_2, y_1+y_2} \\
\downarrow & & \downarrow \\
S_{x_1+x_2, y_1} \otimes S_{x_1+x_2, y_2} & \to & S_{x_1, y_1} \otimes S_{x_2, y_1} \otimes S_{x_2, y_2} \otimes S_{x_2, y_2}
\end{array}
\] (5.1)

commutes.

Proof. Line bundles over \(\mathbb{T}^2 \cong \mathbb{R}^2/\mathbb{Z}^2\) are naturally identified with \(\mathbb{Z}^2\) equivariant line bundles over the universal cover, \(\mathbb{R}^2\). We equip the trivial bundle \(\mathbb{R}^2 \times \mathbb{C}\) with the \(\mathbb{Z}^2\) action covering translation via

\[(n, m) \cdot (x, y, z) = (x + n, y + m, e^{2\pi i m x} z), \quad (n, m) \in \mathbb{Z}^2, \quad (x, y, z) \in \mathbb{R}^2 \times \mathbb{C}\]

and let \(\mathbb{S} \to \mathbb{T}^2\) be the quotient bundle.

The vertical and horizontal identifications in (5.1) correspond to the invariance of the bilinear maps

\[\begin{align*}
(x_1, y_1, z_1) \times (x_2, y_2, z_2) &\mapsto (x_1, y_1 + y_2, z_1 z_2), \\
(x_1, y_1, z_1) \times (x_2, y_2, z_2) &\mapsto (x_1 + x_2, y, z_1 z_2)
\end{align*}\]
which commute under the $\mathbb{Z}_2$ actions. Moreover, the $\mathbb{Z}_2$-action corresponds to parallel transport along the circles with respect to the invariant connection $d + 2\pi i y dx$, the curvature of which is the fundamental class in $H^2(\mathbb{T}^2; \mathbb{Z})$.

**Proposition 5.2.** For a decomposed 4-class $\alpha_1 \cup \alpha_2 \in H^4(X; \mathbb{Z})$, with the $\alpha_i \in H^2(X; \mathbb{Z})$ represented by circle bundles $Y_i \to X$, the pullback under the product of the difference maps $\chi_i : Y_i \to X$, the pullback under $\pi : X \to \mathbb{C}^{\ast}$ corresponding to a homotopy class of maps $\tilde{\chi}_i : \mathbb{T}^2 \to X$ with characteristic class $G(L) = \alpha_1 \cup \alpha_2$.

**Proof.** The bimultiplicative relations of Lemma 5.1 correspond under pullback by $\chi_1 \times \chi_2$ to the bisimplicial conditions for $L$, and that $G(L) = \alpha_1 \cup \alpha_2$ is a consequence of Lemma 2.13.

Similarly if $\rho \in H^1(X; \mathbb{Z})$ and $\alpha \in H^3(X; \mathbb{Z})$ the representability condition is satisfied by the fiber product square given by any locally split maps $\pi_i : Y_i \to X$, $i = 1, 2$ such that $\pi_i^\ast \rho = 0 \in H^1(Y_i; \mathbb{Z})$ and $\pi_2^\ast \alpha = 0 \in H^3(Y_2; \mathbb{Z})$.

Taking the ‘logarithmic’ covering $\tilde{X} \to X$ corresponding to $\rho$, meaning the pullback of the universal cover of $U(1)$ by a homotopy class of maps $X \to U(1)$ representing $\rho$ and a bundle gerbe $(L, Y, X)$ with $\text{DD}(L) = \alpha \in H^3(X; \mathbb{Z})$, there is again a direct construction of a biggerbe for the fiber product square.

**Proposition 5.3.** If $(L, Y, X)$ is a bundle gerbe with Dixmier-Douady class $\alpha \in H^3(X; \mathbb{Z})$ and $\tilde{X} \to X$ is the logarithmic cover corresponding to a class $[\rho] \in H^1(X; \mathbb{Z})$ represented by $\rho : X \to U(1)$ then the line bundle

$$L^x \to \tilde{X}^{[2]} \times_X Y^{[2]},$$

where $\chi : \tilde{X} \times_X \tilde{X} \to \mathbb{Z}$ is the fiber-shift map, defines a biggerbe

$$(L^x, \tilde{X} \times_X Y, \tilde{X}, X) \quad \text{with} \quad G(L^x) = \rho \cup \alpha \in H^4(X; \mathbb{Z}).$$

**Proof.** We view the covering space $\tilde{X} \to X$ as a principal $\mathbb{Z}$ bundle, and then the shift map

$$\chi : \tilde{X}^{[2]} \to \mathbb{Z}$$

defines the collective bundle $L^x$ on $\tilde{X}^{[2]} \times_X Y^{[2]}$ given by the tensor product $L^n$ over $\chi^{-1}(n)$.

The bisimplicial space is

$$W^{[m, n]} = \tilde{X}^{[m]} \times_X Y^{[n]}.$$

and the line bundle $L^x$ is simplicial in the $d_2$ direction, with trivializing section of $d_2(L^x) = (d_2L)^x$ over $W^{[2, 3]}$ given by $s^x$, and the $d_1$ differential of $L^x$ is given by

$$d_1(L^x) = L^{d_1x} = L^0.$$
so is canonically trivial. Thus this is indeed a bigerbe.

To see that $G(L^X) = ρ ∪ α$, observe that representative cocycles $c(L) ∈ \hat{C}^2(Y^2; Z)$ and $χ ∈ \hat{C}^0(\tilde{X}^2; Z)$ pull back to $\hat{C}^\bullet(\tilde{X}^2 × X Y^2; Z)$ by the fiber product projections, and their cup product $χ ∪ c(L) ∈ \hat{C}^2(\tilde{X}^2 × X Y^2; Z)$ represents the transgression image of $ρ ∪ α$ from $X$. Then $χ ∪ c(L) = nc(L) = [L^n]$ locally on components $χ^{-1}(n)$, so the result follows.

5.2. Doubling for bigerbes. As in §3, we may incorporate an additional simplicial structure with respect to the space of products $EX_\bullet = X_\bullet$ in order to promote examples of bigerbes involving based loop spaces to examples involving free loop spaces.

**Definition 5.4.** A bigerbe $L$ on $X^2$ will be said to be **double** if the bigerbe

$$\partial L = \pi_0^4 L \otimes \pi_1^4 L^{-1} \otimes \pi_2^4 L$$

is trivial on $X^3$ with respect to the three projection maps $π_1 : X^3 → X^2$. In the absence of additional data, $\partial L$ is defined with respect to the bisimplicial space over $X^3$ obtained by the fiber products of the three pullbacks of the bisimplicial space $W^2_2[\bullet, \bullet]$ over $X^2$. However, as for gerbes above, it will typically be the case that $X^3$ carries a natural split square and induced bisimplicial space $W^3[\bullet, \bullet]$ along with maps $W^3[\bullet, \bullet] \to W^2_2[\bullet, \bullet]$ over the projections $X^3 \to X^2$. In this case, it suffices that $\partial L → W^3_3[2,2]$, defined by pulling back along the three maps $W^3_3[2,2] → W^2_2[2,2]$ and taking the alternating product, admits a bigerbe trivialization.

As a special case relevant in our primary example, a bigerbe is a double if $\partial L$ is itself trivial as a line bundle over $W^3_3[2,2]$. Naturality of the bigerbe characteristic class and exactness of the sequence (3.5) together lead to the following analogue of Proposition 3.2.

**Proposition 5.5.** The characteristic 4-class, $G(L)$, of a double bigerbe $L$ on $X^2$, descends from $H^4(X; Z)$ to $H^4(X; Z)$.

In §5.5 we will also consider a similar condition with respect to the bisimplicial space of products $X^{m,n} = X^{mn}$, with the two sets of projections $π^j_1 : X^{m,n} → X^{m-1,n}$ and $π^j_2 : X^{m,n} → X^{m,n-1}$. A bigerbe $L$ over $X^4 = X^{2,2}$ is quadruple if $\partial_1 L$ and $\partial_2 L$ are respectively trivial over $X^{1,2} = X^2$ and $X^{2,1} = X^2$. The natural differentials $∂_1$ and $∂_2$, defined on cohomology $H^4(X^{\bullet, \bullet}; A)$, commute, and from exactness of these we obtain the following result.

**Proposition 5.6.** For a quadruple bigerbe $L$ on $X^4$, the characteristic 4-class $G(L)$ descends from $H^4(X^4; Z)$ to $H^4(X; Z)$. 
5.3. Brylinski-McLaughlin bigerbes. Next we turn to our main application. The loop space of a principal $G$ bundle over a manifold is a principal bundle over the loop space with structure group the loop group of $G$. The Brylinski-McLaughlin bigerbe captures the obstruction to lifting this bundle to a (loop-fusion) principal bundle for a central extension of the loop group. While the version involving based path and loop spaces is simpler, we focus from the beginning on the doubled version involving free path and loop spaces, as this gives the results of primary interest. Note that this theory most naturally involves $U(1)$ principal bundles in place of line bundles, which we shall use for the remainder of the section without further comment.

Let $G$ be a compact, simple, connected and simply connected group. As is well-known (see for instance [19]), there is a classification of $U(1)$ central extensions

$$1 \to U(1) \to \hat{LG} \to LG \to 1$$

of the loop group $LG$ by $H^3(G;\mathbb{Z}) \cong H^3_G(G;\mathbb{Z}) \cong \mathbb{Z}$. These extensions descend to the quotient $LG/G \cong \Omega G$ and so the classification of central extensions of the based loop group $\Omega G$ is equivalent.

Forgetting the group structure for the moment, such a central extension may be viewed as a circle bundle over $LG \cong I^G$, the Chern class $c(\hat{LG}) \in H^2(LG;\mathbb{Z})$ of which is the transgression of the defining class in $H^3(G;\mathbb{Z})$. As such, it follows from Theorem 3.6 that $c(\hat{LG})$ has a loop-fusion refinement.

In the equivalent language of the loop-fusion structures of Definition 3.4, we may restate this as follows.

**Theorem 5.7.** As a $U(1)$-bundle, $\hat{LG} \to LG$ has a canonical loop-fusion structure, meaning a trivialization of $d\hat{LG} \to I^G$ inducing the canonical trivialization of $d^2\hat{LG} \to I^G$ and a trivialization of $\partial \hat{LG} \to L_{S^G}$.

**Remark.** In fact, the additional structure that promotes a general $U(1)$-principal bundle over $LG$ to a central extension is also a simplicial one. Indeed, as noted by Brylinski and McLaughlin in [4] and attributed to Grothendieck, a $U(1)$ central extension of any group $H$ is equivalent to a simplicial circle bundle with respect to the simplicial space $BH_\bullet$ defined by $BH_k = H^{k-1}$ with the face maps $H^{k+1} \to H^k$ given by

$$\pi_i : (h_0, h_1, \ldots, h_k) \mapsto \begin{cases} (h_1, \ldots, h_k), & i = 0 \\ (h_0, \ldots, h_{i-1} h_i, h_{i+1}, \ldots, h_k), & 1 \leq i \leq k \\ (h_0, \ldots, h_{k-1}), & i = k. \end{cases}$$

Thus, given a circle bundle $Q \to H = BH_2$, a trivialization of $\partial Q = \pi_0^* Q \otimes \pi_1^* Q^{-1} \otimes \pi_2^* Q$ inducing the canonical trivialization of $d^2 Q \to H^3$ equips $Q$ with the (associative) multiplicative structure of a $U(1)$ central extension of $H$ and vice versa.

For the groups under consideration, we believe it can be shown that the classes in $H^3_G(G;\mathbb{Z}) = H^3(|BG|;\mathbb{Z})$ are represented by cohomology classes
\[ \alpha \in H^3(G = BG_2; \mathbb{Z}) \] satisfying \( \partial \alpha = 0 \in H^3(G^2 = BG_3; \mathbb{Z}) \), and that the corresponding gerbe \((W, PG, G)\), with circle bundle \( W \rightarrow \Omega G \), admits a simplicial structure with respect to \( B\Omega G \), and thus a central extension of \( \Omega G \). Further considering a doubled structure with respect to \( G_\bullet = G^\bullet \) gives rise to the central extensions of \( LG \). For reasons of space, and since the theory of central extensions of \( LG \) is already well-known, we will not elaborate further on this point.

To define the Brylinski-McLaughlin bigerbe, let \( X \) be a connected manifold with principal \( G \)-bundle \( E \rightarrow X \). One case of particular interest is the spin frame bundle over a spin manifold of dimension \( \geq 5 \).

**Lemma 5.8.** With vertical maps projections and evaluation at end- and mid-points in the horizontal directions, the diagrams

\[
\begin{array}{ccc}
E^k & \xleftarrow{\pi} & I_k E \\
\downarrow & & \downarrow \\
X^k & \xleftarrow{\pi} & I_k X
\end{array}
\]

for \( k \geq 2 \) are locally split squares.

**Proof.** The maps are locally trivial fiber bundles, so all maps are locally split, and \( I_k E \) is likewise a fiber bundle over the fiber product \( I_k X \times_{X^k} E^k \), which is the space of paths in \( X \) along with prescribed points in \( E \) over the endpoints (for \( k = 2 \)) and midpoint (for \( k = 3 \)) of the path. A connection on \( E \) gives a horizontal lift of each path segment in \( X \) given an initial point in \( E \), and from the connectedness of \( G \) this can be concatenated with a path in the fiber from the endpoint of the lifted path segment to any other prescribed point in the same fiber; this can be done for each segment defined between the \( k \) marked points of the path on \( X \). This construction can be carried out locally continuously, so giving a local section of \( I_k E \) over \( E^k \times_{X^k} I_k X \). \( \Box \)

In the resulting bisimplicial diagrams we may write \( IE^{[2]} \), etc., without risk of confusion in light of the canonical isomorphisms

\[
\begin{align*}
(I E)^{[2]} &= I E \times_{IX} I E \cong I(E^{[2]}) = I(E \times_X E), \\
(L E)^{[2]} &= L E \times_{LX} L E \cong L(E^{[2]}) = L(E \times_X E), \end{align*}
\]
Filling out the bisimplicial space for \( k = 2 \) by fiber products leads to the diagram

\[
\begin{array}{cccccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
E^{[3]} & \xleftarrow{IE^{[3]}} & LE^{[3]} & \equiv & I^{[3]}E^{[3]} & \quad & \quad & \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
E^{[2]} & \xleftarrow{IE^{[2]}} & LE^{[2]} & \equiv & I^{[2]}E^{[2]} & \quad & \quad & \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
E^2 & \xleftarrow{IE} & LE & \equiv & I^{[3]}E & \quad & \quad & \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
X^2 & \xleftarrow{IX} & LX & \equiv & I^{[3]}X & \quad & \quad & \\
\end{array}
\]

(5.4)

The third column of (5.4) is the simplicial space generated by the fibration \( LE \rightarrow LX \), itself a principal bundle with structure group \( LG \), and thus supports a lifting bundle gerbe

\[
Q = \chi^*\widehat{LG} \rightarrow LE^{[2]},
\]

where

\[
\chi : LE^{[2]} \ni (l_1(\theta), l_2(\theta)) \mapsto \ell(\theta) \in LG, \quad l_2(\theta) = \ell(\theta)l_1(\theta)
\]

is the shift map of the principal bundle, and we consider \( \widehat{LG} \rightarrow LG \) as a \( U(1) \)-bundle. The other columns are likewise the simplicial spaces of principal bundles, with structure groups \( G \) and \( I^kG \) for \( k \geq 1 \), and we denote their associated shift maps by the same letter.

**Theorem 5.9.** Given a central extension (5.2) of level \( \ell \in \mathbb{Z} = H^1_G(G; \mathbb{Z}) \), the lifting bundle gerbe \( Q \rightarrow LE^{[2]} \) is the double bigerbe \( (Q, IE, E^2, IX, X^2) \) with characteristic class

\[
G(Q) = \ell p_1(E) \in H^4(X; \mathbb{Z}),
\]

where \( p_1(E) \) is the first Pontryagin class of \( E \).

The bigerbe \( (Q, IE, E^2, IX, X^2) \) will be called the **Brylinski-McLaughlin** bigerbe.

**Proof.** It follows immediately from the lifting gerbe construction that \( Q \) is vertically simplicial, and the simplicial condition in the horizontal direction follows from naturality of the shift map and Theorem 5.7.

Indeed, unwinding the definitions reveals that \( d_1\chi^*\widehat{LG} = \chi^*d\widehat{LG} \), which admits the trivialization noted in Theorem 5.7 inducing the canonical trivialization of \( \chi^*d^2\widehat{LG} = d_1^2\chi^*\widehat{LG} \).
For doubling, we define $\partial Q$ with respect to the locally split square (5.3) for $k = 3$, the induced bisimplicial space of which sits in the diagram

$$
\begin{array}{cccc}
(E^{[3]})^3 & \rightarrow & I E^{[3]} & \rightarrow & L_8 E^{[3]} & \rightarrow & I^{[3]} E^{[3]} & \rightarrow \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
(E^{[2]})^3 & \rightarrow & I E^{[2]} & \rightarrow & L_8 E^{[2]} & \rightarrow & I^{[3]} E^{[2]} & \rightarrow \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
E^3 & \rightarrow & I E & \rightarrow & L_8 E & \rightarrow & I^{[3]} E & \rightarrow \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
X^3 & \rightarrow & I X & \rightarrow & L_8 X & \rightarrow & I^{[3]} X & \rightarrow \\
\end{array}
$$

(5.6)

with the obvious maps to (5.4), and where we have omitted the subscript 3 on the path spaces and made the identification $L_8 X \cong I^{[3]} X$, etc. Once again, it follows that $\partial Q = \partial \chi^* \hat{LG} = \chi^* \partial LG$ with respect to the shift map for the principal $L_8 G$-bundle $L_8 E \rightarrow L_8 X$, which admits a trivialization in light of Theorem 5.7. Thus $Q$ is product simplicial in the sense of Definition 5.4 and its characteristic class descends to $H^4(X; \mathbb{Z})$.

Observe that this characteristic class can be obtained in two steps, first by regressing the Chern class $c(Q)$ from $H^2_Z(LE^{[2]}; \mathbb{Z})$ to $H^3_Z(LX; \mathbb{Z})$ and then to $H^4(X; \mathbb{Z})$. The image of $c(Q)$ in $H^2_Z(LX; \mathbb{Z})$ is, essentially by definition, the Dixmier-Douady class of the lifting bundle gerbe (or more precisely, its loop-fusion refinement), and it is well-known that this class is the transgression of the Pontryagin class of $E$ on $X$ (multiplied by $\ell$ in the case of higher levels), so from Theorem 3.6 we obtain (5.5).

□

Without the vertical simplicial condition, the gerbe $(Q, LE^{[2]}, LX)$ represents the obstruction to lifting the $LG$ bundle $LE \rightarrow LX$ to an $\hat{LG}$ bundle $\hat{LE} \rightarrow LX$. The enhancement of this data to a bigerbe carries additional information, which is formalized in the following definition.

**Definition 5.10.** Let $E \rightarrow X$ be a principal $G$-bundle for $G$ a simple, connected and simply connected Lie group, and fix a central extension (5.2) of $LG$ of level $\ell \in \mathbb{Z} = H^3_G(G; \mathbb{Z})$. A loop-fusion $\hat{LG}$ lift of $LE \rightarrow LX$ is a principal $\hat{LG}$-bundle $\hat{LE} \rightarrow LX$ lifting $LE$ with the property that $\hat{LE} \rightarrow LE$ is loop-fusion as a $U(1)$-bundle; in other words, $d\hat{LE} \rightarrow I^{[3]} E$ has a trivialization inducing the canonical trivialization of $d^2\hat{LE} \rightarrow I^{[4]} E$ and $\partial \hat{LE} \rightarrow L_8 E$ admits a trivialization.

Without the additional figure-of-eight structure, such fusion lifts have been considered by Waldorf in [29], and with stronger conditions (high regularity and equivariance with respect to diffeomorphisms of $S^1$) by the authors in [11].
Lemma 5.12. \( \partial \) commutes with the product simplicial operator an isomorphism in this case, so the trivializations are classified simply by \( \mathbb{H} \) by the Serre spectral sequence for \( \mathcal{L} \) lifts of \( \mathcal{L} \) to those \( \mathcal{I} \) \( \mathcal{M} \) in particular the condition that \( \partial \mathcal{P} \) is trivial for \( \mathcal{P} \rightarrow \mathcal{I} \mathcal{E} \) means that \( \mathcal{P} \) itself is trivial. Thus doubled trivializations for the biggerbe in question are reduced to loop-fusion line bundles \( \mathcal{P} \rightarrow \mathcal{L} \mathcal{E} \) satisfying \( d_2 \mathcal{P} \cong \mathcal{Q} \). On the one hand, these are clearly equivalent to loop-fusion \( \mathcal{L} \mathcal{G} \) lifts of \( \mathcal{L} \mathcal{E} \rightarrow \mathcal{L} \mathcal{X} \), and on the other, they are classified by those classes in \( \mathcal{H}^3 \mathcal{G}(\mathcal{L} \mathcal{E}; \mathbb{Z}) \) with image \( \mathcal{C}(\mathcal{Q}) \) under \( d_2 \).

By Lemma 5.12 below, the difference of any two such classes descends to a class in \( \mathcal{H}^3 \mathcal{G}(\mathcal{L} \mathcal{X}; \mathbb{Z}) \), and so doubled trivializations form a torsor for the image of \( \mathcal{H}^3 \mathcal{G}(\mathcal{L} \mathcal{E}; \mathbb{Z}) \), which by Theorem 3.6 is equivalent to the image of \( \mathcal{H}^3(\mathcal{X}; \mathbb{Z}) \) in \( \mathcal{H}^3(\mathcal{E}; \mathbb{Z}) \). Finally, given the conditions on \( \mathcal{G} \), it follows by the Serre spectral sequence for \( \mathcal{E} \rightarrow \mathcal{X} \) that \( \mathcal{H}^3(\mathcal{X}; \mathbb{Z}) \rightarrow \mathcal{H}^3(\mathcal{E}; \mathbb{Z}) \) is an isomorphism in this case, so the trivializations are classified simply by \( \mathcal{H}^3(\mathcal{X}; \mathbb{Z}) \).

\[ \square \]

It remains to show that the exactness of \( d_2 \) in the Čech-simplicial double complex is consistent with \( \partial \), which is a consequence of the following.

Lemma 5.12. The homotopy chain contraction for \( d_2 \) in the triple complex \( (\mathbb{Z}^*W^{k*}), \delta, d_1, d_2) \) for the locally split squares \( (\mathcal{I}k \mathcal{E}, \mathcal{E}^k, \mathcal{I}k \mathcal{X}, \mathcal{X}^k) \) commutes with the product simplicial operator \( \partial \).

Proof. This follows ultimately from the existence of local sections of \( \mathcal{E}^k \rightarrow \mathcal{X}^k \) (resp. \( \mathcal{I} \mathcal{E} \rightarrow \mathcal{I} \mathcal{X} \)) which are intertwined by the three projection maps \( \mathcal{E}^3 \rightarrow \mathcal{E}^2 \) and \( \mathcal{X}^3 \rightarrow \mathcal{X}^2 \) (resp. \( \mathcal{I}3 \mathcal{E} \rightarrow \mathcal{I}2 \mathcal{E} \) and \( \mathcal{I}3 \mathcal{X} \rightarrow \mathcal{I}2 \mathcal{X} \)), which we proceed to demonstrate. In the first case, we may fix an admissible pair of covers \( (\mathcal{V}, \mathcal{U}) \) for \( (\mathcal{E}, \mathcal{X}) \) and then equip \( \mathcal{E}^k \) and \( \mathcal{X}^k \) with the covers \( \mathcal{V}^k \) and \( \mathcal{U}^k \) along with the induced sections \( \mathbb{E}(\mathcal{U}^k) \rightarrow \mathbb{E}(\mathcal{V}^k) \), which are then automatically intertwined by the projection maps.

For the path spaces, we begin with the fact that \( (\mathcal{I} \mathcal{E}, \mathcal{I} \mathcal{X}, \mathcal{E}^3, \mathcal{X}^3) \) is a locally split square, so \( \mathcal{I}3 \mathcal{E} \) and \( \mathcal{I}3 \mathcal{X} \) admit covers \( \mathcal{W} \) and \( \mathcal{Y} \) and sections \( \mathbb{E}(\mathcal{Y}) \rightarrow \mathbb{E}(\mathcal{W}) \) lying over the sections \( \mathbb{E}(\mathcal{U}^3) \rightarrow \mathbb{E}(\mathcal{V}^3) \). In general, the reparameterization maps \( \mathcal{I}3 \mathcal{Y} \cong \mathcal{I}2 \mathcal{Y} \) are open, so we may equiv \( \mathcal{I}2 \mathcal{E} \) and \( \mathcal{I}2 \mathcal{X} \) with the union of the three image covers \( \mathcal{W}' = \mathcal{P}0(\mathcal{W}) \cup \mathcal{P}1(\mathcal{W}) \cup \mathcal{P}2(\mathcal{W}) \) and \( \mathcal{Y}' = \mathcal{P}0(\mathcal{Y}) \cup \mathcal{P}1(\mathcal{Y}) \cup \mathcal{P}2(\mathcal{Y}) \), along with the induced section

\[ \mathcal{S}' : \mathbb{E}(\mathcal{Y}') \rightarrow \mathbb{E}(\mathcal{W}) , \]
giving a set of local sections of $I_2E \to I_2X$ which is compatible with the three reparameterization maps $\bar{\pi}_i$, and which covers the local sections of $E^2 \to X^2$. □

There is a simpler version of this biggerbe using based path and loop spaces, starting with the locally split square $(PE, E, PX, X)$, pulling back a central extension $\tilde{\Omega}G \to \Omega G$ to $\Omega E[2]$, and omitting the doubling conditions. We leave the details as an exercise to the reader.

5.4. Loop spin structures. There is a well-known relationship between string structures on a spin manifold $X$ of dimension $2n > 4$, and (loop) spin structures on its loop space $LX$.

Here, a string structure is a lift of the principal Spin$(2n)$ bundle $E \to X$ to a principal bundle with structure group String$(2n)$, a 3-connected topological group covering Spin$(2n)$ in the sequence of ever more connected groups that form the Whitehead tower for $O(2n)$, see for instance [23]. The string group cannot be a finite dimensional Lie group (having a subgroup with the topology of $K(\mathbb{Z}, 2)$), though there are various realizations as a 2-group [2, 21]. The obstruction to lifting the structure group is $\frac{1}{2}p_1(X) \in H^4(X; \mathbb{Z})$ (the Pontryagin class of the Spin bundle being a refinement of the Pontryagin class of the oriented frame bundle), and if unobstructed, string structures are classified by $H^3(X; \mathbb{Z})$ [25].

As originally defined by Killingback in [10] and further developed by McLaughlin in [15], a spin structure on $LX$ is a lift of the $L$Spin bundle $LE \to LX$ to the structure group $L$Spin, the fundamental U$(1)$ central extension of $L$Spin. (By analogy, as originally suggested by Atiyah in [1], an orientation on $LX$ is a refinement of the $L$SO$(2n)$ bundle $LE_{SO} \to LX$ to have structure group the connected component of the identity, $L_+SO(2n) \cong L$Spin$(2n)$, and therefore is typically related to a spin structure on $X$.) The obstruction to this lift is the 3-class on $LX$ obtained by transgression of $\frac{1}{2}p_1(X) \in H^4(X; \mathbb{Z})$.

As defined, string structures on $X$ and spin structures on $LX$ are not necessarily in bijection [18]. In fact, it was Stolz and Teichner in [24] who first noted the importance of the fusion structure on $LX$ and showed that string structures on $X$ were in correspondence with what they called ‘stringor bundles’ on (the piecewise smooth loop space) $LX$, essentially bundles associated to a lift $\tilde{LE}$ along with a fusion condition. It was further proved by the authors in [11] that string structures in the sense of Redden [20] correspond with spin structures on the smooth loop space $LX$ which are both fusion and equivariant for the group Diff$^+(S^1)$ of oriented diffeomorphisms of the loop parameter, and it was independently proved by Waldorf in [29] that string structures on $X$ exist if and only if fusion spin structures on (piecewise smooth) $LX$ exist, using a transgression theory relating the 2-gerbe obstructing string structures of [6] and fusion gerbes on loop space. Waldorf did not obtain a complete correspondence between string structures
and fusion loop spin structures, noting that this would necessitate additional conditions such as equivariance with respect to thin homotopy; in the version considered here, it is the figure-of-eight (i.e., doubling) condition that provides the remedy.

In any case, the bigerbe formulation here leads to the following result.

**Corollary 5.13.** There are natural bijections between the following sets:

(i) the set of string structures on a spin manifold $X$ of dimension $2n > 4$,

(ii) the set of loop-fusion spin structures on $LX$, meaning lifts of $LE \to LX$ to the structure group $\tilde{L}\text{Spin}$ such that the resultant $U(1)$ bundle $\tilde{L}E \to LE$ is a loop-fusion bundle according to Definition 3.4, and

(iii) the set of doubled trivializations of the Brylinski-McLaughlin bigerbe $(Q, IE, E^2, IX, X^2)$.

The sets are empty unless $\frac{1}{2}p_1(E) = 0 \in H^4(X; \mathbb{Z})$ and otherwise are torsors for $H^3(X; \mathbb{Z})$.

**5.5. Path bigerbes.** If $X$ is a path-connected and simply connected space with basepoint $b$, from the based double path space

$$QX = PPX = \{ u : [0,1]^2 \to X : u|_{\{0\} \times [0,1]} = u|_{[0,1] \times \{0\}} = b \}$$

there are two surjective restriction maps

$$f_i : QX \to PX, \quad f_1 u = u|_{[0,1] \times \{1\}} \quad \text{and} \quad f_2 u = u|_{\{1\} \times [0,1]}$$

**Theorem 5.14.** On a connected, simply connected and locally contractible space the end-point maps and restriction maps in (5.8) form a locally split square

$$\begin{array}{ccc}
PX & \xleftarrow{f_1} & QX \\
\downarrow & & \downarrow f_2 \\
X & \leftarrow & PX
\end{array}$$

and any class $\gamma \in H^4(X, \mathbb{Z})$ arises from a bigerbe corresponding to (5.9).

**Proof.** The fiber product of the two copies of $PX$ is the based loop space of $X$. The simple connectedness of $X$ implies the fiber product of the two $f_i$ is surjective and from local contractibility it is locally split. Since $PX$ and $QX$ are both contractible, Theorem 4.17 applies to any 4-class on $X$. \qed

Since the Eilenberg-Mac Lane spaces can be represented by CW complexes Theorem 5.14 applies in particular to $K(\mathbb{Z}, 4)$.

**Theorem 5.15.** There exists a universal bigerbe over $K(\mathbb{Z}, 4)$ with respect to the locally split square (5.9) with $X = K(\mathbb{Z}, 4)$.
Note the structure of the bisimplicial space in this case, in which $\ast$ represents a contractible space:

\[
\begin{array}{c}
K(\mathbb{Z}, 3) \leftarrow \ast \leftrightarrow K(\mathbb{Z}, 2) \quad \cdots \\
\downarrow & \downarrow & \downarrow \\
\ast & \leftrightarrow & \ast \leftrightarrow \ast \quad \cdots \\
\downarrow & & \downarrow \\
K(\mathbb{Z}, 4) & \leftarrow & \ast \leftrightarrow K(\mathbb{Z}, 3) \quad \cdots 
\end{array}
\]

Finally, incorporation of a product-bisimplicial condition allows any 4-class to be represented as a bigerbe on any connected, locally contractible space $X$, whether simply connected or not. Indeed, consider the locally split square $(IIX, IX^2, IX^2, X^4)$, where $IIX = \{ u : [0,1]^2 \to X \}$ is the free double path space and the projection maps are given by evaluation at both endpoints of a given path factor. The induced bisimplicial space becomes

\[
\begin{array}{c}
LX^2 \leftarrow ILX \leftrightarrow LLX \quad \cdots \\
\downarrow & \downarrow & \downarrow \\
IX^2 & \leftarrow IIIX \leftrightarrow LIX \quad \cdots \\
\downarrow & \downarrow & \downarrow \\
X^4 & \leftarrow IX^2 \leftrightarrow LX^2 \quad \cdots 
\end{array}
\] (5.10)

and in particular $W^{[2,2]} = LLX$ is the double free loop space of $X$. We may view $X^4$ at the bottom as the factor $X^{2,2}$ in the bisimplicial space $X^{m,n} = X^{mn}$ of products as discussed in §5.2. Over $X^{3,2}$ consider the locally split square and induced bisimplicial space

\[
\begin{array}{c}
LX^3 \leftarrow I_3LX \leftrightarrow L_3LX \quad \cdots \\
\downarrow & \downarrow & \downarrow \\
I_2X^3 & \leftarrow I_3I_2X \leftrightarrow L_3I_2X \quad \cdots \\
\downarrow & \downarrow & \downarrow \\
X^{3,2} & \leftarrow I_3X^2 \leftrightarrow L_3X^2 \quad \cdots 
\end{array}
\] (5.11)

and likewise for $X^{2,3}$ with factors reversed. The bisimplicial spaces (5.11) map to (5.10) over the product maps $X^{3,2} \cong X^{2,2}$ and $X^{2,3} \cong X^{2,2}$, and there are associated operators $\partial_1$ and $\partial_2$ on line bundles.

**Theorem 5.16.** For a connected, locally contractible space $X$, every class in $H^4(X; \mathbb{Z})$ is represented by a product-bisimplicial bigerbe with respect to (5.10), that is, a bigerbe $(L, IIX, IX^2, IX^2, X^4)$ having in addition trivializations of the line bundles $\partial_1 L \to L_8LX$ and $\partial_2 L \to LL_8X$. 
Proof. Such a bigerbe has characteristic class $G(L) \in H^4(X^{2,2}; \mathbb{Z})$ satisfying $\partial_1 G(L) = 0 \in H^4(X^{3,2}; \mathbb{Z})$ and $\partial_2 G(L) = 0 \in H^4(X^{2,3}; \mathbb{Z})$, hence by Proposition 5.6 this descends to a well-defined class

$$G(L) \in H^4(X; \mathbb{Z}).$$

Conversely, given any $\alpha \in H^4(X; \mathbb{Z})$, let $\beta = \partial_1 \partial_2 \alpha \in H^4(X^{2,2}; \mathbb{Z})$. This evidently satisfies $\partial_i \beta = 0$ for $i = 1, 2$, and moreover, denoting by $\Delta_i : X^2 \hookrightarrow X^{2,2}$ for $i = 1, 2$ the diagonal inclusions, satisfies $\Delta_i^* \beta = 0 \in H^4(X^2; \mathbb{Z})$. Under the deformation retractions $IX^2 \simeq X^2$, the evaluation maps $IX^2 \to X^{2,2}$ become identified with these diagonal inclusions, so it follows that $\beta$ lifts to vanish in $H^4(IX^2; \mathbb{Z})$. Then since $IIX \simeq \ast$ is contractible, Theorem 4.17 applies and it follows that $\beta$ is represented by a bigerbe $(L, IIX, IX^2, IX^2)$, such that $\partial_i L$ is trivial (as a bigerbe) for $i = 1, 2$. As a consequence of Lemma 3.3, which applies to both horizontal and vertical directions in the diagram (5.11), bigerbe triviality of $\partial_i L$ is equivalent to triviality of $\partial_i L$ as a line bundle. □

6. Multigerbes

We end by sketching out the theory of multigerbes, the higher degree generalization of bigerbes. By contrast to bundle gerbes, this generalization to higher degree is straightforward, with symmetry of the simplicial conditions replacing the need for higher and ever more complicated associativity conditions.

Fix a degree $n \in \mathbb{N}$, where $n = 1$ and $n = 2$ correspond to bundle gerbes and bigerbes, respectively. To establish notation, let $e_j = (0, \ldots, 1, \ldots, 0)$ denote the $j$th standard basis vector, and for each integer $k$ let $\underline{k} = (k, \ldots, k)$ denote the vector with constant entries. For a multiindex $\alpha \in \mathbb{N}^n$ we let $|\alpha| = \alpha_1 + \cdots + \alpha_n \in \mathbb{N}$, and we use the partial order on multiindices where $\beta < \alpha$ if $\beta_i < \alpha_i$ for each $1 \leq i \leq n$. We distinguish the sets of natural numbers starting at 1 and 0 respectively by $\mathbb{N}_0$ and $\mathbb{N}_1$.

**Definition 6.1.** Fix a set of spaces $X_\alpha$ indexed by $\alpha = (\alpha_1, \ldots, \alpha_n) \in \{0, 1\}^n$ along with continuous surjective maps

$$X_\alpha \to X_{\alpha-e_j} \quad \text{whenever } \alpha_j = 1.$$  \hfill (6.1)

For each $\alpha \in \mathbb{N}^n$ define

$$X_{<\alpha} = \lim_{\beta<\alpha} X_\beta$$

to be the limit space of the diagram formed by all spaces to which $X_\alpha$ maps along with the maps between them.

Then the set $\{X_\alpha : \alpha \in \{0, 1\}^n\}$ is a **locally split $n$-cube** provided each of the induced maps $X_\alpha \to X_{<\alpha}$ is locally split.

In particular, a locally split 1-cube is just a locally split map $X_1 \to X_0$ and a locally split 2-cube is a locally split square. It follows, in particular by the proof of Lemma 6.6, that all of the maps (6.1) are locally split, and
that each subcube determined by freezing some of the coefficients of $\alpha$ is likewise locally split.

**Lemma 6.2.** A locally split $n$-cube extends naturally by taking fiber products to a set of spaces $\{X_\alpha : \alpha \in \mathbb{N}_0^n\}$ such that $X_{\bullet \geq 1} = \{X_\alpha : \alpha \in \mathbb{N}_1^n\}$ is an $n$-fold multisimplicial space over $X := X_0$; in particular for each fixed $\alpha = (\alpha_1, \ldots, 0, \ldots, \alpha_n)$ with vanishing $j$th coordinate the sequence

$$X_\alpha \leftarrow X_{\alpha + e_j} \leftrightarrow X_{\alpha + 2e_j} \leftrightarrow X_{\alpha + 3e_j} \ldots$$

is the simplicial space of fiber products of the map $X_{\alpha + e_j} \to X_\alpha$.

**Proof.** The proof is by induction on $n$, the case $n = 2$ having been proved as Proposition 4.4. Assuming therefore that the result holds for $n - 1$, the ‘hypersurfaces’ $\{X_\alpha : \alpha_j \equiv 0\}$ are well-defined for $1 \leq j \leq n$, and for general $\alpha \in \mathbb{N}_1^n$ define $X_\alpha$ as a subspace of $X_{\alpha 1 \cdots \alpha n}$ as follows. For each $j$, there are $\alpha_j$ projection maps $X_{\alpha 1 \cdots \alpha n} \to X_{\alpha 1 \cdots \alpha_{j-1} \alpha_{j+1} \cdots \alpha n}$, where the caret denotes omission, and these may be composed with the structure map

$$X_{\alpha 1 \cdots \alpha_{j-1} \alpha_{j+1} \cdots \alpha n} \to X_{\alpha 1 \cdots \alpha_{j-1} \alpha_j \cdots \alpha n}.$$

Introducing the notation $\alpha(j)$ to mean the multiindex obtained from $\alpha$ by setting $\alpha_j = 0$, we may view $X_{\alpha(j)}$ as a subspace of $X_{\alpha 1 \cdots \alpha_{j-1} \alpha_{j+1} \cdots \alpha n}$, and then $X_\alpha$ is well-defined as the subspace of $X_{\alpha 1 \cdots \alpha n}$ in the mutual preimage of $X_{\alpha(j)}$ under the $\alpha_j$ maps $X_{\alpha 1 \cdots \alpha_n} \to X_{\alpha 1 \cdots \alpha_{j-1} \alpha_{j+1} \cdots \alpha n}$ for each $j$. \qed

We denote the face maps of the multisimplicial space by $\pi_k^j$, for $1 \leq j \leq n$. Then for each $\alpha \in \mathbb{N}_0^n$, there are $n$ simplicial differentials defined on a line bundle $L \to X_\alpha$ by

$$d_j L = \bigotimes_{k=0}^{\alpha_j} (\pi_k^j)^* L^{(-1)^k} \to X_{\alpha + e_j}, \quad 1 \leq j \leq n,$$

with the property that $d_j^2 L \to X_{\alpha + 2e_j}$ is canonically trivial.

**Definition 6.3.** A bundle $n$-multigerbe, or simply multigerbe, defined with respect to a locally split $n$-cube $X_\alpha$ is a multisimplicial line bundle $L \to X_\alpha$, meaning $d_j L$ is given a trivializing section $s_j$ inducing the canonical trivialization of $d_j^2 L$ for each $j$, and the induced trivializations of $d_j d_k L \equiv d_k d_j L$ are consistent for each pair $j \neq k$.

A set $(P_1, \ldots, P_n)$ of line bundles $P_j \to X_{2-e_j}$ such that each $P_j$ is multisimplicial with respect to $d_i$ for $i \neq j$ determines a multisimplicial line bundle $\bigotimes_{j=1}^n d_j P_j^{(-1)^j} \to X_2$ in the obvious way, and a trivialization of an $n$-multigerbe $L$ consists of a set $(P_1, \ldots, P_n)$ as above along with an isomorphism

$$L \cong \bigotimes_{j=1}^n d_j P_j^{(-1)^j}.$$
Pullbacks, products and morphisms of multigerbes are defined by generalizing in the obvious way those same operations for bigerbes, and making use of the following result, which follows immediately from Lemma 4.7.

**Lemma 6.4.** The pullback by a continuous map \( X' \to X_0 \) of a locally split \( n \)-cube \( X_\alpha \) is a locally split \( n \)-cube over \( X' \). Likewise, if \( X_\alpha \) and \( X'_\alpha \) are locally split \( n \)-cubes over the same base \( X := X_0 = X'_0 \), then the fiber products \( X_\alpha \times_X X'_\alpha \) form a locally split \( n \)-cube.

The characteristic class is defined as before in terms of the total cohomology of a \( \check{\text{C}} \)ech-simplicial multicomplex.

**Definition 6.5.** An admissible set of covers for a locally split \( n \)-cube is a set of covers \( \mathcal{U}_{X_\alpha} \to X_\alpha \) such that each pair \( (\mathcal{U}_{X_\alpha - e_j}, \mathcal{U}_{X_\alpha - e_i}) \) is admissible and each set \( (\mathcal{U}_{X_\alpha - e_j - e_i}, \mathcal{U}_{X_\alpha - e_i}, \mathcal{U}_{X_\alpha - e_j}, \mathcal{U}_{X_\alpha}) \) is an admissible set of covers for the associated split square.

**Lemma 6.6.** Given any covers for the spaces in a locally split \( n \)-cube, there exists an admissible set refining them.

**Proof.** The proof is by induction on \( n \) of the stronger statement that covers of adjacent \( n \)-cubes rooted at \( X = X_0 \) can be refined to a set which is simultaneously admissible for each cube. Here by adjacent we mean a finite set of \( n \)-cubes rooted at \( X \) which may share subcubes. The cases \( n = 1 \) and \( n = 2 \) are furnished by Lemmas 1.4 and 4.3.

For the inductive case, we consider, for each \( n \)-cube, \( X_1 \) at the top mapping to the limit space \( X_{< 1} := \lim_{\beta < 1} X_\beta \), and we fix a cover \( \mathcal{U}_{X_{< 1}} \) supporting sections, denoted \( t \), of the split map \( p : X_1 \to X_{< 1} \). The topmost spaces in the diagram over which the limit is taken, namely \( X_{1 - e_j} \), are the topmost spaces of a set of adjacent \( (n - 1) \)-cubes rooted at \( X \), so by the inductive hypothesis there are covers \( \mathcal{U}_{X_{< 1}}^0 \cdot < 1 \) refining the given covers, which constitute simultaneous admissible sets for the \( (n - 1) \)-cubes. In addition, at this point the covers \( \mathcal{U}_{X_{1 - e_j}}^0 \) of the penultimate spaces support sections, \( \sigma_j \), of the natural maps \( \text{pr}_j : X_{< 1} \to X_{1 - e_j} \) from the limit space, which participate in commutative squares along with the maps of covers involving the \( \mathcal{U}_{X_{1 - e_j}}^0 \). Finally, we set \( \mathcal{U}_{X_1}^0 \) to be the given cover of \( X_1 \) and this setup constitutes step 0.

**Step 1:** We define \( \mathcal{U}_{X_{< 1}}^1 \) to be the given cover of \( X_{< 1} \) and then subsequently set \( \mathcal{U}_{X_{< 1}}^1 = \mathcal{U}_{X_{< 1}}^0 \cap t^{-1}(\mathcal{U}_{X_1}^1) \), and \( \mathcal{U}_{X_{1 - e_j}}^1 = \mathcal{U}_{X_{1 - e_j}}^0 \cap \sigma_j^{-1}(\mathcal{U}_{X_{< 1}}^1) \); if there are multiple \( n \)-cubes in which \( X_{1 - e_j} \) participates, then we take \( \mathcal{U}_{X_{1 - e_j}}^1 \) to be the mutual refinement by the pullback of the step 1 covers of the associated limit spaces for each such \( n \)-cube.

**Step 2:** Appealing to the inductive step again gives refinements, denoted \( \mathcal{U}_{X_1}^2 \), of all the spaces up to the \( X_{1 - e_j} \), and then we refine the cover of the
limit space by
\[ U_{X_{< \perp}}^2 = U_{X_{< \perp}}^1 \cap \lim_{\beta \leq 1} U_{X_{\beta}}^2, \]
the limit here taken over the maps of covers lifting the \( n \)-cube maps downstairs, and set \( U_{X_{\perp}}^2 = U_{X_{\perp}}^1 \cap p^{-1}(U_{X_{< \perp}}^1) \) for the space at the top of each \( n \)-cube. That this furnishes an admissible set of covers for the \( n \)-cube follows along the lines of the proof of Lemma 4.3, and completes the induction. \( \square \)

**Lemma 6.7.** For a fixed admissible set of covers, the simplicial differentials
\[ d_j = \sum_{k=0}^{\alpha_j} (-1)^k (\pi^j)_k^* : \tilde{C}^*_{\ell}(X_\alpha; A) \rightarrow \tilde{C}^*_{\ell}(X_{\alpha+e_j}; A) \]
on \( \check{\text{C}} \)ech cochains form, along with the \( \check{\text{C}} \)ech differential \( \delta \), an \((n+1)\)-multicomplex \((\tilde{C}^*_{\ell}(X_\bullet; A), \delta, d_1, \ldots, d_n)\) with the following properties:

(i) For each \( j \), fixing all other indices, the complex \((\tilde{C}^*_{\ell}(X_\bullet; A), d_j)\) is exact, and for each \( k \neq j \) admits a homotopy chain contraction commuting with \( d_k \).

(ii) The total cohomology of \((\tilde{C}^*_{\ell}(X_\bullet; A), \delta, d_1, \ldots, d_n)\) is isomorphic to \( H^*_\ell(X; A) \), where \( X = X_{\perp} \).

(iii) The Chern class of the line bundle \( L \rightarrow X_{\perp} \) of a multigerbe is represented by a cocycle \( c(L) \in \tilde{C}^*_{\ell}(X_\perp; \mathbb{C}^*) \) for some admissible set of covers, with \( d_j c(L) = 0 \) for each \( j \), and such a multigerbe is trivial if and only if \( c(L) \) is a coboundary in the total \((\delta, d_1, \ldots, d_n)\) complex.

**Proof.** Here (i) is a consequence of Proposition 4.11, since for each pair \( j \neq k \), the bisimplicial space obtained from \( X_\alpha \) by freezing all but the \( j \)th and \( k \)th indices is equivalent to the one obtained from a locally split square.

Part (ii) follows by induction, rolling up the \((n+1)\) multicomplex into the double complex \((d_n, D_{n-1})\) where \( D_{n-1} \) denotes the total differential associated to \((\delta, d_1, \ldots, d_{n-1})\). By exactness of \( d_n \), this total cohomology is isomorphic to the total \( D_{n-1} \) cohomology of the complex \( \tilde{C}^*_{\ell}(X_{(\alpha); \perp}; A) \) where again \( \alpha(n) \) denotes the index obtained from \( \alpha \) by setting \( \alpha_n = 0 \).

Finally, part (iii) is proved by a straightforward generalization of the proof of Lemma 4.13. \( \square \)

**Definition 6.8.** The characteristic class of a multigerbe \((L, X_\alpha)\) is the class
\[ G(L) \in H^{n+3}(X; \mathbb{Z}), \quad X = X_{\perp}, \]
given by the Bockstein image of \([c(L)] \in H^{n+2}(\tilde{C}^*_{\ell}(X_{(\alpha); \perp}; \mathbb{C}^*), \delta, d_1, \ldots, d_n)\) in \( H^{n+2}_{\ell}(X; \mathbb{C}^*) \) with respect to the isomorphism of Lemma 6.7.(ii).

**Proposition 6.9.** The characteristic class is natural with respect to pullback, product and inverse operations on multigerbes, and a morphism of multigerbes induces an equality of the (pulled back) characteristic classes on the base spaces. It vanishes if and only if the multigerbe admits a trivialization. Moreover, \( G(L) \) transforms according to the sign representation of the
symmetric group $\Sigma_n$ acting by permutation of the indices of the locally split $n$-cube $X_\alpha$.

Proof sketch. As for (bi)gerbes, the naturality of the characteristic class is a consequence of the naturality of the multicomplex $(C^*_U(X_\bullet; A), \delta, d_1, \ldots, d_n)$ and the naturality of the Chern class of the line bundle $L \to X_2$, and the equivalence between vanishing of $G(L)$ and multigerbe triviality of $L$ follows from Lemma 6.7.(iii). Finally, that $G(L)$ is odd with respect to permutations of the $n$-cube is a consequence of the sign convention, Convention 2.2, since changing the order of the differentials in the multicomplex by a permutation $\sigma$ involves multiplying the complex by powers of $-1$, and in particular the sign $(-1)^{sgn(\sigma)}$ on the term $C^1_U(X_2; \mathbb{C}^*)$.

The question of representability of a given $(n+2)$-class by a multigerbe supported by a given locally split $n$-cube can be addressed along similar lines as for bigerbes in §4.4. Consider the multicomplex $(C^*_U(X_\bullet; \mathbb{Z}), \delta, d_1, \ldots, d_n)$ truncated to involve only the spaces in the $n$-cube, so the $X_\alpha$ with $\alpha \in \{0,1\}^n$. The $(\delta, D_{1\ldots n})$ spectral sequence of this complex (with the $d_i$ rolled up into a single differential) has $E_1$ page consisting of the cohomology complexes

$$H^k(X; \mathbb{Z}) \xrightarrow{D_{1\ldots n}} \bigoplus_{|\alpha|=1} H^k(X_\alpha; \mathbb{Z}) \xrightarrow{D_{1\ldots n}} \bigoplus_{|\alpha|=2} H^k(X_\alpha; \mathbb{Z}) \xrightarrow{D_{1\ldots n}} \cdots$$

for each $k \in \mathbb{N}$. At the bottom level, the $D_{1\ldots n}$ differential of a class in $H^{n+2}(X; \mathbb{Z})$ is just the sum of the pullbacks along the $n$-cube face maps to $\bigoplus_{|\alpha|=1} H^{n+2}(X_\alpha; \mathbb{Z})$, and if this vanishes, then we say the class survives to the $E_2$ page. In this case the $E_2$ differential maps the class into the quotient $\bigoplus_{|\alpha|=2} H^{n+1}(X_\alpha; \mathbb{Z}) / \bigoplus_{|\alpha|=1} H^{n+1}(X_\alpha; \mathbb{Z})$ (the cohomology of the $E_1$ page), and we say the class survives to the $E_3$ page if this $E_2$ differential vanishes and so on. Provided the class survives to the $E_n$ page, the associated differential maps it into the quotient of $\bigoplus_{|\alpha|=n} H^2(X_\alpha; \mathbb{Z}) = H^2(X_1; \mathbb{Z})$ by some complicated subgroup, and this is the last nontrivial differential of the spectral sequence, which therefore stabilizes at $E_{n+1} = E_\infty$. We say the class survives to $E_\infty$, or simply stabilizes, if it survives to $E_n$ and has vanishing $E_n$ differential.

**Proposition 6.10.** A given locally split $n$-cube $X_\alpha$ supports an $n$-multigerbe representing a given class $\alpha \in H^{n+2}(X; \mathbb{Z})$ if and only if $\alpha$ stabilizes in the above sense.

We leave the details of the proof, which we claim is a relatively straightforward generalization of the proof of Theorem 4.17, as an exercise.

6.1. Examples. We end with some simple examples of multigerbes which are straightforward generalizations of the $3 + 1$ decomposable bigerbes of §5.1 and the path bigerbes of §5.5.

First, suppose $(L, X_\alpha)$ is an $n$-multigerbe over $X = X_0$ with characteristic class $\alpha = G(L) \in H^{n+2}(X; \mathbb{Z})$, and let $[\rho] \in H^1(X; \mathbb{Z}) \cong \check{H}^0(X; U(1))$ be a
given 1-class represented by a homotopy class of maps $\rho : X \to U(1)$. We proceed to construct a ‘decomposable’ $(n + 1)$-multigerbe representing the class $[\rho] \cup G(L)$. With $\tilde{X} \to X$ the ‘logarithmic’ $\mathbb{Z}$-covering of $X$ associated to $\rho$ as in §5.1, define the $(n + 1)$-cube $\tilde{X}_\beta$ by

$$\tilde{X}_{(\alpha, 0)} = X_\alpha, \quad \tilde{X}_{(\alpha, 1)} = \tilde{X} \times_X X_\alpha.$$ 

Then in the induced multisimplicial simplicial space, $\tilde{X}_2 = \tilde{X}^{[2]} \times_X X_2$ and we define the line bundle by

$$L^\chi = (\text{pr}_2^* L) \otimes \text{pr}_1^* \chi \to \tilde{X}^{[2]} \times_X X_2$$

where $\chi : \tilde{X}^{[2]} \to \mathbb{Z}$ is the fiber shift map with $\tilde{X} \to X$ thought of as a principal $\mathbb{Z}$-bundle.

**Proposition 6.11.** With notation as above, $(L^\chi, \tilde{X}_\beta)$ is an $(n+1)$-multigerbe with characteristic class

$$G(L^\chi) = [\rho] \cup G(L) \in H^{n+3}(X; \mathbb{Z}).$$

For the generalization of the path bigerbes of §5.5, let $X$ be a locally contractible space with a chosen basepoint.

**Lemma 6.12.** With notation $P^1 Y = PY$ and $P^0 Y = Y$, the iterated (based) path spaces

$$X_\alpha = P^{\alpha_1} \cdots P^{\alpha_n} X = P^{[\alpha]} X$$

with evaluation maps $X_\alpha \to X_{\alpha - \epsilon_j}$ form a locally split $n$-cube over $X$, provided $X$ is $(n-1)$-connected.

**Proof.** The obstruction to the locally split condition is the surjectivity of the maps $X_\alpha \to X_{<\alpha}$. However, it can be checked that for each $[\alpha] \in \mathbb{N}$, the limit $X_{<\alpha} \cong C_*(S^{[\alpha]-1}; X)$ can be identified with the set of basepointed maps from the $(|\alpha|-1)$-sphere into $X$. Indeed, the $X_{\alpha - \epsilon_j} \cong C_*([0,1]^{[\alpha]-1}; X)$ are cube mapping spaces of the $(|\alpha|-1)$-cube, and the limit can be realized as the subspace of the product of these in which the cubes are identified along their boundaries in a way that assembles into an $(|\alpha|-1)$-sphere, with the image of an element in $X_\alpha = C_*([0,1]^{[\alpha]}, X)$ in $X_{<\alpha}$ identified with the restriction map to the boundary of the disk $[0,1]^{[\alpha]} \cong D^{[\alpha]}$. Since $X$ is $(n-1)$ connected, each of these maps is surjective. \[\square\]

Since in this case all the $X_\alpha$ in the $n$-cube for $\alpha \neq 0$ are contractible spaces, every class in $H^{n+2}(X; \mathbb{Z})$ survives to the $E_\infty$ page in the $(\delta, D_1 \cdots D_n)$ spectral sequence of the $n$-cube, so in light of Proposition 6.10 we conclude the following.

**Proposition 6.13.** For $X$ $(n-1)$-connected and locally contractible, every class in $H^{n+2}(X; \mathbb{Z})$ is represented by an $n$-multigerbe supported on the iterated path $n$-cube $X_\alpha = P^{[\alpha]} X$; in particular, the multisimplicial line bundle of the multigerbe lives on the iterated loop space $X_2^n = \Omega^n X$. 
Finally, as in §5.5 there is a free path/loop version of this multigerbe obtained at the cost of imposing product-multisimplicial conditions. Indeed, the set \( \{ X^{m_1,\ldots,m_n} = X^{m_1 \cdots m_n} : (m_1,\ldots,m_n) \in \mathbb{N}^n \} \) of \( n \)-fold iterated products of \( X \) along with projections forms a multisimplicial space, with induced “differentials” \( \partial_1,\ldots,\partial_n \) defined on functions, line bundles, gerbes, multi-gerbes, etc. A \textit{product-multisimplicial} multigerbe is a multigerbe \( L \) over \( X^{2,\ldots,2} = X^{2n} \) such that \( \partial_i L \) is a trivial multigerbe for \( 1 \leq i \leq n \), and then its characteristic class descends from \( H^{n+2}(X^{2n};\mathbb{Z}) \) to \( H^{n+2}(X;\mathbb{Z}) \). Again leaving the details of the generalization of Theorem 5.16 as an exercise, we claim the following result.

**Proposition 6.14.** If \( X \) is connected and locally contractible, then every class in \( H^{n+2}(X;\mathbb{Z}) \) is represented by a product-multisimplicial \( n \)-multigerbe supported by the iterated free path \( n \)-cube \( X_\alpha = I^{|\alpha|} X^{(2-|\alpha|)^n} \) with \( X_0 = X^{2,\ldots,2} = X^{2n} \); in particular, the line bundle of the multigerbe lives on the free loop space \( L^n X \) where it satisfies an \( n \)-fold fusion condition as well as the multi-figure-of-eight condition that \( \partial_i L \longrightarrow L \cdots L_8 \cdots LX \) are trivial for each \( i \).

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