ZIMMER’S CONJECTURE FOR LATTICE ACTIONS: THE SL(n, C)-CASE

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ABSTRACT. We prove Zimmer’s conjecture for co-compact lattices in SL(n, C): for any co-compact lattice in SL(n, C), n ≥ 3, any Γ-action on a compact manifold M with dimension: (I) less than 2n − 2 if n ≠ 4, (II) less than 5 if n = 4, by C^{1+\epsilon} diffeomorphisms factors through a finite action.

1. INTRODUCTION

Motivated by a sequence of results on the rigidity of linear representations including [23, 25, 21, 22], Margulis’ superrigidity theorem [20], and the extension to cocycles, Zimmer’s cocycle superrigidity theorem [27], R. Zimmer proposed the following conjecture.

CONJECTURE 1. Let G be a connected, semisimple Lie group with finite center, all of whose almost-simple factors have real-rank at least 2. Let Γ < G be a lattice. Let M be a compact manifold. If\( \dim M < \min(n(G), d(G), v(G)) \) then any homomorphism \( \alpha : \Gamma \rightarrow \text{Diff}(M) \) has finite image.

In the above conjecture, number \( n(G) \) denotes the minimal dimension of a non-trivial real representation of the Lie algebra \( \mathfrak{g} \) of \( G \); number \( v(G) \) denotes the minimal codimension of a maximal (proper) parabolic subgroup of \( Q \) of \( G \); and number \( d(G) \) denotes the minimal dimension of all non-trivial homogeneous space \( K/C \) as \( K \) varies over all compact real-forms of all simple factors of the complexification of \( G \). There are also Zimmer’s conjectures for volume-preserving actions. We refer the readers to [2, Conjecture 1.2] for the statement of the full Zimmer’s conjecture as extended by Farb and Shalen. We refer the readers to [12, 13] for the history of Zimmer’s program as well as recent developments.

In a recent breakthrough [2], Brown, Fisher and Hurtado have proved the non-volume preserving case of Zimmer’s conjecture for co-compact lattices in higher-rank split simple Lie groups as well as

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certain volume preserving cases (under $C^2$ regularity assumption). In [3], the authors proved Zimmer’s conjecture for the non-uniform lattice $SL(n, \mathbb{Z})$. In [7], the authors replaced the regularity assumption $C^2$ in [2] by $C^1$ under a stronger dimensional constrain. We also mention [26] for $SL(n, \mathbb{Z})$ actions by homeomorphisms under a topological condition on the manifold.

For many non-split Lie groups, the results in [2] also give dimensional bounds that are comparable to the optimal bounds. For instance, for $n \geq 5$, the dimensional bound in [2] for $SL(n, \mathbb{C}), SL(n, \mathbb{H})$ are respectively one half and one quarter of the optimal bounds. In this paper, we improve the bound for $SL(n, \mathbb{C})$ to the optimal level for co-compact lattices. The following are the main results of this paper.

**Theorem 1.** Let $n \geq 3$ be an integer, and let $\Gamma < SL(n, \mathbb{C})$ be a co-compact lattice. Let $M$ be a connected, compact manifold satisfying: (I) $\dim M < 2n - 2$ if $n \neq 4$, (II) $\dim M < 5$ if $n = 4$. Then any group homomorphism $\alpha : \Gamma \rightarrow \text{Diff}^{1+\epsilon}(M)$ factors through a finite group.

**Theorem 2.** Let $n \geq 3$ be an integer, and let $\Gamma < SL(n, \mathbb{C})$ be a co-compact lattice. Let $M$ be a connected, compact manifold satisfying: 1. $\dim M < 2n - 2$. Then any group homomorphism $\alpha : \Gamma \rightarrow \text{Diff}^{2}(M)$ preserves a Riemannian metric.

1.1. **Further extensions.** The method of this paper can be generalized to other simple complex Lie group as well. In an on-going joint work with Jinpeng An, we will address Conjecture 1 for all simple complex Lie groups. This will appear as a second version of this paper.

**Notation.** For any positive integer $m$, we denote by $[m]$ the set $\{1, \ldots, m\}$. For any metric space $Z$, we use $B_Z$ to denote the Borel $\sigma$-algebra of $Z$, and use $\mathcal{M}(Z)$ to denote the set of Radon measures on $Z$. Given a measurable partition $\xi$, we denote by $B_\xi$ the $\sigma$-algebra generated by $\xi$.

2. **Preliminary**

Let $M$ be a connected, compact manifold.

Let $G = SL(n, \mathbb{C})$ and let $\mathfrak{g} = sl(n, \mathbb{C})$.

Let $H$ be the standard Cartan subgroup of $G$, i.e., $H$ is the subgroup of diagonal matrices in $G$. We have $H = MA$ where $A$ is the subgroup consisted of positive real diagonal matrices in $G$; and $M$ is
the subgroup consisted of diagonal matrices in $G$ with unit complex numbers on the diagonal.

For each $1 \leq i, j \leq n$, let $E_{i,j}$ denote the $n \times n$-matrix whose entry at $i$-th row $j$-th column equals 1, and 0 at all other places. We can see that the Lie algebra of $A$ and $M$ are respectively,

$$a = \left\{ \sum_{i=1}^{n} a_i E_{i,i} \mid \sum_{i=1}^{n} a_i = 0, a_i \in \mathbb{R} \right\} \quad \text{and} \quad m = ia.$$

For any linear functional $\ell$ on $a$, we denoted by $[\ell]$ the set of linear functionals on $a$ which are positively proportional to $\ell$. We let $\Sigma$ be the set of coarse restricted roots of $G$. In our case, the coarse restricted roots are in bijection with the restricted roots. We will however adopt this notion in [2] to facilitate the citation of certain theorems. We can show that $\Sigma = \{ [\gamma_{i,j}] \mid 1 \leq i < j \leq n \}$ where we set $\gamma_{i,j} = E_{i,i} - E_{j,j}$. When there is no confusion, we slightly abuse the notion and write $\chi$ instead of $[\chi]$, for instance, we say that the root space for $\gamma_{i,j}$ equals $CE_{i,j}$, which we denote by $g_{\chi_{i,j}}$. For each $\chi \in \Sigma$, we denote by $G_\chi$ the root subgroup of $\chi$, and denote by $\nu_{G_\chi}$ the Haar measure on $G_\chi$. Also we denote $L_\chi = \text{Ker}(\chi)$, and let $H_{\chi}$ denote the subgroup of $A$ corresponding to $L_{\chi}$. We denote $\Sigma^+ = \{ \gamma_{i,j} \mid 1 \leq i < j \leq n \}$ and $\Sigma^- = \{ \gamma_{i,j} \mid 1 \leq j < i \leq n \}$. We let $P$ denote the Borel subgroup of $G$ relative to our choice of $\Sigma^+$, i.e., the subgroup consisted of upper triangular matrices. It is clear that $P$ is generated by $A, M$ and $G_\chi, \chi \in \Sigma^+$.

2.1. Suspension space. Let $\Gamma$ be a co-compact lattice in $G$. Let $a : \Gamma \to \text{Diff}^{1+\epsilon}(M)$ be an right action, i.e., $a(gh) = a(h)a(g)$. As in [2], we consider the right $\Gamma$-action

$$(g, x) \cdot \gamma = (g \gamma, a(\gamma)(x))$$

and the left $G$-action

$$a \cdot (g, x) = (ag, x).$$

Let $M^a = (G \times M)/\Gamma$, and let $\check{a}$ denote the left $G$-action on $M^a$. To simply notation, we will abbreviate $\check{a}(\exp(k))$ as $\check{a}(k)$ for every $k \in a$. We denote the canonical projection from $M^a$ to $G/\Gamma$ by $\pi$.

Let $\mu$ be an $A$-invariant $A$-ergodic measure on $M^a$. For any $k \in a$, for $\mu$-a.e. $x$, we denote by $W^{-}_{\check{a}(k)}(x)$, resp. $W^{+}_{\check{a}(k)}(x)$, the stable manifold, resp. unstable manifold, through $x$ for the map $\check{a}(k)$.
For each $\chi \in \Sigma$, we define $E^\chi, E^\chi_F, E^\chi_G, W^\chi, W^\chi_F$ and $W^\chi_G$ as in [5]. For example, we have

$$W^\chi(x) = \bigcap_{k \in a, \chi(k) < 0} W^-_{\tilde{\alpha}(k)}(x).$$

It is clear that $\dim E^\chi_F$ is $\mu$-a.e. constant.

2.2. Conditional measure. In this section, we define a collection of equivalence classes of measures $\{[\mu^\xi_x]\}_{x \in M^a}$ where each $\mu^\xi_x$ is a measure defined up to a scalar with the property that $\mu^\xi_x$ is supported on $W^\chi(x)$. Moreover, this collection is invariant under the $A$-action.

Let $\xi$ be a measurable partition subordinate to $W^\chi$. We let $\{[\mu^\xi_x]\}_{x \in M^a}$ denote the conditional measure associate to $\xi$. Let $\xi_1, \xi_2$ be two measurable partitions subordinate to $W^\chi$. Then for $\mu$-a.e. $x$, the restrictions of $\mu^\xi_1$ and $\mu^\xi_2$ to $\xi_1(x) \cap \xi_2(x)$ coincides up to a factor.

Take $k_0 \in a$ such that $\chi(k_0) > 0$, and take $f = \tilde{\alpha}(k_0)$. We take $\xi$, an $f$-increasing measurable partition subordinate to $W^\chi$. Then for any precompact open neighborhood of $x$ in $W^\chi$, denoted by $U$. For $\mu$-a.e. $x$, we define

$$\mu^W_x = \lim_{n \to \infty} [\mu^f_x(\xi) - 1]^{-1} \mu^f_x(\xi).$$

It is direct to verify that the definition of $\mu^W_x$ is independent of the choice of $\xi$. We say that two Radon measures $\xi_1, \xi_2$ on $W^\chi$ are equivalent if there is $c > 0$ such that $\xi_2 = c\xi_1$. Given a Radon measure $\xi$ on $W^\chi$, we denote by $[\xi]$ the equivalence class of $\xi$. We notice that $[\mu^W_x]$ is independent of the choice of $U$.

By the $A$-invariance of $\mu$, we claim that for any $k \in a$, for $\mu$-a.e. $x$, we have

$$(2.1) \quad [D\tilde{\alpha}(k) \ast \mu^W_x] = [\mu^W_{\tilde{\alpha}(k)(x)}].$$

We define $\{[\mu^W_F_x]\}_{x \in M^a}$ in an analogous way. We can see that $\{[\mu^W_F_x]\}_{x \in M^a}$ is also $A$-invariant.
2.3. Coarse restricted root. Given an $A$-invariant, $A$-ergodic measure $\mu$, we consider the following subsets of $\Sigma$:

\[
\begin{align*}
\Sigma_{\text{out}} &= \{ \chi \in \Sigma \mid E^\chi_F \neq \{0\} \}, \\
\Sigma_{\text{out}}^1 &= \{ \chi \in \Sigma_{\text{out}} \mid \dim E^\chi_F \geq 2 \}, \\
\Sigma_{\text{out}}^2 &= \{ \chi \in \Sigma_{\text{out}} \mid \dim E^\chi_F = 1, \dim E^{-\chi} \geq 1 \}, \\
\Sigma_{\text{out}}^3 &= \Sigma_{\text{out}} \setminus (\Sigma_{\text{out}}^1 \cup \Sigma_{\text{out}}^2).
\end{align*}
\]

We notice that the above subsets can also be defined for any $H$-ergodic measure $\mu$. Indeed, we can define the above subsets of $\Sigma$ for each $A$-ergodic component of $\mu$. As $M$ is compact and commutes with $A$, $\dim E^\chi_F$ and $\Sigma_{\text{out}}^\star$ are the same for all $A$-ergodic components of $\mu$ (see the paragraph below [2, Theorem 5.8]).

It is clear that

\[2|\Sigma_{\text{out}}^1 \cup \Sigma_{\text{out}}^2| + |\Sigma_{\text{out}}^3| \leq \dim M.\]  

(2.2)

Given a closed subgroup $Q \subset G$ containing $H$. We define

\[\Sigma_Q = \{ \chi \in \Sigma \mid G^\chi \subset Q \}.\]  

(2.3)

By [5, Proposition 5.1], we have

\[\Sigma \setminus \Sigma_{\text{out}} \subset \Sigma_Q\]  

(2.4)

for $Q = \{ g \in G \mid g_+ \mu = \mu \}$.

The following proposition plays an important role in our proof.

**Lemma 2.1.** Let $Q$ be a closed subgroup of $G$ such that $H \subset Q$. If $n \geq 2$ and we have

\[|\Sigma \setminus \Sigma_Q| < 2n - 2,\]

then $Q$ is a parabolic subgroup of $G$.

**Proof.** In view of [6, Page 92, Prop. 11], it suffices to verify $\Sigma_Q \cup (-\Sigma_Q) = \Sigma$, i.e., for every $\gamma_{i,j} \in \Sigma$, either $\gamma_{i,j}$ or $\gamma_{j,i}$ lies in $\Sigma_Q$. To show this, consider the following $2n - 2$ mutually disjoint sets:

\[\{ \gamma_{i,j}, \gamma_{j,i}, \gamma_{i,k}, \gamma_{k,i}, \gamma_{j,k}, \gamma_{k,j} \}, \quad k \in [n] \setminus \{i, j\}.\]

It follows from the assumption that at least one of such sets is contained in $\Sigma_Q$. If $\{ \gamma_{i,j} \}$ or $\{ \gamma_{j,i} \}$ is contained in $\Sigma_Q$, then there is nothing to prove. If $\{ \gamma_{i,k}, \gamma_{k,j} \} \subset \Sigma_Q$, then $\gamma_{i,j} = \gamma_{i,k} + \gamma_{k,j} \in \Sigma_Q$. Similarly, if $\{ \gamma_{j,k}, \gamma_{k,i} \} \subset \Sigma_Q$, then $\gamma_{j,i} = \gamma_{j,k} + \gamma_{k,i} \in \Sigma_Q$. \[\square\]

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1Lemma 2.1 is proved by Jinpeng An.
DEFINITION 2.1. Let $Q$ be a subgroup of $G$ containing $H$. We let $\Sigma_Q^{\text{non}}$ be the set of $\chi \in \Sigma_Q$ such that there exist $\chi_1, \chi_2 \in \Sigma_Q \setminus \{\pm \chi\}$ such that $\chi = \chi_1 + \chi_2$.

PROPOSITION 2.1. Let $Q$ be a parabolic subgroup of $G$ with $|\Sigma \setminus \Sigma_Q| < 2n - 2$. Then for any subset $I \subset \Sigma \setminus \Sigma_Q$ such that $|I| < 2n - 2 - |\Sigma| + |\Sigma_Q|$, there exists $\chi \in \Sigma \setminus (\Sigma_Q \cup I)$ such that there exists $\chi' \in \Sigma_Q$ satisfying $\chi + \chi' \in \Sigma \setminus \Sigma_Q$. In particular, $-\chi \in \Sigma_Q^{\text{non}}$.

Proof. We first notice that if there exists $\chi' \in \Sigma_Q$ satisfying $\chi''' := \chi' + \chi \in \Sigma \setminus \Sigma_Q$.

Then as $Q$ is parabolic, $\chi'' := -\chi''' \in \Sigma_Q$. Consequently, we have $-\chi = \chi' + \chi''$. It is clear that $\chi', \chi'' \notin \{\pm \chi\}$. Thus $-\chi \in \Sigma_Q^{\text{non}}$.

Without loss of generality, we may assume that $P \subset Q$. By [17, V. 7, Proposition 5.90] (see also [5, Section 2.1]), there exist constants $1 \leq i_1 < \cdots < i_p \leq n - 1$ such that

$$\Sigma \setminus \Sigma_Q = \bigcup_{l=1}^{p} \{\chi_{u,v} \mid v \leq i_p < u\}.$$ 

The case where $n = 3$ can be verified directly. In the following we assume that $n \geq 4$.

We first assume that there exists $1 \leq p \leq l$ such that $2 \leq i_p \leq n - 2$. Then we have

$$|\Sigma \setminus \Sigma_Q| \geq 2(n - 2).$$

Hence $|I| \leq 1$.

We notice that both $\chi_{n,1}, \chi_{n,2}$ belongs to $\Sigma \setminus \Sigma_Q$. Thus there exists $\chi \in \{\chi_{n,1}, \chi_{n,2}\}$ such that $\chi \in \Sigma \setminus (\Sigma_Q \cup I)$. Notice that $\chi_{n-1,n} \in \Sigma_Q$. Then we have

$$\chi + \chi_{n-1,n} \in \{\chi_{n-1,1}, \chi_{n-1,2}\} \subset \Sigma \setminus \Sigma_Q.$$ 

This concludes the proof in this case.

If there exists no such $\chi_p$, then there are only three possibilities for $\Sigma \setminus \Sigma_Q$:

1. $\{n\} \times [n - 1]$;
2. $([n] \setminus [1]) \times [1]$;
3. the union of the above two.

In each of the above cases, we can verify the proposition directly. \(\Box\)

3. PROOF OF THE MAIN THEOREM

3.1. Review of BFH. The first step in the proof of Theorem [1] is to show that $\alpha$ has uniform subexponential growth of derivatives.
DEFINITION 3.1. Let $\alpha : \Gamma \to \text{Diff}^1(M)$ be an action of $\Gamma$ on a compact manifold $M$ by $C^1$ diffeomorphisms. We fix an arbitrary $C^\infty$ Riemannian metric on $M$. We say that $\alpha$ has uniform subexponential growth of derivatives if for every $\varepsilon > 0$ there is a constant $C_\varepsilon > 0$ such that for all $\gamma \in \Gamma$ we have
$$\|D\alpha(\gamma)\| \leq C_\varepsilon e^{\varepsilon \ell(\gamma)}.$$ It is clear that the above definition is independent of the choice of the metric on $M$.

PROPOSITION 3.1. Let $n \geq 3$ be an integer, and let $\Gamma < \text{SL}(n, \mathbb{C})$ be a co-compact lattice. Let $M$ be a connected, compact manifold satisfying $\dim M < 2n - 2$. Then $\alpha$ has uniform exponential growth of derivatives.

We now start with the proof of Proposition 3.1.

We assume to the contrary that $\alpha$ does not have uniform subexponential growth of derivatives. Then by combining [2, Proposition 3.6], [2, Claim 3.5] and [2, Proof of Proposition 3.7], we have

PROPOSITION 3.2. There exists an $s \in A$ and an $H$-invariant $H$-ergodic Borel probability measure $\mu$ on $M^\alpha$ with $\lambda^F_+(s, \mu) > 0$ such that $\pi_* \mu$ is the Haar measure on $G/\Gamma$. Here $\lambda^F_+(s, \mu)$ is the maximal fiberwise Lyapunov exponent for $s \in A$ with respect to $\mu$ given by the formula
$$\lambda^F_+(s, \mu) = \inf_{n \to \infty} \frac{1}{n} \int \log \|D\tilde{\alpha}(s^n)\|_{E_F(x)} |d\mu(x)|.$$

3.2. From $H$ to $G$. To complete the proof of Proposition 3.1 it remains to show the following.

PROPOSITION 3.3. Assume that $\dim M < 2n - 2$. Let $\mu$ be an $H$-invariant $H$-ergodic measure on $M^\alpha$ such that $\pi_* \mu$ is the Haar measure on $G/\Gamma$. Then $\mu$ is $G$-invariant.

The main technical proposition of our paper is the following.

PROPOSITION 3.4. Let $Q$ be a parabolic subgroup and let $\mu$ be an $Q$-invariant $H$-ergodic measure on $M^\alpha$. Then for any $\chi \in \Sigma^\text{out}_3 \cap (-\Sigma^\text{non}_Q)$, the conditional measure $\mu^G_x$ is non-atomic for $\mu$-a.e. $x$.

The proof of Proposition 3.4 is divided into two parts which occupy the next two sections.

Proof of Proposition 3.3. Assume that $\mu$ is not $G$-invariant. We set
$$Q = \{ g \in G \mid g_* \mu = \mu \}.$$ By hypothesis, $H \subset Q \subsetneq G$. 
Define $E^X_F, E^X_G, E^X, \Sigma_Q, \Sigma_1, \ldots$ with respect to $\mu$. We claim that

$$|\Sigma \setminus \Sigma_Q| \leq \dim M \leq 2n - 3.$$ 

Indeed, if this was not the case, then there would exist $\chi \in \Sigma \setminus \Sigma_Q$ such that $\chi$ is fiberwise non-resonant, i.e., $E^X_\chi = \{0\}$. By [5, Proposition 5.1], we would deduce that $\mu$ is in fact $G^X$-invariant. This would contradict the definitions of $Q$ and $\Sigma_Q$.

By Lemma 2.1, we see that $Q$ is a parabolic subgroup.

We set

$$I := (\Sigma_1 \cup \Sigma_2) \setminus \Sigma_Q.$$ 

By definition, (2.4) and (2.2), it is clear that $I \subset \Sigma \setminus \Sigma_Q$ and

$$|I| = |I| + |\Sigma_1 \cup \Sigma_2| + |\Sigma_3| - |\Sigma_Q| 
\leq 2|\Sigma_1 \cup \Sigma_2| + |\Sigma_3| - |\Sigma_Q| 
\leq \dim M - |\Sigma_Q| 
\leq \dim M - |\Sigma| + |\Sigma_Q| 
< 2n - 2 - |\Sigma| + |\Sigma_Q|.
$$

By Proposition 2.1, there exists

$$\chi \in \Sigma \setminus (\Sigma_Q \cup I) \subset \Sigma_Q \setminus (\Sigma_Q \cup I) \subset \Sigma_Q$$

such that there exists $\chi' \in \Sigma_Q$ satisfying

$$(3.1) \quad \chi'' := \chi + \chi' \in \Sigma \setminus \Sigma_Q.$$ 

In particular, $-\chi \in \Sigma_Q^{non}$.

By Proposition 3.4, the conditional measure $\mu_{\chi''}^G$ is non-atomic for $\mu$-a.e. $x$. By the $Q$-invariance of $\mu$, we see that $\mu_{\chi''}^G$ is Haar for $\mu$-a.e. $x$. Then by the method in [2, 8] for noncommuting foliations along with (3.1), we see that $\mu_{\chi''}^G$ is Haar for $\mu$-a.e. $x$. But this is a contradiction as this would imply that $\mu$ is $G^{X''}$-invariant, and consequently $\chi'' \in \Sigma_Q$. □

Proof of Proposition 3.1 Assume that $\alpha$ fails to have uniform subexponential growth of derivatives. By Proposition 3.2 there is a $s \in A$ and an $H$-invariant $H$-ergodic measure $\mu$ with $\lambda^F_+ (s, \mu) > 0$, and $\pi_* \mu$ is the Haar measure on $G/\Gamma$. By Proposition 3.3 we deduce that $\mu$ is $G$-invariant. We deduce that there exists a $\Gamma$-invariant measure $m$ on $M$. By Zimmer’s cocycle superrigidity theorem, the $\Gamma$-action preserves a measurable metric on $M$. But in this case we should have $\lambda^F_+ (s, \mu) = 0$. This is a contradiction. Thus $\alpha$ must have uniform subexponential growth of derivatives. □
Proof of Theorem 1 and 2: By Proposition 3.1, we see that $\alpha$ has uniform subexponential growth of derivatives. When $\alpha$ acts by $C^2$-diffeomorphisms, we conclude the proof of Theorem 2 by the same argument in [2].

By [2, Theorem 2.9] and [7, Proposition 7], we see that there exists a compact Lie group $K$, an injection $\iota: K \rightarrow \text{Homeo}(M)$; and a group homomorphism $\phi: \Gamma \rightarrow K$ such that $\alpha = \iota \phi$.

We conclude the proof of Theorem 1 by Margulis arithmetic theorem following [2, Section 7]. Here we have used the fact that
\[
d(\text{SL}(n, \mathbb{C})) = \begin{cases} 
2n - 2, & n = 3 \text{ or } n > 4, \\
5, & n = 4.
\end{cases}
\]

In the next two sections, we will give the proof of Proposition 3.4. We let $Q$ be a parabolic subgroup of $G$, and let $\mu$ be a $Q$-invariant $H$-ergodic measure; and let $\chi \in \Sigma^\text{out}_3 \cap (-\Sigma^{\text{non}}_Q)$. We also denote by $\chi_F \in \chi$ the Lyapunov functional for $E^\chi_F$, and denote by $\chi_G \in \chi$ the Lyapunov functional for $E^\chi_G$. To simply notation, we denote $E^\chi_F$ by $E_F$.

By our hypothesis that $\chi \in \Sigma^\text{out}_3$, we have $\dim E = 1$.

4. When $\mu_{W_F}^\chi$ is non-atomic

Through out this section, we assume that for $\mu$-a.e. $x$, the support of $\mu_{W_F}^\chi$ is non-discrete with respect to the leafwise metric.

4.1. Time change and measurable Lyapunov foliation. We fix a small constant $\epsilon > 0$. As in [16, Section 5], for any Lyapunov regular point $x \in M^\alpha$, for any $u, v \in E(x)$, we define the standard $\epsilon$-Lyapunov scalar product
\[
\langle u, v \rangle_\epsilon = \int_a \langle D\tilde{\alpha}(s)u, D\tilde{\alpha}(s)v \rangle \exp(-2\chi(s) - 2\epsilon \|s\|)ds.
\]
For any $C > 0$, we define the Pesin set $R(C)$ as in [16, Proposition 2.2]. By [16, Remark below Proposition 5.3], we have
\[
\tilde{\alpha}(s)R(C) \subset R(e^{2\|s\|\epsilon}C), \quad \forall s \in a.
\]
We summarize the time change argument in [16] (more specifically, Proposition 6.2-6.7 in [16]) in the following two lemmata. As in [16], we fix an element $w \in a$ such that $\chi(w) = 1$. 

□
LEMMA 4.1. For $\mu$-a.e. $x \in M^a$ and any $t \in a$ there exists a real number $g(x, t)$ such that the function $g(x, t) = t + g(x, t)w$ satisfies the following property. The measurable map

$$\hat{\beta}(t, x) = \hat{\alpha}(g(x, t))x$$

is an $a$-action preserving a probability measure $\hat{\mu}$ which is absolutely continuous with respect to $\mu$ with positive density, and for any $t \in a$ we have

$$\|D\alpha(g(x, t))|_{E(x)}\| = e^{\kappa(t)}.$$  

The function $g(x, t)$ is measurable and is continuous in $x$ on Pesin set and along the orbits of $\hat{\alpha}$. Moreover, $g(x, t)$ is $C^1$ in $t$ and it satisfies that

$$|g(x, t)| \leq 2\varepsilon \|t\|, \quad |\partial_t g(x, t)| \leq \varepsilon.$$  

LEMMA 4.2. For any $s \in a$ there is a stable “foliation” $\tilde{W}_-^{\hat{\beta}(s)}$ which is contracted by $\hat{\beta}(s)$ and invariant under the new action $\hat{\beta}$. It consists of “leaves” $\tilde{W}_-^{\hat{\beta}(s)}(x)$ defined for $\mu$-a.e. $x$. The “leaf” $\tilde{W}_-^{\hat{\beta}(s)}(x)$ is a measurable subset of the leaf $\hat{\alpha}(\mathbb{R}w)W_-^{\hat{\alpha}(s)}(x)$ of the form

$$\tilde{W}_-^{\hat{\beta}(s)}(x) = \{\hat{\alpha}(\phi_x^s(y))y \mid y \in W_-^{\hat{\alpha}(s)}(x)\}$$

where $\phi_x^s : W_-^{\hat{\alpha}(s)}(x) \to \mathbb{R}$ is a $\mu_x^{W_-^{\hat{\alpha}(s)}}$-almost everywhere defined measurable function. For $x$ in a Pesin set, $\phi_x^s$ is Hölder continuous on the intersection of this Pesin set with any ball of fixed radius on $W_-^{\hat{\alpha}(s)}(x)$ with Hölder exponent $\gamma$ and Hölder constant which depends on the Pesin set and radius.

We have the following observation.

LEMMA 4.3. For $\mu$-a.e. $x$, for any $t \in \mathbb{R}$, for any $k \in a$, we have $\hat{\beta}(k)\hat{\alpha}(tw)x = \hat{\alpha}(sw)\hat{\beta}(k)x$ where $s \in \mathbb{R}$ satisfies $|t|/4 < |s| < |t|$.

Proof. For $\mu$-a.e. $y$, we define function $\phi_y : \mathbb{R} \to \mathbb{R}$ by

$$\phi_y(t) = t + g(y, tw), \quad \forall t \in \mathbb{R}.$$  

Then it is clear that for $\mu$-a.e. $y$,

$$\hat{\beta}(tw)y = \hat{\alpha}(\phi_y(t)w)y.$$

By (4.2), we see that for $\mu$-a.e. $y$, $\phi_y$ is a diffeomorphism of $\mathbb{R}$ with $\|\phi_y\|, \|\phi_y^{-1}\| < 2$. 

By our choices of \(s, t\), we have
\[
\tilde{\beta}(k)\tilde{\alpha}(tw)x = \tilde{\beta}(k)\tilde{\beta}(\phi_x^{-1}(t)w)x = \tilde{\alpha}(sw)\tilde{\beta}(k)x = \tilde{\beta}(\phi_{\tilde{\beta}(k)x}^{-1}(sw))\tilde{\beta}(k)x.
\]
Consequently, we have \(s = \phi_{\tilde{\beta}(k)x}^{-1}(t)\). This concludes the proof. \(\square\)

In [16 Corollary 6.8], the authors gave the existence of coarse Lyapunov foliations for \(\tilde{\beta}\). In the following, we give a detailed account.

**Lemma 4.4.** For any \(\chi' \in \Sigma\), there exists an essentially unique \(\bar{2}\) collection \(\phi_{\chi'} = \{\phi_{\chi'}_x : \mathcal{W}_{\chi'}(x) \to \mathbb{R}\}_{x \in M^x}\) where for \(\mu\)-a.e. \(x\), \(\phi_{\chi'}^x\) is a \(\mu_{\chi}^{\mathcal{W}_{\chi'}}\) a.e. defined function and Hölder continuous on Pesin sets, such that the following is true: for any \(k \in \mathfrak{a}\), for \(\mu\)-a.e. \(x\), for \(\mu_{\chi}^{\mathcal{W}_{\chi'}}\)-a.e. \(y\), set \(z = \tilde{\alpha}(\phi_{\chi'}^x(y)w)\), we have
\[
\tilde{\beta}(k, z) = \tilde{\alpha}(\phi_{\tilde{\beta}(k,x)}^*(\tilde{\alpha}(g(x,k))y)w)\tilde{\alpha}(g(x,k))y.
\]
We have a similar collection of measurable functions for \(\mathcal{W}_{\chi'}^{\mathcal{W}}\).

**Proof.** We claim that for any \(s, k \in \mathfrak{a}\), for \(\mu\)-a.e. \(x\), for \(\mu_{\chi}^{\mathcal{W}_{\chi'}}\)-a.e. \(y\), set \(z = \tilde{\alpha}(\phi_{\chi}^x(y)w)\), we have
\[
\tilde{\beta}(k, z) = \tilde{\alpha}(\phi_{\tilde{\beta}(k,x)}^*(\tilde{\alpha}(g(x,k))y)w)\tilde{\alpha}(g(x,k))y. \tag{4.3}
\]
To prove the claim, we first notice that by definition we have
\[
\tilde{\beta}(k, z) = \tilde{\alpha}(g(z,k))z = \tilde{\alpha}(\tilde{\alpha}(g(z,k)))\tilde{\alpha}(\phi_{\chi}^x(y)w)y = \tilde{\alpha}(g(z,k))\tilde{\alpha}(\phi_{\chi}^x(y)w)\tilde{\alpha}(\phi_{\chi}^x_g(y)w)\tilde{\alpha}(g(x,k))y = \tilde{\alpha}(g(z,k))\tilde{\alpha}(\phi_{\chi}^x(y)w)\tilde{\alpha}(g(x,k))y = \tilde{\alpha}(t'y)y'
\]
where
\[
t' = g(z,k) - g(x,k) + \phi_{\chi}^x(y), \quad y' = \tilde{\alpha}(g(x,k))y.
\]
Notice that we have \(y' \in \mathcal{W}_{\tilde{\alpha}(\tilde{\beta}(k,x))}^{-\mathcal{W}}\).

By Lemma 4.2 and the fact that \(\mathcal{W}_{\tilde{\beta}(\chi)}^{-\mathcal{W}}\) is \(\tilde{\beta}(\mathfrak{a})\)-invariant, we see that \(z \in \mathcal{W}_{\tilde{\beta}(\chi)}^{-\mathcal{W}}(x)\) and \(\tilde{\beta}(k, z) \in \mathcal{W}_{\tilde{\beta}(\chi)}^{-\mathcal{W}}(\tilde{\beta}(k,x))\). Thus there exists

\(\footnotesize\^2\)We say that two collections \(\{\phi_x : \mathcal{W}_{\chi'}(x) \to \mathbb{R}\}_{x \in M^x}\) and \(\{\phi_x : \mathcal{W}_{\chi'}(x) \to \mathbb{R}\}_{x \in M^x}\) are equivalent if for \(\mu\)-a.e. \(x\), for \(\mu_{\chi}^{\mathcal{W}_{\chi'}}\)-a.e. \(y\), we have \(\phi_x(y) = \phi_x(y)\).
where $y'' \in \mathcal{W}^-_{\tilde{\alpha}(s)}(\tilde{\beta}(k, x))$ such that
\[
\tilde{\beta}(k, z) = \tilde{\alpha}(\varphi_{\tilde{\beta}(k, x)}(y'')w)y''.
\]
Consequently, $y'' \in \mathcal{W}^-_{\tilde{\alpha}(s)}(y')$, and there exists $t'' \in \mathbb{R}$ such that
\[
y'' = \tilde{\alpha}(t''w)y'.
\]
Since we have $d(\tilde{\alpha}(ns)y', \tilde{\alpha}(ns)y'') \to 0$ as $n$ tends to infinity, we can show that $t'' = 0$ for a $\mu$-typical $y$. Consequently, $y'' = y'$. This proves our claim.

Fix an arbitrary $s \in a$ such that
\[
(4.4) \quad \chi'(s) < -10\|s\|\varepsilon.
\]
Then by Lemma 4.2 for $\mu$-a.e. $x$, function $\varphi_x^s$ is defined $\mathcal{W}^-_{\tilde{\alpha}(s)}$ almost everywhere. Thus for $\mu_{\mathcal{W}^-_{\tilde{\alpha}(s)}}$-a.e. $y$, $\varphi_x^s(z)$ is defined for $\mu_{\mathcal{W}^x}$-a.e. $z$.

We define $\varphi_x^{\chi'}$ to be the restriction of $\varphi_x^s$ to $\mathcal{W}^{\chi'}(x)$ for $\mu$-a.e. $x$. In the following we abbreviate $\varphi_x^{\chi'}$ as $\varphi_x$.

We also show that $\varphi_x$ defined above is essentially independent of the choice of $s$. Take another $s' \in a$ with $\chi'(s') < 0$. Assume that there exists a set $\Omega \subset M^a$ with $\mu(\Omega) > 0$ such that for every $x \in \Omega$, there exists a subset $\Omega_x \subset \mathcal{W}^{\chi'}(x)$ with positive $\mu_{\mathcal{W}^{\chi'}}$ measure such that for every $y \in \Omega_x$, $\varphi_x(y) = t''w + \varphi_x^{s'}(y)$ for some $t'' \neq 0$. On the other hand, by (4.1), (4.4), (4.3) and by the H"older continuity of $\varphi_x^{s'}$, $\varphi_x$ on Pesin sets, we see that for typical choices of $x, y$, we have
\[
d(\tilde{\beta}(ns)y', \tilde{\beta}(ns)y'') \to 0 \quad \text{as } n \to \infty
\]
where $y' = \tilde{\alpha}(\varphi_x(y)w)y, y'' = \tilde{\alpha}(\varphi_x^{s'}(y)w)y$. By Lemma 4.3, this contradicts $t'' \neq 0$. Consequently, we see that the definition of $\varphi_x$ is independent of the choice of $s$. This concludes the proof. \hfill \Box

From the proof of Lemma 4.4, we can deduce the following.

**COROLLARY A.** For any $\chi' \in \Sigma$, for any $s \in a$ such that $\chi'(s) < 0$, for $\mu$-a.e. $x$, for $\mu_{\mathcal{W}^{\chi'}}$-a.e. $y$, we have
\[
\varphi_x^s(y) = \varphi_x^{\chi'}(y)
\]
where $\varphi_x^s$ is given by Lemma 4.2 and $\varphi_x^{\chi'}$ is given by Lemma 4.4.

By Lemma 4.4 and Corollary A, we can define for each $\chi' \in \Sigma$ a collection of $\tilde{\beta}(a)$-invariant sets $\tilde{\mathcal{W}}^{\chi'}$ by setting
\[
\tilde{\mathcal{W}}^{\chi'}(x) = \{\tilde{\alpha}(\varphi_x(y)w)y \mid y \in \mathcal{W}^{\chi'}(x)\}.
\]
\[
\text{define } \tilde{\mathcal{W}}_{\tilde{\beta}(a)}^{\chi'}, \tilde{\mathcal{W}}_F^{\chi'} \text{ and } \tilde{\mathcal{W}}_G^{\chi'}.
\]

We have the following useful lemma.

**Lemma 4.5.** For any \( \chi' \in \Sigma, \) for \( \mu \)-a.e. \( x, \) the conditional measure \( \tilde{\mu}_{\chi'}^{\tilde{\mathcal{W}}_G^{\chi'}} \) is absolutely continuous with respect to \((y \mapsto \tilde{\alpha}(\varphi_x(y)w)y)\_*\tilde{\mu}_{\chi'}^{\mathcal{W}^{\chi'}}.\) We have analogous statements for \( \mathcal{W}_F^{\chi'}, \mathcal{W}_G^{\chi'} \) and \( \mathcal{W}_{\tilde{\alpha}(a)}^{\chi'}, a \in a.\)

**Proof.** This is proved in the last paragraph of [16, Lemma 7.1]. \( \square \)

**Remark 1.** The proof of Lemma 4.5 uses the following fact. For any \( \chi' \in \Sigma, \) for \( \mu \)-a.e. \( x, \) we have

\[
\tilde{\alpha}(\mathbb{R}w)\tilde{\mathcal{W}}_{\chi'}^{\chi'}(x) = \tilde{\alpha}(\mathbb{R}w)\mathcal{W}_{\star}^{\chi'}(x) \ast \emptyset, F, G.
\]

In particular, when \( \chi' \in \Sigma, \) the conditional measure of \( \tilde{\mu} \) on \( \tilde{\alpha}(\mathbb{R}w)\tilde{\mathcal{W}}_{\chi'}^{\chi'}(x) \) is equivalent to the natural push-forward of the Lebesgue measure on \( \mathbb{R} \times G^{\chi'}. \)

By Remark 1 for every \( \chi' \in \Sigma, \) we can define \( \mathcal{W}_G^{\chi'} \)-holonomy maps between \( \tilde{\alpha}(\mathbb{R}w) \)-orbits within \( \tilde{\alpha}(\mathbb{R}w)\mathcal{W}_{\star}^{\chi'}(x) \) for \( \mu \)-a.e. \( x.\)

**Lemma 4.6.** Let \( x \) be a \( \mu \)-typical point, and let \( h \in G^{\chi'} \) such that the \( \mathcal{W}_G^{\chi'} \)-holonomy map between \( \tilde{\alpha}(\mathbb{R}w)x \) and \( \tilde{\alpha}(\mathbb{R}w)\tilde{\alpha}(h)x \) is defined for Lebesgue almost every point. Then the \( \mathcal{W}_G^{\chi'} \)-holonomy map between \( \tilde{\alpha}(\mathbb{R}w)x \) and \( \tilde{\alpha}(\mathbb{R}w)\tilde{\alpha}(h)x \) is absolutely continuous.

**Proof.** We will show that this \( \mathcal{W}_G^{\chi'} \)-holonomy map extends to a Lipschitz map.

For \( i = 1, 2, \) we take \( t_i, s_i \in \mathbb{R} \) such that \( \tilde{\mathcal{W}}_{\tilde{\beta}(a)}^{\chi'}(\tilde{\alpha}(t_iw)x) \) intersects \( \tilde{\alpha}(\mathbb{R}w)\tilde{\alpha}(h)x \) at \( \tilde{\alpha}(s_iw)\tilde{\alpha}(h)x.\) Take an arbitrary \( a, b \) such that \( \chi'(a) < 0. \) For any integer \( n > 0, \) we denote by \( u_n, v_n \in \mathbb{R} \) constants such that

\[
\tilde{\beta}(na)\tilde{\alpha}(t_1w)x = \tilde{\alpha}(u_nw)\tilde{\beta}(na)\tilde{\alpha}(t_2w)x,
\]

\[
\tilde{\beta}(na)\tilde{\alpha}(s_1w)\tilde{\alpha}(h)x = \tilde{\alpha}(v_nw)\tilde{\beta}(na)\tilde{\alpha}(s_2w)\tilde{\alpha}(h)x.
\]

On one hand, we know that for \( i = 1, 2, \)

\[
d(\tilde{\beta}(na)\tilde{\alpha}(t_iw)x, \tilde{\beta}(na)\tilde{\alpha}(s_iw)\tilde{\alpha}(h)x) \to 0 \quad \text{as} \quad n \to +\infty.
\]

This implies that \( |v_n - u_n| \) tends to 0 as \( n \) tends to infinity. On the other hand, by Lemma 4.3, we see that both \( \frac{t_1 - t_2}{u_n}, \frac{s_1 - s_2}{v_n} \) are bounded from above and from below by constants independent of \( n. \) This implies our lemma. \( \square \)
4.2. The proof for the non-atomic case.

Proof of Proposition 3.4 — the non-atomic case. Our argument is an adaptation of the \(\pi\)-partition trick (see [15, 16]). The main tool is the following lemma.

LEMMA 4.7. For any Pesin set \(R\), there exists a constant \(K > 0\) such that for \(\mu\)-a.e. \(x \in R\), and \(\mu^W_x\)-a.e. \(y \in R \cap W^x_{F,loc}(x)\), there exists a sequence \(\{l_n\}_{n \in \mathbb{N}} \subset a\) satisfying that \(\tilde{a}(l_n)x \rightarrow y\) as \(n \rightarrow \infty\) and \(\|D\tilde{a}(l_n)|_{E(x)}\| < K\).

Proof. In the following, for any \(b \in a\), we let \(\tilde{W}^{b}_{\beta(b)}\) denote the sub-\(\sigma\)-algebra of \(B_{M^a}\) whose elements are unions of subsets of the form \(\tilde{W}^{b}_{\beta(b)}(y)\) with \(y \in M^a\) (this was previously introduced in [16 Section 6.4]). We define \(\tilde{W}^{a\chi}\) and \(\tilde{W}^{a\chi}_b\) analogously. For any \(a \in a\), we denote by \(E\) the sub-\(\sigma\)-algebra of \(B_{M^a}\) formed by \(\tilde{\beta}(a)\)-invariant sets.

We take a singular generic \(a \in L_x\), i.e., \(a \in L_x\) but \(a \notin L_x^c\), \(\forall \chi' \in \Sigma \setminus \{\pm \chi\}\), and some generic \(b \in a\), close to \(a\), such that \(\chi(b) > 0\) and \(\chi'(b), \chi'(a)\) have the same sign for all \(\chi' \in \Sigma \setminus \{\pm \chi\}\). Then by [18], we have

\[
(4.5) \quad [\tilde{W}^{a\chi}_F] \supset [\tilde{W}^{a\chi}] \supset [\tilde{W}^{a\chi}_b] = [\tilde{W}^{a\chi}_{\tilde{\beta}(b)}].
\]

We also have the following inclusion.

LEMMA 4.8. We have \([\tilde{W}^{a\chi}_{\tilde{\beta}(b)}] \supset [\tilde{W}^{a\chi}_{\tilde{\beta}(a)}] \cap [\tilde{W}^{a\chi}]\).

Proof. Take an arbitrary function \(f \in L^2(M^a, \tilde{\mu})\). We set

\[
 f_0 = \mathbb{E}(f | [\tilde{W}^{a\chi}_{\tilde{\beta}(a)}] \cap [\tilde{W}^{a\chi}]).
\]

Then by definition, Lemma 4.5 and Corollary \(\mathbb{A}\) there exist \(\mu\)-conull sets \(\Omega_0, \Omega_1 \subset M^a\) such that:

1. for every \(x \in \Omega_1\), for \(\mu^W_x\)-a.e. \(y\), the point \(z := \tilde{a}(\phi^y_x(y))\) satisfies \(f_0(x) = f_0(z)\), and \(\phi^y_x(y) = \phi^y_x(y)\);

2. for every \(x \in \Omega_0\), for \(\mu^W_x\)-a.e. \(y\) belongs to \(\Omega_1\). Moreover, for every \(y \in \Omega_0 \cap W^a_{\tilde{\alpha}(a)}(x)\), we have \(z := \tilde{a}(\phi^y_x(y))\) satisfies \(f_0(x) = f_0(z)\), and \(\phi^y_x(y) = \phi^y_x(y)\).

We claim that for \(\mu\)-a.e. \(x\), for \(\tilde{\beta}_x\)-a.e. \(z\), we have \(f_0(x) = f_0(z)\). This will imply that \(f_0\) is \([\tilde{W}^{a\chi}_{\tilde{\beta}(b)}]\)-measurable, and conclude the proof.
As $a \in L_{\eta}$, we have $\hat{a}(g)\hat{a}(g) = \hat{a}(g)\hat{a}(g)$ for any $g \in G^\chi$. Thus for any $x \in M^a$, any $y \in \mathcal{W}_{\hat{a}(g)}^-(x)$, any $g \in G^\chi$, we have $\hat{a}(g)y \in \mathcal{W}_{\hat{a}(g)}^-(\hat{a}(g)x)$. By $\chi \in \Sigma^\text{out}_3$, we know that $\mu$ is $G^\chi$-invariant. This implies that for any $g \in G^\chi$, for $\mu$-a.e. $x$, we have

$$[(\hat{a}(g))_*\mu_x] = [\mu_{\hat{a}(g)x}].$$

Thus for $\mu$-a.e. $x \in \Omega_0$, for a $\mu_x^\chi$-typical $y$, there exists $y' \in \Omega_1 \cap \mathcal{W}_{\hat{a}(g)}^-(x)$ such that $y \in \mathcal{W}^-\chi(y')$. Set $z' = \hat{a}(q_y^b(y')w)y'$ and $z'' = \hat{a}(q_y^b(y')w)y \in \mathcal{W}^-\chi(z')$. Then $z' \in \Omega_1$ and $f_0(x) = f_0(z')$. As the holonomy map between $\mathcal{W}^-\chi(y')$ and $\mathcal{W}^-\chi(z')$ along $\hat{a}(\mathcal{R}w)$-orbits is absolutely continuous, a typical choice of $y \in \mathcal{W}^-\chi(y')$ corresponds to a typical choice of $z'' \in \mathcal{W}^-\chi(z')$. Thus for a typical $y$ we have

$$z := \hat{a}(q_y^b(y)w) = \hat{a}(q_y^b(z')w)z''.$$

Consequently, we have $f_0(z) = f_0(z') = f_0(x)$. \hfill $\square$

**Lemma 4.9.** We have $[\tilde{\mathcal{E}}_{\tilde{\beta}(a)}] \subset [\tilde{\mathcal{W}}^-\chi]$.

**Proof.** By $\chi \in \Sigma^\text{out}_3$, we have $\tilde{\mathcal{W}}^-\chi = \tilde{\mathcal{W}}^-\chi$. Thus it suffices to show that $[\tilde{\mathcal{E}}_{\tilde{\beta}(a)}] \subset [\tilde{\mathcal{W}}^-\chi]$.

We fix a continuous function $\theta$ on $M^a$. We define

$$B_{\theta}^\pm = \{x \mid \lim_{n \to \pm \infty} \frac{1}{n} \sum_{i=0}^{n-1} \theta(\tilde{b}(na)x) = \int \theta d\tilde{\mu}_{\tilde{\beta}(a)}x\}$$

where $\tilde{\mu}_{\tilde{\beta}(a)}$ denotes the $\tilde{\beta}(a)$-ergodic component of $\tilde{\mu}$ at $x$.

By Birkhoff’s ergodic theorem, we know that for $\tilde{\mu}$-a.e. $x$, $B_{\theta}^+(x) = B_\theta^-(x)$. In this case, we say that $x$ is regular (with respect to $\theta$) and denote $B_\theta(x) := B_{\theta}^\pm(x)$. Consequently, by the $\hat{a}(\mathcal{R}w)$-invariance of $\mu$ and the fact that $\tilde{\mu} \sim \mu$, the conditional measures of $\tilde{\mu}$ along $\hat{a}(\mathcal{R}w)$-orbits are absolutely continuous with respect to Lebesgue. Thus for $\tilde{\mu}$-a.e. $x$, for Lebesgue almost every $t \in \mathbb{R}$, $\hat{a}(tw)x$ is regular.

We let $W$ be the set of $x \in M^a$ such that for $\eta \in \{-\chi, \chi_1, \chi_2\}$, $x$ satisfies Corollary [A]. We know that $W$ is a $\tilde{\mu}$-conull set. Then for $\tilde{\mu}$-a.e. $x$, for Lebesgue almost every $t \in \mathbb{R}$, $\hat{a}(tw)x$ is regular and $q_t^n_{\hat{a}(tw)x}$ is
defined at $\overset{\sim}{\alpha}(t w)\overset{\sim}{\alpha}(h)x$. We denote the above $\nu_{G^\eta}$-conull set by $\Omega_\eta$, and for every $h \in \Omega_\eta$ we denote by $W_h$ the above $\mu$-conull set of $x$.

By $\chi \in -\Sigma^\eta_{Q_{\text{non}}}$, there exist $\chi_1, \chi_2 \in \Sigma_Q \setminus \{\pm \chi\}$ such that $-\chi = \chi_1 + \chi_2$. Then for $\nu_{G^\chi}$-a.e. $h$, there exist $h_i, h_{i+2} \in \Omega_{\chi_i}, i = 1, 2$ such that $h = h_i h_3 h_2 h_1$.

It is direct to see that $\sim \mu$-a.e. $x$ satisfies that $x \in W_{h_1}, \overset{\sim}{\alpha}(h_1)x \in W_{h_2}, \overset{\sim}{\alpha}(h_2 h_1)x \in W_{h_3}, \overset{\sim}{\alpha}(h_3 h_2 h_1)x \in W_{h_4}$.

By Lemma [4.6], the $\overset{\sim}{\nu}_G^{\chi_1}$-holonomy map between $\overset{\sim}{\alpha}(\mathbb{R}w)x$ and $\overset{\sim}{\alpha}(\mathbb{R}w)\overset{\sim}{\alpha}(h_1)x$ within the leaf $\overset{\sim}{\alpha}(\mathbb{R}w)\overset{\sim}{\nu}_G^{\chi_1}(x)$ is absolutely continuous. More precisely, for Lebesgue almost every $t \in \mathbb{R}$, the intersection between $\overset{\sim}{\nu}_G^{\chi_1}(\overset{\sim}{\alpha}(t w)x)$ and $\overset{\sim}{\alpha}(\mathbb{R}w)\overset{\sim}{\alpha}(h_1)x$ is $\overset{\sim}{\alpha}(\phi(t)w)\overset{\sim}{\alpha}(h)x$ where

$$\phi(t) = \phi_{\overset{\sim}{\alpha}(t w)}^{\chi_1}(\overset{\sim}{\alpha}(t w)\overset{\sim}{\alpha}(h_1)x) + t.$$ 

Lemma [4.6] implies that $\phi$ preserves the Lebesgue class. Consequently, for Lebesgue almost every $t \in \mathbb{R}$, $\overset{\sim}{\alpha}(t w)x, \overset{\sim}{\alpha}(\phi(t)w)\overset{\sim}{\alpha}(h)x$ are both regular.

By iterating the above argument, we see that for Lebesgue almost every $t \in \mathbb{R}$, there exist regular points $x_1, \cdots, x_4$ such that the following is true. Set $x_0 = \overset{\sim}{\alpha}(t w)x$. We have

$$x_1 \in \overset{\sim}{\nu}_G^{\chi_1}(x_0), x_2 \in \overset{\sim}{\nu}_G^{\chi_2}(x_1), x_3 \in \overset{\sim}{\nu}_G^{\chi_1}(x_2), x_4 \in \overset{\sim}{\nu}_G^{\chi_2}(x_3).$$

Moreover, there exists $s \in \mathbb{R}$ such that $x_4 = \overset{\sim}{\alpha}(sw)\overset{\sim}{\alpha}(h)x_0$.

By definition, it is easy to see that $B_\theta(x_0) = B_\theta(x_4)$, and for some $c \in a$ such that both $\chi_1(c), \chi_2(c) < 0$, we have

$$d(\overset{\sim}{\alpha}(nc)x_0, \overset{\sim}{\alpha}(nc)x_4) \to 0 \quad \text{as} \quad n \to \infty.$$ 

This implies that $x_4 \in \overset{\sim}{\nu}_G^{-\chi}(x_0)$ and consequently $s = \phi_{x_0}^{-\chi}(\overset{\sim}{\alpha}(h)x_0)$.

Finally, by Fubini’s lemma, we deduce that for $\mu$-a.e. $x$, for $\overset{\sim}{\nu}_G^{-\chi}$-a.e. $y$, we have $B_\theta(x) = B_\theta(y)$. By take $\theta$ over a dense subset of $L^1(M^x, \overset{\sim}{\mu})$, we conclude the proof.

It is well-known that

$$[\overset{\sim}{\mathcal{E}}_{\overset{\sim}{\beta}(a)}] \subset [\overset{\sim}{\nu}_G^{-\chi}(a)].$$ 

Consequently, by Lemma [4.8] and Lemma [4.9] we have

$$[\overset{\sim}{\mathcal{E}}_{\overset{\sim}{\beta}(a)}] \subset [\overset{\sim}{\nu}_F^{\chi}]$$

Let $R$ be the Pesin set in the lemma. Let $\phi_x$ be given by Lemma [4.4] for $\overset{\sim}{\nu}_F^{\chi}$. By Lemma [4.4], there exists $K_1 > 0$ such that $|\phi_x(y)| < K_1$ for
any $y \in R \cap W^s_{F,loc}(x)$. Then by (4.1) the point $z = \tilde{a}(\varphi_x(y)w)y$ belongs to a Pesin set $R' \supset R$ which depends on $R$, but is independent of $x$ and $y$.

By Lemma 4.5 for $\mu$-a.e. $x$, for $\mu_x$-a.e. $y \in R \cap W^s_{F,loc}(x)$, $\tilde{a}(\varphi_x(y)w)y$ is a $\tilde{\mu}_{\tilde{b}(a)}$ density point of $R'$. Then by Birkhoff’s ergodic theorem, for the above $x, y, z$ there exists a sequence $\{k_n\}_{n \in \mathbb{N}} \subset \mathbb{N}$ such that $\tilde{b}(k_na)x \in R'$ converges to $z$ as $n$ goes to infinity. Let $l_n = g(x, k_na) - \varphi_x(y)w$, then we have

$$\tilde{a}(l_n)_x = \tilde{a}(-\varphi_x(y)w)\tilde{b}(k_na)x \to y \quad \text{as} \quad n \to \infty.$$ 

We have

$$D\tilde{a}(l_n)(x) = D\tilde{a}(-\varphi_x(y)w)(\tilde{b}(k_na)x)D\tilde{a}(g(x, k_na))(x).$$

Moreover

$$\|D\tilde{a}(g(x, k_na))|_{E(x)}\| \leq K^2$$

where $K^2$ is the maximal distortion between $\| \cdot \|_e$ and the background metric on $M^a$ over the Pesin set $R'$. By $|\varphi_x(y)| < K_1$, we can see that there exists $K > 0$ depending only on $R$, such that

$$\|D\tilde{a}(l_n)|_{E(x)}\| < K.$$

This concludes the proof.

By the argument in [15], we see that $\mu_x$ is absolutely continuous with positive density Lebesgue almost everywhere. As $W^s_F$ is $C^1$ foliated by $W^s_F$ and $W^s_G$, by the absolute continuity of the $W^s_F$-holonomy maps between different $W^s_F$-leaves, we deduce that $\mu_x$ is non-atomic for $\mu$-a.e. $x$. Consequently, $\mu^{C\chi}_x$ is non-atomic for $\mu$-a.e. $x$.

5. WHEN $\mu^{W^s_F}$ IS ATOMIC

Through out this section, we assume that for $\mu$-a.e. $x$, $\mu^{W^s_F}_x$ is supported on a discrete set with respect to the leafwise metric. Then the following result is well-known.

LEMMA 5.1. For $\mu$-a.e. $x$, $\mu^{W^s_F}_x$ is the Dirac measure at $x$.

Proof. Assume to the contrary that the lemma fails. For $\mu$-a.e. $x$, we define

$$r(x) := \sup\{\sigma \mid \mu_x^{W^s_F}(B(x, \sigma) \setminus \{x\}) = 0\}.$$
We obtain a contradiction by the $A$-invariance of $\mu$ and Poincaré’s recurrence theorem.

5.1. A local entropy formula. In this subsection, we recall a local entropy formula from [19].

We fix an arbitrary $k \in a$ such that $\chi(k) > 0$. Let us denote $f = \tilde{\alpha}(k)$.

By the construction in [18, Section 3], we can also choose two measurable partitions $\eta_0$ and $\eta_1$ such that

1. $\eta_0$, resp. $\eta_1$, is subordinate to $\mathcal{W}_G^X$, resp. $\mathcal{W}_G^X$;
2. $\eta_0, \eta_1$ are all $f$-increasing and $f$-generating;
3. $\eta_0 \geq \eta_1$.

Moreover, we can also ensure that

1. $\mathcal{W}_{G,loc}^X(y) \cap \eta_1(x) = \eta_0(y)$ for $\mu$-a.e. $x$ and every $y \in \eta_1(x)$;
2. $f^{-1}(\eta_0(x)) = \eta_0(f^{-1}(x)) \cap f^{-1}(\eta_1(x))$ for $\mu$-a.e. $x$ and every $y \in \eta_1(x)$.

LEMMA 5.2. We have

$$h_\mu(f, \eta_1) \leq h_\mu(f, \eta_0) + \chi_F(k).$$

Proof. This follows from [19, Section 11].

REMARK 2. In Ledrappier-Young [18], this was proved in the setting where an invariant subfoliation of the unstable foliation is foliated by strong unstable foliation. Here neither $\mathcal{W}_F^X$ or $\mathcal{W}_G^X$ is not a strong subfoliation of $\mathcal{W}^X$. But in our setting, the local product structure of $M^a$ and the group action allows us to show that $\mathcal{W}^X$ is $C^1$ foliated by both $\mathcal{W}_F^X$ and $\mathcal{W}_G^X$. This suffices for the construction of $\eta_0, \eta_1$.

LEMMA 5.3. We have

$$h_\mu(f, \eta_1) = 2\chi_G(k).$$

Proof. This is a consequence of [4, Theorem 13.6], our hypothesis that $\mu^{\mathcal{W}_F^X}$ is atomic, and the fact that $\pi_* \mu$ is the Haar measure on $G/\Gamma$.

COROLLARY B. If for $\mu$-a.e. $x$, $\mu^{Gx}_x$ is atomic, then there exists a constant $\lambda > 1$ such that $\chi_F = \lambda \chi_G$.

Proof. By the definition of $\chi$, there exists $\lambda > 0$ such that $\chi_F = \lambda \chi_G$. Take an arbitrary $k \in a$ such that $\chi(k) > 0$, and set $f = \tilde{\alpha}(k)$. By the hypothesis in the lemma, we know that $h_\mu(f, \eta_0) = 0$. By Lemma 5.3 and Lemma 5.2 we obtain

$$\chi_F(k) \geq h_\mu(f, \eta_1) \geq 2\chi_G(k).$$

□
5.2. Non-stationary normal form. We recall a result in [15] on the existence of the non-stationary normal form. In our setting, their result states as follows.

**Lemma 5.4.** For $\mu$-a.e. $x \in M^a$, there exists a $C^{1+\epsilon}$ diffeomorphism $h_x : \mathcal{W}_F^X(x) \to \mathbb{R}$ such that

(i) $h_{\tilde{\alpha}(k)x} \circ \tilde{\alpha}(k) = D\tilde{\alpha}(k) \circ h_x$ for every $k \in a$,
(ii) $h_x(x) = 0$ and $D_xh_x$ is an isometry,
(iii) $h_x$ depends continuously on $x$ in the $C^{1+\epsilon}$ topology on a Pesin set.

Let us denote by $\Omega$ the $\mu$-conull subset in Lemma 5.4 on which the non-stationary normal form is defined. For any $x \in \Omega$, the map $h_x$ can be expressed in an explicit manner which we now describe. We fix $x \in \Omega_0$ and an element $k_0 \in a$ such that $\chi(k_0) < 0$, and denote $\tilde{\alpha}(k_0)$. For any $z \in M^a$, we denote

$$Jf(z) = \|Df|_{E(z)}\|.$$ 

For any $y \in \mathcal{W}_F^X(x)$, we have

$$(5.1) \quad |h_x(y)| = \int_x^y \rho_x(z)dz$$

where

$$\rho_x(z) = \prod_{i=0}^{\infty} \frac{Jf(f^i(z))}{Jf(f^i(x))}.$$ 

The integral in (5.1) is defined using the Riemannian metric on $\mathcal{W}_F^X(x)$.

We define

$$\Omega_1 = \bigcup_{x \in \Omega} \mathcal{W}_F^X(x).$$

Then by (5.1), we can define $h_y$ for any $y \in \Omega_1$. We have the following useful observations.

**Lemma 5.5.** For any $x \in \Omega$, for any $y_1, y_2 \in \mathcal{W}_F^X(x)$, the map $h_{y_1}h_{y_2}^{-1}$ is an affine transformation of $\mathbb{R}$.

**Proof.** This is proved in [15] Lemma 3.3].

**Lemma 5.6.** For any $y \in \Omega$, for any $z \in \Omega_1$ such that there exists $g \in G_x$ satisfying $z = \tilde{\alpha}(g)y$, we have

$$\tilde{\alpha}(g)\mathcal{W}_F^X(y) = \mathcal{W}_F^X(z).$$

Moreover, there exists $c \in \{\pm\|D\tilde{\alpha}(g)|_{E(y)}\|\}$ such that

$$h_z\tilde{\alpha}(g)h_y^{-1}(t) = ct, \quad \forall t \in \mathbb{R}. $$
Proof. Take an arbitrary \( w \in \mathcal{W}_F^\chi(y) \), we denote \( u = \tilde{\alpha}(g)w \). Then for any \( k \in a \) such that \( \chi(k) < 0 \), we have

\[
\lim_{n \to \infty} n^{-1} \log d(\tilde{\alpha}(nk)u, \tilde{\alpha}(nk)z) < 0.
\]

Moreover, we have

\[
\pi(u) = g\pi(w) = g\pi(y) = \pi(z).
\]

Thus we have \( u \in \mathcal{W}_F^\chi(z) \). This proves the first statement.

We now prove the last statement. We use the natural parametrisation of \( G^\chi \) by \( \mathbb{R}^2 \). Namely, we define a diffeomorphism \( \theta_\chi : \mathbb{R}^2 \to G^\chi \) by

\[
\theta_\chi(a, b) = \text{Id} + (a + ib)E_\chi
\]

where \( E_\chi = E_{s,t} \) if \( \chi = \chi_{s,t} \). We write \( g = \theta_\chi(v) \) for some \( v \in \mathbb{R}^2 \), and write \( \lambda = e^{\chi_G(k_0)} < 1 \). Notice that

\[
f(\tilde{\alpha}(\theta_\chi(v))) = \tilde{\alpha}(\theta_\chi(\lambda v))f.
\]

Thus we have

\[
Jf(u) = \|D\tilde{\alpha}(\theta_\chi(\lambda v))\|_{E(f(u))} \|Jf(w)\|D\tilde{\alpha}(\theta_\chi(-v))\|_{E(u)}
\]

\[
= Jf(w)\|D\tilde{\alpha}(\theta_\chi(\lambda v))\|_{E(f(w))} \|D\tilde{\alpha}(\theta_\chi(v))\|_{E(w)}^{-1}.
\]

More generally, for every integer \( i \geq 0 \), we have

\[
Jf(f^i(u)) = Jf(f^i(w))\|D\tilde{\alpha}(\theta_\chi(\lambda^{i+1}v))\|_{E(f^{i+1}(w))} \|D\tilde{\alpha}(\theta_\chi(\lambda^iv))\|_{E(f^i(w))}^{-1}.
\]

Analogously, we have

\[
Jf(f^i(z)) = Jf(f^i(y))\|D\tilde{\alpha}(\theta_\chi(\lambda^{i+1}v))\|_{E(f^{i+1}(y))} \|D\tilde{\alpha}(\theta_\chi(\lambda^iv))\|_{E(f^i(y))}^{-1}.
\]

To simplify notation, we set

\[
\tilde{\xi}_{i,w} = \|D\tilde{\alpha}(\theta_\chi(\lambda^iv))\|_{E(f^i(w))},
\]

\[
\tilde{\xi}_{i,y} = \|D\tilde{\alpha}(\theta_\chi(\lambda^iv))\|_{E(f^i(y))}.
\]
Notice that $\xi_{i,w}, \xi_{i,y}$ tend to 1 exponentially fast as $i$ tends to infinity. Then for any $w_\ast \in W^X_F(y)$, denote $u_\ast = \tilde{\alpha}(g)w_\ast$, we have
\[
|h_z(u_\ast)| = \int_z^{u_\ast} \rho_z(u)du
= \int_z^{u_\ast} \prod_{i=0}^{\infty} \frac{f_i(u)}{f_i(z)} du
= \int_z^{u_\ast} \prod_{i=0}^{\infty} \frac{f_i(w)}{f_i(y)} \frac{\xi_{i+1,w} \xi_{i,y}}{\xi_{i+1,y} \xi_{i,w}} du
= \int_z^{u_\ast} \rho_y(w) \frac{\xi_{0,y}}{\xi_{0,w}} du
(\text{ as } u = \tilde{\alpha}(g)w) = \xi_{0,y} \int_y^{w_\ast} \rho_y(w)dw
= \|D\tilde{\alpha}(g)|_{E(y)}\| |h_y(w_\ast)|.
\]
This confirms the last statement. □

5.3. The proof for the atomic case. We use the following parametrisation of $W^X$. For every $x \in \Omega_1$, we define the map $H_x$ from $W^X(x)$ to $\mathbb{R}^3$ by
\[
H_x(p) = (a(p), b(p)) \text{ if we have } p = \tilde{\alpha}(\theta_X(a(p)))h^{-1}_X(b(p))
\]
where $\theta_X$ is defined in (5.2). It is straightforward to verify that $H_x$ is a homeomorphism.

We notice that for any $x \in \Omega_1$, for any $k \in a$, there exists $c \in \{\pm\|D\tilde{\alpha}(k)|_{E(x)}\|\}$ such that
(5.3) $H_{\tilde{\alpha}(k)x}(k)H^{-1}_X(u, v) = (e^{\chi c(k)}u, cv)$, $\forall u \in \mathbb{R}^2, v \in \mathbb{R}$.

Let us define a subgroup of the affine transformations of $\mathbb{R}^3$ as follows,
$A = \{(v_1, v_2, v_3) \mapsto (v_1 + a_1, v_2 + a_2, bv_3 + c) \mid a_1, a_2, c \in \mathbb{R}, b \in \mathbb{R}^*_\}$. For each $T \in A$, we will use $a_1(T), a_2(T), b(T), c(T)$ to denote the coefficients in the expression of $T$. We also set
\[
a(T) := (a_1(T), a_2(T)).
\]

We collect some useful properties of $H_x$.

**Lemma 5.7.** For $\mu$-a.e. $x$, for $\mu^W_X$-a.e. $y$, the map $H_yH^{-1}_X$ belongs to $A$. Moreover, we have
\[
\pi(y) = \tilde{\alpha}(\theta_X(-a(H_yH^{-1}_X)))\pi(x).
\]
Proof. As \( \Omega \) is \( \mu \)-conull, for \( \mu \)-a.e. \( x, \mu_x^{Wx} \)-a.e. \( y \) belongs to \( \Omega \). We fix \( x, y \in \Omega \) as above. Since \( \pi(y) \in G^{x}(\pi(x)) \), we see that there exists \( v \in \mathbb{R}^2 \) such that \( z := \tilde{a}(\theta_x(v))(x) \in \pi^{-1}(\pi(y)) \cap \mathcal{W}^x(x) \). Then it is clear that \( z \in \mathcal{W}^x_{H}(y) \). By Lemma 5.5 we can see that \( H_yH_z^{-1} \in \mathcal{A} \).

Moreover, it is clear that \( a(H_yH_z^{-1}) = (0,0) \).

By Lemma 5.6 we have

\[
(5.4) \quad h_x^{-1}(ct) = \tilde{a}(\theta_x(v))h_x^{-1}(t), \quad \forall t \in \mathbb{R}
\]

where \( c \in \{ \pm \|D\tilde{a}(\theta_x(v))|_{E(x)}\| \} \). As \( H_x, H_z \) are homeomorphisms between \( \mathcal{W}^x(x) \) and \( \mathbb{R}^3 \), for any \( s \in \mathbb{R}^2, t \in \mathbb{R} \), there exists a unique pair \( s' \in \mathbb{R}^2, t' \in \mathbb{R} \) such that \( H_x^{-1}(s,t) = H_z^{-1}(s',t') \). Then by the definitions of \( H_x, H_z \) and by (5.4), we have

\[
\tilde{a}(\theta_x(s))h_x^{-1}(t) = \tilde{a}(\theta_x(s'))h_x^{-1}(t') = \tilde{a}(\theta_x(s'))\tilde{a}(\theta_x(v))h_x^{-1}(c^{-1}t') = \tilde{a}(\theta_x(s'+v))h_x^{-1}(c^{-1}t').
\]

Consequently, we have

\[
s' = s - v, \quad t' = ct.
\]

Thus \( H_zH_x^{-1} \in \mathcal{A} \) and \( a(H_zH_x^{-1}) = -v \). Hence \( H_yH_x^{-1} \in \mathcal{A} \) and \( a(H_yH_x^{-1}) = -v \). This concludes the proof. \( \square \)

We denote

\[
(5.5) \quad \mathcal{A}^0 = \text{Ker}(p) = \{(v_1, v_2, v_3) \mapsto (v_1, v_2, bv_3 + c \mid b, c \in \mathbb{R}) \}.
\]

We denote by \( \mathcal{PM}(\mathbb{R}^3) \) the space of equivalence classes under proportionality of Radon measure on \( \mathbb{R}^3 \). We define

\[
\mathcal{H} := L^0(\mathbb{R}^2, \text{Leb}).
\]

That is, the set of Borel measurable \( \mathbb{R} \)-valued functions on \( \mathbb{R}^2 \) modulo the equivalence

\[
\sim \quad \text{iff } f_1(v) = f_2(v) \text{ for Lebesgue almost every } v.
\]

It is well-known that \( \mathcal{H} \), equipped with the topology given by convergence in measure, is a complete metric space.

Proof of Proposition 3.4 — the atomic case. We assume by contradiction that \( \mu_x^{Gx} \) is atomic for \( \mu \)-a.e. \( x \). Consequently, \( \mu_x^{Wx} \) is atomic for \( \mu \)-a.e. \( x \).
We let \( \{ \mu^x_{\mathcal{W}} \} \) be defined in subsection 2.2 where \( \mu^x_{\mathcal{W}} \) is a Radon measure on \( \mathcal{W}(x) \) determined up to a scalar. For \( \mu \)-a.e. \( x \), we define

\[
\Psi(x) = [(H_x)_*(\mu^x_{\mathcal{W}})] \in \mathcal{P}\mathcal{M}(\mathbb{R}^3).
\]

We have the following.

**Lemma 5.8.** There exists a unique \( r \in \mathcal{H} \) such that the following is true. Fix an arbitrary constant \( u > 0 \) and an arbitrary Radon measure \( \omega \in \Psi(x) \). For every \( c > 0 \), we define

\[
\omega_c := \omega\big|_{B_{\mathbb{R}^2}(0,u) \times (-c,c)} \in \mathcal{M}(\mathbb{R}^3).
\]

Let \( \pi_{1,2} : \mathbb{R}^3 \to \mathbb{R}^2 \) denote the projection onto the first two coordinates of \( \mathbb{R}^3 \). Then the measure

\[
\tilde{\omega}_c := (\pi_{1,2})_* \omega_c \in \mathcal{M}(\mathbb{R}^2)
\]

is absolutely continuous with respect to the Lebesgue measure on \( \mathbb{R}^2 \); and we have

\[
\omega_c = \int_{r^{-1}(-c,c)} \delta_{(v,r(v))} d\tilde{\omega}_c(v).
\]

**Proof.** We assume for simplicity that \( u = 1 \), and we will define \( r \) over \( B_{\mathbb{R}^2}(0,1) \). The general case is similar.

Given \( d > 0 \). We deduce that \( \tilde{\omega}_d \) is absolutely continuous with respect to the Lebesgue measure by the fact that \( \pi_* \mu \) is the Haar measure on \( G/\Gamma \). To simply notation, we set

\[
R_d = \{ v \mid \frac{d\tilde{\omega}_d}{d\text{Leb}}(v) > 0 \}.
\]

We have \( R_d \subset R_{d'} \) for any \( d < d' \), and \( \cup_{d>0} R_d \) coincides with \( B_{\mathbb{R}^2}(0,1) \) up to a Lebesgue null set.

By Rokhlin’s disintegration theorem, we obtain

\[
\omega_d = \int_{\mathbb{R}^2} \omega^{\{v\} \times \mathbb{R}}_d d\tilde{\omega}_d(v),
\]

where \( \omega^{\{v\} \times \mathbb{R}}_d \) denotes the conditional measure of \( \omega_d \) on \( \{ v \} \times \mathbb{R} \).

As we know that \( \mu^x_{\mathcal{W}} \) is the Dirac measure at \( y \) for \( \mu \)-a.e. \( y \); and that for \( \omega \)-a.e. \( (v,s) \in \mathbb{R}^3 \),

\[
\omega^{\{v\} \times \mathbb{R}}_d \leq (H_x)_*(\mu^x_{\mathcal{W}}(v,s)).
\]
we can conclude that $\omega_d^\{v\} \times \mathbb{R}$ is a Dirac measure on $\{v\} \times \mathbb{R}$ for $\tilde{\omega}_d$-a.e. $v \in R_d$. Thus there exists an essentially unique $\tilde{\omega}_d$-a.e. defined measurable function $r_d : R_d \to (-d, d)$ such that

$$\omega_d = \int_{R_d} \delta_{(v, r_d(v))} d\tilde{\omega}_d(v).$$

We extend $r_d$ to a measurable function from $B_{\mathbb{R}^2}(0, 1)$ to $(-d, d)$ by setting

$$r_d|_{B_{\mathbb{R}^2}(0, 1) \setminus R_d} \equiv 0.$$ 

By definition, for every $c \in (0, d)$, we have

$$\omega_c = \int_{r_d^{-1}(-c, c)} \delta_{(v, r_d(v))} d\tilde{\omega}_d(v).$$

Consequently, we have

$$\tilde{\omega}_c = \tilde{\omega}_d|_{r_d^{-1}(-c, c)}$$

and

$$r_c = r_d|_{r_d^{-1}(-c, c)}.$$ 

We let $r : B_{\mathbb{R}^2}(0, 1) \to \mathbb{R}$ be the pointwise limit of $r_d$ as $d$ tends to infinity. It is straightforward to verify that $r$ satisfies the requirement of the lemma.

For a $\mu$-typical $x$, we let $r$ be given by Lemma 5.8, and set $S(x) = r$.

**Corollary C.** For $\mu$-a.e. $x$, for $\mu^W_x$-a.e. $y$, we have

$$\text{Graph}(S(y)) = (H_y H_x^{-1}) \text{Graph}(S(x)).$$

**Proof.** For $x, y$ in the corollary, there exists a constant $c > 0$ such that

$$\mu^W_x = c \mu^W_y.$$ 

Then by (5.6), we have

$$\Psi(y) = (H_y H_x^{-1}) \ast \Psi(x).$$ 

By Lemma 5.8, we see that $\Psi(x)$, resp. $\Psi(y)$, is supported on the graph of $S(x)$, resp. $S(y)$. The corollary then follows suit.

We define for every $c > 0$ that

$$(5.7) \quad P_c(x) := S(x)^{-1}(-c, c) \cap B_{\mathbb{R}^2}(0, 1).$$
REMARK 3. By definition, it is clear that
\begin{equation}
\lim_{c \to +\infty} \text{Leb}(P_c(x)) = \text{Leb}(B_{\mathbb{R}^2}(0,1)) = \pi.
\end{equation}

**Lemma 5.9.** We have
\[ \lim_{c \to 0} \text{Leb}(P_c(x)) = 0. \]

**Proof.** It is clear that we have
\[ \lim_{c \to 0} \text{Leb}(P_c(x)) = \text{Leb}(S(x)^{-1}(0) \cap B_{\mathbb{R}^2}(0,1)). \]
If \( \text{Leb}(S(x)^{-1}(0) \cap B_{\mathbb{R}^2}(0,1)) > 0 \), then we see that for any \( d > 0 \) and any \( \omega \in \Psi(x) \), the conditional measure of \( \omega_d \) on \( \mathbb{R}^2 \times \{0\} \) is not atomic. On the other hand, by definition, we see that for a \( \mu \)-typical \( x \), we have
\[ \omega_d^{\mathbb{R}^2 \times \{0\}} \leq [(H_x)_*(\mu_x^{\chi(x)})]^{\mathbb{R}^2 \times \{0\}} = (H_x)_*\mu_x^{\chi(x)}. \]
By our hypothesis, \( \mu_x^{\chi(x)} \) is atomic. This is a contradiction. \( \square \)

We define \( \lambda : M^a \to \mathbb{R} \) as follows,
\[ \lambda(x) = \inf\{c > 0 \mid \text{Leb}(P_c(x)) \geq \frac{1}{2}\}. \]
By Lemma 5.9, we see that \( \lambda(x) \in (0, \infty) \).
For any real constant \( c \neq 0 \), we define map \( D_c : \mathbb{R}^3 \to \mathbb{R}^3 \) as
\[ D_c(a,b) = (a, c^{-1}b), \quad \forall a \in \mathbb{R}^2, \forall b \in \mathbb{R}. \]
We define
\[ \Phi(x) = (D_{\lambda(x)})_*\Psi(x), \]
\[ \hat{S}(x) = \lambda(x)^{-1}S(x). \]
By definition, for any \( t \in L_\chi \), we have \( \chi_G(t) = 0 \). By (2.1), (5.6), (5.3), for any \( t \in L_\chi \), we have
\begin{equation}
\Psi(\hat{a}(t)x) = (D_d)_*\Psi(x)
\end{equation}
for certain constant \( d \neq 0 \). Then by definition, we have
\[ S(\hat{a}(t)x) = d^{-1}S(x). \]
Take an arbitrary \( \lambda' > \lambda(x) \). Notice that by the definition of \( P_c \) and (5.9), we have
\[ P_c(\hat{a}(t)x) = P_{cd}(x), \quad \forall c > 0. \]
Then by \((5.9)\) we have
\[
P_{d^{-1} \lambda'}(\tilde{\alpha}(t)x) = P_{\lambda'}(x) \geq \frac{1}{2}.
\]
Consequently, we have
\[
d^{-1} \lambda(x) \geq \lambda(\tilde{\alpha}(t)x).
\]
By symmetry, we can also show that
\[
d^{-1} \lambda(x) \leq \lambda(\tilde{\alpha}(t)x).
\]
Thus
\[
\lambda(\tilde{\alpha}(t)x) = d^{-1} \lambda(x).
\]
By definition, we see that
\[
\Phi(x) = \Phi(\tilde{\alpha}(t)x) \text{ and } \hat{S}(x) = \hat{S}(\tilde{\alpha}(t)x).
\]
As \(t\) is an arbitrary element of \(L_\chi\), we see that \(\hat{S}\) is an \(H_\chi\)-invariant function (modulo \(\mu\)). We set
\[
A = \hat{S}^{-1} B_H.
\]
For any closed subgroup \(H \subset G\), we denote by \(\mathcal{E}_H\) the \(\sigma\)-algebra generated by \(H\)-invariant sets modulo \(\mu\). More precisely, we define
\[
\mathcal{E}_H := \{ B \in \mathcal{B}_{M^A} \mid g^{-1} B = B \mod \mu, \quad \forall g \in H \}.
\]
It is well-known that (for example, see [11, Theorem 6.1]), if \(\mu\) is \(H\)-invariant, then for \(\mu\)-a.e. \(x\), the atom \(\mathcal{E}_H(x)\) is the \(H\)-ergodic component of \(\mu\) at \(x\). By \((5.10)\), we have that
\[
A \subset \mathcal{E}_{H_\chi}.
\]
By \(\chi \in \Sigma_3^{\text{out}} \cap (\Sigma_{\text{non}}^\Lambda)\), we see that
\[
[W^{-\chi}] = [W_{H_\chi}^{-\chi}] \supset \mathcal{E}_{H_\chi}.
\]
By the similar argument as in Section 4, we deduce that
\[
[W^\chi] \supset \mathcal{E}_{H_\chi}.
\]
Consequently, for \(\mu\)-a.e. \(x\), for \(\mu^\chi\)-a.e. \(y\), we have \(y \in A(x)\), or in another words,
\[
\hat{S}(x) = \hat{S}(y).
\]
The consideration of \(\hat{S}\) is related to the method presented in [10].

The above discussion shows that for a \(\mu\)-typical point \(x\), the set
\[
U := \{ y \in W^\chi(x) \mid \hat{S}(x) = \hat{S}(y) \}
\]
satisfies that \(\pi_{1,2}U\) is non-discrete. By Corollary [C], for any \(y \in U\) we have
\[
(D_{\lambda(x)^{-1} \lambda(y)} H_y H_x^{-1}) \text{Graph}(S(x)) = \text{Graph}(S(x)).
\]
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We set
\[ A_x = \{ T \in A \mid T \text{Graph}(S(x)) = \text{Graph}(S(x)) \}. \]
We notice that \( A_x \) has a natural factor, denoted by \( p : A_x \to \overline{A}_x \), where
\[ \overline{A}_x = \{ \tilde{T} : \mathbb{R}^2 \to \mathbb{R}^2 \mid \exists T \in A_x \text{ such that } \tilde{T} \pi_{1,2} = \pi_{1,2} T \} \]
and as before \( \pi_{1,2} \) denotes the projection from \( \mathbb{R}^3 \) onto its first two coordinates. We can naturally identify \( \overline{A}_x \) with a subset of \( \mathbb{R}^2 \) by taking the translation vector.

We set \( A_x^0 = A_x \cap A_x^0 \). By definition, \( A_x, A_x^0, A_x^0 \) are closed subgroups of \( A \), and there is an exact sequence
\[ 0 \to A_x^0 \to A_x \to \overline{A}_x \to 0. \]
We notice the following.

**Lemma 5.10.** We have

1. \( A_x^0 = \{ \text{Id} \} \);
2. the map \( A_x \to \overline{A}_x \) is proper. Consequently, \( \overline{A}_x \) is closed.

**Proof.** We denote \( r = S(x) \). Take an arbitrary \( T \in A_x \). By the uniqueness of \( r \) in Lemma 5.8, we can see that
\[ b(T) r(v - a(T)) + c(T) = r(v) \]
for Lebesgue almost every \( v \in \mathbb{R}^2 \).

If \( A_x^0 \neq \{ \text{Id} \} \) and \( \text{Id} \neq T \in A_x^0 \), then \( r \) must equal to a constant Lebesgue almost everywhere. This contradicts the our hypothesis that \( \mu^G \chi \) is atomic almost everywhere. Item (1) follows suit.

As we have seen \( r \) is not almost everywhere constant, there exist disjoint intervals \( I_1, I_2 \subset \mathbb{R} \) such that \( r^{-1}(I_i) \) has positive Lebesgue measure for \( i = 1, 2 \).

Let \( \{ T_n \}_{n \geq 0} \) be a sequence in \( A_x \) such that
\[ \lim_{n \to \infty} a(T_n) = 0. \]
Then for all sufficiently large \( n \), for \( i = 1, 2 \), we may find \( v_{n,i} \in r^{-1}(I_i) \) such that \( v_{n,i} - a(T_n) \in r^{-1}(I_i) \). Thus
\[ b(T_n) (r(v_{n,1} - a(T_n)) - r(v_{n,2} - a(T_n))) = r(v_{n,1}) - r(v_{n,2}). \]
This implies that for all sufficiently large \( n \) we have
\[ |b(T_n)| \leq \text{dist}(I_1, I_2)^{-1} \text{diam}(I_1 \cup I_2). \]
In a similar way, we may bound \( c(T_n) \) for all sufficiently large \( n \). This implies the properness of the map from \( A_x \) to \( \overline{A}_x \).
By Lemma \ref{lem:5.10}(1), we may define \( b(z) := b(T) \) and \( c(z) := c(T) \) for every \( z \in \overline{A}_x \) where \( T \) is the unique element of \( A_x \) with \( a(T) = z \).

By Lemma \ref{lem:5.10}(2), we conclude that \( \overline{A}_x \) is a closed, non-discrete subgroup of the translations on \( \mathbb{R}^2 \). Thus \( \overline{A}_x \) is a linear subspace of \( \mathbb{R}^2 \) of positive dimension.

It is direct to verify that \( b(z_1 + z_2) = b(z_2)b(z_1) \) for any \( z_1, z_2 \in \overline{A}_x \). Then there exists a linear functional \( \ell^x : \overline{A}_x \to \mathbb{R} \) such that \( b(z) = e^{\ell^x(z)} \) for any \( z \in \overline{A}_x \).

Assume that for \( \mu \)-a.e. \( x \), we have \( \ell^x \neq 0 \). We take a \( \mu \)-typical \( x \), and abbreviate \( \ell^x \) as \( \ell \). Take two arbitrary elements \( T_1, T_2 \in A_x \), and some \( v \in \overline{A}_x, u \in \mathbb{R} \). To simply notation, we set \( z_i = a(T_i) \) for \( i = 1, 2 \). Then we have

\[
T_2T_1(v, u) = T_2(v + z_1, e^{\ell(z_1)}u + c(z_1)) \\
= (v + z_1 + z_2, e^{\ell(z_2)}(e^{\ell(z_1)}u + c(z_1)) + c(z_2)) \\
= (v + z_1 + z_2, e^{\ell(z_2+z_1)}u + (e^{\ell(z_2)}c(z_1) + c(z_2))).
\]

We can see that for any \( z_1, z_2 \in \overline{A}_x \),

\[
(5.11) \quad c(z_1 + z_2) = e^{\ell(z_2)}c(z_1) + c(z_2).
\]

Then by (5.11), we obtain

\[
(5.12) \quad c(z) = c_0(e^{\ell(z)} - 1), \quad \forall z \in \overline{A}_x
\]

for some constant \( c_0 \in \mathbb{R} \).

By (5.12), we see that for \( \mu \)-a.e. \( x \), there exists a linear functional \( \ell^x : \overline{A}_x \to \mathbb{R} \), and a constant \( c_0^x \in \mathbb{R} \) such that

\[
c^x(z) = c_0^x(e^{\ell^x(z)} - 1), \quad \forall z \in \overline{A}_x.
\]

For a \( \mu \)-typical \( x \), for any \( k \in a \), and for any \( z \in \overline{A}_x \), we set

\[
C^x_{k, \pm}(v, u) = (e^{\chi_G(k)}v, \pm\|D\tilde{a}(k)|_{E(x)}\|u)
\]

and

\[
T^x_z(v, u) = (v + z, e^{\ell^x(z)}u + c_0^x(e^{\ell^x(z)} - 1)).
\]

By (5.3) and straightforward computations, we deduce that for any \( \sigma \in \{-, +\} \),

\[
C^x_{k, \sigma}T^x_z(C^x_{k, \sigma})^{-1}(v, u) = C^x_{k, \sigma}T^x_z(e^{-\chi_G(k)}v, \sigma\|D\tilde{a}(k)|_{E(x)}\|^{-1}u) \\
= C^x_{k, \sigma}(e^{-\chi_G(k)}v + z, e^{\ell^x(z)}\sigma\|D\tilde{a}(k)|_{E(x)}\|^{-1}u + c_0^x(e^{\ell^x(z)} - 1)) \\
= (v + e^{\chi_G(k)}z, e^{\ell^x(z)}u + \sigma\|D\tilde{a}(k)|_{E(x)}\|c_0^x(e^{\ell^x(z)} - 1)).
\]
It is direct to see that
\[ C_{k,\sigma}^x T_z^x \left( C_{k,\sigma}^x \right)^{-1} \in A^{\tilde{\alpha}(k)x}. \]

Then for some \( \sigma \in \{-, +\} \) we have for any \( z \in \mathbb{A}_x \) that
\[ C_{k,\sigma}^x T_z^x C_{k,\sigma}^x = T_{e^{\chi_G(k)}z}^{\tilde{\alpha}(k)x}. \]

(5.13)

Then we have
\[ \ell^x(z) = \ell_{\tilde{\alpha}(k)x}(e^{\chi_G(k)}z). \]

By this is impossible by Poincaré’s recurrence lemma and our hypothesis that \( \ell^x \neq 0 \) for \( \mu \)-a.e. \( x \). Consequently, for \( \mu \)-a.e. \( x \), we have \( \ell^x \equiv 0 \). Then it is easy to see that \( c \) is a linear functional on \( \mathbb{A}_x \), which we denote by \( c_x \). Again by (5.13), we deduce that for some \( \sigma \in \{\pm 1\} \),
\[ c_{\tilde{\alpha}(k)x} = \sigma \| D_{\tilde{\alpha}(k)}|_{E(x)} \| e^{-\chi_G(k)}c_x. \]

Consequently, we have
\[ \| c_{\tilde{\alpha}(k)x} \| = \| D_{\tilde{\alpha}(k)}|_{E(x)} \| e^{-\chi_G(k)} \| c_x \|. \]

(5.14)

Assume that \( c_x \neq 0 \) for \( \mu \)-a.e. \( x \). Notice that
\[ \lim_{n \to \infty} n^{-1} \log \| D_{\tilde{\alpha}(nk)}|_{E(x)} \| = \chi_F(k), \quad \forall k \in \mathbb{a}. \]

By Corollary [B] we have \( \chi_F = \lambda \chi_G \) for some \( \lambda > 1 \). We get a contradiction by (5.14) and Poincaré’s recurrence theorem.

Thus we have proved that \( c_x \equiv 0 \) for \( \mu \)-a.e. \( x \), and as a result,
\[ \mathbb{A}_x = \{((v, u) \mapsto (v + z, u)) \mid z \in \mathbb{A}_x\}. \]

However, for any Radon measure \( \omega \) on \( \mathbb{R}^3 \) satisfying that
\[ T_{v, u} \omega = \omega, \forall T \in \mathbb{A}_x, \]
we know that for \( \omega \)-a.e. \( (v, u) \in \mathbb{R}^3 \) where \( v \in \mathbb{R}^2 \) and \( u \in \mathbb{R} \), the conditional measure of \( \omega \) on \( \mathbb{R}^2 \times \{u\} \) is nonatomic. While this contradicts our hypothesis that \( \mu^{\chi_G} \) is atomic.

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