CONDITIONAL CARLESON MEASURES AND RELATED OPERATORS ON BERGMAN SPACES

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Abstract. In this paper first we define generalized Carleson measure. Then we consider a special case of it, named conditional Carleson measure on the Bergman spaces. After that we give a characterization of conditional Carleson measures on Bergman spaces. Moreover, by using this characterization we find an equivalent condition to boundedness of weighted conditional expectation operators on Bergman spaces.

1. Introduction and Preliminaries

Let \( C \) be the complex plane, \( \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \} \) and \( T = \{ z \in \mathbb{C} : |z| = 1 \} \). Likewise, we write \( \mathbb{R} \) for the real line. The normalized area measure on \( \mathbb{D} \) will be denoted by \( dA \). As is known \( dA(z) = \frac{1}{\pi} \frac{1}{r} dr \, d\theta \), in which \( z = x + iy = re^{i\theta} \). For \( 0 < p < +\infty \) and \( -1 < \alpha < +\infty \), the (weighted) Bergman space \( L_{p,\alpha} = L_{p,\alpha}(\mathbb{D}) \) of \( \mathbb{D} \), is the space of all analytic functions in \( L^p(\mathbb{D}, dA_{\alpha}) \), where

\[
dA_{\alpha}(z) = (\alpha + 1)(1 - |z|^2)^\alpha dA(z).
\]

Let \( f \in L^p(\mathbb{D}, dA_\alpha) \) and

\[
\|f\|_{p,\alpha} = \left[ \int_{\mathbb{D}} |f(z)|^p dA_{\alpha}(z) \right]^{1/p}.
\]

Then for \( 1 \leq p < \infty \), the space \( L^p(\mathbb{D}, dA_\alpha) \) is a Banach space with respect to the norm \( \|\cdot\|_{p,\alpha} \). In addition, for \( 0 < p < 1 \), the space \( L^p(\mathbb{D}, dA_\alpha) \) is a complete metric space with respect to the metric defined by \( d(f, g) = \|f - g\|_{p,\alpha}^p \).

Recall that for \( 1 < p < \infty \) and \( -1 < \alpha \), the projection \( P_\alpha \) from \( L^{\alpha,p} \) into \( L^{\alpha,p}_a \) is given by

\[
P_\alpha(f)(w) = \int_{\mathbb{D}} \frac{f(z)}{(1 - w \bar{z})^{2+\alpha}} dA_{\alpha}(z), \quad w \in \mathbb{D}.
\]

The projection \( P_\alpha \) is called the (weighted) Bergman projection on \( \mathbb{D} \). Let \( \mu \) be a finite positive Borel measure on \( \mathbb{D} \) and \( 0 < p < \infty \). We say
that $\mu$ is a Carleson measure on the Bergman space $L^p_a$, if there exists a constant $C > 0$ such that
\[
\int_\mathbb{D} |f(z)|^p d\mu(z) \leq C \int_\mathbb{D} |f(z)|^p dA(z),
\]
for all $f \in L^p_a$. In addition, the functions
\[
K_\alpha(\omega, z) = \frac{1}{(1 - w \overline{z})^{2+\alpha}} \quad z, \omega \in \mathbb{D}
\]
are called the (weighted) Bergman kernels of $\mathbb{D}$.

Let $(\mathcal{X}, \mathcal{M}, \mu)$ be a complete sigma-finite measure space and let $\mathcal{A}$ be a sigma-subalgebra of $\mathcal{M}$ such that $(\mathcal{X}, \mathcal{A}, \mu |_{\mathcal{A}})$ is also sigma-finite. The collection of (equivalence classes modulo sets of zero measure) $\mathcal{M}$-measurable complex-valued functions on $\mathcal{X}$ will be denoted $L^p(\mathcal{M})$. Moreover, we let $L^p(\mathcal{M}) = L^p(\mathcal{X}, \mathcal{M}, \mu)$ and $L^p(\mathcal{A}) = L^p(\mathcal{X}, \mathcal{A}, \mu |_{\mathcal{A}})$, for $1 \leq p < \infty$. As a consequence of the Radon-Nikodym theorem we have that for each non-negative function $f \in L^0(\mathcal{M})$ there exists a unique non-negative $\mathcal{A}$-measurable function $E_A(f) \in L^0(\mathcal{A})$ such that $\int_{\Delta} E(f) d\mu = \int_{\Delta} f d\mu$ for all $\Delta \in \mathcal{A}$. The function $E_A(f)$ is called the conditional expectation of $f$ with respect to $\mathcal{A}$.

Define $E_A P_\alpha = P_\alpha E_A$, then $L^p(\mathcal{D}, \mathcal{M}, \mathcal{A}) \subseteq L^p(\mathcal{D})$. Suppose that $\mathcal{M}$ is the sigma-algebra of the Lebesgue measurable sets in $\mathbb{D}$, $\mathcal{A}$ is a subalgebra of $\mathcal{M}$ and $E = E_A$ is the related conditional expectation operator. For a non-constant analytic self-map $\varphi$ on $\mathbb{D}$ and $z \in \mathbb{D}$, put $c_z = \{\zeta \in \mathbb{D}_0 : \varphi(\zeta) = \varphi(z)\}$, where $\mathbb{D}_0 = \{\zeta \in \mathbb{D} : \varphi'(z)(\zeta) \neq 0\}$. The map $\varphi$ has finite multiplicity if we can find some $n \in \mathbb{N}$ such that for each $z \in \mathbb{D}$, the level set $c_z$ consists of at most $n$ points. By $\mathcal{A} = \mathcal{A}(\varphi)$ we denote the sigma-algebra generated by $\{\varphi^{-1}(U) : U \subset \mathbb{C} \text{ is open}\}$. For the Lebesgue measure $m$ on $\mathbb{C}$ we get that $h = \frac{dm \circ \varphi^{-1}}{dm}$ is almost everywhere finite valued, because the finite measure $\mathcal{A} \circ \varphi^{-1}$ is absolutely continuous with respect to $m$. Now we define the weighted conditional expectation operator on $L^p_a$ as follow:

**Definition 1.1.** Let $u$ be an analytic function on $\mathbb{D}$, $0 < p < \infty$ and $\alpha, \beta > -1$. If $\mathcal{A}$ is a sigma-subalgebra of the sigma-algebra of the Lebesgue measurable sets, then the weighted conditional expectation operator $T = M_u E_A$ on $L^p_a$ is defined as $T(f)(z) = u(z) E_A(f)(z)$, for each $f \in L^p_a$ and $z \in \mathbb{D}$ such that $E_A(f)$ is an analytic function.

Joseph Ball in [2] studied conditional expectation on $H^p$ and he find that $H^p$ is invariant under $E_{A(f)}$ for an inner function $f$. After that Aleksandrov in [1] proved that for a complete sigma-subalgebra $\mathcal{A}$ of $\mathcal{M}$, the sigma-algebra of Lebesgue measurable subsets of $\mathbb{T}$. Then
$A = A(f)$ for some inner function $f$ if and only if $PE_A = E_AP$, in which $P$ is the Riesz projection, i.e. the orthogonal projection from $L^2$ onto the Hardy space $H^2$. If we compose the discussions presented in [8], [9] and [13], then we will get a formula for conditional expectation operator corresponding to $A(\varphi)$, in which $\varphi$ is an arbitrary non-constant analytic self-map on $D$.

In [5], the authors investigated some basic properties of some conditional expectation-type Toeplitz operators on Bergman spaces. In this paper we consider the weighted conditional expectation operators on the Bergman spaces and we will give some results based on Carleson measures. Here we recall some basic Lemmas that we need them in the sequel.

**Lemma 1.2.** [6] If $0 < p < \infty$ and $\alpha > -1$. Then

$$|f(z)| \leq \frac{\|f\|_{p,\alpha}}{(1 - |z|^2)^{(2+\alpha)/p}}$$

for all $f \in L_{a,\alpha}^p$ and $z \in \mathbb{D}$.

**Lemma 1.3.** [14] Let $0 < p < \infty$ and $\alpha > -1$. Then for each $z \in \mathbb{D}$ we have

$$\sup\{|f(z)| : f \in L_{a,\alpha}^p, \|f\|_{L_{a,\alpha}^p} \leq 1\} = \frac{1}{(1 - |z|^2)^{(2+\alpha)/p}}.$$ 

Let $a \in \mathbb{D}$ and

$$\varphi_a(z) = \frac{a - z}{1 - \overline{a}z}, \quad z \in \mathbb{D}.$$ 

The analytic transformation $\varphi_a$ is called the involutive Mobius transformation of $\mathbb{D}$, which interchanges the origin and $z$. It is clear for all $z \in \mathbb{D}, \varphi_a(\varphi_a(z)) = z$ or $\varphi_a^{-1} = \varphi_a$ and

$$\varphi_a'(z) = -\frac{1 - |a|^2}{(1 - \overline{a}z)^2}.$$ 

The metric defined on $\mathbb{D}$ by

$$\rho(a, z) = |\varphi_a(z)| = \left|\frac{a - z}{1 - \overline{a}z}\right|, \quad a, z \in \mathbb{D},$$

is called pseudo-hyperbolic distance. Maybe one of important properties of the pseudo-hyperbolic distance is that it is Mobius invariant, that is

$$\rho(w, z) = \rho(\varphi_a(w), \varphi_a(z)).$$

In addition the metric given by

$$\beta(a, z) = \frac{1}{2} \log \frac{1 + \rho(a, z)}{1 - \rho(a, z)} \quad a, z \in \mathbb{D},$$
is called Bergman metric on \( \mathbb{D} \) (or somewhere is called the hyperbolic metric or the Poincaré metric on \( \mathbb{D} \)). Hence we have

\[
\beta(0, z) = \frac{1}{2} \log \frac{1 + |z|}{1 - |z|}, \quad z \in \mathbb{D}.
\]

The Bergman metric is also Mobius invariant. This means that

\[
\beta(w, z) = \beta(\varphi_a(w), \varphi_a(z)).
\]

for all \( \varphi \in \text{Aut} (\mathbb{D}) \) and all \( z, w \in \mathbb{D} \).

For any \( a \in \mathbb{D} \) and \( r > 0 \), let

\[
D(a, r) = \{ z \in \mathbb{D} : \beta(a, z) < r \}
\]

be the Bergman metric disk with "center" \( a \) and "radius" \( r \). It follows from the expression for \( \beta(a, z) \) that \( D(a, r) \) is a Euclidean disk with Euclidean center and radius

\[
C = \frac{1 - s^2}{1 - s^2|a|^2} a, \quad R = \frac{1 - |a|^2}{1 - s^2|a|^2} s,
\]

where \( s = \tanh r \). We have \( D(a, r) = \varphi_a(D(0, r)) \). As is known for each \( z \in \mathbb{D} \), the Riesz representation theorem implies that there exists a unique function \( K_z \) in \( L^2_a(\mathbb{D}) \) such that for all \( f \in L^2_a(\mathbb{D}) \)

\[
f(z) = \int_{\mathbb{D}} f(w)K_z(w)dA(w).
\]

Let \( K(z, a) \) be the function on \( \mathbb{D} \times \mathbb{D} \) defined by \( K(z, a) = K_z(a) \). \( K(z, a) \) is called the Bergman kernel of \( \mathbb{D} \) or the reproducing kernel of \( L^2_a(\mathbb{D}) \) because of the formula

\[
f(z) = \int_{\mathbb{D}} f(w)K(z, w)dA(w)
\]

reproduces each \( f \in L^2_a(\mathbb{D}) \). And it is known that \( K(z, w) = \frac{1}{(1 - zw)^2} \).

Moreover the functions

\[
K_\alpha(\omega, z) = \frac{1}{(1 - wz)^{2+\alpha}}, \quad z, \omega \in \mathbb{D}
\]

are called (weighted) Bergman kernels of \( \mathbb{D} \). If we set

\[
k_\alpha(z) = \frac{K(z, a)}{\sqrt{K(a, a)}} = \frac{1 - |a|^2}{(1 - az)^2},
\]

then \( k_\alpha(z) \in L^2_a(\mathbb{D}) \) and they are called normalized reproducing kernels of \( L^2(\mathbb{D}) \). Easily we get that the derivative of \( \varphi_a \) at \( z \) is equal to \( k_\alpha(z) \). This implies that

\[
|\varphi'_a(z)| = |k_\alpha(z)|
\]

for all \( a, z \in \mathbb{D} \).
Remark 1.4. Let $0 < p < \infty$, $\alpha > -1$, and $a \in \mathbb{D}$. The normalized Bergman kernel function for $L_{p,\alpha}^a$ is defined by

$$(k_a(z))^{\frac{2+\alpha}{p}} = f_{p,\alpha}^a(z) = \frac{(1-|a|^2)^{(2+\alpha)/p}}{(1-\overline{a}z)^{(2+\alpha)/p}}.$$ 

Direct computations shows that

$$\|f_{p,\alpha}^a\|_{L_{p,\alpha}^a} = \left(1 - |a|^2\right)^{2+\alpha} \|1 - \overline{a}z\|_{L_{\alpha}^{2}}^{2-\alpha}.$$ 

Therefore for all $\alpha > 1$ and $p > 0$ we have $\|f_{p,\alpha}^a\|_{L_{p,\alpha}^a} = 1$. Let $f \in L_{p,\alpha}^a$ and $a \in \mathbb{D}$. If we set

$$F(z) = f(\varphi_a(z))f_{p,\alpha}^a(z) \quad z \in \mathbb{D},$$

then $\|F\|_{L_{p,\alpha}^a} = \|f\|_{L_{p,\alpha}^a}$.

Here we recall some results of [10] that are basic and useful for in the sequel.

Lemma 1.5. [10] For each $r > 0$ there exists a positive constant $C_r$ such that

$$C_r^{-1} \leq \frac{1-|a|^2}{1-|z|^2} \leq C_r$$

and

$$C_r^{-1} \leq \frac{1-|a|^2}{1-\overline{a}z} \leq C_r,$$

for all $a$ and $z$ in $\mathbb{D}$ with $\beta(a, z) < r$. Moreover, if $r$ is bounded above, then we may choose $C_r$ to be independent of $r$.

By some elementary calculations we have

$$\frac{(1-|a|^2)^2}{|1-\overline{a}z|^4} \approx \frac{1}{(1-|z|^2)^2} \approx \frac{1}{(1-|a|^2)^2} \approx \frac{1}{|D(a, r)|^2},$$

for $\beta(a, z) \leq R$ and

$$|D(z, r)|_A \approx (1-|z|^2)^2 \approx (1-|\omega|^2)^2 \approx |D(\omega, s)|_A$$

for $\beta(z, \omega) \leq R$.

Note that for all $z, w \in \mathbb{D}$, we have

$$\frac{1}{1-|w|} \leq \frac{1}{|1-z\overline{w}|} \leq \frac{1}{1-|z|} = \frac{1+|z|}{1-|z|^2} \leq \frac{2}{1-|z|^2}.$$ 

The notation $A \approx B$ means that there is a positive constant $C$ independent of $A$ and $B$ such that $C^{-1}B \leq A \leq CB$.

Now by using the results of [10] we obtain that there is a cover of disjoint balls in $A$ for $D$. 
Lemma 1.6. There is a positive integer $N$ such that for any $r \leq 1$, there exists a sequence $\{a_n\}$ in $\mathbb{D}$ such that $D(a_n, r) \in A$ and satisfying the following conditions:

1. $\bigcup_{n=1}^{\infty} D(a_n, r) = \mathbb{D}$
2. $D(a_n, \frac{1}{2}) \cap D(a_m, \frac{1}{2}) = \emptyset$ if $n \neq m$;
3. Any point in $\mathbb{D}$ belongs to at most $N$ of the sets $D(a_n, 2r)$.

Here we recall a theorem concerning of the form of the functions in the range of conditional expectation operators.

Theorem 1.7. Suppose that $A = A(\varphi)$ for some $\varphi \in A(\mathbb{D})$ with finite multiplicity. Suppose that none of the $\zeta_j(w)$ belongs to $\{z : \varphi'(z) = 0\}$ and that $w \notin f(T)$. Then for every $f$ in $L^p(\mathbb{D})$ and $\zeta$ in $\varphi^{-1}(w)$,

$$E_A(f)(\zeta) = \frac{\sum_{\zeta \in \varphi^{-1}(w)} f(\zeta)}{\sum_{\zeta \in \varphi^{-1}(w)} \varphi'(\zeta)^2}.$$ 

Also, the function $\omega$ defined as

$$\omega(\zeta) = \frac{1}{\sum_{\zeta \in \varphi^{-1}(w)} \varphi'(\zeta)^2}$$ 

is constant on each level set. In particular if $E_A P_\alpha = P_\alpha E_A$, then $\omega$ is constant on $\mathbb{D}$.

Every non-constant analytic self-map $\varphi$ on $\mathbb{D}$ has finite multiplicity, since if there is $w \in \varphi(\mathbb{D})$ such that $\varphi^{-1}(\{w\})$ is infinite, then $\varphi(z) = w$ for all $z \in \mathbb{D}$. Therefore the Theorem 1.7 holds for all non-constant analytic self-map $\varphi$ on $\mathbb{D}$. Now we recall some assertions that we will use them in the sequel.

Lemma 1.8. There is a constant $C > 0$ such that

$$|f(z)|^p \leq \frac{C}{(1 - |z|^2)^{\alpha + 2}} \int_{D(z, r)} |f(\omega)|^p dA_\alpha(\omega),$$

for all $f$ analytic, $z \in \mathbb{D}$, $p > 0$, and $r \leq 1$. We especially note that the constant $C$ above is independent of $r$ and $p$. The restriction $r < 1$ above can be replaced by $r < R$ for any positive number $R$.

Lemma 1.9. For any $r > 0$ and $a \in D$, we have the following equalities:

$$|D(a, r)| = \frac{(1 - |a|^2)^2 s^2}{(1 - |a|^2 s^2)^2},$$

$$\inf_{z \in D(a, r)} |k_a(z)|^2 = \frac{(1 - s |a|)^4}{(1 - |a|^2)^2}.$$
where \( s = \tanh r \in (0, 1/762) \) for \( r \in (0, 1) \) and \( |D(a, r)| \) is the normalized area of \( D(a, r) \).

**Lemma 1.10.** Let \( 0 < p \leq q < \infty \) and \( z \in D \). Then for each \( f \in L^p_\alpha(D) \) we have
\[
\left( \int_D |f(z)|^q d\mu(z) \right) \leq (\|f\|_{L^p_\alpha})^q (\mu(D))^{\frac{p-q}{p}}.
\]

**Proof.**
\[
\left( \int_D |f(z)|^q d\mu(z) \right) \leq \left( \int (|f(z)|^p)^{\frac{q}{p}} d\mu(z) \right)^\frac{p}{q} \left( \int d\mu(z) \right)^\frac{q}{p} = \left( \int (|f(z)|^p)^{\frac{q}{p}} d\mu(z) \right)^\frac{p}{q} (\mu(D))^{\frac{q-p}{p}} = (\|f\|_{L^p_\alpha})^q (\mu(D))^{\frac{p-q}{p}}.
\]
\[\square\]

Hence by this lemma we conclude that \( \int_D |k_a(z)|^q dA_\alpha(z) \leq \mu(D)^{\frac{p-q}{p}} \).

2. **Main Results**

In this section first we define Generalized Carleson measures.

**Definition 2.1.** Let \( \mu \) finite positive Borel measure on \( D \) and \( p > 0 \). We say that \( \mu \) is a \( (L^p_\alpha, p) \) -generalized Carleson measure on \( L^p_\alpha(D) \) if there exists a closed subspace \( M \subseteq L^p_\alpha(D) \) such that
\[
\int_D |f(z)|^p d\mu(z) \leq C \|f\|_{L^p_\alpha}^p
\]
for all \( f \in M \). Moreover, if we set \( M = L^p_\alpha(\mathcal{A}) = E_\mathcal{A}(L^p_\alpha(D)) \) for some sigma-subalgebra \( \mathcal{A} \), then we say that \( \mu \) is a \( (L^p_\alpha, p) \) -conditional Carleson measure on \( L^p_\alpha(D) \) if there exists \( C > 0 \) such that
\[
\int_D |E_\mathcal{A}(f)(z)|^p d\mu \leq C \|f\|_{L^p_\alpha}^p
\]
for all \( f \in L^p_\alpha(D) \).

Now we find an upper bound for the evaluation function \( f \to E_\mathcal{A}(f)(z) \).

**Theorem 2.2.** Let \( f \in L^p_\alpha(D) \) such that \( E_\mathcal{A}(f) \in L^p_\alpha(\mathcal{A}) \), as stated in Theorem 1.7, \( 0 < p < \infty \) and \( \alpha > -1 \). If there exists \( r > 0 \) such that \( c_r \subseteq D(z, r) \), then we have
\[
|E_\mathcal{A}(f)(z)| \leq E_\mathcal{A} \|f\| (z) \leq C_r \sup_{z \in D_0} |k_a(z)| \|f\|_{L^p_\alpha}
\]
for all \( z \in D_0 \).
Proof. It is known that for the conditional expectation $E_{\Lambda}$, we have $|E_{\Lambda}(f)| \leq E_{\Lambda}(|f|)$. Moreover, there exists some $\zeta \in c$ such that for each $\zeta_j \in c$ we have $|f(\zeta_j)| \leq |f(\zeta_n)|$. Hence we get that

$$E_{\Lambda}|f| = \left( \sum_{\zeta_j \in c} \omega(\zeta_j)(|f(\zeta_j)|) \right) \leq \left( \sum_{\zeta_j \in c} \omega(\zeta_j)(|f(\zeta_n)|) \right).$$

Also by the Lemma 1.2 we have

$$\left( \sum_{\zeta_j \in c} \omega(\zeta_j)(|f(\zeta_n)|) \right) = \left( \sum_{\zeta_j \in c} \omega(\zeta_j)(|f(\zeta_n)|) \right).$$

Moreover, by Lemmas 1.5 and Lemma 1.3 we conclude that $\|f\|_{L^{p,\alpha}_a} \leq C_r \sup_{b \in D_0} |k_b(z)| \|f\|_{L^{p,\alpha}_a}$. 

for $\beta(z, \zeta_n) \leq r$. This completes the proof. \hfill $\square$

Here we get the next corollary.

Corollary 2.3. Under the assumptions of Theorem 2.2 we have

$$|E_{\Lambda}(k_a)(z)| \leq E_{\Lambda}(|k_a(z)|) \leq C_r \sup_{a \in D_0} |k_a(z)| \quad z \in D_0$$

for all $a \in D_0$.

Let $0 < p < \infty$ and $\alpha > -1$. Then the function $\Psi_a^\alpha(\mu)(z)$ defined as

$$\Psi_a^\alpha(\mu)(z) = \int_D \left( \frac{1 - |a|^2}{1 - \overline{a}z} \right)^{(2+\alpha)} d\mu(z),$$

is well defined. In the sequel we provide some equivalence conditions to conditional- Carleson measure on the Bergman spaces.

Theorem 2.4. Let $u$ be an analytic function and $\mu$ be a finite Borel measure on $\mathbb{D}$, $0 < p \leq q < \infty$ and $\alpha, \beta > -1$. In addition, let $A = A(\varphi)$, in which $\varphi$ is a non-constant analytic self map on $D$. If

$$\sum_{k=1}^{\infty} \frac{1}{(1 - \tanh(r)|a_k|)^2} < \infty$$

for the sequence $\{a_k\}$ in the Lemma 1.6 and some $0 < r < 1$. Then the followings are equivalent:

(1) There exists $C_1 > 0$ such that for every $f \in L^{p,\alpha}_a(D)$ we have

$$\int_D |E_A(f)(z)|^p d\mu(z) \leq C_1 \|f\|_{L^{p,\alpha}_a}^p.$$
There exists $C_2$ such that
\[

\mu(D(a, r)) \leq C_2 \left( \frac{1 - |a|^2}{(1 - \tanh r |a|)^2} \right)^{(\alpha + 2)},
\]
for all $a \in D$ such that $D(a, r) \in \mathcal{A}$.

There exists $C_3$ such that
\[

\Psi_a^\alpha(\mu)(z) \leq C_3,
\]
for all $a \in D$ such that $f_{\alpha}^{p,\alpha}$ is $\mathcal{A}$-measurable.

**Proof.** (3) $\rightarrow$ (1) Let (3) holds, we have
\[
\int_{D_0} |E(f(z))|^p d\mu(z) \leq \int_{D_0} |f(z)|^p d\mu(z)
\leq \int_{D_0} \left( \frac{1}{1 - |z|^2} \right)^{(\alpha + 2)} d\mu(z) \|f\|_{L_p^\alpha}\n\leq \int_{D_0} \left( \frac{1}{|1 - \bar{a}z|^2} \right)^{(\alpha + 2)} d\mu(z) \|f\|_{L_p^\alpha}\n= \Psi_a^\alpha(\mu)(z) \|f\|_{L_p^\alpha}\n\leq C_3 \|f\|_{L_p^\alpha}.
\]

(1) $\rightarrow$ (3) Let $0 < p < \infty$ and $\alpha, \beta > -1$. Since $\|f_{\alpha}^{p,\alpha}\|_{L_p^\alpha} = 1$, then we have
\[

\Psi_a^\alpha(\mu)(z) = \int_{D} \left( \frac{1 - |a|^2}{|1 - \bar{a}z|^2} \right)^{(\alpha + 2)} d\mu(z)
= \int_{D} |f_{\alpha}^{p,\alpha}(z)|^p d\mu(z)
= \int_{D} |E(f_{\alpha}^{p,\alpha}(z))|^p d\mu(z)
\leq C_1.
\]

(2) $\rightarrow$ (3)
Let $0 < p < \infty$ and $\alpha > -1$. First we assume that $a = 0$, hence $\mu(D_0) \leq C_2$. If $|a| \leq \frac{3}{4}(a \in D_0)$, then by [7] we obtain that
\[
\int_{D_0} \left( \frac{1 - |a|^2}{|1 - \bar{a}z|^2} \right)^{(2 + \alpha)} d\mu(\omega) \leq \mu(D_0) \leq C_2.
\]
If $|a| > \frac{3}{4}(a \in D_0)$, then we define $E_n$ as
\[
E_n = \{ z \in D_0 : \frac{z - a}{|a|} < 2^n (1 - |a|) \} \quad n = 1, 2, 3, ...
\]
such that

\[
\mu(E_n) \leq (2^n(1 - |a|))^{(\alpha+2)} \\
\leq (2^n(1 - |a|^2))^{(\alpha+2)}.
\]

Moreover we have

\[
\frac{1 - |a|^2}{|1 - \bar{a}z|^2} \leq \frac{1}{1 - |a|} \leq \frac{2}{1 - |a|^2} \quad a \in E_1,
\]

\[
\frac{1 - |a|^2}{|1 - \bar{a}z|^2} \leq \frac{1}{2^{2n}(1 - |a|)} \leq \frac{2}{2^{2n}(1 - |a|^2)} \quad a \in E_n \setminus E_{n-1}.
\]

Since for each \( r > 0 \) we have \( 0 < \tanh r < 1 \), then we conclude that

\[
\int_{D_0} \left( \frac{1 - |a|^2}{|1 - \bar{a}\omega|^2} \right)^{(2+\alpha)} \, d\mu(\omega) \leq \int_{E_1} \left( \frac{1 - |a|^2}{|1 - \bar{a}\omega|^2} \right)^{(2+\alpha)} \, d\mu(\omega) \\
+ \sum_{n=2}^{\infty} \int_{E_n \setminus E_{n-1}} \left( \frac{1 - |a|^2}{|1 - \bar{a}\omega|^2} \right)^{(2+\alpha)} \, d\mu(\omega) \\
\leq \sum_{n=1}^{\infty} \frac{2}{(2^{2n}(1 - |a|^2))^{(2+\alpha)}} \mu(E_n) \\
\leq C_2 \sum_{n=1}^{\infty} \frac{1}{2^{n(\alpha+2)}} \\
\leq C_3.
\]
Suppose (3) is true then with the Lemma 1.3 and 1.9, we have

\[
\left( \frac{1 - \tanh(r)|a|^2}{1 - |a|^2} \right)^{(\alpha + 2)} \mu(D(a, r))
\]

\[
\leq \int_{D(a, r)} \left( \frac{1 - \tanh(r)|a|^2}{1 - |a|^2} \right)^{(\alpha + 2)} d\mu(z)
\]

\[
\leq \int_{D(a, r)} \left( \frac{1 + \tanh(r)|a|^2}{1 - |a|^2} \right)^{(\alpha + 2)} d\mu(z)
\]

\[
= \int_{D(a, r)} \sup_{z \in D(a, r)} |f_{a, \alpha}^p(z)|^p d\mu(z)
\]

\[
\leq \int_{D_0} \sup_{z \in D_0} |f_{a, \alpha}^p(z)|^p d\mu(z)
\]

\[
= \int_{D_0} \left( \frac{1}{1 - |z|^2} \right)^{(2+\alpha)} d\mu(z)
\]

\[
\leq \Psi_{a}^\alpha(\mu)(z)
\]

\[
\leq C_2.
\]

For all \( a \in \mathbb{D}_0. \)

(2) \( \rightarrow \) (1)

By some properties of conditional expectation, we have

\[
\int_{\mathbb{D}_0} |E(f)(z)|^p d\mu \leq \int_{\mathbb{D}_0} |f(z)|^p d\mu
\]

\[
\leq \sum_{k=1}^{\infty} \mu(D(a_k, r)) \sup_{z \in D(a_k, r)} |f(z)|^p.
\]

Here by the Lemma 1.8, [6] page 60] we have

\[
\sup\{ |f(z)|^p : z \in D(a_k, r) \} \leq \frac{C}{(1 - |a_k|^2)^{\alpha + 2}} \int_{D(a_k, r)} |f(w)|^p dA_\alpha(w).
\]
using Holder's inequality and let \( q \) be the conjugate index of \( q' \), then
\[
\sum_{k=1}^{\infty} \mu(D(a_k, r)) \sup_{z \in D(a_k, r)} |f(z)|^p \\
\leq \sum_{k=1}^{\infty} \mu(D(a_k, r)) \left( \frac{1}{(1 - |a_k|^2)^{\alpha+2}} \right)^{\frac{1}{q'}} \int_{D(a_k, r)} |f(w)|^p dA_\alpha(w) \\
\leq \left( \sum_{k=1}^{\infty} \left( \frac{1}{(1 - |a_k|^2)^{\alpha+2}} \right)^{\frac{1}{q'}} \int_{D(a_k, r)} |f(w)|^p dA_\alpha(w) \right)^{\frac{1}{q'}} \\
\leq \left( \sum_{k=1}^{\infty} \left( \frac{1}{(1 - |a_k|^2)^{\alpha+2}} \right)^{\frac{1}{q'}} \int_{D(a_k, r)} |f(w)|^p dA_\alpha(w) \right)^{\frac{1}{q'}} \\
\leq N \| f \|_{L_{p, \alpha}}^p.
\]

Then
\[
\int_{D_0} |E(f)(z)|^p d\mu \leq C_1 \| f \|_{L_{p, \alpha}}^p.
\]

**Notice** In Theorem 2.4 the notation \( A \approx B \) means that there exists a positive constant \( C \) independent of \( A \) and \( B \) such that \( C^{-1}B \leq A \leq CB \). Moreover, the best constants \( C_1, C_2 \) and \( C_3 \) are in fact comparable, i.e., there exists a positive constant \( M \) such that
\[
\frac{1}{M} C_1 \leq C_2 \leq MC_1,
\]
\[
\frac{1}{M} C_1 \leq C_3 \leq MC_1.
\]

**Theorem 2.5.** Under the assumptions of the Theorem 2.4 we obtain that the weighted Conditional expectation operator \( M_\alpha E \) is bounded from \( L_{p, \alpha}^p \) into \( L_{p, \beta}^p \) if and only if there exists some \( C \) such that
\[
\Psi_\alpha(\mu_\beta)(z) \leq C
\]
for \( a, z \in D_0 = \{ z \in D : \varphi'(z) \neq 0 \} \).

**Proof.** Suppose that \( M_\alpha E \) is bounded. So we can find \( C > 0 \) such that
\[
\| M_\alpha E(f) \|_{L_{p, \alpha}}^p \leq C \| f \|_{L_{p, \alpha}}^p,
\]
and
\[ \| M_u E(f) \|_{L_p}^p = \int_D |E(f)(z)|^p |u(z)|^p dA_\beta(z) = \int_D |E(f)(z)|^p d\mu_\beta^u(z) \leq C \| f(z) \|_{L_p}^p. \]
(2.1)

In which \( \mu_\beta^u = \int_D |u(z)|^p dA_\beta(z) \). This means that \( d\mu_\beta^u \) is an \((L_p^{\alpha})\)-expectation Carleson measure. By theorem 2.4, this is equivalent to
\[ \int_{D_0} \left( \frac{1 - |a|^2}{|1 - \bar{a}z|^2} \right)^{(2+\alpha)\beta} |u(z)|^p dA_\beta(z) \leq \Psi_\alpha^u(\mu_\beta^u)(z) \leq C. \]
This completes the proof. \( \square \)

**Example 2.6.** Let \( u \) be an analytic function on \( \mathbb{D} \), \( 0 < p \leq q < \infty \) and \( \alpha, \beta > -1 \). If the conditional expectation operator \( E \) is identity, then the Multiplication operator \( M_u \) is bounded from \( L_p^{\alpha} \) into \( L_q^{\beta} \) if and only if there exists some \( C \) such that
\[ \int_{D} \left( \frac{1 - |a|^2}{|1 - \bar{a}z|^2} \right)^{(2+\alpha)\beta} |u(z)|^q dA_\beta(z) \leq C. \]

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