The ERA of FOLE: Superstructure

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Abstract. This paper discusses the representation of ontologies in the first-order logical environment FOLE. An ontology defines the primitives with which to model the knowledge resources for a community of discourse. These primitives consist of classes, relationships and properties. An ontology uses formal axioms to constrain the interpretation of these primitives. In short, an ontology specifies a logical theory. This paper continues the discussion of the representation and interpretation of ontologies in the first-order logical environment FOLE. The formalism and semantics of (many-sorted) first-order logic can be developed in both a classification form and an interpretation form. Two papers, “The ERA of FOLE: Foundation”, defining the concept of a structure, and the current paper, defining the concept of a sound logic, represent the classification form, corresponding to ideas discussed in the “Information Flow Framework”. Two papers, “The FOLE Table”, defining the concept of a relational table, and “The FOLE Database”, defining the concept of a relational database, represent the interpretation form, expanding on material found in the paper “Database Semantics”. Although the classification form follows the entity-relationship-attribute data model of Chen, the interpretation form incorporates the relational data model of Codd. A fifth paper “FOLE Equivalence” proves that the classification form is equivalent to the interpretation form. In general, the FOLE representation uses a conceptual structures approach, that is completely compatible with the theory of institutions, formal concept analysis and information flow.

Keywords: formula, constraint, interpretation, satisfaction, consequence.
# Table of Contents

The **ERA of FOLE: Superstructure** ........................................ 1

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1 Introduction ................................................................. 3
   1.1 Philosophy. ......................................................... 3
   1.2 Knowledge Representation ......................................... 3
   1.3 First Order Logical Environment ................................. 4
   1.4 Overview .......................................................... 5

2 Logical Environment ....................................................... 6
   2.1 Formalism. ......................................................... 6
      2.1.1 Formulas. .................................................. 6
      2.1.2 Sequents. .................................................. 7
      2.1.3 Constraints ................................................ 7
   2.2 Semantics .......................................................... 10
      2.2.1 Formula Interpretation ..................................... 11
      2.2.2 Formula Structures ......................................... 13
      2.2.3 Formula Structure Morphisms ............................... 17
   2.3 Satisfaction ........................................................ 20
      2.3.1 Sequent Satisfaction ........................................ 20
      2.3.2 Constraint Satisfaction .................................... 21
      2.3.3 Institutional Aspect ........................................ 22

3 Architectural Components ................................................ 24
   3.1 Specifications ..................................................... 24
      3.1.1 Specifications ............................................... 24
      3.1.2 Entailment and Consequence ................................ 25
      3.1.3 Specification Flow ......................................... 26
      3.1.4 Legacy Notions ............................................. 29
   3.2 Logics .............................................................. 30
      3.2.1 Logics ....................................................... 30
      3.2.2 Logic Flow ................................................ 30
      3.2.3 Legacy Notions ............................................. 31
   3.3 Sound Logics ........................................................ 32
      3.3.1 Residuation ................................................ 32
      3.3.2 Sound Logic Flow .......................................... 33

4 Conclusion and Future Work ............................................... 35
1 Introduction

1.1 Philosophy.

Following the theory of general systems, an information system consists of a collection of interconnected parts called information resources and a collection of part-part relationships between pairs of information resources called constraints. Formal information systems have specifications as their information resources. Semantic information systems have logics as their information resources. A formal information system has an underlying distributed system with languages as component parts (formalism flows along language links). A semantic information system has an underlying distributed system with structures as component parts (formalism flows along structure links). Hence, semantic information systems allow information flow over a semantic multiverse.

The paper “System Consequence” gave a general and abstract solution, at the level of logical environments, to the interoperation of information systems via the channel theory of information flow. Since FOLE is a logical environment, we can apply this approach to interoperability for information systems based on first-order logic and relational databases. In this paper we show that formal FOLE systems interoperate in a general sense (since the context of FOLE languages has all sums), whereas semantic FOLE systems interoperate in a restricted sense (since the context of FOLE structures has sums over fixed universes). However, we show that distributed databases in a semantic multiverse are interoperable when each defines a portal into a common universe.

The ideas of conservative extensions and modular information systems can be formulated in terms of channels and system morphisms at the general and abstract level of logical environments. By illustrating these ideas in the FOLE logical environment, we capture the idea of modular federated databases.

1.2 Knowledge Representation

Many-sorted (multi-sorted) first-order predicate logic represents a community’s “universe of discourse” as a heterogeneous collection of objects by conceptually scaling the universe according to types. The relational model (Codd [4]) is an approach for the information management of a “community of discourse” using the semantics and formalism of (many-sorted) first-order predicate logic. The relational model was initially discussed in two papers: “A Relational Model of Data for Large Shared Data Banks” by Codd [3] and “The Entity-Relationship Model – Toward a Unified View of Data” by Chen [2]. The relational model follows many-sorted logic by representing data in terms of many-sorted relations, subsets of the Cartesian product of multiple domains. All data is represented horizontally in terms of tuples, which are grouped vertically into relations. A database organized in terms of the relational model is a called relational database. The relational model provides a method for modeling the data stored in a relational database and for defining queries upon it.

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1 Examples include: an academic discipline; a commercial enterprise; library science; the legal profession; etc.
1.3 First Order Logical Environment

**Basics.** The first-order logical environment FOLE is a category-theoretic representation for many-sorted (multi-sorted) first-order predicate logic. The relational model can naturally be represented in FOLE. The FOLE approach to logic, and hence to databases, relies upon two mathematical concepts: (1) lists and (2) classifications. Lists represent database signatures and tuples; classifications represent data-types and logical predicates. FOLE represents the header of a database table as a list of sorts, and represents the body of a database table as a set of tuples classified by the header. The notion of a list is common in category theory. The notion of a classification is described in two books: “Information Flow: The Logic of Distributed Systems” by Barwise and Seligman [1] and “Formal Concept Analysis: Mathematical Foundations” by Ganter and Wille [5].

**Architecture.** A series of papers provides a rigorous mathematical basis for FOLE by defining an architectural semantics for the relational data model, thus providing the foundation for the formalism and semantics of first-order logical/relational database systems. This architecture consists of two hierarchies of two nodes each: the classification hierarchy and the interpretation hierarchy.

- Two papers provide a precise mathematical basis for FOLE classification. The paper “The ERA of FOLE: Foundation” [12] develops the notion of a FOLE structure, following the entity-relationship model of Chen [2]. This provides a basis for the current paper “The ERA of FOLE: Superstructure” [13], which develops the notion of a FOLE sound logic.

- Two papers provide a precise mathematical basis for FOLE interpretation. Both of these papers expand on material found in the paper “Database Semantics” [10]. The paper “The FOLE Table” [14], develops the notion of a FOLE table following the relational model of Codd [4]. This provided a basis for the paper “The FOLE Database” [15], which develops the notion of a FOLE relational database.

The architecture of FOLE is pictured briefly on the right and more completely in Fig.1 of the preface of the paper [16]. This consists of two hierarchies of two nodes each. The paper “FOLE Equivalence” [16] proves that FOLE sound logics are equivalent to FOLE databases.

In the relational model there are two approaches for database management: the relational algebra, which defines an imperative language, and the relational...
calculus, which defines a declarative language. The paper “Relational Operations in FOLE” [17] represents relational algebra by expressing the relational operations of database theory in a clear and implementable representation. The relational calculus will be represented in FOLE in a future paper.

1.4 Overview

The first-order logical environment FOLE (Kent [11]) is a framework for defining the semantics and formalism of logic and databases in an integrated and coherent fashion. Institutions in general, and logical environments in particular, give equivalent heterogeneous and homogeneous representations for logical systems. FOLE is an institution, since “satisfaction is invariant under change of notation”. FOLE is a logical environment, since “satisfaction respects structure linkage”. As an institution, the architecture of FOLE consists of languages as indexing components, structures to represent semantic content, specifications to represent formal content, and logics to combine formalism with semantics. FOLE structures are interpreted as relational/logical databases.

This paper, which is concerned with the classification form of FOLE (see Fig. 2), is presented in two parts: the logical environment and the architecture. §1 is an introduction, which gives a brief discussion of the philosophy, knowledge representation, basics, and architecture of FOLE. §2 discusses the FOLE logical environment, where we define formulas, sequents, constraints; we extend interpretation and classification from entity types to formulas; we define satisfaction for sequents and constraints; and we show that FOLE is an institution and logical environment. §3 develops the FOLE architecture, where we define the architectural components of specifications and logics by developing the logical notions of entailment, consequence, residuation and soundness. §4 gives the conclusion and future work. Table 1 lists the figures and tables in this paper.

| §3.2.1 | Fig. 1: Logic Order |
| §3.3.2 | Fig. 2: FOLE Superstructure |
| §3.3.2 | FOLE Superstructure |

| §1.4 | Tbl. 1: Figures and Tables |
| §2.1.1 | Tbl. 2: Syntactic Flow |
| §2.1.3 | Tbl. 4: Axioms |
| §2.2 | Tbl. 5: Semantic Flow |
| §2.2.1 | Tbl. 6: Formula Interpretation |
| §2.2.2 | Tbl. 8: Formula Classification |
| §2.2.2 | Tbl. 8: Formula Classification |

Table 1. Figures and Tables
2 Logical Environment

2.1 Formalism

2.1.1 Formulas

Let $S = \langle R, \sigma, X \rangle$ be a fixed schema with a set of entity types $R$, a set of sorts (attribute types) $X$ and a signature function $R \rightarrow \text{List}(X)$. The set of entity types $R$ is partitioned $R = \bigcup_{(I,s) \in \text{List}(X)} R(I,s)$ into fibers, where $R(I,s) \subseteq R$ is the fiber (subset) of all entity types with signature $(I,s)$. These are called $(I,s)$-ary entity types. Here, we follow the tuple, domain, and relation calculi from database theory, using logical operations to extend the set of basic entity types $R$ to a set of defined entity types $\tilde{R}$ called formulas or queries.

Formulas, which are defined entity types corresponding to queries, are constructed by using logical connectives within a fiber and logical flow along signature morphisms between fibers (Tbl. 2). Logical connectives on formulas express intuitive notions of natural language operations on the interpretation (extent, view) of formulas. These connectives include: conjunction, disjunction, negation, implication, etc. For any signature $(I,s)$, let $\tilde{R}(I,s) \subseteq \tilde{R}$ denote the set of all formulas with this signature. There are called $(I,s)$-ary formulas. The set of $S$-formulas is partitioned as $\tilde{R} = \bigcup_{(I,s) \in \text{List}(X)} \tilde{R}(I,s)$.

fiber: Let $(I,s)$ be any signature. Any $(I,s)$-ary entity type (relation symbol) is an $(I,s)$-ary formula; that is, $R(I,s) \subseteq \tilde{R}(I,s)$. For a pair of $(I,s)$-ary formulas $\varphi$ and $\psi$, there are the following $(I,s)$-ary formulas: meet ($\varphi \land \psi$), join ($\varphi \lor \psi$), implication ($\varphi \rightarrow \psi$) and difference ($\varphi \setminus \psi$). For $(I,s)$-ary formula $\varphi$, there is an $(I,s)$-ary negation formula $\neg \varphi$. There are top/bottom $(I,s)$-ary formulas $\top$ and $\bot$. For an $(I,s)$-ary formula $\varphi$, there are $(I,s)$-ary existentially/universally quantified formulas $\sum_i(\varphi)$ and $\Pi_i(\varphi)$. For an $(I',s')$-ary formula $\varphi'$, there is an $(I,s)$-ary substitution formula $h^*(\varphi') = \varphi'(t)$.

flow: Let $(I',s') \xrightarrow{h} (I,s)$ be any signature morphism. For $(I,s)$-ary formula $\varphi$, there are $(I',s')$-ary existentially/universally quantified formulas $\sum_i(\varphi)$ and $\Pi_i(\varphi)$. For an $(I',s')$-ary formula $\varphi'$, there is an $(I,s)$-ary substitution formula $h^*(\varphi') = \varphi'(t)$.

For any index $i \in I$, quantification for the complement inclusion signature function $\langle I \setminus \{i\}, s' \rangle \xrightarrow{\text{inc}_i} (I,s)$ gives the traditional syntactic quantifiers $\forall i \varphi$, $\exists_i \varphi$.

In general, we regard formulas to be constructed entities or queries (defining views and interpretations; i.e., relations/tables), not assertions. Contrast this with the use of “asserted formulas” below. For example, in a corporation data model the conjunction $(\text{Salaried} \land \text{Married})$ is not an assertion, but a constructed entity type or query that defines the view “salaried employees that are married”. Formulas form a schema $\text{fmla}(S) = \langle \tilde{R}, \tilde{\sigma}, X \rangle$ that extends $S$ with $S$-formulas as entity types: with the inductive definitions above, the set of entity types $\tilde{R}$ is partitioned as $\tilde{R} = \bigcup_{(I,s) \in \text{List}(X)} \tilde{R}(I,s)$.

\textit{Footnotes:}

3 We use concepts and notations presented in the \texttt{FOLE} foundation paper (Kent [12]).
4 This is a slight misnomer, since the signature of $r$ is $\sigma(R) = \langle I,s \rangle$, whereas the arity of $r$ is $\alpha(R) = I$.
5 An $S$-signature morphism $(I',s') \xrightarrow{h} (I,s)$ in $\text{List}(X)$ is an arity function $I' \xrightarrow{h} I$ that preserves signature $s' = h \cdot s$.
6 The full version of \texttt{FOLE} (Kent [11]) defines syntactic flow along term vectors.
Table 2. Syntactic Flow
A constraint in the fiber are sequents are constraints. In the opposite direction, there are enfolding maps \( \phi \) that the interpretation of the \( \psi \) can be identified with the sequent \( \phi \vdash \psi \). Thus, from an entailment viewpoint we can say that “formulas in some sense, this formula/constraint approach to formalism turns the tuple calculus upside down, with atoms in the tuple calculus becoming constraints here.

Given any schema \( S \), an \( S \)-constraint \( \varphi \overset{h}{\rightarrow} \varphi \) has source formula \( \varphi' \) and target formula \( \varphi \). Constraints are closed under composition: \( \varphi'' \overset{h''}{\rightarrow} \varphi' \overset{h}{\rightarrow} \varphi = \varphi'' \overset{h' \cdot h}{\rightarrow} \varphi \). Let \( \text{Cons}(S) \) denote the mathematical context, whose set of objects are \( S \)-formulas and whose set of morphisms are \( S \)-constraints. This context is fibered over the projection passage \( \text{Cons}(S) \rightarrow \text{List}(X) : (\varphi' \overset{h}{\rightarrow} \varphi) \mapsto ((I', s') \overset{h}{\rightarrow} (I, s)) \).

Sequents are special cases of constraints: a sequent \( \varphi' \vdash \varphi \) asserts a constraint \( \varphi \downarrow \varphi' \) that uses an identity signature morphism. Since an asserted formula \( \varphi \) can be identified with the sequent \( \top \vdash \varphi \), it can also be identified with the constraint \( \varphi \downarrow \top \). Thus, from an entailment viewpoint we can say that “formulas are sequents are constraints”. In the opposite direction, there are enfolding maps

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9 In some sense, this formula/constraint approach to formalism turns the tuple calculus upside down, with atoms in the tuple calculus becoming constraints here.

10 A constraint in the fiber \( \text{Cons}_S(I, s) \) uses an identity signature morphism \( \varphi \downarrow \varphi' \), and hence is a sequent \( \varphi' \vdash \varphi \).
that map \( S \)-constraints to \( S \)-formulas: either \((\varphi' \xrightarrow{h} \varphi) \mapsto (\Sigma_h(\varphi) \rightarrow \varphi')\) with
signature \( (I', s') \), or \((\varphi' \xrightarrow{r} \varphi) \mapsto (\varphi \rightarrow h^*(\varphi'))\) with signature \( (I, s) \).

Given any schema morphism \( S_2 \xrightarrow{\text{cons}(r,f)} S_1 \), there is a constraint passage
\( \text{Cons}(S_2) \xrightarrow{\text{cons}(r,f)} \text{Cons}(S_1) \). An \( S_2 \)-formula \( \varphi_2 \in \text{fmla}(S_2) \) is mapped to the
\( S_1 \)-formula \( \hat{r}(\varphi_2) \in \text{fmla}(S_1) \). An \( S_2 \)-constraint \( \varphi'_2 \xrightarrow{h} \varphi_2 \) in \( \text{Cons}(S_2) \) with \( S_2 \)-enfolding \( \varphi_2 \rightarrow h^*(\varphi'_2) \) in \( \text{Cons}_{S_2}(I_2, s_2) \) is mapped to the \( S_1 \)-constraint \( \hat{r}(\varphi_2) \xrightarrow{h} \hat{r}(\varphi_2) \) in \( \text{Cons}(S_1) \) with \( S_1 \)-enfolding \( \hat{r}(\varphi_2) \rightarrow h^*(\hat{r}(\varphi'_2)) = \hat{r}(\varphi_2) \rightarrow h^*(\varphi'_2) \) in \( \text{Cons}_{S_1}(I_2, s_2) \) using Tbl. 3. The passage \( \text{Sch} \xrightarrow{\text{cons}} \text{Cxt} \) forms an indexed context of constraints.

### Table 4. Axioms

| Axioms | Description |
|--------|-------------|
| \( \Sigma_h \)-monotonicity | \( \varphi \vdash \psi \) implies \( \Sigma_h(\varphi) \vdash \Sigma_h(\psi) \) |
| \( h^* \)-monotonicity | \( \varphi' \vdash \psi' \) implies \( h^*(\varphi') \vdash h^*(\psi') \) |
| \( \Pi_h \)-monotonicity | \( \varphi \vdash \psi \) implies \( \Pi_h(\varphi) \vdash \Pi_h(\psi) \) |
| Adjointness | \( \Sigma_h(\varphi) \vdash \varphi' \) if \( \varphi \vdash h^*(\varphi') \) \( \varphi \vdash h^*(\Sigma_h_{\psi}(\varphi)) \) \( \Sigma_h(h^*(\varphi')) \vdash \varphi' \) |
| \( \hat{r} \)-monotonicity | \( \varphi_2 \vdash \psi_2 \) implies \( \hat{r}(\varphi_2) \vdash \hat{r}(\psi_2) \) |
2.2 Semantics.

For any structure \( \mathcal{M} = (\mathcal{E}, \langle \sigma, \tau \rangle, \mathcal{A}) \), the semantics of formulas involves both a formula interpretation function \( I_M \) defined in Tbl. 6 and a formula classification \( \mathcal{F} \) defined in Tbl. 8. Formula interpretation is independently defined, but formula classification depends upon formula interpretation.

Semantic Quantifiers. Both formula interpretation and formula classification use semantic quantifiers (and substitution) in their definitions. Here we give an intuitive expression for these. Let \( \mathcal{A} = (X, \mathcal{Y}_{\mathcal{A}}, \models_{\mathcal{A}}) \) be a type domain (attribute classification) and let \( \mathcal{S} = \langle R, \sigma, X \rangle \) be a schema with common sort set \( X \). If \( \langle I', s' \rangle \xrightarrow{h} \langle I, s \rangle \) is an \( S \)-signature morphism with the associated tuple function \( \text{tup}_A(I', s') \xleftarrow{\text{tup}_A} \text{tup}_A(I, s) \), we have the following adjoint functions.\(^\text{11} \) \(^\text{12} \) \(^\text{13} \)

| Signature Morphism | Tuple Function | Substitution | Quantification |
|--------------------|----------------|--------------|---------------|
| \( \langle I', s' \rangle \xrightarrow{h} \langle I, s \rangle \) | \( \text{tup}_A(I', s') \xleftarrow{\text{tup}_A} \text{tup}_A(I, s) \) | \( \text{Rel}_A(I', s') \xrightarrow{h^{-1}} \text{Rel}_A(I, s) \) | \( \exists h \vdash \forall h \) |

Table 5. Semantic Flow

**Intuitive explanation:** For any tuple subset \( R \in \text{Rel}_A(I, s) \), you can get two tuple subsets \( \exists h(R), \forall h(R) \in \text{Rel}_A(I', s') \) as follows. Given any possible tuple \( t' \in \text{tup}_A(I', s') \), you can ask either an existential or a universal question about it: “Does there exist a tuple \( t \in R \) with image \( t' \)?” (\( t' = \text{tup}_A(t) \)) or “Is it the case that all possible tuples \( t \in \text{tup}_A(I, s) \) with image \( t' \) are present in \( R' \)?” Clearly, the quantification/substitution operators are monotonic.

\(^\text{11} \) Recall from the original and foundation papers on FOLE ([11], [12]) the following definitions. The set of \( A \)-tuples with signature \( \langle I, s \rangle \) is the \( \text{List}(A) \)-extent \( \text{tup}_A(I, s) = \text{ext}_{\text{List}(A)}(I, s) \). The tuple function \( \text{tup}_A(I', s') \xleftarrow{\text{tup}_A} \text{tup}_A(I, s) \) maps \( \langle I, t \rangle \in \text{tup}_A(I, s) \) to \( \langle I', h \cdot t \rangle \in \text{tup}_A(I', s') \). \( \text{Rel}_A(I, s) = \wp_{\text{tup}_A}(I, s) \) is the power set of \( A \)-tuples with signature \( \langle I, s \rangle \).

\(^\text{12} \) The semantic quantifiers (and substitution) are used in the definition of the fibered context \( \text{Rel}(A) \), which is defined in the paper on the FOLE Table ([14]). An object of \( \text{Rel}(A) \), called an \( A \)-relation, is a pair \( \langle I, s, R \rangle \) consisting of an indexing \( X \)-signature \( \langle I, s \rangle \) and a subset of \( A \)-tuples \( R \in \wp_{\text{Ext}_{\text{List}(A)}}(I, s) \).

\(^\text{13} \) Since existential quantification and substitution are direct/inverse operators \( \text{Rel}_A(I', s') \xrightarrow{\exists_h = \wp_{\text{tup}_A}(h)} \text{Rel}_A(I, s) \) along the tuple function \( \text{tup}_A(I', s') \xleftarrow{\text{tup}_A} \text{tup}_A(I, s) \), they are clearly adjoint to each other \( \exists_h(R) \subseteq R' \text{ iff } R \subseteq h^{-1}(R') \).
2.2.1 Formula Interpretation.

Formula Interpretation. The formula interpretation function

\[ I_M : \hat{R} \rightarrow \text{Rel}(A), \]  

which extends the traditional interpretation function \( I_M : R \rightarrow \text{Rel}(A) \) (see the foundation paper on FOLE [12]), is defined by induction on formulas in Tbl. 6. At the base step, it defines the formula interpretation of an entity type \( r \in \hat{R} \) as the traditional interpretation of that type \( I_M(r) = \varphi(\text{ext}_E(r)) \), which is the set of descriptors for entities of that type. At the induction step, it represents the logical operations by their associated boolean operations: intersection of interpretations for conjunction, union of interpretations for disjunction, etc.; and it represents the syntactic flow operators in Tbl. 2 of §2.1 by their associated semantic flow operators in Tbl. 5.

\[
\begin{align*}
\text{fiber: signature } & (I, s) \text{ with extent (tuple set) } \text{tup}_A(I, s) = \prod_{a \in A} A_a \\
\text{operator} & \quad \text{definition} \\
\text{entity type} & \quad I_M(\varphi) \in \text{Rel}_A(I, s) = \varphi \text{tup}_A(I, s) \\
\text{meet} & \quad I_M(\varphi \land \psi) = I_M(\varphi) \cap I_M(\psi) \quad \varphi, \psi \in \hat{R}(I, s) \\
\text{join} & \quad I_M(\varphi \lor \psi) = I_M(\varphi) \cup I_M(\psi) \\
\text{top} & \quad I_M(\top_{(I, s)}) = \text{tup}_A(I, s) \\
\text{bottom} & \quad I_M(\bot_{(I, s)}) = \emptyset \\
\text{negation} & \quad I_M(\neg \varphi) = \neg I_M(\varphi) = \text{tup}_A(I, s) \setminus I_M(\varphi) \\
\text{implication} & \quad I_M(\varphi \rightarrow \psi) = I_M(\varphi) \rightarrow I_M(\psi) = (\neg I_M(\varphi)) \cup I_M(\psi) \\
\text{difference} & \quad I_M(\varphi \setminus \psi) = I_M(\varphi) \setminus I_M(\psi) = I_M(\varphi) \cap (\neg I_M(\psi)) \\
\end{align*}
\]

flow: signature morphism \( \langle I', s' \rangle \xrightarrow{h} \langle I, s \rangle \) with tuple map \( \text{tup}_A(I', s') \xleftarrow{\text{tup}_A(h)} \text{tup}_A(I, s) \)

\[
\begin{align*}
\text{operator} & \quad \text{definition} \\
\text{existential} & \quad I_M(\Sigma_A(\varphi)) = \exists_h(I_M(\varphi)) \quad \varphi \in \hat{R}(I, s) \\
\text{universal} & \quad I_M(\Pi_A(\varphi)) = \forall_h(I_M(\varphi)) \\
\text{substitution} & \quad I_M(h^*(\varphi')) = h^{-1}(I_M(\varphi')) \quad \varphi' \in \hat{R}(I', s') \\
\end{align*}
\]

Table 6. Formula Interpretation

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\(^{14}\) The function \( \hat{R} \xrightarrow{I_M} \text{Rel}_A \) is the parallel combination of its fiber functions

\[ \left\{ \hat{R}(I, s) \xrightarrow{I_M} \text{Rel}_A(I, s) \mid \langle I, s \rangle \in \text{List}(X) \right\}. \]

The definition of \( I_M \) is directly in terms of these fiber functions.
Formal/Semantics Reflection. The logical semantics of a structure \( \mathcal{M} \) resides in its core, which is defined by its formula interpretation function \( \hat{\mathcal{R}} \xrightarrow{\mathcal{L}_M} \text{Rel}_A \). To respect this, the formal flow operators \((\Sigma, \Pi, h^\ast)\) for existential/universal quantification and substitution reflect the semantic flow operator \((\exists, \forall, h^{-1})\) via interpretation (Tbl. 7). \(^{15} \) \(^{16} \)

\[
\begin{align*}
\langle \hat{\mathcal{R}}(I', s') \rangle & \xrightarrow{\Sigma h} \langle \hat{\mathcal{R}}(I, s) \rangle, \\
\langle \hat{\mathcal{R}}(I', s') \rangle & \xrightarrow{h^\ast} \langle \hat{\mathcal{R}}(I, s) \rangle, \\
\langle \hat{\mathcal{R}}(I', s') \rangle & \xrightarrow{\Pi h} \langle \hat{\mathcal{R}}(I, s) \rangle.
\end{align*}
\]

\[
\begin{align*}
\langle \hat{\mathcal{R}}(I', s') \rangle & \xrightarrow{\exists h} \langle \hat{\mathcal{R}}(I, s) \rangle, \\
\langle \hat{\mathcal{R}}(I', s') \rangle & \xrightarrow{h^{-1}} \langle \hat{\mathcal{R}}(I, s) \rangle, \\
\langle \hat{\mathcal{R}}(I', s') \rangle & \xrightarrow{\forall h} \langle \hat{\mathcal{R}}(I, s) \rangle.
\end{align*}
\]

\[\text{Tup}_A(I', s') \xleftarrow{\text{tup}(h)} \text{Tup}_A(I, s)\]

\[\text{Rel}_A(I', s') \xrightarrow{\text{Rel}_A(I, s)} \text{tup}(h)\]

**Table 7.** Formal/Semantics Reflection

\(^{15}\) This tabular/relational reflection is a special case of Prop. 5.59 in the text *Topos Theory* [8], suggesting that all of the development in this paper could be done in an arbitrary topos, or even in a more general setting.

\(^{16}\) The morphic aspect of Tbl. 7 anticipates the definition of sequent satisfaction in §2.3.1: An \( \mathcal{S} \)-structure \( \mathcal{M} \in \text{Struc}(\mathcal{S}) \) satisfies an \( \mathcal{S} \)-sequent \( \varphi \vdash \psi \) when the interpretation widening of views asserted by the sequent actually holds in \( \mathcal{M} \):

\[I_M(\varphi) \subseteq I_M(\psi).\]
2.2.2 Formula Structures. Any structure $\mathcal{M} = \langle \mathcal{E}, \{\sigma, \tau\}, \mathcal{A} \rangle$ has an associated formula structure $\text{fmla}(\mathcal{M}) = \hat{\mathcal{M}} = \langle \hat{\mathcal{E}}, \{\hat{\sigma}, \hat{\tau}\}, \mathcal{A} \rangle$ with the same universe $\mathcal{U} = \langle K, \tau, Y \rangle$ and type domain $\mathcal{A} = \langle X, Y, \models_{\mathcal{A}} \rangle$, but with the formula schema $\text{fmla}(S) = \hat{S} = \langle \hat{R}, \hat{\sigma}, X \rangle$ defined in §2.1 and the formula classification $\hat{\mathcal{E}} = \langle \hat{R}, K, \models_{\hat{\mathcal{E}}} \rangle$ defined here.

**Formula Classification.** The formula classification $\hat{\mathcal{E}} = \langle \hat{R}, K, \models_{\hat{\mathcal{E}}} \rangle$, which extends the entity classification $\mathcal{E} = \langle R, K, \models_{\mathcal{E}} \rangle$, is defined in Tbl. 8 by induction on formulas using formula interpretation.

fiber: signature \(\langle I, s \rangle\) with extent (tuple set) $\text{tup}_{\mathcal{A}}(I, s) = \prod_{i \in I} \mathcal{A}_i$k $\in K$ and $\varphi, \psi \in \hat{R}(I, s)$

| operator | definiendum | definiens |
|----------|-------------|------------|
| entity type | $k \models_{\hat{\mathcal{E}}} r$ | when $\tau(k) \in I_{\mathcal{M}}(r) \subseteq \text{tup}_{\mathcal{A}}(I, s)$ |
| meet | $k \models_{\hat{\mathcal{E}}} (\varphi \land \psi)$ | when $k \models_{\hat{\mathcal{E}}} \varphi$ and $k \models_{\hat{\mathcal{E}}} \psi$ |
| join | $k \models_{\hat{\mathcal{E}}} (\varphi \lor \psi)$ | when $k \models_{\hat{\mathcal{E}}} \varphi$ or $k \models_{\hat{\mathcal{E}}} \psi$ |
| top | $k \models_{\hat{\mathcal{E}}} \top_{(I,s)}$ | when $\tau(k) \in \text{tup}_{\mathcal{A}}(I, s)$ |
| bottom | $k \not\models_{\hat{\mathcal{E}}} \bot_{(I,s)}$ | |
| negation | $k \models_{\hat{\mathcal{E}}} \neg \varphi$ | when $\tau(k) \in \text{tup}_{\mathcal{A}}(I, s)$ and $k \not\models_{\hat{\mathcal{E}}} \varphi$ |
| implication | $k \models_{\hat{\mathcal{E}}} (\varphi \rightarrow \psi)$ | when if $k \models_{\hat{\mathcal{E}}} \varphi$ then $k \models_{\hat{\mathcal{E}}} \psi$ |
| difference | $k \models_{\hat{\mathcal{E}}} (\varphi \setminus \psi)$ | when $k \models_{\hat{\mathcal{E}}} \varphi$ but not $k \models_{\hat{\mathcal{E}}} \psi$ |

flow: signature morphism $\langle I', s' \rangle \xrightarrow{h} \langle I, s \rangle$

with tuple map $\text{tup}_{\mathcal{A}}(I', s') \xleftarrow{\text{tup}_{\mathcal{A}}(h)} \text{tup}_{\mathcal{A}}(I, s)$

| operator | definiendum | definiens |
|----------|-------------|------------|
| existential | $k \models_{\hat{\mathcal{E}}} \Sigma_k(\varphi)$ | when $\tau(k) \in \exists_h(I_{\mathcal{M}}(\varphi))$ |
| universal | $k \models_{\hat{\mathcal{E}}} \Pi_k(\varphi)$ | when $\tau(k) \in \forall_h(I_{\mathcal{M}}(\varphi))$ |
| substitution | $k \models_{\hat{\mathcal{E}}} h^*(\varphi')$ | when $\tau(k) \in h^{-1}(I_{\mathcal{M}}(\varphi'))$ |

**Table 8. Formula Classification**

**Proposition 1.** For any formula $\varphi \in \hat{R}$, $\text{ext}_{\hat{\mathcal{E}}}(\varphi) = \tau^{-1}(I_{\mathcal{M}}(\varphi))$. Hence, $k \models_{\hat{\mathcal{E}}} \varphi$ if and only if $\tau(k) \in I_{\mathcal{M}}(\varphi)$ for all $k \in K$.

**Proof.** By induction, for all $k \in K$, $\varphi, \psi \in \hat{R}(I, s)$, $\varphi' \in \hat{R}(I', s')$:

meet: $k \models_{\hat{\mathcal{E}}} (\varphi \land \psi)$ when $k \models_{\hat{\mathcal{E}}} \varphi$ and $k \models_{\hat{\mathcal{E}}} \psi$ if and only if $\tau(k) \in I_{\mathcal{M}}(\varphi)$ and $\tau(k) \in I_{\mathcal{M}}(\psi)$ if and only if $\tau(k) \in I_{\mathcal{M}}(\varphi \land \psi)$;

Simply put, an entity is in the view of a query exactly when the descriptor of that entity is in the interpretation of the query. But, there may be tuples in the interpretation of the query that are not descriptors of any entity in the view of the query.
join: $k \models_{\mathcal{E}} (\varphi \lor \psi)$ when $k \models_{\mathcal{E}} \varphi$ or $k \models_{\mathcal{E}} \psi$ iff $\tau(k) \in I_M(\varphi)$ or $\tau(k) \in I_M(\psi)$ iff $\tau(k) \in I_M(\varphi) \cup I_M(\psi) = I_M(\varphi \lor \psi)$;

top: $k \models_{\mathcal{E}} \tau_{(I,s)}$ when $\tau(k) \in \text{tup}_A(I,s) = I_M(\tau_{(I,s)})$

bottom: $k \models_{\mathcal{E}} \tau_{(I,s)}$ when $\tau(k) \in \emptyset = I_M(\bot_{(I,s)})$

negation: $k \models_{\mathcal{E}} (\neg \varphi)$ when $\tau(k) \in I_M(\varphi)$ and $k \not\models_{\mathcal{E}} \varphi$ iff $\tau(k) \in \text{tup}_A(I,s) \setminus I_M(\varphi) = \neg I_M(\varphi) = I_M(\neg \varphi)$;

implication: $k \models_{\mathcal{E}} (\varphi \rightarrow \psi)$ when $k \models_{\mathcal{E}} \varphi$ implies $k \models_{\mathcal{E}} \psi$ iff $\tau(k) \in I_M(\varphi)$ implies $\tau(k) \in I_M(\psi)$ iff $\tau(k) \in (I_M(\varphi) \rightarrow I_M(\psi)) = I_M(\varphi \rightarrow \psi)$;

difference: $k \models_{\mathcal{E}} (\varphi \setminus \psi)$ when $k \models_{\mathcal{E}} \varphi$ but not $k \models_{\mathcal{E}} \psi$ iff $\tau(k) \in I_M(\varphi)$ and $\tau(k) \not\in I_M(\psi)$ iff $\tau(k) \in I_M(\varphi) \cap (\neg I_M(\psi)) = I_M(\varphi \setminus \psi)$;

existential: $k \models_{\mathcal{E}} \exists_h(\varphi)$ when $\tau(k) \in \exists_h(I_M(\varphi)) = I_M(\exists_h(\varphi))$;

universal: $k \models_{\mathcal{E}} \forall_h(\varphi)$ when $\tau(k) \in \forall_h(I_M(\varphi)) = I_M(\forall_h(\varphi))$; and

substitution: $k \models_{\mathcal{E}} h^*((\varphi'))$ when $\tau(k) \in h^{-1}(I_M(\varphi')) = I_M(h^*(\varphi'))$.

Lemma 1. The associated formula structure $\widehat{\mathcal{M}} = (\widehat{\mathcal{E}}, (\widehat{\sigma}, \tau), A)$ is well-defined.

Proof. From Prop. 1 above, $\varphi(\text{ext}_{\mathcal{E}}(\varphi)) \subseteq I_M(\varphi) \subseteq \text{tup}_A(I,s) = \text{ext}_{\text{List}(A)}(I,s)$ for any formula $\varphi \in \widehat{\mathcal{E}}(I,s)$. Hence, the condition for the list designation $(\widehat{\sigma}, \tau) : \widehat{\mathcal{E}} \rightarrow \text{List}(A)$ holds: $k \models_{\mathcal{E}} \varphi$ implies $\tau(k) \models_{\text{List}(A)} \widehat{\sigma}(\varphi)$ for all keys $k \in K$.

For all $\varphi \in R$, we have the relationships

$$\varphi(\text{ext}_{\mathcal{E}}(\varphi)) \subseteq I_M(\varphi) \quad \text{ext}_{\mathcal{E}}(\varphi) = \tau^{-1}(I_M(\varphi)) \quad (2)$$

Compare these orderings to those in Eqn. 3 from the FOLE foundation paper [12].

For all $r \in R$, we have the relationships

$$\varphi(\text{ext}_{\mathcal{E}}(r)) = I_M(r) \quad \text{ext}_{\mathcal{E}}(r) \subseteq \tau^{-1}(I_M(r)). \quad (3)$$

Definition 1. A structure $M$ is extensive when the right hand expression in (3) is an equality: $\text{ext}_{\mathcal{E}}(r) = \tau^{-1}(I_M(r))$ for any entity type $r \in R$. Then, the tuple map $K \rightarrow \text{List}(Y)$ restricts to a bijection: $\text{ext}_{\mathcal{E}}(r) \stackrel{\tau}{\longrightarrow} I_M(r)$. \(18\)

Lemma 2. A structure $M$ is extensive when $K \rightarrow \text{List}(Y)$ is injective. \(19\)

Proof. By Eqn. 3, $\varphi(\text{ext}_{\mathcal{E}}(r)) = I_M(r)$. Hence, $\text{ext}_{\mathcal{E}}(r) = \tau^{-1}(I_M(r))$. \(20\)

Any structure $M$ has an associated extensive structure. An example is the key-embedding structure $\mathcal{M}$ (see [12]).

\(18\) For an extensive structure, extent order is equivalent to interpretation order: $\text{ext}_{\mathcal{E}}(r) \subseteq \text{ext}_{\mathcal{E}}(r')$ iff $I_M(r) \subseteq I_M(r')$ for entity types $r, r' \in R$.

\(19\) This corrects an editing error in the FOLE foundation paper (Kent [12]).

\(20\) For any function $A \xrightarrow{f} B$, direct image is left adjoint to inverse image: $\varphi(X) \subseteq Y$ iff $X \subseteq f^{-1}(Y)$ for any subsets $X \subseteq A$ and $Y \subseteq B$. If $f$ is injective and $\varphi(X) = Y$, then $X = f^{-1}(Y)$.
Definition 2. A structure $\mathcal{M}$ is comprehensive\textsuperscript{21} when the left hand expression in Eqn. 2 is an equality: $\varphi \tau(\text{ext}_\mathcal{E}(\varphi)) = I_\mathcal{M}(\varphi)$ for any formula $\varphi \in R$. Then, the tuple map $K \xrightarrow{\tau} \text{List}(Y)$ restricts to a bijection: $\text{ext}_\mathcal{E}(\varphi) \xrightarrow{\tau} I_\mathcal{M}(\varphi)$.\textsuperscript{22}

The condition $\varphi(\tau(\text{ext}_\mathcal{E}(\varphi))) = I_\mathcal{M}(\varphi)$ means that the restricted tuple function $\text{ext}_\mathcal{E}(\varphi) \xrightarrow{\tau} I_\mathcal{M}(\varphi)$ is surjective. Hence, we can choose an injective inverse function $\text{ext}_\mathcal{E}(\varphi) \xleftarrow{\gamma_\mathcal{E}} I_\mathcal{M}(\varphi)$ satisfying $\gamma_\mathcal{E} \cdot \tau_\mathcal{E} = I_\mathcal{M}(\varphi)$. In a comprehensive structure $\mathcal{M}$, we make this choice for each formula $\varphi \in \bar{R}$.

Proposition 2. Let $\mathcal{M} = (\mathcal{E}, \sigma, \tau, \mathcal{A})$ be a structure, whose key set is a subset of $Y$-tuples $K \subseteq \text{List}(Y)$ and whose tuple map is inclusion $K \xrightarrow{\text{inc}} \text{List}(Y)$. Then $\mathcal{M}$ is comprehensive.

Proof. Let $\varphi \in \bar{R}$ be any formula. By Prop. 1, $k \models_{\mathcal{E}} \varphi$ iff $k = \text{inc}(k) \in I_\mathcal{M}(\varphi)$ for all $k \in K$; equivalently, $\varphi(\text{inc}(\text{ext}_\mathcal{E}(\varphi))) = \text{ext}_\mathcal{E}(\varphi) = I_\mathcal{M}(\varphi)$.

Definition 3. A FOLE structure $\mathcal{M}$ has an associated image structure $\hat{\mathcal{M}} = (\mathcal{E}, \sigma, \tau, \mathcal{A})$ with the same schema $\mathcal{S} = \langle R, \sigma, X \rangle$ and typed domain $\mathcal{A} = (X, Y, \models_{\mathcal{A}})$, but with the trivial universe $\hat{\mathcal{U}} = (\text{List}(Y), I_{\text{List}(Y)}, Y)$ and the descriptor entity classification $\hat{\mathcal{E}} = \exists_\tau(\mathcal{E}) = \langle R, \text{List}(Y), \models_{\mathcal{E}} \rangle$, where a tuple serves as its own identifier: $\langle I, t \rangle \models_{\mathcal{E}} r$ for a tuple $\langle I, t \rangle \in \text{List}(Y)$ and an entity type $r \in R$ when $\langle I, t \rangle$ is the descriptor $\tau(k) = \langle I, t \rangle$ for some key $k \in K$ such that $k \models_{\mathcal{E}} r$. Thus, $\text{ext}_\mathcal{E}(r) = \varphi(\text{ext}_\mathcal{E}(r)) \subseteq \text{List}(Y)$ is the direct image of $\text{ext}_\mathcal{E}(r) \subseteq K$ along the tuple map $K \xrightarrow{\tau} \text{List}(Y)$.

Proposition 3. For the image structure, the formula interpretation in $\hat{\mathcal{M}}$ is the formula interpretation in $\mathcal{M}$, and the formula extent in $\hat{\mathcal{M}}$ is this interpretation: $\text{ext}_{\mathcal{E}}(\varphi) = I_{\hat{\mathcal{M}}}(\varphi) = I_\mathcal{M}(\varphi)$ for any formula $\varphi \in \bar{R}$. The image structure $\hat{\mathcal{M}}$ is comprehensive.

Proof. At the base step in Tbl. 6, $I_{\hat{\mathcal{M}}}(r) = \varphi I_{\text{List}(Y)}(\text{ext}_\mathcal{E}(r)) = \text{ext}_\mathcal{E}(r) = \varphi(\text{ext}_\mathcal{E}(r)) = I_\mathcal{M}(r)$ for any entity type $r \in R$ (Def. 3 above). By induction, $I_{\hat{\mathcal{M}}}(\varphi) = I_\mathcal{M}(\varphi)$ for any formula $\varphi \in \bar{R}$. By Prop. 1, $\langle I, t \rangle \models_{\mathcal{E}} \varphi$ iff $\langle I, t \rangle = I_{\text{List}(Y)}(I, t) \in I_{\hat{\mathcal{M}}}(\varphi)$ for any tuple $\langle I, t \rangle \in \text{List}(Y)$; equivalently, $\varphi I_{\text{List}(Y)}(\text{ext}_\mathcal{E}(\varphi)) = \text{ext}_\mathcal{E}(\varphi) = I_{\hat{\mathcal{M}}}(\varphi)$.

Corollary 1. A structure $\mathcal{M}$ is comprehensive when $K \xrightarrow{\tau} \text{List}(Y)$ is injective.

\textsuperscript{21} In the original discussion (Kent [11]) about tabular interpretation, all FOLE structures were assumed to be comprehensive.

\textsuperscript{22} For a comprehensive structure, extent order is equivalent to interpretation order: $\text{ext}_\mathcal{E}(\varphi) \subseteq \text{ext}_\mathcal{E}(\psi)$ iff $I_\mathcal{M}(\varphi) \subseteq I_\mathcal{M}(\psi)$ for formulas $\varphi, \psi \in \bar{R}$.
Proof. By Prop. 3, $\varphi(\text{ext}_E(\varphi)) = \text{ext}_\hat{E}(\varphi) = I_M(\varphi) = I_M(\varphi)$ for any $\varphi \in \hat{R}$.

**Proposition 4.** For any structure $M = \langle E, \sigma, \tau, A \rangle$, the associated “key-embedding” structure $\hat{M} = \langle E, \hat{\sigma}, \hat{\tau}, \hat{A} \rangle$ (see the FOLE Foundation paper [12]) is comprehensive.

Proof. By Cor. 1, since the tuple map $K \xrightarrow{\hat{\tau}} List(\hat{Y})$ of the key-embedding structure $\hat{M}$ is injective.

**Proposition 5.** If $\mathcal{M}$ is comprehensive, then $\hat{\mathcal{M}}$ is comprehensive with the same interpretation and entity extent: $I_{\hat{\mathcal{M}}} = I_M$ and $\text{ext}_{\hat{E}} = \text{ext}_E$.

Proof. At the base step in Tbl. 6, $I_{\hat{\mathcal{M}}}(\varphi) = \varphi(\text{ext}_E(\varphi)) = I_M(\varphi)$ by comprehension. Hence, by induction $I_{\hat{\mathcal{M}}} = I_M$. By Eqn. 2, $\text{ext}_{\hat{E}}(\varphi) = \tau^{-1}(I_{\hat{\mathcal{M}}}(\varphi)) = \tau^{-1}(I_M(\varphi)) = \text{ext}_E(\varphi)$. 
2.2.3 Formula Structure Morphisms. Let
\[ M_2 = \langle \mathcal{E}_2, \langle \sigma_2, \tau_2 \rangle, A_2 \rangle \xrightarrow{(r, k, f, g)} \langle \mathcal{E}_1, \langle \sigma_1, \tau_1 \rangle, A_1 \rangle = M_1 \] 
be any structure morphism. We can define a formula structure morphism with certain qualifications.

**Lemma 3.** Any of the following equivalent conditions hold for Eqn. 4
\[
\begin{align*}
    k(k_1) &\models \hat{\xi}_2 \varphi_2 \iff k_1 \models \hat{\xi}_1 \hat{\tau}(\varphi_2) \\
    k(k_1) &\in \text{ext}_{\hat{\xi}_2}(\varphi_2) \iff k_1 \in \text{ext}_{\hat{\xi}_1}(\hat{\tau}(\varphi_2)) \\
    k^{-1}(\text{ext}_{\hat{\xi}_2}(\varphi_2)) & = \text{ext}_{\hat{\xi}_1}(\hat{\tau}(\varphi_2))
\end{align*}
\]
for any source boolean formula \( \varphi_2 \in \mathcal{R}_2 \) (containing no quantification/substitution) and any target key \( k_1 \in K_1 \). These express entity infomorphism conditions.

**Proof.** Proved by induction. This is clearly true for entity types (relation symbols). Check on all booleans: meets, joins, negations, etc.

**Proposition 6.** There is a boolean formula structure passage
\[
\text{Struc} \xrightarrow{\text{fmla}_0} \text{Struc},
\]
which is idempotent \( \text{fmla}_0(\text{fmla}_0(M)) \cong \text{fmla}_0(M) \): a formula of formulas is another formula.
Lemma 4. Assume the structures $\mathcal{M}_1$ and $\mathcal{M}_2$ are comprehensive and the data value set $Y$ is fixed. Any of the following equivalent conditions hold for Eqn. 4

$$k(k_1) \models_{\mathcal{E}_2} \phi_2 \iff k_1 \models_{\mathcal{E}_1} \tilde{\phi}(\phi_2)$$
$$k(k_1) \in \text{ext}_{\mathcal{E}_2}(\phi_2) \iff k_1 \in \text{ext}_{\mathcal{E}_1}(\tilde{\phi}(\phi_2))$$
$$k^{-1}(\text{ext}_{\mathcal{E}_2}(\phi_2)) \supseteq \text{ext}_{\mathcal{E}_1}(\tilde{\phi}(\phi_2))$$
$$\text{ext}_{\mathcal{E}_2}(\phi_2) \supseteq \psi k(\text{ext}_{\mathcal{E}_1}(\tilde{\phi}(\phi_2)))$$

for any source boolean formula $\phi_2 \in \tilde{R}_2$ and any target key $k_1 \subseteq K_1$. These imply that the following condition holds

$$I_{\mathcal{M}_2}(\phi_2) \supseteq I_{\mathcal{M}_1}(\tilde{\phi}(\phi_2))$$

holds for any source formula $\phi_2 \in \tilde{R}_2$, since the structures are comprehensive and the value set is fixed. $^{23}$

Proof. Proved by induction. True for booleans by proof analogous to Lem. 3.

all: $(I_1, s_1) = \sigma_1(\tilde{\phi}(\phi_2)) = \Sigma_f(\sigma_2(\phi_2)) = \Sigma_f(I_2, s_2)$  

$tup_{\mathcal{A}_2}(I_2, s_2) = tup_{\mathcal{A}_1}(I_1, s_1)$ type domain morphism

**existential:**  
$$k_1 \models_{\mathcal{E}_1} \tilde{\phi}(\Sigma_f(\phi_2))$$

iff $k_1 \models_{\mathcal{E}_1} \Sigma_f(\tilde{\phi}(\phi_2))$  

iff $\tau_1(k_1) \in \exists_k(I_{\mathcal{M}_1}(\tilde{\phi}(\phi_2)))$  

iff $\tau_2(k(k_1)) \in \exists_k(I_{\mathcal{M}_2}(\phi_2))$  

iff $k(k_1) \models_{\mathcal{E}_2} \Sigma_f(\phi_2)$

**substitution:**  
$$k_1 \models_{\mathcal{E}_1} h^*(\tilde{\phi}(\phi_2))$$

iff $\tau_1(k_1) \in h^{-1}(I_{\mathcal{M}_1}(\tilde{\phi}(\phi_2)))$  

iff $\tau_2(k(k_1)) \in h^{-1}(I_{\mathcal{M}_2}(\phi_2))$  

iff $k(k_1) \models_{\mathcal{E}_2} h^*(\phi_2)$

$^{23}$ $I_{\mathcal{M}_2}(\phi_2) = \varphi \tau_2(\text{ext}_{\mathcal{E}_2}(\phi_2)) \supseteq \varphi \tau_2(\psi k(\text{ext}_{\mathcal{E}_1}(\tilde{\phi}(\phi_2)))) = \varphi \tau_1(\text{ext}_{\mathcal{E}_1}(\tilde{\phi}(\phi_2))) = I_{\mathcal{M}_1}(\tilde{\phi}(\phi_2))$. 


Lemma 5. As in Lem. 4, assume the structures $\mathcal{M}_1$ and $\mathcal{M}_1$ are comprehensive and the data value set $Y$ is fixed. In addition, assume the key function $K_2 \leftarrow^k K_1$ is surjective. Any of the following equivalent conditions hold for Eqn. 4:

\[
  k(k_1) \models_{\mathcal{E}_2} \varphi_2 \iff k_1 \models_{\mathcal{E}_1} \hat{r}(\varphi_2) \\
  k(k_1) \in \text{ext}_{\mathcal{E}_2}(\varphi_2) \iff k_1 \in \text{ext}_{\mathcal{E}_1}(\hat{r}(\varphi_2)) \\
  k^{-1}(\text{ext}_{\mathcal{E}_2}(\varphi_2)) = \text{ext}_{\mathcal{E}_1}(\hat{r}(\varphi_2))
\]

for any source boolean formula $\varphi_2 \in \hat{R}_2$ and any target key $k_1 \in K_1$. These imply that the following condition holds:

\[
  \text{ext}_{\mathcal{E}_2}(\varphi_2) = \varphi k(\text{ext}_{\mathcal{E}_1}(\hat{r}(\varphi_2)))
\]

for any source formula $\varphi_2 \in \hat{R}_2$, since the function $K_2 \leftarrow^k K_1$ is surjective.  

This implies that the following condition holds:

\[
  I_{\mathcal{M}_2}(\varphi_2) = I_{\mathcal{M}_1}(\hat{r}(\varphi_2))
\]

for any source formula $\varphi_2 \in \hat{R}_2$, since the structures are comprehensive and the value set is fixed. 

Proof. Proof is similar to that of Lem. 4. 

Lemma 6. With the assumptions of Lem. 5, there is an associated formula structure morphism between comprehensive formula structures:

\[
  \text{fmla}(\mathcal{M}_2) = \langle \mathcal{E}_2, (\hat{\sigma}_2, \tau_2), A_2 \rangle \overset{\text{fmla}(r, f, 1_Y)}{\longrightarrow} \langle \mathcal{E}_1, (\hat{\sigma}_1, \tau_1), A_1 \rangle = \text{fmla}(\mathcal{M}_1)
\]

with schema morphism $\text{fmla}(\mathcal{S}_2) = \langle \hat{R}_2, \hat{\sigma}_2, X_2 \rangle \overset{\text{fmla}(r, f)}{\longrightarrow} \langle \hat{R}_1, \hat{\sigma}_1, X_1 \rangle = \text{fmla}(\mathcal{S}_1)$

and entity infomorphism $\hat{\mathcal{E}}_2 = \langle \hat{R}_2, K_2, \models_{\mathcal{E}_2} \rangle \overset{\hat{r}(k, f)}{\longrightarrow} \langle \hat{R}_1, K_1, \models_{\mathcal{E}_1} \rangle = \hat{\mathcal{E}}_1$.

Proof. Source and target formula structures are comprehensive by Prop. 5. The entity infomorphism condition $k(k_1) \models_{\mathcal{E}_2} \varphi_2 \iff k_1 \models_{\mathcal{E}_1} \hat{r}(\varphi_2)$ holds for any source formula $\varphi_2 \in \hat{R}_2$ and target key $k_1 \in K_1$ by Lem. 5. 

Let $\text{Struc}(Y)$ denote the subcontext of comprehensive structures with fixed data value set $Y$ whose structure morphisms have a surjective key function.

Proposition 7. There is a formula structure passage

\[
  \text{Struc}(Y) \overset{\text{fmla}_Y}{\longrightarrow} \text{Struc}(Y),
\]

which is idempotent $\text{fmla}_Y(\text{fmla}_Y(\mathcal{M})) \cong \text{fmla}_Y(\mathcal{M})$: a formula of formulas is another formula.

---

24 For any function $A \rightarrow B$, direct image is left adjoint to inverse image: $\varphi f(X) \subseteq Y$ iff $X \subseteq f^{-1}(Y)$ for any subsets $X \subseteq A$ and $Y \subseteq B$. If $f$ is surjective and $X = f^{-1}(Y)$, then $\varphi f(X) = Y$.

25 $I_{\mathcal{M}_2}(\varphi_2) = \varphi \tau_2(\text{ext}_{\mathcal{E}_2}(\varphi_2)) = \varphi \tau_2(\varphi k(\text{ext}_{\mathcal{E}_1}(\hat{r}(\varphi_2)))) = \varphi \tau_1(\text{ext}_{\mathcal{E}_1}(\hat{r}(\varphi_2))) = I_{\mathcal{M}_1}(\hat{r}(\varphi_2))$. 

2.3 Satisfaction.

Satisfaction is a fundamental classification between formalism and semantics. The atom of formalism used in satisfaction is the FOLE constraint, whereas the atom of semantics used is the FOLE structure. Satisfaction is defined in terms of formula interpretation (Eqn. 1 of §2.2).  

2.3.1 Sequent Satisfaction. An S-structure $\mathcal{M} \in \text{Struc}(S)$ satisfies an S-sequent $\varphi \vdash \psi$ (§2.1.2) when the interpretation widening of views asserted by the sequent actually holds in $\mathcal{M}$: $\mathcal{I}_M(\varphi) \subseteq \mathcal{I}_M(\psi)$. Satisfaction is symbolized either by $\mathcal{M} \models_S (\varphi \vdash \psi)$ or by $\varphi \vdash \mathcal{M} \psi$. For each S-signature $\langle I, s, \rangle$, satisfaction defines the fiber order $\mathcal{M}S(I, s) = (\mathcal{R}(I, s), \leq_M)$, where $\varphi \leq_M \psi$ when $\varphi \vdash_M \psi$ for any two formulas $\varphi, \psi \in \mathcal{R}(I, s)$. Sequent satisfaction can be expressed in terms of implication as $\models \leq_M (\varphi \rightarrow \psi)$. Thus, satisfaction of sequents is equivalent to satisfaction of formulas.

**Corollary 2.** Satisfaction in $\mathcal{M}$ is equivalent to satisfaction in the image $\hat{\mathcal{M}}$: $\mathcal{M} \models_S (\varphi \vdash \psi)$ iff $\hat{\mathcal{M}} \models_S (\varphi \vdash \psi)$.  

**Proof.** By Prop. 3, the formula interpretation in $\mathcal{M}$ and $\hat{\mathcal{M}}$ are equal.

For any S-structure $\mathcal{M} \in \text{Struc}(S)$, the formula extent order $\text{ord}(\hat{\mathcal{E}}) = (\hat{\mathcal{R}}, \leq_{\hat{\mathcal{E}}})$ is defined by $\varphi \leq_{\hat{\mathcal{E}}} \psi$ when $\text{ext}_{\hat{\mathcal{E}}}(\varphi) \subseteq \text{ext}_{\hat{\mathcal{E}}}(\psi)$. For each signature $\langle I, s \rangle \in \text{List}(X)$, define the suborder $\text{ord}_{\hat{\mathcal{E}}}(I, s) = (\hat{\mathcal{R}}(I, s), \leq_{\hat{\mathcal{E}}})$.

**Corollary 3.** For an arbitrary S-structure the formula interpretation order is as strong as or stronger than the extent order of the formula classification: $\varphi \leq_M \psi$ implies $\varphi \leq_{\hat{\mathcal{E}}} \psi$, but not necessarily the converse.

**Proof.** Extent is the inverse image of interpretation: $\text{ext}_{\hat{\mathcal{E}}}(\varphi) = \tau^{-1}((\mathcal{I}_M(\varphi)))$ for any formula $\varphi \in \hat{\mathcal{R}}$ (Eqn. 2). Since inverse image is monotonic, $\mathcal{I}_M(\varphi) \subseteq \mathcal{I}_M(\psi)$ implies $\text{ext}_{\hat{\mathcal{E}}}(\varphi) \subseteq \text{ext}_{\hat{\mathcal{E}}}(\psi)$.

**Corollary 4.** For a comprehensive S-structure, the entity extent and interpretation fiber orders are identical $\text{ord}_{\hat{\mathcal{E}}}(I, s) = \mathcal{M}S(I, s)$, since $\varphi \leq_{\hat{\mathcal{E}}} \psi$ iff $\varphi \leq_M \psi$.

**Proof.** See footnote to Def. 2.

**Proposition 8.** Formal quantification and substitution are monotonic.

**Proof.** The formal operators $\sum_h$, $h^*$, and $\Pi_h$ are monotonic, since the semantic operators $\exists_h$, $h^{-1}$, and $\forall_h$ are monotonic. We show the proof for formal existential quantification. If $\langle I', s' \rangle \hookrightarrow \langle I, s \rangle$ is a signature morphism, then the existential operator $\mathcal{M}S(I', s') \models_{\mathcal{E}} \mathcal{M}S(I, s)$ is monotonic: $\varphi \leq_M \psi$ iff $\mathcal{I}_M(\varphi) \subseteq \mathcal{I}_M(\psi)$ implies $\mathcal{I}_M(\sum_h(\varphi)) = \exists_h(\mathcal{I}_M(\varphi)) \subseteq \exists_h(\mathcal{I}_M(\psi)) = \mathcal{I}_M(\sum_h(\psi))$ iff $\sum_h(\varphi) \leq_M \sum_h(\psi)$ for any two formulas $\varphi, \psi \in \mathcal{R}(I, s)$.

26 Important definitions follow a logical order: formula interpretation $\Rightarrow$ satisfaction $\Rightarrow$ institution $\Rightarrow$ structure interpretation $\Rightarrow$ sound logic interpretation.

27 Satisfaction in $\mathcal{M}$ and its key-embedding structure $\hat{\mathcal{M}}$ should also be connected.
2.3.2 Constraint Satisfaction. An $\mathcal{S}$-structure $\mathcal{M} \in \text{Struc}(\mathcal{S})$ satisfies an $\mathcal{S}$-constraint $\varphi' \rightarrow \varphi$ when $\mathcal{M}$ satisfies the sequent $\sum_h(\varphi) \vdash \varphi' \iff \varphi' \geq_M \sum_h(\varphi)$ if $I_M(\varphi') \supseteq I_M(\sum_h(\varphi)) = \exists_h(I_M(\varphi))$; equivalently, when $\mathcal{M}$ satisfies the sequent $\varphi \vdash_h(\varphi') \iff h^*(\varphi') \geq_M h^*(\varphi) \iff h^{-1}(I_M(\varphi')) = I_M(h^*(\varphi')) \supseteq I_M(\varphi)$. Satisfaction is symbolized by $\mathcal{M} \models_{\mathcal{S}} (\varphi' \rightarrow \varphi)$. Constraint satisfaction can be expressed in terms of implication as $\top \leq \hat{E}(\sum_h(\varphi) \rightarrow \varphi')$; equivalently, $\top \leq \hat{E}(\varphi \rightarrow h^*(\varphi'))$. Thus, satisfaction of constraints is equivalent to satisfaction of formulas.

Lemma 7. A structure $\mathcal{M}$ determines a mathematical context $\mathcal{M}^S \subseteq \text{Cons}(\mathcal{S})$, called the conceptual intent of $\mathcal{M}$, whose objects are $\mathcal{S}$-formulas and whose morphisms are $\mathcal{S}$-constraints satisfied by $\mathcal{M}$. The $(I, s)^{th}$ fiber is the order $\mathcal{M}^S(I, s)^{op} = \langle \hat{R}(I, s), \geq_M \rangle$.  

Proof. $\mathcal{M}^S$ is closed under constraint identities and constraint composition.
1: $\mathcal{M} \models_{\mathcal{S}} (\varphi') \rightarrow (\varphi)$, and 2: if $\mathcal{M} \models_{\mathcal{S}} (\varphi'' \rightarrow (\varphi'))$ and $\mathcal{M} \models_{\mathcal{S}} (\varphi'' \rightarrow (\varphi))$, then $\mathcal{M} \models_{\mathcal{S}} (\varphi'' \rightarrow h^*(\varphi'))$, since $\varphi'' \geq_M \sum_h(\varphi')$ and $\varphi' \geq_M \sum_h(\varphi)$ implies $\varphi'' \geq_M \sum_h(\sum_h(\varphi)) = \sum_h(\sum_h(\varphi))$ using the monotonicity of the existential operator $\mathcal{M}^S(I'', s'') \xrightarrow{\Sigma_h} \mathcal{M}^S(I', s')$.

There is an intent(ional) order between $\mathcal{S}$-structures. Structure $\mathcal{M}_2$ is more general than structure $\mathcal{M}_1$, symbolically $\mathcal{M}_1 \leq_{\mathcal{S}} \mathcal{M}_2$, when any constraint satisfied by $\mathcal{M}_2$ is also satisfied by $\mathcal{M}_1$: $\mathcal{M}_2 \models_{\mathcal{S}} (\varphi' \rightarrow (\varphi'))$ implies $\mathcal{M}_1 \models_{\mathcal{S}} (\varphi' \rightarrow (\varphi'))$; that is, when $\mathcal{M}_1^S \supseteq \mathcal{M}_2^S$.

28 See the formal/semantics reflection discussed in Sec. 2.2.1 and illustrated in Tbl. 7.
29 The satisfaction relation corresponds to the “truth classification” in Barwise and Seligman [1], where the conceptual intent $\mathcal{M}^S$ corresponds to the “theory of $\mathcal{M}$”. 
2.3.3 Institutional Aspect. For any schema $\mathcal{S}$, the satisfaction classification $\text{Truth}(\mathcal{S}) = \langle \text{Struc}(\mathcal{S}), \text{Cons}(\mathcal{S}), \models_{\mathcal{S}} \rangle$ has $\mathcal{S}$-constraints $(\varphi' \xrightarrow{b} \varphi) \in \text{Cons}(\mathcal{S})$ as types, $\mathcal{S}$-structures $\mathcal{M} \in \text{Struc}(\mathcal{S})$ as instances, and satisfaction as the classification relation $\mathcal{M} \models_{\mathcal{S}} (\varphi' \xrightarrow{b} \varphi)$. In the propositions below, we give the precise meaning of “interpretation in first-order logic” (Barwise and Seligman [1]) in terms of an infomorphism between truth classifications

\[
\langle \text{cons}_{(r,f)}, \text{struc}^\gamma_{(r,f)} \rangle : \text{Truth}(\mathcal{S}_2) = \text{Truth}(\mathcal{S}_1).
\]

**Proposition 9.** The triple $\langle \text{Sch}, \text{cons}, \text{struc} \rangle$ forms an institution (Goguen and Burstall [7]), where $\text{Sch}$ is the context of schemas, $\text{Sch} \xrightarrow{\text{cons}} \text{Cxt}$ is the indexed context of constraints ($\S 2.1.3$), and $\text{Sch}^{\text{op}} \xrightarrow{\text{struc}} \text{Cxt}$ is the indexed context of structures (appendix of Kent [12]).

**Proof.** See the paper “The First-order Logical Environment” (Kent [11]). ■

In an institution “satisfaction is invariant under change of notation”: for any schema morphism $\mathcal{S}_2 \xrightarrow{(r,f)} \mathcal{S}_1$, the following satisfaction condition holds:

\[
\text{struc}^\gamma_{(r,f)}(\mathcal{M}_1) \models_{\mathcal{S}_2} (\varphi'_2 \xrightarrow{b} \varphi_2) \iff \mathcal{M}_1 \models_{\mathcal{S}_1} \text{cons}_{(r,f)}(\varphi'_2 \models \varphi_2).
\]

(5)

Equivalently, (see $\S 3.1.1$ for the definition of specification flow)

\[
\text{struc}^\gamma_{(r,f)}(\mathcal{M}_1) \models_{\mathcal{S}_2} \iff \text{spec}_{(r,f)}(\mathcal{M}_1) \models_{\mathcal{S}_1}.
\]

(6)

the intent of the structure image is the specification image of the intent.

**Proposition 10.** The institution $\langle \text{Sch}, \text{cons}, \text{struc} \rangle$ is a logical environment.

**Proof.** See the paper “The First-order Logical Environment” (Kent [11]). ■

A logical environment is an institution in which “satisfaction respects structure morphisms”: for any vertical structure morphism if $\mathcal{M}_2 \xrightarrow{(k,g)} \mathcal{M}_1$ in the fiber context $\text{Struc}(\mathcal{S})$ of a schema $\mathcal{S}$, the following satisfaction condition holds:

\[
\mathcal{M}_2 \models_{\mathcal{S}_2} (\varphi'_2 \xrightarrow{b} \varphi_2) \implies \mathcal{M}_1 \models_{\mathcal{S}_2} (\varphi'_2 \xrightarrow{b} \varphi_2).
\]

(7)

Equivalently, we have the intent order

\[
\mathcal{M}_2 \models^S \geq_{\mathcal{S}} \mathcal{M}_1^S
\]

(8)

---

\[30\] The satisfaction classification of the schema $\mathcal{S}$ corresponds to the “truth classification” of a first-order language in Barwise and Seligman [1].
Corollary 5. A structure morphism \( M_2 \xrightarrow{\langle r, k, f, g \rangle} M_1 \) determines an intent passage \( M_2 \xrightarrow{\text{int}_{\langle r, k, f, g \rangle}} M_1 \), which is a restriction of the constraint passage \( \text{Cons}(S_2) \xrightarrow{\text{Cons}(r, f)} \text{Cons}(S_1) : (\varphi'_2 \xrightarrow{h} \varphi_2) \mapsto (\overline{\overline{h}}(\varphi'_2) \xrightarrow{h} \overline{\overline{h}}(\varphi_2)) \).

Proof. The structure morphism \( M_2 \xrightarrow{\langle r, k, f, g \rangle} M_1 \) factors as

\[
M_2 \xrightarrow{(k, g)} \text{struc}_{\langle r, f \rangle}(M_1) \xleftarrow{(r, f)} M_1.
\]

Let \( (\varphi'_2 \xrightarrow{h} \varphi_2) \) be any \( S_2 \)-constraint. From Prop. 10 we know that

\[
M_2 \models S_2 (\varphi'_2 \xrightarrow{h} \varphi_2) \quad \text{implies} \quad \text{struc}_{\langle r, f \rangle}(M_1) \models S_2 (\varphi'_2 \xrightarrow{h} \varphi_2).
\]

From Prop. 9 we know that

\[
\text{struc}_{\langle r, f \rangle}(M_1) \models S_2 (\varphi'_2 \xrightarrow{h} \varphi_2) \quad \text{iff} \quad M_1 \models S_1 \text{cons}_{\langle r, f \rangle} (\varphi'_2 \models \varphi_2).
\]

\[\square\]

Definition 4. There is a conceptual intent passage

\[
\text{Struc} \xrightarrow{\text{int}} \text{Cxt} : M \mapsto M^S
\]

from structures to mathematical contexts.

---

Proven in the appendix of Kent [12].
3 Architectural Components

3.1 Specifications.

3.1.1 Specifications.

Consequence Relations. A FOLE consequence relation (Barwise and Seligman [1]) is a pair $\langle S, \vdash \rangle$, where $S$ is a schema and $\vdash \subseteq \widehat{R} \times \widehat{R}$ is a set of $S$-sequents; that is, a binary relation on $S$-formulas. We want each sequent in a consequence relation to assert logical entailment between component formulas. A consequence relation can be used to represent and express all the subtyping relationships of a data model. In the example illustrated in the ERA data model of the FOLE foundation paper [12], we might have the subtyping relationships $(\text{Manager} \vdash \text{Employee})$ and $(\text{Engineering} \vdash \text{Department})$.

Specifications. Consequence relations only connect formulas within fibers; due to the common signature requirement on sequent components, a FOLE consequence relation $\langle S, \vdash \rangle$ partitions into a collection of fiber consequence relations $\{\vdash_{(I, s)} \subseteq \widehat{R}(I, s) \times \widehat{R}(I, s) \mid (I, s) \in \text{List}(X)\}$ indexed by $S$-signatures. We now define a useful notion that connects formulas across fibers.

Given a schema $S$, an $S$-specification is a subgraph $T \sqsubseteq \text{Cons}(S)$, whose nodes are $S$-formulas and whose edges are $S$-constraints. A consequence relation is a specification in which all constraints are sequents. Let $\text{Spec}(S) = \wp\text{Cons}(S)$ denote the set of all $S$-specifications. $^{32}$ Although implicit, we usually include the schema (language) in the symbolism, so that a FOLE specification (presentation) $\mathcal{T} = \langle S, T \rangle$ is an indexed notion consisting of a schema $S$ and a $S$-specification $T \in \text{Spec}(S)$. We can place axiomatic restrictions on specifications (and consequence relations) in various manners. A FOLE specification requires entailment to be a preorder, satisfying reflexivity and transitivity. It also requires satisfaction of sufficient axioms (Tbl. 4) to described the various logical operations (connectives, quantifiers, etc.) used to build formulas in first-order logic.

Specification Satisfaction. An $S$-structure $\mathcal{M} \in \text{Struc}(S)$ satisfies (is a model of) an $S$-specification $T$, symbolized $\mathcal{M} \models_S T$, when it satisfies every constraint in the specification: $\mathcal{M} \models_S T \iff M^S \models T$. Hence, the intent $\mathcal{M}^S$ is the largest and most specialized $S$-specification satisfied by $\mathcal{M}$. $^{33}$ When specification order is defined below, $\mathcal{M}_1 \leq_S \mathcal{M}_2$ (intentional order) is equivalent to $\mathcal{M}_1^S \leq_S \mathcal{M}_2^S$ (intent specification order).

---

$^{32}$ For any graph $\mathcal{G}$, $\wp\mathcal{G} = \langle \wp\mathcal{G}, \sqsubseteq \rangle$ denotes the power preorder of all subgraphs of $\mathcal{G}$.

$^{33}$ $\mathcal{M}^S$ is not just a mathematical context, but also an $S$-specification.
3.1.2 Entailment and Consequence. Let $\mathcal{S} = (R, \sigma, X)$ be a schema. An $\mathcal{S}$-specification $T$ entails an $\mathcal{S}$-constraint $\langle \phi \hookrightarrow \phi' \rangle$, symbolized by $T \vdash_{\mathcal{S}} \langle \phi \hookrightarrow \phi' \rangle$, when any model of the specification satisfies the constraint: $\mathcal{M} \models_{\mathcal{S}} T$ implies $\mathcal{M} \models_{\mathcal{S}} \langle \phi \hookrightarrow \phi' \rangle$ for any $\mathcal{S}$-structure $\mathcal{M}$; that is, when $\mathcal{M}^{\mathcal{S}} \sqsupseteq T$ implies $\mathcal{M}^{\mathcal{S}} \sqsupseteq \langle \phi \hookrightarrow \phi' \rangle$ for any $\mathcal{S}$-structure $\mathcal{M}$.\(^{34}\) The graph

$$T^\ast = \left\{ \langle \phi \hookrightarrow \phi' \rangle \mid T \vdash_{\mathcal{S}} \langle \phi \hookrightarrow \phi' \rangle \right\} = \cap_{\mathcal{S}} \left\{ \mathcal{M}^{\mathcal{S}} \mid \mathcal{M} \in \text{Struc}(\mathcal{S}), \mathcal{M}^{\mathcal{S}} \sqsupseteq T \right\}$$

of all constraints entailed by a specification $T$ is called its consequence. The consequence $T^\ast$ is a mathematical context, since each conceptual intent $\mathcal{M}^{\mathcal{S}}$ is a mathematical context. The consequence operator $(\cdot)^\ast$ is a closure operator on specifications: (increasing) $T \sqsubseteq T^\ast$; (monotone) $T_1 \sqsubseteq T_2$ implies $T_1^\ast \sqsubseteq T_2^\ast$; and (idempotent) $T^{\ast \ast} = T^\ast$. Closure operators can be alternatively described as entailment relations (Mossakowski, Diaconescu and Tarlecki\(^{18}\)).

$\langle \text{Cons}(\mathcal{S}), \vdash_{\mathcal{S}} \rangle$ forms an entailment relation: (reflexive) $\langle \langle \phi \hookrightarrow \phi' \rangle \rangle \vdash_{\mathcal{S}} \langle \phi \hookrightarrow \phi' \rangle$ for any $\mathcal{S}$-constraint $\langle \phi \hookrightarrow \phi' \rangle$; (monotone) if $T_1 \sqsubseteq T_2$ and $T_1 \vdash_{\mathcal{S}} \langle \phi \hookrightarrow \phi' \rangle$ implies $T_1 \vdash_{\mathcal{S}} \langle \phi \hookrightarrow \phi' \rangle$; and (transitive) if $T_1^\ast \sqsubseteq T_2$ and $T_1 \sqcup T_2 \vdash_{\mathcal{S}} \langle \phi \hookrightarrow \phi' \rangle$, then $T_1 \vdash_{\mathcal{S}} \langle \phi \hookrightarrow \phi' \rangle$.

There is an intentional (concept lattice) entailment order between specifications that is implicit in satisfaction: $T_1 \leq_{\mathcal{S}} T_2$ when $T_1^\ast \sqsupseteq T_2^\ast$; equivalently, $T_1^\ast \sqsubseteq T_2$. This is a specialization-generalization order; $T_1$ is more specialized than $T_2$, and $T_2$ is more generalized than $T_1$. We symbolize this preorder by $\text{Spec}(\mathcal{S}) = (\text{Spec}(\mathcal{S}), \leq_{\mathcal{S}})$. Intersections and unions define joins and meets, with the bottom specification being the empty join $\bot_{\mathcal{S}} = \cap_{\mathcal{S}} = \text{Cons}(\mathcal{S})$ and the top specification being the empty meet $\top_{\mathcal{S}} = \cup_{\mathcal{S}} = \emptyset$.\(^{35}\) Any specification $T$ is entailment equivalent to its consequence $T \equiv T^\ast$. A specification $T$ is said to be closed when it is equal to its consequence $T = T^\ast$. An $\mathcal{S}$-specification $T$ is consistent when some $\mathcal{S}$-structure $\mathcal{M}$ satisfies $T$: $\mathcal{M} \models_{\mathcal{S}} T$ or $\bot_{\mathcal{S}} \ll_{\mathcal{S}} \mathcal{M}^{\mathcal{S}} \ll_{\mathcal{S}} T$. It is inconsistent otherwise. Hence, an $\mathcal{S}$-specification $T$ is inconsistent when $T^\ast = \text{Cons}(\mathcal{S}) = \bot_{\mathcal{S}}$.

\(^{34}\)In particular, the conceptual intent entails a constraint iff it satisfies the constraint: $\mathcal{M}^{\mathcal{S}} \vdash_{\mathcal{S}} \langle \phi \hookrightarrow \phi' \rangle$ iff $\mathcal{M} \models_{\mathcal{S}} \langle \phi \hookrightarrow \phi' \rangle$.

\(^{35}\)The nodes (formulas) in a specification can be identified with identity constraints. Identity constraints added to or subtracted from a specification give an equivalent specification. Hence, we can assume the node-set of formulas of a specification is included in the edge-set of constraints of a specification. With that assumption, boolean operations need only work on the edge-set of a specification.
3.1.3 Specification Flow.

**Specification Flow.** Specifications can be moved along schema morphisms. Given \( S_2 \xrightarrow{(r,f)} S_1 \), direct flow is the direct image operator

\[
\overline{\text{spec}}_{(r,f)} = \varphi_{\text{cons}_{(r,f)}} : \text{Spec}(S_2) \rightarrow \text{Spec}(S_1)
\]

and inverse flow is the inverse image operator (with consequence)

\[
\hat{\text{spec}}_{(r,f)} = \text{cons}_{(r,f)}^{-1} : \text{Spec}(S_1) \leftarrow \text{Spec}(S_2)
\]

along the constraint passage \( \text{Cons}(S_2) \xrightarrow{\text{cons}_{(r,f)}} \text{Cons}(S_1) \). Properties satisfied:

- direct images are closed,
- direct image commutes with consequence,
- direct image is monotonic,

\[
\text{spec}_{(r,f)}(\text{spec}(T_1)) \supseteq \text{spec}(\text{spec}_{(r,f)}(T_1));
\]

\[
\text{spec}_{(r,f)}(\text{spec}(T_2)) \supseteq \text{spec}(\text{spec}_{(r,f)}(T_2));
\]

\[
\text{spec}_{(r,f)}(\text{spec}(T_2)) \supseteq \text{spec}(\text{spec}_{(r,f)}(T_2)).
\]

These are adjoint monotonic functions w.r.t. specification order:

\[
\overline{\text{spec}}_{(r,f)}(T_2) \geq_{s_1} T_1 \text{ iff } T_2 \geq_{s_2} \hat{\text{spec}}_{(r,f)}(T_1)
\]

so that direct image \( \overline{\text{spec}}_{(r,f)} \) preserves all lattice meets \( \bigwedge = \bigcup \) and inverse image \( \hat{\text{spec}}_{(r,f)} \) preserves all lattice joins \( \bigvee = \bigcap \).

The constraint passage \( \text{Cons}(S_2) \xrightarrow{\text{cons}_{(r,f)}} \text{Cons}(S_1) \) is a closure operator morphism, since direct image commutes with consequence: \( \overline{\text{spec}}_{(r,f)}(\text{spec}(T_2)) \subseteq \text{spec}_{(r,f)}(\text{spec}(T_2)) \) for any \( S_2 \)-specification \( T_2 \). Morphisms of closure operators can be alternatively described as morphisms of entailment relations (Mossakowski, Diaconescu and Tarlecki [18]). The constraint passage \( \text{Cons}(S_2) \xrightarrow{\text{cons}_{(r,f)}} \text{Cons}(S_1) \)

forms a morphism of entailment relations, since \( T_2 \vdash_{s_2} (\varphi'_2 \xrightarrow{h} \varphi_2) \) implies \( \overline{\text{spec}}_{(r,f)}(\text{spec}(T_2)) \vdash_{s_1} \text{cons}_{(r,f)}(\varphi'_2 \xrightarrow{h} \varphi_2) \) for any \( S_2 \)-specification \( T_2 \) and any \( S_2 \)-constraint \( (\varphi'_2 \xrightarrow{h} \varphi_2) \) by direct image monotonicity.

**Specification Morphisms.** A specification morphism \( \langle S_2, T_2 \rangle \xrightarrow{(r,f)} \langle S_1, T_1 \rangle \) is a schema morphism \( \xrightarrow{(r,f)} \) that preserves entailment:

\[
T_2 \vdash_{s_2} (\varphi'_2 \xrightarrow{h} \varphi_2) \text{ implies } T_1 \vdash_{s_1} \text{cons}_{(r,f)}(\varphi'_2 \xrightarrow{h} \varphi_2)
\]

for any \( S_2 \)-constraint \( (\varphi'_2 \xrightarrow{h} \varphi_2) \); or more concisely,

\[
\overline{\text{spec}}_{(r,f)}(\text{spec}(T_2)) \subseteq T_1 \text{ iff } \overline{\text{spec}}_{(r,f)}(\text{spec}(T_2)) \subseteq T_1.
\]

Equivalently, that maps the source specification to a generalization of the target specification \( \overline{\text{spec}}_{(r,f)}(\text{spec}(T_2)) \geq_{s_1} T_1 \) or that maps the target specification to a specialization of the source specification \( T_2 \geq_{s_2} \overline{\text{spec}}_{(r,f)}(\text{spec}(T_2)) \).

The fibered mathematical context of specifications \( \text{Spec} \) has specifications as objects and specification morphisms as morphisms. Thus, the fibered context
of specifications $\text{Spec}$ is defined in terms of formal information flow. There is an underlying schema passage $\langle S, T \rangle \mapsto S$ from specifications to schemas $\text{Spec} \xrightarrow{\text{sch}} \text{Sch}$. (Fig. 2)

**Extending Order.** We regard a schema morphism $S_2 \xrightarrow{(r,f)} S_1$ to be a translation device: any $S_2$-formula $\varphi$ with signature $\langle I, s \rangle$ is translated to an $S_1$-formula $\hat{r}(\varphi)$ with signature $\Sigma_f(I, s)$. This notion of a translation device is embodied in the direct image operator $\text{Spec}(S_2) \xrightarrow{\text{spec}(r,f)} \text{spec}(S_1)$. A specification morphism $T_2 = \langle S_2, T_2 \rangle \xrightarrow{(r,f)} \langle S_1, T_1 \rangle = T_2$ from source specification $T_2 \in \text{Spec}(S_2)$ to target specification $T_1 \in \text{Spec}(S_1)$ is a schema morphism $S_2 \xrightarrow{(r,f)} S_1$ that translates the source specification to a generalization of the target specification $\text{spec}(r,f)(T_2) \geq S_1 T_1$. Thus, we interpret a specification morphism as a link between two specifications, where the source specification is more general than the target specification, symbolized as $T_2 \succeq T_1$. In this way, we interpret the context of specifications $\text{Spec}$ to be an extension of $\text{fbr}(S) = \text{Spec}(S)^{op}$, the opposite of the specification order at some schema $S$. This idea is used in later papers (mentioned in §1) to motivate the extension of information systems along indexing passages, which is an integral component in the definition of system morphisms and in the extension of the ideas of conservative extension and modularity to the level of systems.

**Lemma 8.** A structure morphism $M_2 \xrightarrow{(r,k,f,g)} M_1$ with schema morphism $S_2 \xrightarrow{(r,f)} S_1$ determines an intent specification morphism $\langle S_2, M_2 \rangle \xrightarrow{(r,f)} \langle S_1, M_1 \rangle$, which is a restriction of the constraint passage $\text{Cons}(S_2) \xrightarrow{\text{cons}(r,f)} \text{Cons}(S_1)$:

\[(\varphi'_2 \xrightarrow{b} \varphi_2) \mapsto (\hat{r}(\varphi'_2) \xrightarrow{b} \hat{r}(\varphi_2))\] (compare this to Cor. 5).

**Proof.** The structure morphism $M_2 \xrightarrow{(r,k,f,g)} M_1$ factors as

\[M_2 \xrightarrow{(k,g)} \text{struc}_1^{(r,f)}(M_1) \xrightarrow{(r,f)} M_1.\]

Hence,

\[M_2 \xrightarrow{\text{spec}(r,f)} \text{spec}(M_1) \xrightarrow{\text{spec}(r,f)} M_1.\]

\[\text{spec}(M_1) \xrightarrow{(r,f)} M_1.\]

**Definition 5.** There is a conceptual intent passage

\[\text{struc}_1^{int} \xrightarrow{\text{struc}_1^{int}} \text{Cxt} \]

---

36 Proven in the appendix of Kent [12].

37 Compare this to the conceptual intent passage $\text{Struc}^{int}, \text{Cxt}$ in Def. 4.
\[
\text{Struc} \xrightarrow{\text{int}} \text{Spec} : M \mapsto (S, M^S)
\]

(Fig. 2)

from structures to specifications, which respects schema \( \text{int} \circ \text{sch} = \text{sch} \).

\[38\] This defines the lower part of the FOLE superstructure (Fig. 2).
3.1.4 Legacy Notions.

Conservative Extensions. In mathematical logic, theory $T_1$ is a conservative extension of theory $T_2$ when (1) the language of $T_1$ extends the language of $T_2$, (2) every theorem of $T_2$ is a theorem of $T_1$, and (3) any theorem of $T_1$ that is in the language of $T_2$ is already a theorem of $T_2$. We translate this definition to FOLE morphisms. Given (1) a schema morphism $S_2 \Rightarrow S_1$ extending the language of $S_2$ to the language of $S_1$, a target specification $T_1 \in \text{Spec}(S_1)$ is a conservative extension of a source specification $T_2 \in \text{Spec}(S_2)$ when the following conditions hold: (2) the image of any constraint entailed by $T_2$ is entailed by $T_1$ with $\overline{\text{spec}}_{\sigma}(T_1) \geq S_1$, so that $T_2 \Rightarrow T_1$ is a specification morphism; and (3) any constraint whose image is entailed by $T_1$ is already entailed by $T_2$ with $T_1 \leq S_2 \Rightarrow \overline{\text{spec}}_{\sigma}(T_1)$, so that $T_2$ is closed along $S_2 \Rightarrow S_1$ with $T_1^\sigma \equiv S_2 T_2$. \(^{39}\)

Conservative extensions are closed under composition.

Consistency. A conservative extension of a consistent specification is consistent, since the inverse image operator preserves concept lattice joins, mapping the empty join to the empty join: $T_1 \equiv S_1 \perp S_1$ implies $T_2 \equiv S_2 \overline{\text{spec}}_{\sigma}(T_1) \equiv S_2 \perp S_2$.

In contrast, the specification component of any structure morphism preserves soundness (specific consistency).

Corollary 6. Structure morphisms preserve soundness (specific consistency) (§3.3): for any structure morphism $M_2 \xrightarrow{(r,f)} M_1$ with schema morphism $S_2 \Rightarrow S_1$, if $S_2$-structure $M_2$ satisfies $T_2$, then $S_1$-structure $M_1$ satisfies $\overline{\text{spec}}_{(r,f)}(T_2)$.

Proof.

$$S_2 \xrightarrow{(r,f)} S_1 \quad M_2 \overset{\text{sound}}{\leq}_{S_2} T_2 \xrightarrow{\text{spec}_{(r,f)}} M_1 \overset{\text{mono}}{\leq}_{S_1} \overline{\text{spec}}_{(r,f)}(M_2^S_2) \overset{\text{mono}}{\leq}_{S_1} \overline{\text{spec}}_{(r,f)}(T_2)$$

Corollary 7. Reductions reflect consistency: if $M_2 = \text{struc}^S_{(r,f)}(M_1)$ is the reduct of $M_1$ with underlying schema morphism $S_2 \Rightarrow S_1$ and $S_1$-structure $M_1$ satisfies $T_1$, then $S_2$-structure $M_2$ satisfies $\overline{\text{spec}}_{(r,f)}(T_1)$.

Proof.

$$M_2 \overset{\text{reduct}}{\leq} \text{struc}^S_{(r,f)}(M_1) \overset{\text{Eqn.}}{=} \overline{\text{spec}}_{(r,f)}(M_1^S_1) \overset{\text{mono}}{\leq}_{S_2} \overline{\text{spec}}_{(r,f)}(T_1).$$

\(^{39}\) Morphic closure is defined at the system level in Kent [9].
3.2 Logics

3.2.1 Logics. A logic $L = (S, M, T)$ consists of a structure $M$ and a specification $T = (S, T)$ that share a common schema $S$. For any fixed structure $M$, the set of all logics $\text{Log}(M)$ with that structure is a preordered set under the specification order: $\langle S, M, T_1 \rangle \leq_M \langle S, M, T_2 \rangle$ when $T_1 \leq_S T_2$. Note that $\bot_M = (S, M, \text{Cons}(S)) \leq_M L \leq_M \langle S, M, \emptyset \rangle = \top_M$ for any logic $L = (S, M, T)$, since $\text{Cons}(S) = \text{Cons}(S)^* \ni \emptyset \ni \emptyset$. For any structure $M$ with underlying schema $\text{sch}(M) = S$, the logic order over $M$ is isomorphic to the specification order over $S$, $\text{Log}(M) \cong \text{Spec}(S)$.

3.2.2 Logic Flow. The semantic molecules (logics) can be moved along structure morphisms. For any structure morphism $M_2 \langle r, k, f, g \rangle \leftrightarrow M_1$ with underlying schema morphism $S_2 \langle r, f \rangle \leftrightarrow S_1$, define the direct/inverse flow operators

$$\text{Log}(M_2) \xrightarrow{\text{log}(r,k,f,g)} \text{Log}(M_1) : (S_2, M_2, T_2) \mapsto (S_1, M_1, \text{spec}_{(r,f)}(T_2))$$

$$\text{Log}(M_2) \xleftarrow{\text{log}(r,k,f,g)} \text{Log}(M_1) : (S_2, M_2, \text{spec}_{(r,f)}(T_1)) \mapsto (S_1, M_1, T_1)$$

These are adjoint monotonic functions w.r.t. logic order:

$$\text{log}(r,k,f,g)(L_2) \geq_{M_1} L_1 \text{ if and only if } L_2 \geq_{M_2} \text{log}(r,k,f,g)(L_1)$$

for all target logics $L_1$ and source logics $L_2$.

Logic Morphisms. A logic morphism $L_2 = (S_2, M_2, T_2) \xrightarrow{(r,k,f,g)} (S_1, M_1, T_1) = L_1$ consists of a structure morphism $M_2 \xrightarrow{(r,k,f,g)} M_1$ and a specification morphism $T_2 = (S_2, T_2) \xrightarrow{(r,f)} (S_1, T_1) = T_1$ that share a common schema morphism $S_2 \xrightarrow{(r,f)} S_1$. A logic morphism $L_2 \xrightarrow{(r,k,f,g)} L_1$ is a structure morphism...
\( \mathcal{M}_2 \xleftrightarrow{(r,k,f,g)} \mathcal{M}_1 \) that maps the source logic to a generalization of the target logic \( \log_{(r,k,f,g)}(L_2) \geq_{\mathcal{M}_1} L_1 \), or equivalently, that maps the target logic to a specialization of the source logic \( L_2 \geq_{\mathcal{M}_2} \log_{(r,k,f,g)}(L_1) \).

The context of logics \( \text{Log} \) has logics as objects and logic morphisms as morphisms. It is the fibered product

\[
\text{Struct} \xleftarrow{\text{struc}} \text{Log} \xrightarrow{\text{spec}} \text{Spec}, \quad (\text{Fig. 2})
\]

of the contexts of structures and specifications \( \text{Struct} \xrightarrow{\text{sch}} \text{Sch} \xleftarrow{\text{sch}} \text{Spec} \). The projective passages satisfy the condition \( \text{struc} \circ \text{sch} = \text{spec} \circ \text{sch} \). 40

### 3.2.3 Legacy Notions.

**Conservative Extensions** Given a structure morphism \( \mathcal{M}_2 \xleftrightarrow{(r,k,f,g)} \mathcal{M}_1 \), a target logic \( L_1 \in \text{Log}(\mathcal{M}_1) \) is a conservative extension of a source logic \( L_2 \in \text{Log}(\mathcal{M}_2) \) when the following conditions hold: (1) the image of any constraint entailed by \( L_2 \) is entailed by \( L_1 \) with \( \log_{(r,k,f,g)}(L_2) \geq_{\mathcal{M}_1} L_1 \), so that \( L_2 \xrightarrow{(r,k,f,g)} L_1 \) is a logic morphism; and (2) any constraint whose image is entailed by \( L_1 \) is already entailed by \( L_2 \) with \( L_2 \leq_{\mathcal{M}_2} \log_{(r,k,f,g)}(L_1) \), so that \( L_2 \) is closed along \( \mathcal{M}_2 \xleftrightarrow{(r,k,f,g)} \mathcal{M}_1 \) with \( L_2^\bullet \xrightarrow{(r,k,f,g)} \equiv_{\mathcal{M}_2} L_2 \). Hence, logic \( L_1 \in \text{Log}(\mathcal{M}_1) \) is a conservative extension of logic \( L_2 \in \text{Log}(\mathcal{M}_2) \) along structure morphism \( \mathcal{M}_2 \xleftrightarrow{(r,k,f,g)} \mathcal{M}_1 \) when the underlying specifications satisfy this property along the underlying schema morphisms. Conservative extensions are closed under composition.

---

40 This defines the upper-right part of the FOLE superstructure (Fig. 2).
3.3 Sound Logics.

A logic $\mathcal{L} = \langle S, M, T \rangle$ is sound when the component structure $M$ satisfies the component specification $T$: $M \models S T$ or $M^S \leq_S T$. Associated with any $S$-structure $M$ is the natural logic $\text{nat}(M) = \langle S, M, M^S \rangle$, whose specification is the conceptual intent of $M$. The natural logic is the least sound logic: $\text{nat}(M) \leq_M \mathcal{L}$ for any sound logic $\mathcal{L} = \langle S, M, T \rangle$, since soundness means $M^S \leq_S T$ (Fig. 1). Any structure morphism $\mathcal{M}_2 \xrightarrow{(r,k,f,g)} \mathcal{M}_1$ induces the natural logic morphism $\text{nat}(\mathcal{M}) \xrightarrow{(r,k,f,g)} \text{nat}(\mathcal{M})$, since $\langle S_2, M_2^S \rangle \xrightarrow{(r,f)} \langle S_1, M_1^S \rangle$ is a specification morphism (Lem. 8). Hence, there is a natural logic passage

$$\text{Struc} \xrightarrow{\text{nat}} \text{Snd}$$

(Fig. 2)

to the subcontext $\text{Snd} \xleftarrow{\text{inc}} \text{Log}$ of sound logics. Structures form a reflective subcontext of sound logics, since the pair $(\text{struc, nat}) : \text{Snd} \xrightarrow{\text{nat}} \text{Struc}$ forms an adjunction $\mathcal{L} \geq \text{nat}(\text{struc}(\mathcal{L}))$ and $\text{nat} \circ \text{struc} = 1_{\text{Struc}}$.  

3.3.1 Residuation. Associated with any logic $\mathcal{L} = \langle S, M, T \rangle$ is its restriction $\text{res}_M(\mathcal{L}) = \mathcal{L} \vee_M \text{nat}(\mathcal{M}) = \langle S, M, M^S \vee_S T^* \rangle = \langle S, M, M^S \cap T^* \rangle$, which is the conceptual join in $\text{Log}(\mathcal{M})$ (Fig. 1) of the logic with the natural logic of its structure component. Clearly, the restriction is a sound logic and $\text{res}_M(\mathcal{L}) \geq_S \mathcal{L}$. There is a restriction passage

$$\text{Log} \xrightarrow{\text{res}} \text{Snd},$$

(Fig. 2)

which maps a logic $\mathcal{L}$ to the sound logic $\text{res}(\mathcal{L})$ and maps a logic morphism $\mathcal{L}_2 = \langle S_2, M_2, T_2 \rangle \xrightarrow{(r,k,f,g)} \langle S_2, M_2, T_2 \rangle = \mathcal{L}_2$ to the morphism of sound logics

$$\text{res}(\mathcal{L}_2) = \langle S_2, M_2, M_2^S \vee_M T_2^* \rangle \xrightarrow{(r,k,f,g)} \langle S_1, M_1, M_1^S \vee_M T_1^* \rangle = \text{res}(\mathcal{L}_1)$$

This is well-defined, since it just couples the specification morphism conditions for the theories of $\mathcal{L}$ and $\text{nat}(\mathcal{M})$

$$M_2^S \geq_S \llbracket \text{spec}_{(r,f)}(M_1^S) \rrbracket \text{ and } T_1^* \geq_S \llbracket \text{spec}_{(r,f)}(T_1^*) \rrbracket \text{ implies } M_2^S \vee_S T_2^* \geq_S \llbracket \text{spec}_{(r,f)}(M_1^S) \vee_S \text{spec}_{(r,f)}(T_1^*) \rrbracket = \llbracket \text{spec}_{(r,f)}(M_1^S \vee_S T_1^*) \rrbracket.$$

The context of sound logics forms a coreflective subcontext of the context of logics, since the pair $(\text{inc, res}) : \text{Log} \xrightarrow{\text{res}} \text{Snd}$ forms an adjunction with $\text{inc} \circ \text{res} \cong 1_{\text{Snd}}$ and $\text{inc}(\text{res}(\mathcal{L})) \geq_M \mathcal{L}$ for any logic $\mathcal{L}$. For any structure $M$, restriction and inclusion on fibers are adjoint monotonic functions $\langle \text{res}_M, \text{inc}_M \rangle : \text{Log}(\mathcal{M}) \rightarrow \text{Snd}(\mathcal{M})$, where $\text{Log}(\mathcal{M})$ is the opposite fiber of logics over $\mathcal{M}$ and $\text{Snd}(\mathcal{M})$ is the opposite fiber of sound logics (Fig. 1).

$\text{inc}$: An adjunction (generalized pair) of passages; that is, a pair of oppositely-directed passages that satisfy inverse equations up to morphism. Any “canonical construction from one species of structure to another” is represented by an adjunction between corresponding categories of the two species (Goguen [6]).

$\text{nat}$: This defines half of the upper-left part of the FOLE superstructure (Fig. 2).
3.3.2 Sound Logic Flow. The movement of sound logics is a modification of logic flow. Direct flow preserves soundness (Cor. 6). Hence, there is no change. Augment inverse flow by restricting to sound logics, via residuation, by joining with structure-intent. For any structure morphism $\mathcal{M}_2 \xrightarrow{(r,k,f,g)} \mathcal{M}_1$ with underlying schema-morphism $\mathcal{S}_2 \xrightarrow{(r,f)} \mathcal{S}_1$, define the direct/inverse flow operators

$$\xrightarrow{\text{Snd}} (r,k,f,g) : \text{Log}(\mathcal{L}_2) \xrightarrow{\text{Log}} \text{Snd}(\mathcal{M}_2) : \langle \langle \mathcal{S}_2, \mathcal{M}_2, T_2 \rangle \rangle \mapsto \langle \langle \mathcal{S}_1, \mathcal{M}_1, \text{spec}(r,f) \rangle \rangle(T_2)$$

$$\text{Snd}(\mathcal{M}_2) : \langle \langle \mathcal{S}_2, \mathcal{M}_2, \mathcal{S}_2^\mathcal{M}_2 \vee \mathcal{M}_2 \text{spec}(r,f) \rangle \rangle \xleftarrow{\text{Res}_{\mathcal{M}_2}} \langle \langle \mathcal{S}_1, \mathcal{M}_1, \mathcal{T}_1 \rangle \rangle$$

These are adjoint monotonic functions w.r.t. sound logic order:

$$\text{Snd}(\mathcal{M}_2) : \langle \langle \mathcal{S}_2, \mathcal{M}_2, \mathcal{S}_2^\mathcal{M}_2 \vee \mathcal{M}_2 \text{spec}(r,f) \rangle \rangle \xleftarrow{\text{Res}_{\mathcal{M}_2}} \langle \langle \mathcal{S}_1, \mathcal{M}_1, \mathcal{T}_1 \rangle \rangle$$

for all sound target logics $\mathcal{L}_1$ and sound source logics $\mathcal{L}_2$.

**Corollary 8.** Any logic morphism $\mathcal{L}_2 \xrightarrow{(r,k,f,g)} \mathcal{L}_2$ between sound logics satisfies sound logic flow adjointness: Eqn. 9 implies Eqn. 10.

**Proof.** $\text{Snd}(\mathcal{M}_2) : \langle \langle \mathcal{S}_2, \mathcal{M}_2, \mathcal{S}_2^\mathcal{M}_2 \vee \mathcal{M}_2 \text{spec}(r,f) \rangle \rangle \xleftarrow{\text{Res}_{\mathcal{M}_2}} \langle \langle \mathcal{S}_1, \mathcal{M}_1, \mathcal{T}_1 \rangle \rangle$.

**Corollary 9.** For any structure morphism $\mathcal{M}_2 \xrightarrow{(r,k,f,g)} \mathcal{M}_1$, the restriction-inclusion adjunction on fibers is compatible with the inverse-direct flow adjunction. This means that the following composite adjoint pairs are equal:

$$\langle \text{Res}_{\mathcal{M}_1}, \text{Inc}_{\mathcal{M}_1} \rangle \cdot \langle \text{Snd}(r,k,f,g), \text{Snd}(r,k,f,g) \rangle = \langle \text{Log}(r,k,f,g), \text{Log}(r,k,f,g) \rangle \cdot \langle \text{Res}_{\mathcal{M}_2}, \text{Inc}_{\mathcal{M}_2} \rangle$$

**Proof.** We show $\text{Res}_{\mathcal{M}_1} : \langle \text{Inc}_{\mathcal{M}_1} \rangle \cdot \langle \text{Snd}(r,k,f,g) \rangle = \langle \text{Log}(r,k,f,g) \rangle \cdot \text{Res}_{\mathcal{M}_2}$.

Since $\mathcal{M}_2^\mathcal{S}_2 \geq \mathcal{M}_2^\mathcal{S}_1 \xrightarrow{\text{Spec}(r,f)} \mathcal{M}_1^\mathcal{S}_1$, we have $\text{Snd}(r,k,f,g)(\text{Res}_{\mathcal{M}_1}(\mathcal{L}_1))$

$$= \langle \langle \mathcal{S}_2, \mathcal{M}_2, \mathcal{S}_2^\mathcal{M}_2 \vee \mathcal{M}_2 \text{Spec}(r,f) \rangle \rangle \langle \mathcal{M}_1, \mathcal{T}_1 \rangle \rangle \mathcal{M}_2^\mathcal{S}_2 \vee \mathcal{M}_2 \text{Spec}(r,f) \rangle \rangle \mathcal{M}_2^\mathcal{S}_1 \rangle \rangle \mathcal{M}_2 \langle \langle \langle \mathcal{S}_2, \mathcal{M}_2, \mathcal{S}_2^\mathcal{M}_2 \vee \mathcal{M}_2 \text{Spec}(r,f) \rangle \rangle \mathcal{M}_1^\mathcal{S}_1 \rangle \rangle \mathcal{T}_1 \rangle \rangle$$

$$\text{Res}_{\mathcal{M}_1}(\mathcal{L}_1)) = \langle \langle \mathcal{S}_2, \mathcal{M}_2, \mathcal{S}_2^\mathcal{M}_2 \vee \mathcal{M}_2 \text{Spec}(r,f) \rangle \rangle \mathcal{M}_2 \langle \langle \langle \mathcal{S}_2, \mathcal{M}_2, \mathcal{S}_2^\mathcal{M}_2 \vee \mathcal{M}_2 \text{Spec}(r,f) \rangle \rangle \mathcal{M}_1^\mathcal{S}_1 \rangle \rangle \mathcal{T}_1 \rangle \rangle.$$
\begin{align*}
\text{inc} \circ \text{res} &\cong I_{\text{Snd}} \\
\text{inc}(\text{res}(L)) \geq_M L &\quad \text{nat} \circ \text{struc} = I_{\text{Struc}} \\
L \geq_M \text{nat}(\text{struc}(L)) &\quad \text{int} \circ \text{sch} = \text{sch}
\end{align*}

Fig. 2. FOLE Superstructure
4 Conclusion and Future Work

The work in this paper consisted of two parts: development of the FOLE logical environment and presentation of the FOLE superstructure. The development of the FOLE logical environment centered on the satisfaction relation between the polar opposition (formalism–semantics). At the upper pole, we defined the formalism of formulas, sequents and constraints; the latter two allow us to specify ontological hierarchies. At the lower pole, we developed semantics through the interpretation and classification of formulas; here we defined the valuable concept of comprehension. Bridging the poles is the satisfaction relation between a structure and a formalism (sequent or constraint). Finally, to finish the work on the FOLE logical environment, we expressed FOLE as an institution; and more particularly, as a logical environment.

The presentation of the FOLE superstructure involved the mathematical contexts, passages and adjunctions illustrated in the FOLE architectural diagram (Fig. 2). This diagram consists of four components: structures, specifications, logics and sound logics. Structures, which represent the semantic aspect of FOLE, were handled in the FOLE foundation paper [12]. In this paper, we present the remaining architectural components: specifications, logics and sound logics. Specifications represent the formal aspect of FOLE; here, we define the notions of entailment, consequence and flow of formalism. Logics combine the formal and semantic aspects of FOLE. Logics are sound when semantics satisfies formalism.

As outlined in the introduction §1, this paper is one of a series of papers that provide a rigorous mathematical representation for ontologies within the first-order logical environment FOLE. The FOLE representation can be expressed in two forms: a classification form and interpretative form. The foundation paper [12] and the current superstructure paper develop the classification form of FOLE. The paper [14] and the paper [15] develop the interpretative form of FOLE as a transformational passage from sound logics [11], thereby defining the formalism and semantics of first-order logical/relational database systems [10].

System interoperability, in the general setting of institutions and logical environments, was defined in the paper “System Consequence” (Kent [9]). This was inspired by the channel theory of information flow presented in the book Information Flow: The Logic of Distributed Systems (Barwise and Seligman [1]). Since FOLE is a logical environment (§2.3.3), in two further papers we apply this approach to interoperability for information systems based on first-order logic and relational databases: one paper discusses integration over a fixed type domain and the other paper discusses integration over a fixed universe.
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