Equilibrium policies when preferences are time inconsistent

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Abstract

This paper characterizes differentiable and subgame Markov perfect equilibria in a continuous time intertemporal decision problem with non-constant discounting. Capturing the idea of non commitment by letting the commitment period being infinitesimally small, we characterize the equilibrium strategies by a value function, which must satisfy a certain equation. The equilibrium equation is reminiscent of the classical Hamilton-Jacobi-Bellman equation of optimal control, but with a non-local term leading to differences in qualitative behavior. As an application, we formulate an overlapping generations Ramsey model where the government maximizes a utilitarian welfare function defined as the discounted sum of successive generations’ lifetime utilities. When the social discount rate is different from the private discount rate, the optimal command allocation is time inconsistent and we retain subgame perfection as a principle of intergenerational equity. Existence of multiple subgame perfect equilibria is established. The multiplicity is due to the successive governments’ inability to coordinate their beliefs and we single out one of them as (locally) renegotiation-proof. Decentralization can be achieved with both age and time dependent lump sum transfers and, long term distorting capital interest income taxes/subsidy.

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1 Introduction

Time inconsistency is present in many dynamic decision making problems. The primary objective of this paper is to develop a new approach to analyze a class of time inconsistent games where time inconsistency is due to non constant discount rates. Using this methodology, we characterize the equilibria of the game and, for a particular specification of the discount function, we establish their existence and report a new type of indeterminacy for this class of games. The framework is the standard deterministic and stationary Ramsey general equilibrium model of growth and capital accumulation (see [54]). Time inconsistency is due to the planner’s non-constant discount rates and there are several good reasons for which this may occur. First, when the planner is an individual, there is experimental evidence from psychology (e.g. Ainslie [1]) which challenges the assumption of constant discount rates. In particular, there is robust evidence that people can indulge in immediate gratification even if the delayed cost is high. This suggest a revealed discount rate which is declining over time which results in a hyperbolic discount function. Second, if the planner is a government, utilitarianism naturally leads to non constant discounting and time inconsistency in models with multiple generations. The source of time inconsistency is the inability of the forward looking governments to account for the preferences of the deceased cohorts. The idea is intuitively discussed in the context of natural ressources management by Sumaila and Walters [59] and, explained more formally in the optimal growth context in a model with altruistic and non-overlapping generations (Bernheim [8]) and, in a model with non altruistic and overlapping generations (Calvo and Obstfeld [14]).

Following the previous literature on the topic, Strotz [58], Pollak [53], Phelps and Pollak [52] and, Peleg and Yaari [50], we search for subgame perfect equilibria of a dynamic game where decision makers cannot bind their choices (non commitment) and are aware of their inconsistency problem. We restrict the planner to choose pure strategies that are continuously differentiable functions of contemporaneous capital stock, making the framework as close as possible to the standard optimal growth model. The notion of non-commitment is not easily defined in continuous time: there is no notion of “successor” because no matter how close in time two planners are, there is always a third planner who precedes one of them and succeeds the other. As

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1In particular the issue is of economic relevance in the monetary and fiscal policy design of a benevolent government (Kydland and Prescott [42], Calvo [13], Fischer [28] and, Chari and Kehoe [19]), the pricing of a durable good for a monopolist (Stokey [57]), the ownership policy for a large shareholder (DeMarzo and Urošević [22]), the long term environmental decision making (Chichilnisky [20] and, Li and Löfgren [47]) and, the consumption saving private decision under hyperbolic discount (Laibson [44]) as well as the Ramsey growth model when the social planner himself exhibits hyperbolic discount (Krusell, Kuruşçu and Smith [40]).

2A time inconsistency induced by the structure of the preferences is also present for capital accumulation models with non overlapping and altruistic generations when the social planner uses the maxi-min Rawlasian criterion (Dasgupta [21]). A similar situation also arises for consumption saving problems when the preferences over consumption streams are non additively separable (Kihlstrom [37]).

3See Gul and Pesendorfer [30] for an alternative interpretation of the Strotz’s model.
a result, the discrete time induction approach cannot be used to solve for the equilibrium and we have to adapt it to define the concept of equilibrium in continuous time. This is achieved with an intuitive construction: we assume that the planner at any point in time can control his immediate successors, thereby forming a coalition of a given small size that clearly separates the current planner from the more distant ones. A strategy is an equilibrium if it is the best strategy for the current planner when the coalition is vanishingly small, so long as the same strategy is expected to be used by the distant planners. The idea conceptually parallels Aumann’s continuum of consumers [6] and it simply reduces to the Barro’s [7] approach when linear equilibria exist. In the first part of this paper (Sections 2, 3 and 4) we provide a precise sense in which the game can be formulated and, a characterization of equilibrium policies. The characterization consists of an instantaneous saving-consumption indifference condition together with a non linear differential equation displaying a non local term (that is a term which depends on the global behavior of the solution). The non local term reflects the strategic motive of investing and, when the discount rate is constant it becomes a local term and the equation reduces to the familiar Hamilton-Jacobi-Bellman equation (HJB) of optimal control. The equation extends the Generalized Euler Equation of Harris and Laibson [32] concerning the case where the planner is facing a countable number of successors to a case where the planner is facing a continuum of successors. Notice that an alternative approach could have been adopted for our class of continuous time dynamic games. It is possible to consider first a discrete time game on a time grid, define the equilibrium by using induction methods on the grid, and consider the equilibrium policy resulting from letting the grid be vanishingly fine. This approach has been successfully adopted by Luttmer and Mariotti [48] for linear equilibria of consumption-saving games under uncertainty. Indeed the linear equilibria of Luttmer and Mariotti [48] are mutually consistent with ours in a deterministic version of their model. We did not pursue this path here because the approach that we undertook uses simple marginal calculations involving standard differential calculus techniques and does not require an extraneous convergence theorem. More importantly, it directly leads us to an equilibrium characterization that mirrors the HJB equation of optimal control which suggests that our method is the “adequate” notion of dynamic games in continuous time.

What appears to be new in this first part of the paper (Sections 2, 3 and 4) relative, for instance, to Barro [7] and, Luttmer and Mariotti [48] is the formal model of the game in continuous time and the resulting general characterization of equilibrium policies. The framework allows for non linear policies and the equilibrium characterization addresses directly the Pollak’s criticism of Strotz’s work (see section 4 of Pollak [53]). More importantly, we believe the general methodology that we develop here can be applied in a broad set of time inconsistent problems including the problem of optimal fiscal policy. To illustrate the usefulness of the methodology, we

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4See also Phelps [51] for an early discussion on the issue.
5Simon and Stinchcombe [56] also used the infinitesimal grid approach to define a more general class of games and history dependent strategies in continuous time.
6Given the convergence result of Luttmer and Mariotti [48] for discrete time linear equilibria to our solution, there is a suspicion that the convergence holds for non linear equilibria as well.
develop an application in the context of the Blanchard [12] model of economic growth with finite life (the perpetual youth model). The Blanchard model is a continuous time version of the overlapping-generations models of Samuelson [55] and Diamond [23] offering more tractability for the aggregation of individual variables. We explore the dynamic allocation problem of a utilitarian government (the planner) maximizing the discounted sum of the surviving and unborn generations’ lifetime utilities. As we mentioned earlier, implementation of utilitarian welfare optima in overlapping generation models is problematic. Except in few special cases, utilitarianism renders the planner’s objective time inconsistent. Time consistency is achieved for example when the planner’s discount rate is equal to the individual discount rate or if the planner’s discount function satisfies some unnatural assumptions (Calvo and Obstfeld [14]). The purpose of our application is to analyze the centrally planned allocation when the social discount function is aligned with the way in which economists usually model utilitarianism that is, an exponential forward looking discount function with a constant rate of growth. The optimal command is then time inconsistent and the planner is forced to approach the allocation decision as a strategic problem. Our principle of intergenerational equity is then anchored in the concept of subgame perfection of the extensive form game of successive planners. This approach has already been taken for Rawlsian welfare (see Dasgupta [21], Lane and Mitra [46] and, Asheim [3]). To our knowledge, our application is the first to address this preference based time inconsistency friction in an overlapping generations economy with a utilitarian government.

We now summarize the main finding from our application (section 5) that we carry out in two steps.

First, we describe how the centrally planned economy operates. The government has to solve two questions: how to allocate consumption across the surviving cohorts and how to allocate aggregate consumption over time. The inspection of this dual decision shows that when the allocation across surviving generations is restricted to be stationary and linear in aggregate consumption, the problem of allocating the aggregate consumption over time is in fact isomorphic to an infinitively lived representative agent growth model with non constant discount of the type analyzed in the first part of the paper. The social discount function takes the form of a mixture of two exponential functions. Due to this special structure we can reduce the equilibrium characterization from the first part of the paper to a system of two coupled ordinary differential equations which do not involve non local terms. Taking advantage of this special structure and using the central manifold theorem (Carr [17]), we then prove the existence of multiple equilibria. The equilibrium policies are continuous and differentiable in the capital stock and they inherit the smoothness (in a sense to be made precise) from the underlying production function. Equilibrium multiplicity results in a continuum of possible steady state level of capital stock within the range of an open and bounded interval. When the planner’s discount rate is equal to the individual discount rate, the interval shrinks to one point, and the capital stock converges to its modified golden rule level. The driving factor for multiplicity is the governments’ inability to coordinate their expectations on any policy. When adapted to our continuous time framework, renegotiation-proofness is restrictive as in Kocherlakota
Under this refinement, all successive governments use in agreement, amongst all equilibrium policies, the one that induces the capital stock to converge to the highest steady state level. The renegotiation-proof steady state level of capital stock depends on the government discount rate, the private discount rate and the individuals’ life expectancy. When the government discount rate converges to 0, the capital stock path resulting from the renegotiation-proof equilibrium converges to its golden rule level.

Second, we discuss how to find a tax schedule that, in a market with actuarially fair annuities (Blanchard [12]) places the economy on the desired disaggregate path of accumulation. If the social discount rate is equal to the private discount rate, distortionary taxation is not required by the second welfare theorem. If, as we suppose in our application the government is more patient than the individuals, the time inconsistency problem creates a wedge between the cost and benefits of saving at the individual level. Under this assumption, the laissez-faire economy cannot decentralize the allocation and we show that such decentralization can be achieved if the government uses a date and age conditioned lump sum taxation and distorting capital income taxation. We provide a closed-form expression for the required path of capital income tax rates that shows that its long term level can be positive or negative. The result is clearest in one specification of the model where the the consumption allocation across the surviving cohorts is egalitarian. Under this specification, as in the infinitely lived identical agents economy, there is no heterogeneity in consumption at any point in time. Yet the government finds it optimal to subsidize the capital income with an age independent rate, including in the steady state. The motive for the subsidy is to give the savings incentives to the private agents who, from the point of view of the government, are not saving sufficiently. The qualitative conclusion from our application is that the preference based time inconsistency friction faced by a utilitarian government creates in its own a role for capital taxation in the long term. The result is in stark contrast with the benchmark infinitely lived agents economy where capital income taxation should be zero in the long term (Judd [35] and Chamley [18]).

Related Literature. Before turning to the model, we summarize how our work relates to the literature on which it builds. Our main result is the existence of multiple equilibria in a dynamic game. We introduce a definition of continuous time games and we prove existence by using a new approach, based on the central manifold theorem (Carr [17]), in a deterministic Ramsey growth model. Our paper naturally relates to the game theoretic literature on Markov-consistent plan (MCP). An MCP is a sub-game perfect equilibrium of the extensive form game between the successive planners in which the strategy is pure and only depends on the payoff relevant variables (capital stock). the MCPs do not always exist in time inconsistent decision making problems. Peleg and Yaari [50] constructed a simple finite horizon intra-personal game where MCPs do not exist. Linear MCPs have however been reported in the hyperbolic

[58].

See Caplin and Leahy [15] for a non paternalistic argument supporting the idea that social planner can be more patient than private individuals.

However, more general history dependent equilibrium strategies still exist as Goldman [29] showed in a finite horizon setting and Harris [31] showed in an infinite horizon setting.
discount literature\textsuperscript{9} in a frictionless consumption saving problems with homothetic time additive utilities and linear production functions. In a deterministic setting, linear MCPs have been reported in Laibson \textsuperscript{43} in a discrete time consumption saving problem as well as in Barro \textsuperscript{7} in the context of an infinite horizon continuous time decentralized version of the Ramsey growth model. Luttmer and Mariotti \textsuperscript{48} also reported linear MCPs in an infinite horizon endowment asset pricing model with uncertainty. The aggregate temporal allocation problem of our overlapping generations growth model may be interpreted as a consumption saving problem with hyperbolic discount and a non linear state dynamic for wealth. With this interpretation, our application can be seen as an extension of Laibson \textsuperscript{43}, Barro \textsuperscript{7} and Luttmer and Mariotti \textsuperscript{48} to non linear MCPs. Our results suggest then that the observational equivalence results implied by the existence of linear MCPs, does not hold when the technology is non linear. In the presence of non linear technology, the MCP dynamic path emerging from our analysis are not possible to reproduce with constant discount.

Non observational equivalence is also present in the infinite horizon buffer stock models with income uncertainty, borrowing constraints and hyperbolic discounting in discrete time (Harris and Laibson \textsuperscript{32}) and continuous time (Harris and Laibson \textsuperscript{34}). These two papers show existence and uniqueness of smooth MCPs when the discount function is sufficiently close to the exponential discount\textsuperscript{10}. Our results do not overlap with theirs since we allow for non linear technology and we do not have a borrowing constraint. Methodologically, our proof of existence is different and does not require the presence of uncertainty or the discount function to be sufficiently close to the exponential function. Our results is also different because multiplicity emerges as a central feature of the equilibrium in our context.

Another type of MCPs existence results is found in the literature on altruistic generations growth economies. In this context, existence of MCP has been established for a finite horizon setting (Bernheim and Rey \textsuperscript{9}) and for an infinite horizon setting (Bernheim and Rey \textsuperscript{11}) (see also Caplin and Leahy \textsuperscript{16} for a recent related MCP’s existence result under uncertainty). In contrast to our approach, the existence result of those papers hinges critically on introducing production uncertainty.

Our paper also relates to the literature on the Ramsey growth model when the central planner himself displays hyperbolic discount. Krusell, Kuru¸s¸cu and Smith \textsuperscript{40} undertook an elegant comparison of a decentralized and a centralized version of the Ramsey model with quasi-geometric discount, in discrete time\textsuperscript{11}. The analysis in Krusell, Kuru¸s¸cu and Smith \textsuperscript{40} does not cover the fundamental issue of multiple

\textsuperscript{9}This literature shows that apparent irrationality of individuals, even in financial markets, can be ascribed to the fact that the psychological discount factor is not exponential; see Laibson \textsuperscript{45}, Harris and Laibson \textsuperscript{33}, Diamond and Koszegi \textsuperscript{25} and others.

\textsuperscript{10}When the discount function is quasi-geometric (Phelps and Pollak \textsuperscript{52}) and sufficiently close to a geometric one, Harris and Laibson \textsuperscript{32} used tools from the bounded variations calculus to show existence and uniqueness of the equilibrium strategies. Using alternative techniques, Harris and Laibson \textsuperscript{34} proved existence and uniqueness of smooth equilibria for a stochastic discount function when it is in some sense sufficiently close to the exponential discount function (this is what they coin the instantaneous gratification model). It appears that both proofs require the presence of income uncertainty.

\textsuperscript{11}See also a related paper by Judd \textsuperscript{35}. 
equilibria because it is assumed that the equilibria must be the limit of finite-horizon equilibria. We do not impose this restriction in our setting and the infinite horizon naturally underscores multiple equilibria. Multiplicity is also discussed in Karp [36] and Krusell and Smith [41]. In a continuous time model, Karp [36] obtained the MCP's necessary conditions in a growth model with non constant discount rate by considering first the equilibrium of a sequence of planners in discrete time and then he took the continuous time limit. While the passage to the limit is not mathematically justified in Karp [36], his equilibrium necessary conditions are consistent with the infinite horizon naturally underscores multiple equilibria. Our incremental contribution relative to his is that we took the necessary theoretical steps to define the novel notion of continuous time game and, as a byproduct, we proved existence of multiple equilibria. Krusell and Smith [41] also report multiple equilibria in a Ramsey growth model with quasi-geometric discount. However, their MCPs are supported by discontinuous consumption policies whereas our MCPs are continuously differentiable policies. Therefore, our analysis suggests that the multiplicity is somewhat more fundamental because it does not need to be structured around discontinuous saving rules.

Finally, our application also relates to the literature on optimal fiscal policy. In the context of a growth model with infinitely-lived individuals, Judd [35] and Chamley [18] established that capital taxation should be zero in the long term. In contrast to these result, we suggest that in overlapping generations economies, the preference based time inconsistency friction faced by the government creates a role for long term capital income taxation. However, our model ignores important aspects of the governments policy tradeoffs that could result in a myriad of alternative motives for capital income taxation. For example, the Mirrlees approach to optimal taxation (see the papers surveyed in Kocherlakota [39]) taught us that taxing capital income can be required if the planner is facing an informational friction due to the unobservable private skills or productivity. In the closer (non Mirrleesian) context of overlapping generation, Erosa and Gervais [26] rationalize capital income tax as an indirect leisure tax. Nonetheless, we aimed to argue that the preferences based time inconsistency faced by the government by itself creates a role for capital income taxation and we hope that the point is clearest in our simple context.

The rest of the paper is organized as follow. The next section presents the basic model and discuss the issue of time inconsistency. We define the continuous time game in Section 3 and provide the equilibrium characterization in Section 4. In Section 5 we undertake our application to the overlapping generations model. The last section concludes.

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12The literature on optimal governmental policy with overlapping generations includes Diamond [24], Atkinson and Sandmo [1], Auerbach and Kotlikoff [5], Erosa and Gervais [26].
2 The model

2.1 Preferences and production

We consider a deterministic stationary environment where time is continuous. A decision maker derives utility from a consumption schedule rate \( c \), and the date \( t \) utility has the representation

\[
\int_t^{\infty} h(s-t)u(c(s))ds
\]

for some utility function \( u \) and some discount function \( h \). We assume that \( u \) is strictly increasing, twice continuously differentiable and strictly concave. The discount function \( h \) is continuously differentiable and positive, with \( h(0) = 1 \), \( h(t) \geq 0 \), and \( \int_0^{\infty} h(s)ds < \infty \). Note that the discount function between the current time \( t \) and the consumption scheduling time \( s \) depends only of \( s-t \). This assumption implies, as in Strotz [58], that the incremental utility of immediate over postponed consumption remains invariant with the passage of time. As a result, the representation (1) is stationary, meaning that

\[
\int_t^{\infty} h(s-t)u(c(s))ds = \int_0^{\infty} h(s)u(c(t+s))ds.
\]

The decision maker strives to maximize the objective (1) under the resource constraint

\[
\frac{dk(s)}{ds} = f(k(s)) - c(s), \quad k(t) = k_t,
\]

where \( k(s) \) is capital at time \( s \) and, where \( f \) is a strictly increasing, concave and continuously differentiable production function.

2.2 Time consistency

Unless the discount function is exponential, the marginal rate of consumption substitution between two future dates will in general change with the mere passage of time. To see this, fix the dates \( t_1 < t_2 < t_3 < t_4 \) and the consumption rates \( c_3, c_4 \) and consider the marginal rate of substitution (MRS) between consuming \( c_3 \) at date \( t_3 \) and consuming \( c_4 \) at date \( t_4 \). When calculated from the perspective of date \( t_1 \), the MRS is \( \frac{h(t_3-t_1)u'(c_3)}{h(t_3-t_2)u'(c_3)} \), whereas the MRS is \( \frac{h(t_3-t_2)u'(c_3)}{h(t_4-t_2)u'(c_4)} \) from the perspective of date \( t_2 \). The two MRS will be identical if the discount function has the multiplicative property \( \frac{h(t_3-t_4)}{h(t_4-t_4)} = \frac{h(t_3-t_2)}{h(t_4-t_2)} \) and this must hold for all \( t_1 < t_2 < t_3 < t_4 \). As Strotz [58] pointed out, a necessary and sufficient condition for the MRS to be time invariant is that the discount function of the exponential form \( h(t) = e^{-\delta t} \) for some constant discount rate \( \delta \geq 0 \). The MRS changes with the mere passage of time with any other discount function, and intertemporal consistency fails. As a result, when the discount function is not of the exponential form, a consumption schedule \( (c_0(s))_{s \geq 0} \) seems optimal for a decision maker who maximizes the objective (1) under the constraint (2) at time...
At time \( t = 0 \) and yet it will not be perceived as such at later time \( t > 0 \). So when time \( t \) comes, there is no reason to expect that the decision-maker will actually consume \( c_0(t) \), as the decision-maker at time 0 expected of her, unless, of course, the latter has a way to commit the former. In other words, for general discount functions, there are a plethora of temporary optimal policies: Each of them will be optimal when evaluated from one particular point in time, but will cease to be so when time moves forward. As a result, none of them can be implemented, unless one of these viewpoints is given a privileged status and the power to lock in policy for all future times (which, incidentally, may be regretted afterwards).

In the absence of a commitment technology, the problem of maximizing (1) under the constraint (2) can no longer be seen as a classical optimization problem. There is no way for the decision-maker at time 0 to achieve what is, from her point of view, the first-best solution of the problem, and she must turn to a second-best policy. Defining and studying such a policy is the first aim of this paper. The path to follow is clear and it is consistent with Strotz [58], Pollak [53] and Peleg and Yaari [50]. The best the decision-maker at time \( t \) can do is to guess what her successors are planning to do, and to plan her own consumption \( c(t) \) accordingly. We will then be looking for a subgame-perfect equilibrium of a certain game.

### 3 Equilibrium strategies: construction and definition

We now proceed to define subgame-perfect equilibrium strategies. Paralleling Aumann [6] we will consider the continuum of decision makers over the time interval \([0, \infty)\). At any time \( t \), there is a decision-maker who decides what current consumption \( c(t) \) shall be. As is readily seen from the equation (2), changing the value of \( c \) at just one point in time will not affect the trajectory. However, the decision-maker at time \( t \) is allowed to form a coalition with her immediate successors, that is with all \( s \in [t, t + \varepsilon] \), and we will derive the definition of an equilibrium strategy by letting \( \varepsilon \to 0 \). In fact, we are assuming that the decision-maker \( t \) can commit her immediate successors (but not, as we said before, her more distant ones), but that the commitment span is vanishingly small. We now construct and define the equilibrium Markov strategies. We analyze the problem from the perspective of the decision maker at time \( t = 0 \) but, given the stationarity of the environment, a similar analysis can be carried out at any time \( t > 0 \).

We restrict our analysis to Markov strategies, in the sense that the policy depends only on a payoff relevant variable, the current capital stock and not on past history, current time or some extraneous factors. Such a strategy is given by \( c = \sigma(k) \), where \( \sigma : R \to R \) is a continuously differentiable function. If we apply the strategy \( \sigma \), the dynamics of capital accumulation from \( t = 0 \) are given by:

\[
\frac{dk}{ds} = f(k(s)) - \sigma(k(s)), \quad k(0) = k_0
\]

We shall say \( \sigma \) converges to \( \bar{k} \), a steady state of \( \sigma \), if \( k(s) \to \bar{k} \) when \( s \to \infty \),
when the initial value $k_0$ is sufficiently close to $\bar{k}$. A strategy $\sigma$ is *convergent* if there is some $\bar{k}$ such that $\sigma$ converges to $\bar{k}$. In that case, the integral (1) is obviously convergent, and its successive derivatives can be computed by differentiating under the integral. This assumption is not required but it will greatly simplify the exposition, and for this reason we will restrict our attention to convergent strategies. Note that if $\sigma$ converges to $\bar{k}$, then we must have $f(\bar{k}) = \sigma(\bar{k})$.

Let us now proceed to the definition of equilibrium strategies. A convergent Markov strategy $c = \sigma(k)$, where $\sigma : R \rightarrow R$ is a continuously differentiable function, has been announced and is public knowledge. The decision maker begins at time $t = 0$ with capital stock $k$. If all future decision-makers apply the strategy $\sigma$, the resulting capital stock $k_0$ future path obeys

$$\frac{dk_0}{dt} = f(k_0(t)) - \sigma(k_0(t)), \quad t \geq 0$$

$$k_0(0) = k.$$  

(3) (4)

We suppose the decision-maker at time 0 can commit all the decision-makers in $[0, \varepsilon]$, where $\varepsilon > 0$. She expects all later ones to apply the strategy $\sigma$, and she asks herself if it is in her own interest to apply the same strategy, that is, to consume $\sigma(k)$. If she commits to another bundle, $c$ say, the immediate utility flow during $[0, \varepsilon]$ is $u(c) \varepsilon$. At time $\varepsilon$, the resulting capital will be $k + (f(k) - c) \varepsilon$, and from then on, the strategy $\sigma$ will be applied which results in a capital stock $k_\varepsilon$ satisfying

$$\frac{dk_\varepsilon}{dt} = f(k_\varepsilon(t)) - \sigma(k_\varepsilon(t)), \quad t \geq \varepsilon$$

$$k_\varepsilon(\varepsilon) = k + (f(k) - c) \varepsilon.$$  

(5) (6)

The capital stock $k_\varepsilon$ can be written as $k_\varepsilon(t) = k_0(t) + k_1(t) \varepsilon$ where

$$\frac{dk_1}{dt} = (f'(k_0(t)) - \sigma'(k_0(t))) k_1(t), \quad t \geq \varepsilon$$

$$k_1(\varepsilon) = \sigma(k) - c$$  

(7) (8)

where $f'$ and $\sigma'$ stand for the derivatives of $f$ and $\sigma$. Summing up, we find that the total gain for the decision-maker at time 0 from consuming bundle $c$ during the interval of length $\varepsilon$ when she can commit, is

$$u(c) \varepsilon + \int_{\varepsilon}^{\infty} h(s) u(\sigma(k_0(t) + \varepsilon k_1(t))) dt,$$

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13In fact, we can work with a larger set of policies $\sigma$ for which the integral (1) is convergent, and for which the successive derivatives of (1) can be computed by differentiating under the integral. The resulting equilibrium characterizations of Section 4 will be identical.

14To see this, plug $k_\varepsilon(t) = k_0(t) + k_1(t) \varepsilon$ into (5) for $t \geq \varepsilon$, keeping only the terms of first order in $\varepsilon$, and get

$$\frac{dk_\varepsilon}{dt} = f(k_0(t)) + \varepsilon f'(k_0(t)) k_1(t) - \sigma(k_0(t)) - \varepsilon \sigma'(k_0(t)) k_1(t).$$

Comparing this with (3) gives (7). Equation (8) is obtained by substituting the expansion $k_0(\varepsilon) = k + \varepsilon \frac{dk_0}{ds}(0) = k + \varepsilon (f(k) - \sigma(k))$ into (5).
and in the limit, when $\varepsilon \to 0$, and the commitment span of the decision-maker vanishes, expanding this expression to the first order leaves us with two terms

$$\int_{0}^{\infty} h(t) u(\sigma(k_{0}(t))) \, dt + \varepsilon \left[ u(c) - u(\sigma(k)) + \int_{0}^{\infty} h(t) u'(\sigma(k_{0}(t))) \sigma'(k_{0}(t)) k_{1}(t) \, dt \right].$$

(9)

where $k_{1}$ solves the linear equation

$$\frac{d k_{1}}{d t} = \left( f'(k_{0}(t)) - \sigma'(k_{0}(t)) \right) k_{1}(t), \quad t \geq 0$$

(10)

$$k_{1}(0) = \sigma(k) - c.$$  

(11)

Note that the first term of (9) does not depend on the decision taken at time 0, but the second one does. This is the one that the decision-maker at time 0 will try to maximize. In other words, given that a strategy $\sigma$ has been announced and that the current state is $k$, the decision-maker at time 0 faces the optimization problem:

$$\max_{c} P_{1}(k,\sigma,c)$$

(12)

where

$$P_{1}(k,\sigma,c) = u(c) - u(\sigma(k)) + \int_{0}^{\infty} h(t) u'(\sigma(k_{0}(t))) \sigma'(k_{0}(t)) k_{1}(t) \, dt.$$  

(13)

In the above expression, $k_{0}(t)$ solves the Cauchy problem (3),(4) and $k_{1}(t)$ solves the linear equation (10),(11).

**Definition 1** A convergent Markov strategy $\sigma : \mathbb{R} \to \mathbb{R}$ is an equilibrium strategy for the intertemporal decision model (7) under the constraint (2) if, for every $k \in \mathbb{R}$, the maximum in problem (12) is attained for $c = \sigma(k)$:

$$\sigma(k) = \arg \max_{c} P_{1}(k,\sigma,c)$$

(14)

The intuition behind this definition is simple. Each decision-maker can commit only for a small time $\varepsilon$, so he can only hope to exert a very small influence on the final outcome. In fact, if the decision-maker at time 0 plays $c$ when he/she is called to bat, while all the others are applying the strategy $\sigma$, the end payoff for him/her will be of the form

$$P_{0}(k,\sigma) + \varepsilon P_{1}(k,\sigma,c)$$

where the first term of the right hand side does not depend on $c$. In the absence of commitment, the decision-maker at time 0 will choose whichever $c$ maximizes the second term $\varepsilon P_{1}(k,\sigma,c)$. Saying that $\sigma$ is an equilibrium strategy means that the decision maker at time 0 will choose $c = \sigma(k)$. Given the stationarity of the problem, if the strategy $c = \sigma(k)$ is chosen at time 0, it will be chosen at any future time $t$ and as a result, the strategy $\sigma$ can be implemented in the absence of commitment.
Conversely, if a strategy \( \sigma \) for the intertemporal decision model (1), (2) is not an equilibrium strategy, then it cannot be implemented unless the decision-maker at time 0 has some way to commit his successors. Typically, an optimally committed strategy will not be an equilibrium strategy. More precisely, a strategy which appears to be optimal at time 0 no longer appears to be optimal at times \( t > 0 \), which means that the decision-maker at time \( t \) feels he can do better than whatever was planned for him to do at time 0. What happens then if successive decision-makers take the myopic view, and each of them acts as if he could commit his successors? At time \( t \), then, the decision-maker would maximize the integral (1) with the usual tools of control theory, thereby deriving a consumption \( c = \sigma_n(t,k) \). This is the naive strategy (O’Donoghue and Rabin [49]); in general it will not be an equilibrium strategy, so that every decision-maker has an incentive to deviate.

4 Characterization of the equilibrium strategies

The equilibrium strategy can be fully specified by a single function, the value function \( v(k) \), which is reminiscent of—although different from—the value function in optimal control. We will show that the value function satisfies two equivalent equations, the integrated equation (IE) and the differentiated equation (19), the latter one resembling the classical Hamilton-Jacobi-Bellman (HJB) equation of optimal control. This similarity is reassuring since it shows how standard methods from control theory can be adapted to analyze the impact of time inconsistency. Unfortunately, the similarity is superficial only, since (19) is a non-local equation (and not a partial differential equation like (HJB)) and we will demonstrate that its solutions exhibit different qualitative behavior. In the knife edge case where the discount rate is constant, the non-local term in (19) collapses, and (19) becomes identical to (HJB). Consequently, when the discount rate is constant, the equilibrium strategies are also optimal from the perspective of all temporal decision makers.

Given a Markov strategy \( \sigma(k) \), continuously differentiable and convergent, we shall be dealing with the Cauchy problem (3), (4). The value \( k_0(t) \) depends on current time \( t \), initial data \( k \), and the strategy \( \sigma \). To stress this dependence, it is convenient to write \( k_0(t) = K(\sigma; t, k) \) where \( K \) is the flow associated with the differential equation (3) defined by

\[
\frac{\partial K(\sigma; t, k)}{\partial t} = f(K(\sigma; t, k)) - \sigma(K(\sigma; t, k)) \\
K(\sigma; 0, k) = k.
\]

The following theorem characterizes the equilibrium strategies and its proof is given in Appendix A. There are two parts in the equilibrium characterization: a functional equation on the value function and an instantaneous optimality condition.

\textsuperscript{15}Under non-constant discounting, the commitment strategy \( \sigma_n \) is non stationary and so we must extend the definition of the equilibrium to a non stationary strategy. Although, we do not report the definition of the non stationary equilibrium, it can easily be done with some additional notations.
determining current consumption. Here $v'$ is the derivative of $v$ and $i$ is the inverse of marginal utility $u'$. As usual, $i \circ v' (x) = i (v' (x))$

**Theorem 2** Let $\sigma : R \to R$ be a continuously differentiable convergent strategy. If $\sigma$ is an equilibrium strategy, then the value function

$$v (k) := \int_0^{\infty} h (t) u (\sigma (\mathcal{K} (\sigma; t, k))) dt \quad (17)$$

satisfies, for all $k$, the functional equation

$$v (k) = \int_0^{\infty} h (t) u (i \circ v' (\mathcal{K} (i \circ v'; t, k))) dt \quad (\text{IE})$$

and the instantaneous optimality condition

$$u' (\sigma (k)) = v' (k), \quad \sigma (k) = i (v' (k)) \quad (18)$$

Conversely, if a function $v$ is twice continuously differentiable, satisfies (IE), and the strategy $\sigma = i \circ v'$ is convergent, then $\sigma$ is an equilibrium strategy.

The instantaneous relation (18) expresses the usual tradeoff between the utility derived from current consumption and the utility value of saving. This is a standard condition in a world where there is one commodity that can be used for investment or consumption. Let us spell out what equation (IE) means. Given a candidate function $v$, we must first solve the Cauchy problem (15), (16) with $\sigma = i \circ v'$. Second, we calculate the right-hand side of equation (IE), which is an integral along the trajectory of capital stock. The final result should be equal to $v (k)$. Equation (IE) is therefore a fundamental characterization of the equilibrium strategies and it takes the form of a functional equation on $v$. In order to contrast the equilibrium dynamics with the dynamics resulting from using the optimal control approach, the following proposition gives an alternative characterization, the differentiated equation, which resembles the usual Euler equation from optimal control and its proof is given in Appendix B.

**Proposition 3** Let $v$ be a $C^2$ function such that the strategy $\sigma = i \circ v'$ converges to $\bar{k}$. Then $v$ satisfies the integrated equation (IE) if and only if it satisfies the following functional equation

$$- \int_0^{\infty} h' (t) u \circ i (v' (\mathcal{K} (i \circ v'; t, k))) dt = \sup_c [u (c) + v' (k) (f (k) - c)] \quad (19)$$

together with the boundary condition

$$v (\bar{k}) = u (f (\bar{k})) \int_0^{\infty} h (t) dt \quad (20)$$
It is useful to rewrite the differentiated equation (19) as

$$\rho(k) = \frac{1}{v(k)} \sup_c [u(c) + v'(k) (f(k) - c)]$$  \hspace{1cm} (21)$$

where

$$\rho(k) = -\int_0^\infty h'(t) u(\sigma(K(\sigma; s, k))) \, dt / \int_0^\infty h(s) u(\sigma(K(\sigma; s, k))) \, ds$$

is interpreted as an effective discount rate. Equation (21) then tells us that, along an equilibrium path, the relative changes in value to the consumer must be equal to the effective discount rate. The effective discount rate is here endogenous to the model and its presence reflects the strategic behavior of the current decision maker resulting from internalizing the behavior of future decision makers. In order to gain some insights into the economic meaning of equation (19), we first consider the exponential discount function

$$h(t) = e^{-\delta_0 t}$$

for \(\delta_0 > 0\). With exponential discounting, \(h'(t) = -\delta_0 h(t)\), and the resulting effective discount rate is just the constant discount rate \(\rho(k) = \delta_0\) for all \(k\). Equation (19) becomes then simply the familiar (HJB) equation

$$\delta_0 v(k) = \sup_c [u(c) + v'(k) (f(k) - c)].$$ \hspace{1cm} (22)$$

Second, we consider the case where \(h\) is piecewise exponential, \(h(t) = e^{-\delta_0 t}\) for \(t \leq \tau\) and \(h(t) = e^{-\delta_1 t}\) for \(t > \tau\), with \(\tau > 0\) and \(\delta_0 > \delta_1\). The discount rate of this discount function is decreasing with time and therefore the willingness to postpone consumption at the margin is increasing over time. When the decision maker is naive in the sense that he acts as if he could commit the future decision makers and does not learn from his past mistakes, his behavior can be described by the (HJB) equation (22). In contrast, a decision maker following the equilibrium strategy recognizes that future decision makers will spend more than he currently hope and, in reaction to that, he may accumulate more wealth than the naive decision maker. Equation (19) reflects exactly this idea since, when the discount function is piecewise exponential, it can be written as

$$\delta_0 v(k) = \sup_c [u(c) + (\delta_0 - \delta_1) g(k) + v'(k) (f(k) - c)]$$  \hspace{1cm} (23)$$

where

$$g(k) = \int_\tau^\infty e^{-\delta_1 t} u \circ i (v'(K(i \circ v'; t, k))) \, dt.$$  \hspace{1cm} (24)$$

The only difference between the naive policy characterization (22) and the equilibrium policy characterization (23) is the extra term \((\delta_0 - \delta_1)g(k)\). Assuming that \(g\) is positive and increasing in \(k\), it is then evident that the presence of the extra term \((\delta_0 - \delta_1)g(k)\) yields additional incentives to save, relative to the naive policy (where \(g = 0\)).

\[^16\text{To be more concrete, assume further that the technology is linear, } f(k) = Ak \text{ and the utility of the form } u(c) = \log(c). \text{ Solving (22) gives the naive policy } \sigma_n(k) = \delta_0 k.\]
Neither equation (12) nor equation (19) are of a classical mathematical type. If it were not for the integral term, equation (19) would be a first-order partial differential equation of known type (Hamilton-Jacobi), but this additional term, which is non-local (an integral along the trajectory of the flow (15) associated with the solution \( v \)), creates a loss of regularity in the functional equation that generates mathematical complications. As a result, existence and uniqueness problems arise as they typically do in dynamic games. The topic requires more scrutiny. The next section applies the method in the context of an overlapping generation model where time inconsistency is typically faced by a utilitarian government.

5 An overlapping generations growth model

5.1 The model

Structure of the population. We consider a continuous time overlapping generations model of growth analysis, along the lines of Blanchard (1985). The economy is composed of overlapping generations of finitely-lived individuals who face a constant rate of death \( \pi > 0 \). At time \( s \), the probability of surviving until time \( t \geq s \) is given by \( e^{-\pi(t-s)} \) and consequently, the expected life is \( \int_s^\infty t\pi e^{-\pi(t-s)} dt = \frac{1}{\pi} \). At each instant a large cohort of identical individuals is born at a normalized rate of 1 so that the total number of individuals born during a small time interval \([t_1, t_2]\) is \( t_2 - t_1 \). Because the cohort is large, there is no uncertainty on how the cohort's size and the total population vary over time. A cohort born at time \( \tau \) (the \( \tau \)-vintage) has a geometrically declining size which, as of time \( t \geq \tau \), is equal to \( e^{-\pi(t-\tau)} \). At each point of time \( t \), the size of the population is constant and it is given by \( \int_{-\infty}^t e^{-\pi(t-s)} ds = \frac{1}{\pi} \). The time-\( t \) expected lifetime utility of a vintage-\( \tau \) individual \((\tau \leq t)\) is, as in Yaari [60],

\[
\Gamma(\tau, t) = \int_t^\infty e^{-(\delta + \pi)(s-t)} \ln(c(\tau, s)) ds,
\]

where \( \delta > 0 \) is the constant pure rate of time preference and \( c(\tau, s) \) is the consumption rate of an individual born at time \( \tau \), as of time \( s \geq \tau \). The utility function of a newly born individual from vintage-\( -\tau \) is then

\[
\Gamma(\tau, \tau) = \int_{\tau}^\infty e^{-(\delta + \pi)(s-\tau)} \ln(c(\tau, s)) ds.
\]

whereas solving (23) gives the function \( g(k) = \frac{1}{\delta_1} e^{-\delta_1 \tau} \log(k) + \varsigma \) where \( \varsigma \) is a constant, and the resulting equilibrium policy is

\[
\sigma_e(k) = \frac{\delta_0}{1 + \frac{\delta_0}{\delta_1} e^{-\delta_1 \tau}} k.
\]

This example illustrates the strategic motive of saving since the equilibrium marginal propensity to consume is low relative to the naive policy.

\( ^{17} \)As mentioned by Blanchard [12], the individual’s rate of death can also be interpreted at the rate of extinction of a dynasty. With this interpretation, the perpetual youth assumption \((p \text{ constant})\) seems more acceptable. The mathematical analysis suggests that the indeterminacy result does not require the perpetual youth assumption.
Technology. The technology is represented by a constant return to scale production function depending on two factors of production, aggregate capital $K$ and aggregate labor. From above, the size of the population is constant and assuming further that labor supply is inelastic, the production function (net of depreciation) is a continuously differentiable and concave function of aggregate capital stock $f(K)$. As in Section 2, since capital and output are the same commodity, capital can be invested or consumed and the investment rate is

$$\frac{dK(t)}{dt} = f(K(t)) - C(t)$$

(24)

where

$$C(t) = \int_{-\infty}^{t} c(\tau, t)e^{-\pi(t-\tau)}d\tau$$

(25)

is the aggregate consumption at time $t$.

The social criterion. We consider a social planner maximizing a utilitarian criterion balancing the lifetime utilities of the current population and the unborn generations. The planner is concerned with the generations’ welfare from the present (time $t_0 = 0$) onward and considers the alive individuals as if they had just been born so that the criterion takes the form

$$\int_{0}^{\infty} e^{-\rho s} \Gamma(\tau, \tau)d\tau + \int_{-\infty}^{0} e^{\pi s} \left( \int_{0}^{\infty} e^{-(\delta+\pi)s} \ln(c(\tau, s))ds \right) d\tau.$$  (26)

The first term of the above criteria discounts back to time 0 the expected lifetime utility of unborn generations using the social discount rate $\rho > 0$. The second term is the remaining expected lifetime utility of the individuals who were born in the past and are still alive at time 0. Notice the asymmetry of the treatment of the unborn cohorts relative the the surviving ones in the criteria (26). The later’s utilities are discounted back to current time whereas the former are discounted back to their birth date. Unlike the criteria of Calvo and Obstfeld [14] where symmetry is assumed, the absence of symmetry in the criteria (26) creates a time inconsistency due to the dependency of the planner’s utility flow on the planning time. To see this, fix the planning time $t_0 = 0$, change the order of integration in (26) and make the change of variable from vintage $\tau$ to age $n = t - \tau$, to get the welfare function

$$\int_{0}^{\infty} e^{-\rho s} \{j_u(c, 0, s) + j_b(c, 0, s)\} ds.$$  (27)

where $j_u(c, 0, s)$ is the time $s$ utility flow attributed to the unborn cohorts defined for any $t \leq s$ by

$$j_u(c, t, s) = \int_{0}^{s-t} e^{-(\delta+\pi-\rho)n} \ln(c(s - n, s))dn,$$

and where $j_b(c, 0, s)$ is the time $s$ utility flow attributed to the surviving cohorts defined for any $t \leq s$ by

$$j_b(c, t, s) = \int_{s-t}^{\infty} e^{-(\delta+\pi-\rho)n} e^{-(\delta-\rho)(s-t-n)} \ln(c(s - n, s))dn.$$
Alternatively, if the planning time is $t_0 = t > 0$, a similar calculation shows that the planner’s criterium becomes

$$\int_t^\infty e^{-\rho(s-t)} \{j_u(c, t, s) + j_b(c, t, s)\} \, ds.$$ 

When $\delta = \rho$, time consistency obtains because the utility flow from the perspective of the planning times $t_0 = t$ becomes

$$j_u(c, t, s) + j_b(c, t, s) = \int_0^\infty e^{-\pi n} \ln(c(s - n, s)) \, dn$$

and therefore it is independent from the planning time. However, so long as $\delta > \rho$ the planner faces a time inconsistency problem because the utility flow $j_u(c, t, s) + j_b(c, t, s)$ depends explicitly on the planning time $t_0 = t$.

### 5.2 The centrally planned economy

Beginning with a level of capital $K(0)$ at the planning time $t_0 = 0$, the planner maximizes the criterion (27) under the budget constraints (24) and (25). It is useful to partition the planner’s problem into two tasks each of which will take the other as given. These two tasks are executed by two fictitious planners, the *intra-period planner* and the *metaplanner*. The intra-period planner takes as given the aggregate consumption and is in charge of allocating the aggregate consumption across the surviving cohorts. The metaplanner on the other hand takes as given the path of intra cohorts’ allocation rules of aggregate consumption and is in charge of the aggregate investment decision over time. Notice that both planners face a dynamic decision problem. Let us now describe more formally the task of the intra-period planner and the metaplanner.

At the planning time is $t_0 = 0$, the intra-period planners commitment allocation of $C(s)$ at time $s \geq 0$ is obtained by maximizing

$$\int_0^s e^{-(\delta + \pi - \rho)n} \ln(c(s - n, s)) \, dn + \int_s^\infty e^{-(\delta + \pi - \rho)n} e^{-(\delta - \rho)(s-n)} \ln(c(s - n, s)) \, dn$$

under the budget constraint $C(s) = \int_0^\infty c(s-n, s) e^{-\pi n} \, dn$. The aggregate consumption expenditure $C(s)$ is exogenous to the intra-period planner and the optimal commitment allocation is

$$c(s - n, s) = \frac{e^{(\rho - \delta)n}}{\pi + (\delta - \rho)e^{-\pi(s-n)}C(s)} \quad \text{for } n \leq s, \quad (28)$$

$$c(s - n, s) = \frac{e^{(\rho - \delta)s}}{\pi + (\delta - \rho)e^{-(\delta + \pi - \rho)s}C(s)} \quad \text{for } n > s. \quad (29)$$

This allocation rule is however not robust to re-optimization because at the planning time $t_0 = t > 0$, the intra-period planner’s new commitment allocation is in general
not aligned with the allocation that the planner committed to at time $t_0 = 0$. For example, at the planning time $t_0 = 0$ the intra-period planner allocation is

$$c_{t_0=0}(t, t) = \pi(\delta + \pi - \rho)\frac{1}{\pi + (\delta - \rho)e^{-(\pi+\delta-\rho)t}}C(t)$$

for the cohort which is born at time $t$, but from the perspective of the planning time $t_0 = t$, the commitment allocation for the same cohort is the egalitarian allocation

$$c_{t_0=t}(t, t) = \pi C(t).$$

In order to focus on a simple class of equilibria, we restrict our analysis to the set of linear and stationary allocation rules for the intra-period planner of the form

$$c(t-n, t) = \varphi(n)C(t).$$

with

$$\int_{0}^{\infty} e^{-\pi n} \varphi(n) dn = 1. \quad (30)$$

Under this assumption, the objective (27) at the planning time $t_0 = 0$ becomes

$$\int_{0}^{\infty} e^{-\rho s} \left\{ \int_{0}^{s} e^{-(\delta + \pi - \rho)n} \ln(\varphi(n)) \, dn + \int_{s}^{\infty} e^{-\pi n} e^{-(\delta - \rho)n} \ln(\varphi(n)) \, dn + L(s) \right\} ds,$$

where $L(s)$ is a function depending only on time $s$ and the aggregate consumption $C(s)$, which are exogenous and therefore, it drops out of the intra-period planner’s problem. Integrating by part this formula, shows that the intra-period planner’s objective at time $t_0 = 0$ is given by

$$\left(\frac{1}{\rho} - \frac{1}{\delta}\right) \int_{0}^{\infty} e^{-(\delta + \pi)s} \ln(\varphi(s)) \, ds + \frac{1}{\delta} \int_{0}^{\infty} e^{-\pi s} \ln(\varphi(s)) \, ds. \quad (31)$$

When undertaken from the perspective of the planning time $t_0 = t$, the same calculations shows that the intra-period planner’s objective is identical with the objective (31). Therefore, when restricted to stationary linear allocation rules, all intra-period planners agree to use the rule $\varphi(\cdot)$ that maximizes (31) under the constraint (30). The solution to this maximization problem gives the allocation rule

$$\varphi(s) = \frac{\pi}{\delta} \left(\frac{\delta}{\rho + \pi} \left((\delta - \rho)e^{-\delta s} + \rho\right)\right). \quad (32)$$

How important is the assumption of stationary and linear allocation rule? The assumption of stationarity of the allocation rule is consistent with our view that the planner’s problem in invariant with the passage of time. The assumption of linearity is more questionable because the equilibrium may require, for instance, to have a less egalitarian consumption sharing rule in prosperous periods. If we permit non linear allocation rules of the form $c(t-n, t) = \varphi(n, C(t))$ we will in principle have to cope with the time inconsistency of the intra-period planner, the time inconsistency of the
metaplanner (which, as we will show shortly, arises even with linear allocation rules) and, moreover, the (sequential) interaction between the intra-period planner and the metaplanner. We do not attempt to address this question fully. Instead, assuming linear and stationary allocation rules allowed us to work out a simple example of second best equilibrium where the successive intra-period planners agree on the best allocation rule and where the intra-period planner’s problem is decoupled from the metaplanner’s problem.

Plugging either the optimal consumption allocation \( c(t, s) \) (or the egalitarian allocation) in the planner’s criterion \( 27 \) and calculating the resulting integrals yields the time \( s \) utility flow to the metaplanner

\[
\frac{e^{-(\delta + \pi - \rho)s}}{\pi} \ln(C(s)) + \frac{1 - e^{-(\delta + \pi - \rho)s}}{\delta + \pi - \rho} \ln(C(s)) + M(s)
\]

where \( M(s) \) is a function depending only on the variables that are exogenous and therefore, it drops out of the metaplanner’s problem. Substituting the utility flow into the criterion \( 27 \) and dropping the function \( M \) allows to express the metaplanner’s criterion at the planning time \( t_0 = 0 \) solely in terms of the aggregate quantities

\[
\int_0^\infty \frac{e^{-(\delta + \pi - \rho)s}}{\pi} \ln(C(s)) \, ds + \int_0^\infty \frac{e^{-\rho s} - e^{-(\delta + \pi)s}}{\delta + \pi - \rho} \ln(C(s)) \, ds
\]

(33)

After rearranging and normalizing the welfare equation (33), the metaplanner’s objective at the planning time \( t_0 = 0 \) is

\[
\int_0^\infty \frac{(\delta - \rho) e^{-(\delta + \pi)s} + \pi e^{-\rho s}}{\delta + \pi - \rho} \ln(C(s)) \, ds.
\]

(34)

A similar calculation shows that the metaplanner’s objective from the perspective of the planning time \( t_0 = t \) is

\[
\int_t^\infty \frac{(\delta - \rho) e^{-(\delta + \pi)(s-t)} + \pi e^{-\rho(s-t)}}{\delta + \pi - \rho} \ln(C(s)) \, ds.
\]

(35)

The metaplanner is therefore facing a time consistency problem because the marginal rates of substitution between consumption at two dates in the future is changing by the mere passage of time. This is exactly the type of time inconsistency that we discussed in the sections 2 to 4. In the present context, time consistency is endogenous.

\[\text{\textsuperscript{[18]} Different assumptions on the behavior of the intra-period planner are also possible and will lead to an identical dynamic problem for the metaplanner. For example, if we assume that the intra-period planners are naive, in the sense that they do not internalize their time inconsistency problem, the egalitarian policy allocation rule } c(t-n, t) = \pi C(t) \text{ will prevail. The metaplanner’s problem is then identical to the one that results from using the allocation rule } 32. \text{ The metaplanner’s problem also remains intact if we assume that the intra-period planner can commit to never change the allocation rule decided at time } t_0 = 0. \text{ Under this assumption, the intra-period planner’s optimally committed allocation rule is the non-stationary and age dependent allocation rule } 28, 29.\]
since it is created from the structure of the criterion (26) which asymmetrically treats the surviving cohorts (whose lifetime utility is discounted back to current date) and the unborn cohorts (whose lifetime utility is instead discounted back to birth date). Our planner is therefore different from the time consistent planner of Calvo and Obstfeld (1998).

5.3 Equilibrium strategies for the metaplanner

As in Section 4, we denote by $v$ the equilibrium value for the metaplanner. We are in the special case when $u(c) = \ln c$ and:

$$h(t) = \frac{\delta - \rho}{\pi + \delta - \rho} e^{-(\delta + \pi)t} + \frac{\pi}{\pi + \delta - \rho} e^{-\rho t} \tag{36}$$

Due to the specific form of discounting in (36), it is possible to rewrite the equilibrium characterization as a system of two differential equations for two functions, the equilibrium value $v$, and a function $w$ which determines how the welfare is split between the surviving cohorts and the unborn cohorts. The next proposition states this result and its proof is given in the appendix.

**Proposition 4** Let $\sigma$ be a continuously differentiable strategy converging to $\bar{k}$. If $\sigma$ is an equilibrium strategy, then the functions $v$ and $w$ defined by

$$v(k) = \int_0^{\infty} \left( \frac{\delta - \rho}{\pi + \delta - \rho} e^{-(\delta + \pi)t} + \frac{\pi}{\pi + \delta - \rho} e^{-\rho t} \right) \ln \left( \sigma(\mathcal{K}(\sigma; t, k)) \right) dt, \tag{37}$$

$$w(k) = \frac{\pi}{\pi + \delta - \rho} \int_0^{\infty} (-e^{-(\delta + \pi)t} + e^{-\rho t}) \ln \left( \sigma(\mathcal{K}(\sigma; t, k)) \right) dt \tag{38}$$

satisfy the system

$$\left( f - \frac{1}{v} \right) v' - \ln (v') = \delta v - (\delta - \rho) w, \tag{39}$$

$$\left( f - \frac{1}{v} \right) w' = -\pi v + (\rho + \pi) w \tag{40}$$

with the boundary conditions

$$v(\bar{k}) = \frac{\rho + \pi}{\rho (\delta + \pi)} \ln f(\bar{k}), \tag{41}$$

$$w(\bar{k}) = \frac{\pi}{\rho (\delta + \pi)} \ln f(\bar{k}) \tag{42}$$

and the strategy $\sigma$ is given by $\sigma(k) = 1/v'(k)$.

Conversely, let $v$ be a $C^2$ function such that the strategy $\sigma(k) = 1/v'(k)$ converges to $\bar{k}$. If there exists a $C^1$ function $w$, such that $(v, w)$ satisfies the system (39)-(40) and the boundary conditions (41)-(42), then $\sigma$ is an equilibrium strategy converging to $\bar{k}$.
Note that, by Proposition 3, the characterization (39)-(40) is equivalent to the more general equation (19). Thus, when the discount function has the special form (36), the non local one dimensional equation (19) becomes a system of two ordinary differential equations without non local terms. This reduction is critical for our existence result in the next theorem. The proof of Proposition 4 is given in the appendix.

Using the welfare decomposition (33), it can be shown that, given the current level of capital \( K(0) = k \), the criterion (34) allocates the welfare \( v(k) - w(k) \) to the surviving cohorts whereas the unborn cohorts’ welfare is \( w(k) \). To build some intuition on the system (39), (40), it is useful to consider the case where \( \rho = \delta \), in which case equation (36) shows that the metaplanner is facing a standard time consistent investment problem with a constant discount rate \( \delta \). As a result, the system \((v, w)\) is uncoupled in the sense that \( v \) can be solved for without knowing \( w \). In this case, differentiating (39), gives the classical autonomous dynamical system describing the evolution of the variables \((K, C)\)

\[
\frac{dK(t)}{dt} = f(K(t)) - C(t),
\]

\[
\frac{1}{C(t)} \frac{dC(t)}{dt} = f'(K(t)) - \delta.
\]

When the metaplanner’s problem is time inconsistent, the system \((K, C)\) is not autonomous anymore and an additional variable \( W(t) = w(K(t)) \) is required to describe the equilibrium dynamics. Using (39), (40) the evolution of the economy can be described by the autonomous system \((K, C, W)\)

\[
\frac{dK(t)}{dt} = f(K(t)) - C(t),
\]

\[
\frac{1}{C(t)} \frac{dC(t)}{dt} = f'(K(t)) - \delta - (\delta - \rho) \frac{\pi}{\delta} C(t) - \frac{\delta - \rho}{\delta} f(K(t)) - C(t) (\pi \ln(C(t)) - \rho (\pi + \delta) W(t)),
\]

\[
\frac{dW(t)}{dt} = -\frac{1}{\delta} \left( \pi \frac{f(K(t)) - C(t)}{C(t)} + \pi \ln(C(t)) - \rho (\pi + \delta) W(t) \right)
\]

The dynamics of interest are the one where the economy naturally tends to a steady state, that is a condition of the economy in which the aggregate level of output, capital and consumption do not change over time. The following theorem provides our main result on existence of multiple steady states

**Theorem 5** Consider any \( \bar{k} \in I \) where \( I = \{k \mid \rho \frac{\pi + \delta}{\pi + \rho} < f'(k) < \delta\} \). Then there exist an equilibrium strategy, defined on some neighbourhood \( \Omega \) of \( \bar{k} \) and converging to \( \bar{k} \).

This result shows the existence of a continuum of steady states of the economy. The result is proved in the appendix and it is obtained by using the Central Manifold Theorem [17]. Associated to the multiple steady states is a continuum of equilibrium strategies each of them generating a path of aggregate capital stock. The equilibrium
multiplicity is expectation driven in the sense that if all metaplanners agree that $k_1 \in I$ then it will and if all metaplanners agree that $k_2 \in I$ is a steady state, then it will as well. The multiplicity occurs because the subgame perfection condition is not sufficient to pin down how the metaplanners coordinate their beliefs on splitting the resources between the surviving cohorts and the unborn ones. When the social and private discount rate are equal, the interval $I$ becomes one point and the economy’s capital stock converges then to its modified golden rule level $\bar{k}^{mg}$ defined by $f'(k^{mg}) = \delta$. The next step is then to evaluate the efficiency of these equilibria. In particular, one may wonder if conservative equilibrium policies (i.e. those generating higher steady states level of capital and consumption) Pareto dominate the less conservative equilibrium policies. The distant unborn generations clearly favor the conservative equilibria since the steady state consumption is larger. However, the preference of the currently surviving cohorts is ambiguous because current consumption must decrease in order to achieve a higher steady state of capital and as a result, welfare may be lower. While evaluating efficiency at this level of generality does not seem obvious, equilibrium ranking is nevertheless possible if one allow the metaplanners to revise their belief at later stages of the game. So far, we implicitly ruled out renegotiation in our definition of equilibrium policies since beliefs were fixed once for all at an ex ante stage of the game. This assumption may be justified when it is very costly to renegotiate but it seems inappropriate in our context since it is conceivable that the metaplanter may reconsider at future stages their coordination with future metaplanners. Following an argument initially introduced by Farell and Maskin [27] and Bernheim and Ray [10] we illustrate in the next subsection how giving the planner the ability to reconsider his beliefs (present and future) - allows to shrink the set of subgame perfect equilibria.

5.4 Renegotiation-proof equilibrium

In order to select a subgame perfect equilibrium, we use the notion of renegotiation-proof equilibrium introduced by Farrell and Maskin [27] in the context of a two players repeated game. The notion has been used in a multiple selves infinite horizon game by Kocherlakota [38] (see also Asheim [2] for an alternative refinement) and we provide a local definition of this notion.

**Definition 6** An equilibrium strategy $\bar{\sigma}$ converging to $\bar{k}$ is locally renegotiation-proof (l.r.p.) if there exists an open neighborhood $\Omega$ of $\bar{k}$ such that for all $k^* \in \Omega$, the equilibrium policy $\sigma^*$ converging to $k^*$ satisfies

$$\int_0^\infty h(t) u(\sigma^* (K^* (t, k^*))) dt \leq \int_0^\infty h(t) u(\bar{\sigma} (K (t, k^*))) dt \quad (46)$$

for all $k_0 \in \Omega$.

This definition says that if an equilibrium policy is locally renegotiation-proof then, from the perspective of the distant metaplanners, a small perturbation of the policy, inducing a shift in the steady state level of capital is dominated by the status
quo. To see this, notice that since the l.r.p. equilibrium policy \( \bar{\sigma} \) is converging to \( \bar{k} \) then, for all \( k_0 \) in the domain of definition of \( \bar{\sigma} \), there exists a time \( T > 0 \) such that the flow \( (K(\bar{\sigma}; t, k_0), t \geq T) \) belongs to \( \Omega \). Therefore equation \( (46) \) with \( k_0 = K(\bar{\sigma}; s, k_0) \) says that, for any \( s \geq T \) the metaplanner at time \( s \) prefers to keep the strategy \( \bar{\sigma} \) rather than switching to a neighboring rule \( \sigma^* \). As a result, if the metaplanners are restricted to switch policies by small increments rather than by more drastic deviations, only l.r.p equilibria are credible. The next proposition says that the local renegotiation-proof condition rules out some subgame perfect equilibria and allows to identify the steady state of the economy.

**Proposition 7** Any equilibrium strategy \( \bar{\sigma} \) converging to some \( \bar{k} \) satisfying \( \rho^{\frac{\pi+\delta}{\pi+\rho}} < f'(\bar{k}) \) is not locally renegotiation-proof: distant metaplanners will always agree to switch to a new equilibrium policy converging to \( k^* \) where \( k^* \) is slightly above \( \bar{k} \).

The implementation of the switch can be done easily. Along an equilibrium path converging to \( \bar{k} \) and induced by the policy \( \bar{\sigma} \), the time \( T \) metaplanner (where \( T \) is sufficiently large) suggest to the successors to switch to a new equilibrium strategy \( \sigma^* \) converging to \( k^* \) where \( k^* \) is slightly larger that \( \bar{k} \). Proposition 7 says that, for \( T \) sufficiently large, the metaplanners have an incentive to play the new equilibrium policy \( \sigma^* \) instead of keeping \( \bar{\sigma} \). In summary, all the future metaplanners (from time \( T \) onward) coordinate their beliefs on a new equilibrium \( \sigma^* \). Since the metaplanner at time 0 is aware that this will happen, it does not make sense for her to start playing \( \bar{\sigma} \) and as a result, she will use the strategy \( \sigma^* \). The same argument can however be made iteratively to \( \sigma^* \) and the future metaplanners will have interest to deviate from \( \sigma^* \) to a neighboring strategy \( \bar{\sigma} \) converging to \( \tilde{k} \) where \( \tilde{k} \) is slightly higher than \( k^* \). The metaplanner will then apply an equilibrium strategy converging to the highest steady state level of capital defined by \( f'(\bar{k}) = \rho^{\frac{\pi+\delta}{\pi+\rho}} \). Consequently, the steady state level of capital depends on the planner’s discount rate and the individual discount rate. The more patient is the planner (lower \( \rho \)), or the consumers (lower \( \delta \)), the higher is the steady state level of aggregate capital stock stock. This result must be contrasted with the Calvo and Obstfeld (1988) time consistent government where the the private rate of time preference is irrelevant for the steady state level of capital stock. Note that when the planner’s discount rate \( \rho \) converges to 0, the capital stock converges to its golden rule level \( k^g \) defined by \( f'(k^g) = 0 \).

### 5.5 Decentralization of the equilibria and fiscal policy

The government begin at time \( t_0 = 0 \) with capital stock \( K(0) \) and tries to implement the second best allocation that we denote \( (K^*(t), C^*(t), W^*(t))_{t \geq 0} \) and such that the steady capital stock is \( K^*(\infty) = \bar{k} \in I \). This centralized allocation path gives rise to the real interest rate \( r_t^* = f'(K^*(t)) \) and the real per capita wage \( \omega_t^* = (f(K^*(t)) - K^*(t)f'(K^*(t)))\pi \). We recall that \( (K^*, C^*, W^*) \) solves the autonomous system \( (43), (44) \) and \( (45) \) and that the resulting disaggregate allocation is given for any \( \tau \leq t \) by

\[
C^*(\tau, t) = \varphi(t - \tau)C^*(t) 
\]  
(47)
where the function \( \varphi \) is defined by (32). Differentiating equation (47) with respect to time and using equation (44), shows that the the disaggregate allocation obeys

\[
\frac{1}{c^*(\tau,t)} \frac{dc^*(\tau,t)}{dt} = r_t^* - \delta + \psi(t) + \frac{\varphi'(t-\tau)}{\varphi(t-\tau)}
\]

(48)

where \( \varphi' \) is the derivative of \( \varphi \) and where

\[
\psi(t) = -(\delta - \rho) \frac{\pi}{\delta} - \frac{\varphi(t)}{f(K^*(t)) - C^*(t)} (\pi \ln(C^*(t)) - \rho (\pi + \delta) W^*(t))
\]

When \( \rho = \delta \), the intra-period planner uses the egalitarian allocation rule and the metaplanner’s problem admits a first best solution since it is time consistent and we have \( \varphi' = 0 \) and \( \psi = 0 \). As a result, when \( \rho = \delta \), the disaggregate consumption obeys

\[
\frac{1}{c^*(\tau,t)} \frac{dc^*(\tau,t)}{dt} = r_t^* - \delta.
\]

The market economy is as the one of Yaari [60] and Blanchard [12]. In the absence of bequest motives, the mortality risk creates a role for annuity contracts. A competitive insurance market supplies the market with actuarially fair annuities that have an instantaneous rate of return of \( r_t + \pi \), where \( r_t \) is the market real interest rate as of time \( t \). An individual born at time \( \tau \) who had accumulated a positive financial wealth \( a(\tau,s) \) at time \( s \geq \tau \) faces the risk of dying before spending it. Investing in the annuity permits this individual to receive a payment \( (r_s + \pi) a(\tau,s) \) if he survived at time \( s \) and, in exchange, the insurance company will collect \( a(\tau,s) \) if the individual dies at time \( s \). On the other hand, if the individual is a net borrower, the lending institution faces the risk that the individual dies before being able to repay the loan. The individual will issue then an annuity on which a flow of interest of \( (r_s + \pi) a(\tau,s) \) is paid to the insurance company but in which the debt will be forgiven if the issuer dies. Individuals will always invest (or borrow) their financial wealth with the insurance company: If they are lenders, the rate of return is higher with the insurance company during their lifetime and if they are borrowers, they have to insure themselves against death because negative bequests are prohibited.

The fiscal instrument at the disposal of the government consists of government, age dependent non distorting lump sum taxation (that is an income tax with no slope) and age dependent distorting capital income taxes (or subsidy). Denoting by \( a(\tau,t) \) the total assets of the vintage \( \tau \) agent at time \( t \), the individual optimization problem is

\[
\sup \int_t^\infty e^{(\delta + \pi)(s-t)} \log(c(\tau,s)) \, ds
\]

under the constraint

\[
\frac{da(\tau,t)}{dt} = ((1 - \eta(\tau,t))r_t + \pi)a(\tau,t) - c(\tau,t) + \omega_t + \beta(\tau,t)
\]

where \( \eta(\tau,t) \) is the tax rate at time \( t \) for an agent born at \( \tau \) and, \( \beta(\tau,t) \) is the transfer flow received by a vintage \( \tau \) agent at time \( t \) and \( r_t \) (resp. \( \omega_t \)) is the real interest (resp. wage) rate prevailing at time \( t \). The individual optimization problem has the first order condition

\[
\frac{1}{c(\tau,t)} \frac{dc(\tau,t)}{dt} = (1 - \eta(\tau,t))r_t - \delta
\]

(49)
for all $\tau \leq t$ and yields the policy

$$\dot{c}(\tau, t) = (\delta + \pi) [a(\tau, t) + h(\tau, t) + b(\tau, t)]$$  \hspace{1cm} (50)$$

where $h(\tau, t)$ is the human wealth of any individual born a time $\tau$ as of time $t$,

$$h(\tau, t) = \int_t^\infty \omega_d e^{-\int_t^s ((1-\eta(\tau,x))r_x + \pi)dx} ds$$

and where $b(\tau, t)$ is the present value of government transfer flows expected by a vintage $\tau$ agent as of time $t$,

$$b(\tau, t) = \int_t^\infty \beta(\tau, s) e^{-\int_t^s ((1-\eta(\tau,x))r_x + \pi)dx} ds.$$  \hspace{1cm} (51)$$

The problem of the government is to credibly commit to a fiscal policy path $\{({\eta}(\tau, t); {\beta}(\tau, t))_{\tau \leq t}\}_{t \geq 0}$, such that, when the shadow prices $\{r^*_t\}_{t \geq 0}$ and $\{\omega^*_t\}_{t \geq 0}$ are expected, the individual optimal consumption paths coincide with the desired allocation that is,

$$\dot{c}(\tau, t) = c^*(\tau, t),$$

for all $t \geq 0$ and $\tau \leq t$.

It is impossible to decentralize the allocation $c^*$ without government intervention because it is a second best inefficient allocation. More precisely, in the absence of fiscal intervention ($\eta = \beta = 0$), summing at any date the individual optimization policies over all surviving cohorts allows to identify the aggregate dynamics. The resulting steady state of the market economy is $\bar{k}_M$ where $\bar{k}_M$ is the unique solution of the equation

$$(f'(\bar{k}_M) - \delta) f(\bar{k}_M) = \pi(\delta + \pi)\bar{k}_M.$$  \hspace{1cm} (52)$$

We refer the reader to Section 1 and 2 of Blanchard [12] for the details. Blanchard [12] showed that that $f'(k_M) < \delta$ and, since $\delta < f'(\bar{k})$, the market economy does not sufficiently accumulate the capital stock relative to the desired centrally planned economy.

In order to decentralize $c^*$, let us first assume that we can find a fiscal policy ($\eta, \beta$) such that the resulting individual consumption matches the desired allocation at time 0, that is, $\dot{c}(\tau, 0) = c^*(\tau, 0)$ for all $\tau \leq 0$. If we want the decentralization to follow through for later times $t > 0$, we must check that $\dot{c}(\tau, t) = c^*(\tau, t)$ for any $\tau < 0$ and $t > 0$. This condition will hold if and only if the optimality condition (49) is identical to the disaggregate allocation dynamics equation (48) which results in

$$(1 - \eta(\tau, t)) r^*_t - \delta = r^*_t - \delta + \psi(t) + \frac{\varphi'(t - \tau)}{\varphi(t - \tau)} \varphi(t - \tau)$$

This condition can be expressed as

$$\eta(t - n, t) = \frac{1}{r^*_t} \left(-\psi(t) - \frac{\varphi'(n)}{\varphi(n)}\right),$$  \hspace{1cm} (52)$$
where \( n = t - \tau \) is the age of the taxable consumer. Equation (52) uniquely identifies the required capital income tax rate since it is the only distortional tax rate prompting the necessary credibility of the taxation policy.

When the planner’s discount rate is equal to the individual discount rate; \( \rho = \delta \), the time inconsistency disappears, the centrally planned allocation becomes first best and the capital income tax rate is zero \( (\psi = \varphi' = 0) \). When \( \delta \neq \rho \), the tax policy is age dependent. However, if the intra-period planner uses the egalitarian allocation rule instead of the allocation rule (32), the tax rate becomes age independent and it is given by \( \eta(t) = -\psi(t)/r^*_t \).

At date \( t = 0 \), the government has no flexibility in the choice of the tax rate policy \( \eta \) because it is given by equation (52) but it will use the lump sum taxation in order to match the initial individual consumption \( \hat{c}(\tau, 0) \) with the disaggregate allocation \( c^*(\tau, 0) \). To see this, use the policy formula (50) for an agent born at time \( \tau = 0 \) to get

\[
\hat{c}(0, 0) = (\delta + \pi) [a(0, 0) + h(0, 0) + b(0, 0)]
\]

and, assuming that \( a(0, 0) = 0 \); people inherit neither capital stock nor debt when they are born, we see that \( \hat{c}(0, 0) = c^*(0, 0) \) if and only if

\[
b(0, 0) = \frac{1}{\delta + \pi} c^*(0, 0) - h(0, 0).
\] (53)

Equation (50) also gives a similar formula for the present value of transfer flows to the consumers born at \( \tau < 0 \),

\[
b(\tau, 0) = \frac{1}{\delta + \pi} c^*(\tau, 0) - h(\tau, 0) - a(\tau, 0)
\] (54)

where \( a(\tau, 0) \) is the total assets (capital stock and government debt) of a consumer born at time \( \tau \) as of time 0.

Finally, the government will face the same decentralization problem for the generations that will be born at \( \tau > 0 \) and equation (50) gives again the formula

\[
b(\tau, \tau) = \frac{1}{\delta + \pi} c^*(\tau, \tau) - h(\tau, \tau).
\] (55)

The right hand sides of (53), (54) and (55) are known to the government whereas the left hand sides are related to the transfers flow \( \beta \) through the integral formula (51). Any lump sum transfers flow \( \beta \) satisfying (53), (54) and (55) decentralizes the desired plan and hence, unlike the capital income taxation, the lump sum taxation is indeterminate.

Notice that the fiscal policy that decentralizes the plan \( c^* \) is time consistent: even if the government is given the opportunity to revise the path of fiscal instruments over time, it has no interest in doing so in the absence of new information because the problem of time inconsistency is sorted out at the ex ante stage of central planning.

Evaluating equation (44) at the steady state of the economy gives \( \lim_{t \to \infty} \psi(t) = \bar{\psi} = \delta - f'(\bar{k}) > 0 \) and therefore, the steady state tax rate becomes stationary and it is given by

\[
\lim_{t \to \infty} \eta(t - n, t) = \frac{1}{f'(\bar{k})} \left( f'(\bar{k}) - \delta - \frac{\varphi'(n)}{\varphi(n)} \right).
\] (56)
This result shows that it is required to tax (or subsidize) capital income in the long term and it is in stark contrast with representative infinitely lived agents models where the capital interest taxation is not an optimal instrument in the long term.

The long term capital income tax rate require to subsidize the old cohorts and tax the young cohorts. In fact, it can be shown that there exist a threshold age $\tilde{n}$ that the government subsidizes (resp. taxes) capital income for cohorts older (resp. younger) than $\tilde{n}$ years. For instance, if the planner decide to decentralize the l.r.p. allocation converging to $\bar{k}$ with $f'(\bar{k}) = \rho \pi + \delta$, the cutoff age is given by

$$\tilde{n} = -\frac{1}{\delta} \ln \left( \frac{\pi}{\delta + \pi} \right)$$

Notice that if the intra period planner decides to use the egalitarian allocation $\varphi(n) = \pi$, there is no need for an age dependent capital income taxation. Under this assumption, the government will simply use the long term uniform capital income subsidy rate

$$\bar{\eta} = \frac{\pi \delta - \rho}{\rho \pi + \rho}$$

and finance this subsidy by lump sum transfers and/or by issuing bonds.

We must emphasize that if the government has access to a commitment technologies, it will adopt an allocation consistent with the asymptotic discount rate of the discount function (36), that is $\rho$. The result of Calvo and Obstfeld [14] suggests then that long term capital income taxation is not required. Therefore, in our model, the capital income taxation is entirely driven by the time inconsistency friction faced by the government.

6 Conclusion

The central concern of this paper has been the attempt to define a methodology to analyze a capital accumulation dynamic game with non constant discount in continuous time and under the assumption of non commitment. The methodology only requires standard differential calculus techniques and the resulting equilibrium characterization is a generalization of the HJB equation. Using a special fixed point theorem, the central manifold theorem (Carr [17]), we were able to prove existence of multiple equilibria for a specification of the discount function. We perceive the connection of the methodology with the central manifold theorem as a good news: In fact the central manifold theorem comes with an approximation method (See Theorem 2 of Carr [17]) that opens the door to computational policy experimentation. Taken overall, the method seems to be applicable to a broad set of time inconsistency problems. We consider an application to the dynamic allocation problem for a forward looking utilitarian government in an overlapping generations economy. When the social discount rate and the private discount rate are distinct, the optimal command is time inconsistent and the government becomes strategic. We find that there are multiple steady states for the economy but that the (local) renegotiation-proofness requirement selects one of them. The strategic allocation can be implemented in a market economy.
if the government deploys distortional capital income taxation. Our main point here is that the time inconsistency friction that arises from the planning problem creates by itself a role for capital income taxation. The analysis of our overlapping generations model has been limited in several important respects. In particular, we did not impose any restriction on the fiscal tools, neither directly as in the Ramsey approach of optimal taxation nor indirectly as in the Mirrlees approach to optimal taxation. The non restrictiveness of the fiscal tools allowed us to achieve a separation between the allocation problem and the taxation problem. As a result, we shut down the time inconsistency generated by the interaction between the government and the private agents. An unresolved question that we plan to investigate in future work is to see how the preferences based time inconsistency interacts with the time inconsistency due instruments insufficiency. This can be done in the context of the Mirrleesian approach to optimal taxation and the main difficulty is to model a game with multiple private agents and multiple successive governments.

A Proof of Theorem 2

A.1 Preliminaries

Before proceeding with the proof of the theorem, let us mention some facts about the flow $K$ defined by (15), (16). To make notations simpler, we will use $K(t, k)$ instead of $K(\sigma; t, k)$ when the omission of the dependency on $\sigma$ causes no ambiguity.

Note first that, since equation (15) is autonomous, i.e. the right-hand side does not depend explicitly on time, the solution which takes the value $k$ at time 0 coincides with the solution which takes the value $K(t, k)$ at time $t \geq 0$. This is the so-called semi-group property, which is stated precisely as follows

$$K(s, K(t, k)) = K(s + t, k)$$

(57)

Next, consider the linearized equation around a prescribed solution $t \rightarrow K(t, k)$ of the nonlinear system (15), namely:

$$\frac{dk_1}{ds} = (f'(K(s, k)) - \sigma'(K(s, k))) k_1(s)$$

(58)

This is a linear equation, so the flow is linear. The value at time $t$ of the solution which takes the value $k$ at time 0 is $R(t)k$, where the function $R : R \rightarrow (0, \infty)$ satisfies:

$$\frac{dR}{dt} = (f'(K(s, k)) - \sigma'(K(s, k))) R(t)$$

(59)

$$R(0) = 1.$$  

(60)

From standard theory, it is well known that, if $f$ and $\sigma$ are $C^k$, then $K$ is $C^{k-1}$,
\[
\frac{\partial K(t,k)}{\partial k} = R(t)
\]
\[
\frac{\partial K(t,k)}{\partial t} = -R(t) (f(k) - \sigma(k)).
\]

Let us now turn to the actual proof of Theorem 2.

### A.2 Necessary condition

Given a Markov equilibrium strategy \( \sigma \), we define the associated value function \( v(k) \) as in formula (17)

\[
v(k) := \int_0^\infty h(t) u(\sigma(K(t,k))) dt
\]  

(61)

Differentiating with respect to \( k \), we find that:

\[
v'(k) = \int_0^\infty h(t) u'(\sigma(K(t,k))) \frac{\partial K(t,k)}{\partial k} dt
\]

Since \( \sigma \) is an equilibrium strategy, the maximum of \( P_1(k,\sigma,c) \) with respect to \( c \) must be attained at \( \sigma(k) \). The function \( P_1 \) itself is given by formula (13), where \( k_0(t) = K(t,k) \) and where \( k_1 \) is defined by (10) and (11), so that \( k_1(t) = R(t) (\sigma(k) - c) \). Substituting into (13), we get:

\[
P_1(k,\sigma,c) = u(c) - u(\sigma(k))
\]

\[
+ \int_0^\infty h(t) u'(\sigma(K(t,k))) \sigma'(K(t,k)) R(t) (\sigma(k) - c) dt
\]

Since \( u \) is concave and differentiable, the necessary and sufficient condition to maximize \( P_1(k,\sigma,c) \) with respect to \( c \) is

\[
u'(c) = \int_0^\infty h(t) u'(\sigma(K(t,k))) \sigma'(K(t,k)) R(t) dt
\]

which is precisely \( v'(t,k) \), as we just saw. Therefore, the equilibrium strategy must satisfy

\[
u'(\sigma(k)) = v'(k)
\]

and, substituting back into equation (61), we get equation (IE).

### A.3 Sufficient condition

Assume now that there exists a function \( v \) satisfying (IE), and consider the strategy \( \sigma = i \circ v' \). Given any consumption choice \( c \in R \), the payoff to the decision-maker at
time 0 is

\[ P_1 (k, \sigma, c) = \underbrace{u (c) - u (\sigma (k))}_{\text{first equality follows from the definition of } R} \]
\[ + \int_0^\infty h (t) \underbrace{u' (\sigma (K (t, k))) \sigma' (K (t, k))}{\text{second equality is obtained by differentiating } v \text{ with respect to } k} \underbrace{R (t)}{\text{third equality follows from the definition of } \sigma} \underbrace{(\sigma (k) - c)}{\text{the last inequality is due to the concavity of } u}. \]

where the first equality follows from the definition of \( R \), the second equality is obtained by differentiating \( v \) with respect to \( k \), the third equality follows from the definition of \( \sigma \), and the last inequality is due to the concavity of \( u \). Observing that \( P_1 (k, \sigma, \sigma (k)) = 0 \), we see that the inequality \( P_1 (k, \sigma, c) \leq 0 \) implies that \( c = \sigma (k) \) achieves the maximum so that \( \sigma \) is an equilibrium strategy.

### B Proof of Proposition 3

Let a function \( v : R \rightarrow R \) be given. Consider the function \( \varphi : R \rightarrow R \) defined by

\[ \varphi (k) = v (k) - \int_0^\infty h(t)u(\sigma (\sigma (t, k))) \, dt \]  \hspace{1cm} (62)

where \( \sigma (k) = i (v' (k)) \). Consider \( \psi (t, k) \), the value of \( \varphi \) along the trajectory \( t \rightarrow K (\sigma; t, k) \) originating from \( k \) at time 0, that is

\[ \psi (t, k) = \varphi (K (t, k)) \]
\[ = v (K (t, k)) - \int_0^\infty h(s)u(\sigma (s, K (t, k))) \, ds \]
\[ = v (K (t, k)) - \int_0^\infty h(s)u(\sigma (s + t, k))) \, ds \]
\[ = v (K (t, k)) - \int_t^\infty h(s - t)u(\sigma (s, k))) \, ds \]

where we have used formula (57).

We compute the derivative of this function with respect to \( t \):

\[ \frac{\partial \psi}{\partial t} (k, t) = v' (K (t, k)) [f (K (t, k)) - i (\sigma (K (t, k)))] \]
\[ + u (\sigma (K (t, k))) + \int_t^\infty h' (s - t)u (\sigma (K (s, k))) \, ds \]

From the definition of \( i \), we have

\[ u (i (v' (K (t, k)))) - v' (K (t, k)) i (v' (K (t, k))) = \sup_c \{ u (c) - v' (K (t, k)) c \} \]
Substituting in the preceding equation, and recalling that $\sigma = i \circ v'$ gives
\[
\frac{\partial \psi}{\partial t} (k, t) = \sup_c \{ u(c) + v'(K(t, k) (f(K(t, k)) - c)) \} + \int_0^\infty h'(s-t) u(\sigma(K(s, k))) ds
\]
\[
= \sup_c \{ u(c) + v'(K(t, k) (f(K(t, k)) - c)) \} + \int_0^\infty h'(s) u(\sigma(K(s, K(t, k)))) ds
\]
where the second equality is obtained by using a change of variable and formula (57).

If (19) holds, then the right-hand side of the last equation is identically zero along the trajectory, so that $\psi(k, t) = \psi(k)$ does not depend on $t$. Letting $t \to \infty$ in the definition of $\psi$, we get:
\[
\psi(k) = \lim_{t \to \infty} \left\{ v(K(t, k)) - \int_0^\infty h(s) u(\sigma(K(s + t, k))) ds \right\}
\]
\[
= v(\bar{k}) - \int_0^\infty h(s) u(\sigma(\bar{k})) ds = v(\bar{k}) - u(f(\bar{k})) \int_0^\infty h(s) ds
\]
and hence, if (20) holds then, $\psi = \varphi = 0$ and equation (IE) holds.

Conversely, if $v(k)$ satisfies equation (IE), then the same lines of reasoning shows that equation (19) and the boundary condition are satisfied.

\section{C Proof or Proposition 4}

The proof uses the following:

\textbf{Lemma 8} Let $\sigma(k)$ be any convergent Markov strategy. Denote its steady state by $\bar{k}$. Let $h_0 : [0, \infty] \to R$ be any $C^1$ function with exponential decay at infinity, that is
\[
h_0(t) \leq Ce^{-vt}
\]
for some positive constants $\nu > 0$ and $C \geq 0$. A $C^1$ function $I$ satisfies
\[
I'(k) (f(k) - \sigma(k)) + \int_0^\infty h_0'(t) \ln(\sigma(K(\sigma; t, k))) dt + h_0(0) \ln(\sigma(k)) = 0 \quad (63)
\]
for all $k$ and, the boundary condition
\[
I(\bar{k}) = \int_0^\infty h_0(t) \ln(f(\bar{k})) \quad (64)
\]
if and only if it satisfies
\[
I(k) = \int_0^\infty h_0(t) \ln(\sigma(K(\sigma; t, k))) dt. \quad (65)
\]

\textbf{Proof.} We argue as in the preceding proof. For any $C^1$ function $I$, consider the function $\psi(k, t)$ defined by
\[
\psi(\sigma; k, t) = I(K(\sigma; t, k)) - \int_t^\infty h_0(s-t) \ln(\sigma(K(\sigma; s, k))) ds.
\]
Differentiating with respect to $t$, we get
\[
\frac{\partial \psi}{\partial t} = I'(\mathcal{K}(\sigma; t, k)) \left( f(\mathcal{K}(\sigma; t, k)) - \sigma(\mathcal{K}(\sigma; t, k)) \right) \\
+ \int_{t}^{\infty} h'_0(s-t) \ln(\sigma(\mathcal{K}(\sigma; s, k))) \, ds + h_0(0) \ln(\sigma(\mathcal{K}(\sigma; t, k))).
\]

Using a change of variable and equation (57), notice that
\[
\psi(\sigma; k, t) = I(\mathcal{K}(\sigma; t, k)) - \int_{0}^{\infty} h_0(s) \ln(\sigma(\mathcal{K}(\sigma; s, \mathcal{K}(\sigma; t, k)))) \, ds.
\]

Therefore, if (65) holds, then $\psi$ is identically zero, and so is its derivative $\frac{\partial \psi}{\partial t}$, so (63) holds. Conversely, if (63) holds, then $\frac{\partial \psi}{\partial t}$ vanishes, and $\psi(k, t) = \psi(k)$ does not depend on $t$, so that
\[
\psi(k) = \lim_{t \to -\infty} \left\{ I(\mathcal{K}(\sigma; t, k)) - \int_{0}^{\infty} h_0(s) \ln(\sigma(\mathcal{K}(\sigma; s, \mathcal{K}(\sigma; t, k)))) \, ds \right\}
\]
\[= I(\bar{k}) - \int_{0}^{\infty} h_0(s) \ln(\sigma(\bar{k})) \, dt.
\]

If in addition, (64) holds then (65) holds. \qed

Now let us turn to the proof of Proposition 4. To simplify the notations, we shall write $h(t)$ in the following way
\[
h(t) = \begin{cases} \theta e^{-(\delta + \pi)t} + (1 - \theta) e^{-\rho t} 
\end{cases}
\]
where
\[
\theta = \frac{\delta - \rho}{\pi - \delta - \rho}, \quad 1 - \theta = \frac{\pi}{\pi + \delta - \rho}
\]

Suppose that $\sigma$ is an equilibrium strategy converging to $\bar{k}$. Proposition 3 shows that the value $v$ defined by (37) satisfies (19), (20). After performing the optimization in $c$ and substituting $u(c) = \ln(c)$, the right hand side of (19) becomes the left hand side of (39). Using the definitions of $v$ and $w$ given by (37) and (38), it can be checked that the left hand side of (19) coincides with the linear combination of $v$ and $w$ given in the right hand side of (39). Therefore, we proved equation (39). The boundary condition (41) is proved by integrating (37) after replacing the consumption flow $\sigma(\mathcal{K}(\sigma; t, k))$ by the steady state consumption level $f(\bar{k})$.

To get equation (40), we use the preceding Lemma with $\sigma(k) = \begin{cases} 1/v'(k) \end{cases}$, $h_0(t) = (1 - \theta) \left( e^{-\rho t} - e^{-(\delta + \pi)t} \right)$ and we get an equation for $I(k) = w(k)$ given by
\[
w'(k) \left( f(k) - \frac{1}{v'(k)} \right) = -(1 - \theta) \int_{0}^{\infty} \left( (\delta + \pi) e^{-(\delta + \pi)t} - \rho e^{-\rho t} \right) u(\sigma(\mathcal{K}(\sigma; t, k))) \, dt
\]
and a boundary condition
\[
w(\bar{k}) = (1 - \theta) \left( \frac{1}{\rho} - \frac{1}{\delta + \pi} \right) \ln f(\bar{k}) = \frac{\pi}{\rho (\delta + \pi)} \ln f(\bar{k})
\]
The right hand side of (66) can be written as the linear combination \(-\pi v(k) + (\rho + \pi)w(k)\) and thus, we proved equation (40).

Conversely, suppose \(v_1\) and \(w_1\) satisfy the equations (39) and (10), together with the boundary conditions (41) and (42), with the strategy \(\sigma_1 = 1/v_1\) converging to \(\bar{k}\), so that
\[
\left(f - \frac{1}{v_1}\right) v_1' - \ln(v_1') = \delta v_1 - (\delta - \rho)w_1
\]
\[
v_1(\bar{k}) = \left(\frac{\theta}{\delta + \pi} + \frac{1-\theta}{\rho}\right) \ln f(\bar{k})
\]
\[
\left(f - \frac{1}{v_1}\right) w_1' = -\pi v_1 + (\rho + \pi)w_1
\]
\[
w_1(\bar{k}) = (1-\theta) \left(\frac{1}{\rho} - \frac{1}{\delta + \pi}\right) \ln f(\bar{k})
\]

(67)

Consider the functions:
\[
v_2(k) = \int_0^\infty \left(\theta e^{-(\delta + \pi)t} + (1-\theta)e^{-\rho t}\right) \ln (\sigma_1(K(\sigma_1; t, k))) \, dt
\]
\[
w_2(k) = (1-\theta) \int_0^\infty \left(-e^{-(\delta + \pi)t} + e^{-\rho t}\right) \ln (\sigma_1(K(\sigma_1; t, k))) \, dt
\]

(68)

(69)

Applying the preceding Lemma with \(I = v_2\) and \(I = w_2\) successively, we have
\[
v_2'(k) (f - \sigma_1) + \int_0^\infty \left(\theta(\delta + \pi)e^{-(\delta + \pi)t} + (1-\theta)\rho e^{-\rho t}\right) \ln (\sigma_1(K(\sigma_1; t, k))) \, dt + \ln (\sigma_1(k)) = 0
\]
\[
v_2(\bar{k}) = \left(\frac{\theta}{\delta + \pi} + \frac{1-\theta}{\rho}\right) \ln f(\bar{k})
\]
\[
w_2'(k) (f - \sigma_1) + (1-\theta) \int_0^\infty \left(-e^{-(\delta + \pi)t} + e^{-\rho t}\right) \ln (\sigma_1(K(\sigma_1; t, k))) \, dt = 0
\]
\[
w_2(\bar{k}) = (1-\theta) \left(\frac{1}{\rho} - \frac{1}{\delta + \pi}\right) \ln f(\bar{k})
\]

and hence
\[
v_2'(k) (f - \sigma_1) + \ln (\sigma_1(k)) = \delta v_2 - (\delta - \rho)w_2
\]
\[
w_2'(f - \sigma_1) = -\pi v_2 + (\rho + \pi)w_2
\]

(70)

Substracting (70) from (67), and setting \(v = v_1 - v_2,\ w = w_1 - w_2\), we get:
\[
\left(f - \frac{1}{v_1}\right) v' = \delta v_2 - (\delta - \rho)w_2
\]
\[
v(\bar{k}) = 0
\]
\[
\left(f - \frac{1}{v_1}\right) w' = -\pi v_2 + (\rho + \pi)w_2
\]
\[
w(\bar{k}) = 0
\]

(71)

Obviously \(v = w = 0\) is a solution. In the next Lemma, we show that it is the only one, so \(v_1 = v_2\) and \(w_1 = w_2\). Equation (68) then becomes:
\[
v_1(k) = \int_0^\infty \left(\theta e^{-(\delta + \pi)t} + (1-\theta)e^{-\rho t}\right) \ln (\sigma_1(K(\sigma_1; t, k))) \, dt
\]
which is precisely equation (IE) for \(h(t) = \left(\theta e^{-(\delta + \pi)t} + (1-\theta)e^{-\rho t}\right)\) and \(u(c) = \ln c\). Since \(v_1\) satisfies (IE), the strategy \(\sigma_1\) is then an equilibrium strategy.
Lemma 9 If \((v, w)\) is a pair of functions, which are continuous on a neighbourhood \(\Omega\) of \(k\), continuously differentiable for \(k \neq \bar{k}\), and which solve (71) for \(k \neq \bar{k}\), then \(v = 0\) and \(w = 0\).

Proof. Set \(f(k) - 1/v_1'(k) = \varphi(k)\). Note that \(\varphi(k) \to 0\) when \(k \to \bar{k}\). Since \(v_1\) is \(C^1\), in fact \(C^2\), and \(v_1'(\bar{k}) \neq 0\), the value \(\varphi(k)\) changes signs when \(k\) crosses \(\bar{k}\). The system can be rewritten as:

\[
\begin{pmatrix}
\varphi v' \\
\varphi w'
\end{pmatrix} =
\begin{pmatrix}
\delta & -(\delta - \rho) \\
-\pi & \rho + \pi
\end{pmatrix}
\begin{pmatrix}
v \\
w
\end{pmatrix}
\]

The characteristic equation of the matrix on the right-hand side is:

\[
\lambda^2 - (\rho + \delta + \pi) \lambda + \rho (\delta + \pi) = 0
\]

and the roots are \(\lambda = \rho\) and \(\lambda = \delta + \pi\). Changing basis, we can rewrite the system as:

\[
\begin{pmatrix}
\varphi V' \\
\varphi W'
\end{pmatrix} =
\begin{pmatrix}
\rho & 0 \\
0 & \delta + \pi
\end{pmatrix}
\begin{pmatrix}
V \\
W
\end{pmatrix}
\]

where \(V\) and \(W\) are suitable linear combinations of \(v\) and \(w\). The solutions are

\[
V(k) = C_1 \exp \left( \rho \int_{k_0}^{k} \frac{1}{\varphi(l)} dl \right), \quad W(k) = C_2 \exp \left( (\delta + \pi) \int_{k_0}^{k} \frac{1}{\varphi(l)} dl \right)
\]

where \(C_1, C_2\) and \(k_0\) are constants. Since \(1/\varphi(k) \to \pm \infty\) when \(k \to \bar{k}\), and both signs occur, one for \(k < \bar{k}\) and the other for \(k > \bar{k}\), the only way we can get \(V(\bar{k}) = W(\bar{k}) = 0\) is by setting \(C_1 = C_2 = 0\).

D Proof of Theorem 5

By Proposition 4, it is enough to show that the system (39)-(40) with boundary conditions (41)-(42), has a solution \((v, w)\) where \(v\) and \(w\) are required to be \(C^2\) and such that the resulting capital stock

\[
\frac{dk}{dt} = f(k) - \frac{1}{v'(k)}, \quad k(0) = k_0
\]

has the property that \(\lim_{t \to \infty} k(t) = \bar{k}\) when \(k_0\) is sufficiently close to \(\bar{k}\).

We will give the proof in several steps. First, we make a sequence of change of variables and change of coordinates that simplify the system (39)-(40). Second, we apply the Central Manifold Theorem (e.g. [17]) to show existence. Third, we conclude the proof by demonstrating the estimate of the steady states \(\bar{k}\) given in Theorem 5.

D.1 Change of variables

Let us begin with the following useful lemma which proof is omitted.

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Lemma 10 The equation \( x - \ln (1 + x) = \mu \) has no solution for \( \mu < 0 \). For \( \mu = 0 \), the solution is \( x = 0 \). For \( \mu > 0 \), there are two solutions \( X_1(\mu) \) and \( X_2(\mu) \) with the following properties:

(a) \( X_1 \) is decreasing, \( X_1(0) = 0 \), and \( X_1(\mu) \to -1 \) when \( \mu \to \infty \)

(b) \( X_2 \) is increasing, \( X_2(0) = 0 \), and \( X_2(\mu) \to \infty \) when \( \mu \to \infty \)

(c) both \( X_1 \) and \( X_2 \) are continuous on \([0, \infty]\) and \( C^\infty \) on \((0, \infty)\)

Note, that neither \( X_1 \) nor \( X_2 \) are differentiable at 0. In fact, if we replace the function \( x - \ln (1 + x) = \mu \) by its Taylor expansion near \( x = 0 \), we find that the equation \( x - \ln (1 + x) = \mu \) is replaced by the equation \( x^2 = \mu \), so that \( X_1(\mu) \) and \( X_2(\mu) \) are approximated by \(-\sqrt{\mu}\) and \(\sqrt{\mu}\) respectively when \( \mu > 0 \) is small.

Introducing now the function \( \mu \) defined by

\[
\mu(k) := \delta v(k) - (\delta - \rho)w(k) - \ln f(k)
\] (72)
equation (39) becomes

\[
fv' - 1 - \ln fv' = \mu
\]
and, using the notation of Lemma 10 this equation becomes

\[
f(k)v'(k) = 1 + X_i(\mu(k)), \quad i = 1, 2
\] (73)

Next, differentiating equation (72) gives

\[
f\mu' = f\left(\delta v'(k) - (\delta - \rho)w' - \frac{f'}{f}\right)
\]
and, using equation (40) to eliminate \( w' \) from the above equation yields

\[
f\mu' = \delta(1 + X(\mu)) - f' + \frac{1 + X(\mu)}{X(\mu)} \left[-\rho(\pi + \delta)v + (\pi + \rho)\mu + (\pi + \rho)\ln f\right]
\] (74)
after replacing \( fv' \) by \( 1 + X(\mu) \).

Our new system is now (73), (74) where \( k \) is the independent variable, the function \( f(k) \) is given, the functions \( X_1(\mu) \) and \( X_2(\mu) \) are defined in Lemma 10 and the unknown functions are \( v(k) \) and \( \mu(k) \). The right-hand sides of the equations (73), (74) are function of \( (k, v, \mu) \), which are defined and \( C^\infty \) for \( k > 0 \) and \( \mu > 0 \).

Suppose now we have a solution \((v, \mu)\), with

\[
\mu(\bar{k}) = 0.
\] (75)

Then \( X(\mu(\bar{k})) = 0 \), so that

\[
v'(\bar{k}) = 1/f(\bar{k})
\]
by equation (73), and the right-hand side of equation (74) blows up unless

$$v(\bar{k}) = \frac{\rho + \pi}{\rho (\delta + \pi)} \ln f(\bar{k})$$

which is precisely our boundary condition (41). If both (41) and (75) hold, the second boundary condition (42) is automatically implied by (72).

Let us rewrite our new system (omitting the index $i = 1$)

$$f(k) \frac{dv}{dk} = 1 + X(\mu)$$
$$f(k) \frac{d\mu}{dk} = \delta(1 + X(\mu)) - f' + \frac{1 + X(\mu)}{X(\mu)} [-\rho (\pi + \delta) v + (\pi + \rho) \mu + (\pi + \rho) \ln f]$$

with the boundary conditions (41) and (75).

We now turn to a change of coordinate by taking $X(\mu(k)) = x$ as the independent variable instead of $k$ along the trajectory. In other words, instead of looking for functions $v(k)$ and $\mu(k)$ satisfying the equations (73), (74), we will be looking for functions $\tilde{v}(x)$ and $\tilde{k}(x)$ satisfying

$$\frac{d\tilde{v}}{dx} = \frac{dv}{dk} = \frac{1 + x \frac{dk}{dx}}{f} \frac{d}{dx},$$
$$\frac{d\tilde{k}}{dx} = \frac{X}{\mu} \frac{d\mu}{dx} = \frac{1}{d\mu/dx} \left\{ \delta(1 + x) - f' + \frac{1 + x}{x} \left[-\rho (\pi + \delta) v + (\pi + \rho) \mu + (\pi + \rho) \ln f \right] \right\}^{-1}$$

$$= \frac{1}{d\mu/dx} \left\{ \frac{x}{1 + x} - \rho (\pi + \delta) v + (\pi + \rho) \mu + (\pi + \rho) \ln f \right\}^{-1}.$$

Plugging in the relations $\mu = x - \ln (1 + x)$, so that $\frac{d\mu}{dx} = \frac{1}{1 + x}$, yields the new system

$$\frac{d\tilde{k}}{dx} = f(\tilde{k}) \frac{x^2}{1 + x} \frac{1}{D(x, \tilde{k}, \tilde{v})} \quad (76)$$
$$\frac{d\tilde{v}}{dx} = x^2 \frac{1}{D(x, \tilde{k}, \tilde{v})} \quad (77)$$

where the function $D$ is given by

$$D(x, k, v) = (1 + x) [-\rho (\pi + \delta) v + (\pi + \delta + \rho) x]$$
$$(1 + x) (\pi + \delta) [\ln f(k) - \ln (1 + x)] - xf'(k). \quad (78)$$

We pick a point $\bar{k}$ and we look for $C^2$ solutions $\left(\tilde{k}(x), \tilde{v}(x)\right)$ of the system (76), (77), defined in a neighbourhood of $x = 0$ and satisfying:

$$\tilde{k}(0) = \bar{k} \quad (79)$$
$$\tilde{v}(0) = \frac{\rho + \pi}{\rho (\delta + \pi)} \ln f(\bar{k}) := \bar{v} \quad (80)$$
We now introduce a new variable \( s \), and we replace the system (76), (77) with the following autonomous system

\[
\frac{dx}{ds} = D(x, k, v), \quad x(0) = 0
\]
\[
\frac{dk}{ds} = \frac{\tilde{d}k}{dx} \frac{dx}{ds} = f(k) \frac{x^2}{1 + x}, \quad k(0) = \bar{k}
\]
\[
\frac{dv}{ds} = \frac{\tilde{d}v}{dx} \frac{dx}{ds} = x^2, \quad v(0) = \bar{v}
\]

where there are now three unknown functions \( (x(s), k(s), v(s)) \), defined near \( s = 0 \).

Note that \( D \) is as smooth as \( f' \) in a neighbourhood of \( (0, \bar{k}, \bar{v}) \) and that \( D(0, \bar{k}, \bar{v}) = 0 \).

Our problem is then to analyze the system (81), (82) and (83) since it is related to the initial system (39), (40) through a sequence of smooth change of variables and coordinates.

### D.2 Existence

The linearized system near \( (0, \bar{k}, \bar{v}) \) is:

\[
\frac{d}{ds} \begin{pmatrix} x \\ k \\ v \end{pmatrix} = A \begin{pmatrix} x \\ k \\ v \end{pmatrix}
\]

with (all derivatives to be computed at \( (0, \bar{k}, \bar{v}) \)):

\[
A := \begin{pmatrix}
\frac{\partial D}{\partial x} & \frac{\partial D}{\partial k} & \frac{\partial D}{\partial v} \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} = \begin{pmatrix}
\delta - f'(\bar{k}) (\pi + \rho) f'(\bar{k}) & -\rho(\pi + \delta) \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

The matrix \( A \) has the eigenvalues \( (\delta - f'(\bar{k}), 0, 0) \), and can obviously be put in diagonal form (in this case, as a matrix with 1 in the upper left corner, all the other coefficients being 0). To make the change of variables explicit, we note that the eigenvector associated with the eigenvalue \( \delta - f'(\bar{k}) \) is

\[
e = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}
\]

and we set \( E_1 := \text{Span} \{ e \} \). We also consider the kernel of \( A \), and denote it by \( E_0 \):

\[
E_0 := \ker A = \left\{ \begin{pmatrix} x \\ k \\ v \end{pmatrix} \mid (\delta - f'(\bar{k})) x + (\pi + \rho) f'(\bar{k}) k - \rho(\pi + \delta)v \right\}
\]

Both \( E_1 \) and \( E_0 \) are invariant subspaces of the linearized system (84) with the corresponding eigenvalues being \( \lambda_1 := \delta - f'(\bar{k}) \neq 0 \) and 0, and the operator \( A \)
is diagonal in any base \((e_1, e_2, e_3)\) with \(e_1 \in E_1\) and \((e_2, e_3) \in E_3\). By the central manifold theorem (see for instance \cite{17}, Theorem 1\textsuperscript{19}), there exists a map \(h(k, v)\), defined in a neighborhood \(\mathcal{O}\) of \((\bar{k}, \bar{v})\) such that

\[
h(\bar{k}, \bar{v}) = 0, \quad \frac{\partial h}{\partial k}(\bar{k}, \bar{v}) = 0, \quad \frac{\partial h}{\partial v}(\bar{k}, \bar{v}) = 0
\]

and the manifold \(\mathcal{M}\) defined by:

\[
\mathcal{M} := \{(\alpha k + \beta v + h(k, v), k, v) \mid (k, v) \in \mathcal{O}\}
\]

is invariant by the flow associated with the equations (81), (82), (83). The map \(h\) and the central manifold \(\mathcal{M}\) are as smooth as \(f'\); they are \(C^2\), for instance, if \(f\) is \(C^3\). Note that \(\mathcal{M}\) is two-dimensional and tangent to \(E_0\) at \((0, \bar{k}, \bar{v})\). Note also that there is another invariant manifold \(N\), which is tangent to \(E_1\) at \((0, \bar{k}, \bar{v})\): it is one-dimensional, and it is stable if \(\lambda_1 = \delta - f'(\bar{k}) < 0\) and unstable if \(\lambda_1 = \delta - f'(\bar{k}) > 0\). Each of these invariant manifolds gives a different type of solution to the system equations (81), (82), (83).

We are interested in the solutions which lie on the central manifold \(\mathcal{M}\). They can be found by substituting \(x = \alpha k + \beta v + h(k, v)\) in equations (82) and (83), yielding

\[
\frac{dk}{ds} = f(k) \left[\frac{\alpha k + \beta v + h(k, v)}{1 + \alpha k + \beta v + h(k, v)}\right]^2, \quad k(0) = \bar{k} \tag{85}
\]

\[
\frac{dv}{ds} = \left[\frac{\alpha k + \beta v + h(k, v)}{1 + \alpha k + \beta v + h(k, v)}\right]^2, \quad v(0) = \bar{v} \tag{86}
\]

while \(x\) is found by using the fact that \(\mathcal{M}\) is invariant

\[
x(s) = \alpha k(s) + \beta v(s) + h(k(s), v(s)).
\]

Eliminating the variable \(s\) from (85) and (86), we get

\[
\frac{dv}{dk} = \frac{f(k)}{1 + \alpha k + \beta v + h(k, v)}, \quad v(\bar{k}) = \bar{v}
\]

The solution of this initial-value problem is \(v(k) = \psi(k)\), where \(\psi(\bar{k}) = \bar{v}\) and \(\psi\) is \(C^2\) if \(h\) is \(C^2\), that is, if \(f\) is \(C^3\)(see above). Substituting in \(x = h(k, v)\), we get \(x(k) = \alpha k + \beta v + h(k, \psi(k))\). Finally, \(\mu(k) = x(k) - \ln(1 + x(k))\) is also \(C^\infty\), so we have found a smooth solution of equations (73) and (74), as desired.

Differentiating equations (39) and evaluating it at \(k = \bar{k}\), taking into account that \(f(\bar{k}) \psi'(\bar{k}) = 1\), yields the following

\[
w'(\bar{k}) = \frac{1}{(\delta - \rho)f(\bar{k})} (\delta - f'(\bar{k}))
\]

\textsuperscript{19}Theorem 1 in Carr \cite{17} requires that the linearized matrix at the fixed point be block diagonal. The matrix \(A\) is not block diagonal but the theorem can still be applied because the matrix \(A\) can be transformed into a block diagonal matrix with some appropriate change of basis.
D.3 Proving the estimate

It remains to prove that the strategy \( \sigma(k) = 1/v'(k) \) is convergent. Recall that this means that the solutions of the equation

\[
\frac{dk}{dt} = f(k) - \frac{1}{v'(k)}, \quad k(0) = k_0
\]

converge to \( \bar{k} \) if \( k_0 \) is sufficiently close to \( \bar{k} \). Since \( f(\bar{k}) = 1/v'(\bar{k}) \), \( \bar{k} \) is a fixed point of the dynamical system, and we want to show that it is an attractor. This means that the linearized system at \( \bar{k} \), namely

\[
\frac{dk}{dt} = \left( f'(\bar{k}) + \frac{v''(\bar{k})}{v'(\bar{k})^2} \right) k
\]

must have \( k = \bar{k} \) as an attractor. In other words, we must have

\[
f'(\bar{k}) + \frac{v''(\bar{k})}{v'(\bar{k})^2} < 0 \quad (87)
\]

To compute the left-hand side of (87), differentiate equation (40) at set \( k = \bar{k} \). We get

\[
\left( f'(\bar{k}) + \frac{v''(\bar{k})}{v'(\bar{k})^2} \right) w'(\bar{k}) = -\pi v'(\bar{k}) + (\rho + \pi) w'(\bar{k}).
\]

To find \( w'(\bar{k}) \), differentiate equation (39) and evaluate at \( k = \bar{k} \), taking into account that \( f(\bar{k}) v'(\bar{k}) = 1 \), to get

\[
w'(\bar{k}) = \frac{1}{(\delta - \rho)f(\bar{k})} (\delta - f'(\bar{k})).
\]

Hence

\[
\left( f'(\bar{k}) + \frac{v''(\bar{k})}{v'(\bar{k})^2} \right) w'(\bar{k}) = \frac{\rho(\delta + \pi) - (\rho + \pi)f'(\bar{k})}{f(\bar{k})(\delta - \rho)}.
\]

and, switching \( w'(\bar{k}) \) from the left hand side to the right hand side of the last equation gives

\[
f'(\bar{k}) + \frac{v''(\bar{k})}{v'(\bar{k})^2} = \frac{\rho(\delta + \pi) - (\rho + \pi)f'(\bar{k})}{\delta - f'(\bar{k})} \quad (88)
\]

This will be negative if the numerator and denominator have opposite signs. Both the numerator and the denominators of the right hand side of the last equation are increasing functions of \( \bar{k} \) and they change sign respectively at \( f'(\bar{k}) = \rho \frac{\delta + \pi}{\pi + \rho} \) and \( f'(\bar{k}) = \delta \). Since \( \rho < \pi \), we have \( \rho \frac{\delta + \pi}{\pi + \rho} < \delta \) and therefore the only interval where

\[
f'(\bar{k}) + \frac{v''(\bar{k})}{v'(\bar{k})^2} < 0 \]

is the open interval \( I = \left(\rho \frac{\delta + \pi}{\pi + \rho}, \delta \right) \).
Proof of Theorem 7

We will need a preliminary result, the proof of which is quite obvious.

Lemma 11 Let \( f(x,y) \) be a \( C^1 \) function of two variables such that:

\[
\begin{align*}
  f(x,x) &= \varphi(x) \\
  \frac{\partial f}{\partial x}(x,x) &= \psi(x)
\end{align*}
\]

Then:

\[
\frac{\partial f}{\partial y}(x,x) = \varphi'(x) - \psi(x)
\]

To prove that \( \bar{\sigma} \) is not l.r.p, we will establish that there exists \( \varepsilon > 0 \) such the inequality (46) is not satisfied for all \( k_0 \in (\bar{k} - \varepsilon, \bar{k} + \varepsilon) \) and \( k^* \in (\bar{k}, \bar{k} + \varepsilon) \). We will now introduce a new notation of the equilibrium value as a function of the initial point \( k_0 \), as before, and of the terminal point \( \bar{k} \). More precisely, we define

\[
V(k_0, \bar{k}) = \int_0^\infty h(t) u(\bar{\sigma}(\mathcal{K}(\bar{\sigma}; t, k_0))) dt
\]

where \( h(t) \) is given by (36) and \( \bar{\sigma} \) is an equilibrium strategy converging to \( \bar{k} \). Assuming that \( V \) is differentiable with respect to \( \bar{k} \), we can apply the preceding Lemma, with

\[
V(\bar{k}, \bar{k}) = \frac{\rho + \pi}{\rho(\delta + \pi)} \ln f(\bar{k}), \quad \frac{\partial V}{\partial k_0}(\bar{k}, \bar{k}) = \frac{1}{f(\bar{k})}
\]

and get

\[
\frac{\partial V}{\partial k}(\bar{k}, \bar{k}) = \frac{1}{f(\bar{k})} \left( \frac{\pi + \rho}{\rho(\pi + \delta)} f'(\bar{k}) - 1 \right)
\]

which is positive on the whole allowable interval \( I \). It follows that there exists \( \varepsilon > 0 \)

\[
\frac{\partial V}{\partial k}(k_0, \bar{k}) > 0
\]

for all \( k_0, \bar{k} \) in the interval \( (\bar{k} - \varepsilon, \bar{k} + \varepsilon) \).

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20 The function \( V \) is defined for any \( k^* \in I \) by

\[
V(k_0, k^*) = \int_0^\infty h(t) u(\sigma^*(\mathcal{K}(\sigma^*; t, k_0))) dt
\]

where \( \sigma^* \) is an equilibrium policy converging to \( k^* \). Of course, there may be multiple equilibria converging to \( k^* \) and in order to define properly \( V(\ldots) \) we need to make an a priori selection of a converging equilibrium policy for each \( k^* \in I \).
Now, consider any initial capital level \( k_0 \in (\bar{k} - \varepsilon, \bar{k} + \varepsilon) \) and steady state capital level \( k^* \in (\bar{k}, \bar{k} + \varepsilon) \). By the mean value theorem, we have

\[
V(k_0, k^*) - V(k_0, \bar{k}) = (k^* - \bar{k}) \frac{\partial V}{\partial k} (k_0, \tilde{k}) > 0
\]

for some \( \tilde{k} \in (\bar{k}, k^*) \) and where the inequality follows from the fact that \( \tilde{k} \in (\bar{k} - \varepsilon, \bar{k} + \varepsilon) \).

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