HOMOLOGICAL BRANCING LAW FOR \((\text{GL}_{n+1}(F), \text{GL}_n(F))\): PROJECTIVITY AND INDECOMPOSABILITY

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ABSTRACT. This paper studies homological properties of irreducible representations restricted from \(\text{GL}_{n+1}(F)\) to \(\text{GL}_n(F)\). We establish the following:

(1) classify irreducible smooth representations of \(\text{GL}_{n+1}(F)\) which are projective when restricted to \(\text{GL}_n(F)\);

(2) prove that each Bernstein component of an irreducible smooth representation of \(\text{GL}_{n+1}(F)\), restricted to \(\text{GL}_n(F)\), is indecomposable.

In appendixes, we study some aspects of Speh representations, and in particular we give an explicit formulae of Ext-groups between Speh representations in terms of symmetric group representations and discuss the related Ext-bracing law.

1. Introduction

Let \(F\) be a non-Archimedean local field. Let \(G_n = \text{GL}_n(F)\). Let \(\text{Alg}(G_n)\) be the category of smooth representations of \(G_n\). This paper is a sequel of [CS18b] in studying homological properties of representations in \(\text{Alg}(G_{n+1})\) restricted to \(G_n\), which is originally motivated from the study of D. Prasad in his ICM proceeding [Pr18]. In [CS18b], we show that for generic representations \(\pi\) and \(\pi'\) of \(G_{n+1}\) and \(G_n\) respectively, the higher Ext-groups

\[ \text{Ext}^i_{G_n}(\pi, \pi') = 0, \text{ for } i \geq 1, \]

which was previously conjectured in [Pr18]. This result gives a hope that there is an explicit homological branching law, generalizing the multiplicity one theorem [AGRS10], [SZ12] and the local Gan-Gross-Prasad conjecture [GGP12].

The main techniques in [CS18b] are utilizing some Hecke algebra structure developed in [CS19] and simultaneously applying left and right Bernstein-Zelevinsky derivatives, based on the classical approach of using Bernstein-Zelevinsky filtration on representations of \(G_{n+1}\) restricted to \(G_n\) [Pr93, Pr18]. We shall extend these methods further, in combination of other things, to obtain new results in this paper.

In [CS18b], we showed that an essentially square-integrable representation \(\pi\) of \(G_{n+1}\) is projective when restricted to \(G_n\). However, those representations do not account for all irreducible representations whose restriction is projective. The first goal of the paper is to classify such representations:

**Theorem 1.1.** Let \(\pi\) be an irreducible smooth representation of \(G_{n+1}\). Then \(\pi|_{G_n}\) is projective if and only if

(1) \(\pi\) is essentially square integrable, or
There are also recent studies of the projectivity under restriction in other settings \[\text{APS17}, \text{La17} \text{ and CS18c}.\]

A main step in our classification is to show that an irreducible smooth representation of \(G_{n+1}\) is projective restricted to \(G_n\) if and only if \(\pi\) is generic and any irreducible quotient of \(\pi|_{G_n}\) is generic, which turns the projectivity into a Hom-branching problem. This is a consequence of two things: (1) the Euler-Poincaré pairing formula of D. Prasad \[\text{Pr18}\] and (2) the Hecke algebra argument used in \[\text{CS18b}\] by G. Savin and the author. Roughly speaking, (1) is used to show non-projectivity while (2) is used to show projectivity.

The second part of the paper studies indecomposability of a restricted representation. It is clear that an irreducible representation (except one-dimensional ones) restricted from \(G_{n+1}\) to \(G_n\) cannot be indecomposable as it has more than one non-zero Bernstein component. However, the Hecke algebra realization in \[\text{CS18b}, \text{CS19}\] of the projective representations in Theorem 1.1 immediately implies that each Bernstein component of those restricted representation is indecomposable. This is a motivation of our study in general case, and precisely we prove:

**Theorem 1.2.** Let \(\pi\) be an irreducible representation of \(G_{n+1}\). Then each Bernstein component of \(\pi|_{G_n}\) is indecomposable.

For a mirabolic subgroup \(M_n\) of \(G_{n+1}\), it is known \[\text{Ze80}\] that \(\pi|_{M_n}\) is indecomposable for an irreducible representation \(\pi\) of \(G_{n+1}\). The approach in \[\text{Ze80}\] uses the Bernstein-Zelevinsky filtration of \(\pi\) to \(M_n\) and that the bottom piece of the filtration is irreducible. We prove that the bottom piece is indecomposable as a \(G_n\)-module, and then make use of left and right derivatives, developed and used to prove main results in \[\text{CS18b}\]. The key fact is that left and right derivatives of an irreducible representation are asymmetric. We now make more precise the meaning of ‘asymmetric’. We say that an integer \(i\) is the level of an irreducible representation \(\pi\) if the left derivative \(\pi^{(i)}\) (and hence the right derivative \((^{(i)}\pi\)) is the highest derivative of \(\pi\).

**Theorem 1.3.** Let \(\pi\) be an irreducible smooth representation of \(G_n\). Let \(\nu(g) = |\det(g)|_F\). Suppose \(i\) is not the level of \(\pi\). Then \(\nu^{1/2} \cdot \pi^{(i)}\) and \(\nu^{-1/2} \cdot (^{(i)}\pi)\) have no isomorphic irreducible quotients whenever \(\nu^{1/2} \cdot \pi^{(i)}\) and \(\nu^{-1/2} \cdot (^{(i)}\pi)\) are non-zero.

Computing the structure of a derivative of an arbitrary representation is a difficult question in general. Our approach is to try to approximate the information of derivatives of irreducible ones by some parabolically induced modules, whose derivatives can be computed via geometric lemma. On the other hand, the Speh representations behave more symmetrically for left and right derivatives, which motivates our proof to involve Speh representations.

In Appendix A, we demonstrate a way to compute Ext-groups by a Koszul-type resolution constructed in \[\text{Ch16}\] and explain the Ext-branching problem. Hom-branching law for Arthur parameter representations, which includes Speh representations, is recently studied in \[\text{GGP}\] and \[\text{Gu18}\]. In Appendix B, we explain
how an irreducible representation appears as the unique submodule of the product of Speh modules.

Section 2 studies derivatives of generic representations, which simplify some computations for Theorem 1.1. The results also give some guiding examples in the study of this paper and [CS18b].

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2. Bernstein-Zelevinsky derivatives of generic representations

2.1. Notations. Let $G_n = GL_n(F)$. Let $\rho$ be a cuspidal representation of $G_1$. Let $a, b \in \mathbb{C}$ with $b - a \in \mathbb{Z}_{\geq 0}$. We have a Zelevinsky segment $\Delta = [\nu^a \rho, \nu^b \rho]$. Denote $a(\Delta) = \nu^a \rho$ and $b(\Delta) = \nu^b \rho$. The relative length of $\Delta$ is defined as $b - a + 1$ and the absolute length of $\Delta$ is defined as $(b - a + 1)$. We can truncate $\Delta$ form each side to obtain two segments of absolute length $r(b - a)$:

$$-\Delta = [\nu^{a+1} \rho, \ldots, \nu^b \rho] \text{ and } \Delta^- = [\nu^a \rho, \ldots, \nu^{b-1} \rho].$$

Moreover, if we perform the truncation $k$-times, the resulting segments will be denoted by $(k)\Delta$ and $(\Delta)^{(k)}$ (We remark that the convention here is different from the previous paper [CS18b] for convenience later). If $i$ is not an integer divisible by $l$, then we set $\Delta^{(i)}$ and $(\Delta)^{(i)}$ to be empty sets. We also denote $\Delta^\vee = [\nu^{-b} \rho^\vee, \nu^{-a} \rho^\vee]$.

For a singleton segment $[\rho, \rho]$, we abbreviate as $[\rho]$. For a representation $\rho$ of $G_1$, define $n(\rho) = l$.

For a Zelevinsky segment $\Delta$, define $\langle \Delta \rangle$ and $St(\Delta)$ to be the (unique) irreducible submodule and quotient of $\nu^a \rho \times \ldots \times \nu^b \rho$ respectively. For a multisegment, we shall mean a collection of Zelevinsky segments. We have

$$St(\Delta)^\vee \cong St(\Delta^\vee) \quad \text{and} \quad \langle \Delta \rangle^\vee \cong \langle \Delta^\vee \rangle.$$

For two cuspidal representations $\rho_1, \rho_2$ of $G_m$, we say that $\rho_1$ precedes $\rho_2$ if $\nu^c \rho_1 \cong \rho_2$ for some $c > 0$. We say that a segment $\Delta$ precedes $\Delta'$ if $b(\Delta)$ precedes $b(\Delta')$.

Let $m = \{\Delta_1, \ldots, \Delta_r\}$ be a multisegment. We relabel the segments in $m$ such that for $i < j$, $\Delta_i$ does not precede $\Delta_j$. The modules defined below are independent of the labeling (up to isomorphisms) [Ze80]. Define $\zeta(m) = \langle \Delta_1 \rangle \times \ldots \times \langle \Delta_r \rangle$. Denote by $m$ the unique irreducible submodule of $\zeta(m)$. Similarly, define $\lambda(m) = St(\Delta_1) \times \ldots \times St(\Delta_r)$. Denote by $St(m)$ the unique quotient of $\lambda(m)$.

Let $U_n$ be the group of unipotent upper triangular matrices in $G_n$. For $i \leq n$, let $P_i$ be the parabolic subgroup of $G_n$ containing the block diagonal matrices $diag(g_1, g_2)$ ($g_1 \in G_i, g_2 \in G_{n-i}$) and the upper triangular matrices. Let $P_i = M_i N_i$ with the Levi $M_i$ and the unipotent $N_i$. Let $N_i^-$ be the opposite unipotent subgroup.
Let $\nu : G_n \to \mathbb{C}$ given by $\nu(g) = |\det(g)|_F$. Let
\[
R_{n-i} = \left\{ \begin{pmatrix} g & x \\ 0 & u \end{pmatrix} \in G_n : g \in \GL_{n-i}(F), u \in U_i, x \in \Mat_{n-i,i}(F) \right\}.
\]
Let $R_{n-i}^\top$ be the transpose of $R_{n-i}$.

We shall use $\Ind$ for normalized induction and $\ind$ for normalized induction with compact support.

For a smooth representation $\pi$ of $G_n$, define $\pi^{(i)}$ and $\pi^{(i)}$ to be the left and right Bernstein-Zelevinsky derivatives of $\pi$ as in [CS18b]. To recall it, let $\psi_i$ be a character on $U_i$ given by $\psi_i(u) = \overline{\psi}(u_{1,2} + \ldots + u_{i-1,i})$, where $\overline{\psi}$ is a nondegenerate character on $F$. Following [CS18b], define $\pi^{(i)}$ to be the left adjoint functor of $\Ind_{R_{n-i}}^{G_n} \pi \boxtimes \psi_i$. Let $\theta_n : G_n \to G_n$ given by $\theta_n(g) = g^{-T}$, the inverse transpose on $g$. Define the left derivative
\[
\theta_{n-i}(\pi^{(i)}),
\]
which is left adjoint to $\Ind_{R_{n-i}}^{G_n} \pi \boxtimes \psi_i$. The level of an admissible representation of $\pi$ is the largest integer $i$ such that $\pi^{(i)} \neq 0$ and $\pi^{(j)} = 0$ for all $j > i$. By using [Ze80] and [22], if $i$ is the level of $\pi$, then $(i)\pi \neq 0$ and $(j)\pi = 0$ for all $j > i$. When $i$ is the level for $\pi$, we shall call $\pi^{(i)}$ and $\pi^{(i)}$ to be the highest left and right derivative of $\pi$ respectively, where we usually drop the term of left and right if no confusion.

We shall often use the following lemma:

**Lemma 2.1.** Let $\pi$ be a smooth representation of $G_{n+1}$ and let $\pi'$ be an admissible smooth representation of $G_n$. Suppose there exists $i$ such that the following conditions hold:

\[
\Hom_{G_{n+1-i}}(\nu^{1/2} \cdot \pi^{(i)}, (i-1)\pi') \neq 0;
\]

and

\[
\Ext_{G_{n+1-j}}^{k}(\nu^{1/2} \cdot \pi^{(j)}, (j-1)\pi') = 0
\]

for all $j = 1, \ldots, i-1$ and all $k$. Then $\Hom_{G_n}(\pi, \pi') \neq 0$.

**Proof.** The Bernstein-Zelevinsky filtration of $\pi$ gives that there exists

\[
0 \subset \pi_n \subset \ldots \subset \pi_0 = \pi
\]
such that $\pi_{i-1}/\pi_i \cong (\nu^{1/2} \cdot \pi^{(i)}) \times \ind_{U_{i-1}}^{G_{n-i}} \psi_i$. Now by [CS18b], we have

\[
\Ext_{G_n}^{k}(\nu^{1/2} \pi^{(j)} \times \ind_{U_{j-1}}^{G_{n-j}} \psi_i, \pi') \cong \Ext_{G_{n+1-j}}^{k}(\nu^{1/2} \cdot \pi^{(j)}, (j-1)\pi')
\]

for all $k$ and $j$. Now a long exact sequence argument gives that

\[
\dim \Hom_{G_{n+1}}(\pi, \pi') \geq \dim \Hom_{G_{n+1-i}}(\nu^{1/2} \cdot \pi^{(i)}, (i-1)\pi') \neq 0.
\]

\[\square\]
2.2. Subrepresentation of a standard representation. For a multisegment $m$, we say that $\lambda(m)$ is a full principle series if all segments in $m$ are singletons.

**Lemma 2.2.** A full principle series has unique irreducible submodule and quotient. Moreover, the unique submodule is generic.

**Proof.** By definition, $\lambda(m) = \zeta(m)$ and hence has unique submodule and quotient. Since all segments in $m$ are singletons, the submodule is generic $\square$.

It is known that $\lambda(m)$ always has a generic representation as the unique submodule. We shall prove a slightly stronger statement (using the Zelevinsky theory).

**Proposition 2.3.** Let $m$ be a multisegment. Then $\lambda(m)$ can be embedded to a full principle series. In particular, $\lambda(m)$ has a unique simple submodule and moreover, the submodule is generic.

**Proof.** Let $\rho$ be a cuspidal representation in the cuspidal support of $\lambda(m)$ such that for any cuspidal representation $\rho'$ in the cuspidal support $\lambda(m)$, $\rho$ does not precede $\rho'$. Let $\Delta$ be a segment in $m$ with the shortest relative length among all segments $\Delta'$ with $b(\Delta') \cong \rho$.

By definition of $\lambda(m)$, we have that

$$\lambda(m) \cong \text{St}(\Delta) \times \lambda(m \setminus \{\Delta\}).$$

Thus we have that

$$\text{St}(\Delta) \hookrightarrow b(\Delta) \times \text{St}(\Delta^-).$$

On the other hand, we have that

$$\lambda(m) \cong b(\Delta) \times \lambda(m \setminus \{\Delta\} + \Delta^-)$$

The above isomorphism follows from the Zelevinsky theory and our choice of $\Delta$.

Now $\lambda(m \setminus \{\Delta\} + \Delta^-)$ embeds to a full principle series $\lambda'$ by induction. This gives that $b(\Delta) \times \lambda(m \setminus \{\Delta\} + \Delta^-)$ embeds to $b(\Delta) \times \lambda'$, which is also a principle series, and so does $\lambda(m)$ by (2.2).

The second assertion follows from Lemma 2.2. $\square$

2.3. Bernstein-Zelevinsky derivatives. Recall that a socle (resp. cosocle) of an admissible representation $\pi$ of $G_n$ is the maximal semisimple submodule (resp. quotient) of $\pi$.

**Lemma 2.4.** Let $\pi$ be an irreducible representation of $G_n$. Then the cosocle of $\pi^{(i)}$ (resp. $^{(i)}\pi$) is isomorphic to the socle of $\pi^{(i)}$ (resp. $^{(i)}\pi$).

**Proof.** This is almost the same as the proof of [CS18b, Lemma 2.2]. More precisely, it follows for an irreducible $G_{n-1}$-representation $\pi$,

$$^{(i)}\pi \cong \theta_{n-1}(\theta_n(\pi)) = \theta_{n-1}(\theta_n(\pi^{(i)})) = \theta_{n-1}(\theta_n((\pi^{(i)} \pi)^{\vee}))$$

and the fact that $\theta_{n-1}(\tau) \cong \tau^{\vee}$ for any irreducible $G_{n-1}$-representation $\tau$. $\square$
**Proposition 2.5.** Let $\pi$ be an irreducible smooth representation of $G_{n+1}$. The socle and cosocle of $\pi^{(i)}$ (and $(i)\pi$) are multiplicity-free.

**Proof.** Let $\pi_0$ be an irreducible quotient of $\pi^{(i)}$. Let $\pi_1$ be a cuspidal representation of $G_{i-1}$ which is not a unramified twist of a cuspidal representation in the cuspidal support of $\pi_0$. Then

$$\text{Ext}^j_{G_{n+1-k}}(\nu^{1/2} \cdot \pi^{(k)}, (k-1)(\pi_0 \times \pi_1)) = 0$$

for all $j$ and $k < i$. A long exact sequence argument using Bernstein-Zelevinsky filtration gives:

$$\dim \text{Hom}_{G_{n+1-i}}(\nu^{1/2} \cdot \pi^{(i)}, \pi_0) = \dim \text{Hom}_{G_{n+1-i}}(\nu^{1/2} \cdot \pi^{(i)}, (i-1)(\pi_0 \times \pi_1)) \leq \dim \text{Hom}_{G_n}(\pi, \pi_0 \times \pi_1).$$

Now the last dimension is at most one by [AGRS10] and so is the first dimension. This implies the cosocle statement by Lemma 2.1 and the socle statement follows from Lemma 2.4. 

A smooth representation $\pi$ of $G_n$ is called *generic* if $\pi^{(n)} \neq 0$. The Zelevinsky classification of irreducible generic representations is in [Ze80], that is $\text{St}(m)$ is generic if and only if any two segments in $m$ are unlinked. With Proposition 2.5, the following result essentially gives a combinatorial description on socle and cosocle of the derivatives of a generic representation.

**Corollary 2.6.** Let $\pi$ be an irreducible generic representation of $G_{n+1}$. Then any simple quotient and submodule of $\pi^{(i)}$ (resp. $(i)\pi$) is generic.

**Proof.** By Lemma 2.4, it suffices to prove the statement for quotient. Let $m = \{\Delta_1, \ldots, \Delta_r\}$ be the Zelevinsky segment $m$ such that

$$\pi \cong \text{St}(m) = \lambda(m) = \text{St}(\Delta_1) \times \ldots \times \text{St}(\Delta_r).$$

Since any two segments in $m$ are unlinked, we can label in any order and so we shall assume that for $i < j$, $b(\Delta_j)$ does not precede $b(\Delta_i)$. Then geometric lemma produces a filtration on $\pi^{(i)}$ whose successive subquotient is isomorphic to

$$\text{St}((i_1)\Delta_1) \times \ldots \times \text{St}((i_r)\Delta_r),$$

where $i_1 + \ldots + i_r = i$. The last module is isomorphic to $\lambda(m')^\vee$, where

$$m' = \left\{(i_1)\Delta_1, \ldots, (i_r)\Delta_r\right\}.$$

If $\pi'$ is a simple quotient of $\pi^{(i)}$, then $\pi'$ is a simple quotient of one successive subquotient in the filtration, or in other words is a simple submodule of $\lambda(m')$ for a multisegment $m'$. Now the result follows from Proposition 2.3. 

**Remark 2.7.** One can formulate the corresponding statement of Proposition 2.5 for affine Hecke algebra level using a sign module in [CS19]. Then it might be interesting to ask for an analogue result for affine Hecke algebra over fields of positive characteristics.

Here we give a consequence to branching law:
Corollary 2.8. Let \( \pi \) be a generic irreducible representation of \( G_{n+1} \). Let \( \pi' \) be an irreducible smooth representation of \( G_n \) and let \( m \) be a Zelevinsky multisegment with \( \pi' \cong \langle m \rangle \). If \( \text{Hom}_{G_n}(\pi, \langle m \rangle) \neq 0 \), then each segment in \( m \) has relative length at most 2.

Proof. Write \( m = \{ \Delta_1, \ldots, \Delta_r \} \) such that \( \Delta_i \) does not precede \( \Delta_j \) if \( i < j \). Let \( \pi_0 = \langle m \rangle \). By using the Bernstein-Zelevinsky filtration, \( \text{Hom}_{G_n}(\pi, \pi_0) \neq 0 \) implies that

\[
\text{Hom}_{G_{n+1}}(\mu^{1/2} \cdot \pi^{(i)}_1, (i-1)\pi_0) \neq 0
\]

for some \( i \geq 1 \) [CS18b]. Hence Corollary 2.6 implies that \( (i-1)\pi_0 \) is generic for some \( i \geq 1 \). Since \( \pi_0 \hookrightarrow \zeta(m) \) we have \( (i-1)\pi_0 \hookrightarrow (i-1)\zeta(m) \) and so \( (i-1)\zeta(m) \) has a generic composition factor. On the other hand, geometric lemma gives that \( (i-1)\pi_0 \) admits a filtration whose successive quotients are isomorphic to \( (i_1)^{(i)}(\Delta_1) \times \cdots \times (i_r)^{(i)}(\Delta_r) \) where \( i_1, \ldots, i_r \) run for all sums equal to \( i-1 \). Then at least one such quotient is non-degenerate and so in that quotient, all \( (i_k)^{(i)}(\Delta_k) \) are cuspidal. Following from the derivatives on \( \langle \Delta \rangle \), \( \Delta \) can have at most of relative length 2.

\( \square \)

Corollary 2.9. Let \( \pi \) be an irreducible generic representation of \( G_{n+1} \). Then the projections of \( \pi^{(i)} \) and \( (i) \pi \) to any cuspidal support component have unique simple quotient and submodule. In particular, the projections of \( \pi^{(i)} \) and \( (i) \pi \) to any cuspidal support component are indecomposable.

Proof. For a fixed cuspidal support, there is an unique irreducible smooth generic representation. Now the result follows from Proposition 2.5 and Corollary 2.6. \( \square \)

3. Projectivity

3.1. Projectivity Criteria. We need the following formula of D. Prasad:

Theorem 3.1. [Pr18] Let \( \pi_1 \) and \( \pi_2 \) be admissible representations of \( GL_{n+1}(F) \) and \( GL_n(F) \) respectively. Then

\[
\sum_{i \in \mathbb{Z}} (-1)^i \dim \text{Ext}^i_{G_n}(\pi_1, \pi_2) = \dim \text{Wh}(\pi_1) \cdot \dim \text{Wh}(\pi_2),
\]

where \( \text{Wh}(\pi_1) = \pi_1^{(n+1)} \) and \( \text{Wh}(\pi_2) = \pi_2^{(n)} \).

Lemma 3.2. Let \( \pi \) be an irreducible \( G_{n+1} \)-representation. If \( \pi^{(i)} \) has a non-generic irreducible submodule or quotient, then there exists a non-generic representation \( \pi' \) of \( G_n \) such that \( \text{Hom}_{G_n}(\pi, \pi') \neq 0 \). The statement still holds if we replace \( \pi^{(i)} \) by \( (i) \pi \).

Proof. By Lemma 2.4 it suffices just to consider that \( \pi^{(i)} \) has a non-generic irreducible quotient, say \( \lambda \). Now let

\[
\pi' = (\mu^{1/2} \lambda) \times \tau,
\]
where $\tau$ is a cuspidal representation such that $\tau$ is not an unramified twist of a cuspidal representation appearing in a segment in $m$. Here $m$ is a multisegment with $\pi \cong \langle m \rangle$. Now

$$\text{Hom}_{G_{n+1}}(\nu^{1/2} \cdot \pi(j), (j-1)\pi') = 0$$

for $j < n(\tau)$ since $\tau$ is in the cuspidal support of $(j-1)\pi'$ whenever it is nonzero while $\tau$ is never in the cuspidal support of $\nu^{1/2} \cdot \pi(j)$. Moreover, $(n(\tau)-1)\pi'$ has a simple quotient isomorphic to $\nu^{1/2}\lambda$. This checks the Hom and Ext conditions in Lemma 2.1 and hence proves the lemma. The proof for $(i)\pi$ is almost identical with switching left and right derivatives in suitable places. □

**Theorem 3.3.** Let $\pi$ be an irreducible smooth representation of $G_{n+1}$. Then the following conditions are equivalent:

1. $\pi|_{G_n}$ is projective
2. $\pi$ is generic and any irreducible quotient of $\pi|_{G_n}$ is generic

**Proof.** For (2) implying (1), it is proved in [CS18b]. We now consider $\pi$ is projective. All higher Exts vanish and so $\text{EP}(\pi, \pi') = \text{dim Hom}_{G_n}(\pi, \pi')$ for any irreducible $\pi'$ of $G_n$. If $\pi'$ is an irreducible quotient of $\pi$, then $\text{EP}(\pi, \pi') \neq 0$ and hence $\pi$ is generic by Theorem 3.1. But Theorem 3.1 also implies $\pi'$ is generic. This proves (1) implying (2). □

**3.2. Classification.**

**Definition 3.4.** We say that an irreducible representation $\pi$ of $G_{n+1}$ is *restricted projective* if either one of the following conditions holds:

1. $\pi|_{G_n}$ is essentially square-integrable;
2. $\pi$ is isomorphic to $\pi_1 \times \pi_2$ for some cuspidal representations of $G_{(n+1)/2}$ with $\pi_1 \not\cong \nu^{\pm 1}\pi_2$.

In particular, a restricted projective representation is generic. The condition $\pi_1 \not\cong \nu^{\pm 1}\pi_2$ is in fact automatic from $\pi$ being irreducible.

We can formulate the conditions (i) and (ii) combinatorially as follows. Let $\pi \cong \text{St}(m)$ for a multisegment $m = \{\Delta_1, \ldots, \Delta_r\}$. Then (i) is equivalent to $r = 1$; and (ii) is equivalent to that $r = 2$, and $\Delta_1$ and $\Delta_2$ are not linked, and the relative lengths of $\Delta_1$ and $\Delta_2$ are both 1. We also call those $m$ to be of restricted projective type.

**Lemma 3.5.** Let $\pi$ be an irreducible generic representation of $G_{n+1}$, which is not restricted-projective. Then there exists an irreducible non-generic representation $\pi'$ of $G_n$ such that $\text{Hom}_{G_n}(\pi, \pi') \neq 0$.

**Proof.** It suffices to construct an irreducible non-generic representation $\pi'$ satisfying the Hom and Ext properties in Lemma 2.1. Let $m = \{\Delta_1, \ldots, \Delta_r\}$ be a multisegment such that $\pi \cong \text{St}(m)$. We divide into few cases.

**Case 1:** $r \geq 3$; or when $r = 2$, each segment has relative length at least 2; or when $r = 2$, $\Delta_1 \cap \Delta_2 = \emptyset$. We choose a segment $\Delta'$ in $m$ with the shortest absolute
length. Now we choose a maximal segment \( \Delta \) in \( m \) with the property that \( \Delta' \subset \Delta \). By genericity, \( \nu^{-1} a(\Delta) \not\in \Delta_k \) for any \( \Delta_k \in m \). Let

\[
m' = \left\{ \nu^{1/2} \Delta, [\nu^{-1/2} a(\Delta)], [\tau] \right\},
\]

where \( \tau \) is a cuspidal representation so that \( \text{St}(m') \) is a representation of \( G_\nu \) and \( \tau \) is not an unramified twist of any cuspidal representation appearing in a segment of \( m \). To make sense of the construction, it needs the choices and the assumptions on this case. Let \( k = n(a(\Delta)), l = n(\tau) \). Let

\[
\pi' = \text{St}(m).
\]

Now as for \( i < k + l + 1 \), either \( \nu^{-1/2} a(\Delta) \) or \( \tau \) appears in the cuspidal support of \( (i-1)\pi \), but not in that of \( \nu^{1/2} \pi(\iota) \),

\[
\text{Ext}^2_{G_{n+1-i}}(\nu^{1/2} \cdot \pi(\iota), (i-1)\pi) = 0
\]

and all \( j \). By Corollary 2.6 \( \nu^{1/2} \cdot \pi(k+i+1) \) has a simple generic quotient isomorphic to \( \nu^{1/2} \text{St}(-\Delta) \). On the other hand, \( (k+i)\pi' \) can be computed as follows:

\[
\begin{align*}
(3.4) &\ 0 \not= \text{Hom}_{G_\nu}(\lambda(m'), \text{St}(m')) \\
(3.5) &\ \cong \text{Hom}_{G_{n-k-i}}(\text{St}(\nu^{1/2} \cdot \Delta) \boxtimes (\nu^{-1/2} a(\Delta) \times \tau), \text{St}(m')_{N_{k+i}}),
\end{align*}
\]

Here the non-zero-ness comes from the fact that \( \text{St}(m') \) is the unique quotient of \( \lambda(m') \), and the isomorphism follows from Frobenius reciprocity. Since taking the derivative is an exact functor, we have that \( \text{St}(\nu^{1/2} \cdot \Delta) \) is a subrepresentation of \( (k+i)\text{St}(m') \). Thus we have

\[
\text{Hom}_{G_{n-k-i}}(\nu^{1/2} \cdot \pi(k+i+1), (k+i)\pi') \not= 0.
\]

**Case 2:** \( r = 2 \) with \( \Delta_1 \cap \Delta_2 \neq \emptyset \) and one segment having relative length 1 (and not both having relative length 1 by the definition of restricted projective type). By switching the labeling on segments if necessary, we assume that \( \Delta_1 \subset \Delta_2 \). Let \( p \) and let \( l \) be the absolute and relative length of \( \Delta_2 \) respectively. Let

\[
m' = \left\{ [\nu^{1/2} a(\Delta_1)], [\nu^{3/2-l} a(\Delta_1)], [\nu^{-1/2} a(\Delta_1)], [\tau] \right\}, \quad \pi' = \text{St}(m'),
\]

where \( \tau \) is a cuspidal representation of \( G_k \) (here \( k \) is possibly zero) so that \( \text{St}(m') \) is a \( G_\nu \)-representation. Note that \( \pi' \) is non-generic. By Corollary 2.6 and geometric lemma, a simple quotient \( \nu^{1/2} \pi(p) \) is isomorphic to \( \nu^{1/2} a(\Delta_1) \). Similar computation as in [3.4] gives that a simple module of \( (p-1)\pi' \) is isomorphic to \( \nu^{1/2} a(\Delta_1) \). This implies the non-vanishing \( \text{Hom} \) between those two \( G_{n+1-p} \)-representations.

We now prove the vanishing Ext-groups in order to apply Lemma 2.1. Now applying the Bernstein-Zelevinsky derivatives \( (j = 1, \ldots, p-1) \), unless \( a(\Delta_1) = b(\Delta_2) \), we have that \( \nu^{3/2-l} a(\Delta_1) \) is a cuspidal support for \( (j-1)\pi' \) whenever \( (j-1)\pi' \) is nonzero and is not a cuspidal support for \( \nu^{1/2} a(\Delta_1) \). It remains to consider \( a(\Delta_1) = b(\Delta_2) \). We can similarly consider the cuspidal support for \( \nu^{-1/2} a(\Delta_1) \) and \( \nu^{1/2} a(\Delta_1) \) to make conclusion.

**Lemma 3.6.** Let \( \pi \) be an irreducible representation of \( G_{n+1} \). If \( \pi \) is (generic) restricted-projective, then \( \pi |_{G_n} \) is projective.
Proof. When $\pi$ is essentially square-integrable, it is proved in [CS18b]. We now assume that $\pi$ is in the case (2) of Definition 3.4. It is equivalent to prove the condition (2) in Theorem 3.3. Let $\pi' \in \text{Irr}(G_n)$ with $\text{Hom}_{G_n}(\pi, \pi') \neq 0$. We have to show that $\pi'$ is generic. Note that the only non-zero derivative of $\pi^{(j)}$ can occur when $i = n + 1$ and $\frac{n+1}{2}$.

Case 1: $\text{Hom}_{G_{n+1-n}}(\nu^{1/2} \cdot \pi^{(n+1)/2}, (n-1)/2 \pi') \neq 0$. For (1), let

$$\pi = \rho_1 \times \rho_2$$

for some irreducible cuspidal representations $\rho_1, \rho_2$ of $G_{(n+1)/2}$ with $\rho_1 \not\cong \nu^{\pm} \rho_2$, and

$$m' = \{\Delta'_1, \ldots, \Delta'_l\} \quad \text{for} \pi' \cong \text{St}(m').$$

By a simple count on dimensions, we must have $\Delta_k' \cong \nu^{1/2} \rho_1$ or $\cong \nu^{1/2} \rho_2$ for some $k$. Using dimensions again, we have for $l \neq k$, $[\rho_1]$ and $[\rho_2]$ are unlinked to $\Delta_k'$ and so

$$\pi' \cong (\nu^{1/2} \cdot \rho_r) \times \text{St}(m' \setminus \{\Delta_k\}),$$

for $r = 1$ or 2. Then

$$((n-1)/2) \pi' \cong \nu^{1/2} \cdot \rho_r,$$

which implies

$$((n-1)/2) \text{St}(m' \setminus \{\Delta_k\}) \neq 0.$$

Thus $\text{St}(m' \setminus \{\Delta_k\})$ is generic and so is $\pi'$.

Case 2: $\text{Hom}_{G_{n+1-n}}(\nu^{1/2} \cdot \pi^{(n+1)/2}, (n-1)/2 \pi') = 0$. We must have

$$\text{Hom}_{G_{n+1-n}}(\nu^{1/2} \cdot \pi^{(n+1)/2}, (n)/\pi') \neq 0$$

and so $(n) \pi' \neq 0$. Hence $\pi'$ is generic. \hfill \Box

Theorem 3.7. Let $\pi$ be an irreducible $G_{n+1}$-representation. Then $\pi|_{G_n}$ is projective if and only if $\pi$ is restricted-projective in Definition 3.4.

Proof. The if direction is proved in Lemma 3.6. The only if direction follows from Lemma 3.5 and Theorem 3.1. \hfill \Box

One advantage for such classification is that those restricted representations admit a more explicit realization as shown in [CS18b]:

Theorem 3.8. Let $\pi, \pi'$ be irreducible smooth representations of $G_{n+1}$. If $\pi$ and $\pi'$ are restricted projective, then $\pi|_{G_n} \cong \pi'|_{G_n}$. In particular, $\pi|_{G_n}$ is isomorphic to the Gelfand-Graev representation $\text{ind}_{U_n}^{G_n} \psi_n$.

Proof. We have shown that $\pi$ and $\pi'$ have to be generic. Then we apply [CS18b] Corollary 5.5 and Theorem 5.6. \hfill \Box

4. Indecomposability of restricted representations

4.1. Affine Hecke algebras.

Definition 4.1. The affine Hecke algebra $\mathcal{H}_t(q)$ of type $A$ is an associative algebra over $\mathbb{C}$ generated by $\theta_1, \ldots, \theta_l$ and $T_w$ ($w \in S_l$) satisfying the relations:

1. $\theta_i \theta_j = \theta_j \theta_i$;
which argument can be applied to other connected quasi split reductive groups. Proposition 4.3. For any submodule $\pi$ of $\mathcal{H}(\Pi_\mathfrak{s})$, the intersection of $\pi_1$ and $\pi_2$ is non-zero. In particular, $(\text{ind}_{U_n}^G \psi_n)_\mathfrak{s}$ is indecomposable.

Proof. This follows from the fact that $\Pi_\mathfrak{s}|_{\mathcal{A}_\mathfrak{s}}$ is isomorphic to $\mathcal{A}_\mathfrak{s}$ and any two $\mathcal{A}_\mathfrak{s}$-submodules of $\mathcal{A}_\mathfrak{s}$ has non-zero intersection. □

Remark 4.4. We give a proof for indecomposability for $\Pi_\mathfrak{s}$ more directly as below, which argument can be applied to other other connected quasisplit reductive groups $G$. For any $\mathfrak{s} \in \mathfrak{R}(G)$, [BH03] showed that $\text{Hom}_G(\Pi_\mathfrak{s}, \Pi_\mathfrak{s})$ is isomorphic to the
Since \( \mathfrak{K}_s(G) \) is an indecomposable category, we also have \( \mathfrak{Z}_s \) is indecomposable as \( \mathfrak{Z}_s \)-module. This implies that \( \text{End}_G(\Pi_s) \) is indecomposable as \( \mathfrak{Z}_s \)-module and hence \( \Pi_s \) is indecomposable in \( \mathfrak{B}_s(G) \).

### 4.3. Preserving indecomposability of Bernstein-Zelevinsky induction.

**Lemma 4.5.** Let \( \pi \) be an irreducible smooth representation of \( G_n \). Then

\[
\text{Hom}_{G_n}(\pi, \text{ind}_{U_n}^G \psi) = 0.
\]

**Proof.** Let \( s \) be the type such that \( \pi \) is an object in \( \mathfrak{B}_s(G_n) \). Now \( \pi_s \) is an irreducible finite-dimensional \( H_s \)-module, but there is no finite-dimensional submodule for \( \text{ind}_{U_n}^G \psi_s \) as \( H_s \)-module (as there is no finite-dimensional submodule of \( A_s \) as \( A_s \)-module). Hence the Hom space is zero. \( \square \)

**Lemma 4.6.** Let \( P = LN \) be the parabolic subgroup containing upper triangular matrices and block-diagonal matrices \( \text{diag}(g_1, \ldots, g_r) \) with \( g_k \in G_{i_k} \), where \( i_1 + \ldots + i_r = n \). Then \( \text{ind}_{U_n}^G \psi_s \cong \text{ind}_{U_{i_1}}^G \psi \otimes \ldots \otimes \text{ind}_{U_{i_r}}^G \psi \).

**Proof.** Let \( w \) be a permutation matrix in \( G_n \). Then \( w(N) \cap U_n \) contains a unipotent subgroup \( \{ I_n + tu_{k,k+1} : t \in F \} \) for some \( k \) if and only if \( w(N) \not\subset U_n \). Here \( u_{k,k+1} \) is a matrix with \( (k,k+1) \)-entry 1 and other entries 0. For any such \( w \), it gives that \( PwB \) is the same open orbit in \( G_n \). Now the geometric lemma in [BZ77, Theorem 5.2] gives the lemma. \( \square \)

We shall use the following criteria of indecomposable representations:

**Lemma 4.7.** Let \( G \) be a reductive \( p \)-adic group. Let \( \pi \) be a smooth representation of \( G \). The only idempotents in \( \text{End}_G(\pi) \) are 0 and the identity (up to automorphism) if and only if \( \pi \) is indecomposable.

**Proof.** If \( \pi \) is not indecomposable, then any projection to a direct summand gives a non-identity idempotent. On the other hand, if \( \sigma \in \text{End}_G(\pi) \) is a non-identity idempotent (i.e. \( \sigma(\pi) \neq \pi \)), then \( \pi \cong \text{im}(\sigma) \oplus \text{im}(1 - \sigma) \). \( \square \)

**Theorem 4.8.** Let \( \pi \) be an admissible indecomposable smooth representation of \( G_{n-i} \). For each \( s \in \mathfrak{B}(G_n) \), the Bernstein component \( \text{ind}_{R'_n}^G \pi \boxtimes \psi_s \) is indecomposable.

Since we did not prove the second adjointness of Bernstein-Zelevinsky induction \( \text{ind}_{R'_n}^G \pi \boxtimes \psi_s \) for all smooth (not necessarily admissible) representations, we shall not use at this point.

**Proof.** We first prove the following:

**Lemma 4.9.** For an admissible indecomposable smooth representation \( \pi \) of \( G_{n-i} \), as endomorphism algebra,

\[
\text{End}_{G_n}(\text{ind}_{R'_n}^G \pi \boxtimes \psi_s) \cong \text{End}_{G_{n-i}}(\pi) \otimes \mathfrak{Z},
\]

where \( \mathfrak{Z} \) is the Bernstein center.
Proof. We write \( \text{ind}^{G_n}_{R_{n-i}} \pi \otimes \psi_i \equiv \pi \times \text{ind}^{G_i}_{U_i} \psi_i \). Frobenius reciprocity gives that
\[
(4.7) \quad \text{End}_{G_n}(\text{ind}^{G_n}_{R_{n-i}} \pi \otimes \psi_i) \cong \text{Hom}_{G_{n-i} \times G_i}((\text{ind}^{G_n}_{R_{n-i}} \pi \otimes \psi_i)_{N_i}, \pi \otimes \text{ind}^{G_i}_{U_i} \psi_i)
\]

Geometric lemma gives that there exists a filtration on \( (\pi \times \text{ind}^{G_i}_{U_i} \psi_i)_{N_i} \) such that successive quotients are isomorphic to
\[
(4.8) \quad \text{ind}^{G_{n-i} \times G_i}((\pi_{N_j} \otimes (\text{ind}^{G_i}_{U_i} \psi_i)_{N_k}))^w,
\]
where \( j + k = i \). Here \( N_j, P_j, N_k, P_k \) are subgroup of \( G_{n-i} \) (resp. \( G_i \), and for a \( G_{n-i-j} \times G_j \times G_{i-k} \times G_k \)-representation \( \pi \), we denote \( \pi^w \) to be a \( G_{n-i-j} \times G_j \times G_{i-k} \times G_k \)-representation whose action is given by
\[
(g_1, g_2, g_3, g_4).\pi^w = (g_1, g_3, g_2, g_4).\pi^v.
\]

Since \( \pi \) is admissible, we have a filtration on \( \pi_{N_j} \) by simple composition factors, and we denote those successive simple quotients of \( \pi_{N_j} \) by \( \tau_1 \otimes \tau'_1, \ldots, \tau_p \otimes \tau'_p \) \([BZ76]\). For notion simplicity, we set \( \Pi_i = \text{ind}^{G_i}_{U_i} \psi_i \). This gives that \( \text{ind}^{G_{n-i} \times G_i}((\pi_{N_j} \otimes (\Pi_i)_{N_k}))^w \) admits a filtration \( \tau_q \times \Pi_j \otimes \tau'_q \times \Pi_k \). Now we have that
\[
\text{Hom}_{G_{n-i} \times G_i}(\tau_q \times \Pi_j \otimes \tau'_q \times \Pi_k, \pi \times \Pi_i)
\]
\[
\cong \text{Hom}_{G_{n-i} \times G_j \times G_{i-j}}(\tau_q \times \Pi_j \otimes \tau'_q \otimes \Pi_k, \pi \otimes (\Pi_i)_{N_k})
\]
\[
\cong \text{Hom}_{G_{n-i} \times G_j \times G_{i-j}}(\tau_q \times \Pi_j \otimes \tau'_q \otimes \Pi_k, \pi \otimes (\Pi_i \otimes \Pi_k))
\]
\[
\cong \text{Hom}_{G_{n-i} \times G_{i-j}}(\tau_q \times \Pi_j \otimes \Pi_k, \pi \otimes \Pi_k) \otimes \text{Hom}_{G_j}(\tau'_q, \Pi_j)
\]
Here the first isomorphism follows from the second adjointness, the second isomorphism follows from Lemma 4.6, and the last isomorphism uses that \( \tau'_q \) is irreducible (and so admissible).

The above isomorphisms imply that
\[
\text{Hom}_{G_{n-i} \times G_i}(\tau_q \times \Pi_j \otimes \tau'_q \times \Pi_k, \pi \times \Pi_i) = 0
\]
whenever \( j \neq 0 \). Thus \( \Box \) gives that for \( j \neq 0 \)
\[
\text{Hom}_{G_{n-i} \times G_i}(\text{ind}^{G_{n-i} \times G_i}_{G_{n-i} \times G_i}((\pi_{N_j} \otimes (\text{ind}^{G_i}_{U_i} \psi_i)_{N_k}))^w, \pi \otimes \text{ind}^{G_i}_{U_i} \psi_i) = 0.
\]

To complete the proof of Lemma 4.9 it remains to show that the isomorphism is also compatible with the algebra structure. This follows from that the isomorphism
\[
\text{End}_{G_{n-i} \times G_i}(\pi \otimes \text{ind}^{G_i}_{U_i} \psi_i) \cong \text{Hom}_{G_{n-i} \times G_i}((\text{ind}^{G_{n-i} \times G_i}_{G_{n-i} \times G_i}((\pi \otimes \text{ind}^{G_i}_{U_i} \psi_i)))^w, \pi \otimes \text{ind}^{G_i}_{U_i} \psi_i)
\]
is obtained by \( \phi \mapsto (f \mapsto \phi \circ f(1)) \) \([BZ77] \text{ Section } 5\), in particular, \([BZ77] \text{ 5.5}\)). From Frobenius reciprocity, the isomorphism
\[
\text{Hom}_{G_{n-i} \times G_i}((\text{ind}^{G_{n-i} \times G_i}_{G_{n-i} \times G_i}((\pi \otimes \text{ind}^{G_i}_{U_i} \psi_i)))^w, \pi \otimes \text{ind}^{G_i}_{U_i} \psi_i)
\]
is given by
\[
\Phi \mapsto (h \mapsto \Phi(g.h)).
\]
Hence, it gives an element in
\[
\text{End}_{G_n}(\text{ind}^{G_n}_{R_{n-i}} \pi \otimes \psi_i)
\]
determined by \( h \mapsto \phi \circ h \).
Lemma 4.11. For each \( \pi \) does not split. With Theorem 4.8 and (4.9), components. Write \( \pi \) and \( j \) is an indecomposable

\[
\pi_n \subset \pi_{n-1} \subset \ldots \subset \pi_1 \subset \pi_0 = \pi
\]
such that

\[
\pi_i / \pi_{i+1} \cong \text{ind}^{G_n}_{R_{n-1}} \nu^{1/2} \cdot \pi^{(i+1)} \otimes \psi_i.
\]
on the other hand, we also have a \( G_n \)-filtration on \( \pi \) with

\[
\left. n\pi \subset \pi_{n-1} \cdot \pi \subset \ldots \subset 1\pi \subset 0\pi = \pi \right.
\]
such that

\[
i\pi / i+1 \pi \cong \text{ind}^{G_n}_{R_{n-1}} \nu^{-1/2} \cdot (i^{+1}) \pi \otimes \psi_i.
\]
Let \( s \in \mathcal{B}(G_n) \) such that \( \pi_s \neq 0 \). In particular, this implies

\[
(\text{ind}^{G_n}_{R_{n-1}} \nu^{-1/2} \cdot (i^{+1}) \pi \otimes \psi_{i+1})_s \neq 0,
\]
where \( i^{+} \) is the level of \( \pi \). It is a standard homological fact that for two indecomposable representations \( \tau_1 \) and \( \tau_2 \) of \( G_n \), and for \( \lambda \) satisfying a short exact sequence

\[
0 \rightarrow \tau_1 \rightarrow \lambda \rightarrow \tau_2 \rightarrow 0,
\]
\( \lambda \) is an indecomposable \( G_n \)-representation if and only if the short exact sequence does not split. With Theorem 4.8 and (4.9), \( \pi_s \) has finitely many indecomposable components. Write \( \pi_s = \tau_1 \oplus \ldots \oplus \tau_r \) such that each \( \tau_k \) is an indecomposable \( G_n \)-representation.

Corollary 4.10. Let \( \pi \) be an irreducible smooth representation of \( M_n \). Then for any \( s \in \mathcal{B}(G_n) \), \( \pi_s \) is indecomposable.

Proof. This follows from [BZ77, Corollary 3.5] and Theorem 4.8.  

4.4. Main lemma. There exists a Bernstein-Zelevinsky \( G_n \)-filtration on \( \pi \) with

\[
(4.9)
\]
such that

\[
\pi_n \subset \pi_{n-1} \subset \ldots \subset \pi_1 \subset \pi_0 = \pi
\]
such that

\[
\pi_i / \pi_{i+1} \cong \text{ind}^{G_n}_{R_{n-1}} \nu^{1/2} \cdot \pi^{(i+1)} \otimes \psi_i.
\]
on the other hand, we also have a \( G_n \)-filtration on \( \pi \) with

\[
\left. n\pi \subset \pi_{n-1} \cdot \pi \subset \ldots \subset 1\pi \subset 0\pi = \pi \right.
\]
such that

\[
i\pi / i+1 \pi \cong \text{ind}^{G_n}_{R_{n-1}} \nu^{-1/2} \cdot (i^{+1}) \pi \otimes \psi_i.
\]
Let \( s \in \mathcal{B}(G_n) \) such that \( \pi_s \neq 0 \). In particular, this implies

\[
(\text{ind}^{G_n}_{R_{n-1}} \nu^{-1/2} \cdot (i^{+1}) \pi \otimes \psi_{i+1})_s \neq 0,
\]
where \( i^{+} \) is the level of \( \pi \). It is a standard homological fact that for two indecomposable representations \( \tau_1 \) and \( \tau_2 \) of \( G_n \), and for \( \lambda \) satisfying a short exact sequence

\[
0 \rightarrow \tau_1 \rightarrow \lambda \rightarrow \tau_2 \rightarrow 0,
\]
\( \lambda \) is an indecomposable \( G_n \)-representation if and only if the short exact sequence does not split. With Theorem 4.8 and (4.9), \( \pi_s \) has finitely many indecomposable components. Write \( \pi_s = \tau_1 \oplus \ldots \oplus \tau_r \) such that each \( \tau_k \) is an indecomposable \( G_n \)-representation.

Lemma 4.11. For each \( k \), let \( i(k) \) be the largest integer such that \( \pi_{i(k)-1} \cap \tau_k \neq 0 \) and \( \pi_{i(k)} \cap \tau_k = 0 \). Let \( j(k) \) be the largest integer such that \( j(k)-1 \pi \cap \tau_k \neq 0 \) and \( j(k) \pi \cap \tau_k = 0 \). Then both of the following statements hold:

1. \( i(k) = j(k) \);
2. \( \nu^{1/2} \cdot \pi^{(i(k))} \) and \( \nu^{-1/2} \cdot \pi^{(j(k))} \) have isomorphic irreducible quotients.
\textbf{Proof.} Suppose \( j(k) < i(k) \). By the definition of \( i(k) \), we have that

\[
(\text{ind}^{G_n}_{R_{n-i(k)+1}})^{1/2} \cdot \pi^{(i(k))} \not\cong \psi_{i(k)-1}s \neq 0
\]

Let \( \omega \) be an indecomposable component of \( \nu^{1/2} \cdot (r(k)) \pi \) such that

\[
(\text{ind}^{G_n}_{R_{n-i+1}} \omega \boxtimes \psi_{i-1}) \cap \tau_k \neq 0.
\]

Let \( \pi' \) be an irreducible quotient of \( \omega \). Let \( \mu \) be a generic representation such that \( \pi' \times \mu \in \mathcal{B}_s(G_n) \) and for any cuspidal representation \( \rho \) in the cuspidal support of \( \mu \) and any integer \( c, \nu'\rho \) does not appear in \( \nu^{1/2}m \) and \( \nu^{-1/2}m \), where \( m \) is the multisegment for \( \pi = \langle m \rangle \). By comparing cuspidal supports, we have that for \( i < i(k) \), and any \( j \)

\[
\text{Ext}^j_G (\text{ind}^{G_n}_{R_{n-i+1}} \omega \boxtimes \psi_{i-1}, \pi' \times \mu) = 0
\]

and

\[
\text{Hom}_G (\text{ind}^{G_n}_{R_{n-i+1}} \omega \boxtimes \psi_{i-1}, \pi' \times \mu) \neq 0
\]

By intersecting the filtration with \( \tau_k \), we then have

\[
\text{Hom}_G (\tau_k, \pi' \times \mu) \cong \text{Hom}_G ((\text{ind}^{G_n}_{R_{n-i+1}})^{1/2} \cdot \pi^{(i(k))} \boxtimes \psi_{i(k)-1} \cap \tau_k, \pi' \times \mu)
\]

\[
\supset \text{Hom}_G ((\text{ind}^{G_n}_{R_{n-i+1}} \omega \boxtimes \psi_{i-1}) \cap \tau_k, \pi' \times \mu)
\]

\[
\neq 0
\]

On the other hand, our construction on \( \pi' \times \mu \) give that for any \( i \leq j(k) \),

\[
\text{Hom}_G (\text{ind}^{G_n}_{R_{n-i+1}} \nu^{-1/2} \cdot (i) \pi \boxtimes \psi_{i-1}, \pi' \times \mu) = 0
\]

by comparing cuspidal supports. This implies that \( \text{Hom}_G (\tau_k, \pi' \times \mu) = 0 \) since \( \tau_k \) embeds to \( \pi'_{/j(k)} \pi \). This gives a contradiction. One similarly proves that \( j(k) < i(k) \) is impossible. Thus \( i(k) = j(k) \). \( \square \)

\section*{4.5. Indecomposability of restricting an irreducible representation.}

We now prove our main result:

\textbf{Theorem 4.12.} Let \( \pi \) be an irreducible representation of \( G_{n+1} \). Then for each \( s \in \mathcal{R}(G_n) \), \( \pi_s \) is indecomposable whenever it is nonzero.

\textbf{Proof.} We use the notations in Section 4.4 and Lemma 4.11. Let \( i^* \) be the level of \( \pi \). Since \( \pi_{i^*-1} \cong \text{ind}^{G_n}_{R_{n-i^*+1}} \nu^{1/2} \cdot (i^*) \boxtimes \psi_{i^*-1} \) and \( (i^*) \) is irreducible, \( (i^*) \) is indecomposable by Theorem 4.1. Then by reindexing if necessary, we assume that \( 0 \neq (i^*) \cap \tau_1 \). If \( \pi \) is not indecomposable, then \( i^* = j(2) \) (in the notation of Lemma 4.11) are smaller than \( i^* \). But then Lemma 4.11 will contradict the following theorem and this completes the proof of Theorem 4.12. \( \square \)

\textbf{Theorem 4.13.} Let \( \pi \) be an irreducible representation of \( G_{n+1} \). If \( i \) is not the level for \( \pi \), then \( \nu^{1/2} \cdot (i) \) and \( \nu^{-1/2} \cdot (i) \) does not have isomorphic irreducible quotient whenever the two derivatives are not zero.

The proof of Theorem 4.13 will be carried out in Section 5. Note that the converse of the above theorem is also true, which follows directly from the well-known highest derivative due to Zelevinsky [Ze00, Theorem 8.1].
5. Asymmetric property of left and right derivatives

We are going to prove Theorem 4.13 in this section. The idea lies in two simple cases: The first one is a generic representation. Since an irreducible generic representation is isomorphic to \( \lambda(m) \cong \text{St}(m) \) for a Zelevinsky multisegment \( m \), a simple counting on cuspidal support on derivatives can show Theorem 4.13 for that case. The second one is an irreducible representation whose Zelevinsky multisegment has segments with relative length strictly greater than 1. In such case, one can narrow down the possibility of irreducible submodule via the embedding \( \langle m \rangle^{(i)} \hookrightarrow \zeta(m)^{(i)} \) and \( \langle m \rangle \hookrightarrow \zeta(m) \), and use geometric lemma to compute the submodules of derivatives of \( \zeta(m)^{(i)} \) and \( \zeta(m) \). The combination of these two cases seems to require some extra work. We shall use the notations and terminology for Speh representations in the appendices.

5.1. Union-intersection operation. Let \( m = \{\Delta_1, \ldots, \Delta_r\} \). For two segments \( \Delta \) and \( \Delta' \) in \( m \) which are linked, the process of replacing \( \Delta \) and \( \Delta' \) by \( \Delta \cap \Delta' \) and \( \Delta \cup \Delta' \) is called union-intersection process. If follows from [Ze80, Chapter 7] that the Zelevinsky multisegment of any irreducible composition factor in

\[
\langle \Delta_1 \rangle \times \ldots \times \langle \Delta_r \rangle
\]

can be obtained by a chain of union-intersection process. For a positive integer \( l \), define \( N(m, l) \) to be the number of segments in \( m \) with relative length \( l \).

The following lemma can be proved by a simple inductive argument:

**Lemma 5.1.** Let \( m = \{\Delta_1, \ldots, \Delta_r\} \) be a Zelevinsky multisegment. Let \( m' \) be a Zelevinsky multisegment obtained from \( m \) by a chain of union-intersection operations. Then there exists a positive integer \( l \) such that

\[
N(m', l) > N(m, l),
\]

and for any \( l' > l \),

\[
N(m', l') \geq N(m, l').
\]

5.2. Proof of Theorem 4.13. By Lemma 2.6 it suffices to prove the same statement for submodules of the derivatives.

Let \( m \) be the Zelevinsky multisegment with \( \pi \cong \langle m \rangle \). We shall assume that the cuspidal representations in each segment of \( m \) is an unramified twist of a fixed cuspidal representation \( \rho \). We shall prove that Theorem 4.13 for such \( \pi \). The general case follows from this by writing an irreducible representation as a product of irreducible representations of such specific form.

Let \( \pi' \) be a common isomorphic irreducible quotient of \( \nu^{1/2} \cdot \pi(i) \) and \( \nu^{-1/2} \cdot (i) \pi \), (assuming that \( \nu^{1/2} \cdot \pi(i) \) and \( \nu^{-1/2} \cdot (i) \pi \) are non-zero). Recall that we have that

\[
\pi \hookrightarrow \zeta(m).
\]

Since taking derivative is an exact functor, \( \pi(i) \) embeds to \( \zeta(m)^{(i)} \) and so does \( \pi' \).

Define

\[
m^{(i_1, \ldots, i_r)} = \{\Delta_1^{(i_1)}, \ldots, \Delta_r^{(i_r)}\}.
\]

There is a filtration on \( \zeta(m)^{(i)} \) given by \( \zeta(m^{(i_1, \ldots, i_r)}) \) for \( i_1 + \ldots + i_r = i \), where \( i_k = 0 \) or \( n(\rho) \). Then \( \pi' \) is isomorphic to the unique submodule of \( \zeta(m^{(i_1, \ldots, i_r)}) \) for
some \((i'_1, \ldots, i'_r)\) and so \(\pi' \cong \langle m^{(i'_1, \ldots, i'_r)} \rangle\). Suppose \(i\) is not the level of \(\pi\). Then there exists at least one \(i'_k = 0\). We shall choose \(i'_k\), such that \(\Delta_k\) has the largest relative length, say \(L\), among all the segments \(\Delta_i\) with \(i_i = 0\).

We write \(m\) as the sum of Speh multisegments

\[
(5.10) \quad m = m'_1 + \ldots + m'_s
\]
satisfying properties in Proposition \([7,3]\).

Let

\[
m_1, \ldots, m_r
\]
be all the Speh multisegments appearing in the sum \(\langle m \rangle\) such that \(L(m_k) = L\).

For each \(m_k\), write \(m_k = a(m_k, \Delta_k)\) and define \(b(m_k) = b(\Delta_k)\). We shall label \(m_k\) in the way that \(b(m_k)\) does not precede \(b(m_l)\) for \(k < l\). Furthermore, the labelling satisfies the property that

\[
(\emptyset) \quad \text{for any } m_p \text{ and } p < q, m_p + \Delta \text{ is not a Speh multisection for any } \Delta \in m_q.
\]

\[
(\emptyset) \quad \text{if } m_p \cap m_q \neq \emptyset \text{ and } p \leq q, \text{ then } m_q \subset m_p.
\]

Let \(n_1\) be the collection of all segments in \(m \setminus \langle m'_1 + \ldots + m'_r \rangle\) such that any segment \(\Delta'\) in \(n_1\) satisfies the property that (1) \(b(m_1)\) precedes \(b(\Delta')\) or (2) \(b(m_1) \cong b(\Delta')\). Define inductively that \(n_k\) is the collection of all segments in \(m \setminus \langle m'_1 + \ldots + m'_{k-1} + n_1 + \ldots n_{k-1} \rangle\) such that any segment \(\Delta'\) in \(n_k\) satisfies the property that (1) \(b(m_k)\) precedes \(b(\Delta)\), or (2) \(b(m_k) \cong b(\Delta')\) (and \(\Delta' \neq \Delta_k\)).

By Lemma \([7,3]\), we have a series of embedding:

\[
\langle m \rangle \hookrightarrow \zeta(n_1) \times \langle m_1 \rangle \times \ldots \times \zeta(n_r) \times \langle m_r \rangle \times \zeta(n_{r+1})
\]

\[
\hookrightarrow \ldots
\]

\[
\hookrightarrow \zeta(n_1) \times \langle m_1 \rangle \times \zeta(n_2) \times \langle m_2 \rangle \times \zeta(n_3 + \ldots + n_r + m_3 + m_r + n_{r+1})
\]

\[
\hookrightarrow \zeta(n_1) \times \langle m_1 \rangle \times \zeta(n_2 + \ldots + n_r + m_2 + m_r + n_{r+1})
\]

\[
\hookrightarrow \zeta(n_1 + \ldots n_{r+1} + m_1 + \ldots + m_r) = \zeta(m)
\]

For simplicity, define, for \(k \geq 0\)

\[
\lambda_k = \zeta(n_1) \times \langle m_1 \rangle \times \ldots \times \zeta(n_k) \times \langle m_k \rangle \times \zeta(n_{k+1} + \ldots + n_{r+1} + m_{k+1} + \ldots + m_r)
\]

Then for each \(k\), we again have an embedding:

\[
\pi' \hookrightarrow \nu^{1/2} \cdot \pi^{(i)} \hookrightarrow \nu^{1/2} \cdot \lambda_k^{(i)}.
\]

As \(\lambda_k\) is a product of representations, we again have a filtration on \(\lambda_k^{(i)}\). This gives that \(\pi'\) embeds to a successive quotient of the filtration:

\[
\pi' \hookrightarrow \nu^{1/2} \cdot (\zeta(n_1)(p_1^k) \times \langle m_1 \rangle(q_1^k) \times \ldots \times \zeta(n_k)(p_k^k) \times \zeta(o_{k+1})(s_k^k))
\]

with \(p_1^k + \ldots + p_k^k + q_1 + \ldots + q_k^k + s^k = i, \quad o_{k+1} = n_{k+1} + \ldots + n_{r+1} + m_{k+1} + \ldots + m_r\).

Now we assume there exists a smallest \(k\) such that at least one of \(q_k^k\) is not equal to the level of \(\langle m_i \rangle\). Now we shall denote such \(l\) by \(l^*\). We use similar strategy to further consider the filtrations on each \(n_a\) by geometric lemma. For that we write

\[
n_a = \{\Delta_{a,1}, \ldots, \Delta_{a,r(a)}\}
\]
and

\[ \sigma_{k+1} = \{ \Delta_{k+1,1}, \ldots, \Delta_{k+1,r(k+1)} \}. \]

Then we again have an embedding

\[ \pi' \hookrightarrow \nu^{1/2} : (\zeta(\bar{n}_k) \times (m_1)^{\langle q_1 \rangle} \times \ldots \times \zeta(\bar{n}_k) \times (m_k)^{\langle q_k \rangle} \times \zeta(o_{k+1})^{\langle s_k \rangle}), \]

where, for \( a = 1, \ldots, k \),

\[ \bar{n}_a = \{ \Delta_{a,1}^{\langle p_a,1 \rangle}, \ldots, \Delta_{a,r(a)}^{\langle p_a,r(a) \rangle} \}. \]

with \( p_a + \ldots + p_a,r(a) = p_a \) and each \( p_a,b = 0 \) or \( n(\rho) \), and

\[ \bar{o}_{k+1} = \{ \Delta_{k+1,1}^{\langle p_{k+1},1 \rangle}, \ldots, \Delta_{k+1,r(k+1)}^{\langle p_{k+1},r(k+1) \rangle} \}. \]

We claim (*) that if \( \Delta_{a,b} \) has an relative length at least \( L + 1 \), then \( p_a,b = n(\rho) \). This indeed follows from Lemma 5.1(1), since \( \pi' \) is a composition factor of \( \zeta(m') \) for some \( m' \) that \( \nu^{1/2}(\bar{n}_1 + \ldots + \bar{n}_k + \bar{o}_{k+1}) \subset m' \).

Now from our choice of \( k \), we have that \( \langle m_{l*} \rangle^{\langle q_k \rangle} \) is not a Speh representation. We can write

\[ m_{l*} = \{ \nu^{-x+1} \Delta^*, \ldots, \Delta^* \} \]

for a certain \( \Delta^* \) with relative length \( L \) and some \( x \). By \( \langle \rangle \), \( \nu \Delta^* \notin m_l \) for any \( l > l^* \) from our labelling on \( m_l \). Rephrasing the statement, we get the following statement:

\[ (** \) \ \nu^{1/2} \Delta^* \notin \nu^{-1/2}m_l \text{ for any } l > l^*. \]

Now with (*), we have that \( \pi' \) is a composition factor of \( \zeta(m'') \) with \( m''' \) contains all the segments \( \Delta^- \) with \( \Delta \) in \( m \) that has relative length at least \( L + 1 \) and a segment \( \nu^{1/2} \Delta^* \), and we shall call the former segments (i.e. the segment in the form of \( \Delta^- \)) to be special for convenience.

We can apply the intersection-union process to obtain the Zelevinsky multisegment for \( \pi' \) from \( m''' \). However in each step of the process, any one of the two segments involved in the intersection-union cannot be special, otherwise, there exists \( l \geq L + 1 \) such that the number of segments in the resulting multisegment with relative length \( l \) is more than the number of segments in \( m^{(i_1, \ldots, i_r)} \). Hence we obtain the following:

\[ (***) \text{ the number of segments } \Delta \text{ in } m^{(i_1, \ldots, i_r)} \text{ such that } \nu^{1/2} \Delta^* \subset \Delta \text{ is at least equal to one plus the number special segments in } m''' \text{ satisfying the same properties} \]

Now we come to the final part of the proof. We now consider \( \nu^{-1/2, (i, \pi)} \). Following the strategy for right derivatives, we have that for each \( k \)

\[ \pi' \hookrightarrow \nu^{-1/2, (i, \pi)} \hookrightarrow \nu^{-1/2, (i, \chi_k)}. \]

This gives that \( \pi' \)

\[ \pi' \hookrightarrow \nu^{-1/2, (u_1)} \zeta(m_1) \times (v_1) \langle m_1 \rangle \times \ldots \times (u_k) \zeta(m_k) \times (v_k) \zeta(o_{k+1}) \]

with \( u_1^k + \ldots + u_k^k + v_1^k + \ldots + v_k^k + w^k = i \). Again assume there exists a smallest \( k \) such that at least one of \( q_l^k \) is not equal to the level of \( \langle m_l \rangle \).
We firstly consider the case that \( \tilde{k} \geq k \). In this case we similarly have that

\[
\pi' \leftrightarrow \nu^{-1/2} \cdot \zeta(\tilde{n}_1) \times (v_1)(m_1) \times \cdots \times \zeta(\tilde{n}_{k-1}) \times (v_k)(m_{k-1}) \times \zeta(\tilde{o}_{k+1}),
\]

where

\[
\tilde{n}_a = \left\{ (u_{a,1}) \Delta_{a,1}, \ldots , (u_{a,r(a)}) \Delta_{a,r(a)} \right\},
\]

with \( u_{a,1} + \cdots + u_{a,r(a)} = u_a \) and each \( u_{a,b} = 0 \) or \( n(\rho) \). Since we assume that \( \tilde{k} \geq k \), we have that \((v_1)(m_1)\) is a highest derivative and so is a Speh representation, and we can apply Lemma 4.13. Hence the unique subrepresentation of

\[
\nu^{-1/2} \cdot \zeta(\tilde{n}_1) \times (v_1)(m_1) \times \cdots \times \zeta(\tilde{n}_{k-1}) \times (v_k)(m_{k-1}) \times \zeta(\tilde{o}_k)
\]
is isomorphic to

\[
(5.11) \quad \nu^{-1/2} \cdot (\tilde{n}_1 + \tilde{m}_1 + \cdots + \tilde{n}_{k-1} + \tilde{m}_{k-1} + \tilde{o}_k),
\]

where \((\tilde{n}_1) = (v_1)(m_1)\). Similar to (*) for right derivatives (but the proof could be easier here), we obtain the analogous statement for those \( \tilde{n}_a \). Now if

\[
\Delta \in \nu^{-1/2}(\tilde{n}_1 + \tilde{m}_1 + \cdots + \tilde{n}_{k-1} + \tilde{m}_{k-1} + \tilde{o}_k)
\]
such that \( \Delta = \nu^{1/2}\Delta^* \), then we must have that \( \Delta = \nu^{-1/2} \cdot \Delta_0 \) or \( \nu^{-1/2} \cdot \Delta_0 \) for some segment \( \Delta_0 \) in \( m \). However, by (**), the possibility \( \Delta = \nu^{-1/2}\Delta_0 \) cannot happen. Thus we must have that \( \Delta \) is a special in the same sense as the discussion in right derivatives. This concludes the following:

(****) The number of segments \( \Delta \) in \( \nu^{-1/2}(\tilde{n}_1 + \tilde{m}_1 + \cdots + \tilde{n}_{k-1} + \tilde{m}_{k-1} + \tilde{o}_k) \) with the property that \( \nu^{1/2}\Delta^* \subset \Delta \) is equal to the number of special segments satisfying the same properties.

Now the above statement contradicts to (***), since both Zelevinsky multisegments give an irreducible representation isomorphic to \( \pi' \).

Now the way to get contradiction in the case \( k \geq \tilde{k} \) is similar by interchanging the role of left and right derivatives. We remark that to prove the analogue of (**), one uses (⋄). And to obtain the similar isomorphism as (5.11), one needs to use Lemma 7.4(2). We can argue similarly to get an analogue of (***) and (****). Hence the only possibility that \( \nu^{1/2} \cdot \pi(\xi) \) and so \( \nu^{-1/2} \cdot \pi(\xi) \) have an isomorphic irreducible quotient only if \( i \) is the level for \( \pi \).

5.3. Another consequence. Here is another consequence on the Hom-branching law in another direction:

**Corollary 5.2.** Let \( \pi' \) be an irreducible smooth representation of \( G_n \). Let \( \pi \) be an irreducible smooth representation of \( G_{n+1} \). Suppose \( \pi \) is not a 1-dimensional representation of \( G_{n+1} \). Then

\[
\text{Hom}_{G_n}(\pi', \pi|_{G_n}) = 0.
\]

**Proof.** Since \( \pi \) is not one-dimensional, the level of \( \pi \) is not 1 by Zelevinsky classification. By Theorem 4.13 and Proposition 2.5, \( \nu^{1/2} \cdot \pi(\xi) \) and \( \nu^{-1/2} \cdot \pi(\xi) \) have no common irreducible submodule if \( \pi(\xi) \neq 0 \) and \( \pi(\xi) \neq 0 \). Then at least one of

\[
\text{Hom}_{G_n}(\pi', \nu^{1/2} \cdot \pi(\xi)) = 0 \quad \text{or} \quad \text{Hom}_{G_n}(\pi', \nu^{-1/2} \cdot \pi(\xi)) = 0.
\]
On the other hand, we have that for all $i \geq 2$,
$$\text{Hom}^i_{G_n}(\pi', \text{ind}^{G_n}_{R_{n-i+1}} \nu^{-1/2} \cdot \nu^{(i)} \otimes \psi_{i-1}) = 0,$$
and
$$\text{Hom}^i_{G_n}(\pi', \text{ind}^{G_n}_{R_{n-i+1}} \nu^{-1/2} \cdot \nu^{(i)} \otimes \psi_{i-1}) = 0,$$
by a similar argument as in the proof of Theorem 4.9 (which uses Frobenius reciprocity and Lemma 4.5). Now a Bernstein-Zelevinsky filtration implies the corollary. \qed

6. Appendix A: Ext-groups for Speh representations

We studied the homological properties for generic representations in [CST18]. We shall discuss another interesting class of representations, namely the Speh representations, under restriction. We remark that in our context, we do not require Speh representations to be unitary. Speh modules will be used in the next section and we make a digression to discuss some related problems. Indeed, the problem of computing the restricted Ext-groups is partially reduced to computing the ordinary Ext-groups between Speh representations. We explain how one can compute such Ext-groups, and the way to do so is to pass to the setting of graded Hecke algebra modules, which makes use of a resolution constructed in [Ch16].

6.1. Speh multisegments.

Definition 6.1. Let $\Delta$ be a segment. Let
$$m(m, \Delta) = \left\{ \nu^{-(m-1)/2}\Delta, \nu^{-(m-1)/2}\Delta, \ldots, \nu^{(m-1)/2}\Delta \right\}.$$  
We shall call $m(m, \Delta)$ to be a Speh multisegment. Define
$$u(m, \Delta) = \langle m(m, \Delta) \rangle.$$  
We define $L(u(m, \Delta))$ to be the relative length of $\Delta$.

We similarly define
$$u_r(m, i, \Delta) = \langle \nu^{-(m-1)/2}\Delta, \ldots, \nu^{-(m-2i+1)/2}\Delta, \nu^{-(m-2i+1)/2}\Delta, \ldots, \nu^{(m-1)/2}\Delta \rangle,$$
and
$$u_l(m, i, \Delta) = \langle \nu^{-(m-1)/2}\Delta, \ldots, \nu^{(m-2i-1)/2}\Delta, \nu^{(m-2i+1)/2}(\Delta), \ldots, \nu^{(m-1)/2}(\Delta) \rangle.$$  

Let $l = n(\rho)$. It follows from [Ta87, LM14, CS19] that
$$u(m, \Delta) \otimes u_r(m, i, \Delta),$$
and $u(m, \Delta)^{\otimes k}$ is zero if $l$ does not divide $k$. Applying (2.1), we have that
$$u_l(m, \Delta) \equiv u_l(m, i, \Delta),$$
and $(k)u(m, \Delta) = 0$ if $l$ does not divide $k$.

Let $\Delta = [\nu^a \rho, \nu^b \rho]$. Let $L = G_1 \times \ldots \times G_l$, where $G_i$ appears $(b - a + 1)m$ times. Set $d = b - a + 1$. This determines the inertial equivalence class $s = [L, \rho \otimes \ldots \otimes \rho]$ such that $a(m, \Delta)$ is in $\mathfrak{R}_s(G_n)$. In [BK93], we have that $\mathfrak{R}_s(G_n)$ is equivalent to the category of representations of Hecke algebra $H_{dm} := H_{dm}(q)$. 

On the other hand, we have that for all $i \geq 2$,
6.2. Ext-groups of Speh representations. Thus \[ \text{[100]} \] gives that
\[
\text{Ext}_G^{i}(u(m, d), u(m', d')) \cong \text{Ext}_H^{i}(u(m, d)_0, u(m', d')_0).
\]
If \( u(m, d) \) and \( u(m', d') \) have different cuspidal support, then for all \( i \)
\[
\text{Ext}_G^{i}(u(m, d), u(m', d')) = 0.
\]
Thus we only have to consider that \( u(m, d) \) and \( u(m', d') \) have the same cuspidal support. In \( H_{dm} \) setting, that is to consider \( u(m, d)_0 \) and \( u(m', d')_0 \) with the same central character of \( H_{md} \). Let \( Z \) be the center of \( H_{dm} \). Let \( J \) be the corresponding ideal in \( Z \) annihilating \( u(m, d) \) and \( u(m', d') \). We need to pass to the graded Hecke algebra.

**Definition 6.2.** The graded Hecke algebra \( \mathbb{H}_{dm} \) is an associative algebra with unit generated by symbols \( t_w \) (\( w \in S_{dm} \)) and \( x_1, \ldots, x_{dm} \) satisfying the relations:

1. \( t_s t_{s+i} t_s = t_{s+i} t_s t_{s+i}; t_s^2 = 1; \)
2. \( x_j x_j = x_j; \)
3. \( x_j t_s = t_s x_{i+1} = \log q; \)
4. \( x_j t_s = t_s x_j, \) where \( j \neq i, i + 1. \)

There exists a corresponding ideal \( J \) in the center \( Z \) of \( \mathbb{H}_{dm} \) such that the category of finite-dimensional \( H_{dm} \)-modules annihilated by some powers of \( J \) is equivalent to the category of finite-dimensional \( \mathbb{H}_{dm} \)-modules annihilated by some powers of \( J \) \[ \text{[109]} \] (also see \[ \text{[CS19, Theorem 6.2]} \]). Let \( v(m, d) \) (resp. \( v(m', d') \)) be the corresponding \( \mathbb{H}_{dm} \)-module \( u(m, d)_0 \) (resp. \( u(m', d')_0 \)) under that equivalence of categories. Thus we also have:

\[
\text{Ext}_H^{i}(u_{\tau}(m, d), u_{\tau}(m', d')) \cong \text{Ext}_{\mathbb{H}_{dm}}^{i}(b(m, d), b(m', d')).
\]

By tracing \[ \text{[BK93, Theorem 7.6.20]} \] and Lusztig reduction \[ \text{[Lu89]} \], we have that \( v(m, d) \) is isomorphic to an \( \mathbb{H}_{dm} \)-module which is irreducible restricted to \( \mathbb{C}[S_{dm}] \). Here \( \mathbb{C}[S_{dm}] \) is generated by \( t_{s_i} \) (\( i = 1, \ldots, dm - 1 \)). According to \[ \text{[BM99, BC14]} \], we shall denote such module by \( b(m, d) \), whose restriction to \( S_{dm} \)-representation is isomorphic to the irreducible Specht module \( \sigma(m, d) \) of \( S_{dm} \) corresponding to the partition \( (d, d, \ldots, d) \), where \( d \) appears \( m \)-times. In particular, \( \sigma(m, 1) \) is the sign representation and \( \sigma(1, d) \) is the trivial representation.

**Theorem 6.3.** Let \( v(m, d) \) and \( v(m', d') \) as above. Let \( V \cong \mathbb{C}^{dm} \). Then
\[
\text{Ext}_H^{i}(u_{\tau}(m, d), u_{\tau}(m', d')) \cong (\sigma(m, d)^V \otimes \sigma(m', d') \otimes \wedge^i V)^{S_{dm}}.
\]

**Proof.** We recall that in \[ \text{[Ch16, Section 3]} \], we have a Koszul resolution of the form, for an \( \mathbb{H}_{dm} \)-module \( \tau \):
\[
0 \rightarrow \mathbb{H}_{dm} \otimes \mathbb{C}[S_{dm}] (\tau \otimes \wedge^{dm} V) \rightarrow \ldots \rightarrow \mathbb{H}_{dm} \otimes \mathbb{C}[S_{dm}] \tau \rightarrow \tau \rightarrow 0
\]
and the differential map
\[
d : \mathbb{H}_{dm} \otimes \mathbb{C}[S_{dm}] (\tau \otimes \wedge^i V) \rightarrow \mathbb{H}_{dm} \otimes \mathbb{C}[S_{dm}] (\tau \otimes \wedge^{i-1} V)
\]
is given by:

\[
\begin{align*}
d(h \otimes x \otimes (v_1 \wedge \ldots \wedge v_i)) &= \sum_j (-1)^{j+1} h \tilde{v}_j \otimes x \otimes v_1 \wedge \ldots \wedge \tilde{v}_j \wedge \ldots \wedge v_i \\
- \sum_j (-1)^{j+1} h \otimes \tilde{v}_j \cdot x \otimes v_1 \wedge \ldots \wedge \tilde{v}_j \wedge \ldots \wedge v_i,
\end{align*}
\]

where \( \tilde{v} \) is defined as \[Ch16\ (3.4)]. The complex in computing the Ext-group \( \operatorname{Ext}_{\mathbb{H}_{dm}}^i(\tau, \tau') \) takes the form

\[
\operatorname{Hom}_{\mathbb{H}_{dm}}(\mathbb{H}_{dm} \otimes \mathbb{C}[S_{dm}], (\tau \otimes \wedge^i V), \tau') \cong \operatorname{Hom}_{\mathbb{C}[S_{dm}]}(\tau \otimes \wedge^i V, \tau').
\]

For \( \tau = \nu(m, d) \) or \( \nu(m', d') \), we have that \( \tilde{v}_j \) acts by the same scalar on \( \tau \) \[CM15\ Proposition 3.1.1] (also see \[Ch18\ Lemmas 2.2 and 4.2]) as \( \tau \) is twisted from a unitary module by a central character. This implies the differential maps using the Koszul resolution in computing \( \operatorname{Ext}_{\mathbb{H}_{dm}}^i(\tau, \tau') \) are zero. This gives the desired Ext-groups.

\[\square\]

**Remark 6.4.** There is a simple consequence on the above Ext computation on the tensor product structure of symmetric group representations. Note that \( \nu(m, d) \) and \( \nu(m', d') \) have the same central character if and only if either \( m = m' \) and \( d = d' \) or \( m = d' \) and \( d = m' \). As a corollary, we have that

\[
(\sigma(m, d)^{\nu} \otimes \sigma(m', d') \otimes \wedge^i V)^{S_{dm}} = 0
\]

if \((m, d) \neq (m', d') \) and \((d', m') \).

### 6.3. Ext-branching law

We now consider the Ext-branching law for Speh modules. The actual meaning in Speh representations is slightly modified to rule out some triviality. The cases can be considered as a simplest case for Arthur parameters considered in Gan-Gross-Prasad conjectures \[GGP\] and \[Gu18\].

**Proposition 6.5.** Let \( \Delta = [\nu^{-(d-1)/2} \rho, \nu^{(d-1)/2} \rho] \). Let \( \pi = u(m, \Delta) \times 1 \times \ldots \times 1 \) with \( l(\pi) = d \), where \( 1 \) appears \( n + 1 - l \) times. Let

1. \( \pi' = u(m, \nu^{1/2} \Delta^-) \times 1 \times \ldots \times 1 \), where \( 1 \) appears \( n - lm(d-1) \) times;
2. \( \pi' = u(d-1, \tilde{\Delta}) \times 1 \times \ldots \times 1 \), where \( 1 \) appears with \( n - lm(d-1) \) times and \( \tilde{\Delta} = [\nu^{-\frac{lm}{d}} \rho, \nu^{-\frac{lm}{d}} \rho] \);
3. \( \pi' = u(m, + \Delta) \times 1 \times \ldots \times 1 \), where \( 1 \) appears \( n - lm(d+1) \) times and \( + \Delta = \nu^{1/2} [\nu^{-\frac{d-1}{2}} \rho, \nu^{-\frac{d-1}{2}} \rho] = [\nu^{-\frac{d}{2}} \rho, \nu^{\frac{d}{2}} \rho] \).
4. \( \pi' = u(d+1, + \tilde{\Delta}) \times 1 \times \ldots \times 1 \), where \( 1 \) appears \( n - lm(d+1) \) times and \( + \tilde{\Delta} = [\nu^{-\frac{lm}{d}} \rho, \nu^{-\frac{lm}{d}} \rho] \).

Then

\[
\operatorname{Ext}_{G_n}^i(\pi, \pi') \cong \begin{cases}
(\sigma(m, d-1)^{\nu} \otimes \sigma(m, d-1) \otimes \wedge^i V)^{S_{(d-1)m}} & \text{for (1)} \\
(\sigma(m, d-1)^{\nu} \otimes \sigma(d-1, m) \otimes \wedge^i V)^{S_{(d-1)m}} & \text{for (2)} \\
(\sigma(m, d)^{\nu} \otimes \sigma(m, d) \otimes \wedge^i V)^{S_{dm}} & \text{for (3)} \\
(\sigma(m, d)^{\nu} \otimes \sigma(d, m) \otimes \wedge^i V)^{S_{dm}} & \text{for (4)}
\end{cases}
\]
Proof. Let $\Delta' = \left[ \nu^{-(d'-1)/2} \rho, \nu^{(d'-1)/2} \rho \right]$ and let

$$\pi' = a(m'; \Delta') \times 1 \times \ldots \times 1.$$  

We shall freely use (6.12) and (6.13). In those four cases, considering the cuspidal support at $\nu^{-(m'+d'-2)/2} \rho$ and $\nu^{1/2}(\nu^{(m'+d'-2)/2} \rho)$, and at $\nu^0, \nu^{1/2}$, we must have that $\nu^{1/2} \cdot \pi^{(i)}$ and $(i) \pi'$ have the same cuspidal support only if $i$ is the level of $\pi$ or $\pi'$.

We only show the computation for (1) and (2). The argument for other cases is quite similar (and standard). Let $j^*$ be the level of $\pi$. We have that for $i \geq 0$ and $j < j^*$,

$$\text{Ext}^i_{G_{n+1}}(\nu^{1/2} \pi^{(j)} , (j-1) \pi') = 0,$$

by a cuspidal support consideration, and hence

$$\text{Ext}^i_{G_n}(\text{ind}_{R_{j-1}}^G \nu^{1/2} \cdot \pi^{(j)} \boxtimes \psi_{j-1}^{(j)} , \pi') \cong \text{Ext}^i_{G_{n+1-j}}(\nu^{1/2} \cdot \pi^{(j)} , (j-1) \pi') = 0.$$

Now using a standard long exact sequence argument on Bernstein-Zelevinsky filtration, we have that:

$$\text{Ext}^i_{G_n}(\pi, \pi') \cong \text{Ext}^i_G(\text{ind}^G_{R_{n-j^*+1}} \nu^{1/2} \cdot \pi^{(j^*)} \boxtimes \psi_{j^*-1}^{(j^*)} , (j^*-1) \pi')$$

$$\cong \text{Ext}^i_{G_{n+1-j^*}}(\nu^{1/2} \cdot \pi^{(j^*)} , \pi')$$

$$\cong \begin{cases} 
\text{Ext}^i_{G_{n+1-j^*}}(u(m, d-1), u(m, d-1)) & \text{for (1)} \\
\text{Ext}^i_{G_{n+1-j^*}}(u(m, d-1), u(d-1, m)) & \text{for (2)} 
\end{cases}$$

Now the result follows from Theorem 6.3. 

7. Appendix B: Speh representation approximation

7.1. Two lemmas. The parabolic induction is studied in [La15, LM16] in a larger context. We give a proof of the following specific cases, using the theory of derivatives.

Lemma 7.1. Let $\Delta = [\nu^a \rho, \nu^b \rho]$. If $\Delta' = [\nu^k \rho, \nu^l \rho]$ for some

$$-\frac{m-1}{2} + a \leq k \leq l \leq -\frac{m-1}{2} + b,$$

then

$$a(m, \Delta) \times \langle \Delta' \rangle \cong \langle \Delta' \rangle \times a(m, \Delta),$$

and is irreducible.

Proof. The statement is easy when $\Delta$ is singleton $[\rho]$ because

$$a(m, \Delta) \cong \text{St}(\nu^{-(m-1)/2} \rho, \nu^{(m-1)/2} \rho)$$

and $\text{St}(\Delta') \times \langle \Delta'' \rangle \cong \langle \Delta'' \rangle \times \text{St}(\Delta')$ whenever $\Delta'' \subset \Delta'$. (Indeed, one can also prove by similar arguments as below by noting that the Zelevinsky multisegment of any simple composition factor contains a segment $\Delta$ with $b(\Delta) \cong \nu^{(m-1)/2} \rho$ and at least one segment $\Delta$ with $b(\Delta) \cong b(\Delta'').$)

We now assume $\Delta$ is not a singleton. We consider two cases:
(1) Case 1: \( l = b + \frac{m-1}{2} \). Suppose \( \tau \) is a composition factor of \( a(m, \Delta) \times \langle \Delta' \rangle \) with Zelevinsky multisection \( m \). Then we know that at least two segments \( \Delta_1, \Delta_2 \) in \( m \) takes the form \( b(\Delta_1) \cong b(\Delta_2) \cong \nu^{\frac{m-1}{2}} b(\Delta) \). If \( \tau^{(i)} \) is the highest derivative of \( \tau \), then we know that the cuspidal support of \( \tau^{(i)} \) does not contain \( \nu^{\frac{m-1}{2}} b(\Delta) \). We also have that \( \tau^{(i)} \) is a composition factor of \( (a(m, \Delta) \times \langle \Delta \rangle)^{(i)} \). The only possibility is that \( i^* = m + 1 \) i.e \( a(m, \Delta) \times \langle \Delta' \rangle)^{(i)} = a(m, \Delta^-) \times \langle \langle \Delta' \rangle^- \rangle \), which the latter one is irreducible by induction. This proves the lemma. Since taking derivative is an exact functor, we have that \( a(m, \Delta) \times \langle \Delta' \rangle \) is irreducible. Using the Gelfand-Kazhdan involution [BZ76, Section 7], we have that \( \langle \Delta' \rangle \times a(m, \Delta) \cong a(m, \Delta) \times \langle \Delta' \rangle \).

(2) Case 2: \( l < b + \frac{m-1}{2} \). The argument is similar. Again suppose \( \tau \) is a composition factor of \( a(m, \Delta) \times \langle \Delta' \rangle \). Using an argument similar to above, we have that the level of \( \tau \) is either \( m + 1 \) or \( m \). However, if the level of \( \tau \) is \( m \), then \( \tau^{(m)} \) would be \( a(m, \Delta^-) \times \langle \Delta' \rangle \), which is irreducible by induction. Then it would imply that the Zelevinsky multisection of \( \tau^{(m)} \) is \( m + 1 \) and contradicts that the number of segments for the Zelevinsky multisection of the highest derivative of an irreducible representation \( \pi \) must be smaller than that for \( \pi \). Hence, the level of \( \tau \) must be \( m + 1 \). Now repeating a similar argument as in (1) and using the induction, we obtain the statements.

\[\square\]

**Lemma 7.2.** Let \( \Delta \) be a segment. Then
\[
a(m, \Delta^-) \times \nu^{(m-1)/2} \langle \Delta \rangle \cong \nu^{(m-1)/2} \langle \Delta \rangle \times a(m, \Delta^-)
\]
is irreducible.

**Proof.** The statement is clear if \( \Delta \) is a singleton. For the general case, we note by simple counting that the Zelevinsky multisection of any composition factor must contain a segment \( \Delta_1 \) with \( b(\Delta_1) \cong \nu^{(m-1)/2} b(\Delta) \) and at least one segment \( \Delta_2 \) with \( b(\Delta_2) \cong \nu^{(m-1)/2} b(\Delta^-) \). Then one proves the statement by a similar argument using highest derivative as in the previous lemma.

\[\square\]

### 7.2. Speh representation approximation.

For a Speh multisection
\[
m = \left\{ \Delta, \nu^{-1} \Delta, \ldots, \nu^{-k} \Delta \right\},
\]
define \( b(m) = b(\Delta) \).

**Proposition 7.3.** Let \( m = \{ \Delta_1, \ldots, \Delta_k \} \). Then there exists Speh multisections \( m_1, \ldots, m_r \) satisfying the following properties:

1. \( m = m_1 + \ldots + m_r \);
2. For each Speh multisection \( m_i \) and any \( j > i \), there is no segment \( \Delta \) in \( m_j \) such that \( m_j + \{ \Delta \} \) is a Speh multisection;
3. \( \langle m \rangle \) is the unique submodule of \( \langle m_1 \rangle \times \ldots \times \langle m_r \rangle \);
4. \( b(m_i) \) does not precede \( b(m_j) \) if \( i < j \);
5. if \( m_i \cap m_j \neq \emptyset \) and \( i \leq j \), then \( m_j \subset m_i \).
Proof. We shall label the segments $m$ in the way that for $i < j$, (i) $b(\Delta_i)$ does not proceed $b(\Delta_j)$ and (ii) if $b(\Delta_i) = b(\Delta_j)$, then $a(\Delta_i)$ does not proceed $a(\Delta_j)$. Let $\Delta = \Delta_1$. Let $k$ be the largest integer ($k \geq 0$) such that $\Delta, \nu^{-1}\Delta, \ldots, \nu^{-k}\Delta$ are segments in $m$. We claim that

$$ (7.14) \quad \langle m \rangle \leftrightarrow \langle m' \rangle \times \langle m \setminus m' \rangle $$

and moreover $\langle m \rangle$ is the unique submodule of $\langle m' \rangle \times \langle m \setminus m' \rangle$. By induction, the claim proves the lemma.

We now prove the claim. We shall prove by induction that for $i = 0, \ldots, k$,

$$ \langle m \rangle \leftrightarrow \langle m_i \rangle \times \zeta(m \setminus m_i), $$

where $m_i = \{ \Delta, \nu^{-1}\Delta, \ldots, \nu^{-i}\Delta \}$. When $i = 0$ the statement is clear from the definition. Now suppose that we have the inductive statement for $i$. To prove the statement for $i + 1$. We now set $n$ to be the collection of all segments $\Delta'$ in $m$ such that $b(\Delta') = b(\Delta), \nu^{-1}b(\Delta), \ldots, \nu^{-i}b(\Delta)$ and $\nu^{-i}a(\Delta)$ proceeds $a(\Delta')$. We also set $\bar{n}$ to be the collection of all segments $\Delta'$ in $m$ such that $b(\Delta') = b(\Delta), \nu^{-1}b(\Delta), \ldots, \nu^{-i}b(\Delta)$ and $a(\Delta')$ precedes $\nu^{-i}a(\Delta)$. By using the Zelevinsky theory, we have that $\zeta(n) \times \zeta(\bar{n}) \cong \zeta(n + \bar{n})$ and

$$ \zeta(m \setminus m_i) \leftrightarrow \zeta(n) \times \zeta(\bar{n}) \times \zeta(m \setminus (n + \bar{n} + m_i)). $$

From our construction (and $i \neq k$), we have that $\nu^{-i-1}\Delta$ is a segment in $m \setminus (n + \bar{n} + m_i)$ and

$$ m \setminus (n + \bar{n} + m_i) = \nu^{-i-1}\Delta \times \zeta(m \setminus (n + \bar{n} + m_{i+1})). $$

On the other hand

$$ \langle m_i \rangle \times \zeta(n) \times \zeta(\bar{n}) \times \nu^{-i-1}\Delta $$

$$ \cong \langle m_i \rangle \times \zeta(n) \times \zeta(\bar{n}) \times \nu^{-i-1}\Delta $$

$$ \cong \langle m_i \rangle \times \zeta(n) \times \nu^{-i-1}\Delta \times \zeta(\bar{n}) $$

$$ \leftrightarrow \zeta(n) \times \zeta(m_{i+1}) \times \zeta(\bar{n}) $$

$$ \cong \langle m_{i+1} \rangle \times \zeta(n) \times \zeta(\bar{n}). $$

The first and last isomorphism is by Lemma 7.3. The injectivity in forth line comes from the uniqueness of submodule in $\zeta(m_{i+1})$. The second isomorphism follows from Ze80. This proves (1) and (3). And (2), (4) and (5) follows from the inductive construction.

We shall need a variation which is more flexible in our application.

Lemma 7.4. Let $m = a(m, \Delta)$ be a Speh multisegment. Let $n_1$ (resp. $n_2$) be a Zelevinsky multisegment such that for any segment $\Delta'$ in $n_1$ (resp. $n_2$) satisfying one of the following properties:

1. $b(\Delta)$ precedes $b(\Delta')$ or $b(\Delta) \cong b(\Delta')$ (resp. $b(\Delta)$ does not precede $b(\Delta')$)
2. $b(\Delta)$ precedes $b(\Delta')$ or $b(\Delta) \cong b(\Delta')$ (resp. $b(\Delta)$ does not precede $b(\Delta')$ or if $b(\Delta)$ precedes $b(\Delta')$, then $(\Delta')^- = \Delta$).

Then

$$ \langle n_1 + m + n_2 \rangle \leftrightarrow \zeta(n_1) \times \langle m \rangle \times \zeta(n_2) \leftrightarrow \zeta(n_1 + m + n_2). $$
Proof. For all cases, we have that
\[
\zeta(n_1) \times \zeta(m + n_2) \cong \zeta(n_1 + m + n_2).
\]
Using (7.14) for (1) we obtain the lemma. For (2), let \(n'\) be all the segments in \(n_2\) with the property that \((\Delta')^- \cong \Delta\). Then we have that
\[
\zeta(n') \times \zeta(m) \times \zeta(n_2 \setminus n') \mapsto \zeta(n') \times \zeta((m + n_2) \setminus n') \mapsto \zeta(m + n_2).
\]
By Lemma 7.2 we have that
\[
\zeta(n') \times \zeta(m) \times \zeta(n_2 \setminus n') \cong \zeta(m) \times \zeta(n') \times \zeta(n_2 \setminus n') \cong \zeta(m) \times \zeta(n_2),
\]
which proves the lemma. \(\square\)

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