Hidden Hodge symmetries and Hodge correlators

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To Don Zagier for his 60-th birthday

1 Hidden Hodge symmetries

There is a well known parallel between Hodge and étale theories, still incomplete and rather mysterious:

| $l$-adic Étale Theory | Hodge Theory |
|-----------------------|--------------|
| Category of $l$-adic Galois modules | Abelian category $\mathcal{MH}_\mathbb{R}$ of real mixed Hodge structures |
| Galois group $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ | Hodge Galois group $G_{\text{Hod}} := \text{Galois group of the category } \mathcal{MH}_\mathbb{R}$ |
| $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on $H^*_a(X, \mathbb{Q}_l)$, where $X$ is a variety over $\mathbb{Q}$ | $H^*(X(\mathbb{C}), \mathbb{R})$ has a functorial real mixed Hodge structure |
| étale site | ?? |
| $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on the étale site, and thus on categories of étale sheaves on $X$, e.g. on the category of $l$-adic perverse sheaves | ?? |
| $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-equivariant perverse sheaves | Saito’s Hodge sheaves |

The current absence of the “Hodge site” was emphasized by A.A. Beilinson [B].

The Hodge Galois group. A weight $n$ pure real Hodge structure is a real vector space $H$ together with a decreasing filtration $F^\bullet H_\mathbb{C}$ on its complexification satisfying

$$H_\mathbb{C} = \oplus_{p+q=n} F^p H_\mathbb{C} \cap \overline{F^q H_\mathbb{C}}.$$ 

A real Hodge structure is a direct sum of pure ones. The category $\mathcal{PH}_\mathbb{R}$ real Hodge structures is equivalent to the category of representations of the real algebraic group $\mathbb{C}^*_\mathbb{R}$. The group of complex points of $\mathbb{C}^*_\mathbb{R}$ is $\mathbb{C}^* \times \mathbb{C}^*$; the complex conjugation interchanges the factors.

A real mixed Hodge structure is given by a real vector space $H$ equipped with the weight filtration $W_n H$ and the Hodge filtration $F^\bullet H_\mathbb{C}$ of its complexification, such that the Hodge filtration induces on $\text{gr}_n^W H$ a weight $n$ real Hodge structure. The category

$$X^\bullet \mathcal{MH}_\mathbb{R} = \prod_{n \geq 0} \text{Hom}(W_n H, \mathcal{MH}_\mathbb{R})$$

is a category of mixed Hodge structures. The category $\mathcal{MH}_\mathbb{R}$ of real mixed Hodge structures is equivalent to the category $\text{Rep}(\mathbb{C}^*_\mathbb{R})$ of representations of the algebraic group $\mathbb{C}^*_\mathbb{R}$.
\( \mathcal{MH}_\mathbb{R} \) of real mixed Hodge structures is an abelian rigid tensor category. There is a fiber functor to the category of real vector spaces

\[
\omega_{\text{Hod}} : \mathcal{MH}_\mathbb{R} \rightarrow \text{Vect}_\mathbb{R}, \quad H \mapsto \oplus_n \text{gr}_n^W H.
\]

The Hodge Galois group is a real algebraic group given by automorphisms of the fiber functor:

\[
G_H := \text{Aut}^\otimes \omega_{\text{Hod}}.
\]

The fiber functor provides a canonical equivalence of categories

\[
\omega_{\text{Hod}} : \mathcal{MH}_\mathbb{R} \xrightarrow{\sim} G_{\text{Hod}} - \text{modules}.
\]

The Hodge Galois group is a semidirect product of the unipotent radical \( U_{\text{Hod}} \) and \( \mathbb{C}^* \mathbb{C}/\mathbb{R} \):

\[
0 \longrightarrow U_{\text{Hod}} \longrightarrow G_{\text{Hod}} \longrightarrow \mathbb{C}^*_\mathbb{C}/\mathbb{R} \longrightarrow 0, \quad \mathbb{C}^*_\mathbb{C}/\mathbb{R} \hookrightarrow G_{\text{Hod}}.
\]

The projection \( G_{\text{Hod}} \twoheadrightarrow \mathbb{C}^*_\mathbb{C}/\mathbb{R} \) is provided by the inclusion of the category of real Hodge structures to the category of mixed real Hodge structures. The splitting \( s : \mathbb{G}_m \rightarrow G_{\text{Hod}} \) is provided by the functor \( \omega_{\text{Hod}} \).

The complexified Lie algebra of \( U_{\text{Hod}} \) has canonical generators \( G_{p,q} \), \( p, q \geq 1 \), satisfying the only relation \( G_{p,q} = -G_{q,p} \), defined in [G1]. For the subcategory of Hodge-Tate structures they were defined in [L]. Unlike similar but different Deligne’s generators [D], they behave nicely in families. So to define an action of the group \( G_{\text{Hod}} \) one needs to have an action of the subgroup \( \mathbb{C}^*_\mathbb{C}/\mathbb{R} \) and, in addition to this, an action of a single operator

\[
G := \sum_{p,q \geq 1} G_{p,q}.
\]

The twistor Galois group. Denote by \( \mathbb{C}^* \) the real algebraic group with the group of complex points \( \mathbb{C}^* \). The extension induced from \( \mathbb{G}_m \) by the diagonal embedding \( \mathbb{C}^* \subset \mathbb{C}^*_\mathbb{C}/\mathbb{R} \) is the twistor Galois group. It is a semidirect product of the groups \( U_{\text{Hod}} \) and \( \mathbb{C}^* \).

\[
0 \longrightarrow U_{\text{Hod}} \longrightarrow G_T \longleftarrow \mathbb{C}^* \longrightarrow 0.
\]

It is not difficult to prove

**Lemma 1.1** The category of representations of \( G_T \) is equivalent to the category of mixed twistor structures defined by Simpson [Si2].

We suggest the following fills the ??-marks in the dictionary related the Hodge and Galois. Below \( X \) is a smooth projective complex algebraic variety.

**Conjecture 1.2** There exists a functorial homotopy action of the twistor Galois group \( G_T \) by \( A_\infty \)-equivalences of an \( A_\infty \)-enhancement of the derived category of perverse sheaves on \( X \) such that the category of equivariant objects is equivalent to Saito’s category real mixed Hodge sheaves.\footnote{We want to have a natural construction of the action first, and get Saito’s category real mixed Hodge sheaves as a consequence, not the other way around.}
Denote by $D^{b}_{\text{sm}}(X)$ the category of smooth complexes of sheaves on $X$, i.e. complexes of sheaves on $X$ whose cohomology are local systems.

**Theorem 1.3** There exists a functorial for pull-backs homotopy action of the twistor Galois group $G_T$ by $A_\infty$-equivalences of an $A_\infty$-enhancement of the category $D^{b}_{\text{sm}}(X)$.

The action of the subgroup $\mathbb{C}^*$ is not algebraic. It arises from Simpson’s action of $\mathbb{C}^*$ on semisimple local systems [Si1]. The action of the Lie algebra of the unipotent radical $U_{\text{Hod}}$ is determined by a collection of numbers, which we call the *Hodge correlators for semisimple local systems*. Our construction uses the theory of harmonic bundles [Si1]. The Hodge correlators can be interpreted as correlators for a certain Feynman integral. This Feynman integral is probably responsible for the “Hodge site”.

For the trivial local system the construction was carried out in [G2]. A more general construction for curves, involving the constant sheaves and delta-functions, was carried out in [G1].

In the case when $X$ is the universal modular curve, the Hodge correlators contain the special values $L(f, n)$ of weight $k \geq 2$ modular forms for $GL_2(\mathbb{Q})$ outside of the critical strip – it turns out that the simplest Hodge correlators in this case coincide with the Rankin-Selberg integrals for the non-critical special values $L(f, k+n)$, $n \geq 0$ – the case $k=2, n=0$ is discussed in detail in [G1].

**2 Hodge correlators for local systems**

**2.1 An action of $G_T$ on the “minimal model” of $D^{b}_{\text{sm}}(X)$.**

Tensor products of irreducible local systems are semisimple local systems. The *category of harmonic bundles* $\text{Har}_X$ is the graded category whose objects are semi-simple local systems on $X$ and their shifts, and morphisms are given by graded vector spaces

$$\text{Hom}_{\text{Har}_X}^\bullet(V_1, V_2) := H^\bullet(X, V_1^\vee \otimes V_2). \quad (3)$$

Here is our main result.

**Theorem 2.1** There is a homotopy action of the twistor Galois group $G_T$ by $A_\infty$-equivalences of the graded category $\text{Har}_X$, such that the action of the subgroup $\mathbb{C}^*$ is given by Simpson’s action of $\mathbb{C}^*$ on semi-simple local systems.

This immediately implies Theorem 1.3. Indeed, given a small $A_\infty$-category $\mathcal{A}$, there is a functorial construction of the triangulated envelope $\text{Tr}(\mathcal{A})$ of $\mathcal{A}$, the smallest triangulated category containing $\mathcal{A}$. Since $D^{b}_{\text{sm}}(X)$ is generated as a triangulated category by semi-simple local systems, the category $\text{Tr}(\text{Har}_X)$ is equivalent to $D^{b}_{\text{sm}}(X)$ as a triangulated category, and thus is an $A_\infty$-enhancement of the latter. On the other hand, the action of the group $G_T$ from Theorem 2.1 extends by functoriality to the action on $\text{Tr}(\text{Har}_X)$.

Below we recall what are $A_\infty$-equivalences of DG categories and then define the corresponding data in our case.
2.2 $A_\infty$-equivalences of DG categories

The Hochshild cohomology of a small dg-category $\mathcal{A}$. Let $\mathcal{A}$ be a small dg category. Consider a bicomplex whose $n$-th column is

\[
\prod_{[X_i]} \text{Hom} \left( \mathcal{A}(X_0, X_1)[1] \otimes \mathcal{A}(X_1, X_2)[1] \otimes \ldots \otimes \mathcal{A}(X_{n-1}, X_n)[1], \mathcal{A}(X_0, X_n)[1] \right),
\]

where the product is over isomorphism classes $[X_i]$ of objects of the category $\mathcal{A}$. The vertical differential $d_1$ in the bicomplex is given by the differential on the tensor product of complexes. The horizontal one $d_2$ is the degree 1 map provided by the composition $\mathcal{A}(X_i, X_{i+1}) \otimes \mathcal{A}(X_{i+1}, X_{i+2}) \rightarrow \mathcal{A}(X_i, X_{i+2})$.

Let $\text{HC}^*(\mathcal{A})$ be the total complex of this bicomplex. Its cohomology are the Hochshild cohomology $\text{HH}^*(\mathcal{A})$ of $\mathcal{A}$. Let $\text{Fun}_{A_\infty}(\mathcal{A}, \mathcal{A})$ be the space of $A_\infty$-functors from $\mathcal{A}$ to itself. Lemma 2.2 can serve as a definition of $A_\infty$-functors considered modulo homotopy equivalence.

**Lemma 2.2** One has

\[
H^0\text{Fun}_{A_\infty}(\mathcal{A}, \mathcal{A}) = \text{HH}^0(\mathcal{A}).
\]

Indeed, a cocycle in $\text{HC}^0(\mathcal{A})$ is the same thing as an $A_\infty$-functor. Coboundaries correspond to the homotopically to zero functors.

The cyclic homology of a small rigid dg-category $\mathcal{A}$. Let $(\alpha_0 \otimes \ldots \otimes \alpha_m)_c$ be the projection of $\alpha_0 \otimes \ldots \otimes \alpha_m$ to the coinvariants of the cyclic shift. So, if $\overline{\alpha} := \deg \alpha$,

\[
(\alpha_0 \otimes \ldots \otimes \alpha_m)_c = (-1)^{\overline{\alpha}_1 + \ldots + \overline{\alpha}_{m-1}} (\alpha_1 \otimes \ldots \otimes \alpha_m \otimes \alpha_0)_c.
\]

We assign to $\mathcal{A}$ a bicomplex whose $n$-th column is

\[
\prod_{[X_i]} \left( \mathcal{A}(X_0, X_1)[1] \otimes \ldots \otimes \mathcal{A}(X_{n-1}, X_n)[1] \otimes \mathcal{A}(X_n, X_0)[1] \right)_c.
\]

The differentials are induced by the differentials and the composition maps on Hom’s. The cyclic homology complex $\text{CC}_*(\mathcal{A})$ of $\mathcal{A}$ is the total complex of this bicomplex. Its homology are the cyclic homology of $\mathcal{A}$.

Assume that there are functorial pairings

\[
\mathcal{A}(X, Y)[1] \otimes \mathcal{A}(Y, X)[1] \rightarrow \mathcal{H}^*.
\]

Then there is a morphism of complexes

\[
\text{HC}^*(\mathcal{A})^* \rightarrow \text{CC}^*_*(\mathcal{A}) \otimes \mathcal{H}.
\]

For the category of harmonic bundles $\text{Har}_X$ there is such a pairing with

$\mathcal{H} := H_{2n}(X)[-2]$.

It provides a map

\[
\varphi : \text{Hom} \left( H_0(\text{CC}^*_s(\text{Har}_X) \otimes \mathcal{H}, \mathbb{C}) \right) \rightarrow \text{HH}^0(\text{Har}_X) \cong H^0\text{Fun}_{A_\infty}(\text{Har}_X, \text{Har}_X).
\]
2.3 The Hodge correlators

Theorem 2.3  a) There is a linear map, the Hodge correlator map

\[
\text{Cor}_{\text{Har}_X} : H_0(\mathbb{C}C_\ast(\text{Har}_X) \otimes \mathcal{H}) \to \mathbb{C}.
\]  (8)

Combining it with (7), we get a cohomology class

\[
\mathbb{H}_{\text{Har}_X} := \phi(\text{Cor}_{\text{Har}_X}) \in H_0^{A_\infty}(\text{Har}_X, \text{Har}_X).
\]  (9)

b) There is a homotopy action of the twistor Galois group \(G_T\) by \(A_\infty\)-autoequivalences of the category \(\text{Har}_X\) such that

- Its restriction to the subgroup \(\mathbb{C}^\ast\) is the Simpson action \([\text{Si1}]\) on the category \(\text{Har}_X\).
- Its restriction to the Lie algebra \(\text{LieU}_{\text{Hod}}\) is given by a Lie algebra map

\[
\mathbb{H}_{\text{Har}_X} : \text{L}_{\text{Hod}} \to H_0^{A_\infty}(\text{Har}_X, \text{Har}_X),
\]  (10)

uniquely determined by the condition that \(\mathbb{H}_{\text{Har}_X}(G) = \mathbb{H}_{\text{Har}_X}\).

c) The action of the group \(G_T\) is functorial with respect to the pull backs.

2.4 Construction.

To define the Hodge correlator map (8), we define a collection of degree zero maps

\[
\text{Cor}_{\text{Hod}_X} : \left( H^\ast(X, V_0^\ast \otimes V_1)[1] \otimes \ldots \otimes H^\ast(X, V_m^\ast \otimes V_0)[1] \right) \otimes \mathcal{H} \to \mathbb{C}. \]  (11)

The definition depends on some choices, like harmonic representatatives of cohomology classes. We prove that it is well defined on \(HC^0\), i.e. its restriction to cycles is independent of the choices, and coboundaries are mapped to zero.

We picture an element in the source of the map (11) by a polygon \(P\), see Fig 1, whose vertices are the objects \(V_i\), and the oriented sides \(V_iV_{i+1}\) are graded vector space \(\text{Ext}^\ast(V_i, V_{i+1})(1)\).

![Figure 1: A decorated plane trivalent tree; \(V_i\) are harmonic bundles.](image)
Green currents for harmonic bundles. Let $V$ be a harmonic bundle on $X$. Then there is a Doulbeaut bicomplex $(A^\bullet(X, V); D', D'')$ where the differentials $D', D''$ are provided by the complex structure on $X$ and the harmonic metric on $V$. It satisfies the $D', D''$-lemma.

Choose a splitting of the corresponding de Rham complex $A^\bullet(X, V)$ into an arbitrary subspace $\mathcal{H}ar^\bullet(X, V)$ isomorphically projecting onto the cohomology $H^\bullet(X, V)$ ("harmonic forms") and its orthogonal complement. If $V = \mathbb{C}^X$, we choose $a \in X$ and take the $\delta$-function $\delta_a$ at the point $a \in X$ as a representative of the fundamental class.

Let $\delta_\Delta$ be the Schwarz kernel of the identity map $V \to V$ given by the $\delta$-function of the diagonal, and $P_{\text{Har}}$ the Schwarz kernel of the projector onto the space $\mathcal{H}ar^\bullet(X, V)$, realized by an $(n, n)$-form on $X \times X$. Choose a basis $\{\alpha_i\}$ in $\mathcal{H}ar^\bullet(X, V)$. Denote by $\{\alpha_i^\vee\}$ the dual basis. Then we have

$$P_{\text{Har}} = \sum \alpha_i^\vee \otimes \alpha_i, \quad \int_X \alpha_i \wedge \alpha_j^\vee = \delta_{ij}.$$

Let $p_i : X \times X \to X$ be the projections onto the factors.

**Definition 2.4** A Green current $G(V; x, y)$ is a $p_1^*V^* \otimes p_2^*V$-valued current on $X \times X$,

$$G(V; x, y) \in D^{2n-2}(X \times X, p_1^*V^* \otimes p_2^*V), \quad n = \dim_{\mathbb{C}}X,$$

which satisfies the differential equation

$$(2\pi i)^{-1} D''D'G(V; x, y) = \delta_\Delta - P_{\text{Har}}. \quad (12)$$

The two currents on the right hand side of (12) represent the same cohomology class, so the equation has a solution by the $D''D'$-lemma.

**Remark.** The Green current depends on the choice of the "harmonic forms". So if $V = \mathbb{C}$, it depends on the choice of the base point $a$. Solutions of equation (12) are well defined modulo $\text{Im}D'' + \text{Im}D' + \mathcal{H}ar^\bullet(X, V)$.

Construction of the Hodge correlators. **Trees.** Take a plane trivalent tree $T$ dual to a triangulation of the polygon $P$, see Fig 1. The complement to $T$ in the polygon $P$ is a union of connected domains parametrized by the vertices of $P$, and thus decorated by the harmonic bundles $V_i$. Each edge $E$ of the tree $T$ is shared by two domains. The corresponding harmonic bundles are denoted $V_{E^-}$ and $V_{E^+}$. If $E$ is an external edge, we assume that $V_{E^-}$ is before $V_{E^+}$ for the clockwise orientation.

Given an internal vertex $v$ of the tree $T$, there are three domains sharing the vertex. We denote the corresponding harmonic bundles by $V_i, V_j, V_k$, where the cyclic order of the bundles agrees with the clockwise orientation. There is a natural trace map

$$\text{Tr}_v : V_i^* \otimes V_j \otimes V_j^* \otimes V_k \otimes V_k^* \otimes V_i = \longrightarrow \mathbb{C}. \quad (13)$$

It is invariant under the cyclic shift.
Decorations. For every edge $E$ of $T$, choose a graded splitting of the de Rham complex
\[ \mathcal{A}^\bullet(X, V_E^* \otimes V_{E+}) = \mathcal{H}ar^\bullet(X, V_E^* \otimes V_{E+}) \bigoplus \mathcal{H}ar^\bullet(X, V_E^* \otimes V_{E+})^\perp. \]

Then a decomposable class in \( \left( \bigotimes_{i=0}^m H^*(X, V_i^* \otimes V_{i+1})[1] \right)_C \) has a harmonic representative
\[ W = (\alpha_{0,1} \otimes \alpha_{1,2} \otimes \ldots \otimes \alpha_{m,0})_C. \]

We are going to assign to $W$ a top degree current $\kappa(W)$ on $X$ (internal vertices of $T$).

Each external edge $E$ of the tree $T$ is decorated by an element
\[ \alpha_E \in \mathcal{H}ar^\bullet(X, V_E^* \otimes V_{E+}). \]

Put the current $\alpha_E$ to the copy of $X$ assigned to the internal vertex of the edge $E$, and pull it back to (14) using the projection $p_{\alpha_E}$ of the latter to the $X$. Abusing notation, we denote the pull back by $\alpha_E$. It is a form on (14) with values in the bundle $p_{\alpha_E}^*(V_E^* \otimes V_{E+})$

Green currents. We assign to each internal edge $E$ of the tree $T$ a Green current
\[ G(V_E^* \otimes V_{E+}; x_-, x_+). \]

The order of $(x_-, x_+)$ agrees with the one of $(V_E^*; V_{E+})$ as on Fig 2, the cyclic order of $(V_E^*, x_-, V_{E+}, x_+)$ agrees with the clockwise orientation. The Green current (15) is symmetric:
\[ G(V_E^* \otimes V_{E+}; x_-, x_+) = G(V_E^* \otimes V_{E-}; x_+, x_-). \]

So it does not depend on the choice of orientation of the edge $E$.

![Figure 2: Decorations of the Green current assigned to an edge $E$.](image)

The map $\xi$. There is a degree zero map
\[ \xi : \mathcal{A}^\bullet(X, V_0)[-1] \otimes \ldots \otimes \mathcal{A}^\bullet(X, V_m)[-1] \rightarrow \mathcal{A}^\bullet(X, V_0 \otimes \ldots \otimes V_m)[-1]; \]

\[ \xi : \mathcal{A}^\bullet(X, V_0)[-1] \otimes \ldots \otimes \mathcal{A}^\bullet(X, V_m)[-1] \rightarrow \mathcal{A}^\bullet(X, V_0 \otimes \ldots \otimes V_m)[-1]; \]
The graded symmetrization in (18) is defined via isomorphisms $V_{\sigma(0)} \otimes \ldots \otimes V_{\sigma(m)} \to V_0 \otimes \ldots \otimes V_m$, where $\sigma$ is a permutation of $\{0, \ldots, m\}$. It is essential that $\text{deg} D^C \varphi = \text{deg} \varphi + 1$.

An outline of the construction. We apply the operator $\xi$ to the product of the Green currents assigned to the internal edges of $T$. Then we multiply on (14) the obtained local system valued current with the one provided by the decoration $W$, with an appropriate sign. Applying the product of the trace maps (13) over the internal vertices of $T$, we get a top degree scalar current on (14). Integrating it we get a number assigned to $T$. Taking the sum over all plane trivalent trees $T$ decorated by $W$, we get a complex number $\text{Cor}_{\text{HarX}}(W \otimes \mathcal{H})$. Altogether, we get the map (8). One checks that its degree is zero. The signs in this definition are defined the same way as in [G2].

Theorem 2.5 The maps (11) give rise to a well defined Hodge correlator map (8).

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