Dynamical Equations of Controlled Rigid Spacecraft with a Rotor

Hong Wang
School of Mathematical Sciences and LPMC,
Nankai University, Tianjin 300071, P.R.China
E-mail: hongwang@nankai.edu.cn
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Abstract. In this paper, we consider the controlled rigid spacecraft with an internal rotor as a regular point reducible regular controlled Hamiltonian (RCH) system. In the cases of coincident and non-coincident centers of buoyancy and gravity, we first give the regular point reduction and the dynamical vector field of the reduced controlled rigid spacecraft-rotor system, respectively. Then, we derive precisely the geometric constraint conditions of the reduced symplectic form for the dynamical vector field of the regular point reducible controlled spacecraft-rotor system, that is, the two types of Hamilton-Jacobi equation for the reduced controlled spacecraft-rotor system by calculation in detail. These researches reveal the deeply internal relationships of the geometrical structures of phase spaces, the dynamical vector fields and controls of the system.

Keywords: controlled rigid spacecraft with a rotor, regular controlled Hamiltonian system, coincident and non-coincident centers, regular point reduction, Hamilton-Jacobi equation.

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1 Introduction

A regular controlled Hamiltonian (RCH) system is a Hamiltonian system with external force and control, which is defined in Marsden et al. [15]. In general, an RCH system, under the actions of
external force and control, is not Hamiltonian, however, it is a dynamical system closely related to a Hamiltonian system, and it can be explored and studied by extending the methods for external force and control in the study of Hamiltonian systems. Thus, we can emphasize explicitly the impact of external force and control in the study for the RCH systems. In particular, in Marsden et al. [15], the authors give the regular point reduction and the regular orbit reduction for an RCH system with symmetry, by analyzing carefully the geometrical and topological structures of the phase space and the reduced phase space of the corresponding Hamiltonian system. These research work not only gave a variety of reduction methods for the RCH systems, but also showed a variety of relationships of controlled Hamiltonian equivalence of these systems. Now, it is a natural problem if there is a practical RCH system and how to show the effect on controls in regular symplectic reductions of the system. In this paper, we consider the rigid spacecraft-rotor system with the control torque acting on an internal rotor as a regular point reducible RCH system on the generalization of a Lie group, and as the application of the above theoretical result, we first give the regular point reduced controlled spacecraft-rotor systems by calculation in detail, in the cases of coincident and non-coincident centers of buoyancy and gravity.

We note also that Hamilton-Jacobi theory for a Hamiltonian system is an important research subject. On the one hand, it provides a characterization of the generating functions of certain time-dependent canonical transformations, such that a given Hamiltonian system in such a form that its solutions are extremely easy to find by reduction to the equilibrium. On the other hand, it is possible in many cases that Hamilton-Jacobi equation provides an immediate way to integrate the equation of motion of system, even when the problem of Hamiltonian system itself has not been or cannot be solved completely, see Abraham and Marsden [1], Arnold [3] and Marsden and Ratiu [14]. In addition, the Hamilton-Jacobi equation is also fundamental in the study of the quantum-classical relationship in quantization, and it also plays an important role in the study of stochastic dynamical systems, see Woodhouse [27], Ge and Marsden [4] and Lázaro-Camí and Ortega [6]. For these reasons it is described as a useful tools in the study of modern applied mathematics and analytical mechanics.

Let $Q$ be a smooth manifold and $TQ$ the tangent bundle, $T^*Q$ the cotangent bundle with the canonical symplectic form $\omega$, and the projection $\pi_Q : T^*Q \to Q$ induces the map $T\pi_Q : TT^*Q \to TQ$. From Abraham and Marsden in [1], we know that the following theorem give a classical description of Hamilton-Jacobi equation from the generating function and the geometrical point of view.

**Theorem 1.1** Assume that the triple $(T^*Q, \omega, H)$ is a Hamiltonian system with Hamiltonian vector field $X_H$, and $W : Q \to \mathbb{R}$ is a given generating function. Then the following two assertions are equivalent:

(i) For every curve $\sigma : \mathbb{R} \to Q$ satisfying $\dot{\sigma}(t) = T\pi_Q(X_H(dW(\sigma(t))))$, $\forall t \in \mathbb{R}$, then $dW \cdot \sigma$ is an integral curve of the Hamiltonian vector field $X_H$.

(ii) $W$ satisfies the Hamilton-Jacobi equation $H(\dot{q}^i, \frac{\partial W}{\partial q^i}) = E$, where $E$ is a constant.

From the above theorem, we know that the assertion (i) with equivalent to Hamilton-Jacobi equation by the generating function gives a geometric constraint condition of the canonical symplectic form on the cotangent bundle $T^*Q$ for Hamiltonian vector field of the system. Thus, the Hamilton-Jacobi equation reveals the deeply internal relationships of the generating function,
the canonical symplectic form and the dynamical vector field of a Hamiltonian system. But, from Marsden et al. [15] we know that, since the symplectic reduced system of a Hamiltonian system with symmetry defined on the cotangent bundle $T^*Q$ may not be a Hamiltonian system on a cotangent bundle, then we cannot give the Hamilton-Jacobi theorem for the Marsden-Weinstein reduced Hamiltonian system just like same as the above theorem. We have to look for a new way. It is worthy of noting that, in Wang [23], the two new types of Hamilton-Jacobi equations for Hamiltonian system and regular reduced Hamiltonian systems are given. By using the (reduced) symplectic forms and the (reduced) dynamical vector fields, which are the development of classical Hamilton-Jacobi equation given by Abraham and Marsden [1].

Since an RCH system defined on the cotangent bundle $T^*Q$, in general, may not be a Hamiltonian system, and it has yet no generating function, we cannot give the Hamilton-Jacobi theorem for the RCH system and its regular reduced systems just like same as the above theorem. But, in Wang [24] the author can give precisely the geometric constraint conditions of the (reduced) symplectic forms for the dynamical vector fields of an RCH system and its regular reduced systems, that is, two types of Hamilton-Jacobi equations, which are the development of the above two types of Hamilton-Jacobi equations for a Hamiltonian system and its Marsden-Weinstein reduced Hamiltonian system given in Wang [23]. In this paper, as the application of the above theoretical result, we derive precisely the geometric constraint conditions of the reduced symplectic forms for the dynamical vector fields of the regular point reducible controlled spacecraft-rotor system, that is, the two types of Hamilton-Jacobi equations for the reduced controlled spacecraft-rotor system by calculation in detail, in the cases of coincident and non-coincident centers of buoyancy and gravity.

A brief of outline of this paper is as follows. In the second section, we first review some relevant basic facts about rigid spacecraft with an internal rotor, and give the Hamiltonian function of the controlled rigid spacecraft-rotor system, in the cases of coincident and non-coincident centers of buoyancy and gravity, respectively, which will be used in subsequent sections. As the application of the theoretical result of regular point reduction of an RCH system given by Marsden et al [15], in the third section we consider the controlled rigid spacecraft-rotor system with the control torque acting on an internal rotor as a regular point reducible RCH system on the generalization of rotation group $\text{SO}(3) \times S^1$ and on the generalization of Euclidean group $\text{SE}(3) \times S^1$, respectively, we give the regular point reduced controlled spacecraft-rotor systems, in the cases of coincident and non-coincident centers of buoyancy and gravity. Moreover, as an application of the Hamilton-Jacobi theoretical result for the regular reduced RCH system given by Wang [24], in the fourth section, we derive precisely the two types of Hamilton-Jacobi equations for the regular point reduced controlled rigid spacecraft-rotor systems by calculation in detail, in the cases of coincident and non-coincident centers of buoyancy and gravity. These research work reveal the deeply internal relationships of the geometrical structures of phase spaces, the dynamical vector fields and controls of the controlled rigid spacecraft-rotor system, and develop the application of the regular symplectic reduction and Hamilton-Jacobi theory for the RCH systems with symmetries, and make us have much deeper understanding and recognition for the structure of Hamiltonian systems and RCH systems.

2 The Rigid Spacecraft with an Internal Rotor

In this section, we first give the Hamiltonian of rigid spacecraft with an internal rotor, in the cases of coincident and non-coincident centers of buoyancy and gravity, respectively. We first review
some relevant basic facts about rigid spacecraft with an internal rotor, which will be used in subsequent sections. We shall follow the notations and conventions introduced in Marsden [10], Marsden and Ratiu [14], and Marsden et al [15]. In this paper, we assume that all manifolds are real, smooth and finite dimensional and all actions are smooth left actions. For convenience, we also assume that all controls appearing in this paper are the admissible controls.

2.1 With Coincident Centers of Buoyancy and Gravity

We first describe a rigid spacecraft carrying an internal "non-mass" rotor, which is called a carrier body, where "non-mass" means that the mass of a rotor is very very small comparing with the mass of the rigid spacecraft. We first assume that the external forces and torques acting on the rigid spacecraft-rotor system are due to buoyancy and gravity. In general, it is possible that the spacecraft’s center of buoyancy may not be coincident with its center of gravity. But, in this subsection we assume that the spacecraft is symmetric and to have uniformly distributed mass, and the center of buoyancy and the center of gravity are coincident. Denote by $O$ the center of mass of the system in the carrier body frame and at $O$ place a set of (orthogonal) body axes. Assume that the body coordinate axes are aligned with principal axes of the carrier body, where "non-mass" means that the mass of a rotor is very very small comparing with the mass of the carrier body. In this case, we consider the configuration space $Q = \text{SO}(3) \times S^1$, with the first factor being the attitude of rigid spacecraft and the second factor being the angle of rotor. The corresponding phase space is the cotangent bundle $T^*Q$ and locally, $T^*Q = T^*\text{SO}(3) \times T^*S^1$, where $T^*S^1 \cong T^*R$ locally, with the canonical symplectic form $\omega_Q$. By using the local left trivialization, locally, $T^*\text{SO}(3) \cong \text{SO}(3) \times \mathfrak{so}^*(3)$ and $T^*R \cong R \times R^*$, then we have that locally, $T^*Q \cong \text{SO}(3) \times \mathfrak{so}^*(3) \times R \times R^*$. For convenience, in the following we denote uniformly that, locally, $Q = \text{SO}(3) \times R$, and $T^*Q = T^*\text{SO}(3) \times R \cong \text{SO}(3) \times \mathfrak{so}^*(3) \times R \times R^*$.

Let $I = \text{diag}(I_1, I_2, I_3)$ be the matrix of inertia moment of the carrier body in the body fixed frame, which is a principal body frame, and $J_3$ be the moment of inertia of rotor around its rotation axis. Let $J_{3k}$, $k = 1, 2$, be the moments of inertia of the rotor around the $k$th principal axis with $k = 1, 2$, and denote by $\bar{I}_k = I_k + J_{3k}$, $k = 1, 2$, $\bar{I}_3 = I_3$. Let $\Omega = (\Omega_1, \Omega_2, \Omega_3)$ be the angular velocity vector of the rigid spacecraft-rotor system computed with respect to the axes fixed in the carrier body and $(\Omega_1, \Omega_2, \Omega_3) \in \mathfrak{so}(3)$. Let $\alpha$ be the relative angle of rotor and $\dot{\alpha}$ the relative angular velocity vector of rotor about the third principal axis with respect to a carrier body fixed frame. For convenience, we assume the total mass of the system $m = 1$.

Now, by the local left trivialization, locally, $T\text{SO}(3) \cong \text{SO}(3) \times \mathfrak{so}(3)$ and $T^*R \cong R \times R$, then we have that locally, $TQ \cong \text{SO}(3) \times \mathfrak{so}(3) \times R \times R$. We consider the Lagrangian of the rigid spacecraft-rotor system $L(A, \Omega, \alpha, \dot{\alpha}) : TQ \cong \text{SO}(3) \times \mathfrak{so}(3) \times R \times R \to R$, which is the total kinetic energy of the rigid spacecraft plus the kinetic energy of rotor, given by

$$L(A, \Omega, \alpha, \dot{\alpha}) = \frac{1}{2} [\bar{I}_1 \Omega_1^2 + \bar{I}_2 \Omega_2^2 + \bar{I}_3 \Omega_3^2 + J_3 (\Omega_3 + \dot{\alpha})^2],$$

where $A \in \text{SO}(3)$, $\Omega = (\Omega_1, \Omega_2, \Omega_3) \in \mathfrak{so}(3)$, $\alpha \in R$, $\dot{\alpha} \in R$. If we introduce the conjugate angular momentum, given by

$$\Pi_k = \frac{\partial L}{\partial \Omega_k} = \bar{I}_k \Omega_k, \ k = 1, 2, \ \Pi_3 = \frac{\partial L}{\partial \Omega_3} = \bar{I}_3 \Omega_3 + J_3 (\Omega_3 + \dot{\alpha}), \ l = \frac{\partial L}{\partial \alpha} = J_3 (\Omega_3 + \dot{\alpha}),$$
and by the Legendre transformation

\[ FL : TQ \cong SO(3) \times \mathfrak{so}(3) \times \mathbb{R} \times \mathbb{R} \to T^*Q \cong SO(3) \times \mathfrak{so}^*(3) \times \mathbb{R} \times \mathbb{R}^*, \]

\[ (A, \Omega, \alpha, \dot{\alpha}) \to (A, \Pi, \alpha, l), \]

where \( \Pi = (\Pi_1, \Pi_2, \Pi_3) \in \mathfrak{so}^*(3), l \in \mathbb{R}^* \), we have the Hamiltonian \( H(A, \Pi, \alpha, l) : T^*Q \cong SO(3) \times \mathfrak{so}^*(3) \times \mathbb{R} \times \mathbb{R}^* \to \mathbb{R} \) given by

\[ H(A, \Pi, \alpha, l) = \Omega \cdot \Pi + \dot{\alpha} \cdot l - L(A, \Omega, \alpha, \dot{\alpha}) = \frac{1}{2} \left[ \frac{\Pi_1^2}{I_1} + \frac{\Pi_2^2}{I_2} + \left( \frac{(\Pi_3 - l)^2}{I_3} + \frac{l^2}{J_3} \right) \right]. \tag{2.1} \]

In this case, in order to give the dynamical vector field and the two types of Hamilton-jacobi equation of the controlled rigid spacecraft-rotor system, we need to consider the regular point reduction of the controlled spacecraft-rotor system and give precisely the \( R_p \)-reduced symplectic form of the cotangent bundle \( T^*Q \cong SO(3) \times \mathfrak{so}^*(3) \times \mathbb{R} \times \mathbb{R}^* \).

### 2.2 With Non-Coincident Centers of Buoyancy and Gravity

Since it is possible that the rigid spacecraft’s center of buoyancy may not be coincident with its center of gravity, in this subsection then we consider the rigid spacecraft-rotor system with non-coincident centers of buoyancy and gravity. We fix an orthogonal coordinate frame to the carrier body with origin located at the center of buoyancy and axes aligned with the principal axes of the carrier body, and the rotor is aligned along the third principal axis, see Marsden [10], and Leonard and Marsden [8]. Moreover, assume that when the carrier body is oriented so that the carrier body frame is aligned with the inertial frame, the third principal axis aligns with the direction of gravity. The vector from the center of buoyancy to the center of gravity with respect to the carrier body frame is \( h \chi \), where \( \chi \) is a unit vector on the line connecting the two centers which is assumed to be aligned along the third principal axis, and \( h \) is the length of this segment. Assume that the total mass of the carrier body \( m = 1 \), and the magnitude of gravitational acceleration is denoted \( g \), and let \( \Gamma \) be the unit vector viewed by an observer moving with the carrier body, and the rotor spins under the influence of a control torque \( u \) acting on the rotor. In this case, the configuration space is \( Q = SO(3) \otimes \mathbb{R}^3 \times S^1 \cong SE(3) \times S^1 \), with the first factor being the attitude of the rigid spacecraft and the drift of the rigid spacecraft in the rotational process and the second factor being the angle of rotor. The corresponding phase space is the cotangent bundle \( T^*Q \) and locally, \( T^*Q = T^*SE(3) \times T^*S^1 \), where \( T^*S^1 \cong T^*\mathbb{R} \) locally, with the canonical symplectic form \( \omega_Q \). By using the local left trivialization, locally, \( T^*SE(3) \cong SE(3) \times \mathfrak{se}^*(3) \) and \( T^*\mathbb{R} \cong \mathbb{R} \times \mathbb{R}^* \), then we have that locally, \( T^*Q \cong SE(3) \times \mathfrak{se}^*(3) \times \mathbb{R} \times \mathbb{R}^* \). For convenience, in the following we denote uniformly that, locally, \( Q = SE(3) \times \mathbb{R} \), and \( T^*Q = T^*(SE(3) \times \mathbb{R}) \cong SE(3) \times \mathfrak{se}^*(3) \times \mathbb{R} \times \mathbb{R}^* \).

Now, by the local left trivialization, locally, \( TSE(3) \cong SE(3) \times \mathfrak{se}(3) \) and \( T\mathbb{R} \cong \mathbb{R} \times \mathbb{R} \), then we have that locally, \( TQ \cong SE(3) \times \mathfrak{se}(3) \times \mathbb{R} \times \mathbb{R} \). We consider the Lagrangian of the rigid spacecraft-rotor system \( L(A, c, \Omega, \Gamma, \alpha, \dot{\alpha}) : TQ \cong SE(3) \times \mathfrak{se}(3) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \), which is the total kinetic energy of the rigid spacecraft plus the kinetic energy of rotor minus potential energy of the rigid spacecraft-rotor system, given by

\[ L(A, c, \Omega, \Gamma, \alpha, \dot{\alpha}) = \frac{1}{2} \left[ I_1 \Omega_1^2 + I_2 \Omega_2^2 + I_3 \Omega_3^2 + J_3 (\Omega_3 + \dot{\alpha})^2 \right] - gh \Gamma \cdot \chi, \]

where \( (A, c) \in SE(3), \Omega = (\Omega_1, \Omega_2, \Omega_3) \in \mathfrak{so}(3), \alpha \in \mathbb{R}, \dot{\alpha} \in \mathbb{R}, \) and the variable \( \Gamma \in \mathbb{R}^3 \) is regarded as a parameter with respect to potential energy of the system, \( (\Omega, \Gamma) \in \mathfrak{se}(3) \). If we
introduce the conjugate angular momentum, given by
\[ \Pi_k = \frac{\partial L}{\partial \Omega_k}, \quad k = 1, 2, \quad \Pi_3 = \frac{\partial L}{\partial \Omega_3} = I_3 \Omega_3 + J_3 (\Omega_3 + \dot{\alpha}), \quad l = \frac{\partial L}{\partial \dot{\alpha}} = J_3 (\Omega_3 + \dot{\alpha}), \]
and by the Legendre transformation with the parameter \( \Gamma \), that is,
\[ FL : TQ \cong SE(3) \times \mathfrak{se}(3) \times \mathbb{R} \times \mathbb{R} \rightarrow T^*Q \cong SE(3) \times \mathfrak{se}^*(3) \times \mathbb{R} \times \mathbb{R}^*, \]
where \( \Pi = (\Pi_1, \Pi_2, \Pi_3) \in \mathfrak{so}^*(3) \), \( (\Pi, \Gamma) \in \mathfrak{se}^*(3) \), \( l \in \mathbb{R}^* \), we have the Hamiltonian
\[ H(A, c, \Pi, \Gamma, \alpha, l) = \Omega \cdot \Pi + \dot{\alpha} \cdot l - L(A, c, \Omega, \Gamma, \alpha, \dot{\alpha}) \]
\[ = \frac{1}{2} \left[ \frac{\Pi_1^2}{I_1} + \frac{\Pi_2^2}{I_2} + \frac{(\Pi_3 - l)^2}{I_3} \right] + gh \Gamma \cdot \chi. \tag{2.2} \]
In this case, in order to give the dynamical vector field and the two types of Hamilton-jacobi equation of the controlled rigid spacecraft-rotor system, we need to consider the regular point reduction of the controlled spacecraft-rotor system and give precisely the \( R_p \)-reduced symplectic form of the cotangent bundle \( T^*Q \cong SE(3) \times \mathfrak{se}^*(3) \times \mathbb{R} \times \mathbb{R}^* \).

3 Symmetric Reduction of the Controlled Rigid Spacecraft-Rotor System

We know that the main goal of reduction theory in mechanics is to use conservation laws and the associated symmetries to reduce the number of dimensions of a mechanical system required to be described. So, such reduction theory is regarded as a useful tool for simplifying and studying concrete mechanical systems. In particular, the Marsden-Weinstein reduction for a Hamiltonian system with symmetry and momentum map is famous work, and great developments have been obtained around the work in the theoretical study and applications of mathematics, mechanics and physics. See Abraham and Marsden [1], Abraham et al. [2], Arnold [3], Libermann and Marle [9], Marsden [10], Marsden et al. [11, 12], Marsden and Perlmutter [13], Marsden and Ratiu [14], Marsden and Weinstein [16], Meyer [17], Nijmeijer and Van der Schaft [18] and Ortega and Ratiu [19].

It is worthy of noting that the authors in Marsden et al. [15] set up the regular reduction theory for the RCH systems with symplectic structures and symmetries on a symplectic fiber bundle, as an extension of the Marsden-Weinstein reduction theory of Hamiltonian systems under regular controlled Hamiltonian equivalence conditions, and from the viewpoint of completeness of regular symplectic reduction, and some developments around the work are give in Wang and Zhang [26], Ratiu and Wang [20], Wang [22]. In this section, as the application of the theoretical result, we shall regard the rigid spacecraft-rotor system with the control torque \( u \) acting on the rotor as a regular point reducible RCH system on the generalization of rotation group \( SO(3) \times \mathbb{R} \) and on the generalization of Euclidean group \( SE(3) \times \mathbb{R} \), respectively. We shall give the \( R_p \)-reduced controlled spacecraft-rotor systems in the two cases by calculation in detail. We also follow the notations and conventions introduced in Marsden et al [15], Wang [21] and Wang [24].
3.1 Spacecraft-Rotor System with Coincident Centers

We first give the regular point reduction of the controlled rigid spacecraft-rotor system with coincident centers of buoyancy and gravity. Assume that Lie group $G = SO(3)$ acts freely and properly on $Q = SO(3) \times \mathbb{R}$ by the left translation on the first factor $SO(3)$, and the trivial action on the second factor $\mathbb{R}$. By using the local left trivialization of $T^*SO(3) = SO(3) \times \mathfrak{so}^*(3)$, then the action of $SO(3)$ on phase space $T^*Q = T^*SO(3) \times T^*\mathbb{R} \cong SO(3) \times \mathfrak{so}^*(3) \times \mathbb{R} \times \mathbb{R}^*$ is by cotangent lift of left translation on $SO(3)$ at the identity, that is, $\Phi_T^* : (SO(3) \times T^*Q \cong SO(3) \times \mathfrak{so}^*(3) \times \mathbb{R} \times \mathbb{R}^*, \Phi_T^*(B, (A, \Pi, \alpha, l)) = (BA, \Pi, \alpha, l)$, for any $A, B \in SO(3)$, $\Pi \in \mathfrak{so}^*(3)$, $\alpha \in \mathbb{R}$, $l \in \mathbb{R}^*$. Assume that the action is free, proper and symplectic, and the orbit space $(T^*Q)/SO(3)$ is a smooth manifold and $\pi : T^*Q \to (T^*Q)/SO(3)$ is a smooth submerision. Since $SO(3)$ acts trivially on $\mathfrak{so}^*(3)$ and $\mathbb{R} \times \mathbb{R}^*$, it follows that $(T^*Q)/SO(3)$ is diffeomorphic to $\mathfrak{so}^*(3) \times \mathbb{R} \times \mathbb{R}^*$.

We know that $\mathfrak{so}^*(3)$ is a Poisson manifold with respect to its rigid body Lie-Poisson bracket defined by

$$\{F, K\}_{\mathfrak{so}^*(3)}(\Pi) = -\Pi \cdot (\nabla_\Pi F \times \nabla_\Pi K), \forall F, K \in C^\infty(\mathfrak{so}^*(3)), \quad \Pi \in \mathfrak{so}^*(3). \quad (3.1)$$

For $\mu \in \mathfrak{so}^*(3)$, the co-adjoint orbit $O_\mu \subset \mathfrak{so}^*(3)$ has the induced orbit symplectic form $\omega_{O_\mu}$, which coincides with the restriction of the Lie-Poisson bracket on $\mathfrak{so}^*(3)$ to the co-adjoint orbit $O_\mu$. From the Symplectic Stratification theorem we know that the co-adjoint orbits $(O_\mu, \omega_{O_\mu})$, $\mu \in \mathfrak{so}^*(3)$, form the symplectic leaves of the Poisson manifold $(\mathfrak{so}^*(3), \{\cdot, \cdot\}_{\mathfrak{so}^*(3)})$.

Let $\omega_\mathbb{R}$ be the canonical symplectic form on $T^*\mathbb{R} \cong \mathbb{R} \times \mathbb{R}^*$, which is given by

$$\omega_\mathbb{R}((\theta_1, \lambda_1), (\theta_2, \lambda_2)) = <\lambda_2, \theta_1> - <\lambda_1, \theta_2>, \quad (3.2)$$

where $(\theta_i, \lambda_i) \in \mathbb{R} \times \mathbb{R}^*$, $i = 1, 2$, $<\cdot, \cdot>$ is the standard inner product on $\mathbb{R} \times \mathbb{R}^*$. It induces a canonical Poisson bracket $\{\cdot, \cdot\}_\mathbb{R}$ on $T^*\mathbb{R}$, which is given by

$$\{F, K\}_\mathbb{R}(\theta, \lambda) = \frac{\partial F}{\partial \theta} \frac{\partial K}{\partial \lambda} - \frac{\partial K}{\partial \theta} \frac{\partial F}{\partial \lambda}. \quad (3.3)$$

See Marsden and Ratiu [14]. Thus, we can induce a symplectic form $\tilde{\omega}_{O_\mu \times \mathbb{R} \times \mathbb{R}^*} = \pi_{O_\mu}^* \omega_{O_\mu} + \pi_\mathbb{R}^* \omega_\mathbb{R}$ on the smooth manifold $O_\mu \times \mathbb{R} \times \mathbb{R}^*$, where the maps $\pi_{O_\mu} : O_\mu \times \mathbb{R} \times \mathbb{R}^* \to O_\mu$ and $\pi_\mathbb{R} : O_\mu \times \mathbb{R} \times \mathbb{R}^* \to \mathbb{R} \times \mathbb{R}^*$ are canonical projections, and can induce a Poisson bracket $\{\cdot, \cdot\}_- = \pi_{\mathfrak{so}^*(3)}^* \{\cdot, \cdot\}_{\mathfrak{so}^*(3)} + \pi_\mathbb{R}^* \{\cdot, \cdot\}_\mathbb{R}$ on the smooth manifold $\mathfrak{so}^*(3) \times \mathbb{R} \times \mathbb{R}^*$, where the maps $\pi_{\mathfrak{so}^*(3)} : \mathfrak{so}^*(3) \times \mathbb{R} \times \mathbb{R}^* \to \mathfrak{so}^*(3)$ and $\pi_\mathbb{R} : \mathfrak{so}^*(3) \times \mathbb{R} \times \mathbb{R}^* \to \mathbb{R} \times \mathbb{R}^*$ are canonical projections, and such that $(O_\mu \times \mathbb{R} \times \mathbb{R}^*, \tilde{\omega}_{O_\mu \times \mathbb{R} \times \mathbb{R}^*})$ is a symplectic leaf of the Poisson manifold $(\mathfrak{so}^*(3) \times \mathbb{R} \times \mathbb{R}^*, \{\cdot, \cdot\}_-)$. On the other hand, from $T^*Q = T^*SO(3) \times T^*\mathbb{R}$ we know that there is a canonical symplectic form $\omega_Q = \pi_{SO(3)}^* \omega_0 + \pi_\mathbb{R}^* \omega_\mathbb{R}$ on $T^*Q$, where $\omega_0$ is the canonical symplectic form on $T^*SO(3)$ and the maps $\pi_{SO(3)} : Q = SO(3) \times \mathbb{R} \to SO(3)$ and $\pi_\mathbb{R} : Q = SO(3) \times \mathbb{R} \to \mathbb{R}$ are canonical projections. Assume that the cotangent lift of left $SO(3)$-action $\Phi_T^* : SO(3) \times T^*Q \to T^*Q$ is symplectic with respect to $\omega_Q$, and admits an associated $Ad^*$-equivariant momentum map $J_Q : T^*Q \to \mathfrak{so}^*(3)$ such that $J_Q \cdot \pi^*_{SO(3)} = J_{SO(3)}$, where $J_{SO(3)} : T^*SO(3) \to \mathfrak{so}^*(3)$ is a momentum map of left $SO(3)$-action on $T^*SO(3)$ and we assume that it exists, and $\pi^*_{SO(3)} : T^*SO(3) \to T^*Q$. If $\mu \in \mathfrak{so}^*(3)$ is a regular value of $J_Q$, then $\mu \in \mathfrak{so}^*(3)$ is also a regular value of $J_{SO(3)}$ and
\( J_Q^{-1}(\mu) \equiv J_{SO(3)}^{-1}(\mu) \times \mathbb{R} \times \mathbb{R}^* \). Denote by \( SO(3)\mu = \{ g \in SO(3) \mid \text{Ad}_g\mu = \mu \} \) the isotropy subgroup of co-adjoint \( SO(3) \)-action at the point \( \mu \in \mathfrak{so}^*(3) \). It follows that \( SO(3)\mu \) acts also freely and properly on \( J_Q^{-1}(\mu) \), the \( R_p \)-reduced space \((T^*Q)_{\mu} = J_Q^{-1}(\mu)/SO(3)(\mu) \cong (T^*SO(3))_{\mu} \times \mathbb{R} \times \mathbb{R}^* \) of \((T^*Q, \omega_Q)\) at \( \mu \), is a symplectic manifold with symplectic form \( \omega_{\mu} \), uniquely characterized by the relation \( \pi_\mu^{\omega_Q} = i_\mu^*\omega_Q = \pi_{SO(3)}^{\omega} \omega_0 + i_\mu^*\pi_{\mathbb{R}}^{\omega_\mathbb{R}} \), where the map \( \pi_\mu : J_Q^{-1}(\mu) \to T^*Q \) is the inclusion and \( \pi_\mu : J_Q^{-1}(\mu) \to (T^*Q)_{\mu} \) is the projection. From Abraham and Marsden [1], we know that \((T^*SO(3))_{\mu} \), \( \omega_{\mu} \) is symplectically diffeomorphic to \((O_\mu, \omega_{\mathbb{R}_\mu})\), and hence we have that \((T^*Q)_{\mu} , \omega_{\mu} \) is symplectically diffeomorphic to \((O_\mu \times \mathbb{R} \times \mathbb{R}^*, \omega_{\mathbb{R}_\mu, \mathbb{R} \times \mathbb{R}^*})\).

From the expression (2.1) of the Hamiltonian, we know that \( H(A, \Pi, \alpha, l) \) is invariant under the cotangent lift of the left \( SO(3) \)-action \( \Phi^* : SO(3) \times T^*Q \to T^*Q \). From the rigid body Lie-Poisson bracket on \( \mathfrak{so}^*(3) \) and the Poisson bracket on \( \mathfrak{so}^*(3) \times \mathbb{R} \times \mathbb{R}^* \), that is, for \( F, K : \mathfrak{so}^*(3) \times \mathbb{R} \times \mathbb{R}^* \to \mathbb{R} \), we have that
\[
\{ F, K \}_{\Pi}(\Pi, \alpha, l) = -\Pi \cdot (\nabla_\Pi F \times \nabla_\Pi K) + \{ F, K \}_{\mathbb{R}}(\alpha, l).
\]
See Krishnaprasad and Marsden [5]. Hence, the Hamiltonian vector field \( X_H \) of rigid spacecraft-rotor system is given by
\[
X_H(\Pi) = \{ \Pi, H \}_{\mathbb{R}} - \Pi \cdot (\nabla_\Pi \Pi \times \nabla_\Pi H) + \{ \Pi, H \}_{\mathbb{R}}
\]
\[
= -\nabla_\Pi \Pi \cdot (\nabla_\Pi H \times \Pi) + (\partial_\alpha \partial H \partial l - \partial H \partial \Pi / \partial \alpha / \partial l)
\]
\[
= (\Pi_1, \Pi_2, \Pi_3) \times \Pi_1 / I_1, \Pi_2 / I_2, (\Pi_3 - l) / I_3,
\]
\[
= (\bar{I}_2 - \bar{I}_3) \Pi_2 \Pi_3 - \bar{I}_2 \Pi_3 l / I_2 I_3, \bar{I}_3 - \bar{I}_1 \Pi_3 \Pi_1 + \bar{I}_1 \Pi_1 l / I_3 I_1, (\bar{I}_1 - \bar{I}_2) \Pi_1 \Pi_2 / I_1 I_2,
\]
since \( \nabla_\Pi \Pi_i = 1, \nabla_\Pi \Pi_j = 0, i \neq j, i, j = 1, 2, 3, \) and \( \nabla_\Pi H = \Pi_k / \bar{I}_k, k = 1, 2, \nabla_\Pi H = (\Pi_3 - l) / \bar{I}_3, \partial_\Pi / \partial \alpha = 0 \).

\[
X_H(\alpha) = \{ \alpha, H \}_{\mathbb{R}} - \Pi \cdot (\nabla_\Pi \alpha \times \nabla_\Pi H) + \{ \alpha, H \}_{\mathbb{R}}
\]
\[
= -\nabla_\Pi \alpha \cdot (\nabla_\Pi H \times \Pi) + (\partial_\alpha \partial H \partial l - \partial H \partial \Pi / \partial \alpha / \partial l)
\]
\[
= - (\Pi_3 - l) / \bar{I}_3 + l / \bar{I}_3,
\]
since \( \nabla_\Pi \alpha = 0, i = 1, 2, 3, \partial_\Pi / \partial \alpha = 1, \partial_\Pi / \partial \alpha = 0, \) and \( \partial_\Pi / \partial l = - (\Pi_3 - l) / \bar{I}_3 + l / \bar{I}_3 \).

\[
X_H(l) = \{ l, H \}_{\mathbb{R}} - \Pi \cdot (\nabla_\Pi l \times \nabla_\Pi H) + \{ l, H \}_{\mathbb{R}}
\]
\[
= -\nabla_\Pi l \cdot (\nabla_\Pi H \times \Pi) + (\partial l \partial H \partial l - \partial H \partial \Pi / \partial \alpha / \partial l)
\]
\[
= 0,
\]
since \( \nabla_\Pi l = 0, i = 1, 2, 3 \) and \( \partial_\Pi / \partial \alpha = 0 \).

Moreover, if we consider the rigid spacecraft-rotor system with a control torque \( u : T^*Q \to W \) acting on the rotor, where the control subset \( W \subset T^*Q \) is a fiber submanifold, and assume that \( u \in W \) is invariant under the cotangent lift \( \Phi^*T^* \) of left \( SO(3) \)-action, and the dynamical vector field of the regular point reducible controlled rigid spacecraft-rotor system \((T^*Q, SO(3), \omega_Q, H, u)\) can be expressed by
\[
\tilde{X} = X_{(T^*Q, SO(3), \omega_Q, H, u)} = X_H + \text{vlift}(u),
\]
where \( \text{vlift}(u) = v\text{lift}(u) \cdot X_H \) is the change of \( X_H \) under the action of the control torque \( u \).

From the above expression of the dynamical vector field of the controlled spacecraft-rotor system \( (T^*Q, \text{SO}(3), \omega_Q, H, u) \), we know that under the actions of the control torque \( u \), in general, the dynamical vector field is not Hamiltonian, and hence the regular point reducible controlled rigid spacecraft-rotor system is not yet a Hamiltonian system. However, it is a dynamical system closed relative to a Hamiltonian system, and it can be explored and studied by extending the methods for the control torque \( u \) in the study of the Marsden-Weinstein reducible Hamiltonian system \( (T^*Q, \text{SO}(3), \omega_Q, H) \), see Marsden et al [15] and Wang [21].

Since the Hamiltonian \( H(A, \Pi, \alpha, l) \) is invariant under the cotangent lift \( \Phi T^* \) of the left \( \text{SO}(3) \)-action, for the point \( \Pi_0 = \mu \in \mathfrak{so}^*(3) \) is the regular value of \( J_Q \), we have the \( R_p \)-reduced Hamiltonian \( h_{\mu}(\Pi, \alpha, l) : \mathcal{O}_\mu \times \mathbb{R} \rightarrow \mathbb{R}^*(\subset \mathfrak{so}^*(3) \times \mathbb{R} \times \mathbb{R}^*) \rightarrow \mathbb{R} \) given by \( h_{\mu}(\Pi, \alpha, l) \cdot \pi_{\mu} = H(A, \Pi, \alpha, l) \). Moreover, for the \( R_p \)-reduced Hamiltonian \( h_{\mu}(A, \Pi, \alpha, l) : \mathcal{O}_\mu \times \mathbb{R} \rightarrow \mathbb{R}^*(\subset \mathfrak{so}^*(3) \times \mathbb{R} \times \mathbb{R}^*) \rightarrow \mathbb{R} \)

we have the Hamiltonian vector field \( X_{h_{\mu}}(K_{\mu}) = \{K_{\mu}, h_{\mu} \} \mathcal{O}_\mu \times \mathbb{R} \times \mathbb{R}^* \), where \( K_{\mu}(\Pi, \alpha, l) \) is the induced symplectic form from the Poisson bracket on \( \mathfrak{so}^*(3) \times \mathbb{R} \times \mathbb{R}^* \), such that Hamiltonian vector field \( X_{h_{\mu}}(K_{\mu}) = \tilde{\omega}_{\mu, \mathbb{R} \times \mathbb{R}^*}, (X_{\mu} : X_{h_{\mu}}(K_{\mu}) = \{K_{\mu}, h_{\mu} \} \mathcal{O}_\mu \times \mathbb{R} \times \mathbb{R}^* \), since \( (\mathcal{O}_\mu \times \mathbb{R} \times \mathbb{R}^*, \tilde{\omega}_{\mu, \mathbb{R} \times \mathbb{R}^*}) \) is a symplectic leaf of the Poisson manifold \( (\mathfrak{so}^*(3) \times \mathbb{R} \times \mathbb{R}^*, \{\cdot, \cdot \}) \). Moreover, assume that the dynamical vector field of the \( R_p \)-reduced controlled spacecraft-rotor system \( (\mathcal{O}_\mu \times \mathbb{R} \times \mathbb{R}^*, \tilde{\omega}_{\mu, \mathbb{R} \times \mathbb{R}^*}, h_{\mu}, u_{\mu}) \) is expressed by

\[
X_{(\mathcal{O}_\mu \times \mathbb{R} \times \mathbb{R}^*, \tilde{\omega}_{\mu, \mathbb{R} \times \mathbb{R}^*}, h_{\mu}, u_{\mu})} = X_{h_{\mu}} + \text{vlift}(u_{\mu}),
\]

where \( \text{vlift}(u_{\mu}) = v\text{lift}(u_{\mu})X_{h_{\mu}} \in T(\mathcal{O}_\mu \times \mathbb{R} \times \mathbb{R}^*) \), is the change of \( X_{h_{\mu}} \) under the action of the \( R_p \)-reduced control torque \( u_{\mu} \). The dynamical vector fields of the controlled spacecraft-rotor system and the \( R_p \)-reduced controlled spacecraft-rotor system satisfy the condition

\[
X_{(\mathcal{O}_\mu \times \mathbb{R} \times \mathbb{R}^*, \tilde{\omega}_{\mu, \mathbb{R} \times \mathbb{R}^*}, h_{\mu}, u_{\mu})} \cdot \pi_{\mu} = T_{\pi_{\mu}}X(T^*Q, \text{SO}(3), \omega_Q, H, u) \cdot i_{\mu}.
\]

See Marsden et al [15] and Wang [21].

To sum up the above discussion, we have the following theorem.

**Theorem 3.1** In the case of coincident centers of buoyancy and gravity, the rigid spacecraft-rotor system with the control torque \( u \) acting on the rotor, that is, the 5-tuple \( (T^*Q, \text{SO}(3), \omega_Q, H, u) \), where \( Q = \text{SO}(3) \times \mathbb{R} \), is a regular point reducible RCH system. For a point \( \mu \in \mathfrak{so}^*(3) \), the regular value of the momentum map \( J_Q : \mathfrak{so}^*(3) \times \mathfrak{so}^*(3) \times \mathbb{R} \times \mathbb{R}^* \rightarrow \mathfrak{so}^*(3) \), the \( R_p \)-reduced controlled rigid spacecraft-rotor system is the 4-tuple \( (\mathcal{O}_\mu \times \mathbb{R} \times \mathbb{R}^*, \tilde{\omega}_{\mu, \mathbb{R} \times \mathbb{R}^*}, h_{\mu}, u_{\mu}) \), where \( \mathcal{O}_\mu \subset \mathfrak{so}^*(3) \) is the co-adjoint orbit, \( \tilde{\omega}_{\mu, \mathbb{R} \times \mathbb{R}^*} \) is the induced symplectic form on \( \mathcal{O}_\mu \times \mathbb{R} \times \mathbb{R}^* \), \( h_{\mu}(\Pi, \alpha, l) \cdot \pi_{\mu} = H(A, \Pi, \alpha, l) \). Moreover, assume that the dynamical vector field of the \( R_p \)-reduced controlled spacecraft-rotor system \( (\mathcal{O}_\mu \times \mathbb{R} \times \mathbb{R}^*, \tilde{\omega}_{\mu, \mathbb{R} \times \mathbb{R}^*}, h_{\mu}, u_{\mu}) \) satisfies \( (3.6) \) and \( (3.7) \).

### 3.2 Spacecraft-Rotor System with Non-coincident Centers

In the following we shall give the regular point reduction of the controlled rigid spacecraft-rotor system with non-coincident centers of buoyancy and gravity. Because the drift in the direction
of gravity breaks the spacecraft-rotor system is no longer SO(3) invariant. In this case, its physical phase space is $T^* SO(3) \times T^* S^1$ and the symmetry group is $S^1$, regarded as rotations about the third principal axis, that is, the axis of gravity. By the semidirect product reduction theorem, see Marsden et al. [11], we know that the reduction of $T^* SO(3)$ by $S^1$ gives a space which is symplectically diffeomorphic to the reduced space obtained by the reduction of $T^* SE(3)$ by left action of $SE(3)$, that is the coadjoint orbit $O_{(\mu, a)}\subset \mathfrak{se}^*(3) \cong T^* SE(3)/SE(3)$. In fact, in this case, we can identify the phase space $T^* SO(3)$ with the reduction of the cotangent bundle of the special Euclidean group $SE(3) = SO(3) \circledast \mathbb{R}^3$ by the Euclidean translation subgroup $\mathbb{R}^3$ and identifies the symmetry group $S^1$ with isotropy group $G_a = \{ A \in SO(3) \mid Aa = a \} = S^1$, which is Abelian and $(G_a)_{\mu a} = G_a = S^1, \forall \mu a \in g_a$, where $a$ is a vector aligned with the direction of gravity and where SO(3) acts on $\mathbb{R}^3$ in the standard way.

Assume that Lie group $G = SE(3)$ acts freely and properly on $Q = SE(3) \times \mathbb{R}$ by the left translation on the first factor $SE(3)$, and the trivial action on the second factor $\mathbb{R}$. By using the local left trivialization of $T^* SE(3) = SE(3) \times \mathfrak{se}^*(3)$, then the action of $SE(3)$ on phase space $T^* Q = T^* SE(3) \times T^* \mathbb{R}$ is by cotangent lift of left translation on $SE(3)$ at the identity, that is, $\Phi^{T*} : SE(3) \times T^* Q \cong SE(3) \times \mathfrak{se}^*(3) \times \mathfrak{se}^*(3) \times \mathfrak{se}^*(3) \times \mathfrak{se}^*(3) \times \mathfrak{se}^*(3)$, given by $\Phi^{T*}((\mu, A), (B, b), (\alpha, l)) = (B, a, A, b, B, c, \alpha, l, l)$ for any $A, B \in SO(3)$, $\alpha \in \mathfrak{so}^*(3)$, $b, c, \alpha \in \mathbb{R}^3$, $(\Pi, \Gamma) \in \mathfrak{se}^*(3)$, $\alpha \in \mathbb{R}$, $l \in \mathbb{R}^3$. Assume that the action is free, proper and symplectic, and the orbit space $(T^* Q)/SE(3)$ is a smooth manifold and $\pi : T^* Q \to (T^* Q)/SE(3)$ is a smooth submersion. Since $SE(3)$ acts trivially on $\mathfrak{se}^*(3)$ and $\mathbb{R} \times \mathbb{R}^3$, it follows that $(T^* Q)/SE(3)$ is diffeomorphic to $\mathfrak{se}^*(3) \times \mathbb{R} \times \mathbb{R}^3$.

We know that $\mathfrak{se}^*(3)$ is a Poisson manifold with respect to its heavy top Lie-Poisson bracket defined by

$$\{F, K\}_{\mathfrak{se}^*(3)}(\Pi, \Gamma) = -\Pi \cdot (\nabla_\Pi F \times \nabla_\Pi K) - \Gamma \cdot (\nabla_\Gamma F \times \nabla_\Gamma K - \nabla_\Pi K \times \nabla_\Gamma F),$$

where $F, K \in C^\infty(\mathfrak{se}^*(3))$, $(\Pi, \Gamma) \in \mathfrak{se}^*(3)$, see Marsden et al. [11]. For $(\mu, a) \in \mathfrak{se}^*(3)$, the co-adjoint orbit $O_{(\mu, a)} \subset \mathfrak{se}^*(3)$ has the induced orbit symplectic form $\omega_{\mathfrak{O}_{(\mu, a)}}$, which coincides with the restriction of the Lie-Poisson bracket on $\mathfrak{se}^*(3)$ to the co-adjoint orbit $O_{(\mu, a)}$. From the Symplectic Stratification theorem we know that the co-adjoint orbits $O_{(\mu, a)} \subset \mathfrak{se}^*(3)$, form the symplectic leaves of the Poisson manifold $(\mathfrak{se}^*(3), \{ \cdot, \cdot \}_{\mathfrak{se}^*(3)})$. Let $\omega_{\mathbb{R}}$ be the canonical symplectic form on $T^* \mathbb{R} \cong \mathbb{R} \times \mathbb{R}$ given by (3.2), and it induces a canonical Poisson bracket $\{ \cdot, \cdot \}_{\mathfrak{se}^*(3)}$ on $T^* \mathbb{R}$ given by (3.3). Thus, we can induce a symplectic form $\tilde{\omega}_{\mathfrak{O}_{(\mu, a)} \times \mathbb{R} \times \mathbb{R}} = \pi_{\mathfrak{O}_{(\mu, a)}} \omega_{\mathfrak{O}_{(\mu, a)}} + \pi_{\mathbb{R}} \omega_{\mathbb{R}}$ on the smooth manifold $O_{(\mu, a)} \times \mathbb{R} \times \mathbb{R}$, where the maps $\pi_{\mathfrak{O}_{(\mu, a)}} : O_{(\mu, a)} \times \mathbb{R} \times \mathbb{R} \to O_{(\mu, a)}$ and $\pi_{\mathbb{R}} : O_{(\mu, a)} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ are canonical projections, and can induce a Poisson bracket $\{ \cdot, \cdot \}_{-} = \pi_{\mathfrak{se}^*(3)} \{ \cdot, \cdot \}_{\mathfrak{se}^*(3)} + \pi_{\mathbb{R}} \{ \cdot, \cdot \}_{\mathbb{R}}$ on the smooth manifold $\mathfrak{se}^*(3) \times \mathbb{R} \times \mathbb{R}$, where the maps $\pi_{\mathfrak{se}^*(3)} : \mathfrak{se}^*(3) \times \mathbb{R} \times \mathbb{R} \to \mathfrak{se}^*(3)$ and $\pi_{\mathbb{R}} : \mathfrak{se}^*(3) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ are canonical projections, and such that $(O_{(\mu, a)} \times \mathbb{R} \times \mathbb{R}, \tilde{\omega}_{\mathfrak{O}_{(\mu, a)} \times \mathbb{R} \times \mathbb{R}})$ is a symplectic leaf of the Poisson manifold $(\mathfrak{se}^*(3) \times \mathbb{R} \times \mathbb{R}, \{ \cdot, \cdot \}_{-})$.

On the other hand, from $T^* Q = T^* SE(3) \times T^* \mathbb{R}$ we know that there is a canonical symplectic form $\omega_Q = \pi_{\mathfrak{se}(3)} \omega_I + \pi_{\mathbb{R}} \omega_{\mathbb{R}}$ on $T^* Q$, where $\omega_I$ is the canonical symplectic form on $T^* SE(3)$ and the maps $\pi_{\mathfrak{se}(3)} : Q = SE(3) \times \mathbb{R} \to SE(3)$ and $\pi_{\mathbb{R}} : Q = SE(3) \times \mathbb{R} \to \mathbb{R}$ are canonical projections. Assume that the cotangent lift of left SE(3)-action $\Phi^{T*} : SE(3) \times T^* Q \to T^* Q$ is symplectic with respect to $\omega_Q$, and admits an associated $\operatorname{Ad}^*$-equivariant momentum map $J_Q : T^* Q \to \mathfrak{se}^*(3)$.
such that $J_T, \pi_T^* = J_{SE(3)}$, where $J_{SE(3)} : T^*SE(3) \to \mathfrak{se}^*(3)$ is a momentum map of left $SE(3)$-action on $T^*SE(3)$ and assume that it exists, and $\pi_{SE(3)}^* : T^*SE(3) \to T^*Q$. If $(\mu, a) \in \mathfrak{se}^*(3)$ is a regular value of $J_T$, then $(\mu, a) \in \mathfrak{se}^*(3)$ is also a regular value of $J_{SE(3)}$ and $J_{Q}^{-1}(\mu, a) \cong J_{SE(3)}^{-1}(\mu, a) \times \mathbb{R} \times \mathbb{R}^\ast$. Denote by $SE(3)(\mu, a) = \{g \in SE(3) | \text{Ad}^*_g(\mu, a) = (\mu, a)\}$ the isotropy subgroup of co-adjoint $SE(3)$-action at the point $(\mu, a) \in \mathfrak{se}^*(3)$. It follows that $SE(3)(\mu, a)$ acts also freely and properly on $J_{Q}^{-1}(\mu, a)$, the $R_p$-reduced space $(T^*Q)(\mu, a) = J_{Q}^{-1}(\mu, a)/SE(3)(\mu, a) \cong (T^*SE(3))(\mu, a) \times \mathbb{R} \times \mathbb{R}^\ast$ of $(T^*Q, \omega_Q)$ at $(\mu, a)$, is a symplectic manifold with symplectic form $\omega_{\mu, a}$ uniquely characterized by the relation $\pi_{SE(3)}^*(\mu, a) = \pi_{SE(3)}^*(\mu, a)\pi_{SE(3)}^*(\mu, a) + \pi_{SE(3)}^*(\mu, a)\pi_{SE(3)}^*(\mu, a)$, where the map $i_{(\mu, a)} : J_{Q}^{-1}(\mu, a) \to T^*Q$ is the inclusion and $\pi_{(\mu, a)} : J_{Q}^{-1}(\mu, a) \to (T^*Q)(\mu, a)$ is the projection. From Abraham and Marsden [1], we know that $((T^*SE(3))(\mu, a), \omega_{\mu, a})$ is symplectically diffeomorphic to $(\mathcal{O}(\mu, a), \omega_{\mathcal{O}(\mu, a)})$, and hence we have that $((T^*Q)(\mu, a), \omega_{\mu, a})$ is symplectically diffeomorphic to $(\mathcal{O}(\mu, a) \times \mathbb{R} \times \mathbb{R}^\ast, \omega_{\mathcal{O}(\mu, a) \times \mathbb{R} \times \mathbb{R}^\ast})$.

From the expression (2.2) of the Hamiltonian, we know that $H(A, c, \Pi, \Gamma, \alpha, l)$ is invariant under the cotangent lift of the left $SE(3)$-action $\Phi^* : SE(3) \times T^*Q \to T^*Q$. Moreover, from the heavy top Lie-Poisson bracket on $\mathfrak{se}^*(3)$ and the Poisson bracket on $T^*\mathbb{R}$, we can get the Poisson bracket on $\mathfrak{se}^*(3) \times \mathbb{R} \times \mathbb{R}^\ast$, that is, for $F, K : \mathfrak{se}^*(3) \times \mathbb{R} \times \mathbb{R}^\ast \to \mathbb{R}$, we have that

$$\{F, K\}_\mathbb{R}(\Pi, \Gamma, \alpha, l) = -\Pi \cdot (\nabla_F F \times \nabla_K K - \nabla_K F \times \nabla_F K) + \{F, K\}_\mathbb{R}(\alpha, l).$$

Hence, the Hamiltonian vector field of the rigid spacecraft-rotor system is given by

$X_H(\Pi) = \{\Pi, H\}_\mathbb{R} = -\Pi \cdot (\nabla_\Pi \Pi \times \nabla_K H - \nabla_\Pi H \times \nabla_K \Pi) + \{\Pi, H\}_\mathbb{R}$

$= (\nabla_\Pi \Pi_1, \Pi_2, \Pi_3) \times (\Pi_1, \Pi_2, \Pi_3 - l) + gh(\Gamma_1, \Gamma_2, \Gamma_3) \times (\chi_1, \chi_2, \chi_3) + (\partial_\Pi \Pi \partial H \partial \alpha - \partial_\Pi \Pi \partial H \partial \alpha) = (\nabla_\Pi \Pi_1, \Pi_2, \Pi_3 - l) + \frac{(\partial_\Pi \Pi \partial H \partial \alpha - \partial_\Pi \Pi \partial H \partial \alpha)}{I_2 I_3}$(3.9)

$X_H(\Gamma) = \{\Gamma, H\}_\mathbb{R} = -\Gamma \cdot (\nabla_\Gamma \Gamma \times \nabla_K H) - \Gamma \cdot (\nabla_\Pi \Gamma \times \nabla_K H - \nabla_\Pi H \times \nabla_K \Gamma) + \{\Gamma, H\}_\mathbb{R}$

$= \nabla_\Gamma \Gamma \cdot (\nabla_\Gamma \Pi \times \nabla_K H) + (\partial_\Gamma \partial \Pi \partial H \partial \alpha - \partial_\Gamma \partial \Pi \partial H \partial \alpha) = (\Gamma_1, \Gamma_2, \Gamma_3) \times (\Pi_1, \Pi_2, \Pi_3 - l) = (\nabla_\Gamma \Gamma_1, \Pi_2, \Pi_3 - l) + \frac{(\partial_\Gamma \partial \Pi \partial H \partial \alpha - \partial_\Gamma \partial \Pi \partial H \partial \alpha)}{I_2 I_3}$(3.9)

$X_H(\alpha) = \{\alpha, H\}_\mathbb{R} = -\Pi \cdot (\nabla_\Pi \alpha \times \nabla_K H) - \Gamma \cdot (\nabla_\Pi \alpha \times \nabla_K H - \nabla_\Pi H \times \nabla_K \alpha) + \{\alpha, H\}_\mathbb{R}$

$= (\partial_\Pi \Pi \partial H \partial \alpha - \partial_\Pi \Pi \partial H \partial \alpha) = -\frac{(\Pi_3 - l)}{I_2} + \frac{l}{I_3}$,
since $\nabla_{l,\alpha} = \nabla_{\Gamma,\alpha} = 0$, $i = 1, 2, 3$, $\frac{\partial l}{\partial \alpha} = 1$, $\frac{\partial H}{\partial \alpha} = 0$, and $\frac{\partial H}{\partial l} = -(C_{3} - l) / I_{3}$.

\[ X_{H}(l) = \{l, H\} = -\Pi \cdot (\nabla_{\Pi} l \times \nabla_{\Pi} H) - \Gamma : (\nabla_{\Pi} l \times \nabla_{\Gamma} H - \nabla_{\Pi} H \times \nabla_{\Gamma} l) + \{l, H\}_{\mathbb{R}} \]

= \left( \frac{\partial l}{\partial \alpha} \frac{\partial H}{\partial l} - \frac{\partial H}{\partial \alpha} \frac{\partial l}{\partial \alpha} \right) = 0,

since $\nabla_{\Pi} l = \nabla_{\Gamma} l = 0$, $i = 1, 2, 3$, and $\frac{\partial l}{\partial \alpha} = \frac{\partial H}{\partial \alpha} = 0$.

Moreover, if we consider the rigid spacecraft-rotor system with a control torque $u : T^{*}Q \to W$ acting on the rotors, where the control subset $W \subset T^{*}Q$ is a fiber submanifold, and assume that $u \in W$ is invariant under the cotangent lift $\Phi^{T^{*}}$ of the left $SE(3)$-action, and the dynamical vector field of the regular point reducible controlled spacecraft-rotor system $(T^{*}Q, SE(3), \omega_{Q}, H, u)$ can be expressed by

\[ \tilde{X} = X_{(T^{*}Q, SE(3), \omega_{Q}, H, u)} = X_{H} + \text{vlift}(u), \]

(3.10)

where vlift$(u) = \text{vlift}(u) \cdot X_{H}$ is the change of $X_{H}$ under the action of the control torque $u$. From the above expression of the dynamical vector field of the spacecraft-rotor system $(T^{*}Q, SE(3), \omega_{Q}, H, u)$, we know that under the actions of the control torque $u$, in general, the dynamical vector field is not Hamiltonian, and hence the regular point reducible controlled rigid spacecraft-rotor system is not yet a Hamiltonian system. However, it is a dynamical system closed relative to a Hamiltonian system, and it can be explored and studied by extending the methods for the control torque $u$ in the study of the Marsden-Weinstein reducible Hamiltonian system $(T^{*}Q, SE(3), \omega_{Q}, H)$, see Marsden et al [15] and Wang [21].

Since the Hamiltonian $H(A, c, \Pi, \Gamma, \alpha, l)$ is invariant under the cotangent lift $\Phi^{T^{*}}$ of the left $SE(3)$-action, for the point $(\Pi, \Gamma, 0) = (\mu, a) \in se^{*}(3)$ is the regular value of $J_{Q}$, we have the $R_{p}$-reduced Hamiltonian $h_{(\mu, a)}(\Pi, \Gamma, \alpha, l) : O_{(\mu, a)} \times \mathbb{R} \times \mathbb{R}^{*}(\subset se^{*}(3) \times \mathbb{R} \times \mathbb{R}^{*}) \to \mathbb{R}$ given by $h_{(\mu, a)}(\Pi, \Gamma, \alpha, l) \cdot \pi_{(\mu, a)} = H(A, c, \Pi, \Gamma, \alpha, l)|_{O_{(\mu, a)} \times \mathbb{R} \times \mathbb{R}^{*}}$. Moreover, for the $R_{p}$-reduced Hamiltonian $h_{(\mu, a)}(\Pi, \Gamma, \alpha, l) : O_{(\mu, a)} \times \mathbb{R} \times \mathbb{R}^{*} \to \mathbb{R}$, we have the Hamiltonian vector field

\[ X_{h_{(\mu, a)}}(K_{(\mu, a)}) = \{K_{(\mu, a)}, h_{(\mu, a)}\}_{-} \mid O_{(\mu, a)} \times \mathbb{R} \times \mathbb{R}^{*}, \]

where $K_{(\mu, a)}(\Pi, \Gamma, \alpha, l) = O_{(\mu, a)} \times \mathbb{R} \times \mathbb{R}^{*} \to \mathbb{R}$. Assume that $u \in W \cap J_{Q}^{-1}(\mu, a)$ and the $R_{p}$-reduced control torque $u_{(\mu, a)} : O_{(\mu, a)} \times \mathbb{R} \times \mathbb{R}^{*} \to W_{(\mu, a)}(\subset O_{(\mu, a)} \times \mathbb{R} \times \mathbb{R}^{*})$ is given by $u_{(\mu, a)}(\Pi, \Gamma, \alpha, l) \cdot \pi_{(\mu, a)} = u(A, c, \Pi, \Gamma, \alpha, l)|_{O_{(\mu, a)} \times \mathbb{R} \times \mathbb{R}^{*}}$, where $\pi_{(\mu, a)} : J_{Q}^{-1}(\mu, a) \to O_{(\mu, a)} \times \mathbb{R} \times \mathbb{R}^{*}$, $W_{(\mu, a)} = \pi_{(\mu, a)}(W \cap J_{Q}^{-1}(\mu, a))$. The $R_{p}$-reduced controlled spacecraft-rotor system is the 4-tuple $(O_{(\mu, a)} \times \mathbb{R} \times \mathbb{R}^{*}, \omega_{O_{(\mu, a)}} \times \mathbb{R} \times \mathbb{R}^{*}, h_{(\mu, a)}, u_{(\mu, a)})$, where $\omega_{O_{(\mu, a)}} \times \mathbb{R} \times \mathbb{R}^{*}$ is the induced symplectic form from the Poisson bracket on $se^{*}(3) \times \mathbb{R} \times \mathbb{R}^{*}$, such that Hamiltonian vector field

\[ X_{h_{(\mu, a)}}(K_{(\mu, a)}) = \omega_{O_{(\mu, a)}} \times \mathbb{R} \times \mathbb{R}^{*}(X_{K_{(\mu, a)}}, X_{h_{(\mu, a)}}) = \{K_{(\mu, a)}, h_{(\mu, a)}\}_{-} \mid O_{(\mu, a)} \times \mathbb{R} \times \mathbb{R}^{*}, \]

since $(O_{(\mu, a)} \times \mathbb{R} \times \mathbb{R}^{*}, \omega_{O_{(\mu, a)}} \times \mathbb{R} \times \mathbb{R}^{*})$ is a symplectic leaf of the Poisson manifold $(se^{*}(3) \times \mathbb{R} \times \mathbb{R}^{*}, \{\cdot, \cdot\}_{-})$. Moreover, assume that the dynamical vector field of the $R_{p}$-reduced controlled spacecraft-rotor system $(O_{(\mu, a)} \times \mathbb{R} \times \mathbb{R}^{*}, \omega_{O_{(\mu, a)}} \times \mathbb{R} \times \mathbb{R}^{*}, h_{(\mu, a)}, u_{(\mu, a)})$ can be expressed by

\[ X_{(O_{(\mu, a)} \times \mathbb{R} \times \mathbb{R}^{*}, \omega_{O_{(\mu, a)}} \times \mathbb{R} \times \mathbb{R}^{*}, h_{(\mu, a)}, u_{(\mu, a)})} = X_{h_{(\mu, a)}} + \text{vlift}(u_{(\mu, a)}), \]

(3.11)
Theorem 2.6 and Theorem 2.7 in Wang [69] with the control torque $u_{(\mu,a)}$. The dynamical vector fields of the controlled spacecraft-rotor system and the $R_p$-reduced controlled spacecraft-rotor system satisfy the condition

$$X_{(O(\mu,a) \times \mathbb{R} \times \mathbb{R}^*, \omega_{O(\mu,a)}, h_{(\mu,a)}, u_{(\mu,a)})} \cdot \pi_{(\mu,a)} = T\pi_{(\mu,a)} \cdot X(T^*Q, SE(3), \omega_Q, H, u) \cdot \dot{i}(\mu,a).$$  \hspace{1cm} (3.12)

See Marsden et al [15] and Wang [21].

To sum up the above discussion, we have the following theorem.

**Theorem 3.2** In the case of non-coincident centers of buoyancy and gravity, the rigid spacecraft-rotor system with the control torque $u$ acting on the rotor, that is, the 5-tuple $(T^*Q, SE(3), \omega_Q, H, u)$, where $Q = SE(3) \times \mathbb{R}$, is a regular point reducible RCH system. For a point $(\mu,a) \in \mathfrak{se}^*(3)$, the regular value of the momentum map $J_Q: T^*Q \cong SE(3) \times \mathfrak{se}^*(3) \times \mathbb{R} \times \mathbb{R}^* \rightarrow \mathfrak{se}^*(3)$, the $R_p$-reduced controlled spacecraft-rotor system is the 4-tuple $(O(\mu,a) \times \mathbb{R} \times \mathbb{R}^*, \omega_{O(\mu,a)}, h_{(\mu,a)}, u_{(\mu,a)})$.

We shall follow the notations and conventions introduced in Marsden [10], Marsden et al [15], and de León and Wang [7]. As an application of the theoretical result, in this section, we first give precisely the geometric constraint conditions of the canonical symplectic form for the dynamical vector field of the controlled rigid spacecraft-rotor system, that is, Type I and Type II of Hamilton-Jacobi equations for the controlled rigid spacecraft-rotor system. Then, for the above $R_p$-reduced controlled rigid spacecraft-rotor systems with coincident and non-coincident centers of buoyancy and gravity, we shall derive precisely the geometric constraint conditions of the $R_p$-reduced symplectic forms for the dynamical vector fields of the regular point reducible controlled rigid spacecraft-rotor systems, respectively, that is, Type I and Type II of Hamilton-Jacobi equations for the $R_p$-reduced controlled rigid spacecraft-rotor systems. We shall follow the notations and conventions introduced in Marsden [10], Marsden et al [15], Wang [23] and Wang [24].

Let $G = SO(3)$ or $SE(3)$, and $Q = SO(3) \times \mathbb{R}$ or $SE(3) \times \mathbb{R}$, and $\omega_Q$ is canonical symplectic form on $T^*Q$. Denote by $\Omega^i(Q)$ the set of all $i$-forms on $Q$, $i = 1, 2$. For any $\gamma \in \Omega^1(Q)$, $q \in Q$, then $\gamma(q) \in T_q^*Q$, and we can define a map $\gamma: Q \rightarrow T^*Q$, $q \rightarrow (q, \gamma(q))$. Hence we say often that the map $\gamma: Q \rightarrow T^*Q$ is an one-form on $Q$. If the one-form $\gamma$ is closed, then $\text{d}x_1 = 0$, $\forall x, y \in TQ$; and the one-form $\gamma$ is called to be closed with respect to $T\pi_Q: TT^*Q \rightarrow TQ$, if for any $v, w \in TT^*Q$, we have $\text{d}x_1(T\pi_Q(v), T\pi_Q(w)) = 0$. Since the rigid spacecraft-rotor system with the control torque $u$ acting on the rotor is a regular point reducible RCH system, from Theorem 2.6 and Theorem 2.7 in Wang [24], we can obtain directly the following Theorem 4.1. For convenience, the maps involved in the following theorem are shown in Diagram-1.
Theorem 4.1 For the controlled rigid spacecraft-rotor system \((T^* Q, \omega_Q, H, u)\) with the canonical symplectic form \(\omega_Q\) on \(T^* Q\), assume that \(\gamma : Q \rightarrow T^* Q\) is an one-form on \(Q\), and \(\lambda = \gamma \cdot \pi_Q : T^* Q \rightarrow T^* Q\), and the map \(\varepsilon : T^* Q \rightarrow T^* Q\) is symplectic. Denote by \(\tilde{X} \cdot \gamma\) the dynamical vector field of the controlled rigid spacecraft-rotor system \((T^* Q, \omega_Q, H, u)\). Then the following two assertions hold:

(i) If the one-form \(\gamma : Q \rightarrow T^* Q\) is closed with respect to \(T^{\pi Q} : TT^* Q \rightarrow TQ\), then \(\gamma\) is a solution of the Type I of Hamilton-Jacobi equation \(T \gamma \cdot \tilde{X} \cdot \gamma\), where \(\tilde{X} = X_{\xi}^{(T^* Q, \omega_Q, H, u)}\) is the dynamical vector field of the controlled rigid spacecraft-rotor system \((T^* Q, \omega_Q, H, u)\).

(ii) The \(\varepsilon\) is a solution of the Type II of Hamilton-Jacobi equation \(T \gamma \cdot \tilde{X} \cdot \varepsilon = H \cdot \varepsilon\), if and only if it is a solution of the equation \(T \gamma \cdot X_{H \cdot \varepsilon} = T \lambda \cdot \tilde{X} \cdot \varepsilon\), where \(X_H \) and \(X_{H \cdot \varepsilon}\) are the Hamiltonian vector fields of the functions \(H\) and \(H \cdot \varepsilon : T^* Q \rightarrow \mathbb{R}\), respectively. ■

4.1 In The Case of Coincident Centers

In the following we shall derive precisely the geometric constraint conditions of the \(R_p\)-reduced symplectic form \(\mathcal{O}_\mu \times \mathbb{R} \times \mathbb{R}^+\) for the dynamical vector field of the regular point reducible controlled rigid spacecraft-rotor system with coincident centers of buoyancy and gravity, that is, Type I and Type II of Hamilton-Jacobi equation for the \(R_p\)-reduced controlled rigid spacecraft-rotor system \((\mathcal{O}_\mu \times \mathbb{R} \times \mathbb{R}^+, \omega_{\mathcal{O}_\mu \times \mathbb{R} \times \mathbb{R}^+}, h_\mu, u_\mu)\).

Assume that \(\gamma : SO(3) \times \mathbb{R} \rightarrow T^*(SO(3) \times \mathbb{R})\) is an one-form on \(SO(3) \times \mathbb{R}\), \(\gamma(A, \alpha) = (\gamma_1, \ldots, \gamma_8)\), and \(\gamma\) is closed with respect to \(T^{\pi_Q} : TT^*(SO(3) \times \mathbb{R}) \rightarrow T(SO(3) \times \mathbb{R})\). For \(\mu \in so^*(3)\) the regular value of \(J_Q\), \(\text{Im}(\gamma) \subset J^{-1}(\mu)\), and it is \(SO(3)\)-invariant, and \(\tilde{\gamma} = \pi_\mu(\gamma) : SO(3) \times \mathbb{R} \rightarrow \mathcal{O}_\mu \times \mathbb{R} \times \mathbb{R}^+\). Denote by \(\tilde{\gamma} = (\tilde{\gamma}_1, \tilde{\gamma}_2, \tilde{\gamma}_3, \tilde{\gamma}_4) \in \mathcal{O}_\mu \times \mathbb{R} \times \mathbb{R}^+(\subset so^*(3) \times \mathbb{R} \times \mathbb{R}^+)\), where \(\pi_\mu : J_Q^{-1}(\mu) \rightarrow \mathcal{O}_\mu \times \mathbb{R} \times \mathbb{R}^+\). We choose that \((\Pi, \alpha, l) \in \mathcal{O}_\mu \times \mathbb{R} \times \mathbb{R}^+\), and \(\Pi = (\Pi_1, \Pi_2, \Pi_3) = (\tilde{\gamma}_1, \tilde{\gamma}_2, \tilde{\gamma}_3),\) \(\alpha = \tilde{\gamma}_4, l = \tilde{\gamma}_5\), then \(h_\mu \cdot \tilde{\gamma} : SO(3) \times \mathbb{R} \rightarrow \mathbb{R}\) is given by

\[
h_\mu(\Pi, \alpha, l) \cdot \tilde{\gamma} = H(A, \Pi, \alpha, l)|_{\mathcal{O}_\mu \times \mathbb{R} \times \mathbb{R}^+} \cdot \tilde{\gamma} = \frac{1}{2} \left( \frac{\tilde{\gamma}_1^2}{I_1} + \frac{\tilde{\gamma}_2^2}{I_2} + \frac{(\tilde{\gamma}_3 - \tilde{\gamma}_5)^2}{I_3} + \frac{\tilde{\gamma}_5^2}{J_3} \right),
\]

and the vector field

\[
X_{h_\mu}(\Pi) \cdot \tilde{\gamma} = \{\Pi, h_\mu\}|_{\mathcal{O}_\mu \times \mathbb{R} \times \mathbb{R}^+} \cdot \tilde{\gamma}
\]

\[
= -\Pi \cdot (\nabla H_\Pi \cdot \nabla (h_\mu)) \cdot \tilde{\gamma} + \{\Pi, h_\mu\}|_{\mathcal{O}_\mu \times \mathbb{R} \times \mathbb{R}^+} \cdot \tilde{\gamma}
\]

\[
= -\nabla H_\Pi \cdot (\nabla (h_\mu) \times \Pi) \cdot \tilde{\gamma} + \left( \frac{\partial c_1}{\partial \alpha} \frac{\partial h_\mu}{\partial l} - \frac{\partial (h_\mu)}{\partial \alpha} \frac{\partial \Pi}{\partial l} \right) \cdot \tilde{\gamma}
\]

\[
= (\Pi_1, \Pi_2, \Pi_3) \times \left( \frac{\Pi_1}{I_1}, \frac{\Pi_2}{I_2}, \frac{\Pi_3 - l}{I_3} \right) \cdot \tilde{\gamma}
\]

\[
= \left( \frac{I_2 - I_3}{I_2 I_3} \right) \tilde{\gamma}_5 \tilde{\gamma}_3 - \left( \frac{I_2 I_3}{I_2} \right) \tilde{\gamma}_5 \tilde{\gamma}_1, \quad \left( \frac{I_3 - I_1}{I_3} \right) \tilde{\gamma}_3 \tilde{\gamma}_1 + \left( \frac{I_1 - I_2}{I_1} \right) \tilde{\gamma}_1 \tilde{\gamma}_2, \quad \left( \frac{I_1 - I_2}{I_1 I_2} \right) \tilde{\gamma}_1 \tilde{\gamma}_2,
\]
since $\nabla_{\Pi} \Pi_i = 1$, $\nabla_{\Pi} \Pi_j = 0$, $i \neq j$, and $\nabla_{\Pi_k} (h_\mu) = \Pi_k/\bar{l}_k$, $\nabla_{\Pi_3} (h_\mu) = (\Pi_3 - l)/\bar{l}_3$, and $\frac{\partial}{\partial l} \frac{\partial (h_\mu)}{\partial \alpha} = 0$, $i, j = 1, 2, 3$, $k = 1, 2$.

$$X_{h_\mu} (\alpha) \cdot \tilde{\gamma} = \{\alpha, h_\mu\} - |\sigma_{\mu} \times \mathbb{R} \times \mathbb{R}^* \cdot \tilde{\gamma}
= -\Pi \cdot (\nabla_{\Pi} \alpha \times \nabla_{\Pi} (h_\mu)) \cdot \tilde{\gamma} + \{\alpha, h_\mu\} |\sigma_{\mu} \times \mathbb{R} \times \mathbb{R}^* \cdot \tilde{\gamma}
= -\nabla_{\Pi} \alpha \cdot (\nabla_{\Pi} (h_\mu) \times \Pi) \cdot \tilde{\gamma} + \{\alpha, h_\mu\} |\sigma_{\mu} \times \mathbb{R} \times \mathbb{R}^* \cdot \tilde{\gamma}
= -\nabla_{\Pi} \alpha \cdot (\nabla_{\Pi} (h_\mu) \times \Pi) \cdot \tilde{\gamma} + \frac{\partial \alpha}{\partial l} \frac{\partial (h_\mu)}{\partial \alpha} \frac{\partial l}{\partial \alpha} \cdot \tilde{\gamma}
= -\left( \frac{\tilde{\gamma}_3 - \tilde{\gamma}_5}{\bar{l}_3} + \frac{\tilde{\gamma}_5}{\bar{l}_3} \right),$$

since $\nabla_{\Pi} \alpha = 0$, $\frac{\partial \alpha}{\partial l} = 1$, $\frac{\partial (h_\mu)}{\partial \alpha} = 0$, and $\frac{\partial (h_\mu)}{\partial l} = - (\Pi_3 - l)/\bar{l}_3 + \frac{1}{\bar{l}_3}$, $i = 1, 2, 3$.

$$X_{h_\mu} (l) \cdot \tilde{\gamma} = \{l, h_\mu\} - |\sigma_{\mu} \times \mathbb{R} \times \mathbb{R}^* \cdot \tilde{\gamma}
= -\Pi \cdot (\nabla_{\Pi} l \times \nabla_{\Pi} (h_\mu)) \cdot \tilde{\gamma} + \{l, h_\mu\} |\sigma_{\mu} \times \mathbb{R} \times \mathbb{R}^* \cdot \tilde{\gamma}
= -\nabla_{\Pi} l \cdot (\nabla_{\Pi} (h_\mu) \times \Pi) \cdot \tilde{\gamma} + \{l, h_\mu\} |\sigma_{\mu} \times \mathbb{R} \times \mathbb{R}^* \cdot \tilde{\gamma}
= -\nabla_{\Pi} l \cdot (\nabla_{\Pi} (h_\mu) \times \Pi) \cdot \tilde{\gamma} + \frac{\partial l}{\partial \alpha} \frac{\partial (h_\mu)}{\partial l} \cdot \tilde{\gamma}
= 0,$$

since $\nabla_{\Pi} l = 0$, and $\frac{\partial l}{\partial \alpha} = \frac{\partial (h_\mu)}{\partial \alpha} = 0$, $i = 1, 2, 3$.

On the other hand, from the expressions of the dynamical vector field $\tilde{X}$ and Hamiltonian vector field $X_H$, we have that

$$\tilde{X}(\Pi, \alpha, l)^\gamma = T \pi_{SO(3) \times \mathbb{R}} \cdot \tilde{X} \cdot \gamma(\Pi, \alpha, l)
= T \pi_{SO(3) \times \mathbb{R}} \cdot (X_H + \text{vlift}(u)) \cdot \gamma(\Pi, \alpha, l)
= T \pi_{SO(3) \times \mathbb{R}} \cdot X_H \cdot \gamma(\Pi, \alpha, l) = X_H \cdot \gamma(\Pi, \alpha, l),$$

that is,

$$\tilde{X}(\Pi)^\gamma = X_H(\Pi) \cdot \gamma
= \left( \frac{\tilde{l}_2 - \tilde{l}_3}{\tilde{l}_2 \tilde{l}_3} \right)^{\gamma_6 \gamma_7 \gamma_8} - \left( \frac{\tilde{l}_3 - \tilde{l}_1}{\tilde{l}_3 \tilde{l}_1} \right)^{\gamma_6 \gamma_7 \gamma_8} + \left( \frac{\tilde{l}_1 - \tilde{l}_2}{\tilde{l}_1 \tilde{l}_2} \right)^{\gamma_4 \gamma_5 \gamma_7},$$

$$\tilde{X}(\alpha)^\gamma = X_H(\alpha) \cdot \gamma = - \left( \frac{\gamma_6 - \gamma_8}{\tilde{l}_3} \right) + \frac{\gamma_8}{\tilde{l}_3}, \quad \tilde{X}(l)^\gamma = X_H(l) \cdot \gamma = 0,$$

Since $\gamma$ is closed with respect to $T \pi_{SO(3) \times \mathbb{R}} : TT^*(SO(3) \times \mathbb{R}) \to T(SO(3) \times \mathbb{R})$, then $\pi^*_{SO(3) \times \mathbb{R}} (d\gamma) = 0$. We choose that $(\gamma_4, \gamma_5, \gamma_6) = (\Pi, (\Pi_1, \Pi_2, \Pi_3) = (\tilde{\gamma}_1, \tilde{\gamma}_2, \tilde{\gamma}_3)$, and $\gamma_7 = \alpha = \tilde{\gamma}_4$, $\gamma_8 = l = \tilde{\gamma}_5$.

Hence

$$T \tilde{\gamma} \cdot \tilde{X}(\Pi)^\gamma = X_{h_\mu}(\Pi) \cdot \tilde{\gamma}, \quad T \tilde{\gamma} \cdot \tilde{X}(\alpha)^\gamma = X_{h_\mu}(\alpha) \cdot \tilde{\gamma}, \quad T \tilde{\gamma} \cdot \tilde{X}(l)^\gamma = X_{h_\mu}(l) \cdot \tilde{\gamma}.$$

Thus, the Type I of Hamilton-Jacobi equation for the $R_p$-reduced controlled rigid spacecraft-rotor system $(O_{\mu} \times \mathbb{R} \times \mathbb{R}^*, \phi_{\mu} \times \mathbb{R} \times \mathbb{R}^* \cdot h_\mu, u_\mu)$ holds.

Next, for $\mu \in so^*(3)$, the regular value of $J_Q$, and a SO$(3, \mu)$-invariant symplectic map $\varepsilon : T^*(SO(3) \times \mathbb{R}) \to T^*(SO(3) \times \mathbb{R})$, assume that $\varepsilon(A, \Pi, \alpha, l) = (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6, \varepsilon_7, \varepsilon_8)$, and $\varepsilon(J_Q^{-1}(\mu)) \subset J_Q^{-1}(\mu)$. Denote by $\bar{\varepsilon} = \pi_{\mu}(\varepsilon) : J_Q^{-1}(\mu) \to O_{\mu} \times \mathbb{R} \times \mathbb{R}^*$, and $\bar{\varepsilon} = (\bar{\varepsilon}_1, \bar{\varepsilon}_2, \bar{\varepsilon}_3, \bar{\varepsilon}_4, \bar{\varepsilon}_5) \in O_{\mu} \times \mathbb{R} \times \mathbb{R}^*(\subset so^*(3) \times \mathbb{R} \times \mathbb{R}^*)$, and $\lambda = \gamma \cdot \pi_{SO(3) \times \mathbb{R}} : T^*(SO(3) \times \mathbb{R}) \to T^*(SO(3) \times \mathbb{R})$, and $\lambda(A, \Pi, \alpha, l) = \lambda \cdot \pi_{SO(3) \times \mathbb{R}}(A, \Pi, \alpha, l)$ holds.
\( (\lambda_1, \cdots, \lambda_8) \), and \( \bar{\lambda} = \pi_\mu(\lambda) : \mathcal{J}_Q^{-1}(\mu) \to \mathcal{O}_\mu \times \mathbb{R} \times \mathbb{R}^* \), and \( \bar{\lambda} = (\bar{\lambda}_1, \bar{\lambda}_2, \bar{\lambda}_3, \bar{\lambda}_4, \bar{\lambda}_5) \in \mathcal{O}_\mu \times \mathbb{R} \times \mathbb{R}^* \). We choose that \( (\Pi, \alpha, l) \in \mathcal{O}_\mu \times \mathbb{R} \times \mathbb{R}^* \), and \( \Pi = (\Pi_1, \Pi_2, \Pi_3) = (\bar{\epsilon}_1, \bar{\epsilon}_2, \bar{\epsilon}_3) \), and \( \alpha = \bar{\epsilon}_4 \), and \( l = \bar{\epsilon}_5 \), then \( h_\mu \cdot \bar{\epsilon} : T^*(\text{SO}(3) \times \mathbb{R}) \to \mathbb{R} \) is given by

\[
h_\mu(\Pi, \alpha, l) \cdot \bar{\epsilon} = H(A, \Pi, \alpha, l)|_{\mathcal{O}_\mu \times \mathbb{R} \times \mathbb{R}^*} \cdot \bar{\epsilon} = \frac{1}{2} \left[ \frac{\bar{\epsilon}_1^2}{I_1} + \frac{\bar{\epsilon}_2^2}{I_2} + \frac{(\bar{\epsilon}_3 - \bar{\epsilon}_5)^2}{I_3} + \frac{\bar{\epsilon}_5^2}{J_3} \right],
\]

and the vector field

\[
X_{h_\mu}(\Pi) \cdot \bar{\epsilon} = \{\Pi, h_\mu\}_{\mathcal{O}_\mu \times \mathbb{R} \times \mathbb{R}^*} \cdot \bar{\epsilon}
= -\Pi \cdot (\nabla_\Pi \Pi \times \nabla_\Pi (h_\mu)) \cdot \bar{\epsilon} + \{\Pi, h_\mu\}_{\mathcal{O}_\mu \times \mathbb{R} \times \mathbb{R}^*} \cdot \bar{\epsilon}
= \left( \frac{(\bar{\epsilon}_2 - \bar{\epsilon}_3 - I_2 \bar{\epsilon}_2 \bar{\epsilon}_5)}{I_2 I_3}, \frac{(I_3 - I_1) \bar{\epsilon}_3 \bar{\epsilon}_1 + I_1 \bar{\epsilon}_1 \bar{\epsilon}_5}{I_3 I_1}, \frac{(I_1 - I_2) \bar{\epsilon}_1 \bar{\epsilon}_2}{I_1 I_2} \right),
\]

\[
X_{h_\mu}(\alpha) \cdot \bar{\epsilon} = \{\alpha, h_\mu\}_{\mathcal{O}_\mu \times \mathbb{R} \times \mathbb{R}^*} \cdot \bar{\epsilon}
= -\Pi \cdot (\nabla_\Pi \alpha \times \nabla_\Pi (h_\mu)) \cdot \bar{\epsilon} + \{\alpha, h_\mu\}_{\mathcal{O}_\mu \times \mathbb{R} \times \mathbb{R}^*} \cdot \bar{\epsilon}
= \left( \frac{(\bar{\epsilon}_3 - \bar{\epsilon}_5)}{I_3} + \frac{\bar{\epsilon}_5}{J_3} \right),
\]

\[
X_{h_\mu}(l) \cdot \bar{\epsilon} = \{l, h_\mu\}_{\mathcal{O}_\mu \times \mathbb{R} \times \mathbb{R}^*} \cdot \bar{\epsilon}
= -\Pi \cdot (\nabla_\Pi l \times \nabla_\Pi (h_\mu)) \cdot \bar{\epsilon} + \{l, h_\mu\}_{\mathcal{O}_\mu \times \mathbb{R} \times \mathbb{R}^*} \cdot \bar{\epsilon}
= 0.
\]

On the other hand, from the expressions of the dynamical vector field \( \tilde{X} \) and Hamiltonian vector field \( \tilde{X}_H \), we have that

\[
\tilde{X}(\Pi, \alpha, l) \cdot \bar{\epsilon} = T_{\pi_{\text{SO}(3) \times \mathbb{R}}} \cdot \tilde{X} \cdot \bar{\epsilon}(\Pi, \alpha, l)
= T_{\pi_{\text{SO}(3) \times \mathbb{R}}} \cdot (X_H + \text{vlift}(u)) \cdot \bar{\epsilon}(\Pi, \alpha, l)
= T_{\pi_{\text{SO}(3) \times \mathbb{R}}} \cdot X_H \cdot \bar{\epsilon}(\Pi, \alpha, l),
\]

that is,

\[
\tilde{X}(\Pi) \cdot \bar{\epsilon} = X_H(\Pi) \cdot \bar{\epsilon}
= \left( \frac{(\bar{\epsilon}_2 - \bar{\epsilon}_3 - I_2 \bar{\epsilon}_2 \bar{\epsilon}_5)}{I_2 I_3}, \frac{(I_3 - I_1) \bar{\epsilon}_3 \bar{\epsilon}_1 + I_1 \bar{\epsilon}_1 \bar{\epsilon}_5}{I_3 I_1}, \frac{(I_1 - I_2) \bar{\epsilon}_1 \bar{\epsilon}_2}{I_1 I_2} \right),
\]

\[
\tilde{X}(\alpha) \cdot \bar{\epsilon} = X_H(\alpha) \cdot \bar{\epsilon} = \left( \frac{\bar{\epsilon}_6 - \bar{\epsilon}_8}{I_3}, \frac{\bar{\epsilon}_8}{J_3} \right), \quad \tilde{X}(l) \cdot \bar{\epsilon} = X_H(l) \cdot \bar{\epsilon} = 0.
\]

Note that

\[
T_{\bar{\gamma}} \cdot \tilde{X}(\Pi) \cdot \bar{\epsilon} = \left( \frac{(\bar{\epsilon}_2 - \bar{\epsilon}_3) \bar{\gamma}_2 \bar{\gamma}_3 - I_2 \bar{\gamma}_2 \bar{\gamma}_5}{I_2 I_3}, \frac{(I_3 - I_1) \bar{\gamma}_3 \bar{\gamma}_1 + I_1 \bar{\gamma}_1 \bar{\gamma}_5}{I_3 I_1}, \frac{(I_1 - I_2) \bar{\gamma}_1 \bar{\gamma}_2}{I_1 I_2} \right),
\]

\[
T_{\bar{\gamma}} \cdot \tilde{X}(\alpha) \cdot \bar{\epsilon} = \frac{(\bar{\gamma}_3 - \bar{\gamma}_5)}{I_3} + \frac{\bar{\gamma}_5}{J_3}, \quad T_{\bar{\gamma}} \cdot \tilde{X}(l) \cdot \bar{\epsilon} = 0,
\]

and

\[
T_{\bar{\lambda}} \cdot \tilde{X} \cdot \bar{\epsilon} = T_{\pi_\mu} \cdot T_{\lambda} \cdot (X_H + \text{vlift}(u)) \cdot \bar{\epsilon} = T_{\pi_\mu} \cdot T_{\gamma} \cdot T_{\pi_{\text{SO}(3) \times \mathbb{R}}} \cdot (X_H + \text{vlift}(u)) \cdot \bar{\epsilon} = T_{\bar{\lambda}} \cdot X_H \cdot \bar{\epsilon},
\]

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that is,

\[ T\bar{\lambda} \cdot \tilde{X}(\Pi) \cdot \varepsilon = T\bar{\lambda} \cdot X_H(\Pi) \cdot \varepsilon \]

\[ = \left( \frac{(I_2 - I_3)\bar{\lambda}_2\lambda_3 - I_2\lambda_2\bar{\lambda}_5}{I_2I_3}, \frac{(I_3 - I_1)\bar{\lambda}_3\lambda_1 + I_1\lambda_3\bar{\lambda}_5}{I_3I_1}, \frac{(I_1 - I_2)\bar{\lambda}_1\lambda_2}{I_1I_2} \right), \]

\[ T\bar{\lambda} \cdot \tilde{X}(\alpha) \cdot \varepsilon = T\bar{\lambda} \cdot X_H(\alpha) \cdot \varepsilon = -\frac{(\bar{\lambda}_3 - \bar{\lambda}_5)}{I_3} + \frac{\bar{\lambda}_5}{J_3}, \]

\[ T\bar{\lambda} \cdot \tilde{X}(l) \cdot \varepsilon = T\bar{\lambda} \cdot X_H(l) \cdot \varepsilon = 0. \]

Thus, when we choose that \((\Pi, \alpha, l) \in \mathcal{O}_\mu \times \mathbb{R} \times \mathbb{R}^*, \) and \(\Pi = (\Pi_1, \Pi_2, \Pi_3) = (\bar{\gamma}_1, \bar{\gamma}_2, \bar{\gamma}_3) = (\bar{\varepsilon}_1, \bar{\varepsilon}_2, \bar{\varepsilon}_3) = (\bar{\lambda}_1, \bar{\lambda}_2, \bar{\lambda}_3)\), and \(\alpha = \bar{\varepsilon}_4 = \bar{\lambda}_4\), and \(l = \bar{\varepsilon}_5 = \bar{\lambda}_5\), we must have that

\[ T\bar{\gamma} \cdot \tilde{X}(\alpha) \cdot \varepsilon = X_{h_{\mu}}(\alpha) \cdot \varepsilon = T\bar{\lambda} \cdot \tilde{X}(\alpha) \cdot \varepsilon, \]

\[ T\bar{\gamma} \cdot \tilde{X}(l) \cdot \varepsilon = X_{h_{\mu}}(l) \cdot \varepsilon = T\bar{\lambda} \cdot \tilde{X}(l) \cdot \varepsilon. \]

Since the map \(\varepsilon : T^*\text{SO}(3) \times \mathbb{R}) \to T^*\text{SO}(3) \times \mathbb{R})\) is symplectic, then \(T\varepsilon \cdot X_{h_{\mu}, \varepsilon} = X_{h_{\mu}} \cdot \bar{\varepsilon}.\)

Thus, in this case, we must have that \(\varepsilon\) and \(\bar{\varepsilon}\) are the solution of the Type II of Hamilton-Jacobi equation \(T\gamma \cdot \tilde{X} \cdot \varepsilon = X_{h_{\mu}} \cdot \bar{\varepsilon}\), for the \(R_p\)-reduced controlled rigid spacecraft-rotor system \((\mathcal{O}_\mu \times \mathbb{R} \times \mathbb{R}^*, \omega_{\mathcal{O}_\mu \times \mathbb{R} \times \mathbb{R}^*}, h_{\mu}, u_{\mu}),\) if and only if they satisfy the equation \(T\varepsilon \cdot (X_{h_{\mu}, \varepsilon}) = T\bar{\lambda} \cdot \tilde{X} \cdot \varepsilon.\)

To sum up the above discussion, we have the following Theorem 4.2. For convenience, the maps involved in the following theorem are shown in Diagram-2.

\[ J_Q^{-1}(\mu) \xrightarrow{i_{\mu}} T^*Q \xrightarrow{\pi_Q} Q \xrightarrow{\gamma} T^*Q \xrightarrow{\pi_{\mu}} \mathcal{O}_\mu \times \mathbb{R} \times \mathbb{R}^* \]

Diagram-2

**Theorem 4.2** In the case of coincident centers of buoyancy and gravity, if the 5-tuple \((T^*Q, \text{SO}(3), \omega_{\text{SO}(3)}, H, u),\) where \(Q = \text{SO}(3) \times \mathbb{R},\) is a regular point reducible rigid spacecraft-rotor system with the control torque \(u\) acting on the rotor, then for a point \(\mu \in \mathfrak{so}^*(3),\) the regular value of the momentum map \(J_Q : \text{SO}(3) \times \mathfrak{so}^*(3) \times \mathbb{R} \times \mathbb{R}^* \to \mathfrak{so}^*(3),\) the \(R_p\)-reduced controlled rigid spacecraft-rotor system is the 4-tuple \((\mathcal{O}_\mu \times \mathbb{R} \times \mathbb{R}^*, \omega_{\mathcal{O}_\mu \times \mathbb{R} \times \mathbb{R}^*}, h_{\mu}, u_{\mu}).\) Assume that \(\gamma : \text{SO}(3) \times \mathbb{R} \to T^*(\text{SO}(3) \times \mathbb{R})\) is an one-form on \(\text{SO}(3) \times \mathbb{R},\) and \(\lambda = \gamma \cdot \pi_{(\text{SO}(3) \times \mathbb{R})} : T^*(\text{SO}(3) \times \mathbb{R}) \to T^*(\text{SO}(3) \times \mathbb{R})\) is a \(\text{SO}(3)_{\mu}\)-invariant symplectic map. Denote \(X^\gamma = T\pi_{(\text{SO}(3) \times \mathbb{R})} \cdot \tilde{X} \cdot \gamma,\) and \(X^\varepsilon = T\pi_{(\text{SO}(3) \times \mathbb{R})} \cdot \tilde{X} \cdot \varepsilon,\) where \(X = X_{T^*Q, \text{SO}(3), \omega_{\text{SO}(3)}, H, u}\) is the dynamical vector field of the controlled rigid spacecraft-rotor system \((T^*Q, \text{SO}(3), \omega_{\text{SO}(3)}, H, u).\) Moreover, assume that \(\text{Im}(\gamma) \subset J_Q^{-1}(\mu),\) and it is \(\text{SO}(3)_{\mu}\)-invariant, and \(\varepsilon(J_Q^{-1}(\mu)) \subset J_Q^{-1}(\mu).\)

Denote \(\tilde{\gamma} = \pi_{\mu}(\gamma) : \text{SO}(3) \times \mathbb{R} \to \mathcal{O}_\mu \times \mathbb{R} \times \mathbb{R}^*,\) and \(\tilde{\lambda} = \pi_{\mu}(\lambda) : T^*(\text{SO}(3) \times \mathbb{R}) \to \mathcal{O}_\mu \times \mathbb{R} \times \mathbb{R}^*,\) and \(\tilde{\varepsilon} = \pi_{\mu}(\varepsilon) : J_Q^{-1}(\mu) \to \mathcal{O}_\mu \times \mathbb{R} \times \mathbb{R}^*.\) Then the following two assertions hold:

1. If the one-form \(\gamma : \text{SO}(3) \times \mathbb{R} \to T^*(\text{SO}(3) \times \mathbb{R})\) is closed with respect to \(T\pi_{(\text{SO}(3) \times \mathbb{R})} : \)
$TT^*(SO(3) \times \mathbb{R}) \to T(SO(3) \times \mathbb{R})$, then $\gamma$ is a solution of the Type I of Hamilton-Jacobi equation $T\dot{\gamma} \cdot \dot{\gamma} = X_{h_n} \cdot \gamma$;

(ii) The $\varepsilon$ and $\bar{\varepsilon}$ satisfy the Type II of Hamilton-Jacobi equation $T\dot{\varepsilon} \cdot \dot{\varepsilon} = X_{h_n} \cdot \varepsilon$, if and only if they satisfy the equation $T\dot{\bar{\varepsilon}} \cdot (X_{h_n} \cdot \bar{\varepsilon}) = T\dot{\lambda} \cdot \dot{\varepsilon}$.

Remark 4.3 When the rigid spacecraft does not carry any internal rotor, in this case the configuration space is $Q = G = SO(3)$, the motion of rigid spacecraft is just the rotation motion of a rigid body, the above $R_p$-reduced controlled spacecraft-rotor system is just the Marsden-Weinstein reduced rigid body system, that is, 3-tuple $(O_\mu, \omega_{\mu, h_{O_\mu}})$, where $O_{\mu} \subset \mathfrak{so}^*(3)$ is the co-adjoint orbit, $\omega_{\mu}$ is the orbit symplectic form on $O_{\mu}$, which is induced by the rigid body Lie-Poisson bracket on $\mathfrak{so}^*(3)$, $h_{O_\mu} \cdot \pi_{O_\mu} = H(A, \Pi) \mid_{O_{\mu}}$. From the above Theorem 4.2 we can obtain the Proposition 5.3 in Wang [23], that is, we give the two types of Lie-Poisson Hamilton-Jacobi equation for the Marsden-Weinstein reduced rigid body system $(O_{\mu}, \omega_{O_{\mu}}, h_{O_{\mu}})$. See Marsden and Ratiu [14], Ge and Marsden [4], and Wang [23].

It is worthy of noting that, for the controlled rigid spacecraft-rotor system $(T^*Q, SO(3), \omega_Q, H, u)$ with the $R_p$-reduced controlled rigid spacecraft-rotor system $(O_\mu \times \mathbb{R} \times \mathbb{R}^*, \tilde{\omega}_{O_\mu}^{\mu} \times \mathbb{R} \times \mathbb{R}^*, h_{O_{\mu}} u_{\mu})$, we know that the Hamiltonian vector fields $X_{\mu}$ and $X_{h_n}$ for the corresponding Hamiltonian system $(T^*Q, SO(3), \omega_Q, H)$ and its $R_p$-reduced system $(O_\mu \times \mathbb{R} \times \mathbb{R}^*, \tilde{\omega}_{O_\mu}^{\mu} \times \mathbb{R} \times \mathbb{R}^*, h_{O_{\mu}} u_{\mu})$, are $\pi_{\mu}$-related, that is, $X_{h_n} \cdot \pi_{\mu} = T\pi_{\mu} \cdot X_{H} \cdot i_{\mu}$. Then we can prove the following Theorem 4.4, which states the relationship between the solutions of Type II of Hamilton-Jacobi equations and the regular point reduction.

Theorem 4.4 In the case of coincident centers of buoyancy and gravity, for the controlled rigid spacecraft-rotor system $(T^*Q, SO(3), \omega_Q, H, u)$ with the $R_p$-reduced controlled rigid spacecraft-rotor system $(O_{\mu} \times \mathbb{R} \times \mathbb{R}^*, \tilde{\omega}_{O_{\mu}} \times \mathbb{R} \times \mathbb{R}^*, h_{O_{\mu}} u_{\mu})$, assume that $\gamma : SO(3) \times \mathbb{R} \to T^*(SO(3) \times \mathbb{R})$ is an one-form on $SO(3) \times \mathbb{R}$, and $\varepsilon : T^*(SO(3) \times \mathbb{R}) \to T^*(SO(3) \times \mathbb{R})$ is a $SO(3)\mu$-invariant symplectic map, $\bar{\varepsilon} = \pi_{\mu}(\varepsilon) : J^{-1}_{Q}(\mu) \to O_{\mu} \times \mathbb{R} \times \mathbb{R}^*$. Under the hypotheses and notations of Theorem 4.2, then we have that $\varepsilon$ is a solution of the Type II of Hamilton-Jacobi equation $T\dot{\varepsilon} \cdot \dot{\varepsilon} = X_{H} \cdot \varepsilon$, for the regular point reducible controlled rigid spacecraft-rotor system $(T^*Q, SO(3), \omega_Q, H, u)$, if and only if $\varepsilon$ and $\bar{\varepsilon}$ satisfy the Type II of Hamilton-Jacobi equation $T\dot{\gamma} \cdot \dot{\gamma} = X_{h_n} \cdot \gamma$, for the $R_p$-reduced controlled rigid spacecraft-rotor system $(O_{\mu} \times \mathbb{R} \times \mathbb{R}^*, \tilde{\omega}_{O_{\mu}} \times \mathbb{R} \times \mathbb{R}^*, h_{O_{\mu}} u_{\mu})$.

Proof: Note that $\text{Im}(\gamma) \subset J^{-1}_{Q}(\mu)$, and it is $SO(3)\mu$-invariant, in this case, $\pi_{\mu}^{*}\bar{\omega}_{O_{\mu}} \times \mathbb{R} \times \mathbb{R}^* = i_{\mu}^{*}\omega_Q = \omega_Q$, along $\text{Im}(\gamma)$. Since the Hamiltonian vector fields $X_{H}$ and $X_{h_n}$ are $\pi_{\mu}$-related, that is, $X_{h_n} \cdot \pi_{\mu} = T\pi_{\mu} \cdot X_{H} \cdot i_{\mu}$, and by using the $R_p$-reduced symplectic form $\tilde{\omega}_{O_{\mu}} \times \mathbb{R} \times \mathbb{R}^*$, for any $w \in TT^*Q$, and $T\pi_{\mu} \cdot w \neq 0$, we have that

\[
\begin{align*}
\tilde{\omega}_{O_{\mu}} \times \mathbb{R} \times \mathbb{R}^* (T\dot{\gamma} \cdot \dot{\gamma} - X_{h_n} \cdot \gamma, T\pi_{\mu} \cdot w) \\
= \tilde{\omega}_{O_{\mu}} \times \mathbb{R} \times \mathbb{R}^* (T\dot{\varepsilon} \cdot \dot{\varepsilon} - X_{h_n} \cdot \varepsilon, T\pi_{\mu} \cdot w) \\
= \tilde{\omega}_{O_{\mu}} \times \mathbb{R} \times \mathbb{R}^* (T\pi_{\mu} \cdot T\dot{\gamma} \cdot \dot{\gamma} - X_{h_n} \cdot \varepsilon, T\pi_{\mu} \cdot w) \\
= \tilde{\omega}_{O_{\mu}} \times \mathbb{R} \times \mathbb{R}^* (T\pi_{\mu} \cdot T\dot{\gamma} \cdot \dot{\gamma} - X_{h_n} \cdot \gamma, T\pi_{\mu} \cdot w) \\
= \pi_{\mu}^{*}\tilde{\omega}_{O_{\mu}} \times \mathbb{R} \times \mathbb{R}^* (T\gamma \cdot \dot{X}_{\varepsilon} \cdot w - \omega_{\mu} \cdot \pi_{\mu} \cdot \varepsilon, T\pi_{\mu} \cdot w) \\
= \pi_{\mu}^{*}\tilde{\omega}_{O_{\mu}} \times \mathbb{R} \times \mathbb{R}^* (T\gamma \cdot \dot{X}_{\varepsilon} \cdot w - \omega_{\mu} \cdot \pi_{\mu} \cdot \gamma, T\pi_{\mu} \cdot w) \\
= \omega_{\mu} (T\gamma \cdot \dot{X}_{\varepsilon} - X_{H} \cdot \varepsilon, w).
\end{align*}
\]

Because both the symplectic form $\omega_Q$ and the $R_p$-reduced symplectic form $\tilde{\omega}_{O_{\mu}} \times \mathbb{R} \times \mathbb{R}^*$ are non-degenerate, it follows that the equation $T\dot{\gamma} \cdot \dot{\gamma} = X_{h_n} \cdot \gamma$, is equivalent to the equation $T\gamma \cdot \dot{X}_{\varepsilon} = X_{h_n} \cdot \varepsilon$. 

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Thus, $\varepsilon$ is a solution of the Type II of Hamilton-Jacobi equation $T\gamma \cdot \bar{X}_\varepsilon = X_H \cdot \varepsilon$, for the regular point reducible controlled rigid spacecraft-rotor system $(T^*Q, \text{SO}(3), \omega_0, H, u)$, if and only if $\varepsilon$ and $\tilde{\varepsilon}$ satisfy the Type II of Hamilton-Jacobi equation $T\gamma \cdot \bar{X}_{\varepsilon} = X_{h,\mu} \cdot \varepsilon$, for the $R_p$-reduced controlled rigid spacecraft-rotor system $(\mathcal{O}_\mu \times \mathbb{R} \times \mathbb{R}^*, \bar{\omega}_{\mathcal{O}_\mu \times \mathbb{R} \times \mathbb{R}^*}, h_{\mu}, u_{\mu})$.

4.2 In The Case of Non-coincident Centers

In the following we shall derive precisely the geometric constraint conditions of the $R_p$-reduced symplectic form $\bar{\omega}_{\mathcal{O}_{\mu(a)} \times \mathbb{R} \times \mathbb{R}^*}$, for the dynamical vector field of the regular point reducible controlled rigid spacecraft-rotor system with non-coincident centers of buoyancy and gravity, that is, Type I and Type II of Hamilton-Jacobi equation for the $R_p$-reduced controlled rigid spacecraft-rotor system $(\mathcal{O}_{\mu(a)} \times \mathbb{R} \times \mathbb{R}^*, \omega_{\mathcal{O}_{\mu(a)} \times \mathbb{R} \times \mathbb{R}^*}, h_{\mu(a)}, u_{\mu(a)})$.

Assume that $\gamma : \text{SE}(3) \times \mathbb{R} \rightarrow T^*(\text{SE}(3) \times \mathbb{R})$ is an one-form on $\text{SE}(3) \times \mathbb{R}$, and $\gamma(A, c, \alpha) = (\gamma_1, \cdots, \gamma_{14})$, and $\gamma$ is closed with respect to $T\pi_{\text{SE}(3) \times \mathbb{R}} : TT^*(\text{SE}(3) \times \mathbb{R}) \rightarrow T(\text{SE}(3) \times \mathbb{R})$. For $(\mu, a) \in \text{se}^*(3)$, the regular value of $J_Q$, $\text{Im}(\gamma) \subset J_Q^{-1}(\mu, a)$, and it is $\text{SE}(3)_{(\mu, a)}$-invariant, and $\tilde{\gamma} = \pi_{\mu(a)}(\gamma) : \text{SE}(3) \rightarrow \mathcal{O}_{\mu(a)} \times \mathbb{R} \times \mathbb{R}^*$. Denote by $\tilde{\gamma} = (\tilde{\gamma}_1, \tilde{\gamma}_2, \tilde{\gamma}_3, \tilde{\gamma}_4, \tilde{\gamma}_5, \tilde{\gamma}_6, \tilde{\gamma}_7, \tilde{\gamma}_8) \in \mathcal{O}_{\mu(a)} \times \mathbb{R} \times \mathbb{R}^*(\subset \text{se}^*(3) \times \mathbb{R} \times \mathbb{R}^*)$, where $\pi_{\mu(a)} : J_Q^{-1}(\mu, a) \rightarrow \mathcal{O}_{\mu(a)} \times \mathbb{R} \times \mathbb{R}^*$. We choose that $(\Pi, \Gamma, \alpha, l) \in \mathcal{O}_{\mu(a)} \times \mathbb{R} \times \mathbb{R}^*$, and $\Pi = (\Pi_1, \Pi_2, \Pi_3) = (\tilde{\gamma}_1, \tilde{\gamma}_2, \tilde{\gamma}_3), \Gamma = (\Gamma_1, \Gamma_2, \Gamma_3) = (\tilde{\gamma}_4, \tilde{\gamma}_5, \tilde{\gamma}_6), \alpha = \tilde{\gamma}_7$, and $l = \tilde{\gamma}_8$. Then $h_{(\mu, a)} \cdot \tilde{\gamma} : \text{SE}(3) \times \mathbb{R} \rightarrow \mathbb{R}$ is given by

$$h_{(\mu, a)}(\Pi, \Gamma, \alpha, l) \cdot \tilde{\gamma} = H(A, c, \Pi, \Gamma, \alpha, l)|_{\mathcal{O}_{\mu(a)} \times \mathbb{R} \times \mathbb{R}^*} \cdot \tilde{\gamma}$$

$$= \frac{1}{2} \left[ \frac{\gamma^2_1}{\bar{I}_1} + \frac{\gamma^2_2}{\bar{I}_2} + \frac{(\tilde{\gamma}_3 - \tilde{\gamma}_8)^2}{\bar{I}_3} + \frac{\gamma^2_8}{\bar{J}_3} \right] + gh(\tilde{\gamma}_4 \cdot \chi_1 + \tilde{\gamma}_5 \cdot \chi_2 + \tilde{\gamma}_6 \cdot \chi_3),$$

and the vector field

$$X_{h_{(\mu, a)}}(\Pi) = \Pi \cdot (\nabla_H \Pi \times \nabla_H (h_{(\mu, a)})) - |\mathcal{O}_{\mu(a)} \times \mathbb{R} \times \mathbb{R}^*| \cdot \tilde{\gamma}$$

$$= -\Pi \cdot (\nabla_H \Pi \times \nabla_H (h_{(\mu, a)})) \cdot \tilde{\gamma} - \Gamma \cdot (\nabla_H \Pi \times \nabla_H (h_{(\mu, a)})) - \nabla_H (h_{(\mu, a)}) \times \nabla_H \Pi \cdot \tilde{\gamma}$$

$$+ \{\Pi, h_{(\mu, a)}\}_R |_{\mathcal{O}_{\mu(a)} \times \mathbb{R} \times \mathbb{R}^*} \cdot \tilde{\gamma}$$

$$= -\nabla_H \Pi \cdot (\nabla_H (h_{(\mu, a)}) - \Pi) \cdot \tilde{\gamma} - \nabla_H \Pi \cdot (\nabla_H (h_{(\mu, a)}) \times \Gamma) \cdot \tilde{\gamma}$$

$$+ \left( \frac{\partial H}{\partial \alpha} \frac{\partial (h_{(\mu, a)})}{\partial \ell} - \frac{\partial (h_{(\mu, a)})}{\partial \alpha} \frac{\partial \Pi}{\partial \ell} \right) \cdot \tilde{\gamma}$$

$$= (\Pi_1, \Pi_2, \Pi_3) \times \left( \frac{\Pi_1}{\bar{I}_1}, \frac{\Pi_2}{\bar{I}_2}, \frac{(\Pi_3 - l)}{\bar{I}_3} \right) \cdot \tilde{\gamma} + gh(\Gamma_1, \Gamma_2, \Gamma_3) \times (\chi_1, \chi_2, \chi_3) \cdot \tilde{\gamma}$$

$$= \left( \frac{\bar{I}_2 - \bar{I}_3}{\bar{I}_2 \bar{I}_3} \tilde{\gamma}_2 \tilde{\gamma}_3 - \bar{I}_2 \tilde{\gamma}_2 \tilde{\gamma}_8 \right) + gh(\tilde{\gamma}_5 \chi_3 - \tilde{\gamma}_6 \chi_2),$$

$$\left( \frac{\bar{I}_3 - \bar{I}_1}{\bar{I}_3 I_1} \tilde{\gamma}_3 \tilde{\gamma}_1 + \bar{I}_1 \tilde{\gamma}_1 \tilde{\gamma}_8 \right) + gh(\tilde{\gamma}_6 \chi_1 - \tilde{\gamma}_4 \chi_3), \quad \left( \frac{\bar{I}_1}{\bar{I}_2 \bar{I}_3} \tilde{\gamma}_1 \tilde{\gamma}_2 + gh(\tilde{\gamma}_4 \chi_2 - \tilde{\gamma}_5 \chi_1) \right),$$

since $\nabla_H \Pi_i = 1, \nabla_H \Pi_j = 0, i \neq j, \nabla_H \Gamma_j = \nabla_H \Gamma_j = 0$ and $\chi = (\chi_1, \chi_2, \chi_3), \nabla_H (h_{(\mu, a)}) = gh \chi_j, \nabla_{\Pi_3}(h_{(\mu, a)}) = (\Pi_3 - l)/\bar{I}_3, \nabla_{\Pi_k}(h_{(\mu, a)}) = \Pi_k/\bar{I}_k, \frac{\partial H}{\partial \alpha} = \frac{\partial (h_{(\mu, a)})}{\partial \alpha} = 0, i, j = 1, 2, 3, k = 1, 2.$
\[ X_{h(a)}(\Gamma) \cdot \tilde{\gamma} = \{ \Gamma, h(a) \} \cdot (\gamma, h(a)) \]
\[ = -\Pi \cdot (\nabla_\Gamma \cdot \nabla_\Pi (h(a))) \cdot \tilde{\gamma} - \Gamma \cdot (\nabla_\Pi \cdot \nabla_\Gamma (h(a))) - \nabla_\Pi (h(a)) \times \nabla_\Gamma) \cdot \tilde{\gamma} + \{ \Gamma, h(a) \} \cdot (\gamma, h(a)) \]
\[ = \nabla_\Gamma \cdot (\Gamma \times \nabla_\Pi (h(a))) \cdot \tilde{\gamma} + \frac{\partial \Gamma}{\partial \alpha} \frac{\partial (h(a))}{\partial \alpha} \cdot \tilde{\gamma} \]
\[ = (\Gamma_1, \Gamma_2, \Gamma_3) \times \left( \frac{\Pi_1}{I_1}, \frac{\Pi_2}{I_2}, \frac{(\Pi_3 - l)}{I_3} \right) \cdot \tilde{\gamma} \]
\[ = \left( \frac{\bar{I}_2 \gamma \bar{\gamma}_2 - \bar{I}_2 \gamma \bar{\gamma}_2 \bar{\gamma}_8}{I_2 I_3}, \frac{\bar{I}_3 \gamma \bar{\gamma}_1 - \bar{I}_3 \gamma \bar{\gamma}_3}{I_3 I_1}, \frac{\bar{I}_1 \gamma \bar{\gamma}_7 - \bar{I}_1 \gamma \bar{\gamma}_5 \bar{\gamma}_1}{I_1 I_2} \right), \]

since \( \nabla_\Gamma \cdot \Gamma_i = 1, \nabla_\Gamma \cdot \Gamma_j = 0, \ i \neq j, \nabla_\Pi \cdot \Gamma_j = 0, \) and \( \nabla_\Pi (h(a)) = (\Pi_3 - l)/I_3, \nabla_\Pi (h(a)) = \Pi_k/\bar{I}_k, \frac{\partial \gamma}{\partial \alpha} = 0, i, j = 1, 2, 3, k = 1, 2. \)

\[ X_{h(a)}(\alpha) \cdot \tilde{\gamma} = \{ \alpha, h(a) \} \cdot (\gamma, h(a)) \]
\[ = -\Pi \cdot (\nabla_\Pi \alpha \times \nabla_\Pi (h(a))) \cdot \tilde{\gamma} - \Gamma \cdot (\nabla_\Pi \alpha \times \nabla_\Gamma (h(a))) - \nabla_\Pi (h(a)) \times \nabla_\Gamma) \cdot \tilde{\gamma} + \{ \alpha, h(a) \} \cdot (\gamma, h(a)) \]
\[ = \left( \frac{\partial \alpha}{\partial \alpha} \frac{\partial (h(a))}{\partial \alpha} \cdot \tilde{\gamma} \right) \]
\[ = \left( \frac{\bar{I}_3 \gamma \bar{\gamma}_3 - \bar{I}_3 \gamma \bar{\gamma}_2 \bar{\gamma}_8}{I_3 I_1}, \frac{\bar{I}_1 \gamma \bar{\gamma}_7 - \bar{I}_1 \gamma \bar{\gamma}_5 \bar{\gamma}_1}{I_1 I_2} \right), \]

since \( \nabla_\Pi \alpha = \nabla_\Gamma \alpha = 0, \ \frac{\partial \alpha}{\partial \alpha} = 0, \) and \( \frac{\partial (h(a))}{\partial \alpha} = -\Pi_3 - l/\bar{I}_3 = \frac{\bar{I}_3}{\bar{I}_3}, i = 1, 2, 3. \)

\[ X_{h(a)}(l) \cdot \tilde{\gamma} = \{ l, h(a) \} \cdot (\gamma, h(a)) \]
\[ = -\Pi \cdot (\nabla_\Pi \times \nabla_\Pi (h(a))) \cdot \tilde{\gamma} - \Gamma \cdot (\nabla_\Pi \times \nabla_\Gamma (h(a))) - \nabla_\Pi (h(a)) \times \nabla_\Gamma) \cdot \tilde{\gamma} + \{ l, h(a) \} \cdot (\gamma, h(a)) \]
\[ = \left( \frac{\partial l}{\partial \alpha} \frac{\partial (h(a))}{\partial \alpha} \cdot \tilde{\gamma} \right) \]
\[ = \left( \frac{\bar{I}_3 \gamma \bar{\gamma}_3 - \bar{I}_3 \gamma \bar{\gamma}_2 \bar{\gamma}_8}{I_3 I_1}, \frac{\bar{I}_1 \gamma \bar{\gamma}_7 - \bar{I}_1 \gamma \bar{\gamma}_5 \bar{\gamma}_1}{I_1 I_2} \right), \]

since \( \nabla_\Pi \cdot l = \nabla_\Gamma \cdot l = 0, \) and \( \frac{\partial l}{\partial \alpha} = 0, i = 1, 2, 3. \)

On the other hand, from the expressions of the dynamical vector field \( \tilde{X} \) and Hamiltonian vector field \( X_H \), we have that
\[ \tilde{X}(\Pi, \Gamma) \cdot \gamma = T \pi_{SE(3)} \cdot \chi \cdot \gamma(\Pi, \Gamma) \]
\[ = T \pi_{SE(3)} \cdot (X_H + vlift(u)) \cdot \gamma(\Pi, \Gamma) \]
\[ = T \pi_{SE(3)} \cdot X_H \cdot \gamma(\Pi, \Gamma) = X_H \cdot \gamma(\Pi, \Gamma), \]
that is,
\[ \tilde{X}(\Pi) = X_H(\Pi) \cdot \gamma \]
\[ = \left( \frac{\bar{I}_2 - \bar{I}_3 \gamma \bar{\gamma}_3}{I_2 I_3}, \frac{\bar{I}_3 - \bar{I}_1 \gamma \bar{\gamma}_7 + \bar{I}_1 \gamma \bar{\gamma}_7 \bar{\gamma}_1}{I_3 I_1} \right) + \frac{\partial}{\partial \alpha} (\gamma_{11} \chi_1 - \gamma_{12} \chi_2) \]
\[ = \left( \frac{\bar{I}_2 - \bar{I}_3 \gamma \bar{\gamma}_3}{I_2 I_3}, \frac{\bar{I}_3 - \bar{I}_1 \gamma \bar{\gamma}_7 + \bar{I}_1 \gamma \bar{\gamma}_7 \bar{\gamma}_1}{I_3 I_1} \right) + \frac{\partial}{\partial \alpha} (\gamma_{11} \chi_1 - \gamma_{12} \chi_2) \]
\[
\dot{X}(\Gamma)^\gamma = X_H(\Gamma) \cdot \gamma \\
= \left( \frac{I_2\gamma_1 \gamma_9 - I_3\gamma_12\gamma_8 - I_2\gamma_17\gamma_4}{I_2 I_3}, \frac{I_3\gamma_12\gamma_7 - I_1\gamma_10\gamma_9 + I_1\gamma_17\gamma_4}{I_3 I_1}, \frac{I_1\gamma_10\gamma_8 - I_2\gamma_17\gamma_7}{I_1 I_2} \right),
\]

\[
\dot{X}(\alpha)^\gamma = X_H(\alpha) \cdot \gamma = -\frac{\gamma_9 - \gamma_14}{I_3} + \frac{\gamma_14}{J_3}, \quad \dot{X}(l)^\gamma = X_H(l) \cdot \gamma = 0.
\]

Since \(\gamma\) is closed with respect to \(T \pi_{(SE(3) \times \mathbb{R})} : T T^*(SE(3) \times \mathbb{R}) \rightarrow T(SE(3) \times \mathbb{R})\), then \(\pi^*_{(SE(3) \times \mathbb{R})}(\mathbf{d}\gamma) = 0\). We choose that \((\gamma_7, \gamma_8, \gamma_9) = \Pi = (\Pi_1, \Pi_2, \Pi_3) = (\gamma_1, \gamma_2, \gamma_3), (\gamma_{10}, \gamma_{11}, \gamma_{12}) = \Gamma = (\Gamma_1, \Gamma_2, \Gamma_3) = (\gamma_4, \gamma_5, \gamma_6)\), and \(\gamma_{13} = \alpha = \gamma_7\), \(\gamma_{14} = l = \gamma_8\). Hence

\[
T \gamma \cdot \dot{X}(\Pi)^\gamma = X_{h(\mu, a)}(\Pi) \cdot \dot{\gamma}, \quad T \gamma \cdot \dot{X}(\Gamma)^\gamma = X_{h(\mu, a)}(\Gamma) \cdot \dot{\gamma}, \quad T \gamma \cdot \dot{X}(l)^\gamma = X_{h(\mu, a)}(l) \cdot \dot{\gamma}.
\]

Thus, the Type I of Hamilton-Jacobi equation for the \(R_p\)-reduced controlled rigid spacecraft-rotor system \((\mathcal{O}(\mu, a) \times \mathbb{R} \times \mathbb{R}^*, \omega_{\mathcal{O}(\mu, a) \times \mathbb{R} \times \mathbb{R}^*}, h(\mu, a), u(\mu, a))\) holds.

Next, for \((\mu, a) \in \mathfrak{se}^*(3)\), the regular value of \(J_Q\), and a \(SE(3, \mu, a)\)-invariant symplectic map \(\varepsilon : T^*(SE(3) \times \mathbb{R}) \rightarrow T^*(SE(3) \times \mathbb{R})\), assume that \(\varepsilon(A, c, \Pi, \Gamma, \alpha, l) = (\varepsilon_1, \cdots, \varepsilon_{14})\), and \(\varepsilon(J^{-1}((\mu, a))) \subseteq J^{-1}((\mu, a)) \subseteq \mathcal{O}(\mu, a) \times \mathbb{R} \times \mathbb{R}^*, \) and \(\bar{\varepsilon} = \left(\varepsilon_1, \cdots, \varepsilon_8\right) \in \mathcal{O}(\mu, a) \times \mathbb{R} \times \mathbb{R}^*(\subset \mathfrak{se}^*(3) \times \mathbb{R} \times \mathbb{R}^*), \) and \(\lambda = \gamma \cdot \pi_{SE(3), \mathbb{R}} : T^*(SE(3) \times \mathbb{R}) \rightarrow T^*(SE(3) \times \mathbb{R}), \) and \(\lambda (A, c, \Pi, \Gamma, \alpha, l) = (\lambda_1, \cdots, \lambda_{14}), \) and \(\dot{\lambda} = \pi_{SE(3), \mathbb{R}}(\lambda) \cdot T^*(SE(3) \times \mathbb{R}) \rightarrow \mathcal{O}(\mu, a) \times \mathbb{R} \times \mathbb{R}^*, \) and \(\lambda = (\lambda_1, \cdots, \lambda_8) \in \mathcal{O}(\mu, a) \times \mathbb{R} \times \mathbb{R}^*(\subset \mathfrak{se}^*(3) \times \mathbb{R} \times \mathbb{R}^*). \) We choose that \((\Pi, \Gamma, \alpha, l) \in \mathcal{O}(\mu, a) \times \mathbb{R} \times \mathbb{R}^*, \) and \(\Pi = (\Pi_1, \Pi_2, \Pi_3) = (\varepsilon_1, \varepsilon_2, \varepsilon_3), \Gamma = (\Gamma_1, \Gamma_2, \Gamma_3) = (\varepsilon_4, \varepsilon_5, \varepsilon_6), \alpha = \varepsilon_7, \) and \(l = \varepsilon_8. \) Then \(h(\mu, a) \cdot \varepsilon : T^*(SE(3) \times \mathbb{R}) \rightarrow \mathbb{R}\) is given by

\[

\frac{1}{2} \left[ \frac{\varepsilon_1^2}{I_1} + \frac{\varepsilon_2^2}{I_2} + \frac{(\varepsilon_3 - \varepsilon_8)^2}{I_3} + \frac{\varepsilon_9^2}{J_3} \right] + gh(\varepsilon_4 \cdot \chi_1 + \varepsilon_5 \cdot \chi_2 + \varepsilon_6 \cdot \chi_3),
\]

and the vector field

\[
X_{h(\mu, a)}(\Pi) \cdot \varepsilon = \{\Pi, h(\mu, a)\} - |\mathcal{O}(\mu, a) \times \mathbb{R} \times \mathbb{R}^*| \cdot \varepsilon
\]

\[
= (\Pi_1, \Pi_2, \Pi_3) \times \left( \frac{\Pi_1}{I_1}, \frac{\Pi_2}{I_2}, \frac{(\Pi_3 - l)}{I_3} \right) \cdot \varepsilon + gh(\Gamma_1, \Gamma_2, \Gamma_3) \times (\chi_1, \chi_2, \chi_3) \cdot \varepsilon
\]

\[
= \left( \frac{I_2 - I_3}{I_2 I_3}, \frac{I_1 - I_2}{I_1 I_2} \right) \cdot \varepsilon + \frac{gh(\varepsilon_5 \chi_3 - \varepsilon_6 \chi_2)}{I_3 I_1} + \frac{gh(\varepsilon_6 \chi_1 - \varepsilon_4 \chi_3)}{I_1 I_2} + \frac{gh(\varepsilon_4 \chi_2 - \varepsilon_5 \chi_1)}{I_1 I_2},
\]

\[
X_{h(\mu, a)}(\Gamma) \cdot \varepsilon = \{\Gamma, h(\mu, a)\} - |\mathcal{O}(\mu, a) \times \mathbb{R} \times \mathbb{R}^*| \cdot \varepsilon
\]

\[
= (\Pi_1, \Pi_2, \Pi_3) \times \left( \frac{\Pi_1}{I_1}, \frac{\Pi_2}{I_2}, \frac{(\Pi_3 - l)}{I_3} \right) \cdot \varepsilon
\]

\[
= (\frac{I_3 \varepsilon_3 - I_2 \varepsilon_2 - I_2 \varepsilon_2 I_8}{I_2 I_3}, \frac{I_3 \varepsilon_6 - I_1 \varepsilon_1 - I_1 \varepsilon_1 E_3 - I_1 \varepsilon_1 I_8}{I_3 I_2}, \frac{I_1 \varepsilon_4 - I_2 \varepsilon_2 E_1}{I_1 I_2} - \frac{I_2 \varepsilon_5}{I_1 I_2}),
\]

\[
X_{h(\mu, a)}(\alpha) \cdot \varepsilon = \{\alpha, h(\mu, a)\} - |\mathcal{O}(\mu, a) \times \mathbb{R} \times \mathbb{R}^*| \cdot \varepsilon
\]

\[
= (\frac{\partial h(\mu, a)}{\partial \alpha} - \frac{\partial h(\mu, a)}{\partial \alpha}) \cdot \varepsilon = \frac{\varepsilon_3 - \varepsilon_8}{I_3} + \frac{\varepsilon_8}{J_3},
\]
\[ X_{h_{(\mu,a)}}(l) \cdot \bar{e} = \{ l, h_{(\mu,a)} \} - \mathcal{O}_{(\mu,a)} \times \mathbb{R}^\ast \cdot \bar{e} = \left( \frac{\partial l}{\partial \alpha} \frac{\partial (h_{(\mu,a)})}{\partial \bar{e}} - \frac{\partial (h_{(\mu,a)})}{\partial \alpha} \frac{\partial l}{\partial \bar{e}} \right) \cdot \bar{e} = 0. \]

On the other hand, from the expressions of the dynamical vector field \( \tilde{X} \) and Hamiltonian vector field \( X_H \), we have that

\[
\tilde{X}(\Pi, \Gamma, \alpha, l)^\varepsilon = T^\pi_{\text{SE}(3) \times \mathbb{R}} \cdot \tilde{X} \cdot \varepsilon(\Pi, \Gamma, \alpha, l) \\
= T^\pi_{\text{SE}(3) \times \mathbb{R}} \cdot (X_H + \text{vlift}(u)) \cdot \varepsilon(\Pi, \Gamma, \alpha, l) \\
= T^\pi_{\text{SE}(3) \times \mathbb{R}} \cdot X_H \cdot \varepsilon(\Pi, \Gamma, \alpha, l) = X_H \cdot \varepsilon(\Pi, \Gamma, \alpha, l),
\]

that is,

\[
\tilde{X}(\Pi)^\varepsilon = X_H(\Pi) \cdot \varepsilon \\
= \left( \frac{(I_2 - I_3)\varepsilon_8\varepsilon_9 - I_2\varepsilon_8\varepsilon_{14}}{I_2 I_3} + gh(\varepsilon_{11}\chi_3 - \varepsilon_{12}\chi_2), \right. \\
\left. \frac{(I_3 - I_1)\varepsilon_9\varepsilon_7 + I_1\varepsilon_7\varepsilon_{14}}{I_3 I_1} + gh(\varepsilon_{12}\chi_1 - \varepsilon_{10}\chi_3), \right. \\
\left. \frac{(I_1 - I_2)\varepsilon_7\varepsilon_8}{I_1 I_2} + gh(\varepsilon_{10}\chi_2 - \varepsilon_{11}\chi_1) \right),
\]

\[
\tilde{X}(\Gamma)^\varepsilon = X_H(\Gamma) \cdot \varepsilon \\
= \left( \frac{I_2\varepsilon_{11}\varepsilon_9 - I_3\varepsilon_{12}\varepsilon_8 - I_2\varepsilon_8\varepsilon_{14}}{I_2 I_3}, \right. \\
\left. \frac{I_3\varepsilon_{12}\varepsilon_7 - I_1\varepsilon_{10}\varepsilon_9 + I_1\varepsilon_7\varepsilon_{14}}{I_3 I_1}, \right. \\
\left. \frac{I_1\varepsilon_{10}\varepsilon_8 - I_2\varepsilon_{11}\varepsilon_7}{I_1 I_2} \right),
\]

\[
\tilde{X}(\alpha)^\varepsilon = X_H(\alpha) \cdot \varepsilon = -\left( \frac{\varepsilon_9 - \varepsilon_{14}}{I_3} \right) + \frac{\varepsilon_{14}}{J_3}, \quad \tilde{X}(l)^\varepsilon = X_H(l) \cdot \varepsilon = 0,
\]

then we have that

\[
T^\gamma \cdot \tilde{X}(\Pi)^\varepsilon = \left( \frac{(I_2 - I_3)\gamma_2\gamma_3 - I_2\gamma_2\gamma_8}{I_2 I_3} + gh(\gamma_5\chi_3 - \gamma_6\chi_2), \right. \\
\left. \frac{(I_3 - I_1)\gamma_3\gamma_1 + I_1\gamma_1\gamma_8}{I_3 I_1} + gh(\gamma_6\chi_1 - \gamma_4\chi_3), \right. \\
\left. \frac{(I_1 - I_2)\gamma_1\gamma_2}{I_1 I_2} + gh(\gamma_4\chi_2 - \gamma_5\chi_1) \right),
\]

\[
T^\gamma \cdot \tilde{X}(\Gamma)^\varepsilon = \left( \frac{I_2\gamma_5\gamma_3 - I_3\gamma_6\gamma_2 - I_2\gamma_2\gamma_8}{I_2 I_3}, \right. \\
\left. \frac{I_3\gamma_6\gamma_1 - I_1\gamma_4\gamma_3 + I_1\gamma_1\gamma_8}{I_3 I_1}, \right. \\
\left. \frac{I_1\gamma_4\gamma_2 - I_2\gamma_5\gamma_1}{I_1 I_2} \right),
\]

\[
T^\gamma \cdot \tilde{X}(\alpha)^\varepsilon = -\left( \frac{\gamma_3 - \gamma_8}{I_3} \right) + \frac{\gamma_8}{J_3}, \quad T^\gamma \cdot \tilde{X}(l)^\varepsilon = 0.
\]

Note that

\[
T\bar{X} \cdot \varepsilon = T\pi_{(\mu,a)} \cdot T\lambda \cdot (X_H + \text{vlift}(u)) \cdot \varepsilon = T\pi_{(\mu,a)} \cdot T\gamma \cdot T^\pi_{\text{SE}(3) \times \mathbb{R}} \cdot (X_H + \text{vlift}(u)) \cdot \varepsilon = T\bar{X} \cdot X_H \cdot \varepsilon,
\]

that is,

\[
\bar{X}(\Pi)^\varepsilon = \bar{T}\lambda \cdot X_H(\Pi) \cdot \varepsilon \\
= \left( \frac{(I_2 - I_3)\bar{\lambda}_2\bar{\lambda}_3 - I_2\bar{\lambda}_2\bar{\lambda}_8}{I_2 I_3} + gh(\bar{\lambda}_5\chi_3 - \bar{\lambda}_6\chi_2), \right. \\
\left. \frac{(I_3 - I_1)\bar{\lambda}_3\bar{\lambda}_1 + I_1\bar{\lambda}_1\bar{\lambda}_8}{I_3 I_1} + gh(\bar{\lambda}_6\chi_1 - \bar{\lambda}_4\chi_3), \right. \\
\left. \frac{(I_1 - I_2)\bar{\lambda}_1\bar{\lambda}_2}{I_1 I_2} + gh(\bar{\lambda}_4\chi_2 - \bar{\lambda}_5\chi_1) \right),
\]
Thus, in this case, we must have that \( T\bar{\lambda} \cdot \bar{X}(\alpha) \cdot \varepsilon = T\bar{\lambda} \cdot X_H(\alpha) \cdot \varepsilon = \frac{(\bar{\lambda}_3 - \bar{\lambda}_8)}{I_3} + \frac{\bar{\lambda}_8}{I_3} \), \( T\bar{\lambda} \cdot \bar{X}(l) \cdot \varepsilon = T\bar{\lambda} \cdot X_H(l) \cdot \varepsilon = 0 \).

Thus, when we choose that \( T\bar{\lambda} \cdot \bar{X}(\Pi) \cdot \varepsilon = T\bar{\lambda} \cdot \bar{X}(\Pi) \cdot \varepsilon = X_{h(\mu,a)}(\Pi) \cdot \varepsilon = T\bar{\lambda} \cdot \bar{X}(\Pi) \cdot \varepsilon \), \( T\bar{\lambda} \cdot \bar{X}(l) \cdot \varepsilon = T\bar{\lambda} \cdot \bar{X}(l) \cdot \varepsilon = X_{h(\mu,a)}(l) \cdot \varepsilon = T\bar{\lambda} \cdot \bar{X}(l) \cdot \varepsilon \).

Since the map \( \varepsilon : T^*(SE(3) \times \mathbb{R}) \to T^*(SE(3) \times \mathbb{R}) \) is symplectic, then \( T\bar{\varepsilon} \cdot X_{h(\mu,a)} \cdot \varepsilon = X_{h(\mu,a)} \cdot \varepsilon \).

Thus, in this case, we must have that \( \varepsilon \) and \( \bar{\varepsilon} \) are the solution of the Type II of Hamilton-Jacobi equation \( T\bar{\gamma} \cdot \bar{X} = X_{h(\mu,a)} \cdot \bar{\varepsilon} \), for the \( R_p \)-reduced controlled rigid spacecraft-rotor system \( (O_{\mu,a} \times \mathbb{R} \times \mathbb{R}^*, \omega_{O_{\mu,a}^*}^{\varepsilon}, h_{(\mu,a)}, u_{(\mu,a)}) \), if and only if they satisfy the equation \( T\bar{\varepsilon} \cdot (X_{h(\mu,a)} \cdot \varepsilon) = T\bar{\lambda} \cdot \bar{X} \cdot \varepsilon \).

To sum up the above discussion, we have the following Theorem 4.5. For convenience, the maps involved in the following theorem are shown in Diagram-3.

**Theorem 4.5** In the case of non-coincident centers of buoyancy and gravity, if the 5-tuple \( (T^*Q, SE(3), \omega_Q, H, u) \), where \( Q = SE(3) \times \mathbb{R} \), is a regular point reducible rigid spacecraft-rotor system with the control torque \( u \) acting on the control torus, then for a point \( (\mu, a) \in \mathfrak{s}\mathfrak{e}^*(3) \), the regular value of the momentum map \( J_Q : SE(3) \times \mathfrak{s}\mathfrak{e}^*(3) \times \mathbb{R} \times \mathbb{R}^* \to \mathfrak{s}\mathfrak{e}^*(3) \), the \( R_p \)-reduced controlled rigid spacecraft-rotor system is the 4-tuple \( (O_{\mu,a} \times \mathbb{R} \times \mathbb{R}^*, \omega_{O_{\mu,a}^*}^{\varepsilon}, h_{(\mu,a)}, u_{(\mu,a)}) \). Assume that \( \gamma : SE(3) \times \mathbb{R} \to T^*(SE(3) \times \mathbb{R}) \) is an one-form on \( SE(3) \times \mathbb{R} \), and \( \lambda = \gamma \cdot T^*\pi_{SE(3) \times \mathbb{R}} \) is an \( SE(3)_{(\mu,a)} \)-invariant symplectic map. Denote \( \tilde{X} = T\pi_{SE(3) \times \mathbb{R}} \cdot \tilde{X} \cdot \varepsilon \), and \( \bar{X} = T\pi_{SE(3) \times \mathbb{R}} \cdot \bar{X} \cdot \varepsilon \), where \( \tilde{X} = X_{(T^*Q, SE(3), \omega_Q, H, u)} \) is the dynamical vector field of the controlled rigid spacecraft-rotor system \( (T^*Q, SE(3), \omega_Q, H, u) \). Moreover, assume that \( \text{Im}(\gamma) \subset J_Q^{-1}(\mu,a) \), and it is \( SE(3)_{(\mu,a)} \)-invariant, and \( \varepsilon(J_Q^{-1}(\mu, a)) \subset J_Q^{-1}(\mu, a) \). Denote \( \bar{\gamma} = \pi_{(\mu,a)}(\gamma) : SE(3) \times \mathbb{R} \to O_{(\mu,a)} \times \mathbb{R} \times \mathbb{R}^* \), and \( \bar{\lambda} = \pi_{(\mu,a)}(\lambda) : T^*(SE(3) \times \mathbb{R}) \to O_{(\mu,a)} \times \mathbb{R} \times \mathbb{R}^* \), and \( \bar{\varepsilon} = \pi_{(\mu,a)}(\varepsilon) : J_Q^{-1}(\mu, a) \to O_{(\mu,a)} \times \mathbb{R} \times \mathbb{R}^* \).
Then the following two assertions hold:

(i) If the one-form $\gamma : SE(3) \times \mathbb{R} \to T^*(SE(3) \times \mathbb{R})$ is closed with respect to $T\pi_{SE(3)\times\mathbb{R}} : T^2*(SE(3) \times \mathbb{R}) \to T(SE(3) \times \mathbb{R})$, then $\tilde{\gamma}$ is a solution of the Type I of Hamilton-Jacobi equation $\tilde{T}\gamma \cdot \tilde{X}\gamma = X_{\tilde{h}(\mu,a)} \cdot \tilde{\gamma}$.

(ii) The $\varepsilon$ and $\tilde{\varepsilon}$ satisfy the Type II of Hamilton-Jacobi equation $T\bar{\varepsilon} \cdot \bar{X}\varepsilon = X_{\bar{h}(\mu,a)} \cdot \bar{\varepsilon}$, if and only if they satisfy the equation $T\bar{\varepsilon} \cdot (X_{\bar{h}(\mu,a)} \cdot \bar{\varepsilon}) = T\bar{\lambda} \cdot \bar{X} \cdot \varepsilon$. ■

Remark 4.6 When the rigid spacecraft does not carry any internal rotor, in this case the configuration space is $Q = G = SE(3)$, the motion of rigid spacecraft is just the rotation motion with drift of a rigid body, the above $R_p$-reduced controlled rigid spacecraft-rotor system is just the Marsden-Weinstein reduced heavy top system, that is, 3-tuple $(O(\mu,a), \omega_{O(\mu,a)}, h_{O(\mu,a)})$, where $O(\mu,a) \subset \mathfrak{se}^*(3)$ is the co-adjoint orbit, $\omega_{O(\mu,a)}$ is orbit symplectic form on $O(\mu,a)$, which is induced by the heavy top Lie-Poisson bracket on $\mathfrak{se}^*(3)$, $h_{O(\mu,a)}(\Pi, \Gamma) \cdot \pi_{O(\mu,a)} = H(A, c, \Pi, \Gamma)|_{O(\mu,a)}$. From the above Theorem 4.5 we can obtain the Proposition 5.5 in Wang [23], that is, we give the two types of Lie-Poisson Hamilton-Jacobi equation for the Marsden-Weinstein reduced heavy top system $(O(\mu,a), \omega_{O(\mu,a)}, h_{O(\mu,a)})$. See Marsden and Ratiu [14], Ge and Marsden [4], and Wang [23].

It is worthy of noting that, for the controlled rigid spacecraft-rotor system $(T^\ast Q, SE(3), \omega_Q, H, u)$ with the $R_p$-reduced controlled rigid spacecraft-rotor system $(O(\mu,a) \times \mathbb{R} \times \mathbb{R}^\ast, \omega_{O(\mu,a)}|_{\mathbb{R} \times \mathbb{R}^\ast}, h_{O(\mu,a)}, u(\mu,a))$, we know that the Hamiltonian vector fields $X_H$ and $X_{h_{O(\mu,a)}}$ for the corresponding Hamiltonian system $(T^\ast Q, SE(3), \omega_Q, H)$ and its $R_p$-reduced system $(O(\mu,a) \times \mathbb{R} \times \mathbb{R}^\ast, \omega_{O(\mu,a)}|_{\mathbb{R} \times \mathbb{R}^\ast}, h_{O(\mu,a)}, u(\mu,a))$, are $\pi(\mu,a)$-related, that is, $X_{h(\mu,a)} \cdot \pi(\mu,a) = T\pi(\mu,a) \cdot X_H \cdot \iota(\mu,a)$. By using the similar way in proof Theorem 4.4, then we can prove the following Theorem 4.7, which states the relationship between the solutions of Type II of Hamilton-Jacobi equations and the regular point reduction.

Theorem 4.7 In the case of non-coincident centers of buoyancy and gravity, for the controlled rigid spacecraft-rotor system $(T^\ast Q, SE(3), \omega_Q, H, u)$ with the $R_p$-reduced controlled rigid spacecraft-rotor system $(O(\mu,a) \times \mathbb{R} \times \mathbb{R}^\ast, \omega_{O(\mu,a)}|_{\mathbb{R} \times \mathbb{R}^\ast}, h_{O(\mu,a)}, u(\mu,a))$, assume that $\gamma : SE(3) \times \mathbb{R} \to T^*(SE(3) \times \mathbb{R})$ is an one-form on $SE(3) \times \mathbb{R}$, and $\varepsilon : T^*(SE(3) \times \mathbb{R}) \to T^*(SE(3) \times \mathbb{R})$ is a $SE(3)(\mu,a)$-invariant symplectic map, $\tilde{\varepsilon} = \pi_{O(\mu,a)}(\varepsilon) : J_{O(\mu,a)}^{-1}(\mu,a) \to O(\mu,a) \times \mathbb{R} \times \mathbb{R}^\ast$. Under the hypotheses and notations of Theorem 4.5, then we have that $\varepsilon$ is a solution of the Type II of Hamilton-Jacobi equation $T\gamma \cdot \bar{X}\varepsilon = X_H \cdot \varepsilon$, for the regular point reducible controlled rigid spacecraft-rotor system $(T^\ast Q, SE(3), \omega_Q, H, u)$, if and only if $\varepsilon$ and $\tilde{\varepsilon}$ satisfy the Type II of Hamilton-Jacobi equation $T\tilde{\gamma} \cdot \tilde{X}\varepsilon = X_{\tilde{h}(\mu,a)} \cdot \tilde{\varepsilon}$, for the $R_p$-reduced controlled rigid spacecraft-rotor system $(O(\mu,a) \times \mathbb{R} \times \mathbb{R}^\ast, \omega_{O(\mu,a)}|_{\mathbb{R} \times \mathbb{R}^\ast}, h_{O(\mu,a)}, u(\mu,a))$.

The theory of controlled mechanical system is a very important subject, following the theoretical and applied development of geometric mechanics, a lot of important problems about this subject are being explored and studied. In this paper, we reveal the deeply internal relationships of the geometrical structures of phase spaces, the dynamical vector fields and controls of the controlled rigid spacecraft-rotor system. It is worthy of noting that, in the cases of coincident and non-coincident centers of buoyancy and gravity, the motions of the controlled rigid spacecraft-rotor system are different, and the configuration spaces, the Hamiltonian functions, the actions of Lie group, the $R_p$-reduced symplectic forms and the $R_p$-reduced systems of the controlled rigid spacecraft-rotor system are also different. But, the two types of Hamilton-Jacobi equations given by calculation in detail are same, that is, the internal rules are same. It is the key thought of the researches of geometrical mechanics of the professor Jerrold E. Marsden to explore and study the deeply internal relationship between the geometrical structure of phase
space and the dynamical vector field of a mechanical system. It is also our goal of pursuing and inheriting.

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