The existence of light-like homogeneous geodesics in homogeneous Lorentzian manifolds

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Abstract

In previous papers, a fundamental affine method for studying homogeneous geodesics was developed. Using this method and elementary differential topology it was proved that any homogeneous affine manifold and in particular any homogeneous pseudo-Riemannian manifold admits a homogeneous geodesic through arbitrary point. In the present paper this affine method is refined and adapted to the pseudo-Riemannian case. Using this method and elementary topology it is proved that any homogeneous Lorentzian manifold of even dimension admits a light-like homogeneous geodesic. The method is illustrated in detail with an example of the Lie group of dimension 3 with an invariant metric, which does not admit any light-like homogeneous geodesic.

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1 Introduction

Let $M$ be a pseudo-Riemannian manifold. If there is a connected Lie group $G \subset I_0(M)$ which acts transitively on $M$ as a group of isometries, then $M$ is called a homogeneous pseudo-Riemannian manifold. It can be naturally identified with the pseudo-Riemannian homogeneous space $(G/H, g)$, where $H$ is the isotropy group of the origin $p \in M$.

If the metric $g$ is positive definite, then $(G/H, g)$ is always a reductive homogeneous space: We denote by $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebras of $G$ and $H$ respectively and consider the adjoint representation $\text{Ad}: H \times \mathfrak{g} \rightarrow \mathfrak{g}$ of $H$ on $\mathfrak{g}$. There exists the reductive decomposition of the form $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ where $\mathfrak{m} \subset \mathfrak{g}$ is a vector subspace such that $\text{Ad}(H)(\mathfrak{m}) \subset \mathfrak{m}$. For a fixed reductive decomposition $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ there is the natural identification of $\mathfrak{m} \subset \mathfrak{g} = T_eG$ with the tangent space $T_pM$ via the projection $\pi: G \rightarrow G/H = M$. Using this natural identification and the scalar product $g_p$ on $T_pM$, we obtain the invariant scalar product $(\cdot, \cdot)$ on $\mathfrak{m}$.

If the metric $g$ is indefinite, the reductive decomposition may not exist (see for instance [7] or [8] for examples of nonreductive pseudo-Riemannian homogeneous spaces). In such a case, we can study the manifold $M$ using a more fundamental affine method, which was proposed in [6] and [4]. It is based on the well known fact that homogeneous pseudo-Riemannian manifold $M$ with the origin $p$ admits $n = \dim M$ Killing vector fields which are linearly independent at each point of some neighbourhood of $p$. 
A geodesic $\gamma(s)$ through the point $p$ is homogeneous if it is an orbit of a one-parameter group of isometries. More explicitly, if $s$ is an affine parameter and $\gamma(s)$ is defined in an open interval $J$, there exists a diffeomorphism $s = \varphi(t)$ between the real line and the open interval $J$ and a nonzero vector $X \in g$ such that $\gamma(\varphi(t)) = \exp(tX)(p)$ for all $t \in \mathbb{R}$. The vector $X$ is called geodesic vector. The diffeomorphism $\varphi(t)$ may be nontrivial only for null curves in a properly pseudo-Riemannian manifold.

In the reductive case, geodesic vectors are characterized by the following geodesic lemma (see [10] for the Riemannian version, [7] for the first formulation in the pseudo-Riemannian case and [5] for the complete mathematical proof).

**Lemma 1** Let $X \in g$. Then the curve $\gamma(t) = \exp(tX)(p)$ is geodesic with respect to some parameter $s$ if and only if

$$\langle[X, Z]_m, X_m \rangle = k\langle X_m, Z \rangle$$

for all $Z \in m$ and for some constant $k \in \mathbb{R}$. If $k = 0$, then $t$ is an affine parameter for this geodesic. If $k \neq 0$, then $s = e^{-kt}$ is an affine parameter for the geodesic. The second case can occur only if the curve $\gamma(t)$ is a null curve in a properly pseudo-Riemannian space.

In the paper [9], it was proved that any homogeneous Riemannian manifold admits a homogeneous geodesic through the origin. The generalization to the pseudo-Riemannian (reductive and nonreductive) case was obtained in [3] in the framework of a more general result, which says that any homogeneous affine manifold $(M, \nabla)$ admits a homogeneous geodesic through the origin. Here the affine method from [6] and [4], based on the study of integral curves of Killing vector fields, was used. The proof is using differential topology, namely the degree of a smooth mapping $S^n \to S^n$ without fixed points.

A homogeneous pseudo-Riemannian manifold all of whose geodesics are homogeneous is called a pseudo-Riemannian *g.o. manifold* or *g.o. space*. Their analogues with noncompact isotropy group are *almost g.o. spaces*. For many results and further references on homogeneous geodesics in the reductive case see for example the survey paper [2].

In pseudo-Riemannian geometry, null homogeneous geodesics are of particular interest. In [7] and [11], plane-wave limits (Penrose limits) of homogeneous spacetimes along light-like homogeneous geodesics were studied. However, it was not known whether any homogeneous pseudo-Riemannian or Lorentzian manifold admits a null homogeneous geodesic.

In [11], an example of a 3-dimensional Lie group with an invariant Lorentzian metric which does not admit light-like homogeneous geodesic was described. Here the standard geodesic lemma was used, because the example is reductive.

In the present paper, the affine method used in [3], [4] and [6] for the study of homogeneous affine manifolds is adapted to the pseudo-Riemannian case. As the main result it is shown that any Lorentzian homogeneous manifold of even dimension admits a light-like homogeneous geodesic through the origin. The calculation is particularly easy in the case of a Lie group $G = M$ with
2 The main result

Let \((M, g)\) be a homogeneous pseudo-Riemannian manifold of dimension \(n\), let \(G\) be a group of isometries acting transitively on \(M\) and let \(p \in M\). Let \(\nabla\) be the induced pseudo-Riemannian connection on \(M\). It is well known that there exist \(n\) Killing vector fields \(K_1, \ldots, K_n\) on \(M\) which are linearly independent at each point of some neighbourhood \(U\) of \(p\). Let \(B = \{K_1(p), \ldots, K_n(p)\}\) be the basis of the tangent space \(T_p M\). Any tangent vector \(X \in T_p M\) has coordinates \((x^1, \ldots, x^n)\) with respect to the basis \(B\) and it determines the Killing vector field \(X^* = x^1 K_1 + \ldots + x^n K_n\) and the integral curve \(\gamma_X\) of \(X^*\) through \(p\). The following Proposition is a standard one.

**Proposition 2** Let \(\phi_X(t)\) be the 1-parameter group of isometries corresponding to the Killing vector field \(X^*\). For all \(t \in \mathbb{R}\), it holds

\[
\phi_X(t)(p) = \gamma_X(t), \quad \phi_X(t)(X^*_p) = X^*_{\gamma_X(t)}.
\]

It is well known that the covariant derivative \(\nabla_X X^*\) depends only on the values of the vector field \(X^*\) along the curve \(\gamma_X(t)\). From the invariance of the metric \(g\) and the connection \(\nabla\) with respect to the group \(G\), we obtain the following:

**Proposition 3** Along the curve \(\gamma_X(t)\), it holds for all \(t \in \mathbb{R}\)

\[
\begin{align*}
g_p(X^*, X^*) &= g_{\gamma_X(t)}(X^*_{\gamma_X(t)}) \cdot X^*_{\gamma_X(t)}, \\
\phi_X(t)_*(\nabla_X X^*|_p) &= \nabla_{X^*} X^*|_{\gamma_X(t)}.
\end{align*}
\]

Now we formulate the crucial feature.

**Proposition 4** Let \((M, g)\) be a homogeneous Lorentzian manifold, \(p \in M\) and \(X \in T_p M\). Then, along the curve \(\gamma_X(t)\), it holds

\[
\nabla_{X^*} X^*|_{\gamma_X(t)} \in (X^*_{\gamma_X(t)})^\perp.
\]

**Proof.** We use the basic property \(\nabla g = 0\) in the form

\[
\nabla_X g(X^*, X^*) = 2 g(\nabla_X X^*, X^*). \tag{1}
\]

According to Proposition \(\Box\) the function \(g(X^*, X^*)\) is constant along \(\gamma_X(t)\). Hence, the left-hand side of the equality \(\Box\) is zero and the right-hand side gives the statement. \(\Box\)

**Theorem 5** Let \((M, g)\) be a homogeneous Lorentzian manifold of even dimension \(n\) and let \(p \in M\). There exist a light-like vector \(X \in T_p M\) such that along the integral curve \(\gamma_X(t)\) of the Killing vector field \(X^*\) it holds

\[
\nabla_{X^*} X^*|_{\gamma_X(t)} = k \cdot X^*_{\gamma_X(t)}, \tag{2}
\]

where \(k \in \mathbb{R}\) is some constant.
Proof. Let us choose the Killing vector fields $K_1, \ldots, K_n$ such that the vectors $K_1(p), \ldots, K_n(p)$ form a pseudo-orthonormal basis of $T_pM$ with $K_n(p)$ timelike. Again, any arithmetic vector $x = (x^1, \ldots, x^n) \in \mathbb{R}^n$ determines the Killing vector field $X^* = \sum_{i=1}^n x^i K_i$. Using the identification of $x$ with $X^*_p$ we identify $\mathbb{R}^n$ with $T_pM$. There is the natural scalar product on $\mathbb{R}^n$ which comes from the scalar product $g_p$ on $T_pM$ and this identification. Let us consider arithmetic vectors of the form $x = (\tilde{x}, 1)$, where $\tilde{x} \in S^{n-2} \subset \mathbb{R}^{n-1}$. For the corresponding vector field $X^*$, we have $g_p(X^*_p, X^*_p) = 0$ and the vectors $\tilde{x} \in S^{n-2}$ determine light-like directions in $\mathbb{R}^n \cong T_pM$.

For each light-like vector $x = (\tilde{x}, 1) \in \mathbb{R}^n \cong T_pM$, we denote $Y_x = \nabla_{X^*} X^* |_p$. With respect to the pseudo-orthonormal basis $B = \{K_1(p), \ldots, K_n(p)\}$, we denote the components of the vector $Y_x$ as $y(x) = (y^1, \ldots, y^n)$. Using Proposition 4 we see that $y(x) \perp x$. We define the new vector $t_x$ as

$$t_x = y(x) - y^n \cdot x.$$

Because $x$ is light-like vector, it holds also $t_x \perp x$. For the components of $t_x$, we have $t_x = (t_x, 0)$, where $t_x \in \mathbb{R}^{n-1}$. We easily see that $t_x \perp \tilde{x}$, with respect to the positive scalar product on $\mathbb{R}^{n-1}$ which is the restriction of the indefinite scalar product on $\mathbb{R}^n$. The assignment $\tilde{x} \mapsto t_x$ defines a smooth tangent vector field on the sphere $S^{n-2}$. If $n$ is even, then according to the well known topological theorems, this vector field must have a zero value. In other words, there exist a vector $\tilde{x} \in S^{n-2}$ such that for the corresponding vector $x = (\tilde{x}, 1)$ it holds $t_x = 0$. For this vector $x$, it holds $y(x) = k \cdot x$ and formula 2 for the corresponding Killing vector field $X^*$ is satisfied at $t = 0$. Using Proposition 6 we obtain the formula for all $t \in \mathbb{R}$. 

Corollary 6 Let $(M, g)$ be a homogeneous Lorentzian manifold of even dimension $n$ and let $p \in M$. There exist a light-like homogeneous geodesic through $p$.

Proof. We consider the vector $X \in T_pM$ which satisfies Theorem 5. The integral curve $\gamma_X(t)$ through $p$ of the corresponding Killing vector field $X^*$ is homogeneous geodesic. 

3 Invariant metric on a Lie group

Let $M = G$ be a Lie group with a left-invariant pseudo-Riemannian metric $g$ acting on itself by left translations and let $p = e$ be the identity. For any tangent vector $X \in T_eM$ and the corresponding Killing vector field $X^*$, we consider the vector function $X^*_\gamma_X(t)$ along the integral curve $\gamma_X(t)$ through $e$. It can be uniquely extended to the left-invariant vector field $L^X$ on $G$. Hence, along $\gamma_X$, we have

$$L^X_{\gamma_X(t)} = X^*_\gamma_X(t). \quad (3)$$

At general points $q \in G$, values of left-invariant vector field $L_X$ do not coincide with the values of the Killing vector field $X^*$, which is right-invariant. However,
as we are interested in calculations along the curve \( \gamma_X(t) \), we can work with respect to the moving frame of left-invariant vector fields and use formula (3).

**Proposition 7** Let \( \{L_1, \ldots, L_n\} \) be a left-invariant moving frame on a Lie group \( G \) with a left-invariant pseudo-Riemannian metric \( g \) and the induced pseudo-Riemannian connection \( \nabla \). Then it holds

\[
\nabla_{L_i} L_j = \sum_{k=1}^{n} \gamma_{ij}^k L_k, \quad i, j = 1, \ldots, n,
\]

where \( \gamma_{ij}^k \) are constants.

**Proof.** It follows from the invariance of the affine connection \( \nabla \).

Now we illustrate the affine method of the previous section with an example of the 3-dimensional Lie group \( E(1, 1) \) with an invariant Lorentzian metric which has no light-like homogeneous geodesic. We choose one of the examples described in the paper [1] by the standard method for reductive pseudo-Riemannian homogeneous manifolds and the geodesic lemma. We construct explicitly the vector field \( \tilde{t}_x \), which has no zero value in this case.

The group \( E(1, 1) \) can be represented by the matrices of the form

\[
\begin{pmatrix}
e^{-w} & 0 & u \\
0 & e^w & v \\
0 & 0 & 1
\end{pmatrix}.
\]

Hence, the manifold \( M = E(1, 1) \) can be identified with the 3-space \( \mathbb{R}^3[u, v, w] \).

The left-invariant vector fields are \( U = e^{-w}\partial_u \), \( V = e^w\partial_v \), \( W = \partial_w \). We choose the new moving frame \( \{E_1, E_2, E_3\} \) given as

\[
E_1 = U - V, \quad E_2 = -W, \quad E_3 = 1/2(U + V).
\]

In this frame, we have the following rules for the Lie bracket

\[
[E_1, E_3] = 0, \quad [E_2, E_1] = 2E_3, \quad [E_2, E_3] = 1/2E_1.
\]

We introduce the pseudo-Riemannian metric \( g \) such that the basis determined by the above frame at any point \( p \in M \) is pseudo-orthonormal basis of \( T_pM \) with \( E_3 \) timelike (we keep the notation from [1] here).

It is straightforward to write down the above metric \( g \) in coordinates in the form

\[
ds^2 = -\frac{1}{4}(3e^{2w}du^2 + 3e^{-2w}dv^2 + 10dudv - 4dw^2)
\]

and to calculate the nonzero Christoffel symbols

\[
\Gamma_{11}^3 = \frac{3}{4}e^{2w}, \quad \Gamma_{13}^1 = -\frac{3}{4}, \quad \Gamma_{13}^2 = \frac{15}{16}e^{2w}, \\
\Gamma_{22}^3 = -\frac{3}{4}e^{-2w}, \quad \Gamma_{23}^2 = \frac{9}{16}, \quad \Gamma_{23}^1 = -\frac{15}{16}e^{-2w}.
\]
However, we can write down the same information in the frame \{E_1, E_2, E_3\}. By definition, we have at any point \(p \in M\)

\[g(E_1, E_1) = g(E_2, E_2) = 1, \quad g(E_3, E_3) = -1, \quad g(E_i, E_j) = 0, \quad i \neq j.\]

By the straightforward calculations, we obtain nonzero covariant derivatives which satisfy Proposition 7:

\[
\begin{align*}
\nabla_{E_1} E_2 &= -\frac{3}{4} E_3, \quad \nabla_{E_1} E_3 = -\frac{3}{4} E_2, \quad \nabla_{E_2} E_3 = \frac{3}{4} E_1, \\
\nabla_{E_2} E_1 &= \frac{3}{4} E_3, \quad \nabla_{E_3} E_1 = -\frac{3}{4} E_2, \quad \nabla_{E_3} E_2 = -\frac{3}{4} E_1.
\end{align*}
\]

(4)

We will perform all calculations in this moving frame, or with respect to the corresponding pseudo-orthonormal basis \(B = \{E_1(e), E_2(e), E_3(e)\}\) of the tangent space \(T_e M \simeq \mathbb{R}^3\) at the origin \(e \in E(1, 1)\). Any arithmetic vector \(x = (x^1, x^2, x^3) \in \mathbb{R}^3\) determines the left-invariant vector field

\[L^x = x^1 E_1 + x^2 E_2 + x^3 E_3.\]

We are interested in light-like vectors \(X \in T_e M\), hence \(x = (\sin(\varphi), \cos(\varphi), 1)\), \(\tilde{x} = (\sin(\varphi), \cos(\varphi)) \in S^1\) for some \(\varphi \in \mathbb{R}\). For the corresponding left-invariant vector field \(L^x\) we calculate using (4) the covariant derivative

\[
\nabla_{L^x} L^x = 2 \cos(\varphi) E_1 - \frac{3}{2} \sin(\varphi) E_2 + \frac{1}{2} \sin(\varphi) \cos(\varphi) E_3,
\]

\[
y(x) = \left(2 \cos(\varphi), -\frac{3}{2} \sin(\varphi), \frac{1}{2} \sin(\varphi) \cos(\varphi)\right).
\]

We see immediately that \(y(x) \perp x\). The projection \(t_x\) is

\[
t_x = y(x) - \frac{1}{2} \sin(\varphi) \cos(\varphi) \cdot x =
\]

\[
= \left(2 \cos(\varphi) - \frac{1}{2} \sin^2(\varphi) \cos(\varphi), -\frac{3}{2} \sin(\varphi) - \frac{1}{2} \sin(\varphi) \cos^2(\varphi), 0\right) = \left[2 - \frac{1}{2} \sin^2(\varphi)\right] \cdot \left(\cos(\varphi), -\sin(\varphi), 0\right).
\]

We see that \(t_x \perp x\) and \(\tilde{t}_x \perp \tilde{x}\). Clearly, \(\tilde{x} \mapsto \tilde{t}_x\) defines the smooth vector field on \(S^1\), which is nonzero everywhere. Hence, there is not any vector \(X \in T_e G\) which satisfies Theorem 5.

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