CONSISTENCY AND ASYMPTOTIC NORMALITY OF STOCHASTIC EULER SCHEMES FOR
ORDINARY DIFFERENTIAL EQUATIONS

JOHANNES T. N. KREBS

Abstract. General stochastic Euler schemes for ordinary differential equations are studied. We give proofs on the consistency, the rate of convergence and the asymptotic normality of these procedures.

1. Introduction

We study the consistency and asymptotic normality of stochastic Euler schemes which are designed to approximate ordinary differential equations. Euler schemes are often used to simulate stochastic differential equations. Fierro and Torres [2007] consider a special kind of Euler approximation for a given ODE. In this paper, we generalize the idea: Let there be given the ODE system \( \dot{x} = f(t, x) \), \( x(0) = x_0 \in \mathbb{R}^d \), \( t \in [0, T] \), \( 0 < T < \infty \). Then we approximate the solution \( x \) on a partition \( \pi^N \) of \([0, T]\) with a stochastic Euler scheme that is based on random variables \( \tilde{F}_k^N \) instead of \( F \). This approach can be useful in applications where one aims at approximating the trajectory of such a solution \( x \) for a function \( F \) which is costly to evaluate, for instance, in the case where \( F \) is the sum of \( (\text{finitely many single functions} \ f_i, i \in I, i.e. \ F = \sum_i f_i \). The paper is organized as follows: In Section 2 we introduce the basic notions and regularity conditions of the model. In Section 3 we give consistency results for our general Euler scheme. We state results on the asymptotic normality of the procedure in Section 4. Appendix A contains some background material.

2. Preliminaries

We denote for \( p \geq 1 \) by \( \| \cdot \|_p \) the \( p \)-norms on the \( d \)-dimensional Euclidean space. Let \( T \in \mathbb{R}_+ \) be a finite time horizon and let \( F = (F_1, \ldots, F_d) : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d \) be a continuous vector valued function. \( F \) fulfills the following growth conditions w.r.t. the first and second coordinate for \( s, t \in [0, T] \) and for \( x, y \in \mathbb{R}^d \):

\[
\| F(s, x) - F(t, y) \| \leq K_1 (1 + \| x \|_1) |s - t|, \quad (2.1)
\]

\[
\| F(s, x) - F(t, y) \| \leq K_2 \| x - y \|, \quad (2.2)
\]

where \( 0 < K_1, K_2 < \infty \) are some positive constants. Let there be given the ODE \( \dot{x} = F(t, x) \) and \( x(0) = x_0 \in \mathbb{R}^d \) on \([0, T] \). Denote the unique global solution of this equation by \( x : [0, T] \to \mathbb{R}^d \), \( x(t) := x(0) + \int_0^t F(s, x(s)) \, ds \). This solution is guaranteed by the global Lipschitz condition (2.2) and is Lipschitz-continuous with a Lipschitz-constant \( 0 < C < \infty \), i.e. \( \| x(s) - x(t) \| \leq C |s - t| \). Next, choose a sequence of partitions, \( \pi^N, N \in \mathbb{N}_+ \), of the interval \([0, T] \) such that \( \pi^N \) consists of the points \( \tau_0^N = 0 < \tau_1^N < \ldots < \tau_{K_N}^N = T \) and such that the mesh of the partition \( \Delta_N := \max_{1 \leq k \leq K_N} \Delta_k^N \) converges to zero as \( N \to \infty \), where \( \Delta_k^N := \tau_k^N - \tau_{k-1}^N \). The stochastic part is introduced via a probability space \((\Omega, \mathcal{A}, \mathbb{P})\) endowed with the following mappings: For \( N \in \mathbb{N}_+ \) and \( k = 1, \ldots, K_N \) the function

\[
\tilde{F}^N_k = (\tilde{F}^N_{k,1}, \ldots, \tilde{F}^N_{k,d}) : [0, T] \times \mathbb{R}^d \times \Omega \to \mathbb{R}^d
\]

is measurable \( \mathcal{B}([0, T] \times \mathbb{R}^d) \otimes \mathcal{A} \otimes \mathcal{B}(\mathbb{R}^d) \) and is Lipschitz-continuous w.r.t. the second coordinate with the same Lipschitz constant as \( F \). Furthermore, for any selection of time-space coordinates \((t_1, y_1), \ldots, (t_{K_N}, y_{K_N}) \in [0, T] \times \mathbb{R}^d \) the random variables \( \tilde{F}^N_k(t_1, y_1), \ldots, \tilde{F}^N_k(t_{K_N}, y_{K_N}) \) are independent and each \( \tilde{F}^N_k \) is an unbiased estimator of \( F \) in the sense that \( \mathbb{E}[\tilde{F}^N_k(t, y)] = F(t, y) \) for \((t, y) \in [0, T] \times \mathbb{R}^d \). In addition, we assume that there exists a constant...
0 < K_3 < \infty such that for each \( t \in [0, T] \), \( a = 1, \ldots, d \), \( k = 1, \ldots, K_N \) and \( N \in \mathbb{N}_+ \), the variance of the approximation is bounded as \( \text{Var} \left( \hat{F}_k^N \left( t, x(t) \right) \right) \leq K_3 \). We generate for each \( N \in \mathbb{N}_+ \), a stochastic sequence \( \hat{x}_N = \{ \hat{x}_N \left( t_i^N \right) : i = 0, \ldots, K_N \} \) according to the rule
\[
\hat{x}_N \left( t_i^N \right) := x(0) \in \mathbb{R}^d \text{ and } \hat{x}_N \left( t_i^N \right) := \hat{x}_N \left( t_i^N \right) + \Delta_i^N \hat{F}_k^N \left( t_i^N, \hat{x}_N \left( t_{i-1}^N \right) \right) \text{ for } i = 1, \ldots, K_N.
\] (2.3)

We pass from this sequence \( \{ \hat{x}_N \left( t_i^N \right) : i = 0, \ldots, K_N \} \) to a right-continuous process which we denote again by \( \hat{x}_N \), namely, we define
\[
\hat{x}_N(t) := \hat{x}_N \left( \hat{t} \right) \text{ for } t \in \left[ t_i^N, t_{i+1}^N \right) \text{ for } i = 0, \ldots, K_N - 1 \text{ and } \hat{x}_N(T) = \hat{x}_N \left( \hat{t}_{K_N} \right).
\] (2.4)

In the following, when speaking of \( \hat{x}_N \), we shall always refer to this càdlàg process. Moreover, \( \{ F^N(\cdot) : N \in \mathbb{N}_+ \} \) is a sequence of filtrations on \( (\Omega, \mathcal{A}, \mathbb{P}) \) such that for each \( N \in \mathbb{N}_+ \) the filtration \( F^N(\cdot) \) is the natural and right-continuous filtration of the process \( \hat{x}_N \) from equation (2.4).

3. Consistency and Rate of Convergence

We come to the first main result of this paper, this is the convergence in mean of the processes \( \hat{x}_N, N \in \mathbb{N}_+ \), namely

**Theorem 3.1** \((L^2\text{-convergence of } \hat{x}_N \text{ to } x)\). Let the sequence of stochastic processes \( (\hat{x}_N : N \in \mathbb{N}_+) \) be defined in equations (2.3) and (2.4). Let \( x \) be the unique global solution to the ordinary differential equation. Then, there exists a constant \( 0 < B < \infty \) such that
\[
\left\| \sup_{t \in [0,T]} \left\| \hat{x}_N(t) - x(t) \right\| \right\|_{L^2(\mathbb{P})} \leq B \sqrt{\Delta_N}.
\]

**Proof.** Throughout the proof we shall write \( \| \cdot \| \) for the Euclidean 1-norm on \( \mathbb{R}^d \). Furthermore, we set \( x_k^N := \hat{x}_N \left( t_k^N \right) \) and \( \hat{x}_k^N := \hat{x}_N \left( t_k^N \right) \) for \( k = 0, \ldots, K_N \). First, we consider \( \hat{x}_N \) at the points \( t_k^N, k = 1, \ldots, K_n \). We derive for the difference \( \hat{x}_k^N - x_k^N \) at each \( k = 1, \ldots, K_N \) the equation
\[
\hat{x}_k^N - x_k^N = \hat{x}_{k-1}^N - x_{k-1}^N + \left\{ F_k^N \left( t_{k-1}^N, \hat{x}_{k-1}^N \right) - F_k^N \left( t_{k-1}^N, x_{k-1}^N \right) \right\} \Delta_k^N + \left\{ F_k^N \left( x_{k-1}^N, x_{k-1}^N \right) - F \left( t_{k-1}^N, x_{k-1}^N \right) \right\} \Delta_k^N
\]
\[
+ \left\{ F \left( x_{k-1}^N, x_{k-1}^N \right) \Delta_k^N - \int_{t_{k-1}^N}^{t_k^N} F(s, x(s)) \, ds \right\}.
\]

Successive iteration down to \( t_0^N \) yields
\[
\hat{x}_k^N - x_k^N = \sum_{j=1}^{k} \left\{ F_j^N \left( t_{j-1}^N, \hat{x}_{j-1}^N \right) - F_j^N \left( t_{j-1}^N, x_{j-1}^N \right) \right\} \Delta_j^N + \sum_{j=1}^{k} \left\{ F_j^N \left( x_{j-1}^N, x_{j-1}^N \right) - F \left( t_{j-1}^N, x_{j-1}^N \right) \right\} \Delta_j^N
\]
\[
+ \sum_{j=1}^{k} \left\{ F \left( x_{j-1}^N, x_{j-1}^N \right) \Delta_j^N - \int_{s_{j-1}^N}^{s_j^N} F(s, x(s)) \, ds \right\}.
\] (3.1)

By the growth condition w.r.t. the time coordinate and the Lipschitz condition w.r.t. the space coordinate, we have for the last term in (3.1)
\[
\left\| \int_{s_{j-1}}^{s_j} F \left( r_{j-1}^N, x_{j-1} \right) - F(s, x(s)) \, ds \right\| \leq \left\{ K_1 \left( 1 + \sup_{t \in [0,T]} \| x(t) \| \right) + K_2 C \right\} \left( \Delta_j^N \right)^2.
\]

We put for short \( L := K_1 \left( 1 + \sup_{t \in [0,T]} \| x(t) \| \right) + K_2 C \). We can estimate the left-hand side of (3.1) using the Lipschitz condition on the stochastic approximations \( F_k^N \) of \( F \) to arrive at the following bound for \( \| \hat{x}_N^N - x_N^N \| \):
\[
\| \hat{x}_N^N - x_N^N \| \leq K_2 \sum_{j=1}^{k} \| \hat{x}_{j-1}^N - x_{j-1}^N \| \Delta_j^N + \sum_{j=1}^{k} \left\{ F_j^N \left( x_{j-1}^N, x_{j-1}^N \right) - F \left( t_{j-1}^N, x_{j-1}^N \right) \right\} \Delta_j^N + \sum_{j=1}^{k} L \left( \Delta_j^N \right)^2.
\] (3.2)

We apply the discrete Gronwall inequality from Lemma [A1] from Appendix [A] to the bound given in (3.2). We get
\[
\| \hat{x}_N^N - x_N^N \| \leq \left\{ \sum_{j=1}^{k} \left( F_j^N \left( t_{j-1}^N, x_{j-1}^N \right) - F \left( t_{j-1}^N, x_{j-1}^N \right) \right) \Delta_j^N \right\} + \sum_{j=1}^{k} L \left( \Delta_j^N \right)^2.
\]
\[ + \sum_{j=1}^{k-1} \left\{ \left\| \sum_{i=1}^{j} \left( F_N \left( \tau_N^{(i-1)}, x_N^{(i)} \right) - F \left( t_N^{(i-1)}, x_N^{(i)} \right) \right) \Delta N_i \right\| + L \sum_{j=1}^{k} \left( \Delta N_j \right) \exp \left( \sum_{j=1}^{k-1} K_2 \Delta N_j \right) \right\} \leq M_1 \Delta N + \left\| \sum_{i=1}^{k} \left( F_N \left( t_N^{(i)}, x_N^{(i)} \right) - F \left( t_N^{(i)}, x_N^{(i)} \right) \right) \Delta N_i \right\| \]
\[ + M_2 \sum_{j=1}^{k-1} \left\| \sum_{i=1}^{j} \left( F_N \left( \tau_N^{(i-1)}, x_N^{(i)} \right) - F \left( t_N^{(i-1)}, x_N^{(i)} \right) \right) \Delta N_i \right\| \Delta N_j \right\}, \quad (3.3) \]

where the constants \( M_1 \) and \( M_2 \) are given by \( M_1 := LT + K_2 LT^2 \exp(K_2 T) \) and \( M_2 := K_2 \exp(K_2 T) \). In particular, for \( A := 2M_1 \) and \( B := 2(1 + M_2 T)^2 \) it holds good that

\[ \sup_{1 \leq j \leq K_N} \left\| \bar{F}^N \left( t, x \right) \right\| \leq A \left( \Delta N \right)^2 + B \sup_{1 \leq j \leq K_N} \left\| \sum_{i=1}^{j} \left( F_N \left( \tau_N^{(i)}, x_N^{(i)} \right) - F \left( t_N^{(i)}, x_N^{(i)} \right) \right) \Delta N_i \right\|^2. \quad (3.4) \]

Next, we use the independence assumptions on the \( \bar{F}^N \) and the assumption that in each point \( \mathbb{E} \left[ \bar{F}^N(t, x) \right] = F(t, x) \). We show that the discrete process

\[ \left\{ t^N_1, \ldots, t^N_{K_N} \right\} \Rightarrow t^* \mapsto \left\| \sum_{i=1}^{K_N} 1_{\left[ \left[ t^N_i \right] \right]} \left( F_N \left( \tau_N^{(i)}, x \left( t^N_i \right) \right) - F \left( t_N^{(i)}, x \left( t^N_i \right) \right) \right) \Delta N_i \right\| \]

constitutes a submartingale \( \mathbb{E}[F_N(\cdot)] \). Indeed, we have for any two \( 1 \leq j \leq K_N \)

\[ \mathbb{E} \left[ \left\| \sum_{i=1}^{j} \left( F_N \left( \tau_N^{(i)}, x_N^{(i)} \right) - F \left( t_N^{(i)}, x_N^{(i)} \right) \right) \Delta N_i \right\| \mathbb{E} \left[ F_N \left( t^N_j \right) \right] \right] = \mathbb{E} \left[ \sum_{i=1}^{j} \sum_{a=1}^{d} \left( F^N_{\tau a} \left( \tau_N^{(i)}, x \left( \tau_N^{(i)} \right) \right) - F_a \left( t_N^{(i)}, x \left( t_N^{(i)} \right) \right) \right) \Delta N_i \right] \left\| \mathbb{E} \left[ F_N \left( t^N_j \right) \right] \right\| \]

Due to the independence assumption on the stochastic family \( \bar{F}^N \), the 1-dimensional processes

\[ \left\{ t^N_1, \ldots, t^N_{K_N} \right\} \Rightarrow t^* \mapsto \sum_{i=1}^{K_N} 1_{\left[ \left[ t^N_i \right] \right]} \left( F^N_{\tau a} \left( \tau_N^{(i)}, x \left( \tau_N^{(i)} \right) \right) - F_a \left( t_N^{(i)}, x \left( t_N^{(i)} \right) \right) \right) \Delta N_i \]

are submartingales \( \mathbb{E}[F_N(\cdot)] \) for each \( a = 1, \ldots, d \). Summation over the index \( a \) proves the statement about the submartingale property. This puts us in position to use Doob’s \( L^p \)-Inequality for equation \( (3.4) \) with \( p = 2 \) applied to the above submartingale

\[ \mathbb{E} \left[ \sup_{1 \leq j \leq K_N} \left\| \bar{F}^N_j - x^N_j \right\|^2 \right] \leq A \left( \Delta N \right)^2 + B \mathbb{E} \left[ \sup_{1 \leq j \leq K_N} \left\| \sum_{i=1}^{j} \left( F_N \left( \tau_N^{(i)}, x_N^{(i)} \right) - F \left( t_N^{(i)}, x_N^{(i)} \right) \right) \Delta N_i \right\|^2 \right] \quad (3.5) \]
\[ \leq A \left( \Delta N \right)^2 + 4B \mathbb{E} \left[ \left\| \sum_{i=1}^{K_N} \left( F_N \left( \tau_N^{(i)}, x_N^{(i)} \right) - F \left( t_N^{(i)}, x_N^{(i)} \right) \right) \Delta N_i \right\|^2 \right] \quad (3.6) \]
\[ \leq A \left( \Delta N \right)^2 + 4Bd \sum_{a=1}^{d} \mathbb{E} \left[ \sum_{i=1}^{K_N} \left( F^N_{\tau a} \left( \tau_N^{(i)}, x_N^{(i)} \right) - F_a \left( t_N^{(i)}, x_N^{(i)} \right) \right) \Delta N_i \right] \quad (3.7) \]
\[ \leq A \left( \Delta N \right)^2 + 4Bd \sum_{a=1}^{d} \sum_{i=1}^{K_N} \left( \Delta N_i \right)^2 \mathbb{E} \left[ F^N_{\tau a} \left( \tau_N^{(i)}, x_N^{(i)} \right) \right] \quad (3.8) \]

The first inequality \( (3.5) \) follows immediately from inequality \( (3.4) \). Inequality \( (3.6) \) stems from Doob’s \( L^p \)-inequality. Equality \( (3.7) \) follows from the indepence of the random variables \( F^N_{\tau a} \left( \tau_N^{(i)}, x \left( \tau_N^{(i)} \right) \right) \) \( \ldots \) \( F^N_{\tau a} \left( \tau_N^{(K_N)}, x \left( \tau_N^{(K_N)} \right) \right) \) \( \ldots \)

The last inequality \( (3.8) \) follows from the condition that the variance of the approximation is uniformly bounded. We are now in position to consider the processes \( \hat{x}^N \) over the entire interval \( [0, T] \). Remember that \( \hat{x}^N(t) = \hat{x}^N \left( \tau_N^{(k)} \right) \) for \( t \in \left[ \tau_N^{(k-1)}, \tau_N^{(k)} \right] \) and \( \hat{x}^N(T) = \hat{x}^N \left( \tau_N^{(K_N)} \right) \), thus,

\[ \sup_{t \in [0,T]} \left\| \hat{x}^N(t) - x(t) \right\|^2 \leq 2 \left\{ \sup_{1 \leq k \leq K_N} \left\| \hat{x}^N \left( \tau_N^{(k)} \right) - x \left( \tau_N^{(k)} \right) \right\|^2 + C^2 \left( \Delta N \right)^2 \right\}. \quad (3.9) \]
All in all, we find that \( \| \sup_{t \in [0,T]} \| \hat{X}^N(t) - x(t) \| \|_{L^2(P)} \leq \text{const } (\Delta N)^{1/2} \) for a sequence of partitions having a mesh \( \Delta N \) which converges to zero. This finishes the proof.

In addition to the \( L^2(P) \)-convergence of the process \( \hat{s}^N \), we can state another result on the pathwise convergence for a special choice of the partitioning sequence \( \{\pi^N : N = 1, \ldots, \infty \} \). It is an application of Kolmogorov’s maximal inequality and follows immediately from the inequality from equation (3.4). We have the following theorem

**Theorem 3.2** (a.s.-convergence of \( \hat{s}^N \)). Let \( \{\pi^N : N = 1, \ldots, \infty \} \) be a partitioning sequence of the interval \([0, T]\) such that \( \sum_{N=1}^{\infty} \Delta N < \infty \). Then \( \sup_{t \in [0,T]} \| \hat{s}^N(t) - x(t) \| \) converges to zero almost surely.

**Proof.** Write again \( X^N_k := x \left( t^N_k \right) \) and \( \hat{X}^N_k := \hat{s}^N \left( t^N_k \right) \) for \( k = 0, \ldots, K_N \). Consider equation (3.4), the maximum on the right-hand side can be bounded as

\[
\max_{1 \leq j \leq K_N} \left\| \sum_{i=1}^{j} \left( \hat{F}_{i,a}^N \left( t^N_{i-1}, x^N_{i-1} \right) - F_a \left( t^N_{i-1}, x^N_{i-1} \right) \right) \right\|_1 \leq \sum_{a=1}^{d} \max_{1 \leq j \leq K_N} \left| \sum_{i=1}^{j} \left( \hat{F}_{i,a}^N \left( t^N_{i-1}, x^N_{i-1} \right) - F_a \left( t^N_{i-1}, x^N_{i-1} \right) \right) \right|^N \Delta_N.
\]

We show that \( \max_{1 \leq j \leq K_N} \left| \sum_{i=1}^{j} \left( \hat{F}_{i,a}^N \left( t^N_{i-1}, x^N_{i-1} \right) - F_a \left( t^N_{i-1}, x^N_{i-1} \right) \right) \right|^N \Delta_N \to 0 \) almost surely for each coordinate \( a = 1, \ldots, d \). An application of Kolmogorov’s maximal inequality yields for \( \varepsilon > 0 \) that

\[
\mathbb{P} \left\{ \max_{1 \leq j \leq K_N} \left| \sum_{i=1}^{j} \left( \hat{F}_{i,a}^N \left( t^N_{i-1}, x^N_{i-1} \right) - F_a \left( t^N_{i-1}, x^N_{i-1} \right) \right) \right| > \varepsilon \right\} \leq \frac{1}{\varepsilon^2} \sum_{a=1}^{d} \text{Var} \left( \sum_{i=1}^{K_N} \hat{F}_{i,a}^N \left( t^N_{j-1}, x^N_{j-1} \right) \right) \Delta_N \leq K_N T \varepsilon^{-2} \Delta N.
\]

Hence, we conclude the a.s.-convergence from the first Borel-Cantelli Lemma by the convergence assumption on the meshes of partitioning sequence \( \{\pi^N : N = 1, \ldots, \infty \} \). The conclusion follows immediately by combining inequality (3.4) and (3.9), as well as the fact that almost sure convergence is unaffected by continuous transformations.

\[\Box\]

4. Asymptotic Normality of Stochastic Approximation Procedures

In this section we prove the asymptotic normality of the stochastic Euler schemes for ODE approximations.

**Theorem 4.1.** Let \( \{\pi^N : N \in \mathbb{N}_1\} \) be the sequence of dyadic partitions of \([0, T]\), i.e. \( \pi^N = \{Tk/2^N : k = 0, 1, \ldots, 2^N \} \). Let \( F = (F_1, \ldots, F_d) \) fulfill the regularity conditions from (2.1) and (2.2). Additionally, let each component of \( F \) be continuously differentiable w.r.t. the space coordinate, i.e. \( (t, x) \mapsto \nabla_x F (t, x) \) is continuous for \( i = 1, \ldots, d \).

Furthermore, let the stochastic approximations \( \hat{F}^N_k \) be regular in that for all \( N \in \mathbb{N}_1 \) and \( k = 1, \ldots, 2^N \) the \( \hat{F}^N_k \) are independent copies of \( \hat{F} \) where the time-space process \( \hat{F} : [0, T] \times \mathbb{R}^d \times \Omega \to \mathbb{R}^d \) is Lipschitz-continuous in the space coordinate with the Lipschitz constant \( K_2 \) as well as continuous in the time coordinate and fulfills the integrability condition

\[
\mathbb{E} \left\{ \sup_{t \in [0,T]} \left\| \hat{F}(t, x(t)) \cdot \hat{F}(t, x(t))' \right\|_1^2 \right\} < \infty.
\]

Then for each \( t \in [0, T] \) in the limit \( \lim_{N \to \infty} \left( \Delta N \right) \left( \Delta N \right) \hat{s}^N(t) \sim N(0, \Sigma(t)) \), where the function \( \Sigma : [0, T] \to \mathbb{R}^{d \times d} \) is defined as

\[
\Sigma(t) = \int_0^t P(s, t) \mathbb{E} \left[ \left( \hat{F}(s, x(s)) - F(s, x(s)) \right) \cdot \left( \hat{F}(s, x(s)) - F(s, x(s)) \right)' \right] P(s, t) ds,
\]

and \( P \) is the uniform limit of the function \( P^N \) on \([0, T]^2\) given by \([0, T]^2 \ni P^N (s, t) = \prod_{s \leq t \leq 1} \left( I + \Delta N \nabla_x F \left( t^N_{j-1}, x \left( t^N_{j-1} \right) \right) \right) \).

**Proof.** We write \( \| \cdot \| \) throughout the proof for the 2-norm; since any two norms on the Euclidean space are equivalent, bounds and estimates w.r.t. the 1-norm can be multiplied with the corresponding equivalence constant and are thus valid w.r.t. the 2-norm, too. For a matrix \( A \), denote by \( \| A \| := \sup_{\| x \| \leq 1} \| Ax \| \) the spectral norm of \( A \). We use the abbreviations

\[
Z^N := \left( \Delta N \right)^{1/2} \left( \hat{s}^N - x \right) \text{ as well as } x^N_k := x \left( t^N_k \right) \text{ and } \hat{s}^N_k := \hat{s}^N \left( t^N_k \right).
\]
for simplicity. Choose $t \in [0, T]$ arbitrary but fix, w.l.o.g. $t \in \left[ t^N_k, t^N_{k+1} \right]$, if we add the virtual point $t^N_{2^{k+1}}$ in case that $t = T$. Then

\[
\begin{align*}
\hat{X}^N(t) - x(t) = \hat{X}^N - x^N + \left( \hat{X}^N - x^N + \Delta^N \tilde{F}^N \left( t^N_{k-1}, \hat{X}^N - x^N \right) \right) - \int_{t^N_{k-1}}^{t^N_k} F(s, x(s)) \, ds - \int_{t^N_k}^{t^N_{k+1}} F(s, x(s)) \, ds \\
= \left[ I + \Delta^N \nabla F \left( t^N_{k-1}, \hat{X}^N - x^N \right) \right] \left( \hat{X}^N - x^N \right) + \Delta^N \left[ \tilde{F}^N \left( t^N_{k-1}, \hat{X}^N - x^N \right) - F \left( t^N_{k-1}, \hat{X}^N - x^N \right) \right] \\
+ \Delta^N \left[ \tilde{F}^N \left( t^N_{k-1}, \hat{X}^N - x^N \right) - F \left( t^N_{k-1}, \hat{X}^N - x^N \right) - \nabla F \left( t^N_{k-1}, \hat{X}^N - x^N \right) \right] \\
+ \int_{t^N_{k-1}}^{t^N_k} F \left( t^N_{k-1}, \hat{X}^N - x^N - \Delta^N F \left( t^N_{k-1}, \hat{X}^N - x^N \right) \right) ds - \int_{t^N_k}^{t^N_{k+1}} F(s, x(s)) \, ds.
\end{align*}
\]

We make the following definitions

\[
\begin{align*}
m^N_k &:= \tilde{F}^N \left( t^N_{k-1}, \hat{X}^N - x^N \right) - F \left( t^N_{k-1}, \hat{X}^N - x^N \right), \\
R^N_{1k} &:= \Delta^N \left[ \tilde{F}^N \left( t^N_{k-1}, \hat{X}^N - x^N \right) - F \left( t^N_{k-1}, \hat{X}^N - x^N \right) - \nabla F \left( t^N_{k-1}, \hat{X}^N - x^N \right) \right], \\
R^N_{2k} &:= \Delta^N \left[ \tilde{F}^N \left( t^N_{k-1}, \hat{X}^N - x^N \right) - F \left( t^N_{k-1}, \hat{X}^N - x^N \right) \right], \\
R^N_{3k} &:= \int_{t^N_{k-1}}^{t^N_k} F \left( t^N_{k-1}, \hat{X}^N - x^N - \Delta^N F \left( t^N_{k-1}, \hat{X}^N - x^N \right) \right) ds - \int_{t^N_k}^{t^N_{k+1}} F(s, x(s)) \, ds.
\end{align*}
\]

Set $R^N := R^N_{1k} + R^N_{2k} + R^N_{3k}$, note that $|R^N_k| \leq C \Delta^N$, for a constant $0 < C < \infty$. Thus, we get $Z^N(t) = Z^N(t^N_k) + (\Delta^N)^{-\frac{1}{2}} R^N_k$ and at the partitioning points, we face the following structure

\[
Z^N_k = \left[ I + \Delta^N \nabla F \left( t^N_{k-1}, \hat{X}^N - x^N \right) \right] Z^N_{k-1} + \left( \Delta^N \right)^{\frac{1}{2}} m^N_k + \left( \Delta^N \right)^{-\frac{1}{2}} R^N_k \text{ for } 0 < t^N_k \leq T \text{ and } Z^N(0) = 0.
\]

Consequently, successive iteration yields

\[
Z^N(t) = \sum_{k=0}^{\infty} \left\{ \prod_{j=k}^{\infty} \left( I + \Delta^N \nabla F \left( t^N_{j-1}, \hat{X}^N - x^N \right) \right) \left( \Delta^N \right)^{\frac{1}{2}} m^N_j + \left( \Delta^N \right)^{-\frac{1}{2}} R^N_j \right\}.
\]

In the sequel, we prove that the sum which involves the $m^N_k$ tends to the desired normal distribution, whereas the sum involving the remainder $R^N_k$ tends to zero in probability. Hence, $Z^N(t)$ is asymptotically normally distributed with the same parameters. Consider the first sum, we use the definitions

\[
U_{N,k} := \prod_{j=k}^{\infty} \left( I + \Delta^N \nabla F \left( t^N_{j-1}, x^N \right) \right) \left( \Delta^N \right)^{\frac{1}{2}} m^N_j \text{ for } 0 < t^N_k \leq t
\]

and $U_N := \prod_{k=0}^{\infty} U_{N,k}$. Note that for $N \in \mathbb{N}$, the random variables $U_{N,1}, \ldots, U_{N,2^N}$ are independent. W.l.o.g., assume that $(\Omega, \mathcal{A}, \mathbb{P})$ is endowed with independent normal distributions $Y_{N,k}$ such that $Y_{N,k} \sim \mathcal{N}(0, \text{Cov} [U_{N,k}, U_{N,k}])$ for $k = 1, \ldots, 2^N$. Set $Y_N := \prod_{k=0}^{2^N} Y_{N,k}$. We prove that the difference of the characteristic functions $\varphi_{U_N} - \varphi_{Y_n}$ convergences pointwise to zero: For a fix $\alpha \in \mathbb{R}^d$, we show that $\varphi_{U_N}(\alpha) - \varphi_{Y_n}(\alpha) \to 0$ as $N \to \infty$. Therefore, we use the fundamental inequality

\[
\left\| \varphi_{U_N}(\alpha) - \varphi_{Y_n}(\alpha) \right\| \leq \sum_{k=0}^{2^N} \left\| \varphi_{U_N}(\alpha) - \varphi_{Y_n}(\alpha) \right\|.
\]

An application of Lemma $[A.2]$ yields

\[
\left(4.2\right) \leq 2 \|\alpha\|^2 \sum_{k=0}^{2^N} \mathbb{E} \left[ \left\| U_{N,k} \right\|^2 \right] + \varepsilon \|\alpha\|^2 \sum_{k=0}^{2^N} \mathbb{E} \left[ \left\| Y_{N,k} \right\|^2 \right]
\]

\[
+ 2 \|\alpha\|^2 \sum_{k=0}^{2^N} \mathbb{E} \left[ \left\| Y_{N,k} \right\|^2 \right] + \varepsilon \|\alpha\|^2 \sum_{k=0}^{2^N} \mathbb{E} \left[ \left\| Y_{N,k} \right\|^2 \right]
\]

We show that the first and the third sum of $\left(4.2\right)$ converge to zero as $N \to \infty$ for any $\varepsilon > 0$. This implies that the second and the fourth sum are bounded, and, when multiplied by $\varepsilon$, become small, too. We intend to
bound \( \|U_{N,k}\| \), it is
\[
\|U_{N,k}\| \leq (\Delta^N)^\frac{1}{2} \prod_{j' < j} \left( 1 + \Delta^N \|\nabla_x F(\cdot, x(\cdot))\|_{\infty} \right) \|m^N_{j'}\| \leq (\Delta^N)^\frac{1}{2} \exp \left( T \|\nabla_x F(\cdot, x(\cdot))\|_{\infty} \right) \|m^N_{j}\|. \tag{4.4}
\]

We consider the first sum of (4.3): Using (4.4), we arrive at
\[
\sum_{k=0}^d \mathbb{E} \left[ \|U_{N,k}\|^2 \mathbb{1}_{\{\|U_{N,k}\| \geq \epsilon\}} \right] = \exp \left( T \|\nabla_x F(\cdot, x(\cdot))\|_{\infty} \right) \sum_{k=0}^d \mathbb{E} \left[ \|m^N_{j}\|^2 \mathbb{1}_{\{\|m^N_{j}\| \geq \epsilon \exp \left( -T \|\nabla_x F(\cdot, x(\cdot))\|_{\infty} \right) \} \right] ^\frac{1}{2} \left( \mathbb{1}_{\{\|U_{N,k}\| \geq \epsilon\}} \right) \tag{4.5}
\]

where \( \epsilon := \exp \left( T \|\nabla_x F(\cdot, x(\cdot))\|_{\infty} \right) \). An application of Lebesgue’s dominated convergence theorem yields that (4.5) converges to zero as \( N \) converges to infinity. We obtain for the third sum in equation (4.3)
\[
\sum_{k=0}^d \sum_{j=1}^d \mathbb{E} \left[ \langle \alpha, Y_{N,k} \rangle \mathbb{1}_{\{|\langle \alpha, Y_{N,k} \rangle | \geq \epsilon\}} \right] \leq \|\alpha\|^2 \sum_{k=0}^d \sum_{j=1}^d \mathbb{E} \left[ \|Y_{N,k}^{(j)}\|^2 \right] \mathbb{P} \left( \{\langle \alpha, Y_{N,k} \rangle \geq \epsilon\} \right). \tag{4.6}
\]

Since, the \( d \) elements of the vector \( Y_{N,k} \) are normally distributed, we achieve with the notation \( \Sigma_{N,k}^{(j)} \) for the covariance matrix \( \text{Cov}(U_{N,k}U_{N,k}') \) that \( \sum_{i=1}^d \mathbb{E} \left[ \|Y_{N,k}^{(j)}\|^2 \right] = \sqrt{\Sigma_i^{(j)}} = \sqrt{\text{Var} \left( \|U_{N,k}\|^2 \right)} \leq \text{const} \Delta^N \), with the help of equation (4.4). In addition, since \( \langle \alpha, Y_{N,k} \rangle \sim N(0, \langle \alpha, \Sigma_{N,k}^{(j)} \alpha \rangle) \), we get
\[
\mathbb{P} \left( \{\langle \alpha, Y_{N,k} \rangle \geq \epsilon\} \right) \leq \sqrt{\text{Var} \left( \langle \alpha, Y_{N,k} \rangle \right)} \leq \text{exp} \left( -e^{-2} / 4 \|\alpha\|^2 \|\Sigma_{N,k}^{(j)}\|^{-1} \right),
\]

with the help of a bound given in Chiani et al. [2003]. And \( \|\Sigma_{N,k}^{(j)}\| \leq \mathbb{E} \left[ \|U_{N,k}\|^2 \right] \leq \text{const} \Delta^N \) from equation (4.4). This proves that (4.6) converges to zero as \( N \) tends to infinity. Consequently, \( \varphi_{U_{N}}(\alpha) - \varphi_{Y_{N}}(\alpha) \rightarrow 0 \), for any \( \alpha \in \mathbb{R}^d \).

Clearly \( Y_N \sim N(0, \Sigma^N) \), where \( \Sigma^N = \text{Cov}(U_N, U_N) \) and can be written as
\[
\Sigma^N = \sum_{k=0}^d \sum_{j=1}^d \mathbb{E} \left[ \|m^N_{j}\|^2 \right] \mathbb{1}_{\{|\langle \alpha, Y_{N,k} \rangle | \geq \epsilon\}} \mathbb{E} \left[ \langle \alpha, Y_{N,k} \rangle \right] \mathbb{P} \left( \{\langle \alpha, Y_{N,k} \rangle \geq \epsilon\} \right)
\]

with the notation \( P^N(s, t) = \prod_{j=x}^d \left( I + \Delta^N \nabla_x F \left( \frac{x(t)}{t}, \frac{x(\cdot)}{\cdot} \right) \right) \). Due to the continuity of \( t \mapsto \nabla_x F(t, x(t)) \), we get with the help of Lemma A.3 that \( P^N \) converges uniformly on \([0, T]\) to a continuous matrix valued function \( P \). An application of Lebesgue’s dominated convergence theorem yields that the map which is defined from the factor in the middle of (4.7) as
\[
\Gamma : [0, T] \rightarrow \mathbb{R}^{d \times d} : t \mapsto \mathbb{E} \left[ \left( \frac{\partial}{\partial t} F(t, x(t)) \right) \left( \frac{\partial}{\partial x} F(t, x(t)) \right) \right]
\]
is continuous, thus, \( \Sigma^N \) converges uniformly on \([0, T]\) to \( \int_0^T P(s, t) \Gamma(s) P(s, t') ds \).

All in all, \( Y_N \) converges to \( N(0, \int_0^T P(x, t) \Gamma(s) P(x, t') ds) \) in law. It remains to prove that the summed error terms in (4.1) converge to zero in probability. We start with the first error term. Note that due to the independence, we have for all \( j \neq k \) that \( \mathbb{E} \left[ \left( P^N \left( \frac{x(t)}{t}, \frac{x(\cdot)}{\cdot} \right) \right) \mathbb{1}_{\{|\langle \alpha, Y_{N,k} \rangle | \geq \epsilon\}} \mathbb{E} \left[ \langle \alpha, Y_{N,k} \rangle \right] \mathbb{P} \left( \{\langle \alpha, Y_{N,k} \rangle \geq \epsilon\} \right) \right] = 0 \). Hence, we obtain
\[
\mathbb{E} \left[ \left( \Delta^N \right)^\frac{1}{2} \sum_{k=0}^d \sum_{j' < j} \left( I + \Delta^N \nabla_x F \left( \frac{x(t)}{t}, \frac{x(\cdot)}{\cdot} \right) \right) \right] \leq \Delta^N \exp \left( T \|\nabla_x F(\cdot, x(\cdot))\|_{\infty} \right) \sum_{k=1}^\infty \mathbb{E} \left[ \left( \left( \tilde{F}_N (\frac{x(t)}{t}, \frac{x(\cdot)}{\cdot}) \right) - \left( \tilde{F}_N (\frac{x(t)}{t}, \frac{x(\cdot)}{\cdot}) \right) \right) \right] ^\frac{1}{2}
\]
Lemma A.1 (Discrete Gronwall Inequality). Let \( \{f_k : k \in \mathbb{N}\}, \{g_k : k \in \mathbb{N}\}, \{y_k : k \in \mathbb{N}\} \) be positive sequences in \( \mathbb{R}_{\geq 0} \) which fulfill \( y_n \leq f_n + \sum_{k=0}^{n-1} g_k y_k \) for every \( n \in \mathbb{N} \). Then, we have \( y_n \leq f_n + \sum_{k=0}^{n-1} g_k \prod_{j=k+1}^{n-1} (1 + g_j) \leq f_n + \sum_{k=0}^{n-1} g_k \exp \left( \sum_{j=k+1}^{n-1} g_j \right) \) for each \( n \in \mathbb{N} \). The first inequality is actually sharp.

Lemma A.2 (Estimates for characteristic functions). Let \( X \) be a d-dimensional real random variable on \( (\Omega, \mathcal{A}, \mathbb{P}) \) and let \( \mathcal{G} \subseteq \mathcal{A} \) be a sub-\( \sigma \)-algebra of \( \mathcal{A} \). Define \( \mu = \mathbb{E}[X \mid \mathcal{G}] \) and \( \Sigma := \mathbb{E}[XX' \mid \mathcal{G}] \). Then for the conditional characteristic function of \( X \) w.r.t. \( \mathcal{G} \), \( \varphi_X |\mathcal{G} \), it holds that for each \( t \in \mathbb{R}^d \)

\[
\varphi_X |\mathcal{G}(t) = \mathbb{E} \left[ e^{i\langle t, \mu \rangle} \big| \mathcal{G} \right] = \left( 1 - \frac{1}{2} \langle t, \Sigma t \rangle \right) e^{i\langle t, \mu \rangle} + o(t),
\]

where the remainder can be bounded as follows

\[
|o(t)| \leq 2 \|t\|^2 \mathbb{E} \left[ \|X\|^2 \mathbb{1}_{\|X\| > \|t\|/2} \big| \mathcal{G} \right] + \varepsilon \|t\|^3 \mathbb{E} \left[ \|X\|^2 1_{\|t\|/\|X\| \leq 1} \big| \mathcal{G} \right] .
\]

Proof. We can decompose the conditional characteristic function in a real and an imaginary function

\[
\varphi_X |\mathcal{G}(t) = \mathbb{E} [\cos(t, X) \mid \mathcal{G}] + i \mathbb{E} [\sin(t, X) \mid \mathcal{G}],
\]

from which we can compute the gradients and the Hessian matrices. We get for the gradient of the sin- and cos-term

\[
t \mapsto - \mathbb{E} [\sin(t, X) \cdot X' \mid \mathcal{G}] \quad \text{and by } t \mapsto - \mathbb{E} [X \cdot \sin(t, X) \cdot X'] .
\]

The Hessian matrices are given by

\[
t \mapsto - \mathbb{E} [X \cdot \cos(t, X) \cdot X'] \quad \text{and by } t \mapsto - \mathbb{E} [X \cdot \sin(t, X) \cdot X'] .
\]
In the first place let $\mathbb{E}[X \mid \mathcal{G}] = 0$ a.s. $[\mathbb{P}]$, then
\[
\mathbb{E}[\cos(t, X) \mid \mathcal{G}] = 1 - \frac{1}{2} \cdot t' \cdot \mathbb{E}[X \cdot X' \mid \mathcal{G}] \cdot t - \frac{1}{2} \cdot t' \cdot \mathbb{E}[X \cdot (\cos(\theta_1t, X) - 1) \cdot X' \mid \mathcal{G}] \cdot t,
\]
\[
\mathbb{E}[\sin(t, X) \mid \mathcal{G}] = \frac{1}{2} \cdot t' \cdot \mathbb{E}[X \cdot \sin(\theta_2t, X) \cdot X' \mid \mathcal{G}] \cdot t,
\]
for suitable $\theta_1, \theta_2 \in [0, 1]$ by the mean value theorem. Hence, $\varphi_X(t) = 1 - \frac{1}{t}(t, \Sigma_t) + r(t)$, where
\[
|r(t)| = \frac{1}{2} \left(n \left[\mathbb{E}[X \cdot (\cos(\theta_1t, X) - 1) \cdot X' \mid \mathcal{G}] \right] + i \mathbb{E}[X \cdot \sin(\theta_2t, X) \cdot X' \mid \mathcal{G}] \right) |t|
\]
\[
\leq \frac{1}{2} \sup_{\theta \in [0, 1]} \left|\mathbb{E}[X \cdot (\exp(i\theta t, X)) - 1 \cdot X' \mid \mathcal{G}] \right| |t|
\]
\[
+ \frac{1}{2} \sup_{\theta \in [0, 1]} \left|\mathbb{E}[X \cdot (\sin(\theta t, X) - \sin(\theta_1t, X)) \cdot X' \mid \mathcal{G}] \right| |t|
\]
Next, we make use of the estimate $|1 - e^{i\alpha}| \leq \min(|\alpha|, 2)$ real $\alpha$ for the first term. For the second term, we use that the real sinus function is Lipschitz continuous with Lipschitz-constant 1, i.e. $|\sin x - \sin y| \leq |x - y|$. Hence, we have the following two estimates,
\[
\left|\mathbb{E}[X \cdot (\exp(i(\theta t, X)) - 1 \cdot X' \mid \mathcal{G}] \right| \leq \|t\|^2 \mathbb{E}[\|X\|^2 \exp(i(t, X)) - 1 \|\mathcal{G}] \leq 2 \|t\|^2 \mathbb{E}[\|X\|^2 1_{\|\|X\|\| > 2} \|\mathcal{G}] + \varepsilon \|t\|^2 \mathbb{E}[\|X\|^2 1_{\|\|X\|\| \leq 2} \|\mathcal{G}].
\]
And,
\[
\left|\mathbb{E}[X \cdot (\sin(\theta t, X) - \sin(\theta_1t, X)) \cdot X' \mid \mathcal{G}] \right| \leq 2 \|t\|^2 \mathbb{E}[\|X\|^2 1_{\|\|X\|\| > 2} \|\mathcal{G}] + \varepsilon \|t\|^2 \mathbb{E}[\|X\|^2 1_{\|\|X\|\| \leq 2} \|\mathcal{G}].
\]
Combining these estimates, the remainder can be bounded as claimed. For general $\mu$, we find that $\varphi_X(t) = \varphi_X(\mu)e^{i(\mu, \mu')}$, hence, we can apply the above analysis once again. This finishes the proof. 

**Lemma A.3.** Let $\pi^N, N \in \mathbb{N}_+$, be the sequence of dyadic partitions of $[0, T]$. $T > 0$, such that $0 = t_0^N < t_1^N < \ldots < t_N^N = T$. Let $\chi$ be a continuous matrix valued mapping valued from $[0, T]$ to $\mathbb{R}^{d \times d}$. Let $||\cdot||$ be a submultiplicative matrix norm on $\mathbb{R}^{d \times d}$. Then there is a continuous map
\[
P : [0, T]^2 \to \mathbb{R}^{d \times d} \text{ such that } \sup_{s \in [0, T]} \left\| \prod_{i \in \{0, \ldots, N\} \left(I + \Delta^N_t \chi(t^N_i - t^N_{i-1})\right) - P(s, t) \right\| \to 0 \text{ as } N \to \infty.
\]

**Proof.** For $0 \leq s \leq t \leq T$ write $P^N(s, t) := \prod_{i \in \{0, \ldots, N\}} \left(I + \Delta^N_t \chi(t^N_i - t^N_{i-1})\right)$. We show that the $(P^N)_{N \in \mathbb{N}}$, are Cauchy w.r.t. $||\cdot||_{\infty}$ on $[0, T]$. Fix some $0 \leq s \leq t \leq T$. Let $M \geq N$. Then for a $k \leq 2^N$, we have $P^N(s, t) = D_0 \cdot D_1 \ldots \cdot D_k$, where each $D_i = I + \Delta^N_M \chi(t^N_i - t^N_{i-1})$ for some $t^N_i \in (s, t)$ and $D_0$ is determined by the unique $t^N_0$ which fulfills $t^N_0 \leq s < t^N_0 \leq t$. Since $\pi^N \leq \pi^M$, we can choose for $i = 1, \ldots, k$ the factors
$\chi = \left(I + \Delta^M_{t^N_i - t^N_{i-1}} \right)$
and $\chi_i = \left(I + \Delta^N_{t^N_i - t^N_{i-1}} \right)$ such that for $i \neq 1$ : $t^N_i - t^N_{i-1} \cap t^N_i - t^N_{i-1} \cap \chi_i$ for $v = 2^N - 1$. Furthermore, there is a factor $F_0$ given by
\[
F_0 = \left(1 + \Delta^M_{t^N_1} \chi(t^M_1 - t^M_{1-1})\right) \ldots \left(I + \Delta^M_{t^N_N} \chi(t^M_N - t^M_{N-1})\right)
\]
for the unique $t^M_i$ which fulfills $t^M_{i-1} \leq s < t^M_i \leq t$ and $\chi \leq 2^M - N$. Then $P^M(s, t) = F_0 \cdot F_1 \ldots \cdot F_k \cdot \text{Res}$, where the residual factors of $P^M(s, t)$ are collected in Res. Hence, we can write
\[
P^M(s, t) - P^N(s, t) = (F_0 - I) \cdot F_1 \ldots \cdot F_k \cdot \text{Res} - D_0 \cdot D_1 \ldots \cdot D_k
\]
\[
= (F_0 - I) \cdot F_1 \ldots \cdot F_k \cdot \text{Res} + F_1 \ldots \cdot F_k \cdot (\text{Res} - I)
\]
\[
+ |F_1 \ldots \cdot F_k \cdot D_1 \ldots \cdot D_k| - (D_0 - I) \cdot D_1 \ldots \cdot D_k.
\]
Firstly, we have $||\text{Res}|| \leq \exp(\Delta^N ||\chi||_{\infty})$, as well as, $\max \{||F_1 \ldots \cdot F_k||, ||D_1 \ldots \cdot D_k||\} \leq \exp(T ||\chi||_{\infty})$ and secondly,
\[
\max\{||F_0 - I||, ||D_0 - I||, ||\text{Res} - I||\} \leq \Delta^N ||\chi||_{\infty} \exp(\Delta^N ||\chi||_{\infty}).
\]
Thirdly, we can write the main term as \( F_1 \cdot \ldots F_k - D_1 \cdot \ldots D_k = \sum_{j=1}^{k} \prod_{i=1}^{j-1} F_i \prod_{i=j}^{k} D_i \cdot \sum_{j=0}^{k-1} \prod_{i=1}^{j} F_i \prod_{i=j+1}^{k} D_i \), which implies for the norm of this term
\[
\|F_1 \cdot \ldots F_k - D_1 \cdot \ldots D_k\| \leq \sum_{j=1}^{k} \max_{1 \leq s \leq k} \|F_j\|^{j-1} \|F_j - D_j\| \max_{1 \leq s \leq k} \|D_s\|^{k-j}.
\]
(A.2)

The factors of each summand in (A.2) can be estimated as follows, we have for the first and the third factor,
\[
\|D_i\| \leq 1 + \Delta^N \|k\|_\infty \leq \exp(\Delta^N \|k\|_\infty)\quad \text{as well as}\quad \|F_j\| \leq (1 + \Delta^M \|k\|_\infty)^{2M-N} \leq \exp(\Delta^N \|k\|_\infty).
\]

For the factor in the middle, we use the definition of the modulus of continuity: Define for \( \delta > 0 \) the function \( w(\delta, \chi) := \sup \|\chi(s) - \chi(t)\| : |s - t| \leq \delta \). Then
\[
\|F_j - D_j\| \leq \Delta^N w(\Delta^N, \chi) + \sum_{j=2}^{2M-N} \binom{2M-N}{j} (\Delta^M \|k\|_\infty)^j \leq \Delta^N w(\Delta^N, \chi) + (\Delta^N)^2 \|k\|_\infty^2 \exp(\Delta^N \|k\|_\infty).
\]

Eventually, we combine these estimates to find that (A.1) can be bounded over all \( s \) and \( t \) by \( c \big( \Delta^N + w(\Delta^N, \chi) \big) \), for a suitable constant \( 0 < c < \infty \) which does not depend on \( N \). Since \( \chi \) is uniformly continuous on \([0, T]\), we have \( \lim_{N \to \infty} w(\Delta^N, \chi) = 0 \). This proves the Cauchy property and consequently the uniform convergence of the sequence \( (P^N)_{N \in \mathbb{N}} \) to a limit function \( P \). It remains to prove the continuity of this \( P \). We have for all \( (s, t), (s_0, t_0) \in [0, T]^2 \) that
\[
\|P(s, t) - P(s_0, t_0)\| \leq \|P(s, t) - P^N(s, t)\| + \|P^N(s, t) - P^N(s_0, t)\| + \|P^N(s_0, t_0) - P(s_0, t_0)\|.
\]
(A.3)

And we can compute, that for \( 0 \leq a < b < c \leq T \), we have
\[
\left\| \prod_{ac \leq i \leq b} \left( 1 + \Delta^N \chi (t_i^N) \right) - \prod_{ac \leq i \leq c} \left( 1 + \Delta^N \chi (t_i^N) \right) \right\| \leq \|k\|_\infty \exp(T \|k\|_\infty) \exp\left( (c - b + \Delta^N) \|k\|_\infty \right) \left( (c - b) + \Delta^N \right)
\]
where the terms involving the \( \Delta^N \) stem from the fact that \( P^N \) is discontinuous at the partitioning points. All in all, the remaining terms in equation (A.3) can be bounded with
\[
\max \left\{ \|P^N(s, t) - P^N(s_0, t)\|, \|P^N(s_0, t) - P^N(s_0, t_0)\| \right\} \leq \text{const} \left( \max(|s - s_0|, |t - t_0|) + \Delta^N \right)
\]
This yields the desired continuity of the limit function \( P \). \( \square \)

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E-mail address: krebs@mathematik.uni-kl.de

University of Kaiserslautern, Erwin-Schrödinger-Strasse, 67663 Kaiserslautern