Block Coordinate Descent Only Converge to Minimizers

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Abstract

Given a non-convex twice continuously differentiable cost function with Lipschitz continuous gradient, we prove that all of block coordinate gradient descent, block mirror descent and proximal block coordinate descent converge to a local minimizer, almost surely with random initialization. Furthermore, we show that these results also hold true even for the cost functions with non-isolated critical points.

Keywords: Block coordinate gradient descent, block mirror descent, proximal block coordinate descent, saddle points, local minimum, non-convex

1. Introduction

A main source of difficulty for non-convex optimization over continuous spaces is the proliferation of saddle points. Actually, it is easy to find some instances where bad initialization of the gradient descent converges to unfavorable saddle point (Nesterov, 2004, Section 1.2.3). Although there exist such worst case instances in theory, many simple algorithms including the first order algorithms and their variants, perform extremely well in terms of the quality of solutions of continuous optimization.

1.1. Related work

Recently, a milestone result of the gradient descent was established by Lee et al. (2016). That is, the gradient descent method converges to a local minimizer, almost surely with random initialization, by resorting to a tool from topology of dynamical systems. Lee et al. (2016) assume that a cost function satisfies the strict saddle property. Equivalently, each critical point $x$ of $f$ is either a local minimizer, or a “strict saddle”, i.e., $\nabla^2 f(x)$ has at least one strictly negative eigenvalue. They demonstrated that if $f : \mathbb{R}^n \to \mathbb{R}$ is a twice continuously differentiable function whose gradient is Lipschitz continuous with constant $L$, then the gradient descent with a sufficiently small constant step-size $\alpha$ (i.e., $x_{k+1} = x_k - \alpha \nabla f(x_k)$ and $0 < \alpha < 1$).
\( \frac{1}{T} \) and a random initialization converges to a local minimizer or negative infinity almost surely.

There is a followup work given by Panageas and Piliouras (2016), which firstly proven that the results in Lee et al. (2016) do hold true even for cost function \( f \) with non-isolated critical points. One key tool they used is that for every open cover there is a countable subcover in \( \mathbb{R}^n \). Moreover, Panageas and Piliouras (2016) have shown the globally Lipschitz assumption can be circumvented as long as the domain is convex and forward invariant with respect to gradient descent. In addition, they also provided an upper bound on the allowable step-size (such that those results hold true).

There are some prior works showing that first-order descent methods can indeed escape strict saddle points with the assistance of near isotropic noise. Specifically, Pemantle (1990) established convergence of the Robbins-Monro stochastic approximation to local minimizers for strict saddle functions and Kleinberg et al. (2009) demonstrated that the perturbed versions of multiplicative weights algorithm can converge to local minima in generic potential games. In particular, Ge et al. (2015) quantified the convergence rate of noise-added stochastic gradient descent to local minima. Note that the aforementioned methods requires the assistance of isotropic noise, which can significantly slowdown the convergence rate when the problem parameters dimension is large. In contrast, our setting is deterministic and corresponds to simple implementations of block coordinate gradient descent, block mirror descent and proximal block coordinate descent.

1.2. Our contribution

In this paper, under the assumption that \( f \) is twice continuously differentiable function with a Lipschitz continuous gradient over \( \mathbb{R}^n \), we prove that all the following block coordinate descent methods with constant step size,

(1) block coordinate gradient descent;
(2) block mirror descent; and
(3) proximal block coordinate descent,

converge to a local minimizer, almost surely with random initialization. This result even holds for cost function with non-isolated critical points.

In other words, we not only affirmatively answer the open questions whether mirror descent or block coordinate descent does not converge to saddle points in Lee et al. (2016), but also show that the same results hold true for block mirror descent (including mirror descent as a special case) and proximal block coordinate descent. In addition, these results also hold true even for the cost functions with non-isolated critical points, which generalizes the results in Panageas and Piliouras (2016) as well.
1.3. Outline of the proof

Recall the main ideas in Lee et al. (2016). Suppose that \( g : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is an iterative mapping of an optimization method and the fixed point of \( g \) is the critical point of \( f \) as well. Then the following key properties of \( g \),

(i) \( g \) is a diffeomorphism;

(ii) \( x^* \) is a fixed point of \( g \), then there is at least one eigenvalue of the Jacobian \( Dg(x^*) \), whose magnitude is strictly greater than one,

imply the results in Lee et al. (2016) hold true.

Although the basic idea of the proof presented in this paper comes from Lee et al. (2016), answering the underlying questions for block coordinate type algorithms is not easy and the existing analysis methods are not applicable. In particular, the eigenvalue analysis of the Jacobian of the iterative mapping needs a nontrivial argument.

As compared to Property (ii), Property (i) is easier to verify since we can decompose the entire iterative mapping into multiple one-block updating case and then using the chain rule of diffeomorphism, we prove the entire updating is a diffeomorphism. In contrast, Property (ii) is very challenging because the Jacobian \( Dg(x^*) \) of block updating fashion at a saddle point is a non-symmetric matrix and a complex polynomial function of the original \( \nabla f(x^*) \) with degree \( p \) (the number of the blocks of the decision variables). We overcome the above difficulties by the following two steps. Precisely, the first step is to transform the original Jacobian \( Dg_{αf}(x^*) \) into a simpler form which can be handled easily. Next, based on the simple form of \( Dg_{αf}(x^*) \), the second step is to prove that \( Dg(x^*) \) has at least one eigenvalue with magnitude strictly greater than one by resorting to Lemma 9.2 in appendices which follows essentially from Rouche’s Theorem in complex analysis.

1.4. Notations and organization

Notations. Denote complex number \( z \) as \( z = a + bi \), where \( a \) and \( b \) are real numbers and \( i \) is the imaginary unit with \( i^2 = -1 \). We also denote \( a = \text{Re}(z) \) and \( b = \text{Im}(z) \) as the real part and the imaginary part of \( z \), respectively. For a matrix \( X \), we denote \( \text{eig}(X) \) as the set of eigenvalues of \( X \), \( X^T \) as the transpose of \( X \), \( X^H \) as the conjugate transpose or Hermitian transpose of \( X \), \( ρ(X) \) as the spectral radius of \( X \) (i.e., the maximum modulus of the eigenvalues of \( X \)), and \( \|X\| \) as the spectral norm of \( X \). When \( X \) is a real symmetric matrix, let \( \lambda_{\text{max}}(X) \) and \( \lambda_{\text{min}}(X) \) denote the maximum and minimum eigenvalues of \( X \), respectively. Moreover, for two real symmetric matrices \( X_1 \) and \( X_2 \), \( X_1 \succeq X_2 \) (resp. \( X_1 \succeq X_2 \)) means \( X_1 - X_2 \) is positive definite (resp. positive semi-definite). We use \( I_n \) to denote the identity matrix with dimension \( n \), and we will simply use \( I \) when it is clear from context what the dimension is. For square matrices \( X_s \in \mathbb{R}^{n_s \times n_s}, s = 1, 2, \ldots, p \), we denote \( \text{Diag}(X_1, X_2, \ldots, X_p) \) as the block-diagonal matrix with \( X_s \) being the \( s \)-th diagonal
block. For square matrices $X_s \in \mathbb{R}^{n \times n}$, $s = 1, \ldots, p$, and $t, k \in \{1, 2, \ldots, p\}$, we use $\prod_{s=t}^{k} X_s$ to denote the continued products $X_t \cdot X_{t+1} \cdots X_{k-1} \cdot X_k$ if $t \leq k$ and $X_t \cdot X_{t-1} \cdots X_{k+1} \cdot X_k$ if $t > k$, respectively. $\mathbb{P}_\nu$ denotes the probability with respect to a prior measure $\nu$, which is assumed to be absolutely continuous with respect to Lebesgue measure.

**Organization.** In section 2, we introduce the basic setting and definitions used throughout the paper. Section 3 provides the main results for block coordinate gradient descent method. The main results for block mirror descent and proximal block coordinate descent are given in Section 4 and Section 5, respectively. Section 6 provides several lemmas. Finally, we conclude this paper in Section 7. The detailed proofs of some lemmas and propositions are presented in Section 9.

## 2. Preliminaries

We consider optimization model

$$
\min \{ f(x) : x \in \mathbb{R}^n \},
$$

(2.1)

where we make the following blanket assumption:

**Assumption 2.1** $f$ is twice continuously differentiable function whose gradient is Lipschitz continuous over $\mathbb{R}^n$, i.e., there exists a parameter $L > 0$ such that

$$
\| \nabla f(x) - \nabla f(y) \| \leq L \| x - y \| \quad \text{for every } x, y \in \mathbb{R}^n.
$$

(2.2)

Throughout this paper, in order to introduce the block coordinate descent method, we assume the vector of decision variables $x$ has the following partition:

$$
x = \begin{pmatrix}
x(1) \\
x(2) \\
\vdots \\
x(p)
\end{pmatrix},
$$

(2.3)

where $x(s) \in \mathbb{R}^{n_s}$, $n_1, n_2, \ldots, n_p$ are $p$ positive integer numbers satisfying $\sum_{s=1}^{p} n_s = n$.

Moreover, we use the notations in [Nesterov (2012)] and define matrices $U_s \in \mathbb{R}^{n \times n_s}$, the $s$-th block-column of $I_n$, $s = 1, \ldots, p$, such that

$$
\begin{pmatrix}
U_1, U_2, \ldots, U_p
\end{pmatrix} = I_n.
$$

(2.4)
Clearly, according to our notations, we have \( x(s) = U_s^T x \) for every \( x \in \mathbb{R}^n \), \( s = 1, \ldots, p \). Consequently, \( x = \sum_{s=1}^{p} U_s x(s) \) and the derivative corresponding variables in the vector \( x(s) \) can be expressed as

\[
\nabla_s f(x) \equiv U_s^T \nabla f(x), \quad s = 1, \ldots, p.
\]

Below we give some necessary definitions as appeared in Lee et al. (2016) and Panageas and Piliouras (2016).

**Definition 2.1**

1. A point \( x^* \) is a critical point of \( f \) if \( \nabla f(x^*) = 0 \). We denote \( C = \{ x : \nabla f(x) = 0 \} \) as the set of critical points (can be uncountably many).

2. A critical point \( x^* \) is isolated if there is a neighborhood \( U \) around \( x^* \), and \( x^* \) is the only critical point in \( U \). Otherwise is called non-isolated.

3. A critical point is a local minimum if there is a neighborhood \( U \) around \( x^* \) such that \( f(x^*) \leq f(x) \) for all \( x \in U \), and a local maximum if \( f(x^*) \geq f(x) \).

4. A critical point is a saddle point if for all neighborhoods \( U \) around \( x^* \), there are \( x, y \in U \) such that \( f(x) \leq f(x^*) \leq f(y) \).

**Definition 2.2 (Strict Saddle)** A critical point \( x^* \) of \( f \) is a strict saddle if \( \lambda_{\min}(\nabla^2 f(x^*)) < 0 \).

**Definition 2.3 (Global Stable Set)** The global stable set \( W^s(x^*) \) of a critical point \( x^* \) is the set of initial conditions of an iterative mapping \( g \) of an optimization method that converge to \( x^* \):

\[
W^s(x^*) = \{ x : \lim_k g^k(x) = x^* \}.
\]

3. The BCGD method

In this section, we will prove that the BCGD method does not converge to saddle points under appropriate choice of step size, almost surely with random initialization. This lays the ground for the analysis of the BMD and PBCD methods.

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3If the critical points are isolated then they are countably many or finite.
3.1. The BCGD method description

For ease of later reference and also for the sake of clarity, based on notations in Section 2, we present a detailed description of the BCGD method (Beck and Tetruashvili, 2013) for problem (2.1) below.

Method 3.1 (BCGD)

Input: \(\alpha < \frac{1}{L}\).

Initialization: \(x_0 \in \mathbb{R}^n\).

General Step \((k = 0, 1, \ldots)\): Set \(x_0^{k+1} = x_k\) and define recursively

\[
x_k^s = x_k^{s-1} - \alpha U_s \nabla_s f(x_k^{s-1}), \quad s = 1, \ldots, p.
\]

Set \(x_{k+1} = x_k^p\).

In what follows, for a given step size \(\alpha > 0\), we use \(g_{s}^{\alpha f}\) to denote the corresponding gradient mapping with respect to \(x(s)\), i.e.,

\[
g_{s}^{\alpha f}(x) \triangleq x - \alpha U_s \nabla_s f(x), \quad s = 1, \ldots, p.
\] (3.5)

It is clear that, given \(x_k\), the above BCGD method generates \(x_{k+1}\) in the following manner,

\[
x_{k+1} = g_{s}^{\alpha f}(x_k),
\] (3.6)

where composite mapping

\[
g_{\alpha f}(x) \triangleq g_{p}^{\alpha f} \circ g_{p-1}^{\alpha f} \circ \cdots \circ g_{2}^{\alpha f} \circ g_{1}^{\alpha f}(x).
\] (3.7)

By simple computation, the Jacobian of \(g_{s}^{\alpha f}\) is given by

\[
Dg_{s}^{\alpha f}(x) = I_n - \alpha \nabla^2 f(x) U_s U_s^T,
\] (3.8)

where \(\nabla^2 f(x) = \left(\frac{\partial^2 f(x)}{\partial x_s \partial x_t}\right)_{1 \leq s, t \leq p}\).

By using the chain rule, we obtain the following Jacobian of the mapping \(g\), i.e.,

\[
Dg_{\alpha f}(x) = Dg_{\alpha f}(y_1) \times Dg_{\alpha f}^2(y_2) \times \cdots \times Dg_{\alpha f}^{p-1}(y_{p-1}) \times Dg_{\alpha f}^p(y_p),
\] (3.9)

where \(y_1 = x\), and \(y_s = g_{s-1}^{\alpha f}(y_{s-1})\), \(s = 2, \ldots, p\).

Given the above basic notations, as mentioned in Section 1, it is sufficient for us to prove that the iterative mapping \(g_{\alpha f}\) admits the following two key properties:

(i) \(g_{\alpha f}\) is a diffeomorphism;

(ii) \(x^*\) is a fixed point of \(g\), then there is at least one eigenvalue of the Jacobian \(Dg_{\alpha f}(x^*)\), whose magnitude is strictly greater than one.

In what follows, we will mainly concern with proving the above two properties of \(g_{\alpha f}\). Specifically, Subsections 3.2 and 3.3 present their proofs, respectively.
3.2. The iterative mapping $g_{\alpha f}$ of BCGD method is a diffeomorphism

In this subsection, we first present the following Lemma 3.1 which shows that $g_{\alpha f}^s, s = 1, \ldots, p$, are diffeomorphisms. Based on this lemma, then we further prove that $g_{\alpha f}$ is a diffeomorphism as well.

**Lemma 3.1** Under Assumption 2.1, if step size $\alpha < \frac{1}{L}$, then the iterative mappings $g_{\alpha f}^s$ defined by (3.5), $s = 1, \ldots, p$, are diffeomorphisms.

**Proof.** The proof of Lemma 3.1 is lengthy and has been relegated to the Appendix.

Given the above lemma, we immediately obtain the following result.

**Proposition 3.1** Under Assumption 2.1, the mapping $g_{\alpha f}$ defined by (3.7) with step size $\alpha < \frac{1}{L}$ is a diffeomorphism.

**Proof.** It follows from Proposition 2.15 in [Lee (2013)] that the composition of two diffeomorphisms is also a diffeomorphism. Using this fact and the previous Lemma 3.1, the proof is completed.

**Remark 3.1** Note that, in order to guarantee invertibility of $Dg_{\alpha f}(x^*)$, Eq. (9.145) in the appendix must hold, which clearly implies that $\alpha < \frac{1}{L}$ is necessary.

3.3. Eigenvalue analysis of the Jacobian of $g_{\alpha f}$ at a strict saddle point

In this subsection, we will analyze the eigenvalues of the Jacobian of $g_{\alpha f}$ at a strict saddle point and show that it has at least one eigenvalue with magnitude greater than one, which is a crucial part in our entire proof. The proof mainly involves two steps.

More specifically, the first step is to transform the original Jacobian $Dg_{\alpha f}(x^*)$ into a simpler form which can be dealt with more easily. Furthermore, based on the simple form of $Dg_{\alpha f}(x^*)$, the second step is to prove that $Dg(x^*)$ has at least one eigenvalue with magnitude strictly greater than one by resorting to Lemma 9.2 in Appendix which follows essentially from Rouche’s Theorem in complex analysis.

In what follows, we assume that $x^* \in \mathbb{R}^n$ is a strict saddle point. Hence, $g_{\alpha f}^s(x^*) = x^*, s = 1, 2, \ldots, p$. By chain rule (3.9), we have

$$Dg_{\alpha f}(x^*) = Dg_{\alpha f}^1(x^*) \times Dg_{\alpha f}^2(x^*) \times \cdots \times Dg_{\alpha f}^{p-1}(x^*) \times Dg_{\alpha f}^p(x^*) .$$

(3.10)

Since eig $(Dg_{\alpha f}(x^*)) = \text{eig} \left( (Dg_{\alpha f}(x^*))^T \right)$, it is sufficient for us to analyze the eigenvalues of matrix $(Dg_{\alpha f}(x^*))^T$. In addition, Eq. (3.10) leads to

$$
(Dg_{\alpha f}(x^*))^T = (Dg_{\alpha f}^p(x^*))^T \times (Dg_{\alpha f}^{p-1}(x^*))^T \times \cdots \times (Dg_{\alpha f}^2(x^*))^T \times (Dg_{\alpha f}^1(x^*))^T \\
= (I_n - \alpha U_p U_p^T \nabla^2 f(x^*)) \times (I_n - \alpha U_{p-1} U_{p-1}^T \nabla^2 f(x^*)) \times \cdots \times (I_n - \alpha U_2 U_2^T \nabla^2 f(x^*)) \times (I_n - \alpha U_1 U_1^T \nabla^2 f(x^*)) ,
$$

(3.11)
where the second equality is due to (3.8). For the sake of simplicity, we denote matrix \( \nabla^2 f(x^*) \) as matrix \( A \in \mathbb{R}^{n \times n} \) with \( p \times p \) blocks. Specifically,

\[
A \triangleq (A_{st})_{1 \leq s, t \leq p},
\]

(3.12)

and its \((s, t)\)-th block is defined as

\[
A_{st} \triangleq \frac{\partial^2 f(x^*)}{\partial x^*(s) \partial x^*(t)}, \quad 1 \leq s, t \leq p.
\]

(3.13)

Furthermore, we denote the \(s\)-th block-row of \( A \) as

\[
A_s \triangleq (A_{st})_{1 \leq t \leq p}, \quad s = 1, \ldots, p.
\]

(3.14)

Based on the above notations, we have

\[
U_s^T \nabla^2 f(x^*) = A_s, \quad 1 \leq s \leq p,
\]

(3.15)

and

\[
A_s U_t = A_{st}, \quad 1 \leq s, t \leq p.
\]

(3.16)

Combining (3.11) and (3.15), we have

\[
(Dg_{\alpha f}(x^*))^T = (I_n - \alpha U_p A_p) \times (I_n - \alpha U_{p-1} A_{p-1}) \times \cdots \times (I_n - \alpha U_2 A_2) \times (I_n - \alpha U_1 A_1).
\]

(3.17)

In order to analyze the eigenvalues of the above matrix, we furthermore define a matrix below:

\[
G \triangleq \frac{1}{\alpha} \left[ I_n - (Dg_{\alpha f}(x^*))^T \right],
\]

(3.18)

or equivalently,

\[
(Dg_{\alpha f}(x^*))^T = I_n - \alpha G.
\]

(3.19)

The above relation (3.19) clearly means that

\[
\lambda \in \text{eig} (G) \iff 1 - \alpha \lambda \in \text{eig} \left( (Dg_{\alpha f}(x^*))^T \right).
\]

(3.20)

In the rest of this subsection, our focus is on analyzing the eigenvalues of \( G \). We first introduce the following Lemma 3.2 and Lemma 3.3 which assert a particular relation between \( G \) and \( A \).

**Lemma 3.2** Assume that \( x^* \in \mathbb{R}^n \) is a strict saddle point, \( G \) is given by (3.18), \( U_s \) and \( A_s \) are defined by (2.4) and (3.14), respectively, \( s = 1, \ldots, p \). Then,

\[
U_s^T G = A_s - \alpha \sum_{t=1}^{s-1} A_{st} U_t^T G, \quad s = 1, \ldots, p.
\]

(3.21)
The proof of Lemma 3.2 is lengthy and has been relegated to the Appendix.

In order to give a clear and simple expression of $G$, we further define the strictly block lower triangular matrix based on $A$ below:

$$
\hat{A} \triangleq (\hat{A}_{st})_{1 \leq s,t \leq p} \quad (3.22)
$$

with $p \times p$ blocks and its $(s,t)$-th block is given by

$$
\hat{A}_{st} = \begin{cases} 
A_{st}, & s > t, \\
0, & s \leq t.
\end{cases} \quad (3.23)
$$

Similarly, we denote the $s$-th block-row of $\hat{A}$ as

$$
\hat{A}_s \triangleq (\hat{A}_{st})_{1 \leq t \leq p}, \quad s = 1, \ldots, p. \quad (3.24)
$$

The following lemma plays an important role in this subsection because it gives a simple expression of $G$ in terms of $A$ and $\hat{A}$, which allows us to analyze the eigenvalues of $G$ more easily.

**Lemma 3.3** Let $x^* \in \mathbb{R}^n$ be a strict saddle point. Assume that $G$, $A$ and $\hat{A}$ are defined by (3.18), (3.12) and (3.22), respectively. Then,

$$
G = (I_n + \alpha \hat{A})^{-1} A. \quad (3.25)
$$

**Proof.** We assume that $G$ has the following partition:

$$
G = \begin{pmatrix} G_1 \\ G_2 \\ \vdots \\ G_p \end{pmatrix},
$$

where $G_s \in \mathbb{R}^{n_s \times n_s}$ is the $s$-th block-row of $G$, $s = 1, 2, \ldots, p$. Consequently, we have

$$
G_s = U_s^T G = A_s - \alpha \sum_{t=1}^{s-1} A_s U_t^T G = A_s - \alpha \sum_{t=1}^{s-1} A_{st} G_t = A_s - \alpha \hat{A}_s G, \quad s = 1, 2, \ldots, p,
$$

where in the second equality we use Lemma 3.2 and the last equality is due to the definition of $\hat{A}_s$ in (3.24). Since the above equality holds for any $s$, it immediately follows that

$$
G = A - \alpha \hat{A} G,
$$

which is equivalent to

$$
(I_n + \alpha \hat{A}) G = A.
$$
Note that $I_n + \alpha \tilde{A}$ is an invertible matrix because $\tilde{A}$ is a block strictly lower triangular matrix. Premultiplying both sides of the above equality by $(I_n + \alpha \tilde{A})^{-1}$, we arrive at (3.25). Therefore, the lemma is proved.

The following proposition plays a central role in this subsection because it provides a sufficiently exact description of the distribution of the eigenvalues of $G$. More importantly, it leads immediately to the subsequent Proposition 3.3 which asserts that, there is at least one eigenvalue of Jacobian of iterative mapping $g_{\alpha f}$ at a strict saddle point, whose magnitude is strictly greater than one.

**Proposition 3.2** Assume that $x^* \in \mathbb{R}^n$ is a strict saddle point, and $G$ is defined by (3.18) with $\alpha \in (0, \frac{1}{L})$, where $L$ is determined by (2.2). Then, there exists at least one eigenvalue of $G$ which lies in closed left half complex plane excluding origin, i.e.,

$$\forall \beta \in \left(0, \frac{1}{L}\right) \Rightarrow \exists \lambda \in \left[\text{eig}(G) \cap \Omega\right],$$

(3.26)

where

$$\Omega \triangleq \{a + bi | a, b \in \mathbb{R}, a \leq 0, (a, b) \neq (0, 0), i = \sqrt{-1}\}. \quad (3.27)$$

**Proof.** It follows from Lemma 3.3 that $G$ has the following expression:

$$G = (I_n + \alpha \tilde{A})^{-1} A,$$

(3.28)

where $A$ and $\tilde{A}$ are defined by (3.12) and (3.22), respectively. Since $A = \nabla^2 f(x^*)$ and $x^*$ is a strictly saddle point, $A$ has at least one negative eigenvalue. Furthermore, by applying Lemma 6.4 in Section 6 with identifications $A \sim B$, $\tilde{A} \sim \tilde{B}$, $\alpha \sim \beta$ and $\rho(A) \sim \rho(B)$, we have

$$\forall \beta \in \left(0, \frac{1}{\rho(A)}\right) \Rightarrow \exists \lambda \in \left[\text{eig}(G) \cap \Omega\right],$$

(3.29)

where $\Omega$ is defined by (3.27).

In addition, Lemma 7 in Panageas and Piliouras (2016) implies that the gradient Lipschitz continuous constant $L \geq \rho(A) = \rho(\nabla f(x^*))$, which amounts to $\alpha \in \left(0, \frac{1}{L}\right) \subseteq \left(0, \frac{1}{\rho(A)}\right)$. Hence, the above (3.29) leads immediately to (3.26). \(\square\)

Based on the above Proposition 3.2, we immediately obtain the key proposition in this subsection.

**Proposition 3.3** Assume Assumption 2.1 holds. Suppose that BCGD iterative mapping $g_{\alpha f}$ is defined by (3.7) with $\alpha \in (0, \frac{1}{L})$, $L$ is determined by (2.2), and $x^* \in \mathbb{R}^n$ is a strict saddle point. Then $Dg_{\alpha f}(x^*)$ has at least one eigenvalue whose magnitude is strictly greater than one.
Proof. Recalling Eq. (3.20), we have

$$\lambda \in \text{eig} (G) \iff 1 - \alpha \lambda \in \text{eig} \left( (Dg_{\alpha f}(x^*))^T \right),$$

(3.30)

where $G = \frac{1}{\alpha} \left[ I_n - (Dg_{\alpha f}(x^*))^T \right]$ is defined by (3.18).

Combined with (3.30), Proposition 3.2 implies that there exists at least one eigenvalue of $(Dg_{\alpha f}(x^*))^T$ which can be expressed as

$$1 - \alpha (a + bi),$$

(3.31)

where $a + bi$ belongs to $\Omega$ defined by (6.130), or equivalently,

$$a \leq 0 \text{ and } (a,b) \neq (0,0).$$

(3.32)

Consequently, its magnitude is

$$|1 - \alpha (a + bi)| = \sqrt{1 - 2\alpha a + \alpha^2 a^2 + \alpha^2 b^2} \geq \sqrt{1 + \alpha^2 (a^2 + b^2)} > 1,$$

where the first inequality is due to $a \leq 0$ and $\alpha > 0$; and the second inequality thanks to $(a,b) \neq (0,0)$. Note that the eigenvalues of $Dg_{\alpha f}(x^*)$ are the same as those of $(Dg_{\alpha f}(x^*))^T$. Thus, the proof is finished.

3.4. Main results of BCGD

We first introduce the following proposition, which asserts that the limit point of the sequence generated by the BCGD method 3.1 is a critical point of $f$.

Proposition 3.4 Under Assumption 2.1, if $\{x_k\}_{k \geq 0}$ is generated by the BCGD method 3.1 with $0 < \alpha < \frac{1}{L}$, $\lim_{k} x_k$ exists and denote it as $x^*$, then $x^*$ is a critical point of $f$, i.e., $\nabla f(x^*) = 0$.

Proof. Since $\{x_k\}_{k \geq 0}$ is generated by the BCGD method 3.1 $x_k = g^k_{\alpha f}(x_0)$, where $g_{\alpha f}$ is defined by (3.5). Hence, $\lim_{k} x_k = \lim_{k} g^k_{\alpha f}(x_0) = x^*$. Since $g_{\alpha f}$ is a diffeomorphism, we immediately know that $x^*$ is a fixed point of $g_{\alpha f}$. It follows easily from the definition (3.7) of $g_{\alpha f}$ that $\nabla f(x^*) = 0$. Thus the proof is finished.

Armed with the results established in previous subsections and the above Proposition 3.4 we now state and prove our main theorems of BCGD. Specifically, similar to the proof of Theorem 4 in Lee et al. (2016), the center-stable manifold theorem in Smale (1967); Shub (1987); Hirsch et al. (1977) is a primary tool because it gives a local characterization of the stable set. Hence, we first rewrite it as follows.

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4 $g^k_{\alpha f}$ denotes the composition of $g_{\alpha f}$ with itself $k$ times.
Theorem 3.1 Let $0$ be a fixed point for the $C^r$ local differomorphism $\phi : U \rightarrow E$ where $U$ is a neighborhood of zero in the Banach space $E$ and $\infty > r \geq 1$. Let $E_{sc} \oplus E_u$ be the invariant splitting of $\mathbb{R}^n$ into the generalized eigenspaces of $Df(0)$ corresponding to eigenvalues of absolute value less than or equal to one, and greater than one. Then there is a local $\phi$ invariant $C^r$ embedded disc $W^{sc}_{loc}$ tangent to $E_{sc}$ at $0$ and a ball $B$ around zero in an adapted norm such that $\phi(W^{sc}_{loc}) \cap B \subset W^{sc}_{loc}$ and $\bigcap \phi^{-k}(B) \subset W^{sc}_{loc}$.

The following theorem is similar to Theorem 4 in Lee et al. (2016).

**Theorem 3.2** Let $f$ be a $C^2$ function and $x^*$ be a strict saddle. If $\{x_k\}$ is generated by the BCGD method with $0 < \alpha < \frac{1}{L}$, then

$$\mathbb{P}_\nu \left[ \lim_{k} x_k = x^* \right] = 0.$$  

**Proof.** Proposition 3.3 implies that, if $\lim x_k$ exists then it must be a critical point. Hence, we consider calculating the Lebesgue measure (or probability with respect to the prior measure $\nu$) of the set $\left[ \lim_{k} x_k = x^* \right] = W^s(x^*)$ (see Definition 2.3).

Furthermore, since Proposition 3.3 means the BCGD iterative mapping $g_{\alpha f}$ is a diffeomorphism, we replace $\phi$ and fixed point by $g_{\alpha f}$ and the strict point $x^*$ in the above Stable Manifold Theorem 3.1 respectively. Then the manifold $W^s_{loc}(x^*)$ has strictly positive codimension because of Proposition 3.3 and $x^*$ being a strict saddle point. Hence, $W^s_{loc}(x^*)$ has measure zero.

In what follows, we are able to apply the same arguments as in Lee et al. (2016) to finish the proof of the theorem. Since the proof follows a similar pattern, it is omitted.

Given the above Theorem 3.2, we immediately obtain the following Theorem 3.3 and its corollary by the same arguments as in the proofs of Theorem 2 and Corollary 12 in Panageas and Piliouras (2016). Therefore, we omit their proofs.

**Theorem 3.3 (Non-isolated)** Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a twice continuously differentiable function and $\sup_{x \in \mathbb{R}^n} \|\nabla f(x)\| \leq L < \infty$. The set of initial points $x \in \mathbb{R}^n$, for each of which the BCGD method with step size $0 < \alpha < \frac{1}{L}$ converges to a strict saddle point, is of (Lebesgue) measure zero, without assumption that critical points are isolated.

A straightforward corollary of Theorem 3.3 is given below:

**Corollary 3.1** Assume that the conditions of Theorem 3.3 are satisfied and all saddle points of $f$ are strict. Additionally, assume $\lim_k g_{\alpha f}^k(x)$ exists for all $x \in \mathbb{R}^n$. Then

$$\mathbb{P}_\nu \left[ \lim_{k} g_{\alpha f}^k(x) = x^* \right] = 1,$$
where $g_{\alpha f}$ is defined by (3.5) and $x^*$ is a local minimum.

4. The BMD method

In this section, we will extend the above results to the BMD method in Beck and Teboulle (2003) andJuditsky and Nemirovski (2014). In other words, the BMD method, based on Bregman’s divergence, converges to minimizers as well, almost surely with random initialization.

4.1. The BMD method description

For clarity of notation, recall the vector of decision variables $x$ has been assumed to have the following partition (see (2.3)):

$$x = \begin{pmatrix} x(1) \\ x(2) \\ \vdots \\ x(p) \end{pmatrix},$$

where $x(t) \in \mathbb{R}^{n_t}$, and $n_1$, $n_2$, ..., $n_p$ are $p$ positive integer numbers satisfying

$$\sum_{t=1}^{p} n_t = n.$$

Correspondingly, we assume a set of variables $x^s_k$ have the following partition as well:

$$x^s_k \triangleq \begin{pmatrix} x^s_k(1) \\ x^s_k(2) \\ \vdots \\ x^s_k(p) \end{pmatrix}, \quad s = 1, \ldots, p; \ k = 0, 1, \ldots,$$

where $x^s_k(t) \in \mathbb{R}^{n_t}$; $t, s = 1, \ldots, p; \ k = 0, 1, \ldots$.

In order to introduce the BMD method, we assume there are $p$ strictly convex and continuously differentiable functions $\varphi_t : \mathbb{R}^{n_t} \to \mathbb{R}, t = 1, 2, \ldots, p$. Furthermore, we make the following assumption.

**Assumption 4.1** $\varphi_t$ is a strongly convex and twice continuously differentiable function with parameter $\mu_t > 0$, i.e., for any $y(t)$ and $x(t) \in \mathbb{R}^{n_t}$,

$$\varphi_t(y(t)) \geq \varphi_t(x(t)) + \langle \nabla \varphi_t(x(t)), y(t) - x(t) \rangle + \frac{\mu_t}{2} \|y(t) - x(t)\|^2, \quad t = 1, 2, \ldots, p.$$

(4.35)

The Bregman divergences of the above strongly convex functions, $B_{\varphi_t} : \mathbb{R}^{n_t} \times \mathbb{R}^{n_t} \to \mathbb{R}^+$, are defined as

$$B_{\varphi_t}(x(t), y(t)) \triangleq \varphi_t(x(t)) - \varphi_t(y(t)) - \langle x(t) - y(t), \nabla \varphi_t(y(t)) \rangle, \quad t = 1, 2, \ldots, p.$$

(4.36)
The Bregman divergence was initially studied by Bregman (1967) and later by many others (see Auslender and Teboulle (2006); Bauschke et al. (2006); Teboulle (1997) and references therein).

**Remark 4.1** We should mention that Bregman divergence is generally defined for a continuously differentiable and strongly convex function which is not necessarily twice differentiable. Here, the twice differentiability of $\varphi_s$ seems a necessary and reasonable condition for our subsequent analysis because the existence of the Jacobian of the iterative mapping depends directly on $\nabla^2 \varphi_t$ (see (4.47)).

Let
\[
\mu = \min \{\mu_1, \mu_2, \ldots, \mu_p\}. \tag{4.37}
\]

Given the above notations, the detailed description of the block mirror descent algorithm for problem (2.1) is given below.

**Method 4.1 (BMD)**

**Input:** $\alpha < \frac{\mu}{L}$.

**Initialization:** $x_0 \in \mathbb{R}^n$.

**General Step** $(k = 0, 1, \ldots)$: Set $x^0_k = x_k$ and define recursively for $s = 1, 2, \ldots, p$:

- $t = 1, 2, \ldots, p$,
- If $t = s$,
  \[
  x^s_k(t) = \arg \min_{x(t)} \left\{ x(t), \nabla_t f \left( x^s_k \right) \right\} + \frac{1}{\alpha} B_{\varphi_t} \left( x(t), x^{s-1}_k(t) \right). \tag{4.38}
  \]

- Else
  \[
  x^s_k(t) = x^{s-1}_k(t). \tag{4.39}
  \]

**End**

Set $x_{k+1} = x^p_k$.

Note that $\varphi_s$ is a strongly convex function. Then it is easily seen from (4.36) that $B_{\varphi_t} (x(s), y(s))$ is a strongly convex function with respect to $x(s)$ if $y(s)$ is fixed. Hence, let $x^*_k(s)$ be the unique solution of problem (4.38). The KKT condition implies that
\[
0 = \nabla_s f \left( x^{s-1}_k \right) + \frac{1}{\alpha} \left( \nabla \varphi_s \left( x^s_k(s) \right) - \nabla \varphi_s \left( x^{s-1}_k(s) \right) \right), \tag{4.40}
\]
which is equivalent to
\[
\nabla \varphi_s \left( x^s_k(s) \right) = \nabla \varphi_s \left( x^{s-1}_k(s) \right) - \alpha \nabla_s f \left( x^{s-1}_k \right). \tag{4.41}
\]

Combined with the assumption about $\varphi_s$, Lemma [9.4] in Appendix leads immediately to the existence of the inverse mapping of $\nabla \varphi_s$. Let $[\nabla \varphi_s]^{-1}$ denote its inverse.
Then $x^s_k(s)$ can be expressed in terms of $x^{s-1}_k$ as

$$x^s_k(s) = [\nabla \varphi_s]^{-1} (\nabla \varphi_s (x^{s-1}_k(s)) - \alpha \nabla f (x^{s-1}_k)) \tag{4.42}$$

It follows from Eqs. (4.34), (4.38), and (4.39) that

$$x^s_k = (I_n - U_sU_s^T) x^{s-1}_k + U_s [\nabla \varphi_s]^{-1} (\nabla \varphi_s (x^{s-1}_k(s)) - \alpha \nabla f (x^{s-1}_k)) \tag{4.43}$$

where $U_s$ is defined by (2.4).

In what follows, we define $\psi_s : \mathbb{R}^{n_s} \rightarrow \mathbb{R}^{n_s}$ as

$$\psi_s (x) \triangleq (I_n - U_sU_s^T) x + U_s [\nabla \varphi_s]^{-1} (\nabla \varphi_s (x(s)) - \alpha \nabla f (x)) \tag{4.44}$$

It is clear that, given $x_k$, the above BMD method generates $x_{k+1}$ in the following manner,

$$x_{k+1} = \psi (x_k) \tag{4.45}$$

where the composite mapping

$$\psi(x) \triangleq \psi_p \circ \psi_{p-1} \circ \cdots \circ \psi_2 \circ \psi_1 (x) \tag{4.46}$$

Additionally, it follows from (4.44) that, for each $s = 1, 2, \ldots, p$, the Jacobian of $\psi_s$ is given below:

$$D \psi_s (x) = (I_n - U_sU_s^T) + D \{ [\nabla \varphi_s]^{-1} (\nabla \varphi_s (x(s)) - \alpha \nabla f (x)) \} U_s^T$$

$$= (I_n - U_sU_s^T)$$

$$+ D \{ \nabla \varphi_s (x(s)) - \alpha \nabla f (x) \} \times \{ [\nabla \varphi_s]^{-1} (\nabla \varphi_s (x(s)) - \alpha \nabla f (x)) \}^{-1} U_s^T$$

$$= (I_n - U_sU_s^T)$$

$$+ \{ U_s \nabla^2 \varphi_s (x(s)) - \alpha \nabla^2 f (x) U_s \} \{ [\nabla \varphi_s]^{-1} (\nabla \varphi_s (x(s)) - \alpha \nabla f (x)) \}^{-1} U_s^T \tag{4.47}$$

where the first equality is due to chain rule; the second equality holds because of chain rule and inverse function theorem in [Spivak (1965)]; the last equality follows from the definition of $U_s$ and $\nabla^2 f (x) = \left( \frac{\partial^2 f (x)}{\partial x(s) \partial x(t)} \right)_{1 \leq s, t \leq p}$.

By using the chain rule, we obtain the following Jacobian of the mapping $\psi$, i.e.,

$$D \psi (x) = D \psi_1 (y_1) \times D g_2 (y_2) \times \cdots \times D g_{p-1} (y_{p-1}) \times D g_p (y_p) \tag{4.48}$$

where $y_1 = x$, and $y_s = \psi_{s-1}(y_{s-1})$, $s = 2, \ldots, p$. 

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4.2. The iterative mapping $\psi$ of BMD is a diffeomorphism

In this subsection, we first present the following Lemma 4.1 which shows that $\psi_s, s = 1, \ldots, p,$ are diffeomorphisms. Based on this lemma, then we further prove that $\psi$ is a diffeomorphism as well.

Lemma 4.1 If the step size $\alpha < \frac{\mu}{L}$, then the mappings $\psi_s$ defined by (4.44), $s = 1, \ldots, p$, are diffeomorphisms.

Proof. The proof is lengthy and has been relegated to the Appendix. \]

Proposition 4.1 The mapping $\psi$ defined by (4.46) with step size $\alpha < \frac{\mu}{L}$ is a diffeomorphism.

Proof. It follows from Proposition 2.15 in Lee (2013) that the composition of two diffeomorphisms is also a diffeomorphism. Using this fact and previous Lemma 4.1, the proof is completed. \]

4.3. Eigenvalue analysis of the Jacobian of $\psi$ at a strict saddle point

In this subsection, we will analyze the eigenvalues of the Jacobian of $\psi$ at a strict saddle point and show it has at least one eigenvalue with magnitude greater than one, which is a crucial part.

Suppose that $x^* \in \mathbb{R}^n$ is a strict saddle point. Then $\nabla f(x^*) = 0$ and $\psi_s(x^*) = x^*, s = 1, 2, \ldots, p$. Hence,

$$D\psi_s(x^*) = \left( I_n - U_s U_s^T \right) +$$

$$\{ U_s \nabla^2 \varphi_s(x^*(s)) - \alpha \nabla^2 f(x^*) U_s \} \left\{ \nabla^2 \varphi_s \left\{ [\nabla \varphi_s]^{-1} (\nabla \varphi_s(x^*(s)) - \alpha \nabla_s f(x^*)) \right\} \right\}^{-1} U_s^T$$

$$= \left( I_n - U_s U_s^T \right) + \{ U_s \nabla^2 \varphi_s(x^*(s)) - \alpha \nabla^2 f(x^*) U_s \} \left\{ \nabla^2 \varphi_s \left\{ [\nabla \varphi_s]^{-1} (\nabla \varphi_s(x^*(s))) \right\} \right\}^{-1} U_s^T$$

$$= \left( I_n - U_s U_s^T \right) + \{ U_s \nabla^2 \varphi_s(x^*(s)) - \alpha \nabla^2 f(x^*) U_s \} \left\{ \nabla^2 \varphi_s (x^*(s)) \right\}^{-1} U_s^T$$

$$= \left( I_n - \alpha \nabla^2 f(x^*) U_s \right) \left\{ \nabla^2 \varphi_s (x^*(s)) \right\}^{-1} U_s^T,$$

(4.49)

where the second equality is due to $\nabla f(x^*) = 0$; the third equality thanks to $[\nabla \varphi_s]^{-1} (\nabla \varphi_s(x^*(s))) = x^*(s)$.

Furthermore, since the eigenvalues of $D\psi(x^*)$ are the same as those of $\{D\psi(x^*)\}^T$,
it suffices to analyze the eigenvalues of \( \{D\psi (x^*)\}^T \). It follows from (4.48) that
\[
\{D\psi (x^*)\}^T = \{D\psi_1(x^*) \times D\psi_2(x^*) \times \cdots \times D\psi_{p-1}(x^*) \times D\psi_p(x^*)\}^T
\]
\[
= \{D\psi_p(x^*)\}^T \times \{D\psi_{p-1}(x^*)\}^T \times \cdots \times \{D\psi_2(x^*)\}^T \times \{D\psi_1(x^*)\}^T
\]
\[
= \prod_{s=p}^1 \left\{ I_n - \alpha \nabla^2 f (x^*) U_s \left\{ \nabla^2 \varphi_s (x^*(s)) \right\}^{-1} U_s^T \right\}^T
\]
\[
= \prod_{s=p}^1 \left\{ I_n - \alpha U_s \left\{ \nabla^2 \varphi_s (x^*(s)) \right\}^{-1} U_s^T \nabla^2 f (x^*) \right\}
\]
\[
= \prod_{s=p}^1 \left\{ I_n - \alpha U_s \left\{ \nabla^2 \varphi_s (x^*(s)) \right\}^{-1} A_s \right\},
\]
where the third equality uses (4.49) and the last equality is due to definition (3.14) of \( A_s \).

Similar to the case of BCGD, we furthermore define
\[
\tilde{G} \triangleq \frac{1}{\alpha} \left[ I_n - \{D\psi(x^*)\}^T \right],
\]
or equivalently,
\[
(D\psi(x^*))^T = I_n - \alpha \tilde{G}.
\]
The above relation (4.52) clearly means that
\[
\lambda \in \text{eig} \left( \tilde{G} \right) \iff 1 - \alpha \lambda \in \text{eig} \left( (D\psi(x^*))^T \right).
\]
In order to achieve a simple form of \( \tilde{G} \), we first introduce the following notations. Define
\[
\Psi \triangleq \text{Diag} \left( \nabla^2 \varphi_1 (x^*(1)), \nabla^2 \varphi_2 (x^*(2)), \ldots, \nabla^2 \varphi_p (x^*(p)) \right).
\]
Then
\[
\Psi^{-1} = \text{Diag} \left( \left\{ \nabla^2 \varphi_1 (x^*(1)) \right\}^{-1}, \left\{ \nabla^2 \varphi_2 (x^*(2)) \right\}^{-1}, \ldots, \left\{ \nabla^2 \varphi_p (x^*(p)) \right\}^{-1} \right).
\]
Note that \( A, A_{st} \) and \( A_s \) have been defined by (3.12), (3.13) and (3.14), respectively. Then
\[
A = (A_{st})_{1 \leq s,t \leq p} = \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_p \end{pmatrix},
\]

where \( A_{st} \in \mathbb{R}^{n_s \times n_t} \) and \( A_s \in \mathbb{R}^{n_s \times n} \) denote the \((s,t)\)-th block and the \(s\)-th block-row of \( A \), respectively. We further define

\[
T \triangleq \Psi^{-1} A = \left( \{ \nabla^2 \varphi_1 (x^*(s)) \}^{-1} A_{st} \right)_{1 \leq s, t \leq p} = \begin{pmatrix}
\{ \nabla^2 \varphi_1 (x^*(1)) \}^{-1} A_1 \\
\{ \nabla^2 \varphi_2 (x^*(2)) \}^{-1} A_2 \\
\vdots \\
\{ \nabla^2 \varphi_p (x^*(p)) \}^{-1} A_p
\end{pmatrix} = \begin{pmatrix}
T_1 \\
T_2 \\
\vdots \\
T_p
\end{pmatrix},
\]

where \( \{ \nabla^2 \varphi_s (x^*(s)) \}^{-1} A_{st} \in \mathbb{R}^{n_s \times n_t} \) and \( \{ \nabla^2 \varphi_s (x^*(s)) \}^{-1} A_s \in \mathbb{R}^{n_s \times n} \) denote the \((s,t)\)-th block and the \(s\)-th block-row of \( T \), respectively. Given the above notations, we further denote the strictly block lower triangular matrix based on \( T \) as

\[
\tilde{T} \triangleq (\tilde{T}_{st})_{1 \leq s, t \leq p}
\]

with \( p \times p \) blocks and its \((s,t)\)-th block is given by

\[
\tilde{T}_{st} = \begin{cases}
T_{st}, & s > t, \\
0, & s \leq t,
\end{cases}
\]

\[
= \begin{cases}
\{ \nabla^2 \varphi_s (x^*(s)) \}^{-1} A_{st}, & s > t, \\
0, & s \leq t.
\end{cases}
\]

Substituting \( T_s \) (see its definition (4.57)) into (4.50), we deduce that

\[
\{ D\psi (x^*) \}^T = \prod_{s=p}^{1} \{ I_n - \alpha U_s T_s \}.
\]

The following Lemma shows that \( \tilde{G} \) still has a form similar to \( G \) defined by (3.18). In fact, it just replaces \( A \) and \( \breve{A} \) in (3.18) by \( T \) and \( \tilde{T} \), respectively.

**Lemma 4.2** Let \( x^* \in \mathbb{R}^n \) be a strict saddle point. Assume that \( \tilde{G}, T \) and \( \tilde{T} \) are defined by (4.51), (4.57) and (4.58), respectively. Then

\[
\tilde{G} = (I_n + \alpha \tilde{T})^{-1} T.
\]

**Proof.** With the identifications (4.51) \( \sim \) (3.18), (4.60) \( \sim \) (3.17), (4.57) \( \sim \) (3.12) and (4.58) \( \sim \) (3.22), we are able to apply the same arguments as in the proof of Lemma 3.3 to obtain (4.61). Since the proof follows a similar pattern, it is therefore omitted.
Based on the above Lemma 4.2, along with definitions (4.55), (4.57), (4.58), (3.12) and (3.22), we further have

\[
\tilde{G} = \left( I_n + \alpha \Psi^{-\frac{1}{2}} \tilde{A} \right)^{-1} \Psi^{-\frac{1}{2}} A
\]

\[
= (\Psi + \alpha \tilde{A})^{-1} A
\]

\[
= \left[ \Psi^\frac{1}{2} \left( I_n + \alpha \Psi^{-\frac{1}{2}} \tilde{A} \Psi^{-\frac{1}{2}} \right) \Psi^\frac{1}{2} \right]^{-1} A
\]

\[
= \Psi^{-\frac{1}{2}} \left( I_n + \alpha \Psi^{-\frac{1}{2}} \tilde{A} \Psi^{-\frac{1}{2}} \right)^{-1} \Psi^{-\frac{1}{2}} A
\]

\[
= \left( \Psi^{-\frac{1}{2}} \right)^T \left[ I_n + \alpha \Psi^{-\frac{1}{2}} \tilde{A} \left( \Psi^{-\frac{1}{2}} \right)^T \right]^{-1} \Psi^{-\frac{1}{2}} A,
\]

where the second equality holds because \( \Psi^{-\frac{1}{2}} \) denotes the unique symmetric, positive definite square root matrix of the symmetric, positive definite matrix \( \Psi^{-1} \).

Switching the order of the products (4.62) by moving the first component to the last, we get a new matrix

\[
\overline{G} \triangleq \left[ I_n + \alpha \Psi^{-\frac{1}{2}} \tilde{A} \left( \Psi^{-\frac{1}{2}} \right)^T \right]^{-1} \Psi^{-\frac{1}{2}} A \left( \Psi^{-\frac{1}{2}} \right)^T.
\]

Note that eig \((XY) = \text{eig} (YX)\) for any two square matrices, thus

\[
\text{eig} \left( \tilde{G} \right) = \text{eig} \left( \overline{G} \right),
\]

which shows that it suffices to analyze eigenvalues of \( \overline{G} \).

Before presenting Lemma 4.4, the grand result of this subsection, we first provide two special properties of \( \Psi^{-\frac{1}{2}} A \left( \Psi^{-\frac{1}{2}} \right)^T \) which will be used in the subsequent analysis.

**Lemma 4.3** Let \( x^* \in \mathbb{R}^n \) be a strict saddle point. Assume that \( A \) and \( \Psi \) are defined by (3.12) and (4.54), respectively. Then,

(i) \( \Psi^{-\frac{1}{2}} A \left( \Psi^{-\frac{1}{2}} \right)^T \) has at least one negative eigenvalue.

(ii) The spectral radius of the symmetric matrix \( \Psi^{-\frac{1}{2}} A \left( \Psi^{-\frac{1}{2}} \right)^T \) is upper bounded by \( \frac{\mu}{L} \).

**Proof.** (i) Since \( \Psi^{-\frac{1}{2}} A \left( \Psi^{-\frac{1}{2}} \right)^T \) is a congruent transformation of \( A \), they have same index of inertia. In addition, \( x^* \) is a strict saddle point implies that \( A = \nabla^2 f(x^*) \) (see its definition (3.12)) has at least one negative eigenvalue. Hence,
\[ \Psi^{-\frac{1}{2}}A \left( \Psi^{-\frac{1}{2}} \right)^T \] has at least one negative eigenvalue as well.

(ii) In what follows, we will prove that

\[ \rho \left( \Psi^{-\frac{1}{2}}A \left( \Psi^{-\frac{1}{2}} \right)^T \right) \leq \frac{L}{\mu}. \]  \hspace{1cm} (4.65)

Obviously, it suffices to prove that

\[ -\frac{L}{\mu} I_n \preceq \Psi^{-\frac{1}{2}}A \left( \Psi^{-\frac{1}{2}} \right)^T \preceq \frac{L}{\mu} I_n \]

holds. It follows easily from (2.2) and Lemma 7 in Panageas and Piliouras (2016) that

\[ -LI_n \preceq \nabla^2 f(x^*) = A \preceq LI_n. \]  \hspace{1cm} (4.66)

In addition, Eqs. (4.35) means that

\[ \nabla^2 \varphi_s(x^*(s)) \succeq \mu_s I_{n_s} \succeq \mu I_{n_s}, \ s = 1, 2, \ldots, p, \]  \hspace{1cm} (4.67)

where the last inequality is due to the definition of \( \mu \) (see it definition in (4.37)). Consequently, combined with the definition (4.54), the above inequalities imply that

\[ \frac{1}{\mu} \Psi \succeq I_n, \]  \hspace{1cm} (4.68)

which, combined with (4.66) further implies

\[ -\frac{L}{\mu} \Psi I_n \preceq A \preceq \frac{L}{\mu} \Psi I_n. \]  \hspace{1cm} (4.69)

Multiplying the left-hand side of the above inequalities by \( \Psi^{-\frac{1}{2}} \) and the right-hand side by \( \left( \Psi^{-\frac{1}{2}} \right)^T \), respectively, we arrive at

\[ -\frac{L}{\mu} I_n \preceq \Psi^{-\frac{1}{2}}A \left( \Psi^{-\frac{1}{2}} \right)^T \preceq \frac{L}{\mu} I_n. \]  \hspace{1cm} (4.70)

Thus, the proof is finished.

Based on the above Lemma 4.3 and Lemma 6.4 in Section 6, we will prove that there exists at least one eigenvalue of \( G \) defined by (4.63) which belongs to \( \Omega \) defined by (3.27).

**Lemma 4.4** Suppose that \( x^* \in \mathbb{R}^n \) is a strict saddle point, and

\[ \overline{G} = \left[ I_n + \alpha \Psi^{-\frac{1}{2}} \hat{A} \left( \Psi^{-\frac{1}{2}} \right)^T \right]^{-1} \Psi^{-\frac{1}{2}}A \left( \Psi^{-\frac{1}{2}} \right)^T \]  \hspace{1cm} (4.71)

is given by (4.63) with \( \alpha \in \left( 0, \frac{\mu}{L} \right) \), where \( L \) is determined by (2.2); \( \mu, A \) and \( \hat{A} \) are given by (4.37), (3.12) and (3.22), respectively; and \( \Psi^{-\frac{1}{2}} \) denotes the unique symmetric, positive definite square root matrix of the symmetric, positive definite matrix \( \Psi^{-1} \) defined by (4.55). Then, there exists at least one eigenvalue of the \( \overline{G} \) which belongs to \( \Omega \) defined by (3.27).
Proof. Since \( x^* \) is a strict saddle point and \( A = \nabla^2 f(x^*) \) is given by (3.12), Lemma 4.3 clearly implies that \( \Psi^{-\frac{1}{2}}A\left(\Psi^{-\frac{1}{2}}\right)^T \) has at least one negative eigenvalue and the spectral radius of the symmetric matrix \( \Psi^{-\frac{1}{2}}A\left(\Psi^{-\frac{1}{2}}\right)^T \) is upper bounded by \( \frac{\mu}{L} \).

Further, note that \( \tilde{G} = \left[I_n + \alpha\Psi^{-\frac{1}{2}}\tilde{A}\left(\Psi^{-\frac{1}{2}}\right)^T\right]^{-1}\Psi^{-\frac{1}{2}}A\left(\Psi^{-\frac{1}{2}}\right)^T \) and \( \psi^{-\frac{1}{2}}A\left(\psi^{-\frac{1}{2}}\right)^T \) is a strictly block lower triangle matrix based on \( \Psi^{-\frac{1}{2}}A\left(\Psi^{-\frac{1}{2}}\right)^T \), where \( \tilde{A} \) is defined by (3.22). Therefore, by applying Lemma 6.4 in Section 6 with identifications \( \Psi^{-\frac{1}{2}}A\left(\Psi^{-\frac{1}{2}}\right)^T \sim B, \psi^{-\frac{1}{2}}A\left(\psi^{-\frac{1}{2}}\right)^T \sim \tilde{B}, \alpha \sim \beta \) and \( \rho\left(\Psi^{-\frac{1}{2}}A\left(\Psi^{-\frac{1}{2}}\right)^T\right) \sim \rho(B) \), we know, for any \( \alpha \in (0, \frac{\mu}{L}) \), there exists at least one eigenvalue of \( \tilde{G} \) belonging to \( \Omega \) defined by (3.27).

In addition, Lemma 4.3 implies that gradient Lipschitz continuous constant \( \frac{L}{\mu} \geq \rho\left(\Psi^{-\frac{1}{2}}A\left(\Psi^{-\frac{1}{2}}\right)^T\right) \), which leads immediately to \( \alpha \in (0, \frac{\mu}{L}) \subseteq \left(0, \frac{1}{\rho\left(\Psi^{-\frac{1}{2}}A\left(\Psi^{-\frac{1}{2}}\right)^T\right)}\right) \).

Then, the above arguments clearly imply that the proposition holds true. \( \Box \)

The following proposition shows that the Jacobian of the BMD iterative mapping \( \psi \) defined by (4.46) at a strict saddle point admits at least one eigenvalue whose magnitude is strictly greater than one.

**Proposition 4.2** Assume that BMD iterative mapping \( \psi \) is defined by (4.46) with \( \alpha \in (0, \frac{\mu}{L}) \), and \( x^* \in \mathbb{R}^n \) is a strict saddle point. Then \( D\psi(x^*) \) has at least one eigenvalue whose magnitude is strictly greater than one.

**Proof.** Note that (4.64) and (4.63) imply that \( \tilde{G} \) and \( \nabla G \) have same eigenvalues. Consequently, Lemma 4.4 means there exists at least one eigenvalue of the \( \tilde{G} \) lying in \( \Omega \) defined by (6.130).

In what follows, with the identifications \( D\psi(x^*) \sim Dg_{\alpha}(x^*), \tilde{G} \sim G \) and \( \frac{L}{\mu} \sim \rho \), we are able to apply the same arguments as in the proof of Proposition 3.3 in Subsection 3.3 to finish the proof. Since the proof follows a similar pattern, it is therefore omitted. \( \Box \)

### 4.4. Main results of BMD method

We first introduce the following proposition, which asserts that the limit point of the sequence generated by the BMD method 4.1 is a critical point of \( f \).

**Proposition 4.3** Under Assumption 2.1, if \( \{x_k\}_{k \geq 0} \) is generated by the BMD method 4.1 with \( 0 < \alpha < \frac{\mu}{L} \), \( \lim_k x_k \) exists and denote it as \( x^* \), then \( x^* \) is a critical point of \( f \), i.e., \( \nabla f(x^*) = 0 \).
Proof. First, since \( \{ x_k \}_{k \geq 0} \) is generated by the BMD method 4.1, then \( x_k = \psi^k(x_0) \), where the BMD iterative mapping \( \psi \) is defined by (4.54). Hence, \( \lim_{k \to \infty} x_k = \lim_{k \to \infty} \psi^k(x_0) = x^* \). Since \( \psi \) is a diffeomorphism, we immediately know that \( x^* \) is a fixed point of \( \psi \).

Second, notice that \( \psi \) and \( \psi_s \) are defined by (4.46) and (4.44), respectively. If \( x^* \) is a fixed point of \( \psi \), then
\[
\psi(x^*) = x^*,
\] (4.72)
which implies that
\[
\psi_s(x^*) = (I_n - U_s U_s^T) x^* + U_s [\nabla \varphi_s]^{-1} (\nabla \varphi_s (x^*(s)) - \alpha \nabla_s f(x^*)) , \quad s = 1, 2, \ldots, p.
\] (4.73)
Consequently,
\[
x^*(s) = [\nabla \varphi_s]^{-1} (\nabla \varphi_s (x^*(s)) - \alpha \nabla_s f(x^*)) , \quad s = 1, 2, \ldots, p.
\] (4.74)
Since Lemma 9.4 in Appendix asserts that \( \nabla \varphi_s \) is a diffeomorphism, then \( [\nabla \varphi_s]^{-1} \) is a diffeomorphism as well. Thus it follows from (4.74) that
\[
\nabla \varphi_s (x^*(s)) = \nabla \varphi_s (x^*(s)) - \alpha \nabla_s f(x^*) , \quad s = 1, 2, \ldots, p.
\] (4.75)
We arrive at
\[
\nabla_s f(x^*) = 0, \quad s = 1, 2, \ldots, p,
\] (4.76)
or equivalently, \( x^* \) is a stationary point of \( f \). Thus the proof is finished. \( \square \)

Armed with the results established in previous subsections, we now state and prove our main theorem of the BMD method, whose proof is similar to that of Theorem 3.2 in Subsection 3.4. However, its proof is still given as follows in detail for the sake of completeness.

**Theorem 4.1** Let \( f \) be a \( C^2 \) function and \( x^* \) be a strict saddle point. If \( \{ x_k \} \) is generated by the BMD method 4.1 with \( 0 < \alpha < \frac{\mu}{L} \), then
\[
\mathbb{P}_\nu \left[ \lim_{k \to \infty} x_k = x^* \right] = 0.
\]

**Proof.** First, Proposition 4.3 implies that, if \( \lim_{k \to \infty} x_k \) exists then it must be a critical point. Hence, we consider calculating the Lebesgue measure (or probability with respect to the prior measure \( \nu \)) of the set \( \left[ \lim_{k \to \infty} x_k = x^* \right] = W^s(x^*) \) (see Definition 2.3).

Second, since Proposition 4.1 means the BMD iterative mapping \( \psi \) is a diffeomorphism, we replace \( \phi \) and fixed point by \( \psi \) and the strict saddle point \( x^* \) in the

\( ^5 \psi^k \) denotes the composition of \( \psi \) with itself \( k \) times.
above Stable Manifold Theorem 3.1 in Subsection 3.4, respectively. Then the manifold \( W^s_{loc}(x^*) \) has strictly positive codimension because of Proposition 4.2 and \( x^* \) being a strict saddle point. Hence, \( W^s_{loc}(x^*) \) has measure zero.

In what follows, we are able to apply the same arguments as in [Lee et al. (2016)] to finish the proof of the theorem. Since the proof follows a similar pattern, it is therefore omitted.

Given the above Theorem 4.1, we immediately obtain the following Theorem 4.2 and its corollary by the same arguments as in the proofs of Theorem 2 and Corollary 12 in [Panageas and Piliouras (2016)]. Therefore, we omit their proofs.

**Theorem 4.2 (Non-isolated)** Let \( f : \mathbb{R}^n \to \mathbb{R} \) be a twice continuously differentiable function and \( \sup_{x \in \mathbb{R}^n} \|\nabla f(x)\|_2 \leq L < \infty \). The set of initial conditions \( x \in \mathbb{R}^n \) so that the BMD method 4.1 with step size \( 0 < \alpha < \frac{\mu}{L} \) converges to a strict saddle point is of (Lebesgue) measure zero, without assumption that critical points are isolated.

A straightforward corollary of Theorem 4.2 is given below:

**Corollary 4.1** Assume that the conditions of Theorem 4.2 are satisfied and all saddle points of \( f \) are strict. Additionally, assume \( \lim_{k} \psi^k(x) \) exists for all \( x \) in \( \mathbb{R}^n \).

Then

\[
P_{\nu} \left[ \lim_{k} \psi^k(x) = x^* \right] = 1,
\]

where \( \psi \) is defined by (4.54) and \( x^* \) is a local minimum.

5. The PBCD method

In this section, we will prove that the PBCD method in [Hong et al. (2017); Fercoq and Richtrik (2013); Hong et al. (2015)] converges to minimizers as well, almost surely with random initialization.

5.1. The PBCD method description

For clarity of notation, recall the vector of decision variables \( x \) has been assumed to have the following partition (see (2.3)):

\[
x = \begin{pmatrix}
x(1) \\
x(2) \\
\vdots \\
x(p)
\end{pmatrix},
\]

(5.77)
where \( x(t) \in \mathbb{R}^{n_t}, n_1, n_2, \ldots, n_p \) are \( p \) positive integer numbers satisfying \( \sum_{s=1}^{t} n_t = n \).

Correspondingly, we assume a set of variables \( x^s_k \) has the following partition as well:

\[
x^s_k \triangleq \begin{pmatrix}
x^s_k(1) \\
x^s_k(2) \\
\vdots \\
x^s_k(p)
\end{pmatrix}, \quad s = 1, \ldots, p; \ k = 0, 1, \ldots.
\] (5.78)

where \( x^s_k(t) \in \mathbb{R}^{n_t}; t, s = 1, \ldots, p; k = 0, 1, \ldots \)

Given the above notations, the detailed description of the PBCD method for problem (2.1) is given below.

### Method 5.1 (PBCD)

**Input:** \( \alpha < \frac{1}{L} \).

**Initialization:** \( x_0 \in \mathbb{R}^n \).

**General Step** \( (k = 0, 1, \ldots) \): Set \( x_0^0 = x_k \) and define recursively for \( s = 1, 2, \ldots, p; \ t = 1, 2, \ldots, p \):

- If \( t = s \),
  \[
  x^s_k(t) = \arg \min_{x(t)} \left\{ f \left( x^s_k(t-1), x^{s-1}(t-1), x(t), x^s_k(t+1), x^{s-1}(p) \right) + \frac{1}{2\alpha} \| x(t) - x^{s-1}(t) \|_2^2 \right\}. \] (5.79)

- Else
  \[
  x^s_k(t) = x^{s-1}_k(t). \] (5.80)

**End**

Set \( x_{k+1} = x^p_k \).

It follows from \( \alpha < \frac{1}{L} \) that

\[
f \left( x^{s-1}_k(1), \ldots, x^{s-1}_k(s-1), x(s), x^{s-1}_k(s+1), x^{s-1}_k(p) \right) + \frac{1}{2\alpha} \| x(s) - x^{s-1}_k(s) \|_2^2
\]

is a strongly convex function with respect to variable \( x(s) \). Hence, let \( x^s_k(s) \) be the unique minimizer of problem (5.79). Then by the KKT condition,

\[
0 = \nabla_s f \left( x^{s-1}_k(1), \ldots, x^{s-1}_k(s-1), x^s_k(s), x^{s-1}_k(s+1), x^{s-1}_k(p) \right) + \frac{1}{\alpha} \left( x^s_k(s) - x^{s-1}_k(s) \right),
\]

which is equivalent to

\[
x^{s-1}_k(s) = x^s_k(s) + \alpha \nabla_s f \left( x^{s-1}_k(1), \ldots, x^{s-1}_k(s-1), x^s_k(s), x^{s-1}_k(s+1), x^{s-1}_k(p) \right). \] (5.81)
Combining Eqs. (5.78), (5.80) and (5.81), we have the following relationships between \(x_{s-1}^k\) and \(x_s^k\),

\[
x_{s-1}^k = x_k^s + \alpha U_s \nabla_s f(x_k^s), \quad s = 1, \ldots, p.
\]  
(5.82)

Recall that, for a given step size \(\alpha > 0\), we have used \(g_{sf}^s\) (see its definition in (3.5)) to denote the gradient mapping of function \(f\) with respect to variable \(x(s)\), i.e.,

\[
g_{sf}^s(x) = x - \alpha U_s \nabla_s f(x), \quad s = 1, \ldots, p.
\]  
(5.83)

By using same notations, for a given step size \(\alpha > 0\), the gradient mapping of function \(-f\) with respect to variable \(x(s)\) is

\[
g_{s(-f)}^s(x) = x - \alpha U_s \nabla_s (-f(x)) = x - \alpha U_s (-\nabla_s f(x)) = x + \alpha U_s \nabla_s f(x).
\]  
(5.84)

It is obvious that Eqs. (5.82) and (5.84) imply that

\[
x_{s-1}^k = g_{s(-f)}^s(x_k^s), \quad s = 1, \ldots, p.
\]  
(5.85)

Note that Lemma 5.2 in Subsection 5.2 implies that \(g_{s(-f)}^s\) is a diffeomorphism with \(\alpha \in (0, \frac{1}{L})\). We denote its inverse as \([g_{s(-f)}^s]^{-1}\) which is a diffeomorphism as well.

Then, (5.85) is further equivalent to

\[
x_k^s = [g_{s(-f)}^s]^{-1}(x_{s-1}^k), \quad s = 1, \ldots, p.
\]  
(5.86)

**Lemma 5.1** Assume that \(g_{s(-f)}^s\) is determined by (5.84), \(s = 1, \ldots, p\). Given \(x_k\), the above PBCD method generates \(x_{k+1}\) in the following manner,

\[
x_{k+1} = [g_{s(-f)}^s]^{-1}(x_k),
\]  
(5.87)

where the composite iterative mapping \([g_{s(-f)}^s]^{-1}\) denotes the inverse mapping of the following mapping

\[
g_{s(-f)} = g_{s(-f)}^1 \circ g_{s(-f)}^2 \circ \cdots \circ g_{s(-f)}^{p-1} \circ g_{s(-f)}^p.
\]  
(5.88)

**Proof.** According to the PBCD method and (5.86), we have \(x_0^0 = x_k\) and

\[
x_{k+1} = [g_{s(-f)}^p]^{-1} \circ [g_{s(-f)}^{p-1}]^{-1} \circ \cdots \circ [g_{s(-f)}^2]^{-1} \circ [g_{s(-f)}^1]^{-1}(x_k)
\]  
\[
= [g_{s(-f)}^1 \circ g_{s(-f)}^2 \circ \cdots \circ g_{s(-f)}^{p-1} \circ g_{s(-f)}^p]^{-1}(x_k).
\]  
(5.89)

Thus the proof is finished.

By simple computation, the Jacobian of \(g_{s(-f)}^s\) is given by

\[
Dg_{s(-f)}^s(x) = I_n + \alpha \nabla^2 f(x)U_s U_s^T,
\]  
(5.90)
where
\[ \nabla^2 f(x) = \left( \frac{\partial^2 f(x)}{\partial x(s) \partial x(t)} \right)_{1 \leq s, t \leq p}. \]  
(5.91)

Since \( g_{\alpha(-f)}(x) \) is defined by (5.88), the chain rule implies that the Jacobian of the mapping \( g \) is
\[ Dg_{\alpha(-f)}(x) = Dg_{\alpha(-f)}^p(y_p) \times Dg_{\alpha(-f)}^{p-1}(y_{p-1}) \times \cdots \times Dg_{\alpha(-f)}^2(y_2) \times Dg_{\alpha(-f)}^1(y_1), \]  
(5.92)

where \( y_1 = x \), and \( y_s = g_{\alpha(-f)}^{s-1}(y_{s-1}), s = 2, \ldots, p \).

5.2. The iterative mapping \( [g_{\alpha(-f)}]^{-1} \) of the PBCD method is a diffeomorphism

In this subsection, we first present the following Lemma 5.2 which shows that \( g_{\alpha(-f)}^s, s = 1, \ldots, p, \) are diffeomorphisms. Based on this lemma, then we further prove that \( [g_{\alpha(-f)}]^{-1} \) is a diffeomorphism as well.

**Lemma 5.2** If step size \( \alpha < \frac{1}{L} \), then the mappings \( g_{\alpha(-f)}^s \) defined by (3.5), \( s = 1, \ldots, p, \) are diffeomorphisms.

**Proof.** Note that \( L \) is also the Lipschitz constant of gradient of \(-f\). By replacing \(-f\) with \( f \), Lemma 3.1 in Subsection 3.2 implies that \( g_{\alpha(-f)}^s \) is a diffeomorphism with \( \alpha \in (0, \frac{1}{L}) \), the proof is completed. 

**Proposition 5.1** The mapping \( [g_{\alpha(-f)}]^{-1} \) determined by (5.88) with step size \( \alpha < \frac{1}{L} \) is a diffeomorphism.

**Proof.** Note that Lemma 5.2 implies that \( g_{\alpha(-f)}^s, s = 1, \ldots, p, \) defined by (3.5) are diffeomorphisms and \( g_{\alpha(-f)} \) is defined (5.88) with \( 0 < \alpha < \frac{1}{L} \). By a similar argument as in the proof of Proposition 3.1 in Subsection 3.2 we know \( g_{\alpha(-f)} \) is a diffeomorphism. Thus its inverse is a diffeomorphism as well, the proof is completed. 

5.3. Eigenvalue analysis of the Jacobian of \( [g_{\alpha(-f)}]^{-1} \) at a strict saddle point

In this subsection, we consider the eigenvalues of the Jacobian of \( [g_{\alpha(-f)}]^{-1} \) at a strict saddle point, which is a crucial part in our entire proof.

If \( x^* \in \mathbb{R}^n \) is a strict saddle point, then \( \nabla f(x^*) = 0 \). Consequently,
\[ x^* = [g_{\alpha(-f)}^s]^{-1}(x^*), \quad s = 1, \ldots, p, \]  
(5.93)

which implies that
\[ x^* = g_{\alpha(-f)}(x^*). \]  
(5.94)
More importantly, the above Eq. (5.94) implies that the Jacobian of \([g_\alpha(-f)]^{-1}\) at \(x^*\) can be expressed as
\[
D[g_\alpha(-f)]^{-1}(x^*) = (Dg_\alpha(-f)(x^*))^{-1},
\]
(5.95)
which is due to inverse function theorem in Spivak (1965). Hence, in order to show that there is at least one eigenvalue of \(D[g_\alpha(-f)]^{-1}(x^*)\) whose magnitude is strictly greater than one, we first argue that \(Dg_\alpha(-f)(x^*)\) still has a similar structure as that of \(Dg_\alpha f(x^*)\) in Section 3.

Specifically, chain rule (5.92) and Eq. (5.94) imply
\[
(Dg_\alpha(-f)(x^*))^T = (Dg_\alpha^1(-f)(x^*))^T \times \cdots \times (Dg_\alpha^p(-f)(x^*))^T,
\]
(5.96)
Moreover, the eigenvalues of \(Dg_\alpha(-f)(x^*)\) are the same as those of its transpose. Hence, we compute
\[
(Dg_\alpha(-f)(x^*))^T = (Dg_\alpha^1(-f)(x^*))^T \times \cdots \times (Dg_\alpha^p(-f)(x^*))^T \times (I_n + \alpha U_1 U_1^T \nabla^2 f(x^*)) \times \cdots \times (I_n + \alpha U_{p-1} U_{p-1}^T \nabla^2 f(x^*)) \times (I_n + \alpha U_p U_p^T \nabla^2 f(x^*)) ,
\]
(5.97)
where the second equality is due to (5.90).

Now we define
\[
H \triangleq \frac{1}{\alpha} \left[ I_n - (Dg_\alpha(-f)(x^*))^T \right],
\]
(5.98)
or equivalently,
\[
(Dg_\alpha(-f)(x^*))^T = I_n - \alpha H.
\]
(5.99)
The above relation (5.99) clearly means that
\[
\lambda \in \text{eig} (H) \Leftrightarrow 1 - \alpha \lambda \in \text{eig} \left( (Dg_\alpha(-f)(x^*))^T \right).
\]
(5.100)
For the sake of clarification, we rewrite \(A\) defined by (3.12) below:
\[
A = \left( A_{st} \right)_{1 \leq s, t \leq p},
\]
(5.101)
and its \((s, t)\)-th block is given by
\[
A_{st} = \frac{\partial^2 f(x^*)}{\partial x^*(s) \partial x^*(t)}, \quad 1 \leq s, t \leq p.
\]
(5.102)
Similarly, we denote the strictly block upper triangular matrix based on \(A\) as
\[
\hat{A} \triangleq \left( \hat{A}_{st} \right)_{1 \leq s, t \leq p},
\]
(5.103)
with \( p \times p \) blocks and its \((s, t)\)-th block is given by
\[
\hat{A}_{st} = \begin{cases} 
A_{st}, & s < t, \\
0, & s \geq t.
\end{cases}
\tag{5.104}
\]

Based on the above notations, we are able to obtain the following Lemma 5.3 which is similar to Lemma 3.3 in Subsection 3.3. It gives a simple expression of \( H \) in terms of \( A \) and \( \hat{A} \).

**Lemma 5.3** Let \( x^* \in \mathbb{R}^n \) be a strict saddle point. Assume that \( H, A \) and \( \hat{A} \) are defined by (5.98), (5.101) and (5.103), respectively. Then
\[
H = - \left( I_n - \alpha \hat{A} \right)^{-1} A. \tag{5.105}
\]

**Proof.** The proof is similar to that of Lemma 3.3 in Subsection 3.3. \( \square \)

In what follows, we proceed now in a way similar to the case of BCGD, although some specific technical difficulties will occur in the analysis.

The following proposition shows that the Jacobian of the PBCD iterative mapping \([g_{\alpha(-f)}]^{-1}\) determined by (5.88) at a strict saddle point admits at least one eigenvalue whose magnitude is strictly greater than one, which plays a key role in this subsection.

**Proposition 5.2** Assume that PBCD iterative mapping \([g_{\alpha(-f)}]^{-1}\) is determined by (5.88) and \( x^* \in \mathbb{R}^n \) is a strict saddle point. Then \( D[g_{\alpha(-f)}]^{-1}(x^*) \) has at least one eigenvalue whose magnitude is strictly greater than one.

**Proof.** First, recall the Jacobian of \([g_{\alpha(-f)}]^{-1}\) at \( x^* \) can be expressed as
\[
D[g_{\alpha(-f)}]^{-1}(x^*) = (Dg_{\alpha(-f)}(x^*))^{-1}. \tag{5.106}
\]
Second, recall Eqs. (5.98)-(5.100) as follows.
\[
H \triangleq \frac{1}{\alpha} \left[ I_n - (Dg_{\alpha(-f)}(x^*))^T \right], \tag{5.107}
\]
or equivalently,
\[
(Dg_{\alpha(-f)}(x^*))^T = I_n - \alpha H. \tag{5.108}
\]
It follows from Lemma 5.3 that \( H \) has the following expression:
\[
H = - \left( I_n - \alpha \hat{A} \right)^{-1} A = \left( I_n + \alpha (-\hat{A}) \right)^{-1} (-A), \tag{5.109}
\]
where \( A \) and \( \hat{A} \) are defined by (5.101) and (5.103). Combining Eqs. (5.106) and (5.108), we have

\[
\text{eig}\left(D\left[g_\alpha(-f)\right]^{-1}(x^*)\right) \\
= \text{eig}\left(\left\{D\left[g_\alpha(-f)\right]^{-1}(x^*)\right\}^T\right) \\
= \text{eig}\left(\left\{(Dg_\alpha(-f)(x^*))^T\right\}^{-1}\right) \\
= \text{eig}\left(\left(I_n - \alpha H\right)^{-1}\right) \\
= \text{eig}\left(\left\{I_n - \alpha \left[I_n + \alpha \left(-\hat{A}\right)\right]^{-1}(-A)\right\}^{-1}\right),
\]

where the last equality is due to (5.109).

Since \( A = \nabla^2 f(x^*) \) and \( x^* \) is a strictly saddle point, \( A \) has at least one negative eigenvalue. Hence, \(-A\) has at least one positive eigenvalue. Consequently, by applying Lemma 6.6 with identifications \(-A \sim B, -\hat{A} \sim \hat{B}, \alpha \sim \beta \) and \( \rho(A) \sim \rho(B) \), we know, for any \( \alpha \in \left(0, \frac{1}{\rho(A)}\right) \subseteq \left(0, \frac{1}{\rho(A)}\right) \). Then, the above arguments and (5.110) imply that the proposition holds true.

5.4. Main results of PBCD

We first introduce the following proposition, which asserts that the limit point of the sequence generated by the PBCD method is a critical point of \( f \).

**Proposition 5.3** Under Assumption 2.1, if \( \{x_k\}_{k \geq 0} \) is generated by the PBCD method with \( 0 < \alpha < \frac{1}{L} \), \( \lim_{k} x_k \) exists and denote it as \( x^* \), then \( x^* \) is a critical point of \( f \), i.e., \( \nabla f(x^*) = 0 \).

**Proof.** Notice that \( \{x_k\}_{k \geq 0} \) is generated by the PBCD method. We clearly have \( x_k = \left\{[g_\alpha(-f)]^{-1}\right\}^k(x_0) \), where \( [g_\alpha(-f)]^{-1} \) denotes the inverse mapping of \( g_\alpha(-f) \) defined by (5.88). Hence, \( \lim_{k} x_k = \lim_{k} \left\{[g_\alpha(-f)]^{-1}\right\}^k(x_0) = x^* \). Since \( [g_\alpha(-f)]^{-1} \) is a diffeomorphism, we immediately know that \( x^* \) is a fixed point of \( [g_\alpha(-f)]^{-1} \).

\[^6\left\{[g_\alpha(-f)]^{-1}\right\}^k \text{ denotes the composition of } [g_\alpha(-f)]^{-1} \text{ with itself } k \text{ times.} \]
Equivalently, it is a fixed point of \( g_{\alpha(-f)} \). It follows easily from the definition (5.88) of \( g_{\alpha(-f)} \) that \( \nabla f(x^*) = 0 \). Thus the proof is finished.

Armed with the results established in previous subsections and the above Proposition 5.3, we now state and prove our main theorem of PBCD, whose proof is similar to that of Theorem 3.2 in Subsection 3.4. However, its proof is still given as follows in detail for the sake of completeness.

**Theorem 5.1** Let \( f \) be a \( C^2 \) function and \( x^* \) be a strict saddle point. If \( \{x_k\}_{k \geq 0} \) is generated by the PBCD method 5.1 with \( 0 < \alpha < \frac{1}{L} \), then
\[
\mathbb{P}_\nu \left[ \lim_{k \to \infty} x_k = x^* \right] = 0.
\]

**Proof.** First, Proposition 5.3 implies that, if \( \lim_{k} x_k \) exists then it must be a critical point. Hence, we consider calculating the Lebesgue measure (or probability with respect to the prior measure \( \nu \)) of the set \( \lim_{k} x_k = x^* \) (see Definition 2.3).

Second, since Proposition 5.1 means PBCD iterative mapping \( [g_{\alpha(-f)}]^{-1} \) is a diffeomorphism, we replace \( \phi \) and fixed point by \( [g_{\alpha(-f)}]^{-1} \) and the strict saddle point \( x^* \) in the above Stable Manifold Theorem 3.1 in Subsection 3.4 respectively. Then the manifold \( W_{\text{loc}}^s(x^*) \) has strictly positive codimension because of Proposition 5.2 and \( x^* \) being a strict saddle point. Hence, \( W_{\text{loc}}^s(x^*) \) has measure zero.

In what follows, we are able to apply the same arguments as in Lee et al. (2016) to finish the proof of the theorem. Since the proof follows a similar pattern, it is therefore omitted.

Given the above Theorem 5.1, we immediately obtain the following Theorem 5.2 and its corollary by the same arguments as in the proofs of Theorem 2 and Corollary 12 in Panageas and Piliouras (2016). Therefore, we omit their proofs.

**Theorem 5.2 (Non-isolated)** Let \( f : \mathbb{R}^n \to \mathbb{R} \) be a twice continuously differentiable function and \( \sup_{x \in \mathbb{R}^n} \|\nabla f(x)\|^2 < \infty \). The set of initial conditions \( x \in \mathbb{R}^n \) so that the PBCD method 5.1 with step size \( 0 < \alpha < \frac{1}{L} \) converges to a strict saddle point is of (Lebesgue) measure zero, without assumption that critical points are isolated.

A straightforward corollary of Theorem 5.2 is given below:

**Corollary 5.1** Assume that the conditions of Theorem 5.2 are satisfied and all saddle points of \( f \) are strict. Additionally, the composite iterative mapping \( [g_{\alpha(-f)}]^{-1} \) denotes the inverse mapping of \( g_{\alpha(-f)} \), assume \( \lim_{k} \{[g_{\alpha(-f)}]^{-1}\}^k(x) \) exists for all \( x \) in \( \mathbb{R}^n \). Then
\[
\mathbb{P}_\nu \left[ \lim_{k \to \infty} \{[g_{\alpha(-f)}]^{-1}\}^k(x) = x^* \right] = 1,
\]
where \( [g_{\alpha(-f)}]^{-1} \) denotes the inverse mapping of \( g_{\alpha(-f)} \) defined by (5.88) and \( x^* \) is a local minimum.
6. Several Technical Lemmas

In what follows, we will provide several technical lemmas (Lemmas 6.1 – 6.6), which provide the basis for proving that Jacobian of iterative mappings including \( g_{\alpha f} \) defined by (3.7) in Section 3, \( \psi \) defined by (4.54) in Section 4 and \([g_{\alpha(-f)}]^{-1}\) defined by (5.87) in Section 5 at a strict saddle point has at least one eigenvalue with magnitude strictly greater than one. In particular, Lemma 6.4 gives a sufficiently exact description of the distribution of the eigenvalues of matrix with the kind of structure as \( G \) which is related to \( g_{\alpha f} \) and \( \psi \), while Lemmas 6.5 and 6.6 give the similar description of that of the eigenvalues of matrix with the kind of structure as \( H \) which is related to \( g_{\alpha(-f)} \).

Before presenting the grand result of this subsection, we first introduce two notations which will be used in what follows. Assume \( B \in \mathbb{R}^{n \times n} \) is symmetric matrix with \( p \times p \) blocks. Specifically,

\[
B \triangleq (B_{st})_{1 \leq s,t \leq p},
\]

and its \((s,t)\)-th block

\[
B_{st} \in \mathbb{R}^{n_s \times n_t}, \quad 1 \leq s, t \leq p,
\]

where \( n_1, n_2, \ldots, n_p \) are \( p \) positive integer numbers satisfying \( \sum_{s=1}^{p} n_s = n \). In addition, we denote the strictly block lower triangular matrix based on \( B \) as

\[
\hat{B} \triangleq (\hat{B}_{st})_{1 \leq s,t \leq p}
\]

with \( p \times p \) blocks and its \((s,t)\)-th block is given by

\[
\hat{B}_{st} = \begin{cases} 
B_{st}, & s > t, \\
0, & s \leq t.
\end{cases}
\]

Similarly, we denote the strictly block upper triangular matrix based on \( B \) as

\[
\check{B} \triangleq (\check{B}_{st})_{1 \leq s,t \leq p}
\]

with \( p \times p \) blocks and its \((s,t)\)-th block is given by

\[
\check{B}_{st} = \begin{cases} 
B_{st}, & s < t, \\
0, & s \geq t.
\end{cases}
\]

Given the above notations, then we have the following Lemma 6.1, which asserts that, for any unit vector \( \eta \in \mathbb{C}^n \), the real parts of \( \eta^H \hat{B} \eta \) and \( \eta^H \check{B} \eta \) are both bounded by the spectral radius of \( B \).
Lemma 6.1 Assume that $B$, $\tilde{B}$ and $\hat{B}$ are defined by (6.111), (6.113) and (6.115), respectively. Then for an arbitrary $n$ dimensional vector $\eta \in \mathbb{C}^n$ with $\|\eta\| = 1$, we have

$$-\rho(B) \leq Re(\eta^H \tilde{B} \eta) \leq \rho(B),$$

and

$$-\rho(B) \leq Re(\eta^H \hat{B} \eta) \leq \rho(B).$$

Proof. We first define a block diagonal matrix based on $B$ below:

$$\tilde{B} \triangleq \text{Diag}(B_{11}, B_{22}, \ldots, B_{pp}),$$

whose main diagonal blocks are the same as those of $B$. Therefore, $B$ has the following decomposition, i.e.,

$$B = \tilde{B} + \breve{B} + \check{B}^T.$$

In addition, it is obvious that

$$-\rho(B)I_n \preceq B \preceq \rho(B)I_n,$$

which, combined with Theorem 4.3.15 in \cite{Horn1986}, means that

$$-\rho(B)I_n \preceq \tilde{B} \preceq \rho(B)I_n.$$

Assume that $\eta \in \mathbb{C}^n$ and $\|\eta\| = 1$. On the one hand,

$$2\rho(B) \geq \eta^H (\rho(B)I_n + B) \eta$$

$$= \eta^H \left( \rho(B)I_n + (\tilde{B} + \breve{B} + \check{B}^T) \right) \eta$$

$$= \eta^H \left( \rho(B)I_n + \tilde{B} \right) \eta + \eta^H (\breve{B} + \check{B}^T) \eta$$

$$= \eta^H \left( \rho(B)I_n + \tilde{B} \right) \eta + \eta^H (\tilde{B} + \tilde{B}^H) \eta$$

$$= \eta^H \left( \rho(B)I_n + \tilde{B} \right) \eta + 2\text{Re} (\eta^H \tilde{B} \eta)$$

$$\geq 2\text{Re} (\eta^H \tilde{B} \eta),$$

where the first inequality is due to (6.121); the first equality holds because of (6.120); the third equality follows from the fact that $\tilde{B}$ is a real matrix; and (6.122) amounts to the last inequality.
On the other hand, we also have

\[
2\rho(B) \geq \eta^H \left( \rho(B)I_n - B \right) \eta
\]

\[
= \eta^H \left[ \rho(B)I_n - \left( \tilde{B} + \tilde{B}^T \right) \right] \eta
\]

\[
= \eta^H \left( \rho(B)I_n - \tilde{B} \right) \eta - \eta^H \left( \tilde{B} + \tilde{B}^T \right) \eta
\]

\[
= \eta^H \left( \rho(B)I_n - \tilde{B} \right) \eta - \eta^H \left( \tilde{B} + \tilde{B}^H \right) \eta
\]

\[
= \eta^H \left( \rho(B)I_n - \tilde{B} \right) \eta - 2 \text{Re} \left( \eta^H \tilde{B} \eta \right)
\]

\[
\geq -2 \text{Re} \left( \eta^H \tilde{B} \eta \right)
\]

(6.124)

where the first inequality is due to (6.121); the first equality holds because of (6.120); the third equality follows from the fact that \( \tilde{B} \) is a real matrix; and (6.122) amounts to the last inequality.

Clearly, Eqs. (6.123) and (6.124) lead to (6.117). By using the same argument, we know (6.118) holds true.

The following lemma states that, if \( B \) is invertible, the real part of the eigenvalues of \( B^{-1} \left( I_n + \beta \tilde{B} \right) \) does not equal zero for any \( \beta \in \left( 0, \frac{1}{\rho(B)} \right) \).

**Lemma 6.2** Assume that \( B \) and \( \tilde{B} \) are defined by (6.111) and (6.113), respectively. Moreover, suppose \( B \) is invertible. For any \( \beta \in \left( 0, \frac{1}{\rho(B)} \right) \) and \( t \in [0, 1] \), if \( \lambda \) is an eigenvalue of \( B^{-1} \left( I_n + t\beta \tilde{B} \right) \), then \( \text{Re}(\lambda) \neq 0 \).

**Proof.** According to assumptions, \( B \) is invertible. Combining with \( I_n + t\beta \tilde{B} \) being invertible with any \( t \in [0, 1] \), we know \( B^{-1} \left( I_n + t\beta \tilde{B} \right) \) is an invertible matrix. Let \( \lambda \) be an eigenvalue of \( B^{-1} \left( I_n + t\beta \tilde{B} \right) \) and \( \xi \) be the corresponding eigenvector of unit length. Then \( \lambda \neq 0 \) and

\[
B^{-1} \left( I_n + t\beta \tilde{B} \right) \xi = \lambda \xi,
\]

(6.125)

which is clearly equivalent to equation:

\[
(I_n + t\beta \tilde{B}) \xi = \lambda B \xi.
\]

(6.126)

Premultiplying both sides of the above equality by \( \xi^H \), we arrive at

\[
1 + t\beta \xi^H \tilde{B} \xi = \lambda \xi^H B \xi.
\]

(6.127)

Note that \( 0 < \beta < \frac{1}{\rho(B)} \) and \( t \in [0, 1] \). Lemma 6.1 implies that \( 0 < \text{Re} \left( 1 + t\beta \xi^H \tilde{B} \xi \right) \) < 2, i.e., the real part of \( 1 + t\beta \xi^H \tilde{B} \xi \) is a positive real number. We denote \( 1 + t\beta \xi^H \tilde{B} \xi \) as \( a + bi \) where \( 0 < a < 2 \). If \( \lambda = ci \) \((c \neq 0)\) is a purely imaginary number, then...
\((\xi^H B \xi) ci\) is also a purely imaginary number because of \(B\) being a real symmetric matrix. Hence, (6.127) becomes

\[
1 + t \beta \xi^H \hat{B} \xi = a + bi = (\xi^H B \xi) ci, \tag{6.128}
\]

which is a contradiction. Hence, \(\text{Re}(\lambda) \neq 0\) is proved.

The following Lemma states that, if \(B\) is invertible, the real part of the eigenvalues of \((\beta B)^{-1} \left( I_n + t \beta \hat{B} \right)\) is strictly larger than \(\frac{1}{2}\) for any \(\beta \in \left( 0, \frac{1}{\rho(B)} \right)\) and \(t \in [0, 1]\).

**Lemma 6.3** Assume that \(B\) and \(\hat{B}\) are defined by (6.111) and (6.115), respectively. Moreover, suppose \(B\) is invertible. For any \(\beta \in \left( 0, \frac{1}{\rho(B)} \right)\) and \(t \in [0, 1]\), if \(\lambda\) is an eigenvalue of \((\beta B)^{-1} \left( I_n + t \beta \hat{B} \right)\) and \(\text{Re}(\lambda) > 0\), then \(\text{Re}(\lambda) \geq \frac{1}{2} + \frac{1 - \beta \rho(B)}{\beta \rho(B)} > \frac{1}{2}\).

**Proof.** The proof is lengthy and has been relegated to the Appendix.

The following Lemma plays a key role because it provides a sufficiently exact description of the distribution of the eigenvalues of \((I_n + \beta \hat{B})^{-1} B\), which has the the same structure as \(G\) defined by (3.18).

**Lemma 6.4** Assume that \(B\) and \(\hat{B}\) are defined by (6.111) and (6.113), respectively. Furthermore, if \(\lambda_{\min}(B) < 0\), then, for an arbitrary \(\beta \in \left( 0, \frac{1}{\rho(B)} \right)\), there is at least one eigenvalue \(\lambda\) of \((I_n + \beta \hat{B})^{-1} B\) which lies in closed left half complex plane excluding origin, i.e.,

\[
\forall \beta \in \left( 0, \frac{1}{\rho(B)} \right) \Rightarrow \exists \lambda \in \left[ \text{eig} \left( (I_n + \beta \hat{B})^{-1} B \right) \right] \cap \Omega, \tag{6.129}
\]

where

\[
\Omega \triangleq \{ a + bi | a, b \in \mathbb{R}, a \leq 0, (a, b) \neq (0, 0), i = \sqrt{-1} \}. \tag{6.130}
\]

**Proof.** The proof is lengthy and has been relegated to the Appendix.

Similar to Lemma 6.4, the following lemma plays a key role in this case because it provides a sufficiently exact description of the distribution of the eigenvalues of \(\beta \left( I_n + \beta \hat{B} \right)^{-1} B\), which has the the same structure as \(H\) defined by (5.98).

**Lemma 6.5** Assume that \(B\) and \(\hat{B}\) are defined by (6.111) and (6.115), respectively. Furthermore, if \(\lambda_{\max}(B) > 0\), then, for an arbitrary \(\beta \in \left( 0, \frac{1}{\rho(B)} \right)\), there is at least one nonzero eigenvalue \(\lambda\) of \(\beta \left( I_n + \beta \hat{B} \right)^{-1} B\) such that

\[
\frac{1}{\lambda} \in \Xi(\beta, B), \tag{6.131}
\]
where
\[ \Xi(\beta, B) \triangleq \left\{ a + bi \mid a, b \in \mathbb{R}, \frac{1}{2} + \frac{1 - \beta \rho(B)}{\beta \rho(B)} \leq \frac{i}{a}, i = \sqrt{-1} \right\}. \] (6.132)

**Proof.** The proof is lengthy and has been relegated to the Appendix.

Based on the above Lemma, we directly obtain the following Lemma, which shows there is at least one nonzero eigenvalue of \( \left[ I_n - \beta \left( I_n + \beta \hat{B} \right)^{-1} B \right]^{-1} \) with the same structure as \( \{ D [g_{\alpha_{(-i)}}]^{-1} (x^*) \}^T \), whose magnitude is strictly greater than one.

**Lemma 6.6** Assume that \( B \) and \( \hat{B} \) are defined by (6.111) and (6.115), respectively. Furthermore suppose \( \lambda_{\text{max}}(B) > 0 \) and \( I_n - \beta \left( I_n + \beta \hat{B} \right)^{-1} B \) is invertible, then, for an arbitrary \( \beta \in \left( 0, \frac{1}{\rho(B)} \right) \), there is at least one nonzero eigenvalue of \( \left[ I_n - \beta \left( I_n + \beta \hat{B} \right)^{-1} B \right]^{-1} \), whose magnitude is strictly greater than one.

**Proof.** According to assumptions, \( I_n - \beta \left( I_n + \beta \hat{B} \right)^{-1} B \) is invertible. Then
\[
\lambda \in \text{eig} \left( \beta \left( I_n + \beta \hat{B} \right)^{-1} B \right) \iff \frac{1}{1 - \lambda} \in \text{eig} \left( \left[ I_n - \beta \left( I_n + \beta \hat{B} \right)^{-1} B \right]^{-1} \right).
\] (6.133)

Hence, the statement of the lemma is equivalent to that there is at least one eigenvalue \( \lambda \) of \( \beta \left( I_n + \beta \hat{B} \right)^{-1} B \) such that
\[
\left| \frac{1}{1 - \lambda} \right| > 1 \iff \left| \frac{1}{\lambda} - 1 \right| > 1.
\] (6.134)

The above inequality is clearly equivalent to
\[
\left| \frac{1}{\lambda} \right| > \left| \frac{1}{\lambda} - 1 \right| \iff \frac{1}{2} < \text{Re} \left( \frac{1}{\lambda} \right).
\] (6.135)

In addition, under the same assumptions, Lemma 6.5 shows that there exists at least one eigenvalue \( \lambda \) of \( \beta \left( I_n + \beta \hat{B} \right)^{-1} B \) satisfying
\[
\frac{1}{2} < \left( \frac{1}{2} + \frac{1 - \beta \rho(B)}{\beta \rho(B)} \right) \leq \text{Re} \left( \frac{1}{\lambda} \right).
\] (6.136)

Thus, the proof is finished. \( \square \)
7. Conclusion

In this paper, given a non-convex twice continuously differentiable cost function with Lipschitz continuous gradient, we prove that all of BCGD, BMD and PBCD converge to a local minimizer, almost surely with random initialization.

As a by-product, it affirmatively answers the open questions whether mirror descent or block coordinate descent does not converge to saddle points in [Lee et al. (2016)]. More importantly, our results also hold true even for the cost functions with non-isolated critical points, which generalizes the results in [Panageas and Piliouras (2016)] as well.

By using the similar arguments, it is interesting to further research whether other methods, such as ADMM, BCGD, BMD, PBCD and their variations for the problems with some specially structured constraints, admit similar results or not.

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9. Appendix

9.1. Proof of Lemma 3.1

Proof. Firstly, we prove that $g^1_{\alpha f}$ with step size $\alpha < \frac{1}{L}$ is a diffeomorphism. The proof is given by the following four steps.

(a) We prove that $g^1_{\alpha f}$ is injective from $\mathbb{R}^n \to \mathbb{R}^n$ for $\alpha < \frac{1}{L}$. Suppose that there exist $x$ and $y$ such that $g^1_{\alpha f}(x) = g^1_{\alpha f}(y)$. Then $x - y = \alpha U_1 (\nabla_1 f(x) - \nabla_1 f(y))$ and

\[
\|x - y\| = \alpha \|U_1 (\nabla_1 f(x) - \nabla_1 f(y))\| \\
= \alpha \|\nabla_1 f(x) - \nabla_1 f(y)\| \leq \alpha \|\nabla f(x) - \nabla f(y)\| \leq \alpha L \|x - y\|, \tag{9.137}
\]

where the second equality is due to $U_1^T U_1 = I_{n_1}$. Since $\alpha L < 1$, (9.137) means $x = y$.

(b) To show $g^1_{\alpha f}$ is surjective, we construct an explicit inverse function. Given a point $y$ in $\mathbb{R}^n$, suppose it has the following partition,

\[
y = \begin{pmatrix} y^{(1)} \\ y^{(2)} \\ \vdots \\ y^{(p)} \end{pmatrix}.
\]

Then we define $n - n_1$ dimensional vector

\[
y_{-1}^{(1)} \triangleq \begin{pmatrix} y^{(2)} \\ \vdots \\ y^{(p)} \end{pmatrix} \tag{9.138}
\]
and define a function $\hat{f}(\cdot; y_{-1}) : \mathbb{R}^{n_1} \to \mathbb{R}$,

$$\hat{f}(x(1); y_{-1}) \triangleq f\left( \begin{pmatrix} x(1) \\ y_{-1} \end{pmatrix} \right),$$

which is determined by function $f$ and the remained block coordinate vector $y_{-1}$ of $y$. Then, the proximal point mapping of $-\hat{f}(\cdot; y_{-1})$ centered at $y(1)$ is given by

$$x_y(1) = \arg \min_{x(1)} \frac{1}{2} \|x(1) - y(1)\|^2 - \alpha \hat{f}(x(1); y_{-1}) \quad (9.139)$$

For $\alpha < \frac{1}{L}$, the function above is strongly convex with respect to $x(1)$, so there is a unique minimizer. Let $x_y(1)$ be the unique minimizer, then by the KKT condition,

$$y(1) = x_y(1) - \alpha \nabla \hat{f}(x_y(1); y_{-1}) = x_y(1) - \alpha \nabla_1 f\left( \begin{pmatrix} x_y(1) \\ y_{-1} \end{pmatrix} \right), \quad (9.140)$$

where the second equality is due to the definition of function $\hat{f}(\cdot; y_{-1})$. Let $x_y$ be defined as

$$x_y \triangleq \begin{pmatrix} x_y(1) \\ y_{-1} \end{pmatrix}, \quad (9.141)$$

where $x_y(1)$ is defined by (9.139). Accordingly,

$$y = \begin{pmatrix} y(1) \\ y_{-1} \end{pmatrix} = \begin{pmatrix} x_y(1) - \alpha \nabla_1 f\left( \begin{pmatrix} x_y(1) \\ y_{-1} \end{pmatrix} \right) \\ y_{-1} \end{pmatrix} = x_y - \alpha U_1 \nabla_1 f(x_y) = g^{1}_{\alpha f}(x_y),$$

where the first equality is due to the definition of $y_{-1}$ (see (9.138)); the second equality thanks to (9.140); and the third equality holds because of the definition of $x_y$ (see (9.141)).

Hence, $x_y$ is mapped to $y$ by the mapping $g^{1}_{\alpha f}$.

(c) In addition, combined with $\nabla^2 f(x) U_1 U_1^T \in \mathbb{R}^{n \times n}$ and $U_1^T \nabla^2 f(x) U_1 \in \mathbb{R}^{n_1 \times n_1}$, Theorem 1.3.20 in [Horn and Johnson, 1986] means

$$\text{eig} \left( \nabla^2 f(x) U_1 U_1^T \right) \subseteq \text{eig} \left( U_1^T \nabla^2 f(x) U_1 \right) \cup \{0\}. \quad (9.142)$$

Moreover, since

$$U_1^T \nabla^2 f(x) U_1 = \frac{\partial^2 f(x)}{\partial x(1) \partial x(1)} \quad (9.143)$$

which is the $n_1$-by-$n_1$ principal submatrix of $\nabla^2 f(x)$, it follows from Theorem 4.3.15 in [Horn and Johnson, 1986] that

$$\text{eig} \left( \frac{\partial^2 f(x)}{\partial x(1)^2} \right) \subseteq \left[ \lambda_{\min} \left( \nabla^2 f(x) \right), \lambda_{\max} \left( \nabla^2 f(x) \right) \right] \subseteq [-L, L], \quad (9.144)$$
where the last relation holds because of Eq. (2.2) and Lemma 7 in Panageas and Piliouras (2016). Since \( \alpha < \frac{1}{T} \), Eqs. (9.142), (9.143), and (9.144) imply that
\[
eig (\alpha \nabla^2 f(x) U_1 U_1^T) \subseteq (-1, 1). \tag{9.145}
\]
Hence, \( Dg^1_{\alpha f}(x) = I - \alpha \nabla^2 f(x) U_1 U_1^T \) is invertible for \( \alpha < \frac{1}{T} \).

(d) Note that we have shown that \( g^1_{\alpha f} \) is bijection, and continuously differentiable. Since \( Dg^1_{\alpha f}(x) \) is invertible for \( \alpha \leq \frac{1}{T} \), the inverse function theorem guarantees \( [g^1_{\alpha f}]^{-1} \) is continuously differentiable. Thus, \( g^1_{\alpha f} \) is a diffeomorphism.

Secondly, it is obvious that similar arguments can be applied to verify that, \( g^s_{\alpha f}, s = 2, \ldots, p \), are also diffeomorphisms. \( \square \)

9.2. Proof of Lemma 3.2

Proof. First, we define recursively
\[
G[s] \triangleq \frac{1}{\alpha} \left[ I_n - \prod_{t=s}^{1} (I_n - \alpha U_t A_t) \right], \quad 1 \leq s \leq p, \tag{9.146}
\]
which, combined with (3.17) and (3.18), implies that \( G = G[p] \). In addition, it is easily seen that
\[
U_s^T (I_n - \alpha U_t A_t) = U_s^T - \alpha U_s^T U_t A_t = U_s^T \tag{9.147}
\]
when \( s \neq t \). If \( k < s \), then the above (9.147) follows that
\[
U_s^T G[k] = \frac{1}{\alpha} U_s^T \left[ I_n - \prod_{t=k}^{1} (I_n - \alpha U_t A_t) \right] = \frac{1}{\alpha} \left[ U_s^T - \alpha U_s^T \prod_{t=k}^{1} (I_n - \alpha U_t A_t) \right] = 0. \tag{9.148}
\]
Consequently, if \( 1 \leq s < q \leq p \), then
\[
U_s^T (\alpha G[q]) = U_s^T \left[ I_n - \prod_{t=q}^{s+1} (I_n - \alpha U_t A_t) \right] = U_s^T I_n - \prod_{t=q}^{s+1} (I_n - \alpha U_t A_t) \prod_{t=s}^{1} (I_n - \alpha U_t A_t)
\]
\[
= U_s^T I_n - U_s^T \prod_{t=q}^{s+1} (I_n - \alpha U_t A_t) \prod_{t=s}^{1} (I_n - \alpha U_t A_t)
\]
\[
= U_s^T I_n - U_s^T \prod_{t=s}^{1} (I_n - \alpha U_t A_t)
\]
\[
= U_s^T \left[ I_n - \prod_{t=s}^{1} (I_n - \alpha U_t A_t) \right] = U_s^T \alpha G[s], \tag{9.149}
\]

where the fourth equality is due to (9.147) and the last equality uses the definition (9.146) of \( G[s] \). Particularly if \( q = p \), the above equation becomes
\[
U_s^T \alpha G[s] = U_s^T \alpha G[p] = U_s^T \alpha G. \tag{9.150}
\]
From equality (9.150), we further have
\[
U_s^T (\alpha G) = U_s^T [I_n - (I_n - \alpha G[s])] = \alpha U_s^T G[s - 1] + \alpha A_s - \alpha^2 A_s G[s - 1]
\]
where (a) uses the definitions of \( G[s] \) in (9.146); (b) is due to (9.148); (c) thanks to the definition (2.4) of \( U_t \); (d) uses (9.148) again; (e) holds because of (9.149); (f) follows from \( G[p] = G \). Dividing both sides of the above equation by \( \alpha \), we have (3.21).

9.3. Proof of Lemma 6.4

Proof. We divide the proof into two cases.

Case 1: \( B \) is an invertible matrix. In this case, we clearly have,
\[
\left( (I_n + \beta \bar{B})^{-1} B \right)^{-1} = B^{-1} (I_n + \beta \bar{B}). \tag{9.151}
\]
In what follows, we will prove that (6.149) is true by using Lemma 9.2 in Appendix. Fristly, we define an analytic function with \( t \) as a parameter:
\[
\chi_t(z) \triangleq \det \{ z I_n - [(1 - t)B^{-1} + tB^{-1} (I_n + \beta \bar{B})] \}, \quad 0 \leq t \leq 1
\]
\[
= \det \{ z I_n - B^{-1} (I_n + t \beta \bar{B}) \}, \quad 0 \leq t \leq 1. \tag{9.152}
\]
Moreover, define a closed rectangle in the complex plane as
\[ \mathcal{D} \triangleq \{ a + bi \mid -2\nu \leq a \leq 0, \ -2\nu \leq b \leq 2\nu \}, \quad (9.153) \]
with \( \nu \) being defined below:
\[ \nu \triangleq \| B^{-1} \| + \frac{1}{\rho (B)} \| B^{-1} \hat{B} \| \geq \| B^{-1} \| + t\beta \| B^{-1} \hat{B} \| \geq \| B^{-1} (I_n + t\beta \hat{B}) \|, \quad \forall t \in [0, 1], \quad (9.154) \]
where the first inequality holds because of \( \beta \in \left( 0, \frac{1}{\rho (B)} \right) \) and \( t \in [0, 1] \). Note that the boundary \( \partial \mathcal{D} \) of \( \mathcal{D} \) consists of a finite number of smooth curves. Specifically, define
\[
\begin{align*}
\gamma_1 &\triangleq \{ a + bi \mid a = 0, \ -2\nu \leq b \leq 2\nu \}, \\
\gamma_2 &\triangleq \{ a + bi \mid a = -2\nu, \ -2\nu \leq b \leq 2\nu \}, \\
\gamma_3 &\triangleq \{ a + bi \mid -2\nu \leq a \leq 0, \ b = 2\nu \}, \\
\gamma_4 &\triangleq \{ a + bi \mid -2\nu \leq a \leq 0, \ b = -2\nu \}, \quad (9.155)
\end{align*}
\]
then
\[ \partial \mathcal{D} = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4. \quad (9.156) \]

In order to apply Lemma 9.2, we will show that
\[ \chi_t (z) \neq 0, \quad \forall t \in [0, 1], \ \forall z \in \partial \mathcal{D}. \quad (9.157) \]
On the one hand, since the spectral norm of a matrix is larger than or equal to its spectral radius, the above inequality (9.154) yields that, for any \( t \in [0, 1] \), every eigenvalue of \( B^{-1} (I_n + t\beta \hat{B}) \) has a magnitude less than \( \nu \). Note that for an arbitrary \( z \in \gamma_2 \cup \gamma_3 \cup \gamma_4 \), then \( |z| \geq 2\nu \). Consequently,
\[ \chi_t (z) \neq 0, \quad \forall t \in [0, 1], \ \forall z \in \gamma_2 \cup \gamma_3 \cup \gamma_4. \quad (9.158) \]
On the other hand, it follows from Lemma 6.2 that, for any \( t \in [0, 1] \), if \( \lambda \) is an eigenvalue of \( B^{-1} (I_n + t\beta \hat{B}) \), then \( \text{Re} (\lambda) \neq 0 \). Hence, \( \lambda \notin \gamma_1 \). As mentioned at the beginning of the proof, \( B^{-1} (I_n + \beta \hat{B}) \) is invertible. Hence, there are no zero eigenvalues of \( B^{-1} (I_n + \beta \hat{B}) \) in \( \gamma_1 \), i.e.,
\[ \chi_t (z) \neq 0, \quad \forall t \in [0, 1], \ \forall z \in \gamma_1. \quad (9.159) \]
Combining (9.158) and (9.159), we obtain (9.157).

As a result, it follows from Lemma 9.2 in Appendix, Eqs. (9.152), (9.153) and (9.157) that \( \chi_0 (z) = \det \{ zI_n - B^{-1} \} \) and \( \chi_1 (z) = \det \{ zI_n - B^{-1} (I_n + \beta \hat{B}) \} \) have the same number of zeros in \( \mathcal{D} \). Note that \( \lambda_{\min} (B) < 0 \) implies that there is at least
one negative eigenvalue $\frac{1}{\lambda_{\min}(B)}$ of $B^{-1}$. Recalling the definition (9.154) of $\nu$, we know $\left|\frac{1}{\lambda_{\min}(B)}\right| \leq \nu$. Thus $\frac{1}{\lambda_{\min}(B)}$ must lie inside $\mathcal{D}$. In other words, the number of zeros of $X_0(z)$ inside $\mathcal{D}$ is at least one, which in turn shows the number of zeros of $X_1(z)$ inside $\mathcal{D}$ is at least one as well. Thus, there must exist at least one eigenvalue of $B^{-1}(I_n + \beta\hat{B})$ lying inside $\mathcal{D}$. We denote it as $x + yi$, then $-2\nu < x < 0$ and $-2\nu < y < 2\nu$. Consequently, $\frac{1}{x + yi} = \frac{x - yi}{x^2 + y^2}$ is an eigenvalue of $(I_n + \beta\hat{B})^{-1} B$ with real part $\frac{x}{x^2 + y^2} < 0$. Hence, $\frac{1}{x + yi}$ lies in $\Omega$ defined by (6.130) and the proof is finished in this case.

**Case 2: $B$ is a singular matrix.** In this case, we will apply perturbation theorem based on the results in **Case 1** to prove (6.129).

Suppose the multiplicity of zero eigenvalue of $B$ is $m$ and $B$ has an eigenvalue decomposition in the form of

$$B = V \begin{pmatrix} \Theta & 0 \\ 0 & 0 \end{pmatrix} V^T = V_1 \Theta V_1^T,$$

(9.160)

where $\Theta = \text{Diag}(\theta_1, \theta_2, \ldots, \theta_{n-m})$, $\theta_s$, $s = 1, \ldots, n - m$, are the nonzero eigenvalues of $B$ and

$$V = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix},$$

(9.161)

is an orthogonal matrix and $V_1$ consists of the first $(n - m)$ columns of $V$.

Denote

$$\delta \doteq \min \{|\theta_1|, |\theta_2|, \ldots, |\theta_{n-m}|\}.$$  

(9.162)

For any $\epsilon \in (0, \delta)$, we define

$$B(\epsilon) \doteq B + \epsilon I_n,$$

(9.163)

then,

$$\text{eig}(B(\epsilon)) = \{\theta_1 + \epsilon, \theta_2 + \epsilon, \ldots, \theta_{n-m} + \epsilon, \epsilon\} \neq 0, \quad \forall \epsilon \in (0, \delta),$$

(9.164)

and

$$\lambda_{\min}(B(\epsilon)) = \lambda_{\min}(B) + \epsilon \leq -\delta + \epsilon < 0, \quad \forall \epsilon \in (0, \delta),$$

(9.165)

where the first inequality is due to the definition of $\delta$ and $\min \{\theta_1, \theta_2, \ldots, \theta_{n-m}\} = \lambda_{\min}(B) < 0$.

Since $B$ is defined by (6.111), $B(\epsilon)$ has $p \times p$ blocks form as well. Specifically,

$$B(\epsilon) = (B(\epsilon)_{st})_{1 \leq s, t \leq p},$$

(9.166)

and its $(s, t)$-th block is given by

$$B(\epsilon)_{st} = \begin{cases} B_{st} + \epsilon I_n, & s = t, \\ B_{st}, & s \neq t, \end{cases}$$

(9.167)
where \( n_1, n_2, \ldots, n_p \) are \( p \) positive integer numbers satisfying \( \sum_{s=1}^{p} n_s = n \). Similar to the definition (6.113) of \( \tilde{B} \), we denote the strictly block lower triangular matrix based on \( B (\epsilon) \) as

\[
\tilde{B} (\epsilon) \triangleq (\tilde{B} (\epsilon)_{st})_{1 \leq s, t \leq p} \tag{9.168}
\]

with \( p \times p \) blocks and its \((s, t)\)-th block is given by

\[
\tilde{B} (\epsilon)_{st} = \begin{cases} 
B (\epsilon)_{st}, & s > t, \\
0, & s \leq t,
\end{cases} \tag{9.169}
\]

where the second equality holds because of (9.167); the last equality is due to (6.114).

It follows easily from Eqs. (6.113), (6.114), (9.168) and (9.169) that

\[
\tilde{B} (\epsilon) = \tilde{B}. \tag{9.170}
\]

Consequently,

\[
(I_n + \beta \tilde{B} (\epsilon))^{-1} B (\epsilon) = (I_n + \beta \tilde{B})^{-1} B (\epsilon) = (I_n + \beta \tilde{B})^{-1} (B + \epsilon I_n), \tag{9.171}
\]

where the first equality is due to (9.170) and the second equality holds because of (9.163). For simplicity, let

\[
\lambda^\beta_s (\epsilon), \ s = 1, \ldots, n, \tag{9.172}
\]

be the eigenvalues of \((I_n + \beta \tilde{B} (\epsilon))^{-1} B (\epsilon)\) which lies in \( \Omega \) defined by (6.130). Taking into account definition (9.163), we have \( \rho (B (\epsilon)) \leq \rho (B) + \epsilon \). Hence, for any \( \epsilon \in (0, \delta) \) and \( \beta \in \left( 0, \frac{1}{\rho (B (\epsilon))} \right) \subseteq \left( 0, \frac{1}{\rho (B)} \right) \), there exists at least one eigenvalue of \((I_n + \beta \tilde{B} (\epsilon))^{-1} B (\epsilon)\) which lies in \( \Omega \) defined by (6.130). Taking into account definition (9.163), we have \( \rho (B (\epsilon)) \leq \rho (B) + \epsilon \). Hence, for any \( \epsilon \in (0, \delta) \) and \( \beta \in \left( 0, \frac{1}{\rho (B (\epsilon))} \right) \subseteq \left( 0, \frac{1}{\rho (B)} \right) \), there exists at least one index denoted as \( s (\epsilon) \in \{1, 2, \ldots, n\} \) such that

\[
\lambda^\beta_{s(\epsilon)} (\epsilon) \in \Omega. \tag{9.173}
\]

Furthermore, it is well known that the eigenvalues of a matrix \( M \) are continuous functions of the entries of \( M \). Therefore, for any \( \beta \in \left( 0, \frac{1}{\rho (B)} \right) \), \( \lambda^\beta_s (\epsilon) \) is a continuous function of \( \epsilon \) and

\[
\lim_{\epsilon \to 0^+} \lambda^\beta_s (\epsilon) = \lambda^\beta_s (0), \ s = 1, \ldots, n, \tag{9.174}
\]

where \( \lambda^\beta_s (0) \) is the eigenvalue of \((I_n + \beta \tilde{B})^{-1} B\).
In what follows, we will prove that (6.129) holds true by contradiction.

Suppose for sake of contradiction that, there exists a \( \beta^* \in (0, \frac{1}{\rho(B)}) \) such that,
for any \( s \in \{1, \ldots, n\} \), we have
\[
\lambda^\beta_s(0) \notin \Omega, \tag{9.175}
\]
where \( \lambda^\beta_s(0) \) is the eigenvalue of \( (I_n + \beta^* \bar{B})^{-1} B \).

According to Lemma 9.3 in Appendix and the assumption that the multiplicity of zero eigenvalue of \( B \) is \( m \), we know that the multiplicity of eigenvalue 0 of \( (I_n + \beta^* \bar{B})^{-1} B \) is \( m \) as well. Then there are exactly \( m \) eigenvalue functions of \( \epsilon \) whose limits are 0 as \( \epsilon \) approaches zero from above. Without loss of generality, we assume
\[
\lim_{\epsilon \to 0^+} \lambda^\beta_s(\epsilon) = \lambda^\beta_s(0) = 0, \quad s = 1, \ldots, m, \tag{9.176}
\]
and
\[
\lim_{\epsilon \to 0^+} \lambda^\beta_s(\epsilon) = \lambda^\beta_s(0) \neq 0, \quad s = m + 1, \ldots, n. \tag{9.177}
\]

Subsequently, under the assumption (9.175), we will first prove that there exists a \( \delta^*_1 > 0 \) such that, for any \( \epsilon \in (-\delta^*_1, 0) \), then \( \beta^* \in \left(0, \frac{1}{\rho(B) + \epsilon}\right) \subseteq \left(0, \frac{1}{\rho(B(\epsilon))}\right) \) and there exists no \( s \in \{1, \ldots, n\} \) such that \( \lambda^\beta_s(\epsilon) \) belongs to \( \Omega \). This would contradict (9.173). The proof is given by the following four steps.

**Step (a):** Under the assumption (9.175), we first prove that there exists a \( \delta^*_1 > 0 \) such that, for any \( \epsilon \in (-\delta^*_1, 0) \), \( \beta^* \in \left(0, \frac{1}{\rho(B) + \epsilon}\right) \subseteq \left(0, \frac{1}{\rho(B(\epsilon))}\right) \) and there does not exist any \( s \in \{m + 1, \ldots, n\} \) such that \( \lambda^\beta_s(\epsilon) \) belongs to \( \Omega \).

Taking into account the definition of \( \Omega \), Eq. (9.175) and \( \beta^* \in \left(0, \frac{1}{\rho(B)}\right) \), we imply that, there exists a \( \bar{\delta} \) such that, for any \( \epsilon \in (0, \bar{\delta}) \),
\[
\beta^* \in \left(0, \frac{1}{\rho(B) + \epsilon}\right) \subseteq \left(0, \frac{1}{\rho(B(\epsilon))}\right), \tag{9.178}
\]
and
\[
\text{Re} \left(\lambda^\beta_s(0)\right) > 0, \quad \forall \ s \in \{m + 1, \ldots, n\}. \tag{9.179}
\]

Moreover, note that \( \lambda^\beta_s(\epsilon) \) is a continuous function of \( \epsilon \) and (9.177) holds. Combining with the above inequalities, we know that there exists a \( \delta^*_1 > 0 \) with \( \delta^*_1 \leq \bar{\delta} \) such that
\[
|\lambda^\beta_s(\epsilon) - \lambda^\beta_s(0)| < \frac{1}{3} \text{Re} \left(\lambda^\beta_s(0)\right), \quad \forall \ s \in \{m + 1, \ldots, n\}, \ \forall \ \epsilon \in [0, \delta^*_1], \tag{9.180}
\]
which further means that
\[
0 < \frac{2}{3} \text{Re} \left(\lambda^\beta_s(0)\right) < \text{Re} \left(\lambda^\beta_s(\epsilon)\right), \quad \forall \ s \in \{m + 1, \ldots, n\}, \ \forall \ \epsilon \in [0, \delta^*_1]. \tag{9.181}
\]
Hence, we arrive at
\[ \lambda_s^\beta(\epsilon) \notin \Omega, \; \forall s \in \{m + 1, \ldots, n\}, \; \forall \epsilon \in [0, \delta^*_1]. \] (9.182)

**Step (b):** In this step, we will prove that there exists a \( \delta^*_2 > 0 \) such that, for any \( \epsilon \in (0, \delta^*_2] \) and \( s \in \{1, \ldots, m\} \),
\[ \operatorname{Re}\left(\lambda_s^\beta(\epsilon)\right) > 0, \] (9.183)
which immediately implies that
\[ \lambda_s^\beta(\epsilon) \notin \Omega, \; \forall s \in \{1, \ldots, m\}, \; \forall \epsilon \in (0, \delta^*_2]. \] (9.184)

For simplicity, let
\[ \dot{C}_{ij} \triangleq V_i^T \dot{\bar{B}} V_j, \; 1 \leq i, j \leq 2, \] (9.185)
where \( V_1 \) and \( V_2 \) are given by (9.161).

In what follows, we take an arbitrary \( s \in \{1, \ldots, m\} \). Since we assume that \( \lambda_s^\beta(\epsilon) \) is the eigenvalue of \((I_n + \beta^* \dot{\bar{B}})^{-1} (B + \epsilon I_n)\), then, for any \( \epsilon \in (0, \delta) \),
\[
\det \left\{ \left( I_n + \beta^* \dot{\bar{B}} \right)^{-1} (B + \epsilon I_n) - \lambda_s^\beta(\epsilon) I_n \right\} = 0,
\] which is clearly equivalent to
\[
\det \left\{ \left( I_n + \beta^* \dot{\bar{B}} \right) \right\} \det \left\{ \left( I_n + \beta^* \dot{\bar{B}} \right)^{-1} (B + \epsilon I_n) - \lambda_s^\beta(\epsilon) I_n \right\} = 0, \forall \epsilon \in (0, \delta). \] (9.186)

It is easily seen from the above Eq. (9.186) that, for any \( \epsilon \in (0, \delta) \),
\[
0 = \det \left\{ \left( I_n + \beta^* \dot{\bar{B}} \right) \right\} \det \left\{ \left( I_n + \beta^* \dot{\bar{B}} \right)^{-1} (B + \epsilon I_n) - \lambda_s^\beta(\epsilon) I_n \right\} = 0, \forall \epsilon \in (0, \delta). \] (9.187)

where the second equality is due to (9.160) and the last equality holds because of Eqs. (9.161) and (9.185).

Besides, recalling
\[
\lim_{\epsilon \to 0^+} \lambda_s^\beta(\epsilon) = \lambda_s^\beta(0) = 0, \] (9.188)
we have
\[
\lim_{\epsilon \to 0^+} \left[ \Theta + \epsilon I_{n-m} - \lambda_s^\beta(\epsilon) \left( I_{n-m} + \beta^* \dot{C}_{11} \right) \right] = \Theta, \] (9.189)
which is an invertible matrix because $\Theta$ is given by (9.160). Clearly, the above further implies that, there exists a $\delta_1 > 0$ such that, for any $\epsilon \in [0, \delta_1]$,

$$\Theta + \epsilon I_{n-m} - \lambda_s^\beta(\epsilon) (I_{n-m} + \beta^* \hat{C}_{11})$$  \hspace{1cm} (9.190)

is an invertible matrix as well, and

$$\lim_{\epsilon \to 0^+} \left[ \Theta + \epsilon I_{n-m} - \lambda_s^\beta(\epsilon) (I_{n-m} + \beta^* \hat{C}_{11}) \right]^{-1} = \Theta^{-1}. \hspace{1cm} (9.191)$$

Since the inverse of a matrix $M$, $M^{-1}$, is a continuous function of the elements of $M$, there exists a $\delta_2 > 0$ with $\delta_2 \leq \delta_1$, such that, for any $\epsilon \in [0, \delta_2]$,

$$\left\| \left[ \Theta + \epsilon I_{n-m} - \lambda_s^\beta(\epsilon) (I_{n-m} + \beta^* \hat{C}_{11}) \right]^{-1} \right\| \leq 2 \| \Theta^{-1} \|. \hspace{1cm} (9.192)$$

Consequently, for any $\epsilon \in [0, \delta_2]$, it follows easily from (9.187) and (9.190) that

$$0 = \det \left\{ \Theta + \epsilon I_{n-m} - \lambda_s^\beta(\epsilon) (I_{n-m} + \beta^* \hat{C}_{11}) \right\} \times \det \left\{ \epsilon I_m - \lambda_s^\beta(\epsilon) (I_m + \beta^* \hat{C}_{22}) \right\}$$

$$- (\beta^*)^2 \left( \lambda_s^\beta(\epsilon) \right)^2 \hat{C}_{21} \left[ \Theta + \epsilon I_{n-m} - \lambda_s^\beta(\epsilon) (I_{n-m} + \beta^* \hat{C}_{11}) \right]^{-1} \hat{C}_{12} \right\}, \hspace{1cm} (9.193)$$

which, combined with the fact that $\Theta + \epsilon I_{n-m} - \lambda_s^\beta(\epsilon) (I_{n-m} + \beta^* V_1^T \hat{B} V_1)$ is an invertible matrix again (see (9.190)), means that

$$\epsilon I_m - \lambda_s^\beta(\epsilon) (I_m + \beta^* \hat{C}_{22}) - (\beta^*)^2 \left( \lambda_s^\beta(\epsilon) \right)^2 \hat{C}_{21} \left[ \Theta + \epsilon I_{n-m} - \lambda_s^\beta(\epsilon) (I_{n-m} + \beta^* \hat{C}_{11}) \right]^{-1} \hat{C}_{12}$$

is a singular matrix. Therefore, there exists one nonzero vector $v(\epsilon) \in \mathbb{C}^m$ with $\|v(\epsilon)\| = 1$ satisfying

$$\begin{cases} \epsilon I_m - \lambda_s^\beta(\epsilon) (I_m + \beta^* \hat{C}_{22}) \\ - (\beta^*)^2 \left( \lambda_s^\beta(\epsilon) \right)^2 \hat{C}_{21} \left[ \Theta + \epsilon I_{n-m} - \lambda_s^\beta(\epsilon) (I_{n-m} + \beta^* \hat{C}_{11}) \right]^{-1} \hat{C}_{12} \end{cases} v(\epsilon) = 0. \hspace{1cm} (9.194)$$

Moreover, for any $\epsilon \in [0, \delta_2]$, premultiplying both sides of the above equality by $(v(\epsilon))^H$, we have

$$(v(\epsilon))^H \begin{cases} \epsilon I_m - \lambda_s^\beta(\epsilon) (I_m + \beta^* \hat{C}_{22}) \\ - (\beta^*)^2 \left( \lambda_s^\beta(\epsilon) \right)^2 \hat{C}_{21} \left[ \Theta + \epsilon I_{n-m} - \lambda_s^\beta(\epsilon) (I_{n-m} + \beta^* \hat{C}_{11}) \right]^{-1} \hat{C}_{12} \end{cases} v(\epsilon) = 0, \hspace{1cm} (9.195)$$

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or equivalently,
\[
\epsilon - \lambda_s^\beta (\epsilon) \left( 1 + \beta^* (v(\epsilon))^H \tilde{C}_{22} v(\epsilon) \right) \\
= (\beta^*)^2 (\lambda_s^\beta (\epsilon))^2 (v(\epsilon))^H \tilde{C}_{21} \left[ \Theta + \epsilon I_{n-m} - \lambda_s^\beta (\epsilon) (I_{n-m} + \beta^* \tilde{C}_{11}) \right]^{-1} \tilde{C}_{12} v(\epsilon).
\]
(9.196)

Dividing both sides of the above expression by \(\lambda_s^\beta (\epsilon)\), we have, for any \(\epsilon \in (0, \delta_2]\),
\[
0 \leq \left| \frac{\epsilon}{\lambda_s^\beta (\epsilon)} - \left( 1 + \beta^* (v(\epsilon))^H \tilde{C}_{22} v(\epsilon) \right) \right| \\
= \left| (\beta^*)^2 \lambda_s^\beta (\epsilon)(v(\epsilon))^H \tilde{C}_{21} \left[ \Theta + \epsilon I_{n-m} - \lambda_s^\beta (\epsilon) (I_{n-m} + \beta^* \tilde{C}_{11}) \right]^{-1} \tilde{C}_{12} v(\epsilon) \right| \\
\leq (\beta^*)^2 \left| \lambda_s^\beta (\epsilon) \right| \left\| \tilde{C}_{21} \right\| \left\| \left[ \Theta + \epsilon I_{n-m} - \lambda_s^\beta (\epsilon) (I_{n-m} + \beta^* \tilde{C}_{11}) \right]^{-1} \right\| \left\| \tilde{C}_{12} \right\| \\
\leq 2 (\beta^*)^2 \left| \lambda_s^\beta (\epsilon) \right| \left\| \tilde{C}_{21} \right\| \left\| \Theta^{-1} \right\| \left\| \tilde{C}_{12} \right\|, \\
(9.197)
\]
where the second inequality is due to \(\|v(\epsilon)\| = 1\) and Cauchy–Schwartz inequality; the last inequality follows from (9.192). As \(\lim_{\epsilon \to 0^+} \lambda_s^\beta (\epsilon) = \lambda_s^\beta (0) = 0\), the right-hand limit in the above expression is equal to zero. Hence, we have,
\[
\lim_{\epsilon \to 0^+} \left| \frac{\epsilon}{\lambda_s^\beta (\epsilon)} - \left( 1 + \beta^* (v(\epsilon))^H \tilde{C}_{22} v(\epsilon) \right) \right| = 0. \\
(9.198)
\]

Recall that
\[
\text{Re} \left( 1 + \beta^* (v(\epsilon))^H \tilde{C}_{22} v(\epsilon) \right) \\
= \text{Re} \left( 1 + \beta^* (v(\epsilon))^H V_2^T \tilde{B} V_2 v(\epsilon) \right) \\
= \text{Re} \left( 1 + \beta^* (v(\epsilon))^H V_2^H \tilde{B} V_2 v(\epsilon) \right) \\
= \text{Re} \left( 1 + \beta^* (V_2 v(\epsilon))^H \tilde{B} V_2 v(\epsilon) \right) \\
\geq 1 - \beta^* \rho(B) > 0,
\]
where the first equality follows from (9.185); the second equality holds because \(V_2\) is a real matrix (see eq. (9.160)); the first inequality follows easily from Lemma 6.2 and \(\|V_2 v(\epsilon)\| = (v(\epsilon))^H V_2^H V_2 v(\epsilon) = (v(\epsilon))^H v(\epsilon) = 1\); and the last inequality thanks to \(\beta^* \in \left( 0, \frac{1}{\rho(B)} \right)\). Consequently, there exists a \(\delta_3\) with \(\delta_3 \leq \delta_2\) such that,
for any $\epsilon \in (0, \delta_3]$,  
\[
\frac{1}{3} (1 - \beta^* \rho (B)) \geq \left| \frac{\epsilon}{\lambda_s^\beta (\epsilon)} - \left( 1 + \beta^* (v (\epsilon))^H \hat{C}_{22} v (\epsilon) \right) \right|
\geq \left| \text{Re} \left( \frac{\epsilon}{\lambda_s^\beta (\epsilon)} - \left( 1 + \beta^* (v (\epsilon))^H \hat{C}_{22} v (\epsilon) \right) \right) \right|
= \left| \text{Re} \left( \frac{\epsilon}{\lambda_s^\beta (\epsilon)} \right) - \text{Re} \left( 1 + \beta^* (v (\epsilon))^H \hat{C}_{22} v (\epsilon) \right) \right|.
\]  
(9.200)

The above inequalities (9.199) and (9.200) imply that, for any $\epsilon \in (0, \delta_3]$,  
\[
0 < \frac{2}{3} (1 - \beta^* \rho (B)) \leq \text{Re} \left( \frac{\epsilon}{\lambda_s^\beta (\epsilon)} \right) = \frac{\epsilon}{\left| \lambda_s^\beta (\epsilon) \right|^2} \text{Re} \left( \lambda_s^\beta (\epsilon) \right).
\]  
(9.201)

Since the above argument is applied to any $s \in \{1, \ldots, m\}$, there exists a $\delta^*_3 > 0$ such that,  
\[
0 < \text{Re} \left( \lambda_s^\beta (\epsilon) \right), \ \forall s \in \{1, \ldots, m\}, \ \forall \epsilon \in (0, \delta^*_3],
\]  
(9.202)

which further implies that  
\[
\lambda_s^\beta (\epsilon) \notin \Omega, \ \forall s \in \{1, \ldots, m\}, \ \forall \epsilon \in (0, \delta^*_3].
\]  
(9.203)

**Step (c):** Combining (9.182) and (9.203), we arrive at,  
\[
\lambda_s^\beta (\epsilon) \notin \Omega, \ \forall s \in \{1, \ldots, n\}, \ \forall \epsilon \in (0, \delta^*_3],
\]  
(9.204)

where $\delta^*_3 = \min \{\delta^*_1, \delta^*_2\}$.

**Step (d):** Let  
\[
\delta^* = \min \left\{ \frac{1}{2} \delta^*_1, \delta^*_3 \right\}.
\]  
(9.205)

Then, for any $\epsilon \in (0, \delta^*)$, we have  
\[
\epsilon \in (0, \delta), \tag{9.206}
\]

\[
\beta^* \in \left( 0, \frac{1}{\rho (B) + \epsilon} \right) \subseteq \left( 0, \frac{1}{\rho (B (\epsilon))} \right)
\]  
(9.207)

and  
\[
\lambda_s^\beta (\epsilon) \notin \Omega, \ \forall s \in \{1, \ldots, n\},
\]  
(9.208)

where (9.206) uses the definition (9.205) of $\delta^*$; (9.207) is due to the definition (9.205) of $\delta^*$ (i.e., $\delta^* \leq \delta^*_3 \leq \delta^*_1 \leq \delta$) and (9.178); and (9.208) thanks to the definition (9.205) of $\delta^*$ and (9.204). Clearly, this contradicts (9.173).

Hence, we conclude that (6.129) holds true.
9.4. Proof of Lemma 4.1

Proof. We first prove that \( g_1 \) with step size \( \alpha < \frac{\mu}{L} \) is a diffeomorphism. The proof is given by the following four steps.

(a) We first prove that \( \psi_1 \) is injective from \( \mathbb{R}^{n_1} \rightarrow \mathbb{R}^{n_1} \) for \( \alpha < \frac{\mu}{L} \). Suppose that there exist \( x \) and \( y \) such that \( \psi_1(x) = \psi_1(y) \), which implies that

\[
\left\{ \begin{array}{l}
[\nabla \varphi_1^{-1}(\nabla \varphi_1(x(t)) - \alpha \nabla_1 f(x)) = [\nabla \varphi_1^{-1}(\nabla \varphi_1(y(t)) - \alpha \nabla_1 f(y)), \ t = 1, \\
x(t) = y(t), \ \\
\end{array} \right.
\]

Since Lemma 9.4 in Appendix asserts that \( \nabla \varphi_1 \) is a diffeomorphism, then \( [\nabla \varphi_1^{-1}] \) is a diffeomorphism as well. Hence, the above equality (9.209) is equivalent to

\[
\left\{ \begin{array}{l}
\nabla \varphi_1(x(t)) - \alpha \nabla_1 f(x) = \nabla \varphi_1(y(t)) - \alpha \nabla_1 f(y), \ t = 1, \\
x(t) = y(t), \ \\
\end{array} \right.
\]

In particular, \( \nabla \varphi_1(x(1)) - \alpha \nabla_1 f(x) = \nabla \varphi_1(y(1)) - \alpha \nabla_1 f(y) \) further implies that

\[
\|x(1) - y(1)\| \leq \frac{1}{\mu} \|\nabla \varphi_1(x(1)) - \nabla \varphi_1(y(1))\| \\
= \frac{\alpha}{\mu} \|\nabla_1 f(x) - \nabla_1 f(y)\| \\
\leq \frac{\alpha}{\mu} \|\nabla f(x) - \nabla f(y)\| \leq \frac{\alpha L}{\mu} \|x - y\| \\
= \frac{\alpha L}{\mu} \|x(1) - y(1)\|, \tag{9.211}
\]

where the first inequality is due to strong convexity (see (4.35)); the third inequality thanks to (2.2); the last equality holds because of (9.210). Since \( \alpha L < 1 \), (9.211) means \( x(1) = y(1) \). Combining with (9.210), we have \( x = y \).

(b) To show \( \psi_1 \) is surjective, we construct an explicit inverse function. Given a point \( y \) in \( \mathbb{R}^n \), suppose it has the following partition,

\[
y = \begin{pmatrix}
y(1) \\
y(2) \\
\vdots \\
y(p)
\end{pmatrix}, \tag{9.212}
\]

Then we define \( n - n_1 \) dimensional vector

\[
y_-(1) \triangleq \begin{pmatrix}
y(2) \\
\vdots \\
y(p)
\end{pmatrix}, \tag{9.213}
\]

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and define a function $\bar{f}(\cdot; y_{-}(1)) : \mathbb{R}^{n_1} \to \mathbb{R}$,

$$\bar{f}(x(1); y_{-}(1)) \triangleq f \left( \begin{pmatrix} x(1) \\ y_{-}(1) \end{pmatrix} \right), \quad (9.214)$$

which is determined by function $f$ and the remained block coordinate vector $y_{-}(1)$ of $y$. Consider the following problem,

$$\min_{x(1)} B_{\varphi_1} (x(1), y(1)) - \alpha \bar{f}(x(1); y_{-}(1)) \quad (9.215)$$

For $\alpha < \frac{\mu}{L}$, the function above is strongly convex with respect to $x(1)$, so there is a unique minimizer of the problem $(9.215)$. Let $x_{y(1)}$ be the unique minimizer, then by the KKT condition,

$$\nabla \varphi_1 (y(1)) = \nabla \varphi_1 (x_{y(1)}) - \alpha \nabla \bar{f}(x_{y(1)}; y_{-}(1)), \quad (9.216)$$

which is equivalent to

$$y(1) = [\nabla \varphi_1]^{-1} \left( \nabla \varphi_1 (x_{y(1)}) - \alpha \nabla \bar{f}(x_{y(1)}; y_{-}(1)) \right). \quad (9.217)$$

Let $x_y$ be defined as

$$x_y \triangleq \begin{pmatrix} x_{y(1)} \\ y_{-}(1) \end{pmatrix}, \quad (9.218)$$

where $x_{y(1)}$ is determined by $(9.217)$. Accordingly,

$$y = \begin{pmatrix} y(1) \\ y_{-}(1) \end{pmatrix} = \begin{pmatrix} [\nabla \varphi_1]^{-1} \left( \nabla \varphi_1 (x_{y(1)}) - \alpha \nabla \bar{f}(x_{y(1)}; y_{-}(1)) \right) \\ y_{-}(1) \end{pmatrix} \quad (9.219)$$

$$= \left( I_n - U_1 U_1^T \right) x_y + U_1 [\nabla \varphi_1]^{-1} \left( \nabla \varphi_1 (x_{y(1)}) - \alpha \nabla \bar{f}(x_{y(1)}; y_{-}(1)) \right)$$

$$= \psi_1(x_y),$$

where the first equality is due to the definition of $y_{-}(1)$ (see $(9.213)$); the second equality thanks to $(9.217)$; and the third equality holds because of definition of $U_1$ (see $(2.21)$); since $\psi_1(x_y)$ is defined by $(4.44)$, the last equality holds true.

Hence, $x_y$ is mapped to $y$ by the mapping $\psi_1$. 

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(c) In addition, recalling (1.17), we have

\[ D\psi_1(x) = (I_n - U_1U_1^T) + \]

\[
\begin{pmatrix}
0 & 0 & \cdots & 0 \\
0 & I_{n_2} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & I_{n_p}
\end{pmatrix}
\]

\[
\begin{pmatrix}
\{\nabla^2\varphi_1(x) - \alpha A_{11}\} \left\{ [\nabla\varphi_1]^{-1} (\nabla\varphi_1(x) - \alpha \nabla f(x)) \right\}^{-1} & 0 & \cdots & 0 \\
A_{21} \left\{ [\nabla\varphi_1]^{-1} (\nabla\varphi_1(x) - \alpha \nabla f(x)) \right\}^{-1} & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
A_{p1} \left\{ [\nabla\varphi_1]^{-1} (\nabla\varphi_1(x) - \alpha \nabla f(x)) \right\}^{-1} & 0 & \cdots & I_{n_p}
\end{pmatrix}
\]

where the second equality is due to the definitions of \( U_1 \) and \( A_{11} \) which are given by (2.4) and (3.13), respectively. The above equality means that

\[ \text{eig} (D\psi_1(x)) = \left\{ \{\nabla^2\varphi_1(x) - \alpha A_{11}\} \left\{ [\nabla\varphi_1]^{-1} (\nabla\varphi_1(x) - \alpha \nabla f(x)) \right\}^{-1} \right\}. \]

Moreover,

\[
\nabla^2\varphi_1(x) - \alpha A_{11} \succeq \nabla^2\varphi_1(x) - \alpha LI_{n_1} \succeq \nabla^2\varphi_1(x) - \frac{\mu}{L} LI_{n_1}
\]

\[
= \nabla^2\varphi_1(x) - \mu I_{n_1} \succeq 0,
\]

where the first inequality holds because of \( A_{11} = \nabla^2_{11}f(x), \) (2.2) and Lemma 7 in Panageas and Piliouras (2016); the second inequality is due to \( \alpha < \frac{\mu}{L} \); the last inequality thanks to (2.13). Hence, \( \nabla^2\varphi_1(x) - \alpha A_{11} \) is an invertible matrix. Consequently, \( D\psi_1(x) \) is an invertible matrix as well.

(d) Note that we have shown \( \psi_1 \) is bijective, and continuously differentiable. Since \( D\psi_1(x) \) is invertible for \( \alpha < \frac{\mu}{L} \), the inverse function theorem guarantees that \( [\psi_1]^{-1} \) is continuously differentiable. Thus, \( \psi_1 \) is a diffeomorphism.

Secondly, it is obvious that similar arguments can be applied to verify that, \( \psi_s, s = 2, \ldots, p, \) are also diffeomorphisms. Thus, the proof is completed. \( \blacksquare \)
9.5. Proof of Lemma 6.3

Proof. Let $\lambda$ be an eigenvalue of $(\beta B)^{-1} \left( I + t\beta \hat{B} \right)$ and $\xi$ be the corresponding eigenvector of unit length, then $\lambda \neq 0$ and

$$(\beta B)^{-1} \left( I_n + t\beta \hat{B} \right) \xi = \lambda \xi,$$  \tag{9.221}

which is clearly equivalent to equation:

$$\left( I_n + t\beta \hat{B} \right) \xi = \lambda (\beta B) \xi.$$ \tag{9.222}

Premultiplying both sides of the above equality by $\xi^H$, we arrive at

$$1 + t\beta \xi^H \hat{B} \xi = \lambda \xi^H (\beta B) \xi,$$ \tag{9.223}

or equivalently,

$$\lambda = \frac{1 + t\beta \xi^H \hat{B} \xi}{\beta \xi^H B \xi}.$$ \tag{9.224}

Recalling that $0 < \beta < \frac{1}{\rho(B)}$ and $t \in [0, 1]$, then Lemma 6.1 implies that $0 < \text{Re} \left( 1 + \beta \xi^H \hat{B} \xi \right) < 2$. Combining with the assumptions that $\text{Re} (\lambda) > 0$ and $B$ is a symmetric matrix, we have

$$\beta \xi^H B \xi > 0.$$ \tag{9.225}

We rewrite $\hat{B}$ defined by (6.119) below:

$$\hat{B} = \text{Diag} \left( B_{11}, B_{22}, \ldots, B_{pp} \right),$$ \tag{9.226}

whose main diagonal blocks are the same as those of $B$. Therefore, $B$ has the following decomposition:

$$B = \hat{B} + \tilde{B} + \tilde{B}^T.$$ \tag{9.227}

In addition, Theorem 4.3.15 in Horn and Johnson (1986) means that

$$-\rho(B) \leq \tilde{B} \leq \rho(B).$$ \tag{9.228}
Hence, if \( t \in \left[ \frac{1}{2}, 1 \right] \), then

\[
\text{Re} (\lambda) - \frac{1}{2} = \frac{\text{Re} \left( 1 + t \beta \xi^H \hat{B} \xi \right)}{\beta \xi^H B \xi} - \frac{1}{2}
\]

\[
= \frac{2 \text{Re} \left( 1 + t \beta \xi^H \hat{B} \xi \right) - \beta \xi^H B \xi}{2 \beta \xi^H B \xi}
\]

\[
= \frac{2 + \beta \text{Re} \left( \xi^H \hat{B} \xi \right) - 2 \beta \text{Re} \left( \xi^H \hat{B} \xi \right) - \beta \xi^H \hat{B} \xi}{2 \beta \xi^H B \xi}
\]

\[
= \frac{2 + 2(t - 1) \beta \text{Re} \left( \xi^H \hat{B} \xi \right) - \beta \xi^H \hat{B} \xi}{2 \beta \xi^H B \xi}
\]  

\[
= \frac{2 + 2(t - 1) \beta \rho (B) - \beta \rho (B)}{2 \beta \xi^H B \xi}
\]

\[
\geq \frac{2 - 2 \beta \rho (B)}{2 \rho (B)}
\]

\[
= \frac{1 - \beta \rho (B)}{\beta \rho (B)}
\]

\[
> 0,
\]

where the third equality is due to (9.227); the first inequality thanks to Eqs. (6.118), (9.225) and (9.228); the second inequality holds because of \( t \in \left[ \frac{1}{2}, 1 \right] \); the last
inequality holds because of $\beta \in (0, \frac{1}{\rho(B)})$. If $t \in [0, \frac{1}{2}]$, then

$$\text{Re}(\lambda) - \frac{1}{2} = \frac{\text{Re}\left(1 + t\beta\xi^H\hat{B}\xi\right)}{\beta\xi^H\xi} - \frac{1}{2}$$

$$= \frac{2\text{Re}\left(1 + t\beta\xi^H\hat{B}\xi\right) - \beta\xi^H\xi}{2\beta\xi^H\xi}$$

$$= \frac{2 + 2t\beta\text{Re}\left(\xi^H\hat{B}\xi\right) - \beta\xi^H\xi}{2\beta\xi^H\xi}$$

$$\geq \frac{2 - 2t\beta\rho(B) - \beta\rho(B)}{2\beta\rho(B)}$$

$$\geq \frac{2 - 2\beta\rho(B)}{2\beta\rho(B)}$$

$$= \frac{1 - \beta\rho(B)}{\beta\rho(B)}$$

$$> 0,$$

where the first inequality is due to Eqs. (6.118) and (9.225); the second inequality holds because of $t \in [0, \frac{1}{2}]$; $\beta \in \left(0, \frac{1}{\rho(B)}\right)$ implies the last inequality.

Thus, the proof is finished. \[\square\]

### 9.6. Proof of Lemma 6.5

**Proof.** We divide the proof into two cases.

**Case 1: B is an invertible matrix.** Therefore, we have

$$\left(\beta\left(I_n + \beta\hat{B}\right)^{-1}B\right)^{-1} = (\beta B)^{-1}\left(I_n + \beta\hat{B}\right),$$

(9.231)

which implies that

$$\lambda \in \text{eig}\left(I_n + \beta\hat{B}\right) \iff \frac{1}{\lambda} \in \text{eig}\left((\beta B)^{-1}\left(I_n + \beta\hat{B}\right)\right).$$

(9.232)

For clarity of notation, we use $\sigma$ to denote the eigenvalue of $(\beta B)^{-1}\left(I_n + \beta\hat{B}\right)$. Hence, it is sufficient for us to prove that, for an arbitrary $\beta \in \left(0, \frac{1}{\rho(B)}\right)$, there is at least one nonzero eigenvalue $\sigma$ of $(\beta B)^{-1}\left(I_n + \beta\hat{B}\right)$ such that

$$\sigma \in \Xi(\beta, B).$$

(9.233)

Subsequently, we will prove that relation (9.233) is true by using Lemma 9.2 in Appendix.
We first define an analytic function with $t$ as parameter:

$$
X_t(z) \triangleq \det \left\{ zI_n - \left[ (1 - t) (\beta B)^{-1} + t (\beta B)^{-1} \left( I_n + \beta \hat{B} \right) \right] \right\}, \quad 0 \leq t \leq 1
$$

$$
= \det \left\{ zI_n - (\beta B)^{-1} \left( I_n + t\beta \hat{B} \right) \right\}, \quad 0 \leq t \leq 1.
$$

(9.234)

In order to construct a closed region, we define

$$
\nu \triangleq \| (\beta B)^{-1} \| + \frac{1}{\rho(\hat{B})} \left\| (\beta B)^{-1} \hat{B} \right\| \geq \left\| (\beta B)^{-1} \| + t\beta \right\| (\beta B)^{-1} \hat{B} \right\|, \quad \forall t \in [0, 1]
$$

$$
\geq \left\| (\beta B)^{-1} \left( I_n + t\beta \hat{B} \right) \right\|, \quad \forall t \in [0, 1]
$$

(9.235)

where the first inequality holds because of $\beta \in \left( 0, \frac{1}{\rho(B)} \right)$ and $t \in [0, 1]$. In addition, let $t = 0$, the above equation also means that

$$
\nu \geq \| (\beta B)^{-1} \| \geq \rho ((\beta B)^{-1}) \geq \frac{1}{\beta \rho(B)} > \frac{1}{\beta \rho(B)} - \frac{1}{2} = \frac{1}{2} + \frac{1 - \beta \rho(B)}{\beta \rho(B)},
$$

(9.236)

where the second equality is due to the definitions of spectral norm and spectral radius; the third inequality thanks to property of spectral radius; the last inequality holds because of $\beta \in \left( 0, \frac{1}{\rho(B)} \right)$.

Thus given the above $\nu$ satisfying (9.235) and (9.236), we can define a closed rectangle as

$$
\mathcal{D} \triangleq \left\{ a + bi \left| \frac{1}{2} \leq a \leq 2\nu, \quad -2\nu \leq b \leq 2\nu \right. \right\},
$$

(9.237)

which is a closed region in the complex plane. Note its boundary $\partial \mathcal{D}$ consists of a finite number of smooth curves. Specifically, define

$$
\gamma_1 \triangleq \left\{ a + bi \left| a = \frac{1}{2}, \quad -2\nu \leq b \leq 2\nu \right. \right\},
$$

$$
\gamma_2 \triangleq \left\{ a + bi \left| a = 2\nu, \quad -2\nu \leq b \leq 2\nu \right. \right\},
$$

$$
\gamma_3 \triangleq \left\{ a + bi \left| \frac{1}{2} \leq a \leq 2\nu, \quad b = 2\nu \right. \right\},
$$

$$
\gamma_4 \triangleq \left\{ a + bi \left| \frac{1}{2} \leq a \leq 2\nu, \quad b = -2\nu \right. \right\},
$$

(9.238)

then

$$
\partial \mathcal{D} = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4.
$$

(9.239)
In order to apply Lemma 9.2, we will show that
\[ X_t(z) \neq 0, \; \forall t \in [0, 1], \; \forall z \in \partial \mathcal{D}. \] (9.240)

On the one hand, since the spectral norm of a matrix is larger than or equal to its spectral radius, the above inequality (9.235) yields that, for any \( t \in [0, 1] \), every eigenvalue of \( B^{-1} (I + t\beta \hat{B}) \) has a magnitude less than \( \nu \). Note that for an arbitrary \( z \in \gamma_2 \cup \gamma_3 \cup \gamma_4 \), then \( |z| \geq 2\nu \). Consequently,
\[ X_t(z) \neq 0, \; \forall t \in [0, 1], \; \forall z \in \gamma_2 \cup \gamma_3 \cup \gamma_4. \] (9.241)

On the other hand, if \( \sigma \) is an eigenvalue of \( (\beta B)^{-1} \left( I_n + t\beta \hat{B} \right) \) with any \( t \in [0, 1] \), and \( \text{Re}(\sigma) > 0 \), then Lemma 6.3 implies that,
\[ \text{Re}(\sigma) > \frac{1}{2}, \] (9.242)
which immediately implies that
\[ \sigma \notin \gamma_1. \] (9.243)

Hence, we have
\[ X_t(z) \neq 0, \; \forall t \in [0, 1], \; \forall z \in \gamma_1. \] (9.244)

Combining (9.241) and (9.244), we obtain (9.240).

As a result, it follows from Lemma 9.2 in Appendix, (9.234), (9.237) and (9.240)
that \( X_0(z) = \det \{ zI_n - (\beta B)^{-1} \} \) and \( X_1(z) = \det \left\{ zI_n - \left( \beta B \right)^{-1} \left( I_n + \beta \hat{B} \right) \right\} \) have the same number of zeros in \( \mathcal{D} \). Note that \( \lambda_{\max}(B) > 0 \) implies that there is at least one positive eigenvalue \( \frac{1}{\beta \lambda_{\max}(B)} \) of \( (\beta B)^{-1} \). Note that \( \frac{1}{\beta \lambda_{\max}(B)} \geq \frac{1}{\beta \rho(B)} > 1 \).

Recalling the definition (9.235) of \( \nu \), we know \( \left| \frac{1}{\beta \lambda_{\max}(B)} \right| \leq \nu \). Thus \( \frac{1}{\beta \lambda_{\max}(B)} \) must lie inside \( \mathcal{D} \). In other words, the number of zeros of \( X_0(z) \) inside \( \mathcal{D} \) is at least one, which in turn shows the number of zeros of \( X_1(z) \) is at least one as well. Thus, there must exist at least one eigenvalue of \( (\beta B)^{-1} \left( I_n + \beta \hat{B} \right) \) lying inside \( \mathcal{D} \). We denote it as \( \sigma \), then \( \text{Re}(\sigma) > \frac{1}{2} \). Moreover, Lemma 6.3 means that \( \text{Re}(\sigma) \geq \frac{1}{2} + \frac{1 - \beta \rho(B)}{\beta \rho(B)} \). Hence, \( \sigma \) lies in \( \Xi(\beta, B) \) defined by (6.132) and the proof is finished in this case.

Case 2: B is a singular matrix. In this case, we will apply perturbation theorem based on the results in Case 1 to prove (6.131).

Suppose the multiplicity of zero eigenvalue of \( B \) is \( m \). For clarity of notation, we rewrite the eigen decomposition (9.160) of \( B \) below:
\[ B = V \begin{pmatrix} \Theta & 0 \\ 0 & 0 \end{pmatrix} V^T = V_1 \Theta V_1^T, \] (9.245)
where $\Theta = \text{Diag}(\theta_1, \theta_2, \ldots, \theta_{n-m})$, $\theta_s$, $s = 1, \ldots, n-m$, are the nonzero eigenvalues of $B$ and
\[
V = \begin{pmatrix} V_1 & V_2 \end{pmatrix}
\] (9.246)
is an orthogonal matrix and $V_1$ consists of the first $(n-m)$ columns of $V$.
Denote
\[
\delta \triangleq \min \{|\theta_1|, |\theta_2|, \ldots, |\theta_{n-m}|\}.
\] (9.247)
For any $\epsilon \in (-\delta, 0)$, we define
\[
B(\epsilon) \triangleq B + \epsilon I_n,
\] (9.248)
then,
\[
\text{eig}(B(\epsilon)) = \{\theta_1 + \epsilon, \theta_2 + \epsilon, \ldots, \theta_{n-m} + \epsilon, \epsilon\} \not\ni 0, \quad \forall \epsilon \in (-\delta, 0),
\] (9.249)
and
\[
\lambda_{\max}(B(\epsilon)) = \lambda_{\max}(B) + \epsilon \geq \delta + \epsilon > 0, \quad \forall \epsilon \in (-\delta, 0),
\] (9.250)
where the first inequality is due to the definition of $\delta$ and $\max\{\theta_1, \theta_2, \ldots, \theta_{n-m}\} = \lambda_{\max}(B) > 0$.

Since $B$ is defined by (6.111), $B(\epsilon)$ has $p \times p$ blocks form as well. Specifically,
\[
B(\epsilon) = (B(\epsilon)_{st})_{1 \leq s, t \leq p},
\] (9.251)
and its $(s,t)$-th block is given
\[
B(\epsilon)_{st} = \begin{cases} B_{st} + \epsilon I_{n_s}, & s = t, \\ B_{st}, & s \neq t, \end{cases}
\] (9.252)
where $n_1, n_2, \ldots, n_p$ are $p$ positive integer numbers satisfying $\sum_{s=1}^{p} n_s = n$. Similar to definition (6.115), we denote the strictly block upper triangular matrix based on $B(\epsilon)$ as
\[
\hat{B}(\epsilon) \triangleq \left(\hat{B}(\epsilon)_{st}\right)_{1 \leq s, t \leq p},
\] (9.253)
with $p \times p$ blocks and its $(s,t)$-th block is given by
\[
\hat{B}(\epsilon)_{st} = \begin{cases} B(\epsilon)_{st}, & s < t, \\ 0, & s \geq t, \end{cases}
= \begin{cases} B_{st}, & s < t, \\ 0, & s \geq t, \end{cases}
\] (9.254)
where the second equality holds because of (9.252); the last equality is due to (6.116).
It follows easily from Eqs. (6.115), (6.116) (9.253) and (9.254) that
\[ \hat{B}(\epsilon) = \hat{B}. \] (9.255)

Consequently,
\[ \beta \left( I_n + \beta \hat{B}(\epsilon) \right)^{-1} B(\epsilon) = \beta \left( I_n + \beta \hat{B} \right)^{-1} (B + \epsilon I_n), \]
where the first equality is due to (9.255) and the second equality holds because of (9.248). For simplicity, let
\[ \lambda_{\beta}^s(\epsilon), \ s = 1, \ldots, n, \] (9.256)
be the eigenvalues of \( \beta \left( I_n + \beta \hat{B}(\epsilon) \right)^{-1} (B + \epsilon I_n). \)

Note that for any \( \epsilon \in (-\delta, 0), B(\epsilon) \) is invertible and \( \lambda_{\text{max}}(B(\epsilon)) > 0 \) (see (9.250)). According to the definitions of \( B(\epsilon) \) and \( \hat{B}(\epsilon) \), a similar argument in Case 1 can be applied with the identifications \( B(\epsilon) \sim B, \hat{B}(\epsilon) \sim \hat{B}, \beta \sim \beta \) and \( \rho(B(\epsilon)) \sim \rho(B) \), to prove that, for any \( \beta \in \left( 0, \frac{1}{\rho(B)} \right) \), there must exist at least one eigenvalue of \( \left( I_n + \beta \hat{B}(\epsilon) \right)^{-1} B(\epsilon) \) which lies in \( \Xi(\beta, B(\epsilon)) \) defined by the following (9.258). Taking into account definition (9.248), we have \( \rho(B(\epsilon)) \leq \rho(B) + \epsilon \). Hence, for any \( \epsilon \in (-\delta, 0) \) and \( \beta \in \left( 0, \frac{1}{\rho(B)+\epsilon} \right) \subseteq \left( 0, \frac{1}{\rho(B)} \right) \), there exists at least one index denoted as \( s(\epsilon) \in \{1, 2, \ldots, n\} \) such that
\[ \frac{1}{\lambda_{\beta}^s(\epsilon)} \in \Xi(\beta, B(\epsilon)), \] (9.257)
where
\[ \Xi(\beta, B(\epsilon)) \equiv \left\{ a + bi \mid a, b \in \mathbb{R}, \frac{1}{2} + \frac{1 - \beta \rho(B(\epsilon))}{\beta \rho(B(\epsilon))} \leq a, i = \sqrt{-1} \right\}. \] (9.258)

Furthermore, it is well known that the eigenvalues of a matrix \( M \) are continuous functions of the entries of \( M \). Therefore, for any \( \beta \in \left( 0, \frac{1}{\rho(B)} \right), \lambda_{\beta}^s(\epsilon) \) is a continuous function of \( \epsilon \) and
\[ \lim_{\epsilon \to 0^-} \lambda_{\beta}^s(\epsilon) = \lambda_{\beta}^s(0), \ s = 1, \ldots, n, \] (9.259)
where \( \lambda_{\beta}^s(0) \) is the eigenvalue of \( \beta \left( I_n + \beta \hat{B} \right)^{-1} B. \)

In what follows, we will prove that (6.131) holds true by contradiction.

Suppose for sake of contradiction that, there exists a \( \beta^* \in \left( 0, \frac{1}{\rho(B)} \right) \) such that, for any \( s \in \{1, \ldots, n\}, \) if \( \lambda_{\beta^*}^s(0) \neq 0, \) then
\[ \frac{1}{\lambda_{\beta^*}^s(0)} \notin \Xi(\beta^*, B(0)) = \Xi(\beta^*, B), \] (9.260)
where $\lambda^\beta_s(0)$ is the eigenvalue of $\beta^* \left( I_n + \beta^* \hat{B} \right)^{-1} B$.

According to Lemma 9.3 in Appendix and the assumption that the multiplicity of zero eigenvalue of $B$ is $m$, we know that the multiplicity of eigenvalue 0 of $\beta^* \left( I_n + \beta^* \hat{B} \right)^{-1} B$ is $m$ as well. Then there are exactly $m$ eigenvalue functions of $\epsilon$ whose limits are 0 as $\epsilon$ approaches zero from above. Without loss of generality, we assume

$$
\lim_{\epsilon \to 0^+} \lambda^\beta_s(\epsilon) = \lambda^\beta_s(0) = 0, \quad s = 1, \ldots, m,
$$

and

$$
\lim_{\epsilon \to 0^+} \lambda^\beta_s(\epsilon) = \lambda^\beta_s(0) \neq 0, \quad s = m + 1, \ldots, n.
$$

Subsequently, under the assumption (9.260), we will prove that there exists a $\delta^* > 0$ with $\delta^* \leq \delta$ such that, for any $\epsilon \in (-\delta^*, 0)$, then $\beta^* \in \left(0, \frac{1}{\rho(B) + \epsilon}\right) \subseteq \left(0, \frac{1}{\rho(B)}\right)$. This would contradict (9.277) and there does not exist any $s \in \{1, \ldots, n\}$ such that $\frac{1}{\lambda^\beta_s(\epsilon)}$ belongs to $\Xi(\beta^*, B(\epsilon))$. The proof is given by the following four steps.

**Step (a):** Under the assumption (9.260), we first prove that there exists a $\delta^*_1 > 0$ such that, for any $\epsilon \in (-\delta^*_1, 0)$, $\beta^* \in \left(0, \frac{1}{\rho(B) + \epsilon}\right) \subseteq \left(0, \frac{1}{\rho(B)}\right)$ and there does not exist any $s \in \{m + 1, \ldots, n\}$ such that $\frac{1}{\lambda^\beta_s(\epsilon)}$ belongs to $\Xi(\beta^*, B(\epsilon))$.

Taking into account the definition of $\Xi(\beta^*, B(\epsilon))$, Eq. (9.260) and $\beta^* \in \left(0, \frac{1}{\rho(B)}\right)$, we imply that, there exists a $\bar{\delta}$ such that, for any $\epsilon \in (\bar{\delta}, 0)$,

$$
\beta^* \in \left(0, \frac{1}{\rho(B) + \epsilon}\right) \subseteq \left(0, \frac{1}{\rho(B)}\right)
$$

and

$$
\text{Re}\left(\frac{1}{\lambda^\beta_s(\epsilon)}\right) < \frac{1}{2} + \frac{1 - \beta^* \rho(B(\epsilon))}{\beta^* \rho(B(\epsilon))}, \quad \forall s \in \{m + 1, \ldots, n\}.
$$

Moreover, note that $\frac{1}{\lambda^\beta_s(\epsilon)}$ is a continuous function of $\epsilon$ and (9.262) holds. Combining with the above inequalities, we know that there exists a $\delta^*_1 > 0$ with $\delta^*_1 \leq \bar{\delta}$ such that

$$
\left| \frac{1}{\lambda^\beta_s(\epsilon)} - \frac{1}{\lambda^\beta_s(0)} \right| < \frac{1}{3} \left[ \frac{1}{2} + \frac{1 - \beta^* \rho(B(\epsilon))}{\beta^* \rho(B(\epsilon))} - \text{Re}\left(\frac{1}{\lambda^\beta_s(0)}\right) \right], \quad \forall s \in \{m + 1, \ldots, n\}, \forall \epsilon \in (-\delta^*_1, 0),
$$

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which further means that, for any \( s \in \{m + 1, \ldots, n\} \) and \( \epsilon \in (-\delta^*_1, 0) \),
\[
\text{Re} \left( \frac{1}{\lambda_s^\beta(\epsilon)} \right) < \text{Re} \left( \frac{1}{\lambda_s^\beta(0)} \right) + \frac{1}{3} \left[ \frac{1}{2} + \frac{1 - \beta^* \rho(B(\epsilon))}{\beta^* \rho(B(\epsilon))} - \text{Re} \left( \frac{1}{\lambda_s^\beta(0)} \right) \right],
\]
\[
= \frac{1}{2} + \frac{1 - \beta^* \rho(B(\epsilon))}{\beta^* \rho(B(\epsilon))} - \frac{2}{3} \left[ \frac{1}{2} + \frac{1 - \beta^* \rho(B(\epsilon))}{\beta^* \rho(B(\epsilon))} - \text{Re} \left( \frac{1}{\lambda_s^\beta(0)} \right) \right],
\]
\[
< \frac{1}{2} + \frac{1 - \beta^* \rho(B(\epsilon))}{\beta^* \rho(B(\epsilon))}.
\]

Hence, we arrive at
\[
\frac{1}{\lambda_s^\beta(\epsilon)} \notin \Xi (\beta^*, B(\epsilon)), \ \forall \ s \in \{m + 1, \ldots, n\}, \ \forall \epsilon \in (-\delta^*_1, 0). \quad (9.265)
\]

**Step (b):** In this step, by the same arguments as in **Step (b)** in the proof of Lemma 6.4 (see (9.185) – (9.203)), we can prove that there exists a \( \delta^*_2 > 0 \) such that, for any \( \epsilon \in (-\delta^*_2, 0) \) and \( s \in \{1, \ldots, m\} \),
\[
\text{Re} \left( \lambda_s^\beta(\epsilon) \right) < 0, \quad (9.266)
\]
or equivalently,
\[
\text{Re} \left( \frac{1}{\lambda_s^\beta(\epsilon)} \right) < 0, \quad (9.267)
\]
which immediately implies that
\[
\frac{1}{\lambda_s^\beta(\epsilon)} \notin \Xi (\beta^*, B(\epsilon)), \ \forall \ s \in \{m + 1, \ldots, n\}, \ \forall \epsilon \in (-\delta^*_2, 0). \quad (9.268)
\]

**Step (c):** Combining (9.265) and (9.268), we have the following conclusion
\[
\frac{1}{\lambda_s^\beta(\epsilon)} \notin \Xi (\beta^*, B(\epsilon)), \ \forall \ s \in \{1, \ldots, n\}, \ \forall \epsilon \in (-\delta^*_3, 0), \quad (9.269)
\]
where \( \delta^*_3 = \min \{\delta^*_1, \delta^*_2\} \).

**Step (d):** Let
\[
\delta^* = \min \left\{ \frac{1}{2} \delta, \delta^*_3 \right\}. \quad (9.270)
\]

Then, for any \( \epsilon \in (-\delta^*, 0) \), we have
\[
\epsilon \in (-\delta, 0), \quad (9.271)
\]
\[
\beta^* \in \left( 0, \frac{1}{\rho(B) + \epsilon} \right) \subseteq \left( 0, \frac{1}{\rho(B(\epsilon))} \right) \quad (9.272)
\]
and

\[ \frac{1}{\lambda_s^{\beta}(\epsilon)} \notin \Xi(\beta^*, B(\epsilon)), \ \forall s \in \{1, \ldots, n\}, \] (9.273)

where (9.271) uses the definition (9.270) of \( \delta^* \); (9.272) is due to the definition (9.270) of \( \delta^* \) (i.e., \( \delta^* \leq \delta_3^* \leq \delta_1^* \leq \bar{\delta} \)) and (9.263); and (9.273) thanks to the definition (9.270) of \( \delta^* \) and (9.269). Clearly, this contradicts (9.257).

Hence, we conclude that Eq. (6.131) holds true.

Lemma 9.1 (Rouche’s Theorem: Conway (1973)) Suppose \( f \) and \( g \) are meromorphic in a neighborhood of \( \bar{B}(a; R) \) with no zeros or poles on the circle \( \gamma = \{z : |z - a| = R\} \). If \( Z_f \) and \( Z_g \) (\( P_f \) and \( P_g \)) are the number of zeros (poles) of \( f \) and \( g \) inside \( \gamma \) counted according to their multiplicities and if

\[ |f(z) + g(z)| < |f(z)| + |g(z)| \]
on \( \gamma \), then

\[ Z_f - P_f = Z_g - P_g. \]

Lemma 9.2 (Ostrowski (1958)) Assume \( M, N \in \mathbb{C}^{n \times n} \) and define

\[ X_t(z) \triangleq \det \{zI_n - [(1 - t)M + tN]\}, \ 0 \leq t \leq 1. \] (9.274)

Moreover, suppose that \( \mathcal{D} \) is a closed region and \( \partial\mathcal{D} \) consists of a finite number of smooth curves. If

\[ X_t(z) \neq 0, \ \forall t \in [0, 1], \ \forall z \in \partial\mathcal{D}, \] (9.275)
then, for any \( t_1, t_2 \in [0, 1] \), \( X_{t_1}(z) \) and \( X_{t_2}(z) \) have the same number of zeros in \( \mathcal{D} \) counted according to their multiplicities.

Proof. For any \( t_0 \in [0, 1] \), define

\[ u(z) \triangleq X_{t_0}(z) \] (9.276)

and

\[ p \triangleq \min_{z \in \partial\mathcal{D}} |X_{t_0}(z)| > 0. \]
(9.277)

Moreover, for any \( t_0 + \epsilon \in [0, 1] \), let

\[ v(z) \triangleq X_{t_0 + \epsilon}(z) - X_{t_0}(z). \] (9.278)

Since \( X_{t_0}(z) \) is a continuous function on the close set \([0, 1] \times \partial\mathcal{D}\); the following inequality holds true

\[ \max_{z \in \partial\mathcal{D}} |v(z)| < p. \] (9.279)
Hence $|\epsilon|$ is sufficiently small. Therefore, two single-valued analytic functions $u(z)$ and $v(z)$ on region $\mathcal{D}$ satisfy

$$|u(z)| > |v(z)|, \forall z \in \partial \mathcal{D}. \quad (9.280)$$

According to Rouche’s theorem (see Lemma 9.1), $u(z) + v(z) = X_{t_0+\epsilon}(z)$ and $u(z) = X_{t_0}(z)$ have the same number of zeros in $\mathcal{D}$. Hence, given any $t$ in the $|\epsilon|$ neighborhood of $t_0$, the number of zeros of $X_t(z)$ in the region $\mathcal{D}$ denoted as $N(t)$, is a constant. Thereby, $N(t)$ is a continuous function defined on $[0,1]$. However, the function $N(t)$ can only be nonnegative integers. Therefore, $N(t)$ is a constant function on $[0,1]$.

**Lemma 9.3** Assume that $B$ and $\hat{B}$ are defined by (6.111) and (6.113), respectively, then, for any $\beta \in \left(0, \frac{1}{\rho(B)}\right)$, the multiplicity of zero eigenvalue of $\left( I_n + \beta \hat{B} \right)^{-1} B$ is the same as that of zero eigenvalue of $B$.

**Proof.** Without a loss of generality, suppose the multiplicity of zero eigenvalue of $B$ is $m$ and $B$ has an eigen decomposition in the form of (9.160). Therefore, in order to investigate the eigenvalues of $\left( I_n + \beta \hat{B} \right)^{-1} B$, its characteristic polynomial is given below:

$$\det \left\{ \left( I_n + \beta \hat{B} \right)^{-1} B - \lambda I_n \right\}$$

$$= \det \left\{ B - \lambda \left( I_n + \beta \hat{B} \right) \right\}$$

$$= \det \left\{ V \begin{pmatrix} \Theta & 0 \\ 0 & 0 \end{pmatrix} V^T - \lambda \left( I_n + \beta \hat{B} \right) \right\}$$

$$= \det \left\{ \begin{pmatrix} \Theta & 0 \\ 0 & 0 \end{pmatrix} - \lambda \left( I_n + \beta V^T \hat{B} V \right) \right\}$$

$$= \det \left\{ \begin{pmatrix} \Theta - \lambda (I_{n-m} + \beta \hat{C}_{11}) & -\lambda \beta \hat{C}_{12} \\ -\lambda \beta \hat{C}_{21} & -\lambda (I_m + \beta \hat{C}_{22}) \end{pmatrix} \right\}$$

$$= \det \left\{ \Theta - \lambda \left( I_{n-m} + \beta \hat{C}_{11} \right) \right\}$$

$$\times \det \left\{ -\lambda (I_m + \beta \hat{C}_{22}) - \lambda^2 \beta^2 \hat{C}_{21} \left[ \Theta - \lambda \left( I_{n-m} + \beta \hat{C}_{11} \right) \right]^{-1} \hat{C}_{12} \right\}$$

$$= \lambda^n \det \left\{ \Theta - \lambda \left( I_{n-m} + \beta \hat{C}_{11} \right) \right\}$$

$$\times \det \left\{ - (I_m + \beta \hat{C}_{22}) - \lambda \beta^2 \hat{C}_{21} \left[ \Theta - \lambda \left( I_{n-m} + \beta \hat{C}_{11} \right) \right]^{-1} \hat{C}_{12} \right\},$$

where the second equality holds because of the eigen decomposition form (9.160) of $B$ and the fourth equality is due to the definitions (9.185) of $C_{ij}, i, j = 1, 2$. Note that $\Theta$ is invertible, we know zero is not a root of polynomial $\det \left\{ \Theta - \lambda \left( I_{n-m} + \beta \hat{C}_{11} \right) \right\}$. 

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In addition, assume that \( \lambda \) is an eigenvalue of \( I_m + \beta \hat{C}_{22} \) and \( v \in \mathbb{C}^m \) is its eigenvector of length one, then

\[
\text{Re} (\lambda) = \text{Re} \left( v^H (I_m + \beta \hat{C}_{22}) v \right) \\
= \text{Re} \left( 1 + \beta v^H \hat{C}_{22} v \right) \\
= \text{Re} \left( 1 + \beta v^H V_2^T \hat{B} V_2 v \right) \\
= \text{Re} \left( 1 + \beta v^H V_2^H \hat{B} V_2 v \right) \\
= \text{Re} \left( 1 + \beta (V_2 v)^H \hat{B} V_2 v \right) \\
\geq 1 - \frac{\beta}{\rho(B)} = \frac{\rho(B) - \beta}{\rho(B)} > 0,
\]

where the first equality follows from (9.185); the second equality is because \( V_1 \) is a real matrix in equation (9.160); it follows the last equality from Lemma 6.2 and \( \|V_1 v\| = (V_1 v)^H V_1 v = v^H v = 1 \). The above inequality also shows that the real part of the eigenvalues of \( I_m + \beta \hat{C}_{22} \) is not zero. Hence, zero is not one of its eigenvalues.

The above analysis shows that zero is not one of the eigenvalues of \( \Theta - \lambda (I_{n-m} + \beta \hat{C}_{11}) \) nor \( I_m + \beta \hat{C}_{22} \). Combined with the expression (9.281), the multiplicity of zero eigenvalue of \( (I_n + \beta \hat{B})^{-1} B \) is exactly \( m \).

**Lemma 9.4** Suppose that \( \phi(x) \) is a strongly convex twice continuously differentiable function with parameter \( \sigma > 0 \), i.e., for any \( y \) and \( x \in \mathbb{R}^n \),

\[
\phi(x) \geq \phi(y) + \langle x - y, \nabla \phi(y) \rangle + \frac{\sigma}{2} \|x - y\|^2,
\]

then its gradient mapping \( \nabla \phi : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is a diffeomorphism.

**Proof.** Since \( \phi \) is a \( \sigma \)-strongly convex function, it follows easily from (9.283) that

\[
\langle \nabla \phi(y) - \nabla \phi(x), y - x \rangle \geq \sigma \|y - x\|^2.
\]

We first check that \( \nabla \phi \) is injective from \( \mathbb{R}^n \rightarrow \mathbb{R}^n \). Suppose that there exist \( x \) and \( y \) such that \( \nabla \phi(x) = \nabla \phi(y) \). Then we would have

\[
\|y - x\|^2 \leq \frac{1}{\sigma} \langle \nabla \phi(y) - \nabla \phi(x), y - x \rangle = 0,
\]

which means \( x = y \).

To show the gradient map \( \nabla \phi \) is surjective, we construct an explicit inverse function \( (\nabla \phi)^{-1} \) given by

\[
x_y = \arg \min_x \phi(x) - \langle y, x \rangle.
\]
Since $\phi$ is a $\sigma$-strongly convex function, the function above is strongly convex with respect to $x$. Hence, there is a unique minimizer and we denote it as $x_y$. Then by the KKT condition,

$$\nabla \phi(x_y) = y.$$ 

Thus, $x_y$ is mapped to $y$ by the mapping $\nabla \phi$. Consequently, $\nabla \phi$ is bijection. The assumptions also mean it is continuously differentiable.

In addition, for any $x \in \mathbb{R}^n$,

$$D\nabla \phi(x) = \nabla^2 \phi(x) \succeq \mu I_n \succ 0,$$  

(9.285)

where the last but one inequality holds because of $\sigma$ strong convexity of $\phi$ (Nesterov, 2004). Therefore, $D\nabla \phi(x)$ is invertible, and the inverse function theorem guarantees $(\nabla \phi)^{-1}$ is continuously differentiable. Thus $\nabla \phi$ is a diffeomorphism.

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