Directed sandpile models with different toppling rules are studied by means of numerical simulations in two dimensions, with the purpose of determining the different universality classes. It is concluded that the random-threshold directed model is in the same universality of the Manna directed model where multiple toppling events plays a determinant role. The BTW model with uniform and noisy driving are found to display the same critical behavior. Moreover it is observed that the Zhang model does not satisfy simple finite size scaling.

64.60.Lx, 05.70.Ln

I. INTRODUCTION

The classification of sandpile models in their different universality class has been a topic of intensive research in the last decade \[1\]. However, most of the work has been devoted to models with undirected toppling while their corresponding directed variants has been less studied. From the theoretical side we count with the exact solution for the directed BTW model \[2\] obtained by Dhar and Ramaswamy \[2\], which can be taken as a reference for numerical simulations. On the other hand, Pastor-Satorras and Vespignani \[3\] have recently reported numerical simulations for the Manna model \[5\], which puts clear evidence about the classification of the BTW and Manna directed models in different universality classes. Other studies include a directed model with a probabilistic toppling \[2\] and a recent report concerning the effect of local dissipation on the SOC state \[4\].

The analysis of other directed models, the non Abelian Zhang model for instance, is of great interested and may give light to the study of their corresponding undirected variants. With this scope three different directed models are investigated by means of numerical simulations. This includes the well known Zhang model \[8\], the random threshold model (RT) \[9\] and the BTW model under an uniform driving. The influence of a uniform driving in the Zhang model has been already discussed in the literature \[10\] although some aspects are still not clear. However, the same analysis has not been made for models with a discrete toppling rule, like the BTW model, where the energy transfer always takes place in discrete units. In this direction Narayan and Middleton \[11\] suggested that the BTW model under noisy and uniform driving has the same critical behavior.

From the numerical data it is concluded that the Manna and RT directed models belong to the same universality class, in agreement with the general believe for the corresponding undirected variants \[10,12\]. Moreover, the data obtained for the Zhang model does not satisfy finite-size scaling which suggest that this model is in different universality class from that of the models mentioned above.

In the case of the BTW model under uniform driving it is shown, after some algebra with the toppling operators \[2\], that its evolution is periodic in time with a period scaling linearly with system size. In spite of this periodicity, which is not present in the original model with noisy driving \[3\], the statistics of the avalanches is found to be practically identical to its noisy driving counterpart.

II. MODELS AND SIMULATIONS

A. Models with noisy driving

Consider a square lattice of $L^2$ sites labeled by index $(i,j)$ ($i,j = 1,\ldots,N$) and assign a variable $z_{ij}$ to each of them. $z_{ij}$ can be continuous or discrete and may have different interpretations depending on the system one is modeling. It will be referred here as the energy storage at the corresponding site. The geometry used is shown in fig. \[i\], in which a site can transfer energy only to its three downward nearest neighbors (nn). In the horizontal direction periodic boundary conditions are considered while the downward boundary is taken open. This geometry allows a natural implementation of the Manna toppling rule \[4\].

One motivation for the use of this geometry was given in \[1\], it is just introduced to allow the implementation of the Manna toppling rule.

To completely define a model one should specify the initial condition and the evolution rules (addition of energy and toppling) of the sandpile cellular automaton model. A threshold $z_c$ is considered in such a way that sites with $z < z_c$ are say to be stable and their energy remains constant, while those with $z \geq z_c$ are say to be active and topple transferring energy to their downward nn. First the usual noisy addition of energy is considered. In this case, if all sites are stable a unit of energy is added
to a site selected at random. Then the system is updated
in parallel using the toppling rule until all sites are stable.
The number of toppling events required to drive the sys-
tem to an stable configuration is the size of the avalanche
and is denoted by \( s \). On the other hand, the number of
steps (parallel updates) required is its duration and is
denoted by \( T \). Since the driving acts at random after
some avalanches the system "forget" its initial condition
and reaches a stationary state. In other words the initial
condition is irrelevant.

Models will differ from each other depending on the
specific toppling rule one implements. Here the following
toppling rules are considered,

- **BTW**: \( z_c = 3 \), \( z_{ij} \rightarrow z_{ij} - 3 \) and \( z_{k,j-1} \rightarrow z_{k,j-1} + 1 \)
  \( (k = i - 1, i, i + 1) \);
- **Manna**: \( z_c = 2 \), \( z_{ij} \rightarrow z_{ij} - 2 \) and \( z_{k,j-1} \rightarrow z_{k,j-1} + \delta_k \)
  \( (k = i - 1, i, i + 1) \), where \( \delta_k \) can take the
  values 0, 1, 2 at random but with the constraint of
  conservation \( \sum_k \delta_k = 2 \);
- **Zhang**: \( z_c = 1 \), \( z_{ij} \rightarrow 0 \) and \( z_{k,j-1} \rightarrow z_{k,j-1} + z_{ij}/3 \)
  \( (k = i - 1, i, i + 1) \);
- **RT**: \( z_c \) takes the values 3 and 4 at random after
  each toppling, \( z_{ij} \rightarrow z_{ij} - 3 \) and \( z_{k,j-1} \rightarrow z_{k,j-1} + 1 \)
  \( (k = i - 1, i + 1) \).

### B. BTW model under uniform driving

In the noisy driving described above the addition of
energy takes place at one site selected at random. How-
ever, there are many situations where a uniform driving
in which the energy at all sites increases in the same
amount becomes more realistic. Examples can be found
in earthquake dynamics \( [10] \), interface depinning \( [11] \) and
also in some experimental setups to for granular materi-
als \( [12] \). This type of driving has been investigated in
models with a toppling rule similar to that of the Zhang
models but with local dissipation \( [11] \).

In the case of the BTW model we should be careful
when introducing a global driving. If as usual \( z_{ij} \) is an
integer variable and the energy at all sites is increased at
a constant rate \( c \) then many sizes will reach the threshold
energy at the same time and, therefore, many avalanches
will start at different points of the lattice leading to the
superposition of avalanches.

This problem can be solved considering a continuum
energy profile. This is still not enough because if all sites
start with a discrete energy it will remain discrete for-
ever. We are thus forced to consider a continuum initial
profile \( z_{ij}(0) \). Then as it was already shown by Narayan
and Middleton \( [12] \) the continuum addition of energy
can be replaced by a sequential addition of energy. For sim-
plicity consider the low disorder regime where \( z_{ij}(0) < 1 \)
for all sites. For the analysis below is irrelevant if the
initial energy profile is displaced uniformly at all sites.

Now, suppose that the energy increases at rate \( c \) at all
sites. An example is shown in fig. 3 for a lattice made of
a horizontal line of three sites. Notice that in this case one
has only input of energy coming from the driving field
and output dissipation under toppling, a simplification
considered for illustrative purposes.

In the continuum time scale the energy increases lin-
early until it reaches the threshold where the site topples
and its energy decreases by 3 (BTW toppling rule). But
the system can be also monitored in a discrete time and
energy scales. In these scales, at step \( t = 0 \) all sites
have energy 0. In step 1, 2 and 3 sites 1, 2 and 3 receives
one unit of energy. Then in subsequent steps the same se-
quence of addition is repeated. The order in the sequence
of addition is clearly determined by the initial condition
and all sites should receive a unit of energy before the
first site of the second receives the second grain.

Now consider an square lattice, where sites can also
receive energy from nearest neighbors in the layer above.
The picture will not change in relation to the addition
of energy from the external field. In the BTW model
the energy is transferred in discrete units and, therefore,
the toppling only modifies the integer part of \( z \) with no
modification of the sequence of addition. This is a fund-
amental difference with the Zhang toppling rule which
not only involve the integer part of the energy but on top-
pling all the energy at the active site is transferred. The
consequences derived from this periodic sequential driv-
ing is investigated below, using the formalism introduced
by Dhar et al. \( [11] \).

Let be \( a_{ij} \) the operator which add a particle at site
\((i,j)\) and lead the system relax to an equilibrium posi-
tion \( [11] \). After \( N \) steps all sites receives one, and only
one, unit of energy in certain order determined at \( t = 0 \).
Thus, if at time \( t \) we have a configuration \( C(t) \) then at
time \( t + N \) we will obtain the configuration

\[
C(t + N) = \prod_{i=1}^{L} \prod_{j=1}^{L} a_{ij} C(t). \tag{1}
\]

The order in which the string of operators appears in this
equation is irrelevant because the operators \( a_{ij} \) commute
among them self.

Applying these string three times it results that

\[
C(t + 3N) = \prod_{i=1}^{L} \prod_{j=1}^{L} a_{ij}^3 C(t). \tag{2}
\]

This expression can be simplified using the following
property of the toppling operators \( [11] \)

\[
a_{ij}^3 = \begin{cases} a_{i-1,j-1} a_{i,j-1} a_{i+1,j-1}, & \text{for } j < L, \\ 1, & \text{for } j = L. \end{cases} \tag{3}
\]

The first equality express the fact that the addition of
three grains at a site \((i,j)\), with \( j < L \), makes this site
active transferring one grain to each of its downward nn.
The second one applies for the boundary sites which after receiving three grains become active dissipating these three grains through the boundary and, therefore, leaving the energy configuration invariant.

Starting at layer \( j = 1 \) all the operators \( a_{i,j} \) are eliminated using eq. (3). This will increase the power of operators \( a_{i,j} \) in eq. (2) by 3. The same procedure is applied to the second, third ... \( L - 1 \) layer finally resulting in

\[
\mathcal{C}(t + 3N) = \prod_{i=1}^{L} a_{i+1}^{3N} \mathcal{C}(t). \tag{4}
\]

The application of the operator \( a_{i,L} \) three consecutive times lead its energy invariant and, therefore, eq. (4) is reduced to

\[
\mathcal{C}(t + 3N) = \mathcal{C}(t). \tag{5}
\]

Hence the evolution of the energy profile is periodic with period \( 3N \).

This property is not observed in the noisy driving case where the randomness introduced by the driving field makes the dynamics Markovian [3]. Nevertheless, as it is shown in the next section, the statistical properties of the avalanches in the BTW directed model are independent of the driving mechanism.

**C. Numerical simulations and discussion**

Numerical simulations of the BTW, Manna, Zhang and RT models with directed toppling were performed. In all cases lattice sizes ranging from \( L = 64 \) to \( L = 2048 \) where used. The numerical results obtained for the BTW and Manna models are taken only as a reference because we count with the largest scale simulations reported in [4] (up to \( L = 6400 \)).

**Noisy driving**: Starting from an initial flat profile all systems were updated until they reach the stationary state. After that statistics over \( 10^8 \) avalanches was taken, recording the avalanche sizes and durations.

**Uniform driving**: the evolution in time of the energy profile is periodic and, therefore, average was taken over the period \( 3N \). Different initial conditions where simulated using different permutations of the sequence of addition of energy.

To extract the scaling exponents we use the moment analysis technique [13]. The \( q \) moment of the probability density \( p_x(x) \) of a magnitude \( x \) is defined by

\[
\langle x^q \rangle = \int dx p_x(x)x^q. \tag{6}
\]

where \( x = s, T \). As defined above \( s \) and \( T \) are the avalanche size and duration, i.e. the number of toppling events and parallel updates, respectively, required to drive the system to an stable configuration.

If the hypothesis of finite-size scaling is satisfied, that is the distribution of avalanche size and duration can be written in the form \( p_x(x) = x^{-\tau_s}f_x(x/L^{\beta_s}) \), then the \( q \) moment scales with system size according to the power law

\[
\langle x^q \rangle \sim L^{\sigma_x(q)}, \tag{7}
\]

with

\[
\sigma_x(q) = \beta_x(1 - \tau_x) + \beta_x q, \tag{8}
\]

where \( \beta_s = D \) and \( \beta_T = z \) are effective dimensions which characterize how the cutoffs of the distribution of avalanche sizes and durations, respectively, scales with system size. On the other hand \( \tau_x \) is the power low exponent which can be measured in the scaling region before the finite-size cutoffs.

The plot of \( \sigma_s(q) \) and \( \sigma_T(q) \) vs. \( q \) is shown in figs. 3 for different directed models. and 4 respectively. If two models belong to the same universality class then the linear part of the plot should overlap. Based on this argument it is then concluded that the RT model belong to the same universality class of the Manna model. A more quantitative comparison can be seen in table I where the exponents computed here for the RT model are compared with those reported in [4] for the Manna model. The scaling exponents are found in very good agreement within the numerical error.

If the hypothesis of finite size scaling is valid then one can take the scaling exponents obtained from the moment analysis and plot the different distributions on rescaled variables in such a way that curve for different system sizes overlap. This is done in figs. 7 and 8 for the RT model resulting in a very good data collapse, as it has been also observed for the directed Manna model [4].

On the other hand, one cannot distinguish between the curve for the BTW model with noisy or uniform driving, leading to the same scaling exponents. Thus, the periodicity introduced by the uniform driving carry no consequence for the critical behavior of the BTW model. Hence, the noisy driving can be substituted by a uniform driving together with an initial random energy profile. This will correspond in an interface depinning description, the number of toppling events playing the role of the interface height, to a columnar disorder. A similar conclusion was obtained by Lauritsen and Alava using a different argument [13].

The things becomes less clear when analyzing the Zhang model. In this case from the moment technique it results that \( D \approx 1.55, z \approx 1.03, \tau_x \approx 1.31 \) and \( \tau_T \approx 1.53 \). These exponents define by them self a new universality class. However, the moment analysis technique is based on the hypothesis of finite size scaling which in this case is not satisfied. This fact becomes clear in figs. 7 and 8, where the data collapse is shown, revealing that in this case the finite-size scaling is not satisfied. Deviations are observed not only for the smallest avalanches but also for the largest avalanches where the finite size scaling is expected to be better.
The anomalies observed for the Zhang model are associated with the existence of huge avalanches which practically empties the system. After one of these huge avalanches the system needs some time to reach again the critical state. This means that the mean energy of the system displays strong fluctuations and, therefore, the overall avalanche statistics is given by the small avalanches taking place during the accumulation of new grains and these huge avalanches. This picture is illustrated in fig. 9, where the fraction of avalanches of size $s$ is plotted. It is characterized by a rounded peak at the largest avalanche sizes which shifts with lattice size. On the other hand, the other part of the distribution cannot be fitted by a single power law.

The classification of the Manna and RT directed models in the same universality class is in agreement with a similar report for the corresponding undirected variants [13]. Thus, there should be some common element in these models, which is off course not present in the BTW model. A clue was given in [1] related with the possibility of multiple toppling events. In this final part of this section we discuss this statement in more detail.

In the directed BTW model the cluster of sites which topples within an avalanche is compact and these sites topple only once. On the contrary, Pastor-Satorras and Vespignani [4] observed that in the directed Manna model the cluster of sites touched by the avalanche is still compact but each site participating in the avalanche can topple more than once. If the existence of multiple toppling events is the property that puts the Manna model in a different universality class then a similar behavior should be observed in the present simulations of the directed RT model.

A decomposition of the sites participating in an avalanche based on the number of toppling events performed at those sites is shown in fig. 9, for the case of the directed random-threshold model. In this particular realization the cluster of sites touched by the avalanche is decomposed in three sub clusters where sites have toppled one, two and three times. The fraction of sites toppling three times is small but the one with two toppling is comparable with that of one. In general it was observed that in large avalanches the fraction of sites which topple one and two times are of the same order and, therefore, multiple toppling events are relevant.

In the case of undirected models it is known that multiple toppling events are present even in the BTW model, which leads to the decomposition of the avalanches in waves [14]. However, their origin is different than in directed models. In the undirected BTW model a site may topple more than once because after a first toppling (let say at step $t$) it is possible that all its neighbors become active and topple (at step $t + 1$) and, therefore, the site will again be active and topple (at step $t + 2$). In the decomposition of waves one apply the toppling rule to all sites until they are stable before toppling the initial active site a second, third, ... time, generating in this way the first, second, ... wave.

One may think in applying a similar approach for the avalanches in the Manna and RT directed models, decomposing the avalanche as a superposition of waves. A fundamental property of the waves is that within it sites can toppling only once, otherwise the concept is useless. Below its is shown that such a decomposition is not possible in the Manna and RT directed models, a least not in such a simple way.

Let us analyze in detail how a multiple toppling event can be generated in the RT directed model. Suppose the lattice has a configuration where a site has height 3 and threshold 4 and its three upward nearest neighbors are active. Then in the next step the site will receive three grains, one per active neighbor, taking an energy $3 + 3 = 6 > 4$, becoming active. After toppling the energy will decrease to $6 - 3 = 3$ and a new threshold is assigned. But the new threshold can be either 3 or 4. If it is 4 the site will be stable but if it is 3 it will be still active and topple in the next step. Since in the particular model considered here the two threshold are selected with equal probability then the multiple toppling can take place with the same probability than the single one, which explains the previous observation that in large avalanche the fraction of two-toppling events at the same site are of the order of the one-toppling one.

During the evolution of an avalanche which started at layer $j_0$ is possible that a site at a layer below $j_1 > j_0$ needs two consecutive toppling to be stable. Thinking in a decomposition in waves one can delay the second toppling until all the sites below are stable (first wave) and then topple the site the second time generating the second wave. However, during the first wave is possible that a site at a deeper layer $j_2 > j_1 > j_0$ also needs two toppling to become stable and, therefore, the first wave has to be decomposed in sub-waves where sites topple only once. The same process may occur even at deeper layer thus generating a hierarchical structure of sub-waves. Hence, the decomposition of avalanche in waves in these models lead to a more complex structure which nevertheless may be exploited to obtain some estimate of the scaling exponents. This is nevertheless out of the scope of this work.

III. SUMMARY AND CONCLUSIONS

Directed sandpile models with different toppling rule has been studied by means of numerical simulations, with the purpose of determining the different universality classes. To extract the scaling exponents the moment analysis technique was used and the resulted exponents were latter corroborated by finite size scaling of the distribution of avalanche size and duration.

The numerical analysis reveals that the introduction of a uniform driving in the BTW directed model does not change the critical properties. The evolution in time of the energy profile is in this case periodic with a pe-
period which scales linearly with the system size. In spite of this periodicity the avalanche distributions are practically identical to that obtained for the same model but with the usual noisy driving.

It is concluded that the Manna and RT models are in the same universality, where multiple toppling events appear to be a fundamental property. The existence of multiple toppling events leads to a decomposition of the avalanche in a hierarchical structure of waves which may be a starting point for future research.

Finally, it is observed that the avalanches in the directed Zhang model displays a complex structure which does not satisfy the finite-size scaling hypothesis. It is given by the superposition of huge avalanches involving a large dissipation of energy through the boundary a small avalanches taking place during the accumulation of energy.

ACKNOWLEDGEMENTS

I thanks R. Pastor Satorras and A. Vespignani for useful comments and discussion during the elaboration of this manuscript. The numerical simulations where performed using the computing facilities at the ICTP.

[1] A. Ben-hur and O. Biham, Phys. Rev. E 53, R1317 (1996); S. Lübeck and K. D. Usadel, Phys. Rev. E 55, 4095 (1997); ibid. 56, 5138 (1997); A. Chessa, H. E. Stanley, A. Vespignani and S. Zapperi, Phys. Rev. E 59, R12 (1999); C. Tebaldi, M. De Menech, and A. L. Stella, Phys. Rev. Lett. 83, 3952 (1999); S. Lübeck, Phys. Rev. E 61, 204 (2000).
[2] P. Bak, C. Tang, and K. Wiesenfeld, Phys. Rev. Lett. 59, 381 (1987); Phys. Rev. A 38, 364 (1988).
[3] D. Dhar and R. Rammaswamy, Phys. Rev. Lett. 63, 1659 (1989); for a review see cond-mat/9909009.
[4] R. Pastor-Satorras and A. Vespignani, J. Phys. A 33, L33 (2000).
[5] S. S. Manna, J. Phys. A 24, L363 (1991).
[6] B. Tadić and D. Dhar, Phys. Rev. Lett. 79, 1519 (1997).
[7] S. S. Manna, A. D. Chakrabarti, and R. Cafiero, Phys. Rev. E 60, R5005 (1999).
[8] Y.-C. Zhang, Phys. Rev. Lett. 63, 470 (1989).
[9] K. Christensen, A. Corral, V. Frette, J. Feder, and T. Tossang, Phys. Rev. Lett. 77, 107 (1996).
[10] Z. Olami, H. J. S. Feder, and K. Christensen, Phys. Rev. Lett 68, 1244 (1992); PRE
[11] O. Narayan and A. A. Middleton, Phys. Rev. B 49, 244 (1994).
[12] A. Vázquez and O. Sotolongo-Costa, cond-mat/9811417.
[13] K. B. Lauritsen and M. J. Alava, cond-mat/9903346.
[14] H. M. Jaeger and S. R. Nagel, Science 255, 1522 (1992).
[15] M. De Menech, A. L. Stella, and C. Tebaldi, Phys. Rev. E 58, R2677 (1998).
[16] V. B. Priezzhev, D. V. Kitarev, and E. V. Ivashkevich, Phys. Rev. Lett. 76, 2093 (1996); D. V. Kitarev and V. B. Priezzhev, Phys. Rev. E 58, 2883 (1998).

FIG. 1. Geometry used in the simulations. A square lattice oriented from top to bottom with each site having three upward and three downward nearest neighbors.

FIG. 2. Schematic mapping of the BTW model under uniform driving with a continuum energy profile to the same model with a sequential driving and a discrete energy profile.
FIG. 3. The exponents $\sigma_s(q)$ as a function of $q$. From top to bottom appear the curve for the Manna, RT, Zhang and BTW directed models. In all cases except the one labelled by CD (continuous driving) a noisy driving was used. In the case of the BTW directed model one can not distinguish the curves for noisy and uniform or continuous driving.

FIG. 4. The exponents $\sigma_T(q)$ as a function of $q$. From top to bottom appear the curve for the Zhang, BTW, RT and Manna directed models. In the case of the BTW directed model one can not distinguish the curves for noisy and uniform driving (CD) models.

FIG. 5. Finite size scaling of the integrated distribution of avalanche sizes (avalanches of size greater than $s$) of the RT directed model, using the scaling exponents reported in tab. I.

FIG. 6. Finite size scaling of the integrated distribution of avalanche durations (avalanches of duration larger than $T$) of the RT directed model, using the scaling exponents reported in tab. I.

FIG. 7. Finite size scaling of the distribution of avalanche sizes of the Zhang directed model, using $D = 1.55$ and $\tau_s = 1.31$.

FIG. 8. Finite size scaling of the distribution of avalanche durations of the Zhang directed model, using $z = 1.03$ and $\tau_t = 1.53$. 
FIG. 9. Fraction of avalanches of size $s$ in the Zhang model for three different lattice sizes. Notice the existence of a rounded peak for the largest avalanche sizes with shift with system size.

FIG. 10. Number of toppling events at each lattice size in an avalanche taking place in a system of size $L = 64$, using the RT toppling rule. Toppling takes place from small to large $j$. Three layers are clearly observed, corresponding to one, two and three toppling.

| Model | $D$ | $z$     | $\tau_s$ | $\tau_t$ |
|-------|-----|---------|-----------|-----------|
| BTW   | 3/2 | 1       | 4/3       | 3/2       |
| Manna | 1.75(1) | 0.99(1) | 1.43(1)   | 1.74(4)   |
| RT    | 1.73(2) | 0.99(3) | 1.44(2)   | 1.70(4)   |

TABLE I. The scaling exponents for the BTW, Manna and RT directed models. Those for the BTW model are exact while the others are numerical estimates.