Superfluid Hydrodynamic Equations of Multi-component Bosonic systems

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The superfluid hydrodynamic equations of a bosonic gas at zero temperature are often derived from the Gross-Pitaevskii equation (GPE), which is valid in the dilute region when the Bose-Einstein-Condensation dominates and the effects beyond the mean-field are negligible. In this paper, we generally show that the zero-temperature superfluid hydrodynamic equation of a multi-component bosonic system with short-ranged interactions can be obtained in the path-integral formalism under the local equilibrium assumption, which is even valid beyond the dilute region. Our approach can be extended to systems with long-range interactions such as dipole-dipole interactions by treating the Hartree energy properly. The Andreev-Bashkin effect of a multi-component superfluid can be described in this formalism. The implication of our results on quantum droplets is discussed.

I. INTRODUCTION

About ninety years ago, shortly after the discovery of superfluidity [1], Tisza [2] and Landau [3] proposed a two-fluid model to describe the hydrodynamics of a bosonic superfluid. Based on macroscopic conservation laws, the two-fluid model gives a set of superfluid hydrodynamic equations in a phenomenological way [4]. At zero temperature, according to the two-fluid model, all particles participate in superfluid flow and the superfluid density is equal to the total local density. The superfluid hydrodynamic equations of a single-component Bose system are given by [4]

\[ \partial_t n = - \nabla \cdot (n \mathbf{v}), \]
\[ m \partial_t \mathbf{v} = - \nabla \left( \frac{1}{2} m v^2 + \mu \right), \]

where \( n \) is the superfluid density, \( m \) is the boson mass, \( \mathbf{v} \) is the superfluid velocity, and \( \mu \) is the chemical potential.

For a dilute Bose gas, this set of hydrodynamic equations can be derived from the Gross-Pitaevskii equation (GPE), [5, 6] given by

\[ i \hbar \partial_t \psi_0 = \left( - \frac{\hbar^2}{2m} \nabla^2 + V + g |\psi_0|^2 \right) \psi_0, \]

where \( \psi_0(\mathbf{r}, t) \) is the condensate wave-function, \( V(\mathbf{r}) \) is the external potential, and \( g = 4 \hbar^2 \pi a_s / m \) is the s-wave coupling constant. The GPE is valid in the dilute region when the s-wave scattering length \( a_s \) is much less than the mean interparticle spacing \( n^{-1/3} \) [7]. The GPE is obtained in the mean-field approximation which assumes every boson is in the condensate. By writing the condensate wavefunction \( \psi_0 \) in terms of the condensate density \( n_0 \) and its phase \( \phi, \psi_0 = \sqrt{n_0} e^{i \phi}, \) and substituting it into Eq. (2), one can obtain the superfluid hydrodynamic equations from the GPE,

\[ \partial_t n_0 = - \nabla \cdot (n_0 \mathbf{v}), \]
\[ \partial_t \mathbf{v} = - \nabla \left( \frac{1}{2} m v^2 + \tilde{\mu} \right), \]

where

\[ \mathbf{v} = \frac{\hbar}{m} \nabla \phi, \]
\[ \tilde{\mu} = \mu_0 - \frac{\hbar^2 \nabla^2 \sqrt{n_0}}{2m \sqrt{n_0}}, \]

and \( \mu_0 = V + gn_0 \) is the local mean-field chemical potential. However, there are two differences between Eq. (4) and Eq. (2). The first one is the last term on the r.h.s. of Eq. (4), called quantum pressure, absent in Eq. (2). The second one is that in Eq. (4) the condensate density \( n_0 \) appears rather than the total density \( n \). These two differences are negligible for a dilute Bose gas close to uniform.

There are some superfluid systems that cannot be described in this simple GPE approach. For example, Helium II is a dense fluid with strong interactions [9], and only a very small fraction of atoms are in the condensate. Another example is the quantum droplet system, extensively studied in recent years, where the mean-field effect is suppressed to the extent that the beyond-mean-field effect becomes crucial even in the dilute region [10]. For quantum droplets, the ordinary GPE is no longer applicable, and the Lee-Huang-Yang (LHY) energy due to quantum fluctuations must be taken into account, leading to the extended Gross-Pitaevskii equation (EGPE) [10, 13].

There has been a lot of work in the past to microscopically derive the superfluid hydrodynamic equations beyond the GPE approach. These approaches are restricted to either the dilute system where the condensate dominates or the uniform system [14, 26]. The phenomenological derivations of the superfluid hydrodynamic equations are based on macroscopic conservation laws [27, 31], with the quantum-pressure term missing. In this paper, we obtain the general form of the superfluid hydrodynamic equations of a multi-component bosonic system in the path-integral formalism under the local equilibrium assumption. Our approach is valid for bosons with short-ranged interactions as well as with long-ranged interactions. The superfluid hydrodynamic equations fully describe the motion of atoms in and outside the condensate in contrast to the GPE-type equations which are in

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terms of the condensate wavefunctions. The Andreev-Bashkin effect of a multi-component superfluid can be described in our approach. The relevance of our results to quantum droplets is discussed.

\[ S = \int dt dr \sum_{\sigma} \Psi_\sigma^*(r, t) [i\hbar \partial_t + \frac{\hbar^2 \nabla^2}{2m_\sigma} - V_\sigma(r, t)] - \frac{1}{2} \int dr' \sum_{\sigma, r'} U_{\sigma r'}(r - r') \Psi_{\sigma r'}^*(r', t) \Psi_{\sigma r'}(r', t) \Psi_\sigma(r, t), \]

where \( \Psi_\sigma \) and \( m_\sigma \) are the boson field and mass of the \( \sigma \)-component, \( U_{\sigma r'}(r - r') \) is the short-ranged interaction, and \( V_\sigma(r, t) \) is the external potential. The boson field \( \Psi_\sigma \) can be written in terms of the density \( \rho_\sigma \) and phase \( \Phi_\sigma \), \( \Psi_\sigma = \sqrt{\rho_\sigma} e^{i\Phi_\sigma} \). In the superfluid phase, the density and phase fields can be separated into their mean values and fluctuations,

\[ \rho_\sigma = n_\sigma + \delta n_\sigma, \quad \Phi_\sigma = \phi_\sigma + \delta \phi_\sigma, \]

where \( n_\sigma \) is the average density of the \( \sigma \)-component, which is also the superfluid density, \( \phi_\sigma \) is the mean phase of the \( \sigma \)-component, and \( \delta n_\sigma \) and \( \delta \phi_\sigma \) are the density fluctuation and phase fluctuation [8, 29, 32, 33].

We consider the dynamical process with temporal and spatial scales of variances much larger than the intrinsic scales of the system so that the local equilibrium assumption (LEA) can be applied. In this case, the system can be divided into many small blocks in space with scales much larger than the intrinsic lengths and much smaller than the spatial-variance scale. The evolution time can be divided into many small intervals which are much larger than the intrinsic time and much smaller than the temporal-variance scale. Within each time interval, the subsystem within each block can be approximated as a macroscopic and homogeneous system in its local equilibrium, with constant densities and linear phases. The fluctuations in each block are decoupled from other blocks and can be integrated out, leading to the effective local action,

\[ S_j = \Delta \{- \sum_{\sigma} n_{j\sigma} [\hbar \partial_t \phi_{j\sigma} + V_{j\sigma}] + \frac{\hbar^2}{2m_\sigma} \left( \frac{|\nabla n_{j\sigma}|^2}{4n_{j\sigma}} + n_{j\sigma} |\nabla \phi_{j\sigma}|^2 \right) - \mathcal{E}(\{n_{j\sigma}\}, \{\nabla \phi_{j\sigma}\}) \}, \]

leading to the superfluid hydrodynamic equations

\[ \partial_t n_\sigma = -\nabla \cdot (n_\sigma v_\sigma + j_\sigma'), \]

\[ m_\sigma \partial_t v_\sigma = -\nabla \left( \frac{1}{2} m_\sigma v_\sigma^2 + \frac{\partial \mathcal{E}}{\partial n_\sigma} + V_\sigma - \frac{\hbar^2 \nabla^2}{2m_\sigma \sqrt{n_\sigma}} \right), \]

where \( v_\sigma = \hbar \nabla \phi_\sigma / m_\sigma \) is the \( \sigma \)-component superfluid velocity, \( j_\sigma' = \partial \mathcal{E} / (\hbar \partial \phi_\sigma) \), and \( \phi_\sigma = \partial_\sigma \phi_\sigma \). The anomalous current \( j' \) is the origin of the Andreev-Bashkin effect of a multi-component superfluid, which will be discussed in the last section of our paper. In our approach, the superfluid hydrodynamic equations are obtained without the restrictions of uniformness or diluteness and are applicable to strongly interacting bosonic systems as well as those with nonnegligible beyond-mean-field effects. The quantum-pressure term shows up on r.h.s. of Eq. (10), consistent with the quantum potential in systems with nonuniform probabilities [34].

In the single-component case, due to Galilean invariance, the energy density due to interaction, \( \mathcal{E}(n, \nabla \phi) \), is exactly the energy density of uniform ground-state \( \mathcal{E}_0(n) \). The superfluid hydrodynamic equations can be simplified...
to the familiar form,
\[ \partial_t n = -\nabla \cdot (n \mathbf{v}), \]
\[ m \partial_t \mathbf{v} = -\nabla \left( \frac{1}{2} m v^2 + \mu - \frac{\hbar^2 \nabla^2}{2 m \sqrt{n}} \right), \]
where \( \mu = V + \partial \mathcal{E} / \partial n \) is the local chemical potential. If we define the superfluid order parameter \( \psi = \sqrt{n} e^{i \phi} \), it is easy to obtain its equation of motion from the superfluid hydrodynamic equations,
\[ i \hbar \partial_t \psi (\mathbf{r}, t) = \left[ -\frac{\hbar^2 \nabla^2}{2m} + \mu \right] \psi (\mathbf{r}, t), \]
which looks similar to the GPE, but it is obtained without the assumption that every atom is in the condensate. Although the superfluid order parameter and the condensate wavefunction have the same phase, it relates to the total density \( n \), not just the condensate density \( n_0 \). In the dilute limit, to the leading order of the gas parameter, the superfluid density is equal to the condensate density, \( n = n_0 \). The rest steps are essentially the same as in the above section. Under LEA, we divide the system into many blocks. Neighboring blocks are now coupled by the Hartree energy. In each block, the Hartree energy is treated as a constant, and fluctuations can be integrated out locally. Thus we obtain the effective action given by
\[ S_{eff} = \int dt \int d\mathbf{r} \sum_{\sigma} n_{\sigma} \left[ -\hbar \partial_t \phi_{\sigma} - V_{\sigma} - \frac{\hbar^2}{2 m_{\sigma}} \left( \nabla \phi_{\sigma} \right)^2 + n_{\sigma} |\nabla \phi_{\sigma}|^2 \right] + \frac{1}{2} \sum_{\sigma} \int d\mathbf{r}' U_{\sigma \sigma'}(\mathbf{r} - \mathbf{r}') n_{\sigma}(\mathbf{r}') \right] - \mathcal{E} \right], \]
where \( \mathcal{E} \) is now the interaction-energy density of the lowest-energy state for constant densities and phase gradients with the Hartree energy excluded and can be calculated in the thermodynamic limit by adding a proper background potential to cancel out the Hartree energy explicitly as in the jellium model.

The superfluid hydrodynamic equations can be obtained from the equation of motion,
\[ \partial_t n_{\sigma} = -\nabla \cdot (n_{\sigma} \mathbf{v}_{\sigma} + j_{\sigma}'), \]
\[ m_{\sigma} \partial_t \mathbf{v}_{\sigma} = -\nabla \left( \frac{1}{2} m_{\sigma} v_{\sigma}^2 + \frac{\partial \mathcal{E}}{\partial n_{\sigma}} + V_{\sigma}' - \frac{\hbar^2 \nabla^2 \sqrt{n_{\sigma}}}{2 m_{\sigma} \sqrt{n_{\sigma}}} \right), \]
where \( V_{\sigma}'(\mathbf{r}, t) = V_{\sigma}(\mathbf{r}, t) + \sum_{\sigma'} \int d\mathbf{r}' U_{\sigma \sigma'}(\mathbf{r} - \mathbf{r}') n_{\sigma'}(\mathbf{r}', t) \). Thus the superfluid hydrodynamic equations have the same form as those in the case with the short-ranged interactions except for the treatment of the Hartree energy.
IV. DISCUSSIONS AND CONCLUSION

Quantum droplets of ultracold atoms are formed in the mean-field unstable region and stabilized by quantum fluctuations. They are perfect examples to demonstrate the beyond-mean-field effects. So far most experiments on quantum droplets are performed on the nonmagnetic binary boson mixture, such as the homonuclear mixture of $^{39}$K [37, 39], and the heteronuclear $^{39}$K-$^{87}$Rb [40], and the single-component dipolar Bose gas, such as $^{16}$Dy [41–43], and $^{166}$Er [44] atoms. In the binary boson mixture, the mean-field energy is small and attractive, and the system is stabilized by the LHY energy [10]. In the Bogoliubov theory, the LHY energy not only has a dominant real part, but also a small imaginary part due to phonon instabilities which sparks the research on the ground state of the droplet [45, 46]. In a Beliaev approach [49], it was found that higher-order fluctuations restore the phonon stability removing the imaginary part of the LHY energy, which was also confirmed in a path-integral approach [50]. For this binary droplet without phase separation [51], the density is approximately given by the condensate, $n_\sigma \approx n_\sigma_0$, and the interaction-energy density of the droplet is dominated by the mean-field and LHY energies [10],

$$\mathcal{E} = \sum_{\sigma \sigma'} \frac{g_{\sigma \sigma'}}{2} n_\sigma n_{\sigma'} + \frac{8m^{3/2}}{15\pi^2\hbar^2} (g_{11}n_1 + g_{22}n_2)^{5/2}, \quad (20)$$

where $m = m_1 = m_2$, and $a_{\sigma \sigma'} = \frac{m_{\sigma \sigma'}}{4\hbar^2}$ is the scattering length between a $\sigma$-component atom and a $\sigma'$-component atom. From Eq. (GP4), we obtain the EGPE of the two components as in Ref. [40],

$$i\hbar \partial_t \psi_1(r, t) = \left[ -\frac{\hbar^2\nabla^2}{2m} + \frac{\partial E}{\partial n_1} \right] \psi_1(r, t), \quad (21)$$

$$i\hbar \partial_t \psi_2(r, t) = \left[ -\frac{\hbar^2\nabla^2}{2m} + \frac{\partial E}{\partial n_2} \right] \psi_2(r, t),$$

which in the single-mode approximation reduce to

$$i\hbar \partial_t \psi = \left[ -\frac{\hbar^2\nabla^2}{2m} + \frac{2\sqrt{g_{11}g_{22}}}{} \right] \psi \approx \left[ -\frac{4m^{3/2}}{3\pi^2\hbar^2} \right] |\psi|^2 \psi, \quad (22)$$

consistent with the result in Ref. [10], where $\delta g = g_{12} + \sqrt{g_{11}g_{22}}$. A similar situation occurs in a single-component dipolar Bose gas, where the quantum droplet is formed in the region with the DD1 strength larger than that of the repulsive $s$-wave interaction, and stabilized by the LHY energy as demonstrated from EGPE [11, 13, 52]. In the Bogoliubov theory, the LHY energy of the dipolar droplet also has an imaginary part due to phonon instability in certain propagating directions which is removed by higher-order fluctuations [53]. In this case, the EGPE can be also constructed from the superfluid hydrodynamic equations. In both cases, the EGPE is successful in the dilute region by taking into account the Lee-Huang-Yang energy which is the dominant beyond-mean-field effect. In general, however, it is more advantageous to study the superfluid hydrodynamic equations for its broader range of validity, especially when the quantum depletion is significant.

Andreev and Bashkin predicted that in a 3He-4He mixture the supercurrent of one component will drag the other component into superflow [54]. Although it has not been observed so far, there are theoretical proposals [55, 56] that such effect is also present in a two-component Bose superfluid. Here we show that the Andreev-Bashkin effect of a multi-component Bose superfluid can be generally described by the superfluid hydrodynamic equations given by Eq. (10). We consider the case with small supervelocities in the linear-response regime so that the interaction-energy density can be expanded to the quadratic order in phase gradients,

$$\mathcal{E} \{ n_\sigma \}, \{ \nabla \phi_\sigma \} \approx \mathcal{E}_0 \{ n_\sigma \} + \frac{1}{2} \sum_{\sigma, \sigma'} \chi_{\sigma, \sigma'} \nabla \phi_\sigma \cdot \nabla \phi_{\sigma'}, \quad (23)$$

where $\mathcal{E}_0 \{ n_\sigma \}$ is the interaction-energy density of the uniform ground state, and the expansion coefficient $\chi_{\sigma, \sigma'}$ is related to the static current-current correlation function $\chi_{\sigma, \sigma'} = \chi_{\sigma', \sigma}$. Note that we assume the inversion symmetry which eliminates the transverse coupling between phase gradients in the quadratic order. The superfluid hydrodynamic equations can be further written as

$$\partial_t n_\sigma = -\nabla \cdot (m_\sigma \nabla \phi_\sigma) + \sum_{\sigma'} \frac{m_{\sigma \sigma'}}{\hbar^2} \chi_{\sigma, \sigma'} \nabla \phi_{\sigma'}, \quad (24)$$

$$m_\sigma \partial_t \phi_\sigma = -\nabla \left( \frac{1}{2} m_\sigma \phi_\sigma^2 + \mu_\sigma - \frac{\hbar^2}{2m_\sigma} \sqrt{n_\sigma} \right) + \sum_{\sigma_1, \sigma_2} \frac{m_{\sigma_1, \sigma_2}}{2\hbar^2} \nabla \phi_{\sigma_1} \cdot \nabla \theta_{\nabla \chi_{\sigma_1, \sigma_2}}, \quad (25)$$

where $\mu_\sigma = \partial \mathcal{E}_0 / \partial n_\sigma + V_\sigma$ is the local chemical potential, and the last term on r.h.s. of Eq. (25) clearly shows the drag force. The continuity equation (24) shows that generally the current $j_\sigma$ is different from $n_\sigma v_\sigma$ due to the anomalous current, $j_\sigma = \sum_{\sigma'} m_{\sigma \sigma'} \chi_{\sigma, \sigma'} v_{\sigma'} / \hbar^2$. For the case of a uniform system with the same superfluid velocity $v$ for all the components, from the Galilean invariance, we obtain

$$\mathcal{E} \{ n_\sigma \}, \{ \nabla \phi_\sigma = \frac{m_\sigma}{\hbar} v \} = \mathcal{E}_0 \{ n_\sigma \}, \quad (26)$$

which leads to the identity

$$\sum_{\sigma, \sigma'} m_{\sigma \sigma'} \chi_{\sigma, \sigma'} = 0. \quad (27)$$

In this case, the total current is still given by $j = \sum_n n_\sigma v$, in agreement with the two-component case [57].

In conclusion, we obtain the superfluid hydrodynamic equations of a multi-component bosonic system in the path-integral formalism under the local equilibrium assumption. Our method is valid for nonuniform and strongly interacting systems with short-ranged interactions as well as with long-ranged interactions. The superfluid hydrodynamic equations are superior to the GPE-type equations with the condensate wavefunctions as
variables, as the latter cannot fully describe the dynamics of quantum depletion. Our approach provides a general description of the Andreev and Bashkin effect of a multi-component Bose superfluid. The implications on quantum droplets are discussed. We would like to thank Z. Q. Yu for his helpful discussions.

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