General-type discrete self-adjoint Dirac systems: explicit solutions of direct and inverse problems, asymptotics of Verblunsky-type coefficients and stability of solving inverse problem.

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Abstract

We consider discrete self-adjoint Dirac systems determined by the potentials (sequences) \( \{C_k\} \) such that the matrices \( C_k \) are positive definite and \( j \)-unitary, where \( j \) is a diagonal \( m \times m \) matrix and has \( m_1 \) entries 1 and \( m_2 \) entries \(-1\) \((m_1 + m_2 = m)\) on the main diagonal. We construct systems with rational Weyl functions and explicitly solve inverse problem to recover systems from the contractive rational Weyl functions. Moreover, we study the stability of this procedure. The matrices \( C_k \) (in the potentials) are so called Halmos extensions of the Verblunsky-type coefficients \( \rho_k \). We show that in the case of the contractive rational Weyl functions the coefficients \( \rho_k \) tend to zero and the matrices \( C_k \) tend to the identity matrix \( I_m \).

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1 Introduction

General-type discrete self-adjoint Dirac systems have the form:

\[ y_{k+1}(z) = (I_m + izjC_k)y_k(z) \quad (k \in \mathbb{N}_0), \quad (1.1) \]

where \( \mathbb{N}_0 \) stands for the set of non-negative integers, \( I_m \) is the \( m \times m \) identity matrix, \( "i" \) is the imaginary unit \( (i^2 = -1) \) and the \( m \times m \) matrices \( \{C_k\} \) are positive and \( j \)-unitary:

\[ C_k > 0, \quad C_kjC_k = j, \quad j := \begin{bmatrix} I_{m_1} & 0 \\ 0 & -I_{m_2} \end{bmatrix} \quad (m_1 + m_2 = m; \ m_1, m_2 \neq 0). \quad (1.2) \]

First, we will consider (in Section 2) explicit solutions of the direct and inverse problems for system (1.1), (1.2) in terms of Weyl-Titchmarsh (or simply Weyl) functions. General-type direct and inverse problems for this system were studied (in terms of Weyl functions) in [5] and explicit solutions in the case \( m_1 = m_2 \) were dealt with in [4]. Our Section 2 (and Appendix) complete the results from [5] by adding the properties of the Weyl functions in the lower half-plane and generalize the explicit results from [4] for the case when \( m_1 \) does not necessarily equal \( m_2 \). We will often reduce our proofs in Section 2 and Appendix and refer to the more detailed proofs in [4, 5]. However, a complete procedure of explicitly solving the inverse problem from Section 2 is missing in [4] (and so it is new for \( m_1 = m_2 \) as well).

The case of explicit solutions of direct and inverse problems corresponds to the rational Weyl functions. The results in Section 2 are based on our generalized Bäcklund-Darboux (GBDT) approach, which was initiated by the seminal book [14] by V.A. Marchenko. For various versions of Bäcklund-Darboux transformations and related commutation methods see, for instance, [1, 2, 7, 9, 11, 15, 17, 20] and references therein.

Section 3 is dedicated to the asymptotics of the potentials (sequences) \( \{C_k\} \) corresponding to rational Weyl functions. For this purpose, we first derive the asymptotics of the so called [19] Verblunsky-type coefficients.

Finally, in Section 4 we study stability of our method of explicit solving inverse problem for system (1.1), (1.2), and these results are new even in the cases \( m_1 = m_2 \) and \( m_1 = m_2 = 1 \). We note that various important
early results on the stability of solving inverse problems were obtained by V.A. Marchenko (see, e.g., [13]). In the paper, \( \mathbb{N} \) denotes the set of natural numbers, \( \mathbb{R} \) denotes the real axis, \( \mathbb{C} \) stands for the complex plane, and \( \mathbb{C}_+ (\mathbb{C}_-) \) stands for the open upper (lower) half-plane. The spectrum of a square matrix \( A \) is denoted by \( \sigma(A) \).

2 GBDT and direct and inverse problems

1. The fundamental \( m \times m \) solution \( \{W_k\} \) of (1.1) is normalized by

\[
W_0(z) = I_m. \tag{2.1}
\]

For the case \( z \in \mathbb{C}_+ \), the definition of the Weyl function \( \varphi(z) \) of Dirac system (1.1), (1.2) was given in [5] in terms of \( W_k(z) \). Below we define the Weyl function in \( \mathbb{C}_- \), which is somewhat more convenient for our purposes. Clearly, this Weyl function has similar properties to those in [5, Theorem 3.8].

**Definition 2.1** The Weyl function of the Dirac system (1.1) (which is given on the semi-axis \( 0 \leq k < \infty \) and satisfies (1.2)) is an \( m_1 \times m_2 \) matrix function \( \varphi(z) \) in the lower half-plane, such that the following inequalities hold:

\[
\sum_{k=0}^{\infty} q(z)^k \left[ \varphi(z)^* \right] W_k(z)^* C_k W_k(z) \left[ \varphi(z) \right]^{\ast} I_{m_2} \leq \sum_{k=0}^{\infty} q(z)^k \left[ \varphi(z)^* \right] W_k(z)^* C_k W_k(z) \left[ \varphi(z) \right]^{\ast} I_{m_2} < \infty \quad (z \in \mathbb{C}_-), \tag{2.2}
\]

\[
q(z) := (1 + |z|^2)^{-1}. \tag{2.3}
\]

The properties of the Weyl function are described in the theorem below, which is proved in Appendix (using the standard Weyl disk procedure).

**Theorem 2.2** There is a unique Weyl function of the discrete Dirac system (1.1), which is given on the semi-axis \( 0 \leq k < \infty \) and satisfies (1.2). This Weyl function \( \varphi \) is analytic and contractive (i.e., \( \varphi^{\ast} \varphi \leq I_{m_2} \)) on \( \mathbb{C}_- \).

In the proof of Theorem 2.2 in Appendix, we will need the inequalities

\[
C_k \geq j, \tag{2.4}
\]

which (together with the inequalities \( C_k \geq -j \)) immediately follow from [5, Proposition 2.2].
Another way to prove Theorem 2.2 and the uniqueness of the solution of the inverse problem, which we will need further, is to consider Dirac systems

\[ \tilde{y}_{k+1}(z) = (I_m + iz \tilde{j} \tilde{C}_k)\tilde{y}_k(z) \quad (k \in \mathbb{N}_0), \tag{2.5} \]

\[ \tilde{j} := -J \tilde{j} J^* = \begin{bmatrix} I_{m_2} & 0 \\ 0 & -I_{m_1} \end{bmatrix}, \quad J := \begin{bmatrix} 0 & I_{m_2} \\ I_{m_1} & 0 \end{bmatrix}, \quad \tilde{C}_k := JC_k J^*. \tag{2.6} \]

Systems (2.5), (2.6) are dual to the systems (1.1), (1.2), and it is immediate from (1.2), (2.6) that the relations

\[ J^* J = I_m, \quad \tilde{C}_k > 0, \quad \tilde{C}_k \tilde{j} \tilde{C}_k = \tilde{j} \tag{2.7} \]

are valid. Hence, systems (2.5) are again self-adjoint Dirac systems. Similar to \( \tilde{j} \) and \( \tilde{C}_k \) we use “tilde” in other notations (introduced for self-adjoint Dirac systems), when it goes about systems (2.5). For instance, clearly we have \( \tilde{m}_1 = m_2, \tilde{m}_2 = m_1 \). It is easy to see that the fundamental solution \( \{\tilde{W}_k(z)\} \) of the system (2.5) is connected with the fundamental solution \( \{W_k(z)\} \) of (1.1) by the equality

\[ \tilde{W}_k(z) = W_k(-z). \tag{2.8} \]

Thus, according to (2.2) and (2.8) the function

\[ \tilde{\varphi}(z) = \varphi(-z), \tag{2.9} \]

where \( \varphi \) is the Weyl function of the system (1.1), satisfies the inequalities

\[ \sum_{k=0}^{\infty} q(z)^k [I_{m_2} \tilde{\varphi}(z) J^* \tilde{W}_k(z)^* J \tilde{C}_k \tilde{W}_k(z)] < \infty \quad (z \in \mathbb{C}_+). \tag{2.10} \]

Therefore, by virtue of [5, Definition 3.6], the matrix function \( \tilde{\varphi}(z) \) is the Weyl function (on \( \mathbb{C}_+ \)) of the dual system (2.5). Moreover, we see that there is a one to one correspondence (2.6), (2.9) between systems (1.1) and (2.5) and their Weyl functions (on \( \mathbb{C}_- \) and \( \mathbb{C}_+ \), respectively). Hence, [5, Corollary 4.7] yields the theorem below.

**Theorem 2.3** *Dirac system (1.1), (1.2) is uniquely recovered from its Weyl function \( \varphi(z) \) (\( z \in \mathbb{C}_- \)) introduced by (2.2).*
In order to consider the case of rational Weyl functions, we introduce generalized Bäcklund-Darboux transformation (GBDT) of discrete Dirac systems. Each GBDT of the initial discrete Dirac system is determined by a triple \( \{A, S_0, \Pi_0\} \) of parameter matrices. Here, we take a trivial initial system and choose \( n \in \mathbb{N} \) \((n > 0)\), two \( n \times n \) parameter matrices \( A \) \((\det A \neq 0)\) and \( S_0 > 0 \), and an \( n \times m \) parameter matrix \( \Pi_0 \) such that

\[
AS_0 - S_0A^* = i\Pi_0 j \Pi_0^*.
\]

(2.11)

Define recursively the sequences \( \{\Pi_k\} \) and \( \{S_k\} \) \((k > 0)\) by the relations

\[
\Pi_{k+1} = \Pi_k + iA^{-1}\Pi_k j, \tag{2.12}
\]

\[
S_{k+1} = S_k + A^{-1}S_k(A^*)^{-1} + A^{-1}\Pi_k \Pi_k^*(A^*)^{-1}. \tag{2.13}
\]

From (2.11)–(2.13), the validity of the matrix identity

\[
AS_r - S_rA^* = i\Pi_r j \Pi_r^* \quad (r \geq 0)
\]

(2.14)

follows by induction.

**Definition 2.4** The triple \( \{A, S_0, \Pi_0\} \), where \( \det A \neq 0 \), \( S_0 > 0 \) and (2.11) holds, is called admissible.

In view of (2.13), for the admissible triple we have \( S_k > 0 \) \((k \geq 0)\). Thus, the sequence

\[
C_k := I_m + \Pi_k^* S_k^{-1}\Pi_k - \Pi_{k+1}^* S_{k+1}^{-1}\Pi_{k+1}
\]

(2.15)

is well-defined. We say that the sequence \( \{C_k\} \) is determined by the admissible triple \( \{A, S_0, \Pi_0\} \). We will need also the matrix function \( w_A \), which for each \( k \geq 0 \) is a so called transfer matrix function in Lev Sakhnovich form [20, 22] and is defined by the relation

\[
w_A(k, \lambda) := I_m - ij\Pi_k^* S_k^{-1}(A - \lambda I_n)^{-1}\Pi_k.
\]

(2.16)

Now, similar to [4,9], we obtain the theorem below.

**Theorem 2.5** Let the triple \( \{A, S_0, \Pi_0\} \) be admissible and assume that the recursions (2.12) and (2.13) are valid. Then, the matrices \( C_k \) given by (2.15)
(i.e., determined by \( \{ A, S_0, \Pi_0 \} \)) are well-defined and satisfy (1.2). Moreover, in this case the fundamental solution \( \{ W_k \} \) of the Dirac system (1.1) admits the representation

\[
W_k(z) = w_A(k, -1/z)(I_m + izj)^k w_A(0, -1/z)^{-1} \quad (k \geq 0),
\]

where \( w_A \) is defined in (2.16).

**Proof.** Recall that since \( S_0 > 0 \), relation (2.13) yields by induction that \( S_k > 0 \), and so the sequence \( \{ C_k \} \) is well-defined.

Next, formula (2.17) easily follows from the equality

\[
w_A(k + 1, \lambda)(I_m - i\lambda j) = (I_m - i\lambda j C_k)w_A(k, \lambda) \quad (k \geq 0),
\]

which is proved quite similar to the proof of [4, (2.24)] (and so we omit this proof here).

It remains to prove (1.2). The second equality in (1.2), that is, \( C_k j C_k = j \) follows from (2.18) and the equalities

\[
w_A(k, \lambda) j w_A(k, \lambda)^* = j,
\]

which may be found in [21] (see also [20, (1.84)]). Indeed, we easily check that

\[
(I_m - i\lambda j) j (I_m + i\lambda j) = \left(1 + \frac{1}{\lambda^2}\right) j,
\]

and formulas (2.18)–(2.20) imply that

\[
(I_m - i\lambda j C_k) j (I_m + i\lambda C_k j) = \left(1 + \frac{1}{\lambda^2}\right) j.
\]

Clearly, the second equality in (1.2) is immediate from (2.21).

Finally, the first equality in (1.2) is proved in the same way as [4, Proposition 3.1].

**3.** It is convenient to partition \( \Pi_0 \) into the \( n \times m_i \) blocks \( \vartheta_i \) and to partition \( w_A(0, \lambda) \) in the four blocks of the same orders as for \( j \) in (1.2):

\[
\Pi_0 = [\vartheta_1 \quad \vartheta_2], \quad w_A(0, \lambda) = \begin{bmatrix} a(\lambda) & b(\lambda) \\ c(\lambda) & d(\lambda) \end{bmatrix}.
\]
Theorem 2.6 Let a sequence \( \{C_k\} \) and so Dirac system (1.1), (1.2) be determined by some admissible triple \( \{A, S_0, \Pi_0\} \). Then, the unique Weyl function of this system is given by the formula

\[
\varphi(z) = -iz\vartheta_1^* S_0^{-1}(J_n + zA^\times)^{-1}\vartheta_2, \quad A^\times = A + i\vartheta_2\vartheta_1^* S_0^{-1}. \tag{2.23}
\]

Proof. Recall the definition (2.2) of the Weyl function \( \varphi(z) \), where

\[
q(z) = (1 + |z|^2)^{-1}.
\]

First, let us show that the summation formula

\[
\sum_{k=0}^{r} q(z)^k W_k(z)^* C_k W_k(z) = i(1 + |z|^2) \left( (q(z)^r + W_{r+1}(z)^* j W_{r+1}(z) - j) \right) \tag{2.24}
\]

is valid. Indeed, according to (1.1) and (1.2) we have

\[
W_{k+1}(z)^* j W_{k+1}(z) = W_k(z)^* (I_m - i\varpi C_k) j (I_m + iz C_k) W_k(z) = q(z)^{-1} W_k(z)^* j W_k(z) + i(z - \varpi) W_k(z)^* C_k W_k(z),
\]

that is,

\[
q(z)^k W_k(z)^* C_k W_k(z) = \frac{iq(z)^{k-1}}{(\varpi - z)} \times (q(z) W_{k+1}(z)^* j W_{k+1}(z) - W_k(z)^* j W_k(z)), \tag{2.25}
\]

and (2.24) is immediate from (2.25).

Next, we will need the inequality

\[
w_A \left( k; -\frac{1}{z} \right)^* j w_A \left( k; -\frac{1}{z} \right) \leq j \quad (z \in \mathbb{C}_-), \tag{2.26}
\]

which together with (2.19) follows from a more general formula (see, e.g., [20, (1.88)]) of the form

\[
w_A (k, \lambda)^* j w_A (k, \lambda) = j - i(\lambda - \overline{\lambda})\Pi_k^\times (A^* - \overline{\lambda} I_n)^{-1} S_k^{-1} (A - \lambda I_n)^{-1} \Pi_k. \tag{2.27}
\]

Formulas (2.17) and (2.26) yield (in \( \mathbb{C}_- \)) the inequality

\[
W_{r+1}(z)^* j W_{r+1}(z) \leq (w_A(0, -1/z)^{-1})^* (I_m - i\varpi j)^{r+1} j \times (I_m + iz j)^{r+1} w_A(0, -1/z)^{-1}. \tag{2.28}
\]
Setting

$$\phi(z) = b(-1/z)d(-1/z)^{-1}$$  \hspace{1cm} (2.29)

and taking into account (2.22) and (2.29), we derive

$$(I_m + izj)^{r+1}w_A(0, -1/z)^{-1} \begin{bmatrix} \phi(z) \\ I_{m_2} \end{bmatrix} = (I_m + izj)^{r+1} \begin{bmatrix} 0 \\ I_{m_2} \end{bmatrix} d(-1/z)^{-1}$$

$$= (1 - iz)^{r+1} \begin{bmatrix} 0 \\ I_{m_2} \end{bmatrix} d(-1/z)^{-1}. \hspace{1cm} (2.30)$$

It is immediate from (2.28) and (2.30) that

$$\left[ \phi(z)^* I_{m_2} \right] W_{r+1}(z)^*jW_{r+1}(z) \left[ \phi(z) \right] \leq 0 \ (z \in \mathbb{C}_-). \hspace{1cm} (2.31)$$

For $\phi(z)$ given by (2.29), relations (2.24) and (2.31) imply that (2.2) holds, and so this $\phi(z)$ is the Weyl function. (We did not discuss the singularities of $d(-1/z)$ and $d(-1/z)^{-1}$ but $\phi(z)$ is analytic in $\mathbb{C}_-$ because it is meromorphic and it is the Weyl function.)

It remains to show that the right-hand sides of (2.23) and (2.29) coincide. By virtue of (2.16) and (2.22), using inversion formula from system theory (see, e.g., [20, Appendix B] and references therein), we obtain

$$b(\lambda)d(\lambda)^{-1} = -i\vartheta_1^* S_0^{-1}(A - \lambda I_n)^{-1}\vartheta_2 (I_m + i\vartheta_2^* S_0^{-1}(A - \lambda I_n)^{-1}\vartheta_2)^{-1}$$

$$= -i\vartheta_1^* S_0^{-1}(A - \lambda I_n)^{-1}\vartheta_2 (I_m - i\vartheta_2^* S_0^{-1}(A^\times - \lambda I_n)^{-1}\vartheta_2),$$

where $A^\times = A + i\vartheta_2^* \vartheta_2 S_0^{-1}$. Since $i\vartheta_2^* \vartheta_2 S_0^{-1} = A^\times - A = (A^\times - \lambda I_n) - (A - \lambda I_n)$, we essentially simplify the right-hand side in the formula above:

$$b(\lambda)d(\lambda)^{-1} = -i\vartheta_1^* S_0^{-1}(A^\times - \lambda I_n)^{-1}\vartheta_2. \hspace{1cm} (2.32)$$

Hence, the right-hand sides of (2.23) and (2.29), indeed, coincide.  ■

4. We note that the Weyl function $\phi(z)$ in (2.23) is rational and contractive on $\mathbb{C}_-$. Moreover, $\phi(-1/z)$ is strictly proper rational and contractive. It is
well-known (see, e.g., [10, 12]) that each strictly proper rational $m_1 \times m_2$ matrix function $\psi(z)$ admits a representation (so called realization)

$$\psi(z) = C(zI_n - A)^{-1}B,$$  \hspace{1cm} (2.33)

where $A$ is an $n \times n$ matrix, $C$ is an $m_1 \times n$ matrix and $B$ is an $n \times m_2$ matrix. Further in the text we assume that the realization (2.33) is a minimal realization, that is, the value of $n$ in (2.33) is minimal (among the corresponding values in different realizations of $\psi$). The following proposition is immediate from [18, Lemma 3.1] (and is based on several theorems from [12], see the details in [18]).

**Proposition 2.7** Assume that a strictly proper rational $m_1 \times m_2$ matrix function $\psi(z)$ is contractive on $\mathbb{C}$ and that (2.33) is its minimal realization. Then, there is a unique Hermitian solution $X$ of the Riccati equation

$$XBB^*X - i(A^*X - XA) + C^*C = 0,$$  \hspace{1cm} (2.34)

such that the relation

$$\sigma(A - iBB^*X) \subset (\mathbb{C}_+ \cup \mathbb{R})$$  \hspace{1cm} (2.35)

holds. Moreover, this solution $X$ is positive.

Next, we give an explicit procedure of solving the inverse problem to recover Dirac system from its Weyl function.

**Theorem 2.8** Let $\varphi(z)$ be a rational $m_1 \times m_2$ matrix function such that $\psi(z) = \varphi(-1/z)$ is a strictly proper rational matrix function, which is contractive on $\mathbb{R}$ and has no poles on $\mathbb{C}_-$. Assume that (2.33) is a minimal realization of $\psi$ and that $X > 0$ is a solution of (2.34).

Then, $\varphi(z)$ is the Weyl function of the Dirac system (1.1), (1.2), the potential $\{C_k\}$ of which is determined by the admissible triple

$$A = A - iBB^*X, \quad S_0 = X^{-1}, \quad \vartheta_1 = iX^{-1}C^*, \quad \vartheta_2 = B.$$  \hspace{1cm} (2.36)

**Proof.** Since $\psi(z)$ is contractive on $\mathbb{R}$ and has no poles on $\mathbb{C}_-$, it is contractive on $\mathbb{C}_-$. Thus, according to Proposition 2.7 a positive definite solution $X$
of (2.34) exists. In view of (2.36), choosing \(X > 0\) we have \(S_0 > 0\). Moreover, relations (2.34) and (2.35) yield the equality

\[ \vartheta_2 \vartheta^*_2 + i \left( (A + i \vartheta_2 \vartheta^*_2 S_0^{-1}) S_0 - S_0 (A + i \vartheta_2 \vartheta^*_2 S_0^{-1})^* \right) + \vartheta_1 \vartheta^*_1 = 0, \tag{2.37} \]

which is equivalent to (2.11). Hence, the triple \(\{A, S_0, \Pi_0\}\) is admissible.

It remains to show that for the Weyl function \(\varphi(z)\) of the Dirac system (determined by this triple), the function \(\psi(z) = \varphi(-1/z)\) coincides with \(\psi(z)\) admitting the realization (2.33). Taking into account Theorem 2.6 and equalities (2.36), we see that \(\psi(z)\) determined by our triple has the form

\[ \psi(z) = i \vartheta^*_1 S_0^{-1} (zI_n - A)^{-1} \vartheta_2 = C (zI_n - A)^{-1} B, \tag{2.38} \]

and the right-hand sides of (2.33) and (2.38), indeed, coincide. ■

3 Verblunsky-type coefficients and asymptotics of the potentials

Recall that the matrices \(C_k\) from the potential (sequence) \(\{C_k\}\) are positive definite and \(j\)-unitary (i.e., they satisfy (1.2)). According to [5, Proposition 2.4] it means that they admit representations

\[ C_k = D_k H_k, \quad D_k := \text{diag} \left\{ \left( I_{m_1} - \rho_k \rho_k^* \right)^{-\frac{1}{2}}, \left( I_{m_2} - \rho_k \rho_k^* \right)^{-\frac{1}{2}} \right\}, \tag{3.1} \]

\[ H_k := \begin{bmatrix} I_{m_1} & \rho_k \\ \rho_k^* & I_{m_2} \end{bmatrix} \quad (\rho_k^* \rho_k < I_{m_2}). \tag{3.2} \]

Here, the \(m_1 \times m_2\) matrices \(\rho_k\) are so called Verblunsky-type coefficients, which were studied in detail in [19]. It is well-known (see, e.g., [3]) that \(D_k H_k = H_k D_k\). Clearly, \(\rho_k^* \rho_k < I_{m_2}\) yields \(\rho_k^* \rho_k < I_{m_1}\) and vice versa.

In this section, we show that

\[ \lim_{k \to \infty} \begin{bmatrix} I_{m_1} & 0 \end{bmatrix} C_k \begin{bmatrix} I_{m_1} \\ 0 \end{bmatrix} = I_{m_1}, \tag{3.3} \]

and so \(\rho_k \to 0\) and \(C_k \to I_m\). More precisely, we prove the following statement.
**Theorem 3.1** Let the triple \( \{A, S_0, \Pi_0\} \) be admissible and assume that \(-i \not\in \sigma(A)\). Then, for the potential \( \{C_k\} \) (of the Dirac system (1.1)) determined by this triple the asymptotic relations

\[
\lim_{k \to \infty} \rho_k = 0, \quad \lim_{k \to \infty} C_k = I_m
\] (3.4)

are valid.

**Proof.** Consider the equality

\[
S_{k+1} - (I_n + iA^{-1}) S_k (I_n - i(A^*)^{-1}) = S_{k+1} - S_k - A^{-1} S_k (A^*)^{-1} + iA^{-1} (AS_k - S_k A^*)(A^*)^{-1}. \quad (3.5)
\]

Using (2.13) and (2.14), we rewrite (3.5):

\[
S_{k+1} - (I_n + iA^{-1}) S_k (I_n - i(A^*)^{-1}) = A^{-1} \Pi_k(I_m - j) \Pi_k^*(A^*)^{-1}. \quad (3.6)
\]

Now, we partition \( \Pi_k \) and, taking into account (2.12) and (2.22), write it down in the form

\[
\Pi_k = [(I_n + iA^{-1})^k \vartheta_1 (I_n - iA^{-1})^k \vartheta_2]. \quad (3.7)
\]

In view of (3.6) and (3.7), setting

\[
R_r := (I_n + iA^{-1})^{-r} S_k(I_n - i(A^*)^{-1})^{-r} \quad (3.8)
\]

we have

\[
R_{k+1} - R_k = 2(I_n + iA^{-1})^{-k-1} A^{-1} (I_n - iA^{-1})^k \vartheta_2 \vartheta_2^* ((I_n - iA^{-1})^k)^*(A^{-1})^* \times ((I_n + iA^{-1})^{-k-1})^* \geq 0. \quad (3.9)
\]

Since \( R_0 = S_0 > 0 \), relations (3.9) imply that there is a limit

\[
\lim_{k \to \infty} R_k^{-1} \geq \kappa_R \geq 0. \quad (3.10)
\]

On the other hand, from (3.7) and (3.8) we derive

\[
[I_{m_1} \ 0] \Pi_k^* S_k^{-1} \Pi_k \begin{bmatrix} I_{m_1} \\ 0 \end{bmatrix} = \vartheta_1^* R_k^{-1} \vartheta_1, \quad (3.11)
\]
and so (3.10) yields

$$\lim_{k \to \infty} \begin{bmatrix} I_{m_1} & 0 \\ 0 & \Pi_k^* S_k^{-1} \Pi_k \end{bmatrix} = \vartheta_k^* \mathcal{X}_R \vartheta_1,$$

(3.12)

The definition (2.15) of $C_k$ and the existence of the limit in (3.12) show that (3.3) holds. It is easy to see that the first equality in (3.4) follows from (3.1)–(3.3). Finally, the second equality in (3.4) is immediate from (3.1), (3.2) and the first equality in (3.4). ■

Remark 3.2 According to Theorems 2.6, 2.8 and 2.3 and to Proposition 2.7, given a potential \{C_k\} determined by some admissible triple we may recover another admissible triple \{A, S_0, \Pi_0\}, which determines the same sequence \{C_k\} and has additional property \(\sigma(A) \subset (\mathbb{C}_+ \cup \mathbb{R})\). Namely, we construct first the Weyl function using the initial triple and the procedure from Theorem 2.6. Next, we recover another admissible triple \{A, S_0, \Pi_0\} such that \(\sigma(A) \subset (\mathbb{C}_+ \cup \mathbb{R})\) in the process of solving inverse problem.

Thus, we may assume \(\sigma(A) \subset (\mathbb{C}_+ \cup \mathbb{R})\) without loss of generality, and so the condition \(-i \not\in \sigma(A)\) in Theorem 3.1 may be deleted.

We note that in the case of \{C_k\} determined by some admissible triple, Verblunsky-type coefficients may be expressed explicitly. Indeed, in view of (3.1) and (3.2) we have

$$\rho_k = \left( \begin{bmatrix} I_{m_1} & 0 \\ 0 & 0 \end{bmatrix} C_k \begin{bmatrix} I_{m_1} \\ 0 \end{bmatrix} \right)^{-1} \begin{bmatrix} I_{m_1} & 0 \end{bmatrix} C_k \begin{bmatrix} 0 \\ I_{m_2} \end{bmatrix}. \quad (3.13)$$

Hence, taking into account (2.15) and (3.11) we derive

$$\rho_k = \left( I_{m_1} + \vartheta_k^* R_k^{-1} \vartheta_1 - \vartheta_1^* R_k^{-1} \vartheta_1 \right)^{-1}$$

$$\times \begin{bmatrix} I_{m_1} & 0 \\ 0 & (\Pi_k^* S_k^{-1} \Pi_k - \Pi_k^* S_{k+1}^{-1} \Pi_{k+1}) \end{bmatrix} \begin{bmatrix} 0 \\ I_{m_2} \end{bmatrix}. \quad (3.14)$$

4 Stability of solving inverse problem

It is easy to see that the procedure (given in Theorem 2.8) to recover system (1.1), (1.2) consists from two steps. The first step is the construction of
X > 0 and the second step is the construction of the potential \( \{ C_k \} \) using this \( X \).

We start with the matrix function \( \varphi(z) \) such that \( \psi(z) = \varphi(-1/z) \) is a strictly proper rational \( m_1 \times m_2 \) matrix function, which is contractive on \( \mathbb{C}_- \). More precisely, we start with a minimal realization (2.33) of \( \psi \) (or, equivalently, with the triple \( \{ A, B, C \} \)) and consider the stability in recovery of \( X > 0 \) satisfying additional condition (2.35). The existence and uniqueness of \( X > 0 \) satisfying (2.35) follows from Proposition 2.7.

**Definition 4.1** By \( G_n \) we denote the class of triples \( \{ \tilde{A}, \tilde{B}, \tilde{C} \} \) which determine minimal realizations \( \tilde{\psi}(z) = \tilde{C}(zI_n - \tilde{A})^{-1}\tilde{B} \) of \( m_1 \times m_2 \) matrix functions \( \tilde{\psi}(z) \) contractive on \( \mathbb{C}_- \).

The recovery of \( X > 0 \) satisfying (2.34), (2.35) from the minimal realization (2.33) of \( \psi(z) \) (where \( \{ A, B, C \} \in G_n \) is called stable if for any \( \varepsilon > 0 \) there is \( \delta > 0 \) such that for each \( \{ \tilde{A}, \tilde{B}, \tilde{C} \}, \) satisfying conditions

\[
\{ \tilde{A}, \tilde{B}, \tilde{C} \} \in G_n, \quad \|A - \tilde{A}\| + \|B - \tilde{B}\| + \|C - \tilde{C}\| < \delta,
\]

there is a solution \( \tilde{X} = \tilde{X}^* \) of the equation

\[
\tilde{X} \tilde{B} \tilde{B}^* \tilde{X} - i(\tilde{A}^* \tilde{X} - \tilde{X} \tilde{A}) + \tilde{C}^* \tilde{C} = 0
\]

in the neighbourhood \( \|X - \tilde{X}\| < \varepsilon \) of \( X \).

The stability of the recovery of \( X \) follows (similar to the case of continuous Dirac system) from [18, Theorem 3.3] based on [16, Theorem 4.4]. Namely, applying [18, Theorem 3.3] to the triples \( \{-A, B, -C\} \) and \( \{-\tilde{A}, \tilde{B}, -\tilde{C}\} \) we obtain our next statement.

**Proposition 4.2** The recovery of \( X > 0 \), satisfying (2.34), (2.35) from the minimal realization (2.33) (with \( \{ A, B, C \} \in G_n \)) is stable.

**Remark 4.3** Note that (according to [16, Theorem 4.4]) we may consider a wider than \( G_n \) class of perturbed triples \( \{ \tilde{A}, \tilde{B}, \tilde{C} \} \), that is, such perturbed triples that (4.2) has a Hermitian solution \( \tilde{X} = \tilde{X}^* \).
Recall that given the triple \( \{A, B, C\} \) and \( X > 0 \) we construct the matrices \( A, S_k, R_k, \ldots \). For the matrices constructed in a similar way in the case of the triple \( \{\breve{A}, \breve{B}, \breve{C}\} \) and of \( \breve{X} > 0 \) satisfying
\[
\breve{X}\breve{B}\breve{B}^*\breve{X} - i(\breve{A}^*\breve{X} - \breve{X}\breve{A}) + \breve{C}^*\breve{C} = 0,
\]
we use the notations with “tilde”: \( \breve{A}, \breve{S}_k, \breve{R}_k, \ldots \).

The stability of the second step of solving inverse problem one can prove under additional condition \( \kappa_R = 0 \) or, equivalently,
\[
\lim_{k \to \infty} R_k = +\infty,
\]
which means that all the eigenvalues of \( R_k \) tend to infinity. Unlike the skew-self-adjoint case \([6]\), the equality (4.4) is not fulfilled automatically.

Sufficient condition of stability may be expressed also in terms of matrices \( Q_r \), which are introduced by the relations
\[
Q_r := (I_n - iA^{-1})^{-r}S_r(I_n + i(A^*)^{-1})^{-r}.
\]
Clearly, we assume in (4.5) that \( i \not\in \sigma(A) \). Similar to the equality (3.6), from (2.13) and (2.14) we have
\[
S_{k+1} - (I_n - iA^{-1})S_k(I_n + i(A^*)^{-1}) = A^{-1}\Pi_k(I_m + j)\Pi_k^*(A^*)^{-1}.
\]
Hence, taking into account (3.7) (in analogy with the relation (3.9) for \( R_r \)) we derive
\[
Q_{k+1} - Q_k = 2(I_n - iA^{-1})^{-k-1}A^{-1}(I_n + iA^{-1})^k\vartheta_1^*(I_n + iA^{-1})^k(A^{-1})^* \\
\times ((I_n - iA^{-1})^{-k-1})^* \geq 0.
\]
Since \( Q_0 = S_0 > 0 \), relations (4.7) imply that there is a limit
\[
\lim_{k \to \infty} Q_k^{-1} = \kappa_Q \geq 0.
\]
Moreover, (3.7) and (4.5) yield
\[
\lim_{k \to \infty} \begin{bmatrix} 0 & I_{m_1} \end{bmatrix} \Pi_k^*S_k^{-1}\Pi_k \begin{bmatrix} 0 & I_{m_1} \end{bmatrix} = \vartheta_2^*\kappa_Q\vartheta_2.
\]
Formula (4.9) implies that
\[
\lim_{k \to \infty} \begin{bmatrix} 0 & I_{m_2} \end{bmatrix} C_k \begin{bmatrix} 0 & I_{m_2} \end{bmatrix} = I_{m_2},
\]
which gives another way to prove Theorem 3.1. The cases when (4.4) or the equality
\[
\lim_{k \to \infty} Q_k = +\infty
\]
hold are considered in the stability theorem below. (Recall that the sequence \(\{R_k\}\) is given by (3.8) or, equivalently, by (3.9) together with (2.36) and \(R_0 = S_0\).) In Proposition 4.5 at the end of this section we present a wide class, where (4.11) is valid.

**Theorem 4.4** Consider the procedure (from Theorem 2.8) of the unique recovery of the potential \(\{C_k\}\) of the discrete self-adjoint Dirac system (1.1), (1.2) from a minimal realization (2.33), where \(\psi(z) = \varphi(-1/z)\) and \(\varphi(z)\) is the Weyl function of the system (1.1), (1.2). Assume that \(X\) in this procedure is chosen so that (2.35) holds (which is always possible). Assume also that either the sequence \(\{R_k\}\) satisfies (4.4) or \(i \notin \sigma(A)\) and the sequence \(\{Q_k\}\) satisfies (4.11).

Then, this procedure of the recovery of the potential \(\{C_k\}\) is stable in the class of the triples from \(G_n\).

**Proof.** The recovery of \(X > 0\) satisfying (2.34), (2.35) is possible according to Proposition 2.7 and is stable according to Proposition 4.2.

Now, in order to show that the recovery of \(\{C_k\}\) is stable under condition (4.4), we choose some small \(\tilde{\varepsilon} > 0\) and such a large \(N > 0\) and a small neighbourhood of \(\{A, B, C\}\) that \(\|R^{-1}_k\| < \tilde{\varepsilon}\) and \(\|\tilde{R}^{-1}_k\| < 2\tilde{\varepsilon}\) for \(X > 0\) satisfying (2.34), (2.35), for \(k > N\), and for the matrices \(\tilde{X} > 0\) satisfying (4.3) (where the triples \(\tilde{A}, \tilde{B}, \tilde{C}\) belong to the mentioned above neighbourhood of \(\{A, B, C\}\) and \(\tilde{X}\) are those solutions of (4.3) which belong to the neighbourhood of \(X\)). Here, we use the fact that the sequence \(\{\tilde{R}_k\}\) is monotonically increasing and if \(\tilde{R}_r\) is sufficiently large, then \(\tilde{R}_r\) \((r > r_0)\) is sufficiently large as well.
In view of (2.15) and (3.11), we see that for sufficiently small $\hat{\varepsilon}$ the matrices
\[
\begin{bmatrix}
I_{m_1} & 0
\end{bmatrix}
C_k
\begin{bmatrix}
I_{m_1} & 0
\end{bmatrix},
\begin{bmatrix}
I_{m_1} & 0
\end{bmatrix}
\tilde{C}_k
\begin{bmatrix}
I_{m_1} & 0
\end{bmatrix},
\]
(4.12)
are sufficiently close to $I_{m_1}$. This, in turn, means that (in view of (3.1) and (3.2)) the matrices $\rho_k$, $\tilde{\rho}_k$ are sufficiently small, and so $C_k$ and $\tilde{C}_k$ are sufficiently close to $I_m$. Therefore, for any $\varepsilon > 0$ we may choose $\hat{\varepsilon}$ such that
\[
\|C_k - \tilde{C}_k\| < \varepsilon \quad \text{for all} \quad k > N(\hat{\varepsilon}).
\]
Moreover, for any $\varepsilon > 0$ we may choose a neighbourhood of $X$ and of \{A, B, C\} such that for \{\tilde{A}, \tilde{B}, \tilde{C}\} from this neighbourhood the inequalities
\[
\|C_k - \tilde{C}_k\| < \varepsilon \quad (0 \leq k \leq N(\hat{\varepsilon}))
\]
are valid as well. Thus, the recovery of \{C_k\} is stable, indeed.

The stability of the recovery of \{C_k\} under condition (4.11) is proved in a similar way. ■

Now, consider the case when $A$ is similar to a diagonal matrix $D$ ($A$ is diagonalisable):
\[
A = UDU^{-1}.
\]
(4.13)
Relations (2.35), (2.36) and (4.13) yield $\sigma(D) \in (\mathbb{C}_+ \cup \mathbb{R})$ or, equivalently:
\[
i(D^* - D) \geq 0.
\]
(4.14)

Proposition 4.5 Let the sequence \{Q_k\} be given by (4.5), where $A$ and \{S_k\} are constructed using the procedure from Theorem 4.4, $A$ is diagonalisable (i.e., the representation (4.13) holds) and $i \not\in \sigma(A)$. Then, (4.11) is valid.

Proof. According to (4.7) we have
\[
Q_{k+n} - Q_k = 2(A - iI_n)^{-n-k}(A + iI_n)^k F A^* - iI_n)^k (A^* + iI_n)^{-n-k},
\]
(4.15)
\[
F := \sum_{\ell=1}^n (A - iI_n)^{n-\ell} (A + iI_n)^{\ell-1} \partial_1 \partial_1^* (A^* - iI_n)^{\ell-1} (A^* + iI_n)^{n-\ell},
\]
(4.16)
where $F$ does not depend on $k$. Let us show that $F$ is strictly positive, that is, $F > 0$. Indeed, it is easy to see (more details are given in the similar part of the proof of [6, Proposition 4.10]) that

$$\text{Span} \bigcup_{\ell=1}^{n} (A - iI_n)^n - \ell (A + iI_n)^{\ell-1} \vartheta_1 = \text{Span} \bigcup_{\ell=1}^{n} A^{\ell-1} \vartheta_1,$$

and so we need only to prove that the pair $\{A, \vartheta_1\}$ is controllable.

Since the realization (2.33) is minimal, the pair $\{A^*, C^*\}$ is controllable. In view of (2.36), the controllability of the pair $\{X^{-1}A^*X, \vartheta_1\}$ follows from the controllability of $\{A^*, C^*\}$. Hence, the equality

$$X^{-1}A^*X = A - i\vartheta_1\vartheta_1^*X \quad (4.17)$$

(which we derive below) implies that the pair $\{A, \vartheta_1\}$ is controllable as well.

Finally, using (2.36) we rewrite (2.11) in the form

$$AX^{-1} - X^{-1}A^* = i(\vartheta_1\vartheta_1^* - \vartheta_2\vartheta_2^*).$$

This yields in turn that $X^{-1}A^*X = A + i\mathcal{B}\mathcal{B}^*X - i\vartheta_1\vartheta_1^*X$. Applying now the first equality in (2.36), we obtain (4.17), and so $\{A, \vartheta_1\}$ is controllable and the inequality $F > 0$ is proved.

Next, we show that

$$(D - iI_n)^{-1}(D + iI_n)((D - iI_n)^{-1}(D + iI_n))^* \geq I_n. \quad (4.18)$$

The inequality (4.18) is equivalent to the inequality

$$(D + iI_n)(D^* - iI_n) \geq (D - iI_n)(D^* + iI_n),$$

which follows from (4.14).

Now, formula (4.13), representation (4.13) and inequalities $F > 0$ and (4.18) imply that

$$Q_{k+n} - Q_k \geq \varepsilon I_n \quad (4.19)$$

for some $\varepsilon > 0$, which does not depend on $k$. The asymptotics (4.11) is immediate from (4.19).

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5 Appendix

Proof of Theorem 2.2. It is easy to see that
\[(I_m + i\overline{z}C_k)^s j(I_m + izC_k) = (1 + z^2)^j,\]  \hspace{1cm} (5.20)
and so both \((I_m + izC_k)\) and \(W_r(z) = \prod_{k=0}^{r-1}(I_m + izC_k)\) are invertible for \(z \neq \pm i\). Now, let us consider the sets \(N_r\) of the linear fractional transformations
\[\phi_r(z, \mathcal{P}) = [I_{m_1} \ 0] \ W_r(z)^{-1} \mathcal{P}(z) \left( \begin{bmatrix} 0 & I_{m_2} \end{bmatrix} \ W_r(z)^{-1} \mathcal{P}(z) \right)^{-1},\]  \hspace{1cm} (5.21)
where \(\mathcal{P}(z)\) are nonsingular \(m \times m_2\) matrix functions with property-\(j\). That is, \(\mathcal{P}(z)\) are meromorphic on \(\mathbb{C}_-\) matrix functions such that the inequalities
\[\mathcal{P}(z)^* \mathcal{P}(z) > 0, \quad \mathcal{P}(z)^* j \mathcal{P}(z) \leq 0\]  \hspace{1cm} (5.22)
hold for all the points in \(\mathbb{C}_-\) (excluding, possibly, discrete sets of points). The sets \(N_r\) are well-defined because the inequality
\[\text{det} \left( \begin{bmatrix} 0 & I_{m_2} \end{bmatrix} \ W_r(z)^{-1} \mathcal{P}(z) \right) \neq 0\]  \hspace{1cm} (5.23)
follows from (5.22). Indeed, since relations (1.2) and (2.4) yield
\[(I_m + izC_k)^s j(I_m + izC_k) = (1 + |z|^2)^j + i(z - \overline{z})C_k \geq \tilde{q}(z)^j,\]  \hspace{1cm} (5.24)
where \(\tilde{q}(z) := 1 + |z|^2 + i(z - \overline{z}) > 0,\) we have
\[W_r(z)^* j W_r(z) \geq \tilde{q}(z)^* j, \quad \text{i.e.,} \quad (W_r(z)^{-1})^* j W_r(z)^{-1} \leq \tilde{q}(z)^{-r} j.\]  \hspace{1cm} (5.25)
Thus, the inequalities
\[\mathcal{P}(z)^* (W_r(z)^{-1})^* j W_r(z)^{-1} \mathcal{P}(z) \leq 0, \quad [0 \ I_{m_2}] \ j \left[ \begin{bmatrix} 0 \\ I_{m_2} \end{bmatrix} \right] < 0\]  \hspace{1cm} (5.26)
are valid, and (5.23) is immediate from [20 Proposition 1.43].

In view of (5.21) we have
\[\phi_{r+1}(z, \mathcal{P}) = [I_{m_1} \ 0] \ W_r(z)^{-1} \tilde{\mathcal{P}}(z) \left( \begin{bmatrix} 0 & I_{m_2} \end{bmatrix} \ W_r(z)^{-1} \tilde{\mathcal{P}}(z) \right)^{-1},\]  \hspace{1cm} (5.27)
where
\[ \tilde{\mathcal{P}}(z) = (I_m + i z C_r)^{-1} \mathcal{P}(z). \] (5.29)

Relations (5.24), (5.25) and (5.29) imply that
\[ \tilde{\mathcal{P}}(z)^* j \tilde{\mathcal{P}}(z) \leq 0. \] (5.30)

Compare (5.21), (5.22) with (5.28), (5.30) to see that the sets (Weyl disks) \( N_r \) are embedded:
\[ N_{r+1} \subseteq N_r. \] (5.31)

Clearly, formulas (5.28)–(5.30) remain valid when we put there \( r = 0 \). For that case, we partition \( \tilde{\mathcal{P}} \) and (in view of (2.1)) we rewrite (5.28) in the form
\[ \varphi_1(z, \mathcal{P}) = \tilde{\mathcal{P}}_1(z) \tilde{\mathcal{P}}_2(z)^{-1}, \quad \tilde{\mathcal{P}} =: \begin{bmatrix} \tilde{\mathcal{P}}_1 \\ \tilde{\mathcal{P}}_2 \end{bmatrix}, \] (5.32)

where (according to (5.23) with \( r = 1 \)) we have \( \det \tilde{\mathcal{P}}_2(z) \neq 0 \). It follows from (5.30) and (5.32) that the functions from \( N_1 \) are contractive. Hence, (5.31) implies that all the functions \( \varphi_r(z, \mathcal{P}) \) given by (5.21) are analytic and contractive in \( \mathbb{C}_- \).

Next, using Montel’s theorem and arguments from the Step 1 in the proof of [5, Theorem 3.8] one may easily show that there is an analytic and contractive in \( \mathbb{C}_- \) matrix function \( \varphi_\infty(z) \) such that
\[ \varphi_\infty \in \bigcap_{r \geq 1} N_r. \] (5.33)

(We note the functions \( \begin{bmatrix} I_{m_1} \\ \varphi \end{bmatrix} \) in the proof of [5, Theorem 3.8] should be substituted by \( \begin{bmatrix} \varphi \\ I_{m_2} \end{bmatrix} \) for our case of Weyl functions in \( \mathbb{C}_- \).) Taking into account (5.21) and (5.33) we write the representations
\[ \begin{bmatrix} \varphi_\infty(z) \\ I_{m_2} \end{bmatrix} = W_{r+1}(z) \mathcal{P}(z, r + 1) \quad (r \geq 0), \] (5.34)
where \( P(z, r + 1) \) are nonsingular with property-\( j \). Using the summation formula (2.24) and representation (5.34), we derive

\[
\begin{bmatrix}
\varphi^*_\infty \\
I_{m_2}
\end{bmatrix}
\sum_{k=0}^{r} q(z)^k W_k(z)^* C_k W_k(z)
\begin{bmatrix}
\varphi^*_\infty \\
I_{m_2}
\end{bmatrix}
\leq \frac{i(1 + |z|^2)}{(\overline{z} - z)} I_{m_2}.
\tag{5.35}
\]

Compare (5.35) with the Definition 2.1 of the Weyl function in order to see that \( \varphi^*_\infty \) is a Weyl function of (1.1), (1.2). Moreover, this Weyl function analytic and contractive in \( \mathbb{C}_- \). It remains to show that the Weyl function is unique.

First notice that (2.25) yields

\[
q(z) W_{k+1}(z)^* j W_{k+1}(z) \geq W_k(z)^* j W_k(z) \quad (k \geq 0).
\tag{5.36}
\]

Thus, we have \( q(z)^{k+1} W_{k+1}(z)^* j W_{k+1}(z) \geq j \), and so (2.4) implies that

\[
\begin{bmatrix}
I_{m_1} \\
0
\end{bmatrix}
\sum_{k=0}^{r} q(z)^k W_k(z)^* C_k W_k(z)
\begin{bmatrix}
I_{m_1} \\
0
\end{bmatrix}
\geq (r + 1) I_{m_1}.
\tag{5.37}
\]

Therefore, there is an \( m_1 \)-dimensional subspace of vectors \( g \in \mathbb{C}^m \) such that

\[
\sum_{k=0}^{\infty} g^* q(z)^k W_k(z)^* C_k W_k(z) g = \infty.
\tag{5.38}
\]

The further proof of the uniqueness of the values, which the Weyl function may take at any fixed \( z \in \mathbb{C}_- \) is easy and coincides with the arguments in [3, Theorem 3.8].

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