Extremal basic frequency of non-homogeneous plates

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Abstract. In this paper we propose two numerical algorithms to derive the extremal principal eigenvalue of the bi-Laplacian operator under Navier boundary conditions or Dirichlet boundary conditions. Consider a non-homogeneous hinged or clamped plate $\Omega$, the algorithms converge to the density functions on $\Omega$ which yield the maximum or minimum basic frequency of the plate.

Key Words: Bi-Laplacian; Eigenvalue Optimization; Rearrangement

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1 Introduction

Eigenvalue problems for elliptic partial differential equations have many applications in engineering and applied sciences and these problems have been intensively attractive to mathematicians in the past decades [16].

This paper is concerned with a fourth-order elliptic eigenvalue problem modeling the vibration of a non-homogeneous plate $\Omega$ which is either hinged or clamped along the boundary $\partial\Omega$. Several materials with $m$ different kinds of densities $0 < c_1 < c_2 < \ldots < c_m$ are given where the area of the domain with density $c_i$ is $S_i > 0, i = 1..m$. The problem involves geometrical constraints that can be described as $\sum_{i=1}^{m} S_i$ should be equal to the area of $\Omega$. We investigate the location of these materials throughout $\Omega$ in order to optimize the basic frequency in the vibration of the corresponding plate.

Motivated by the above explanation, we introduce the mathematical equations governing the structure and associated optimization problems. Let $\Omega$ be a...
bounded smooth domain in $\mathbb{R}^N$ and let $\rho_0(x) = c_1 \chi_{D_1} + \cdots + c_m \chi_{D_m}$, the density function, be a measurable function such that $|D_i| = S_i > 0$, $(i = 1..m)$ and $\sum_{i=1}^m S_i = |\Omega|$ where $|.|$ stands for Lebesgue measure. Define $\mathcal{P}$ as the family of all measurable functions which are rearrangement of $\rho_0$. For $\rho \in \mathcal{P}$, consider eigenvalue problems

$$\Delta^2 u = \lambda \rho u, \quad \text{in } \Omega, \quad u = 0, \Delta u = 0, \quad \text{on } \partial \Omega, \quad (1.1)$$

$$\Delta^2 v = \Lambda \rho v, \quad \text{in } \Omega, \quad v = 0, \frac{\partial v}{\partial n} = 0, \quad \text{on } \partial \Omega, \quad (1.2)$$

where $\lambda = \lambda_\rho$, $\Lambda = \Lambda_\rho$ are the first eigenvalues or the basic frequencies and $u = u(x)$, $v = v(x)$ are the corresponding eigenfunctions or the lateral displacements. The operator $\Delta^2$ stands for usual bi-Laplacian, that is $\Delta^2 u = \Delta(\Delta u)$. The principal eigenvalue $\lambda$ of problem (1.1) is obtained by minimizing the associate Rayleigh quotient

$$\lambda = \inf\left\{ \frac{\int_{\Omega} (\Delta w)^2 dx}{\int_{\Omega} \rho w^2 dx} : w \in H^2(\Omega) \cap H^1_0(\Omega), w \neq 0 \right\}, \quad (1.3)$$

and the first eigenvalue $\Lambda$ of problem (1.2) is obtained by minimizing the associate Rayleigh quotient

$$\Lambda = \inf\left\{ \frac{\int_{\Omega} (\Delta w)^2 dx}{\int_{\Omega} \rho w^2 dx} : w \in H^2_0(\Omega), w \neq 0 \right\}, \quad (1.4)$$

where it is well known [28] that the inferior is attained in both cases. By regularity results the solutions to problems (1.1) and (1.2) belongs to $H^4_{\text{loc}}(\Omega)$ and these equations hold a.e. in $\Omega$ [1].

To determine the system’s profile which gives the maximum and minimum principal eigenvalues, Cuccu et al have verified the following optimization problems

$$\max_{\rho \in \mathcal{P}} \lambda_\rho, \quad (1.5)$$

$$\min_{\rho \in \mathcal{P}} \lambda_\rho, \quad (1.6)$$

$$\max_{\rho \in \mathcal{P}} \Lambda_\rho, \quad (1.7)$$

$$\min_{\rho \in \mathcal{P}} \Lambda_\rho, \quad (1.8)$$

in [10, 11, 4]. The existence of solutions for problems (1.5) - (1.6) and (1.8) have been proved for general domain $\Omega$. But, the existence of a solution for problem (1.7) has been established when $\Omega$ is a positivity preserving domain for $\Delta^2 u$ under homogenous Dirichlet boundary conditions. For instance, the ball is a domain that enjoys such a property. In spite of these existence results, the precise identifications of the maximums and minimums were found only in case $\Omega$ is a ball.
In eigenvalue optimization for elliptic partial differential equations, one of challenging mathematical problems after the problem of existence is an exact formula of the optimizer or optimal shape design. Most papers in this field answered this question just in case $\Omega$ is a ball. For other domains qualitative properties of solutions were investigated and partial answers were given [6, 8, 12, 13, 23, 7, 24, 25]. From the physical point of view, it is important to know the shape of the optimal density functions in case $\Omega$ is not a ball.

This class of problems is difficult to solve because of the lack of the topology information of the optimal shape. There must be numerical approaches to determine the optimal shape design. The mostly used methods now are the homogenization method [2] and the level set method [27]. The level set method is well known for its capability to handle topological changes, such as breaking one component into several, merging several components into one and forming sharp corners. This approach has been applied to the study of extremum problems of eigenvalues of inhomogeneous structures including the identification of composite membranes with extremum eigenvalues [26, 15, 30], design of composite materials with a desired spectral gap or maximal spectral gap [19], finding optical devices that have a high quality factor [20] and principle eigenvalue optimization in population biology [18].

Recently, Kao and Su [21] proposed an efficient rearrangement algorithm based on the Rayleigh quotient formulation of eigenvalues. They have solved minimization and maximization problem for the $k$-th eigenvalue ($k \geq 1$) and maximization of spectrum ratios of the second order elliptic differential operator in $\mathbb{R}^2$. We extend the approach to solve a fourth order partial differential equation. Most of the previous results are for second order operators with Dirichlet boundary conditions and the geometric constraint have been considered for $m = 2$. Here we study a partial differential equation with different boundary conditions and we develop our algorithms for general $m \geq 2$. It is common in the literature that $\Omega$ have been considered as a rectangle [22]. To show the capacity and efficiency of the numerical method, we apply it to solve problems (1.5)-(1.8) when $\Omega$ has a more complicated geometrical structure than a rectangle. The algorithms start from a given density $\rho_0$ and converge to the optimizers.

Throughout this paper we shall write increasing instead of non-decreasing, and decreasing instead of non-increasing.

This section is closed with some definitions from the rearrangement theory related to our optimization problems. The reader can refer to [5, 3] for further information about rearrangement theory.

**Definition 1.1.** Two Lebesgue measurable functions $\rho : \Omega \to \mathbb{R}$, $\rho_0 : \Omega \to \mathbb{R}$, are said to be rearrangements of each other if

$$|\{x \in \Omega : \rho(x) \geq r\}| = |\{x \in \Omega : \rho_0(x) \geq r\}| \quad \forall r \in \mathbb{R}. \quad (1.9)$$

The notation $\rho \sim \rho_0$ means that $\rho$ and $\rho_0$ are rearrangements of each other.
Consider \( \rho_0 : \Omega \to \mathbb{R} \), the class of rearrangements generated by \( \rho_0 \), denoted \( \mathcal{P} \), is defined as follows

\[
\mathcal{P} = \{ \rho : \rho \sim \rho_0 \}.
\]

2 Numerical Algorithms

In this section we describe the algorithms that start from a given initial density function and converge to the optimizers. Consider the case that \( m \) different materials with densities \( 0 < c_1 < c_2 < \ldots < c_m \) are distributed in arbitrary pairwise disjoint measurable subsets \( D_i \), \( i = 1 \ldots m \), respectively of \( \Omega \) so that \( \bigcup_1^m D_i = \Omega \) and \( |D_i| = S_i \). Then, set \( \rho_0(x) = c_1 \chi_{D_1} + \ldots + c_m \chi_{D_m} \) and we have the following technical assertion.

Lemma 2.1. Function \( \rho \) belongs to the rearrangement class \( \mathcal{P} \) if and only if 
\[
\rho = c_1 \chi_{D'_1} + \ldots + c_m \chi_{D'_m}
\]
where \( D'_i \), \( i = 1 \ldots m \), are pairwise disjoint subsets of \( \Omega \) such that \( |D'_i| = S_i \) and \( \bigcup_1^m D'_i = \Omega \).

Proof. Assume \( \rho \in \mathcal{P} \). In view of definition 1.1,

\[
|\{x \in \Omega : \rho_0(x) = r\}| = |\cap_1^\infty \{x \in \Omega : r \leq \rho_0(x) < r + 1/n\}|
\]

\[= \lim_{n \to \infty} |\{x \in \Omega : \rho_0(x) \geq r\}| - |\{x \in \Omega : \rho_0(x) \geq r + 1/n\}|\]

\[= \lim_{n \to \infty} |\{x \in \Omega : \rho(x) \geq r\}| - |\{x \in \Omega : \rho(x) \geq r + 1/n\}|\]

\[= |\cap_1^\infty \{x \in \Omega : r \leq \rho(x) < r + 1/n\}| = |\{x \in \Omega : \rho(x) = r\}|,
\]

where it means that the level sets of \( \rho \) and \( \rho_0 \) have the same measures and this yields the assertion. The other part of the theorem is concluded from definition 1.1.

Now, we propose algorithms to derive the solutions of problems (1.5)-(1.8) respectively. For the maximization problems, we start from a given initial density functions \( \rho_0 \) and extract new density functions \( \rho_1 \) using the eigenfunction of equations (1.1)- (1.2) such that the first eigenvalues are increased, i.e.

\[
\lambda_{\rho_0} \leq \lambda_{\rho_1}, \quad \Lambda_{\rho_0} \leq \Lambda_{\rho_1}.
\]

In the same spirit as in the maximization cases, we initiate from a given density functions \( \rho_0 \) and extract another density functions \( \rho_1 \) such that principal eigenvalues of (1.1)- (1.2) are decreased, i.e.

\[
\lambda_{\rho_0} \geq \lambda_{\rho_1}, \quad \Lambda_{\rho_0} \geq \Lambda_{\rho_1}.
\]
The algorithms strongly based on the Rayleigh quotients in formulas (1.3)-(1.4). Such algorithms have been applied successfully to minimize eigenvalues of some second order elliptic operators [6, 17, 21]. They rely on the variational formulation of the eigenvalues and use level sets of the eigenfunctions or gradient of them. Employing the level sets of the eigenfunctions, we need some results of the rearrangement theory with an eye on our problem [5].

**Lemma 2.2.** Let \( \mathcal{P} \) be the set of rearrangements of a fixed function \( \rho_0 \in L^r(\Omega) \), \( r > 1, \rho_0 \not\equiv 0 \), and let \( q \in L^s(\Omega) \), \( s = r/(r - 1), q \not\equiv 0 \). If there is an increasing function \( \xi : \mathbb{R} \to \mathbb{R} \) such that \( \xi(q) \in \mathcal{P} \), then
\[
\int_\Omega \rho q dx \leq \int_\Omega \xi(q) q dx \quad \forall \rho \in \mathcal{P},
\]
and the function \( \xi(q) \) is the unique maximizer relative to \( \mathcal{P} \). Furthermore, if there is a decreasing function \( \eta : \mathbb{R} \to \mathbb{R} \) such that \( \eta(q) \in \mathcal{P} \), then
\[
\int_\Omega \rho q dx \geq \int_\Omega \eta(q) q dx \quad \forall \rho \in \mathcal{P},
\]
and the function \( \eta(q) \) is the unique minimizer relative to \( \mathcal{P} \).

Next two lemmas provide our main tool for constructing the numerical algorithms.

**Lemma 2.3.** Let \( f(x) \) be a nonnegative function in \( L^1(\Omega) \) such that its level sets have measure zero. Then the minimization problem
\[
\inf_{\rho \in \mathcal{P}} \int_\Omega \rho f dx,
\]
is uniquely solvable by \( \hat{\rho}(x) = c_1 \chi_{\hat{D}_1} + \ldots + c_m \chi_{\hat{D}_m} \) where \( |\hat{D}_i| = S_i \), \( i = 1..m \), and
\[
\hat{D}_1 = \{ x \in \Omega : f(x) \geq t_1 \},
\]
\[
t_1 = \sup \{ s \in \mathbb{R} : \{|x \in \Omega : f(x) \geq s\} \geq S_1 \},
\]
\[
\hat{D}_i = \{ x \in \Omega \setminus \bigcup_{j=1}^{i-1} \hat{D}_j : f(x) \geq t_i \},
\]
\[
t_i = \sup \{ s \in \mathbb{R} : \{|x \in \Omega \setminus \bigcup_{j=1}^{i-1} \hat{D}_j : f(x) \geq s\} \geq S_i \},
\]
for \( i = 2..m - 1 \).

**Proof.** Let us rearrange the super level sets of the function \( f(x) \) with an eye on the sets \( \hat{D}_i, i = 1..m \). We should mention here that the pairwise disjoint sets \( \hat{D}_i, i = 1..m \) are determined uniquely since the level sets of the function \( f(x) \) have measure zero. In addition,
\[
t_1 > t_2 > \ldots > t_{m-1},
\]
since we have \( S_i > 0, \ i = 1..m \). This motivates the following decreasing function

\[
\eta(t) = \begin{cases} 
  c_1 & t \geq t_1, \\
  c_2 & t_2 \leq t < t_1, \\
  \vdots \\
  c_{m-1} & t_{m-1} \leq t < t_{m-2}, \\
  c_m & \text{otherwise},
\end{cases}
\]

where it yields

\[
\eta(f(x)) = c_1 \chi_{\hat{D}_1} + \ldots + c_m \chi_{\hat{D}_m}.
\]

Invoking lemma 2.1, this function belongs to the rearrangement class \( \mathcal{P} \). According to lemma 2.2, \( \eta(f(x)) \) is the unique minimizer of the problem (2.1).

**Lemma 2.4.** Let \( f(x) \) be a nonnegative function in \( L^1(\Omega) \) such that its level sets have measure zero. Then the maximization problem

\[
\sup_{\rho \in \mathcal{P}} \int_{\Omega} \rho f \, dx,
\]

is uniquely solvable by \( \hat{\rho}(x) = c_1 \chi_{\hat{D}_1} + \ldots + c_m \chi_{\hat{D}_m} \) where \( |\hat{D}_i| = S_i, \ i = 1..m \), and

\[
\hat{D}_1 = \{ x \in \Omega : f(x) \leq t_1 \},
\]

\[
t_1 = \inf \{ s \in \mathbb{R} : |\{ x \in \Omega : f(x) \leq s \}| \geq S_1 \},
\]

\[
\hat{D}_i = \{ x \in \Omega \setminus \bigcup_{j=1}^{i-1} \hat{D}_j : f(x) \leq t_i \},
\]

\[
t_i = \inf \{ s \in \mathbb{R} : |\{ x \in \Omega \setminus \bigcup_{j=1}^{i-1} \hat{D}_j : f(x) \leq s \}| \geq S_i \},
\]

for \( i = 2..m - 1 \).

**Proof.** The proof is similar to that of lemma 2.3 and is omitted.

Lemmas 2.3 and 2.4 allow us to derive sequences of density functions \( \rho_n \) such that corresponding eigenvalues \( \lambda(\rho_n) \) and \( \Lambda(\rho_n) \) are monotone sequences of eigenvalues. To do this, we need the following theorem.

**Theorem 2.5.** Assume \( \rho_0(x) = c_1 \chi_{D_1} + \ldots + c_m \chi_{D_m} \) is a member of \( \mathcal{P} \). Then, there exist two functions \( \rho_1 \) and \( \rho_1' \) in the rearrangement class \( \mathcal{P} \) such that

\[
\lambda(\rho_0) \geq \lambda(\rho_1), \quad \Lambda(\rho_0) \geq \Lambda(\rho_1').
\]
Proof. We prove the assertion for equation (1.1). For (1.2), the proof is similar to that of equation (1.1) and it is omitted. Inserting $\rho_0$ as the density function into (1.1), we find $u_0$ as the eigenfunction of the equation corresponding to $\lambda_0 = \lambda(\rho_0)$.

We claim that the level sets of $u_0$ have measure zero. As we mentioned, $u_0$ satisfies equation (1.1) a.e. in $\Omega$. This means

$$\Delta^2 u_0 = \lambda \rho u_0, \quad \text{a.e. in } \Omega.$$ 

If the set $E = \{ x \in \Omega : u_0(x) = 0 \}$ has positive measure then $u_0$ is identically zero by theorem 4.4 of [11]. This yields that $\Delta^2 u_0 \neq 0$ a.e. in $\Omega$. Consequently, the level sets of $u_0$ have measure zero applying lemma 7.7 of [14].

Set $f(x) = u_0^2(x)$ in lemma 2.4, then one can achieve $\rho_1$ in $\mathcal{P}$ such that

$$\int_\Omega \rho_0 u_0^2 dx \leq \int_\Omega \rho_1 u_0^2 dx.$$ 

Therefore,

$$\frac{\int_\Omega (\Delta u_0)^2 dx}{\int_\Omega \rho_0 u_0^2 dx} \geq \frac{\int_\Omega (\Delta u_0)^2 dx}{\int_\Omega \rho_1 u_0^2 dx},$$

then

$$\lambda(\rho_0) \geq \lambda(\rho_1),$$

based on (1.3).

Utilizing theorem 2.5, we can derive two decreasing sequences of eigenvalues

$$\lambda(\rho_{n-1}) \geq \lambda(\rho_n), \quad \Lambda(\rho_{n-1}) \geq \Lambda(\rho_n).$$

Obviously, these decreasing sequences are bounded below by zero and so they converge. In view of theorem 2.5, we can propose an iterative procedure to find the minimal density configurations for the eigenvalue minimization problems (1.6) and (1.8).

The maximization problems are more complicated and an iterative method cannot be derived by arguments similar to those in the minimization cases [21]. If we consider $\rho_0$ as an arbitrary function in $\mathcal{P}$ and $u_0$ as an associated eigenfunction of (1.1), then one can find a density function $\rho_1$ in $\mathcal{P}$ regarding lemma 2.3 such that

$$\int_\Omega \rho_0 u_0^2 dx \geq \int_\Omega \rho_1 u_0^2 dx,$$

and then

$$\lambda(\rho_0) \leq \frac{\int_\Omega (\Delta u_0)^2 dx}{\int_\Omega \rho_0 u_0^2 dx} \leq \frac{\int_\Omega (\Delta u_0)^2 dx}{\int_\Omega \rho_1 u_0^2 dx} \geq \lambda(\rho_1).$$

The same argumentation is valid for the principal eigenvalues of (1.2) as well. Hence, we cannot produce an increasing sequence of eigenvalues since the next generated eigenvalue may be less than the previous one. Inspired by the method introduced in [21], the strategy to guarantee a monotone increasing sequence is to add
an acceptance rejection method. If this new eigenvalue increases the eigenvalue, the density function will be accepted. Otherwise, the partial swapping method will be used. Indeed, in the partial swapping method we use \( \rho \) where \( \delta \rho = \rho_1 - \rho_0 \) is small enough.

**Theorem 2.6.** Let \( \rho_0, \rho_1 \) be functions in \( \mathcal{P} \) where

\[
\int_{\Omega} \rho_0 u_0^2 dx > \int_{\Omega} \rho_1 u_0^2 dx, \tag{2.3}
\]

and \( \| \delta \rho \|_{L^2(\Omega)} \) is small enough. Then,

\[
\lambda(\rho_0) < \lambda(\rho_1), \quad \Lambda(\rho_0) < \Lambda(\rho_1).
\]

**Proof.** We prove the assertion for equation (1.1). For (1.2), the proof is similar to that of equation (1.1) and is omitted. Assume \( u_0 \) and \( u_1 \) are eigenfunctions of (1.1) corresponding to \( \rho_0 \) and \( \rho_1 \) respectively where normalized so that \( \int_{\Omega} \Delta u_0^2 = \int_{\Omega} \Delta u_1^2 = 1 \). Applying (1.3), we have

\[
\lambda(\rho_0) = \frac{1}{\int_{\Omega} \rho_0 u_0^2 dx} \leq \frac{1}{\int_{\Omega} \rho_1 u_1^2 dx},
\]

which it yields

\[
\int_{\Omega} \rho_0 u_0^2 dx \leq \int_{\Omega} \rho_1 u_1^2 dx. \tag{2.4}
\]

Then,

\[
\frac{1}{\lambda(\rho_1)} - \frac{1}{\lambda(\rho_0)} = \int_{\Omega} \rho_0 u_0^2 dx - \int_{\Omega} \rho_1 u_1^2 dx
\]

\[
= \int_{\Omega} \rho_1 u_1^2 - \rho_1 u_0^2 + \rho_1 u_1^2 - \rho_0 u_0^2 dx
\]

\[
= \int_{\Omega} (\rho_0 + \delta \rho)(u_1^2 - u_0^2) dx + \int_{\Omega} \delta \rho u_0^2 dx
\]

\[
= \int_{\Omega} \rho_0 (u_1^2 - u_0^2) dx + \int_{\Omega} \delta \rho (u_1^2 - u_0^2) dx + \int_{\Omega} \delta \rho u_0^2 dx.
\]

On the right hand side of the last equality, we have three integrals where the first and the last one from the left are negative according to (2.3) and (2.4). We claim that \( \| u_1 - u_0 \|_{L^2(\Omega)} \to 0 \) as \( \| \delta \rho \|_{L^2(\Omega)} \to 0 \). Then, one can observe that the second integral converges to zero with the rate of convergence \( O(\| \delta \rho \|_{L^2(\Omega)}^{1+s}) \) for some \( s > 0 \) and the third integral converges to zero with the rate of convergence \( O(\| \delta \rho \|_{L^2(\Omega)}) \). Hence if \( \| \delta \rho \|_{L^2(\Omega)} \) is small enough, we can infer that the right hand side of the last equality is negative and

\[
\lambda(\rho_0) < \lambda(\rho_1).
\]

It remains to prove the claim. Applying the dominate convergence theorem, we deduce that if \( \rho_0 \to \rho_1 \) strongly in \( L^2(\Omega) \) then \( \rho_0 \to \rho_1 \) weakly in \( L^\infty(\Omega) \). By the same reasoning, used in the proof of lemma 3.1 in [10], it can be concluded that \( u_0 \to u_1 \) weakly in \( H^2(\Omega) \) and \( u_0 \to u_1 \) strongly in \( L^2(\Omega) \).
Remark 2.1. Let \( \rho_0(x) = c_1 \chi_{D_1} + \ldots + c_m \chi_{D_m} \) be a function in \( P \). We derive \( \rho_1 \) stated in (2.3) in the following form
\[
\rho_1(x) = c_1 \chi_{D_1} + \ldots + c_j \chi_{D_j'} + \ldots + c_i \chi_{D_i'} + \ldots + c_m \chi_{D_m},
\]
such that \( D_j' = (D_j - A) \cup B \) and \( D_i' = (D_i - B) \cup A \) where \( A \) is a subset of \( D_j \) and \( B \) is a subset of \( D_i \) so that \( |A| = |B| \). Then,
\[
\delta \rho = \rho_1 - \rho_0 = c_j (\chi_B - \chi_A) + c_i (\chi_A - \chi_B),
\]
and functions \( \rho_1, \rho_0 \) satisfies (2.3) if
\[
\int_B u_0^2 > \int_A u_0^2.
\]
It is noteworthy that \( ||\delta \rho||_{L^2(\Omega)} \) will be small enough if one adjusts \( |A| = |B| \) small enough. The sets \( A \) and \( B \) are selected by trail and error.

Remark 2.2. Utilizing lemma (2.3) and theorem 2.6, we can derive an increasing sequences of eigenvalues
\[
\lambda(\rho_{n-1}) \leq \lambda(\rho_n), \quad \Lambda(\rho_{n-1}) \leq \Lambda(\rho_n).
\]
Using variational formulation (1.3) and (1.4), it can be said that these sequences are bounded above. Consider an entire density function \( \rho \) in \( P \). Then from equation (1.3) or (1.4) and lemma 2.3 we conclude
\[
\lambda_{\rho}, \Lambda_{\rho} \leq \int_\Omega (\Delta \psi)^2 dx \int_\Omega \rho \psi^2 dx \leq \int_\Omega (\Delta \psi)^2 dx \int_\Omega \bar{\rho} \psi^2 dx,
\]
where \( \psi \) is the eigenfunction associated with the principal eigenvalue of the Laplacian with Dirichlet boundary conditions and \( \bar{\rho} \) is the minimizer stated in lemma 2.3 for \( f = \psi^2 \). Consequently, the increasing sequences of eigenvalues are bounded above and are convergent.

3 Implementation of the numerical algorithms

This section provides us with the details of the implementation for algorithms introduced in the previous section and some examples are chosen to illustrate the numerical solutions. The algorithms work for both equations (1.1) and (1.2) similarly and so we state the procedures just for (1.1).

At iteration step \( n \), there is a guess for the configuration of the optimal density function where it is denoted by \( \rho_n \). We use the finite element method with piecewise linear basis functions to discretize equation (1.1) with \( \rho_n \). Let \( u_n \) be an eigenfunction of (1.1) associated with eigenvalue \( \lambda_n = \lambda(\rho_n) \). For minimization problem (1.6), we should extract a new density function \( \rho_{n+1} \) based upon the level
Table 1

| Algorithm 2. Eigenvalue minimization |
|-------------------------------------|
| **Data:** An initial density function \(\rho_0\) |
| **Result:** A sequence of decreasing eigenvalues \(\lambda(\rho_n)\) |
| 1. Set \(n = 0\); |
| 2. Compute \(u_n\) and \(\lambda(\rho_n)\); |
| 3. Compute \(\rho_{n+1}\) applying lemma 2.4; |
| 4. If \(\|\delta\rho\|_{L^2(\Omega)} < TOL\) then stop; |
| \hspace{1cm} else Set \(n = n + 1\); |
| \hspace{1cm} Go to step 2; |

sets of eigenfunction \(u_n\) where it belongs to \(\mathcal{P}\) and \(\lambda(\rho_n) > \lambda(\rho_{n+1})\). To derive this \(\rho_{n+1}\), we make use of lemma 2.4 and identify \(\rho_{n+1}\) by setting \(f(x) = u_n^2(x)\). According to theorem 2.5, we have \(\lambda(\rho_n) > \lambda(\rho_{n+1})\) and the generated sequence is convergent. The resulting algorithm is shown in table 1. There is a stopping criterion in this method. The algorithm stops when \(\|\delta\rho\|_{L^2(\Omega)} = \|\rho_{n+1} - \rho_n\|_{L^2(\Omega)}\) is less than a prescribed tolerance \(TOL\).

Table 2

| Algorithm 1. Eigenvalue maximization |
|-------------------------------------|
| **Data:** An initial density function \(\rho_0\) |
| **Result:** A sequence of increasing eigenvalues \(\lambda(\rho_n)\) |
| 1. Set \(n = 0\); |
| 2. Compute \(u_n\) and \(\lambda(\rho_n)\); |
| 3. Compute \(\rho_{n+1}\) applying lemma 2.3; |
| 4. Compute \(\lambda(\rho_{n+1})\); |
| 5. If \(\lambda(\rho_n) < \lambda(\rho_{n+1})\) then go to step 6; |
| \hspace{1cm} else Compute \(\rho_{n+1}\) applying remark 2.1; |
| 6. If \(\|\delta\rho\|_{L^2(\Omega)} < TOL\) then stop; |
| \hspace{1cm} else Set \(n = n + 1\); Go to step 2; |

For maximization problem (1.5), we should extract a new density function \(\rho_{n+1}\) where it belongs to \(\mathcal{P}\) and \(\lambda(\rho_n) < \lambda(\rho_{n+1})\). To derive this \(\rho_{n+1}\), we make use of lemma 2.3 and identify \(\rho_{n+1}\) by setting \(f(x) = u_n^2(x)\). If \(\lambda(\rho_n) < \lambda(\rho_{n+1})\), the derived function \(\rho_{n+1}\) is accepted. Otherwise, the partial swapping method introduced in theorem 2.6 and remark 2.1 is used to generate the new \(\rho_{n+1}\). The resulting algorithm is shown in table 2. According to remark 2.2, this increasing sequence is convergent.
In the above algorithms we need to calculate parameters \( t_1, t_{m-1} \) which is mentioned in lemma 2.3 or 2.4. In order to find \( t_1, t_{m-1} \), two algorithms are developed which they apply the idea of the bisection method. Both algorithms are the same in essence and we only state the algorithm related to the maximization problem. Introducing the distribution function \( F(s) = \left| \{ x \in \Omega : u^2_n(x) > s \} \right| \), we state algorithm 3 in table 3 to compute \( t_1, t_{m-1} \). Again, we should consider a tolerance \( TOL \) in this algorithm since it is meaningless computationally to find a set \( D_k \) satisfying \( |D_k| = S_k \) exactly.

Table 3

| Algorithm 3. Bisection method for \( t_1, t_{m-1} \) |
|--------------------------------------------------|
| **Data:** Eigenfunction \( u_n \) on the domain \( \Omega \) |
| **Result:** The level \( t_1, t_{m-1} \) |
| **For** \( i = 1 \ldots m - 1 \) **do** |
| 1. Set \( L = 0 \), \( U = \max_{x \in \Omega} u^2_n(x) \) |
| 2. Set \( \theta = (L + U)/2 \); |
| 3. If \( |F(\theta) - S_i| < TOL \) then set \( t_i = \theta \) and \( \Omega = \Omega \setminus D_i \); |
| else |
| If \( F(\theta) < S_i \) then |
| Set \( U = \theta \); Go to step 2; |
| else |
| Set \( L = \theta \); Go to step 2; |

![Figure 1](image)

(a) \( \lambda_{\text{max}} = 1.51 \)  (b) \( \lambda_{\text{max}} = 6.28 \)

Figure 1: The maximizer sets in black

Let us present some results in dimensions \( N = 2 \) based on algorithms 1 and 2.

**Example 1.** Consider a hinged non-homogeneous plate \( \Omega \) which it is made of two different materials. We want to find the solutions of optimization problems (1.5)- (1.6). Set \( c_1 = 1 \) and \( c_2 = 2 \), we illustrate the optimum sets in cases rectangle, circle, ellipse and crescent such that \( |\Omega| = 16.00, 6.28, 16.49, 6.28 \) and \( S_2 = 4.00, 3.14, 4.00, 0.78 \) respectively. Remember that our aim is to locate these two materials throughout \( \Omega \) so to optimize the first eigenvalue in the vibration of
the corresponding plate. Distribution of these materials as the level sets of the maximal density functions are plotted in figures 1 and 2 for various geometries \( \Omega \). The sets with the highest density are depicted in black. These shapes reveal that the maximal distribution of the materials consists of putting the material with the highest density in a neighborhood of the boundary. In figures 3 and 4, distribution of these materials as the level sets of the minimal density functions are plotted. Physically speaking, in order to minimize the basic frequency of the hinged non-homogeneous plates it is best to place the material with the highest density in a region in the center of the domain.

**Example 2.** Consider a clamped non-homogeneous plate \( \Omega \) that is made of two different materials with densities \( c_1 = 1 \) and \( c_2 = 2 \). This means that we should deal with the solutions of optimization problems (1.7)- (1.8). The optimum sets are illustrated when \( \Omega \) is a circle and a rectangle with a hole such that \( |\Omega| = 6.28, 2.52 \) and \( S_2 = 3.14, 1.12 \) respectively. The distribution of these materials as the level sets of the maximal and the minimal density functions are plotted in figures 5 and 6. The sets with the highest density are depicted in black. These shapes reveal that the maximal distribution of the materials consists of putting the material with the highest density in a neighborhood of the boundary and the minimal distribution consists of putting the material with the highest density in the center of the region.
Example 3. Let \( \Omega \) be a hinged plate which is made of three different materials with densities \( c_1 = 1, c_2 = 2 \) and \( c_3 = 3 \). The optimum sets are depicted when \( \Omega \) is a rectangle with \( |\Omega| = 6 \) and \( S_i = 2, i = 1..3 \). The locations of these materials in \( \Omega \) as the level sets of the maximal and minimal density functions are shown in figure 7a and 7b. According to this optimum density functions, one can discover that the maximal distribution of the materials consists of putting the material with the highest density around the boundary and the material with the lowest density in the center of \( \Omega \). In addition, the minimal distribution of the materials consists of putting the material with the lowest density around the boundary and the material with the highest density in the center of \( \Omega \).

Remark 3.1. In our numerical tests, the procedures typically converge to the global optimal set of the respective problems, although this has not been established theoretically that the derived sets are the global optimizers. It is noteworthy that the algorithms may stick to a local minimizer. This is, the main drawback of such algorithms, see [21, 7]. To overcome this problem, a usual way is to run the algorithms with different initializers. Then, one can compare the derived optimizers and choose the best one.
When $\Omega$ is a ball, the numerical algorithm converges to the Schwartz increasing rearrangement of $\rho_0$ in the maximization problem. For the minimization problem, the algorithm converges to the Schwartz decreasing rearrangement of $\rho_0$. Indeed, both procedures converge to the optimal solutions derived analytically in [10, 11]. Of course, our results derived in the above examples agree with physical intuition.

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