Heavy Quark Effective Field Theory
at $O(1/m_Q^2)$. I.
QCD Corrections to the Lagrangian

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Abstract

We present a new calculation of the renormalized HQET Lagrangian
at order $O(1/m_Q^2)$ and discuss the consequences of the BRST invari-
ance of QCD and the reparameterization invariance of HQET. Our
result corrects earlier, conflicting calculations and sets the stage for
the calculation of the renormalized currents at order $O(1/m_Q^2)$.
1 Introduction

Heavy Quark Effective Field Theory (HQET) [1] has been established as the theoretical tool of choice for the description of mesons and baryons containing heavy quarks [2]. This derives from the fact that it is a systematic expansion in inverse powers of the heavy quark with well defined and calculable coefficients. Furthermore, its realization of the spin and flavor symmetry of the low energy theory is a phenomenologically powerful tool.

However, since the expansion parameter $\Lambda_H/m_Q$ is about 0.12 (using a hadronic scale $\Lambda_H \approx 600\text{MeV}$ and $m_Q = m_b$) leading order calculations are not sufficient for precision calculations. Terms of order $\mathcal{O}(1/m_Q^2)$ including leading QCD corrections have to be under control.

An indispensable prolegomenon to the calculation of renormalized matrix elements of currents is the renormalization of the Lagrangian. Unfortunately, two calculations with conflicting results [3, 4] have been reported. In this note we present the result of a new calculation of the renormalized Lagrangian, which differs from the previous two. We will demonstrate that our result satisfies important consistency conditions that are violated by the earlier calculations.

This note is organized as follows: in section 2 we introduce our operator basis. Our result for the anomalous dimensions is presented in section 3. In section 4 we will discuss the consistency of this result and compare it in section 5 with earlier calculations. Finally, we present the renormalization group flow in section 6 and conclude in section 7.

The renormalized currents will be presented in a subsequent note [5] and phenomenological applications will appear later [6]. A more detailed discussion of technical matters will be presented in [7].

2 Operator basis

The Lagrangian of HQET is defined by a systematic expansion of QCD in inverse powers of the heavy quark mass

$$\mathcal{L}_{\text{HQET}} = \bar{h}_v(i\sigma^D)h_v + \frac{1}{2m_Q} \sum_i \tilde{C}^{(1)}_i \mathcal{O}^{(1)}_i + \frac{1}{(2m_Q)^2} \sum_i \tilde{C}^{(2)}_i \mathcal{O}^{(2)}_i + \mathcal{O}(1/(2m_Q)^3).$$

At order $\mathcal{O}(1/m_Q^0)$ there is only one operator $\bar{h}_v(i\sigma^D)h_v$, which is independent of the spin and flavor of the quark, resulting in the celebrated spin-flavor symmetry of HQET.
At order $\mathcal{O}(1/m_Q)$ there are three independent operators. We shall use the conventional basis

\begin{align}
\mathcal{O}_1^{(1)} &= h_v(iD)^2h_v \\
\mathcal{O}_2^{(1)} &= g \frac{h_v}{2} \sigma_{\mu\lambda} F_{\mu\lambda} h_v \\
\mathcal{O}_3^{(1)} &= h_v (ivD)^2 h_v.
\end{align}

Below, the operators $\mathcal{O}_1^{(1)}$ and $\mathcal{O}_2^{(1)}$ will also be called kinetic and chromomagnetic respectively. The operator $\mathcal{O}_3^{(1)}$ vanishes by the equations of motion (EOM)

\[ ivDh_v|Q\rangle = 0 \] (3)

for heavy quark states $|Q\rangle$. For the renormalization of the Lagrangian, it is not necessary to include this operator, if the EOM are used consistently. It is however needed as a counterterm in the renormalization of the heavy quark currents and will be included here. In addition, inclusion of the operators vanishing by the EOM allows to make the relations following from reparameterization invariance explicit. Finally, the extraction of coefficients with the aid of symbolic manipulation programs is more straightforward in the full basis, while the work induced by the additional operators is insignificant.

At order $\mathcal{O}(1/m_Q^2)$ there are thirteen independent operators. They are grouped in four classes. Two of the local operators do not vanish by the EOM. We will denote them collectively by $\bar{\mathcal{O}}^{(2)}$ and choose them as

\begin{align}
\mathcal{O}_1^{(2)} &= \bar{h}_v i D_\mu (ivD)i D^\mu h_v \\
\mathcal{O}_2^{(2)} &= \bar{h}_v i \sigma^{\mu\lambda} i D_\mu (ivD)i D_\lambda h_v.
\end{align}

The five remaining local operators vanish by the EOM. We denote them by $\bar{\mathcal{O}}_{\text{EOM}}^{(2)}$ and choose the basis

\begin{align}
\mathcal{O}_3^{(2)} &= \bar{h}_v (ivD)(iD)^2 h_v \\
\mathcal{O}_4^{(2)} &= \bar{h}_v (iD)^2 (ivD) h_v \\
\mathcal{O}_5^{(2)} &= \bar{h}_v (ivD)^3 h_v \\
\mathcal{O}_6^{(2)} &= -\frac{g}{2} \bar{h}_v (ivD)\sigma_{\mu\lambda} F_{\mu\lambda} h_v \\
\mathcal{O}_7^{(2)} &= -\frac{g}{2} \bar{h}_v \sigma^{\mu\lambda} F_{\mu\lambda}(ivD) h_v.
\end{align}
In addition to the local operators, there are the time-ordered products of the lower dimensional operators. There are three of them that do not vanish by the EOM. They will be denoted \( \vec{T}^{(2)} \).

\[
\begin{align*}
\mathcal{T}^{(2)}_{11} &= \frac{i}{2} T \left\{ [\bar{h}_v (iD)^2 h_v] \left[ \bar{h}_v (iD)^2 h_v \right] \right\} \quad (4h) \\
\mathcal{T}^{(2)}_{12} &= \frac{i g}{2} T \left\{ [\bar{h}_v (iD)^2 h_v] \left[ \bar{h}_v \sigma^\mu \lambda F_{\mu \lambda} h_v \right] \right\} \quad (4i) \\
\mathcal{T}^{(2)}_{22} &= \frac{ig^2}{8} T \left\{ [\bar{h}_v \sigma^\mu \lambda F_{\mu \lambda} h_v] \left[ \bar{h}_v \sigma^\mu \lambda F_{\mu \lambda} h_v \right] \right\} \quad (4j)
\end{align*}
\]

Below, the operators \( \mathcal{T}^{(2)}_{11} \) and \( \mathcal{T}^{(2)}_{22} \) will also be called double-kinetic and double-chromo-magnetic respectively. Finally there are three more time-ordered products \( \mathcal{T}^{(2)}_{\text{EOM}} \) that vanish by the EOM

\[
\begin{align*}
\mathcal{T}^{(2)}_{13} &= i T \left\{ [\bar{h}_v (iD)^2 h_v] \left[ \bar{h}_v (ivD)^2 h_v \right] \right\} \quad (4k) \\
\mathcal{T}^{(2)}_{23} &= \frac{ig}{2} T \left\{ [\bar{h}_v \sigma^\mu \lambda F_{\mu \lambda} h_v] \left[ \bar{h}_v (ivD)^2 h_v \right] \right\} \quad (4l) \\
\mathcal{T}^{(2)}_{33} &= i \frac{1}{2} T \left\{ [\bar{h}_v (ivD)^2 h_v] \left[ \bar{h}_v (ivD)^2 h_v \right] \right\} \quad (4m)
\end{align*}
\]

Below we shall refer to the operators vanishing by the equations of motion as EOM operators, for short.

For calculational convenience, we have not chosen a manifestly hermitian basis, but the results presented below show that indeed only hermitian linear combinations show up in counter terms.

# 3 Anomalous dimensions

The most convenient approach to the calculation of anomalous dimensions uses the background field gauge \( \square \) with gauge fixing term

\[
-\frac{1}{2} \xi (D_\mu (V) A^\mu)^2 , \quad (5)
\]

because only the renormalization constants of those operators that are manifestly invariant under gauge transformations of the background field have to be calculated. Therefore, only the divergent three-point functions have to be calculated to derive the anomalous dimensions. Furthermore, the Ward identities for the classical background fields are particularly simple and provide a powerful tool for checking our results.
The background field gauge with an arbitrary gauge parameter allows an independent test of the consistency of our results by comparing the \(\xi\) dependence with general results [9] for the one-loop effective action. Below we shall write \(\bar{\xi} = 1 - \xi\) for the gauge parameter.

The anomalous dimensions at order \(O(1/m_Q)\) are well known [10] (the gauge parameter dependence has been calculated in [11]):

\[
\hat{\gamma}^{(1)} = \begin{pmatrix} 0 & 0 & 2C_F^\xi \\ 0 & -\frac{1}{2}C_A & 0 \\ 0 & 0 & -C_F^\xi \end{pmatrix},
\]

where we have introduced the shorthand \(C_F^\xi = C_F(1 + \bar{\xi}/2)\). Here and below, we have extracted the common loop factor \(\alpha/\pi\) from (6).

The matrix of anomalous dimensions can naturally be written in block form, separating local operators from non-local operators and separating EOM operators from the rest:

\[
\hat{\gamma}^{(2)} = \begin{pmatrix} \tilde{\mathcal{O}}^{(2)} & \tilde{\mathcal{O}}^{(2)}_{\text{EOM}} & \tilde{\mathcal{F}}^{(2)} & \tilde{\mathcal{F}}^{(2)}_{\text{EOM}} \\ \tilde{\mathcal{O}}^{(2)}_{\text{EOM}} & \tilde{\mathcal{O}}^{(2)}_{\text{EOM}} & \tilde{\mathcal{F}}^{(2)} & \tilde{\mathcal{F}}^{(2)}_{\text{EOM}} \\ \tilde{\mathcal{F}}^{(2)} & \tilde{\mathcal{F}}^{(2)}_{\text{EOM}} & \tilde{\mathcal{F}}^{(2)} & \tilde{\mathcal{F}}^{(2)}_{\text{EOM}} \\ \tilde{\mathcal{F}}^{(2)}_{\text{EOM}} & \tilde{\mathcal{F}}^{(2)}_{\text{EOM}} & \tilde{\mathcal{F}}^{(2)}_{\text{EOM}} & \tilde{\mathcal{F}}^{(2)}_{\text{EOM}} \end{pmatrix}.
\]

The upper right block has to vanish, because Weinberg’s theorem [12] guarantees that the renormalization of the local operators does not require counterterms from the time-ordered products. The other three blocks display the block triangular structure required by the fact that the renormalization of EOM operators only induces counterterms that are EOM operators themselves.

The anomalous dimensions have been calculated manually. These calculations have been verified with the help of FORM [13] as a warm-up for the renormalization of the currents [4], which requires the use of symbolic manipulation programs for economical reasons.

We start the presentation of the results with the renormalization of the local operators

\[
\hat{\gamma}^{(2)}_{\nu} = \begin{pmatrix} -\frac{1}{3}C_A & 0 \\ 0 & 0 \end{pmatrix}.
\]
\[ \hat{\gamma}_{1,1}^{(2) \text{EOM}} = \begin{pmatrix} \frac{1}{6}C_A & \frac{1}{6}C_A & -2C_F(1 + \bar{\xi}) & 0 & 0 \\ 0 & 0 & 0 & -\frac{2}{3}C_A & -\frac{2}{3}C_A \end{pmatrix} \]  
(8b)

\[ \hat{\gamma}_{1,2}^{(2) \text{EOM}} = \begin{pmatrix} 0 & 0 & -C_F(1 + 2\bar{\xi}) \\ 0 & 0 & -C_F(1 + 2\bar{\xi}) & 0 \\ 0 & 0 & -C^\xi_F \\ 0 & 0 & 0 & -C_A \\ 0 & 0 & 0 & 0 & -C_A \end{pmatrix} \]  
(8c)

The renormalization of the time-ordered products requires local counterterms as well. The corresponding anomalous dimensions are

\[ \hat{\gamma}_{m}^{(2)} = \begin{pmatrix} -\frac{1}{6}C_A & \frac{2}{3}C_F & 0 \\ 0 & 0 & -C_A \\ -\frac{5}{6}C_A & 0 \end{pmatrix} \]  
(8d)

\[ \hat{\gamma}_{m,1}^{(2) \text{EOM}} = \begin{pmatrix} C^\xi_{AF} & C^\xi_{AF} & -8C^\xi_F \\ 0 & 0 & 0 & C_A - 2C^\xi_F & C_A - 2C^\xi_F \\ \frac{5}{12}C_A & \frac{5}{12}C_A & 2C_F & -\frac{3}{4}C_A & -\frac{3}{4}C_A \end{pmatrix} \]  
(8e)

using the shorthand \( C^\xi_{AF} = \frac{1}{12}C_A + C_F(\frac{10}{3} + \bar{\xi}) \).

\[ \hat{\gamma}_{m,2}^{(2) \text{EOM}} = \begin{pmatrix} -C^\xi_F & -C^\xi_F & 6C_F(1 + \bar{\xi}) & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2}C_A + C^\xi_F & -\frac{1}{2}C_A + C^\xi_F \\ 0 & 0 & -C^\xi_F & 0 & 0 \end{pmatrix} \]  
(8f)

The blocks \( \hat{\gamma}_m^{(2)} \), \( \hat{\gamma}_{m,1}^{(2) \text{EOM}} \) and \( \hat{\gamma}_{m,2}^{(2) \text{EOM}} \) are given by the sum of the appropriate anomalous dimensions from \( \hat{\gamma}^{(1)} \).

4 Consistency of the results

The symmetries of HQET entail relations among the anomalous dimensions that can be used to check the result (8). Such consistency checks are useful in the present case and are of vital importance in the considerably more involved renormalization of the HQET currents [5].
4.1 Gauge invariance

As alluded to in the previous section, we have used the Ward identities for the background fields and the $\xi$-dependence of the one-loop effective action for verifying our results.

From the simple QED-like Ward identity for the background field

$$q_{\alpha} \tilde{\Gamma}^{\alpha}(p, q) = \tilde{S}(p + q) - \tilde{S}(p),$$

(9)

with $\tilde{S}$ and $\tilde{\Gamma}^{\alpha}$ denoting the one-particle-irreducible two- and three-point functions with one operator insertion respectively, follows that the counterterms proportional to $C_A$ (i.e. the non-abelian contributions) have to be transversal. Our result (8) passes this consistency check. The technical details will be presented in [7].

In background field gauge, the $\xi$-dependence of the renormalized effective action at one loop order is known to have the following form [9]

$$-2\xi \frac{\partial}{\partial \xi} \tilde{\Gamma}^{\alpha} = \hat{\Gamma}^{(0)}_{A^\alpha Q} \ast \tilde{\Gamma}^{(1)}_{JLh_vh_v} + \hat{\Gamma}^{(0)}_{h_vA^\alpha h_v} \ast \Gamma^{(1)}_{MLh_v} + \hat{\Gamma}^{(0)}_{h_vA^\alpha h_v} \ast \tilde{\Gamma}^{(1)}_{MLh_v} + \ldots$$

(10)

where $\Gamma^{(n)}$ denotes the effective action at $n$-loop order, $\hat{\Gamma}^{(n)}$ the same effective action with the gauge fixing term (5) subtracted and $\tilde{\Gamma}^{(n)}$ the effective action with an operator inserted. Finally, subscripts denote functional differentiation and $\ast$ integration over the corresponding space-time argument.

With the help of power counting we can identify the possible contributions to the right hand side of (10) that have the correct tensor structure. It can be shown [8] for all operators, with the exception of $O^{(2)}_6$ and $O^{(2)}_7$, that such contributions have to be proportional to $C_F$. The explicit calculation shows that this feature remains true for $O^{(2)}_6$ and $O^{(2)}_7$ as well. Furthermore, there are no $\xi$-dependent counter terms that do not vanish by the EOM. Our result (8) passes these consistency checks as well.

4.2 Reparameterization invariance

The HQET Lagrangian is a reparameterized form of the QCD Lagrangian, therefore the matching coefficients $\tilde{C}$ of different orders in $1/m_Q^2$ are related [14]. For example the matching coefficient of the chromo-magnetic operators in $O(1/m_Q^2)$ can be derived from the coefficient in $O(1/m_Q)$

$$\tilde{C}_2^{(2)} = 2\tilde{C}_2^{(1)} - 1.$$ 

(11)
On the other hand, we know that the product of the matching coefficients and the renormalization constants $\vec{C} \hat{Z}_{\text{MS}}^{-1}$ is finite and we can derive relations between the $O(\alpha)$ matching coefficients, the tree-level matching coefficients and the anomalous dimensions:

$$\vec{C}^{(\alpha)} + \vec{C}^{(\text{tree})} \hat{\gamma}^{(2)} = 0. \quad (12)$$

Since there are reparameterization invariance relations among the $O(\alpha)$ matching coefficients, (12) induces reparameterization invariance relations among the anomalous dimensions [7]. These relations are satisfied by our result (8).

5 Comparison with earlier calculations

Two calculations of the renormalized HQET Lagrangian at order $O(1/m_Q^2)$ have been circulated as preprints in the past [3, 4]. Their results are not consistent with each other and our result differs from both. Therefore a brief discussion of the errors in these calculations seems to be in order:

- After transforming the result of [3] to our basis, it turns out that the coefficients $\hat{\gamma}_m^{(2)}$ of the local counter terms for the double insertions are incorrect. In particular, the $-C_A$ entry in (8d) is fixed by reparameterization invariance, which is therefore violated by the result in [3]. The argument is unfortunately technically involved and will be presented elsewhere [7].

- The operator basis used in [4] is inconsistent. While these authors have used the EOM in their calculations, they do include an operator

$$\frac{ig}{2} \left( h_v \sigma^{\alpha\nu} T^a h_v \right) D_\alpha F^a_{\mu\nu} v^\mu = -\frac{g}{4} h_v \sigma^{\mu\nu} \left[ F_{\mu\nu}, ivD \right] h_v \quad (13)$$

in their basis, which vanishes by this equation of motion. On the other hand, they have missed a spin-symmetric operator that is required as a counter term for the double insertions of the kinetic and chromomagnetic operators.

6 Renormalization group flow

An analytical solution of the renormalization group equation using (8) in the full basis seems to be impracticable. We can however restrict ourselves to
the basis of the operators $\bar{\mathcal{O}}^{(2)}$ that do not vanish by the EOM:

$$\hat{\gamma}_{\text{phys}}^{(2)} = \left(\begin{array}{cccccc}
-\frac{1}{3}C_A & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
-\frac{1}{6}C_A - \frac{8}{3}C_F & 0 & 0 & 0 & 0 \\
0 & -C_A & 0 & -\frac{1}{2}C_A & 0 \\
-\frac{5}{6}C_A & 0 & 0 & 0 & -C_A \\
\end{array}\right). \tag{14}
$$

This reduced renormalization group equation

$$\frac{d}{d\ln \mu} \bar{\mathcal{C}}^{(2)}(\mu) + \frac{\alpha}{\pi} \hat{\gamma}_{\text{phys}}^{(2)} \bar{\mathcal{C}}^{(2)}(\mu) = 0 \tag{15}\label{15}$$

with the initial (matching) conditions

$$C_1^{(2)}(m_Q) = -1 \tag{16a}\label{16a}$$
$$C_2^{(2)}(m_Q) = 1 \tag{16b}\label{16b}$$
$$C_{11}^{(2)}(m_Q) = C_1^{(1)}(m_Q)C_1^{(1)}(m_Q) = 1 \tag{16c}\label{16c}$$
$$C_{12}^{(2)}(m_Q) = C_1^{(1)}(m_Q)C_2^{(1)}(m_Q) = 1 \tag{16d}\label{16d}$$
$$C_{22}^{(2)}(m_Q) = C_2^{(1)}(m_Q)C_2^{(1)}(m_Q) = 1 \tag{16e}\label{16e}$$

can be solved analytically

$$C_1^{(2)}(\mu) = \left(\frac{8C_F}{C_A} - \frac{7}{4}\right) \left(\frac{\alpha(\mu)}{\alpha(m_Q)}\right) - \frac{C_A}{6\beta^{(1)}} + \frac{5}{4} \left(\frac{\alpha(\mu)}{\alpha(m_Q)}\right) - \frac{C_A}{2\beta^{(1)}} - \frac{8C_F}{C_A} - \frac{1}{2} \tag{17a}\label{17a}$$

$$C_2^{(2)}(\mu) = 2 \left(\frac{\alpha(\mu)}{\alpha(m_Q)}\right) - \frac{C_A}{4\beta^{(1)}} - 1 \tag{17b}\label{17b}$$
$$C_{11}^{(2)}(\mu) = C_1^{(1)}(\mu)C_1^{(1)}(\mu) = 1 \tag{17c}\label{17c}$$
$$C_{12}^{(2)}(\mu) = C_1^{(1)}(\mu)C_2^{(1)}(\mu) = \left(\frac{\alpha(\mu)}{\alpha(m_Q)}\right) - \frac{C_A}{4\beta^{(1)}} \tag{17d}\label{17d}$$
$$C_{22}^{(2)}(\mu) = C_2^{(1)}(\mu)C_2^{(1)}(\mu) = \left(\frac{\alpha(\mu)}{\alpha(m_Q)}\right) - \frac{C_A}{2\beta^{(1)}} \tag{17e}\label{17e}$$


Flavor threshold at which $\beta^{(1)}$ changes have been ignored in [17]. It is straightforward to recover them by pasting solutions together at the thresholds.

The solution of the renormalization group equation in the full basis can be obtained numerically for specific values of the gauge parameter $\xi$ and a specific gauge group (i.e. SU(3)). This will be done for the renormalization of the currents $\nabla$.

7 Conclusions

We have presented a new calculation of the renormalized HQET Lagrangian at order $\mathcal{O}(1/m_Q^2)$. Our result corrects previous calculations and obeys the Ward identities imposed by the BRST invariance of QCD and the reparameterization invariance of HQET.

The new renormalized Lagrangian has been used in a calculation of the renormalized HQET currents at order $\mathcal{O}(1/m_Q^2)$. The results will be published in a sequel [3] to this note. A more detailed discussion of the consistency checks provided by BRST and reparameterization invariance will be presented elsewhere [7], together with technical details of the calculation of renormalized Lagrangian and currents.

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