UNCONDITIONAL UNIQUENESS IN THE CHARGE CLASS FOR
THE DIRAC-KLEIN-GORDON EQUATIONS IN TWO SPACE
DIMENSIONS

SIGMUND SELBERG AND ACHENEF TESFAHUN

Abstract. Recently, A. Grünrock and H. Pecher proved global well-posedness
of the 2d Dirac-Klein-Gordon equations given initial data for the spinor and
scalar fields in \( H^s \) and \( H^{s+1/2} \times H^{s-1/2} \), respectively, where \( s \geq 0 \), but
uniqueness was only known in a contraction space of Bourgain type, strictly
smaller than the natural solution space \( C([0, T]; H^s \times H^{s+1/2} \times H^{s-1/2}) \). Here
we prove uniqueness in the latter space for \( s \geq 0 \). This improves a recent result
of H. Pecher, where the range \( s > 1/30 \) was covered.

1. INTRODUCTION

We consider the Dirac-Klein-Gordon system (DKG) in two space dimensions,
which reads

\[
\begin{cases}
-i(\partial_t + \alpha \cdot \nabla)\psi = -M\beta\psi + \phi\beta\psi, \\
(-\Box + m^2)\phi = \langle \beta\psi, \psi \rangle, \\
(\Box = -\partial_t^2 + \Delta)
\end{cases}
\]

with initial data

\[
\psi|_{t=0} = \psi_0 \in H^s, \quad \phi|_{t=0} = \phi_0 \in H^s, \quad \partial_t\phi|_{t=0} = \phi_1 \in H^{s-1},
\]

where \( \psi(t, x) \) is the Dirac spinor, regarded as a column vector in \( \mathbb{C}^2 \), and \( \phi(t, x) \)
is real-valued. Here \( t \in \mathbb{R}, \ x \in \mathbb{R}^2; \ M, m \geq 0 \) are constants; \( \nabla = (\partial_{x_1}, \partial_{x_2}) \);
\( \langle u, v \rangle := \langle v^\dagger u \rangle_{\mathbb{C}^2} = v^\dagger u \) for column vectors \( u, v \in \mathbb{C}^2 \),
where \( v^\dagger \) is the complex conjugate transpose of \( v; \ H^s = (1 + \sqrt{-\Delta})^{-s}L^2(\mathbb{R}^2) \) is the standard Sobolev space
of order \( s \). The Dirac matrices \( \alpha^j, \beta \) satisfy

\[
\beta^\dagger = \beta, \quad (\alpha^j)^\dagger = \alpha^j, \quad \beta^2 = (\alpha^j)^2 = I, \quad \alpha^j\beta + \beta\alpha^j = 0.
\]

A particular representation is

\[
\alpha^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \alpha^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]
Local well-posedness of this problem with low-regularity data was first studied in [1]. In [3], P. D’Ancona, D. Foschi and S. Selberg proved the existence of a local solution

\[(\psi, \phi, \partial_t \phi) \in C([0, T]; H^s \times H^r \times H^{r-1})\]

for the Cauchy problem (1.1), (1.2) whenever

\[s > -\frac{1}{5}, \quad \max \left(\frac{1}{4} + \frac{|s|}{2}, \frac{3}{4}s + 2s + 1 + s\right) < r < \min \left(\frac{3}{4} + 2s, 1 + s\right),\]

but uniqueness was only obtained in the contraction space, which is strictly smaller than the natural solution space \(C([0, T]; H^s \times H^r \times H^{r-1})\). The question therefore arises whether uniqueness holds in the latter space; if this is the case, one says that \textit{unconditional uniqueness} holds at the regularity \((s, r)\).

The question of unconditional uniqueness is especially interesting for \(s \geq 0\) and \(r = s + 1/2\), since for that range it was proved by A. Grünrock and H. Pecher [5] that the local solution extends globally in time. Recently, H. Pecher [6] proved unconditional uniqueness for \(s > 1/30, r = s + 1/2\). In this paper we improve this to \(s \geq 0\).

\textbf{Theorem 1.1.} \textit{If} \(s \geq 0\), \textit{uniqueness of the solution to} (1.1), (1.2) \textit{holds in}

\[(\psi, \phi, \partial_t \phi) \in C\left([0, T]; H^s \times H^{s+1/2} \times H^{s-1/2}\right).\]

To prove this we rely on the null structure of DKG, found in [3], and on bilinear space-time estimates of Klainerman-Machedon type proved in [4]. Using also an idea of Zhou [7] we iteratively improve the known regularity of the solution, until we reach a space where uniqueness is known by the results in [3].

Some notation: In estimates we shall use \(X \lesssim Y\) as shorthand for \(X \leq CY\), where \(C \gg 1\) is a constant. We write \(X \approx Y\) for \(X \lesssim Y \lesssim X\). Throughout, \(\varepsilon\) is understood to be a sufficiently small positive number. The space-time Fourier transform of a function \(u(t, x)\) is denoted \(\mathcal{F}u(\tau, \xi) = \tilde{u}(\tau, \xi)\).

2. Null structure and bilinear estimates

To uncover the null structure of the system we follow [3] and use the Dirac projections

\[P_\pm(\xi) = \frac{1}{2} \left( I \pm \frac{\xi}{|\xi|} \cdot \alpha \right)\]
to split $\psi = \psi_+ + \psi_-$, where $\psi_{\pm} = P_{\pm}(D)\psi$. Here $D = -i\nabla$, which has Fourier symbol $\xi$. Applying $P_{\pm}(D)$ to the Dirac equation in (1.1), and using the identities $\alpha \cdot D = |D|P_+(D) - |D|P_-(D)$, $P_\pm^2(D) = P_\pm(D)$, $P_\pm(D)P_\mp(D) = 0$ and $P_\pm(D)\beta = \beta P_\pm(D)$, the Dirac equation becomes

$$(-i\partial_t \pm |D|)\psi_{\pm} = -M\beta\psi_{\mp} + P_{\pm}(D)(\phi\beta\psi).$$

On the other hand, splitting $\phi = \phi_+ + \phi_-$, where

$$\phi_{\pm} = \frac{1}{2}(\phi \pm i\langle D \rangle_{m}^{-1}\phi_0), \quad \langle D \rangle_{m} = \sqrt{m^2 + |D|^2},$$

the Klein-Gordon part of the system becomes

$$(-i\partial_t \pm \langle D \rangle_{m})\phi_{\pm} = \mp(2\langle D \rangle_{m})^{-1}\langle \beta\psi, \psi \rangle.$$

Thus, the DKG system has been rewritten as

$$(-i\partial_t \pm |D|)\psi_{\pm} = -M\beta\psi_{\mp} + P_{\pm}(D)(\phi\beta\psi),$$

$$(-i\partial_t \pm \langle D \rangle_{m})\phi_{\pm} = \mp(2\langle D \rangle_{m})^{-1}\langle \beta\psi, \psi \rangle,$$

with initial data

$$\psi_{\pm}(0) = P_{\pm}(D)\psi_0 \in H^s, \quad \phi_{\pm}(0) = \frac{1}{2}(\phi_0 \pm i\langle D \rangle_{m}^{-1}\phi_1) \in H^{s+\frac{1}{2}}.$$

Theorem 1.1 then reduces to the following.

**Theorem 2.1.** If $s \geq 0$, uniqueness of the solution to (2.1), (2.2) holds in

$$(\psi_{\pm}, \phi_{\pm}) \in C([0, T]; H^s \times H^{s+1/2}).$$

The null structure in the second equation in (2.1) is quantified by the following estimate from [3]: For Schwartz functions $\psi, \psi' : \mathbb{R}^{1+2} \to \mathbb{C}^2$ and independent signs $\pm_1$ and $\pm_2$,

$$|F(\beta P_{\pm_1}(D)\psi, P_{\pm_2}(D)\psi')(\tau, \xi)| \lesssim \int_{\mathbb{R}^{1+2}} \theta(\pm_1\eta, \pm_2(\eta - \xi))|\tilde{\psi}(\lambda, \eta)||\tilde{\psi}'(\lambda - \eta, \tau, \eta - \xi)|d\lambda d\eta,$$

where $\theta(\xi, \eta)$ denotes the angle between nonzero $\xi, \eta \in \mathbb{R}^2$. The null structure in the first equation in (2.1) is seen to be of the same type via a duality argument (see the next section). To exploit this structure we need the following estimate:
Lemma 2.2. Let \( a, b, c \in [0, \frac{1}{2}] \). For all signs \((\pm_1, \pm_2)\), all \( \lambda, \mu \in \mathbb{R} \) and all nonzero \( \eta, \zeta \in \mathbb{R}^2 \),

\[
\theta(\pm_1 \eta, \pm_2 \zeta) \lesssim \left( \frac{|\lambda - \mu| - |\eta - \zeta|}{\min(|\eta|, |\zeta|)} \right)^a + \left( \frac{|\lambda \pm_1 |\eta|}{\min(|\eta|, |\zeta|)} \right)^b + \left( \frac{|\mu \pm_2 |\zeta|}{\min(|\eta|, |\zeta|)} \right)^c.
\]

Proof. In [2, Section 5.1] it is shown that this holds when \( a = b = c = 1/2 \), and since \( \theta(\pm_1 \eta, \pm_2 \zeta) \lesssim 1 \) the lemma follows. 

Next, we define some function spaces. We remark that the symbols \( \tau \pm |\xi| \) and \( \tau \pm \langle \xi \rangle \) associated with the operators \(-i\partial_t \pm |D|\) and \(-i\partial_t \pm \langle D \rangle\) appearing in (2.1) are comparable, in the sense that \( \langle \tau \pm |\xi| \rangle \approx \langle \tau \pm \langle \xi \rangle \rangle \). Hence, the \( X_{\pm}^{s,b} \) spaces corresponding to the two operators are in fact equivalent. Specifically, the space \( X_{\pm}^{s,b} \) in question is defined (given \( s,b \in \mathbb{R} \)) as the completion of \( \mathcal{S}(\mathbb{R}^{1+2}) \) with respect to the norm

\[
\|u\|_{X_{\pm}^{s,b}} = \left\| \langle \xi \rangle^s \langle \tau \pm |\xi| \rangle^b \tilde{u}(\tau,\xi) \right\|_{L^2_{\tau,\xi}}.
\]

We also need the wave-Sobolev space \( H_{\pm}^{s,b} \) with norm

\[
\|u\|_{H_{\pm}^{s,b}} = \left\| \langle \xi \rangle^s \langle |\tau| - |\xi| \rangle^b \tilde{u}(\tau,\xi) \right\|_{L^2_{\tau,\xi}}.
\]

For \( T > 0 \), let \( X_{\pm}^{s,b}(S_T) \) and \( H_{\pm}^{s,b}(S_T) \) be the respective restriction spaces to the slab \( S_T = (0, T) \times \mathbb{R}^2 \).

Clearly, for \( b \geq 0 \),

\[
\|u\|_{H_{\pm}^{s,b}} \leq \|u\|_{X_{\pm}^{s,b}}.
\]

We also recall that for \( b > 1/2 \), \( H_{\pm}^{s,b} \) is a proper subspace of \( C(\mathbb{R}; H^s) \), and the inclusion is continuous. Thus,

\[
X_{\pm}^{s,b} \subset H_{\pm}^{s,b} \subset C(\mathbb{R}, H^s) \quad \text{for } b > 1/2.
\]

Moreover, it is well known that the initial problem

\[
 \left(-i\partial_t \pm |D| \right) u = F \in X_{\pm}^{s,b-1}, \quad u|_{t=0} = f \in H^s,
\]

for any \( s \in \mathbb{R} \) and \( b > \frac{1}{2} \), has a unique solution \( u \in C([0, T]; H^s) \), and \( u \) satisfies

\[
\|u\|_{X_{\pm}^{s,b}(S_T)} \leq C \left( \|f\|_{H^s} + \|F\|_{X_{\pm}^{s-1,b-1}(S_T)} \right),
\]

where \( C \) depends on \( b \) and \( T \). See, e.g., [3].
The final key ingredient that we need for the proof of uniqueness is the following product estimate for the spaces $H^{s,b}$.

**Theorem 2.3.** Assume

\begin{align*}
  (2.7) & \quad b_0 \leq 0 < b_1, b_2 \\
  (2.8) & \quad b_0 + b_1 + b_2 > \frac{1}{2} \\
  (2.9) & \quad b_0 + b_1 > 0, \quad b_0 + b_2 > 0 \\
  (2.10) & \quad s_0 + s_1 + s_2 > \frac{3}{2} - (b_0 + b_1 + b_2) \\
  (2.11) & \quad s_0 + s_1 + s_2 > 1 - (b_0 + b_1) \\
  (2.12) & \quad s_0 + s_1 + s_2 > 1 - (b_0 + b_2) \\
  (2.13) & \quad s_0 + s_1 + s_2 > \frac{1}{2} - b_0 \\
  (2.14) & \quad s_0 + s_1 + s_2 > \frac{3}{4} \\
  (2.15) & \quad s_0 + b_0 + 2(s_1 + s_2) > 1 \\
  (2.16) & \quad s_1 + s_2 \geq -b_0, \quad s_0 + s_2 \geq 0, \quad s_0 + s_1 \geq 0.
\end{align*}

Then the product estimate

\begin{equation}
(2.17) \quad \|uv\|_{H^{s_0 - b_0}} \lesssim \|u\|_{H^{s_1, b_1}} \|v\|_{H^{s_2, b_2}}
\end{equation}

holds for all $u, v \in S(\mathbb{R}^{1+2})$.

This follows from Theorems 4.1 and 6.1 in [4].

3. **Proof of Uniqueness**

We now prove Theorem 2.1. Without loss of generality take $s = 0$. From [3] it is known that the solution is unique in the iteration space

\[(\psi_\pm, \phi_\pm) \in X_{\pm}^{-1/4 - \frac{\delta}{2}, 0} (S_T) \times X_{\pm}^{-1/4 - \frac{\delta}{2}, 0} (S_T),\]

where we use the notation $a_\pm := a \pm \delta$ for sufficiently small $\delta > 0$. Thus, it suffices to show that if

\begin{equation}
(3.1) \quad (\psi_\pm, \phi_\pm) \in C \left( [0, T], L^2 \times H^{1/2} \right),
\end{equation}

 Then the product estimate

\begin{equation}
(2.17) \quad \|uv\|_{H^{s_0 - b_0}} \lesssim \|u\|_{H^{s_1, b_1}} \|v\|_{H^{s_2, b_2}}
\end{equation}

holds for all $u, v \in S(\mathbb{R}^{1+2})$.

This follows from Theorems 4.1 and 6.1 in [4].

3. **Proof of Uniqueness**

We now prove Theorem 2.1. Without loss of generality take $s = 0$. From [3] it is known that the solution is unique in the iteration space

\[(\psi_\pm, \phi_\pm) \in X_{\pm}^{-1/4 - \frac{\delta}{2}, 0} (S_T) \times X_{\pm}^{-1/4 - \frac{\delta}{2}, 0} (S_T),\]

where we use the notation $a_\pm := a \pm \delta$ for sufficiently small $\delta > 0$. Thus, it suffices to show that if

\begin{equation}
(3.1) \quad (\psi_\pm, \phi_\pm) \in C \left( [0, T], L^2 \times H^{1/2} \right),
\end{equation}

Then the product estimate

\begin{equation}
(2.17) \quad \|uv\|_{H^{s_0 - b_0}} \lesssim \|u\|_{H^{s_1, b_1}} \|v\|_{H^{s_2, b_2}}
\end{equation}

holds for all $u, v \in S(\mathbb{R}^{1+2})$.
is a solution of (2.1), (2.2), then

\[(\psi_{\pm}, \phi_{\pm}) \in X_{\pm}^{-\frac{1}{16}, \frac{3}{16}+}(S_T) \times X_{\pm}^{\frac{1}{16}, \frac{3}{16}+}(S_T).\]

To this end, we split \(\psi_{\pm}\) as

\[\psi_{\pm} = \psi_{\pm}^h + \psi_{\pm}^l + \Psi_{\pm},\]

where \(\psi_{\pm}^h\) is the homogeneous part while \(\psi_{\pm}^l\) and \(\Psi_{\pm}\) are the inhomogeneous parts corresponding to the linear and bilinear terms, respectively, in the right-hand side of the first equation in (2.1). Similarly, we split \(\phi_{\pm}\) as

\[\phi_{\pm} = \phi_{\pm}^h + \Phi_{\pm},\]

where \(\phi_{\pm}^h\) and \(\Phi_{\pm}\) are the homogeneous and inhomogeneous parts of \(\phi_{\pm}\).

First note that by (2.6) and assumption (3.1),

\[\psi_{\pm}^h \in X_{\pm}^{0, \frac{1}{2}+}(S_T) \quad \text{and} \quad \phi_{\pm}^h \in X_{\pm}^{\frac{1}{2}, \frac{1}{2}+}(S_T).\]

Moreover, (3.1) implies

\[\psi_{\pm} \in X_{\pm}^{0,0}(S_T),\]

so (2.6) gives

\[\psi_{\pm}^l \in X_{\pm}^{0,1}(S_T).\]

Thus it only remains to show that the pair \((\Psi_{\pm}, \Phi_{\pm})\) satisfies (3.2). To this end we start with (3.1) and use the null structure and product estimates to successively improve the regularity.

Note that by the above, \(\Psi_{\pm} = \psi_{\pm} - \psi_{\pm}^h - \psi_{\pm}^l \in X_{\pm}^{0,0}(S_T)\), and a similar argument shows that \(\Phi_{\pm} = \phi_{\pm} - \phi_{\pm}^h \in X_{\pm}^{\frac{1}{2},0}\).

### 3.1. First estimate for \(\Psi_{\pm}\).

We first claim

\[(\Psi_{\pm}) \in X_{\pm}^{-\frac{1}{2}, 0, \alpha}(S_T) \quad \text{for} \quad \alpha \in [0, 1].\]

Indeed, using (2.6),

\[
\|\Psi_{\pm}\|_{X_{\pm}^{-\frac{1}{2}, 0, \alpha}(S_T)} \lesssim \|\phi \beta \psi\|_{X_{\pm}^{\frac{1}{2}, 0}(S_T)} \lesssim \|\phi \beta \psi\|_{L_t^\infty H_x^{\frac{1}{2}}(S_T)} \lesssim T^\frac{1}{2} \|\phi\|_{L_t^{\infty} L_x^2(S_T)} \|\psi\|_{L_t^{\infty} L_x^2(S_T)},
\]

where in the last line we used the product Sobolev inequality in the $x$-variable (see, e.g., the introduction in [4]). Interpolating $\Psi_\pm \in X^{\frac{1}{2}}_{\pm} (S_T)$ with $\Psi_\pm \in X^{0,0}_{\pm} (S_T)$, we get (3.3).

3.2. First estimate for $\Phi_\pm$. We show that

(3.4) \quad \Phi_\pm \in X^{\frac{1}{2}-\varepsilon, \frac{1}{2}+\varepsilon}_{\pm} (S_T).

Using (3.3) with $\alpha = \frac{1}{4} + 2\varepsilon$ and (2.6), the claim reduces to the bilinear estimate

$$\| \langle \beta P_{\pm1} (D) \psi, P_{\pm2} (D) \psi' \rangle \|_{X^{\frac{1}{2}}_{\pm0} \rightarrow X^{\frac{1}{2}}_{\pm1}} \lesssim \| \psi \|_{X^{\frac{1}{2}}_{\pm1}} \| \psi' \|_{X^{\frac{1}{2}}_{\pm2}}.$$

Without loss of generality, take $\pm_1 = +, \pm_2 = \pm$. By Plancherel and using (2.3) we reduce to

$$I \lesssim \| u \|_{X^{\frac{1}{2}+\varepsilon, \frac{1}{2}+2\varepsilon}_{\pm}} \| v \|_{X^{\frac{1}{2}-\varepsilon, \frac{1}{2}+2\varepsilon}_{\pm}},$$

where

$$I = \left\| \iint \frac{\theta(\eta, \pm (\eta - \xi))}{(\xi)^{\frac{1}{2}+\varepsilon} (|\tau| - |\xi|)\xi^{\frac{1}{2}}} \hat{u}(\lambda, \eta) \hat{v}(\lambda - \tau, \eta - \xi) \, d\lambda \, d\eta \right\|_{L^2_{\tau,\xi}}.$$

Applying Lemma 2.2 with $a = \frac{1}{2} - \varepsilon$ and $b = c = \frac{1}{4} + 2\varepsilon$, and using also (2.5), the estimate for $I$ reduces to the following three estimates:

(3.5) \quad \begin{cases} 
\| uv \|_{H^{-\varepsilon,0} \rightarrow H^{\varepsilon,0}} \lesssim \| u \|_{H^{\frac{1}{2}+2\varepsilon, \frac{1}{2}+2\varepsilon}} \| v \|_{H^{-\varepsilon, \frac{1}{2}+2\varepsilon}} \\
\| uv \|_{H^{-\varepsilon, \frac{1}{2}+\varepsilon} \rightarrow H^{\varepsilon, \frac{1}{2}+2\varepsilon}} \lesssim \| u \|_{H^{\frac{1}{2}+\varepsilon,0}} \| v \|_{H^{-\varepsilon, \frac{1}{2}+2\varepsilon}} \\
\| uv \|_{H^{-\varepsilon, \frac{1}{2}+\varepsilon} \rightarrow H^{\varepsilon, \frac{1}{2}+2\varepsilon}} \lesssim \| u \|_{H^{-\varepsilon, \frac{1}{2}+\varepsilon}} \| v \|_{H^{\frac{1}{2}+\varepsilon, \frac{1}{2}+2\varepsilon}}. 
\end{cases}

All these hold by Theorem 2.2 (via duality for the last two), proving (3.4). Interpolating with $\Phi_\pm \in X^{\frac{1}{2},0}_{\pm}$ we get moreover

(3.6) \quad \Phi_\pm \in X^{\frac{1}{2}+\varepsilon, \frac{1}{2}+\varepsilon}_{\pm} (S_T),

which we now use to improve (3.3).

3.3. Second estimate for $\Psi_\pm$. We show that

(3.7) \quad \Psi_\pm \in X^{-\frac{1}{2}+\varepsilon, \frac{1}{2}+\varepsilon}_{\pm} (S_T).

Applying (2.6) and using (3.3) (with $\alpha = \frac{1}{2} - 2\varepsilon$) and (3.6), the claim reduces to the bilinear estimate

$$\| P_{\pm2} (D) (\phi \beta P_{\pm1} (D) \psi) \|_{X^{\frac{1}{2}, 0}_{\pm2}} \lesssim \| \phi \|_{X^{\frac{1}{2}, \varepsilon}_{\pm0}} \| \psi \|_{X^{\frac{1}{2}+\varepsilon, \frac{1}{2}+2\varepsilon}_{\pm2}}.$$
Duality reduces this to
\[ \| \langle \beta P_{\pm 1}(D)\psi, P_{\pm 2}(D)\psi' \rangle \|_{X_{\pm 0}^0 \cap \frac{1}{2}, \cdot \cdot \cdot , \frac{1}{4}} \lesssim \| \psi' \|_{X_{\pm 1}^0 \cap \frac{1}{2}, \cdot \cdot \cdot , \frac{1}{4}} \| \psi \|_{X_{\pm 2}^{t+\varepsilon, \cdot \cdot \cdot , \frac{1}{4}}}. \]
Taking again \( \pm_1 = +, \pm_2 = \pm \) without loss of generality, applying Plancherel and using (2.3), we reduce further to
\[ J \lesssim \| u \|_{X_{\pm 1}^0 \cap \frac{1}{2}, \cdot \cdot \cdot , \frac{1}{4}} \| v \|_{X_{\pm 2}^{t+\varepsilon, \cdot \cdot \cdot , \frac{1}{4}}}, \]
where
\[ J = \left\| \iint \frac{\theta(\eta, \pm(\eta - \xi))}{(\xi)^{\frac{1}{4}} + \frac{1}{4} (|\tau| - |\xi|)^{\frac{1}{4}} + \frac{1}{4}} \hat{\nu}(\tau, \eta) \hat{\nu}(\lambda - \tau, \eta - \xi) \, d\lambda \, d\eta \right\|_{L^2_{\tau, \xi}}. \]
Applying Lemma 2.2 with \( a = \frac{1}{2}, b = \frac{1}{2} - \varepsilon \) and \( c = \frac{1}{2} - 2\varepsilon \), and using (2.5), we reduce to the following six estimates:
\[ (3.8) \]
All these hold by Theorem 2.3 (via duality for the last four), proving (3.7). Interpolation with \( \Psi_\pm \in X_{\pm}^{0,0} \) yields
\[ (3.9) \]
We now use this to improve (3.4).

### 3.4. Second estimate for \( \Phi_\pm \)
We show that
\[ (3.10) \]
Applying (2.6) and using (3.9) we reduce to
\[ \| \langle \beta P_{\pm 1}(D)\psi, P_{\pm 2}(D)\psi' \rangle \|_{X_{\pm 0}^{t+\varepsilon, \cdot \cdot \cdot , \frac{1}{4}}} \lesssim \| \psi' \|_{X_{\pm 1}^0 \cap \frac{1}{2}, \cdot \cdot \cdot , \frac{1}{4}} \| \psi \|_{X_{\pm 2}^{t+\varepsilon, \cdot \cdot \cdot , \frac{1}{4}}}. \]
Proceeding as in subsection 3.2, applying Lemma 2.2 with \( a = \frac{1}{2} - \varepsilon \) and \( b = c = \frac{1}{2} + \varepsilon \), and using (2.5), we further reduce to the following three estimates:

\[
\begin{align*}
\|uv\|_{H^{-\frac{1}{4}+\varepsilon}} & \lesssim \|u\|_{H^{-\frac{1}{4}+\varepsilon}} \|v\|_{H^{-\frac{1}{4}+\varepsilon}} \\
\|uv\|_{H^{-\frac{1}{4}+\varepsilon}} & \lesssim \|u\|_{H^{-\frac{1}{4}+\varepsilon}} \|v\|_{H^{-\frac{1}{4}+\varepsilon}} \\
\|uv\|_{H^{-\frac{1}{4}+\varepsilon}} & \lesssim \|u\|_{H^{-\frac{1}{4}+\varepsilon}} \|v\|_{H^{-\frac{1}{4}+\varepsilon}}.
\end{align*}
\]

(3.11)

All these are seen to hold by Theorem 2.3 (via duality for the last two).

Interpolating (3.10) with \( \Phi_{\pm} \in X_{\pm}^{\frac{1}{4},0} \), we further get

\[
\Phi_{\pm} \in X_{\pm}^{-\frac{3}{2},\frac{3}{2}+\varepsilon}(S_T).
\]

(3.12)

3.5. Third estimate for \( \Psi_{\pm} \). We prove

\[
\Psi_{\pm} \in X_{\pm}^{-\frac{3}{2},\frac{3}{2}+\varepsilon}(S_T).
\]

(3.13)

Applying (2.6) and using (3.7) and (3.12), the claim reduces (after duality) to

\[
\|\langle \beta P_{\pm 1}(D)\psi, P_{\pm 2}(D)\psi' \rangle\|_{X_{\pm 0}^{-\frac{3}{2},\frac{3}{2}+\varepsilon}} \lesssim \|\psi\|_{X_{\pm 1}^{\frac{1}{2},\frac{3}{2}+\varepsilon}} \|\psi\|_{X_{\pm 2}^{\frac{1}{2},\frac{3}{2}+\varepsilon}}.
\]

Proceeding as in subsection 3.3, applying Lemma 2.2 with \( a = \frac{1}{2}, b = \frac{1}{2} - \varepsilon \) and \( b = \frac{1}{2}, \) and using (2.5), we reduce to

\[
\begin{align*}
\|uv\|_{H^{-\frac{1}{4}+\varepsilon}} & \lesssim \|u\|_{H^{\frac{1}{4}+\varepsilon}} \|v\|_{H^{-\frac{1}{4}+\varepsilon}} \\
\|uv\|_{H^{-\frac{1}{4}+\varepsilon}} & \lesssim \|u\|_{H^{\frac{1}{4}+\varepsilon}} \|v\|_{H^{-\frac{1}{4}+\varepsilon}} \\
\|uv\|_{H^{-\frac{1}{4}+\varepsilon}} & \lesssim \|u\|_{H^{\frac{1}{4}+\varepsilon}} \|v\|_{H^{-\frac{1}{4}+\varepsilon}} \\
\|uv\|_{H^{-\frac{1}{4}+\varepsilon}} & \lesssim \|u\|_{H^{\frac{1}{4}+\varepsilon}} \|v\|_{H^{-\frac{1}{4}+\varepsilon}} \\
\|uv\|_{H^{-\frac{1}{4}+\varepsilon}} & \lesssim \|u\|_{H^{\frac{1}{4}+\varepsilon}} \|v\|_{H^{-\frac{1}{4}+\varepsilon}} \\
\|uv\|_{H^{-\frac{1}{4}+\varepsilon}} & \lesssim \|u\|_{H^{\frac{1}{4}+\varepsilon}} \|v\|_{H^{-\frac{1}{4}+\varepsilon}}.
\end{align*}
\]

(3.14)

all of which hold by Theorem 2.3 (via duality for the last four).

3.6. Third estimate for \( \Phi_{\pm} \). We finally prove that

\[
\Phi_{\pm} \in X_{\pm}^{\frac{1}{4},\frac{3}{2}+\varepsilon}(S_T).
\]

(3.15)

By (2.6) and (3.13) we reduce to

\[
\|\langle \beta P_{\pm 1}(D)\psi, P_{\pm 2}(D)\psi' \rangle\|_{X_{\pm 0}^{-\frac{3}{2},\frac{3}{2}+\varepsilon}} \lesssim \|\psi\|_{X_{\pm 1}^{\frac{1}{2},\frac{3}{2}+\varepsilon}} \|\psi\|_{X_{\pm 2}^{\frac{1}{2},\frac{3}{2}+\varepsilon}}.
\]
Proceeding as in subsection 3.2, applying Lemma 2.2 with \( a = \frac{1}{2} - \varepsilon \) and \( b = c = \frac{1}{2} \), the estimate reduces to

\[
\begin{align*}
\|uv\|_{H^{\frac{5}{32} - 3\varepsilon, 0}} & \lesssim \|u\|_{H^{\frac{11}{32} - 4\varepsilon, \frac{1}{2} + \varepsilon}} \|v\|_{H^{\frac{5}{32} - 3\varepsilon, \frac{1}{2} + \varepsilon}} \\
\|uv\|_{H^{\frac{5}{32} - 3\varepsilon, -\frac{1}{2} + \varepsilon}} & \lesssim \|u\|_{H^{\frac{11}{32} - 3\varepsilon, 0}} \|v\|_{H^{\frac{5}{32} - 3\varepsilon, \frac{1}{2} + \varepsilon}} \\
\|uv\|_{H^{\frac{5}{32} - 3\varepsilon, -\frac{1}{2} + \varepsilon}} & \lesssim \|u\|_{H^{\frac{5}{32} - 3\varepsilon, 0}} \|v\|_{H^{\frac{11}{32} - 3\varepsilon, \frac{1}{2} + \varepsilon}}.
\end{align*}
\]

(3.16)

All these are true by Theorem 2.3.

This finishes the proof of Theorem 2.1.

REFERENCES

[1] N. Bournaveas, Low regularity solutions of the the Dirac-Klein-Gordon equations in two space dimensions, Comm. Partial Differential Equations 26(7-8) (2001), 1345–1366.

[2] P. D’Ancona, D. Foschi, and S. Selberg, Null structure and almost optimal local regularity of the Dirac-Klein-Gordon system, Journal of the EMS 9 (2007) no. 4, 877–898.

[3] P. D’Ancona, D. Foschi, and S. Selberg, Local well-posedness below the charge norm for the Dirac-Klein-Gordon system in two space dimensions, Journal of Hyperbolic Differential Equations (2007), no. 2, 295–330.

[4] P. D’Ancona, D. Foschi, and S. Selberg, Product estimates for wave-Sobolev spaces in 2+1 and 1+1 dimensions, In ”Nonlinear Partial Differential Equations and Hyperbolic Wave Phenomena”, Contemporary Mathematics, vol. 526, Amer. Math. Soc., Providence, RI, 2010, pp. 125–150.

[5] A. Grünrock and H. Pecher, Global solutions for the Dirac-Klein-Gordon system in two space dimensions, Comm. Partial Differential Equations 35 (2010), no. 1, 89-112.

[6] H. Pecher, Unconditional well-posedness for the Dirac-Klein-Gordon system in two space dimensions, Preprint, arXiv:1001.3065

[7] Y. Zhou, Uniqueness of generalized solutions to nonlinear wave equations, Amer. J. Math. 122 (2000), no. 5, 939-965