Herein, two different types of confinement in bilayer graphene by top and bottom gating with symmetrical microelectrodes are discussed and compared. Trivial confinement corresponds to the same polarity of all top gates, which is opposed to that of all bottom ones. Topological confinement requires the polarity of part of the top–bottom pairs of gates to be reversed. It is shown that the main qualitative difference between trivial and topological bound states manifests itself in the magnetic field dependence. The finding is illustrated with an explicit calculation of the energy spectrum for quantum dots and rings. Trivial confinement shows bunching of levels into degenerate Landau bands, with a non-centered gap, while topological confinement shows no field-induced gap and a sequence of states branches always crossing zero energy.

1. Introduction

The topological nature of bound states is a research question of ongoing interest.[1,2] Partly due to the many applications that these states might have for quantum computation,[3] it is crucial to propose reliable tests that help differentiate between bound states arising from trivial confinement and those arising from topological confinement. The latter show more robustness against weak disorder but it is not easy to detect their presence,[4] which is a prerequisite for their subsequent manipulation and measurement in topological quantum information tasks.

Bilayer graphene (BLG) in a Bernal stacking structure is especially a suitable platform for creating solid-state electronic qubits due to its long decoherence times.[5] Confinement in BLG is achieved by means of a pair of gates applied to the graphene sheets. In this manner, an energy gap opens around the Fermi level, which is the building block for tunnel barriers.[7,8] The only requirement is that the potential applied to a sheet has the sign opposite to that of the second sheet. This mechanism has been proven successfully in the recent years when serially connecting tunnel barriers.[9–12] This confinement is dubbed trivial because the electronic wave functions vanish asymptotically deep in the barriers. Calculations of trivial dots and rings can be found, for example, in refs. [13–18].

A totally different confinement mechanism arises when the gate potential changes sign on the same sheet (with the corresponding sign reversal on the second sheet).[19] This can be done with inhomogeneous potentials such as those occurring across a domain wall. As a consequence, there appear kink electronic states (two per valley) bounded along the wall. They are topological because their motion is along the edge and exhibit valley-momentum locking.[19–21] When the wall forms a closed loop, one can create quantum dots and rings supporting topological bound states.[19,22] In this article, we show that these states could be unambiguously detected by examining their behavior with an external magnetic field, which allows us to distinguish between both (trivial and topological) binding mechanisms.

Figure 1 shows a sketch of a BLG system with top–bottom gating defining circular nanostructures. The distribution of gates is symmetric, that is, the same for top and bottom sheets while the applied potentials $V_g$ on the top gates is sign reversed with respect to the bottom gates. This way, a potential difference is created between the two graphene layers sandwiched by a given pair of top–bottom gates. Due to the proximity of both graphene layers, much smaller than the gates separation, the effective interlayer field will be smaller than the inter-gate field, but it may be tuned in a proportional way.

We call trivial confinement the case when all gates on a given side of the BLG planes, top or bottom, have the same sign. This corresponds to an electric field always in the same direction across the BLG planes, say upward, confinement being caused by the preference of electrons to attach to the regions of low or vanishing electric field. This is indicated by the white region in Figure 1a, and in the 0 meV region of Figure 1b (trivial dot) and Figure 1c (trivial ring).

Topological confinement requires gates on a given side, top or bottom, to have potentials of different signs. This way, the BLG interlayer field changes direction creating a topological domain wall able to bind electrons. Figure 1d sketches the
situation of a topological circular ring. It would correspond to the top gates of blue and red color in Figure 1a having potentials 10 and \(-10\) meV, respectively, with a sign change in the white region. We also stress that, differently to the trivial confinement, the topological confinement does not require an extended region of vanishing field; even an abrupt domain wall Figure 1d creates bound states. This is already indicating the predominantly 1d character of topological states in BLG, as opposed to the predominant 2d character of the trivial confinement. As shown later, this results in conspicuous physical differences regarding the spectrum dependence on magnetic field.

We can now present the main finding of this work. In the presence of a perpendicular magnetic field, trivial confinement spectra show a bunching of levels indicating the emergence of the 2d physics of Landau levels in BLG. The larger the 2d region of trivial states, the smaller the finite-size spectrum that has attracted much attention is the circularly

Figure 1. a) Generic distribution of top–bottom gate pairs inducing the formation of bound states in bilayer graphene (BLG). Only the top gates are shown (red and blue), with a similar distribution of bottom gates hidden under the two graphene layers (gray planes). The potentials \(V_x, V_y\) are reversed in the bottom gates. b–e) Selected configurations for bound states in BLG nanostructures induced by the indicated gate potentials: b) trivial dot of radius \(R\), c) trivial ring of radius \(R\) and width \(w\), and d) topological rings of radius \(R\) with vanishing width and e) finite width \(w\).

2. Model

We consider a 2d \((xy)\) continuum model for the low-energy excitations of BLG, already used in our previous works, and also by many other authors (see work by Edward and Mikito and Rozhkov et al. for reviews). The Hamiltonian reads

\[
H = v_F \left( p_x - \frac{\hbar}{2\ell_x} \right) \tau_x \sigma_x + v_F \left( p_y + \frac{\hbar}{2\ell_y} \right) \sigma_y + \frac{1}{2} \left( \lambda_x \sigma_x + \lambda_y \sigma_y \right) + V_{bl}(x,y) \lambda_z
\]

(1)

Here, \(\hbar v_F = 660\) meV nm and \(t = 380\) meV are the Fermi velocity and interlayer coupling, respectively, which are BLG intrinsic parameters. The sublattice, layer and valley twofold discrete degrees of freedom of BLG are represented by the \(\sigma_{x,y,z}\), \(\lambda_{x,y,z}\), and \(\tau_{x,y,z}\) sets of Pauli matrices, respectively. The influence of a vertical magnetic field \(B\) is included by means of the magnetic length parameter \(\ell_x = \sqrt{\hbar/eB}\), in a symmetric gauge affecting the \(p_x\) and \(p_y\) operators. Notice that \(H\) contains only linear momentum terms, a characteristic of Dirac or relativistic-like Hamiltonians.

A remarkable property of BLG is that confinement to nanostructures can be achieved with the Hamiltonian of Equation (1) by space modulation of the layer-asymmetry potential \(V_{bl}(x,y)\). This is a potential imbalance between the two graphene layers that can be tuned by top and bottom gating, as sketched in Figure 1. Physically, the electrons have a preference to stay in regions where this potential imbalance is lower and this can be exploited to confine them in nanostructures whose shape is controlled by the shape of the gates. A simplest geometry of confinement that has attracted much attention is the circularly
symmetric shape, both as quantum dots and rings. Noncircular shapes have been discussed in ref. [23].

In this work, we consider a layer-asymmetry potential of circular shape, parameterized as

\[
V_a(r) = \frac{V_a^{(\text{in})}}{1 - e^{i \phi}} + \frac{V_a^{(\text{out})}}{1 - e^{i (\phi - \theta)}}
\]

where \( V_a^{(\text{in})} \) represents an inner saturation value for \( r < R \) and \( V_a^{(\text{out})} \) an outer saturation value for \( r > R + w \). The potential \( V_a(r) \) is vanishing for \( r \in [R, R + w] \) and \( s \) is a small diffusivity suggested by realistic electrostatic simulations of straight kinks,[25] which is also convenient for numerical stability. Appropriately choosing parameters \( R, w, \) and \( V_a^{(\text{in, out})} \), it is possible to model the different types of confinements sketched in Figure 1. These radial potentials can be created by disk-like microelectrodes, as suggested in Figure 1.

As mentioned in Section 1, the aim of this work is to compare two qualitatively different types of confinement in BLG: trivial confinement corresponding to saturation potentials of the same sign, \( \text{sgn}(V_a^{(\text{in})}) = \text{sgn}(V_a^{(\text{out})}) \), and topological confinement corresponding to saturation potentials of different signs, \( \text{sgn}(V_a^{(\text{in})}) \neq \text{sgn}(V_a^{(\text{out})}) \). Our parameterization allows a flexible modeling of trivial dots and rings. It also allows us to describe topological rings of zero or finite widths. Previous works have investigated trivial dots and rings as well as topological rings, but the latter only with \( w = 0.22,23 \). Here, we will also explore the case of a topological ring with a finite \( w \), where potential \( V_a(r) \) vanishes and the electrons are essentially free to move.

In the presence of a vertical magnetic field, electron states in bulk BLG, with \( V_a(r) = 0 \), are characterized by the emergence of discrete Landau levels with energies.

\[
\begin{align*}
E_0 &= 0, \\
E_1 &= 0, \\
E_{\ell \pm} &= \pm \hbar \omega_c \sqrt{\ell (\ell - 1)}, \quad \ell = 2, 3, 4 \ldots
\end{align*}
\]

There exists a twofold degenerate Landau level at zero energy and a sequence of field-dispersing levels at both positive and negative energies. The cyclotron frequency in Equation (3) is \( \omega_c = eB/m_0 \), with a mass parameter given by the BLG intrinsic parameters, \( m_0 = 1/2m_0^\ast \). The spectrum of Equation (3) is degenerate for both valleys.

In finite structures like those in Figure 1, we can expect the emergence of Landau levels for high enough fields provided that the system contains a 2d-like region, that is, a region with \( V_a = 0 \) where locally electron motion is free and the system resembles bulk BLG. We will show later that, indeed, our calculations indicate Landau level formation in trivial dots (Figure 1b), trivial rings (Figure 1c), and topological rings of finite width (Figure 1e), but not in topological rings of vanishing width (Figure 1d). The latter only contain 1d-like loop states, whose energies show \( B \)-periodic repetitions of linearly dispersing branches reflecting the Aharonov–Bohm periodicities in the flux piercing the loop.

A practical advantage of the circular symmetry is that we can define subspaces of fixed angular momentum \( m \), performing independent diagonalizations in each \( m \) subspace. Notice, first, that valley subspaces are always independent in the Hamiltonian of Equation (1), irrespectively of the spatial circular or noncircular symmetry. We can then assume \( \tau_z \equiv 1 \), with the reversed valley \( \tau_z \equiv -1 \) eigenvalues being given by symmetry arguments reversing the energy signs of the \( \tau_z \equiv 1 \) eigenvalues. In the remaining sublattice and layer subspaces, the spatial wave function for angular momentum \( m \) can be written as a four component spinor

\[
\begin{pmatrix}
\psi^{(m-1)\theta} C_4(r) \\
\psi^{\text{ind}} C_2(r) \\
\psi^{\text{ind}} C_1(r) \\
\psi^{(m+1)\theta} C_4(r)
\end{pmatrix}
\]

with \((r, \theta)\) the polar coordinates.

It can be shown that, with the spinor wave function of Equation (4) and Hamiltonian \( H \) of Equation (1), one can fully remove the \( \theta \)-dependencies and diagonalize a purely radial Hamiltonian \( H_m(r, p) \) for each angular momentum \( m \)

\[
H_m(r, p) = \psi_p \sigma_x + \hbar v_F \left( \frac{\pi^2}{2} + \frac{c^2}{2} \right) \sigma_y - \hbar v_F \sigma_z \lambda_z + \frac{\hbar^2}{2} (\lambda_x \sigma_x + \lambda_y \sigma_y) + V_a(r) \lambda_z
\]

with \( p = -i \hbar \partial/\partial r \) the radial momentum.

Hamiltonian \( H_m \) can be diagonalized for a given layer-asymmetry potential \( V_a(r) \) using finite differences in a radial grid and imposing the zero condition at the boundaries. Notice that with \( m \neq 0 \), there is a 1/r divergence at the origin in Equation (5), which is compensated by the behavior of the wave function. Numerically, this is more easily taken into account by including a finite value of \( R \) (even if quite small) as in trivial and topological rings. An important aspect to bear in mind in the grid diagonalization of Equation (5) is the possible appearance of spurious solutions due to the known problem of Fermion doubling for Dirac Hamiltonians. We have carefully considered this, filtering out spurious solutions by defining grid-average wave functions and eliminating those solutions whose norm is affected by such grid averaging.[21]

As mentioned in Section 1, we have also performed in some test cases the diagonalization of the Hamiltonian given by Equation (1) without separating in subspaces of angular momentum. However, this is much more demanding computationally and we have only checked the agreement of both methods in a few selected cases.

3. Results

Figure 2 presents selected results for trivial and topological circular systems as a function of size, in zero magnetic field. Notice that we associate the trivial or topological character to the type of confinement and, therefore, this character is shared by the whole low-energy spectrum of eigenstates associated with that particular type of confinement. A trivial ring (Figure 2a) is characterized by a conspicuous gap in the spectrum around zero energy, due to the energy quantization induced by the finite \( w \). In trivial dots or
rings with smaller \( w \)'s, this energy gap is much reduced or totally closed for large values of the radius (Figure 2b). In sharp contrast, the topological systems present a qualitatively different behavior (Figure 2c,d). Intersecting energy branches of positive and negative slopes, always crossing zero energy, is the main characteristic of the topological systems. In \( w = 0 \) topological loops (Figure 2d), the pattern of crossings is very regular and, as discussed in ref. [23], can be explained with a quantization rule for 1d closed orbits, similar to the Bohr–Sommerfeld one. The topological ring of finite width (Figure 2c) presents a remarkable behavior, simultaneously showing the zero-energy intersecting branches and also a merging of horizontal branches at energies \( \approx \pm 2 \text{meV} \), clearly reminiscent of the gap in a trivial ring of the same size (Figure 2a).

We consider next the role of a perpendicular magnetic field. As discussed earlier, in systems with regions of 2d electronic motion, there is a competition of finite size and B-field discretization into Landau levels; the latter being eventually dominant for large enough fields. Figure 3a,b shows the results for trivial rings with a large and a small width \( w \), respectively; the latter resembling a quantum dot. Figure 3 is for one valley \( (r_z = +1) \), with the spectrum for the complementary valley \( (r_z = -1) \) being given by reversing the energy signs. The energy states have an exact symmetry by inversion of both magnetic field and valley. This implies complete valley degeneracy at zero field, which is the time-reversal-invariant limit. The most noticeable feature in Figure 3 is the evolution of the energy gap position. Zero energy is the gap center for \( B = 0 \) but it evolves into a gap edge at large fields. This is a clear indication of Landau band discretization. For instance, with positive magnetic fields, the bulk Landau gap \([0, \sqrt{2} \hbar \omega_c] \) at 1.5 T is \([0, 7.5 \text{meV}] \), in good qualitative agreement with the results of Figure 3. This figure also shows how for the same value of \( w \) the spectrum of a ring with a larger \( R \) contains more bands and a cleaner merging into Landau levels at large fields.

The spectra for topological systems in magnetic field are shown in Figure 4. As anticipated in Section 1, the topological loop of zero width (Figure 4a) shows no signs of Landau level physics. Instead, its level spectrum is a sequence of almost parallel branches with similar B-slopes. Figure 4 is for one valley, the reversed valley having similar branches but with the opposite B-slopes. The case of a topological loop of finite width (Figure 4b) is again most remarkable (as in Figure 2c), showing the mentioned almost parallel branches and, also, signatures of
Landau level discretization. The latter is hinted by the level bunching around zero energy for large (positive and negative) fields, as well as by the new branches emerging at large (positive and negative) energies in Figure 4b.

4. Discussion and Conclusions

We have discussed two types of confinement in BLG nanostructures induced by top and bottom gating. Trivial and topological confinement differ in the potential signs applied to the gates. The regions with zero layer-asymmetry potential correspond to locally free electronic motion.

Confronting Figure 3 and 4, we observe the sharp differences in the eigenvalue spectra of trivial and topological systems. The $B$-increasing gap (for a given valley), with one gap edge pinned at $E = 0$, is a characteristic that permits to intrinsically differentiate trivial and topological states in BLG systems. Landau level physics requires a 2d region of vanishing layer-asymmetry potential, where electron motion is locally free. On the contrary, topological rings of vanishing width behave as purely 1d loops and do not show signatures of Landau level formation. Instead, they manifest $B$-periodicities indicating Aharonov–Bohm physics reminiscent of the flux periodicities of rings built with metals or semiconductor 2d electron gases.

At $B = 0$, trivial rings and dots have an energy gap centered around zero energy, larger for the case of rings than for dots. Switching on a magnetic field, this gap evolves into the mentioned non-centered gap of Landau level physics. Quite remarkably, topological rings of finite width manifest both Aharonov–Bohm periodicities and signatures of Landau level discretization in magnetic field.

Our numerical estimates suggest that the magnetic spectrum of both confinement types could be detected using today’s experimental techniques, which would represent a significant step toward topological quantum computation in graphene systems.

Figure 3. Spectra as a function of perpendicular magnetic field of trivial rings (Figure 1c) with a) $R = 50$ nm, $w = 50$ nm; b) $R = 10$ nm, $w = 50$ nm. Panel (b) is similar to a trivial dot. We used the same $V_{in}^{(n)}$, $V_{out}^{(n)}$, and $s$ of Figure 2.

Figure 4. Spectra of topological rings (Figure 1d,e) with a) $R = 50$ nm, $w = 0$ nm; b) $R = 50$ nm, $w = 50$ nm. We used the same $V_{in}^{(n)}$, $V_{out}^{(n)}$, and $s$ of Figure 2.
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Conflict of Interest
The authors declare no conflict of interest.

Data Availability Statement
The data that support the findings of this study are available from the corresponding author upon reasonable request.

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bilayer graphene, low dimensional systems, magnetic effects, topological states

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