Entanglement in a Dimerized Antiferromagnetic Heisenberg Chain

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Abstract

The entanglement properties in an antiferromagnetic dimerized Heisenberg spin-1/2 chain are investigated. The entanglement gap, which is the difference between the ground-state energy and the minimal energy that any separable state can attain, is calculated to detect the entanglement. It is found that the entanglement gap can be increased by varying the alternation parameter. Through thermal energy, the witness of the entanglement can determine a characteristic temperature below that an entangled state can be obtained. The entanglement detected by the energy can provide a lower bound for that determined by the concurrence. If the alternation parameter is smaller than a critical value, there is always no inter-dimer entanglement in the chain.

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I. INTRODUCTION

The quantum entanglement is considered as key resources of quantum information processing [1, 2]. The property of entanglement plays an essential role in understanding and quantifying the physical systems [3-9]. In recent years, some useful measures of entanglement have been proposed. The entropy of entanglement is used to qualify the entanglement of pure states [3] while the entanglement of formation is one measure for mixed states [4]. The entanglement in optical systems has been theoretically analyzed and observed [5-9]. For further developing the experimental detection, the separability criterion [10, 11] was suggested. On this basis, the thermal energy [12, 13], the magnetization [14] and the susceptibility [15] have been used as entanglement witness [16] to detect the entanglement. If the expected value of the witness is negative, an entangled state could be obtained. These measurements can provide an intuitive way to evaluate the quantum entanglement. With the help of these measurements, it is possible to find some entangled systems suitable for quantum computation and quantum communication [17]. Recently, much interest has been focused on the entanglement in solid-state systems [18-27]. The spin chain is one kind of entangled systems in condensed matters. The entanglement has been observed experimentally in Heisenberg spin chains [18]. The dependence of entanglement on the external magnetic field [19] and anisotropy [20] has been investigated. The frustration [21, 22] and arbitrary spin-s Heisenberg chain [23, 24] were considered. Some interesting effects were discussed concerning the relation of entanglement with correlation [25] and quantum phase transition [26, 27]. The dimerized spin chain is another essential sort of spin models in many real solids. It is often used as a model to explain thermodynamical properties of many substances [28]. It is interesting to note that there is the same structure of phase diagrams in the antiferromagnetic dimerized Heisenberg spin-1/2 and uniform spin-1 chains [29]. The model belongs to quasi one-dimensional magnets of spin-ladder systems with even number of coupled spin chains [30]. Therefore, it is necessary to evaluate and detect the quantum entanglement in alternating spin chains.

In this paper, the entanglement in an antiferromagnetic dimerized Heisenberg spin chain is investigated. In Sec. II, the energy is introduced as a witness for the detection of the dimer entanglement. The characteristic temperature for the presence of the dimer entanglement is derived. The entanglement gap is introduced to evaluate the entanglement for a chain
with large number of spin dimers. The upper bound to the entanglement gap is derived. In Sec. III, the relation of the entanglement witness and concurrence is deduced. As an example, the four-spin dimerized chain with two dimers is analyzed in detail. The limit cases are discussed when the alternation parameter equals to zero or one. A brief discussion concludes the paper.

II. WITNESS FOR DIMER ENTANGLEMENT

The Hamiltonian of the dimerized Heisenberg spin chain can be written as \[ H = J \sum_{i=1}^{L} [1 - (-1)^i \delta] \vec{S}_i \cdot \vec{S}_{i+1} \] (1)

where \( \vec{S}_i = \frac{1}{2} \vec{\sigma}_i \) is the \( i \)th spin vector. The number of spins \( L \) is even and the periodic boundary condition \( L + 1 = 1 \) is assumed. At temperature \( T \), the thermal equilibrium state is \( \rho = \exp(-H\beta)/Z \) where \( Z = \text{Tr}[\exp(-H\beta)] \) is the partition function and \( \beta = 1/kT \). For the convenience, the Boltzmann constant \( k \) is assumed to be one. The values of the exchange coefficient \( J > 0 \) and \( J < 0 \) correspond to the antiferromagnetic and ferromagnetic cases respectively. The parameter \( \delta \) denotes the alternating ratio of exchange interactions. There is an equivalent expression of the Hamiltonian \[ H = J' \sum (\vec{S}_{2i-1} \cdot \vec{S}_{2i} + \alpha \vec{S}_{2i} \cdot \vec{S}_{2i+1}) \] with \( J = J' \frac{1+\alpha}{2} \) and \( \delta = \frac{1-\alpha}{1+\alpha} \) where \( \alpha \) is the alternation parameter. Such dimerized Heisenberg spin chain is schematically illustrated in Fig. 1. The spin chain is constructed by the dimers of the number \( d = L/2 \). Thus, the Hamiltonian can be described as \( H = \sum_{d=1}^{L/2} H_d \).

In Fig. 1, the spin dimer is labelled by an elliptical box. The solid line represents the dimer interaction \( J' \) and the dash line denotes the inter-dimer interaction \( \alpha J' \). In the following discussions, the antiferromagnetic case of \( J' = 1 \) and \( 0 \leq \alpha \leq 1 \) is considered. It is easily found that the ground state at \( \alpha = 0 \) is a tensor product of a singlet state. When \( \alpha \neq 0 \), the ground state cannot be expressed as a tensor product of each dimer state. For the simplest case of \( d = 2 \), the ground state at \( \alpha = 1 \) can be written as \( |\psi_0\rangle = \frac{1}{\sqrt{12}}[(|1100\rangle + |0011\rangle + |1001\rangle + |0110\rangle) - 2(|1010\rangle + |0101\rangle)] \). Here \( |1\rangle, |0\rangle \) are assumed to be the eigenstates of the pauli operator \( \sigma^z \) with the eigenvalues \( \pm 1 \). Therefore, it is clear that there exists the transition of the ground state \( |\psi_0\rangle \) if \( \alpha \) is varied from zero.

On the basis of the separability criterion, there exists a minimal separable energy \( E_{sep} \) \[12, 13\]. That is, \( E_{sep} \) is the minimal energy that any separable state can arrive at. For
the antiferromagnetic dimerized Heisenberg chain, the pure separable state for the minimal energy can be analyzed by the standard symmetry methods [24, 31] and expressed as $|\psi_{sep}\rangle = \frac{1}{2^L} \prod_{i=1}^{L} (|0\rangle_i \otimes |0\rangle_{i+1} + |1\rangle_i \otimes |1\rangle_{i+1})$. The minimal separable energy is $E_{sep} = \langle \psi_{sep}|H|\psi_{sep}\rangle = -\frac{L(1+\alpha)}{8}$. For the detection of the dimer entanglement, the entanglement gap $g_E$, which is the difference between the minimal separable energy $E_{sep}$ and the ground-state energy $E_0$, can be introduced. The upper bound of the entanglement gap can be derived. The lower bound to the ground state energy of the Hamiltonian

$$H = J' \sum (\vec{S}_{2i-1} \cdot \vec{S}_{2i} + \alpha \vec{S}_{2i} \cdot \vec{S}_{2i+1})$$

(2)
can be estimated. By the variation principle, one can show that the ground state energy of the Hamiltonian $H$ can be written as

$$E_0(H) \geq E_0(H_1) + \alpha E_0(H_2).$$

(3)

Where $E_0(H_1)$ is the ground state energy of the Hamiltonian $H_1 = \sum (\vec{S}_{2i-1} \vec{S}_{2i}) = \vec{S}_1 \vec{S}_2 + \vec{S}_3 \vec{S}_4 + \ldots$, while $E_0(H_2)$ is the ground state energy of the Hamiltonian $H_2 = \sum (\vec{S}_{2i} \vec{S}_{2i+1}) = \vec{S}_2 \vec{S}_3 + \vec{S}_4 \vec{S}_5 + \ldots$. Since

$$\vec{S} \vec{S} = \frac{1}{4} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

(4)

one has $E_0(H_1) = E_0(H_2) = -\frac{3}{8} L$. Thus for the ground state $E_0(H) \geq -\frac{3}{8} L(\alpha + 1)$. The expression of the entanglement gap $g_E$ and its upper bound can be given by

$$g_E = \frac{E_{sep} - E_0}{L} \leq \frac{1}{4}(\alpha + 1).$$

(5)

If the entanglement gap is nonzero, the entanglement could be detected experimentally below a certain temperature. To some degree, the entanglement gap can be regarded as a useful indicator of the existence of entanglement. The entanglement at a finite temperature can be detected more easily if the entanglement gap is large [12]. For the dimerized chain with a large number of dimers, the ground-state energy $E_0$ can be numerically calculated. The dependence of the entanglement gap on the alternation parameter is plotted in Fig. 2. It is shown that the entanglement gap is decreased with the increase of the alternation $\alpha$ and reaches a minimal value at about $\alpha = 0.7$, and then increased again until $\alpha = 1.0$. Here
the dimer interaction is chosen as \( J' = 1 \). It is found that the entanglement gap \( g_E \) can be increased when \( \alpha \neq 0.7 \). Meanwhile, the entanglement witness at finite temperature \( T \) can be introduced as \([12, 13]\)

\[
W = E - E_{\text{sep}}
\]

where \( E = \sum_d \text{Tr}(\rho_H) \) is the energy at the thermal state \( \rho \). If the value \( W < 0 \), an entangled state is obtained. The value of \( W > 0 \) means that it is a separable (unentangled) state. The entanglement witness \( W \) as a function of the temperature \( T \) and the alternating parameter \( \alpha \) is plotted in Fig. 3. Figure 3(a) is a three-dimensional plot of the entanglement witness \( W \) as a function of \( \alpha \) and \( T \). From Fig. 3(a), it is seen that the value of \( W \) is increased to a maximum and then decreased when \( \alpha \) is increased. The value of \( W \) is increased from negative to the positive when the temperature \( T \) is increased. That is, the thermal states are changed from the entangled states to the separable states as \( T \) is increased. The contour of \( W \) as a function of the characteristic temperature \( T_c \) and the alternation parameter \( \alpha \) is illustrated in Fig. 3(b). The value of \( T_c \) is located on the contour map of \( W = 0 \). The contour map of \( W < 0 \) corresponds to the entangled thermal states. It is found that both entanglement-detecting methods are almost equivalent to each other.

III. RELATION OF ENTANGLEMENT WITNESS AND CONCURRENCE

Although some of the real solids is composed of a large number of spins, many properties, like thermal and magnetic properties can be efficiently studied by the model of small number of spins. It is necessary to analyze the entanglement properties of the dimerized chain with small number of dimers. In the Hilbert space \( \{|11\}, |10\}, |01\}, |00\} \), the reduced density matrix of one dimer \( \rho_d \) can be written as

\[
\rho_d = \begin{pmatrix}
  u_d & 0 & 0 & 0 \\
  0 & w_d & t_d & 0 \\
  0 & t_d & w_d & 0 \\
  0 & 0 & 0 & u_d
\end{pmatrix}
\]

where the elements are expressed as \( u_d = \frac{1}{4} + \frac{1}{12} \text{Tr}[\rho_d \sigma \cdot \sigma] \), \( w_d = \frac{1}{4} - \frac{1}{12} \text{Tr}[\rho_d \sigma \cdot \sigma] \) and \( t_d = \frac{1}{6} \text{Tr}[\rho_d \sigma \cdot \sigma] \). The concurrence of the system is given by \( C = \max\{0, 2\lambda_1 - \sum_i \lambda_i\} \) where \( \lambda_i \) are the square roots of eigenvalues of the matrix \( R = \rho(\sigma^y \otimes \sigma^y)\rho(\sigma^y \otimes \sigma^y) \) in
decreasing order and $\lambda_1$ is the maximum one [4]. From the definition of the concurrence $C$, the dimer entanglement $C$ of the alternating Heisenberg chain can be calculated as $C = 2 \max \{0, |u_d| - |t_d|\} = \frac{1}{\pi} \max \{0, -\text{Tr} [\rho_d \vec{\sigma} \cdot \vec{\sigma}] - 1\}$. Here $\text{Tr} [\rho_d \vec{\sigma} \cdot \vec{\sigma}] < 0$ is the correlation function which is monotonically increased with the temperature $T$ [23]. According to Eq. (6), the witness $W$ can also be expressed by the correlation function

$$W = \frac{L(1 + \alpha)}{8} \left[ 1 + \text{Tr} [\rho_d \vec{\sigma} \cdot \vec{\sigma}] \right]$$

Thus, the relation of the entanglement witness $W$ and concurrence $C$ can be expressed as

$$C \geq \max \{0, -\frac{4W}{(1 + \alpha)L}\}.$$  

Here the equality can be achieved if the alternation parameter $\alpha = 0$ or 1. The item of $\max \{0, -\frac{4W}{(1 + \alpha)L}\}$ denotes the entanglement witnessed by the thermal energy. From Eq. (9), the detection of the entanglement can offer a lower bound for the concurrence $C$.

In the limit case of $\alpha = 0$ or $\alpha = 1$, the dimerized Heisenberg spin chain is reduced to a single dimer or an isotropic Heisenberg chain. For the case of $\alpha = 1$, the concurrence for any one dimer is just the entanglement between two nearest neighboring spins. It can be written by $C_{i,i+1}(\alpha = 1) = \frac{1}{2} \max \{0, -K_{i,i+1} - 1\}$, where $K_{i,i+1} = \text{Tr} [\rho_i \vec{\sigma}_i \cdot \vec{\sigma}_{i+1}]$ is the correlation function for any two nearest neighboring spins. According to Eq. (6), the entanglement witness can be expressed by $W(\alpha = 1) = E + \frac{L}{4}$. Here the minimal separable energy is $E_{\text{sep}} = -\frac{L}{4}$. For such an isotropic spin chain, the thermal energy is $E(\alpha = 1) = \sum_i^L \text{Tr} [\rho_i \vec{S}_i \cdot \vec{S}_{i+1}] = \frac{L}{4} K_{i,i+1}$. It is easily seen that the concurrence $C_{i,i+1}(\alpha = 1) = \max \{0, -\frac{4W}{L}\}$ [24]. When $\alpha = 0$, it is easily seen that there is no inter-dimer interaction, i. e., the correlation function $K_{2i,2i+1}(\alpha = 0) = 0$. The correlation function $K_{2i-1,2i}$ can still exist in the chain. The entanglement witness for $\alpha = 0$ can also be written by $W(\alpha = 0) = \sum_i^{L/2} \text{Tr} [\rho_{2i-1} \vec{S}_{2i-1} \cdot \vec{S}_{2i}] + \frac{L}{8} = \frac{L}{8} (K_{2i-1,2i} + 1)$. Similarly, the concurrence for a single dimer is $C_{2i-1,2i}(\alpha = 0) = \max \{0, -\frac{1}{2}(K_{2i-1,2i} + 1)\} = \max \{0, -\frac{4W}{L}\}$. Therefore, the equality in Eq. (9) is obtained when the alternation parameter is in the limit of $\alpha = 0$ or $\alpha = 1$.

As a simple example, the antiferromagnetic dimerized Heisenberg chain with two dimers is analyzed. Because the Hamiltonian satisfies $[H, \sum_i S_i^z] = 0$, the total spin number $S$ is
conserved. The eigenvalues $E_i$ of the Hamiltonian can be written as

$$E_1 = E_2 = E_3 = E_{10} = E_{13} = \epsilon_1 = (1 + \alpha)/2,$$

$$E_4 = E_9 = E_{14} = \epsilon_2 = -(1 + \alpha)/2,$$

$$E_5 = E_7 = E_{15} = \epsilon_3 = (1 - \alpha)/2,$$

$$E_6 = E_8 = E_{16} = \epsilon_4 = -(1 - \alpha)/2,$$

$$E_{11} = \epsilon_5 = -(1 + \alpha)/2 + \sqrt{\alpha^2 - \alpha + 1},$$

$$E_{12} = \epsilon_6 = -(1 + \alpha)/2 - \sqrt{\alpha^2 - \alpha + 1},$$

The corresponding eigenstates $|\psi_i\rangle$ can also be expressed by

$$|\psi_1\rangle = |1111\rangle; |\psi_2\rangle = |0000\rangle;$$

$$|\psi_3\rangle = \frac{1}{2}((|0111\rangle + |1101\rangle + |1011\rangle + |1110\rangle); |\psi_4\rangle = \frac{1}{2}(|0111\rangle + |1101\rangle - |1011\rangle - |1110\rangle);$$

$$|\psi_5\rangle = \frac{1}{2}(|0111\rangle - |1101\rangle + |1011\rangle - |1110\rangle); |\psi_6\rangle = \frac{1}{2}(|0111\rangle - |1101\rangle - |1011\rangle + |1110\rangle);$$

$$|\psi_7\rangle = \frac{1}{\sqrt{2}}(|0011\rangle - |1100\rangle); |\psi_8\rangle = \frac{1}{\sqrt{2}}(|1001\rangle - |0110\rangle); |\psi_9\rangle = \frac{1}{\sqrt{2}}(|1010\rangle - |0101\rangle);$$

$$|\psi_{10}\rangle = \frac{1}{\sqrt{6}}(|1010\rangle + |0101\rangle) + |1001\rangle + |0110\rangle + |1100\rangle + |0011\rangle;$$

$$|\psi_{11}\rangle = \frac{1}{\sqrt{2}}(x_+ (|1100\rangle + |0011\rangle) + y_+ (|1010\rangle + |0101\rangle) + |1001\rangle + |0110\rangle);$$

$$|\psi_{12}\rangle = \frac{1}{\sqrt{2}}(x_- (|1100\rangle + |0011\rangle) + y_- (|1010\rangle + |0101\rangle) + |1001\rangle + |0110\rangle);$$

$$|\psi_{13}\rangle = \frac{1}{2}(|1000\rangle + |0010\rangle + |0100\rangle + |0001\rangle); |\psi_{14}\rangle = \frac{1}{2}(|1000\rangle + |0010\rangle - |0100\rangle - |0001\rangle);$$

$$|\psi_{15}\rangle = \frac{1}{2}(|1000\rangle - |0010\rangle + |0100\rangle - |0001\rangle); |\psi_{16}\rangle = \frac{1}{2}(|1000\rangle - |0010\rangle - |0100\rangle + |0001\rangle);$$

where the parameters of $x_\pm$ and $y_\pm$ are given by $x_\pm = -(1 - \alpha) \pm \sqrt{\alpha^2 - \alpha + 1}$ and $y_\pm = -\alpha \pm \sqrt{\alpha^2 - \alpha + 1}$. At finite temperature $T$, the correlation function $K = Tr(\rho_4 \vec{\sigma} \cdot \vec{\sigma})$ can be calculated as

$$K = -3 + \frac{1}{Z}(20e^{-\epsilon_1\beta} + 6e^{-\epsilon_2\beta} + 12e^{-\epsilon_3\beta} + 6e^{-\epsilon_4\beta} + \frac{6x_-^2}{1 + x_-^2 + y_-^2}e^{-\epsilon_5\beta} + \frac{6x_+^2}{1 + x_+^2 + y_+^2}e^{-\epsilon_6\beta}) \quad (12)$$

From Eqs. (8)-(12), the concurrence $C$ and the entanglement witness $W$ of one dimer can be easily calculated.

It is also known that there is inter-dimer interaction in the antiferromagnetic dimerized Heisenberg spin-1/2 chain. The concurrence $C_{2i,2i+1}$ between two spins $2i$ and $2i+1$ denotes
the inter-dimer entanglement. It is induced from the the inter-dimer exchange coupling $\alpha J'$. The reduced density matrix between any two spins has the same structure as that given by Eq. (7). Therefore, the entanglement $C_{i,j}$ between any two spins $i$ and $j$ is also dependent on the correlation function $K_{i,j}$, which can be expressed by

$$C_{i,j} = \frac{1}{2} \max\{0, -K_{i,j} - 1\}.$$  (13)

The inter-dimer correlation function is $K_{2i,2i+1} = \text{Tr}[\rho_{2i,2i+1} \bar{\sigma} \cdot \bar{\sigma}]$, where $\rho_{2i,2i+1}$ is the reduced density matrix for inter-dimer two spins. It is clearly seen that there is the inter-dimer entanglement if the correlation function $K_{2i,2i+1} < -1$. For such spin chain, the correlation function between any two spins is increased with the increase of the temperature [23]. Thus, the inter-dimer entanglement of the concurrence $C_{2i,2i+1}$ is always decreased from that of the ground state. When the correlation for the ground state $K^0_{2i,2i+1} \geq -1$, there is no inter-dimer entanglement at any temperature $T$. As an example, for the case of $L = 4$, the ground state is given by the state $|\psi_{12}\rangle$. The inter-dimer correlation function for the ground state is obtained by $K^0_{2,3} = -3(1 - \frac{2}{1+x^2+y^2})$. In the limit of $\alpha = 0$, $K^0_{2,3} = 0$, there is no inter-dimer entanglement. When the value of $\alpha$ is increased from zero, the value of inter-dimer correlation function is also decreased from zero. If $\alpha > 0.5$, $K^0_{2,3} < -1$, the inter-dimer entanglement exists. In the limit of $\alpha = 1$, $K^0_{2,3} = -2$, the inter-dimer entanglement reaches the maximum value. The inter-dimer entanglement of the concurrence $C_{2,3}$ can be calculated by the correlation function of $K^0_{2,3}$. By numerical calculations, it is found that there is no inter-dimer entanglement when the alternation parameter is below a critical value of $\alpha_c$. The critical alternation parameter $\alpha_c$ is plotted in Fig. 4(a) as a function of spin number $L$. It is seen that the value of $\alpha_c$ increases and then saturates to a constant of $\alpha_c^* \sim 0.78$ at $L = 12$. To show the saturation effect, the characteristic temperature $T_c$ of $W = 0$ is numerically calculated as a function of the alternation parameter $\alpha$ and is plotted in Fig. 4(b) when the number of spins $L$ is varied. It is seen that the value $T_c$ of $L = 10$ (the bullet) keeps almost the same value as that of $L = 12$ (the triangle) for relatively large value of the alternation parameter $\alpha$.

It is very interesting to note that the entanglement between two spins depends on the separable distance $|j-i|$ in the antiferromagnetic dimerized Heisenberg spin-1/2 chain. This can be illustrated by the correlation function $K^0_{1,j}$ between two spins 1 and $j$ for the ground state. For the chain with spin number $L = 12$, the correlation function $K^0_{1,j} = \text{Tr}[\rho^0_{1,j} \bar{\sigma} \cdot \bar{\sigma}]$
is numerically calculated and plotted in Fig. 5 as a function of separable distance $|j - 1|$. Here $\rho_{1,j}^{0}$ is the reduced density matrix between spins 1 and $j$ for the ground state. In Fig. 5, the dashed horizontal line denotes the value of $K_{1,j}^{0} = -1$. Below the horizontal line, the entanglement exists. From Fig. 5, it is clear that the values of the correlation function is $K_{1,j}^{0} > -1$ when the separable distance is $|j - 1| > 2$. According to the relation of the correlation function and the concurrence, the concurrence $C_{1,j}$ is always zero for $|j - 1| > 2$. In the Heisenberg chain, only $K_{1,2}^{0}$ is smaller than $-1$ while the others $K_{1,j(j>2)}^{0} > -1$. The entanglement always declines from the ground state at $T = 0$. Therefore, at any temperature $T$, there is no entanglement between two spins $i$ and $j$ when the separable distance is $|j - i| > 2$ in the antiferromagnetic dimerized Heisenberg spin-1/2 chain.

IV. DISCUSSION

The entanglement in the antiferromagnetic dimerized Heisenberg spin-1/2 chain is analyzed. The entanglement gap $g_{E}$ can be increased if the alternation parameter $\alpha \neq 0.7$. If there is a large number of dimers in the chain, the characteristic temperature $T_{c}$ determined by the entanglement witness is decreased to a constant value. The entanglement can be detected below a certain temperature in real solids. The relation of the witness and the concurrence is also derived. The energy as the entanglement witness can provide a lower bound for the dimer entanglement. In the limit of $\alpha = 0$ or 1, both of the witness and the concurrence are equivalent to detect the entanglement. It is also found that there is no inter-dimer entanglement when the alternation parameter is smaller than a critical value. These results may be helpful for the further study of solid-state quantum communication and computation.

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**Figure Captions**

**Fig. 1.**
The antiferromagnetic alternating Heisenberg spin-1/2 chain can be described by spin dimers. The spin dimer is labelled by an elliptical box. The solid line represents the dimer interaction and the dash line denotes the inter-dimer interaction.

**Fig. 2.**
The entanglement gap $g_E$ is plotted as a function the alternation parameter. The minimal gap is located at about $\alpha = 0.7$.

**Fig. 3.**
The witness $W$ is plotted as a function of the alternation parameter $\alpha$ and the temperature $T$.

(a). The three dimensional plot of $W$ as a function of $\alpha$ and $T$.

(b). The contour plot of $W$ on the plane of $(T - \alpha)$.

**Fig. 4.**
The characteristic alternation parameter $\alpha_c$ and temperature $T_c$ are plotted.

(a). $\alpha_c$ is plotted as a function of the number of spins $L$.

(b). $T_c$ is plotted as a function of $\alpha$ with $L = 6(\circ), 10(\bullet), 12(\triangle)$.

**Fig. 5**
The correlation function of the ground state $K_{1,j}^0$ is plotted when the spin separable distance $|j - 1|$ is increased from 1 to $L - 1$. Here spin number $L = 12$ and the alternation parameter $\alpha$ is varied from $\alpha = 0.3(\circ)$ to $\alpha = 0.8(\bullet)$ to $\alpha = 1(\triangle)$. The dash line denotes the value $-1$. 


