Confinement without a center: the exceptional group $G(2)$

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We discuss theories with the exceptional centerless gauge group $G(2)$, paying attention to confinement and the pattern of chiral symmetry breaking. Exploiting the Higgs mechanism to break the symmetry down to $SU(3)$, we also present how the familiar features of confinement and chiral symmetry breaking of $SU(3)$ gauge theories reemerge. $G(2)$ gauge theories show up as an unusual theoretical framework to study $SU(3)$ gauge theories without the “luxury” of a center.

1. Introduction and motivations

In the last few years, accumulating numerical evidence of the relevance of center vortices in the effective mechanism of confinement in non–Abelian gauge theories has been collected. Center vortices and twist sectors$^i$ are present in a pure gauge theory with symmetry group $G$ if $\Pi_1(G/\text{center}(G)) \neq 0$, $\Pi_1$ being the first homotopy group. The exceptional group $G(2)$ is the simplest group without center vortices and twist sectors. Thus, it is interesting to investigate how confinement can show up in a theory with such a gauge group. Moreover, a property making $G(2)$ particularly interesting is that it has $SU(3)$ as a subgroup. In our study we have focused our attention on the way confinement and the pattern of chiral symmetry breaking show up in $G(2)$ gauge theories and how they change into the more familiar $SU(3)$ case as the symmetry gets broken.

2. $G(2)$: basic generalities

$G(2)$ is a subgroup of $SO(7)$. Its fundamental representation (rep) $\{7\}$ is 7 dimensional and a matrix $\Omega$ satisfies the following constraints:

$$\det \Omega = 1 \quad ; \quad \delta_{ab} = \delta_{a'b'} \Omega_{aat} \Omega_{bb'}$$

$$T_{abc} = T_{a'b'c'} \Omega_{aat} \Omega_{bb'} \Omega_{cc'} \quad ; \quad T = \text{antisym.}$$

In addition to the two defining $SO(7)$ properties$^e$, $G(2)$ leaves invariant a completely antisymmetric three-index tensor $T$ and is generated by 14 of the 21 $SO(7)$ generators. $G(2)$ has rank 2 and so its reps can be drawn on a plane. For instance, this is the diagram of the fundamental one $\{7\}$:

![Figure 1. The weight diagram of the 7-dimensional fundamental representation of $G(2)$.

$G(2)$ has $SU(3)$ as a subgroup in the real reducible rep $\{3\} \oplus \{\bar{3}\} \oplus \{1\}$. In a suitable basis, 8 of the 14 $G(2)$ generators can be written in the following way

$$\Lambda_a = \frac{1}{\sqrt{2}} \begin{pmatrix} \lambda_a & 0 & 0 \\ 0 & -\lambda^*_a & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$
and splits up into the sum $\Pi_3(G(2)) = \mathbb{Z} \oplus \mathbb{Z}$, and two kinds of monopoles $\Pi_2(G(2)/U(1)^2) = \mathbb{Z}^2$.

3. $G(2)$ Yang–Mills

Let us first consider $G(2)$ Yang–Mills (YM) theory. There are 14 $G(2)$ “gluons” which transform according to the adjoint rep $\{14\}$. Restricting to the $SU(3)$ subgroup, this rep is reducible and splits up into the sum $\{8\} \oplus \{3\} \oplus \{\overline{3}\}$. Thus, 8 of the 14 $G(2)$ “gluons” are related among themselves as gluons and the remaining 6 $G(2)$ “gluons” form a triplet and an antitriplet. These last 6 $G(2)$ “gluons” behave like quarks and antiquarks w.r.t. the $SU(3)$ color degrees of freedom. The $G(2)$ YM theory is asymptotically free and we expect it to be confining in the low energy regime but with a vanishing string tension (defined as the slope of the heavy quark potential when the distance goes to infinity), since the string breaks by pair production of dynamical $G(2)$ “quarks”. So, contrary to $SU(3)$ YM theory, in $G(2)$ YM theory we expect confinement to resemble more closely that in QCD but without the complications related to dynamical fermions. Due to the screening of $G(2)$ “quarks” by $G(2)$ “gluons”, the Wilson loop is not a good order parameter for confinement. At $T = 0$, one can consider the Fredenhagen–Marcu operator $\hat{B}$ as an order parameter for confinement. By strong coupling computation, one obtains confining behaviour in this regime. $SU(N)$ YM theory has a deconfinement phase transition at finite temperature. Confined and deconfined phases differ by the way the center symmetry is realized. In $G(2)$ YM, since the center is trivial, it is unclear how to define an order parameter to investigate the issue of a finite temperature phase transition.

4. $G(2)$ Yang–Mills + Higgs $\{7\}$

Let us now break the $G(2)$ gauge symmetry to $SU(3)$. This can be accomplished by adding to the $G(2)$ YM theory a Higgs field in the fundamental rep $\{7\}$ of $G(2)$. The 8 $G(2)$ “gluons” related among themselves as gluons stay massless while the remaining 6 get a mass proportional to the v.e.v. of the Higgs field. If the mass of these $G(2)$ “gluons” is not too high, they participate in the dynamics but, as the v.e.v. of the Higgs field increases, they progressively decouple and, in the end, we are left with an $SU(3)$ gauge theory. Thus, a Higgs field in the rep $\{7\}$ gives us a handle to smoothly interpolate between $G(2)$ and $SU(3)$. The breaking of the string between two heavy $G(2)$ “quarks” happens for the pair production of these 6 massive $G(2)$ “gluons” and so the breaking scale is related to their mass. As this mass increases, the breaking scale gets larger as well. When the 6 massive $G(2)$ “gluons” completely decouple, it is sent to infinity and we recover the familiar picture of the unbreakable $SU(3)$ string.

5. $G(2)$ QCD

Let us now add to the $G(2)$ YM theory $N_f$ flavours of Majorana fermions in the fundamental rep $\{7\}$. In this way, we obtain a theory like QCD but with gauge group $G(2)$. As above, we will exploit the Higgs mechanism to smoothly interpolate between $G(2)$ and $SU(3)$. When we break the $G(2)$ symmetry to $SU(3)$, the rep $\{7\}$ reduces to the sum of $\{3\}, \{\overline{3}\}$ and a color singlet. Thus, reexpressing the Majorana degrees of freedom in the following way

$$\begin{pmatrix}
\psi^{(1)}_M \\
\psi^{(2)}_M \\
\psi^{(3)}_M \\
\psi^{(4)}_M \\
\psi^{(5)}_M \\
\psi^{(6)}_M \\
\psi^{(7)}_M
\end{pmatrix}
= \begin{pmatrix}
\psi^{(1)}_D + i\psi^{(4)}_M \\
\psi^{(2)}_D + i\psi^{(3)}_M \\
\psi^{(3)}_D - i\psi^{(2)}_M \\
\psi^{(4)}_D - i\psi^{(1)}_M \\
\psi^{(5)}_D + i\psi^{(6)}_M \\
\psi^{(6)}_D - i\psi^{(5)}_M \\
\psi^{(7)}_M
\end{pmatrix}$$

we see that, by the Higgs mechanism, we are interpolating between a $G(2)$ QCD-like theory with $N_f$ Majorana fermions and QCD with $N_f$ Dirac quark flavours plus one Majorana fermion. However, this last particle is an $SU(3)$ color singlet and so does not feel the $SU(3)$ strong interactions. We now consider the issue of the pattern
fundamental bosons, one for every broken generator. As we start from the QCD case – i.e. we assume a very small value of the v.e.v. of the Higgs field becomes large. Let us discuss separately the cases \( N_f = 1 \) and \( N_f \geq 2 \).

\* \( N_f = 1 \). Consider first the case of one flavour. In QCD the baryon number symmetry is \( U(1)_{L=+} \). It stays unbroken and there is no Goldstone particle. In \( G(2) \) QCD, due to the reality of the \( G(2) \) reps, \( G(2) \) “quarks” and \( G(2) \) “antiquarks” are indistinguishable. Left (\( L \)) and right (\( R \)) components do not transform independently but \( L = R^* \). So the baryonic \( U(1)_{L=R} \) symmetry of QCD becomes \( U(1)_{L=R=R^*} = Z_B(2) \) in \( G(2) \) QCD and the number of \( G(2) \) “quarks” is conserved only modulo two. Thus we have two kinds of states: those with an odd (\( uGGG \), \( uuu \), ...) and those with an even (\( uu \), ...) number of \( G(2) \) “quarks” \( u \), bound or not with \( G(2) \) “gluons” \( G \). If we now add the Higgs field to the dynamics, \( 6 \) \( G(2) \) “gluons” become massive and one can show that the states \( uGGG \) start to become heavy. The mixing between \( G(2) \) “quarks” and \( G(2) \) “antiquarks” – which is mediated by the massive \( G(2) \) “gluons” – becomes weaker and baryon number violating processes are rare. Then the \( U(1)_{L=R} = U(1)_B \) symmetry of QCD reemerges as an approximate symmetry, becoming exact when the 6 massive \( G(2) \) “gluons” decouple from the dynamics.

\* \( N_f \geq 2 \). Let us now consider the case of two or more flavours. The Abelian part of the chiral symmetry is that discussed in the \( N_f = 1 \) case, so in the following we will only take into account the non–Abelian part. In QCD the pattern of chiral symmetry breaking is \( SU(N_f)_L \otimes SU(N_f)_R \rightarrow SU(N_f)_{L=R} \) with \( (N_f^2 - 1) \) Goldstone bosons. Again, in \( G(2) \) QCD, left and right components are not independent and are related by \( L = R^* \). So the unbroken chiral symmetry is \( SU(N_f)_{L=R^*} \) which breaks down to the vector subgroup \( SU(N_f)_{L=R} = SO(N_f) \). As a consequence, there are \( N_f(N_f+1)/2 - 1 \) Goldstone bosons, one for every broken generator. As before, we add a Higgs field in the fundamental rep \( \{ 7 \} \) in order to smoothly interpolate between \( G(2) \) QCD and QCD. It is now better to start from the QCD case – i.e. we assume a very large v.e.v. for the Higgs field – and move to \( G(2) \) QCD by decreasing this value. \((N_f - 1)\) of the \( (N_f^2 - 1) \) Goldstone bosons of QCD are self-conjugate (e.g. \( \pi^0 \) for \( N_f = 2 \)) and the remaining \( 2[N_f(N_f - 1)/2 - 1] \) ones are pairwise conjugate (e.g. \( \pi^+ - \pi^- \)) under charge conjugation. The odd states are invariant under \( SO(N_f) \) and so acquire a mass as the v.e.v. of the Higgs field becomes smaller and smaller, that is as the broken symmetry reduces from \( SU(N_f)_{L=R} \) to \( SU(N_f)_{L=R=R^*} \). The other \((N_f - 1) + N_f(N_f - 1)/2 = N_f(N_f + 1)/2 - 1 \) ones stay instead massless.

6. Conclusions

We have studied \( G(2) \) gauge field theories with and without fermions. We have focused our attention on confinement and the pattern of chiral symmetry breaking. Adding a Higgs field in the fundamental rep \( \{ 7 \} \), we can smoothly interpolate between \( G(2) \) and \( SU(3) \). In particular, we have discussed how the familiar confinement and pattern of chiral symmetry breaking of \( SU(3) \) gauge theories reemerge as the \( G(2) \) gauge symmetry gets broken. In conclusion, we have considered \( G(2) \) theories as a theoretical laboratory to study \( SU(3) \) gauge theories in an unusual context and without the “luxury” of a center. In a forthcoming paper, we will report on our study with more details. In that paper, we will also discuss the \( G(2) \) YM theory with 1 Majorana fermion flavour in the adjoint rep \( \{ 14 \} \) (SUSY–\( G(2) \)) and present analytic results in the strong coupling approximation to support our heuristic investigations.

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