Poly-analytic Functions and Representation Theory

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Abstract
We propose the Lie-algebraic interpretation of poly-analytic functions in $L_2(\mathbb{C}, d\mu)$, with the Gaussian measure $d\mu$, based on a flag structure formed by the representation spaces of the $\mathfrak{sl}(2)$-algebra realized by differential operators in $z$ and $\bar{z}$. Following the pattern of the one-dimensional situation, we define poly-Fock spaces in $d$ complex variables in a Lie-algebraic way, as the invariant spaces for the action of generators of a certain Lie algebra. In addition to the basic case of the algebra $\mathfrak{sl}(d+1)$, we consider also the family of algebras $\mathfrak{sl}(m_1 + 1) \otimes \ldots \otimes \mathfrak{sl}(m_n + 1)$ for tuples $m = (m_1, m_2, \ldots, m_n)$ of positive integers whose sum is equal to $d$.

Keywords Fock space · Poly-analytic functions · Lie algebras

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1 Introduction

The paper deals with some aspects of the theory of the so-called poly-analytic functions (see e.g. [1,3,8] and the literature cited therein). Recall that, in case of one complex variable, the poly-analytic functions of order $k$ are those that satisfy the iterated Cauchy-Riemann equation.
One of the important and interesting questions here is as follows. Given a domain $\Omega \subset \mathbb{C}^d$, $d \geq 1$, consider the corresponding (weighted) Hilbert space $L_2(\Omega, d\mu)$. How the poly-analytic functions are located inside $L_2(\Omega, d\mu)$, and whether and how the Hilbert space $L_2(\Omega, d\mu)$ can be made of the sets of poly-analytic function of different orders (see e.g. [9,19–21]). This question is more challenging in the case of several complex variables, as by now there is no commonly accepted understanding how to define poly-analytic functions in this situation.

In the paper our Hilbert space is the standard $L_2(\mathbb{C}^d, d\mu)$, $d \geq 1$, with the Gaussian measure. The essence of our results is that the spaces of poly-analytic functions (both in one- and multidimensional cases) can be identified with the invariant subspaces under the action of certain Lie algebras.

In Sect. 3 we consider in detail the one-dimensional case of [20, Section 2] and show that for each $k \in \mathbb{N}$ the poly-analytic space of order $k$, $k$-poly-Fock space $F^2_k(\mathbb{C})$, can be alternatively defined as the common invariant subspace for the action the operators $J^+_k, J^0_k, J^-_k$ in $L_2(\mathbb{C}, d\mu)$,

$$
J^+_k = \left( \frac{\partial}{\partial z} - \frac{\partial}{\partial \overline{z}} \right) \left( \overline{z} \frac{\partial}{\partial z} - \frac{\partial^2}{\partial z \partial \overline{z}} - (k - 1)I \right),
$$

$$
J^0_k = \overline{z} \frac{\partial}{\partial z} - \frac{\partial^2}{\partial z \partial \overline{z}} - \frac{k-1}{2}I,
$$

$$
J^-_k = \frac{\partial}{\partial \overline{z}},
$$

cf. [17, Formulas (A.1.5)], obeying the $\mathfrak{sl}(2)$-algebra commutation relations.

It turns out that the situation of Sect. 3 is in fact a particular case of a general construction based on the Fock space formalism extended to the case of infinite dimensional space generated by vacuum vectors. These results are presented in Sect. 2.

In final Sect. 4 we define poly-Fock spaces in $d$ complex variables as the representation spaces for $\mathfrak{sl}(d + 1)$ (Sect. 4.1), and for $\mathfrak{sl}(m_1 + 1) \otimes \cdots \otimes \mathfrak{sl}(m_n + 1)$ for a given tuple $\mathbf{m} = (m_1, m_2, \ldots, m_n)$ of natural numbers whose sum is equal to $d$, $m_1 + m_2 + \ldots + m_n = d$ (Sect. 4.2).

### 2 Fock Space Formalism

Introduce the three-dimensional Heisenberg algebra $\mathbb{H}_3 = \{a, b, 1\}$ with commutator $[a, b] = 1$ and $[a, 1] = [b, 1] = 0$.

A formal construction of the Fock space, see e.g. [6, Chapter 5, Section 5.2], is given by the representation of the algebra $\mathbb{H}_3$ on a separable Hilbert space $\mathcal{H}$. That is, there are operators $a$ and $b$ acting in $\mathcal{H}$ that satisfy the above commutation relations, and there also is a normalized element $|0\rangle := \Phi_0 \in \mathcal{H}$, $\|\Phi_0\| = 1$, called the vacuum vector, such that $a\Phi_0 = 0$ and the linear span of elements $b^n\Phi_0$ with $n \in \mathbb{Z}_+$ is dense in $\mathcal{H}$. The Hilbert space $\mathcal{H}$ obeying the above properties can be called the Fock space related to $a$ and $b$, or $(a, b)$-Fock space.
The problems of complex analysis related to the study of the so-called poly-analytic functions, i.e., those smooth functions that satisfy the equation
\[ \left( \frac{\partial}{\partial \bar{z}} \right)^{k} f = 0, \quad k \in \mathbb{N}, \]
motivate us to deal with a more general situation: There is a space \( L_1 \) spanned by vacuum vectors, i.e., \( ah = 0 \), for all \( h \in L_1 \), and the linear span of all elements from all spaces \( L_n := b^{n-1}L_1, n \in \mathbb{N} \) is dense in \( \mathcal{H} \). Note that generically the elements from different spaces \( L_n \) are not orthogonal. The previous situation corresponds to the case when \( L_1 \) is one-dimensional and is generated by \( \Phi_0 \).

Again, motivated by the study of the poly-analytic functions, we define the \( k \)-poly-\( \alpha \)-space \( \mathcal{H}_k \) as the closure of all elements of \( \mathcal{H} \) that satisfy the equation
\[ \alpha^k h = 0. \] (2.1)

Easily verified relation
\[ [a, b^{n-1}] = (n - 1)b^{n-2} \] (2.2)
implies that for each \( h_1 \in L_1 \) the element \( h_n \equiv b^{n-1}h_1 \) belongs to \( \mathcal{H}_n \). This follows from
\[
\alpha^k h_n = \alpha^k b^{n-1} h_1 = \alpha^{k-1} \left( ab^{n-1} \right) h_1 = \alpha^{k-1} \left( b^{n-1}a + (n - 1)b^{n-2} \right) h_1 \\
= (n - 1)\alpha^{k-1}b^{n-2}h_1 = \ldots = (n - 1)\ldots(n - (k - 1))ab^{n-k}h_1
\]
for \( k = n \). Thus \( \mathcal{H}_k = \text{closure}(L_1 + L_2 + \ldots + L_k) \). Note that (2.1) implies
\[ b^k \alpha^k h = 0. \] (2.3)

Observation 2.1 In the case when \( b = \alpha^\dagger \) is adjoint to \( \alpha \), considered hereinafter, statements (2.1) and (2.3) are equivalent. Indeed, \( (\alpha^\dagger)^k \alpha^k h = 0 \), together with \( h \neq 0 \), implies
\[
0 = \langle (\alpha^\dagger)^k \alpha^k h, h \rangle = \langle \alpha^k h, \alpha^k h \rangle = \| \alpha^k h \|^2
\]
yielding \( \alpha^k h = 0 \).

By induction on \( k \) one can show, see [15], that
\[
b^k \alpha^k = \prod_{m=0}^{k-1} (ba - m),
\]
where the right hand side is a product of the so-called Euler-Cartan operators

\[ I_0^{(m)} = ba - m, \quad m = 0, 1, \ldots, k - 1. \]

This reveals a Lie-algebraic nature of the $k$-poly-$\alpha$-space $\mathcal{H}_k$. Indeed, consider three (generically unbounded densely defined in $\mathcal{H}$) operators, see [17, Formulas (A.1.5)]

\[
J^+_k = b^2 a - (k - 1) b, \\
J^0_k = ba - \frac{k-1}{2}, \\
J^-_k = a.
\] (2.4)

For all $k \in \mathbb{C}$ these operators obey the $sl(2)$-algebra commutation relations

\[ [J^-_k, J^+_k] = 2J^0_k \quad \text{and} \quad [J^\pm_k, J^0_k] = \mp J^\pm_k. \]

Restricting $k$ to positive integers, it is easily seen that the space

\[ V_k := L_1 + L_2 + \ldots + L_k \]

is invariant under the action of the operators $J^+_k, J^0_k$ and $J^-_k$.

So the $k$-poly-$\alpha$-spaces $\mathcal{H}_k$ can be defined alternatively as the closure of the invariant subspaces $V_k$ for the action the operators $J^+_k, J^0_k, J^-_k$ in $\mathcal{H}$, obeying the $sl(2)$-algebra commutation relations.

The situation becomes much more transparent and substantial in case when the operators $a, b$ are identified with the lowering and raising operators $a$ and $a^\dagger$, i.e., when additionally the operator $a^\dagger$ is adjoint to $a$.

Before we proceed with its description let us give three examples, starting with a very classical one.

**Example** (See e.g. [6, Chapter 5, Section 5.2]) Take $\mathcal{H} = L_2(\mathbb{R})$, choose $a = \frac{1}{\sqrt{2}}(x + \frac{d}{dx}), \quad a^\dagger = \frac{1}{\sqrt{2}}(x - \frac{d}{dx})$ as the lowering and raising operators, and the vacuum vector $\Phi_0 = (\pi)^{1/4}e^{-x^2/2}$ being the ground state of the harmonic oscillator. The linear span of functions $(a^\dagger)^n \Phi_0 = c_n H_n(x) e^{-x^2/2}, \quad n \in \mathbb{Z}_+$, with an appropriate constant $c_n$, is dense in $L_2(\mathbb{R})$, and the $k$-poly-$\alpha$-space $\mathcal{H}_k$ coincides with the finite dimensional space of weighted Hermite polynomials $H^k_p(x)$,

\[ e^{-x^2/2} \sum_{p=0}^{k-1} a_p H_p(x). \]

**Example** (See e.g. [6, Chapter 5, Section 5.2]) Take $\mathcal{H} = F_2(\mathbb{C})$ being the space of all anti-analytic functions $f(\bar{z})$ endowed with the scalar product

\[ \langle f, g \rangle = \frac{1}{\pi} \int_{\mathbb{C}} f(\bar{z})g(\bar{z})e^{-|z|^2} dx dy, \quad \bar{z} = x - iy. \]

Then $a = \frac{\partial}{\partial z}, \quad a^\dagger = \bar{z}, \quad \text{and} \quad \Phi_0 = 1$. The $k$-poly-$\alpha$-spaces $\mathcal{H}_k$ coincides with the finite dimensional space of polynomials on $\bar{z}$ of degree not greater that $k - 1$. 
Example Take \( \mathcal{H} = L_2(\mathbb{C}, d\mu) \) of square-integrable functions on \( \mathbb{C} \) with the Gaussian measure
\[
d\mu(z) = \pi^{-1} e^{-z \bar{z}} dv(z), \quad dv(z) = dx dy.
\]
Then \cite[Formula (2.4)]{20}, take \( a = \frac{\partial}{\partial z}, \ a^\dagger = -\frac{\partial}{\partial z} + \bar{z} \), while the role of the vacuum vector \( \Phi_0 \) can be played by any normalized analytic function from \( L_2(\mathbb{C}, d\mu) \). This example is considered in more detail in the next section.

So we are now in the following setup. There is a separable Hilbert space \( \mathcal{H} \), two mutually adjoint lowering and raising operators \( a \) and \( a^\dagger \) having common domain dense in \( \mathcal{H} \), and a linear subspace \( L_1 \) being the kernel of \( a \). Furthermore, the operators \( a \) and \( a^\dagger \) obey the relation
\[
[a, a^\dagger] = I, \tag{2.5}
\]
and the union of all spaces \( L_n := (a^\dagger)^{n-1}L_1 \) is dense in \( \mathcal{H} \).

Then the \( k \)-poly-a-space \( \mathcal{H}_k \), defined through Eq. \eqref{2.1}, coincides with closure \( (L_1 + L_2 + \ldots + L_k) \), where \( L_n = (a^\dagger)^{n-1}L_1 \). A simple application of the relation \( [a, (a^\dagger)^n] = n(a^\dagger)^{n-1} \) yields the following proposition.

**Proposition 2.2** Different subspaces \( L_n \) and \( L_m \) are orthogonal. For each \( k = 2, 3, \ldots \), the raising operator
\[
\frac{1}{\sqrt{k-1}} a^\dagger |_{\mathcal{T}_{k-1}} : \mathcal{T}_{k-1} \rightarrow \mathcal{T}_k
\]
is an isometric isomorphism, and the lowering operator
\[
\frac{1}{\sqrt{k-1}} a |_{\mathcal{T}_k} : \mathcal{T}_k \rightarrow \mathcal{T}_{k-1},
\]
is its inverse. Here \( \mathcal{T}_n := \text{closure}(L_n) \).

**Proof** Take \( h_n = (a^\dagger)^{n-1}h_1 \in L_n, \ g_m = (a^\dagger)^{m-1}g_1 \in L_m \), and let \( n > m \). Then
\[
\langle h_n, g_m \rangle = \langle (a^\dagger)^{n-1}h_1, (a^\dagger)^{m-1}g_1 \rangle = \langle (a^\dagger)^{n-m-1}h_1, a^m (a^\dagger)^{m-1}g_1 \rangle = \langle (a^\dagger)^{n-m-1}h_1, (m-1)! a g_1 \rangle = 0.
\]
Take now \( h_{k-1} = (a^\dagger)^{k-2}h_1 \in L_{k-1} \) and calculate
\[
\left\langle \frac{1}{\sqrt{k-1}} a^\dagger h_{k-1}, \frac{1}{\sqrt{k-1}} a^\dagger h_{k-1} \right\rangle = \frac{1}{k-1} \langle (a^\dagger)^{k-1}h_1, (a^\dagger)^{k-1}h_1 \rangle \\
= \frac{1}{k-1} \langle (a^\dagger)^{k-2}h_1, a (a^\dagger)^{k-1}h_1 \rangle = \frac{1}{k-1} \langle (a^\dagger)^{k-2}h_1, (k-1)(a^\dagger)^{k-2}h_1 \rangle \\
= \langle h_{k-1}, h_{k-1} \rangle.
\]
That is, the operator \( \frac{1}{\sqrt{k-1}} a^\dagger \) is an isometric isomorphism of \( L_{k-1} \) onto \( L_k \), which extends by continuity onto the closure of these spaces.
Finally, take \( h_k = (a^\dagger)^{k-1} h_1 \in L_k \) and calculate

\[
\left\langle \frac{1}{\sqrt{k-1}} ah_k, \frac{1}{\sqrt{k-1}} a h_k \right\rangle = \frac{1}{k-1} \langle a(a^\dagger)^{k-1} h_1, a(a^\dagger)^{k-1} h_1 \rangle \\
= \frac{1}{k-1} \langle (a^\dagger)^{k-1} h_1, a^\dagger a(a^\dagger)^{k-1} h_1 \rangle \\
= \frac{1}{k-1} \langle (a^\dagger)^{k-1} h_1, (k-1)(a^\dagger)^{k-1} h_1 \rangle = \langle h_k, h_k \rangle,
\]

and the result follows. \( \square \)

The proposition implies that the \( k \)-poly-a-space \( \mathcal{H}_k \) admits the representation

\[
\mathcal{H}_k = L_k \oplus L_{k-1} \oplus \cdots \oplus L_1 = L_k \oplus \mathcal{H}_{k-1}.
\]

Following the notion introduced in the theory of the poly-analytic spaces [19,20], we define the true-\( k \)-poly-a-space \( \mathcal{H}_{(k)} \) as

\[
\mathcal{H}_{(k)} := \mathcal{H}_k \ominus \mathcal{H}_{k-1} = (a^\dagger)^{k-1} \text{closure}(L_1) \quad \text{and} \quad \mathcal{H}_{(1)} := \mathcal{H}_1 = \text{closure}(L_1),
\]

in other words

\[
\mathcal{H}_k = \bigoplus_{\ell=1}^k \mathcal{H}_{(\ell)} \quad \text{with} \quad \mathcal{H}_{(\ell)} = (a^\dagger)^{\ell-1} \mathcal{H}_1. \tag{2.6}
\]

Iterating the statements of Proposition 2.2 we come to the following

**Corollary 2.3** For each \( k \in \mathbb{N} \), the operator

\[
A_{(k)} := \frac{1}{\sqrt{(k-1)!}} (a^\dagger)^{k-1}|_{\mathcal{H}_1} : \mathcal{H}_1 \rightarrow \mathcal{H}_{(k)}, \tag{2.7}
\]

gives an isometric isomorphism between \( \mathcal{H}_1 \) and the true-\( k \)-poly-a-space \( \mathcal{H}_{(k)} \), and the operator

\[
A_{(k)}^{-1} := \frac{1}{\sqrt{(k-1)!}} a^{k-1}|_{\mathcal{H}_{(k)}} : \mathcal{H}_{(k)} \rightarrow \mathcal{H}_1, \tag{2.8}
\]

is an inverse isomorphism.

Thus, the true-\( \ell \)-poly-a-spaces are nothing but the components of the orthogonal decomposition (2.6) of the \( k \)-poly-a-spaces \( \mathcal{H}_k \), each component is isomorphic to the closure of the space of vacuum vectors and is obtained by the “lifting up” this closure by the normalised powers of the raising operator \( a^\dagger \). Furthermore, the density in \( \mathcal{H} \) of the union of all spaces \( L_k \) implies
Proposition 2.4  The following direct sum decomposition of $\mathcal{H}$ holds

$$\mathcal{H} = \bigoplus_{k=1}^{\infty} \mathcal{H}(k).$$

In the classical case of one-dimensional space $L_1$, generated by a vacuum vector $\Phi_0$, the statement of proposition is nothing but the well known fact that the infinite system of elements $\frac{1}{\sqrt{(k-1)!}} (a^\dagger)^{k-1} \Phi_0, k \in \mathbb{N}$ forms an orthonormal basis on $\mathcal{H}$.

The isomorphisms of Corollary 2.3 imply the following isomorphisms

$$\mathcal{H} \cong \ell_2(\mathbb{N}, \mathcal{H}_1) = \ell_2(\mathbb{N}, \mathbb{R}) \otimes \mathcal{H}_1,$$

$$\mathcal{H}_k \cong \mathcal{H}_1 \oplus \mathcal{H}_1 \ldots \oplus \mathcal{H}_1 = \mathbb{R}^k \otimes \mathcal{H}_1.$$  \hspace{1cm} (2.9)

An equivalent to Proposition 2.4 statement can be formulated in terms of the $k$-poly-a-spaces $\mathcal{H}_k$ as follows.

Corollary 2.5  The set of $k$-poly-a-subspaces $\mathcal{H}_k, k \in \mathbb{N}$, of the space $\mathcal{H}$ forms an infinite flag in $\mathcal{H}$

$$\mathcal{H}_1 \subset \mathcal{H}_2 \subset \ldots \subset \mathcal{H}_k \subset \ldots \subset \mathcal{H},$$  \hspace{1cm} (2.10)

and

$$\mathcal{H} = \bigcup_{k=1}^{\infty} \mathcal{H}_k.$$

The generators (2.4) of the Lie algebra $\mathfrak{sl}(2)$ in the case (2.5) have the form [17]

$$J^+_k = (a^\dagger)^2 a - (k - 1) a^\dagger,$$

$$J^0_k = a^\dagger a - \frac{k-1}{2} I,$$

$$J^-_k = a,$$$$

(2.11)

and they act on elements $h(\ell) \in \mathcal{H}(\ell)$ as follows

$$J^+_k h(\ell) = \begin{cases} 
(a^\dagger (\ell - k) h(\ell) \in \mathcal{H}(\ell+1), & \ell \neq k, \\
0, & \ell = k \end{cases}, \quad J^0_k h(\ell) = \frac{2\ell-k-1}{2} h(\ell),$$

$$J^-_k h(\ell) = \begin{cases} 
(a h(\ell) \in \mathcal{H}(\ell-1), & \ell > 1 \\
0, & \ell = 1 \end{cases},$$

which implies that for positive integer $k$ the $k$-poly-a-space $\mathcal{H}_k$ is a common invariant subspace for the action of $J^+_k, J^0_k,$ and $J^-_k$, where they act boundedly.

We describe now the isomorphic isomorphism (2.9) in more detail.
Observation 2.6 By (2.6) each element \( h_k \in \mathcal{H}_k \) can be represented as a sum of its mutually orthogonal components \( h_k = h_{(1)} + h_{(2)} + \ldots + h_{(k)} \) from different true-poly-a-spaces \( \mathcal{H}_\ell \), and which we will thus identify with a column vector \( h_k = (h_{(1)}, h_{(2)}, \ldots, h_{(k)})^T \). Each element \( g \in \mathbb{R}^k \otimes \mathcal{H}_1 \) we also represent as a column vector \( g = (g_1, g_2, \ldots, g_k)^T \). Then the isomorphism \( A_k : \mathbb{R}^k \otimes \mathcal{H}_1 \to \mathcal{H}_k \) is given by \( h_k = A_k g \), in matrix form

\[
\begin{bmatrix}
  h_{(1)} \\
  h_{(2)} \\
  \vdots \\
  h_{(k)}
\end{bmatrix} =
\begin{bmatrix}
  A_{(1)} & 0 & \cdots & 0 \\
  0 & A_{(2)} & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \cdots & A_{(k)}
\end{bmatrix}
\begin{bmatrix}
  g_1 \\
  g_2 \\
  \vdots \\
  g_k
\end{bmatrix},
\]

the inverse isomorphism \( A_k^{-1} : \mathbb{R}^k \otimes \mathcal{H}_1 \to A_k \) is given by

\[
\begin{bmatrix}
  g_1 \\
  g_2 \\
  \vdots \\
  g_k
\end{bmatrix} =
\begin{bmatrix}
  A_{(1)}^{-1} & 0 & \cdots & 0 \\
  0 & A_{(2)}^{-1} & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \cdots & A_{(k)}^{-1}
\end{bmatrix}
\begin{bmatrix}
  h_{(1)} \\
  h_{(2)} \\
  \vdots \\
  h_{(k)}
\end{bmatrix},
\]

where the operators \( A_\ell \) and \( A_\ell^{-1} \) are given by (2.7) and (2.8) respectively.

Observation 2.7 We note that unless the dimension of the vacuum vector \( \mathcal{H}_1 \) space is one, the action of the operators (2.11) on \( \mathcal{H}_k \) is reducible. The general form of all closed invariant subspaces for this action is as follows. Take any closed subspace \( \tilde{\mathcal{H}} \) of \( \mathcal{H}_1 \), then the space \( A_k(\mathbb{R}^k \otimes \tilde{\mathcal{H}}) \subset \mathcal{H}_k \) is invariant for the action of (2.11). The space \( \mathcal{H}_k \) is maximal (under inclusion) among all those invariant subspaces.

Given a Hilbert space \( H \), we denote by \( \mathcal{L}(H) \) the set of all bounded linear operators acting on \( H \).

Observation 2.6 implies that there is one-to-one correspondence between elements of \( \mathcal{L}(\mathcal{H}_k) \) and elements of

\[
\mathcal{L}(\mathbb{R}^k \otimes \mathcal{H}_1) = \mathcal{L}(\mathbb{R}^k) \otimes \mathcal{L}(\mathcal{H}_1) = \text{Mat}_k(\mathbb{R}) \otimes \mathcal{L}(\mathcal{H}_1) = \text{Mat}_k(\mathcal{L}(\mathcal{H}_1)),
\]

i.e., each element \( T \in \mathcal{L}(\mathcal{H}_k) \) is unitarily equivalent to a certain element \( \tilde{T} \in \mathcal{L}(\mathbb{R}^k \otimes \mathcal{H}_1) \), which are related by \( T = A_k \tilde{T} A_k^{-1} \) or \( \tilde{T} = A_k^{-1} T A_k \). That is, each bounded linear operator acting on \( \mathcal{H}_k \) is unitarily equivalent to a \( (k \times k) \)-matrix operator with entries from \( \mathcal{L}(\mathcal{H}_1) \).

It is a matter of a simple calculation to see that the operators \( J^+_k, J^0_k, \) and \( J^-_k \), acting on \( \mathcal{H}_k \), are unitarily equivalent to the following operators, acting on \( \mathbb{R}^k \otimes \mathcal{H}_1 \),

\[
\tilde{J}^+_k = M^+_k \otimes I, \quad \tilde{J}^0_k = M^0_k \otimes I, \quad \tilde{J}^-_k = M^-_k \otimes I,
\]
where the \((k \times k)\)-matrices \(M^+_k\), \(M^0_k\), and \(M^-_k\) are given by

\[
M^+_k = -\begin{bmatrix}
0 & 0 & 0 & \cdots & 0 & 0 \\
k - 1 & 0 & 0 & \cdots & 0 & 0 \\
0 & \sqrt{2}(k - 2) & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \sqrt{k - 1} & 0
\end{bmatrix},
\]

\[
M^0_k = \begin{bmatrix}
\frac{1-k}{2} & 0 & \cdots & 0 \\
0 & \frac{3-k}{2} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \frac{k-1}{2}
\end{bmatrix},
\]

\[
M^-_k = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & \sqrt{2} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \sqrt{k - 1} \\
0 & 0 & 0 & \cdots & 0
\end{bmatrix},
\]

and form the \((k \times k)\)-matrix representation of the generators of the algebra \(\mathfrak{sl}(2)\). Note that the matrices \(M^+_k\), \(M^0_k\), and \(M^-_k\) have no non-trivial common invariant subspace in \(\mathbb{R}^k\). Thus they generate the algebra \(\text{Mat}_k(\mathbb{R})\), equivalently, each \((k \times k)\)-matrix is a polynomial of \(M^+_k\), \(M^0_k\), and \(M^-_k\). Note that by dimension reasoning there can not be more then \(k^2\) linearly independent such polynomials. At the same time there are \(k^2\) linearly independent polynomials \(P_{m,n}(M^+_k, M^0_k, M^-_k) = E_{m,n}\) realizing matrix-units of \((k \times k)\)-matrix. This leads us to the following proposition.

**Proposition 2.8** Each bounded linear operator acting on \(\mathbb{R}^k \otimes \mathcal{H}_1\) can be uniquely represented in the form

\[
\tilde{T} = \sum_{n,m=1}^{k} P_{m,n}(\tilde{J}^+_k, \tilde{J}^0_k, \tilde{J}^-_k) S_{m,n},
\]

where \(S_{m,n} \in \mathcal{L}(\mathcal{H}_1)\), \(m, n = 1, 2, \ldots, k\).

Thus each bounded linear operator acting on the \(k\)-poly-a-space \(\mathcal{H}_k\) can be uniquely represented in the form

\[
T = A_k \tilde{T} A_k^{-1} = \sum_{n,m=1}^{k} P_{m,n}(J^+_k, J^0_k, J^-_k) A_{(m)} S_{m,n} A_{(n)}^{-1},
\]

with \(S_{m,n} \in \mathcal{L}(\mathcal{H}_1)\), \(m, n = 1, 2, \ldots, k\).

Restricting the class of operators from \(\mathcal{L}(\mathcal{H}_1)\) to just scalar operators we come to
Corollary 2.9 Each operator \( P(J_0^k, J_-^k) \) being a polynomial of \( J_0^k \) and \( J_-^k \) with complex coefficients acts invariantly on each \( \mathcal{H}_k \), is bounded there, and preserves the flag (2.10).

Additionally, each polynomial \( P(a^\dagger a) = P(J_0^k + \frac{k-1}{2}I) \) with complex coefficients preserves all true-poly-\( \alpha \)-spaces \( \mathcal{H}_n \) and acts boundedly on each of them.

Note that in the standard case of the one-dimensional \( \mathcal{H}_1 \) (single vacuum vector) the above statements are fundamental in the study of the so-called exactly-solvable and quasi-exactly-solvable problems, see [14–17], for review see also [18].

3 Poly-analytic Functions in One Variable Revisited

We start with \( \mathcal{H} \) being the space \( L_2(\mathbb{C}, d\mu) \) of square-integrable on \( \mathbb{C} \) with the Gaussian measure

\[
d\mu(z) = \pi^{-1} e^{-z^2} dv(z),
\]

where \( dv(z) = dx dy \) is the Euclidean volume measure on \( \mathbb{C} = \mathbb{R}^2 \), and the following [20, Section 2] lowering and raising operators

\[
a = \frac{\partial}{\partial z}, \quad a^\dagger = -\frac{\partial}{\partial z} + z.
\] (3.1)

Recall that the classical Fock [5,7] (or Segal-Bargmann [4,13]) space \( F^2(\mathbb{C}) \) is the closed subspace of \( L_2(\mathbb{C}, d\mu) \), which consists of all analytic in \( \mathbb{C} \) functions.

Alternatively, it can be defined as the (closed) subspace of all smooth functions satisfying the Cauchy–Riemann equation

\[
a \varphi = \frac{\partial \varphi}{\partial \bar{z}} = 0,
\] (3.2)

implying that each normalized analytic function can serve as a vacuum vector.

We recall now some results from [20], where all proofs and details can be found. Alternatively some of them yields from the considerations in the previous section.

Besides the Fock space \( F^2(\mathbb{C}) \), we additionally introduce the poly-Fock spaces, i.e., for each \( k \in \mathbb{N} \), the \( k \)-Fock (or \( k \)-poly-Fock) space \( F_k^2(\mathbb{C}) \) \((k\)-poly-\( \alpha \)-space, as defined by (2.1)) is the closed set of all smooth functions from \( L_2(\mathbb{C}, d\mu) \) satisfying the equation

\[
a^k \varphi = \left(\frac{\partial}{\partial \bar{z}}\right)^k \varphi = 0.\]
(3.3)

It is convenient to introduce the spaces

\[
F_{(k)}^2(\mathbb{C}) = F_k^2(\mathbb{C}) \ominus F_{k-1}^2(\mathbb{C}), \quad \text{for } k > 1,
\]
\[ F_{(1)}^2(\mathbb{C}) = F_1^2(\mathbb{C}) = F^2(\mathbb{C}), \quad \text{for} \; k = 1. \]

We call the space \( F_{(k)}^2(\mathbb{C}) \) the true-\( k \)-Fock space. It is evident that

\[ F_k^2(\mathbb{C}) = \bigoplus_{p=1}^{k} F_{(p)}^2(\mathbb{C}). \]  

(3.4)

Then we have

**Proposition 3.1** [20, Corollary 2.4] *The space \( L_2(\mathbb{C}, d\mu) \) admits the following decomposition*

\[ L_2(\mathbb{C}, d\mu) = \bigoplus_{k=1}^{\infty} F_{(k)}^2(\mathbb{C}). \]

An equivalent to Proposition 3.1 statement can be formulated in terms of the \( k \)-Fock spaces as follows.

**Corollary 3.2** The set of \( k \)-Fock subspaces \( F_{(k)}^2(\mathbb{C}), k \in \mathbb{N} \), of the space \( L_2(\mathbb{C}, d\mu) \) forms an infinite flag in \( L_2(\mathbb{C}, d\mu) \)

\[ F_1^2(\mathbb{C}) \subset F_2^2(\mathbb{C}) \subset \ldots \subset F_k^2(\mathbb{C}) \subset \ldots \subset L_2(\mathbb{C}, d\mu), \]  

(3.5)

and

\[ L_2(\mathbb{C}, d\mu) = \bigcup_{k=1}^{\infty} F_k^2(\mathbb{C}). \]

Among other properties of the true-\( k \)-Fock spaces we mention

**Proposition 3.3** [20, Theorem 2.9] *For each \( k = 2, 3, \ldots \), the operator*

\[ \frac{1}{\sqrt{k-1}} a^\dagger |_{F_{(k-1)}^2(\mathbb{C})} : F_1^2(\mathbb{C}) \longrightarrow F_{(k)}^2(\mathbb{C}) \]  

(3.6)

*is an isometric isomorphism, and the operator*

\[ \frac{1}{\sqrt{k-1}} a |_{F_{(k)}^2(\mathbb{C})} : F_{(k)}^2(\mathbb{C}) \longrightarrow F_1^2(\mathbb{C}) \]  

(3.7)

*is its inverse.*

And for each \( k \in \mathbb{N} \), the operator

\[ A_{(k)} := \frac{1}{\sqrt{(k-1)!}} (a^\dagger)^{k-1} |_{F^2(\mathbb{C})} : F^2(\mathbb{C}) \longrightarrow F_{(k)}^2(\mathbb{C}) \].
gives an isometric isomorphism between the Fock space $F^2(\mathbb{C})$ and the true-poly-Fock space $F^2_{(k)}(\mathbb{C})$, and the operator
\[
A^{-1}_{(k)} := \frac{1}{\sqrt{(k-1)!}} a^{k-1} \big|_{F^2_{(k)}(\mathbb{C})} : F^2_{(k)}(\mathbb{C}) \rightarrow F^2(\mathbb{C}),
\]
is an inverse isomorphism.

It is also worth recalling that $F^2_1(\mathbb{C}) = F^2_{(1)}(\mathbb{C}) = F^2(\mathbb{C})$ is the kernel of the operator $\alpha$.

As a direct corollary we have

**Corollary 3.4** [20, Corollary 2.10] Each function $\psi(z, \bar{z})$ from the true-$k$-Fock space $F^2_{(k)}(\mathbb{C})$ is uniquely defined by a function $\varphi(z) \in F^2(\mathbb{C})$ and has the form
\[
\psi(z) = \psi(z, \bar{z}) = \sum_{m=0}^{k-1} (-1)^m \frac{\sqrt{(k-1)!}}{m! (k-1-m)!} \bar{z}^{k-1-m} \varphi^{(m)}(z),
\tag{3.8}
\]
where $\varphi^{(m)}$ is the $m$-th derivative of the (analytic) function $\varphi$, and
\[
\|\psi\|_{F^2_{(k)}(\mathbb{C})} = \|\varphi\|_{F^2(\mathbb{C})}.
\]

The above proposition permits us to characterize explicitly elements of the $k$-Fock space $F^2_k(\mathbb{C})$.

**Theorem 3.5** Each function $\varphi(z, \bar{z}) \in F^2_k(\mathbb{C})$ is uniquely defined by $k$ functions $f_1(z)$, ..., $f_k(z)$ from the Fock space $F^2(\mathbb{C})$ and admits the representation
\[
\varphi(z, \bar{z}) = \sum_{\ell=1}^{k} \bar{z}^{\ell-1} \cdot \varphi_{\ell}(z),
\tag{3.9}
\]
where the analytic in $\mathbb{C}$ functions $\varphi_{\ell}(z)$ have the form
\[
\varphi_{\ell}(z) = \sum_{p=\ell}^{k} (-1)^{p-\ell} \frac{\sqrt{(p-1)!}}{(p-\ell)! (\ell-1)!} f_p^{(p-\ell)}(z).
\]

**Proof** By (3.4), each function $\varphi(z, \bar{z}) \in F^2_k(\mathbb{C})$ admits the unique representation
\[
\varphi(z, \bar{z}) = \sum_{p=1}^{k} \psi(p)(z, \bar{z}),
\tag{3.10}
\]
and, by (3.8),
\[
\psi(p)(z, \bar{z}) = \sum_{m=0}^{p-1} (-1)^m \frac{\sqrt{(p-1)!}}{m! (p-1-m)!} \bar{z}^{p-1-m} f_p^{(m)}(z).
\]
for a certain $f_p(z) \in F^2(\mathbb{C})$. Substituting now the above expression for $\psi(p)(z, \bar{z})$ to (3.10), and collecting terms with $\bar{z}^{\ell-1}$, we obtain that

$$\varphi(z, \bar{z}) = \sum_{\ell=1}^{k} \bar{z}^{\ell-1} \cdot \varphi_\ell(z),$$

where

$$\varphi_\ell(z) = \sum_{p=\ell}^{k} (-1)^{p-\ell} \frac{\sqrt{(p-1)!}}{(p-\ell)! (\ell-1)!} f_p^{(p-\ell)}(z).$$

Note that $\varphi_k(z) = \frac{f_k(z)}{\sqrt{(k-1)!}} \in F^2(\mathbb{C})$, while the others $\varphi_\ell(z)$, with $\ell = 1, 2, \ldots, k-1$, generically do not belong to $F^2(\mathbb{C})$.

**Observation 3.6** For a function $\varphi(z, \bar{z}) \in F^2_k(\mathbb{C})$ represented in form (3.9), the following recursive formulas allow us to recover all functions $f_1, f_2, \ldots, f_k$ from $F^2(\mathbb{C})$ generated it

$$f_\ell(z) = \sqrt{(\ell-1)!} \varphi_\ell(z), \ldots,$$

$$f_\ell(z) = \sqrt{(\ell-1)!} \left[ \varphi_\ell(z) - \sum_{p=\ell+1}^{k} (-1)^{p-\ell} \frac{\sqrt{(p-1)!}}{(p-\ell)! (\ell-1)!} f_p^{(p-\ell)}(z) \right],$$

for all $\ell < k$.

Further, each true-$k$-Fock space is a reproducing kernel Hilbert space and

**Theorem 3.7** [20, Lemma 3.2 and Theorem 3.3] The operator

$$(P_k f)(z) = \langle f(\xi), q_\xi^{[k]}(\xi) \rangle = \int_{\mathbb{C}} f(\xi) q_\xi^{[k]}(z) \, d\mu(\xi)$$

is the orthogonal Bargmann projection of $L^2_2(\mathbb{C}, d\mu)$ onto the true-$k$-Fock space $F^2_k(\mathbb{C})$, where the reproducing kernel $q_\xi^{[k]}(\xi)$ is given by

$$q_\xi^{[k]}(\xi) = \frac{1}{(k-1)!} \left( -\frac{\partial}{\partial \xi} + \bar{\xi} \right)^{k-1} \left( -\frac{\partial}{\partial \bar{\xi}} + \xi \right)^{k-1} e^{\xi \bar{\xi}}, \quad \xi, \bar{\xi} \in \mathbb{C}.$$
For each \( z \in \mathbb{C} \), the function \( q^{(k)}_z(\zeta) \) belongs to \( F^2_{(k)}(\mathbb{C}) \), has the form

\[
q^{(k)}_z(\zeta) = e^{\zeta \bar{\zeta}} p_{k-1}(z - \zeta)(\bar{\zeta} - \bar{z}),
\]

for certain real coefficient polynomial \( p_{k-1}(\lambda) \), and

\[
q^{(k)}_{\overline{z}}(z) = \overline{q^{(k)}_z(\zeta)}.
\]

By Observation 2.1, condition (3.3) is equivalent to

\[
(a^\dagger)^k a^k \varphi = \left( -\frac{\partial}{\partial z} + \overline{z} \right)^k \left( \frac{\partial}{\partial \overline{z}} \right)^k \varphi = 0
\]

with

\[
(a^\dagger)^k a^k = \prod_{m=0}^{k-1} \left( a^\dagger a - m \right),
\]

where the right hand side is a product of the so-called Euler-Cartan operators

\[
I^{(m)}_0 = a^\dagger a - m - \left( -\frac{\partial}{\partial z} + \overline{z} \right) \frac{\partial}{\partial \overline{z}} - m, \quad m = 0, 1, \ldots, k - 1. \tag{3.12}
\]

corresponding to the representation marked by \( m \) of the algebra \( \mathfrak{sl}(2) \). Formulas (3.6) and (3.7) imply that \( I^{(m)}_0 \) boundedly acts on each \( F^2_{(p)} \) and has \( F^2_{(m)} \) as its kernel. We note as well that each of the operators

\[
J^+_k = (a^\dagger)^2 a - (k - 1) a^\dagger = \left( \overline{z} - \frac{\partial}{\partial \overline{z}} \right) \left( \overline{\partial}_{\overline{z}} - \frac{\partial^2}{\partial z \partial \overline{z}} - (k - 1) I \right),
\]

\[
J^0_k = a^\dagger a - \frac{k-1}{2} I = \overline{z} \frac{\partial}{\partial \overline{z}} - \frac{\partial^2}{\partial z \partial \overline{z}} - \frac{k-1}{2} I,
\]

\[
J^-_k = a = \frac{\partial}{\partial \overline{z}}.
\]

is bounded on each poly-Fock space \( F^2_{(p)} \).

It is straightforward that for all positive integers \( k \) the above operators \( J^+_k, J^0_k \) and \( J^-_k \) act invariantly on \( k \)-Fock space \( F^2_k \). Thus, the poly-Fock (poly-analytic) spaces \( F^2_{(k)}(\mathbb{C}) \), \( k = 1, 2, \ldots \) can be defined alternatively as the maximal (in a sense of Observation 2.7) invariant subspaces for the action the operators \( J^+_k, J^0_k, J^-_k \) in \( L^2(\mathbb{C}, d\mu) \), obeying the \( \mathfrak{sl}(2) \)-algebra commutation relations.

We characterize now some invariance properties of the \( k \)-poly-Fock spaces.

As well known any motion of the complex plane \( \mathbb{C} \), a one-to-one mapping of \( \mathbb{C} \) that preserves distances and does not changes the orientation, is a combination of rotations \((z \mapsto w = az, |a| = 1)\) and parallel translations \((z \mapsto w = z + a, a \in \mathbb{C})\), generating the group \( E_2 \) of motions in \( \mathbb{C} \). The corresponding operators, that act unitarily on both \( L^2(\mathbb{C}, d\mu) \) and \( F^2(\mathbb{C}) \), have the form of rotation \((U_\alpha f)(w) = f(\overline{\alpha} w)\) and the Weyl
operator (see e.g. [23, Section 2]) \((W_a f)(w) = e^{\pi w - \frac{1}{2}|a|^2} f(w - a)\), being a shift operator with a gauge factor.

**Theorem 3.8** For each \(k\), the true \(k\)-Fock space \(F^2_{(k)}(\mathbb{C})\) is invariant under the action of the operators \(U_\alpha, |\alpha| = 1\) and \(W_a, a \in \mathbb{C}\), where they act isometrically.

**Proof** We mention first that a routine verification gives

\[ U_\alpha a^\dagger U_\alpha^{-1} = \alpha a^\dagger \quad \text{and} \quad W_a a^\dagger W_a^{-1} = a^\dagger \]

We proceed then by induction on \(k\). For \(k = 1\), i.e. for the Fock space, the result is already known. Assume that for each \(k - 1, k = 2, 3, \ldots\), the result is valid: for each \(\psi \in F^2_{(k-1)}(\mathbb{C})\)

\[ U_\alpha \psi = \psi_\alpha \in F^2_{(k-1)}(\mathbb{C}) \quad \text{and} \quad W_a \psi = \psi_a \in F^2_{(k-1)}(\mathbb{C}). \]

Then, using (3.6) and assuming that \(\varphi = \frac{1}{\sqrt{k-1}} a^\dagger \psi \in F^2_{(k)}(\mathbb{C})\),

\[ U_\alpha \varphi = U_\alpha \frac{1}{\sqrt{k-1}} a^\dagger U_\alpha^{-1} (U_\alpha \psi) = \alpha \frac{1}{\sqrt{k-1}} a^\dagger \psi_\alpha \in F^2_{(k)}(\mathbb{C}), \]

\[ W_a \psi = W_a \frac{1}{\sqrt{k-1}} a^\dagger W_a^{-1} (W_a \psi) = \frac{1}{\sqrt{k-1}} a^\dagger \psi_a \in F^2_{(k)}(\mathbb{C}). \]

The isometric action of \(U_\alpha\) and \(W_a\) follows from the isometric action of \(\frac{1}{\sqrt{k-1}} a^\dagger\) on each \(F^2_{(k-1)}(\mathbb{C})\), for all \(k = 2, 3, \ldots\), and \(|\alpha| = 1\).

**Corollary 3.9** The unitary operators associated with motions of the plane \(\mathbb{C}\) preserve flag (3.5) of \(k\)-Fock spaces.

The general description of operators boundedly acting and preserving \(k\)-poly-Fock spaces \(F^2_k(\mathbb{C})\), flag (3.5) of these spaces, or all true-\(k\)-poly-Fock spaces \(F^2_{(k)}(\mathbb{C})\) is given by Proposition 2.8 and Corollary 2.9.

Note that among the operators acting on each \(F^2_{(k)}(\mathbb{C})\) there is well known and important for physics applications operator.

**Example Landau magnetic Hamiltonian.** Let us consider the operator

\[
\widetilde{\Delta} = -\frac{\partial^2}{\partial z \partial \bar{z}} + \bar{z} \frac{\partial}{\partial \bar{z}},
\]

which (in suitable units and up to an additive constant) is a realization on \(L^2(\mathbb{C}, d\mu)\) of the similarity-transformed Schrödinger operator (which we call here the Landau magnetic Hamiltonian) describing the transverse motion of a charged particle evolving in the complex plane \(\mathbb{C}\) subject to a normal uniform constant magnetic field in asymmetric (Landau) gauge, see e.g. [10, Chapter XV, §112]. The first term in (3.14) has a meaning of the kinetic energy.
It is immediately recognized that (3.14) is the Euler-Cartan generator of the algebra \( \mathfrak{sl}(2) \) (3.13) at \( k = 1 \) and can be rewritten as follows

\[
\tilde{\Delta} = J_1^0 = a^\dagger a,
\]

with the operators \( a^\dagger \) and \( a \) given by (3.1). By Corollary 2.9 the operator \( \tilde{\Delta} \) acts invariantly on each \( k \)-poly-Fock space \( F^2_k(\mathbb{C}) \) and preserves the infinite flag of poly-Fock spaces. Furthermore, by (3.6) and (3.7), each true-poly-Fock space \( F^2_{(k)}(\mathbb{C}) \) is invariant subspace for the operator \( \tilde{\Delta} = a^\dagger a \) as well and

\[
\tilde{\Delta} |_{F^2_{(k)}(\mathbb{C})} = (k - 1)I, \quad k \in \mathbb{N}.
\]

Hence, the spectrum of the operator \( \tilde{\Delta} \) consists of infinitely many equidistant eigenvalues, each of infinite multiplicity (Landau levels), they are of the form

\[
\lambda_k = k - 1, \quad k \in \mathbb{N},
\]

and the corresponding eigenspaces are nothing but the true-\( k \)-poly-Fock spaces \( F^2_{(k)}(\mathbb{C}) \), cf. [12]. Hence, (3.14) is the exactly-solvable operator (see [15,18]), whose eigenvalues depend linearly on \( k \), and are of infinite multiplicity.

Let us mention an alternative way to get the same result. Again, by Corollary 2.9 the operator \( \tilde{\Delta} \) acts invariantly on each true-\( k \)-poly-Fock space \( F^2_{(k)}(\mathbb{C}) \), which consists of all functions \( \psi = (a^\dagger)^{k-1} f \), with \( f \in F^2(\mathbb{C}) \). Consider then the corresponding spectral problem on \( F^2_{(k)}(\mathbb{C}) \)

\[
\tilde{\Delta} \psi = \lambda \psi \quad \text{or} \quad (a^\dagger a)(a^\dagger)^{k-1} f = \lambda (a^\dagger)^{k-1} f.
\]

Making use of (2.2) and \( a f = 0 \) we arrive at

\[
(k - 1)(a^\dagger)^{k-1} f = \lambda (a^\dagger)^{k-1} f \quad \text{or} \quad \tilde{\Delta} \psi = (k - 1)\psi, \quad \text{forall } \psi \in F^2_{(k)}(\mathbb{C}).
\]

**Observation 3.10** This leads us to the following approach to the spectral problems for the operators \( D = P(J^+_k, J^0_k, J^-_k) \), with \( P \) being a polynomial with complex coefficients. By Proposition 2.8 the operator \( D \) acts invariantly on the \( k \)-poly-Fock space \( F^2_k(\mathbb{C}) \), which implies the following form of the spectral problem for \( D \) restricted to \( F^2_k(\mathbb{C}) \)

\[
D \psi = \lambda \psi, \quad \text{where } \psi \in F^2_k(\mathbb{C}),
\]

or

\[
\left( P(J^+_k, J^0_k, J^-_k) - \lambda \right) \sum_{j=0}^{k-1} (a^\dagger)^j f_j = 0,
\]
where \( f_j \in F^2(\mathbb{C}) \) for all \( j = 0, 1, \ldots, k - 1 \).

Using (2.2) and \( \ker a = F^2(\mathbb{C}) \), we rearrange then the operator on the left-hand side to its so-called Wick normal form (powers of \( a^\dagger \) to the left and powers of \( a \) to the right). This results that a certain \( k \)-poly-analytic function, being the sum of its mutually orthogonal true-\((j + 1)\)-poly-analytic functions, \( j = 0, 1, \ldots, k - 1 \), is identically zero. Setting these true-poly-analytic components to zero we come to a system of \( k \) equations for \( f_j \in F^2(\mathbb{C}) \) which can be solved then by the linear algebra means.

We illustrate this on a modified Landau magnetic Hamiltonian, where we change the “mark” 1 in \( J^0_2 \) to mark 2 (for simplicity) with extra constant term and by adding the generators \( J^\pm_2 \) (3.13).

Example Let us introduce the operator

\[
\widetilde{\Delta}_2 = J^0_2 + \frac{1}{2} I + \alpha J^+_2 + \beta J^-_2 = a^\dagger a + \alpha a^\dagger (a^\dagger a - I) + \beta a,
\]

with parameters \( \alpha \) and \( \beta \). The operator \( \widetilde{\Delta}_2 \) is a differential operator of the third order which becomes of the second order if \( \alpha = 0 \). By Proposition 2.8 the \( 2 \)-poly-Fock space \( F^2_2(\mathbb{C}) \) is its invariant subspace, while if \( \alpha = 0 \), by Corollary 2.9, \( \widetilde{\Delta}_2 \) acts invariantly on each poly-Fock space preserving the flag (3.5).

The corresponding spectral problem on \( F^2_2(\mathbb{C}) \) has the form

\[
\widetilde{\Delta}_2 \psi = \lambda \psi, \quad \text{where } \psi \in F^2_2(\mathbb{C}),
\]

or

\[
\left( a^\dagger a + \alpha a^\dagger (a^\dagger a - I) + \beta a - \lambda \right) (a^\dagger f_1 + f_0) = 0.
\]

Let first \( \alpha \neq 0 \). Then rearranging to the Wick normal form and making all cancellations, we have

\[
a^\dagger ((1 - \lambda) f_1 - \alpha f_0) + (\beta f_1 - \lambda f_0) = 0.
\]

Equations

\[
(1 - \lambda) f_1 - \alpha f_0 = 0 \quad \text{and} \quad \beta f_1 - \lambda f_0 = 0
\]

imply that \( f_1 \) and \( f_0 \) are related as \( \alpha f_0 = (1 - \lambda) f_1 \) and the following formula for eigenvalues

\[
\lambda_{1,2} = \frac{1}{2} \left( 1 \pm \sqrt{1 - 4 \alpha \beta} \right).
\]

Finally, the corresponding infinite dimensional eigenspaces consist of all \( F^2_2(\mathbb{C}) \)-functions of the form

\[
\psi_{1,2} = \left( a^\dagger + \frac{1}{\alpha} (1 - \lambda_{1,2}) \right) f, \quad f \in F^2(\mathbb{C}).
\]
Hence, $\tilde{\Delta}_2$ is the quasi-exactly-solvable operator [15,18] with two algebraically known eigenvalues, each of infinite multiplicity.

Let now $\alpha = 0$, then

$$\tilde{\Delta}_2 = J_2^0 + \frac{1}{2} I + \beta J_2^- = a^\dagger a + \beta a.$$  \hspace{1cm} (3.15)

It is clear that the second order operator $\tilde{\Delta}_2$ is isospectral to $\tilde{\Delta}$: the spectra remains unchanged, at the same time the corresponding eigenspaces are quite different. A straightforward calculation shows that the infinite dimensional eigenspace that corresponds to the eigenvalue $\lambda_k = k - 1$ consists of all $F_k^2(\mathbb{C})$-functions of the form

$$\psi = \left(a^\dagger + \beta\right)^{k-1} f, \quad f \in F^2(\mathbb{C}).$$

That is, the operator (3.15) acts on each $k$-poly-Fock space $F_k^2(\mathbb{C})$ invariantly, has there $k$ eigenvalues $\lambda_j = j - 1, \ j = 1, 2, \ldots, k$, whose corresponding eigenspaces are of the form

$$\left(a^\dagger + \beta\right)^{j-1} f, \quad \text{for all } \ f \in F^2(\mathbb{C}), \quad j = 1, 2, \ldots, k.$$

4 Poly-analytic Functions of Several Complex Variables

Consider now the following spaces over $\mathbb{C}^d$

$$L_2(\mathbb{C}^d, d\mu_d) = L_2(\mathbb{C}, d\mu) \otimes L_2(\mathbb{C}, d\mu) \otimes \cdots \otimes L_2(\mathbb{C}, d\mu),$$

$$F^2(\mathbb{C}^d) = F^2(\mathbb{C}) \otimes F^2(\mathbb{C}) \otimes \cdots \otimes F^2(\mathbb{C}) \quad \text{d times} \ ,$$  \hspace{1cm} (4.1)

where the Fock space $F^2(\mathbb{C}^d)$ is the closed subspace of $L_2(\mathbb{C}^d, d\mu_d)$, which consists of analytic functions in $d$ complex variables. Given a multi-index $\mathbf{k} = (k_1, k_2, \ldots, k_d)$, introduce the true-$\mathbf{k}$-Fock space $F^2_{(\mathbf{k})}(\mathbb{C}^d)$ as the following tensor product of the true-poly-Fock spaces over $\mathbb{C}$

$$F^2_{(\mathbf{k})}(\mathbb{C}^d) = F^2_{(k_1)}(\mathbb{C}) \otimes \cdots \otimes F^2_{(k_d)}(\mathbb{C}) \quad \text{d times}.$$  

In particular, for the multi-index $\mathbf{1} = (1, 1, \ldots, 1)$, we have

$$F^2_{\mathbf{1}}(\mathbb{C}^d) = F^2_{(1)}(\mathbb{C}^d) = F^2(\mathbb{C}^d).$$
The above implies that the orthogonal projection $P_{(k)} : L_2(\mathbb{C}^d, d \mu_d) \rightarrow F^{2}_{(k)}(\mathbb{C}^d)$ has the form

$$P_{(k)} = P_{k_1} \otimes \ldots \otimes P_{k_d},$$

(4.2)

where each $P_{k_j}$ is given by (3.11).

**Theorem 4.1** ([20]) The space $L_2(\mathbb{C}^d, d \mu_d)$ admits the following decomposition

$$L_2(\mathbb{C}^d, d \mu_d) = \bigoplus_{|k| = 1}^{\infty} F^{2}_{(k)}(\mathbb{C}^d).$$

Introduce the $(2d + 1)$-dimensional Heisenberg algebra $\mathbb{H}_{2d+1} = \{a, a^\dagger, 1\}$ with commutator $[a_i, a_j^\dagger] = \delta_{ij} I$, $i, j = 1, 2, \ldots, d$, $[a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0$ and $[a_i, 1] = [a_i^\dagger, 1] = 0$. Its representation on $L_2(\mathbb{C}^d, d \mu_d)$ is given by $d$ pairs of raising and lowering operators (3.1) related to different $z_j$ in $z = (z_1, z_2, \ldots, z_d) \in \mathbb{C}^d$:

$$a_j^\dagger = \bar{z}_j - \frac{\partial}{\partial \bar{z}_j}, \quad a_j = \frac{\partial}{\partial z_j},$$

the identity operator $I$, with $[a_j, a_j^\dagger] = I$, $j = 1, 2, \ldots, d$, and the closed infinite dimensional space of vacuum vectors $\mathcal{H}_1$ which is given by

$$\mathcal{H}_1 = \bigcap_{j=1}^{d} \ker a_j = F^2(\mathbb{C}^d).$$

For each multi-index $k = (k_1, k_2, \ldots, k_d)$ the true-$k$-Fock space $F^{2}_{(k)}(\mathbb{C}^d)$ again is isomorphic to the space of vacuum vectors $\mathcal{H}_1 = F^2(\mathbb{C}^d)$ and is the result of its “lifting up” by the normalised product of the raising operators $(a_1^\dagger)^{k_1-1}(a_2^\dagger)^{k_2-1} \ldots (a_d^\dagger)^{k_d-1}$. Corollary 3.4 implies now

**Lemma 4.2** Given a multi-index $k = (k_1, k_2, \ldots, k_d)$, each function $\psi(z, \bar{z})$ from the true-$k$-Fock space $F^{2}_{(k)}(\mathbb{C}^d)$ is uniquely defined by a function $\varphi(z) \in F^2(\mathbb{C}^d)$ and has the form

$$\psi(z, \bar{z}) = \left( \prod_{p=1}^{d} \sum_{m_p=0}^{k_p-1} (-1)^{m_p} \frac{\sqrt{(k_p - 1)!}}{m_p!(k_p - 1 - m_p)!} \frac{\partial^{m_p}}{\partial z_p^{m_p}} \right) \varphi(z),$$

(4.3)

moreover

$$\|\psi\|_{F^{2}_{(k)}(\mathbb{C}^d)} = \|\varphi\|_{F^2(\mathbb{C}^d)}.$$
True-k-Fock spaces, being the “elementary pieces” in the construction of the forthcoming poly-Fock spaces, can be obviously rearranged (in infinitely many ways) to various sets of poly-analytic spaces, so that the closure of their union will give \( L_2(\mathbb{C}^d, d\mu_d) \). At this stage the question on the most appropriate such rearrangements of the true-k-Fock spaces naturally appears.

Following the pattern of the one-dimensional situation of the previous section, it is quite natural to define each “distinguished” set of poly-Fock spaces in a Lie-algebraic way, i.e., as the infinite system of the representation spaces for the action of generators of a certain Lie algebra \( g \).

### 4.1 Primary Case: Homogeneous Poly-analytic Functions

In this case the algebra \(\mathfrak{sl}(d+1)\) plays a role of the algebra \(g\). The simplest (symmetric) representation of the \(\mathfrak{sl}(d+1)\)-algebra given by the following combination of the raising and lowering operators \(a_i^+\) and \(a_i, i = 1, 2, \ldots, d\), see [17],

\[
J_i^- = a_i = \frac{\partial}{\partial \bar{z}_i}, \quad i = 1, 2, \ldots, d, \\
J_{i,j}^0 = a_i^+ a_j = \left( \bar{z}_i - \frac{\partial}{\partial z_i} \right) \frac{\partial}{\partial \bar{z}_j}, \quad i, j = 1, 2, \ldots, d, \\
J_i^+ = \left( \bar{z}_i - \frac{\partial}{\partial z_i} \right) \left( \sum_{j=1}^d \left( \bar{z}_j - \frac{\partial}{\partial z_j} \right) \frac{\partial}{\partial \bar{z}_j} - kI \right), \quad i = 1, 2, \ldots, d. \tag{4.4}
\]

For any real \(k\) these operators obey \(\mathfrak{sl}(d+1)\)-algebra commutation relations. Restricting \(k\) to positive integers, we define the \(k\)-homogeneous-Fock space \(F_{k,\text{hom}}^2(\mathbb{C}^d)\) as the (closed) subspace of the smooth functions in \(L_2(\mathbb{C}^d, d\mu_d)\) satisfying the equation

\[
\prod_{m=0}^{k-1} \left( \sum_{i=1}^d \left( \bar{z}_i - \frac{\partial}{\partial z_i} \right) \frac{\partial}{\partial \bar{z}_i} - mI \right) f = 0. \tag{4.5}
\]

As in section 3, we see that alternatively \(F_{k,\text{hom}}^2(\mathbb{C}^d)\) is the maximal (in a sense of Observation 2.7) closed subspace of \(L_2(\mathbb{C}^d, d\mu_d)\) being invariant under the action of the operators \(J_i^+, J_{i,j}^0, J_i^-, i, j = 1, 2, \ldots, d\) obeying \(\mathfrak{sl}(d+1)\)-algebra commutation relations.

As is easily seen,

\[
F_{k,\text{hom}}^2(\mathbb{C}^d) = \bigoplus_{\mathbf{p}} F_{(\mathbf{p})}^2(\mathbb{C}^d) : \mathbf{p} = (p_1, \ldots, p_d) \in \mathbb{N}^d \ \text{with} \ |\mathbf{p}| \leq k \bigg\}. \tag{4.6}
\]
Then each function \( \varphi(z, \bar{z}) \in F^2_{k,\text{hom}}(\mathbb{C}^d) \) is uniquely defined by \( \frac{c^{k-1}}{d^{k-1}} \) functions from \( F^2(\mathbb{C}^d) \) and admits the representation

\[
\varphi(z, \bar{z}) = \sum_{|m|=0}^{k-1} z^m f_m(z_1, \ldots, z_d),
\]

where all \( f_m(z_1, \ldots, z_d) \) are analytic functions on \( z_1, \ldots, z_d \) and their explicit form in terms of \( f_m \) can be given using (4.3).

Furthermore there is an isometric isomorphism

\[
F^2_{k,\text{hom}}(\mathbb{C}^d) \cong \mathbb{R}^{C^k_{d+k-1}} \otimes F^2(\mathbb{C}^d),
\]

and the orthogonal projection \( P_{k,\text{hom}} : L_2(\mathbb{C}^d, d\mu_d) \to F^2_{k,\text{hom}}(\mathbb{C}^d) \) has the form

\[
P_{k,\text{hom}} = \bigoplus P_{\mathbf{p}} : \mathbf{p} = (p_1, \ldots, p_d) \in \mathbb{N}^d \text{ with } |\mathbf{p}| \leq k,
\]

where each \( P_{\mathbf{p}} \) is given by (4.2).

Note that alternatively the \( k \)-homogeneous-Fock space \( F^2_{k,\text{hom}}(\mathbb{C}^d) \) can be defined as the (closed) subspace of all smooth functions in \( L_2(\mathbb{C}^d, d\mu_d) \) satisfying the equations

\[
a_1^{p_1} \cdots a_d^{p_d} f = \frac{a|\mathbf{p}|}{\partial z_1^{p_1} \cdots \partial z_d^{p_d}} f = 0, \quad \text{forall } |\mathbf{p}| = k.
\]

An analog of Corollary 3.2 reads now as follows

**Corollary 4.3** The set of \( k \)-homogeneous-Fock subspaces \( F^2_{k,\text{hom}}(\mathbb{C}^d) \), \( k \in \mathbb{N} \), of the space \( L_2(\mathbb{C}^d, d\mu_d) \) forms an infinite flag in \( L_2(\mathbb{C}^d, d\mu_d) \)

\[
F^2_{1,\text{hom}}(\mathbb{C}^d) \subset F^2_{2,\text{hom}}(\mathbb{C}^d) \subset \ldots \subset F^2_{k,\text{hom}}(\mathbb{C}^d) \subset \ldots \subset L_2(\mathbb{C}^d, d\mu_d),
\]

and

\[
L_2(\mathbb{C}^d, d\mu) = \bigcup_{k=1}^{\infty} F^2_{k,\text{hom}}(\mathbb{C}^d).
\]

Let us describe now the invariance properties of the \( k \)-homogeneous-Fock spaces \( F^2_{k,\text{hom}}(\mathbb{C}^d) \).

Recall that any motion of \( \mathbb{C}^d \) is a combination of complex rotations \( (z \mapsto w = uz) \), where \( u \in U(d) \) is a unitary matrix) and parallel translations \( (z \mapsto w = z + a, a = (a_1, \ldots, a_d) \in \mathbb{C}^d) \). The corresponding operators, that act unitarily both on \( L_2(\mathbb{C}^d, d\mu_d) \) and \( F^2(\mathbb{C}^d) \), have the form \( (U_u f)(w) = f(u^{-1} w) \) and the Weyl operator \( (W_a f)(w) = e^{a \cdot w - \frac{1}{2}|a|^2} f(w - a) \).
Theorem 4.4 The operators $U_u, u \in U(d)$, and $W_a, a \in \mathbb{C}^d$ act unitarily on each $k$-homogeneous-Fock space $F_{k,\text{hom}}^2(\mathbb{C}^d)$.

Proof It is sufficient to check that the operators $U_u$ and $W_a$ preserve the spaces $F_{k,\text{hom}}^2(\mathbb{C}^d)$. For the operator $U_u$ it follows from the easily verified connection between the operators $J_{0}^{(m)}, m = 0, 1, \ldots, k - 1$, defining $F_{k,\text{hom}}^2(\mathbb{C}^d)$ (see (4.5)):

$$U_u \left( \sum_{i=1}^{d} \left( \bar{z}_i - \frac{\partial}{\partial \bar{z}_i} \right) \frac{\partial}{\partial z_i} - mI \right) U_u^{-1} = \sum_{i=1}^{d} \left( \bar{w}_i - \frac{\partial}{\partial \bar{w}_i} \right) \frac{\partial}{\partial w_i} - mI.$$

For the operator $W_a$ it follows from equality (4.6) and Theorem 3.8. \hfill \square

That is, the unitary operators associated with a motion of the space $\mathbb{C}^d$ preserve flag (4.7) of $k$-homogeneous-Fock spaces $F_{k,\text{hom}}^2(\mathbb{C}^d)$. Moreover the $k$-homogeneous-Fock spaces $F_{k,\text{hom}}^2(\mathbb{C}^d)$ do not depend on the choice of the Cartesian coordinates in $\mathbb{C}^d$. This justifies the adjective homogeneous in their definition.

4.2 $m$-Quasi-Homogeneous-Poly-analytic Functions

We start with a tuple

$$m = (m_1, m_2, \ldots, m_n)$$

of natural numbers whose sum is equal to $d$, $m_1 + m_2 + \ldots + m_n = d$. Among all possible tuples $m$ there are two extreme cases: $m = (1, 1, \ldots, 1)$ with $n = d$, and $m = (d)$ with $n = 1$.

For a given tuple $m$ the complex space $\mathbb{C}^d$ can be written as

$$\mathbb{C}^d = C^{m_1} \oplus C^{m_2} \oplus \ldots \oplus C^{m_n},$$

whose points $z = (z_1, \ldots, z_d) \in \mathbb{C}^d$ are arranged in $n$ groups

$$z = (z_{(1)}, z_{(2)}, \ldots, z_{(n)}), \quad \text{where} \quad z_{(j)} = (z_{j,1}, \ldots, z_{j,m_j}) \in \mathbb{C}^{m_j}.$$

We will use the same arrangement for

$$a = (a_1, a_2, \ldots, a_d) = (a_{(1)}, a_{(2)}, \ldots, a_{(n)}),$$

$$a^\dagger = (a_1^\dagger, a_2^\dagger, \ldots, a_d^\dagger) = (a_{(1)}^\dagger, a_{(2)}^\dagger, \ldots, a_{(n)}^\dagger).$$
Correspondingly, the $d$-dimensional Fock space $F^2(\mathbb{C}^d)$ is decomposed as

$$F^2(\mathbb{C}^d) = F^2(\mathbb{C}^{m_1}) \otimes F^2(\mathbb{C}^{m_2}) \otimes \cdots \otimes F^2(\mathbb{C}^{m_n}).$$

Then for each tuple as $\mathbf{k} = (k_1, \ldots, k_n)$, where $k_j$’s are positive integers, we define $\mathbf{k}$-$\mathbf{m}$-quasi-homogeneous-Fock space $F^2_{k \cdot m \cdot q \cdot h o m}(\mathbb{C}^d)$ as the set of all functions satisfying the equations

$$\prod_{\ell=0}^{k_j-1} \left( \sum_{i=1}^{m_j} a_{j,i}^+ a_{j,i} - \ell I \right) f = 0, \quad j = 1, 2, \ldots, n. \quad (4.8)$$

The spaces thus defined are invariant for the action of the algebra

$$\mathfrak{g} = \mathfrak{sl}(m_1 + 1) \otimes \cdots \otimes \mathfrak{sl}(m_n + 1),$$

where the representation of each $\mathfrak{sl}(m_j + 1), j = 1, 2, \ldots, n,$ is given by the following combination of $a_{j,i}^+$ and $a_{j,i}$, $i = 1, 2, \ldots, m_j$

$$J^-_{j,i} = a_{j,i} = \frac{\partial}{\partial z_{j,i}}, \quad i = 1, 2, \ldots, m_j,$$

$$J^0_{j,i,\ell} = \left( z_{j,i} - \frac{\partial}{\partial z_{j,i}} \right) \frac{\partial}{\partial z_{j,\ell}}, \quad i, \ell = 1, 2, \ldots, m_j,$$

$$J^+_{j,i} = \left( z_{j,i} - \frac{\partial}{\partial z_{j,i}} \right) \left( \sum_{\ell=1}^{m_j} \left( z_{j,\ell} - \frac{\partial}{\partial z_{j,\ell}} \right) \frac{\partial}{\partial z_{j,\ell}} - k_j I \right), \quad i = 1, 2, \ldots, m_j.$$

For the case $\mathbf{m} = (1, 1, \ldots, 1)$ the algebra $\mathfrak{g} = \mathfrak{sl}(2)^{\otimes d}$ occurs, while for the another extreme case $\mathbf{m} = (d)$ the corresponding algebra is $\mathfrak{g} = \mathfrak{sl}(d + 1)$, and this is the case considered in Subsection 4.1. Each intermediate case, defined by a tuple $\mathbf{m} = (m_1, m_2, \ldots, m_n)$, corresponds to the algebra $\mathfrak{g} = \mathfrak{sl}(m_1 + 1) \otimes \cdots \otimes \mathfrak{sl}(m_n + 1)$.

Alternative to (4.8) equations that define the $\mathbf{k}$-$\mathbf{m}$-quasi-homogeneous-Fock space $F^2_{k \cdot m \cdot q \cdot h o m}(\mathbb{C}^d)$ are as follows

$$a_{j,1}^{p_1} \cdots a_{j,m_j}^{p_{kj}} f = \frac{\partial |p|}{\partial z_{j,1}^{p_1} \cdots \partial z_{j,m_j}^{p_{kj}}} f = 0, \quad \text{forall } |p| = k_j, \quad j = 1, 2, \ldots, n.$$

Note that the classes of $\mathbf{m}$-poly-analytic spaces for both extreme cases were defined in different contexts in [2,3] for $\mathbf{m} = (1, 1, \ldots, 1)$, and in [11,22] for $\mathbf{m} = (d)$, respectively.

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