Discrete structures in gravity

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Abstract

Discrete approaches to gravity, both classical and quantum, are reviewed briefly, with emphasis on the method using piecewise-linear spaces. Models of 3-dimensional quantum gravity involving 6j-symbols are then described, and progress in generalising these models to four dimensions is discussed, as is the relationship of these models in both three and four dimensions to topological theories. Finally, the repercussions of the generalisations are explored for the original formulation of discrete gravity using edge-length variables.

I Introduction to discrete gravity

A Basic formalism

The original motivation for the development of a discrete formalism for gravity\textsuperscript{1} arose from a number of problems with the continuum formulation of general relativity. These included the difficulty of solving Einstein’s equations for general systems without a large degree of symmetry, the problems of representing complicated topologies and the need for considerable geometric insight and capacity for visualisation. It turned out, as we shall see, that the discretisation scheme to be described not only helped with these problems but also found a vital rôle in numerical relativity and in attempts at a formulation of quantum gravity.

The related branches of mathematics which found their application to physics in this formulation of gravity are those of piecewise-linear spaces and topology and the geometric notion of intrinsic curvature on polyhedra. The immediate aim was to develop an approach to general relativity which avoided the
use of coordinates, since the physical predictions of the theory are coordinate-independent. The basic idea of the approach, which has subsequently become known as Regge calculus, is as follows. Rather than considering spaces (or space-times) with continuously varying curvature, we deal with spaces where the curvature is restricted to subspaces of codimension two. This is achieved by considering collections of n-dimensional blocks, which are glued together by identification of their flat (n-1)-dimensional faces. The curvature lies on the (n-2)-dimensional subspaces, known as hinges or bones. For technical reasons, it is convenient to use blocks which are simplices (triangles, tetrahedra and their higher dimensional analogues).

Consider first the realisation of these ideas in two dimensions. Here we have examples in everyday life, geodesic domes; these consist of networks of flat triangles which are fitted together to approximate curved surfaces, usually parts of a sphere. Since two triangles with a common edge can be flattened out without distortion, there is no curvature on the edges. However, when a collection of triangles meeting at a vertex is flattened, there will be a gap, indicating the presence of curvature at the vertex. The amount of curvature there depends simply on the size of the gap or deficit angle.

It is relatively simple to visualise the generalisation of a triangulated surface to three dimensions, where a collection of flat tetrahedra are glued together on their flat triangular faces. In general, the tetrahedra at an edge will not fit together exactly in flat space, so there will be a deficit angle at that edge giving a measure of the curvature there. In four dimensions, the curvature is restricted to the triangles between the tetrahedra where the four-simplices meet. And so on in higher dimensions. Thus we have a set of flat simplices glued together to approximate a curved space.

There is another way of viewing the scheme that has just been described. Piecewise-flat spaces are interesting in their own right, so in addition to using them as an approximation scheme for some curved “reality”, we may also study such spaces for their own sake. It has been argued (for example by Friedberg and Lee) that space-time is actually discrete at the smallest scales, so one could also regard curved spaces as approximations to a discrete reality. À chacun ses goûts!

In order for the piecewise-flat spaces to be of any practical use in relativity, beyond ease of visualisation, it must be possible to calculate geometric quantities like curvature and volume, and in particular to evaluate the Einstein action of such a space. In (1) it was shown heuristically that the analogue of the Einstein action

\[ I = \frac{1}{2} \int R \sqrt{g} d^nx, \]

is given by

\[ I_R = \sum_{\text{hinges}} |\sigma^i| \epsilon_i \]

where \(|\sigma^i|\) is the measure of a hinge \(\sigma^i\) and \(\epsilon_i\) is the deficit angle there, equal to \(2\pi\) minus the sum of the dihedral angles between the faces of the simplices.
meeting at that hinge. Rigorous justification for this formula followed in [3], where it was shown that it converges to the continuum form of the action, in the sense of measures, provided that certain conditions on the fatness of the simplices are satisfied. Friedberg and Lee [4] approached the problem from the opposite direction, deriving the Regge action from a sequence of continuum spaces approaching a discrete one.

The reason for choosing the building blocks to be simplices is that the geometry of a flat simplex is completely determined by the specification of its edge lengths, so a simplicial space may be described exactly by these lengths without the need for any further variables like angles. This means that the simplest choice of variables for the discrete theory is the edge lengths; clearly the action may be calculated once they are specified and they are also the obvious analogues of the metric tensor, which serves as variable in the continuum theory. There, an elegant way of deriving Einstein’s equations is from the principle of stationary action, varying $I$ with respect to the metric. The analogue in Regge calculus is to vary $I_R$ with respect to the edge lengths, giving the simplicial equivalent of Einstein’s equations:

$$\sum_i \frac{\partial |\sigma^i|}{\partial l_j} \epsilon_i = 0,$$

(3)

where we have used the result in [1], that the variation of the angular terms gives zero when summed over each simplex (Schläfli’s differential identity).

At first sight, it appears that there is one equation for each variable, promising the possibility of a complete solution for the edge lengths. However the situation is not as simple as that; there are analogues of the Bianchi identities in Regge calculus [1, 5, 6, 7, 8], which in the case of flat space provide exact relations between sets of equations, and approximate relations in the nearly-flat case, so the equations may not provide sufficient information for a complete solution. In that case there is freedom to specify certain variables, in analogy with the freedom to specify lapse and shift in the 3+1 version of continuum general relativity.

**B Classical applications**

In the ten years after its formulation, Regge calculus was applied almost exclusively to problems in classical relativity, in particular to the time development of simple model universes. (Rather than give a complete list of references here, we refer the reader to the bibliography [9] which contains a comprehensive list for the first 20 years.) The basic idea was really 3+1 in nature: take a triangulation of a 3-dimensional surface (usually closed but not necessarily so) to represent a hypersurface at a particular moment of time and join its vertices to the corresponding vertices of a second 3-dimensional triangulation, representing the same hypersurface at a later time. The edges used to join these vertices are taken to be timelike and the slice of 4-dimensional space-time between the two triangulations is then divided into 4-simplices by inserting appropriate diagonals. Given the edge lengths on the first 3-d triangulation, and specifying
the timelike edge lengths, the Regge equations may in principle be solved for the edge lengths on the second 3-d triangulation. By repetition of this process, the classical evolution of the initial spacelike surface may be calculated. This sounds simple enough, but unless quite strong assumptions of symmetry are made, the numerical calculation, involving large sets of simultaneous equations for the edge lengths, can be very time-consuming and complicated.

Significant progress with this approach was made in the early nineties when, based on an idea of Sorkin [10], it was realised that in general, the Regge equations decouple into a collection of much smaller groups. These groups of equations can then be solved in parallel, which means that the computer time required for an equivalent calculation is much less. This parallelisable implicit evolution scheme is described in detail in [11] and the basic mechanism is as follows. Consider a single vertex in a triangulated 3-dimensional spacelike hypersurface and introduce a new vertex “above” this. Connect the new vertex by a “vertical” edge to the chosen vertex, and by “diagonal” edges to all the vertices in the original hypersurface to which the chosen vertex was joined. Each tetrahedron in the original surface now has based on it a 4-simplex, with apex at the new vertex. Note that there is one diagonal corresponding to each edge in the original vertex radiating from the chosen vertex. We now use the Regge equations for these edges in the original surface and for the vertical edge; the only unknown edges which these equations involve are the new vertical edge and the diagonal edges, and there is precisely the same number of equations as unknowns. Thus, in principle, we can solve exactly for the unknown edge lengths. (In practice, because of the approximate relationship between the equations from the Bianchi identities, it is often more convenient to ignore some of the equations and instead specify conditions equivalent to the lapse and shift.)

We have described how to evolve vertices one-by-one in the Sorkin evolution scheme, and the entire hypersurface can be evolved in this way. The method is very general and can be used for a hypersurface with arbitrary topology. However, advancing the vertices one-by-one will not ordinarily be the most efficient way of evolving a hypersurface. If any two vertices in a hypersurface are not connected by an edge, then they can be evolved to the next surface at the same time without interfering with each other, which is why the method is obviously parallelisable.

C Some quantum applications

The earliest application of Regge calculus to quantum gravity was in three dimensions [12] and involved 6j-symbols. This work, and subsequent developments along those lines, will be the subject of the next two main sections and we shall not discuss it further here.

From the early eighties onwards, there have been many attempts to formulate a theory of quantum gravity based on Regge calculus, and we shall summarise the salient features of some of those approaches, both analytic and numerical.

The first work on quantum Regge calculus in four dimensions involved using
a study of small perturbations about a flat background to relate the discrete variables with their continuum counterparts \[13\]. The discrete propagator was derived in the Euclidean case and shown to agree with the continuum propagator in the weak field limit. (More details of this calculation will be given in the section on area Regge calculus.) The technique of weak field approximation has proved to be very useful not only for comparisons with the continuum theory but also as a guide in numerical calculations.

The difficulties of analytic calculations in quantum Regge calculus, coupled with the need for a non-perturbative approach and also the availability of sophisticated techniques developed in lattice gauge theories, have combined to stimulate numerical work in quantum gravity, based on Regge calculus. One approach is to start with a Regge lattice for, say, flat space, and allow it to evolve using a Monte Carlo algorithm (see for example \[14, 15, 16\]). Random fluctuations are made in the edge lengths and the new configuration is rejected if it increases the action, and accepted with a certain probability if it decreases the action. The system evolves to some equilibrium configuration, about which it makes quantum fluctuations, and expectation values of various operators can be calculated. It is also possible to study the phase diagram and search for phase transitions, the nature of which will determine the vital question of whether or not the theory has a continuum limit. Many of the simulations have involved an action with an extra term, quadratic in the curvature, to avoid problems of convergence of the functional integral; some have included scalar fields coupled to gravity \[17\]. Recent work by Riedler and collaborators in four dimensions describes evidence for a new continuous phase transition, essential for a continuum limit, at negative gravitational coupling \[18\].

The choice of measure in the functional integral is still a matter for controversy, depending both on attitude to simplicial diffeomorphisms and also on the stage at which translation from the continuum to the discrete takes place. The numerical simulations just described mainly use a simple scale invariant measure \[19\]. Menotti and Peirano have derived an expression for the functional measure in 2-dimensional Regge gravity, starting from the DeWitt supermetric, and giving exact expressions for the Fadeev-Popov determinant for both $S^2$ and $S^1 \times S^1$ topologies (see \[20\] and references therein).

A rather different approach to numerical simulations of quantum gravity is that of dynamical triangulations. (For a review containing an extensive set of references, see \[21\]). This also uses Regge lattices and the Regge action, but there are important differences. In the traditional approach, we are effectively integrating over the edge lengths in the functional integral, but in dynamical triangulations, the lattice is taken to be equilateral, with a certain length scale, and the summation is over different triangulations, which are generated by a set of $(k,l)$ moves \[22\]. In two dimensions, there are just two possible moves (and their inverses): the reconnection of vertices in two triangles with a common edge, and the insertion of a vertex and edges in a triangle to divide it into three triangles (2-2 and 1-3 moves). There are straightforward generalisations of these moves to higher dimensions. The moves are ergodic in the sense that any combinatorially equivalent triangulation can be generated by a finite succession
of these moves. It is argued that the restriction to equilateral triangulations is a way of avoiding over-counting gauge-related configurations. The approach has been very successful in two dimensions, where there are analytic results with which to compare the calculations. In three and four dimensions, there has been progress in, for example, deriving the crucially important exponential bound on the number of triangulations for a given number of vertices \(23\), but there are still open questions on the continuum limit, since the phase transition appears to be first order (see the review by Loll \(24\)). Recently a Lorentzian version of dynamical triangulations has been formulated in \((1+1)\)-dimensions \(25\). Numerical simulations have revealed a new universality class for pure gravity, with Hausdorff dimension two.

Discrete gravity has also proved very useful in calculations of the wave function of the universe \(26\). According to the Hartle-Hawking prescription, the wave function for a given 3-geometry is obtained by a path integral over all 4-geometries which have the given 3-geometry as a boundary. To calculate such an object in all its glorious generality is impossible, but one can hope to capture the essential features by integrating over those 4-geometries which might, for whatever reason, dominate the sum over histories. This has led to the concept of \textit{minisuperspace} models, involving the use of a single 4-geometry (or perhaps several). In the continuum theory, the calculation then becomes feasible if the chosen geometry depends only on a small number of parameters, but anything more complicated soon becomes extremely difficult. For this reason, Hartle \(27\) introduced the idea of summing over \textit{simplicial} 4-geometries as an approximation tool in quantum cosmology. Although this is an obvious way of reducing the number of integration variables, there are still technical difficulties: the unboundedness of the Einstein action (which persists in the discrete Regge form) leads to convergence problems for the functional integral, and it is necessary to rotate the integration contour in the complex plane to give a convergent result \(28, 29\).

In principle, the sum over 4-geometries should include not only a sum over metrics but also a sum over manifolds with different topologies. One then runs into the problem of classifying manifolds in four and higher dimensions, which led Hartle \(30\) to suggest a sum over more general objects than manifolds, \textit{unruly topologies}. Schleich and Witt \(31\) have explored the possibility of using conifolds, which differ from manifolds at only a finite number of points, and this has been investigated in some simple cases \(32, 33\). However, a sum over topologies is still very far from implementation.

Yet another area of application of Regge calculus in quantum gravity involves the study of the simplicial supermetric, the metric on the space of 3-geometries. Its signature is crucial for determining spacelike surfaces in superspace, which are important in Dirac quantisation and in quantum cosmology. In the continuum, there are limited results on the signature and this led to the possibility of investigating it in the discrete case \(34\) where the analogue is the Lund-Regge supermetric \(37\). This supermetric was constructed for some simple manifolds (\(S^3\) and \(T^3\)) and its signature calculated. The results agreed with the continuum predictions and also showed that the supermetric can become degenerate. We
still do not have a complete understanding of the division of the modes into “vertical” (corresponding to metrics related by diffeomorphisms) and “horizontal” ones.

D Other approaches to discrete gravity

Of course Regge calculus is not the only way of setting up a theory of discretised general relativity. In this subsection, we shall describe some alternatives.

One important class of schemes involves treating gravity as a gauge theory. For example, Mannion and Taylor [36] defined a theory of gravity on a fixed hypercubic lattice, and Kaku [37] used a fixed random lattice. However a dynamical lattice seems more appropriate in a theory aiming to describe the quantum fluctuations of space-time, and this was used in much earlier work by Weingarten [38].

In an approach closely related to Regge calculus, Caselle, D’Adda and Magnea [39] defined a theory of gravity on the dual lattice, giving both first- and second-order formulations. The action they obtained was a compactified form of the Regge action, involving the sine of the deficit angle. D’Adda and Gionti showed [40] that Regge calculus is a solution of the first order formulation in the limit of small deficit angles. The action of Caselle, D’Adda and Magnea was also used by Kawamoto and Nielsen [41] in their version of lattice gravity with fermions.

Immirzi investigated the links between canonical general relativity in the continuum, loop quantum gravity, and spin networks, in an attempt to formulate a quantised version of discrete gravity in the spirit of Regge calculus but ran into problems over hermiticity [42].

A totally different approach to discrete gravity is ’t Hooft’s polygon model in (2+1)-dimensions [43]. This was introduced as a way to refute Gott’s claim of acausality in (2+1) gravity coupled to point particles [44]. ’t Hooft’s method is to split space-time into the direct product of cosmological time and a Cauchy surface tessellated by flat polygons. The local flatness of space-time in the pure gravity regime and the cone-like structure introduced by particles, as in Regge calculus [45], are expressed in terms of conditions on the edges and vertices of the polygons. A local Lorentz frame is attached to each polygon and two constraints imposed; these are firstly that time runs at the same rate in each polygon (which corresponds to a partial gauge fixing) and secondly that all vertices are trivalent (which is acceptable because higher order vertices can always be split into trivalent vertices connected by edges of zero length). The consequences of these conditions are that the length and velocity of an edge are the same in both polygons to which it belongs, and that the velocity of each edge is orthogonal to it in both frames. These facts result in transition rules for the vertices in the tessellation.

The method for evolving such a space-time is as follows. Initial data (lengths and velocities), subject to consistency conditions, are assigned to the edges on a polygonally-tessellated hypersurface. The configuration evolves linearly until an edge collapses to zero length or a vertex crosses another edge. A transition, gov-
erned by the vertex conditions, then takes place to another configuration which
will, in general, have different numbers of vertices, edges and polygons. The
new data will still satisfy the consistency conditions and the process is then re-
peted. When there are particles present at the vertices, there are deficit angles
proportional to their masses, and the transition rules are modified accordingly.

It is not an easy task to follow the time-evolution of a (2+1)-dimensional
model with particles, even though the system has a finite number of degrees of
freedom. 't Hooft did numerical simulations on a small computer, with some
unexpected predictions. His big-bang and big-crunch hypotheses were based on
the evolution of a Cauchy surface with $S^2$ or $S^1 \times S^1$ topology, tessellated by a
single polygon [16]. It would be interesting to test these predictions for more
complex initial configurations, and as a means to this, there has been recent
work [17] in which the constraint equations have been interpreted in terms of
hyperbolic geometry (see also [18]), and various consistent sets of initial data
set up, but the evolution calculations have not yet been completed. Part of the
motivation for this work is to compare 't Hooft’s method with other approaches
to (2+1) gravity, in particular Regge calculus. A (2+1)-dimensional code has
been set up for Regge space-times and a number of calculations performed [19],
with a view to making detailed comparisons with the 't Hooft method. The ul-
timate aim is to understand the exact relationship between the two approaches,
which seem rather different but have many concepts in common.

Based on the polygon approach, various toy models of (2+1)-dimensional
gravity have been constructed [46, 50], issues of topology been ad-
ressed [51] and particle decay and space-time kinematics been investigated [52]. 't
Hooft himself has proposed quantised models of (2+1)-dimensional space-time
[53], showing that gravitating particles live on a space-time lattice. For an
$S^2 \times S^1$ topology, first quantisation of Dirac particles is possible. Waelbroeck has
suggested a similar approach, using canonical quantisation in (2+1) dimensions
[54].

Back at the classical level, Brewin has formulated [55] a discretisation of
gravity which he feels is closer to the original theory of general relativity. Pre-
liminary calculations are encouraging. For other important work on lattice
gravity by Bander, Jevicki and Ninomiya, Khatsymovsky and Lehto, Nielsen
and Ninomiya, we refer the reader to the Regge calculus review and biblio-
graphy [9]. We emphasise again that this paper is not meant to be an exhau-
stive review of the subject.

After this rather rapid survey of applications of Regge calculus, and some
other approaches to discrete gravity, we shall now concentrate on one particular
approach and show how it has led to exciting new developments in the search
for a quantum theory of gravity.

2. 6j-symbols in 3-dimensional quantum gravity

As promised, we now look in detail at the earliest link forged between Regge
calculus and quantum gravity, now known as the Ponzano-Regge model [12].
This emerged from a paper on 6j-symbols and we will first give the background to these.

A 6j-symbols

6j-symbols, which are generalisations of the more well-known Clebsch-Gordan coefficients, first arose as tools for the computation of matrix elements in the theory of complex spectra [56], and are now used routinely by atomic physicists and theoretical chemists in quantum mechanical calculations involving angular momentum. In particular, they relate the possible basis wave functions when three angular momentum are added:

\[
|j_1, (j_2, j_3), j_{23}, J > = \sum_{j_{12}} \sqrt{(2j_{12} + 1)(2j_{23} + 1)}(-1)^{j_1 + j_2 + j_3 + J} \begin{pmatrix}
  j_1 & j_2 & j_3 \\
  j_{12} & j_{23} & J
\end{pmatrix} |(j_1, j_2, j_3, J >.
\]

(4)

An alternative and useful definition involves the *recoupling* diagram:

\[
|j_1, j_2, j_3, J > = \sum_j \begin{pmatrix}
  j_1 & j_2 & j_3 \\
  j_{12} & j_{23} & J
\end{pmatrix} |(j_1, j_2, j_3, J >.
\]

(4)

A graphical representation is obtained by associating a 6j-symbol with a tetrahedron:

(5)

with the arguments of the 6j-symbol corresponding to the edge lengths of the tetrahedron. (For technical reasons, it turns out to be more accurate to associate a symbol with arguments \( a, b, ... \) to a tetrahedron with edge lengths \( a + 1/2, b + ... \).)
For the 6j-symbol to be non-zero, the arguments have to satisfy the analogue of the triangle inequalities for each face of the tetrahedron:

\[ j_3 \leq j_1 + j_2 \quad etc \]  

They can be evaluated from the formula

\[
\begin{vmatrix} a & b & c \\ d & e & f \end{vmatrix} = \sqrt{\Delta(a, b, c)\Delta(a, e, f)\Delta(c, d, e)\Delta(b, d, f)}
\]

\[
\sum_x (-1)^x (x + 1)![(a + b + d + e - x)!(a + c + d + f - x)!(b + c + e + f - x)!
\]

\[
(x - a - b - c)!(x - a - e - f)!(x - c - d - e)!(x - b - d - f)!]^{-1}
\]  

where

\[
\Delta(a, b, c) = (a + b - c)!(b + c - a)!(c + a - b)!(a + b + c + 1)!^{-1}.
\]  

These 6j-symbols are based on the group \( SU(2) \), but as we shall see, it is also possible to have q-deformed 6j-symbols based on quantum groups. For example, define \[ 57 \]

\[ q = \exp(2\pi i/r) \]  

and

\[ [n] = \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}}. \]  

Then the q-deformed 6j-symbol for \( SU_q(2) \) is defined in the same way as the undeformed one, with \( n \) replaced everywhere by \([n]\). Note that \([n] \to n\) as \( q \to 1 \) and \( r \to \infty \).

**B The Ponzano-Regge model**

The main purpose of the paper by Ponzano and Regge \[ 12 \] was to derive asymptotic formulae for classical (ie undeformed) 6j-symbols in the limit when certain arguments became large. The case most relevant to the exposition here is when all six parameters become large. The edge lengths of the corresponding tetrahedron are really related to \( j_i\hbar \) and these quantities are kept finite as \( j_i \to \infty \) while \( \hbar \to 0 \) so this process corresponds to the semi-classical limit. This asymptotic behaviour is given by

\[
\begin{vmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{vmatrix} \sim \frac{1}{12\pi V} \cos(\sum_i j_i\theta_i + \pi/4)
\]  

where \( V \) is the volume of the tetrahedron and \( \theta_i \) is the exterior dihedral angle at edge \( i \) (ie the angle between the outward normals to the faces meeting there). This was recently proved rigorously by Roberts \[ 58 \].
To see the connection between this formula and quantum gravity, consider the following state sum defined in [12]. Take a closed 2-dimensional surface, triangulate it and divide its interior into tetrahedra, possibly inserting internal vertices. Label the internal edges by $x_i$ and the external ones by $l_i$. Define

$$S(\{l_i\}) = \sum_{x_i \text{ tetrahedra}} \prod (6j)(-1)^X \prod (2x_i + 1)$$

(12)

where the $X$ in the phase factor is a function of the edge lengths.

Although this expression is infinite in many cases, it has some extremely interesting properties. In particular, noting that in the sum over the internal edges, the large values dominate, we can replace the sum by an integral with respect to those edge lengths and use the asymptotic formula stated above. Then the dominant contribution to the integral comes from the points of stationary phase, which are given by

$$\sum_{\text{tetrahedra } k \text{ meeting on edge } i} (\pi - \theta_i^k) = 2\pi.$$  

(13)

This means that the sum of the dihedral angles at each edge is $2\pi$, which is precisely the condition for local flatness in a 3-dimensional simplicial space.

What is more, the state sum is given approximately by

$$S \approx \frac{1}{\sqrt{12\pi}} \int \prod_i dx_i (2x_i + 1)(-1)^X \prod \frac{1}{\sqrt{V_k}} \cos \left( \sum_{l \in \text{tet } k} j_l \theta_i^k + \frac{\pi}{4} \right).$$

(14)

Now this contains a term of the form

$$\int \prod_i dx_i (2x_i + 1) \exp(i \sum_{\text{edges } l} j_l (2\pi - \sum_{\text{tet } k \ni i} (\pi - \theta_i^k))) = \int \prod_i dx_i (2x_i + 1) \exp(i \sum j_l \epsilon_l),$$

(15)

which looks precisely like a Feynman sum over histories with the Regge calculus action in three dimensions:

$$\int \prod_i d\mu(x_i) \exp(i I_R)$$

(16)

with

$$I_R = \sum_l j_l \epsilon_l,$$

(17)

where $\epsilon_l$ is the deficit angle at edge $l$ and $d\mu(x_i)$ is the measure on the space of edge lengths.

This result was rather puzzling and, although Hasslacher and Perry [59] emphasised the connection between spin networks and simplicial gravity, its significance was not fully appreciated until much later, when a very similar expression was written down in a different context.
The Turaev-Viro model

In the late eighties and early nineties, mathematicians put a lot of effort into searching for invariants of manifolds, the hope being, at least in part, that such quantities would help with the classification of manifolds. Without being aware of the Ponzano-Regge work, Turaev and Viro [60] defined a state sum for triangulated 3-manifolds, which in many aspects was identical to that of Ponzano and Regge. The main differences were that they gave formulae for closed manifolds as well as those with boundary, they showed explicitly that the quantity obtained was independent of triangulation, and finally, they used 6j-symbols for the quantum group $SL_q(2)$. Only some of the irreducible representations of this group, the ones with $j$ taking finite values, have suitable algebraic properties, which means that the edge lengths are not summed up to infinite values; $j_i$ can take only integer and half-integer values from the set $(0, 1/2, 1, ..., (r - 2)/2)$, with $r \geq 3$. A very important consequence of this is that the answer obtained is finite, and so the model appears to be a regularised version of the Ponzano-Regge model.

The obvious question to ask is how the Turaev-Viro state sum is connected to quantum gravity. Witten [61] conjectured that it was equivalent to a Feynman path integral with the Chern-Simons action for $SU_k(2) \otimes SU_{-k}(2)$, and this and equivalent results were proved by a number of people [62, 63, 64]. To see how this works [65, 66], consider the Chern-Simons Lagrangian for this group product:

$$L = \frac{k}{4\pi} \int_M Tr(A_+ \wedge dA_+ + \frac{2}{3} A_+ \wedge A_+ \wedge A_+)$$

$$- \frac{k}{4\pi} \int_M Tr(A_- \wedge dA_- + \frac{2}{3} A_- \wedge A_- \wedge A_-)$$

(18)

where

$$A_\pm = A_\pm^a T_a dx_i$$

(19)

with $T_a$ a basis of the $SU(2)$ Lie algebra. Making the change of variables

$$A_\pm^a = \omega_\pm^a = \omega_i^a \pm \frac{1}{k} e_i^a$$

(20)

where $e_i^a$ is the dreibein and

$$\omega_i^a = \frac{1}{2} \epsilon^{abc} \omega_{abc}$$

(21)

with $\omega_{abc}$ being the connection 2-form, we obtain

$$\int (e \wedge R + \frac{\lambda_k}{3} e \wedge e \wedge e)$$

(22)

which is the Einstein-Hilbert action for gravity with cosmological constant given by
\[ \lambda_k = \left( \frac{4\pi}{k} \right)^2. \]  

(Note that the \( k \) here is equal to \( r - 2 \), where \( r \) appears in the definition of \( q \).) By taking the limit as \( k \to \infty \), we obtain 3-dimensional gravity with zero cosmological constant i.e. the theory represented by the Ponzano-Regge model. This result is consistent with the fact that the \( q \to 1 \) limit of the Turaev-Viro model is the Ponzano-Regge model.

The properties of the Turaev-Viro state sum show that the formalism is an example of a topological quantum field theory (see e.g. [67]). This is perfectly appropriate for a theory of gravity in three dimensions where there are no local degrees of freedom. As for the Ponzano-Regge theory, the dominant classical configurations are locally flat (recall that in Chern-Simons gravity, the solutions involve the space of flat connections.)

The relationship between the Turaev-Viro invariant and 3-dimensional quantum gravity is an extremely important one. It means that in three dimensions, we have in principle a way of calculating the partition function for triangulated manifolds. This has been done for many of the simpler 3-manifolds (see [68, 69] for example). The Turaev-Viro expression can also be used for calculating topology-changing amplitudes in 3-dimensional gravity; the method here is to construct a cobordism between two 2-dimensional triangulated surfaces and then use the Turaev-Viro expression for a manifold with boundaries to evaluate the transition probability [70].

D Spin networks

The Turaev-Viro expression is not the only method of calculating this particular invariant of 3-manifolds. Various other prescriptions have been written down, and one that is worth describing at this stage is that using spin networks. These were invented by Penrose [71] who wanted to formulate a purely combinatorial approach to space-time. His networks had trivalent vertices and the edges of the graphs were labelled by spins. He developed a method of calculating the value of an arbitrary spin network and was able to show that this led to the usual angles of 3-dimensional space.

Penrose’s spin networks were later generalised in a number of ways. The edges were labelled by representations of quantum groups and it was necessary to introduce intertwining operators or intertwiners at the vertices [72]. In some cases a framing was introduced and the graphs became “ribbon graphs” [73]. Kauffman [74] showed how to calculate the Turaev-Viro invariant by taking the graph dual to a triangulation to be a spin network; the edges of the graph inherit the labels of the triangulation edges which they cross. Spin networks have also been introduced into loop quantum gravity [75], where they are an important calculational tool, for instance in the derivation of the spectrum of the area and volume operators [76]. (Note that Freidel and Krasnov also obtained a discrete spectrum for the volume operator in BF theory by differentiating the Turaev-Viro amplitude with respect to the cosmological constant [77].) As we
shall see, spin networks also play a rôle in recent attempts at formulations of 4-dimensional quantum gravity.

### III Extensions to four dimensions

After it was realised that the Turaev-Viro state sum provides a finite theory of 3-dimensional gravity, the search began for a generalisation to four dimensions. Before this is described, we shall stop to ask what we hope to achieve by this. In classical general relativity, there are enormous qualitative differences between gravity in three and four dimensions. In particular, there are gravitons in four dimensions, but not in three, so although it seems reasonable to describe 3-dimensional gravity by a topological invariant of a manifold, it seems likely that an invariant of a 4-manifold might describe only some topological sector of gravity. We shall return to this point later.

The obvious way of setting about extending the 3-dimensional model, based on 6j-symbols, to four dimensions is by using some 3nj-symbol for a value of \( n \) larger than 2. The 3nj-symbols in the state sum would then be expanded in terms of 6j-symbols, and the Ponzano-Regge formula for their asymptotic values inserted, the hope being that this would give an expression looking like a path integral with the 4-dimensional Regge action. The problem with this is that the asymptotic formula involves the 3-dimensional dihedral angles and it is very difficult to relate these to 4-dimensional angles. This indicates that a more radical generalisation may be needed.

We shall now describe some of the attempts at generalisation, leading up to some recent work which seems very promising.

#### A The Ooguri model

A source of inspiration for some generalisations of the Ponzano-Regge and Turaev-Viro models was Boulatov’s generalised matrix model \([77]\), which involved a scheme for generating 3-dimensional simplicial complexes as terms in a perturbative expansion. The contribution from each simplicial-complex was weighted by its Ponzano-Regge or Turaev-Viro invariant, depending on the value of \( q \). Boulatov’s model was formulated in a way that it could be extended to higher dimensions, and the 4-dimensional case for \( q = 1 \) was worked out by Ooguri \([78]\).

The essential ingredients in Ooguri’s model are the assigning of group variables to the tetrahedra and spin \( j \) labels to the triangles in the triangulated 4-manifold. The terms in Ooguri’s action are of two types: the first is a product of two functions of the group variables, and this represents two glued tetrahedra; the second is a product of five functions and represents the tetrahedra in a 4-simplex. A Fourier decomposition is performed in terms of rotation matrices and the group variables are then integrated out, using the standard relationship between rotation matrices and 3j-symbols, and the invariant Haar measure normalised to unity. The resulting expression has four 3j-symbols associated to
each tetrahedron; these may then be divided between the 4-simplices meeting on that tetrahedron, and then each 4-simplex ends up with ten 3j-symbols which can be combined to give a 15j-symbol. At first sight, it seems odd to associate a 15j-symbol with a 4-simplex which has only 10 triangles labelled by spin values. The way to interpret the symbol is to consider the dual graph, which has ten edges and five 4-valent vertices (corresponding to each tetrahedron in the original triangulation). Each of these 4-valent vertices can be split into two trivalent ones, and an extra spin label can be assigned to the edge joining them. This splitting sounds rather arbitrary but different splittings are related by 6j-symbols (see the second diagram in the section on 6j-symbols) and when all summations are performed, the result is independent of splitting.

The partition function is calculated by integrating the exponential of minus the action over the Fourier coefficients, and the resulting expression is

\[ Z = \sum_C \frac{1}{N_{symm}(C)} \lambda^{N_4(C)} Z(C), \]  

with \( Z(C) \) given by

\[ Z(C) = \sum \prod_{\text{triangles}} (2j_t + 1) \prod_{\text{tetrahedra}} \{6j\} \prod_{\text{4-simplices}} \{15j\}. \]

The summation in \( Z \) is over simplicial complexes \( C \), with \( N_{symm} \) being the rank of the symmetry group of \( C \), and \( N_4 \) the number of 4-simplices in \( C \).

By writing the contributions from all the tetrahedra meeting on a particular triangle in terms of rotation matrices, one can show that the holonomy around any triangle is trivial. This ties up with the proposed link between Ooguri’s model and BF theory, as we shall see later.

**B The Archer, Crane-Yetter and Roberts models**

The extension of the Ooguri model to general values of \( q \) was worked out by various people. Archer \[79\] showed how to construct a \( q \)-deformed topological quantum field theory in general dimension, giving realisations in three and four dimensions based on the quantum group \( U_q(SL_N) \), and suggesting that his theory corresponded to BF theory with a cosmological constant.

Crane and Yetter \[80\] outlined the construction of a \( q \)-deformed version of Ooguri’s model and recognised its relationship with the work of Roberts \[64\], who had defined a 4-dimensional generalisation of his own “chain-mail” formulation of the Turaev-Viro invariant. Roberts showed that his invariant for a 4-manifold \( M \) depended on two simple functions of \( r \), one raised to the power of \( \sigma(M) \), the signature, and the other to the power of \( \chi(M) \), the Euler character.

The result of Roberts was disappointing but instructive for those trying to construct a theory of 4-dimensional quantum gravity by this method. Since the models do not give any new information about 4-manifolds, it showed that a more radical generalisation was needed.
C The Barrett-Crane model

An important step forward in these generalisation attempts has been taken recently with the formulation of the Barrett-Crane model. (Although the details of some aspects of the model, and other related models, have yet to be worked out, we consider the ideas sufficiently important to include in this review.) First came the realisation that it made sense to generalise spin networks to relativistic spin networks appropriate to four dimensions \[^{[81]}\]. The symmetry group \(SO(3)\) in three dimensions is replaced by \(SO(4)\) in four dimensions, which has spin covering \(SU(2) \otimes SU(2)\). Barrett and Crane therefore label the triangles by two spin labels rather than one. Thus in a relativistic spin network, the edges (dual to the triangles in the 4-complex) carry labels \((j_1, j_2)\) and the vertices (dual to tetrahedra) carry the appropriate intertwiners. Barrett and Crane suggested that the two labels \(j_1\) and \(j_2\) should be equal to satisfy the constraints at the vertices, and Reisenberger \[^{[82]}\] showed that this solution is unique. Thus the Barrett-Crane model is a constrained doubling of the earlier attempts described in the previous subsections, which can thus be regarded as just describing the self-dual section of gravity.

We now describe the Barrett-Crane model in a little more detail. Consider a single 4-simplex, draw its dual graph and then split the vertices as described for the Ooguri model. The first expression written down by Barrett and Crane for the amplitude of a 4-simplex was of the form

\[
I_1 = \sum_{\text{extra edges}} c_j \{15j\}^2, \tag{26}
\]

where \(c_j\) is a weight factor and the \(15j\)-symbol is squared because of the \((j, j)\) labelling on each edge of the dual graph. It turned out to be very difficult to evaluate the asymptotic value of this expression, so Barrett and Crane tried a second approach.

Label the five tetrahedra in a 4-simplex by \(k\); the spin label on the triangle where tetrahedra \(k\) and \(l\) meet is then denoted by \(j_{kl}\). The matrix representing the element \(g \in SU(2)\) in the irreducible representation of spin \(j_{kl}\) is denoted by \(\rho_{kl}(g)\). Variables \(h_k \in SU(2)\) are assigned to the tetrahedra and the invariant \(I_2\) (the second Barrett-Crane model) is obtained by integrating a function of these variables over each copy of \(SU(2)\):

\[
I_2 = (-1)^{\sum_{k<l} 2j_{kl}} \int_{h \in SU(2)^5} \prod_{k<l} \text{Tr} \rho_{kl}(h_k h_l^{-1}). \tag{27}
\]

The measure used is the Haar measure normalised to unity.

The next step is to relate this expression to the geometry of the 4-simplex \[^{[83]}\]. Using the fact that \(SU(2)\) is isomorphic to \(S^3\), and embedding \(S^3\) in \(R^4\), we can regard the element \(h_k \in SU(2)\) as a unit vector in \(R^4\), normal to the 3-dimensional hyperplane in which tetrahedron \(k\) lies. Then according to a well-known formula in representation theory,
Tr(\rho h_k h_l^{-1}) = \frac{\sin(2j + 1)\phi}{\sin \phi} \quad (28)

where \cos \phi = h_k h_l and \phi is the angle between the normals and thus the exterior angle between the two hyperplanes.

Note that the five hyperplanes define a 4-simplex up to translation and an overall scale. Thus integration over the elements \( h_k \) may be interpreted as integration over all possible 4-simplices.

Recalling the equivalence of the asymptotic value of the Ponzano-Regge model to a path integral with the 3-dimensional Regge calculus action, we now look for a similar result here [84]. We write \( \sin(2j + 1)\phi \) in terms of exponentials and, for large \( j \), use the method of stationary phase to find the asymptotic value of the integral. Setting \( \epsilon_{kl} = \pm 1 \), we write \( I_2 \) as

\[
I_2 = \frac{(-1)^{\sum c_l < l 2j_{kl}}}{(2l)!} \sum_{\epsilon_{kl}=\pm 1} \prod_{h \in SU(2)} \frac{\epsilon_{kl}}{\sin \phi_{kl}} \exp(i \sum_{s \leq l} \epsilon_{kl}(2j_{kl} + 1)\phi_{kl}), \quad (29)
\]

which makes it clear that we need the stationary points of

\[
I = \sum_{k < l} \epsilon_{kl}(2j_{kl} + 1)\phi_{kl}. \quad (30)
\]

Now the \( \phi_{kl} \)'s for a 4-simplex are not independent variables; as is shown in the original formulation of Regge calculus [1], their variations are related by

\[
\sum_{k < l} A_{kl} d\phi_{kl} = 0. \quad (31)
\]

Adding this constraint to \( I \) with a Lagrange multiplier \( \mu \), we find that for each triangle,

\[
\epsilon_{kl}(2j_{kl} + 1) = \mu A_{kl}. \quad (32)
\]

The overall scale can then be fixed by taking \( \mu = \pm 1 \).

What has been established is that for a stationary phase point, then firstly, the angles \( \phi_{kl} \) are those of a geometric 4-simplex with triangle areas

\[
A_{kl} = 2j_{kl} + 1, \quad (33)
\]

and secondly, the integrand is \( \exp(i\mu I_R) \), with

\[
I_R = \sum_{\text{triangles } k l} A_{kl} \phi_{kl}, \quad (34)
\]

the Regge calculus version of the Einstein action for a 4-simplex, with \( \mu = \pm 1 \).

The formulation of this model is by no means complete. The next step is to sum over 4-simplices, which is likely to be more difficult than for the first model,
where the extra labels on tetrahedra could provide links between neighbouring 4-simplices. The resulting expression will need to be regularised by passing to representations of the quantum group $U_q(SL_2)$, as in the transition from the Ponzano-Regge state sum to that of Turaev and Viro. This analogy is not precise because the Barrett-Crane amplitude is not independent of triangulation. The covariance lost here may perhaps be restored by summing over triangulations using a generalised matrix model approach, as suggested by De Pietri, Freidel, Krasnov and Rovelli [85]. (Note that these authors refer to what we have called the “second Barrett-Crane model” as their “first version”.)

We shall return to the interpretation of this model in the next subsection, but first note that the formulation described so far is Euclidean. There have been Lorentzian models proposed recently: in (2+1) dimensions, Freidel [86] has set up a version in which $SU(2)$ is replaced by $SL(2, R)$, for which both discrete and continuous representations are used. This results in a model in which time is discrete and space continuous. The partition function requires summation over causal structures, which obviously has no analogue in the Euclidean case. The 6j-symbols for the discrete series representation of $SL(2, R)$ were defined first by Davids [87], who also obtained the analogous Ponzano-Regge formula, which here involves $exp(iI_L)$, where $I_L$ is the Lorentzian Regge action. In (3+1) dimensions, Barrett and Crane [88] have proposed versions based on the classical Lorentz group and on the quantum Lorentz algebra, but the second of these is still at a preliminary stage.

\section*{D Relation to BF theory}

This is not the place for a review of BF theory, but let us briefly mention its relevant properties. It is a gauge theory which can be defined in any dimension and is “background-free” in the sense that no pre-existing metric or other geometrical structure on space-time is needed. It is a theory with no local degrees of freedom.

The action for BF theory in four dimensions is

$$I_{BF} = \int_M Tr(B \wedge F),$$

(35)

where $B$ is a Lie algebra-valued 2-form, and $F = dA + A \wedge A$, with $A$ the connection 1-form. It gives rise to the constraint $F = 0$, which means that the connection $A$ is flat. This ties up with the trivial holonomy around triangles in the Ooguri model. The other constraint, $d_AB = 0$, is the statement of a particular type of gauge symmetry in BF theory.

To understand the relationship between general relativity and BF theory in four dimensions [89], consider the Palatini formulation of general relativity, which has action

$$I_P = \int_M Tr(e \wedge e \wedge F),$$

(36)
with \( e \) a 1-form on the manifold \( M \), and \( F \) defined in terms of the connection as for BF theories. It is immediately apparent that there is a relationship between this Palatini formulation, and BF theory with \( B \) constrained to be of the form \( e \wedge e \). There is a subtle difference between the equations of motion derived from the two actions: for general relativity, we have

\[
e \wedge F = 0, \quad d_A B = 0
\]  

(37)

as compared with the BF equations

\[
F = 0, \quad d_A B = 0.
\]  

(38)

Thus the equations of general relativity are weaker here than those for BF theory, which, heuristically, is why general relativity in four dimensions is more general than a topological theory. We see that general relativity in four dimensions is equivalent to BF theory with an extra constraint \( (B = e \wedge e) \) (giving rise to the paradoxical statement that adding a constraint produces a less restricted theory!)

We see now a further justification of why, in the Barrett-Crane model, the two spin labels on each triangle should be equal (ie we see the parallel between \((j, j)\) and \( e \wedge e \)). Thus the constraint which Reisenberger \(\text{[82]}\) derived may be interpreted as equivalent to the constraint which relates BF theory to general relativity in four dimensions.

Reisenberger \(\text{[90]}\) has explored further the relationship between the Barrett-Crane model and continuum theories, showing that the model corresponds to an \( SO(4) \) BF theory in which the right- and left-handed areas, defined by the self-dual and anti-self-dual components of \( B \), are constrained to be equal.

Before considering an extension of BF theory in four dimensions, let us return to the case of three dimensions. It can be shown that 3-dimensional general relativity without matter is a special case of BF theory, where the equations of motion give simply that the connection is torsion-free and flat. Adding an extra term to the BF Lagrangian has a very interesting effect. Starting from the modified action

\[
I'_{BF} = \int_M \text{Tr}(B \wedge F + \frac{\lambda}{6} B \wedge B \wedge B)
\]  

(39)

and making the transformation

\[
A_\pm = A \pm \sqrt{\lambda} B,
\]  

(40)

we can show that \( I'_{BF} \) is equal to the difference of the two Chern-Simons actions as in section 2. It was shown there that this was equivalent to 3-dimensional general relativity with a cosmological constant \( \lambda \) related to the deformation parameter \( q \), which gives a finite theory of quantum gravity in that dimension \(\text{[2]}\). Thus a rôle of the cosmological constant is to regularise the theory.
In four dimensions, the extra term that we need seems to be slightly different. The proposed modified action is

\[ I'_{BF} = \int_M \text{Tr}(B \wedge F + \frac{\lambda}{12} B \wedge B). \] (41)

The form of this extra term was first suggested by Archer [79], whose contribution is described earlier. It has been discussed more recently by Baez [89, 92], who gives a very comprehensive discussion of BF theory and the discrete models of quantum gravity in three and four dimensions. (Reference [89] is recommended strongly for fuller details of these issues.) Imposing the constraint \( B = e \wedge e \) as before, the action becomes that for the Palatini formulation of general relativity with cosmological constant,

\[ I'_p = \int_M \text{Tr}(e \wedge e \wedge F + \frac{\lambda}{12} e \wedge e \wedge e \wedge e). \] (42)

This suggests the possibility of finding a regularised version of 4-dimensional quantum gravity by constructing a q-deformed version of the Barrett-Crane model, satisfying the relationship

\[ \lambda \rightarrow 0 \text{ as } q \rightarrow 1. \] (43)

Another possible (and related) way forward is through spin foam models, as described briefly in the next subsection.

### E Spin foam

As mentioned in the section on 3-dimensional gravity, spin networks have played an important role in calculations of invariants of 3-manifolds, and in loop quantum gravity, where they provide a gauge-invariant basis of states [75, 93]. If we wish to describe space-time by this type of method, we need, as we have already remarked, an extension of the concept of spin networks. An alternative to the idea of relativistic spin networks is provided by what has been called spin foam [94], because one can think of a spin foam as a soap film connecting two spin networks at different times. “Sums over surfaces” formulations of loop quantum gravity have been given by Reisenberger and Rovelli [95], and Iwaski [96] has formulated the Ponzano-Regge model in terms of surfaces. Turaev and Viro [60] formulated their theory not only in terms of a triangulation of the 3-manifold but also in terms of simple 2-polyhedra forming a 2-complex embedded in the manifold, and we can interpret this second method as the first example of a spin foam model! The relationship between the evolution of spin networks and the approach using triangulated manifolds has been explored and illuminated by Markopoulou [97].

The theory of spin foam is a way of formalising the calculation of the partition function in BF theory by triangulating manifolds. Recall that a spin network is a graph with edges labelled by irreducible representations and vertices by
intertwiners. Imagine moving such an object through space, or rather spacetime, so that it traces out a 2-dimensional surface, a generic slice through which would be a spin network; this, heuristically, is what we mean by a spin foam. It is a 2-complex, the faces of which are labelled by irreducible representations and the edges by intertwiners. The dual triangulation of a manifold is an example of such an object.

Baez [89] has outlined how to calculate transition amplitudes in BF theory using sums over spin foams, and the derivation of the spin foam model from the classical action principle based on BF theory has been discussed by Freidel and Krasnov [8]. It has already been shown [8] that a particular type of spin network may be evaluated as a Feynman graph, and the idea in the evaluation of spin foam sums is to use Feynman’s sum over histories approach, with BF theory playing the rôle of the free theory and spin foams as 2-dimensional analogues of Feynman diagrams. These techniques have produced agreement with the lowest order terms in the known state sum models [8]. Markopoulou and Smolin [100] have defined a model of the time evolution of spin networks based on local causality rules, which are equivalent to those for spin foams.

Recently Smolin [101] has suggested a connection between evolving spin networks, spin foam and such approaches related to loop quantum gravity, and string theory, where there are clearly intuitive similarities in the evolution of strings and membranes. Any precise equivalence still needs to be worked out, but Smolin’s suggestion is typical of recent ideas in which a number of apparently unrelated approaches to quantum gravity seem at last to be coming together.

IV Area Regge calculus

It seems that those attempts at formulating a theory of quantum gravity in four dimensions described in the last section all need one ingredient to be at all successful; this is the assignment of labelling to the triangles instead of (or possibly as well as) the edges. (This fits in with work by Birmingham and Rakowski [102] who constructed state sum models based on \( \mathbb{Z}_p \) for 4-dimensional triangulated manifolds. When the colourings from \( \mathbb{Z}_p \) were assigned only to the edges, the invariant depended only on the 3-dimensional boundary manifold, but when colourings were assigned also to the triangles, the invariant depended on the 4-dimensional structure.) Even the spin foam description fits into this pattern when one considers the triangulation to which it is dual. By considering the asymptotic value of the amplitude of a 4-simplex, we have seen that in this case, it appears to be related to the path integral with the Regge calculus action but with the triangle areas playing the most important rôle, rather than the edge lengths.

A Problems with the basic idea

The idea that, in four dimensions, the triangle areas could be regarded as the basic variables in a modified form of Regge calculus was first suggested by Rovelli
and the possibility was discussed in some detail in [104]. In this section, we shall consider the advantages and disadvantages of the approach, and report on some progress in understanding the relationship between the two types of variable.

A 4-simplex not only has ten edges, it also has ten triangles. Thus at first sight, the change from edge lengths to triangles areas as basic variables looks very straightforward, but there are actually a number of problems [104].

Consider first a single 4-simplex. It is simple to express the triangle areas in terms of the edge lengths. However, to express the Regge action in terms of the new variables, we need to invert the relationship between areas and edge lengths to be able to calculate the deficit angles. Unfortunately the Jacobian is singular in cases where a number of triangles are right-angled and there is not necessarily a unique set of edge lengths corresponding to a given set of areas [105, 106]. This means that, right from the start, certain regions in the space of edge lengths must be avoided.

Secondly, for a collection of 4-simplices joined together, there will not in general be equal numbers of edges and triangles so there may be ambiguity about which is the correct number of variables.

Thirdly, by considering two 4-simplices meeting on a tetrahedron with all triangle areas assigned, we can envisage the following bizarre situation. Solve for the edge lengths of one of the 4-simplices in terms of its triangle areas. Repeat this for the other 4-simplex. It is possible that the edge lengths of the common tetrahedron will differ according to the 4-simplex where the calculation was done (see [104] for an example). Clearly there are difficulties in interpreting the edge lengths as real physical quantities in the usual sense.

In this section, we shall now discuss possible theories in terms of equations of motion and then investigate the dynamical content of area Regge calculus by studying the weak-field expansion about a flat background in terms of variations in the areas.

B Equations of motion

The counting of degrees of freedom in a discrete theory is never completely straightforward. In a simplicial theory, the usual argument is that in $n$ dimensions, an $n$-simplex has $n(n+1)/2$ edges, which corresponds to the number of independent degrees of freedom of the metric tensor in $n$ dimensions. If one thinks of these variables as being at some chosen point in each simplex, the counting becomes somewhat less clear when one realises that each of the edges is shared by a number of other simplices, so the number of variables per point is quite obscure.

Given this ambiguity, we can take two attitudes to the counting problem in area Regge calculus. Either we can take the areas as the fundamental variables, worrying about the different numbers of edge lengths only insofar as we need them to calculate deficit angles or volumes, or we can regard some of the areas as redundant variables and aim to reduce their number to the number of edge lengths in the simplicial complex.
In the theory where the areas are taken seriously as variables (which is our principal interest here since we aim thereby to understand the the models described as 4-dimensional generalisations of the Turaev-Viro theory), we concentrate on the restricted class of metrics where the Jacobian is non-singular. Then the hyperdihedral angles are well-defined and the Regge action may be written as

$$I_R(A_s) = \sum_t A_t \epsilon_t(A_s)$$ (44)

where the sum is over triangles $t$ and $\epsilon_t$ is the deficit angle at triangle $t$. Variation of the action with respect to the area $A_u$, use of the chain rule, an interchange of the orders of summation and use of the Regge identity [1] leads to

$$\epsilon_u = 0 \quad \text{for all } u.$$ (45)

For details, see [104]. Since all deficit angles vanish, the space is locally flat; the holonomy round any triangle is trivial. This agrees with Ooguri’s state sum model for BF theory [78]. The interpretation of this result is not obvious and the investigation of such spaces using parallel transport is under way.

The other possibility, that of regarding some of the areas as redundant variables, has been investigated by Mäkelä [107]. Clearly in order to recover the conventional view of simplicial gravity where the edge lengths are real physical quantities, it is necessary to impose the condition that a given edge has the same length in whichever 4-simplex that length is calculated. This leads to a large number of constraints: for each edge, there is a constraint for each pair of 4-simplices meeting there. For a simplicial complex with $N_1$ edges and $N_2$ triangles, a total of $N_2 - N_1$ of these constraints will be independent, but it is not easy to give any general rule for picking out which these are. (An ad hoc rule has been formulated for a particular model and it is likely that there is some group-theoretic basis for the rule [108]). Mäkelä has shown that if the variations of the constraints are added in with Lagrange multipliers to the variation of the Regge action expressed in area variables, then the usual Regge calculus equations of motion are recovered.

C Dynamics

Restricting our attention now to the area variable theory without constraints, we investigate its dynamical content by performing a weak field expansion about a flat background [109]. This is in analogy with the weak field expansion for edge length variables [13], which we now describe briefly.

In the original calculation, a 4-dimensional hypercubic lattice is divided into simplices by drawing in various diagonals, giving fifteen edges per vertex. Small variations of the edge lengths about their flat space values are made by setting

$$l_i = l_i^{(0)}(1 + \delta_i)$$ (46)
with $\delta_i \ll 1$. The second variation of the Regge action (the first non-vanishing term) is evaluated as a quadratic expression in the $\delta$'s, written as

$$\delta^2 S = \delta_i M_{ij} \delta_j,$$  \hfill (47)

with $M_{ij}$ a sparse infinite dimensional matrix. A Fourier transform is then performed by relating $\delta$ in the $n$ direction and based at the lattice point $(i, j, k, l)$ steps in the $(1, 2, 4, 8)$ directions from the origin (see [13] for details of the binary notation) to the corresponding $\delta$ at the origin by

$$\delta_n^{(i,j,k,l)} = \omega_1^i \omega_2^j \omega_4^k \omega_8^l \delta_n^{(0)},$$  \hfill (48)

with $\omega_{n} = \exp(2\pi i/n_{n})$, where $n_{n}$ is the period in the $\mu$-direction. Acting on periodic modes, $M$ reduces to a block diagonal matrix with 15x15 dimensional blocks, $M_{\omega}$. This matrix $M_{\omega}$ has four zero modes, corresponding to periodic translations of points of the lattice, and a fifth zero mode corresponding to periodic fluctuations of the hyperbody diagonal. Block diagonalising $M_{\omega}$ decouples four further modes; they enter without $\omega$'s and so do not contribute to the dynamics at all. Their equations of motion constrain them to vanish. We see from this that an apparent mismatch in the number of components (fifteen per vertex) is corrected by the dynamics of the theory, leaving ten degrees of freedom per vertex, as would be expected from the continuum theory. (The zero modes correspond of course to gauge fluctuations.)

We now perform the analogous calculation with area variables. In this case, it is necessary to use a “distorted” hypercubic lattice because the original one contains many right angles which lead to vanishing of the Jacobian when transforming between areas and edge lengths. This is obtained by squeezing each unit hypercube along its hyperbody diagonal until it has length 1 in lattice units, like the edges originally along the coordinate axes. The face and body diagonals then all have length $\sqrt{(3/2)}$. Small variations of these edge lengths about their flat space values are then made and the second variation of the action within each 4-simplex calculated. These variations in edge lengths induce changes in the triangle areas represented by

$$A_i = A_i^{(0)}(1 + \Delta_i),$$  \hfill (49)

with $\Delta_i \ll 1$. Within each 4-simplex, the expressions for the $\Delta_i$'s in terms of the $\delta_i$'s are inverted (uniquely) and the second variation of the action written in terms of the $\Delta_i$'s. Adding together the contributions from all 4-simplices gives

$$\delta^2 S = \Delta_i N_{ij} \Delta_j,$$  \hfill (50)

with $N_{ij}$ again a sparse infinite dimensional matrix. A Fourier transform is then performed as in the edge-length variable case, and $N$ reduces to a block diagonal matrix with 50x50 dimensional blocks $N_{\omega}$ (note that there are 50 triangles based at each vertex). The size of $N_{\omega}$ makes it necessary to investigate the modes numerically, and, somewhat contrary to our original expectations, it turns out that the number of dynamical modes is exactly the same as in
the edge length case. There are again four zero modes, corresponding to periodic fluctuations of the lattice, and six further modes scaling with $k^2$, where $k$ is the momentum in the Fourier transform. The remaining forty modes enter non-dynamically (they are massive and do not scale with momentum) and are constrained to vanish by their equations of motion.

Thus the theory with area variables is equivalent to the edge length variable theory from the point of view of dynamical content. This is very encouraging and gives impetus to the search for the exact correspondence between the variables in models like that of Barrett and Crane, the variables of Regge calculus and ultimately the variables of conventional general relativity. That search continues.

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References

[1] T. Regge, Nuovo Cimento 19, 558 (1961).
[2] R. Friedberg and T. D. Lee, Nucl. Phys. B 225, 1 (1983).
[3] J. Cheeger, W. Müller and R. Schrader, Commun. Math. Phys. 92, 405 (1984).
[4] R. Friedberg and T. D. Lee, Nucl. Phys. B 242, 145 (1984).
[5] M. Roček and R. M. Williams, “Introduction to quantum Regge calculus”, in Quantum Structure of Space and Time, edited by M. J. Duff and C. J. Isham (Cambridge University Press, 1982).
[6] W. A. Miller, “Geometric computation: null-strut geometrodynamics and the inchworm algorithm”, in Dynamical Spacetimes and Numerical Relativity, edited by J. Centrella (Cambridge University Press, 1986).
[7] L. Brewin, Class. Quantum Grav. 5, 839 (1988).
[8] P. A. Tuckey, “Approaches to 3+1 Regge calculus”, Ph.D. thesis, University of Cambridge, 1988.
[9] R. M. Williams and P. A. Tuckey, Class. Quantum Grav. 9, 1409 (1992).
[10] R. D. Sorkin, Phys. Rev. D 12, 385 (1975).
[11] J. W. Barrett, M. Galassi, W. A. Miller, R. D. Sorkin, P. A. Tuckey and R. M. Williams, Int. J. Theor. Phys. 36, 809 (1997).
[12] G. Ponzano and T. Regge, “Semi-classical limit of Racah coefficients”, in *Spectroscopic and Group Theoretical Methods in Physics*, edited by F. Block, S. G. Cohen, A. DeShalit, S. Sambursky and I. Talmi (North Holland, Amsterdam, 1968).

[13] M. Roček and R. M. Williams, Phys. Lett. 104B, 31 (1981); Z. Phys. C 21, 371 (1984).

[14] H. W. Hamber, “Simplicial quantum gravity”, in *Critical Phenomena, Random Systems, Gauge Theories*, edited by K. Osterwalder and R. Stora (North Holland, Amsterdam, 1986); Nucl. Phys. A (Proc. Suppl.) 25, 150 (1991).

[15] H. W. Hamber and R. M. Williams, Nucl. Phys. B 248, 392 (1984); Phys. Lett. 157B, 368 (1985); Nucl. Phys. B 267, 482 (1986); Nucl. Phys. B 269, 712 (1986).

[16] B. Berg, Phys. Rev. Lett. 55, 904 (1985); Phys. Lett. 176B, 39 (1986).

[17] H. W. Hamber and R. M. Williams, Nucl. Phys. B 415, 463 (1994).

[18] J. Riedler, W. Beirl, E. Bittner, A. Hauke, P. Homolka and H. Markum, Class. Quantum Grav. 16, 1163 (1999).

[19] H. W. Hamber and R. M. Williams, Phys. Rev. D 59, 06014 (1999).

[20] P. Menotti and R. P. Peirano, Nucl. Phys. B (Proc. Suppl.) 57, 82 (1997).

[21] J. Ambjørn, M. Carfora and A. Marzuoli, *The Geometry of Dynamical Triangulations*, Lecture Notes in Physics (Springer-Verlag, Berlin, 1997).

[22] U. Pachner, Eur. J. Comb. 12, 129 (1991).

[23] M. Carfora and A. Marzuoli, J. Math. Phys. 36, 6353 (1995).

[24] R. Loll, “Discrete approaches to quantum gravity in four dimensions”, gr-qc/9805049, to appear in *Living Reviews in Relativity*.

[25] J. Ambjørn and R. Loll, Nucl. Phys. B 536, 407 (1999).

[26] J. B. Hartle and S. W. Hawking, Phys. Rev. D 28, 2960 (1983).

[27] J. B. Hartle, J. Math. Phys. 26, 804 (1985).

[28] J. B. Hartle, J. Math. Phys. 30, 452 (1989).

[29] J. Louko and P. A. Tuckey, Class. Quantum Grav. 9, 41 (1991).

[30] J. B. Hartle, Class. Quantum Grav. 2, 707 (1985).

[31] K. Schleich and D. M. Witt, Nucl. Phys. B 402, 411, 469 (1983).

[32] D. Birmingham, Phys. Rev. D 52, 5760 (1995).
[33] C. L. B. Correia da Silva and R. M. Williams, Class. Quantum Grav. 16, 2197 (1999); Class. Quantum Grav. 16, 2681 (1999).

[34] J. B. Hartle, W. A. Miller and R. M. Williams, Class. Quantum Grav. 14, 2137 (1997).

[35] F. Lund and T. Regge, "Simplicial approximation to some homogeneous cosmologies", (1984) unpublished.

[36] C. L. Mannion and J. G. Taylor, Phys. Lett. 100, 261 (1981).

[37] M. Kaku, “Generally covariant lattices, the random calculus and the strong coupling approach to the renormalisation of gravity”, in Quantum Field Theory and Quantum Statistics, edited by I. A. Batalin, C. J. Isham and G. A. Vilkovisky (Adam Hilger, Bristol, 1987).

[38] D. Weingarten, J. Math. Phys. 18, 165 (1977); Nucl. Phys. B 210, 229 (1982).

[39] M. Caselle, A. D’Adda and L. Magnea, Phys. Lett. 232B, 457 (1989).

[40] G. Gionti, “Discrete approaches towards the definition of a quantum theory of gravity”, Ph.D. thesis, SISSA, Trieste, 1989.

[41] N. Kawamoto and H. B. Nielsen, “Lattice gauge gravity with fermions”, preprint, 1990.

[42] G. Immirzi, Nucl. Phys. B (Proc. Suppl.) 57, 65 (1997).

[43] G. ’t Hooft, Class. Quantum Grav. 9, 1335 (1992).

[44] J. R. Gott, Phys. Rev. Lett. 66, 1126 (1991).

[45] M. Roček and R. M. Williams, Class. Quantum Grav. 2, 701 (1985).

[46] G. ’t Hooft, Class. Quantum Grav. 10, 1023 (1993).

[47] H. R. Hollmann and R. M. Williams, Class. Quantum Grav. 16, 1503 (1999).

[48] G. ’t Hooft, Class. Quantum Grav. 10, S79 (1993).

[49] A. P. Gentle and R. M. Williams, in preparation.

[50] M. Welling, Class. Quantum Grav. 14, 929 (1997).

[51] R. Franzosi and E. Guadagnini, Class. Quantum Grav. 13, 433 (1996).

[52] R. Franzosi and E. Guadagnini, Nucl. Phys. B 450, 327 (1995).

[53] G. ’t Hooft, Class. Quantum Grav. 10, 1653 (1993); Class. Quantum Grav. 13, 1023 (1996).
[54] H. Waelbroeck, Phys. Rev. D 50, 4982 (1994).

[55] L. Brewin, Class. Quantum Grav. 15, 2427 (1998).

[56] G. Racah, Phys. Rev. 61, 186 (1942).

[57] A. N. Kirillov and N. Yu. Reshetikhin, “Representations of the algebra $U_q(sl(2))$, $q$-orthogonal polynomials and invariants of links”, in Infinite Dimensional Lie Algebras and Groups, edited by V. G. Kac (World Scientific, Singapore, 1989).

[58] J. D. Roberts, Geometry and Topology 3, 21 (1999).

[59] B. Hasslacher and M. J. Perry, Phys. Lett. B103, 21 (1981).

[60] V. G. Turaev and O. Y. Viro, Topology 31, 865 (1992).

[61] E. Witten, private communication.

[62] V. G. Turaev, Lect. Notes In Math. 1510, 363 (1992); “Topology of shadows”, preprint, 1992.

[63] K. Walker, “On Witten’s 3-manifold invariants”, preprint, 1991.

[64] J. D. Roberts, Topology, 34, 771 (1995).

[65] H. Ooguri and N. Sasakura, Mod. Phys. Lett. A 6, 3591 (1991).

[66] R. M. Williams, Int. J. Mod. Phys. B 6, 2097 (1992).

[67] D. Birmingham, M. Blau, M. Rakowski and G. Thompson, Phys. Rep. 209, 129 (1991).

[68] L. H. Kauffman and S. Lins, Manuscripta Math. 72, 81 (1991).

[69] R. Ionicioiu and R. M. Williams, Class. Quantum Grav. 15, 3469 (1998).

[70] R. Ionicioiu, Class. Quantum Grav. 15, 1885 (1998).

[71] R. Penrose, “Angular momentum: an approach to combinatorial space-time”, in Quantum Theory and Beyond, edited by T. Bastin (Cambridge University Press, 1971).

[72] J. W. Barrett and B. W. Westbury, Trans. Amer. Math. Soc. 348, 3997 (1996).

[73] N. Yu. Reshetikhin and V. G. Turaev, Commun. Math. Phys. 127, 1, (1990).

[74] L. H. Kauffman, Knots and Physics (World Scientific, Singapore, 1991).

[75] C. Rovelli and L. Smolin, Phys. Rev. D 52, 5743 (1995): Nucl. Phys. B 442, 593 (1995).
[76] L. Freidel and K. Krasnov, Class. Quantum Grav. 16, 351 (1999).

[77] D. V. Boulatov, Mod. Phys. Lett. A 7, 1629 (1992).

[78] H. Ooguri, Mod. Phys. Lett. A 7, 2799 (1992).

[79] F. J. Archer, “A simplicial approach to topological quantum field theory”, Ph.D. thesis, University of Cambridge, 1993; J. Geom. Phys. 16, 39 (1995).

[80] L. Crane and D. Yetter, “A categorical construction of 4-D topological quantum field theories”, in Quantum Topology, edited by L. H. Kauffman and R. Baadhio (World Scientific, Singapore, 1993).

[81] J. W. Barrett and L. Crane, J. Math. Phys. 39, 3296 (1998).

[82] M. Reisenberger, J. Math. Phys. 40, 2046 (1999).

[83] J. W. Barrett, Adv. Theor. Math. Phys. 2, 593 (1998).

[84] J. W. Barrett and R. M. Williams, Adv. Theor. Math. Phys. 3, 1 (1999).

[85] R. De Pietri, L. Freidel, K. Krasnov and C. Rovelli, “Barrett-Crane model from a Boulatov-Ooguri field theory over a homogeneous space”, hep-th/9907154.

[86] L. Freidel, Lecture at The Third International Conference on Constrained Dynamics and Quantum Gravity, Villasimius, Italy, September 1999.

[87] S. Davids, “Semiclassical limits of extended Racah objects”, gr-qc/9807061, to appear in J. Math. Phys..

[88] J. W. Barrett and L. Crane, “A Lorentzian signature model for quantum general relativity”, gr-qc/9904025.

[89] J. Baez, “An introduction to spin foam models of BF theory and quantum gravity”, gr-qc/9905087, to appear in Geometry and Quantum Physics, edited by H. Gausterer and H. Grosse, Lecture Notes in Physics, (Springer-Verlag, Berlin).

[90] M. Reisenberger, Class. Quantum Grav. 16, 1357 (1999).

[91] J. W. Barrett, J. Math. Phys. 36, 6161 (1995).

[92] J. Baez, Lett. Math. Phys. 38, 129 (1996).

[93] J. Baez, Adv. Math. 117, 253 (1996).

[94] J. Baez, Class. Quantum Grav. 15, 1827 (1998).

[95] M. Reisenberger and C. Rovelli, Phys. Rev. D 56, 3490 (1997).

[96] J. Iwasaki, J. Math. Phys. 36, 6288 (1995).
[97] F. Markopoulou, “Dual formulation of spin network evolution”, gr-qc/9704013.

[98] L. Freidel and K. Krasnov, Adv. Theor. Math. Phys. 2, 1221 (1998).

[99] L. Freidel and K. Krasnov, “Simple spin networks as Feynman graphs”, hep-th/9903192.

[100] F. Markopoulou and L. Smolin, Nucl. Phys. B 508, 409 (1997); Phys. Rev. D 58, 084032 (1998).

[101] L. Smolin, “Strings as perturbations of evolving spin networks”, hep-th/9801022; “Towards a background independent approach to M-theory”, hep-th/9808192.

[102] D. Birmingham and M. Rakowski, J. Mod. Phys. A 10, 1329 (1995); “State sum models and simplicial cohomology”, hep-th/9405108.

[103] C. Rovelli, Phys. Rev. D 48, 2702 (1993).

[104] J. W. Barrett, M. Roček and R. M. Williams, Class. Quantum Grav. 16, 1373 (1999).

[105] J. W. Barrett, Class. Quantum Grav. 11, 2723 (1994).

[106] P. A. Tuckey, private communication.

[107] J. Mäkelä, “Variation of area variables in Regge calculus”, gr-qc/9801022.

[108] J. Mäkelä and R. M. Williams, in preparation.

[109] M. Roček and R. M. Williams, in preparation.