Friedmann Limit of a point mass universe (is not Friedmann)
— Gravitational Lens Effect —

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We study the distance-redshift relation in a universe filled with point particles, and discuss what the universe looks like when we make the number of particles \(N\) very large, while fixing the averaged mass density. Using the Raychaudhuri equation and a simple analysis of the probability of strong lensing effects, we show that the statistical nature of the amplification is independent of \(N\), and clarify the appearance of the point particle universe.

I. INTRODUCTION

Over the last several decades, a great deal of interest has been paid on the (cumulative) gravitational lensing effect on distant sources due to inhomogeneities in the matter distribution of the universe. This problem has been studied using various methods \([1,2,3,4,5,6,7,8,9,10,11]\). In some cases, the lens objects can be treated as point particles. The point particles may be galaxies for cosmological lenses, or stars for microlensing events. While surveying these papers, one question arises to us out of purely theoretical interest: what happens when we bring the number of particles \(N\) very large, while fixing the mass density? Does it look like a Friedman–Lemaître (FL) universe, or a completely different universe?

In this article, we discuss what the universe looks like when we take the large \(N\) limit of a universe filled with point particles by studying the distance-redshift relation \(†\).

II. DISTANCE-REDSHIFT RELATION IN A POINT MASS UNIVERSE

We distribute point particles of the same mass \(m\) uniformly throughout the universe with a mean separation \(l\). We assume that on large scales, the spacetime is described by an isotropic homogeneous metric (Robertson–Walker metric). The energy density parameter \(\rho\) is of order of \(m/l^3\). Condier a photon beam which is emitted from a distant source which we also treat as a point source. We observe the redshift and the luminosity of this source. During the propagation, the luminosity of the photon beam may be amplified by the gravitational lensing effect. We can consider two types of lensing effect:

- Strong lensing effect: when the beam passes very near to a point particle, it suffers a strong amplification.
- Cumulative weak lensing effect: the beam does not pass very near to any particle, but travels through the “ripples” of gravitational potential, and suffers a weak amplification many times.

The cumulative amplification of the weak lensing effect is estimated as follows \([12,13]\).

The expansion, \(\theta = \frac{1}{2}k_{\alpha\beta}k^{\alpha\beta}\), of the null geodesic satisfies

\[
\frac{d\theta}{d\lambda} = \theta^2 - |\sigma|^2
\]

(2.1)

where \(\lambda\) is the affine parameter and \(\sigma\) is the shear; \(|\sigma|^2 = \frac{1}{2}k_{(\alpha\beta)}k^{\alpha\beta} - \theta^2\). We neglected the vorticity. Since we assume that the beam does not pass very near the particles, the evolution of the shear \(\sigma\) is estimated as

\[
\frac{d\sigma}{d\lambda} = -2\theta \sigma + \mathcal{O}(m/l^3).
\]

(2.2)

Thus, the change in \(\sigma\) during passing by one particle is approximately given by

\[
\Delta \sigma \sim l(m/l^3).
\]

(2.3)

Then, the “random walk” for distance \(L\) results in

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\(\dagger\) The analysis made in this article is valid for any compact object whose mass is contained within its Einstein radius.
\[ \sigma(L) = \mathcal{O}[(L/l)^{1/2}l(m/l^3)]. \] \hspace{1cm} (2.4)

This leads to
\[ \theta(L) = \mathcal{O}[L^2l(m/l^3)^2]. \] \hspace{1cm} (2.5)

Since we assume \( m/l^3 \sim \rho = 3\Omega_0 H_0^2 / 8\pi \) and \( L \sim H_0^{-1} \) where \( H_0 \) is the Hubble constant, we obtain
\[ \sigma(L)L \sim \mathcal{O}[(l/L)^{1/2}] < < 1, \] \hspace{1cm} (2.6)
\[ \theta(L)L \sim \mathcal{O}[(l/L)] < < 1. \] \hspace{1cm} (2.7)

In the FL case, on the contrary, the shear term vanishes and the Ricci focusing term determines the evolution of the expansion;
\[ \frac{d\theta}{d\lambda} = \theta^2 + \mathcal{O}(m/l^3). \] \hspace{1cm} (2.8)

This leads to \( \theta(L)L \sim 1 \). Thus, the cumulative amplification is negligible in the point mass universe when the number of particles is large enough. Note that we have assumed that the relation between the affine parameter and the redshift coincides with that of FL model. The proof that the difference is negligible is given in [1].

As we cannot tell that the source is strongly amplified (i.e., \( P \sim 10 \)) when the impact parameter is \( b \),
\[ \rho \] becomes of order of \( 1 \) when the impact parameter is \( b \).

Next, we estimate the probability that the photon beam suffers the strong lensing effect as follows. We adopt the thin lens approximation. The Einstein radius of the lens whose mass is \( m \) is given by [4]
\[ r_E = 2\sqrt{mR} \] \hspace{1cm} (2.9)

where \( R \) is the distance from the observer to the source. The magnification factor \( A \), the ratio of the flux density in the observed image to the flux density in the absence of the lens, is
\[ A = \frac{1 + 2r_E^2/b^2}{(1 + 4r_E^2/b^2)^{1/2}} \] \hspace{1cm} (2.10)

where \( b \) is the impact parameter. From this expression, we can tell that the source is strongly amplified (i.e., \( A \sim 1 \) becomes of order of 10%) when the impact parameter is close to the Einstein radius; \( b \sim r_E \). We can estimate the probability \( P \) that the source is strongly amplified, by considering the probability of hitting any of \( N \) discs of radius \( r_E \) when a photon beam travels in a tube of area \( S \) (see Fig.1);

\[ P = \frac{\pi r_E^2 N m}{S L} = \frac{\pi r_E^2 L}{m} = 4\pi \rho LR \] \hspace{1cm} (2.11)

where \( L \) is the distance to the source, and \( \rho \) is the mean mass density of the point mass universe. Since \( \rho = 3\Omega_0 H_0^2 / 8\pi \) and \( L, R \sim H_0^{-1} \) where \( H_0 \) is the Hubble constant, the probability \( P \) is much smaller than unity for sources at low redshifts, or in a low density universe, and \( P \sim 1 \) even at rather high redshifts \( z \sim 1 \).

When \( P(z) \) is smaller than unity, part of the beams \( 1 - P(z) \) will reach us without hitting any disc. The distance (which is estimated from the observed flux) to such sources is obtained by following the evolution of the flux in an empty spacetime. That is, we can regard the distance to these sources as so-called Dyer-Roeder distance [3]. Therefore, when we observe the luminosity and redshift of distant point sources in the point mass universe, we would obtain the following:

- Part of sources \( 1 - P(z) \) follow the distance-redshift relation of Dyer-Roeder distance;
- Other sources \( P(z) \) are strongly amplified.

The fraction \( 1 - P(z) \), of the sources which never hit a discs is invariant when we change the value of \( N \). Also, the statistical nature of the distribution of amplification factor is clearly independent of \( N \) when \( P(z) \) is enough smaller than unity. Therefore, we can say

- these natures are independent of the number of particles \( N \) if the mass density \( \rho \) is fixed.

Actually, this is the well known fact “the optical depth of gravitational lens is independent of the mass of the lens.” We here point out that the statement is valid even in the large \( N \) limit. That is, even if the universe is filled not with stars but with much smaller point particles, the distance-redshift relation of point sources satisfies the above features, as long as we can keep our assumptions, such as geometrical optics treatment.

The fraction \( P(z) \) increases for higher redshifts and a high density universe. Then, \( P(z) \) becomes larger than unity. The number of beams which never hit a disc becomes very small, and multi-scattered events dominate. However, we expect that the resulting distribution of luminosity of distant point sources is insensitive to \( N \) as long as the distribution of point particles is random and the thin lens approximation holds. This is because we

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\(^1\) We here do not take into account of the expansion of the universe, but it would not change the essential point. See [4] for discussion including the cosmic expansion.

\(^2\) The discussion in this article holds when the whole mass of the object is contained within the Einstein radius. However, the Einstein radius for a galaxy is usually smaller than its size. The Einstein radius is \( \sim 20\sqrt{z} \) Kpc when \( m = 10^{12} M_\odot \), and \( \sim 10^5 \sqrt{z} \) times the solar radius when \( m = M_\odot \), where \( z \) is the redshift of the lens. Thus, a star is usually smaller, and a galaxy is larger, than its Einstein radius.

\(^3\) We do not consider a case there are two (or more) images; we assume that we cannot resolve them or the flux of one image is much stronger than the other. This may be justified from the fact that the ratio of flux densities is \( 7 \) when \( b = r_E \), and becomes larger as \( b \) increases.
III. DISCUSSION

The above analysis is valid for any matter as long as they are compact enough and interact with photons only through gravity. The lensing objects need not to be galaxies or stars; they may be sands or elementary (but dark) particles. Moreover, the thin lens approximation we adopted above seems to become better as we decrease the mass of lens, since the ratio of Einstein radius $r_E$ to the mean separation $l$ becomes smaller; $l \propto m^{1/3}, r_E \propto m^{1/2}$. Clearly, the effect of cumulative weak lensing effect becomes negligible as we can see from equations (2.6) and (2.7). Thus, we conclude that the behaviour of distance-redshift relation of the point mas universe does not agree with that of a FL model when we take the large $N$ limit.

Holz and Wald [11] studied the lensing effect when the matter distribution of the universe is not homogeneous but the masses are concentrated into compact objects. They commented that the probability distributions of lensing effect are indistinguishable between the cases where the masses of lenses are $M = 10^{12} M_\odot, M = 10^{13} M_\odot$ and $M = M_\odot$. Their results may support the correctness of the above analysis. Related with this, one can show that, in their formalism, the observed distance-redshift relation for any point source follows that of a FL model in the case of uniform (not discrete) distribution of matter, though they do not give an explicit statement. We give a rough proof in the appendix.

In Sugiura et al. [14], it was shown that the discreteness of matter distribution is harmless when we consider the distance-redshift relation in a spherically symmetric space. It suggests that, in the point mass universe, if we average the luminosity of the sources of the same redshift over the whole sky and calculate the distance-redshift relation with the averaged luminosity, it should agree with the FL relation. Holz and Wald [11] state that the averaged luminosity of the beams agrees with that of a FL model. These statements justify the result we obtained in [14].

We also notice that, we would obtain the FL relation if we take the average of sources over a region larger than the mean separation of the intervening lens objects, i.e., over the region which includes enough strong lensing events. That is, when the source is much larger than the intervening lens objects, we can safely calculate its distance using a FL model.

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APPENDIX

We show that the distance–redshift relation obtained by the method in [11] agrees with that of a FL model in the case of uniform and continuous density field.

We start from the geodesic deviation equation. Let $\eta^a$ be the deviation vector, and define matrix $A^{ab}$ by

$$\eta^a(\lambda) = A^a_b(\lambda) \frac{dy^b}{d\lambda}(0)$$  \hspace{1cm} (A1)

where $\lambda$ is the affine parameter. Then the geodesic deviation equation is written as

$$\frac{d^2 A^a_b}{d\lambda^2} = -R_{ede}{}^ak^e \kappa^d A^d_b,$$  \hspace{1cm} (A2)

where $k^a$ is the tangent vector of the null geodesic. In a Robertson-Walker spacetime, this equation takes the form

$$\frac{d^2 A^a_b}{d\lambda^2} = -4\pi\omega^2 \rho A^a_b,$$  \hspace{1cm} (A3)

where $\omega$ is the frequency of the photon and $\rho$ is the energy density of the universe. Then, after traveling for small $\Delta \lambda$,

$$\frac{dA^a_b}{d\lambda}(\lambda + \Delta \lambda) = \frac{dA^a_b}{d\lambda}(\lambda) - 4\pi\omega^2 \rho \Delta \lambda A^a_b(\lambda).$$  \hspace{1cm} (A4)

Consider a ball of radius $R$ whose density is uniform, and a bundle of light ray which passes through this ball at a distance $b$ from the center of the ball. By direct calculation, they show that

$$\frac{dA^a_b}{d\lambda}(\lambda + \Delta \lambda) = \frac{dA^a_b}{d\lambda}(\lambda) - \omega J A^a_b(\lambda)$$  \hspace{1cm} (A5)

where

$$J = 6M(1 - b^2/R^2)^{1/2}/R^2,$$  \hspace{1cm} (A6)

where $M$ is the total mass of the matter inside the ball. From the relations

$$\omega \Delta \lambda = 2(R^2 - b^2)^{1/2}$$  \hspace{1cm} (A7)

and

$$\rho = \left(\frac{4\pi}{3}\frac{R^3}{M}\right)^{-1},$$  \hspace{1cm} (A8)

we can see that equation (A5) agrees with (A4).

[1] R. Kantowski, Astrophys. J. 155, 89 (1969)
[2] C.C. Dyer and R.C. Roeder, Astrophys. J. 189, 167 (1974)
[3] M. Sasaki, Mon. Not. R. Astron. Soc. 228, 653 (1987)
[4] P. Schneider and A. Weiss, Astrophys. J. 330, 1 (1988)
FIG. 1. Photon beam traveling through a tube filled with point mass particles. Each particle is regarded as a disc (or sphere) of radius $r_E$. If the beam hits a disc, it will be strongly amplified.
