Zero-point excitation of a circularly moving detector in an atomic condensate and phonon laser dynamical instabilities

Jamir Marino,1,* Gabriel Menezes,2,* and Iacopo Carusotto3

1Department of Physics, Harvard University, Cambridge MA 02138, USA
2Department of Quantum Matter Physics, University of Geneva, 1211, Geneva, Switzerland
3INO-CNR BEC Center and Department of Physics, University of Trento, I-38123 Povo, Italy

(Dated: January 24, 2020)

Introduction — Since Unruh’s pioneering proposal in 1981 [1], the last decades have witnessed the surge of a new field of research which aims at investigating concepts of general relativity and of quantum field theories in curved backgrounds in the so-called condensed matter analog models of gravity [2]. As a most celebrated example, acoustic analogs of black holes were investigated in trans-sonically flowing atomic Bose-Einstein condensates: the acoustic black hole horizon corresponds to the interface between regions of respectively sub- and super-sonic flow and was anticipated to emit a thermal radiation of phonons via Hawking processes [3]. The first experimental observations of such phenomena [4] were instrumental in triggering the on-going explosion of the field, with a revived interest in using analog models to investigate a variety of different effects of quantum field theories in curved space times, from the dynamical Casimir effect [5], to acceleration radiation [6], to vacuum friction and Casimir forces [1, 7].

The subject of the present work is the phenomenon of rotational superradiance [9, 10], namely the amplification of classical waves reflected by a fast rotating body. In the simplest formulation, superradiance may occur whenever the linear velocity of the object (or of parts of it) exceeds the phase velocity of the waves, so that the wave frequency seen in the comoving frame turns negative. Such negative-energy modes then provide the energy that is required to amplify positive-energy waves via superradiant effects.

In cylindrical geometries, a mode of frequency \(\omega\) and azimuthal quantum number \(n\) can be superradiantly amplified when the angular velocity \(\Omega\) of the rotating body satisfies \(\Omega > \omega/n\). Being a consequence of basic kinematical arguments, superradiance is an ubiquitous phenomenon in physics. Its first incarnation was the discovery of amplification of acoustic waves hinging upon a supersonically moving boundary [11], or the amplification of cylindrical electromagnetic waves interacting with a rotating material [12]. In an astrophysical rotating black hole, the superradiant amplification of waves is a consequence of Hawking’s area theorem and of the space-like character of the generator of time translations inside the ergosphere [10, 13], and it is at the root of the several instability phenomena of the quantum vacuum in Kerr black holes [14, 15]. In the framework of analog models, a number of recent studies have investigated superradiant phenomena in rotating classical and quantum fluids [16] and have established useful relations to basic hydrodynamic phenomena [17].

In this Letter, we investigate novel forms of superradiance amplification and superradiant instabilities that take place in the different configuration where the quantum fluid is at rest but a neutral impurity moves (classically) along a circular trajectory with constant angular speed \(\Omega\) across it, as sketched in Fig. 1. In close analogy with the concept of detector in quantum field theories, the neutral impurity serves as a detector for the phonons in the condensate: it does not possess a static analog of the charge, but its internal dynamics is coupled to the density fluctuations in the same way as the electric dipole of a polarizable object does [1, 6]. In what follows we will con-

Figure 1: A two-level detector with internal frequency, \(\omega_0\), rotates with uniform angular velocity \(\Omega\) at a distance \(R\) from its rotational axis. The impurity couples to the density fluctuations, \(\delta \rho(r)\), of a weakly interacting Bose gas, which is trapped inside a cylindrical cavity of radius \(a\); the cavity is modeled via Dirichlet boundary conditions on the density fluctuations, \(\delta \rho(r)|_{r=a} = 0\).
sider impurities with either a two-level or a harmonic nature of its internal degree of freedom.

Beyond the usual amplification of incident phonon waves from the rotating detector, the signature of superradiance consists here of a spontaneous excitation of the impurity’s internal degrees of freedom in response to zero-point quantum fluctuations in the condensate. In what follows, we will concentrate on the realistic case where the quantum fluid is confined along the radial direction, which provides a perfectly reflecting cylindrical cavity for phonon modes. Then, the conditions for the onset of dynamical instabilities are highlighted in a two-dimensional geometry with discrete Bogolyubov modes, envisioning prospects for a phonon laser operation triggered by the fast moving impurity.

A circularly moving impurity in a uniform condensate — Consider a two-level phonon detector with internal frequency \( \omega_0 \) in circular motion with constant angular velocity \( \Omega \) at distance \( R \) from its rotation axis (see Fig. 1 for an illustration). As originally proposed in [1, 6], the detector is assumed to be coupled to density fluctuations of a weakly interacting three dimensional Bose gas. Such a detector can be realized by means of an atomic quantum dot [18], namely an impurity atom immersed in the condensate and collisionally coupled to the Bose gas via two terms, one reminiscent of the interaction of a charged particle to an electric field, and the other resembling an electric dipole coupling mediated by a coupling constant \( g \). If the first term is cancelled via proper tuning of the interaction constants (e.g. via Feshbach resonances), the Hamiltonian of the system acquires the form

\[
H = \frac{\omega_0}{2} \sigma_z + g \sigma_x \delta \rho(\mathbf{r}) + H_B,
\]

where the field operator \( \delta \rho(\mathbf{r}) \) couples to the detector only along its trajectory and \( \sigma_x, \sigma_z \) are quantum operators (proportional to the Pauli matrices) associated with the two-level detector. It is immediate to recognize how this Hamiltonian closely resemble the dipole coupling between a two-level system and the electromagnetic field. To facilitate the discussion of dynamical instabilities, in the last part of this work we will make use of a generalized model where the two-level detector is replaced by a harmonic-oscillator; physically, such a harmonic oscillator model may be realized with atoms with large spin.

As usual, density fluctuations are treated in Bogolyubov theory [5], and the Bose gas Hamiltonian is thus given by the usual creation \( H_B = \int d\mathbf{q} \omega_q b^\dagger_q b_q \), where \( b^\dagger_q, b_q \) are the usual creation and annihilation operators of Bogolyubov quanta with dispersion relation \( \omega_q \), and momentum \( q \). As customary, we refer with \( m \) to the mass of the condensate’s particles, with \( \mu \) to the condensate’s chemical potential, with \( c \) to its speed of sound, and with \( \xi \) to its healing length.

Vacuum excitation rate — Since the motion of the detector is non-inertial, we can expect a non-vanishing transition probability \( A_\uparrow \) for the detector to jump from the ground to the excited state even for a condensate initially in its ground state. This effect is due to the zero-point quantum fluctuations in the phonon quantum vacuum and results in the emission of a phonon. It can be viewed as a superfluid analog of Zeldovich amplification of electromagnetic waves by a rotating dielectric [12]. A related excitation mechanism was discussed for a detector in uniform super-sonic motion along a rectilinear trajectory in standard electromagnetism, the so-called Ginzburg effect [20], as well as in a BEC analog model [1].

While this effect shares analogies with the well-known Unruh effect [21], it is important to highlight important differences. A particle moving circularly with relativistic acceleration is not expected to emit thermal radiation at the Unruh temperature, contrary to its linear counterpart. Even though one can still try to define an effective temperature for such a particle [22], the rotating Killing vector \( \partial_t - \Omega \partial_\theta \) is not a globally timelike Killing field, which hinders an interpretation in terms of particle content of the quantum vacuum: the dynamics of populations of the energy levels will in general violate detailed balance, thus breaking the thermal nature of the emission.

Taking inspiration from previous works [1], the transition probability \( A_\uparrow \) can be calculated making use of second-order perturbation theory (see ‘Supplemental Material’ for details). In particular, we focus on the case of a circularly moving detector in a cylindrically-shaped condensate, as sketched in

---

**Figure 2:** The dimensionless excitation rate, \( \Gamma_\uparrow = A_\uparrow \omega_0 / P(\omega_0, R) \) (with \( P(\omega_0, R) = g^2 \rho_0/(2m\omega_0R^2) \)), evaluated for \( \tilde{y} = \xi/\alpha = 0.4, \tilde{R} = R/\alpha = 0.6 \) and \( \omega_0/\Omega = 0.3 \), as function of the rescaled rotation speed \( \tilde{v} = (\Omega \alpha)/c \) of the detector. For these specific values of parameters, the rate drops to zero for \( \tilde{v} \lesssim 3.73 \). The first threshold corresponds to setting \( n = 4 \) and \( \nu = 1 \) in Eq. (3), and it is indicated by a dotted green line in the main panel. Each peak represents the resonant contribution associated to the excitation of one eigenmodes of the cylindrical cavity. In figure we indicate some of the resonance channels that open upon increasing \( \tilde{v} \), labelled by the pair of quantum numbers \((n, \nu)\). The inset shows that the minimal velocity for the onset of spontaneous excitation of the ground state occurs at \( n = 4 \) for \( \nu = 1 \) (green dotted line), in agreement with the main panel. Higher values of \( \nu \) yield resonances falling outside the range of \( \tilde{v} \) plotted in the main panel (for instance, \( \tilde{v}_{(n=1, \nu=2)} \simeq 9.64 \)).

---

\[ (\text{Equation number}) \]
Fig. 1. The condensate is assumed to be confined in a cylinder of radius $a$ which we model by imposing Dirichlet boundary conditions, $\delta \rho(r)|_{r=a} = 0$, on the density perturbation (see 'Supplemental Material' and Ref. [21] for further details). This geometry implies quantization of the Bogolyubov cylindrical waves in the BEC, with spatial mode profiles proportional to $J_n(\xi_{n}\nu r/a) e^{in\theta} e^{ikz}$. For each value $n$ of the angular momentum $n$, the radial momenta $q_{n\nu} \equiv \xi_{n\nu}/a$ are determined by the $n$th zero $\xi_{n\nu}$ of the Bessel function $J_n(\cdot)$.

An instance of the dimensionless ground-state excitation rate, $\Gamma_n$, as a function of the rescaled detector speed $\bar{v} = \Omega v_0/c$, is reported in Fig. 2. For the Bogolyubov mode of azimuthal and radial quantum numbers $(n,\nu)$, such excitation rate is non-vanishing only for

$$n\Omega > q_{n\nu}c\left(1 + \frac{q_{n\nu}^2}{2}\right)^{1/2} + \omega_0.$$  

Eq. (2) is a generalised superradiant condition [12] involving the cut-off frequency of the quantized Bogolyubov modes propagating along the axis of the 'cigar', given by $\Omega_{n\nu,k=0} = q_{n\nu}c[1 + (q_{n\nu}/2)^2]^{1/2}$ and shifted by the frequency $\omega_0$ of the detector. Eq. (2) imposes a necessary condition on the angular velocity to excite modes with a given angular momentum $n$. A further trend in the strength of the emission is due to the Bessel factor $J_n(\xi_{n\nu} R/a)$ appearing in the mode profile, that suppresses the coupling of the detector to the angular momentum phonon modes. This Bessel factor and, in particular, its strong suppression at short radii $R$ (the fixed radial coordinate of the detector), puts on rigorous grounds the usual qualitative reasoning based on the local dispersion relation of the waves in the rotating frame and on the need for a local super-sonic motion [23].

In the geometry under consideration here, $\Gamma_n$ displays peaks whenever the condition (2) holds as an equality. For each $n,\nu > 0$, this condition determines the cut-off velocities where new emission channels open

$$\bar{v}_{n,\nu} = \xi_{n\nu}/\sqrt{4 + (\xi_{n\nu}^2/2)}.$$  

In the vicinity of the cut-off, the excitation rate can be written as (see 'Supplemental Material' for the details)

$$\Gamma_n \propto \sum_{n=1}^{\infty} \sum_{\nu=1}^{\infty} \gamma_{n\nu}(\bar{v})\theta(n - \omega_0/\Omega)\theta(\bar{v} - \bar{v}_{n,\nu})$$  

where the square-root divergence

$$\gamma_{n\nu}(\bar{v}) \propto \frac{1}{\sqrt{\bar{v} - \bar{v}_{n,\nu}}}$$  

that is visible in Fig. 2 follows from the effective one-dimensional density of states of each radial-azimuthal branch of Bogolyubov eigenmodes in the cigar-shaped condensate. Interestingly, the dependence of $\bar{v}_{n,\nu}$ on $n$ is non-monotonic for a fixed value of $\nu$, as illustrated in the inset of Fig. 2: this feature explains the non-monotonous labelling of the quantum numbers associated to the opening of the resonance channels in the main panel of Fig. 2.

Even though we have restricted our attention on the experimentally accessible zero-point excitation of the detector, it is useful to recall that this process is always strictly associated to the emission of phonons in the condensate that may also be experimentally detected [4]. As a final remark, it is worth highlighting that the emission mechanisms studied in this work differ in nature from synchrotron radiation from circularly moving charges in classical electrodynamics [28] in exactly the same way as Ginzburg emission from superluminally moving polarizable objects [20] is conceptually different from Cherenkov radiation by moving charges or static dipoles [29].

**Dynamical instabilities** — In the astrophysical context, rotational superradiance can give rise to different kinds of dynamical instabilities depending on the specific geometry, from black hole bombs to ergoregion instabilities [10]. Analog effects are also at play in rotating superfluids [17]. In this final section, we explore dynamical instability mechanisms induced by the circularly moving detector. In order to have discrete modes, we now focus on a two-dimensional cylindrical geometry and we restrict to the lowest excitation mode along the $\hat{z}$ direction, spatial profile proportional to $J_n(\xi_{n\nu} R/a) e^{in\theta}$ in the radial and azimuthal directions.

In order for the instability not to be disturbed by saturation of the two-level detector, we assume that the internal structure of this latter is a harmonic oscillator coupled to the density fluctuations of the condensate. The total Hamiltonian in the frame co-moving with the rotating detector is given by

$$H = \omega_0 d^\dagger d + \sum_{n,\nu} \bar{\omega}_{n\nu} \bar{b}_{n\nu}^\dagger \bar{b}_{n\nu} + \frac{g}{N} \sum_{n,\nu} \delta \rho(\mathbf{r}_D),$$  

where $d, d^\dagger$ are the harmonic oscillator destruction and creation operators for the impurity and we have defined $x = (2\omega_0)^{-1/2}(d + d^\dagger)$. Moreover, we have set $\bar{\omega}_{n\nu} = \omega_{n\nu} - n\Omega$ equal to the Bogolyubov mode frequency in the rotating frame and we indicate with $r_D$ the radial position of the detector. In practical calculations, the sums will be restricted to $-n_0 \leq n \leq n_0$ and $0 \leq \nu \leq \nu_0$. For convenience, a factor $\mathcal{N} = (2n_0 + 1)\nu_0$ has been included to count the number of cylindrical Bogolyubov modes, so to ensure a proper scaling of the coupling in the multi-mode limit $\mathcal{N} \gg 1$.

In order to identify dynamical instabilities, we consider the corresponding equation of motions,

$$\dot{d} = -i \left[ \frac{\omega_0}{\Omega} d + \sum_{n=1}^{n_0} \sum_{\nu=1}^{\nu_0} g_{n\nu}(\bar{b}_{n\nu}^\dagger + \bar{b}_{n\nu}) \right]$$  

$$\dot{\bar{b}}_{n\nu} = -i \left[ \bar{\omega}_{n\nu} \bar{b}_{n\nu} + g_{n\nu}(d + d^\dagger) \right]$$  

where we have introduced the dimensionless coupling be-
with and Hamiltonian can display are physically understood considering allow for direct inspection for the conditions of stability. If those of the second kind (akin to parametric instabilities), a crucial contribution is due to the spatial profile of the mode via the $J_n(\xi_n, R)$ factor. Physically, the onset of instabilities will be observable visible as an exponential growth of the amplitude of some Bogolyubov mode, associated to a correspondingly growing amplitude in the detector’s internal oscillation mode. It is thus apparent how two-level detectors can hardly give rise to instabilities, as their internal motion would be quickly saturated. The choice of restricting to a two-dimensional geometry with discrete Bogolyubov modes removes subtleties related to the propagation of the excitation along the unbounded $\bar{z}$-direction [17].

Triggering these dynamical instabilities of the Bogolyubov modes using detectors moving at large circular velocities is of great interest as a new example of phonon lasing. Even though spontaneous oscillations are a common feature in classical acoustics as well as in laser operation, non-trivial mechanisms for mechanical oscillation accompanied by the onset of quantum fluctuations, were studied in the last decade in a broad range of platforms, e.g. driven-dissipative coupled microcavities [24], ion-traps [25], nanomagnets [26], and optically driven quantum dots [27].

**Perspectives** — To summarize, we have shown in this Letter that a circularly moving impurity immersed in an atomic condensate at rest constitutes a promising avenue to investigate quantum superradiance effects in a novel context. If the interaction of the impurity is tuned in a way to serve as a phonon detector, signatures of superradiance include the excitation of the impurity by zero-point quantum fluctuations of the phonon field in the condensate and the onset of dynamical instabilities for the Bogolyubov modes that provide a new avenue for phonon lasing. A intriguing future direction of study consists of analysing the impact of such superradiant effects on higher order quantum vacuum processes such as the Casimir-Polder forces between a pair of circularly rotating impurities. In view of experimental realizations of this idea in photonic quantum simulators, the open challenge is to find a viable implementation of the moving detector concept in quantum fluids of light.

**Acknowledgements** — IC is grateful to Luca Giacomelli and Andrea Vinante for stimulating discussions on the subject of superradiance.

JM is supported by the European Union’s Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No 745608 (QUAKE4PRELIMAT). GM is supported by Conselho Nacional de Desenvolvimento Científico e Tecnológico – CNPq under grant 310291/2018-6, and Fundação Carlos Chagas Filho de Amparo à Pesquisa do Estado do Rio de Janeiro – FAPERJ under grant E-26/202.725/2018. IC acknowledges financial support from the Provincia Autonoma di Trento and from the FET-Open Grant MIR-BOSE (737017) and employed in the plot of Fig. 2. The resonant condition underlying each instability window is specified in the caption. Different strengths are found for instabilities of the two types. For other strengths is satisfied when equation (2) holds as an equality – an occurrence possible only because of the rotational Doppler shift experienced by the Bogolyubov modes.

This physics is illustrated in Fig. 3 where the imaginary parts of the eigenvalues of the linear system (7) for the fully multi-mode problem are evaluated for the same parameters.

![Figure 3: Positive imaginary parts of the eigenvalues of (7) as a function of $\bar{v}$, evaluated for $g = 0.4$, $R = 0.6$, $\omega_0/\Omega = 0.3$, $\bar{g}/N \simeq 1.53$; numerical cut-off at $n_0 = 9$ and $\nu_0 = 1$ have been used ($N = 19$). The non-vanishing imaginary parts visible in the $\bar{v}$ interval plotted in the figure correspond to the parametric resonances $\bar{\omega}_n + \omega_0 = 0$ for respectively $n = 4, 5, 6$. The parametric resonance for $n = 3$ triggers a small imaginary part of order $10^{-3}$ (not visible in the figure) around $\bar{v} \simeq 3.82$. The dynamical instabilities corresponding to the other peaks of Fig. 2 fall at larger $\bar{v}$ values. Other types of instabilities arise when the conditions $\bar{\omega}_n + \omega_0 \simeq 0$ are satisfied; the one occurring at $n = 3$ and $\nu = 1$ is illustrated in the inset.](image-url)
Quantum Flagship Grant PhoQuS (820392) of the European Union.

* These two authors contributed equally

[1] Unruh W. G., Phys. Rev. Lett., 46 (1981) 1351.
[2] C. Barceló, S. Liberati, and M. Visser, Living Reviews in Relativity, 8, 12 (2005).
[3] Garay L. J., Anglin J. R., Cirac J. I. and Zoller P., Phys. Rev. Lett. 85 (2000) 4643; Balbiniot R., Fabbri A., Fagnocchi S., Recati A. and Carusotto I., Phys. Rev. A 78 (2008) 021603; I. Carusotto et al., New J. Phys. 10 103001 (2008). Recati A., Pavloff N. and Carusotto I., Phys. Rev. A 80, 043603 (2009).
[4] Lahav O., Itah A., Blumkin A., Gordon C., Rinott S., Zayats A.
[5] J.-C. Jaskula, G. B. Partridge, M. Bonneau, R. Lopes, J. Ruaduel, D. Boiron, and C. I. Westbrook, Phys. Rev. Lett. 109, 220401 (2012).
[6] A. Retzker, J. I. Cirac, M. B. Plenio, and B. Reznik, Phys. Rev. Lett. 101, 110402 (2008).
[7] A. Recati, J. N. Fuchs, C. S. Peca, W. Zwerger, Phys. Rev. A 72, 023616 (2005); J. N. Fuchs, A. Recati, W. Zwerger, Phys. Rev. A 75, 043615 (2007); A. Klein, and M. Fleischhauer, Phys. Rev. A 71, 033605 (2005).
[8] J. Marino, A. Recati, and I. Carusotto, Phys. Rev. Lett. 118, 045301 (2017).
[9] J. Bekenstein, M. Schiffer, Phys. Rev. D 58 (1998) 064014.
[10] R. Brito, V. Cardoso, and P. Pani, Superradiance: Energy Extraction, Black-Hole Bombs and Implications for Astrophysics and Particle Physics (Springer, London, 2015).
[11] N. N. Andreev and I. Rusakov, Acoustic of a Moving Medium, GTTI, 1934.
[12] Ya. B. Zeldovich, Zh. Eksp. Teor. Fiz. Pisma 14, 270 (1971) [JETP Letters 14, 180 (1971)]; Zh. Eksp. Teor. Fiz. 62, 2076 (1971) [JETP 35, 1085 (1971)].
[13] G. Menezes, Phys. Rev. D 95, 065015 (2017); Erratum, Phys. Rev. D 97, 029901(E) (2018).
[14] A. Starobinskii, Zh. Eksp. Teor. Fiz. 64, 48 (1973) [Sov. Phys. JETP 87, 28 (1973)]; W. G. Unruh, Phys. Rev. D 10, 3194 (1974).
[15] A. L. Matacz, P. C. W. Davies, and A. C. Ottewill Phys. Rev. D 47, 1557 (1993); G. Kang, Phys. Rev. D 55, 7563 (1997); A. C. Ottewill and E. Winstanley, Phys. Rev. D 62, 084018 (2000).
[16] S. Basak and P. Majumdar, Classical and Quantum Gravity 20, 3907 (2003); T. R. Slatyer and C. Savage, Classical and Quantum Gravity 22, 3833 (2005); F. Federici, C. Cherubini, S. Succi, and M. Tosi, Physical Review A 73, 033604 (2006); M. Richartz, A. Prain, S. Liberati, and S. Weinfurter, Phys. Rev. D 91, 124018 (2015); V. Cardoso, A. Coutant, M. Richartz, and S. Weinfurter, Physical Review Letters 117, 271101 (2016); L. Giacomelli and S. Liberati Phys. Rev. D 96, 064014 (2017); T. Torres, S. Patrick, A. Coutant, M. Richartz, E. W. Tedford, and S. Weinfurter, Nature Physics 13, 833 (2017).
[17] L. Giacomelli and I. Carusotto, arXiv:1905.02447 (2019).
[18] A. Recati, P. O. Fedichev, W. Zwerger, J. von Delft, and P. Zoller, Phys. Rev. Lett. 94, 040404 (2005).
[19] L. Pitaevskii and S. Stringari, Bose-Einstein Condensation and Superfluidity (Oxford Science Publications, Oxford, 2016); C. J. Pethick, H. Smith, Bose-Einstein Condensation in Dilute Gases (Cambridge University Press, Cambridge, 2008).
[20] V. L. Ginzburg and V. P. Frolov, Zh. Eksp. Teor. Fiz. 43, 6, 265 (1961); V. L. Ginzburg, Physics-Uspekhi 39, 973 (1996).
[21] P. C. W. Davies, T. Dray, and C. A. Manogue, Phys. Rev. D 53, 4382 (1996); L. C. B. Crispino, A. Higuchi, and G. E. A. Matsas, Rev. Mod. Phys. 80, 787 (2008).
[22] J. Audretsch, R. Müller and M. Holzmann, Class. Quant. Grav. 12, 2927 (1995).
[23] A. Calogeracos, G. E. Volovik, JETP Lett. 69 (1999) 281-287; Pisma Zh. Eksp. Teor. Fiz. 69 (1999) 257-262.
[24] H. Jing, S. K. Özdemir, X-Y. Lü, J. Zhang, L. Yang, and F. Nori, Phys. Rev. Lett. 113, 053604 (2014); H. Lü, S. K. Özdemir, L.-M. Kuang, Franco Nori, and H. Jing, Phys. Rev. Applied 8, 044020 (2019).
[25] S. Knünz, M. Herrmann, V. Batteiger, G. Saathoff, T. W. Hänsch, K. Vahala, and Th. Udem, Phys. Rev. Lett. 105, 013004 (2010); I. S. Grudinin, Hansuek Lee, O. Painter, and K. J. Vahala, Phys. Rev. Lett. 104, 083901 (2010).
[26] E. M. Chudnovsky and D. A. Garanin Phys. Rev. Lett. 93, 257205 (2004).
[27] J. Kabuss, A. Carmele, T. Brandes, and A. Knorr, Phys. Rev. Lett. 109, 05430 (2012).
[28] J. D. Jackson, Classical Electrodynamics (John Wiley and Sons, New York, 1999).
[29] J. V. Jelley, Čerenkov radiation, and its applications (Pergamon Press, 1958).
Supplemental Material:
Zero-point excitation of a circularly moving detector in an atomic condensate and phonon laser dynamical instabilities

The model

As discussed in Ref. [1], the detector couples to the density fluctuations of a Bose gas via the following Hamiltonian

\[ H_I = g_- \sigma_z \delta \rho(r_A) \]  

(11)

which resembles the dipole coupling in quantum electrodynamics. The Hamiltonian that governs the time evolution of the detector is given by

\[ H_{AA} = \frac{\hbar \omega_0}{2} \sigma_z \]  

(12)

In the above expressions \( \sigma_{x,z} \) are the usual Pauli matrices, which can be written in terms of the atomic two-level states \(|+\rangle\) (with energy \( \hbar \omega_0/2 \)) and \(|-\rangle\) (with energy \( \hbar \omega_0/2 \)), namely

\[ \sigma_x = |+\rangle\langle-| + |-\rangle\langle+| = \sigma_+ + \sigma_- \]

\[ \sigma_z = |+\rangle\langle+| - |-\rangle\langle-|. \]  

(13)

In order to implement our approach, we switch to Heisenberg picture. One gets

\[ H_{AA}(t) = \frac{\hbar \omega_0}{2} \sigma_z(t) \]  

(14)

and

\[ H_I(t) = g_- \sigma_z(t) \delta \rho(t, r_A). \]  

(15)

Notice that we have a coupling between atomic operators and the field operator \( \delta \rho \) which is effective only on the trajectory of the atoms. Henceforth we employ units such that \( \hbar = 1 \).

In order to describe the dynamics associated with the scalar field \( \delta \rho \) one may resort to the usual Bogoliubov theory of weakly interacting Bose gas [4]. Hence one has the following effective Hamiltonian for the scalar field which generates the time evolution with regard to \( t \) (neglecting an overall constant term)

\[ H_F(t) = \int d\mathbf{k} \omega_k b^\dagger_k(t) b_k(t) \]  

(16)

where \( b^\dagger_k, b_k \) are the usual creation and annihilation operators of the scalar field. In addition, \( \mathbf{k} \) labels the wave vector of the field modes. In the next section we will discuss the field quantization and present explicitly the associated modes. Within the Bogoliubov theory of dilute Bose gas, one has the dispersion relation (in the laboratory frame where the condensate is at rest)

\[ \omega_k = ck \sqrt{1 + \left( \frac{k \xi}{c} \right)^2}, \]  

(17)

where \( \xi = 1/\sqrt{m \mu} \) is the healing length and \( c = \sqrt{\mu/m} \) is the local velocity of sound [5]. The quantity \( \mu = \lambda \rho_0 \) is the chemical potential of the condensate.

The method we employ here consists in identifying two different contributions to the time evolution of an arbitrary atomic observable, namely the vacuum fluctuations and radiation reaction. Afterwards, we rewrite these contributions at a given order in perturbation theory as quantum evolutions given by two effective Hamiltonians and then we compute each of such contributions to the atomic energy level shift. As discussed in Refs. [2, 3], one should recall that the vacuum-fluctuation term should contain the free part of the field in addition to the contribution of the field to the atomic observable (the source part). In turn, the radiation-reaction contribution should comprise the free part of the atomic observable and, likewise, the source part of the field which emerges due to the presence of the atom itself.
Density correlations

Our aim in this section is to evaluate the aforementioned density correlation functions. For a dilute Bose gas, small fluctuations on top of the condensate can be described by the Bogoliubov theory of dilute condensates. The time evolution of the macroscopic wavefunction is described by the Gross-Pitaevski equation:

\[ i\frac{\partial}{\partial t}\psi = \left[ -\frac{1}{2m}\nabla^2 + \mu + \lambda|\psi|^2 \right]\psi \]  

(18)

with chemical potential \( \mu \). In the usual formulation to describe elementary excitations in the condensate, one takes a steady state \( \psi_0 \) as the mean field solution,

\[ \psi(t, r) = \psi_0(r)[1 + \delta\psi(t, r)]e^{-i\mu t}. \]  

(19)

From the above equation one may write the atomic Bose gas density as

\[ \rho(x) = |\psi_0(r)|^2[1 + \delta\psi(t, r) + \delta\psi^\dagger(t, r)] \]  

(20)

where we are keeping only terms that are linear in \( \delta\psi \). The operator \( \delta\psi \) describing fluctuations satisfies the Bogoliubov-de Gennes equation,

\[ i\frac{\partial}{\partial t}\delta\psi = -\left[ \frac{1}{2m}\nabla^2 + \frac{\nabla\psi_0}{m}\nabla \right]\delta\psi + \lambda\rho_0 \left( \delta\psi + \delta\psi^\dagger \right), \]  

(21)

where we assumed an uniform Bose gas, \( V_{\text{ext}} = 0 \). Henceforth we assume that \( \rho_0 \) is uniform. We search for solutions in cylindrical coordinates of the form

\[ \delta\psi = (\sqrt{\rho_0})^{-1}\sum_{n=-\infty}^{\infty}\frac{1}{2\pi}\int_0^\infty dk\int_0^\infty dq(qn_k(r)b_{qn_k}(t) + v_{qn_k}^*(r)b_{qn_k}^\dagger(t)) \]  

(22)

with commutation relations given by \((s = (qnk))\)

\[ [b_s, b_s^\dagger] = \frac{(2\pi)}{q}\delta(q - q')\delta(k - k')\delta nn' \]

\[ b_{qnk}(t) = b_{qnk}(0)e^{-i\omega_k t} \]  

(23)

where \( K^2 = q^2 + k^2 \) and all other commutators vanish. The mode functions have the form

\[ u_{qnk}(r) = u_K J_n(qr)e^{i\theta}e^{ikz} \]

\[ v_{qnk}(r) = v_K J_n(qr)e^{i\theta}e^{ikz} \]  

(24)

where \( J_n(z) \) is the Bessel function of the first kind, with a normalization condition given by

\[ \int dr \left( u_{qn}(r)u_{n'}(r) - u_{n}(r)u_{n'}^*(r) \right) = \frac{(2\pi)^2}{q}\delta(q - q')\delta(k - k')\delta nn' \]  

(25)

Employing the closure equation for Bessel functions, one finds that

\[ |u_K|^2 - |v_K|^2 = 1. \]  

(26)

Using the expression for the Laplacian in cylindrical coordinates, one finds that the dispersion relation is given by

\[ \omega_K = cK\sqrt{1 + \left( \frac{K\xi}{2} \right)^2}. \]  

(27)

Without loss of generality, one may consider the coefficients \( u_K, v_K \) as real quantities. One finds

\[ u_K^2 = \frac{1}{2}\left( \frac{\xi_K}{\omega_K} + 1 \right) \]

\[ v_K^2 = \frac{1}{2}\left( \frac{\xi_K}{\omega_K} - 1 \right), \]  

(28)
where $\zeta_K = E_K + \mu$, $E_K = K^2/2m$. Moreover, $(u_K + v_K)^2 = E_K/\omega_K$. Hence, the density fluctuations can be written in cylindrical coordinates as

$$\delta \rho(t, r) = \sqrt{\rho_0} \sum_{n=-\infty}^{\infty} \sum_{\nu=1}^{\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \int_{0}^{\infty} dq q(u_K + v_K)(\phi_{qK} b_{qK}(t) + \phi^*_{qK} b^\dagger_{qK}(t))$$

where $\phi_{qK} = J_n(qr)e^{in\theta}e^{ikz}$. As usual, the associated vacuum state is defined as $b_{qK}|0\rangle = 0$.

Now let us suppose that the system is confined to a cylinder of radius $a$, on which the fields satisfy Dirichlet boundary conditions. The mode functions are given by

$$u_{n\nu K}(r) = u_{n\nu K} J_n \left( \frac{\xi_{n\nu} r}{a} \right) e^{i\nu \theta} e^{ikz}$$

$$v_{n\nu K}(r) = v_{n\nu K} J_n \left( \frac{\xi_{n\nu} r}{a} \right) e^{i\nu \theta} e^{ikz}$$

where $\xi_{n\nu}$ is the $\nu$th zero of the Bessel function $J_n(x)$. Now the normalization condition reads ($p = (n\nu K)$)

$$\int dr \left( u^*_p(r) u_{p'}(r) - v^*_p(r) v_{p'}(r) \right) = (2\pi)^2 \delta(k - k') \delta_{nn'} \delta_{\nu\nu'}.$$  

Using standard relations coming from integrals of Bessel functions, one obtains that

$$|u_{n\nu K}|^2 - |v_{n\nu K}|^2 = \frac{2}{a^2[J_{n+1}(\xi_{n\nu})]^2}.$$ 

Now we find the following dispersion relation

$$\omega_{n\nu K} = c K_{n\nu K} \sqrt{1 + \left( \frac{K_{n\nu K} \xi_{n\nu}}{2} \right)^2}$$

where $K^2_{n\nu K} = \xi^2_{n\nu}/a^2 + k^2$. One also has that

$$\delta \psi = (\sqrt{\rho_0})^{-1} \sum_{n=-\infty}^{\infty} \sum_{\nu=1}^{\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \left( u_{n\nu K}(r) b_{n\nu K}(t) + v^*_{n\nu K}(r) b^\dagger_{n\nu K}(t) \right)$$

and the commutation relations become

$$[b_p, b^\dagger_{p'}] = (2\pi) \delta(k - k') \delta_{nn'} \delta_{\nu\nu'}$$

$$b_{n\nu K}(t) = b_{n\nu K}(0)e^{-i\omega_{n\nu K} t}.$$  

with all other commutators being zero. As above we take the coefficients $u_{n\nu K}, v_{n\nu K}$ to be real. Hence, with an analogous calculation as before, one finds that

$$u^2_{n\nu K} = \frac{1}{2} \left( \frac{\xi_{n\nu}}{\omega_{n\nu K}} + 1 \right) \frac{2}{a^2[J_{n+1}(\xi_{n\nu})]^2}$$

$$v^2_{n\nu K} = \frac{1}{2} \left( \frac{\xi_{n\nu}}{\omega_{n\nu K}} - 1 \right) \frac{2}{a^2[J_{n+1}(\xi_{n\nu})]^2}$$

where $\xi_{n\nu} = E_{n\nu K} + \mu$, $E_{n\nu K} = K^2_{n\nu K}/2m$. Moreover

$$(u_{n\nu K} + v_{n\nu K})^2 = \frac{2}{a^2[J_{n+1}(\xi_{n\nu})]^2} \frac{E_{n\nu K}}{\omega_{n\nu K}}.$$ 

Hence, the density fluctuations become

$$\delta \rho(t, r) = \sqrt{\rho_0} \sum_{n=-\infty}^{\infty} \sum_{\nu=1}^{\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} dk (u_{n\nu K} + v_{n\nu K})(\psi_{n\nu K} b_{n\nu K}(t) + \psi^*_{n\nu K} b^\dagger_{n\nu K}(t)).$$

$$\delta \rho(t, r) = \sqrt{\rho_0} \sum_{n=-\infty}^{\infty} \sum_{\nu=1}^{\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} dk (u_{n\nu K} + v_{n\nu K})(\psi_{n\nu K} b_{n\nu K}(t) + \psi^*_{n\nu K} b^\dagger_{n\nu K}(t)).$$
In this case, the vacuum state is defined as $b_{\nu k}|0_{C}\rangle = 0$.

A cylindrical coordinate system $(t, r, \theta, z)$ rigidly rotating at a fixed angular velocity $\Omega$ is related to the cylindrical inertial coordinate system $(t, r, \theta, z)$ by the usual transformation $\bar{\theta} = \theta - \Omega t$. In this case, the Bogoliubov-de Gennes equation for a uniform Bose gas (with $\rho_0$ uniform) should be properly modified. One finds

$$
\dot{\psi} \left( \frac{\partial}{\partial \theta} - \Omega \frac{\partial}{\partial t} \right) \delta \psi = - \frac{1}{2m} \nabla^2 \delta \psi + \lambda \rho_0 \left( \delta \psi + \delta \psi^\dagger \right),
$$

where the Laplacian $\nabla^2$ is written in cylindrical coordinates. Now one proceeds with analogous considerations as in the case of the cylindrical inertial coordinate system, with the replacements $\bar{\theta} \rightarrow \theta$ and $\omega_{\nu k} \rightarrow \bar{\omega}_{\nu k}$, where $\bar{\omega}_{\nu k} = \omega_{\nu k} - n\Omega$, and we are assuming again that the system is confined to a cylindrical mirror of radius $a$. It is evident that the associated rotating modes are not generally of positive frequency. The generalization for this rotating coordinate system is then straightforward. One obtains

$$
\delta \rho(t, r) = \sqrt{\rho_0} \sum_{n=-\infty}^{\infty} \sum_{\nu=1}^{\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} dk (u_{\nu k} + v_{\nu k}) \left( \bar{\psi}_{\nu k} b_{\nu k}(t) + \bar{\psi}^*_{\nu k} \bar{b}_{\nu k}(t) \right),
$$

where $\bar{\psi}_{\nu k} = J_n \left( \frac{\xi_{\nu r}}{\bar{a}} \right) e^{i\bar{\theta} e^{ikz}}$, $\bar{b}_{\nu k}(t) = \bar{b}_{\nu k}(0)e^{-i\bar{\omega}_{\nu k} t}$, and

$$
(u_{\nu k} + v_{\nu k})^2 = \frac{2}{a^2[J_n(\xi_{\nu r})]^2 (\bar{\omega}_{\nu k} + n\Omega)}
$$

with $\bar{\omega}_{\nu k} + n\Omega > 0$. The creation and annihilation operators $b_{\nu k}, \bar{b}_{\nu k}$ satisfy identical commutation relations to those involving canonically conjugated observables in an inertial coordinate system. An analogous situation is discussed in Ref. [6]. The authors argue that, due to the fact that the Bogoliubov transformation between the rotating and inertial vacuum states is trivial, the inertial and rotating vacuum should produce identical results. This means that the rotating vacuum $|\bar{0}\rangle$, defined as $b_{\nu k}|\bar{0}\rangle = 0$ should be identical to the vacuum state $|0_{C}\rangle$ defined above. For more discussion regarding stationary coordinate systems on flat spacetime, we also refer the reader to the Refs. [7, 8].

Let us present the associated expressions for the correlation function of the density fluctuations. One finds, for the inertial vacuum in an unbounded space

$$
\langle 0_M|\delta \rho(t, r)\delta \rho(t', r')|0_M\rangle = \frac{\rho_0}{2m(2\pi)} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} dk \int_{0}^{\infty} dq q \left( q^2 + k^2 \right) \frac{E_{\nu k}}{\omega K}
$$

$$
\times J_n(qr)J_n(q'r)e^{in(\theta-\theta')}e^{ik(z-z')e^{-i\omega_{\nu k}(t-t')}.
$$

Here the detector’s trajectory is then given by $r = R = \text{const}$, $z = \text{const}$ and $\theta = \Omega t$. On the other hand, if one wishes to switch to a frame co-moving with the rotating detector, one should consider the above results in a rotating coordinate system. This is most easily achieved by performing the calculations in the rotating coordinate system discussed above. As argued above, in this case we consider that the system is confined inside a cylindrical mirror of radius $a$. One finds that

$$
\langle \bar{0}|\delta \rho(t, r)\delta \rho(t', r')|\bar{0}\rangle = \frac{\rho_0}{m(2\pi)} \sum_{n=-\infty}^{\infty} \sum_{\nu=1}^{\infty} \frac{1}{a^2[J_n(\xi_{\nu r})]^2} \int_{-\infty}^{\infty} dk \left( \frac{\xi_{\nu r}^2}{a^2 + k^2} \right) \frac{E_{\nu k}}{(\bar{\omega}_{\nu k} + n\Omega)}
$$

$$
\times J_n \left( \frac{\xi_{\nu r}}{a} \right) J_n \left( \frac{\xi_{\nu r}'}{a} \right) e^{in(\bar{\theta}-\bar{\theta}')} e^{ik(z-z')} e^{-i\bar{\omega}_{\nu k}(t-t')}.
$$

In this case, the detector’s trajectory is simply given by $r = R = \text{const}$, $z = \text{const}$ and $\bar{\theta} = \text{const}$. Given the correlation function of the field $\delta \rho$, we can construct the relevant quantities of the formalism employed here. For instance, the normalized symmetric correlation function of the field is given by

$$
D(t, t'; r, r') = \langle 0_a|\{\delta \rho(t, r), \delta \rho(t', r')\}|0_a\rangle,
$$

where $a = M, C$, depending on the situation that is being under study, whereas the normalized linear susceptibility of the field reads

$$
\Delta(t, t'; r, r') = \langle 0_a|\{\delta \rho(t, r), \delta \rho(t', r')\}|0_a\rangle.
$$

One should consider the free part of the field when evaluating such expressions.
Rate of variation of the atomic energy

Now one is ready to evaluate the contributions from vacuum fluctuations and radiation reaction to the rate of change of the atom excitation energy. The sum of such contributions produces the total rate. One finds, for large enough $\Delta t = t - t_0$ the following expression for the rate of change of the mean atomic excitation energy in the case of the rotating atom coupled with an inertial vacuum state (in an unbounded space)

$$
\left\langle \frac{dH_{AA}}{dt} \right\rangle = -g^2 \frac{\rho_0 \omega_0}{2m} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} dk \int_{0}^{\infty} dq \frac{(q^2 + k^2)}{\omega_K} (J_n(qR))^2 \delta(\omega_K - \Delta \omega - n\Omega).
$$

(44)

where $\Delta \omega = \omega - \omega'$ and $A(\omega, \omega') = |\langle \omega|\sigma_x(0)|\omega' \rangle|^2$, $|\omega\rangle$ and $|\omega'\rangle$ being atomic states (this quantity comes from the atomic correlation functions). The superscript $f$ here means the free part of the atomic operator $\sigma_x$, as discussed in Sec.. Observe that, for the rate of change of the atomic excitation energy, when $\Delta \omega > 0$ one has spontaneous emission; otherwise, for $\Delta \omega < 0$ one has spontaneous excitation.

Let us consider that the atom was initially prepared in the state $|+\rangle$. In this case, $\Delta \omega = -\omega_0$ and $A(\omega, \omega') = 1$. Hence

$$
\left\langle \frac{dH_{AA}}{dt} \right\rangle = g^2 \frac{\rho_0 \omega_0}{2m} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} dk \frac{q_n^-}{k^2 + (q_n^-)^2} F(k, q_n^-) (J_n(q_n^- R))^2 \theta(-\omega_0 + n\Omega)
$$

where

$$
F(k, q) = \frac{K^3}{\omega_K}, \quad K = \sqrt{q^2 + k^2}
$$

$$
G(k, q) = \frac{c q (\xi^2 (k^2 + q^2) + 2)}{\sqrt{k^2 + q^2} \sqrt{\xi^2 (k^2 + q^2) + 4}}
$$

$$
q_n^\pm = q_n^\pm(k) = \frac{1}{\xi} \left[ \frac{2\sqrt{c^2 + \xi^2 (\omega_0 \pm n\Omega)^2}}{c} - (k^2 \xi^2 + 2) \right]^{1/2}.
$$

(46)

Observe that $q_n^-$ is real (and hence spontaneous excitation may occur) only if

$$
n\Omega > k c \sqrt{1 + \left( \frac{k \xi}{2} \right)^2} + \omega_0
$$

(47)

where we used the fact that $n$ must be a positive integer. This implies that

$$
n\Omega/c > k \sqrt{1 + \left( \frac{k \xi}{2} \right)^2}.
$$

(48)

On the other hand, if the atom was initially prepared in the state $|-\rangle$, one would find that $\Delta \omega = \omega_0$ and $A(\omega, \omega') = 1$. Therefore

$$
\left\langle \frac{dH_{AA}}{dt} \right\rangle = -g^2 \frac{\rho_0 \omega_0}{2m} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} dk \frac{q_n^+}{k^2 + (q_n^+)^2} F(k, q_n^+) (J_n(q_n^+ R))^2 \theta(\omega_0 + n\Omega).
$$

(49)

For a positive integer $n$, $q_n^+$ is real if the following condition is met:

$$
\omega_0 > -n\Omega + k c \sqrt{1 + \left( \frac{k \xi}{2} \right)^2}
$$

(50)

which implies that

$$
n\Omega/c < k \sqrt{1 + \left( \frac{k \xi}{2} \right)^2}.
$$

(51)
Let us recast our results in terms of dimensionless variables, defined as \( X_n^\pm = \frac{g_n^\pm}{R}, z = c/(\omega_0 R), W = \omega_0/\Omega, \ell(R) = kR, y(R) = \xi/R, \) and \( \tilde{v}(R) = \Omega R/c. \) Notice that for \( \tilde{v} > 1 \) we are in the supersonic regime. In terms of such variables, the spontaneous excitation rate can be rewritten as

\[
\left< \frac{dH_{AA}}{dt} \right> = P(\omega_0, R)W^2 \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} dt \frac{X_n^-}{\ell^2 + (X_n^-)^2} \frac{\bar{F}(\ell, X_n^-)}{G(\ell, X_n^-)} (J_n(X_n^-))^2 \theta(-W + n)
\]

(52)

where

\[
P(\omega_0, R) = \frac{g_0^2 \rho_0}{2m\omega_0 R^5}
\]

(53)

and

\[
\omega_\tilde{v}(\ell, X_n^\pm) = \left( \frac{\ell^2 + (X_n^\pm)^2}{\tilde{v}} \right)^{1/2} \sqrt{1 + \left( \frac{(\ell^2 + (X_n^\pm)^2)}{2} \right)^2}
\]

\[
\bar{F}(\ell, X_n^\pm) = \frac{(\ell^2 + (X_n^\pm)^2)^{3/2}}{\omega_\tilde{v}(\ell, X_n^\pm)}
\]

\[
\bar{G}(\ell, X_n^\pm) = \frac{X_n^\pm \left[ y^2 (\ell^2 + (X_n^\pm)^2) + 2 \right]}{\tilde{v} \sqrt{\ell^2 + (X_n^\pm)^2} y (\ell^2 + (X_n^\pm)^2) + 4}
\]

\[
X_n^\pm = X_n^\pm(\ell) = \frac{1}{y} \left[ 2\sqrt{1 + \tilde{v}^2y^2(W \pm n)^2} - (\ell^2y^2 + 2) \right]^{1/2}.
\]

(54)

For spontaneous emission, one has that

\[
\left< \frac{dH_{AA}}{dt} \right> = -P(\omega_0, R)W^2 \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} dt \frac{X_n^+}{\ell^2 + (X_n^+)^2} \frac{\bar{F}(\ell, X_n^+)}{G(\ell, X_n^+)} (J_n(X_n^+))^2 \theta(W + n).
\]

(55)

For clarity we have omitted the dependence on \( R \) for the variables \( \ell, y, \tilde{v} \) in the above expressions.

Finally, let us consider the rate of variation of the atomic energy with respect to the co-moving frame of the rotating detector. In this situation we confine the system inside a cylindrical mirror of radius \( a \). With an almost identical calculation as before, one finds, for a large enough \( \Delta t \)

\[
\left< \frac{dH_{AA}}{dt} \right> = -g_0^2 \rho_0 \sum_{n=-\infty}^{\infty} \sum_{\nu=1}^{\infty} \frac{\Delta \omega A(\omega, \omega')}{\omega_n^0(\xi_n \nu)} \left[ J_n \frac{(\xi_n R \nu)}{a} \right]^2 \int_{-\infty}^{\infty} dk \frac{(\xi_n^2/a^2 + k^2)}{(\omega_n^r \nu + \nu \Omega)} \left[ J_n \frac{(\xi_n R \nu)}{a} \right]^2 \delta(\omega_n^r - \Delta \omega - n\Omega).
\]

(56)

For spontaneous excitation, we find that

\[
\left< \frac{dH_{AA}}{dt} \right> = g_0^2 \rho_0 \sum_{n=-\infty}^{\infty} \sum_{\nu=1}^{\infty} \frac{\Delta \omega A(\omega, \omega')}{\omega_n^0(\xi_n \nu)} \left[ J_n \frac{(\xi_n R \nu)}{a} \right]^2 \frac{(\xi_n^2/a^2 + (ak_n^-)^2)}{(\omega_n^r \nu + \nu \Omega)} \theta(-\omega_0 + \nu \Omega) \frac{G_n^r R_n^r}{G_n^r R_n^r + 2}
\]

(57)

where

\[
G_{n^r k^r} = \frac{ck(\xi_n^2 k_n^2 + 2)}{K_n k_n^r \sqrt{\xi_n^2 k_n^2 + 4}}
\]

(58)

and

\[
k_0^\pm = \frac{1}{\xi} \left[ 2\sqrt{c^2 + \xi^2(\omega_0 \pm \nu \Omega)^2} - \left( \frac{\xi^2 \xi \nu}{a^2} + 2 \right) \right]^{1/2}.
\]

(59)

Observe that \( k_0^\pm \) is real (and hence spontaneous excitation may occur) only if

\[
n\Omega > \frac{c \xi \nu}{a} \sqrt{1 + \left( \frac{\xi \nu}{2a} \right)^2} + \omega_0
\]

(60)
where we used the fact that $n$ must be a positive integer. This implies

$$\frac{n\Omega a}{c} > \xi_{n\nu} \sqrt{1 + \left(\frac{\xi_{n\nu}}{2a}\right)^2}. \quad (61)$$

The lowest bound is obtained for $\nu = 1$. In particular, since the zeros of the Bessel function obey $\xi_{n\nu} > n$, we obtain that the atom remains inert unless $\Omega a > c$. Hence we obtain a similar conclusion as the one in Ref. [6]. In other words, if the cylindrical mirror has a radius greater than $c/\Omega$, then spontaneous excitation can occur.

In terms of the dimensionless variables defined above, one has that

$$\nu$$

On the other hand, for spontaneous emission, we find that

$$\nu$$

one obtains that

$$\nu$$

where

$$\nu$$

$$\nu$$

$$\nu$$

On the other hand, for spontaneous emission, we find that

$$\nu$$

For a negative integer $n$, the condition for $k_0^+$ to be real is similar to the one derived above for $k_0^-$. On the other hand, for a positive integer $n$, $k_0^+$ is real if the following condition is met:

$$\omega_0 > -n\Omega + \frac{c\xi_{n\nu}}{a} \sqrt{1 + \left(\frac{\xi_{n\nu}}{2a}\right)^2} \quad (65)$$

which implies that

$$\frac{n\Omega a}{c} < \xi_{n\nu} \sqrt{1 + \left(\frac{\xi_{n\nu}}{2a}\right)^2}. \quad (66)$$

In terms of the dimensionless variables defined above, one has that

$$\nu$$

\textbf{Einstein coefficients}

It is not difficult to display the Einstein coefficients for spontaneous emission, denoted by $A_{\uparrow}$, and spontaneous excitation, given by $A_\downarrow$. Using a similar approach as employed in Ref. [9], one finds

$$A_{\uparrow} = P(\omega_0, R) \frac{\omega_0^2}{\Omega^2} \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} d\ell \frac{X_n^-}{\ell^2 + (X_n^-)^2} \frac{F(\ell, X_n^-)}{G(\ell, X_n^-)} \left(J_n(X_n^-)\right)^2 \theta(-W + n) \quad (68)$$
and

$$A_\downarrow = P(\omega_0, R) \frac{\omega_0}{\Omega^2} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} d\ell \frac{X_n^+}{\sqrt{\ell^2 + (X_n^+)^2}} \frac{\tilde{F}(\ell, X_n^+)}{G(\ell, X_n^+)} (J_n(X_n^+))^2 \theta(W + n).$$  (69)

One can define the following dimensionless Einstein coefficients from the above expressions:

$$\Gamma_\uparrow = \frac{\omega_0}{P(\omega_0, R)} A_\uparrow = W^2 \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} d\ell \frac{X_n^-}{\sqrt{\ell^2 + (X_n^-)^2}} \frac{\tilde{F}(\ell, X_n^-)}{G(\ell, X_n^-)} (J_n(X_n^-))^2 \theta(-W + n)$$

(70)

and

$$\Gamma_\downarrow = \frac{\omega_0}{P(\omega_0, R)} A_\downarrow = W^2 \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} d\ell \frac{X_n^+}{\sqrt{\ell^2 + (X_n^+)^2}} \frac{\tilde{F}(\ell, X_n^+)}{G(\ell, X_n^+)} (J_n(X_n^+))^2 \theta(W + n).$$

(71)

Defining the excitation rate per mode as $d\Gamma/d\ell$, one finds, for large $\bar{\nu}$ (at leading order)

$$\frac{d\Gamma_\uparrow}{d\ell} \approx \bar{\nu}^{-3/2} \mathcal{F}(\bar{\nu})$$

(72)

where

$$\mathcal{F}(\bar{\nu}) = 4W^2 \sum_{n=1}^{\infty} \frac{\theta(-W + n)}{\pi (2(-W + n)^3)^{1/2}} \cos^2 \left( \frac{B^-}{4} \right)$$

$$B^\pm = B^\pm(\ell, \bar{\nu}) = \sqrt{2} \left( \ell^2 y^2 + 2 \right) \sqrt{(-W + n)\bar{\nu}} - 4\sqrt{2} \frac{(-W + n)\bar{\nu}}{y} + 2\pi n + \pi.$$  (73)

Finally, for the case of the rotating vacuum (with the system confined inside a cylindrical mirror), one finds that

$$A_\uparrow = \frac{4\omega_0}{\Omega^2} P(\omega_0, a) \sum_{n=1}^{\infty} \sum_{\nu=1}^{\infty} \frac{J_n(\xi_{n\nu}R)}{J_{n+1}(\xi_{n\nu})} \frac{X_{n\nu\bar{\nu}}^2}{\omega_{n\nu\bar{\nu}}^2} \frac{\theta(-W + n)}{G_{n\nu\bar{\nu}}(\bar{\nu})}$$

(74)

and

$$A_\downarrow = \frac{4\omega_0}{\Omega^2} P(\omega_0, a) \sum_{n=-\infty}^{\infty} \sum_{\nu=1}^{\infty} \frac{J_n(\xi_{n\nu}R)}{J_{n+1}(\xi_{n\nu})} \frac{X_{n\nu\bar{\nu}}^2}{\omega_{n\nu\bar{\nu}}^2} \frac{\theta(W + n)}{G_{n\nu\bar{\nu}}(\bar{\nu})}.$$  (75)

Accordingly, the associated dimensionless coefficients are given by

$$\Gamma_\uparrow = \frac{\omega_0}{P(\omega_0, a)} A_\uparrow = 4W^2 \sum_{n=1}^{\infty} \sum_{\nu=1}^{\infty} \frac{J_n(\xi_{n\nu}R)}{J_{n+1}(\xi_{n\nu})} \frac{X_{n\nu\bar{\nu}}^2}{\omega_{n\nu\bar{\nu}}^2} \frac{\theta(-W + n)}{G_{n\nu\bar{\nu}}(\bar{\nu})}$$

(76)

and

$$\Gamma_\downarrow = \frac{\omega_0}{P(\omega_0, a)} A_\downarrow = 4W^2 \sum_{n=-\infty}^{\infty} \sum_{\nu=1}^{\infty} \frac{J_n(\xi_{n\nu}R)}{J_{n+1}(\xi_{n\nu})} \frac{X_{n\nu\bar{\nu}}^2}{\omega_{n\nu\bar{\nu}}^2} \frac{\theta(W + n)}{G_{n\nu\bar{\nu}}(\bar{\nu})}.$$  (77)

For large $\bar{\nu}$ one obtains, at leading order

$$\Gamma_\uparrow \approx \bar{\nu}^{-3/2} C^-_{n\nu}$$

(78)

where

$$C^-_{n\nu} = 8W^2 \sum_{n=1}^{\infty} \sum_{\nu=1}^{\infty} \frac{J_n(\xi_{n\nu}R)}{J_{n+1}(\xi_{n\nu})} \frac{\theta(\pm W + n)}{\sqrt{2(\pm W + n)y^2}}.$$  (79)
∗ These two authors contributed equally

[1] J. Marino, A. Recati and I. Carusotto, Phys. Rev. Lett. 118, 045301 (2017).
[2] J. Dalibard, J. Dupont-Roc, and C. Cohen-Tannoudji, J. Phys. (Paris) 43, 1617 (1982).
[3] J. Dalibard, J. Dupont-Roc, and C. Cohen-Tannoudji, J. Phys. (Paris) 45, 637 (1984).
[4] Y. Castin, in Coherent Atomic Matter Waves, Lecture Notes of Les Houches Summer School, edited by R. Kaiser, C. Westbrook, and F. David (EDP Sciences and Springer-Verlag, Berlin-Heidelberg, 2001).
[5] L. Pitaevskii and S. Stringari, Bose-Einstein Condensation and Superfluidity (Oxford Science Publications, Oxford, 2016); C. J. Pethick, H. Smith, Bose-Einstein Condensation in Dilute Gases (Cambridge University Press, Cambridge, 2008).
[6] P. C. W. Davies, T. Dray and C.A. Manogue, Phys. Rev. D 53, 4382 (1996).
[7] J. R. Letaw and J. D. Pfautsch, Phys. Rev. D 24, 1491 (1981).
[8] G. Duffy and A.C. Ottewell, Phys. Rev. D 67, 044002 (2003).
[9] J. Audretsch and R. Müller, Phys. Rev. A 50, 1755 (1994).
[10] A. Pais and G. E. Uhlenbeck, Phys. Rev. 79, 145 (1950).
[11] M. Pavsic, Int. J. Geom. Methods Mod. 13, 1630015 (2016).
[12] M. V. Ostrogradski, Mem. Acad. Imp. Sci. St. Petersbg. 6, 385 (1850).