WEAKLY CONTRACTIVE ITERATED FUNCTION SYSTEMS AND BEYOND: 
A MANUAL

KRZYSZTOF LEŚNIAK, NINA SNIGIREVA, FILIP STROBIN

Abstract. We give a systematic account of iterated function systems (IFS) of weak contractions of different types (Browder, Rakotch, topological). We show that the existence of attractors and asymptotically stable invariant measures, and the validity of the random iteration algorithm ("chaos game"), can be obtained rather easily for weakly contractive systems. We show that the class of attractors of weakly contractive IFSs is essentially wider than the class of classical IFSs’ fractals. On the other hand, we show that, in reasonable spaces, a typical compact set is not an attractor of any weakly contractive IFS. We explore the possibilities and restrictions to break the contractivity barrier by employing several tools from fixed point theory: geometry of balls, average contractions, remetrization technique, ordered sets, and measures of noncompactness. From these considerations it follows that while the existence of invariant sets and invariant measures can be assured rather easily for general iterated function systems under mild conditions, to establish the existence of attractors and unique invariant measures is a substantially more difficult problem. This explains the central role of contractive systems in the theory of IFSs.

Contents

1. Introduction 2
2. Metrically contractive iterated function systems 4
  2.1. Taming a plethora of weak contractions 4
  2.2. Attractors for weakly contractive IFSs 6
  2.3. Invariant measures for weakly contractive IFSs 8
3. Topologically contractive iterated function systems 9
  3.1. Notation and basic definitions 9
  3.2. Existence of attractors and the coding map 10
  3.3. Existence of an invariant measure 12
  3.4. Remetrization of topologically contractive IFSs 15
4. Baire genericity of attractors 16
5. Deterministic chaos game for weakly contractive IFSs 18
6. Between contractive and non-contractive realm 22
  6.1. Eventual contractions 22
  6.2. Average contractive IFSs 23
7. Beyond contractivity 25
  7.1. Multivalued IFSs 26
  7.2. The Birkhoff theorem on invariant set for compact IFSs 27
  7.3. The Birkhoff theorem on invariant set for condensing IFSs 28
  7.4. Condensing maps vs compact maps, IFSs with condensation and weak contractions 30
  7.5. Global maximal attractor of the IFS 31
  7.6. Invariant measures. The Krylov-Bogolyubov theorem for IFSs 33

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1. Introduction

The aim of the article is to show that invariant sets and invariant measures exist in iterated function systems (IFS) under fairly general conditions. Special attention is paid to systems of weakly contractive maps, because invariant sets and invariant measures in such systems turn out to be unique and attracting. A topological characterization of weakly contractive systems is discussed which leads to the notion of topologically contractive IFS. The prevalence of topological fractals (attractors of topologically contractive IFSs) in the sense of Baire category is exhibited. A simple proof of a derandomized (a.k.a. deterministic a.k.a. disjunctive) chaos game for weakly contractive and topologically contractive IFSs has been supplied. Limitations to various generalizations of contractivity for IFSs, as well as employment of metric fixed point theory methods, have been highlighted.

Large part of this article, its core, can be summarized as follows: The sequences of successive iterates of sets and measures are convergent to invariant sets and invariant measures. While this premise comes as no surprise within the realm of contractive IFSs, it is less obvious for non-contractive IFSs, yet true if the convergence is understood in a suitably weakened manner.

In this survey, we omit the theory of fractal dimension (cf. [32]), fractal interpolation (cf. [67]), fractal compression (cf. [56]) and analysis on fractals (cf. [91]), and concentrate on fundamental questions of existence and uniqueness of attractors and invariant measure. The presentation of the chosen aspects of fractal geometry is self-contained. Therefore, we elaborate on some aspects of topology, measure and order.

To facilitate proper understanding of the level of generality of the presented results, we employ the Vietoris topology instead of the Hausdorff metric whenever needed (both yield the same convergence of compacta) and we use Radon measures instead of Borel measures (which is indifferent when measures live on a complete separable metric space). The standard notions such as, Vietoris topology and Monge-Kantorovich metric, are recalled in the appendix for the convenience of a reader.

Our survey complements the existing surveys on IFSs such as [43], [90], [14]. Among many books devoted to fractal geometry we choose to cite [30] and [56] as they best cover the aspects of IFSs we are interested in in this survey.

Let us begin with some necessary definitions.

Definition 1.1. If $X$ is a set and $f : X \to X$, then we say that $x_* \in X$ is a fixed point of $f$ provided $f(x_*) = x_*$. If additionally $X$ is a metric (or topological) space, then $x_*$ is called a contractive fixed point, CFP for short, if for every $x \in X$, the sequence of iterations $(f^k(x))$ converges to $x_*$. 

Remark 1.2. (1) A map having CFP is sometimes called a Picard operator, cf. [84].

(2) A CFP is unique if the underlying space $X$ is at least Hausdorff. Notably, for the considerations of the weak convergence of the iterations of the Markov operator this may not be the case.
(3) If \( x_* \) is a CFP of the \( p \)-fold composition \( f^p \) of \( f \), then \( x_* \) is also a CFP of \( f \). Indeed, let \( x_* \in X \) be a CFP of \( f^p \). Then for every \( x \in X \) and \( i = 0, ..., p - 1 \), we have \( f^{bk+i}(x) = (f^p)^k(f^i(x)) \to x_* \), which implies that \( f^k(x) \to x_* \) (see also [38] chap.I §1.6 (A.1) or [24] Remark 2.4 p.13.)

**Definition 1.3.** An iterated function system \( \mathcal{F} = \{w_1, ..., w_N\} \), IFS in short, is a finite family of maps \( w_i : X \to X \) acting on a (topological or metric) space \( X \). The Hutchinson operator induced by an IFS \( \mathcal{F} \), denoted without ambiguity again by \( \mathcal{F} \), is the map \( \mathcal{F} : 2^X \to 2^X \) acting on the power set \( 2^X \) of subsets of \( X \) and given by the formula

\[
\mathcal{F}(S) = \bigcup_{i=1}^{N} w_i(S) \text{ for every } S \subseteq X.
\]

A set \( A_* \subseteq X \) is called \( \mathcal{F} \)-invariant, if it is a fixed point of \( \mathcal{F} \), i.e., \( \mathcal{F}(A_*) = A_* \). (When \( \mathcal{F} \) is clear from the context we just speak of invariant sets instead of \( \mathcal{F} \)-invariant sets.)

**Remark 1.4.** (1) If not stated otherwise, for simplicity we will assume that if \( \mathcal{F} \) is an IFS, then it consists of maps \( w_1, ..., w_N \). We also associate with \( \mathcal{F} \) an alphabet of symbols \( I = \{1, ..., N\} \) representing maps \( w_i \).

(2) Closedness of an invariant set is built-in its definition.

(3) Invariant sets are often called self-similar, especially when the IFS consists of similarities.

Usually the Hutchinson operator is considered on some hyperspace of \( X \). For most of the work we will consider \( \mathcal{F} : \mathcal{K}(X) \to \mathcal{K}(X) \) acting on the hyperspace \( \mathcal{K}(X) \) of nonempty compact subsets of \( X \), equipped either with the Hausdorff distance \( d_H \) or the Vietoris topology. In case \( X \) is a metric space, the Hausdorff distance topology and the Vietoris topology on \( \mathcal{K}(X) \) coincide (see Appendix, Theorem 9.2). The restriction of \( \mathcal{F} \) from \( 2^X \) to \( \mathcal{K}(X) \) is possible when each \( w_i \) is continuous and then also \( \mathcal{F}(K) = \bigcup_{i=1}^{N} w_i(K) \) for \( K \in \mathcal{K}(X) \).

**Definition 1.5.** The probabilistic IFS \((\mathcal{F}, p)\) is an IFS \( \mathcal{F} = \{w_1, ..., w_N\} \) of continuous maps acting on a Hausdorff topological space \( X \), together with a vector \( p = (p_1, ..., p_N) \) of positive weights, that is \( \sum_{i=1}^{N} p_i = 1 \), \( p_i > 0 \). The Markov operator associated with \((\mathcal{F}, p)\), is a map \( M : \mathcal{M}_\pm(X) \to \mathcal{M}_\pm(X) \) defined on the space \( \mathcal{M}_\pm(X) \) of signed Radon measures on \( X \), according to the formula

\[
M(\mu) = \sum_{i=1}^{N} p_i \cdot \mu \circ w_i^{-1} \text{ for every } \mu \in \mathcal{M}_\pm(X),
\]

where \( \mu \circ w_i^{-1} \) is the push-forward of \( \mu \) through \( w_i \) (see Appendix, Section 9.4). A measure \( \mu_* \in \mathcal{M}_\pm(X) \) is called invariant, if it is a fixed point of the Markov operator \( M \) induced by the probabilistic IFS \((\mathcal{F}, p)\), i.e., \( M\mu_* = \mu_* \).

**Remark 1.6.** (1) To underline that the Markov operator is induced by the probabilistic IFS \((\mathcal{F}, p)\) we write \( M_{(\mathcal{F}, p)} \) in place of simple \( M \).

(2) Invariant measures are also called self-similar, or stationary.

(3) For future use, it is worth to note here that, if \( \mathcal{F} \) consists of continuous maps \( w_i : X \to X \), then for all bounded Borel measurable functions \( g : X \to \mathbb{R} \) and \( p_i > 0 \), \( i = 1, ..., N \), the following holds:

\[
\int_X g \, dM_{(\mathcal{F}, p)}\mu = \sum_{i=1}^{N} p_i \int_X g \circ w_i \, d\mu.
\]

(See Appendix, Proposition 9.8)

The space of signed Radon measures is endowed with the weak topology. Usually the action of the Markov operator \( M \) is considered on some subspace of \( \mathcal{M}_\pm(X) \): \( \mathcal{P}(X) \) comprising Radon probability measures or (if \( X \) is a metric space) \( \mathcal{P}_1(X) \) comprising Radon probability measures with integrable distance. For more information see Appendix, Section 9.4.
Given \( I = \{1,\ldots,N\}, \ N \geq 1 \), the code space \( I^\infty \) is the Tikhonov product of countably many copies of \( I \). We endow it with the Baire metric \( d_B \) (see Section 9.5 in Appendix for further information).

**Definition 1.7.** Consider the code space \( I^\infty \), where \( I = \{1,\ldots,N\} \). For each \( i \in I \), let \( \tau_i : I^\infty \to I^\infty \) be defined by \( \tau_i((\alpha_1,\alpha_2,\ldots)) := (i,\alpha_1,\alpha_2,\ldots) \). Then we call \( T = \{\tau_1,\ldots,\tau_N\} \) the canonical IFS on \( I^\infty \).

## 2. Metrically contractive iterated function systems

### 2.1. Taming a plethora of weak contractions.

In this section we make a short overview of generalizations of the Banach fixed point theorem for weak contractions.

The classical Banach fixed point theorem from 1922 states that if \( X \) is a complete metric space and \( f : X \to X \) is a Banach contraction (that is, the Lipschitz constant \( \text{Lip}(f) < 1 \)), then \( f \) has the CFP. Since the beginning of 1960’s there has been an effort to weaken the contractive assumptions in Banach’s theorem.

**Definition 2.1.** Given a function \( \varphi : \mathbb{R}_+ \to \mathbb{R}_+ \), we say that \( f \) is a \( \varphi \)-contraction (and that \( \varphi \) is a modulus of continuity or a comparison function for \( f \)), if

\[
d(f(x), f(y)) \leq \varphi(d(x, y)) \quad \text{for } x, y \in X.
\]

Clearly, \( f : X \to X \) is a Banach contraction iff \( f \) is a \( \varphi \)-contraction for \( \varphi(t) := \lambda t \), where \( \lambda \in (0, 1) \). It turned out that much less can be assumed on \( \varphi \).

**Definition 2.2.** Given a metric space \( X \), we say that \( f : X \to X \) is

(i) a Rakotch contraction, provided \( f \) is a \( \varphi \)-contraction for some comparison function \( \varphi \) of the form \( \varphi(t) := \lambda(t)t \), where \( \lambda : \mathbb{R}_+ \to [0,1] \) is nonincreasing and \( \lambda(t) < 1 \) for \( t > 0 \);

(ii) a Browder contraction, provided \( f \) is a \( \varphi \)-contraction, where \( \varphi \) is nondecreasing right continuous and \( \varphi(t) < t \) for \( t > 0 \);

(iii) an Edelstein contraction, provided \( f \) satisfies

\[
d(f(x), f(y)) < d(x, y) \quad \text{for all } x, y \in X, \ x \neq y.
\]

**Remark 2.3.** As we develop the theory of weakly contractive iterated function systems we will introduce some additional types of contractions which are broad generalizations of weak contractions. Namely, we will introduce Matkowski contractions in Remark 2.5, eventual contractions and Tarafdar contractions in Section 6.1 as well as generalizations of weakly contractive IFSs: topologically contractive in Definition 3.1, average contractive in Remark 6.6, and average Rakotch contractive in Definition 6.7.

Clearly, each Banach contraction is a Rakotch contraction and each Browder contraction is Edelstein’s. Also, as we show in Lemma 2.10 below, each Rakotch contraction is a Browder contraction. The converse implications do not hold. For example, the map \( f(x) = \sin x \) on \( X = \mathbb{R} \) is a Rakotch contraction but not Banach’s.

F. Browder in 1968 proved that each Browder contraction on a complete metric space has the CFP, generalizing an earlier result of E. Rakotch from 1962, who proved the CFP for Rakotch contractions; e.g., [38] chap.I §1.6 (B2). On the other hand, M. Edelstein in 1962 showed that each Edelstein contraction on a compact space \( X \) has the CFP; e.g. [38] chap.I §1.6 (A.7) and (B.7). Although Edelstein’s result fails on arbitrary complete metric spaces (e.g., for \( f(x) := x + e^{-x} \) on \( X = [0,\infty) \), see also [38] chap.I §1.6 (A.7) (c)), it looks like a generalization of the Browder theorem on compact spaces. This impression breaks down according to the following folklore observation (the proof can be found in [44]; it relies on a characterization of Rakotch contractions given in Lemma 2.10).

**Proposition 2.4.** Assume that \( X \) is compact and \( f : X \to X \). Then \( f \) is an Edelstein contraction iff \( f \) is a Rakotch contraction.
Remark 2.5. (1) It is worth to observe that if $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is nondecreasing, right continuous and $\varphi(t) < t$ for $t > 0$, then
\begin{equation}
\lim_{k \to \infty} \varphi^k(t) = 0 \text{ for } t > 0;
\end{equation}
e.g., [38] chap.I §1.6 (B.2). A map $f : X \to X$ is called a Matkowski contraction, if $f$ is a $\varphi$-contraction for some nondecreasing function $\varphi$ which satisfies (3). In 1975 J. Matkowski proved that each Matkowski contraction on a complete metric space has the CFP, generalizing the result of Browder. However, as was proved by J. Jachymski (see [46]), the second iteration of a Matkowski contraction is Browder’s, so Matkowski theorem follows from Browder’s (recall Remark 1.2). In fact, the proof given in [45] Theorem 3 shows more: if $f, g : X \to X$ are Matkowski contractions, then their composition $f \circ g$ is a Browder contraction.

(2) Monotonicity of $\varphi$ is an important ingredient in the definition of Matkowski’s contraction. Wicks in his definition of a reduction ([92]) assumes only (3), without monotonicity, but in his proofs he actually uses monotonicity. An example of a $\varphi$-contraction, for $\varphi$ satisfying only (3), on a complete space without a fixed point can be found in [69].

In the literature we can find many other (and essentially weaker) contractive-type conditions which guarantee the existence of the CFP of a map - we refer the interested reader to the survey [46] by J. Jachymski and I. Jóźwik, in which a detailed discussion on various types of contractive conditions and mutual relationships between them is given (if not stated otherwise, presented results in this section can also be found there). We restricted ourselves here to probably the most important ones, which in addition can be defined in a simple and natural way. In fact, Proposition 2.4 shows that for compact spaces $X$, all these weak contractions are Rakotch contractions. Moreover, as was proved by J. Matkowski and R. Węgrzyk [68], if the underlying space $X$ is metrically convex (in particular, if $X$ is a Banach space), then $f$ is a Rakotch contraction if and only if $f$ is a $\varphi$-contraction for a comparison function that satisfies $\varphi(t) < t$ for $t > 0$. Thus, Rakotch contractions are sufficient to explain other fixed point theorems in many natural cases.

A bit surprisingly, allowing for a change of the underlying metric, most such generalizations can be deduced from the Banach theorem, as was shown, for example, by P. Meyers in 1967 (e.g., [38] chap.I §1.7 p.24; note that there is a misprint in the formulation of Meyers’ theorem: neighbourhoods $V$ and $U$ are misplaced in condition (ii) below).

Theorem 2.6 (Meyers’ remetrization theorem). Let $f$ be a continuous selfmap of a completely metrizable space $X$. Assume that there exists $x_0 \in X$ such that

(i) for any $x \in X$, $\lim_{k \to \infty} f^k(x) = x_0$ (i.e., $x_0$ is the CFP of $f$);

(ii) there exists an open neighbourhood $U$ of $x_0$ such that for every open neighbourhood $V$ of $x_0$, there is $n \in \mathbb{N}$ such that $f^k(U) \subset V$ for all $k \geq n$.

Then there is an admissible complete metric $\rho$ on $X$ such that $f$ is a Banach contraction with respect to $\rho$.

Remark 2.7. Each Browder contraction satisfies (i) and (ii) in Theorem 2.6 (just take $U := B(x_0, 1)$ to be the open ball). Thus the Browder fixed point theorem essentially follows from the Banach fixed point principle.

Remark 2.8. Conditions (i) and (ii) of Theorem 2.6 are satisfied, if $X$ is compact and the intersection $\bigcap_{k \in \mathbb{N}} f^k(X)$ is a singleton, i.e., when $f$ is Tarafdar’s topological contraction ([24] chap.2.7.1 p.88); see also Section 6.1.

Remark 2.9. We note that Meyers remetrization theorem is only applicable to an iterated function system (IFS) consisting of a single map. In this case, we obtain an IFS consisting of a Banach contraction. In general, one cannot remetrize a given space so that several maps are Banach contractions with respect to the same metric (see [50] Section 1.4). To deal with the situation when an IFS consists of more than one map we will prove remetrization theorem for topologically contractive IFSs (TIFS) which we will discuss in Section 3. In this case, we obtain an IFS consisting of weak contractions.
Therefore, for an IFS consisting of a single map Meyers remetrization theorem gives a stronger result. We also note that the remetrization theorem for TIFS is only used in Section 3.4. In order to obtain results for TIFS from a weakly contractive case via the remetrization theorem for TIFS one needs to assume that the underlying space \( X \) is metrizable (cf. Remark 5.7). However, in general, TIFS can be defined on nonmetrizable spaces (Example 3.8).

We end this section with two lemmas which will be useful later, when dealing with iterated function systems consisting of weak contractions. The first one gives a nice characterization of Rakotch contractions (see [46, Theorem 1] for more characterizations like this one).

**Lemma 2.10.** Let \( X \) be a complete metric space and \( f : X \to X \). The following conditions are equivalent:

(i) \( f \) is a Rakotch contraction;
(ii) \( f \) is a \( \varphi \)-contraction for some concave and strictly increasing \( \varphi \) such that \( \varphi(t) < t \) for \( t > 0 \);
(iii) for every \( \delta > 0 \), there exists \( \lambda < 1 \) such that if \( x, y \in X \) satisfy \( d(x, y) \geq \delta \), then \( d(f(x), f(y)) \leq \lambda d(x, y) \).

The second lemma is technical and states some relationship between different types of comparison functions \( \varphi \) suitable for Rakotch contractions. It follows directly from [46, Lemma 1].

**Lemma 2.11.** Let \( \varphi : \mathbb{R}_+ \to \mathbb{R}_+ \) be defined by \( \varphi(t) = \lambda(t)t \) for some nonincreasing \( \lambda : \mathbb{R}_+ \to [0,1] \) with \( \lambda(t) < 1 \) for \( t > 0 \). Then there exists a concave strictly increasing function \( \tilde{\varphi} : \mathbb{R}_+ \to \mathbb{R}_+ \) such that \( \varphi(t) \leq \tilde{\varphi}(t) < t \) for \( t > 0 \).

### 2.2. Attractors for weakly contractive IFSs.

**Definition 2.12.** We say that an IFS \( \mathcal{F} \) on a metric space \( X \) is **Banach (Rakotch, Browder, Edelstein respectively)** contractive if it consists of Banach (Rakotch, Browder, Edelstein respectively) contractions. We will refer to all such IFSs as **weakly contractive IFSs**.

The classical Hutchinson-Barnsley theorem states that:

**Theorem 2.13.** If \( \mathcal{F} = \{w_1, \ldots, w_N\} \) is a Banach contractive IFS on a complete metric space \( X \), then the Hutchinson operator \( \mathcal{F} : \mathcal{K}(X) \to \mathcal{K}(X) \) has the CFP, i.e., there is a unique set \( A_\mathcal{F} \subseteq \mathcal{K}(X) \) satisfying two conditions

(i) invariance: \( A_\mathcal{F} = \mathcal{F}(A_\mathcal{F}) = \bigcup_{i=1}^N w_i(A_\mathcal{F}) \), and
(ii) attractivity: for every \( K \subseteq \mathcal{K}(X) \), the sequence of iterations \( \mathcal{F}^k(K) \) converges to \( A_\mathcal{F} \) (with respect to the Hausdorff metric).

The above theorem suggests the following definition:

**Definition 2.14.** Given an IFS \( \mathcal{F} \) on a Hausdorff topological space \( X \) (in particular, a metric space \( X \)), a nonempty compact set \( A_\mathcal{F} \), which is the CFP (contractive fixed point) of the Hutchinson operator \( \mathcal{F} : \mathcal{K}(X) \to \mathcal{K}(X) \), will be called an **attractor** generated by \( \mathcal{F} \). In other words, \( A_\mathcal{F} \) fulfills conditions (i) and (ii) from Theorem 2.13. The convergence of sets in (ii) has to be understood in the Vietoris topology on \( \mathcal{K}(X) \).

**Remark 2.15.** If the Hutchinson operator is continuous (see Proposition 7.2), then the invariance condition (i) follows from the attractivity condition (ii) in Definition 2.14.

**Example 2.16.** Let \( \mathcal{T} \) be the canonical IFS on the code space \( I^\infty \), cf. Definition 1.7. It is easy to see that \( \mathcal{T} \) is a Banach contractive IFS and \( I^\infty \) is its attractor.

Theorem 2.13 follows easily from the Banach fixed point theorem, as the Hutchinson operator turns out to be a Banach contraction on the hyperspace \( (\mathcal{K}(X), d_H) \), provided that the underlying IFS is Banach contractive. Many known generalizations of this result are based on the same idea, that is, they rely on proving that the Hutchinson operator satisfies the same contractive condition as maps from the IFS. In many papers we can find particular cases of the following:
Lemma 2.17. Let $\mathcal{F}$ be an IFS on a metric space $X$ that consists of $\varphi$-contractions for some non-decreasing function $\varphi$. Then the Hutchinson operator $\mathcal{F} : \mathcal{K}(X) \to \mathcal{K}(X)$ is a $\varphi$-contraction.

Proof. Assume first that $w : X \to X$ is a $\varphi$-contraction and choose $K, S \in \mathcal{K}(X)$ and $x_0 \in K$. There is $y_0 \in S$ such that $d(x_0, y_0) = \inf \{d(x_0, y) : y \in S\}$. We have

$$\inf_{y \in S} d(w(x_0), w(y)) \leq d(w(x_0), w(y_0)) \leq \varphi(d(x_0, y_0)) = \varphi\left(\inf_{y \in S} d(x_0, y)\right) \leq \varphi(d_H(K, S)).$$

Hence

$$e(w(K), w(S)) = \sup_{x \in K, y \in S} d(w(x), w(y)) \leq \varphi(d_H(K, S)).$$

By symmetry $e(w(S), w(K)) \leq \varphi(d_H(K, S))$, which implies $d_H(w(K), w(S)) \leq \varphi(d_H(K, S))$.

Now, if $\mathcal{F} = \{w_1, \ldots, w_N\}$ consists of $\varphi$-contractions, then by the above and known properties of the Hausdorff metric, we have

$$d_H(\mathcal{F}(K), \mathcal{F}(S)) = d_H\left(\bigcup_{i=1}^{N} w_i(K), \bigcup_{i=1}^{N} w_i(S)\right) \leq \max_{i=1,\ldots,N} d_H(w_i(K), w_i(S)) \leq \varphi(d_H(K, S)) \text{ for } K, S \in \mathcal{K}(X).$$

□

As an immediate corollary, we get the following extension of Theorem 2.13:

Theorem 2.18. If $\mathcal{F}$ is an IFS on a complete metric space $X$ that consists of Browder [Banach, Rakotch, Matkowski, respectively] contractions, then the Hutchinson operator $\mathcal{F} : \mathcal{K}(X) \to \mathcal{K}(X)$ is a Browder [Banach, Rakotch, Matkowski, respectively] contraction and hence it has the CFP, i.e., $\mathcal{F}$ generates the unique compact invariant set which is an attractor.

Proof. In view of Lemma 2.17, we only have to observe that if $\mathcal{F} = \{w_1, \ldots, w_N\}$ consists of Browder [Banach, Rakotch, Matkowski, respectively] contractions $w_i$ with respective comparison functions $\varphi_i$, then all $w_i$’s are $\varphi$-contractions for the same function $\varphi$ with suitable properties. Namely, $\varphi := \max\{\varphi_1, \ldots, \varphi_N\}$ meets the desired conditions. □

Remark 2.19. In the literature there are given many further generalizations of Theorem 2.13 in the spirit of Theorem 2.13. Here we want to point out two further directions.

1. We can extend the thesis of Theorem 2.13 by showing that for any nonempty closed and bounded set $B \subset X$ (denote by $\mathcal{CB}(X)$ the family of such sets; note that the Hausdorff metric on $\mathcal{CB}(X)$ is complete provided $X$ is complete - see Appendix), the sequence of iterations $\mathcal{F}^k(B)$ converges to $A_{\mathcal{F}}$. To see it, observe that we can easily extend Lemma 2.17 by considering the Hutchinson operator $\mathcal{F} : \mathcal{CB}(X) \to \mathcal{CB}(X)$, under additional assumption that $\varphi$ is right continuous. Thus $\mathcal{F} : \mathcal{CB}(X) \to \mathcal{CB}(X)$ is a Browder contraction if the IFS $\mathcal{F}$ is (at least) Browder contractive and we are done. If $\mathcal{F}$ is Matkowski contractive, then using the mentioned result from [15], the second iteration $\mathcal{F}^2 : \mathcal{CB}(X) \to \mathcal{CB}(X)$ is Browder’s, so has the CFP and hence also $\mathcal{F} : \mathcal{CB}(X) \to \mathcal{CB}(X)$ has the CFP.

2. Let $\mathcal{F}$ be an IFS consisting of infinitely many weak contractions. Moreover, let us assume that the weak contractivity of $\mathcal{F}$ is uniform in the sense that all mappings from $\mathcal{F}$ are $\varphi$-contractions for a common comparison function $\varphi$ with appropriate properties. Assume additionally that the induced Hutchinson operator $\mathcal{F}$ transforms compact sets onto compact sets, i.e., $\mathcal{F}(\mathcal{K}(X)) \subseteq \mathcal{K}(X)$. Then the same reasoning as in Lemma 2.17 shows that $\mathcal{F}$ is a $\varphi$-contraction and thus it has the CFP. Note that the assumption that all mappings are $\varphi$-contractions for the same function $\varphi$ is important. For instance the IFS $\mathcal{F} := \{w_t : t \in (0, 1)\}$, where $w_t(x) = tx$, $x \in [0, 1]$, does not generate a unique invariant set (each set $[0, a]$, where $a \in (0, 1]$, is $\mathcal{F}$-invariant). See also [92] and [56] chap.2.6.4.1.

3. Finally, we can mix the above two approaches and obtain a result for closed and bounded sets for infinite IFSs consisting of weak contractions.
Remark 2.20. While a weakly contractive IFS induces a weakly contractive Hutchinson operator (Theorems 2.13 and 2.18), an IFS comprising non-contractive maps can induce a Hutchinson operator with CFP, see Example 7.34. This opens the gate to a whole world of non-contractive IFSs, e.g., [14].

2.3. Invariant measures for weakly contractive IFSs. The existence of a unique invariant measure for weakly contractive IFSs has been established in 1996 independently by A. Fan [33], who employed ergodic theory techniques, and A. Edalat [29], who used order theory with the aim of studying some aspects of the computation theory for IFSs. Since then it has been rediscovered several times, e.g., [5], [82].

We provide an elementary proof of this fact for Rakotch contractive IFSs. Our proof is much like the one given by K. Okamura in [82]. However, he restricted the discussion to measures supported on compact \( F \)-invariant set, so our result is more general. Let us also note that a more straightforward proof for Banach contractive IFSs can be found in [56], but it seems to be hard to adjust that approach to weakly contractive IFSs (see also the discussion in [82]).

Let \( (X,d) \) be a complete metric space and let \( \mathcal{P}_1(X) \) be the set of all Radon probability measures on \( X \) with integrable distance \( d \), i.e., measures \( \mu \) such that for some (equivalently - for any) \( x_0 \in X \), the integral \( \int_X d(x,x_0) \, d\mu(x) < \infty \). We endow \( \mathcal{P}_1(X) \) with the Monge-Kantorovitch metric \( d_{MK} \), which is complete and has the property that the convergence of measures with respect to \( d_{MK} \) implies their weak convergence (see Appendix, Lemma 9.13 (i) and (ii)).

Theorem 2.21. Let \( (\mathcal{F},\tilde{p}) \) be a probabilistic IFS consisting of Rakotch contractions which act on a complete metric space \( X \). Let \( M : \mathcal{M}(X) \to \mathcal{M}(X) \) be the Markov operator induced by \( (\mathcal{F},\tilde{p}) \). Then

(a) \( M(\mathcal{P}_1(X)) \subseteq \mathcal{P}_1(X) \);

(b) \( M : \mathcal{P}_1(X) \to \mathcal{P}_1(X) \) is a Rakotch contraction with respect to the Monge-Kantorovitch metric;

(c) \( M \) has the CFP in \( \mathcal{P}_1(X) \), i.e., there exists a unique \( \mu_* \in \mathcal{P}_1(X) \) s.t. \( M(\mu_*) = \mu_* \) and for every \( \mu \in \mathcal{P}_1(X) \), the sequence of iterations \( M^k(\mu) \) converges weakly to \( \mu_* \).

Proof. (a) Using equality (2) from Remark 1.6 and Proposition 9.8 we easily infer that since \( \mathcal{F} \) consists of Lipschitz maps, then \( M(\mu) \in \mathcal{P}_1(X) \) for every \( \mu \in \mathcal{P}_1(X) \).

(b) Let \( \mathcal{F} = \{w_1,...,w_N\} \) and \( \tilde{p} = (p_1,...,p_N) \). By Lemma 2.11 we can assume that all maps \( w_i \) are \( \varphi_i \)-contractions for concave strictly increasing functions \( \varphi_i \) which satisfy \( \varphi_i(t) < t \) for \( t > 0 \). It is easy to see that then \( \varphi(t) := \sum_{i=1}^N p_i \varphi_i(t), t \geq 0 \), is also strictly increasing and concave, and \( \varphi(t) < t \) for \( t > 0 \).

By Lemma 9.13 the Monge-Kantorovitch metric \( d_{MK} \) has the following description:

\[
d_{MK}(\mu,\eta) = \min \left\{ \int_{X \times X} d(x,y) \, d\lambda : \lambda \in \Lambda(\mu,\eta) \right\},
\]

where \( \Lambda(\mu,\eta) \) consists of all Radon probability measures \( \lambda \) on \( X \times X \) such that the projection of \( \lambda \) on the first and the second coordinate equals \( \mu \) and \( \eta \), respectively.

Take \( \lambda \in \Lambda(\mu,\eta) \) so that \( d_{MK}(\mu,\eta) = \int_{X \times X} d(x,y) \, d\lambda(x,y) \) and let \( \tilde{\lambda} := \sum_{i=1}^N p_i \lambda \circ (w_i,w_i)^{-1} \).

Then for every Borel \( B \subset X \), we have

\[
\tilde{\lambda}(B \times X) = \sum_{i=1}^N p_i \lambda(w_i^{-1}(B) \times w_i^{-1}(X)) = \\
= \sum_{i=1}^N p_i \mu(w_i^{-1}(B) \times X) = \sum_{i=1}^N p_i \mu(w_i^{-1}(B)) = M(\mu)(B).
\]

Similarly we can show that \( \tilde{\lambda}(X \times B) = M(\eta)(B) \). Hence \( \tilde{\lambda} \in \Lambda(M(\mu),M(\eta)) \).

Observe now that \( \tilde{\lambda} = M(\mathcal{F},\tilde{p})(\lambda) \), where \( M(\mathcal{F},\tilde{p}) \) is the Markov operator induced by the probabilistic IFS \( \tilde{\mathcal{F}} = \{(w_i,w_i) : i = 1,...,N\} \) with vector of weights \( \tilde{p} \). Therefore we can employ equality (2) from
Remark 1.6 alongside the concavity of \( \varphi_i \)'s to obtain
\[
\left| \int_{\mathcal{K}} dMK(M(\mu), M(\eta)) \, d\lambda \right| =
\leq \sum_{i=1}^{N} p_i \varphi_i \left( \int_{X \times X} d(x, y) \, d\lambda \right) = \sum_{i=1}^{N} p_i \varphi_i \left( dMK(\mu, \eta) \right) = \varphi \left( dMK(\mu, \eta) \right).
\]

Item (c) is immediate from (b). \(\square\)

Remark 2.22. It turns out that the support of the CFP \( \mu_* \) equals to the attractor of \( F \); see Theorem 3.13 (ii) and Theorem 7.31.

3. Topologically contractive iterated function systems

3.1. Notation and basic definitions. Let \( F = \{w_1, ..., w_N\} \) be an IFS on \( X \). We find the following notational conventions very useful for the whole section. We fix the alphabet \( I = \{1, ..., N\} \). Given a finite word \( \alpha = (\alpha_1, ..., \alpha_k) \in I^k \), \( k \in \mathbb{N} \), we set
\[
w_\alpha = w_{\alpha_1} \circ ... \circ w_{\alpha_k}.
\]
(Note carefully the order of composition typical for symbolic dynamics.) In particular, we can use that notation for a composition of \( w_i \)'s along a finite prefix \( \alpha_{|k} \) of an infinite word \( \alpha \in I^\infty \) and write \( w_{\alpha_{|k}} \).

Definition 3.1. We say that an IFS \( F \) on a topological space \( X \) is a topologically contractive iterated function system (TIFS), if it consists of continuous maps and the following two conditions hold:
(i) \( F \) is compactly dominated in the sense that for every \( K \in \mathcal{K}(X) \), there exists \( C \in \mathcal{K}(X) \) such that \( K \subseteq C \) and \( F(C) \subseteq C \);
(ii) for every \( C \in \mathcal{K}(X) \) with \( F(C) \subseteq C \) and every sequence \( \alpha \in I^\infty \), the set \( \cap_{k \in \mathbb{N}} w_{\alpha_{|k}}(C) \) is a singleton.

Remark 3.2. If the space \( X \) is compact or if \( F \) has an attractor \( A_F \), then condition (i) in Definition 3.1 is fulfilled for free. To see it in the second case, observe that for every \( K \in \mathcal{K}(X) \), the set
\[
C := A_F \cup K \bigcup_{n \in \mathbb{N}} \bigcup_{n \in \mathbb{N}} F^n(K) = \bigcup_{n=0}^{\infty} F^n(K)
\]
is compact, \( K \subseteq C \) and \( F(C) \subseteq C \).

A careful reader will notice that the compact dominance property appears implicitly in many discussions about IFSs. It is a form of localizing (or, in other words, trapping) an attractor. It allows to reduce the discussion about the dynamics of the IFS to a compact set.

Remark 3.3. Definition 3.1 was introduced by A. Mihail in 2012 (cf. [75]) under the name topologically iterated function system and, independently, by A.V. Tetenov in 2010 (see [85] and the references therein) under the name self-similar topological structure satisfying condition (P). However, its particular versions were considered earlier. In the case when \( X \) is a compact metric space, TIFSs were called weakly hyperbolic IFSs by A. Edalat [29] and point fibred IFSs by B. Kieninger [51]. A. Kameyama [50] called a compact topological space as a topological self similar set, if there exists an IFS \( F \) on \( X \) comprising continuous maps together with a continuous surjection \( \pi : I^\infty \to X \) such that \( \pi \circ \tau_i = w_i \circ \pi \) for every \( i \in I \). Propositions 3.9 and 3.10 explain how the notions of a topological self similar set, point-fibred IFS and TIFS are interrelated.

Kameyama’s definition of a topologically self similar set ([50]) involves a very important concept of the coding map. We restate it as a separate definition.
Definition 3.4. Let $\mathcal{F} = \{w_1, ..., w_N\}$ be an IFS on a metric (or topological) space $X$ and let $\mathcal{T} = \{\tau_1, ..., \tau_N\}$ be the canonical IFS on the code space $I^\infty$; see Definition 3.7. The continuous map $\pi : I^\infty \to X$ satisfying $w_i \circ \pi = \pi \circ \tau_i$ for every $i \in I$, will be called the coding map.

Remark 3.5. The name “coding map” for $\pi$, used frequently in the literature (e.g., [70]), has not been standardized so far. Some other names are the address map ([56]), the coordinate map ([51]), and the projection from the code space (e.g., J. Geronimo, Ch. Bandt — personal communication).

Remark 3.6. Assume that $X$ is a Hausdorff topological space. According to Kameyama’s definition, $X$ is topological self similar set if there exists a surjective coding map for some IFS $\mathcal{F}$ consisting of continuous selfmaps of $X$ (as $\pi$ is continuous and $I^\infty$ is compact, it automatically implies compactness of $X$). On the other hand, if a coding map $\pi$ is not surjective, then its image $\pi(I^\infty)$ is a topological self similar set. Indeed, as $\pi(I^\infty)$ is $\mathcal{F}$-invariant:

$$\pi(I^\infty) = \pi\left(\bigcup_{i=1}^{N} \tau_i(I^\infty)\right) = \bigcup_{i=1}^{N} \pi \circ \tau_i(I^\infty) = \bigcup_{i=1}^{N} w_i \circ \pi(I^\infty) = \bigcup_{i=1}^{N} w_i(\pi(I^\infty)),$$

we just have to consider the IFS $\tilde{\mathcal{F}} := \{w_i|_{\pi(I^\infty)} : i = 1, ..., N\}$ on $\pi(I^\infty)$ consisting of restrictions of $w_i$’s to $\pi(I^\infty)$.

3.2. Existence of attractors and the coding map. The following result (essentially given in [75]) shows that TIFSs generate attractors which are projections of the code space via coding maps.

Theorem 3.7. Assume that $\mathcal{F} = \{w_1, ..., w_N\}$ is a topologically contractive IFS on a Hausdorff topological space $X$.

(i) (Symbolic conjugation) There exists a coding map $\pi : I^\infty \to X$, i.e., a continuous map for which $w_i \circ \pi = \pi \circ \tau_i$ for $i \in I = \{1, ..., N\}$.

(ii) The set $\pi(I^\infty)$ is the attractor of $\mathcal{F}$ (i.e., the contractive fixed point of the operator $\mathcal{F}$).

(iii) For every $\alpha \in I^\infty$ and $C \in \mathcal{K}(X)$ with $\mathcal{F}(C) \subseteq C$, the value $\pi(\alpha)$ is the only element of the intersection $\bigcap_{k \in \mathbb{N}} w_{\alpha_k}(C)$.

(iv) For every $K \in \mathcal{K}(X)$ and $\alpha \in I^\infty$, the sequence $(w_{\alpha_k}(K))$ converges to $\{\pi(\alpha)\}$ with respect to the Vietoris topology.

(v) (Williams’ formula) The attractor $\pi(I^\infty)$ is the closure of the set of all fixed points of maps $w_{\alpha}$, $\alpha \in I^{<\infty}$.

(vi) (Metrizability) The attractor $\pi(I^\infty)$ is metrizable.

Proof. We first prove (i) and (iii). Take $C \in \mathcal{K}(X)$ so that $\mathcal{F}(C) \subseteq C$ and $\alpha \in I^\infty$. By definition, the set $\bigcap_{k \in \mathbb{N}} w_{\alpha_k}(C)$ is singleton. Its unique element, denoted by $\pi(\alpha)$, does not depend on the choice of $C$, because if $C_1, C_2 \in \mathcal{K}(X)$ satisfy $\mathcal{F}(C_i) \subseteq C_i$ for $i = 1, 2$, then by the compact dominance we can find $\tilde{C} \in \mathcal{K}(X)$ such that $C_1, C_2 \subseteq \tilde{C}$ and $\mathcal{F}(\tilde{C}) \subseteq \tilde{C}$.

Now we show that the constructed map $\pi : I^\infty \to X$ is continuous. Fix $\alpha \in I^\infty$ and take any open neighborhood $U$ of $\pi(\alpha)$. Since the sequence $(w_{\alpha_k}(C))$ is a decreasing sequence of compact sets whose intersection contains just $\pi(\alpha)$, we see that there exists $k \in \mathbb{N}$ such that $w_{\alpha_k}(C) \subseteq U$. Then if $\beta \in I^\infty$ agrees with $\alpha$ on the first $k$ coordinates, we get

$$\pi(\beta) \in w_{\alpha_k}(C) \subseteq U$$

and $\pi$ is continuous.

Now take any $\alpha \in I^\infty$ and observe that

$$\{w_i(\pi(\alpha))\} = w_i\left(\bigcap_{k \in \mathbb{N}} w_{\alpha_k}(C)\right) \subseteq \bigcap_{k \in \mathbb{N}} w_i \circ w_{\alpha_k}(C) = \{\pi(i\alpha)\} = \{\pi(\tau_i(\alpha))\},$$

so we get $w_i(\pi(\alpha)) = \pi(\tau_i(\alpha))$ and the proofs of (i) and (iii) are finished.

Now we move to (ii). Similarly, as in Remark 3.6 we can show that $\pi(I^\infty)$ is $\mathcal{F}$-invariant and compact.
We will show that $A := \pi(I^\infty)$ is the CFP of $\mathcal{F}$. Choose any nonempty and compact set $K$ and let $C \subseteq \mathcal{K}(X)$ be such that $\mathcal{F}(C) \subseteq C$ and $K \subseteq C$. Fix any nonempty and open sets $U_0, U_1, \ldots, U_k \subseteq X$ with $A \subseteq U_0$ and $A \cap U_i \neq \emptyset$ for $i = 1, \ldots, k$ (that is, we establish a neighbourhood of $A$ in the Vietoris topology). We have to find $n_0$ such that for $n > n_0$, $\mathcal{F}^n(K) \subseteq U_0$ and $\mathcal{F}^n(K) \cap U_i \neq \emptyset$ for $i = 1, \ldots, k$. Given $n \in \mathbb{N}$, set
\[ P_n := \{ \alpha \in I^\infty : w_{\alpha_n}(C) \subseteq U_0 \}. \]

Clearly, $(P_n)$ is an increasing sequence of open sets and (by the inclusion $A \subseteq U_0$ and definition of a TIFS) we have that $I^\infty = \bigcup_{n \in \mathbb{N}} P_n$. Hence by compactness of $I^\infty$, there is $n_0 \in \mathbb{N}$ such that $I^\infty = P_{n_0}$ and thus for $j \geq n_0$,
\[ \mathcal{F}^j(K) \subseteq \mathcal{F}^{n_0}(C) \subseteq \bigcup_{\alpha \in I^{n_0}} w_{\alpha}(C) \subseteq U_0. \]

Now choose $i = 1, \ldots, k$ and $\alpha \in I^\infty$ with $\pi(\alpha) \in U_i$. There exists $n_i \in \mathbb{N}$ such that $w_{\alpha_{n_i}}(C) \subseteq U_i$ for $n \geq n_i$ and hence $w_{\alpha_{n_i}}(K) \subseteq \mathcal{F}^{n_i}(K) \cap U_i$. All in all, it is enough to take $k_0 := \max\{n_0, n_1, \ldots, n_k\}$.

To see (iv), choose any $\alpha \in I^\infty$ and $K \subseteq \mathcal{K}(X)$, and find $C \subseteq \mathcal{K}(X)$ with $K \subseteq C$ and $\mathcal{F}(C) \subseteq C$. By (iii), $w_{\alpha_{k_0}}(C) \rightarrow \{\pi(\alpha)\}$. As for any $k \in \mathbb{N}$, $w_{\alpha_{k}}(K) \subseteq w_{\alpha_{k_0}}(C)$, it also holds $w_{\alpha_{k}}(K) \rightarrow \{\pi(\alpha)\}$.

Now we observe (v). Choose $\beta \in I^{<\infty}$ and let $x_\beta$ be the unique fixed point of $w_\beta$. Using (iv) for a compact set $\{x_\beta\}$ and a sequence $\alpha := \beta^{\beta^{\beta^{\cdots}}}$, we have that $w_{\alpha_k}(x_\beta) \rightarrow \pi(\alpha)$. But the latter sequence has a constant subsequence whose unique element equals $x_\beta$, and therefore $x_\beta = \pi(\alpha) \in A_F$. Now choose any $\beta \in I^\infty$ and any open neighbourhood $U$ of $\pi(\beta)$. By the continuity of $\pi$, there is $k$ such that if $\alpha \in I^\infty$ agrees with $\beta$ on the first $k$ coordinates, then $\pi(\alpha) \in U$. Hence taking $\alpha := (\beta^k)^{(\beta^k)^{\cdots}}$, we get $\pi(\alpha) \in U$. By the earlier observation, $\pi(\alpha)$ is the fixed point of $w_{\beta^k}$ and the result follows.

Finally, the attractor $\pi(I^\infty)$ is metrizable as a continuous image of a compact metric space (cf. [31] chap.3.7 p.182 and Theorem 4.4.17), so we get (vi). \qed

By Theorem 3.7, attractors of topologically contractive IFS are always metrizable. On the other hand, the underlying space $X$ for a TIFS can be nonmetrizable.

**Example 3.8 (Nonmetrizable TIFS).** Let us endow $[0,1]$ with the Euclidean topology and $(1,2]$ with any nonmetrizable topology. Let $X$ be the disjoint union of $[0,1]$ and $(1,2]$. Define $w_1(x) = \frac{1}{2} \cdot \min\{1,x\}$ and $w_2(x) = w_1(x) + \frac{1}{2}$. Then $w_1, w_2$ are continuous (as they are so on each of disjoint subspaces $[0,1]$ and $(1,2]$). For every compact $K \subseteq X$, the set $C := K \cup [0,1]$ is compact, contains $K$ and $w_1(C) \cup w_2(C) \subseteq C$. This verifies point (i) of Definition 3.1. Furthermore, for every $\alpha \in \{1,2\}^\infty$, the intersection $\bigcap_{k \in \mathbb{N}} w_{\alpha_k}(X) = \bigcap_{k \in \mathbb{N}} w_{\alpha_k}([0,1])$ is a singleton. This verifies point (ii) of Definition 3.1. Thus $\mathcal{F} = \{w_1, w_2\}$ is a TIFS.

The next two results show that Kameyama’s topological self similar sets and Kieninger’s point-fibred attractors are nothing else but attractors of TIFSs.

**Proposition 3.9.** Let $X$ be a compact Hausdorff topological space and $\mathcal{F}$ be an IFS on $X$ consisting of continuous maps. The following conditions are equivalent:

(i) there exists a surjective coding map $\pi : I^\infty \to X$ adjusted to $\mathcal{F}$ (i.e., $X$ is Kameyama’s topological self similar set);

(ii) $\mathcal{F}$ is a topologically contractive IFS and $X$ is its attractor.

**Proof.** Implication (ii)⇒(i) follows directly from Theorem 3.7. To see (i)⇒(ii), assume that $\mathcal{F}$ is an IFS on $X$ for which there exists a surjective coding map and choose $\alpha \in I^\infty$. By induction we can show that for every $k \in \mathbb{N}$, $w_{\alpha_{k+}} \circ \pi = \pi \circ \tau_{\alpha_{k+}}$. Hence
\[
\bigcap_{k \in \mathbb{N}} w_{\alpha_{k+}}(X) = \bigcap_{k \in \mathbb{N}} w_{\alpha_{k+}}(\pi(I^\infty)) = \bigcap_{k \in \mathbb{N}} \pi(\tau_{\alpha_{k+}}(I^\infty)) = \pi \left( \bigcap_{k \in \mathbb{N}} \tau_{\alpha_{k+}}(I^\infty) \right) = \pi(\{\alpha\}),
\]
where the penultimate equality holds as \((\tau_{\alpha_{ik}}(I^\infty))\) is a decreasing sequence of compact sets. The result follows. \(\square\)

**Proposition 3.10.** Let \(F\) be an IFS on a Hausdorff topological space \(X\) consisting of continuous maps. The following conditions are equivalent:

(a) \(F\) is a TIFS;
(b) \(F\) has the attractor \(A_F\) which is point-fibred in the sense of Kieninger, that is point (ii) of Definition 3.1 holds true for \(C = A_F\).

**Proof.** Implication (a)\(\Rightarrow\)(b) follows from Theorem 3.7. We will prove the opposite one.

First note that, by Remark 3.2, \(F\) is compactly dominated, because it has the attractor.

Now, let \(F = \{w_1, ..., w_N\}\). Choose any \(C \in \mathcal{K}(X)\) so that \(F(C) \subseteq C\) and \(\alpha = (\alpha_i)_{i=1}^\infty \in I^\infty\). Since \(A_F\) is the attractor of \(F\), we have \(F^k(C) \rightarrow A_F\) and, in turn, \(A_F = \bigcap_{k \in \mathbb{N}} F^k(C)\). Now if \(k > j > 1\), then

\[
w_{\alpha_{ik}}(C) = w_{\alpha_{ij}}(w_{(\alpha_{j+1}, ..., \alpha_k)}(C)) \subseteq w_{\alpha_{ij}}(F^{k-j}(C)).
\]

Hence

\[
\bigcap_{k \in \mathbb{N}} w_{\alpha_{ik}}(C) = \bigcap_{k > j} w_{\alpha_{ij}}(F^{k-j}(C)) \subseteq \bigcap_{k > j} w_{\alpha_{ij}}(F^{k-j}(C)) = w_{\alpha_{ij}}(A_F).
\]

Since \(j\) was taken arbitrarily and \(A_F \subseteq C\), we get \(\bigcap_{k \in \mathbb{N}} w_{\alpha_{ik}}(C) = \bigcap_{k \in \mathbb{N}} w_{\alpha_{ik}}(A_F)\). As \(A_F\) is a point fibred attractor, the proof is finished. \(\square\)

Finally, we show that Browder contractive IFSs on complete spaces are topologically contractive, whence the thesis of Theorem 3.7 holds for such IFSs.

**Corollary 3.11.** Let \(F\) be a Browder contractive IFS on a complete metric space \(X\). Then \(F\) is a TIFS.

**Proof.** Condition (i) of Definition 3.1 follows, due to Remark 3.2, from the existence of an attractor of \(F\).

Now, if \(C \in \mathcal{K}(X)\) satisfies \(F(C) \subseteq C\) and \(\alpha \in I^\infty\), then the sequence \((w_{\alpha_{ik}}(C))\) is a decreasing sequence of compact sets such that \(\text{diam}(w_{\alpha_{ik}}(C)) \leq \varphi^k(\text{diam}(C))\) for all \(k \in \mathbb{N}\), where \(\varphi\) is a comparison function common for all maps comprising \(F\). Since \(\varphi^k(\text{diam}(C)) \rightarrow 0\), the Cantor theorem assures that the intersection \(\bigcap_{k \in \mathbb{N}} w_{\alpha_{ik}}(C)\) is a singleton. Thus condition (ii) of Definition 3.1 is satisfied. \(\square\)

**Remark 3.12.** It is worth pointing out that although Theorem 3.7 implies Theorem 2.18 (point (ii) states that \(\pi(I^\infty)\) is the CFP of \(F\), we needed to establish Theorem 2.18 directly. Theorem 2.18 is employed to justify the compact dominance of \(F\) in the course of proving Theorem 3.7. If the space \(X\) was already compact, we could have omitted such a detour.

### 3.3. Existence of an invariant measure

The next result extends Theorem 2.21 to the setting of TIFSs. Let \((\mathcal{F}, \mathcal{P})\) be a probabilistic IFS consisting of continuous maps acting on a Hausdorff topological space \(X\). Let \(M_{(\mathcal{F}, \mathcal{P})}\) be the Markov operator corresponding to \((\mathcal{F}, \mathcal{P})\). Let \(\mathcal{P}(X)\) be the set of all Radon probability measures on \(X\). By Proposition 9.8, \(M_{(\mathcal{F}, \mathcal{P})}(\mu) \in \mathcal{P}(X)\) for every \(\mu \in \mathcal{P}(X)\). Thus we can consider the Markov operator restricted to \(\mathcal{P}(X)\).

**Theorem 3.13.** Let \((\mathcal{F}, \mathcal{P})\) be a probabilistic topologically contractive IFS on a Hausdorff topological space \(X\). Then

(i) the Markov operator \(M : \mathcal{P}(X) \rightarrow \mathcal{P}(X)\) corresponding to \((\mathcal{F}, \mathcal{P})\) has the (not necessarily unique) CFP \(\mu_\ast\), i.e., \(M(\mu_\ast) = \mu_\ast\) is an invariant Radon probability measure and \(M^n(\mu)\) converges weakly to \(\mu_\ast\) for all \(\mu \in \mathcal{P}(X)\);

(ii) \(\mu_\ast\) supports the attractor \(A_F\) of \(F\), i.e., \(\text{supp}(\mu_\ast) = A_F\);
(iii) $\mu_*$ is unique up to Radon probability measures with compact supports and, if additionally $X$ is normal, $\mu_*$ is unique up to all Radon probability measures.

Proof. Step 1. Existence of an invariant measure for the canonical IFS on a code space.

Set $I := \{1, ..., N\}$ and let $\mu_b$ be the Bernoulli measure on $I^\infty$ corresponding to the vector of weights $\vec{p} = (p_1, ..., p_N)$; cf. Appendix, Section [9.4]. This means that $\mu_b$ is the unique Borel measure such that for every cylinder $A_{(\alpha_1, ..., \alpha_k)} := \{ (\alpha_1, ..., \alpha_k) \} \times I^\infty$, $\alpha_1, ..., \alpha_k \in I$, the following holds

$$\mu_b(A_{(\alpha_1, ..., \alpha_k)}) = p_{\alpha_1} \cdots p_{\alpha_k}.$$ 

Letting $M_{(T, \vec{p})}$ to be the Markov operator induced by a probabilistic canonical IFS $(T, \vec{p})$ on $I^\infty$, for every $(\alpha_1, ..., \alpha_k) \in I^k$, $k \geq 2$, we have

$$M_{(T, \vec{p})}(\mu_b(A_{(\alpha_1, ..., \alpha_k)})) = \sum_{i=1}^N p_i \cdot \mu_b(\tau_i^{-1}(A_{(\alpha_1, ..., \alpha_k)})) = p_{\alpha_1} \cdot \mu_b(A_{(\alpha_2, ..., \alpha_k)}) = \mu_b(A_{(\alpha_1, ..., \alpha_k)}).$$

This implies that $\mu_b = M_{(T, \vec{p})}(\mu_b)$.

Step 2. Existence of an invariant measure for arbitrary TIFS.

Fix a probabilistic TIFS $(F, \vec{p})$, where $F = \{w_1, ..., w_N\}$. Define the measure $\mu_* := \mu_b \circ \pi^{-1}$, where $\pi : I^\infty \to X$ is the coding map and $\mu_b$ is the Bernoulli measure on the code space. As $\mu_b$ is Radon, so is $\mu_*$ (see Proposition [9.8]). By the invariance of $\mu_b$ established in Step 1 and the definition of $\pi$, for every Borel $B \subseteq X$ the following holds

$$M(\mu_*)(B) = \sum_{i=1}^N p_i \cdot \mu_*(w_i^{-1}(B)) = \sum_{i=1}^N p_i \cdot \mu_b(\pi^{-1}(w_i^{-1}(B))) = \sum_{i=1}^N p_i \cdot \mu_b(\tau_i^{-1}(\pi^{-1}(B))) = \mu_b(\pi^{-1}(B)) = \mu_*(B).$$

Thus $\mu_*$ is an invariant Radon probability measure for $(F, \vec{p})$. Moreover, Proposition [9.11] together with Theorem [3.7](ii) imply that

$$\text{supp}(\mu_*) = \pi(\text{supp}(\mu_b)) = \pi(I^\infty) = A_F.$$

Step 3. The measure $\mu_*$ is CFP.

We show that the invariant measure $\mu_*$ defined in Step 2 is the CFP of the Markov operator $M : \mathcal{P}(X) \to \mathcal{P}(X)$ corresponding to $(F, \vec{p})$. Take any measures $\mu, \eta \in \mathcal{P}(X)$ and a continuous and bounded function $g : X \to \mathbb{R}$. It is enough to show that

$$\lim_{n \to \infty} \left| \int_X g \, dM^n(\mu) - \int_X g \, dM^n(\eta) \right| = 0. \quad (5)$$

Indeed, choosing $\mu_*$ for $\eta$, and using its invariance, we see that (5) implies that $M^n(\mu) \to \mu_*$ weakly.

Fix $\varepsilon > 0$. Since $\mu, \eta$ are Radon, there is a compact set $C$ such that $\mu(X \setminus C), \eta(X \setminus C) < \varepsilon$,$\frac{1}{\varepsilon}$, where $P := \sup\{|g(x)| : x \in X\} + 1$. Switching if needed to some bigger compact set, we can assume that $F(C) \subseteq C$. As $g$ is continuous at each point of $C$, we can find a finite open cover $U_1, ..., U_m$ of $C$ such that for every $i = 1, ..., m$ and $x, y \in U_i$, we have $|g(x) - g(y)| < \frac{\varepsilon}{4}$. Now if $k \in \mathbb{N}$, then we set

$$A_k := \{ \alpha \in I^\infty : w_{\alpha_k}(C) \subseteq U_i \text{ for some } i = 1, ..., m \}.$$ 

Clearly, each $A_k$ is open and, by properties of a TIFS, the family $A_k$, $k \in \mathbb{N}$, is a cover of $I^\infty$. By compactness of $I^\infty$ and the fact that the sequence $(A_k)$ is increasing, we can find $k_0 \in \mathbb{N}$ such that $I^\infty = A_k$ for $k \geq k_0$. Now let $\alpha \in I^k$ and find $i = 1, ..., m$ so that $w_i(C) \subseteq U_i$. Choosing any $x_0 \in U_i$,
we have
\[
\left| \int_X g \circ w_\alpha \, d\mu - \int_X g \circ w_\alpha \, d\eta \right| \leq \left| \int_X g \circ w_\alpha \, d\mu - g(x_0) \right| + \left| g(x_0) - \int_X g \circ w_\alpha \, d\eta \right| = \\
= \left| \int_X g \circ w_\alpha \, d\mu - \int_X g(x_0) \, d\mu \right| + \left| \int_X g(x_0) \, d\eta - \int_X g \circ w_\alpha \, d\eta \right| \\
\leq \int_X |g \circ w_\alpha - g(x_0)| \, d\mu + \int_X |g(x_0) - g \circ w_\alpha| \, d\eta \leq \\
\leq \int_{X \setminus C} 2P \, d\mu + \int_C \frac{\varepsilon}{4} \, d\mu + \int_{X \setminus C} 2P \, d\eta + \int_C \frac{\varepsilon}{4} \, d\eta \leq \varepsilon.
\]
Hence by formula (2) in Remark 1.9 and an easy inductive argument:
\[
\left| \int_X g \, dM^n(\mu) - \int_X g \, dM^n(\eta) \right| = \left| \sum_{\alpha \in I^k} p_\alpha \int_X g \circ w_\alpha \, d\mu - \sum_{\alpha \in I^k} p_\alpha \int_X g \circ w_\alpha \, d\eta \right| \leq \\
\leq \sum_{\alpha \in I^k} p_\alpha \left| \int_X g \circ w_\alpha \, d\mu - \int_X g \circ w_\alpha \, d\eta \right| < \varepsilon,
\]
where \( p_\alpha = p_{\alpha_1} \cdots p_{\alpha_k}, \alpha = (\alpha_1, \ldots, \alpha_k) \in I^k \).

Step 4. Uniqueness of \( \mu_* \).

If \( X \) is normal, then the weak topology on \( \mathcal{P}(X) \) is Hausdorff (see Appendix, Proposition 9.5). Therefore, the CFP \( \mu_* \) of \( M_{(\mathcal{F}, \mathcal{B})} \) is necessarily unique.

Now let us consider the case of a general, not necessarily normal space \( X \). Assume that \( \mu \neq \eta \) are two invariant measures with compact supports. By the compact dominance of \( \mathcal{F} \), we can find a compact set \( C \subseteq X \) so that \( \mathcal{F}(C) \subseteq C \) and
\[
\text{supp}(\mu) \cup \text{supp}(\eta) \subseteq C.
\]
Let \( \mu_C, \eta_C \) be the restrictions of \( \mu, \eta \), respectively, to the Borel \( \sigma \)-algebra \( \mathcal{B}(C) \). Consider the restriction of \( \mathcal{F} \) to \( C \), \( \mathcal{F}|_C := \{w_1|C, \ldots, w_N|C\} \) and the induced Markov operator \( M_{(\mathcal{F}|_C, \mathcal{B})} \). Then for every \( B \in \mathcal{B}(C) \),
\[
M_{(\mathcal{F}|_C, \mathcal{B})}(\mu_C)(B) = \sum_{i=1}^N p_i \cdot \mu_C((w_i|C)^{-1}(B)) = \sum_{i=1}^N p_i \cdot \mu(C \cap w_i^{-1}(B)) = \\
= \sum_{i=1}^N p_i \cdot \mu(w_i^{-1}(B)) = \mu(B) = \mu_C(B),
\]
because of Lemma 9.10(ii), inclusion (3) and simple set-algebra
\[
(w_i|C)^{-1}(B) = C \cap w_i^{-1}(B).
\]
This means that \( \mu_C \) is (clearly Radon) invariant measure for \( \mathcal{F}|_C \). By symmetry, \( \eta_C \) is also an invariant measure for \( \mathcal{F}|_C \). It remains to show that \( \mu_C \neq \eta_C \) as this will give a contradiction with the "normal" case. By our assumption that \( \mu \neq \eta \), there is a Borel set \( B \subseteq X \) such that \( \mu(B) \neq \eta(B) \). Thanks to Lemma 9.10(ii) and (6), we have
\[
\mu_C(B \cap C) = \mu(B \cap C) = \mu(B \cap C) + \mu(B \setminus C) = \mu(B) \neq \eta(B) = \eta_C(B \cap C).
\]
Hence \( \mu_C \neq \eta_C \) and the result follows.

Remark 3.14. For a metric version of Theorem 3.13 see [56] Theorem 2.67.
3.4. Remetrization of topologically contractive IFSs. Turning back to remetrization Theorem 2.6, it is natural to ask whether, having a TIFS $\mathcal{F}$ on a metrizable space $X$, there exists an admissible complete metric $d$ in $X$ making $\mathcal{F}$ a Banach contractive IFS. Such a question for topological self similar sets was considered by Kameyama in [50]. He proved that given a topological self similar set $X$ and an appropriate IFS $\mathcal{F}$, we can always define a family of pseudometrics $d_a^2$, for $a = (a_1,...,a_N) \in (0,1)^N$, with the property that $\text{Lip}_{d_{a_i}}(w_i) \leq a_i$ for $i = 1,...,N$, and the following conditions are equivalent:

(K-1) $d_a^2$ is admissible metric for some $a \in (0,1)^N$;
(K-2) there is an admissible metric $d$ on $X$ such that $\text{Lip}_d(w_i) < 1$ for every $i = 1,...,N$.

Kameyama also gave an example of a topological self similar system for which condition (K-1) does not hold ([50] Section 1.4). Hence the answer to his question is negative. However, unexpectedly, it turns out that things are different if we do not insist that the remetrized IFS has to comprise Banach contractions and we agree to use weak contractions in place of Banach ones (below $w_{id} = \text{id}_X$ stands for the identity function).

Theorem 3.15. Let $\mathcal{F} = \{w_1,...,w_N\}$ be a topologically contractive IFS on a metric space $(X,d)$. Let $(a_n)_{n=0}^{\infty}$ be a strictly increasing sequence of reals such that $1 \leq a_n \leq 2$ for all $n \geq 0$. Define

$$d(x,y) := \max\{a_kd(w_\alpha(x), w_\alpha(y)) : k = 0,1,2,..., \alpha \in I^k\} \text{ for } x,y \in X.$$ 

Then

(i) $\hat{d}$ is admissible metric on $X$;
(ii) $\hat{d}$ is complete provided $d$ is complete;
(iii) $\mathcal{F}$ is Edelstein contractive with respect to $\hat{d}$;
(iv) $\mathcal{F}$ is Rakotch contractive with respect to $\hat{d}$ provided that, additionally, for every $\alpha \in I^\infty$, the set $\bigcap_{k \in \mathbb{N}} w_{a_k}(X)$ is a singleton.

Before we give a proof, let us observe that the above result yields a natural characterization of topological self similar sets.

Corollary 3.16. A topological space $X$ is a topological self similar set if and only if $X$ is homeomorphic to the attractor of some weakly contractive IFS.

Proof. (of Theorem 3.15) We first observe that $\hat{d}$ is well defined. Take distinct points $x,y \in X$. By definition, we can find a compact set $C$ containing $x,y$ and such that $\mathcal{F}(C) \subseteq C$. Then, using similar reasonings to those in the proof of Theorem 3.13, we can find $k_0$ such that for every $k \geq k_0$ and $\alpha \in I^k$, it holds $\text{diam}(w_\alpha(C)) < \frac{1}{2}d(x,y)$. (Indeed, we just have to put $A_k := \{\alpha \in I^\infty : \text{diam}(w_{a_{\alpha k}}(C)) < \frac{1}{3}d(x,y)\}$). Hence for every $k \geq k_0$ and $\alpha \in I^k$, it holds

$$a_kd(w_\alpha(x), w_\alpha(y)) \leq a_k\text{diam}(w_\alpha(C)) < \frac{2}{3}d(x,y) \leq \frac{2}{3}\hat{d}(x,y) < \hat{d}(x,y).$$

Thus $\hat{d}$ is well defined.

Now we will prove (i). It is easy to see that $\hat{d}$ is a metric. Symmetry and triangle inequality are immediate. A less trivial implication $\hat{d}(x,y) = 0 \Rightarrow x = y$ follows from the inequality $d \leq \hat{d}$. From the same inequality we see that to prove admissibility of $\hat{d}$, it is enough to show that if $d(x_n,x) \to 0$, then $d(x_n,x) \to 0$. Fix any $\varepsilon > 0$, and choose $k_0$ such that for any $k \geq k_0$, $n \in \mathbb{N}$ and $\alpha \in I^k$, we have $d(w_\alpha(x_n), w_\alpha(x)) < \frac{1}{2}\varepsilon$ (this can be done in a similar way as earlier; observe that $\{x_n : n \in \mathbb{N}\} \cup \{x\}$ is compact). Now since the family $\{w_\alpha : \alpha \in I^k, k \leq k_0\}$ is finite and consists of continuous functions, we can find $k_1 \in \mathbb{N}$ with the property that $d(w_\alpha(x_n), w_\alpha(x)) < \frac{1}{2}\varepsilon$ for all $\alpha \in I^k, k \leq k_0$ and $n \geq k_1$. Hence for every $n \geq k_1$, $d(x_n,x) < \varepsilon$ which proves that $d(x_n,x) \to 0$.

Now we move to (ii). Assume that $d$ is complete and let $(x_n)$ be a $\hat{d}$-Cauchy sequence. As $d \leq \hat{d}$, it is also $d$-Cauchy, hence $d$-convergent. Since $d$ and $\hat{d}$ are equivalent, the sequence $(x_n)$ is also $\hat{d}$-convergent.
We postpone (iii) for a moment and jump to prove (iv). We will show that each \( w_i \) satisfies condition (iii) from Lemma 2.10. Take \( \delta > 0 \) and choose \( k_0 \in \mathbb{N} \) such that for all \( k \geq k_0 \) and \( \alpha \in I^k \), the following inequality holds: \( \text{diam}(w_\alpha(X)) < \frac{1}{4} \delta \). Set

\[
\lambda := \max \left\{ \frac{a_k}{a_{k+1}} : k \leq k_0 \right\} < 1.
\]

Now choose \( x, y \in X \) with \( \hat{d}(x, y) \geq \delta \) and \( i = 1, \ldots, N \). For \( \alpha \in I^k \), we see that if \( k \geq k_0 \), then

\[
a_k d(w_\alpha(w_i(x)), w_\alpha(w_i(x))) \leq 2 \cdot \frac{1}{4} \delta \leq \frac{1}{2} \hat{d}(x, y),
\]

and if \( k \leq k_0 \), then

\[
a_k d(w_\alpha(w_i(x)), w_\alpha(w_i(y))) = \frac{a_k}{a_{k+1}} \cdot a_{k+1} d(w_\alpha(w_i(x)), w_\alpha(w_i(y))) \leq \frac{a_k}{a_{k+1}} \hat{d}(x, y) \leq \lambda \hat{d}(x, y).
\]

Therefore

\[
\hat{d}(w_i(x), w_i(y)) \leq \max \left\{ \frac{1}{2} \lambda \right\} \hat{d}(x, y).
\]

Overall \( w_i \) is a Rakotch contraction with respect to \( \hat{d} \). This gives (iv).

Finally we show (iii). Let \( x, y \in X, x \neq y \). By compact dominance of \( \mathcal{F} \) we can find a compact set \( C \) such that \( x, y \in C \) and \( \mathcal{F}(C) \subseteq C \). Then using (iv) for the IFS \( \mathcal{F}|C = \{w_1|C, \ldots, w_N|C\} \) we deduce that \( \hat{d}(w_i(x), w_i(y)) < \hat{d}(x, y) \). The result follows.

**Remark 3.17.** The above remetrization theorem shows that Theorem 3.7 for completely metrizable spaces can be deduced from its version for Rakotch contractive IFSs (which is given for example in [41]). Indeed, although IFSs consisting of Edelstein contractions may fail to generate attractors, we can always switch to some suitably large compact set \( C \) with \( \mathcal{F}(C) \subseteq C \) and use the fact that an Edelstein contraction on a compact space is always Rakotch.

Formula (7) was conceived by R. Miculescu and A. Mihail in [73]. They employed it to metrize Kameyama’s topological self-similar sets. Their technique of proof required some additional properties of the sequence \( (a_n) \). In [9], Banakh et al. considered also several more restrictive topological-type contractive conditions on \( \mathcal{F} \) and proved that \( \hat{d} \) has better properties in such cases. For example, IFSs which satisfy the sufficient condition in point (iv) of Theorem 3.15 are called globally contractive. Moreover, they extended the whole discussion to Tikhonov (a.k.a completely regular) spaces. It is known that the Tikhonov topology is generated by an appropriate family of pseudometrics, say \( \mathcal{D} \), and Banakh et.al. obtained counterpart of Theorem 2.18 for IFSs consisting of weakly contractive maps with respect to pseudometrics from \( \mathcal{D} \). Additionally they proved that the formula (7) can be successfully adjusted to that general pseudometric setting and hence Theorem 3.7 (at least the existence of the CFP) can be explained by its pseudometric version. Other remetrization results in this spirit can be found for example in [74].

4. Baire genericity of attractors

A natural question arises whether the class of attractors of weakly contractive IFSs is essentially wider than the class of attractors of Banach contractive ones. Since attractors of Banach contractive IFSs have necessarily finite Hausdorff dimension, it is enough to find a weakly contractive IFS which admits an infinite dimensional attractor. As shown by D. Dumitrut [28], each Peano continuum \( X \) with a free arc (i.e., with an arc which is open) is the attractor of some TIFS. Hence, by Corollary 3.16 \( X \) is homeomorphic to the attractor of some weakly contractive IFS. On the other hand, if \( X \) has also infinite topological dimension, then it has infinite Hausdorff dimension with respect to any admissible metric, so it is not homeomorphic to an attractor of a Banach contractive IFS.

Interestingly, one can look for suitable examples in the finite dimension as we do below.
We will show that the first category in the hyperspace iterated function system. Formally, the collection of attractors of weakly contractive IFSs on $X$, the Polish space. Typical nonempty and compact subset of $X$ is the attractor of some weakly contractive IFS but it is not an attractor of any Banach contractive IFS. (The latter observation follows from a nice criterion given in [86].)

Although many continua turn out to be attractors of IFSs if we allow to employ weak contractions instead of restricting ourselves to Banach contractions, not all compact sets are attractors of weakly contractive IFSs. Indeed, every connected attractor must be locally connected (see Theorem 7.32 or [31]). Even more, attractors of weakly contractive IFSs are exceptional sets among compacta.

**Theorem 4.2.** Let $X$ be a complete separable metric space without isolated points (a.k.a. perfect Polish space). Typical nonempty and compact subset of $X$ is not an attractor of any weakly contractive iterated function system. Formally, the collection of attractors of weakly contractive IFSs on $X$ is of the first category in the hyperspace $K(X)$ of compacta.

**Proof. Step 1.** Construction of nowhere dense sets. For any nonempty open set $D \subset X$, define

$$C_D := \{K \in K(X) : \emptyset \neq K \cap D \subseteq f(K \setminus D) \text{ for some } f : K \to X \text{ with Lip}(f) \leq 1\}.$$  

We will show that $C_D$ is nowhere dense in $K(X)$.

Take $r > 0$ and $K \in C_D$. Since $X$ has no isolated points and $D$ is open, we can find a finite set $K' \subseteq X$ such that $d_H(K',K) < \frac{r}{2}$, $D \cap K' = \{x_1,...,x_k\}$ and $K' \setminus D = \{y_1,...,y_j\}$ for some distinct points $x_1,...,x_k,y_1,...,y_j$. Set

$$\varepsilon := \min \left\{r, \delta, \min\{d(x_i,x_l) : i \neq l\} \right\} > 0,$$

$$\varepsilon' := \frac{\varepsilon}{4},$$

where $\delta > 0$ is chosen so that $B(x_i,\delta) \subseteq D$ for $i = 1,...,k$. It remains to prove that

$$B_H(K',\varepsilon') \subseteq B_H(K,r) \setminus C_D,$$

where $B_H(\cdot,\cdot)$ stands for an open ball in the hyperspace ($K(X),d_H$).

Clearly, if $S \in K(X)$ satisfies $d_H(S,K') < \varepsilon'$, then $d_H(S,K) \leq d_H(S,K') + d_H(K',K) < r$. Further we will see that also $S \notin C_D$.

First observe that $S \setminus D \subseteq \bigcup_{i=1}^j B(y_i,\varepsilon')$. Indeed, take $x \in S \setminus D$. If $x \in B(x_i,\varepsilon')$ for some $i = 1,...,k$, then $x \in D$ by the choice of $\varepsilon'$ and $\delta$, which is a contradiction. Thus $x \in B(y_i,\varepsilon')$ for some $i$ and we are done.

Next, take any $f : S \to X$ with $\text{Lip}(f) \leq 1$ and for each $i = 1,...,k$, find a point $\tilde{x}_i \in S$ such that $d(x_i,\tilde{x}_i) < \varepsilon'$. In particular $\tilde{x}_1,...,\tilde{x}_k \in D$. Moreover, for every $i \neq l$

$$d(\tilde{x}_i,\tilde{x}_l) > d(x_i,x_l) - 2\varepsilon' \geq 2\varepsilon'.$$

Since $\{\tilde{x}_1,...,\tilde{x}_k\} \subseteq S \cap D$, $S \setminus D \subseteq \bigcup_{i=1}^j B(y_i,\varepsilon')$, and $k > j$, it is enough to show that each $f(B(y_i,\varepsilon') \cap S)$ can contain at most one $\tilde{x}_i$. Supposing this is not the case, we would get

$$2\varepsilon' < d(\tilde{x}_i,\tilde{x}_l) \leq \text{diam}(f(B(y_n,\varepsilon') \cap S)) \leq \text{diam}(B(y_n,\varepsilon')) \leq 2\varepsilon'$$

for some $i \neq l$ and $n$, which is a contradiction. All in all, $S \cap D$ is not contained in $f(S \setminus D)$ and $S \notin C_D$. Thus we have proved that $C_D$ is nowhere dense in $K(X)$.

**Step 2.** Construction of a residual set $A$.

Let us fix a countable basis $B$ of the topology of the space $X$ and consider the collection of sets $\{C_D : D \in B\}$ as defined in Step 1. Define

$$A := \text{Perf}(X) \setminus \bigcup_{D \in B} C_D,$$
for any Edelstein (hence Rakotch) contraction \( f : K \to K \), the image \( f(K) \) is nowhere dense in \( K \). Actually, it suffices to show that \( f(K) \), being closed in \( K \), has empty interior in \( K \).

Choose any open set \( U \subset X \) so that \( U \cap K \neq \emptyset \). Then \( \text{diam}(U \cap K) > 0 \). By Lemma \ref{lem:subsetU} we can find \( \lambda < 1 \) so that \( d(f(x), f(y)) \leq \lambda d(x, y) \) whenever \( d(x, y) \geq \delta := \frac{1}{2} \text{diam}(U \cap K) \). Hence

\[
\text{diam}(f(K \cap U)) \leq \max \{\delta, \lambda \text{diam}(K \cap U)\} < \text{diam}(K \cap U).
\]

Thus there exists \( D \in \mathcal{B} \) such that \( \emptyset \neq D \cap K \subseteq (K \cup U) \setminus f(K \cap U) \). By definition, \( K \notin C_D \), so we have that \( (K \cap D) \notin f(K \setminus D) \). Therefore

\[
(K \cup U) \setminus f(K) = [(K \cup U) \setminus f(K \cap U)] \cap [(K \cup U) \setminus f(K \setminus U)] \supseteq (K \cap D) \cap [(K \cap D) \setminus f(K \setminus D)] = (K \cap D) \setminus f(K \setminus D) \neq \emptyset
\]

and hence \( f(K) \) has empty interior. The result follows. \( \square \)

Remark 4.3. The proof of Theorem \ref{thm:IFS} given above extends some ideas from D’Aniello and Steele’s paper \cite{DAnielloSteele}, where it was assumed that \( X \) is a finite dimensional unit cube.

Remark 4.4. An alternative proof of Theorem \ref{thm:IFS} was given by Balka and Máté in \cite{Balka}. They defined the so-called balanced sets and proved that typical sets are balanced while a balanced set cannot be an attractor of a weakly contractive IFS. Moreover, they extended the thesis of Theorem \ref{thm:IFS} by showing that if \( X \) is separable and complete, then typical nonempty and compact set is either finite or it is not an attractor of any weakly contractive IFS. Finally, let us note that to learn how much prevalent are fractals (attractors with possibly fractional dimension), one is led to study typical dimension of sets and measures, e.g., \cite{Balka, DAnielloSteele, Falcinelli} and \cite{Seuren}.

Theorem \ref{thm:IFS} stated that it is rare for a set to be an attractor. Quite opposite is true when we demand only that the set at hand is homeomorphic to an attractor. Lemma \ref{lem:subsetU} implies that typical subset of a complete metric space without isolated point is homeomorphic to the Cantor ternary set, a classical example of an attractor of a Banach contractive IFS. We state this observation as a separate theorem.

Theorem 4.5. Typical nonempty and compact subset of a complete perfect metric space is homeomorphic to the attractor of a Banach contractive iterated function system.

For a list of examples that distinguish different classes of IFS attractors we refer the reader to the survey paper \cite{PerisSierpinski} on counterexamples in the IFS theory.

5. Deterministic chaos game for weakly contractive IFSs

In the present section we exhibit a simple proof of the deterministic version of the chaos game algorithm. The main tool is the coding map. Derandomization of the algorithm is due to contractivity and employment of algorithmically random sequences — disjunctive sequences.

Definition 5.1. (\cite{GOLDBERG}). Let \( I \) be a finite set of symbols (alphabet). A sequence \( \sigma = (\sigma_n)_{n=1}^\infty \in I^\infty \) is called disjunctive, whenever each finite word \( \alpha \in I^k \), \( k \geq 1 \), appears in \( \sigma \), i.e., \( (\sigma_n, ..., \sigma_{n+k-1}) = \alpha \) for some \( n \geq 1 \).

Example 5.2. (Champernowne sequence). Let \( I = \{1, 2, ..., N\} \), \( 1 < 2 < ... < N \), \( N \geq 1 \). The following sequence is disjunctive:

\[
1, 2, ..., N, 1, 1, 1, 2, ..., 1, N, 2, 1, 2, 2, ..., 2, N, ..., N, 1, N, 2, ..., N, N, ...
\]
It is created by writing in the lexicographic order first all symbols from $I$, then all 2-letter words over $I$, then 3-letter words etc. This simple sequence is very regular in a probabilistic manner — it is Borel normal (in base $N$), cf. [53] Corollary 2.9.2.

A disjunctive sequence is random in the qualitative manner: it contains all possible finite sequences. A disjunctive sequence is also chaotic, because: (i) it is not almost periodic (a.k.a. uniformly recurrent), cf. [70]; (ii) the Bernoulli shift $\vartheta : I^\infty \to I^\infty$, $\vartheta(\sigma_1,\sigma_2,\sigma_3...) = (\sigma_2,\sigma_3...)$, generates a dense orbit in the code space $I^\infty$ when the orbit $\{\vartheta^k(\sigma)\}_k=0$ starts at a disjunctive sequence $\sigma = (\sigma_1,\sigma_2,...)$, cf. [14].

Disjunctive sequences are prevalent in both topological and measure-theoretic sense. The set of sequences which are not disjunctive is: (i) $\sigma$-porous, in particular it is of the first Baire category, in the code space $(I^\infty,d_B)$ endowed with the Baire metric; (ii) null with respect to the Bernoulli measure $\mu_b$ in $I^\infty$; e.g., [20]. Moreover, many discrete stochastic processes, like nondegenerate Bernoulli and Markov processes, generate disjunctive outcomes with probability 1. A sufficient condition for a general chain to yield disjunctive sequences almost surely provides

**Theorem 5.3** ([61] Theorem 3.4, Lemma 3.5). Let $(Z_n)_{n=1}^\infty$ be a (not necessarily stationary) stochastic process with states in a finite set $I$. Suppose that $Z_n$ satisfies

$$\Pr(Z_n = \sigma_n | Z_{n-1} = \sigma_{n-1}, ..., Z_1 = \sigma_1) \geq p_n > 0$$

for all $\sigma_1,...,\sigma_n \in I$ and $n \geq 1$. Let the minorant $(p_n)_{n=1}^\infty$ satisfy

$$\lim_{n \to \infty} \frac{p_n}{n^c} = 0 \text{ for every } c > 0.$$  \tag{8}

Then $Z_n$ generates a disjunctive sequence $\sigma_n$ with probability 1.

(In the above $\Pr(Z_n = \sigma_n | Z_{n-1} = \sigma_{n-1}, ..., Z_1 = \sigma_1)$, understood as $\Pr(Z_1 = \sigma_1)$ when $n = 1$, denotes the conditional probability that $Z_n = \sigma_n$ occurs if earlier have occured already $Z_{n-1} = \sigma_{n-1},...,Z_1 = \sigma_1$; cf. [53].)

**Example 5.4** ([14] Definition 4.1; [61] Examples 3.2 and 3.6). The following sequences fulfill (8): $p_n \equiv \text{const.} > 0$, $p_n = (\log n)^{-b}$ for some $b > 0$. The following sequences do not fulfill (8): $p_n = n^{-b}$, $p_n = \sin(n^{-b})$, where $b > 0$.

Let $F = \{w_1,...,w_N\}$ be an IFS on a Hausdorff topological space $X$. The sequence $(x_n)_{n=0}^\infty$ given by iteration

$$\begin{cases} x_0 \in X, \\
x_n = w_{\sigma_n}(x_{n-1}), n \geq 1, \\
\end{cases}$$

is called an orbit starting at $x_0$ and driven by $(\sigma_n)_{n=1}^\infty \in I^\infty$. The $\omega$-limit set of the orbit $x_n$ is defined to be

$$\omega((x_n)) := \bigcap_{m=0}^\infty \{x_n : n \geq m\}.$$  \tag{9}

**Theorem 5.5** (Deterministic chaos game). Let $F = \{w_1,...,w_N\}$ be an IFS consisting of Brouwer contractions on a complete metric space $(X,d)$. Let $A$ be an attractor of $F$. Then for every $x_0 \in X$ and any disjunctive driver $\sigma \in \{1,...,N\}^\infty$ the orbit $x_n$ starting at $x_0$ and driven by $\sigma$ (according to (9)) recovers $A$, i.e., $\omega((x_n)) = A$.

**Proof.** For a given driver $\sigma$ and two starting points $x_0 \in X$ and $a_0 \in A$, the orbits $x_n = w_{\sigma_n}(x_{n-1})$ and $a_n = w_{\sigma_n}(a_{n-1})$ are getting closer to each other, i.e.,

$$d(x_n,a_n) \leq \varphi^n(d(x_0,a_0)) \to 0,$$

where $\varphi$ is a module of continuity common for all maps $w_i$, $\varphi^n$ stands for the $n$-fold composition of $\varphi$ and $\lim_{n \to \infty} \varphi^n(t) = 0$ due to [38] chap.1 §1.6 (B.2) p.19 (see also Remark 2.5). In consequence

$$\omega((x_n)) = \omega((a_n)).$$  \tag{10}
The conjugation is established by the coding map \( \pi \). Theorem 3.7, the rest follows as in the proof of Theorem 5.5. It is sufficient to prove (10). Since \( \sigma \) is disjunctive, for any given \( m \geq 1 \), the prefixes of addresses
\[
\varsigma_n := (\sigma_1, \ldots, \sigma_1)^\alpha, \ n \geq m,
\]
see also Figure 1. Since \( \sigma \) is disjunctive, for any given \( m \geq 1 \), the prefixes of addresses
\[
\omega((a_n)) = \bigcap_{m=0}^{\infty} \{a_n : n \geq m\} = \bigcap_{m=0}^{\infty} \pi(\{\varsigma_n : n \geq m\}) = \pi(I^\infty) = A,
\]
where the second equality follows from the closedness of the continuous map \( \pi \) on a compact set. \( \square \)

Deterministic chaos game is also valid for topologically contractive IFSs.

**Theorem 5.6.** Let \( \mathcal{F} = \{w_1, \ldots, w_N\} \) be a TIFS on a Hausdorff topological space \( X \). Let \( A \) be an attractor of \( \mathcal{F} \). Then for every \( x_0 \in X \) and any disjunctive driver \( \sigma \in \{1, \ldots, N\}^\infty \) the orbit \( x_n \) starting at \( x_0 \) and driven by \( \sigma \) (according to (10)) recovers \( A \), i.e., \( \omega((x_n)) = A \).

**Proof.** It is sufficient to prove (10). Since TIFS admits a compact attractor and a conjugation with a canonical IFS via coding map (Theorem 3.7), the rest follows as in the proof of Theorem 5.5.

Let \( x_n, a_n \) be two orbits with the same driver \( (\sigma_n)_{n=1}^\infty \), one starting at \( x_0 \), and the other at \( a_0 \in A \). We are going to show that \( \omega((x_n)) \subseteq \omega((a_n)) \). The reverse inclusion will follow by symmetry. By Definition 3.4 (i):
\[
\omega((x_n)) \cup \omega((a_n)) \subseteq C.
\]
In particular, \( \omega((x_n)) \cup \omega((a_n)) \subseteq C \). Therefore, by restricting \( \mathcal{F} \) to \( C \) if necessary, we can assume further that \( X \) is compact. (The \( \omega \)-limit set of an orbit of \( \mathcal{F} \) which lies in \( C \) coincides with the \( \omega \)-limit set of that orbit understood as an orbit of the restricted system \( \mathcal{F}|C = \{w_1|C, \ldots, w_N|C\} \).)

Let \( y \in \omega((x_n)) \). Fix open \( U \ni y \) and arbitrarily large number \( m \in \mathbb{N} \). Shrink \( U \) to open \( V \ni y, \nabla \subseteq U \). Let \( C \) be chosen according to (12). By Definition 3.4 (ii) there exists \( k_0 \) s.t. for all \( (\alpha_1, \ldots, \alpha_k) \in \{1, \ldots, N\}^k, k \geq k_0 \), it holds
\[
\omega_{\alpha_1} \circ \ldots \circ \omega_{\alpha_k}(C) \subseteq U \text{ or } \omega_{\alpha_1} \circ \ldots \circ \omega_{\alpha_k}(C) \subseteq X \setminus \nabla.
\]
Remark 5.8. The generation of the random orbit (9) can be viewed as an action of a skew-product, making a system (a.k.a. cocycle a.k.a. random dynamical system) over the Bernoulli shift in such a way that to recover the attractor 

\[ w_1 \circ w_1(x_0) \quad w_2 \circ w_1(x_0) \quad w_1 \circ w_2(x_0) \quad w_2 \circ w_2(x_0) \]

\[ \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \]

\[ w_1(x_0) \quad w_2(x_0) \]

\[ x_0 \]

Figure 2. Part of a full tree of iterations of \( \mathcal{F} = \{w_1, w_2\} \) up to a second level: \( \bigcup_{n=0}^{2} \mathcal{F}^n(\{x_0\}) \).

(Inward composed maps yield a singleton.) Since \( y \in \omega((x_n)) \), we can pick \( k \geq \max\{k_0, m\} \) so that \( x_k \in V \). Recalling that \( x_k = w_{\sigma_k} \circ \ldots \circ w_{\sigma_1}(x_0), x_0 \in C \), this implies

\[ V \cap w_{\sigma_k} \circ \ldots \circ w_{\sigma_1}(C) \neq \emptyset. \]

Therefore the second alternative in (13) is false, and we have \( w_{\sigma_k} \circ \ldots \circ w_{\sigma_1}(C) \subseteq U \). Recalling that \( a_k = w_{\sigma_k} \circ \ldots \circ w_{\sigma_1}(a_0), a_0 \in C \), shows that

\[ a_k \in w_{\sigma_k} \circ \ldots \circ w_{\sigma_1}(C) \subseteq U \]

for some \( k \geq m \).

Therefore, \( y \in \omega((a_n)) \), because \( m \) and \( U \) were arbitrary. \( \square \)

**Remark 5.7.** If \( X \) is metrizable, then the proof of Theorem 5.6 can be simplified on the basis of Theorem 3.15 on remetrizability. One just takes a suitable compact \( C \supseteq A \) with \( \mathcal{F}(C) \subseteq C \) and considers the restriction of \( \mathcal{F} \) to \( C \), \( \mathcal{F}|C = \{w_1|C, \ldots, w_N|C\} \). Then there is a complete metric \( d \) so making \( w_i|C \) Rakotch contractive all at once.

**Remark 5.8.** The generation of the random orbit (9) can be viewed as an action of a skew-product system (a.k.a. cocycle a.k.a. random dynamical system) over the Bernoulli shift \( \vartheta : I^\infty \to I^\infty \) with fiber \( X \); cf. [23, chap.14], [6, chap.2.1] (see also [15] and [51, Remark 4.1.5(b)] with the footnote on p.84).

Let us comment upon the essence and history of the chaos game. The chaos game algorithm works in such a way that to recover the attractor

\[ A = \bigcap_{m=0}^{\infty} \bigcup_{n=m}^{\infty} \mathcal{F}^n(\{x_0\}), \]

instead of building the full tree \( T = \bigcup_{n=0}^{\infty} \mathcal{F}^n(\{x_0\}) \) of iterations of the Hutchinson operator \( \mathcal{F} \),

\[ \mathcal{F}^n(\{x_0\}) = \{w_{\sigma_n} \circ \ldots \circ w_{\sigma_1}(x_0) : (\sigma_1, \ldots, \sigma_n) \in I^n\}, \]

see Figure 2. It is enough to climb along a single, yet sufficiently complex branch of the tree \( T \); namely, it is enough to follow a disjunctive orbit \( x_n = w_{\sigma_n} \circ \ldots \circ w_{\sigma_1}(x_0) \). (Note the order of composition along a word, which is different from that in (11).)

The proof of the deterministic chaos game which we have provided above was not presented anywhere so far. Nevertheless, it is much in the spirit of [30, Theorem 5.1.3; see also [66]. The ideas behind it are buried in some papers from 1990s, cf. [37] and [71]. Formally, the deterministic chaos game was stated for the first time in [14] (without a proof, for strongly-fibred systems).

Recently it was shown that the probabilistic chaos game algorithm (i.e., when the orbit is driven by a stochastic process) works for very general noncontractive IFSs, cf. [12]. Since many stochastic processes generate disjunctive sequences, the probabilistic version of the chaos game readily follows from its deterministic version. One could hope that it is always the case and the probabilistic chaos game is just a corollary to the deterministic chaos game. Notably, the deterministic chaos game
algorithm is valid for two large classes of systems: IFSs with a strongly-fibred attractor (e.g., [14]) and IFSs comprising nonexpansive maps (cf. [63]). Unfortunately, there exist IFSs for which the probabilistic chaos game works while deterministic — fails, cf. [15].

6. BETWEEN CONTRACTIVE AND NON-CONTRACTIVE REALM

We are going to exhibit some types of contractive iterated function systems (IFS) which suffer various deficiencies making them less flexible proposals for generalized contractive IFSs than topologically contractive IFSs (TIFS).

6.1. Eventual contractions. Given a complete metric space $X$, we say that a continuous map $w : X \to X$ is an eventual contraction if for some $p \in \mathbb{N}$, the $p$-fold composition $w^p$ is a Banach contraction (e.g., [38] chap.I §1.6 (A.1)). This concept has proven to be handy in some circumstances, e.g. in the theory of integral equations [54] chap.II §6.15.

Clearly, if $X$ is compact, then an eventual contraction is a topological contraction in the sense of Tarafdar, i.e., the intersection $\bigcap_{k \in \mathbb{N}} w^k(X)$ is a singleton; cf. [24] Definition 2.6 and Theorem 2.36 on p.88. Noting that in such cases $\mathcal{F} = \{w\}$ is a topologically contractive IFS, it is natural to ask whether all IFSs consisting of Tarafdar’s contractions or eventual contractions are TIFS or, at least, they generate unique invariant sets. The answer is negative as the following example shows.

**Example 6.1** (Non-TIFS of topological contractions). Let $w_i : [0, 1] \to [0, 1], i = 1, 2$, be defined by $w_1(x) = \max\left\{\frac{1}{2}, 1 - x\right\}$ and $w_2(x) = \min\left\{\frac{1}{2}, 1 - x\right\}$. Then $w_1, w_2$ are continuous and $w_2^2([0, 1]) = w_2^2([0, 1]) = \{\frac{1}{2}\}$, so they are continuous Tarafdar’s contractions (eventual contractions) acting on a compact space. On the other hand, for every $t \in \left[0, \frac{1}{2}\right]$, we have

$$w_1([t, 1 - t]) \cup w_2([t, 1 - t]) = \left[\frac{1}{2}, 1 - t\right] \cup \left[t, \frac{1}{2}\right] = [t, 1 - t].$$

Hence the IFS $\mathcal{F} := \{w_1, w_2\}$ does not generate a unique nonempty compact invariant set and, in particular, $\mathcal{F}$ is not topologically contractive.

**Remark 6.2.** Observe that in Example 6.1 we have $|w_1(x_1) - w_1(x_2)| = |x_1 - x_2|$ for some $x_1 \neq x_2$ (namely $\frac{1}{2} \leq x_1 < x_2 \leq 1$). This means that a topologically contractive map need not be a weak contraction under any good comparison function $\varphi$ for the original metric despite it is a $\varphi$-contraction after suitable change of a metric (by either the Miculescu-Mihail or the Meyer remetrization theorem; cf. Theorems 2.6 and 3.15).

Whether an IFS comprising eventually contractive maps yields a topologically contractive IFS depends on the interaction between the maps, not only upon the individual maps. We look closer at this phenomenon below.

A map $u : X \to X$ on a complete metric space $(X, d)$ is called bi-Lipschitz provided there exist two coefficients $\lambda(u) \geq \kappa(u) > 0$ s.t.

$$\kappa(u) \cdot d(x_1, x_2) \leq d(u(x_1), u(x_2)) \leq \lambda(u) \cdot d(x_1, x_2) \text{ for } x_1, x_2 \in X.$$

In other words $\lambda(u)$ is a (not necessarily minimal) Lipschitz constant of $u$, and $1/\kappa(u)$ is a Lipschitz constant of $u^{-1} : u(X) \to X$, the inverse of $u$. Obviously $\lambda(u_1 \circ u_2) \leq \lambda(u_1) \cdot \lambda(u_2)$ and $\kappa(u_1 \circ u_2) \geq \kappa(u_1) \cdot \kappa(u_2)$ for bi-Lipschitz $u_1, u_2 : X \to X$.

Let us define for $t_1, t_2 \geq 0$, $\ell^p(t_1, t_2) = (\ell_1^p + \ell_2^p)^{\frac{1}{p}}$ when $1 \leq p < \infty$ and $\ell^\infty(t_1, t_2) = \max\{t_1, t_2\}$ when $p = \infty$. The function $\ell^p : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is continuous, monotone in each variable, and obeys many other properties; for instance $\ell^p(t_1, t_2) = \ell^p(t_2, t_1) \geq \ell^\infty(t_1, t_2)$.

**Example 6.3** (Eventually contractive IFS). Let $(X, d)$ be a complete metric space containing at least two points. We endow $X \times X$ with the $\ell^p$-product metric

$$d_{X \times X}((x_1, y_1), (x_2, y_2)) = \ell^p(d(x_1, x_2), d(y_1, y_2)) \text{ for } x_1, y_1, x_2, y_2 \in X.$$

(See also [59] for more general product metrics.) So metrized product $X \times X$ is a complete space.
The class of such weights can be easily determined. For each $i$ assume for simplicity that $c_i$ is a positive weight.

A separable metric space has CFP (in other words, $w_i$ stands for the Lipschitz constant of $d_{MK}$).

By induction

(i) $\kappa(u_i) \geq 1$ for $i = 1, 2, \ldots$, yet

(ii) $\kappa(u_i) \geq 1, \kappa(u_i) \cdot \kappa(v_i) > 1, \lambda(u_i) \cdot \lambda(v_i) < 1$ for all $i = 1, 2$. This means that $w_i$'s are eventually contractive ($w_i \circ w_i$ is contractive), but $F$ does not admit an attractor. Suppose, a contrario, that $A \subseteq X \times X$ is a compact attractor of $F$. Pick $(a_x, a_y) \in A, (x, y) \neq (a_x, a_y)$. Then, using (14), we obtain

$$d_{X \times X}((w_2 \circ w_1)(x, y), (w_2 \circ w_1)(x, y)) = \max_{i,j \in \{1, 2\}} \lambda(v_i) \cdot \lambda(u_j) > 0.$$

**Remark 6.4.** Putting $X = \mathbb{R}, u_1 = u_2 = \text{id}, v_i(x) = \frac{x}{2}$, and $v_2(x) = \frac{x - 1}{2}$ in Example 6.3 yields [56] Exercise 2.28.

**Remark 6.5.** For $X = \mathbb{R}$ Example 6.3 takes particularly nice form. One can plug various values of $\tilde{v}_i, \tilde{u}_i, \tilde{u}_i, i = 1, 2$, and play with matrices

$$w_i(x, y) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} \tilde{v}_i & 0 \\ 0 & \tilde{u}_i \end{bmatrix} x + \begin{bmatrix} \tilde{v}_i \\ \tilde{u}_i \end{bmatrix}, x, y \in \mathbb{R}.$$

Then $v_i(x) = \tilde{v}_i \cdot x + \tilde{v}_i$, $\kappa(v_i) = \lambda(v_i) = \tilde{v}_i$, and similarly $u_i(x) = \tilde{u}_i \cdot x + \tilde{u}_i$, $\kappa(u_i) = \lambda(u_i) = \tilde{u}_i$.

**6.2. Average contractive IFSs.** A probabilistic iterated function system $(F, \vec{p})$ comprising Lipschitz maps $w_1, \ldots, w_N$ is said to be average contractive, provided $\sum_{i=1}^{N} \lambda(p_i) \cdot \lambda(w_i) < 1$, where $\lambda(w_i)$ stands for the Lipschitz constant of $w_i$.

**Remark 6.6.** Let $F = \{w_1, \ldots, w_N\}$ be an IFS consisting of Lipschitz maps. There exists a vector of positive weights $\vec{p}$ such that $(F, \vec{p})$ is average contractive if and only if $\min \{\lambda(w_i) : i = 1, \ldots, N\} < 1$. The class of such weights can be easily determined. For each $i = 1, \ldots, N$, set $c_i := \lambda(w_i)$ and assume for simplicity that $c_1 = \min\{c_1, \ldots, c_N\}$. Observe that $p_1 + \ldots + p_N = 1$ is equivalent to $p_1 = 1 - p_2 - \ldots - p_N$. Hence our aim is to find all positive values $p_2, \ldots, p_N$ so that $p_2 + \ldots + p_N < 1$ and $p_{N-1} < 1$. All in all, the family of all desired vectors consists of all $N$-tuples $(p_1, p_2, \ldots, p_N)$ so that

$$p_2 \in (0, 1), \quad p_2(c_2 - c_1) < 1 - c_1, \quad p_3 \in (0, 1 - p_2), \quad p_3(c_3 - c_1) < 1 - c_1 - p_2(c_2 - c_1), \quad \ldots$$

$$p_N \in (0, 1 - p_2 - \ldots - p_{N-1}), \quad p_N(c_N - c_1) < 1 - c_1 - p_2(c_2 - c_1) - \ldots - p_{N-1}(c_{N-1} - c_1),$$

$$p_1 = 1 - p_2 - \ldots - p_N.$$

The interest in this kind of systems stems from image processing. The basis for the applications is the observation that the Markov operator $M$ associated with an average contractive IFS on a complete separable metric space has CFP (in other words, $M$ is asymptotically stable); cf. [79] Fact 3.2. Even more, under suitable assumptions, $M$ is contractive with respect to the Monge-Kantorovitch metric $d_{MK}$; cf. [56] Theorem 2.60. Having this in mind and recalling that $M$ arising from the Kakutani contractive IFS is weakly contractive with respect to $d_{MK}$, one can look for a hybrid generalization of an average contractive and a weakly contractive IFS. We realize this idea below.
Definition 6.7. We say that a probabilistic IFS $(\mathcal{F}, \vec{p})$ is average Rakotch contractive, if there exist positive numbers $c_1, \ldots, c_N$ such that

(i) $\sum_{i=1}^{N} p_i c_i \leq 1$;
(ii) each $w_i$ is $\varphi_i$-contraction for some comparison function of the form $\varphi_i(t) = \lambda_i(t)t$, where $\lambda_i : \mathbb{R}_+ \to \mathbb{R}_+$ is nonincreasing and $\lambda_i(t) < c_i$ for $t > 0$.

Let us exhibit a probabilistic IFS $(\mathcal{F}, \vec{p})$ which is neither average contractive nor Rakotch contractive, yet it is average Rakotch contractive.

Example 6.8. Let $w_1, w_2 : [0, \frac{\pi}{2}] \to [0, \frac{\pi}{2}]$ be defined by

$$w_1(x) := 2 \sin(x), \quad w_2(x) := \frac{1}{2} \sin(x).$$

For every $0 \leq x < y \leq \frac{\pi}{2}$, we have

$$\sin(y) - \sin(x) \leq \sin(y - x) - \sin(0) = \sin(y - x) = \frac{\sin(y - x)}{y - x} (y - x).$$

From this we see that $w_1$ is a $\varphi_1$-contraction for $\varphi_1(t) = \lambda_1(t)t$, where $\lambda_1(t) = \frac{2\sin(t)}{t}$, and $w_2$ is a $\varphi_2$-contraction for $\varphi_2(t) = \lambda_2(t)t$, where $\lambda_2(t) = \frac{\sin(t)}{2t}$. In particular, $\text{Lip}(w_1) = 2$ and $\text{Lip}(w_2) = \frac{1}{2}$.

Hence the IFS $\mathcal{F} = \{w_1, w_2\}$ is not weakly contractive and the probabilistic IFS $(\mathcal{F}, \vec{p})$ is not average contractive for $\vec{p} = \left(\frac{1}{3}, \frac{2}{3}\right)$. Nevertheless, $(\mathcal{F}, \vec{p})$ is average Rakotch contractive.

Careful examination of the IFS $\mathcal{F}$ from Example 6.8 reveals that although $(\mathcal{F}, \vec{p})$ is not average contractive for $\vec{p} = \left(\frac{1}{3}, \frac{2}{3}\right)$, it is average contractive for many other vectors of weights (precisely for those $\vec{p} = (p_1, p_2)$ which satisfy $2 \cdot p_1 + \frac{1}{3} \cdot p_2 < 1$.) This is not a coincidence.

Remark 6.9. Let $\mathcal{F} = \{w_1, \ldots, w_N\}$ be an IFS consisting of Lipschitz maps. The following conditions are equivalent:

(i) $(\mathcal{F}, \vec{p})$ is average Rakotch contractive for some $\vec{p}$;
(ii) either $\mathcal{F}$ is Rakotch contractive, or $(\mathcal{F}, \vec{p})$ is average contractive for some $\vec{p}$.

Indeed, Definition 6.7 (i) implies that either $c_i < 1$ for some $i$ or $c_i = 1$ for all $i$. The first alternative is related to average contractivity due to Remark 6.6. The second alternative is related to Rakotch contractivity.

The following result is an extension of Theorem 2.21 for average Rakotch contractive IFSs.

Theorem 6.10. Let $(\mathcal{F}, \vec{p})$ be an average Rakotch contractive IFS on a complete metric space $X$. Then the Markov operator $M : P_1(X) \to P_1(X)$ induced by $(\mathcal{F}, \vec{p})$ is a Rakotch contraction with respect to the Monge-Kantorovich metric on the space $P_1(X)$ of Radon probability measures with integrable distance. In particular, $M$ admits a unique invariant measure $\mu_* \in P_1(X)$ which is the CFP of $M$, i.e., the iterations $M^n(\mu) \to \mu_*$ converge weakly.

Proof. The proof is similar to the proof of Theorem 2.21. Hence we will only outline the additional steps required to complete the proof.

First we need to find strictly increasing, concave functions $\psi_1, \ldots, \psi_N : \mathbb{R}_+ \to \mathbb{R}_+$ such that each $w_i$ is a $\psi_i$-contraction and for every $t > 0$, $\sum_{i=1}^{N} p_i \psi_i(t) < t$. This can be done with the aid of Lemma 2.11. Indeed, we apply this lemma to $\tilde{\psi}_i(t) = \frac{\varphi(t)}{c_i}$, obtaining maps $\tilde{\psi}_i(t)$ which satisfy $\frac{\varphi(t)}{c_i} \leq \tilde{\psi}_i(t) < t$ for $t > 0$. Finally, we set $\psi_i(t) := c_i \tilde{\psi}_i(t)$.

Then we proceed as in the proof of Theorem 2.21 and show that that $M$ is a $\varphi$-contraction for $\varphi(t) := \sum_{i=1}^{N} p_i \psi_i(t)$. As $\varphi$ is strictly increasing, concave and $\varphi(t) < t$, we infer that $M$ is a Rakotch contraction. The result follows. \hfill $\Box$

Average contractive IFSs are amenable for techniques of ergodic theory of Markov processes, yet allow for non-contractive maps to be employed. This makes them a good proposal for the marriage
of theory and applications. We will see that there are some clouds on the horizon in this picturesque landscape.

It is a common belief that average contractivity explains behaviour of several non-contractive IFSs experimented with in computer graphics, e.g., [64], [56]. However, it should be stressed that average contractive IFSs may lack attractors and that running the random iteration may produce artifacts which can be confirmed theoretically, so they do not occur simply because of poor numerics. The example below offers some insight into this phenomenon.

**Example 6.11** (Lasota–Myjak semiattractor). Let $\mathcal{F} = \{w_1, w_2\}$ be an IFS on the real line $\mathbb{R}$, $w_1(x) = \frac{x}{2}$, $w_2(x) = 2x$. The IFS $\mathcal{F}$

(i) does not have an attractor;
(ii) is average contractive;
(iii) induces for the vector of weights $\vec{p} = (p_1, p_2)$, $0 < p_2 < \frac{1}{3}$, $p_1 = 1 - p_2$, a Markov operator $M = M_{(\mathcal{F}, \vec{p})} : \mathcal{P}(\mathbb{R}) \rightarrow \mathcal{P}(\mathbb{R})$ which has CFP and the attracting invariant probability measure is $\mu_* = \delta_0$;
(iv) has a unique compact $\mathcal{F}$-invariant set $A_* = \operatorname{supp} \mu_* = \{0\}$, but $A_*$ cannot be recovered by running the chaos game algorithm.

To see (i) one simply notes that
\begin{equation}
\mathcal{F}^n(\{x\}) \ni w_2^n(x) = 2^n \cdot x \rightarrow \pm \infty \text{ for } x \neq 0.
\end{equation}

Item (ii) is obvious by Remark 6.6 (Lip($\vec{p}$)).

Item (iii). Denote by $s \cdot B = \{s \cdot b : b \in B\}$ the scaling of $B \subseteq \mathbb{R}$ by a factor $s > 0$. Then the Markov operator $M$ takes the following form
\[ M\mu(B) = p_1 \cdot \mu(2 \cdot B) + p_2 \cdot \mu(\frac{1}{2} \cdot B) \text{ for } B \in \mathcal{B}(\mathbb{R}), \mu \in \mathcal{P}(\mathbb{R}). \]

Since $0 \in B$ iff $0 \in s \cdot B$, we have $M(\delta_0) = \delta_0$. (Actually it is true for all weights $p_i$.) The weak convergence $M^n(\mu) \rightarrow \delta_0$ for any initial probability measure $\mu \in \mathcal{P}(\mathbb{R})$ is ensured by Fact 3.2 in [79]. Indeed, $p_1 \cdot \frac{1}{2} + p_2 \cdot 2 < 1$.

Finally we address (iv). That $A_*$ is the unique compact invariant set is obvious from (15).

Let $x_0 \neq 0$. Define $Z = \{2^q \cdot x_0 : q \in \mathbb{Z}) \}; Z \cap \mathbb{Z} - \text{integer numbers.}$. By regulating the number of repetitions of a given symbol, it is not hard to construct a disjunctive sequence $\sigma \in \{1, 2\}^\infty$ so that the orbit $x_n$ starting at $x_0$ and driven by $\sigma$ has a “diffused” omega-limit
\begin{equation}
\omega((x_n)) = Z \cup \{0\} \neq \{0\}.
\end{equation}

Therefore the deterministic chaos game algorithm fails to recover $A_*$. Further, from the elementary theory of random walk (on the lattice $Z$; e.g. [58, chap. 3.10]), it follows that the equation (16) holds almost surely for any sequence driving an orbit $x_n$, which is generated by a Bernoulli scheme (see also Theorem 5.3). Therefore random orbits starting nearby $A_*$ do not recover it.

**Remark 6.12.** The unique invariant set in the example above is called a semiattractor in the Lasota–Myjak sense; e.g. [79].

### 7. Beyond contractivity

We overview some possible extensions of the theory of iterated function systems to systems comprising maps which are far from contractive. Our choice is very subjective and should be treated as an element of a larger landscape. For instance, we omit the Lasota-Myjak theory of semiattractors (e.g., [79]), the Conley theory for IFSs (e.g., [13]), limit sets of Kleinian groups (e.g., [77]), limit sets of parabolic IFSs (e.g., [70]), quantum IFSs (e.g., [47]) and non-conformal IFSs (e.g., [10]).

In the present section we will show that the existence of invariant sets for IFSs on compact spaces or, more generally, for condensing IFSs, can be inferred rather easily from general principles of set theory and nonlinear functional analysis. Very similar approach makes possible to mimic the theory
of global maximal attractors for semiflows on Banach spaces in the case of iterated function systems. Finally, general principles of functional analysis are also suitable to prove the existence of invariant measures for Markov operators induced by IFSs.

In the course of our further presentation we will use the tool from nonlinear functional analysis, called the measure of noncompactness, e.g., [1]. We will also employ the language of set-valued analysis, e.g. [16]. We find it more natural than then traditional language of relations employed in topological dynamics, e.g., [2].

Before we move further, let us address shortly the limitations to the naive approach to extend the Hutchinson-Barnsley theory of IFSs by trying to apply the metric fixed point theory to the Hutchinson operator. The extension of the Banach fixed theorem to nonexpansive maps on Banach spaces employs the geometry of balls. It turns out that given a metric space, its natural hyperspaces of subsets, metrized with the Hausdorff (\(\ell^\infty\)-type), Pompeiu (\(p\)-type) or the Borsuk metric of continuity, are very "pokey" (as expressed by Ch. Bandt). For instance, two closed balls \(B_1, B_2\) in a hyperspace have wide intersection, like the balls in a Banach space with the \(\ell^\infty\)-norm; that is diam(\(B_1 \cap B_2\)) = diam(\(B_1\)) = 2r, when \(B_1\) and \(B_2\) have equal radii \(r\) and have their centers at the distance \(r\). Formally, the so-called Lifshitz constant is equal 1 for any practically interesting hyperspace; cf. [59].

Another imaginable trick to harness the Hutchinson operator could be an isometric embedding of the hyperspace into a Banach space and then to apply the metric fixed point theory on Banach spaces. It turns out that classic embedding theorems for hyperspaces are of no use. The Radström-Hörmander embedding ([16] Theorem 3.2.9) works for the hyperspace of convex sets which is against the philosophy of fractal geometry where jugged sets are privalent. A less restrictive Kuratowski-Wojdyslawski embedding ([51] Problem 4.5.23(f)) allows for jugged sets. It works for the hyperspace of closed bounded subsets of any metric space. Unfortunately, the obstacle to apply the embedding theorems is of purely geometric nature. The hyperspace is always embedded into a metric space with the geometry of \(\ell^\infty\)-space.

Overall, there is no obvious way to apply to the Hutchinson operator, fixed point theorems for nonexpansive maps like the Browder-Goehde-Kirk theorem, e.g., [38] chap.I §4.6 (C.1) p.76 and (C.5) p.77. In a sense, however, some sort of useful embedding of a hyperspace into a linear space is possible. Namely, viewing sets as supports of measures turns out to be a powerful technique of turning the nonlinear problem of finding fixed points of the Hutchinson operator into a linear problem of finding invariant measures of the Markov operator; see section 7.6.

7.1. Multivalued IFSs. Let \(X\) be a Hausdorff topological space. A mapping \(W: X \to 2^X\) is called a multfunction, or a multivalued map, or a set-valued map. The set \(W(x)\) is called a value of \(W\) at \(x \in X\). The set
\[
W(S) := \bigcup_{x \in S} W(x)
\]
is called an image of \(S \subseteq X\) via \(W\). Thus, nomen omen, \(W(x) = W(\{x\})\).

A multfunction \(W: X \to 2^X\) with nonempty values defines a multivalued IFS. The Hutchinson operator \(\mathcal{F}: 2^X \to 2^X\) induced by \(W\) is given by the formula
\[
\mathcal{F}(S) := \bigcup_{x \in S} W(x) = \overline{W(S)} \text{ for } S \subseteq X;
\]

cf. [51] Definition 3.1.1 p.64.

Let \(\mathcal{F} = \{w_1, ..., w_N\}\) be an IFS comprising functions \(w_i: X \to X, i = 1, ..., N\). Define the multfunction \(W: X \to 2^X\) by
\[
W(x) = \{w_i(x) : i = 1, ..., N\} \text{ for } x \in X.
\]

Then the Hutchinson operator induced by the IFS \(\mathcal{F}\) coincides with the Hutchinson operator induced by \(W\). We can replace IFSs with multifunctions and speak within the realm of multifunctions about all concepts already defined for IFSs in terms of the Hutchinson operator, like for example an \(\mathcal{F}\)-invariant set, e.g., [14], [51], [56]. However, restricting the study of IFSs merely to study of the
dynamics of the Hutchinson operator has its drawbacks. For instance, symbolic dynamics for fractals relies on the existence of the coding map, which is enabled by the representation of multifunction via single-valued contractive maps as given in equation (17).

It should be underlined that no continuity assumption about a multifunction $W$ is made unless stated explicitly. In particular, Birkhoff type theorems on invariant sets in sections 7.2 and 7.3 hold for general IFSs, regardless of whether the maps comprising an IFS are continuous or discontinuous. That is possible because the definition of the Hutchinson operator involves the closure operator and the Hutchinson operator has good order-theoretic properties.

Lemma 7.1. The Hutchinson operator $F : 2^X \rightarrow 2^X$ induced by a multifunction $W : X \rightarrow 2^X$ is

(a) order-monotone with respect to the inclusion, i.e.,
if $S \subseteq S'$ then $F(S) \subseteq F(S')$, for all $S, S' \subseteq X$;

(b) set-additive

$$F(S \cup S') = F(S) \cup F(S')$$

for $S, S' \subseteq X$.

Since some crucial theorems about IFSs involve continuity in their assumptions (e.g., the general statement of the chaos game algorithm, or invariance of the attractor), we note the following.

Proposition 7.2 (Continuity of $F$; [11]). Let $X$ be a normal topological space. Let $F = \{w_1, ..., w_N\}$ be an IFS comprising continuous functions $w_i : X \rightarrow X$, $i = 1, ..., N$. The associated Hutchinson operator $F : K(X) \rightarrow K(X)$ acting on the hyperspace $K(X)$ of nonempty compact subsets of $X$ is continuous with respect to the Vietoris topology. In particular, if $X$ is a metric space, then $F : K(X) \rightarrow K(X)$ is continuous with respect to the Hausdorff distance.

Remark 7.3. The Hutchinson operator $F$ is Vietoris continuous on the hyperspace of closed sets. When $X$ is a metric space and $F$ is considered on the hyperspace of nonempty closed bounded subsets of $X$, $F$ typically fails to be continuous with respect to the Hausdorff distance; cf. [11].

7.2. The Birkhoff theorem on invariant set for compact IFSs. The material in the present section is based on [51, 51, 60]; see also [24] Theorem 2.41 p.92.

Let $X$ be a Hausdorff topological space. We say that a multifunction $W : X \rightarrow 2^X$ is compact, provided the closure of the image $\overline{W(X)} \subseteq X$ is a compact set.

Theorem 7.4 (Birkhoff theorem for compact IFSs). Let $W : X \rightarrow 2^X$ be a compact multifunction with nonempty values, which acts on a Hausdorff topological space $X$. Then $W$ admits:

(a) the greatest invariant set, and
(b) a minimal invariant set,

which are compact.

Proof. Let $F : K(X) \rightarrow K(X)$ be the Hutchinson operator induced by $W$. Since $F(X)$ is compact, without loss of generality we may assume that $X$ is compact. (Any set $C \subseteq X$ with $F(C) = C$ is a subset of $F(X)$.)

Proof via order-theoretic fixed point principles. It is enough to recall the following: (i) any chain in $(K(X), \subseteq)$ admits an infimum, (ii) the Hutchinson operator $F$ is order-monotone, (iii) $F(X) \subseteq X$, and then apply the Kleene principle to get (a) and the Knaster-Tarski principle to get (b).

Proof via order-theoretic consequences of the axiom of choice. Let

$$S = \{S \subseteq X : F(S) \subseteq S = \overline{S} \neq \emptyset\}.$$ 

The poset $(S, \subseteq)$ has the greatest element $X \in S$. Moreover, any chain $C$ in $(S, \subseteq)$ admits an infimum. Namely, $M = \bigcap C \in S$ is the greatest lower bound for $C$. (Indeed, for all $C \in C$: $M \subseteq C$ and $F(M) \subseteq F(C) \subseteq C$. Hence $F(M) \subseteq M$.) Therefore, we can define a transfinite sequence

$$S_0 = X,$$

$$S_{\alpha+1} = F(S_{\alpha})$$

for an isolated ordinal number $\alpha$,

$$S_\beta = \bigcap_{\alpha < \beta} S_\alpha$$

for a limit ordinal number $\beta$, ...
where $\alpha$ runs through all ordinals less than $\chi$, the first ordinal of cardinality $(\text{card } X)^+$ (the successor of the cardinal number of $X$). Obviously $S_{\alpha_2} \subseteq S_{\alpha_1}$ for $\alpha_1 < \alpha_2$. It is impossible that $S_{\alpha+1} \neq S_{\alpha}$ for all $\alpha < \chi$. Thus $F(S_{\alpha^*}) = S_{\alpha^*+1} = S_{\alpha^*}$ for some $\alpha^* < \chi$. By the construction, $S_{\alpha^*}$ is nonempty and compact, and it is the greatest $F$-invariant set. We have established (a).

For (b) it is enough to note that any chain in $S$ admits a lower bound (namely its intersection), so the Zorn lemma gives the existence of a minimal element in $S$, denote it $S_*$. The definition of $S$ says that $F(S_*) \subseteq S_*$. Since $F(F(S_*)) \subseteq F(S_*)$, due to order monotonicity of $F$, and $S_* \neq \emptyset$, we also have that $F(S_*) \in S$. Finally, minimality of $S_*$, enforces that $F(S_*) = S_*$. That is, $S_*$ is nonempty and compact and it is the minimal $F$-invariant set. We have established (b).

**Remark 7.5.** One could try to employ the Kantorovitch fixed point theorem instead of the Knaster–Tarski theorem, but it is a more demanding approach to the question of existence of invariant sets. In such a case one has to ensure order-continuity in addition to order-monotonicity of the Hutchinson operator, cf. [24] chap.3.6 (and the references therein).

**Example 7.6.** Let $X = [-1, 1] \subseteq \mathbb{R}$ and $W: X \to 2^X, W(x) = \{-x\}$ for $x \in X$. Every set of the form $\{a, -a\}, a \in X$, is a minimal invariant set. The whole space $X$ is the greatest invariant set.

**7.3. The Birkhoff theorem on invariant set for condensing IFSs.** The material in the present section is based on [81], [4], [60].

**Definition 7.7.** An extended-valued nonnegative functional $\gamma: 2^X \to [0, \infty]$, defined on subsets of a Hausdorff topological space $X$, is called a measure of noncompactness (shortly MNC), if it satisfies the following axioms:

- $\gamma(\emptyset) = 0$;
- $\gamma(S) = \gamma(\overline{S})$;
- regularity: if $\gamma(S) = 0$, then $\overline{S}$ is a compact set;
- monotonicity: if $S \subseteq S'$, then $\gamma(S) \leq \gamma(S')$;
- nonsingularity: $\gamma(S \cup \{x\}) = \gamma(S)$;

where $S, S' \subseteq X, x \in X$.

Monotonicity (γ-3) explains why an abstract functional $\gamma$ deserves the name of a measure (in the spirit of Choquet’s capacities), while regularity (γ-2) explains why $\gamma$ measures noncompactness.

**Remark 7.8.** The set of axioms (γ-0)—(γ-4) is strong enough to yield the Kuratowski intersection property for $\gamma$ ([60] Theorem 3.4): if $S_n, n \geq 1$, is a decreasing sequence of nonempty closed subsets of a metric space $X$ such that $\gamma(S_n) \to 0$ as $n \to \infty$, then the intersection $S_\infty = \bigcap_{n=1}^{\infty} S_n$ is a nonempty compact set, and $\lim_{n \to \infty} d_H(S_n, S_\infty) = 0$.

**Example 7.9.** (Trivial MNC). Let $X$ be a Hausdorff topological space. Fix any $e \in (0, \infty]$. For $S \subseteq X$ put $\gamma(S) = 0$ if $\overline{S}$ is a compact set, and $\gamma(S) = e$ otherwise. Then $\gamma: 2^X \to [0, \infty]$ is an MNC.

**Example 7.10.** (Hausdorff MNC). Let $(X, d)$ be a complete metric space. The functional

$$
\gamma(S) := \inf \left\{ r > 0 : S \subseteq \bigcup_{j=1}^{k} B(x_j, r) \text{ for some } x_j \in X, k \in \mathbb{N} \right\} = \inf_{K \in K(X)} d_H(K, S),
$$

is called the Hausdorff MNC. If $X$ is not complete, then $\gamma$ is not an MNC in the sense of Definition 7.7. Indeed, if $x_n$ is a Cauchy sequence which is not convergent, then for $S = \{x_n\}_{n=1}^{\infty}$ we have that $\gamma(S) = 0$ and $\overline{S}$ is not compact. Note that $\gamma(S) < \infty$ if and only if $S \subseteq X$ is bounded.

**Remark 7.11.** The most important application of MNCs is the common generalization of two fixed point principles on Banach spaces: the Banach and Schauder theorem, due to Darbo and Sadovskii; cf. [1] Theorem 1.5.11 or [38] chap.II §6.9.C p.133. This however involves an additional property of an MNC $\gamma$,

- Darbo formula: $\gamma(\overline{\text{conv} S}) = \gamma(S)$ for $S \subseteq X$. 

The Darbo formula may be viewed as a quantitative generalization of the Mazur theorem on compactness of the convex hull. The Hausdorff MNC in a Banach space obeys (γ-D).

**Definition 7.12.** (Condensing multifunction). Let $X$ be a Hausdorff space and $\gamma$ an MNC in it. A multifunction $W : X \to 2^X$ is said to be condensing with respect to $\gamma$, if

$$
\gamma(W(S))\begin{cases} < \gamma(S), & \text{when } 0 < \gamma(S) < \infty \\ = 0, & \text{when } \gamma(S) = 0 \end{cases}
$$

for $S \subseteq X$.

**Remark 7.13.** If $W : X \to 2^X$ is condensing with respect to an MNC $\gamma$ satisfying (γ-0)—(γ-4), then its values $W(x), x \in X$, are relatively compact. Indeed,

$$
\gamma(\{x\}) = \gamma(\emptyset \cup \{x\}) = \gamma(\emptyset) = 0,
$$

so $\gamma(W(x)) = 0$.

**Remark 7.14.** Sometimes it is assumed that the MNC $\gamma$ is additive in the sense of max-plus algebra. Formally

(γ-3') *ultra-additivity*: $\gamma(S \cup S') = \max\{\gamma(S), \gamma(S')\}$ for $S, S' \subseteq X$.

Ultra-additivity (γ-3') is a stronger property than monotonicity (γ-3). The Hausdorff MNC is an example of an MNC satisfying (γ-3'). If $W_1, W_2 : X \to 2^X$ are two multifunctions condensing with respect to $\gamma$ which is ultra-additive, then their set-theoretic union $W_1 \cup W_2 : X \to 2^X$, $(W_1 \cup W_2)(x) := W_1(x) \cup W_2(x), x \in X$, is also condensing with respect to $\gamma$.

The following is a Birkhoff theorem on minimal invariant set for IFSs.

**Theorem 7.15** (Birkhoff theorem for condensing IFSs). Let $X$ be a Hausdorff topological space and $\gamma$ an MNC in it. Let $W : X \to 2^X$ be a multifunction with nonempty values, condensing with respect to $\gamma$. Assume that there exists a nonempty closed set $B \subseteq X$ such that $W(B) \subseteq B$ and $\gamma(W(B)) < \infty$. Then $W$ admits a nonempty minimal invariant set which is compact.

**Proof.** Without loss of generality we may assume that $\gamma(B) < \infty$. For, if not, then we can replace $B$ with $\overline{W(B)}$.

Denote by $\mathcal{F} : 2^X \to 2^X$ the Hutchinson operator induced by $W$. Pick anyhow $b_0 \in B$. Consider the following family of sets

$$
S = \{S \subseteq X : \mathcal{F}(S) \subseteq S = \overline{S}, b_0 \in S \subseteq B\}.
$$

Obviously $B \in S$, so $S \neq \emptyset$.

Let $S_* = \bigcap S$. It turns out that: (i) $S_* \in S$, (ii) $S_*$ is the least element of $(S, \subseteq)$, and (iii) $S_*$ is compact.

Let us verify (i). It is evident that

$$
b_0 \in S_* = \overline{S_e} \subseteq B.
$$

Moreover, for all $S \in S$, we have

$$
\mathcal{F}(S_e) \subseteq F(S) \subseteq S,
$$

so $\mathcal{F}(S_e) \subseteq \bigcap S = S_*$. Property (ii) is obvious from the definition of the intersection.

It left to verify (iii). Put $S_0 = \mathcal{F}(S_e) \cup \{b_0\}$. Then

$$
\mathcal{F}(S_0) = \mathcal{F}(\mathcal{F}(S_e)) \cup \mathcal{F}(\{b_0\}) \subseteq \mathcal{F}(S_e) \cup \mathcal{F}(S_e) \subseteq \mathcal{F}(S_e) \cup \{b_0\} = S_0,
$$

because $b_0 \in S_e$ and $\mathcal{F}(S_e) \subseteq S_e$ (thanks to (ii)). Thus $S_0 \in S$. Since $S_*$ is the least element of $S$ (due to (ii)), we get that $S_0 = \mathcal{F}(S_e) \cup \{b_0\} \supseteq S_*$. Therefore we have

$$
\gamma(S_*) \leq \gamma(\mathcal{F}(S_e) \cup \{b_0\}) = \gamma(\mathcal{F}(S_e)) \leq \gamma(S_e) \leq \gamma(B) < \infty.
$$

From the assumption that $W$ is condensing with respect to $\gamma$ it follows that $\gamma(S_e) = 0$, so $S_e = \overline{S_e}$ is compact.
Summarizing, $F(S_* \subset S_*$ and $S_*$ is a nonempty compact subset of $X$. We are in position to restrict the action of $F$ from $2^X$ to $K(S_*)$. Application of the Birkhoff theorem for compact IFSs finishes the proof.

7.4. Condensing maps vs compact maps, IFSs with condensation and weak contractions. We explain in this section that IFSs of condensing maps embrace IFSs on compact spaces and weakly contractive IFSs, as well as a mix of both: IFSs with condensation; see also Remark 7.11. While the case of compact maps is readily embraced by condensing maps, the case of weakly contractive maps needs more elaboration. Then the case of IFSs with condensation follows smoothly.

Proposition 7.16. If $W : X \to 2^X$ is a compact multifunction, then it is condensing both with respect to the trivial MNC from Example 7.9 and with respect to the Hausdorff MNC.

It left to discuss weakly contractive multifunctions.

Definition 7.17. A multifunction $W : X \to 2^X$ is a multivalued Browder contraction, if

$$d_H(W(x_1), W(x_2)) \leq \varphi(d(x_1, x_2)) \text{ for } x_1, x_2 \in X,$$

where $d_H$ stands for the Hausdorff distance and $\varphi$ is a modulus of continuity; cf. Definition 2.2

A multivalued Banach contraction (i.e., $\varphi(t) = \lambda \cdot t, t \in \mathbb{R}_+$) is often called the Nadler contraction.

Lemma 7.18. Let $W : X \to 2^X$ be a Browder contraction with modulus of continuity $\varphi$. Then for every $r > 0$, $x \in X$, $\epsilon > 0$

$$W(D(x, r)) \subseteq B(W(x), \varphi(r) + \epsilon).$$

Proposition 7.19 (\cite{13}, \cite{14} Proposition 2). Let $X$ be a complete metric space.

(a) Let $F = \{w_1, \ldots, w_N\}$ be an IFS comprising Browder weak contractions on $X$. Then the multifunction $W : X \to 2^X$, associated with $F$ according to formula (17), is a multivalued Browder contraction.

(b) If a multifunction $W : X \to 2^X$ is a Browder contraction with compact values, then it is condensing with respect to the Hausdorff MNC.

Proof. Part (a) is just a particular case of Theorem 2.18. Indeed, the Hutchinson operator $F$ agrees with $W$ on singletons: $F(\{x\}) = W(x)$ for $x \in X$.

Part (b) follows from Lemma 7.18 by handling carefully neighbourhoods and the definition of the Hausdorff MNC $\gamma$.

If $\gamma(S) = 0$, then $F$ is compact. Since $W$ is continuous with compact values, we have that $W(S) \subseteq W(S) = W(S)$ are compact; cf. \cite{16} Theorem 6.2.9, Proposition 6.2.11. Hence $\gamma(W(S)) = 0$.

Suppose now that $0 < \gamma(S) < r$. By the definition of $\gamma$ there exists a finite set $\{x_j\}_j$ with $\bigcup_j B(x_j, r) \subseteq S$. Fix $\epsilon > 0$. Let $K = \bigcup_j W(x_j)$. Compactness of $K$ ensures that there exists a finite set $\{y_l\}_l$ with $\bigcup_j B(y_l, \epsilon) \supseteq K$. Therefore we have

$$W(S) \subseteq \bigcup_j W(B(x_j, r)) \subseteq \bigcup_j B(W(x_j), \varphi(r) + \epsilon) = B(K, \varphi(r) + \epsilon) \subseteq \bigcup_l B(y_l, \varphi(r) + 2\epsilon).$$

Hence $\gamma(W(S)) \leq \varphi(r)$ because $\epsilon > 0$ was arbitrary. Further,

$$\gamma(W(S)) \leq \lim_{r \to \gamma(S)+} \varphi(r) = \varphi(\gamma(S)) < \gamma(S).$$

The result below allows to apply the theory of condensing IFSs to weakly contractive IFSs. Namely, one has to restrict a weakly contractive system to a sufficiently large closed ball $D(x_0, r)$. Then the system is condensing with respect to the Hausdorff MNC $\gamma$ and $\gamma(D(x_0, r)) < \infty$. The same
observation can be used to localize an attractor (and may be viewed as a “distant relative” of the so-called collage theorem).

**Proposition 7.20** ([4] Proposition 3). Let \( W : X \to 2^X \) be a multivalued Browder contraction with bounded values and the modulus of continuity \( \varphi \) satisfying
\[
\lim_{r \to \infty} (r - \varphi(r)) = \infty
\]
(in particular, \( \varphi \) can be taken as in the definition of the Rakotch contraction). Then for each \( x_0 \in X \) there exists sufficiently large radius \( r_0 > 0 \), so that
\[
W(D(x_0, r)) \subseteq D(x_0, r)
\]
for all \( r \geq r_0 \).

**Proof.** Denote by \( \varphi \) the modulus of continuity of \( W \). Fix \( \varepsilon > 0 \). Thanks to the boundedness of values of \( W \), there exists \( \rho > 0 \) such that \( W(x_0) \subseteq B(x_0, \rho) \). Thanks to (18) we can find \( r_0 \) so that
\[
(r - \varphi(r)) > \rho + \varepsilon \quad \text{for all } r \geq r_0.
\]
Combining Lemma 7.18 with (19) gives:
\[
W(D(x_0, r)) \subseteq B(W(x_0), \varphi(r) + \varepsilon) \subseteq B(B(x_0, \rho), \varphi(r) + \varepsilon) \subseteq B(x_0, \eta(r) + \varepsilon + \rho) \subseteq D(x_0, r).
\]
\( \square \)

Let us recall that an **IFS with condensation** \( \mathcal{F}_K \) is a weakly contractive IFS \( \mathcal{F} = \{w_1, \ldots, w_N\} \) on a complete metric space \( X \) with a given nonempty compact subset \( K \subseteq X \); e.g., [56] chap.2.6.1, [3]. The Hutchinson operator for \( \mathcal{F}_K \) is defined by \( \mathcal{F}_K : 2^X \to 2^X, \mathcal{F}_K(S) = \bigcup_{i=1}^N w_i(S) \cup K \). The IFS \( \mathcal{F}_K \) is a multivalued IFS induced by a multifunction \( W_K : X \to 2^X, W_K(x) = W(x) \cup K \) for \( x \in X \), where \( W \) is a multifunction associated with \( \mathcal{F} \) according to (17). Then we can check that \( W_K \) is a multivalued weak contraction (as a union of single-valued weak contractions and a constant multifunction) and we land in the realm of weakly contractive multivalued IFSs. Another approach could exploit the observation that weakly contractive and compact maps are condensing with respect to the Hausdorff MNC and so are their set-theoretic unions by Remark 7.14.

### 7.5. Global maximal attractor of the IFS

In the present subsection we are going to present for IFSs an adaptation of the classic theory of global maximal attractors for semigroups (e.g. [23], [25], [57]). The adaptation of definitions is not faithful, but it is done so that to avoid some technicalities with the so-called absorbing sets and trapping regions. Throughout the subsection we assume that \( X \) is a complete metric space.

**Definition 7.21.** Let \( W : X \to 2^X \) be a multivalued IFS and \( \mathcal{F} : 2^X \to 2^X \) the Hutchinson operator induced by \( W \). We say that \( A \subseteq X \) attracts \( S \subseteq X \) under \( \mathcal{F} \), provided \( \lim_{n \to \infty} \epsilon(\mathcal{F}^n(S), A) = 0 \); putting that other way, for every \( \varepsilon > 0 \) there exists \( n_0 \in \mathbb{N} \) s.t. \( \mathcal{F}^n(S) \subseteq B(A, r) \) for all \( n \geq n_0 \). A nonempty closed set \( A^* \subseteq X \) is called a **global maximal attractor**, when \( A^* \) is a minimal nonempty closed set attracting all subsets \( S \subseteq X \).

**Remark 7.22.** The set \( A \subseteq X \) is attracting all subsets of \( X \) if and only if \( A \) attracts \( X \).

**Example 7.23.** Let \( \mathcal{F} = \{w_1, \ldots, w_N\} \) be an IFS acting on \( X \). Let \( W : X \to 2^X \) be the multifunction associated with \( \mathcal{F} \) according to (17). If \( W(X) = X \), then \( X \) is the global maximal attractor of \( W \). This is the case, when \( w_i \) are affine maps on the euclidean space \( X \), which could make the theory of global attractors not interesting for the fractal geometry. However, the following is true. If \( \mathcal{F} = \{w_1, \ldots, w_N\} \) is a contractive IFS on a complete metric space \( X \) and \( W(D(x_0, r)) \subseteq D(x_0, r) \) for some \( x_0 \in X \) and \( r > 0 \) (Proposition 7.20), then the attractor of \( \mathcal{F} \) is precisely the global maximal attractor of \( \mathcal{F} \) restricted to \( D(x_0, r) \), \( \mathcal{F}|_{D(x_0, r)} = \{w_1|_{D(x_0, r)}, \ldots, w_N|_{D(x_0, r)}\} \).

Basic properties of the global attractor are collected below.
Proposition 7.24. Let $A^*$ be a global maximal attractor of $W : X \to 2^X$. Let $F : 2^X \to 2^X$ be the Hutchinson operator induced by $W$. Then the following hold:

(a) $A^*$ is the smallest nonempty closed set attracting all $S \subseteq X$;
(b) $A^*$ has the form $A^* = \bigcap_{n=1}^{\infty} F^n(X)$;
(c) $F(A^*) \subseteq A^*$;
(d) $\lim_{n \to \infty} d_H(F^n(X), A^*) = 0$;
(e) if $\bigcap_{n=1}^{\infty} F^n(X) \supseteq S$ and $S \subseteq X$ is a nonempty closed set, in particular, if $S$ is an $F$-invariant set, then $S \supseteq A^*$;
(f) if $A^*$ is compact and $W : X \to 2^X$ is an upper semicontinuous multifunction \cite[Definition 6.2.4 p.193]{16}, i.e., for each $x_0 \in X$ and every open $V \supseteq W(x_0)$ there exists an open $U \ni x_0$ s.t. $W(U) \subseteq V$, then $A^*$ is the greatest $F$-invariant set.

Proof. Item (a). Let $A$ be a nonempty closed set attracting $X$ under $F$. Fix $r > 0$, $0 < \varepsilon \leq r$. Then there exists $n_0$ s.t. $F^n(X) \subseteq B(A^*, \varepsilon) \cap B(A, \varepsilon)$ for all $n \geq n_0$. Now observe that

$$B(A^*, \varepsilon) \cap B(A, \varepsilon) \subseteq B(A^* \cap B(A, 2r), \varepsilon).$$

Hence, as $\varepsilon > 0$ was arbitrary, $A^* \cap B(A, 2r)$ is attracting $X$. Since $A^*$ is a minimal attracting nonempty closed set, we have that $A^* \cap B(A, 2r) = A^*$. So $A^* \subseteq B(A, 2r)$ for all $r > 0$. Overall $A^* \subseteq \bigcap_{r>0} B(A, 2r) = A$; $A^*$ is contained in every nonempty closed attracting set $A$.

Item (b). Since $\lim_{n \to \infty} e(F^n(X), A^*)$, we have $\bigcap_{n=1}^{\infty} F^n(X) \subseteq \bigcap_{\varepsilon > 0} B(A^*, \varepsilon) = \overline{A^*} = A^*$. On the other hand, each $F^n(X)$, $m \in \mathbb{N}$, is attracting $X$ (as the sequence $F^n(X)$ is decreasing with respect to $\subseteq$). Recalling that $A^*$ is the smallest nonempty closed set attracting $X$ (due to (a)), we arrive at $A^* \subseteq F^n(X)$ for each $m$.

Item (c) follows immediately from (b):

$$F(A) \subseteq F \left( \bigcap_{n=1}^{\infty} F^n(X) \right) \subseteq F^{n+1}(X) \subseteq F^n(X) \text{ for all } n,$$

so $F(A^*) \subseteq \bigcap_{n=1}^{\infty} F^n(X) = A^*$.

Item (d) can be seen by observing that $\varepsilon(A^*, F^n(X)) = 0$, because of (c).

Item (e). If $S \subseteq F(S)$, then $S \subseteq F^n(S) \subseteq F^n(X)$ for all $n$, so $S \subseteq A^*$ thanks to (b).

Item (f). Thanks to (c) and (e) it is enough to check that $F(A^*) \supseteq A^*$. Fix $\varepsilon > 0$. By the upper semicontinuity of $W$ to every $x \in A^*$ there exists $\delta_x > 0$ s.t. $W(B(x, \delta_x)) \supseteq B(W(x), \varepsilon)$. The open cover $\bigcup_{x \in A^*} B(x, \delta_x) \supseteq A^*$ of a compact set has a Lebesgue number $\delta > 0$, i.e., for each $a \in A^*$ there exists $x \in A^*$ s.t. $B(a, \delta) \subseteq B(x, \delta_x)$; cf. \cite[Theorem 2.3.1 p.54]{16}, \cite[Theorem 4.3.31]{31}. Hence

$$W(B(a^*, \delta)) \subseteq \bigcup_{x \in A^*} W(B(a, \delta)) \subseteq \bigcup_{x \in A^*} W(B(x, \delta_x)) \subseteq \bigcup_{x \in A^*} B(W(x), \varepsilon) = B(W(A^*), \varepsilon).$$

Since $A^*$ attracts $X$, we have $F^n(X) \subseteq B(A^*, \delta)$ for large $n$. Taking into account (20) yields

$$F^{n+1}(X) \subseteq W(B(A^*, \delta)) \subseteq W(B(W(A^*), \varepsilon)) \subseteq W(W(A^*), 2\varepsilon) \text{ for large } n.$$

Overall $F(A^*) = W(A^*)$ attracts $X$. Since $A^*$ is the smallest closed nonempty set attracting $X$, due to (a), we finally have $F(A^*) \supseteq A^*$. \hfill $\Box$

The main criterion for the existence of global attractors in IFSs provides

Theorem 7.25. Let $\gamma$ be an MNC in a complete metric space $X$. Let $W : X \to 2^X$ be a multifunction. Assume that $\gamma(W(X)) < \infty$ and either of the conditions holds:

(i) $W$ is a set-contraction with respect to $\gamma$, i.e., there exists $\lambda < 1$ s.t.

$$\gamma(W(S)) \leq \lambda \cdot \gamma(S) \text{ for } S \subseteq X;$$

(ii) $\gamma$ is the Hausdorff MNC and $W$ is condensing with respect to $\gamma$.

Then $\bigcap_{n=1}^{\infty} F^n(X)$ is a compact global maximal attractor of $W$, where $F$ is the Hutchinson operator associated with $W$. 

Proof. Denote \( A^* = \bigcap_{n=1}^\infty F^n(X) \). It is enough to observe that
\[
\lim_{n \to \infty} \gamma(F^n(X)) = 0.
\]
Then by the Kuratowski intersection theorem (Remark 7.8) we have
\[
e(F^n(X), A^*) \leq d_H(F^n(X), A^*) \to 0.
\]
Thus \( A^* \) is a compact global maximal attractor of \( W \).

It left to ensure (21). Under assumption (i) we have:
\[
\gamma(F^{n+1}(X)) \leq \lambda \cdot \gamma(F^n(X)) \leq \lambda^n \cdot \gamma(F(X)) \to 0.
\]
Under assumption (ii) the property (21) is covered by technical Lemma 5 in [4] (see also [1] Lemma 1.6.11).

7.6. Invariant measures. The Krylov-Bogolyubov theorem for IFSs. Let \( F = \{w_1, ..., w_N\} \) be an IFS of continuous maps acting on a Hausdorff topological space \( X \). Given a vector \( \vec{p} = (p_1, ..., p_N) \) of positive weights \( p_i > 0, \sum_{i=1}^N p_i = 1 \), we can form a probabilistic IFS \((F, \vec{p})\) (with constant probabilities). The Markov operator induced by \((F, \vec{p})\), \( M(F, \vec{p}) : \mathcal{M}_\pm(X) \to \mathcal{M}_\pm(X) \), acts on signed Radon measures on \( X \), according to the formula (1), i.e.,
\[
M(F, \vec{p}) = \sum_{i=1}^N p_i \cdot (w_i)^\#,
\]
where \((w_i)^\#\) is the push-forward of measures (see Appendix 9.4).

Proposition 7.26. Let \( M : \mathcal{M}_\pm(X) \to \mathcal{M}_\pm(X) \) be the Markov operator induced by a probabilistic IFS on a normal topological space \( X \). Then
(a) \( M \) is linear;
(b) \( M \) is continuous with respect to the weak topology;
(c) \( M \) sends probability measures to probability measures, that is \( M(\mathcal{P}(X)) \subseteq \mathcal{P}(X) \).

Proof. Items (a) and (c) are readily verified. Item (b) follows from Proposition 9.8. □

Theorem 7.27 (Krylov-Bogolyubov theorem for IFSs). Let \( X \) be a compact topological space. Let \( M \) be a Markov operator associated with a probabilistic IFS comprising continuous maps on \( X \). Then there exists an invariant probability measure \( \mu_* = M(\mu_*) \).

Proof. The Markov operator \( M : \mathcal{M}_\pm(X) \to \mathcal{M}_\pm(X) \) is linear and weakly continuous, the simplex of probability measures \( \mathcal{P}(X) \subseteq \mathcal{M}_\pm(X) \) is a nonempty convex weakly compact set, and \( M(\mathcal{P}(X)) \subseteq \mathcal{P}(X) \). Hence there are several ways to establish the theorem.

Proof via Schauder–Tikhonov principle. The set up allows for a direct application of the Schauder-Tikhonov fixed point principle on topological vector spaces; e.g., [38] chap.II §7.1.e Theorem (1.13) p.148.

Proof via Markov–Kakutani theorem. The set up allows for a direct application of the Markov-Kakutani fixed point theorem on topological vector spaces; e.g., [38] chap.I §3.3 Theorem (3.2) p.43.

Proof via Mann iteration. Fix \( \mu_0 \in \mathcal{P}(X) \). Consider the orbit \( M^n(\mu_0) \) and its averages
\[
\nu_n = \frac{1}{n} \sum_{k=0}^{n-1} M^k(\mu_0), \quad n \geq 1.
\]
The sequence \( \nu_n \in \mathcal{P}(X) \) admits a weakly convergent subnet (not necessarily a subsequence unless \( \mathcal{P}(X) \) is a Fréchet sequential space, cf. [31] Exercise 1.6.D):
\[
\nu_{n_k} \to \mu_* \in \mathcal{P}(X).
\]
Theorem 7.31

rich and well-developed.

invariant measures, which is often a preferred approach, as the theory of Markov processes is very

supports of invariant measures are invariant sets. Thus, instead of studying sets one can study

relatively compact.

Using Proposition 9.11 we have

Proof.

Thus \( M(\mu_*) = \mu_* \) (because weak limit is unique).

\[ \square \]

Remark 7.28. The Mann iteration is a general iterative scheme for finding fixed points; cf. [17] chap.4. The method of proof via Mann iteration is employed for instance in [79], Theorem 4.3.

Remark 7.29. Another way of proving Theorem 7.27 could be an application of the Hahn–Banach theorem along the lines of the proof of the Markov–Kakutani theorem in [38] (chap.I §3.3 Theorem (3.1) p.43).

Remark 7.30. Theorem 7.27 can be viewed as a measure counterpart of the Birkhoff theorem for compact IFSs (Theorem 7.34). One could ask for a measure counterpart of the Birkhoff theorem for condensing IFSs (Theorem 7.15). Theorems of this kind take some sort of compactness of orbits \( \{M^n(\mu)\}_{\infty=1}^{\infty} \) as an assumption, instead of the condensation with respect to an MNC, e.g., [79] Theorems 4.3 and 4.11, [83], [35], [89]. For comparison, note that if \( w : X \to X \) is a single-valued map condensing with respect to some MNC \( \gamma \), then every orbit \( S = \{w^n(x)\}_{n=0}^{\infty} \) with \( \gamma(S) < \infty \) is relatively compact.

The Krylov-Bogolyubov theorem can be used to establish the existence of invariant sets, because supports of invariant measures are invariant sets. Thus, instead of studying sets one can study invariant measures, which is often a preferred approach, as the theory of Markov processes is very rich and well-developed.

Theorem 7.31 ([56] Exercise 2.64, [52] Proposition 5). Let \( F = \{w_1, \ldots, w_N\} \) be an IFS of continuous maps on a Hausdorff topological space \( X \). Let \( \vec{p} = (p_1, \ldots, p_N) \) be a fixed vector of positive weights \( p_i > 0 \). Let \( M : \mathcal{P}(X) \to \mathcal{P}(X) \) be the Markov operator corresponding to the probabilistic IFS \( (F, \vec{p}) \). If \( \mu_* = M\mu_* \) is an invariant Radon probability measure, then \( A_* = \text{supp} \mu_* \) is a closed invariant set for \( F \), i.e., \( A_* = F(A_*) \).

Proof. Using Proposition 9.11 we have

\[
F(A_*) = \bigcup_{i=1}^{N} w_i(\text{supp} \mu_*) = \\
= \bigcup_{i=1}^{N} w_i(\text{supp} \mu_*) = \bigcup_{i=1}^{N} \text{supp}((w_i)_{\sharp} \mu_*) = \text{supp} \left( \sum_{i=1}^{N} p_i \cdot (w_i)_{\sharp} \mu_* \right) = \\
= \text{supp} M\mu_* = \text{supp} \mu_* = A_*.
\]

The assumption that \( p_i > 0 \) is needed to have \( \text{supp}((w_i)_{\sharp} \mu_*) = \text{supp}(p_i \cdot (w_i)_{\sharp} \mu_*) \).

\[ \square \]

7.7. Attractors of non-contractive IFSs. In Section 7.5 we have presented the theory of global maximal attractors. These are minimal limits with respect to the upper Hausdorff metric topology (described by the excess functional, cf. [16] chap.4.2 p.114) of the Hutchinson iterates. Still one can ask about attractors of non-contractive IFSs understood as CFP’s of the Hutchinson operator, see Definition 2.14. Recall that a nonempty compact set \( A_F \subseteq X \) is an attractor of the IFS \( F \) comprising continuous maps acting on a topological space \( X \), provided \( F^n(S) \to A_F \) with respect to the Vietoris topology for all nonempty compact \( S \subseteq X \). (As usual, \( F \) stands for both — the IFS and the induced Hutchinson operator.) Let us remark that there exists a more refined notion of the attractor, called strict attractor, which was proposed by Barnsley and Vince, and could be roughly described as a local
attractor. More precisely, a nonempty compact subset $A \subseteq X$ is a **strict attractor** of $F$, if there exists an open neighbourhood $U \supseteq A$ with $F(U) \subseteq U$ s.t. $A = A_{F|U}$ is an attractor of the IFS $F|U$, i.e., $F$ restricted to $U$. (If $F = \{w_1, \ldots, w_N\}$, $w_i : X \to X$, then $F|U = \{w_1|U, \ldots, w_N|U\}$, $w_i|U(x) = w_i(x)$ for $x \in U$.)

The theory of (strict) attractors of non-contractive IFSs is rather general and not many results can be lifted from the contractive case. For instance the following theorem has no counterpart in the realm of non-contractive IFSs.

**Theorem 7.32** (Hata’s connectedness principle; [13] Theorem 1). If $F$ is a TIFS and its attractor $A_F$ is connected, then $A_F$ is necessarily locally connected and arcwise connected.

Despite these kind of phenomena met outside the contractive realm, some fundamental results on attractors of contractive IFSs are also valid for attractors of non-contractive IFSs. The most notable case seems to be the chaos game algorithm on the representation of the attractor by orbits. Probabilistic version of the chaos game holds for any IFS on a topological space; cf. [12]. Derandomization of the chaos game is more demanding; see the comments at the end of Section 5.

Since the theory of (strict) attractors of non-contractive IFSs is still under initial development, we present only a couple of characteristic examples.

First example explains why Hata’s principle does not work outside contractive realm.

**Example 7.33** (Connected non-arcwise connected attractor; [13] Example 2). Let $X = \{0\} \times [-1,1] \cup \{(t, \sin\left(\frac{1}{t}\right)) : t \in (0,1]\} \subset \mathbb{R}^2$ be the Warsaw sine curve. It is a connected but not arcwise connected set. It turns out that there exists a continuous map $w_1 : X \to X$ and a point $x_0 \in X$ s.t. the orbit $\{w_1^n(x_0)\}_{n=0}^\infty$ is dense in $X$, e.g., [85] Example 12. Putting $w_2 : X \to X$ to be the constant map $w_2(x) = x_0$ for all $x \in X$, yields an IFS $\{w_1, w_2\}$ for which the Warsaw sine is an attractor; cf. [12] Example 4.

Second example shows that CFP of the Hutchinson operator corresponding to a collection of maps does not enforce the CFP for individual maps.

**Example 7.34** (Non-contractive IFS with attractor). Let $X = [0,1]/\{0,1\}$ be the circle (unit interval with glued ends). Let $w_1 : X \to X$ be the irrational rotation, i.e., $w_1(x) = x + r \hspace{1pt} \text{mod} \hspace{1pt} 1$ for $x \in X$ and a fixed irrational $r$. Let $w_2 : X \to X$ be the identity map, i.e., $w_2(x) = x$ for $x \in X$. Then the IFS of isometries $\{w_1, w_2\}$ induces the Hutchinson operator with the CFP being the whole circle.

Third example shows that unlike in the case of attractors of TIFS, attractors in general IFSs need not be metrizable.

**Example 7.35** (Non-metrizable attractor; [12] Example 6). Let $X = \bigcup_{j \in \{0,1\}} ([0,1] \setminus \{j\}) \times \{j\} \subset \mathbb{R}^2$ be the Alexandrov two arrows space, e.g., [18] Example 6.1.20 or [31] Exercise 3.10.C p.212. The space $X$ is a compact non-metrizable Hausdorff topological space. Define continuous maps $w_i : X \to X$, $i = 1, 2, 3$, according to the formulas

$$w_1(t, j) := \left(\frac{t}{2}, j\right),$$

$$w_2(t, j) := \left(\frac{t + 1}{2}, j\right),$$

$$w_3(t, j) := (1 - t, 1 - j),$$

for $j = 0, 1$, $t \in [0,1]$, $t \neq j$. Then the IFS $\{w_1, w_2, w_3\}$ admits the double arrow space as an attractor.

It is known that the Hilbert cube $[0,1]^\mathbb{N}$ is not homeomorphic to a topologically contractive IFS. On the other hand, it is an open problem whether Hilbert cubes $[0,1]^\mathbb{N}$ and $[0,1]^\mathbb{R}$ (infinite product of either countable or continuum number of copies of the unit interval $[0,1]$) are attractors of some IFS. We know only that a Hilbert cube of weight higher than continuum is not separable, so it cannot be an attractor of an IFS ([12] Proposition 5).
Infinite iterated function systems have been not discussed in our survey, except Remark 2.19. We would like to make some further remarks in connection with multivalued IFSs. We do it in such a manner that, hopefully, a more unified view on various matters will be achieved by the reader. The basic observation is that some aspects of the dynamics of infinite IFSs can be captured by turning an infinite IFS into a multivalued IFS.

If \( \{w_i : i \in I\} \) is an infinite system of maps (i.e., \( I \) is infinite) acting on a topological space \( X \), then it induces a multifunction \( W : X \rightarrow 2^X \), \( W(x) := \{w_i(x) : i \in I\} \), and we end up in the framework of multivalued IFSs (see Section 7.1). But there is a price to pay for this reduction. Often it is too bold to yield sufficient insight. For instance, to study the structure of the invariant set and, in particular its dimension (e.g., [12], [49], [70], [65]) or the chaos game algorithm ([13]), one needs to access individual maps comprising an infinite IFS, rather than look at the rough description of a collective dynamical behaviour of the IFS encoded by a single multifunction. Although selection theorems allow to decompose a multifunction into individual single-valued maps, these decompositions (whenever exist) suffer several drawbacks. For instance, Lipschitz constants of selectors depend not only on the Lipschitz constant of a multifunction under decomposition, but also upon the dimension of the ambient space, cf. [7] chap.1.9 (see also [63] Proposition 9 where the role of equicontinuity is addressed). Furthermore, it should be stressed out that the so-called inverse problem of fractal geometry has trivial solution for infinite and multivalued IFSs (cf. [56] chap. 2.6.4.1).

Let us suppose that we accept all the aforementioned drawbacks and we reduce infinite IFSs to multivalued IFSs. Another question arises: when an infinite IFS of weakly contractive or condensing maps gives rise to a weakly contractive or, respectively, condensing multivalued IFS? For the weakly contractive case we discussed this in Remark 2.19. For the condensing case the following formula for a measure of noncompactness of an infinite union of sets addresses the raised question (at least partially):

\[
\sup_{t \in T} \gamma(S_t) + \gamma^2(\{S_t\}_{t \in T}) \leq \gamma \left( \bigcup_{t \in T} S_t \right) \leq \sup_{t \in T} \gamma(S_t) + 2 \cdot \gamma^2(\{S_t\}_{t \in T}),
\]

where \( X \) is a metric space, \( \gamma \) is the Hausdorff MNC in \( X \), \( \{S_t\}_{t \in T} \) is an arbitrary family of subsets \( S_t \subseteq X \), and \( \gamma^2 \) is the Hausdorff MNC with respect to the Hausdorff distance \( d_H \) in the power set \( 2^X \). (Be aware that \( d_H \) is only an extended-valued semimetric in \( 2^X \).) Thus, in a suitable function space, a relatively compact collection of maps which are condensing with respect to the Hausdorff MNC gives rise to a condensing set-theoretic union of maps; cf. [58].

Another problem is related to weakly contractive multifunctions with non-compact values. Such multifunctions may come from bounded infinite IFSs (see Remark 2.19 (3)). Since condensing multifunctions have relatively compact values, weakly contractive multifunctions are not reducible to the condensing case in general. To override this obstacle we are led to consider hyper-condensing multifunctions, that is, multifunctions which are condensing with respect to an MNC in the hyperspace; cf. [62].

Usually it is demanded that attractors are compact sets. Nevertheless, it is worth to consider non-compact attractors. Already infinite and multivalued IFSs comprising weakly contractive maps lead to non-compact closed bounded attractors (e.g., [3], [92]). Quite extraordinarily, another proposal, the Mauldín–Urbański limit set, can be non-closed invariant set with a complicated descriptive topology (cf. [70] Example 5.2.2 p.140). Finally, unbounded fractal sets offer an interesting and fruitful excursion outside the realm of compacta. These include the Lasota–Myjak semiattractors like the Sierpiński chessboard ([64] Example 6.2), the Barnsley–Vince fast basins like the Kigami web ([13] Section 5 Fig.4) and fractal tilings (e.g., [14] Section 10).

New extensions of the framework of IFSs are constantly proposed, e.g., generalized IFSs (GIFS) which comprise mappings defined on a finite Cartesian product \( X^m \) (or even infinite product) with values in \( X \), cf. [72] and [48]. Numerous approaches to self-similarity and frameworks related to IFSs are explored by researchers to this day: cocycles and non-autonomous dynamical systems (e.g., [39],...
23, 2), abstract self-similarity via topology, category theory and algebra (e.g., 92, 22, 57, 36, 34, 55), infinite products of matrices and chains (e.g., 40, 19), which is just a small sample to move the imagination of the reader. Iterated function systems constitute only one — though sparkling creativity — view on the rich landscape of dynamics.

9. Appendix: Topology and measure

We collect in this section the rudimentary notation and terminology from topology and measure theory.

Let $(X, d)$ be a metric space. For $x \in X$, $S \subset X$ and $r > 0$, we define

- an open ball, $B(x, r) = \{ y \in X : d(y, x) < r \}$;
- a closed ball, $\bar{B}(x, r) = \{ y \in X : d(y, x) \leq r \}$;
- an $r$-neighbourhood of $S$, $B(S, r) = \bigcup_{x \in S} B(x, r)$;
- diameter of $S$, $\text{diam}(S) = \sup_{x,x' \in S} d(x, x')$.

The closure of a subset $S \subseteq X$ of a metric or topological space $X$ is denoted by $\overline{S}$.

A set which is both closed and open is shortly called clopen. A set is perfect if it is closed and has no isolated points.

9.1. Nets. In topological spaces the convergence of countable sequences is not enough to describe topology. For this reason, the notion of nets (a.k.a. Moore–Smith sequences) was introduced; e.g. Appendix: Preliminaries B p.593.

A directed set is a pair $(N, \succeq)$, where $N$ is a nonempty set and $\succeq$ is a binary relation satisfying:

- (i) (reflexivity) $n \succeq n$ for all $n \in N$;
- (ii) (transitivity) for every $n_1, n_2, n_3 \in N$, if $n_3 \succeq n_2 \succeq n_1$, then $n_3 \succeq n_1$;
- (iii) (direction) for every $n_1, n_2 \in N$ there exists $n_3 \in N$ s.t. $n_3 \succeq n_1, n_3 \succeq n_2$.

A net of elements from $X$ is a function $x : (N, \succeq) \to X$, denoted $(x_n)_{n \in N}$ or simply $x_n$. The net $(x_n)_{n \in N}$ of elements from a topological space $X$ is convergent to $x \in X$, written $x_n \to x$, if for every open neighbourhood $U \ni x$ there exists $n_0 \in N$ such that $x_n \in U$ for all $n \succeq n_0$.

Let $(N, \succeq)$ and $(K, \succ)$ be directed sets. Let $\bar{n} : K \to N$ satisfy:

- (i) (monotonicity) for every $k_1, k_2 \in K$, if $k_2 \succ k_1$, then $\bar{n}(k_2) \succeq \bar{n}(k_1)$;
- (ii) (cofinality) for every $n \in N$ there exists $k \in K$ such that $\bar{n}(k) \succeq n$.

Given a net $(x_n)_{n \in N}$ and $\bar{n} : K \to N$, the net $(x_{n_k})_{k \in K}$, where $n_k := \bar{n}(k)$, is called a subnet of $x_n$.

We have the following properties:

- every sequence $(x_n)_{n=1}^{\infty}$ is a net with a directed set of indices $(N, \succeq)$;
- every subsequence is a subnet, although there exist on some compact spaces countable sequences with plenty of convergent subnets yet no convergent subsequence;
- in a Hausdorff topological space a limit of the net is unique;
- if a net $x_n$ converges to $x$, then all its subnets $x_{n_k}$ converge to $x$;
- a subset $S \subseteq X$ is compact precisely when every net over $S$, $x_n \in S$, admits a convergent subnet $x_{n_k} \to x \in S$;
- a map $w : X \to Y$ is continuous precisely when from the convergence of the net $x_n \to x$ it follows that the net $w(x_n) \to w(x)$ converges.

9.2. Baire category. If $X$ is a metric (or topological) space, then we say that $M \subseteq X$ is called:

- nowhere dense, provided $\text{Int}(\overline{M}) = \emptyset$, where $\text{Int}$ stands for the interior; equivalently, under assumption that $X$ is a metric space: for every $x \in X$ and $R > 0$, there exist $y \in X$ and $r > 0$ such that

\begin{equation}
B(y, r) \subseteq B(x, R) \setminus M;
\end{equation}

- of the first Baire category or meager, if $M$ is a countable union of nowhere dense sets;
- residual, if $X \setminus M$ is of the first Baire category.
A famous Baire category theorem states that if $X$ is a complete metric space, then each meager set has empty interior. Also, there are sets which have empty interior but are not meager (for example the set of irrationals on the real line $\mathbb{R}$). Meager sets in a complete metric space are considered to be small, and, in turn, residual sets are big. In particular, it is common to say that a typical element from a complete space $X$ has some property, say (P), if the set $\{x \in X : x$ has property (P)$\}$ is residual.

Finally, let us note that, at least within metric spaces, there exist approaches to topologic smallness other than the Baire category. Namely, in the “ball characterization” of nowhere density [23] one can require that the smaller ball $B(y, r)$ is not too small with respect to a bigger one $B(x, R)$. Thus we obtain sets with sufficiently large holes, called porous. This idea can be formalized in many ways, cf. [94]. By substituting meager sets with countable unions of porous sets, called $\sigma$-porous, we get a strengthened analogue of the Baire category (which could be termed the Denjoy category).

9.3. **Hyperspaces.** Let $X$ be a metric or topological space. We distinguish the following families of sets:

- the family of all subsets of $X$, denoted $2^X$;
- the family of all nonempty compact subsets of $X$, denoted $\mathcal{K}(X)$;
- the family of all nonempty closed bounded subsets of $X$, denoted $\mathcal{CB}(X)$ (provided $X$ is a metric space).

Once a family of sets is topologized, we call it a hyperspace.

If $(X, d)$ is a metric space, then for $S, S' \subseteq X$ we define

- $e(S, S') = \sup_{x \in S} \inf_{x' \in S'} d(x, x')$;
- the Hausdorff distance: $d_H(S, S') = \max\{e(S, S'), e(S', S)\}$.

(Congruously inf $\emptyset = \infty$.) The following geometric description.

**Proposition 9.1.** Let $X$ be a metric space. For $S, S' \subseteq X$ it holds:

(a) $e(S, S') = \inf\{r > 0 : S \subseteq B(S', r)\}$;
(b) $d_H(S, S') = \inf\{r > 0 : S \subseteq B(S', r), S' \subseteq B(S, r)\}$.

If $X$ is a Hausdorff topological space, then in $\mathcal{K}(X)$ (or more generally in the family of all nonempty closed subsets of $X$) we introduce the Vietoris topology by declaring that the following sets form its open subbase:

$$V^+ = \{K \in \mathcal{K}(X) : K \subseteq V\}, V^- = \{K \in \mathcal{K}(X) : K \cap V \neq \emptyset\}$$

where $V$ runs over open subsets of $X$. The Vietoris topology in $\mathcal{K}(X)$ satisfies the Hausdorff separation; cf. [31] Problem 3.12.27(b).

**Theorem 9.2** ([16] Definition 3.2.1, Theorem 3.2.4, Exercise 3.2.9, or [31] Problem 4.5.23). Let $X$ be a metric space.

(a) The pair $(\mathcal{CB}(X), d_H)$ is a metric space, while $(\mathcal{K}(X), d_H)$ is its closed subspace.
(b) If $X$ is a complete space, then $\mathcal{CB}(X)$ and $\mathcal{K}(X)$ are complete with respect to $d_H$.
(c) If $X$ is a compact space, then $\mathcal{K}(X) = \mathcal{CB}(X)$ is compact with respect to $d_H$.
(d) The Vietoris topology in $\mathcal{K}(X)$ coincides with the Hausdorff topology (that is, induced by $d_H$). In particular, topologically equivalent metrics in $X$ yield topologically equivalent Hausdorff metrics in $\mathcal{K}(X)$.

We end this section with a lemma which seems to be a folklore (see [78]). Recall that a Cantor space is a topological space homeomorphic to the Cantor ternary set. Equivalently (e.g., [31], Exercise 6.2.A (c) and Theorems 6.2.1 and 6.2.9), it is any compact metrizable topological space without isolated points which is totally disconnected in the sense that it has no nontrivial connected subsets.

**Lemma 9.3.** If $X$ is a complete metric space without isolated points, then the set

$$\mathcal{C}(X) := \{K \in \mathcal{K}(X) : K \text{ is a Cantor space}\}$$

is residual in $\mathcal{K}(X)$. 

Proof. For every \( n \in \mathbb{N} \), let
\[
G_n := \left\{ K \in \mathcal{K}(X) : \forall x \in K \ \exists y \in K \ 0 < d(x, y) < \frac{1}{n} \right\}.
\]
It is easy to see that arbitrarily close to any \( K \in \mathcal{K}(X) \) we can find a finite set which belongs to \( G_n \) (multiplying, if needed, some points in initially chosen finite set). Therefore each \( G_n \) is dense in \( \mathcal{K}(X) \). Also, it is open. Indeed, choose any \( K \in G_n \) and for any \( 0 < \alpha < \beta < \frac{1}{n} \), let
\[
K_{\alpha, \beta} := \left\{ x \in K : \exists y \in K \ \alpha < d(x, y) < \beta \right\}.
\]
Then each set \( K_{\alpha, \beta} \) is open in \( K \) and \( K = \bigcup_{\alpha, \beta} K_{\alpha, \beta} \). By compactness of \( K \), we can find \( 0 < \alpha_0 < \beta_0 < \frac{1}{n} \) so that \( K = K_{\alpha_0, \beta_0} \). Then \( G_n \) contains an open ball around \( K \) with radius \( \frac{1}{2} \min\{\alpha_0, \frac{1}{n} - \beta_0\} \). Hence \( G_n \) is open in \( (\mathcal{K}(X), d_H) \).

Now for every \( n \in \mathbb{N} \), let
\[
D_n := \left\{ K \in \mathcal{K}(X) : \text{diam}(P) < \frac{1}{n} \text{ for any connected } P \subseteq K \right\}.
\]
Each set \( D_n \) is dense because, similarly as for \( G_n \), arbitrarily close to any \( K \in \mathcal{K}(X) \) we can find a finite set, which clearly belongs to \( D_n \). We will prove that it is open by showing that \( \mathcal{K}(X) \setminus D_n \) is closed.

Choose a sequence \( K_k \subseteq \mathcal{K}(X) \setminus D_n \) convergent to some \( K \in \mathcal{K}(X) \), as \( k \to \infty \). Then for every \( k \in \mathbb{N} \), we can find a connected set \( P_k \subseteq K_k \) such that \( \text{diam}(P_k) \geq \frac{1}{n} \). Since the set
\[
C := K \cup \bigcup_{k \in \mathbb{N}} K_k = \bigcup_{k \in \mathbb{N}} K_k
\]
is compact, the sequence \( P_k \in \mathcal{K}(C) \), admits a convergent subsequence \( P_{m_k} \to P \in \mathcal{K}(C) \) thanks to Theorem 9.2 (c). Now it is routine to check that its limit \( P \) is a subset of \( K \), \( \text{diam}(P) \geq \frac{1}{n} \) and that \( P \) is connected, i.e., \( K \not\subseteq D_n \). All in all, the set \( D_n \) is open.

Taking the above onto account, we infer that the set \( (\bigcap_{n \in \mathbb{N}} G_n) \cap (\bigcap_{n \in \mathbb{N}} D_n) \) is residual. Obviously, it equals \( C(X) \) and the result follows.

9.4. Measures. Let \( X \) be a Hausdorff topological space. The Borel \( \sigma \)-algebra, denoted \( \mathcal{B}(X) \), is the smallest \( \sigma \)-algebra containing open sets in \( X \). Its elements are called Borel sets. A function \( w : X \to Y \) between two Hausdorff topological spaces \( X, Y \) is called Borel measurable, if \( w^{-1}(B) \in \mathcal{B}(X) \) for all \( B \in \mathcal{B}(Y) \). In particular a continuous map \( w \) is such.

A functional \( \mu : \mathcal{B}(X) \to (-\infty, \infty) \), for which \( \mu(\emptyset) = 0 \), is called
- a signed Borel measure if it is countably additive, i.e., \( \mu(\bigcup_{k=1}^{\infty} B_k) = \sum_{k=1}^{\infty} \mu(B_k) \) and \( \sum_{k=1}^{\infty} |\mu(B_k)| < \infty \) for every countable family of disjoint Borel sets \( B_k \in \mathcal{B}(X) \), \( B_k \cap B_m = \emptyset \) for \( k \neq m \), \( k, m \in \mathbb{N} \);
- a Borel measure if \( \mu \) is a signed Borel measure and \( \mu(B) \geq 0 \) for all \( B \in \mathcal{B}(X) \); we write then \( \mu : \mathcal{B}(X) \to [0, \infty) \);
- a Radon measure if \( \mu : \mathcal{B}(X) \to [0, \infty) \) is a Borel measure and
\[
\mu(B) = \sup\{\mu(K) : K \subseteq B, K \in \mathcal{K}(X) \cup \{\emptyset\} \} \text{ for all } B \in \mathcal{B}(X);
\]
equivalently, for every \( \varepsilon > 0 \), \( B \in \mathcal{B}(X) \), there exists a compact subset \( K \subseteq B \) such that \( \mu(B \setminus K) < \varepsilon \);
- a signed Radon measure, if \( \mu = \mu_1 - \mu_2 \) for two Radon measures \( \mu_1, \mu_2 : \mathcal{B}(X) \to [0, \infty) \);
- a probability measure, if \( \mu \) is a Radon measure for which \( \mu(\{\{\}) = 1 \).

We consider only finite (signed) measures, so the adjective finite is usually omitted.

Note that on a complete separable metric space, every (signed) Borel measure is necessarily a (signed) Radon measure (18 Theorem 7.1.7), while there exists a Borel measure on a compact topological space which is not a Radon measure (18 Example 7.1.3).

We denote the following collections of measures:
• the collection of all signed Radon measures on \( X, \mathcal{M}_\pm(X) \);
• the collection of Radon probability measures on \( X, \mathcal{P}(X) \).

Overall we have \( \delta_x \in \mathcal{P}(X) \subseteq \mathcal{M}_\pm(X) \), where \( \delta_x \) is the Dirac measure at \( x \in X \).

**Proposition 9.4.** Signed Radon measures form a vector space \( \mathcal{M}_\pm(X) \) under the addition of measures, multiplication of a measure by a scalar, and with the null measure \( 0 \in \mathcal{M}_\pm(X) \) as a zero vector. Radon probability measures \( \mathcal{P}(X) \) form a convex subset of \( \mathcal{M}_\pm(X) \).

**Proof.** It is enough to check that if \( \mu, \nu \) are (nonnegative!) Radon measures and \( p > 0 \), then \( \mu + \nu \) and \( p \cdot \mu \) are Radon measures. The rest is obvious.

Fix \( \varepsilon > 0 \), \( B \in \mathcal{B}(X) \) and compact subsets \( K, K' \subseteq B \) s.t. \( \mu(B \setminus K) < \varepsilon, \nu(B \setminus K') < \varepsilon \). Then \( K \cup K' \subseteq B \) is compact, \( (\mu + \nu)(B \setminus (K \cup K')) \leq \mu(B \setminus K) + \nu(B \setminus K') < 2\varepsilon, p \cdot \mu(B \setminus K) < p \cdot \varepsilon \). \( \square \)

Every signed Borel measure \( \mu : \mathcal{B}(X) \to (-\infty, \infty) \) is a difference \( \mu = \mu_1 - \mu_2 \) of two nonnegative Borel measures \( \mu_1, \mu_2 : \mathcal{B}(X) \to [0, \infty) \). In particular, \( \int_X f \, d\mu = \int_X f \, d\mu_1 - \int_X f \, d\mu_2 \) for a bounded Borel measurable function \( f : X \to \mathbb{R} \). The Jordan-Hahn decomposition \( \mu = \mu^+ - \mu^- \), \( \mu^+, \mu^- : \mathcal{B}(X) \to [0, \infty) \), is the difference being minimal in the sense that for any other decomposition \( \mu = \mu_1 - \mu_2 \) into nonnegative Borel measures \( \mu_1, \mu_2 \), it holds \( \mu_1 \geq \mu^+, \mu_2 \leq \mu^- \); e.g., [18] chap.3.1. Given the Jordan-Hahn decomposition \( \mu = \mu^+ - \mu^- \) of a signed measure \( \mu \) we can define the total variation measure of \( \mu, |\mu| : \mathcal{B}(X) \to [0, \infty) \), \( |\mu| := \mu^+ + \mu^- \). It should be remarked that a signed Borel measure \( \mu \) is Radon if and only if its total variation measure \( |\mu| \) is Radon. We also have

\[
(24) \quad \left| \int_X f \, d\mu \right| \leq \sup_{x \in X} |f(x)| \cdot |\mu|
\]

for a bounded Borel function \( f : X \to \mathbb{R} \).

Let \( X \) be a Hausdorff topological space. Denote by \( \mathcal{C}_b(X) \) the space of all bounded continuous real-valued functions \( f : X \to \mathbb{R} \). We endow the vector space \( \mathcal{M}_\pm(X) \) of signed Radon measures on \( X \) with the weak topology generated by the following subbasis of open sets

\[
V_{f,\varepsilon}(\mu) := \left\{ \nu \in \mathcal{M}_\pm(X) : \left| \int_X f \, d\nu - \int_X f \, d\mu \right| < \varepsilon \right\},
\]

where \( f \in \mathcal{C}_b(X), \varepsilon > 0, \mu \in \mathcal{M}_\pm(X) \). So endowed \( \mathcal{M}_\pm(X) \) constitutes a locally convex topological vector space: the addition of measures and multiplication of a measure by a scalar are continuous in the weak topology and the subbasic neighbourhoods are convex.

In terms of functional analysis, the weak topology in \( \mathcal{M}_\pm(X) \) is the weak* topology transported from the predual space \( (\mathcal{C}_b(X), \| \cdot \|_{\infty}) \) via duality pairing

\[
\mathcal{C}_b(X) \times \mathcal{M}_\pm(X) \ni (f, \mu) \mapsto \int_X f \, d\mu.
\]

It is a standard fact that the weak* topology in the space of continuous linear functionals \( (\mathcal{C}_b(X))^* \) obeys Hausdorff separation without any assumptions on \( X \). To identify measures with functionals requires that some sort of the Riesz-Skorokhod representation theorem holds true, which is quite restrictive. (If \( X \) is not locally compact, then not all elements of \( (\mathcal{C}_b(X))^* \) arise as integrals with respect to measures and distinct measures may yield the same functional; say all functions in \( \mathcal{C}_b(X) \) are constant, so \( \int_X f \, dB_{x_1} = \int_X f \, dB_{x_2} \) for \( x_1 \neq x_2 \in X, f \in \mathcal{C}_b(X) \).) Therefore we deliver a direct proof that \( \mathcal{M}_\pm(X) \) is Hausdorff when \( X \) is normal.

**Proposition 9.5.** If \( X \) is a normal topological space, then the space \( \mathcal{M}_\pm(X) \) of signed Radon measures on \( X \) equipped with the weak topology is a Hausdorff space.

**Proof.** Let \( \mu, \eta \in \mathcal{M}_\pm(X), \mu \neq \eta \). We shall find \( \delta > 0 \) and \( f \in \mathcal{C}_b(X) \) s.t. \( V_{f,\delta}(\mu) \cap V_{f,\delta}(\eta) = \emptyset \).

Since \( \mu \) and \( \eta \) are Radon, there exists a compact set \( K \subseteq X \) s.t. \( \mu(K) \neq \eta(K) \). Set \( \varepsilon := |\mu(K) - \eta(K)| > 0 \) and let \( \delta := \frac{\varepsilon}{4} \). By the definition of a Radon measure, we can find compact sets \( S', S'' \subseteq X \setminus K \) such that \( |\mu|(X \setminus K \setminus S') < \delta \), and \( |\eta|(X \setminus K \setminus S'') < \delta \).
Define $f(x) := 0$ for $x \in S' \cup S''$, and $f(x) := 1$ for $x \in K$. Then $f : S' \cup S'' \cup K \to [0,1]$ is continuous. By the Tietze-Urysohn theorem (31 Theorem 2.1.8), $f$ admits a continuous extension $f : X \to [0,1]$. Let us observe that using standard properties of the integral, in particular inequality (24), we have

$$\varepsilon = |\mu(K) - \eta(K)| = \left| \int_K f \, d\mu - \int_K f \, d\eta \right| =$$

$$\leq \left| \int_X f \, d\mu - \int_X f \, d\eta \right| + \left| \int_{(X \setminus K)} f \, d\mu + \int_{(X \setminus K)} f \, d\eta \right| <$$

$$\leq \left| \int_X f \, d\mu - \int_X f \, d\eta \right| + |\mu|((X \setminus K) \setminus S') + |\eta|((X \setminus K) \setminus S'') <$$

$$< \left| \int_X f \, d\mu - \int_X f \, d\eta \right| + 2\delta.$$ 

Hence

$$\left| \int_X f \, d\mu - \int_X f \, d\eta \right| \geq \varepsilon - 2\delta.$$

Suppose that there is $\nu \in V_{f,\delta}(\mu) \cap V_{f,\delta}(\eta)$. Then

$$\varepsilon - 2\delta \leq \left| \int_X f \, d\mu - \int_X f \, d\eta \right| \leq \left| \int_X f \, d\mu - \int_X f \, d\nu \right| + \left| \int_X f \, d\nu - \int_X f \, d\eta \right| < 2\delta,$$

which yields a contradiction with $\varepsilon = 4\delta$. $\square$

If a net (or a sequence) $\mu_n \in \mathcal{M}_\pm(X)$ converges to $\mu \in \mathcal{M}_\pm(X)$ in the weak topology, that is $\int_X f \, d\mu_n \to \int_X f \, d\mu$ for all $f \in C_b(X)$, then we say that $\mu_n$ weakly converges to $\mu$. The weak limit is unique when $X$ is normal. The weak convergence is usually not interesting when $X$ is not normal. For instance, if $C_b(X)$ merely consists of constant functions (e.g., 31 Problem 2.7.17 p.119), then any given sequence of probability measures weakly converges to all probability measures at once.

**Lemma 9.6.** Let $\Delta \subseteq \mathcal{M}_\pm(X)$ be a set of signed Radon measures which is compact with respect to the weak topology. Then

(a) $\Delta' = \{\int_X f \, d\mu : \mu \in \Delta\} \subseteq \mathbb{R}$ is compact (hence bounded) for each $f \in C_b(X)$;

(b) every sequence $\mu_n \in \Delta$ has the property that $\frac{1}{n} \cdot \mu_n \to 0$ in the weak topology.

**Proof.** For (a) observe that for each fixed $f \in C_b(X)$ the functional

$$\mathcal{M}_\pm(X) \ni \mu \mapsto \int_X f \, d\mu \in \mathbb{R}$$

is continuous in the weak topology. Hence $T(\Delta) = \Delta'$ is compact.

Item (b) follows from (a), because for every $f \in C_b(X)$ one has

$$\left| \int_X f \, d\left(\frac{1}{n} \cdot \mu_n\right) \right| = \frac{1}{n} \cdot \left| \int_X f \, d\mu_n \right| \leq \frac{1}{n} \cdot \sup_{\delta \in \Delta'} |\delta| \to 0.$$

$\square$

**Theorem 9.7** (Alexandrov–Prokhorov; [18] Theorem 8.9.3 (i)). If $X$ is a compact topological space, then the set $\mathcal{P}(X)$ of probability measures on $X$ is a compact subset of $\mathcal{M}_\pm(X)$ with respect to the weak topology.
Given a Borel measurable \( w : X \to Y \) and a signed Borel measure \( \mu : \mathcal{B}(X) \to (-\infty, \infty) \), we define the push forward measure through \( w \) to be \( w_\# \mu : \mathcal{B}(Y) \to (-\infty, \infty) \),

\[
w_\# \mu(B) := \mu(w^{-1}(B)) \quad \text{for all } B \in \mathcal{B}(Y).
\]

We also write \( \mu \circ w^{-1} \) for \( w_\# \mu \). Obviously the operator \( w_\# \) is linear:

\[
w_\#(p_1 \cdot \mu_1 + p_2 \cdot \mu_2) = p_1 \cdot w_\# \mu_1 + p_2 \cdot w_\# \mu_2,
\]

for all measures \( \mu_i \) and scalars \( p_i \), \( i = 1, 2 \).

**Proposition 9.8** (Theorem 9.1.1(i), Theorem 3.6.1, chap.8.10(v)). Let \( w : X \to Y \) be a continuous map between Hausdorff topological spaces \( X, Y \). Then

(a) \( w_\# \mu \) is a (signed or nonnegative) Radon measure in \( Y \) whenever \( \mu \) is a (signed or respectively, nonnegative) Radon measure in \( X \);

(b) for all bounded Borel measurable functions \( g : Y \to \mathbb{R} \)

\[
\int_X g \, d(w_\# \mu) = \int_X g \circ w \, d\mu,
\]

where \( \mu \) is a signed Borel measure in \( X \);

(c) the transport of measure \( w_\# : \mathcal{M}_\pm(X) \to \mathcal{M}_\pm(Y) \), \( w_\#(\mu) = \mu \circ w^{-1} \) for \( \mu \in \mathcal{M}_\pm(X) \), is continuous in the weak topology.

**Proof.** For (a) fix \( \varepsilon > 0 \), \( B \in \mathcal{B}(Y) \) and a nonnegative Radon measure \( \mu \). Since \( \mu \) is Radon there exists a compact subset \( K \subseteq w^{-1}(B) \) s.t. \( \mu(w^{-1}(B) \setminus K) < \varepsilon \). Simple set-algebra shows that

\[
w^{-1}(B \setminus w(K)) \subseteq w^{-1}(B) \setminus K.
\]

Hence \( w_\# \mu(B \setminus K') < \varepsilon \) for the compact set \( K' = w(K) \subseteq B \). For the case of signed \( \mu \) it is enough to use the Jordan-Hahn decomposition \( w_\# \mu = w_\# \mu^+ - w_\# \mu^- \).

We check (b) for \( g = \chi_B \), the characteristic function of the set \( B \in \mathcal{B}(Y) \). Recall that \( \chi_B(y) = 1 \) if \( y \in B \), and 0 otherwise. Observe that \( \chi_B \circ w = \chi_{w^{-1}(B)} \). Thus we have

\[
\int_X \chi_B \, d(w_\# \mu) = w_\# \mu(B) = \mu(w^{-1}(B)) = \int_X \chi_{w^{-1}(B)} \, d\mu = \int_X \chi_B \circ w \, d\mu.
\]

One readily extends (b) to simple functions \( g \), their monotone limits and differences of these limits.

For (c) it is enough to check that for every \( \mu \in \mathcal{M}_\pm(X) \), \( \varepsilon > 0 \) and \( g \in \mathcal{C}_b(Y) \) there exists \( f \in \mathcal{C}_b(X) \) s.t.

\[
\int_X g \, d(w_\# \nu) - \int_X g \, d(w_\# \mu) = \int_X g \circ w \, d\nu - \int_X g \circ w \, d\mu < \varepsilon.
\]

\[\square\]

**Definition 9.9.** For a nonnegative Borel measure \( \mu : \mathcal{B}(X) \to [0, \infty) \) on a Hausdorff topological space \( X \) we define the support of \( \mu \) to be

\[
\text{supp } \mu = \{ x \in X : \mu(U) > 0 \text{ for all open neighbourhoods } U \ni x \}.
\]

Alternatively, \( C = \text{supp } \mu \) is the largest closed set \( C \) s.t. \( \mu(U) > 0 \) for every open \( U \) with \( U \cap C \neq \emptyset \).

Supports, especially of Radon measures, obey some nice properties.

**Lemma 9.10.** Let \( \mu : \mathcal{B}(X) \to [0, \infty) \) be a nonnegative Radon measure on a Hausdorff topological space \( X \).

(i) If \( B \in \mathcal{B}(X) \) is such that \( B \subseteq X \setminus \text{supp } \mu \), then \( \mu(B) = 0 \). Conversely, if \( \mu(B) > 0 \), then \( B \cap \text{supp } \mu \neq \emptyset \).


Proposition 9.11. Let $f$ be the minimal Lipschitz constant of $X$. Then

Remark 9.12. (i) If $X$ is bounded, then $\mu(X) \neq 0$.

Lemma 9.13. (a) $\supp(C \mu) = \supp(C \mu)$, 
(b) $\supp(C \mu + \nu) = \supp(C \mu) \cup \supp(C \nu)$,
(c) $\supp(C \mu) = \supp(C \mu)$.

Proof. Parts (a) and (b) are easy to check. For (c) put $C = \supp\mu$. We shall verify that $\supp(C \mu) = \sup(S(C \mu))$.

First we check that $\supp(C \mu) \supseteq \sup(S(C \mu))$. Take an open $V \subseteq Y$ s.t. $V \cap \sup(S(C \mu)) \neq \emptyset$. Then $\sup(S(C \mu))$ is open and $\sup(S(C \mu)) \cap C = \emptyset$. Hence $\sup(S(C \mu)) = \sup(S(C \mu)) > 0$ by definition of $\supp\mu$.

Now, by contradiction, we suppose that there exists $y \in \sup(S(C \mu)) \setminus \sup(S(C \mu))$. We can separate $y$ from $\sup(S(C \mu))$ by some open neighbourhood $V \supseteq y$, $V \cap \sup(S(C \mu)) = \emptyset$. (Namely $V := X \setminus \sup(S(C \mu))$.) Put $U := \sup(S(C \mu))$. Then $U$ is open and $U \cap C = \emptyset$. (Indeed, $x \in \sup(S(C \mu)) \cap C$ implies $w(x) \in V \cap \sup(S(C \mu)) = \emptyset$.) Hence $U \subseteq X \setminus \sup(S(C \mu))$. Since $\mu$ is Radon we can apply Lemma 9.10. Finally, we get $\sup(S(C \mu)) = \sup(S(C \mu)) = \mu(U) = 0$. As $V \supseteq y$, this means that $\not\exists y \in \sup(S(C \mu))$.

We now discuss the special case when $(X, d)$ is a complete metric space. By $\mathcal{P}_1(X)$ we denote the collection of Radon probability measures with integrable distance, i.e., measures $\mu \in \mathcal{P}(X)$ such that for some (equivalently – for all) $x_0 \in X$, the integral $\int_X d(x, x_0) \, d\mu(x) < \infty$. We endow the space $\mathcal{P}_1(X)$ with

- the Monge-Kantorovich metric (cf. [56] chap. B.5.1)

  \[ d_{MK}(\mu, \nu) := \sup \left\{ \int_X f \, d\mu - \int_X f \, d\nu : f : X \to \mathbb{R}, \text{Lip}(f) \leq 1 \right\}; \]

- the Fortet-Mourier metric (cf. [79])

  \[ d_{FM}(\mu, \nu) := \sup \left\{ \int_X f \, d\mu - \int_X f \, d\nu : f : X \to \mathbb{R}, \text{Lip}(f) \leq 1, \sup_{x \in X} |f(x)| \leq 1 \right\}. \]

In the above, $\mu, \nu \in \mathcal{P}_1(X)$, and

\[ \text{Lip}(f) = \sup_{x_1 \neq x_2} \frac{|f(x_1) - f(x_2)|}{d(x_1, x_2)} \]

is the minimal Lipschitz constant of $f : X \to \mathbb{R}$.

Remark 9.12. (i) If $X$ is bounded, then $\mathcal{P}_1(X) = \mathcal{P}(X)$.

(ii) If $X$ is a separable complete metric space, then $\mathcal{P}_1(X)$ constitutes the space of all Borel probability measures $\mu$ on $X$ which satisfy $\int_X d(x, x_0) \, d\mu < \infty$ for some $x_0 \in X$ (see [56]).

Lemma 9.13. Let $\mathcal{P}_1(X)$ be the set of Radon probability measures with integrable distance on a complete metric space $X$.

(i) The Monge-Kantorovich metric $d_{MK}$ is a complete metric on $\mathcal{P}_1(X)$. 

Lemma 9.14. A subset \( \Omega \subset I^\infty \) is dense with respect to \( d_B \) if prefixes of elements from \( \Omega \) exhaust all possible finite words, i.e., for every \( k \in \mathbb{N} \) and \( \alpha \in I^k \) there exists \( \omega \in \Omega \) s.t. \( \omega|_k = \alpha \).

The Borel \( \sigma \)-algebra of \( I^\infty \) is a countable product of full \( \sigma \)-algebras of \( I \), \( \mathcal{B}(I^\infty) = \bigotimes_{n \in \mathbb{N}} 2^I \). (Recall that if \( I \) is discrete, then \( \mathcal{B}(I) = 2^I \).)
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References

1. R. R. Akhmerov, M. I. Kamenskiǐ, A. S. Potapov, A. E. Rodkina, B. N. Sadovskiǐ, Measures of Noncompactness and Condensing Operators, Birkhäuser Verlag, Basel, 1992.
2. E. Akin, The General Topology of Dynamical Systems, American Mathematical Society, 1993.
3. J. Andres, J. Fiser, Metric and topological multivalued fractals, Int. J. Bifurcat. Chaos 14 (2004), no. 4, 1277–1289.
4. J. Andres, J. Fiser, G. Gabor, K. Leśniak, Multivalued fractals, Chaos Solitons Fract. 24 (2005), no. 3, 665–700.
5. A. Abrieto, A. Junqueira, S. Santiago, On weakly hyperbolic iterated function systems, Bull. Braz. Math. Soc. (N.S) 48 (2017), 111–140.
6. L. Arnold, Random Dynamical Systems (2nd Printing), Springer 2003.
7. J.-P. Aubin, A. Cellina, Differential Inclusions, Springer, 1984.
8. R. Bal, A. Máthé, Generalized Hausdorff measure for generic compact sets, Ann. Acad. Sci. Fenn. Math. 38 (2013), no. 2, 797–804.
9. T. Banakh, W. Kubiś, M. Nowak, F. Strobin, Contractive function systems, their attractors and metrization, Topol. Methods Nonlinear Anal. 46 (2015), no. 2, 1029–1066.
10. B. Bárany, M. Rams, K. Simon, Dimension Theory of some non-Markovian repellers. Part I: A gentle introduction arXiv: 1901.04035 (2019).
11. M. F. Barnsley, K. Leśniak, On the continuity of the Hutchinson operator, Symmetry (Basel) 7 (2015), no. 4, 1831–1840.
12. M. F. Barnsley, K. Leśniak, M. Rypka, Chaos game for IFSs on topological spaces, J. Math. Anal. Appl. 435 (2016), no. 2, 1458–1466.
13. M. F. Barnsley, K. Leśniak, M. Rypka, Basic topological structure of fast basins, Fractals 26 (2018), no. 01, 1850011/1-11.
14. M. F. Barnsley, A. Vince, Developments in fractal geometry, Bull. Math. Sci. 3 (2013), no. 2, 299–348.
15. P. G. Barrientos, F. H. Ghane, D. Malicet, A. Sarizadeh, On the chaos game of iterated function systems, Topol. Methods Nonlinear Anal. 49 (2017), no. 1, 105–132.
16. G. A. Beer, Topologies on Closed and Closed Convex Sets, Kluwer Academic Publishers, 1993.
17. V. Berinde, Iterative Approximation of Fixed Points, Springer, 2007.
18. V. Bogachev, Measure Theory, Springer, 2007.
19. D. Buraczewski, E. Damek, T. Mikosch, Stochastic Models with Power-Law Tails, Springer, 2016.
20. C. S. Calude, L. Staiger, Generalisations of disjunctive sequences, MLQ Math. Log. Quart. 51 (2005), 120–128.
21. S. Carl, S. Heikkilä, Fixed Point Theory in Ordered Sets and Applications, Springer, 2007.
22. W. J. Charatonik, A. Dilks, On self-homeomorphic spaces, Topology Appl. 55 (1994), no. 3, 215–238.
23. D. N. Cheban, Global Attractors Of Non-autonomous Dynamical And Control Systems (2nd Edition), World Scientific, 2014.
24. M. Chowdhury, E. Tafasd, Topological Methods for Set-Valued Nonlinear Analysis, World Scientific Publishing, Singapore, 2008.
25. I. D. Chueshov, Introduction to the Theory of Infinite-Dimensional Dissipative Systems, Acta Scientific Publishing House, Kharkov 1999.
26. E. D'Aniello, T. H. Steele, Attractors for iterated function schemes on $[0, 1]^N$ are exceptional, J. Math. Anal. Appl. 541 (2015), no. 1, 537–541.
27. E. D'Aniello, T. H. Steele, Attractors for classes of iterated function systems, European J. Math. 5 (2019), 116–137.
28. D. Dumitru, Attractors of topological iterated function system, Annals of Spiru Haret University: Mathematics-Informatics series 8 (2012), no. 2, 11–16.
29. A. Edalat, Power domains and iterated function systems, Inf. Comput. 124 (1996), no. 2, 182–197.
30. G. A. Edgar, Integral, Probability, and Fractal Measures, Springer, 1998.
31. R. Engelking, General Topology. Revised and completed version, Heldermann Verlag, 1989.
32. K. Falconer, Techniques in Fractal Geometry, Wiley, 1997.
33. A. Fan, Ergodicity, unidimensionality and multifractality of self-similar measures, Kyushu J. Math. 50 (1996), no. 2, 541–574.
34. M. Fernández-Martínez, M. A. Sánchez-Granero, A new fractal dimension for curves based on fractal structures, Topology Appl. 203 (2016), 108–124.
35. S. R. Fougé, Existence of invariant measures for Markov processes. II, Proc. Amer. Math. Soc. 17 (1966), 387–389.
36. H. Furstenberg, Ergodic Theory and Fractal Geometry, American Mathematical Society CBMS, Providence RI, 2014.
37. G. S. Goodman, A probabilist looks at the chaos game, Fractals in the Fundamental and Applied Sciences, pp. 159–168, North-Holland (1991).
38. A. Granas, J. Dugundji, Fixed Point Theory, Springer, 2003.
39. G. Guzik, Asymptotic stability of discrete cocycles, J. Difference Equ. Appl. 21 (2015), no. 11, 1044–1057.
40. D. J. Hartfiel, Nonhomogeneous Matrix Products, World Scientific, 2002.
41. M. Hata, On the structure of self-similar sets, Japan J. Appl. Math. 2 (1985), 381–414.
M. Hille, Remarks on limit sets of infinite iterated function systems, Monatsh. Math. 168 (2012), no. 2, 215-237.

M. Iosifescu, Iterated function systems. A critical survey, Math. Rep., Buchar. 11(61) (2009), no. 3, 181–229.

J. Jachymski, An extension of A. Ostroffski’s theorem on the round-off stability of iterations, Aequationes Math. 53 (1997), 242–253.

J. R. Jachymski, On iterative equivalence of some classes of mappings, Ann. Math. Sil. 13 (1999), 149–165.

J. Jachymski, I. Jóźwik, Nonlinear contractive conditions: a comparison and related problems, Banach Center Publ. 77 (2007), 123–146.

A. Jadczyk, Quantum Fractals, World Scientific, 2014.

O. Knill, Iterated function systems. A critical survey

K. Kuhlmann, The structure of spaces of

A. N. Kolmogorov, S. V. Fomin, Elements of the Theory of Functions and Functional Analysis

J. Jachymski, I. J´ o´ zwik, Quantum Fractals

P. Jaros, L. Mašlanka, F. Strobin, Algorithms generating images of attractors of generalized iterated function systems, Numer. Algorithms 73 (2016), 477–499.

A. Käänemaker, H. W. J. Reeve, Multifractal analysis of Birkhoff averages for typical infinitely generated self-affine sets, J. Fractal Geom. 1 (2014), no. 1, 83–152.

A. Kameyama, Distances on topological self-similar sets and the kneading determinants, J. Math. Kyoto Univ. 40 (2000), no. 4, 601–672.

B. Kieninger, Iterated Function Systems on Compact Hausdorff Spaces, PhD thesis, University of Augsburg, Aachen: Shaker-Verlag, 2002.

V. A. Kleptsyn, M. B. Nalskii, Contraction of orbits in random dynamical systems on the circle, Funct. Anal. Appl. 38 (2004), no. 4, 267–282.

O. Knill, Probability Theory and Stochastic Processes with Applications, Overseas Press, 2009.

A. N. Kolmogorov, S. V. Fomin, Elements of the Theory of Functions and Functional Analysis, Dover, 1999.

K. Kuhlmann, The structure of spaces of R-places of rational function fields over real closed fields, Rocky Mt. J. Math. 46 (2016), no. 2, 533–557.

H. Kunze, D. La Torre, F. Mendivil, E. R. Vrscay, Fractal-Based Methods in Analysis, Springer, 2012.

T. Leinster, A general theory of self-similarity, Adv. Math. 226 (2011), no. 4, 2935–3017.

K. Leśniak, Infinite iterated function systems: a multivalued approach, Bull. Pol. Acad. Sci. Math. 52 (2004) no. 1, 1–8.

K. Leśniak, On the Lifshits constant for hyperspaces, Bull. Pol. Acad. Sci. Math. 55 (2007) no. 2, 155–160.

K. Leśniak, Invariant sets and Knaster-Tarski principle, Cent. Eur. J. Math. 10 (2012), no. 6, 2077–2087.

K. Leśniak, On discrete stochastic processes with disjunctive outcomes, Bull. Aust. Math. Soc. 90 (2014), 149–159.

K. Leśniak, Note on multifunctions condensing in the hyperspace, Fixed Point Theory 16 (2015), no. 2, 343–352.

K. Leśniak, Random iteration for infinite nonexpansive iterated function systems, Chaos 25 (2015), 083117-1–5.

A. Lasota, J. Myjak, Semifractals, Bull. Pol. Acad. Sci., Math. 44 (1996), no. 1, 5–21.

G. Mantica, R. Peirone, Attractors of iterated function systems with uncountably many maps and infinite sums of Cantor sets, J. Fractal Geom. 4 (2017), no. 3, 215–256.

T. Martyn, The chaos game revisited: Yet another, but a trivial proof of the algorithm’s correctness, Appl. Math. Lett. 25 (2012), no. 2, 206–208.

P. Massopust, Interpolation and Approximation with Splines and Fractals, Oxford University Press, 2010.

J. Matkowski, R. Wegrzyk, On equivalence of some fixed point theorems for selfmappings of metrically convex space, Boll. Un. Mat. Ital. A (5) 15 (1978), 359–369.

J. Matkowski, J. Miś, Examples and remarks to a fixed point theorem, Facta Univ. Ser. Math. Inform. No. 1 (1986), 53–56.

R. D. Mauldin, M. Urbański, Graph Directed Markov Systems, Cambridge University Press, 2003.

I. McFarlane, S. G. Hoggar, Optimal drivers for the ‘random’ iteration algorithm, Comput. J. 37 (1994), no. 7, 629–640.

R. Miculescu, A. Mihail, Generalized IFSs on noncompact spaces, Fixed Point Theory Appl. Volume 2010, Article ID 584215, 11 pp.

R. Miculescu, A. Mihail, On a question of A. Kameyama concerning self-similar metrics, J. Math. Anal. Appl. 422 (2015), no. 1, 265–271.

R. Miculescu, A. Mihail, A sufficient condition for a finite family of continuous functions to be transformed into ϕ-contractions, Ann. Acad. Sci. Fenn., Math. 41 (2016), 51–65.

A. Mihail, A topological version of iterated function systems, An. Ştiinţ. Univ. Al. I. Cuza, Iaşi, (S.N.), Matematica 58 (2012), 105–120.

A. Muchnik, A. Semenov, M. Ushakov, Almost periodic sequences, Theoret. Comput. Sci. 304 (2003), no. 13, 1–33.

D. Mumford, C. Series, D. Wright, Indra’s Pearls, Cambridge University Press, 2002.

J. Myjak, Some typical properties of dimensions of sets and measures, Abstr. Appl. Anal. (2005), no. 3, 329–333.

J. Myjak, T. Szarek, Attractors of iterated function systems and Markov operators, Abstr. Appl. Anal. (2003), no. 8, 479–502.

M. Nowak, M. Fernández-Martínez, Countereamples for IFS-attractors, Chaos Solitons Fractals 89 (2016), 316–321.

E. A. Ok, Fixed set theory for closed correspondences with applications to self-similarity and games, Nonlinear Anal., Theory Methods Appl., Ser. A, 56 (2004), no. 3, 309–330.
K. Okamura, *Self-similar measures for iterated function systems driven by weak contractions*, Proc. Japan Acad. Ser. A Math. Sci. 94 (2018), no. 4, 31–35.

L. Olsen and N. Snigireva, *Multifractal spectra of in-homogeneous self-similar measures*, Indiana U. Math. J. 57 (2008), 1789–1844.

I. A. Rus, A. Petruşel, G. Petruşel, *Fixed Point Theory*, Cluj University Press, 2008.

M. Samuel, A. V. Tetenov, *On attractors of iterated function systems in uniform spaces*, Sib. Élektron. Mat. Izv. 14 (2017), 151–155.

M. J. Sanders, *Non-attractors of iterated function systems*, Texas Project NexT e-Journal 1 (2003), 1–9.

G. R. Sell, Y. You, *Dynamics of Evolutionary Equations*, Springer, 2013.

A. G. Sivak, *On the structure of transitive ω-limit sets for continuous maps*, Qual. Theory Dyn. Syst. 4 (2003), no. 1, 99–113.

A. V. Škorokhod, *Topologically recurrent Markov chains: Ergodic properties*, Theory Probab. Appl. 31 (1987), no. 4, 563–571.

Ö. Stenflo, *Survey of average contractive iterated function systems*, J. Difference Equ. Appl. 18 (2012), no. 8, 1355–1380.

R. S. Strichartz, *Differential Equations on Fractals*, Princeton University Press, 2006.

K. R. Wicks, *Fractals and Hyperspaces*, Springer, 1991.

A. Wiśnicki, *On a nonstandard approach to invariant measures for Markov operators*, Ann. Univ. Mariae Curie-Skłodowska, Sect. A 64 (2010), no. 2, 73–80.

L. Zajáček, *On σ-porous sets in abstract spaces*, Abstract Appl. Analysis 5 (2005), 509–534.