Entropy per Baryon in Strong Coupling QCD

Neven Bilić* and Jean Cleymans
Department of Physics, University of Cape Town
Rondebosch, South Africa

December 1, 2021

Abstract

The entropy per baryon is studied in the strong coupling large dimension $d$ limit of lattice QCD with staggered fermions. The partition function is calculated for non-zero chemical potential and temperature using the $1/d$ expansion. It is found that the entropy per baryon ratio is almost continuous across the transition from the quark-gluon to the hadronic phase. The relevance of these results for heavy ion collisions is discussed.

Relativistic heavy ion collisions test the thermodynamic properties of QCD. It is well known that lattice gauge theory is the most suitable method for studying these properties theoretically, however Monte Carlo simulations at non-zero chemical potential $\mu$ are notoriously difficult [1, 2, 3], and it is necessary to investigate separately analytical methods based on the strong-coupling, $1/d$ expansions of QCD at both $T \neq 0$ and $\mu \neq 0$. This is the subject matter of the present investigation. This method has been successfully applied to the calculation of the hadronic spectrum [4] and leads to a

*On leave of absence from Rudjer Bošković Institute, Zagreb, Croatia
quantitative description of the phase diagram in the temperature and chemical potential plane [1, 2, 3, 4, 5]. It yields a second order phase transition at vanishing $\mu$ and predicts a first order transition for any non-zero value of $\mu$. However, unlike the weak coupling regime, where the dimensionless lattice parameters are related to the lattice scale via the renormalization group equation, such a relation does not exist in the strong coupling regime. Consequently, a physical interpretation of the strong coupling results is not straightforward. In this paper we propose a natural solution to this problem.

We calculate the thermodynamic properties of QCD and contrast the results to those obtained in the hadronic gas model. The difference in results is very remarkable. In the ideal gas approximation the entropy to baryon ratio ($S/B$) is almost always larger in a quark-gluon plasma than in a hadronic resonance gas, if we compare them at the same temperature and chemical potential [9, 10, 11]. Our results provide no support for the contention that experimental results on the charge asymmetry ratio show evidence for quark-gluon plasma formation [12].

As is usual in these considerations, we work on an asymmetric lattice with $N_t$ points in the time and $N_s$ in the $d$ space directions with an anisotropy $\gamma$. The finite-temperature behavior can be studied in the infinite volume limit ($N_s \to \infty$) as a function of $\gamma$ and $N_t$. In the naive continuum limit $\gamma$ can be interpreted as a ratio of the space and time lattice spacing, $a_s/a_t$. As we shall shortly demonstrate, this naive relation does not hold in the strong coupling regime. We shall propose instead a new relation based on general physical considerations.

We start from the effective partition function derived at infinite coupling and in the large-$d$ limit [13]

$$Z_\infty = \int [d\lambda] \exp\{-\frac{N_c}{d} \sum_{x,y} (\lambda_x - m)V_{x,y}^{-1}(\lambda_y - m)\} \prod_x \int dU \det \mathbf{D}(U), \quad (1)$$

with

$$V_{x,y} = \frac{1}{2d} \sum_{k=1}^{d} (\delta_{y,x+k} + \delta_{y,x-k}). \quad (2)$$
$D(U)$ is an $N_t N_c \times N_t N_c$ matrix

$$
D(U) = \begin{pmatrix}
  u_1 \mathbf{1} & a_1 \mathbf{1} & 0 & \cdots & b_{N_t} U^\dagger \\
  -b_1 \mathbf{1} & u_2 \mathbf{1} & a_2 \mathbf{1} & 0 & \cdots \\
  0 & -b_2 \mathbf{1} & u_3 \mathbf{1} & 0 & \cdots \\
  \vdots & \ddots & \ddots & \ddots & \ddots \\
  -a_{N_t} U & 0 & 0 & \cdots & u_{N_t} \mathbf{1}
\end{pmatrix}
$$

(3)

with $u_i = 2(\lambda_i + m)/\gamma$. In the final expressions the parameters $a_x$ and $b_x$ at each space-time point $x$ are set equal to the constants $a = e^\mu$ and $b = e^{-\mu}$ which are related to the chemical potential. The explicit $\bar{x}$-dependence of the $a_i$, $b_i$ and $\lambda_i$ in (3) has been omitted. The integration over $dU$ can be performed by exploiting the appropriate SU($N_c$) group integral. Details are given in appendices of refs. [8, 13].

The $1/d$ expansion begins with a saddle-point approximation of the $\lambda$-integral, or equivalently by minimizing the mean-field free energy

$$
F = \frac{N_t N_c}{d} \lambda^2 - \ln Z_0,
$$

(4)

with $Z_0 = \int dU \det D(U)$ being a “zero-dimensional” partition function. We improve infinite-coupling mean-field calculations by replacing the free energy (4) by

$$
F \rightarrow F - \sum_{i=1}^{5} \delta F_i,
$$

(5)

with corrections $\delta F_i$ of order $1/g^2$ or $1/d$ [7, 8].

Provided we know how to fix the scale $a_s$ and $a_t$ we can relate the physical thermodynamical observables to the corresponding lattice quantities derived from the free energy (4,5)

$$
P = \frac{1}{a_t a_s^3} P_{\text{latt}}, \quad n_B = \frac{1}{a_s^3} n_{\text{latt}},
$$

$$
\mathcal{E} = \frac{1}{a_t a_s^3} \mathcal{E}_{\text{latt}}, \quad s = \frac{1}{a_s^3} s_{\text{latt}},
$$

(6)
$P_{\text{latt}} = - \frac{F}{N_t} - P_{\text{vac}}$, $n_{\text{latt}} = - \frac{1}{N_t N_c} \frac{\partial F}{\partial \mu}$.

$\mathcal{E}_{\text{latt}} = \left. \frac{\partial F}{\partial N_t} \right|_{\mu N_t} - \mathcal{E}_{\text{vac}}$,

$s_{\text{latt}} = N_t (P_{\text{latt}} + \mathcal{E}_{\text{latt}} - \mu N_c n_{\text{latt}})$.

$P_{\text{vac}}$ and $\mathcal{E}_{\text{vac}}$ are the vacuum pressure and energy density calculated in the limit $N_t \to \infty$. $n_B$ and $s$ refer to the baryon density and the entropy density respectively. The entropy per baryon ratio $S/B$ is then given by $s/n_B$.

At $\mu = 0$ the mean-field calculation predicts a second-order chiral phase transition. Including $1/d$ and $1/g^2$ corrections yields the critical anisotropy $\gamma_2$:

$$\gamma_2^2 = \gamma_0^2 - \frac{\gamma_c}{g^2 N_c} (2d - \frac{N_c^2 - 1}{N_c})$$

where the critical anisotropy at infinite coupling takes on the value

$$\gamma_0^2 = \frac{d(N_c + 1)(N_c + 2)}{6(N_c + 3)} N_t$$

The anisotropy $\gamma$ is a measure of the temperature. However, unlike the weak coupling regime, where $\gamma$ can directly be related to the spatial and temporal lattice spacings [14] allowing one to establish the connection with the temperature $T = \gamma/N_0 a_s$, such a direct relation does not exist in the strong coupling limit. In other words, the physical anisotropy of the lattice $a_s/a_t$ is not necessarily equal to $\gamma$. If one insists on using this relation in the strong coupling regime the critical temperature, owing to (8,9), would strongly depend on $N_t$ which is physically unacceptable. We must, therefore, alter the naive relation between $\gamma$ and the physical anisotropy (or temperature), in such a way as to maintain the critical temperature independent of unphysical lattice parameters, such as $N_t$. Quite generally we may set

$$a_t(\gamma, g) = \frac{a_s}{f(\gamma, g)}$$

where the function $f$ can be determined from the requirements:

a) $T_c$ should not depend on $N_t$,
b) at $\gamma = 1$ we have $a_t(1, g) = a_s$. From the first requirement it follows that the $N_t$ dependence of $f(\gamma_c, g)$ must be of the form $N_t \varphi(g)$. Furthermore, in the limit $g \to \infty$ owing to (3) we must have $f(\gamma_0, \infty) = \gamma_0^2$ up to a constant which may be absorbed in the lattice scale $a_s$. Thus $f(\gamma_c, g) = \gamma_0^2 \varphi(g)$, with $\varphi$ being an arbitrary function of $g$. From the functional relationship between $\gamma_c$ and $\gamma_0$ in (8) it immediately follows

$$f(\gamma, g) = \gamma^2 + \frac{\gamma}{g^2 N_c} (2d - \frac{N_c^2 - 1}{N_c}).$$

(11)

up to an arbitrary multiplicative function of $g$. The second requirement then implies

$$a_t(\gamma, g) = a_s \frac{f(1, g)}{f(\gamma, g)}.$$  

(12)

We note that the correction to the critical anisotropy does not depend on $N_t$ although $\gamma_0^2$ itself is proportional to $N_t$. The temperature can now be defined in the usual way $T = 1/N_t a_t(\gamma, g)$. Thus, for example, at infinite coupling we have $T = \gamma^2 / N_t a_s$ in contrast to the standard $T = \gamma / N_t a_s$. The physical chemical potential is related to the dimensionless lattice parameter $\mu$ as usual $\mu_{phys} = \mu a_t(\gamma, g)$.

The scale $a_s$ in (12) remains completely arbitrary. Unlike in the weak coupling regime, where the scale can be related to the coupling constant via the renormalization group equation, such a direct relation does not exist in the strong coupling regime. One can, nevertheless, fix the scale by fixing the value of some physical observable, such as $T_c$ or $\mu_c$, at a chosen value of $g$.

In Fig 1 we show the critical temperature as a function of $6 / g^2$ for a fixed $\mu_{phys}$. The reason for a slight $N_t$ dependence is that the numerical evaluation of the transition temperature includes higher powers of $1 / g^2$ which are not present in the analytic formula (8) used to derive (11). The scale is fixed by requiring that the strong coupling $T_c$ at $\mu = 0$ and $6 / g^2 = 5$ coincides with the Monte Carlo value $T_c = 130$ MeV [15].

The critical entropy per baryon in the hadronic phase increases rapidly with decreasing $g$ (Fig 2). If we fix the coupling at the point where the strong-coupling curve crosses the experimental region (hatched area) [16] we can plot the entropy per baryon as a function of temperature on both sides of the transition point and compare this with the quark-gluon plasma and hadronic gas model (Fig 3). In contrast to this simple phenomenological
model, strong coupling lattice thermodynamics predicts a continuous $s/n_B$ across the phase transition. In the infinite coupling limit ($6/g^2 = 0$) the entropy per baryon in the chiral-broken (hadronic) phase is greater than in the symmetric (quark-gluon) phase.

Preliminary data from the emulsion experiment EMU05 cited in [16] show that in S-Pb collisions at 200 GeV the charge asymmetry ratio

$$\frac{N^+ - N^-}{N^+ + N^-}$$

is approximately $0.085 \pm 0.01$ in the central rapidity region. To relate this to the entropy per baryon ratio one can proceed as follows. Assuming that this region is symmetric in isospin (for S-S collisions this would be exact), the numerator in (13) equals the baryon number divided by 2, assuming furthermore that the denominator is approximately $2/3$ of the total particle number (neutral particles are not included), the ratio becomes:

$$\frac{N^+ - N^-}{N^+ + N^-} \approx \frac{3B}{4N}$$

To relate this to the entropy we use as a guess that the entropy per particle is 4 (as is the case of a massless Boltzmann gas). We thus conclude that

$$S/B \approx 36 \pm 5$$

This estimate is compatible with the more detailed analysis presented in [16, 17]. This result was used above in Fig. 2.

Contrary to the predictions of quark-gluon plasma - hadronic gas models we find in the strong coupling regime that the entropy per baryon is almost continuous across the phase transition. Moreover, including the strong coupling corrections increases the entropy per baryon content in the hadronic phase near the phase transition to the order close to the estimates obtained from the measured charge asymmetry ratio.

In conclusion, we have calculated the thermodynamic properties of QCD in the strong coupling large $d$ limit at finite temperature and chemical potential. We have introduced a new relation between the lattice anisotropy parameter $\gamma$ and the physical anisotropy $a_s/a_t$ allowing us to establish a physical interpretation of the dimensionless parameters $\gamma$ and $\mu$ in terms of temperature and the chemical potential. Our results show that the entropy
per baryon ratio $S/B$ is almost continuous across the phase transition point.

Acknowledgments

We acknowledge useful discussions with Krzysztof Redlich. We furthermore acknowledge financial support from the Foundation for Research and development (FRD), Pretoria.

References

[1] I. Barbour, N.-E. Behilil, E. Dagotto, F. Karsch, A. Moreo, M. Stone and H.W. Wyld, Nucl. Phys. B275 (1986) 296.

[2] A. Gocksch, Phys. Rev. Lett. 61 (1988) 2054

[3] F. Karsch and K.-H. Mütter, Nucl. Phys. B313 (1989) 541

[4] T. Jolicoeur, H. Kluberg-Stern, A. Morel, M. Lev and B. Petersson, Nucl. Phys. B235 [FS11] (1984) 455.

[5] P. Damgaard, N. Kawamoto and K. Shigemoto, Nucl. Phys. B264 (1986) 1

[6] N. Bilić and K. Demeterfi, Phys. Lett. B212 (1988) 83

[7] N. Bilić, K. Demeterfi and B. Petersson, Nucl. Phys. B377 (1992) 651.

[8] N. Bilić, F. Karsch and K. Redlich, Phys. Rev. D45 (1992) 3228.

[9] P.R. Subramanian, H. Stöcker and W. Greiner, Phys. Lett. B173 (1986) 468.

[10] K.S. Lee, M.J. Rhoades-Brown and U. Heinz, Phys. Rev C37 (1988) 1459.

[11] A. Leonidov, K. Redlich, H. Satz, E. Suhonen and G. Weber, Phys. Rev. D(to be published).
[12] Preliminary results have been reported by N. Bilić at the 12th International Symposium on Lattice Field Theory, LATTICE 94, Bielefeld, to appear in Nucl. Phys. B (Proc. Suppl.)

[13] G. Fäldt and B. Petersson, Nucl. Phys. B264 [FS15] (1986) 197.

[14] F. Karsch and I.O. Stamatescu, Phys. Lett. B227 (1989) 153.

[15] R. Altmeyer et al. (MTc collaboration), Nucl. Phys. B389 (1993) 445.

[16] J. Letessier, A. Tounsi, U. Heinz, J. Sollfrank and J. Rafelski, Phys. Rev. Lett. 70 (1993) 3530.

[17] U. Heinz, Nucl. Phys. A566 (1994) 205c.
Figure 1: Critical temperature at fixed $\mu = 0$ and $\mu = 250$ MeV for $N_t = 4$ and 6.

Figure 2: Critical entropy per baryon in hadronic phase at fixed $\mu = 250$ MeV for $N_t = 4$ and 6.

Figure 3: Entropy per baryon calculated in strong coupling QCD compared to quark-gluon plasma (QGP) and hadronic gas model at fixed $\mu = 250$ MeV.
This figure "fig1-1.png" is available in "png" format from:

http://arxiv.org/ps/hep-lat/9501019v1
This figure "fig1-2.png" is available in "png" format from:

http://arxiv.org/ps/hep-lat/9501019v1
This figure "fig1-3.png" is available in "png" format from:

http://arxiv.org/ps/hep-lat/9501019v1