Exact sequences of semistable vector bundles on algebraic curves

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Abstract

Let $X$ be a smooth complex projective curve of genus $g \geq 1$. If $g \geq 2$, assume further that $X$ is either bielliptic or with general moduli. Fix integers $r, s, a, b$ with $r > 1$, $s > 1$ and $as \leq br$. Here we prove the existence of an exact sequence

$$0 \to H \to E \to Q \to 0$$

of semistable vector bundles on $X$ with $\text{rk}(H) = r$, $\text{rk}(Q) = s$, $\deg(H) = a$ and $\deg(Q) = b$.

Introduction

Let $X$ be a smooth projective curve on an algebraically closed field $k$. A vector bundle $E$ is said stable (resp. semistable) if for every proper subbundle $A$ of $F$ it holds $\mu(A) < \mu(E)$ (resp. $\mu(A) < \mu(E)$). If $\text{MCD}(\text{rk}(E), \deg(E)) = 1$ then $E$ is stable if and only if it is semistable. Since almost forty years these bundles are an active subject of study. In this paper we present a method to construct a new semistable bundle, $E$, as representative of an extension of the type

$$0 \to H \to E \to Q \to 0$$

with $H$ and $Q$ semistable. As one can easily see, a necessary condition for the stability (resp. semistability) of $E$ fitting in the exact sequence (1) is

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\( \mu(H) < \mu(Q) \) (resp. \( \mu(H) \leq \mu(Q) \)). If \( \text{rk}(H) = \text{rk}(Q) = 1 \) it is known that these are also sufficient conditions (see e.g. the properties of rank 2 stable vector bundles given in [3]). Our main results are the following theorems.

**Theorem 0.1** Let \( X \) be an elliptic curve. Fix integers \( r, s, a, b \) such that \( r \geq 1, s \geq 1 \) and \( as \leq br \). Then there exists an exact sequence (1) of vector bundles on \( X \) with \( \text{rk}(H) = r, \text{rk}(Q) = s, \text{deg}(H) = a, \text{deg}(Q) = b, \) and \( H, E \) and \( Q \) semistable.

**Theorem 0.2** Let \( X \) be a smooth complex projective curve of genus \( g \geq 2 \) such that there is a degree 2 map \( f : X \to Y \) with \( Y \) an elliptic curve. Fix integers \( r, s, a, b \) such that \( r \geq 1, s \geq 1 \) and \( as \leq br \). Then there exists an exact sequence (1) of vector bundles on \( X \) with \( \text{rk}(H) = r, \text{rk}(Q) = s, \text{deg}(H) = a, \text{deg}(Q) = b, \) and \( H, E \) and \( Q \) semistable.

**Theorem 0.3** Let \( X \) be a general smooth complex projective curve of genus \( g \geq 2 \). Fix integers \( r, s, a, b \) such that \( r \geq 1, s \geq 1 \) and \( as \leq br \). Then there exists an exact sequence (1) of vector bundles on \( X \) with \( \text{rk}(H) = r, \text{rk}(Q) = s, \text{deg}(H) = a, \text{deg}(Q) = b, \) and \( H, E \) and \( Q \) semistable.

In section 1 we will give a few general lemmata and deduce Theorem 0.3 from Theorem 0.2. In section 2 we will prove Theorem 0.1. In section 3 we will prove Theorem 0.1.

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## 1 General remarks

We work in characteristic 0; in this case we may use the fact that, if \( h : U \to W \) is a finite morphism of smooth curves and \( F \) is a semistable vector bundle on \( W \), then \( h^*(F) \) is semistable.

**Remark 1.1** Let \( X \) be a smooth projective curve of genus \( g \). It is well known that, for any bounded family of vector bundles on \( X \) of fixed rank and degree, there exists a family of vector bundles on \( X \) containing all members
of the given family and parametrised by an irreducible variety. For reader’s sake we will give the following short proof; up to a twist by a line bundle we may assume that every such bundle is spanned; since $\dim(X) = 1$, any such $r$–bundle, $E$, is spanned by $r + 1$ global sections, i.e. there is a surjection $h : \mathcal{O}_X^{r+1} \to E$; the vector bundle $\ker(h)$ is isomorphic to $\text{det}(E)^*$ and hence any such bundle $E$ appears as cokernel of a family of embeddings of a family of line bundles of fixed degree into the trivial bundle of rank $r + 1$. If $g > 0$ there exist semistable vector bundles on $X$ of any rank and any determinant. Hence, by the openness of semistability, it follows that every vector bundle on $X$ is the flat limit of a family of semistable bundles with the same determinant. If $g \geq 2$, the same holds with ‘semistable’ replaced by ‘stable’.

**Lemma 1.2** Fix a flat family $\{X(t), t \in T\}$ of smooth genus $g$ curves and flat families $H(t), Q(t)$ of vector bundles on the family $\{X(t), t \in T\}$ with $T$ integral. Assume the existence of $0 \in T$ such that $\text{Hom}(Q(0), H(0)) = 0$. Then any extension of $Q(0)$ by $H(0)$ is the flat limit of extensions of $Q(t)$ by $H(t)$ for $t$ in some open subset of $T$ containing $0$.

**Proof.** By semicontinuity we can find an open subset $T_0$ of $T$ containing $0$ such that $\text{Hom}(Q(t), H(t)) = 0$ for all $t \in T_0$. Hence by Riemann–Roch $h^1(\text{Hom}(Q(t), H(t)))$ is independent of $t \in T_0$. It follows that there exists a vector bundle $V$ over $T_0$ whose fibre over $t$ can be identified with $H^1(\text{Hom}(Q(t), H(t)))$; moreover $V$ is the base of a family parametrising all extensions of $Q(t)$ by $H(t)$ for $t \in T_0$ (see [3], Cor. 3.3 ). The result follows.

**Corollary 1.3** Fix a flat family of curves as in Lemma 1.2 and let $H, Q$ be vector bundles on $X$ such that $\text{Hom}(Q, H) = 0$. Then any extension is the limit of a flat family of extensions

$$0 \to H(t) \to E(t) \to Q(t) \to 0$$

with $H(t)$ and $Q(t)$ semistable. Moreover, if $E$ is semistable, we can suppose that $E(t)$ is semistable.

**Proof.** This follows from remark 1.1, Lemma 1.2 and the openness of semistability.

**Proof of Theorem 0.3 assuming Theorem 0.2.** If $as = br$, we can choose $H$ and $Q$ semistable and then take $E = H \oplus Q$. If $as < br$, choose a
flat family of curves as in Lemma 1.2 such that $X(t)$ is a general curve of genus $g$ for $t \neq 0$, while $X(0)$ is bielliptic. Choose an extension $[1]$ on $X(0)$ and note that, since $H$ and $Q$ are semistable and $\mu(H) < \mu(Q)$, we have $\text{Hom}(Q, H) = 0$. The result now follows by corollary 1.3.

**Lemma 1.4** Let $F$ and $G$ be vector bundles on $X$ and set

$t := \max_{f \in \text{Hom}(F,G)} \text{rk}(\text{Im}(f))$ \quad $h := \max_{f \in \text{Hom}(F,G), \text{rk}(\text{Im}(f)) = t} \deg(\text{Im}(f))$.

Then the subset of $\text{Hom}(F,G)$

$U := \{ f \in \text{Hom}(F,G) \mid \text{rk}(\text{Im}(f)) = t, \deg(\text{Im}(f)) = h \}$

is open and dense in $\text{Hom}(F,G)$ and $\text{det}(\text{Im}(f))$ is constant for $f \in U$.

**Proof.** Let $U' := \{ f \in \text{Hom}(F,G) \mid \text{rk}(\text{Im}(f)) = t \}$. By the semicontinuity of the function ‘rank’, $U'$ is open and dense in $\text{Hom}(F,G)$. Now, the restriction to $U'$ of the function ‘degree’ is semicontinuous; hence $U$ is open and dense in $\text{Hom}(F,G)$. Since $\text{Hom}(F,G)$ is rational, the morphism from $U$ to $\text{Pic}^h(X)$ mapping $f$ into $\text{det}(\text{Im}(f))$ is constant.

## 2 Elliptic curves; proof of Theorem 0.1

We suppose throughout this section that $X$ is an elliptic curve. For general facts about vector bundles on $X$, see [1].

**Remark 2.1** Let $E$ be a vector bundle on the elliptic curve $X$ with $\deg(E) > 0$. Then, by Riemann-Roch, $h^0(E) > 0$.

**Remark 2.2** Recall that a polystable vector bundle on a smooth curve is a direct sum of stable vector bundles with the same slope. Note that, for any rank $r$ and degree $d$, there exist polystable bundles on $X$. In fact, if $(r, d) = l$, it follows from the results of [1], theorem 7 and 10, pg. 433 and 442, that there exist non-isomorphic stable bundles $E_1, \ldots, E_l$ of rank $1$ and degree $d$. Then $E_1 \oplus \ldots \oplus E_l$ is polystable and no two among its indecomposable factors are isomorphic. Moreover $E_1, \ldots, E_l$ can be chosen such that $E_1 \oplus \ldots \oplus E_l$ has any given determinant. Note also that any subbundle of $E$ with the same slope as $E$ is a direct factor of it.

In order to prove theorem 0.1 we need the following fundamental lemma on polystable vector bundles.
Proposition 2.3  Let $F$, $G$ be polystable vector bundle on an elliptic curve $X$ with $rk(F) \geq rk(G)$ and $\mu(F) < \mu(G)$. Assume that no two among the indecomposable factors of $F$ (resp. of $G$) are isomorphic. Let $f \in U$, where $U$ is defined as in Lemma 1.4. Then

(a) We have $rk(\text{Im}(f)) = rk(G)$.
(b) If $rk(F) > rk(G)$, then $f$ is surjective.

Let us notice that (a) and (b) have the following ‘dual’ version

(a') If $rk(\text{F}) \leq rk(G)$ a general $f \in \text{Hom}(F, G)$ is injective.
(b') If $rk(F) < rk(G)$ then a general $f \in \text{Hom}(F, G)$ is injective and $\text{coker}(f)$ is torsion free.

Proof.  We will use the same notation as in Lemma 1.4. Since $\text{Hom}(F, G)$ has slope $\mu(G) - \mu(F) > 0$, we have $\text{Hom}(F, G) \neq 0$ by remark 2.1.

First we prove that if (a) is true then (b) follows. Let us fix $F$ and an integer $s$. Then consider vector bundles $G$ of rank $s$. Since (a) holds we may suppose $rk(\text{Im}(F)) = rk(G) = s$. Then choose $d_0 \in \mathbb{Z}$ such that $(d_0 - 1)/s \leq \mu(F) < d_0/s$. We prove first that (b) holds if $\text{deg}(G) = d_0$. In this case, we have, since $F$ is polystable,

$$\frac{(d_0 - 1)}{rk(G)} \leq \mu(F) \leq \mu(\text{Im}(f)) \leq \frac{d_0}{rk(G)}.$$ 

If $\mu(\text{Im}(F)) = d_0/rk(G)$, we are done. Otherwise, we have $(d_0 - 1)/rk(G) = \mu(F) = \mu(\text{Im}(f))$; hence by remark 2.2 $\text{Im}(f)$ is a direct factor of $F$, say $F \cong \text{Im}(f) \oplus B$. By the assumption on the indecomposable factors of $F$ we have $\text{Hom}(B, \text{Im}(f)) = 0$, while by remark 2.1 $\text{Hom}(B, G) \neq 0$. Hence there is $u : B \to G$ with $\text{Im}(u)$ not contained in $\text{Im}(f)$. Define $v : F \to G$ with $v|B = u$ and sending the factor $\text{Im}(f)$ of $F$ to the subsheaf $\text{Im}(f)$ of $G$ via the identity. By construction $\text{Im}(v)$ strictly contains $\text{Im}(f)$. Since $rk(G) = s = rk(\text{Im}(v))$ then $\text{deg}(\text{Im}(v)) > \text{deg}(\text{Im}(f))$ which is absurd because $f \in U$. This proves (b) in the case $\text{deg}(G) = d_0$. To complete the proof of (b), we check by induction that the set

$$B := \{ d \in \mathbb{N} \mid d \geq d_0 \text{ and } \forall \text{ polystable vector bundle } G \text{ of rank } s \text{ and degree } d, \text{ a general } f : F \to G \text{ is surjective}\}.$$ 

is the set of all the integers greater than or equal to $d_0$. By the previous argument $d_0 \in B$. Then we assume $d-1 \in B$ and prove that $d \in B$. Suppose $d \notin B$. By the inductive step, for any $L \in \text{Pic}^0(X)$ and for every polystable vector bundle, $T$, of rank $s$ and degree $d - 1$ the general map $F \to T \otimes L$
is surjective. Since (a) and (a') hold then the general map $T \otimes L \to G$ is injective. Then the composition $f_L : F \to G$ has image of rank $s$ and degree $d - 1$. If $h$ is the maximum defined in lemma [14], then $h = d - 1$ and hence $f_L \in U$. Now we can vary $L \in \text{Pic}^0(X)$ (which is infinite) and find images of maps $f_L \in U$ with non-isomorphic determinants. This contradicts Lemma [14].

Finally let us prove part (a). We will use induction on $\text{rk}(G)$. If $\text{rk}(G) = 1$, the first assertion is obvious. Suppose $\text{rk}(G) > 1$ and the inductive step. In order to obtain a contradiction we assume $t < \text{rk}(G)$. Now the proof is divided in two cases. First assume $\frac{h}{t} > \mu(F)$. By remark [2.2] we can fix a polystable bundle $M$ with $\text{rk}(M) = t$, $\text{deg}(M) = h$ and factors which are mutually non-isomorphic. Then by the inductive step, for any $L \in \text{Pic}^0(X)$ the rank of the image of the general map $F \to M \otimes L$ is $t$. Since (a) implies (b), the image is, in fact, all of $M \otimes L$. If $\frac{h}{t} < \mu(G)$, using the dual form of the statement of the theorem, the inductive step applies to the pair $(M \otimes L, G)$.

Then we get by composition a map $f_L : F \to G$ which has image in $G$ with maximal degree and rank. Varying $L \in \text{Pic}^0(X)$ we get a contradiction. If $\frac{h}{t} = \mu(G)$, we can choose $f \in U$ so that $\text{Im}(f)$ is a direct factor of $G$. Let $l = (h, t)$. Then we can write $\text{Im}(f) = G_1 \oplus \ldots \oplus G_l$ and $G = G_1 \oplus \ldots \oplus G_m$ with $l < m$ and all $G_j$ stable and non-isomorphic. By [4], theorem 7 and 10, pg. 434 and 442], the bundles $\text{det}(G_j)$ are all non-isomorphic. Hence $G' = G_1 \oplus \ldots \oplus G_{l-1} \oplus G_{l+1}$ has a different determinant from $G_1 \oplus \ldots \oplus G_l$. But, by the inductive hypothesis there exists a surjective homomorphism $f' : F \to G'$. Since $f$ and $f'$ both belong to $U$, this contradicts Lemma [14].

Now assume $\frac{h}{t} = \mu(F)$. Since $F$ is polystable, then $\text{Im}(f)$ is a direct factor of $F$, say $F \simeq \text{Im}(f) \oplus B$. By the assumption on the indecomposable factors of $F$ and the semistability of its factors, we have $\text{Hom}(B, \text{Im}(f)) = 0$, while $\text{Hom}(B, G) \neq 0$ by remark [2.1]. Hence there is $f' : F \to G$ with $f'|B \neq 0$ and $f'|\text{Im}(f) = f$. Thus $\text{rk}(f') > \text{rk}(f)$, a contradiction.

**Corollary 2.4** Let $F$ and $G$ be polystable vector bundles with non-isomorphic factors, $\text{rk}(F) = \text{rk}(G)$ and $\text{deg}(F) = \text{deg}(G) + 1$. Then $G$ is a subsheaf of $F$. Furthermore a general positive elementary transformation of $G$ is semistable.

**Proof.** Apply the previous proposition to the pair $(G, F)$. The last assertion follows from the openness of semistability.

**Remark 2.5** Applying twice corollary [2.4] one may obtain for every pair $G'$ and $F'$ of polystable vector bundles with non-isomorphic factors, the same rank and $\text{deg}(G') - \text{deg}(F') = 2$, that the generic map, $G' \to F'$, is an inclusion.
Proof of Theorem 0.1. If \( as = br \), take any \( H \) and \( Q \) semistable and set 
\[ E := H \oplus Q. \]
If \( as < br \), by Proposition 2.3 there exist polystable vector bundles 
\( E', Q' \) with non-isomorphic factors and a surjection \( u : E' \to Q' \) 
with \( \text{rk}(E') = r + s, \deg(E') = a + b, \text{rk}(Q') = r, \deg(Q') = b. \)
To prove the theorem it is sufficient to show that for general \((E', Q', u)\), \( \text{Ker}(u) \) is semistable. Since \( E' \) is semistable and 
\( \mu(E') < \mu(Q') \), \( \text{Hom}(Q', \text{Ker}(u)) = 0. \)
The result follows from corollary 1.3.

Remark 2.6 Since the condition for a bundle to be polystable with non-
isomorphic factors is preserved for general deformation, what we get at the 
end of the proof of Theorem 0.1 is an exact sequence in which the last two 
terms are polystable vector bundles with non-isomorphic factors while the 
first one is semistable.

3 Proof of theorem 0.2

Let \( X \) be a curve of genus \( g \) and \( f : X \to Y \) a 2 : 1 morphism on an elliptic 
curve \( Y \). Let \( \sigma : X \to X \) be the involution corresponding to the morphism 
\( f \), i.e. with \( Y = X/\sigma \). First let us notice that if \( as = br \), we can choose \( H \) 
and \( Q \) semistable and set \( E := H \oplus Q. \) Hence from now on, we may assume 
\( as < br \). In the proof the following remark will turn out to be quite useful.

Remark 3.1 Let \( U \) be a vector bundle on \( X \). By the descent theory \( U \) is 
of the form \( f^*(F) \) with \( F \) a vector bundle on \( Y \) if and only if the following 
two conditions are satisfied:

i) \( U \) is \( \sigma \) invariant,

ii) for every ramification point \( Q \) of \( f/\sigma \) acts as the identity on the fiber 
\( U_Q \).

Notice that a saturated subbundle of \( f^*(F) \) satisfies condition ii). Hence 
a saturated subbundle \( A \) of \( f^*(F) \) is of the form \( f^*(B) \) with \( B \) a vector 
bundle on \( Y \) if and only if \( A \) is \( \sigma \)-invariant. Furthermore, the saturation in 
\( f^*(B) \) of a \( \sigma \)-invariant subsheaf of \( f^*(B) \) has even degree.

In order to prove the theorem we need the following fundamental lemmata.

Lemma 3.2 Let \( M \) and \( N \) be semistable vector bundles on \( Y \) with \( \text{rk}(M) = \text{rk}(N), \deg(M) = \deg(N) + 1 \) and \( N \hookrightarrow M. \) Set \( G = f^*(N) \) and \( T = f^*(M). \) 
Then every vector bundle \( E \) with \( G \hookrightarrow E \hookrightarrow T \) and \( \text{length}(E/G) = 1 \), is 
semistable.
Lemma 3.3 Let $M$ and $N$ be semistable vector bundles on $Y$ with $rk(M) = rk(N) = deg(M) = deg(N) + 2$ and $N \hookrightarrow M$. Set $G = f^*(N)$ and $T = f^*(M)$. Then every vector bundle $E$ with $G \hookrightarrow E \hookrightarrow T$ and length $(E/G) = 2$, is semistable.

The proof of lemma 3.2 turns out to be a special case of the proof of lemma 3.3. Several situations of the latter do simplify in the form $er$ (for example $A$ is always saturated in the proof of lemma 3.2). Hence we will present only the proof of lemma 3.3.

Proof of 3.3. Since the field characteristic is 0, then $G$ and $T$ are semistable. Let $E$ be a subsheaf of $T$ containing $G$ and with length $(T/E) = 2$. In order to find a contradiction we assume that $E$ is not semistable. Then there exists a stable proper subbundle $A$ of $E$ with $\mu(A) > \mu(E)$. Since $\mu(G) = \mu(E) - 2/(r + s)$ and $E''$ is semistable, we see that $A \cap E'' \neq A$. Hence there exists an inclusion $A \hookrightarrow T$. Moreover note that $A \neq \sigma(A)$ otherwise $A = f^*(N)$ would be contained in $E \cap \sigma(E)$ which is impossible because the latter is contained in $E''$. Now our intent is to prove that $A \cap \sigma(A)$ and $A + \sigma(A)$ are saturated in $T$. In order to prove this, we distinguish two cases depending if $A$ is saturated in $T$ or not.

If $A$ is saturated in $T$ one has $A \cap \sigma(A)$ saturated in $T$. In fact the saturation $F$ of $A \cap \sigma(A)$ is contained in $A$ and since it is $\sigma$-invariant, it is also contained in $\sigma(A)$. Then the saturation coincides with $A \cap \sigma(A)$. Anyway in this case one might still have that $A + \sigma(A)$ is not saturated in $T$. Then one observes that by the saturation of $A \cap \sigma(A)$, $deg(A + \sigma(A))$ is even. Hence setting $K$ the saturation of $deg(A + \sigma(A))$, one gets

$$
\mu(K)rk(K) = deg(K) \geq \mu(A + \sigma(A)) + 2 = \mu(A + \sigma(A))rk(A + \sigma(A)) + 2
$$

$$
\geq \mu(A)rk(A + \sigma(A)) + 2 > \mu(E)rk(A + \sigma(A)) + 2 = \mu(T)rk(A + \sigma(A)) -
$$

$$
\frac{2rk(A + \sigma(A))}{rk(E)} + 2,
$$

which contradicts the semistability of $T$.

Suppose now that $A$ is not saturated in $T$. Let $A''$ be its saturation in $T$. Call $A'$ the first term of the Harder-Narasimhan filtration of $A''$. If $A''$ is not semistable, then $A'$ is a proper subbundle of $A''$. In any case we get a semistable subbundle $A'$ of $T$ of slope $\mu(A') \geq \mu(A) + \frac{1}{rk(A)} > \mu(E) + \frac{1}{rk(A)}$. Then the inequalities

$$
\mu(A') \geq \mu(A) + \frac{1}{rk(A)} > \mu(E) + \frac{1}{rk(A)}.
$$
Proof of the Claim: get a contradiction and the theorem is proven.

Assume the claim and consider the exact sequence

\[ f \]

By construction we have

\[ H \]

Hence degree of \( B \) may assume \( d \) and the projection formula we assume that \( \operatorname{rk}(A) = 2\) and \( \operatorname{rk}(A) = 2 \).

We have

\[ \text{Claim:} \quad \mu(K) \operatorname{rk}(K) = \deg(K) \geq \deg(A' + \sigma(A')) + 1 \geq \mu(A) \operatorname{rk}(A + \sigma(A)) + \]

\[ + \frac{\operatorname{rk}(A + \sigma(A))}{\operatorname{rk}(A)} + 1 > \mu(T) \operatorname{rk}(A + \sigma(A)) - 2 \frac{\operatorname{rk}(A + \sigma(A))}{\operatorname{rk}(E)} + \frac{\operatorname{rk}(A + \sigma(A))}{\operatorname{rk}(A)} + 1 \]

\[ \geq \mu(T) \operatorname{rk}(A + \sigma(A)) + 2 \frac{\operatorname{rk}(E) - \operatorname{rk}(A)}{\operatorname{rk}(E)} \]

contradict the semistability of \( T \) and they imply the saturation of \( A + \sigma(A) \). Hence degree of \( A \cap \sigma(A) \) is even. Therefore

\[ \mu(F) \operatorname{rk}(F) = \deg(F) \geq \deg(A' \cap \sigma(A')) + 2 = 2 \mu(A') \operatorname{rk}(A) - \]

\[ - \mu(A' + \sigma(A')) \operatorname{rk}(A + \sigma(A)) + 2 \geq 2 \mu(A) \operatorname{rk}(A) + 2 - \mu(T) \operatorname{rk}(A + \sigma(A)) + \]

\[ + 2 > 2 \mu(E) \operatorname{rk}(A) - \mu(T) \operatorname{rk}(A + \sigma(A)) + 4 = \mu(T) \operatorname{rk}(A \cap \sigma(A)) - \frac{\operatorname{rk}(A)}{\operatorname{rk}(E)} + 4 \]

which again contradicts the semistability of \( T \) implying the saturation of \( A \cap \sigma(A) \). At this point we know that even if \( A \) is not saturated we have \( A' \cap \sigma(A') \simeq f^*(U) \), \( A' + \sigma(A') \simeq f^*(B) \). Observe that \( A + \sigma(A) \) is saturated whenever \( A \cap \sigma(A) \) is zero or not. Once we know this we can prove that \( A \cap \sigma(A) \neq 0 \), indeed. In fact in order to prove a contradiction we assume that \( \operatorname{rk}(A + \sigma(A)) = 2 \operatorname{rk}(A) \). Thus \( f^*(B) \) splits. We may assume \( B \) indecomposable because \( \operatorname{rk}(A) \) is minimal. Note that the branch locus \( B(f) \) of \( f \) is not empty because \( g > 1 \). By the projection formula we have \( f^*(\operatorname{End}(f^*(B))) \simeq \operatorname{End}(B) \oplus \operatorname{End}(B)(-B(f)) \). Since \( \deg(B(f)) > 0 \) and \( \operatorname{End}(B)(-B(f)) \) is semistable of degree less then zero, we have \( H^0(f^*(\operatorname{End}(f^*(B))) \simeq H^0(\operatorname{End}(B)) \), contradicting the indecomposability of \( B \).

Claim: We have \( h^0(X, \operatorname{End}(f^*(B/U))) = h^0(Y, \operatorname{End}(B/U)) \).

Assume the claim and consider the exact sequence

\[ 0 \to A \cap \sigma(A) \to A + \sigma(A) \to A + \sigma(A)/A \cap \sigma(A). \]

By construction \( f^*(B/U) = A + \sigma(A)/A \cap \sigma(A) \) splits into two direct factors \( A/A \cap \sigma(A) \) and \( \sigma(A)/A \cap \sigma(A) \). The projection of \( f^*(B/U) \) into its factor \( A/A \cap \sigma(A) \) does not come from an element of \( H^0(Y, \operatorname{End}(B/U)) \). Hence we get a contradiction and the theorem is proven.

Proof of the Claim: By the projection formula we have

\[ h^0(X, \operatorname{End}(f^*(B/U))) = h^0(Y, \operatorname{End}(B/U)) + h^0(Y, \operatorname{End}(B/U)(-B(f))). \]
Hence it is sufficient to show the vanishing of $h^0(Y, \text{End}(B/U)(-B(f)))$. Since $Y$ is elliptic and \text{card}(B(f)) = 2g-2$ by Riemann-Hurwitz theorem, it is sufficient to show that either $B/U$ is indecomposable or that the difference, $t$, between the highest and the lowest slope of its direct factors is less than $2g-2$. Suppose $B/U$ decomposable. Let us prove that $t < 2$ and hence $t < 2g-2$ for $g \geq 2$. In order to obtain a contradiction we assume the existence of a direct factor $A'$ of $B/U$ of slope $\geq \mu(A)/2+2$. Define $B' = B/U = A' \oplus B'$. Since $A$ is stable and $f^*(B/U)$ is a quotient of $A \oplus \sigma(A)$, every direct factor of $f^*(B/U)$ has slope $\geq \mu(A)$. Hence our assumption implies that $\text{deg}(B/U) = \mu(A) \text{rk}(A')/2 + 2 \text{rk}(A') + \mu(A) \text{rk}(B')/2 \geq \mu(A) \text{rk}(B/U)/2 + 2$. Moreover

$$
\text{deg}(A + \sigma(A)) = \text{deg}(f^*(B/U)) + \text{deg}(A \cap \sigma(A)) = \text{deg}(f^*(B/U)) + 2 \text{deg}(A) - \text{deg}(A + \sigma(A)).
$$

Hence

$$
\text{deg}(A + \sigma(A)) = \frac{\text{deg}(f^*(B/U))}{2} + \text{deg}(A) \geq \frac{\mu(A) \text{rk}(B/U)}{2} + 2 + \mu(A) \text{rk}(A) = \mu(A) \text{rk}(A + \sigma(A)) + 2.
$$

Since $\mu(A) > \mu(T) - 2/\text{rk}(T)$, and $T$ is semistable, we obtain a contradiction of the semistability of $T$. Hence every direct factor $A''$ of $B/U$ has $\mu(A)/2 < \mu(A'') < \mu(A)/2 + 2$. Hence $t < 2$, as wanted.

**Corollary 3.4** Let $N$ be a polystable bundle on $Y$. Set $G = f^*(N)$. Then there exists a semistable bundle $E$ containing $G$ with $\text{length}(E/G) = 1$.

**Proof of 3.4.** By remark 2.2 and corollary 2.4, we can find a polystable bundle $M$ on $Y$, with non-isomorphic factors, containing $N$, with $\text{length}(M/N) = 1$. Hence we can find a semistable bundle $T = f^*(M)$ and an inclusion $G \rightarrow T$ compatible with $\sigma$ and with $\text{length}(T/G) = 2$. Set $\text{Supp}(T/G) = \{P, \sigma(P)\}$. Since semistability is an open condition, we may assume $\sigma(P) \neq P$. Then there exists a vector bundle $E$ with $G \hookrightarrow E \hookrightarrow T$, $\text{Supp}(E/G) = \{P\}$ and $\text{Supp}(T/E) = \{\sigma(P)\}$. Hence we may apply lemma 3.2. The same proof gives the following corollary

**Corollary 3.5** Let $N$ and $M$ be polystable bundles on $Y$ with $N \hookrightarrow M$ and $\text{length}(M/N) = 1$. Set $G = f^*(N)$ and $T = f^*(M)$. Then there exists a semistable bundle $E$ with $G \hookrightarrow E \hookrightarrow T$ and $\text{length}(E/G) = 1$. 

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**Corollary 3.6** Let $F'$ a polystable bundle on $Y$. Set $T = f^*(F')$. Then there exists a semistable bundle $E$ contained in $T$ with $\text{length}(T/E) = 1$.

**Proof.** The statement is just the dual version of corollary 3.4 or it derives directly by lemma 3.2.

**Proof of Theorem 0.1**: Unless otherwise specified, every exact sequence of vector bundles on $X$ will consist of a subbundle of rank $r$, a middle term of rank $r + s$ and a quotient of rank $s$.

The proof is divided into four cases according to the parity of the two integers $a$ and $b$.

i) Here we assume $a$ even, $b$ even. By theorem 0.1, we can find an exact sequence

$$0 \to H' \to E' \to Q' \to 0$$

of semistable vector bundles on $Y$ with $\deg(H') = a/2$, $\deg(Q') = b/2$. Set $H := f^*(H')$, $E := f^*(E')$ and $Q := f^*(Q')$. The pull-back of (2) gives the desired exact sequence.

ii) Here we assume $a$ even, $b$ odd. By remark 2.6 and corollary 2.4 we have on $Y$ the following diagram

$$
\begin{array}{ccc}
0 & 0 & 0 \\
0 & H' & F' \\
& \downarrow & \downarrow \\
& H' & E' \\
0 & 0 & 0 \\
\end{array}
$$

where $H'$ is a semistable vector bundle with $\deg(H') = a/2$, while $F'$, $Z'$, $E'$, $Q'$ are polystable bundles with non-isomorphic factors, $\deg(E') = \frac{a+b+1}{2}$, $\deg(Q') = \frac{b-1}{2}$, $\deg(F') = \frac{a+b-1}{2}$, $\deg(Z') = \frac{b+1}{2}$. Pulling back this diagram and applying corollary 3.5 to $j : F' \to E'$ and $i : Z' \to Q'$, one gets the desired exact sequence:

$$0 \to H \to E \to Q \to 0$$

of semistable bundles on $X$ with $\deg(H) = a$, $\deg(Q) = b$.

iii) Here we assume $a$ odd, $b$ even. Just dualize the previous result.

iv) Here we assume $a$ odd, $b$ odd. By remark 2.6, corollary 2.4, remark 2.5 and the snake lemma for the injectivity of the first vertical map, we have on
Y the following diagram

\[
\begin{array}{ccc}
0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow \\
0 & \rightarrow U' & \rightarrow F' & \rightarrow Z' & \rightarrow 0 \\
\downarrow & \downarrow & \downarrow \\
0 & \rightarrow H' & \rightarrow E' & \rightarrow Q' & \rightarrow 0
\end{array}
\]

where \( H' \) and \( U' \) are semistable bundles with \( \text{deg}(H') = \frac{a+1}{2} \) and \( \text{deg}(U') = \frac{a-1}{2} \), while \( F' \), \( Z' \), \( E' \) and \( Q' \) are polystable bundles with non-isomorphic factors and degrees \( \text{deg}(Z') = \frac{b-1}{2} \), and \( \text{deg}(Q') = \frac{b+1}{2} \). Then pullback the diagram to \( X \). Use corollary 3.5 for the vertical map on the right, finding a semistable bundle, \( Q \), with \( f^*(Z') \hookrightarrow Q \hookrightarrow f^*(z) \).

Now set \( \text{Supp}(f^*(E')/f^*(F')) = \{P, Q, \sigma(P), \sigma(Q)\} \). Then as in the proof of corollary 3.4, there exists a vector bundle \( E \) with \( f^*(F') \hookrightarrow E \hookrightarrow f^*(E') \) with \( \text{Supp}(E/f^*(F')) = \{P, Q\} \) and \( \text{Supp}(f^*(E')/E) = \{\sigma(P), \sigma(Q)\} \). Hence we may apply lemma 3.3.

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