A Closed-Form Localization Method Utilizing Pseudorange Measurements From Two Nonsynchronized Positioning Systems

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Abstract—In a time of arrival (TOA) or pseudorange-based positioning system, user location is obtained by observing multiple anchor nodes (ANs) at known positions. Utilizing more than one positioning systems, e.g., combining global positioning system (GPS) and BeiDou navigation satellite system (BDS), brings better positioning accuracy. However, ANs from two systems are usually synchronized to two different clock sources. Different from single-system localization, an extra user-to-system clock offset needs to be handled. Existing dual-system methods either have high computational complexity or suboptimal positioning accuracy. In this article, we propose a new closed-form dual-system localization (CDL) approach that has low complexity and optimal localization accuracy. We first convert the nonlinear problem into a linear one by squaring the distance equations and employing intermediate variables. Then, a weighted least-squares (WLS) method is used to optimize the positioning accuracy. We prove that the positioning error of the new method reaches Cramér–Rao lower bound (CRLB) in far-field conditions with small measurement noise. Simulations on 2-D and 3-D positioning scenes are conducted. Results show that, compared with the iterative approach, which has high complexity and requires a good initialization, the new CDL method does not require initialization and has lower computational complexity with comparable positioning accuracy. The numerical results verify the theoretical analysis on positioning accuracy, and show that the new CDL method has superior performance over the state-of-the-art closed-form method. Experiments using real GPS and BDS data verify the applicability of the new CDL method and the superiority of its performance in the real world.

Index Terms—Closed-form localization, dual systems, global navigation satellite system (GNSS), pseudorange, time of arrival (TOA).

I. INTRODUCTION

POSITION information is becoming more and more pivotal to many modern applications, including smart cities, autonomous vehicles, Internet of Things (IoT), emergency rescues, [1]–[4]. Among those pervasive positioning techniques, wireless localization systems are usually comprised of anchor nodes (ANs) at known locations and user devices (UDs) that need to be localized. Several measurement techniques, including time of arrival (TOA), time of flight (TOF), angle of arrival (AOA), received signal strength (RSS), etc., can be adopted for localization [5]–[15]. The TOF requires perfect synchronization between the UD and the AN, which may be costly to obtain. The TOA or pseudorange measurement does not need such synchronization and is currently one of the most widely adopted methods to determine the UD position due to its relative device simplicity and localization accuracy. A typical example of such a scheme is the widely used global positioning system (GPS).

Positioning techniques based on pseudorange measurements from a single system are extensively studied in the literature. They can be mainly categorized into two types, iterative methods and closed-form methods. Iterative methods based on Taylor series expansion are widely adopted [16]–[21]. They provide optimal positioning results that reach Cramér–Rao lower bound (CRLB). However, they require proper initialization and have high computational complexity.

A variety of closed-form localization methods, which have low computational complexity and require no initial guess, are developed. Schau and Robinson [22] employed the squared user distance as intermediate variable, and solved a quadratic equation to obtain the localization result. Smith and Abel [23] used the same intermediate variable and equations and it only applies for over-determined cases. Chan and Ho [24] proposed a two-step weighted least-squares (WLSs) estimator that achieves CRLB at the small noise level. Bancroft [25] employed the squared difference of the user position and clock offset as intermediate variable and obtains the localization result by finding the root of a quadratic equation containing this intermediate variable as unknown. Closed-form localization method based on the multidimensional scaling technique that utilizes a squared distance matrix is proposed in [26] and [27]. However, all of the above methods and their improved versions such as [28]–[34] are only applicable with measurements from synchronized ANs within a single system.

Utilizing more than one positioning systems provides navigation users with more measurements, and thus better availability and higher accuracy [35]–[38]. For example, combining other global navigation satellite systems (GNSSs) with GPS, such as Glonass, Galileo, and BeiDou navigation satellite...
system (BDS), which are under development or becoming available, can provide better positioning services. However, these systems have different designs and thus have different clock bases [39]. It causes positioning with multiple systems more challenging than in the single-system case. For the dual-system positioning case, iterative methods, which are modified from the single-system case by adding another clock offset term, are commonly adopted [16], [40], [41]. However, they still require proper initialization and have high complexity. A closed-form dual-system positioning algorithm is proposed by Juang and Tsai [42], in which the positioning problem is converted to finding the solution of the two clock offset terms. However, this method does not provide optimal localization result (as will be shown later in this article). Teng et al. [43] modified this method to simplify computation by reducing an unknown clock offset term. However, its estimate result is not optimal either. In addition, an extra measurement is required to reduce the clock term, making this method only applicable to over-determined cases, i.e., five instead of four measurements for 2-D and six instead of five for 3-D cases.

In this article, we propose a new closed-form dual-system localization (CDL) method. We first difference the pseudorange measurements with a reference AN from the same system to remove the clock offset term and form the time-difference-of-arrival (TDOA) measurements. Squaring operation is taken on the distance equations, and two intermediate variables containing the distances between the unknown user position and the reference ANs are employed, to convert the nonlinear problem into a linear one. After finding the solution of the two intermediate variables by solving a quadratic equation set, a WLS method is applied to obtain the user location. The covariance of the localization result is analyzed theoretically to evaluate its positioning accuracy. We prove that the analytic form of the localization error covariance is identical with CRLB under small measurement noise and far-field assumption. Simulations are conducted to compare the localization accuracy of the proposed new CDL method against existing representative methods. The numerical results show that the localization accuracy of the proposed CDL algorithm reaches CRLB under small noise and far-field conditions, and is better than that of the state-of-the-art method in [42]. Furthermore, we conduct experiments using real GPS and BDS data. Results show the feasibility and performance of the new method in the real world. Compared with the iterative method, the new CDL method does not require initialization, and the computational time reduces by about 40% with similar positioning accuracy.

This article is organized as follows. In Section II, the localization problem model for two nonsynchronized systems is formulated. A new localization algorithm for the dual-system case named CDL is proposed in detail in Section III. Then the position error covariance is analyzed and compared with CRLB in Section IV. Simulations and real-data experiment are conducted to evaluate the performance of the new CDL method compared with other methods in Section V. Finally, Section VI draws the conclusion of this article.

The main notations used in this article are summarized in Table I.

| lower case x | scalar         |
|--------------|----------------|
| bold lower case x | vector          |
| bold uppercase X  | matrix          |
| \( \tilde{x} \) | noisy version of a variable |
| \( \hat{x}, \tilde{X} \) | estimate of a variable |
| \( ||\tilde{x}|| \) | Euclidean norm of a vector |
| tr(\(X\)) | trace of a matrix |
| \( [X]_{i,j} \) | the i-th row and the j-th column of a matrix, respectively |
| \( \{x\}_i \) | entry at the i-th row and the j-th column of a matrix |
| diag(\{\}) | diagonal matrix with the elements inside |
| \( M,N \) | numbers of ANs of system A and B, respectively |
| \( O_{M \times N} \) | \( M \times N \) matrix with all-zero entries |
| \( 1_M \) | \( M \)-element vector filled with ones |
| \( 0_M \) | \( M \)-element vector filled with zeros |
| \( p_{A_i}, p_{B_j} \) | known position vectors of the i-th or j-th AN in system A and B, respectively |
| \( p_u \) | unknown position vector of the UD |
| \( \rho_{A_1}, \rho_{B_j} \) | pseudorange measurements between the UD and the i-th and j-th ANs in system A and B, respectively |
| \( r \) | physical distance between the UD and AN |
| \( b \) | clock offset caused distance between the UD and AN |
| \( \epsilon \) | pseudorange measurement noise |
| \( \sigma^2 \) | pseudorange measurement noise variance |
| \( l \) | unit line-of-sight (LOS) direction vector from the UD to AN |
| \( \mathcal{P} \) | Fisher information matrix |
| \( W \) | weighting matrix for CDL |
| \( Q \) | covariance matrix of TDOA measurements |

Fig. 1. Dual positioning systems setup. Systems A and B have independent clock sources. ANs have known positions. UD receives signals from ANs to localize itself.

II. PROBLEM STATEMENT

We consider two positioning systems denoted as system A and B, respectively, as depicted in Fig. 1. System A contains \( M \) ANs and system B contains \( N \) ANs, i.e., \( M \) and \( N \) pseudorange measurements can be obtained from systems A and B, respectively. The coordinates of all the ANs are known, which are denoted as \( p_{A_i} \) and \( p_{B_j} \) for the i-th AN in system A and the j-th AN in system B, respectively.
and the $j$th AN in system $B$, respectively, $i = 1, \ldots, M$ and $j = 1, \ldots, N$. The location of a UD, denoted as $p_u$, is the unknown to be determined. The dimension of all the position vectors is $K$ (e.g., $K = 2$ in 2-D case and $K = 3$ in 3-D case), i.e., $p_{A1}, p_{B1}, p_u \in \mathbb{R}^K$. Therefore, the distance between UD and AN are expressed by

$$ r_{Ai} = ||p_u - p_{Ai}|| $$

(1)

and

$$ r_{Bj} = ||p_u - p_{Bj}|| $$

(2)

where $r_{Ai}$ represents the true distance between UD and the $i$th AN in system $A$, and $r_{Bj}$ represents the true distance between UD and the $j$th AN in system $B$.

The ANs in both systems are synchronized to their own system clock source, i.e., clock sources $A$ and $B$ in Fig. 1. However, the two systems are not synchronized, i.e., the two system clock sources are independent to each other.

For the system shown in Fig. 1, there are usually two schemes to obtain TOA or pseudorange measurements. One is that the ANs transmit signals and the TOA is measured upon the UD reception. Another is a reverse to the first one, i.e., the UD transmits signal and ANs receive and measure the TOAs. Either way works and has real-world applications.

Without loss of generality, in this article, we suppose that the TOAs. Either way works and has real-world applications. Therefore, a natural idea is to remove them by differencing the pseudorange measurements with a common reference to form TDOA measurements.

Without loss of generality, the first AN in system $A$ and the first AN in system $B$ are selected as references. The differenced pseudorange measurement between the reference distance and other distances in system $A$ is given by

$$ \rho_{Ai} - \rho_{A1} = r_{AiA1} + \epsilon_{A,i} $$

(5)

where

$$ r_{AiA1} = r_{Ai} - r_{A1} $$

(6)

and $\epsilon_{A,i} = \epsilon_{A,i} - \epsilon_{A,1}$.

The relationship between the UD coordinates and the distance as given by (1) and (2) is nonlinear. In order to convert it to a linear relation, we take a square on (1) and (2), and then come to

$$ r_{Ai}^2 = r_{AiA1}^2 + 2r_{AiA1}r_{A1} + r_{A1}^2 = ||p_u||^2 - 2p_u^T p_{Ai} + ||p_{Ai}||^2. $$

(7)

In order to remove the squared term of the UD coordinates $||p_u||^2$, we substitute $i = 1$ into (7), and obtain

$$ r_{A1}^2 = ||p_u||^2 - 2p_u^T p_{A1} + ||p_{A1}||^2. $$

(8)

By subtracting (8) from (7), the squared UD coordinates are removed and it reads

$$ r_{AiA1}^2 + 2r_{AiA1}r_{A1} = ||p_{Ai}||^2 - ||p_{A1}||^2 + 2(p_{Ai}^T p_{A1} - p_{Ai}^T p_{A1}) = 0. $$

(9)

We put the unknown UD position to the left and all the rest terms to the right, and (9) becomes

$$ (p_{A1}^T - p_{Ai}^T)p_u = r_{AiA1}r_{A1} + \frac{1}{2} \left( r_{AiA1}^2 + ||p_{Ai}||^2 - ||p_{A1}||^2 \right). $$

(10)

Similarly, for system $B$, by replacing the subscript of “$A$” to “$B$,” we have

$$ (p_{B1}^T - p_{Bj}^T)p_u = r_{BjB1}r_{B1} + \frac{1}{2} \left( r_{BjB1}^2 + ||p_{Bj}||^2 - ||p_{B1}||^2 \right). $$

(11)

We use matrices and vectors to rewrite (10) and (11) into the collective form

$$ Gp_u = C[r_{A1}, r_{B1}]^T + h $$

(12)
where

\[
G = \begin{bmatrix}
    p_{A_1}^T - p_{A_2}^T \\
    \vdots \\
    p_{A_M}^T - p_{A_1}^T \\
    p_{B_1}^T - p_{B_2}^T \\
    \vdots \\
    p_{B_N}^T - p_{B_1}^T
\end{bmatrix}, \quad C = \begin{bmatrix}
    r_{A_2A_1} & 0 \\
    \vdots & \vdots \\
    r_{A_MA_1} & 0 \\
    0 & r_{B_1B_1} \\
    \vdots & \vdots \\
    0 & r_{B_NB_1}
\end{bmatrix}
\]

and

\[
h = \frac{1}{2} \begin{bmatrix}
    r_{2A_2A_1}^2 + \|p_{A_1}\|^2 - \|p_{A_2}\|^2 \\
    \vdots \\
    r_{2A_MA_1}^2 + \|p_{A_1}\|^2 - \|p_{A_M}\|^2 \\
    r_{2B_1B_2}^2 + \|p_{B_1}\|^2 - \|p_{B_2}\|^2 \\
    \vdots \\
    r_{2B_NB_1}^2 + \|p_{B_1}\|^2 - \|p_{B_N}\|^2
\end{bmatrix}
\]

At this stage, the linear relation of the unknown UD position \( \hat{p}_u \) with the distance variables \( r_{A_1} \) and \( r_{B_1} \) is obtained in (12). The AN positions \( p_{A_1} \) and \( p_{B_1} \) are known and the difference distances \( r_{A_MA_1} \) and \( r_{B_MB_1} \) can be approximated by \( \rho_{A_1} - \rho_{A_1} \) and \( \rho_{B_1} - \rho_{B_1} \), respectively. We treat these two distances of \( r_{A_1} \) and \( r_{B_1} \) as intermediate variables and find the solution of these, then the UD position can be computed using this linear relation of (12).

B. Identification of Intermediate Variables

The intermediate variables \( r_{A_1} \) and \( r_{B_1} \) will be solved in this section. First, by observing (8), we note that if \( \hat{p}_u \) is replaced by \( r_{A_1} \) and \( r_{B_1} \), an equation set with respect to the intermediate variables can be formed and solved. To this end, we then express the UD position by

\[
\hat{p}_u = (G^T G)^{-1} G^T \left( C[r_{A_1}, r_{B_1}]^T + h \right) \tag{13}
\]

where \( G \) has full column rank which is usually satisfied when there are sufficient amount of ANs with a proper geometry.

By substituting \( p_{u} \) from (13) into (8), a quadratic equation with the two intermediate variables is formed. We replace the subscript of \( A_1 \) in (8) with \( B_1 \), and substitute \( p_{u} \) from (13) into it again, another quadratic equation with the same two variables is obtained. These two quadratic equations are given by

\[
a_1 r_{A_1}^2 + b_1 r_{A_1} r_{B_1} + c_1 r_{B_1}^2 + d_1 r_{A_1} + e_1 r_{B_1} + f_1 = 0 \tag{14}
\]

and

\[
a_2 r_{A_1}^2 + b_2 r_{A_1} r_{B_1} + c_2 r_{B_1}^2 + d_2 r_{A_1} + e_2 r_{B_1} + f_2 = 0 \tag{15}
\]

where

\[
\begin{align*}
    a_1 &= [S]^T_{[1]} [S]_{[1]} - 1, & b_1 &= b_2 = 2[S]^T_{[1]} [S]_{[2]} \\
    c_1 &= [S]^T_{[2]} [S]_{[2]}, & d_1 &= [S]_{[1]} (g - p_{A_1}) \\
    e_1 &= 2[S]_{[2]} (g - p_{A_1}), & f_1 &= (g - p_{A_1})^T (g - p_{A_1}) \\
    a_2 &= [S]^T_{[1]} [S]_{[1]}, & c_2 &= [S]^T_{[2]} [S]_{[2]} - 1 \\
    d_2 &= 2[S]_{[1]} (g - p_{B_1}), & e_2 &= 2[S]_{[2]} (g - p_{B_1}) \\
    f_2 &= (g - p_{B_1})^T (g - p_{B_1})
\end{align*}
\]

in which the matrix \( S \) and vector \( g \) are defined as

\[
S \triangleq (G^T G)^{-1} G^T C
\]

and

\[
g \triangleq (G^T G)^{-1} G^T h.
\]

The quadratic equation set of (14) and (15) can be solved analytically. The approach is given in Appendix A. There are at most 4 roots of sets. We know that the intermediate variables \( r_{A_1} \) and \( r_{B_1} \) represent the distances between the UD and the ANs. They are thereby real and nonnegative values. Select these real and nonnegative roots as reasonable solutions to \( r_{A_1} \) and \( r_{B_1} \).

C. WLS Localization

After obtaining the intermediate variables \( r_{A_1} \) and \( r_{B_1} \), we can estimate \( \hat{p}_u \) in the expression of the intermediate variables by applying a WLS method to (12), and it comes to

\[
\hat{p}_u = (G^T W^{-1} G)^{-1} G^T W^{-1} \left(C[\tilde{r}_{A_1}, \tilde{r}_{B_1}]^T + h \right) \tag{16}
\]

where \( \tilde{r}_{A_1} \) and \( \tilde{r}_{B_1} \) represent the solutions from (14) and (15), \( \hat{p}_u \) represents the position result estimated from \( \tilde{r}_{A_1} \) and \( \tilde{r}_{B_1} \), and \( W \) is the weighting matrix.

Theorem 1: Under the condition of far field and small measurement noise, i.e., the squared error term \( o(e^2) \) is negligible, the weighting matrix \( W \) in (16) has the form of

\[
W = DQD \tag{17}
\]

where \( D = \text{diag}(r_{A_1}, \ldots, r_{A_M}, r_{B_1}, \ldots, r_{B_N}) \)

\[
Q_A = \begin{bmatrix}
    \sigma^2_{A_1} + \sigma^2_{A_2} & \sigma^2_{A_1} & \cdots & \sigma^2_{A_1} \\
    \sigma^2_{A_1} & \sigma^2_{A_1} + \sigma^2_{A_2} & \cdots & \sigma^2_{A_1} \\
    \vdots & \vdots & \ddots & \vdots \\
    \sigma^2_{A_1} & \sigma^2_{A_1} & \cdots & \sigma^2_{A_1} + \sigma^2_{A_M}
\end{bmatrix}
\]

and

\[
Q_B = \begin{bmatrix}
    \sigma^2_{B_1} + \sigma^2_{B_2} & \sigma^2_{B_1} & \cdots & \sigma^2_{B_1} \\
    \sigma^2_{B_1} & \sigma^2_{B_1} + \sigma^2_{B_2} & \cdots & \sigma^2_{B_1} \\
    \vdots & \vdots & \ddots & \vdots \\
    \sigma^2_{B_1} & \sigma^2_{B_1} & \cdots & \sigma^2_{B_1} + \sigma^2_{B_N}
\end{bmatrix}
\]

Proof: See Appendix B.

Theorem 1 gives the construction method for the weighting matrix \( W \) in (16). We note that this weighting matrix is related to the distances between UD and ANs, which form the matrix \( D \) in (17). A natural way to compute these distances in matrix \( D \) is using the UD position. However, at this stage, the UD position has not been found yet. Instead, the intermediate variables representing the distances from the UD to the reference ANs have been solved from the previous step. Under the condition of far field and small measurement noise, the measurement noise term in (5) is at least one order of magnitude smaller than
Algorithm 1 CDL

1. Input pseudorange measurements $\rho_A$ and $\rho_B$, and AN positions $p_A^i, i = 1, \ldots, M$, and $p_B^j, j = 1, \ldots, N$.
2. Linearization: Form matrix $G$ and $C$ and vector $h$ based on (5) and (12).
3. Identification of intermediate variables: Solve quadratic equations (14) and (15) and select the real and nonnegative roots $(\epsilon)$. Therefore, the distance errors by $\rho_{A1}$ and $\rho_{B1}$ can be approximated by the roots of the quadratic equations.
4. WLS localization: Compute candidate position results using (16). Select position result $\hat{p}_u$, that minimizes (20).
5. Output the selected position result.

The entire procedure of the proposed new method is identical with the CRLB using TDOA. It is proved in Appendix C.

The CRLB is usually used to evaluate the error variance of the position estimate that minimizes this sum is selected as the final result.

IV. ERROR ANALYSIS

The CRLB is usually used to evaluate the error variance of an unbiased estimator. In this section, we derive the CRLB of the dual-system localization case and compare the covariance of the localization error from the proposed new CDL method against CRLB.

A. CRLB for Dual-System Localization

The CRLB of the dual-system localization problem using TDOA measurements is derived as a benchmark. The CRLB relating to the parameter vector $\theta$ is defined as

$$\text{CRLB}(\theta) \triangleq F^{-1}(\theta)$$

where $F$ is the Fisher information matrix (FIM), and in the dual-system localization case, the parameter to be estimated is the user position, i.e., $\theta = p_u$.

The entry of FIM is expressed by

$$[F(\theta)]_{i,v} = -\mathbb{E} \left[ \frac{\partial^2 \ln p(\rho_D|\theta)}{\partial \theta_i \partial \theta_v} \right]$$

in which $p(\rho_D|\theta)$ is the likelihood function, and $\rho_D$ is a vector containing all TDOA measurements.

Therefore, when using TDOA measurements from dual systems as given by (5), the likelihood function is written as

$$p(\rho_D|\theta) = \exp\left(\frac{-1}{2} f(\theta)^T Q^{-1} f(\theta)\right)$$

where

$$f(\theta)_i = \begin{cases} \rho_{A1}-\rho_{A1} - \|\theta - p_{A1}\| + \|\theta - p_{A1}\|, & i = 1, \ldots, M - 1 \\ \rho_{B1}-\rho_{B1} - \|\theta - p_{B1}\| + \|\theta - p_{A1}\|, & i = M, \ldots, M + N - 2 \end{cases}$$

Then, we have

$$-\mathbb{E} \left[ \frac{\partial^2 \ln p(\rho_D|\theta)}{\partial \theta_i \partial \theta_v} \right] = \left(\frac{\partial f(\theta)}{\partial \theta} \right)^T Q^{-1} \frac{\partial f(\theta)}{\partial \theta}.$$
and

$$\hat{h} = \frac{1}{2} \begin{bmatrix} (r_{A2:B1} + \epsilon_{A2:B1})^2 + \|p_{A1}\|^2 - \|p_{A2}\|^2 \\
\vdots \\
(r_{B2:A1} + \epsilon_{B2:A1})^2 + \|p_{B1}\|^2 - \|p_{B2}\|^2 \\
(r_{B2:B1} + \epsilon_{B2:B1})^2 + \|p_{B1}\|^2 - \|p_{B2}\|^2 \end{bmatrix}.$$  

Without loss of generality, the first element of (26) can be derived as

$$[G(p_u + \Delta p_u)]_1 = \begin{bmatrix} \hat{c}_1 \end{bmatrix} + \begin{bmatrix} r_{A1} + \Delta r_{A1}, r_{B1} + \Delta r_{B1} \end{bmatrix}^T + \begin{bmatrix} \hat{h} \end{bmatrix}.$$  

By subtracting

$$[Gp_u]_1 = r_{A1}r_{A2}\hat{A}_1 + \frac{1}{2}(r_{A2:A1}^2 + \|p_{A1}\|^2 - \|p_{A2}\|^2)$$  

which is the first element of (12), from (27), we come to

$$[G\Delta p_u]_1 = r_{A2:A1}\Delta r_{A1} + r_{A2}\epsilon_{A2:A1} + \epsilon_{A2:A1}\Delta r_{A1} + \frac{1}{2}\epsilon_{A2:A1}^2.$$  

(28)

Given the condition of small measurement noise and far field, the distance error $\Delta r_{A1}$ is equal to the projection of the UD position error $\Delta p_u$ onto the LOS direction. This relationship is given by

$$\Delta r_{A1} = -\hat{t}_{A1}\Delta p_u.$$  

(29)

We substitute (29) into (28), expand $[G\Delta p_u]_1$, ignore the quadratic error terms, move all the $\Delta p_u$ terms to the left of the equation, and then come to

$$[G\Delta p_u]_1 + r_{A2:A1}\hat{t}_{A1}\Delta p_u = (p_{A1}^T - p_{A2}^T)\Delta p_u + r_{A2} - r_{A1})\hat{t}_{A1}\Delta p_u$$

$$= (p_{A1}^T - p_{A2}^T - p_{A1}^T)\Delta p_u + r_{A2} - r_{A1})\hat{t}_{A1}\Delta p_u$$

$$= (r_{A1}\hat{t}_{A1} - r_{A2}\hat{A}_1)\Delta p_u + (r_{A2} - r_{A1})\hat{t}_{A1}\Delta p_u$$

$$= r_{A2}\hat{t}_{A1} - r_{A2}\hat{A}_1)\Delta p_u$$

$$= \epsilon_{A2:A1}\hat{t}_{A1}.$$  

(30)

The distance error $\Delta r_{B1}$ is treated similarly as (29), and thus the other elements of $G\Delta p_u$ can be processed similarly as (30). After eliminating the $\hat{A}_1$ term on both sides of (30), we then write it in vector form as

$$H\Delta p_u = \epsilon_D.$$  

(31)

where

$$H = \begin{bmatrix} I_{A1} - \hat{t}_{A1}^T \\
I_{A2} - \hat{t}_{A2}^T \\
I_{B1} - \hat{t}_{B1}^T \\
I_{B2} - \hat{t}_{B2}^T \end{bmatrix}, \quad \epsilon_D = \begin{bmatrix} \epsilon_{A2:A1} \\
\epsilon_{A2:B1} \\
\epsilon_{B2:A1} \\
\epsilon_{B2:B1} \end{bmatrix}.$$  

We note that the covariance of $\epsilon_D$ is given by $Q$ as defined in (17). Hence, the covariance of $\Delta p_u$ is written as

$$\mathbb{E}[\Delta p_u\Delta p_u^T] = (H^T Q^{-1} H)^{-1}.$$  

(32)

It can be observed that (32) is identical with the inverse of (24). Thus, we have proved that the proposed new CDL method reaches CRLB under the condition of far field and small measurement noise.

V. PERFORMANCE EVALUATION

In this section, after the evaluation metrics are briefly introduced, simulation tests as well as real data experiment are carried out to evaluate the performance of the new CDL method. The iterative method using the TOAs [19], which is commonly adopted in many applications such as GNSS receivers, is selected as one of the comparison methods. The state-of-the-art closed-form dual-system method proposed by Juang and Tsai [42] (referred to as Juang’s method hereinafter) is implemented as another comparison. The computational platform running the following simulations is MATLAB R2019b on a PC with Intel Core i5-4590 CPU @3.3 GHz and 32-GB RAM.

A. Localization Performance Metrics

The root-mean-square error (RMSE) of the localization results is used to evaluate the positioning accuracy in the simulation tests. It is given by

$$\text{RMSE} = \sqrt{\frac{1}{N_s} \sum_{u=1}^{N_s} \|p_u - \hat{p}_u\|^2}.$$  

(33)

where $N_s$ is the total number of simulation runs.

CRLB is used as a benchmark to assess the localization accuracy. In this 2-D scene, the position error lower bound derived from CRLB is written as

$$\text{error}_{\text{CRLB}} = \sqrt{\text{CRLB}(p_u) + \text{CRLB}(p_{u2})}.$$  

(34)

For 3-D cases, the position error bound is similar to (34) but the term representing the third axis is added.

B. 2-D Simulation

We first create a 2-D simulation scene with four ANs from system $A$ and four ANs from system $B$. As shown in Fig. 2, the ANs are placed on a plain at the sides and corners of a square area with a side length of 200 m. All the positions of ANs are known without error. UD is placed randomly in a square region with a side length of 40 m. To ensure the far-field assumption
for the proposed method, the UD area is placed in the middle of the area. We set the σ of the pseudorange measurement noise varying from 0.1 m to 10 m with a step of 0.9 m. Thus, there are 12 steps in total. At every step, we conduct 1500 Monte Carlo simulations with uniformly distributed random positions of UD inside the gray region.

The position error result with varying measurement noise is depicted in Fig. 3. The errors from Juang’s method and iterative method are included for comparison. Theoretical position error lower bound from CRLB is computed based on (34). It can be observed that, under the condition of small noise and far field, the positioning accuracy of the proposed method reaches CRLB. The position error of the proposed method is close to that of the iterative method and is smaller than that of Juang’s method throughout the measurement noise varying range. When the measurement noise increases, all three methods show degraded positioning accuracy. The proposed method performs slightly worse than the iterative method in terms of positioning accuracy, but still outperforms Juang’s method since its RMSE is closer to CRLB. This result validates the feasibility of the proposed method in dual-system case and verifies the theoretical error analysis in the previous section.

To evaluate computational complexity, we compare the running time of the new CDL method, the conventional iterative approach, and Juang’s method. The Monte Carlo simulation consists of 18000 calls for each algorithm. The total running times for the Monte Carlo simulation run of the new CDL method, Juang’s method, and the iterative method are 3.38, 3.48, and 6.49 s, respectively. The computation time of the new CDL method is the least among the three methods. Compared with the iterative method, the low complexity of the new CDL method mainly attributes to the noniterative feature of the proposed method. Additional Monte Carlo simulations give consistent results showing that the new CDL method has the least computational complexity. Thus, from Fig. 3, it can be seen that the CDL method can obtain similar positioning accuracy with much lower computational complexity compared with the conventional iterative method.

C. 3-D Simulation

A 3-D simulation scene is created to evaluate the positioning performance of the new CDL method in 3-D case. There are four ANs from system A and six ANs from system B. UD is placed randomly in a cubic region with a size of 40 m × 40 m × 40 m centered at (100, 100, 20) m. The locations of ANs and UD are shown in Fig. 4. The range measurement noise σ is varying from 0.1 m to 10 m in this simulation. 1500 Monte Carlo simulations with a random position of UD inside the UD region are conducted for each step.

The position errors of the CDL method are illustrated in Fig. 5. It can be seen that, with small measurement noise, the localization error of the new CDL method reaches CRLB. The localization error of the CDL method is similar to that of the iterative method and is closer to CRLB than that of Juang’s method. This numerical localization result also matches the error analysis in the previous section.

The computation times for the new CDL method, Juang’s method, and the iterative method are 3.51, 3.72, and 7.95 s,
The new CDL and the iterative method reach CRLB with small noise. The new CDL outperforms Juang’s method in terms of positioning accuracy. The computation time of the new CDL method is the least among the three methods, identical with the result in the 2-D simulation. This indicates a significant reduction in complexity of the new CDL algorithm compared with the iterative method.

To summarize the 3-D simulation, the numerical results also verify that the positioning accuracy of the new CDL method reaches CRLB under small noise and far-field condition. With increasing measurement noise, its performance degrades but is still closer to CRLB than Juang’s method. Besides, its computational complexity is smaller than that of the iterative method.

D. Real GPS+BDS Data Experiment

In order to evaluate the performance in real-world applications, we implement the CDL algorithm to process real GNSS pseudorange observation data. The iterative method is also realized as comparison. A 24-h consecutive GPS and BDS real observation data set with a 30-s sampling interval from IGS Site TOW2, Cape Ferguson, Australia, is used. The observation period starts from 0:00, October 1, and ends at 0:00, October 2, 2018 (Universal Time Coordinated). These observation data are available on BKG Data Center website [44]. The navigation message data covering the same period from the crustal dynamics data information system (CDDIS) website [45] are used to calculate the satellite positions. The sky view of the visible GPS and BDS satellites at one epoch of the data is depicted in Fig. 6.

The 3-D positioning results in the earth-centered, earth-fixed (ECEF) coordinate of both the CDL algorithm and the iterative method are shown in Fig. 7. It can be seen that 3-axis positioning results, including both the mean coordinate and the standard deviation (STD) of both methods are almost identical. The localization result curves for all three axes of both methods have an identical epoch-by-epoch pattern, showing that the two methods have almost the same localization accuracy. The similarity of the localization performance between the CDL method and the conventional iterative approach is consistent with the simulated 2-D and 3-D results in the above sections. This validates the feasibility and performance of the new CDL method in the real world.

The computational complexity is also evaluated. The real-world data set has 2880 epochs in total. That means the new CDL algorithm and the iterative method are respectively called 2880 times when processing the real data. The computation time cost of the new CDL method is 1.93 s compared with 3.64 s for the iterative method, about 40% improvement. This shows a complexity reduction with comparable positioning accuracy of the CDL method in the real-world application compared with the conventional iterative method.

With the fast development of IoT, more and more new applications such as drone control, vehicle positioning and navigation, and location-based services require higher accuracy and better availability. Dual localization systems, such as GPS and BDS, can be used to meet such requirements by adopting the new CDL method for these novel applications. Besides, low computation complexity of the new CDL method as shown in the experiment can benefit these applications on size and power-constrained electronics systems such as cellphones, digital bracelets, and drone platforms.

VI. Conclusion

In this article, a new CDL algorithm, namely, CDL, for two nonsynchronized pseudorange-based systems is developed. In this method, the nonlinear relationship between the user position and the pseudorange measurements is converted to a linear one by taking a square on the distances. Solving for the user position is then reduced to finding the roots to two intermediate distance variables in a closed form. After analytically solving a quadratic equation set to identify the intermediate variables, the user position is computed by applying a WLS method. Theoretical analysis on the localization error covariance of the new CDL method is conducted. We prove that the positioning accuracy reaches CRLB under small noise and far-field condition. Compared with the iterative method, the new CDL method does not require initial guess and has lower complexity with similar positioning accuracy. The localization
In this subcase, the problem reduces to solving the equation of

\[ a_1 x^2 + b_1 xy + c_1 y^2 + d_1 x + e_1 y + f_1 = 0 \] (14)

\[ a_2 x^2 + b_2 xy + c_2 y^2 + d_2 x + e_2 y + f_2 = 0. \] (15)

After reorganizing, we obtain

\[ (t_1 x + t_2) y = t_3 x^2 + t_4 x + t_5 \] (37)

where

\[ t_1 = b_1 c_2 - b_2 c_1, \quad t_2 = e_1 c_2 - e_2 c_1 \]
\[ t_3 = -a_1 c_2 + a_2 c_1, \quad t_4 = -d_1 c_2 + d_2 c_1 \]
\[ t_5 = -f_1 c_2 + f_2 c_1. \]

Here, are two cases. One is \( t_1 x + t_2 = 0 \) and the other is \( t_1 x + t_2 \neq 0 \).

**Case 1** (\( t_1 x + t_2 = 0 \)): If \( t_1 = 0 \), then \( t_2 \) must equal to zero.

In this subcase, the problem reduces to solving the equation of

\[ t_3 x^2 + t_4 x + t_5 = 0. \] (38)

After substituting the root of \( x \) from (38) into (36), the root of \( y \) can be found.

If \( t_1 \neq 0 \), then we need to test if \( -t_2/t_1 \) is the root of \( x \) by substituting it into (38). If it satisfies (38), then the root of \( y \) can be found by substituting \( x \) into (36). Otherwise, there is no solution.

**Case 2** (\( t_1 x + t_2 \neq 0 \)): In this case, we have

\[ y = \left( t_3 x^2 + t_4 x + t_5 \right) / (t_1 x + t_2). \] (39)

By substituting (39) into (36), we come to a quartic equation of \( x \) as

\[ ax^4 + \beta x^3 + \gamma x^2 + \lambda x + \mu = 0 \] (40)

where

\[ a = a_1 t_1^2 + b_1 t_1 t_3 + c_1 t_3^2 \]
\[ \beta = d_1 t_1^2 + 2a_1 t_2 t_3 + b_1 t_2 t_4 + b_3 t_3 + 2c_1 t_3 t_4 + e_1 t_3 t_5 + f_1 t_1 t_5 \]
\[ \gamma = c_1 (t_1^2 + 2t_3 t_5) + a_1 t_2^2 + f_1 t_1^2 + b_1 t_1 t_5 \]
\[ + b_1 t_2 t_4 + 2d_1 t_1 t_2 + e_1 t_1 t_4 + e_1 t_2 t_3 \]
\[ \lambda = d_1 t_2^2 + b_1 t_2 t_5 + 2c_1 t_4 t_5 + e_1 t_1 t_5 + e_1 t_2 t_4 + 2f_1 t_1 t_5 \]
\[ + b_1 t_2 t_4 + e_1 t_1 t_4 + e_1 t_2 t_3 \]
\[ \mu = f_1 t_2^2 + e_1 t_2 t_5 + c_1 t_5^2. \]

The closed-form solution of the quartic equation can be found in mathematical literature, such as [46], [47]. We simply write the solution as follows in this section without derivation so that interested readers are able to grasp the final result without diving into literature. There are at most four roots for this equation, either real or complex values. The general form of the roots is given by

\[ x(1), x(2) = -\frac{\beta}{4\alpha} - s \pm \frac{1}{2} \sqrt{-4s^2 - 2p + \frac{q_0}{s}} \] (41)

\[ x(3), x(4) = -\frac{\beta}{4\alpha} + s \pm \frac{1}{2} \sqrt{-4s^2 - 2p - \frac{q_0}{s}} \]
with the variables expressed as follows:
\[
\begin{align*}
  s & = 2 \sqrt{-\frac{2p}{3} + \frac{1}{3\alpha} \left(q_1 + \Delta_0 \right)^2} , \\
  p & = \frac{8\alpha \gamma - 3 \beta^2}{8\alpha^2} , \\
  q_0 & = \frac{\beta^3 - 4\alpha \beta \gamma + 8 \alpha^2 \lambda}{8\alpha^3} , \\
  q_1 & = \sqrt{\frac{\Delta_1 + \sqrt{-2\Delta}}{2}} , \\
  \Delta & = \frac{\Delta_1^2 - 4 \Delta_0^3}{27} , \\
  \Delta_0 & = \gamma^2 - 3 \beta \lambda + 12 \alpha \mu , \\
  \Delta_1 & = 2 \gamma^3 - 9 \beta \gamma \lambda + 27 \beta^2 \mu + 27 \alpha \lambda^2 - 72 \alpha \gamma \mu .
\end{align*}
\]

**APPENDIX B**

**PROOF OF THEOREM 1**

Let \( \hat{C} \) and \( \hat{h} \) be the noisy versions of \( C \) and \( h \) in (12), respectively. The error vector \( \psi \) is then defined as
\[
\psi = \hat{C}[r_{A_1}, r_{B_1}] + \hat{h} - Gp_n .
\]

The weighting matrix \( W \) can be written in terms of the covariance of the error vector \( \psi \) as
\[
W = E[\psi \psi^T] .
\]

The first row of \( \hat{C} \) and the first element of \( \hat{h} \) in (42) are given by
\[
\begin{align*}
  [\hat{C}]_1 & = [r_{A_2 A_1} + \epsilon_{A_2 A_1} , 0] , \\
  [\hat{h}]_1 & = \frac{1}{2} \left( (r_{A_2 A_1} + \epsilon_{A_2 A_1})^2 + \|p_{A_1}\|^2 - \|p_{A_2}\|^2 \right) .
\end{align*}
\]

The first element of \( \psi \) is then given by
\[
[\psi]_1 = [\hat{C}]_1 \cdot [r_{A_1}, r_{B_1}]^T + [\hat{h}]_1 - [G]_1 p_n \\
+ \epsilon_{A_2 A_1} (r_{A_1} + r_{A_2 A_1}) + \frac{1}{2} \epsilon_{A_2 A_1}^2
= \epsilon_{A_2 A_1} r_{A_2} + \frac{1}{2} \epsilon_{A_2 A_1} .
\]

Given the condition that UD is far from ANs and the measurement noise is small, the second squared error term in (46) can be ignored, i.e.,
\[
[\psi]_1 = \epsilon_{A_2 A_1} r_{A_2} .
\]

The vector form of \( \psi \) is then written as
\[
\psi = \begin{bmatrix}
  r_{A_2} (\epsilon_{A_2} - \epsilon_{A_1}) \\
  \vdots \\
  r_{A_M} (\epsilon_{A_M} - \epsilon_{A_1}) \\
  r_{B_2} (\epsilon_{B_2} - \epsilon_{B_1}) \\
  \vdots \\
  r_{B_N} (\epsilon_{B_N} - \epsilon_{B_1})
\end{bmatrix} = D \xi
\]

where \( D = \text{diag}(r_{A_2}, \ldots, r_{A_M}, r_{B_2}, \ldots, r_{B_N}) \),

\[
\xi = \begin{bmatrix}
  \epsilon_{A_2} - \epsilon_{A_1} \\
  \vdots \\
  \epsilon_{A_M} - \epsilon_{A_1} \\
  \epsilon_{B_2} - \epsilon_{B_1} \\
  \vdots \\
  \epsilon_{B_N} - \epsilon_{B_1}
\end{bmatrix} .
\]

Then, the covariance of \( \psi \) is given by
\[
E[\psi \psi^T] = D E[\xi \xi^T] D .
\]

We note that the pseudorange measurement noises for all ANs are independent identically distributed and follow a Gaussian distribution. Therefore
\[
E[\epsilon_i^2] = \sigma_i^2 , \quad E[\epsilon_i \epsilon_j] = 0 , \quad i \neq m , j \neq n .
\]

We denote the expectation term on the right side of (49) as \( Q \), which then has the form of
\[
Q = \begin{bmatrix}
  Q_A \\
  O_{(M-1) \times (N-1)} \\
  Q_B
\end{bmatrix} ,
\]

where
\[
Q_A = \begin{bmatrix}
  \sigma_{A_2}^2 + \sigma_{A_2}^2 & \sigma_{A_2}^2 & \ldots & \sigma_{A_2}^2 \\
  \sigma_{A_2}^2 & \sigma_{A_2}^2 + \sigma_{A_2}^2 & \ldots & \ldots \\
  \vdots & \vdots & \ddots & \vdots \\
  \sigma_{A_2}^2 & \ldots & \ldots & \sigma_{A_2}^2 + \sigma_{A_2}^2
\end{bmatrix}
\]

and
\[
Q_B = \begin{bmatrix}
  \sigma_{B_2}^2 + \sigma_{B_2}^2 & \sigma_{B_2}^2 & \ldots & \sigma_{B_2}^2 \\
  \sigma_{B_2}^2 & \sigma_{B_2}^2 + \sigma_{B_2}^2 & \ldots & \ldots \\
  \vdots & \vdots & \ddots & \vdots \\
  \sigma_{B_2}^2 & \ldots & \ldots & \sigma_{B_2}^2 + \sigma_{B_2}^2
\end{bmatrix} .
\]

As a result, the covariance matrix of \( \psi \) is written as
\[
E[\psi \psi^T] = D Q D .
\]

Finally, based on (43) and (50), we have obtained the expression of the weighting matrix \( W \), which is identical with (17). Thus, we have finished the proof of Theorem 1.

**APPENDIX C**

**DERIVATION OF SIMPLIFIED SOLUTION SELECTION FORM**

Based on (17), matrix \( Q \) is rewritten as
\[
Q = \text{diag} \left( \sigma_{A_2}^2 , \ldots , \sigma_{A_M}^2 , \sigma_{B_2}^2 , \ldots , \sigma_{B_N}^2 \right) + \sigma_{A_2}^2 J_{(M-1) \times (M-1)} + \sigma_{B_2}^2 J_{(N-1) \times (N-1)} .
\]

where \( J \) is a matrix with all entries being one.
According to [48], the inverse of $Q$ is written as

$$Q^{-1} = \text{diag}\left(\frac{1}{\sigma_{A_1}^2}, \ldots, \frac{1}{\sigma_{A_M}^2}, \frac{1}{\sigma_{B_2}^2}, \ldots, \frac{1}{\sigma_{B_N}^2}\right)$$

and

$$X_A = \frac{1}{\sum_{i=1}^{M} \sigma_{A_i}} \begin{bmatrix} \frac{1}{\sigma_{A_1}^2} & \frac{1}{\sigma_{A_2}^2} & \cdots & \frac{1}{\sigma_{A_M}^2} \\ \frac{1}{\sigma_{A_1}^2} & \frac{1}{\sigma_{A_2}^2} & \cdots & \frac{1}{\sigma_{A_M}^2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\sigma_{A_1}^2} & \frac{1}{\sigma_{A_2}^2} & \cdots & \frac{1}{\sigma_{A_M}^2} \end{bmatrix}$$

$$X_B = \frac{1}{\sum_{i=1}^{N} \sigma_{B_i}} \begin{bmatrix} \frac{1}{\sigma_{B_1}^2} & \frac{1}{\sigma_{B_2}^2} & \cdots & \frac{1}{\sigma_{B_N}^2} \\ \frac{1}{\sigma_{B_1}^2} & \frac{1}{\sigma_{B_2}^2} & \cdots & \frac{1}{\sigma_{B_N}^2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\sigma_{B_1}^2} & \frac{1}{\sigma_{B_2}^2} & \cdots & \frac{1}{\sigma_{B_N}^2} \end{bmatrix}$$

(52)

The second matrix term of (52) can be ignored if the number of ANs, i.e., $M$ and $N$, are large. Furthermore, if all the measurement noise variances are identical and denoted by $\sigma^2$, then $Q^{-1} \approx (1/\sigma^2)I_{(M+N-2)\times(M+N-2)}$, where $I$ is an identity matrix. Therefore, (20) can reduce to the form of $\min_{r^p} d^T_r d_r$.

APPENDIX D

PROOF OF EQUIVALENCE FOR POSITION RELATED CRLB

USING TOA AND TDOA FROM DUAL SYSTEMS

In the dual-system positioning case, when using TDOA measurements, the FIM is written as

$$F_{\text{TDOA}} = H^T Q^{-1} H$$

(53)

where $H$ and $Q$ have the same definitions as in (31) and (17), respectively.

If we divide $H$ column wisely into two blocks relating to systems $A$ and $B$, respectively, then we have

$$H = \begin{bmatrix} H^T_A & H^T_B \end{bmatrix}$$

(54)

where

$$H_A = \begin{bmatrix} I^T_{\text{A}_1} - I^T_{\text{A}_2} \\ \vdots \\ I^T_{\text{A}_1} - I^T_{\text{A}_M} \end{bmatrix}, \quad H_B = \begin{bmatrix} I^T_{\text{B}_1} - I^T_{\text{B}_2} \\ \vdots \\ I^T_{\text{B}_1} - I^T_{\text{B}_N} \end{bmatrix}$$

We note that $Q$ is divided into blocks in (17). Then, the FIM is rewritten as

$$F_{\text{TDOA}} = H_A^T Q_A^{-1} H_A + H_B^T Q_B^{-1} H_B.$$  

(55)

When using TOA or pseudorange measurements for the dual-system positioning case, the FIM denoted by $F_{\text{TOA}}$ is written as

$$F_{\text{TOA}} = H_{\text{TOA}}^T Q_{\text{TOA}}^{-1} H_{\text{TOA}} = \begin{bmatrix} F_{11} & F_{12} \\ F_{12}^T & F_{22} \end{bmatrix}$$

(56)

where

$$H_{\text{TOA}} = \begin{bmatrix} H_{\text{A}_{\text{TOA}}} & -I M & 0_M \\ H_{\text{B}_{\text{TOA}}} & 0_N & -I_N \end{bmatrix}$$

$$Q_{\text{TOA}} = \begin{bmatrix} Q_{\text{A}_{\text{TOA}}} O_{M\times N} \\ O_{N\times M} Q_{\text{B}_{\text{TOA}}} \end{bmatrix}$$

$$Q_{\text{A}_{\text{TOA}}} = \text{diag}(\sigma_{A_1}^2, \ldots, \sigma_{A_M}^2)$$

$$Q_{\text{B}_{\text{TOA}}} = \text{diag}(\sigma_{B_1}^2, \ldots, \sigma_{B_N}^2)$$

$$F_{11} = H_{\text{A}_{\text{TOA}}}^T Q_{\text{A}_{\text{TOA}}}^{-1} H_{\text{A}_{\text{TOA}}} + H_{\text{B}_{\text{TOA}}}^T Q_{\text{B}_{\text{TOA}}}^{-1} H_{\text{B}_{\text{TOA}}}$$

$$F_{12} = -H_{\text{A}_{\text{TOA}}}^T Q_{\text{A}_{\text{TOA}}}^{-1} 1_M - H_{\text{B}_{\text{TOA}}}^T Q_{\text{B}_{\text{TOA}}}^{-1} 1_N$$

$$F_{22} = \text{diag}(\text{tr}(Q_{\text{A}_{\text{TOA}}}^{-1}), \text{tr}(Q_{\text{B}_{\text{TOA}}}^{-1}))$$

The upper left square submatrix (either $2 \times 2$ in 2-D cases or $3 \times 3$ in 3-D cases) in the inverse of the TOA FIM ($F^{-1}_{\text{TOA}}$) contains the CRLB relating to the position errors. We denote it by $J_{\text{pos}}$. According to the inverse of a partitioned matrix [49], we come to

$$J^{-1}_{\text{pos}} = F_{11} - F_{12} F_{22}^{-1} F_{12}^T$$

$$= H_{\text{A}_{\text{TOA}}}^T Q_{\text{A}_{\text{TOA}}}^{-1} H_{\text{A}_{\text{TOA}}}$$

$$- H_{\text{A}_{\text{TOA}}}^T Q_{\text{B}_{\text{TOA}}}^{-1} 1_M \text{tr}(Q_{\text{B}_{\text{TOA}}}^{-1})^{-1} 1_N^T Q_{\text{A}_{\text{TOA}}}^{-1} H_{\text{A}_{\text{TOA}}}$$

$$+ H_{\text{B}_{\text{TOA}}}^T Q_{\text{B}_{\text{TOA}}}^{-1} H_{\text{B}_{\text{TOA}}}$$

$$- H_{\text{B}_{\text{TOA}}}^T Q_{\text{B}_{\text{TOA}}}^{-1} 1_N \text{tr}(Q_{\text{B}_{\text{TOA}}}^{-1})^{-1} 1_M^T Q_{\text{A}_{\text{TOA}}}^{-1} H_{\text{B}_{\text{TOA}}}.$$  

(57)

The problem then boils down to the proof of equivalence of $F_{\text{TDOA}}$ and $J^{-1}_{\text{pos}}$. By observing (55) and (57), we note that these two matrices are both the sum of system $A$ related terms and system $B$ related terms. If the terms of system $A$ (and $B$) in $F_{\text{TDOA}}$ is equal to the $A$ (and $B$) related terms in $J^{-1}_{\text{pos}}$, then the proof will be done. In other words, we need to prove that in the single-system case, the positioning CRLB using TOA measurements is identical with the one using TDOA measurements. This proof is presented in [48] and [50], and interested readers are referred to their mathematical derivations.

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