Covariant Spin Kinetic Theory I: Collisionless Limit

Yu-Chen Liu, Kazuya Mameda, and Xu-Guang Huang

1Physics Department and Center for Particle Physics and Field Theory, Fudan University, Shanghai 200433, China
2RIKEN Nishina Center for Accelerator-Based Science, Wako 351-0198, Japan
3Key Laboratory of Nuclear Physics and Ion-beam Application (MOE), Fudan University, Shanghai 200433, China

We develop a covariant kinetic theory for massive fermions in curved spacetime and external electromagnetic field based on quantum field theory. We derive four coupled semi-classical kinetic equations accurate at $O(\hbar)$, which describe the transports of particle number and spin degrees of freedom. The relation with the chiral kinetic theory is discussed. As an application, we study the spin polarization in the presence of finite Riemann curvature and electromagnetic field in both local and global equilibrium states.

I. INTRODUCTION

Kinetic theory is widely used to study the transport phenomena in many-particle systems. The classical Boltzmann kinetic theory is established as the framework to describe the evolution of the distribution function in phase space. To study the effects of spin, one has to use the quantum kinetic theory and single distribution function is usually insufficient for such a purpose. For massless Dirac fermions, the leading spin effects appear at $O(\hbar)$, and we need two distribution functions, i.e., one for right-hand chirality and the other for left-hand chirality. Such established framework is the so-called chiral kinetic theory (CKT) [1–3], which has been under intensive investigations recently [4–14]. The out-of-equilibrium dynamics of anomaly-induced phenomena, e.g., the chiral magnetic effect [15, 16] and the chiral vortical effect [17–19], have been commendably investigated in the framework of CKT.

Unlike the massless case, the spin of a massive fermion is independent of the kinetic momentum. As a result, the dynamical evolution of massive fermions is specified with more than two degrees of freedom. Suppose that $\theta^\mu$ is the unit spacelike vector specifying the spin quantizing orientation. Then the dynamical variables are two parameters to determine $\theta^\mu$, and two distribution functions, $f_{\pm}$, for particles with spin parallel and anti-parallel to $\theta^\mu$, respectively. Therefore the kinetic theory of the massive fermion is more complicated than the CKT, and we need extensive concerns of this framework [20–23].

One of the motivations to develop the aforementioned kinetic theory is the spin polarization in heavy-ion collision, which is an important probe of the hot and dense quark gluon matter [24–27]. The first signal of the global spin polarization of $\Lambda$ hyperons (hereafter, $\Lambda$ polarization) [28] indicates the existence of a very strong fluid vorticity [29–31]. The subsequent measurements reveals very nontrivial features that cannot be understood based on the simple vorticity interpretation of the spin polarization [32, 33]. For example, the measured longitudinal and transverse $\Lambda$ polarization showed the opposite azimuthal angle dependence compared with the so-called thermal vorticity [34–41]. This strongly indicates that the spin polarization has independent dynamical evolution, in a non-equilibrium state, rather than being enslaved to the fluid vorticity. The covariant kinetic theory for spin transport (hereafter, spin kinetic theory for short) would be one of the promising tools to capture the dynamics of the spin polarization.

In this paper, we derive the collisionless spin kinetic theory at $O(\hbar)$ in curved background spacetime and external electromagnetic field. As an application, we investigate the spin polarization from the spin kinetic theory. We give the general expression for the spin polarization in terms of $f_{\pm}$ and $\theta^\mu$. We then specify the equilibrium conditions from the spin kinetic theory and derive the spin polarization at both local and global equilibrium. We stress that the present study differs from the earlier works [20–23] in the following aspects: (i) We include curved geometric background spacetime as well as electromagnetic field. Such a general formulation should be applicable to the spin transport not only in the heavy-ion collisions, but also in deformed materials and the thermal gradient system, which attracts huge attentions in condensed matter physics. (ii) We show that the frame choosing vector disappears in the covariant kinetic theory of the massive fermions, unlike the massless case. (iii) We discuss the underlying physics in the Clifford components and their constraint equations. (iv) We write down the kinetic equations in a more transparent way, including the physical contexts of them. Especially, we verify that in the classical limit, these equations are correctly reduced to the Vlasov equation, the Bargmann–Michel–Telegdi (BMT) equation [42], and Mathisson–Papapetrou–Dixon (MPD) equations [43–45]. (v) We discuss the global equilibrium in terms of the spin vector $\theta^\mu$. The validity of this equilibrium state is ensured by the resulting spin polarization, which is consistent with that in Refs. [46, 47].

This paper is organized as follows: In Sections II and III, we introduce the Wigner function and discuss the physical meaning of the dynamical equation for each Clifford component of the Wigner function. In Section IV, we derive the semi-classical kinetic theory for massive fermions. In Section V, we derive the kinetic representation of the spin polarization for both massive and massless fermions and investigate the spin polarization at local and global equilibrium. In this paper, we adopt the same notations and conventions as those of

\footnote{1 A massless particle and antiparticle with its spin parallel (anti-parallel) to its momentum is called to have right-(left-)handed chirality and the left-(right-)handed one, respectively.}
Ref. [13]; for instance, $\nabla \mu$ denotes the covariant derivative in terms of diffeomorphism and the local Lorentz transformation, and $p_\mu (y^\mu)$ is the momentum variable (its conjugate one).

II. WIGNER FUNCTIONS

The Wigner operator covariant under the U(1) gauge, local Lorentz transformations, and diffeomorphism is defined as [13]

$$\hat{W}(x, p) = \int d^3y \sqrt{-g(x)} e^{-ip \cdot y / \hbar} \hat{\rho}(x, y),$$

$$\hat{\rho}(x, y) = \hat{\psi}(x) e^{y} \frac{D}{2} \otimes e^{-y} D/2 \psi(x),$$

where $\hat{\psi}(x)$ is the Dirac spinor operator. Here we introduce the following notations: $\hat{\psi}(x) \equiv \psi^+(x) \gamma^0$ and $\hat{\psi}^0 \equiv [O \psi]^+ \gamma^0$ for an operator $\hat{O}$, and $[\hat{\psi} \otimes \hat{\psi}]_{ab} = \hat{\psi}_b \psi_a$ with $a, b$ being the spinor indices. The derivative $D_\mu$ is called the horizontal lift of $\nabla_\mu$ (including the gauge field if necessary): $D_\mu = \nabla_\mu - \Gamma^\lambda_\mu_\nu y^\nu \partial^\lambda_\nu$ in the tangent bundle [i.e., the $(x, y)$-space]. Similarly the horizontal lift in the cotangent bundle [i.e., the $(x, p)$-space] is given by

$$D_\mu = \nabla_\mu + \Gamma^\lambda_\mu_\nu p_\lambda \partial^\nu_\nu.$$  

This $D_\mu$ gives us a great advantage of the analysis due to the property $[D_\mu, y^\mu] = [D_\mu, p_\mu] = 0$. We note that the gauge field $A_\mu$ is involved in $D_\mu$ acting on the Dirac spinor: $D_\mu \psi(x, y) = (\nabla_\mu - \Gamma^\lambda_\mu_\nu y^\nu \partial^\lambda_\nu + iA_\mu / \hbar) \psi(x, y)$ with $\psi(x, y) \equiv e^{y} D/2 \psi(x)$.

The Wigner function is defined by replacing the operator $\hat{\rho}(x, y)$ with the ensemble average $\rho(x, y) \equiv (\hat{\rho}(x, y))$ in Eq. (1). In this paper, we focus on the collisionless limit, and thus we impose the spinor field obeying the Dirac equation $(i\hbar \gamma^\mu D_\mu - m) \psi(x) = \hat{\psi}(x) (i\hbar D_\mu \gamma^\mu + m) = 0$. In this case, we derive the kinetic theory of massive fermions in the same manner as that in Ref. [13] (especially, see Section III and Appendices C, D, and E therein). After the semiclassical expansion 2, and the decomposition in terms of the Clifford algebra as $W = \frac{1}{4} [\mathcal{F} + i\gamma^5 \mathcal{P} + \gamma^\mu \mathcal{Y}_\mu + \gamma^\sigma \gamma^\mu A_\mu + \frac{1}{2} \sigma^{\mu\nu} S_{\mu\nu}]$. we arrive at the following system of equations:

$$\Delta_\mu \psi^\mu = \frac{\hbar^2}{24} (\nabla_\mu R_{\nu\rho}) \partial^\mu_\nu \partial^\rho_\nu \psi^\rho;$$  

$$\hbar \Delta_\mu A^\mu = -2 m \mathcal{P},$$  

$$\hbar \Delta_\mu [A_\mu] - \epsilon_{\mu\nu\rho\sigma} \Pi^\sigma_\nu = -\frac{\hbar^2}{8} \tilde{R}_{\rho\nu\sigma\mu} \partial^\rho_\nu \psi^\sigma;$$  

$$\Pi_\mu \psi^\mu = m \mathcal{F} + \frac{\hbar^2}{8} R_{\nu\rho} \partial^\nu_\rho \psi^\mu;$$  

$$\Pi_\mu A^\mu = \frac{\hbar^2}{8} R_{\nu\rho} \partial^\nu_\rho A^\mu;$$  

$$\hbar \Delta_\mu [\mathcal{V}_\nu] - \epsilon_{\mu\nu\rho\sigma} \Pi^\sigma_\nu = m \mathcal{S}_{\mu\nu} - \frac{\hbar^2}{8} \tilde{R}_{\rho\nu\sigma\mu} \partial^\rho_\nu \psi^\sigma;$$  

$$\Pi_\mu \mathcal{F} + \frac{\hbar}{2} \Delta^\mu \mathcal{S}_{\mu\nu} = m \mathcal{Y}_\nu;$$  

$$\frac{\h}{2} \Delta_\mu \mathcal{F} - \Pi_\nu \mathcal{S}_{\mu\nu} = -\frac{\h^2}{16} (R_{\nu\rho\mu\alpha} \partial^\nu_\rho \psi^\alpha + 2 R^\phi_\rho \partial^\phi_\rho \psi^\nu).$$

In the above equations (4)-(13), the spacetime curvature enters at least at $O(\hbar^2)$. We note that the Clifford coefficients $\mathcal{F}$, $\mathcal{P}$, $\mathcal{Y}_\mu$, $A_\mu$, and $\mathcal{S}_{\mu\nu}$ are not totally independent. To proceed, we choose $\mathcal{V}_\nu$, and $A_\mu$ as the independent variables 3. Then $\mathcal{P}$, $\mathcal{F}$, and $\mathcal{S}_{\mu\nu}$ are expressed in terms of $\mathcal{V}_\nu$, and $A_\mu$ through Eqs. (5), (7), and (9). In Minkowski spacetime, the same set of equations up to $O(\hbar)$ was first derived in Ref. [48].

In the kinetic description, various physical quantities are built from $W$, which is (the Wigner transformation of) a two-point correlator of Dirac fermions. For instance, the vector

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2 We employ the power counting scheme as $p_\mu = O(1)$ and $y^\mu \sim i\hbar \partial^\mu_\nu = O(\hbar)$.

3 In fact, only two (for massless case) or four (for massive case) components of $(\mathcal{V}_\mu, A_\mu)$ are independent.
and axial current are computed as

\[ J^\mu = \int_p \text{tr} \left( \gamma^\mu W \right) = \int_p \mathcal{V}^\mu, \quad (15) \]

\[ J_5^\mu = \int_p \text{tr} \left( \gamma^\mu \gamma^5 W \right) = \int_p \mathcal{A}^\mu \quad (16) \]

with \( \int_p \equiv \int \frac{d^4 p}{(2\pi)^4} \delta(D_0 - m) \). From these, one identifies the Clifford coefficients \( \mathcal{V}^\mu \) and \( \mathcal{A}_\mu \) as the corresponding current densities in phase space (see more discussions in Sec. V). In a similar way, one finds that \( \mathcal{F} \) is the scalar condensate density (which in the classical limit is also interpreted as the distribution function of the vector charge); \( \mathcal{P} \) is the axial condensate density; and \( S_{\mu\nu} \) is the electromagnetic dipole moment density, up to a factor of \( m \) [see Eqs. (50)-(54)]. For latter convenience, we further represent the canonical energy-momentum tensor, spin current, and total angular momentum current as

\[ T^{\mu\nu} = \int_p \text{tr} \left[ \frac{i\hbar}{2} \gamma^\mu \delta^{\nu}_\lambda \right] \hat{x}_\lambda = \int_p \mathcal{V}^{\mu\nu}, \quad (17) \]

\[ S^{\mu\nu\rho} = \int_p \text{tr} \left[ \frac{\hbar}{4} (\gamma^\mu, \sigma^{\nu\rho}) \right] W = -\frac{\hbar}{2} \int_p \epsilon^{\mu\nu\rho\lambda} \mathcal{A}_\lambda, \quad (18) \]

\[ M^{\lambda,\mu\nu} = x^\mu T^{\lambda\nu} - x^\nu T^{\lambda\mu} + S^{\lambda,\mu\nu}, \quad (19) \]

where we define \( \hat{D}_\mu (\bar{\psi} \otimes \psi_a) \equiv \bar{\psi} \otimes D_\mu \psi_a - \bar{\psi}_b \hat{D}_\mu \otimes \psi_a \).

It is worthwhile to point out that \( S^{\lambda,\mu\nu} \) is not anticipated to be an observable for spin because there exists the Belinfante–Rosenfeld type pseudo-gauge ambiguity [49]; \( S^{\lambda,\mu\nu} \) in Eq. (19) can be absorbed into the definition of the energy-momentum tensor once one introduces the Belinfante tensor. Instead, an unambiguous way to represent particles spin is to employ the Pauli-Lubanski (PL) vector operator:

\[ \mathcal{W}^{\mu} \equiv -\frac{1}{\hbar} \epsilon^{\mu\nu\rho\sigma} P_\nu \hat{M}_{\rho\sigma}, \quad (20) \]

where the hat symbol denotes quantum mechanical operators, \( P_\nu \) and \( \hat{M}_{\rho\sigma} \) are the canonical momentum and total angular momentum operators, respectively, and the prefactor \(-1/\hbar\) is introduced as our convention. It is important to notice that the orbital part of the canonical angular momentum does not contribute the above equation.

Following Eq. (20), we may define the PL vector in our kinetic theory as

\[ \mathcal{W}^{\mu}(x, p) \equiv -\frac{1}{1/p \cdot \nu} \epsilon^{\mu\nu\rho\sigma} p_\nu M_{\rho\sigma}, \quad (21) \]

where we define \( M_{\rho\sigma} \equiv \nu^\lambda \hat{M}_{\lambda,\rho\sigma} \) with \( \nu^\mu \) being a unit time-like vector perpendicular to the hypersurface, and the factor \( 1/p \cdot \nu \) is introduced for normalization. One can readily check that the above definition of \( \mathcal{W}^{\mu} \) excludes the orbital angular momentum part, and it is reduced as

\[ \mathcal{W}^{\mu}(x, p) = \mathcal{A}^{\mu}(x, p). \quad (22) \]

The coincidence between \( \mathcal{W}^{\mu} \) and \( \mathcal{A}^{\mu} \) is expectable; e.g., for massless fermions, the magnetic-field induced spin polarization can be considered as the axial current generation, which is the chiral separation effect [51, 52]. The spin polarization density defined with the PL vector is hence equivalent to the axial current:

\[ \mathcal{Q}^{\mu}(x) \equiv \int_p \mathcal{W}^{\mu}(x, p) = \int_p \mathcal{A}^{\mu}(x, p) = J_5^{\mu}(x). \quad (23) \]

### III. PHYSICAL INTERPRETATION

In the following we discuss the physical meanings of Eqs. (4)-(13). To show this, we perform the integration over momentum space, which makes much simpler expressions; most of total derivatives terms vanishes as surface integral [an exception in Eq. (5) is discussed later]. For simplicity, hereafter we only focus on \( \mathcal{O}(\hbar) \) terms so that the Riemann curvature is neglected and Eq. (14) is reduced to \( \Pi_\mu = p_\mu \) and \( \Delta_\mu = D_\mu - F_{\lambda\mu} \partial_\lambda \).

First, we demonstrate that Eqs. (4) and (6) leads to fundamental Ward identities. After integrating over momentum, Eq. (4) gives the vector current conservation law:

\[ \nabla_\mu J^\mu = 0. \quad (24) \]

It is obvious that from this that Eq. (4) is the kinetic equation of the vector charged particle. Integrating Eq. (4) after multiplying by \( p^\nu \), we obtain the energy-momentum conservation law in the presence of the external field:

\[ \nabla_\mu (T^{\mu\nu} + T_{\mu\nu}^{\text{ext}}) = 0, \quad (25) \]

where \( T_{\mu\nu}^{\text{ext}} = -F^{\mu\lambda} F_{\lambda\nu} + \frac{1}{2} \delta^{\mu\rho} F_{\rho\sigma} F_{\sigma\nu} \) is the energy-momentum tensor of electromagnetic field. Here we have used Maxwell’s equation \( \nabla_\mu F^{\mu\nu} = J^\nu \) and the Bianchi identity \( \nabla_\mu F^{\mu\nu} = 0 \). Also integrating Eq. (6) over momentum, we obtain

\[ \nabla_\mu S^{\mu\nu\rho\sigma} = T^{\sigma\rho} - T^{\rho\sigma}. \quad (26) \]

This, combined with Eq. (25), gives the conservation law of the canonical total angular momentum:

\[ \nabla_\mu \left( M^{\mu,\rho\sigma} + M_{\rho\sigma}^{\mu,\text{ext}} \right) = 0 \quad (27) \]

with \( M_{\rho\sigma}^{\mu,\text{ext}} = x^\mu T^{\lambda\nu}_{\text{ext}} - x^\nu T^{\lambda\mu}_{\text{ext}} \) being the angular momentum of electromagnetic field. This reflects the absence of the Lorentz anomaly [53].

Next, we consider Eqs. (8)-(13), whose physical contents are less transparent than Eqs. (4) and (6). Equation (8) involves \( \mathcal{A}^{\mu} \), so it is a subsidiary condition for \( \mathcal{A}^{\mu} \). Up to \( \mathcal{O}(\hbar) \), it reduces to

\[ p_\mu \mathcal{A}^{\mu} = 0. \quad (28) \]

\[ \text{This is not equivalent to the ensemble average of the PL vector operator } \mathcal{W}^{\mu}. \]
Based on the identification $\mathcal{W}^\mu = A^\mu$ (22), the above equation implies the following facts; the spin must be either perpendicular to the momentum (i.e., for massive fermions) or parallel to the momentum (i.e., for massless fermions so that $p^2 = 0$ on-shell classically). In Section IV we discuss the details even with quantum corrections. The electromagnetic dipole moment is derived from Eq. (9):

$$m \int p \mathcal{S}_{\mu\nu} = - \int \epsilon_{\mu\nu\rho\sigma} p^\rho A^\sigma + \hbar \nabla_{[\mu} J_{\nu]} ,$$

(29)

where the first (second) term on the right-hand side represents the spin (orbital) contributions. Equations (10) and (11) are Gordon decompositions for the vector and axial currents. Upon integration over momentum, they separate the convection and the gradient currents:

$$m J^\mu = \int p^\rho F + \frac{\hbar}{2} \nabla_\nu \int p \mathcal{S}^{\mu\nu} ,$$

(30)

$$m J^5_\mu = - \int p^\rho \mathcal{S}^{\mu\rho} + \frac{\hbar}{2} \nabla_\nu \int p \mathcal{F} .$$

(31)

We note that the second term in Eq. (30) is the covariant form of the well-known magnetization current. Similarly, Eqs. (12) and (13) give

$$0 = \int p^\rho p^\nu + \frac{\hbar}{2} \nabla_\nu \int p \mathcal{S}^{\mu\nu} ,$$

(32)

$$0 = - \int p^\rho p^\nu \mathcal{S}^{\mu\nu} + \frac{\hbar}{2} \nabla_\nu \int p \mathcal{F} ,$$

(33)

where the right-hand sides are dual to those of Eqs. (30) and (31). We note that $\mathcal{F}$ is regarded as the source of spin [see Eq. (5)]. In this sense, we novelty understand that Eqs. (32) and (33) imply as the following fact: there never exist vector and axial currents carrying ‘magnetic charges’.

Finally we mention quantum anomaly related to Eqs. (5) and (7). The momentum integral of Eq. (5) generates a non-vanishing surface term. After a technical evaluation of such a term, we derive the anomalous axial Ward identity:

$$\nabla_{\mu} J^\mu_5 = \alpha - \frac{2m}{\hbar} \int p \mathcal{P} ,$$

(34)

where $\alpha$ is the anomaly originated from the surface integral (see Appendix A). Also from Eq. (7), we obtain

$$T^\mu_\mu = m \int p \mathcal{F} ,$$

(35)

which represents the Ward identity in terms of the dilatation. We emphasize that up to $O(\hbar)$, no surface integral contributes to Eq. (35). As a result, the trace anomaly does not emerge here, while the chiral anomaly does in Eq. (34). Indeed one can confirm from dimensional analysis that the trace anomaly is $O(\hbar)$ higher than the chiral anomaly \footnote{5}. For the same reason, the chiral anomaly in Eq. (34) is not involved in the gravitational contribution, which is $O(\hbar^2)$ higher than the electromagnetic one [13]. In the kinetic theory involving $O(\hbar^2)$ or $O(\hbar^3)$ terms, these additional contributions should enter in the right-hand side of Eqs. (34) and (35). We will leave the discussion of the higher order kinetic theory in future publication.

IV. KINETIC EQUATIONS AT $O(\hbar)$

In the kinetic theory up to $O(\hbar)$, the general solutions for $\mathcal{V}_\mu$ and $A_\mu$ take the following forms (see Appendix B):

$$\mathcal{V}^\mu = 4\pi \left[ \delta(\xi) \left( \epsilon^{\mu\nu\rho\sigma} p^\rho n_\nu \Delta_\rho \tilde{A}_\sigma \right) + \delta'(\xi) \hbar \tilde{F}^{\mu\nu} \left( \tilde{A}_\nu - \frac{p \cdot A}{p^2} n_\nu \right) \right] ,$$

(36)

$$A^\mu = 4\pi \left[ \delta(\xi) \tilde{A}^\mu + \delta'(\xi) \hbar \tilde{F}^{\mu\nu} p_\nu f \right] ,$$

(37)

where we utilize $\delta'(x) = -\delta(x)$, and denote $\tilde{F}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$, $\Delta_\mu = D_\mu - F_{\mu\lambda} \partial^\lambda$, and

$$\xi = p^2 - m^2 .$$

(38)

The scalar function $f = f(x, p)$ is to be considered as the distribution function of vector charge, and $n^\mu$ is a unit timelike vector that satisfies $p \cdot n \neq 0$. According to Eq. (8), the vector $\tilde{A}^\mu$ must satisfy the condition

$$p_\mu \tilde{A}^\mu \delta(\xi) = 0 .$$

(39)

Here $\tilde{A}^\mu$ is not necessarily perpendicular to the momentum due to the presence of the delta function. To proceed, we decompose $\tilde{A}^\mu$ as

$$\tilde{A}^\mu = p_\mu f + \tilde{A}^\mu_\perp ,$$

(40)

where $\tilde{A}^\mu_\perp$ is perpendicular to the momentum: $p \cdot \tilde{A}_\perp = 0$.

A. Massless case

In the massless limit, plugging Eqs. (36) and (37) into Eq. (9) we identify

$$\tilde{A}^\mu_\perp = \hbar \Sigma^{\mu\nu}_n \Delta_\nu f , \quad \Sigma^{\mu\nu}_n = \frac{\epsilon^{\mu\nu\rho\sigma} p_\rho n_\sigma}{2 p \cdot n} .$$

(41)

turn phase space integral. Counting such an additional power of $\hbar$, one can write the well-known anomalous Ward identities for massless fermions in Minkowski spacetime: $\delta_\mu J^\mu_5 = \hbar^2 \frac{1}{16\pi^2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} F_{\alpha\beta} \delta_{\alpha\beta}$ and $T^\mu_\mu = \hbar^{-1} \beta \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} F_{\alpha\beta}$ with $\beta$ being the $\beta$-function (we take $c = 1$ but recover $e$ explicitly).
Then the solutions (36) and (37) are reduced to
\[
(\mathcal{V}, A)^\mu = 4\pi \left[ \delta(p^2) \left\{ p^\mu (f, f_3) + h\Sigma^{\mu\nu}_{n_1} \Delta_\nu (f, f_3) \right\} + h\delta'(p^2) \mathcal{F}^{\mu\nu} p_\nu (f, f_3) \right].
\]  
(42)

This indicates that \(f_5\) plays the role of the axial charge distribution function. The second term in the above equation is called the side-jump term (or the magnetization current), and \(\Sigma^{\mu\nu}_{n_1}\) is known as the spin tensor at the spin-defining vector \(n^\mu\) [6, 7]; e.g., \(\Sigma^{\mu\nu}_{n_1} = \epsilon^{jk\lambda} p_k / 2p_0\) at the rest frame \(n^\mu = (1, 0)\). Besides, it is important to notice from the above \(\mathcal{A}^\mu\) that \(\Sigma^{\mu\nu}_{n_1}\) is connected to the spin current (18) through
\[
S^{\lambda,\mu\nu}_{n_1} = h \int p \cdot n f_5 \Sigma^{\mu\nu}_{n_1}.
\]  
(43)

This relation more transparently accounts for why \(\Sigma^{\mu\nu}_{n_1}\) characterizes the particle spin, and \(n^\mu\) represents the frame of the spin.

Note that Eq. (42) correctly reproduces the solution in the CKT, with the replacement as \(R^\mu / L^\mu = \frac{1}{2} (\mathcal{V} + A)\mu\) and \(f_{R/L} = \frac{1}{2}(f + f_5)\) [13]. Accordingly, the chiral kinetic equations are also obtained from Eqs. (4) and (5) with the above solutions (42):
\[
0 = \delta(p^2 + h F_{\alpha\beta} \Sigma^{\alpha\beta}_{n_1}) \left\{ p_\mu \Delta_\mu f_{R/L} \right\} \pm \frac{h}{p \cdot n} \tilde{F}_{\mu\nu} n^\mu \Delta_\nu f_{R/L} \pm h\Delta_\mu \left( \Sigma^{\mu\nu}_{n_1} \Delta_\nu f_{R/L} \right). 
\]  
(44)

More discussions about the CKT are found, e.g., in Refs. [8, 11, 13].

Now we mention the chiral anomaly in the kinetic theory. Using the \(O(h)\) solution \(\mathcal{A}^\mu\) in Eq. (42), we derive the anomalous Ward identity
\[
\nabla_\mu J^\mu_5 = \mathcal{A},
\n\mathcal{A} = -\frac{h}{8} F^{\mu\nu} \tilde{F}_{\mu\nu} \int f(p) \partial_\mu \frac{p_\mu}{|p|^2} \left[ \frac{p_\mu}{|p|^2} \right]
\]  
(45)

with \(\int_p = \int \frac{d^4p}{(2\pi)^4}\). This reproduces the usual chiral anomaly when \(f\) is (the twice of) the Fermi-Dirac distribution (see details in Appendix A). The important fact is that \(\mathcal{A}\) receives the contribution only from the singular term at \(p^2 = 0\), which generates the Berry monopole \(\partial_\mu \frac{p_\mu}{|p|^2} = 4\pi \delta^3(p)\). The above covariant expression hence manifests that the chiral anomaly is a topological nature of massless fermion [1, 2].

B. Massive case

Let us now focus on the massive case, in which we can perform two reductions for the solutions (36) and (37). First, Eq. (39) for \(m \neq 0\) implies
\[
f_5 \delta(\xi) = 0,
\]  
(46)

with which one can remove the parallel part in \(\bar{A}^\mu\) from the solutions (36) and (37). Second, the frame vector \(n^\mu\) in Eqs. (36) and (37) can be absorbed into the distribution function \(f\) redefined as (see Appendix C):
\[
f \to f + \frac{1}{m^2} \epsilon_{\mu\nu\rho\sigma} p^\mu n^\nu \Delta_\rho \bar{A}_{\perp}^\sigma.
\]  
(47)

We emphasize that this redefinition is equivalent to identify the frame \(n^\mu\) as the particle’s rest frame \(n_{\text{rest}}^\mu = p^\mu / m\). The frame vector \(n^\mu\) can be removed since in the massive case there is a special choice of \(n^\mu\), i.e., the rest frame \(n_{\text{rest}}^\mu\). Thus, we can always redefine the scalar distribution function \(f\) from \(n^\mu\) to \(n_{\text{rest}}^\mu\) through a local Lorentz boost. This procedure does not work for massless fermions due to the lack of such a special frame, and it is inevitable to introduce \(n^\mu\).

Due to the constraint \(p \cdot \bar{A}_{\perp} = 0\), there are three degrees of freedom in \(\bar{A}_{\perp}^\mu\). One of three is interpreted as the axial charged distribution, which specifies the norm of \(\bar{A}_{\perp}^\mu\). Since the axial current density is identified as the particle spin as shown in Eq. (22), the other two correspond to the parameters of the spin direction. In the massive case, we hence parametrize \(\bar{A}_{\perp}^\mu\) as
\[
\bar{A}_{\perp}^\mu = m \theta^\mu f_A,
\]  
(48)

where \(f_A\) is the axial distribution function, and \(\theta^\mu\) is a unit vector (with two degrees of freedom). Note that \(\theta^\mu\) is normalized with the spacelike condition \(\theta^\mu \theta_\mu = -1\) and \(p_\mu \theta^\mu = 0\). In addition, it is useful for later discussion to introduce the following tensor:
\[
\Sigma_{S}^{\mu\nu} = \frac{1}{2m} \epsilon^{\mu\nu\rho\sigma} \theta_\rho p_\sigma,
\]  
(49)

which may be regarded as the spin tensor of massive fermions, as so is \(\Sigma_{n_1}^{\mu\nu}\) for massless fermions (41). Indeed it is readily checked that \(\Sigma_{S}^{\mu\nu} = \epsilon^{jk\lambda} \theta_k / 2\) for the rest particle with \(p_\mu = (m, 0)\).

Collecting the discussions above, we present the solutions

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6 Strictly speaking, \(n^\mu\) in Eq. (43) is in a subset of the \(n^\mu\)’s allowed to enter \(\Sigma^{\mu\nu}_{n_1}\). The latter is defined in phase space while the former depends on \(x\) only.
of the Clifford coefficients $F$, $\mathcal{P}$, $\mathcal{V}_\mu$, $A_\mu$ and $S_{\mu\nu}$, as follows:

$$F = 4\pi \left[ \left( \delta(\xi) m f - \delta'(\xi) \hbar m F^{\mu\nu} S^{\mu\nu} f_A \right) \right],$$  

(50)

$$\mathcal{P} = 4\pi \delta(\xi) \left( -\frac{\hbar}{2} \Delta_\mu (\theta^\mu f_A) \right),$$  

(51)

$$Y^\mu = 4\pi \delta(\xi) \left[ \left( \frac{p^\mu f + \hbar \epsilon^{\mu\nu\rho\sigma} \frac{p_\nu p_\sigma}{2m} p_\tau \Delta_\rho (\theta_\sigma f_A) \right) + \delta'(\xi) \hbar m F^{\mu\nu} \theta_\nu f_A \right],$$  

(52)

$$A^\mu = 4\pi \delta(\xi) m \theta^\mu f_A + \delta'(\xi) \hbar m F^{\mu\nu} \theta_\nu f_A,$$  

(53)

$$S^{\mu\nu} = 4\pi \delta(\xi) \left( 2m f_A S^{\mu\nu} + \frac{\hbar}{m} \Delta^{\mu\nu} (p^\rho f^\rho) \right) - \delta'(\xi) \hbar m F^{\mu\nu} f_A \right],$$  

(54)

with $\xi = p^2 - m^2$. In Eqs. (50)-(54), there are four independent variables: two for the distribution functions $f$ and $f_A$, and the other two for the spatial orientation of the spin vector $\theta^\mu$. Therefore, the covariant spin kinetic theory up to $O(\hbar)$ is described by the following four independent evolution equations:

$$0 = \delta(\xi) \hbar^2 S_{\alpha\beta} C_{\alpha\beta},$$  

(55)

$$\times \left[ \left( p^\mu \Delta_\mu \mp \frac{\hbar}{2} S_{\mu\nu} \left( \nabla_\rho F^{\rho\mu} - \frac{\hbar}{2} S_{\rho\sigma} R^{\rho\sigma}_{\mu\nu} \right) \theta_\nu \right) f_{\pm} ight] \right] + \frac{\hbar}{2} (f_+ - f_-) \left( \nabla_\rho F^{\rho\mu} - \frac{\hbar}{2} S_{\rho\sigma} R^{\rho\sigma}_{\mu\nu} \right) \theta_\nu S^{\mu\nu},$$  

(56)

with $f_{\pm} \equiv \frac{1}{2} (f \pm f_A)$. In Appendix D, we present the derivation of the above kinetic equations. With given initial conditions, Eqs. (55) and (56) determine the time evolutions of $f_{\pm}$ and $\theta^\mu$ for massive fermions at collisionless limit. The flat-space counterparts of Eqs. (55) and (56) were discussed recently in Refs. [20–22].

We give some comments about Eqs. (55) and (56):

1. $S_{\mu\nu}$ is related to $\theta^\mu$ through its definition (49). Thus in Eq. (55), it is sufficient to keep only the $O(1)$ order contribution in $S_{\mu\nu}$, which is always accompanied by an additional $\hbar$ factor.

2. The delta function in Eq. (55) shows that the onshell condition is shifted by $\pm \hbar S_{\alpha\beta} F_{\alpha\beta}$. This term should be regarded as the magnetization coupling, similarly to $\pm \hbar S_{\alpha\beta} F_{\alpha\beta}$ in the massless kinetic equation (44).

3. Note that $f_+$ ($f_-$) represents the distribution for fermions whose spin is parallel (antiparallel) to $\theta^\mu$. Indeed, the particle number of such spin-aligned fermions are written with the Wigner function, as follows:

$$N_{\pm} \equiv \int f_{\pm} W = \int 4\pi \delta(\xi) \hbar S_{\alpha\beta} F_{\alpha\beta} m f_{\pm},$$  

(57)

where $\mathcal{P}_{\pm} \equiv \frac{1}{2} (1 \pm \gamma^5 \alpha^\mu \theta_\mu)$ is the spin projection operator in terms of $\theta^\mu$ [54]. Moreover, this observation of $f_{\pm}$ is consistent with Eq. (55); the two kinetic equations of $f_{\pm}$ degenerate to the same Vlasov equation $\delta(\xi) m f + \frac{\hbar}{2} S_{\mu\nu} \left( \nabla_\rho F^{\rho\mu} - \frac{\hbar}{2} S_{\rho\sigma} R^{\rho\sigma}_{\mu\nu} \right) f_{\pm} = 0$, in the classical limit, where spin-up-down particles are indistinguishable.

4. The third term in Eq. (56) is of $O(\hbar)$ order, as we check by substituting Eq. (55). Therefore, in the classical limit, Eq. (56) is reduced to $p \cdot \Delta \theta^\mu = F^{\mu\nu} \theta_\nu$ with on-shell condition $p^2 = m^2$. This is nothing but the Bargmann–Michel–Telegdi (BMT) equation, which describes the Larmor-Thomas precession of the spin [42]; in Minkowski spacetime, the BMT equation for a rest particle under a magnetic field $B$ is written as the well-known form of the usual Larmor precession: $m \theta = B \times \theta$.

5. From Eq. (55), we extract the following single-particle equations of motion:

$$\frac{Dx^\mu}{Dt} = \frac{p^\mu}{m},$$  

(58)

$$\frac{Dp^\mu}{Dt} = F^{\mu\lambda} p_\lambda \mp \frac{\hbar}{2m} S^{\mu\beta} \left( \nabla_\nu F_{\alpha\beta} - \frac{\hbar}{2m} S_{\alpha\beta} R^{\nu\alpha\beta} \right).$$  

(59)

Here $D/Dt$ is the covariant derivative in terms of $\tau$ to be the proper time along the trajectory of the particle, and the on-shell condition $\xi = \mp \hbar S_{\alpha\beta} F_{\alpha\beta} = 0$ is implicitly applied. Equation (59) is known as the first Mathisson–Papapetrou–Dixon (MPD) equation [43–45]. The first, second, and third term in Eq. (59) represent the Coulomb-Lorentz force, the Zeeman force, and the spin curvature coupling, respectively.

6. By multiplying $\epsilon_{\alpha\beta\mu\nu} p^\nu$, one recasts Eq. (56) into $p \cdot \Delta S^{\mu\nu} = 2F_\sigma [\sigma S^{\mu\nu}] + O(\hbar)$. Combing this with Eq. (58), one derives the following equation of motion:

$$\frac{Dh S_{\mu\nu}}{Dt} = \frac{1}{m} F_\sigma [\sigma h S_{\mu\nu}] + 2p^\mu \frac{DX^\nu}{Dt}.$$  

(60)

This is the second MPD equation, which determines the spin motion in gravitational backgrounds [43–45]. Note that the Tulczyjew-Dixon condition [55, 56] is automatically satisfied: $p^\mu S_{\mu\nu} = 0$.

V. APPLICATION: SPIN POLARIZATION

A. General state

As an application of our spin kinetic theory, we calculate the spin polarization of Dirac fermions, which is an intensive topic of heavy-ion collisions. As we have already mentioned, an unambiguous definition of the spin polarization is the PL vector; $W^\mu = A^\mu$ in Eq. (22) and $W^\mu (x) = \int p W^\mu = \int \mathcal{P}^\mu$ in Eq. (23). Combined with Eqs. (42) and (53), this polariza-
tion vector are given by
\[
\mathcal{W}^\mu(x) = \begin{cases} 
4\pi \delta(p^2) \left[ p^\mu f_5 + h \Sigma_{n}^{\mu\nu} \Delta_{\nu} f - \frac{\hbar}{2} \tilde{F}^{\mu\nu} \partial_\nu f \right] & \text{(massless)}, \\
4\pi \delta(\xi) \left[ m\theta^\mu f_A - \frac{\hbar}{2} \tilde{F}^{\mu\nu} \partial_\nu f \right] & \text{(massive)}.
\end{cases}
\] (61)

For later use, we also define the polarization per particle in the phase space:
\[
w^\mu(x, p) = \frac{\mathcal{W}^\mu(x, p)}{4\pi f(x, p)} = A^\mu(x, p). \tag{62}
\]

These expressions are available in nonequilibrium state. The last terms in each case stem from the Zeeman coupling, which gives an additional \( O(h) \) contribution. On top of magnetic field, other sources, e.g., the fluid vorticity (or rotation), also induces the spin polarization. In Eq. (61), such contributions are found, only after the concrete forms of the distribution functions are determined. For this analysis, one needs the collision terms, which we will leave in a subsequent paper. In global equilibrium state, however, we can identify the vorticity-dependence of the distribution functions, as shown below.

B. Equilibrium state

In the following, we study the spin polarization in the equilibrium state. In kinetic theory, the local equilibrium state is specified by the distribution functions that eliminate the collision kernel. This implies that the distribution functions must depend only on the linear combination of the collisional conserved quantities: the particle number, the energy and momentum, and the angular momentum. Therefore, we consider the following ansatz for the local equilibrium distributions, \( f_{\pm} = n_F(g_{\pm}) \) with \( g_{\pm} = p \cdot \beta + \alpha_{\pm} \pm h \Sigma_{n}^{\mu\nu} \omega_{\mu\nu} \) for the massive fermions (we have absorbed the orbital angular momentum into a redefinition of \( \beta \) field), and \( f_{R/L} = n_F(g_{R/L}) \) with \( g_{R/L} = p \cdot \beta + \alpha_{R/L} \pm h \Sigma_{n}^{\mu\nu} \omega_{\mu\nu} \) for the massless fermions. The coefficients \( \beta_{\mu}, \alpha_{\mu} \), \( \omega_{\mu\nu} \) (called spin chemical potential) depend only on \( x \), where \( \beta_{\mu} \) is assumed to be timelike. Although the actual functional form of \( n_F \) is not essential, let us take it to be the Fermi-Dirac function for demonstration.

1. Massive case

In the massive case, at local equilibrium, the spin-polarization vectors are readily computed from Eqs. (61) and (62), as follows:
\[
w_{LE}^\mu(x, p) = -\delta(\xi) m\theta^\mu (\alpha_A + h\theta \cdot \Omega) n_F + h \delta'(\xi) \tilde{F}^{\mu\nu} p_\nu, \tag{63}
\]
\[
\mathcal{W}_{LE}^\mu(x) = 4\pi \int_p \delta(\xi) \left[ 2m\theta^\mu (\alpha_A + h \Omega \cdot \theta) - h \tilde{F}^{\mu\nu} \beta_\nu \right] n_F', \tag{64}
\]

with \( n_F = n_F(p \cdot \beta + \alpha) \), \( \tilde{n}_F = 1 - n_F, \alpha = (\alpha_+ + \alpha_-)/2 \), and \( \alpha_A = (\alpha_+ - \alpha_-)/2 \), and \( \Omega^\mu = \epsilon^{\mu\nu\sigma \tau} p_\nu \omega_{\sigma \tau}/(2m) \). Note that \( \alpha_A \) is assumed to be of \( O(h) \); otherwise a finite spin would be generated even in the classical limit.

It is more important to discuss the polarization at global equilibrium. For this purpose, let us find necessary constraints imposed by the kinetic equations (55) and (56). Substituting \( f_{LE} \) into Eq. (55), one can show that the following conditions can fulfill Eq. (55) up to \( O(h) \) for arbitrary spin vector \( \theta^\mu \):
\[
\nabla_\mu \beta_\nu + \nabla_\nu \beta_\mu = 0, \tag{65}
\]
\[
\nabla_\mu \beta_\nu - 2\omega_{\mu\nu} = 0, \tag{66}
\]
\[
\nabla_\mu \alpha_{\pm} = F_{\mu\nu} \beta^\nu. \tag{67}
\]

Furthermore, we verify that under the conditions (65)-(67), one possible choice of \( \alpha_A \) and \( \theta^\mu \) to fulfill Eq. (56) is given by (see Appendix E):
\[
\alpha_A = 0, \quad \theta^\mu = -\frac{1}{2m\Gamma} \epsilon^{\mu\nu\rho\sigma} p_\nu \nabla_\rho \beta_\sigma, \tag{68}
\]

where \( \Gamma = (\frac{1}{4} \nabla_{[\beta_\mu]} \Lambda^{\mu\rho} \Lambda^{\nu\sigma} \nabla_{[\beta_\sigma]} \beta_\nu)^{1/2} \) with \( \Lambda^{\mu\nu} = g^{\mu\nu} - p^{\mu} p^{\nu}/m^2 \). We call the state specified by the conditions (65)-(67) and (68) the global equilibrium state, and denote \( f_{ GE } \) as the corresponding distribution function. At global equilibrium the thermal vorticity \( \nabla_{[\beta_\nu]} \) determines both the spin chemical potential \( \omega_{\mu\nu} \) and the spin vector \( \theta^\mu \). We emphasize that the finite Riemann curvature or the external electromagnetic field is necessary to derive Eq. (66). Without the external field and the curved background geometry, the spin degree of freedom is inactive in the collisionless kinetic theory and we cannot link \( \omega_{\mu\nu} \) to \( \nabla_{[\beta_\nu]} \). Besides, in Appendix F, we derive the conditions (65)-(67) for massive fermions [and (72)-(74) for massless fermions] based on the density operator.

At global equilibrium, the spin polarization vectors read
\[
w_{GE}^\mu(x, p) = \frac{h}{2} \delta(\xi) \tilde{\omega}_{\mu\nu} p_\nu n_F + h \delta'(\xi) \tilde{F}^{\mu\nu} p_\nu, \tag{69}
\]
\[
\mathcal{W}_{GE}^\mu(x) = 4\pi h \int_p \delta(\xi) \left[ -\tilde{\omega}_{\mu\nu} p_\nu - \tilde{F}^{\mu\nu} \beta_\nu \right] n_F'. \tag{70}
\]

with \( \tilde{\omega}_{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} \omega_{\sigma\tau} \). On top of \( \mathcal{W}^\mu \) and \( w^\mu \), at global equilibrium, it is also practically useful to compute the space-integrated polarization. Suppose that the fermions are frozen out on a space-like hypersurface \( \Sigma^\mu(x) \). Then the average spin polarization per particle after freeze-out is given by
\[
\mathcal{W}_{GE}^\mu(p) \equiv \frac{\int d\Sigma^\nu \lambda \Pi_F \int_0^\infty d(p \cdot u) \mathcal{W}_{GE}^\mu(x, p)}{4\pi \int d\Sigma^\nu \lambda \Pi_F \mathcal{W}_{GE}^\mu(x, p)} = \frac{\int d\Sigma^\nu \lambda \Pi_F \frac{h}{2} \left[ -\tilde{\omega}_{\mu\nu} p_\nu - \tilde{F}^{\mu\nu} \beta_\nu \right] n_F'}{\int d\Sigma^\nu \lambda \Pi_F n_F}. \tag{71}
\]

\[7\] Here we pick up the particle-branch contribution. The anti-particle-branch contribution are similarly obtained by replacing \( f_0^\infty d(p \cdot u) \) by \( f_{-\infty}^0 d(p \cdot u) \).
If we set $F_{\mu\nu} = 0$, the above equation is consistent with the result derived in Refs. [46, 47], which has been widely used for the calculation of the hadron spin polarization. In the above $u^\mu = T^\beta \beta$ is the fluid velocity and the momentum in the second line is on-shell; in Minkowski spacetime and in the local rest frame of the fluid, $p^\mu = (E_p = \sqrt{p^2 + m^2}, p)$ with $p$ the three momentum.

2. Massless case

In the same manner, Eq. (44) with $f^{\text{LE}}_{R/L}$ yields the following global equilibrium conditions [13]:

\[ \nabla_\mu \beta_\nu + \nabla_\nu \beta_\mu = \phi(x) g_{\mu\nu}, \quad (72) \]

\[ \nabla_{\mu} \beta_{\nu} - 2 \omega_{\mu\nu} = 0, \quad (73) \]

\[ \nabla_{\mu} \alpha_{R/L} = F_{\mu\nu} \beta^\nu. \quad (74) \]

Unlike the massive case, the first conditions has an arbitrary function $\phi(x)$, which arises due to the conformal invariance in the massless case, so $\beta^\mu$ is a conformal Killing vector. At global equilibrium, analogously to Eqs. (69)-(71), we calculate

\[ w^\mu_{\text{GE}}(x, p) = \frac{\hbar \delta(p^2)}{2} (-2p^\mu \alpha_5 / \hbar + \omega^\mu_{\nu} p_\nu) n_F + \hbar \delta(p^2) \tilde{F}^{\mu\nu} p_\nu, \quad (75) \]

\[ \mathcal{W}_\text{GE}^\mu(x) = 4\pi \hbar \int p \delta(p^2) \left[ 2p^\mu \alpha_5 / \hbar - \omega^\mu_{\nu} p_\nu - \tilde{F}^{\mu\nu} \beta_\nu \right] n_F, \quad (76) \]

\[ \mathcal{V}_\text{GE}^\mu(p) = \frac{\int d\Sigma^\lambda p_\lambda n_F}{\int d\Sigma^\lambda p_\lambda n_F} \left[ 2p^\mu \alpha_5 / \hbar - \omega^\mu_{\nu} p_\nu - \tilde{F}^{\mu\nu} \beta_\nu \right] n_F, \quad (77) \]

with $\alpha_5 = \alpha_R - \alpha_L$, which is of $O(\hbar)$ as well as $\alpha_A$. In the second equation, the on-shell condition is implicitly applied and we define $E_p = u \cdot p$; in Minkowski spacetime and at the rest frame of the fluid, $E_p = |p|$. Note that spin polarization induced by the thermal vorticity and the electromagnetic field take the same form for both the massless and massive cases at global equilibrium, up to the difference in the on-shell conditions. Also the results are independent of choice of the frame vector $n^\mu$, as they should be.

VI. SUMMARY AND OUTLOOK

In this paper, we derive the collisionless covariant spin kinetic theory at $O(\hbar)$ for Dirac fermions in curved spacetime and external electromagnetic field. We start with deriving the dynamical equation for each Clifford component of the Wigner function up to $O(\hbar^2)$. We discuss the physical meaning of each such dynamical equation. We then pick up $\mathcal{V}^\mu$ and $\mathcal{A}^\mu$ as independent dynamical variables and derive two evolution equations for massless fermions (44), and four evolution equations for massive fermions (55) and (56), respectively. We introduce a timelike unit frame-choosing vector $n^\mu$ to solve the Wigner function. In the massless case, $n^\mu$ is necessary because it represents the frame in which the spin for the massless particle is defined. In the massive case, we show that the vector $n^\mu$ can be removed by redefining the vector distribution function through a boost from the frame $n^\mu$ to the rest frame of the particle.

As an application, we analyze the spin polarization from the approach of the kinetic theory. We derive the global equilibrium conditions from the kinetic equations, and find that the finite Riemann curvature or the external electromagnetic field is necessary to determine the spin-thermal vorticity coupling. At global equilibrium, we given the expression of the spin polarization induced by the electromagnetic field and the thermal vorticity, which is consistent with the results in the early literatures.

We expect the spin kinetic theory to be useful for the study of both the electromagnetic plasma and quark-gluon plasma in heavy-ion collisions. Furthermore, formulating the kinetic theory in curved spacetime may find fundamental applications in astrophysics and condensed matter physics. For example, our present theory may be used to study the deformed crystal or an material with the temperature gradient, which is described as the electron system in a fictitious gravity [57, 58]. Numerical works to solve the kinetic theory and to simulate the evolution of spin polarization in heavy-ion collisions are also important tasks. Once the collision term is included, it would be interesting to derive the covariant spin hydrodynamics [59–61] from the covariant spin kinetic theory.

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Appendix A: Derivation of chiral anomaly

In this Appendix we derive the chiral electromagnetic anomaly from the solutions of the Wigner function. We consider the massless case for demonstration. Plugging $\mathcal{A}^\mu$ in...
Eq. (42) into the kinetic equation (5), and integrating it, we get $\nabla_\mu J_0^\mu = \mathcal{A}$ with

$$\mathcal{A} = \int_p 4\pi F^{\mu\lambda} \partial_\lambda \left[ h \delta'(p^2) \tilde{F}_{\mu\nu} p^\nu f \right] + \delta(p^2) \left( p_\mu f_\nu + \frac{\epsilon_{\mu\nu\rho\sigma}}{2p\cdot n} n^\rho \Delta^\sigma f \right)$$

$$= \frac{h}{4} F^{\mu\nu} \tilde{F}_{\mu\nu} \int_p 4\pi \delta'(p^2) p^\lambda \partial_\lambda f,$$

where we employ the Schouten identity and $x \delta''(x) = -2\delta'(x)$, and drop the surface terms without the singularity at $p^2 = 0$. As the integral is scalar, the following computation can be performed in the local Lorentz coordinate. The roots of $p^2 = 0$ is given by $p_0 = \pm |p|$, with which the delta function is reduced to

$$\delta(p^2) = \frac{1}{2|p|} \delta(p_0 - |p|) + \delta(p_0 + |p|).$$

Furthermore, when we carry out the $p_0$-integration, we need the replacement of the momentum derivatives, as follows:

$$\partial_p f(\pm|p|, p_0) = \left( \partial_\rho f(p_0, p) \right)_{p_0 = \pm|p|} = \left( \partial_\rho f(p_0, p) \right)_{p_0 = \pm|p|} + \partial_\rho \left( \frac{p}{2p_0} \right) f(p_0, p_0)|_{p_0 = \pm|p|} = \bar{\partial}_\rho \left( \frac{p}{2p_0} \right) f(p_0, p_0)|_{p_0 = \pm|p|}.$$}

Then the integral in Eq. (A1) is cast into

$$\int_p 4\pi \delta'(p^2) p^\lambda \partial_\lambda f$$

$$= \int_p 4\pi \frac{1}{2} \left[ \partial_\lambda \delta(p^2) \right] p^\lambda f = \frac{1}{2} \int_p 4\pi \delta(p^2) \partial_\lambda f$$

$$= -\frac{1}{2} \int_p 4\pi \delta(p^2) \left[ \bar{\partial}_\rho \partial_\rho f + \frac{2}{p_0} \partial_\rho f + \frac{2}{p_0} p^\rho \bar{\partial}_\rho f \right].$$

In the last line, the second and third terms in the integrand cancel out; performing the integration by parts, we rewrite the third term as

$$\int_p \delta(p^2) \bar{\partial}_\rho f = \int_p \delta(p^2) \left[ \bar{\partial}_\rho f + \frac{p}{p_0} \partial_\rho f - \bar{\partial}_\rho f \right]$$

$$= - \int_p \delta(p^2) \frac{1}{p_0} \partial_\rho f.$$}

Finally, $\mathcal{A}$ in Eq. (A1) is calculated as

$$\mathcal{A} = -\frac{h}{8} F^{\mu\nu} \tilde{F}_{\mu\nu} \int_p \frac{1}{p} \bar{\partial}_\rho \bar{\partial}_\rho f$$

$$= -\frac{h}{16\pi^2} F^{\mu\nu} \tilde{F}_{\mu\nu} f(p = 0),$$

where we utilize

$$\partial_\rho \left. \frac{p_\rho}{|p|^2} \right|_{p = 0} = 4\pi \delta(p).$$

The usual chiral anomaly relation is recovered, hence, once we take $f(p = 0) = 2$, where the factor 2 accounts for the spin degeneracy of Dirac fermions.

**Appendix B: General solutions at $O(\hbar)$**

We parametrize the perturbative solutions as

$$\gamma^\mu = \gamma^\mu(0) + \hbar \gamma^\mu(1) + O(\hbar^2), \quad A^\mu = A^\mu(0) + \hbar A^\mu(1) + O(\hbar^2).$$

According to Eqs. (7), (8) and (10), the general solutions in the classical limit are given by

$$\gamma^\mu(0) = 4\pi \gamma^\mu(f(0)) \delta(\xi),$$

$$A^\mu(0) = 4\pi A^\mu(0) \delta(\xi).$$

with $\xi = p^2 - m^2$. Here $f(0) = f(0)(x, p)$ is the classical vector charge distribution function and the vector $A^\mu(0)$ satisfies the condition $p_\mu A^\mu(0) \delta(\xi) = 0$. Substituting Eqs. (B2) and (B3) into Eqs. (6)–(13), we obtain the solutions at $O(\hbar)$:

$$\gamma^\mu(1) = 4\pi \left[ \left( p^\mu f(1) + \frac{1}{2p\cdot n} \epsilon^{\mu\nu\rho\sigma} n_\nu \Delta_\rho A^\sigma(0) \right) \delta(\xi) \right]$$

$$+ \tilde{F}_{\mu\nu} \left( A^\nu(0) - \frac{p_\nu A^\nu(0)}{p\cdot n} n_\nu \right) \delta(\xi),$$

$$A^\mu(1) = 4\pi \left[ A^\mu(1) \delta(\xi) + \tilde{F}_{\mu\nu} \gamma^\nu(0) \delta(\xi) \right],$$

where $f(1) = f(1)(x, p)$ is the first order quantum correction to the vector distribution function and $A^\mu(1)$ satisfies the same condition as $A^\mu(0)$. $p_\mu A^\mu(1) \delta(\xi) = 0$. Defining $f \equiv f(0) + \hbar f(1)$ and $A^\mu \equiv A^\mu(0) + \hbar A^\mu(1)$, we get Eqs. (36) and (37).

**Appendix C: Elimination of $n^\mu$**

In this Appendix, we show that with the redefinition of the distribution function $f$ in Eq. (47), we can remove the frame vector $n^\mu$ from the spin kinetic theory for massive fermions. The discussion is kept at $O(\hbar)$. Acting $\Delta_\alpha$ to Eq. (9), we derive

$$p \cdot \Delta A^\beta = F^{\alpha\beta} A_\alpha - p^\beta \Delta_\alpha A^\alpha$$

$$= \frac{m}{2} \epsilon^{\alpha\beta\rho\sigma} \Delta_\alpha S^\rho_{\sigma} - \frac{h}{2} \epsilon^{\alpha\beta\rho\sigma} \Delta_\alpha \Delta_\rho \gamma^\sigma .$$

Using Eqs. (5) and (12), we obtain

$$p^\beta \Delta_\alpha A^\alpha = \frac{m}{2} \epsilon^{\alpha\beta\rho\sigma} \Delta_\alpha S^\rho_{\sigma} .$$

Combining the above two equations, we find

$$p \cdot \Delta A_\mu = F_{\mu\nu} A^\nu + \frac{h}{2} \epsilon_{\mu\rho\sigma} \Delta^\nu \Delta^\rho \gamma^\sigma .$$

Next, we substitute the redefined distribution function in Eq. (47) into the solution of $\gamma^\mu$ in Eq. (36), and obtain

$$\gamma^\mu = 4\pi \delta(\xi) \left[ p^\mu \left( f - \frac{\hbar \epsilon^{\alpha\beta\rho\sigma} p_\rho n_\beta \Delta_\rho A^\sigma(0)}{2m^2 p \cdot n} \right) \right]$$

$$+ \frac{\hbar \epsilon_{\mu\nu\rho\sigma} n_\nu \Delta_\rho A^\sigma(0)}{2p \cdot n} + 4\pi \delta(\xi) h \tilde{F}_{\mu\nu} \tilde{A}^\nu .$$
In order to reduce the above equation, we utilize the Schouten identity:

\[ p^{\mu} \epsilon^{\alpha\beta\rho\sigma} p_\alpha n_\beta \Delta_\rho A^\perp_{\perp \sigma} = - \left( p^2 \epsilon^{\beta\rho\mu} n_\beta \Delta_\rho + p \cdot n \epsilon^{\rho\sigma\mu} p_\alpha \Delta_\rho \right. \]

\[ + \epsilon^{\sigma\rho\beta\mu} p_\alpha n_\beta p_\rho \cdot \Delta + \epsilon^{\mu\rho\beta\sigma} p_\alpha n_\beta p^\rho \Delta_\rho \left. \right) \tilde{A}^\perp_{\perp \sigma}, \tag{C5} \]

where the last two terms cancel according to Eq. (C3). Then we rewrite Eq. (C4) as the \( n^\mu \) independent from:

\[ \mathcal{V}^\mu = 4 \pi \delta(\xi) \left[ p^\mu f + \frac{\hbar \epsilon^{\mu\rho\sigma} p_\rho}{2 m^2} \Delta_\rho \tilde{A}^\perp_{\perp \sigma} - \frac{\hbar}{\xi} F^{\mu\nu} \tilde{A}^\perp_{\perp \nu} \right], \tag{C6} \]

The frame vector \( n^\mu \) is also removed from the solution of \( \mathcal{A}^\mu \).

Comparing the above equation with Eq. (36), we find that the redefinition of \( f \) is equivalent to replacing \( n^\mu \) with \( p^\mu/m \) in Eq. (36).

**Appendix D: Derivation of Eqs. (55) and (56)**

Here, we derive the kinetic equations for \( f_{\perp} \). Substituting Eq. (52) into Eq. (4), one obtains the following equation:

\[ 0 = p^\mu \Delta_\mu f \delta(\xi) - \hbar \Sigma^\alpha_\beta F_{\alpha\beta} p^\mu \Delta_\mu f \delta(\xi) \]

\[ \quad + \frac{\hbar}{2} \left( \nabla^\alpha F_{\mu\nu} \partial^\mu_p + [D_\mu, D_\nu] \right) (\Sigma^\mu_\nu f_A) \delta(\xi). \tag{D1} \]

Besides contracting Eq. (C3) with \( \theta^\mu \) and inserting Eqs. (52) and (53), we get

\[ 0 = p^\mu \Delta_\mu f \delta(\xi) - \hbar \Sigma^\alpha_\beta F_{\alpha\beta} p^\mu \Delta_\mu f \delta(\xi) \]

\[ \quad + \frac{\hbar}{2} \Sigma^\mu_\nu \left( \nabla_\rho F_{\mu\nu} \partial^\rho_p + [D_\mu, D_\nu] \right) f \delta(\xi). \tag{D2} \]

The addition and subtraction of the above two equations result in Eq. (55). Also the kinetic equation to determine \( \theta^\mu \) is obtained from Eq. (C3) with the solutions (52) and (53).

**Appendix E: Global equilibrium condition from kinetic theory**

In the massless case, the discussion of the global equilibrium conditions (65)-(67) was already given in Ref. [13]. Following the similar strategy, one can show that, for the massive case, the conditions (65)-(67) can fulfill Eq. (55) for arbitrary \( \theta^\mu \) and for \( \alpha_A = O(\hbar) \). Also we verify that the condition (68) fulfills Eq. (56) under the conditions (65)-(67), as follows. The leading order of \( f_{\perp}^{LE} \) is written as

\[ f_{\perp}^{LE} = 2(\alpha_A + \hbar \Sigma^\alpha_\beta \omega^{\alpha\beta}) n_{\perp}(\beta \cdot p + \alpha) + O(\hbar^2). \]

Eqs. (65)-(68) and inserting \( f^{LE} \) and \( f_{\perp}^{LE} \), we obtain

RHS of Eq. (56)

\[ = 2 \delta(\xi) n_{\perp} \left[ \hbar p \cdot \Delta (\theta^\mu \Sigma^\alpha_\beta \omega^{\alpha\beta}) - (\alpha_A + \hbar \Sigma^\alpha_\beta \omega^{\alpha\beta}) \right] F^{\mu\nu} \theta_{\nu} \]

\[ - \frac{\hbar}{4m} \epsilon^{\mu\rho\sigma\nu} p_\alpha \left( \nabla_\sigma F_{\nu\rho} - p_\lambda R^\lambda_{\sigma\nu\rho} \right) \beta^\sigma \]

\[ = - \frac{1}{2m} \delta(\xi) \left[ \frac{p \cdot \Delta (\epsilon^{\mu\rho\sigma\nu} p_\alpha \nabla_\rho \beta_\nu) - F^{\mu\nu} \epsilon_{\lambda\rho\sigma\nu} p^\lambda \nabla_\rho \beta^\sigma + \epsilon^{\mu\rho\sigma\nu} p_\alpha \left( \nabla_\sigma F_{\nu\rho} - p_\lambda R^\lambda_{\sigma\nu\rho} \right) \beta^\sigma} \right]. \tag{E1} \]

In the above equation, the second equality follows from \( \alpha_A = 0 \) and

\[ \theta^\mu \Sigma^\alpha_\beta \omega^{\alpha\beta} = \frac{1}{2} \theta^\mu \Gamma = - \frac{1}{4m} \epsilon^{\mu\rho\sigma\nu} p_\alpha \nabla_\rho \beta_\nu, \tag{E2} \]

which stems from Eq. (68). One finds that the above three terms in Eq. (E1) totally vanishes, as follows:

\[ p \cdot \Delta (\epsilon^{\mu\rho\sigma\nu} p_\alpha \nabla_\rho \beta_\nu) = p_\lambda \left( \epsilon^{\mu\rho\sigma\nu} F_{\nu\rho} + 2 \epsilon^{\mu\rho\sigma\nu} F_{\nu\rho} \right) \nabla_\rho \beta_\nu \]

\[ + \epsilon^{\mu\rho\sigma\nu} p_\alpha R^\lambda_{\sigma\nu\rho} \beta^\sigma \tag{E3} \]

\[ = F^{\mu\nu} P_\alpha R^\lambda_{\nu\mu\rho} \beta_\nu \]

\[ + \epsilon^{\mu\rho\sigma\nu} p_\alpha R^\lambda_{\sigma\nu\rho} \beta^\sigma. \]

In the above equation, we use the Schouten identity and the equilibrium condition (65) with \( \nabla_\mu \nabla_{[\mu \beta]} = - \beta^\lambda R_{\lambda \mu \rho}. \)

**Appendix F: Global equilibrium condition from density operator**

We discuss the global equilibrium using the maximum entropy principle, following Refs. [62-64]. The density operator for local equilibrium state is written as

\[ \hat{\rho}_{LE} = \frac{1}{Z} e^{-\frac{1}{\hbar} \int d\Xi_\mu \left( \bar{T}^{\mu\nu} \beta_\nu + \bar{S}^{\mu\lambda\nu} \omega_{\lambda\nu} + \alpha J^\mu \right)}, \tag{F1} \]

with \( Z \equiv Tr \left[ e^{-\frac{1}{\hbar} \int d\Xi_\mu \left( \bar{T}^{\mu\nu} \beta_\nu + \bar{S}^{\mu\lambda\nu} \omega_{\lambda\nu} + \alpha J^\mu \right)} \right] \). Here \( \Xi_\mu \) is a spacelike hypersurface, \( \beta^\mu, \alpha, \) and \( \omega_{\mu\nu} \) have the same meanings as in the main text. The entropy is defined as

\[ S = - (\ln \rho_{LE}) = - Tr (\rho_{LE} \ln \rho_{LE}) \tag{F2} \].

We denote \( \ln z = \int d\Xi_\mu \phi^\mu, \) where \( \phi^\mu \) is (minus of) the thermodynamic potential density current. Then the entropy is represented as

\[ S = \int d\Xi_\mu s^\mu \tag{F3} \]

\[ = \phi^\mu + T^{\mu\nu} \beta_\nu + S^{\mu\lambda\nu} \omega_{\lambda\nu} + \alpha J^\mu. \]

The global equilibrium is given by the condition that the local thermodynamic potential and entropy are maximized so
that $\nabla_\mu \phi^\mu = 0$ and $\nabla_\mu s^\mu = 0$. After some straightforward calculations, we arrive at

$$0 = T_{xy}^{\mu\nu} \nabla_\mu \beta_\nu + T_{as}^{\mu\nu} (\nabla_\mu \beta_\nu - 2\omega_{\mu\nu}) + S^{\mu\nu,\lambda\nu} \nabla_\mu \omega_{\lambda\nu} + J^{\mu} (\nabla_\mu \alpha - F_{\mu\nu} \beta^\nu), \quad (F4)$$

where $T_{xy}^{\mu\nu}$ is the symmetric/antisymmetric part of $T^{\mu\nu}$. In the massless case, $T^{\mu\nu} = 0$, we obtain Eqs. (72)-(74) (with $\alpha_\mu = \alpha_L = \alpha$); In the massive case, we obtain Eqs. (65)-(67). Note that one further constraint from Eq. (F4), $\nabla_{[\mu} \omega_{\nu\lambda]} = 0$, is automatically fulfilled.
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