Abstract

The entanglement of clouds of $N=10^{11}$ atoms recently experimentally verified is expressed in terms of the fluctuation algebra introduced by Verbeure et al. A mean field hamiltonian describing the coupling to a laser beam leads to different time evolutions if considered on microscopic or mesoscopic operators. Only the latter creates non trivial correlations that finally after a measurement lead to entanglement between the clouds.

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1 Introduction

In [1] the entanglement of macroscopic objects namely of two atomic clouds with $N \sim 10^{11}$ atoms has been observed. It is not that just a few atoms have been entangled among themselves but the entanglement took place on the level of a collective coordinate $\vec{J}_N = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \vec{j}_i$ if one thinks about the individual atoms as spins or more generally as angular momenta $\vec{j}_i$. The limit $N \to \infty$ of such a quantity was studied in [2], [3] and was called fluctuation algebra. It shows some mathematically tricky aspects and in this paper we shall investigate their relevance for the phenomenon discovered in [1].

In a mean field theory one considers the mean magnetizations $\vec{m}_N = \frac{1}{N} \sum_{i=1}^{N} \vec{j}_i$ which are a norm bounded set and thus they have weak accumulation points. Further more $[m_N, j_i]$ converges in norm to 0, thus the accumulation points have to lie in the center of the representation. This implies that though the existence of $\lim_{N \to \infty} \vec{m}_N$ is representation dependent in an irreducible representation $\vec{m}_N$ must contain subsequences which converge to a multiple of 1 (a c-number). In contradistinction $\|\vec{J}_N\| \sim \sqrt{N}$ and thus $\vec{J}_N$ do not converge in any operator topology. Of course the unitaries $e^{i\vec{\alpha}(\vec{J}_N - <\vec{J}_N>)}$ are also norm bounded and by the previous argument their weak accumulation points are also c-numbers. This c-number is the expectation value of $\vec{m}_N$, thus it is unique in the representation and corresponds to the measured mean magnetization. Under appropriate assumptions on the state $\omega$ [2], [3], [5] these limits are $e^{-<\vec{a}|A|\vec{a}>}$ with some $\omega$-depending $A$ and we will give an explicit example when the limit is $e^{-|\vec{a}|^2}$. Thus there is no strong convergence of the unitaries to a c-number (which would necessarily have norm 1). Nevertheless one can find a state dependent map of the limiting elements onto a Weyl algebra with a distinguished Gaussian state such that some properties of expectation values carry over. In particular the characterization of entanglement as having smaller square fluctuations than separable states [10] also applies to the limiting algebra.

We consider $\vec{m}_N$ as a macroscopic quantity and call $\vec{J}_N - <\vec{J}_N>$ mesoscopic in the sense that it is between the macroscopic and the microscopic level. By the limiting procedure this mesoscopic quantity is well defined and preserves some of the quantum structure of the underlying microscopic system.

The tool used in [1] for producing entanglement is the coupling of the $\vec{j}_i$ to a laser beam. This produces a time evolution $\tau^t_N$ which depends on $N$. For finite $N$ it certainly exists $\forall t \in R$ but in the limit $N \to \infty$ various problems arise. First of all the question arises whether the limit exists or whether the motion keeps getting faster. For the microscopic quantities the limit is state dependent but exists for reasonable states in the same sense as the mean magnetization does. To carry $\tau^t_N$ over to the Weyl algebra $\mathcal{W}$ one has to watch out for discontinuities $\lim_{N \to \infty} \tau^t_N(\vec{J}_N) \neq \lim_{N \to \infty} \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \lim_{M \to \infty} \tau^t_M(\vec{j}_i)$. The latter exists under the above conditions, the former also for appropriate states but is different from the latter. Nevertheless it is the relevant time evolution $\tilde{\tau}_t$ and in contradistinction to the other it leads in combination with a measurement of the radiation field (and this measurement is necessary!) to entanglement. What happens is that we have three parties, two clouds of atoms and a laser beam, that get mixed by $\tilde{\tau}_t$. We start with a product state, after some time it will not be a product state any more but nevertheless
reduces on the two clouds to a separable state. If however we measure the radiation field and then reduce the state it becomes entangled. Clearly the amount of entanglement depends on the precision of the measurement and only if the quantum state of the radiation field is completely specified we can reach optimal entanglement.

2 The Fluctuation Algebra

First we repeat the facts that are known about the fluctuation algebra. It was introduced in [2] and studied in more detail in [3]. Variations in its definitions adjusted to varies cluster properties of the underlying state were presented in [4]. Recently the idea was taken up again and the definition was generalized from strictly local operators to exponentially localized operators, provided the underlying state also clusters exponentially [5],[6]. The main definitions and results are the following:

Definition 1: Let \( A = \bigotimes_{l \in \mathbb{Z}} A_l \) be an algebra on a one dimensional lattice where the \( A_l \approx M_d \) are finite dimensional matrix algebras and the closure is taken in norm. \( A \) contains the local algebra \( A_{loc} = \bigvee_{|\Lambda| < \infty} \otimes_{l \in \Lambda} A_l \). The shift \( \alpha_j : A_l \to A_{l+j} \) is an automorphism of \( A \) as well as of \( A_{loc} \). For shift invariant states \( \omega = \omega \circ \alpha \) and \( q \in A_{loc} \) we define
\[
q_{<N>} = \frac{1}{\sqrt{2N+1}} \sum_{|j| \leq N} (\alpha^j(q) - \omega(q))
\]

\[
s(q_1, q_2) = \lim_{N \to \infty} \omega(\sum_{|j| \leq N} [q_1, \alpha^j(q_2)]_+).
\]

If \( \omega \) is clustering in the sense that
\[
|\omega(q_1 \alpha^j(q_2)) - \omega(q_1)\omega(q_2)| \leq \|q_1\|\|q_2\|m_j
\]
with \( \sum_j |j|m_j < \infty \) then we can also define
\[
t(q_1, q_2) = \lim_{N \to \infty} \omega(\sum_{|j| \leq N} [q_1 - \omega(q_1), \alpha^j(q_2) - \omega(q_2)]_+).
\]

In the corresponding GNS representation it follows ([2], [3]) that

Proposition: \( \forall q_k \in A_{loc}, k = 1, 2, ..r \)
\[
w \lim_{N \to \infty} \prod e^{iq_{<N>}} = e^{-i/2 \sum_{k<l} s(q_k, q_l) - \frac{i}{2}(\sum_k q_k, \sum_l q_l)}
\]

Remarks: The weak convergence refers to the representation \( \Pi_\omega \). The weak limit of \( e^{iq_{<N>}} \) depends on \( \omega \), the strong limit does not exist and \( q_{<N>} \) does not converge even weakly.

The remarkable fact is, that though we only have weak convergence nevertheless we can assign to the limits an algebraic structure:
**Corollary:** To a set \( q_k \in A_{loc}, k = 1, 2, \ldots, r \) we can associate by a map \( W_\omega \) unitaries from a Weyl-algebra \( W = (e^{iQ_1}, e^{iQ_2}, \ldots, e^{iQ_r}) \) with symplectic form \([Q_k, Q_l] = s(q_k, q_l)\). The state \( \omega \) over \( A \) implements a state \( \omega \) over \( W \)

\[
\omega(\prod_{k=1}^r e^{i\beta_k q_k}) = \omega(\prod_{k=1}^r W_\omega(q_k)) = w \lim_{N \to \infty} \prod_{k=1}^r e^{i\beta_k q_k < N>}
\]

We can illustrate this convergence in the special

**Example:** We take \( A_j = M_2 \) and \( \omega(\cdot) = \langle \uparrow \uparrow ... \uparrow | \uparrow \uparrow ... \uparrow \rangle \). Then \( \omega(e^{i \sum \beta_k \sigma_k}) = \omega(\cos(|\beta|)) + i \sum \frac{\beta_k \sigma_k}{|\beta|} \sin|\beta| \) so that

\[
\lim_{N \to \infty} \omega(e^{i \sum \beta_k \sigma_k < N>}) = \left[ \cos \left( \frac{|\beta|}{\sqrt{2N+1}} \right) + i \frac{\beta_z}{|\beta|} \sin \left( \frac{|\beta|}{\sqrt{2N+1}} \right) \right] e^{-i \frac{\beta_z}{\sqrt{2N+1}} (2N+1)} =
\]

\[
= \lim_{N \to \infty} \left( 1 - \frac{|\beta|^2 - \beta_z^2}{2(2N+1)} \right)^{2N+1} = e^{-1/2(\beta^2 - \beta_z^2)}
\]

which equals the standard Gaussian state. In this sense \( e^{i \beta_x} \to 1, e^{i \beta_x} \to e^{i \beta_x}, e^{i \beta_y} \to e^{i \beta_y} \).

**Remark:** Two states lead to the same Weyl algebra resp. to the same map \( W_\omega \) if they yield the same \( s \) and it remains to see how far \( s \) determines \( \omega \). But since in the definition of the map \( W_\omega \) from the quasilocal algebra into the Weyl algebra the expectation value is included the state on the Weyl algebra for a given map is unique. Though other states on the Weyl algebra exist they cannot be constructed from a quasilocal state by a central limit theorem.

If on the quasilocal algebra \( A \) there exists an automorphisms \( \gamma \) for which \( \gamma A_{loc} \subset A_{loc} \) then we can assign to this automorphism an automorphism \( \gamma A \) on the Weyl-algebra \( W \) by \( \gamma W_\omega(q) = W_\omega(\gamma q) \) provided \( \omega \circ \gamma = \omega \) so that the symplectic form remains unchanged. In this result is generalized to automorphisms \( \gamma \) for which only \( \gamma A_{exp} \subset A_{exp} \) i.e. for typical time evolutions with short range interaction, provided also the state clusters exponentially. (For the detailed definition see [3])

But examining another famous example, the BCS-model, we can observe, that the time evolution on the fluctuation algebra is not inherited from the time evolution of the quasilocal algebra and \( \gamma \) may be misleading. As a simpler example we consider the following:

**Example:** We stay in the previous example and consider the time evolution given by a mean field hamiltonian

\[
H_N = \frac{1}{N} \sum_{i,j}^N \left( \sigma_{x,i} \sigma_{x,j} + \sigma_{y,i} \sigma_{y,j} + \sigma_{z,i} \sigma_{z,j} \right)
\]
For this Hamiltonian

\[
\text{st} - \lim_{N \to \infty} \frac{d}{dt} \tau_N \sigma_{ak} = \epsilon^{\alpha\beta\gamma} m_\beta \sigma_{\gamma k}, \quad m_\beta = \lim_{N \to \infty} \sum_{i=1}^{N} \frac{\omega(\beta_i)}{N}
\]

gives on the local level a rotation of the spins around the mean magnetization and is therefore state dependent. But for the fluctuation algebra we have the operator identity

\[
[H_N, \sum_k \frac{1}{\sqrt{N}} \sigma_{x,k}] = \sum_{i \neq k} \frac{1}{\sqrt{N}} (\sigma_{y,i} \sigma_{z,k} - \sigma_{z,i} \sigma_{y,k}) = 0.
\]

Therefore \( \tau_N \sigma_{<N>} = \sigma_{<N>} \). In the experiment [1] one studies the dynamics \( \tau_N \) over \( \sigma_{<N>} \) for \( N \sim 10^{10} \), therefore in the corresponding limit in the above example the fluctuation algebra is invariant under the time evolution.

### 3 The Model and its Time Evolution

In [1] two bulks of atoms are considered, independent from one another in the sense that the state factorizes, but one bulk is oppositely oriented to the other. The two bulks are influenced by the same laser beam. By measuring the laser beam after the interaction the two bulks become entangled. In this paper we want to study in detail when and how the entanglement emerges. In [3] the atoms in the bulks together with their interaction with the laser beam are described in such a way that it is justified to assign to the laser beam as well as to the bulks quasilocal algebras on a linear chain where the local algebra of the laser beam corresponds to the spin algebra describing the polarization of the laser whereas the local algebra of the bulks correspond to a finite representation of the angular momentum describing the different eigenstates of the individual atoms. The interaction between laser beam and bulk is given by a mean field Hamiltonian. Therefore we are in the framework where we can use the theory of the fluctuation algebra, but the passage from the mean field time evolution on the quasilocal algebra to the mean field time evolution of the fluctuation algebra does not work and the situation is similar to the BCS-example. The resulting entanglement resides in long range correlation that are too weak to be observed on a local level but only emerge in the fluctuation algebra.

**The Model:** We consider the algebra \( \mathcal{A} \otimes \mathcal{B} \otimes \mathcal{C} \) where

\[
\mathcal{A} = \otimes_i \mathcal{A}_i, \mathcal{A}_i = \{ \hat{\sigma}_i \}
\]

\[
\mathcal{B} = \otimes_i \mathcal{B}_i, \mathcal{B}_i = \{ \hat{j}_i \}
\]

\[
\mathcal{C} = \otimes_i \mathcal{C}_i, \mathcal{C}_i = \{ \sigma_i \}
\]

where \( \hat{\sigma}_i, \hat{j}_i \) are sets of independent Pauli matrices (or angular momenta). We assume that the state factorizes \( \omega = \omega_A \otimes \omega_B \otimes \omega_C \) with

\[
\omega_B(j_{i,y}^+) = \omega_B(j_{i,z}^+) = 0
\]
It is not necessary to specify expectation values of products, but we assume that the state is exponentially clustering in space. The influence of the laser beam on the bulks is given by a Hamiltonian

$$H_N = \frac{1}{2N} \sum_{l,i,k=1}^{N} \left[ a_x \sigma_{l,x}(j_{i,x}^+ + j_{k,x}^-) + a_y \sigma_{l,y}(j_{i,y}^+ + j_{k,y}^-) + a_z \sigma_{l,z}(j_{i,z}^+ + j_{k,z}^-) \right]$$

The $a_m \in \mathbb{R}$ include the possibility that $a_x \neq a_y \neq a_z$ so that we cover the time evolution in [1], but to simplify calculations we will sometimes assume rotation invariance. For fixed size of the samples, i.e. $N$ finite and equal for $\mathcal{A} , \mathcal{B} , \mathcal{C}$ the Hamiltonian determines a time evolution

$$\frac{d\sigma_{l,\alpha}}{dt} = a_\beta \epsilon^{\alpha\beta\gamma} \sigma_{l,\gamma} \sum_k \frac{j_{k,\beta}^+ + j_{k,\beta}^-}{N}.$$  
$$\frac{dj_{i,\alpha}^+}{dt} = a_\beta \epsilon^{\alpha\beta\gamma} \sum_l \frac{\sigma_{l,\beta} j_{i,\gamma}^+}{N}.$$  
$$\frac{dj_{i,\alpha}^-}{dt} = a_\beta \epsilon^{\alpha\beta\gamma} \sum_l \frac{\sigma_{l,\beta} j_{i,\gamma}^-}{N}.$$  

In the isotropic situation $a_x = a_y = a_z$ the automorphism $\tau_t$ for fixed $N$ can be written down explicitly. There the total angular momentum $\sum_k (\sigma_{\alpha,k} + \tilde{j}_{\alpha,k}^+ + \tilde{j}_{\alpha,k}^-) = D_\alpha$ is a constant and the motion is a rotation around it. Exponentiating the operator valued $3 \times 3$ matrix $(\epsilon D)_{\alpha\beta} = \epsilon_{\alpha\beta\gamma} D_\gamma$ we can write

$$\tau_t \sigma_{\alpha,k} = (e^{t \epsilon D})_{\alpha\beta} \sigma_{\beta,k}, \tau_t \tilde{j}_{\alpha,k}^\pm = (e^{t \epsilon D})_{\alpha\beta} \tilde{j}_{\beta}^\pm$$

The limit $N \to \infty$ has to be taken with care. On the level of the quasilocal algebra we know that

$$st - \lim \sum_l \frac{\sigma_{l,x}}{N} = s$$  
$$st - \lim \sum_l \frac{\sigma_{l,y}}{N} = st - \lim \sum_l \frac{\sigma_{l,z}}{N} = 0$$  
$$st - \lim \sum_k \frac{j_{k,x}^+ + j_{k,x}^-}{N} = \gamma - \gamma = 0$$  
$$st - \lim \sum_k \frac{j_{k,y}}{N} = st - \lim \sum_k \frac{j_{k,z}}{N} = 0$$
With the iterative solution e.g.

\[ \tau_{t,N}^{(n)} \sigma_{t,\alpha} = \sigma_{t,\alpha} + i \int_0^t dt' [H_N, \tau_{t',N}^{(n-1)} \sigma_{t',\alpha}] \]

we can conclude

\[ \text{st-} \lim_{N \to \infty} \tau_{t,N} \sigma_{t,\alpha} = \sigma_{t,\alpha}, \quad \alpha = x, y, z \]

\[ \text{st-} \lim_{N \to \infty} j_{k,x}^+ = \text{st-} \lim_{N \to \infty} j_{k,x}^- = 0 \]

and obtain an automorphism \( \tau_t \) that acts on the local algebra and satisfies

\[ \omega_A \otimes \omega_B \otimes \omega_C \circ \tau_t = \omega_A \otimes \omega_B \otimes \omega_C. \]

Our assumptions on the state guarantee that we can construct the fluctuation algebra. With the notation

\[ W_\omega(e^{i\alpha \vec{\sigma}}) = e^{i\alpha \vec{S}}, \quad W_\omega(e^{i\alpha \vec{j}^\pm}) = e^{i\alpha \vec{J}^\pm} \]

the commutation relations read

\[ [S_y, S_z] = \sum_j \omega([\sigma_{i,y}, \sigma_{j,z}]) = i\omega(\sigma_{i,z}) = is \]

\[ [S_y, S_x] = \omega(\sigma_{i,z}) = 0 = [S_z, S_x] \]

and similarly

\[ [J_y^+, J_z^+] = i\gamma = [J_z^-, J_y^-] \]

whereas all other commutators vanish. According to our previous consideration the time automorphism on the local algebra allows to construct a time automorphism on the Weyl algebra of fluctuations which acts as a rotation

\[ \tau\tau_t J_y^\pm = \cos \text{st} \tau_x J_y^\pm \pm \sin \text{st} \tau_x J_z^\pm. \]

But this time evolution is not the one given by \( \lim_{N \to \infty} \tau_{t,N}^{(n)} \sigma_{<N>} \). In fact we can rewrite

\[ \frac{d}{dt} \sum_{l} \frac{\sigma_{l,x} - s}{\sqrt{N}} = a_y \sum_{l} \frac{\sigma_{l,z}}{\sqrt{N}} \sum_{k} \frac{j_{k,y}^+ + j_{k,y}^-}{N} - a_z \sum_{l} \frac{\sigma_{l,y}}{\sqrt{N}} \sum_{k} \frac{j_{k,z}^+ + j_{k,z}^-}{N} \]

Again we can keep in mind that

\[ \text{st-} \lim \sum j_{k,x}^+ + j_{k,x}^- = \text{st-} \lim \sum \frac{j_{k,y}^+}{N} + \text{st-} \lim \sum \frac{j_{k,z}^+}{N} = 0 \]

Therefore, using \( [H, e^{i\alpha \vec{j}}] = \int_{0}^{\alpha} d\alpha' e^{i\alpha' j[H, j]} e^{i(\alpha-\alpha')j} \) we get

\[ \frac{d}{dt} \omega(e^{i\alpha \sigma_{<N>}(t)}) = \int_{0}^{\alpha} d\alpha' \omega(e^{i\alpha' \sigma_{<N>}(t)} \sigma_{<N>}(t)) \sum \frac{j_{k,y}^+(t) + j_{k,y}^-(t)}{N} e^{i(\alpha-\alpha')\sigma_{<N>}(t)} + y \leftrightarrow z \]

with time dependent operators. We solve the evolution equation by using the fact that the fluctuation algebra is not influenced by local perturbations so that the strong convergence
of the mean values appearing in the differential equation together with the fact that 
\( \omega_t(\sigma^z_{<N>}) \) can be controled on the quasilocal level and is uniformly bounded leads to

\[
\lim_{N \to \infty} \frac{d}{dt} \omega_t(e^{i\alpha \sigma^z_{<N>}}) = 0.
\]

For \( S_y \) we have to split

\[
\frac{d}{dt} \sum_l \frac{\sigma_{ly}}{\sqrt{N}} = -a_x \sum_l \frac{\sigma_{lx}}{\sqrt{N}} \sum_k \frac{j_{k,x}^+ + j_{k,x}^-}{N} + a_z \sum_l \frac{\sigma_{lx}}{N} \sum_k \frac{j_{k,z}^+}{\sqrt{N}}
\]

so that we can use the strong convergence of the mean values we obtain

\[
\lim_{N \to \infty} \omega_t(e^{i\alpha \sigma^y_{<N>}}) = \lim_{N \to \infty} \omega_C(e^{i\alpha \sigma^y_{<N>}}) \omega_{A \otimes B,t}(e^{i\alpha sa_z(j_{z<}^+, j_{z>}^-)})
\]

With a similar argument we obtain

\[
\frac{d}{dt} \sum_k \frac{j_{k,y}^+ - \gamma}{\sqrt{N}} = a_y \sum_l \frac{\sigma_{ly,y}}{\sqrt{N}} \sum_k \frac{j_{k,y}^+}{N} - a_z \sum_l \frac{\sigma_{lx}}{N} \sum_k \frac{j_{k,y}^+}{\sqrt{N}} = 0
\]

\[
\frac{d}{dt} \sum_k \frac{j_{k,y}^+}{\sqrt{N}} = -a_x \sum_l \frac{\sigma_{lx,y}}{\sqrt{N}} \sum_k \frac{j_{k,x}^+}{N} + a_z \sum_k \frac{j_{k,x}^+}{\sqrt{N}} \sum_l \frac{\sigma_{lx}}{\sqrt{N}}
\]

\[
\frac{d}{dt} \sum_k \frac{j_{k,y}^+ + j_{k,y}^-}{\sqrt{N}} = -a_x \sum_l \frac{j_{k,x}^+ + j_{k,x}^-}{\sqrt{N}} \sum_k \frac{\sigma_{lx}}{N} + a_z \sum_k \frac{j_{k,x}^+ + j_{k,x}^-}{\sqrt{N}} \sum_l \frac{\sigma_{lx}}{\sqrt{N}}
\]

These differential equations define a time evolution \( \tilde{\tau}_t \) on the Weyl-algebra

\[
\tilde{\tau}_t S_x = S_x
\]

\[
\tilde{\tau}_t S_y = S_y + sa_z \int_0^t dt' \tilde{\tau}^\prime_t (J^+_z + J^-_z)
\]

\[
\tilde{\tau}_t S_z = S_z - sa_y \int_0^t dt' \tilde{\tau}^\prime_t (J^+_y + J^-_y)
\]

\[
\tilde{\tau}_t J^+_x = J^+_x, \; \tilde{\tau}_t J^-_x = J^-_x
\]

\[
\tilde{\tau}_t (J^+_y + J^-_y) = \cos sta_x(J^+_y + J^-_y) + \sin sta_x(J^+_z + J^-_z)
\]

\[
\tilde{\tau}_t (J^+_z + J^-_z) = -\sin sta_x(J^+_y + J^-_y) + \cos sta_x(J^+_z + J^-_z)
\]

This can be seen by evaluating e.g.

\[
\lim_{N \to \infty} \frac{d}{dt} \omega(e^{iH_N t} e^{i\tilde{\tau}_t \sigma^z_{<N>}} e^{-iH_N t}) = 0
\]

Altogether we observe that the time automorphisms \( \tilde{\tau} \) and \( \bar{\tau} \) coincide on \( J^+ + J^- \) but they do not on \( S \). The automorphism \( \tilde{\tau}_t \) is on the level of the Weyl-algebra implemented by the operator

\[
\tilde{H} = sa_z(J^+_x + J^-_x)S_x + sa_y(J^+_y + J^-_y)S_y + \frac{sa_x}{\gamma}(J^+_z + J^-_z)(J^+_z - J^-_z) + \frac{sa_x}{\gamma}(J^+_y + J^-_y)(J^+_y - J^-_y).
\]
Generalizing these considerations to other operators of the Weyl-algebra we first consider as a typical example the operator $\sum_k \frac{1}{\sqrt{N}} j^+_{k,y} j^+_{k+l,z}$. If we start with an even state $\omega$ then for $q_1$ even and $q_2$ odd $s(q_1, q_2)$ vanishes. Hence for even $q$ as in our example the resulting Weyl-operator commutes with $e^{i\alpha \vec{S}}$, $e^{i\beta \vec{J}}$. Neglecting as before terms that strongly tend to 0 the time evolution is determined by

$$
\lim_{N \to \infty} \frac{d}{dt} \tau_{t,N} \sum_k \frac{j^+_{k,y} j^+_{k+l,z}}{\sqrt{N}} = \lim_{N \to \infty} (s a_x \sum_k \frac{j^+_{k,z} j^+_{k+l,z}}{\sqrt{N}} - s a_x \sum_k \frac{j^+_{k,y} j^+_{k+l,y}}{\sqrt{N}})
$$

so that this part of the Weyl-algebra inherits the automorphism of the quasilocal algebra. For another typical candidate we get

$$
\lim_{N \to \infty} \sum \frac{j^+_{y,i+l,y} j^+_{y,i+l,y}}{\sqrt{N}} = \lim_{N \to \infty} (-s \sum \frac{j^+_{i,x} j^+_{i+k,y} j^+_{i+l,y}}{\sqrt{N}} + \ldots) + \sum \frac{\sigma_{m,z} j^+_{i,x} j^+_{i+k,y} j^+_{i+l,y}}{\sqrt{N}}.
$$

Here we can use

$$
w - \lim_{N \to \infty} \sum_i \frac{j^+_{i,x} j^+_{i+k,y} j^+_{i+l,y}}{\sqrt{N}} = \lim_{N \to \infty} \omega(\sum_i \frac{j^+_{i,x} j^+_{i+k,y} j^+_{i+l,y}}{\sqrt{N}}, \frac{j^+_{m,z}}{\sqrt{N}}) = \gamma_{kl}
$$

with $\gamma_{kl}$ a c-number. If we therefore calculate the time evolution of the operator $\sum \frac{j^+_{i,x} j^+_{i+k,y} j^+_{i+l,y}}{\sqrt{N}} - \gamma_{kl} j^+_{i+y}$ that in the limit commutes with $\vec{S}$ and $\vec{J}$ then again the time evolution coincides with the time evolution inherited from the quasilocal algebra $\tau_t$. Generalizing our observation we take some $q \in B_{loc}$ where we assume, if necessary by replacing $q$ by $q + \alpha j^+_{i,y} + \beta j^+_{i,z}$ so that $[Q, J^\pm] = 0$. Then with $q_i = \alpha_i q$

$$
\lim \frac{1}{N} \sum_{i,k} \omega(\{H_N, q_i, j_k\}) = \lim \frac{1}{N} \sum_{i,k} \{\omega(\{j_k, H_N\}, q_i) + \omega(\{q_i, j_k\}, H_N)\}.
$$

Now we know on the one hand that $\sum_k \frac{1}{\sqrt{N}} [j_k, H_N]$ remains in this Weyl-algebra which commutes with the Weyl algebra to which $\sum_k \frac{1}{\sqrt{N}} q_k$ belongs so that the first contribution vanishes. In the second contribution $[q_i, j_k]$ reduces to a local operator. On local operators we have already proven that the time evolution corresponds to a local rotation. Together with the rotation invariance of the state this term vanishes, too. Altogether the time evolution in the complement of the Weyl algebra $\mathcal{W}(\vec{S}, J^\pm)$ remains in this complement, and on this complement $\tau = \tilde{\tau}$.

It remains to interpret the difference of $\tau \neq \tilde{\tau}$ on $\mathcal{W}(\vec{S}, J^\pm)$. It results from the coupling of the parameter $N$ in the mean field hamiltonian and in the fluctuation algebra. Whether $\tau$ or $\tilde{\tau}$ describes correctly the situation is determined by the experimental setup. In the experiment in $[\text{ref}]$ all atoms are influenced by the laser beam and therefore we have to choose $\tilde{\tau}$. But this has severe consequences on the interpretation of the fluctuation algebra. Whereas for the quasilocal state $\omega_t = \omega \circ \tau_t = \omega$ now

$$
\tilde{\omega}_t = \tilde{\omega} \circ \tilde{\tau}_t \neq \tilde{\omega}.
$$
Here we have an example where the commutation relations determined by \( s \) remain unchanged, whereas different to the considerations in [3] and [6] the state evolves in time and will not be reachable by quasilocal states. If we want to construct a quasilocal state that produces \( \tilde{\omega}_t \) we fail as can be seen in a counterexample:

Assume the initial state is a pure product state with \( \omega(\sigma_x) = \omega(j_x^+ + x) = 1 \) and also the \( \vec{j} \) are given in a two dimensional representation. Then

\[
\begin{align*}
\tilde{\omega}(e^{iS_x}) &= \tilde{\omega}(e^{iJ_x^+}) = \tilde{\omega}(e^{-iJ_x^-}) = 1 \\
\tilde{\omega}(e^{i\alpha S_y}) &= \tilde{\omega}(e^{i\alpha J_y^+}) = \tilde{\omega}(e^{i\alpha J_y^-}) = e^{-\alpha^2}.
\end{align*}
\]

The expectation values of \( S_x \) and \( J_x^\pm \) remain unchanged, therefore also the corresponding state over the quasilocal algebra remains unchanged, which is in contradiction to \( \tilde{\tau}_t(S_y) = S_y + a_t(J_y^+ + J_y^-) \) and hence

\[
\tilde{\omega}_t(e^{i\alpha S_y}) = e^{-\alpha^2(1+a_t^2)}.
\]

We can explain this effect by considering a sequence of states on the quasilocal algebra, where the fluctuations of \( S_x \) remain of order \( 1/N \) but the fluctuations of \( S_y \) are larger and are correlated over large distances according to the long range effect in the mean field hamiltonian.

### 4 The Entanglement in the Fluctuation algebra

Since on the fluctuation algebra our state is Gaussian good characterizations of entanglement are available [10]. We are interested in the entanglement of the tensor product of two fluctuations algebras resp. two Weyl-algebras. The basic facts are

\[
\delta A^2 + \delta B^2 \geq | <A, B>| \\
\delta_{\omega_A \otimes \omega_B} (A \otimes 1 + 1 \otimes B)^2 = \delta_{\omega_A} A^2 + \delta_{\omega_B} B^2
\]

If we therefore consider the variance of our operators \( \tilde{J}_k^\pm \) with \([J_y^\pm, J_z^\pm] = \pm i\) in appropriate units, then \( \delta(J^\pm_y)^2 + \delta(J^\pm_z)^2 \geq 1 \). Since by convex combinations of states the square fluctuations become greater or equal the convex combinations of the square fluctuations it follows that in all separable states

\[
\delta(J_y^+ + J_y^-)^2 + \delta(J_z^+ + J_z^-)^2 \geq \delta(J_y^+)^2 + \delta(J_y^-)^2 + \delta(J_z^+)^2 + \delta(J_z^-)^2 \geq 2.
\]

However the general inequality allows that the above fluctuations of two commuting operators can approach 0. In order to be sure that a resulting state is entangled it suffices to calculate that the fluctuations are sufficiently small. Exactly this consideration was the basis of the experiment in [1]: The bulk is influenced by a laser beam, and after a measurement on the laser beam the fluctuations in the bulk are examined and are so small, that the entanglement of the bulks is proven.
We want to examine in more detail, whether it is the time evolution or the measurement that is responsible for the entanglement. To simplify the calculation we assume that we start with a state that is invariant under the time evolution $\mathcal{T}_t$. As a consequence

$$\tilde{\omega} \circ \tilde{\tau}_t = \tilde{\omega} \circ \mathcal{T}_{-t} \circ \tilde{\tau}_t = \tilde{\omega} \circ \tilde{\tau}_t$$

where

$$\tilde{\tau}_t S_y = S_y + a_t (J_y^+ + J_y^-) + b_t (J_z^+ + J_z^-)$$

is generated by $e^{i(J_y^+ + J_y^-)S_y + i(J_z^+ + J_z^-)S_y}$ whereas $J_{y,z}^\pm$ are $\tilde{\tau}_t$ independent. Here $a_t$ and $b_t$ are some numbers that vary periodically with $t$ and therefore also can become 0, but with the appropriate choice of the mean field Hamiltonian also can become arbitrarily large.

We have to prove that the time evolution introduced by the mean field Hamiltonian though, as shown before, mixes the factors, it does not create entanglement in the sense that taking the partial trace over the laser algebra $\mathcal{A}$ results in a state over $\mathcal{B} \otimes \mathcal{C}$, that is not entangled. The state over $\mathcal{B} \otimes \mathcal{C}$ is determined by the expectation values of the Weyl operators and we have to remember that the Weyl-algebra inherits Gaussian states from the local algebra. Therefore we calculate with omitting or not specifying all unnecessary parameters

$$\text{Tr} e^{-i(J_y^+ + J_y^-)S_y - i(J_z^+ + J_z^-)S_y} \rho_{\mathcal{A}} \otimes \rho_{\mathcal{B}} \otimes \rho_{\mathcal{C}} e^{i(J_y^+ + J_y^-)S_y + i(J_z^+ + J_z^-)S_y} e^{i\alpha J_y^+ + i\beta J_y^- + i\gamma J_z^+ + i\delta J_z^-} =$$

$$\text{Tr} \rho_{\mathcal{A}} \otimes \rho_{\mathcal{B}} \otimes \rho_{\mathcal{C}} e^{i\alpha J_y^+ + i\beta (J_y^+ + S_y) + i\gamma (J_z^+ - S_y) + i\delta (J_z^- + S_y)} =$$

$$\text{Tr}_{\mathcal{B} \otimes \mathcal{C}} \rho_{\mathcal{B}} \otimes \rho_{\mathcal{C}} e^{i\alpha J_y^+ + i\beta J_y^- + i\gamma J_z^+ + i\delta J_z^-} \text{Tr}_{\mathcal{A}} \rho_{\mathcal{A}} e^{i(\alpha - \beta)S_y + i(\gamma + \delta)S_y} =$$

$$\int \text{Tr}_{\mathcal{B} \otimes \mathcal{C}} \rho_{\mathcal{B}} \otimes \rho_{\mathcal{C}} e^{i\alpha J_y^+ + i\beta J_y^- + i\gamma J_z^+ + i\delta J_z^-} e^{-c(\alpha - \beta)^2 - d(\gamma + \delta)^2} du dv.$$

In this way we have written the expectation value as an integral over factorizing states and therefore we see that in fact the time evolution does not produce entanglement.

It remains to examine whether measurements are able to destroy separability. First we consider only the maximally abelian subalgebra $\mathcal{M}$ generated by $S_y, J_y^+ + J_y^- =: J_y, J_z^+ + J_z^- =: J_z$. We start with a product state which reduced to

$$\text{Tr} e^{-\frac{1}{2}(aS_y^2 + b J_y^2 + c J_z^2)}$$

where $a, b, c$ are parameters that vary periodically with $t$ and therefore also can become 0, but with the appropriate choice of the mean field Hamiltonian also can become arbitrarily large.

$$\text{Tr} e^{-\frac{1}{2}(aS_y^2 + b J_y^2 + c J_z^2)}$$

with $c$ and $s$ abbreviating cos and sin. If we measure $S_y$ this corresponds to multiplication with a characteristic function. For convenience we replace it by a Gaussian $e^{-\frac{1}{2}dS_y^2}$ and our probability distribution becomes

$$e^{-\frac{1}{2}(a + d) + \frac{\alpha^2}{a + d} \chi_0} - \frac{ad}{a + d} \alpha^2 \chi_0^2 - \frac{1}{2} (J_y^2 + J_z^2).$$

Reduction to $J_y, J_z$ is obtained by integration $\int dS_y$ such that we finally get

$$e^{-\frac{1}{2}(a + d + \frac{\alpha^2}{a + d} (c_1 - 1)^2)} - \frac{ad}{a + d} \alpha^2 - J_y J_z \frac{ad}{a + d} \alpha^2 (1 - c_1) \chi_t.$$
This probability distribution corresponds to fluctuations

\[ \delta J^2_x = \frac{(a + d)a_x^2 + ada_z^2(c_t - 1)^2}{(a + d)a_x^2 + 2ada_z^2(1 - c_t)} \]

\[ \delta J^2_y = \frac{(a + d)a_y^2 + ada_z^2s_t^2}{(a + d)a_y^2 + 2ada_z^2(1 - c_t)} \]

\[ \delta J^2_x + \delta J^2_y = \frac{2(a + d)a_x^2 + 2ada_z^2(1 - c_t)^2}{(a + d)a_x^2 + 2ada_z^2(1 - c_t)} \]

For \( a_x << a_z \) and \( c_t \to \pm 1 \) the individual fluctuations can be made arbitrarily small but their sum is always \( > 1 \). This means that we can go below the limit 2 and thus generate entanglement, but the limit 0 cannot be reached by this kind of measurement.

Nevertheless we know that \( J_y \) and \( J_z \) commute hence \( \delta J^2_y + \delta J^2_z \) arbitrarily small is compatible with the algebraic structure. Since any function on \( J_y \) can only project on an infinite dimensional subspace we next try whether an optimal measurement, i.e. a measurement corresponding to a one dimensional projector can produce arbitrarily small fluctuations for both \( J_y, J_z \). In this case we cannot work only with the maximal abelian subalgebra but really have to project in the Hilbert space \( \mathcal{H}_A \). We assume in addition that we start with a pure state over \( A \) given by a Gaussian vector and let it evolve with \( e^{-iHt} \). We start with the Gauss function \( |e^{-\frac{i}{2}(J_y^2 - \frac{1}{2}(J_y^2 + J_z^2))} \).

The \( J \) part is invariant under the generator \( H_M \) of the microscopic evolution so that we can use only the unitaries \( e^{-iHt}e^{iH_Mt} \). Up to an irrelevant phase factor this unitary has the form \( e^{iS_y\frac{a_y}{a_z}[c_t(c_t - 1)]J_y + \frac{a_y}{a_z}[c_t(c_t - 1)]J_y + s_tJ_z + iS_z\frac{a_z}{a_x}[c_t(c_t - 1)]J_z - s_tJ_y} \). It transforms \( e^{-\frac{i}{4}S_y} \) into \( e^{iS_y\frac{a_y}{a_z}[c_t(c_t - 1)]J_y + s_tJ_z} \). Performing a measurement corresponding to the projector onto a Gauss function \( |e^{-\frac{i}{4}S_y^2} \) implies that we have to take the square of the scalar product of the two Gauss function in the \( \mathcal{H}_A \) space:

\[ \int dS_y e^{-\frac{a_y}{4}(S_y + \frac{a_y}{a_z}[c_t(c_t - 1)]J_y + s_tJ_z) + \frac{1}{2}(\frac{a_y}{a_z}[c_t(c_t - 1)]J_y - s_tJ_y)^2 + i\delta J^2_y} \]

\[ \int dS_y e^{-\frac{a_z}{4}(S_z + \frac{a_z}{a_x}[c_t(c_t - 1)]J_z + s_tJ_y) + \frac{1}{2}(\frac{a_z}{a_x}[c_t(c_t - 1)]J_z - s_tJ_y)^2 + i\delta J^2_z} \]

the imaginary part drops out and we remain with

\[ e^{-\frac{1}{2}(\frac{a_y}{a_z}[c_t(c_t - 1)]J_y + s_tJ_z)^2 + \frac{a_z}{a_x}(\frac{a_z}{a_x}[c_t(c_t - 1)]J_z - s_tJ_y)^2} \]

Together with the rest we get \( e^{-\frac{a_y}{2}J_y^2 - \frac{a_z}{2}J_z^2 - \beta J_yJ_z} \) where

\[ \alpha = 1 + \frac{ada_z^2s_t^2 + a_y^2(c_t - 1)^2}{a_x^2(a + d)} \]

\[ \gamma = 1 + \frac{ada_z^2(c_t - 1)^2 + a_y^2s_t^2}{a_x^2(a + d)} \]
\[
\beta = \frac{s_l(c_l - 1)ad(a_y^2 + a_y^2)}{a_x^2(a + d)}
\]

Now the square fluctuations become

\[
\delta J_y^2 + \delta J_z^2 = \frac{\alpha + \gamma}{\alpha \gamma - \beta^2} = \frac{2a_x^2(a + d) + 2(ada_y^2 + a_y^2)(1 - c_t)}{a_x^2(1 - c_t) + 4a_x^2a_y^2c_t(1-c_t)^2}
\]

Since for \( a_x \to 0 \) we can make the last term in the denominator arbitrarily big in a moment where \( c_t \neq 0,1 \) the sum of the square fluctuations can become arbitrarily small.

Collecting these observations we see, that we are in a similar situation as for the GHZ state \( \uparrow\uparrow\uparrow + \downarrow\downarrow\downarrow \): we deal with three systems, Alice for the laser and Bob and Charles for the two bulks. The initial pure product state evolves in time to a state that similar to the GHZ state is no product state any more but reduced to Bob and Charles is separable (the tracial state in the GHZ-example). But this state can be transformed into a state that is optimally entangled for Bob and Charles and decoupled from Alice by a pure local manipulation of Alice (not of Bob and Charles). For the GHZ state this is

\[
| \uparrow + \downarrow \rangle \langle \uparrow + \downarrow | \otimes 1 | \uparrow\uparrow\uparrow + \downarrow\downarrow\downarrow \rangle = | \uparrow + \downarrow \rangle \otimes | \uparrow\uparrow + \downarrow\downarrow \rangle
\]

5 **Microscopic Effects on the Entanglement**

Having created entanglement via a measurement we can still wonder which are the possible entanglement witnesses and how stable the entanglement is with respect to the time evolution of the bulk. The latter question has interesting experimental consequences [12].

We have so far localized the entanglement in the Weyl-algebra \( \mathcal{W}(J_y^+, J_z^+, J_y^-, J_z^-) \). From [11] we know that the entanglement can be observed by a violation of a [CHSH] inequality or by violating [12] the positivity criterium [13]. Enlarging the Weyl-algebra can not increase the entanglement, because we have already observed that in the Weyl algebra of fluctuations outside of \( \mathcal{W}(J_y^+, J_z^+, J_y^-, J_z^-) \) the mean field hamiltonian reduces to a strictly local time evolution. Also on the quasilocal level the entanglement can not be observed, because here the relevant time evolution \( \tau \) does not even change the state. Correspondingly the entanglement sits in long range correlations, that do not appear on the local level. If we now assume that on the quasilocal level the time evolution after the interaction with the laser beam is given by an automorphism \( \gamma_t \) such that \( \gamma_t \mathcal{B} = \mathcal{B} \) and \( \gamma_t \mathcal{C} = \mathcal{C} \) and \( \gamma_t \mathcal{B}_{exp} \subset \mathcal{B}_{exp} \) and satisfying \( \omega \circ \gamma_t = \omega \) and \( \gamma_t \circ \alpha^j = \alpha^j \circ \gamma_t \) then the symplectic form of the Weyl algebra is stable under \( \gamma_t \). Therefore \( \gamma_t \) can be defined on the Weyl-algebra

\[
\tilde{\gamma}_t \mathcal{W}(q) = \mathcal{W}(\gamma_t q).
\]
\( \tilde{\gamma}_t \) respects the tensor product structure of the Weyl-algebra, though it will transfer the entanglement to other witnesses. Of course we should keep in mind that the state \( \hat{\omega}_t = \omega \circ \tilde{\tau}_t \) does not correspond to a quasi local state. Therefore the passage from \( \gamma \) to \( \tilde{\gamma} \) is not justified by the considerations in chapter 2. We have to expect that the effect of microscopic time evolution on the fluctuations might influence the atypical long range correlations and could sweep out the entanglement, though it is not implausible, that this sweeping effect appears only in a different order of magnitude in time.

6 Conclusion

Based on the experiment described in [1] and on the considerations offered there we clarify that the time evolution has to be expressed on the fluctuation algebra, i.e. on a mesoscopic level, whereas it does not produce any change of the state on the microscopic level. As a consequence the spatial correlations decay differently than before though not in a way that would be observable on a microscopic level. Examining the time evolved state of the bulks after the interaction with the laser beam is switched of the state of the two fluctuation bulks does not factorize any more but it remains separable. Measuring the laser beam appropriately one can produce entanglement of the two bulks, similar as in the GHZ experiment. Therefore it is not necessary to expose the two bulks to two different laser beams as it is done in [1], the second laser beam is only a tool to observe the entanglement. On the other hand it does not suffice to expose the bulks to a laser beam, a measurement on the laser beam is necessary to produce entanglement.

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