Abstract

We discuss the problem of finding "marginal" distributions within different tomographic approaches to quantum state measurement, and we establish analytical connections among them.

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1 Introduction

Recently the problem of measuring quantum states in quantum optics and quantum mechanics found experimental realization. Different procedures for measuring quantum states were suggested, i.e., to measure an observable and from the experimental result of the observations to reconstruct the density matrix of the quantum system in terms, for example, of its Wigner function, or its quasidistribution function like the Husimi Q-function and the Glauber–Sudarshan P-distribution.
The special interest to the problem of measuring the quantum states is related to the experimental realization of specific nonclassical quantum states \[9, 10\]. Such nonclassical states, as one-mode even and odd coherent states \[11\] (or Schrödinger cat states) were created in high-quality cavity (see, for example, \[12\]). It was also shown \[13\] that three-mode even and odd coherent states might be created in such experiments. For an ion in a Paul trap, the even and odd coherent states may be realized \[2, 14, 15\], as well as the new type of nonclassical states like nonlinear coherent states \[16, 17\].

Thus, the problem of measuring such states, i.e., reconstructing their density matrix in any representation, is actual. For trapped ion several methods were indeed suggested for such a reconstruction \[18\]. Instead to determine the state of a radiation field there exists a dominant method, called optical tomography method \[1\], which is based on a relation of the Wigner function to the distribution function of a homodyne observable \[19, 20\], which may be derived from the general scheme of s-ordered quasidistributions \[21\].

Recently, the optical tomography method was extended into symplectic tomography \[22\], in which the basic observable is the generic squeezed and rotated quadrature. It was used to find classical-like description of the quantum dynamic evolution \[23\]. If in optical tomography the Wigner function is reconstructed from measurable marginal distribution of the homodyne observable through Radon transform, in the symplectic tomography the Fourier transform is applied to reconstruct the density matrix by measuring the marginal distribution of the squeezed and rotated quadrature. In this context, the symplectic tomography is similar to the field-strength method of Ref. \[24\]. The symplectic tomography was generalized for multimode case as well \[25\].

Recently, it was also suggested the photon number tomography method \[26, 27, 28\], for which the measurable observable is the discrete photon number of the measurable field superposed with the local oscillator field by scanning its complex amplitude. The photon number distribution, depending on the complex amplitude of the local oscillator field, may be converted into the density matrix of the measurable field state. The symplectic tomography of nonclassical states (namely, squeezed states and Schrödinger cat states) of an ion in a Paul trap was discussed in Ref. \[29\].
The reconstruction of the density matrix of the squeezed vacuum, by using reproducible measurements [30] of the distribution for discrete photon numbers, was recently reported in Ref. [4].

Thus, in the theoretical and experimental framework of measuring quantum states there exist different methods yielding the density matrix from the corresponding measurable probability distributions. On the other hand, till now the relations among all these distributions were not clarified in details, to our knowledge. The aim of this work is to establish the invertible transformations of the symplectic quadrature distribution to the photon number distribution as well as to study the connection of the symplectic tomography scheme to the optical tomography. Some aspects of this problem were discussed in Ref. [31]. The relation of the Radon transform to the symplectic tomography procedure was discussed in Ref. [32]. By finding the explicit relations among the different measurable probability distributions one is aimed to make prediction of measurements for different observables, from measurements, for example, of the marginal distribution of the generic squeezed and rotated quadrature in the symplectic tomography scheme.

The paper is organized as follows. In Section 2, we review the properties of the marginal distributions extending the concept of "marginalizations" from the optical tomography to the symplectic, and photon number tomography. In Section 3, the connections among the various "marginal" distributions are given. The main results obtained are summarized in the Conclusion.

2 Marginal Distributions

By referring to the standard definitions given in the literature [5, 20], by "marginalization" one should intend a line integral in the phase space \( \{ q, p \} \) of the Wigner function \( W(q, p) \), i.e.

\[
 w(x, \theta) = \int \frac{dqdp}{(2\pi)^2} W(q, p) \delta(x - \cos \theta q - \sin \theta p),
\]

where \( \theta \) is the angle orientation of the line. On the other hand, the marginal distribution \( w(x, \theta) \) can be considered as well as the Fourier transform of the characteristic function \( \chi(k) \)

\[
 w(x, \theta) = \int \frac{dk}{2\pi} \chi(k) e^{-ikx},
\]
with
\[ \chi(k) = \langle e^{ik(\cos \theta q + \sin \theta p)} \rangle. \] (3)

The marginal distribution represents a true measurable probability, at least in the field of optics, by means of homodyne detection methods. Furthermore, recently has been shown [19] that from a collection of marginal distributions it is possible to recover the phase space pseudo-disturbances (tomography) or, even better, the density operator [33] representing the quantum state of the system under study.

Our purpose is now to generalize the previous definition of the marginalization in order to include all measurable probabilities related to observables which allow a tomographic reconstruction of the quantum state [31].

In fact, the tomographic principle may be generalized as follows: given a density operator \( \rho \) and a group element \( \mathcal{G} \) (or to better say an operator belonging to representation of a group, which acts in the space of quantum states), one can create different types of tomography if, by knowing the matrix elements
\[ w(y, \mathcal{G}) = \langle y | \mathcal{G} \rho \mathcal{G}^{-1} | y \rangle \] (4)
from measurements, is able to invert the formula expressing the density operator in terms of the above “marginal” distribution \( w(y, \mathcal{G}) \), where \( y \) may denote either continuous or discrete eigenvalues. The positive distribution \( w(y, \mathcal{G}) \) is normalized
\[ \int w(y, \mathcal{G}) \, dy = 1. \] (5)

It is to remark that, to simplify the terminology, from now on we refer indistinctly to any positive, measurable, and normalized probability distribution like of Eq. (4) as marginal distribution.

To get the density operator from a marginal distribution an inversion procedure should be employed, and one can use the properties of summation or integration over group parameters \( \mathcal{G} \). Thus, the marginal distribution of the observable \( y \) depends on extra parameters determining the operator \( \mathcal{G} \). It is worth noting that the operator \( \mathcal{G} \) may belong not only to a group but also to
other algebraic construction, for example, to quantum group. The only problem is mathematical one, to make the inversion, and/or physical one, to realize the transformation $G$ the in laboratory.

Along this line the “symplectic tomography” has been introduced in Ref. [25] by means of the transformation

$$\hat{x} = \mu \hat{q} + \nu \hat{p}$$

with the real parameters $\mu, \nu$ generalizing the homodyne rotation. In this case the marginal distribution becomes

$$w(x, \mu, \nu) = \int \frac{dq \, dp}{(2\pi)^2} W(q, p) \delta(x - \mu q - \nu p),$$

while the density operator can be written in the invariant form as

$$\hat{\rho} = \int dx \, d\mu \, d\nu \, w(x, \mu, \nu) \hat{K}_{\mu, \nu},$$

with the kernel operator

$$\hat{K}_{\mu, \nu} = \frac{1}{2\pi} \frac{z^2 e^{-izx} e^{-z(\nu - i\mu)\hat{a}^\dagger / \sqrt{2} e^{z(\nu + i\mu)\hat{a} / \sqrt{2}} e^{-z^2(\mu^2 + \nu^2)/4}}.}$$

Here $z$ represents the variable conjugate to $x$ by the Fourier transform. The fact that $\hat{K}_{\mu, \nu}$ depends on the $z$ variable as well (i.e., each Fourier component gives a selfconsistent kernel) shows the overcompleteness of information achievable by measuring the observable of Eq. (6). Due to this, one could set $z = 1$ in (9).

Beside that, a “photon number tomography” was developed in Ref. [28] parallely to Refs. [26, 27]. It is based on the possibility to measure the number of photons in the state displaced over the phase space. Hence, the marginal distribution will be

$$w(n, \alpha) = \text{Tr} \{ \hat{D}(\alpha) \hat{\rho} \hat{D}^{-1}(\alpha) |n\rangle \langle n| \} = \langle n | \hat{D}(\alpha) \hat{\rho} \hat{D}^{-1}(\alpha) |n\rangle,$$

and it also corresponds to a ”propensity“ [34] obtained via filtering the original state described by $\hat{\rho}$ with the filter in a Fock state. The propensities have been also studied in connection with a quantum state reconstruction [33]. In some sense the probability $w(n, \alpha)$ differs from the other distribution $w(x, \mu, \nu)$ since it is intrinsically a phase space distribution (in the case of $n = 0$ it
corresponds to the Husimi-Q function). However, for convenience, we still refer to it as a marginal distribution in the sense previously explained.

The invariant expression for the density operator in terms of the distribution (10) is
\[
\hat{\rho} = \sum_{n=0}^{\infty} \int \frac{d^2\alpha}{\pi} w(n, \alpha) \hat{K}_s(n, \alpha),
\]
(11)
with the kernel given by
\[
\hat{K}_s(n, \alpha) = \frac{2}{1-s} \left( \frac{s+1}{s-1} \right)^n \hat{T}(-\alpha, -s).
\]
(12)
The marginal distribution is normalized as
\[
\sum_{n=0}^{\infty} w(n, \alpha) = 1.
\]
(13)
The operator \(\hat{T}\) represents the complex Fourier transform of the \(s\)-ordered displacement operator \(\hat{D}(\xi, s) = \hat{D}(\xi)e^{s|\xi|^2/2}\), which can also be written as \[21\]
\[
\hat{T}(\alpha, s) = \frac{2}{1-s} \hat{D}(\alpha) \left( \frac{s+1}{s-1} \right)^{\hat{a}^\dagger \hat{a}} \hat{D}^{-1}(\alpha).
\]
(14)

3 Connection among marginal distributions

Here we consider the possibility to express the above marginal distributions as function one of other.

First, since the operator (11) is proportional to the displacement operator generating the coherent state from the vacuum, one can use the known matrix elements of the displacement operator in the Fock basis, expressed in terms of Laguerre polynomials \[21\], to get the density matrix in the photon number basis as a convolution of the marginal distribution \(w(x, \mu, \nu)\)
\[
\langle m|\rho|n \rangle = \sqrt{\frac{n!}{m!} \frac{2^{(n-m)/2}}{2\pi}} \int e^{ix-(\mu^2+\nu^2)/4} w(x, \mu, \nu) (\nu - i\mu)^{m-n} L_{m-n}^{m-n} \left( \frac{\nu^2 + \mu^2}{2} \right) dx d\mu d\nu; \quad m > n.
\]
(15)
For \(m < n\), we use the hermiticity property of the density matrix \(\langle m|\rho|n \rangle = \langle n|\rho|m \rangle^*\). The diagonal matrix elements of the density matrix in the Fock basis give the photon distribution in
terms of the marginal distribution
\[
\langle n | \rho | n \rangle = P(n) = \frac{1}{2\pi} \int e^{ix-(\mu^2+\nu^2)/4} w(x, \mu, \nu) L_n \left( \frac{\nu^2 + \mu^2}{2} \right) dx \, d\mu \, d\nu.
\] (16)

Let us now rewrite Eq. (10) as
\[
w(n, \alpha) = \sum_{k,l} \langle n | D(\alpha) | k \rangle \langle k | \rho | l \rangle \langle l | D^{-1}(\alpha) | n \rangle,
\] (17)
then we can use the density matrix elements of Eq. (15) and the number representation of the displacement operator given in Ref. [21], to get
\[
w(n, \alpha) = \frac{1}{2\pi} \int w(x, \mu, \nu) \exp \left\{ ix - \frac{\mu^2 + \nu^2}{4} + \frac{\alpha(\nu + i\mu)}{\sqrt{2}} - \frac{\alpha^*(\nu - i\mu)}{\sqrt{2}} \right\}
\times L_n \left( \frac{\mu^2 + \nu^2}{2} \right) dx \, d\mu \, d\nu.
\] (18)

For \( \alpha = 0 \), Eq. (18) gives \( w(n, 0) = P(n) \), where \( P(n) \) is the photon distribution (16).

To derive the inverse relation, we start with an oscillator in the thermal equilibrium state. One can easily obtain the quadrature marginal distribution for the thermal equilibrium state of the harmonic oscillator, which is interacting with a heat bath at dimensionless temperature \( T \), in the form
\[
w_T(x, \mu, \nu) = \left[ \pi (\mu^2 + \nu^2) \coth [(2T)^{-1}] \right]^{-1/2} \exp \left[ -\frac{x^2}{(\mu^2 + \nu^2) \coth [(2T)^{-1}]} \right].
\] (19)

It means that for the non normalized state density operator of the harmonic oscillator of the form
\[
\hat{\rho}_{\text{non}} = \exp \left( -\frac{a^\dagger a}{T} \right)
\] (20)
we have the corresponding non normalized marginal distribution of the form \( \hat{\rho}_{\text{non}} \rightarrow w_{\text{non}} \)
\[
w_{\text{non}}^T(x, \mu, \nu) = \frac{1}{1 - \exp (-1/T)} \, w_T(x, \mu, \nu).
\] (21)

If the density operator corresponds to the shifted equilibrium position in the driven oscillator phase space, i.e.,
\[
\hat{\rho}_{\text{shift}} = D(\alpha) \hat{\rho}_{\text{non}} D(-\alpha) = \exp \left( -\frac{(a^\dagger - \alpha^*) (a - \alpha)}{T} \right),
\] (22)
we have the non normalized marginal distribution of the driven oscillator

$$w_{\alpha,s}(x, \mu, \nu) = w_T^{\text{non}} \left( x - \mu \frac{\alpha + \alpha^*}{\sqrt{2}} - \nu \frac{\alpha - \alpha^*}{i \sqrt{2}}, \mu, \nu \right).$$  \hspace{1cm} (23)

One can see that the temperature $T$ is

$$T = \left[ \ln \frac{s - 1}{s + 1} \right]^{-1}$$  \hspace{1cm} (24)

and the operator $[(1 - s)/2] \hat{T}(\alpha, s)$ has exactly the form of the operator (22). It means that the non normalized marginal distribution corresponding to the operator $\hat{T}(\alpha, s)$ is

$$w_{\alpha,s}(x, \mu, \nu) = \frac{1 - s}{2} \frac{1}{1 - \exp(-1/T)} \left[ \pi (\mu^2 + \nu^2) \coth[(2T)^{-1}] \right]^{-1/2} \exp \left\{ - \left( x - \left[ \frac{\mu (\alpha + \alpha^*)}{\sqrt{2}} \right] - \left[ \frac{\nu (\alpha - \alpha^*)}{i \sqrt{2}} \right] \right)^2 \right\},$$  \hspace{1cm} (25)

where the temperature $T$ is given by (24). Expression (25) gives the connection of the marginal distribution $w(x, \mu, \nu)$ with the photon number marginal distribution $w(n, \alpha)$. In fact, by taking the expectation value of both side of Eq. (11) over the eigenket $|x\rangle$ of the quadrature operator (4), one obtains

$$w(x, \mu, \nu) = \int \frac{d^2\alpha}{\pi} \sum_{m=0}^{\infty} \frac{2}{1 - s} \left( \frac{s + 1}{s - 1} \right)^m w(m, \alpha) w_{-\alpha,-s}(x, \mu, \nu),$$  \hspace{1cm} (26)

where

$$w_{-\alpha,-s}(x, \mu, \nu) = \frac{1 + s}{2} \frac{1}{1 - \exp(-1/T)} \left[ \pi (\mu^2 + \nu^2) \coth[(2T)^{-1}] \right]^{-1/2} \exp \left\{ - \left( x + \left[ \frac{\mu (\alpha + \alpha^*)}{\sqrt{2}} \right] + \left[ \frac{\nu (\alpha - \alpha^*)}{i \sqrt{2}} \right] \right)^2 \right\}$$  \hspace{1cm} (27)

and the “temperature” $T$ is given by (24) with $s \to -s$, i.e.,

$$T = \left( \ln \frac{s + 1}{s - 1} \right)^{-1}.$$  \hspace{1cm} (28)

The equation (26), along with (27) and (28), is the inverse relation for (18). Thus, measuring the quadrature marginal distribution $w(x, \mu, \nu)$ in homodyne experiments one can predict the
results of measurement of the photon marginal distribution and, vice versa, measuring the photon marginal distribution one can predict the results of measurements of the quadrature marginal distribution.

Let us now address the question how to find the rotated quadrature marginal distribution \( w(x, \theta) \) of the optical tomography scheme if one knows the squeezed and rotated quadrature distribution \( w(x, \mu, \nu) \) of symplectic tomography. The answer is obvious, namely,

\[
w(x, \cos \theta, \sin \theta) = w(x, \theta).
\]

Formula (29) may be rewritten in the integral form

\[
w(x, \theta) = \int w(x, \mu, \nu) \delta(\mu - \cos \theta) \delta(\nu - \sin \theta) \, d\mu \, d\nu.
\]

So, given the function \( w(x, \mu, \nu) \) one gets the homodyne marginal distribution \( w(x, \theta) \). The inverse problem may also be solved, namely, how to find the marginal distribution of the symplectic tomography \( w(x, \mu, \nu) \), if one knows the homodyne marginal distribution \( w(x, \theta) \). It means that knowing the distribution function of two variables one has to reconstruct the distribution function of three variables.

To make the reconstruction, we first decompose the periodic function \( w(x, \theta) \) into the Fourier series

\[
w(x, \theta) = \sum_{n=0}^{\infty} \left[ c_n(x) \cos n\theta + d_n(x) \sin n\theta \right].
\]

Then, in view of the known expressions for \( \cos n\theta, \sin n\theta \) as polynomials in \( \sin \theta, \cos \theta \), we make in (31) the replacement

\[
\cos \theta \to \mu; \sin \theta \to \nu.
\]

Thus, we get the expression of the marginal distribution \( w(x, \mu, \nu) \) in the form

\[
w(x, \mu, \nu) = \frac{1}{2\pi} \int_0^{2\pi} w(x, \theta) \, d\theta + \frac{1}{2\pi} \sum_{n=1}^{\infty} \left[ (\mu + i\nu)^n \frac{1}{\pi} \int_0^{2\pi} e^{-in\theta} w(x, \theta) \, d\theta + c.c. \right].
\]

Finally, by knowing the marginal distribution of the optical tomography scheme we directly obtain the marginal distribution of symplectic tomography.
Using (26) one can find the connection of the photon number tomography to the optical tomography formalism. The marginal distribution of the rotated quadrature is expressed in terms of the photon number marginal distribution as follows

\[
w(x, \theta) = \int \frac{d^2\alpha}{\pi} \sum_{m=0}^{\infty} \frac{2}{1-s} \left( \frac{s+1}{s-1} \right)^m w(m, \alpha) w_{-\alpha, -s}(x, \cos \theta, \sin \theta),
\]

where

\[
w_{-\alpha, -s}(x, \cos \theta, \sin \theta) = \frac{1 + s}{2} \frac{1}{1 - \exp(-1/T)} \left[ \pi \coth((2T)^{-1}) \right]^{-1/2} \\
\times \exp \left\{-\left(x + \frac{\cos \theta (\alpha + \alpha^*)/\sqrt{2}}{\coth((2T)^{-1})} + \frac{\sin \theta (\alpha - \alpha^*)/(i\sqrt{2})}{\coth((2T)^{-1})}\right)^2\right\}
\]

(33)

and the “temperature” \( T \) is given by (28).

4 Conclusion

We have provided analytical relations which connect the ”marginal” distributions of different tomographic processes. Physically they will give predictions on how to make cross checking of the state measurements or how, by using one sort of measurement, to reconstruct another sort, i.e., by measuring an observable one gets information on the distribution of other observables.

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