Estimates of anisotropic Sobolev spaces with mixed norms for the Stokes system in a half-space

Tongkeun Chang$^1$ and Kyungkeun Kang$^2$

1) Department of Mathematics, Yonsei University, Seoul 120-749, South Korea
2) Department of Mathematics, Yonsei University, Seoul 120-749, South Korea

ABSTRACT

We are concerned with the non-stationary Stokes system with non-homogeneous external force and non-zero initial data in $\mathbb{R}^n_+ \times (0, T)$. We obtain new estimates of solutions including pressure in terms of mixed anisotropic Sobolev spaces. As an application, some anisotropic Sobolev estimates are presented for weak solutions of the Navier-Stokes equations in a half-space in dimension three. To be more precise, we study the non-stationary Stokes system in a half-space $\mathbb{R}^n_+ \times [0, T)$, $n \geq 3$

\[ v_t - \Delta v + \nabla p = f, \quad \text{div} \, v = 0 \quad \text{in} \, Q^+_T := \mathbb{R}^n_+ \times [0, T), \quad (1) \]

where $v : Q^+_T \to \mathbb{R}^n$ is the velocity field and $p : Q^+_T \to \mathbb{R}$ is the pressure. We consider the initial and boundary value problem of (1), whereby no slip boundary conditions are assigned, that is

\[ v(x, 0) = v_0(x) \quad \text{and} \quad v(x, t) = 0, \quad x \in \partial \mathbb{R}^n_+ = \mathbb{R}^{n-1}. \quad (2) \]

One of main results reads as follows:

**Theorem 0.1** Let $0 \leq \alpha \leq 2$, $0 < T \leq \infty$ and $1 < p, q < \infty$. Suppose that $f \in H^{\alpha-2, \frac{1}{2} \alpha-1}_{pq, 0} (\mathbb{R}^n_+ \times (0, T))$ and $v_0 \in B^{\alpha-2, -\frac{1}{2} \alpha}_{pq, 0} (\mathbb{R}^n_+)$ with $\text{div} \, f = 0$ and $\text{div} \, v_0 = 0$ in the sense of distributions. Then, there exists a unique solution $v$ in $H^{\alpha-2, \frac{1}{2} \alpha}_{pq, 0} (\mathbb{R}^n_+ \times (0, T))$ of (1) such that the following estimate is satisfied:

\[ \|v\|_{H^{\alpha-2, \frac{1}{2} \alpha}_{pq, 0} (\mathbb{R}^n_+ \times (0, T))} \leq c\|f\|_{H^{\alpha-2, \frac{1}{2} \alpha-1}_{pq, 0} (\mathbb{R}^n_+ \times (0, T))} + c\|v_0\|_{B^{\alpha-2, -\frac{1}{2} \alpha}_{pq, 0} (\mathbb{R}^n_+)}. \quad (3) \]

Moreover, if $1 + 1/p < \alpha \leq 2$, then $p \in L_q(0, T; H^{\alpha-2, \frac{1}{2} \alpha-1}_{p, 0} (\mathbb{R}^n_+))$ such that

\[ \|p\|_{L_q(0, T; H^{\alpha-2, \frac{1}{2} \alpha-1}_{p, 0} (\mathbb{R}^n_+))} \leq c\|f\|_{H^{\alpha-2, \frac{1}{2} \alpha-1}_{pq, 0} (\mathbb{R}^n_+ \times (0, T))} + c\|v_0\|_{B^{\alpha-2, -\frac{1}{2} \alpha}_{pq, 0} (\mathbb{R}^n_+)}. \quad (4) \]

If $f \in \dot{H}^{\alpha-2, \frac{1}{2} \alpha-1}_{pq, 0} (\mathbb{R}^n_+ \times (0, T))$ and $v_0 \in \dot{B}^{\alpha-2, -\frac{1}{2} \alpha}_{pq, 0} (\mathbb{R}^n_+)$, then $v \in \dot{H}^{\alpha-2, \frac{1}{2} \alpha}_{pq, 0} (\mathbb{R}^n_+ \times (0, T))$ and satisfies

\[ \|v\|_{\dot{H}^{\alpha-2, \frac{1}{2} \alpha}_{pq, 0} (\mathbb{R}^n_+ \times (0, T))} \leq c\|f\|_{\dot{H}^{\alpha-2, \frac{1}{2} \alpha-1}_{pq, 0} (\mathbb{R}^n_+ \times (0, T))} + c\|v_0\|_{\dot{B}^{\alpha-2, -\frac{1}{2} \alpha}_{pq, 0} (\mathbb{R}^n_+)}. \quad (5) \]

Furthermore, if $1 + 1/p < \alpha \leq 2$, then $p \in L_q(0, T; \dot{H}^{\alpha-1}_{p, 0} (\mathbb{R}^n_+))$ such that

\[ \|p\|_{L_q(0, T; \dot{H}^{\alpha-1}_{p, 0} (\mathbb{R}^n_+))} \leq c\|f\|_{\dot{H}^{\alpha-2, \frac{1}{2} \alpha-1}_{pq, 0} (\mathbb{R}^n_+ \times (0, T))} + c\|v_0\|_{\dot{B}^{\alpha-2, -\frac{1}{2} \alpha}_{pq, 0} (\mathbb{R}^n_+)}. \quad (6) \]