Global Existence for Coupled Systems of Nonlinear Wave and Klein-Gordon Equations in Three Space Dimensions

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Abstract

We consider the Cauchy problem for coupled systems of wave and Klein-Gordon equations with quadratic nonlinearity in three space dimensions. We show global existence of small amplitude solutions under certain condition including the null condition on self-interactions between wave equations. Our condition is much weaker than the strong null condition introduced by Georgiev for this kind of coupled system. Consequently our result is applicable to certain physical systems, such as the Dirac-Klein-Gordon equations, the Dirac-Proca equations, and the Klein-Gordon-Zakharov equations.

1 Introduction

We consider the Cauchy problem for the following system:

\[(\Box + m_i^2) u_i = F_i(u, \partial u, \partial_x \partial u), \quad i = 1, 2, \ldots, N\]

in \((0, \infty) \times \mathbb{R}^3\) with initial data

\[u(0, x) = \varepsilon f(x), \quad (\partial_t u)(0, x) = \varepsilon g(x) \quad \text{for} \quad x = (x_1, x_2, x_3) \in \mathbb{R}^3, \]

where \(\Box = \partial_t^2 - \Delta_x\), \(u = (u_j)_{1 \leq j \leq N}\), \(m_i \geq 0\) for \(1 \leq i \leq N\), and \(\varepsilon\) is a small and positive parameter. Here each component \(u_j\) of \(u\) is supposed to be a real-valued unknown function of \((t, x) \in (0, \infty) \times \mathbb{R}^3\), and \(\Delta_x\) denotes the Laplacian in \(x\)-variables. In the above, \(\partial u\) and \(\partial_x \partial u\) are given by

\[\partial u = (\partial_a u_j)_{1 \leq j \leq N, 0 \leq a \leq 3}, \quad \partial_x \partial u = (\partial_k \partial_a u_j)_{1 \leq j \leq N, 1 \leq k \leq 3, 0 \leq a \leq 3},\]

respectively, with the notation

\[\partial_0 = \partial_t = \frac{\partial}{\partial t} \quad \text{and} \quad \partial_k = \frac{\partial}{\partial x_k} \quad \text{for} \quad 1 \leq k \leq 3.\]

Here, by writing \((\partial_a u_j)_{j,a}\), we mean that \(\partial_a u_j\)'s are arranged in dictionary order with respect to \((j, a)\). Similarly, \((\partial_k \partial_a u_j)_{j,k,a}\) means that \(\partial_k \partial_a u_j\)'s are arranged...
in dictionary order with respect to \((j,k,a)\). Similar convention will be used throughout this paper. We assume that, for \(1 \leq i \leq N\), each \(F_i = F_i(\xi, \xi', \xi'')\) is a real-valued smooth function of \((\xi, \xi', \xi'') \in \mathbb{R}^N \times \mathbb{R}^{4N} \times \mathbb{R}^{12N}\), where \(\xi, \xi', \text{ and } \xi''\) are independent variables for which \(u, \partial u, \text{ and } \partial_x \partial u\) are substituted in \((1.1)\); more precisely, if we write

\[
\xi = (\xi_j)_{1 \leq j \leq N} \in \mathbb{R}^N, \quad \xi' = (\xi'_{j,a})_{1 \leq j \leq N, 0 \leq a \leq 3} \in \mathbb{R}^{4N}, \\
\xi'' = (\xi''_{j,k,a})_{1 \leq j \leq N, 1 \leq k \leq 3, 0 \leq a \leq 3} \in \mathbb{R}^{12N},
\]

then \(\xi, \xi'_{j,a}, \text{ and } \xi''_{j,k,a}\) are the independent variables for which \(u_j, \partial_a u_j, \text{ and } \partial_k \partial_a u_j\) are substituted in \((1.1)\), respectively. We assume that \(F = (F_i)_{1 \leq i \leq N}\) vanishes of second order at the origin \((\xi, \xi', \xi'') = (0, 0, 0)\), namely

\[F(\xi, \xi', \xi'') = O(|\xi|^2 + |\xi'|^2 + |\xi''|^2)\text{ around } (\xi, \xi', \xi'') = (0, 0, 0).
\]

For simplicity, we also assume that the system is quasi-linear. In other words, we assume that

\[F_i(\xi, \xi', \xi'') = \sum_{j=1}^{N} \sum_{k=1}^{3} \sum_{a=0}^{3} \gamma^{ij}_{ka}(\xi, \xi') \xi''_{j,k,a} + G_i(\xi, \xi'), \quad 1 \leq i \leq N, \tag{1.3}\]

where \(\gamma^{ij}_{ka} = \gamma^{ij}_{ka}(\xi, \xi')\) and \(G_i = G_i(\xi, \xi')\) are some functions vanishing of first and second order at the origin, respectively. Moreover, to assure the hyperbolicity of the system, we always assume the symmetricity condition

\[\gamma^{ij}_{ka}(\xi, \xi') = \gamma^{ij}_{ka}(\xi, \xi') \text{ and } \gamma^{ij}_{kl}(\xi, \xi') = \gamma^{ij}_{lk}(\xi, \xi') \quad \tag{1.4}\]

for any \((\xi, \xi') \in \mathbb{R}^N \times \mathbb{R}^{4N}, 1 \leq i, j \leq N, 1 \leq k, l \leq 3, \text{ and } 0 \leq a \leq 3\) (note that the last half of \((1.4)\) is no restriction as far as we consider smooth solutions).

For a while, we suppose that \(f = (f_i)_{1 \leq i \leq N}, g = (g_i)_{1 \leq i \leq N} \in C^0(\mathbb{R}^3; \mathbb{R}^N)\). Under the conditions \((1.3)\) and \((1.4)\), the classical theory of nonlinear hyperbolic systems implies local existence of smooth solutions to the Cauchy problem \((1.1) - (1.2)\) for sufficiently small \(\varepsilon\). Hence we are interested in the sufficient condition for global existence of small amplitude solutions. We recall the known results briefly. If the nonlinearity \(F\) vanishes of third order at the origin, \((1.1) - (1.2)\) admits a global solution for small \(\varepsilon\). For arbitrary quadratic nonlinearity \(F\), we also have global existence of small solutions if \((1.1)\) is a system of nonlinear Klein-Gordon equations, namely \(m_i > 0\) for all \(i = 1, \ldots, N\) (see Klainerman \[16\] and Shatah \[25\]; see also Bachelot \[2\] and Hayashi-Naumkin-Ratno Bagus Edy Wibowo \[4\]). On the other hand, this is not true for a system of wave equations, namely the case where \(m_1 = m_2 = \cdots = m_N = 0\), and the solution to \((1.1) - (1.2)\) with certain quadratic nonlinearity \(F\) may blow up in finite time no matter how small \(\varepsilon\) is (see John \[10\], \[11\]). Thus we need to put some condition on quadratic nonlinearity, in order to obtain global solutions for wave equations. The null condition introduced by Klainerman \[17\] is one of such conditions. Before
describing the null condition, we introduce the following notation: For a smooth function \( \Phi = \Phi(z) \) \((z \in \mathbb{R}^d)\), we write \( \Phi^{(2)} \) for the quadratic part of \( \Phi \); more precisely, for a smooth function \( \Phi = \Phi(z) \), we define
\[
\Phi^{(2)}(z) = \sum_{|\alpha| = 2} \frac{(\partial^\alpha \Phi)(0)}{\alpha!} z^\alpha, \quad z = (z_1, \ldots, z_d) \in \mathbb{R}^d,
\]
where \( \partial_z = (\partial_{z_1}, \ldots, \partial_{z_d}) \), \( \alpha \) is a multi-index, and we have used the standard notation of multi-indices. The null condition can be stated as follows:

**Definition 1.1 (The null condition)** We say that a function \( F = (F_i)_{1 \leq i \leq N} \) of \((\xi, \xi', \xi'') \in \mathbb{R}^N \times \mathbb{R}^{4N} \times \mathbb{R}^{12N} \) satisfies the null condition if each \( F_i \) \((1 \leq i \leq N)\) satisfies
\[
F_i^{(2)}(\lambda, (X_{a_k})_{0 \leq a \leq 3}, (X_{k_a}X_{a_k})_{1 \leq j \leq N, 1 \leq k, \leq 3, 0 \leq a \leq 3}) = 0 \quad (1.6)
\]
for any \( \lambda = (\lambda_j)_{1 \leq j \leq N} \), \( \mu = (\mu_j)_{1 \leq j \leq N} \), \( \nu = (\nu_j)_{1 \leq j \leq N} \) \( \in \mathbb{R}^N \), and any \( X = (X_{a_k})_{0 \leq a \leq 3} \in \mathbb{R}^4 \) satisfying \( X_0^2 - X_1^2 - X_2^2 - X_3^2 = 0 \), where \( F_i^{(2)} = F_i^{(2)}(\xi, \xi', \xi'') \) is the quadratic part of \( F_i \) given by \((1.5)\) with \( \Phi = F_i \) and \( z = (\xi, \xi', \xi'') \).

If \( F \) satisfies the null condition, then we have global existence of small solutions for systems of wave equations \((1.1)\) with \( m_1 = \cdots = m_N = 0 \) (see Klainerman \([17]\) and Christodoulou \([3]\)).

Klainerman used the so-called vector field method in \([16]\) and \([17]\). But his method is not directly applicable to systems consisting of both wave and Klein-Gordon equations, because the scaling operator \( S = t\partial_t + \sum_{k=1}^3 x_k \partial_k \) is compatible with the wave equations, but not with the Klein-Gordon equations. This causes some difficulty in the treatment of the null condition, and hence Georgiev \([4]\) introduced the strong null condition to obtain global existence of small solutions for coupled systems of nonlinear wave and Klein-Gordon equations (see Section 4 below for the detail), where \( F \) is said to satisfy the strong null condition if \((1.6)\) holds for any \( \lambda, \mu, \nu \in \mathbb{R}^N \) and any \( X \in \mathbb{R}^4 \) not necessarily satisfying \( X_0^2 - X_1^2 - X_2^2 - X_3^2 = 0 \).

Our aim in this paper is to establish a global existence theorem for systems of the nonlinear wave and Klein-Gordon equations under more natural and weaker condition than the strong null condition, so that it can cover the previous results for wave equations and the Klein-Gordon equations, as well as some important examples from physics.

## 2 The Main Result and Examples

First we introduce some notation. Suppose that we can take some natural number \( N_1 \) such that we have
\[
m_i > 0 \text{ for } 1 \leq i \leq N_1, \text{ and } m_i = 0 \text{ for } N_1 + 1 \leq i \leq N \quad (2.1)
\]
in (1.1). We set
\[ v = (v_j)_{1 \leq j \leq N_1} := (u_j)_{1 \leq j \leq N_1} \quad \text{and} \quad w = (w_j)_{1 \leq j \leq N_2} := (u_{N_1+j})_{1 \leq j \leq N_2}, \]  
where \( N_2 = N - N_1 \), so that \( u = (u_1, \ldots, u_{N_1}, u_{N_1+1}, \ldots, u_N) = (v, w) \). Note that each \( v_j (= u_j) \) satisfies a nonlinear Klein-Gordon equation, while each \( w_j (= u_{N_1+j}) \) is governed by a nonlinear wave equation. In accordance with (2.2), we introduce independent variables \((\eta, \zeta) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}, (\eta', \zeta') \in \mathbb{R}^{4N_1} \times \mathbb{R}^{4N_2}, \) and \((\eta'', \zeta'') \in \mathbb{R}^{12N_1} \times \mathbb{R}^{12N_2} \) to write
\[ \xi = (\xi_j)_{1 \leq j \leq N} =: ((\eta_j)_{1 \leq j \leq N_1}, (\zeta_j)_{1 \leq j \leq N_2}) = (\eta, \zeta), \]
\[ \xi' = (\xi'_j,\zeta'_j)_{1 \leq j \leq N, 0 \leq a \leq 3} =: ((\eta'_j,\zeta'_j)_{1 \leq j \leq N, 0 \leq a \leq 3}, (\zeta''_j,a)_{1 \leq j \leq N_2, 0 \leq a \leq 3}) = (\eta', \zeta'), \]
\[ \xi'' = (\xi''_{j,k,a})_{1 \leq j \leq N, 1 \leq k \leq 3, 0 \leq a \leq 3} \]
\[ =: ((\eta''_{j,k,a})_{1 \leq j \leq N_1, 1 \leq k \leq 3, 0 \leq a \leq 3}, (\zeta''_{j,k,a})_{1 \leq j \leq N_2, 1 \leq k \leq 3, 0 \leq a \leq 3}) = (\eta'', \zeta''). \]
Correspondingly, we write \( \partial u = (\partial v, \partial w) \) and \( \partial_{\xi} \partial u = (\partial_x \partial v, \partial_x \partial w) \). For a smooth function \( \Phi = \Phi(\xi, \xi', \xi'') \), we define
\[ \Phi^{(W)}(\xi, \xi', \xi'') = \Phi^{(2)}((\eta, \zeta), (\eta', \zeta'), (\eta'', \zeta'')) \bigg|_{(\eta,\eta',\eta'')=(0,0,0)} \]  
for \((\xi, \xi', \xi'') \in \mathbb{R}^{N_2} \times \mathbb{R}^{4N_2} \times \mathbb{R}^{12N_2}, \) where \( \Phi^{(2)} \) is the quadratic part of \( \Phi \) given by (1.3) with \( z = (\xi, \xi', \xi'') \).

Now we are in a position to state our main result.

**Theorem 2.1** Suppose that (1.3), (1.4), and (2.1) are fulfilled. Assume that the following two conditions (a) and (b) hold:

(a) \( (F_i^{(W)})_{N_1+1 \leq i \leq N} \) satisfies the null condition,

(b) There exist two (empty or non-empty) sets \( \mathcal{I}_1 \) and \( \mathcal{I}_2 \) satisfying
\[ \mathcal{I}_1 \cup \mathcal{I}_2 = \{1, \ldots, N_2\}, \quad \mathcal{I}_1 \cap \mathcal{I}_2 = \emptyset, \]  
and the following:

(b–i) For any \( k \in \mathcal{I}_1 \), we have
\[ \frac{\partial F_i^{(2)}}{\partial \xi_k}((\eta, \zeta), (\eta', \zeta'), (\eta'', \zeta'')) \left( \frac{\partial F_i^{(2)}}{\partial \xi_{N_1+k}}(\xi, \xi', \xi'') \right) = 0 \]
for all \( i = 1, \ldots, N \), and all \((\xi, \xi', \xi'') \in \mathbb{R}^N \times \mathbb{R}^{4N} \times \mathbb{R}^{12N} \).

(b–ii) For any \( k \in \mathcal{I}_2 \), there exist some functions \( \mathcal{G}_{k,a} = \mathcal{G}_{k,a}(\xi, \xi') \) with \( 0 \leq a \leq 3 \) such that
\[ F_{N_1+k}(\phi, \partial \phi, \partial_x \partial \phi) = \sum_{a=0}^{3} \partial_{a} \{ \mathcal{G}_{k,a}(\phi, \partial \phi) \} \]  
(2.6)
holds for any \( \phi = \phi(t,x) \in C^2 \left( (0,\infty) \times \mathbb{R}^3; \mathbb{R}^N \right) \), and

\[
\frac{\partial G^{(2)}_{k,a}}{\partial \xi_l}((\eta,\zeta), (\eta',\zeta')) = \frac{\partial G^{(2)}_{k,a}}{\partial \xi_{N_1+l}}((\xi,\xi')) = 0, \quad 0 \leq a \leq 3 \tag{2.7}
\]

holds for all \( l \in \mathcal{I}_1 \), and all \((\xi,\xi') \in \mathbb{R}^N \times \mathbb{R}^{4N} \), where \( G^{(2)}_{k,a} \) is the quadratic form of \( G_{k,a} \).

Then, for any \( f, g \in \mathcal{S}(\mathbb{R}^3;\mathbb{R}^N) \), there exists a positive constant \( \varepsilon_0 \) such that for any \( \varepsilon \in (0,\varepsilon_0] \) the Cauchy problem \((1.1)-(1.2)\) admits a unique global solution \( u \in C^{\infty}(\mathbb{R}^3;\mathbb{R}^N) \). Here \( \mathcal{S} \) denotes the Schwartz class, the class of rapidly decreasing functions.

Here and hereafter, we say that \( (F_i^{(W)})_{N_1+1 \leq i \leq N} \) satisfies the null condition, if each \( F_i^{(W)} = F_i^{(W)}(\zeta,\zeta',\zeta''), \) with \( N_1 + 1 \leq i \leq N \) satisfies

\[
F_i^{(W)}(\lambda, (X_a \nu_j)_{1 \leq j \leq N_2}, \nu_{j})_{1 \leq j \leq N_2}, X_k X_a \nu_j)_{1 \leq j \leq N_2, 1 \leq k \leq 3, 0 \leq a \leq 3} = 0
\]

for any \( \lambda = (\lambda_j)_{1 \leq j \leq N_2}, \mu = (\mu_j)_{1 \leq j \leq N_2}, \nu = (\nu_j)_{1 \leq j \leq N_2} \in \mathbb{R}^{N_2}, \) and for any \( X = (X_a)_{0 \leq a \leq 3} \in \mathbb{R}^4 \) satisfying \( X_2^2 - X_1^2 - X_2^2 - X_3^2 = 0 \). Notice that the null condition is required only for \( F_i^{(W)} \) with \( N_1 + 1 \leq i \leq N \) in Theorem 2.1. We define the null forms \( Q_0 \) and \( Q_{ab} \) by

\[
Q_0(\varphi, \psi) = (\partial_1 \varphi)(\partial_1 \psi) - (\nabla_x \varphi) \cdot (\nabla_x \psi), \tag{2.8}
\]

\[
Q_{ab}(\varphi, \psi) = (\partial_a \varphi)(\partial_b \psi) - (\partial_b \varphi)(\partial_a \psi), \quad 0 \leq a < b \leq 3, \tag{2.9}
\]

where \( \nabla_x = (\partial_1, \partial_2, \partial_3) \), and \( \cdot \) denotes the inner product in \( \mathbb{R}^3 \). Then we can easily check that the assumption (a) in Theorem 2.1 is equivalent to the following condition (refer to [17] for instance):

(a') There exist some constants \( A_{i,j,k}^{\alpha\beta} \) and \( B_{i,j,k}^{ab,\alpha\beta} \) such that

\[
F_i^{(W)}(w, \partial w, \partial_x \partial w) = \sum_{1 \leq j, k \leq N_2} \sum_{0 \leq |\alpha|, |\beta| \leq 1} A_{i,j,k}^{\alpha\beta} Q_0(\partial^\alpha w_j, \partial^\beta w_k)
\]

\[
+ \sum_{1 \leq j, k \leq N_2} \sum_{0 \leq a < b \leq 3} \sum_{0 \leq |\alpha|, |\beta| \leq 1} B_{i,j,k}^{ab,\alpha\beta} Q_{ab}(\partial^\alpha w_j, \partial^\beta w_k) \tag{2.10}
\]

holds for any \( i = N_1 + 1, \ldots, N \), and any \( C^2 \)-function \( w = (w_1, \ldots, w_{N_2}) \), where \( \partial^\alpha = \partial_0^\alpha \partial_1 \partial_2 \partial_3 \) for a multi-index \( \alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3) \).

The condition (b) in Theorem 2.1 is assumed in order to compensate for bad behavior of the solutions to the wave equations, as compared with their derivatives: Let \( u = (v, w) \) be the solution to \((1.1)\). The condition (b-i) says that if \( k \in \mathcal{I}_1 \), then all of \( F_i^{(2)}(u, \partial u, \partial_x \partial u) \) can depend on \( (\partial w_k, \partial_x \partial w_k) \), but
not on \( w_k \) itself (remember that \( \zeta_k (= \xi_{N_1 + k}) \) is the variable corresponding to \( w_k (= u_{N_1 + k}) \)), while the divergence structure in the condition (b–ii) assures that for each \( k \in I_2 \), \( w_k \) behaves better than we can expect in general (see Lemmas 3.6 and 3.7 below; observe that the equation for \( w_k (= u_{N_1 + k}) \) with \( k \in I_2 \) in (1.1) is \( \Box w_k = F_{N_1 + k}(u, \partial u, \partial^2 u) = \sum_{a=0}^{3} \partial_a \{ \mathcal{G}_{k,a}(u, \partial u) \} \). Here we remark that the condition (b–i) does not imply (2.7) for \((l, k) \in I_1 \times I_2 \) in general, because we have

\[
2Q_{a,b}(\varphi, \psi) = \partial_a \{ \varphi(\partial_b \psi) - (\partial_b \varphi)\psi \} + \partial_b \{ (\partial_a \varphi)\psi - \varphi(\partial_a \psi) \} \tag{2.11}
\]

for example.

To help the understanding of our condition, we give a typical example here. In what follows, for a finite family of functions \( \{ \phi_\lambda \}_{\lambda \in \Lambda} \) and a function \( \psi \), we write \( \psi = \sum_{\lambda \in \Lambda} \phi_\lambda \) if there exists a family of constants \( \{ c_\lambda \}_{\lambda \in \Lambda} \) such that \( \psi = \sum_{\lambda \in \Lambda} c_\lambda \phi_\lambda \). Let \( u = (v, w) = (v, w_1, w_2) \), and let \( m \) be a positive constant. Then the assumption in Theorem 2.1 is fulfilled with \( I_1 = \{ 1 \} \) and \( I_2 = \{ 2 \} \) for the following semilinear system:

\[
(\Box + m^2)v = \sum_{|\alpha|,|\beta| \leq 1} (\partial^\alpha v)(\partial^\beta v) + \sum_{|\alpha| \leq 1, 0 \leq b \leq 3} (\partial^\alpha v)(\partial_b w_1) + \sum_{|\alpha|,|\beta| \leq 1} (\partial^\alpha v)(\partial^\beta w_2) \\
+ \sum_{0 \leq a, b \leq 3} (\partial_a w_1)(\partial_b w_1) + \sum_{0 \leq a \leq 3, |\beta| \leq 1} (\partial_a w_1)(\partial^\beta w_2) \\
+ \sum_{|\alpha|,|\beta| \leq 1} (\partial^\alpha w_2)(\partial^\beta w_2) + H_1(u, \partial u), \tag{2.12}
\]

\[
\Box w_1 = \sum_{|\alpha|,|\beta| \leq 1} (\partial^\alpha v)(\partial^\beta v) + \sum_{|\alpha| \leq 1, 0 \leq b \leq 3} (\partial^\alpha v)(\partial_b w_1) + \sum_{|\alpha|,|\beta| \leq 1} (\partial^\alpha v)(\partial^\beta w_2) \\
+ \sum_{j,k=1,2} Q_{0}(w_j, w_k) + \sum_{j,k=1,2} Q_{ab}(w_j, w_k) + H_2(u, \partial u), \tag{2.13}
\]

\[
\Box w_2 = \sum_{a=0}^{3} \partial_a \left( C_{1,a} v^2 + C_{2,a} v w_2 + H_{3,a}(u) \right), \tag{2.14}
\]

where \( C_{j,a} \)'s are real constants, while \( H_1, H_2, \) and \( H_{3,a} \) are smooth functions in their arguments satisfying \( H_1(u, \partial u), H_2(u, \partial u) = O(|u|^3 + |\partial u|^3) \) near \((u, \partial u) = 0\), and \( H_{3,a}(u) = O(|u|^3) \) near \( u = 0 \).

Now we would like to see the relation between our theorem and the previous results. When \( m_i > 0 \) for all \( i = 1, \ldots, N \), by regarding \( v = u \), and by neglecting the meaningless conditions (a) and (b), Theorem 2.1 covers the previous results in [16] and [25] for nonlinear Klein-Gordon equations. Similarly, when \( m_i = 0 \) for all \( i = 1, \ldots, N \), by regarding \( w = u \) and \( N_1 = 0 \) (thus \( F_i^{(W)} \) is regarded as \( F_i^{(2)} \) for \( 1 \leq i \leq N \)), it also covers the previous results in [3] and [17] for nonlinear wave equations; note that the condition (b) for this case is automatically satisfied under
the condition (a), because (2.10) implies (b) with $I_1 = \{1, \ldots, N\}$ and $I_2 = \emptyset$. It is easy to show that the strong null condition is satisfied if and only if each $F_i^{(2)}$ $(1 \leq i \leq N)$ is a linear combination of $Q_{ab}(\partial^a u_j, \partial^b u_k)$ with $1 \leq j, k \leq N$, $|\alpha|, |\beta| \leq 1$ and $0 \leq a < b \leq 3$. Hence our conditions (a) and (b) are much weaker than the strong null condition in [4]. Note that some case of variable coefficients is also treated in [4], but we can easily modify our conditions (a) and (b) to treat such case.

The main difficulty in the proof of Theorem 2.1 lies in the fact that we can only use the vector fields which are compatible with both wave and Klein-Gordon equations. To prove Theorem 2.1 instead of the weighted $L^2-L^\infty$ estimate derived in [4], we use weighted $L^\infty-L^\infty$ estimates for wave equations (see Lemma 3.4 below), which require a smaller set of vector fields than the admissible set of vector fields for the Klein-Gordon equations. We also need some estimates for null forms without using the scaling operator $S$, which will be given in Lemma 4.1 below. To treat $F_i^{(W)}$ with $1 \leq i \leq N_1$, for which the null condition is not assumed, we adopt a technique used in Y. Tsutsumi [29], where the Dirac-Proca equations are considered (see (2.18)–(2.19) below). This technique is motivated by Bachelot [2] and Kosecki [19], and it is closely related to the normal form technique in Shatah [25]. We will prove Theorem 2.1 in Section 5.

We conclude this section with some examples from physics which can be treated by Theorem 2.1. Note that all the following examples are semilinear (or can be regarded as semilinear), and the conditions (1.3) and (1.4) in Theorem 2.1 are trivially satisfied. Thus we only have to check the conditions (a) and (b).

**Example 1 (The Dirac-Klein-Gordon equations)** Let us consider the Dirac equation coupled with the Klein-Gordon or wave equation:

\[-\sqrt{-1} \sum_{a=0}^{3} \gamma_a \partial_a \psi + M \psi = \sqrt{-1} c \varphi \gamma_5 \psi, \tag{2.15}\]
\[(\Box + m^2) \varphi = \psi^* H \psi\tag{2.16}\]

in $(0, \infty) \times \mathbb{R}^3$, where $\sqrt{-1}$ denotes the imaginary unit, $M, m \geq 0$, $c$ is a real constant, $H$ is a $4 \times 4$ Hermitian matrix, $\psi$ is a $\mathbb{C}^4$-valued function, $\varphi$ is a real valued function, and $\psi^*$ denotes the complex conjugate transpose of $\psi$. $\gamma_a$ $(0 \leq a \leq 3)$ in the above are $4 \times 4$ matrices satisfying $\gamma_a \gamma_b + \gamma_b \gamma_a = 2 g_{ab} I$ for $0 \leq a, b \leq 3$, where $I$ is the $4 \times 4$ identity matrix, and $(g_{ab})_{0 \leq a, b \leq 3} = \text{diag}(1, -1, -1, -1)$; $\gamma_5$ is defined by $\gamma_5 = -\sqrt{-1} \gamma_0 \gamma_1 \gamma_2 \gamma_3$. We set $D_M^\pm = \pm \sqrt{-1} \sum_{a=0}^{3} \gamma_a \partial_a + MI$. Since we have $\gamma_a \gamma_5 = -\gamma_5 \gamma_a$ for
0 \leq a \leq 3$, we get $D_M^+ \gamma_5 = \gamma_5 D_M^-$. Therefore, operating $D_M^+$ to $(2.15)$, we get
\[
(\Box + M^2) \psi = \sqrt{-1} c D_M^+ (\varphi \gamma_5 \psi) = -c \sum_{a=0}^{3} (\partial_a \varphi) \gamma_a \gamma_5 \psi + \sqrt{-1} c \varphi \gamma_5 (D_M^- \psi)
\]
\[= -c \sum_{a=0}^{3} (\partial_a \varphi) \gamma_a \gamma_5 \psi - c^2 \varphi^2 \psi,
\]
where we have used $(2.15)$ and $\gamma_5 \gamma_5 = I$ to obtain the last identity. If $M > 0$ and $m > 0$, the system $(2.16)-(2.17)$ is a system of the nonlinear Klein-Gordon equations, and we have the global solution. When $M > 0$ and $m = 0$, putting $u = (v, w)$ with $v = (\text{Re} \psi, \text{Im} \psi)$ and $w(= w_1) = \varphi$, we see that the conditions (a) and (b) are satisfied for the system $(2.16)-(2.17)$, and thus Theorem 2.1 is applicable; more precisely the condition (b) is satisfied with $\mathcal{I}_1 = \{1\}$ and $\mathcal{I}_2 = \emptyset$. The global existence result for this case where $M > 0$ and $m = 0$, with compactly supported initial data, has been already obtained by Bachelot [2]. Moreover, we can also treat the case where $M = 0$ and $m > 0$. Indeed, the first identity in $(2.17)$ with $M = 0$ can be read as $\Box \psi = -c \sum_{a=0}^{3} \gamma_a \partial_a (\varphi \gamma_5 \psi)$, and putting $u = (u_i)_{1 \leq i \leq 9} = (v, w)$ with $v(= v_1) = \varphi$ and $w = (w_k)_{1 \leq k \leq 8} = (\text{Re} \psi, \text{Im} \psi)$, we can verify the conditions (a) and (b) with $\mathcal{I}_1 = \emptyset$ and $\mathcal{I}_2 = \{1, \ldots, 8\}$. This last case is closely connected to the next example, the Dirac–Proca equations.

**Example 2 (The Dirac-Proca equations)** Y. Tsutsumi [29] proved the global existence of small solutions to the Dirac-Proca equations, which can be reduced to the following coupled system of the massless Dirac and the Klein-Gordon equations:

\[
-\sqrt{-1} \sum_{a=0}^{3} \gamma_a \partial_a \psi = -\frac{1}{2} \sum_{a=0}^{3} g_{aa} A_a (I + \gamma_5) \psi,
\]

\[
(\Box + m^2) A_a = \frac{1}{2} \psi^* \gamma_0 \gamma_a (I + \gamma_5) \psi, \quad a = 0, 1, 2, 3
\]

with the constraint $\sum_{a=0}^{3} \partial_a A_a = 0$ at $t = 0$, where $m > 0$, $\psi$ is a $\mathbb{C}^4$-valued function, and $A_a$ for $0 \leq a \leq 3$ are real-valued functions. $I$, $\gamma_a$ ($a = 0, 1, 2, 3, 5$) and $(g_{ab})$ are as in the Dirac-Klein-Gordon equations. In a similar manner to $(2.17)$ with $M = 0$, from $(2.18)$ we obtain

\[
\Box \psi = -\frac{1}{2} \sqrt{-1} \sum_{b=0}^{3} \partial_b \left( \sum_{a=0}^{3} g_{aa} A_a \gamma_b \gamma_a (I + \gamma_5) \psi \right).
\]

Putting $u = (v, w) \in \mathbb{R}^4 \times \mathbb{R}^8$ with $v = (A_a)_{0 \leq a \leq 3}$, and $w = (\text{Re} \psi, \text{Im} \psi)$, we find that the conditions (a) and (b) hold for the system $(2.19)-(2.20)$, and thus Theorem 2.1 is applicable; more precisely $(2.20)$ implies that (b) is satisfied with $\mathcal{I}_1 = \emptyset$ and $\mathcal{I}_2 = \{1, \ldots, 8\}$. 

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Example 3 (The Klein-Gordon-Zakharov equations) Ozawa-Tsutaya-Tsutsumi [23] and Tsutaya [28] proved the global existence of small solutions to the Klein-Gordon-Zakharov equations:

\[
\begin{align*}
(\Box + 1)\tilde{u} &= -\tilde{n}\tilde{u}, \\
\Box \tilde{n} &= \Delta_x |\tilde{u}|^2
\end{align*}
\] (2.21)

in \((0, \infty) \times \mathbb{R}^3\), where \(\tilde{u} = (\tilde{u}_1, \tilde{u}_2, \tilde{u}_3)\) is a \(\mathbb{C}^3\)-valued function, and \(\tilde{n}\) is a real valued function (see also Ozawa-Tsutaya-Tsutsumi [24] for the multiple speed case).

By setting \(v_i = \operatorname{Re} \tilde{u}_i, v_{i+3} = \operatorname{Im} \tilde{u}_i (1 \leq i \leq 3)\), \(v_{3i+3+k} = \partial_k v_i (1 \leq i \leq 6, 1 \leq k \leq 3)\), and \(w (= w_1) = \tilde{n}\), we see that solving (2.21) is equivalent to solving

\[
\begin{align*}
(\Box + 1) v_i &= -w v_i, & 1 \leq i \leq 6, \\
(\Box + 1) v_{3i+3+k} &= -w(\partial_k v_i) - (\partial_k w) v_i, & 1 \leq i \leq 6, 1 \leq k \leq 3, \\
\Box w &= \sum_{j=1}^3 \partial_j \sum_{i=1}^3 2(v_{i+3} v_{3i+3+j} + v_{i+3} v_{3i+12+j}).
\end{align*}
\] (2.22)

Note that the system (2.22) is a semilinear system of

\[ u = (u_1, \ldots, u_{25}) = (v_1, \ldots, v_{24}, w_1) = (v, w). \]

The conditions (a) and (b) (with \(I_1 = \emptyset\) and \(I_2 = \{1\}\)) are satisfied for (2.22). Hence we can apply Theorem 2.1 to show the global existence of small amplitude solutions to (2.21).

Example 4 The last example is not from physics, as far as the author knows. This example shows that some change of unknowns may help us to apply our theorem. Consider

\[
\begin{align*}
(\Box + 1) v &= w^2, \\
\Box w &= v^2
\end{align*}
\] (2.23)

in \((0, \infty) \times \mathbb{R}^3\). We can treat this example in the following way, though it does not explicitly satisfy the assumption of Theorem 2.1. Set \(\tilde{v} = v - w^2\) (cf. (5.45) below). Then we get

\[
\begin{align*}
(\Box + 1) \tilde{v} &= -2Q_0(w, w) - 2w(\tilde{v} + w^2)^2, \\
\Box w &= (\tilde{v} + w^2)^2
\end{align*}
\] (2.24)

(cf. (5.51) below). This system (2.24) satisfies the assumption in Theorem 2.1 with \(u = (\tilde{v}, w) = (\tilde{v}_1, w_1), I_1 = \{1\}, I_2 = \emptyset\). Thus we get a global solution \((\tilde{v}, w)\) to (2.24) for small data, and accordingly we obtain a global solution \((v, w)\) to the original system (2.23).
3 Preliminary Results

In this section, we gather the known estimates for the wave and Klein-Gordon equations. Throughout this paper, we write \( \langle z \rangle = \sqrt{1 + |z|^2} \) for \( z \in \mathbb{R}^d \), where \( d \) is a positive integer.

We start this section with the well-known energy inequality for hyperbolic systems.

**Lemma 3.1** Let \( m_i \geq 0 \) for \( 1 \leq i \leq N \), and \( T > 0 \). Suppose that \( \tilde{\gamma} = (\tilde{\gamma}_{ij}^{kl}) \) be a smooth function satisfying

\[
\tilde{\gamma}_{ij}^{kl}(t, x) = \tilde{\gamma}_{ij}^{kl}(t, x), \quad \tilde{\gamma}_{ij}^{kl}(t, x) = \tilde{\gamma}_{ik}^{lj}(t, x), \quad (t, x) \in (0, T) \times \mathbb{R}^3
\]

for \( 1 \leq i, j \leq N \), \( 1 \leq k, l \leq 3 \), and \( 0 \leq a \leq 3 \). We also assume that

\[
\left| \sum_{1 \leq i,j \leq N} \sum_{1 \leq k,l \leq 3} \tilde{\gamma}_{ij}^{kl} (t, x) \xi_i^{k} \xi_j^{l} \right| \leq \frac{1}{2} \sum_{1 \leq i \leq N} \sum_{1 \leq k \leq 3} |\xi_i^{k}|^2, \quad (t, x) \in (0, T) \times \mathbb{R}^3
\]

for any \((\xi_i^{k})_{1 \leq i \leq N, 1 \leq k \leq 3} \in \mathbb{R}^{3N} \).

Let \( \varphi = (\varphi_1, \ldots, \varphi_N) \) be the solution to

\[
(\Box + m_i^2) \varphi_i - \sum_{1 \leq j \leq N} \sum_{1 \leq k \leq 3} \tilde{\gamma}_{ij}^{kl} (\partial_k \partial_a \varphi_j) = \Phi_i \quad \text{in} \quad (0, T) \times \mathbb{R}^3,
\]

\[
\varphi_i(0, x) = \varphi_i^0(x), \quad (\partial_t \varphi_i)(0, x) = \varphi_i^1(x), \quad x \in \mathbb{R}^3
\]

for \( 1 \leq i \leq N \), where \( \varphi^0 = (\varphi_i^0)_{1 \leq i \leq N} \in H^1(\mathbb{R}^3; \mathbb{R}^N) \), \( \varphi^1 = (\varphi_i^1)_{1 \leq i \leq N} \in L^2(\mathbb{R}^3; \mathbb{R}^N) \), and \( \Phi = (\Phi_i)_{1 \leq i \leq N} \in L^1((0, T); L^2(\mathbb{R}^3; \mathbb{R}^N)) \).

Then, there exists a positive constant \( C \), which is independent of \( T \), such that

\[
\sum_{i=1}^{N} (\| \partial \varphi_i(t) \|_{L^2} + m_i \| \varphi_i(t) \|_{L^2})
\]

\[
\leq C \left( \| \varphi^0 \|_{H^1} + \| \varphi^1 \|_{L^2} + \int_0^t \| \partial \tilde{\gamma}(\tau) \|_{L^\infty} \| \partial \varphi(\tau) \|_{L^2} d\tau + \int_0^t \| \Phi(\tau) \|_{L^2} d\tau \right)
\]

for \( 0 \leq t < T \).

Before we proceed to the decay estimates of the solutions to the Klein-Gordon and wave equations, we introduce the vector fields \( \Omega_j \) and \( L_j \) for \( 1 \leq j \leq 3 \) by

\[
\Omega_j = (\Omega_1, \Omega_2, \Omega_3) = x \times \nabla_x = (x_2 \partial_3 - x_3 \partial_2, x_3 \partial_1 - x_1 \partial_3, x_1 \partial_2 - x_2 \partial_1), \quad \text{for} \quad 1 \leq j \leq 3
\]

\[
L_j = (L_1, L_2, L_3) = x \partial_t + t \nabla_x = (x_1 \partial_t + t \partial_1, x_2 \partial_t + t \partial_2, x_3 \partial_t + t \partial_3), \quad \text{for} \quad 1 \leq j \leq 3
\]

where \( \nabla_x = (\partial_1, \partial_2, \partial_3) \), and \( \times \) is the external product in \( \mathbb{R}^3 \). Writing \( \partial = (\partial_a)_{0 \leq a \leq 3} \), we set

\[
Z = (Z_1, \ldots, Z_{10}) = ((\Omega_j)_{1 \leq j \leq 3}, (L_j)_{1 \leq j \leq 3}, (\partial_a)_{0 \leq a \leq 3}) = (\Omega, L, \partial).
\]
Note that we have

\[ [L_j, \Box + m^2] = [\Omega_j, \Box + m^2] = [\partial_a, \Box + m^2] = 0, \quad 1 \leq j \leq 3, \quad 0 \leq a \leq 3 \quad (3.4) \]

for \( m \geq 0 \), where \([A, B] = AB - BA\) for operators \( A \) and \( B \). Hence the vector fields \( Z = (\Omega, L, \partial)\) are compatible with the Klein-Gordon equations, as well as the wave equations. Here we note that we have

\[ [Z_j, \partial_a] = \sum_{b=0}^{3} C^{ja}_b \partial_b, \quad [Z_j, Z_k] = \sum_{l=1}^{10} D^{jk}_l Z_l, \quad 1 \leq j, k \leq 10, \quad 0 \leq a \leq 3 \quad (3.5) \]

with appropriate constants \( C^{ja}_b \) and \( D^{jk}_l \). For a multi-index \( \alpha = (\alpha_1, \ldots, \alpha_{10}) \), we define \( Z^\alpha = Z_{\alpha_1}^{1} \cdots Z_{\alpha_{10}}^{10} \). For a function \( \varphi(t, x) \) and a nonnegative integer \( s \), we define

\[ |\varphi(t, x)|_s = \sum_{|\alpha| \leq s} |Z^\alpha \varphi(t, x)|, \quad \|\varphi(t)\|_s = \sum_{|\alpha| \leq s} \|Z^\alpha \varphi(t, \cdot)\|_{L^2(\mathbb{R}^3)}. \quad (3.6) \]

Using these vector fields \( Z = (\Omega, L, \partial) \), Klainerman [16] obtained the decay estimate for the solutions to the Klein-Gordon equations. This estimate has been modified and extended by many authors (for instance, see Bachelot [2], Sideris [26], Hörmander [8], and Georgiev [5]). Here we state the estimate obtained in [5]. Let \( \chi_j(j \geq 0) \) be nonnegative \( C^\infty_0(\mathbb{R})\)-functions satisfying

\[ \sum_{j=0}^{\infty} \chi_j(\tau) = 1 \text{ for } \tau \geq 0, \]

\[ \text{supp } \chi_j = [2^{j-1}, 2^{j+1}] \text{ for } j \geq 1, \text{ and supp } \chi_0 \cap [0, \infty) = [0, 2]. \quad (3.8) \]

**Lemma 3.2** Let \( m > 0 \), and \( v \) be a smooth solution to

\[ (\Box + m^2) v(t, x) = \Phi(t, x), \quad (t, x) \in (0, \infty) \times \mathbb{R}^3. \]

Then there exists a positive constant \( C = C(m) \) such that we have

\[ \langle t + |x| \rangle^{3/2} |v(t, x)| \leq C \sum_{j=0}^{\infty} \sum_{|\alpha| \leq 4} \sup_{\tau \in [0, t]} \chi_j(\tau) \|\tau + |\cdot|\| Z^\alpha \Phi(\tau, \cdot) \|_{L^2(\mathbb{R}^3)} \]

\[ + C \sum_{j=0}^{\infty} \sum_{|\alpha| \leq 5} \|\cdot\|^{3/2} \chi_j(|\cdot|) Z^\alpha v(0, \cdot) \|_{L^2(\mathbb{R}^3)} \quad (3.9) \]

for \( (t, x) \in (0, \infty) \times \mathbb{R}^3 \), provided that the right-hand side of (3.9) is finite.
For the proof, see Georgiev [5, Theorem 1].

Now we turn our attention to the wave equations. In [17], a weighted $L^1 – L^\infty$ estimate for the wave equation is derived (see also Hörmander [7]), where the scaling operator $S = t\partial_t + x \cdot \nabla_x$ as well as $Z = (\Omega, L, \partial)$ is used. Since we have $[S, \Box] = -2\Box$, the scaling operator $S$ is applicable to the wave equations, but it is incompatible with the Klein-Gordon equations. Therefore Georgiev ([4]) developed a weighted $L^2 – L^\infty$ estimate involving only $Z$. On the other hand, there is also a large literature on the study of systems of nonlinear wave equations with multiple speeds of the form

$$\Box c_i u_i = F_i(u, \partial u, \partial_x u), \quad 1 \leq i \leq N,$$

where $c_i > 0$ ($1 \leq i \leq N$) and $\Box c = \partial_t^2 - c^2 \Delta_x$ (see, for example, [13] and the references cited therein). In the study of this kind of system, the vector field method using only $(S, \Omega, \partial)$ has been developed, because $L = (L_j)_{1 \leq j \leq 3}$ is incompatible with such system (observe that $[L_j, \Box] = 2(c^2 - 1)\partial_t \partial_j$ has no good property when $c \neq 1$). Especially, in Yokoyama [30] and Kubota-Yokoyama [22] (see also the author [12]), weighted $L^\infty – L^\infty$ estimates requiring only $(\Omega, \partial)$ are adopted to prove some global existence results under the null condition (the origin of these estimates can be found in John [10] and Kovalyov [20]; see also Kovalyov-Tsutaya [21]). We will employ these $L^\infty – L^\infty$ estimates in the proof of Theorem 2.1 because they require only $(\Omega, \partial)$ and are easily applicable to the coupled system of the wave and Klein-Gordon equations. Here we note that $S$ is still used in the arguments in [30], [22] and [12] to treat the null forms (see (4.11) below).

To state the weighted $L^\infty – L^\infty$ estimates, we define

$$W_\rho(t, r) = \begin{cases} 
\langle t + r \rangle^\rho & \text{if } \rho < 0, \\
\log \left( 2 + \langle t + r \rangle \langle t - r \rangle^{-1} \right) \langle t - r \rangle^{-1} & \text{if } \rho = 0, \\
\langle t - r \rangle^\rho & \text{if } \rho > 0.
\end{cases} \quad (3.10)$$

We also introduce

$$W_-(t, r) = \min \{ \langle r \rangle, \langle t - r \rangle \}. \quad (3.11)$$

For the homogeneous wave equations we have the following:

**Lemma 3.3** Let $w$ be a solution to

$$\Box w(t, x) = 0, \quad (t, x) \in (0, \infty) \times \mathbb{R}^3$$

with initial data $w = w^0$, $\partial_t w = w^1$ at $t = 0$.

Let $\kappa > 0$. Then, there exists a positive constant $C = C(\kappa)$ such that

$$\langle t + |x| \rangle W_{\kappa - 1}(t, |x|) |w(t, x)|$$

$$\leq C \sup_{|y - x| \leq t} \langle y \rangle^\kappa \left( \langle y \rangle \sum_{|\alpha| \leq 1} \left| \langle \partial_x^\alpha w^0 \rangle (y) \right| + |y| \langle w^1(y) \rangle \right) \quad (3.12)$$
for \((t, x) \in (0, \infty) \times \mathbb{R}^3\), provided that the right-hand side of (3.12) is finite. Here \(\partial = (\partial_1, \partial_2, \partial_3)\), and we have used the standard notation of multi-indices.

This is essentially proved in Asakura [1, Proposition 1.1] (see also [15, Lemma 3.1] for the expression above).

The following weighted \(L^\infty-L^\infty\) estimates are the special cases of the estimates obtained in Kubota-Yokoyama [22, Lemma 3.2] (see also Katayama-Kubo [14, Lemma 3.4] for the expression below):

**Lemma 3.4** Let \(w\) be a solution to
\[\Box w(t, x) = \Psi(t, x), \quad (t, x) \in (0, \infty) \times \mathbb{R}^3\]
with initial data \(w = \partial w = 0\) at \(t = 0\).

Suppose that \(\rho \geq 0\), \(\kappa \geq 1\), and \(\mu > 0\). Then there exists a positive constant \(C = C(\rho, \kappa, \mu)\) such that
\[
\begin{align*}
\langle t + |x| \rangle^{1-\rho} W_{\kappa-1}(t, |x|)|w(t, x)| &\leq C \sup_{\tau \in [0, t]} \sup_{y \neq x \leq t-\tau} \langle y \rangle^\kappa \mu W_\tau(y) |\Psi(\tau, y)|, \\
\langle t + |x| \rangle^{-\rho} \langle t - |x| \rangle^\kappa |\partial w(t, x)| &\leq C \sup_{\tau \in [0, t]} \sup_{y \neq x \leq t-\tau} \langle y \rangle^\kappa \mu W_\tau(y) |\Psi(\tau, y)|
\end{align*}
\]
for \((t, x) \in (0, \infty) \times \mathbb{R}^3\), provided that the right-hand sides of (3.13) and (3.14) are finite. Here \(\partial = (\partial_0, \partial_1, \partial_2, \partial_3)\), and \(\Omega\) is given by (3.1).

The following Sobolev type inequality will be used to combine decay estimates with the energy estimates (see Klainerman [18] for the proof):

**Lemma 3.5** For a smooth function \(\varphi\) on \(\mathbb{R}^3\), we have
\[
\sup_{x \in \mathbb{R}^3} \langle x \rangle |\varphi(x)| \leq C \sum_{|\alpha| + |\beta| \leq 2} \|\partial_\alpha^\beta \varphi\|_{L^2(\mathbb{R}^3)},
\]
provided that the right-hand side of (3.15) is finite. Here \(C\) is a universal positive constant.

We conclude this section with some observation on the wave equations of the following type:
\[
\begin{align*}
\Box \psi(t, x) &= \sum_{a=0}^3 \partial_a \Psi_a(t, x), \quad (t, x) \in (0, \infty) \times \mathbb{R}^3, \\
\psi(0, x) &= \psi^0(x), \quad (\partial_t \psi)(0, x) = \psi^1(x), \quad x \in \mathbb{R}^3.
\end{align*}
\]
For \(0 \leq a \leq 3\), let \(\psi_a = \psi_a(t, x)\) be the solution to \(\Box \psi_a = \Psi_a\) with initial data \(\psi_a = \partial_t \psi_a = 0\) at \(t = 0\), and let \(\psi_t(t, x)\) be the solution to \(\Box \psi_t = 0\) with initial data \(\psi_t = \psi^0\) and \((\partial_t \psi_t) = \psi^1 - \Psi_0(0, \cdot)\) at \(t = 0\). It is easy to verify that the solution \(\psi\) to (3.16) can be written as \(\psi = \sum_{a=0}^3 \partial_a \psi_a + \psi_t\). Therefore, we can essentially regard \(\psi\) as derivatives of solutions to some wave equations, and \(\psi\) enjoys better estimates than we can expect in general.
Lemma 3.6 Let $\psi$ be the solution to (3.16). Then we have
\[
\|\psi(t, \cdot)\|_{L^2(\mathbb{R}^3)} \leq C \left( \|\psi^0\|_{L^2(\mathbb{R}^3)} + \|\psi^1\|_{L^{6/5}(\mathbb{R}^3)} + \|\Psi^0(0, \cdot)\|_{L^{6/5}(\mathbb{R}^3)} \right) + C \sum_{a=0}^{3} \int_{0}^{t} \|\Psi_a(\tau, \cdot)\|_{L^2(\mathbb{R}^3)} d\tau,
\]  
(3.17)
provided that the right-hand side of (3.17) is finite.

Proof. We have $\|\psi\|_{L^2} \leq \sum_{a=0}^{3} \|\partial_\tau \psi_a\|_{L^2} + \|\psi_t\|_{L^2}$. Hence (3.17) follows from the energy inequality (cf. Lemma 3.1) for $\psi_a$ ($0 \leq a \leq 3$), and the $L^2$-estimate for $\psi_t$ (see Strauss [27] for example).

Lemma 3.7 Let $\psi$ be the solution to (3.16). Suppose that $\rho \geq 0$, $\kappa \geq 1$ and $\mu > 0$. Then we have
\[
\langle t + |x| \rangle^{-\rho} \langle x \rangle \langle t - |x| \rangle^{\kappa} \|\psi(t, x)\| \leq C \sup_{\tau \in [0, t]} \sup_{|y - z| \leq t - \tau} |y| \langle \tau + |y| \rangle^{\kappa - \rho + \mu} W_-(\tau, |y|)^{1-\mu} \sum_{\beta \leq 1, 0 \leq a \leq 3} |\partial_\beta \Omega^a \Psi_a(\tau, y)|
\]
\[
+ C \sup_{|y - z| \leq t} \langle y \rangle^{\kappa + 1-\rho} \left( \langle y \rangle \sum_{|\alpha| \leq 1} \left| \left( \partial_\alpha \psi^0 \right)(y) \right| + |y| \left| \psi^1(y) \right| \right).
\]  
(3.18)

Proof. The estimates for $\psi_a$ corresponding to (3.18) are proved in [22] (in fact, (3.14) is obtained as a corollary to these estimates for $\psi_a$ in [22]). Using Lemma 3.3 to estimate $\psi_t$, and noting that we have
\[
\langle y \rangle^{\kappa + 1-\rho} |y| |\Psi^0(0, y)| = \left| \langle y \rangle^{\kappa + 1-\rho} |y| \langle \tau + |y| \rangle^{\kappa - \rho + \mu} W_-(\tau, |y|)^{1-\mu} |\Psi^0(\tau, y)| \right|_{\tau=0},
\]
we obtain the desired result immediately.

4 Estimates for the Null Forms

In this section, we will derive some estimates for the null forms. We set $r = |x|$, $\omega = (\omega_1, \omega_2, \omega_3)$ with $\omega_j = x_j/r$ for $1 \leq j \leq 3$, and $\partial_r = \omega \cdot \nabla_x = \sum_{j=1}^{3} \omega_j \partial_j$. Then we have
\[
\nabla_x = \omega \partial_r - r^{-1}(\omega \times \Omega),
\]  
(4.1)
where $\Omega$ is defined by (3.1). Since (3.1) and (3.2) yield $tr^{-1}\Omega = \omega \times L$, the expression (4.1) implies
\[
(t + r)(\nabla_x - \omega \partial_r) = -\omega \times (\Omega + \omega \times L).
\]  
(4.2)
From \( (4.2) \), we obtain
\[
|Q_0(\varphi, \psi) - Q_0^\text{rad}(\varphi, \psi)| + \sum_{k=1}^3 |Q_{0k}(\varphi, \psi) - \omega_k Q_{0r}^\text{rad}(\varphi, \psi)| \\
+ \sum_{1 \leq j < k \leq 3} |Q_{jk}(\varphi, \psi)| \leq C \langle t + r \rangle^{-1} (|Z\varphi| \langle \partial \psi \rangle + |\partial \varphi| \langle Z\psi \rangle) \tag{4.3}
\]
at \((t, x) \in [0, \infty) \times \mathbb{R}^3\), where
\[
Q_0^\text{rad}(\varphi, \psi) = (\partial_t \varphi)(\partial_t \psi) - (\partial_r \varphi)(\partial_r \psi), \quad Q_{0r}^\text{rad}(\varphi, \psi) = (\partial_t \varphi)(\partial_r \psi) - (\partial_r \varphi)(\partial_t \psi),
\]
\(Z\) is given by \( (3.3) \), and \( C \) is a positive constant. Putting \( L_r = \omega \cdot L = r \partial_t + t \partial_r \), we get
\[
(t + r)Q_{0r}^\text{rad}(\varphi, \psi) = (\partial_t \varphi - \partial_r \varphi)(L_r \psi) - (L_r \varphi)(\partial_t \psi - \partial_r \psi). \tag{4.4}
\]
From \( (4.3) \) and \( (4.4) \), we obtain
\[
|Q_{ab}(\varphi, \psi)| \leq C \langle t + r \rangle^{-1} (|Z\varphi| \langle \partial \psi \rangle + |\partial \varphi| \langle Z\psi \rangle) \tag{4.5}
\]
for \( 0 \leq a < b \leq 3 \) at \((t, x) \in [0, \infty) \times \mathbb{R}^3\), where \( C \) is some positive constant. Thus we only need the vector fields \( Z = (\Omega, L, \partial) \) to obtain the extra decay factor \( \langle t + r \rangle^{-1} \) for terms satisfying the strong null condition.

To treat the null form \( Q_0 \), we introduce \( \partial_\pm = \partial_t \pm \partial_r \). Then we get
\[
Q_0^\text{rad}(\varphi, \psi) = \frac{1}{2} ((\partial_+ \varphi)(\partial_- \psi) + (\partial_- \varphi)(\partial_+ \psi)). \tag{4.6}
\]
As we will see below, estimates of \( \partial_+ \varphi \) and \( \partial_+ \psi \) are important in deriving enhanced decay for \( Q_0 \). Note that we also have
\[
Q_{0r}^\text{rad}(\varphi, \psi) = (\partial_+ \varphi)(\partial_r \psi) - (\partial_r \varphi)(\partial_+ \psi). \tag{4.7}
\]
Rewriting \( \partial_+ \) as
\[
\partial_+ = (t + r)^{-1}(S + L_r) \tag{4.8}
\]
with \( S = t \partial_t + x \cdot \nabla_x = t \partial_t + r \partial_r \), from \( (4.3) \) and \( (4.6) \) we obtain
\[
|Q_0(\varphi, \psi)| \leq C \langle t + r \rangle^{-1} (|\Gamma \varphi| \langle \partial \psi \rangle + |\partial \varphi| \langle \Gamma \psi \rangle), \tag{4.9}
\]
where \( \Gamma = (S, Z) = (S, L, \Omega, \partial) \). The estimate \( (4.9) \) was used in Klainerman \( [17] \), and the usage of \( S \) in \( (4.9) \) makes it difficult to treat the null form \( Q_0 \) included in coupled systems of the wave and Klein-Gordon equations, and this is the reason why the notion of the strong null condition was introduced in \( [1] \).

Before we proceed to our new estimate for \( Q_0 \), we introduce another kind of known estimate for the null forms here. If we only use \( (4.1) \), then we find that the left-hand side of \( (4.3) \) is bounded from above by \( C \langle r \rangle^{-1} (|Z\varphi| \langle \partial \psi \rangle + |\partial \varphi| \langle Z\psi \rangle) \), where \( Z' = (\Omega, \partial) \). Hence, writing
\[
\partial_+ = r^{-1}(S - (t - r)\partial_t), \tag{4.10}
\]
we get
\[ |Q_0(\varphi, \psi)| \leq C (r)^{-1} \left( |\Gamma' \varphi| |\partial \psi| + |\partial \varphi| |\Gamma' \psi| + \langle t-r \rangle |\partial \varphi| |\partial \psi| \right), \quad (4.11) \]
where \( \Gamma' = (S, \Omega, \partial) \). Similar estimate can be obtained for \( Q_{ab} \) in view of (4.7).

These estimates are used in the study of systems of wave equations with multiple speeds because \( L \) is incompatible with such systems (see Hoshiga-Kubo [9] and Yokoyama [30] for example). As we have mentioned in the previous section, the estimate (4.11) is the point where \( S \) comes in the arguments of [30], [22] and [12], though the weighted \( L^\infty - L^\infty \) estimates (cf. Lemma 3.4) are free of \( S \).

In Katayama-Kubo [13], for the \( \partial_+ \)-derivative of the solution to the wave equation, a weighted \( L^\infty - L^\infty \) estimate with a better decay factor than (3.14) is directly obtained through an explicit expression of the solution (without rewriting \( \partial_+ \) by the other vector fields), and the null forms are treated using only \( Z' = (\Omega, \partial) \) (see also [14]). We can adopt this approach in [13] to systems of the wave and Klein-Gordon equations because the required vector fields \( \Omega \) and \( \partial \) are admissible.

However we take another approach here since we can use the vector field \( L \); motivated by (4.10), we rewrite \( \partial_+ \) as
\[ \partial_+ = (t+r)^{-1} \left( 2Lr + (t-r)\partial_t - (t-r)\partial_r \right). \quad (4.12) \]
Then, by (4.13), (4.6) and (4.12), we obtain
\[ |Q_0(\varphi, \psi)| \leq C (t+r)^{-1} \left( |Z\varphi| |\partial\psi| + |\partial\varphi| |Z\psi| + \langle t-r \rangle |\partial\varphi| |\partial\psi| \right) \quad (4.13) \]
at \((t,x) \in [0, \infty) \times \mathbb{R}^3\).

For any multi-index \( \alpha \), we can easily check that \( Z^\alpha Q_0(\varphi, \psi) \) can be written as a linear combination of the null forms \( Q_0(Z^\beta \varphi, Z^\gamma \psi) \) and \( Q_{cd}(Z^\beta \varphi, Z^\gamma \psi) \) with \( |\beta| + |\gamma| \leq |\alpha| \) and \( 0 \leq c < d \leq 3 \). The same is true for \( Z^\alpha Q_{ab}(\varphi, \psi) \) \((0 \leq a < b \leq 3)\). Therefore, (4.5) and (4.13) yield the following:

**Lemma 4.1** Let \( k \) be a nonnegative integer, and let \( Q \) be one of the null forms \( Q_0 \) and \( Q_{ab} \) with \( 0 \leq a < b \leq 3 \). Then we have
\[ \langle t+|x| \rangle |Q(\varphi, \psi)| \leq C \left( |\varphi|_{[k/2]+1} |\partial\psi|_k + |\varphi|_{k+1} |\partial\psi|_{[k/2]} \right) \]
\[ + C \left( |\partial\varphi|_{[k/2]} |\psi|_{k+1} + |\partial\varphi|_k |\psi|_{[k/2]+1} \right) \]
\[ + C \langle t-|x| \rangle \left( |\partial\varphi|_{[k/2]} |\partial\psi|_k + |\partial\varphi|_k |\partial\psi|_{[k/2]} \right) \]
at \((t,x) \in [0, \infty) \times \mathbb{R}^3\) for any smooth functions \( \varphi \) and \( \psi \). Here \( C \) is a positive constant depending only on \( k \), \( \cdot |_s \) is given by (3.6) for a nonnegative integer \( s \), and \([m]\) denotes the largest integer not exceeding the number \( m \).

## 5 Proof of Theorem 2.1

In this section, we will prove Theorem 2.1.
Suppose that all the assumptions in Theorem 2.1 are fulfilled. We can easily obtain the local existence of the classical solutions to (1.1)-(1.2) for small $\varepsilon$. Moreover, we see that the local solution $u$ exists as far as $\sum_{|\alpha| \leq 2} \| \partial^\alpha u(t, \cdot) \|_{L^\infty(\mathbb{R}^3)}$ stays finite (see Hörmander [8] for instance). Hence what we need for the proof of Theorem 2.1 is such an $a$ priori estimate. Let $u = (u_i)_{1 \leq i \leq N} = (v, w)$ be the local solution to (1.1)-(1.2) for $0 \leq t < T_0$ with some $T_0 > 0$, where $v$ and $w$ are given by (2.2). If both $\mathcal{I}_1$ and $\mathcal{I}_2$ in the condition (b) are non-empty, without loss of generality we may assume that $\mathcal{I}_1 = \{1, \ldots, N_3\}$, and $\mathcal{I}_2 = \{N_3 + 1, \ldots, N_2\}$ with some positive integer $N_3$. Correspondingly, we write
\[
 w = (w_k)_{1 \leq k \leq N_2} = ((w_k^{(i)})_{1 \leq k \leq N_3}, (w_k^{(ii)})_{1 \leq k \leq N_4}) = (w^{(i)}, w^{(ii)}),
\]
where $N_4 = N_2 - N_3$. If $\mathcal{I}_2$ (resp. $\mathcal{I}_1$) is empty, then we put $w^{(i)} = w$ (resp. $w^{(ii)} = w$), and $w^{(ii)}$ (resp. $w^{(i)}$) should be neglected in what follows.

For a nonnegative integer $\sigma$, and a positive constant $p$, we define
\[
 d_{\sigma,p}(t, x) = \langle t + |x|^2 \rangle^{3/2} |v(t, x)|_{\sigma+2} + \langle t - |x|^2 \rangle |\partial w(t, x)|_{\sigma+1} + \langle t + |x|^2 \rangle (W_0(t, |x|) |w^{(i)}(t, x)|_{\sigma+2} + W_-(t, |x|)^{1-p} |w^{(ii)}(t, x)|_{\sigma+2})
\]
for $(t, x) \in [0, T_0) \times \mathbb{R}^3$, where $| \cdot |_s$, $W_0$, and $W_-$ are given by (3.6), (3.10), and (3.11), respectively.

For a smooth function $\varphi = \varphi(x)$ and a nonnegative integer $s$, we set
\[
 \| \varphi \|_{X^s} = \sum_{|\alpha| \leq s} \left( \left( \int_{\mathbb{R}^3} |\langle x \rangle^{s+2} \partial^\alpha_x \varphi(x)|^2 dx \right)^{1/2} + \left( \int_{\mathbb{R}^3} \langle x \rangle^s |\partial^\alpha_x \varphi(x)|^{6/5} dx \right)^{5/6} \right).
\]
Note that the Sobolev embedding theorem implies that
\[
 \sup_{x \in \mathbb{R}^3} \sum_{|\alpha| \leq s-2} \langle x \rangle^{s+4} |\partial^\alpha_x \varphi(x)| \leq \sup_{x \in \mathbb{R}^3} \sum_{|\alpha| \leq s-2} \langle x \rangle^{s+2} |\partial^\alpha_x \varphi(x)| \leq C_s \| \varphi \|_{X^s}
\]
for $s \geq 2$, where $C_s$ is a positive constant depending only on $s$.

Our aim here is to show the following:

**Proposition 5.1**  Fix some $\sigma \geq 19$, and $0 < p < 1/100$, say. Suppose that all the assumptions in Theorem 2.7 are fulfilled. Assume that $\|f\|_{X^{2s+1}} + \|g\|_{X^{2s}} \leq M_0$ with some positive constant $M_0$. Let $u = (v, w)$ be the local solution for $0 \leq t < T_0$, and suppose $0 < T \leq T_0$. Then there exists a positive constant $A_0 = A_0(M_0)$ having the following property: For any $A \geq A_0$, there exists a positive constant $\varepsilon_0 = \varepsilon_0(A)$ such that
\[
 \sup_{0 \leq t < T} \|d_{\sigma,p}(t, \cdot)\|_{L^\infty(\mathbb{R}^3)} \leq A \varepsilon \tag{5.1}
\]
implies
\[
 \sup_{0 \leq t < T} \|d_{\sigma,p}(t, \cdot)\|_{L^\infty(\mathbb{R}^3)} \leq \frac{A}{2} \varepsilon, \tag{5.2}
\]
provided that $0 < \varepsilon \leq \varepsilon_0$. Here $A_0$ and $\varepsilon_0$ are independent of $T$ and $T_0$.  

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Once Proposition 5.1 is established, by the continuity argument (or the bootstrap argument), we find that \( \|d_{\sigma,p}(t, \cdot)\|_{L^\infty(\mathbb{R}^3)} \) stays bounded as far as the solution exists, provided that \( \varepsilon \) is small enough. Indeed, suppose that \( f \) and \( g \) belong to \( \mathcal{C}_0^\infty(\mathbb{R}^3; \mathbb{R}^N) \) at first. Then, taking the support of \( u \) into account, we see that \( \|d_{\sigma,p}(t, \cdot)\|_{L^\infty(\mathbb{R}^3)} \) is continuous in \( t \). Noting that we have \( \|d_{\sigma,p}(0, \cdot)\|_{L^\infty(\mathbb{R}^3)} < A\varepsilon \) for some positive constant \( A \geq A_0 \), we see that (5.1) is true for some small \( T \). Let \( T_0(>0) \) be the supremum of \( T \in (0, T_0) \) for which (5.1) holds. If \( \varepsilon \in (0, \varepsilon_0) \), then we have \( \text{sup}_{0 \leq t < T_0} \|d_{\sigma,p}(t, \cdot)\|_{L^\infty} \leq A\varepsilon \), provided that \( \varepsilon \in (0, \varepsilon_0) \). We see that the same is true for general \( f, g \in \mathcal{S}(\mathbb{R}^3; \mathbb{R}^N) \) through the approximation by \( \mathcal{C}_0^\infty \)-functions. This \textit{a priori} estimate implies Theorem 2.1 immediately.

Now we are going to prove Proposition 5.1. We assume that (5.1) holds. In the following, various positive constants, being independent of \( A(>0), \varepsilon(\leq 1), T(>0), \) and \( M_0 \), are indicated just by the same letter \( C \). Thus the practical value of \( C \) may change line by line. Similarly, \( C_\delta \) stands for various positive constants depending only on \( M_0 \) and finite numbers of derivatives of \( F \). We always assume that \( \varepsilon \) is small enough to satisfy \( A\varepsilon \leq 1 \), say.

First we remark that for any nonnegative integer \( s \), there exists a positive constant \( C_s \) such that

\[
C_s^{-1} |\partial \varphi(t, x)| \leq \sum_{|\alpha| \leq s} |\partial Z^\alpha \varphi(t, x)| \leq C_s |\partial \varphi(t, x)| \tag{5.3}
\]

holds for any smooth function \( \varphi \), because of (3.5). We also note that we have

\[
\langle x \rangle^{-1} (t - |x|)^{-1} \leq C \langle t + |x| \rangle^{-1} W_-(t, |x|)^{-1}, \quad (t, x) \in [0, \infty) \times \mathbb{R}^3. \tag{5.4}
\]

We fix some small and positive constant \( \delta \). Then we have

\[
W_0(t, |x|) \leq C \langle t + |x| \rangle^\delta \langle t - |x| \rangle^{-\delta}, \quad (t, x) \in [0, \infty) \times \mathbb{R}^3. \tag{5.5}
\]

We will use (5.4) and (5.5) repeatedly in the following.

The proof of Proposition 5.1 is divided into several steps.

**Step 1: Energy Estimate.** Let \( 0 < \lambda < p/4 \). In this step, we are going to prove that

\[
\sup_{0 \leq t < T} \left( 1 + t \right)^{-\lambda} \left( \|v(t)\|_{2\sigma} + \|w^{(i)}(t)\|_{2\sigma} + \|\partial u(t)\|_{2\sigma} \right) \leq C_s \varepsilon \tag{5.6}
\]

holds for small \( \varepsilon \), where \( \|\cdot\|_s \) is given by (3.6). The difficulty here is the lack of a natural estimate for \( \|w^{(i)}(t)\|_{2\sigma} \) (cf. Lemma 3.1). To overcome this difficulty, we will use the following lemma that is easily obtained from the definition of \( Z \) and (5.3):

**Lemma 5.1** For any \( s \geq 1 \), there exists a positive constant \( C = C(s) \) such that we have

\[
|\varphi(t, x)| \leq C \left( |\varphi(t, x)| + (t + |x|) \|\partial \varphi(t, x)\|_{s-1} \right) \tag{5.7}
\]

for any smooth function \( \varphi = \varphi(t, x) \).
In fact, for $s \geq 1$, we have
\[
|\varphi(t, x)|_s \leq C \left( |\varphi(t, x)| + \sum_{|\alpha|=1} \sum_{|\beta| \leq s-1} |Z^\alpha Z^\beta \varphi(t, x)| \right)
\leq C \left( |\varphi(t, x)| + \langle t + |x| \rangle \sum_{|\beta| \leq s-1} |\partial(Z^\beta \varphi)(t, x)| \right),
\]
which leads to (5.7), thanks to (5.3).

Let $|\alpha| = s \leq 2\sigma$. We set
\[
F_{i,\alpha} = Z^\alpha \{ F_i(u, \partial u, \partial_x \partial u) \} - \sum_{j,k,a} \gamma_{i,j,k,a}^{ij}(u, \partial u) \partial_k \partial_a (Z^\alpha u_j), \tag{5.8}
\]
where $\gamma = (\gamma_{i,j,k,a}^{ij})$ is from (1.3). Then we have
\[
(\square + m_i^2)(Z^\alpha u_i) - \sum_{j,k,a} \gamma_{i,j,k,a}^{ij}(u, \partial u) \partial_k \partial_a (Z^\alpha u_j) = F_{i,\alpha}, \ 1 \leq i \leq N. \tag{5.9}
\]
Note that we have $|[Z^\alpha, \partial_k \partial_a] u_j| \leq C |\partial u|_s$ by (3.5). Hence, in view of (1.3), (3.5), and (5.8), from the condition (b–i), (5.4), and (5.5), we get
\[
|F_{i,\alpha}| \leq C (|v|_{s/2} + |w^{(ii)}|_{s/2} + |\partial u|_{s+1}) (|v|_s + |w^{(ii)}|_s + |\partial u|_s)
+ C |u|_{s+2} (|v|_s + |\partial u|_s)
\leq C A \varepsilon (1 + t)^{-1} (|v|_s + |w^{(ii)}|_s + |\partial u|_s)
+ C A^2 \varepsilon^2 (t + |x|)^{-2+2\delta} \langle t - |x| \rangle^{-2\delta}
\times \left( A \varepsilon \langle t + |x| \rangle^{-1+\delta} \langle t - |x| \rangle^{-\delta} + (t + x) |\partial u|_{s-1} + |\partial u|_s \right) \tag{5.10}
\]
at $(t, x) \in [0, T) \times \mathbb{R}^3$. Here we have also used (5.7) to estimate $|u|_s$. Thus the term $(t + |x|) |\partial u|_{s-1}$ on the right-hand side of (5.10) should be neglected when $s = 0$. Since we have
\[
\left\| (t + |\cdot|)^{-3+3\delta} \langle t - |\cdot| \rangle^{-3\delta} \right\|_{L^2(\mathbb{R}^3)} \leq C (1 + t)^{-3/2}
\]
for $\delta < 1/6$, (5.10) yields
\[
\|F_{i,\alpha}\|_{L^2(\mathbb{R}^3)} \leq C A \varepsilon (1 + t)^{-1} (|v|_s + |w^{(ii)}|_s + |\partial u|_s)
+ C A^2 \varepsilon^2 (1 + t)^{-1+2\delta} |\partial u|_{s-1} + C A^3 \varepsilon^3 (1 + t)^{-3/2}, \tag{5.11}
\]
where the term $C A^2 \varepsilon^2 (1 + t)^{-1+2\delta} |\partial u|_{s-1}$ should be neglected when $s = 0$. In view of the condition (b–i), from (5.1), (5.4), and (5.5), we also get
\[
|\gamma_i| \leq C (|v|_1 + |w^{(ii)}|_1 + |\partial u|_1 + |u|_1^2 + |\partial u|_1^2) \leq C A \varepsilon (1 + t)^{-1} \tag{5.12}
\]
at $(t, x) \in [0, T) \times \mathbb{R}^3$. Because of (1.4) and (5.12), we can apply Lemma 3.1 to (5.9) for small $\varepsilon$. 

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For $N_3 + 1 \leq k \leq N_2$ and $0 \leq a \leq 3$, let $G_{k,a}$ be from the condition (b–ii). Because of (3.5), we get

$$\square \left( Z^{\alpha} w^{(ii)}_j \right) = \sum_{a,b=0}^{3} \sum_{|\beta| \leq |\alpha|} C^{ab\beta}_a \partial_\alpha \left( Z^{\beta} G_{N_3+j,b}(u, \partial u) \right), \quad 1 \leq j \leq N_4 \quad (5.13)$$

with appropriate constants $C^{ab\beta}_a$. Remember that each $G^{(2)}_{k,a}$ is independent of $w^{(i)}$ itself. Thus, going similar lines to (5.10) and (5.11), we get

$$|G_{k,a}(u, \partial u)|_s \leq C \lambda/(1 + t)^{-1} (\|v\|_s + \|w^{(ii)}\|_s + \|\partial u\|_s) + C A^2 \epsilon^2 (1 + t)^{-1+2\delta} \|\partial u\|_{s-1} + C A^3 \epsilon^3 (1 + t)^{-3/2} \quad (5.14)$$

at $(t, x) \in [0, T) \times \mathbb{R}^3$ for $s \leq 2\sigma$. As before, the term including $\|\partial u\|_{s-1}$ on the right-hand side should be neglected when $s = 0$.

We put

$$E_s(t) = \|v(t)\|_s + \|w^{(ii)}(t)\|_s + \|\partial u(t)\|_s$$

for $s \geq 0$. In view of (5.11), (5.12), and (5.14), applying Lemma 3.1 to (5.9) with $|\alpha| = s = 0$, and applying Lemma 3.6 to (5.13) with $|\alpha| = s = 0$, we obtain

$$E_0(t) \leq C \epsilon + C A^3 \epsilon^3 + C A \epsilon \int_0^t (1 + \tau)^{-1} E_0(\tau) d\tau. \quad (5.15)$$

The Gronwall lemma yields

$$E_0(t) \leq \left( C \epsilon + C A^3 \epsilon^3 \right) (1 + t)^{C A \epsilon} \leq C \epsilon (1 + t)^{C A \epsilon}, \quad (5.15)$$

provided that $\epsilon$ is small enough to satisfy $A^3 \epsilon^2 \leq 1$. From this, we can inductively obtain

$$E_s(t) \leq C \epsilon (1 + t)^{2s\delta + C A \epsilon} \quad (5.16)$$

for $0 \leq s \leq 2\sigma$, where $C \epsilon$s are positive constants depending on $M_0$ and the nonlinearity $F$. In fact, if (5.16) with $s$ replaced by $s - 1$ is true for some $s \geq 1$, then applying Lemmas 3.1 and 3.6 to (5.9) and (5.13) with $|\alpha| = s$, respectively, and using (5.11), (5.12) and (5.14), we obtain

$$E_s(t) \leq C \epsilon + C \left( A^3 \epsilon^3 + C \epsilon^{s-1} A^2 \epsilon^3 (1 + t)^{2s\delta + C A \epsilon} \right) + C A \epsilon \int_0^t (1 + \tau)^{-1} E_s(\tau) d\tau,$$

and the Gronwall lemma leads to (5.16).

Finally, we obtain (5.16) from (5.16) with $s = 2\sigma$, provided that $\delta$ in (5.5) is chosen to satisfy $4\sigma \delta \leq \lambda/2$, and $\epsilon$ is small enough to satisfy $A^3 \epsilon^2 \leq 1$ and $C A \epsilon \leq \lambda/2$.

\textbf{Step 2: Decay Estimates, Part 1.} By Lemma 3.5 and (5.6), we get

$$\langle x \rangle \left( |v(t, x)|_{2\sigma-2} + |w^{(ii)}(t, x)|_{2\sigma-2} + |\partial u(t, x)|_{2\sigma-2} \right) \leq C \epsilon (1 + t)^{\lambda} \quad (5.17)$$
for \((t,x) \in [0,T) \times \mathbb{R}^3\).

Similarly to (5.10), we get

\[
|F_i|_s \leq C \left( |v|_s + |w^{(i)}|_s + |\partial u|_{s+1} \right) \left( |v|_s + |w^{(i)}|_s + |\partial u|_{s+1} \right) \\
+ C|u|_{s+1}^2 \left( |u|_s + |\partial u|_{s+1} \right)
\]

\[
\leq C A \varepsilon \left( t + |x| \right)^{-3/2} + \left( t + |x| \right)^{-1} W_-(t,|x|)^{-1+p} \left( |v|_s + |w^{(i)}|_s + |\partial u|_{s+1} \right)
\]

\[
+ C A^2 \varepsilon^2 \left( t + |x| \right)^{-2+2\delta} W_-(t,|x|)^{-2\delta} \left( |v|_s + |w|_s + |\partial u|_{s+1} \right)
\]

(5.18)

for \(s \leq 2\sigma\). For \(\rho \geq 0\) and a nonnegative integer \(s\), we set

\[
M_{\rho,s} = \sup_{(t,x) \in [0,T) \times \mathbb{R}^3} \langle t + |x| \rangle^{1-\rho} \mathcal{W}_0(t,|x|) |w^{(i)}(t,x)|_s.
\]

Then, using (5.17), we get

\[
\langle x \rangle |F_i|_{2\sigma-3} \leq C_s A \varepsilon^2 \left( t + |x| \right)^{-1+\lambda} W_-(t,|x|)^{-1/2}
\]

\[
+ C_s A^2 \varepsilon^3 \left( t + |x| \right)^{-2+2\delta+\lambda} W_-(t,|x|)^{-2\delta}
\]

\[
+ C A^2 \varepsilon^2 M_{\lambda+(1/2),2\sigma-3}^{(i)} \left( t + |x| \right)^{-3/2+3\delta+\lambda} W_-(t,|x|)^{-3\delta}
\]

\[
\leq \left( C_s A \varepsilon^2 + C A^2 \varepsilon^2 M_{\lambda+(1/2),2\sigma-3}^{(i)} \right) \left( t + |x| \right)^{-1/2+\lambda-\mu} W_-(t,|x|)^{-1+\mu},
\]

where \(\mu\) is a small and positive constant. Hence, by Lemma 3.3 and also by (3.13) of Lemma 3.4 with \((\rho,\kappa) = (\lambda + (1/2), 1)\), we get

\[
M_{\lambda+(1/2),2\sigma-3}^{(i)} \leq C_s \left( \varepsilon + A \varepsilon^2 \right) + C A^2 \varepsilon^2 M_{\lambda+(1/2),2\sigma-3}^{(i)}
\]

Therefore, if \(\varepsilon\) is small enough to satisfy \(C A^2 \varepsilon^2 \leq 1/2\), then we obtain

\[
\sup_{(t,x) \in [0,T) \times \mathbb{R}^3} \langle t + |x| \rangle^{-\lambda+(1/2)} \mathcal{W}_0(t,|x|) |w^{(i)}(t,x)|_{2\sigma-3} = M_{\lambda+(1/2),2\sigma-3}^{(i)}
\]

\[
\leq C_s \varepsilon.
\]

(5.20)

Using (5.20), we have

\[
\langle x \rangle |F_i|_{2\sigma-3} \leq C_s (A \varepsilon^2 + A^2 \varepsilon^3) \left( t + |x| \right)^{-(1/2)+\lambda-\mu} W_-(t,|x|)^{-1+\mu}.
\]

(5.21)

Hence, similarly to (5.20), Lemma 3.3 and (3.14) of Lemma 3.4 yield

\[
\sup_{(t,x) \in [0,T) \times \mathbb{R}^3} \langle t + |x| \rangle^{-\lambda-(1/2)} \langle x \rangle \langle t - |x| \rangle |\partial w(t,x)|_{2\sigma-4} \leq C_s \varepsilon,
\]

(5.22)

provided that \(\varepsilon\) is small enough. Going similar lines to (5.18)–(5.21), we get

\[
\langle x \rangle |G_{k,a}(u,\partial u)|_{2\sigma-3} \leq C_s (A \varepsilon^2 + A^2 \varepsilon^3) \left( t + |x| \right)^{-(1/2)+\lambda-\mu} W_-(t,|x|)^{-1+\mu},
\]

and applying Lemma 3.7 to (5.13) with \(|\alpha| \leq 2\sigma - 4\), we get

\[
\sup_{(t,x) \in [0,T) \times \mathbb{R}^3} \langle t + |x| \rangle^{-\lambda-(1/2)} \langle x \rangle \langle t - |x| \rangle |w^{(ii)}(t,x)|_{2\sigma-4} \leq C_s \varepsilon,
\]

(5.23)

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provided that $\varepsilon$ is small enough.

Using (5.6), by (5.8), (5.10), and (5.12), we obtain
\[
\| t + |t| \|_{F_1|2\sigma - 1} \leq C_\varepsilon A\varepsilon^2 (1 + t)^\lambda + C_\varepsilon A^2\varepsilon^3 (1 + t)^{\lambda + 2\delta} + CA^3\varepsilon^3 \left\| t + |t| \right\|_{2}^{2+3\delta} \left\| t - |t| \right\|_{2}^{-3\delta} \leq C_\varepsilon A\varepsilon^2 (1 + t)^{2\lambda}
\]
for sufficiently small $\delta$, and $0 < \varepsilon < A^{-1}$. Hence Lemma 3.2 leads to
\[
\langle t + |t| \rangle^{3/2} \| v(t, x) \|_{2\sigma - 5} \leq C_\varepsilon \left( \varepsilon + A\varepsilon^2 \sum_{j=0}^{\infty} \sup_{\tau \in [0, t]} \chi_j(\tau) (1 + \tau)^{2\lambda} \right).
\]
Let $2^{J-1} \leq t < 2^J$ with some nonnegative integer $J$. Then we have
\[
\sum_{j=0}^{\infty} \sup_{\tau \in [0, t]} \chi_j(\tau) (1 + \tau)^{2\lambda} = \sum_{j=0}^{J} \sup_{\tau \in [0, t]} \chi_j(\tau) (1 + \tau)^{2\lambda} \leq \sum_{j=0}^{J} 2^{2(j+2)\lambda} = \frac{2^{4\lambda} (2^{2\lambda(J+1)} - 1)}{2^{2\lambda} - 1} \leq C(1 + t)^{2\lambda}.
\]
A similar estimate for $0 \leq t < 1$ is trivially obtained. Now (5.24) leads to
\[
\langle t + |t| \rangle^{(3/2)-2\lambda} \| v(t, x) \|_{2\sigma - 5} \leq C_\varepsilon \varepsilon,
\]
provided that $\varepsilon$ is small enough.

**Step 3: Decay Estimates, Part 2.** We make use of the detailed structure of the nonlinearity from now on. For a smooth function $\Phi = \Phi(\xi, \xi', \xi'')$, we define
\[
\Phi^{(K)}(\eta, \eta', \eta'') = \Phi^{(2)}((\eta, \xi), (\eta', \xi'), (\eta'', \xi''))\big|_{(\xi', \xi'', \xi'') = (0, 0, 0)};
\]
\[
\Phi^{(KW)}(\xi, \xi', \xi'') = \Phi^{(2)}((\eta, \xi), (\eta', \xi'), (\eta'', \xi'')) - \Phi^{(K)}(\eta, \eta', \eta''),
\]
\[
\Phi^{(H)}(\xi, \xi', \xi'') = \Phi(\xi, \xi', \xi'') - \Phi^{(2)}(\xi, \xi', \xi''),
\]
where $(\xi, \xi', \xi'') = ((\eta, \xi), (\eta', \xi'), (\eta'', \xi''))$ as before, and $\Phi^{(W)}$ is given by (2.3).

Since $(F^{(W)}_i)_{N_1 + 1 \leq i \leq N}$ satisfies the null condition, by (2.11), Lemma 4.1 and (5.1), we get
\[
\left| F^{(W)}_i \right|_s \leq C \langle t + |t| \rangle^{-1} \left( |w|_{[s/2]+2} |\partial w|_{s+1} + |\partial w|_{[s/2]+1} |w|_{s+1} \right) + C \langle t + |t| \rangle^{-1} \left( t - |t| \right) |\partial w|_{[s/2]+1} |\partial w|_{s+1} \leq C A \varepsilon \langle t + |t| \rangle^{-2+\delta} \left( t - |t| \right)^{-\delta} |\partial w|_{s+1} + C A \varepsilon \langle t + |t| \rangle^{-1} \left( t - |t| \right)^{-1} |w|_{s+1} + |\partial w|_{s+1}
\]
at $(t, x) \in [0, T] \times \mathbb{R}^3$ for $N_1 + 1 \leq i \leq N$, provided that $s \leq 2\sigma$. 

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For $1 \leq i \leq N$, it is easy to see that each $F_i^{(K)}(v, \partial v, \partial_x \partial v)$ is a linear combination of $(\partial^\alpha \psi_j) (\partial^\beta \psi_k)$ with $|\alpha|, |\beta| \leq 2$, and $1 \leq j, k \leq N_1$. On the other hand, each $F_i^{(KW)}(u, \partial u, \partial_x \partial u)$ is a linear combination of $(\partial^\alpha \psi_j) (\partial^\beta \psi_k)$ and $(\partial^\alpha \psi_j) w^{(ii)}_i$ with $|\alpha| \leq 2$, $1 \leq |\beta| \leq 2$, $1 \leq j \leq N_1$, $1 \leq k \leq N_2$, and $1 \leq l \leq N_4$, because of the condition (b-i). Therefore (5.1) yields

$$
|F_i^{(K)}|_s \leq C |v|_{s/2}^2 |v|_{s+2} \leq CA \varepsilon (t + |x|)^{-3/2} |v|_{s+2},
$$

(5.30)

$$
|F_i^{(KW)}|_s \leq C (|w^{(ii)}|^{|s/2} + |\partial w|_{s/2+1}) |v|_{s+2} + |w|_{s/2+2} (|w^{(ii)}|_s + |\partial w|_{s+1})

\leq CA \varepsilon (t + |x|)^{-1} W_- (t, |x|)^{-1+\lambda} |v|_{s+2}

+ CA \varepsilon (t + |x|)^{-3/2} (|w^{(ii)}|_s + |\partial w|_{s+1})
$$

(5.31)

at $(t, x) \in [0, T) \times \mathbb{R}^3$ for $1 \leq i \leq N$, provided that $s \leq 2\sigma$.

Since we have $F_i^{(H)}(u, \partial u, \partial_x \partial u) = O (|u|^3 + |\partial u|^3 + |\partial_x \partial u|^3)$, we get

$$
|F_i^{(H)}|_s \leq C |u|^2 + |w|_s + |\partial w|_{s+1})

\leq CA \varepsilon^2 (t + |x|)^{-2+2\delta} (t - |x|)^{-2\delta} (|v|_{s+2} + |w|_s + |\partial w|_{s+1})
$$

(5.32)

at $(t, x) \in [0, T) \times \mathbb{R}^3$. Similarly, by (5.32), we get

$$
\langle x \rangle \left| F_i^{(W)} \right|_{2\sigma - 7} \leq C_4 A \varepsilon^2 \langle t + |x| \rangle^{-(3/2)+\lambda} W_- (t, |x|)^{-1-\delta}
$$

(5.33)

$$
\langle x \rangle \left| F_i^{(K)} \right|_{2\sigma - 7} \leq C_4 A \varepsilon^2 \langle t + |x| \rangle^{-2+2\lambda}
$$

(5.34)

$$
\langle x \rangle \left| F_i^{(KW)} \right|_{2\sigma - 7} \leq C_4 A \varepsilon^2 \langle t + |x| \rangle^{-1+\lambda} W_- (t, |x|)^{-1}
$$

(5.35)

at $(t, x) \in [0, T) \times \mathbb{R}^3$. Similarly, by (5.32), we get

$$
\langle x \rangle \left| F_i^{(H)} \right|_{2\sigma - 7} \leq C_4 A^2 \varepsilon^3 \langle t + |x| \rangle^{-(3/2)+2\delta+2\lambda} \langle t - |x| \rangle^{-1-2\delta}

+ CA^2 \varepsilon^2 M^{(i)}_{3\lambda, 2\sigma - 7} \langle t + |x| \rangle^{-2+3\delta+3\lambda} \langle t - |x| \rangle^{-3\delta}
$$

(5.36)

at $(t, x) \in [0, T) \times \mathbb{R}^3$, where $M^{(i)}_{3\lambda, 2\sigma - 7}$ is given by (5.19). Since we can choose $\delta$ as small as we wish, (5.33) – (5.36) lead to

$$
\langle x \rangle \left| F_i \right|_{2\sigma - 7} \leq \left( C_4 A \varepsilon^2 + CA^2 \varepsilon^2 M^{(i)}_{3\lambda, 2\sigma - 7} \right) \langle t + |x| \rangle^{-1+3\lambda-\mu} W_- (t, |x|)^{-1+\mu}
$$

(5.37)

for small $\mu > 0$. Therefore, by (3.13) with $(\rho, \kappa) = (3\lambda, 1)$, and by Lemma 3.3, we obtain

$$
M^{(i)}_{3\lambda, 2\sigma - 7} \leq C_4 \left( \varepsilon + A \varepsilon^2 \right) + CA^2 \varepsilon^2 M^{(i)}_{3\lambda, 2\sigma - 7},
$$

which leads to

$$
\sup_{(t, x) \in [0, T) \times \mathbb{R}^3} \langle t + |x| \rangle^{1-3\lambda} W_0 (t, |x|) |w^{(i)}(t, x)|_{2\sigma - 7} = M^{(i)}_{3\lambda, 2\sigma - 7} \leq C_4 \varepsilon,
$$

(5.38)
provided that \( \varepsilon \) is sufficiently small. Now (5.37) and (5.38) yield
\[
\langle x \rangle |F_i|_{2\sigma-7} \leq C_s A \varepsilon^2 \langle t + |x| \rangle^{-1+3\lambda-\mu} W_{-}(t, |x|)^{-1+\mu}. 
\] (5.39)
Hence Lemma 3.3 and (3.14) imply
\[
\sup_{(t,x) \in [0,T) \times \mathbb{R}^3} \langle t + |x| \rangle^{-3\lambda} \langle x \rangle \langle t - |x| \rangle |\partial w(t,x)|_{2\sigma-8} \leq C_s \varepsilon, 
\] (5.40)
provided that \( \varepsilon \) is sufficiently small.

Now we are going to estimate \( w^{(ii)} \). Suppose that \( N_3 + 1 \leq k \leq N_2 \), and \( 0 \leq a \leq 3 \). Let \( G_{k,a}(\zeta, \zeta') \), \( G_{k,a}(\eta, \eta') \), \( G_{k,a}(\xi, \xi') \), and \( G_{k,a}(\xi, \xi') \) be given by (2.3), (5.26), (5.28), and (5.28), respectively, with \( \Phi = G_{k,a}(\xi, \xi') \). It is easy to see that \( G_{k,a}(\zeta, \zeta') \), \( G_{k,a}(\eta, \eta') \), and \( G_{k,a}(\xi, \xi') \) have similar structures to \( F^{(K)}_i \), \( F^{(KW)}_i \), and \( F^{(H)}_i \), respectively. On the other hand, in view of (2.11), we find that \( G_{k,a}(\xi, \xi') \) may not be written in terms of the null forms. Hence we divide \( w^{(ii)} \) into two parts: For \( 1 \leq l \leq N_4 \), let \( w^{(iii)}_l \) and \( w^{(iv)}_l \) be the solutions to
\[
\begin{align*}
\Box w^{(iii)}_l &= \sum_{a=0}^{3} \partial_a \left\{ G_{N_3+l,a}(u, \partial u) - G_{N_3+l,a}(w, \partial w) \right\}, \\
w^{(iii)}_l(0,x) &= w^{(ii)}_l(0,x), \quad \partial_t w^{(iii)}_l(0,x) = \partial_t w^{(ii)}_l(0,x),
\end{align*}
\]
and
\[
\begin{align*}
\Box w^{(iv)}_l &= F^{(W)}_{N_1+N_3+l}(w, \partial w, \partial_x \partial w), \\
w^{(iv)}_l(0,x) &= \partial_t w^{(iv)}_l(0,x) = 0,
\end{align*}
\]
respectively. Since we have
\[
F^{(W)}_{N_1+N_3+l}(w, \partial w, \partial_x \partial w) = \sum_{a=0}^{3} \partial_a \left( G^{(W)}_{N_3+l,a}(w, \partial w) \right), \quad 1 \leq l \leq N_4,
\]
we get \( w^{(ii)}_l = w^{(iii)}_l + w^{(iv)}_l \). We put \( w^{(iii)} = (w^{(iii)}_l) \) and \( w^{(iv)} = (w^{(iv)}_l) \) with \( 1 \leq l \leq N_4 \).

Note that we have \( G_{k,a} - G^{(W)}_{k,a} = G^{(K)}_{k,a} + G^{(KW)}_{k,a} + G^{(H)}_{k,a} \). It is easy to see that \( G^{(K)}_{k,a} \), \( G^{(KW)}_{k,a} \), and \( G^{(H)}_{k,a} \) enjoy the estimates corresponding to (5.30), (5.31) and (5.32), respectively. Hence, similarly to (5.39), we get
\[
\langle x \rangle |G_{k,a} - G_{k,a}^{(W)}|_{2\sigma-7} \leq C_s A \varepsilon^2 \langle t + |x| \rangle^{-1+3\lambda-\mu} W_{-}(t, |x|)^{-1+\mu}, 
\] (5.41)
and Lemma 3.7 yields
\[
\sup_{(t,x) \in [0,T) \times \mathbb{R}^3} \langle t + |x| \rangle^{-3\lambda} \langle x \rangle \langle t - |x| \rangle |w^{(iii)}(t,x)|_{2\sigma-8} \leq C_s \varepsilon. 
\] (5.42)
On the other hand, in view of (5.33), using (3.13) with \( (\rho, \kappa) = (0, (3/2) - 2\lambda) \), we get
\[
\sup_{(t,x) \in [0,T) \times \mathbb{R}^3} \langle t + |x| \rangle \langle t - |x| \rangle^{(1/2) - 2\lambda} |w^{(iv)}(t,x)|_{2\sigma-8} \leq C_s \varepsilon, 
\] (5.43)
since we may assume $\mu + \delta - \lambda < 0$ for small $\mu > 0$. Summing up, we obtain

$$\sup_{(t,x) \in [0,T) \times \mathbb{R}^3} \langle t + |x| \rangle^{1-3\lambda} W_-(t, |x|) |w^{(i)}(t, x)|_{2\sigma-8} \leq C_* \varepsilon, \quad (5.44)$$

provided that $\varepsilon$ is sufficiently small.

**Step 4: Decay Estimates, Part 3.** Let $1 \leq i \leq N_1$ in this step. Motivated by the technique in [2], [19] and [29], we introduce

$$\tilde{v}_i = v_i - m_i^{-2} F_i^{(W)}(w, \partial w, \partial_x \partial w) \quad (5.45)$$

in order to treat $F_i^{(W)}$. Then we get

$$\Box + m_i^2 \tilde{v}_i = \left\{ \Box + m_i^2 v_i - F_i^{(W)}(w, \partial w, \partial_x \partial w) \right\} - m_i^{-2} \Box \left\{ F_i^{(W)}(w, \partial w, \partial_x \partial w) \right\}. \quad (5.46)$$

From the condition (b–i), we can write

$$F_i^{(W)}(w, \partial w, \partial_x \partial w) = \sum_{|\alpha|,|\beta| \leq 2} \sum_{1 \leq j, k \leq N_2} P_i^{jk\alpha \beta}(\partial^\alpha w_j)(\partial^\beta w_k) \quad (5.47)$$

with appropriate constants $P_i^{jk\alpha \beta}$, where $P_i^{jk\alpha \beta}$ vanishes either when $1 \leq j \leq N_3$ and $|\alpha| = 0$, or when $1 \leq k \leq N_3$ and $|\beta| = 0$. Since we have

$$\Box(\varphi \psi) = 2Q_0(\varphi, \psi) + (\Box \varphi)\psi + \varphi(\Box \psi)$$

for any smooth functions $\varphi$ and $\psi$, from (5.47) we get

$$\Box F_i^{(W)} = \tilde{F}_i^{(W)} + \tilde{F}_i^{(H)}, \quad (5.48)$$

where

$$\tilde{F}_i^{(W)} = 2 \sum_{|\alpha|,|\beta| \leq 2} \sum_{1 \leq j, k \leq N_2} P_i^{jk\alpha \beta} Q_0(\partial^\alpha w_j, \partial^\beta w_k), \quad (5.49)$$

$$\tilde{F}_i^{(H)} = \sum_{|\alpha|,|\beta| \leq 2} \sum_{1 \leq j, k \leq N_2} P_i^{jk\alpha \beta} \left\{ (\partial^\alpha F_{N_1+j})(\partial^\beta w_k) + (\partial^\alpha w_j)(\partial^\beta F_{N_1+k}) \right\}. \quad (5.50)$$

Note that each $\tilde{F}_i^{(W)}$ is written in terms of the null forms, and we can expect extra decay for $\tilde{F}_i^{(W)}$. On the other hand, each $\tilde{F}_i^{(H)}$ is a function of cubic order with respect to $\partial^\alpha u_j$ with $|\alpha| \leq 4$ and $1 \leq j \leq N$. From (5.46) and (5.48), we obtain

$$\Box + m_i^2 \tilde{v}_i = F_i^{(K)} + F_i^{(KW)} - m_i^{-2} \tilde{F}_i^{(W)} + (F_i^{(H)} - m_i^{-2} \tilde{F}_i^{(H)}), \quad (5.51)$$

By (5.30) and (5.6), we have

$$\left\| \langle t + | \cdot | \rangle \left| F_i^{(K)} \right|_{2\sigma-10} \right\|_{L^2(\mathbb{R}^3)} \leq CA \varepsilon (1 + t)^{-1/2} \|v\|_{2\sigma-8}$$

$$\leq C_* A \varepsilon^2 (1 + t)^{\lambda-(1/2)}. \quad (5.52)$$
Similarly to (5.33) and (5.35), but using (5.38), (5.40) and (5.44) instead of (5.20), (5.22) and (5.23), we obtain

\[
\langle t + |x| \rangle \left| \hat{F}_i^{(W)} \right|_{2\sigma-10} \leq C_* A \varepsilon^2 \langle x \rangle^{-1} (t + |x|)^{-1+\delta+3\lambda} W_-(t, |x|)^{-1-\delta}, \tag{5.53}
\]

\[
\langle t + |x| \rangle \left| F_i^{(KW)} \right|_{2\sigma-10} \leq C_* A \varepsilon^2 (t + |x|)^{-(3/2)+3\lambda} W_-(t, |x|)^{-(1/2)-\lambda} . \tag{5.54}
\]

at \((t, x) \in [0, \infty) \times \mathbb{R}^3\) (for the later usage, we note that (5.54) is also true for \(1 \leq i \leq N\)). Since we may assume \(\delta < 1/2\), from (5.53) and (5.54) we obtain

\[
\left\| \langle t + \cdot \rangle \left( \left| \hat{F}_i^{(W)} \right|_{2\sigma-10} + \left| F_i^{(KW)} \right|_{2\sigma-10} \right) \right\|_{L^2} \\
\leq C_* A \varepsilon^2 \left\| \langle \cdot \rangle^{-1} (t + |\cdot|)^{-(1/2)+3\lambda} W_-(t, |\cdot|)^{-(1/2)-\lambda} \right\|_{L^2} \\
\leq C_* A \varepsilon^2 (1 + t)^{-(1/2)+3\lambda}. \tag{5.55}
\]

Going similar lines to (5.32), and then using (5.6) and (5.38), we obtain

\[
\left\| \langle t + |\cdot| \rangle \left| F_i^{(H)} \right|_{2\sigma-10} + m_i^{-2} \hat{F}_i^{(H)} \right\|_{L^2} \\
\leq C A^2 \varepsilon^3 (1 + t)^{-1+2\delta} (\|v\|_{2\sigma-10} + \|w^{(ii)}\|_{2\sigma-10} + \|\partial u\|_{2\sigma-7}) \\
+ C_* A^2 \varepsilon^3 \left\| \langle t + |\cdot| \rangle^{-2+3\delta+3\lambda} \langle t - |\cdot| \rangle^{-3\delta} \right\|_{L^2} \\
\leq C_* A^2 \varepsilon^3 (1 + t)^{-1+\lambda+2\delta} + (1 + t)^{-(1/2)+3\lambda}. \tag{5.56}
\]

Summing up, we have proved

\[
\left\| \langle t + |\cdot| \rangle \left( \Box + m_i^2 \right) \hat{v}_i \right\|_{L^2} \leq C_* A \varepsilon^2 (1 + t)^{-1/4}, \tag{5.57}
\]

because \(\lambda \leq 1/12\) and \(\delta \ll 1\). Now Lemma 3.2 implies

\[
\sup_{(t, x) \in [0, T) \times \mathbb{R}^3} \langle t + |x| \rangle^{3/2} \left| \hat{v}(t, x) \right|_{2\sigma-14} \\
\leq C_* \left( \varepsilon + A \varepsilon^2 \sum_{j=0}^{\infty} \sup_{\tau \in (0, t)} \chi_j(\tau)(1 + \tau)^{-1/4} \right) \leq C_* \varepsilon, \tag{5.58}
\]

because we have \(\sup_{\tau \in (0, t)} \chi_j(\tau)(1 + \tau)^{-1/4} \leq 2^{-(j-1)/4}\) for \(j \geq 1\). From (5.40), (5.44) and (5.47), we get

\[
\left| \hat{F}_i^{(W)} \right|_{2\sigma-14} \leq C \left( \|\partial w\|_{\sigma+1} + \|w^{(ii)}\|_{\sigma+2} \right) \left( \|\partial w\|_{2\sigma-13} + \|w^{(ii)}\|_{2\sigma-14} \right) \\
\leq C_* A \varepsilon^2 \langle t + |x| \rangle^{3\lambda-2},
\]

which, together with (5.45) and (5.58), yields

\[
\sup_{(t, x) \in [0, T) \times \mathbb{R}^3} \langle t + |x| \rangle^{3/2} |v(t, x)|_{2\sigma-14} \leq C_* \varepsilon \tag{5.59}
\]
for small $\varepsilon$.

**Step 5: Decay Estimates, the Final Part.** Let $N_1 + 1 \leq i \leq N$. Similarly to (5.53), we get

$$
\langle x \rangle |F_i^{(W)}|_{2\sigma-9} \leq C_* A\varepsilon^2 \langle t + |x| \rangle^{\tilde{\delta} + 3\delta} W^\ast (t, |x|) - \tilde{\delta}.
$$

(5.60)

Therefore, using (3.13) with $(\rho, \kappa) = (0, 2 - 4\lambda)$, we get

$$
\sup_{(t,x) \in [0,T) \times \mathbb{R}^3} \langle t + |x| \rangle \langle t - |x| \rangle^{1-4\lambda} \left| u^{(iv)}(t, x) \right|_{2\sigma-9} \leq C_* \varepsilon,
$$

(5.61)

which, together with (5.42), yields

$$
\sup_{(t,x) \in [0,T) \times \mathbb{R}^3} \langle t + |x| \rangle^{1-3\lambda} W^\ast (t, |x|)^{-1-\lambda} \left| u^{(ii)}(t, x) \right|_{2\sigma-9} \leq C_* \varepsilon.
$$

(5.62)

By (5.30) and (5.59), we obtain

$$
\langle x \rangle |F_i^{(K)}|_{2\sigma-16} \leq C_* A\varepsilon^2 \langle t + |x| \rangle^{-2} \leq C_* A\varepsilon^2 \langle t + |x| \rangle^{-1-\mu} \langle t - |x| \rangle^{-\mu-1}
$$

(5.63)

at $(t, x) \in [0, T) \times \mathbb{R}^3$ for small $\mu > 0$. From (5.31), (5.40), (5.59), and (5.62), we get

$$
\langle x \rangle |F_i^{(KW)}|_{2\sigma-16} \leq C_* A\varepsilon^2 \langle t + |x| \rangle^{3\lambda - 3/2} W^\ast (t, |x|)^{-1+\tilde{\lambda}}
$$

$$
\leq C_* A\varepsilon^2 \langle t + |x| \rangle^{-1-\mu} \langle t - |x| \rangle^{-\mu-1}
$$

(5.64)

for small $\mu > 0$. On the other hand, (5.32), (5.40), (5.59), and (5.62) yield

$$
\langle x \rangle |F_i^{(H)}|_{2\sigma-16} \leq C_* A^2 \varepsilon^3 \langle t + |x| \rangle^{-(5/2) + 2\tilde{\delta}} \langle t - |x| \rangle^{-2\tilde{\delta}}
$$

$$
+ C A^2 \varepsilon^2 M_{0,2\sigma-16} \langle t + |x| \rangle^{-3+\delta} \langle t - |x| \rangle^{-3\delta}
$$

$$
+ C_* A^2 \varepsilon^3 \langle t + |x| \rangle^{-2+2\tilde{\delta}+3\lambda} \langle t - |x| \rangle^{-\lambda-2\tilde{\delta}-1}
$$

$$
\leq (C_* A^2 \varepsilon^3 + C A^2 \varepsilon^2 M_{0,2\sigma-16}^{(i)}) \langle t + |x| \rangle^{-\tilde{\mu} - \mu} \langle t - |x| \rangle^{-\mu-1}
$$

(5.65)

at $(t, x) \in [0, T) \times \mathbb{R}^3$. Since we may assume $\delta + 3\lambda \leq 1/4$, say, from (5.60), (5.64), (5.63), and (5.65), we obtain

$$
\langle x \rangle |F_2|_{2\sigma-16} \leq (C_* A^2 + C A^2 \varepsilon^2 M_{0,2\sigma-16}^{(i)}) \langle t + |x| \rangle^{-1-\mu} W^\ast (t, |x|)^{-\mu-1}.
$$

(5.66)

Now Lemma 3.3 and (3.13) with $(\rho, \kappa) = (0, 1)$ imply

$$
M_{0,2\sigma-16}^{(i)} \leq C_(\varepsilon + A\varepsilon^2) + C A^2 \varepsilon^2 M_{0,2\sigma-16}^{(i)},
$$

which leads to

$$
\sup_{(t,x) \in [0,T) \times \mathbb{R}^3} \langle t + |x| \rangle |\mathcal{W}_0(t, |x|)|w^{(i)}(t, x)|_{2\sigma-16} = M_{0,2\sigma-16}^{(i)} \leq C_* \varepsilon,
$$

(5.67)
provided that \( \varepsilon \) is small enough. By (5.66) and (5.67), using Lemma 3.3 and (3.14), we obtain
\[
\sup_{(t,x) \in (0,T) \times \mathbb{R}^3} \langle x \rangle (t - |x|) |\partial w(t, x)|_{2\sigma-17} \leq C_* \varepsilon, \tag{5.68}
\]
provided that \( \varepsilon \) is small enough.

Going similar lines to (5.63)–(5.66), and using (5.67), we obtain
\[
\langle x \rangle \left| G_{k,a} - G_{k,a}^{(W)} \right|_{2\sigma-16} \leq C_* A \varepsilon^2 \langle t + |x| \rangle^{-1-\mu} W_-(t, |x|)^{\mu-1}, \tag{5.69}
\]
and Lemma 3.7 yields
\[
\sup_{(t,x) \in (0,T) \times \mathbb{R}^3} \langle x \rangle (t - |x|) |w^{(iii)}(t, x)|_{2\sigma-17} \leq C_* \varepsilon. \tag{5.70}
\]
Now, (5.61) and (5.70) imply
\[
\sup_{(t,x) \in (0,T) \times \mathbb{R}^3} \langle t + |x| \rangle W_-(t, |x|)^{1-p} |w^{(ii)}(t, x)|_{2\sigma-17} \leq C_* \varepsilon, \tag{5.71}
\]
provided that \( \varepsilon \) is small enough, since \( 4\lambda \leq p \).

**Step 6: Conclusion.** Finally, (5.59), (5.67), (5.68), and (5.71) yield
\[
\sup_{0 \leq t \leq T} \|d_{\sigma,p}(t, \cdot)\|_{L^\infty(\mathbb{R}^3)} \leq C_0 \varepsilon \tag{5.72}
\]
for \( \varepsilon \leq \varepsilon_0(A) \), where \( \varepsilon_0(A) \) is a positive constant depending on \( A \), and \( C_0 \) is some positive constant which depends on \( M_0 \) and \( F \), but is independent of \( A, \varepsilon \) and \( T \). We put \( A_0 = 2C_0 \). Now (5.72) implies (5.2) for \( A \geq A_0 \) and \( \varepsilon \leq \varepsilon_0(A) \). This completes the proof of Proposition 5.1.

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