On OWA Linear Operators for Decision Making

E. H. Cables Pérez\textsuperscript{a}, M. T. Lamata\textsuperscript{b}, D. Pelta\textsuperscript{b} and J. L. Verdegay\textsuperscript{b}

\textsuperscript{a}Facultad de Ingeniería de Sistemas, Universidad Antonio Nariño, Bogotá, Colombia; \textsuperscript{b}Departamento de CCIA, Universidad de Granada, Granada, Spain

ABSTRACT
When we deal with problems of decision making where we need to give an order of alternatives and the rewards (number of points earned in a race) are not associated with the criteria, but with positions, it makes sense to deal with ordered weighted averaging (OWA) operators. Within the class of OWA operators we can consider those that are based on a linear function. This paper is related to a study of the properties that verify the linear functions set. Taking these properties into account we shall see how the final classification for drivers in Formula One changes when the number of points earned in a race is given by means of a linear function as opposed to the current assignments.

ARTICLE HISTORY
Received 3 January 2018
Revised 1 February 2018
Accepted 11 February 2018

KEYWORDS
OWA operators; linear functions; decision making

1. Introduction

Decision making consists in choosing the best alternative from a given set of alternatives, options, candidates, etc., or in obtaining the ranking among them. A decision-making problem takes into account criteria, alternatives and weights.

Consider a set of candidates such as Formula One drivers and let us see what happens if instead of assigning the following points \{25, 18, 15, 12, 10, 8, 6, 4, 2, 1\}, we make an linear assignment: \{10, 9, 8, 7, 6, 5, 4, 3, 2, 1\}.

Our work is based on making a study of those linear functions \(f_i = ai + b\) that give origin to ordered weighted averagings (OWAs) operators. Among those that have a linear behaviour the Borda–Kendal law can be considered as an example:

\[
 w_i = \frac{2(n + 1 - i)}{n(n + 1)} = \frac{-2}{n(n + 1)} i + \frac{2}{n + 1} n(n + 1) \tag{1}
\]

or

\[
 w_i = \frac{2(n - i + 1)}{n^2} = \frac{-2}{n^2} i + \frac{2n + 1}{n^2} \tag{2}
\]

For whatever \(n\) is selected, it is satisfied that all \(w_i\) are aligned and is satisfied that \(w_1 = c\), \(w_i = w_{i-1} - c\), \(\forall i\). This situation served as the basis to study the behaviour of other linear functions which constitute a straight line and contain Equations (5) and (2), with the only
condition that its behaviour is linear. In this way, we may obtain a function class that contains a set of infinite linear functions $F_L$ that allows the $w_i$ to be obtained, thus completing the properties of the Ordered Weighted Averaging (OWA). The idea is to assign values to each of the positions depending on the OWA operator obtained.

This paper is organised as follows. Section 2 provides a brief summary about the OWA operator. Section 3 shows the obtaining of the weight vector and its properties associated with an OWA linear operator. In Section 4, the Formula One results are evaluated according to different OWA operators and, finally, the 25 conclusions are presented.

2. OWA Operator: Definition and Properties

An OWA operator [1] of dimension $n$ is given by a function $F : R^n \rightarrow R$ that has an associated weighting vector $W = [w_1, w_2, \ldots, w_n]^T$ such that $w_i \in [0, 1]$ and $\sum_i w_i = 1$.

Furthermore, $F(a_1, a_2, \ldots, a_n) = \sum_j w_j b_j$ where $b_j$ is the largest $j$th of the $a_i$. In our case $b_j$ is the element that represents the best finishing position in the race. In particular, a weight $w_i$ is not associated with a specific argument but with an ordered position in the aggregation.

A number of properties can be associated with these operators. It is first noted that the OWA aggregation is commutative, that is, the aggregation is indifferent to the initial indexing of the argument. A second characteristic associated with these operators is that it is monotonic. Thus, if $\hat{a}_i \geq a_i \forall i$, then $F(\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n) \geq F(a_1, a_2, a_n)$. Another characteristic associated with these operators is the idempotency. In particular, if $\forall i, a_i = a$ then $F(a_1, a_2, \ldots, a_n) = a$.

The satisfaction of these three conditions, as noted in [2], ensures that these operators belong to the class of operators called mean operators; the fact that an OWA operator is a mean operator can then be shown. It is known that

- The weight vector $(1, 0, \ldots, 0)$ enables the Max that emphasises the largest element in the argument bag to be obtained.
- The weight vector $(0, 0, \ldots, 1)$ enables the Min that emphasises solely the smallest element in the bag to be obtained.
- The weight vector $(1/n, 1/n, \ldots, 1/n)$ enables the simple average to be obtained.

Additionally, for any OWA operator $F$, $\text{Min}_i[a_i] \leq F(a_1, a_2, \ldots, a_n) \leq \text{Max}_i[a_i]$. In [3] a number of families of these operators are discussed. The various different mean operators are implemented by appropriate selection of the weights in the associated weighting vector.

3. Obtaining the Weight Vector of an OWA with Linear behaviour

To calculate the weights vector $W = [w_i], \forall i$ there are several functions, some of which have a linear behaviour [4–6] and others are non-linear [7,8]. This paper deals with a study of the properties that verify the linear functions set $F_L$ and $F_L^*$ as shown in the results obtained by Cables and Lamata [4]. Because of that they identify the linear function that delimits the
Figure 1. Region of the linear functions set $F_L$ that verifies the OWAs properties.

functions set, which is defined by the region $F_L$:

\[ f_{\text{sup}}(x) = \frac{-2}{n(n-1)}x + \frac{2}{n-1}, \]  

\[ f_{\text{inf}}(x) = \frac{1}{n}. \]  

And obtaining the region represented in Figure 1.

We can observe that the bundle of lines contains the point $P_m(\frac{1+n}{2}, \frac{1}{n})$ and all functions have negative slopes, therefore all those $w_i$ obtained starting from any linear function belonging to this region will be organised in descending order ($w_1 > w_2 > \cdots > w_n$).

Moreover, it is verified that the linear functions set that allows to obtain $w_i$ verifies the properties of the OWA, has a slope $a \in ]-2/n(n-1), 0[,$ that depends on the value of $n$ [4]. In the study carried out, another region is also identified, $F^*_L$ (see Figure 2) of linear functions that allows to obtain $w_i$ in decreasing order and to verify the OWAs properties. In addition, the linear functions set also contains the point $P_m((1+n)/2, 1/n)$, which is delimited by the following functions (see Figure 2):

\[ f_{\text{sup}}(x) = \frac{2}{n(n-1)}x - \frac{2}{n-1}, \]  

\[ f_{\text{inf}}(x) = \frac{1}{n}. \]
Finally, the authors conclude that the general form of the linear functions belong to these regions, and that verify the OWAs properties are given by (see Figure 3):

\[ f_g(x) = ax + \frac{1}{n} - a \frac{1+n}{2} = a \left( x - \frac{1+n}{2} \right) + \frac{1}{2}. \]  \hspace{1cm} (7)
With regard to the slope and the orness measure, we present theorems, the relation between them in the case of the OWA operators obtained by means of linear functions.

**Theorem 3.1:** Given the function \( f_{\inf}(x) = 1/n \), that contends the point \( P_m \) and slope \( a = 0 \), it represents an OWA and their \( \alpha = 0.5 \).

**Proof:**

\[
\alpha = \frac{1}{n-1} \sum_{i=1}^{n} (n - i) f_{\inf}(i) = \frac{1}{n-1} \sum_{i=1}^{n} (n - i) = \frac{n - 1}{2(n - 1)} \rightarrow \alpha = \frac{1}{2}.
\]

because \( \sum_{i=1}^{n} (n) = n^2 \) and \( \sum_{i=1}^{n} (i) = n(n + 1)/2 \).

**Theorem 3.2:** Let the linear functions set be \( f_g(x) \) given in Equation (7), with slope \( a \rightarrow 0 \). Then the satisfaction degree \( \alpha \rightarrow 0 \).

**Proof:** By definition, it is known that

\[
\alpha = \frac{1}{n-1} \sum_{i=1}^{n} (n - i) f_g(i) = \frac{1}{n-1} \sum_{i=1}^{n} (n - i) \left( ai + \frac{1}{n} - a \left( \frac{1+n}{2} \right) \right).
\]

Taking into account that \( \sum_{i=1}^{n} (i) = n(n + 1)/2 \) and \( \sum_{i=1}^{n} (i^2) = n(n + 1)(2n + 1)/6 \), it is obtained that

\[
\alpha = \frac{-an(n^2 - 1) + 6n - 6}{12(n - 1)} = \frac{-an(n - 1)(n + 1)}{12(n - 1)} + \frac{6(n - 1)}{12(n - 1)} = \frac{-an(n + 1) + 1}{2}.
\]

And when it is calculate

\[
\lim_{n \to \infty} \frac{-an(n + 1) + 1}{2} = \frac{1}{2}
\]

then it is verified that \( \alpha = 0.5 \).

**Theorem 3.3:** Let the linear functions set be \( f_g(x) \), with a value of slope \( a < 0 \). Then the satisfaction degree is \( \alpha > 0.5 \).

**Proof:** Starting from the result obtained in Theorem 3.2, it is known that the biggest slope in the linear functions that constitutes the OWA is \( \alpha_1 = -2/(n(n - 1)) \) then, substituting in the value of \( \alpha \), the following is obtained:

\[
\alpha_1 = \frac{1}{n-1} \sum_{i=1}^{n} (n - i) \left[ \frac{-2}{n(n - 1)} \left( i - \frac{1+n}{2} \right) + \frac{1}{n} \right] = \frac{2n(n + 1)}{12(n - 1)} + \frac{1}{2} = \frac{2(n + 1)}{6(n - 1) + \frac{1}{2}} = \frac{2n + 3n - 1}{6(n - 1)} \Rightarrow \alpha_1 = \frac{2n - 1}{3(n - 1)}.
\]

And then we calculate

\[
\lim_{n \to \infty} \frac{2n - 1}{3(n - 1)} = \frac{2}{3}
\]

then it is verified that \( \alpha > 0.5 \).
**Theorem 3.4:** Let the linear functions set be \( f_g(x) \), which represents an OWA and if the value of the slope is \( a > 0 \). Then the satisfaction degree is \( \alpha < 0.5 \).

**Proof:** As in the previous theorem, \( a = 2/(n(n-1)) \) is substituted in \( f_g(x) \) and it is obtained

\[
\alpha_2 = \frac{1}{n-1} \sum_{i=1}^{n} (n-i) \left[ \frac{2}{n(n-1)} \left( i - \frac{1+n}{2} \right) + \frac{1}{n} \right] \\
= \frac{-2n(n+1)}{12n(n-1)} + \frac{1}{2} = \frac{-(n+1)}{6(n+1)} + \frac{1}{2} = \frac{-n-1+3(n-2)}{6(n-1)} = \frac{2n-4}{6(n-1)} \\
\Rightarrow \alpha_2 = \frac{xn-2}{3(n-1)}.
\]

And then we calculate

\[
\lim_{n \to \infty} \frac{n-2}{3(n-1)} = \frac{1}{3},
\]

then it is verified that \( \alpha < 0.5 \).

In Figure 4 the behavior of \( \alpha \) can be seen.

**Theorem 3.5:** Let the linear functions set be \( f_g(x) \) which represents an OWA with the slope \( a \in ]-2/(n(n-1)),0[ \). Then the satisfaction degree is \( \alpha \in \left[ \frac{1}{2}, \frac{2n-1}{3n-3} \right] \).

**Proof:** It is known that if \( a = 0 \) then the function \( f_{\text{inf}}(x) \) is obtained, and by Theorem 3.1 it is verified that \( \alpha = \frac{1}{2} \), and if \( a = -2/(n(n-1)) \) the function \( f_{\text{sup}}(x) \) is obtained and by the results of Theorem 3.3 it is verified that \( \alpha = (2n-1)/(3n-3) \). Therefore, it is demonstrated that the value of \( \alpha \in \left[ \frac{1}{2}, \frac{2n-1}{3n-3} \right] \).

![Figure 4. Behaviour of the \( \alpha \) value.](image-url)
Table 1. The $\alpha$ intervals for different values of $n$, depending on the positive slope or negative slope.

| Values of $n$ | Negative slope | Positive slope |
|---------------|----------------|---------------|
| 2             | $[\frac{1}{2}, 0]$ | $[0, \frac{1}{2}]$ |
| 3             | $[\frac{1}{2}, \frac{2}{3}]$ | $[\frac{2}{3}, \frac{1}{2}]$ |
| 4             | $[\frac{1}{2}, \frac{5}{9}]$ | $[\frac{5}{9}, \frac{1}{2}]$ |
| ...           | ...            | ...           |
| 10            | $[\frac{1}{2}, \frac{19}{27}]$ | $[\frac{8}{27}, \frac{1}{2}]$ |

Theorem 3.6: Let the linear functions set be $f_g(x)$ if the slope $a \in ]0, 2/(n(n - 1))[$. Then the satisfaction degree of $\alpha \in [(n - 2)/(3n - 3), \frac{1}{2}]$.

Proof: It is known that if $a = 0$ then the function $f_{\text{inf}}(x)$ is obtained and by Theorem 3.1, it is verified that $\alpha = \frac{1}{2}$, also, if $a = 2/(n(n - 1))$ the function $f_{\text{sup}}(x)$ is obtained and by the results of Theorem 3.4 it is verified that $\alpha = (n - 2)/(3n - 3)$.

Therefore, it is demonstrated that the value of $\alpha \in [(n - 2)/(3n - 3), \frac{1}{2}]$.

Starting from the results obtained in Theorems 3.5 and 3.6, we may determine the interval of the value of $\alpha$ for a given $n$. Next, the interval of the value of $\alpha$ is shown for $n = 2, \ldots, 10$, for negative and positive slope functions (see Table 1).

Theorem 3.7: All the linear functions that verify the OWAs properties, such that the values of the satisfaction degree $\alpha \in [\frac{1}{2}, (2n - 1)/(3n - 3)]$ or $\alpha \in [(n - 2)/(3n - 3), \frac{1}{2}]$ have a slope $a = (6 - 12\alpha)/(n(n + 1))$ and vice versa.

Proof: It is known by Theorem 3.2 that

$$\alpha = \frac{-an(n + 1)}{12} + \frac{1}{2},$$

then

$$\alpha - \frac{1}{n} = \frac{-an(n + 1)}{12} \iff 12 \left(\alpha - \frac{1}{2}\right) = -an(n + 1)$$

$$\implies -a = \frac{12\alpha - 6}{n(n + 1)} \iff a = \frac{6 - 12\alpha}{n(n + 1)}.$$

Case 1. $\alpha \in [\frac{1}{2}, (2n - 1)/(3n - 3)]$, then

- if $\alpha = \frac{1}{2}$ the value of $a = 0$,
- if $\alpha = \frac{2n - 1}{3n - 3}$, then $a = \frac{6 - 12(\frac{2n - 1}{3n - 3})}{n(n + 1)} \iff a = \frac{18n - 18 - 12(2n - 1)}{3n - 3} \frac{3}{n(n + 1)}$.

$$a = \frac{18n - 18 - 24n + 12}{3(n - 1)n(n + 1)} \iff a = \frac{-6n - 6}{3(n - 1)n(n + 1)}$$

$$\iff a = \frac{-6(n + 1)}{3(n - 1)n(n + 1)} \iff a = \frac{-2}{n(n - 1)}.$$
Therefore, it is verified that the value of the slope $a \in \left] -2/(n(n + 1)), 0 \right[$. Case 2 $\alpha \in [(n - 2)/(3n - 3), \frac{1}{2}]$, then

\[
\text{if } \alpha = \frac{1}{2} \text{ the value of } a = 0, \\
\text{if } \alpha = \frac{n - 2}{3n - 3} \text{ then } a = \frac{6 - 12(\frac{n-1}{3n-3})}{n(n+1)} \iff a = \frac{18n-18-12(n-2)}{3n-3} \cdot \frac{n}{n(n+1)}, \\
a = \frac{18n - 18 - 12n + 24}{3(n-1)n(n+1)} \iff a = \frac{6(n+1)}{3(n-1)n(n+1)} \iff a = \frac{2}{n(n-1)}.
\]

Therefore, it is verified that the value of the slope of $a \in \left] 0, 2/n(n + 1) \right[.$

The procedure to obtain the linear functions are:

- Select the quantity of weights $n$ to calculate.
- Select the value of $\alpha$, such that $\alpha \in [\frac{1}{2}, (2n - 1)/(3n - 3), \frac{1}{2}]$ or $\alpha \in [(n - 2)/(3n - 1), \frac{1}{2}]$.
- The slope of the linear function is calculated through the formula $a = (6 - 12\alpha)/(n(n + 1))$.
- Substitute $\alpha$ in the general function of the linear functions class $f_{\alpha}(x)$ (Equation (7)) which is an OWA.

### Remark 3.1:

On the other hand, by Theorem 3.7 we have that the slope of the straight line can be expressed on the value of $\alpha$.

According to the last theorem we can use the following expression:

\[
a = \frac{6 - 12\alpha}{n(n + 1)}.
\]

And the general expression for $f_{\alpha}(x)$ in function of $\alpha$ is given by

\[
f_{\alpha}(x) = \frac{6 - 12\alpha}{n(n + 1)} \left( x - \frac{1 + n}{2} \right) + \frac{1}{n},
\]

where the value of $\alpha$ must belong to the following intervals: $\alpha \in \left[ \frac{1}{2}, (2n - 1)/(3n - 3) \right]$ for negative slopes (decreasing function) and $\alpha \in [(n - 2)/(3n - 3), \frac{1}{2}]$ for positive slopes (increasing function).

### Remark 3.2:

Consider the official scores \{25, 18, 15, 12, 10, 8, 6, 4, 2, 1\} assigned to the first 10 finishing positions in Formula One. Table 2 shows the values in the corresponding weights vector with $\alpha = 0, 721672167$.

But for $n = 10$, \( \Rightarrow \alpha \in \left[ \frac{1}{2}, \frac{2(10-1)}{3(10-3)} \right] \) and $\alpha = 0, 721672167 \notin \left[ \frac{1}{2}, \frac{19}{27} \right]$ and it is therefore not possible to obtain a linear function instead of an OWA operator.

### Table 2. Weights according to F1, the Borda law $\alpha = 2/3$ and for and $\alpha = 0.7$.

|      | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   | 10   |
|------|-----|-----|-----|-----|-----|-----|-----|-----|-----|------|
| F1   | 25  | 18  | 12  | 10  | 8   | 6   | 4   | 2   | 1   |      |
|      | 1/7 | 1/7 | 1/7 | 1/7 | 1/7 | 1/7 | 1/7 | 1/7 | 1/7 |      |
| BORDA|  5  |  5  |  5  |  5  |  5  |  5  |  5  |  5  |  5  |  5   |
|      |  2  |  2  |  2  |  2  |  2  |  2  |  2  |  2  |  2  |  2   |
| $\alpha = 0.7$ | 109 | 97  | 85  | 73  | 61  | 49  | 37  | 25  | 13  | 1    |
|      | 550 | 550 | 550 | 550 | 550 | 550 | 550 | 550 | 550 | 550  |
Thus, we will take the values nearest to $\frac{19}{27} = 0.7037$ which will be $\alpha = 0.70$.

Therefore, it is possible to obtain the weights for $n = 10$ and $\alpha = 0.7$.

Starting from:

\[
\begin{align*}
    w_i &= \frac{6 - 12\alpha}{n(n+1)} \left( 1 - \frac{1 + n}{2} \right) + \frac{1}{n}, \\
    w_1 &= \frac{6 - 12\alpha}{n(n+1)} \left( 1 - \frac{1 + n}{2} \right) + \frac{1}{n} = \frac{-2.4}{110} \ast \frac{-9}{2} + \frac{1}{10} = \frac{109}{550}, \\
    w_2 &= \frac{6 - 12\alpha}{n(n+1)} \left( 2 - \frac{1 + n}{2} \right) + \frac{1}{n} = \frac{-2.4}{110} \ast \frac{-7}{2} + \frac{1}{10} = \frac{97}{550}, \\
    \vdots \\
\end{align*}
\]

and so on.

The complete results can be seen in Table 2.

4. Results

In a Formula One competition only the first 10 drivers who cross the finish line obtain points. These scores are reflected in the first row of Table 3, which correspond to the weights of the first row of Table 2. The following lines of Table 3 (Table 2) represent the scores or weights.

| Table 3. Scores according to F1, the Borda law of $\alpha = 2/3$, and for $\alpha = 0.7$. |
|---------------------------------------------|
| $F1$ | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 |
| BORDA | 25 | 18 | 15 | 12 | 10 | 8  | 6  | 4  | 2  | 1  |
| $\alpha = 0.7$ | 109 | 97 | 85 | 73 | 61 | 49 | 37 | 25 | 13 | 1  |

| Table 4. The scores of the 20 F1 races. |
|---------------------------------------------|
| $F1$ | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 |
| $c_1$ | 18 | 25 | 15 | 12 | –  | 10 | 6  | 1  | 4  | –  |
| $c_2$ | 25 | 18 | 8  | 10 | 12 | 15 | 2  | 1  | 6  | –  |
| $c_3$ | 18 | 25 | 15 | 12 | 10 | –  | 6  | 1  | –  | 2  |
| $c_4$ | 12 | 18 | 25 | 15 | –  | 10 | 8  | 6  | 1  | 4  |
| $c_5$ | 25 | 18 | –  | –  | 15 | –  | 10 | 6  | 8  | –  |
| $c_6$ | 6  | 25 | 12 | 18 | 15 | 10 | –  | –  | 8  | –  |
| $c_7$ | 25 | 12 | 18 | 6  | 15 | –  | 10 | 8  | –  | 4  |
| $c_8$ | 10 | 12 | 18 | –  | 25 | –  | –  | 8  | 4  | –  |
| $c_9$ | 12 | 18 | 25 | 10 | 15 | –  | 6  | 4  | –  | –  |
| $c_{10}$ | 25 | 6  | 18 | 15 | 10 | 12 | 2  | 4  | –  | 8  |
| $c_{11}$ | 12 | 25 | 15 | 18 | –  | 10 | 4  | 2  | 6  | –  |
| $c_{12}$ | 25 | 18 | 10 | 12 | 15 | –  | 2  | 1  | 8  | –  |
| $c_{13}$ | 25 | 15 | 18 | 10 | 12 | 1  | 2  | 8  | –  | –  |
| $c_{14}$ | 25 | –  | 15 | –  | 18 | –  | 10 | 1  | 12 | –  |
| $c_{15}$ | 18 | 12 | 10 | –  | 15 | 25 | 8  | 1  | –  | –  |
| $c_{16}$ | 25 | –  | 12 | 10 | 15 | 18 | 6  | 8  | –  | –  |
| $c_{17}$ | 25 | 18 | 10 | 15 | –  | 12 | 4  | 8  | 6  | –  |
| $c_{18}$ | 2  | 12 | 18 | 15 | –  | 25 | 6  | 10 | –  | –  |
| $c_{19}$ | 12 | 25 | 18 | 15 | 8  | 10 | 2  | –  | –  | 1  |
| $c_{20}$ | 18 | 15 | 25 | 12 | –  | 10 | 6  | 6  | –  | 8  |
| $SUM$ | 363 | 317 | 305 | 205 | 200 | 168 | 100 | 87 | 54 | 43 |
Table 5. The scores of the 20 races according to the Borda law.

|   | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|----|
| $c_1$ | 9 | 10 | 8 | 7 | – | 6 | 4 | 1 | 3 | – |
| $c_2$ | 10 | 9 | 5 | 6 | 7 | 8 | 2 | 1 | 4 | – |
| $c_3$ | 9 | 10 | 8 | 7 | 6 | – | 4 | 1 | – | 2 |
| $c_4$ | 7 | 9 | 10 | 8 | – | 6 | 5 | 4 | 1 | 3 |
| $c_5$ | 10 | 9 | – | – | 8 | – | 7 | 6 | 4 | 5 |
| $c_6$ | 4 | 10 | 7 | 9 | 8 | 6 | – | – | 5 | – |
| $c_7$ | 10 | 7 | 9 | 4 | 8 | – | 6 | 5 | – | 3 |
| $c_8$ | 6 | 7 | 9 | – | 10 | – | – | 5 | 3 | – |
| $c_9$ | 7 | 9 | 10 | 6 | 8 | – | 4 | 3 | – | – |
| $c_{10}$ | 10 | 4 | 9 | 8 | 6 | 7 | 2 | 3 | – | 5 |
| $c_{11}$ | 7 | 10 | 8 | 9 | 7 | 8 | – | 6 | 3 | 2 |
| $c_{12}$ | 10 | 9 | 6 | 7 | 8 | – | 6 | 3 | 2 | 4 |
| $c_{13}$ | 10 | 8 | 9 | 6 | 7 | 1 | 2 | 5 | – | – |
| $c_{14}$ | 10 | – | 8 | – | 9 | – | 6 | 1 | 7 | – |
| $c_{15}$ | 9 | 7 | 6 | – | 8 | 10 | 5 | 1 | – | – |
| $c_{16}$ | 10 | – | 7 | 6 | 8 | 9 | 4 | 5 | – | – |
| $c_{17}$ | 10 | 9 | 6 | 8 | – | 7 | 3 | 5 | 4 | – |
| $c_{18}$ | 2 | 7 | 9 | 8 | – | 10 | 4 | 6 | – | – |
| $c_{19}$ | 7 | 10 | 9 | 8 | 5 | 6 | 2 | – | – | 1 |
| $c_{20}$ | 9 | 8 | 10 | 7 | – | 6 | 4 | 3 | – | 5 |
| SUM | 166 | 152 | 153 | 114 | 106 | 88 | 67 | 59 | 36 | 29 |

Table 6. Final scores for F1, Borda law and $\alpha = 0.7$.

|   | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|----|
| SUM | 363 | 317 | 305 | 205 | 200 | 168 | 100 | 87 | 54 | 43 |
| F1 | 166 | 152 | 153 | 114 | 106 | 88 | 67 | 59 | 36 | 29 |
| BORDA | 1772 | 1626 | 1627 | 1192 | 1118 | 913 | 617 | 510 | 322 | 260 |
| $\alpha = 0.7$ | 1772 | 1626 | 1627 | 1192 | 1118 | 913 | 617 | 510 | 322 | 260 |

that the pilots would have received if the allocation had been considered by linear OWAs with values of $\alpha = \frac{2}{3}$ (Borda) and an $\alpha = 0.70$, respectively (Tables 4 and 5).

If we consider the results (see Table 6) associated with an $\alpha = 0.7$, the results are totally analogous to those of Borda’s law.

5. Conclusion

Linear functions giving rise to OWA operators have been described and studied in this paper. It can be seen that there are infinite functions from which weights can be calculated with different slopes and orness. The relationships between the slope of the straight line and the orness measure of the $\alpha$ aggregation are shown to be interesting properties.

Finally, taking the classification of F1 in the year 2017, the number of points to be assigned to the first 10 finishing positions has been calculated for $\alpha = \frac{2}{3}$ and $\alpha = 0.70$ and an $\alpha = 0.70$ using the proposed method and it was observed that in all three cases the first place has not changed, but that when the scores are linear, the second and third positions have been inverted.

Disclosure statement

No potential conflict of interest was reported by the authors.
**Funding**

Research partially funded by projects TIN2014-55024-P, TIN2017-86647-P and P11-TIC-8001 (all of them including FEDER funds). Also the support provided by the Antonio Nariño University (Colombia) is also acknowledged.

**References**

[1] Yager RR. On ordered weighted averaging aggregation operators in multicriteria decisionmaking. IEEE Trans Syst Man Cybern. 1988;18:183–190.

[2] Dubois D, Prade H. A review of fuzzy sets aggregation connectives. Inf Sci. 1985;36(1):85–121.

[3] Yager RR. Families of OWA operators. Fuzzy Sets Syst. 1993;59(2):125–148.

[4] Cables E, Lamata M. Owa weights determination by means of linear functions. Mathware Soft Comput. 2009;16:107–122.

[5] Liu X. On the properties of equidifferent OWA operator. Int J Approx Reason. 2006;43(1):90–107.

[6] Ahn B. On the properties of OWA operator weights functions with constant level of orness. IEEE Trans Fuzzy Syst. 2006;14(4):511–515.

[7] O'Hagan M. Aggregating template or rule antecedents in real-time expert systems with fuzzy set. Proceedings of the 22nd Annual IEEE Asilomar Conference on Signals, Systems, and Computers; 1988, Pacific Grove, California.

[8] Filev D, Yager RR. Analytic properties of maximum entropy OWA operators. Inf Sci. 1995;85(1–3):11–27.