On Structural Controllability of Symmetric (Brain) Networks

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Abstract

The question of controllability of natural and man-made network systems has recently received considerable attention. In the context of the human brain, the study of controllability may not only shed light into the organization and function of different neural circuits, but also inform the design and implementation of minimally invasive yet effective intervention protocols to treat neurological disorders. While the characterization of brain controllability is still in its infancy, some results have recently appeared and given rise to scientific debate. Among these, [1] has numerically shown that a class of brain networks constructed from DSI/DTI imaging data are controllable from one brain region. That is, a single brain region is theoretically capable of moving the whole brain network towards any desired target state. In this note we provide evidence supporting controllability of brain networks from a single region as discussed in [1], thus contradicting the main conclusion and methods developed in [2].

We consider brain networks modeled by a weighted graph $G = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, \ldots, n\}$ and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ are the vertex and edge sets, respectively. Let $A = [a_{ij}]$ be the weighted adjacency matrix of $G$, where $a_{ij} = 0$ if $(i, j) \notin \mathcal{E}$ and $a_{ij} \in \mathbb{R}_{\geq 0}$ if $(i, j) \in \mathcal{E}$. We assume that $A$ is symmetric and that the graph $G$ has no self loops, forcing the diagonal entries of $A$ to zero. These assumptions are dictated by the use of DSI/DTI scans to reconstruct brain networks [1]. Let $x : \mathbb{N} \rightarrow \mathbb{R}^n$ be the vector containing the state of the brain regions over time. The network dynamics with control region $i \in \mathcal{V}$ read as

$$x(t + 1) = Ax(t) + b^i u(t),$$

where $u : \mathbb{N} \rightarrow \mathbb{R}$ is the control input injected into the $i$-th brain region, and the input vector $b^i$ satisfies $b^i_j = 0$ if $j \neq i$ and $b^i_i = 1$. The network (1) is controllable if and only if the controllability matrix $C(A, b^i)$ is invertible [3], where

$$C(A, b^i) = [b^i \ A b^i \ \cdots \ A^{n-1} b^i].$$

Assessing controllability of network systems is numerically difficult because the controllability matrix typically becomes ill-conditioned as the network cardinality increases; e.g., see [4], [5]. Because different controllability tests suffer similar numerical difficulties, a convenient tool to study controllability of networks is to resort to the theory of structural systems. To formalize this discussion, notice that the determinant $\det(C(A, b^i)) = \phi(a_{ij})$ is a polynomial function of the nonzero entries of the adjacency matrix. The network (1) is uncontrollable when the weights are chosen so that $C(A, b^i)$ is not invertible or, equivalently, when $\phi(a_{ij}) = 0$. Let $\mathcal{S}$ contain the choices of weights that render the network (1) uncontrollable, that is,

$$\mathcal{S} = \{a_{ij} : (i, j) \in \mathcal{E} \text{ and } \phi(a_{ij}) = 0\}.$$

Notice that each element of $\mathcal{S}$ can be represented as a point in $\mathbb{R}^d$, where $d = |\mathcal{E}|$ is the number of nonzero entries of $A$. Formally, the set $\mathcal{S}$ defines an algebraic variety of $\mathbb{R}^d$ [6]. This implies that controllability of (1) is a generic property because it fails to hold on an algebraic variety of the parameter space [7]. Thus, when assessing controllability of (1) as a function of the network weights, only two mutually exclusive cases are possible:

(i) either there is no choice of weights $a_{ij}$, with $(i, j) \in \mathcal{E}$, rendering the network (1) controllable, or

(ii) the network (1) is controllable for all choices of weights $a_{ij}$ except, possibly, those lying in a proper algebraic variety of the parameter space $\mathbb{R}^d$ (see Example 1 below).

Loosely speaking the above discussion implies that, if one can find a choice of weights $a_{ij}$ such that (1) is controllable, then almost all choices of weights $a_{ij}$ yield a controllable network. In this case, the network is said to be structurally controllable [6], [8], [9]. In what follows we show that brain networks are structurally controllable from one single region, thus providing theoretically-validated and numerically-reliable support to the result in [1]. This further shows that the result in [2] is likely incorrect and misleading. In fact, even in an unfortunate choice of weights that prevents controllability, that is, a choice of weights that lies in a proper algebraic variety, a random and arbitrarily small deviation of network weights due to perturbation or uncertainty in estimating neural connections would guarantee controllability.

Classic results on structural controllability cannot be directly applied to symmetric (brain) networks. In fact, these results assume that the network weights can be selected arbitrarily and independently from one another, a condition that cannot be satisfied when the weights need to be symmetric. To overcome this limitation we proceed as follows: first we show that network controllability remains a generic property when the weights are symmetric; then, we find a choice of symmetric weights that guarantees controllability. This ensures that brain networks are structurally controllable from one single node, even with symmetric weights, and that almost all choices of symmetric edge weights yield controllable networks.

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(Generic controllability of networks with symmetric weights) Let \( d = |E| \) and notice that a network with symmetric weights is uniquely specified by \( d/2 \) parameters, which can be represented as a point in the Euclidean space \( \mathbb{R}^{d/2} \). With the symmetry constraint, the determinant of the controllability matrix is a polynomial function of \( d/2 \) parameters and can be obtained, for instance, from the determinant \( \phi(a_{ij}) \) in (3) by substituting \( a_{ij} \) with \( a_{ji} \) whenever \( i > j \). Thus, even for symmetric networks, the determinant of the controllability matrix is a polynomial function, and the weights that render the network uncontrollable lie on an algebraic variety of the parameter space \( \mathbb{R}^{d/2} \). We conclude that controllability of networks with symmetric weights remains a generic property: either there exists no choice of symmetric weights \( a_{ij} \) that makes the network controllable, or the network is controllable for almost all choices of symmetric weights.

(A controllable realization of a brain network) Because controllability of symmetric networks is a generic property, it is sufficient to construct a controllable symmetric network to show that almost all choices of weights yield a controllable network. To do this, we construct a Hamiltonian path starting from the control node, select the weights of the edges in the path equal to one, and set all other weights equal to zero. Notice that the determinant of the controllability matrix associated with the constructed network has unit magnitude, proving that the network is structurally controllable with symmetric weights.

We remark that not all networks admit a Hamiltonian path, and that the existence of such a path is only a sufficient condition for structural controllability with symmetric weights. Yet, as we show in Fig. 1 for one brain network and one control region, all the networks in our dataset admit a Hamiltonian path from every region, showing that they are structurally controllable from every region even with symmetric weights. We conclude this discussion with an academic example.

Example 1: (Structural controllability of a network with symmetric weights) Consider a network with adjacency matrix

\[
A = \begin{bmatrix}
0 & a_{12} & a_{13} \\
a_{12} & 0 & a_{23} \\
a_{13} & a_{23} & 0
\end{bmatrix},
\]  

(4)

control node \{1\}, and input vector \( b^T = [1 \ 0 \ 0]^T \). The network is represented in Fig. 2(a). Following our analysis, the network is structurally controllable even with symmetric weights because it admits a Hamiltonian path (see Fig. 2(b)). That is, the network is controllable for almost all choices of symmetric weights. To see this, compute the controllability matrix

\[
\mathcal{C}(A, b^T) = \begin{bmatrix}
1 & 0 & a_{12}^2 + a_{13}^2 \\
a_{12} & 0 & a_{13}a_{23} \\
a_{13} & a_{23} & 0
\end{bmatrix},
\]  

(5)

and the determinant \( \det(\mathcal{C}(A, b^T)) = a_{23}a_{12}^2 - a_{23}a_{13}^2 \). Thus, the network is controllable (i.e., \( \det(\mathcal{C}(A, b^T)) \neq 0 \)) for all symmetric choices of weights \( a_{12}, a_{13}, \) and \( a_{23} \), except those lying on the proper algebraic variety shown in Fig. 2(c). □

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1A path in a graph is Hamiltonian if it visits all the vertices exactly once.
Fig. 2. (a) Network with symmetric weights considered in Example 1. (b) Network induced by a Hamiltonian path starting from the control node. (c) Algebraic variety containing the weights for which the network is not controllable. The network is controllable for all weights outside of this hypersurface.

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