BIASES IN THE GRAVITATIONAL LENS POPULATION INDUCED BY HALO AND GALAXY TRIAXIALITY

EDUARDO ROZO\textsuperscript{1}, JACQUELINE CHEN\textsuperscript{2}, ANDREW R. ZENTNER\textsuperscript{3,4,5}

Draft version February 2, 2008

ABSTRACT

The lensing cross section of triaxial halos depends on the relative orientation between a halo’s principal axes and its line of sight. Consequently, a lensing subsample of randomly oriented halos is not, in general, randomly oriented. Using an isothermal mass model for the lensing galaxies and their host halos, we show that the lensing subsample of halos that produces doubles is preferentially aligned along the lines of sight, whereas halos that produce quads tend to be projected along their middle axes. These preferred orientations result in different projected ellipticity distributions for quad, doubles, and random galaxies. We show that \( \approx 300 \) lens systems must be discovered to detect this effect at the 95\% confidence level. We also investigate the importance of halo shape for predicting the quad-to-double ratio and find that the latter depends quite sensitively on the distribution of the short-to-long axis ratio, but is otherwise nearly independent of halo shape. Finally, we estimate the impact of the preferred orientation of lensing galaxies on their projected substructure mass fraction, and find that the observed alignment between the substructure distribution and the mass distribution of halos result in a negligible bias.

Subject headings: galaxies, halos, lensing

1. INTRODUCTION

Statistics of lensing galaxies have been used as cosmological and galaxy formation probes since early in the modern history of gravitational lensing (Turner et al. 1984). Lensing rates can be used to constrain dark energy (Fukugita et al. 1992; Chae 2003; Mitchell et al. 2005; Chae et al. 2007; Oguri et al. 2007), to probe the structure of lensing galaxies (Keeton 2001; Kochanek & White 2001; Chae 2005), and to probe galaxy evolution (Chae & Mao 2003; Ofek et al. 2003; Rusin & Kochanek 2005). While the use of lensing statistics as a cosmological probe has had mixed success, particularly early on, it remains a unique probe with entirely different systematics from more traditional approaches. Consequently, lensing statistics are likely to remain a fundamental cross-check of our understanding of cosmology and galaxy evolution.

One of the difficulties that confronts the study of lensing statistics is that, in general, the halo population that produces gravitational lenses can in fact be a highly biased subsample of the general halo population. For instance, it has long been known that while early type galaxies compose only \( \approx 30\% \) of all luminous galaxies, the majority of lensing galaxies are in fact early type since these tend to be more massive and reside in more massive halos than their late counterparts. By the same token, lensing early type galaxies tend to have higher luminosity and velocity dispersions than non-lensing early type galaxies (Moeller et al. 2006; Bolton et al. 2006). Overall, when interpreting lensing statistics, one ought to always remember that by selecting lensing galaxies one is automatically introducing an important selection effect that can significantly bias the distribution of any galaxy observable that has an impact on the lensing probabilities. Here, we consider one such source of bias, the triaxiality of galaxy halos.\footnote{Throughout this work, we will be using the term galaxy and halo more or less interchangeably. The reason for this is that we are primarily focused on the impact of halo triaxiality on the lensing cross section, and the latter depends only on the total matter density. Consequently, differentiating between halo and galaxy would only obfuscate presentation and introduce unnecessary difficulties. For instance, while modeling the total matter distribution as isothermal is a reasonable approximation, neither the baryons nor the dark matter by itself is isothermally distributed. Thus, it is much simpler to adopt an isothermal model, and refer to the baryons plus dark matter as a single entity, than to try to differentiate between the two. Likewise, when discussing triaxiality, what is important in this work is the triaxiality of the total matter distribution.}

That halo triaxiality can have important consequences for lensing statistics has been known for several years. For instance, Oguri & Keeton (2004) have shown that triaxiality can significantly enhance the optical depth of large image separation lenses. Similar conclusions have been reached concerning the formation of giant arcs by lensing clusters (see e.g. Oguri et al. 2003; Rozo et al. 2006a; Hennawi et al. 2007, and references therein). Curiously, however, little effort has gone into investigating how observational properties of lensing galaxies can be different from those of the galaxy population as a whole due to the triaxial structure of galactic halos. This work addresses this omission.

The first observable we consider is the projected axis ratio of lensing galaxies. Roughly speaking, given that non-zero ellipticities are needed in order to produce quad systems, one would generically expect lenses that lead to this image configuration to be more elliptical than the overall galaxy population. Likewise, lensing galaxies that produce doubles should, on average, be slightly more circular than a random galaxy. There can, however, be complications for these simple predictions due to halo triaxiality. For instance, given a prolate halo, projections along the long axis of the lens will result in highly concentrated, very circular profiles. Will the increase in Einstein radius of such projections compensate for the lower ellipticity of the system, implying most quads will be projected along their long axis, or will it be the other way

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\textsuperscript{1} Center for Cosmology and Astro-Particle Physics (CCAPP), The Ohio State University, Columbus, OH, USA
\textsuperscript{2} Argelander-Institut für Astronomie, University of Bonn, Auf dem Hügel 71, 53121 Bonn, Germany
\textsuperscript{3} Department of Physics and Astronomy, University of Pittsburgh, Pittsburgh, PA 15260, USA
\textsuperscript{4} Kavli Institute for Cosmological Physics and Department of Astronomy, Chicago, IL 60637, USA
\textsuperscript{5} The Enrico Fermi Institute, The University of Chicago, Chicago, IL 60637, USA
around? Clearly, the relation between ellipticity and lensing cross sections is not straightforward once triaxiality of the lensing galaxies is taken into account, but it seems clear that there should be some observable difference between the ellipticity distribution of lensing galaxies and that of all early types. Interestingly, no such difference has been observed (Keeton et al. 1997; Rusin & Tegmark 2001), which seems to fly in the face of our expectations (though see also the discussion in Keeton et al. 1998). Is this actually a problem, or will a quantitative analysis show that the consistency of the two distributions is to be expected? Here, we explicitly resolve this question, and demonstrate that current lens samples are much too small to detect the expected differences.

Having considered the ellipticity distribution of random and lensing galaxies, it is then a natural step to investigate the impact of halo triaxiality on predictions of the quad-to-double ratio. Specifically, it is well known that the quad-to-double ratio is sensitive to the ellipticity distribution of lensing galaxies (Keeton et al. 1997), so if lensing can bias the distribution of ellipticities in lensing galaxies, then it should also affect the predicted quad-to-double ratios. This is an important point because it has been argued that current predictions for the quad-to-double ratio are at odds with observations. More specifically, the predicted quad-to-double ratio for the CLASS (Cosmic Lens All-Sky Survey, Myers et al. 2003; Browne et al. 2003) sample of gravitational lenses is too low relative to observations (Rusin & Tegmark 2001; Huterer et al. 2005). Curiously, however, recent work on the quad-to-double ratio observed in the SQLS (Sloan Digital Sky Survey Quasar Lens Search, Oguri et al. 2006; Inada et al. 2007) suggests that the exact opposite is true for the latter sample, namely, theoretical expectations are too high relative to observations (Oguri 2007). In either case, it is of interest to determine how exactly does triaxiality affects theoretical predictions, especially since the aforementioned difficulties with the CLASS sample has led various authors to offer possibilities as to how one might boost the expected quad-to-double ratios. Specifically, one can boost the quad-to-double ration in the class sample either from the effect of massive satellite galaxies near the lensing galaxies (Cohn & Kochaneck 2004), or through the large-scale environment of the lensing galaxy (Keeton & Zabludoff 2004). Clearly, we should determine whether halo triaxiality can be added to this list.

This brings us then to the final problem we consider here, namely whether the substructure population of lensing galaxies is different from that of non-lensing galaxies. Specifically, we have argued that lensing galaxies will not be isotropically distributed in space. Since the substructure distribution of a dark matter halo is typically aligned with its parent halo’s long axis (Zentner et al. 2005; Libeskind et al. 2005; Agustsson & Brainerd 2006; Azzaro et al. 2006), it follows that the projected distribution of substructures for lensing galaxies may in fact be different for lensing halos than for non-lensing halos. Such an effect could be quite important given the claimed tension between the Cold Dark Matter (CDM) predictions for the substructure mass fraction of halos (see Mao et al. 2004) and their observed values (Dalal & Kochaneck 2002; Kochaneck & Dalal 2004). Likewise, such a bias would impact the predictions for the level of astrometric and flux perturbations produced by dark matter substructures in gravitational lenses (Roso et al. 2006b; Chen et al. 2007). Here, we wish to estimate the level at which the projected substructure mass fraction of lensing halos could be affected due to lensing biasing.

The paper is organized as follows: in section 2 we derive the basic equations needed to compute how observable quantities will be biased in lensing galaxy samples due to halo triaxiality. Section 3 presents the model used in this work to quantitatively estimate the level of these biases, and discusses how lensing halos are oriented relative to the line of sight as a function of the halos’ axes ratios. Section 4 investigates the projected axis ratio distributions of lensing versus non-lensing galaxies, and demonstrates that present day lensing samples are too small to detect the triaxiality induced biases we have predicted. Section 5 discusses the problem of the quad to double ratio, and section 6 demonstrates that halo triaxiality biases the projected substructure mass fraction in lensing halos by a negligible amount. Section 7 discusses a few of the effects we have ignored in our work and how these may alter our results, and finally section 8 summarizes our work and presents our conclusions.

2. Lens Biases Induced by Triaxiality

We begin by deriving the basic expressions on which we rely to estimate the effects of halo triaxiality on the observed properties of lensing galaxies. In particular, we show that since the lensing cross section for triaxial lenses is in general not spherically symmetric, this implies that a population of randomly oriented halos produces a non-random lens population. Finally, we show that the induced non-randomness of the lensing halo population can alter the mean observational properties of these halos relative to the general halo population.

2.1. The Lensing Cross Section

Let \( p \) be a set of parameters that characterizes the projected gravitational potential of a halo. For instance, \( p \) can be the Einstein radius of the lens, its ellipticity, and so on. Given a background source density \( n_s(z_s) \) and a halo density \( n_h(p, z_h) \), and in the absence of a flux limit, the mean number of lensing events per unit redshift per area is given by

\[
\frac{dN_{\text{lenses}}}{dz_s dz_h d\Omega} = n_s(z_s)n_h(p, z_h) \frac{d\chi}{dz_s} \frac{d\chi}{dz_h} \sigma(p, z_s, z_h)
\]  

(1)

where \( \chi \) is the comoving distance to the appropriate halo or source redshift, and

\[
\sigma(p, z_s, z_h) = \int_{\text{lensing}} d^2 y.
\]

(2)

The integral is over all regions of the source plane that produce lensed images of interest. For instance, if one were interested in quadruply imaged sources, the integral would be over all source positions that result in four image lenses. The quantity \( \sigma \) is called the lensing cross section, and of particular interest to us will be the cross sections \( \sigma^{(N)} \) for producing \( N \)-image systems.

In reality, one always has some flux limit \( F_{\text{min}} \) which corresponds to a minimum source luminosity \( L_{\text{min}} \). Fortunately, the above argument is easily generalized: let \( dn_s(L, z_s)/dL \) be the number density of background sources with luminosity \( L \). Then, the mean number of lensing events becomes

\[
\frac{dN_{\text{lenses}}}{dz_s dz_h d\Omega} = n_h \frac{d\chi}{dz_s} \frac{d\chi}{dz_h} \int d^2 y \int_{L_{\text{min}}/\mu(y)}^{\infty} dl \frac{dn_s(L, z_s)}{dL}.
\]

(3)

If the source luminosity function can be approximated by a power law \( dn_s(L, z_s)/dL \propto L^{-\alpha} \) (note both \( A \) and \( \alpha \) can depend on \( z_s \)), the above expression reduces to

\[
\frac{dN_{\text{lenses}}}{dz_s dz_h d\Omega} = n_h > L_{\text{min}} \frac{d\chi}{dz_s} \frac{d\chi}{dz_h} \sigma(p, z_s, z_h)
\]

(4)

where

\[
\sigma(p, z_s, z_h, \alpha) = \int_{\text{lensing}} d^2 y.
\]
where \( n_{\text{th}}(> L_{\text{min}}, z_s) \) is the number density of sources above the flux limit in the absence of lensing, and \( \sigma_B \) is given by

\[
\sigma_B(p, z_h, z_s, \alpha) = \int d^2y \mu(y)^{\alpha-1}
\]

where \( \mu(y) \) is the total magnification of a source at position \( y \). Following Huterer et al. (2005), we call \( \sigma_B \) the biased cross section. Indeed, since the distribution of magnifications \( p(\mu) \) among all lensing events is given by

\[
p(\mu) = \frac{1}{\sigma} \int d^2y \, \delta(\mu(y) - \mu)
\]

where \( \sigma \) is the (unbiased) lensing cross section defined in Eq. (2) then we can rewrite Eq. (4) as

\[
\sigma_B = \langle \mu^{\alpha-1} \rangle \sigma,
\]

where

\[
\langle \mu^{\alpha-1} \rangle = \int d\mu \, \mu(\mu)^{\alpha-1}.
\]

Thus, the net effect of gravitational magnification on the frequency of lensing events can be summarized as a biasing factor \( \langle \mu^{\alpha-1} \rangle \) that multiplies the unbiased lensing cross section \( \sigma \).

### 2.2. Triaxiality and Lensing Biasing

Let \( P \) characterize the mass distribution of a triaxial halo, and let \( \hat{n} \) be the orientation of the halo’s long axis relative to the line of sight. The halo’s two dimensional potential is then characterized by a new set of parameters \( p(P, \hat{n}) \) which depend on the halo properties \( P \) and the particular line of sight \( \hat{n} \) along which the halo is being viewed. For instance, the vector \( P \) can include such halo properties as halo mass and axis ratios, whereas \( \hat{n} \) could include parameters such as the Einstein radius of the projected mass distribution as well as the projected axis ratio.

As discussed above, the mean number of lensing events per unit redshift by a halo along a given line of sight is given by Eq. (4). For convenience, we define the halo and source surface densities \( d\Sigma_h/dP \) and \( d\Sigma_s/dz_s \) via

\[
\frac{d\Sigma_h}{dP d\chi} = \frac{dn_h}{dp} \frac{d\chi}{d\chi_s}
\]

and

\[
\frac{d\Sigma_s}{d\chi_s} = n_{\text{th}}(> L_{\text{min}}) \frac{d\chi}{dz_s}.
\]

In terms of these surface densities, and assuming a randomly-oriented distribution of halos, the mean number of lenses per unit area as a function of their orientation \( \hat{n} \) is given by

\[
\frac{dN_{\text{lenses}}}{dP d\hat{n} d\chi_d d\chi_s} = \frac{1}{2\pi} \frac{d\Sigma_h}{dP d\chi} \sigma_B(p(p, \hat{n}), \chi_d, \chi_s, \alpha).
\]

The prefactor of \( 1/(2\pi) \) arises from the fact that \( dn_h/dP d\hat{n} = (dn_h/dp)/(2\pi) \) due to our assumption of randomly oriented halos.\(^7\) We emphasize that Eq. (11) characterizes the number of lenses as a function of the relative orientation \( \hat{n} \) between the halo’s major axis and the line of sight. Thus, to compute the total number of lenses irrespective of halo orientation, we would simply integrate the above expression over all lines of sight \( \hat{n} \).

There is an absolutely key point to be made concerning Eq. (11) which provides the motivation behind this work. Specifically, we note that the number of lenses is proportional to \( \sigma_B(p(p, \hat{n})) \). This implies that even though the overall halo population does not have a preferred orientation in space, the lens population is not randomly oriented, a fact which can have observable consequences. In particular, given an observable halo property \( f(P, \hat{n}) \) that depends on the line of sight projection (e.g. the projected axis ratio or projected substructure mass fraction), the mean value of \( f \) over all \( P \) halos is simply

\[
\langle f(P) \rangle_{\text{lenses}} = \frac{1}{\langle \sigma_B \rangle} \int d^2\hat{n} f(P, \hat{n}),
\]

whereas the mean value of \( f \) over all lenses is given by

\[
\langle f(P) \rangle_{\text{lenses}} = \frac{1}{\langle \sigma_B \rangle} \int d^2\hat{n} \sigma_B(p(p, \hat{n})) f(P, \hat{n})
\]

where \( \langle \sigma_B \rangle \) is the average value of \( \sigma_B \) over all lines of sight,

\[
\langle \sigma_B \rangle = \int d^2\hat{n} \sigma_B(p(p, \hat{n})).
\]

Thus, in general, one expects that the mean value of \( f \) over lenses and over all halos will be different. In the next few sections, we identify a few halo properties that depend on line of sight projection, and determine whether lensing biases induced by triaxiality are likely to be significant.

### 3. The Model

We estimate the impact of halo triaxiality on the properties of lenses by considering a triaxial isothermal profile. The merit of this approach is its simplicity: because of the simple form of the matter distribution in this model, we can compute all of the relevant quantities in a semi-analytic fashion, and the main features of the model can be easily understood, thereby providing an important reference point for investigating more elaborate models. Moreover, by working out in detail a simple analytic model, our results provide an ideal test bed for more involved numerical codes, which would then allow us to investigate how our conclusions are changed as more complicated models are allowed (Chen et al. 2007, in preparation).

#### 3.1. Semi-Analytical Modeling

Our analytical halo model is that of a simple triaxial isothermal profile of the form

\[
\rho(\chi) = N(q_1, q_2) \frac{\sigma_a^2}{2\pi G} \frac{1}{x^2 + y^2 + z^2} (q_1^2 + q_2^2)
\]

where \( q_1 \) and \( q_2 \) are the axis ratios of the profile and we have chosen a coordinate system that is aligned with the halo’s principal axes, and such that \( 1 \geq q_1 \geq q_2 \).\(^8\) The normalization constant \( N(q_1, q_2) \) is chosen to ensure that the mass contained within a sphere of radius \( r \) is independent of the axis ratios for fixed velocity dispersion \( \sigma_a \), the latter being the velocity dispersion of the Singular Isothermal Sphere (SIS) obtained when \( q_1 = q_2 = 1 \).

Let then \( (\theta, \phi) \) denote a line of sight. In appendix B we show that the corresponding projected surface mass density

\[
\frac{d\Sigma}{dz_s} = n_{\text{th}}(> L_{\text{min}}) \frac{d\chi}{dz_s}.
\]

\(^7\) If \( \hat{n} \) denotes the angle between the line of sight and a specified halo axis, and given that \( \hat{n} \) and \( -\hat{n} \) correspond to the same line of sight, then it is evident that the space of all lines of sight is simply \( S^2/Z_2 \) - a sphere with its diametrically opposed points identified. The volume of such a space with the usual metric is thus simply \( 2\pi \).

\(^8\) i.e. \( q_1 \) is the ratio of medium to long axis of the halo, whereas \( q_2 \) is the ratio of the short to long axis. The motivation behind our particular choice of axis labeling will be made clear momentarily.
profile is that of a Singular Isothermal Ellipsoid (SIE) which, following [Kormann et al. 1994], we write as

$$\Sigma(x, y) = \frac{1}{2G} \frac{\sqrt{q_\alpha^2 \eta^2 + q_\beta^2 \eta^2}}{x^2 + q_\alpha^2 \eta^2}$$  \hspace{1cm} (16)$$

where both $q$ and $\sigma$ are known functions of $q_1$, $q_2$, and, in the case of $\sigma_\alpha$, of $N(q_1, q_2)\sigma^2_\alpha$ (see Appendix B for details). In the above expression, $\sigma$ and $q$ are the effective velocity dispersion and axis ratio respectively of the projected SIE profile. As shown by [Kormann et al. 1994], the lensing cross section for an SIE scales trivially with the Einstein radius $b^9$

$$b = 4\pi \frac{\sigma^2_\alpha}{c^2} \frac{D_D D_L}{D_s}$$  \hspace{1cm} (17)$$

of the profile. Consequently, the distribution of halo orientations for a lens sample, $\rho(\hat{n}) = \sigma(\hat{n}) / \langle \sigma(\hat{n}) \rangle$, is independent of the velocity dispersion $\sigma_v$ of the halo.

There is one last important element of the model that needs to be specified, namely the luminosity function of the sources being lensed. Here, we take the luminosity function to be a mean lensing cross section of a halo averaged over all lines of sight. For a detailed description of our calculations, we refer the reader to the Appendices.

Before we end, however, it is important to remark here that, despite its simplicity, we expect our model is more than adequate to investigate the qualitative trends that we would expect to observe in the data, and for providing order of magnitude estimates of the impact of triaxiality. Specifically, elliptical isothermal profiles appear to be excellent approximations to the true matter distribution in real lens systems [e.g. Gerhard et al. 2001; Rusin & Me 2001; Rusin et al. 2003; Rusin & Kochanek 2003; Treu et al. 2006; Koopmans et al. 2006; Gavazzi et al. 2007], so the triaxial isothermal mass distribution considered here should provide a reasonably realistic model for order of magnitude estimates. While more sophisticated models are certainly possible (see e.g. Jiang & Kochanek 2007), it is our view that the simplicity of the isothermal model more than justifies our choice of profile for a first pass at the problem.

3.2. The Distribution of Halo Orientations for Triaxial Isothermal Profiles

9 By trivially, we mean $\sigma \propto b^2$. Before we look at the distribution of halo orientations, it is worth taking a minute to orient ourselves in the coordinate system we have chosen. Consider first Eq. 15. The distance from the center of the halo to the intercept of a constant density contour is maximized for the $y$ axis, and minimized for the $z$ axis, while the $x$ axis is intermediate between the two. If we then parameterize the line of sight using the circular coordinates $\theta$ and $\phi$ where $\theta$ is the angle with the $z$ axis and $\phi$ is the projected angle with the $x$ axis, then our coordinate system is such that it has the following properties.

- The $x, y,$ and $z$ axis of our coordinate system correspond to the middle, long, and short axis of the halo respectively.
- Projections along $\cos(\theta) = 1$ are along the short axis of the halo.
- Projections along $\cos(\theta) = 0$, $\phi = 0$ are along the middle axis of the halo.
- Projections along $\cos(\theta) = 0$, $\phi = \pi/2$ are along the long axis of the halo.

The nice thing about this particular choice of coordinates is that in the $\cos(\theta) - \phi$ plane, both the long and the middle axis are represented by a single point, whereas the short axis is represented by an entire line. As we shall see, projections along the middle and long axis maximize the lensing cross section of a halo for quad and double lenses respectively, so having that maximum be a single point in the space of lines of sight is a desirable quality of our chosen coordinate system.

Figure 1 shows the ratio $b(\hat{n})/b_0$ where $b_0$ is the Einstein radius of an SIS with velocity dispersion $\sigma_v$, as well as the projected axis ratio $q(\hat{n})$, for an isothermal ellipsoid with axis ratios $q_1 = 0.75$, $q_2 = 0.5$. We can see the Einstein radius of the projected profile is maximized when projecting along the long axis of the halo, whereas the ellipticity is maximized when projecting along the middle axis of the halo, as it should be. Note we have only considered the range $\theta \in [0, \pi/2]$ and $\phi \in [0, \pi/2]$ rather than the full range of possible lines of sight $\theta \in [0, \pi/2]$ and $\phi \in [0, 2\pi]$. This is due to the symmetry of our model; all eight of the octants defined by the symmetry planes of the ellipsoids are identical.

Let us now go back and study the distribution of line of sights for both doubles and quads. Figure 2 shows these distributions for three types of halos: a prolate halo, an oblate halo, and a halo that is neither strongly oblate nor strongly prolate. As is customary, we parameterize the halo shape in terms of the shape parameter $T$ which is defined as

$$T = \frac{1 - q_1^2}{1 - q_2^2}.$$  \hspace{1cm} (18)$$

Note that a perfectly prolate halo ($q_1 = q_2$) has $T = 1$, whereas a perfectly oblate halo ($q_1 = 1$) has $T = 0$. From top to bottom, the halo shape parameters used to produce Figure 2 are $T = 0.9$ (cigar shape), $T = 0.5$ (neither strongly oblate nor strongly prolate), and $T = 0.1$ (pancake shape). The axis ratio $q_2$ was held fixed at $q_2 = 0.5$. Finally, the left column is the distribution of lines of sight for double systems, whereas the right column is the distribution for quads. For ease of comparison, the color scale has been kept fixed in all plots.

Let us begin by looking in detail at the doubles column first. As is to be expected, the distribution of lines of sight is peaked for projections along the long axis of the lens, as this line of
the increase in ellipticity more than offsets the slightly smaller Einstein radii for the purposes of enhancing the lensing cross section for producing quad systems. It is also interesting to note that while the peak of the distribution is always clearly about the middle axis of the lens, the shape of the distribution varies considerably in going from prolate halos to oblate halos. In particular, note that for prolate halos the peak about the middle axis is relatively narrow. What is more, projections along the short axis of the lens are more likely than projections along the long axis because the latter minimizes the ellipticity of the projected profile. For oblate halos, on the other hand, projections along the long axis of the lens are almost as likely as projections along the middle axis. This is simply because for such halos, there is little difference in the ellipticity of the projected profile between projections along the middle and long axis of the halos. Consequently, both axes result in highly effective quad lenses. Note too that for pancake-like halos, projections along the short axis are strongly avoided, since this projection minimizes both the Einstein radius and the projected axis ratio of the lens.

In short, then, prolate halos and oblate halos will have very different orientation distributions: for prolate halos, nearly all doubles will be due to projections along the long axis of the lens, while most quads will be due to projections along the middle axis of the lens, followed by projections along the short axis. For oblate halos, however, all halo orientations are almost equally likely in the case of doubly imaged systems, whereas quads strongly avoid projections along the short axis of the halo.

The remainder of the paper will explore whether these results have a significant impact on the statistical properties of the halo population. Specifically, we will first consider the ellipticity distribution of lensing galaxies compared to that of galaxies as a whole. We will then discuss how these results affect the predicted quad-to-double ratio, and finally, we will investigate whether lensing halos are expected to have a significantly biased projected substructure mass fraction.

4. THE PROJECTED AXIS RATIOS OF LENSing HALOS

As mentioned in the introduction, if one assumes that the ellipticity of the light and that of the mass are monotonically related, then one would naively expect that lensing galaxies that produce quads ought to be more elliptical than the average galaxy because the lensing cross section for quads increases with increasing ellipticity. Similarly, galaxies that produce doubles should tend to be more spherical. In this section, we discuss the impact of halo triaxiality on the distribution of axis ratios for double and quad lenses.

Given a line of sight \( \hat{\mathbf{n}} \), we can compute the axis ratio \( q(\hat{\mathbf{n}}) \) of the projected mass distribution (see Eq. 19). Using the distribution of lines of sight \( \rho(\hat{\mathbf{n}}) \), one can then easily compute the distribution of projected axis ratios \( q \) for a sample of lenses via

\[
\rho(q_1, q_2) = \int \frac{d\hat{\mathbf{n}}}{2\pi} \rho(\hat{\mathbf{n}}) \delta_p(q(\hat{\mathbf{n}}|q_1, q_2) - q).
\]

Figure 1 shows the distribution of the projected axis ratio of both quad and double systems for the sample pancake-like (oblate, \( T = 0.1 \)) and cigar-like (prolate, \( T = 0.9 \)) halos from Figure 2. As is to be expected, the distribution for quad systems is considerably skewed towards high ellipticity systems, whereas the distribution for doubles is much flatter. Moreover, the quads distribution is significantly more skewed for prolate (cigar-like) systems than for oblate (pancake-like) halos.
Fig. 2.— Contours of the orientation distribution $\rho(\hat{n}) = \sigma_B/\langle \sigma_B \rangle$ for triaxial isothermal profiles. The left and right columns show the distributions for double and quads respectively, while the three rows are for three different halos: the top row is for prolate (cigar-like) halos ($T = 0.9$), the middle row if a for a triaxial halo that is neither strongly oblate nor prolate ($T = 0.5$), and the bottom row is for oblate (pancake-like) halos ($T = 0.1$). The short to long axis ratio $q_2$ was held fixed at $q_2 = 0.5$. For ease of comparison, the color scale is the same in all plots. Note that for doubles, the distribution of lines of sight is peaked about projections along the long axis of the lens, as we would expect. For quad systems, however, the distribution peaks for projections along the middle axis of the lens, which corresponds to maximizing the ellipticity of the resulting projected profile (see Figure 1).
los. Based on Figure 3, we have attempted to distill the difference between quads and doubles into a single number. We define the axis ratio $q_{0.75}$ as the axis ratio for which 75% of the lenses have axis ratios $q \leq q_{0.75}$. The value $q_{0.75}$ for quads and doubles for both sample halos is also shown in Figure 3 as lines along the top axis of the plot. It is clear that the projected axis ratio $q_{0.75}$ for doubles and quads is very different, with $\Delta q_{0.75} > 0.1$ for both oblate and prolate halos.

Figure 4 shows the difference $\Delta q_{0.75}$ between doubles and quads as a function of the small-to-large axis ratio $q$ (solid line), in which case values as low as $\Delta q_{0.75} = 0.05$ are possible. Turning now to the comparison between doubles and random halos, we see that the difference in $q_{0.75}$ for these two halo populations becomes negligible in the case of oblate halos, reflecting the near uniform distribution of lines of sights for doubles for oblate halos (see Figure 2). On the other hand, the fact that most prolate doubles are seen along the long axis of the halo implies that $\Delta q_{0.75}$ between doubles and random halos must be significant, and thus doubles tend to be more circular than the typical halo.

In short, then, the quantity $\Delta q_{0.75}$ between doubles and quads and between doubles and random halos can, at least in principle, help determine whether most halos are oblate or prolate. If halos are prolate, the difference $\Delta q_{0.75}$ between doubles and random halos is large. If this difference is small, we can then look at the difference $\Delta q_{0.75}$ between doubles and quads. If this last difference is large, then halos are typically oblate, whereas if the difference is small, then halos are neither strongly oblate nor strongly prolate and $q_{2} \approx q_{1}^{7}$.

In practice, however, the above test is difficult to execute. In particular, while lens modeling can provide some measure of the axis ratio $q$ in quad systems, there remains a fair amount of uncertainty due to the approximate degeneracy between galaxy ellipticity and external shear (see e.g. Keeton et al. 1997). This degeneracy is even stronger for doubly-imaged systems, and worse, there is no way of determining the axis ratio of the mass for non-lensing galaxies. Fortunately, at the scales relevant for strong lensing (≲ 5 kpc), baryons dominate the total matter budget in early type galaxies (Rusin et al. 2003), so one expects that the dark matter distribution in these systems will have the same ellipticity and orientation as the baryons. Observationally, Keeton et al. 1998 (see also Keeton et al. 1997) compared the projected ellipticity of the light in lensing galaxies to the ellipticity recovered from explicit lens modeling, and found that the light and the mass tend to be very closely aligned, though the magnitude of the ellipticities is not clearly correlated and the modest quality of the photometry available at the time made their ellipticity measurements difficult. Moreover, the galaxy sample Keeton et al. 1998 included many galaxies that had non-negligible environments that were not incorporated into the model. More recently, a detailed study of the Sloan Lens ACS Survey (SLACS; Bolton et al. 2006) with more isolated galaxies supports the hypothesis that the ellipticity of the light is in fact extremely well matched to the ellipticity of the projected mass, at least on scales comparable to the Einstein radii of...
the galaxies (Koopmans et al. 2006). Thus, for the purposes of this work, we simply take the isophotal axis ratio of lensing galaxies to be identical to the total matter axis ratio for the purposes of investigating whether lens biasing can be detected in current lensing samples.

Figure 5 shows the cumulative distribution of isophotal axis ratios for quad lenses (solid) and double lenses (dashed) for all lensing galaxies in the CASTLES database with isophotal axis ratios measurements. Of course, the selection function for this sample is impossible to quantify objectively, but our intent is simply to see whether any differences between lensing galaxies and random galaxies can be found. Also shown in the figure are the axis ratio distributions of early type galaxies as reported by two different groups: the dotted line shown is the fit used by Rusin & Tegmark (2001) to model the distribution of axis ratios in early type galaxies based on measurements by Jorgensen & Franx (1994), and is also quite close to the distribution recovered by Lambas et al. (1992). The dashed-dotted line is the axis ratio distribution obtained by Hao et al. (2006) using the SDSS Data Release 4 photometric catalog, and is a very close match to the distribution recovered by Fasano & Vio (1991). Hao et al. (2006) noted that it is unclear why these two distributions differ, though Keeton et al. (1997) noted that such a difference can easily arise depending on whether SO galaxies are included in the galaxy sample or not (with SO galaxies being more elliptical). Here, we simply consider both distributions.

Given that the axis ratio distribution for both quads and doubles largely fall in between the two model distributions we considered, it is immediately obvious that no robust results can be obtained at this time. Specifically, uncertainties in the details of the selection function of the galaxies used to construct the isophotal axis ratios are a significant systematic. More formally, using a KS-test, we find that the isophotal axis ratio distributions of both quad and double lens galaxies are consistent with that of the early type galaxy population as a whole (irrespective of which model distribution we choose) and with each other as well. Interestingly, whether or not we restrict ourselves to galaxies that are isolated or whether we include all lensing galaxies does not appear to change the result in any way. Naively, then, the consistency of the axis ratio distributions suggests that halos are typically neither strongly oblate nor prolate, but rather somewhere in between, where the quantity $\Delta q_{0.75}$ exhibits a minimum, which occurs at $q_2 \approx q_1^*$. Given that current lens samples are too small for detecting any difference on the ellipticities of quadruply and doubly imaged systems, it is worth asking whether or not a detection is possible in principle. That is, how many lenses must one have in order to detect quad systems as being more elliptical than doubles? To answer this question, we need to first assume a simple model for the distribution of axis ratios $Q(q_1, q_2)$, with which one could then compute the resulting projected axis ratio distributions for doubles, quads, and the galaxy population at large. We should note here, however, that in detail our results will depend on the adopted distribution $Q(q_1, q_2)$, which is not known.

It is not immediately obvious what the most correct model distribution $Q(q_1, q_2)$ should be. While there have been many studies that have investigated the distribution of axis ratios of dark matter halos in simulations (see e.g. Warren et al. 1992; Jing & Suto 2002; Bailin & Steinmetz 2003), it has become clear that the distribution itself depends on many variables, including halo mass (Kasun & Evrard 2003; Bett et al. 2007), radius at which the shape of the halo is measured (Hayashi et al. 2007), halo environment (Hahn et al. 2007), and whether the halo under consideration is a parent halo or a subhalo of a larger object (Kuhlen et al. 2007). Adding to these difficulties is the fact that different authors use different definitions and methods for measuring the shapes of halos, which forces one to go to great lengths in order to ensure a fair comparison of the results from different groups (see for example Allgood et al. 2006). Even more problematic that all of these difficulties, however, is the fact that not only can the distributions of baryons have a different shape from the dark matter (Gottlöber & Yepes 2007), baryons dominate the mass budget in the halo regions where strong lensing occurs, and can therefore dramatically impact halo shapes at those scales (Kazantzidis et al. 2004; Bailin et al. 2005; Gustafsson et al. 2006). Since our intent here is simply to provide a rough estimate of the number of lenses required to detect a significant difference in the ellipticities of quad and double systems, we simply adopt a fiducial model that is based primarily on the results of Allgood et al. (2006) and Kazantzidis et al. (2004), and use it to estimate the number of lenses necessary to detect the larger ellipticity of quad systems. Specifically, Allgood et al. (2006) obtain that for an $M_\star$ halo the distribution of the short-to-long axis ratio of dark matter halos is Gaussian with a mean of $\langle q_2 \rangle = 0.54$ and a standard deviation $\sigma_{q_2} = 0.1$. As noted by Kazantzidis et al. (2004), baryonic cooling tends to circularize the mass profiles of halos, so we adopt instead a somewhat larger ratio $\langle q_2 \rangle = 0.65$, but retain the dispersion $\sigma_{q_2} = 0.1$. The adopted value for $\langle q_2 \rangle$ is larger than that obtained from dissipationless simulations, but smaller than that found in the simulations of Kazantzidis et al. (2004), as the latter suffer from the well known over-cooling problem and therefore overestimate the impact of baryons on the profiles. In addition, we truncate...
Since the projected axis ratio of a halo plays a key role in the expected quad-to-double ratio of lensing galaxies, it is easy to see that triaxiality should also affect this statistical observable. This is the problem we wish to consider now: how does triaxiality affect the quad-to-double ratio of lensing galaxies?

Consider first equation (11). For our semi-analytic case, the halo parameters $P$ that determine the mass distribution of the halo are simply the halo velocity dispersion and its two axis ratios $q_1$ and $q_2$. What is more, we saw that if we define $b_{0}(\sigma)$ as the Einstein radius of an SIS of velocity dispersion $\sigma_v$, then the ratio $\sigma/\sigma_v$ depends only on the axis ratios $q_1$ and $q_2$. If we make the further assumption that the distribution of halo parameters is separable, i.e. that

$$\frac{dn_{halos}}{dz_h d\sigma dq_1 dq_2} = \frac{dn_{halos}}{dz_h d\sigma} Q(q_1, q_2),$$

then it is easy to see that the ratio of the total number of quad systems to double systems depends only on the distribution of axis ratios $Q(q_1, q_2)$ because the overall scaling of the lensing cross sections for both doubles and quads just factors out of the problem. Thus, the ratio of quad-to-doubles is given simply by

$$r(q_1, q_2) = \frac{\text{No. of quads}}{\text{No. of doubles}} = \frac{\langle \sigma_B^{(4)} \rangle_{q_1, q_2}}{\langle \sigma_B^{(2)} \rangle_{q_1, q_2}}.$$  

The top panel of Figure 7 shows the dimensionless mean biased lensing cross section $\langle \sigma_B \rangle / \sigma_v^2$ for both doubles and quads averaged over lines of sight for a population of randomly oriented halos. Also shown in the bottom panel is the quad-to-double ratio. As expected, large (\geq 0.3) quad-to-doubles ratios require strong deviations from spherical symmetry, so $q_2$ needs to be small. Interestingly, however, all of the contours in both the top and bottom panel of Figure 7 are nearly vertical: lensing cross sections are nearly independent of halo shape. We can understand this qualitatively as follows. In the case of doubles, there is a tradeoff between two competing effects: for $1 \lesssim q_1 \lesssim q_2$, there are many lines of sight that enhance the Einstein radius of the lens, but only moderately so. For $1 \gtrsim q_1 \gtrsim q_2$ on the other hand, there are only a few lines of sight that enhance the Einstein radius of the lens (i.e. projections along the long axis of the halo), but the enhancement is much greater. Thus, the overall boost to the Einstein radius is offset by the reduced “volume” of lines of sight available for forming doubles and vice versa. A similar effect occurs for quads: oblate halos make effective lenses when projected along either the long or middle axis of the lens, but strongly avoid the short axis, so the “volume” of lines of sight available to oblate halos is small. Prolate halos, on the other hand, are not quite as effective as oblate halos at making quads, but can produce quads over a larger range of possible lines of sight.

At any rate, one thing that is clear from Figure 7 is that halo shape does not have a significant impact on the expected quad-to-double ratio. One extremely interesting consequence of this results is that it implies that halo triaxiality can be properly incorporated into lensing statistics studies without greatly increasing the number of degrees of freedom in the problem. More explicitly, traditional lens statistics studies use as input the observed two dimensional ellipticity distribution of early type galaxies, and approximate the effects of triaxiality by multiplying the usual isothermal ellipsoidal profiles with a normalization factor computed assuming halos are either all perfectly oblate, or perfectly prolate (see e.g. Chae...
The range of the axis ratios for quads (solid) and doubles (dotted) as a function of the axis ratios $q_1$ and $q_2$ of the halo population. Bottom panel: quad-to-double ratio as a function of the axis ratios $q_1$ and $q_2$ for triaxial isothermal profiles. We only consider the range $q_1 \geq 0.4$ because below this value our semi-analytical calculations based on Kormann et al. (1994) break down. Note the observed ratio of about 0.4 can only be obtained for halos that deviate strongly from spherical symmetry. Interestingly, all contours in both panels are very nearly horizontal, so whether halos are prolate (cigar-like) or oblate (pancake-like) has almost no impact on the lensing optical depths or the quad-to-double ratio.

Fig. 7.— Top panel: The dimensionless biased lensing cross section $\sigma_B/b_c^2$ for quads (solid) and doubles (dotted) as a function of the axis ratios $q_1$ and $q_2$ of the halo population. Bottom panel: quad-to-double ratio as a function of the axis ratios $q_1$ and $q_2$ for triaxial isothermal profiles. We only consider the range $q_1 \geq 0.4$ because below this value our semi-analytical calculations based on Kormann et al. (1994) break down. Note the observed ratio of about 0.4 can only be obtained for halos that deviate strongly from spherical symmetry. Interestingly, all contours in both panels are very nearly horizontal, so whether halos are prolate (cigar-like) or oblate (pancake-like) has almost no impact on the lensing optical depths or the quad-to-double ratio.

The main reason this is done, rather than considering triaxial halos and averaging over lines of sight, is that in order to do the latter calculation, one needs to know something about the distribution of axis ratios. We have shown, however, that such a calculation would in fact be nearly independent of assumptions made about the intermediate axis $q_1$. In other words, a proper calculation that weights lines of sight according to their biased lensing cross section rather than uniform weighting (as implicitly done when taking the ellipticity distribution to be that of early type galaxies as a whole) effectively involves no more freedom than the usual approach, the main difference being that the assumptions made will involve not the ellipticity distribution, but rather the distribution of the short-to-long axis ratio $q_2$, which can itself be constrained using the projected ellipticity distribution (e.g. Lambas et al. 1992).

**6. THE SUBSTRUCTURE MASS FRACTION IN THE INNER REGIONS OFLENSEING HALOS**

One of the important predictions of the CDM paradigm of structure formation is that galactic halos contain a large amount of bound substructure within them (see e.g. White & Rees 1978, Blumenthal et al. 1984). Observationally, however, both our own galaxy and M31 have an order of magnitude less luminous companions than is predicted if one assumes substructures have a fixed mass to light ratio (Kauffmann et al. 1993, Klypin et al. 1999, Moore et al. 1999). Currently, the favored explanation for this discrepancy is that the mass to light ratio of such small structures depends strongly on the history of the objects, and therefore only a select subset of the substructures within the halo become luminous (e.g. Somerville & Primack 1999, Benson et al. 2002, Kravtsov et al. 2004, Sales et al. 2007). While such scenarios appear to be in good agreement with the data, it would still be desirable to provide as direct detection as possible of the remaining dark substructures.

Motivated by the fact that dark substructures can only be discovered via their gravitational signal, Dalal & Kochanek (2002a) investigated whether the well known flux anomalies problem could be explained as the action of dark substructures embedded within the halo of the lensing galaxy. Using a relatively simple model, they found that in order to explain the observed flux anomalies, one requires a projected substructure mass fraction $f_s$ in the range $7\% > f_s > 0.6\%$ at the 90% confidence level. It was then argued by Mao et al. (2004) that such a substructure mass fraction was slightly larger than the mass fraction obtained from simulations $f_s \approx 0.5\%$.

Recently, it has become clear that the distribution of substructures in dark matter halos is not spherically symmetric, but is instead triaxial, and aligned with the major axis of the halo. Since lensing halos are not randomly oriented in space, the mean projected substructure mass fraction for all halos - the $f_s \approx 0.5\%$ value obtained by Mao et al. (2004) - need not be the same as the mean substructure mass fraction for lensing halos, which would in turn affect theoretical predictions (e.g. Rozo et al. 2006b, Chen et al. 2007). Here, we use the results on substructure alignments in numerical simulations to estimate the dependence of the projected substructure mass fraction $f_s$ on the projection axis. More specifically, assuming that substructures do not significantly alter the biased lensing cross sections for the halos, we compute the mean substructure mass fraction for doubles and quad lenses as a function...
Overall, though, it is clear that for quad systems - which are the only kind of systems for which \( f_s \) may be estimated using the methods of Dalal & Kochanek (2002a) - the substructure mass fraction in the inner regions of a halo cannot be significantly enhanced due to lens biasing if the impact of substructures on the lensing cross section of galactic halos can be neglected. Thus, lens biasing does little to soften the slight (and in these authors’ opinion, not terribly significant) discrepancy between the values of \( f_s \) recovered by Dalal & Kochanek (2002a) and those from numerical simulations.

7. CAVEATS AND SYSTEMATICS

Before we finish, we believe it is important to mention two systematics that could significantly affect the conclusions presented in this work. Specifically, throughout we have assumed that the lensing cross section is dominated by the smooth mass distribution of lensing galaxies, and we presented in section 3 several studies that suggest that our model for the mass distribution of early type galaxies is a reasonable one. As mentioned in the introduction, the possible discrepancy between theory and observation concerning the quad-to-double ratio of the CLASS lenses has raised the possibility that lensing cross sections are in fact heavily influenced by the environment of the halo or possibly by substructures with it. We briefly discuss each of these in turn.

We begin by discussing halo environments. In our calculations above, and in most of the lensing statistics literature, the effect of halo environment on lensing statistics is neglected. This is not an entirely ad hoc assumption. Theoretical estimates of the amount of shear that the typical lens experiences are quite small (\( \gamma \approx 0.02 \)), see e.g. Keeton et al. (1997), Faltenbacher et al. (2007), implying the second solution to the quad-to-double ratio problem in-...
have shown that the lensing cross section of galaxies is severely affected by substructures. If this is indeed the case, the way in which lensing galaxies are biased relative to the overall galaxy population depend not only on the smooth component of its mass distribution, but also on the spatial distribution of substructures within the galaxy halo. Interestingly, in such a scenario halo triaxiality would impact the orientation of halos relative to the line of sight now only through the biasing due to the smooth matter component, but also because of the previously mentioned alignment between the substructure distributions and the smooth mass distributions. We leave the question of exactly how such a population of halos would be biased to future work (Chen et al., in preparation).

8. SUMMARY AND CONCLUSIONS

The triaxial distribution of mass in galactic halos implies that the probability that a galaxy becomes a lens is dependent on the relative orientation of the galaxy’s major axis to the line of sight. Consequently, a subsample of randomly oriented galaxies that act as strong lenses will not be randomly oriented in space. The relative orientation and the strength of the alignment depends on the shape of the matter distribution, and on the type of lens under consideration: prolate doubles have a high probability of being project along their long axis, whereas the distribution of oblate doubles is nearly isotropic. Prolate quads are most often projected along their middle axis, but interestingly, highly prolate quads are also more likely to be projected along their short axis than along the other two axis are almost equally likely. An important consequence of the differences in the distribution of halo orientations for quadruples, double lenses, and that the ellipticity distribution of doubles is very slightly more circular than that of random galaxies. While current data do not show any indication of these trends, we have shown that ≈ 300 (1, 400) lenses are necessary to obtain a 2σ (5σ) detection of the effect. The fact that halo triaxiality affects the ellipticity distribution of lensing galaxies also means that halo triaxiality needs to be properly taken into account in lensing statistics. Consequently, we estimate how the biased lensing cross sections of galaxies depend on halo shape, and find that they are nearly independent of the halo shape parameter $T$. Instead, the mean biased cross section of a lens depends almost exclusive on the distribution on the short-to-long axis ratio $q_2$ (often denoted by $s$).

Finally, given that the distribution of substructures in numerical simulations is observed to be preferentially aligned with the long axis of the host halos, we estimate how the aligned orientation of lensing galaxies affects their predicted substructure mass fraction. We find that biases due to non-isotropic distribution of halos relative to the line of sight have an insignificant impact on the mean substructure mass fraction of lensing galaxies.

Acknowledgements: ER would like to thank Christopher Kochanek for numerous discussions and valuable comments on the manuscript which have greatly improved both the form and content of this work. The authors would also like to thank to Emilio Falco for kindly providing the isophotal axis ratio data that was needed for producing Figure S and to Charles Keeton for a careful reading of the manuscript. ER was funded by the Center for Cosmology and Astro-Particle Physics (CCAPP) at The Ohio State University. ARZ has been funded by the University of Pittsburgh, the National Science Foundation (NSF) Astronomy and Astrophysics Post-doctoral Fellowship program through grant AST 0602122, and by the Kavli Institute for Cosmological Physics at the University of Chicago. This work made use of the National Aeronautics and Space Administration Astrophysics Data System.

REFERENCES

Agustsson, I. & Brainerd, T. G. 2006, ApJ, 650, 550
Allgood, B., Flores, R. A., Primack, J. R., Kravtsov, A. V., Wechsler, R. H., Faltenbacher, A., & Bullock, J. S. 2006, MNRAS, 367, 1781
Azzaro, M., Zentner, A. R., Prada, F., & Klypin, A. A. 2006, ApJ, 645, 228
Bailin, J., Kaswata, D., Gibson, B. K., Steinmetz, M., Navarro, J. F., Brook, C. B., Gill, S. P. D., Ibata, R. A., Knebe, A., Lewis, G. F., & Okamoto, T. 2005, ApJ, 627, L17
Bailin, J. & Steinmetz, M. 2005, ApJ, 627, 647
Benson, A. J., Frenk, C. S., Lacey, C. G., Baugh, C. M., & Cole, S. 2002, MNRAS, 333, 177
Bett, P., Eke, V., Frenk, C. S., Jenkins, A., Helly, J., & Navarro, J. 2007, MNRAS, 376, 215
Blumenthal, G. R., Faber, S. M., Primack, J. R., & Rees, M. J. 1984, Nature, 311, 517
Bolton, A. S., Burles, S., Koopmans, L. V. E., Treu, T., & Moustakas, L. A. 2006, ApJ, 638, 703
Brownie, I. W. A., Wilkinson, P. N., Jackson, N. J. F., Myers, S. T., Fassnacht, C. D., Koopmans, L. V. E., Marlow, D. N., Norbury, D., Rusin, D., Sykes, C. M., Biggs, A. D., Blandford, R. D., de Bruyn, A. G., Chae, K.-H., Helbig, P., King, L. J., McKean, J. P., Pearson, T. J., Phillips, P. M., Readhead, A. C. S., Xanthopoulos, E., & York, T. 2003, MNRAS, 341, 13
Chae, K.-H. 2003, MNRAS, 346, 746
—. 2005, ApJ, 630, 764
—. 2007, ApJ, 658, L71
Chae, K.-H. & Mao, S. 2003, ApJ, 599, L61
Chen, J., Roxo, E., Dalal, N., & Taylor, J. E. 2007, ApJ, 659, 52
Cohn, J. D. & Kochanek, C. S. 2004, ApJ, 608, 25
Dalal, N. & Kochanek, C. S. 2002a, ApJ, 572, 52
Dalal, N. & Watson, C. 2004, ArXiv Astrophysics e-prints
Donoso, E., O’Mill, A., & Lambas, D. G. 2006, MNRAS, 369, 479
Faltenbacher, A., Li, C., Mao, S., van den Bosch, F. C., Yang, X., Jing, Y. P., Pasquali, A., & Mo, H. J. 2007, ApJ, 662, L71
Fasano, G. & Vio, R. 1991, MNRAS, 249, 629
Fukugita, M., Futamase, T., Kasai, M., & Turner, E. L. 1992, ApJ, 393, 3
Gavazzi, R., Treu, T., Rhodes, J. D., Koopmans, L. V. E., Bolton, A. S., Burles, S., Massey, R., & Moustakas, L. A. 2007, ArXiv Astrophysics e-prints
Gerhard, O., Kronawitter, A., Saglia, R. P., & Bender, R. 2001, AJ, 121, 1936
Gottlöber, S. & Turchaninov, V. 2006, in Engineering and Science, Vol. 20, EAS Publications Series, ed. G. A. Mamon, F. Combes, C. Deffayet, & B. Fort, 25–28
Gottlöber, S. & Yepes, G. 2007, ApJ, 664, 117
Gustafsson, M., Fairbairn, M., & Sommer-Larsen, J. 2006, Phys. Rev. D, 74, 123522
Hahn, O., Porciani, C., Carollo, C. M., & Dekel, A. 2007, MNRAS, 375, 489
Hao, C. N., Mao, S., Deng, Z. G., Xia, Y. Y., & Wu, H. 2006, MNRAS, 370, 1339
Hayashi, E., Navarro, J. F., & Springel, V. 2007, MNRAS, 377, 50
Hennawi, J. F., Dalal, N., Bode, P., & Ostriker, J. P. 2007, ApJ, 654, 714
Huterer, D., Keeton, C. R., & Ma, C.-P. 2005, ApJ, 624, 34

The biasing due to non-isotropic distribution of halos relative to the line of sight now only through the biasing due to the smooth matter component, but also because of the previously mentioned alignment between the substructure distributions and the smooth mass distributions. We leave the question of exactly how such a population of halos would be biased to future work (Chen et al., in preparation).
The Singular Isothermal Ellipsoid (SIE) is one of the simplest lens models that can produce quadruply imaged sources. Kormann et al. (1994) performed a detailed study of the lensing properties of SIE lenses, and, in particular, derived simple expressions for the total area contained within the tangential and radial caustics of such lenses. Specifically, given an SIE profile

\[ \Sigma = \frac{\sqrt{q} \sigma_t^2}{2G} \frac{1}{x^2 + \frac{3}{4}y^2}, \]  

Kormann et al. (1994) found that the area \( \sigma_r \) and \( \sigma_t \) contained inside the radial and tangential caustics is given by

\[ \sigma_r = 4q \int_0^{1/q} dx \cos^{-1}(x) \sqrt{x^2 - q}, \]  

and

\[ \sigma_t = 4q \int_0^{1/q} dx \left( \frac{\sqrt{1-x^2}}{x} - \cos^{-1}(x) \right) \sqrt{x^2 - q^2} \]

respectively. Moreover, the latter cross section requires one to compute the magnification distribution \( p(\mu) \) for double and quad lenses, for which there are no closed form expressions. In this appendix, we numerically compute the magnification distribution \( p(\mu) \), and its first moment \( \langle \mu \rangle \) for both doubles and quads, and use them to compute the biased lensing cross section \( B = \langle \mu \rangle \sigma_t \), for a source luminosity function \( n_s(L) \propto L^{-2} \).
The left panel of Figure 10 shows the magnification distribution for doubly and quadruply image systems for SIE profiles with axis ratios $q = 0.4$ and $q = 0.8$. Note that the magnification distribution for doubles is very rich in features. The magnification distribution for quads, on the other hand, is relatively simple, and we can provide a simple fitting formula for it. To do so, first note that we know that in the limit $\mu \to \infty$, $p(\mu) \propto \mu^{-\frac{3}{2}}$, so we expect that $p(\mu) \approx N \vec{x}^{-3} f(\vec{x})$ where $x = \mu/\mu_{\text{min}}$ and $\mu_{\text{min}}$ is the minimum magnification for quad lenses, $N$ is a normalization constant, and $f(\vec{x})$ is a function which asymptotes to unity and deviates from unity only for $x \approx 1$. Consequently, we expand $f(\vec{x})$ in a power series in terms of $x^{-1}$, of which we expect only the first few terms would be necessary to produce a good fit. As it turns out, we found that $f(\vec{x})$ needs only one non-constant term to result in excellent fits to $p(\mu)$, and our final fitting function for $p(\mu)$ is thus

$$p(\mu) \approx \frac{2}{\mu_{\text{min}}^3} \left(1 + \frac{a}{x^2}\right).$$

(A4)

A priori, we would expect that the best fit value of the $a$ coefficient in the above expression would be a function of the axis ratio $q$ of the profile. While there does appear to be some such dependence, it is extremely mild, so we have opted for keeping $a$ fixed to the value $a = 0.83$. We found that this expression is accurate to better than 5% for $\mu_{\text{min}} \leq \mu \leq 20\mu_{\text{min}}$ and $q \geq 0.4$.

The right panel of Figure 10 shows the actual quantities we are interested in, the biased lensing cross sections $\sigma_B = \langle \mu \rangle \sigma$. As is obvious from the figure, the form of these biased cross sections is very simple, so even a simple quadratic fit results in quite good fits (of order a few percent). Since we wish our empirical fit to be accurate, we fit the numerically computed cross sections with a cubic, which is enough to obtain sub-percent level accuracy. Our best fit curves (in a least square sense) are

$$\sigma_{B}^{(2)} = -6.902 + 42.937q - 33.240q^2 + 9.736q^3$$

(A5)

$$\sigma_{B}^{(4)} = 11.409 - 20.833q + 13.256q^2 - 3.816q^3.$$  

(A6)

Of course, we could have just as easily splined the numerically estimated values to compute the lensing cross section at any axis ratio $q$. We opted to fit the cross sections with a simple form both for simplicity, and in the chance that the fitting formulae provided here will be useful for other works.

PROJECTED SURFACE DENSITY PROFILES OF TRIAXIAL ISOTHERMAL HALOS

Consider an SIS profile

$$\rho_{\text{SIS}}(r) = \frac{\sigma_v^2}{2\pi G r^2}. \quad (B1)$$

Its triaxial generalization takes the form

$$\rho_{\text{SIE}}(\bar{x}) = N(q_1,q_2) \frac{\sigma_v^2}{2\pi G \bar{x}/q_1^2 + 1/\bar{x}^2 + 1/q_2^2} \quad \text{(B2)}$$

where $q_1$ and $q_2$ are the halo’s axis ratios, and we have chosen a coordinate system that is aligned the halo’s principal axis and such that $1 \geq q_1 \geq q_2$. $p_2$ remains the ratio of the small to large axis. The prefactor $N(q_1,q_2)$ represents a relative normalization for halos of varying axis ratios which we will compute shortly. First however, since [Kormann et al. 1994] use the notation where the axis ratios multiply rather than divide the coordinates, we rewrite the mass density as

$$\rho_{\text{SIE}}(\bar{x}) = \bar{N}(q_1,q_2) \frac{\sigma_v^2}{2\pi G p_1^2 \bar{x}^2 + p_2^2 \bar{x}^2 + 1} \quad \text{(B3)}$$

where $p_1 = q_1/q_2$ and $p_2 = q_2/q_1$.
where \( p_1 = q_2 / q_1, \) \( p_2 = q_2, \) and \( \tilde{N}(p_1, p_2) = q_2^2 N(q_1, q_2) \). Note \( p_1 \) is the ratio of the small to middle axis, while \( p_2 \) remains the ratio of the small to large axis. We choose the normalization function \( \tilde{N}(p_1, p_2) \) such that the mass contained within a radius \( r \) is independent of the axis ratios \( p_1 \) and \( p_2 \), as appropriate if one wishes to investigate the impact of triaxiality on lensing cross sections at fixed mass with the latter defined using spherical overdensities. Integrating the above profiles and setting \( M_{\text{SIS}}(r) = M_{\text{SIE}}(r) \) results in \(^{15}\)

\[
\tilde{N}(p_1, p_2) = \left\{ \frac{2}{\pi} \int_0^{\pi/2} d\phi \tan^{-1} \left( \frac{\sqrt{(1-a)/a}}{\sqrt{a(1-a)}} \right) \right\}^{-1}
\]

where we have defined \( a(\phi; q_1, q_2) \) via

\[
a(\phi; q_1, q_2) = p_1^2 \cos^2(\phi) + p_2^2 \sin^2(\phi).
\]

We wish to project \( \rho_{\text{SIE}} \) along an arbitrary line of sight. Let \( \mathbf{x} \) be a coordinate system such that the \( z \) axis is aligned with the line of sight. We choose the \( x \) and \( y \) axis to be such that a rotation by an angle \( \theta \) along the \( y \) axis followed by a rotation along the \( z \) axis by an angle \( \phi \) recovers the coordinate system \( \mathbf{x} \) from Eq. [B3]. The corresponding rotation matrix is given by

\[
R = \begin{pmatrix}
\cos \theta \cos \phi & -\sin \phi & \sin \theta \cos \phi \\
\cos \theta \sin \phi & \cos \phi & \sin \theta \sin \phi \\
-\sin \theta & 0 & \cos \theta
\end{pmatrix}.
\]

By construction, the corresponding projected surface density \( \Sigma(x, y) \) is given simply by

\[
\Sigma(x, y) = \int_{-\infty}^{\infty} dz \, \rho_{\text{SIE}}(Rz)
\]

which has the form

\[
\Sigma(x, y) = \tilde{N}(p_1, p_2) \frac{\sigma_v^2}{2\pi G} \int_{-\infty}^{\infty} dz \, \frac{1}{A + Bz + Cz^2}
\]

where

\[
A = A_{xx} x^2 + A_{xy} xy + A_{yy} y^2
\]

\[
B = B_x x + B_y y
\]

\[
C = p_1^2 \sin^2 \theta \cos^2 \phi + p_2^2 \sin^2 \theta \sin^2 \phi + \cos^2 \theta
\]

and

\[
A_{xx} = p_1^2 \cos^2 \theta \cos^2 \phi + p_2^2 \cos^2 \theta \sin^2 \phi + \sin^2 \theta
\]

\[
A_{xy} = \sin(2\phi) \cos(\theta)(-p_1^2 + p_2^2)
\]

\[
A_{yy} = p_1^2 \sin^2 \theta \cos^2 \phi + p_2^2 \cos^2 \theta
\]

\[
B_x = \sin(2\theta)(p_1^2 \cos^2 \phi + p_2^2 \sin^2 \phi - 1)
\]

\[
B_y = \sin(\theta)(\sin(2\phi)(-p_1^2 + p_2^2)).
\]

Note that if \( q_1 = q_2 = 1 \), then \( A_{xx} = A_{yy} = C = 1 \) and \( A_{xy} = B_x = B_y = 0 \), exactly as it should. Performing the integral in Eq. [B8] we find

\[
\Sigma(x, y) = \tilde{N}(p_1, p_2) \frac{\sigma_v^2}{2G} \frac{1}{\sqrt{AC - B^2/4}}
\]

which has the generic form

\[
\Sigma(x, y) = \tilde{N}(p_1, p_2) \frac{\sigma_v^2}{2G} \frac{1}{\sqrt{(\alpha_{xx} x^2 + \alpha_{xy} xy + \alpha_{yy} y^2)^2}}
\]

where

\[
\alpha_{xx} = A_{xx} C - B_x^2/4
\]

\[
\alpha_{xy} = A_{xy} C - B_x B_y/2
\]

\[
\alpha_{yy} = A_{yy} C - B_y^2/4
\]

For \( q_1 = q_2 = 1 \), the above expressions reduce to \( \alpha_{xx} = \alpha_{yy} = 1 \) and \( \alpha_{xy} = 0 \) as appropriate for a SIS profile. For the more general case it is evident from equationation [B13] that using an additional rotation of the \( x-y \) plane we can diagonalize the projected mass density \( \Sigma(x, y) \). We find that the required rotation angle \( \psi \) is given by

\[
\tan 2\psi = \frac{\alpha_{xy}}{\alpha_{xx} - \alpha_{yy}}.
\]

\(^{15}\) To obtain the expressions above, we perform first the radial integral and then the \( \theta \) integral where \( \theta \) is the azimuthal angle.
Using a ∼ to denote the new coordinate system, we can thus write

$$\Sigma(\tilde{x}, \tilde{y}) = \frac{\sqrt{q\tilde{\sigma}_v^2}}{2G} \frac{1}{(\tilde{x}^2 + q\tilde{y}^2)^{1/2}}$$  \hspace{1cm} (B23)

where

$$q^2 = \frac{\tilde{\alpha}_{yy}}{\tilde{\alpha}_{xx}}$$  \hspace{1cm} (B24)

$$\tilde{\sigma}_v^2 = \frac{\tilde{N}(p_1, p_2)}{q\tilde{\alpha}_{xx}}\sigma_v^2$$  \hspace{1cm} (B25)

and we have defined

$$\tilde{\alpha}_{xx} = \alpha_{xx} \cos^2 \psi + \alpha_{xy} \sin \psi \cos \psi + \alpha_{yy} \sin^2 \psi$$  \hspace{1cm} (B26)

$$\tilde{\alpha}_{yy} = \alpha_{xx} \sin^2 \psi - \alpha_{xy} \sin \psi \cos \psi + \alpha_{yy} \cos^2 \psi.$$  \hspace{1cm} (B27)

As expected, the above expression for $q$ reduces to $q = p_2/p_1 = q_1$ when we project along the $z$ axis (i.e. the short axis), to $q = p_2$ when projecting along the $y$ axis (i.e. the long axis), and to $q = p_1 = q_2/q_1$ when projecting along the $x$ axis (i.e. the middle axis). The particular form of the parameterization of the surface density in Eq. [B23] is meant to match the conventions in Kormann et al. (1994), which was chosen to ensure the mass contained within a given density contour be independent of $q$ for fixed $\sigma_v$. and the