Infrared structure of $e^+e^- \to 3$ jets at NNLO: QED-type contributions

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The NNLO QCD corrections to the $e^+e^- \to 3$ jets can be decomposed according to their colour factors. Out of the seven colour factors, three are of QED-type: $1/N^2$, $N_F/N$ and $N_F^2$. We use the antenna subtraction method to compute these contributions, providing complete expressions for the subtraction terms in $N_F/N$ and $N_F^2$.

1. Introduction

Electron-positron annihilation into three jets provides one of the most precise experimental tests of QCD dynamics. The extraction of the strong coupling constant $\alpha_s$ from data on this observable [1] is currently limited [2] by the uncertainty on the available theoretical next-to-leading order (NLO) calculations. To improve upon this situation, it is mandatory to compute next-to-next-to-leading order (NNLO) QCD corrections to $e^+e^- \to 3$ jets.

This calculation contains three ingredients [3]: matrix elements for $\gamma^* \to 5$ partons at tree-level, for $\gamma^* \to 4$ partons at one-loop and $\gamma^* \to 3$ partons at two loops. The four-parton and five-parton contributions contain infrared singularities due to real radiation of massless final state partons. To extract these singularities, which cancel with explicit infrared singularities in the virtual corrections for any infrared-safe observable, one needs to define a subtraction scheme at NNLO. We proposed a formulation of the antenna subtraction method [4,5] to NNLO in [6].

2. Antenna subtraction

In the antenna subtraction method, antenna functions describe the colour-ordered radiation of unresolved partons between a pair of hard (radiation) partons. All antenna functions at NLO and NNLO can be derived systematically from physical matrix elements, as shown in [6,7]. They can be integrated over the factorized antenna phase space [8] using loop integral reduction techniques extended to phase space integrals [9,10], and then combined with virtual corrections to partonic processes with lower multiplicity.

3. Colour structure of $e^+e^- \to 3$ jets

Decomposing the three-jet cross section (and related event shape observables) at NNLO according to the QCD colour structures one finds seven colour factors: $N^2$, $N_0$, $1/N^2$, $N_F/N$, $N_F/N^2$ and $N_F\gamma$. The last colour factor $N_F\gamma$ arises from the interference of amplitudes with two independent quark lines coupling to the exchanged vector boson. This colour factor is infrared finite and numerically small in the three-parton channel [11], and contributes only a negligible amount to four-jet observables at NLO [12]. It can therefore be neglected safely.

Among the remaining six colour factors, three do not receive contributions from diagrams with gluon self-coupling, and are therefore of QED-type: $1/N^2$, $N_F/N$ and $N_F^2$. In [8,6], we described the calculation of NNLO corrections in the $1/N^2$ colour factor using the antenna subtraction method. In the present paper, we describe the calculation of NNLO corrections in the $N_F/N$ and $N_F^2$ colour factors, thus completing the corrections to all QED-type colour factors. Our notation for partonic matrix elements, antenna functions and subtraction terms follows the notation introduced in [6].
4. Subtraction in the $N_F/N$ colour factor

The $N_F/N$ colour factor receives contributions from five-parton tree-level $\gamma^* \rightarrow q\bar{q}q'\bar{q}'g$, four-parton one-loop $\gamma^* \rightarrow q\bar{q}q'g^*$ and $\gamma^* \rightarrow q\bar{q}gg$ at subleading colour as well as tree-level two-loop $\gamma^* \rightarrow q\bar{q}gg$. The gluon emissions are all photon-like.

4.1. Five-parton contribution

The NNLO radiation term appropriate for the three jet final state is given by

\[ \frac{d\sigma_{NNLO,N_F/N}}{dN_F} = N_5 \frac{N_F}{N} \frac{d\Phi_5(p_1,\ldots,p_5;q)}{2} \left\{ \begin{array}{c}
-A^0_5(1,q,5,g,2q_2) B^0_5(15,2g_3,5g,4g_4) J^{(5)}_3(p_1,\ldots,p_5) \\
-A^0_5(i,q',5,q_3) B^0_5(1,q,2g_3,5g,4g_4) J^{(4)}_3(p_1,\ldots,p_5) \\
-\frac{1}{2} \bigg\{ E^0_3(1,q,i,q',2g_2) A^0_3(15,5g,2g_3,4g_4) J^{(5)}_3(p_1,\ldots,p_5) + (1 \leftrightarrow 2) \\
- \left( B^0_3(1,q,i,q',2g_2) - \frac{1}{2} \left( E^0_3(1,q,i,q',2g_2) A^0_3(15,5g,2g_3,4g_4) J^{(5)}_3(p_1,\ldots,p_5) + (1 \leftrightarrow 2) \right) \right) \\
\times A^0_3((1i)q,5g,2i) J^{(3)}_3(p_1i,\ldots,p_5i) \bigg\} + (1 \leftrightarrow 2) \\
+ \frac{1}{2} \bigg\{ E^0_3(1,q,i,q',2g_2) A^0_3((1i)q,5g,2g_3,4g_4) J^{(5)}_3(p_1,\ldots,p_5) + (1 \leftrightarrow 2) \\
- \frac{1}{2} \bigg\{ \tilde{E}^0_3(1,q,i,q',5g) - A^0_3(i,q',5g,2g_3,4g_4) E^0_3(15,5g,2g_3,4g_4) J^{(5)}_3(p_1i,\ldots,p_5i) + (1 \leftrightarrow 2) \right) \\
\times A^0_3((1i)q,5g,2g_2) J^{(3)}_3(p_1i,\ldots,p_5i) + (1 \leftrightarrow 2) \right) \bigg\} \\
\end{array} \right\} \quad (1) \]

where the symmetrisation over the momenta of the secondary quark-antiquark pair exploits the fact that the jet algorithm does not distinguish quarks and antiquarks. This symmetrisation reduces the number of non-vanishing unresolved limits considerably, since the interference term in $B^0_5$ is odd under this interchange. The subtraction term reads:

\[ \frac{d\sigma^R_{NNLO,N_F/N}}{dN_F} = N_5 \frac{N_F}{N} \frac{d\Phi_5(p_1,\ldots,p_5;q)}{2} \sum_{(i,j)\in(3,4)} \right\{ \\
\end{array} \right\} \quad (2) \]

4.2. Four-parton contribution

The four parton contribution to the $N_F/N$ colour factor reads:

\[ \frac{d\sigma^V_{NNLO,N_F/N}}{dN_F} = N_4 \frac{N_F}{N} \frac{d\Phi_4(p_1,\ldots,p_4;q)}{2\pi} \left\{ \begin{array}{c}
- \frac{1}{2} \sum_{(i,j)\in(3,4)} \left( B^1_4(1,q,i,q',2g_2) \right) \\
\end{array} \right\} \]

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\[ + 2C_4^{ij}(1_{\epsilon}, i_{\epsilon}, j_{\epsilon}, 2_{\epsilon}) + \tilde{A}_4^{ec}(1_{\epsilon}, i_{\epsilon}, j_{\epsilon}, 2_{\epsilon}) \} \cdot J_3^{(4)}(p_1, \ldots, p_4). \] (3)

The expression is symmetrised over the momenta (3) and (4) to remove terms which are antisymmetric under charge conjugation, and can not be accounted for properly by the quark-gluon antenna functions.

The corresponding subtraction term is:

\[
d\sigma_{N N L O, N F/N}^{V S, 1} = N_4 \frac{N_F}{N} \left( \frac{\alpha_s}{2\pi} \right) \int d\Phi_4(p_1, \ldots, p_4; q)
\times \left\{ A_3^0(1_{\epsilon}, i_{\epsilon}, j_{\epsilon}, 2_{\epsilon}) B_3^0(1_{\epsilon}, i_{\epsilon}, j_{\epsilon}, 2_{\epsilon}) J_3^{(4)}(p_1, \ldots, p_4)
\right.
\]
\[
+ \frac{1}{4} \left[ E_3^0(s_{11}) + E_3^0(s_{12}) + E_3^0(s_{21}) + E_3^0(s_{22}) \right] \tilde{A}_3^0(1_{\epsilon}, i_{\epsilon}, j_{\epsilon}, 2_{\epsilon}) J_3^{(4)}(p_1, \ldots, p_4)
\] \[\left. - \frac{1}{2} \left( E_3^0(1_{\epsilon}, i_{\epsilon}, j_{\epsilon}, 2_{\epsilon}) \tilde{A}_3^0((13)_q, (34)_g, 2_{\epsilon}) + A_3^0((13)_q, (34)_g, 2_g) \cdot J_3^{(3)}(\tilde{p}_{13}, \tilde{p}_{34}, p_2) + (1 \leftrightarrow 2) \right) \right\}
\]
\[
- \frac{1}{2} \sum_{(i,j) \in (3,4)} \left( A_3^0(1_{\epsilon}, i_{\epsilon}, j_{\epsilon}, 2_{\epsilon}) \tilde{A}_3^0((11)_q, j_{\epsilon}, (2i)_q) + \tilde{A}_3^0(1_{\epsilon}, i_{\epsilon}, j_{\epsilon}, 2_{\epsilon}) \right)
\]
\[
+ \frac{1}{2} \left( E_3^0(s_{11}) + E_3^0(s_{12}) + E_3^0(s_{21}) + E_3^0(s_{22}) \right) A_3^0(1_{\epsilon}, i_{\epsilon}, j_{\epsilon}, 2_{\epsilon}) J_3^{(3)}(\tilde{p}_{11}, \tilde{p}_{21}, p_j)
\] \[+ 2b_{0,F} \log^2 \frac{q^2}{s_{12}} \left( A_3^0(1_{\epsilon}, i_{\epsilon}, j_{\epsilon}, 2_{\epsilon}) A_3^0((11)_q, j_{\epsilon}, (2i)_q) \right) J_3^{(3)}(\tilde{p}_{11}, \tilde{p}_{21}, p_j) \] (4)

The terms in the second and third line of this equation, as well as in the second-last line are obtained by integrating terms from the five-parton subtraction term (2). All remaining terms are integrated to yield three-parton contributions.

4.3. Three-parton contribution

The three parton contribution to the $N_F/N$ colour factor consists of the three-parton virtual two-loop correction and the integrated five-parton tree-level and four-parton one-loop subtraction terms, which read

\[
d\sigma_{N N L O, N F/N}^{SS} = \frac{N_F}{N} \int d\Phi_3(p_1, p_2, p_3)
\times \left\{ - B_3^0(s_{12}) - \frac{1}{2} E_4^0(s_{13}) - \frac{1}{2} E_4^0(s_{23}) - \frac{1}{2} A_3^0(s_{12}) \left( E_3^0(s_{13}) + E_3^0(s_{23}) \right)
\right.
\]
\[
- \left( A_3^0(1_{\epsilon}, 3_g, 2_g) + \frac{1}{2} E_3(1_{\epsilon}, 3_g, 2_g) \right)
\]
\[
- A_3^0(s_{12}) A_3^0(1_{\epsilon}, 3_g, 2_g) - b_{0,F} \cdot \log \frac{q^2}{s_{12}} A_3^0((s_{12})^{-\epsilon} - (s_{123})^{-\epsilon}) A_3^0(1_{\epsilon}, 3_g, 2_g) \right\} d\sigma_3. \] (5)

Taking the infrared pole part of this expression, we obtain cancellation of all infrared poles in this
channel:
\[
\mathcal{Poles} \left( d\sigma^{V,1}_{NNLO,N_F/N} \right) + \mathcal{Poles} \left( d\sigma^{V,2}_{NNLO,N_F/N} \right) + \mathcal{Poles} \left( d\sigma^{V,3}_{NNLO,N_F/N} \right) = 0.
\] (6)

5. Subtraction in the \( N_F^2 \) colour factor

The \( N_F^2 \) colour factor receives contributions only from the four-parton one-loop process \( \gamma^* \rightarrow q\bar{q}q'\bar{q}' \) and from the three-parton two-loop process \( \gamma^* \rightarrow q\bar{q}g \).

5.1. Four-parton contribution

The four-parton one-loop contribution to this colour factor is
\[
d\sigma^{V,1}_{NNLO,N_F} = N_F^2 \left( \frac{O_5}{2\pi} \right) d\Phi_4(p_1,\ldots,p_4;q) B_4^{1,V}(1,q,3,q',4,2) J_3^{(1)}(p_1,\ldots,p_4).
\] (7)

This contribution is free of explicit infrared poles (as can be inferred from the absence of a five-parton contribution to this colour structure).

The subtraction term appropriate to this contribution is
\[
d\sigma^{V,3}_{NNLO,N_F} = N_F^2 \left( \frac{O_5}{2\pi} \right) d\Phi_4(p_1,\ldots,p_4;q) \frac{1}{2} \left\{ \left[ \left( E_3^1(1,q,3,q') + 2 b_0,F \log \frac{q^2}{s_{134}} E_3^0(1,q,3,q') \right) A_3^0(\langle 13 \rangle_q, \langle 34 \rangle_g, 2) \right.ight.
\[
+ E_3^0(1,q,3,q') A_3^1(\langle 13 \rangle_q, \langle 34 \rangle_g, 2) \right\} J_3^{(3)}(p_\bar{1},p_\bar{3},p_4,p_2)
\]
\[
+ \left[ \left( E_3^1(2,q,3,q') + 2 b_0,F \log \frac{q^2}{s_{234}} E_3^0(2,q,3,q') \right) A_3^0(1,q,\langle 34 \rangle_g, \langle 23 \rangle_q) \right.
\[
+ E_3^0(2,q,3,q') A_3^1(1,\langle 34 \rangle_g, \langle 23 \rangle_q) \right\} J_3^{(3)}(p_1,p_\bar{3},p_4,p_\bar{2}) \right\}.
\] (8)

Although \( E_3^1 \) and \( A_3^1 \) contain explicit infrared poles, these cancel in their sum, as can be seen from (5.16) and (6.32) of [6]. \( d\sigma^{V,3}_{NNLO,N_F} \) is therefore free of explicit infrared poles.

5.2. Three-parton contribution

The three parton contribution to the \( N_F^2 \) colour factor consists of the three-parton virtual two-loop correction and the integrated four-parton one-loop subtraction term, which reads
\[
d\sigma^{V,1}_{NNLO,N_F} = N_F^2 \left( \frac{O_5}{2\pi} \right) \times \left\{ \left( \hat{E}^1_3(s_{13}) + \hat{E}^1_3(s_{23}) \right) A^0_3(1,q,3,q,2) + \left( e^0_3(s_{13}) + e^0_3(s_{23}) \right) \hat{A}^1_3(1,q,3,q,2) \right.
\]
\[
+ \frac{b_0,F}{\epsilon} \left[ \epsilon^0_3(s_{13}) (\langle s_{13} \rangle^{-\epsilon} - \langle s_{13} \rangle^{-\epsilon}) + \epsilon^0_3(s_{23}) (\langle s_{23} \rangle^{-\epsilon} - \langle s_{23} \rangle^{-\epsilon}) \right] A^0_3(1,q,3,q,2) \right\} d\sigma_3.
\] (9)

Taking the infrared pole part of this expression, we obtain cancellation of all infrared poles in this channel:
\[
\mathcal{Poles} \left( d\sigma^{V,1}_{NNLO,N_F} \right) + \mathcal{Poles} \left( d\sigma^{V,2}_{NNLO,N_F} \right) = 0.
\] (10)
6. Numerical implementation

Starting from the program EERAD2 [4], which computes the four-jet production at NLO, we implemented the NNLO antenna subtraction method for all QED-type colour factor contributions to $e^+e^- \rightarrow 3$ jets. EERAD2 already contains the five-parton and four-parton matrix elements relevant here, as well as the NLO-type subtraction terms.

The implementation contains three channels, classified by their partonic multiplicity: (a) in the five-parton channel, we integrate $d\sigma^{R}_{NNLO} - d\sigma^{S}_{NNLO}$; (b) in the four-parton channel, we integrate $d\sigma^{V,1}_{NNLO} - d\sigma^{V,S,1}_{NNLO}$; (c) in the three-parton channel, we integrate $d\sigma^{V,2}_{NNLO}$ + $d\sigma^{S}_{NNLO}$ + $d\sigma^{V,S,1}_{NNLO}$. The numerical integration over these channels is carried out by Monte Carlo methods.

By construction, the integrands in the four-parton and three-parton channel are free of explicit infrared poles. In the five-parton and four-parton channel, we tested the proper implementation of the subtraction by various checks. To verify the local convergence of the subtraction terms and to verify that cancellations among individual contributions to the subtraction terms take place as expected, we examined the behaviour of matrix element and subtraction terms along various trajectories approaching the unresolved limits. Moreover, we checked the correctness of the subtraction by introducing a lower cut (slicing parameter) on the phase space variables, and observing that our results are independent of this cut (provided it is chosen small enough). This behaviour indicates that the subtraction terms ensure that the contribution of potentially singular regions of the final state phase space does not contribute to the numerical integrals, but is accounted for analytically. Finally, distributions in double and triple invariants of the five-parton or four-parton phase space illustrate the proper onset of the subtraction terms towards the single and double unresolved edges of phase space.

7. Outlook

In this paper, we described the calculation of NNLO corrections to $e^+e^- \rightarrow 3$ jets in all QED-type colour factors: $1/N^2$, $N_F/N$ and $N_F^2$, using the antenna subtraction method [6]. We documented explicit expressions for the subtraction terms for $N_F/N$ and $N_F^2$ ($1/N^2$ was documented in [6] already) and described their implementation in a numerical program computing three-jet cross sections and related event shape observables. Work on the remaining colour factors is ongoing.

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