On the structure of 1-perfectly orientable graphs∗

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Abstract

We study the class of 1-perfectly orientable (1-p.o.) graphs, that is, graphs having an orientation in which every out-neighborhood induces a tournament. 1-p.o. graphs form a common generalization of chordal graphs and circular arc graphs. Even though 1-p.o. graphs can be recognized in polynomial time, little is known about their structure. In this paper, we prove several structural results about 1-p.o. graphs and characterizations of 1-p.o. graphs in special graph classes. This includes: (i) a characterization of 1-p.o. graphs in terms of edge clique covers, (ii) identification of several graph transformations preserving the class of 1-p.o. graphs, (iii) a complete characterization of 1-p.o. cographs and of 1-p.o. complements of forests, and (iv) an infinite family of minimal forbidden induced minors for the class of 1-p.o. graphs.

Keywords: structural characterization of families of graphs, 1-perfectly orientable graph, fraterinally orientable graph, in-tournament digraph, cographs, forests

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1 Introduction

Many graph classes can be defined with the existence of orientations satisfying certain properties (see, e.g., [1]). In this paper, we study graphs having an orientation that is an out-tournament, that is, a digraph in which the out-neighborhood of every vertex induces an orientation of a complete graph. (An in-tournament is defined similarly.) Following the terminology of Kammer and Tholey [24], we say that an orientation of a graph is 1-perfect if the out-neighborhood of every vertex induces a tournament, and that a graph is 1-perfectly orientable (1-p.o. for short) if it has a 1-perfect orientation. In [24], the authors introduced a hierarchy of graph classes, with 1-p.o. graphs being the first member of the family. Namely, they defined a graph to be k-perfectly orientable if in

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some orientation of it every out-neighborhood induces a disjoint union of at most \( k \) tournaments. The paper [24] develops several approximation algorithms for optimization problems on \( k \)-perfectly orientable graphs and related classes.

The notion of 1-p.o. graphs was introduced by Skrien [36] (under the name \( \{B_2\} \)-graphs), where the problem of characterizing 1-p.o. graphs was posed. By definition, 1-p.o. graphs are exactly the graphs that admit an orientation that is an out-tournament. A simple arc reversal argument shows that that 1-p.o. graphs are exactly the graphs that admit an orientation that is an in-tournament. Such orientations were called \textit{fraternal orientations} in several papers [14–17,31,32,38]. 1-p.o. graphs are also exactly the underlying graphs of the so-called locally in-semicomplete digraphs (which are defined similarly as in-tournaments, except that pairs of oppositely oriented arcs are allowed), see [1].

As observed in [2, Theorem 5.1], 1-p.o. graphs can be recognized in polynomial time via a reduction to 2-SAT. A polynomial time algorithm for recognizing 1-p.o. graphs that works directly on the graph was given by Urrutia and Gavril [38]. Bang-Jensen et al. [2] (see also [34]) proved a topological property of 1-p.o. graphs (stating that every 1-p.o. graph is 1-homotopic), and that every graph representable as the intersection graph of connected subgraphs of unicyclic graphs is 1-p.o. This implies that all chordal graphs and all circular arc graphs are 1-p.o., as observed already in [38] and in [36], respectively. (In fact, a graph \( G \) is chordal if and only if it admits a 1-perfect \textit{acyclic} orientation.) It was also shown in [2] that every graph having a unique induced cycle of order at least 4 is 1-p.o.

Since 1-p.o. graphs form a generalization of chordal graphs and of circular arc graphs, two well studied graph families for which many structural and algorithmic results are known, a further understanding of structural and algorithmic properties of 1-p.o. graphs is of interest. Known polynomial time recognition algorithms of 1-p.o. graphs [2,38] do not give much insight into the structure of 1-p.o. graphs. Neither a constructive structural characterization of 1-p.o. graphs nor the list of minimal forbidden induced subgraphs are known. A characterization of 1-p.o. line graphs and of 1-p.o. triangle-free graphs appears in [2]. Also, the subclass of 1-p.o. graphs consisting of graphs that admit an orientation that is both an in-tournament and an out-tournament was characterized [36] (see also [22]): this class coincides with the class of proper circular arc graphs.

In Section 3, we give a characterization of 1-p.o. graphs in terms of edge clique covers, similar to a known characterization of squared graphs due to Mukhopadhyay.

In Section 4 we exhibit several examples of 1-p.o. and non-1-p.o. graphs. The examples of non-1-p.o. graphs consist of three specific graphs on 6, 7, and 10 vertices, respectively, and two infinite families: the complements of even cycles of length at least 6, and the complements of the graphs obtained from odd cycles by adding a component consisting of a single edge.

In Section 5 we identify several graph transformations preserving the class of 1-p.o. graphs. We also study the behavior of 1-p.o. graphs under some operations that in general do not preserve the class, such as pasting along a clique and the join. The result for the join (Proposition 11) motivates the problem of characterizing the 1-p.o. co-bipartite graphs.

Our main results are developed in Section 6, where, based on results developed in earlier sections, we obtain a complete characterization of 1-p.o. graphs within the classes of cographs and of complements of forests.

The list of graph operations preserving the class of 1-p.o. graphs derived in Section 5 implies
that the class of 1-p.o. graphs is closed under taking induced minors. We conclude the paper in Section 7 by showing that all the examples of non-1-p.o. graphs from Section 4 are minimal forbidden induced minors of the class of 1-p.o. graphs.

2 Preliminaries

All graphs in this paper are simple and finite, but may be either directed (in which case we will refer to them as digraphs) or undirected (in which case we will refer to them as graphs). We use standard graph and digraph terminology. In this section, we recall the definitions of some of the most used notions in this paper. For further background on graphs, we refer to [7], on graph classes, to [5,20], and on digraphs, to [1].

An edge in a graph (resp. an arc in a digraph) connecting vertices $u$ and $v$ will be denoted by $\{u,v\}$ or simply $uv$ (resp., $(u,v)$ or simply $uv$). We will also use the notation $u \rightarrow v$ to denote the fact that an edge $uv$ of a graph $G$ is oriented from $u$ to $v$ in an orientation of $G$. The set of all vertices adjacent to a vertex $v$ in a graph $G$ will be denoted by $N_G(v)$, and its cardinality, the degree of $v$ in $G$, by $d_G(v)$. The closed neighborhood of $v$ in $G$ is the set $N_G(v) \cup \{v\}$, denoted by $N_G[v]$. An orientation of a graph $G = (V,E)$ is a digraph $D = (V,A)$ obtained by assigning a direction to each edge of $G$. A tournament is an orientation of a complete graph. Given a digraph $D$, the in-neighborhood of a vertex $v$ in $D$, denoted by $N_D^-(v)$, is the set of all vertices $w$ such that $(w,v) \in A$. Similarly, the out-neighborhood of $v$ in $D$ is the set of all vertices $w$ such that $(v,w) \in A$. The cardinalities of the in- and the out-neighborhood of $v$ are the in-degree and the out-degree of $v$ and are denoted by $d_D^-(v)$ and $d_D^+(v)$, respectively.

Given two graphs $G$ and $H$, their disjoint union is the graph $G \cup H$ with vertex set $V(G) \cup V(H)$ (disjoint union) and edge set $E(G) \cup E(H)$. We write $2G$ for $G + G$. The join of two graphs $G$ and $H$ is the graph denoted by $G \ast H$ and obtained from the disjoint union of $G$ and $H$ by adding to it all edges joining vertex of $G$ with a vertex of $H$. Given a graph $G$ and a subset $S$ of its vertices, we denote by $G[S]$ the subgraph of $G$ induced by $S$, that is, the graph with vertex set $S$ and edge set $\{uv \in E(G) \mid u,v \in S\}$. By $G - S$ we denote the subgraph of $G$ induced by $V(G) \setminus S$, and when $S = \{v\}$ for a vertex $v$, we also write $G - v$. The graph $G/e$ obtained from $G$ by contracting an edge $e = uv$ is defined as $G/e = (V,E)$ where $V = (V(G) \setminus \{u,v\}) \cup \{w\}$ with $w$ a new vertex and $E = E(G - \{u,v\}) \cup \{wx \mid x \in N_G(u) \cap N_G(v)\}$.

A clique (resp., independent set) in a graph $G$ is a set of pairwise adjacent (resp., non-adjacent) vertices of $G$. The complement of a graph $G$ is the graph $\overline{G}$ with the same vertex set as $G$ in which two distinct vertices are adjacent if and only if they are not adjacent in $G$. The fact that two graphs $G$ and $H$ are isomorphic to each other will be denoted by $G \cong H$. In this paper we will often not distinguish between isomorphic graphs. The path, the cycle, and complete graph on $n$ vertices will be denoted by $P_n$, $C_n$, and $K_n$, respectively, and the complete bipartite graph with parts of size $m$ and $n$ by $K_{m,n}$.

A connected graph is said to be unicyclic if it has exactly one cycle. The following proposition from [2] will be used in some out our proofs.

Proposition 1. Every unicyclic graph is 1-p.o.
3 A characterization of 1-p.o. graphs in terms of edge clique covers

A graph $G$ is said to have a square root if there exists a graph $H$ with $V(H) = V(G)$ such that for all $u,v \in V(G)$, we have $uv \in E(G)$ if and only if the distance in $H$ between $u$ and $v$ is either 1 or 2. An edge clique cover in a graph $G$ is a collection of cliques $\{C_1, \ldots, C_k\}$ in $G$ such that every edge of $G$ belongs to some clique $C_i$. In this section, we characterize 1-p.o. graphs in terms of edge clique covers, in a spirit similar to the well known Mukhopadhyay’s characterization of graphs admitting a square root, which we now recall.

**Theorem 2** (Mukhopadhyay [30]). A graph $G$ with $V(G) = \{v_1, \ldots, v_n\}$ has a square root if and only if $G$ has an edge clique cover $\{C_1, \ldots, C_n\}$ such that the following two conditions hold:

(a) $v_i \in C_i$ for all $i$,

(b) for every edge $v_iv_j \in E(G)$, we have $v_i \in C_j$ if and only if $v_j \in C_i$.

In the original statement of the theorem, the second condition is required for all $i \neq j$, but since $v_iv_j \notin E(G)$ clearly implies $v_i \notin C_j$ and $v_j \notin C_i$, the equivalence in condition (b) automatically holds for all non-adjacent vertex pairs.

**Theorem 3.** For every graph $G$ with $V(G) = \{v_1, \ldots, v_n\}$, the following conditions are equivalent:

(i) $G$ is a 1-p.o. graph.

(ii) $G$ has an edge clique cover $\{C_1, \ldots, C_n\}$ such that the following two conditions hold:

(a) $v_i \in C_i$ for all $i$,

(b) for every edge $v_iv_j \in E(G)$, we have $v_i \in C_j$ or $v_j \in C_i$, but not both.

(iii) $G$ has an edge clique cover $\{C_1, \ldots, C_n\}$ such that the following two conditions hold:

(a) $v_i \in C_i$ for all $i$,

(b) for every edge $v_iv_j \in E(G)$, we have $v_i \in C_j$ or $v_j \in C_i$.

Three remarks are in order before we proceed to the proof of Theorem 3. First, note that the difference between Theorem 2 and the equivalence of conditions (i) and (iii) in Theorem 3 consists in replacing the equivalence in condition (b) of Theorem 2 with disjunction. This seemingly minor difference is in sharp contrast with the fact that recognizing graphs admitting a square root is NP-complete [29], while the recognition of 1-p.o. graphs can be done in polynomial time. Second, recall that a pointed set is a pair $(S,v)$ where $S$ is a nonempty set and $v \in S$. To every family $\mathcal{S} = \{(S_1, v_1), \ldots, (S_n, v_n)\}$ of pointed sets, one can associate a graph, the so called catch graph of $\mathcal{S}$ by setting $V(G) = \{v_1, \ldots, v_n\}$ and joining two distinct vertices $v_i$ and $v_j$ if and only if $v_i \in S_j$ or $v_j \in S_i$ (see, e.g. [28]). The equivalence between (i) and (iii) in the above theorem implies that every 1-p.o. graphs is the catch graph of a family of pointed sets. Third, conditions (a) and (b) in (iii) of Theorem 3 imply that the condition that $\{C_1, \ldots, C_n\}$ is a clique cover could be replaced by the seemingly weaker condition asserting that each $C_i$ is a clique (the condition that every edge belongs to one of the cliques would then follow).
Proof of Theorem 3. First, we show that (i) implies (ii). Given a 1-perfect orientation $D$ of a 1-p.o. graph $G$ with $V(G) = \{v_1, \ldots, v_n\}$, we define an edge clique cover $\{C_1, \ldots, C_n\}$ of $G$ by setting $C_i = \{v_i\} \cup N_D^+(v_i)$. By definition, each $C_i$ contains $v_i$, and, since $D$ is 1-perfect, is a clique in $G$. Note that for all $i \neq j$, we have $v_j \in C_i$ if and only if $(v_i, v_j) \in A(D)$. In particular, if $v_j \in C_i$ and $v_i \in C_j$, since for every edge $v_iv_j \in E(G)$, we have either $(v_i, v_j) \in A(D)$ or $(v_j, v_i) \in A(D)$ but not both, the second condition in (iii) follows.

The implication (ii) $\Rightarrow$ (iii) is trivial.

Now, we show that (iii) implies (i). Suppose that $G$ has an edge clique cover $\{C_1, \ldots, C_n\}$ such that $v_i \in C_i$ for all $i$, and for every edge $v_iv_j \in E(G)$, $v_i \in C_j$ or $v_j \in C_i$. We define an orientation of $D$ as follows: for $1 \leq i < j \leq n$ such that $v_iv_j \in E(G)$, set $(v_i, v_j) \in A(D)$ if $v_j \in C_i$, and $(v_j, v_i) \in A(D)$, otherwise. By definition, for every vertex $v_i \in V(G)$ we have
\[
N_D^+(v_i) = \{v_j \mid j < i \land v_i \notin C_j\} \cup \{v_j \mid j > i \land v_j \in C_i\}
\subseteq \{v_j \mid j < i \land v_j \in C_i\} \cup \{v_j \mid j > i \land v_j \in C_i\} \subseteq C_i,
\]
where the first inclusion relation holds due to the second condition in (iii). Hence, $D$ is a 1-perfect orientation of $G$, and $G$ is 1-p.o.

\[\square\]

Corollary 4. For every graph $G$ with $V(G) = \{v_1, \ldots, v_n\}$, the following conditions are equivalent:

(i) $G$ is the complement of a 1-p.o. graph.

(ii) $G$ has a collection of independent sets $\{I_1, \ldots, I_n\}$ such that the following two conditions hold:

(a) $v_i \in I_i$ for all $i$,

(b) for every non-adjacent vertex pair $v_iv_j \in E(G)$, we have $v_i \in I_j$ or $v_j \in I_i$.

Corollary 5. The edges of every 1-p.o. graph with $n$ vertices can be covered by $n$ cliques.

Note that the converse of Corollary 5 does not hold. For example, the complement of the 10-vertex graph $S_3$ (see Fig. 1 on p. 6) is not 1-p.o. (see Proposition 8), but can be edge-covered with (at most) 9 cliques. For a characterization of graphs with $n$ vertices the edges of which can be covered by $n$ cliques (the recognition of which is NP-complete), see [9].

4 Examples of 1-p.o. and non-1-p.o. graphs

Recall that all graphs containing at most one induced cycle of order at least four are 1-p.o., as are all circular arc graphs [36,38]. The fact that every circular arc graph is 1-p.o. has the following consequence.

Proposition 6. The complement of every odd cycle is 1-p.o.

Proof. Recall that the $k$-th power of a graph $G$ is the graph with the same vertex set as $G$ in which two distinct vertices are adjacent if and only if their distance in $G$ is at most $k$. It is easy to see (and also follows from the fact that the class of circular arc graphs is closed under taking powers) that all powers of cycles are circular arc graphs. Therefore, the fact that the complement of every odd cycle is 1-p.o. follows from two facts: (i) that the complement of $C_3$ is 1-p.o., and (ii) for every
$k \geq 2$, the complement of the odd cycle $C_{2k+1}$ is isomorphic to a power of a cycle, namely to $C_{2k+1}^{k-1}$.
(An explicit isomorphism from $C_{2k+1}$ to $C_{2k+1}^{k-1}$, with vertices of $C_{2k+1}$ labeled $0, 1, \ldots, 2k$ in the cyclic order, is given by the map $x \mapsto kx \pmod{2k+1}$.)

In the rest of the section, we identify some non-1-p.o. graphs. In Section 7, we will prove that each of this graphs is a minimal forbidden induced minor for the class of 1-p.o. graphs.

We start with a simple necessary condition for the 1-p.o. property. We say that a chordless cycle $C$ in a graph $G$ is oriented cyclically (see Zou [40]) in an orientation $D$ of $G$ if every vertex of the cycle has exactly one out-neighbor on the cycle.

Lemma 7. In every 1-perfect orientation $D$ of a 1-p.o. graph $G$, every chordless cycle of length at least four is oriented cyclically.

Proof. Suppose that a chordless cycle $C$ in $G$ is not oriented cyclically in some 1-perfect orientation $D$ of $G$. Let $C'$ be the orientation of $C$ induced by $D$. By assumption, $C$ contains a vertex $v$ with $d_{C'}^+(v) \neq 1$. Since $\sum_{u \in V(C)} d_{C'}^+(u) = |A(C')| = |E(C)| = |V(C)|$, it is not possible that $d_{C'}^+(u) \leq 1$ for all $u \in V(C)$, as this would imply $d_{C'}^+(v) = 0$ and consequently $\sum_{u \in V(C)} d_{C'}^+(u) < |V(C)|$. Thus, $C$ contains a vertex $v$ with $d_{C'}^+(v) = 2$. Since $C$ is of length at least 4 and chordless, the out-neighborhood of $v$ in $C'$, and hence in $D$, is not a clique in $G$, contradicting the fact that $D$ is a 1-perfect orientation of $G$.

Proposition 8. Let $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \{F_1, F_2, F_3\}$, where:

- $\mathcal{F}_1 = \{C_{2k} \mid k \geq 3\}$, the set of complements of even cycles of length at least 6,
- $\mathcal{F}_2 = \{K_2 + C_{2k+1} \mid k \geq 1\}$, the set of complements of the graphs obtained as the disjoint union of $K_2$ and an odd cycle,
- graphs $F_1, F_2$ are depicted in Fig. 1 and
- $F_3$ is the complement of the graph $S_3$, depicted in Fig. 1.

Then, every graph in $\mathcal{F}$ is not 1-p.o.

![Figure 1: Four non-1-p.o. graphs and a complement of a non-1-p.o. graph. The leftmost two graphs are the smallest members of families $\mathcal{F}_1$ and $\mathcal{F}_2$, respectively.](image)

Proof. First consider the graphs $F_1$ and $F_2$. Since they are both triangle-free, every edge clique cover of $F_i$ (for $i \in \{1, 2\}$) contains all edges of $F_i$ and hence has at least $|E(F_i)| > |V(F_i)|$ members. Hence, Corollary 5 implies that $F_1$ and $F_2$ are not 1-p.o.
Now, consider the graph $F_3$ with vertex set $\{0, 1, \ldots, 9\}$ labeled as in Fig. 2 where its complement is depicted.

Suppose for a contradiction that $F_3 = \overline{S_3}$ has a 1-perfect orientation $D$. By symmetry, we may assume without loss of generality that $(1, 4) \in A(D)$. We will derive a contradiction by listing a sequence $C_1, \ldots, C_{10}$ of induced 4-cycles in $F_3$, the orientation of which will be forced in $D$ using the assumption that $(1, 4)$ is an arc and Lemma 7. We will denote the two possible cyclic orientations of a 4-cycle $C$ induced by vertex set $\{a, b, c, d\}$ by $abcd$ or by $dcba$ (or by cyclic permutations of them). The sequence of cycles, their orientations, and reasons for given orientations are displayed in the following table:

| $i$ | $V(C_i)$ | $\vec{C_i}$, orientation of $C_i$ in $D$ | reason for $\vec{C_i}$ |
|-----|----------|------------------------------------------|------------------------|
| 1   | $\{1, 2, 4, 5\}$ | 1425 | $(1, 4) \in A(D)$ |
| 2   | $\{0, 2, 3, 4\}$ | 0342 | compatibility with $\vec{C_1}$ (since $(4, 2) \in A(D)$) |
| 3   | $\{0, 2, 3, 7\}$ | 0372 | compatibility with $\vec{C_2}$ (since $(0, 3) \in A(D)$) |
| 4   | $\{1, 2, 7, 8\}$ | 1728 | compatibility with $\vec{C_3}$ (since $(7, 2) \in A(D)$) |
| 5   | $\{0, 1, 5, 6\}$ | 0516 | compatibility with $\vec{C_4}$ (since $(5, 1) \in A(D)$) |
| 6   | $\{0, 5, 6, 7\}$ | 0576 | compatibility with $\vec{C_5}$ (since $(0, 5) \in A(D)$) |
| 7   | $\{5, 6, 7, 8\}$ | 5768 | compatibility with $\vec{C_6}$ (since $(5, 7) \in A(D)$) |
| 8   | $\{4, 5, 8, 9\}$ | 4859 | compatibility with $\vec{C_7}$ (since $(8, 5) \in A(D)$) |
| 9   | $\{0, 4, 8, 9\}$ | 0948 | compatibility with $\vec{C_8}$ (since $(4, 8) \in A(D)$) |
| 10  | $\{0, 1, 8, 9\}$ | 0918 | compatibility with $\vec{C_9}$ (since $(0, 9) \in A(D)$) |

The contradiction now comes from the fact that the orientations $\vec{C_4}$ and $\vec{C_{10}}$ are incompatible, since they orient oppositely the edge $\{1, 8\}$. Thus, we conclude that $F_3$ is not 1-p.o.

Now let $F \in F_1$, that is, $F = \overline{C_{2k}}$ for some $k \geq 3$. Suppose we name the vertices as $1, 2, \ldots, 2k$ along the cycle. Every two consecutive vertices (in the cyclic order) together with their two diametrically opposite vertices in the cycle induce a $2K_2$ in the cycle, hence a $C_4$ in $F$. Suppose that $F$ admits a 1-perfect orientation $D$. By Lemma 7, every induced 4-cycle in $F$ has to be oriented cyclically. Without loss of generality we can assume that the $C_4$ induced by the first two vertices 1 and 2 and their two opposite vertices, $k + 1$ and $k + 2$, is oriented in $D$ as $1 \to k \to 2 \to k + 1 \to 1$. Since the next $C_4$, induced by vertices 2, 3, $k + 1$, $k + 2$ shares edge $\{2, k + 1\}$ with the first $C_4$, its orientation will be forced, and the same will happen to the following $C_4$’s, each of which shares an edge with the previous one. The orientation of the $C_4$’s must always be as that of the first one: from the smallest vertex, say $i$, to its opposite vertex $(k + i)$, from that one to the second vertex, and then to its opposite and back to the beginning. Hence, this forces the following sequence of
arcs to be present in $D$: $(1, k), (2, k + 1), (3, k + 2), \ldots, (k, 1)$, a contradiction. We conclude that the graph cannot have a 1-perfect orientation.

Finally, let $F \in \mathcal{F}_2$, that is, $F = \overline{K_2 + C_{2k+1}}$ for some $k \geq 1$. Let the vertices of the cycle component of $F$ be named $1, \ldots, 2k + 1$, according to a cyclic order of $C_{2k+1}$. Also, let the two vertices of the $K_2$ component of $F$ be named $w_1$ and $w_2$. Suppose that $F$ admits a 1-perfect orientation $D$. For each two consecutive vertices from the cycle we have an induced $C_4$ given by these two vertices plus the vertices in $2K_1$, $w_1$ and $w_2$. By Lemma 7, every such $C_4$ must be oriented cyclically. We consider the first $C_4$ given by vertices $\{1, 2, w_1, w_2\}$. Without loss of generality we may assume that it is oriented as: $w_1 \rightarrow 1 \rightarrow w_2 \rightarrow 2 \rightarrow w_1$. Now, this will determine the orientation of the second $C_4$ given by $2, 3, w_1, w_2$. Since the edge $w_12$ is oriented as $2 \rightarrow w_1$ the edge $w_13$ must be oriented as $w_1 \rightarrow 3$, and so on. But in the final step we will get to a contradiction since according to the final $C_4$, the edge $w_11$ would need to be oriented as $1 \rightarrow w_1$ but it was already oriented the opposite way. Therefore, $F$ is not 1-p.o.  

\section{Operations preserving 1-p.o. graphs}

In this section, we identify several operations preserving 1-p.o. graphs, which we now describe. Two distinct vertices $u$ and $v$ in a graph $G$ are said to be true twins if $N_G[u] = N_G[v]$, and false twins if $N_G(u) = N_G(v)$. We say that a vertex $v$ is simplicial if its neighborhood forms a clique. A vertex is called universal if it is adjacent to every other vertex of the graph. The operations of adding a true twin, a simplicial vertex or a universal vertex to a given graph are defined in the obvious way.

The operation of duplicating a 2-branch in the complement of a graph $G$ is defined as follows. A 2-branch in a graph $G$ is a path $(a, b, c)$ such that $d_G(b) = 2$ and $d_G(c) = 1$. We say that such a 2-branch is rooted at $a$. Duplicating a 2-branch $G$ results in a graph $H$ where $(a, b, c)$ is a 2-branch in $G$, $V(H) = V(G) \cup \{b', c'\}$, where $b'$ and $c'$ are new vertices, $H - \{b', c'\} = G$, and $(a, b', c')$ is a 2-branch in $H$. Finally, the result of duplicating a 2-branch in the complement of a graph $G$ is the complement of a graph obtained by duplicating a 2-branch in $\overline{G}$.

\textbf{Theorem 9.} The class of 1-p.o. graphs is closed under each of the following operations:

(a) Disjoint union.

(b) Adding a universal vertex (that is, join with $K_1$).

(c) Adding a true twin.

(d) Adding a simplicial vertex.

(e) Duplicating a 2-branch in the complement.

(f) Vertex deletion.

(g) Edge contraction.

\textbf{Proof.} For a 1-p.o. graph $G$, let us denote by $D(G)$ an arbitrary (but fixed) 1-perfect orientation of $G$.

(a) If $G = G_1 + G_2$ is the disjoint union of two 1-p.o. graphs $G_1$ and $G_2$, then the disjoint union of $D(G_1)$ and $D(G_2)$ is a 1-perfect orientation of $G$.  

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(b) Suppose we have a 1-p.o. graph $G$ with orientation $D(G)$ and we add a universal vertex $v$ to $G$, thus obtaining a graph $G'$. We define an orientation $D'$ of $G'$ by orienting an edge $xy \in E(G)$ from $x$ to $y$ if the edge is oriented from $x$ to $y$ in $D(G)$, and orienting the edges of the form $uv$ from $u$ to $v$. Now, for any vertex $w \in V(G)$ its out-neighborhood in $D'$ is $N^+_D(w) = N^+_D(u) \cup \{v\}$. Since $N^+_D(w)$ is a clique in $G$, and $v$ is adjacent to every vertex in $G$, it follows that $N^+_D(u)$ will be a clique in the new graph. Moreover, $N^+_D(v) = \emptyset$. This shows that $D'$ is a 1-perfect orientation of $G'$.

(c) Let $w$ be a vertex in a 1-p.o. graph $G$. If we add to the graph a true twin of $w$, say $v$, then the new graph $G'$ is also 1-perfectly orientable. We maintain the same orientation as in $D(G)$ for the edges in $G$ and orient the new edges (incident with $v$) as $vu$ if $u \in N^+_D(w)$, and $wv$ if $u \in N^+_D(w)$. We also orient the edge between $w$ and $v$ as $wv$. Denoting by $D'$ the so obtained orientation, we have $N^+_D(w) = N^+_D(u) \cup \{v\}$, and since $v$ shares the closed neighborhood with $w$, this is a clique in $G'$. Furthermore, $N^+_D(v)$ is a clique in $G'$, since it equals $N^+_D(w)$. For all $u \in N^+_D(w)$, the set $N^+_D(u)$ is a clique in $G'$, and since $w \in N^+_D(u)$, then $N^+_D(w)$ is formed only by neighbors of $w$. Therefore, $N^+_D(u) = N^+_D(u) \cup \{v\}$ is also a clique in $G'$. For all remaining vertices $u$, that is, $u \in V(G') \setminus (\{w, v\} \cup N^+_D(w))$, the set $N^+_D(u)$ is a clique in $G'$, since it equals $N^+_D(u)$.

(d) If we add a simplicial vertex $v$ to a 1-p.o. graph $G$, then extending $D(G)$ by orienting all edges incident with $v$ away from $v$ results in an orientation $D'$ of the new graph, say $G'$, such that $N^+_D(v)$ is a clique in $G'$. The other out-neighborhoods were not changed, so they are cliques as well.

(e) Let $V(G) = \{v_1, \ldots, v_n\}$. If $G$ is 1-p.o., then Corollary 1 applies to $\overline{G}$. Hence, $\overline{G}$ has a collection of independent sets $\{I_1, \ldots, I_n\}$ such that $v_i \in I_i$ for all $i$, and for every edge $v_iv_j \in E(\overline{G})$, we have $v_i \in I_j$ or $v_j \in I_i$. Let $H$ be the graph resulting from duplicating a 2-branch $(a, b, c)$ in $\overline{G}$; without loss of generality, we may assume that $(a, b, c) = (v_1, v_2, v_3)$; furthermore, let the two new vertices $b'$ and $c'$ be labeled as $v_{n+1}$ and $v_{n+2}$, respectively. It suffices to prove that $H$ has a collection of independent sets $\{J_1, \ldots, J_{n+2}\}$ such that $v_k \in J_k$ for all $k$, and for every edge $v_iv_k \in E(\overline{G})$, we have $v_i \in J_k$ or $v_k \in J_i$. We may assume without loss of generality that the sets $I_j$ are maximal independent sets in $\overline{G}$, which in particular implies that each $I_j$ contains exactly one of the vertices $b$ and $c$. We define the sets $J_k$ for $k \in \{1, \ldots, n+2\}$ with the following rule:

- For all $v_k \in V(G)$, set
  $$J_k = \left\{ \begin{array}{ll} 
  I_k \cup \{b'\}, & \text{if } b \in I_k; \\
  I_k \cup \{c'\}, & \text{if } c \in I_k.
  \end{array} \right.$$ 

- For $k = n+1$ (that is, $v_k = b'$), set $J_k = (I_2 \setminus \{b\}) \cup \{b', c\}$.

- For $k = n+2$ (that is, $v_k = c'$), set $J_k = (I_3 \setminus \{a, c\}) \cup \{b, c'\}$.

Clearly, each $J_k$ is an independent set in $H$. Let $v_iv_k \in E(\overline{G})$. Since $b'c' \notin E(\overline{G})$, we may assume that $v_i \in V(G)$. We analyze three cases according to where is $v_k$.

If $v_k \in V(G)$, then $v_iv_k \in E(\overline{G})$ and hence $v_i \in I_k$ or $v_k \in I_i$, implying $v_i \in J_k$ or $v_k \in J_i$.

If $v_k = b'$, then either $v_i \in J_k$ (in which case we are done), or $v_i \notin J_k = (I_2 \setminus \{b\}) \cup \{b', c\}$, in which case either $v_i = b$ or $v_i \notin I_2$. In the former case, we have $i = 2$ and $v_k = b' \in J_2$, while in the latter case, we have $b = v_2 \in I_i$, which implies $v_k = b' \in J_i$. 

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If \( v_k = c' \), then either \( v_i \in J_k \) (in which case we are done), or \( v_i \notin J_k = (I_3 \setminus \{a, c\}) \cup \{b, c'\} \), in which case either \( v_i \in \{a, c\} \) or \( v_i \notin I_3 \). In the former case, we have \( c \in I_i \) (if \( v_i = a \) this follows from the maximality of \( I_i \)) and consequently \( v_k = c' \in J_i \). In the latter case, we have \( c = v_3 \in I_i \), which implies \( v_k = c' \in J_i \).

Hence, by Corollary 4 \( H \) is the complement of a 1-p.o. graph, which establishes item (e).

(f) Closure under vertex deletions follows immediately from the fact that the class of complete graphs is closed under vertex deletions.

(g) Let \( e = uv \) be an edge of a 1-p.o. graph \( G \), and let \( D \) be a 1-perfect orientation of \( G \). Without loss of generality, we may assume that \( uv \in A(D) \). Let \( G' = G/e \) be the graph obtained by contracting the edge \( e \), and let \( w \) be the vertex replacing \( u \) and \( v \). Let \( D' \) be an orientation of \( G' \) defined as follows:

1. Set \( X = N_G(u) \setminus N_G(v) \),
2. \( Y = \{ x \in N_G(u) \cap N_G(v) \mid xv \in A(D) \} \),
3. \( U = \{ x \in N_G(u) \cap N_G(v) \mid vx \in A(D) \} \),
4. \( W = \{ x \in N_G(v) \setminus N_G(u) \mid xv \in A(D) \} \),
5. \( Z = \{ x \in N_G(v) \setminus N_G(u) \mid vx \in A(D) \} \),
6. \( R = V(G) \setminus (X \cup Y \cup U \cup W \cup Z \cup \{u, v\}) \).

(2) For all edges \( e \in E(G') \) whose endpoints are not incident with \( w \), orient \( e \) the same way as it is oriented in \( D \).

(3) For all \( x \in X \), orient the edge \( xw \) as \( x \to w \).

(4) For all \( x \in Y \), orient the edge \( xw \) as \( x \to w \).

(5) For all \( x \in U \), orient the edge \( xw \) as \( w \to x \).

(6) For all \( x \in W \), orient the edge \( xw \) as \( w \to x \).

(7) For all \( x \in Z \), orient the edge \( xw \) as \( w \to x \).

We will show that \( D' \) is a 1-perfect orientation of \( G' \). Note that \( X \cup Y \cup U \cup W \cup Z \cup \{w\} \cup R \) is a partition of \( V(G') \). We will consider all possible cases, according to where a vertex \( x \) of \( V(G') \) is:

1. \( x \in X \). In this case, \( N_{D'}^+(x) = (N_D^+(x) \setminus \{u\}) \cup \{w\} \). Note that since \( uv \in A(D) \) and \( D \) is a 1-perfect orientation of \( G \), we have \( u \in N_D^+(x) \). Consequently, since \( N_D^+(x) \) is a clique in \( G \) containing \( u \), it contains no vertex from \( R \cup Z \), and thus \( N_{D'}^+(x) = (N_D^+(x) \setminus \{u\}) \cup \{w\} \) is a clique in \( G' \).

2. \( x \in W \). In this case, \( v \in N_{D'}^+(x) \), and a similar reasoning as in Case (1) shows that \( N_{D'}^+(x) = (N_D^+(x) \setminus \{v\}) \cup \{w\} \) is a clique in \( G' \).

3. \( x \in Z \). In this case, \( N_{D'}^+(x) = N_D^+(x) \) and this set is a clique in \( G \) and hence in \( G' \).

4. \( x \in Y \). In this case, we have two possibilities, either \( u \in N_D^+(x) \) or not. In the first case, we have \( N_{D'}^+(x) = (N_D^+(x) \setminus \{u, v\}) \cup \{w\} \) which is a clique in \( G' \), since \( N_D^+(x) \) is a clique in \( G \) containing \( u \).
and $v$, and every neighbor of $w$ in $G'$ is a neighbor of either $u$ or of $v$ in $G$. In the second case, we have $N^+_D(x) = (N^+_D(x) \setminus \{v\}) \cup \{u\}$, which is again a clique in $G'$ by a similar argument.

(5) $x \in U$. Now, $N^+_D(x) = N^+_D(x) \setminus \{u\}$, which is a clique in $G$ not containing $u$ or $v$, and hence a clique in $G'$.

(6) $x \in R$. Since the edges with endpoints in $R$ have no endpoint in $\{u, v\}$, the edges which have $x$ as an endpoint will maintain the same orientation as in $D$. Therefore, $N^+_D(x) = N^+_D(x)$ is a clique in $G'$.

(7) $x = w$. In this final case, $N^+_D(x) = N^+_D(v)$, therefore it forms a clique in $G'$.

Since we showed that for every $x \in V(G')$, the set $N^+_D(x)$ is a clique in $G'$, we conclude that $D'$ is a $1$-perfect orientation of $G'$. Hence $G'$ is 1-p.o. □

We now analyze the behavior of 1-p.o. graphs under two more well known graph operations, the operation of pasting along a clique and the join. Given two vertex-disjoint graphs $G_1$ and $G_2$, we say that a graph $G$ is obtained by pasting $G_1$ and $G_2$ along a clique if there exists a pair of cliques $C_1$ and $C_2$ in $G_1$ and $G_2$, respectively, and a bijection $\varphi : C_1 \to C_2$ such that $G$ is obtained from the disjoint union of $G_1$ and $G_2$ by identifying each vertex $x \in C_1$ with vertex $\varphi(x) \in C_2$.

While the class of 1-p.o. graphs is in general not closed under pasting along a clique (as shown by the non-1-p.o. graphs $F_1$ and $F_2$, see Fig. 1), we show in the next proposition that the class of 1-p.o. graphs is closed under the special case of this operation in which one of the two graphs is chordal.

**Proposition 10.** If $G$ is 1-p.o. and $H$ is chordal, then pasting $G$ and $H$ along a clique results in a 1-p.o. graph.

**Proof.** Let $K$ denote the graph resulting from pasting $G$ and $H$ along a clique, where $G$ is 1-p.o. and $H$ is chordal. Let $C$ be the clique in $K$ which we obtain by identifying vertices in two cliques of $G$ and $H$. We will show that $K$ is 1-p.o. by induction on $|V(H)|$. If $H$ is a complete graph, then the conclusion follows from the fact that the class of 1-p.o. graphs is closed under adding simplicial vertices and true twins. Suppose now that $H$ is not complete. Then $H$ contains a pair of non-adjacent simplicial vertices. Since $C$ is a clique, at least one simplicial vertex of $H$, say $v$, is not in $C$. Thus, the graph $H' = H - v$ is a chordal graph with fewer vertices than $H$ that contains $C$. By the inductive hypothesis, the graph obtained by pasting $G$ and $H'$ along a clique (namely, $C$) results in a 1-p.o. graph $K'$. The conclusion now follows from the fact that $K$ can be obtained from $K'$ by adding to it a simplicial vertex. □

Since a graph $G$ is 1-p.o. if and only if each connected component of $G$ is 1-p.o., in the study of 1-p.o. graphs we may restrict our attention to connected graphs. It is a natural question whether we may also assume that $G$ is co-connected, that is, that its complement is connected, or, equivalently, that $G$ is not the join of two smaller graphs. As the following proposition shows, this problem reduces to the problem characterizing 1-p.o. co-bipartite graphs, where a graph is said to be co-bipartite of its complement is bipartite.

**Proposition 11.** Suppose that a graph $G$ is the join of two graphs $G_1$ and $G_2$. Then, $G$ is 1-p.o. if and only if one of the following conditions hold:

(i) $G_1$ is a complete graph and $G_2$ is a 1-p.o. graph, or vice-versa.
Each of $G_1$ and $G_2$ is a co-bipartite 1-p.o. graph.

In particular, the class of co-bipartite 1-p.o. graphs is closed under join.

Proof. Suppose first that $G$ is 1-p.o. Clearly, both $G_1$ and $G_2$ are 1-p.o. graphs. If one of $G_1$ or $G_2$ is complete or both are co-bipartite, we are done. So suppose that neither of them is complete and $G_1$, say, is not co-bipartite. Then, $G_1$ contains the complement of an odd cycle, $\overline{C_{2k+1}}$ for some $k \geq 1$, as induced subgraph. Since $G_2$ is not complete, it contains $2K_1$ as induced subgraph. Consequently, $G$ contains the join of $\overline{C_{2k+1}}$ and $2K_1$ as induced subgraph. This graph is isomorphic to the complement of $C_{2k+1} + K_2$, which by Proposition 8 is not 1-p.o., implying that $G$ is not 1-p.o. This contradiction establishes the forward direction.

For the converse direction, suppose first that $G_1$ is complete and $G_2$ is 1-p.o., or vice-versa. In this case $G$ is 1-p.o., since it can be obtained from a 1-p.o. graph by a sequence of universal vertex additions, and Theorem [9] applies. Suppose now that $G_1$ and $G_2$ are two co-bipartite 1-p.o. graphs with bipartitions of their respective vertex sets into cliques $\{A_1, B_1\}$ and $\{A_2, B_2\}$, respectively (one of the two cliques in each graph can be empty). Fixing a 1-perfect orientation $D_i$ of each $G_i$ (for $i = 1, 2$), we can construct a 1-perfect orientation, say $D$, of $G = G_1 \ast G_2$, as follows. Every edge of $G$ that is an edge of some $G_i$ is oriented as in $D_i$. Orient the remaining edges of the join from $A_1$ to $A_2$, from $B_1$ to $B_2$, from $A_2$ to $B_1$ and from $B_2$ to $A_1$. Let us verify that the out-neighborhood of a vertex $x \in A_1$ with respect to $D$ forms a clique in $G$ (the other cases follow by symmetry). We have $N^+_D(x) = N^+_D(x) \cup A_2$, and since $N^+_D(x)$ is a clique in $G_1$, $A_2$ is a clique in $G$ and there are all edges between $G_1$ and $A_2$, the set $N^+_D(x)$ is indeed a clique in $G$. This shows that $G$ is 1-p.o. Since the class of bipartite graphs is closed under disjoint union, the class of co-bipartite graphs is closed under join. Consequently, the the set of co-bipartite 1-p.o. graphs is closed under join. \qed

6 1-p.o. graphs in particular graph classes

In this section, we characterize 1-p.o. graphs in two particular graph classes: in the class of cographs and in the class of complements of forests. In the case of cographs, we obtain a composition result for the 1-p.o. graphs in the class. In the case of complements of forests, we show that the complement of the 10-vertex graph $S_3$ (see Fig. 1) is the unique obstruction for the 1-p.o. property in the case of complements of forests. The proofs of the results in this section are based on the results obtained in earlier sections.

6.1 1-p.o. cographs

The class of cographs is defined recursively by stating that $K_1$ is a cograph, the disjoint union of two cographs is a cograph, the join of two cographs is a cograph, and there are no other cographs. It is well known (see, e.g., [5]) that cographs can be recognized in linear time and characterized in terms of forbidden induced subgraphs by a single obstruction, namely the 4-vertex path $P_4$. We start with two lemmas, the first of which will also be used in Section 7.

Lemma 12. Every graph the complement of which is a disjoint union of paths is 1-p.o.

Proof. Note that every disjoint union of paths is an induced subgraph of a sufficiently large odd cycle. Therefore, the lemma follows by Proposition 6 and Theorem 9. \qed

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Lemma 13. Let $G$ be a 1-p.o. cograph. Then $G$ is either disconnected, has a universal vertex, has a pair of true twins, or $G \cong mK_2$ for some $m \geq 2$.

Proof. Let $G$ be a cograph that is not disconnected and does not have a universal vertex or a pair of true twins. We want to see that $G = mK_2$. Since $G$ is not disconnected and $G \neq K_1$, its complement $\overline{G}$ is disconnected. Let $m \geq 2$ denote the number of co-components of $G$ (subgraphs of $G$ induced by the vertex sets of components of $\overline{G}$).

If one of the co-components has exactly one vertex, then that vertex is universal in $G$, which is a contradiction. Therefore, each co-component has at least two vertices. The recursive structure of cographs implies that each co-component of $G$ is disconnected. In particular, it has independence number at least 2. Also, note that Proposition 8 and Theorem 9 imply that $G$ is $K_{2,3}$-free (the graph $K_{2,3}$ is the complement of $K_2 + C_3$). Therefore, each co-component of $G$ has independence number at most 2. This implies that each co-component is the disjoint union of two complete graphs. Since $G$ has no true twins, each co-component is isomorphic to $2K_1$, in other words, $G \cong mK_2$.

Theorem 9 and Lemma 13 imply the following composition theorem for 1-p.o. cographs:

Theorem 14. A cograph $G$ is 1-p.o. if and only if one of the following conditions holds:

- $G$ is the disjoint union of two smaller 1-p.o. cographs.
- $G$ is obtained from a 1-p.o. cograph by adding to it a universal vertex.
- $G$ is obtained from a 1-p.o. cograph by adding to it a true twin.
- $G \cong mK_2$ for some $m \geq 2$.
- $G \cong K_1$.

Proof. If a cograph $G$ is 1-p.o., then Lemma 13 applies and the conclusion follows from the fact that the set of 1-p.o. graphs is closed under vertex deletions.

Conversely, if $G$ is a cograph such that one of the four conditions holds, then the conclusion follows from Theorem 9 and the observation that for every $m \geq 2$, the graph $mK_2$, that is, the complete $m$-partite graph with parts of size 2, is 1-p.o. This follows, e.g., from Lemma 12.

6.2 1-p.o. complements of forests

Now, we characterize 1-p.o. complements of forests, showing that the complement of $S_3$ (cf. Fig. 2) is the unique minimal obstruction (with respect to the induced subgraph relation). As a side remark, let us note that the $S_3$-free forests are also exactly the forests that admit a so-called tolerance representation, which means that every vertex $x$ can be assigned a closed interval $I_x$ and a positive integer $t_x$ such that two distinct vertices $x$ and $y$ are adjacent if and only if $|I_x \cap I_y| \geq \min\{t_x, t_y\}$ [21]. A 2-caterpillar is a tree that has a path $P$ such that every vertex not in $P$ is at distance at most two from $P$.

Theorem 15. For every forest $F$, the following conditions are equivalent:

1. The complement of $F$ is 1-p.o.
2. $F$ is $S_3$-free, where $S_3$ is the graph depicted in Fig. 2.
3. Each connected component of $F$ is a 2-caterpillar.

Proof. If $F$ is 1-p.o., then $F$ is $S_3$-free by Proposition 8 and Theorem 9. This establishes the implication 1 $\Rightarrow$ 2.

Now, let $F$ be $S_3$-free. We will argue that each connected component $T$ of $F$ is a 2-caterpillar. Let $P$ be a longest path in $T$. For every vertex $x \in V(P)$, let $V_x$ denote the set of vertices $y$ not in $P$ such that $x$ is the closest vertex to $y$ in $P$. Since $T$ is $S_3$-free, every vertex $x$ in $P$ that is at distance at least three from both endpoints of $P$ has the property that every vertex $w \in V_x$ is at distance at most two from $x$. Moreover, the maximality of $P$ implies that if $x$ is a vertex in $P$ at distance $d \leq 2$ from its endpoints, then has the property that every vertex $w \in V_x$ is at distance at most $d$ from $x$. Therefore, $T$ is a 2-caterpillar. This establishes the implication 2 $\Rightarrow$ 3.

Suppose finally that each connected component of $F$ is a 2-caterpillar. We will show that the complement of $F$ is 1-p.o. By Proposition 11, it is enough to consider the case when $F$ is connected. (If $F$ is disconnected, then the complement of $F$ is the join of at least two co-trees. Proposition 11 together with an inductive argument on the number of components of $F$ then establish the statement.)

Let us first simplify $F$ using Theorem 9. Since the class of 1-p.o. graphs is closed under adding true twins, we may assume without loss of generality that $F$ has no pairs of false twins, that is, that every leaf is the unique leaf adjacent to its unique neighbor. Moreover, since the class of 1-p.o. graphs is also closed under duplicating a 2-branch in the complement, we may also assume that no two branches in $F$ are rooted at the same vertex. Recall that given a vertex $a \in V(F)$, a 2-branch rooted at $a$ in $F$ is path $(a, b, c)$ such that $d_F(b) = 2$ and $d_F(c) = 1$.

After this simplification, every vertex not in path $P$ is either a leaf with a neighbor on $P$ or a part of a 2-branch rooted at a vertex in $P$. Moreover, every vertex in $P$ is adjacent to at most one leaf, and is the root of at most one 2-branch. Let us say that a 2-caterpillar of such a form is reduced. Since the set of 1-p.o. graphs is closed under vertex deletions, it suffices to show that for every $n$, the complement of the following “canonical” 2-caterpillar $CP_n$, is 1-p.o., where $CP_n$ for $n \geq 1$ consists of a path with $n$ vertices, each adjacent to a unique leaf and the root of a unique 2-branch. The conclusion for the general case will follow from the observation that every reduced 2-caterpillar is an induced subgraph of some $CP_n$.

To show that the complement of each $CP_n$ is 1-p.o., we will apply the characterization given by Corollary 3. Thus, we need to show that to every vertex $x \in V(CP_n)$ we can assign an independent set $I_x$ in $CP_n$ containing $x$ such that for every non-adjacent vertex pair $xy \in E(CP_n)$, we have $x \in I_y$ or $y \in I_x$. First, let us name the vertices of $CP_n$ as follows: the vertices in $P$ are named $a_1, \ldots, a_n$ along the path, the unique 2-branch rooted at each $a_i$ is $(a_i, b_i, c_i)$, and the unique leaf adjacent to $a_i$ is named $d_i$, for each $i = 1, \ldots, n$. We define the independent sets $I_{a_i}$, $I_{b_i}$, $I_{c_i}$, $I_{d_i}$ (for $i = 1, \ldots, n$) as follows:

\[
I_{a_i} = \{a_j \mid j \leq i - 3, i - j \text{ odd}\} \cup \{a_j \mid j \geq i, j - i \text{ even}\} \\
\cup \{b_j \mid j \leq i - 2, i - j \text{ even}\} \cup \{b_j \mid j \geq i + 1, j - i \text{ odd}\} \\
\cup \{c_j \mid j \leq i - 1, i - j \text{ odd}\} \cup \{c_j \mid j \geq i, j - i \text{ even}\} \\
\cup \{d_j \mid j \leq i - 2, i - j \text{ even}\} \cup \{d_j \mid j \geq i + 1, j - i \text{ odd}\}
\]

See Fig. 5 for an example with $n = 20$ and $i = 10$. 

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Figure 3: The black vertices form the set $I_{a_{10}}$.

$I_{b_i} = \{a_j \mid j \leq i-2, i-j \text{ even}\} \cup \{a_j \mid j \geq i+1, j-i \text{ odd}\} \\
\cup \{b_j \mid j \leq i-1, i-j \text{ odd}\} \cup \{b_j \mid j \geq i, j-i \text{ even}\} \\
\cup \{c_j \mid j \leq i-2, i-j \text{ even}\} \cup \{c_j \mid j \geq i+1, j-i \text{ odd}\} \\
\cup \{d_j \mid j \leq i-1, i-j \text{ odd}\} \cup \{d_j \mid j \geq i, j-i \text{ even}\}$

See Fig. 4 for an example with $n = 20$ and $i = 10$.

Figure 4: The black vertices form the set $I_{b_{10}}$.

$I_{c_i} = \{a_j \mid j \leq i-1, i-j \text{ odd}\} \cup \{a_j \mid j \geq i+2, j-i \text{ even}\} \\
\cup \{b_j \mid j \leq i-2, i-j \text{ even}\} \cup \{b_j \mid j \geq i+1, j-i \text{ odd}\} \\
\cup \{c_j \mid j \leq i-1, i-j \text{ odd}\} \cup \{c_j \mid j \geq i, j-i \text{ even}\} \\
\cup \{d_j \mid j \leq i, i-j \text{ even}\} \cup \{d_j \mid j \geq i+1, j-i \text{ odd}\}$

See Fig. 5 for an example with $n = 20$ and $i = 10$.

Figure 5: The black vertices form the set $I_{c_{10}}$. 
\[ I_{d_i} = \{a_j \mid j \leq i - 2, i - j \text{ even}\} \cup \{a_j \mid j \geq i + 1, j - i \text{ odd}\} \]
\[ \cup \{b_j \mid j \leq i - 1, i - j \text{ odd}\} \cup \{b_j \mid j \geq i, j - i \text{ even}\} \]
\[ \cup \{c_j \mid j \leq i - 2, i - j \text{ even}\} \cup \{c_j \mid j \geq i + 1, j - i \text{ odd}\} \]
\[ \cup \{d_j \mid j \leq i - 1, i - j \text{ odd}\} \cup \{d_j \mid j \geq i, j - i \text{ even}\} \]

See Fig. 6 for an example with \( n = 20 \) and \( i = 10 \).

Figure 6: The black vertices form the set \( I_{d_{10}} \).

It is clear that each of the sets \( I_v \) is an independent set in \( F \) containing \( v \). Consider now an arbitrary pair \( x, y \) of distinct non-adjacent vertices in \( F \). We need to show that \( x \in I_y \) or \( y \in I_x \).

Suppose that \( \{x, y\} = \{a_i, a_j\} \) for some \( i, j \). Then we may assume \( i \geq j + 2 \), and we have either \( a_i \in I_{a_j} \) or \( a_j \in I_{a_i} \), depending on whether \( i - j \) is even or odd. A similar argument can be used to settle the cases when \( \{x, y\} = \{z_i, z_j\} \) for some \( i, j \) and \( z \in \{b, c, d\} \).

Suppose that \( \{x, y\} = \{a_i, b_j\} \) for some \( i, j \). Then \( i \neq j \). If \( b_j \notin I_{a_i} \), then one of the following happens: (i) \( j < i \) and \( i - j \) is odd, or (ii) \( j > i \) and \( j - i \) is even. In either case, we have \( a_i \in I_{b_j} \).

Suppose that \( \{x, y\} = \{a_i, c_j\} \) for some \( i, j \). If \( c_j \notin I_{a_i} \), then one of the following happens: (i) \( j < i \) and \( i - j \) is even, or (ii) \( j > i \) and \( j - i \) is odd. In either case, we have \( a_i \in I_{c_j} \).

Suppose that \( \{x, y\} = \{a_i, d_j\} \) for some \( i, j \). Then \( i \neq j \). If \( d_j \notin I_{a_i} \), then one of the following happens: (i) \( j < i \) and \( i - j \) is odd, or (ii) \( j > i \) and \( j - i \) is even. In either case, we have \( a_i \in I_{d_j} \).

Suppose that \( \{x, y\} = \{b_i, c_j\} \) for some \( i, j \). Then \( i \neq j \). If \( c_j \notin I_{b_i} \), then one of the following happens: (i) \( j < i \) and \( i - j \) is odd, or (ii) \( j > i \) and \( j - i \) is odd. In either case, we have \( b_i \in I_{c_j} \).

Suppose that \( \{x, y\} = \{b_i, d_j\} \) for some \( i, j \). If \( d_j \notin I_{b_i} \), then one of the following happens: (i) \( j < i \) and \( i - j \) is odd, or (ii) \( j > i \) and \( j - i \) is odd. In either case, we have \( b_i \in I_{d_j} \).

Thus, in all cases, we have either \( x \in I_y \) or \( y \in I_x \). This completes the proof.

7 Some minimal forbidden induced minors for the class of 1-p.o. graphs

Theorem \([9]\) implies that the class of 1-p.o. graphs is closed under vertex deletions and edge contractions. Hence, it is also closed under taking induced minors. Recall that a graph \( H \) is said to be an induced minor of a graph \( G \) if \( H \) can be obtained from \( G \) by a series of vertex deletions or edge contractions. Many interesting graph classes arising from different application domains (for instance minor closed classes, chordal graphs, and classes studied in papers \([6, 23, 25, 26, 37]\), to mention just a few) are closed under taking induced minors. Notions related to the induced minor
relation are a topic of active investigation from both structural and algorithmic points of view, see, e.g., [3,4,8,10,13,18,19,27,39]. Since the class of 1-p.o. graphs is closed under induced minors, it can be characterized in terms of minimal forbidden induced minors. That is, there exists a unique minimal set of graphs $\bar{F}$ such that a graph $G$ is 1-p.o. if and only if $G$ is $\bar{F}$-induced-minor-free. In other words, no induced minor of $G$ is isomorphic to a member of $\bar{F}$, and every proper induced minor of every graph in $\bar{F}$ is 1-p.o. In this section we identify some minimal forbidden induced minors for the class of 1-p.o. graphs.

Recall that $\mathcal{F}$ denotes the family of graphs defined in Proposition 8 on p. 6.

**Proposition 16.** Every graph in set $\mathcal{F}$ is a minimal forbidden induced minor for the class of 1-p.o. graphs.

*Proof.* We need to show that each $F \in \mathcal{F}$ is not 1-p.o., but every proper induced minor of $F$ is. The fact that each $F \in \mathcal{F}$ is not 1-p.o. was already established in Proposition 8; thus we only need to argue minimality.

First consider the graphs $F_1$ and $F_2$. Deleting any vertex of either $F_1$ or $F_2$ results in either a chordal graph or in a unicyclic graph, hence in a 1-p.o. graph (cf. Proposition 1). Contracting any edge of $F_1$ results in a graph that is either chordal, or is obtained from a cycle by adding to it a simplicial vertex, hence in either case a 1-p.o. graph. Contracting any edge of $F_2$ results in a graph that can be reduced to a cycle by removing true twins and simplicial vertices, hence this graph is also 1-p.o.

We are left with graphs that are defined in terms of their complements. To argue minimality for them, it will be convenient to understand the effect of performing the operation of edge contraction on a given graph on its complement. It can be seen that if $G_1$ is the graph obtained from a graph $G$ by contracting an edge $uv$, then $\overline{G_1}$ is the graph obtained from $\overline{G}$ identifying a pair of non-adjacent vertices (namely, $u$ and $v$) and making the new vertex adjacent exactly to the common neighbors in $\overline{G}$ of $u$ and $v$. For simplicity, let us refer to this operation as co-contracting a non-edge.

Consider first the graph $F_3$. Using the above observation, it follows that deleting any vertex or contracting any edge of $F_3$ results in the complement of an $S_3$-free forest, hence in a 1-p.o. graph (by Theorem 15).

Now let $F \in \mathcal{F}_1$, that is, $F = \overline{C_{2k}}$ for some $k \geq 3$. Deleting a vertex from $F$ results in the complement of a path, which is 1-p.o. by Lemma 12. Similarly, one can verify that co-contracting a non-edge of $F$ results in a disjoint union of paths. Thus, every proper induced minor of $F$ is 1-p.o.

Finally, let $F \in \mathcal{F}_2$, that is, $F = \overline{C_{2k+1} + K_2}$ for some $k \geq 1$. Deleting a vertex in the cycle component of $F$ from $F$ results in the complement of a disjoint union of paths, which is 1-p.o. by Lemma 12. Deleting a vertex in the $K_2$ component of $F$ from $F$ results in the graph that consists of the join of $K_1$ and the complement of an odd cycle, which is 1-p.o. by Theorem 3 and Proposition 6. Furthermore, co-contracting a non-edge of $F$ results in a disjoint union of paths, and Lemma 12 applies again. Thus, every proper induced minor of $F$ is 1-p.o. This completes the proof.

We do not know if $\mathcal{F}$ is a complete list of forbidden induced minors for the class of 1-p.o. graphs, and hence conclude the paper with the following problem.

**Open problem.** Determine the set of minimal forbidden induced minors for the class of 1-p.o. graphs.
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References

[1] J. Bang-Jensen and G. Gutin. *Digraphs*. Springer Monographs in Mathematics. Springer-Verlag London, Ltd., London, second edition, 2009. Theory, algorithms and applications.

[2] J. Bang-Jensen, J. Huang, and E. Prisner. In-tournament digraphs. *J. Combin. Theory Ser. B*, 59(2):267–287, 1993.

[3] R. Belmonte, P. A. Golovach, P. Heggernes, P. van ’t Hof, M. Kamiński, and D. Paulusma. Finding contractions and induced minors in chordal graphs via disjoint paths. In *Algorithms and computation*, volume 7074 of *Lecture Notes in Comput. Sci.*, pages 110–119. Springer, Heidelberg, 2011.

[4] R. Belmonte, P. A. Golovach, P. Heggernes, P. van ’t Hof, M. Kamiński, and D. Paulusma. Detecting fixed patterns in chordal graphs in polynomial time. *Algorithmica*, 69(3):501–521, 2014.

[5] A. Brandstädt, V. B. Le, and J. P. Spinrad. *Graph classes: a survey*. SIAM Monographs on Discrete Mathematics and Applications. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1999.

[6] F. Cicalese and M. Milanič. Graphs of separability at most 2. *Discrete Appl. Math.*, 160(6):685–696, 2012.

[7] R. Diestel. *Graph theory*, volume 173 of *Graduate Texts in Mathematics*. Springer, Heidelberg, fourth edition, 2010.

[8] G. Ding. Chordal graphs, interval graphs, and wqo. *J. Graph Theory*, 28(2):105–114, 1998.

[9] R. D. Dutton and R. C. Brigham. A characterization of competition graphs. *Discrete Appl. Math.*, 6(3):315–317, 1983.

[10] M. R. Fellows, D. Hermelin, and F. A. Rosamond. Well quasi orders in subclasses of bounded treewidth graphs and their algorithmic applications. *Algorithmica*, 64(1):3–18, 2012.

[11] M. R. Fellows, J. Kratochvíl, M. Middendorf, and F. Pfeiffer. The complexity of induced minors and related problems. *Algorithmica*, 13(3):266–282, 1995.

[12] J. Fiala, M. Kamiński, and D. Paulusma. Detecting induced star-like minors in polynomial time. *J. Discrete Algorithms*, 17:74–85, 2012.

[13] J. Fiala, M. Kamiński, and D. Paulusma. A note on contracting claw-free graphs. *Discrete Math. Theor. Comput. Sci.*, 15(2):223–232, 2013.

[14] H. Galeana-Sánchez. Normal fraternally orientable graphs satisfy the strong perfect graph conjecture. *Discrete Math.*, 122(1-3):167–177, 1993.
[15] H. Galeana-Sánchez. A characterization of normal fraternally orientable perfect graphs. *Discrete Math.*, 169(1-3):221–225, 1997.

[16] F. Gavril. Intersection graphs of proper subtrees of unicyclic graphs. *J. Graph Theory*, 18(6):615–627, 1994.

[17] F. Gavril and J. Urrutia. Intersection graphs of concatenable subtrees of graphs. *Discrete Appl. Math.*, 52(2):195–209, 1994.

[18] P. A. Golovach, M. Kamiński, and D. Paulusma. Contracting a chordal graph to a split graph or a tree. In *Mathematical foundations of computer science 2011*, volume 6907 of *Lecture Notes in Comput. Sci.*, pages 339–350. Springer, Heidelberg, 2011.

[19] P. A. Golovach, D. Kratsch, and D. Paulusma. Detecting induced minors in AT-free graphs. *Theoret. Comput. Sci.*, 482:20–32, 2013.

[20] M. C. Golumbic. *Algorithmic graph theory and perfect graphs*, volume 57 of *Annals of Discrete Mathematics*. Elsevier Science B.V., Amsterdam, second edition, 2004. With a foreword by Claude Berge.

[21] M. C. Golumbic, C. L. Monma, and W. T. Trotter, Jr. Tolerance graphs. *Discrete Appl. Math.*, 9(2):157–170, 1984.

[22] P. Hell, J. Bang-Jensen, and J. Huang. Local tournaments and proper circular arc graphs. In *Algorithms (Tokyo, 1990)*, volume 450 of *Lecture Notes in Comput. Sci.*, pages 101–108. Springer, Berlin, 1990.

[23] C. R. Johnson and T. A. McKee. Structural conditions for cycle completable graphs. *Discrete Math.*, 159(1-3):155–160, 1996.

[24] F. Kammer and T. Tholey. Approximation algorithms for intersection graphs. *Algorithmica*, 68(2):312–336, 2014.

[25] J. Kratochvíl. String graphs. I. The number of critical nonstring graphs is infinite. *J. Combin. Theory Ser. B.*, 52(1):53–66, 1991.

[26] H.-J. Lai. Super-Eulerian graphs and excluded induced minors. *Discrete Math.*, 146(1-3):133–143, 1995.

[27] J. Matoušek, J. Nešetřil, and R. Thomas. On polynomial time decidability of induced-minor-closed classes. *Comment. Math. Univ. Carolin.*, 29(4):703–710, 1988.

[28] T. A. McKee and F. R. McMorris. *Topics in intersection graph theory*. SIAM Monographs on Discrete Mathematics and Applications. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1999.

[29] R. Motwani and M. Sudan. Computing roots of graphs is hard. *Discrete Appl. Math.*, 54(1):81–88, 1994.

[30] A. Mukhopadhyay. The square root of a graph. *J. Combinatorial Theory*, 2:290–295, 1967.
[31] J. Nešetřil and P. Ossona de Mendez. Fraternal augmentations of graphs, coloration and minors. In 6th Czech-Slovak International Symposium on Combinatorics, Graph Theory, Algorithms and Applications, volume 28 of Electron. Notes Discrete Math., pages 223–230.

[32] J. Nešetřil and P. Ossona de Mendez. Fraternal augmentations, arrangeability and linear Ramsey numbers. European J. Combin., 30(7):1696–1703, 2009.

[33] R. J. Opsut. On the computation of the competition number of a graph. SIAM J. Algebraic Discrete Methods, 3(4):420–428, 1982.

[34] E. Prisner. Familien zusammenhängender Teilgraphen eines Graphen und ihre Durchschnittsgraphen. 1988.

[35] A. Raychaudhuri. On powers of strongly chordal and circular arc graphs. Ars Combin., 34:147–160, 1992.

[36] D. J. Skrien. A relationship between triangulated graphs, comparability graphs, proper interval graphs, proper circular-arc graphs, and nested interval graphs. J. Graph Theory, 6(3):309–316, 1982.

[37] A. Tucker. Characterizing circular-arc graphs. Bull. Amer. Math. Soc., 76:1257–1260, 1970.

[38] J. Urrutia and F. Gavril. An algorithm for fraternal orientation of graphs. Inform. Process. Lett., 41(5):271–274, 1992.

[39] P. van ’t Hof, M. Kamiński, D. Paulusma, S. Szeider, and D. M. Thilikos. On graph contractions and induced minors. Discrete Appl. Math., 160(6):799–809, 2012.

[40] T. Zou. Cyclical orientations of graphs. Sichuan Daxue Xuebao, 47(1):21–26, 2010.