Spinor vacuum and $C$, $P$, $T$ inversions

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Abstract. We have developed the theory of Clifford reflections and extended spacetime inversions. This extended spacetime has two additional dimensions associated with the presence of internal degrees of freedom of spinors. Inversions $C$, $P$, and $T$ contain not only reflections of the basis Clifford vectors and transformations of basis spinors, but also transformations of the components of vector and spinor quantities. The research is carried out on the basis of algebraic quantum field theory using the superalgebraic representation of spinors as well as the 8-component matrix representation of spinors. We have proved that due to the presence of internal degrees of freedom of spinors, there are two vacua, the vacuum of our Universe and an alternative vacuum. The inversion operators $C$ and $T$ transform the vacuum into an alternative one, and therefore cannot be operators of the exact symmetry of our Universe.

1. Introduction
Modern theories of spacetime are increasingly based not only on the idea of geometrization proposed by A. Einstein, but also on the idea of algebraization. Modern generalizations of Einstein's general theory of relativity use the theory of spinor bundles taking into account torsion and nonmetricity. However, the quantum theory of the spinor vacuum is still in its infancy. It is hardly possible to construct a realistic quantum theory of spacetime and gravity without understanding the properties of the spinor vacuum.

The modern theory of spinors is the algebraic theory of spinors. It is based on the theory of Clifford algebras [1]-[4]. We develop quantum field theory based on the superalgebraic representation of spinors [5]-[10]. It is an algebraic quantum field theory in which the Dirac spinors are a superposition of Grassmann densities in momentum space and their derivatives [6]-[10]. This theory is a development of the algebraic theory of spinors. Ordinary algebraic theory of spinors [1]-[4] is not based on the canonical anticommutation relations (CAR), although the modern formulation of the algebraic quantum field theory is based on them. In ordinary algebraic spinor theory, the creation and annihilation operators of spinors are elements of a CAR algebra [11] that has nothing to do with the Clifford algebra, of which the spinors are two-valued representations [1]-[4]. And in our theory, the creation and annihilation operators of spinors are elements of the same Clifford algebra as the basis Clifford vectors of spacetime (gamma operators, which are algebraic analogs of Dirac matrices). Moreover, in our theory, we used CAR as a basis for deriving generalized Dirac conjugation formulas [7] and for constructing an explicit expression for the spinor vacuum state vector [6].

Our study of the properties of the spinor vacuum [9]-[10] showed that they differ from the earlier assumptions in the framework of quantum field theory. It turned out that the spinor vacuum is not invariant under the operations of charge conjugation $C$ and time inversion $T$. Moreover, it turned out to be invariant with respect to the operation of space inversion $P$ (spatial parity) and combinations $CT$ and $CPT$. In this article, we refine the results of articles [9]-[10].
2. Superalgebraic representation of spinors

In the theory of superalgebraic spinors, the analogs of Dirac gamma matrices $\gamma^\mu_\alpha$ are gamma operators $\dot{\gamma}^\mu$, formulas which are given in (1).

\[
\begin{align*}
\dot{\gamma}^0 &= \int d^3 p \left[ -\frac{\partial}{\partial \theta^1(p)} \theta^1(p) + \frac{\partial}{\partial \theta^2(p)} \theta^2(p) + \frac{\partial}{\partial \theta^3(p)} \theta^3(p) + \frac{\partial}{\partial \theta^4(p)} \theta^4(p) \right], \\
\dot{\gamma}^1 &= \int d^3 p \left[ -\frac{\partial}{\partial \theta^1(p)} \theta^1(p) - \theta^4(p) \theta^1(p) + \frac{\partial}{\partial \theta^2(p)} \theta^2(p) - \theta^3(p) \theta^2(p) \right], \\
\dot{\gamma}^2 &= \int d^3 p \left[ -\frac{\partial}{\partial \theta^1(p)} \theta^1(p) - \theta^3(p) \theta^1(p) + \frac{\partial}{\partial \theta^2(p)} \theta^2(p) + \theta^4(p) \theta^2(p) \right], \\
\dot{\gamma}^3 &= \int d^3 p \left[ -\frac{\partial}{\partial \theta^1(p)} \theta^1(p) - \theta^2(p) \theta^1(p) - \frac{\partial}{\partial \theta^2(p)} \theta^2(p) + \theta^4(p) \theta^2(p) \right], \\
\dot{\gamma}^4 &= \int d^3 p \left[ -\frac{\partial}{\partial \theta^1(p)} \theta^1(p) + \frac{\partial}{\partial \theta^2(p)} \theta^2(p) + \frac{\partial}{\partial \theta^3(p)} \theta^3(p) + \frac{\partial}{\partial \theta^4(p)} \theta^4(p) \right].
\end{align*}
\]

(1)

Operator $\hat{A} = [A, \cdot]$ denotes commutator $\hat{A} \Psi = [A, \cdot] \Psi = [A, \Psi] = A \Psi - \Psi A$. Unlike Dirac gamma matrices, gamma operators are commutators. Moreover, they are composite and consist of a combination of Grassmann densities $\theta^i(p)$, $i=1,2,3,4$, and their derivatives $\frac{\partial}{\partial \theta^i(p)}$ [6]. These densities are generators of CAR-algebra with anticommutation relations (2)

\[
\begin{align*}
\{ \frac{\partial}{\partial \theta^i(p)}, \theta^j(p') \} &= \delta^i_j \delta(p-p'), \\
\{ \frac{\partial}{\partial \theta^i(p)}, \frac{\partial}{\partial \theta^j(p')} \} &= \{ \theta^i(p), \theta^j(p') \} = 0.
\end{align*}
\]

(2)

Compared to Dirac's theory, there are two additional gamma operators, $\dot{\gamma}^5$ and $\dot{\gamma}^7$ [7]. They do not refer to spacetime and set the internal degrees of freedom of the spinors [12]-[13].

We obtain the Dirac conjugated spinor $\overline{\Psi}$ from the spinor $\Psi$ using the generalized Dirac conjugation operation [7]

\[
\overline{\Psi} = (\gamma^0 \Psi)^+.
\]

(3)

The Grassmann densities $\frac{\partial}{\partial \theta^1(0)}$ and $\frac{\partial}{\partial \theta^2(0)}$ are the annihilation operators of spinors with the projections of the spin $s_3 = +\frac{1}{2}$ and $s_3 = -\frac{1}{2}$, respectively. Grassmann densities $\theta^3(0) \text{ and } \theta^4(0)$ are antispinor creation operators with projections of the spin $s_3 = +\frac{1}{2}$ and $s_3 = -\frac{1}{2}$, respectively.
Lorentz boost $\exp(\gamma^0 \varphi_k)$ changes operator $b_\alpha(0) = \frac{\partial}{\partial \vartheta^\alpha(0)}$ of annihilation of a spinor (or antispinor) and the operator $\bar{b}_\alpha(0) = \theta^\alpha(0)$ of creation of an antispinor (or spinor) with the momentum $p=0$ to the operators of annihilation $b_\alpha(p)$ and creation $\bar{b}_\alpha(p)$ with the momentum $p$

$$b_\alpha(p) = \exp(\gamma^0 \varphi_k) \frac{\partial}{\partial \vartheta^\alpha(0)} |_{0 \to p},$$

$$\bar{b}_\alpha(p) = \exp(\gamma^0 \varphi_k) \theta^\alpha(0) |_{0 \to p}.$$  (4)

Annihilation and creation operators (4) are also generators of the CAR-algebra

$$\{ b_i(p), \bar{b}_j(p') \} = \delta^j_i \delta(p - p'),$$

$$\{ b_i(p), b_j(p') \} = \{ \bar{b}_i(p), \bar{b}_j(p') \} = 0.$$  (5)

The vacuum state vector

$$\Psi_v = \prod_i \Psi_v(p_i)$$  (6)

is given via the factors $\Psi_v(p_i)$ local in the momentum space

$$\Psi_v(p_i) = (\hat{A} p_i)^4 b_1(p_i) b_2(p_i) b_3(p_i) b_4(p_i) \bar{b}_1(p_i) \bar{b}_2(p_i) \bar{b}_3(p_i) \bar{b}_4(p_i),$$  (7)

where $\hat{A} p_i$ is infinitesimal volume in the momentum space corresponding to the discrete momentum $p_i$ [6], [8], [9].

It is known from the theory of CAR-algebras that there are an infinite number of spinor vacuum state vectors [11]. Moreover, they are all physically equivalent. But only one of them is the vacuum of the Fock representation, for which the operator of the number of particles is defined. At first glance, these statements are contradictory, but they are not. A change in the formula (6) the order of the product of any annihilation operator $b_\alpha(p_i)$ and the corresponding creation operator $\bar{b}_\alpha(p_i)$ gives us another vacuum state vector. But for this vacuum, the operator, which was the annihilation operator for the normal vacuum, becomes the creation operator, and vice versa.

In addition to the spinor vacuum $\Psi_v$ of the Fock representation, there is another spinor vacuum that is important for the theory of the $C$, $P$, $T$ inversions. It is an alternative spinor vacuum

$$\Psi_{av} = \prod_i \Psi_{av}(p_i),$$  (8)

where

$$\Psi_{av}(p_i) = (\hat{A} p_i)^4 b_1(p_i) b_2(p_i) b_3(p_i) b_4(p_i) \bar{b}_1(p_i) \bar{b}_2(p_i) \bar{b}_3(p_i) \bar{b}_4(p_i).$$  (9)

In it, in contrast to the spinor vacuum $\Psi_v$, operators $b_\alpha(p_i)$ are creation operators, and operators $\bar{b}_\alpha(p_i)$ are annihilation operators [9], [10].

3. 8-component matrix representation of spinors

Matrix representation of Grassmann densities [6] is

$$\frac{\partial}{\partial \vartheta^k(p)} \equiv w_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}; \quad \frac{\partial}{\partial \vartheta^2(p)} \equiv w_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}; \quad \vartheta^3(p) \equiv w_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}; \quad \vartheta^4(p) \equiv w_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$  (10)

where $w_1, w_2, w_3, w_4$ are basis matrix spinors. In what follows, for brevity, we will sometimes omit the momentum $p$ as a parameter.

Consider the action of gamma operators (1) on Grassmann densities (10). With the arrow, we indicate the expression into which the gamma operator transforms the corresponding Grassmann density.
Consider the Dirac matrices in the Dirac representation.

$$\gamma^0 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}, \quad \gamma^1 = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad \gamma^2 = \begin{pmatrix}
0 & 0 & 0 & -i \\
0 & 0 & i & 0 \\
0 & i & 0 & 0 \\
-i & 0 & 0 & 0
\end{pmatrix} \quad (12)$$

In accordance with (12) and (10), we find that under the action of gamma matrices on the columns, the following correspondence to the transformations of Grassmann densities should be observed

$$\gamma^0 \cdot w_1 \rightarrow w_1; \quad w_2 \rightarrow w_2; \quad w_3 \rightarrow -w_3; \quad w_4 \rightarrow -w_4;$$

$$\gamma^1 \cdot w_1 \rightarrow -w_1; \quad w_2 \rightarrow w_2; \quad w_3 \rightarrow w_3; \quad w_4 \rightarrow w_4;$$

$$\gamma^2 \cdot w_1 \rightarrow -iw_1; \quad w_2 \rightarrow iw_2; \quad w_3 \rightarrow w_3; \quad w_4 \rightarrow iw_4;$$

$$\gamma^3 \cdot w_1 \rightarrow -iw_1; \quad w_2 \rightarrow w_2; \quad w_3 \rightarrow w_3; \quad w_4 \rightarrow -w_4;$$

$$\gamma^4 \cdot w_1 \rightarrow w_1; \quad w_2 \rightarrow w_2; \quad w_3 \rightarrow w_3; \quad w_4 \rightarrow w_4. \quad (13)$$

Comparison of (13) with (11) shows that the matrix representation of gamma operators under their action on Grassmann densities (10) is the Dirac matrices (12)

$$\gamma^\lambda \equiv \gamma_D^\lambda, \quad \lambda = 0,1,2,3,5. \quad (14)$$

The situation is more complicated when it comes to finding matrix representations of conjugate spinors. In Dirac's theory, spinors are matrix columns, and Dirac's conjugate spinors are matrix rows. But they cannot be elements of the same algebra. In the theory of Clifford algebras, the so-called algebraic spinors are introduced, whose matrix representations are square matrices [1]-[4]. However, there are a number of problems in the theory of algebraic spinors, and one of them is the problem of conjugate spinors. Since spinors are left ideals of the Clifford algebra, and conjugate spinors are right ideals [3]. Therefore, as in the case of Dirac matrix spinors, they exist in different spaces. Moreover, these spinors have nothing to do with the CAR-algebra of creation and annihilation operators. In the case of a superalgebraic representation of spinors, spinors and conjugated spinors are elements of the CAR-algebra of creation and annihilation operators.
We will consider spinor $\Psi_1$ and conjugated spinor $\Phi_2$ together as an 8-component column

$$\Psi = \begin{pmatrix} \Psi_1 \\ \Phi_2 \end{pmatrix}. \quad (15)$$

This representation of the Dirac spinors is often used to study the properties of solutions of the Dirac equation [14], [15].

First, let us try to assign $\Phi_2 = \overline{\Psi}_2$ [14]. With such an arrangement of the spinor components, we obtain a correspondence to the basis conjugated spinors-columns

$$\theta^1(p) \cong w_1' = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}; \quad \theta^2(p) \cong w_2' = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}; \quad -\frac{\partial}{\partial \theta^3(p)} \cong w_3' = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}; \quad -\frac{\partial}{\partial \theta^4(p)} \cong w_4' = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \quad (16)$$

Then the Dirac matrices will perform the following transformations

$$\gamma^0_D : w_1' \to w_1' ; \quad w_2' \to w_2' ; \quad w_3' \to -w_3' ; \quad w_4' \to -w_4' ;$$

$$\gamma^1_D : w_1' \to -w_4' ; \quad w_2' \to -w_3' ; \quad w_3' \to w_2' ; \quad w_4' \to w_1' ;$$

$$\gamma^2_D : w_1' \to -iw_4' ; \quad w_2' \to iw_3' ; \quad w_3' \to iw_2' ; \quad w_4' \to -iw_1' ;$$

$$\gamma^3_D : w_1' \to -w_3' ; \quad w_2' \to w_4' ; \quad w_3' \to w_1' ; \quad w_4' \to -w_2';$$

$$\gamma^5_D : w_1' \to w_3' ; \quad w_2' \to w_4' ; \quad w_3' \to w_1' ; \quad w_4' \to w_2'. \quad (17)$$

Comparison of (17) with (11) shows that the matrix representation of gamma operators under their action on Grassmann densities (16) is

$$\gamma^0 \cong \left( \begin{array}{cc} \gamma^0_D & 0 \\ 0 & -\gamma^0_D \end{array} \right); \quad \gamma^1 \cong \left( \begin{array}{cc} \gamma^1_D & 0 \\ 0 & -\gamma^1_D \end{array} \right); \quad \gamma^2 \cong \left( \begin{array}{cc} \gamma^2_D & 0 \\ 0 & -\gamma^2_D \end{array} \right); \quad \gamma^3 \cong \left( \begin{array}{cc} \gamma^3_D & 0 \\ 0 & -\gamma^3_D \end{array} \right); \quad \gamma^5 \cong \left( \begin{array}{cc} \gamma^5_D & 0 \\ 0 & -\gamma^5_D \end{array} \right). \quad (18)$$

Second, we can try to use CPT-conjugate spinors as columns of conjugated spinors. Let us try to assign $\Phi_2 = \Psi_{2,CPT}$

$$\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_{2,CPT} \end{pmatrix}. \quad (19)$$

According to [9] and [10],

$$\Psi_{CPT} = \pm R_{-q} R_{-e} \gamma^{26} \Psi^*, \quad (20)$$

where $R_{-q}$ changes to the opposite value $q$ of the electric charge, operator $R_{-e}$ changes to opposite coordinates $\chi'$, and $\gamma^{26} = \frac{1}{2}(\gamma^2 \gamma^6 - \gamma^6 \gamma^2)$. Wherein

$$\gamma^{26} = \int d^3 p \left[ -\frac{\partial}{\partial \theta^3(p)} \theta^1(p) + \frac{\partial}{\partial \theta^4(p)} \theta^2(p) - \frac{\partial}{\partial \theta^1(p)} \theta^3(p) - \frac{\partial}{\partial \theta^2(p)} \theta^4(p) \right], \quad (21)$$

$$\gamma^{26} \cdot \frac{\partial}{\partial \theta^1} \to \frac{\partial}{\partial \theta^3}; \quad \frac{\partial}{\partial \theta^2} \to -\frac{\partial}{\partial \theta^3}; \quad \theta^3 \to -\theta^3; \theta^4 \to -\theta^4; \quad (22)$$

Let us choose the minus sign in (20). We obtain a correspondence to the basis conjugated spinors-columns

...
\[- \frac{\partial}{\partial \theta^2(p)} \approx w'_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} ; \quad - \frac{\partial}{\partial \theta^4(p)} \approx w'_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} ; \quad \theta^1(p) \approx w'_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} ; \quad \theta^2(p) \approx w'_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \quad (23)\]

Similar to the previous consideration, we get the matrix representation of gamma operators under their action on Grassmann densities (23)

\[\tilde{\gamma}^0 \approx \begin{pmatrix} \gamma^0_D \\ 0 \\ 0 \end{pmatrix} ; \quad \tilde{\gamma}^1 \approx \begin{pmatrix} 0 \\ \gamma^1_D \\ 0 \end{pmatrix} ; \quad \tilde{\gamma}^2 \approx \begin{pmatrix} 0 \\ \gamma^2_D \\ -\gamma^2_D \end{pmatrix} ; \quad \tilde{\gamma}^3 \approx \begin{pmatrix} \gamma^3_D \\ 0 \\ 0 \end{pmatrix} ; \quad \tilde{\gamma}^5 \approx \begin{pmatrix} \gamma^5_D \\ 0 \\ 0 \end{pmatrix}. \quad (24)\]

At first glance, it seems that from relations (17) and (23) it follows that conjugated spinors in the matrix representation are transformed according to a law that differs from the law of transformation of spinors. However, it is not. It can be noted that the difference between the basis conjugated spinors in the two considered conjugation variants is only in the order of the basis spinors. However, it is not. It can be noted that the difference between the basis conjugated spinors in the two considered conjugation variants is only in the order of the basis spinors in the basis. As the basis conjugated spinors, we can choose any linear combination of them that does not violate the positive frequency of the annihilation operators and the negative frequency of the creation operators. Let us choose as the conjugated spinor a linear combination corresponding to the 4-component column

\[i\gamma^5_D\gamma^2_{D,2,\text{CPT}} = \Psi^\prime_{2,\text{CPT}}. \]

Then the basis spinors of the conjugated spinors-columns are

\[- \frac{\partial}{\partial \theta^2(p)} \approx w'_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} ; \quad - \frac{\partial}{\partial \theta^4(p)} \approx w'_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} ; \quad -\theta^1(p) \approx w'_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} ; \quad -\theta^2(p) \approx w'_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad (25)\]

and the 8-component spinor

\[\Psi' = \begin{pmatrix} \Psi'_1 \\ i\gamma^5_D\gamma^1_D\Psi'_{2,\text{CPT}} \end{pmatrix}. \quad (26)\]

is transformed by gamma operators in accordance with the relations

\[\tilde{\gamma}^0 \approx \begin{pmatrix} \gamma^0_D \\ 0 \\ 0 \end{pmatrix} ; \quad \tilde{\gamma}^1 \approx \begin{pmatrix} 0 \\ \gamma^1_D \\ 0 \end{pmatrix} ; \quad \tilde{\gamma}^2 \approx \begin{pmatrix} 0 \\ \gamma^2_D \\ -\gamma^2_D \end{pmatrix} ; \quad \tilde{\gamma}^3 \approx \begin{pmatrix} \gamma^3_D \\ 0 \\ 0 \end{pmatrix} ; \quad \tilde{\gamma}^5 \approx \begin{pmatrix} \gamma^5_D \\ 0 \\ 0 \end{pmatrix}. \quad (27)\]

Thus, when choosing the Grassmann densities (25) as the basis conjugated spinors, we obtain for the conjugated spinors the same transformation formulas as for the spinors themselves, but in this case the matrix \(\gamma^5_D\) changes sign. That is, the "right" spinor \(\Psi'_{1,R} = \frac{1 + \gamma^5_D}{2}\Psi'_1\) corresponds to the "left" antispinor \(\Psi'_{2,L} = \frac{1 - \gamma^5_D}{2}i\gamma^5_D\gamma^1_D\Psi'_{2,\text{CPT}}\).

Note that in [15] the 8-component spinor satisfying the Dirac equation was written in the form

\[\Psi' = \begin{pmatrix} \Psi'_1 \\ \Psi'_2 \end{pmatrix} = \begin{pmatrix} \Psi'_1 \\ i\gamma^2_D\Psi'_{2,\text{CPT}} \end{pmatrix}. \quad (28)\]

The formula for the charge conjugated spinor in the matrix representation

\[\Psi'^* = i\gamma^5_D\Psi'_{2,\text{CPT}}(x) \quad (29)\]

differs by a factor \(\gamma^5_D\) from the CPT-conjugated spinor formula in the matrix representation

\[\Psi'^*_{2,\text{CPT}}(x) = i\gamma^5_D\gamma^1_D\Psi'_{2,\text{CPT}}(x'). \quad (30)\]

This factor arises due to the fact that during CPT conjugation, the sign of spatial coordinates and coordinates along the time axis changes, and therefore \(x'^\mu = -x^\mu\) and \(\partial'^\mu = -\partial^\mu\).
Note the fundamental difference between the operators $\gamma^\mu$ and $\gamma_D^\mu$ [10]. Operators $\gamma_D^\mu$ have two different causes. First, there are matrix representations of the operators $\gamma^\mu$. Second, these are matrix operators arising from the replacement of the basis of conjugate spinors, which does not change the physical quantities themselves. In our case, the choice of basis (25) consists of a change in the order of the terms and a change in the sign of two of the four basis spinors. Spinors are specified up to a sign, but the relative signs of their components matter.

Therefore, when considering matrix representations of superalgebraic spinors, it is necessary to distinguish between matrices that represent gamma operators and matrices associated with the choice of basis spinors. In the superalgebraic representation of spinors, in addition to gamma operators $\gamma^\mu$, there are Lorentz-invariant operators $\tilde{\gamma}^\mu$ composed of the creation and annihilation operators of spinors and antispinors [8]. As shown in [10], it is these operators that transform the basis of spinors and conjugated spinors. It is their matrix representation that corresponds to the matrices associated with the choice of basis spinors.

4. Action of spatial parity $P$, time inversion $T$, and charge conjugation $C$ operators on the spinor vacuum

In [9] and [10], we derived formulas for operators of reversal operator $R$, spatial parity (inversion) $P$, time inversion (reversal) $T$ and charge conjugation $C$

$$R = R_{\mu\rho} R^\mu_{\rho\alpha} (\bullet)^\dagger,$$

$$P = R_{\mu\rho} R_{\mu\rho}^\dagger R_{\rho\lambda} = R_{\mu\lambda} R_{\rho\lambda}^\dagger,$$

$$T = R R_{\mu\rho} R_{\mu\rho}^\dagger (\bullet)^* = R_{\mu\lambda} R_{\mu\lambda}^\dagger,$$

$$C = R R_{\mu\rho} R_{\rho\lambda} = R_{\mu\lambda} R_{\rho\lambda}^\dagger.$$

In [9], we made a statement about the action of operators (31) - (34) on the spinor vacuum (6) and the alternative vacuum (8)

$$R \Psi_V = \Psi_{\text{altV}},$$

$$R \Psi_{\text{altV}} = \Psi_{V},$$

$$P \Psi_V = \Psi_{N},$$

$$T \Psi_V = \Psi_{\text{altV}},$$

$$C \Psi_V = \Psi_{\text{altV}}.$$

However, the proofs of these statements, with the exception of formula (35), were either absent or were not completely correct. Below we will correctly prove formulas (35)-(38). First, let us indicate how the operators $R$, $P$, $T$, $C$ and the factors $R_{\mu\nu}$, $R_{\mu\lambda}$, $R_{\rho\mu}$ included in them change the spatial momentum $p$, on which the Grassmann densities depend [9]

$$R : p \rightarrow p,$$

$$P, R_{\mu\lambda} : p \rightarrow -p,$$

$$T, R_{\mu\rho} : p \rightarrow -p,$$

$$C, R_{\rho\mu} : p \rightarrow p.$$

Consider the action of the operator $P$ on each factor $b_{\alpha\beta}(p_1)\bar{b}_{\gamma\delta}(p_1)$ in $\Psi_V(p_1)$ given in (7). The brackets $<\cdot\cdot\cdot>$ mean that there is no automatic summation at index $I$. The proof is very close to the insufficiently correct proof we made in [9]. But, in contrast to [9], the action on $b_{\alpha\beta}(p_1)\bar{b}_{\gamma\delta}(p_1)$ the operator $R_{\mu\lambda} R_{\rho\mu}$, not the operator $R_{\mu\rho}$, is considered.
\[ P_{b_{d>}}(p_i)b_{d>}(p_i) = R_{-\delta} R_{\varphi} (b_{d>}(p_i)b_{d>}(p_i)) = (R_{-\delta}^\varphi \hat{Q} b_{d>}(p_i))(R_{-\delta}^\varphi \hat{Q} b_{d>}(p_i)) = \]
\[ = -(R_{-\delta}^\varphi \hat{Q} b_{d>}(p_i))R_{-\delta}^\varphi (\hat{Q} b_{d>}(p_i)) = -(\hat{Q} R_{-\delta}^\varphi b_{d>}(p_i))(\hat{Q} R_{-\delta}^\varphi b_{d>}(p_i)) = \]
\[ = \left\{ \hat{\gamma}^0 \exp(\hat{\gamma}^0 \varphi_k) \frac{\partial}{\partial \theta^{\text{d>}}(p_i)} \right\} \left( \hat{\gamma}^0 \exp(\hat{\gamma}^0 \varphi_j) \frac{\partial}{\partial \theta^{\text{d>}}(p_i)} \right)^\dagger = \]
\[ = \left\{ \exp(-\hat{\gamma}^0 \varphi_k) \frac{\partial}{\partial \theta^{\text{d>}}(p_i)} \right\} \left( \hat{\gamma}^0 \exp(-\hat{\gamma}^0 \varphi_j) \frac{\partial}{\partial \theta^{\text{d>}}(p_i)} \right)^\dagger = \]
\[ = \left\{ \exp(-\hat{\gamma}^0 \varphi_k) \right\} \left( \hat{\gamma}^0 \exp(-\hat{\gamma}^0 \varphi_j) \right)^\dagger = \]
\[ b_{d>}(p_i)b_{d>}(p_i) = P \Psi_V(p_i) = \Psi_V(-p_i). \]

It follows from (40) that \( P \Psi_V(p_i) = \Psi_V(-p_i). \) Then
\[ P \Psi_V = \prod_i (P \Psi_V(p_i)) = \prod_i \Psi_V(-p_i) = \prod_i \Psi_V(p_i) = \Psi_V. \] (41)

Statement (36) is proved.

Now consider the action of the time inversion operator (33) on \( \Psi_V. \) To do this, first consider the action of the operator \( T \) on \( b_{d>}(p_i)b_{d>}(p_i). \)
\[ T b_{d>}(p_i)b_{d>}(p_i) = R_{-\delta}^\varphi (b_{d>}(p_i)b_{d>}(p_i))^\dagger = \]
\[ = (R_{-\delta}^\varphi \hat{\gamma}^\varphi \hat{Q} b_{d>}(p_i))(R_{-\delta}^\varphi \hat{\gamma}^\varphi \hat{Q} b_{d>}(p_i))^\dagger = \]
\[ = \left[ R_{-\delta}^\varphi \hat{\gamma}^\varphi \hat{\gamma}^\varphi \exp(\hat{\gamma}^0 \varphi_k) \frac{\partial}{\partial \theta^{\text{d>}}(p_i)} \right] \left[ R_{-\delta}^\varphi \hat{\gamma}^\varphi \exp(\hat{\gamma}^0 \varphi_j) \frac{\partial}{\partial \theta^{\text{d>}}(p_i)} \right]^\dagger = \]
\[ = \left[ R_{-\delta}^\varphi \hat{\gamma}^\varphi \exp(\hat{\gamma}^0 \varphi_k) \frac{\partial}{\partial \theta^{\text{d>}}(p_i)} \right] \left[ R_{-\delta}^\varphi \hat{\gamma}^\varphi \exp(-\hat{\gamma}^0 \varphi_j) \theta^{\text{d>}}(p_i) \right]^\dagger = \]
\[ = \left[ R_{-\delta}^\varphi \exp(-\hat{\gamma}^0 \varphi_k) \hat{\gamma}^\varphi \hat{\gamma}^\varphi \frac{\partial}{\partial \theta^{\text{d>}}(p_i)} \right] \left[ R_{-\delta}^\varphi \exp(-\hat{\gamma}^0 \varphi_j) \hat{\gamma}^\varphi \theta^{\text{d>}}(p_i) \right]^\dagger = \]
\[ = \left[ \exp(-\hat{\gamma}^0 \varphi_k) \hat{\gamma}^\varphi \hat{\gamma}^\varphi \frac{\partial}{\partial \theta^{\text{d>}}(p_i)} \right] \left( \exp(-\hat{\gamma}^0 \varphi_j) \hat{\gamma}^\varphi \theta^{\text{d>}}(p_i) \right)^\dagger = \]

According to (11), operator \( \hat{\gamma}^\varphi \) transforms \( \frac{\partial}{\partial \theta^{\text{a}}(-p_i)} \) into \( \pm \theta^{\text{a}}(p_i) \) and \( \theta'(p_i) \) into \( \pm \frac{\partial}{\partial \theta^{\text{a}}(-p_i)} \), where the plus sign for \( a \) equal to 1 and 3, and the minus sign for \( a \) equal to 2 and 4. Therefore
\[ T b_{d>}(p_i)b_{d>}(p_i) = \pm \left[ \exp(-\hat{\gamma}^0 \varphi_j) \theta^{\text{a>}}(p_i) \right] \left[ \exp(-\hat{\gamma}^0 \varphi_k) \frac{\partial}{\partial \theta^{\text{a>}}(p_i)} \right] = \]
\[ = \pm b_{\text{d>}}(-p_i)b_{\text{d>}}(-p_i). \]

In \( \Psi_V(p_i), \) the number of factors with a minus sign is even, and we get that \( \Psi_V(p_i) \) is transformed by the operator \( T \) into \( \Psi_{\text{d>}}(p_i) \). As a result
\[ T\Psi_v = \prod_i (T\Psi_v(p_i)) = \prod_i \Psi_{avV}(-p_i) = \prod_i \Psi_{avV}(p_i) = \Psi_{avV}. \] (44)

Statement (37) is proved.

The action of the charge conjugation operator (34) on \( \Psi_v \) is considered in a similar way. First, consider the action of the operator \( C \) on \( \tilde{b}_{\downarrow>} (p_i) \).

\[
Cb_{\downarrow>} (p_i) \tilde{b}_{\downarrow>} (p_i) = RR_q R_{\tilde{q}, \gamma} b_{\downarrow>} (p_i) \tilde{b}_{\downarrow>} (p_i) = (i\hat{\gamma}^{56} \tilde{b}_{\downarrow>} (p_i))(i\hat{\gamma}^{56} b_{\downarrow>} (p_i)) =
\]

\[
= \left( i\hat{\gamma}^{56} \exp(\hat{\gamma}^{01}) \varphi_j \theta^{\downarrow>} (p_i) \right) \left( i\hat{\gamma}^{56} \exp(\hat{\gamma}^{0k}) \varphi_k \frac{\partial}{\partial \theta^{\downarrow>} (p_i)} \right)
= \left( \exp(\hat{\gamma}^{01}) \varphi_j i\hat{\gamma}^{56} \theta^{\downarrow>} (p_i) \right) \left( \exp(\hat{\gamma}^{0k}) i\hat{\gamma}^{56} \frac{\partial}{\partial \theta^{\downarrow>} (p_i)} \right).
\] (45)

From (11) we obtain that

\[ i\hat{\gamma}^{56} \frac{\partial}{\partial \theta^1} \to \frac{\partial}{\partial \theta^1}; \quad i\hat{\gamma}^{56} \frac{\partial}{\partial \theta^2} \to \frac{\partial}{\partial \theta^1}; \quad i\hat{\gamma}^{56} \frac{\partial}{\partial \theta^3} \to -\theta^2; \quad i\hat{\gamma}^{56} \frac{\partial}{\partial \theta^4} \to -\theta^3; \quad i\hat{\gamma}^{56} \frac{\partial}{\partial \theta^2} \to \theta^2; \quad i\hat{\gamma}^{56} \frac{\partial}{\partial \theta^3} \to \theta^3. \] (46)

It follows from (46) that \( i\hat{\gamma}^{56} \theta^j (p_i) = \pm \theta^j (p_i) \) and \( i\hat{\gamma}^{56} \frac{\partial}{\partial \theta^k} (p_i) = \pm \frac{\partial}{\partial \theta^k} (p_i) \). Therefore

\[ C\Psi_v (p_i) = \Psi_{avV} (p_i). \] (47)

Taking into account (6)-(9), we obtain

\[ C\Psi_v = \Psi_{avV}. \] (48)

Statement (38) is proved.

5. Conclusions

We have derived formulas for 8-component matrix representation of spinors and carried out a correct proof that the spatial parity operator \( P \) leaves the spinor vacuum invariant, and the operators of time inversion \( T \) and charge conjugation \( C \) transform the spinor vacuum into an alternative one.

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