Tensorial perturbations in a brane with induced gravity

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Abstract. We calculate the amplitude of gravitational waves produced by inflation on a de Sitter brane embedded in five-dimensional anti-de Sitter bulk space-time, extending previous calculations in Randall-Sundrum type cosmology to include the effect of induced gravity corrections on the brane. These corrections arise via a term in the brane action that is proportional to the brane Ricci scalar. We find that, as in the Randall-Sundrum case, there is a mass gap between the discrete zero-mode and a continuum of massive bulk modes, which are too heavy to be excited during inflation. We give the normalisation of the zero-mode as a function of the Hubble rate on the brane and are thus able to calculate the high energy correction to the spectrum of gravitational wave (tensor) modes excited on large scales during inflation from initial vacuum fluctuations on small scales.

1. Introduction
String theory and M theory have motivated interest in brane-world models that describe extra-dimensional gravity. In these models, the observable four-dimensional (4D) universe is a brane hyper-surface embedded in a higher-dimensional bulk space. A simple but rich phenomenology for cosmology is provided by the Randall-Sundrum (RS) model [1] with a single brane in a five-dimensional (5D) anti de Sitter (AdS$_5$) space-time (for a review, see for example Ref. [2]). Although the fifth dimension is infinite, the zero-mode of the 5D graviton, corresponding to 4D gravitational waves, is localised at low energies on the brane due to the warped geometry of the bulk. This allows one to recover general relativity in the low-energy limit.

Quantum corrections to the RS model are expected to arise from the induced coupling of brane matter to bulk gravitons. This is the so-called induced gravity (IG) effect, that leads to the appearance of terms proportional to the 4D Ricci scalar in the brane action [3, 4, 5]. Hence, the IG effect can appear as a correction to the RS brane-world at high energy (see Refs. [6, 7, 8, 9, 10]). In fact, the IG effect will operate at high energies, above the brane tension, which is the threshold energy for RS modifications to general relativity. As a consequence high-energy inflation will be subject to the IG effect, and we seek to determine the consequent changes to the gravitational waves generated during inflation.

In the RS brane-world, the zero-mode of the 5D graviton is a massless spin-2 field on the brane that represents 4D gravitational waves. But there are also massive modes on the brane that arise via the projection onto the brane of the 5D graviton. These massive modes could qualitatively alter the spectrum of gravitational waves generated during inflation. It turns out that there is a mass gap between the zero-mode and the the massive modes [11], so that the massive
modes are too heavy to be excited and the spectrum is qualitatively the same as in general relativity. However, the amplitude of the zero-mode is boosted at high energies, compared with the standard 4D case [12].

We show that significant changes to the RS case are introduced by the IG term at high energies: the IG correction acts to limit the growth of amplitude, relative to the 4D case [12].

2. Gravitational field equations

We assume that the 5D bulk contains only negative vacuum energy, and the gravitational field obeys Einstein gravity. The 4D brane has a positive brane tension, and an induced gravity term localised on it. Then the gravitational action is

$$S = \frac{1}{2\kappa_5^2} \int d^5x \sqrt{-(5)g} \left[ (5)R - 2\Lambda_5 \right] + \int_{\text{brane}} d^4x \sqrt{-g} \left[ -\sigma + \frac{\gamma}{2\kappa_4^2} R + \mathcal{L}_{\text{matter}} \right],$$

where $\gamma$ is a dimensionless constant controlling the strength of the IG correction, with $\gamma = 0$ giving the RS model. The brane tension is $\sigma (\geq 0)$ and $\Lambda_5 (\leq 0)$ is the bulk cosmological constant. The fundamental energy scale of gravity is the 5D scale $M_5$, where $\kappa_5^2 = \frac{8\pi}{M_5^3}$. The 4D Planck scale $M_4 \sim 10^{19}$ GeV is an effective scale, fixed by the effective gravitational coupling constant $\kappa_4^2 = \frac{8\pi G}{M_4^2} = \frac{8\pi}{M_5^2}$ on the brane at low energies (typically $M_4 \gg M_5$). Note that for the low- and high-energy limits to yield a positive effective gravitational coupling constant, we require $0 \leq \gamma < 1$. The 5D field equations following from the bulk action are unchanged from the RS case:

$$^{(5)}G_{\mu\nu} = -\Lambda_5^{(5)}g_{\mu\nu}. \quad (2)$$

The IG correction term enters via the junction conditions at the brane. Assuming mirror ($Z_2$) symmetry about the brane, these are [7]

$$K_{\mu\nu} - K g_{\mu\nu} = -\frac{\kappa_5^2}{2} \left( T_{\mu\nu} - \sigma g_{\mu\nu} - \frac{\gamma}{\kappa_4^2} G_{\mu\nu} \right), \quad (3)$$

where $K_{\mu\nu}$ is the extrinsic curvature and $T_{\mu\nu}$ is the energy-momentum tensor of matter on the brane, which is conserved on the brane.

The length scale $\ell$ and energy scale $\mu$ associated with the bulk curvature are given by

$$\Lambda_5 = -\frac{6}{\ell^2} = -6\mu^2. \quad (4)$$

If we impose the RS fine-tuning for the brane tension, $\kappa_5^2\sigma = \sqrt{-6\Lambda_5}$, this sets the brane cosmological constant to zero, and then we have [7, 10]

$$\kappa_4^2 \equiv \kappa_5^2\mu(1 - \gamma). \quad (5)$$

2.1. Background bulk metric

We are interested in 4D tensor metric perturbations about a de Sitter brane (modelling extreme slow-roll inflation) in an AdS$_5$ bulk. Such a brane is a solution of the junction equations (3) with a constant energy density on the brane, $\rho > 0$, and hence a constant Hubble rate on the brane, $H$. The background space-time metric, i.e., an AdS$_5$ bulk sliced into 4D slices with de Sitter geometry, may be written in the Gaussian normal form

$$ds^2 = n^2(y) \left[ -dt^2 + \exp(2Ht)dx^2 \right] + dy^2. \quad (6)$$

There is another solution which corresponds to the Dvali, Gabadadze and Porrati (DGP) model [4]. We disregard this solution as it does not correspond to a perturbation of the RS model. Indeed, the DGP model produces a low energy modifications to general relativity.
The warp factor in the bulk is
\[ n(y) = \frac{H}{\mu} \sinh [\mu (y_- - |y|)] . \] (7)

The brane is fixed at \( y = 0 \) and we choose \( n(0) = 1 \) on the brane, so that \( y_- \) is given by
\[ y_- = \frac{1}{\mu} \arcsinh \frac{\mu}{H} . \] (8)

The Hubble parameter \( H \) is given by the modified Friedman equation
\[ H^2 = \frac{\kappa}{3\gamma} \rho + \sigma + \frac{6\kappa^2}{\gamma\kappa^4} \left[ 1 - \sqrt{1 + \frac{\gamma^2}{3\kappa^2} \left( \rho + \sigma + \frac{\gamma\kappa^2}{12\kappa^4} \right) } \right] , \] (9)
where we have imposed Eq. (5). In the following we take the energy density, \( \rho \), to be constant, corresponding to the inflaton energy density in the extreme slow roll limit. We recover the Randall-Sundrum case \([12]\) in the limit \( \gamma \to 0 \). The bulk coordinate is bounded such that \( |y| \leq y_- \), where \( |y| = y_- \) corresponds to the location of the Cauchy horizon, where \( n(y_-) = 0 \). The warp factor has its maximum at the brane.

2.2. Perturbed bulk metric

We now consider tensor metric perturbations. The perturbed metric reads \([12]\)
\[ ds^2 = n^2(y) \left[ -dt^2 + e^{2Ht}(\delta_{ij} + h_{ij}) \ dx^i dx^j \right] + dy^2, \] (10)
where \( h_{ij} \) is transverse and traceless. The wave equation in the bulk for the perturbations is
\[ \delta \left( ^{(5)}G_{ab} \right) = 0 , \] (11)
the same as in the RS case. This means that the bulk mode solutions for metric perturbations will be the same as in the RS case \([12]\), but the IG junction conditions will introduce changes to the normalisation and amplitudes of the modes. Indeed, the IG junction condition at the brane is very different from the RS case. The tensor part of Eq. (3) implies
\[ \delta K^\mu_{\nu} = \frac{\gamma\kappa^2}{2\kappa^4} \delta G^\mu_{\nu} = \frac{\gamma}{2\mu(1-\gamma)} \delta G^\mu_{\nu} , \] (12)
where we have neglected any tensor contributions of the anisotropic stress exerted by matter on the brane, and the second equality follows from Eq. (5). We recover RS case for \( \gamma \to 0 \).

The general metric perturbation \( h_{ij} \) can be decomposed as
\[ h_{ij}(t, \bar{x}, y) = \int dm \psi_m(t) \mathcal{E}_m(y) e_{ij}(\bar{x}) , \] (13)
where \( e_{ij} \) is a transverse traceless harmonic on the spatially flat 3-space, i.e., \( \vec{\nabla}^2 e_{ij} = -k^2 e_{ij} \).

Then the perturbed Einstein equation (11) implies the following equation for the Kaluza-Klein (KK) modes
\[ \ddot{\psi}_m + 3H \dot{\psi}_m + \left[ k^2 e^{-2Ht} + m^2 \right] \psi_m = 0 , \] (14)
\[ (n^4 \mathcal{E}'_m) + m^2 n^2 \mathcal{E}_m = 0 , \] (15)
where a dot denotes a derivative with respect to $t$ and a prime a derivative with respect to $y$. The junction condition (12) imposes a boundary condition for $E_m$

$$E'_m(0) = -\frac{m^2 \gamma}{2\mu(1-\gamma)} E_m(0).$$

(16)

It is important to note that this boundary condition depends on the mass of the modes, $m^2$, due to the IG term. Only the zero-mode, $m = 0$, has the same boundary condition as in the RS case. As a result of this new feature, the scalar product of the eigenmode functional space has to include suitable boundary terms (this is qualitatively the same as the case of Gauss-Bonnet modifications to the RS model [13, 14].). In fact, the eigenmodes resulting from Eqs. (15) and (16) are orthonormal with respect to the following scalar product:

$$\langle E_m, E_{\tilde{m}} \rangle = 2 \int_0^y dy n^2 E_m E_{\tilde{m}} + \frac{\gamma}{\mu(1-\gamma)} E_m(0) E_{\tilde{m}}(0) = \delta(m, \tilde{m}),$$

(17)

where $\delta(m, \tilde{m})$ denotes a Kronecker symbol for the discrete modes and a Dirac distribution for the continuous ones. For $\gamma \to 0$ the scalar product reduces to the one used in the RS case.

The spectrum of modes resulting from Eqs. (15) and (16) has a zero-mode solution as in the RS case [12]

$$E_0 = C \quad \text{or} \quad E_{\tilde{m}} = 0$$

(18)

where $C$ is a constant. The zero mode is normalizable. Using Eqs. (17) and (18), the condition $\langle E_0, E_0 \rangle = 1$ gives the normalisation constant $C = C(H/\mu)$ as a function of the Hubble rate relative to the AdS scale, $\mu$, where

$$C^{-2}(x) = \frac{\gamma}{\mu(1-\gamma)} + \frac{1}{\mu} \left[ \sqrt{1 + x^2} - x \arcsinh \frac{1}{x} \right].$$

(19)

This reduces to the RS result [12] when $\gamma = 0$. Furthermore, the massless mode $E_0$ is the only normalizable discrete ($m^2 < \frac{9}{4} H^2$) mode. On the other hand, there is a continuous spectrum of normalizable modes with $m^2 > \frac{9}{4} H^2$ (for details see Ref. [9]). Consequently, the spectrum of KK modes is similar to the RS case: we have only one discrete light mode with $m^2 = 0$ and then a mass-gap before the continuum of heavy modes.

3. Amplitude of gravitational waves on the brane

We will now estimate the spectrum of graviton fluctuations generated in de Sitter inflation on the brane. We treat each normalizable KK mode as a quantum field in four dimensions, as in the RS case [12]. Taking these 4D fields to be in an initial vacuum state on small scales is consistent with taking an incoming AdS vacuum state in the five-dimensional viewpoint [15].

Massive modes with $m^2 > \frac{9}{4} H^2$ remain underdamped even on large scales, and fluctuations are strongly suppressed on super-horizon scales. They remain in their vacuum state [12, 16]. However initial vacuum fluctuations in the zero-mode become over-damped as they are stretched beyond the Hubble scale. The zero-mode thus acquires a spectrum of classical perturbations on super-horizon scales. For $m^2 = 0$, the effective action has the standard form of 4D general relativity, except for the overall factor $\kappa_5^2$ instead of $\kappa_4^2$ [17], which rescales the amplitude of quantum fluctuations accordingly [12]. The amplitude of the zero-mode metric fluctuations on the brane ($E_0 = C(H/\mu)$), where $C(H/\mu)$ is given in Eq. (19), then introduces a further rescaling relative to the 4D result, which is dependent on the Hubble scale relative to the AdS scale. The amplitude of gravitational waves produced on super-horizon scales on the brane is thus given by

$$A_{2T}^2 = \frac{2\kappa_5^2}{25} \left( \frac{H}{2\pi} \right)^2 F_\gamma^2(H/\mu),$$

(20)
where the correction to standard 4D general relativity is given by

\[ F_\gamma^2(H/\mu) = \frac{\kappa_5^2}{\kappa_4^2} C^2(H/\mu). \]  

(21)

From Eq. (19), we have

\[ F^{-2}_\gamma(x) = \gamma + (1 - \gamma) \left[ \sqrt{1 + x^2} - x^2 \text{arcsinh} \frac{1}{x} \right]. \]  

(22)

This correction depends on the IG coupling (through the parameter \( \gamma \)) and on the energy scale at which inflation occurs, relative to the 5D curvature scale \( \mu \). It reduces to the result of Ref. [12] for the RS case (\( \gamma \to 0 \)).

When \( x \equiv H/\mu \to 0 \), we have \( F_\gamma \to 1 \) and we recover the standard 4D result [18]. The amplitude of the normalised zero-mode on the brane gives the ratio between the effective 4D Newton constant at low energies, \( \kappa_4^2 \), and the 5D constant, \( \kappa_5^2 \).

For \( 0 < \gamma < 1 \) we find that tensor fluctuations are enhanced relative to the 4D result (\( F_\gamma > 1 \)) at high energies. However, unlike the RS case, the amplitude of the tensor zero-mode relative to the 4D general relativity result is bounded, \( 1 \leq F_\gamma^2 < 1/\gamma \). For \( H \gg \mu \) and \( 0 < \gamma < 1 \), we have

\[ F_\gamma^2(H/\mu) \approx \frac{1}{\gamma}, \]  

(23)

while the RS case (\( \gamma = 0 \)) yields [12]

\[ F_0^2(H/\mu) \approx \frac{3}{2} \frac{H}{\mu}. \]  

(24)

The qualitative behaviour is illustrated in Fig. 1 for \( \gamma = 0.1 \).

**Figure 1.** The dimensionless amplitude \( F_\gamma^2 \) of the tensor zero-mode relative to the 4D general relativity result, plotted against the dimensionless energy scale of inflation, \( H/\mu \), for \( \gamma = 0.1 \).

4. Conclusions

We have studied the tensorial metric perturbations on a brane-world model with an IG term on the brane action, where the background geometry corresponds to a de Sitter brane embedded in AdS5 bulk. The 5D wave equation and its fundamental solutions are not changed by the
IG term. The spectrum contains a normalizable zero-mode and a continuous tower of massive modes after a mass gap, \( m > \frac{3}{2} H \), as in the RS case. The massive modes are not excited during inflation, as in the RS case. However, the IG term changes the boundary conditions at the brane, and therefore changes the normalisation of the zero-mode, as shown by Eq. (19). This has the consequence of enhancing the amplitude of tensor perturbations (gravitational waves) produced by de Sitter inflation on the brane, Eq.(20), relative to the standard 4D result. Unlike the RS case, the relative enhancement at high energies is bounded in the presence of an IG correction term.

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