Non-Linear Oscillations of the Channel Wall Filled with a Viscous Liquid Induced by the Vibrating Foundation

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Abstract. The paper considers the issues of mathematical modelling nonlinear vibrations for the wall of a narrow parallel walled channel filled with a viscous incompressible liquid. The upper channel wall is a rigid rectangular plate supported by nonlinear spring with a cubic restoring force, while the bottom one is a fixed rigid rectangular plate. The case of nonlinear oscillations of the upper channel wall due to channel’s foundation vibration has been investigated in the frame of hydroelasticity problem. The main attention is paid to the consideration of steady-state nonlinear oscillations for the upper channel wall and the creeping motion of the liquid in the channel. The liquid layer reaction acting on the upper channel wall is found and the channel’s wall nonlinear oscillation equation is obtained taking into account the energy dissipation due to the liquid viscosity. It was shown that this equation coincided with the Duffing one. The solution of the nonlinear oscillation equation was carried out by the harmonic balance method. Based on the found solution, the hydroelastic response of the channel wall on a nonlinear elastic suspension to the channel foundation vibration was determined.

Keywords Non-linear oscillations, Hydroelastic response, Foundation vibration, Spring with a cubic nonlinearity, Viscous liquid, Harmonic balance method.

1. Introduction

Traditionally, when calculating the vibrations of the mechanical systems elements, they are limited to the linear theory for elastic elements [1,2]. For example, in this approach, the hydroelastic vibrations of plates and beams interacting with a liquid [3-11], as well as elastically fixed channel walls [12-15], are studied. In particular, in paper [3], a model was developed for studying the natural vibrations of a clamped circular plate, on one side of which there is an unlimited volume of an ideal liquid. Studies [4, 5] generalize the model proposed in [3], namely, in [4] the coupled hydroelasticity problem for a circular plate was solved, and in [5] the viscosity of the liquid was taken into account. The interaction of the elastic channel wall with the opposite vibrating rigid one via a viscous liquid filling the channel was investigated in [6]. Modelling of vibrations for a rectangular plate immersed in an ideal incompressible liquid with a free surface was carried out in [7]. Resonant vibrations of a membrane, which is part for the bottom of a reservoir filled with an ideal incompressible liquid, were considered in [8], as well as, vibrations of a rectangular plate on the ideal incompressible liquid surface were studied in [9]. The dynamics interaction simulation for a circular plate with an ideal liquid as applied to a pressure sensor installed at the pipe end was performed in [10]. Reference [11] is devoted to the
study of forced vibrations of a three-layer beam interacting with a layer of viscous liquid. In paper [12], the vibrations of a die supported by a linear spring and located at the channel bottom, during its interaction with an ideal liquid with a free surface, were studied. The forced vibrations simulation for the end seal flexibly restrained at the edge of a narrow channel filled with a viscous liquid was performed in [13, 14]. Longitudinal and transverse vibrations of a wall suspended on linear springs, as applied to a wedge-shaped channel filled with a viscous liquid, were studied in [15]. On the other hand, at present, elastic elements with significantly nonlinear characteristics are increasingly being used, which requires consideration of nonlinear oscillations [16]. For studying these oscillations, the simplest model is a single-mass oscillator with a nonlinear restoring force [16, 17]. However, there are practically no studies devoted to the use of this model to study the oscillations of the channel walls interacting with a viscous liquid. In the proposed paper, a model is developed for studying the vibrations of a rigid wall suspended on a spring with the cubic hardening nonlinearity, as applied to a channel filled with a viscous liquid and mounted on a vibrating foundation. This model is reduced to the Duffing equation and, as a result, a nonlinear hydroelastic response of the channel wall is determined.

2. Statement of the Problem
The schematic diagram of the considered oscillatory system is shown in Fig. 1. In particular, it shows a channel with rigid plane wall parallel to each other. The walls of the channel are enclosed in a common body, conventionally shown by a dotted line. We assume that the bottom wall is rigidly fixed to the body and the horizontal foundation. The foundation performs harmonic oscillations in the vertical direction with an amplitude $z_0$. The upper wall is suspended by a spring, which is fixed to the body and has a characteristic with a cubic hardening nonlinearity. The suspension is made so that the upper wall vibrates only in the direction perpendicular to its plane. The gap $\delta_0$ between the channel walls filled with a viscous incompressible liquid. The ends of the channel, on the left and on the right, are adjacent to the end cavities of the body, which contain the same liquid. The pressure in it has a constant component, taken as the initial level of pressure reading. In addition, the vibrating foundation acceleration acts on the liquid; the effect of gravity is neglected. Further, we assume that the dimensions of the channel walls size in the plan view are $b \times 2\ell$ and $b >> 2\ell$. The last assumption allows us to proceed to the study of the plane problem. We introduce the Cartesian coordinate system Oxz, the center of which coincides with the symmetry center of the bottom wall inner surface. We direct the Oz axis perpendicular to the surface of the bottom wall and vertically upward, and the Ox axis to the right along the $2\ell$ direction. The upper wall on an elastic suspension performs nonlinear oscillations with an amplitude $z_m$, induced by the foundation vibration, while the liquid can freely flow from the channel into the end cavities. We will study the forced steady-state nonlinear (anharmonic) oscillations of the upper channel wall due to the foundation vibration.

Figure 1. The scheme of narrow parallel walled channel filled with a viscous incompressible liquid and mounted on a vibrating foundation:
1 is the upper rigid wall of the channel attached by a massless nonlinear spring to the channel body, 2 is the bottom rigid wall of the channel, which is fixed to the channel body and the vibrating foundation, 3 is a viscous incompressible liquid, 4 is the channel body.
The law of motion of the vibrating base is assumed to be given as:

\[ z_0 = z_{0m} f_0(\omega t), \quad f_0(\omega t) = \sin(\omega t), \]  

then its vibration acceleration can be expressed in units of gravitational acceleration

\[ z''_0 = z_{0m} \frac{d^2 f_0(\omega t)}{dt^2} = -z_{0m} \omega^2 f_0(\omega t) = -kg \sin(\omega t), \]  

where \( t \) is time, \( \omega \) is the angular frequency, \( z_{0m} \) is the amplitude of the vibrating foundation, which presented as \( z_{0m} = kg/\omega^2 \), \( k \) is the vibration overload coefficient.

Due to the narrowness of the channel, we consider the dynamics equations for a viscous incompressible liquid within the framework of the hydrodynamic theory of lubrication, assuming its motion to be creeping [18], that is, we represent them in the form:

\[
\rho \left( \frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_z}{\partial z^2} \right) = \frac{\partial p}{\partial x},
\]

\[
\rho \left( \frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_z}{\partial z^2} \right) = \rho \ddot{z}_0 + \frac{\partial p}{\partial z},
\]

\[
\frac{\partial u_x}{\partial x} + \frac{\partial u_z}{\partial z} = 0.
\]

Here \( u_x, u_z \) are the projections of the liquid velocity on the coordinate axes, \( \rho \) is the liquid density, \( \nu \) is the kinematic coefficient of liquid viscosity, \( p \) is the liquid pressure.

Let us formulate the boundary conditions for Eqs. (3) taking into account that a viscous liquid adheres to the channel walls, i.e. we represent them as

\[ u_x = 0, \quad u_z = 0 \quad \text{at} \quad z = 0, \]  

\[ u_x = 0, \quad u_z = \frac{dz}{dt} \quad \text{at} \quad z = \delta_0 + z_m f(\eta t), \]  

in addition, we write down the conditions for equality of pressure in the cross sections at the ends of the channel and end cavities on the right and left

\[ p = -\rho \ddot{z}_0 (z - \delta_0/2) \quad \text{at} \quad x = \pm \ell. \]  

here \( z_m f(\eta t) \) is the motion law of the upper wall supported by a nonlinear spring, \( \eta \) is the oscillation frequency of the upper wall. When writing this condition, we take into account that the constant level of liquid pressure in the end cavities is taken to be zero (initial level of pressure reading). The second component of pressure due to the acceleration of the vibrating foundation is similar to the hydrostatic pressure due to gravity.

The equation of motion for a wall supported by a spring with cubic hardening nonlinearity has the form:

\[ m(\ddot{z} + \ddot{z}_0) + F(z) = h \int_{x_0}^{l} p|_{\eta = \delta_0 + z_m f(\eta t)} \, dx \]  

Here \( m \) is the mass of the wall, \( F(z) \) is a nonlinear restoring force.

The restoring force acting on the upper channel wall from the side of the supporting spring with cubic hardening nonlinearity can be represented as [16]

\[ F(z) = n_1 z + n_2 z^3. \]
where \( n_1 \) is the linear stiffness coefficient of the spring, \( n_3 \) is the cubic stiffness coefficient of the spring with hardening nonlinearity, i.e. \( n_3 > 0 \).

### 3. Determining of Non-Linear Hydroelastic Response of the Oscillatory System

According to the formulation problem under consideration, it is possible to single out small parameters for the hydroelasticity problem and introduce the following dimensionless variables

\[
\begin{align*}
\psi &= \frac{x}{\ell} \ll 1, \\
\lambda &= \frac{z_n}{\delta_0} \ll 1, \\
\xi &= \frac{z}{\delta_0}, \\
\zeta &= \frac{\eta}{U_\xi}, \\
\eta &= \frac{U_n}{\eta U_\xi}, \\
p &= -\rho \varepsilon_\eta (z - \delta_0/2) + \frac{\rho \varepsilon_\eta}{\delta_\eta} |p|.
\end{align*}
\]

Substituting (8) into (3) - (6) and neglecting small terms [19], we obtain

- the equation of motion for the upper channel wall supported by a spring with cubic hardening nonlinearity

\[
m \ddot{z} + n_n x + n_n x^3 = -\dot{z}_0 (m + b \ell \bar{\delta}_z \rho) + b \ell \dot{\varepsilon}_\eta \frac{\rho \varepsilon_\eta}{\delta_\eta} \int \varepsilon p \xi, (9)
\]

- dimensionless hydrodynamic equations for a thin layer of a viscous incompressible liquid

\[
\begin{align*}
\frac{\partial^2 U_\xi}{\partial \xi^2} &= \frac{\partial P}{\partial \xi}, \\
\frac{\partial P}{\partial \xi} &= 0, \\
\frac{\partial U_\eta}{\partial \xi} + \frac{\partial U_\xi}{\partial \xi} &= 0,
\end{align*}
\]

and the corresponding boundary conditions

\[
U_\xi = U_{\xi} = 0 \text{ at } \xi = 0, \\
U_\xi = 0, \quad U_\xi = \frac{1}{\eta} \frac{df}{dt} \text{ at } \xi = 1, \\
P = 0 \text{ at } \xi = \pm 1.
\]

Implementing the solution (10) taking into account the boundary conditions (11), we found an expression for the liquid pressure in the channel

\[
P = \frac{6(\xi^2 - 1) \frac{df}{dt}}{\eta dt}. (12)
\]

Taking into account (2), (12) in (9), we obtain

\[
m \frac{d^2 z}{dt^2} + K \frac{dz}{dt} + n_n z + n_n z^3 = (m + b \ell \bar{\delta}_z \rho) \kappa g \sin(\omega t), (13)
\]

where \( K = 8 \ell b \rho \nu (\bar{\delta}_z)^3 \) is the damping coefficient of a viscous liquid in the channel when it interacts with the channel walls.

As a result, we obtained the equation for nonlinear oscillations of the upper channel wall (13), which coincides with the Duffing equation [16, 20]. Let us solve this equation by the harmonic
balance method [20]. Following this method, we assume that the forced oscillation frequency of the upper channel wall practically coincides with the frequency of the driving force (vibrating foundation), and we seek the solution as harmonic one, i.e. \( \eta \approx \omega \) and \( z = z_m \sin(\omega t + \phi) \). After that, we linearize the nonlinear term (13) by performing it as a Fourier series and keeping only the first term of the expansion. Next, we will find the primary hydroelastic response of the considered oscillatory system.

As a result, carrying out linearization Eq. (12) by the harmonic balance method, we obtain

\[
\frac{d^2 z}{dt^2} + K \frac{dz}{dt} + n_1 z + \frac{3}{4} z_m^3 n_2 z = (m + b/\delta_\rho)kg \sin(\omega t) .
\]  

(14)

Solution Eq. (14) for steady-state forced nonlinear oscillations has the form

\[
z = z_m \sin(\omega t + \phi),
\]

\[
z_m = \frac{(m + b/\delta_\rho)kg/m}{\sqrt{\left[\omega^2(z_m) - \omega^2\right]^2 + (K\omega/m)^2}},
\]

(15)

where \( \omega^2(z_m) = (n_1/m) + \left(3/4\right) z_m^2 n_2/m \).

4. Results and Discussion

Based on the solution of the hydrodynamics problem for a thin viscous liquid layer, an expression was determined for the damping coefficient of this layer, which determines the friction in the considered oscillatory system. The above made it possible to obtain a Duffing-type equation for studying nonlinear oscillations of the upper channel wall. As a result of solving this equation by the harmonic balance method, the primary hydroelastic response of the upper channel wall to the vibration of channel foundation was obtained. This response is the expression for \( z_m \) in (15), i.e. the amplitude-frequency characteristic of the upper wall nonlinear vibrations for the channel under consideration. This characteristic can be constructed and investigated numerically [21]. In particular, by examining the nature of the change in \( z_m \) from frequency, it is possible to identify the characteristic zones of a jump-like change in the nonlinear oscillation amplitudes caused by the cubic hardening nonlinearity of the supporting spring, as well as to determine the oscillation amplitudes corresponding to these frequencies.

5. Summary and Conclusion

The article develops a mathematical model for the study of forced nonlinear vibrations of a rigid channel wall having an elastic suspension with a cubic hardening nonlinearity, excited by the vibration of the channel foundation. Within the framework of this model, an analytical expression is determined for the primary hydroelastic response of the channel wall, i.e., the amplitude-frequency characteristic of its nonlinear oscillations. This response allows us to conduct computational experiments to assess the influence of the viscosity and density of the liquid, the channel geometric dimensions, the stiffness coefficients of the supporting spring on the nonlinear vibrations of the channel wall. This model can be used to study the dynamic characteristics of elastically fixed elements of hydraulic drive systems, lubrication, cooling and fuel supply.

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