Policy Learning with Competing Agents

Roshni Sahoo
rsahoo@stanford.edu
Stanford University

Stefan Wager
swager@stanford.edu

Abstract

Decision makers often aim to learn a treatment assignment policy under a capacity constraint on the number of agents that they can treat. When agents can respond strategically to such policies, competition arises, complicating the estimation of the effect of the policy. In this paper, we study capacity-constrained treatment assignment in the presence of such interference. We consider a dynamic model where the decision maker allocates treatments at each time step and heterogeneous agents myopically best respond to the previous treatment assignment policy. When the number of agents is large but finite, we show that the threshold for receiving treatment under a given policy converges to the policy’s mean-field equilibrium threshold. Based on this result, we develop a consistent estimator for the policy effect. In simulations and a semi-synthetic experiment with data from the National Education Longitudinal Study of 1988, we demonstrate that this estimator can be used for learning capacity-constrained policies in the presence of strategic behavior.

1 Introduction

Decision makers often aim to learn policies for assigning treatments to human agents under capacity constraints [Athey and Wager, 2021, Bhattacharya and Dupas, 2012, Kitagawa and Tetenov, 2018, Manski, 2004]. These policies map an agent’s observed individual characteristics to a treatment assignment. For example, a school may use a policy to decide which applicants to admit, under the capacity constraint that they can only accept a small fraction of the applicant pool. Similarly, an employer may use a policy to decide which candidates should be extended offers, under the capacity constraint that they have a fixed number of positions to fill. To enforce the capacity constraint, a decision maker uses a selection criterion, such as a machine learning model, to score agents and assigns treatments to agents who score above a threshold, given by a quantile of the score distribution [Bhattacharya and Dupas, 2012]. When the decision maker has a capacity constraint, an agent’s treatment assignment depends on how their score ranks relative to that of other agents.

In policy learning, practitioners often assume that the observed data is exogenous to the treatment assignment policy. However, when human agents are being considered for the treatment and the agents have knowledge of the assignment policy, the observed data is not exogenous because agents may change their behavior in response to the policy. For example, in college admissions, students enroll in test preparation services and take advanced courses to improve their chances of getting accepted to college [Bound et al., 2009], and in job hiring, job candidates may join intensive bootcamps to improve their career prospects [Thayer and Ko, 2017]. A growing body of work focuses on policy learning in the presence of strategic human behavior [Björkegren et al., 2020, Frankel and Kartik, 2019a, Munro, 2020]. However, these works implicitly assume that the decision maker does not have a capacity constraint because they model an agent’s treatment assignment as only depending on their own strategic behavior, unaffected by the behavior of others in the population.

In this work, we study the problem of capacity-constrained treatment assignment in the presence of strategic behavior. This setting can be distinguished from previous works on policy learning with strategic agents because competition for the treatment arises due to the combination of agents’ strategic behavior and the decision maker’s capacity constraint. We frame the problem in a dynamic setting, where agents report

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their covariates to the decision maker and the decision maker assigns treatments at each time step. Suppose a decision maker deploys a fixed selection criterion for all time. At time step $t$, agents react to the policy from time step $t-1$, which depends on the fixed selection criterion and the threshold for receiving treatment at time step $t-1$. To enforce the capacity constraint, the decision maker sets the threshold for receiving treatment at time step $t$ to the appropriate quantile of the score distribution observed at time step $t$. So, the threshold for receiving treatment depends on agents’ strategic behavior. At an equilibrium induced by a fixed selection criterion, the threshold for receiving treatment is fixed over time. The goal of the decision maker, and the main goal of this work, is to find a selection criterion that obtains high equilibrium policy value, which is the policy value obtained at the equilibrium induced by the selection criterion.

The goal of learning a policy that maximizes the equilibrium policy value is motivated by prior works that estimate policy effects or treatment effects at equilibrium [Heckman et al., 1998, Munro et al., 2021, Wager and Xu, 2021]. Heckman et al. [1998] estimate the effect of a tuition subsidy program on college enrollment by accounting for the program’s impact on the equilibrium college skill price. Munro et al. [2021] estimate the effect of a binary intervention in a marketplace setting by accounting for the impact of the intervention on the resulting supply-demand equilibrium. Wager and Xu [2021] estimate the effect of supply-side payments on a platform’s utility in equilibrium.

In Section 2, we outline a dynamic model for capacity-constrained treatment assignment in the presence of generic strategic behavior. A key element of our model is that agents are myopic, so the covariates they report to the decision maker at time step $t$ depend only on the state of the system in time step $t-1$. Also, as in the aggregative games literature [Acemoglu and Jensen, 2010, 2015, Corchón, 1994], we assume that agents respond to an aggregate of other agents’ actions. In particular, at time step $t$, agents will react to the threshold for receiving treatment from time step $t-1$, which is an aggregate of agents’ strategic behavior from time step $t-1$. Finally, as in, e.g., Frankel and Kartik [2019a,b], we assume that agents are heterogenous in their raw covariates (covariates prior to modification) and in their ability to deviate from their raw covariates in their reported covariates.

In Section 3, we give conditions on our model that guarantee existence and uniqueness of equilibria in the mean-field regime, the limiting regime where at each time step, an infinite number of agents are considered for the treatment. Furthermore, we show that under additional conditions, the mean-field equilibrium arises via fixed-point iteration. In Section 4, we translate these results to the finite regime, where a finite number of agents, sampled i.i.d. at each time step, are considered for treatment. We show that as the number of agents grows large, the behavior of the system converges to the equilibrium of the mean-field model in a stochastic version of fixed-point iteration.

In Section 5, we propose a method for learning the selection criterion that maximizes the equilibrium policy value. Following Wager and Xu [2021], we take the approach of optimizing the selection criterion via gradient descent. To this end, we develop a consistent estimator for the policy effect, the gradient of the equilibrium policy value. To estimate the policy effect without disturbing the equilibrium, we adapt the approach of Munro et al. [2021], Wager and Xu [2021]. We recover components of the policy effect by applying symmetric, mean-zero perturbations to the selection criterion and the threshold for receiving treatment for each unit and running regressions from the perturbations to outcomes of interest. In Section 6, we validate that our policy effect estimator can be used to learn policies in presence of competition in simulations and a semi-synthetic experiment with data the National Education Longitudinal Study of 1988 [Ingels, 1994].

1.1 Related Work

The problem of learning optimal treatment assignment policies has received attention in econometrics, statistics, and computer science [Athey et al., 2018, Bhattacharya and Dupas, 2012, Kallus and Zhou, 2021, Kitagawa and Tetenov, 2018, Manski, 2004]. Treatments can be discrete-valued (typically, binary) or continuous-valued, and the policy may be subject to budget or capacity constraints. Most related to our work, Bhattacharya and Dupas [2012] study the problem of optimal capacity-constrained treatment assignment, where the decision maker can only allocate treatments to $1 - q$ proportion of the population, where $q \in (0, 1)$. They show that the welfare-maximizing assignment policy is a threshold rule on the agents’ scores, where agents who score above $q$-th quantile of the score distribution are allocated treatment. Our work differs from Bhattacharya and Dupas [2012] because we do not assume that the distribution of observed
covariates is exogenous to the treatment assignment policy.

Björkegren et al. [2020], Frankel and Kartik [2019a], Munro [2020] study policy learning in the presence of strategic behavior. Björkegren et al. [2020] propose a structural model for manipulation, estimate the parameters of this model with data from a field experiment, and compute the optimal policy under the estimated model. Frankel and Kartik [2019a] demonstrate that optimal policies that account for strategic behavior will underweight manipulable data. Munro [2020] studies the optimal unconstrained assignment of binary-valued treatments in the presence of strategic behavior, without parametric assumptions on agent behavior. The main difference between our work and these previous works is that we account for the equilibrium effects of strategic behavior that arise via competition.

The area of strategic classification in computer science is also related to our work [Brückner et al., 2012, Chen et al., 2020, Dalvi et al., 2004, Dong et al., 2018, Hardt et al., 2016, Jagadeesan et al., 2021, Levanon and Rosenfeld, 2022, Liu et al., 2021]. These works propose models for the interaction between the classifier and the strategic agent and methods for training classifiers that are robust to gaming. In addition, other works in this area investigate how decision makers can design classifiers that incentivize agents to invest effort in improving, instead of gaming [Ahmadi et al., 2022, Kleinberg and Raghavan, 2020]. Nevertheless, the setting of strategic classification implicitly assumes that an agent’s classification does not depend on the behavior of others in the population, limiting the applicability of these methods to our setting of policy learning with capacity constraints.

To the best of our knowledge, Liu et al. [2021] is the only existing work that studies capacity-constrained allocation in the presence of strategic behavior. Liu et al. [2021] introduces the problem of strategic ranking, where agents’ rewards depend on their ranks after investing effort in modifying their covariates. They consider a setting where agents are heterogenous in their raw covariates but homogenous in their ability to modify their covariates. Under these assumptions, the authors find that agents’ post-effort ranking preserves their original ranking by raw covariates and analyze the implications this has on decision maker, agent, and societal utility. We note, however, that the assumption of homogeneity in ability to modify covariates is very strong, and may not be credible in some applications; for example, in the context of college admissions, students with high socioeconomic status may be more readily able to improve their test scores by investing in tutoring services than students with lower socioeconomic status. Our work differs from Liu et al. [2021] because we allow agents to be heterogenous in both their raw covariates and ability to modify their reported covariates. When agents can be heterogenous across both dimensions, ranks are not necessarily preserved after the agents have modified their covariates. In our model, the selection criterion modulates how the equilibrium post-effort ranks change from the pre-effort ranks; in other words, strategic behavior changes who receives treatment, and thus fundamentally alters the nature of the resulting policy learning problem.

The problem of estimating the effect of an intervention in a marketplace setting is also relevant to our work. Marketplace interventions can impact the resulting supply-demand equilibrium, introducing interference and complicating estimation of the intervention’s effect [Blake and Coey, 2014, Heckman et al., 1998]. We find that our setting yields analogous challenges to estimating the effect of a marketplace intervention because when agents are strategic and the decision maker is capacity-constrained, the selection criterion impacts the equilibrium threshold for receiving treatment. To estimate an intervention’s effect without disturbing the market equilibrium, Munro et al. [2021], Wager and Xu [2021] propose a local experimentation scheme, motivated by mean-field modeling. Methodologically, we adapt their mean-field modeling and estimation strategies to estimate the effect of a policy in our setting.

Finally, we note that our dynamic model draws on game theory concepts, such as the myopic best response and dynamic aggregative games. Our assumption that agents are myopic, or will take decisions based on information from short time horizons, is a standard heuristic used in many previous works [Cournot, 1838, Kandori et al., 1993, Monderer and Shapley, 1996]. In addition, our assumption that agents account for the behavior of other agents through an aggregate quantity of their actions is a paradigm borrowed from aggregative games [Acemoglu and Hansen, 2010, 2015, Corchón, 1994]. Most related to our work, Acemoglu and Jensen [2015] consider a dynamic setting where the market aggregate at time step \( t \) is an aggregate function of all the agents’ best responses from time step \( t \), and an agent’s best response at time step \( t \) is selected from a constraint set determined by the market aggregate from time step \( t - 1 \). Analogously, in our work, the “market aggregate” is the threshold for receiving treatment. The threshold for receiving treatment is a particular quantile of the agents’ scores, so we can view it as a function of agents’ reported covariates (agents’ best responses). Furthermore, the covariates that agents report in time step \( t \) depend on the value
of the market aggregate, or the threshold for receiving treatment, in time step \( t - 1 \).

2 Model

In this section, we first define a dynamic model for capacity-constrained treatment assignment in the presence of strategic behavior and define the decision maker’s equilibrium policy value in terms of this model. We then propose a model for agent behavior in terms of myopic utility maximization and provide conditions under which the resulting best response functions vary smoothly in problem parameters.

2.1 Dynamic Model

A decision maker aims to allocate treatments to \( 1 - q \) proportion of a population of agents \( i = 1, 2, \ldots \) at each time step \( t \in \{1, 2, \ldots \} \). Agents’ observed covariates are denoted \( X^t_i \in \mathbb{R}^d \). At time step \( t \), the decision maker observes covariates \( X^t_i \) and assigns treatments \( W^t_i \in \{0, 1\} \) using a policy \( \pi: \mathbb{R}^d \to \{0, 1\} \). We consider linear threshold rules with coefficients \( \beta \in B = \mathbb{R}^{d-1} \) and a threshold \( S^t \in \mathbb{R} \), i.e.

\[
\pi(X; \beta, S^t) = \mathbb{I}(\beta^T X > S^t).
\]  

(2.1)

The decision maker fixes the selection criterion \( \beta \) for all time steps \( t \), while the threshold \( S^t \) varies with \( t \) to ensure that the capacity constraint is satisfied at each time step. In time step \( t \), agents will respond strategically to the coefficients \( \beta \) and the threshold from the previous time step \( t - 1 \), which is \( S^{t-1} \). So, we have that

\[
X^t_i = X_i(\beta, S^{t-1}).
\]  

(2.2)

Note that the function \( X_i \) may be stochastic. Additional structure for agents’ strategic behavior is assumed in Section 2.3. Let \( P(\beta, s)(\cdot) \) denote the distribution over scores \( \beta^T X_i(\beta, s) \) that results when agents report covariates in response to a policy with parameters \( \beta \) and \( s \). So, the distribution over scores at time step \( t \) is given by

\[
P^t = P(\beta, S^{t-1}).
\]  

(2.3)

Following Bhattacharya and Dupas [2012], the threshold \( S^t \) is set to \( q(P^t) \), which is the \( q \)-th quantile of \( P^t \), ensuring that only \( 1 - q \) proportion of agents are treated. So, we can write that

\[
S^t = q(P^t)
\]  

(2.4)

\[
W^t_i = \pi(X^t_i; \beta, S^t).
\]  

(2.5)

After treatment assignment, the decision maker observes individual outcomes for each agent \( Y^t_i \). These outcomes may depend on the treatment received. The potential outcomes of agent \( i \) are given by \( Y_i(0), Y_i(1) \), where \( Y_i(0) \) is the outcome observed when the agent is not assigned to treatment and \( Y_i(1) \) is the outcome observed when the agent is assigned to treatment.

\[
Y^t_i = Y_i(W^t_i).
\]  

(2.6)

In the context of college admissions, we can imagine the outcome \( Y_i \) represents the number of months student \( i \) enrolls in the college. We note that \( Y_i(1) \) represents the number of months student \( i \) would enroll in the college if admitted, and \( Y_i(0) = 0 \) because the student cannot enroll for any months if they are rejected. Let \( V^t \) be the value of a policy at time step \( t \). We define the policy value to be the mean outcome of the agents after treatments are allocated.

\[
V^t(\beta) = \mathbb{E}[Y_i(W^t_i)] = V(\beta, S^{t-1}, S^t),
\]  

(2.7)

where

\[
V(\beta, s, r) = \mathbb{E}[Y_i(\pi(X_i(\beta, s); \beta, r))].
\]  

(2.8)

Note that the previous equation makes explicit the dependence of \( V^t \) on \( S^{t-1} \) and \( S^t \). The argument \( s \) represents the previous threshold for treatment allocation, which agents respond to strategically in the current time step. The argument \( r \) represents the realized threshold of the current time step, which enforces the capacity constraint in the current time step.
2.2 Equilibrium

At an equilibrium induced by the policy parameter $\beta$, the threshold for receiving treatment is fixed over time. In other words, the previous and realized thresholds for receiving treatment are equal. Let $s(\beta)$ be the equilibrium threshold induced by the fixed selection criterion $\beta$. If $S^t = s(\beta)$, then we have that $S^{t+1}, S^{t+2} \ldots$ is a constant sequence where each term is $s(\beta)$. In the following definition, we express the decision maker’s policy value at equilibrium.

**Definition 1 (Equilibrium Policy Value).** Given a fixed selection criterion $\beta \in B$, let $s(\beta)$ be an equilibrium threshold. The decision maker’s equilibrium policy value is given by

$$V_{eq}(\beta) = V(\beta, s(\beta), s(\beta)),$$

where $V$ is as defined in (2.8).

Under conditions where the equilibrium is guaranteed to exist and is unique, the decision maker aims to find $\beta$ such that $V_{eq}(\beta)$ is maximized. Such an objective is motivated by the observation that it may not be feasible for the decision maker to change their selection criterion at each time step. Instead, the decision maker aims to select $\beta$ that performs well with respect to the equilibrium behavior of the system.

2.3 Agent Behavior

Next, we specify a flexible model for agent behavior and establish when agent behavior exhibits useful properties, such as continuity and contraction. In our model, agents are heterogenous in their raw covariates and ability to modify their covariates, myopic in that they choose their reported covariates based on the previous policy, and imperfect in that their reported covariates are subject to noise.

For every agent $i$, unobservables $(Z_i, G_i, \epsilon_i, Y_i(0), Y_i(1))$ are sampled i.i.d. from a distribution $F$, where $\epsilon_i$ is independent from the other unobservables and has distribution $N(0, \sigma^2 I_d)$. Following Frankel and Kartik [2019a,b], we suppose that $Z_i$ denotes the agent’s covariates prior to modification and $G_i$ denotes the agent’s ability to change their raw covariates. We suppose that $Z_i \in \mathcal{X}$, where $\mathcal{X}$ is a convex, compact set in $\mathbb{R}^d$. Let the function $c_i(\cdot) : \mathbb{R}^d \rightarrow \mathbb{R}$ capture agent $i$’s cost deviating from their raw covariates $Z_i$. The agent is myopic in that they modify their covariates with knowledge of only the policy coefficients $\beta \in B$ and the previous threshold for receiving treatment $s \in \mathbb{R}$. In addition, we suppose the agent has imperfect control over the realized value of their modified covariates. For example, an agent can influence their performance on an exam by changing the number of hours they study but cannot perfectly control their exam score. To capture this uncertainty, the agent’s modified covariates are subject to noise $\epsilon_i$.

We incorporate these properties of the agent behavior into the agent’s utility function. The utility function of agent $i$ takes the following form

$$U_i(x; \beta, s) = -c_i(x - Z_i; G_i) + \pi(x + \epsilon_i; \beta, s).$$

(2.10)

The left term is the cost to the agent of deviating from their raw covariates. The right term is the reward from receiving the treatment. Taking the expectation over the noise yields the following expected utility function. Let $\Phi_{\sigma}$ denote the Normal c.d.f. with variance $\sigma^2$ and let $\phi_{\sigma}$ denote the corresponding density.

$$\mathbb{E}_\epsilon [U_i(x; \beta, s)] = -c_i(x - Z_i; G_i) + 1 - \Phi_{\sigma}(s - \beta^T x).$$

(2.11)

We show an example expected utility function.

**Example 2 (Expected Utility Function with Quadratic Cost).** This expected utility function has a quadratic cost of deviating from the raw covariates. Let $G_i \in (\mathbb{R}^+)^d$.

$$\mathbb{E}_\epsilon [U_i(x; \beta, s)] = -(x - Z_i)^T \text{Diag}(G_i)(x - Z_i) + 1 - \Phi_{\sigma}(s - \beta^T x),$$

(2.12)

where $\text{Diag}(G_i)$ is a diagonal matrix in $\mathbb{R}^{d \times d}$ with diagonal equal to $G_i$. We note that the cost function $c_i(x - Z_i; G_i) = (x - Z_i)^T \text{Diag}(G_i)(x - Z_i)$ is $2 \cdot \lambda_{\min}(\text{Diag}(G_i))$-strongly convex, where $\lambda_{\min}(A)$ denotes the minimum eigenvalue of a matrix $A$.
Under Assumption 1, if conditions, the best response mapping is continuously differentiable in $\beta, s$ exists and is unique. Furthermore, using the Implicit Function Theorem, we can show that under the same best response mapping may not have a fixed point. Right: In cases where the best response mapping is discontinuous, the score of the best response mapping may not have a fixed point.

The best response mapping for an agent is obtained by finding the covariates $x \in X$ that maximize the expected utility function, as follows

$$x^*_i(\beta, s) = \arg\max_{x \in X} \mathbb{E}_x[U_i(x; \beta, s)].$$  \hspace{1cm} (2.13)

The covariates that an agent reports to the decision maker is the agent’s best response subject to noise $\epsilon_i$

$$X_i(\beta, s) = x^*_i(\beta, s) + \epsilon_i.$$  \hspace{1cm} (2.14)

2.4 Properties of Agent Best Response

Using the following assumption, we establish a condition on the variance $\sigma^2$ of noise that guarantees that the agent best response is a well-defined function and is continuously differentiable in $\beta, s$. We also establish a related condition on $\sigma^2$ which guarantees that the score of the agent best response is a contraction mapping.

Assumption 1. The cost function $c_i : \mathbb{R}^d \rightarrow \mathbb{R}$ is twice continuously differentiable. In addition, $c_i$ is also $\alpha_i$-strongly convex function for $\alpha_i > 0$ and $c_i(0)$ is its minimum.

Assumption 1 provides structure to the agent’s cost of covariate modification $c_i$ by requiring that it is an $\alpha_i$-strongly convex function. The cost is minimized when the agent does not deviate from their raw covariates $Z_i$.

In the following lemma, we give a condition on the variance of the noise $\sigma^2$ under which the agent best response exists and is unique. This is essential so that we can treat the best response mapping as a well-defined function of $\beta$ and $s$.

Lemma 1. Under Assumption 1, the best response $x^*_i(\beta, s)$ exists. Furthermore, if $\sigma^2 > \frac{1}{\alpha_i \sqrt{2\pi \epsilon}}$, then the best response $x^*_i(\beta, s)$ is uniquely defined. Proof in Appendix C.1.

We can loosely interpret Lemma 1 as follows: if $\sigma^2$ is sufficiently high, then the best response mapping exists and is unique. Furthermore, using the Implicit Function Theorem, we can show that under the same conditions, the best response mapping is continuously differentiable in $\beta, s$.

Lemma 2. Under Assumption 1, if $\sigma^2 > \frac{1}{\alpha_i \sqrt{2\pi \epsilon}}$ and the best response $x^*_i(\beta, s) \in \text{Int}(X)$, then the best response is continuously differentiable in $\beta$ and $s$. Proof in Appendix C.2.

Given a slightly stronger bound on the variance $\sigma^2$, we can strengthen our result and verify that the score of the agent best response mapping is a contraction in $s$, i.e., there is $\kappa \in (0, 1)$ such that, for any fixed $\beta \in \mathcal{B}$

$$|\beta^T x^*_i(\beta, s) - \beta^T x^*_i(\beta, s')| \leq \kappa |s - s'| \quad \forall s, s' \in \mathbb{R}.$$
The contraction property is useful because fixed-point iteration is known to converge for functions that are contractions (see Theorem 23).

**Lemma 3.** Under Assumption 1, if \( \sigma^2 > \frac{2}{\alpha_i \sqrt{2\pi}} \) and a best response \( x^*_i(\beta, s) \in \text{Int}(\mathcal{X}) \), then for fixed \( \beta \in \mathcal{B} \), the score of an agent’s best response \( \beta^T x^*_i(\beta, s) \) is a contraction mapping in \( s \). Proof in Appendix C.3.

In short, Lemma 3 gives that if \( \sigma^2 \) is sufficiently high (twice as high as that required for continuity of the best response), then the score of the best response mapping is a contraction.

We end this section by numerically investigating the role of noise on an agent’s best response function, and verify that in the absence of sufficient noise, unstable behaviors may occur. Qualitatively, the reason why instability may arise is that in a zero-noise setting, there are two modes of agent behavior. In one mode, the agent does not deviate from their raw covariates at all, so \( \beta^T x^*_i(\beta, s) = \beta^T Z_i \). This is either because the threshold is low enough that the agent expects to receive the treatment without deviating from their raw covariates or because the threshold is so high that the benefit of receiving the treatment does not outweigh the cost of modifying their covariates. In the other mode, the threshold takes on intermediate values, so the agent will invest the minimum effort to ensure that they receive the treatment under the previous policy, meaning that \( \beta^T x^*_i(\beta, s) = s \). A discontinuity in the best response arises when an agent no longer finds modifying their covariates beneficial. The presence of noise increases the agent’s uncertainty in whether they will receive the treatment, which causes agents to be less reactive to the previous policy and smooths the agent best response.

Under different noise levels \( \sigma^2 \), we analyze the score of an agent’s best response with a fixed selection criterion \( \beta \) while the threshold \( s \) varies. We consider agent \( i \) with \( Z_i = [3, 0]^T \) and \( G_i = [0.1, 1]^T \). We suppose the decision maker’s model is \( \beta = [1, 0]^T \). Let the agent have an expected utility function with quadratic cost of covariate modification, as given in (2.12).

In Figure 1, we visualize the score of the agent best response, \( \beta^T x^*_i(\beta, s) \), as a function of \( s \), the previous threshold for receiving treatment. We plot \( \beta^T x^*_i(\beta, s) \) vs. \( s \) at four different noise levels \( \sigma^2 \). In the left plot of Figure 1, we have that \( \sigma^2 > \frac{2}{\alpha_i \sqrt{2\pi}} \), so Lemma 3 is applicable, and we observe that the score of the best response is a contraction in \( s \). In the middle left plot, we have that \( \sigma^2 > \frac{1}{\alpha_i \sqrt{2\pi}} \), so Lemma 2 is applicable, and the score of the best response is continuous. In the plots on the right of Figure 1, we have that \( \sigma^2 \leq \frac{1}{\alpha_i \sqrt{2\pi}} \). In such cases, the best response mapping may be discontinuous and may not necessarily have a fixed point.

The lack of a fixed point in the score of an agent best response in low-noise regimes (rightmost plot, Figure 1) implies that there are distributions over unobservables for which there is no equilibrium in our dynamic model in low-noise regimes. As a result, when the noise condition for continuity of the agent best response does not hold, an equilibrium of our dynamic model may not exist. In Section 3, when we establish uniqueness and existence of the equilibrium in our dynamic model, we assume a noise condition that guarantees continuity properties of the agents’ best response mappings.

## 3 Mean-Field Results

Thus far, we have presented a dynamic model for capacity-constrained treatment assignment in the presence of generic strategic behavior and specified the form of strategic behavior we consider in this work. Recall that the decision maker’s objective, as outlined in Section 2, is to find a selection criterion \( \beta \) that maximizes the equilibrium policy value \( V_{eq}(\beta) \). This is a sensible goal in settings where an equilibrium exists and is unique for each selection criterion \( \beta \) in consideration. In this section, we give conditions for existence and uniqueness of an equilibrium in the mean-field regime, where there are an infinite number of agents. We describe a plausible mechanism through which the equilibrium will arise in the mean-field regime. Finally, we show that the mean-field equilibrium threshold is differentiable with respect to \( \beta \), which is crucial for defining the policy effect in Section 5.

We instantiate the dynamic model from Section 2 in the mean-field regime. An infinite population of agents with unobservables sampled from \( F \) is considered for the treatment at each time step \( t \). Let \( \beta \) be the decision maker’s fixed selection criterion. At time step \( t \), suppose agents report covariates with knowledge of policy parameter \( \beta \) and previous threshold for receiving treatment \( S_{t-1} \). Recall that \( P^t \) is the distribution
over scores that is induced and \(q(P^t)\) denotes the \(q\)-th quantile of \(P^t\). Then, at time step \(t\), the agents who score above threshold \(S^t = q(P^t)\) will receive treatment. Since \(P^t = P(\beta, S^{t-1})\), iterating this procedure gives a fixed-point iteration process

\[
S^t = q(P(\beta, S^{t-1})) \quad t = 1, 2, \ldots
\]  

(3.1)

As described in Section 2, the system at equilibrium if the threshold for receiving treatment is fixed over time. The equilibrium induced by \(\beta\) is characterized by an equilibrium threshold \(s(\beta)\) for which \(s(\beta)\) is equal to the \(q\)-th quantile of the distribution over scores \(\beta^T X_i(\beta, s(\beta))\).

To give conditions under which the equilibrium is unique, we use the following two assumptions.

**Assumption 2.** The \((Z, G)\)-marginal of \(F\) has finite support. Agents with the same unobservable \((Z_i, G_i)\) have the same cost function \(c_i\).

Assumption 2 is made for convenience. In combination with Assumption 1, we have that \(\alpha_*(F)\), as defined below, is positive.

\[
\alpha_*(F) = \inf_{i \in \mathbb{N}} \alpha_i.
\]  

(3.2)

We will omit the dependence of \(\alpha_*(F)\) on \(F\) when it is clear that there is only one distribution over unobservables of interest.

**Assumption 3.** We assume that

\[
x^*_i(\beta, s) \in \text{Int}(X).
\]

In other words, we require that agent best responses fall in the interior of the set \(X\).

We require Assumption 3 to ensure that the best response mapping for each agent \(i\) is uniquely defined, so that \(P(\beta, s)(\cdot)\) is a valid distribution function. These assumptions, along with a noise condition that \(\sigma^2 > \frac{1}{\alpha_* \sqrt{2 \pi e}}\), guarantee uniqueness of the equilibrium. Note that this condition ensures that for all unobservables \((Z_i, G_i) \in \text{supp}(F_{Z,G})\), the corresponding agent best response mappings are well-defined (Lemma 1) and continuously differentiable in \(\beta, s\) (Lemma 2).

**Theorem 4.** Fix \(\beta \in B\). Under Assumption 1, 2, and 3, if \(\sigma^2 > \frac{1}{\alpha_* \sqrt{2 \pi e}}\) and \(q(P(\beta, s))\) has a fixed point, then the fixed point must be unique. Proof in Appendix D.1.

The proof of uniqueness relies on exhibiting useful properties of the distribution function \(P(\beta, s)(\cdot)\), namely that it is continuously differentiable in \(\beta, s\), and its argument. Also, \(P(\beta, s)(\cdot)\) has a well-defined inverse function. When this holds, we have that a fixed point of \(q(P(\beta, s))\) is given by a value of \(s\) that solves the equation \(P(\beta, s)(s) = q\). Finally, we observe that \(P(\beta, s)(s)\) is a monotonically increasing function, so it can intersect the horizontal line \(y = q\) in at most one point, yielding uniqueness.

Under the same assumptions, we can also show that the \(q(P(\beta, s))\) is continuously differentiable in \(\beta\) and \(s\). This result follows from the Implicit Function Theorem.

**Lemma 5.** Under Assumption 1, 2, and 3, if \(\sigma^2 > \frac{1}{\alpha_* \sqrt{2 \pi e}}\), then \(q(P(\beta, s))\) is continuously differentiable in \(\beta\) and \(s\). Proof in Appendix D.2.

With the result that \(q(P(\beta, s))\) is continuous, we can establish the existence of the equilibrium in the mean-field model through an application of Intermediate Value Theorem.

**Theorem 6.** Fix \(\beta \in B\). Under Assumption 1, 2, and 3, if \(\sigma^2 > \frac{1}{\alpha_* \sqrt{2 \pi e}}\), then there exists a threshold \(s\) such that \(q(P(\beta, s)) = s\). In other words, \(q(P(\beta, s))\) has at least one fixed point. Proof in Appendix D.3.

The next two results give conditions under which the equilibrium arises via fixed-point iteration \((3.1)\). Corollary 7 is a direct application of Banach’s Fixed-Point Theorem.

**Corollary 7.** Fix \(\beta \in B\). Under Assumptions 1, 2, and 3, if \(q(P(\beta, s))\) is a contraction mapping in \(s\), then fixed-point iteration \((3.1)\) converges to the unique fixed point of \(q(P(\beta, s))\). Proof in Appendix D.4.

In Corollary 8, we give a sufficient condition for ensuring that \(q(P(\beta, s))\) is a contraction.
Corollary 8. Fix $\beta \in \mathcal{B}$. Under Assumptions 1, 2, and 3, if $\sigma^2 > \frac{2}{\alpha \sqrt{2 \pi e}}$, then $q(P(\beta, s))$ is a contraction in $s$ and fixed-point iteration (3.1) converges to the unique fixed point of $q(P(\beta, s))$. Proof in Appendix D.5.

The sufficient condition given in Corollary 8 is equivalent to ensuring that all agents with unobservables in the support of $F$ have best response mappings that are contractions in $s$. In the proof of this corollary, we use the fact that the derivative of $q(P(\beta, s))$ with respect to $s$ is a convex combination of the derivatives of the scores of agents’ best response mappings with respect to $s$. We note that a univariate differentiable function is a contraction if and only if its derivative is bounded between $-1$ and $1$ (Lemma 22). So, ensuring that each agent’s best response mapping is a contraction guarantees that $q(P(\beta, s))$ is a contraction. We note that this condition is sufficient but not necessary for $q(P(\beta, s))$ to be a contraction.

Thus far, we have demonstrated that under sufficient regularity conditions, for a fixed selection criterion $\beta$, an equilibrium exists and is unique in the mean-field limit, and fixed-point iteration is a mechanism through which this equilibrium arises. Crucially, the existence and uniqueness of equilibria induced by fixed selection criteria allows us to define a function $s(\beta) : \mathcal{B} \to \mathbb{R}$ that maps a selection criterion $\beta \in \mathcal{B}$ to the equilibrium threshold $s(\beta) \in \mathbb{R}$ that characterizes the criterion’s mean-field equilibrium. The following theorem establishes the differentiability of $s$.

Corollary 9. Under Assumption 1, 2, and 3, if $\sigma^2 > \frac{1}{\alpha \sqrt{2 \pi e}}$, then we can define a function $s : \mathcal{B} \to \mathbb{R}$ that maps a selection criterion $\beta$ to the unique fixed point of $q(P(\beta, s))$, i.e.,

$$s(\beta) = q(P(\beta, s(\beta))),$$

and the function $s$ is continuously differentiable in $\beta$. Proof in Appendix D.6.

We conclude this section by observing that a selection criterion impacts the level of competition that agents experience through its impact on the equilibrium threshold for receiving the treatment. Although different selection criteria will induce different equilibrium thresholds in the mean-field limit, under regularity conditions the equilibrium thresholds will vary smoothly with the selection criteria. These results make it possible to define and estimate policy effects in Section 5.

4 Finite Sample Approximation

Understanding equilibrium behavior of our dynamic model in the finite regime is of interest because our ultimate goal is to learn optimal equilibrium policies in finite samples. In this section, we instantiate the dynamic model from Section 2 in the regime where a finite number of agents are considered for the treatment at each time step. A difficulty of the finite regime is that deterministic equilibria do not exist. Instead, we give conditions under which stochastic equilibria arise and show that, in large samples, these stochastic equilibria sharply approximate the mean-field limit derived above.

Let $\beta$ be the decision maker’s fixed selection criterion. At each time step, $n$ new agents with unobservables sampled i.i.d. from $F$ are considered for the treatment. For example, in the context of college admissions, the sampled agents at each time step represent a class of students applying for admission each year. At time step $t$, the $n$ agents who are being considered for the treatment best respond with knowledge of the sampled agents at each time step. For example, in the context of college admissions, the random operator $q(P_n(\beta, \cdot))$ given some initial threshold $S^t_0$ yields a stochastic process $(S^t_n)_{t \geq 0}$. We note that for any fixed $\beta$, the random operator $q(P_n(\beta, \cdot))$ approximates the deterministic function $q(P(\beta, \cdot))$. Let $S^t_n$ be the empirical score distribution when $n$ agents best respond to a policy $\beta$ and threshold for receiving treatment $s$. So, the distribution over scores at time step $t$ is given by

$$P^n_{t} = P^n(\beta, S^t_n).$$

Let $q(P^n_{t})$ denote the $q$-th quantile of $P^n_{t}$. Then, agents who score above $S^t_n = q(P^n_{t})$ will receive the treatment. Due to (4.1), iterating this procedure gives a stochastic version of fixed-point iteration

$$S^t_n = q(P^n_{t}) = q(P^n(\beta, S^{t-1}_n)), \quad t = 1, 2, \ldots$$

Since new agents are sampled at each time step, $q(P^n(\beta, \cdot))$ is a random operator. Iterating the random operator $q(P^n(\beta, \cdot))$ given some initial threshold $S^0_n$ yields a stochastic process $(S^t_n)_{t \geq 0}$. We note that for any fixed $\beta$, the random operator $q(P^n(\beta, \cdot))$ approximates the deterministic function $q(P(\beta, \cdot))$. The sufficient condition given in Corollary 8 is equivalent to ensuring that all agents with unobservables in the support of $F$ have best response mappings that are contractions in $s$. In the proof of this corollary, we use the fact that the derivative of $q(P(\beta, s))$ with respect to $s$ is a convex combination of the derivatives of the scores of agents’ best response mappings with respect to $s$. We note that a univariate differentiable function is a contraction if and only if its derivative is bounded between $-1$ and $1$ (Lemma 22). So, ensuring that each agent’s best response mapping is a contraction guarantees that $q(P(\beta, s))$ is a contraction. We note that this condition is sufficient but not necessary for $q(P(\beta, s))$ to be a contraction.

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$$s(\beta) = q(P(\beta, s(\beta))),$$

and the function $s$ is continuously differentiable in $\beta$. Proof in Appendix D.6.

We conclude this section by observing that a selection criterion impacts the level of competition that agents experience through its impact on the equilibrium threshold for receiving the treatment. Although different selection criteria will induce different equilibrium thresholds in the mean-field limit, under regularity conditions the equilibrium thresholds will vary smoothly with the selection criteria. These results make it possible to define and estimate policy effects in Section 5.

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$$P^n_{t} = P^n(\beta, S^t_n).$$

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Since new agents are sampled at each time step, $q(P^n(\beta, \cdot))$ is a random operator. Iterating the random operator $q(P^n(\beta, \cdot))$ given some initial threshold $S^0_n$ yields a stochastic process $(S^t_n)_{t \geq 0}$. We note that for any fixed $\beta$, the random operator $q(P^n(\beta, \cdot))$ approximates the deterministic function $q(P(\beta, \cdot))$.
Figure 2: Given a fixed distribution $F$ over unobservables, we consider the finite model for various $n$ and the mean-field model. In this example, the conditions of Theorem 7 are satisfied, so fixed-point iteration in the mean-field model (3.1) converges to its unique fixed point. Fixed-point iteration in the finite models (4.2) oscillates about the fixed point of the mean-field model. For large $n$, we observe that the iterates $\{\hat{S}_t^n\}$ are more concentrated about the fixed point of the mean-field model.

In Section 3, we showed that there are conditions under which fixed-point iteration of the mean-field model’s deterministic operator $q(P(\beta, \cdot))$ converges to $s(\beta)$, the mean-field equilibrium threshold. In the finite model, if $q(P^n(\beta, \cdot))$ closely approximates $q(P(\beta, \cdot))$, we may expect that there are conditions under which the stochastic process $\{\hat{S}_t^n\}_{t \geq 0}$ will eventually oscillate in a small neighborhood about $s(\beta)$. This is illustrated in Figure 2.

We define a constant that will be used in our concentration inequality and convergence result for the behavior of the finite system. For $\epsilon > 0$ and $\beta \in \mathcal{B}$, define

$$M_\epsilon = \inf_{s \in \mathbb{R}} \{P(\beta, s)(q(P(\beta, s)) + \epsilon) - q, q - P(\beta, s)(q(P(\beta, s)) - \epsilon)\}. \quad (4.3)$$

The following result guarantees that $M_\epsilon$ is positive and gives a finite-sample concentration inequality for the behavior of $q(P^n(\beta, s))$.

**Lemma 10.** Fix $\beta \in \mathcal{B}$. Under Assumptions 1, 2, 3, if $\sigma^2 > \frac{1}{\alpha \sqrt{2\pi e}}$, then $M_\epsilon > 0$ and

$$P(\{q(P(\beta, s)) - q(P^n(\beta, s))\} < \epsilon) \geq 1 - 4e^{-2nM^2_\epsilon}.$$

*Proof in Appendix E.1.*

Notably, the bound in the concentration inequality does not depend on the particular choice of $s$. We use this lemma to characterize the behavior of the system of $n$ agents for sufficiently large iterates $t$ and number of agents $n$ in Theorem 11. Theorem 11 shows that under the same conditions that enable fixed-point iteration in the mean-field model to converge to the mean-field equilibrium threshold (3.1), sufficiently large iterates of the stochastic fixed-point iteration in the finite model (4.2) will lie in a small neighborhood about the mean-field equilibrium threshold with high probability. We can view these iterates as stochastic equilibria of the finite system.

**Theorem 11.** Fix $\beta \in \mathcal{B}$. Suppose Assumptions 1, 2, 3 hold. Let $\epsilon \in (0, 1), \delta \in (0, 1)$, and $s(\beta)$ is the mean-field equilibrium threshold induced by selection criterion $\beta$. Let $\kappa$ is the Lipschitz constant of $q(P(\beta, \cdot))$. Let $\epsilon_\delta = \epsilon^{(1-\kappa)}$. Let $C = |\hat{S}_0^n - s(\beta)|$. If $\sigma^2 > \frac{2}{\alpha \sqrt{2\pi e}}$, then for $t$ such that

$$t \geq \left\lceil \frac{\log\left(\frac{\epsilon_\delta}{2C}\right)}{\log \kappa} \right\rceil$$

and $n$ such that

$$n \geq \frac{1}{2M^2_{\epsilon_\delta}} \log\left(\frac{4t}{\delta}\right),$$
we have that
\[ P(|\hat{S}_n^t - s(\beta)| \geq \epsilon) \leq \delta. \]

\textit{Proof in Appendix E.2.}

The main idea of the proof of this result is at each time step the quantity \(|\hat{S}_n^t - s(\beta)|\) can be decomposed into two terms,
\[ |\hat{S}_n^t - s(\beta)| \leq |q(P^n(\beta, \hat{S}_n^{t-1}))-q(P(\beta, \hat{S}_n^{t-1}))| + |q(P(\beta, \hat{S}_n^{t-1})) - s(\beta)|. \]

The first term on the right side is a noise term that arises due to the difference between an empirical quantile and a population quantile. The second term on the right side can be upper bounded by \(\kappa|\hat{S}_n^{t-1} - s(\beta)|\) because \(q(P(\beta, \cdot))\) is assumed to be a contraction with Lipschitz constant \(\kappa\). Recursively applying this decomposition \(t\) times leaves a vanishing series of dependent noise terms and a term that depends on \(C\), the distance of the initial iterate \(\hat{S}_n^0\) from the fixed point \(s(\beta)\). Analyzing the series of noise terms is difficult due to the dependence between the noise terms. We sidestep this challenge by introducing a sequence of independent random variables, each of which stochastically dominates the corresponding noise terms in our series of interest. Analysis of the series of the independent random variables yields our result.

Also, the following corollary is a building block for our consistency results in Section 5.

\textbf{Corollary 12.} Fix \(\beta \in \mathcal{B}\). Let \(\{t_n\}\) be a sequence such that \(t_n \uparrow \infty\) as \(n \to \infty\) and \(t_n \prec \exp(n)\) (\(t_n\) grows slower than exponentially fast in \(n\)). Under the conditions of Theorem 11, \(\hat{S}_n^t \overset{P}{\to} s(\beta)\), where \(s(\beta)\) is the unique fixed point of \(q(P(\beta, \cdot))\). \textit{Proof in Appendix E.3.}

5 Learning Policies via Gradient-Based Optimization

In this section, we apply the equilibrium concepts developed in Sections 3 and 4 to define and estimate the policy effect, the gradient of the equilibrium policy value with respect to the selection criterion. To enable learning of the optimal policy, we rely on estimation of the derivative of the policy value, a method that is motivated by Wager and Xu [2021]. First, we give conditions under which the policy value is continuously differentiable as a function of \(\beta\) and define the policy effect in terms of the mean-field equilibrium threshold. Next, using results from Section 4, we give methods for estimating these effects in finite samples in a unit-level randomized experiment as in Munro et al. [2021], Wager and Xu [2021]. Finally, we propose a method for learning the optimal policy by using the policy effect estimator.

5.1 Policy Effect

Recall the equilibrium policy value defined in Section 2.

\textbf{Lemma 13.} Under the conditions of Corollary 9, \(V_{\text{eq}}(\beta)\) is continuously differentiable in \(\beta\). \textit{Proof in Appendix F.1.}

From the definition of \(V_{\text{eq}}(\beta)\) in Definition 1, we have that the total derivative of \(V_{\text{eq}}(\beta)\) can be written as
\[ \frac{dV_{\text{eq}}}{d\beta}(\beta) = \frac{\partial V}{\partial \beta}(\beta, s(\beta), s(\beta)) + \left( \frac{\partial V}{\partial s}(\beta, s(\beta), s(\beta)) + \frac{\partial V}{\partial \tau}(\beta, s(\beta), s(\beta)) \right) \cdot \frac{\partial s}{\partial \beta}(\beta). \] (5.1)

We decompose the total derivative of \(V_{\text{eq}}(\beta)\), or the policy effect, into two parts. The first term corresponds to the model effect and the second term corresponds to the equilibrium effect.

\textbf{Definition 3 (Model Effect).} Let \(\tau_{\text{ME}}\) denote the model effect of deploying selection criterion \(\beta\) on the equilibrium policy value the decision maker incurs.
\[ \tau_{\text{ME}}(\beta) = \frac{\partial V}{\partial \beta}(\beta, s(\beta), s(\beta)). \]
The selection criterion $\beta$ impacts the policy value because the criterion is used to score the agents and agents modify their covariates in response to the criterion. Both of these influence the treatments that the agents receive and the policy value. The approach of maximizing the policy value by gradient-based optimization with the model effect is related to the policy learning approach of Munro [2020] because both approaches aim to maximize the policy value by accounting for agents’ strategic behavior.

Due to the decision maker’s capacity constraint, the equilibrium threshold for receiving treatment also depends on the selection criterion. So, we must also account for how policy value changes with respect to the equilibrium threshold and how the equilibrium threshold changes with respect to the selection criterion. Following notation from (2.8), we write $\partial V/\partial s$ and $\partial V/\partial r$ for the partial derivatives of $V$ in its second and third arguments respectively.

**Definition 4** (Equilibrium Effect). Let $\tau_{EE}$ denote the equilibrium effect of deploying selection criterion $\beta$ on the equilibrium policy value the decision maker incurs.

$$
\tau_{EE}(\beta) = (\frac{\partial V}{\partial s}(\beta, s(\beta)) + \frac{\partial V}{\partial r}(\beta, s(\beta))) \cdot \frac{\partial s}{\partial \beta}(\beta).
$$

The previous threshold for receiving treatment $s$ impacts the policy value because agents modify their covariates in response to $s$. This influences the treatments that agents receive and thus the policy value. The realized threshold for receiving treatment $r$ impacts the policy value because it determines agents’ treatment assignments, which influences the policy value, as well. At equilibrium, we have that $s = r = s(\beta)$, so we can account for both of these effects simultaneously.

**Definition 5** (Policy Effect). Let $\tau_{PE}$ denote the policy effect of deploying selection criterion $\beta$ on the equilibrium policy value the decision maker incurs.

$$
\tau_{PE}(\beta) = \tau_{ME}(\beta) + \tau_{EE}(\beta).
$$

The decomposition of the policy effect into the model and equilibrium effect is related to the decomposition theorem of Hu et al. [2021]. Hu et al. [2021] consider a Bernoulli trial, where treatments are allocated according to $W_i \sim \text{Bernoulli}(p_i)$ for $p \in (0, 1)^n$, and they demonstrate that the effect of a policy intervention that infinitesimally increases treatment probabilities can be decomposed into the average direct and indirect effects. The average direct effect captures how the outcome $Y_i$ of a unit is affected by its own treatment $W_i$ on average. The average indirect effect is the term that captures the responsiveness of outcome $Y_i$ to the treatment of other units $j \neq i$ on average, thus measuring the effect of cross-unit interference. When there is no cross-unit interference, the average direct effect matches the usual average treatment effect [Hu et al., 2021, Sävje et al., 2021]. In our setting, the policy effect consists of the model effect, which captures how changes in the selection criterion directly impact the policy value through scoring the agents and agents’ strategic behavior, and the equilibrium effect, which captures how changes in the selection criterion indirectly impact the policy value by modulating the level of competition for the treatment. Furthermore, in the absence of capacity constraints, the policy effect reduces to just the model effect.

### 5.2 Estimation of Policy Effect

We derive estimators for the model, equilibrium, and policy effects through a unit-level randomized experiment in a finite samples. In a system consisting of $n$ agents, we apply symmetric, mean-zero perturbations to the parameters of the policy that each agent responds to. Let $R$ represent the distribution of Rademacher random variables and let $R^d$ represent a distribution over $d$-dimensional Rademacher random variables. For agent $i$, we perturb the policy parameters as follows

$$
\beta_i = \beta + b\xi_i, \quad \zeta_i \sim R^d,
$$
$$
s_i = s + b\xi_i, \quad \xi_i \sim R.
$$

In practice, the perturbation to the selection criterion can be implemented by telling agent $i$ that they will be scored according to $\beta_i$ instead of $\beta$. The perturbation to the threshold $s$ could be implemented by publicly reporting the previous threshold $s$ but telling agent $i$ that a small shock of size $-b\xi_i$ will be added to the
agent’s score in the next time period. We extend our myopic agent model and assume that an agent $i$ will report covariates in response to a policy $\pi(x; \beta_i, s_i)$ as follows:

$$ x^*_i(\beta_i, s_i) = \arg\max_{x \in \mathcal{X}} \mathbb{E}_x [U_i(x; \beta_i, s_i)], \quad (5.2) $$

so

$$ X_i(\beta_i, s_i) = x^*_i(\beta_i, s_i) + \epsilon_i. $$

The prescribed perturbations are applied to determine the agents’ scores and treatment assignments. For clarity, we contrast the form of a score in the perturbed setting with the unperturbed setting. In the unperturbed setting, an agent with unobservables $(Z, G_i)$ who best responds to $\beta, s$ will obtain a score $\beta^T X_i(\beta, s)$. In the perturbed setting, an agent with unobservables $(Z, G_i)$ who best responds to a perturbed version of $\beta, s$ will obtain a score $\beta^T X_i(\beta_i, s_i) - b\xi_i$. Let $P(\beta, s, b)(\cdot)$ denote the distribution over scores in the perturbed setting. We define the threshold for receiving treatment as

$$ r = q(P^n_{\beta, s, b}). $$

In addition, we can define an agent-specific threshold for receiving treatment as follows

$$ r_i = r + b\xi_i. $$

So, in the experiment, the treatments are allocated as follows

$$ W_i = \pi(X_i(\beta_i, s_i); \beta_i, r_i). $$

As in Munro et al. [2021], Wager and Xu [2021], the purpose of applying these perturbations is so that we can recover relevant gradient terms without disturbing the equilibrium behavior of the system. We compute the gradients by running regressions from the perturbations to outcomes of interest, which include the policy value and the proportion of agents whose score exceeds the threshold $r$. To construct the estimators of the model and equilibrium effects, we rely on gradient estimates of the policy value function $V(\beta, s, r)$ and gradient estimates of the complementary CDF of the score distribution $\Pi(\beta, s)(\cdot)$, which is defined as

$$ \Pi(\beta, s)(r) = 1 - P(\beta, s)(r). \quad (5.3) $$

In this experiment, we suppose that thresholds evolve by the stochastic fixed-point iteration process below. Note that it differs slightly from the process given in (4.2).

$$ \begin{cases} 
q(P^n(\beta, \hat{S}_n^{t-1}, b)) & \text{if } q(P^n(\beta, \hat{S}_n^{t-1}, b)) \in [-D, D] \\
-D & \text{if } q(P^n(\beta, \hat{S}_n^{t-1}, b)) < -D \\
q(P^n(\beta, \hat{S}_n^{t-1}, b)) > D & \text{if } q(P^n(\beta, \hat{S}_n^{t-1}, b)) > D \\
\end{cases} \quad (5.4) $$

This process differs from (4.2) because it includes perturbations of size $b$ to the selection criterion and threshold and restricts the threshold to a bounded set $S = [-D, D]$ where $D$ is sufficiently large constant so that $s(\beta) \in S$. Such a set exists because it can be shown there exists $D > 0$ such that $|q(P(\beta, s))| < D$ for all $s \in \mathbb{R}$.

Analyzing the stochastic process $\{\hat{S}_n^{t}\}_{t \geq 0}$ generated by (5.4) above presents two technical challenges. First, the above stochastic process truncates the threshold values so that they lie in $S$, whereas the results from Section 4 do not involve truncation. Nevertheless, the truncation is a contraction map to the equilibrium threshold, so the results of Section 4 also apply to stochastic fixed point iteration with truncation. The other challenge is that the results from Section 3 and Section 4 focus on the setting where all agents best respond to the same policy $\pi(x; \beta, s)$, whereas in (5.4), each agent $i$ best responds to a different perturbed policy $\pi(x; \beta_i, s_i)$. Nevertheless, under the following assumption, we can show that for sufficiently small $b$, analogous results hold under unit-level perturbations.

**Assumption 4.** For any $(Z_i, G_i, \epsilon_i, Y_i(0), Y_i(1)) \sim F$, we have that $Z_i \in \text{Int}(\mathcal{X})$. 

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To show that results from Section 4 transfer to the setting with unit-level perturbations, we can define a new distribution over agent unobservables $F$ and new cost functions that are related to the original distribution over agent unobservables $F$ and original cost functions. When agents with unobservables sampled from $F$ best respond to $\pi(x; \beta, s)$ according to the new cost functions, the score distribution that results equals $P(\beta, s, b)$.

First, we define the model effect estimator. We consider an experiment where $n$ agents are considered for the treatment. Let $b$ be the size of the perturbation. Let each row of $M_\beta \in \mathbb{R}^{n \times d}$ correspond to $b\zeta_i^T$, the perturbation applied to the selection criterion $\beta$ observed by the $i$-th agent. Since the $n$ agents will best respond to these perturbations as in (5.2), we observe an empirical distribution over scores $P^n(\beta, s, b)$. Let each entry of $Y \in \mathbb{R}^n$ correspond to the outcome of the $i$-th agent

$$Y_i = Y_i(W_i).$$

Let $\hat{\Gamma}_{b,n}(\beta, s, r)$ be the regression coefficient that is obtained by running OLS of $Y$ on $M_\beta$. In particular,

$$\hat{\Gamma}_{b,n}(\beta, s, r) = b^{-1} \left( \frac{1}{n} \sum_{i=1}^n \zeta_i \zeta_i^T \right)^{-1} \left( \frac{1}{n} \sum_{i=1}^n \zeta_i Y_i \right).$$

The model effect estimator with sample size $n$, perturbation size $b$, and iteration $t$ as

$$\hat{\tau}_{ME,b,n}(\beta) = \hat{\Gamma}_{b,n}(\beta, \hat{S}_{b,n}, \hat{S}_{b,n}),$$

where $\hat{S}_{b,n}$ is given by (5.4).

One of the challenges in analyzing this estimator is that $\hat{\Gamma}_{b,n}$ is a stochastic function, and in the model effect estimator, the arguments for $s$ and $r$ are also stochastic. To establish consistency, we first show that the arguments converge in probability to the equilibrium threshold induced by $\beta, s(\beta)$. To demonstrate consistency of the overall estimator, we must establish stochastic equicontinuity for $\hat{\Gamma}_{b,n}$. As a result, we require the following assumption on the outcomes $Y_i(w)$.

**Assumption 5.** Let $m(z, g; w) = \mathbb{E}[Y_i(w) \mid Z_i = z, G_i = g]$. The potential outcomes $Y_i(w)$ can be decomposed as

$$Y_i(w) = m(Z_i, G_i; w) + \delta_i,$$

where $\delta_i$ is a mean-zero random variable and $m(z, g; w)$ is continuous with respect to $(z, g)$ and bounded.

**Theorem 14.** Fix $\beta \in \mathcal{B}$. Let $\{t_n\}$ be a sequence such that $t_n \uparrow \infty$ as $n \to \infty$ and $t_n \prec \exp(n)$. Let $\mathcal{S} = [-D, D]$ for a sufficiently large constant $D > 0$, so that the equilibrium threshold $s(\beta) \in \mathcal{S}$. Under Assumptions 1, 2, 3, 4, and 5, if $\sigma^2 > \frac{2}{\alpha, \sqrt{2\pi e}}$, then there exists a sequence $\{b_n\}$ such that $b_n \to 0$ so that $\hat{\tau}_{ME,b,n}(\beta) \overset{p}{\to} \tau_{ME}(\beta)$. *Proof in Appendix F.2.*

Second, we define the equilibrium effect estimator. Although the same approach applies, estimating the equilibrium effect is more complicated than estimating the model effect. We estimate the equilibrium effect by estimating the two components of the equilibrium effect, $\frac{\partial V}{\partial s}$ and $\frac{\partial V}{\partial \tilde{g}}$.

We consider an experiment where $n$ agents are considered for the treatment. Let $b$ be the size of the perturbation. Let each row of $M_\beta \in \mathbb{R}^{n \times d}$ and of $M_s \in \mathbb{R}^{n \times 1}$ correspond to the perturbation applied to the selection criterion $\beta$ and threshold $s$, respectively for the $i$-th agent. Since the $n$ agents will best respond to these perturbations as in (5.2), we observe an empirical distribution over scores $P^n_{\beta, s, b}$. Let each entry of $Y, I \in \mathbb{R}^n$ correspond to the following outcomes for the $i$-th agent

$$Y_i = Y_i(W_i),$$

$$I_i = \pi(X_i(\beta_i, s_i); \beta_i, r).$$

Let $\hat{\Gamma}_{b,n}(\beta, s, r), \tilde{\Gamma}_{b,n}(\beta, s, r)$, and $\hat{\Gamma}_{b,n}(\beta, s, r)$ correspond to the regression coefficients from running OLS
of $Y$ on $M_s$, $I$ on $M_B$, and $I$ on $M_s$, respectively. In particular,

$$
\hat{\Gamma}^{b,n}_{Y,s,r}(\beta, s, r) = b^{-1}\left(\frac{1}{n} \sum_{i=1}^{n} \xi_i \xi_i^T\right)^{-1}\left(\frac{1}{n} \sum_{i=1}^{n} \xi_i Y_i\right),
$$

$$
\hat{\Gamma}^{b,n}_{II,\beta}(\beta, s, r) = b^{-1}\left(\frac{1}{n} \sum_{i=1}^{n} \xi_i \xi_i^T\right)^{-1}\left(\frac{1}{n} \sum_{i=1}^{n} \xi_i I_i\right),
$$

$$
\hat{\Gamma}^{b,n}_{II,s}(\beta, s, r) = b^{-1}\left(\frac{1}{n} \sum_{i=1}^{n} \xi_i \xi_i^T\right)^{-1}\left(\frac{1}{n} \sum_{i=1}^{n} \xi_i I_i\right).
$$

Let $\{h_n\}$ be a sequence such that $h_n \to 0$ and $nh_n \to \infty$. Let $p^n(\beta, s, b)(r)$ denote a kernel density estimate of $p(\beta, s, b)(r)$ with kernel function $k(z) = I(z \in [-\frac{1}{2}, \frac{1}{2}])$ and bandwidth $h_n$. We define the equilibrium effect estimator with sample size $n$ and iteration $t$ as

$$
\hat{\tau}^{t}_{EE,b,n}(\beta) = \hat{\Gamma}^{b,n}_{Y,s,r}(\beta, \hat{S}_{b,n}^{t-1}, \hat{S}_{b,n}^{t}) \cdot \left(\frac{1}{p^n(\beta, \hat{S}_{b,n}^{t}, b)(\hat{S}_{b,n}^{t}) - \hat{\Gamma}^{b,n}_{II,\beta}(\beta, \hat{S}_{b,n}^{t}, \hat{S}_{b,n}^{t})} \cdot \hat{\Gamma}^{b,n}_{II,s}(\beta, \hat{S}_{b,n}^{t}, \hat{S}_{b,n}^{t})\right).
$$

(5.6)

In Theorem 15, we show that our equilibrium effect estimator is consistent for the equilibrium effect.

**Theorem 15.** Fix $\beta \in \mathcal{B}$. Let $\{t_n\}$ be a sequence such that $t_n \uparrow \infty$ as $n \to \infty$ and $t_n \prec \exp(n)$. Let $\mathcal{S} = [-D, D]$ for a sufficiently large constant $D > 0$, so that the equilibrium threshold $s(\beta) \in \mathcal{S}$. Under the conditions of Theorem 14, there exists a sequence $\{b_n\}$ such that $b_n \to 0$ so that $\hat{\tau}^{t}_{EE,b,n}(\beta) \overset{D}{\to} \tau_{EE}(\beta)$. **Proof in Appendix F.3.**

Finally, we can sum the estimators of the model and equilibrium effects to estimate the policy effect.

**Corollary 16.** Fix $\beta \in \mathcal{B}$. Let $\{t_n\}$ be a sequence such that $t_n \uparrow \infty$ as $n \to \infty$ and $t_n \prec \exp(n)$. Let $\mathcal{S} = [-D, D]$ for a sufficiently large constant $D > 0$, so that the equilibrium threshold $s(\beta) \in \mathcal{S}$. We consider the sequence of approximate policy effects given by

$$
\hat{\tau}^{t}_{PE,b,n}(\beta) = \hat{\tau}^{t}_{ME,b,n}(\beta) + \hat{\tau}^{t}_{EE,b,n}(\beta).
$$

Under the conditions of Theorem 14, there exists a sequence $\{b_n\}$ such that $b_n \to 0$ so that $\hat{\tau}^{t}_{PE,b,n}(\beta) \overset{D}{\to} \tau_{PE}(\beta)$. **Proof in Appendix F.4.**

### 5.3 Learning the Optimal Policy

We now describe an algorithm (see Algorithm 1) for learning the optimal policy. Following Wager and Xu [2021], the algorithm entails first learning equilibrium-adjusted gradients of the policy value as discussed above and then updating the selection criterion with the gradient to maximize the policy value. In this paper, we will only investigate empirical properties of this approach, and refer to Wager and Xu [2021] for formal results for this type of gradient-based learning.

The decision maker runs the algorithm for $J$ epochs. In Section 2, we describe that it may be infeasible for the decision maker to update the selection criterion at each time step. This algorithm requires the decision maker to deploy an updated selection criterion at each epoch $j$. In other words, updates to the selection criterion are necessary but infrequent. We emphasize that deploying different selection criteria is only necessary for the learning procedure, and ultimately, we aim to learn a fixed selection criterion that maximizes the equilibrium policy value.

In epoch $j$, the decision maker deploys a policy with coefficients $\beta^j$. Through the stochastic fixed-point iteration process with perturbations (5.4), a stochastic equilibrium induced by $\beta^j$ emerges, yielding the threshold for receiving treatment $s^j$. Each agent best responds to their perturbed policy and the decision maker observes their reported covariates. Following the procedure from Section 5.2, the decision maker can then use the outcomes and the perturbations to estimate the policy effect of $\beta^j$ on the equilibrium policy value (Algorithm 2). The decision maker can set $\beta^{j+1}$ by taking a gradient step from $\beta^j$ using the policy effect estimator as the gradient. Note that because we aim to maximize the policy value, we update $\beta$ by moving in the direction of the gradient.
6 Numerical Experiments

In this section, we empirically evaluate our proposed policy learning approach and existing baselines. First, we consider a one-dimensional toy example, where we suppose that the distribution over agent unobservables $F$ contains cross-types, which are pairs of agent unobservables where one agent has higher ability to modify their covariates and the other has more favorable raw covariates. Second, we consider a simulation where agents have high-dimensional ($d = 10$) covariates. Lastly, in semi-synthetic experiment using data from the National Education Longitudinal Study of 1988 (NELS:88) [Ingels, 1994], we instantiate our model in the context of college admissions and discuss the implications of using competition-aware policy on the distribution of accepted agents. Across our experiments, we demonstrate that the policy effect estimator defined Section 5 can be used to learn a competition-aware policy that achieves higher equilibrium policy value compared to approaches that only account for capacity constraints or only account for strategic behavior.

6.1 Methods

We describe the two baselines and our proposed estimator that we evaluate in these experiments. The methods we consider rely on finite samples and do not have access to unobservables.

1. Capacity-Aware Policy Learning. This approach accounts for the decision maker’s capacity constraint. However, it naively assumes that agent covariates are exogenous to the policy, so it does not take into account that agents will react strategically to the deployed policy. Following Bhattacharya and Dupas [2012], the decision maker runs a randomized controlled trial to obtain a model for the conditional average treatment effect (CATE) $\tau(x) = \mathbb{E}[Y_i(1) - Y_i(0) \mid X = x]$ and at deployment, computes an estimate of the CATE for each agent using their observed covariates and assigns treatment to the

Algorithm 1: Gradient Descent with $\hat{\tau}_{PE}$

\begin{algorithm}
  \begin{algorithmic}
    \STATE \textbf{while} $j \leq J$ \textbf{do}
    \STATE Decision maker deploys $\beta_j$ ;
    \STATE Stochastic fixed-point iteration for sufficiently many iterations with unit-level perturbations (see (5.4)) until $s_j$ is reached ;
    \FOR {$i \in \{1 \ldots n\}$}
    \STATE Sample random perturbation $\zeta_i \sim R^d$ and $\xi_i \sim R$ ;
    \STATE $\beta_i^j \leftarrow \beta_i^j + b_n \zeta_i$ ;
    \STATE $s_i^j \leftarrow s_i^j + b_n \xi_i$ ;
    \STATE Agent $i$ best responds to $\beta_i^j, s_i^j$ ;
    \STATE Decision maker observes best response $x_i^j$ ;
    \ENDFOR
    \STATE Given the scores $\{\beta_i^j x_i^j - b_n \xi_i^j\}_{i=1}^n$, the decision maker computes the $q$-th quantile of the scores $q^j$ and density of scores at $q^j$, yields $\rho^j$ ;
    \FOR {$i \in \{1 \ldots n\}$}
    \STATE Observes outcome $Y_i^j$ and measures \newline $I_i^j \leftarrow \mathbb{I}(\langle \beta_i^j \rangle^T x_i^j > s_i^j)$ ;
    \ENDFOR
    \STATE $M^j_\beta \leftarrow b_n \zeta^j$ is the $n \times d$ matrix of perturbations $\zeta$ ;
    \STATE $M^j_\xi \leftarrow b_n \zeta^j$ is the $n \times 1$ matrix of perturbations $\xi$ ;
    \STATE $Y^j$ is the $n$-length vector of outcomes $Y_i^j$ ;
    \STATE $P^j$ is the $n$-length vector of indicators $I_i^j$ ;
    \STATE Construct gradient estimate $\Gamma^j$ from $M^j_\beta, M^j_\xi, Y^j, P^j, \rho^j$ (See Algorithm 2) ;
    \STATE Take a projected gradient step $\beta_\phi^{j+1} \leftarrow \text{Proj}_B(\beta_\phi^j + a \cdot \Gamma^j)$ ;
  \ENDWHILE
\end{algorithmic}
\end{algorithm}
The decision maker’s equilibrium policy value $V$ modify their observed covariates. We suppose that $Y$.

Define $\Theta$. We suppose that the capacity constraint limits the decision maker to accept only 30% of the agent population.

For a one-dimensional example, we consider policies with the following parametrization.

### Algorithm 2: Construct gradient estimates

- Run OLS of $Y$ on $M_{\beta}^j$: $\Gamma_{Y,\beta}^j \leftarrow (M_{\beta}^j)^T(M_{\beta}^j)^{-1}(M_{\beta}^j)^TY$.
- Run OLS of $Y$ on $M_{\beta}^j$: $\Gamma_{Y,s}^j \leftarrow (M_{\beta}^j)^T(M_{\beta}^j)^{-1}(M_{\beta}^j)^TY$.
- Run OLS of $P$ on $M_{\beta}^j$: $\Gamma_{p,\beta}^j \leftarrow (M_{\beta}^j)^T(M_{\beta}^j)^{-1}(M_{\beta}^j)^TY$.
- Run OLS of $P$ on $M_{\beta}^j$: $\Gamma_{p,s}^j \leftarrow (M_{\beta}^j)^T(M_{\beta}^j)^{-1}(M_{\beta}^j)^TP$.
- For each $j$, $\Gamma_{s,\beta}^j \leftarrow \frac{1}{\rho_j} - \Gamma_{p,s}^j \cdot \Gamma_{p,\beta}^j$.
- $\Gamma^j \leftarrow \Gamma_{Y,\beta}^j + \Gamma_{Y,s}^j \cdot \Gamma_{s,\beta}^j$.

agents with CATE above the q-th quantile. Note that treatment assignment is random in the randomized controlled trial, so agents are not strategic in the trial. In our implementation, we estimate $E[Y_i(1) | X = x], E[Y_i(0) | X = x]$ via linear regression and subtract the models to obtain a model for the CATE of the form $\beta^TX + \beta_0$. The decision maker’s deployed selection criterion is $\beta_{cap} = \text{Proj}_{B}(\beta_1)$, which is a projection of the parameters of the CATE onto the allowed policy class $B$. The polar coordinate representation of the policy is $\theta_{cap}$.

2. **Strategy-Aware Gradient-Based Optimization.** This approach assumes that agents will react strategically to the policy parameters $\beta, s$ but it does not account for how the equilibrium threshold changes with respect to $\beta$. Each epoch, the decision maker runs a unit-level experiment as described in Section 5.2 to estimate the model effect. Using the model effect estimator as the gradient, the decision maker then updates the selection criterion $\beta$ by taking a step in the direction of the gradient. Recall that the model effect accounts for agents’ strategic behavior but does not account for the equilibrium effect. We refer to the learned solution of this method as $\beta_{strat}$, or its polar coordinate representation $\theta_{strat}$.

3. **Competition-Aware Gradient-Based Optimization.** Our proposed approach accounts for agents’ strategic behavior and the decision maker’s capacity constraint. Each epoch, the decision maker runs a unit-level experiment as described in Section 5.2 to estimate the policy effect. Using the policy effect estimator as the gradient, the decision maker then updates the selection criterion $\beta$ by taking a step in the direction of the gradient. Recall that the policy effect captures both the model effect and the equilibrium effect. We refer to the learned solution of this method as $\beta_{comp}$, or its polar coordinate representation $\theta_{comp}$.

We emphasize that the first baseline, capacity-aware policy learning, considers capacity constraints but ignores strategic behavior, and the second baseline, strategy-aware gradient-based optimization, accounts for strategic behavior but ignores capacity constraints. Lastly, our proposed method accounts for both the agents’ strategic behavior and the decision maker’s capacity constraints.

### 6.2 Toy Example

For a one-dimensional example, we consider policies with the following parametrization

$$\beta = [\cos \theta, \sin \theta]^T, \quad \text{where } \theta \in [0, 2\pi).$$

We suppose that the capacity constraint limits the decision maker to accept only 30% of the agent population. Define $Y_i(1)$ and $Y_i(0)$ as follows

$$Y_i(1) = Z_{i,1},$$
$$Y_i(0) = 0.$$  

The decision maker’s equilibrium policy value $V_{eq}(\beta)$ is given by Definition 1.

We consider an agent distribution where agents are heterogeneous in their raw covariates and ability to modify their observed covariates. We suppose that

$$\mathcal{X} = [0, 10]^2, \quad G_i \in [0.01, 20]^2.$$
The variance of the noise distribution $\sigma^2$ is set to ensure the continuous differentiability property of the quantile mapping of the score distribution; we set $\sigma = 3.30$. Agents optimize the quadratic utility function in (2.12). So, the entry $G_{i,j}$ quantifies the cost to agent $i$ of modifying $X_{i,j}$ from $Z_{i,j}$.

Motivated by Frankel and Kartik [2019b], we consider an agent distribution with two groups of agents in the population of equal proportion, the naturals and the gamers. The naturals have

$$Z_{i,1}, Z_{i,2} \sim \text{Uniform}[5, 7], \quad G_{i,1}, G_{i,2} \sim \text{Uniform}[10, 20].$$

In contrast, the gamers have

$$Z_{i,1}, Z_{i,2} \sim \text{Uniform}[3, 5], \quad G_{i,1} \sim \text{Uniform}[0.01, 0.02], \quad G_{i,2} \sim \text{Uniform}[10, 20].$$

In this simulation, there are 10 unique agent unobservable pairs $(Z_{i}, G_{i})$, 5 naturals and 5 gamers. The naturals and gamers are cross types as in Frankel and Kartik [2019b] because the naturals have higher values of $Z_{i,1}, Z_{i,2}$ compared to the gamers and the gamers have lower cost to modifying $Z_{i,1}$ compared to the naturals. Note that accepting a natural yields a higher policy value compared to a gamer because naturals have higher $Z_{i,1}$.

If the decision maker ignores the presence of strategic behavior, they may use data from an RCT to estimate the CATE and deploy the thresholded CATE as the selection criterion, which corresponds to our first baseline of capacity-aware policy learning. We refer to such a policy as $\beta_{\text{cap}}$. We expect a naive application of $\beta_{\text{cap}}$ to yield suboptimal policy value because $\beta_{\text{cap}}$ likely places substantial weight on the first covariate but gamers have high ability to deviate from $Z_{i,1}$ when reporting $X_{i,1}$.

Intuitively, there should exist a better policy in this setting. We note that all agents are relatively homogenous in their ability to deviate from $Z_{i,2}$ when reporting $X_{i,2}$ because $G_{i,2} \sim \text{Uniform}[10, 20]$ for all agents. At the same time, $Z_{i,2}$ is correlated with $Z_{i,1}$. So, a selection criterion that places high weight on the second covariate should yield higher policy value by accepting more naturals.

We plot the equilibrium policy value of decision maker as a function of $\theta = \arctan(\beta_{1}/\beta_{0})$ in (Figure 3, left plot). As expected, we observe that deploying $\beta_{\text{cap}}$, which corresponds to $\theta_{\text{cap}}$, is suboptimal for maximizing the equilibrium policy value. Under $\beta_{\text{cap}}$, we observe that only 30% of agents who score above the threshold for acceptance under $\beta_{\text{cap}}$ are naturals (Figure 3, middle plot). The policy $\beta^{*} = [0.345, 0.938]^T$ achieves the optimal equilibrium policy value, and as expected it places considerable weight on the second covariate. When $\beta = \beta^{*}$, we observe that 69% of agents who score above the threshold for acceptance are naturals (Figure 3, right plot).

We compare the solutions obtained by the two baselines and our proposed approach. For the capacity-aware baseline, we learn $\theta_{\text{cap}}$ using data from an RCT where treatment is allocated randomly among $n = 1000000$ agents. For the second baseline, strategy-aware gradient-based optimization, and our proposed method, competition-aware gradient-based optimization, we optimize the selection criterion via vanilla stochastic gradient descent on $\theta$, the polar-coordinate representation of $\beta$. We initialize gradient descent...
Table 1: In our toy experiment, we observe that \( \theta_{\text{comp}} \) converges to the optimal policy \( \theta^* \). However, \( \theta_{\text{strat}} \) and \( \theta_{\text{cap}} \) are suboptimal.

| Method               | \( |V_{eq}(\hat{\beta}) - V_{eq}(\beta^*)| \) | \( |\hat{\theta} - \theta^*| \) |
|----------------------|----------------------------------|-----------------|
| Capacity-Aware (\( \theta_{\text{cap}} \)) | 0.16 ± 0.11                      | 2.53 ± 2.10     |
| Strategy-Aware (\( \theta_{\text{strat}} \)) | 0.04 ± 0.05                      | 0.32 ± 0.31     |
| Competition-Aware (\( \theta_{\text{comp}} \)) | 0.00 ± 0.00                      | 0.04 ± 0.05     |

Figure 4: We plot the equilibrium policy value obtained by strategy-aware and competition-aware gradient-based optimization in our high-dimensional simulation (\( d = 10 \)). We find that competition-aware gradient-based optimization converges to a solution that obtains higher equilibrium policy value. Note that the capacity-aware policy learning baseline is not depicted because it is not a gradient-based method.

with \( \theta = 0 \) and run it for 100 iterations, and we assume that \( n = 1000000 \) agents are observed by the decision maker at each iteration. We report the equilibrium policy value obtained by \( \theta_{\text{cap}}, \theta_{\text{strat}}, \theta_{\text{comp}} \). We report the absolute difference between the equilibrium policy value of the optimal solution and the learned solutions, and we report the absolute difference between the polar coordinate representations of the optimal solution and the learned solutions.

Across 10 random trials (where the randomness is over the sampled unobservable values and sampled agents), we observe that \( \theta_{\text{comp}} = \theta^* \) and \( \theta_{\text{comp}} \) achieves optimal equilibrium policy value (Table 1), demonstrating that accounting for competition is beneficial. Meanwhile, \( \theta_{\text{cap}} \) and \( \theta_{\text{strat}} \) obtain suboptimal equilibrium policy value (Table 1). Nevertheless, we note that strategy-aware gradient-based optimization is a relatively strong baseline because it accounts for the impact of the strategic behavior due to agents’ knowledge of the policy on the policy value.

6.3 High-Dimensional Simulation

We consider \( d \)-dimensional linear policies \( \beta \in \mathcal{B} = \mathbb{S}^{d-1} \) for \( d = 10 \). We suppose the capacity constraint only allows the decision maker to accept 30% of the agent population. We specify the outcomes \( Y_i(w) \) as follows

\[
Y_i(1) = Z_{i,1} \\
Y_i(0) = 0.
\]

The decision maker’s equilibrium policy value \( V_{eq}(\beta) \) is given by Definition 1.

We suppose that

\[
\mathcal{X} = [0, 10]^d, \quad G_i \in [0.05, 5]^d.
\]
Table 2: Across 10 random trials, we find that competition-aware gradient-based optimization outperforms the other baselines (\(d = 10\)). A one-sided paired \(t\)-test, where we compare the final policy value of \(\beta_{\text{comp}}\) and the final policy value of \(\beta_{\text{strat}}\) with the same random seed, yields a \(p\)-value of 3e-4.

| Method                | Equilibrium Policy Value \(\hat{V}_{\text{eq}}(\bar{\beta})\) |
|-----------------------|---------------------------------------------------------------|
| Capacity-Aware (\(\beta_{\text{cap}}\)) | 1.65 ± 0.12                                                   |
| Strategy-Aware (\(\beta_{\text{strat}}\)) | 1.81 ± 0.14                                                   |
| Competition-Aware (\(\beta_{\text{comp}}\)) | 2.06 ± 0.12                                                   |

Figure 5: We cluster the agent unobservables computed from the NELS:88 dataset into \(K = 8\) clusters with \(K\)-means clustering. Using t-SNE, we visualize two-dimensional embeddings of unobservables \(\{(Z_i, G_i, Y_{i,1}(1), Y_{i,2}(1))\}_{i=1}^{m}\) from the NELS data. Each point represents an two-dimensional embedding of \((Z_i, G_i, Y_{i,1}(1), Y_{i,2}(1))\) the and the color of the point corresponds to the cluster that the agent belongs to.

The variance of the noise distribution \(\sigma^2\) is set to ensure the continuous differentiability property of the quantile mapping of the score distribution; we set \(\sigma = 1.10\). Agents optimize the quadratic utility function in (2.12), and \(G_{i,j}\) quantifies the cost of modifying agent \(i\)'s \(j\)-th covariate \(X_{i,j}\) from \(Z_{i,j}\). We suppose that the \((Z,G)\)-marginal of \(F\) is supported on 10 points \((Z_i, G_i)\), where the entries are distributed following

\[
Z_{i,j} \sim \text{Uniform}[3,8], \quad G_{i,j} \sim \text{Uniform}[0.05,5], \quad j \in \{1, \ldots, d\}.
\]

For the capacity-aware baseline, we learn \(\theta_{\text{cap}}\) using data from an RCT where treatment is allocated randomly among \(n = 1000000\) agents. For the strategy-aware and competition-aware methods, we optimize \(\beta\) via projected stochastic gradient descent (in our case, ascent because we aim to maximize the policy value), initialized with

\[
\bar{\beta} = \left[\frac{1}{\sqrt{d}}, \frac{1}{\sqrt{d}}, \ldots, \frac{1}{\sqrt{d}}\right]^T.
\]

We assume that \(n = 1000000\) agents are observed by the decision maker at each iteration. We plot the equilibrium policy value obtained by iterates of the gradient-based methods from Section 6.1 in Figure 4. Across 10 random trials (where the randomness is over the sampled unobservable values and the sampled agents), we observe that \(\beta_{\text{comp}}\) finds a solution with higher equilibrium policy value than \(\beta_{\text{strat}}\) and \(\beta_{\text{cap}}\) (Table 2).

6.4 Semi-synthetic Experiment with NELS:88

We simulate policy learning for college admissions. First, we use data from the National Educational Longitudinal Study of 1988 (NELS:88) [Ingels, 1994] to construct a realistic distribution over agent unobservables. Then, we draw agent unobservables from this distribution in a simulation to evaluate the performance of the three baselines from Section 6.1.

NELS:88 is a nationally representative, longitudinal study that followed eighth graders in 1988 throughout their secondary and postsecondary years. The study data includes results from student assessments in
Figure 6: We plot the equilibrium policy value obtained from iterates of strategy-aware and competition-aware gradient-based optimization in our semi-synthetic experiment with the NELS dataset \((d = 9)\) and find that competition-aware gradient-based optimization converges to a solution that obtains higher equilibrium policy value. **Left:** We plot \(V_{eq, 1}(\beta)\) vs. iteration for the gradient-based methods. **Right:** We plot \(V_{eq, 2}(\beta)\) vs. iteration for the gradient-based methods. Note that the capacity-aware policy learning baseline is not depicted in either figure because it is not a gradient-based method.

Figure 7: We plot the score distributions that are induced by \(\beta_{comp}, \beta_{strat}, \beta_{cap}\), respectively when we apply the baselines in Section 6.1 to maximize \(V_{eq, 1}(\beta)\). We plot a histogram of scores for each agent with a distinct unobservable \((Z_i, G_i)\). Agents are color-coded on the spectrum of blue to red depending on whether their unobservable corresponds to low SES (blue) or high SES (red). The selection criterion \(\beta_{comp}\) favors students with high SES, compared to \(\beta_{strat}\) and \(\beta_{cap}\).

Figure 8: We plot the score distributions that are induced by \(\beta_{comp}, \beta_{strat}, \beta_{cap}\), respectively when the baselines in Section 6.1 are trained to maximize \(V_{eq, 2}(\beta)\). We plot a histogram of scores for each agent with a distinct unobservable \((Z_i, G_i)\). Agents are color-coded on the spectrum of blue to red depending on whether their unobservable corresponds to low SES (blue) or high SES (red). The selection criterion \(\beta_{comp}\) accepts a higher proportion of agents with low SES.
Under these assumptions, we can compute each student’s raw covariates \(G_m\) where \(m = 14915\). We assume that the reported test scores and grades from the NELS:88 reflect student \(i\)’s best response \(x_i^*\). So, \(x_i^* \in \mathbb{R}^d\) for \(d = 9\). We use the subscripts “test scores” and “grades” to specify the covariates and unobservables that correspond to test scores and grades, respectively. Student \(i\)’s gaming ability \(G_i\) is given by

\[G_{i,\text{test scores}} = 5, \quad G_{i,\text{grades}} \propto 1 - \gamma_i,\]

where \(\gamma_i\) is student \(i\)’s socioeconomic status (SES) percentile reported in the NELS:88 dataset. So, all agents have the same cost to improving their standardized test scores, but students with high SES can more easily improve their grades compared to students with low SES.

Under a simple model for student behavior, this information is sufficient for us to estimate the unobservables \((Z_i, G_i)\) for each student \(i\) that are consistent with the NELS data. We assume that each student’s utility is determined by a quadratic utility function as in (2.12). The variance \(\sigma^2\) of the noise distribution assumed to be \(\sigma^2 = 4.92\), which ensures that students’ best responses are continuous. In addition, we assume that \(x_i^*\) is the best response to a particular policy with parameters \((\bar{\beta}, \bar{s})\), where

\[
\bar{\beta} = \begin{bmatrix} \frac{1}{\sqrt{d}} & \frac{1}{\sqrt{d}} & \ldots & \frac{1}{\sqrt{d}} \end{bmatrix}^T, \quad \bar{s} = 19.5.
\]

Under these assumptions, we can compute each student’s raw covariates

\[Z_{i,j} = x_{i,j}^* - \frac{1}{2G_{i,j}} \phi_\sigma(\bar{s} - \bar{\beta}^T x_i^*) \bar{\beta}_j, \quad j \in \{1, 2, \ldots, d\}.
\]

Finally, we consider two outcomes \(Y_{i,1}(w)\) and \(Y_{i,2}(w)\). The outcome \(Y_{i,1}(w)\) represents the number of months student \(i\) will enroll in the college if accepted \((w = 1)\) and if rejected \((w = 0)\). We derive \(Y_{i,1}(1)\) from the number of months student \(i\) was enrolled in postsecondary school from June 1992 - August 1994, reported in the NELS:88 dataset and have \(Y_{i,1}(0) = 0\) for all \(i\) because a rejected student cannot enroll in the college. The other outcome \(Y_{i,2}(w)\) is student \(i\)’s college grades if accepted \((w = 1)\) and if rejected \((w = 0)\). In our simulation, we define a student \(i\)’s college grades if admitted to be a function of the student’s raw test scores

\[Y_{i,2}(1) = \bar{Z}_{i,\text{test scores}},\]

and we have \(Y_{i,2}(0) = 0\) because a student cannot receive college grades if they are rejected. So, for each student \(i\) in the NELS:88 dataset, we have the unobservables \((Z_i, G_i, Y_{i,1}(1), Y_{i,2}(1))\). To simulate agent best responses to policy parameters, we must run Newton’s method to compute the best response.

| Method               | \(\dot{V}_{1,eq}(\dot{\beta})\) | \(\dot{V}_{2,eq}(\dot{\beta})\) |
|----------------------|---------------------------------|---------------------------------|
| Capacity-Aware \((\beta_{cap})\) | 3.69 ± 0.00                      | 1.92 ± 0.00                      |
| Strategy-Aware \((\beta_{strat})\) | 4.35 ± 0.02                      | 2.49 ± 0.01                      |
| Competition-Aware \((\beta_{comp})\) | 5.05 ± 0.01                      | 2.59 ± 0.00                      |

Table 3: We find that the competition-aware gradient-based optimization outperforms strategy-aware gradient-based optimization in terms of equilibrium policy value in our semi-synthetic experiment based on the NELS:88 dataset. A one-sided paired \(t\)-test, where we compare \(\dot{V}_{eq}(\beta_{comp})\) and \(\dot{V}_{eq}(\beta_{strat})\) from trials with the same random seed, yields a \(p\)-value of less than 1e-5. Similarly, a one-sided pair \(t\)-test, where we compare \(\dot{V}_{eq}(\beta_{comp})\) and \(\dot{V}_{eq}(\beta_{cap})\) from trials with the same random seed, yields a \(p\)-value of less than 1e-5.

In our simulation, we suppose that the covariates consist of twelfth grade standardized test scores in reading, math, science, and history and average grades in English, math, science, social studies, and foreign language from twelfth grade. NELS:88 contains twelfth grade and postsecondary data on students \(i = 1, 2, \ldots, m\), where \(m = 14915\). We assume that the reported test scores and grades from the NELS:88 reflect student \(i\)’s best response \(x_i^*\). So, \(x_i^* \in \mathbb{R}^d\) for \(d = 9\). We use the subscripts “test scores” and “grades” to specify the covariates and unobservables that correspond to test scores and grades, respectively. Student \(i\)’s gaming ability \(G_i\) is given by

\[G_{i,\text{test scores}} = 5, \quad G_{i,\text{grades}} \propto 1 - \gamma_i,\]

where \(\gamma_i\) is student \(i\)’s socioeconomic status (SES) percentile reported in the NELS:88 dataset. So, all agents have the same cost to improving their standardized test scores, but students with high SES can more easily improve their grades compared to students with low SES.

Under a simple model for student behavior, this information is sufficient for us to estimate the unobservables \((Z_i, G_i)\) for each student \(i\) that are consistent with the NELS data. We assume that each student’s utility is determined by a quadratic utility function as in (2.12). The variance \(\sigma^2\) of the noise distribution assumed to be \(\sigma^2 = 4.92\), which ensures that students’ best responses are continuous. In addition, we assume that \(x_i^*\) is the best response to a particular policy with parameters \((\bar{\beta}, \bar{s})\), where

\[
\bar{\beta} = \begin{bmatrix} \frac{1}{\sqrt{d}} & \frac{1}{\sqrt{d}} & \ldots & \frac{1}{\sqrt{d}} \end{bmatrix}^T, \quad \bar{s} = 19.5.
\]

Under these assumptions, we can compute each student’s raw covariates

\[Z_{i,j} = x_{i,j}^* - \frac{1}{2G_{i,j}} \phi_\sigma(\bar{s} - \bar{\beta}^T x_i^*) \bar{\beta}_j, \quad j \in \{1, 2, \ldots, d\}.
\]

Finally, we consider two outcomes \(Y_{i,1}(w)\) and \(Y_{i,2}(w)\). The outcome \(Y_{i,1}(w)\) represents the number of months student \(i\) will enroll in the college if accepted \((w = 1)\) and if rejected \((w = 0)\). We derive \(Y_{i,1}(1)\) from the number of months student \(i\) was enrolled in postsecondary school from June 1992 - August 1994, reported in the NELS:88 dataset and have \(Y_{i,1}(0) = 0\) for all \(i\) because a rejected student cannot enroll in the college. The other outcome \(Y_{i,2}(w)\) is student \(i\)’s college grades if accepted \((w = 1)\) and if rejected \((w = 0)\). In our simulation, we define a student \(i\)’s college grades if admitted to be a function of the student’s raw test scores

\[Y_{i,2}(1) = \bar{Z}_{i,\text{test scores}},\]

and we have \(Y_{i,2}(0) = 0\) because a student cannot receive college grades if they are rejected. So, for each student \(i\) in the NELS:88 dataset, we have the unobservables \((Z_i, G_i, Y_{i,1}(1), Y_{i,2}(1))\). To simulate agent best responses to policy parameters, we must run Newton’s method to compute the best response.
for each unique choice of agent type \((Z_i, G_i)\) and policy parameters \((\beta, s)\) because (2.13) does not have a closed form. Due to computational constraints of the data generation, we consider a distribution over \(K\) representative unobservables derived from the NELS:88 data instead of the direct approach of using the empirical distribution over all unobservables.

We determine the \(K\) representative unobservables \(\{(Z_k, G_k, Y_{k,1}(1), Y_{k,2}(1))\}_{k=1}^{K}\) by clustering the dataset \(\{(Z_i, G_i, Y_{i,1}(1), Y_{i,2}(1))\}_{i=1}^{m}\) into \(K = 8\) clusters via \(K\)-means clustering. A visualization of this clustering is provided in Figure 5. After that, we compute the proportion \(p_k\) of agent unobservables in each cluster \(k\) for \(k = 1, 2, \ldots, K\). Under our simulated distribution, an agent with unobservables \((Z_k, G_k)\) and outcomes \((Y_{k,1}(1), Y_{k,2}(1))\), is sampled with probability \(p_k\). Note that \(Y_{k,1}(0) = 0\) and \(Y_{k,2}(0) = 0\) for all representative outcomes.

### 6.4.2 Learning

We consider learning policies when student types and outcomes are sampled from the distribution over representative unobservables derived from the NELS:88 dataset. We consider a hypothetical decision maker who can only accept 30\% of the student population. Based on our definition of \(V_{\text{eq}}(\beta)\) in Definition 1, we can analogously define \(V_{\text{eq},1}(\beta), V_{\text{eq},2}(\beta)\) as the equilibrium policy value when the policy value is defined in terms of \(Y_1\) and \(Y_2\), respectively. We conduct two sets of simulations; in the first set of simulations, we evaluate the performance of the three baseline methods in maximizing \(V_{\text{eq},1}\), and in the second set of simulations, we evaluate the performance of the three baseline methods in maximizing \(V_{\text{eq},2}\). We also compare how the learned solutions denoted \(\beta_{\text{comp}}, \beta_{\text{strat}}, \text{ and } \beta_{\text{cap}}\) rank the agents.

For the capacity-aware baseline, we consider an RCT where treatments are assigned to \(n = 1000000\) agents in order to estimate the CATE. For the gradient-based methods, we initialize the policy at \(\bar{\beta}\) and optimize \(\beta\) via projected stochastic gradient descent using Algorithm 1 (in this case, ascent because we aim to maximize the policy value). We assume that \(n = 1000000\) students are observed by the decision maker at each iteration, and we run the algorithm for 60 iterations. We run 10 random trials, where the randomness is over the sampled agents.

### 6.4.3 Results

Across both sets of simulations, we observe that \(\beta_{\text{comp}}\) obtains higher equilibrium policy value than \(\beta_{\text{strat}}\) and \(\beta_{\text{cap}}\) (Table 3). Figure 6 visualizes iterates of the gradient-based methods for maximizing \(V_{\text{eq},1}(\beta)\) (left plot) and for maximizing \(V_{\text{eq},2}(\beta)\) (right plot).

Qualitatively, we find that in the simulation where the objective is \(V_{\text{eq},1}(\beta)\), the most performant selection criterion \(\beta_{\text{comp}}\) favors students with high SES, which can be seen in the first plot in Figure 7. Note that Figure 7 stratifies the score distribution according to unique agent type \((Z_i, G_i)\) by overlaying histograms of the score distribution for each unique agent type. The histograms are color-coded by the SES of the agent (which corresponds to the unobservable \(G_i\)), where blue represents low SES and red represents high SES. We hypothesize \(\beta_{\text{comp}}\) favors students with high SES because \(Y_{i,1}(1)\), which is defined to be the number of months a student \(i\) is enrolled in postsecondary education, is positively correlated with SES. In the NELS:88 dataset, we observe that the correlation between these two variables is \(r = 0.44\). This also implies that the outcome \(Y_{i,1}(1)\) is positively correlated with a student’s ability to modify their covariates \(G_i\). To favor students with high SES, \(\beta_{\text{comp}}\) places high weight on covariates that are less costly to modify for agents with high \(G_i\).

In contrast, in the simulation where the objective is \(V_{\text{eq},2}(\beta)\), we find that \(\beta_{\text{comp}}\) accepts agents with varied socioeconomic background, including a high proportion of students with low SES. We hypothesize that this solution arises because \(Y_{i,2}(1)\) is not strongly correlated with SES. In the NELS:88 dataset, the outcome \(Y_{i,2}(1)\) and SES have correlation \(r = 0.08\). We find the \(\beta_{\text{comp}}\) places zero or negative weights on grades, which are the covariates that can be more easily modified, and high weight on test scores, which are more costly to modify.

Here, we have illustrated that our policy effect estimator can be used to optimize different possible objectives, and the distribution of agents that are treated under the different policies will vary depending on the correlation between the outcome of interest with the agents’ types \((Z_i, G_i)\). An additional insight from our empirical analysis is that the outcome of interest has a large impact on the learned policy. To learn a decision rule that is robust to strategic modification of the covariates, the decision maker cannot simply
deploy gradient-based optimization with the policy effect with any outcome measure. As demonstrated in
the simulation where the objective is $V_{eq,1}$, if the outcome measure is positively correlated with agents’ ability
to modify their covariates, a value-maximizing decision rule will place high weight on covariates that are less
costly to modify.

7 Discussion

In many settings, a decision maker may rely on a policy to determine the allocation of a treatment among
a population of agents. If the treatment assignment policy is known to the agents, agents may behave
strategically to get a more desirable treatment. Although many recent works have considered the problem
of learning policies or models in the presence of strategic behavior [Ahmadi et al., 2022, Björkegren et al.,
2020, Brückner et al., 2012, Chen et al., 2020, Dalvi et al., 2004, Dong et al., 2018, Frankel and Kartik,
2019a, Hardt et al., 2016, Jagadeesan et al., 2021, Kleinberg and Raghavan, 2020, Levanon and Rosenfeld,
2022, Liu et al., 2021, Munro, 2020], the problem of learning policies in the presence of capacity constraints
and strategic behavior has not previously been studied in depth. However, this problem is of practical
importance because many of the motivating applications for learning in the presence of strategic behavior,
such as college admissions and hiring, are precisely settings where the decision maker is capacity-constrained.
In this work, we demonstrate that the combination of strategic behavior and capacity constraints leads to a
type of interference which manifests itself as competition.

In the literature on learning in the presence of strategic behavior, agents are assumed to invest effort
to modify their covariates to get a more desirable treatment. We adopt a flexible model where agents
are heterogenous in their ability to modify covariates and their raw features. Depending on the context,
strategic behavior may be harmful, beneficial, or neutral for the decision maker. In some applications,
strategic behavior may be a form of “gaming the system,” e.g. cheating on exams in the context of college
admissions, and the decision maker may not want to assign treatment to agents who have high ability to
modify their covariates. In contrast, in other applications, the decision maker may want to accept such
agents because the agents who would benefit the most from the treatment are those who can invest effort to
make themselves look desirable. Lastly, as demonstrated by Liu et al. [2021], in the setting where all agents
have identical ability to modify their covariates, the strategic behavior may be neutral for the decision maker
because it does not affect which agents are assigned treatment. Our model permits all of these interpretations
because we allow for the potential outcomes of an agent to be flexibly related to the agent’s ability to modify
their covariates and their raw features.

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A Experiment Details

A.1 Toy Experiment

We use a learning rate of $a = 0.5$ in vanilla SGD with $\hat{\tau}_{PE}$. We use a learning rate of $a = 0.25$ in vanilla SGD with $\hat{\tau}_{ME}$. We use a perturbation size $b = 0.025$ for $\beta$ and $b = 0.2$ for $s$.

A.2 High-Dimensional Experiment

We use a learning rate of $a = 0.5$ in projected SGD for all three baselines. For $\hat{\tau}_{ME}$, and $\hat{\tau}_{PE}$, we use a perturbation size $b = 0.025$ for $\beta$ and $b = 0.2$ for $s$. 
A.3 NELS:88 Semi-Synthetic Experiment

A.3.1 Dataset Information

National Education Longitudinal Study of 1988 (NELS:88) is nationally representative, longitudinal study of eighth graders in 1988. The cohort of students is followed throughout secondary and postsecondary years; the first followup occurs in 1990 (tenth grade for students who continue school), the second followup occurs in 1992 (twelfth grade for students who continue school), and the third occurs in 1994 (postsecondary years).

In this experiment, we aim to simulate a college admissions process, so we focus on the data collected in 1992 followup, where members of the initial cohort who remained in school are in 12th grade. We use the following publicly-available variables in Table 4 from the NELS dataset to construct the agent types, the agent covariates, and the agents’ outcomes based on whether they are admitted to the college or not.

Note that for the grades, 1. represents the highest grade (A+) and 13. represents the lowest grade and lower standardized scores represent higher performance. For simplicity, we negate these quantities in our preprocessing step, so that a higher score is more desirable. Let \( z \) be a variable that represents socioeconomic percentile (F2SES1). The remaining variables form the student’s best response \( x^* \).

| Variable     | Meaning                                      | Imputed Value | Range       |
|--------------|----------------------------------------------|---------------|-------------|
| F2SES1       | Socio-economic status composite              | -0.088        | [-3.243, 2.743] |
| F22XRSTD     | Reading standardized score                   | 63.81         | [29.01, 68.35] |
| F22XMSTD     | Mathematics standardized score               | 63.96         | [29.63, 71.37] |
| F22XSSTD     | Science standardized score                   | 64.01         | [29.70, 70.81] |
| F22XHSTD     | History standardized score                   | 64.30         | [25.35, 70.26] |
| F2RHENG2     | Average grade in English                     | 7.07          | [1., 13.]    |
| F2RHMAG2     | Average grade in mathematics                 | 7.61          | [1., 13.]    |
| F2RHSCG2     | Average grade in science                     | 7.43          | [1., 13.]    |
| F2RHSOG2     | Average grade in social studies              | 7.01          | [1., 13.]    |
| F2RHFOG2     | Average grade in foreign language            | 6.58          | [1., 13.]    |
| F3ATTEND     | Number of months attended postsecondary      | 19.21         | [1., 27.]    |
|              | institutions 06/1992-08/1994                 |               |             |

Table 4: NELS:88 variables used in semi-synthetic experiment.

A.3.2 Learning

We use a learning rate of \( a = 0.1 \) in projected SGD for all three baselines. We use a perturbation size \( b = 0.025 \) for \( \beta \) and \( b = 0.2 \) for \( s \).

B Standard Results

**Lemma 17.** Let \( \mathcal{X} \subset \mathbb{R}^d \) is a convex set. Let \( f : \mathcal{X} \to \mathbb{R} \) be a strictly concave function. If \( f \) has a global maximizer, then the maximizer is unique (Boyd et al. [2004]).

**Lemma 18.** Let \( f : \mathcal{X} \to \mathbb{R} \), where \( \mathcal{X} \subset \mathbb{R}^d \) is a convex set, be a twice-differentiable function. If \( f \) is a strictly concave function and \( x^* \) is in the interior of \( \mathcal{X} \), then \( x^* \) is the unique global maximizer of \( f \) on \( \mathcal{X} \) if and only if \( \nabla f(x^*) = 0 \) (Boyd et al. [2004]).

**Theorem 19** (Implicit Function Theorem). Suppose \( f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m \) is continuously differentiable in an open set containing \((x_0, y_0)\) and \( f(x_0, y_0) = 0 \). Let \( M \) be the \( m \times m \) matrix

\[
D_{x,y}f(x, y) = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix}
\]

If \( \det(M) \neq 0 \), then there is an open set \( X \subset \mathbb{R}^n \) containing \( x_0 \) and an open set \( Y \subset \mathbb{R}^m \) containing \( y_0 \), with the following property: for each \( x \in X \) there is a unique \( g(x) \in Y \) such that \( f(x, g(x)) = 0 \). The function \( g \) is continuously differentiable.
Theorem 20 (Sherman-Morrison Formula). Suppose $A \in \mathbb{R}^{d \times d}$ is an invertible square matrix, $u, v \in \mathbb{R}^d$. Then $A + uv^T$ is invertible iff $1 + v^T A^{-1} u \neq 0$. In this case,

$$(A + uv^T)^{-1} = A^{-1} - \frac{A^{-1}uv^TA^{-1}}{1 + v^T A^{-1} u}.$$ 

Lemma 21. Suppose $A, B \in \mathbb{R}^{d \times d}$ are positive definite matrices. If $A - B$ is positive semidefinite, then $B^{-1} - A^{-1}$ is positive semidefinite (Dhrymes [1978]).

Lemma 22. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable real function. The function $f$ is a contraction with modulus $\kappa \in (0, 1)$ if and only if $|f'(x)| \leq \kappa$ for all $x \in \mathbb{R}$ (Ortega [1990]).

Theorem 23 (Banach’s Fixed-Point Theorem). Let $(X, d)$ be a non-empty complete metric space with a contraction mapping $T : X \rightarrow X$. Then $T$ admits a unique fixed-point $x^*$ such that $T(x^*) = x^*$. Furthermore, for any number $x_0 \in X$, the sequence defined by $x_n = T(x_{n-1}), n \geq 1$ converges to the unique fixed point $x^*$.

Theorem 24. Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables from a c.d.f. $F$. Let $\theta_p$ be the $p$-th quantile of $F$ and let $\hat{\theta}_p$ be the $p$-th quantile of $F_n$. Suppose $F$ satisfies $p < F(\theta_p + \epsilon)$ for any $\epsilon > 0$. Then for every $\epsilon > 0$, then

$$P(|\hat{\theta}_p - \theta_p| > \epsilon) \leq 4e^{-2nM_e^2},$$

where $M_e = \min\{F(\theta_p + \epsilon) - p, p - F(\theta_p - \epsilon)\}$ (Theorem 5.9, Shao [2003]).

Theorem 25 (Bernoulli’s Inequality). For every $r \geq 0$ and $x \geq -1$, $(1 + x)^r \geq 1 + rx$.

Lemma 26. If the $u_i$ i.i.d., $\Theta$ is compact, $a(\cdot, \theta)$ is continuous at each $\theta \in \Theta$ with probability one, and there is $d(u)$ with $||a(u, \theta)|| \leq d(u)$ for all $\theta \in \Theta$ and $\mathbb{E}[d(u)] < \infty$, then $\mathbb{E}[a(u, \theta)]$ is continuous and

$$\sup_{\theta \in \Theta} \left| \frac{1}{n} \sum_{i=1}^n a(u_i, \theta) - \mathbb{E}[a(u, \theta)] \right| \overset{p}{\rightarrow} 0$$

(Lemma 2.4, Newey and McFadden [1994]).

Lemma 27. Suppose $\Theta$ is compact and $f(\theta)$ is continuous. Then $\sup_{\theta \in \Theta} |\hat{f}_n(\theta) - f(\theta)| \rightarrow 0$ if and only if $\hat{f}_n(\theta) \overset{p}{\rightarrow} f(\theta)$ for all $\theta \in \Theta$ and $\{\hat{f}_n(\theta)\}$ is stochastically equicontinuous (Lemma 2.8, Newey and McFadden [1994]).

Lemma 28. Suppose $\{Z_n(t)\}$ is a collection of stochastic processes indexed by $t \in T$. Suppose $\{Z_n(t)\}$ is stochastically equicontinuous at $t_0 \in T$. Let $\tau_n$ be a sequence of random elements of $T$ known to satisfy $\tau_n \overset{p}{\rightarrow} \tau_{t_0}$. It follows that $Z_n(\tau_n) \overset{p}{\rightarrow} Z_n(\tau_{t_0})$, (Pollard [2012]).

Lemma 29. Let $f_n : \mathcal{X} \rightarrow \mathbb{R}$ where $\mathcal{X} \subset \mathbb{R}$ is a compact set. Let $\{f_n\}$ be a sequence of continuous, monotonic functions that converge pointwise to a continuous function $f$. Then $f_n \rightarrow f$ uniformly (Buchanan and Hildebrandt [1968]).

Lemma 30. Let $f_n : \mathcal{X} \rightarrow \mathbb{R}$ where $\mathcal{X} \subset \mathbb{R}^d$ is a compact set. Let $\{f_n\}$ be a sequence of continuous, concave functions that converge pointwise to $f$. Furthermore, assume that $f$ is continuous. Then $f_n \rightarrow f$ uniformly (Rockafellar [1970]).

Lemma 31. Let $f_n : \mathcal{X} \rightarrow \mathbb{R}$ where $\mathcal{X} \subset \mathbb{R}$ is a compact set. Let $\{f_n\}$ be a sequence of continuous functions that converge uniformly to $f$. Suppose each $f_n$ has exactly one root $x_n \in \mathcal{X}$ and $f$ has exactly one root $x^* \in \mathcal{X}$. Then $x_n \rightarrow x^*$. Proof in Appendix G.1.

Lemma 32. Let $f_n : \mathcal{X} \rightarrow \mathbb{R}$ where $\mathcal{X} \subset \mathbb{R}^d$ is a compact set. Let $\{f_n\}$ be a sequence of continuous functions that converge uniformly to $f$. Suppose each $f_n$ has exactly one maximizer $x_n \in \mathcal{X}$ and $f$ has exactly one maximizer $x^* \in \mathcal{X}$. Then $x_n \rightarrow x^*$. Proof in Appendix G.2.

Theorem 33. Let us assume the following:

1. $K$ vanishes at infinity, and $\int_{-\infty}^{\infty} K^2(x)dx < \infty$, 

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Lemma 35. Under Assumption 1, if $\sigma^2 > \frac{1}{\alpha_i \sqrt{2\pi} e}$, then $\nabla^2_x \mathbb{E}_x[U_i(x; \beta, s)]$ is negative definite. Proof in Appendix G.4.

Lemma 36. Under Assumption 1, if $\sigma^2 > \frac{1}{\alpha_i \sqrt{2\pi} e}$ and $x \in \text{Int}(\mathcal{X})$, then $x = x_1^*(\beta, s)$ if and only if $\nabla_x \mathbb{E}_x[U_i(x; \beta, s)] = 0$. Proof in Appendix G.5.

Lemma 37. Under Assumption 1, if $\sigma^2 > \frac{1}{\alpha_i \sqrt{2\pi} e}$ and $x_1^*(\beta, s) \in \text{Int}(\mathcal{X})$, then

$$
\beta^T \nabla_s x_1^* = \phi'(s - \beta^T x_1^*(s)) \beta^T H^{-1} \beta - \left( \phi'(s - \beta^T x_1^*(s)) \beta^T H^{-1} \beta \right)^2,
$$

(C.1)

where $(Z, G) \sim F_{Z,G}, x_1^*(s) := x_1^*(\beta, s)$, and $H := \nabla^2 \psi_i(x_1^*(s) - Z; G)$. Proof in Appendix G.6.

Lemma 38. Let $H = \nabla^2 \psi_i(y)$ for some $y \in \mathbb{R}^d$. Under Assumption 1, we have that $H$ is positive definite, $H^{-1}$ is positive definite, and

$$
\sup_{\beta' \in \mathcal{B}} \beta'^T H^{-1} \beta' \leq \frac{1}{\alpha_i}.
$$

(C.2)

Proof in Appendix G.7.

Lemma 39. Under Assumptions 1, if $\sigma^2 > \frac{1}{\alpha_i \sqrt{2\pi} e}$ and $x_1^*(\beta, s) \in \text{Int}(\mathcal{X})$, then the function $h_i(s; \beta) = s - \beta^T x_1^*(\beta, s)$ is strictly increasing in $s$. Proof in Appendix G.8.

Lemma 40. Let $(Z, G) \sim F_{Z,G}$. Consider $\omega_i(s; \beta) = \beta^T x_1^*(\beta, s)$. Under Assumption 1, if $\sigma^2 > \frac{1}{\alpha_i \sqrt{2\pi} e}$ and $x_1^*(\beta, s) \in \text{Int}(\mathcal{X})$,

$$
\lim_{s \to -\infty} \omega_i(s; \beta) = \beta^T Z_i.
$$

(C.3)

$$
\lim_{s \to \infty} \omega_i(s; \beta) = \beta^T Z_i.
$$

(C.4)

Proof in Appendix G.9.

Lemma 41. Let $(Z, G) \sim F_{Z,G}$. Consider $\omega_i(s; \beta) = \beta^T x_1^*(\beta, s)$. Under Assumption 1, if $\sigma^2 > \frac{1}{\alpha_i \sqrt{2\pi} e}$ and $x_1^*(\beta, s) \in \text{Int}(\mathcal{X})$, then $\omega_i(s; \beta)$ is maximized at a point $s^*$, $\omega_i(s; \beta)$ is increasing when $s < s^*$ and is decreasing on $s > s^*$. Proof in Appendix G.10.

C.1 Proof of Lemma 1

We can apply Lemma 34 to show that the expected utility (2.11) is twice continuously differentiable in $x$, and thus continuous in $x$. Since $\mathcal{X}$ is compact, the expected utility attains a maximum value on $\mathcal{X}$ because a continuous function attains a maximum value on a compact set. Thus, there exists $x_1^* \in \mathcal{X}$ that maximizes the expected utility.

From Lemma 35, $\nabla^2_x \mathbb{E}_x[U_i(x; \beta, s)]$ is negative definite everywhere. This implies that the expected utility is strictly concave. Since $\mathcal{X}$ is a convex set and the expected utility is strictly concave, we can apply Lemma 17 to conclude that the best response is the unique maximizer of the expected utility on $\mathcal{X}$.
C.2 Proof of Lemma 2

We use the following abbreviations for the expected utility and best response

\[ E_s[U_i(x; s)] := E_s[U_i(x; \beta, s)] \]

\[ x_i^*(s) := x_i^*(\beta, s), \]

where \( \beta \) is fixed. We aim to show that if a best response \( x_i^*(s) \in \text{Int}(X) \), then \( x_i^* \) is continuously differentiable in \( s \). By Lemma 36, if a best response \( x_i^*(s) \in \text{Int}(X) \), then it satisfies \( \nabla_x E_s[U_i(x; s)] = 0 \). Our goal is to apply the Implicit Function Theorem (Theorem 19) to show that \( x \) that satisfies \( \nabla_x E_s[U_i(x; s)] = 0 \) can be written as a continuously differentiable function of \( s \).

Now, we verify the conditions of the Implicit Function Theorem. The conditions include \( \nabla_x E_s[U_i(x; s)] \) is continuously differentiable in its arguments and that at the point \((x_0, s_0)\) where the theorem is applied, we have \( \det(\nabla^2_x E_s[U_i(x_0; s_0)]) \neq 0 \). The first condition follows from Lemma 34, which in fact states that \( E_s[U_i(x; s)] \) is twice continuously differentiable in its arguments. For the second condition, we note that \( \nabla^2_x E_s[U_i(x; s)] \) is always negative definite everywhere from Lemma 35.

As a result, the conditions of the Implicit Function Theorem are satisfied. Let \((x_0, s_0)\) be any point that satisfies \( \nabla_x E_s[U_i(x; s)] = 0 \). In an open neighborhood \( A \times B \subset \mathbb{R}^d \times \mathbb{R} \) of \((x_0, s_0)\), for each \( s \in B \) there is a unique \( g(s) \in A \) such that \( \nabla_x E_s[U_i(g(s); s)] = 0 \) and \( g \) is a continuously differentiable function of \( s \). If \( g(s) \in \text{Int}(X) \), then \( g(s) \) coincides with the unique best response \( x_i^*(s) \) by Lemma 36. This implies that for \( x_i(s) \in \text{Int}(X) \), then \( x_i^* \) is continuously differentiable in \( s \).

An analogous proof can be used to show that the best response \( x_i^* \) is continuously differentiable in \( \beta \).

C.3 Proof of Lemma 3

Without loss of generality, we fix \( \beta \). We abbreviate

\[ x_i^*(s) := x_i^*(\beta, s). \]

To show that \( \beta^T x_i^*(s) \) is a contraction, it is sufficient to show that \( |\beta^T \nabla_x x_i^*| < 1 \) (Lemma 22). We show this result in two steps. First, we use Lemma 39 to show that \( \beta^T \nabla_x x_i^* < 1 \). Second, we can use our assumption that \( \sigma^2 > \frac{2}{\alpha_i \sqrt{2\pi e}} \) to show \( \beta^T \nabla_x x_i^* > -1 \).

We first show that \( \beta^T \nabla_x x_i^* < 1 \). Since we assume that \( \sigma^2 > \frac{2}{\alpha_i \sqrt{2\pi e}} \), we certainly have that \( \sigma^2 > \frac{1}{\alpha_i \sqrt{2\pi e}} \), so we can apply Lemma 39. This gives us that \( h_i(s; \beta) = s - \beta^T x_i^*(\beta, s) \) is strictly increasing. We can apply Lemma 2 to establish the differentiability of the best response, which consequently gives the differentiability of \( h_i \). Since \( h_i \) is also strictly increasing, we have that

\[ \frac{dh_i}{ds} = 1 - \beta^T \nabla_x x_i^*(s) > 0. \]

This gives us that \( \beta^T \nabla_x x_i^*(s) < 1 \).

Now, we establish that \( \beta^T \nabla_x x_i^* > -1 \). We use Lemma 37 to get an expression \((C.1)\) for \( \beta^T \nabla_x x_i^* \).

We can simplify \((C.1)\) as follows,

\[ \beta^T \nabla_x x_i = \phi_x(s - \beta^T x_i^*) \beta^T H^{-1} \beta \]

\[ = \phi_x'(s - \beta^T x_i^*) \beta^T H^{-1} \beta \]

\[ = \phi_x'(s - \beta^T x_i^*) \beta^T H^{-1} \beta \]

We study the numerator of the term on the right side of \((C.6)\).

\[ \phi_x'(s - \beta^T x_i^*) \beta^T H^{-1} \beta \]

\[ \geq \inf_{y \in \mathbb{R}} \phi_x'(y) \sup_{\beta \in \mathbb{B}} \beta^T H^{-1} \beta' \]

\[ \geq \left( -\frac{1}{\alpha_i} \right) \cdot \frac{1}{\alpha_i} \]

\[ > -\frac{\alpha_i}{2} \]

\[ = -\frac{1}{2}. \]
Lemma 44. Let \( \sigma \) be the distribution over \( Z_1, Z_2, \ldots, Z_M \) with \( \mathbf{1}^\top \mathbf{1} = 1 \). If \( \sigma^2 > \frac{1}{\alpha \sqrt{2\pi}} \), then \( \sigma \) is a well-defined function. Furthermore, it is strictly increasing in \( r \), continuously differentiable in \( \beta, s, r \), and has a unique continuous inverse distribution function. Proof in Appendix G.11.

Lemma 43. Fix \( \beta \in \mathcal{B} \). Suppose Assumptions 1, 2, and 3 hold. If \( \sigma^2 > \frac{1}{\alpha \sqrt{2\pi}} \), then \( \frac{\partial q(P(\beta, s))}{\partial s} < 1 \). If \( \sigma^2 > \frac{2}{\alpha \sqrt{2\pi}} \), then \( \frac{\partial q(P(\beta, s))}{\partial s} \) is positive definite. Proof in Appendix G.12.

Lemma 44. Let \( \beta \in \mathcal{B} \). Suppose Assumptions 1, 2, and 3 hold. If \( \sigma^2 > \frac{1}{\alpha \sqrt{2\pi}} \), then \( \sigma \) is a well-defined function. Furthermore, it is strictly increasing in \( r \), continuously differentiable in \( \beta, s, r \), and has a unique continuous inverse distribution function. Proof in Appendix G.13.
D.1 Proof of Theorem 4

Our main goal is show that \( P(\beta, s)(s) \) is a continuous and strictly increasing function of \( s \). A continuous and strictly increasing function can intersect a horizontal line in at most one point, so if a fixed point of \( q(P(\beta, s)) \) exists, then it must be unique.

From Lemma 42, \( P(\beta, s) \) has a unique inverse. So, if there is a fixed point \( s = q(P(\beta, s)) \), then the fixed point satisfies \( P(\beta, s)(s) = q \).

Applying (D.1) from Lemma 42, we have

\[
P(\beta, s)(s) = \int \Phi_\sigma(s - \beta^T x^*_i(\beta, s))dF
= \int \Phi_\sigma(h_i(s; \beta))dF;
\]

where \( h_i(\beta; \beta) = s - \beta^T x^*_i(\beta, s) \).

The continuity of \( P(\beta, s)(s) \) in \( s \) follows from the continuity of \( P(\beta, s)(r) \) in \( (s, r) \) (Lemma 42).

We show that \( P(\beta, s)(s) \) is strictly increasing in \( s \). From Lemma 39, we have that \( h_i(s; \beta) \) is strictly increasing in \( s \) for any agent with unobservables in the support of \( F \). Since \( \Phi_\sigma \) is a strictly increasing CDF, so we have that \( \Phi_\sigma(h_i(s; \beta)) \) is also strictly increasing. Finally, the sum of strictly increasing functions is strictly increasing, which gives that the integral is also a strictly increasing function of \( s \).

Since \( P(\beta, s)(s) \) is continuous and strictly increasing in \( s \), there is at most one point where it can equal \( q \). Thus, if a fixed point of \( q(P(\beta, s)) \) exists, then it is unique.

D.2 Proof of Lemma 5

To show that \( q(P(\beta, s)) \) is continuously differentiable in \( s \), we first show that \( q(P(\beta, s)) \) can be expressed implicitly as a the value of \( r \) that solves

\[
h(s, r) = P(\beta, s)(r) - q = 0. \tag{D.2}
\]

Second, we verify that the Implicit Function Theorem (Theorem 19) can be applied to \( h(s, r) = 0 \), so that \( r \) can be expressed as a continuously differentiable function of \( s \). Since \( r = q(P(\beta, s)) \), we can conclude that \( q(P(\beta, s)) \) is continuously differentiable in \( s \).

Applying Lemma 42 yields the first step–we have that \( P(\beta, s) \) has a unique inverse distribution function, so there exists a unique \( r \) such that \( r = q(P(\beta, s)) \). Equivalently, \( P(\beta, s)(r) = q \), which yields (D.2).

In the second step, we aim to apply Implicit Function Theorem to \( h(s, r) = 0 \) at any point \((s_0, r_0)\) that satisfies \( h(s, r) = 0 \) to show that \( r \) can be expressed as a continuously differentiable function of \( s \). Since \( r = q(P(\beta, s)) \), this is sufficient for showing that \( q(P(\beta, s)) \) is continuously differentiable function of \( s \).

The conditions of the Implicit Function Theorem include that \( h(s, r) \) is continuously differentiable in its arguments and that \( \frac{\partial h}{\partial r}(s_0, r_0) \neq 0 \). We verify that these conditions hold as follows. Both of these conditions follow from Lemma 42, which gives that \( P(\beta, s)(r) \) is continuously differentiable in \( (s, r) \) and strictly increasing in \( r \). We have that

\[
\frac{\partial h}{\partial r} = \frac{\partial P(\beta, s)(r)}{\partial r} > 0.
\]

So, for any \((s_0, r_0)\), we have that \( \frac{\partial h}{\partial r}(s_0, r_0) \neq 0 \).

As a result, the conditions of the Implicit Function Theorem are satisfied. Let \((s_0, r_0)\) be any point that satisfies \( h(s, r) = 0 \). In an open neighborhood \( A \times B \subset \mathbb{R} \times \mathbb{R} \) of \((s_0, r_0)\), for each \( s \in A \) there is a unique \( g(s) \in B \) such that \( h(s, g(s)) = 0 \) and \( g \) is a continuously differentiable function of \( s \). Since \( r = q(P(\beta, s)) \) satisfies \( h(s, r) = 0 \), we must have that \( q(P(\beta, s)) \) is a continuously differentiable function of \( s \).

An analogous proof can be used to show that \( q(P(\beta, s)) \) is continuously differentiable in \( \beta \).

D.3 Proof of Theorem 6

We aim to apply the Intermediate Value Theorem to the function \( g(s) = s - q(P(\beta, s)) \) to show that \( q(P(\beta, s)) \) has at least one fixed point. We note that by Lemma 5 that \( g(s) \) is continuous. It remains to show that there
exists $s_l$ such that $g(s_l) < 0$ and there exists $s_h$ such that $s_h > s_l$ and $g(s_h) > 0$. Then, by the Intermediate Value Theorem, there must be $s \in [s_l, s_h]$ for which $g(s) = 0$, which gives that $q(P(\beta, s))$ has at least one fixed point.

First, by Lemma 44, we have that

$$\lim_{s \to \infty} q(P(\beta, s)) = q(P)$$

$$\lim_{s \to -\infty} q(P(\beta, s)) = q(P).$$

So, for some $s \geq S_1$ where $S_1 < \infty$, we have that $|q(P(\beta, s)) - q(P)| < \delta$. Let $s_h = \max(q(P) + \delta, S_1)$. Then we have for $s \geq s_h$,

$$g(s) = s - q(P(\beta, s))$$

$$> s - q(P) - \delta$$

$$\geq q(P) + \delta - q(P) - \delta$$

$$= 0.$$

Similarly, for some $s \leq S_2$ where $S_2 > -\infty$, we have that $|q(P(\beta, s)) - q(P)| < \delta$. Let $s_l = \min(q(P) - \delta, S_2)$. Then we have for $s \leq s_l$,

$$g(s) = s - q(P(\beta, s))$$

$$< s - q(P) + \delta$$

$$\leq q(P) - \delta - q(P) + \delta$$

$$= 0.$$

So, by the Intermediate Value Theorem there must be $s \in [s_l, s_h]$ for which $g(s) = 0$, which gives that $q(P(\beta, s))$ has at least one fixed point.

### D.4 Proof of Corollary 7

Since we assumed that $q(P(\beta, s))$ is a contraction in $s$ and for fixed $\beta \in \mathcal{B}$, $q(P(\beta, s)) : \mathbb{R} \to \mathbb{R}$, then we can apply Banach’s Fixed Point Theorem (Theorem 23) to conclude that the process in (3.1) converges to the unique fixed point of $q(P(\beta, s))$.

### D.5 Proof of Corollary 8

If $\sigma^2 > \frac{2}{\alpha \sqrt{2\pi}}$, we can apply the second part of Lemma 43 to conclude that $|\frac{\partial q(P(\beta, s))}{\partial s}| < 1$. By Lemma 22, $q(P(\beta, s))$ is a contraction in $s$. As a consequence of Theorem 7, we can conclude that fixed point iteration (3.1) converges to the unique fixed point of $q(P(\beta, s))$.

### D.6 Proof of Corollary 9

First, we show that we can define a function that maps linear coefficient $\hat{\beta}$ to the equilibrium threshold $s^*$ induced by $\hat{\beta}$. Second, we give an equation that implicitly expresses this function. We verify that this equation satisfies the conditions of the Implicit Function Theorem at any point $(\hat{\beta}, s^*)$, where $\hat{\beta}$ is the linear coefficient and $s^*$ is equilibrium threshold induced by $\hat{\beta}$, and apply the Implicit Function Theorem to arrive at the desired result. To verify one of the conditions of the Implicit Function Theorem, we will use the first part of Lemma 43.

Recall that for every $\beta \in \mathcal{B}$, there exists a fixed point $s^*$ that satisfies $q(P(\beta, s^*)) = s^*$ (Theorem 6), and it is unique (Theorem 4). As a result, we can define a function $s : \mathcal{B} \to \mathbb{R}$ that maps $\beta$ to the fixed point induced by $\beta$.

Note that we can implicitly represent $s(\beta)$ by $s$ in the following equation

$$h(\beta, s) = s - q(P(\beta, s)) = 0.$$
We aim to apply the Implicit Function Theorem to \( h(\beta, s) \) at any point \((\beta_0, s_0)\) where \( h(\beta_0, s_0) = 0 \). We verify that the conditions of the Implicit Function Theorem are satisfied, which include that \( h(\beta, s) \) must be continuously differentiable in its arguments and \( \frac{\partial h(\beta, s)}{\partial s} (\beta_0, s_0) \neq 0 \).

For the first condition, Lemma 5 gives us that \( q(P(\beta, s)) \) is continuously differentiable in its arguments. As a result, \( h(\beta, s) \) is also continuously differentiable in \( \beta, s \).

For the second condition, we note that

\[
\frac{\partial h(\beta, s)}{\partial s} = 1 - \frac{\partial q(P(\beta, s))}{\partial s}.
\]

From Lemma 43, we have that \( \frac{\partial q(P(\beta, s))}{\partial s} < 1 \), so \( \frac{\partial h(\beta, s)}{\partial s} > 0 \). Thus, the conditions of the Implicit Function Theorem are satisfied.

Let \((\beta_0, s_0)\) be a point that yields \( h(\beta_0, s_0) = 0 \). In an open neighborhood \( A \times B \subset \mathbb{R}^d \times \mathbb{R} \) of \((\beta_0, s_0)\), for every \( \beta \in A \), there is a unique \( g(\beta) \in B \) such that \( h(\beta, g(\beta)) = 0 \) and \( g \) is a continuously differentiable function of \( \beta \). We note that such \( g(\beta) \) must correspond to the unique equilibrium threshold induced by \( \beta \), so \( s(\beta) = g(\beta) \). Thus, \( s(\beta) \) is a continuously differentiable function of \( \beta \).

### E Proofs of Finite Approximation Results

#### Lemma 45
Suppose the conditions of Theorem 11 hold. Let \( \{z^t\} \) be a sequence of random variables where

\[
z^t = \begin{cases} 
\epsilon_g & \text{w.p. } p_n(\epsilon_g) \\
C_k & \text{w.p. } \frac{1-p_n(\epsilon_g)}{2^k}, k \geq 1,
\end{cases}
\]

where \( p_n(\epsilon_g) \) is the bound from Lemma 10 and

\[
C_k = \sqrt{\frac{1}{2nM_{\epsilon_g}^2}} \cdot \log \left( \frac{2^{k+1}}{1-p_n(\epsilon_g)} \right).
\]

For any \( s \in \mathbb{R} \), \( z^t \) stochastically dominates \( |q(P^n(\beta, s)) - q(P(\beta, s))| \). Proof in Appendix G.14.

#### Lemma 46
Suppose the conditions of Theorem 11 hold. Let \( \{\hat{S}^t_n\}_{t \geq 0} \) be a stochastic process generated via (4.2). Let \( C \) be as defined in Theorem 11. Let \( \{z^t\}_{t \geq 0} \) be a sequence of random variables where

\[
z^t = \begin{cases} 
\epsilon_g & \text{w.p. } p_n(\epsilon_g) \\
C_k & \text{w.p. } \frac{1-p_n(\epsilon_g)}{2^k}, k \geq 1,
\end{cases}
\]

where \( p_n(\epsilon_g) \) is the bound from Lemma 10 and \( C_k \) is as defined in Lemma 45. Let \( \kappa \) be the Lipschitz constant of \( q(P(\beta, s)) \). Then \( \sum_{t=1}^{\lambda^*} z^{t-i} \kappa^{i} + \kappa^{t} \cdot C \) stochastically dominates \( |\hat{S}^t_n - s(\beta)| \). Proof in Appendix G.15.

#### E.1 Proof of Lemma 10
We define notation that will be used in the rest of the proof. Let \( s^*_L = \arg\max_{s \in \mathbb{R}} x^*_L(\beta, s) \). Let \( s_L = \inf_{(z_i, G_i) \in \text{supp}(F_{Z,G})} s^*_L \) and let \( s_H = \sup_{(Z_i, G_i) \in \text{supp}(F_{Z,G})} s^*_L \). We also can define functions

\[
f_1(s) := P(\beta, s)(q(P(\beta, s)) + \epsilon) - q
\]

\[
f_2(s) := q - P(\beta, s)(q(P(\beta, s)) - \epsilon).
\]

We define

\[
M_{\epsilon} = \inf_{s \in \mathbb{R}} \{f_1(s), f_2(s)\},
\]

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and we aim to show that $M_e > 0$. We note that

$$M_e = \min\{\inf_{s \in \mathbb{R}} f_1(s), \inf_{s \in \mathbb{R}} f_2(s)\}.$$  

We note that

$$\inf_{s \in \mathbb{R}} f_1(s) = \min\{\inf_{s < s_L} f_1(s), \inf_{s > s_H} f_1(s), \inf_{s \in [s_L, s_H]} f_1(s)\}.$$  

The differentiability of $P(\beta, s)$ and $q(P(\beta, s))$ in $s$ is given by Lemma 42 and Lemma 5, respectively. Thus, we can write that

$$\frac{df_1}{ds} = p_{\beta, s}(q(P(\beta, s)) + \epsilon) \cdot \frac{\partial q(P(\beta, s))}{\partial s}.$$  

By Lemma 41, the score of the best response $\beta^T x^*_i(\beta, s)$ for each agent $i$ is increasing on $s \leq s_L$ and decreasing on $s \geq s_H$. By Lemma 43, $\frac{\partial q(P(\beta, s))}{\partial s}$ is a convex combination of $\beta^T \nabla_s x^*_i$. Thus, $\frac{\partial q(P(\beta, s))}{\partial s}$ is positive for $s < s_L$ and negative on $s > s_H$. This implies that $\frac{df_1}{ds}$ is positive for $s < s_L$ and negative on $s > s_H$. Thus, $f_1(s)$ is increasing on $(-\infty, s_L)$ and $f_1(s)$ is decreasing on $(s_H, \infty)$. So,

$$\inf_{s < s_L} f_1(s) = \lim_{s \to -\infty} f_1(s)$$

$$= \lim_{s \to -\infty} P(\beta, s)(q(P(\beta, s)) + \epsilon) - q$$

$$= P(q(P) + \epsilon) - q,$$

where the last line follows from Lemma 44 and $P$ is the distribution defined that lemma. Similarly, we can show that

$$\inf_{s > s_H} f_1(s) = \lim_{s \to -\infty} f_1(s) = P(q(P) + \epsilon) - q.$$  

Finally, because $[s_L, s_H]$ is a compact set, there is some $s_1 \in [s_L, s_H]$ for which $f_1(s)$ achieves its infimum on the interval. Thus,

$$\inf_{s \in \mathbb{R}} f_1(s) = \min\{P(q(P) + \epsilon) - q, P(\beta, s_1)(q(P(\beta, s_1)) + \epsilon) - q\}.$$  

Thus, $\inf_{s \in \mathbb{R}} f_1(s) > 0$.

Similarly, we can compute $\inf_{s \in \mathbb{R}} f_2(s)$. We note that

$$\inf_{s \in \mathbb{R}} f_2(s) = \min\{\inf_{s < s_L} f_2(s), \inf_{s > s_H} f_2(s), \inf_{s \in [s_L, s_H]} f_2(s)\}.$$  

We can write that

$$\frac{df_2}{ds} = -p_{\beta, s}(q(P(\beta, s)) - \epsilon) \cdot \frac{\partial q(P(\beta, s))}{\partial s}.$$  

From this result, we can see that $f_2(s)$ is decreasing on $(-\infty, s_L)$ and $f_2(s)$ is increasing on $(s_H, \infty)$. So,

$$\inf_{s < s_L} f_2(s) = \lim_{s \to s_L} f_2(s)$$

$$= f_2(s_L)$$

$$\inf_{s > s_H} f_2(s) = \lim_{s \to s_H} f_2(s)$$

$$= f_2(s_H).$$

In addition, because $[s_L, s_H]$ is a compact set, there is some $s_2 \in [s_L, s_H]$ for which $f_2(s)$ achieves its infimum on the interval. Thus,

$$\inf_{s \in \mathbb{R}} f_2(s) = \min_{s \in \{s_L, s_H, s_2\}} f_2(s),$$

so $\inf_{s \in \mathbb{R}} f_2(s) > 0$. Thus, $M_e > 0$.

Now, we proceed to show the second component of the lemma. From Theorem 24, we have that

$$P(|q(P(\beta, s)) - q(P^n(\beta, s))| < \epsilon) \geq 1 - 4e^{-2nM_e^2 \epsilon}.$$
where \( M_{\epsilon,s} = \min\{f_1(s), f_2(s)\} \). We can obtain a bound that is uniform over \( s \) by realizing that \( M_{\epsilon} = \inf_{s \in \mathbb{R}} \min\{f_1(s), f_2(s)\} \) and \( M_{\epsilon} > 0 \). So, we have that

\[
P(|q(P(\beta, s)) - q(P^n(\beta, s))| < \epsilon) \geq 1 - 4e^{-2nM^2_{\epsilon}}.
\]

E.2 Proof of Theorem 11

Let \( \{z^t\}_{t \geq 1} \) be a sequence of random variables where

\[
z^t = \begin{cases} \epsilon_g & \text{w.p. } p_n(\epsilon_g) \\ C_k & \text{w.p. } \frac{1 - p_n(\epsilon_g)}{2}, k \geq 1 \end{cases},
\]

where \( \epsilon_g = \frac{\epsilon(1 - \kappa)}{2} \) and \( p_n(\epsilon_g) \) is the bound from Lemma 10. From Lemma 46, we have that

\[
|\hat{S}_n^t - s(\beta)| \leq \text{SD} \sum_{i=1}^{t} z^{t-i} \kappa^i + \kappa^t \cdot C. \tag{E.2}
\]

We note that

\[
\sum_{i=1}^{t} \epsilon_g \kappa^i < \sum_{i=0}^{\infty} \epsilon_g \kappa^i = \frac{\epsilon_g}{1 - \kappa} = \frac{\epsilon(1 - \kappa)}{2} \cdot \frac{1}{1 - \kappa} = \frac{\epsilon}{2}.
\]

In addition, let \( t \geq \left\lceil \frac{\log(\frac{\epsilon}{2C})}{\log \kappa} \right\rceil \). For such \( t \), we have that

\[
t \geq \frac{\log(\frac{\epsilon}{2C})}{\log \kappa}.
\]

Rearranging the above inequality gives

\[
\kappa^t C \leq \frac{\epsilon}{2}.
\]

As a result, we have that

\[
\sum_{i=0}^{t} \epsilon_g \kappa^i + \kappa^t C < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\]

By (E.2) and the definition of stochastic dominance, we have that

\[
P(|\hat{S}_n^t - s(\beta)| \leq \epsilon) \geq P\left( \sum_{i=1}^{t} z^{t-i} \kappa^i + \kappa^t C \leq \epsilon \right) \\
\geq P(z^{t-i} = \epsilon_g \text{ for } i = 1, 2 \ldots t) \\
\geq (p_n(\epsilon_g))^t
\]

If we have that

\[
n \geq \frac{1}{2M_{\epsilon}^2} \log \left( \frac{4t}{\delta} \right), \tag{E.3}
\]

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then we can show that $p_n(\epsilon_g) \geq 1 - \frac{\delta}{t}$. We can rearrange (E.3)

$$e^{-2nM^2_{\epsilon_g}} \leq \frac{\delta}{4t}.$$  

Rearranging again,

$$1 - 4e^{-2nM^2_{\epsilon_g}} \geq 1 - \frac{\delta}{t}.$$  

Thus, we have that

$$p_n(\epsilon_g) \geq 1 - \frac{\delta}{t}.$$  

So, $(p_n(\epsilon_g))^t \geq (1 - \frac{\delta}{t})^t$. Applying Theorem 25 gives that $(p_n(\epsilon_g))^t \geq 1 - \delta$. Therefore, we conclude that if $t \geq \lceil \log(\frac{\epsilon^2}{2C}) \rceil$ and $n \geq \frac{1}{2M^2_{\epsilon_g}} \log \left( \frac{4t}{\delta} \right)$, then

$$P(|\hat{S}^t_n - s(\beta)| \leq \epsilon) \geq 1 - \delta,$$

as desired.

### E.3 Proof of Corollary 12

To show that $\hat{S}^t_n \xrightarrow{P} s(\beta)$, we must show that

$$\lim_{n \to \infty} P(|\hat{S}^t_n - s(\beta)| > \epsilon) = 0.$$  

It is sufficient to show that for any $\delta > 0$, there exists $N$ such that for $n \geq N$,

$$P(|\hat{S}^t_n - s(\beta)| > \epsilon) \leq \delta.$$  

As in the statement of Theorem 11, let $C = |\hat{S}^t_n - s(\beta)|$. Let $N_1 \in \mathbb{N}$ be the smallest value of $n$ such that

$$t_n \geq \lceil \frac{\log(\frac{\epsilon}{\delta})}{\log \kappa} \rceil.$$  

We have that $t_n \sim \exp(n)$. So, there exists $N_2 \in \mathbb{N}$ such that for $n \geq N_2$,

$$t_n + 1 \leq \frac{\delta}{4} \exp \left( 2nM^2_{\epsilon_g} \right).$$  

Rearranging this equation, we have that for $n \geq N_2$,

$$\exp(2nM^2_{\epsilon_g}) \geq \frac{4(t_n + 1)}{\delta}.$$  

Taking log of both sides yields for $n \geq N_2$

$$2nM^2_{\epsilon_g} \geq \log \left( \frac{4(t_n + 1)}{\delta} \right).$$  

So, for $n \geq N_2$, we have that

$$n \geq \frac{1}{2M^2_{\epsilon_g}} \log \left( \frac{4(t_n + 1)}{\delta} \right).$$  

We can take $N = \max\{N_1, N_2\}$. By Theorem 11, we have that for $n \geq N$, $P(|\hat{S}^t_n - s(\beta)| > \epsilon) < \delta$. Thus, we have that $\hat{S}^t_n \xrightarrow{P} s(\beta)$.  

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F Proofs of Learning Results

We state technical lemmas that will be used in many of our learning results.

Lemma 47. Let $\beta \in \mathcal{B}$. Let $s(\beta)$ be the mean-field equilibrium threshold. Define a truncated stochastic fixed point iteration process

$$
\hat{S}_n^t = \begin{cases} 
q(P_{\beta,s_n^{-1}}^n) & q(P_{\beta,s_n^{-1}}^n) \in \mathcal{S} \\
-D & q(P_{\beta,s_n^{-1}}^n) < -D \\
D & q(P_{\beta,s_n^{-1}}^n) > D 
\end{cases}
$$

Under Assumptions 1, 2, 3, if $\sigma^2 > \frac{2}{\alpha \sqrt{2n}e}$, then for any sequence $\{t_n\}$ such that $t_n \uparrow \infty$ as $n \to \infty$ and $t_n \prec \exp(n)$, we have that $\hat{S}_n^t \xrightarrow{p} s(\beta)$. Proof in Appendix G.16.

Lemma 48. Recall that under Assumption 5, $m(z,g;w) = \mathbb{E}[Y_i(w) \mid Z_i = z, G_i = g]$. Let $\tilde{j}, \tilde{y}, \tilde{k}$ be functions

$$
\tilde{j}(Z_i, G_i, \epsilon_i, \beta, s, r) = \pi(X_i(\beta, s, \beta, r)), \\
\tilde{y}(Z_i, G_i, \epsilon_i, \beta, s, r) = m(Z_i, G_i; \pi(X_i(\beta, s, \beta, r)), \\
\tilde{k}(Z_i, G_i, \epsilon_i, \beta, s, r) = \mathbb{I}
$$

for any unobservables $(Z_i, G_i, \epsilon_i) \sim F_{Z,G,e}$. Under the conditions of Theorem 14, $\tilde{j}, \tilde{y}, \tilde{k}$ satisfy the requirements on the function $a(u; \theta)$ from Lemma 26, where the data $u$ is given by $(Z_i, G_i, \epsilon_i) \sim F_{Z,G,e}$ and the parameter $\theta$ is given by $(\beta, s, r)$. Proof in Appendix G.17.

Lemma 49. Let $\beta \in \mathcal{B}, s \in \mathcal{S}, \zeta \in \{-1,1\}^d, \xi \in \{-1,1\}$, and $b > 0$. Define $T$ to be a mapping from an agent $i$ and cost function $c_i$ to an agent $i'$ with unobservables $(Z_{i'}, b, \zeta, \xi, G_{i'})$ and cost function $c_{i'}$, where $(Z_i, G_i) \in \text{supp}(F_{Z,G})$ and cost function $c_i$ satisfies Assumption 1. Let

$$
x_1 := x_i^*(\beta + b \zeta, s + b \xi) \\
r := (\beta + b \zeta)^T x_1 - b \zeta.
$$

Define $Z_{i', b, \zeta, \xi}$ and $c_{i'}$ as follows.

$$
Z_{i', b, \zeta, \xi} := Z_i + \beta \cdot b \cdot (\zeta^T x_1 - \xi) \\
G_{i', b, \zeta, \xi} := G_i \\
c_{i'}(y) := c_i(y) - \phi(\epsilon; s - r) \beta^T y.
$$

If $Z_i, x_1 \in \text{Int}(\mathcal{X})$ and $b$ sufficiently small, then $Z_{i', b, \zeta, \xi} \in \mathcal{X}$, $c_{i'}$ is $\alpha_i$-strongly convex,

$$
x_i^*(\beta, s) = x_1 + b \cdot \beta (\zeta^T x_1 - \xi),
$$

$x_i^*(\beta, s) \in \text{Int}(\mathcal{X})$, and $\beta^T x_i^*(\beta, s) = r$. In other words, when the agent $i'$ with unobservables and cost function given by $T(i)$ best responds to the unperturbed model $\beta$ and threshold $s$, they obtain the same raw score (without noise) as the agent $i$ who responds to a perturbed model $\beta + b \zeta$ and threshold $s + b \xi$. Proof in Appendix G.18.

Lemma 50. Suppose the conditions of Theorem 14 hold. Fix $\beta \in \mathcal{B}, s \in \mathcal{S}$. For sufficiently small $b$, there exists a distribution over unobservables, $\tilde{F}$ and corresponding cost functions $c_{i'}$ for each agent $i'$ with unobservables $(Z_{i'}, G_{i'}) \in \text{supp}(\tilde{F}_{Z,G})$ such that when agents with unobservables $(Z_{i'}, G_{i'}) \sim \tilde{F}_{Z,G}$ and cost functions $c_{i'}$ best respond to the unperturbed model $\beta$ and threshold $s$ the induced score distribution is equal to $P(\beta, s, b)$. Furthermore, the support of $\text{supp}(\tilde{F}_{Z,G}) \subset \mathcal{X}$, each $c_{i'}$ satisfies Assumption 1, $\tilde{F}$ has a finite number of agent types, $\alpha_i(F) = \alpha_i(F)$, and for any agent $i'$ with unobservables $(Z_{i'}, G_{i'}) \sim \tilde{F}$, we have $x_i^*(\beta, s) \in \text{Int}(\mathcal{X})$. Proof in Appendix G.19.
Lemma 51. Fix $\beta \in \mathcal{B}$. Suppose the conditions of Theorem 14 hold. If $b$ is sufficiently small, then $q(P(\beta, s, b))$ has a unique fixed point $s(\beta, b)$. As $b \to 0$, $s(\beta, b) \to s(\beta)$, where $s(\beta)$ is the unique fixed point of $q(P(\beta, s))$. Proof in Appendix G.20.

Lemma 52. Fix $\beta \in \mathcal{B}$. Let $\{t_n\}$ be a sequence such that $t_n \uparrow \infty$ as $n \to \infty$. Under the conditions of Theorem 14, if $b$ sufficiently small,

$$\hat{S}_{b/n}^{t_n} \Rightarrow s(\beta, b), \quad \hat{S}_{b/n}^{t_n-1} \Rightarrow s(\beta, b)$$

where $s(\beta, b)$ is the unique fixed point of $q(P(\beta, s, b))$. Proof in Appendix G.21.

F.1 Proof of Lemma 13
Let $\Delta_i = Y_i(1) - Y_i(0)$. We have that

$$V_{eq}(\beta) = V(\beta, s(\beta), s(\beta))$$

$$= \mathbb{E}_P [Y_i(\pi(X_i(\beta, s(\beta)); \beta, s(\beta)))]$$

$$= \mathbb{E}_P [Y_i(\pi(X_i(\beta, s(\beta)) \geq s(\beta)))]$$

$$= \mathbb{E}_{Z,G} [\mathbb{E}_{i|Z,G} [Y_i(1) \cdot \mathbb{I}(\beta^T X_i(\beta, s(\beta)) \geq s(\beta))]]$$

$$= \mathbb{E}_{Z,G} [Y_i(1)(1 - \Phi_x(s(\beta) - \beta^T x_i^*(\beta, s(\beta))))]$$

$$+ \mathbb{E}_{Z,G} [Y_i(0)\Phi_x(s(\beta) - \beta^T x_i^*(\beta, s(\beta)))]$$

$$= \mathbb{E}_{Z,G} [Y_i(1) - \Delta_i \cdot \Phi_x(s(\beta) - \beta^T x_i^*(\beta, s(\beta)))] .$$

Under the assumed conditions, we have that $x_i^*$ is continuously differentiable in its first and second arguments by Lemma 2. We also have that $s$ is continuously differentiable in $\beta$ by Corollary 9. Thus, $V_{eq}(\beta)$ continuously differentiable in $\beta$.

F.2 Proof of Theorem 14
Let $s(\beta)$ be the equilibrium threshold induced by $\beta$. We introduce the following quantities.

$$\hat{Y}_i(\beta, s, r) = Y_i(\pi(X_i(\beta, s); \beta, r))$$

$$\hat{V}_n(\beta, s, r) := \frac{1}{n} \sum_{i=1}^n \hat{Y}_i(\beta, s, r).$$

By Lemma 13, we recall that $V(\beta, s, r)$ is continuously differentiable in $s$ and $r$.

The model effect estimator $\hat{\tau}_{ME,b,n}^t(\beta, s, r)$ is the regression coefficient obtained by running OLS of $Y$ on $M_\beta$. The regression coefficient must have the following form.

$$\hat{\tau}_{ME,b,n}^t(\beta, s, r) = \left(S_{zz}^n\right)^{-1}S_{zy}^n, \text{ where } S_{zz}^n := \frac{1}{b^2 n}M_\beta^TM_\beta, \quad s_{zy}^n := \frac{1}{b^2 n}M_\beta^TY.$$  (F.5)

In this proof, we establish convergence in probability of the two terms above separately. The bulk of the proof is the first step, which entails showing that

$$S_{zy}^n \Rightarrow \frac{\partial V}{\partial \beta}(\beta, s(\beta), s(\beta)).$$

Due to $Y$’s dependence on the stochastic processes $\{\hat{S}_{b,n}^{t_n-1}\}$ and $\{\hat{S}_{b,n}^{t_n}\}$, the main workhorse of this result is Lemma 28. To apply this lemma, we must establish stochastic equicontinuity for the collection of stochastic processes $\{\hat{V}_n(\beta, s, r)\}$. Second, through a straightforward application of the Weak Law of Large Numbers, we show that

$$S_{zz}^n \Rightarrow I_d.$$
Finally, we use Slutsky’s Theorem to establish the convergence of the model effect estimator.

We proceed with the first step of establishing convergence of $s^n_{zy}$. We have that

$$s^n_{zy} = \frac{1}{b^n n} M^n Y$$

$$= \frac{1}{b^n n} \sum_{i=1}^{n} b_n \zeta_i Y_i$$

$$= \frac{1}{b_n} \cdot \frac{1}{n} \sum_{i=1}^{n} \zeta_i Y_i.$$

We fix $j$ and $b_n = b$ where $b > 0$ and is small enough to satisfy the hypothesis of Lemma 52. For each $\zeta \in \{-1, 1\}^d$ and $\xi \in \{-1, 1\}$, let

$$n_{\zeta, \xi} = \sum_{i=1}^{n} \mathbb{1}(\zeta_i = \zeta, \xi_i = \xi).$$

Let $z(\zeta)$ map a perturbation $\zeta \in \{-1, 1\}^d$ to the identical vector $\zeta$, except with $j$-th entry set to 0. So, if the $j$-th entry of $\zeta$ is 1, then $\zeta = e_j + z(\zeta)$. If the $j$-th entry of $\zeta$ is -1, then $\zeta = -e_j + z(\zeta)$. So, we have that

$$Y_i = \bar{Y}_i(\beta + b\zeta_i, S_{n}^{i-1} + b\xi_i, \hat{S}_{b,n}^{i-1} + b\xi_i)$$

$$= \bar{Y}_i(\beta + b\zeta_{i,j} e_j + b \cdot z(\zeta_i), \hat{S}_{b,n}^{i-1} + b\xi_i, \hat{S}_{b,n}^{i-1} + b\xi_i).$$

As a result, we have that

$$\frac{1}{n} \sum_{i=1}^{n} \zeta_{i,j} Y_i$$

$$= \frac{1}{n} \sum_{i=1}^{n} \zeta_{i,j} \cdot \bar{Y}_i(\beta + b\zeta_{i,j} e_j + b \cdot z(\zeta_i), \hat{S}_{b,n}^{i-1} + b\xi_i, \hat{S}_{b,n}^{i-1} + b\xi_i)$$

$$= \sum_{\zeta \in \{-1, 1\}^d \text{ s.t. } \zeta_j = 1} \sum_{\xi \in \{-1, 1\}} \frac{n_{\zeta, \xi}}{n} \sum_{i=1}^{n} \bar{Y}_i(\beta + b e_j + b \cdot z(\zeta), \hat{S}_{b,n}^{i-1} + b\xi, \hat{S}_{b,n}^{i-1} + b\xi)$$

$$- \sum_{\zeta \in \{-1, 1\}^d \text{ s.t. } \zeta_j = -1} \sum_{\xi \in \{-1, 1\}} \frac{n_{\zeta, \xi}}{n} \sum_{i=1}^{n} \bar{Y}_i(\beta - b e_j + b \cdot z(\zeta), \hat{S}_{b,n}^{i-1} + b\xi, \hat{S}_{b,n}^{i-1} + b\xi)$$

To establish convergence properties of each term in the double sum in (F.6) and (F.7), we must establish stochastic equicontinuity of the collection of stochastic processes $\{\hat{V}_n(\beta, s, r)\}$ indexed by $(s, r) \in \mathcal{S} \times \mathcal{S}$. Because $\mathcal{S} \times \mathcal{S}$ compact and $V(\beta, s, r)$ is continuous in $s$ and $r$, we can show that $\{\hat{V}_n(\beta, s, r)\}$ is stochastically equicontinuous by establishing that $\hat{V}_n(\beta, s, r)$ converges uniformly in probability (with respect to $(s, r)$) to $V(\beta, s, r)$ (Lemma 27).

We show that $\hat{V}_n(\beta, s, r)$ converges uniformly (with respect to $(s, r)$) in probability to $V(\beta, s, r)$. We
realize that

\[
\sup_{(s,r) \times \mathcal{S} \times \mathcal{S}} |\hat{V}_n(\beta, s, r) - V(\beta, s, r)|
\]

(F.10)

\[
= \sup_{(s,r) \times \mathcal{S} \times \mathcal{S}} \left| \frac{1}{n} \sum_{i=1}^{n} \tilde{Y}_i(\beta, s, r) - V(\beta, s, r) \right|
\]

(F.11)

\[
= \sup_{(s,r) \times \mathcal{S} \times \mathcal{S}} \left| \frac{1}{n} \sum_{i=1}^{n} m(Z_i, G_i; \pi(X_i(\beta, s); \beta, r) + \delta_i) - V(\beta, s, r) \right|
\]

(F.12)

\[
\leq \sup_{(s,r) \times \mathcal{S} \times \mathcal{S}} \left| \frac{1}{n} \sum_{i=1}^{n} m(Z_i, G_i; \pi(X_i(\beta, s); \beta, r)) - V(\beta, s, r) \right| + \sup_{(s,r) \times \mathcal{S} \times \mathcal{S}} \left| \frac{1}{n} \sum_{i=1}^{n} \delta_i \right|.
\]

(F.13)

(F.12) follows from Assumption 5. By Lemma 48, we have that \(m\) satisfies the conditions of Lemma 26. So, we have that

\[
\sup_{(s,r) \in \mathcal{S} \times \mathcal{S}} \left| \frac{1}{n} \sum_{i=1}^{n} m(Z_i, G_i; \pi(X_i(\beta, s); \beta, r)) - \mathbb{E}_F[m(Z_i, G_i; \pi(X_i(\beta, s); \beta, r))] \right|
\]

\[
= \sup_{(s,r) \in \mathcal{S} \times \mathcal{S}} \left| \frac{1}{n} \sum_{i=1}^{n} m(Z_i, G_i; \pi(X_i(\beta, s); \beta, r)) - V(\beta, s, r) \right| \xrightarrow{p} 0.
\]

Furthermore, under Assumption 5, \(\delta_i\) is mean-zero, so by the Weak Law of Large Numbers, \(\frac{1}{n} \sum_{i=1}^{n} \delta_i \xrightarrow{p} 0\). Because \(\delta_i\) does not depend on \((s, r)\), this convergence also holds uniformly in \((s, r)\). Combining these results yields that \(\hat{V}_n(\beta, s, r)\) converges uniformly (with respect to \((s, r)\)) in probability to \(V(\beta, s, r)\).

As a consequence, the collection of stochastic processes \(\{\hat{V}_n(\beta, s, r)\}\) is stochastically equicontinuous. In particular, \(\hat{V}_n(\beta, s, r)\) is stochastically equicontinuous at \((s(\beta, b), s(\beta, b))\), where \(s(\beta, b)\) is the unique fixed point of \(q(P(\beta, s, b))\) (see Lemma 51). By Lemma 52, we have that

\[
\hat{S}_{b,n}^{t_{1,n}-1} \xrightarrow{p} s(\beta, b)
\]

\[
\hat{S}_{b,n}^{t_n} \xrightarrow{p} s(\beta, b).
\]

Now, we can apply Lemma 28 to establish convergence in probability for each term in the double sum of (F.8), (F.9). As an example, for a perturbation \(\zeta \in \{-1, 1\}^d\) with \(j\)-th entry equal to 1 and arbitrary \(\xi \in \{-1, 1\}\), Lemma 28 gives that

\[
\hat{V}_{\alpha_{\xi,\zeta}}(\beta + b e_j + b \cdot z(\zeta), \hat{S}_{b,n}^{t_{1,n}-1} + b \xi, \hat{S}_{b,n}^{t_n} + b \xi) \xrightarrow{p} \hat{V}_{\alpha_{\xi,\zeta}}(\beta + b e_j + b \cdot z(\zeta), s(\beta, b) + b \xi, s(\beta, b) + b \xi),
\]

and by the Weak Law of Large Numbers, we have that

\[
\hat{V}_{\alpha_{\xi,\zeta}}(\beta + b e_j + b \cdot z(\zeta), s(\beta, b) + b \xi, s(\beta, b) + b \xi) \xrightarrow{p} V(\beta + b e_j + b \cdot z(\zeta), s(\beta, b) + b \xi, s(\beta, b) + b \xi).
\]

Analogous statements hold for the remaining terms in (F.8) and (F.9). Also,

\[
\frac{n_{\xi,\zeta}}{n} \xrightarrow{p} \frac{1}{2^{d+1}}, \quad \zeta \in \{-1, 1\}^d, \xi \in \{-1, 1\}.
\]

By Slutsky’s Theorem, when any \(j\) and \(b\) fixed, we have

\[
S_{z_{2y,j}}^{t_n} \xrightarrow{p} \sum_{\zeta \in \{-1, 1\}^d \text{ s.t. } \zeta_j = 1} V(\beta + b e_j + b \cdot z(\zeta), s(\beta, b) + b \xi, s(\beta, b) + b \xi)
\]

\[
- \sum_{\zeta \in \{-1, 1\}^d \text{ s.t. } \zeta_j = -1} V(\beta - b e_j + b \cdot z(\zeta), s(\beta, b) + b \xi, s(\beta, b) + b \xi),
\]

(F.14)

(F.15)
Let \( R_b \) denote the expression on the right side of the above equation. If there is a sequence \( \{ b_n \} \) such that \( b_n \to 0 \), then by Lemma 51, \( s(\beta, b_n) \to s(\beta) \), where \( s(\beta) \) is the unique fixed point of \( q(P(\beta, s)) \). Furthermore, by the continuity of \( V \), we have that

\[
R_{b_n} \to \frac{\partial V}{\partial \beta_j}(\beta, s(\beta), s(\beta)).
\]

Using the definition of convergence in probability, we show that there exists such a sequence \( \{ b_n \} \). From (F.14) and (F.15), we have that for each \( \epsilon, \delta > 0 \) and \( b > 0 \) and sufficiently small, there exists \( n(\epsilon, \delta, b) \) such that for \( n \geq n(\epsilon, \delta, b) \)

\[
P(\{ s_{z,y,j}^n - R_b \mid \leq \epsilon \} \geq 1 - \delta).
\]

So, we can fix \( \delta > 0 \). For \( k = 1, 2, \ldots \), let \( N(k) = n(\frac{1}{k}, \delta, \frac{1}{k}) \). Then, we can define a sequence such that \( b_n = \epsilon_n = \frac{1}{k} \) for all \( N(k) \leq n \leq N(k + 1) \). So, we have that \( \epsilon_n \to 0 \) and \( b_n \to 0 \). Finally, this gives that

\[
s_{z,y,j}^n \xrightarrow{P} \frac{\partial V}{\partial \beta_j}(\beta, s(\beta), s(\beta)).
\]

Considering all indices \( j \),

\[
s_{z,y}^n \xrightarrow{P} \frac{\partial V}{\partial \beta}(\beta, s(\beta), s(\beta)).
\]

It remains to establish convergence in probability for \( S_{zz}^n \). We have that

\[
S_{zz}^n = \frac{1}{b_n^2} M_{zz}^T M_{\beta}
\]

\[
= \frac{1}{b_n^2} \sum_{i=1}^{n} (b_n \zeta_i)^T (b_n \zeta_i).
\]

\[
= \frac{1}{n} \sum_{i=1}^{n} \zeta_i^T \zeta_i.
\]

We note that

\[
E_{\zeta_i \sim \mathcal{R}^d} [\zeta_i \cdot \zeta_i] = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}
\]

because \( \zeta_i \) is a vector of independent Rademacher random variables. So, \( E[\zeta_i^T \zeta_i] = I_d \). By the Weak Law of Large Numbers, we have that

\[
S_{zz}^n \xrightarrow{P} I_d.
\]

Finally, we can use Slutsky’s Theorem to show that

\[
\hat{r}_{ME,b_n,n}^t(\beta) = (S_{zz}^n)^{-1} s_{z,y}^n \xrightarrow{P} (I_d)^{-1} \frac{\partial V}{\partial \beta}(\beta, s(\beta), s(\beta)) = \frac{\partial V}{\partial \beta}(\beta, s(\beta), s(\beta)) = r_{ME}(\beta).
\]

### F.3 Proof of Theorem 15

Let \( s(\beta) = s(\beta) \). Let \( \hat{r}_{b,n}^{Y,s,r}, \hat{r}_{b,n}^{\beta,s}, \hat{r}_{b,n}^{\beta,t} \) be the regression coefficients defined in Section 5.2. Let \( p^n(\beta, s, b)(s) \) be the density estimate defined in Section 5.2. In this proof, we rely on the results on the following convergence results for these estimators.

**Corollary 53.** Let \( \{ t_n \} \) be a sequence such that \( t_n \uparrow \infty \) as \( n \to \infty \). Let \( M_{s,Y}, \hat{r}_{b,n}^{Y,s,r} \) be as defined in Section 5.2. Under the conditions of Theorem 14, there exists a sequence \( \{ b_n \} \) such that \( b_n \to 0 \) so that

\[
\hat{r}_{b,n}^{Y,s,r}(\beta, \hat{r}_{b,n}^{t,n}, \hat{r}_{b,n}^{s,n}) \xrightarrow{P} \frac{\partial V}{\partial \beta}(\beta, s(\beta), s(\beta)) + \frac{\partial V}{\partial r}(\beta, s(\beta), s(\beta)).
\]

*Proof in Appendix G.22.*
Lemma 54. Let \( \{t_n\} \) be a sequence such that \( t_n \uparrow \infty \) as \( n \to \infty \). Let \( M_{\alpha}, I, \hat{\Gamma}_{b,n}^{\beta,s,r}(\beta,s,r) \) be as defined Section 5.2. Under the conditions of Theorem 14, there exists a sequence \( \{b_n\} \) such that \( b_n \to 0 \) so that

\[
\hat{\Gamma}_{b,n}^{\beta,s} (\beta, \hat{S}_{b,n}^t, \hat{S}_{b,n}^n) \xrightarrow{p} \frac{\partial \Pi}{\partial \beta}(\beta, s(\beta); s(\beta)).
\]  

(F.17)

Proof in Appendix G.29.

Corollary 55. Let \( \{t_n\} \) be a sequence such that \( t_n \uparrow \infty \) as \( n \to \infty \). Let \( M_{\alpha}, I, \hat{\Gamma}_{b,n}^{\beta,s,r}(\beta,s,r) \) be as defined in Section 5.2. Under the conditions of Theorem 14, there exists a sequence \( \{b_n\} \) such that \( b_n \to 0 \) so that

\[
\hat{\Gamma}_{b,n}^{\beta,s} (\beta, \hat{S}_{b,n}^t, \hat{S}_{b,n}^n) \xrightarrow{p} \frac{\partial \Pi}{\partial s}(\beta, s(\beta); s(\beta)).
\]  

(F.18)

Proof in Appendix G.24.

Lemma 56. Fix \( \beta \in \mathcal{B} \). Let \( \{h_n\} \) be a sequence such that \( h_n \to 0 \) and \( nh_n \to \infty \). Let \( p_n(\beta,s,b)(r) \) denote a kernel density estimate of \( p(\beta,s,b)(r) \) with kernel function \( k(z) = \mathbb{I}(z \in [-\frac{1}{2}, \frac{1}{2}]) \) and bandwidth \( h_n \). Let \( \{t_n\} \) be a sequence such that \( t_n \uparrow \infty \) as \( n \to \infty \). Under the conditions of Theorem 14, there exists a sequence \( \{b_n\} \) such that \( b_n \to 0 \) so that

\[
p_n^{b_n,S_{b,n}^t,b_n}(\hat{S}_{b,n}^n) \xrightarrow{p} p(\beta,s(\beta))(s(\beta)),
\]  

(F.19)

where \( s(\beta) \) is the unique fixed point of \( q(P(\beta,s)) \). Proof in Appendix G.25.

Finally, we use the following lemma to show that we recover the equilibrium effect.

Lemma 57. Let \( \Pi(\beta,s;r) \) be defined as in (5.3). Under Assumptions 1, 2, and 3, if \( \sigma^2 > \frac{1}{\alpha \sqrt{2 \pi \epsilon}} \) then

\[
\frac{\partial s}{\partial \beta} = \frac{1}{p(\beta,s(\beta))(s(\beta))} \cdot \frac{\partial \Pi}{\partial \beta}(\beta, s(\beta); s(\beta)),
\]  

(F.20)

where \( s(\beta) = s(\beta) \), the unique fixed point induced by the model \( \beta \). Proof in Appendix G.26.

We proceed with the main proof. The equilibrium effect estimator in (5.6) consists of two terms. We see that the convergence of the first term is immediately given by (F.16) above. It remains to show that the second term converges in probability to \( \frac{\partial s}{\partial \beta}(\beta) \). We have that

\[
\frac{\hat{\Gamma}_{b,n}^{\beta,s} (\beta, \hat{S}_{b,n}^t, \hat{S}_{b,n}^n)}{p_n^{b_n,S_{b,n}^t,b_n}(\hat{S}_{b,n}^n) - \hat{\Gamma}_{b,n}^{\beta,s} (\beta, S_{b,n}^t, S_{b,n}^n)} \xrightarrow{p} \frac{1}{p(\beta,s(\beta))(s(\beta))} \cdot \frac{\partial \Pi}{\partial \beta}(\beta, s(\beta); s(\beta))
\]  

(F.21)

\[
\frac{\partial s}{\partial \beta}(\beta).
\]  

(F.22)

(F.21) follows by Slutsky’s Theorem given (F.18), (F.17), and (F.19). (F.22) follows from Lemma (57). Combining (F.16) and (F.22) using Slutsky’s Theorem, yields

\[
\hat{v}_{EE,n}(\beta) = \hat{v}_{Y,s,r}(\beta, S_{b,n}^t, S_{b,n}^n) \cdot \hat{\Gamma}_{b,n}^{\beta,s} (\beta, \hat{S}_{b,n}^t, \hat{S}_{b,n}^n) \frac{p_n^{\beta,s,b_n}(S_{b,n}^n) - \hat{\Gamma}_{b,n}^{\beta,s,b_n}(\beta, S_{b,n}^t, S_{b,n}^n)}{p_n^{\beta,b_n}(S_{b,n}^n) - \hat{\Gamma}_{b,n}^{\beta,b_n}(\beta, S_{b,n}^t, S_{b,n}^n)} \xrightarrow{p} \left( \frac{\partial V}{\partial s}(\beta, s(\beta)) + \frac{\partial V}{\partial r}(\beta, s(\beta), s(\beta)) \right) \cdot \frac{\partial s}{\partial \beta}(\beta)
\]  

\[
= \tau_{EE}(\beta).
\]

F.4 Proof of Corollary 16

This result follows from applying Slutsky’s Theorem to the results of Theorem 14 and Theorem 15.
G Proofs of Technical Results

G.1 Proof of Lemma 31
Since $f_n \to f$ uniformly and $f_n$‘s are defined on a compact domain, then $f$ must be continuous. By assumption, $f$ has only one zero $x_i^*$ in $[a, b]$. We can choose
\[
\epsilon = \inf \{|f(x)| : |x - x_i^*| > \delta\}.
\]
By uniform convergence, there exists $N$ such that for $n \geq N$, $\sup_{x \in X} |f_n(x) - f(x)| < \frac{\epsilon}{2}$. By the triangle inequality we have that
\[
|f(x)| = |f(x) - f_n(x) + f_n(x)| \\
\leq |f(x) - f_n(x)| + |f_n(x)| \\
\leq |f_n(x)| + |f(x) - f_n(x)|.
\]

For $n \geq N$ and $x$ such that $|x - x_i^*| > \delta$, we realize that $|f_n(x)| > \frac{\epsilon}{2}$. So $x$ cannot be a fixed point of $f_n$. Thus, if $x$ is a fixed point of $f_n$, then we have that $|x_n - x_i^*| < \delta$. This implies that $x_n \to x_i^*$.

G.2 Proof of Lemma 32
By uniform convergence, we have that for any $\epsilon > 0$, for every $x \in X$, there is $n \geq N$ so that
\[
f(x) - \epsilon < f_n(x) < f(x) + \epsilon.
\]
So, for all $x \in X$, we have that $f_n(x) < f(x) + \epsilon < x_i^* + \epsilon$. In addition, for all $x \in X$, we have that $f(x) - \epsilon < x_n$. We realize that this implies that $x_n \leq x + \epsilon$ and $x_i^* - \epsilon \leq x_n$. Thus, we have that $|x_n - x| < \epsilon$.

So, we have that $x_n \to x_i^*$.

G.3 Proof of Lemma 34
From Assumption 1, we have that $c_i$ is twice continuously differentiable. Since $\Phi_\sigma$ is the Normal CDF, we have that it is twice continuously differentiable. Since the composition and sum of twice continuously differentiable functions is also twice continuously differentiable, we have that $E_e[U_i(x; \beta, s)]$ is twice continuously differentiable in $x, \beta, s$.

G.4 Proof of Lemma 35
We abbreviate
\[
E_e[U_i(x)] := E_e[U_i(x; \beta, s)].
\]
From Lemma 34, we have that $E_e[U_i(x)]$ is twice continuously differentiable in $x$, so
\[
\nabla_x E_e[U_i(x)] = -\nabla c_i(x - Z_i; G_i) + \phi_\sigma(s - \beta^T x) \beta^T.
\]
\[
\nabla_x^2 E_e[U_i(x)] = -\nabla^2 c_i(x - Z_i; G_i) - \beta \phi'_\sigma(s - \beta^T x) \beta^T.
\]

To show that $\nabla^2_x E_e[U_i(x)]$ is negative definite at any point $(x, s)$, we can show that $-\nabla^2_x E_e[U_i(x)]$ is positive definite at any point $(x, s)$. Let $z \in \mathbb{S}^{d-1}$,
\[
z^T(-\nabla^2_x E_e[U_i(x)])z = z^T[\nabla^2 c_i(x - Z_i; G_i) + \beta \phi'_\sigma(s - \beta^T x) \beta^T]z
\]
\[
= z^T \nabla^2 c_i(x - Z_i; G_i) z + \phi'_\sigma(s - \beta^T x) \cdot z^T \beta \beta^T z
\]
\[
\geq \inf_{y \in \mathbb{R}^d} z^T \nabla^2 c_i(y) z + \inf_{y \in \mathbb{R}} \phi'_\sigma(y) \cdot z^T \beta \beta^T z
\]
\[
\geq \alpha_i + \left(-\frac{1}{2\sigma^2 \sqrt{2\pi} \epsilon}\right) \cdot z^T \beta \beta^T z
\]
\[
\geq \alpha_i - \alpha_i \cdot z^T \beta \beta^T z
\]
\[
> 0
\]
We check the above inequality as follows. By Assumption 1, $c_i$ is $\alpha_i$-strongly convex and twice differentiable. So, we can lower bound the first term in (G.2). (G.4) holds because $-\frac{1}{\sigma^2 \sqrt{2\pi \sigma}} \leq \phi'_\sigma(y) \leq \frac{1}{\sigma^2 \sqrt{2\pi \sigma}}$. The assumption that $\sigma^2 > \frac{1}{\alpha \sqrt{2\pi \alpha}}$ yields (G.5). Finally, $z^T \beta T z = (z^T \beta)^2$, and the dot product of two unit vectors, $\beta$ and $z$, must be between -1 and 1, so $0 \leq z^T \beta T z \leq 1$. Thus, $-\nabla_z^2 \mathbb{E}_\epsilon[U_i(x)]$ is positive definite.

G.5 Proof of Lemma 36

Without loss of generality, we fix $\beta, s$. So, we abbreviate

$$\mathbb{E}_\epsilon[U_i(x)] := \mathbb{E}_\epsilon[U_i(x; \beta, s)].$$

We apply Lemma 18 to $\mathbb{E}_\epsilon[U_i(x)]$ to establish the claim. The conditions of Lemma 18 include twice-differentiability in $x$ and strict concavity in $x$ of $\mathbb{E}_\epsilon[U_i(x)]$. Twice-differentiability follows from Lemma 34 and strict concavity follows from Lemma 35, which establishes that $\nabla^2 \mathbb{E}_\epsilon[U_i(x)]$ is negative definite everywhere. Since $\mathbb{E}_\epsilon[U_i(x)]$ satisfies the conditions of Lemma 18, we have that if $x \in \text{Int}(\mathcal{X})$, then $x$ is the unique global maximizer of $\mathbb{E}_\epsilon[U_i(x)]$ on $\mathcal{X}$ if and only if $\nabla_x \mathbb{E}_\epsilon[U_i(x)] = 0$. Under these conditions, if $x \in \text{Int}(\mathcal{X})$, then $x = x^*_i(\beta, s)$ if and only if $\nabla_x \mathbb{E}_\epsilon[U_i(x)] = 0$, as desired.

G.6 Proof of Lemma 37

Without loss of generality, we fix $\beta$. We use the following abbreviations

$$\mathbb{E}_\epsilon[U_i(x; s)] := \mathbb{E}_\epsilon[U_i(x; \beta, s)],$$

$$x^*_i(s) := x^*_i(\beta, s),$$

$$h_i(s) := h_i(s; \beta).$$

We state an additional lemma that will be used in the proof of Lemma 37.

Lemma 58. Let $x \in \mathcal{X}, \beta \in \mathcal{B}, s \in \mathbb{R}$. For an agent $i$ with unobservables $(Z_i, G_i) \sim F_{Z_iG_i}$, define $H = \nabla^2 c_i(x - Z_i; G_i)$. Under Assumption 1, we have that

$$(H + \phi'_\sigma(s - \beta^T x)\beta \beta^T)^{-1} = H^{-1} - \frac{\phi'_\sigma(s - \beta^T x)\beta \beta^T H^{-1}}{1 + \phi'_\sigma(s - \beta^T x)\beta \beta^T H^{-1}\beta}.$$

Proof in Appendix G.27.

Now, we proceed with the main proof. We compute $\nabla_s x^*_i$ by using the implicit expression for $x^*_i(s)$ given by the first-order condition in Lemma 36. From Lemma 36, we note that $x^*_i(s)$ must satisfy $\nabla_x \mathbb{E}_\epsilon[U_i(x; s)] = 0$. We have that

$$\nabla_x \mathbb{E}_\epsilon[U_i(x; s)] = -\nabla c_i(x - Z_i; G_i) + \phi_\sigma(s - \beta^T x)\beta^T.$$

So, the best response $x^*_i(s)$ satisfies

$$-\nabla c_i(x^*_i(s) - Z_i; G_i) + \phi_\sigma(s - \beta^T x^*_i(s))\beta^T = 0.$$

From Lemma 2, we have that the best response $x^*_i(s)$ is continuously differentiable in $s$, so we can differentiate the above equation with respect to $s$. This yields the following equation

$$(\nabla^2 c_i(x^*_i(s) - Z_i) + \phi'_\sigma(s - \beta^T x^*_i(s))\beta \beta^T)\nabla_s x^*_i = \phi'_\sigma(s - \beta^T x^*_i(s))\beta.$$

Let $H = \nabla^2 c_i(x^*_i(s) - Z_i)$. The above equation can be rewritten as

$$(H + \phi'_\sigma(s - \beta^T x^*_i(s))\beta \beta^T)\nabla_s x^*_i = \phi'_\sigma(s - \beta^T x^*_i(s))\beta.$$

From Lemma 58, we realize that the matrix term on the left side of the equation is invertible. We multiply both sides of the equation by the inverse of the matrix to compute $\nabla_s x^*_i$. 45
\[ \nabla_s x_i^* = (H + \phi'_\sigma(s - \beta^T x_i^*(s))\beta\beta^T)^{-1}\phi'_\sigma(s - \beta^T x_i^*(s))\beta. \]

We can substitute the expression for \((H + \phi'_\sigma(s - \beta^T x_i^*(s))\beta\beta^T)^{-1}\) from Lemma 58 into the above equation.

\[
\nabla_s x_i^* = \left( H^{-1} - \frac{\phi'_\sigma(s - \beta^T x_i^*(s))H^{-1}\beta\beta^TH^{-1}}{1 + \phi'_\sigma(s - \beta^T x_i^*(s))\beta\beta^TH^{-1}\beta} \right) \phi'_\sigma(s - \beta^T x_i^*(s))\beta.
\]

This gives us that
\[
\beta^T \nabla_s x_i^* = \beta^T \left( H^{-1} - \frac{\phi'_\sigma(s - \beta^T x_i^*(s))H^{-1}\beta\beta^TH^{-1}}{1 + \phi'_\sigma(s - \beta^T x_i^*(s))\beta\beta^TH^{-1}\beta} \right) \phi'_\sigma(s - \beta^T x_i^*(s))\beta
\]
\[
= \phi'_\sigma(s - \beta^T x_i^*(s))\beta^T H^{-1}\beta - \left( \frac{\phi'_\sigma(s - \beta^T x_i^*(s))\beta\beta^TH^{-1}\beta}{1 + \phi'_\sigma(s - \beta^T x_i^*(s))\beta\beta^TH^{-1}\beta} \right).
\]

as desired.

**G.7 Proof of Lemma 38**

By Assumption 1, \(H = \nabla^2 c_i(y)\) is positive definite and
\[
H \succeq \alpha_i I.
\]

Since \(H\) is positive definite, it is invertible and its inverse \(H^{-1}\) is also positive definite. Since \(H\) and \(\alpha_i I\) are positive definite and \(H - \alpha_i I\) is positive semidefinite, Lemma 21 gives us that \((\alpha_i I)^{-1} = H^{-1}\) is positive semidefinite. As a result,
\[
\frac{1}{\alpha_i} I \succeq H^{-1}.
\]

We conclude that (C.2) holds because
\[
\sup_{\beta' \in B} \beta'^T H^{-1} \beta' \leq \frac{1}{\alpha_i} \sup_{\beta' \in B} \beta'^T \beta' \leq \frac{1}{\alpha_i}.
\]

The last line follows because \(B = S^{d-1}\).

**G.8 Proof of Lemma 39**

Without loss of generality, we fix \(\beta\). We use the following abbreviations
\[
x_i^*(s) := x_i^*(\beta, s)
\]
\[
h_i(s) := h_i(s; \beta).
\]

First, we establish that \(h_i\) is differentiable and compute \(\frac{dh_i}{ds}\) as follows
\[
\frac{dh_i}{ds} = 1 - \beta^T \nabla_s x_i^*.
\]  

Second, we use the expression for \(\beta^T \nabla_s x_i^*\) from Lemma 37 to show that under our conditions, \(\beta^T \nabla_s x_i^* < 1\). Finally, using this fact along with (G.9), we conclude that \(h_i\) has a positive derivative, so it must be strictly increasing.

Now, we proceed with the main proof. First, we observe that \(h_i\) is differentiable in \(s\) because Lemma 2 gives that \(x_i^*(s)\) is continuously differentiable in \(s\). Differentiating with respect to \(s\) yields (G.9).
Next, we use (C.1) to upper bound $\beta^T \nabla_s x_i^*$. We show that the second term on the right side of (C.1) is nonnegative. Let $N$ and $D$ be the numerator and denominator of the term, respectively. In particular,

$$
N := (\phi'_{\sigma}(s - \beta^T x_i^*(s))\beta^T H^{-1}\beta)^2 \\
D := 1 + \phi'_{\sigma}(s - \beta^T x_i^*(s))\beta^T H^{-1}\beta
$$

Clearly, we must have $N \geq 0$ because it consists of a squared term. We show that $D > 0$, as well.

$$
D = 1 + \phi'_{\sigma}(s - \beta^T x_i^*(s))\beta^T H^{-1}\beta \\
\geq 1 + \inf_{y \in \mathbb{R}} \phi'_{\sigma}(y) \sup_{\beta' \in \mathcal{B}} \beta'^T H^{-1}\beta' \\
\geq 1 + \left( -\frac{1}{\sigma^2 \sqrt{2\pi e}} \right) \cdot \frac{1}{\alpha_i} \\
> 1 + (-\alpha_i) \cdot \frac{1}{\alpha_i} \\
> 0
$$

(G.11) follows from the observation that $H^{-1}$ is positive definite (Lemma 38), so $\beta^T H^{-1}\beta > 0$, while $\phi'_{\sigma}(y)$ may take negative values. In (G.12), we apply Lemma 38 and we note that $-\frac{1}{\sigma^2 \sqrt{2\pi e}} \leq \phi'_{\sigma}(y) \leq \frac{1}{\sigma^2 \sqrt{2\pi e}}$. In (G.13), we use the condition that $\sigma^2 > \frac{1}{\alpha_i \sqrt{2\pi e}}$.

Since $D > 0$ and $N \geq 0$, we have that the second term on the right side of (C.1) is nonnegative. As a result, we can upper bound $\beta^T \nabla_s x_i^*$ as follows

$$
\beta^T \nabla_s x_i^* = \phi'_{\sigma}(s - \beta^T x_i^*(s))\beta^T H^{-1}\beta - \frac{(\phi'_{\sigma}(s - \beta^T x_i^*(s))\beta^T H^{-1}\beta)^2}{1 + \phi'_{\sigma}(s - \beta^T x_i^*(s))\beta^T H^{-1}\beta} \\
\leq \phi'_{\sigma}(s - \beta^T x_i^*(s))\beta^T H^{-1}\beta \\
\leq \sup_{\beta' \in \mathcal{B}} \beta'^T H^{-1}\beta' \cdot \sup_{y} \phi'_{\sigma}(y) \\
\leq \sup_{\beta' \in \mathcal{B}} \beta'^T H^{-1}\beta' \cdot \frac{1}{\sigma^2 \sqrt{2\pi e}} \\
< \frac{1}{\alpha_i} \cdot \alpha_i \\
< 1.
$$

We can apply $\beta^T \nabla_s x_i^* < 1$ to (G.9) to find that

$$
\frac{dh_i}{ds} = 1 - \beta \nabla_s x_i^* > 0.
$$

Thus, $h$ has a positive derivative, so it must be strictly increasing.

**G.9 Proof of Lemma 40**

Define $\bar{u}_i(x; \beta) = \lim_{s \to \infty} E_\epsilon [U_i(x; \beta, s)]$. Note that

$$
\bar{u}_i(x; \beta) = -c_i(x - Z_i; G_i) + 1.
$$

We realize that

$$\arg\max_{x \in \mathcal{N}} \bar{u}_i(x; \beta) = Z_i.$$
To show that (C.3) holds, we establish that $\mathbb{E}_x [U_i(x; \beta, s)] \to \bar{u}(x; \beta)$ uniformly in $x$ for $x \in \mathcal{X}$ as $s \to \infty$. Then, we show that the maximizer of $\mathbb{E}_x [U_i(x; \beta, s)]$ must converge to the maximizer of $\bar{u}_i(x; \beta)$ as $s \to \infty$, which gives the desired result.

First, we verify the conditions of Lemma 30 to establish the uniform convergence of $\mathbb{E}_x [U_i(x; \beta, s)]$. We note that $\mathcal{X}$ is compact. In addition, for every $s$, we have that the expected utility is continuous (Lemma 34). In addition, for every $s$, Lemma 35 gives that the expected utility’s second derivative is negative definite, which implies that it is strictly concave. Also, $\bar{u}(x; \beta)$ is continuous and concave. We note that $\mathbb{E}_x [U_i(x; \beta, s)] \to \bar{u}(x; \beta)$ pointwise in $x$ as $s \to \infty$. Thus, Lemma 30 implies that $\mathbb{E}_x [U_i(x; \beta, s)] \to \bar{u}(x; \beta)$ converges uniformly in $x$ for $x \in \mathcal{X}$ as $s \to \infty$.

Second, we verify the conditions of Lemma 32 to show that

$$\lim_{s \to \infty} x_i^*(\beta, s) \to Z_i.$$  \hfill (G.15)

We note that $\mathbb{E}_x [U_i(x; \beta, s)]$ has a unique maximizer $x_i^*(\beta, s)$ for every $s$ (Lemma 1), and $\bar{u}_i(x; \beta)$ is uniquely maximized at $Z_i$. As shown in the previous part, $\mathbb{E}_x [U_i(x; \beta, s)]$ converges uniformly in $x$ as $s \to \infty$. So, we can apply Lemma 32 to conclude that (G.15). This implies (C.3). An identical argument implies (C.4).

**G.10 Proof of Lemma 41**

We use the following abbreviations

$$\omega_i(s) := \omega_i(s; \beta)$$
$$x_i^*(s) := x_i^*(\beta, s)$$
$$h_i(s) := h_i(s; \beta).$$

We will use the following lemma in this proof.

**Lemma 59.** Consider $\omega_i(s; \beta) = \beta^T x_i^*(\beta, s)$. Under Assumption 1, if $\sigma^2 > \frac{1}{\alpha_i \sqrt{2\pi e}}$ and $x_i^*(\beta, s) \in \text{Int}(\mathcal{X})$, $\omega_i(s; \beta)$ has a unique fixed point. \textit{Proof in Appendix G.28.}

By Lemma 59, $\omega_i(s)$ has a unique fixed point. Let the unique fixed point be $s(\beta)$. We show that $\nabla_s \omega_i(s(\beta)) = 0$ by an application of Lemma 37. Let $H = \nabla^2 c(x_i^*(s(\beta)) - Z_i; G_i)$.

$$\nabla_s \omega_i(s(\beta)) = \beta^T \nabla_s x_i^*(s(\beta))$$
$$= \phi'_\sigma(s(\beta)) - \beta^T x_i^*(s(\beta)) \beta^T H^{-1} \beta - \left( \frac{(\phi'_\sigma(s(\beta)) - \beta^T x_i^*(s(\beta))) \beta^T H^{-1} \beta}{1 + \phi'_\sigma(s(\beta)) - \beta^T x_i^*(s(\beta))) \beta^T H^{-1} \beta} \right)$$
$$= \phi'_\sigma(0) \cdot (\beta^T H^{-1} \beta) - \left( \frac{(\phi'_\sigma(0) \beta^T H^{-1} \beta)^2}{1 + (\phi'_\sigma(0) \beta^T H^{-1} \beta)} \right)$$
$$= 0.$$

The last line follows from the fact that $\phi'_\sigma(0) = 0$. Thus, we have that $\nabla_s \omega_i(s(\beta)) = 0$.

When $s < s(\beta)$, we have that $h_i(s; \beta) < h_i(s(\beta); \beta)$ because by Lemma 39, $h_i(s; \beta)$ is strictly increasing in $s$. Since $h_i(s(\beta); \beta) = 0$, this implies that $h_i(s; \beta) < 0$. Thus, we have that

$$\nabla_s \omega_i(s) = \beta^T \nabla_s x_i^*(s)$$
$$= \phi'_\sigma(h_i(s)) \beta^T H^{-1} \beta - \left( \frac{(\phi'_\sigma(h_i(s)) \beta^T H^{-1} \beta)^2}{1 + \phi'_\sigma(h_i(s)) \beta^T H^{-1} \beta} \right)$$
$$= \frac{\phi'_\sigma(h_i(s)) \beta^T H^{-1} \beta}{1 + \phi'_\sigma(h_i(s)) \beta^T H^{-1} \beta} > 0.$$

The last line follows because $\phi'_\sigma(h_i(s)) > 0$ when $h_i(s) < 0$. Thus, $\omega_i(s)$ is increasing when $s < s(\beta)$.}

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When \( s > s(\beta) \), we have that \( h_i(s(\beta); \beta) < h_i(s; \beta) \), again because \( h_i \) is strictly increasing. This implies that \( h_i(s; \beta) > 0 \). So, when \( s > s(\beta) \), \( \phi'_s(h_i(s)) < 0 \). Meanwhile,

\[
1 + \phi'_s(h_i(s))\beta^TH^{-1}\beta \geq 1 + \inf_{y \in \mathbb{R}} \phi'_s(y) \cdot \sup_{\beta^T \in B} \beta^TH\beta' \\
\geq 1 + \left( -\frac{1}{\sigma^2\sqrt{2\pi}e} \right) \cdot \frac{1}{\alpha_i} \\
\geq 1 + (-\alpha_i) \cdot \frac{1}{\alpha_i} \\
> 0.
\]

This means that for \( s > s(\beta) \), we have that \( \nabla_s \omega_i(s) < 0 \). So, \( \omega_i(s) \) is decreasing when \( s > s(\beta) \).

Since \( \omega_i(s) \) is increasing when \( s < s(\beta) \) and is decreasing when \( s > s(\beta) \), then \( \omega_i(s) \) is maximized when \( s = s(\beta) \).

G.11 Proof of Lemma 42

Recall that \( P(\beta, s) \) denotes the distribution over scores, and the score for an agent \( i \) is denoted by

\[
\beta^TX_i(\beta, s) = \beta^Tx_i^*(\beta, s) + \beta^T\epsilon_i,
\]

where the randomness in the score comes from the unobservables \( (N_i, G_i, \epsilon_i) \sim F \). Note that \( \beta^T\epsilon_i \sim \Phi_\sigma \) because \( \epsilon_i \sim N(0, \sigma^2I_d) \). We have that

\[
P(\beta, s)(r) = P(\beta^Tx_i^*(\beta, s) + \beta^T\epsilon_i \leq r) \\
= \int P(\beta^T\epsilon_i \leq r - \beta^Tx_i^*(\beta, s))dF_{Z,G} \\
= \int \Phi_\sigma(r - \beta^Tx_i^*(\beta, s))dF_{Z,G}.
\]

Thus, \( P(\beta, s)(r) \) has the form given in (D.1). Under our conditions, the best response for each agent type exists and is unique via Lemma 1, so \( P(\beta, s)(r) \) is a well-defined function.

First, we establish that \( P(\beta, s) \) is strictly increasing because we know that \( \Phi_\sigma \) is strictly increasing, and the sum of strictly increasing functions is also strictly increasing.

Second, we establish that \( P(\beta, s) \) is continuously differentiable in \( r \) because \( \Phi_\sigma \) is continuously differentiable. \( P(\beta, s) \) is continuously differentiable in \( \beta \), because \( \Phi_\sigma \) is continuously differentiable and the best response mappings \( x_i^*(\beta, s) \) are continuously differentiable (Lemma 2).

The combination of the above two properties is sufficient for showing that \( P(\beta, s) \) has a continuous inverse distribution function.

G.12 Proof of Lemma 43

First, we compute \( \frac{\partial q(P(\beta, s))}{\partial s} \) via implicit differentiation. We note that our expression for \( \frac{\partial q(P(\beta, s))}{\partial s} \) consists of a convex combination of terms of the form \( \beta^T\nabla_s x_i^*(\beta, s) \). Finally, we can bound each term in the convex combination and bound \( \frac{\partial q(P(\beta, s))}{\partial s} \).

From Lemma 42, we have that \( P(\beta, s) \) has an inverse distribution function, so \( q(P(\beta, s)) \) is uniquely defined. Thus, \( P(\beta, s)(q(P(\beta, s))) = q \) implicitly defines \( q(P(\beta, s)) \). Using the expression for \( P(\beta, s)(r) \) from (D.1), we have

\[
\int \Phi_\sigma(q(P(\beta, s)) - \beta^Tx_i^*(\beta, s))dF = q \tag{G.16}
\]

From Lemma 5, we have that \( q(P(\beta, s)) \) is differentiable in \( s \). So, we can differentiate both sides of (G.16) with respect to \( s \).
\[
\frac{\partial}{\partial s} \int \Phi_\sigma(q(P(\beta, s)) - \beta^T x_i^*(\beta, s))dF_{Z,G} \\
= \int \frac{\partial}{\partial s} \left( \Phi_\sigma(q(P(\beta, s)) - \beta^T x_i^*(\beta, s) \right)dF_{Z,G} \\
= \int \Phi_\sigma(q(P(\beta, s)) - \beta^T x_i^*(\beta, s)) \cdot \left( \frac{\partial q(P(\beta, s))}{\partial s} - \beta^T \nabla_s x_i^*(\beta, s) \right)dF_{Z,G} \\
= 0.
\]

Rearranging the last two lines to solve for \( \frac{\partial q(P(\beta, s))}{\partial s} \) yields

\[
\frac{\partial q(P(\beta, s))}{\partial s} = \int \beta^T \nabla_s x_i^*(\beta, s) \cdot \frac{\Phi_\sigma(q(P(\beta, s)) - \beta^T x_i^*(\beta, s))}{\Phi_\sigma(q(P(\beta, s)) - \beta^T x_i^*(\beta, s))}dF_{Z,G}.
\]

We can define

\[
\lambda_i(\beta, s) = \frac{\Phi_\sigma(q(P(\beta, s)) - \beta^T x_i^*(\beta, s))}{\Phi_\sigma(q(P(\beta, s)) - \beta^T x_i^*(\beta, s))}dF_{Z,G}.
\]

where \( 0 \leq \lambda_i(\beta, s) \leq 1 \) and \( \int \lambda_i(\beta, s)dF_{Z,G} = 1 \). As a result, \( \frac{\partial q(P(\beta, s))}{\partial s} \) is a convex combination of \( \beta^T \nabla_s x_i^*(\beta, s) \) terms:

\[
\frac{\partial q(P(\beta, s))}{\partial s} = \int \beta^T \nabla_s x_i^*(\beta, s) \cdot \lambda_i(\beta, s)dF_{Z,G}.
\]

We can upper bound each term \( \beta^T \nabla_s x_i^*(\beta, s) \). When \( \sigma^2 > \frac{1}{\alpha \sqrt{2\pi\epsilon}} \) Lemma 39 gives us that for any agent type \((Z_i, G_i) \in \text{supp}(F_{Z,G})\), the function \( h_i(s; \beta = s - \beta^T x_i^*(\beta, s) \) is strictly increasing. Since \( h_i(s; \beta) \) is strictly increasing and differentiable, we have that

\[
\frac{dh_i}{ds} = 1 - \beta^T \nabla_s x_i^*(\beta, s) > 0.
\]

As a result, each term satisfies \( \beta^T x_i^*(\beta, s) < 1 \). Since \( \frac{\partial q(P(\beta, s))}{\partial s} \) is a convex combination of such terms, we also have that \( \frac{\partial q(P(\beta, s))}{\partial s} \) < 1. When \( \sigma^2 > \frac{\beta^2}{\alpha \sqrt{2\pi\epsilon}} \) Lemma 3 gives us that for any agent \( i \) with unobservables sampled from \( F \), the function \( \beta^T x_i^*(\beta, s) \) is a contraction in \( s \), so \( |\beta^T \nabla_s x_i^*| < 1 \). As a result, since \( \frac{\partial q(P(\beta, s))}{\partial s} \) is a convex combination of such terms \( \beta^T \nabla_s x_i^* \), then \( |\frac{\partial q(P(\beta, s))}{\partial s}| < 1 \).

\textbf{G.13 Proof of Lemma 44}

The cdf of \( P \) must be

\[
P(r) = \int \Phi_\sigma(r - \beta^T Z_i)dF_Z. \tag{G.17}
\]

We have that

\[
\lim_{s \to \infty} P(\beta, s)(r) = \lim_{s \to \infty} \int \Phi_\sigma(r - \beta^T x_i^*(\beta, s))dF_{Z,G} \\
= \int \Phi_\sigma(r - \beta^T Z_i)dF_{Z,G} \\
= P(r).
\]

The first line follows from the definition of \( P(\beta, s) \) from (D.1). The second line follows from Lemma 40. The third line follows from (G.17). An identical proof can be used to show that

\[
\lim_{s \to -\infty} P(\beta, s)(r) = P(r).
\]

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Since $P(\beta, s)(r) \to P(r)$ pointwise in $r$ as $s \to \infty$, $P(\beta, s)$ and $P$ are continuous and invertible and have continuous inverses (Lemma 42), then we also have that

$$\lim_{s \to \infty} q(P(\beta, s)) = q(P).$$

Similarly, we also have that

$$\lim_{s \to -\infty} q(P(\beta, s)) = q(P).$$

### G.14 Proof of Lemma 45

First, we verify that $z^t$ is a valid random variable. We note that

$$P(z^t = \epsilon_g) + \sum_{k=1}^{\infty} P(z^t = C_k) = p_n(\epsilon_g) + \sum_{k=1}^{\infty} \frac{(1 - p_n(\epsilon_g))}{2^k}$$

$$= p_n(\epsilon_g) + \frac{1}{1 - \frac{1}{2}} \frac{1}{2},$$

$$= p_n(\epsilon_g) + (1 - p_n(\epsilon_g))$$

$$= 1.$$

Now, we show that $z^t$ stochastically dominates $|q(P^n_{\beta,s}) - q(P(\beta, s))|$. So, for $b \in \mathbb{R}$, we show that

$$P(|q(P^n_{\beta,s}) - q(P(\beta, s))| \geq b) \leq P(z^t \geq b), \quad (G.18)$$

which is equivalent to the condition that $z^t$ stochastically dominates $|q(P^n_{\beta,s}) - q(P(\beta, s))|$. From Lemma 10 we realize that for $b \in \mathbb{R}$,

$$P(|q(P^n_{\beta,s}) - q(P(\beta, s))| \geq b) \leq 1 - p_n(b).$$

In addition, we have that

$$P(z^t \geq b) = \begin{cases} 1 & \text{if } b \leq \epsilon_g \\ 1 - p_n(\epsilon_g) & \text{if } \epsilon_g < b \leq C_1 \\ 1 - p_n(\epsilon_g) & \text{if } C_{k-1} < b \leq C_k, k \geq 2. \end{cases}$$

We show that (G.18) holds for the three cases 1) $b \leq \epsilon_g$, 2) $\epsilon_g < b \leq C_1$, and 3) $C_{k-1} < b \leq C_k$ for $k \geq 2$. When $b \leq \epsilon_g$, we have that

$$P(|q(P^n_{\beta,s}) - q(P(\beta, s))| \geq b) \leq 1 - p_n(b) \leq 1 = P(z^t \geq b).$$

When $\epsilon_g < b \leq S_1$, we have that $p_n(b) \geq p_n(\epsilon_g)$ because $p_n(y)$ is increasing in $y$. So, we note that $1 - p_n(b) \leq 1 - p_n(\epsilon_g)$. This yields

$$P(|q(P^n_{\beta,s}) - q(P(\beta, s))| \geq b) \leq 1 - p_n(b) \leq 1 - p_n(\epsilon_g) = P(z^t \geq b).$$

To prove the result in the case where $C_{k-1} < b \leq C_k, k \geq 2$, we first show that the definition of $C_k$ in (E.1) implies that

$$1 - p_n(C_{k-1}) = \frac{1 - p_n(\epsilon_g)}{2^{k-1}}, \quad (G.19)$$

as follows. First, we consider the definition of $C_{k-1}$ below

$$C_{k-1} = \sqrt{\frac{1}{2nM_{\epsilon_g}} \log \left( \frac{2^k}{1 - p_n(\epsilon_g)} \right)},$$

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and we square both sides:

\[ C_{k-1}^2 = \frac{1}{2nM_{\epsilon_g}^2} \cdot \log \left( \frac{2^k}{1 - p_n(\epsilon_g)} \right). \]

Multiplying by \(-2nM_{\epsilon_g}^2\) and exponentiating both sides yields

\[ \exp(-2nM_{\epsilon_g}^2 C_{k-1}^2) = \frac{1 - p_n(\epsilon_g)}{2^k}. \quad \text{(G.20)} \]

Finally, we realize that (G.19) holds because

\[
1 - p_n(C_{k-1}) = 2 \exp(-2nM_{\epsilon_g}^2 C_{k-1}^2) \\
= 2 \cdot \frac{1 - p_n(\epsilon_g)}{2^k} \\
= \frac{1 - p_n(\epsilon_g)}{2^{k-1}},
\]

and the second line follows by (G.20). Using (G.19), we observe that

\[
P(|q(P_{\beta,s}^n) - q(P(\beta,s))| \geq b) = 1 - p_n(b) \leq 1 - p_n(C_{k-1}) \\
= \frac{1 - p_n(\epsilon_g)}{2^{k-1}} \\
= P(z^t \geq C_k) \\
= P(z^t \geq b).
\]

Thus, we conclude that (G.18) holds, yielding the desired result.

### G.15 Proof of Lemma 46

We observe that

\[
|\hat{S}_n^t - s(\beta)| = |q(P_{\beta,s}^n) - q(P_{\beta,s}^n) + q(P_{\beta,s}^n) - s(\beta)| \\
= |q(P_{\beta,s}^n) - q(P_{\beta,s}^n) + q(P_{\beta,s}^n) - s(\beta)| \\
\leq |q(P_{\beta,s}^n) - q(P_{\beta,s}^n)| + |q(P_{\beta,s}^n) - s(\beta)| \\
\leq |q(P_{\beta,s}^n) - q(P_{\beta,s}^n)| + \kappa |\hat{S}_n^t - s(\beta)| \\
\leq \kappa |\hat{S}_n^t - s(\beta)|.
\]

We have (G.22) because \(\hat{S}_n^t\) is generated via (4.2). (G.24) holds because \(q(P(\beta,s))\) is a contraction mapping in \(s\) and \(s(\beta)\) is the unique fixed point of \(q(P(\beta,s))\). (G.25) follows from Lemma 45.

Using recursion, we find that

\[
|\hat{S}_n^t - s(\beta)| \leq SD \sum_{i=1}^{t} z^{t-i} \kappa^i + \kappa^t |\hat{S}_n^0 - s(\beta)| = \sum_{i=1}^{t} z^{t-i} \kappa^i + \kappa^t \cdot C.
\]

### G.16 Proof of Lemma 47

Let \(\{z^t\}_{t \geq 1}\) be a sequence of random variables where

\[
z^t = \begin{cases} 
\epsilon_g & \text{w.p. } p_n(\epsilon_g) \\
C_k & \text{w.p. } \frac{1 - p_n(\epsilon_g)}{2^k},
\end{cases}
\]

where \(\epsilon_g = \frac{\epsilon(1-\kappa)}{2}\) and \(p_n(\epsilon_g)\) is the bound from Lemma 10.
Since $s(\beta) \in \mathcal{S}$, we have that

\[
|\hat{S}_n^t - s(\beta)| \leq |q(P^t_{\beta, S_n^{t-1}}) - s(\beta)| \leq |q(P^t_{\beta, S_n^{t-1}}) - q(P_{\beta, S_n^{t-1}})| + |q(P_{\beta, S_n^{t-1}}) - s(\beta)| \leq_{SD} z^t + \kappa |\hat{S}_n^{t-1} - s(\beta)|.
\]

(G.26)

(G.27)

(G.28)

where (G.26) holds because the truncation is contraction map to the equilibrium threshold $s(\beta)$ and (G.28) holds because $q(P(\beta, s))$ is a contraction in $s$ and $s(\beta)$ is equilibrium threshold. So, as in Lemma 46, we can show that

\[
|\hat{S}_n^t - s(\beta)| \leq_{SD} \sum_{i=1}^{k} z^{t-i} \kappa^i + \kappa^k |\hat{S}_n^{t-k} - s(\beta)|.
\]

Let $C = |\hat{S}_n^t - s(\beta)|$. Thus, an identical argument as the proof of Theorem 11 can be used to show that if

\[
t \geq \left\lfloor \frac{\log(2T)}{\log \kappa} \right\rfloor,
\]

\[n \geq \frac{1}{2M^2} \log \left(\frac{4T}{\delta} \right),\]

we have that

\[P(|\hat{S}_n^t - s(\beta)| \geq \epsilon) \leq \delta.
\]

Using an identical argument as the proof of Corollary 12, it can be shown that for any sequence $\{t_n\}$ such that $t_n \uparrow \infty$ as $n \to \infty$ and $t_n \prec \exp(n)$, $\hat{S}_n \overset{P}{\to} s(\beta)$, as desired.

**G.17 Proof of Lemma 48**

We can use the following abbreviations

\[u := (Z_i, G_i, \epsilon_i)\]

\[\theta := (\beta, s, r)\]

\[\tilde{j}(u, \theta) := \tilde{j}(Z_i, G_i, \epsilon_i, \beta, s, r)\]

\[\tilde{y}(u, \theta) := \tilde{y}(Z_i, G_i, \epsilon_i, \beta, s, r)\]

\[\hat{k}(u, \theta) := \hat{k}(Z_i, G_i, \epsilon_i, \beta, s, r).
\]

The conditions on $a(u; \theta)$ in Lemma 26 include that

1. $\theta \in \Theta$, where $\Theta$ is compact.
2. $a(u; \theta)$ is continuous with probability 1 for each $\theta \in \Theta$.
3. $|a(u; \theta)| \leq d(u)$ and $\mathbb{E}[d(u)] < \infty$.

First, for all of $\tilde{y}, \tilde{j}, \hat{k}$, we have that the parameter space $\mathcal{B} \times \mathcal{S} \times \mathcal{S}$ is compact.

Second, we can verify that for fixed parameters, $\tilde{y}, \tilde{j},$ and $\hat{k}$ are continuous with probability 1. By Assumption 5, we have that for each $w \in \{0, 1\}$, $m(Z_i, G_i; w) + \delta_i$ is continuous w.r.t. to the data, so the only discontinuity of $\tilde{y}$ occurs at the threshold when $w$ flips from 0 to 1. Similarly, $\tilde{j}$ is an indicator function, so its only discontinuity occurs at the threshold. Thus, $\tilde{y}(\cdot; \theta)$ and $\tilde{j}(\cdot; \theta)$ are discontinuous on the set

\[A = \{(Z_i, G_i, \epsilon_i) : \beta^T X_i(\beta, s) = r\}
\]

but are otherwise continuous. The probability that $(Z_i, G_i, \epsilon_i) \sim F_{Z_i, G_i, \epsilon}$ satisfies the condition of set $A$ is equal to the probability that a score $z \sim P(\beta, s)$ takes value exactly $r$. We note that a singleton subset $\{r\}$ will have measure 0, so the probability that a score takes value $r$ is 0. Thus, $A$ must also have measure 0. Since $\tilde{y}$ and $\tilde{j}$ are continuous except on a set of measure 0, $\tilde{y}$ and $\tilde{j}$ are continuous with probability 1. We realize that $\hat{k}$ is continuous except for on the following set

\[A' = \{(Z_i, G_i, \epsilon_i) : \beta^T X_i(\beta, s) = r + \frac{h}{2}\} \cup \{(Z_i, G_i, \epsilon_i) : \beta^T X_i(\beta, s) = r - \frac{h}{2}\}.
\]
The probability that \((Z_i, G_i, \epsilon_i) \sim F_{Z,G,\epsilon}\) satisfies the condition of set \(A'\) is equal to the probability that a score \(z \sim P(\beta, s)\) takes value exactly \(r + \frac{2}{3}\) or value \(r - \frac{2}{3}\). Since the sets \(\{r + \frac{2}{3}\}\) and \(\{r - \frac{2}{3}\}\) have measure zero and a countable union of measure zero sets has measure zero, then \(A'\) has measure zero. Thus, \(k\) is continuous with probability 1.

Third, we note that \(\hat{y}, \hat{j}\), and \(\hat{k}\) are dominated. For \(\hat{y}\), Assumption 5 gives us that \(\hat{y}\) is bounded, so any constant function \(d(u) = c\) for \(c = \sup_{(z, g) \in \text{supp}(F_{Z,G})), w \in (0, 1)} Y_i(w, z, g)\) dominates \(\hat{y}\). Since \(\hat{j}\) and \(\hat{k}\) are indicators, they only take values \(\{0, 1\}\), so any constant function \(d(u) = c\) where \(c > 1\) satisfies the required condition. Thus, \(\hat{y}, \hat{j}\), and \(\hat{k}\) satisfy the conditions of Lemma 26.

**G.18 Proof of Lemma 49**

In this proof, we first verify that the agent with unobservables \((Z_i, G_i)\) has \(Z_i, G_i \in \mathcal{X}\) and has a strongly-convex cost function \(c_i\). Second, we verify the value of the best response for this agent and show that it lies in \(\text{Int}(\mathcal{X})\). Lastly, we show that this agent’s raw score (without noise) matches that of the agent with unobservables \((\hat{Z}_i, \hat{G}_i)\) and cost function \(c_i\) under the perturbed model.

By the following lemma, we realize that with \(Z_i, G_i\) as defined in (F.2), \(Z_i, G_i \in \mathcal{X}\) as long as the perturbation magnitude \(b\) is sufficiently small.

**Lemma 60.** Let \(\xi \in \{−1, 1\}^d\) and \(\xi \in \{−1, 1\}\) denote perturbations. Let \(b\) be the magnitude of the perturbations. Let \(\beta \in \mathcal{B}\). If \(y \in \text{Int}(\mathcal{X})\) and \(x \in \mathcal{X}\), then for any \(b\) sufficiently small and

\[
y' = y + \beta \cdot b \cdot (\zeta^T x - \xi),
\]

we have that \(y' \in \text{Int}(\mathcal{X})\). *Proof in Appendix G.29.*

With the above lemma, we define \(b_1\) so that \(Z_i, G_i \in \mathcal{X}\) for \(b < b_1\).

We verify that \(c_i\) satisfies Assumption 1. We note that \(c_i\) is twice continuously differentiable because it is the sum of twice continuously differentiable functions. Second, we show that \(c_i\) is strongly convex. Since \(c_i\) satisfies Assumption 1, then \(c_i\) is \(\alpha_i\)-strongly convex for \(\alpha_i > 0\) and twice continuously differentiable. In addition, \(\phi(s - r)\beta y\) is differentiable and convex in \(y\). By the strong convexity of \(c_i\) and the convexity of \(\phi(s - r)\beta y\) in \(y\), we have that \(c_i(y)\) is \(\alpha_i\)-strongly convex, satisfying Assumption 1.

Let

\[
x_2 = x_1 + b \cdot \beta (\zeta^T x_1 - \xi).
\]

Note that by Lemma 60, for sufficiently small \(b\), we have that \(x_2 \in \text{Int}(\mathcal{X})\). Suppose that \(x_2 \in \text{Int}(\mathcal{X})\) for \(b < b_2\). We will show two useful facts about \(x_2\) that will enable us to show that the best response of the agent with unobservables \((Z_i, G_i, \xi)\) to the model \(\beta\) and threshold \(s\) is given by \(x_2\). For the first fact, we see that

\[
x_2 - Z_i = x_1 + \beta (b\zeta^T x_1 - b\xi) - Z_i = x_1 + \beta (b\zeta^T x_1 - b\xi) - Z_i - \beta \cdot b \cdot (\zeta^T x_1 - \xi)
\]

For the second fact, we have that

\[
\beta^T x_2 = \beta^T \left( \beta (b\zeta^T x_1 - b\xi) + x_1 \right)
\]

\[
= (\beta \beta^T) \cdot \beta (b\zeta^T x_1 - b\xi) + \beta^T x_1
\]

\[
= (\beta + b\zeta)^T x_1 - b\xi
\]

Now, we show that \(x_2 = x_i^* (\beta + b\zeta, s + b\xi)\). Since Assumption 1 holds, by Lemma 36 it is sufficient to check \(\nabla_x \mathbb{E}_\epsilon [U_i(x_2; \beta, s)] = 0\) to verify that \(x_2\) is the best response:

\[
\nabla_x \mathbb{E}_\epsilon [U_i(x_2; \beta, s)] = -\nabla c_i (x_2 - Z_i) + \phi(s - \beta^T x_2) \beta^T
\]

\[
= -\nabla c_i (x_1 - Z_i) + \phi(s - r) \beta^T
\]

\[
= -\nabla c_i (x_1 - Z_i) + \phi(s - r) b\zeta^T + \phi(s - r) \beta^T.
\]
To further simplify the above equation, we have that $x_1 = x_1^*(\beta + b\zeta, s + b\xi)$ and $x_1 \in \text{Int}(\mathcal{X})$. By Lemma 36, this implies that $
abla_X \mathbb{E}_\nu[U_i(x_1; \beta + b\zeta, s + b\xi)] = 0$. This gives that

$$
\nabla_X \mathbb{E}_\nu[U_i(x_1; \beta + b\zeta, s + b\xi)] = -\nabla c_i(x_1 - Z_i) + \phi_c(s + b\xi - (\beta + b\zeta)^T x_1)(\beta + b\zeta)^T
$$

$$
= -\nabla c_i(x_1 - Z_i) + \phi_c(s - r)(\beta + b\zeta)^T
$$

$$
= 0.
$$

So, we have that $\nabla c_i(x_1 - Z_i) = \phi_c(s - r)(\beta + b\zeta)^T$.

Substituting this result into (G.35) yields

$$
\nabla_X \mathbb{E}_\nu[U_i(x_2; \beta, s)] = -\nabla_X c_i(x_1 - Z_i) + \phi_c(s - r)b\zeta^T + \phi_c(s - r)\beta^T
$$

$$
= -\phi_c(s - r)(\beta + b\zeta)^T + \phi_c(s - r)b\zeta^T + \phi_c(s - r)\beta^T
$$

$$
= 0.
$$

We note that if $b < \min(b_1, b_2)$, then we have that $Z_{i';b,\zeta,\xi}, x_2 \in \text{Int}(\mathcal{X})$. Under such conditions, we conclude that $x_2 = x_1^*(\beta, s)$. The score obtained by the agent with type $(Z_i', G_i', c_{i'})$ and cost function $c_{i'}$ under the model $\beta$ and threshold $s$ is $\beta^T x_2$. As we showed earlier in (G.32), this quantity is equal to $r$. Thus, for sufficiently small perturbations, the agent with unobservables $(Z_i, G_i)$ under perturbations obtains the same raw score as the agent with unobservables $(Z_{i'}, G_{i'}, c_{i'})$ in the unperturbed setting.

G.19 Proof of Lemma 50

Note that by Assumption 2, $F_{Z,G}$ has finite support. We denote elements of the support as $\nu$. Let $f$ be the probability mass function of $F_{Z,G}$, so $f(\nu)$ gives the probability of that an agent has unobservables $\nu$.

We construct the probability mass function $\tilde{f}_b$ of the distribution $\tilde{F}$. Let $\nu \sim F$ and $c_1$ be its cost function. Under Assumptions 3 and 4, we use the transformation $T$ to map an agent with unobservables $\nu_i = (Z_i, G_i)$ and cost function $c_i$ to an agent with unobservables $\nu_{i';b,\zeta,\xi} = (Z_{i'}, G_{i'}, c_{i'})$ and cost function $c_{i'}$, as defined in Lemma 49, for each perturbation $\zeta \sim \{-1,1\}^d$ and $\xi \sim \{-1,1\}$ and unobservable $\nu \sim F_{Z,G}$.

We assign $\tilde{f}(\nu'_{i';b,\zeta,\xi}) = \frac{1}{2d+1} f(\nu)$. Since there are finitely many unobservables that occur with positive probability in $F_{Z,G}$ and finitely many perturbations ($2^{d+1}$ possible perturbations), there exists $b > 0$ such that this transformation is possible simultaneously for all types in the support of $F$ and all perturbations. Note that this is a valid probability mass function because $\sum_{\nu \sim \text{supp}(F)} f(\nu) = 1$, so

$$
\sum_{\nu'_{i';b,\zeta,\xi} \in \text{supp}(\tilde{F})} \tilde{f}(\nu'_{i';b,\zeta,\xi}) = \sum_{\nu \sim \text{supp}(F)} \sum_{\zeta \in \{-1,1\}^d} \sum_{\xi \in \{-1,1\}} \frac{1}{2d+1} f(\nu) = 1.
$$

In addition, note that the transformation yields

$$
\beta^T x_1^*(\beta, s) = (\beta + b\zeta)^T x_1^*(\beta + b\zeta, s + b\xi) - b\xi.
$$

The transformation given in Lemma 49 also provides other desirable properties. For instance, $Z_{i';b,\zeta,\xi} \in \mathcal{X}$, which means that the support of $\tilde{F}_{Z}$ is contained in $\mathcal{X}$. The cost function of the $c_{i'}$ satisfies Assumption 1 with $\alpha_{i'} = \alpha_i$, which means that $\alpha_{i'}(\tilde{F}) = \alpha_{i}(F)$. Lastly, for $\nu_{i';b,\zeta,\xi} \sim \tilde{F}$, the best responses of the agents lie in $\text{Int}(\mathcal{X})$. 

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Additionally, we have that

\[
P(\beta, s, b)(r) = \frac{1}{2d+1} \sum_{\zeta \in \{-1,1\}^d} \int \Phi_\sigma \left( r - (\beta + b\zeta)^T x^*_i(\beta + b\zeta, s + b\xi) - b\xi \right) dF_{Z,G}
\]

\[
= \frac{1}{2d+1} \sum_{\zeta \in \{-1,1\}^d} \sum_{\nu \in \text{supp}(F_{Z,G})} \Phi_\sigma \left( r - (\beta + b\zeta)^T x^*_i(\beta + b\zeta, s + b\xi) - b\xi \right) f(\nu)
= \sum_{\zeta \in \{-1,1\}^d} \sum_{\nu \in \text{supp}(F_{Z,G})} \Phi_\sigma \left( r - (\beta + b\zeta)^T x^*_i(\beta + b\zeta, s + b\xi) - b\xi \right) f(\nu)
\]

\[
= \sum_{\zeta \in \{-1,1\}^d} \sum_{\nu \in \text{supp}(F_{Z,G})} \Phi_\sigma \left( r - \beta^T x^*_i(\beta, s) \right) f_b(\nu)
= \int \Phi_\sigma \left( r - \beta^T x^*_i(\beta, s) \right) dF_{Z,G}.
\]

The final line matches the form of the score distribution’s CDF given in Lemma 42 assuming that the unobservables are distributed according to \( F \).

G.20 Proof of Lemma 51

We define the sequence of functions \( \{h_b(s)\} \) where \( h_b : \mathcal{S} \to \mathcal{S} \). Let \( h_b(s) := s - q(P(\beta, s, b)) \) and \( h(s) := s - q(P(\beta, s)) \).

We aim to apply Lemma 31 to this sequence of functions. We realize that the requirements on \( h(s) \) are given by our results from Section 3. Theorem 4 and Theorem 6 give us that \( h(s) \) has a unique root, which is the unique fixed point of \( q(P(\beta, s)) \) called \( s(\beta) \). Also, we note that \( h_b(s) \) and \( h(s) \) are defined on the compact set \( \mathcal{S} \). It remains to check that

1. Each \( h_b(s) \) is continuous,
2. Each \( h_b(s) \) has a unique root, which is the fixed point of \( q(P(\beta, s, b)) \) called \( s(\beta, b) \),
3. As \( b \to 0 \), \( h_b(s) \to h(s) \) is uniformly.

To verify the first two properties from the above list, we apply the transformation provided in Lemma 50 to \( P(\beta, s, b) \). This transformation enables us to apply the results from Section 3 directly to expressions involving \( P(\beta, s, b) \).

Since the transformation maintains all of our assumptions and

\[
\sigma^2 > \frac{1}{\alpha_s(F)^2} = \frac{1}{\alpha_s(F_b)^2}.
\]

we can apply Lemma 5 to see that \( q(P(\beta, s, b)) \) is continuous in \( s \). This gives the continuity of \( h_b(s) \). In addition, we can apply Theorem 4 and Theorem 6 to find that \( q(P(\beta, s, b)) \) has a unique fixed point in \( \mathcal{S} \). We can call the fixed point \( s(\beta, b) \), and \( s(\beta, b) \) is also the unique root of \( h_b(s) \).

Finally, we must check the third point, which is uniform convergence of \( h_b(s) \) to \( h(s) \). We aim to apply Lemma 29. First, we note that the continuity of \( h(s) \) is given by Lemma 5. Second, we check that each \( h_b(s) \) is monotonically increasing. Under the transformation from Lemma 50, we can apply Lemma 43 to observe that under our conditions, \( \frac{\partial q(P(\beta, s, b))}{\partial s} < 1 \), so \( h_b(s) \) is strictly increasing. Third, we show that \( h_b(s) \) converges pointwise to \( h(s) \) as follows.
To show \( h_b(s) \to h(s) \) pointwise, we show \( q(P(\beta, s, b)) \to q(P(\beta, s)) \) pointwise. Note that by Lemma 42, \( P(\beta, s) \) is strictly increasing, so we can let a lower bound on its density be \( d \) for \( s \in \mathcal{S} \), i.e.
\[
d = \inf_{r \in \mathcal{S}} p(\beta, s)(r).
\]

Then, we realize that
\[
|q(P(\beta, s, b)) - q(P(\beta, s))| \leq \frac{1}{d} \cdot \sup_{r \in \mathcal{S}} |P(\beta, s, b)(r) - P(\beta, s)(r)|.
\]

The following lemma gives us the required uniform convergence in \( r \).

**Lemma 61.** Under Assumptions 1, 2, 3, and 4, if \( \sigma^2 > \frac{1}{\alpha_t(F) \sqrt{2 \pi}} \) then \( P(\beta, s, b)(r) \to P(\beta, s)(r) \) uniformly in \( r \) as \( b \to 0 \).\textit{Proof in Appendix G.30.}

So, we have that \( q(P(\beta, s, b)) \to q(P(\beta, s)) \) pointwise in \( s \). This implies \( h_b(s) \to h(s) \) pointwise. Thus, we have that \( h_b(s) \) and \( h(s) \) satisfy the conditions of Lemma 29, which implies that \( h_b(s) \to h(s) \) uniformly.

Thus, the conditions of Lemma 31 are satisfied, so we have that \( s(\beta, b) \to s(\beta) \) as \( b \to 0 \).

### G.21 Proof of Lemma 52

For sufficiently small \( b \), we can apply Lemma 50 to show that \( P(\beta, s, b) \) is equal to the score distribution generated when agents with unobservables \( \nu'_{b,\xi,\zeta} = (Z_{i,b,\xi,\zeta}, G_{i,b,\xi,\zeta}) \sim F_{Z,G} \) and cost functions \( c_{\nu'} \) best respond to a model \( \beta \) and threshold \( s \). The conditions assumed when unobservables \( \nu = (Z, G) \sim F_{Z,G} \) and cost functions are \( c_{\nu} \) also hold when unobservables are distributed \( \nu'_{b,\xi,\zeta} \sim \tilde{F} \) and cost functions are \( c_{\nu'} \).

In particular, \( \alpha_t(\tilde{F}^b) = \alpha_t(F) \), so we have that \( \sigma^2 > \frac{2}{\alpha_t(F)^2 \sqrt{2 \pi}} \), so we have that \( q(P(\beta, s, b)) \) is a contraction in \( s \) by Corollary 8. Furthermore, the conditions of Lemma 47 are satisfied by the assumed conditions, the results of Lemma 50, and the fact that \( q(P(\beta, s, b)) \) is a contraction. So, we have that
\[
\hat{S}_{b,n}^n \xrightarrow{p} s(\beta, b),
\]

where \( s(\beta, b) \) is the unique fixed point of \( q(P(\beta, s, b)) \). In addition, since \( \{t_n\} \) is a sequence such that \( t_n \uparrow \infty \) as \( n \to \infty \) and \( t_n \prec \exp(n) \), we certainly have that \( \{t_n - 1\} \) is a sequence such that \( t_n - 1 \uparrow \infty \) as \( n \to \infty \) and \( t_n - 1 \prec \exp(n) \), so again by Lemma 47, we have that
\[
\hat{S}_{b,n}^{t_n - 1} \xrightarrow{p} s(\beta, b).
\]

### G.22 Proof of Corollary 53

The proof of this result is analogous to Theorem 14.

### G.23 Proof of Lemma 54

To simplify notation, we use the following abbreviations. Let \( s(\beta) \) be the unique fixed point of \( q(P(\beta, s)) \).
\[
\hat{I}_i(\beta, s, r) := \pi(\beta^TX_i(\beta, s); \beta, r)
\]
\[
\hat{\Pi}_n(\beta, s, r) := \frac{1}{n} \sum_{i=1}^{n} \hat{I}_i(\beta, s, r).
\]

The regression coefficient obtained by running OLS of \( I \) on \( M_\beta \) is denoted by \( \hat{\Gamma}^{b_{n,n}}_{\Pi,\beta}(\beta, \hat{S}_{b,n}^n, \hat{S}_{b,n}^n) \). The regression coefficient must have the following form.
\[
\hat{\Gamma}^{b_{n,n}}_{\Pi,\beta}(\beta, \hat{S}_{b,n}^n, \hat{S}_{b,n}^n) = (S_{zz}^n)^{-1}S_{zy}^n, \text{ where } S_{zz}^n := \frac{1}{b_n^2 n} M_{zz}^T M_\beta, \quad S_{zy}^n := \frac{1}{b_n^2 n} M_{zy}^T I. \tag{G.36}
\]
In this proof, we establish convergence in probability of the two terms above separately. The bulk of the proof is the first step, which entails showing that
\[
\mathbf{s}_{xy}^n \xrightarrow{P} \frac{\partial \Pi}{\partial \beta}(\beta, s(\beta); s(\beta)).
\]
Due to \(\tilde{I}_s\)'s dependence on the stochastic process \(\{\hat{S}_{b,n}^t\}\), the main workhorse of this result is Lemma 28. To apply this lemma, we must establish stochastic equicontinuity for the collection of stochastic processes \(\{\hat{j}_n(\beta, s, r)\}\). Second, through a straightforward application of the Weak Law of Large Numbers, we show that
\[
\mathbf{s}_{zz}^n \xrightarrow{P} \mathbf{I}_d.
\]
Finally, we use Slutsky’s Theorem to establish the convergence the regression coefficient.

We proceed with the first step of establishing convergence of \(s_{xy}\). We have that
\[
\mathbf{s}_{xy}^n = \frac{1}{b_{n}^2} \mathbf{M}_{\beta}^T \mathbf{I}
\]
\[
= \frac{1}{b_{n}^2} \sum_{i=1}^{n} b_n \zeta_i I_i
\]
\[
= \frac{1}{b_n} \cdot \frac{1}{n} \sum_{i=1}^{n} \zeta_i I_i
\]

We fix \(j\) and \(b_n = b\) where \(b > 0\) and is small enough to satisfy the hypothesis of Lemma 52. For each \(\zeta \in \{-1, 1\}^d\) and \(\xi \in \{-1, 1\}\), let
\[
n_{\zeta, \xi} = \sum_{i=1}^{n} \delta(\zeta_i = \zeta, \xi_i = \xi).
\]
Let \(z(\zeta)\) map a perturbation \(\zeta \in \{-1, 1\}^d\) to the identical vector \(\zeta\), except with \(j\)-th entry set to 0. So, if the \(j\)-th entry of \(\zeta\) is 1, then \(\zeta = \mathbf{e}_j + z(\zeta)\). If the \(j\)-th entry of \(\zeta\) is -1, then \(\zeta = -\mathbf{e}_j + z(\zeta)\). So, we have that
\[
I_i = \hat{I}_i(\beta + b\zeta_i, \hat{S}_{b,n}^t + b\xi_i, \hat{S}_{b,n}^t) = \hat{I}_i(\beta + b\zeta_i, \mathbf{e}_j + b \cdot z(\zeta_i), \hat{S}_{b,n}^t + b\xi_i, \hat{S}_{b,n}^t).
\]

As a result, we have that
\[
\frac{1}{n} \sum_{i=1}^{n} \zeta_{i,j} I_i = \frac{1}{n} \sum_{i=1}^{n} \zeta_{i,j} \cdot \hat{I}_i(\beta + b\zeta_i, \mathbf{e}_j + b \cdot z(\zeta_i), \hat{S}_{b,n}^t + b\xi_i, \hat{S}_{b,n}^t) (G.37)
\]
\[
= \frac{1}{n} \sum_{i=1}^{n} \sum_{\xi \in \{-1, 1\}^d} \frac{n_{\zeta, \xi}}{n} \hat{I}_i(\beta + b\zeta_i, \mathbf{e}_j + b \cdot z(\zeta_i), \hat{S}_{b,n}^t + b\xi_i, \hat{S}_{b,n}^t) (G.38)
\]
\[
= \sum_{\zeta \in \{-1, 1\}^d} \sum_{\xi \in \{-1, 1\}} \frac{n_{\zeta, \xi}}{n} \hat{I}_i(\beta + b\zeta_i, \mathbf{e}_j + b \cdot z(\zeta_i), \hat{S}_{b,n}^t + b\xi_i, \hat{S}_{b,n}^t) (G.39)
\]
\[
- \sum_{\zeta \in \{-1, 1\}^d} \sum_{\xi \in \{-1, 1\}} \frac{n_{\zeta, \xi}}{n} \hat{I}_i(\beta - b\zeta_i, \mathbf{e}_j + b \cdot z(\zeta_i), \hat{S}_{b,n}^t + b\xi_i, \hat{S}_{b,n}^t) (G.40)
\]

To establish convergence properties of terms in the double sums in (G.39) and (G.40), we must establish stochastic equicontinuity of the collection of stochastic processes \(\{\tilde{\Pi}_n(\beta, s, r)\}\) indexed by \((s, r) \in \mathcal{S} \times \mathcal{S}\). Because \(\mathcal{S} \times \mathcal{S}\) is compact and \(\Pi(\beta, s; r)\) is continuous in \((s, r)\), then we can show that \(\{\tilde{\Pi}_n(\beta, s, r)\}\) by
showing that  \( \tilde{\Pi}_n(\beta, s, r) \) converges uniformly in probability to  \( \Pi(\beta, s; r) \) (Lemma 27). We can use Lemma 26 to show the necessary uniform convergence result.

By Lemma 48, we have that  \( \hat{I} \) satisfies the conditions of Lemma 26. Thus, we can apply Lemma 26 to establish uniform convergence in probability of  \( \Pi_n(\beta, s, r) \) with respect to  \( (s, r) \). As a consequence, the collection of stochastic processes \( \{ \Pi_n(\beta, s, r) \} \) is stochastically equicontinuous. In particular,  \( \Pi_n(\beta, s, r) \) is stochastically equicontinuous at  \( (s(\beta, b), s(\beta, b)) \), where  \( s(\beta, b) \) is the unique fixed point of  \( q(P(\beta, s, b)) \) (see Lemma 51). By Lemma 52, we have that

\[
\hat{S}_{b,n} \xrightarrow{P} s(\beta, b).
\]

Now, we can apply Lemma 28 to establish convergence in probability for the terms in (G.39) and (G.40). As an example, for a perturbation  \( \zeta \in \{-1,1\}^d \) with  \( j \)-th entry equal to 1 and arbitrary  \( \xi \in \{-1,1\} \), Lemma 28 gives that

\[
\hat{\Pi}_{n,\zeta}(\beta + b\xi, \zeta, \hat{S}_{b,n}) = \hat{\Pi}_{n,\zeta}(\beta + b\xi + b\cdot z(\zeta), s(\beta, b) + b\xi, s(\beta, b)),
\]

and by the Weak Law of Large Numbers, we have that

\[
\hat{\Pi}_{n,\zeta}(\beta + b\xi + b\cdot z(\zeta), s(\beta, b) + b\xi, s(\beta, b)) \xrightarrow{P} \Pi(\beta + b\xi + b\cdot z(\zeta), s(\beta, b) + b\xi, s(\beta, b)).
\]

Analogous results for the remaining terms in (G.39) and (G.40). Also,

\[
\frac{n_{\zeta,\xi}}{n} \xrightarrow{P} \frac{1}{2d+1}, \quad \zeta \in \{-1,1\}^d, \xi \in \{-1,1\}.
\]

By Slutsky’s Theorem, when any  \( j \) and  \( b \) fixed, we have

\[
s_{z,y,j}^n \xrightarrow{P} \sum_{\zeta \in \{-1,1\}^d \text{ s.t. } \zeta_j = 1} \frac{\Pi(\beta + b\xi + b\cdot z(\zeta), s(\beta, b) + b\xi, s(\beta, b))}{2^{d+1} \cdot b} \tag{G.41}
\]

\[
- \sum_{\zeta \in \{-1,1\}^d \text{ s.t. } \zeta_j = -1} \frac{\Pi(\beta - b\xi + b\cdot z(\zeta), s(\beta, b) + b\xi, s(\beta, b))}{2^{d+1} \cdot b}. \tag{G.42}
\]

Let  \( R_b \) denote the expression on the right side of the above equation. If there is a sequence  \( \{b_n\} \) such that  \( b_n \to 0 \), then by Lemma 51,  \( s(\beta, b_n) \to s(\beta) \), where  \( s(\beta) \) is the unique fixed point of  \( q(P(\beta, s)) \). By continuity of  \( \Pi \),

\[
R_{b_n} \to \frac{\partial \Pi}{\partial \beta_j}(\beta, s(\beta); s(\beta)).
\]

Using the definition of convergence in probability, we show that there exists such a sequence  \( \{b_n\} \). From (G.41) and (G.42), we have that for each  \( \epsilon, \delta > 0 \) and  \( b > 0 \) and sufficiently small, there exists  \( n(\epsilon, \delta, b) \) such that for  \( n \geq n(\epsilon, \delta, b) \)

\[
P(|s_{z,y,j}^n - R_b| \leq \epsilon) \geq 1 - \delta.
\]

So, we can fix  \( \delta > 0 \). For  \( k = 1, 2, \ldots \), let  \( N(k) = \min\{\frac{1}{\delta}, \frac{1}{k}\} \). Then, we can define a sequence such that  \( b_n = \epsilon_n = \frac{1}{k} \) for all  \( N(k) \leq n \leq N(k + 1) \). So, we have that  \( \epsilon_n \to 0 \) and  \( b_n \to 0 \). Thus, we have that

\[
s_{z,y,j}^n \xrightarrow{P} \frac{\partial \Pi}{\partial \beta_j}(\beta, s(\beta); s(\beta)).
\]

Considering all indices  \( j \),

\[
s_{z,y}^n \xrightarrow{P} \frac{\partial \Pi}{\partial \beta}(\beta, s(\beta); s(\beta)).
\]
It remains to establish convergence in probability for $S_{zz}$. We have that

$$S_{zz}^n = \frac{1}{b_n^2} \mathbf{M}_p^T \mathbf{M}_p$$

$$= \frac{1}{b_n^2} \sum_{i=1}^n (b_n \zeta_i)^T (b_n \zeta_i).$$

$$= \frac{1}{n} \sum_{i=1}^n \zeta_i^T \zeta_i.$$

We note that

$$\mathbb{E}_{\zeta_i \sim \mathbb{R}^d} [\zeta_i, j \zeta_i, k] = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}$$

because $\zeta_i$ is a vector of independent Rademacher random variables. So, $\mathbb{E} [\zeta_i^T \zeta_i] = \mathbf{I}_d$. By the Weak Law of Large Numbers, we have that

$$S_{zz}^n \xrightarrow{p} \mathbf{I}_d.$$

Finally, we can use Slutsky’s Theorem to show that

$$\hat{\Gamma}_{\beta}^{b,n} (\beta, \hat{S}_{t,n} ; \hat{S}_{t,n}) = \mathbb{E}_{\zeta_i \sim \mathbb{R}^d} [\zeta_i, j \zeta_i, k] = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}$$

because $\zeta_i$ is a vector of independent Rademacher random variables. So, $\mathbb{E} [\zeta_i^T \zeta_i] = \mathbf{I}_d$. By the Weak Law of Large Numbers, we have that

$$S_{zz}^n \xrightarrow{p} \mathbf{I}_d.$$

G.24 Proof of Corollary 55

The proof of this result is analogous to Lemma 54.

G.25 Proof of Lemma 56

We study the convergence of the kernel density estimate $p^\beta (\beta, \hat{S}_{t,n} ; b_n)$. Let $p^\beta (\beta, s, b)(r)$ is a kernel density estimate of density of $P(\beta, s, b)$ at a point $r$. Let $\beta_i = \beta + b \zeta_i, s_i = s + b \xi_i$, where $\zeta \sim \mathbb{R}^d$ and $\xi \sim \mathbb{R}$. We can write the explicit form of $p^\beta (\beta, s, b)(r)$ as follows

$$p^\beta (\beta, s, b)(r) = \frac{1}{h_n} \sum_{i=1}^n k \left( r - \beta X_i (\beta, s_i) + b \xi_i \right).$$

For sufficiently small $b$, we can apply Lemma 50 to map the unobservables $\nu_i = (Z_i, G_i) \sim \mathbb{F}_{Z,G}$ and cost functions $c_i$ to types $\nu_{c_i} = (Z_i, c_i, Z_i, c_i) \sim \mathbb{F}_b$ and cost functions $c_i$, so that when the agent with unobservables $\nu_{c_i}$ best responds to the unperturbed model and threshold, they obtain the same raw score (without noise) as the agent with unobservables $\nu$ who responds to a perturbed model and threshold. So, we can write

$$p^\beta (\beta, s, b)(r) = \frac{1}{h_n} \sum_{i=1}^n k \left( r - \beta X_i (\beta, s) \frac{1}{h_n} \sum_{i=1}^n k \left( r - \beta X_i (\beta, s) - \beta \epsilon_i \right).$$

We make the following abbreviations.

$$\tilde{k} (\nu_{c_i}, \zeta_i, \epsilon_i, \beta, s, r ; h) := k \frac{r - \beta X_i (\beta, s)}{h}$$

$$\tilde{k}_n (\beta, s, r ; h) := \frac{1}{n} \sum_{i=1}^n \tilde{k} (\nu_{c_i}, \zeta_i, \epsilon_i, \beta, s, r ; h)$$

$$K (\beta, s, r ; h) := \mathbb{E}_{\hat{G}_h} \left[ \tilde{k} (\nu_{c_i}, \zeta_i, \epsilon_i, \beta, s, r ; h) \right].$$

We can write

$$p^\beta (\beta, \hat{S}_{t,n} ; b) (\hat{S}_{t,n}) = \frac{1}{h_n} \tilde{k}_n (\beta, \hat{S}_{t,n} ; \hat{S}_{t,n} ; h_n).$$
Due to the density estimate’s dependence on the stochastic process $\tilde{S}_{b,n}^{t_n}$, we first must establish the stochastic equicontinuity of the collection of stochastic processes $\{\tilde{k}_{n}(\beta, s, r)\}$ indexed by $(s, r) \in S \times S$. We show stochastic equicontinuity via uniform convergence in probability (Lemma 26). The remainder of the proof follows by the Weak Law of Large Numbers and taking standard limits.

We fix $h_n = h$. Since $k$ satisfies the conditions of Lemma 26, we can apply Lemma 26 to realize that $\tilde{k}_{n}(\beta, s, r; h)$ converges uniformly in probability to $K(\beta, s, r; h)$ with respect to $(s, r) \in S \times S$. As a result, the collection of stochastic processes $\{\tilde{k}_{n}(\beta, s, r; h)\}$ indexed by $(s, r) \in S \times S$ are stochastically equicontinuous. In particular, $\{\tilde{k}_{n}(\beta, s, r; h)\}$ is stochastically equicontinuous at $(s(\beta, b), s(\beta, b))$. By Lemma 52, we have that

$$\tilde{S}_{b,n}^{t_n} \xrightarrow{p} s(\beta, b),$$

where $s(\beta, b)$ is the unique fixed point of $g(P(\beta, s, b))$. We can apply Lemma 28 to see that

$$k_{n}(\beta, \tilde{S}_{b,n}^{t_n}, \tilde{S}_{b,n}^{t_n}; h) - k_{n}(\beta, s(\beta, b), s(\beta, b); h) \xrightarrow{p} 0.$$

Furthermore, by the Weak Law of Large Numbers, we have that

$$k_{n}(\beta, s(\beta, b), s(\beta, b); h) \xrightarrow{P} K(\beta, s(\beta, b), s(\beta, b)); h).$$

Given our definition of the kernel function $k$ and for fixed $h$, we have that

$$p^n(\beta, \tilde{S}_{b,n}^{t_n}, b)(\tilde{S}_{b,n}^{t_n}) \xrightarrow{P} K(\beta, s(\beta, b), s(\beta, b); h) \xrightarrow{h} \frac{P(\beta, s(\beta, b), b)(s(\beta, b) + \frac{bh}{2}) - P(\beta, s(\beta, b), b)(s(\beta, b) - \frac{bh}{2})}{h}.$$

Given that our sequence $h_n \to 0$ and $nh_n \to \infty$ and $k$ satisfies the assumptions of Theorem 33, we can apply Theorem 33 to see that for each fixed $b$, we obtain a consistent density estimate.

$$p^n(\beta, \tilde{S}_{b,n}^{t_n}, b)(\tilde{S}_{b,n}^{t_n}) \xrightarrow{p} \lim_{h_n \to 0} \frac{P(\beta, s(\beta, b), b)(s(\beta, b) + \frac{bh}{2}) - P(\beta, s(\beta, b), b)(s(\beta, b) - \frac{bh}{2})}{h_n}$$

$$= p(\beta, s(\beta, b), b)(s(\beta, b)).$$

Let $R_b$ denote the right side of the above equation. Suppose there exists a sequence such that $b_n \to 0$. By Lemma 51, this gives us that $s(\beta, b_n) \to s(\beta)$, where $s(\beta)$ is the unique fixed point of $q(P(\beta, s))$. We can show that $R_{b_n} \to p(\beta, s(\beta))(s(\beta))$ as follows.

$$|R_{b_n} - p(\beta, s(\beta))(s(\beta))| \leq |p(\beta, s(\beta, b_n), b_n)(s(\beta, b_n)) - p(\beta, s(\beta, b_n))(s(\beta, b_n))|$$

$$+ |p(\beta, s(\beta, b_n))(s(\beta, b_n)) - p(\beta, s(\beta))(s(\beta))|$$

$$\leq \sup_{s, r \in S} |p(\beta, s, b_n)(r) - p(\beta, s)(r)|$$

$$+ |p(\beta, s(\beta, b_n))(s(\beta, b_n)) - p(\beta, s(\beta))(s(\beta))|.$$ 

Since $p(\beta, s)(r)$ is continuous in $s$ and $r$ (Lemma 42), there exists $N$ such that for $n \geq N$, the second term is less than $\epsilon$. To bound the first term, we require the following lemma.

**Lemma 62.** Under Assumptions 1, 2, 3, and 4, if $\sigma^2 > \frac{2}{a \sqrt{2\pi c}}$, then $p(\beta, s, b)(r) \to p(\beta, s)(r)$ uniformly in $s$ and in $r$ as $b \to 0$. *Proof in Appendix G.31.*

Due to the uniform convergence result, we have that if there exists a sequence $\{b_n\}$ such that $b_n \to 0$, then

$$R_{b_n} \to p(\beta, s(\beta))(s(\beta)).$$

It remains to show that there exists such a sequence $\{b_n\}$ where $b_n \to 0$. Using the definition of convergence in probability, we show that there exists such a sequence $\{b_n\}$. From (G.44), we have that for each $\epsilon, \delta > 0$ and $b > 0$ and sufficiently small, there exists $n(\epsilon, \delta, b)$ such that for $n \geq n(\epsilon, \delta, b)$

$$P(|p^n(\beta, \tilde{S}_{b,n}^{t_n}, b_n)(\tilde{S}_{b,n}^{t_n}) - R_b| \leq \epsilon) \geq 1 - \delta.$$

So, we can fix $\delta > 0$. For $k = 1, 2, \ldots$, let $N(k) = n(\frac{1}{k}, \delta, \frac{1}{k})$. Then, we can define a sequence such that $b_n = \epsilon_n = \frac{1}{k}$ for all $N(k) \leq n \leq N(k + 1)$. So, we have that $\epsilon_n \to 0$ and $b_n \to 0$. Finally, this gives that

$$p^n(\beta, \tilde{S}_{n}^{t_n}, b_n)(\tilde{S}_{n}^{t_n}) \xrightarrow{p} p(\beta, s(\beta))(s(\beta)).$$
G.26 Proof of Lemma 57

First, we note that $s(\beta)$ is continuously differentiable in $\beta$ by Corollary 9, so we can use implicit differentiation to compute the following expression for $\frac{\partial s}{\partial \beta}$

$$
\frac{\partial s}{\partial \beta} = \frac{1}{1 - \frac{\partial q(P(\beta, s(\beta)))}{\partial \beta}} \cdot \frac{\partial q(P(\beta, s(\beta)))}{\partial \beta}.
$$

(G.45)

After that, we apply the lemma below to express the partial derivatives of the quantile mapping $q(P(\beta, s))$ in terms of partial derivatives of the complementary CDF $\Pi(\beta, s; r)$.

**Lemma 63.** Let $\beta \in B, s \in S$. Under Assumption 1, 2, 3, if $\sigma^2 > \frac{2}{\alpha \cdot \sqrt{2\pi e}}$ then for $\beta^t, s^t$ sufficiently close to $\beta, s$, the derivative of $q(P(\beta, s))$ with respect to a one-dimensional parameter $\theta$ is given by

$$
\frac{\partial q(P(\beta, s))}{\partial \theta} = \frac{1}{p(\beta^t, s^t)(r^t)} \cdot \frac{\partial \Pi(\beta, s; r^t)}{\partial \theta},
$$

where $r^t = q(P(\beta^t, s^t))$. *Proof in Appendix G.32.*

Since $s(\beta)$ is the fixed point induced by $\beta$, we have that $s(\beta) - q(P(\beta, s(\beta))) = 0$.

From Corollary 9, we have that $s(\beta)$ is continuously differentiable in $\beta$. Differentiating both sides of the above equation with respect to $\beta$ yields

$$
\frac{\partial s}{\partial \beta} - \left( \frac{\partial q(P(\beta, s(\beta)))}{\partial \beta} + \frac{\partial q(P(\beta, s(\beta)))}{\partial s} \cdot \frac{\partial s}{\partial \beta} \right) = 0.
$$

Rearranging the above equation yields (F.20), which shows that $\frac{\partial s}{\partial \beta}$ in terms of $\frac{\partial q(P(\beta, s))}{\partial s}$ and $\frac{\partial q(P(\beta, s))}{\partial \beta}$.

From Lemma 63, we have that for $\beta^t, s^t$ sufficiently close to $\beta, s$, we have that

$$
\frac{\partial q(P(\beta, s))}{\partial \beta} = \frac{1}{p(\beta^t, s^t)(r^t)} \cdot \frac{\partial \Pi(\beta, s; r^t)}{\partial \beta},
$$

where $r^t = q(P(\beta^t, s^t))$. Let $s(\beta) = s(\beta)$. Suppose that we aim to estimate the derivative when the model parameters are $\beta$ and the threshold is $s(\beta)$. If we consider $\beta^t = \beta, s^t = s(\beta)$, then $r^t = s(\beta)$. So, we have that

$$
\frac{\partial q(P(\beta, s(\beta)))}{\partial s} = \frac{1}{p(\beta, s(\beta))(s(\beta))} \cdot \frac{\partial \Pi(\beta, s(\beta); s(\beta))}{\partial s},
$$

(G.46)

$$
\frac{\partial q(P(\beta, s(\beta)))}{\partial \beta} = \frac{1}{p(\beta, s(\beta))(s(\beta))} \cdot \frac{\partial \Pi(\beta, s(\beta); s(\beta))}{\partial \beta}.
$$

(G.47)

Substituting (G.46) and (G.47) into (G.45) yields

$$
\frac{\partial s}{\partial \beta} = \frac{1}{p(\beta, s(\beta))(s(\beta))} \cdot \frac{\partial \Pi(\beta, s(\beta); s(\beta))}{\partial \beta}.
$$
G.27 Proof of Lemma 58

By Lemma 38, $H$ is positive definite and invertible. As a result, we can apply the Sherman-Morrison Formula (Theorem 20) to

$$(H + \phi'_\sigma(s - \beta^T x)\beta \beta^T)^{-1} = H^{-1} - \frac{H^{-1}(\phi'_\sigma(s - \beta^T x)\beta \beta^T)H^{-1}}{1 + \beta^T H^{-1}(\phi'_\sigma(s - \beta^T x)\beta)}$$

$$= H^{-1} - \frac{\phi'_\sigma(s - \beta^T x)\beta \beta^T H^{-1}}{1 + \phi'_\sigma(s - \beta^T x)\beta \beta^T H^{-1}}.$$ 

G.28 Proof of Lemma 59

In the first part of the proof, we establish existence of a fixed point of $\omega_i(s; \beta)$. In the second part of the proof, we show that if a fixed point exists, then it must be unique.

First, we use the Intermediate Value Theorem (IVT) to show existence of a fixed point. We apply the IVT to the function $h_i(s) = s - \omega_i(s; \beta)$. We note that by Lemma 2 that $\omega_i(s)$ is continuous. It remains to show that there exists $s_l$ such that $h_i(s_l) < 0$ and there exists $s_2$ such that $s_2 > s_1$ and $h_i(s_2) > 0$. Then, by the Intermediate Value Theorem, there must be $s \in [s_1, s_2]$ for which $h_i(s) = 0$, which gives that $\omega_i(s)$ has at least one fixed point.

Let $\delta > 0$. By Lemma 40, we have that there exists $S_{l,1}$ so that for all $s \leq S_l$, we have that

$$|\beta^T x_i^*(\beta, s) - \beta^T Z_i| < \delta.$$ 

Let $S_{l,2} = \beta^T Z_i - \delta$. Let $s_1 < \min(S_{l,1}, S_{l,2})$. Then we have that

$$h_i(s_1) = s_1 - \beta^T x_i^*(\beta, s_1)$$

$$\leq s_1 - \beta^T Z_i + \delta$$

$$< (\beta^T Z_i - \delta) - \beta^T Z_i + \delta.$$ 

$$< 0.$$ 

Second, by Lemma 40, we have that there exists $S_{h,1}$ so that for all $s \geq S_l$, we have that

$$|\beta^T x_i^*(\beta, s) - \beta^T Z_i| < \delta.$$ 

Let $S_{h,2} = \beta^T Z_i + \delta$. Let $s_2 > \max(S_{h,1}, S_{h,2})$. Then we have that

$$h_i(s_2) = s_2 - \beta^T x_i^*(\beta, s_2)$$

$$\geq s_2 - \beta^T Z_i - \delta$$

$$> 0.$$ 

We have that $s_1 < \beta^T Z_i - \delta < \beta^T Z_i + \delta < s_2$. By the IVT, there must be some $s \in [s_1, s_2]$ so that $h_i(s) = 0$. Second, we show that if a fixed point exists, then the fixed point must be unique. By Lemma 39, $h_i(s)$ is strictly increasing in $s$. There can be only one point at which $h_i(s) = 0$. So, there is only one $s$ such that $s - \omega_i(s; \beta) = 0$. Thus, $\omega_i(s; \beta)$ has a unique fixed point.
G.29 Proof of Lemma 60

Since $y$ is in the interior of $X$, then there exists some $\epsilon > 0$ such that the open ball of radius $\epsilon$ about $y$ is a subset of $X$. We note that

$$|y' - y| = \left| \beta \cdot (b\zeta^T x - b\xi) \right|$$

$$\leq ||\beta|| \cdot |b\zeta^T x - b\xi|$$

$$\leq |b\zeta^T x - b\xi|$$

$$\leq b|\zeta^T x - \xi|$$

$$\leq b(|\zeta| |x| + |\xi|)$$

$$\leq b(\sqrt{d} \cdot \sup_{x \in X} |x| + 1)$$

Since $X$ is compact, we can say that the supremum in the above equation is achieved on $X$ and we can call its value $m$. So, if $b < \frac{\epsilon}{(m \sqrt{d} + 1)}$, then $y' \in \text{Int}(X)$.

G.30 Proof of Lemma 61

We first show that $P(\beta, s, b)(r) \to P(\beta, s)(r)$ uniformly in $r$ as $b \to 0$. We aim to apply Lemma 29. First, note that the continuity of $P(\beta, s)$ in $r$ is given by Lemma 42. We recall that

$$P(\beta, s, b)(r) = \frac{1}{2^d + 1} \sum_{\zeta \in \{-1, 1\}^d} \int \Phi_\sigma \left( r - (\beta + b\zeta)^T x_i^*(\beta + b\zeta, s + b\xi) \right) dF.$$ 

$\Phi_\sigma$ is strictly increasing, so $P(\beta, s, b)(r)$ is strictly increasing because the sum of strictly increasing functions is also strictly increasing. By continuity of $x$ in $\beta$ and $s$ (Lemma 2), we have that $P(\beta, s, b)(r) \to P(\beta, s)(r)$ pointwise in $r$. By Lemma 29, as $b \to 0$, we have that

$$\sup_{r \in S} |P(\beta, s, b)(r) - P(\beta, s)(r)| \to 0.$$

G.31 Proof of Lemma 62

We show that $p(\beta, s, b)(r) \to p(\beta, s)(r)$ uniformly in $s$ and $r$ as $b \to 0$. We prove the claim in two steps. First, we rewrite $p(\beta, s)(r)$ and $p(\beta, s, b)(r)$ as a finite sum of terms that align by type and perturbation. Second, we can show uniform convergence for pairs of terms in the sums, which gives that the aggregate quantity $p(\beta, s, b)(r) \to p(\beta, s)(r)$ uniformly.

First, we rewrite $p(\beta, s)$ as follows

$$p(\beta, s)(r) = \int \phi_\sigma(r - \beta^T x_i^*(\beta, s)) dF$$

$$= \sum_{\nu \in \text{supp}(F_{x, \sigma})} \sum_{\zeta \in \{-1, 1\}^d} \phi_\sigma(r - \beta^T x_i^*(\beta, s)) \frac{f(\nu)}{2^d + 1}.$$ 

To rewrite $p(\beta, s, b)$, recall that for sufficiently small $b$, we can use Lemma 50 to express $P(\beta, s, b)$ as the score distribution induced by agents with unobservables $\nu_{i'v} \sim F_{i'}$ and cost functions $c_{i'}$ who best respond to a model $\beta$ and threshold $s$. The type and cost function is given by transformation $T$ from Lemma 60. Recall that $T(i; b, \zeta, \xi)$ maps an agent $i$ with unobservables $(Z_i, G_i) \sim F$ and cost function $c_i$ to an agent $i'$ with unobservables $i'$ with unobservables $(Z_{i'v}, b, \zeta, \xi, G_{i'v}, b, \zeta, \xi)$ and cost function $c_{i'v}$.
Since our assumed conditions also transfer to $\tilde{F}_b$, we have that $P(\beta, s, b)(r)$ is continuously differentiable in $r$ with density $p(\beta, s, b)$ (Lemma 42). Using the function $T_1$, we have that

$$p(\beta, s, b)(r) = \int \phi_\sigma(r - \beta^T x^*_i(\beta, s)) d\tilde{F}_{b,Z,G}$$

(G.50)

$$= \sum_{\nu, \theta, b, \zeta, \xi \in \text{supp}(\tilde{F}_{b,Z,G})} \phi_\sigma(r - \beta^T x^*_i; b, \zeta, \xi(\beta, s)) \cdot \tilde{f}_{b,Z,G}(\nu, \theta, b, \zeta, \xi)$$

(G.51)

$$= \sum_{\nu \in \text{supp}(F_{Z,G})} \sum_{\zeta \in \{ -1, 1 \}^d} \sum_{\xi \in \{ -1, 1 \}^d} \phi_\sigma(r - \beta^T x^*_i; b, \zeta, \xi(\beta, s)) \cdot \tilde{f}_{b,Z,G}(\nu, \theta, b, \zeta, \xi)$$

(G.52)

$$= \sum_{\nu \in \text{supp}(F_{Z,G})} \sum_{\zeta \in \{ -1, 1 \}^d} \sum_{\xi \in \{ -1, 1 \}^d} \phi_\sigma(r - \beta^T x^*_i; b, \zeta, \xi(\beta, s)) \cdot \frac{f_{Z,G}(\nu)}{2^{d+1}}.$$  

(G.53)

The last line follows from Lemma 50. Therefore, the terms of $p(\beta, s)$ in (G.49) align with the terms of $p(\beta, s, b)$ in (G.53) by unobservable and perturbation. We have that

$$|p(\beta, s, b)(r) - p(\beta, s)(r)|$$

$$= \left| \sum_{\nu \in \text{supp}(F_{Z,G})} \sum_{\zeta \in \{ -1, 1 \}^d} \sum_{\xi \in \{ -1, 1 \}^d} \phi_\sigma(r - \beta^T x^*_i; b, \zeta, \xi(\beta, s)) - \phi_\sigma(r - \beta^T x^*_i(\beta, s)) \cdot \frac{f(\nu)}{2^{d+1}} \right|$$

$$\leq \sum_{\nu \in \text{supp}(F_{Z,G})} \sum_{\zeta \in \{ -1, 1 \}^d} \sum_{\xi \in \{ -1, 1 \}^d} \left| \phi_\sigma(r - \beta^T x^*_i; b, \zeta, \xi(\beta, s)) - \phi_\sigma(r - \beta^T x^*_i(\beta, s)) \right| \cdot \frac{f(\nu)}{2^{d+1}}.$$  

Since the sum in the above inequality is finite, we can show $p(\beta, s, b)(r) \to p(\beta, s)(r)$ uniformly in $s, r$ if we can show that for every unobservable $\nu$ and perturbation $(\zeta, \xi)$, we have that

$$\sup_{(s, r) \in S \times S} \left| \phi_\sigma(r - \beta^T x^*_i; b, \zeta, \xi(\beta, s)) - \phi_\sigma(r - \beta^T x^*_i(\beta, s)) \right| \to 0.$$

Now, we can use the following lemma to show uniform convergence (in $s$ and $r$) of the arguments to $G'$ in the above expression.

**Lemma 64.** Suppose Assumption 1 and 4 hold. Let agent $i$ be an agent with unobservables $\nu_i \sim F$ and cost function $c_i$. Let $T(i; b, \zeta, \xi)$ yield an agent $i'$ with unobservables $\nu_{i'; b, \zeta, \xi}$ and cost function $c_{i'}$, as defined in Lemma 49 for any $\zeta \in \{ -1, 1 \}^d, \xi \in \{ -1, 1 \}$, and $b > 0$ and sufficiently small. As $b \to 0$, $\beta^T x^*_i; b, \zeta, \xi(\beta, s) \to \beta^T x^*_i(\beta, s)$ uniformly in $s$. **Proof in Appendix G.33.**

We observe that

$$\sup_{(s, r) \in S \times S} |(r - \beta^T x^*_i; b, \zeta, \xi(\beta, s)) - (r - \beta^T x^*_i(\beta, s))|$$

$$= \sup_{s \in S} |(\beta^T x^*_i; b, \zeta, \xi(\beta, s)) - (\beta^T x^*_i(\beta, s))|$$

$$\to 0$$

where the uniform convergence in the last line follows from Lemma 64. Since the argument to $\phi_\sigma$ in (G.53) converges uniformly in $s$ and $r$, the argument to $\phi_\sigma$ is uniformly bounded. So, we can restrict the domain of $\phi_\sigma$ to an closed interval on which it is uniformly continuous. As a result, we also have that

$$\sup_{s \in S} |\phi_\sigma(r - \beta^T x^*_i; b, \zeta, \xi(\beta, s)) - \phi_\sigma(r - \beta^T x^*_i(\beta, s))| \to 0,$$

which concludes the proof.
G.32 Proof of Lemma 63

For simplicity, we can write

\[ r^t = q(P(\beta^t, s^t)) = P(\beta^t, s^t)^{-1}(q), \quad r = q(P(\beta, s)) = P(\beta, s)^{-1}(q). \]

From Lemma 5, we have the \( q(P(\beta, s)) \) is continuous in \( \beta, s \). In addition, we note that the density of the scores \( p(\beta, s)(y) \) is continuous with respect to \( \beta, s, y \) (Lemma 42). By the continuity of the density of the scores and the quantile mapping, we can choose \( \beta^t, s^t \) sufficiently close to \( \beta, s \) such that \( |r - r^t| < \epsilon \) and \( |p(\beta, s)(r^t) - p(\beta^t, s^t)(r^t)| < \epsilon \).

From Lemma 42, we have that \( P(\beta, s) \) and \( P(\beta^t, s^t) \) have unique inverses. So, the quantile mapping is uniquely defined, which means

\[ P(\beta^t, s^t)(r^t) = q, \quad P(\beta, s)(r) = q. \]

As a result, we have that \( P(\beta^t, s^t)(r^t) = P(\beta, s)(r) \). Without loss of generality, suppose that \( r > r^t \),

\[
P(\beta^t, s^t)(r^t) - P(\beta, s)(r^t) = P(\beta, s)(r) - P(\beta, s)(r)
= \int_{-\infty}^{r} p(\beta, s)(y)dy - \int_{-\infty}^{r^t} p(\beta, s)(y)dy
= \int_{r^t}^{r} p(\beta, s)(y)dy
= (r - r^t)p(\beta, s)(r^t) + o(|r^t - r|)
= (r - r^t)p(\beta^t, s^t)(r^t) + o((r - r^t)|p(\beta, s)(r^t) - p(\beta^t, s^t)(r^t)|) + o(|r - r^t|)
= (r - r^t)p(\beta^t, s^t)(r^t) + o(|r - r^t| · |p(\beta, s)(r^t) - p(\beta^t, s^t)(r^t)|) + o(|r - r^t|)
= (r - r^t)p(\beta^t, s^t)(r^t) + o(\epsilon^2) + o(\epsilon)
= (q(P(\beta^t, s^t)) - q(P(\beta, s)))p(\beta^t, s^t)(r^t) + o(\epsilon^2) + o(\epsilon)
\]

We can differentiate both sides of the above equation with respect to a one-dimensional parameter \( \theta \).

\[
-\frac{\partial P(\beta, s)(r^t)}{\partial \theta} = \frac{\partial q(P(\beta, s))}{\partial \theta} \cdot p(\beta^t, s^t)(r^t).
\]

From the Definition of \( \Pi(\beta, s; r) \) in (5.3), we observe that

\[
\frac{\partial P(\beta, s)(r)}{\partial \theta} = -\frac{\partial \Pi}{\partial \theta}(\beta, s; r).
\]

Solving for \( \frac{\partial q(P(\beta, s))}{\partial \theta} \), we find that

\[
\frac{\partial q(P(\beta, s))}{\partial \theta} = \frac{1}{p(\beta^t, s^t)(r^t)} \cdot \frac{\partial \Pi}{\partial \theta}(\beta, s; r^t).
\]

G.33 Proof of Lemma 64

Consider \( b \) sufficiently small so that the transformation in Lemma 49 is possible. Let \( Z_{i:1:b,c,\xi} \) be as defined in (F.2). Let \( h_b : S \rightarrow \mathbb{R} \), where \( h_b(s) := s - \beta^T x^*_i:1:b,c,\xi(\beta, s) \) and \( h(s) := s - \beta^T x^*_i(\beta, s) \). It is sufficient to show that \( h_b(s) \rightarrow h(s) \) uniformly in \( s \) because

\[
\sup_{s \in S} |\beta^T x^*_i:1:b,c,\xi(\beta, s) - \beta^T x^*_i(\beta, s)| = \sup_{s \in S} |s - \beta^T x^*_i:1:b,c,\xi(\beta, s) - s + \beta^T x^*_i(\beta, s)|
= \sup_{s \in S} |h_b(s) - h(s)|.
\]

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We aim to apply Lemma 29 to show \( h_b \to h \) uniformly. We have that \( \mathcal{S} \) compact. By Lemma 2, we have that \( h(s) \) is continuous. By Lemma 39, we have that each \( h_b \) strictly increasing in \( s \). In addition, we have the following pointwise convergence

\[
\lim_{b \to 0} h_b(s) = \lim_{b \to 0} s - \beta^T x^*_{i:b,\zeta,\xi}(\beta, s) \\
= \lim_{b \to 0} s - \beta^T \left( x^*_i(\beta + b\zeta, s + b\xi) + b \cdot \beta(\zeta^T x^*_i(\beta + b\zeta, s + b\xi) - \xi) \right) \\
= \lim_{b \to 0} s - \beta^T x^*_i(\beta + b\zeta, s + b\xi) - b \cdot (\zeta^T x^*_i(\beta + b\zeta, s + b\xi) - \xi) \\
= s - \beta^T x^*_i(\beta, s) \\
= h(s). 
\]

(G.55) follows from Lemma 49, which gives an explicit expression for \( x^*_{i:b,\zeta,\xi}(\beta, s) \). (G.57) follows from continuity of the best response mapping in \( \beta, s \) (Lemma 2). Thus, \( h_b \to h \) uniformly, so we have that \( \beta^T x^*_{i:b,\zeta,\xi}(\beta, s) \to \beta^T x^*_i(\beta, s) \) uniformly in \( s \).