THE BRUCE-ROBERTS NUMBERS OF A FUNCTION ON AN ICIS

B. K. LIMA-PEREIRA, J.J. NUÑO-BALLESTEROS, B. ORÉFICE-OKAMOTO, J.N. TOMAZELLA

Abstract. We give formulas for the Bruce-Roberts number $\mu_{BR}(f, X)$ and its relative version $\mu_{\tilde{BR}}(f, X)$ of a function $f$ with respect to an ICIS $(X, 0)$. We show that $\mu_{\tilde{BR}}(f, X) = \mu(f^{-1}(0) \cap X, 0) + \mu(X, 0) - \tau(X, 0)$, where $\mu$ and $\tau$ are the Milnor and Tjurina numbers, respectively, of the ICIS. The formula for $\mu_{BR}(f, X)$ is more complicated and also involves $\mu(f)$ and some lengths in terms of the ideals $I_X$ and $J_f$. We also consider the logarithmic characteristic variety, $LC(X)$, and its relative version, $LC(X)^-$. We show that $LC(X)^-$ is Cohen-Macaulay and that $LC(X)$ is Cohen-Macaulay at any point not in $X \times \{0\}$. We generalize previous results presented by the authors when $(X, 0)$ has codimension one and by Bruce and Roberts when it is weighted homogeneous of any codimension.

1. Introduction

An important invariant of the germ of an analytic function $f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$ is its Milnor number, $\mu(f)$, which is equal to $\dim_\mathbb{C} \mathcal{O}_n/J_f$, where $\mathcal{O}_n$ is the ring of germs of analytic functions on $(\mathbb{C}^n, 0)$, and $J_f = \langle \partial f/\partial x_i \rangle$ is the Jacobian ideal of $f$.

Let $(X, 0) \subset (\mathbb{C}^n, 0)$ be the germ of an analytic variety and let $\Theta_X$ be the $\mathcal{O}_n$-submodule of vectors fields that are tangent to $(X, 0)$, that is,

$$\Theta_X = \{ \xi \in \Theta_n | dh(\xi) \in I_X, \forall h \in I_X \},$$

where $\Theta_n$ is the $\mathcal{O}_n$-module of germs of vector fields on $(\mathbb{C}^n, 0)$ and $I_X \subset \mathcal{O}_n$ is the ideal that defines $(X, 0)$. The Bruce-Roberts number and the relative Bruce-Roberts number are, respectively,

$$\mu_{BR}(f, X) = \dim_\mathbb{C} \frac{\mathcal{O}_n}{d(f(\Theta_X))}, \quad \mu_{\tilde{BR}}(f, X) = \dim_\mathbb{C} \frac{\mathcal{O}_n}{d(f(\Theta_X)) + I_X},$$

where $d(f(\Theta_X))$ is the image of $\Theta_X$ by the differential of $f$. These numbers are defined in [4] and may be considered as generalizations of the Milnor number of the function germ, because when $X = \mathbb{C}^n$ then $\Theta_X = \Theta_n$ and $d(f(\Theta_X)) = J_f$.

In general, the computation of both invariants is not easy since the submodule $\Theta_X$ is a complicated object and usually it requires the use of a computer algebra system like SINGULAR [4]. So, it is interesting to obtain formulas which give $\mu_{BR}(f, X)$ or $\mu_{\tilde{BR}}(f, X)$ in terms of other known invariants. The case where $(X, 0)$ is an isolated hypersurface singularity IHS was considered previously in [16, 17, 19]. In this paper we extend the
formulas to the case that \((X, 0)\) is an isolated complete intersection singularity ICIS. Our main results are

\[
\mu_{BR}(f, X) = \mu(X \cap f^{-1}(0), 0) - \mu(X, 0) + \tau(X, 0),
\]

\[
\mu_{BR}(f, X) = \mu_{BR}(f, X) + \mu(f) - \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{J_f + I_X} + \dim_{\mathbb{C}} \frac{I_X \cap J_f}{I_X J_f},
\]

where \(\mu\) and \(\tau\) are the Milnor and Tjurina numbers respectively of an ICIS. We remark that both formulas extend the ones obtained in [16, 17, 19] when \((X, 0)\) is an IHS and that the formula for \(\mu_{BR}(f, X)\) also appears in [4] in the particular case that \((X, 0)\) is a weighted homogeneous ICIS.

Another important property is that, like the Milnor number, the Bruce-Roberts numbers \(\mu_{BR}(f, X)\) and \(\mu_{BR}(f, X)\) may be calculated in terms of the number of stratified critical points of a Morsification of \(f\) with respect to the logarithmic stratification of \(X\). This happens when the logarithmic characteristic variety \(LC(X)\) and its relative version \(LC(X)\) are Cohen-Macaulay. In fact, the Cohen-Macaulayness of \(LC(X)\) and \(LC(X)\) implies the conservation of both numbers in any deformation of \(f\). Many authors have recent papers about these issues [1, 2, 12, 13, 16, 17, 19, 20]. Here we show that if \((X, 0)\) is any ICIS, then \(LC(X)\) is Cohen-Macaulay and \(LC(X)\) is also Cohen-Macaulay at any point not in \(X \times \{0\}\). Again, this extends previous results of [17, 16, 19] when \((X, 0)\) is an IHS and of [14] when \((X, 0)\) is a weighted homogeneous ICIS. We remark that when \((X, 0)\) has codimension > 1, \(LC(X)\) is not Cohen-Macaulay at any point in \(X \times \{0\}\) (see [14 5.10]).

As a byproduct of our process, we also prove that the Tjurina number \(\tau(X, 0)\) of an ICIS \((X, 0)\) can be computed as

\[
\tau(X, 0) = \dim_{\mathbb{C}} \frac{\Theta_X}{\Theta^T_X},
\]

where \(\Theta^T_X\) is the submodule of \(\Theta_X\) of trivial vector fields. This was proved in [16, 20] for IHS.

2. The relative Bruce-Roberts number

When \((X, 0)\) is a weighted homogeneous ICIS, the generators of \(\Theta_X\) are exhibited in [21]. For the non weighted homogeneous case we resort to the trivial vector fields instead, which are defined as

\[
\Theta^T_X = \langle \xi \in \Theta_X; d\phi_i(\xi) = 0, \forall i = 1, \ldots, k \rangle + \langle \phi_j \partial / \partial x_i; j = 1, \ldots, k; i = 1, \ldots, n \rangle.
\]

**Proposition 2.1.** Let \((X, 0)\) be the ICIS determined by \(\phi = (\phi_1, \ldots, \phi_k) : (\mathbb{C}^n, 0) \to (\mathbb{C}^k, 0)\), then

\[
\Theta^T_X = I_{k+1} \begin{pmatrix}
\frac{\partial}{\partial x_1} & \cdots & \frac{\partial}{\partial x_n} \\
\frac{\partial}{\partial x_1} & \cdots & \frac{\partial}{\partial x_n} \\
\vdots & \ddots & \vdots \\
\frac{\partial}{\partial x_1} & \cdots & \frac{\partial}{\partial x_n}
\end{pmatrix} + \left\langle \phi_i \frac{\partial}{\partial x_j}, i = 1, \ldots, k, j = 1, \ldots, n \right\rangle,
\]

where the first term in the right hand side is the submodule of \(\Theta_X\) generated by the \((k+1)\)-minors of the matrix.
To prove the proposition we use the generalized Koszul complex introduced by Buchsbaum and Rim [5]. Let $R$ be a commutative Noetherian ring, $g : R^m \to R^l$ an $R$-homomorphism and 

$$\gamma(g) : R^m \times R^{l*} \to R$$

$$(b, a) \mapsto a(g(b))$$

with $R^{l*} = \text{Hom}(R^l, R)$.

The generalized Koszul complex, $K(p \wedge g)$, for each $p$, is defined by:

\[ \ldots \to \sum_{s_0 \geq l+1-p} R^{l*} \otimes \bigwedge^{p+s_0} R^m \to \sum_{s_0 \geq l+1-p} \bigwedge^{p+s_0} R^l \otimes \bigwedge^p R^l \to \bigwedge^p R^m \to 0 \]

with $s_i \geq 1$ for all $i \geq 1$. The differential

$$d : \sum_{s_0 \geq l+1-p} \bigwedge^{p+s_0} R^l \otimes \bigwedge^p R^m \to \bigwedge^p R^m$$

is defined by:

$$d(\alpha \otimes \beta) = \sum_{1 \leq j_1 < \ldots < j_p \leq p+1} (-1)^{j_k} \det(\gamma(a_{i}, b_{j_k})) b_1 \wedge \ldots \wedge \hat{b}_{j_1} \wedge \ldots \wedge \hat{b}_{j_p} \wedge \ldots \wedge b_{p+s_0},$$

where $\alpha = a_1 \wedge \ldots \wedge a_{s_0} \in \bigwedge^{p+s_0} R^l$ and $\beta = b_1 \wedge \ldots \wedge b_{p+s_0} \in \bigwedge^p R^m$.

If $\text{coker}(g) \neq 0$, $K(p \wedge g)$ is a free resolution of $\text{coker}(p \wedge g)$ for some $p$, $1 \leq p \leq l$ (or for all $p$, $1 \leq p \leq l$) if and only if $\text{depth}(I(g), R) = m - l + 1$, where $I(g)$ is the annihilator of $\text{coker}(\bigwedge g)$ ([5, Corollary 2.6]).

**Proof of Proposition 2.1.** We consider the previous complex with the following homomorphism of $\mathcal{O}_n$-modules

$$d\phi = \left( \frac{\partial \phi_1}{\partial x_1} \quad \ldots \quad \frac{\partial \phi_k}{\partial x_1} \quad \ldots \quad \frac{\partial \phi_1}{\partial x_n} \quad \ldots \quad \frac{\partial \phi_k}{\partial x_n} \right) : \mathcal{O}_n^k \to \mathcal{O}_n^k.$$

Here $I(d\phi)$ is the annihilator of $\text{coker}(\bigwedge^k d\phi) = \mathcal{O}_n^k / J(\phi_1, \ldots, \phi_k)$, that is, $I(d\phi)$ is the ideal generated by the minors of maximum order of the Jacobian matrix of $\phi = (\phi_1, \ldots, \phi_k)$ and

$$\dim \frac{\mathcal{O}_n^k}{I(d\phi)} = k - 1 = n - (n - k + 1)(k - k + 1).$$

Therefore, $K(p \wedge d\phi)$ is a free resolution of $\text{coker}(p \wedge d\phi)$, for all $p$, $1 \leq p \leq k$.

Considering $p = 1$, $K(1 \wedge d\phi) = K(d\phi)$ is an exact sequence and the final part of (1) in this case is equal to

$$\bigwedge^k \mathcal{O}_n^{k*} \otimes \bigwedge^{1+k} \mathcal{O}_n^k \to \mathcal{O}_n^k \to \mathcal{O}_n^k.$$
and
\[
\text{Im}(d) = I_{k+1} \left( \begin{array}{cccc}
\frac{\partial}{\partial x_1} & \ldots & \frac{\partial}{\partial x_n} \\
\frac{\partial}{\partial \phi_1} & \ldots & \frac{\partial}{\partial \phi_n} \\
\vdots & \ddots & \vdots \\
\frac{\partial}{\partial x_1} & \ldots & \frac{\partial}{\partial x_n}
\end{array} \right) = \ker(d\phi).
\]

As a consequence of the previous proposition, for any function germ \( f : (\mathbb{C}^n, 0) \to \mathbb{C} \),
\[
(2) \quad df(\Theta_X^T) = J(f, \phi) + \langle \phi_j \frac{\partial}{\partial \phi_i} \rangle_{i = 1, \ldots, n; \ j = 1, \ldots, k},
\]
where \( J(f, \phi) \) is the ideal in \( \mathcal{O}_n \) generated by the maximal minors of the Jacobian matrix of \( (f, \phi_1, \ldots, \phi_k) \).

**Theorem 2.2.** Let \((X, 0) \subset (\mathbb{C}^n, 0)\) be an ICIS and \( f \in \mathcal{O}_n \) such that \( \mu_{BR}(f, X) < \infty \), then \((X \cap f^{-1}(0), 0)\) defines an ICIS and
\[
\mu(X \cap f^{-1}(0), 0) = \mu_{BR}(f, X) - \mu(X, 0) + \tau(X, 0).
\]

**Proof.** Let \( \phi = (\phi_1, \ldots, \phi_k) : (\mathbb{C}^n, 0) \to (\mathbb{C}^k, 0) \) be the map which defines \((X, 0)\). From the equality \([11]\) and the exact sequence
\[
0 \to df(\Theta_X^T) + I_X \to \mathcal{O}_n \to df(\Theta_X) + I_X \to 0,
\]
we have
\[
\mu(X \cap f^{-1}(0), 0) = \mu_{BR}(f, X) + \dim_{\mathbb{C}} \frac{df(\Theta_X^T) + I_X}{df(\Theta_X) + I_X} - \mu(X, 0).
\]
Then we need to prove
\[
\dim_{\mathbb{C}} \frac{df(\Theta_X^T) + I_X}{df(\Theta_X) + I_X} = \tau(X, 0) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n^k}{\text{Im} d\phi + I_X \mathcal{O}_n},
\]
where the second equality is a characterization for Tjurina number, (see \([11]\) Theorem 1.16)). Let us consider the following exact sequence
\[
0 \to \ker(\alpha) \xrightarrow{i} \mathcal{O}_n \xrightarrow{\pi} \mathcal{O}_{n+1} \xrightarrow{\pi} \mathcal{O}_n \to 0,
\]
in which \( i \) is the inclusion and \( \pi \) and \( \overline{\pi} \) (respectively) are induced by
\[
\pi : \mathcal{O}_{n+1} \to \mathcal{O}_n \quad \text{and} \quad \overline{\pi} : \mathcal{O}_n \to \mathcal{O}_{n+1}
\]
given by \( \pi(a_1, a_2, \ldots, a_k) = (a_2, \ldots, a_k) \) and \( \alpha(a) = (a, 0, \ldots, 0) \).

Since this ring \( \mathcal{O}_n/I_X \) is Cohen-Macaulay and the Jacobian matrix of \((f, \phi)\) is a parameter matrix in the sense of \([5]\) for this ring, we have
\[
\dim_{\mathbb{C}} \frac{\mathcal{O}_n}{df(\Theta_X^T) + I_X} = \dim_{\mathbb{C}} \frac{\mathcal{O}_{n+1}}{\text{Im}(d(f, \phi)) + I_X \mathcal{O}_n},
\]
see \([5]\) p. 224]. Hence \( \dim_{\mathbb{C}} \ker(\overline{\pi}) = \tau(X, 0) \) and we claim that
\[
\ker(\overline{\pi}) = \frac{df(\Theta_X) + I_X}{df(\Theta_X^T) + I_X}.
\]
Let \( \overline{\alpha} \in \ker(\overline{\pi}) \), therefore there exists \( \xi \in \Theta_n \) such that
\[
(a, 0, \ldots, 0) = (df(\xi), d\phi_1(\xi), \ldots, d\phi_k(\xi)) + (\alpha_1, \ldots, \alpha_{k+1}),
\]
with $\alpha_i \in I_X \forall i = 1, \ldots, k + 1$. Thus $a - a_1 = df(\xi)$ and hence $a = df(\Theta_X) + I_X$.

For the other inclusion, if $\xi \in \Theta_X$, then $df(\xi) \in (df(\Theta_X) + I_X)/(df(\Theta_X^0) + I_X)$ and
$$\alpha(df(\xi)) = (df(\xi), 0, \ldots, 0) = (df(\xi), d\phi_1(\xi), \ldots, d\phi_k(\xi)) \in \text{Im}(d(f, \phi)) .$$

It is curious that by the previous result the dimension of the quotient $(df(\Theta_X) + I_X)/(df(\Theta_X^0) + I_X)$ as a $\mathbb{C}$-vector space does not depend of the function germ $f$ such that $(X \cap f^{-1}(0), 0)$ defines an ICIS.

**Corollary 2.3.** Let $(X, 0)$ be a weighted homogeneous ICIS, then
$$\mu(X, 0) = \tau(X, 0) .$$

**Proof.** It follows of Theorem 2.2 and [4] Proposition 7.7. □

**Corollary 2.4.** The relative Bruce-Roberts number is a topological invariant for a family of functions germs over a fixed ICIS.

**Corollary 2.5.** If $(X, 0)$ is an ICIS and $f$ is an $\mathcal{R}_X$-finitely determined function germ, then $\mu_{BR}(f, X) = \dim_{\mathbb{C}} \mathcal{O}_n/(J(f, \phi) + I_X) - \tau(X, 0)$.

From the previous corollary, if $(X, 0)$ is an ICIS of codimension $k$ in $\mathbb{C}^n$,
$$\mu_{BR}(p, X) = m_{n-k}(X, 0) - \tau(X, 0) ,$$
where $m_{n-k}$ is the $(n-k)$th polar multiplicity as defined in [4] and $p : \mathbb{C}^n \to \mathbb{C}$ is a generic linear projection. Hence
$$m_{n-k}(X, 0) = \mu_{BR}(p, X) + \tau(X, 0) = \mu_{BR}(p, X) + \tau(X, 0) ,$$
because $\mu_{BR}(p, X) = \mu_{\bar{BR}}(p, X)$.

Moreover, proceeding as in [16], by [14] and [15],
$$(-1)^{n-k-1} - \mu(X, 0) = \text{Eu}(X, 0) - m_{n-k}(X, 0) ,$$
where $\text{Eu}(X, 0)$ is the local Euler obstruction of $(X, 0)$.

**Corollary 2.6.** Let $(X, 0)$ be an ICIS of codimension $k$ and $p : \mathbb{C}^n \to \mathbb{C}$ a generic linear projection, then

a) $m_{n-k}(X, 0) = \mu_{BR}(p, X) + \tau(X, 0)$;
b) $\text{Eu}(X, 0) = \mu_{BR}(p, X) + \tau(X, 0) - \mu(X, 0) + (-1)^{n-k-1} .$$

3. THE RELATIVE LOGARITHMIC CHARACTERISTIC VARIETY

We recall the definition of the logarithmic characteristic variety of $(X, 0)$. Assume that the module $\Theta_X$ is generated over $\mathcal{O}_n$ by vector fields $\xi_1, \ldots, \xi_k$ and put
$$\xi_i = \sum_{j=1}^n a_{ij} \frac{\partial}{\partial x_j}, \ a_{ij} \in \mathcal{O}_n .$$

We denote by $x_1, \ldots, x_n, p_1, \ldots, p_n$ the coordinates of the cotangent bundle $T^*\mathbb{C}^n$. For each $i = 1, \ldots, k$ we define $\xi_i = \sum_{j=1}^n a_{ij} p_j \in \mathcal{O}_n[p_1, \ldots, p_n]$. The logarithmic characteristic variety $LC(X)$ is the complex subspace of $T^*\mathbb{C}^n$ given by the ideal $I$ in $\mathcal{O}_n[p_1, \ldots, p_n]$ generated by $\xi_1, \ldots, \xi_k$. Observe that $LC(X)$ is a germ in $T^*\mathbb{C}^n$ along the subset $T_X^0 \mathbb{C}^n$. The definition of $LC(X)$ does not depend on the choice of generators of $\Theta_X$ (see [4]).
Another property of $LC(X)$ is that $\dim LC(X) = n$ if and only if $(X, 0)$ is holonomic (see [4, Proposition 1.14]). In fact, if $X_0, \ldots, X_r$ are the logarithmic strata, then, as set germs,

$$LC(X) = \bigcup_{i=0}^{r} N^*X_i,$$

where $N^*X_i$ is the closure of the conormal bundle $N^*X_i$ of $X_i$ in $T^*\mathbb{C}^n$. In particular, if we assume $X_0 = \mathbb{C}^n \setminus X$, then $N^*X_0 = \mathbb{C}^n \times \{0\}$, the zero section given by the ideal $P = \langle p_1, \ldots, p_n \rangle$. Obviously $I \subseteq P$ and the relative logarithmic characteristic variety $LC(X)^- = \pi^{-1}(X)$ is the complex subspace of $T^*\mathbb{C}^n$ given by the quotient ideal $I: P$. As set germs,

$$LC(X)^- = \bigcup_{i=1}^{r} N^*X_i = LC(X) \cap \pi^{-1}(X),$$

where $\pi : T^*\mathbb{C}^n \to \mathbb{C}^n$ is the projection.

In [4, Proposition 5.10] they show that if $(X, 0)$ has codimension $> 1$ in $(\mathbb{C}^n, 0)$, then $LC(X)$ is not Cohen-Macaulay at any point in $X \times \{0\} \subset LC(X)$. However, they also show in [4, Proposition 7.3] that if $(X, 0)$ is is a weighted homogeneous ICIS then $LC(X)^-$ is Cohen-Macaulay at any point and $LC(X)$ is Cohen-Macaulay at any point not in $X \times \{0\}$. We extend both results for any ICIS, non necessarily weighted homogeneous.

**Theorem 3.1.** If $(X, 0)$ is an ICIS, then $LC(X)^-$ is Cohen-Macaulay.

**Proof.** This proof is motivated by [16, Theorem 5.4]. We suppose that $(X, 0)$ is determined by $\phi : (\mathbb{C}^n, 0) \to (\mathbb{C}^k, 0)$. Let $(0, p) \in LC(X)^-$. Since $LC(X)^- \subset LC(X)$ there exists $f \in \mathcal{O}_n$ a $\mathcal{R}_X$-finitely determined function germ such that $df(0) = p$. By Corollary 2.5,

$$\mu_{BR}(f, X) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{J(f, \phi) + I_X} - \tau(X, 0).$$

Let $F : (\mathbb{C} \times \mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be given by $F(t, x) = f_t(x)$ a Morsification of $f$. We consider $R = \mathcal{O}_{n+1}/I_X$ and $I = (J(f_t, \phi) + I_X)/I_X$. The ring $R$ is Cohen-Macaulay of dimension $n - k + 1$ and $f$ is generated by the minors of order $k + 1$ of a matrix of size $(k + 1) \times n$. Since $\dim R/I = 1 = \dim R - (n - (k + 1) + 1)(k + 1 - (k + 1) + 1)$, then by Eagon-Hochster, $R/I$ is determinantal and therefore Cohen-Macaulay. By conservation of multiplicity, for all $t \neq 0$,

$$\dim_{\mathbb{C}} \frac{\mathcal{O}_n}{J(f_t, \phi) + I_X} = \sum_{i=1}^{s+1} \sum_{x \in X_i} \dim_{\mathbb{C}} \frac{\mathcal{O}_{n,x}}{J(f_t, \phi) + I_X},$$

where $X_i$ are the logarithmic strata of $(X, 0)$ as defined in the beginning of the section.

When $i = 1, \ldots, s$ and $x \in X_i$, $X$ is smooth at $x$, so

$$\sum_{x \in X_i} \dim_{\mathbb{C}} \frac{\mathcal{O}_{n,x}}{J(f_t, \phi) + I_X} = \sum_{x \in X_i} \mu_{BR}(f_t, X)_x^{(*)} \sum_{x \in X_i \cap \Sigma f_t} m_i = n_i m_i.$$

The equality $(*)$ is consequence of [17, Lemma 4.4]. For $i = s + 1$, we have just one critical point $x = 0$ in $X_{s+1} = \{0\}$, thus

$$\dim_{\mathbb{C}} \frac{\mathcal{O}_n}{J(f_t, \phi) + I_X} = \mu_{BR}(f_t, X) + \tau(X, 0) = n_{s+1} m_{s+1} + \tau(X, x).$$
Summing up for all $i = 1, \ldots, s + 1$ we have
\[
\dim_{\mathbb{C}} \frac{\mathcal{O}_n}{J(f, \phi) + I_X} = \sum_{i=1}^{s+1} n_im_i + \tau(X,0)
\]
and hence $\mu_{BR}(f, X) = \sum_{i=1}^{s+1} n_im_i$. Therefore, $LC(X)^-\!$ is Cohen-Macaulay by \cite{4} Proposition 5.11] □

**Corollary 3.2.** If $(X, 0)$ is an ICIS, then $LC(X)$ is Cohen-Macaulay at all points not in $X \times \{0\}$.

**Proof.** Let $(x, p) \in LC(X)$ such that $(x, p) \notin X \times \{0\}$. If $x \in X$ then $p \neq 0$ and $LC(X)$ coincides, as a complex space, with $LC(X)^-\!$ on an open neighbourhood of $(x, p)$ in $T^*\mathbb{C}^n$. Hence, $LC(X)$ is Cohen-Macaulay at $(x, p)$ by Theorem 3.1.

Otherwise, if $x \notin X$ then $LC(X)$ coincides, as a complex space, with $(\mathbb{C}^n \setminus X) \times \{0\}$ on an open neighbourhood of $(x, p)$ in $T^*\mathbb{C}^n$. Again, $LC(X)$ is also Cohen-Macaulay at $(x, p)$. □

The following corollaries are also motivated by \cite{4} Proposition 7.4 and Corollary 7.6]

**Corollary 3.3.** Let $(X, 0)$ be an ICIS and $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ a function germ. If $f$ has an isolated critical point at $x$, then
\[
\mu_{BR}(f, X)_x \geq \mu_{BR}(f, X)_x + \mu(f)_x,
\]
with equality if either $x \in \mathbb{C}^n \setminus X$ or $df(x) \neq 0$.

Moreover, if $x \in \mathbb{C}^n \setminus X$ then $\dim_{\mathbb{C}} \mathcal{O}_{n,x}/df(\Theta^-_{X,x}) = 0$ while if $df(x) \neq 0$, $\mu(f)_x = 0$.

If $(X, 0)$ is not an isolated hypersurface singularity then the above sufficient conditions for the equality are also necessary.

**Proof.** By \cite{4} Corollary 5.8 and Propositions 5.11, 5.14]
\[
\mu_{BR}(f, X)_x \geq \sum_{i=0}^{k+1} m_in_i = \mu_{BR}(f, X)_x + m_0n_0 = \mu_{BR}(f, X)_x + \mu(f)_x.
\]
The other statements are consequences of the previous corollary and \cite{4} Proposition 5.8]. □

**Corollary 3.4.** Let $(X, 0)$ be an ICIS and $f \in \mathcal{O}_n$ be a function germ $\mathcal{R}_X$-finitely determined. If $x$ is a critical point of $f$ then
\[
\sum_{i=1}^{k} n_i + m_{k+1} = \mu_{BR}(f, X)_x
\]
\[
\sum_{i=0}^{k} n_i + m_{k+1} \leq \mu_{BR}(f, X)_x.
\]
Let \((X, 0)\) be an ICIS determined by \(\phi : (\mathbb{C}^n, 0) \to (\mathbb{C}^k, 0)\). Bruce and Roberts define in \([4]\) \(LC(X)^T\) as the complex subspace of \(T^*\mathbb{C}^n\) given by \(\phi_i, i = 1, ..., k\) and

\[
I_{k+1} = \begin{pmatrix}
\frac{\partial}{\partial x_1} & \cdots & \frac{\partial}{\partial x_n} \\
\frac{\partial \phi_1}{\partial x_1} & \cdots & \frac{\partial \phi_k}{\partial x_n}
\end{pmatrix}
\]

As observed in \([4]\), the logarithmic stratification obtained by integration of the vectors fields given by the minors

\[
I_{k+1} = \begin{pmatrix}
\frac{\partial}{\partial x_1} & \cdots & \frac{\partial}{\partial x_n} \\
\frac{\partial \phi_1}{\partial x_1} & \cdots & \frac{\partial \phi_k}{\partial x_n}
\end{pmatrix}
\]

is still holonomic. Actually it is the same as that given by \(\Theta_X\), so \(LC(X)^T\) is \(n\)-dimensional with the same irreducible components as \(LC(X)^-\). Let \(Y_i, i = 1, ..., k\) the irreducible components of \(LC(X)^T\), then \(Y_i\) has multiplicity \(m_i = 1, i = 1, ..., k\) and \(Y_{k+1}\) has multiplicity denoted by \(m(X, 0)^T\). In general \(m(X, 0)^T\) is greater than than the multiplicity of \(Y_{k+1}\) in \(LC(X)^-, m_{k+1}\). The principal advantage of considering \(LC(X)^T\) is that it is Cohen-Macaulay for any ICIS (see \([4]\) Proposition 7.10]).

**Proposition 3.5.** Let \((X, 0)\) be an ICIS and \(f \in \mathcal{O}_n\) an \(R_X\)-finitely determined function germ, then

\[m(X, 0)^T - m_{k+1} = \tau(X, 0).\]

**Proof.** It is a consequence of Corollary \([3, \text{Corollary 3.4}]\) and \([4, \text{Corollary 7.11}]\).

Let \(f \in \mathcal{O}_n\) be an \(R_X\)-finitely determined function germ and \(F : (\mathbb{C}^n \times \mathbb{C}, 0) \to (\mathbb{C}, 0)\), \(F(x, t) = f_t(x)\), a 1-parameter deformation of \(f\). The polar curve of \(F\) with respect to \((X, 0)\) is

\[C = \{(x, t) \in \mathbb{C}^n \times \mathbb{C}; df_t(\delta_i) = 0 \forall i = 1, ..., m\},\]

where \(\Theta_X = \langle \delta_1, ..., \delta_m \rangle\). The relative polar curve is the set of points \((x, t)\) in \(C\) such that \(x \in X\). As a consequence of Theorem 3.4 (see \([17, \text{Proposition 4.3}]\)), \(C^-\) is Cohen-Macaulay and

\[\mu^-_{BR}(f, X) = \sum_{x \in \mathbb{C}^n} \mu^-_{BR}(f_t, X)_x.\]

4. **The Bruce-Roberts number**

In \([16]\), we prove that if \((X, 0)\) is an IHS and \(f\) an \(R_X\)-finitely determined then the Bruce-Roberts number of \(f\) with respect to \(X\) and the Milnor number of \(f\) are related by

\[\mu_{BR}(f, X) = \mu(f) + \mu(X \cap f^{-1}(0), 0) + \mu(X, 0) - \tau(X, 0).\]

In this section, we study how these invariants are related when \((X, 0)\) is an ICIS.

**Proposition 4.1.** Let \((X, 0)\) be an ICIS and \(f \in \mathcal{O}_n\) a function germ. Then

\[\mu_{BR}(f, X) < \infty\text{ if, and only if, } \dim_{\mathbb{C}} \mathcal{O}_n/df(\Theta_X^T) < \infty.\]

The proof of this result follows the same idea of Lemma 2.3 in \([16]\).
Proposition 4.2. Let \((X, 0)\) be an ICIS determined by \((\phi_1, \ldots, \phi_k) : (\mathbb{C}^n, 0) \to (\mathbb{C}^k, 0)\) and \(f \in \mathcal{O}_n\) an \(\mathcal{R}_X\)-finitely determined function germ then

\[
\mu_{BR}(f, X) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n^k}{Jf \mathcal{O}_n^k + \langle \phi_i e_j - \phi_j e_i \rangle} + \mu(X \cap f^{-1}(0), 0) + \mu(X, 0) - \dim_{\mathbb{C}} \frac{df(\Theta_X^T)}{df(\Theta_X^T)}.
\]

Proof. We consider the following sequence

\[
0 \to \frac{\mathcal{O}_n^k}{Jf \mathcal{O}_n^k + \langle \phi_i e_j - \phi_j e_i \rangle} \xrightarrow{\alpha} \frac{\mathcal{O}_n}{df(\Theta_X^T)} \xrightarrow{\pi} \frac{\mathcal{O}_n}{df(\Theta_X^T) + \langle \phi_1, \ldots, \phi_k \rangle} \to 0
\]

with \(\pi\) the projection and \(\alpha((a_1, \ldots, a_k) + Jf \mathcal{O}_n^k + \langle \phi_i e_j - \phi_j e_i \rangle) = \Sigma_{i=1}^k a_i \phi_i + df(\Theta_X^T)\). Obviously, \(\pi\) is an epimorphism and \(\text{Im} \alpha = \ker \pi\). To see the exactness of the sequence it only remains to see that \(\alpha\) is a monomorphism.

Since \(f\) is \(\mathcal{R}_X\)-finitely determined, \((f, \phi_1, \ldots, \phi_k)\) defines an ICIS \([2\text{ Proposition 2.8}]\), which implies

\[
\dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\langle \phi_1, \ldots, \phi_k \rangle + J(f, \phi)} = 0.
\]

This gives \(\dim \mathcal{O}_n/J(f, \phi) \leq n - k\). Since \(J(f, \phi)\) is the ideal generated by the maximal minors of a matrix of size \(n \times (k + 1)\), \(\mathcal{O}_n/J(f, \phi)\) is determinantal, and hence, Cohen-Macaulay of dimension \(n - k\). Again by \([3]\) we conclude that \(\phi_1 + J(f, \phi), \ldots, \phi_k + J(f, \phi)\) is a regular sequence in \(\mathcal{O}_n/J(f, \phi)\).

We prove now that \(\alpha\) is a monomorphism. Let \((a_1, \ldots, a_k) \in \mathcal{O}_n^k\) be such that \(\Sigma_{i=1}^k a_i \phi_i \in df(\Theta_X^T)\). By \([2]\), \(df(\Theta_X^T) = Jf \langle \phi_1, \ldots, \phi_k \rangle + J(f, \phi)\), so there exist \(\alpha_1, \ldots, \alpha_k \in Jf\) such that

\[
\sum_{i=1}^k a_i \phi_i - \sum_{i=1}^k \alpha_i \phi_i = \sum_{i=1}^k (a_i - \alpha_i) \phi_i \in J(f, \phi).
\]

By the regularity of the classes of \(\phi_1, \ldots, \phi_k\) in \(\mathcal{O}_n/J(f, \phi)\),

\[
(a_1 - \alpha_1, \ldots, a_k - \alpha_k) \in \langle \phi_i e_j - \phi_j e_i \rangle + J(f, \phi) \mathcal{O}_n^k,
\]

which implies \((a_1, \ldots, a_k) \in \langle \phi_i e_j - \phi_j e_i \rangle + J(f) \mathcal{O}_n^k\).

Finally, the exactness of the sequence implies

\[
\mu_{BR}(f, X) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{df(\Theta_X^T)} - \dim_{\mathbb{C}} \frac{df(\Theta_X^T)}{df(\Theta_X^T)}
\]

\[
= \dim_{\mathbb{C}} \frac{\mathcal{O}_n^k}{Jf \mathcal{O}_n^k + \langle \phi_i e_j - \phi_j e_i \rangle} + \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{df(\Theta_X^T) + \langle \phi_1, \ldots, \phi_k \rangle} - \dim_{\mathbb{C}} \frac{df(\Theta_X^T)}{df(\Theta_X^T)}
\]

\[
= \dim_{\mathbb{C}} \frac{\mathcal{O}_n^k}{Jf \mathcal{O}_n^k + \langle \phi_i e_j - \phi_j e_i \rangle} + \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{J(f, \phi) + \langle \phi_1, \ldots, \phi_k \rangle} - \dim_{\mathbb{C}} \frac{df(\Theta_X^T)}{df(\Theta_X^T)}
\]

\[
= \dim_{\mathbb{C}} \frac{\mathcal{O}_n^k}{Jf \mathcal{O}_n^k + \langle \phi_i e_j - \phi_j e_i \rangle} + \mu(X \cap f^{-1}(0), 0) + \mu(X, 0) - \dim_{\mathbb{C}} \frac{df(\Theta_X^T)}{df(\Theta_X^T)},
\]

where the last equality follows from the Lé-Greuel formula.

We need another characterization for the Tjurina number which is a generalization for the IHS case obtained in \([16, 20]\).
A vector field $\xi$ belongs to $\Theta_X$ if and only if there exist a matrix $[\lambda_{ij}]$ in the set of the matrices of order $k$ with elements in $O_n$, $M_k(O_n)$, such that

$$d\phi(\xi) = [\lambda_{ij}][\phi]^T$$

(4)

The matrix $[\lambda_{ij}]$ is not unique, so we consider the class $[\lambda_{ij}]$ in $M_k(O_n)/H$, where $H$ is the submodule generated by the matrices whose rows belong to syzygy($\phi_1, \ldots, \phi_k$).

**Lemma 4.4.** Let $(X, 0)$ be an ICIS determined by $\phi = (\phi_1, \ldots, \phi_k) : (\mathbb{C}^n, 0) \to (\mathbb{C}^k, 0)$. Then $\xi \in \Theta_X$ if and only if there exist a matrix $[\lambda_{ij}] \in (T + H)/H$ which satisfies (4), where $T$ is the submodule generated by the matrices $T_{lm}$, $l = 1, \ldots, k$; $m = 1, \ldots, n$, such that $l$-th column is equal to $m$-th column of the jacobian matrix of $\phi$ and the other columns are null.

**Proof.** Let $\xi \in \Theta_n$ and $[\lambda_{ij}] + H \in (T + H)/(H)$, such that (4) holds, so there exist $\alpha_{ij} \in O_n$ and matrices $[h_{lm}] \in H$ such that

$$[\lambda_{ij}] = \sum_{j=1}^n \sum_{i=1}^p \alpha_{ij}T_{ij} + [h_{lm}].$$

Therefore,

$$d\phi(\xi) = [\lambda_{ij}][\phi]^T = \left(\sum_{j=1}^n \sum_{i=1}^p \alpha_{ij}T_{ij} + [h_{lm}]\right)[\phi]^T = \left(\sum_{j=1}^n \sum_{i=1}^p \alpha_{ij}T_{ij}\right)[\phi]^T = d\phi(\eta),$$

with $\eta = (\sum_{i=1}^k \alpha_{i1}\phi_1, \ldots, \sum_{i=1}^k \alpha_{in}\phi_l) \in \Theta_X^T$, and $\xi \in \Theta_X^T$.

The converse is immediate. $\square$

When $(X, 0)$ is an IHS and $f$ is an $R_X$-finitely determined function germ, any $\xi \in \Theta_X$ such that $df(\xi) \in I_X$ is trivial. We will use this result for any ICIS, in order to prove it we need the following lemma.

**Lemma 4.4.** Let $(X, 0)$ an ICIS determined by $\phi = (\phi_1, \ldots, \phi_k) : (\mathbb{C}^n, 0) \to (\mathbb{C}^k, 0)$. Then $\phi_1, \ldots, \phi_{k-1}$ is a regular sequence in $O_n^k/\text{Im}(d\phi)$.

**Proof.** We denote $M = O_n^k/\text{Im}(d\phi)$. The sequence

$$O_n \xrightarrow{d\phi} O_n^k \to M \to 0,$$

is a presentation of $M$, the 0-th fitting ideal of $M$ is $F_0(M) = J(\phi_1, \ldots, \phi_k)$ and

$$\dim(M) = \dim \frac{O_n}{\text{Ann}(M)} = \dim \frac{O_n}{J(\phi_1, \ldots, \phi_k)} = k - 1.$$

Hence, $M$ is a $O_n$-module Cohen-Macaualy by [5].

The module $M_X = O_X^k/\text{Im}(d\phi) \approx O_n^{k+1}/(\text{Im}(d\phi) + \langle \phi_1, \ldots, \phi_k \rangle O_n^k)$ has the following presentation

$$O_X \xrightarrow{d\phi} O_X^k \to M_X \to 0,$$

and the 0-th fitting ideal is $J(\phi_1, \ldots, \phi_k)$. Thus,

$$\dim(M_X) = \dim \frac{O_X}{\text{Ann}(M_X)} = \dim \frac{O_n}{\langle \phi_1, \ldots, \phi_{k-1} \rangle + J(\phi_1, \ldots, \phi_k)} = 0,$$

and therefore $\phi_1, \ldots, \phi_{k-1}$ is a regular sequence in $O_n^k/\text{Im}(d\phi)$. $\square$
Theorem 4.5. Let $(X, 0)$ be an ICIS determined by $(\phi_1, ..., \phi_k) : (\mathbb{C}^n, 0) \to (\mathbb{C}^k, 0)$ and $f \in \mathcal{O}_n$ an $\mathcal{R}_X$-finitely determined function germ. If $\xi \in \Theta_X$ and $df(\xi) \in I_X$, then $\xi \in \Theta_X^f$. In particular the evaluation map $E : \Theta_X \to df(\Theta_X)$ given by $E(\xi) = df(\xi)$ induces an isomorphism

$$\overline{E} : \frac{\Theta_X}{\Theta_X^f} \to \frac{df(\Theta_X)}{df(\Theta_X^f)}.$$ 

Proof. Since $\xi \in \Theta_X$ and $df(\xi) \in I_X$, there exist $\lambda_{ij}, \mu_j \in \mathcal{O}_n$ with $i, j = 1, ..., k$ such that $d\phi_i(\xi) = \sum_{j=1}^k \lambda_{ij} \phi_j$ and $df(\xi) = \sum_{j=1}^k \mu_j \phi_j$. This gives

$$d(f, \phi)(\xi) = \sum_{j=1}^k \phi_j \begin{pmatrix} \mu_j \\ \lambda_{1j} \\ \vdots \\ \lambda_{kj} \end{pmatrix} = \sum_{j=1}^k \phi_j v_j^T,$$

where $v_j^T$ is the transpose of $v_j = (\mu_j, \lambda_{1j}, ..., \lambda_{kj})$. Therefore, $\sum_{j=1}^k \phi_j v_j^T \in \text{Im} d(f, \phi)$ and

$$\phi_k v_k \in \text{Im} d(f, \phi) + \langle \phi_1, ..., \phi_{k-1} \rangle \mathcal{O}_n^{k+1}.$$ 

However, since $f$ is $\mathcal{R}_X$-finitely determined, $(f, \phi)$ defines an ICIS. By Lemma 4.4, $\phi_1, ..., \phi_k$ is a regular sequence in $\mathcal{O}_n^{k+1}/\text{Im} d(f, \phi)$, so

$$v_k \in \text{Im} d(f, \phi) + \langle \phi_1, ..., \phi_{k-1} \rangle \mathcal{O}_n^{k+1}.$$ 

That is, there exist $w_k \in \text{Im} d(f, \phi)$ and $a_{ij}^k \in \mathcal{O}_n$, such that

$$v_k = w_k + \sum_{i=1}^{k-1} \sum_{j=1}^{k+1} a_{ij}^k \phi_i e_j.$$ 

Therefore,

$$\sum_{j=1}^k \phi_i v_i = \phi_1 v_1 + ... + \phi_k \left( w_k + \sum_{i=1}^{k-1} \sum_{j=1}^{k+1} a_{ij}^k \phi_i e_j \right)$$ 

$$= \phi_1 \left( v_1 + \sum_{j=1}^{k+1} a_{1j}^k \phi_k e_j \right) + ... + \phi_{k-1} \left( v_{k-1} + \sum_{j=1}^{k+1} a_{(k-1)j}^k \phi_k e_j \right) + \phi_k w_k.$$ 

Hence,

$$\phi_{k-1} \left( v_{k-1} + \sum_{j=1}^{k+1} a_{(k-1)j}^k \phi_k e_j \right) \in \text{Im} d(f, \phi) + \langle \phi_1, ..., \phi_{k-2} \rangle \mathcal{O}_n^{k+1},$$

and thus,

$$v_{k-1} + \sum_{j=1}^{k+1} a_{(k-1)j}^k \phi_k e_j \in \text{Im} d(f, \phi) + \langle \phi_1, ..., \phi_{k-2} \rangle \mathcal{O}_n^{k+1}.$$ 

That is, there exist $w_{k-1} \in \text{Im} d(f, \phi)$ and $a_{ij}^{k-1} \in \mathcal{O}_n$ such that

$$v_{k-1} + \sum_{j=1}^{k+1} a_{(k-1)j}^k \phi_k e_j = w_{k-1} + \sum_{i=1}^{k-2} \sum_{j=1}^{k+1} a_{ij}^{k-1} \phi_i e_j.$$
We have
\[
\phi_1 \left( v_1 + \sum_{j=1}^{k+1} a_{1j}^k \phi_k e_j \right) + \cdots + \phi_{k-1} \left( v_{k-1} + \sum_{j=1}^{k+1} a_{(k-1)j}^k \phi_k e_j \right) + \phi_k w_k = \\
\phi_1 \left( v_1 + \sum_{j=1}^{k+1} a_{1j}^k \phi_k e_j \right) + \cdots + \phi_{k-1} \left( w_{k-1} + \sum_{j=1}^{k-2} \sum_{i=1}^{k+1} a_{ij}^{k-1} \phi_i e_j \right) + \phi_k w_k = \\
\phi_1 \left( v_1 + \sum_{j=1}^{k+1} a_{1j}^k \phi_k e_j \right) + \cdots + \phi_{k-1} \left( v_{k-2} + \sum_{j=1}^{k+1} a_{(k-2)j}^k \phi_k e_j + \sum_{j=1}^{k+1} a_{(k-2)j}^{k-1} \phi_{k-1} e_j \right) + \phi_{k-1} w_{k-1} + \phi_k w_k
\]
and again
\[
\phi_{k-2} \left( v_{k-2} + \sum_{j=1}^{k+1} a_{(k-2)j}^k \phi_k e_j + \sum_{j=1}^{k+1} a_{(k-2)j}^{k-1} \phi_{k-1} e_j \right) \in \text{Im} \, d(f, \phi) + \langle \phi_1, ..., \phi_{k-3} \rangle \mathcal{O}_n^{k+1}.
\]
Proceeding like this, we show that
\[
\phi_1 \left( v_1 + \sum_{j=1}^{k+1} a_{1j}^k \phi_k e_j + \cdots + \sum_{j=1}^{k+1} a_{3j}^3 \phi_3 e_j \right) + \phi_2 \left( v_2 + \sum_{j=1}^{k+1} a_{2j}^k \phi_k e_j + \cdots + \sum_{j=1}^{k+1} a_{2j}^3 \phi_3 e_j \right)
\]
is in Im \( d(f, \phi) \). This implies
\[
\phi_2 \left( v_2 + \sum_{j=1}^{k+1} a_{2j}^k \phi_k e_j + \cdots + \sum_{j=1}^{k+1} a_{2j}^3 \phi_3 e_j \right) \in \text{Im} \, d(f, \phi) + \langle \phi_1 \rangle \mathcal{O}_n^{k+1}.
\]
That is, there exist \( w_2 \in \text{Im} \, d(f, \phi_1, ..., \phi_k) \) and \( a_{ij}^2 \in \mathcal{O}_n \) such that
\[
v_2 + \sum_{j=1}^{k+1} a_{2j}^k \phi_k e_j + \cdots + \sum_{j=1}^{k+1} a_{2j}^3 \phi_3 e_j = w_2 + \sum_{j=1}^{k+1} a_{1j}^2 \phi_1 e_j.
\]
We arrive to
\[
\phi_1 \left( v_1 + \sum_{j=1}^{k+1} a_{1j}^k \phi_k e_j + \cdots + \sum_{j=1}^{k+1} a_{3j}^3 \phi_3 e_j \right) + \phi_2 \left( v_2 + \sum_{j=1}^{k+1} a_{2j}^k \phi_k e_j + \cdots + \sum_{j=1}^{k+1} a_{2j}^3 \phi_3 e_j \right) = \\
\phi_1 \left( v_1 + \sum_{j=1}^{k+1} a_{1j}^k \phi_k e_j + \cdots + \sum_{j=1}^{k+1} a_{3j}^3 \phi_3 e_j \right) + \phi_2 \left( w_2 + \sum_{j=1}^{k+1} a_{1j}^2 \phi_1 e_j \right) \in \text{Im} \, d(f, \phi).
\]
Therefore,
\[
\phi_1 \left( v_1 + \sum_{j=1}^{k+1} a_{1j}^k \phi_k e_j + \cdots + \sum_{j=1}^{k+1} a_{3j}^3 \phi_3 e_j + \sum_{j=1}^{k+1} a_{1j}^2 \phi_2 e_j \right) \in \text{Im} \, d(f, \phi),
\]
and there exist $w_1 \in \text{Im} \, d(f, \phi)$ such that
\[
v_1 + \sum_{j=1}^{k+1} a_{ij}^k \phi_k e_j + \ldots + \sum_{j=1}^{k+1} a_{ij}^3 \phi_3 e_j + \sum_{j=1}^{k+1} a_{ij}^2 \phi_2 e_j = w_1.
\]

We conclude that there are $w_1, \ldots, w_k \in \text{Im} \, d(f, \phi)$ such that
\[
v_1 = w_1 - \sum_{j=1}^{k+1} a_{ij}^k \phi_k e_j - \ldots - \sum_{j=1}^{k+1} a_{ij}^3 \phi_3 e_j - \sum_{j=1}^{k+1} a_{ij}^2 \phi_2 e_j
\]
\[
v_2 = w_2 + \sum_{j=1}^{k+1} a_{ij}^2 \phi_1 e_j - \sum_{j=1}^{k+1} a_{ij}^2 \phi_k e_j - \ldots - \sum_{j=1}^{k+1} a_{ij}^3 \phi_3 e_j
\]
\[
\vdots
\]
\[
v_{k-1} = w_{k-1} + \sum_{j=1}^{k+1} a_{(k-2)j}^{k-1} \phi_{k-2} e_j + \ldots + \sum_{j=1}^{k+1} a_{ij}^1 \phi_1 e_j - \sum_{j=1}^{k+1} a_{(k-1)j}^1 \phi_k e_j
\]
\[
v_k = w_k + \sum_{j=1}^{k+1} a_{(k-1)j}^k \phi_{k-1} e_j + \ldots + \sum_{j=1}^{k+1} a_{ij}^1 \phi_1 e_j.
\]

Thus,
\[
\phi_1 (v_1 - w_1) + \ldots + \phi_k (v_k - w_k) = 0.
\]

Since $v_i^T = (\mu_i, \lambda_{i1}, \ldots, \lambda_{ik})^T$ and $w_i = (\alpha_i, \alpha_{i1}, \ldots, \alpha_{ik})^T \in \text{Im} \, d(f, \phi)$ for each $i = 1, \ldots, k$, we have $[\lambda_{ij}] - [\alpha_{ij}] \in H$, and consequently $\xi \in \Theta_X^T$, by Lemma 4.3.

The previous Theorem shows that the quotient $df(\Theta_X)/df(\Theta_X^T)$ does not depend on the $\mathcal{R}_X$-finitely determined function germ $f$. In fact, we show in the next proposition that its dimension as a $\mathbb{C}$-vector space is equal to the Tjurina number of $(X, 0)$. This extends the IHS case considered in [16, 20].

**Proposition 4.6.** Let $(X, 0)$ be an ICIS and $f \in \mathcal{O}_n$ an $\mathcal{R}_X$-finitely determined function germ, then
\[
\dim_{\mathbb{C}} \Theta_X / \Theta_X^T = \dim_{\mathbb{C}} \frac{df(\Theta_X)}{df(\Theta_X^T)} = \tau(X, 0).
\]

**Proof.** As the dimension $\dim_{\mathbb{C}} df(\Theta_X)/df(\Theta_X^T)$ does not depend of $f$ we consider $p \in \mathcal{O}_n$ a generic linear projection and the following exact sequence
\[
0 \longrightarrow \ker(\overline{\pi}) \overset{i}{\longrightarrow} \frac{\mathcal{O}_n}{dp(\Theta_X^T)} \overset{\pi}{\longrightarrow} \frac{\mathcal{O}_n^{k+1}}{\text{Im}(dp(p, \phi)) + I_X \mathcal{O}_n^k} \overset{\pi}{\longrightarrow} \frac{\mathcal{O}_n^k}{\text{Im}(d(\phi)) + I_X \mathcal{O}_n^k} \longrightarrow 0,
\]
where $i$ is the inclusion, $\pi$ is induced by the projection $\pi : \mathcal{O}_n^{k+1} \rightarrow \mathcal{O}_n^k$ given by $\pi(a_0, \ldots, a_k) = (a_1, \ldots, a_k)$ and $\overline{\pi}$ is the map induced by $\alpha : \mathcal{O}_n \rightarrow \mathcal{O}_n^{k+1}$ given by $\alpha(a) = (a, 0, \ldots, 0)$.

Since $p$ is a generic projection, $dp(\Theta_X) = dp(\Theta_X) + I_X$ and $dp(\Theta_X^T) = dp(\Theta_X^T) + I_X$. Now we proceed as in the proof of Theorem 2.2. □
In the proof of Theorem 4.5 we observe that
\[
\frac{df(\Theta^{-1})}{df(\Theta)} \approx \frac{O_n^k}{JfO_n + \text{syzygy}(\phi_1, ..., \phi_k)} \approx \frac{I_X}{JfI_X}.
\]

Therefore from Propositions 4.2 and 4.6 we have
\[
\mu_{BR}(f, X) = \dim C \frac{I_X}{I_X Jf I_X} + \mu(f^{-1}(0) \cap X, 0) + \mu(X, 0) - \tau(X, 0).
\]

**Theorem 4.7.** Let \((X, 0)\) be an ICIS and \(f \in \mathcal{O}_n\) an \(\mathcal{R}_X\)-finitely determined germ, then
\[
\mu_{BR}(f, X) = \mu(f) + \mu(X \cap f^{-1}(0), 0) + \mu(X, 0) - \tau(X, 0) - \dim C \frac{O_n}{Jf + I_X} + \dim C \frac{I_X \cap Jf}{I_X Jf}.
\]

**Proof.** We consider the following exact sequence,
\[
0 \rightarrow \frac{I_X \cap Jf}{I_X Jf} \rightarrow \frac{I_X}{I_X Jf} \rightarrow \frac{I_X}{I_X \cap Jf} \rightarrow 0,
\]

hence,
\[
\dim C \frac{I_X}{I_X Jf} = \dim C \frac{I_X \cap Jf}{I_X Jf} + \dim C \frac{I_X}{I_X \cap Jf} = \dim C \frac{I_X \cap Jf}{I_X Jf} + \dim C \frac{I_X + Jf}{Jf}.
\]

Using the equality (5) we conclude the proof. \(\square\)

In general, to calculate the dimension \(\dim C(I_X \cap Jf)/(I_X Jf)\) is not easy. In order to improve the formula for \(\mu_{BR}(f, X)\) in Theorem 4.7 we observe
\[
\frac{I_X \cap Jf}{I_X Jf} \approx \mathrm{Tor}_1^\mathcal{O}_n (\frac{O_n}{I_X}, \frac{O_n}{Jf}),
\]
(see [10]).

Comparing the formula for \(\mu_{BR}(f, X)\) in the previous theorem with the formula of [16] in the IHS case, we get the following:

**Corollary 4.8.** Let \((X, 0) \subset (\mathbb{C}^n, 0)\) be an isolated hypersurface singularity and \(f\) an \(\mathcal{R}_X\)-finitely determined function germ. Then,
\[
\dim C \mathrm{Tor}_1^\mathcal{O}_n (\frac{O_n}{I_X}, \frac{O_n}{Jf}) = \frac{O_n}{I_X + Jf}.
\]

In order to improve our formula we conjecture:

**Conjecture 4.9.** Let \(R\) be a regular local ring of dimension \(n\), \(I\) an ideal in \(R\) generated by a regular sequence of length \(k \leq n\) and \(J\) an ideal in \(R\) generated by a regular sequence of length \(n\). Then,
\[
\text{length} \left( \mathrm{Tor}_i^R \left( \frac{R}{I}, \frac{R}{J} \right) \right) = \binom{k}{i} \text{length} \left( \frac{R}{J+I} \right).
\]
We observe that the conjecture is known to be true in the particular case that $J \subset I$, see [3, Lemma 3.4.1].

If the conjecture is true, then for any ICIS $(X, 0)$ of codimension $k$ and any $\mathcal{R}_X$-finitely determined function germ $f$,

$$\mu_{BR}(f, X) = \mu(f) + \mu(X \cap f^{-1}(0), 0) + \mu(X, 0) + \tau(X, 0) + (k - 1) \dim_\mathbb{C} \frac{\mathcal{O}_n}{Jf + I_X}.$$ 

4.1. ICIS of codimension 2. The following proposition gives a proof of Conjecture 4.9 when $R = \mathcal{O}_n$ and $k = 2$.

**Proposition 4.10.** Let $I \subset \mathcal{O}_n$ be an ideal generated by a regular sequence of length 2 and $J \subset \mathcal{O}_n$ an ideal generated by a regular sequence of length $n$, then

$$\dim_\mathbb{C} \text{Tor}_1^{\mathcal{O}_n} \left( \frac{\mathcal{O}_n}{I}, \frac{\mathcal{O}_n}{J} \right) = 2 \dim_\mathbb{C} \frac{\mathcal{O}_n}{J + I}.$$ 

**Proof.** Assume $I$ is generated by the regular sequence $\phi_1, \phi_2$ and consider the Koszul complex,

$$0 \to \mathcal{O}_n \xrightarrow{\psi_2} \mathcal{O}_n^2 \xrightarrow{\psi_1} \mathcal{O}_n \to 0,$$

where $\psi_2(\alpha) = \alpha(\phi_2, -\phi_1)$ and $\psi_1(\alpha, \beta) = \alpha \phi_1 + \beta \phi_2$. Tensoring with $R := \mathcal{O}_n/J$, we obtain

$$0 \to R \xrightarrow{\psi_2} R^2 \xrightarrow{\psi_1} R \to 0,$$

and $\text{Tor}_1^{\mathcal{O}_n}(\mathcal{O}_n/I, R) = \ker(\Psi_1)/\text{Im}(\Psi_2)$, where $\Psi_1$ and $\Psi_2$ are induced maps by $\psi_1$ and $\psi_2$, respectively.

The image of $\Psi_1$ is equal to $\bar{I} := (I + J)/J$, hence

$$\dim_\mathbb{C} \ker \Psi_1 = 2 \dim_\mathbb{C} R - \dim_\mathbb{C} \bar{I} = \dim_\mathbb{C} R + \dim_\mathbb{C} \frac{R}{\bar{I}}.$$ 

The kernel of $\Psi_2$ is $\text{Ann}(\bar{I})$, the annihilator of $\bar{I}$ in $R$, so

$$\dim_\mathbb{C} \text{Im} \Psi_2 = \dim_\mathbb{C} R - \dim_\mathbb{C} \text{Ann}(\bar{I}) = \dim_\mathbb{C} \frac{R}{\text{Ann}(\bar{I})}.$$ 

By [13, Proposition 11.4], there exists a perfect pairing on $R$, that is, a symmetric non degenerate bilinear form

$$\langle \cdot, \cdot \rangle : R \times R \to \mathbb{C}$$

such that $\sigma : R \to R^* \text{ defined by } \sigma(a) = \langle a, \cdot \rangle$ is an isomorphism.

Let $g_1, ..., g_\mu$ be a basis over $\mathbb{C}$ of $R$ such that $g_1, ..., g_r$ is basis of $\bar{T}$. By using the dual basis in $R^*$ and the isomorphism $\sigma$ we get a basis of $R$, $h_1, ..., h_\mu$, such that

$$\langle g_i, h_j \rangle = \delta_{ij}.$$ 

Let $I^\perp = \{ a \in R ; \langle a, b \rangle = 0, \forall b \in \bar{T} \}$. Then $I^\perp$ is generated over $\mathbb{C}$ by $h_{r+1}, ..., h_\mu$. By [8, Proposition 3.2(i)], $I^\perp = \text{Ann}(\bar{T})$.

Therefore $\dim_\mathbb{C} \text{Ann}(\bar{T}) = \mu - r$ and

$$\dim_\mathbb{C} \text{Im} \Psi_2 = \dim_\mathbb{C} \frac{R}{\text{Ann}(\bar{T})} = r = \dim_\mathbb{C} \bar{T}.$$
To conclude,
\[
\dim C \text{Tor}_1^{O_n} \left( \frac{O_n}{I}, \frac{O_n}{J} \right) = \dim C \ker \Psi_2 - \dim C \text{Im} \Psi_1 = \dim C R + \dim C \frac{R}{I} - \dim C \bar{T} = 2 \dim C \frac{R}{I} = 2 \dim C \frac{O_n}{J + I}.
\]

\[\square\]

**Corollary 4.11.** Let \((X, 0)\) be an ICIS of codimension 2 and \(f\) an \(R_X\)-finitely determined function germ, then
\[
\mu_{BR}(f, X) = \mu(f) + \mu(f^{-1}(0) \cap X, 0) + \mu(X, 0) - \tau(X, 0) + \dim C \frac{O_n}{Jf + \langle \phi_1, \phi_2 \rangle}.
\]

**References**

[1] I. Ahmed, M. A. S. Ruas, J. N. Tomazella, *Invariants of topological relative right equivalences*, Mathematical Proceedings of the Cambridge Philosophical Society, 155 (2013), no. 2, 307–315.

[2] C. Bivià-Ausina, M. A. S. Ruas, *Mixed Bruce-Roberts number*, Proc. Edinb. Math. Soc. (2), 63, (2020), no. 2, 456–474.

[3] K. Borna, *Betti numbers of modules over Noetherian rings with applications to local cohomology*, Thesis, Tehran, Iran, 2008.

[4] J. W. Bruce, R. M. Roberts, *Critical points of functions on analytic varieties*, Topology 27 (1988), No. 1, 57–90.

[5] D. A. Buchsbaum, D. S. Rim, *A generalized Koszul complex. II. Depth and multiplicity*, Trans. Amer. Math. Soc. 111 (1964), 197–224.

[6] W. Decker, G. M. Greuel, G. Pfister, H. Schönemann, *SINGULAR 4-3-0 – A computer algebra system for polynomial computations*. https://www.singular.uni-kl.de (2022).

[7] J. A. Eagon, M. Hochster, *Cohen–Macaulay rings, invariant theory, and the generic perfection of determinantal loci*, Amer. J. Math. 93 (1971), 1020–1058.

[8] D. Eisenbud, H. I. Levine, *An algebraic formula for the degree of a \(C^\infty\) map germ*, Ann. of Math. (2) 106, 1977, no. 1, 19–44.

[9] T. Gaffney, *Multiplicities an equisingularity of ICIS germs*, Invent. Math. 123 (1996), No. 2, 209-220.

[10] G. M. Greuel, G. Pfister, *A Singular introduction to commutative algebra*, Second, extended edition. With contributions by Olaf Bachmann, Christoph Lossen and Hans Schönemann. Springer, Berlin, 2008.

[11] G. M. Greuel, C. Lossen, E. Shustin, *Introduction to singularities and deformations*. Springer Monographs in Mathematics. Springer, Berlin, 2007.

[12] N. G. Grulha Júnior, *The Euler Obstruction and Bruce–Roberts’ Milnor Number*, Quarterly Journal of Mathematics, v. 60 (2009), 291–302.

[13] K. Kourliouros, *The Milnor-Palamodov Theorem for Functions on Isolated Hypersurface Singularities*. Bull Braz Math Soc, New Series (2021), no. 2, 405–413.

[14] V. H. Jorge Pérez, M. J. Saia, *Euler obstruction, polar multiplicities and equisingularity of map germs in \(O_{(n,p)}, n < p\)*, Internat. J. Math. 17 (2006), No. 8, 887–903.

[15] D. T. Lê, B. Tessier, *Varietes polaires locales et classes de Chern des varietes singulieres*, Ann. of Math. 114 (1981), 457–491.

[16] B. K. Lima-Pereira, J. J. Nuño-Ballesteros, B. Oréfice-Okamoto, J. N. Tomazella, *The Bruce–Roberts number of a function on a hypersurface with isolated singularity*. Q. J. Math. 71 (2020), no. 3, 1049–1063.
[17] B. K. Lima-Pereira, J. J. Nuño-Ballesteros, B. Orféice-Okamoto, J. N. Tomazella, *The relative Bruce-Roberts number of a function on a hypersurface*. Proc. Edinb. Math. Soc. (2) 64 (2021), no. 3, 662–674.

[18] D. Mond, J. J. Nuño Ballesteros, *Singularities of mappings*, volume 357 of Grundlehren der mathematischen Wissenschaften. Springer, Cham, 2020.

[19] J. J. Nuño Ballesteros, B. Orféice-Okamoto, J. N. Tomazella, *The Bruce-Roberts number of a function on a weighted homogeneous hypersurface*, Q. J. Math. 64 (2013), no. 1, 269-280.

[20] S. Tajima, *On Polar Varieties, Logarithmic Vector Fields and Holonomic D-modules*, Recent development of micro-local analysis for the theory of asymptotic analysis, 41–51, RIMS Kökyüroku Bessatsu, B40, Res. Inst. Math. Sci. (RIMS), Kyoto, 2013.

[21] J. M. Wahl, *Derivations, automorphisms and deformations on quasi-homogeneous singularity*, Singularities, Part 2 (Arcata, Calif., 1981), 613–624, Proc. Sympos. Pure Math., 40, Amer. Math. Soc., Providence, RI, 1983.

Departamento de Matemática, Universidade Federal de São Carlos, Caixa Postal 676, 13560-905, São Carlos, SP, BRAZIL
Email address: barbarapereira@estudante.ufscar.br

Departament de Matemàtiques, Universitat de València, Campus de Burjassot, 46100 Burjassot SPAIN.

Departamento de Matemática, Universidade Federal da Paraíba CEP 58051-900, João Pessoa - PB, BRAZIL
Email address: Juan.Nuno@uv.es

Departamento de Matemática, Universidade Federal de São Carlos, Caixa Postal 676, 13560-905, São Carlos, SP, BRAZIL
Email address: brunaorefice@ufscar.br

Departamento de Matemática, Universidade Federal de São Carlos, Caixa Postal 676, 13560-905, São Carlos, SP, BRAZIL
Email address: jntomazella@ufscar.br