THE FINITE-TIME RUIN PROBABILITY OF A RISK MODEL WITH A GENERAL COUNTING PROCESS AND STOCHASTIC RETURN

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ABSTRACT. This paper considers a general risk model with stochastic return and a Brownian perturbation, where the claim arrival process is a general counting process and the price process of the investment portfolio is expressed as a geometric Lévy process. When the claim sizes are pairwise strong quasi-asymptotically independent random variables with heavy-tailed distributions, the asymptotics of the finite-time ruin probability of this risk model have been obtained.

1. Introduction. In this paper, we consider a risk model perturbed by a Brownian diffusion with stochastic return. In this risk model, the successive claim sizes, \(X_i, i \geq 1\), form a sequence of identically distributed random variables (r.v.s) with common distribution \(F\). The inter-arrival times, \(Y_i, i \geq 1\), form another sequence of r.v.s. Denote the times of successive claims by \(\tau_n = \sum_{i=1}^{n} Y_i, n \geq 1\). The claim arrival process \(\{N(t), t \geq 0\}\) is a general counting process, satisfying \(0 < \lambda(t) = EN(t) < \infty, t > 0, \lambda(0) = 0\) and

\[N(t) = \sum_{n=1}^{\infty} 1_{\{\tau_n \leq t\}}, \ t \geq 0,\]

where \(1_A\) is the indicator function of a set \(A\). In this risk model, the insurer is allowed to make risk free and risky investments. The price process of the investment portfolio is described as a geometric Lévy process \(\{e^{R(t)}, t \geq 0\}\), where \(\{R(t), t \geq 0\}\)
0) is a Lévy process, which starts from zero and has independent and stationary increments. The discounted aggregate claims up to time $t \geq 0$ is denoted by

$$D(t) = \sum_{i=1}^{N(t)} X_i e^{-R(t_i)}.$$  

The total amount of premium accumulated up to time $t \geq 0$, denoted by $C(t)$ with $C(0) = 0$ and $C(t) < \infty$ almost surely for every $t \geq 0$, is a nonnegative and nondecreasing stochastic process. The process $\{B(t), t \geq 0\}$ is a standard Brownian motion and $\sigma \geq 0$ is the volatility factor. Let $r \geq 0$ be the constant interest rate and $x \geq 0$ be the initial asset of an insurance company. Then the discounted value of the surplus process is denoted by

$$U(x, t) = x + \int_0^t e^{-R(s)} dC(s) - D(t) + \sigma \int_0^t e^{-rs} dB(s), t \geq 0.$$  

(1)

Hence, the finite-time ruin probability up to time $t \geq 0$ is defined by

$$\psi(x, t) = P \left( \inf_{0<s\leq t} U(x, s) < 0 \; | \; U(x, 0) = x \right)$$

$$= P \left( \sup_{0<s\leq t} \left( D(t) - \int_0^t e^{-R(s)} dC(s) - \sigma \int_0^t e^{-rs} dB(s) \right) > x \right).$$  

(2)

As usual, we assume that $\{X_i, i \geq 1\}, \{Y_i, i \geq 1\}, \{R(t), t \geq 0\}, \{C(t), t \geq 0\}$ and $\{B(t), t \geq 0\}$ are mutually independent. For every $0 \leq t \leq \infty$, let

$$C(t) = \int_0^t e^{-R(s)} dC(s) < \infty.$$  

It is noted that for every $0 \leq t \leq \infty$, $B(t) = \sigma \int_0^t e^{-rs} dB(s) < \infty$ almost surely.

This risk model with $R(t) = rt, t \geq 0$ and $\sigma = 0$ is called as a general risk model by [29]. This paper will consider a general risk model with stochastic return and a Brownian perturbation as (1), and investigate the finite-time ruin probability $\psi(x, t)$ for heavy-tailed claim sizes, $X_i, i \geq 1$, with a dependence structure. In the following, we will introduce some heavy-tailed distribution classes and dependence structures.

1.1. Heavy-tailed distribution classes. Throughout this paper, all limit relations are for $x \to \infty$, unless stated otherwise. For two positive functions $a(x)$ and $b(x)$, we write $a(x) \prec b(x)$ if $\lim \sup a(x)/b(x) \leq 1$; write $a(x) \succ b(x)$ if $\lim \inf a(x)/b(x) \geq 1$; write $a(x) \sim b(x)$ if $\lim a(x)/b(x) = 1$; write $a(x) \prec\prec b(x)$ if $\lim a(x)/b(x) = 0$; write $a(x) = O(b(x))$ if $\lim \sup a(x)/b(x) < \infty$; write $a(x) \asymp b(x)$ if $0 < \lim \inf a(x)/b(x) \leq \lim \sup a(x)/b(x) < \infty$.

For a proper distribution $V$ on $(-\infty, \infty)$, denote its tail by $V(x) = 1 - V(x), x \in (-\infty, \infty)$. Say that a distribution $V$ on $(-\infty, \infty)$ is heavy-tailed, if for any $\beta > 0$, $\int_{-\infty}^{\infty} e^{\beta y} V(dy) = \infty$; otherwise, say that $V$ is light-tailed. An important heavy-tailed distribution class is the long-tailed distribution class, denoted by $L$. Say that a distribution $V$ on $(-\infty, \infty)$ belongs to the class $L$, if for any $y > 0$,

$$V(x+y) \sim V(x).$$

It is well known that the above relation holds uniformly for every compact set of $y$. A related heavy-tailed distribution class is the dominated varying distribution class, denoted by $\mathcal{D}$. Say that a distribution $V$ on $(-\infty, \infty)$ belongs to the class $\mathcal{D}$, if for any $0 < y < 1$,

$$V(xy) = O(V(x)).$$
Another heavy-tailed distribution class is the consistently varying tail distribution class, denoted by \( \mathcal{C} \). Say that a distribution \( V \) on \((-\infty, \infty)\) belongs to the class \( \mathcal{C} \), if
\[
\lim_{y \to 1^+} \limsup_{x \to \infty} \frac{V(xy)}{V(x)} = 1, \quad \text{or equivalently,} \quad \lim_{y \to 1^-} \liminf_{x \to \infty} \frac{V(xy)}{V(x)} = 1.
\]
A subclass of the class \( \mathcal{C} \) is the regularly varying tail distribution class, denoted by \( \mathcal{R}^{-\alpha} \) for some \( 0 \leq \alpha < \infty \). Say that a distribution \( V \) on \((-\infty, \infty)\) belongs to the class \( \mathcal{R}^{-\alpha} \) for some \( 0 \leq \alpha < \infty \), if for any \( y > 0 \),
\[
V(xy) \sim y^{-\alpha} V(x).
\]
Let \( \mathcal{R} \) to denote the union of all \( \mathcal{R}^{-\alpha} \) over the range \( 0 \leq \alpha < \infty \). Then the above distribution classes have the following proper relation:
\[
\mathcal{R} \subset \mathcal{C} \subset \mathcal{L} \cap \mathcal{D} \subset \mathcal{L},
\]
see, e.g. [4], [6], [7], [13].

1.2. \textbf{Dependence structures.} In the studies of risk theory, more and more literatures focus on dependent risk models. Some dependence structures are used to model the dependence structure among the claim sizes. In the following, some common dependence structures are presented.

Adopting the definition in [12] (see also [15]), real-valued r.v.s \( \xi_i, i \geq 1 \) are said to be pairwise strong quasi-asymptotically independent (pSQAI) if, for any \( i \neq j \geq 1 \),
\[
\lim \min \{x_i, x_j\} \to \infty \quad P(\xi_i > x_i, \xi_j > x_j) = 0. \tag{3}
\]
Actually, r.v.s \( \xi_i, i \geq 1 \) are pSQAI if they are mutually independent or pairwise negatively dependent. [12] and [15] also pointed out that if \( \xi_i, i \geq 1 \) are pairwise FGM (Farlie-Gumble-Morgenstern) distributed, i.e. for any \( i \neq j \) and \( x_i, x_j \in (-\infty, \infty) \),
\[
P(\xi_i \leq x_i, \xi_j \leq x_j) = V_i(x_i)V_j(x_j)(1 + a_{ij}V_i(x_i)V_j(x_j)), \tag{4}
\]
where \( V_i \) is the distribution of \( \xi_i, i \geq 1 \) and \( a_{ij} \) is a real number such that the right-hand side of (4) is a proper distribution. Example 3.1 of [18] also gave two r.v.s satisfying (3).

For original ideas and analogues of pSQAI, one can refer to [20], [24], [25]. This dependence structure has been investigated by many researchers, such as [1], [11], [15], [34], among others.

[30] introduced the widely (upper/lower) orthant dependence (WUOD/WLOD) structure, when they investigated the finite-time ruin probability of a dependent
risk model. For the real-valued r.v.s $\xi_i, i \geq 1$, if there exists a sequence of positive numbers $\{g_U(i), i \geq 1\}$ satisfying for each $n \geq 1$ and for all $x_i \in (-\infty, \infty), 1 \leq i \leq n$,

$$P\left(\bigcap_{i=1}^{n}\{\xi_i > x_i\}\right) \leq g_U(n) \prod_{i=1}^{n} P(\xi_i > x_i),$$

then we say that r.v.s $\{\xi_i, i \geq 1\}$ are widely upper orthant dependent (WUOD) with dominating coefficients $g_U(i), i \geq 1$; if there exists a sequence of positive numbers $\{g_L(i), i \geq 1\}$ satisfying for each $n \geq 1$ and for all $x_i \in (-\infty, \infty), 1 \leq i \leq n$,

$$P\left(\bigcap_{i=1}^{n}\{\xi_i \leq x_i\}\right) \leq g_L(n) \prod_{i=1}^{n} P(\xi_i \leq x_i),$$

then we say that r.v.s $\{\xi_i, i \geq 1\}$ are widely lower orthant dependent (WLOD) with dominating coefficients $g_L(i), i \geq 1$.

Actually, r.v.s $\xi_i, i \geq 1$ are WUOD/WLOD if they are mutually independent or negatively upper orthant dependent (NUOD)/negatively lower orthant dependent (NLOD) (see, e.g. [5] or [3]). In Section 3 of [30] some examples of WUOD and WLOD r.v.s are given. We note that r.v.s $\xi_i, i \geq 1$ are pSQAI if they are WUOD. In addition, Example 3.1 of [18] pointed out that there exist random variables which are pSQAI, but don’t satisfy WUOD structure.

1.3. Main results. For the above risk model without a Brownian perturbation (i.e. $\sigma = 0$ in (1)), when the process $\{N(t), t \geq 0\}$ is a renewal counting process, i.e. the inter-arrival times, $Y_i, i \geq 1$, are independent and identically distributed (i.i.d.), [27] obtained the ruin probabilities for the independent claim sizes with regularly varying tails. [35] considered the pSQAI claim sizes and obtained the asymptotics of the finite-time ruin probability for the consistently varying claim sizes. [14] studied a time-dependent risk model with extended regularly varying claim sizes, in which there was a dependence structure between the claim sizes and their corresponding inter-arrival times. For this dependent risk model in [14], [9] and [37] extended the results of [14] to the dominated varying claim sizes. Some other dependent cases about this risk model also have been investigated such as [21], [22], [32] and so on. A bidimensional risk model of the risk model (1) has been discussed by [8], [10], [16], [36], [38] and so on.

For the above risk model with a Brownian perturbation (i.e. $\sigma > 0$ in (1)), [31] investigated the asymptotics of the finite-time ruin probability for i.i.d subexponential claim sizes, $X_i, i \geq 1$ and i.i.d inter-arrival times, $Y_i, i \geq 1$.

It is worth pointing out that many existing literatures on the risk model (1) require the assumption of independence (or dependence) among the inter-arrival times, $Y_i, i \geq 1$. That is to say, the claim arrival process $\{N(t), t \geq 0\}$ is a renewal (or quasi-renewal) counting process. In this paper, we will consider a general risk model, in which the process $\{N(t), t \geq 0\}$ is a general counting process and the assumption of independence or dependence among the inter-arrival times is not needed. When investigating the general risk model, we will consider the effect of a Brownian perturbation on the finite-time ruin probability.

The following theorem is the main result of this paper.

**Theorem 1.1.** Consider the risk model (1). Assume that the claim sizes, $X_i, i \geq 1$, are pSQAI r.v.s with common distribution $F \in \mathcal{L} \cap \mathcal{D}$, and $R(t) \geq 0$ almost surely.
for any \( t > 0 \). Suppose that for some \( t > 0 \), there exists some \( p > J_F^+ \) such that

\[
E(N(t))^{p+1} < \infty.
\]  

(5)

Then

\[
\psi(x,t) \sim \int_0^t P(X_1 e^{-R(s)} > x) \lambda(ds).
\]  

(6)

Remark 1. (1) In [35], they considered the risk model (1.1) for the case that \( \sigma = 0 \), the premium process \( C(t) \) has a bounded density function and the inter-arrival times, \( Y_i, i \geq 1 \) are i.i.d. r.v.s. For the pSQAI claim sizes, \( X_i, i \geq 1 \) with distribution \( F \in \mathcal{C} \), Corollary 2.1 of [35] obtained (6), which holds uniformly for \( t \) in finite time interval. It is well-known that (5) is satisfied for i.i.d. \( Y_i, i \geq 1 \). Therefore, to a certain extent, Theorem 1.1 extends the risk model and Corollary 2.1 of [35].

(2) [19] considered a general risk model with a constant interest without a perturbation, i.e. \( \sigma = 0 \) and \( R(t) = rt, t \geq 0 \) in (1). When the claim size, \( X_i, i \geq 1 \) are WUOD r.v.s with distribution \( F \in \mathcal{L} \cap \mathcal{D} \), Theorem 1.1 of [19] gave the asymptotics of the finite-time ruin probability \( \psi(x,t) \) as (6). Since we know that \( X_i, i \geq 1 \) are pSQAI if they are WUOD, the result of [19] can be obtained from Theorem 1.1.

(3) For a general risk model with \( \sigma = 0 \) and \( R(t) = rt, t \geq 0 \) in (1), when the claim sizes \( X_i, i \geq 1 \) are i.i.d. r.v.s with distribution \( F \in \mathcal{L} \cap \mathcal{D} \) and for some \( t > 0 \), there exists some \( \eta = \eta(t) > 0 \) such that

\[
E(1 + \eta)^{N(t)} < \infty,
\]  

(7)

Theorem 2.2 of [29] obtained that (6) holds. Since (5) can be got from (7), Theorem 2.2 of [29] can be obtained from Theorem 1.1.

Remark 2. Note that if \( \{N(t), t \geq 0\} \) is a homogeneous Poisson process then (5) is satisfied. If \( \{N(t), t \geq 0\} \) is a negative binomial process, i.e. for each \( t > 0 \),

\[
P(N(t) = n) = \binom{\gamma + n - 1}{n} \left( \frac{\beta}{\beta + t} \right)^\gamma \left( \frac{t}{\beta + t} \right)^n, n \geq 1, \beta > 0, \gamma > 0,
\]

then (5) holds according to Example 1.3.11 of [6].

Remark 3. If the inter-arrival times, \( Y_i, i \geq 1 \) are identically distributed and WLOD r.v.s with finite positive mean \( \lambda^{-1} \) and dominating coefficients \( g_L(n), n \geq 1 \) satisfying

\[
\lim_{n \to \infty} g_L(n)n^{-\alpha} = 0
\]  

(8)

for some constant \( \alpha > 0 \), then the counting process \( \{N(t), t \geq 0\} \) generated by \( \{Y_i, i \geq 1\} \) satisfies (5).

Indeed, by Lemma 4.2 of [33], we get that for any positive integer \( r \),

\[
\lim_{t \to \infty} E(N(t))^r(\lambda t)^{-r} = 1.
\]

Hence, taking some \( r - 1 = p > J_F^+ \), for any \( \varepsilon > 0 \) there exists \( t_0 > 0 \), such that for all \( t \geq t_0 \),

\[
E(N(t))^{p+1} \leq (1 + \varepsilon)(\lambda t)^{p+1}.
\]

When \( 0 \leq t < t_0 \),

\[
E(N(t))^{p+1} \leq E(N(t_0))^{p+1} \leq (1 + \varepsilon)(\lambda t_0)^{p+1}.
\]

Thus, for each \( t \geq 0 \),

\[
E(N(t))^{p+1} < \infty.
\]
Therefore, we can obtain the following corollary.

**Corollary 1.** Consider the risk model (1). Assume that the claim sizes, \(X_i, i \geq 1\), are \(p\)SQAI r.v.s with common distribution \(F \in \mathcal{L} \cap \mathcal{D}\). The inter-arrival times, \(Y_i, i \geq 1\) are WLOD r.v.s with finite positive mean and dominating coefficients \(g_L(n), n \geq 1\). If \(R(t) \geq 0\) almost surely for any \(t > 0\) and there exists a constant \(\alpha > 0\) such that (8) holds, then for each \(t > 0\), (6) holds.

2. **Proof of main results.** Before giving the proof of Theorem 1.1, we first present some lemmas.

**Lemma 2.1.** a. Let \(V\) be a distribution on \((-\infty, \infty)\), belonging to the class \(\mathcal{D}\).

(1) For any \(\beta > J^+_V\), \(x^{-\beta} = o(V(x))\).

(2) For any \(\beta > J^+_V\), there exist two positive constants \(A\) and \(B\) such that when \(x \geq y \geq B\),

\[
\frac{V(y)}{V(x)} \leq A \left(\frac{x}{y}\right)^\beta.
\]

b. Let \(\xi\) and \(\theta\) be two nonnegative independent r.v.s. The r.v \(\theta\) is not degenerate at zero.

(1) If the distribution of \(\xi\) belongs to the class \(\mathcal{L}\) and \(P(\theta > u\xi) = o(P(\theta \xi > x))\) for all \(u > 0\), then the distribution of \(\theta \xi\) belongs to the class \(\mathcal{L}\).

(2) If the distribution of \(\xi\) belongs to the class \(\mathcal{D}\), then the distribution of \(\theta \xi\) belong to the class \(\mathcal{D}\).

c. For any \(t > 0\), it holds that

\[
P\left(\sup_{0 \leq s \leq t} \left(-\tilde{B}(s)\right) > x\right) = P\left(\sup_{0 \leq s \leq t} \tilde{B}(s) > x\right) \sim 2\tilde{\Phi}\left(\frac{\sqrt{2t}}{s \sqrt{1 - e^{-2rt}}} x\right),
\]

where \(\tilde{\Phi}(x) \sim \frac{1}{\sqrt{2\pi}x} e^{-x^2/2}\) is the tail of the standard Gaussian distribution and \(\sqrt{2t}/\sqrt{1 - e^{-2rt}}\) is understood as \(1/\sqrt{t}\) (i.e. its limit as \(r \to 0\)) when \(r = 0\).

*Proof.* The result a is from Proposition 2.2.1 of [2] and Lemma 3.5 of [26]. The result b is closure properties of the classes \(\mathcal{L}\) and \(\mathcal{D}\), which comes from Lemma 2 of [28] and Theorem 3.3 (ii) of [4]. The result c is Theorem D.3 (ii) of [23] (or see (3.6) of [17]). \(\Box\)

**Lemma 2.2.** Let \(\xi_i, 1 \leq i \leq n\) be \(n\) real-valued and \(p\)SQAI r.v.s with distributions \(V_i, 1 \leq i \leq n\), respectively.

(1) If \(V_i \in \mathcal{L} \cap \mathcal{D}, 1 \leq i \leq n\), then for every \(0 < a \leq b < \infty\), it holds uniformly for \((c_1, \cdots, c_n) \in [a, b]^n\) that

\[
P\left(\sum_{i=1}^n c_i \xi_i > x\right) \sim \sum_{i=1}^n P(c_i \xi_i > x).
\]

(2) If \(\xi_i, 1 \leq i \leq n\) are nonnegative r.v.s and \(V_i \in \mathcal{D}, 1 \leq i \leq n\), then

\[
\sum_{i=1}^n \nabla_i(x) < P\left(\sum_{i=1}^n \xi_i > x\right) < \sum_{i=1}^n L^{-1}_{\nabla_i} \nabla_i(x) \quad (9)
\]

*Proof.* The result of (1) is Theorem 2.1 of [15]. We will prove (9). When \(n = 1\), by \(V_1 \in \mathcal{D}\), we know that (9) holds obviously. In the following we assume that
\( n \geq 2 \). Denote \( S_n = \sum_{k=1}^{n} \xi_k \). Firstly, we estimate an asymptotic upper bound for \( P(S_n > x) \). For any \( 0 < \varepsilon < 1 \) and \( x > 0 \),

\[
P(S_n > x) \leq P\left( \bigcup_{k=1}^{n} (\xi_k > (1 - \varepsilon)x) \right) + P\left( S_n > x, \bigcap_{k=1}^{n} (\xi_k \leq (1 - \varepsilon)x) \right) =: I_1(x) + I_2(x). \tag{10}
\]

For \( I_1(x) \), we have that for any \( x > 0 \),

\[
I_1(x) \leq \sum_{k=1}^{n} V_k((1 - \varepsilon)x).
\]

For any \( 1 \leq k \leq n \), from the definition of \( L_{V_k} \), we have

\[
\lim_{y \uparrow 1} \limsup_{x \to \infty} \frac{V_k(xy)}{V_k(x)} = \left( \lim_{y \downarrow 1} \liminf_{x \to \infty} \frac{V_k(xy \cdot y^{-1})}{V_k(xy)} \right)^{-1} = L_{V_k}^{-1}.
\]

Thus, for any \( \varepsilon_1 > 0 \), there exist some \( 0 < \delta_1 = \delta_1(\varepsilon_1, n) < 1 \) and \( x_1 = x_1(\varepsilon_1, n) > 0 \), such that for all \( \delta_1 \leq y \leq 1 \), \( x \geq x_1 \) and \( 1 \leq k \leq n \),

\[
\frac{V_k(xy)}{V_k(x)} \leq L_{V_k}^{-1} + \varepsilon_1 \tag{11}
\]

and for all \( 1 \leq i \neq j \leq n \) and \( x \geq x_1 \),

\[
P\left( \xi_j > \frac{\varepsilon x}{n} \left| \xi_i > \frac{x}{n} \right. \right) \leq \frac{\varepsilon_1}{n}. \tag{12}
\]

Therefore, for the above \( \varepsilon_1 \), taking \( 0 < \varepsilon \leq 1 - \delta_1 \) in (10), since \( 0 < L_{V_k} \leq 1, 1 \leq k \leq n \), by (11), for all \( x \geq x_1 \), it holds that

\[
I_1(x) \leq \sum_{k=1}^{n} \left( L_{V_k}^{-1} + \varepsilon_1 \right) V_k(x)
\]

\[
\leq (1 + \varepsilon_1) \sum_{k=1}^{n} L_{V_k}^{-1} V_k(x).
\]

Hence,

\[
\limsup_{x \to \infty} \frac{I_1(x)}{\sum_{k=1}^{n} L_{V_k}^{-1} V_k(x)} \leq \lim_{\varepsilon_1 \downarrow 0} (1 + \varepsilon_1) = 1.
\tag{13}
\]

For \( I_2(x) \), by (12) it holds for \( x > x_1 \) that

\[
I_2(x) \leq \sum_{i=1}^{n} P\left( S_n > x, \xi_i > \frac{x}{n}, \bigcap_{k=1}^{n} (\xi_k \leq (1 - \varepsilon)x) \right)
\]

\[
\leq \sum_{i=1}^{n} P\left( \xi_i > \frac{x}{n}, S_n - \xi_i > \varepsilon x \right)
\]

\[
\leq \sum_{i=1}^{n} \sum_{1 \leq j \neq i \leq n} P\left( \xi_i > \frac{x}{n}, \xi_j > \frac{\varepsilon x}{n-1} \right)
\]

\[
= \sum_{i=1}^{n} \sum_{1 \leq j \neq i \leq n} P\left( \xi_j > \frac{\varepsilon x}{n-1} \left| \xi_i > \frac{x}{n} \right. \right) P\left( \xi_i > \frac{x}{n} \right)
\]
\[
\leq \varepsilon_1 \sum_{k=1}^{n} V_k \left( \frac{x}{n} \right).
\]  
(14)

Since \( V_k \in D, 1 \leq k \leq n \), by Lemma 2.1 a(2), for any \( \rho > \max \{ J_{V_k}^+, 1 \leq k \leq n \} \), there exist constants \( A > 0 \) and \( B \geq x \) such that for all \( 1 \leq k \leq n \) and \( x > y \geq B \),
\[
\frac{V_k(y)}{V_k(x)} \leq A \left( \frac{x}{y} \right)^\rho.
\]  
(15)

It follows from (14) and (15) that for all \( x \geq nB \)
\[
I_2(x) \leq \varepsilon_1 An^\rho \sum_{k=1}^{n} V_k(x) \leq \varepsilon_1 An^\rho \sum_{k=1}^{n} L_{V_k}^{-1} V_k(x).
\]

Therefore,
\[
\limsup_{x \to \infty} \frac{I_2(x)}{\sum_{k=1}^{n} L_{V_k}^{-1} V_k(x)} \leq \lim_{\varepsilon_1 \to 0} \varepsilon_1 An^\rho = 0.
\]  
(16)

By (10), (13) and (16), we get that
\[
P(S_n > x) \leq \sum_{k=1}^{n} L_{V_k}^{-1} V_k(x).
\]

Now we prove the asymptotic lower bound for \( P(S_n > x) \). Since \( \{ \xi_i, 1 \leq i \leq n \} \) are pSQAI, for any \( 0 < \varepsilon_2 < 1 \), there exists \( x_2 = x_2(\varepsilon_2, n) > 0 \) such that for all \( 1 \leq i \neq j \leq n \) and \( x > x_2 \),
\[
P(\xi_i > x | \xi_j > x) \leq \frac{\varepsilon_2}{n}.
\]

Hence, for all \( x > x_2 \),
\[
\sum_{1 \leq i < j \leq n} P(\xi_i > x, \xi_j > x) = \sum_{1 \leq i < j \leq n} P(\xi_i > x | \xi_j > x) P(\xi_j > x)
\leq \varepsilon_2 \sum_{k=1}^{n} V_k(x).
\]

Therefore, since \( \{ \xi_k, 1 \leq k \leq n \} \) are nonnegative, for all \( x > x_2 \)
\[
P(S_n > x) \geq P \left( \bigcup_{k=1}^{n} \{ \xi_k > x \} \right)
\geq \sum_{k=1}^{n} V_k(x) - \sum_{1 \leq i < j \leq n} P(\xi_i > x, \xi_j > x)
\geq (1 - \varepsilon_2) \sum_{k=1}^{n} V_k(x),
\]

which means that
\[
\liminf_{x \to \infty} \frac{P(S_n > x)}{\sum_{k=1}^{n} V_k(x)} \geq \lim_{\varepsilon_2 \to 0} (1 - \varepsilon_2) = 1.
\]

This completes the proof of this lemma. \( \Box \)
For the result of the following lemma, Lemma 3.2 of [17] and Lemma 2.2 of [31] considered the independent random variables. The following lemma investigates the pSQAI random variables, which can be proved following the line of the proof of Lemma 3.2 of [17] by using Lemma 2.2 (1).

**Lemma 2.3.** Let $\xi_i, 1 \leq i \leq n$ be $n$ real-valued and pSQAI r.v.s with distribution $V_i \in L, 1 \leq i \leq n$. $\xi_0$ is another real-valued r.v. with distribution $V_0$ and independent of $\{\xi_i, 1 \leq i \leq n\}$. If there exists a distribution $V \in \mathcal{L} \cap D$ such that $V_i(x) \approx \bar{V}(x), 1 \leq i \leq n$ and $V_0(x) = o(\bar{V}(x/c))$ for some $c > 0$ then for any fixed $d > c$, it holds uniformly for all $(c_1, \cdots, c_n) \in [c, d]^n$ that

$$P\left(\sum_{i=1}^{n} c_i \xi_i + \xi_0 > x\right) \sim \sum_{i=1}^{n} P(c_i \xi_i > x).$$

The first result of the following lemma is a closure property of pSQAI random variables about products, which can be obtained from Theorem 2.2 of [15]. The second result is about random weighted sums of pSQAI r.v.s, which can be proved by using the first result and Lemmas 2.2 and 2.1 b. We omit the details.

**Lemma 2.4.** Let $\xi_i, 1 \leq i \leq n$ be $n$ nonnegative and pSQAI r.v.s with distributions $V_i, 1 \leq i \leq n$, respectively. Let the random weights $\theta_i, 1 \leq i \leq n$, be nonnegative, arbitrarily dependent on each other and not degenerate at zero, but independent of $\xi_i, 1 \leq i \leq n$. Assume that $\theta_i, 1 \leq i \leq n$ are bounded above.

1. If $V_i \in D, 1 \leq i \leq n$ then $\theta_i \xi_i, 1 \leq i \leq n$ are pSQAI.
2. If $V_i \in L \cap D, 1 \leq i \leq n$ then

$$P\left(\sum_{i=1}^{n} \theta_i \xi_i > x\right) \sim \sum_{i=1}^{n} P(\theta_i \xi_i > x).$$

**Lemma 2.5.** Let $\xi_i, 1 \leq i \leq n$ be $n$ nonnegative and pSQAI r.v.s with distributions $V_i \in L, 1 \leq i \leq n$. $\xi_0$ is another real-valued r.v. with distribution $V_0$ and independent of $\{\xi_i, 1 \leq i \leq n\}$. Let the random weights $\theta_i, 1 \leq i \leq n$, be nonnegative, arbitrarily dependent on each other and not degenerate at zero, but independent of $\xi_i, 0 \leq i \leq n$. If there exists a distribution $V \in \mathcal{L} \cap D$ such that $\bar{V}(x) \approx \bar{V}(x), 1 \leq i \leq n$ and $\bar{V}(x) = o(\bar{V}(x/c))$ for all $c > 0$, and $\theta_i, 1 \leq i \leq n$ are bounded above then

$$P\left(\sum_{i=1}^{n} \theta_i \xi_i + \xi_0 > x\right) \sim \sum_{i=1}^{n} P(\theta_i \xi_i > x).$$

**Proof.** The result can be proved by following the line of the proof of Lemma 2.5 of [31]. (or the proof of Theorem 3.1 of [28]) and using Lemmas 2.1 b, 2.3 and 2.4. We omit the details.

The next lemma presents an upper bound of the tail of a sum of real-valued random variables with dominated varying tails.

**Lemma 2.6.** Let $\{\xi_i, i \geq 1\}$ be a sequence of real-valued r.v.s with common distribution $V \in D$. Then for any $p > J_V^+$, there exists some constant $M = M(p) > 0$ such that for all $x \in (-\infty, \infty)$ and $n \geq 1$,

$$P\left(\sum_{i=1}^{n} \xi_i > x\right) \leq Mn^{p+1} \bar{V}(x).$$
Proof. Since $V \in D$, by Lemma 2.1 a, for any $p > J_V^+$, there exists $x_0 > 0$ such that for all $x \geq x_0$
\[ x^{-p} \leq \overline{V}(x) \] (17)
and there exist positive constants $A$ and $B > x_0$ such that for all $x \geq y \geq B$,
\[ \frac{\overline{V}(y)}{\overline{V}(x)} \leq A \left( \frac{x}{y} \right)^p. \] (18)
Thus, for the above $p > J_V^+$ and for all $x \geq x_0$ and $n \geq 1$, by (17) and (18), it holds that
\[ \overline{V} \left( \frac{x}{n} \right) \leq 1_{\{x_0 \leq x \leq nB\}} + \overline{V} \left( \frac{x}{n} \right) 1_{\{x > nB\}} \]
\[ \leq \left( \frac{nB}{x} \right)^p + A n^p \overline{V}(x) \]
\[ \leq (B^p + A) n^p \overline{V}(x). \] (19)
Hence, for all $x \geq x_0$ and $n \geq 1$, by (19) we have that
\[ P \left( \sum_{i=1}^{n} \xi_i > x \right) \leq \sum_{i=1}^{n} P \left( \xi_i > \frac{x}{n} \right) \]
\[ \leq (B^p + A) n^{p+1} \overline{V}(x). \] (20)
For all $x < x_0$ and $n \geq 1$, it holds that
\[ P \left( \sum_{i=1}^{n} \xi_i > x \right) \leq \frac{\overline{V}(x)}{\overline{V}(x_0)} \leq \frac{1}{\overline{V}(x_0)} n^{p+1} \overline{V}(x). \] (21)
Taking $M = \max \left\{ B^p + A, \frac{1}{\overline{V}(x_0)} \right\}$, then for all $x \in (-\infty, \infty)$ and $n \geq 1$, by (20) and (21), we get that
\[ P \left( \sum_{i=1}^{n} \xi_i > x \right) \leq M n^{p+1} \overline{V}(x). \]
This completes the proof of this lemma. \(\square\)

Lemma 2.7. Let $\{\xi_i, i \geq 1\}$ be a sequence of real-valued r.v.s with common distribution $V \in D$. $\xi_0$ is another real-valued r.v. with distribution $V_0$. Assume that $\xi_0$ is independent of $\{\xi_i, i \geq 1\}$ and $V_0(x) = O(\overline{V}(x))$. Then for any $p > J_V^+$, there exists some constant $M = M(p) > 0$ such that for all $x \in (-\infty, \infty)$ and $n \geq 1$,
\[ P \left( \sum_{i=1}^{n} \xi_i + \xi_0 > x \right) \leq M n^{p+1} \overline{V}(x). \]

Proof. We will use the line of the proof of Lemma 3.4 of [17]. It follows from $V_0(x) = O(\overline{V}(x))$ that there exist constants $M_0 \geq 1$ and $x_0 > 0$ such that for all $x \geq x_0$,
\[ \overline{V}_0(x) \leq M_0 \overline{V}(x) \leq 1. \] (22)
Define r.v.s $\eta_i, i = 1, 2$ with the following tail
\[ P(\eta_i > x) = \begin{cases} M_0 \overline{V}(x), & x \geq x_0, \\ 1, & x < x_0 \end{cases}, \quad i = 1, 2. \] (23)
Then $\eta_1$ and $\eta_2$ are identically distributed with a distribution belonging to the class $D$. It follows from (22) and (23) that for all $x \in (-\infty, \infty)$,

$$P(\eta_1 > x) \geq P(\xi_0 > x) \quad \text{and} \quad P(\eta_2 > x) \geq P(\xi_1 > x).$$

Thus

$$\xi_0 \leq \text{st} \eta_1 \quad \text{and} \quad \xi_1 \leq \text{st} \eta_2. \tag{24}$$

By Lemma 2.6, for any $p > J^+_1$, there exists a constant $M_1 = M_1(p) > 0$ such that for all $x \in (-\infty, \infty)$ and $n \geq 1$,

$$P\left(\sum_{i=1}^{n} \xi_i > x\right) \leq M_1 n^{p+1} \mathbb{V}(x) \tag{25}$$

and

$$P(\eta_1 + \eta_2 > x) \leq M_1 2^{p+1} P(\eta_1 > x). \tag{26}$$

For the above $p > J^+_1$,

$$P\left(\sum_{i=1}^{n} \xi_i + \xi_0 > x\right) = \int_{-\infty}^{\infty} P\left(\sum_{i=1}^{n} \xi_i > x - y\right) P(\xi_0 \in dy) \leq M_1 n^{p+1} \int_{-\infty}^{\infty} P(\xi_1 > x - y) P(\xi_0 \in dy) = M_1 n^{p+1} P(\xi_0 + \xi_1 > x) \leq M_1 n^{p+1} P(\eta_1 + \eta_2 > x) \leq M_1 2^{p+1} n^{p+1} P(\eta_1 > x).$$

Hence, when $x \geq x_0$, by (23) it holds for all $n \geq 1$ that

$$P\left(\sum_{i=1}^{n} \xi_i + \xi_0 > x\right) \leq M_1 M_2 2^{p+1} n^{p+1} \mathbb{V}(x).$$

When $x < x_0$, for all $n \geq 1$, it holds that

$$P\left(\sum_{i=1}^{n} \xi_i + \xi_0 > x\right) \leq n^{p+1} \frac{\mathbb{V}(x)}{\mathbb{V}(x_0)}$$

Taking $M = \max\big\{M_1 M_2 2^{p+1}, 1/\mathbb{V}(x_0)\big\}$, then for all $x \in (-\infty, \infty)$ and $n \geq 1$, it holds that

$$P\left(\sum_{i=1}^{n} \xi_i + \xi_0 > x\right) \leq M n^{p+1} \mathbb{V}(x).$$

This completes the proof of this lemma.

**Proof of Theorem 1.1.** We first prove the upper bound of the finite-time ruin probability $\psi(x,t)$. For each fixed $t > 0$, it follows from (2) and Lemma 2.1c that for any positive integer $m$ and $x > 0$,

$$\psi(x,t) \leq P\left(D(t) + \sup_{0 \leq s \leq t} (-\overline{B}(s)) > x\right) = P\left(\sum_{i=1}^{N(t)} X_i e^{-R(\tau_i)} + \sup_{0 \leq s \leq t} \overline{B}(s) > x\right)$$
Thus, for all $\epsilon > 0$ such that for all $x, t > 0$

$$P \left( \sup_{0 \leq s \leq t} \tilde{B}(s) > x \right) = o \left( \mathcal{F}(x/c) \right).$$

By Lemma 2.5 it holds for sufficiently large $x$ that

$$I_1(x, t) \leq \sum_{n=1}^{m} P \left( \sum_{i=1}^{n} X_i e^{-R(t_i)} 1_{\{N(t) = n\}} + \sup_{0 \leq s \leq t} \tilde{B}(s) > x, N(t) = n \right)$$

$$\sim \sum_{n=1}^{m} \sum_{i=1}^{n} P \left( X_i e^{-R(t_i)} > x, N(t) = n \right)$$

$$\leq \sum_{n=1}^{m} \sum_{i=1}^{n} P \left( X_i e^{-R(t_i)} > x, N(t) = n \right)$$

$$= \int_{0^-}^{t} P \left( X_1 e^{-R(s)} > x \right) \lambda(ds).$$

By Lemma 2.7, for $p > J_p^+$, there exists a constant $M = M(p) > 0$ such that for all $x \geq 0$ and $n \geq 1$,

$$P \left( \sum_{i=1}^{n} X_i + \sup_{0 \leq s \leq t} \tilde{B}(s) > x \right) \leq M n^{p+1} \mathcal{F}(x).$$

Hence, by (29) it holds for all $x \geq 0$ that

$$I_2(x, t) = \sum_{n=m+1}^{\infty} P \left( \sum_{i=1}^{n} X_i e^{-R(t_i)} + \sup_{0 \leq s \leq t} \tilde{B}(s) > x, N(t) = n \right)$$

$$\leq \sum_{n=m+1}^{\infty} P \left( \sum_{i=1}^{n} X_i + \sup_{0 \leq s \leq t} \tilde{B}(s) > x, N(t) = n \right)$$

$$= \sum_{n=m+1}^{\infty} P \left( \sum_{i=1}^{n} X_i + \sup_{0 \leq s \leq t} \tilde{B}(s) > x \right) P(N(t) = n)$$

$$\leq \sum_{n=m+1}^{\infty} M n^{p+1} \mathcal{F}(x) P(N(t) = n)$$

$$= M \mathcal{F}(x) E \left( N(t)^{p+1} 1_{\{N(t) \geq m+1\}} \right).$$

Taking $0 < \varepsilon_0 < 1$ such that for sufficiently large $x$,

$$\mathcal{F} \left( \frac{x}{\varepsilon_0} \right) > 0$$

and

$$\int_{0^-}^{t} P \left( e^{-R(s)} > \varepsilon_0 \right) \lambda(ds) > 0.$$

Thus, for all $x \geq 0$,

$$\int_{0^-}^{t} P \left( X_1 e^{-R(s)} > x \right) \lambda(ds)$$

$$= \int_{0^-}^{t} \int_{0}^{1} \mathcal{F} \left( \frac{x}{u} \right) P \left( e^{-R(s)} \in du \right) \lambda(ds).$$
By (30) and (31), since $F \in \mathcal{D}$ and $EN(t)^{p+1} < \infty$, it holds that

\begin{align*}
\lim_{m \to \infty} \limsup_{x \to \infty} & \int_0^t \int_{\varepsilon_0}^1 F \left( \frac{x}{u} \right) P \left( e^{-R(s)} \in du \right) \lambda(ds) \\
\ge & \ F \left( \frac{x}{\varepsilon_0} \right) \int_0^t \int_{\varepsilon_0}^1 P \left( e^{-R(s)} > \varepsilon_0 \right) \lambda(ds) \\
\ge & \ F \left( \frac{x}{\varepsilon_0} \right) \int_0^t P \left( e^{-R(s)} > \varepsilon_0 \right) \lambda(ds) (31)
\end{align*}

By (27), (28) and (32), it holds that

\begin{align*}
\lim_{m \to \infty} \limsup_{x \to \infty} & \int_0^t \int_{\varepsilon_0}^1 P \left( X_1 e^{-R(s)} > x \right) \lambda(ds) \\
\le & \ \lim_{m \to \infty} \limsup_{x \to \infty} \frac{M \mathcal{F}(x) E \left( N(t)^{p+1} \mathbb{1}_{\{N(t) \geq m+1\}} \right)}{\mathcal{F} \left( \frac{x}{\varepsilon_0} \right) \int_0^t P \left( e^{-R(s)} > \varepsilon_0 \right) \lambda(ds)} \\
\le & \ \lim_{m \to \infty} \int_0^t P \left( e^{-R(s)} > \varepsilon_0 \right) \lambda(ds) \limsup_{x \to \infty} \frac{\mathcal{F}(x)}{\mathcal{F} \left( \frac{x}{\varepsilon_0} \right)} \lim_{m \to \infty} E \left( N(t)^{p+1} \mathbb{1}_{\{N(t) \geq m+1\}} \right) \\
= & \ 0. (32)
\end{align*}

By (27), (28) and (32), it holds that

$$\psi(x, t) < \int_0^t P \left( X_1 e^{-R(s)} > x \right) \lambda(ds).$$

Now we estimate the lower bound of $\psi(x, t)$. For each fixed $t > 0$ and all $x > 0$,

$$\psi(x, t) \geq P \left( D(t) - \sup_{0 \leq s \leq t} \tilde{B}(s) - \tilde{C}(t) > x \right)$$

$$= \sum_{n=1}^\infty P \left( \sum_{i=1}^n X_i e^{-R(\tau_i)} - \sup_{0 \leq s \leq t} \tilde{B}(s) - \tilde{C}(t) > x, N(t) = n \right)$$

$$\geq \sum_{n=1}^\infty P \left( \sum_{i=1}^n X_i e^{-R(\tau_i)} \mathbb{1}_{\{N(t) = n\}} - \sup_{0 \leq s \leq t} \tilde{B}(s) - \tilde{C}(t) > x \right).$$

For any fixed integer $m > 0$ and for all $x > 0$, since $- \sup_{0 \leq s \leq t} \tilde{B}(s) - \tilde{C}(t) \leq 0$, by Lemma 2.5, it holds that

$$\psi(x, t) \geq \sum_{n=1}^m P \left( \sum_{i=1}^n X_i e^{-R(\tau_i)} \mathbb{1}_{\{N(t) = n\}} - \sup_{0 \leq s \leq t} \tilde{B}(s) - \tilde{C}(t) > x \right).$$

$$\sim \sum_{n=1}^m \sum_{i=1}^n P \left( X_i e^{-R(\tau_i)} > x, N(t) = n \right).$$

$$= \left( \sum_{n=1}^m - \sum_{n=m+1}^\infty \right) \sum_{i=1}^n P \left( X_i e^{-R(\tau_i)} > x, N(t) = n \right)$$

$$= J_1(x, t) - J_2(x, t). (33)$$

For all $x > 0$,

$$J_1(x, t) = \sum_{n=1}^\infty \sum_{i=1}^n P \left( X_i e^{-R(\tau_i)} > x, N(t) = n \right)$$

$$= \int_0^t P \left( X_1 e^{-R(s)} > x \right) \lambda(ds) (34)$$
and
\[
J_2(x, t) = \sum_{n=m+1}^{\infty} \sum_{i=1}^{n} P \left( X_i e^{-R(t_i)} > x, N(t) = n \right)
\leq \sum_{n=m+1}^{\infty} \sum_{i=1}^{n} P(X_i > x) P(N(t) = n)
= \bar{F}(x) E[N(t)] 1_{\{N(t) \geq m+1\}}.
\]
Similarly to (32), by (31), \( F \in \mathcal{D} \) and \( E[N(t)] < \infty \), it holds that
\[
\lim_{m \to \infty} \limsup_{x \to \infty} \frac{J_2(x, t)}{\int_{0}^{t} P \left( X_1 e^{-R(s)} > x \right) \lambda(ds)}
\leq \left( \int_{0}^{t} P \left( e^{-R(s)} > \epsilon_0 \right) \lambda(ds) \right)^{-1} \limsup_{x \to \infty} \frac{\bar{F}(x)}{\bar{F}(\epsilon_0)} \lim_{m \to \infty} E[N(t)] 1_{\{N(t) \geq m+1\}}
= 0,
\]
which combining with (33) and (34) yields that
\[
\psi(x, t) > \int_{0}^{t} P \left( X_1 e^{-R(s)} > x \right) \lambda(ds).
\]
This completes the proof of Theorem 1.1. \( \square \)

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