GIT-EQUIVALENCE AND DIAGONAL ACTIONS

POLINA YU. KOTENKOVA

Abstract. We describe the GIT-equivalence classes of linearized ample line bundles for the diagonal actions of the linear algebraic groups SL(V) and SO(V) on \( \mathbb{P}(V)^{m_1} \times \mathbb{P}(V^*)^{m_2} \) and \( \mathbb{P}(V)^m \) respectively.

1. Introduction

Let \( G \) be a complex reductive algebraic group, \( X \) a projective \( G \)-variety, and \( L \) an ample \( G \)-linearized line bundle over the variety \( X \). Classical Mumford’s construction, see [4], takes these objects to the open subset of semi-stable points

\[
X_L^{ss} = \{ x \in X : F(x) \neq 0 \text{ for some } m > 0 \text{ and } F \in \Gamma(X, L^{\otimes m})^G \}
\]

and the categorical quotient \( X_L^{ss} \rightarrow X_L^{ss}/G \). Two \( G \)-linearized line bundles \( L_1 \) and \( L_2 \) over the \( G \)-variety \( X \) are called GIT-equivalent if \( X_L^{ss} = X_{L_2}^{ss} \). The papers [3], [6] and [7] are devoted to the study of GIT-equivalence. It is shown that the GIT-equivalence classes define the fan structure on the cone of \( G \)-linearized ample line bundles. The main approach used to describe the cones in this fan is the Hilbert-Mumford criterion [4, Chapter 2]; also see [3, Example 3.3.24], where the GIT-equivalence classes are described for the diagonal action of the group \( SL(V) \) on the variety \( \mathbb{P}(V)^m \).

In [2] an elementary description of GIT-equivalence classes for algebraic torus actions is obtained. The authors use so-called orbit cones. Using the Cox construction, in [1] this description is adopted for a large class of \( G \)-varieties, compare [7, Section 3]. In [1, Theorem 6.2] there is also a description of the GIT-equivalence classes for the diagonal action of the symplectic group \( Sp(V) \) on \( \mathbb{P}(V)^m \).

The aim of this paper is to find the GIT-equivalence classes for the diagonal actions of other classical groups. In Section 2 we give some necessary information from works [1] and [2]. In Section 3 we describe the GIT-fan for the diagonal action of the special orthogonal group \( SO(V) \), and in Section 4 for the diagonal action of the group \( SL(V) \) on the variety \( \mathbb{P}(V)^{m_1} \times \mathbb{P}(V^*)^{m_2} \).

These results are based on the description of generators of the algebra of invariants (The First Fundamental Theorem of the Classical Invariant Theory).

The author is grateful to I.V. Arzhantsev for posing the problem and his permanent support.

2. Orbit cones and the GIT-fan

Let \( G \subseteq GL(V) \) be a complex algebraic group acting diagonally on the space \( V = V^{m_1} \oplus (V^*)^{m_2} \), \( m_1 + m_2 = m \). Denote by \( P(a_1, \ldots, a_m) \subset \mathbb{C}[V] \), \( a_i \in \mathbb{Z}_{\geq 0} \), the subspace consisting of homogeneous polynomials of multidegree \( (a_1, \ldots, a_m) \). To each vector \( a = (a_1, \ldots, a_m) \in \mathbb{Z}_{\geq 0}^m \) assign the open subset

\[
U(a) = \{ v \in V \mid \exists k \in \mathbb{N}, F \in P(ka_1, \ldots, ka_m)^G : F(v) \neq 0 \},
\]

where \( P(ka_1, \ldots, ka_m)^G \subseteq P(ka_1, \ldots, ka_m) \) is the subspace of \( G \)-invariants. Note that the subset \( U(a) \) corresponds to the set of semistable points \( X_L^{ss} \), where \( X = \mathbb{P}(V)^{m_1} \times \mathbb{P}(V^*)^{m_2} \). Here the line bundle \( L \) is represented by the point \( a \in \mathbb{Z}^m \cong \text{Pic}(X) \).

Two points \( a \) and \( b \in \mathbb{Z}_{\geq 0}^m \) are called GIT-equivalent if \( U(a) = U(b) \).

Assume that the algebra of invariants \( \mathbb{C}[V]^G \) is finitely generated and \( F_1, \ldots, F_r \) are its homogeneous generators. Denote by \( a(1), \ldots, a(r) \in \mathbb{Z}_{\geq 0}^m \) the multidegrees of \( F_1, \ldots, F_r \).
**Proposition 1.** The weight cone $\Omega \subset \mathbb{Q}^m$ is the cone generated by $a(1), \ldots, a(r)$.

**Lemma 1.** The set $U(a)$ is non-empty if and only if $a \in \Omega$.

**Proof.** Suppose $U(a) \neq \emptyset$. Then there exist $k \in \mathbb{N}$ and $F \in P_1(k a_1, \ldots, k a_m)^G$ such that $F \neq 0$. Since $F \in \mathbb{C}[F_1, \ldots, F_r]$, we have

$$F = \sum_{p_1, \ldots, p_r} c_{p_1, \ldots, p_r} F_{p_1}^{d_1} \ldots F_{p_r}^{d_r}.$$ 

If $c_{p_1, \ldots, p_r} \neq 0$, then we obtain $ka = p_1 a(1) + \ldots + p_r a(r)$. Hence $a \in \Omega$.

Conversely, assume that $a \in \Omega$. Then

$$a = \lambda_1 a(1) + \ldots + \lambda_r a(r),$$

where $\lambda_1, \ldots, \lambda_r \in \mathbb{Q}_{\geq 0}$. Multiplying this equation by the common denominator of $\lambda_1, \ldots, \lambda_r$, we get

$$ka = c_1 a(1) + \ldots + c_r a(r),$$

where $k, c_1, \ldots, c_r \in \mathbb{Z}_{\geq 0}$. Then $F_{p_1}^{d_1} \ldots F_{p_r}^{d_r} \in P(k a)^G$, $F_{p_1}^{d_1} \ldots F_{p_r}^{d_r} \neq 0$. Hence $U(a) \neq \emptyset$. \hfill $\square$

Let $v \in V$. The orbit cone associated to $v$ is the rational cone

$$\omega(v) = \text{cone}(a | \exists F \in P(a)^G : F(v) \neq 0).$$

**Lemma 2.** One has $\omega(v) = \text{cone}(a(i) | F_i(v) \neq 0)$.

**Proof.** It is evident that $\text{cone}(a(i) | F_i(v) \neq 0)$ is contained in $\omega(v)$.

Consider the point $a \in \mathbb{Z}_{\geq 0}^m$ and the polynomial

$$F = \sum_{p_1, \ldots, p_r} c_{p_1, \ldots, p_r} F_{p_1}^{d_1} \ldots F_{p_r}^{d_r} \in P(a)^G,$$

such that $F(v) \neq 0$. Then there is a summand $c_{p_1, \ldots, p_r} F_{p_1}^{d_1} \ldots F_{p_r}^{d_r}$ not vanishing at $v$. Further, if $p_i \neq 0$, then $F_i(v) \neq 0$. Therefore $a = p_i a(i_1) + \ldots + p_s a(i_s)$, where $F_i(v) \neq 0$, $l = 1, \ldots, s$. Hence we obtain the inverse inclusion $\text{cone}(a(i) | F_i(v) \neq 0) \supseteq \omega(v)$. \hfill $\square$

**Corollary.** The set of cones $\{\omega(v) | v \in V\}$ is finite.

**Proposition 1.** Two points $a$ and $b$ are GIT-equivalent if and only if for any $v \in V$ either $a \in \omega(v)$ and $b \in \omega(v)$, or $a \not\in \omega(v)$ and $b \not\in \omega(v)$.

**Proof.** Suppose $U(a) = U(b)$ and $a \in \omega(v)$. It follows from Lemma 2 that

$$a = \lambda s a(i) + \ldots + \lambda r a(i_r),$$

where $F_{i_j}(v) \neq 0$, $j = 1, \ldots, s$, $\lambda_j \in \mathbb{Q}_{\geq 0}$. Multiplying this equation by the common denominator of $\lambda_1, \ldots, \lambda_r$, we get

$$ka = p s a(i) + \ldots + p r a(i_r),$$

where $k, p_1, \ldots, p_r \in \mathbb{Z}_{\geq 0}$. Then $F_{p_1}^{d_1} \ldots F_{p_r}^{d_r} \in P(k a)^G$ does not vanish at $v$, and hence $v \in U(a) = U(b)$. Therefore there exist $l \in \mathbb{N}$ and $F \in P(lb)^G$ such that $F(v) \neq 0$. Thus $lb \in \omega(v)$ and $b \in \omega(v)$. Similarly if $b \in \omega(v)$, then $a \in \omega(v)$.

It can easily be checked that $a \in \omega(v)$ if and only if $v \in U(a)$. If for any $v \in V$ either $a \in \omega(v)$ and $b \in \omega(v)$, or $a \not\in \omega(v)$ and $b \not\in \omega(v)$ hold, then for any $v \in V$ we have either $v \in U(a)$ and $v \in U(b)$, or $v \not\in U(a)$ and $v \not\in U(b)$. Hence $U(a) = U(b)$. \hfill $\square$

The GIT-cone of a point $a \in \mathbb{Z}_{\geq 0}^m$ is the cone $\tau(a) = \bigcap_{a \in \omega(v)} \omega(v)$.

Recall that a finite set $\{\sigma_i\}$ of cones in $\mathbb{Q}^m$ is called a fan, if

1. each face of a cone in $\{\sigma_i\}$ is also a cone in $\{\sigma_i\}$;
2. the intersection of two cones in $\{\sigma_i\}$ is a face of each.
Theorem 1. The set of the cones \( \Psi = \{ \tau(a) \mid a \in \Omega \} \) is a fan.

The proof may be found in [2, Theorem 2.11].

The fan \( \Psi \) is called the GIT-fan. It follows from Proposition 1 that the classes of GIT-equivalence are relative interiors of GIT-cones.

Let \( T = (\mathbb{C}^\times)^m \) be a torus. It acts on the space \( V \) as

\[
t \circ (v_1, \ldots, v_m, l_1, \ldots, l_m) = (t_1 v_1, \ldots, t_m v_m, s_1 l_1, \ldots, s_m l_m),
\]

where \( t = (t_1, \ldots, t_m, s_1, \ldots, s_m) \in T, (v_1, \ldots, v_m, l_1, \ldots, l_m) \in V \). This action commutes with the action of \( G \), hence the action of \( T \) on the categorical quotient \( V//G := \text{Spec} \mathbb{C}[V]^G \) is well defined.

Consider a point \( v \in V \). It is not hard to see that

\[
\dim \omega(v) + \dim T_{\pi(v)} = \dim T,
\]

where \( \pi : V \longrightarrow V//G \) is the quotient morphism, and \( T_{\pi(v)} \) is the stabilizer of the point \( \pi(v) \).

Note that our definition of the orbit cone agrees with [2, Definition 2.1] for the action of the torus \( T \) on the variety \( V//G \).

3. The case of \( \text{SO}(V) \)

Consider \( G = \text{SO}(V), \dim V \geq 3 \). Let \( (\ldots) \) be a non-degenerate symmetric bilinear form on \( V \) preserved by \( \text{SO}(V) \). Since \( V \) is \( G \)-isomorphic to its dual \( V^* \), we can assume that \( m_2 = 0, m = m_1, \) and \( V = V^m \). Let us construct the GIT-fan for the diagonal action \( \text{SO}(V) \) on the variety \( \mathbb{P}(V)^m \).

The algebra of invariants for the action \( \text{SO}(V) \) on \( V \) is generated by \( u_{ij} = (v_i, v_j) \), where \( (v_1, \ldots, v_m) \in V^m \) [5, § 9.3]. Here the multidegrees are \( f_{ii} := (0, \ldots, 0, 1, 0, \ldots, 0, 0, \ldots, 0) \) and \( f_{ij} := (0, \ldots, 0, 1_i, 0, \ldots, 1_j, 0, \ldots, 0) \), \( i, j = 1, \ldots, m \). The morphism \( \pi : V \longrightarrow V//G \) sends \((v_1, \ldots, v_m)\) to the symmetric matrix \((v_i, v_j)_{i,j=1}^m\).

The GIT-fan is contained in \( \mathbb{Q}^m \). Let \( x_1, \ldots, x_m \) be the coordinates in this space.

Clearly, the weight cone \( \Omega \) generated by \( f_{ij} \) is given by inequalities

\[
x_i \geq 0, \ i = 1, \ldots, m.
\]

Proposition 2. Each \((m - 1)\)-dimensional orbit cone lies in some hyperplane

\[
\sum_{i \in I} x_i = \sum_{j \in J} x_j,
\]

where \( I, J \subset \{1, \ldots, m\}, I \neq \emptyset, J \neq \emptyset, I \cap J = \emptyset \).

Proof. The torus \( T = (\mathbb{C}^\times)^m \) acts on \( V^m: \ t \circ (v_1, \ldots, v_m) = (t_1 v_1, \ldots, t_m v_m) \). Then \( t \circ (v_i, v_j) = t_i t_j (v_i, v_j) \). The orbit cone associated to \( v = (v_1, \ldots, v_m) \) is \((m - 1)\)-dimensional if and only if the stabilizer \( T_{\pi(v)} \) of the point \( \pi(v) \) is one-dimensional.

Consider a graph \( \Gamma_v \) with the set of vertices \( \{v_1, \ldots, v_m\} \). By definition, \( v_i \) and \( v_j \) are joined by an edge in \( \Gamma_v \) if and only if \( (v_i, v_j) \neq 0 \). Assume that \( (v_i, v_j) \neq 0 \). Then any \( t \in T_{\pi(v)} \) satisfies \( t_i = t_j^{-1} \).

Let \( \Gamma_v = \Gamma_1 \sqcup \ldots \sqcup \Gamma_l \) be the decomposition into connected components. If \( \Gamma_k \) contains a cycle of odd length or a loop (type A), then \( t_i^2 = 1 \) for all \( v_i \in \Gamma_k \) and \( t \in T_{\pi(v)} \). In other case (type B), it is possible to divide the set of vertices of \( \Gamma_k \) into two subsets. For a point of the first subset \( t_i = s_k \) holds, and for a point of the second subset we have \( t_i = (s_k)^{-1} \), where \( s_k \in \mathbb{C}^\times \). The stabilizer is one-dimensional if and only if there is only one component of type B in the graph \( \Gamma_v \).

Denote by \( I \) and \( J \) the sets of vertices in the first and the second subsets of this component. The weight \( f_{ij} \) lies in \( \omega(v) \) if and only if \( i \in I, j \in J \) or \( j \in I, i \in J \). Hence the orbit cone is contained in hyperplane (1).
It follows from Proposition 2 that if dimension of the orbit cone $\omega(v)$ is less than $m - 1$, then $\omega(v)$ lies in the intersection of some $(m - 1)$-dimensional orbit cones. Thus two points are GIT-equivalent if and only if they lie in the same $m$-dimensional and $(m - 1)$-dimensional orbit cones.

**Theorem 2.** For the diagonal action of the group $SO(V)$ on the variety $\mathbb{P}(V)^m$ the GIT-fan is obtained by cutting of the cone

$$\Omega = \{(x_1, \ldots, x_m) | x_i \geq 0\}$$

by hyperplanes

$$\sum_{i \in I} x_i = \sum_{j \in J} x_j,$$

where $I, J \subset \{1, \ldots, m\}$, $I \neq \emptyset$, $J \neq \emptyset$, $I \cap J = \emptyset$.

**Proof.** It is sufficient to find all $(m - 1)$-dimensional orbit cones. It follows from Proposition 2 that we should only prove that the intersection of each hyperplane (1) with the cone $\Omega$ is the orbit cone for some point $v$.

Let $v_k = (1, i, 0, \ldots, 0)$ for $k \in I$, $v_j = (1, -i, 0, \ldots, 0)$ for $j \in J$, and $v_l = (0, 0, 1, 0, \ldots, 0)$ for $l \notin I \cup J$. (Here $i^2 = -1$). The orbit cone associated to $v$ is generated by the weights $f_{kj}(k \in I, j \in J)$ and $f_{il}(l \notin I \cup J)$. Hence $\omega(v)$ is $(m - 1)$-dimensional and lies in hyperplane (1). It is easy to check that the rays $(f_{kj})(k \in I, j \in J)$ and $(f_{il})(l \notin I \cup J)$ are precisely the edges of the intersection of $\Omega$ with hyperplane (1). This completes the proof of Theorem 2.

**Example.** Consider the action of $SO_3$ on the space $\mathbb{C}^3 \oplus \mathbb{C}^3 \oplus \mathbb{C}^3$. The weight cone is the cone $\Omega = \{(x_1, x_2, x_3) \in \mathbb{Q}^3 | x_1 \geq 0, x_2 \geq 0, x_3 \geq 0\}$. The GIT-fan is obtained by cutting of the cone $\Omega$ by hyperplanes

$$x_1 = x_2, \quad x_1 = x_3, \quad x_2 = x_3,$$

$$x_1 + x_2 = x_3, \quad x_1 + x_3 = x_2, \quad x_2 + x_3 = x_1.$$

The intersection of the GIT-fan with the hyperplane $x_1 + x_2 + x_3 = 1$ looks like:

![ GIT-fan intersection diagram ]

There are 33 classes of GIT-equivalence: 12 classes are three-dimensional, 21 classes are two-dimensional, and 10 classes are one-dimensional.

4. **The case of $SL(V)$**

Consider $G = SL(V)$. Let us construct the GIT-fan for the diagonal action $SL(V)$ on the variety $\mathbb{P}(V)^{m_1} \times \mathbb{P}(V^*)^{m_2}$.

The algebra of invariants $\mathbb{C}[V]^G$, where $V = V^{m_1} \oplus (V^*)^{m_2}$, is generated by $\det(v_{i_1}, \ldots, v_{i_n})$, $\det(l_{j_1}, \ldots, l_{j_n})$, and $l_j(v_i)$, where $(v_1, \ldots, v_{m_1}, l_1, \ldots, l_{m_2}) \in V^{m_1} \oplus (V^*)^{m_2}$ [5, § 9.3]. Here the multidegrees are

$$f_{i_1 \ldots i_n} = (\alpha^{m_1}_{i_1 \ldots i_n}, \ldots, \alpha^{m_2}_{i_1 \ldots i_n}, 0, \ldots, 0), \quad g_{j_1 \ldots j_n} = (0, \ldots, 0, \beta^{m_1}_{j_1 \ldots j_n}, \ldots, \beta^{m_2}_{j_1 \ldots j_n}),$$

and $h_{ij} = (\varepsilon^{m_1}_{ij}, \varepsilon^{m_2}_{ij}, \delta_{ij}, \ldots, \delta_{ij}^{m_2})$.
Proof. If there is a zero vector or a zero function in the set \( v \), assume that all the components of the associated \( \text{cone} \) of \( v \) are zero. Consider the orbit \( \Gamma \) of \( v \) and the orbit \( \text{cone} \) associated to \( v \). The case \( 1 \) occurs in the decomposition of \( v \). Further, we assume that all the components of \( v \) are zero.

The GIT-fan is contained in \( \mathbb{Q}^m \). Let \( x_1, \ldots, x_{m_1}, y_1, \ldots, y_{m_2} \) be the coordinates in this space. First, suppose that \( m_1 \geq n \) or \( m_2 \geq n \).

**Proposition 3.** Each \((m-1)\)-dimensional orbit cone lies in one of the hyperplanes

\[
\begin{align*}
(2) & \quad x_i = 0, \ i = 1, \ldots, m_1, \\
(3) & \quad y_j = 0, \ j = 1, \ldots, m_2, \\
(4) & \quad x_1 + \ldots + x_{m_1} = y_1 + \ldots + y_{m_2}, \\
(5) & \quad (n - k) \sum_{i \in I} x_i - k \sum_{i \notin I} x_i = (n - k) \sum_{j \in J} y_j - k \sum_{j \notin J} y_j,
\end{align*}
\]

where \( 1 \leq k \leq n - 1 \), \( I \subset \{1, \ldots, m_1\} \), \( J \subset \{1, \ldots, m_2\} \), and either \( k \leq |I| \leq m_1 - n + k \) or \( k \leq |J| \leq m_2 - n + k \).

**Proof.** If there is a zero vector or a zero function in the set \( \{v_1, \ldots, v_{m_1}, l_1, \ldots, l_{m_2}\} \), then the orbit \( \text{cone} \) associated to \( v = (v_1, \ldots, v_{m_1}, l_1, \ldots, l_{m_2}) \) lies in hyperplane of type (2) or (3). Further, we assume that all the components of \( v \) are nonzero.

The torus \( T = (\mathbb{C}^*)^m \) acts on \( V^{m_1} \oplus (V^*)^{m_2} \) as above. Then for any \( t \in T \) we have

\[
\begin{align*}
& t \circ l_j(v_i) = t_is_jl_j(v_i), \\
& t \circ \det(v_1, \ldots, v_n) = t_{i_1} \cdots t_{i_n} \det(v_1, \ldots, v_n), \\
& t \circ \det(l_{j_1}, \ldots, l_{j_n}) = s_{j_1} \cdots s_{j_n} \det(l_{j_1}, \ldots, l_{j_n}),
\end{align*}
\]

and the orbit \( \text{cone} \) associated to \( v \) is \((m-1)\)-dimensional if and only if the stabilizer \( T_{\pi(v)} \) of the point \( \pi(v) \) is one-dimensional.

Consider a graph \( \Gamma_v \) with \( \{v_1, \ldots, v_{m_1}, l_1, \ldots, l_{m_2}\} \) as the set of vertices. By definition, \( v_i \) and \( l_j \) are joined by an edge in \( \Gamma_v \) if and only if \( l_j(v_i) \neq 0 \). If vertices \( v_1, v_{i_2}, l_1, l_{j_2} \) lie in the same connected component of the graph \( \Gamma_v \), then any \( t \in T_{\pi(v)} \) satisfies \( t_{i_1} = t_{i_2}, s_{j_1} = s_{j_2}, t_{i_1} = s_{j_1}^{-1} \).

**Case 1:** \( \dim(v_1, \ldots, v_{m_1}) < n, \dim(l_1, \ldots, l_{m_2}) < n. \)

In this case all the determinants are zero. If the stabilizer \( T_{\pi(v)} \) is one-dimensional, then the graph \( \Gamma_v \) is connected. The orbit \( \text{cone} \) is generated by the weights \( \{h_{ij}\} \). Their span is \((m-1)\)-dimensional and lies in the hyperplane \( x_1 + \ldots + x_{m_1} = y_1 + \ldots + y_{m_2} \). Hence the orbit \( \text{cone} \) lies in hyperplane of type (4).

**Case 2:** \( \dim(v_1, \ldots, v_{m_1}) = n, \dim(l_1, \ldots, l_{m_2}) = n. \)

Suppose \( \det(v_1, \ldots, v_n) \neq 0. \) Then \( v_1, \ldots, v_n \) is a basis of \( V \). For any \( t \in T_{\pi(v)} \) the equation \( t_{i_1} \ldots t_{i_n} = 1 \) holds. If \( v_1 \) occurs in the decomposition of \( v \) with respect to the basis \( v_{i_1}, \ldots, v_{i_n} \), then \( \det(v_1, v_{i_2}, \ldots, v_{i_n}) \neq 0 \) and \( t_it_{i_2} \cdots t_{i_n} = t_{i_1}t_{i_2} \cdots t_{i_n} = 1 \). Hence \( t_i = t_{i_1}. \) Similarly consider other \( v_{i_j}, \) where \( j = 2, \ldots, n. \) Thus the space \( V \) decomposes into the sum \( V = V_{k_1} \oplus \cdots \oplus V_{k_n} \). The torus \( T \) multiplies any \( V_{k_i} \) by \( t_j = t_{i_1} \), for some \( j \). In the same way \( V^* \) decomposes into the sum \( V^* = \tilde{W}_{k_1} \oplus \cdots \oplus \tilde{W}_{k_n} \), the torus \( T \) acts on any \( \tilde{W}_{k_i} \) as multiplication by \( \tilde{h}_{ij}. \) Thus any element \( t \in T_{\pi(v)} \) satisfies the conditions \( (\tilde{t}_j)_{i \in V_{k_1}} \cdots (\tilde{t}_j)_{i \in V_{k_n}} ). \)

Consider a new graph \( \Gamma_v' \) with \( V_{k_1}, \ldots, V_{k_n}, W_{k_1}, \ldots, W_{k_n} \) as the set of vertices. The vertices \( V_k \) and \( W_k \) are joined by an edge in \( \Gamma_v' \) if and only if there exist \( v_i \in V_k \) and \( l_j \in W_k \) such that \( l_j(v_i) \neq 0. \)

Denote by \( H_1, \ldots, H_p \) the connected components of the graph \( \Gamma_v'. \) Let

\[
V_i' = \bigoplus_{V_k \in H_i} V_k, \quad W_i' = \bigoplus_{W_k \in H_i} W_k.
\]
Then $T_{\pi(v)}$ multiples $V'_i$ by $t'_i$ and $W'_i$ by $(t'_i)^{-1}$. The stabilizer is given by the equations

$$(t'_1)^{\dim V'_1} \cdots (t'_p)^{\dim V'_p} = 1,$$

$$(t'_1)^{\dim W'_1} \cdots (t'_p)^{\dim W'_p} = 1.$$ 

By construction of $H_i$, all linear functions of $W'_i$ vanish at all vectors of $V'_j$ for $i \neq j$, hence $\dim W'_i \leq \dim V'_i$. But $\sum_{i=1}^{p} \dim W'_i = \sum_{i=1}^{p} \dim V'_i$, hence $\dim W'_i = \dim V'_i$.

Thus the stabilizer is given by the equation

$$(t'_1)^{\dim V'_1} \cdots (t'_p)^{\dim V'_p} = 1,$$

and is one-dimensional if and only if $p = 2$.

So if the orbit cone associated to $\nu = (v_1, \ldots, v_{m_1}, l_1, \ldots, l_{m_2})$ is $(m - 1)$-dimensional, then $V = V_1 \oplus V_2$, $V' = W_1 \oplus W_2$, $\dim V_1 = \dim W_1 = k$, $1 \leq k \leq n - 1$; any vector $v_i$ lies in $V_1$ or in $V_2$; any linear function $l_j$ lies in $W_1$ or in $W_2$; any linear function from $W_j$ is zero on any vector from $V_i$ for $i \neq j$.

Let $I$ be the set of numbers of vectors $v_i$ from $V_1$, $J$ be the set of numbers of linear functions $l_j$ from $W_1$. Then the orbit cone associated to $\nu$ lies in hyperplane given by the equation

$$(n - k) \sum_{i \in I} x_i - k \sum_{i \notin I} x_i = (n - k) \sum_{j \in J} y_j - k \sum_{j \notin J} y_j.$$ 

Here the inequalities $k \leq |I| \leq m_1 - n + k$ and $k \leq |J| \leq m_2 - n + k$ are satisfied.

**Case 3:** $\dim \langle v_1, \ldots, v_{m_1} \rangle = n$, $\dim \langle l_1, \ldots, l_{m_2} \rangle < n$ or $\dim \langle v_1, \ldots, v_{m_1} \rangle < n$, $\dim \langle l_1, \ldots, l_{m_2} \rangle = n$. In this case we have one equation on the stabilizer and the graph $\Gamma_v$ should have two connected components. We obtain hyperplanes given by equations (5).

**Proposition 4.** The weight cone $\Omega$ is given by inequalities

$$(6) \quad x_l \geq 0, \quad l = 1, \ldots, m_1, \quad y_p \geq 0, \quad p = 1, \ldots, m_2,$$

$$(n - k) \sum_{j=1}^{m_2} y_j - \sum_{i \in I} x_i + k \sum_{i \notin I} x_i \geq 0, \quad (n - k) \sum_{i=1}^{m_1} x_i - \sum_{j \notin J} y_j + k \sum_{j \in J} y_j \geq 0,$$

where $1 \leq k \leq n - 1$, $I \subset \{1, \ldots, m_1\}$, $J \subset \{1, \ldots, m_2\}$, $|I| = |J| = k$.

**Proof.** It is sufficient to find hyperplanes (2)--(5) which contain facets of the cone $\Omega$. It is clear that hyperplanes (2) and (3) do. Since the weights $\{f_{j_1, \ldots, j_n}\}$ and $\{g_{j_1, \ldots, j_n}\}$ lie on different sides of hyperplane (4), this hyperplane intersects the interior of the cone $\Omega$.

Consider equation (5). First suppose that $0 < |I| < m_1$, $0 < |J| < m_2$. Let $i_1 \in I$, $i_2 \notin I$, $j_1 \in J$, $j_2 \notin J$. The weights $h_{i_1, j_2}$ and $h_{i_2, j_1}$ lie on different sides of the hyperplane. Now let $|J| = m_2$. If $|I| > k$, then there exist numbers $i_1, \ldots, i_{k+1} \in I$, $i_{k+2}, \ldots, i_n \notin I$. Weights $f_{i_1, \ldots, i_n}$ and $g_{j_1, \ldots, j_n}$ lie on different sides from the hyperplane. For $|I| = k$ we obtain inequalities (6). The cases $|I| = 0$, $m_1$ and $|J| = 0$ are analyzed similarly.

It follows from the proof of Proposition 4 that hyperplanes (4) and (5) intersect the interior of the cone $\Omega$. Further, if dimension of the orbit cone $\omega(\nu)$ is less than $m - 1$, then $\omega(\nu)$ lies in the intersection of some $(m - 1)$-dimensional orbit cones. Thus two points are GIT-equivalent if and only if they lie in the same $m$-dimensional and $(m - 1)$-dimensional orbit cones.
Theorem 3. For the diagonal action of the group $SL(V)$ on the variety $\mathbb{P}(V)^{m_1} \times \mathbb{P}(V)^{m_2}$, where $m_1$ or $m_2$ does not exceed $n = \dim V$, the GIT-fan is obtained by cutting the cone $\Omega$ given by inequalities (6) by hyperplanes

\[(4) \quad x_1 + \ldots + x_{m_1} = y_1 + \ldots + y_{m_2},\]

\[(5) \quad (n-k) \sum_{i \in I} x_i - k \sum_{i \notin I} x_i = (n-k) \sum_{j \in J} y_j - k \sum_{j \notin J} y_j,\]

where $1 \leq k \leq n-1$, $I \subset \{1, \ldots, m_1\}$, $J \subset \{1, \ldots, m_2\}$, $k \leq |I| \leq m_1 - n + k$ or $k \leq |J| \leq m_2 - n + k$.

Proof. Let us find all $(m-1)$-dimensional orbit cones. It follows from Proposition 4 that we should only prove that the intersection $\Pi$ of any hyperplane of type (4) or (5) with the cone $\Omega$ is the orbit cone for some point $v$.

For the case of hyperplane (4) let $v = (e_1, \ldots, e_1, e^1, \ldots, e^1)$. The orbit cone associated to $v$ lies in this hyperplane and its dimension equals $m - 1$. Note that the inequalities

\[(n-k)(\sum_{j=1}^{m_2} y_j - \sum_{i \in I} x_i) + k \sum_{i \in I} x_i \geq 0\]

\[(n-k)(\sum_{i=1}^{m_1} x_i - \sum_{j \in J} y_j) + k \sum_{j \notin J} y_j \geq 0,\]

for the points of hyperplane (4) become

\[\sum_{i \notin I} x_i \geq 0 \quad \sum_{j \notin J} y_j \geq 0.\]

Hence the cone $\Pi$ is the intersection of hyperplane (4) with the positive orthant. Finally, the weights which generate the orbit cone $\omega(v)$ and edges of the cone $\Pi$ lie on the same rays.

Denote by $H$ the hyperplane (5). Without loss of generality it can be assumed that equation (5) is of the form

\[(n-k) \sum_{i=1}^{m_1} x_i - k \sum_{i=|I|+1}^{m_1} x_i = (n-k) \sum_{j=1}^{m_2} y_j - k \sum_{j=|J|+1}^{m_2} y_j,\]

where $k \leq |I| \leq m_1 - n + k$.

The orbit cone $\omega(v)$ associated to the point

\[v = (e_1, \ldots, e_k, e_{k+1}, \ldots, e_n, e_n, \ldots, e_n, e^1, \ldots, e^1, e^1, \ldots, e^n, e^{k+1}, \ldots, e^{k+1}, \ldots, e^n),\]

is $(m-1)$-dimensional and lies in hyperplane $H$. It is easy to check that this cone lies in $\Pi$. It remains to prove the converse implication.

Let $A = (x_1, \ldots, x_{m_1}, y_1, \ldots, y_{m_2}) \in \Pi$. Then

\[A = (x_1 - \sum_{j=2}^{l+1} y_j - \alpha)h_{11} + \sum_{i=2}^{k} (x_i - \alpha)h_{i1} + \sum_{i=k+1}^{l} x_i h_{i1} + (x_{|I|+1} - \sum_{j=|J|+2}^{m_2} y_j - \alpha)h_{|I|+1,|J|+1} + \sum_{i=|I|+2}^{l+n-k} (x_i - \alpha)h_{i,|J|+1} + \sum_{i=|I|+n-k+1}^{l+n-k} x_i h_{i,|J|+1} + \sum_{j=|J|+2}^{m_2} y_j h_{|I|+1,|J|+1} + \alpha f_1 \ldots k_{|I|+1,|J|+1} + \alpha f_1 \ldots k_{|I|+1,|J|+1} + \ldots + \alpha f_1 \ldots k_{|I|+1,|J|+1} + \ldots + \alpha f_1 \ldots k_{|I|+1,|J|+1},\]
where $\alpha = \frac{1}{k}(\sum_{i=1}^{|I|} x_i - \sum_{j=1}^{|J|} y_j)$. It follows from inequalities (6), that coefficients of this decomposition are positive. Therefore $A \in \omega(v)$, and $\Pi$ lies in $\omega(v)$. Hence $\Pi$ coincides with $\omega(v)$. This completes the proof of Theorem 3.

Now let $m_1 < n$ and $m_2 < n$. In this case the weight cone is $(m-1)$-dimensional.

**Theorem 4.** For the diagonal action of the group $SL(V)$ on the variety $\mathbb{P}(V)^{m_1} \times \mathbb{P}(V^*)^{m_2}$, where $m_1, m_2 < n = \dim V$, the GIT-fan is obtained by cutting of the cone

$$
\Omega = \{(x_1, \ldots, x_{m_1}, y_1, \ldots, y_{m_2}) \mid x_1 + \ldots + x_{m_1} = y_1 + \ldots + y_{m_2}; x_i, y_j \geq 0\}
$$

by hyperplanes

$$
\sum_{i \in I} x_i = \sum_{j \in J} y_j,
$$

where $I \subset \{1, \ldots, m_1\}$, $J \subset \{1, \ldots, m_2\}$, $I \neq \emptyset$, $\{1, \ldots, m_1\}$, $J \neq \emptyset$, $\{1, \ldots, m_2\}$.

**Proof.** In this case the weight cone is generated by the weights $\{h_{ij}\}$. It is clear that the weight cone is contained in the cone $\Omega$. On the other hand, edges of the cone $\Omega$ are precisely the generators of the weight cone.

We need to find all $(m-2)$-dimensional orbit cones. As above let us construct the graph $\Gamma_v$ for any vector $v$. The stabilizer $T_{\pi(v)}$ should be of dimension two. Hence the graph $\Gamma_v$ has two connected components. In this case the orbit cone $\omega(v)$ is contained in the intersection of the weight cone $\Omega$ with hyperplane (7), where $I$ and $J$ are sets of numbers: $i \in I$ and $j \in J$ if $v_i$ and $l_j$ lie in the first connected component of the graph $\Gamma_v$. Finally it is necessary to prove that intersection of the cone $\Omega$ with hyperplane (7) is the orbit cone associated to some vector $v$. For this let the vector $v$ be $(v_1, \ldots, v_{m_1}, l_1, \ldots, l_{m_2})$, where $v_i = e_1$, $l_j = e^1$, if $i \in I, j \in J$ and $v_i = e_2, l_j = e^2$, if $i \notin I, j \notin J$. This completes the proof of Theorem 4.

**References**

[1] I.V. Arzhantsev, J. Hausen, Geometric Invariant Theory via Cox rings. J. Pure Appl. Algebra 213 (2009), no. 1, 154-172

[2] F. Berchtold, J. Hausen, GIT-equivalence beyond the ample cone. Michigan Math. J. 54 (2006), no. 3, 483-516

[3] I.V. Dolgachev, Y. Hu, Variation of Geometric Invariant Theory quotients. (With an appendix: "An example of a thick wall" by N. Ressayre). Publ. Math., Inst. Hautes Etud. Sci. 87 (1998), 5-56

[4] D. Mumford, J. Fogarty, F. Kirwan, Geometric Invariant Theory. 3rd Edition, in: Ergebnisse der Mathematik und ihrer Grenzgebiete, Springer-Verlag, Berlin, 1993

[5] V.L. Popov, E.B. Vinberg, Invariant theory. – Itogi Nauki i Tekhniki. VINITI. Sovrem. Probl. Mat. Fund. Naprav. – 1989 – Vol. 55, 137-309. English Transl.: Algebraic Geometry IV, Encyclopedia of Math. Science, vol. 55, Springer-Verlag, Berlin, 1994

[6] N. Ressayre, The GIT-equivalence for G-line bundles. Geom. Dedicata 81 (1-3) (2000), 295-324

[7] M. Thaddeus, Geometric Invariant Theory and flips. J. Amer. Math. Soc. 9 (1996), 691-723

**Department of Higher algebra, Faculty of Mechanics and Mathematics, Moscow State Lomonosov University, Lennskie Gory 1, Moscow, 119991, Russia**

**E-mail address:** kotpy@mail.ru