New Upper Bounds on Sizes of Permutation Arrays

Lizhen Yang, Ling Dong, Kefei Chen

Abstract

A permutation array (or code) of length $n$ and distance $d$, denoted by $(n, d)$ PA, is a set of permutations $C$ from some fixed set of $n$ elements such that the Hamming distance between distinct members $x, y \in C$ is at least $d$. Let $P(n, d)$ denote the maximum size of an $(n, d)$ PA. New upper bounds on $P(n, d)$ are given. For constant $\alpha, \beta$ satisfying certain conditions, whenever $d = \beta n^\alpha$, the new upper bounds are asymptotically better than the previous ones.

Index Terms

permutation arrays (PAs), permutation code, upper bound.

I. INTRODUCTION

Let $\Omega$ be an arbitrary nonempty infinite set. Two distinct permutations $x, y$ over $\Omega$ have distance $d$ if $xy^{-1}$ has exactly $d$ unfixed points. A permutation array (permutation code, PA) of length $n$ and distance $d$, denoted by $(n, d)$ PA, is a set of permutations $C$ from some fixed set of $n$ elements such that the distance between distinct members $x, y \in C$ is at least $d$. An $(n, d)$ PA of size $M$ is called an $(n, M, d)$ PA. The maximum size of an $(n, d)$ PA is denoted as $P(n, d)$.

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PAs are somewhat studies in the 1970s. A recent application by Vinck \cite{1}, \cite{2}, \cite{3}, \cite{4} of PAs to a coding/modulation scheme for communication over power lines has created renewed interest in PAs. But there are still many problems unsolved in PAs, e.g. one of the essential problem is to compute the values of \( P(n, d) \). It’s known that determining the exactly values of \( P(n, d) \) is a difficult task, except for special cases, it can be only to establish some lower bounds and upper bounds on \( P(n, d) \). In this correspondence, we give some new upper bounds on \( P(n, d) \), which are asymptotically better than the previous ones.

A. Concepts and Notations

We introduce concepts and notations that will be used throughout the correspondence.

Since for two sets \( \Omega, \Omega' \) of the same size, the symmetric groups \( \text{Sym}(\Omega) \) and \( \text{Sym}(\Omega') \) formed by the permutations over \( \Omega \) and \( \Omega' \) respectively, under compositions of mappings, are isomorphic, we need only to consider the PAs over \( Z_n = \{0, 1, \ldots, n-1\} \) and write \( S_n \) to denote the special group \( \text{Sym}(Z_n) \). In the rest of the correspondence, without special pointed out, we always assume that PAs are over \( Z_n \). We also write a permutation \( a \in S_n \) as an \( n \)-tuple \( (a_0, a_1, \ldots, a_{n-1}) \), where \( a_i \) is the image of \( i \) under \( a \) for each \( i \). Especially, we write the identical permutation \( (0, 1, \ldots, n-1) \) as \( 1 \) for convenience. The Hamming distance \( d(a, b) \) between two \( n \)-tuples \( a \) and \( b \) is the number of positions where they differ. Then the distance between any two permutations \( x, y \in S_n \) is equivalent to their Hamming distance.

Let \( C \) be an \( (n, d) \) PA. For an arbitrary permutation \( x \in S_n \), \( d(x, C) \) stands for the Hamming distance between \( x \) and \( C \), i.e., \( d(x, C) = \min_{c \in C} d(x, c) \). A permutation in \( C \) is also called a codeword of \( C \). For convenience for discussion, without loss of generality, we always assume that \( 1 \in C \), and the indies of an \( n \)-tuple (vector, array) are started by 0. The support of a binary vector \( a = (a_0, a_1, \ldots, a_{n-1}) \in \{0, 1\}^n \) is defined as the set \( \{i : a_i = 1, i \in Z_n\} \), and the weight of \( a \) is the size of its support, namely the number of ones in \( a \). The support of a permutation \( x = (x_0, x_1, \ldots, x_{n-1}) \in S_n \) is defined as the set of the points not fixed by \( x \), namely \( \{i \in Z_n : x_i \neq i\} = \{i \in Z_n : x(i) \neq i\} \), and the weight of \( x \), denoted as \( wt(x) \), is defined as the size of its support, namely the number of points in \( Z_n \) not fixed by \( x \).

A derangement of order \( k \) is an element of \( S_k \) with no fixed points. Let \( D_k \) be the number of derangements of order \( k \), with the convention that \( D_0 = 1 \). Then \( D_k = k! \sum_{i=0}^{k} \frac{(-1)^k}{k!} = \left[ e^{1/e} \right] \), where \( \lceil x \rceil \) is the nearest integer function, and \( e \) is the base of the natural logarithm. The ball in
$S_n$ of radius $r$ with center $x$ is the set of all permutations of distance $\leq r$ from $x$. The volume of such a ball is

$$V(n, r) = \sum_{i=0}^{r} \binom{n}{i} D_i.$$ (1)

An $(n, d, w)$ constant-weight binary code is a set of binary vectors of length $n$, such that each vector contains $w$ ones and $n-w$ zeros, and any two vectors differ in at least $d$ positions. The largest possible size of an $(n, d, w)$ constant-weight binary code is denoted as $A(n, d, w)$. Similarly, we define an $(n, d, w)$ constant-weight PA as an $(n, d)$ PA such that each permutation is of weight $w$, and denote the largest possible size of an $(n, d, w)$ constant-weight PA as $P(n, d, w)$.

The concept of $P(n, d)$ can be further generalized. Let $\Omega \subseteq S_n$, then $P_\Omega(n, d)$ denotes the maximum size of an $(n, d)$ PA $C$ such that $C \subseteq \Omega$. For trivial case $\Omega = S_n$, $P(n, d) = P_\Omega(n, d)$.

B. Previous Results

The most basic upper bound on $P(n, d)$ is given by Deza and Vanstone [?].

Theorem 1: [?].

$$P(n, d) \leq \frac{n!}{(d-1)!}$$ (2)

We call the PAs which attain the Deza-Vanstone bound perfect PAs and the known perfect PAs are

- $(n, n, n)$ PAs for each $n \geq 1$;
- $(n, n!, 2)$ PAs for each $n \geq 1$;
- $(n, n!/2, 3)$ PAs for each $n \geq 1$ [?];
- $(q, q(q-1), q-1)$ PAs for each prime power $q$ [?];
- $(q+1, (q+1)q(q-1), q-1)$ PAs for each prime power $q$ [?];
- $(11, 11 \cdot 10 \cdot 9 \cdot 8, 8)$ PA [?];
- $(12, 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8, 8)$ PA [?].

The Deza-Vanstone bound can be derived by recursively applying the following inequality.

Proposition 1: [?].

$$P(n, d) \leq nP(n-1, d).$$ (3)

Then for $d \leq m < n$, if we know $P(m, d) \leq M < \frac{m!}{(d-1)!}$, we can get a stronger upper bound on $P(n, d)$:

$$P(n, d) \leq \frac{n!P(m, d)}{m!} \leq \frac{n!M}{m!}. $$
Another nontrivial upper bound on $P(n, d)$ is the sphere packing bound obtained by considering the balls of radius $\lfloor (d - 1)/2 \rfloor$ [?].

**Theorem 2:**

$$P(n, d) \leq \frac{n!}{V(n, \lfloor (d - 1)/2 \rfloor)}.$$  \hfill (4)

For small values of $n$ and $d$, still stronger upper bounds are founds in Tarnanen [?] by the method of linear programming.

### C. Organization and New Results

The correspondence is organized as follows. In Section II, we first prove a relation between $P(n, d)$ and $P_\Omega(n, d)$ that is the inequality

$$P(n, d) \leq \frac{n!P_\Omega(n, d)}{|\Omega|}.$$  \hfill (5)

Next, we give some elementary properties of $P(n, d, w)$, and then use them to show a new upper bound on $P(n, d)$ for $d$ is even and a new upper bound on $P(n, d)$ for $d$ is odd. They are given by the following inequalities:

$$P(n, 2k) \leq \frac{n!}{V(n, k - 1) + \binom{n}{k}D_k}, \text{ for } 2 \leq k \leq \lfloor n/2 \rfloor;$$

$$P(n, 2k + 1) \leq \frac{n!}{V(n, k) + \binom{n}{k + 1}D_{k+1} - A(n-k, 2k, k+1)\binom{n}{k}}A_{A(n, 2k, k+1)}^{-1}, \text{ for } 2 \leq k \leq \lfloor (n - k - 1)/2 \rfloor.$$  \hfill (6)

In Section III, we compare the upper bounds on $P(n, d)$ and show for constant $\alpha, \beta$ satisfying certain conditions, whenever $d = \beta n^\alpha$, the new upper bounds are asymptotically better than the previous ones.

### II. The New Upper Bounds

**Theorem 3:** Let $\Omega$ be a subset of $S_n$. Then

$$P(n, d) \leq \frac{n!P_\Omega(n, d)}{|\Omega|}.$$  \hfill (7)

**Proof:** Suppose $C$ is an $(n, P(n, d), d)$ PA. For any $x \in S_n$, let $xC = \{xc : c \in C\}$. Then

$$\sum_{x \in S_n} |xC \cap \Omega| = \sum_{c \in C} \sum_{\omega \in \Omega} |\{x \in S_n : xc = \omega\}|$$

$$= \sum_{c \in C} \sum_{\omega \in \Omega} |\omega c^{-1}|$$

$$= P(n, d)|\Omega|.$$  \hfill (8)
On the other hand, there must exist $x' \in S_n$ such that $n!|x'C \cap \Omega| \geq \sum_{x \in S_n} |xC \cap \Omega|$. Then $n!|x'C \cap \Omega| \geq P(n, d)|\Omega|$, in other words, $P(n, d) \leq \frac{n!|x'C \cap \Omega|}{|\Omega|}$. This in conjunction with $|x'C \cap \Omega| \leq P_c(n, d)$ results the theorem. QED.

Since $S_d$ can be considered as a subset of $S_n$ for $d \leq n$, Theorem 1 is also a directly result of the above theorem, in fact

$$P(n, d) \leq \frac{n!|x'C \cap \Omega|}{|\Omega|} \leq n! \binom{d}{d} \frac{n!}{(d-1)!}.$$ 

The following is also obtained immediately by Theorem 3.

**Corollary 1:**

$$P(n, d) \leq n! \frac{P(n, d, w)}{\binom{n}{w} D_w}.$$ 

The following are well-known elementary properties of $A(n, d, w)$, which will be applied to the proof of the properties of $P(n, d, w)$.

**Lemma 1:**

- $A(n, d, w) = 1$, if $d > 2w$;
- $A(n, 2w, w) = \left\lfloor \frac{n}{w} \right\rfloor$;
- $A(n, 2k, k+1) \leq \left\lfloor \frac{n}{k+1} \frac{n-1}{k} \right\rfloor$.

**Theorem 4:**

1. $P(n, d, w) \leq A(n, 2d - 2w, w)$, for $d > w$;
2. $P(n, d, w) = 1$, for $d > 2w, w \neq 1, d \geq 1$;
3. $P(n, 2k, k) = \left\lfloor \frac{n}{k} \right\rfloor$, for $2 \leq k \leq \lfloor n/2 \rfloor$;
4. $P(n, 2k + 1, k + 1) = A(n, 2k, k+1)$, for $1 \leq k \leq \lfloor (n-1)/2 \rfloor$;
5. $P(n, 4, 3) \leq \frac{2}{3}$, for $n \geq 4$.

**Proof:** Part (I) Let $C$ be an $(n, d, w)$ constant-weight PA with maximal size $P(n, d, w)$, where $d > w$. Define $f : S_n \mapsto \{0, 1\}^n$ such that for any $a = (a_0, a_1, \ldots, a_{n-1}) \in S_n$ with support $A$, $f(a) = a' = (a'_0, a'_1, \ldots, a'_{n-1}) \in \{0, 1\}^n$, where

$$a'_i = \begin{cases} 1, & \text{for } i \in A, \\ 0, & \text{for } i \notin A. \end{cases}$$

Then $C' = \{f(a) : a \in C\}$ is an $(n, 2d - 2w, w)$ constant-weight code with size $P(n, d, w)$ and this means $P(n, d, w) \leq A(n, 2d - 2w, w)$. To prove this fact we need only to prove that $C'$ have mutual distances $\geq 2d - 2w$. 

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Let \(a, b \in C\), \(a \neq b\), and let \(A\) and \(B\) be the supports of \(a\) and \(b\) respectively. Suppose \(a' = f(a), b' = f(b)\). (6) implies
\[
d(a', b') = |(A/B) \cup (B/A)|
\[
= |A| + |B| - 2|A \cap B|
\[
= 2w - 2|A \cap B|
\]
(7)

On the other hand, we have
\[
d \leq d(a, b)
\[
\leq |A \cup B|
\[
= |A| + |B| - |A \cap B|
\[
= 2w - |A \cap B|,
\]
namely \(|A \cap B| \leq 2w - d\). Putting this into (7) we obtain
\[
d(a', b') \geq 2d - 2w.
\]
Since \(f\) is an onto mapping, we complete the proof of Part (I).

Part (II) For \(d > 2w, w \neq 1\) and \(d \geq 1\), since
\[
2d - 2w > 2 \cdot 2w - 2w = 2w,
\]
\(A(n, 2d - 2w, w) = 1\) (by Lemma 1). This in conjunction with part (I) yields \(P(n, d, w) = 1\).

Part (III) For \(2 \leq k \leq \lfloor n/2 \rfloor\), by part (I) and Lemma 1 we have
\[
P(n, 2k, k) \leq A(n, 2k, k) = \lfloor n/k \rfloor.
\]
On the other hand, we can construct an \((n, 2k, k)\) constant-weight PA as follows:
\[
C = \{c_i = (c_{i,0}, c_{i,1}, \ldots, c_{i,n-1}) | i = 0, 1, \ldots, \lfloor n/k \rfloor - 1\},
\]
where
\[
c_{i,j} = \begin{cases} 
  j + 1, & \text{for } j = ik, ik + 1, \ldots, ik + k - 2 \\
  ik, & \text{for } j = ik + k - 1 \\
  j, & \text{others.}
\end{cases}
\]
Then we conclude \(P(n, 2k, k) = \lfloor n/k \rfloor\).

Part (IV) For case \(1 \leq k \leq \lfloor (n - 1)/2 \rfloor\), by part (I) we have
\[
P(n, 2k + 1, k + 1) \leq A(n, 2k, k + 1).
\]
Let $C'$ be an $(n, 2k, k + 1)$ constant-weight binary code with maximal size $A(n, 2k, k + 1)$, then there exists $C \subseteq S_n$ such that for each member of $C$ there have one and only one member of $C'$ with same support. We will prove that $C$ is an $(n, 2k + 1, k + 1)$ constant-weight PA, which implies $P(n, 2k+1, k+1) \geq A(n, 2k, k+1)$ and then results $P(n, 2k+1, k+1) = A(n, 2k, k+1)$.

Let $x, y \in C$, $x \neq y$ with corresponding supports $X$ and $Y$. For case $X \cap Y = \emptyset$, $d(x, y) = |X| + |Y| = 2k + 2$. So we need only to discuss the case $X \cap Y \neq \emptyset$. Let $x', y' \in C'$ be the corresponding binary codewords with supports $X, Y$. Since $d(x', y') = |X| + |Y| - 2|X \cap Y| = 2(k + 1) - 2|X \cap Y| \geq 2k$, $|X \cap Y| \leq 1$. Therefore, if $X \cap Y \neq \emptyset$, then $|X \cap Y| = 1$. Suppose $X \cap Y = \{a\}$. Then $x(a) \neq y(a)$, otherwise $x(a) = y(a) = a$ and it leads to a contradiction. Hence for this case, $d(x, y) = |A/B \cup B/A| + |\{a\}| = |A/B| + |B/A| + 1 = 2k + 1$. Now we conclude that $C$ is an $(n, 2k + 1, k + 1)$ constant-weight PA of size $A(n, 2k, k + 1)$, which completes the proof of Part (IV).

Part (VI) Suppose $C$ is an $(n, 4, 3)$ constant-weight PA. For any pair $\{i, j\} \in Z_n \times Z_n$ with $i \neq j$, let $C_{i,j} \subseteq C$ be the maximal set such that for each $x \in C_{i,j}$ with support $X$, $\{i, j\} \subseteq X$. We are now ready to prove $|C_{i,j}| \leq 2$. Assume the contrary, i.e., that $|C_{i,j}| \geq 3$ and $x, y, z$ are distinct elements of $C_{i,j}$. W.l.o.g, $(x(i), x(j)) = (k, i)$, where $k \neq i, j$. Then $(y(i), y(j)) = (j, k')$, where $k' \neq i, j, k$, otherwise $d(x, y) < 4$, which is a contradiction. Similarly, $(z(i), z(j)) = (j, k'')$, where $k'' \neq i, j, k, k'$. Thus $d(y, z) < 4$, which is a contradiction. Therefore $|C_{i,j}| \leq 2$.

Since there are $\binom{n}{2}$ pairs of $(i, j) \in Z_n \times Z_n$ with $i \neq j$,

$$\sum_{i,j, i \neq j} |C_{i,j}| \leq 2 \binom{n}{2}. \tag{9}$$

On the other hand, for each member of $C$, there are exactly 3 $C_{i,j}$ containing it, hence

$$\sum_{i,j, i \neq j} |C_{i,j}| = 3|C|. \tag{10}$$

Substituting (10) into (9) yields $|C| \leq \frac{2\binom{n}{2}}{3}$, this means $P(n, 4, 3) \leq \frac{2\binom{n}{2}}{3}$. QED.

**Theorem 5:** For $2 \leq k \leq \lfloor n/2 \rfloor$,

$$P(n, 2k) \leq \frac{n!}{V(n, k - 1) + \frac{(n)D_k}{\lfloor n/k \rfloor}} \tag{11}$$

**Proof:** Let there be $N_k$ permutations in $S_n$ which have distance $k$ to the $(n, M, d)$ PA $C$. Then

$$MV(n, k - 1) + N_k \leq n! \tag{12}$$
In order to estimate $N_k$ we consider an arbitrary codeword $c$ which we can take to be $1$ (w.l.o.g.). Then all permutations of weight $k$ has distance $k$ to $C$. Since there are $\binom{n}{k}$ $D_k$ permutations of weight $k$, there must have $\binom{n}{k}$ $D_k$ permutations that have distance $k$ to $C$. By varying $c$ we thus count $M \binom{n}{k}$ $D_k$ permutations in $S_n$ that have distance $k$ to the PA. How often has each of these permutations been counted. Take one of them; again w.l.o.g. we call it $1$. The codewords with distance $k$ to $1$ form an $(n, 2k, k)$ constant-weight PA since they have mutual distances $\geq 2k$ and weight $k$. Hence there are at most $P(n, 2k, k) = \lfloor n/k \rfloor$ (by part $(III)$ of Theorem 4) such codewords. This gives $N_k \geq \frac{M \binom{n}{k}}{\lfloor n/k \rfloor}$. Substituting this lower bounds on $N_k$ into (12) implies the Theorem.

**Theorem 6:** For $2 \leq k \leq \lfloor (n - k - 1)/2 \rfloor$, 

$$P(n, 2k + 1) \leq \frac{n!}{V(n, k) + \binom{n}{k} D_{k+1} - A(n-k, 2k,k+1) \binom{n}{k} D_k}$$  \hspace{1cm} (13) 

**Proof:** Let $C$ be an $(n, M, 2k + 1)$ PA. For any $x \in S_n$, let $B_i(x) = |\{c : c \in C, d(c, x) = i\}|$. The proof relies on the following lemma.

**Lemma 2:** 

$$A(n, 2k, k + 1) \sum_{i<k} B_i(x) + (A(n, 2k, k + 1) - A(n - k, 2k, k + 1)) B_k(x) + B_{k+1}(x)$$ 

$$\leq A(n, 2k, k + 1)$$ 

**Proof:** Without loss of generality, we can take $x = 1$, then $B_i(x)$ is the number of codewords with weight $i$. Clearly, a permutation with weight $w_1$ has distance $\leq w_1 + w_2$ to that with weight $w_2$. Hence $\sum_{i<k} B_i(x) \leq 1$. If $B_i(x) > 0$ for any $i < k$, then $B_k(x) = B_{k+1}(x) = 0$ and all the other summands are zeros, and there is nothing to prove. Assume, therefore, that $B_i(x) = 0$ for all $i < k$. We know that $B_k(x) \leq P(n, 2k + 1, k) = 1$ (by part $(II)$ of Theorem 4), in other words $B_k(x)$ is either 0 or 1: if it is 0, then the claim becomes $B_{k+1}(x) \leq A(n, 2k, k + 1) = P(n, 2k + 1, k + 1)$ (by part $(IV)$ of Theorem 4), which is clear; if it is 1, then the claim becomes $B_{k+1}(x) \leq A(n - k, 2k, k + 1) = P(n - k, 2k + 1, k + 1)$, which is correct for there are no points moved by both codewords of weight $k$ and of weight $k + 1$. QED.

We are now ready to complete the proof of the theorem. It follows from Lemma 2 that 

$$\sum_{x \in S_n} (A(n, 2k, k + 1) \sum_{i<k} B_i(x) + (A(n, 2k, k + 1) - A(n - k, 2k, k + 1)) B_k(x) + B_{k+1}(x)) \leq n! A(n, 2k, k + 1)$$  \hspace{1cm} (14)
The left side of the above inequality can be also written as
\[
A(n, 2k, k + 1) \sum_{i < k} \sum_{x \in S_n} B_i(x) + (A(n, 2k, k + 1) - A(n - k, 2k, k + 1)) \sum_{x \in S_n} B_k(x) + \sum_{x \in S_n} B_{k+1}(x).
\]

(15)

Now we shall give an expression in term of \(M\) for \(\sum_{x \in S_n} B_i(x)\). Since each codeword \(x \in C\) has exactly \(\binom{n}{i} D_i\) permutations in \(S_n\) which have distance \(i\) to \(x\), each codeword is counted exactly \(\binom{n}{i} D_i\) times by \(\sum_{x \in S_n} B_i(x)\), which means
\[
\sum_{x \in S_n} B_i(x) = M \binom{n}{i} D_i.
\]

(16)

Finally, the theorem is given by putting (16) into (15), and rewriting (14) after replaced its left side by the new expression of (15).

QED.

Using the upper bound on \(A(n, 2k, k + 1)\) in Lemma 1, we get a determined upper bound on \(P(n, 2k + 1)\).

**Corollary 2:** For \(2 \leq k \leq \lfloor (n - k - 1)/2 \rfloor\),
\[
P(n, 2k + 1) \leq \frac{n!}{V(n, k) + \binom{n}{k+1} D_{k+1} - \binom{n}{k+1} \frac{n!}{V(n, \lfloor (d-1)/2 \rfloor)} D_k}.
\]

III. COMPARISON OF UPPER BOUNDS

In this section, we will prove that for constant \(\alpha, \beta\) satisfying certain conditions, whenever \(d = \beta n^\alpha\), the new upper bounds on \(P(n, d)\) are stronger than the previous ones when \(n\) large enough.

For large \(n\) and \(d\), the previous upper bounds on \(P(n, d)\) have Deza-Vanstone bound and sphere packing bound. Let \(DV(n, d)\) denote the Deza-Vanstone upper bound on \(P(n, d)\) and \(SP(n, d)\) denote the sphere packing upper bound on \(P(n, d)\), i.e.
\[
DV(n, d) = \frac{n!}{(d - 1)!},
\]
\[
SP(n, d) = \frac{n!}{V(n, \lfloor (d-1)/2 \rfloor)}.
\]

Although we can get more upper bounds on \(P(n, d)\) by recursively applying inequality (3) and using the sphere packing bounds as the initial bound, namely, for \(d \leq m < n\),
\[
P(n, d) \leq \frac{n! SP(m, d)}{m!},
\]

(17)
these bounds are not stronger than the best bounds given by $DV(n, d)$ and $SP(n, d)$, which is shown as follows.

**Lemma 3:** For $d \leq m < n$,
\[
\frac{n!SP(m, d)}{m!} \geq \min\{DV(n, d), SP(n, d)\}.
\]

**Proof:** If $SP(m, d) \geq DV(m, d)$, \[
\frac{n!SP(m, d)}{m!} \geq \frac{n!}{m!} \cdot \frac{m!}{(d-1)!} = DV(n, d),
\] and there is nothing to prove. Therefore, assume $SP(m, d) < DV(m, d)$. The claim is also correct since
\[
SP(n, d) = \frac{SP(n, d)}{SP(m, d)} \cdot SP(m, d) = \frac{n!}{m!} \cdot \sum_{i=0}^{\lfloor (d-1)/2 \rfloor} \binom{m}{i} D_i \cdot SP(m, d) < \frac{n!SP(m, d)}{m!}. \tag{18}
\]

QED.

Let $ME(n, k)$ denote the new upper bound on $P(n, 2k)$ and $MO(n, k)$ denote the new upper bound on $P(n, 2k + 1)$, i.e.
\[
ME(n, k) = \frac{n!}{V(n, k - 1) + \binom{n}{k} D_k},
\]
and
\[
MO(n, k) = \frac{n!}{V(n, k) + \binom{n}{k+1} D_{k+1} - A(n-k, 2k, k+1) \binom{n}{k} D_k}. \tag{19}
\]

**Lemma 4:** For constants $\alpha, \beta$ satisfying either $0 < \alpha < 1/2$, $\beta > 0$ or $\alpha = 1/2, 0 < \beta < e$, whenever $d = \beta n^\alpha$,
\[
\lim_{n \to \infty} \frac{DV(n, d)}{SP(n, d)} = \infty.
\]

**Proof:** Let $k = \lfloor (d-1)/2 \rfloor$. We have
\[
\lim_{n \to \infty} \frac{DV(n, d)}{SP(n, d)} = \lim_{n \to \infty} \frac{V(n, k)}{\binom{n}{k} D_k} \geq \lim_{n \to \infty} \frac{n!k!}{ek!(n-k)!(d-1)!} = \lim_{n \to \infty} \frac{\sqrt{2\pi n(n/e)^n}}{e \sqrt{2\pi(n-k)((n-k)/e)^{n-k}} \sqrt{2\pi(d-1)((d-1)/e)^{d-1}}}.
\]
where the last equation is followed by Stirling’s formula \( \lim_{n \to \infty} \frac{n!}{\sqrt{2\pi n \left( \frac{n}{e} \right)^n}} = 1 \). By (18),

\[
\lim_{n \to \infty} \frac{DV(n, d)}{SP(n, d)} \geq \frac{1}{\sqrt{2\pi}} \lim_{n \to \infty} e^{d-k-2} n^{n+1/2} (n-k)^{-k+1/2} (d-1)^{d-1/2}.
\]

(19)

Let \( c \) be a constant such that \( c < 1 \). Since

\[
\lim_{n \to \infty} \left( \frac{n}{n-k} \right)^{n-k} = \lim_{n \to \infty} \left( 1 + \frac{1}{n/k-1} \right)^{n/k-1} = e,
\]

for \( n \) large enough, \( \left( \frac{n}{n-k} \right)^{n-k} \geq e^c \), i.e.

\[
(n-k)^{n-k} \leq e^{-ck} n^{-k}.
\]

(20)

Putting (20) into the right side of (19), and multiplying the right side of (19) by \( \lim_{n \to \infty} \frac{(n-k)^{1/2}}{n^{1/2}} = 1 \) and

\[
\lim_{n \to \infty} \frac{(d-1)^{d-1/2}}{e^{-1} (\beta n^a)^{d-1/2}} = \lim_{d \to \infty} e^{(1-\beta) d-1/2} \lim_{d \to \infty} e(1-1/d)^{d-1/2} = 1,
\]

we obtain

\[
\lim_{n \to \infty} \frac{DV(n, d)}{SP(n, d)} \geq \frac{1}{\sqrt{2\pi}} \lim_{n \to \infty} e^{d-k-2} n^{n+1/2} (n-k)^{-k+1/2} (d-1)^{d-1/2} e^{-ck} n^{-k} n^{-k} \beta^a (d-1/2) n^{-k} \alpha d + \alpha/2
\]

\[
\geq \frac{1}{\sqrt{2\pi}} \lim_{n \to \infty} e^{d-k-(1/2) \ln \beta - 1 + \alpha/2} n^{-k} (d-1/2) \alpha + \alpha/2
\]

(21)

where the last inequality follows from \((c-1)k \geq (c-1)d/2\) and

\[
k - \alpha d + \alpha/2 \geq (d/2 - 1) - \alpha d + \alpha/2 = (1/2 - \alpha) d - 1 + \alpha/2.
\]

To see the limit of right side of (21), we discuss in two cases:

Case I:) \( 0 < \alpha < 1/2 \). Since the coefficient \( 1/2 - \alpha > 0 \), the limit is determined by exponent \( n^{(1/2 - \alpha) d - 1 + \alpha/2} \), and then the statement holds for this case.

Case II:) \( \alpha = 1/2, 0 < \beta < e \). The right side of (21) is equal to

\[
\frac{1}{\sqrt{2\pi}} \lim_{n \to \infty} e^{d - (1/2) \ln \beta - 1 + \alpha/2} n^{-3/4}.
\]
The statement holds also, since for $0 < \beta < e$ we can take $c$ such that $2 \ln \beta - 1 < c < 1$, in other words,

$$0 < \frac{1 + c}{2} - \ln \beta < 1 - \ln \beta.$$ 

QED.

**Lemma 5:** For $k \geq 5$,

$$SP(n, 2k) - ME(n, k) > \frac{2(n-k+1)!}{n(k-1)}.$$ 

**Proof:** Since

$$V(n, k - 1) + \binom{n}{k} D_k = V(n, k),$$

$$SP(n, 2k) - ME(n, k) = \frac{n!}{V(n, k-1)} - \frac{n!}{V(n, k-1) + \binom{n}{k} D_k}$$

$$= \frac{n! \binom{n}{k} D_k}{V(n, k-1) \left( V(n, k-1) + \binom{n}{k} D_k \right)}$$

$$\geq \frac{n! \binom{n}{k} D_k}{[n/k] V(n, k-1) V(n, k)}.$$ 

When $k \geq 5$, $V(n, k - 1) \leq (k-1) \binom{n}{k-1} D_{k-1}$ and $V(n, k) \leq k \binom{n}{k} D_k$, thereby

$$SP(n, 2k) - ME(n, k) \geq \frac{n! \binom{n}{k} D_k}{[n/k] k \binom{n}{k} D_k \left( k \binom{n}{k-1} D_{k-1} \right)}$$

$$\geq \frac{n!}{n(k-1) \binom{n}{k-1} D_{k-1}}.$$ 

(22)

When $k \geq 5$, $D_{k-1} = [(k-1)!/e] < (k-1)!/2$, putting this into (22) we have

$$SP(n, 2k) - ME(n, k) > \frac{n!}{n(k-1) \cdot \frac{n!}{(n-k+1)!(k-1)!} \cdot \frac{(k-1)!}{2}}$$

$$= \frac{2(n-k+1)!}{n(k-1)!}.$$ 

QED.

**Theorem 7:** For constants $\alpha, \beta$ satisfying either $0 < \alpha < 1/2$, $\beta > 0$ or $\alpha = 1/2$, $0 < \beta < e$, whenever $2k = \beta n^\alpha$,

$$\lim_{n \to \infty} \left( \min\{ DV(n, 2k), SP(n, 2k) \} - ME(n, k) \right) = \infty.$$
Proof: By Lemma 5, we have
\[
\lim_{n \to \infty} SP(n, 2k) - ME(n, k) \geq \lim_{n \to \infty} \frac{2(n - k + 1)!}{n(k - 1)} = \lim_{n \to \infty} \frac{2(n - (\beta n^a)/2 + 1)!}{n((\beta n^a)/2 - 1)} = \infty.
\]

By Lemma 4, we have
\[
\lim_{n \to \infty} (DV(n, 2k) - SP(n, 2k)) = \lim_{n \to \infty} SP(n, 2k) \left(\frac{DV(n, 2k)}{SP(n, 2k)} - 1\right) = \infty,
\]
hence \(\lim_{n \to \infty} (DV(n, 2k) - ME(n, k)) = \infty\), and then follows the theorem. QED.

As a simple example of the superiority of the new bound \(ME(n, k)\) over \(DV(n, 2k)\) and \(SP(n, 2k)\) we can compare them for small values of \(d\) and \(n\).

Example 1: \(ME(20, 4) < 0.218 \cdot 10^{15}\), \(DV(20, 8) > 0.482 \cdot 10^{15}\), \(SP(20, 8) > 0.984 \cdot 10^{15}\), then \(ME(20, 4)\) provides the best upper bound on \(P(20, 8)\).

Lemma 6: For \(k \geq 4\),
\[
SP(n, 2k + 1) - MO(n, k) > \frac{2(n - k)!}{(k + 1)n(n - 1)} \left(1 + k - \frac{n - 1}{k}\right).
\]
Proof: We have
\[
SP(n, 2k + 1) - MO(n, k) = \frac{n!}{V(n, k)} - \frac{n!}{V(n, k)} = \frac{n!}{V(n, k)} \left(\binom{n}{k+1}D_{k+1} - A(n - k, 2k, k + 1)\binom{n}{k}D_k\right)
\]
\[
= \frac{n!}{V(n, k)} \left(\binom{n}{k+1}D_{k+1} - A(n - k, 2k, k + 1)\binom{n}{k}D_k\right)
\]
\[
= \frac{n!}{V(n, k)} \left(\binom{n}{k+1}D_{k+1} - A(n - k, 2k, k + 1)\binom{n}{k}D_k\right)
\]
\[
\geq \frac{n!}{V(n, k)V(n, k + 1)} \left(n(n-1)(k+1)^2\right)\]
where the last inequality is followed by \(A(n - k, 2k, k + 1) \leq \frac{(n-k)(n-k-1)}{(k+1)k}\), \(A(n, 2k, k + 1) \leq \frac{n(n-1)}{(k+1)k}\) (by Lemma 1) and
\[
V(n, k) + \frac{n}{k+1}D_{k+1} - A(n - k, 2k, k + 1)\left(\binom{n}{k}D_k\right) \leq V(n, k) + \left(\binom{n}{k+1}\right)D_{k+1} = V(n, k + 1).
\]
When \( k \geq 4 \), \( V(n, k) \leq k \binom{n}{k} D_k \) and \( V(n, k + 1) \leq (k + 1) \binom{n}{k+1} D_{k+1} \), then

\[
SP(n, 2k + 1) - MO(n, k) \geq \frac{n! \left( \frac{(n-k)!}{(k+1)k} \right)}{k \binom{n}{k} D_k (k + 1) \binom{n}{k+1} D_{k+1}}
\]

\[
= \frac{(n-2)! \left( \frac{D_{k+1} - n-k-1}{D_k} \right)}{\binom{n}{k} D_{k+1}}
\]

Since for \( k \geq 4 \), \( \frac{D_{k+1}}{D_k} \geq \frac{(k+1)!/e}{k!/e+1} = k + 1 - \frac{k+2}{k! / e + 1} > k \), and \( D_{k+1} \leq \frac{(k+1)!}{e} + 1 \leq (k+1)!/2 \),

\[
SP(n, 2k + 1) - MO(n, k) \geq \frac{(n-2)! \left( k \right) (n-k-1)!/2}{\binom{n}{k} (k+1)!} \left( 1 + k - \frac{n-1}{k} \right).
\]

**Theorem 8:** For constant \( \beta \) such that \( 2 < \beta < e \), whenever \( 2k + 1 = \beta n^{1/2} \),

\[
\lim_{n \to \infty} \left( \min\{DV(n, 2k + 1), SP(n, 2k + 1)\} - MO(n, k) \right) = \infty.
\]

**Proof:** Since

\[
1 + k - \frac{n-1}{k} \geq 1 + \frac{2\sqrt{n}-1}{2} - \frac{n-1}{2\sqrt{n}-1}
\]

\[
= 1 + \left( \sqrt{n} - 1 \right) - \left( \sqrt{n} - 1 + \frac{\sqrt{n} - \frac{5}{4}}{\sqrt{n} - \frac{1}{2}} \right)
\]

\[
= \frac{3}{4\sqrt{n} - 2},
\]

by Lemma 6 we have

\[
\lim_{n \to \infty} SP(n, 2k + 1) - MO(n, k) \geq \lim_{n \to \infty} \frac{2(n-k)!}{(k+1)n(n-1)} \left( 1 + k - \frac{n-1}{k} \right)
\]

\[
\geq \lim_{n \to \infty} \frac{2(n-k)!}{(k+1)n(n-1)} \cdot \frac{3}{4\sqrt{n} - 2}
\]

\[
= \infty.
\]

By Lemma 4, we have

\[
\lim_{n \to \infty} (DV(n, 2k + 1) - SP(n, 2k + 1)) = \lim_{n \to \infty} SP(n, 2k + 1) \left( \frac{DV(n, 2k + 1)}{SP(n, 2k + 1)} - 1 \right)
\]

\[
= \infty,
\]

hence \( \lim_{n \to \infty} (DV(n, 2k + 1) - MO(n, k)) = \infty \), and then follows the theorem. QED.
As a simple example of the superiority of the new bound $M_O(n, k)$ over $D_V(n, 2k + 1)$ and $S_P(n, 2k + 1)$ we can compare them for small values of $d$ and $n$.

**Example 2:** $M_O(20, 4) < 0.380 \cdot 10^{14}$ by Corollary 2, $S_P(20, 9) > 0.528 \cdot 10^{14}$, $D_V(20, 9) > 0.603 \cdot 10^{14}$, then $M_O(20, 4)$ provide the best upper bound on $P(20, 9)$.

mds

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