Probability Ranking in Vector Spaces

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Abstract

The Probability Ranking Principle states that the document set with the highest values of probability of relevance optimizes information retrieval effectiveness given the probabilities are estimated as accurately as possible. The key point of the principle is the separation of the document set into two subsets with a given level of fallout and with the highest recall. The paper introduces the separation between two vector subspaces and shows that the separation yields a more effective performance than the optimal separation into subsets with the same available evidence, the performance being measured with recall and fallout. The result is proved mathematically and exemplified experimentally.

1 Introduction

Information Retrieval (IR) systems decide about the relevance. The decision about relevance is subject to uncertainty. A probability theory provides a measure of uncertainty. To this end, a probability theory defines the event space and the probability distribution. The research in probabilistic IR is based on the classical theory, which describes events and probability distributions using, respectively, sets and set measures, according to Kolmogorov’s axioms [1].

Ranking is perhaps the most crucial IR task and the Probability Ranking Principle (PRP) reported in [2] is by far the most important theoretical result to date. Although IR systems reach good results thanks to (classical) probability theory and parameter tuning, ranking is far from being perfect because useless units are often ranked at the top of, or useful units are missed from the retrieved document list.
The paper investigates whether an alternative probability theory may achieve further improvement. We propose Vector Probability to describe events and probability distributions using vectors, matrices and operators on them. The adoption of Vector Probability means a radical change: Vector Probability is based on vector subspaces whereas classical probability is based on sets such that the regions of acceptance and rejection of a hypothesis system are sets. We express Vector Probability by means of the mathematical apparatus used in Quantum Theory. However, the use of the mathematical apparatus of Quantum Theory does not end in an investigation or assertion of quantum phenomena in IR. Rather, we argue the superiority of vector probability for ranking documents over the classical probability theory. Every reflection on Quantum Theory and IR is out of the scope of the paper.

The paper shows that ranking in accordance with Vector Probability is more effective than ranking by classical probability given that the same evidence is available for probability estimation. The effectiveness is measured in terms of probability of correct decision or, equivalently, of probability of error. We propose a decision rule based on vector subspaces such that, in the long run, the IR system will deem a document relevant correctly at a higher recall than the recall measured in the event of ranking as a result of classical probability when the fallout is not more than a given threshold. So, the decision rule minimizes the probability of error.

We organize the paper as follows. Section 2 provides the definitions used in the paper: this section can be skipped if the reader has knowledge about Quantum Theory and Probability; further definitions can be found in 3. Section 3 introduces the aspects of the probability of relevance related to the subsequent sections. Section 4 explains Vector Probability. Section 5 describes an instance of the vector probability of relevance when the Poisson distribution is used for an observable of a document. Section 6 introduces the optimal observable vectors. Section 7 shows that ranking by means of the optimal observable vectors is more effective than ranking by means of the best subsets of observed values. Section 8 describes an experimental study that confirms the theory. Section 9 is about the measurement of observable vectors and makes some remarks about the actual use. Section 10 refers to the main related publications and Section 11 concludes the paper.
2 Definitions

We report and compare some definitions to IR concepts in Table 1.

**Definition 1 (Observable)** An observable is a property that can be measured from an entity.

**Definition 2 (Probability Distribution)** A probability distribution is a function that maps observable values to the real range \([0, 1]\). As usual, the probabilities sums to 1.

**Definition 3 (Classical Probability Distribution)** A classical probability distribution admits only sets of observable values.

The subsets of observable values can be defined by means of the set operations (i.e., intersection, union, complement).

**Definition 4 (State)** A state, or hypothesis, is a condition of the measured entity and molds the probability distribution of the measurement.

In classical probability, “hypothesis” is more used than “state”. We use “state” because it is used in the formalism of Quantum Theory. We correspond the null state to non-relevance and the alternative state to relevance. An IR system decides between the relevance state and the non-relevance provided an observable value.

**Definition 5 (Probability of Detection)** It is the probability that the system decides for relevance when relevance is true; it is also called power.

| Definition                  | IR Example or Corresponding Concept                                      |
|-----------------------------|--------------------------------------------------------------------------|
| Observable                  | Term frequency, relevance, color, aboutness                              |
| Probability Distribution    | Distribution of probability of term frequency                           |
| State                       | Relevance, aboutness, utility                                            |
| Probability of Detection    | Recall                                                                   |
| Probability of False Alarm  | Precision                                                                |
| Region of Acceptance        | Term frequencies that are higher than a threshold                        |
| Observable Vector           | Term frequency, relevance, color, aboutness                              |
| State Vector                | Distribution of Relevance, aboutness, utility probability                |

Table 1: Definitions, IR examples and concepts.
**Definition 6 (Probability of False Alarm)** It is the probability that the system decides for relevance when relevance is false; it is also called size or level.

As the probability of detection and the probability of false alarm cannot be simultaneously optimized, the decision rules maximize the probability of detection when the probability of false alarm is fixed.

**Definition 7 (Region of Acceptance)** A region of acceptance consists of the observable values that induce the system to decide for relevance. The most powerful region of acceptance yields the maximum power for a fixed size.

Note that “acceptance” does often refer to the null state in Statistics.

The Neyman-Pearson lemma states that the maximum likelihood (ML) ratio test defines the most powerful region of acceptance \[4\].

**Definition 8 (Probability of Correct Decision)**

\[ P_c = \xi(1 - P_0) + (1 - \xi)P_d \]  

(1)

provided that \(\xi\) the prior probability of the null state, \(P_0\) is the probability of false alarm and \(P_d\) is the probability of detection.

**Definition 9 (Probability of Error)**

\[ P_e = \xi P_0 + (1 - \xi)(1 - P_d) \]  

(2)

From \[1\] and \[2\] \(P_e + P_c = 1\).

**Definition 10 (Vector Space)** A vector space over a field \(\mathcal{F}\) is a set of vectors subject to linearity, namely, a set such that, for every vector \(|u\rangle\), there are three scalars \(a, b, c \in \mathcal{F}\) and three vectors \(|v\rangle, |x\rangle, |y\rangle\) of the same space such that \(|u\rangle = a|v\rangle\) and \(|u\rangle = b|x\rangle + c|y\rangle\). If \(|u\rangle\) is a vector, \(\langle u|\) is its transpose, \(\langle v|u\rangle\) is the inner product with \(|v\rangle\) and \(|v\rangle\langle u|\) is the outer product with \(|v\rangle\). If \(|\langle x|x\rangle|^2 = 1\), the vector is normal. If \(|\langle x|y\rangle = 0\), the vectors are mutually orthogonal.

We adopt the Dirac notation to write vectors so that the reader may refer to the literature on Quantum Theory; a brief illustration of the Dirac notation is in \[3\].
Definition 11 (Observable and Observable Vector) An observable is a collection of values and of vectors. The observable vectors are mutually orthonormal and 1:1 correspondence with the values.

An observable corresponds to a random variable in Statistics whereas the observable vectors correspond to the indicator functions.

Definition 12 (State Vector) A state vector defines a vector probability distribution. The possible states (or hypotheses) correspond to state vectors.

A state vector plays the role of parameters in Statistics.

Definition 13 (Vector Probability) The vector probability that \( x \) is observed given the state \( m \) is \( |\langle x|m \rangle|^2 \).

Vector probability is axiomatically defined in [5] and is applied to IR in [3]; the generalization to state matrices or density matrices is not necessary in this paper.

3 Probability of Relevance

Suppose that a document is represented through the random variable \( X \) such that \( X = x \) means that a term has frequency \( x \). The decision is between relevance and non-relevance. Thus, the probability of detection is the probability that the observed frequency belongs to the region of acceptance when a document is relevant and the probability of false alarm is the probability that the observed frequency belongs to the region of acceptance when the document is not relevant.

The very general form of Probability Ranking Principle (PRP) and then the BM25 reported in [6, page 340] are based on the ML ratio test. The PRP states that, if a cut-off is defined for the fallout (i.e., the probability of false alarm), we would clearly optimize (i.e., maximize recall, namely, probability of detection, or equivalently, precision) if we included in the retrieved set those documents with the highest probability of relevance [2, page 297] which is the probability that \( X = x \) when the document is relevant.

Therefore, the PRP and the Neyman-Pearson lemma state that, given a region of acceptance and then a probability of false alarm (i.e., fallout), the document ranking as a result of probability of relevance is optimal because the recall is maximum.
4 Vector Probability of Relevance

Suppose that $X$ is an observable (e.g., term frequency) and $x$ a value of the set $\{0, 1, \ldots, N\}$. The orthonormal observable vectors that correspond to the values are $|0\rangle, |1\rangle, \ldots, |N\rangle$; the actual implementation of these vectors is not essential. A observable vector $|x\rangle$ correspond to $x$ and

$$X = \sum_{x=0}^{N} x|x\rangle\langle x|$$

(3)

Suppose that $p(x; m)$ is the probability that $X = x$ given a parameter $m$. In the event of binary relevance, $m$ is either $m_0$ (non-relevance) or $m_1$ (relevance). Note that $m$ may refer to more than one parameter. However, we assume that $m$ is scalar for the sake of clarity.

Two relevance state vectors represent binary relevance: a relevance state vector $|m_0\rangle$ represents non-relevance state and an orthogonal relevance state vector $|m_1\rangle$ represents relevance state. Relevance state vectors and the observable vectors belong to an finite-dimensional vector space. Thus, either state vectors can be defined in terms of a given orthonormal basis of that space and, in particular, the observable vectors are a basis. The following expressions

$$|m_0\rangle = \sum_{x=0}^{N} a(x; m_0)|x\rangle$$

$$|m_1\rangle = \sum_{x=0}^{N} a(x; m_1)|x\rangle$$

$$a(x; m) = \pm \sqrt{p(x; m)}$$

(4)

establish the relationship between parametric distributions and vector spaces, namely, between the parameters $m_0, m_1$, the relevance state vectors $|m_0\rangle, |m_1\rangle$ and the observable $X$. The sign of $a(x; m)$ is chosen so that the orthogonality between the state vectors is retained. Equations (4) are instances of superposition. In Physics, superposition models observables that are known only if they are measured. In IR, the event that an observable exists only if

\footnote{In Quantum Theory, the vector spaces are complex Hilbert spaces. For the sake of simplicity, we do not consider the field.}
observed is a much debated hypothesis. Moreover, the orthogonality of the relevance state vectors and the following expression

\[|\langle y|m \rangle|^2 = \left| \sum_{x=0}^{N} a(x;m)\langle y|x \rangle \right|^2 \]

\[= |a(y;m)|^2 \text{ due to orthogonality} \quad (5)\]

\[= p(y;m) \quad (6)\]

establish the relationship between probability distribution and vector-based representation of relevance.

## 5 Poisson-Based Probability of Relevance

The Poisson distribution is used because we want to make the illustration of the theory accessible in the remainder of the section and consistent with the past literature, e.g., [7, 8, 9, 10]. Moreover, the Poisson distribution is asymptotically derived from the Binomial distribution and approximates the Normal distribution.

The observable \( X \) is the frequency of a term in a document. Thus, \( X = x \) means that the term occurs \( x \) times in a document. The Poisson probability distribution gives the probability that \( X = x \), that is,

\[ p(x;m) = e^{-m} \frac{m^x}{x!} \quad (8)\]

provided that \( m \) is the expected term frequency. \( X \) is defined in the set of natural number. However, we assume that \( N \) is finite, large and equal to the maximum observable term frequency in the collection; indeed, the estimated probability that a term frequency is greater than \( N \) is zero.

Two distinct parameters \( m_0, m_1 \) encode non-relevance and relevance, respectively, in parametric Statistics. Thus,

\[ p(x;m_0) \quad (9)\]

is the probability that the term occurs \( x \) times in a *non-relevant* document and

\[ p(x;m_1) \quad (10)\]

is the probability that the term occurs \( x \) times in a *relevant* document.
|     | \( p(0; m) \) | \( p(1; m) \) |
|-----|-------|-------|
| \( m_0 \) | \( \frac{1}{5} \) | \( \frac{4}{5} \) |
| \( m_1 \) | 0     | 1     |

For example, consider the probability distributions in Table 5. We have that

\[
X = 0|0\rangle\langle 0| + 1|1\rangle\langle 1| = |1\rangle\langle 1|
\]

\[
|m_0\rangle = \frac{1}{\sqrt{5}}|0\rangle + \frac{2}{\sqrt{5}}|1\rangle \quad |m_1\rangle = |1\rangle
\]

\[
p(0; m_0) = |a(0; m_0)|^2 = \frac{1}{5} \quad p(0; m_1) = |a(0; m_1)|^2 = 0
\]

The Poisson-based probabilities of detection and false alarm are, respectively,

\[
P_d = \sum_{x=x_\alpha}^{N} p(x; m_1) \quad P_0 = \sum_{x=x_\alpha}^{N} p(x; m_0) \tag{11}
\]

assuming that \( N \) is so large that \( p(x; m) = 0, x > N \) and \( \{x_\alpha, \ldots, N\} \) is the region of acceptance of the state of relevance at size \( \alpha \).

When the states are equiprobable and \( m_1 > m_0 \), the Poisson-based probability of error and the Poisson-based probability of correct decision are

\[
P_e = \frac{1}{2} \left( \sum_{x=x_\alpha}^{N} p(x; m_0) + \sum_{x=0}^{x_\alpha-1} p(x; m_1) \right) \tag{12}
\]

\[
= \frac{1}{2} \left( 1 - \int_{m_0}^{m_1} \frac{t^{x_\alpha-1}e^{-t}}{\Gamma(x_\alpha)} dt \right) \tag{13}
\]

\[
P_c = \frac{1}{2} \left( \sum_{x=x_\alpha}^{N} p(x; m_1) + \sum_{x=0}^{x_\alpha-1} p(x; m_0) \right) \tag{14}
\]

\[
= \frac{1}{2} \left( 1 + \int_{m_0}^{m_1} \frac{t^{x_\alpha-1}e^{-t}}{\Gamma(x_\alpha)} dt \right) \tag{15}
\]

The greater the difference between \( m_0 \) and \( m_1 \), the greater \( P_c \) and the smaller \( P_e \). If \( m_1 < m_0 \), the superscript and the subscript of the integral function of (12) swap. If \( m_1 = m_0 \), error and correct decision are equiprobable, i.e.,
Figure 1: Polygonal curve that minimizes the probability of error. The polygonal curve is obtained from the lowest segments connected at the intersection points.

the decision is ruled through coin tossing. In other words, the discrimination power increases when $|m_1 - m_0|$ increases. Note that the increase of $|m_1 - m_0|$ corresponds to making the relevance state vectors orthogonal.

Probability of error and probability of correct decision provide an alternative form of the decision procedure. From Equations 11 and 12, the maximum $P_d$ and the minimum $P_0$ minimize $P_e$ and maximize $P_c$. Suppose we have three size values: $\alpha_0 \leq \alpha_1 \leq \alpha_2$, thus yielding three power values $\beta_0 \leq \beta_1 \leq \beta_2$. The prior probability minimizes the probability of error as follows:

$$\min P_e = \begin{cases} 
\xi \alpha_0 + (1 - \xi)(1 - \beta_0) & 0 \leq \xi < \xi_1 \\
\xi \alpha_1 + (1 - \xi)(1 - \beta_1) & \xi_1 \leq \xi < \xi_2 \\
\xi \alpha_2 + (1 - \xi)(1 - \beta_2) & \xi_2 \leq \xi < 1 
\end{cases} \quad (16)$$

such that

$$\xi_1 = \frac{\beta_1 - \beta_0}{\alpha_1 - \alpha_0 + \beta_1 - \beta_0} = \frac{4}{7}$$

$$\xi_2 = \frac{\beta_2 - \beta_1}{\alpha_2 - \alpha_1 + \beta_2 - \beta_1} = \frac{8}{23}$$

Figure 1 shows an example of the polygonal curve with $\alpha_0 = \frac{1}{4}, \alpha_1 = \ldots$
1/2, \alpha_2 = \frac{3}{4} and \beta_0 = \frac{1}{3}, \beta_1 = \frac{2}{3}, \beta_2 = \frac{4}{3}. The abscissas of the intersection points are \xi_1, \xi_2.

6 Optimal Observable Vectors

Let us recapitulate some facts. Neyman-Pearson’s lemma states that the set of term frequencies can be partitioned into two disjoint regions: one region includes all the frequencies such that relevance will be accepted; the other region denotes rejection. If a term is observed from documents and only presence/absence is observed, the set of the observable values is \{0, 1\} and each region is one out of possible subsets, i.e., \emptyset, \{0\}, \{1\}, \{0, 1\}.

If term frequency is observed instead, the observable values are \{0 \ldots N\} and each region is one out of possible subsets of \{0 \ldots N\}. Note that the ML ratio test yields two subsets, i.e., \{0 \ldots x_\alpha - 1\} and \{x_\alpha \ldots N\}. An alternative region can be defined with only set operations (intersection, complement, union). However, set operations cannot define more powerful regions than that dictated by dint of the Neyman-Pearson lemma.

The subspaces are equivalent to subsets and they then can be subject to set operations if they are mutually orthogonal \[11\]. The subsets yielded by dint of the ML ratio test become \{\langle 0\rangle \langle 0\rangle, \ldots, \langle x_\alpha - 1\rangle \langle x_\alpha - 1\rangle\} and \{\langle x_\alpha\rangle \langle x_\alpha\rangle, \ldots, \langle N\rangle \langle N\rangle\} and can be subjected to set operations because they are mutually orthogonal.

Suppose that the vector subspaces that correspond to the subsets yielded by dint of the ML ratio test, are rotated so that the new subspaces are oblique to them. The new oblique subspaces cannot be reformulated in terms of the observable vectors through set operations and thus they represent something different.

Figure 2 shows three-dimensional vector space spanned by \{|e_1\rangle, |e_2\rangle, |e_3\rangle\} to make the difference between subsets and subspaces. The ray (i.e. one-dimensional subspace) \(L_x\) is spanned by \(x\), the plane (i.e. two-dimensional subspace) \(L_{x,y}\) is spanned by \(|x\rangle, |y\rangle\). Note that \(L_{e_1,e_2} = L_{x,y} = L_{e_1,y}\) and so on. According to \[12\] page 191, consider the subspace \(L_{e_2} \wedge (L_y \vee L_x)\) provided that \wedge means “intersection” and \vee means “span” (and not set union). Since \(L_y \vee L_x = L_{x,y} = L_{e_1,e_2}, L_{e_2} \wedge (L_y \vee L_x) = L_{e_2} \wedge L_{e_1,e_2} = L_{e_2}\). However, \((L_{e_2} \wedge L_y) \vee (L_{e_2} \wedge L_x) = 0\) because \(L_{e_2} \wedge L_y = 0\) and \(L_{e_2} \wedge L_x = 0\), therefore

\[L_{e_2} \wedge (L_y \vee L_x) \neq (L_{e_2} \wedge L_y) \vee (L_{e_2} \wedge L_x)\]
thus meaning that the distributive law does not hold, hence, set operations cannot be applied to subspaces.

The key point is that, if the subspaces are rotated in an optimal way, we can achieve the most powerful regions; these regions cannot be ascribed to the subsets yielded by dint of the ML ratio test. The following Helstrom’s theorem is the rule to compute the most powerful regions of a vector space provided two state vectors.

**Theorem 1** Let $|m_1\rangle, |m_0\rangle$ be the state vectors. The region of acceptance at the highest probability of detection at every probability of false alarm is given by the eigenvectors of

$$|m_1\rangle\langle m_1| - |m_0\rangle\langle m_0|$$

whose eigenvalues are positive.

**Proof** See [13]. (The $|m_i\rangle\langle m_i|$ are defined in Section 2)
Definition 14 An optimal observable vector is a vector that divides the region of acceptance from the region of rejection as stated by Theorem 14.

The optimal observable vectors always exist due to the Spectral Decomposition theorem.

The angle between the relevance state vectors $|m_1\rangle, |m_0\rangle$ determines the geometry of the decision between the two states. Suppose $|\mu_1\rangle, |\mu_0\rangle$ are two observable vectors. They are mutually orthogonal because are eigenvectors of (17) and can be defined in the space spanned by the relevance state vectors. The probability of correct decision and the probability of error are given by the angle between the two relevance state vectors and by the angles between the observable vectors and the relevance state vectors; how the geometry defines the optimal ranking is described in the next section.

7 Optimal Probability Ranking

Figure 3(a) illustrates the geometry of decision and the observable vectors. (The figure is in the two-dimensional space for the sake of clarity, but the reader should generalize to higher dimensionality than two.) The angles $\eta_0, \eta_1$ between the observable vectors and the relevance state vectors $|m_0\rangle, |m_1\rangle$ are related with the angle $\gamma$ between $|m_0\rangle, |m_1\rangle$ because the observable vectors are always mutually orthogonal and then the angle is $\frac{\pi}{2} = \eta_0 + \gamma + \eta_1$.

The optimal observable vectors are achieved when the angles between an observable vector and a relevance state vector are equal to

$$\theta = \frac{\frac{\pi}{2} - \gamma}{2}$$

(18)

The rotation of the non-optimal observable vectors such that (18) holds, yields the optimal observable vectors $|\mu_1\rangle, |\mu_0\rangle$ as Figure 3(b) illustrates: the optimal observable vectors are “symmetrically” located around the relevance state vectors.

The replacement of the angle between an observable vector and a relevance state vector with the angle of (18) yields the minimal probability of error and the maximal probability of correct decision, that is,

$$Q_e = \frac{1}{2} \left(1 - \sqrt{1 - 4\xi(1 - \xi)|X|^2}\right) \quad Q_c = 1 - Q_e$$

(19)
given that
\[ |X|^2 = |\langle m_0|m_1 \rangle|^2 = \left| \sum_{x=0}^{N} e^{-m_0} \frac{m_0^x}{x!} \sqrt{e^{-m_1} \frac{m_1^x}{x!}} \right|^2 \]  \hspace{1cm} (20)

is the squared cosine of the angle between the relevance state vectors if the Poisson distribution is used. Figure 4 superposes the polygonal curve plotted for \( P_e \) and the bell-shaped curves plotted for \( Q_e \) with \( |X|^2 = 0.90 \) and \( |X|^2 = 0.50 \). \( |X|^2 \) measures the degree to which the distributions of probability of relevance and non-relevance are distinguishable. The less they are distinguishable, the higher \( Q_e \). Indeed, the probability of error increases when the distribution of probability of relevance is very similar to the distributions of probability of non-relevance.

We prove the following

**Proposition 1** For all \( m_0, m_1, x_\alpha, \)
\[ Q_c \geq P_c \quad Q_e \leq P_e \]  \hspace{1cm} (21)

**Proof** Let \( x_\alpha \geq 0 \) and let \( m_1 \geq m_0 \) – the complement is proved in similar way.

\[ Q_c \geq P_c \text{ if and only if } \sqrt{1 - |X|^2} \geq \int_{m_0}^{m_1} \frac{t^{x_\alpha-1}e^{-t}}{\Gamma(x_\alpha)} \, dt \]  \hspace{1cm} (22)

because (1) and (2) also hold for \( Q_c, Q_e \). Moreover, (22) holds if

\[ 1 - |X|^2 \geq \int_{m_0}^{m_1} \frac{t^{x_\alpha-1}e^{-t}}{\Gamma(x_\alpha)} \, dt \]  \hspace{1cm} (23)

because the sides of (22) lies between 0 and 1. The calculation of the angle between the relevance state vectors yields

\[ |X|^2 = e^{-|m_1-m_0|} \]  \hspace{1cm} (24)

The relationships between the Poisson distribution and the Gamma function allows us to state that

\[ 1 - e^{-|m_1-m_0|} = e^{-m_1} \sum_{x=0}^{\infty} \frac{m_1^x}{x!} - e^{-m_0} \sum_{x=0}^{\infty} \frac{m_0^x}{x!} \]  \hspace{1cm} (25)
We split the summations in (25), thus achieving that (23) holds if and only if

\[ 2 \sum_{x=0}^{x_\alpha-1} e^{-m_1} \frac{m_1^x}{x!} + e^{-m_1} \sum_{x=x_\alpha}^{N} \left( \frac{m_1^x}{x!} - \frac{m_0^x}{x!} \right) + (e^{-m_0} - e^{-m_1}) \sum_{x=0}^{x_\alpha-1} \frac{m_0^x}{x!} \geq 0 \]  

(26)

Every operand of the sum (26) is not negative, thus proving the left side of (21). The right side is proved in symmetric way due to (2) when applied to \( Q_e, Q_c \).

Proposition 1 tells us that the decision as to whether a document is relevant is most effective when the test is function of the optimal observable vectors even if the Poisson-based probability is estimated as accurately as possible.

Corollary 1 For all \( m_0, m_1, x_\alpha \) and if \( Q_0 = P_0 \),

\[ Q_d \geq P_d \]  

(27)

Proof Let \( Q_0 = P_0 \), recall (11), (2) and the left side of (21). Thus,

\[ 0 \leq Q_c - P_c = (1 - \xi)(Q_d - P_d) \]  

(28)

We have that \( Q_c \geq P_c \) because \( Q_d \geq P_d \) and \( 0 \leq \xi \leq 1 \).  

Corollary 1 tells us that, once the probability of false alarm is fixed at an arbitrary size, the state that a document is relevant is correctly accepted with vector probability that is higher than any Poisson-based probability.

The key point is that the region of acceptance induced by the optimal observable vectors \( \mu_1, \mu_0 \) is more powerful than the region of acceptance of the PRP, when the Poisson distribution measures the probability of term frequency, all the other things being equal. A distribution different from Poisson’s or a different estimation of the probability values might revert the outcome of Corollary 1. Does the power of the region of acceptance induced
through the optimal observable vectors and then Corollary 1 depend on the
probability of term frequency? In the remainder of the section, we generalize the result for either distribution of probability of term frequency.

The probability of term frequency is given by two items: (i) the probability values estimated for each \( x \); (ii) the distribution used to compute the probability of term frequency.

As for (i), note that the vector probability and the classical probability of relevance are functions of the same probability distributions \( p(x, m_0), p(x, m_1) \) for every \( m_0, m_1 \). Thus, the power of the region of acceptance induced through the optimal observable vectors and then Corollary 1 do not depend on the probability values calculated for each \( x, m \).

As for (ii), what distinguishes the probability of detection (and false alarm) computed with the optimal observable vectors from those computed with the region of acceptance given as a result of the PRP (i.e., the ML ratio test) is the region of acceptance. We prove that the region of acceptance given through the optimal observable vectors is always more effective than the region of acceptance given as a result of the PRP, that is, independently of the probability distribution of the observable.

Suppose that \( p(x; m_j), j = 0, 1 \) are two arbitrary probability distributions indexed by the parameters \( m_0, m_1 \), the latter indicating the probability distribution of term frequency in non-relevant documents and in relevant documents, respectively.

**Theorem 2** For every \( p(x; m_j), j = 0, 1 \) and \( x_\alpha \)

\[
Q_e \geq P_e \quad Q_e \leq P_e \quad (29)
\]

**Proof** Consider Figures 3(a) and 3(b). A probability of detection \( p_d \) and a probability of false alarm \( p_0 \) defines the coordinates of \( |m_0\rangle \) and \( |m_1\rangle \) with a given orthonormal basis \( |e_0\rangle, |e_1\rangle \) (that is, an observable):

\[
|m_0\rangle = \sqrt{1 - p_d}|e_0\rangle + \sqrt{p_d}|e_1\rangle \quad (30)
\]

\[
|m_1\rangle = \sqrt{p_0}|e_0\rangle + \sqrt{1 - p_0}|e_1\rangle \quad (31)
\]

The coordinates are expressed in terms of angles:

\[
1 - p_d = \sin^2 \eta_1 \quad p_0 = \sin^2 \eta_0 \quad (32)
\]

provided that \( \eta_i \) is the angle between \( |m_i\rangle \) and \( |e_1\rangle \).
The probability of error is

\[ p_e = \xi p_0 + (1 - \xi)(1 - p_d) = \xi \sin^2 \eta_0 + (1 - \xi) \sin^2 \eta_1 \]  

(33)

The probability of error is minimum when \( \eta_0 = \eta_1 = \theta \) as shown in [13, page 99].

But, \( \theta \) is exactly the angle between \( |m_i\rangle, i = 0, 1 \) and \( |\mu_i\rangle \) and is defined as a result of Equation (18). The probability of error is then minimized when the observable vectors are the \( |\mu_i\rangle, i = 0, 1 \).

Therefore, \( Q_e \leq P_e \) for all \( P_e \), that is, for all the observable vectors. As \( Q_e = 1 - Q_c \), the probability of detection is also maximum.

Suppose, for example, that \( X \) is a binary observable, that is, \( x \in \{0, 1\} \). The probability distributions are in Table 5. Suppose that the size of the test is \( \alpha = \frac{1}{5} \). Thus, relevance is accepted when \( X = 1, P_d = \frac{4}{5}, P_0 = \frac{1}{5} \) and, if the states are equiprobable, \( P_e = \frac{1}{2} (P_0 + 1 - P_d) = \frac{1}{5} \) and \( P_c = \frac{1}{2} (1 - P_0 + P_d) = \frac{4}{5} \).

The optimal observable vectors are

\[
|\mu_1\rangle = \begin{pmatrix} 0.97 \\ -0.23 \end{pmatrix} 
|\mu_0\rangle = \begin{pmatrix} 0.23 \\ 0.97 \end{pmatrix}
\]  

(34)

These vectors can be computed in compliance with [15]. Hence, the region of acceptance is the subspace spanned by \( |\mu_1\rangle \) and, if the states are equiprobable, \( Q_e = \frac{1}{2} (Q_0 + 1 - Q_d) = 0.05 \) and \( Q_c = \frac{1}{2} (1 - Q_0 + Q_d) = 0.95 \). Hence, if we were able to find the optimal observable vector and to actually measure it, retrieval performance would be much higher than the performance achieved through the classical region of acceptance.

8 Experimental Study

We have tested the theory illustrated in the previous sections through experiments based on the TIPSTER test collection, disks 4 and 5. The experiments aimed at measuring the difference between \( P_e \) and \( Q_e \) by means of a realistic test collection. To this end, we have used the TREC-6, 7, 8 topic sets. The experimental algorithm is explained in Figure 5.

We have implemented the following test: \( p(x; m) \) has been computed for each topic word and for each \( m \) by means of the usual relative frequency
\( f_w(x; m) / \sum_x f_w(x; m) \) assuming that \( f_w(x; m) \) is the frequency of \( w \) in the documents with state \( m \). Note that we do not aim at measuring effectiveness; rather, we aim at measuring the difference between probabilities of error given a document ranking.

Consider Figure 6; \( Q_e \) is always not greater than \( P_e \) for every size \( \alpha \) and for every prior probability \( \xi \). The superposition of linear curves, one curve for each \( \alpha \), yields a polygonal curve like Figure 11.

Some linear curves are secant because they cut a bell-shaped curve in two parts. However, they refer to different words: given a word, the linear curve is never secant of the bell-shaped curve.

Figures 7, 8, 9 illustrate the plots for other three topics; these topics are representative of the main types of plot – the plots of all the 450 topics exhibit the similar pattern.

9 Discussion

The precedent example points out the issue of the measurement of the optimal observable vectors. Measurement means the actual finding of the presence / absence of the optimal observable vectors via an instrument or device. The measurement of term frequency is straightforward because term occurrence is a physical property measured through an instrument or device. (A program that reads texts and writes frequencies is sufficient.) The measurement of the optimal observable vectors is much more difficult because no physical property does correspond to them and cannot be expressed in terms of term frequencies. Thus, the question is: what should we observe from a document so that the outcome corresponds to the optimal observable vector? The question is not futile because the answer(s) would effect automatic indexing and retrieval. In particular, if we were able to give an interpretation to the optimal observable vectors, retrieval and indexing algorithms could measure those vectors.

Following [3], three interpretations of the optimal observable vectors can be provided:

- Geometrically, each vector is a superposition of other two independent vectors. Figure 10 depicts the way the vectors interact and shows that the observation of a binary feature places the observer upon either \( |0\rangle \) or \( |1\rangle \) whenever he measure 0 or 1, respectively. There is no way to move upon \( |\mu_0\rangle \) or \( |\mu_1\rangle \) because \( \mu_0, \mu_1 \) cannot be measured.
• Probabilistically, the observable vectors and the state vectors are related as a result of the rule of Equation 5. (Also known as trace rule because the general form of the equation is the trace of two matrices.) As the observable vectors are mutually orthonormal by definition, they induce a valid probability distribution.

• Logically, the observable vectors are assertions, e.g., $X = x$ corresponds to $|x\rangle$. The basic difference between subspaces and subsets is that the vectors belong to a subspace if and only if they are spanned by a basis of the subspace. However, the logic to combine subspaces cannot be the set-based logic used to combine subsets.

In classical probability theory, if we observe 1, we say that every document described by 1 is either relevant or not relevant, when relevance is measured. In general, we say that it either possesses a property or does not, when a property is measured. Hence, if an observable is described as sets of values (e.g., the set of documents indexed by a term), we can always describe relevance as a set. That is, the union of the set of relevant documents indexed by a term with the set of relevant documents not indexed by the term.

The orthogonality between the observable vectors implements the mutually exclusiveness between the observable values. Hence, if we observe 0, we can say only that we do not observe 1, but cannot say anything about $\mu_1$ because $|\mu_1\rangle$ is oblique to $|0\rangle$, $|1\rangle$ and vector subspace complement, union and intersection are not the same as subset complement, union and intersection [3].

At the present time, an IR system is capable of measuring observable vectors like $|0\rangle$, $|1\rangle$ which correspond to term occurrence. The documents can be ranked as specified by the PRP, thus achieving $P_e$, which is the current lower bound provided that the probabilities are estimated as accurately as possible [2].

$Q_e$ and $Q_c$ can be achieved if and only if an IR system is capable of observing the optimal observable vectors (Theorem 1). If an IR system observed $\mu_0$ or $\mu_1$ in a document, the system would decide whether the document is relevant with probability of error $Q_e$.

The open problem is due to the difficulty of observing the optimal observable vectors in a document; if a system is given a textual document as input, how can it decide if the document would provide either $|\mu_0\rangle$ or $|\mu_1\rangle$ (or the corresponding eigenvalues) if the optimal observable vectors were measured? We shall pay a great deal of attention to the question because, if the
problem were solved, the solution would give a significant breakthrough in IR research.

10 Related Work

Van Rijsbergen’s book [3] is the point of departure of our work. Helstrom’s book [13] provides the theoretical foundation for the vector probability and the optimal observabl vectors. Eldar and Forney’s paper [15] gives an algorithm for the optimal observable vectors. Hughes’ book [12] is an excellent introduction to Quantum Theory. An introduction to Quantum Theory and Information Retrieval can be found in [16]. In [17] the authors propose quantum formalism for modeling some IR tasks and information need aspects. The paper does not limit the research to the application of an abstract formalism, but exploits the formalism to illustrate how the optimal observable vectors significantly improve effectiveness. In [18], the authors propose $|X|^2$ for modifying probability of relevance; $|X|^2$ is intended to model quantum correlation (also known as interference) between relevance assessments.

11 Conclusions

The research in IR has been traditionally concentrated on extracting and combining evidence as accurately as possible in the belief that the observed features (e.g., term occurrence, word frequency) have to ultimately be scalars or structured objects. The quest for reliable, effective, efficient retrieval algorithms requires to implement the set of features as best one can. The implementation of a set of features is thus an “answer” to an implicit “question”, that is, which is the best set of features for achieving effectiveness as high as possible? We suggest to ask another “question” to achieve an even better answer: Which is the best subspace?

References

[1] A.N. Kolmogorov. *Foundations of the theory of probability*. Chelsea Publishing Company, New York, second edition, 1956.
[2] S.E. Robertson. The probability ranking principle in information retrieval. *Journal of Documentation*, 33(4):294–304, 1977.

[3] C.J. van Rijsbergen. *The geometry of information retrieval*. Cambridge University Press, UK, 2004.

[4] J. Neyman and E.S. Pearson. On the problem of the most efficient tests of statistical hypotheses. *Philosophical Transactions of the Royal Society, Series A*, 231:289–337, 1933.

[5] J. von Neumann. *Mathematical foundations of quantum mechanics*. Princeton University Press, 1955.

[6] S.E. Robertson and H. Zaragoza. The probabilistic relevance framework: BM25 and beyond. *Foundations and Trends in Information Retrieval*, 3(4):333–389, 2009.

[7] S.P. Harter. A probabilistic approach to automatic keyword indexing: part 1: On the distribution of specialty words in a technical literature. *Journal of the American Society for Information Science*, 26(4):197–206, 1975.

[8] S.P. Harter. A probabilistic approach to automatic keyword indexing: part 2: An algorithm for probabilistic indexing. *Journal of the American Society for Information Science*, 26(5):280–289, 1975.

[9] S.E. Robertson and S. Walker. Some simple effective approximations to the 2-Poisson model for probabilistic weighted retrieval. In *Proceedings of the ACM International Conference on Research and Development in Information Retrieval (SIGIR)*, pages 232–241, Dublin, Ireland, 1994.

[10] S.E. Robertson, C.J. van Rijsbergen, and M.F. Porter. *Probabilistic models of indexing and searching*, chapter 4, pages 35–55. Butterworths, 1981.

[11] R. B. Griffiths. *Consistent quantum theory*. Cambridge University Press, 2002.

[12] R.I.G. Hughes. *The structure and interpretation of quantum mechanics*. Harvard University Press, 1989.
[13] C. W. Helstrom. *Quantum detection and estimation theory*. Academic Press, 1976.

[14] P.R. Halmos. *Finite-dimensional vector spaces*. Undergraduate Texts in Mathematics. Springer, 1987.

[15] Y.C. Eldar and G.D. Forney. On quantum detection and the square-root measurement. *IEEE Transactions on Information Theory*, 47(3):858–872, 2001.

[16] Anonymous.

[17] B. Piwowarski, I. Frommholz, M. Lalmas, and K. van Rijsbergen. What can quantum theory bring to information retrieval. In *Proceedings of the 19th ACM international conference on Information and knowledge management*, CIKM ’10, pages 59–68, New York, NY, USA, 2010. ACM.

[18] G. Zuccon and L. Azzopardi. Using the quantum probability ranking principle to rank interdependent documents. In *Proceedings of the European Conference on Information Retrieval Research (ECIR)*, pages 357–369, 2010.
(a) Non-optimal observable vectors are mutually orthogonal and *asymmetrically* placed around $|m_0\rangle, |m_1\rangle$

(b) Optimal observable vectors are mutually orthogonal and *symmetrically* placed around $|m_0\rangle, |m_1\rangle$

Figure 3: Geometry of decision and observable vectors
Figure 4: Probabilities of error: $P_e$ vs. $Q_e$. Figure 1 describes the polygonal curve. The shortest bell-shaped curve corresponds to $\xi = 0.50$, whereas the other bell-shaped curve corresponds to $\xi = 0.90$.

sort data by increasing $f_w(x; m)$ and by $m$

for all topic $t$ do
extract title and description of $t$

for all topic word $w$ of $t$ do
compute $p_m(w; x)$ for every $x$
compute $|X|^2$

for all size $\alpha \in \{0.25, 0.50, 0.75\}$ do
for all prior $\xi \in \{0.01, \ldots, 0.99\}$ do
compute $P_e, P_c$
compute $Q_e, Q_c$
end for
end for
end for

Figure 5: The experimental algorithm
Figure 6: $P_e$ and $Q_e$ plotted against $\xi$ for each word of topic 439 and for each $\alpha \in \{0.25, 0.50, 0.75\}$; each curve corresponds to a word: $+$ labels classical probability of error curves, $\times$ labels vector probability of error curves.

Figure 7: $P_e$ and $Q_e$ plotted against $\xi$ for each word of topic 301 (Figures 7(a), 7(b) and 7(c)) and for each $\alpha \in \{0.25, 0.50, 0.75\}$; each curve corresponds to a word: $+$ labels classical probability of error curves, $\times$ labels vector probability of error curves.

Figure 8: $P_e$ and $Q_e$ plotted against $\xi$ for each word of topic 303 (Figures 8(a), 8(b) and 8(c)) and for each $\alpha \in \{0.25, 0.50, 0.75\}$; each curve corresponds to a word: $+$ labels classical probability of error curves, $\times$ labels vector probability of error curves.
Figure 9: $P_e$ and $Q_e$ plotted against $\xi$ for each word of topic 426 (Figures 9(a), 9(b) and 9(c)) and for each $\alpha \in \{0.25, 0.50, 0.75\}$; each curve corresponds to a word: $+$ labels classical probability of error curves, $\times$ labels vector probability of error curves.

Figure 10: A geometric view of incompatible observable vectors