ON SOME AUTOMORPHISM RELATED PARAMETERS IN GRAPHS

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ABSTRACT. In this paper, we deduce some properties of $f$-sets of connected graphs. We introduce the concept of fixing share of each vertex of a fixing set $D$ to see the participation of each vertex in fixing a connected graph $G$. We also define a parameter, called the fixing percentage, by using the concept of fixing share, which is helpful in determining the measure of the amount of fixing done by the elements of a fixing set $D$ in $G$. It is shown that for every positive integer $N$, there exists a graph $G$ with $dtr(G) - Det(G) \geq N$, where $dtr(G)$ is the determined number and $Det(G)$ is the determining number of $G$.

1. Preliminaries

Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$. If two vertices $u$ and $v$ share an edge, then they are called adjacent, otherwise they are called non-adjacent. The open neighborhood of a vertex $u$ is $N(u) = \{v \in V(G) : v$ is adjacent to $u$ in $G\}$, and the closed neighborhood of $u$ is $N[u] = N(u) \cup \{u\}$. For a subset $U$ of $V(G)$, the set $N_G(U) = \{v \in V(G) : v$ is adjacent to some $u \in U\}$ is the open neighborhood of $U$ in $G$. Two distinct vertices $u, v$ are adjacent twins if $N[u] = N[v]$ and non-adjacent twins if $N(u) = N(v)$. A set $U \subseteq V(G)$ is called a twin-set of $G$ if $u, v$ are twins in $G$ for every pair of distinct vertices $u, v \in U$. The distance $d(u, v)$ between two vertices $u, v \in V(G)$ is the length of a shortest path between them and the diameter $diam(G)$ of $G$ is $\max_{u,v \in V(G)} d(u, v)$. We refer to the book [3] for the general graph theoretic notations and terminology not described in this paper.

For a graph $G$, an automorphism of $G$ is a bijective mapping $f$ on $V(G)$ such that $f(u)f(v) \in E(G)$ if and only if $uv \in E(G)$. The set of all automorphisms of a graph $G$ forms a group, denoted by $\Gamma(G)$, under the operation of composition of functions. For a vertex $v$ of $G$, the set $\{f(v) : f \in \Gamma(G)\}$ is the orbit of $v$ under $\Gamma(G)$, denoted by $\mathcal{O}(v)$, and two vertices in the same orbit are similar. If $u$ and $v$ are similar, then we write $u \sim v$, if they are not similar, then we write $u \not\sim v$. We define $S(G) = \{v \in V(G) : v \sim u \text{ for some } u(\neq v) \in V(G)\}$ (set of all vertices of $G$ which are more than one in their orbits). Also consider $V_s(G) = \{(u, v) : u \sim v (u \neq v)$
and \(u, v \in V(G)\). Note that if \(G\) is a rigid graph (a graph with \(\Gamma(G) = id\)), then \(V_s(G) = \emptyset\). A well-established fact is that every automorphism is also an isometry, that is, for \(u, v \in V(G)\) and \(g \in \Gamma(G)\), \(d(u, v) = d(g(u), g(v))\).

An automorphism \(g \in \Gamma(G)\) is said to fix a vertex \(v \in V(G)\) if \(g(v) = v\). The set of automorphisms that fix a vertex \(v \in V(G)\) is a subgroup of \(\Gamma(G)\). It is called the stabilizer of \(v\) and is denoted by \(\Gamma_v(G)\). For \(D \subseteq V(G)\) and \(g \in \Gamma(G)\), an automorphism \(g\) is said to fix the set \(D\) if for every \(v \in D\), we have \(g(v) = v\). The set of automorphisms that fix \(D\) is a subgroup \(\Gamma_D(G)\) of \(\Gamma(G)\) and \(\Gamma_D(G) = \cap_{v \in D} \Gamma_v(G)\). If \(D\) is a set of vertices for which \(\Gamma_D(G) = \{id\}\), then \(D\) fixes the graph \(G\) and we say that \(D\) is a fixing set of \(G\). Erwin and Harary introduced the fixing number, \(fix(G)\), of a graph \(G\) in [6] and it is defined as the minimum cardinality of a set of vertices that fixes \(G\). A fixing set containing \(fix(G)\) number of vertices is called a minimum fixing set of \(G\). The notion of fixing set has equivalence with another notion, determining set, introduced by Boutin in [1]. A set \(E \subseteq V(G)\) is said to be a determining set for a graph \(G\) if whenever \(g, h \in \Gamma(G)\) so that \(g(x) = h(x)\) for all \(x \in E\), then \(g(v) = h(v)\) for all \(v \in V(G)\), i.e., if two automorphisms \(g\) and \(h\) agree on \(E\), then they must agree on \(V(G)\). The equivalence between the definitions of a fixing set and a determining set was given in [8]. The minimum cardinality of a determining set of a graph \(G\), denoted by \(Det(G)\), is called the determining number of \(G\). In this paper, we will use both the terms, fixing and determining. Term ‘fixing’ is used to fix vertices and ‘determining’ is used to determine automorphisms. However, the term ‘fixing set’ and ‘determining set’ can be used interchangeably.

A vertex \(x \in V(G)\) is called a fixed vertex if \(g(x) = x\) for all \(g \in \Gamma(G)\), i.e., \(\Gamma_x(G) = \Gamma(G)\). A vertex \(x \in V(G)\) is said to fix a pair \((u, v) \in V_s(G)\), if \(h(u) \neq v\) or \(h(v) \neq u\) whenever \(h \in \Gamma_x(G)\). If \((u, v) \notin V_s(G)\), then \(u \not\sim v\) and hence question of fixing pair \((u, v)\) by any vertex of \(G\) does not arise. Let \((u, v) \in V_s(G)\) and the set \(fix(u, v) = \{x \in V(G) : g(u) \neq v\} \cap \{x \in V(G) : g(v) \neq u\}\) for all \(g \in \Gamma_x(G)\) is called the fixing set (or an f-set) relative to the pair \((u, v)\). It is also further assumed that if \((u, v) \notin V_s(G)\), then \(fix(u, v) = \emptyset\). Hence, \(\{u, v\} \subseteq fix(u, v) \subseteq V(G)\). Let \(x \in V(G)\) and the set \(F(x) = \{(u, v) \in V_s(G) : h(u) \neq v\} \cap \{x \in V(G) : h(v) \neq u\}\) for all \(h \in \Gamma_x(G)\) is called the fixed neighborhood of \(x\). Also, if \(x \in V(G)\) is a fixed vertex, then \(F(x) = \emptyset\). The fixing graph, \(F(G)\), of a graph \(G\) is a bipartite graph with bipartition \((S(G), V_s(G))\) and a vertex \(x \in S(G)\) is adjacent to a pair \((u, v) \in V_s(G)\) if \(x\) fixes \((u, v) \in V_s(G)\). Let a set \(D \subseteq S(G)\), then \(N_{F(G)}(D) = \{(u, v) \in V_s(G) : x\} \cap \{x \in D\}\) fixes \((u, v)\) for some \(x \in D\). In the fixing graph, \(F(G)\), the minimum cardinality of a subset \(D\) of \(V(G)\) such that \(N_{F(G)}(D) = V_s(G)\) is the fixing number of \(G\).

An upper bound on \(fix(G)\) was given by Erwin and Harary by using another well-studied invariant, metric dimension, defined in the following way: Let \(W = \{v_1, v_2, \ldots, v_k\}\) be a \(k\)-subset of \(V(G)\) and, for each vertex \(v \in V(G)\), define \(r(v|W) = \)
(d(v, v_1), d(v, v_2), \ldots, d(v, v_k)). A k-set W is called a resolving set for G if for every pair u, v of distinct vertices of G, r(u|W) \neq r(v|W). The metric dimension, dim(G), is the smallest cardinality of a resolving subset W. A resolving set of minimum cardinality is a metric basis for G. Following lemma and theorem were given in [6]:

**Lemma 1.1.** If W is a metric basis for G, then \( \Gamma_W(G) \) is trivial.

**Theorem 1.2.** For every connected graph G, \( fix(G) \leq dim(G) \).

Considering the fact that the metric dimension is greater than or equal to fixing number and automorphisms preserve distances, metric dimension and fixing number are closely related notions [3, 6]. Cáceres et al. studied this relation and answered the following question which appeared first in [1]: Can the difference between both parameters of a graph of order \( n \) be arbitrarily large? The resolving number, \( res(G) \), of a graph G is the minimum \( k \) such that every \( k \)-set of vertices is a resolving set of G. Resolving number of a graph G gives natural upper bound to the metric dimension of G, i.e., \( dim(G) \leq res(G) \). For every pair \( a, b \) of integers with \( 2 \leq a \leq b \), existence of a connected graph G with \( dim(G) = a \) and \( res(G) = b \), was given in [4]. Motivated by definition of resolving number and resemblance between parameters metric dimension and determining number, we defined determined number of a graph G in [11]. The determined number of a graph G, \( dtr(G) \), is the minimum \( k \) such that every \( k \)-set of vertices is a determining set for a graph G. It may be noted that \( 0 \leq Det(G) \leq dtr(G) \leq |V(G)| - 1 \). In [11], we found determined number of some well known graphs and discussed some of its properties.

In the next section, we study some properties of \( f \)-sets following the study of \( R \)-sets by Tomescu and Imran [13]. To see the contribution of each vertex in resolving a graph, the concepts of resolving share and resolving percentage were introduced in [12]. In the second section, we introduce the concept of the fixing share which tells the participation of each vertex of a fixing set in fixing a graph G. We also define the fixing percentage in G, by using the concept of fixing share of each element of a fixing set of G, which is the measure of the amount of fixing done by a fixing set in G. Then we compute the fixing share and the fixing percentage in paths and cycles. In the third section, we will prove existence of a graph G for which \( dtr(G) - Det(G) \geq N \) for a given positive integer \( N \).

### 2. Properties of \( f \)-sets

**Lemma 2.1.** If there exists an automorphism \( g \in \Gamma(G) \) such that \( g(u) = v, u \neq v \) and if \( d(u, x) = d(v, x) \) for some \( x \in V(G) \), then \( x \notin fix(u, v) \).

**Proof.** Since automorphism \( g \) is an isometry, so \( d(u, x) = d(g(u), g(x)) = d(v, g(x)) \) and by hypothesis \( d(v, g(x)) = d(v, x) \). Thus \( g(x) = x \) and \( g \in \Gamma_x(G) \). Hence, \( x \notin fix(u, v) \). \( \square \)
Corollary 2.2. Let $G$ be a cycle of order $n$ and let $u, v \in V_s(G) = V(G)$ be a pair of distinct vertices in $G$,

(i) $\text{fix}(u, v) = V(G)$, if $n$ is even and $d(u, v)$ is odd.

(ii) $\text{fix}(u, v) = V(G) \setminus \{x_1, x_2\}$, if $n$ is even and $d(u, v)$ is even, where $x_1, x_2 \in V(G)$ are the antipodal vertices with $d(x_i, u) = d(x_i, v)$, $1 \leq i \leq 2$.

(iii) $\text{fix}(u, v) = V(G) \setminus \{x\}$, if $n$ is odd and $x \in V(G)$ is the vertex with $d(x, u) = d(x, v)$.

Proposition 2.3. Let $G$ be a path of order $n$ and $V(G) = \{u_1, ..., u_n\}$ where $u_i$ is adjacent to $u_{i+1}$ with $(1 \leq i \leq n-1)$, then $V_s(G) = \{(u_i, u_{n+1-i}) : 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \}$ and

(i) $\text{fix}(u_i, u_{n+1-i}) = V(G)$, if $n$ is even.

(ii) $\text{fix}(u_i, u_{n+1-i}) = V(G) \setminus \{u_{\frac{n+1}{2}}\}$, if $n$ is odd.

Two vertices $u$ and $v$ in a graph $G$ are said to be distance similar if $d(u, w) = d(v, w)$ for all $w \in V(G) \setminus \{u, v\}$.

Theorem 2.4. $\text{fix}(u, v) = \{u, v\}$ if and only if $u, v$ are distance similar.

Proof. If $\text{fix}(u, v) = \{u, v\}$, then this implies that $u$ can be fixed either by fixing $u$ or by fixing $v$. Now if $u$ and $v$ are not distance similar, then there exists a vertex $w \in V(G)$ such that $d(u, w) \neq d(v, w)$. If we fix $w$ by an automorphism $f \in \Gamma_w(G)$, then $f(u) = v$ implies $d(w, f(u)) = d(w, v)$. Now $w = f(w)$ implies $d(f(w), f(u)) = d(w, v)$ and fact that $f$ is an isometry implies $d(w, u) = d(w, v)$, a contradiction. Thus $u, v$ are distance similar.

Conversely, let $u, v$ are distance similar and $\{u, v\} \subset \text{fix}(u, v)$, then there exists at least one vertex $w(\neq u, v) \in V(G)$ such that $f(u) \neq v$ and $f(v) \neq u$ for all $f \in \Gamma_w(G)$. Since $f$ is an isometry and $f(u) \neq v$, so $d(u, w) = d(f(u), f(w)) \neq d(v, w)$,

a contradiction that $u, v$ are distance similar. \square

In a complete graph, every pair is distance similar. Therefore, we have the following corollary:

Corollary 2.5. Let $G$ be a complete graph of order $n$ and $u, v$ be a pair of distinct vertices, then $(u, v) \in V_s(G)$ and $\text{fix}(u, v) = \{u, v\}$.

Corollary 2.6. Let $G = K_{m,n}$ be a complete bipartite graph,

(i) $\text{fix}(u, v) = \{u, v\}$, if both $u, v$ are in same partite sets.

(ii) $\text{fix}(u, v) = V(G)$, if both $u, v$ are not in same partite sets and $m = n$.

(iii) $\text{fix}(u, v) = \emptyset$, if both $u, v$ are not in same partite set and $m \neq n$.

Proposition 2.7. Let $G$ be a graph of order $n \geq 2$ and fixing number $k$, then

$$|E(F(G))| \leq n\left(\binom{n}{2} - k + 1\right).$$

Proof. Let $|S(G)| = r$ and $|V_s(G)| = s$, then $r \leq n$ and $s \leq \binom{n}{2} \leq \binom{r}{2}$. Let $v \in S(G)$. We will prove that $\text{deg}_{F(G)}(v) \leq s - k + 1$. Suppose $\text{deg}_{F(G)}(v) \geq s - k + 2$, then there
are at most \( k - 2 \) pairs in \( V_s(G) \) which are not adjacent to \( v \). Let \( V_s(G) \setminus N_{F(G)}(v) = \{(u_1, v_1), (u_2, v_2), ..., (u_t, v_t)\} \), where \( t \leq k - 2 \). Note that, \( u_i \) fixes \((u_i, v_i)\) for each \( i, 1 \leq i \leq t \). Hence, \( u_i \) is adjacent to pair \((u_i, v_i)\) in \( F(G) \) for each \( i, 1 \leq i \leq t \). Therefore, \( N_{F(G)}(\{v, u_1, u_2, ..., u_t\}) = V_s(G) \). Hence, \( fix(G) \leq t + 1 \leq k - 1 \), which is a contradiction. Thus, \( \deg_{F(G)}(v) \leq s - k + 1 \leq \binom{n}{2} - k + 1 \) and consequently, \( |E(F(G))| \leq n\left(\binom{n}{2} - k + 1\right) \). \( \square \)

3. Fixing share in graphs

If \( u, v \) are twins in a connected graph \( G \), then \( d(u, x) = d(v, x) \) for every vertex \( x \in V(G) \setminus \{u, v\} \). From this we have the following remarks:

**Remark 3.1.** If \( u, v \) are twins in a connected graph \( G \) and \( D \) is a fixing set for \( G \), then \( u \) or \( v \) is in \( D \). Moreover, if \( u \in D \) and \( v \notin D \), then \( (D \setminus \{u\}) \cup \{v\} \) is also a fixing set for \( G \).

**Remark 3.2.** If \( U \) is a twin-set in a connected graph \( G \) of order \( n \) with \( |U| = m \geq 2 \), then every fixing set for \( G \) contains at least \( m - 1 \) vertices from \( U \).

**Definition 3.3.** (Sole fixer) Let \( G \) be a connected graph and \( D \) be a minimum fixing set of \( G \). Let \( (u, v) \in V_s(G) \). If \( fix(u, v) \cap D = \{x\} \), then \( x \) is called sole fixer for the pair \((u, v)\).

![Graph G1](image)

**Figure 1.** Graph \( G_1 \)

Consider the graph \( G_1 \) in Figure 1 with vertex set \( V(G_1) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\} \) and \( V_s(G_1) = \{(v_1, v_3), (v_1, v_5), (v_3, v_5), (v_2, v_4), (v_2, v_6), (v_4, v_6)\} \). A fixing set of \( G_1 \) with minimum cardinality is \( D_1 = \{v_1, v_3\} \). Since \( fix(v_2, v_4) = \{v_1, v_5\} \) and \( fix(v_2, v_6) = \{v_3, v_5\} \), so \( v_1 \) is the sole fixer of the pair \((v_2, v_4)\) and \( v_3 \) is the sole fixer of the pair \((v_2, v_6)\) with respect to \( D_1 = \{v_1, v_3\} \).
Remark 3.4. Let $G$ be a connected graph and $D$ be a minimum fixing set of $G$. Let $(u, v) \in V_s(G)$ be a pair and $x \in D$ be the sole fixer of $(u, v)$ with respect to $D$. Then, $x$ is considered to have 1-share in fixing pair $(u, v)$. Further, if $(u, v)$ is also fixed by, say $m < |D|$ other vertices of $D \setminus \{x\}$, then $x$ is considered to have $\frac{1}{m+1}$ share in fixing the pair $(u, v)$.

For example in $G_1$, the vertex $v_1$ is considered to have 1-share in fixing the pair $(v_2, v_4)$ with respect to $D_1$. However, the vertex $v_1$ is considered to have $\frac{1}{2}$-share in fixing the pair $(v_4, v_6)$, as $fix(v_4, v_6) \cap D_1 = \{v_1, v_3\}$.

If $D$ is a fixing set with minimum cardinality for a connected graph $G$ and $x \in D$, then the fixing share of $x$ in $D$ is defined as a measure of the amount of fixing done by $x$ in $G$. Formally, we have the following definition:

Definition 3.5. (fixing share) Let $G$ be a connected graph and $D$ be a minimum fixing set of $G$. Let $x \in D$ and $(u, v) \in F(x)$. We define a set $F(u, v) = \{F(y) : y \in D \text{ and } (u, v) \in F(y)\}$. The fixing share, $f(x, D)$ of an $x \in D$ in $D$ is defined as

$$f(x; D) = \sum_{(u, v) \in F(x)} \frac{1}{|F(u, v)|}$$

In example of $G_1$ we have

$F(v_1) = \{(v_1, v_5), (v_1, v_3), (v_2, v_4), (v_4, v_6)\}$ and $F(v_3) = \{(v_1, v_3), (v_2, v_6), (v_3, v_5), (v_4, v_6)\}$.

Also $F(v_1, v_3) = \{F(v_1), F(v_3)\}$, $F(v_1, v_3) = \{F(v_1), F(v_3)\}$, $F(v_2, v_4) = \{F(v_1)\}$, $F(v_4, v_6) = \{F(v_1), F(v_3)\}$. Thus $f(v_1; D_1) = 1 + \frac{1}{2} + 1 + \frac{1}{2} = 3$ and similarly $f(v_3; D_1) = \frac{1}{2} + 1 + 1 + \frac{1}{2} = 3$.

Definition 3.6. (fixing sum and percentage) Let $D$ be a minimum fixing set for a connected graph $G$ and let $F_{sum}(G) = \sum_{x \in D} f(x; D)$, called the fixing sum in $G$. Then the quantity $\frac{|D|}{F_{sum}(G)}$, denoted by $F\% (G)$, is the measure of the amount of fixing done by $D$ in $G$, and we call it the fixing percentage of $D$ in $G$.

For graph $G_1$, $F_{sum}(G_1) = 6$ and $F\% (G_1) = \frac{2}{6} = \frac{1}{3}$.

Theorem 3.7. Let $G$ be a complete graph of order $n \geq 3$ and $D$ be a minimum fixing set of $G$. Let $v \in D$, then $f(v; D) = \frac{n}{2}$. Therefore $F_{sum}(G) = (n - 1)\frac{n}{2}$ and $F\% (G) = \frac{2}{n}$.

Proof. Let $V(G) = \{v_1, ..., v_n\}$. Since $G$ is a complete graph, so $V_s(G) = \{(v_i, v_j) : i \neq j \text{ and } 1 \leq i, j \leq n\}$. Also cardinality of a minimum fixing set of $G$ is $n - 1$. Let $D = \{v_1, v_2, ..., v_{n-1}\} \subset V(G)$ be a minimum fixing set of $G$, also $D$ is a twin-set in $G$. Then for each $i$, $1 \leq i \leq n - 1$, $F(v_i) = \{(v_i, v_j)\}$, where $j \neq i$ and $1 \leq j \leq n$. It can be seen that for each $j$ where $j \neq i$, $1 \leq j \leq n - 1$, $|F(v_i, v_j)| = 2$ (because $(v_i, v_j)$ appears in exactly two $F(v_i)$, where $1 \leq i \leq n - 1$). Also $|F(v_i, v_n)| = 1$ (because $(v_i, v_n)$ appears in exactly one $F(v_i)$, where $1 \leq i \leq n - 1$). So $f(v_i; D) = (n - 2)\frac{1}{2} + 1 = \frac{n}{2}$. Hence, $F_{sum}(G) = (n - 1)\frac{n}{2}$ and $F\% (G) = \frac{2}{n}$. □
Theorem 3.8. Let $G$ be a path of order $n \geq 2$ and $D = \{v\}$ be a minimum fixing set of $G$. Then fixing share $f(v; D) = \left(\frac{n}{2}\right) = F_{\text{sum}}(G)$ and $F\% (G) = \frac{1}{\left\lfloor \frac{1}{2}\right\rfloor}$.

Proof. As $\text{fix}(G) = 1$ and one of the end vertices of $G$ forms a fixing set $D$. Also $|V_s(G)| = \left\lfloor \frac{n}{2}\right\rfloor$ and $v$ is the sole fixer of $(\frac{n}{2})$ pairs in $V_s(G)$. Thus $|F(v)| = \left\lfloor \frac{n}{2}\right\rfloor$ and $|\mathcal{F}(u,v)| = 1$ for all $(u,v) \in F(v)$ and hence, $f(v; D) = \left\lfloor \frac{n}{2}\right\rfloor = F_{\text{sum}}(G)$. Thus $F\% (G) = \frac{1}{\left\lfloor \frac{1}{2}\right\rfloor}$ for all $n \geq 2$.

Remark 3.9. Since, a path $P_n$ is a graph with fixing number 1 and a complete graph $K_n$ is the graph with fixing number $n - 1$, hence we can deduce that for a connected graph $G$ of order $n \geq 2$, $1 \leq F_{\text{sum}}(G) \leq \binom{n}{2}$ and $\frac{2}{n^2 - n} \leq F\% (G) \leq \frac{2}{n}$.

Two vertices $u, v$ in a connected graph $G$ of order $n$ are said to be antipodal vertices if $d(u, v) = \frac{n}{2}$.

Theorem 3.10. Let $G$ be a cycle graph of order $n \geq 4$ and $D$ be a minimum fixing set of $G$. Let $v \in D$, then the fixing share $f(v; D) = \frac{1}{2} \binom{n}{2}$. Therefore $F_{\text{sum}}(G) = \binom{n}{2}$ and $F\% (G) = \frac{4}{n^2 - n}$.

Proof. For all $n \geq 4$, let $D = \{u, v\}$ be a minimum fixing set of $G$ consisting of two non-antipodal vertices $u, v$. Then $|V_s(G)| = \binom{n}{2}$. Two cases arise:

Case I (when $n$ is even)
We notice that $|F(v)| = \binom{n}{2} - \frac{n-2}{2}$. There are two types of pairs $(x, y)$ in $F(v)$. (i) There are $(\frac{n-2}{2})$ pairs $(x, y)$ in $F(v)$ such that $|\mathcal{F}(x,y)| = 1$ (as $v$ is the sole fixer for $(\frac{n-2}{2})$ pairs in $F(v)$). (ii) There are remaining $\binom{n}{2} - (n - 2)$ pairs $(x, y)$ in $F(v)$ such that $|\mathcal{F}(x,y)| = 2$ (as $u$ and $v$ equally participate to fix the remaining $\binom{n}{2} - (n - 2)$ pairs in $F(v)$). Thus, $f(v; D) = 1 \cdot \frac{n-2}{2} + \frac{1}{2} \left( \binom{n}{2} - (n - 2) \right) = \frac{1}{2} \binom{n}{2}$ for each $v \in D$.

Case II (when $n$ is odd)
We notice that $|F(v)| = \binom{n}{2} - \frac{n-1}{2}$. There are two types of pairs $(x, y)$ in $F(v)$. (i) There are $(\frac{n-1}{2})$ pairs $(x, y)$ in $F(v)$ such that $|\mathcal{F}(x,y)| = 1$ (as $v$ is the sole fixer for $(\frac{n-1}{2})$ pairs in $F(v)$). (ii) There are remaining $\binom{n}{2} - (n - 1)$ pairs $(x, y)$ in $F(v)$ such that $|\mathcal{F}(x,y)| = 2$ (as $u$ and $v$ equally participate to fix the remaining $\binom{n}{2} - (n - 1)$ pairs in $F(v)$). Thus, $f(v; D) = 1 \cdot \frac{n-1}{2} + \frac{1}{2} \left( \binom{n}{2} - (n - 1) \right) = \frac{1}{2} \binom{n}{2}$ for each $v \in D$.

So $F_{\text{sum}}(G) = \binom{n}{2}$ and hence $F\% (G) = \frac{4}{n^2 - n}$.

A vertex of degree at least 3 in a tree is called a major vertex of tree. An end vertex $u$ of a tree $T$ is said to be a terminal vertex of a major vertex $v$, if $d(u, v) < d(u, w)$ for every other major vertex $w$ of $T$. The terminal degree, $\text{ter}(u)$, of a major vertex $u$ of $T$ is the number of terminal vertices of $u$.

Theorem 3.11. Let $T$ be a tree which is not a path and every major vertex of $T$ has different terminal degree and all terminal vertices rooted at a major vertex are at the same distance from that major vertex, $D$ is a fixing set of $T$, $v \in D$ be a terminal vertex of a major vertex of $T$, then $f(v; D) = \frac{\text{ter}(v)}{2}$.
Proof. Let $v$ be a terminal vertex of a major vertex $u$ of $T$ and $ter(u)$ is different from terminal degrees of remaining major vertices of $T$, so $\mathcal{O}(v)$ contains remaining terminal vertices of major vertex $u$ of $T$. Let $|\mathcal{O}(v)| = p$ and $\mathcal{O}(v) = \{v, v_1, v_2, \ldots, v_{p-1}\}$. Now, $D$ being a fixing set of $T$ contains $p-1$ vertices of $\mathcal{O}(v)$, say $A = \{v, v_1, v_2, \ldots, v_{p-2}\}$. The only pairs of vertices fixed by vertex $v$ are pairs of vertices in $\mathcal{O}(v)$. Thus fixing share of $v$ has equal contribution from $\{v, v_1, v_2, \ldots, v_{p-1}\}$.

$$F(v) = \{(v, v_1), (v, v_2), \ldots, (v, v_{p-1}), (v, v_{p-1})\}$$
$$F(v_1) = \{(v_1, v), (v_1, v_2), \ldots, (v_1, v_{p-1}), (v_1, v_{p-1})\}$$
$$\cdots$$
$$F(v_{p-2}) = \{(v_{p-2}, v), (v_{p-2}, v_1), \ldots, (v_{p-2}, v_{p-1})\}$$

As no vertex in $D \setminus A$ can fix pairs $(v, v_i) \in F(v)$, $1 \leq i \leq p-1$. Thus each $F(w)$ where $w \in D \setminus A$, does not contain any pair $(v, v_i) \in F(v)$, $1 \leq i \leq p-1$. Thus $f(v; D) = (p-2)\frac{1}{2} + 1 = \frac{p}{2}$ \hfill \Box

4. Determined Number of graphs

In the next theorem, we will prove the existence of a graph $G$ for a given positive integer $N$ such that $dtr(G) - Det(G) \geq N$.

Theorem 4.1. For every positive integer $N$, there exists a graph $G$ such that $dtr(G) - Det(G) \geq N$.

Proof. We choose $k \geq \max\{3, \frac{N+3}{2}\}$. Let $V(G) = U \cup W$ be a bipartite graph, where $U = \{u_1, \ldots, u_{2^k-2}\}$ and ordered set $W = \{w_1, w_2, \ldots, w_{k-1}\}$ and both $U$ and $W$ are disjoint. Before defining adjacencies, we assign coordinates to each vertex of $U$ by expressing each integer $j$ ($1 \leq j \leq 2^k - 2$) in its base $2$ (binary) representation. We assign each $u_j$ ($1 \leq j \leq 2^k - 2$) the coordinates $(a_{k-1}, a_{k-2}, \ldots, a_0)$ where $a_m$ ($0 \leq m \leq k - 1$) is the value in the $2^m$ position of binary representation of $j$. For integers $i$ ($1 \leq i \leq k - 1$) and $j$ ($1 \leq j \leq 2^k - 2$), we join $w_i$ and $u_j(a_{k-1}, a_{k-2}, \ldots, a_0)$ if and only if $i = \sum_{m=0}^{k-1} a_m$. This completes the construction of graph $G$.

Next we will prove that $\text{Det}(G) = 2^k - (k+1)$. We denote $N(w_i) = \{u_j \in U : u_j$ is adjacent to $w_i, 1 \leq j \leq 2^k - 2\}$ and it is obvious to see that $N(w_i) \cap N(w_j) = \emptyset$ as if $u(a_{k-1}, a_{k-2}, \ldots, a_0) \in N(w_i) \cap N(w_j)$, then $i = \sum_{m=0}^{k-1} a_m = j$. Number of vertices in each $N(w_i)$ is the permutation of $k$ digits in which digit $1$ appears $i$ times and digit $0$ appears $(k - i)$ times, hence $|N(w_i)| = \binom{k}{i}$. As $N(w_i) \cap N(w_j) = \emptyset$, so minimum determining set $E$ must have $\binom{k}{i} - 1$ vertices from each $N(w_i)$, $1 \leq i \leq k - 1$, for otherwise if $u, v \in N(w_i)$ and $u, v \notin E$ for some $i$, then there exists an automorphism $g \in \Gamma(G)$ such that $g(u) = v$ because $u$ and $v$ have only one common neighbor $w_i$, which is a contradiction that $E$ is a determining set. Moreover $E \subseteq U$ as each $w_i$,
\[ 0 \leq i \leq k - 1, \text{ is fixed while fixing at least } \binom{k}{i} - 1 \text{ vertices in each } N(w_i). \text{ Hence,} \]

\[ \text{Det}(G) = \sum_{i=1}^{k-1} \binom{k}{i} - (k - 1) = 2^k - (k + 1) \]

Next we will find \( dtr(G) \). As order of \( G \) is \( 2^k + k - 3 \) and set of all vertices of \( G \) except one vertex forms a determining set of \( G \). It can be seen that \( dtr(G) = 2^k + k - 4 \), for otherwise if \( dtr(G) < 2^k + k - 4 \) and \( u, v \in N(w_i) \) for some \( i \), then the set \( E = W \cup U \setminus \{u, v\} \) consisting of \( 2^k + k - 5 \) is not a determining set, which implies that \( dtr(G) = 2^k + k - 4 \). Hence for the graph \( G \), we have \( dtr(G) - \text{Det}(G) = 2k - 3 \geq N \) as required. \( \square \)

5. Summary

In this paper, we have described some properties of \( f \)-sets in graphs. We have found bound on cardinality of edge set of the fixing graph of a graph \( G \). We have defined fixing share of a vertex in a fixing set \( D \) of a graph \( G \) and studied it for vertices in fixing sets of some common classes of graphs. Finally we have proved existence of a graph \( G \) for a given positive integer \( N \) for which \( dtr(G) - \text{Det}(G) \geq N \).

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