WEIGHTED LOCAL ESTIMATES
FOR SINGULAR INTEGRAL OPERATORS

JONATHAN POELHUIS AND ALBERTO TORCHINSKY

In remembrance of Björn Jawerth (1952-2013) who believed in local sharp maximal functions

Abstract. A local median decomposition is used to prove that a weighted mean of a function is controlled locally by the weighted mean of its local sharp maximal function. Together with the estimate $M^\#_0, s(Tf)(x) \leq c Mf(x)$ for Calderón-Zygmund singular integral operators, this allows us to express the local weighted control of $Tf$ by $Mf$. Similar estimates hold for $T$ replaced by singular integrals with kernels satisfying Hörmander-type conditions or integral operators with homogeneous kernels, and $M$ replaced by an appropriate maximal function $M_T$. Using sharper bounds in the local median decomposition we prove two-weight, $L^p_v - L^q_w$ estimates for the singular integral operators described above for $1 < p < q < \infty$ and a range of $q$. The local nature of the estimates leads to results involving weighted generalized Orlicz-Campanato and Orlicz-Morrey spaces.

1. Introduction

An underlying principle of the Calderón-Zygmund theory, first expressed by Cotlar for the Hilbert transform [11], is that the Hardy-Littlewood maximal function controls the Calderón-Zygmund singular integral operators. Coifman formulated this principle in the weighted setting as follows. We say that a continuous function $\Phi$ satisfies condition $C$ if it is increasing on $[0, \infty)$ with $\Phi(0) = 0$ and $\Phi(2t) \leq c \Phi(t)$, all $t > 0$. Then, if $\Phi$ satisfies condition $C$, $w$ is an $A_\infty$ weight, and $T$ is a Calderón-Zygmund singular integral operator,

\begin{equation}
\int_{\mathbb{R}^n} \Phi(|Tf(x)|) w(x) \, dx \leq c \int_{\mathbb{R}^n} \Phi(Mf(x)) w(x) \, dx,
\end{equation}

where $Mf$ denotes the Hardy-Littlewood maximal function of $f$. In the setting of the $A_p$ weights, extending the result of Hunt, Muckenhoupt and Wheeden for the Hilbert transform [25], Coifman and Fefferman [9] proved that

\begin{equation}
\int_{\mathbb{R}^n} |Tf(x)|^p w(x) \, dx \leq c \int_{\mathbb{R}^n} |f(x)|^p w(x) \, dx,
\end{equation}

provided that $1 < p < \infty$ and $w \in A_p$. The proof of both of these results makes use of the good-$\lambda$ inequalities of Burkholder and Gundy [4].

Along similar lines, and with the additional purpose of considering vector-valued singular integrals, Córdoba and Fefferman [10] proved that for a weight $w$, i.e., a nonnegative locally integrable function $w$, and $1 < p < \infty$,

\begin{equation}
\int_{\mathbb{R}^n} |Tf(x)|^p w(x) \, dx \leq c \int_{\mathbb{R}^n} |f(x)|^p M_w(x) \, dx,
\end{equation}

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where \( M_r w(x) = M(w^r)(x)^{1/r} \) denotes the Hardy-Littlewood maximal function of order \( r, 1 < r < \infty \). Their proof is based on the pointwise inequality
\[
M^2(Tf)(x) \leq c M_r f(x),
\]
where \( M^2 \) denotes the sharp maximal function. Since this inequality does not hold for \( r = 1 \) and \( T \) the Hilbert transform, it is of interest that for Calderón-Zygmund singular integral operators the pointwise control can be expressed in terms of the local sharp maximal function \( M_{0,s}^2 \) by means of an estimate proved by Jawerth and Torchinsky \cite{27} that preserves the weak-type information, to wit,
\[
M_{0,s}^2(Tf)(x) \leq c M f(x).
\]

In the first part of this paper we revisit this inequality, recast it in local terms, and establish weighted local estimates for Calderón-Zygmund singular integral operators. In particular, our results include that these operators satisfy
\[
M_{0,s,Q_0}^2(Tf)(x) \leq c \sup_{x \in Q, Q \subset Q_0} \inf_{y \in Q} M f(y),
\]
where \( M_{0,s,Q_0}^2 \) denotes the local sharp maximal function restricted to the cube \( Q_0 \). Combined with the inequality
\[
\int_{Q_0} \Phi(|f(x) - m_f(t, Q_0)|) w(x) dx \leq c \int_{Q_0} \Phi(M_{0,s,Q_0}^2 f(x)) v(x) dx,
\]
where \( \Phi \) is any function satisfying condition \( C \), \( m_f(t, Q_0) \) denotes the (maximal) median of \( f \) with parameter \( t \), and \( v = w \) when \( w \in A_\infty \) and \( v = M_r w \) when \( w \) is an arbitrary weight, it readily follows that
\[
\int_{Q_0} \Phi(|Tf(x) - m_{Tf}(t, Q_0)|) w(x) dx \leq c \int_{Q_0} \Phi(M f(x)) v(x) dx,
\]
where \( c \) is independent of the cube \( Q_0 \) and \( f \).

Furthermore, if \( \lim_{Q_0 \to \mathbb{R}^n} m_{Tf}(t, Q_0) = 0 \), then
\[
\int_{\mathbb{R}^n} \Phi(|Tf(x)|) w(x) dx \leq c \int_{\mathbb{R}^n} \Phi(M f(x)) v(x) dx.
\]
We also prove that (1.6) and (1.7) hold for appropriate non-\( A_\infty \) weights \( w \) and \( v \).

Estimates such as (1.6) represent a local version of (1.1) and (1.3) and fail for singular integrals when \( M f \) is replaced by \( |f| \) on the right-hand side, even with \( \lambda Q_0 \) in place of \( Q_0 \) for any \( \lambda > 1 \) there.

As for the integral inequality (1.7), it includes all three estimates, (1.1), (1.2), and (1.3), and implies the following one. A function \( \Phi \) that satisfies condition \( C \) which is convex and such that \( \Phi(t) \to \infty \) as \( t \to \infty \), or, more generally, such that \( \Phi(t)/t \to \infty \) as \( t \to \infty \), is called a Young function. Let \( w \in A_\infty \) and let \( \Phi, \Psi \) be Young functions such that
\[
\int_0^t \frac{\Phi(s)}{s^2} ds \leq c \frac{\Psi(t)}{t}, \quad t > 0,
\]
and \( \Phi(t)/t^q \) decreases for some \( q > 1 \). Then
\[
\int_{\mathbb{R}^n} \Phi(|Tf(x)|) w(x) dx \leq c \int_{\mathbb{R}^n} \Psi(|f(x)|) M w(x) dx.
\]
with
\[ M_p^\sharp(g)(x) = \sup_{x \in Q} \inf_c \left( \frac{1}{|Q|} \int_Q |g(y) - c|_p^p \, dy \right)^{1/p}, \]
cp \to \infty as \( p \to 1 \), and
\[ (1.9) \quad M^\sharp(Tf)(x) \leq c M_{L \log L} f(x). \]

Now, by (1.9), if \( \Phi(t)/t^p \) increases and \( \Phi(t)/t^q \) decreases for some \( 1 < p < q < \infty \), and \( w \) is an arbitrary weight, by Theorem 1.7 in [42] it readily follows that
\[ (1.10) \quad \int_{\mathbb{R}^n} \Phi(M^\sharp(Tf)(x)) \, w(x) \, dx \leq c \int_{\mathbb{R}^n} \Phi(|f(x)|) \, Mw(x) \, dx. \]

Note that the weaker inequality with \( M_{0,\infty}^\sharp \) in place of \( M^\sharp \) on the left-hand side above holds for an even wider class of \( \Phi \) by (1.5) and the Fefferman-Stein maximal inequality. Now, these inequalities are of interest because they do not hold for an arbitrary weight \( w \) for all singular integral operators \( T \) with \( |Tf(x)| \) in place of \( M^\sharp(Tf)(x) \) on the left-hand side of (1.10). This observation follows from Theorem 1.1 in [40]: if \( T \) is a singular integral operator and \( 1 < p < \infty \), there exists a constant \( c \) such that for each weight \( w \),
\[ (1.11) \quad \int_{\mathbb{R}^n} |Tf(x)|^p \, w(x) \, dx \leq c \int_{\mathbb{R}^n} |f(x)|^p \, M^{[p] + 1}_w(x) \, dx. \]

Furthermore, the result is sharp since it does not hold for \( M^{[p]} \) in place of \( M^{[p] + 1} \) in (1.11).

And, as illustrated below in the case of singular integral operators of Dini type, or with kernels satisfying Hörmander-type conditions, and integral operators with homogeneous kernels, our approach applies in other instances as well.

We then take a closer look at the integral inequalities (1.2) and (1.11) which, being \( L^p \)-specific, are of a different nature. The question of determining weights \( (w, v) \) so that Calderón-Zygmund singular integral operators map \( L^p_v \) continuously into \( L^p_w \) was pioneered by Muckenhoupt and Wheeden [38] for the Hilbert transform and continues to attract considerable attention. For weights that satisfy some additional property, such as being radial, interesting results are proved and referenced, for instance, in [22]. Interestingly, Reguera and Scurry have shown that there is no a priori relationship between the Hilbert transform and the Hardy-Littlewood maximal function in the two-weight setting [44].

To deal with singular integral operators in the two-weight context we consider sharper local median decompositions and the weights naturally associated to them. The first scenario corresponds to the \( W_p \) classes of Fujii [19], where estimates with the flavor of extrapolation are obtained. And, a further refined decomposition, like the one used by Lerner in his proof of the \( A_2 \) conjecture [34], coupled with weights satisfying the Orlicz “bump” condition introduced by Pérez [40] and used by Lerner [32], gives the estimate for Calderón-Zygmund singular integrals, including those of Dini type, or with kernels satisfying a Hörmander-type condition, from \( L^p_v(\mathbb{R}^n) \) into \( L^p_w(\mathbb{R}^n) \) for \( 1 < p \leq q < \infty \) and a range of \( q \). The \( A_2 \) conjecture asks for the precise dependence of the norm of a Calderón-Zygmund operator from \( L^2_w(\mathbb{R}^n) \) into itself in terms of the Muckenhoupt \( A_2 \) norm of \( w \) and was solved by Hytönen [26] using results of Nazarov, Treil and Volberg [39], and Lacey, Petermichl and Reguera [30].
Finally, the local estimates are well suited to the generalized weighted Orlicz-Morrey spaces $M_{w}^{\Phi,\phi}$ and generalized weighted Orlicz-Campanato spaces $L_{w}^{\Phi,\phi}$, defined in Section 7. Indeed, if $T$ is a Calderón-Zygmund singular integral operator, from (1.6) it readily follows that for a Young function $\Phi, w \in A_{\infty}$, and every appropriate $\phi$,

$$
\|Tf\|_{L_{w}^{\Phi,\phi}} \leq c \|Mf\|_{M_{w}^{\Phi,\phi}}.
$$

(1.12)

Moreover, suppose that $S$ is a sublinear operator that satisfies

$$
\int_{\mathbb{R}^n} \Phi(|Sf(y)|) w(y) \, dy \leq c \int_{\mathbb{R}^n} \Phi(|f(y)|) w(y) \, dy
$$

and such that for any cube $Q$, if $x \in Q$ and $\text{supp}(f) \subset \mathbb{R}^n \setminus 2Q$, then

$$
|Sf(x)| \leq \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^n} \, dy.
$$

Then, if $u_{\Phi}$ denotes the upper index of $L_{w}^{\Phi}$, $w \in A_{p}$ where $p = 1/u_{\Phi}$, and for all $x \in \mathbb{R}^n$ and $l > 0$, $\phi(x,t)$ and $\psi(x,l)$ satisfy

$$
\psi(x,l) \int_{l}^{\infty} \frac{1}{\phi(x,t)} \frac{dt}{t} \leq c,
$$

we have that

$$
\|Sf\|_{M_{w}^{\Phi,\psi}} \leq c \|f\|_{M_{w}^{\Phi,\phi}}.
$$

This result applies to the Hardy-Littlewood maximal function, Calderón-Zygmund singular integrals, and other operators [22].

The paper is organized as follows. The first two sections contain the essential ingredients in what follows. The local median decomposition of an arbitrary measurable function is given in Section 2 and the local control of a weighted mean of a function by the weighted mean of its corresponding local maximal function is done in Section 3. In Section 4 we prove a local version of the estimate $M_{0,w}^{\Phi}(Tf)(x) \leq c Mf(x)$ for Calderón-Zygmund singular integral operators and recast similar estimates with $T$ replaced by singular integral operators with kernels satisfying Hörmander-type conditions or integral operators with homogeneous kernels, and $M$ by an appropriate maximal function $M_{T}$. In Section 5 we use these estimates to express the weighted local integral control of $Tf$ in terms of $M_{T}f$; the case $M_{T} = M_{r}$ is done in some detail. In Section 6 we use variants of the local median decomposition obtained in Section 2 to prove two-weight, $L_{v}^{p} - L_{w}^{q}$ estimates for singular integral operators for $1 < p \leq q < \infty$, for a range of $q$. And finally, in Section 7 we consider the weighted generalized Orlicz-Morrey and Orlicz-Campanato spaces.

Some closely related topics are not addressed here. Because we concentrate on integral inequalities, weak-type inequalities are not considered. Neither are commutator estimates, which can, for instance, be treated as in [2], nor homogeneous spaces, the foundation for which has been laid in [24, 52, 55] and [3]. And, for the various definitions or properties that the reader may find unfamiliar, several treatises in the area may be helpful, including [55, 57].

2. Local median decomposition

The decomposition of a measurable function presented here was first considered in terms of averages by Carleson [6] and Garnett and Jones [21], and suggested in terms of medians by Fujii [19]. It complements Lerner’s “local mean oscillation” decomposition [32,33], which corresponds to the case \( t = 1/2, s = 1/4 \) in Theorem 2.1. Although the bound below is larger than his, it holds for arbitrarily small values of \( s \), which are necessary for the applications of interest to us. Also, the proof relies on medians and is somewhat more geometric.

In what follows, we adopt the notations of [43] and [54]. In particular, all cubes have sides parallel to the axes. Also, for a cube \( Q \subset \mathbb{R}^n \) and \( 0 < t < 1 \), we say that
\[
m_f(t,Q) = \sup \{ M : |\{ y \in Q : f(y) < M \}| \leq t|Q| \}
\]
is the (maximal) median of \( f \) over \( Q \) with parameter \( t \). For a cube \( Q_0 \subset \mathbb{R}^n \) and \( 0 < t < 1/2 \), the local sharp maximal function restricted to \( Q_0 \) of a measurable function \( f \) at \( x \in Q_0 \) is
\[
M_{0,s,Q_0}^2 f(x) = \sup_{x \in Q, Q \subset Q_0} \inf_c \{ \alpha \geq 0 : |\{ y \in Q : |f(y) - c| > \alpha \}| < s|Q| \},
\]
and the local sharp maximal function of a measurable function \( f \) at \( x \in \mathbb{R}^n \) is
\[
M_{0,s}^2 f(x) = \sup_{x \in Q} \inf_c \{ \alpha \geq 0 : |\{ y \in Q : |f(y) - c| > \alpha \}| < s|Q| \}.
\]

Additionally, we consider the maximal function \( m_{t,Q_0}^{t,\Delta} \) defined as follows. Let \( D \) be the family of dyadic cubes in \( \mathbb{R}^n \). For a cube \( Q \subset \mathbb{R}^n \), let \( D(Q) \) denote the family of dyadic subcubes relative to \( Q \); that is to say, those formed by repeated dyadic subdivision of \( Q \) into \( 2^n \) congruent subcubes. Then
\[
m_{Q_0}^{t,\Delta}(x) = \sup_{x \in Q, Q \in D(Q_0)} |m_f(t, Q)|.
\]

A related nonsyndetic maximal function was introduced by A. P. Calderón in order to exploit cancellation to obtain estimates for singular integrals in terms of maximal functions [5].

Finally, \( \hat{Q} \) denotes the dyadic parent of a cube \( Q \).

**Theorem 2.1.** Let \( f \) be a measurable function on a fixed cube \( Q_0 \subset \mathbb{R}^n \), \( 0 < s < 1/2 \), and \( 1/2 \leq t < 1 - s \). Then there exists a (possibly empty) collection of subcubes \( \{Q_j^v\} \subset D(Q_0) \) and a family of collections of indices \( \{I_j^v\}_v \) such that

(i) for a.e. \( x \in Q_0 \),
\[
|f(x) - m_f(t, Q_0)| \leq 4M_{0,s,Q_0}^2 f(x) + \sum_{v=1}^{\infty} \sum_{j \in I_j^v} a_j^v 1_{Q_j^v}(x),
\]
where
\[
a_j^v \leq 10n \inf_{y \in Q_j^v} M_{0,s,Q_j^v}^2 f(y) + 2 \inf_{y \in Q_j^v} M_{0,s,Q_j^v}^2 f(y)
\]
\[
\leq (10n + 2) \inf_{y \in Q_j^v} M_{0,s,Q_j^v}^2 f(y); \tag{2.1}
\]

(ii) for fixed \( v \), the \( \{Q_j^v\} \) are nonoverlapping;

(iii) if \( \Omega^v = \bigcup_j Q_j^v \), then \( \Omega^{v+1} \subset \Omega^v \); and

(iv) for all \( j \), \( |\Omega_j^{v+1} \cap Q_j^v| \leq (s/(1-t)) |Q_j^v| \).
Proof. Let \( E^1 = \{ x \in Q_0 : |f(x) - m_f(t, Q_0)| > 2 \inf_{y \in Q_0} M^{t \Delta}_0, s, Q_0 f(y) \} \). If \(|E^1| = 0\), the decomposition halts – trivially, for a.e. \( x \in Q_0 \),

\[
|f(x) - m_f(t, Q_0)| \leq 2 \inf_{y \in Q_0} M^t_{0, s, Q_0} f(y).
\]

So suppose that \(|E^1| > 0\). Recall that by Lemma 4.1 in [43], for \( \eta > 0 \),

\[
|\{ x \in Q_0 : |f(x) - m_f(t, Q_0)| \geq 2 \inf_{y \in Q_0} M^t_{0, s, Q_0} f(y) + \eta \}| < s|Q_0|.
\]

Thus, picking \( \eta_k \to 0^+ \), by continuity from below it readily follows that

\[
\text{(2.2)} \quad |\{ x \in Q_0 : |f(x) - m_f(t, Q_0)| > 2 \inf_{y \in Q_0} M^t_{0, s, Q_0} f(y) \}| \leq s|Q_0|.
\]

Now let \( f^0 = (f - m_f(t, Q_0)) \mathbb{1}_{Q_0} \) and

\[
\Omega^1 = \{ x \in Q_0 : m^t_{Q_0} (f^0)(x) > 2 \inf_{y \in Q_0} M^t_{0, s, Q_0} f(y) \}.
\]

Then by Theorem 2.1 in [43], \( E^1 \subset \Omega^1 \) and \(|\Omega^1| > 0 \) as well. Write \( \Omega^1 = \bigcup_j Q_j^1 \) where the \( Q_j^1 \) are nonoverlapping maximal dyadic subcubes of \( Q_0 \) such that

\[
\text{(2.3)} \quad |m_f^0(t, Q_j^1)| > 2 \inf_{y \in Q_0} M^t_{0, s, Q_0} f(y), \quad \text{and} \quad |m_f^0(t, \hat{Q_j}^1)| \leq 2 \inf_{y \in Q_0} M^t_{0, s, Q_0} f(y).
\]

Since \( m_f^0(t, Q_j^1) = 0, Q_j^1 \neq Q_0 \) for any \( j \).

Now since \( t \geq 1/2 \), from (1.10) in [43] it follows that

\[
2 \inf_{y \in Q_0} M^t_{0, s, Q_0} f(y) < |m_f^0(t, Q_j^1)| \leq m_f^0(t, Q_j^1),
\]

and therefore by the definition of median

\[
\text{(2.4)} \quad |\{ x \in Q_j^1 : |f^0(x)| > 2 \inf_{y \in Q_0} M^t_{0, s, Q_0} \}| \geq (1 - t)|Q_j^1|.
\]

When (2.4) is summed over \( j \), we have by (2.2) that

\[
(1 - t) \sum_j |Q_j^1| \leq \sum_j |\{ x \in Q_j^1 : |f^0(x)| > 2 \inf_{y \in Q_0} M^t_{0, s, Q_0} f(y) \}|
\]

\[
\leq |\{ x \in Q_0 : |f^0(x)| > 2 \inf_{y \in Q_0} M^t_{0, s, Q_0} f(y) \}| \leq s|Q_0|,
\]

so that

\[
\text{(2.5)} \quad \sum_j |Q_j^1| \leq \frac{s}{1 - t}|Q_0|,
\]

where by the choice of \( s \) and \( t \), \( s/(1 - t) < 1 \).

Let \( a_j^1 = m_f^0(t, Q_j^1) \). By Lemma 4.3 in [43] we see that

\[
\text{(2.6)} \quad |m_f^0(t, Q_j^1) - m_f^0(t, \hat{Q_j}^1)| \leq 10n \inf_{y \in Q_j^1} M^t_{0, s, \hat{Q_j}^1} f(y),
\]

and therefore by (2.3) and (2.6)

\[
|a_j^1| \leq |m_f^0(t, Q_j^1) - m_f^0(t, \hat{Q_j}^1)| + |m_f^0(t, \hat{Q_j}^1)|
\]

\[
\leq 10n \inf_{y \in Q_j^1} M^t_{0, s, \hat{Q_j}^1} f(y) + 2 \inf_{y \in Q_0} M^t_{0, s, Q_0} f(y).
\]
The first iteration of the local median oscillation decomposition of \( f \) when \( |E^1| > 0 \) is then as follows: for a.e. \( x \in Q_0 \), with \( g^1 = f^0 1_{Q_0 \setminus \Omega^1} \),

\[
f^0(x) = g^1(x) + \sum_j a_j^1 1_{Q_j^1}(x) + \sum_j (f^0(x) - m_{f^0}(t, Q_j^1)) 1_{Q_j^1}(x) .
\]

Note that \( g^1 \) has support off \( \Omega^1 \), and clearly for a.e. \( x \in Q_0 \),

\[
|g^1(x)| \leq 2 \inf_{y \in Q_0} M^2_{0,s, Q_0} f(y) .
\]

Now focus on the second sum. Since \( f^0(x) - m_{f^0}(t, Q) = f(x) - m_f(t, Q) \) for all cubes \( Q \) and functions \( f \) supported in \( Q \), this sum equals

\[
\sum_j (f(x) - m_f(t, Q_j^1)) 1_{Q_j^1}(x) .
\]

The idea is to repeat the above argument for each of the functions \( f_j = (f - m_f(t, Q_j^1)) 1_{Q_j^1} \), and so on.

We now describe the iteration. Assuming that \( \{Q_j^{k-1}\} \) are the dyadic cubes corresponding to the \((k - 1)\)st generation of subcubes of \( Q_0 \) obtained as above, let

\[
f_j^{k-1} = (f - m_f(t, Q_j^{k-1})) 1_{Q_j^{k-1}}
\]

and

\[
E_j^k = \{ x \in Q_j^{k-1} : f_j^{k-1}(x) > 2 \inf_{y \in Q_j^{k-1}} M^2_{0,s, Q_j^{k-1}} f(y) \} .
\]

If \( |E_j^k| = 0 \), we write \( s_j^k = f_j^{k-1} \) which satisfies

\[
|s_j^k(x)| \leq 2 \inf_{y \in Q_j^{k-1}} M^2_{0,s, Q_j^{k-1}} f(y) .
\]

(2.8) for a.e. \( x \in Q_j^{k-1} \). These are the “s” functions since the decomposition “stops” at \( Q_j^{k-1} \); clearly \( s_j^k \) has its support on \( Q_j^{k-1} \), which contains no further subcubes of the decomposition.

If \( |E_j^k| > 0 \), we define

\[
\Omega_j^k = \{ x \in Q_j^{k-1} : m_{f^{k-1}}(x) (f_j^{k-1})(x) > 2 \inf_{y \in Q_j^{k-1}} M^2_{0,s, Q_j^{k-1}} f(y) \} \supset E_j^k .
\]

Note that the \( Q_j^{k-1} \), and thus the \( \Omega_j^k \), are nonoverlapping. Then \( |\Omega_j^k| > 0 \) as well and

\[
\Omega_j^k = \bigcup_i Q_i^k ,
\]

where the \( Q_i^k \)'s are nonoverlapping maximal dyadic subcubes of \( Q_j^{k-1} \) such that

\[
|m_{f^{k-1}}(t, Q_i^k)| > 2 \inf_{y \in Q_j^{k-1}} M^2_{0,s, Q_j^{k-1}} f(y) ,
\]

and

(2.9) \[
|m_{f^{k-1}}(t, Q_i^k)| \leq 2 \inf_{y \in Q_j^{k-1}} M^2_{0,s, Q_j^{k-1}} f(y) .
\]

Then define

\[
\Omega^k = \bigcup_i \Omega_i^k .
\]

Let

\[
a_{i,j}^{k-1} = m_{f^{k-1}}(t, Q_i^k) ,
\]

\[
a_{j,i}^{k-1} = m_{f^{k-1}}(t, Q_j^k) .
\]
and note that by (2.6) and (2.9)

\[
|\alpha^{k,j}_i| \leq |m_{f^j_{k-1}}(t, Q^k_i) - m_{f^j_{k-1}}(t, \hat{Q}^k_i)| + |m_{f^j_{k-1}}(t, \hat{Q}^k_i)|
\]

(2.10) \[ \leq 10n \inf_{y \in Q^k_i} M^2_{0,s, Q^k_i} f(y) + 2 \inf_{y \in Q^k_{i-1}} M^2_{0,s, Q^k_{i-1}} f(y). \]

We then have

\[
f^{k-1}_{j}(x) = g^{k}_{j}(x) + \sum_i \alpha^{k,j}_i \mathbb{1}_{Q^k_i}(x) + \sum_i (f(x) - m_{f(t, Q^k_i)}) \mathbb{1}_{Q^k_i}(x)
\]

for a.e. \( x \in Q^k_{j-1} \), where \( g^{k}_{j} = f^{k-1}_{j} \mathbb{1}_{Q^k_{j-1} \setminus \Omega^k_j} \) is readily seen to satisfy

(2.11) \[ |g^{k}_{j}(x)| \leq 2 \inf_{y \in Q^k_{j-1}} M^2_{0,s, Q^k_{j-1}} f(y) \]

for a.e. \( x \in Q^k_{j-1} \). These are the "\( g \)" functions since the decomposition "goes on" or continues, into \( Q^k_{j-1} \); \( g^{k}_{j} \) has support on \( Q^k_{j-1} \) away from \( \Omega^k_j \), which are the next subcubes in the decomposition.

We separate the \( Q^k_{j-1} \) into two families. One family, indexed by \( I^k_1 \), contains those cubes where the decomposition stops, and the other, indexed by \( I^k_2 \), where it continues. Specifically, let

\[
I^k_1 = \{ j : \Omega^k \cap Q^k_{j-1} = \emptyset \}, \quad I^k_2 = \{ j : \Omega^k \cap Q^k_{j-1} \neq \emptyset \}.
\]

Now we group the \( Q^k_i \) based on which \( Q^k_{j-1} \) contains them: if \( j \in I^k_2 \), let

\[
J^k_j = \{ i : Q^k_i \subset Q^k_{j-1} \}.
\]

These definitions give that

\[
\Omega^k_j = \bigcup_{i \in J^k_j} Q^k_i.
\]

Note that, as in (2.5),

(2.12) \[ |\Omega^k_j \cap Q^k_{j-1}| = \sum_{i \in J^k_j} |Q^k_i| \leq \left( \frac{s}{1-t} \right) |Q^k_{j-1}| \]

so that

\[
|\Omega^k| = \sum_j |\Omega^k_j \cap Q^k_{j-1}| \leq \left( \frac{s}{1-t} \right) \sum_j |Q^k_{j-1}|
\]

(2.13) \[ = \left( \frac{s}{1-t} \right) |\Omega^{k-1}| \leq \left( \frac{s}{1-t} \right)^k |Q_0|.
\]

In fact, we claim that for all \( j \) and \( 1 \leq v < k \),

(2.14) \[ |\Omega^k \cap Q^v_j| \leq \left( \frac{s}{1-t} \right)^{k-v} |Q^v_j|,
\]

an estimate that is useful in what follows.
To see this, for a given \( k \), let \( 1 \leq v \leq k - 1 \); if \( v = k - 1 \) the conclusion is (2.12). Next, if \( v = k - 2 \) note that

\[
|\Omega^k \cap Q_j^{k-2}| = \sum_{Q_i^k \subset Q_j^{k-2}} |Q_i^k| = \sum_{Q_i^{k-1} \subset Q_j^{k-2}} \sum_{Q_i^k \subset Q_j^{k-1}} |Q_i^k| = \sum_{Q_i^{k-1} \subset Q_j^{k-2}} |\Omega^k_i \cap Q_j^{k-1}| \leq \left( \frac{s}{1-t} \right) \sum_{Q_i^{k-1} \subset Q_j^{k-2}} |Q_i^{k-1}| = \left( \frac{s}{1-t} \right) |\Omega_j^{k-1} \cap Q_j^{k-2}| \leq \left( \frac{s}{1-t} \right)^2 |Q_j^{k-2}|,
\]

where the inequalities follow by (2.12). Continuing recursively, we have (2.14).

The \( k \)-th iteration of the local median oscillation decomposition of the function \( f \) is as follows: for a.e. \( x \in Q_0 \),

\[
f(x) - m_f(t, Q_0) = \sum_{v=1}^{k} \left( \sum_{j \in I_1^v} s_j^v + \sum_{j \in I_2^v} g_j^v \right) + \sum_{v=1}^{k} \sum_{j \in I_2^v} \sum_{i \in J_j^v} \alpha_i^{v,j} 1_{Q_i^v}(x) + \psi^k(x),
\]

where

\[
\psi^k = \sum_{j \in I_2^k} \sum_{i \in J_j^k} (f - m_f(t, Q_i^k)) 1_{Q_i^k}.
\]

Since \( \psi^k \) is supported in \( \Omega^k \), by (2.13) it readily follows that \( \psi^k \to 0 \) a.e. in \( Q_0 \) as \( k \to \infty \), and therefore

\[
f(x) - m_f(t, Q_0) = \sum_{v=1}^{\infty} \left( \sum_{j \in I_1^v} s_j^v + \sum_{j \in I_2^v} g_j^v \right) + \sum_{v=1}^{\infty} \sum_{j \in I_2^v} \sum_{i \in J_j^v} \alpha_i^{v,j} 1_{Q_i^v}(x)
\]

\[
= S_1(x) + S_2(x),
\]

say.

In order to bound \( |f(x) - m_f(t, Q_0)| \), consider first \( S_1 \). Of course, for all \( v \) and \( j \) the \( s_j^v \) have nonoverlapping support. This is also true for the \( g_j^v \). Furthermore, the support of any \( g_j^v \) is nonoverlapping with that of any \( s_j^v \). So for every \( v, j, \) and a.e. \( x \in Q_0 \), by (2.8) and (2.11) it readily follows that

\[
\left| \sum_{v=1}^{\infty} \left( \sum_{j \in I_1^v} s_j^v + \sum_{j \in I_2^v} g_j^v \right) \right| \leq \max \left\{ \sup_{j \in I_1^v} \| f_{j+1}^{v-1} \|_{L^{\infty}}, \sup_{j \in I_2^v} \| f_{j+1}^{v-1} 1_{Q_j^{v-1} \setminus Q_j^v} \|_{L^{\infty}} \right\}
\]

\[
\leq \max \left\{ \sup_{j \in I_1^v} \left( 2 \inf_{y \in Q_j^{v-1}} M_0^{M_0, Q_j^{v-1}} f(y) \right), \sup_{j \in I_2^v} \left( 2 \inf_{y \in Q_j^{v-1}} M_0^{M_0, Q_j^{v-1}} f(y) \right) \right\}
\]

(2.15)

\[
\leq 2 M_0^{M_0, Q_0} f(x).
\]
We consider $S_2$ next. The summand for $v = 1$ is distinguished, so we deal with it separately. By (2.7) above,
\[
\left| \sum_j a^v_j \mathbb{1}_{Q_j^v} (x) \right| \leq \sum_j |a^v_j| \mathbb{1}_{Q_j^v} (x)
\]
\[
\leq \sum_j \left( 10n \inf_{y \in Q_j^v} M_{0, s, Q_j^v}^2 f(y) + 2 \inf_{y \in Q_0} M_{0, s, Q_0}^2 f(y) \right) \mathbb{1}_{Q_j^v} (x)
\]
(2.16)
\[
\leq \sum_j \left( 10n \inf_{y \in Q_j^v} M_{0, s, Q_j^v}^2 f(y) \right) \mathbb{1}_{Q_j^v} (x) + 2 \inf_{y \in Q_0} M_{0, s, Q_0}^2 f(y).
\]

As for the other terms of the sum, by (2.10) we have
\[
\left| \sum_{v=2}^{\infty} \sum_{j \in I_j^v} \sum_{i \in J_j^v} a^v_{i,j} \mathbb{1}_{Q_i^v} (x) \right| \leq \sum_{v=2}^{\infty} \sum_{j \in I_j^v} \sum_{i \in J_j^v} |a^v_{i,j}| \mathbb{1}_{Q_i^v} (x)
\]
\[
\leq \sum_{v=2}^{\infty} \sum_{j \in I_j^v} \sum_{i \in J_j^v} \left( 10n \inf_{y \in Q_i^v} M_{0, s, Q_i^v}^2 f(y) + 2 \inf_{y \in Q_j^v} M_{0, s, Q_j^v}^2 f(y) \right) \mathbb{1}_{Q_i^v} (x)
\]
\[
\leq \sum_{v=2}^{\infty} \sum_{j \in I_j^v} \sum_{i \in J_j^v} \left( 10n \inf_{y \in Q_i^v} M_{0, s, Q_i^v}^2 f(y) \right) \mathbb{1}_{Q_i^v} (x)
\]
(2.17)
\[
+ \sum_{v=2}^{\infty} \sum_{j \in I_j^v} \left( 2 \inf_{y \in Q_j^v} M_{0, s, Q_j^v}^2 f(y) \right) \mathbb{1}_{Q_i^v} (x).
\]

We combine (2.16) and (2.17) and note that since the sum is infinite and the families $I_j^v$ are nested,
\[
\left| \sum_{v=1}^{\infty} \sum_{j \in I_j^v} \sum_{i \in J_j^v} a^v_{i,j} \mathbb{1}_{Q_i^v} (x) \right|
\]
\[
\leq \sum_{v=1}^{\infty} \sum_{j \in I_j^v} \sum_{i \in J_j^v} \left( 10n \inf_{y \in Q_i^v} M_{0, s, Q_i^v}^2 f(y) \right) \mathbb{1}_{Q_i^v} (x)
\]
\[
+ \sum_{v=1}^{\infty} \sum_{j \in I_j^v} \left( 2 \inf_{y \in Q_j^v} M_{0, s, Q_j^v}^2 f(y) \right) \mathbb{1}_{Q_j^v} (x)
\]
(2.18)
\[
+ 2 \inf_{y \in Q_0} M_{0, s, Q_0}^2 f(y).
\]

Combining (2.15) and (2.18), finally we get that for a.e. $x \in Q_0$,
\[
|f(x) - m_f(t, Q_0)| \leq 4 M_{0, s, Q_0}^2 f(x)
\]
\[
+ \sum_{v=1}^{\infty} \sum_{j \in I_j^v} \left( 10n \inf_{y \in Q_j^v} M_{0, s, Q_j^v}^2 f(y) + 2 \inf_{y \in Q_j^v} M_{0, s, Q_j^v}^2 f(y) \right) \mathbb{1}_{Q_j^v} (x),
\]
and we have finished. \qed
3. Weighted local mean estimates for local maximal functions

In this section we consider the control of a weighted local mean of a function by the weighted local mean of its local maximal function. In \( \mathbb{R}^n \), for the sharp maximal function, this result was first established in the unweighted case by Fefferman and Stein [17], and in the weighted case by several authors, including Fujii [18]. For the local sharp maximal function and an \( A_\infty \) weight \( w \), it follows from the fact that there exists a constant \( 0 < s_1 < 1 \) with the following property: given \( 0 < s \leq s_1 \), there exist constants \( c, c_1 \) such that for all cubes \( Q \),

\[
w(\{ x \in Q : |f(x) - m_f(s, Q)| > \lambda, M_{0,s}^q f(x) < \alpha \}) \leq c e^{-c_1 \lambda^{\alpha/\alpha} w(Q)}
\]

for all \( \lambda, \alpha > 0 \). This is proved in Chapter III of [55].

We are interested in the weighted local version of these results involving weights that are not necessarily \( A_\infty \).

**Definition 3.1.** We say that the weights \((w, v)\) satisfy condition \( F \) if there exist positive constants \( c_1, \alpha, \beta \) with \( 0 < \alpha < 1 \), such that for any cube \( Q \) and measurable subset \( E \) of \( Q \) with \( |E| \leq \alpha |Q| \),

\[
(3.1) \quad \int_E w(x) \, dx \leq c_1 \left( \frac{|E|}{|Q|} \right)^\beta \int_{Q \setminus E} v(x) \, dx.
\]

Fujii observed that if \((w, v)\) satisfy condition \( F \), then \( w(x) \leq c v(x) \) a.e., and that for \( w = v \), (3.1) is equivalent to the \( A_\infty \) condition for \( w \). He also gave a simple example of a pair \((w, v)\) that satisfy condition \( F \) so that neither of them is an \( A_\infty \) weight and no \( A_\infty \) weight can be inserted between them: let \( w(x) = 0 \) if \( 0 < x < 1 \) and \( w(x) = 1 \) otherwise, and \( v(x) = 0 \) if \( 1/3 < x < 2/3 \) and \( v(x) = 1 \) otherwise [18].

Along similar lines, if \( w \) is in weak \( A_\infty \), i.e., there exist positive numbers \( c, \beta \) such that for any cube \( Q \) and measurable subset \( E \) of \( Q \),

\[
w(E) \leq c \left( \frac{|E|}{|Q|} \right)^{\beta} w(2Q),
\]

a simple computation gives that \((w, Mw)\) satisfy condition \( F \). In fact, by an observation of Sawyer [50] this also follows from the next example.

We say that a weight \( w \) is in the Muckenhoupt class \( C_p \) if there exist positive constants \( \beta, c \) such that

\[
\int_E w(x) \, dx \leq c \left( \frac{|E|}{|Q|} \right)^{\beta} \int_{\mathbb{R}^n} M(\mathbb{1}_E)(x)^p \, w(x) \, dx
\]

whenever \( E \) is a subset of a cube \( Q \subset \mathbb{R}^n \); clearly \( A_\infty \subset C_p \), \( 1 < p < \infty \), but \( C_p \) contains weights not in \( A_\infty \), as the example \( w(x) = \mathbb{1}_{(0, \infty)}(x) v(x) \) with \( v \in A_\infty \) of the line shows. Now, \( C_p \) is necessary for the integral inequality (1.1) to hold with \( \Phi(t) = t^p \), \( 1 < p < \infty \), and \( C_q \) with \( q > p \) is sufficient for (1.1) to hold ([37], [50], and [55]).

Note that for a fixed \( 0 < \alpha < 1 \), we have \( M(\mathbb{1}_Q)(x) \leq c M(\mathbb{1}_{Q \setminus E})(x) \), where \( E \subset Q \) is such that \( |E| \leq \alpha |Q| \) and \( c \) depends on \( \alpha \) but is independent of \( E \) and \( Q \). Then by the Fefferman-Stein maximal inequality,

\[
\int_{\mathbb{R}^n} M(\mathbb{1}_Q)(x)^p \, w(x) \, dx \leq c \int_{\mathbb{R}^n} M(\mathbb{1}_{Q \setminus E})(x)^p \, w(x) \, dx \leq c \int_{Q \setminus E} Mw(x) \, dx.
\]
Thus, if $w$ satisfies the condition $C_p$ for some $p$, $1 < p < \infty$, 
\[
\int_E w(x) \, dx \leq c \left( \frac{|E|}{|Q|} \right)^{\beta} \int_{Q \setminus E} Mw(x) \, dx
\]
and $(w, Mw)$ satisfy condition $F$.

A word of caution: at the end of this section we show indirectly that for some weight $w$, $(w, Mw)$ do not satisfy condition $F$. Along these lines, for a Young function $A$ let 
\[
\|f\|_{L^A(Q)} = \inf \left\{ \lambda > 0 : \frac{1}{|Q|} \int_Q A\left( \frac{|f(y)|}{\lambda} \right) dy \leq 1 \right\}
\]
and 
\[
M_A f(x) = \sup_{x \in Q} \|f\|_{L^A(Q)}.
\]
It then readily follows that for a weight $w$, $M_A w \in A_1$ if 
\[
\int_0^t \frac{A(s)}{s^2} \, ds \leq c \frac{A(t)}{t}.
\]
Hence, for any weight $w$ and $1 < r < \infty$, $(w, M_w, w)$ satisfy condition $F$.

On the other hand, for an integer $k = 0, 1, \ldots$, let $A_k(t) = t \log^k(1 + t)$. Then, if $M^{k+1}$ denotes the $k + 1$ composition of the Hardy-Littlewood maximal function operator with itself, $M^{k+1}$ is pointwise comparable to the maximal operator $M_{A_k}$, and by the comments after Theorem 5.4, for every $k$ there exists a weight $w$ such that $(w, M_{A_k} w)$ do not satisfy condition $F$. In particular, for such a weight $w$, $M_{A_k} w \notin A_\infty$.

Two remarks are in order before we proceed to prove the main result in this section. First, the choice of the parameters $s$ and $t$ in (3.2) below remains fixed throughout the paper unless otherwise noted, and second, note that in the proof below the constant is linear with respect to the constant $c_1$ of the weights that satisfy condition $F$, and in particular, linear in the $A_\infty$ norm of $w$.

**Theorem 3.1.** Let $\Phi$ satisfy condition $C$ with doubling constant $c_0$, $(w, v)$ weights on $\mathbb{R}^n$ satisfying condition $F$ with constants $\beta, c_1$, and pick $s, t$ such that $0 < s \leq 1/2$, $1/2 < t < 1 - s$, and 
\[
c_0 \left( \frac{s}{1 - t} \right)^\beta < 1.
\]

Then for any measurable function $f$ and a cube $Q_0 \subset \mathbb{R}^n$, with a constant $c$ independent of $\Phi, Q_0$, and $f$, 
\[
\int_{Q_0} \Phi(|f(x) - m_f(t, Q_0)|) w(x) \, dx \leq c \int_{Q_0} \Phi(M_{0, s, Q_0}^t f(x)) v(x) \, dx.
\]

Furthermore, if $f$ is such that $\lim_{Q \to \mathbb{R}^n} m_f(t, Q) = 0$, we also have 
\[
\int_{\mathbb{R}^n} \Phi(|f(x)|) w(x) \, dx \leq c \int_{\mathbb{R}^n} \Phi(M_{0, s}^t f(x)) v(x) \, dx.
\]

**Proof.** Fix a cube $Q_0$. Then by Theorem 2.1, for a.e. $x \in Q_0$, 
\[
|f(x) - m_f(t, Q_0)| \leq 4M_{0, s, Q_0}^t f(x) + \sum_{s=1}^{\infty} \sum_{j \in I_s^t} a_j^{1/4} Q_j^c(x),
\]
where by (2.1),
\[ a_j^v \leq (10n + 2) \inf_{y \in Q_j^k} M_x^{u,L} f(y) \leq (10n + 2) \inf_{y \in Q_j^k} M_{0,s,Q_j^k} f(y). \]
Then
\[
\Phi(|f(x) - m_f(t,Q_0)|) \leq \Phi \left( 4M_{0,s,Q_0} f(x) + \sum_{v=1}^{\infty} \sum_{j \in I_2^v} a_j^v 1_{Q_j^v}(x) \right)
\leq c_0^3 \Phi(M_{0,s,Q_0} f(x)) + c_0 \Phi \left( \sum_{v=1}^{\infty} \sum_{j \in I_2^v} a_j^v 1_{Q_j^v}(x) \right),
\]
and therefore,
\[
\int_{Q_0} \Phi(|f(x) - m_f(t,Q_0)|) w(x) \, dx \leq c \int_{Q_0} \Phi(M_{0,s,Q_0} f(x)) w(x) \, dx 
+ c \int_{Q_0} \Phi \left( \sum_{v=1}^{\infty} \sum_{j \in I_2^v} a_j^v 1_{Q_j^v}(x) \right) w(x) \, dx
= c I + c J,
\]
say. Now, since \( w(x) \leq c_3 v(x) \) a.e., \( I \) is of the right order.

As for \( J \), since \( \Phi(0) = 0 \) and the \( \Omega^k \) are nested, the domain of integration extends to \( \Omega^k = \bigcup_{k=1}^{\infty} (\Omega^k \setminus \Omega^{k+1}) \). Then we may write \( J = \sum_{k=1}^{\infty} J_k \) where
\[
J_1 = \int_{\Omega^2 \setminus \Omega^1} \Phi \left( \sum_{j \in I_2^1} a_j^v 1_{Q_j^v}(x) \right) w(x) \, dx
\]
and for \( k \geq 2 \), since only cubes of up to the \( k \)th generation enter in \( \Omega^k \setminus \Omega^{k+1} \),
\[
J_k = \int_{\Omega^k \setminus \Omega^{k+1}} \Phi \left( \sum_{v=1}^{k} \sum_{j \in I_2^v} a_j^v 1_{Q_j^v}(x) \right) w(x) \, dx.
\]

Focusing on the \( J_k \) for \( k \geq 2 \), the integrand is bounded by
\[
\Phi \left( \sum_{v=1}^{k} \sum_{j \in I_2^v} a_j^v 1_{Q_j^v}(x) \right) \leq \sum_{v=1}^{k} c_0^{k-v+1} \Phi \left( \sum_{j \in I_2^v} a_j^v 1_{Q_j^v}(x) \right)
\leq c_0 \Phi \left( \sum_{j \in I_2^k} a_j^k 1_{Q_j^k}(x) \right) + \sum_{v=1}^{k-1} c_0^{k-v+1} \Phi \left( \sum_{j \in I_2^v} a_j^v 1_{Q_j^v}(x) \right),
\]
and therefore \( J_k \) does not exceed
\[
c_0 \left( \int_{\Omega^k \setminus \Omega^{k+1}} \Phi \left( \sum_{j \in I_2^k} a_j^k 1_{Q_j^k}(x) \right) w(x) \, dx \right)
+ \sum_{v=1}^{k-1} c_0^{k-v} \int_{\Omega^k} \Phi \left( \sum_{j \in I_2^v} a_j^v 1_{Q_j^v}(x) \right) w(x) \, dx
= c_0 (J_k^1 + J_k^2),
\]
say.
Note that the $J_k^1$ and $J_k^1$ are essentially of the same form, and with $c \leq c_0 \log(10n+2)$ their total contribution is

$$J_1 + \sum_{k=2}^{\infty} J_k^1 \leq c_0 \sum_{k=1}^{\infty} \int_{\Omega^k \setminus \Omega^{k+1}} \Phi \left( \sum_{i \in I_2^k} a_i^k \mathbb{1}_{Q_i^k}(x) \right) w(x) \, dx$$

$$\leq c_0 \sum_{k=1}^{\infty} \sum_{j \in I_2^k} \int_{Q_j^k \setminus \Omega^{k+1}} \Phi(a_j^k) w(x) \, dx$$

$$\leq c_0 c \sum_{k=1}^{\infty} \sum_{j \in I_2^k} \int_{Q_j^k \setminus \Omega^{k+1}} \Phi(M_{0,s,Q_0}^2 f(x)) w(x) \, dx$$

$$\leq c_0 c \int_{Q_0} \Phi(M_{0,s,Q_0}^2 f(x)) w(x) \, dx.$$

As for the $J_k^2$, we claim that each $J_k^2$ satisfies, for $1 \leq v \leq k - 1$,

$$\int_{\Omega^k} \Phi \left( \sum_{j \in I_2^k} a_j^v \mathbb{1}_{Q_j^v}(x) \right) w(x) \, dx \leq c_1 \left( \frac{s}{1 - t} \right)^{\beta(k-v)} \sum_{j \in I_2^k} \int_{Q_j^v \setminus \Omega^k} \Phi(a_j^v) v(x) \, dx.$$

Indeed, since $\{Q_j^v\}_j$ are pairwise disjoint and $\Phi(0) = 0$, by condition $F$ we have

$$\int_{\Omega^k} \Phi \left( \sum_{j \in I_2^k} a_j^v \mathbb{1}_{Q_j^v}(x) \right) w(x) \, dx = \sum_{j \in I_2^k} \Phi(a_j^v) \int_{\Omega^k} \mathbb{1}_{Q_j^v}(x) w(x) \, dx$$

$$= \sum_{j \in I_2^k} \Phi(a_j^v) \int_{\Omega^k \cap Q_j^v} w(x) \, dx$$

$$\leq c_1 \sum_{j \in I_2^k} \Phi(a_j^v) \left( \frac{\Omega^k \cap Q_j^v}{|Q_j^v|} \right)^{\beta} \int_{Q_j^v \setminus \Omega^k} v(x) \, dx$$

$$\leq c_1 \left( \frac{s}{1 - t} \right)^{\beta(k-v)} \sum_{j \in I_2^k} \int_{Q_j^v \setminus \Omega^k} \Phi(a_j^v) v(x) \, dx,$$

where the last inequality follows from (2.14) in Theorem 2.1.

Therefore, for $k \geq 2$, with

$$\alpha = c_0 \left( \frac{s}{1 - t} \right)^{\beta} < 1,$$

we have

$$J_k^2 \leq c_1 \sum_{v=1}^{k-1} \alpha^{k-v} \sum_{j \in I_2^v} \int_{Q_j^v \setminus \Omega^k} \Phi(a_j^v) v(x) \, dx,$$

so that with $c \leq c_0 \log(10n+2)$,

$$\sum_{j \in I_2^k} \int_{Q_j^v \setminus \Omega^k} \Phi(a_j^v) v(x) \, dx \leq c \sum_{j \in I_2^v} \int_{Q_j^v \setminus \Omega^k} \Phi(M_{0,s,Q_0}^2 f(x)) v(x) \, dx$$

$$\leq c \int_{\Omega^v \setminus \Omega^k} \Phi(M_{0,s,Q_0}^2 f(x)) v(x) \, dx.$$
It only remains to bound \( \sum_{k=2}^{\infty} J_k^2 \). By (3.5) we have
\[
\sum_{k=2}^{\infty} J_k^2 \leq c \sum_{k=2}^{\infty} k^{-v} \int_{\Omega^v \setminus Q_k} \Phi(M_{0,s,Q_k}^2 f(x)) v(x) \, dx
\]
\[
= c \sum_{k=2}^{\infty} \sum_{v=1}^{\infty} k^{-v} \int_{\Omega^v \setminus Q_k} \Phi(M_{0,s,Q_k}^2 f(x)) v(x) \, dx
\]
\[
= c \sum_{v=1}^{\infty} \sum_{k=1}^{\infty} \alpha^k \int_{\Omega^v \setminus Q_k} \Phi(M_{0,s,Q_k}^2 f(x)) v(x) \, dx
\]
\[
= c \sum_{k=1}^{\infty} \alpha^k \int_{Q_0} \Phi(M_{0,s,Q_0}^2 f(x)) v(x) \, dx.
\]
(3.7)

Now, since the \( \Omega^v \) are nested and contained in \( Q_0 \), and for fixed \( k \), a set \( \Omega^v \setminus \Omega^{v+k} \) overlaps at most \( k \) of the other sets \( \{ \Omega^v \setminus \Omega^{v+k} \}_{v=1}^{\infty} \), we have
\[
\sum_{v=1}^{\infty} \sum_{k=1}^{\infty} \alpha^k \int_{\Omega^v \setminus \Omega^{v+k}} \chi_{\Omega^v \setminus \Omega^{v+k}}(x) \Phi(M_{0,s,Q_0}^2 f(x)) v(x) \, dx
\]
\[
\leq c \int_{Q_0} \Phi(M_{0,s,Q_0}^2 f(x)) v(x) \, dx.
\]

We thus have
\[
\int_{Q_0} \Phi(|f(x) - m_f(t, Q_0)|) w(x) \, dx \leq c \int_{Q_0} \Phi(M_{0,s,Q_0}^2 f(x)) v(x) \, dx.
\]

Furthermore, if \( \lim_{Q_0 \to \mathbb{R}^n} m_f(t, Q_0) = 0 \), by Fatou’s lemma
\[
\int_{\mathbb{R}^n} \Phi(|f(x)|) w(x) \, dx \leq c \int_{\mathbb{R}^n} \Phi(M_{0,s}^2 f(x)) v(x) \, dx.
\]

This completes the proof. \( \square \)

A couple of comments. First, observe that if the right-hand side of (3.4) above is finite, as in Chapter III of [55], \( \lim_{Q_0 \to \mathbb{R}^n} m_f(t, Q_0) = m_f \) exists along a sequence of \( Q_0 \to \mathbb{R}^n \), and the conclusion then reads
\[
\int_{\mathbb{R}^n} \Phi(|f(x) - m_f|) w(x) \, dx \leq c \int_{\mathbb{R}^n} \Phi(M_{0,s}^2 f(x)) v(x) \, dx.
\]

As Lerner observed, \( m_f = 0 \) if \( f^*(+\infty) = 0 \), where \( f^* \) denotes the nonincreasing rearrangement of \( f \), which in turn holds if and only if \( \{|x \in \mathbb{R}^n : |f(x)| > \alpha| \} < \infty \) for all \( \alpha > 0 \) [31]. In particular, this holds if the support of \( f \) has finite measure or if \( f \) is in weak-\( L^p(\mathbb{R}^n) \) for some \( 0 < p < \infty \).

Second, (3.5), (3.6) and (3.7) hold if (3.1) above is replaced by
\[
\int_E w(x) \, dx \leq c_1 \psi \left( \frac{|E|}{Q} \right) \int_{Q \setminus E} v(x) \, dx,
\]
where \( \sum_{k=1}^{\infty} k c_1^k \psi(\alpha^k) < \infty \). Thus, the class of weights that satisfy condition \( F \) could be extended to include these general \( \psi \) as well.

Our next result, essentially due to Lerner, holds for concave \( \Phi \), including \( \Phi(t) = t \), with \( v = Mw \) on the right-hand side of (3.3) [32,33].
Theorem 3.2. Let $\Phi$ be a concave function with $\Phi(0) = 0$, $f$ a measurable function on $\mathbb{R}^n$, and $0 < s < 1/2$ and $1/2 \leq t < 1 - s$. Then for any weight $w$ and cube $Q_0 \subset \mathbb{R}^n$,
\[
\int_{Q_0} \Phi(|f(x) - m_f(t, Q_0)|) w(x) \, dx \leq c \int_{Q_0} \Phi(M_{0,s}^2 f(x)) Mw(x) \, dx.
\]
Furthermore, if $f$ is such that $m_f(t, Q_0) \to 0$ as $Q_0 \to \mathbb{R}^n$, then
\[
(3.8) \quad \int_{\mathbb{R}^n} \Phi(|f(x)|) w(x) \, dx \leq c \int_{\mathbb{R}^n} \Phi(M_{0,s}^2 f(x)) Mw(x) \, dx.
\]

Proof. By Theorem 2.1 and the concavity of $\Phi$ we have
\[
\int_{Q_0} \Phi(|f(x)| - m_f(t, Q_0)|) w(x) \, dx \leq c \int_{Q_0} \Phi(M_{0,s}^2 f(x)) w(x) \, dx + c \sum_{v=1}^{\infty} \sum_{j \in I^2_v} \Phi(a_j^v) \int_{Q_j^v} w(x) \, dx,
\]
where
\[
a_j^v \leq (10n + 2) \inf_{y \in Q_j^v} M_{0,s}^2 f(y).
\]

Now, since $w(x) \leq Mw(x)$, the integrals term is of the right order. Next, by construction, the $Q_j^v$ are nonoverlapping over fixed $v$, but since each $Q_j^v$ is a subcube of some $Q_i^{v-1}$, they are not nonoverlapping over all $v$. So we define $F_j^v = Q_j^v \setminus \Omega^{v+1}$, which are pairwise disjoint over all $v$ and $j$. Note that by (2.12) we have
\[
|F_j^v| \geq \left(1 - \frac{s}{1 - t}\right)|Q_j^v| = c_{s,t} |Q_j^v|.
\]

We then estimate
\[
\Phi(a_j^v) \int_{Q_j^v} w(x) \, dx \leq c \Phi \left( \inf_{y \in Q_j^v} M_{0,s}^2 f(y) \right) \int_{Q_j^v} w(x) \, dx
\]
\[
\leq c c_{s,t} |F_j^v| \Phi \left( \inf_{y \in Q_j^v} M_{0,s}^2 f(y) \right) \frac{1}{|Q_j^v|} \int_{Q_j^v} w(x) \, dx
\]
\[
\leq c \left( \int_{F_j^v} \Phi(M_{0,s}^2 f(x)) \, dx \right) \inf_{y \in F_j^v} Mw(y)
\]
\[
\leq c \int_{F_j^v} \Phi(M_{0,s}^2 f(x)) Mw(x) \, dx.
\]

Then summing, by the disjointness of the $F_j^v$ this gives
\[
\sum_{v=1}^{\infty} \sum_{j \in I^2_v} \inf_{y \in Q_j^v} M_{0,s}^2 f(y) \int_{Q_j^v} w(x) \, dx \leq c \int_{Q_0} \Phi(M_{0,s}^2 f(x)) Mw(x) \, dx,
\]
and the desired conclusion follows in this case.

Finally, if $m_f(t, Q_0) \to 0$ as $Q_0 \to \mathbb{R}^n$, Fatou’s lemma gives (3.8). $\square$

Now, for an arbitrary weight $w$, note that (3.8) cannot hold for arbitrary $\Phi$. Indeed, suppose that it holds for $\Phi(t) = t^p$ for some $p > 2$. Then if $f = Tg$,
where $T$ is a singular integral operator, from (3.8), Theorem 4.1 below, and the Fefferman-Stein maximal inequality it readily follows that
\[
\int_{\mathbb{R}^n} |Tg(x)|^p w(x) \, dx \leq c \int_{\mathbb{R}^n} |g(x)|^p M_{L \log L} w(x) \, dx,
\]
which contradicts (1.11). Thus for arbitrary $w$, $(w, Mw)$ gives (3.8) for some but not all $\Phi$, and therefore for some $w$, $(w, Mw)$ do not satisfy condition $F$.

4. **Pointwise inequalities revisited**

We prove here a local version of the estimate $M^\sharp_0, s (Tf)(x) \leq c Mf(x)$ for Calderón-Zygmund singular integral operators established in [27]. We also recast similar estimates with $T$ replaced by a singular integral operator with kernel satisfying Hörmander-type conditions and an integral operator with a homogeneous kernel, and $M$ by an appropriate maximal function $M_T$.

We start with the singular integral case. First an observation of a geometric nature: there exists a dimensional constant $c_n$ such that for every cube $Q$ in $\mathbb{R}^n$, if $x, x' \in Q$ and $y \in (2Q)^c$ for some $m \geq 1$, then
\[
(4.1) \quad \frac{|x - x'|}{|x - y|} \leq c_n 2^{-m}.
\]

We then have

**Theorem 4.1.** Let $T$ be a singular integral operator defined by
\[
(4.2) \quad Tf(x) = \text{p.v.} \int_{\mathbb{R}^n} k(x, y) f(y) \, dy
\]
such that

1. for some $c > 0$, $k(x, y)$ satisfies
\[
|k(x, y) - k(x', y)| \leq c \frac{1}{|x - y|^n} \omega \left( \frac{|x - x'|}{|x - y|} \right)
\]
whenever $x, x' \in Q$ and $y \in (2Q)^c$ for any cube $Q$, where $\omega(t)$ is a nondecreasing function on $(0, \infty)$ such that
\[
\int_0^1 \omega(c_n t) \, \frac{dt}{t} < \infty;
\]

and

2. for some $1 \leq r < \infty$, $T$ is of weak-type $(r, r)$.

Then for $0 < s \leq 1/2$, any cube $Q_0$, and $x \in Q_0$,
\[
(4.3) \quad M^\sharp_0, s, Q_0 (Tf)(x) \leq c \sup_{x \in Q, Q \subset Q_0} \inf_{y \in Q} M_r f(y).
\]

Moreover, if we also have that $T(1) = 0$ and
\[
\int_0^1 \omega(c_n t) \ln(1/t) \, \frac{dt}{t} < \infty,
\]
then
\[
M^\sharp_0, s, Q_0 (Tf)(x) \leq c \sup_{x \in Q, Q \subset Q_0} \inf_{y \in Q} M^\sharp_r f(y).
\]
In particular, if \( Q_0 = \mathbb{R}^n \), then
\[
M_{0,s}^q(Tf)(x) \leq c M_r f(x) \quad \text{and} \quad M_{0,s}^q(Tf)(x) \leq c M_r^2 f(x),
\]
respectively.

**Proof.** We consider the case when \( T(1) = 0 \) first. Fix a cube \( Q_0 \subset \mathbb{R}^n \) and take \( x \in Q_0 \). Let \( Q \subset Q_0 \) be a cube containing \( x \) with center \( x_Q \) and sidelength \( l_Q \). Let \( 1/2 \leq t \leq 1-s \), \( f_1 = (f - m_f(t, Q))1_{2Q} \), and \( f_2 = (f - m_f(t, Q))1_{(2Q)^c} \). Then by the linearity of \( T \), \( Tf(z) - Tf_2(x_Q) = Tf_1(z) + Tf_2(z) - Tf_2(x_Q) \) for \( z \in Q \).

We claim that there exist constants \( c_1, c_2 > 0 \) independent of \( f \) and \( Q \) such that
\[
\{z \in Q : |Tf_1(z)| > c_1 \inf_{y \in Q} M_r f(y)\} \subset \{z \in Q : |Tf_2(z)| > c_2 \inf_{y \in Q} M_r f(y)\}.
\]
(4.4)

and
\[
||Tf_2 - Tf_2(x_Q)||_{L^\infty(Q)} \leq c_2 \inf_{y \in Q} M_r f(y).
\]
(4.5)

We prove (4.5) first. For any \( z \in Q \), by (4.1),
\[
|Tf_2(z) - Tf_2(x_Q)| \leq \int_{(2Q)^c} |k(z, y) - k(x_Q, y)| |f(y) - m_f(t, Q)| dy
\]
\[
\leq c \int_{(2Q)^c} \frac{1}{|z-y|^m} \omega\left(\left|\frac{|x_Q-z|}{|y-z|}\right|\right) |f(y) - m_f(t, Q)| dy
\]
\[
= c \sum_{m=1}^{\infty} \int_{2^{m+1}Q \setminus 2^m Q} \frac{1}{|z-y|^m} \omega\left(\left|\frac{|x_Q-z|}{|y-z|}\right|\right) |f(y) - m_f(t, Q)| dy
\]
\[
\leq c \sum_{m=1}^{\infty} \omega\left(c_n/2^m\right) \frac{1}{2^n |2^n Q|} \int_{2^n Q} |f(y) - m_f(t, Q)| dy.
\]
(6.6)

It readily follows from Proposition 1.1 in [43] that for any cube \( Q' \),
\[
|m_f(t, Q') - f_Q| \leq \frac{1}{|Q'|} \int_{Q'} |f(y) - f_Q| dy,
\]
and consequently, with \( c = c_s \),
\[
\frac{1}{|Q'|} \int_{Q'} |f(y) - m_f(t, Q')| dy
\]
\[
\leq \frac{1}{|Q'|} \int_{Q'} |f(y) - f_Q| dy + |f_Q - m_f(t, Q')|
\]
\[
\leq c \frac{1}{|Q'|} \int_{Q'} |f(y) - f_Q| dy
\]
\[
\leq c \inf_{y \in Q'} M_r^2 f(y).
\]
(4.7)

Also, from (4.7) and the triangle inequality we have
\[
|m_f(t, 2Q') - m_f(t, Q')| \leq c \inf_{y \in Q'} M_r^2 f(y).
\]
(4.8)
Then (4.7) and (4.8) give that
\[
\int_{2^mQ} |f(y) - m_f(t, Q)| \, dy \leq \int_{2^mQ} |f(y) - m_f(t, 2^mQ)| \, dy \\
+ \int_{2^mQ} \sum_{j=1}^{m} |m_f(t, 2^jQ) - m_f(t, 2^{j-1}Q)| \, dy
\]
\[
\leq c |2^mQ| \inf_{y \in Q} M^2f(y) + c |2^mQ| \sum_{j=1}^{m} \inf_{y \in 2^{j-1}Q} M^2f(y)
\]
(4.9)
\[
\leq c |2^mQ| (1 + m) \inf_{y \in Q} M^2f(y).
\]

Using (4.9), we bound (4.6) as
\[
|T f_2(z) - T f_2(x_Q)| \leq c \left( \sum_{m=1}^{\infty} (1 + m) \omega(c_n/2^m) \right) \inf_{y \in Q} M^2f(y)
\]
\[
\leq c \left( \int_0^1 \omega(c_n t) \ln(1/t) \frac{dt}{t} \right) \inf_{y \in Q} M^2f(y),
\]
and so
\[
||T f_2 - T f_2(x_Q)||_{L^\infty(Q)} \leq c_2 \inf_{y \in Q} M^2f(y) \leq c_2 \inf_{y \in Q} M^2f(y).
\]

As for (4.4), since \( T \) is of weak-type \((r, r)\), by (4.7) and (4.8) we have that for any \( \lambda > 0 \),
\[
\lambda \cdot \{|z \in Q : |T f_1(z)| > \lambda|\} \leq c \int_{2^Q} |f(y) - m_f(t, Q)|^r \, dy
\]
\[
\leq c \int_{2^Q} |f(y) - m_f(t, 2Q)|^r \, dy \\
+ c |m_f(t, Q) - m_f(t, 2Q)|^r \, |2Q|
\]
\[
\leq c \inf_{y \in 2Q} M^2f(y)^r |Q| + c \inf_{y \in Q} M^2f(y)^r |Q|
\]
\[
\leq c \inf_{y \in Q} M^2f(y)^r |Q|,
\]
and (4.4) follows by picking \( \lambda = c_1 \inf_{y \in Q} M^2f(y) \) for an appropriately chosen \( c_1 \).

Then, with \( c > \max\{c_1, c_2\} \), (4.4) and (4.5) give
\[
|\{z \in Q : |T f(z) - T f_2(x_Q)| > 2c \inf_{y \in Q} M^2f(y)\}|
\]
\[
\leq |\{z \in Q : |T f_2(z) - T f_2(x_Q)| > c_2 \inf_{y \in Q} M^2f(y)\}|
\]
\[
+ |\{z \in Q : |T f_2(z)| > c_1 \inf_{y \in Q} M^2f(y)\}|
\]
\[
< s|Q|.
\]
Whence
\[
\inf_{c'} \inf_{\alpha \geq 0} |\{z \in Q : |T f(z) - c'| > \alpha\}| < s|Q| \leq c \inf_{y \in Q} M^2f(y),
\]
and consequently, since this holds for all \( Q \subset Q_0, x \in Q \),
\[
M^2_{0,s,Q_0} T f(x) \leq c \sup_{x \in Q, Q \subset Q_0} \inf_{y \in Q} M^2f(y).
\]
To prove the case where \( T(1) \neq 0 \), let \( f_1 = f \mathbb{1}_{2Q} \) and \( f_2 = f \mathbb{1}_{(2Q)^c} \), and proceed as above. Then, for any \( z \in Q \), as in the proof of (4.6),

\[
|Tf_2(z) - Tf_2(x_Q)| \leq c \int_{(2Q)^c} \frac{1}{|z - y|^n} \omega\left(\frac{|x_Q - z|}{|y - z|}\right) |f(y)| \, dy
\]

\[
\leq c \sum_{m=1}^{\infty} \omega\left(c_n/2^m\right) \frac{1}{|2^m Q|} \int_{2^m Q} |f(y)| \, dy
\]

\[
\leq c \left( \int_0^1 \omega(c_n t) \frac{dt}{t} \right) \inf_{y \in Q} M_f(y)
\]

\[
\leq c \inf_{y \in Q} M_r f(y).
\]

(4.10)

And, as in the proof of (4.4),

\[
\lambda^r \{ z \in Q : |Tf_1(z)| > \lambda \} \leq c \int_{2Q} |f(y)|^r \, dy \leq c \inf_{y \in Q} M_r f(y)^r |Q|.
\]

(4.11)

Then as before,

\[
M^2_{0,s,Q_0} T f(x) \leq c \sup_{x \in Q, Q \subset Q_0} \inf_{y \in Q} M_r f(y).
\]

The proof is thus complete.

That \( M_r \) is relevant on the right-hand side of (4.3) for all \( r, 1 \leq r < \infty \), is clear from (4.11) above, and is useful when \( T \) is not known to be of weak-type (1,1). Also, there are operators of weak-type (1,1) where \( M_r \) is necessary on the right-hand side of (4.10), and hence on the right-hand side of (4.3), for \( 1 < r < \infty \). These are the Calderón-Zygmund convolution operators of Dini type, i.e., \( k(x) = \Omega(x')/|x|^n \), \( x \neq 0 \), where \( \Omega \) is a function on \( S^{n-1} \) that satisfies \( \int_{S^{n-1}} \Omega(x') dx' = 0 \) and an \( L^q \)-Dini condition for some \( 1 \leq q \leq \infty \) [20]. Because of their similarity with the singular integral operators with kernels satisfying Hörmander-type conditions considered in Theorem 4.3 below, the analysis of this case is omitted.

Theorem 4.1 is the prototype of the following general principle.

**Theorem 4.2.** Let \( T \) be a linear operator with the following property: There exists a mapping \( M_T \) with the property that for every fixed cube \( Q_0 \), for any \( Q \subset Q_0 \), there exist \( x_Q \in Q \) and constants \( c_1, c_2 > 0 \) such that every \( f \) in a dense class of functions of the domain of \( T \) can be written as \( f = f_1 + f_2 \) so that

\[
\{|z \in Q : |Tf_1(z)| > c_1 \inf_{y \in Q} M_T f(y)\}| < s|Q|,
\]

(4.12)

and

\[
||Tf_2 - Tf_2(x_Q)||_{L^\infty(Q)} \leq c_2 \inf_{y \in Q} M_T f(y).
\]

(4.13)

Then, there exists a constant \( c \) independent of \( f \) and \( Q_0 \) such that

\[
M^2_{0,s,Q_0}(T f)(x) \leq c \sup_{x \in Q, Q \subset Q_0} \inf_{y \in Q} M_T f(y).
\]

In particular, if \( Q_0 = \mathbb{R}^n \),

\[
M^2_{0,s}(T f)(x) \leq c M_T f(x).
\]
We exploit Theorem 4.2 – whose proof is similar to that of Theorem 4.1, and is therefore omitted – in the results that follow.

Our first observation is that singular integral operators with kernels satisfying Hörmander-type conditions similar to the convolution operators considered by Lorente et al. [36] satisfy local-type estimates with an appropriate $L^\infty$-Hörmander condition similar to the convolution operators considered by Lorente et al. [36] satisfy local-type estimates with an appropriate $M_T$. $M_T$ could be, as we noted above, $M_r$, or more generally $M_A$, where $A$ is a Young function; we denote by $\overline{A}$ its conjugate function, given by

$$\overline{A}(t) = \sup_{s>0} (st - A(s)).$$

We then have

**Theorem 4.3.** Let $T$ be a Calderón-Zygmund singular integral operator of weak-type $(1,1)$ such that for a Young function $A$, every cube $Q$, and $u,v \in Q$,

$$\sum_{m=1}^{\infty} |2^{m+1}Q| \|1_{2^{m+1}Q \setminus 2^mQ} (k(u,\cdot) - k(v,\cdot))\|_{L^A(2^{m+1}Q)} \leq c_A < \infty.$$

Then, with $c$ independent of $x$, $Q_0$, and $f$,

$$M^2_{0, s, Q_0} (T f)(x) \leq c \sup_{x \in Q, Q \subset Q_0} \inf_{y \in Q} M_{\overline{A}} f(y).$$

**Proof.** Let $f = f_1 + f_2$, where $f_1 = f 1_{2Q}$. Then, since $T$ is of weak-type $(1,1)$, as in the proof of (4.11),

$$\lambda \{|z \in Q : |T f_1(z)| > \lambda\} \leq c \int_{2Q} |f(y)| \ dy \leq c \inf_{y \in Q} M f(y) |Q|,$$

and since $M_{\overline{A}} f(y) \leq M_{\overline{A}} f(y)$ for all $y$, (4.12) holds for an appropriately chosen $c_1$.

Next,

$$\int_{\mathbb{R}^n \setminus 2Q} |k(u, y) - k(v, y)||f(y)| \ dy = \sum_{m=1}^{\infty} \int_{2^{m+1}Q \setminus 2^mQ} |k(u, y) - k(v, y)||f(y)| \ dy,$$

where, with

$$\lambda_m = \sup_{u,v \in Q} |2^{m+1}Q| \|1_{2^{m+1}Q \setminus 2^mQ} (k(u,\cdot) - k(v,\cdot))\|_{L^A(2^{m+1}Q)},$$

each summand above is bounded by Hölder’s inequality for $A$ and its conjugate $\overline{A}$ by $2 \lambda_m |2^{m+1}Q| \|f\|_{L^A(2^{m+1}Q)}$ and, therefore, the sum is bounded by

$$2 \left( \sum_{m=1}^{\infty} \lambda_m \right) \|f\|_{L^A(2^{m+1}Q)} \leq 2 c_A \inf_{y \in Q} M_{\overline{A}}(f).$$

Hence

$$|T f_2(u) - T f_2(v)| \leq 2 c_A \inf_{y \in Q} M_{\overline{A}} f(y),$$

and therefore (4.13) holds with $c_2 = 2 c_A$. The conclusion then follows from Theorem 4.2 with $M_T = M_{\overline{A}}$. 

The idea of the proof is essentially that of [29] and [36], where $T$ is assumed to be of convolution type. In that case, if $k$ satisfies the $L^A$-Hörmander condition for any Young function $A$, it also satisfies the usual $L^1$-Hörmander condition, and $T$ is of weak-type $(1,1)$. 

Finally, we consider the integral operators with homogeneous kernels defined as follows \[46\]. If \( A_1, \ldots, A_m \) are invertible matrices such that \( A_j - A_{j'} \) is invertible for \( j \neq j' \), \( 1 \leq j, j' \leq m \), and \( \alpha_j > 0 \) for all \( j \) and \( \alpha_1 + \cdots + \alpha_m = n \), then

\[
T f(x) = \int_{\mathbb{R}^n} |x - A_1 y|^{-\alpha_1} \cdots |x - A_m y|^{-\alpha_m} f(y) \, dy .
\]

For these operators we have

**Theorem 4.4.** For \( T \) defined as in \( (4.14) \), any cube \( Q_0 \subset \mathbb{R}^n \), and \( x \in Q_0 \), we have

\[
M^\sharp_{0,s,Q_0}(Tf)(x) \leq c \sum_{j=1}^m \sup_{x \in Q, Q \subset Q_0} \inf_{y \in Q} Mf(A_j^{-1}y) .
\]

**Proof.** Let \( Q_0 \) be a fixed cube and \( x \in Q_0 \). As in the proof of Theorem 2.1 in \[46\], for a cube \( Q \) containing \( x \), for an appropriate dimensional constant \( \lambda \) and \( 1 \leq j \leq m \), let \( Q_j = A_j^{-1}(\lambda Q) \), and put

\[
f_1(y) = f(y) 1_{\bigcup_{j=1}^m Q_j}(y)
\]

and \( f_2 = f - f_1 \).

First, by Theorem 3.2 in \[46\], \( T \) is of weak-type \((1,1)\). Moreover, since by the inequality following (2.2) in the proof of Theorem 2.1 in \[46\], for all integrable functions \( g \),

\[
\sum_{j=1}^m \int_{Q_j} |g(y)| \, dy \leq c \left( \sum_{j=1}^m \inf_{y \in Q} Mg(A_j^{-1}y) \right) |Q| ,
\]

taking \( \lambda > (c/s) \sum_{j=1}^m \inf_{y \in Q} Mf(A_j^{-1}y) \), it follows that

\[
|\{y \in Q : |Tf_1(y)| > \lambda\}| < s|Q| .
\]

And, concerning the \( Tf_2 \) term, for any \( y \in Q \) we have

\[
|Tf_2(y) - Tf_2(x_Q)| \leq \int_{\mathbb{R}^n \setminus \bigcup_{1 \leq j \leq m} Q_j} |k(y,z) - k(x_Q,z)| |f(z)| \, dz .
\]

Now, by breaking up \( \mathbb{R}^n \setminus \bigcup_{1 \leq j \leq m} Q_j \) into regions as in (2.4) in the proof of Theorem 2.1 of \[46\], we have that

\[
|Tf_2(y) - Tf_2(x_Q)| \leq c \sum_{j=1}^m \inf_{y \in Q} Mf(A_j^{-1}y) .
\]

The conclusion then follows as indicated in Theorem 4.2. \( \square \)

The weighted local estimates obtained below also follow using the full strength of the result in \[46\], namely,

\[
M^\sharp(|Tf|^{\delta})(x)^{1/\delta} \leq c \sum_{j=1}^m Mf(A_j^{-1}x) ,
\]

where \( 0 < \delta < 1 \).
5. Weighted local estimates

The weighted local estimates in the previous section can be used to express the local integral control of $Tf$ in terms of $M_Tf$. Specifically, we have

**Theorem 5.1.** Let $T, M_T$ be operators such that the conditions of Theorem 4.2 hold. Then for any $\Phi$ satisfying condition $C$, cube $Q_0$, and weights $(w, v)$ satisfying condition $F$,

$$
\int_{Q_0} \Phi(\|Tf(x) - m_{Tf}(t, Q_0)\|) w(x) \, dx \leq c \int_{Q_0} \Phi(M_{Tf}(x)) v(x) \, dx,
$$

where $c$ is independent of $Q_0$ and $f$.

Furthermore, if $\lim_{Q_0 \to \mathbb{R}^n} m_{Tf}(t, Q_0) = 0$,

$$
\int_{\mathbb{R}^n} \Phi(\|Tf(x)\|) w(x) \, dx \leq c \int_{\mathbb{R}^n} \Phi(M_{Tf}(x)) v(x) \, dx.
$$

**Proof.** The proof follows immediately from Theorem 3.1 and Theorem 4.2. Thus, the result holds with $M_T = M_r$ or $M_T^2$ for singular integral operators, $M_T = M_{Tr}$ for singular integral operators with kernels satisfying Hörmander-type conditions, and $M_Tf(x) = \sum_{j=1}^{n} Mf(A_j^{-1} x)$ for integral operators with homogeneous kernels. □

We discuss now the case $M_T = M_r$. Note that, in particular, Theorem 5.1 (with $r = 1$) gives (1.1) as well as (1.3), which are then one and the same result.

And, concerning (1.4), we have the following observation.

**Theorem 5.2.** Let $T$ be an operator such that the conditions of Theorem 4.2 hold with $M_T = M_r$, $1 \leq r < \infty$. Then, for $0 < p < r$, and any cube $Q_0$,

$$
M^2_{p, Q_0}(Tf)(x) \leq c_p \sup_{x \in Q, Q \subset Q_0} \inf_{y \in Q} M_r f(y),
$$

with $c_p \to \infty$ as $p \to r$, and, consequently,

$$
M^2_p(Tf)(x) \leq c_p M_r f(x),
$$

with the same $c_p$ as above.

**Proof.** Fix a cube $Q_0$ and $x \in Q_0$. By Theorem 5.1 with $w = v = 1$, $\Phi(t) = t^p$, and $0 < p < r$, since $(M_r f)^p = (M(|f|^r))^{p/r} \in A_1$, for a cube $Q \subset Q_0$ containing $x$ we have

$$
\frac{1}{|Q|} \int_Q |Tf(y) - m_{Tf}(t, Q)|^p \, dy \leq c \frac{1}{|Q|} \int_Q M_r f(y)^p \, dy \leq c_p \inf_{y \in Q} M_r f(y)^p.
$$

Therefore taking the supremum over $Q \subset Q_0$ containing $x$, it follows that

$$
M^2_{p, Q_0}(Tf)(x) \leq c_p \sup_{x \in Q, Q \subset Q_0} \inf_{y \in Q} M_r f(y),
$$

where $c_p \to \infty$ as $p \to r$. Furthermore, taking the supremum over all cubes $Q_0$ containing $x$ it readily follows that

$$
M^2_p(Tf)(x) \leq c_p M_r f(x)
$$

with the same $c_p$ as before. This gives (5.1).
Next, again fix a cube $Q_0$ and $x \in Q_0$. By Theorem 5.1 with $w = v = 1$ and $\Phi(t) = t^r$, for a cube $Q \subset Q_0$ containing $x$ we have
\[
\frac{1}{|Q|} \int_Q |Tf(y) - m_{Tf}(t, Q)|^r dy \leq c \frac{1}{|Q|} \int_Q M_r f(y)^r dy.
\]
Therefore, taking the supremum over $Q \subset Q_0$ containing $x$ it follows that
\[
M_{r, Q_0}^2(Tf)(x) \leq c \sup_{x \in Q, Q \subset Q_0} \left( \frac{1}{|Q|} \int_Q M_r f(y)^r dy \right)^{1/r}.
\]
Hence, taking the supremum over those cubes $Q_0$ containing $x$ it readily follows that
\[
M_r^2(Tf)(x) \leq c M_r(M_r(f))(x)
\]
and since $M_r \circ M_r$ is pointwise comparable to the maximal operator $M_{L^\infty \log L}$ \cite{7}, (5.2) follows.

Of course, it is of interest to remove the maximal function on the right-hand side of Theorem 5.1. The answer is precise for $A_p$ weights. Given a Young function $\Phi$ such that it and its conjugate $\Phi^*$ satisfy the $\Delta_2$ condition, recall that the upper index $u_{\Phi}$ of $L^\Phi$ is given by
\[
u_{\Phi} = \lim_{s \to 0^+} -\frac{\ln h(s)}{\ln s}, \quad h(s) = \sup_{t>0} \frac{\Phi^{-1}(s)}{\Phi^{-1}(st)}.
\]
Then the integral inequality
\[
\int_{\mathbb{R}^n} \Phi(Mf(x)) w(x) dx \leq c \int_{\mathbb{R}^n} \Phi(|f(x)|) w(x) dx
\]
holds if and only if $w \in A_p$ where $p = 1/u_{\Phi}$ \cite{25}.

Then, (1.2) can be formulated as follows.

**Theorem 5.3.** Let $T$ and $\Phi$ be as in Theorem 5.1 with $M_T = M$. Then, if $w \in A_p$ where $p = 1/u_{\Phi}$,
\[
\int_{\mathbb{R}^n} \Phi(|Tf(x)|) w(x) dx \leq c \int_{\mathbb{R}^n} \Phi(|f(x)|) w(x) dx.
\]

As for general weights, we have, as described in (1.8),

**Theorem 5.4.** Let $T$ be a Calderón-Zygmund singular integral operator of weak-type $(1,1)$ with $M_T = M$, $\Phi, \Psi$ Young functions such that
\[
\int_0^t \frac{\Phi(s)}{s^2} ds \leq c \frac{\Psi(t)}{t}, \quad t > 0,
\]
and $\Phi(t)/t^q$ decreases for some $1 < q < \infty$, and $(w, v)$ satisfy condition $F$. Then
\[
\int_{\mathbb{R}^n} \Phi(|Tf(x)|) w(x) dx \leq c \int_{\mathbb{R}^n} \Psi(|f(x)|) M v(x) dx.
\]
Moreover, if $w \in C_q$ for some $1 < q < \infty$,
\[
\int_{\mathbb{R}^n} \Phi(|Tf(x)|) w(x) dx \leq c \int_{\mathbb{R}^n} \Psi(|f(x)|) M_{L^\infty \log L} v(x) dx.
\]
Proof. Fefferman and Stein observed that for a weight \( u \), \( M \) is bounded from \( L^\infty(Mu) \) to \( L^\infty(u) \) and it maps \( L^1(Mu) \) weakly into \( L^1(u) \) \([16]\). Then for \( \Phi, \Psi \) as above a simple interpolation argument \([51, 56]\) gives that

\[
(5.6) \quad \int_{\mathbb{R}^n} \Phi(Mf(x)) u(x) \, dx \leq c \int_{\mathbb{R}^n} \Psi(|f(x)|) Mu(x) \, dx.
\]

Therefore, if \( T \) satisfies the conditions of Theorem 4.2 and \((w, v)\) satisfy condition \( F \),

\[
\int_{\mathbb{R}^n} \Phi(|Tf(x)|) w(x) \, dx \leq \int_{\mathbb{R}^n} \Phi(Mf(x)) v(x) \, dx \leq c \int_{\mathbb{R}^n} \Psi(|f(x)|) Mv(x) \, dx.
\]

Moreover, if \( w \in C_q \) for some \( 1 < q < \infty \), since \((w, Mw)\) satisfy condition \( F \) and \( M \circ M \sim M_{L \log L} \), by (5.6),

\[
\int_{\mathbb{R}^n} \Phi(|Tf(x)|) w(x) \, dx \leq c \int_{\mathbb{R}^n} \Psi(|f(x)|) M_{L \log L} w(x) \, dx.
\]

This completes the proof. \( \square \)

In the case of \( A_\infty \) weights the result is of interest when \( p = 1/u_\Phi \) and \( w \in A_q \), with \( 1 < p < q < \infty \). Similarly, if \( p = 1/u_\Phi \), the result is of interest when \( w \in C_q \) where \( q > p \).

Note that Theorem 5.4 implies that for every integer \( k \) there exists a weight \( w \) such that \((w, M_{A_k}w)\) do not satisfy condition \( F \), where the \( A_k \) are as in Section 3. For the sake of argument suppose that \((w, M_{A_k}w)\) satisfy condition \( F \) for all weights \( w \). Then by (5.4),

\[
\int_{\mathbb{R}^n} |Tf(x)|^q w(x) \, dx \leq c \int_{\mathbb{R}^n} |f(x)|^q M(M_{A_k}w)(x) \, dx \\
\leq c \int_{\mathbb{R}^n} |f(x)|^q M_{A_{k+1}}w(x) \, dx
\]

for all singular integrals \( T \) and all \( q \), which contradicts (1.11) for sufficiently large \( q \).

And, if \( M_T = M_r, 1 \leq r < \infty \), we have

**Theorem 5.5.** Let \( T \) be an operator that satisfies the conditions of Theorem 4.2 with \( M_T = M_r, 1 \leq r < \infty \). Let \( \Phi \) satisfy condition \( C \) such that \( \Phi(t)/t^p \) increases and \( \Phi(t)/t^q \) decreases for some \( r < p < q < \infty \). Then if \( \Psi(t) = \Phi(t^{1/r}) \) is convex, \( p = 1/u_\Phi \), and \( w \in A_{p/r} \),

\[
\int_{\mathbb{R}^n} \Phi(|Tf(x)|) w(x) \, dx \leq c \int_{\mathbb{R}^n} \Phi(|f(x)|) w(x) \, dx.
\]

**Proof.** By Theorem 5.1, for any \( \Phi \) satisfying condition \( C \) and \( w \in A_\infty \),

\[
\int_{\mathbb{R}^n} \Phi(|Tf(x)|) w(x) \, dx \leq c \int_{\mathbb{R}^n} \Phi(M_r f(x)) w(x) \, dx.
\]
Now, since $\Psi^{-1}(t) = \Phi^{-1}(t)^r$, by a simple computation $u_\Psi = ru_\Phi = r/p$. Thus, if $w \in A_{p/r}$, by (5.3),
\[
\int_{\mathbb{R}^n} \Phi(M_f f(x)) w(x) \, dx = \int_{\mathbb{R}^n} \Psi(M(|f|^r)(x)) w(x) \, dx
\]
\[
\leq c \int_{\mathbb{R}^n} \Psi(|f(x)|^r) w(x) \, dx
\]
\[
= c \int_{\mathbb{R}^n} \Phi(|f(x)|) w(x) \, dx,
\]
and we have finished.

That this result is essentially sharp when $\Phi$ is a power is discussed in [29].

Now, (5.5) is reminiscent of the estimate
\[
\int_{\mathbb{R}^n} |Tf(x)|^p w(x) \, dx \leq c \int_{\mathbb{R}^n} |f(x)|^p M_A w(x) \, dx
\]
established by Pérez, which holds for any weight $w$ if $A \in B_p$, i.e., the doubling Young function $A$ is such that
\[
\int_c^\infty \frac{A(t) \, dt}{t^p} < \infty
\]
for some $c > 0$ [12].

Since $B_p$ implies $B_q$ for $p < q < \infty$, if $A$ satisfies condition $B_p$, $T$ maps continuously $L^p(M_A w)$ into $L^p(w)$ and $L^q(M_A w)$ into $L^q(w)$ for every $q > p$. Then, if $\Phi$ is such that $\Phi(t)/t^p$ increases and $\Phi(t)/t^q$ decreases for some $q > p$, by essentially the same interpolation argument as before
\[
\int_{\mathbb{R}^n} \Phi(|Tf(x)|) w(x) \, dx \leq c \int_{\mathbb{R}^n} \Phi(|f(x)|) M_A w(x) \, dx.
\]

We now consider the case $M_T = M_A$.

**Theorem 5.6.** Let $T$ be a singular integral operator as in Theorem 5.1 with $M_T = M_A$, weights $(w, v)$ that satisfy property $F$, and $A \in B_p$. Then, if $\Phi(t)/t^p$ increases and $\Phi(t)/t^q$ decreases for some $q > p$, we have
\[
\int_{\mathbb{R}^n} \Phi(|Tf(x)|) w(x) \, dx \leq c \int_{\mathbb{R}^n} \Phi(|f(x)|) M_A v(x) \, dx.
\]

**Proof.** First, by Theorem 5.1,
\[
\int_{\mathbb{R}^n} \Phi(|Tf(x)|) w(x) \, dx \leq c \int_{\mathbb{R}^n} \Phi(M_A f(x)) v(x) \, dx.
\]

Next, recall that by Theorem 1.7 in [12], $A \in B_p$ if and only if for all weights $u$,
\[
\int_{\mathbb{R}^n} M_A f(x)^p u(x) \, dx \leq c \int_{\mathbb{R}^n} |f(x)|^p M u(x) \, dx.
\]

Now, $M_A$ maps continuously $L^p(M_v)$ into $L^p(v)$ and $L^q(M_v)$ into $L^q(v)$ for every $q > p$. Then if $\Phi$ is such that $\Phi(t)/t^p$ increases and $\Phi(t)/t^q$ decreases for some $q > p$, interpolating we have
\[
\int_{\mathbb{R}^n} \Phi(M_A f(x)) v(x) \, dx \leq c \int_{\mathbb{R}^n} \Phi(|f(x)|) M_A v(x) \, dx.
\]

The conclusion follows combining (5.7) and (5.8). Note that, in particular, if $w \in A_\infty$, $v = w$. 
\[\square\]
Lastly, we discuss the integral operators with homogeneous kernels.

**Theorem 5.7.** Let $T$ be an integral operator with homogeneous kernel as in (4.14), $\Phi$ a Young function, $p = 1/u_\Phi$, and $w \in A_p$, $1 < p < \infty$, such that $w(A_j x) \leq c w(x)$ for a.e. $x \in \mathbb{R}^n$, all $j$. Then, if $\lim_{Q_0 \to \mathbb{R}^n} m_{Tf}(t, Q_0) = 0$,

$$\int_{\mathbb{R}^n} \Phi(|Tf(x)|) w(x) \, dx \leq c \int_{\mathbb{R}^n} \Phi(|f(x)|) w(x) \, dx.$$  

**Proof.** By Theorem 5.1,

$$\int_{\mathbb{R}^n} \Phi(|Tf(x)|) w(x) \, dx \leq c \sum_{j=1}^m \int_{\mathbb{R}^n} \Phi(Mf(A_j^{-1} x)) w(x) \, dx$$

$$\leq c \sum_{j=1}^m \int_{\mathbb{R}^n} \Phi(Mf(x)) w(A_j x) \, dx$$

$$\leq c \sum_{j=1}^m \int_{\mathbb{R}^n} \Phi(Mf(x)) w(x) \, dx$$

$$\leq c \int_{\mathbb{R}^n} \Phi(|f(x)|) w(x) \, dx,$$

and we have finished. \qed

A simple computation shows that if $w(x) \in A_p$, then $w(\lambda x)$ is in $A_p$ with the same constant for $\lambda > 0$. So, when the $A_j$ are diagonal matrices with diagonal element $a_j > 0$, $1 \leq j \leq m$, as in [15], without any additional assumptions on $w$ the conclusion is

$$\int_{\mathbb{R}^n} \Phi(|Tf(x)|) w(x) \, dx \leq c \sum_{j=1}^m \int_{\mathbb{R}^n} \Phi(|f(A_j^{-1} x)|) w(x) \, dx.$$  

6. $L^p_w - L^q_w$ estimates for singular integral operators, $1 < p \leq q < \infty$

In this section we consider two-weight $L^p$ estimates for a fixed $1 < p < \infty$ and two-weight, $L^p_w - L^q_w$ estimates with $1 < p \leq q < \infty$ that apply directly to a Calderón-Zygmund singular integral operator and where the control exerted by a maximal function is not apparent. The strategy to deal with these operators follows the ideas developed so far: we consider local median decompositions and weights naturally associated to them. The first scenario corresponds to the $W_p$ classes of Fujii [19].

**Definition 6.1.** Fix $1 < p < \infty$. We say that the weights $(w, v)$ satisfy condition $W_p$ if there exist positive constants $\alpha, \beta, c_0$, with $\alpha < 1$, so that for every cube $Q$ and for all measurable $E, E' \subset Q$ with $E \cap E' = \emptyset$ and $|E'| \geq \alpha |Q|$,

$$\int_E w(x) \, dx \left( \frac{1}{|Q|} \int_{c(n, \alpha)Q} v(x)^{1-p'} \, dx \right)^p \leq c_0 \left( \frac{|E|}{|Q|} \right)^\beta \int_{E'} v(x)^{1-p'} \, dx < \infty,$$

where $c(n, \alpha) > 1$ is increasing with respect to $\alpha$. For such weights we write $(w, v) \in W_p$.

Fujii notes that for $w = v$, $W_p$ is equivalent to the $A_p$ condition. He also shows that $W_p$ implies Sawyer’s testing condition for the two-weight, $(p, p)$ boundedness of $M$ [19]. Also, note that if $(w, v) \in W_p$, $1 < p < \infty$, $(w, v)$ satisfy condition $F$. 

\[ \]
Indeed, observe that for a fixed $0 < \alpha < 1$, for all $E \subset Q$ with $|E| \leq (1 - \alpha)|Q|$, by Hölder’s inequality
\[
1 \leq c \left( \frac{\int_{Q \setminus E} v(x) \, dx}{\int_{Q \setminus E} v(x)^{1-p'} \, dx} \right) \left( \frac{1}{|Q|} \int_{c(n,\alpha)Q} v(x)^{1-p'} \, dx \right)^p,
\]
and therefore
\[
\int_E w(x) \, dx \leq c \left( \frac{|E|}{|Q|} \right)^\beta \left( \int_{Q \setminus E} v(x)^{1-p'} \, dx \right) \left( \frac{\int_{Q \setminus E} v(x) \, dx}{\int_{Q \setminus E} v(x)^{1-p'} \, dx} \right) \leq c \left( \frac{|E|}{|Q|} \right)^\beta \int_{Q \setminus E} v(x) \, dx,
\]
which gives condition $F$.

The variant of the decomposition of Section 2 that corresponds to these weights is sketched below and is referred to as the annular decomposition.

**Definition 6.2.** For $0 < s \leq 1/2$ and a measurable function $f$, we define
\[
m_f^s(1-s,Q) = \inf_c m_{|f-c|}(1-s,Q).
\]
Recall that by (4.3) of [43] we have that
\[
m_f^s(1-s,Q) \leq m_{|f-m_f(1-s,Q)|}(1-s,Q) \leq 2 m_f^s(1-s,Q).
\]

The annular decomposition is obtained by establishing the analogues to Lemma 4.1 and Lemma 4.3 in [43] in this context, and then the decomposition follows entirely as Theorem 2.1, but with different bounds on the constants $a_{j,v}^s$.

**Theorem 6.1.** Let $f$ be a measurable function on a fixed cube $Q_0 \subset \mathbb{R}^n$, $0 < s < 1/2$, and $1/2 \leq t < 1 - s$. Then (ii)-(iv) of Theorem 2.1 hold, and for a.e. $x \in Q_0$,
\[
|f(x) - m_f(t,Q_0)| \leq 4 M_{0,s} f(x) + \sum_{v=1}^{\infty} \sum_{j \in I_{Q_v}(x)} a_{j,v}^s \chi_{Q_{j,v}}(x),
\]
where
\[
a_{j,v}^s \leq (10n + 2) \sup_{Q_0 \supset Q \supset Q_{j,v}} m_f^s(1-s,Q).
\]

Then, there is the corresponding pointwise inequality similar to Theorem 4.1.

**Theorem 6.2.** Let $T$ be a singular integral operator satisfying the conditions of Theorem 4.1 with $r = 1$. Then for $0 < s \leq 1/2$ and any cubes $Q_1 \supset Q_0$,\[
\sup_{Q_1 \supset Q \supset Q_0} m_{Tf}^s(1-s,Q) \leq c \sup_{Q_1 \supset Q \supset Q_0} \frac{1}{|Q|} \int_Q |f(y)| \, dy.
\]
Moreover, if we also have that $T(1) = 0$, then
\[
\sup_{Q_1 \supset Q \supset Q_0} m_{Tf}^s(1-s,Q) \leq c \sup_{Q_1 \supset Q \supset Q_0} \frac{1}{|Q|} \int_Q |f(y) - f_Q| \, dy.
\]
The proof of this result is based on what are by now familiar ideas and is therefore omitted.

We can now prove the two-weight $L^p$ boundedness.
Theorem 6.3. Let $1 < p < \infty$. Suppose that $(w, v) \in W_p$. Let $T$ be a singular integral operator satisfying the conditions of Theorem 4.1 with $r = 1$. Then if $f$ has support contained in a cube $Q_0$,

$$
\int_{Q_0} |Tf(x) - m_{Tf}(t, Q_0)|^p \, w(x) \, dx \leq c \int_{Q_0} |f(x)|^p \, v(x) \, dx.
$$

Furthermore, if $m_{Tf}(t, Q_0) \to 0$ as $Q_0 \to \mathbb{R}^n$, then

$$
\int_{\mathbb{R}^n} |Tf(x)|^p \, w(x) \, dx \leq c \int_{\mathbb{R}^n} |f(x)|^p \, v(x) \, dx.
$$

Proof. We only sketch the proof, which follows the blueprint of the proof of Theorem 5.1 and [19]. By Theorem 6.1, for a.e. $x \in Q_0$,

$$
|Tf(x) - m_{Tf}(t, Q_0)| \leq 4M_{0, s, Q_0}^p(Tf)(x) + \sum_{v=1}^{\infty} \sum_{j \in I_n^s} a_j^v \mathbb{1}_{Q_j^v}(x),
$$

where

$$
a_j^v \leq (10n + 2) \sup_{Q_0 \supset Q \supset Q_j^v} m_{Tf}^p(1 - s, Q).
$$

Then

$$
|Tf(x) - m_{Tf}(t, Q_0)|^p \leq c_p (a(x, Q_0, p) + b(x, Q_0, p)),
$$

where we define

$$
a(x, Q_0, p) = M_{0, s, Q_0}^p(Tf)(x) + \sum_{k=1}^{\infty} \sum_{j \in I_n^k} |a_j^k|^p \mathbb{1}_{Q_j^k}(x)
$$

and

$$
b(x, Q_0, p) = \sum_{k=2}^{\infty} \mathbb{1}_{\Omega^k \setminus \Omega^{k+1}}(x) \sum_{v=1}^{k-1} c_p^{k-v} \sum_{j \in I_n^s} |a_j^v|^p \mathbb{1}_{Q_j^v}(x).
$$

Thus it suffices to prove that

$$
\int_{Q_0} a(x, Q_0, p) \, w(x) \, dx \leq c \int_{Q_0} |f(x)|^p \, v(x) \, dx
$$

and a similar estimate with $b(x, Q_0, p)$ in place of $a(x, Q_0, p)$ above. By (4.3) and (6.1) we have that $a(x, Q_0, p) \leq c Mf^p(x)$, and since supp$(f) \subset Q_0$, that $W_p$ implies Sawyer’s testing condition allows us to handle the first inequality. As for the second, we use (6.1) and that $(w, v) \in W_p$ as in the proof of the Theorem in [19]. That (6.2) holds follows immediately from Fatou’s lemma. \qed

Condition $W_p$ also gives continuity results for singular integral operators for values of $q \neq p$. Indeed, we have

Theorem 6.4. Let $1 < p < \infty$. Suppose that $(w, v) \in W_p$. Let $T$ be a singular integral operator satisfying the conditions of Theorem 4.1 with $r = 1$.

Then, if $1 < q < p$, and $0 < \eta < p' / q'$,

$$
\int_{\mathbb{R}^n} |Tf(x)|^q \, w(x) \, dx \leq c \int_{\mathbb{R}^n} |f(x)|^q \left( \frac{v(x)}{Mv(x)} \right)^\eta Mv(x) \, dx.
$$

And, if $p < q < \infty$ and $0 < 1 - \eta < p / q < 1$,

$$
\int_{\mathbb{R}^n} |Tf(x)|^q \, w(x) \, dx \leq c \int_{\mathbb{R}^n} |f(x)|^q \left( \frac{Mv(x)}{v(x)} \right)^\eta v(x) \, dx.
$$
Proof. Since \((w,v)\) satisfy property \(F\), by Theorem 5.4,
\[
\int_{\mathbb{R}^n} |Tf(x)|^r w(x) \, dx \leq c_r \int_{\mathbb{R}^n} |f(x)|^r Mv(x) \, dx,
\]
for all \(1 < r < \infty\). Now, if \(1 < q < p\) and \(0 < \eta < p'/q'\), the index \(r\) defined by the relation
\[
\frac{1}{r} = \frac{1}{1 - \eta} \left( \frac{1}{q} - \frac{\eta}{p} \right)
\]
satisfies \(1 < r < q < p\) and \(1/q = (1 - \eta)/r + \eta/p\), and since \(T\) maps \(L^r(Mv)\) into \(L^r(w)\) and \(L^p(v)\) into \(L^p(w)\), the conclusion follows by the Stein-Weiss theorem of interpolation with change of measure.

Now, in case \(p < q\) and \(0 < \eta < p'/q'\), the index \(r\) given by
\[
\frac{1}{r} = \frac{1}{\eta} \left( \frac{1}{q} - \frac{1 - \eta}{p} \right)
\]
satisfies \(p < q < r < \infty\) and \(1/r = (1 - \eta)/q + \eta/p\), and since \(T\) maps \(L^p(v)\) into \(L^p(w)\) and \(L^r(Mv)\) into \(L^r(w)\), the conclusion also follows by the Stein-Weiss theorem of interpolation with change of measure. \(\Box\)

While (6.4) is reminiscent of extrapolation estimates \([12]\), the estimate (6.5) for values \(q < p\) is not.

So far, we have produced two median function decompositions leading to two-weight continuity results for Calderón-Zygmund singular integral operators. Now, each decomposition generates families of cubes that share certain properties, and it is these properties and not the specific cubes that are of interest. In particular, the bounds on the respective \(a^v_j\) are related, and in this way the annular decomposition is stronger than the first decomposition. Indeed, it is readily seen that for a cube \(Q^v_j\) generated by the annular decomposition,
\[
\sup_{Q_0 \supset Q \supset Q^v_j} m_f^t(1 - s, Q) \leq \inf_{y \in Q^v_j} M^t_{0,s,Q_0} f(y).
\]
(This right-hand side is the bound for the \(a^v_j\) used when invoking the first decomposition, as in the proof of Theorem 3.1.)

If we are more deliberate in the construction of the local median decomposition, an even sharper bound for the \(a^v_j\) results. This third decomposition is needed for other applications, including Lerner’s proof of the \(A_2\) conjecture \([34]\).

**Theorem 6.5.** Let \(f\) be a measurable function on a fixed cube \(Q_0 \subset \mathbb{R}^n\), \(0 < s < 1/2\), and \(1/2 \leq t < 1 - s\). Then (ii)-(iv) of Theorem 2.1 hold, and for a.e. \(x \in Q_0\),
\[
|f(x) - m_f(t,Q_0)| \leq 8M^t_{0,s} f(x) + \sum_{v=1}^{\infty} \sum_{j \in I^v_2} a^v_j \mathbb{1}_{Q^v_j}(x),
\]
where
\[
a^v_j \leq m_{|f - m_f(t,Q_0)|}(1 - (1 - t)/2^n, \hat{Q}^v_j).
\]

**Proof.** We follow the proof of Theorem 2.1 in form, with a few definitional changes. First note that for any cube \(Q\),
\[
(6.5) \quad m_{|f - m_f(t,Q)|}(t,Q) \leq 4 \inf_{y \in Q} M^t_{0,s,Q} f(y).
\]
To see this, from Proposition 1.1 and (4.3) in [43],
\[ m_{|f-m_{f(t,Q)}}(t, Q) \leq m_{|f-m_{f(1-s,Q)}} + m_{f(1-s,Q)-m_{f(t,Q)}}(t, Q) \]
\[ \leq 2 m_{|f-m_{f(1-s,Q)}}(t, Q) \leq 2 m_{|f-m_{f(1,s,Q)}}(1 - s, Q) \]
\[ \leq 4 \inf_{y \in Q} M_{0,s,Q} f(y) . \]

We define \( E^1 = \{ x \in Q_0 : |f(x) - m_{f(t,Q_0)}| > m_{|f-m_{f(t,Q_0)}}(t, Q_0) \} \). If \(|E^1| = 0\), the decomposition halts, just as in Theorem 2.1. So we suppose \(|E^1| > 0\). We then define
\[ \Omega^1 = \{ x \in Q_0 : m_{t,Q} f^0(x) > m_{|f-m_{f(t,Q_0)}}(t, Q_0) \} . \]

Proceeding as before, we have that \( \Omega^1 = \bigcup_j Q_j^1 \) so that (as in (2.3))
\[ |m_{f_0(t,Q_j^1)}| > m_{|f_0|}(t,Q_0) \quad \text{and} \quad |m_{f(t,Q_j^1)}| \leq m_{|f_0|}(t,Q_0) . \]
Furthermore, we also have that
\[ \sum_j |Q_j^1| \leq \frac{s}{1 - t} |Q_0| . \]

Before continuing, observe that for any cube \( Q \),
\[ m_{f(t,Q)} \leq m_{f(1 - (1-t)/2^n, Q)} . \]

To see this, note that
\[ |\{ y \in \hat{Q} : f(y) \geq m_{f(t,Q)} \}| \geq |\{ y \in Q : f(y) \geq m_{f(t,Q)} \}| \]
\[ \geq (1-t)|Q| = \frac{1-t}{2^n} |\hat{Q}| , \]
so taking complements in \( \hat{Q} \) we have
\[ |\{ y \in \hat{Q} : f(y) < m_{f(t,Q)} \}| \leq \left( 1 - \frac{1-t}{2^n} \right) |\hat{Q}| . \]

Note also that by our choice of \( t \), it follows that \( 1 - (1-t)/2^n \geq 1/2 \).

Let \( \alpha_j^1 = m_{f_0(t,Q_j^1)} \). By (6.6) and (6.7) we have
\[ |\alpha_j^1| \leq |m_{f_0(t,Q_j^1)} - m_{f_0(t,Q_j^1)}| + |m_{f_0(t,Q_j^1)}| \]
\[ \leq m_{f-m_{f_0}(t,\hat{Q}_j^1)}(t,\hat{Q}_j^1) + m_{|f_0|}(t,Q_0) \]
\[ \leq m_{f-m_{f_0}(t,\hat{Q}_j^1)}(1 - (1-t)/2^n, \hat{Q}_j^1) + m_{|f_0|}(t,Q_0) . \]

This gives the first iteration of the decomposition of \( f \) when \(|E^1| > 0\): for a.e. \( x \in Q_0 \), with \( g^1 = f^0\mathbb{1}_{Q_0\setminus\Omega^1} \),
\[ f^0(x) = g^1(x) + \sum_j \alpha_j^1 \mathbb{1}_{Q_j^1}(x) + \sum_j (f(x) - m_{f_0(t,Q_j^1)}) \mathbb{1}_{Q_j^1}(x) . \]

Clearly by (6.5)
\[ |g^1(x)| \leq m_{|f-m_{f(t,Q_0)}}(t,Q_0) \leq 4 \inf_{y \in Q_0} M_{0,s,Q_0} f(y) \leq 4M_0^2 f(x) \]
a.e. on \( Q_0 \setminus \Omega^1 \).

By proceeding as in the proof of Theorem 2.1, the result follows. □
This decomposition generates families of cubes sharing the same properties as those from the first and the annular decompositions. And, as anticipated, for some parameters $s, t$ the bound on the $a^v_j$ from this decomposition is even smaller. Indeed, for any $c$, since $1/2 \leq t \leq 1 - (1 - t)/2^n$,

$$m_{|f-m_f(t,Q^v_j)}|(1 - (1 - t)/2^n, Q^v_j) \leq m_{|f-c_1}(1 - (1 - t)/2^n, Q^v_j) + |c - m_f(t, Q^v_j)| \leq m_{|f-c_1}(1 - (1 - t)/2^n, Q^v_j) + m_{|f-c_1}((1 - 1)/2^n, Q^v_j).$$

Thus

$$m_{|f-m_f(t,Q^v_j)}|(1 - (1 - t)/2^n, Q^v_j) \leq 2 m_f((1 - (1 - t)/2^n, Q^v_j).$$

Then for any $0 < s < 1/2^{n+1}$ and $1/2 \leq t \leq 1 - 2^n s$,

$$m_{|f-m_f(t,Q^v_j)}|(1 - (1 - t)/2^n, Q^v_j) \leq c \sup_{Q_0 \supset Q \supset Q^v_j} m_f(1 - s, Q).$$

Before we proceed to prove our next theorem, we need a couple of preliminary results. The first is an extension of a property given in Lemma 4.8 in [27] and the comments that follow it.

**Lemma 6.1.** Let $T$ be a Calderón-Zygmund singular integral operator defined by (4.2) and $Q$ a cube of $\mathbb{R}^n$. If $T$ satisfies the assumptions of Theorem 4.1 with $1 \leq r < \infty$, let

$$\lambda_m = \omega(c_n/2^m), \quad m \geq 1.$$ 

If $T$ satisfies the assumptions of Theorem 4.3 for the Young function $A(t) = t^r$, $1 \leq r < \infty$, let $\lambda_m$ be as defined there. In either case, we have $\sum_{m=1}^{\infty} \lambda_m < \infty$ and

$$m^r_{T_f}(1 - s, Q) \leq c \sum_{m=1}^{\infty} \lambda_m \left( \frac{1}{|2^m Q|} \int_{2^m Q} |f(y)|^r \, dy \right)^{1/r}. \tag{6.8}$$

**Proof.** Fix $Q$, let $x_Q \in Q$, and put $f = f_1 + f_2$ where $f_1 = f 1_{2Q}$. We claim that there exist constants $c_1, c_2 > 0$ independent of $f$ and $Q$ such that

$$|\{z \in Q : |Tf_1(z)| > c_1 I\}| < s |Q|, \tag{6.9}$$

and

$$\|Tf_2 - Tf_2(x_Q)\|_{L^\infty(Q)} \leq c_2 I, \tag{6.10}$$

where

$$I = \sum_{m=1}^{\infty} \lambda_m \left( \frac{1}{|2^m Q|} \int_{2^m Q} |f(y)|^r \, dy \right)^{1/r}.$$ 

Now, if $T$ satisfies the assumptions of Theorem 4.1, (6.9) follows since $T$ is of weak-type $(r, r)$, and (6.10) follows as in the proof of (4.10) followed by Hölder’s inequality when $r > 1$.

And, when $T$ satisfies the assumptions of Theorem 4.3, (6.10) holds automatically. (6.9) follows using that $T$ is of weak-type $(1, 1)$ and Hölder’s inequality when $r > 1$.

Then, in either case, with $c > \max\{c_1, c_2\}$, (6.9) and (6.10) give

$$|\{z \in Q : |Tf(z) - Tf_2(x_Q)| > 2c I\}| < s |Q|. $$
Whence for all $Q$,
\[
m_{T_f}^s(1-s,Q) = \inf_{c'} \inf_{\alpha \geq 0} \{ |\{z \in Q : |Tf(z) - c'| > \alpha\}| < s|Q| \}
\leq c \sum_{m=1}^{\infty} \lambda_m \left( \frac{1}{2^m Q} \int_{2^m Q} |f(y)|^r dy \right)^{1/r}
\]
and (6.8) holds.

Note that Lemma 6.1 also applies to the Calderón-Zygmund singular integral operators of Dini type. In that case, using Lemma 5 in [29], the $\lambda_m$ can be estimated in terms of the $\omega_{\alpha r}$ modulus of continuity of $\Omega$.

Next, we collect some properties of Young functions in the classes $B_p$ and $B_p^\alpha$. The latter class was introduced by Cruz-Uribe and Moen and for $0 < \alpha < 1$ and $1 < p < 1/\alpha$, it consists of those Young functions $A$ such that with $1/q = 1/p - \alpha$,
\[
\|A\|_{\alpha,p} = \left( \int_c^\infty A(t)^{q/p} \frac{dt}{t^q} \right)^{1/q} < \infty .
\]

As they point out, if $\alpha > 0$, $B_p^\alpha$ is weaker than $B_p$. The result of interest to us, Theorem 3.3 in [14], is that for $A \in B_p^\alpha$ the maximal function
\[
M_{\alpha,A} f(x) = \sup_{x \in Q} |Q|^\alpha \|f\|_{L^A(Q)}
\]
maps $L^p(\mathbb{R}^n)$ continuously into $L^q(\mathbb{R}^n)$ with norm $\leq c \|A\|_{\alpha,p}$.

We also have

**Proposition 6.1.** Let $A$ be a Young function and $C(t) = A(t^r)$ for $1 < r < \infty$.

(i) If $A \in B_p$, then $C \in B_{rp}$. Furthermore, for all cubes $Q$,
\[
\|g\|_{L^C(Q)} = \| |g|^r \|_{L^{A}(Q)}^{1/r} .
\]

(ii) If $\overline{A} \in B_{p^\prime}$, there exists a positive constant $c$ independent of $g$ and $Q$ such that
\[
\|g\|_{L^r(Q)} \leq c \|g\|_{L^A(Q)} .
\]

(iii) If $A \in B_{p^r}^\alpha$, for $r < p$, then $C \in B_p^\alpha$.

(iv) If $\overline{A} \in B_p^\alpha$, then $\overline{C} \in B_p^\alpha$. In particular, if $A \in B_p$, then $C \in B_p$.

**Proof.** The proof of (i) is a straightforward computation and is therefore omitted. As for (ii), recall that if $\overline{A} \in B_{p^\prime}$, for some constant $c > 0$,
\[
\int_c^\infty \left( \frac{t^r}{A(t)} \right)^{p^\prime - 1} \frac{dt}{t} < \infty ,
\]
and therefore, there exist positive constants $c_0, c_1$ such that $A(t) \geq c_0 \ t^p$ for $t \geq c_1$. Then, by a direct computation or the closed graph theorem, there exists a positive constant $c$ independent of $g$ and $Q$ such that (6.13) holds.

Now, for (iii), let $q$ be given by the relation $1/q = 1/p - \alpha$; then $r/q = r/p - \alpha r$ and the value of $q$ in (6.11) for membership in the class $B_{p^r}^\alpha$ is $q/r$. Then, since $(q/r)/(p/r) = q/p$, we have
\[
\int_c^\infty \frac{C(t)^{q/p}}{t^q} \frac{dt}{t} = \int_c^{c_1} \frac{A(t)^{q/r}(p/r)}{t^{q/r}} \frac{dt}{t} < \infty ,
\]
and $C \in B_p^\alpha$.
Finally, (iv); since the proof for $\alpha = 0$ follows by setting $p = q$ in the proof for the case $\alpha > 0$, we do the latter. Taking inverses, $C^{-1}(t) = A^{-1}(t)^{1/r}$, and therefore it readily follows that $\overline{C}^{-1}(t) \sim t^{1/r'} \overline{A}^{-1}(t)^{1/r}$. Then,

$$
\int_{c}^{\infty} \frac{\overline{C}(t)^{q/p}}{t^{q}} \frac{dt}{t} \leq c_{1} \int_{c_{2}}^{\infty} \frac{t^{q/p-1}}{\overline{C}(t)^{q}} \frac{dt}{t} \sim c_{3} \int_{c_{2}}^{\infty} \frac{t^{q/p-1}}{\overline{A}(t)^{q/r} t^{q/r}} \frac{dt}{t} \sim c_{4} \int_{c_{5}}^{\infty} \frac{1}{\overline{A}(t)^{q/r}} \frac{t}{t^{q/r}} \frac{dt}{t},
$$

which, since $\overline{A}(t)/t$ increases, is bounded by

$$
c_{6} \int_{c_{5}}^{\infty} \frac{\overline{A}(t)^{q/p}}{t^{q((1/r) + 1/r')}} \frac{dt}{t} = c_{6} \int_{c_{5}}^{\infty} \frac{\overline{A}(t)^{q/p}}{t^{q}} \frac{dt}{t} < \infty.
$$

This completes the proof.

We will also rely on the following result of Pérez, Theorem 2.11 in [11] or Theorem 3.5 in [20]. Let $p, q$ with $1 < p \leq q < \infty$, and $(w, v)$ a pair of weights such that for every cube $Q$,

$$
|Q|^{1/q-1/p} \|w^{1/q}\|_{L^{q}(Q)} \|v^{-1/p}\|_{L^{q}_{B}(Q)} \leq c,
$$

where $B$ is a Young function with $\overline{B} \in B_{p}$. Then, the Hardy-Littlewood maximal function $M$ maps $L^{p}_{w}(\mathbb{R}^{n})$ continuously into $L^{q}_{v}(\mathbb{R}^{n})$, i.e.,

$$
\|Mf\|_{L^{q}_{v}} \leq c \|f\|_{L^{p}_{w}}.
$$

We are now ready to prove our result.

**Theorem 6.6.** Let $T$ be a Calderón-Zygmund singular integral operator that satisfies the assumptions of Theorem 4.1 with $1 \leq r < \infty$ or Theorem 4.3 with the Young function $t^{r'}$ there and $1 \leq r < \infty$. Let $r < p \leq q < \infty$, define $0 \leq \alpha < 1$ by the relation $\alpha = 1/p - 1/q$, and let $\alpha_{1}, \alpha_{2} \geq 0$ be such that $\alpha = \alpha_{1} + \alpha_{2}$. Further, suppose that the Young functions $A, B$ are so that $A \in B_{q/r} \cap B^{\alpha_{1}}_{q}$ and $\overline{B} \in B^{\alpha_{2}}_{p/r}$, and $w, v$ are weights such that for all cubes $Q$,

$$
|Q|^{r/q-1/p} \|w^{r/q}\|_{L^{q}(Q)} \|v^{-r/p}\|_{L^{q}_{B}(Q)} \leq c < \infty.
$$

Then, if the $\lambda_{m}$ as defined in Lemma 6.1 satisfy

$$
\sum_{m=1}^{\infty} \lambda_{m} 2^{mn/q} < \infty,
$$

we have

$$
\left(\int_{\mathbb{R}^{n}} |Tf(x)|^{q} w(x) \, dx\right)^{1/q} \leq c \left(\int_{\mathbb{R}^{n}} |f(x)|^{p} v(x) \, dx\right)^{1/p}
$$

for those $f$ such that $\lim_{Q_{0} \to \mathbb{R}^{n}} m_{Tf}(t, Q_{0}) = 0$.

**Proof.** We begin by considering the local version of (6.17). Fix a cube $Q_{0}$ and note that by Theorem 6.5,

$$
|Tf(x) - m_{Tf}(t, Q_{0})| \leq 8 M_{0,s,Q_{0}}^{r}(Tf)(x) + c \sum_{v, j} m_{Tf}^{r}(1 - (1 - t)/2^{n}, \hat{Q}_{v}^{j}) 1_{Q_{v}^{j}}(x),
$$

where

$$
M_{0,s,Q_{0}}^{r}(Tf)(x) = \sup_{Q_{0}} \int_{Q_{0}} |Tf(y)| \chi_{Q_{0}}(y) \, dy.
$$

We then have

$$
\left(\int_{\mathbb{R}^{n}} |Tf(x)|^{q} w(x) \, dx\right)^{1/q} \leq c \left(\int_{\mathbb{R}^{n}} |f(x)|^{p} v(x) \, dx\right)^{1/p}
$$

for those $f$ such that $\lim_{Q_{0} \to \mathbb{R}^{n}} m_{Tf}(t, Q_{0}) = 0$.
and therefore to estimate the $L^q_w(Q_0)$ norm of $Tf(x) - m_{Tf}(t, Q_0)$ it suffices to estimate the norm of each summand separately. Since by Theorem 4.1 or Theorem 4.3 we have

$$M^2_{0,s,Q_0}(Tf)(x) \leq c M_{r,f}(x) = c M(|f|^r)(x)^{1/r},$$

the first term above can be estimated by $\|M(|f|^r\|_L^q_w = \|M(|f|^r\|_L^q_{w^{q/r}}$. Now, since $A \in B_{(q/r)}$, by (6.17), $\|w^r/q\|_{L^{q/r}(Q)} \leq c \|w^{r/q}\|_{L^q(A(Q)}$ for all cubes $Q$, and therefore (6.16) implies (6.14) with indices $p/r$ and $q/r$ there. Thus,

$$\|M(|f|^r)\|_{L^q_w}^{1/r} \leq c \|f\|^{1/r}_{L^p_w} = c \|f\|_{L^p_w}$$

and

$$\|M^2_{0,s,Q_0}(Tf)\|_{L^q_w} \leq c \|f\|_{L^p_w}.$$

Next, note that by a geometric argument, if $Q$ is any of the cubes $Q_j^v$, there is a dimensional constant $c$ such that

$$\sum_{m=1}^\infty \lambda_m \left( \frac{1}{|2^m Q|} \int_{2^m Q} |f(y)|^r \, dy \right)^{1/r} \leq c \sum_{m=1}^\infty \lambda_m \left( \frac{1}{|2^m Q|} \int_{2^m Q} |f(y)|^r \, dy \right)^{1/r}.

To estimate the norm of the sum by duality, let $h$ be such that $\text{supp}(h) \subset Q_0$ and $\|h\|_{L^{q'}(Q_0)} = 1$, and note that by (6.8) and (6.18),

$$\int_{Q_0} \left( \sum_{v,j} m^2_{Tf}(1 - (1 - t)/2^n, Q_j^{v}\big|_{Q_j^v}(x) \right) w(x)^{1/q} h(x) \, dx \leq c \sum_{m=1}^\infty \lambda_m \sum_{v,j} \left( \frac{1}{|2^m Q_j^v|} \int_{2^m Q_j^v} |f(y)|^r \, dy \right)^{1/r} \int_{Q_j^v} w(x)^{1/q} h(x) \, dx.

We consider each term in the inner sum of (6.19) separately. First, let $D$ be the Young function defined by $D(t) = BD(t)$, and note that by H"older's inequality for the conjugate Young functions $B, \overline{B}$ and (6.12),

$$\left( \frac{1}{|2^m Q_j^v|} \int_{2^m Q_j^v} |f(y)|^r \, dy \right)^{1/r} = \left( \frac{1}{|2^m Q_j^v|} \int_{2^m Q_j^v} |f(y)|^r v(y)^{r/p} v(y)^{-r/p} \, dy \right)^{1/r} \leq 2 \left( ||f|^{r/p}\|_{L^r(2^m Q_j^v)} ||v^{-r/p}\|_{L^{r/p}(2^m Q_j^v)} \right)^{1/r} = 2 \left( ||f|^{r/p}\|_{L^p(2^m Q_j^v)} ||v^{-r/p}\|_{L^{r/p}(2^m Q_j^v)} \right)^{1/r}.$$

Next, let $C$ be the Young function defined by $C(t) = A(t^r)$ and note that by H"older’s inequality for the conjugate Young functions $C, \overline{C}$ and (6.12),

$$\int_{Q_j^v} w(x)^{1/q} h(x) \, dx \leq 2^{mn} |Q_j^v| \frac{1}{|2^m Q_j^v|} \int_{2^m Q_j^v} w(x)^{1/q} h(x) \, dx = 2 \cdot 2^{mn} \left( ||w^r/q\|_{L^r(2^m Q_j^v)} ||h|^{1/r}\|_{L^r(2^m Q_j^v)} |Q_j^v| \right) \leq 2 \cdot 2^{mn} \left( ||w|^{1/r}\|_{L^r(2^m Q_j^v)} ||h|^{1/r}\|_{L^r(2^m Q_j^v)} |Q_j^v| \right).$$

Moreover, since for each $\lambda > 1$ and each cube $Q$ we have

$$\|g|Q\|_{L^\infty(Q)} \leq \|g\|_{L^\infty(\lambda Q)},$$
it follows that 
\[
\int_{Q_j} w(x)^{1/q} h(x) dx \leq 2 \cdot 2^{mn} \|w^{r/q}\|_{L^1(2^m Q_j)} \|h\|_{L^{\infty}(2^m Q_j)} |Q_j^v|.
\]

Therefore, since by (6.16) with \(1/p - 1/q = \alpha\),
\[
\|w^{r/q}\|_{L^1(2^m Q_j)} \|v^{-r/p}\|_{L^p(2^m Q_j)} \leq c |2^m Q_j^v|^\alpha,
\]
each term in the inner sum of (6.23) is bounded by
\[
c 2^{mn} |2^m Q_j^v|^\alpha \|f v^{1/p}\|_{L^p(2^m Q_j)} \|h\|_{L^{\infty}(2^m Q_j)} |Q_j^v|,
\]
and consequently the sum itself does not exceed
\[(6.20) \quad c \sum_{m=1}^{\infty} \lambda_m 2^{mn} \sum_{v,j} |2^m Q_j^v|^\alpha \|f v^{1/p}\|_{L^p(2^m Q_j)} ^\alpha \|h\|_{L^{\infty}(2^m Q_j)} |Q_j^v|.
\]

Let \(F_j^v = Q_j^v \setminus \Omega^v + 1\); as in Theorem 3.2 it follows that the \(F_j^v\) are pairwise disjoint and \(|F_j^v| \geq c|Q_j^v|\), where \(c\) depends on \(s\) and \(t\) but is independent of \(v\) and \(j\). Then, with \(\alpha = \alpha_1 + \alpha_2\), the innermost sum in (6.20) is bounded by
\[
J = c \sum_{v,j} |2^m Q_j^v|^\alpha_1 \|f v^{1/p}\|_{L^p(2^m Q_j)} |2^m Q_j^v|^\alpha_2 \|h\|_{L^{\infty}(2^m Q_j)} |F_j^v|,
\]
and, since
\[
|2^m Q_j^v|^\alpha_1 \|f v^{1/p}\|_{L^p(2^m Q_j)} \leq \inf_{x \in F_j^v} M_{\alpha_1,D}(f v^{1/p})(x)
\]
and similarly
\[
|2^m Q_j^v|^\alpha_2 \|h\|_{L^{\infty}(2^m Q_j)} \leq \inf_{x \in F_j^v} M_{\alpha_2,\overline{\mathcal{C}}}(h)(x),
\]
we have that
\[
J \leq c \sum_{v,j} \int_{F_j^v} M_{\alpha_1,D}(f v^{1/p})(x) M_{\alpha_2,\overline{\mathcal{C}}}(h)(x) dx
\]
\[
\leq c \int_{Q_0} M_{\alpha_1,D}(f v^{1/p})(x) M_{\alpha_2,\overline{\mathcal{C}}}(h)(x) dx.
\]

Now, since \(1/q' - 1/p' = \alpha\), if \(0 \leq \alpha_1, \alpha_2 \leq \alpha\) are so that \(\alpha_1 + \alpha_2 = \alpha\), there exist \(1 < s_1, s_2 < \infty\) such that
\[
1/p - \alpha_1 = 1/s_1 \quad \text{and} \quad 1/q' - \alpha_2 = 1/s_2.
\]
Since
\[
1/s_1 + 1/s_2 = 1/p - \alpha_1 + 1 - 1/q - \alpha_2 = 1/p - \alpha - 1/q + 1 = 1,
\]
s_1, s_2 are conjugate exponents, and, therefore, by Hölder’s inequality,
\[
\int_{Q_0} M_{\alpha_1,D}(f v^{1/p})(x) M_{\alpha_2,\overline{\mathcal{C}}}(h)(x) dx \leq \|M_{\alpha_1,D}(f v^{1/p})\|_{L^{s_1}} \|M_{\alpha_2,\overline{\mathcal{C}}}(h)\|_{L^{s_2}}.
\]
Now, by (iii) and (iv) in Proposition 6.1, \(D \in B_p^{\alpha_1}\) and \(\overline{\mathcal{C}} \in B_q^{\alpha_2}\), respectively, and, therefore, by (6.11) and Theorem 3.3 in [13],
\[
\int_{Q_0} M_{\alpha_1,D}(f v^{1/p})(x) M_{\alpha_2,\overline{\mathcal{C}}}(h)(x) dx \leq c \|f v^{1/p}\|_{L^p} 2^{-mn/q'} \|h\|_{L^{q'}(Q_0)}.
\]
and the right-hand side of (6.19) is bounded by
\[ c \left( \sum_{m=1}^{\infty} \lambda_m 2^{m(1-1/q')} \right) \|f\|_{L^p_t} \leq c \|f\|_{L^p_t}. \]
Hence, combining the above estimates,
\[ \|Tf - m_{Tf}(t, Q_0)\|_{L^{q'}_t(Q_0)} \leq c \|f\|_{L^p_t}. \]
Finally, by Fatou’s lemma, (6.17) follows for functions \( f \) such that \( m_{Tf}(t, Q_0) \to 0 \) as \( Q_0 \to \mathbb{R}^n \). □

Observe that \( B_{q'} \subset B_{(q/r)'} \cap B_{q''}^{q'} \) and that, as the function \( \Phi(t) = \left( \frac{t}{\log(t)^{(1+\epsilon)q'/s}} \right) \left( 1 + \frac{\epsilon}{q'/s} \right) \), where \( \epsilon < q'/s - 1 \) and \( 1 < q' \leq s < \infty \) is defined as \( 1/s = 1/q' - \alpha_2 \), shows that the inclusion is proper [14]. Now, in the case \( \omega(t) = t \), Theorem 6.6 holds in the full range for \( n < q < \infty \). This is in line with the case \( p = q \), where the result builds on Theorem 1.3 in [32], where Lerner proves that for \( n < p < \infty \), the full validity of a result anticipated by Cruz-Uribe and Pérez [15] holds for singular integrals with \( \omega(t) = t \); the sharpness of this result is discussed in [12,15] and Lerner has completed the case \( 1 < p \leq n \) in [35]. Still for \( p = q \), Theorem 6.6 in particular gives that if \( r = 1 \) and \( \omega(t) = t^\eta \) with \( 0 < \eta < 1 \), then the continuity holds for \( n/\eta < p < \infty \), and that in the case of kernels that satisfy a Dini condition, whenever \( \int_0^1 \omega_{c_n}(c_n t) \frac{1}{tn/p} \frac{dt}{t} < \infty \),
where \( 1 \leq r < \infty \).

7. Morrey spaces

For a Young function \( \Phi \) and a positive continuous function \( \phi(x, t) \) on \( \mathbb{R}^n \times \mathbb{R}^+ \) with \( \phi(x, 0) = 0 \) that increases for \( t \) in \( [0, \infty) \) for each \( x \in \mathbb{R}^n \), with \( Q = Q(x_Q, l_Q) \), let
\[ \|f\|_{\mathcal{M}^{\Phi, \phi}} = \sup_{Q(x_Q, l_Q) \subset \mathbb{R}^n} \phi(x_Q, l_Q) \inf_{c} \|f - c\|_{L^p_t(Q)}. \]
Note that if \( w = 1 \), \( \Phi(t) = t^p \), and \( \phi(x, t) = t^{n/p} \) for \( 1 \leq p \leq p_0 \), then \( \mathcal{M}^{\Phi, \phi} = \mathcal{M}^{p, p_0} \), the familiar Morrey space.

As for the Campanato spaces \( \mathcal{L}^{\Phi, \phi}_w \), consider the seminorms
\[ \|f\|_{\mathcal{L}^{\Phi, \phi}_w} = \sup_{Q(x_Q, l_Q) \subset \mathbb{R}^n} \phi(x_Q, l_Q) \inf_{c} \|f - c\|_{L^p_t(Q)}. \]
Although a priori the functions \( \Phi \) and \( \phi \) are unrelated, even in the simplest case there are some limitations [28]. In the unweighted case and when \( \phi \) is independent of \( x \), in order that the characteristic function of the unit cube belongs to \( \mathcal{M}^{\Phi, \phi} \) we assume that
\[ \sup_{t>1} \frac{\phi(t)}{t^{-1}(t^n)} < \infty. \]
As pointed out in (1.12) in the introduction, if $T$ is a Calderón-Zygmund singular integral operator that satisfies the conditions of Theorem 5.1 with $r = 1$, for every $w \in A_\infty$ and Young function $\Phi$,

$$
\|Tf\|_{L^p_w} \leq c \|Mf\|_{M^p_w}.
$$

The question is then to remove the maximal function on the right-hand side of the above inequality. Let $S$ be a sublinear operator such that for a weight $w$,

$$
\int_{\mathbb{R}^n} \Phi(|Sf(y)|) w(y) \, dy \leq c \int_{\mathbb{R}^n} \Phi(|f(y)|) w(y) \, dy,
$$

and for any cube $Q$, if $x \in Q$ and $\text{supp}(f) \subset \mathbb{R}^n \setminus 2Q$, then

$$
|Sf(x)| \leq \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^n} \, dy.
$$

Such operators are considered, for instance, in [23], and they include the Hardy-Littlewood maximal function as well as a variety of singular integral operators.

Let then $\Phi$ be a Young function with $p = 1/u_\Phi$ and $w \in A_p$. These weights satisfy condition $A_p$, namely, for every $\varepsilon > 0$ and cube $Q$, with $\Phi(t) = \int_0^t a(s) \, ds$,

$$
\left( \frac{1}{|Q|} \int_Q \varepsilon w(y) \, dy \right)a \left( \frac{1}{|Q|} \int_Q \varepsilon^{-1} \left( \frac{1}{\varepsilon w(y)} \right) \, dy \right) \leq c,
$$

which is equivalent to the integral inequality (7.1) for $S = M$, the Hardy-Littlewood maximal function [28].

Now, if $w \in A_p$, by Hölder’s inequality,

$$
\frac{1}{|Q|} \int_Q |f(y)| \, dy = \varepsilon \frac{w(Q)}{|Q|} \frac{1}{w(Q)} \int_Q |f(y)| \frac{1}{\varepsilon w(y)} w(y) \, dy
$$

$$
\leq 2 \varepsilon \frac{w(Q)}{|Q|} \|1/\varepsilon w\|_{L^\infty(Q)} \|f\|_{L^p_w(Q)}.
$$

We claim that for the choice $\varepsilon = \|1/w\|_{L^\infty(Q)}$,

$$
\varepsilon \frac{w(Q)}{|Q|} \leq c,
$$

with a constant $c$ independent of $Q$.

Indeed, since $\Phi(s) \sim s a^{-1}(s)$, we have

$$
1 \sim \frac{1}{w(Q)} \int_Q \Phi \left( \frac{1}{\varepsilon w(y)} \right) w(y) \, dy \sim \frac{1}{w(Q)} \int_Q a^{-1} \left( \frac{1}{\varepsilon w(y)} \right) \frac{1}{\varepsilon} \, dy,
$$

and therefore,

$$
a \left( \frac{1}{|Q|} \int_Q a^{-1} \left( \frac{1}{\varepsilon w(y)} \right) \, dy \right) \sim a \left( \varepsilon \frac{w(Q)}{|Q|} \right),
$$

which, by (7.3) gives

$$
\Phi \left( \frac{w(Q)}{|Q|} \right) \sim a \left( \varepsilon \frac{w(Q)}{|Q|} \right) \sim \varepsilon \frac{w(Q)}{|Q|} a \left( \frac{1}{|Q|} \int_Q a^{-1} \left( \frac{1}{\varepsilon w(y)} \right) \, dy \right) \leq c,
$$

and (7.5) holds.

Thus, (7.4) gives

$$
\frac{1}{|Q|} \int_Q |f(y)| \, dy \leq c \|f\|_{L^p_w(Q)}.
$$
We then have

**Theorem 7.1.** Let $S$ be a sublinear operator that satisfies (7.1) and (7.2), $\Phi$ a Young function so that $0 < u_\Phi = 1/p < 1$, $w \in A_p$, and $\phi(x,t), \psi(x,t)$ such that for all $x \in \mathbb{R}^n$ and $l > 0$,

\[
\psi(x,l) \int_l^\infty \frac{1}{\phi(x,t)} \frac{dt}{t} \leq c.
\]

Then

\[
\|Sf\|_{M^\Phi_w} \leq c \|f\|_{M^\Phi_w}.
\]

**Proof.** Fix a cube $Q = Q(x_Q, l_Q)$ of $\mathbb{R}^n$, and for a function $f \in M^\Phi_w$, let $f_1 = f1_{2Q}$ and $f_2 = f - f_1$. Then

\[
\|Sf\|_{L^\Phi_w(Q)} \leq \|Sf_1\|_{L^\Phi_w(Q)} + \|Sf_2\|_{L^\Phi_w(Q)}.
\]

Now, by (7.1),

\[
\int_Q \Phi(|Sf_1(y)|) w(y) \, dy \leq c \int_{\mathbb{R}^n} \Phi(|f_1(y)|) w(y) \, dy = c \int_{2Q} \Phi(|f(y)|) w(y) \, dy,
\]

and so

\[
\frac{1}{w(Q)} \int_Q \Phi(|Sf_1(y)|) w(y) \, dy \leq c \frac{1}{w(2Q)} \int_{2Q} \Phi(|f(y)|) w(y) \, dy
\]

which readily gives

\[
\|Sf_1\|_{L^\Phi_w(Q)} \leq c \|f\|_{L^\Phi_w(2Q)} \leq c \int_{2l_Q} \|f\|_{L^\Phi_w(Q(x_Q,t))} \frac{dt}{t}.
\]

We deal with the term with $f_2$ next. Note that for $x \in Q$ and $y \notin 2Q$, $|x - y| \sim |x_Q - y|$, and therefore

\[
|Sf_2(x)| \leq c \int_{\mathbb{R}^n \setminus 2Q} \frac{|f(y)|}{|x_Q - y|^n} \, dy.
\]

Now, by Fubini’s theorem,

\[
\int_{\mathbb{R}^n \setminus 2Q} \frac{|f(y)|}{|x_Q - y|^n} \, dy \leq c \int_{\mathbb{R}^n \setminus 2Q} |f(y)| \int_{|x_Q - y|}^\infty \frac{1}{t^n} \frac{dt}{t} \, dy
\]

\[
\leq c \int_{2l_Q}^\infty \int_{Q(x_Q,t) \setminus Q(x_Q,2l_Q)} |f(y)| \, dy \frac{1}{t^n} \frac{dt}{t}
\]

\[
\leq c \int_{2l_Q}^\infty \frac{1}{|Q(x_Q,t)|} \int_{Q(x_Q,t)} |f(y)| \, dy \frac{dt}{t},
\]

and consequently by (7.6),

\[
|Sf_2(x)| \leq c \int_{2l_Q}^\infty \|f\|_{L^\Phi_w(Q(x_Q,t))} \frac{dt}{t}.
\]

Moreover, since for every $\Phi, w, Q$, $\|g\|_{L^\Phi_w(Q)} \leq c \|g\|_{L^\infty(Q)}$, it follows that

\[
\|Sf_2\|_{L^\Phi_w(Q)} \leq c \int_{2l_Q}^\infty \|f\|_{L^\Phi_w(Q(x_Q,t))} \frac{dt}{t},
\]

which combined with (7.8) gives

\[
\|Sf\|_{L^\Phi_w(Q)} \leq c \int_{2l_Q}^\infty \|f\|_{L^\Phi_w(Q(x_Q,t))} \frac{dt}{t}.
\]
Therefore by (7.7),
\[ \|Sf\|_{L^p_w(Q)} \leq c \int_{2Q}^\infty \frac{1}{\phi(x_Q, t)} \, dt \|f\|_{L^p_w(Q(x_Q, t))} \frac{1}{\phi(x_Q, t)} \, dt \]
\[ \leq c \left( \int_{2Q}^\infty \frac{1}{\phi(x_Q, t)} \, dt \right) \|f\|_{M^p_w, \phi} \]
\[ \leq c \|f\|_{M^p_w, \phi} \psi(x_Q, 2t_Q) \cdot \]
\[ (7.9) \]

Since \( \psi \) is increasing, from (7.9) it follows that \( \psi(x_Q, t_Q) \|Sf\|_{L^p_w(Q)} \leq c \|f\|_{M^p_w, \phi} \) and so, taking the supremum over \( Q \),
\[ \|Sf\|_{M^p_w, \phi} \leq c \|f\|_{M^p_w, \phi}. \]

The proof is thus complete. \( \Box \)

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Department of Mathematics, Indiana University, Bloomington, Indiana 47405
E-mail address: jpoelhui@indiana.edu

Department of Mathematics, Indiana University, Bloomington, Indiana 47405
E-mail address: torchins@indiana.edu