CHAOTIC DYNAMICS IN A TRANSPORT EQUATION ON A NETWORK

PROSCOVIA NAMAYANJA
School of Mathematics, Statistics & Computer Sciences
University of KwaZulu-Natal
Private Bag X54001, Durban 4001, South Africa

Abstract. We show that for a system of transport equations defined on an infinite network, the semigroup generated is hypercyclic if and only if the adjacency matrix of the line graph is also hypercyclic. We further show that there is a range of parameters for which a transport equation on an infinite network with no loops is chaotic on a subspace $X_e$ of the weighted Banach space $\ell_1$. We relate these results to Banach-space birth-and-death models in literature by showing that when there is no proliferation, the birth-and-death model is also chaotic in the same subspace $X_e$ of $\ell_1$. We do this by noting that the eigenvalue problem for the birth-and-death model is in fact an eigenvalue problem for the adjacency matrix of the line graph (of the network on which the transport problem is defined) which controls the dynamics of the the transport problem.

1. Introduction. Chaotic dynamics in birth-and-death models has been the subject of several papers in the recent past, see [1], [3], [2]. In these papers, topological chaos was investigated in Kolmogorov birth-and-death models with proliferation. In particular, the dynamics is described by infinite first order differential equations with constant coefficients

$$f'_1 = af_1 + df_2, \quad f'_n = af_n + bf_{n-1} + df_{n+1}, \quad n \geq 2.$$  \hspace{1cm} (1)

In [3], conditions on $a, b, d$ were determined for existence of chaos in (1) and these results were later generalised in [1]. Here, matter from state $i$ can move to state $i + 1$ at a rate given by $d$, or to state $i - 1$ at a rate $b$ or remain in state $i$ at a rate $a$. The birth-and-death process described by Equation (1) can be described graphically as follows, with the exception that in the graph, there is no proliferation (otherwise the graph would have loops at each vertex). Models of the form (1) have

![Figure 1. A graph of the Birth-death model with no proliferation](image-url)

**Figure 1.** A graph of the Birth-death model with no proliferation
applications in biology as well as in traffic control. Occurrence of distributional chaos in a traffic flow model related to the system in (1) was studied in [7], and in another related model in [8]. We now define some of the important matrices which will be used in this study.

**Definition 1.1.** Let $G$ be a directed graph. The weighted outgoing incidence matrix of $G$, $\Phi^{-w}$, and the incoming incidence matrix of $G$, $\Phi^{+}$, are matrices given by

$$
\begin{align*}
(\phi^{-w})_{ij} &= \begin{cases} 
w_{ij} & \text{if } v_i \xrightarrow{e_j} 0 \\
0 & \text{otherwise}
\end{cases} \\
\phi^{+}_{ij} &= \begin{cases} 
1 & \text{if } e_j \xrightarrow{e_i} v_i \\
0 & \text{otherwise}
\end{cases}
\end{align*}
$$

The weighted adjacency matrix of graph $G$, labeled $L$, is then given by $L = (\Phi^{+}(\Phi^{-w})^{T})$.

We note that the coefficient matrix, $L$, associated with (1) is the weighted adjacency matrix of a directed graph similar to Figure 1, with the graph containing self loops at all the vertices if $a \neq 0$. This matrix was essential in proving chaotic dynamics in system (1) in literature (see [1] and [3]). We define another matrix which will be specifically used in studying the dynamics of system of transport equations on some graph $G$.

**Definition 1.2.** The line graph of $G$ is the graph $L(G) = (V(L), E(L))$, where $V(L) = E(G)$, the edge set of $G$ and the edge set of $L(G)$ is

$$
E(L) = \{e_{ij} \mid \text{there exist } e_i, e_j \text{ in } G \text{ such that the head of } e_i \text{ coincides with the tail of } e_j\}.
$$

The adjacency matrix, $B$, of $L(G)$ is the matrix defined as $B = (b_{ij})$, where

$$
b_{ij} = \begin{cases} 
1 & \text{if there exists } v_k \in V(G) \text{ such that } e_j \xrightarrow{e_i} v_k \xrightarrow{e_j}, \\
0 & \text{otherwise}.
\end{cases}
$$

The line graph $L(G)$ has a weighted adjacency matrix $B_w = (\Phi^{-w})^{T} \Phi^{+}$.

1.1. **Statement of the problem.** In this paper, we investigate existence of chaos in a flow problem on an infinite directed graph $G$, where the flow of material along the edges of a graph is described by a simple transport equation with a Kirchoff-type boundary condition, below

$$
\begin{align*}
\partial_t (u_j)(x,t) &= \partial_x u_j(x,t), \\
u_j(x,0) &= f_j(x), \\
\phi_{ij} u_j(1,t) &= w_{ij} \sum_{k \in N} \phi_{kj} u_j(0,t) \forall i,j \in N.
\end{align*}
$$

Here, $u_j(x,t)$ represents the density of matter flowing along edge $e_j$ at time $t \geq 0$ and at position $x \in [0,1]$. We will assume that our graphs are simple, that is, they have no multiple edges or loops, and that every edge has been normalised to have a unit length (hence $0 \leq x \leq 1$), and we follow notation in [9], [10], so that the head of an edge is at position $x = 0$ and the tail is at $x = 1$. In other words, if a substance is flowing along edge $e_j$, it will start at position $x = 1$, the tail of $e_j$, and leave the edge at position $x = 0$ when it enters another vertex. We will also assume that there is no build up of matter at the vertices, meaning that it is immediately redistributed through the outgoing edges as soon as it enters a vertex from the incoming edges.

Existence and uniqueness of solutions in the space $\ell^1$ to this problem has been studied in [9] (on an infinite graph), and in [10], [13] on finite graphs, with the
weights having the restriction $\sum_{j \in N} w_{ij} = 1$, and the explicit formula for the solution was given to be

$$T(t)f(x) = B^n_w f(t + x - n), \quad t > 0, n \in \mathbb{N}_0, 0 < t + x - n < 1,$$

(4)

where $B_w$ is the weighted adjacency matrix of the line graph of Figure 1. If this condition on the weights is maintained, the semigroup generated by the flow problem cannot be chaotic (in the sense of Devaney) as the related matrix, $B_w$, is Markovian.

In [4], amplification and/or destruction of matter in the vertices was allowed, which is similar to dropping the restriction $\sum_{j} w_{ij} = 1$ on the weights. Existence and uniqueness of solutions in $W^1([0,1])^m$, where $m<\infty$ was the number of edges, for this more general model was proved.

In this paper, we will also drop the assumption that $\sum_{j \in N} w_{ij} = 1$ and we will consider dynamics in the space $\ell^1$ with norm $\|f\|_1 = \sum_{i \in N} |f_i| < \infty$, and the weighted space $\ell^1_s$ of infinite sequences summable with weight $s_n = (s_n)_{n \in N}$ with $s_n > 0$ for all $n$. The norm for elements in $\ell^1_s$ is defined as

$$\|f\|_s = \sum_{n \in N} |f_n| s_n < \infty.$$  

(5)

As previously shown ([4], [9], [10]), the flow problem above can be written in an abstract way as

$$\begin{aligned}
\begin{cases}
  u'(t) = Au(t) \\
  u(0) = f,
\end{cases}
\end{aligned}$$

(6)

where

$$D(A) = \{ u \in W^{1,1}([0,1], \ell^1)|u(1, t) = B u(0, t) \},$$

where $A$ is the differential operator on the domain $D(A)$. We will show that the flow problem in 3 is chaotic on a subspace of $\ell^1$ but not on the whole space. This then would imply that the birth-and-death model in (1) with no proliferation is also chaotic on the same subspace, since $B_w$ has a close relationship with the coefficient matrix $L$ of system (1).

**Remark 1.** We note that the semigroup in Equation (4) is also the solution (under certain conditions), to the abstract Cauchy problem (6) where the matrix in the domain ($B$) is non-negative but is not the adjacency matrix of a line graph (see [5]). In such a case, the flow problem (3) is still considered to be on a non-physical network, as is the case in the Rotenberg-Rubinov-Lebowitz model, citeBFN16. The results of the next section would still apply to such networks as well.

2. **Hypercyclicity and chaos.** We begin by noting that the semigroup generated by the transport problem in (3) on the space $\ell^1_s$ is given by (4), and the proof is similar to that given in [9], [10] and in [4]. Here, we restrict ourselves to the space $\ell^1_s$ to ensure that the iterates $B^n_w f(\cdot)$ are contained in the space $\ell^1_s$. We relate chaoticity and hypercyclicity of the $C_0$ semigroup generated by (6) to the chaotic behaviour of $B_w$ in the following results. It was shown in [9], [10] that $T(1)f(x) = B_w x$, where $T(t), t > 0$ is the semigroup generated by the flow problem and the (point) spectrum of $B_w$ is the same as that of $T(1)$. We will use this result to prove the following two results.

**Theorem 2.1.** The semigroup $T(t)_{t \geq 0}$ in (4) is hypercyclic on $L^1([0,1], \ell^1_s)$ if and only if $B_w$ is hypercyclic.
Proof. Suppose that $\mathbb{B}_w$ is hypercyclic. Then there exists a nonzero vector $y \in \ell_1^+$ such that the closure of the orbit
\begin{equation}
\{y, \mathbb{B}_w y, \mathbb{B}_w^2 y, \ldots\}
\end{equation}
is all $\ell_1$. This, in particular, means that $T(1)$ is hypercyclic since $T(1)f(x) = \mathbb{B}_w f(x)$, [9]. Next, divide the interval $[0,1]$ into non-overlapping subintervals $I_0, I_1, \ldots, I_k$ (for some large but finite $k$) such that $\bigcup_{i=0}^k I_i = [0,1]$ and define $g$ as
\begin{equation}
g(x) := \mathbb{B}_w^j y \text{ if } x \in I_j \text{ for } j \in [0,k];
\end{equation}
where $y$ is the vector in (7). So $g(\cdot)$ is a step function depending on $x$ whose space derivative is 0. With $g$ defined this way, we claim that the closure of the set
\begin{equation}
\{g(x), \mathbb{B}_w g(x), \mathbb{B}_w^2 g(x), \ldots\}
\end{equation}
is the whole of $L^1([0,1], \ell_1^+)$, Indeed, for any fixed $x$, the density of the set in (8) in $\ell_1$ is inherited from the density of the set in (7) and any function in $L^1([0,1])$ can be approximated by an arbitrary collection of step functions. So the set in (8) is dense in $L^1([0,1])$. Since $\{T(1)f(x) = \mathbb{B}_w f(x)\}$, and going back to the formula (4), we see that whenever $t \in \mathbb{N}$, then $n = t$ so that the formula reduces to
\begin{equation}
(T(t)f)(x) = T(n)f(x) = \mathbb{B}_w^n f(x) = T(1)^n f(x).
\end{equation}
Thus, the orbit of $T(1)$, given by
\begin{equation}
\text{Orb}(g(x), T(1)) = \{g(x), T(1)g(x), T(1)^2 g(x), \ldots\} = \{g(x), \mathbb{B}_w g(x), \mathbb{B}_w^2 g(x), \ldots\}
\end{equation}
is dense in $\ell_1^+$ and in $L^1([0,1])$. Also note that the orbit of $T(t), t \in \mathbb{N}_2$ is
\begin{equation}
\text{Orb}(g, T(t)) = \{g(x), T(t)g(x), (T(t))^2 g(x), \ldots\}
\end{equation}
\begin{equation}
= \{g(x), \mathbb{B}_w^t g(x), \mathbb{B}_w^2 g(x), \mathbb{B}_w^3 g(x), \ldots\}
\end{equation}
which means that for $t \in \mathbb{N}_2$, $\text{Orb}(g, T(t)) = \text{Orb}(g, T(1)) - \{\mathbb{B}_w^l : l = 1, \ldots, t - 1\}$ is also dense in $L^1([0,1], \ell_1^+)$. Thus,
\begin{equation}
\text{Orb}(g(x), T(t) : t \in \mathbb{N}_2) \subseteq \text{Orb}(g(x), T(1)) \subset \text{Orb}(g(x), T(t) : t \in \mathbb{R}^+)
\end{equation}
and since the former is dense in $L^1([0,1], \ell_1^+)$, we see that $\text{Orb}(g(x), T(t) : t \in \mathbb{R}^+)$ is dense in $L^1([0,1], \ell_1^+)$. Thus the semigroup $(T(t))_{t \geq 0}$ is hypercyclic.

Conversely, suppose that $(T(1))_{t \geq 0}$ in (4) is hypercyclic on $L^1([0,1], \ell_1^+)$. Then it is topologically transitive. That is, there exist nonempty sets $U, V \subseteq L^1([0,1], \ell_1^+)$ such that
\begin{equation}
(T^m(V)) \cap U \neq \emptyset; \forall t \geq 0
\end{equation}
for some $m \in \mathbb{N}$. In particular, this is true for $t = 1$. Hence, $(T^m(1)V) \cap U \neq \emptyset$. But this last step implies that $\mathbb{B}_w$ is topologically transitive, hence hypercyclic. \qed

Theorem 2.2. The semigroup $(T(t))_{t \geq 0}$ is chaotic if matrix operator $\mathbb{B}_w$ is chaotic.

Proof. Suppose that $\mathbb{B}_w$ has a dense set of periodic points and define
\begin{equation}
\text{Per}(\mathbb{B}_w) := \{y_i \in \ell_1 : \mathbb{B}_w^n y_i = y_i, n_i \in \mathbb{N}\}.
\end{equation}
Then $\text{Per}(\mathbb{B}_w) = \ell_1^+$. Let $I_1, I_2, \ldots, I_k$ be open subintervals of $[0,1]$ with $I_i \cap I_j = \emptyset$, whenever $i \neq j$. Define $U$ to be a subset of $X = L^1([0,1], \ell_1^+)$ such that for every $f \in U, f(x) = y_i$ if $x \in I_i$ for some $y_i \in \text{Per}(\mathbb{B}_w)$. We claim that every $f \in U$ is periodic for $T(t)$. In particular, we note that if $n = n_1 n_2 \cdots n_k$, and letting $t = 1$, we find that
\begin{equation}
T^n(1)f(x) = (T(n_1 n_2 \cdots n_k) f)(x) = \mathbb{B}_w^{n_1 n_2 \cdots n_k} f(x) = f(x), \quad \forall f(x) \in U.
\end{equation}
Hence every \( f \in U \) is periodic for \( T(1) \) with natural period, hence periodic for every \( t \geq 0 \). Next, we show that such a set \( U \) is dense in \( L^1([0,1], \ell^1) \). From the construction of the space \( L^1 \), we immediately see that \( U \) is dense in \( L^1 \) since any function in \( L^1 \) can be approximated by an arbitrary collection of step functions.  

We could not make the result in Theorem (2.2) stronger but a quick check (from the definition of a periodic point) enables one to see that if the \( C_0 \) semigroup in (4) is periodic with integer period, then the operator matrix \( B_w \) is periodic as well, with the same period and with the same periodic points.

3. An example of chaos in a simple transport model. In this section, we restrict ourselves to the flow problem on the network shown in figure 1, which represents birth-and-death. Since matrix \( B_w \) is influential in network flow of the form (3), we will take a closer look at this matrix. Drawing \( L(G) \), we find that the adjacency matrix \( B \) of \( L_G \) is given by

\[
B = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & \cdots \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & \cdots \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & \cdots \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & \cdots \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & \cdots \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\
\end{pmatrix}
\]

Since the nonzero pattern of \( B \) is the same as that of \( B_w \), we will have a closer look at \( B \), the unweighted matrix first.

**Proposition 1.** The point spectrum of the infinite matrix operator \( B \) in \( \ell^1 \) is contained in the unit circle.

**Proof.** Solving the eigenvalue equation \( Bf = \lambda f \), we find that the eigenvector \( f = (f_1, f_2, \ldots) \), if it exists, must satisfy the difference equations \( \lambda f_{k} = f_{k-1} + f_{k+2} \) for \( k \geq 2 \), \( k \) even and \( \lambda f_{k} = f_{k+1} + f_{k-2} \), with \( k > 1 \), \( k \) odd and \( \lambda f_1 = f_2 \). These are third order linear difference equations with complex parameter \( \lambda \) for which the direct sum of their solutions gives the eigenvector \( f \). We seek for conditions on \( \lambda \), if any exist, for which solutions to the difference equations exist in \( \ell^1 \). We notice from these equations that \( f_2 = f_3, f_4 = f_5, f_6 = f_7 \) and so on. Setting \( f_1 = 1 \), we find that \( f_2 = \lambda = f_3 \), and substituting these in the difference equations and simplifying gives vector \( f \) whose terms are

\[
f_{nk} = \lambda^{2k} - \binom{2n-1}{1} n \in \mathbb{N} \lambda^{2k-2} + \binom{2n}{2} n \in \mathbb{N} \lambda^{2k-4} - \binom{2n-1}{3} n \in \mathbb{N}_2 \lambda^{2k-6} + \binom{2n}{4} n \in \mathbb{N}_2 \lambda^{2k-8} - \binom{2n-1}{5} n \in \mathbb{N}_3 \lambda^{2k-10} + \binom{2n}{6} n \in \mathbb{N}_3 \lambda^{2k-12} + \cdots + \lambda^0,
\]

where \((n_k) = 4k + 1, k \in \mathbb{N}_0\). Defining the sequence \((n_t)_{t \in \mathbb{N}} = 4t - 1\), we obtain the other part of the solution as

\[
f_{nt} = \lambda^{2t-1} - \binom{2n}{1} n \in \mathbb{N} \lambda^{2t-3} + \binom{2n+1}{2} n \in \mathbb{N} \lambda^{2t-5} - \binom{2n}{3} n \in \mathbb{N}_2 \lambda^{2t-7}
\]
Hence, the subsequence \( f \) as we can see, the sum of the terms in the sequence will diverge when \( |\lambda| \geq 1 \). Convergence of the eigenvector \( \lambda = (f_n)_{n \in \mathbb{N}} \) is now analyzed using the subsequences \( f_{n_k} \) and \( f_{n_l} \) above. For \( (f_n)_{n \in \mathbb{N}} \) to be in \( \ell^1 \), we require that \( \sum_{i \in \mathbb{N}} |f_i| < \infty \). We also note that both subsequences

\[
(f_{a_{n_k}})_{k \in \mathbb{N}_0, n \neq N_0} : a_{n_k} = 4k + 1 \quad \text{and} \quad (f_{a_{n_l}})_{l \in \mathbb{N}, n \neq N_0} : a_{n_l} = 4l + 1
\]

must also be convergent in \( \ell^1 \). Now if

\[
\sum_{t = 4n-1; n \in \mathbb{N}} f_t \not\to \infty
\]

then the sequence \( (f_n)_{n \in \mathbb{N}} \) must be divergent in \( \ell^1 \), which would imply that \( \mathbb{B} \) has an empty point spectrum. Summing the terms of the subsequence \( f_3, f_7, f_{11}, f_{15}, \cdots \) and simplifying, we get

\[
\sum_{a_{n_t} = 4t+1} f_{a_{n_t}} = \sum_{n \in \mathbb{N}_0} \lambda^{2n+1} - \sum_{n \in \mathbb{N}_1} 2n\lambda^{2n} + \sum_{n \in \mathbb{N}} \left( \frac{2n+1}{2} \right) \lambda^{2n-1} - \sum_{n \in \mathbb{N}_2} \left( \frac{2n+1}{4} \right) \lambda^{2n-3} - \cdots
\]

\[
= \sum_{n \in \mathbb{N}_0} \lambda^{2n+1} + \sum_{n \in \mathbb{N}} \left( \frac{2n+1}{2} \right) \lambda^{2n-1} + \sum_{n \in \mathbb{N}_2} \left( \frac{2n+1}{4} \right) \lambda^{2n-3} + \sum_{n \in \mathbb{N}_3} \left( \frac{2n+1}{6} \right) \lambda^{2n-5} + \cdots
\]

\[
= \sum_{n \in \mathbb{N}_0} \lambda^{2n+1} + \sum_{n \in \mathbb{N}} \frac{2}{2!} n(2n-1) \lambda^{2n-1} + \sum_{n \in \mathbb{N}_2} \frac{2^2}{4!} n(n-1)(2n-1)(2n-3) \lambda^{2n-3} + \sum_{n \in \mathbb{N}_3} \frac{2^3}{6!} (2n-5)(2n-3)(2n-1)(n-1)(n-2) \lambda^{2n-5} + \sum_{n \in \mathbb{N}_4} \frac{2^4}{8!} \prod_{k=1}^4 (2n - (2k-1)) n \prod_{l=1}^3 (n-l) \lambda^{2n-7} + \cdots
\]

As we can see, the sum of the terms in the sequence will diverge when \( |\lambda| \geq 1 \). Hence, the subsequence \( f_3, f_7, f_{11}, \cdots = (f_{a_{n_t}})_{t \in \mathbb{N}, n \neq N_0} : a_{n_t} = 4t - 1 \) is not in \( \ell^1 \) when \( |\lambda| \geq 1 \), implying that the entire sequence \( (f_{a_n})_{n \in \mathbb{N}_0} \) cannot be in \( \ell^1 \). A similar argument holds when we consider sums of terms of the subsequence \( f_1, f_5, f_9, f_{13}, \cdots \).
Consequently, the point spectrum of $B$, if non-empty, is contained in the unit circle.

**Lemma 3.1.** The point spectrum of $B$ is non-empty. Furthermore, $\lambda = m$, $m \in \mathbb{R}$ such that $|m| < 1$ is an eigenvalue of $B$.

**Proof.** We point out that $\lambda = 0$ is an eigenvalue of $B$. Solving $Bf = 0$, we note that vector $f$ must be of the form

$$f = (f_1, 0, f_3, -f_1, f_5, -f_3, f_7, -f_5, f_9, -f_7, \ldots).$$

Setting $f_{2n-1} = \frac{1}{n^2}$, $n \in \mathbb{N}$, we see right away that $f \in \ell^1$ since

$$\sum_{n \in \mathbb{N}} |f_n| = 2 \sum_{n \in \mathbb{N}} \frac{1}{n^2} < \infty,$$

hence $f$ is an eigenvector of $B$. This clearly shows that the point spectrum of $B$ is non-empty. Indeed, there are infinitely many eigenvectors of $B$ corresponding to $\lambda = 0$, for example, choosing $r > 1$, and setting $f_{2n-1} = (\frac{1}{r})^n$ gives an eigenvector of $B$. Similarly, we can set $f_{2n-1} = e^{-n}$, $n \in \mathbb{N}$, or $f_{2n-1} = e^{r\pi}$, $n \in \mathbb{N}$, and all these will give other eigenvectors of $B$ in $\ell^1$ corresponding to $\lambda = 0$.

Furthermore, substituting $\lambda = m$ in Equations (9) and (10), we find that the series

$$\sum_{t=4n-1}^{4n+1} f_t \text{ and } \sum_{k=4n+1}^{4n+3} f_k$$

converge absolutely and note that since all powers of $\lambda$ in Equation (9) are even, the terms of the subsequence $f_{ak} \in \mathbb{R}$, $a_k = 4k + 1$, $k \in \mathbb{N}_0$ are all real while $f_{a_t}, a_t = 4t - 1$, $t \in \mathbb{N}_0$ are all purely imaginary. Hence vector $f$ which satisfies $Bf = mf$ exists in $\ell^1$, for $|m| < 1$, $m \in \mathbb{R}$. \hfill \Box

### 3.1. Point spectrum of the weighted matrix operator $B_w$.

Now suppose that we placed weights $w_{ij}$ on the outgoing edges of the graph in Figure 1. That is, if $e_j$ is an outgoing edge of vertex $v_i$, it is assigned a weight $w_{ij}$. The weighted adjacency matrix of $L(G)$, $B_w$, is then given by

$$B_w = \begin{pmatrix}
0 & w_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
0 & 0 & w_{22} & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & w_{23} & 0 & 0 & 0 & 0 & 0 & \ldots \\
0 & 0 & w_{34} & 0 & 0 & w_{34} & 0 & 0 & 0 & \ldots \\
0 & 0 & w_{35} & 0 & 0 & w_{35} & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & 0 & w_{46} & 0 & 0 & w_{46} & 0 & \ldots \\
0 & 0 & 0 & 0 & w_{47} & 0 & 0 & w_{47} & 0 & \ldots \\
0 & 0 & 0 & 0 & 0 & w_{58} & 0 & 0 & w_{58} & \ldots \\
0 & 0 & 0 & 0 & 0 & 0 & w_{59} & 0 & 0 & w_{59} & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots 
\end{pmatrix}.$$ 

Solving $B_w f = \lambda f$ gives the (coupled) difference equations

$$w_{n,2n-1}(f_{2n-3} + f_{2n}) = \lambda f_{2n-1} \quad \forall n \geq 2,$$

and

$$w_{n,2n-2}(f_{2n-3} + f_{2n}) = \lambda f_{2n-2} \quad \forall n \geq 2,$$

subject to the initial condition

$$w_{11} f_2 = \lambda f_1.$$ 

(14)
We note that these are third order difference equations whose characteristic equations are, respectively,

\[ z^3 - \sigma(n)z^2 + 1 = 0 \] \hspace{1cm} (15)
\[ z^3 - \delta(n)z + 1 = 0, \] \hspace{1cm} (16)

where \( \sigma(n) \) and \( \delta(n) \) are complex parameters related by the equation

\[ \sigma(n)w_{n,2n-1} = \delta(n)w_{n,2n-2}. \] \hspace{1cm} (17)

Solving the difference equations above using the characteristic equations in (15) and (16) is quite easy, thanks to the formula for the third order polynomial derived in the 1500s by Descartes. However the expressions are cumbersome and studying the convergence of the solution in \( \ell_1 \), even with fixed \( \delta(n) = \delta \) and \( \sigma(n) = \sigma \) for all \( n \), becomes a problem. So we try another approach by noticing that the solution to the difference equations is quite easy to state, thanks to the special nature of the zero pattern of \( \mathcal{B} \), as follows. Dividing Equation (12) by (13) and rearranging, we get

\[ f_{2n-1} = \frac{w_{n,2n-1}}{w_{n,2n-2}} f_{2n-2}; \forall n \geq 2. \]

For simplicity, we set \( w_{n,2n-1} = \mu_1 \), for \( n \in \mathbb{N} \) and \( w_{n,2n-2} = \mu_2, \forall n \in \mathbb{N}_2 \). Then Equations (13) and (14) give the simplified solutions \( f_2 = \frac{\lambda}{\mu_1} f_1 \),

\[ f_4 = \left( \frac{\mu_2}{\mu_1} \right)^2 \left( \frac{\mu_2}{\mu_1} \right)^{-1}, f_6 = \frac{\mu_2}{\mu_1} \left( \frac{\lambda}{\mu_1} \right)^3 - 2\lambda \left( \frac{\mu_2}{\mu_1} \right), \]
\[ f_8 = \frac{\mu_2}{\mu_1} \left( \frac{\lambda}{\mu_1} \right)^4 - 3 \left( \frac{\lambda}{\mu_1} \right)^2 + \frac{\mu_2}{\mu_1} f_{10} = \frac{\mu_2}{\mu_1} \left( \frac{\lambda}{\mu_1} \right)^5 - 4 \left( \frac{\lambda}{\mu_1} \right)^3 + 3 \frac{\mu_2}{\mu_1} \frac{\lambda}{\mu_2}, \cdots \]

This can be greatly simplified to the equations already seen in (9) and (10) by using the map \( \tilde{\lambda} \mapsto \frac{\lambda}{\sqrt{\mu_1 \mu_2}} \). Then for \( n = 4k, k \in \mathbb{N} \), we have

\[ f_{4k} = \left( \frac{\mu_1}{\mu_2} \right)^{\frac{3}{2} - 1} \left[ \lambda^{2k} - \binom{2n-1}{1} \sum_{n \in \mathbb{N}} \binom{2n}{2} \sum_{n \in \mathbb{N}} \lambda^{2k-4} \right. \]
\[ - \binom{2n-1}{3} \sum_{n \in \mathbb{N}} \binom{2n}{4} \sum_{n \in \mathbb{N}} \lambda^{2k-8} \]
\[ - \binom{2n-1}{5} \sum_{n \in \mathbb{N}} \binom{2n}{6} \sum_{n \in \mathbb{N}} \lambda^{2k-12} \cdots + \lambda^{0} \], \hspace{1cm} (18)

and now performing the transformation \( \tilde{f}_{4k} \mapsto \left( \frac{\mu_2}{\mu_1} \right)^{\frac{3}{2} - 1} f_{4k} \), we find that \( (\tilde{f}_{4k})_{k \in \mathbb{N}} \) satisfies Equations (9). Likewise, for \( n = 4t - 2, t \in \mathbb{N} \), we apply the map \( \tilde{\lambda} \mapsto \frac{\lambda}{\sqrt{\mu_1 \mu_2}} \) followed by \( \tilde{f}_{4t-2} \mapsto \left( \frac{\mu_2}{\mu_1} \right)^{1 - \frac{3}{2}} \tilde{f}_{4t-2} = \left( \frac{\mu_2}{\mu_1} \right)^{\frac{1}{2} - \frac{3}{2}} \tilde{f}_{4t-2} \), we find that \( \tilde{f}_{4t-2}, t \in \mathbb{N} \) satisfies Equation (10).

**Remark 2.** There is a relationship between the eigenvalue problem \( \mathcal{B}_w f = \lambda f \) with \( w_{n,2n-1} = \mu_1, w_{n,2n-2} = \mu_2 \) and the eigenvalue problem for the birth-and-death model in system (1). To see this, set \( a = 0, b = \mu_1 \) and \( d = \mu_2 \) in system (1) and note that the coefficient matrix of (1) is the weighted adjacency matrix of Figure 1, which can also be expressed in terms of matrices \( \Phi^+, \Phi^- \) as \( L = \Phi^+ (\Phi^-)^T \).

With this in mind, we then state that \( \mathcal{B}_w \) and \( L \) have the same point spectrum. Indeed suppose that \( y \neq 0 \) is such that \( L y = \alpha y \) for some \( \alpha \in \mathbb{C}, y \in \ell_1 \). Then \( (\Phi_w)^T \Phi^+ (\Phi^-)^T y = \mathcal{B}_w ((\Phi^-)^T y) = \alpha (\Phi^-)^T y \), thus, \( \alpha \) is also an eigenvalue of
We immediately note that if $B_m$ where $\sum$ has exactly one non-zero entry in every row, hence $(\Phi - \ell B_y)$ whenever $y \in \ell^1$. This realisation would then imply that if $B_w$ is chaotic on $\ell^1$ or in $\ell^1$, then the birth-and-death model in (1) with no proliferation would also be chaotic on $\ell^1$ ($\ell^1$, respectively).

**Lemma 3.2.** Suppose that $\mu_1, \mu_2 > 0$ and let $\mu_1 < \mu_2$ and $|\lambda| < \mu_2$. Then the matrix operator $B_w$ has a non-empty point spectrum in $\ell^1$ of the form $\sigma_p(B_w) := |\mu_2(\epsilon^{-k}) \cup \{0\}$, where $k$ is any complex number with positive real part.

**Proof.** Using similar arguments as in the previous section, we note that

$$
\sum_{i=4t+1, t \in \mathbb{N}_0} f_i = \sum_{n \in \mathbb{N}_0} \lambda^{2n} - \frac{\mu_1}{\mu_2} \sum_{n \in \mathbb{N}_1} (2n-1) \left( \frac{\lambda}{\mu_2} \right)^{2n-2} \\
+ \left( \frac{\mu_1}{\mu_2} \right)^2 \sum_{n \in \mathbb{N}} \left( \frac{2n}{2} \right) \left( \frac{\lambda}{\mu_2} \right)^{2n-2} \\
- \left( \frac{\mu_1}{\mu_2} \right)^3 \sum_{n \in \mathbb{N}_2} \left( \frac{2n-1}{3} \right) \left( \frac{\lambda}{\mu_2} \right)^{2n-4} \\
+ \left( \frac{\mu_1}{\mu_2} \right)^4 \sum_{n \in \mathbb{N}_2} \left( \frac{2n-1}{4} \right) \left( \frac{\lambda}{\mu_2} \right)^{2n-4} + \cdots
$$

$$
= \sum_{n \in \mathbb{N}_0} \left( \frac{\lambda}{\mu_2} \right)^{2n} + \frac{\mu_1}{2! \mu_2} \sum_{n \in \mathbb{N}} (2n-1) \left( \frac{2n \mu_1}{\mu_2} - 2 \right) \left( \frac{\lambda}{\mu_2} \right)^{2n-2} \\
+ \frac{1}{4!} \left( \frac{\mu_1}{\mu_2} \right)^3 \sum_{n \in \mathbb{N}_2} \prod_{k=1}^3 (2n-k) \left( \frac{2n \mu_1}{\mu_2} - 4 \right) \left( \frac{\lambda}{\mu_2} \right)^{2n-4} \\
+ \frac{1}{6!} \left( \frac{\mu_1}{\mu_2} \right)^5 \sum_{n \in \mathbb{N}_2} \prod_{k=1}^5 (2n-k) \left( \frac{2n \mu_1}{\mu_2} - 6 \right) \left( \frac{\lambda}{\mu_2} \right)^{2n-6} + \cdots
$$

We immediately note that if $|\lambda| > \mu_2$, the sum $f_1 + f_3 + f_5 + \cdots$ diverges; hence $\sum |f_i|$ will also diverge. The same is true if $\mu_1 > \mu_2$. Therefore, $B_w$ has an empty point spectrum in $\ell^1$ if the conditions in the lemma are not satisfied. Now, note that the $(m+1)^{th}$ term of

$$
\sum_{i=4t+1, t \in \mathbb{N}_0} f_i
$$

is given by

$$
\frac{1}{(2m)!} \left( \frac{\mu_1}{\mu_2} \right)^{2m-1} \sum_{n \in \mathbb{N}_m} \left( 2n \frac{\mu_1}{\mu_2} - 2m \right) (2n-1)(2n-2) \cdots (2n-2m+1) \left( \frac{\lambda}{\mu_2} \right)^{2n-2m}
$$

(19)

where $m \in \mathbb{N}$. Hence,

$$
\sum_{i=4t+1, t \in \mathbb{N}_0} f_i = \sum_{n \in \mathbb{N}_0} \left( \frac{\lambda}{\mu_2} \right)^{2n} \\
+ \sum_{m \in \mathbb{N}} \frac{1}{(2m)!} \left( \frac{\mu_1}{\mu_2} \right)^{2m-1} \sum_{n \in \mathbb{N}_m} \left( 2n \frac{\mu_1}{\mu_2} - 2m \right) \prod_{s=1}^{2m-1} (2n-s) \left( \frac{\lambda}{\mu_2} \right)^{2n-2m}
$$

(20)
Now suppose that $\lambda = \mu_2 \Theta(e^{-k})$, where $k = a + ib, a > 0, b \in \mathbb{R}$. Then clearly, Expression (20) converges since the terms

\[
\left( \frac{\lambda}{\mu_2} \right)^{2n-2m} \to 0 \text{ faster (exponential time) than }
\]

\[
0 < \left( 2n \frac{\mu_1}{\mu_2} - 2m \right)^{2m-1} \prod_{s=1}^{n} (2n - s) \to \infty \text{ as } n \to \infty \text{ (in polynomial time)}.\]

However, convergence of (20) does not guarantee existence of eigenvector $f$ in $\ell^1$. For this to happen, let us examine convergence of a related sequence of terms. Consider $\tilde{f}_i = f_1, f_5, f_9, \ldots$, where $\tilde{f}_i, i = 4t + 1, t \in \mathbb{N}$ is $f_i$ with all the coefficients made positive. That is,

\[
\tilde{f}_5 = \left( \frac{\lambda}{\mu_2} \right)^2 + \frac{\mu_1}{\mu_2}, \tilde{f}_9 = \left( \frac{\lambda}{\mu_2} \right)^4 + 3 \left( \frac{\lambda}{\mu_2} \right)^2 \frac{\mu_1}{\mu_2} + \left( \frac{\mu_1}{\mu_2} \right)^2, \cdots
\]

Then, we can see that (with $\lambda = \mu_2 \Theta(e^{-k})$)

\[
\tilde{f}_i \geq |f_i| \forall i = 4t + 1, t \in \mathbb{N}_0,
\]

so that if $\sum_i \tilde{f}_i$ converges in $\ell^1$, then so does $\sum_i |f_i|$. Following the arguments above, we write

\[
\sum_{i=4t+1,t \in \mathbb{N}_0} \tilde{f}_i = \sum_{m \in \mathbb{N}_0} \left( \frac{\lambda}{\mu_2} \right)^{2n} + \sum_{m \in \mathbb{N}_0} \frac{1}{(2m)!} \left( \frac{\mu_1}{\mu_2} \right)^{2m-1} \sum_{n \in \mathbb{N}_m} \left( 2n \frac{\mu_1}{\mu_2} + 2m \right)^{2m-1} \prod_{s=1}^{n} (2n - s) \left( \frac{\lambda}{\mu_2} \right)^{2n-2m}.
\]

Again, the term $\left( 2n \frac{\mu_1}{\mu_2} + 2m \right) (2n - 1)(2n - 2) \cdots (2n - 2m + 1)$ diverges in polynomial time but the hypothesis that $\lambda = \mu_2 \Theta(e^{-k})$, where $k = a + ib, a > 0, b \in \mathbb{R}$ guarantees convergence of (21). Therefore, $B_w$ has non-empty spectrum in $\ell^1$.

Conversely, if $B_w$ has non-empty point spectrum in $\ell^1$, then $\sum_{i \in \mathbb{N}} |f_i| < \infty$ and indeed $\sum_{i \in \mathbb{N}} f_i$ must converge too. But from (20), we see that convergence is only possible if and only if $\lambda = \mu_2 \Theta(e^{-k})$, where $k = a + ib, a > 0, b \in \mathbb{R}$. We end the proof by noting that $\lambda = 0$ is an eigenvalue of $B_w$ with the same eigenvectors as $B$, hence point spectrum $\sigma_p(B_w)$ touches the imaginary axis.

We can see from the result above that $\mu_1, \mu_2$ can be any finite positive numbers provided the restrictions in the Lemma are satisfied. With this in mind, we also note that the spectral circle of $B_w$ can be bigger than the unit circle. To ensure this, we simply pick $\mu_2 > 1$ sufficiently big.

3.2. Sub-chaos on a network in birth-and-death model. Using the results of the two theorems above, we show that the weighted adjacency matrix $B_w$, and consequently, the abstract problem (6) is subspace chaotic but not chaotic on $\ell^1$ on the graph given in Figure 1. We used ideas from [1] to prove the result.

**Theorem 3.3.** Let $|\mu_1| < |\mu_2|, |\mu_2| > 1$. Then the flow problem in (3) on the graph in Figure 1 is subspace chaotic on $\ell^1$.
Proof. Consider a submatrix $B_{wo}$ of $B_w$ containing rows 1,3,5,... and a submatrix $B_{we}$ of $B_w$ containing rows 2,4,6,... Then, $f = (f_1,0,f_3,0,f_5,0,...) + (0,f_2,0,f_4,0,0,...) = f_o + f_e$. Let $X = X_0 \oplus X_e$, where $X_e$ consists of $\ell^1_s$ vectors of the form $(0,f_2,0,f_4,0,0,...)$ and $x_o$ consists of $\ell^1_s$ vectors of the form $(f_1,0,f_3,0,f_5,0,...)$. If we simplify notation so that $f_o = (f_1,f_3,f_5,...)$ and $f_e = (f_2,f_4,...)$, we will then have to simplify matrices $B_{wo}$ and $B_{we}$ (by dropping even (odd)columns, respectively) so that

$$
\bar{B}_{wo} = \mu_1 \begin{pmatrix}
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots
\end{pmatrix}, \quad \bar{B}_{we} = \mu_2 \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\vdots & \ddots & \ddots & \ddots
\end{pmatrix}.
$$

We can see that $B_{wo}$ induces an operator which we will also denote $B_{wo}$ on $\ell^1_s$ and $B_{we}$ also induces an operator on $\ell^1_s$, which is the weighted backward shift operator while $B_w$ is the weighted forward shift operator. The operator $B_{we}$ is chaotic on the spaces $\ell^p, 1 \leq p < \infty$ for $|\mu_2| > 1$, see ([6], [12]). In light of this, we see that $B_{we}$ and indeed $B_w$ have a dense set of periodic points on the space $X_e$. However, the operator $B_{wo}$ is not hypercyclic. Indeed, this operator has empty point spectrum and so it is not chaotic. For this reason, $B_w$ is not hypercyclic on $X_o$ and therefore not chaotic on $X = \ell^1_s$ but it is chaotic on the subspace $X_e$.

From this result, we note that in general, whenever $w_{n,2n-2} > 1, \forall n \in \mathbb{N}$ matrix $B_w$ will always be chaotic on the subspace $X_e$ regardless of the coefficients $w_{n,2n-1}$. To see this, it is enough to note that $B_{we}$ (with general coefficients $w_{n,2n-2}$) satisfies $B_{we} \geq \tau B$, where $B$ is the backward shift operators and $\tau := \min_n w_{n,2n-2}$. With this relationship, we note that since $\tau B$ has dense orbits in $X_e$, $B_{we}$ also has dense orbits as well which implies hypercyclicity.

If loops are allowed in Figure 1, calculations will be a bit more involved in the computation of the point spectrum (of both $B$ and $B_w$), but one may still show that it is non-empty in $\ell^1_s$, provided $w_{n,n}$ satisfy certain conditions. Following results from [2], [3] and [7], one may suspect that with loops allowed, chaos may result in the whole space $\ell^1_s$.

4. Conclusion. We have shown that the flow problem on the edges on an infinite directed graph is chaotic provided the matrix operator $B_w$ is chaotic. This indicates that the graph structure has a big role in determining overall dynamics of the system. In particular, we have shown that the flow problem in (3) on the network in Figure 1 with arbitrary weights is chaotic whenever the coefficients $w_{n,2n-2} > 1$, and $|w_{n,2n-1}| < |w_{n,2n-2}|$ since the matrix operator $B_w$ is chaotic on a subspace $X_e$ of the space $\ell^1_s$. We have also related the birth-and-death model in (1) to the flow problem in (3) and noted that when, there is no proliferation and $|d| = |\mu_2| > 1$, $|b| = |\mu_1| < |d|$, then the birth-and-death process (1) is chaotic on a subspace $X_e$ of $\ell^1_s$ but not on the entire space $X = \ell^1_s$. Indeed, on the subspace $X_e$, the birth-and-death model with no proliferation is simply reduced to a pure death model which was shown to be chaotic, as it represents the backward shift.

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E-mail address: Namayanja@ukzn.ac.za