Bounces/Dyons in the Plane Wave Matrix Model 
and SU(N) Yang-Mills Theory

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Abstract
We consider SU(N) Yang-Mills theory on the space \( \mathbb{R} \times S^3 \) with Minkowski signature \((-+++)\). The condition of SO(4)-invariance imposed on gauge fields yields a bosonic matrix model which is a consistent truncation of the plane wave matrix model. For matrices parametrized by a scalar \( \phi \), the Yang-Mills equations are reduced to the equation of a particle moving in the double-well potential. The classical solution is a bounce, i.e. a particle which begins at the saddle point \( \phi = 0 \) of the potential, bounces off the potential wall and returns to \( \phi = 0 \). The gauge field tensor components parametrized by \( \phi \) are smooth and for finite time both electric and magnetic fields are nonvanishing. The energy density of this non-Abelian dyon configuration does not depend on coordinates of \( \mathbb{R} \times S^3 \) and the total energy is proportional to the inverse radius of \( S^3 \). We also describe similar bounce dyon solutions in SU(N) Yang-Mills theory on the space \( \mathbb{R} \times S^2 \) with signature \((-++)\). Their energy is proportional to the square of the inverse radius of \( S^2 \). From the viewpoint of Yang-Mills theory on \( \mathbb{R}^{1,1} \times S^2 \) these solutions describe non-Abelian (dyonic) flux tubes extended along the \( x^3 \)-axis.

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1 Introduction

The idea of AdS/CFT correspondence [1, 2, 3] is one of the most important concepts in studying nonperturbative aspects of string, gravity and gauge theories. In particular, it was recently shown that large $N$ weakly coupled SU($N$) gauge theory on $S^1 \times S^3$ demonstrates a confinement-deconfinement transition at temperatures $T$ proportional to the inverse radius of $S^3$ and the corresponding black hole-string transition was discussed (see [4, 5, 6] and references therein). Note that for finite radius $R$ of $S^3$ and nonzero temperature $T = 1/L$, where $L$ is the circumference of $S^1$, there are finite-action non-BPS solutions of Yang-Mills equations on $S^1 \times S^3$ describing $n$ instanton-antistanton pairs [7]. Furthermore, they exist only if $L \geq L_n = 2\pi R n$, i.e. for $T \leq T_n = 1/(2\pi R n)$ or $TR \leq 1/(2\pi n)$. Maybe it is not coincidence that the number $n$ of such instanton-antiinstanton pairs increases for $T \to 0$; in other words, $n \to \infty$ if $T \to 0$.

Recently, new proposals for gauge/gravity correspondence were formulated [8]. In particular, the Lin-Maldacena method works for the plane wave matrix model [9], for maximally supersymmetric Yang-Mills (SYM) theory on $\mathbb{R} \times S^2$ [8, 10] and for $\mathcal{N} = 4$ SYM theory on $\mathbb{R} \times S^3/\mathbb{Z}_k$ on the gauge side (see [11]-[13] and references therein). Using this method, one can describe smooth gravity solutions corresponding to vacua of the above SYM models and study the gravity description of instantons interpolating between gauge vacua [8, 11, 12].

Instantons in the plane wave matrix model and SU($N$) Yang-Mills theory on $\mathbb{R} \times S^2$ were described e.g. in [14, 7]. It would be of interest to construct some other explicit solutions in these models for further checking nonperturbative aspects of gauge/gravity correspondence. In this paper, we describe finite-energy dyon configurations in Yang-Mills theories on the spaces $\mathbb{R} \times S^3$ and $\mathbb{R} \times S^2$ with Minkowski signature. In fact, imposing SO($4$)-invariance we reduce Yang-Mills theory on $\mathbb{R} \times S^3$ to a bosonic matrix model which is the $\mathcal{N} = 0$ subsector of the $\mathcal{N} = 4$ plane wave matrix model in conformity with the previous results [15, 16, 17]. We briefly discuss perspectives of integrability of $\mathcal{N} = 4$ SYM and plane wave matrix theories. Then we reduce Yang-Mills theory on $\mathbb{R} \times S^2$ to a non-Abelian (matrix) analog of the $\phi^4$ kink model. In the simplest case the solution of both matrix models is a bounce, i.e. a particle in the double-well potential which begins at $\phi = 0$ for $t = -\infty$, bounces off the potential wall at $t = 0$ and returns to $\phi = 0$ for $t = +\infty$ (cf. [18]). In this case, matrices in the above models are constant matrices multiplied by $\phi$, and we obtain finite-energy dyon solutions of Yang-Mills equations on $\mathbb{R} \times S^3$ and $\mathbb{R} \times S^2$. Furthermore, solutions on $\mathbb{R} \times S^2$ can be uplifted to configurations with finite energy (dyons) on the space $\mathbb{R} \times S^2 \times S^1$ and to configurations with finite energy per unit length along $x^3$-axis (vortex tube) on the space $\mathbb{R} \times S^2 \times \mathbb{R}$.

2 Dyon configurations in Yang-Mills theory on $\mathbb{R} \times S^3$

Manifold $\mathbb{R} \times S^3$. Let us consider the space $\mathbb{R} \times S^3$ with Minkowski signature $(---+)$. On the standard three-sphere $S^3$ of the constant radius $R$ we consider one-forms $\{e^a\}$ satisfying the Maurer-Cartan equations
\begin{equation}
\frac{1}{\sqrt{2R}} e^a_\alpha e^\alpha_\beta \wedge e^\beta = 0 ,
\end{equation}
where $a, b, \ldots = 1, 2, 3$. Introducing $e^0 := dx^0 = dt$, we can write the metric on $\mathbb{R} \times S^3$ in the form
\begin{equation}
ds^2 = -(e^0)^2 + \delta_{ab} e^a e^b .
\end{equation}

\footnote{See e.g. [12, 7] for the explicit expression.}
Note that our choice of $e^a$’s differs from the standard one by the factor $\sqrt{2}$. The standard choice can be restored e.g. by rescaling $R$ in (2.1) and other formulae.

**SO(4)-invariance.** On $\mathbb{R} \times S^3$ we consider a gauge potential $A$ and the gauge field $F = dA + A \wedge A$ taking values in the Lie algebra $su(N)$. Since $S^3 = SO(4)/SO(3)$ is a homogeneous SO(4)-space, we can impose on $A$ and $F$ a condition of SO(4)-invariance defined up to gauge transformations (cf. [19]) which in the ‘temporal gauge’ $A_t = 0$ yields

$$A = \frac{1}{2} X_a e^a, \quad F = \frac{1}{2} \dot{X}_a dt \wedge e^a + \frac{1}{2} \left( \frac{1}{\sqrt{2} R} \varepsilon^c_{ab} X_c + \frac{1}{4} [X_a, X_b] \right) e^a \wedge e^b,$$

(2.3)

where the overdot denotes differentiation with respect to time.

Substitution of (2.3) into the standard Yang-Mills equations on $\mathbb{R} \times S^3$ leads to the second order equations on $X_a$:

$$\ddot{X}_a + \frac{2}{R^2} X_a + \frac{3}{2\sqrt{2} R} \varepsilon_{abc} [X_b, X_c] + \frac{1}{4} [X_b, [X_a, X_b]] = 0.$$

(2.4)

These equations describe a bosonic matrix model which is a consistent truncation from $\mathcal{N} = 4$ to $\mathcal{N} = 0$ of the plane wave matrix model [9]. In particular, (2.4) coincide with the proper equations in [16] after the redefinition $X_a \mapsto -2iX_a$ and $R \mapsto R/\sqrt{2}$. For more details on the reduction of $\mathcal{N} = 4$ SYM theory to the plane wave matrix model and further studies see e.g. [15, 16, 17, 12].

**Bounces in the plane wave matrix model.** Let us consider the ansatz (cf. [7])

$$X_a = \left( \phi - \frac{\sqrt{2}}{R} \right) T_a,$$

(2.5)

where $\phi = \phi(t)$ is a real-valued function of $t$ and $T_a$’s are generators of $N$-dimensional representation of SU(2). Substituting (2.5) into (2.4), we obtain the equation

$$\ddot{\phi} - \frac{1}{R^2} \phi + \frac{1}{2} \phi^3 = 0$$

(2.6)

which can also be obtained as the Euler-Lagrange equation from the reduced Yang-Mills action after integration over $S^3$. This action describes a particle with the kinetic energy density

$$T(\phi) = \dot{\phi}^2$$

(2.7)

moving in the double-well potential

$$U(\phi) = \frac{1}{4} \left( \frac{2}{R^2} - \phi^2 \right)^2.$$

(2.8)

The solution of eq. (2.6) is known as a bounce (see e.g. [18] and references therein)

$$\phi = \frac{2}{R \cosh\left(\frac{\pi}{R}\right)} \Rightarrow \dot{\phi} = -\frac{2 \sinh\left(\frac{\pi}{R}\right)}{R^2 \cosh^2\left(\frac{\pi}{R}\right)}$$

(2.9)

since

$$\phi(\pm \infty) = 0, \quad \phi(0) = \frac{2}{R}, \quad \dot{\phi}(\pm \infty) = 0 \quad \text{and} \quad \dot{\phi}(0) = 0.$$

(2.10)
i.e. the particle starts at the saddle point $\phi = 0$ of the potential (2.8), bounces off the potential wall on the right at $\phi = \frac{2}{R}$, and returns to $\phi = 0$ at $t = +\infty$. Of course, in (2.9) one can shift $t \rightarrow t - t_0$ due to translational invariance.

For the solution (2.9) we have

$$T(\phi) = \frac{4 \sinh^2(\frac{t}{2R})}{R^4 \cosh^4(\frac{t}{2R})}, \quad U(\phi) = \frac{(\cosh^2(\frac{t}{R}) - 2)^2}{R^4 \cosh^4(\frac{t}{R})}$$

(2.11)

and the energy density is

$$\mathcal{E}(\phi) = T(\phi) + U(\phi) = \frac{1}{R^4}.$$  

(2.12)

**Dyons in Yang-Mills theory.** Substituting (2.9) into (2.3), we obtain a dyon configuration

$$\mathcal{A} = \frac{1}{\sqrt{2R}} \left( \frac{\sqrt{2}}{\cosh^2(\frac{t}{2R})} - 1 \right) T_a e^a,$$

(2.13a)

$$\mathcal{F} = -\frac{\sinh(\frac{t}{2R})}{R^2 \cosh^2(\frac{t}{2R})} T_c \, dt \wedge e^c + \frac{1}{4R^2} \left( \frac{2}{\cosh^2(\frac{t}{2R})} - 1 \right) \varepsilon_{abc} T_c e^a \wedge e^b.$$  

(2.13b)

In fact, we have

$$E_a = -\frac{\sinh(\frac{t}{2R})}{R^2 \cosh^2(\frac{t}{2R})} T_a \quad \text{and} \quad B_a = \frac{1}{2R^2} \left( \frac{2}{\cosh^2(\frac{t}{2R})} - 1 \right) T_a$$

(2.14)

for the non-Abelian electric and magnetic fields $E_a = E_c T_c = F_{0a}$ and $B_a = B_c T_c = \frac{1}{2} \varepsilon_{abc} F_{bc}$. Note that $E_a \rightarrow 0$ and $B_a \rightarrow -\frac{1}{2R^2} T_a$ for $t \rightarrow \pm \infty$, i.e. the asymptotic fields are nonvanishing and on the 3-spheres $S^3$ at $t = \pm \infty$ we have

$$\mathcal{A} = -\frac{1}{\sqrt{2R}} e^a T_a = \frac{1}{2} g^{-1} dg,$$

(2.15)

where $g$ is a smooth map $g : S^3 \rightarrow SU(N)$ of winding number one.

For the energy density of the bounce dyon configuration we obtain

$$\mathcal{E} = -\text{tr}(E_a E_a + B_a B_a) = \frac{1}{16R^4} \sum_k p_k (p_k^2 - 1) \leq \frac{1}{16R^4} N(N^2 - 1),$$

(2.16)

where it is assumed that $T_a$’s are generators of reducible $N$-dimensional representation $(p_1, ..., p_K)$ of $SU(2)$ such that $\sum_k p_k = N$. For irreducible representation we simply have $p_1 = N$. For the energy we have

$$E = \int_{S^3} d^3x \sqrt{g} \mathcal{E} = \frac{\pi^2}{8R} \sum_k p_k (p_k^2 - 1)$$

(2.17)

with $\text{Vol}(S^3) = 2\pi^2 R^3$. Note that the dependence on $p_k$’s in (2.16), (2.17) will disappear for another usual choice of $T_a$’s such that $\text{tr}(T_a T_b) = -\frac{1}{2} \delta_{ab}$. 

3
Towards integrability of the plane wave matrix model. For the matrix model (2.4) the instanton subsector is described by the first order BPS equations (see e.g. [14, 7]) with \( t \mapsto \tau = \text{i} t \).

These first order equations are transformed to the standard Nahm equations by the redefinitions

\[
X_a \mapsto Y_a = \frac{1}{2} \exp\left( -\frac{\sqrt{2}}{R \tau} \right) X_a \quad \text{and} \quad \tau \mapsto r = \frac{R}{\sqrt{2}} \exp\left( \frac{\sqrt{2}}{R \tau} \right) \Rightarrow \tau = \frac{R}{\sqrt{2}} \log\left( \frac{\sqrt{2}}{R r} \right) \tag{2.18}
\]

discussed in [21] along with their integrability, Lax pair, charges and the finite-dimensional moduli space. Note that Nahm’s equations can be algebraically reduced to the (periodic) Toda chain equations (see e.g. [22]), solutions of which are known explicitly. In fact, the integrability of the BPS subsector of the model (2.4) follows from the integrability of the self-dual Yang-Mills equations on \( \mathbb{R} \times S^3 \) after imposing SO(4)-invariance. Furthermore, the redefinitions in (2.18) correspond to the well-known conformal transformations from the space \( \mathbb{R} \times S^3 \) to the space \( \mathbb{R}^4 \setminus \{0\} \),

\[
d\tau^2 + \frac{R^2}{2} d\Omega_3^2 = \frac{R^2}{r^2}(dr^2 + r^2 d\Omega_3^2), \tag{2.19}
\]

so that translations in \( \tau \) (and \( t \)) correspond to dilatations in \( \mathbb{R}^4 \setminus \{0\} \) (see e.g. [15, 16] and references therein).

The second order equations (2.4), corresponding to the SO(4)-invariant subsector of the full Yang-Mills theory on \( \mathbb{R} \times S^3 \), are not integrable. However, it is well known that the \( \mathcal{N} = 3 \) SYM equations (equivalent to \( \mathcal{N} = 4 \) ones) in Minkowski signature can be represented in the twistor approach as the compatibility conditions of some linear equations on an auxiliary function \( \psi \) [23] (see [24] for recent reviews and references). In fact, this function \( \psi \) depending on an extra ‘spectral’ parameter encodes all the information about the \( \mathcal{N} = 4 \) SYM multiplet and the above ‘zero curvature’ representation hints on integrability of \( \mathcal{N} = 4 \) SYM theory and of the plane wave matrix model which is its SO(4)-reduction. For deriving the system of linear equations for the matrix model one should write them down on the \( \mathcal{N} = 3 \) superambitwistor space for \( \mathbb{R} \times S^3 \) and impose the condition of SO(4)-equivariance on all fields. It is expected that, similar to the case of vortex equations on Riemann surfaces of genus \( g > 1 \) [25], these linear differential equations on \( \psi \) will keep derivatives along \( S^3 \) in spite of the fact that the plane wave matrix model is formulated in \( 0 + 1 \) dimensions. Note that we are speaking about nonperturbative integrability which might stem from ‘Lax representation’ of nonlinear equations of motion and not about the perturbative one discussed e.g. in [17].

3 Dyons in Yang-Mills theory on \( \mathbb{R} \times S^2 \)

Here we want to construct bouncing dyon configurations in Yang-Mills theory on \( \mathbb{R} \times S^2 \) similar to the case of gauge theory on \( \mathbb{R} \times S^3 \). Namely, imposing the condition of SO(3)-invariance, we reduce the Yang-Mills equations on \( \mathbb{R} \times S^2 \) to bosonic matrix equations in \( 0 + 1 \) dimensions. Note that this matrix model can be obtained via some limit and truncation from the plane wave matrix model as discussed e.g. in [8, 11, 12]. For discussion of gravity dual to maximally supersymmetric Yang-Mills theory on \( \mathbb{R} \times S^2 \) and \( \mathbb{R} \times S^3 \) see [8], [11]-[13] and references therein. However, we will not discuss these correspondences here. Instead, we embed the model into Yang-Mills theory in \( 3 + 1 \) dimensions and describe its dyon solutions.

\[2\]It is of interest to generalize noncommutative instantons on \( \mathbb{R}^4 \) (see e.g. [20] and references therein) to the space \( \mathbb{R} \times S^3 \). Note that one can consider a quantum group type deformation of \( \mathbb{R} \times S^3 \) since this space is a Lie group.
**Manifold** $\mathbb{R} \times S^2 \times \mathbb{T}$. We consider the space $\mathbb{R} \times S^2 \times \mathbb{T}$ with Minkowski signature $(-+++)$, where $\mathbb{T}$ is $S^1$ or $\mathbb{R}$. We choose local real coordinates $x^\mu$ with indices $\mu, \nu, \ldots$ running through $0, 1, 2, 3$ so that $x^1, x^2$ are local coordinates on $S^2$ and $x^3$ is a coordinate on $\mathbb{T}$. We also denote by $x^i$ the coordinates $x^0, x^3$ and introduce on $\mathbb{R} \times \mathbb{T}$ the Minkowski metric $\eta = (\eta_{ij}) = \text{diag}(-1, +1)$ with $i, j, \ldots = 0, 3$. On $S^2 \cong \mathbb{C}P^1$ we introduce the local complex coordinate $\tilde{y} = \frac{1}{2}(x^1 + ix^2)$ related with angle coordinates $0 \leq \theta < \pi$, $0 \leq \varphi \leq 2\pi$ by

$$y = R\tan\left(\frac{\theta}{2}\right)\exp(-i\varphi) \quad \text{and} \quad \tilde{y} = R\tan\left(\frac{\theta}{2}\right)\exp(i\varphi), \quad (3.1)$$

where bar denotes complex conjugation. In these coordinates the metric on $\mathbb{R} \times S^2 \times \mathbb{T}$ has the form

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = ds_{\Sigma}^2 + ds_{\mathbb{CP}^1}^2 = \eta_{ij}dx^i dx^j + 2g_{\tilde{y}\tilde{y}}d\tilde{y}d\bar{\tilde{y}} = -(dx^0)^2 + (dx^3)^2 + R^2(\sin^2\theta \, d\varphi^2) = -(dx^0)^2 + (dx^3)^2 + \frac{4R^4}{(R^2 + \tilde{y}\bar{\tilde{y}})^2} \, d\tilde{y}d\bar{\tilde{y}}, \quad (3.2)$$

where $\Sigma^{1,1} = \mathbb{R} \times \mathbb{T}$.

**SO(3)-invariance.** The question of invariance of the gauge fields on manifolds $X \times S^2$ under the action of the isometry group $\text{SO}(3) \cong \text{SU}(2)$ of $S^2$ was discussed e.g. in [26, 27]. Here we restrict ourselves to a particular $\text{SU}(2)$-invariant ansatz for fields on $\Sigma^{1,1} \times S^2$ described in [7]. It has the form

$$\mathcal{A} = a\mathcal{Y}_m + \frac{1}{2}\Phi_m^\beta - \frac{1}{2}\Phi_m^\dagger \beta, \quad (3.3)$$

where

$$a = \frac{1}{2(R^2 + \tilde{y}\bar{\tilde{y}})}(y \, dy - \tilde{y} \, d\bar{\tilde{y}}), \quad \beta = \frac{\sqrt{2}R^2}{R^2 + \tilde{y}\bar{\tilde{y}}} \, dy, \quad (3.4)$$

$$\mathcal{Y}_m = \text{diag}(m, \ldots, m - 2\ell, \ldots, -m) \quad \text{for} \quad \ell = 0, \ldots, m \quad (3.5)$$

and

$$\Phi_m = \begin{pmatrix} 0 & \phi_1 & \ldots & 0 \\ \vdots & 0 & \ddots & \vdots \\ \vdots & \ddots & \phi_m \\ 0 & \ldots & \ldots & 0 \end{pmatrix}. \quad (3.6)$$

Here $\phi_\ell = \bar{\phi}_\ell$ for $\ell = 1, \ldots, m$ are real scalar fields, $a$ in (3.3) and (3.4) is the gauge potential on the Dirac one-monopole line bundle over $\mathbb{C}P^1$ and $\beta$ is the $(1,0)$ type form on $\mathbb{C}P^1$.

For the gauge field tensor components we have

$$\mathcal{F}_{ij} = 0, \quad \mathcal{F}_{\tilde{y}\bar{\tilde{y}}} = -\frac{1}{4}g_{\tilde{y}\bar{\tilde{y}}} \left( \frac{2}{R^2}\mathcal{Y}_m - [\Phi_m, \Phi_m^\dagger] \right), \quad (3.7a)$$

$$\mathcal{F}_{i\bar{\tilde{y}}} = \frac{1}{2} \rho \partial_i \Phi_m \quad \text{and} \quad \mathcal{F}_{\bar{\tilde{y}}y} = -\frac{1}{2} \rho \partial_i \Phi_m^\dagger, \quad (3.7b)$$

where

$$\rho = (g_{\tilde{y}\bar{\tilde{y}}})^{1/2} = \frac{4\sqrt{2}R^2}{4R^2 + (x^1)^2 + (x^2)^2}. \quad (3.8)$$

5
Matrix $\Phi^4$ type model. Substituting (3.3)-(3.7) into the Yang-Mills action on $M := \mathbb{R} \times S^2 \times \mathbb{T}$ and integrating over $\mathbb{C}P^1$, we obtain

$$S = -\frac{1}{4\pi} \int_M \text{tr}(\mathcal{F} \wedge * \mathcal{F}) = R^2 \int_{\Sigma_{\ell,1}} d^2 \sigma \text{tr}\left\{ \partial_i \Phi_m \partial^i \Phi_m^\dagger + \frac{1}{8} \left( \frac{2}{R^2} \chi_m - [\Phi_m, \Phi_m^\dagger] \right)^2 \right\},$$

(3.9)

where $*$ is the Hodge operator. From (3.9) we obtain the matrix field equations

$$\partial_i \partial^i \Phi_m + \frac{1}{R^2} \Phi_m - \frac{1}{4} [\Phi_m, \Phi_m^\dagger], \Phi_m = 0,$$

(3.10)

which is equivalent to the linked equations

$$\partial_i \partial^i \phi_\ell + \frac{1}{R^2} \phi_\ell + \frac{1}{4} (\phi_{\ell-1}^2 - 2\phi_\ell^2 + \phi_{\ell+1}^2) \phi_\ell = 0$$

(3.11)

with $\ell = 1, ..., m$ and $\phi_0 := - \phi_{m+1}$. $\Phi^4$ bounces. We impose the condition $\partial_3 \phi_\ell = 0$ reducing (3.7)-(3.11) to the Yang-Mills model on the space $\mathbb{R} \times S^2$ with Minkowski signature ($- + +$). From (3.11) we obtain the equations

$$\ddot{\phi_\ell} - \frac{1}{R^2} \phi_\ell - \frac{1}{4} (\phi_{\ell-1}^2 - 2\phi_\ell^2 + \phi_{\ell+1}^2) \phi_\ell = 0.$$

(3.12)

As solution of these equations we have

$$\Phi_m = \frac{2}{R \cosh(\frac{t}{R})} \Phi_m^0,$$

(3.13)

where $\Phi_m^0$ are given by

$$\Phi_m^0 : \phi_\ell^0 = \pm \sqrt{\ell(\ell - 1)} \quad \text{for} \quad \ell = 1, ..., m,$$

(3.14)

so that $\frac{\sqrt{2}}{R} \Phi_m^0$ are the vacua of the model.

As more general SU$(N)$ solutions we can take

$$m = 2r : \phi_1 = \frac{2}{R \cosh(\frac{t - a_1}{R})}, \quad \phi_2 = 0, \quad \phi_3 = \frac{2}{R \cosh(\frac{t - a_3}{R})},$$

$$\cdots \phi_{m-1} = \frac{2}{R \cosh(\frac{t - a_{m-1}}{R})}, \quad \phi_m = 0,$$

(3.15a)

$$m = 2r + 1 : \phi_1 = \frac{2}{R \cosh(\frac{t - a_1}{R})}, \quad \phi_2 = 0, \quad \phi_3 = \frac{2}{R \cosh(\frac{t - a_3}{R})},$$

$$\cdots \phi_{m-1} = 0, \quad \phi_m = \frac{2}{R \cosh(\frac{t - a_m}{R})},$$

(3.15b)

where each $\phi_\ell \neq 0$ describes a bounce with different moduli $a_\ell$.

Dyons. To obtain dyon configurations in SU$(N)$ Yang-Mills theory we should substitute the bounce solution (3.13) or (3.15) into (3.3)-(3.7). Note that from the non-Abelian bounces in 0+1 dimensions we obtain dyon solutions of Yang-Mills equations in 3+1 dimensions, on $\mathbb{R} \times S^2 \times \mathbb{T}$, similar to monopoles on the same space obtained from $\Phi^4$ kinks [7]. Furthermore, in 3+1 dimensions we can
compare them by applying to both a Lorenz rotation, substituting $t \mapsto \gamma(t - vx^3)$ for bounces and $x^3 \mapsto \gamma(x^3 - vt)$ for kinks, where $\gamma = (1 - v^2)^{-1/2}$ and $-1 < v < 1$.

We will write down here the explicit form of $A$ and $F$ only for the $su(2)$-bounce (3.13) with $m = 1$. Namely, substituting (3.13) into (3.3) and (3.7), we obtain

$$A = a\sigma_3 + \frac{1}{R\cosh\left(\frac{t}{R}\right)}(\beta\sigma_+ - \beta\sigma_-) \quad \text{with} \quad \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \sigma_+^\dagger,$$

$$F = \sqrt{2}\sinh\left(\frac{t}{R}\right)\frac{R^2}{R^2\cosh^2\left(\frac{t}{R}\right)}\beta^0 \wedge \left(\beta^1\sigma_2^a - \beta^3\sigma_1^a\right) - \frac{1}{R^2}\left(1 - \frac{2}{\cosh^2\left(\frac{t}{R}\right)}\right)\frac{\sigma_3}{2i}\beta^1 \wedge \beta^2,$$

where

$$\beta^0 := dt, \quad \beta^1 := \frac{1}{\sqrt{2}}(\beta + \bar{\beta}), \quad \beta^2 := -\frac{i}{\sqrt{2}}(\beta - \bar{\beta}) \quad \text{and} \quad \beta^3 := dx^3 \quad (3.18)$$

form the nonholonomic basis of one-forms on $\mathbb{R} \times S^2 \times \mathbb{T}$ such that for (3.2) we have

$$ds^2 = -(\beta^0)^2 + (\beta^1)^2 + (\beta^2)^2 + (\beta^3)^2 \quad (3.19)$$

Thus, from the viewpoint of $3 + 1$ dimensions we have

$$E_1^2 = \frac{\sqrt{2}\sinh\left(\frac{t}{R}\right)}{R^2\cosh^2\left(\frac{t}{R}\right)} = -E_2^1, \quad E_3^a = 0 \quad \text{and} \quad B_1^a = 0 = B_2^a, \quad B_3^3 = \frac{1}{R^2}\left(\frac{2}{\cosh^2\left(\frac{t}{R}\right)} - 1\right) \quad (3.20)$$

with the energy density

$$E = -\text{tr}(E_aE_a + B_aB_a) = \frac{1}{2}(E_a^cE_a^c + B_a^cB_a^c) = \frac{1}{2R^4}. \quad (3.21)$$

Therefore, in $2 + 1$ dimensions we have

$$E_{S^2} = \int_{S^2}d^2x \sqrt{g} E = \frac{2\pi}{R^2}, \quad (3.22)$$

which can be considered as the energy density per unit length along the space $\mathbb{T}$. For $\mathbb{T} = S^1$ we can integrate further obtaining

$$E_{S^2 \times S^1} = \frac{2\pi L}{R^2}, \quad (3.23)$$

where $L$ is the circumference of $S^1$. For $\mathbb{T} = \mathbb{R}$ we obtain the dyonic vortex tube extended along the $x^3$-axis.

4 Concluding remarks

In the paper [7] we have shown how kinks in the $\phi^4$ type models in $1 + 0$ dimensions can be uplifted to instantons of Yang-Mills theories on $S^3 \times \mathbb{R}$ and $S^2 \times \mathbb{R}$. Similarly, sphalerons in the same $\phi^4$ type models uplifted to chains of instanton-antiinstanton pairs on $S^3 \times S^1$ and $S^2 \times S^1$. Note that instanton configurations in Yang-Mills theory on the space $S^2 \times \mathbb{T}$ can be interpreted as static monopole configurations on $\mathbb{R} \times S^2 \times \mathbb{T}$, where $\mathbb{T}$ is $\mathbb{R}$ or $S^1$. In this paper, we have shown that bounces in $\phi^4$ type models in $0 + 1$ dimensions can be uplifted to dyons of Yang-Mills theory on the spaces $\mathbb{R} \times S^3$, $\mathbb{R} \times S^2$ and $\mathbb{R} \times S^2 \times \mathbb{T}$ with Minkowski signatures in each case. These dyons
have finite energy densities on the spaces $S^3 \subset \mathbb{R} \times S^3$, $S^2 \subset \mathbb{R} \times S^2$ and $S^2 \times S^1 \subset \mathbb{R} \times S^2 \times S^1$. In fact, dyons on $\mathbb{R} \times S^3$ are bounce solutions of the plane wave matrix model and dyons in Yang-Mills theory on $\mathbb{R} \times S^2$ are bounce solutions of the related matrix model.

It would be of interest

- to study gravity dual description of the above-mentioned monopole, dyon and instanton configurations following [8, 11, 13]
- to construct supersymmetric generalizations of our exact solutions
- to consider quantum effects in the nonperturbative background defined by these monopole, dyon and instanton configurations
- to construct noncommutative generalizations of the above-mentioned solutions to the Yang-Mills equations

As a more general perspective, it is of interest to study classical and quantum integrability of maximally supersymmetric Yang-Mills theories on the spaces $\mathbb{R} \times S^2$ and $\mathbb{R} \times S^3$ and of the related matrix models in $0 + 1$ dimensions.

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