SYMBOLIC BLOWUP ALGEBRAS AND INVARIANTS ASSOCIATED TO CERTAIN MONOMIAL CURVES IN $\mathbb{P}^3$

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ABSTRACT. In this paper we explicitly describe the symbolic powers of the ideal defining the curve $C(q, m)$ in $\mathbb{P}^3$ parametrized by $(x^{d+2m}, x^{d+m}y^m, x^d y^{2m}, y^{d+2m})$, where $q, m$ are positive integers, $d = 2q + 1$ and $\gcd(d, m) = 1$. We show that the symbolic blowup algebra is Noetherian and Gorenstein. An explicit formula for the resurgence and the Waldschmidt constant of the prime ideal $p := p_{C(q, m)}$ defining the curve $C(q, m)$ is computed. We also give a formula for the Castelnuovo-Mumford regularity of the symbolic powers $p^{(n)}$ for all $n \geq 1$.

1. Introduction

Let $k$ be a field, $A = \mathbb{k}[x_1, \ldots, x_t]$ a polynomial ring and $I$ a homogeneous ideal in $A$ with no embedded components. For every $n \geq 1$, the $n$-th symbolic power of $I$ is defined as $I^{(n)} := \bigcap_{p \in \text{Ass}(A/I)} (I^n A_p \cap A)$. By a classical result of Zariski and Nagata $n$-th symbolic power of a given prime ideal consists of the elements that vanish up to order $n$ on the corresponding variety. However, describing the generators of symbolic powers is not easy. One can verify that $I^n \subseteq I^{(n)}$ and in fact for $0 \neq I \subset A$, $I^r \subseteq I^{(n)}$ holds if and only if $r \geq n$. It is a challenging problem to determine for which $n$ and $r$ the containment $I^{(n)} \subseteq I^r$ holds true. The results in [10] and [14] show that $I^{(n)} \subseteq I^r$ for $n \geq (t-1)r$. In the direction of comparing the symbolic powers and ordinary powers of ideals, B. Harbourne raised the following conjecture in [1, Conjecture 8.4.3]: For any homogeneous ideal $I \subset A$, $I^{(n)} \subseteq I^r$ if $n \geq r(t-1)-(t-2)$. It is of interest to study the least integer $n$ for which $I^{(n)} \subseteq I^r$ holds for a given ideal $I$ and for an integer $r$. To answer this question C. Bocci and B. Harbourne defined an asymptotic quantity called resurgence which is defined as $\rho(I) = \sup \{m/r \mid I^{(m)} \not\subseteq I^r \}$ (see [3]). From the results in [10] and [14] it follows that this quantity exists for radical ideals. In fact, $1 \leq \rho(I) \leq t-1$ (see [3]). Since it is hard to compute the exact value of resurgence, in the same paper [3], they defined another invariant was first introduced by Waldschmidt in [21]. They call this invariant the Waldschmidt constant and denote it by $\gamma(I)$ in [3]. This invariant is defined as $\gamma(I) = \lim_{n \to \infty} \frac{\alpha(I^{(n)})}{n}$, where $\alpha(I)$ denotes the least degree of a homogeneous generator of
They showed that if $I$ is an homogeneous ideal, then $\alpha(I)/\gamma(I) \leq \rho(I)$ and in addition if $I$ is a zero dimensional subscheme in a projective space, then $\alpha(I)/\gamma(I) \leq \rho(I) \leq \text{reg}(I)/\gamma(I)$, where $\text{reg}(I)$ denotes the Castelnuovo-Mumford regularity \cite[Theorem 1.2.1]{3}. Hence, if $\alpha(I) = \text{reg}(I)$, then $\rho(I) = \alpha(I)/\gamma(I)$. Later, in \cite[Conjecture 2.1]{13} Harbourne and Huneke raised the following Conjecture: Let $I$ be an ideal of fat points in $A$ and $m = (x_1, \ldots, x_t)$. Then $I^{(n(t-1))} \subseteq m^{n(t-2)}I^n$ holds true for all $I$ and $n$. In the same paper they showed that the conjecture is true for fat point ideals arising as symbolic powers of radical ideals generated in a single degree in $\mathbb{P}^2$.

The resurgence and the Waldschmidt constant has been studied in a few cases: for certain general points in $\mathbb{P}^2$ \cite[2], smooth subschemes \cite[12], fat linear subspaces \cite[11], special point configurations \cite[8] and monomial ideals \cite[4]. The behaviour of Castelnuovo-Mumford regularity of symbolic powers is not easy to predict. From a result of Cutkosky, Herzog and Trung, it follows that if $I$ is an ideal of points in a projective space and the symbolic Rees algebra $\bigoplus_{n \geq 0} I^{(n)}$ is Noetherian, then $\text{reg}(I^{(n)})$ is a quasi-polynomial \cite[Theorem 4.3]{6}). Moreover, $\lim_{n \to \infty} \left( \frac{\text{reg}(I^{(n)})}{n} \right)$ exists and can even be irrational \cite[7]{7}.

Though there are several results available for the the resurgence, the Waldschmidt constant and the Castelnuovo-Mumford regularity of symbolic powers, there is no precise result for monomial curves in a projective space. Though it is well known that $p^{(2)}/p^2$ is a cyclic module (for example see \cite[Lemma 2.1]{19}), the explicit description of the generator has been crucial in our study of the various invariants. In this paper we focus on the ideal defining the monomial curve $C(q, m)$ in $\mathbb{P}^3$ parametrized by $(x^{d+2m}, x^{d+m}y^m, x^dy^{2m}, y^{d+2m})$, where $q, m$ are positive integers, $d = 2q + 1$ with $\gcd(d, m) = 1$.

Another topic of interest is the Gorenstein property of the symbolic blowup algebras. If $q = 1$, i.e., $d = 3$, then the monomial curves we consider coincide with the curves in \cite[Theorem 3.2(v)(2)(a)]{19}. However, our proof here is different. We use properties of monomial ideals to obtain our results. These computations are also useful in computing the Castelnuovo-Mumford regularity of the symbolic powers. The Gorenstein property of monomial curves in $\mathbb{P}^3$ have also been studied by Schenzel in \cite[20]{20}. However, their curves do not overlap with the monomial curves we consider in this paper.

We now briefly describe the contents of our paper. Let $\phi : R = \mathbb{k}[x_1, x_2, x_3, x_4] \longrightarrow S = \mathbb{k}[x, y]$ be a homomorphism given by $\phi(x_1) = x^{d+2m}, \phi(x_2) = x^{d+m}y^m, \phi(x_3) = x^dy^{2m}$ and $\phi(x_4) = y^{d+2m}$. For the rest of this paper $p := \text{p}^{(q, m)} := \text{ker} \phi$. In Section 2, we prove a few preliminary results. In Section 3, we prove some results on monomial ideals which will be used in the subsequent sections. In Section 4, we explicitly describe the generators of $p^{(n)}$ (Theorem 4.6) and show that the symbolic Rees algebra $\bigoplus_{n \geq 0} p^{(n)}$ is Noetherian. As computing symbolic powers is not easy we use a simple trick. We consider the ring $T = R/(x_1, x_4)$. Then $pT$ is a monomial ideal and eventually for all $n \geq 1$, $p^{(n)}T$ is a monomial ideal (Proposition 4.4). In Section 5, we show that $\mathcal{R}_n(p)$ is Gorenstein.
Moreover, the symbolic fiber cone \( F_s(p) := \mathcal{R}(p) \otimes_R R/m = \bigoplus_{n \geq 0} p^{(n)}/m^{n} \) is Cohen-Macaulay (Theorem 5.5).

Section 6 is devoted to study certain invariants associated to \( p \), namely the resurgence, the Waldschmidt constant and the Castelnuovo-Mumford regularity. We verify that Conjecture 8.4.3 in [1] and Conjecture 2.1 in [13] is true for \( p \) (Corollary 6.8, Corollary 6.8). We express the resurgence for the monomial curve \( C(q,m) \) in terms of the degree of the curve \( C(q,m) \). In particular we show that \( \rho(p) = \frac{e(R/p) - 1}{e(R/p) - 2} \), where \( e(R/p) \) is the degree of \( C(q,m) \) (Theorem 6.10). The Waldschmidt constant is calculated for the same (Theorem 6.13). We next give an explicit formula for the Castelnuovo-Mumford regularity for all the symbolic powers \( p^{(n)} \) and show that it is a quasi-polynomial (Theorem 6.28). As a consequence we show that \( \lim_{n \to \infty} \text{reg} \left( \frac{R}{p^{(n)}} \right) = \frac{e(R/p)}{2} \) (Corollary 6.29). We end this paper by comparing all these invariants and show that there exist monomial curves for which Theorem 1.2.1(b) in [3] may not hold true (Lemma 6.31).

2. Basic results

In this section we prove several results which are probably well known in literature. We provide proofs for the sake of convenience. For the rest of this paper \( q, m \) and \( d \) are as in the introduction and \( p := p_{C(q,m)} \subseteq R \).

It is well known that the generators for \( p \) are the \( 2 \times 2 \) minors of the matrix

\[
\begin{pmatrix}
x_1 & x_2 & x_3^{q+m} \\
x_2 & x_3 & x_4^{q+m}
\end{pmatrix}
\]

[18]. In particular, if

\[g_1 = x_1^q x_2 x_4^m - x_3^{q+m+1}, \quad g_2 = x_1^{q+1} x_4^m - x_2 x_3^{q+m} \quad \text{and} \quad g_3 = x_1 x_3 - x_2^2\]

then \( p = (g_1, g_2, g_3) \).

**Lemma 2.2.**

(1) \( R/p \) is Cohen-Macaulay. In particular \( x_1, x_4 \) is a regular sequence in \( R/p \).

(2) \( e \left( \frac{R}{p(x_1, x_4)} \right) = 2(q + m) + 1 = d + 2m \).

**Proof.** (1) From the Hilbert-Burch theorem it follows that the minimal free resolution of \( p \) is of the form

\[
0 \to R[\pm (q + m + 2)]^2 \xrightarrow{\phi} R[\pm (q + m + 1)]^2 \oplus R[-2] \xrightarrow{\psi} p \to 0
\]

where \( \phi = \begin{pmatrix} x_2 & x_1 \\ -x_3 & -x_2 \end{pmatrix} \) and \( \psi = \begin{pmatrix} g_1 & g_2 & g_3 \end{pmatrix} \). Hence \( \text{depth}(R/p) = 2 = \text{dim}(R/p) = 2 \). This implies that \( R/p \) is Cohen-Macaulay. As \( x_1, x_4 \) is a system of parameters for \( R/p \), it is a regular sequence by [15, Corollary 11.12].
(2) Since \(x_1, x_4\) is a regular sequence,  
\[ e \left( (x_1, x_4); \frac{R}{\mathfrak{p}} \right) = \ell \left( \frac{R}{(\mathfrak{p} + (x_1, x_4))} \right) = \ell \left( \frac{R}{(x_1, x_4, x_2, x_3^{2(q+m)+1})} \right) = 2(q + m) + 1. \]

\[ \square \]

Put
\[ f := x_3^{q+m} g_1 - x_1^q x_4^m g_2 + x_1^{q-1} x_3^{q+m-1} x_2 x_4^m g_3 \]
\[ = -x_3^{2(q+m)+1} - x_1^{q-1} x_2 x_3^{q+m-1} x_4^m + 3 x_1^q x_2 x_3^{q+m} x_4^m - x_1^{2q+1} x_4^{2m}. \] (2.4)

**Lemma 2.6.** (1) For \(i = 1, 2, 3\), \(x_i f \in \mathfrak{p}^2\).
(2) \(f \in \mathfrak{p}^{(2)}\).
(3) For all \(n = 1, \ldots, q + m + 1\), \(f^n \in \mathfrak{p}^{2n-1}\).

**Proof.** (1) As \(g_i \in \mathfrak{p}\) for \(i = 1, 2, 3\),
\[ x_1 f = x_3^{q+m-1} g_1 g_3 - g_2^2 \in \mathfrak{p}^2 \]
\[ x_2 f = -x_1^{q-1} x_3^{q+m-1} x_4^m g_2 - g_1 g_2 \in \mathfrak{p}^2 \]
\[ x_3 f = -x_1^{q-1} x_4^m g_2 g_3 - g_1^2 \in \mathfrak{p}^2. \] (2.7)

(2) From (1) it follows that \(x_1 f \in \mathfrak{p}^2 \subseteq \mathfrak{p}^{(2)}\). As \(x_1 \notin \mathfrak{p}\), \(f \in \mathfrak{p}^{(2)}\).

(3) Let \(1 \leq n \leq q + m + 1\). By the definition of \(f\),
\[ f^n = (x_3^{q+m} g_1 - x_1^q x_4^m g_2 + x_1^{q-1} x_3^{q+m-1} x_2 x_4^m g_3) f^{n-1} \]
\[ = (x_3^{q+m} f^{n-1}) g_1 - (x_1 x_4^m f^{n-1}) g_2 + (x_1^{q-1} x_3^{q+m-1} x_2 x_4^m f^{n-1}) g_3 \]
\[ \in \mathfrak{p}^{2(n-1)} \mathfrak{p} \quad \text{[from (1)]} \]
\[ = \mathfrak{p}^{2n-1}. \] (2.8)

\[ \square \]

3. Computations with monomial ideals

In general, symbolic powers are not easy to compute. Hence, we first consider the ring \(T := R/(x_1, x_4) \cong \mathbb{k}[x_2, x_3]\). Since \(\mathfrak{p} T\) is a monomial ideal, \(\mathfrak{p}^n T\) is also. Consider
\[ \mathfrak{p} T = (x_2^2, x_2 x_3^{q+m}, x_3^{q+m+1}), \quad (f) T = (x_3^{2(q+m)+1}), \quad I_n := \sum_{n_1 + 2n_2 = n} (f^{n_2} T)(\mathfrak{p} T)^{n_1} \subseteq \mathfrak{p}^n T. \] (3.1)

Our aim in this section is to compute \(\ell(T/I_n)\). For this, we first need to show that \((I_n : x_3^{q+m}) \subseteq I_{n-1}\).
Next we will compute \(\ell(I_{n-1}/(I_n : x_3^{q+m}))\).

**Lemma 3.2.** For all \(n \geq 2\), \((I_n : x_3^{q+m}) \subseteq I_{n-1}\).
Proof. From the definition of $I_n$ we get
\[
(I_n : x_3^{q+m}) = \sum_{a_1+2a_2=n; a_2 \neq 0} (f_{a_2} (pT)^{a_1} : x_3^{q+m}) + ((x_2^2, x_2 x_3^{q+m}, x_3^{q+m+1})^n : x_3^{q+m})
\]
\[
= \sum_{a_1+2a_2=n; a_2 \neq 0} (x_3^{q+m+1}) f_{a_2-1} (pT)^{a_1} + (x_2^{2n}) + \sum_{i=1}^n (x_3^{q+m}(i-1)(x_2, x_3)^{i-1} x_2^{2(n-i)})(x_2, x_3)
\]
\[
\subseteq I_{n-1}.
\]

Our next step is to describe the generating set of $I_n$ modulo $(I_n : x_3^{q+m})$.

Lemma 3.3. The minimal set of generators of $I_{n-1}/(I_n : x_3^{q+m})$ form a vector space basis over $\mathbb{k}$.

Proof. Put $M = I_{n-1}/(I_n : x_3^{q+m})$ and $m' = (x_2, x_3)$. Since $x_3^{q+m} m' I_{n-1} \subseteq (pT) I_{n-1} \subseteq I_n$, we get $m' I_{n-1} \subseteq (I_n : x_3^{q+m})$. Hence $m' M = 0$ which implies that $M/m' M \cong M$. By graded Nakayama’s Lemma the generators of $M$ form a vector space basis over $T/m' \cong \mathbb{k}$.

In Lemma 3.7 we explicitly describe the generating set of $I_{n-1}$ modulo $(I_n : x_3^{q+m})$. We state a result on monomial ideals which follows from [9, Proposition 1.14] and will be consistently used in all the proofs which involve monomial ideals.

Proposition 3.4. Let $I = (u_1, \ldots, u_r)$ and $J = (v)$ be monomial ideals in a polynomial ring over a field $\mathbb{k}$. Then $I : J = \{u_i/\gcd(u_i, v) : i = 1, \ldots, r\}$.

Lemma 3.5. For all $n \geq 1$, $p^n T \subseteq x_2^{2n-1} (x_2, x_3^{q+m}) + (I_{n+1} : x_3^{q+m})$

Proof. We prove by induction on $n$. If $n = 1$, then
\[
p T = x_2 (x_2, x_3^{q+m}) + (x_3^{q+m+1}) \subseteq x_2 (x_2, x_3^{q+m}) + (I_2 : x_3^{q+m}).
\]
Hence the claim is true for $n = 1$. Let $n > 1$. Then
\[
p^n T
\]
\[
= (p T)(p^{n-1} T)
\]
\[
\subseteq ((x_2^2, x_2 x_3^{q+m}), x_3^{q+m+1}) (x_2^{2n-3} (x_2, x_3^{q+m}) + (I_n : x_3^{q+m}))
\]
\[
= x_2^{2n-1} (x_2, x_3^{q+m}) + (x_2^{2n-2} x_3^{2(q+m)}) + (x_2^2 x_3^{q+m}) (I_n : x_3^{q+m})
\]
\[
+ (f T) (x_2^{2n-3} (x_2, x_3^{q+m}) + (I_n : x_3^{q+m}))
\]

(3.6)
We now verify that all the terms except \( x_2^{2n-1} (x_2, x_3^{q+m}) \) are in \( (I_{n+1} : x_3^{q+m}) \).

\[
x_3^{q+m} \left( (x_2^{2n-2}) x_3^{2(q+m)} \right) = (x_2^{2(n-1)}) (x_3^{2(q+m)+1} x_3^{q+m-1}) \subseteq f(p^{n-1} T) \subseteq I_{n-1+2} = I_{n+1}
\]

\[
(x_2^2, x_2 x_3^{q+m}) (I_n : x_3^{q+m}) \subseteq (pT) (I_n : x_3^{q+m}) \subseteq (I_{n+1} : x_3^{q+m})
\]

\[
x_3^{q+m} \left( x_2^{q+m+1} x_2^{2n-3} (x_2, x_3^{q+m}) \right) = x_2^{2(n-2)} \cdot x_2 (x_2, x_3^{q+m}) \cdot x_3^{2(q+m)+1} \subseteq f(p^{n-1} T) \subseteq I_{n-1+2} = I_{n+1}
\]

\[
f(I_n : x_3^{q+m}) \subseteq (I_{n+1} : x_3^{q+m})
\]

\[
\square
\]

We are now ready to describe the generators of \( I_{n-1} \) modulo \( (I_n : x_3^{q+m}) \).

**Lemma 3.7.** For all \( n \geq 2 \),

\[
I_{n-1} = \left\{ \begin{array}{ll}
\sum_{a_2=0}^{n-2} x_3^{2q+m+1} a_2 x_2^{2(n-1-2a_2)-1} (x_2, x_3^{q+m}) + (I_n : x_3^{q+m}) & \text{if } 2 \nmid (n-1) \\
(2q+m) (a_2 + 1) x_3^{2q+m+1} + \sum_{a_2=0}^{n-3} f^{a_2} (pT)^{n-1-2a_2} & \text{if } 2 \mid (n-1)
\end{array} \right.
\]

**Proof.** From (3.1) we get

\[
I_{n-1} = \left\{ \begin{array}{ll}
\sum_{a_2=0}^{n-2} f^{a_2} (pT)^{n-1-2a_2} & \text{if } 2 \nmid (n-1) \\
(2q+m) (a_2 + 1) x_3^{2q+m+1} + \sum_{a_2=0}^{n-3} f^{a_2} (pT)^{n-1-2a_2} & \text{if } 2 \mid (n-1)
\end{array} \right.
\]

\[
\subseteq \left\{ \begin{array}{ll}
\sum_{a_2=0}^{n-2} f^{a_2} (x_2^{2(n-1-2a_2)-1} (x_2, x_3^{q+m}) + (I_{n-2a_2} : x_3^{q+m})) & \text{if } 2 \nmid (n-1) \\
(2q+m+1) (a_2 + 1) x_3^{2q+m+1} + \sum_{a_2=0}^{n-3} f^{a_2} (x_2^{2(n-1-2a_2)-1} (x_2, x_3^{q+m}) + (I_{n-2a_2} : x_3^{q+m})) & \text{if } 2 \mid (n-1)
\end{array} \right.
\]

\[
\subseteq \left\{ \begin{array}{ll}
\sum_{a_2=0}^{n-2} x_2^{2(n-1-2a_2)-1} (x_3^{2q+m+1}) a_2 (x_2, x_3^{q+m}) + (I_n : x_3^{q+m}) & \text{if } 2 \nmid (n-1) \\
(2q+m+1) (a_2 + 1) x_3^{2q+m+1} + \sum_{a_2=0}^{n-3} x_2^{2(n-1-2a_2)-1} (x_3^{2q+m+1}) a_2 (x_2, x_3^{q+m}) + (I_n : x_3^{q+m}) & \text{if } 2 \mid (n-1)
\end{array} \right.
\]

This implies that \( I_{n-1} \subseteq \text{RHS}. \) The other inclusion follows from Lemma 3.2 and checking element-wise. \( \square \)

**Proposition 3.8.** For all \( n \geq 1 \),

\[
\ell \left( \frac{I_{n-1}}{(I_n : x_3^{q+m})} \right) = n.
\]
Proof. From Lemma 3.3, \( \ell \left( \frac{I_{n-1}}{(I_n : x_3^{q+m})} \right) = \dim_k \left( \frac{I_{n-1}}{(I_n : x_3^{q+m})} \right) \), which is the number of minimal set of generators of \( I_{n-1}/(I_n : x_3^{q+m}) \). From Lemma 3.7, we observe that in the generators of \( I_{n-1} \) modulo \((I_n : x_3^{q+m})\), the terms which are of even degree in \( x_2 \) are \( x_2^{2(n-1-2a_2)} x_3^{(2(q+m)+1)a_2} \) where \( a_2 \leq (n-1)/2 \) and they are all distinct. Hence they are all linearly independent. Similarly, the generators which are of odd degree in \( x_2 \) are all distinct and form a linearly independent set. Hence

\[
\dim_k \left( \frac{I_{n-1}}{(I_n : x_3^{q+m})} \right) = \begin{cases} 2(n/2) & \text{if } 2 \nmid n - 1 \\ 1 + [2(n - 1)/2] & \text{if } 2 | n - 1 \end{cases} = n. \tag{3.9}
\]

Proposition 3.11. For all \( n \geq 1 \),

\[
\ell \left( \frac{T}{I_n} \right) = (2(q + m) + 1) \binom{n + 1}{2}.
\]

Proof. We prove by induction on \( n \). If \( n = 1 \), then

\[
\ell \left( \frac{T}{I_n} \right) = \ell \left( \frac{k[x_2, x_3]}{(x_2^q, x_2^m + q, x_3^{q+m+1})} \right) = 1 + 2(q + m).
\]

Now let \( n > 1 \). From the exact sequence

\[
0 \to \frac{T}{(I_n : x_3^{q+m})} \to \frac{T}{I_n} \to \frac{T}{I_n + (x_3^{q+m})} \to 0
\]

we get

\[
\ell \left( \frac{T}{I_n} \right) = \ell \left( \frac{T}{I_n + (x_3^{q+m})} \right) + \ell \left( \frac{T}{(I_n : x_3^{q+m})} \right) \tag{3.10}
\]

[Lemma 3.2]

\[
= 2(q + m)n + (2(q + m) + 1) \binom{n}{2} + n \quad \text{[by induction hypothesis and Proposition 3.8]}
\]

\[
= (2(q + m) + 1) \binom{n + 1}{2}.
\]
4. The symbolic powers

In this section we explicitly describe the symbolic powers \( p^{(n)} \). Using the fact \( x_1, x_4 \) is a regular sequence in \( R \), we get the results we are interested in for the symbolic powers \( p^{(n)} \) (Proposition 4.4, Theorem 4.6). Let

\[
\mathcal{I}_n := \sum_{n_1+2n_2=n} f^{n_2}p^{n_1} \subset p^{(n)}.
\] (4.1)

**Proposition 4.2.** Let \( n \geq 1 \). Then

1. \( \mathcal{I}_n \subset p^{(n)} \).
2. Let \( m = (x_1, x_2, x_3, x_4) \). Then \( (\mathcal{I}_n + (x_1, x_4)) \) is an \( m \)-primary ideal.

**Proof.** (1) As \( f \subset p^{(2)} \) (Lemma 2.6(3)),

\[
\sum_{n_1+2n_2=n} f^{n_2}p^{n_1} \subset \sum_{n_1+2n_2=n} p^{2n_2+n_1} = p^{(n)}.
\] (4.3)

(2) By (4.1), \( p^n \subset \mathcal{I}_n \) and \( (p^n + (x_1, x_4)) = ((x_2^2, x_2x_3^{q+m}, x_3^{q+m+1})^n, x_1, x_4) \) which implies that \( m = (\sqrt{p^n} + (x_1, x_4)) \subset (\sqrt{\mathcal{I}_n} + (x_1, x_4)) \subset m \). \( \square \)

**Proposition 4.4.** For all \( n \geq 1 \),

\[
e(\frac{R}{(x_1, x_4); p^{(n)}}) = \ell_R\left(\frac{R}{p^{(n)} + (x_1, x_4)}\right) = \ell_R\left(\frac{R}{\mathcal{I}_n + (x_1, x_4)}\right) = \ell(T_{\mathcal{I}_n})
\]

\[
= (2(q + m) + 1)\left(\begin{array}{c} n+1 \\ 2 \end{array}\right).
\]

**Proof.** From Proposition 4.2(1), \( \mathcal{I}_n \subset p^{(n)} \). Hence,
\[ e\left((x_1, x_4); \frac{R}{p^{(n)}}\right) = \ell_R\left(\frac{R}{p^{(n)} + (x_1, x_4)}\right) \quad \text{[as } R/p^{(n)} \text{ is Cohen-Macaulay]} \]
\[ \leq \ell_R\left(\frac{R}{I_n, x_1, x_4}\right) = \ell_R\left(\frac{T}{I_n}\right) = \ell_R\left(\frac{T}{I_n}\right) \quad \text{[as } (x_1, x_4) \subseteq \text{Ann}(T/I_n)] \]
\[ = \ell_R\left(\frac{T}{I_n}\right) \quad \text{[(3.1)]} \]
\[ = (2(q + m) + 1)^{\left(\frac{n + 1}{2}\right)} \quad \text{[Proposition 3.11]} \]
\[ = e\left((x_1, x_4); \frac{R}{p}\right) \ell_{R_p}\left(\frac{R_p}{p^n R_p}\right) \quad \text{[Lemma 2.2(2)]} \]
\[ = e\left((x_1, x_4); \frac{R}{p}\right) \ell_{R_p}\left(\frac{R_p}{p^{(n)} R_p}\right) \quad \text{[since } p^{(n)} R_p = p^n R_p] \]
\[ = e\left((x_1, x_4); \frac{R}{p^{(n)}}\right) \quad \text{[by [15, 1.8], (4.5)]} \]

Thus equality holds in (4.5) which proves the theorem. \(\square\)

We end this section by explicitly describing the generators of \(p^{(n)}\) for all \(n \geq 1\).

**Theorem 4.6.** For all \(n \geq 0\),

1. \(p^{(n)} = \mathcal{I}_n\).
2. \(p^{(2n)} = (p^{(2)})^n\) and \(p^{(2n+1)} = p^{(2n)}\).

**Proof.** (1) By Proposition 4.4 we get \((p^{(n)} + (x_1, x_4)) = \mathcal{I}_n + (x_1, x_4)\). Localizing at \(m\) we get \((p^{(n)} + (x_1, x_4)) R_m = (\mathcal{I}_n + (x_1, x_4)) R_m\). From Lemma 2.2(1), we conclude that \(x_1 R_m, x_4 R_m\) is a regular sequence on \(R_m/p^{(n)} R_m\). Hence

\[ (p^{(n)}, x_1) R_m = (\mathcal{I}_n, x_1) R_m + x_4((p^{(n)}, x_1) : x_4) R_m = (\mathcal{I}_n, x_1) R_m + x_4(p^{(n)}, x_1) R_m. \]

By Nakayama’s Lemma, \((p^{(n)}, x_1) R_m = (\mathcal{I}_n, x_1) R_m\). This implies that

\[ p^{(n)} R_m = \mathcal{I}_n R_m + x_1(p^{(n)} : x_1) R_m = \mathcal{I}_n R_m + x_1 p^{(n)} R_m. \]

Once again by Nakayama’s lemma, \(p^{(n)} R_m = \mathcal{I}_n R_m\). This implies that \(p^{(n)} R_m/\mathcal{I}_n R_m = (0) R_m\). As this is a graded module, \(p^{(n)}/\mathcal{I}_n = (0)\) which implies that \(p^{(n)} = \mathcal{I}_n\).
(2) For all \( n \geq 3 \), applying Proposition 4.2(1) we get
\[
p^{(n)} = I_n = \sum_{a_1+2a_2=n} f^{a_1} p_1^{a_1} \subseteq \sum_{a_1+2a_2=n} p^{a_1}(p^{(2)})^{a_2} \subseteq p^{(n)}.
\]
Hence equality holds and \( p^{(n)} = \sum_{a_1+2a_2=n} p^{a_1}(p^{(2)})^{a_2} \). Thus
\[
p^{(2n)} = \sum_{a_1+2a_2=2n} p^{a_1}(p^{(2)})^{a_2} = \sum_{a_2=0}^n p^{2n-a_2}(p^{(2)})^{a_2} \leq \sum_{a_2=0}^n (p^{(2)})^{2n} = p^{(2n)}
\]
\[
p^{(2n+1)} = \sum_{a_1+2a_2=2n+1} p^{a_1}(p^{(2)})^{a_2} = \sum_{a_2=0}^n p^{2n+1-a_2}(p^{(2)})^{a_2} \leq \sum_{a_2=0}^n p(p^{(2)})^{2n} = pp^{(2n)} \subseteq p^{(2n+1)}.
\]

\[\square\]

**Corollary 4.7.** For all \( n \geq 1 \), \( R/p^{(n)} \) is Cohen-Macaulay

**Proof.** As \( x_1, x_4 \) is a system of parameters in \( R/p^{(n)} \) and \( e((x_1, x_4); R/p^{(n)}) = \ell \left( R/p^{(n)}+(x_1, x_4) \right) \) (Theorem 4.4), \( R/p^{(n)} \) is Cohen-Macaulay. \(\square\)

## 5. Gorenstein Property of Symbolic Blowup Algebras

In this section we discuss the Gorenstein property of symbolic blowup algebras. If \( q = 1 \), then the curves we are interested in has been studied in [19]. Our proof here is different.

Throughout this section \( U := \mathbb{k}[x_1, x_2, x_3, x_4, u_1, u_2, u_3, v] \) and \( K := (w_1, w_2, z_1, z_2, z_3) \) where

\[
\begin{align*}
w_1 &= x_1u_1 - x_2u_2 + x_3^{q+m}u_3 \\
w_2 &= x_2u_1 - x_3u_2 + x_1^{q+m}u_3 \\
z_1 &= x_1v - x_3^{q+m-1}u_1u_3 + u_2^2, \\
z_2 &= x_2v + x_1^{q-1}x_3^{q+m-1}x_4^{m}u_3 + u_1u_2, \\
z_3 &= x_3v + x_1^{q-1}x_4^{m}u_2u_3 + u_1^2.
\end{align*}
\]

Before we prove our main result we prove some preliminary results.

**Lemma 5.1.** \( U/K \) is Gorenstein.

**Proof.** Using the Buchsbaum-Eisenbud criterion one can check that minimal free resolution of \( U/K \) is
\[
0 \rightarrow U \xrightarrow{\phi_3} U^5 \xrightarrow{\phi_2} U^5 \xrightarrow{\phi_1} U \rightarrow \frac{U}{K} \rightarrow 0 \quad (5.2)
\]
where
\[
\phi_1 = (w_1 \ w_2 \ z_1 \ z_2 \ z_3),
\]
\[
\phi_2 = \begin{pmatrix}
0 & v & -x_1^{q-1}x_4^m u_3 & -u_1 & u_2 \\
-v & 0 & -u_1 & u_2 & -x_3^{q+m-1} u_3 \\
x_1^{q-1}x_4^m u_3 & u_1 & 0 & x_3 & -x_2 \\
u_1 & -u_2 & -x_3 & 0 & x_1 \\
-u_2 & x_3^{q+m-1} u_3 & x_2 & -x_1 & 0
\end{pmatrix},
\phi_3 = \begin{pmatrix} w_1 \\ w_2 \\ z_1 \\ z_2 \\ z_3 \end{pmatrix}.
\]
Moreover, \(K\) is generated by the Pfaffians of order 4 of the anti-symmetric matrix \(\phi\) and \(R/K\) is Gorenstein.

**Proposition 5.3.** Let \(\tau : \mathbb{k}[x_1, x_2, x_3, x_4, u_1, u_2, u_3, v] \to R_s(p)\) be an homomorphism given by \(\tau(x_i) = x_i\) (1 \(\leq i \leq 4\), \(\tau(u_i) = g_it\), 1 \(\leq j \leq 3\) and \(\tau(v) = ft^2\)). Then \(\ker \tau = K\).

**Proof.** By Lemma 5.1 all associated primes of \(K\) are minimal primes. As \(K \subseteq \ker \tau \) and \(\text{ht}(K) = \text{ht}(\ker \tau) = 3\), \(\ker \tau\) is a minimal prime of \(K\).

We claim that there exists \(a_1, a_2, a_3 \in \mathbb{k}\), \((a_1, a_2, a_3) \neq (0, 0, 0)\) such that \(\alpha = a_1x_1 + a_2x_2 + a_3x_3 \not\in \bigcup_{P \in \text{Ass}(K)} P\). Otherwise \((x_1, x_2, x_3) \subseteq \bigcup_{P \in \text{Ass}(K)} P\) which implies that \((x_1, x_2, x_3) + K \subseteq \bigcup_{P \in \text{Ass}(K)} P\) and hence \((x_1, x_2, x_3, u_1, u_2) \subseteq \bigcup_{P \in \text{Ass}(K)} P\). Consequently, \((x_1, x_2, x_3, u_1, u_2) \subseteq Q\) for some \(Q \in \text{Ass}(K)\).

This implies that \(\text{ht}(Q) \geq 5\) which leads to a contradiction and proves the claim.

Fix \(\alpha = a_1x_1 + a_2x_2 + a_3x_3 \not\in \bigcup_{P \in \text{Ass}(K)} P\). Then
\[
\alpha v = a_1x_1v + a_2x_2v + a_3x_3v
= a_1z_1 + a_2z_2 + a_3z_3
- [a_1(-x_3^{q+m-1}u_1u_3 + u_2^2) + a_2(x_1^{q-1}x_3^{q+m-1}x_4^mu_3^2 + u_1u_2) + a_3(x_1^{q-1}x_4^mu_2u_3 + u_1^2)]
= a_1z_1 + a_2z_2 + a_3z_3 - \beta
\]
where \(\beta = a_1(-x_3^{q+m-1}u_1u_3 + u_2^2) + a_2(x_1^{q-1}x_3^{q+m-1}x_4^mu_3^2 + u_1u_2) + a_3(x_1^{q-1}x_4^mu_2u_3 + u_1^2)\). Then
\[
\frac{U[1/\alpha]}{(w_1, w_2, v + \beta/\alpha)} \approx \frac{\mathbb{k}[x_1, x_2, x_3, x_4, u_1, u_2, u_3][1/\alpha]}{(w_1, w_2)} \approx R(p)[1/\alpha].
\]
Recall that \(p = (g_1, g_2, g_3)\) and \(g_1, g_2, g_3\) form a d-sequence [17] and
\[
R(p) = \bigoplus_{n \geq 0} p^n t^n \approx \frac{\mathbb{k}[x_1, x_2, x_3, x_4, u_1, u_2, u_3]}{(w_1, w_2)}.
\]
Moreover \(R(p)\) is a domain [16, Theorem 3.1] and \(\dim R(p) = 5\). Hence \((w_1, w_2)\) is a prime ideal of height 2. Since \(\alpha \not\in (w_1, w_2)\), \(\text{ht}(w_1, w_2)U[1/\alpha] = 2\). This implies that \((w_1, w_2, v + \beta/\alpha)U[1/\alpha]\) is
a prime ideal and \( \text{ht}(w_1, w_2, v + \beta/\alpha)U[1/\alpha] = 3 \). In the ring \( U[1/\alpha] \),
\[
(w_1, w_2, v + \beta/\alpha)U[1/\alpha] = (w_1, w_2, (a_1 z_1 + a_2 z_2 + a_3 z_3)/\alpha)U[1/\alpha]
\subseteq KU[1/\alpha]
\subseteq (\ker \tau)U[1/\alpha].
\]
Since \( \alpha \not\in \ker \tau \), \( \text{ht}(w_1, w_2, v + \beta/\alpha)U[1/\alpha] = 3 \) and
\[
(w_1, w_2, v + \beta/\alpha)U[1/\alpha] = KU[1/\alpha] = (\ker \tau)U[1/\alpha].
\] (5.4)

By our choice of \( \alpha \) and (5.4)
\[
K = KU[1/\alpha] \cap U = (\ker \tau)U[1/\alpha] \cap U = \ker \tau.
\]

\[\square\]

**Theorem 5.5.** (1) \( \mathcal{R}_s(p) = R[pt, ft^2] \).

(2) \( \mathcal{R}_s(p) \) is Cohen-Macaulay.

(3) \( \mathcal{R}_s(p) \) is Gorenstein.

(4) The symbolic fiber cone \( F_s(p) = \bigoplus_{n \geq 0} p^{(n)}/mp^{(n)} \) is Cohen-Macaulay.

**Proof.** (1) The proof follows from Theorem 4.6.

(2) and (3) follows from Lemma 5.1 and Proposition 5.3.

(4) Let \( m = (x_1, x_2, x_3, x_4) \). Then \( F_s(p) \cong U/(K + m) \cong k[u_1, u_2, u_3, v]/(u_1^2, u_1 u_2, u_2^2) \). Since \( \dim(F_s(p)) = 2 \) and the images of \( u_3 \) and \( v \) form a regular sequence in \( F_s(p) \), \( F_s(p) \) is Cohen-Macaulay. \[\square\]

6. **Invariants associated to symbolic powers**

In this section we compute certain invariants namely the resurgence, the Waldschmidt constant, regularity associated to the symbolic powers of \( p \). Finally we compare these invariants.

6.1. **Containment.**

In order to compare the symbolic powers and ordinary powers C. Bocci and B. Harbourne in [3] defined the resurgence of an ideal \( I \) in \( R \) as
\[
\rho(I) := \sup \left\{ \frac{n}{r} : I^{(n)} \nsubseteq I^r \right\}.
\]
We can also compute the resurgence in the following way. For any ideal \( I \subseteq R \) let \( \rho_n(I) := \min\{r : I^{(n)} \nsubseteq I^r\} \). Then
\[
\rho(I) := \sup \left\{ \frac{n}{\rho_n(I)} : n \geq 1 \right\}.
\]
Lemma 6.1. For all $k \geq 1$ and $j = 0, 1$, $p^{(k(2q+2m)+j)} \subseteq p^{k(2q+2m-1)+j}$ and $p^{(k(2q+2m)+j)} \not\subseteq p^{k(2q+2m-1)+j+1}$.

Proof. For all $k \geq 1$ and $j = 0, 1$, by Lemma 2.6(3) and Theorem 4.6 we get

$$p^{(k(2q+2m))} = ((p^{2} + f)^{q+m})^k = \left( \sum_{i=0}^{q+m} f^{q+m-i} p^{2i} \right)^k \subseteq (p^{2(q+m-1)} p^{2i})^k = p^{k(2q+2m-1)+j}.$$

Let $j = 0$. We will show that $p^{(k(2q+2m))} \not\subseteq p^{k(2q+2m-1)+1}$. By Lemma 2.6(2) and Theorem 4.6 we get $f^{k(q+m)} \in p^{(k(2q+2m))}$. Observe that, $f^{k(q+m)} \equiv (x_1^{2q+1}x_4^{2q+1})^{k(q+m)} \mod (x_3)$. Also

$$p^{(2q+2m-1)+1} = (x_1^{q} x_2^{q+1} x_4^{q+1} x_3^{2})^{k(2q+2m-1)+1} \mod (x_3)$$

and hence $(x_1^{q+1}x_4^{m})^{k(2q+2m-1)+1}$ is a minimal generator of $p^{k(2q+2m-1)+1} \mod (x_3)$. Since

$$k(2q+1)(q+m) = k(2q^2 + 2qm + q + m)$$

$$< k(2q^2 + 2qm - q + 2q + 2m - 1) + q + 1$$

$$= k(q + 1)(2q + 2m - 1) + q + 1,$$

comparing the powers of $x_1$ we get $f^{k(q+m)} \not\subseteq p^{k(2q+2m-1)+1}$. Hence the lemma is true for $j = 0$. Using the similar argument we can show that $f^{k(q+m)} g_2 \in p^{((k(2q+2m))+1) \setminus p^{k(2q+2m-1)+2}}$. This shows that the lemma is true for $j = 1$.

Lemma 6.2. For all $k \geq 0$ and $j = 2, \ldots, 2q + 2m - 1$, $p^{(k(2q+2m)+j)} \subseteq p^{k(2q+2m-1)+j-1}$ and $p^{(k(2q+2m)+j)} \not\subseteq p^{k(2q+2m-1)+j}$.

Proof. Let $k = 0$. If $j = 2j'$ and $j' = 1 \ldots q + m - 1$, then by Lemma 2.6(3) and Theorem 4.6 we get

$$p^{(2j')} = (p^{2} + f)^{j'} = \sum_{i=0}^{j'} f^i p^{2(j'-i)} \subseteq p^{2i-1} + 2j'-2i = p^{2j'-1} = p^{j-1}. \quad (6.3)$$

If $j = 2j' + 1$ and $j' = 1, \ldots, q + m - 1$, then from Theorem 4.6(2) and (6.3) we get

$$p^{(2j'+1)} = p^{(2j')} p \subseteq p^{2j'-1} p = p^{2j'} = p^{j-1}. \quad (6.4)$$

Hence the lemma is true for $k = 0$.

Now let $k \geq 1$. Then by Theorem 4.6(2), induction hypothesis and (6.3) we get

$$p^{(k(2q+2m)+j)} = (p^{(2q+2m)})^k p^{(j)} \subseteq p^{k(2q+2m-1)} p^{j-1} = p^{k(2q+2m-1)+j-1}.$$

We now show that $p^{(k(2q+2m)+j)} \not\subseteq p^{k(2q+2m-1)+j}$. Let $j = 2j'$ where $j' = 1, \ldots, q + m - 1$. By Lemma 2.6(2) and Theorem 4.6(2) we get

$$f^{k(q+m)+j'} \in (p^{2})^{(k(q+m)+j')} = p^{(k(2q+2m)+2j')}$$
and
\[ f^{k(q+m)+j'} = (x_1^{2q+1}x_4^{2m})^{k(q+m)+j'} \mod (x_3). \] (6.5)

Since \( p = (x_1^q x_2 x_4^m, x_1^{q+1} x_4^m, x_2^2) \mod (x_3), \)
\[ (x_1^{q+1} x_4^m)^{k(2q+2m-1)+2j'} \in p^{k(2q+2m-1)+2j'} \mod (x_3) \] (6.6)
and is a minimal generator. Comparing the power of \( x_1 \) in (6.5) and (6.6) we get \( f^{k(q+m)+j'} \not\in p^{k(2q+2m-1)+2j'} \mod (x_3) \) since
\[ (2q + 1)(k(q + m) + j') - (q + 1)(k(2q + 2m - 1) + 2j') = -k(q + m) + (q + 1) - 1 - j' < 0 \]
Hence \( f^{k(q+m)+j'} \not\in p^{k(2q+2m-1)+2j'} \).

Using the above argument we can show that if \( k > 1 \) and \( j = 2j' + 1 \) where \( j' = 1, \ldots, q + m - 1 \), then
\[ f^{k(q+m)+j'}g_2 \in p^{(k(2q+2m)+j)} \setminus p^{k(2q+2m-1)+j}. \]

As a consequence of Lemma 6.1 and Lemma 6.2 we verify Conjecture 8.4.3 on [1] for \( p \).

**Lemma 6.7.** Let \( r \geq 1 \). Then for all \( n \geq 2r - 1 \), \( p^{(n)} \subseteq p^r \).

**Proof.** If \( n \geq 2r - 1 \), then \( p^{(n)} \subseteq p^{(2r-1)} \). Hence it is enough to show that \( p^{(2r-1)} \subseteq p^r \). By Theorem 4.6 we have \( p^{(2r-1)} = p(p^{(2)})^{r-1} \subseteq p^r \). \( \square \)

We now verify that Conjecture 2.1 in [13] holds true in our case.

**Corollary 6.8.** \( p^{(3n)} \subseteq m^{2n}p^n \).

**Proof.** Let \( r \geq 1 \). If \( n = 2r \), then by repeatedly applying Theorem 4.6 we get \( p^{(3(2r))} = (p^{(2)})^{3r} = p^{(2r)}p^{(4r)} \subseteq m^4p^{2r} = m^{2n}p^n \) as \( p^{(2)} \subseteq m^4 \) and \( p^{(4r)} = (p^{(2)})^{2r} \subseteq p^{2r} \).

If \( n = 2r - 1 \), then \( p^{(3n)} = p^{(6(r-1))}p^{(3)} \). Using the even case argument, \( p^{(6(r-1))} \subseteq m^{2(2r-2)}p^{2r-2} \). Hence \( p^{(3n)} = p^{(6(r-1))}p^{(3)} \subseteq m^{2(2r-2)}p^{2r-2}p^{(2)}p \subseteq m^{2(2r-1)}p^{2r-1} = m^{2n}p^n \). \( \square \)

We have an improved version of Corollary 6.8.

**Corollary 6.9.** For all \( n \geq 1 \), \( p^{(2n)} \subseteq m^np^n \) and \( p^{(2n+1)} \subseteq m^{n+1}p^n \).

**Proof.** Let \( r \geq 1 \). Then by applying Theorem 4.6(2) and (2.4) we get
\[ p^{(2n)} = (p^{(2)})^n = (p^2 + (f))^n \subseteq (mp)^n = m^n p^n \]
\[ p^{(2n+1)} = pp^{(2n)} \subseteq m^np^{n+1} \subseteq m^np^n. \]

\( \square \)
As a consequence of Lemma 6.1 and Lemma 6.2 we give the exact value for the resurgence $\rho(p)$.

**Theorem 6.10.** For all $q, m \geq 1$, $\rho(p) = \frac{e(R/p) - 1}{e(R/p) - 2}$.

**Proof.** Let $j = 0, 1$, $k \geq 1$ and $n_{k,j} = k(2q + 2m) + j$. Then by Lemma 6.1, for all $k \geq 1$, $\rho_{k(2q+2m)+j}(p) = k(2q + 2m - 1) + j + 1$. Hence

$$\sup_k \left\{ \frac{n_{k,j}}{\rho_{n_{k,j}}(p)} \right\} = \sup_k \left\{ \frac{k(2q + 2m) + j}{k(2q + 2m - 1) + j + 1} \right\} = \frac{2q + 2m}{2q + 2m - 1}. \quad (6.11)$$

Let $j = 2, \ldots, 2q + 2m - 1$, $k \geq 0$ and $n_{k,j} = k(2q + 2m) + j$. Then by Lemma 6.2, for all $k \geq 0$, $\rho_{k(2q+2m)+j}(p) = k(2q + 2m - 1) + j$. Hence

$$\sup_k \left\{ \frac{n_{k,j}}{\rho_{n_{k,j}}(p)} \right\} = \sup_k \left\{ \frac{k(2q + 2m) + j}{k(2q + 2m - 1) + j} \right\} = \frac{2q + 2m}{2q + 2m - 1}. \quad (6.12)$$

From (6.11) and (6.12) we get

$$\rho(p) = \sup_k \left\{ \frac{n_{k,j}}{\rho_{n_{k,j}}(p)} : j = 0, \ldots, q + m - 1 \right\} = \frac{2q + 2m}{2q + 2m - 1}.$$

As $e(R/p) = 2(q + m) + 1$, the result follows. $\square$

6.2. Waldschmidt Constant.

For a homogeneous ideal $I \subset R$, let $\alpha(I)$ denote the least generating degree of $I$. The Waldschmidt constant of $I$ is defined as

$$\gamma(I) = \lim_{s \to \infty} \frac{\alpha(I^{(s)})}{s}.$$

Here we will compute the Waldschmidt constant for $p$.

**Theorem 6.13.** $\gamma(p) = \alpha(p) = 2$.

**Proof.** As $\deg(g_1) = \deg(g_2) \geq \deg(g_3) = 2$, $\alpha(p) = 2$. By Theorem 4.6(2), $p^{(2n)} = (p^{(2)})^n$ and $p^{(2n+1)} = p(p^{(2)})^n$. Since $p^{(2)} = (p^2 + f)$ and $\deg(f) = 2(q + m) + 1 \geq \alpha(p^2) = 2 \cdot \alpha(p) = 4$. Thus

$$\frac{\alpha(p^{(2n)})}{2n} = \frac{4n}{2n} = 2 \text{ and } \frac{p^{(2n+1)}}{2n+1} = \frac{4n + 2}{2n + 1} = 2. \text{ Hence } \gamma(p) = 2. \quad \square$$

6.3. Regularity.

In this subsection we compute the regularity of the symbolic powers of $p$. Let $pT$, $fT$ and $I_n$ as in (3.1). We first prove a preliminary result (Lemma 6.14) which indicates that it is enough to compute the regularity of $T/I_n$.

**Lemma 6.14.** $\text{reg}(R/p^{(n)}) = \text{reg}(T/I_n)$.
Proof. As \( x_1, x_4 \) is a regular sequence in \( R/p \), by [5, Remark 4.1],

\[
\text{reg}(\frac{R}{\mathfrak{p}(n)}) = \text{reg}(\frac{R}{\mathfrak{p}(n) + (x_1)}) = \text{reg}(\frac{R}{\mathfrak{p}(n) + (x_1, x_4)}) = \text{reg}(\frac{T}{I_n}).
\]

\[\square\]

Let \( G(\mathcal{F}) := \bigoplus_{n \geq 0} I_n/I_{n+1} \) be the associated graded ring corresponding to the filtration \( \mathcal{F} := \{I_n\}_{n \geq 0} \). We show \( G(\mathcal{F}) \) is Cohen-Macaulay and this result is very useful in computing the regularity. We first prove a preliminary lemma.

**Lemma 6.15.** For all \( n \geq 1 \),

1. \( p^nT : (x_2^2) \subseteq p^{n-1}T \).
2. \( (p^nT : x_3^{2(q+m)+1}) \subseteq p^{n-2}T \).

**Proof.** (1) By Proposition 3.4,

\[
(p^nT : x_2^2) = \sum_{i=0}^{n-1} (x_2^{2(n-i)}x_3^{(q+m)i}(x_2, x_3)^i : x_2^2) + (x_3^{(q+m)n}(x_2, x_3)^n : x_2^2) \\
= \sum_{i=0}^{n-1} (x_2^{2(n-i)}x_3^{(q+m)i}(x_2, x_3)^i) + (x_3^{(q+m)n}(x_2, x_3)^n-2) \\
\subseteq p^{n-1}T, \\
(p^nT : x_3^{2(q+m)+1}) \\
= \sum_{i=0}^{n-1} (x_2^{2(n-i)}x_3^{(q+m)i}(x_2, x_3)^i) + \sum_{i=3}^{n}(x_2^{2(n-i)}x_3^{(q+m)i}(x_2, x_3)^i : x_3^{2(q+m)+1}) \\
= x_2^{2n}(x_2, x_3) + \sum_{i=3}^{n+2} (x_2^{2(n+2-i)}x_3^{(q+m)(i-3)}(x_2, x_3)^{i-3}(x_3^{q+m-1}, x_2x_3^{q+m}, x_2x_3^{q+m+1}, x_3^{q+m+2}) \\
\subseteq (x_2^{2n}) + \sum_{i=3}^{n} (x_2^{2(n+2-i)}x_3^{(q+m)(i-3)}(x_2, x_2)^{i-3}(x_3^{q+m}, x_3^{q+m+1}) \\
\subseteq p^{n-2}T. 
\]

(6.17)

For any element \( r \in T \), let \( r^* \) denote the image in \( G(\mathcal{F}) \).

**Theorem 6.18.** \( G(\mathcal{F}) \) is Cohen-Macaulay.
Proof. We first show that \((x_2^*)^*\) is a regular element in \(G(F)\). We claim that \((I_n : x_2^2) = I_{n-1}\) for all \(n \geq 1\). Clearly \(x_2^2 I_{n-1} \subseteq (pT) I_{n-1} \subseteq I_n\). For the other inclusion, from Lemma 6.15(1) we get

\[
(I_n : x_2^2) = \sum_{a_1 + 2a_2 = n} (f^{a_2} T : x_2^2) \subseteq \sum_{a_1 + 2a_2 = n} (f^{a_2} T) (p^{a_1 - 1} T) \subseteq I_{n-1}.
\]

Let \(\overline{\text{denote}}\) the image in \(T/(x_2^2)\). Then

\[
\frac{G(F) \langle x_2^2 \rangle^*}{G(F)} \cong \bigoplus_{n \geq 0} \frac{I_n}{I_{n-1} + x_2^2 I_{n-1}} = G(F).
\]

To show that \(x_3^{3(q+m)+1}\) is a regular element in \(G(F)\), we need to verify that

\[
((I_{n+2} + x_2^2 I_n) : (x_3^{3(q+m)+1})) = I_n + x_2^2 I_{n-2}.
\]

One can verify that \(x_3^{3(q+m)+1} (I_{n+2} + x_2^2 I_n) \subseteq fT (I_n + x_2^2 I_{n-2}) \subseteq I_{n+2}\). For the other inclusion, for all \(n \geq 0\)

\[
\begin{align*}
((I_{n+2} + x_2^2 I_n) : (x_3^{3(q+m)+1})) &= \sum_{a_1 + 2a_2 = n + 2} (f^{a_2} p^{a_1} T : x_3^{3(q+m)+1}) + \sum_{a_1 + 2a_2 = n} (x_2^2 f^{a_2} p^{a_1} T : (x_3^{3(q+m)+1})) \\
&= \sum_{a_1 + 2a_2 = n + 2; a_2 \neq 0} (f^{a_2 - 1} p^{a_1} T) + (p^{n+2} T : x_3^{3(q+m)+1}) \\
&\quad + \sum_{a_1 + 2a_2 = n; a_2 \neq 0} (x_2^2 f^{a_2 - 1} p^{a_1} T) + (x_2^2 p^n T : x_3^{2(q+m)+1}) \\
&\subseteq \sum_{a_1 + 2a_2 = n + 2; a_2 \neq 0} (f^{a_2 - 1} p^{a_1} T) + p^n T + \sum_{a_1 + 2a_2 = n; a_2 \neq 0} (x_2^2 f^{a_2 - 1} p^{a_1} T) + x_2^2 p^{n-2} T \quad \text{[by (6.17)]} \\
&\subseteq I_n + x_2^2 I_{n-2}.
\end{align*}
\]

As \(x_2^* \in [G(F)]_1\) is a regular element, we can use it to determine \(\text{reg}(T/(I_n + x_2^2))\).

Lemma 6.21. Let \(n \geq 1\). Then

\[
\text{reg} \left( \frac{T}{I_n + (x_2^2)} \right) = \begin{cases} 2r \left( q + m + \frac{1}{2} \right) & \text{if } n = 2r, \\ (2r - 1) \left( q + m + \frac{1}{2} \right) - \frac{1}{2} & \text{if } n = 2r - 1. \end{cases}
\]

Proof. By Theorem 4.6,

\[
I_{2r} + (x_2^2) = \left( I_2 + (x_2^2) \right)^r + (x_2^2) = (x_2^2, x_3^{2(q+m)+1})^r + (x_2^2) = (x_2^2, x_3^{r(2(q+m)+1)}).
\]
the minimal free resolution of $T/(I_{2r} + (x_3^2))$ is

$$0 \to T[-(r(2(q + m) + 1) + 2)] \xrightarrow{T[-2]} T[-(r(2(q + m) + 1))] \xrightarrow{T} I_{2r} + (x_3^2) \to 0$$

This implies that

$$\text{reg} \left( \frac{T}{I_{2r} + (x_3^2)} \right) = r(2(q + m) + 1) = 2r \left( q + m + \frac{1}{2} \right).$$

Let $n = 2r - 1$. Then by Theorem 4.6,

$$I_{2r-1} + (x_3^2) = (I_1, x_2^2)(I_{2(r-1)}, x_2^2) + (x_3^2) = (x_2^2, x_2x_3^{(q+m)}, x_3q^{m+1})(x_2^2, x_3^{(r-1)(2(q+m)+1)}) + (x_2^2)$$

$$= (x_2^2, x_2x_3^{(2r-1)(q+m)+r-1}, x_3^{(2r-1)(q+m)+r})$$

$$= (x_2^2, x_2x_3^{n(q+m)+r-1}, x_3^{n(q+m)+r}).$$

By Hilbert-Burch Theorem, the minimal free resolution of $T/I_n$ is

$$0 \to T[-n(q + m) - r - 1]^2 \xrightarrow{T[-2]} T[-n(q + m) - r]^2 \xrightarrow{T} I_n + (x_3^2) \to 0.$$

Hence $\text{reg}(T/I_n + (x_3^2)) = n(q + m + 1) + r - 1 = n(q + m + 1) - \frac{1}{2}$. \hfill \square

**Lemma 6.24.** For all $n \geq 1$, $\text{reg}(T/I_n + (x_3^{2(q+m)+1})) = 2n + 2(q + m) - 3$.

**Proof.** As $(fT) = (x_3^{2(q+m)+1})$, from Theorem 4.6, for all $n \geq 1$,

$$I_n + (x_3^{2(q+m)+1}) = (pT)^n + (x_3^{2(q+m)+1}) = (x_2^{2n}, x_2^{2n-1}x_3^{q+m}, x_2^{2n-2}x_3^{q+m+1}, x_3^{2(q+m)+1}).$$

By Hilbert-Burch Theorem the minimal free resolution of $I_n + (x_3^{2(q+m)+1})$ is

$$0 \to T[-(2n + q + m)]^2 \xrightarrow{T[-(2n)]} T[-(2n - 1 + q + m)]^2 \xrightarrow{T} I_n + (x_3^{2(q+m)+1}) \to 0.$$

Hence $\text{reg}(T/I_n + (x_3^{2(q+m)+1})) = 2n + 2(q + m) - 3$. \hfill \square
We now use the fact that \((x_3^{2(q+m)+1})^* \in [G(F)]_2\) is a regular element to compute \(\text{reg}(T/I_n)\).

**Proposition 6.25.** Let \(n \geq 1\). Then

\[
\text{reg} \left( \frac{T}{I_n} \right) = \begin{cases} 
2r \left( q + m + \frac{1}{2} \right) & \text{if } n = 2r, \\
(2r - 1) \left( q + m + \frac{1}{2} \right) - \frac{1}{2} & \text{if } n = 2r - 1.
\end{cases}
\]

**Proof.** Let \(n = 2r\). We prove the Proposition by induction on \(r\). If \(r = 1\), then the result follows from Lemma 6.24. Let \(r > 1\). By Theorem 6.18, \((x_3^{2(q+m)+1})^* \) is a regular element in \(G(F)\). Hence, we have the exact sequence,

\[
0 \to \frac{T}{I_{2r-2}} [-2(q + m) - 1] \xrightarrow{x_3^{2(q+m)+1}} \frac{T}{I_{2r}} \to \frac{T}{I_{2r} + (x_3^{2(q+m)+1})} \to 0. \tag{6.26}
\]

Then from the exact sequence (6.26) we get

\[
\text{reg} \left( \frac{T}{I_{2r}} \right) = \max \left\{ \text{reg} \left( \frac{T}{I_{2r-2}} \right) + 2(q + m) + 1, \text{reg} \left( \frac{T}{I_{2r} + (x_3^{2(q+m)+1})} \right) \right\} \\
= \max \left\{ (2r - 2) \left( q + m + \frac{1}{2} \right) + 2(q + m + \frac{1}{2}), 4r + 2(q + m) - 3 \right\} \quad \text{[Lemma 6.24]} \\
= \max \left\{ 2r \left( q + m + \frac{1}{2} \right), 4r + 2(q + m) - 3 \right\} \\
= 2r \left( q + m + \frac{1}{2} \right).
\]

Let \(n = 2r - 1\) and \(r \geq 1\). If \(r = 1\), then the result follows from Corollary 6.21. Let \(r > 1\). As \((x_2^2)^* \) is a nonzerodivisor in \(G(F)\), we have the exact sequence

\[
0 \to \frac{T}{I_{2r-2}} [-2] \xrightarrow{x_2^2} \frac{T}{I_{2r-1}} \to \frac{T}{I_{2r-1} + (x_2^2)} \to 0. \tag{6.27}
\]

As all the modules in (6.27) are Artinian,

\[
\text{reg} \left( \frac{T}{I_{2r-1}} \right) = \max \{ \text{reg} \left( \frac{T}{I_{2r-2}} [-2] \right), \text{reg} \left( \frac{T}{I_{2r-1} + (x_2^2)} \right) \} \\
= \max \{ (2r - 2) \left( q + m + \frac{1}{2} \right) + 2, (2r - 1) \left( q + m + \frac{1}{2} \right) - \frac{1}{2} \} \\
= (2r - 1) \left( q + m + \frac{1}{2} \right) - \frac{1}{2}.
\]

\(\square\)

**Theorem 6.28.** Let \(n \geq 1\). Then \(\text{reg} \left( \frac{R}{p^n} \right) = n(e(R/p)/2) + \theta\) where

\[
\theta = \begin{cases} 
0 & \text{if } n \text{ is even} \\
-1/2 & \text{if } n \text{ is odd.}
\end{cases}
\]
Proof. From Lemma 6.14 and Proposition 6.25 we get

\[
\text{reg} \left( \frac{R}{p^{(n)}} \right) = \begin{cases} 
2r \left( q + m + \frac{1}{2} \right) & \text{if } n = 2r, \\
(2r-1) \left( q + m + \frac{1}{2} \right) - \frac{1}{2} & \text{if } n = 2r-1.
\end{cases}
\]

Since \( e(R/p) = 2q + 1 + 2m \), the result follows.

As an immediate corollary we have:

Corollary 6.29. \( \lim_{n \to \infty} \frac{\text{reg} \left( \frac{R}{p^{(n)}} \right)}{n} = \frac{e(R/p)}{2} \).

6.4. Comparing invariants.

In this subsection we compare the various invariants. We verify that Theorem 1.2.1(b) does not always hold true if the scheme defined by ideal \( I \) is not zero-dimensional.

Lemma 6.30. \( \rho(p) \geq \frac{\alpha(p)}{\gamma(p)} \)

Proof. As \( \alpha(p)/\gamma(p) = 1 \) and \( \rho(p) \geq 1 \) the result follows.

Lemma 6.31. If \( q = m = 1 \), then \( \rho(p) \geq \text{reg}(p)/\gamma(p) \). If either \( q > 1 \) or \( m > 1 \), then \( \rho(p) < \text{reg}(p)/\gamma(p) \).

Proof. From (2.3) it follows that \( \text{reg}(R/p) = q + m \). Hence

\[
\rho(p) - \frac{\text{reg}(p)}{\gamma(p)} = \frac{2q + 2m}{2q + 2m - 1} - \frac{q + m}{2} = \frac{(q + m)(5 - 2q - 2m)}{2(2q + 2m - 1)}.
\]

If \( q = m = 1 \), then \( 5 - 2q - 2m = 1 \). If either \( q > 1 \) or \( m > 1 \), then \( 5 - 2q - 2m < 0 \).

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