Transmission-Constrained Consensus of Multiagent Networks

Xiaotian Wang and Housheng Su

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functions are distributed in a specific range (i.e., satisfy the conditions of Theorems 4 or 5), the systems will reach an asymptotically stable equilibrium even though that intersection of constraints is empty.

Compared with the existing works about the consensus of networked systems with constraints, the contributions of this work are as follows.

1) This work first studies transmission-constrained consensus problems of multiagent networks. The transmission-constrained consensus model studied in this article does not have a definite form, so that it can be regarded as a paradigm. The MASs that can be translated to this model are able to make the system achieve consensus under the necessary conditions, such as interval consensus [6]. Unlike traditional constrained consensus problems, the transmission-constrained consensus problem has the following features:

a) each link in the interaction network is limited by an individual constraint function, which is more general in reality;

b) constraint functions do not have a uniform type, and they can be various functions, such as trigonometric function, saturation function, and Sigmoid function, etc.

Those features make the transmission-constrained model have a wide range of applications but they also bring heterogeneity into the dynamics, which increases the difficulty of analysis.

2) For the transmission-constrained consensus problem, we obtain some consensus conditions, in which a necessary and sufficient condition limits the distribution of constraint functions. As a more general case than the consensus case, the equilibrium of MAS is seldom studied. We investigate this phenomenon and obtain conditions of the equilibrium’s existence, uniqueness, and stability.

Due to the novel model, where the unknown transmission constraints make the dynamics nonlinear, the analysis of MAS’s stability is quite a challenge. We design some linear boundaries and propose the corresponding lemmas to analyze the boundedness of dynamics. Then, we analyze the limit points of multiple solutions to prove the convergence of MASs. By coordinate transformations, another Lyapunov function is constructed to study the equilibrium’s stability and uniqueness.

II. PRELIMINARIES

A. Graph Theory

Consider an MAS with \( n \) agents, and denote \( \mathbb{N} = \{1, 2, \ldots, n\} \). The finite vertex set is denoted by \( \mathcal{V} = \{v_1, \ldots, v_n\} \), and \( \mathcal{E} \subseteq \mathcal{V} \times \mathcal{V} \) denotes edge set, where \((v_j, v_i) \in \mathcal{E}\) means that there exists a communication link from agent \( j \) to agent \( i \). The adjacency weight matrix \( \mathcal{A} \in \mathbb{R}^{n \times n} \) is defined as \( a_{ij} > 0 \) if and only if \((v_j, v_i) \in \mathcal{E}\), and \( a_{ij} = 0 \), otherwise. Then, the underlying interaction network of MAS is described by a (weighted) graph \( \mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A}) \), which is a triple. \((v_j, v_i)\) is defined as the directed edge from agent \( j \) to agent \( i \), and \( \mathcal{N}_i = \{v_j \in \mathcal{V} : (v_j, v_i) \in \mathcal{E}\} \) denotes the neighbor set of agent \( i \). Denote \( \alpha_i = \sum_{j=1}^{n} a_{ij} \) as the row sum of \( \mathcal{A} \), and \( \bar{a} = \max \alpha_i \).

B. Problem Statement

For any \( i \in \mathbb{N} \), denote the state of \( v_i \) by \( x_i(t) \in \mathcal{R} \). Then consider the continuous-time dynamics of single-integrator MAS with \( n \) agents: \( \dot{x}_i(t) = u_i(t), i \in \mathcal{N} \), where \( u_i(t) \in \mathcal{R} \) is the control input.

The problem studied in this work is different from the general MAS dynamics. In this problem, the information transmissions between agents are disturbed by interference functions (or attenuation functions), i.e., the transmission of state \( x_i(t) \) is replaced by a transmission constraint function \( f_{ij}(x_i(t)) \). \( f_{ij} \) represents the transmission constraint imposed on the link from \( i \) to \( j \). Then, the transmission-constrained consensus algorithm of \( x_i(t) \) is

\[
\dot{x}_i(t) = u_i(t) = \sum_{j \in \mathcal{N}_i} a_{ij} \left[ f_{ji}(x_j(t)) - x_i(t) \right].
\]  

**Assumption 1:** \( \forall i, j \in \mathbb{N} \), the transmission constraint function \( f_{ij}(x) \) are piecewise continuous.

**Remark 1:** Since \( \forall i, j \in \mathbb{N} \), \( f_{ij}(x) \) are piecewise continuous, the autonomous system (1) may have multiple solutions. However, the following theorems and corollaries apply to both unique and multiple solutions.

**Assumption 2:** The directed graph \( \mathcal{G} \) is strongly connected.

**Assumption 3:** There exists an interval \([\partial_m, \partial_M]\), a value \( \partial \in [\partial_m, \partial_M] \), and two rays

\[
L_1(x) = k_1(x - \partial) + \partial, \quad x \in (-\infty, \partial]
\]

\[
L_2(x) = k_2(x - \partial) + \partial, \quad x \in [\partial, +\infty)
\]

where \( k_1, k_2 < 0 \), such that \( \forall j \in \mathbb{N}, i \in \mathcal{N}_j \)

\[
x \leq f_{ij}(x) < L_1(x), \quad x \in (-\infty, \partial_m)
\]

\[
L_2(x) < f_{ij}(x) \leq x, \quad x \in (\partial_m, +\infty).
\]

**Assumption 4:** For any \( x' \in (-\infty, \partial_m] \cup (\partial_M, +\infty) \), there exist \( j \in \mathbb{N} \) and \( i \in \mathcal{N}_j \) such that \( f_{ij}(x) \) is continuous on \( x' \) and \( f_{ij}(x') \neq f_{ij}(x) \).

This article aims to find which transmission constraints could make MAS stable and obtain the consensus conditions for MAS (1).

C. Applications and Motivating Examples

The distortion (attenuation or saturation) in information transmission or detection is an actual embodiment of transmission constraints. Those transmission constraints may be caused by objective physical constraints, or those constraints are added on purpose.

1) **Objective Constraints:** There are three kinds of objective constraints to show that those transmission constraints are common in real-world scenarios.

1) **Information distortion caused by transmission:** Energy loss exists during the signal transmission, which may cause information distortion. The voltage drop on wires is an inevitable phenomenon during signal transmission. If states of the agent (or device) are represented by signal
voltage and this signal is transmitted on wires, then we can obtain information distortion
\[
f_{ij}(x_i) = \frac{R_r}{R_L + R_r} x_i
\]
where \(R_L\) is the resistance of wires, and \(R_r\) is the equivalent resistance of the port. For example, in a decentralized control scheme for microgrids [10], the distributed generation (agent) acquires the information from the bus line of the microgrid to estimate others’ states, where information may be distorted by the voltage drop.

2) Information distortion caused by detection: In real-world scenarios, agents use sensors to get themselves or neighbors’ states. For instance, due to weak communication capabilities, underwater vehicles use sonar to actively determine the location and velocity of their neighbors to maintain formation [11]. However, except for noise interference, state information cannot be obtained precisely due to the inherent characteristics of sensors. For example, temperature offset leads to signal fluctuation in ultrasonic distance measurements [12]. Likewise, the saturation characteristic of the Hall sensor may cause information distortion, i.e., \(f_{ij}(x_i) = \text{sat}(x_i)\) [13].

3) Information distortion caused by privacy protection: In social networks, individuals may express an opinion that is different from his/her private opinion, probably due to the pressure of conforming to a group standard or norm [14], [15]. Hence, \(x_i\) could represent the private opinion, and \(f_{ij}(x_i)\) is the expressed opinion.

2) Subjective Constraints: As a class of constraints, transmission constraints could make agent’s states converge into the expected set (see Theorem 2 and Remark 3). This is especially the case that since we do not specify the formula of transmission constraints, different transmission constraints can be designed to suit different scenarios, such as interval consensus [6] and discarded consensus [16]. Related discussions are in Remarks 5 and 6.

Those above examples show that information distortion during transmission is a common phenomenon in the real world. Hence, study consensus under transmission constraints is necessary.

D. Notations and Some Definitions

Notations: The set of positive integers is denoted by \(\mathbb{N}^+\). Consider a matrix \(B = [b_{ij}] \in M_{m,n}\) and denote \(|B| = |[b_{ij}]|\) (i.e., elementwise absolute value of matrix \(B\)), \(d^+ Z(t)\) denotes the upper right Dini derivative of \(Z(t)\). The arrow \(\Rightarrow\) means “implies,” and the arrow \(\iff\) means “if and only if.” Denote sign function

\[
\text{sign}(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -1, & \text{if } x < 0. \end{cases}
\]

The distance between interval \([\partial_m, \partial_M]\) and vector \(x(t)\) is denoted by

\[
\text{distance}(\partial_x, x(t)) = \min_{c \in [\partial_m, \partial_M]} \|x(t) - c\|.
\]

Denote \(e = \{e_1, \ldots, e_n\}^T\) to be an equilibrium of MAS (1), it can then be concluded that for all \(i \in \mathbb{N}\)
\[
\dot{x}_i(t)\big|_{e_i} = \sum_{j \in \mathcal{N}_i} a_{ij} (f_{ij}(e_j) - e_i) = 0.
\]

Denote the error between \(x(t)\) and \(e\) by \(\varepsilon_i(t) = x_i(t) - e_i\ \forall i \in \mathbb{N}\).

Introduce the definition of consensus zone, and this work can be divided into two parts: 1) the part of nonempty consensus zone and 2) the part of empty consensus zone.

Definition 1: For MAS (1), denote \(\Theta_{ij} = \{x : f_{ij}(x) = x\}\), and consensus zone \(\Phi = \bigcap_{(i,j) \in \Theta_{ij}}\). Consensus zone means the transmission constraints vanish when all the states of MAS are in this consensus zone.

Theorem 2 shows that under the given conditions, MAS (1) reaches consensus and its states fall into consensus zone.

Remark 2: For any time, if an agent’s state is in consensus zone, then the information it transmits to its neighbors is without transmission constraints. If all agents’ initial states are in consensus zone, then the MAS becomes standard consensus dynamics. In that case, under a strongly connected digraph (or the digraph has a spanning tree), the MAS will reach consensus and the consensus value is in the consensus zone. That is why we name it consensus zone.

III. MAIN RESULTS

In this section, the transmission-constrained consensus problems are studied. Initially, we analyze the convergence of MAS in Section III-A. Second, for the nonempty consensus zone, we obtain the consensus conditions in Section III-B. Third, for the empty consensus zone, the system’s states may achieve an equilibrium, and the existence, stability, and uniqueness of equilibrium are studied in Section III-C.

A. Convergence Analysis

The following theorem states the conditions where the states of the MAS are bounded and gives the boundary. Furthermore, the conclusion of Theorem 1 plays an important role in the proofs of Theorems 2 and 3.

Theorem 1: Along the system (1), suppose Assumptions 1, 2, 3, and 4 hold, and \(k_1 k_2 = 1\). Then, for any MAS (1) and initial state \(x(t_0) \in \mathbb{R}^n\)
\[
\lim_{t \to \infty} \text{distance}(\partial_m, \partial_M) = 0
\]
if and only if for any MAS (1), initial state \(x(t_0) \in \mathbb{R}^n\), and \(j \in \mathcal{N}_i\), \(\partial_m \leq f_{ij}(x) \leq \partial_M, x \in [\partial_m, \partial_M]\).

B. Nonempty Consensus Zone: Consensus

Then, we introduce conditions of transmission-constrained consensus. The following theorem states the consensus conditions for MAS, and it is worth noting that the condition about the range of constraint functions is necessary and sufficient.

Theorem 2: Along system (1), suppose Assumptions 1–4 hold, and \(f_{ij}(x) = x, x \in [\partial_m, \partial_M]\). Then, for any MAS (1), \(i \in \mathbb{N}\), \(\lim_{t \to \infty} x_i(t) = v^*, v^* \in [\partial_m, \partial_M]\) if and only if for any MAS (1), \(k_1 k_2 \leq 1\). 
Remark 3: Theorem 2 shows that although the initial state is not in the consensus zone, the transmission constraints will limit the final state of the system. Hence, it shows that imposing constraints on links of the interaction networks can indirectly limit agents’ states.

Remark 4: Under the conditions in Theorem 2, the constraint functions can be various functions. For example, constraint functions can be a Sigmoid function or tanh function, which have many applications such as activation function in artificial neural networks, logistic function in biology, and many more. More candidate constraint functions are given in the numerical example section.

Remark 5: In [6], the interval consensus problem is studied. We propose a smooth interval consensus model

\[ \dot{x}_i(t) = \sum_{j \in N_i} a_{ij} [T_j(x_j(t)) - x_i(t)], \quad i \in \mathbb{N} \]

\[ T(x) = \begin{cases} \rho x + (1 - \rho)q, & \text{if } x > q \\ x, & \text{if } p \leq x \leq q \\ \rho x + (1 - \rho)p, & \text{if } x < p \end{cases} \]

where \( \rho \in (0, 1) \) is a constant. By Theorem 2, it can be concluded that the system will reach interval consensus.

Remark 6: The discarded consensus problem is studied in [16], but the initial states of the system must be in the constraint set. Another discarded consensus model can be proposed

\[ \dot{x}_i(t) = \sum_{j \in N_i} a_{ij} \sin (x_j(t) + \pi) - x_i(t), \quad i \in \mathbb{N} \]

(2)

where \( \Omega_{c_i} = [-c_i, c_i] \) is the constraint interval of agent \( i \) [16]. By Theorem 2, we can get that the MAS (2) will reach discarded consensus with arbitrarily initial states.

Remark 7: Consider the following MAS:

\[ \dot{x}_i(t) = \sum_{j \in N_i} a_{ij} \sin (x_j(t) + \pi) - x_i(t), \quad i \in \mathbb{N} \]

(3)

Due to the consensus zone of MAS (3) being \( \Phi = \{0\} \), Theorem 2 shows that the agents’ states will converge to 0 when the underlying directed graph \( \mathcal{G} \) is strongly connected.

Corollary 1: Along the system (1), suppose the following conditions hold:

1) Assumption 2 holds;
2) the consensus zone \( \Phi \neq \emptyset \);
3) for any \( j \in \mathbb{N} \), \( i \in N_j \) and \( \omega \neq 0 \)

\[ -1 < f_{ij}(x + \omega) - f_{ij}(x) \leq 1, \quad x \in \mathbb{R}. \]

Then, \( \forall i \in \mathbb{N}, \lim_{t \to \infty} x_i(t) = v^*, v^* \in \Phi \).

C. Empty Consensus Zone: Existence, Stability, and Uniqueness of Equilibria

In this part, the existence, stability, and uniqueness of equilibria are discussed and proved.

Theorem 3 gives the existence conditions of equilibrium, which is a prerequisite for Theorem 5.

Theorem 3: Suppose \( \forall j \in \mathbb{N}, i \in N_j, f_{ij}(x) \) is a continuous function. If there exists an interval \( [\partial_m, \partial_M] \) and \( \forall j \in \mathbb{N}, i \in N_j \)

\[ \partial_m \leq f_{ij}(x) \leq \partial_M, \quad x \in [\partial_m, \partial_M] \]

then the system (1) exists at least one equilibrium. In fact, all equilibria of the system lie within \( [\partial_m, \partial_M]^n \) if the following conditions hold:

1) Assumptions 2, 3, and 4 hold;
2) \( k_1k_2 = 1 \), and \( \partial_m \leq f_{ij}(x) \leq \partial_M, x \in [\partial_m, \partial_M] \);
3) \( f_{ij}(x) \) is a continuous function \( \forall j \in \mathbb{N}, i \in N_j \).

Theorem 3 establishes the existence of equilibrium and gives the region where all equilibria exist. But it does not illustrate whether the MAS will reach equilibria, not to mention the stability of equilibria. The following theorem indicates that the system will converge to an asymptotically stable equilibrium, if some conditions hold.

Theorem 4: Along the system (1), suppose the following conditions hold.

1) Assumption 2 holds.
2) There is an equilibrium \( e = \{e_1, \ldots, e_n\}^T \), i.e.,

\[ \dot{x}_i(t) \bigg|_{e_i} = \sum_{j \in N_i} a_{ij} (f_{ji}(e_j) - e_i) = 0 \quad \forall i \in \mathbb{N}. \]

3) There exist two rays

\[ L_{e1}(\varepsilon) = k_{e1} \varepsilon, \quad \varepsilon \in (-\infty, 0] \]

\[ L_{e2}(\varepsilon) = k_{e2} \varepsilon, \quad \varepsilon \in [0, +\infty) \]

where \( k_{e1}, k_{e2} < 0 \) and \( k_{e1}k_{e2} = 1 \), such that \( \forall j \in \mathbb{N}, i \in N_j \)

\[ \varepsilon \leq f_{ij}(e_i + \varepsilon) - f_{ij}(e_i) < L_{e1}(\varepsilon), \quad \varepsilon \in (-\infty, 0] \]

\[ L_{e2}(\varepsilon) < f_{ij}(e_i + \varepsilon) - f_{ij}(e_i) \leq \varepsilon, \quad \varepsilon \in (0, +\infty). \]

4) For any \( \varepsilon' \neq 0 \), there exist \( j \in \mathbb{N}, i \in N_j \), such that \( f_{ij}(e_i + \varepsilon') \) is continuous on \( \varepsilon' \) and \( f_{ij}(e_i + \varepsilon') - f_{ij}(e_i) \neq \varepsilon' \).

Then, the equilibrium \( e \) is unique and asymptotically stable, i.e., \( \lim_{t \to \infty} x_i(t) = e_i \forall i \in \mathbb{N} \).

Remark 8: Unlike Theorem 2, the boundary rays in Theorem 4 are two clusters of parallel lines with the same slopes, but the endpoints may be different. The auxiliary lines in Fig. 6 are the illustrations of two clusters of parallel lines.

Remark 9: The unique equilibrium’s values are only decided by the network structure and transmission constraint functions but not related to the initial states of MAS (1).

The following theorem can be regarded as a combination of Theorems 3 and 4, which gives the conditions for the system to converge to an asymptotically stable equilibrium.

Theorem 5: Along the system (1), suppose the following conditions hold:

1) Assumption 2 holds;
2) the consensus zone \( \Phi = \emptyset \);
3) for any $j \in \mathbb{N}$, $i \in N_j$ and $\omega \neq 0$

$$-1 < \frac{f_{ij}(x + \omega) - f_{ij}(x)}{\omega} < 1, \quad x \notin \Theta_{ij}.$$ 

Then there is a unique, asymptotically stable equilibrium of the MAS (1).

**Remark 10:** Theorem 4 requires a known equilibrium of MAS. In contrast, Theorem 5 relaxes the condition that the equilibrium is known, i.e., we just need to know the constraints’ functions and the connectivity of interaction networks, then we can predict the trajectory of MAS. In conclusion, Theorem 5 states the existence, stability, and uniqueness of Equilibria.

**Corollary 2:** Suppose the directed graph $G$ is strongly connected and the consensus zone $\Phi = \emptyset$. If for all $j \in \mathbb{N}$, $i \in N_j$, $f_{ij}(x_i) = k_{ij}x_i + m_{ij}$ is a continuous and piecewise linear function with its slopes $k_{ij} \in (-1, 1]$ and $m_{ij} = 0$ when $k_{ij} = 1$, then the MAS (1) has a unique, asymptotically stable equilibrium.

**IV. NUMERICAL EXAMPLE**

In this section, four numeral examples are presented to illustrate the theorems and corollaries proposed in this article. Examples 1 and 2 illustrate the consensus theorem. Additionally, Example 3 illustrates the theorem for stability, uniqueness of equilibrium, i.e., Theorem 5.

In all of the following examples, the interaction networks are strongly connected and the number of agents $n = 5$. For simplicity, in examples 2 and 3, let $f_{ij}(x) = f_i(x) \forall j \in \mathbb{N}$.

**Example 1:** The adjacency matrix in this example is

$$A_1 = \begin{bmatrix}
0 & 0 & 3.6 & 0 & 0 \\
0 & 0 & 4.6 & 1.3 & 6.5 \\
3.6 & 0 & 0 & 0 & 7.6 \\
0.5 & 1.4 & 2.1 & 0 & 0 \\
2.9 & 6.5 & 0 & 0 & 0
\end{bmatrix}.$$ 

Fig. 1 shows candidates for the constraint function imposed on the information transmissions, and the configuration of constraint functions is given in Table I. It is easy to know that the consensus zone $\Phi = \{0\}$. Fig. 2 shows that MAS achieves transmission-constrained consensus with $\lim_{t \to \infty} x(t) = 0$.

**TABLE I**

| Transmission Constraints $f_{ij}$ |
|-----------------------------------|
| $f_{ij}$ | 1 | 2 | 3 | 4 | 5 |
| $f_{ij}$ | $f_i$ | $f_i$ | $(f_i + f_d)/2$ | $f_i$ | $f_i$ |
| $f_{ij}$ | $f_i$ | $f_i$ | $f_i$ | $f_i$ | $f_i$ |
| $f_{ij}$ | $f_i$ | $f_i$ | $f_i$ | $f_i$ | $f_i$ |

**Fig. 1.** Transmission constraint functions in Example 1.

**Fig. 2.** Trajectories of $x(t)$ in Example 1.

**Fig. 3.** Constraint functions in Example 2.

**Fig. 4.** Trajectories of $x(t)$ in Example 2.
In this example, the constraint function \( f_c(x) \) is a piecewise continuous function similar to a sawtooth wave. And the constraint function \( f_c(x) \) can be chosen approximately as the boundary rays since it satisfies the Condition 2) of Theorem 2 and \( k_1 k_2 = 0.8 < 1 \).

**Example 2:** The adjacency matrix in this example is

\[
A_2 = \begin{bmatrix}
0 & 2.5 & 0.6 & 0 & 0 \\
0 & 0 & 0 & 0 & 4.5 \\
0 & 5.6 & 0 & 3.3 & 0 \\
0.5 & 0 & 0 & 0 & 0 \\
1.9 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

In Fig. 3, we can obtain consensus zone \( \Phi = [-1, 1]^5 \). The auxiliary line in Fig. 3 represents the boundary rays with \( k_1 k_2 = 0.64 < 1 \). Fig. 4 shows that \( x(t) \to \Phi \) as \( t \to \infty \).

**Example 3:** The adjacency matrix in this continuous-time example is \( A_2 \). Fig. 5 shows that regardless which initial states the agents are, MAS will reach the same equilibrium, i.e., unique equilibrium (see the points \( (e_i, f_i(e_i)) \) shown by circles in Fig. 6). It is easy to know that for any equilibrium \( e \), there exist two clusters of rays (see auxiliary lines in Fig. 6) satisfying the Condition 3) of Theorem 4 or the Condition 3) of Theorem 5, which means that the system will converge to a unique, asymptotically stable equilibrium even though we do not know the value of equilibrium.

**V. Conclusion**

This article studies the transmission-constrained consensus problem of multiagent networks, where information transmissions between agents are affected by irregular constraint functions. We obtain the necessary and sufficient conditions about the range of transmission constraint functions where agents’ states can converge to consensus. Due to the piecewise continuous constraint functions, the LaSalle invariance principle is not applicable in those proofs. We construct a sophisticated Lyapunov function and discuss the boundaries of multiple limit points of MAS states to facilitate the convergence analysis. Meanwhile, in some cases where the system cannot achieve consensus, there is an asymptotically stable and unique equilibrium independent of the initial values of agents’ states. Finally, the numerical simulations are presented to verify the effectiveness of the proposed theoretical results.

**Appendix A**

**Proof of Theorem 1**

First of all, we introduce some technical lemmas.

**Lemma 1** (See [17, Lemma 2.2]): If for all \( i \in \mathbb{N} \), \( Z_i(x) : \mathbb{R}^d \to \mathbb{R} \) is of class \( C^1 \), and denote \( Z(y) = \max_{i \in \mathbb{N}} Z_i(y) \). Denote \( N_m(t) = \{i \in \mathbb{N} : Z_i(y) = Z_i(y)\} \), the indices set in which the maximum is reached at time \( t \). Then it turns out that \( d^+ Z(y(t)) = \max_{i \in N_m(t)} \dot{Z}_i(y(t)) \).

**Lemma 2:** If \( \partial \in [\partial_m, \partial_M] \) and \( k_1 k_2 = 1 \), then

\[
\partial_M - \partial_m \geq \min\{1 - k_2\} (\partial_M - \partial_m) \}
\]

**Proof:** When \( \partial_M - \partial_m = 0 \) or \( \partial_M - \partial_m \geq 0 \) or \( \partial_m - \partial_M \leq 0 \), the conclusion is obvious.

When \( \partial_M - \partial_m > 0 \) and \( (\partial_M - \partial_m)(\partial_m - \partial_M) > 0 \), we use a contradiction argument to prove it. Let \( \partial_M - \partial_m < \min\{1 - k_2\} (\partial_M - \partial_m) \}
\]

and denote \( N_m(t) \), we let \( \partial = \partial \partial_M + (1 - \rho) \partial_m \) with \( \rho \in (0, 1) \). Then, (4) can be rewritten as

\[
\begin{aligned}
\partial_M - \partial_m &< (1 - k_2) (\partial_M - \partial_m) \\
\partial_M - \partial_m &< (1 - k_2) (\partial_m - \partial_M)
\end{aligned}
\]

Then, we can get that \( -k_2 > \frac{\partial_m}{\partial_M} > 0 \) and \( -k_1 > \frac{\partial_m}{\partial_M} > 0 \), which implies that \( k_1 k_2 > 1 \). We get a contradiction.

**The following lemma shows the implicit inequality from the given condition, and it helps us discuss the boundedness of transmission-constrained consensus dynamics.**

**Lemma 3:** Denote \( x_M(t) = \min_{i \in \mathbb{N}} x_i(t), x_M(t) = \max_{i \in \mathbb{N}} x_i(t) \). \( L_1(x_i(t)) = k_1 x_i(t) + (1 - k_1) \partial \) \( L_2(x_i(t)) = k_2 x_i(t) + (1 - k_2) \partial \) and

\[
Y(t) = \max\{L_1(x_M(t)) - x_M(t), x_M(t) - L_2(x_M(t))\}
\]

where \( \partial \) is a constant value. If \( k_1, k_2 < 0 \) and \( k_1 k_2 = 1 \), then \( \forall i \in \mathbb{N} \)

1) \( Y(t) = x_M(t) - L_2(x_M(t)) \to x_M(t) \geq L_1(x_i(t)) \)
2) \( Y(t) = L_1(x_m(t)) - x_m(t) \Rightarrow x_m(t) \leq L_2(x_i(t)) \).

**Proof:** We first discuss Item 1, i.e., the case where \( Y(t) = x_m(t) - L_2(x_m(t)) \), which implies that \( x_m(t) \geq \partial \).

Note that for any \( i \in \mathbb{N} \), if \( x_i(t) \geq \partial \), it is easy to know that
\[
x_m(t) \geq \partial \geq \partial + (1 - k_1)(\partial - x_i(t)) = L_1(x_i(t)).
\]

For any \( i \in \mathbb{N} \), if \( x_i(t) < \partial \)
\[
Y(t) = x_m(t) - L_2(x_m(t)) \\
\iff (1 - k_2)(x_m(t) - \partial) \geq (1 - k_1)(\partial - x_i(t)) \\
\iff x_m(t) - \partial \geq \frac{1 - k_1}{1 - k_2}(\partial - x_i(t)) \\
\iff x_m(t) - \partial \geq k_1(x_i(t) - \partial).
\]

Then we get \( x_m(t) - k_1x_i(t) - (1 - k_1)\partial \geq 0 \), which implies
\[
x_m(t) \geq L_1(x_i(t)) = k_1x_i(t) + (1 - k_1)\partial \quad \forall i \in \mathbb{N}.
\]

Therefore, the proof of Item 1 is completed. The proof method of Item 2 is similar to that of Item 1 and, hence, is omitted here.

**Lemma 4:** For the MAS (1), if there is an interval \([\partial_m, \partial_M]^n\) such that for all \( j \in \mathbb{N}, i \in \mathcal{N}_j \)
\[
\partial_m \leq f_{ij}(x) \leq \partial_M, \quad x \in [\partial_m, \partial_M]
\]
then \([\partial_m, \partial_M]^n\) is a positively invariant set.

**Proof:** The dynamics of MAS (1) can be rewritten as
\[
\dot{x}(t) = h(x(t)) = (h_1(x(t)), \ldots, h_n(x(t)))^T
\]
where \( h_i(x(t)) = \sum_{j \in \mathbb{N}} a_{ij} [f_{ji}(x_j(t)) - x_i(t)] \).

The initial state of (5) is \( x_0(t_0) \). Assume that \( x(t_0) \in [\partial_m, \partial_M]^n \). Since the vector field \( h \) is pointing inward \([\partial_m, \partial_M]^n\); which is an \( n \)-dimensional cube, it concludes that
\[
x(t) \in [\partial_m, \partial_M]^n \quad \forall t \geq t_0.
\]

It shows that \([\partial_m, \partial_M]^n\) is a positively invariant set.

**Lemma 5:** Along the system (1), suppose there exist an interval \([\partial_m, \partial_M]\), a value \( \partial \in [\partial_m, \partial_M]\), and two rays
\[
L_1(x) = k_1(x - \partial) + \partial, \quad x \in (-\infty, \partial] \\
L_2(x) = k_2(x - \partial) + \partial, \quad x \in [\partial, +\infty)
\]
where \( k_1k_2 = 1 \) such that \( \forall j \in \mathbb{N}, i \in \mathcal{N}_j \)
\[
x \leq f_{ij}(x) \leq L_1(x), \quad x \in (-\infty, \partial_m) \\
\partial_m \leq f_{ij}(x) \leq \partial_M, \quad x \in [\partial_m, \partial_M] \\
L_2(x) \leq f_{ij}(x) \leq x, \quad x \in (\partial_M, +\infty).
\]

Denote \( x_m(t) = \min_{i \in \mathbb{N}} x_i(t) \), \( x_m(t) = \max_{i \in \mathbb{N}} x_i(t) \) and
\[
Y(t) = \max \{\partial_m - x_m(t), x_m(t) - \partial_m, \partial_M - x_m(t), x_m(t) - L_2(x_m(t)), L_1(x_m(t)) - x_m(t)\}.
\]
If \( \partial_m - x_m(t) \geq \max\{(1 - k_2)(\partial_M - \partial), (1 - k_1)(\partial - x_m(t))\} \),
then \( Y(t) \) is a nonincreasing function for any initial state \( x_s \in \mathbb{R}^n \).

**Proof:** By the structure of \( Y(t) \), there exists five cases:

1) \( Y(t) = Y_1(t) = x_m(t) - L_2(x_m(t)) \);
2) \( Y(t) = Y_2(t) = x_m(t) - \partial_m \);
3) \( Y(t) = Y_3(t) = L_1(x_m(t)) - x_m(t) \);
4) \( Y(t) = Y_4(t) = \partial_M - x_m(t) \);
5) \( Y(t) = Y_5(t) = \partial_M - \partial_m \).

At first, we analyze Case 1. Since
\[
x_m(t) - L_2(x_m(t)) \geq \partial_M - \partial_m \geq (1 - k_2)(\partial_M - \partial)
\]
it means that \( x_m(t) \geq \partial_M \).
Denote \( \mathcal{I}_M(t) = \{k : x_k(t) = \max_{i \in \mathbb{N}} x_i(t)\} \). By Lemma 1, we have
\[
d^+ Y_1(t) = d^+ \max_{i \in \mathcal{I}_M(t)} \{(1 - k_2)(x_i(t) - \partial)\}
= \max_{i \in \mathcal{I}_M(t)} \{(1 - k_2) \sum_{j \in \mathcal{N}_i(t)} a_{ij} (f_{ji}(x_j(t)) - x_i(t))\}.
\]
For all \( i \in \mathcal{I}_M(t) \), which implies that \( x_i(t) = x_m(t) \), we conduct the following analysis:
\[
\begin{align*}
x_m(t) & \geq L_1(x_i(t)) > f_{ji}(x_j), \quad \text{if } x_j \leq \partial_m \\
x_m(t) & \geq \partial_M > f_{ji}(x_j), \quad \text{if } x_j \in [\partial_m, \partial_M] \\
x_m(t) & \geq x_j > f_{ji}(x_j), \quad \text{if } x_j > \partial_M
\end{align*}
\]
where the first inequality follows from Lemma 3 and the fact that \( x_m(t) - L_2(x_m(t)) \geq L_1(x_m(t)) - x_m(t) \).
Then, it can be concluded that \( d^+ Y_1(t) \leq 0 \) when \( Y(t) = Y_1(t) = x_m(t) - L_2(x_m(t)) \).
Second, we discuss Case 2. Since \( x_m(t) - \partial_m \geq \partial_M - \partial_m \),
then it shows that \( x_m(t) \geq \partial_M \).
Since \( x_m(t) - \partial_m \geq L_1(x_m(t)) - x_m(t) \), we can get that
\[
\begin{align*}
x_m(t) & \geq k_1x_m(t) + (1 - k_1)\partial + \partial_M - x_m(t) \\
& = L_1(x_m(t)) + \partial_M - x_m(t) \\
& \geq L_1(x_i(t)) + \partial_M - x_m(t) \quad \forall i \in \mathbb{N}.
\end{align*}
\]
It turns out that
\[
\begin{align*}
x_m(t) & \geq L_1(x_i(t)) > f_{ji}(x_j), \quad \text{if } x_j \leq \partial_m \\
x_m(t) & \geq \partial_M > f_{ji}(x_j), \quad \text{if } x_j \in [\partial_m, \partial_M] \\
x_m(t) & \geq x_j > f_{ji}(x_j), \quad \text{if } x_j > \partial_M
\end{align*}
\]
where the first inequality follows from \( x_M \geq L_1(x_i) + \partial_m - x_m \geq L_1(x_i) \) when \( x_m \leq \partial_m \).
Then, it shows that
\[
d^+ Y_2(t) = d^+ \max_{i \in \mathcal{I}_M(t)} \left\{ \sum_{j \in \mathcal{N}_i(t)} a_{ij} (f_{ji}(x_j(t)) - x_i(t)) \right\} \leq 0.
\]
The analyses of Cases 3 and 4 are symmetric to those of Cases 1 and 2; hence, they are omitted. As for Case 5, the conclusion is obvious.
Therefore, by the above five cases, it can be concluded that \( Y(t) \) is a nonincreasing function.

**Lemma 6:** Suppose the MAS (1) satisfies the conditions in Lemma 5. Denote the initial time \( t_0 \). Then for any \( t \geq t_0 \), we have
1) \( Y(t_0) = \partial_M - \partial_m \Rightarrow \partial_m \leq x_i(t) \leq \partial_M \quad \forall i \in \mathbb{N} \);
2) \( Y(t_0) = x_M(t_0) - L_2(x_M(t_0)) \Rightarrow x_m(t) \geq L_2(x_M(t_0)); \)
3) \( Y(t_0) = L_1(x_m(t_0)) - x_m(t_0) \Rightarrow x_m(t) \leq L_1(x_m(t_0)); \)
4) \( Y(t_0) = x_M(t_0) - \partial M \Rightarrow x_m(t) \geq \min\{x_m(t_0), \partial M\}; \)
5) \( Y(t_0) = \partial M - x_m(t_0) \Rightarrow x_m(t) \leq \max\{x_M(t_0), \partial M\}. \)

**Proof:** \( Y(t_0) = [\partial M - \partial m] \) means that \( x(t_0) \in [\partial m, \partial M]\).

By Lemma 4, we can get that \([\partial m, \partial M]\) is a positively invariant set, and this case is proven trivially.

When \( Y(t_0) = x_M(t_0) - L_2(x_M(t_0)), \) if \( x_m(t) < L_2(x_M(t_0)), \) we have

\[
Y(t) \geq L_1(x_m(t)) - x_m(t) > (1 - k_1)(\partial - L_2(x_M(t_0)))
\]

\[
= -k_2(1 - k_1)(x_M(t_0) - \partial) = (1 - k_2)(x_M(t_0) - \partial)
\]

\[
Y(t_0)
\]

which contradicts Lemma 5. By symmetry, the case where \( Y(t_0) = L_1(x_m(t_0)) - x_m(t_0) \) is also proven.

When \( Y(t_0) = x_M(t_0) - \partial M, \) it means that \( L_2(x_M(t_0)) \geq \partial M. \)

Since \( x_M(t) \leq x_M(t_0) \) \( \forall t \geq t_0, \) we have \( L_2(x_M(t)) \geq L_2(x_M(t_0)), \) \( \forall i \in \mathbb{N}, t \geq t_0. \) Hence, it can be concluded that \( f_{ji}(x_i(t)) \geq \min\{x_i(t), \partial, M, L_2(x_j(t))\} \geq \min\{x_m(t), \partial m\} \) \( \forall i, j \in \mathbb{N}. \) Therefore, we can get that \( x_m(t) \geq \min\{x_m(t_0), \partial m\}. \) By symmetry, the case where \( Y(t_0) = \partial M - x_m(t_0) \) is also proven.

When \( Y(t_0) = \partial M - x_m(t_0), \) it means that \( L_1(x_m(t_0)) \leq \partial M. \)

Since \( x_m(t) \geq x_m(t_0) \) \( \forall t \geq t_0, \) it can be concluded that \( f_{ji}(x_i(t)) \leq \max\{x_i(t), \partial, M, L_1(x_j(t))\} \leq \max\{x_M(t), \partial M\} \) \( \forall i, j \in \mathbb{N}. \) Then, use a contradiction argument to prove that \( x_M(t) \leq \max\{x_M(t_0), \partial M\} \) \( \forall t \geq t_0. \) Assume that \( \exists t_s \geq t_0, x_M(t_s) > \max\{x_M(t_0), \partial M\}. \) Hence, there is a \( T \geq t_0 \) such that for any \( t \in [t_0, T], \)

\[
d^+x_M(T) = \max_{i \in \mathbb{N}_i} \dot{x}_i(T)
\]

\[
= \max_{i \in \mathbb{N}_i} \sum_{j \in \mathbb{N}_j(t)} a_{ij} (f_{ji}(x_j(T)) - x_i(T))
\]

\[
\leq \max_{i \in \mathbb{N}_i} \sum_{j \in \mathbb{N}_j} a_{ij} \left( \max\{x_i(T), \partial, M\} - x_i(T) \right)
\]

\[
= \max_{i \in \mathbb{N}_i} \sum_{j \in \mathbb{N}_j} a_{ij} \left( \max\{x_M(t_0), \partial M\} - \min\{x_M(t_0), \partial M\} \right)
\]

\[
= 0
\]

which leads to a contradiction and it shows that \( x_M(t) \leq \max\{x_M(t_0), \partial M\} \) \( \forall t \geq t_0. \) By symmetry, the case where \( Y(t_0) = \partial M - x_m(t_0) \) is also proven.

**B. Proof of Theorem 1**

1) **Necessity:** Proof: We use a contradiction argument.

For simplicity, we assume that there are only two agents in MAS (1), i.e., \( N = \{1, 2\}. \) Since \( G \) is strongly connected, it turns out that \( N_1 = \{2\} \) and \( N_2 = \{1\}. \)

Denote the initial time \( t_0 \geq 0. \) Suppose there exist \( j \in \mathbb{N}, i \in \mathbb{N}_j, \) and \( x_j(t_0) \in [\partial m, \partial M] \) such that

\[
f_{ji}(x_j(t_0)) = \partial M + \omega, \quad \omega > 0.
\]

Without loss of generality, assume that \( j = 1 \) and \( i = 2, \) i.e., \( f_{21}(x_j(t_0)) = \partial M + \omega \) in which \( x_j(t_0) \in [\partial m, \partial M]. \) Let \( x_2(t_0) = f_{21}(x_1(t_0)) \) and \( f_{21}(x_2(t_0)) = x_1(t_0), \) then it can be concluded that for all \( t \geq t_0 \)

\[
\dot{x}_1(t) = a_{12} (f_{21}(x_2(t)) - x_1(t)) = 0
\]

\[
\dot{x}_2(t) = a_{21} (f_{21}(x_1(t)) - x_2(t)) = 0
\]

which implies that \( x_1(t) = x_1(t_0), x_2(t) = x_2(t_0) \) \( \forall t \geq t_0. \) Moreover, because

\[
\begin{cases}
    x_1(t_0) \in [\partial m, \partial M] \\
    f_{21}(x_1(t_0)) = \partial M + \omega > \partial M \\
    x_2(t_0) = f_{21}(x_1(t_0)) = \partial M + \omega > \partial M \\
    f_{21}(x_2(t_0)) = x_1(t_0) \leq \partial M
\end{cases}
\]

it is easy to find two rays \( L_1 \) and \( L_2 \) satisfying the Assumptions 3 and 4, and the condition \( k_1k_2 = 1 \) is also satisfied.

Since \( \forall t \geq t_0, x_2(t) = \partial M + \omega > \partial M, \) it shows that

\[
\lim_{t \to \infty} (\partial m, \partial M, x(t) \neq 0.
\]

Hence, we get a contradiction.

2) **Sufficiency:** We prove it in three steps.

**Step 1:** Since \( \partial \in [\partial m, \partial M] \) and \( k_1k_2 < 1, \) by Lemma 2, there exist the following two possibilities:

1) \( \partial M - \partial m \geq \max\{(1 - k_2) (\partial M - \partial), (1 - k_1) (\partial - \partial m)\}; \)
2) \( \partial M - \partial m < (1 - k_2) (\partial M - \partial) \) or \( \partial M - \partial m < (1 - k_1) (\partial - \partial m). \)

We discuss the Possibility 1) in the rest of Step 1, and the Possibility 2) is analyzed in Step 2.

Assume that \( \partial M - \partial m \geq \max\{(1 - k_2)(\partial M - \partial), (1 - k_1)(\partial - \partial m)\}. \) From Lemma 5, we have that \( Y(t) \) is nonincreasing.

Let \( n \) be the number of agents. Since

\[
Y(t) = \max\{\partial M - \partial m, x_M(t) - \partial M, \partial M - x_m(t), x_M(t) - L_2(x_M(t)), L_1(x_m(t)) - x_m(t)\}
\]

we continue this proof case by case.

**Case 1:** \( Y(t_0) = Y(t_0) = x_M(t_0) - L_2(x_M(t_0)). \)

By Lemma 6, we have for any \( t \geq t_0 \)

\[
L_1(x_m(t)) = k_1 (x_m(t) - \partial) + \partial
\]

\[
\leq k_1k_2x_m(t) + k_1(1 - k_2) \partial + (1 - k_1) \partial = x_m(t_0).
\]

Choose \( i_0 \in I_0 := \{i : x_i(t_0) = x_m(t_0)\}. \) For any \( j \in N_{i_0}, \)

we can get that

\[
f_{j_{i_0}}(x(t)) \leq \max\{x_M(t), L_1(x_m(t)), \partial M\} \leq x_m(t_0).
\]

It turns out that

\[
\dot{x}_{i_0}(t) = \sum_{j \in N_{i_0}} a_{i_0j} [f_{ji}(x_j(t)) - x_{i_0}(t)]
\]

\[
\leq a_{i_0j} [x_M(t) - x_{i_0}(t)]
\]
which implies that
\[ x_{i_0}(t) \leq e^{-\alpha_0 (t-t_0)} x_m(t_0) + \left[ 1 - e^{-\alpha_0 (t-t_0)} \right] x_M(t_0). \]
If \( t \in [t_0, t_0 + \tau] \), then we have for any \( i_0 \in I_0 \)
\[ x_{i_0}(t) \leq \gamma_0 x_m(t_0) + (1 - \gamma_0) x_M(t_0) \quad (6) \]
where \( \gamma_0 = e^{-\tau \alpha_0}. \)
Choose \( i_0 \in I_0 := \{ i : \exists j \in I_0, j \in N_i \} \). By the conditions of Theorem 1 and (6), it is trivial to get that \( f_{i_0i_1}(x_{i_0}(t)) \leq x_M(t_0) \). Hence, for any \( t \in [t_0, t_0 + \tau \omega / n] \), there exists a constant \( \gamma_0' \in (0, 1) \) such that
\[ f_{i_0i_1}(x_{i_0}(t)) \leq \gamma_0' x_m(t_0) + (1 - \gamma_0') x_M(t_0). \]
Then, we can get that
\[

x_{i_1}\left(t_0 + \frac{\tau}{n}\right) \\
\leq e^{-\alpha_0 \frac{\tau}{n}} x_{i_0}(t_0) + [a_{i_0i_1} \gamma_0' (x_m(t_0) - x_M(t_0))] \\
+ \alpha_1 x_M(t_0) \int_{t_0}^{t_0 + \frac{\tau}{n}} e^{-\alpha_0 (t_0 + \frac{\tau}{n} - s)} ds \\
\leq e^{-\alpha_0 \frac{\tau}{n}} x_M(t_0) + (1 - e^{-\alpha_0 \frac{\tau}{n}}) x_M(t_0) \\
+ a_{i_0i_1} \gamma_0' (x_m(t_0) - x_M(t_0)) \int_{t_0}^{t_0 + \frac{\tau}{n}} e^{-\alpha_0 (t_0 + \frac{\tau}{n} - s)} ds \\
= x_M(t_0) + \frac{a_{i_0i_1}}{\alpha_1} (1 - e^{-\alpha_0 \frac{\tau}{n}}) \gamma_0' (x_m(t_0) - x_M(t_0)).
\]
Since there is a constant \( \rho_1 > 0 \) such that for any \( i_1 \in I_1, i_0 \in N_i_1, \rho_1 \leq \frac{a_{i_0i_1}}{\alpha_1} (1 - e^{-\alpha_0 \frac{\tau}{n}}) \). Therefore, it turns out that
\[ x_{i_1}(t_0 + \frac{\tau}{n}) \leq \rho_1 \gamma_0' x_m(t_0) + (1 - \rho_1 \gamma_0') x_M(t_0). \] Similar to (6), we can get that for any \( i_1 \in I_1, i_0 \in N_i_1, t_0 \in [t_0, t_0 + \tau + \tau \omega / n] \)
\[ x_{i_1}(t) \leq \gamma_1 x_m(t_0) + (1 - \gamma_1) x_M(t_0) \quad (7) \]
where \( \gamma_1 = \rho_1 \gamma_0' \gamma_0. \) Continuing the above mentioned analysis over \([t_0 + \frac{\tau}{m}, t_0 + \tau + \tau \omega / n] \) \( m = 1, 2, \ldots, n - 1 \), it can be concluded that for all \( i \in N \)
\[ x_i(t_0 + \tau) \leq \gamma_{n-1} x_m(t_0) + (1 - \gamma_{n-1}) x_M(t_0) \]
where \( \gamma_{n-1} = \rho_1 - 1 \gamma_{n-2} \gamma_0. \)
If \( x_m(t_0) < x_M(t_0) \), then there exists a constant and \( \omega \in (0, 1] \) such that \( x_m(t_0) \leq \omega \partial (1 - \omega) x_M(t_0) \). Here, we have
\[ Y_1(t_0 + \tau) = (1 - k_2)(x_m(t_0 + \tau) - \partial) \leq (1 - \omega \gamma_{n-1}) Y_1(t_0). \]
If \( x_m(t_0) = x_M(t_0) > \partial M \), we use the Assumption 4 to get the convergence. Since there exists \( i_0, i_1 \) such that \( f_{i_0i_1}(x_{i_0}(t_0)) \) is continuous on \( x_{i_0}(t_0) = x_M(t_0) \) and \( f_{i_0i_1}(x_{i_0}(t_0)) < x_M(t_0) \). Hence, there exists \( T(\omega') \) such that \( \forall t \in [t_0, t_0 + T(\omega')] \)
\[ f_{i_0i_1}(x_{i_0}(t)) \leq \omega \partial + (1 - \omega') x_M(t_0), \] where the constant \( \omega' \in (0, 1] \). Similar to (7), we have \( \forall t \in [t_0 + T(\omega'), t_0 + nT(\omega')] \)
\[ x_{i_1}(t) \leq \omega_1 \partial + (1 - \omega_1) x_M(t_0) \]
where \( \omega_1 = \rho \omega \gamma_0. \) Furthermore, it shows that
\[ Y_1(t_0 + nT(\omega')) \leq (1 - \omega_{n-1}) Y_1(t_0) \]
where \( \omega_{n-1} = \rho_{n-1} \omega_{n-2} \gamma_0. \)

The analysis of Case 1 is completed. The case \( Y(t_0) = Y_3(t_0) = L_1(x_m(t_0)) - x_m(t_0) \) is symmetric to Case 1, so we omit its analysis.

**Case 2:** \( Y(t_0) = Y_2(t_0) = x_M(t_0) - \partial M. \)
By Lemma 6, it turns that \( \forall i \in N, t \geq t_0, x_M(t_0) \geq L_1(x_i(t)). \) Hence, it is trivial to get that
\[ x_i(t_0 + \tau) \leq \gamma_{n-1} x_m(t_0) + (1 - \gamma_{n-1}) x_M(t_0) \quad \forall i \in N. \]
Furthermore, we can get that \( Y_2(t_0 + \tau) \leq (1 - \omega \gamma_{n-1}) Y_2(t_0) \) or \( Y_2(t_0 + nT(\omega')) \leq (1 - \omega \gamma_{n-1}) Y_2(t_0). \) The case \( Y(t_0) = Y_3(t_0) = \partial M - x_m(t_0) \) is symmetric to this case, so we omit its analysis.

Finally, it concludes that \( x(t) \rightarrow [\partial_m, \partial_M] \) as \( t \rightarrow \infty \), and the proof of the Possibility 1 is completed.

**Step 2:** In this step, we will complete the proof of Possibility 2, i.e., \( \partial_M - \partial_m < 1 - k_2 \) \( \partial_M - \partial_m < 1 - k_1 \) \( \partial_M - \partial_m > k_2 \) \( \partial_M - \partial_m > k_1 \) \( \partial_M - \partial_m \). By symmetry, we let \( \partial_M - \partial_m < 1 - k_1 \) \( \partial_M - \partial_m > k_2 \) \( \partial_M - \partial_m \). Denote
\[ Y'(t) = \max \{ \partial_M - \partial_m, x_M(t) - \partial_m, \partial_M - x_m(t) \} \]
\( (1 - k_2)(x_M(t) - \partial_m), (1 - k_1)(\partial - x_m(t)) \) \( \partial_M - \partial_m \).

Similar to the proof in Step 1, it concludes that \( x(t) \rightarrow [\partial_m, \partial_M] \) as \( t \rightarrow \infty \). Then, we use a contradiction argument to prove that for any solution \( x(t) \rightarrow [\partial_m, \partial_M] \) as \( t \rightarrow \infty \).

Assume that there exist a solution \( \hat{x}(t) \) and \( \hat{x}'(t) \) such that \( \hat{x}_i(t) \rightarrow [\partial_M, \partial_M] \) as \( t \rightarrow \infty \). Then, there is a \( T^* \) such that for any \( t > T^* \) and \( j \in N, f_{i_0j}(\hat{x}_{i_0j}(t)) > \partial_m \).

Since the directed graph \( \mathcal{G} \) is strongly connected, we can get that there exists a \( T^* \) such that \( \forall t > T^*, \forall \mathcal{G} \rightarrow \mathcal{G}_i \).

Denote \( Z(t) = \max \{ \hat{x}_M(t), \partial_M \}. \) Repeat the analysis of Step 1, it shows that \( \lim_{t \to \infty} Z(t) = \partial_M \), which implies that \( \limsup_{t \to \infty} \hat{x}_i(t) \leq \partial_M \).

Therefore, the trajectory of \( \hat{x}_i(t) \) cannot converge to \( \partial_M, \partial_M \) as \( t \rightarrow \infty \). Here, we prove that for any solution \( x(t) \rightarrow [\partial_m, \partial_M] \) as \( t \rightarrow \infty \).

**Remark 11:** In the sufficiency proof, the idea of constructing auxiliary variables to analyze the boundedness of MAS is inspired by Fontan et al. [6]. However, since our dynamics is not Lipschitz continuous, the LaSalle invariance principle is not applicable. We design some linear boundaries and propose the corresponding lemmas to eliminate nonlinearity and analyze the states’ tending to obtain the boundedness of agents’ states.

**APPENDIX B**

**PROOF OF** **THEOREM 2**

**A. Technical Lemma**

**Lemma 7** (See [3, Prop. 4.10]): Let graph \( \mathcal{G} \) has a directed spanning tree, and consider the dynamics of MAS defined over \( \mathcal{G} \)
\[ \dot{x}_i(t) = \sum_{j=1}^{N} a_{ij} (x_j(t) - x_i(t)) + \theta_i(t), \quad i = N \]
in which \( \theta_i(t) \) is piecewise continuous on \([t_0, \infty)\) and is finite. If
\[
\lim_{t \to \infty} \theta_i(t) = 0 \quad \forall i \in \mathbb{N},
\]
then \( \lim_{t \to \infty} x_i(t) - x_j(t) = 0 \quad \forall i, j \in \mathbb{N} \).

\section*{B. Proof of Theorem 2}

1) \textbf{Necessity: Proof:} An antagonistic interaction means that the underlying edges between agents have negative weights. A signed graph \( G_A \) is \textit{structurally balanced} if there exists a bipartition \( \{V_1, V_2\} \) of the nodes, where \( V_1 \cup V_2 = V \), \( V_1 \cap V_2 = \emptyset \) such that \( a_{ij} \geq 0 \quad \forall v_i, v_j \in V_m \quad (m \in \{1, 2\}) \) and \( a_{ij} \leq 0 \quad \forall v_i \in V_1, v_j \in V_2, \quad m \neq l \quad (m, l \in \{1, 2\}) \) [18]. We use a contradiction argument. Consider the MAS with antagonistic interactions

\[
\dot{x}_i(t) = \sum_{j=1}^{N} [a_{ij}^s \cdot \text{sgn}(a_{ij}) \cdot x_j(t) - x_i(t)]
\]

(8)
in which the signed graph \( G_s = \{V_s, E_s, A_s = [a_{ij}^s]\} \), and suppose the signed graph \( G_s \) is strongly connected and structurally balanced. By the bipartite consensus theorem (see [18, Th. 2]), it shows that the system (8) reaches bipartite consensus but not consensus, in which agents’ state values are the same except for the sign.

Let \( A^* = \{A_s\} \) and \( G^* = \{V_s, E_s, A^*\} \), i.e., \( G^* \) is a strongly connected graph with only cooperative interactions. Assume the MAS (1) is under the graph \( G^* \), and for all \( i \in \mathbb{N}, j \in \mathcal{N}_i, \quad f_{ji}(x_j(t)) = \text{sgn}(a_{ij}^s) \cdot x_j(t) \), which implies that \( k_1 k_2 > 1 \). With the above assumption, the dynamics of MAS (1) is equivalent to the dynamics of MAS (8). Therefore, it turns out that MAS (1) cannot achieve consensus.

On the other hand, it is obvious that under the above assumption, the system (1) satisfies all conditions in Theorem 2. By Theorem 2, the states of agents will converge to a consensus value. Hence, we get a contradiction.

2) \textbf{Sufficiency:} Applying Theorem 1, we have

\[
x(t) \to [\partial_m, \partial_M]^n, \quad \text{as } t \to \infty.
\]

(9)

Note that if \( \partial_m = \partial_M \), then it turns out that \( \lim_{t \to \infty} x_i(t) = \partial_m = \partial_M = v^* \quad \forall i \in \mathbb{N} \).

Hence, we continue our proof in the condition that \( \partial_m < \partial_M \).

Assume \( \partial_m < \partial_M \) in the following.

\textbf{Step 1:} In this step, it is shown that the states of agents tend to achieve consensus.

Denote \( \theta_i(t) = \sum_{j \in \mathcal{N}_i} a_{ij} (f_{ji}(x_j(t)) - x_i(t)) \), and the dynamics of MAS (1) can be rewritten as

\[
\frac{d}{dt} x_i(t) = \sum_{j \in \mathcal{N}_i} a_{ij} (x_j(t) - x_i(t)) + \theta_i(t).
\]

(10)

Then, we use a contradiction argument to prove that \( \lim_{t \to \infty} \theta_i(t) = 0 \quad \forall i \in \mathbb{N} \). Without loss of generality, assume that there exists a time sequence \( \{t'_n\} \) such that \( \forall i \in \mathbb{N} \), \( \lim_{t \to t'_n} \theta_i(t) > 0 \) and \( \lim_{n \to \infty} x_i(t'_n) = \infty \) and \( \lim_{n \to \infty} x_i(t'_n) = \chi_1 \), and \( \{t'_n\} \) with \( \lim_{n \to \infty} t'_n = \infty \) and \( \lim_{n \to \infty} x_i(t'_n) = \chi_2 \), etc. Since \( \lim \inf_{t \to t'_n} \theta_i(t) > 0 \), it turns out that \( \exists i' \in \mathcal{N}_i' \) such that

\[
\sup_{x_i \in \mathcal{P}} \left( \limsup_{x_i \to \chi_1} \left( f_{i'i'}(x_i) - x_i \right) \right) > 0.
\]

Furthermore, since \( x_i(t) \to [\partial_m, \partial_M] \) as \( t \to \infty \), it can be concluded that \( \lim \sup_{x_i \to \chi_1} f_{i'i'}(x_i) > \partial_M \).

Therefore, there is a time sequence \( \{t'_n\} \to \infty \) with \( \{x_i(t'_n)\} \to \partial_m \) and \( \lim_{n \to \infty} f_{i'i'}(x_i(t'_n)) > \partial_M \). It is clear that \( \partial_m \in \mathcal{P} \).

Since \( f_{i'i'}(x) \) is continuous in \( [\partial_m, \partial_M] \) and has finite breaks in \((-\infty, \partial_m) \), there exists a time sequence \( \{t''_n\} \) and \( f_{i'i'}(x) \) is continuous on \( x_i(t) \quad \forall t \in \{t'_n\} \), which implies that \( f_{i'i'}(x_i(t)) \) is continuous on \( \{t''_n\} \).

Further, there exist \( \epsilon > 0 \) and \( T(\epsilon) \) such that \( f_{i'i'}(x_i(t)) > \partial_m \) and \( f_{i'i'}(x_i(t)) \) is continuous in \( (t - \epsilon, t + \epsilon) \), for all \( t > T(\epsilon) \) and \( t \in \{t''_n\} \). Denote closed and connected interval \( [t_k - \frac{\epsilon}{2}, t_k + \frac{\epsilon}{2}] \), where \( t_k > T(\epsilon) \) and \( t_k \in \{t''_n\}, k \in \mathbb{N}^+ \).

Since \( \lim \inf_{t \to \infty} \theta_i(t) > 0 \quad \forall i \in \mathbb{N} \), it can be concluded that \( \lim \inf_{t \to \infty} f_{i'i'}(x_i(t)) \geq \partial_m \forall i, j \in \mathbb{N} \).

Notice that \( f_{i'i'}(x_i(t)) > \partial_m \) and repeat the analysis of Theorem 1, we can get that \( \lim \inf_{t \to \infty} x_i(t) > \partial_m \). Then, it turns out that \( \partial_m \notin \mathcal{P} \). Here, we find a contradiction. Hence, we have proven that \( \lim_{t \to \infty} \theta_i(t) = 0 \quad \forall i \in \mathbb{N} \).

Applying Lemma 7, we can get that

\[
\lim_{t \to \infty} x_i(t) - x_j(t) = 0 \quad \forall i, j \in \mathbb{N}.
\]

(11)

\textbf{Step 2:} In Step 1, it shows that the states of agents will converge to consensus. In this step, by the fact that \( x(t) \to [\partial_m, \partial_M]^n \), we prove that for any \( i \in \mathbb{N}, \lim_{t \to \infty} x_i(t) = v^* \) and \( v^* \in [\partial_m, \partial_M] \).

By (9), it turns out that for any \( \omega_2 > 0 \), there exists a finite \( T_1 > 0 \), which holds the following equation:

\[
\partial_m - \omega_1 \leq x_i(t) \leq \partial_M + \omega_1 \quad \forall t \geq T_1, i \in \mathbb{N}.
\]

(12)

Without loss of generality, assume that \( \frac{\partial_m - \partial_M}{2} \leq x_k(T_k) \leq \partial_M + \omega_1 \), where \( k \) is a fixed node.

Similarly, by (11), \( \exists T_2 > 0 \) which is finite, there holds

\[
|v_i(t) - x_k(t)| \leq \omega_2 \quad \forall t \geq T_2, i \in \mathbb{N}.
\]

(13)

According to (12) and (13), let \( \omega_1 \) and \( \omega_2 \) be sufficiently small, and we get that \( \partial_m < x_i(T_k) < \partial_M + \omega_1 + \omega_2 \forall i \in \mathbb{N}, \) where \( T_k > \max(T_1, T_2) \).

Depending on whether \( \exists i \in \mathbb{N}, x_i(T_k) = \partial_m \) or not, there are two cases in the following proof.

1) \( \exists i \in \mathbb{N}, x_i(T_k) = \partial_m \). Repeating the analysis in Step 1 in the proof of Theorem 1, we can get that

\[
\bar{Y}(t) = \max_{i \in \mathbb{N}} \{(1 - k_2) (x_i(t) - \partial)\}
\]

is nonincreasing for \( \geq T_k \). Since \( 1 - k_2 > 0 \), it turns out that \( \max_{i \in \mathbb{N}} \bar{Y}(t) \) is nonincreasing for \( t \geq T_k \).

2) \( \forall i \in \mathbb{N}, \partial_m < x_i(T_k) < \partial_M \). It is easy to get

\[
\bar{Y}(x_i(T_k)) = x_i(T_k) \quad \forall i, j \in \mathbb{N}, \] and the system degenerates into a standard MAS at time \( T_k \). Therefore, it is easy to know that \( \max_{i \in \mathbb{N}} \bar{Y}(t) \) is nonincreasing for \( t \geq T_k \).
Combining the above mentioned analyses, we can conclude that \( \max_{i \in I} x_i(t) \) is nonincreasing for \( t \geq T_c \). Furthermore, \( \max_{i \in I} x_i(t) \) converges to a finite limit value (denote the value by \( \overline{x} \)). According to (11), \( \min_{i \in I} x_i(t) \) must converge to the same limit value \( \overline{x} \). Since \( \min_{i \in I} x_i(t) \leq x_j(t) \leq \max_{i \in I} x_i(t) \) \( \forall j \in \mathbb{N} \), it is trivial to get that \( \lim_{t \to \infty} x_i(t) = \overline{v} = v^* \forall i \in \mathbb{N} \).

Using (9), we can conclude that \( v^* \in [\partial_m, \partial_M] \). Based on the above analysis, it is shown that all \( x_i(t) \) will converge to a finite limit \( v^* \) and \( v^* \in [\partial_m, \partial_M] \).

**Remark 12:** The robust consensus idea is inspired by Fontan et al. [6]. However, since our dynamics is not Lipschitz continuous, the system may have multiple solutions. We analyze the limit points of multiple solutions and integrate the relevant variables over a short period to analyze the system’s convergence.

**APPENDIX C**

**PROOF OF THEOREM 3**

**Proof:** By Lemma 4, it shows \( [\partial_m, \partial_M]^n \) is a positively invariant set. By the Brouwer fixed point Theorem extended to dynamical systems [19], we can conclude that there exists an equilibrium in \( [\partial_m, \partial_M]^n \). Hence, along the system (1), the existence of equilibria is proven. By Theorem 1, it shows that

\[
\lim_{t \to \infty} \text{distance} ([\partial_m, \partial_M]^n, x(t)) = 0
\]

which implies that every equilibrium \( e \in [\partial_m, \partial_M]^n \). ■

**APPENDIX D**

**PROOF OF THEOREM 4**

**Proof:** Denote

\[
V(t) = \max_{i \in \mathbb{N}} \left\{ (1 - k_{e1}) (e_i - x_i(t)), (1 - k_{e2}) (x_i(t) - e_i) \right\}
\]

\[
= \max_{i \in \mathbb{N}} \left\{ (1 - k_{e1}) (-\varepsilon_i(t)), (1 - k_{e2}) \varepsilon_i(t) \right\}
\]

and clearly \( V(t) \) is Lipschitz continuous. At first, we will prove that \( V(t) \) is a nonincreasing function.

Denote \( \varepsilon_m(t) = \min_{i \in \mathbb{N}} \varepsilon_i(t) \), \( \varepsilon_M(t) = \max_{i \in \mathbb{N}} \varepsilon_i(t) \). By the structure of \( V(t) \), there exists two cases:

1) \( V(t) = (1 - k_{e2}) \varepsilon_M(t); \)
2) \( V(t) = (1 - k_{e1}) (-\varepsilon_m(t)). \)

We first consider the Case 1, which exists at least one intersection between functions \( f_j \) and \( f_i \) for \( i, j \in \mathbb{N} \), \( i \neq j \). Denote \( X_M = \max \{ x \in \bigcup \Theta_{ij} \} \) and \( X_m = \min \{ x \in \bigcup \Theta_{ij} \} \). Since \( \Theta_{ij} \neq \emptyset \), let \( x_{ij} \in \Theta_{ij} \), i.e., \( f_i(x_{ij}) = x_{ij} \).

By (15), it shows that

\[
\begin{align*}
&f_i(x_{ij} + \omega) \leq f_i(x_{ij}) + \omega = x_{ij} + \omega, \quad \omega > 0 \\
&f_i(x_{ij} + \omega) \geq f_i(x_{ij} + \omega) = x_{ij} + \omega, \quad \omega < 0
\end{align*}
\]

which implies that

\[
\left\{ \begin{array}{l}
-f_i(x) \leq x, \quad x \geq X_M \\
-f_i(x) \geq x, \quad x \leq X_m.
\end{array} \right.
\]

There are two parallel lines with slope \( k^* \in (-1, 0) \)

\[
\begin{align*}
&L_M(x) = k^* x + (1 - k^*) X_M \\
&L_m(x) = k^* x + (1 - k^*) X_m
\end{align*}
\]

and it can be concluded that

\[
\left\{ \begin{array}{l}
f_i(x) \leq L_M(x), \quad x \leq X_M \\
f_i(x) \geq L_m(x), \quad x \geq X_m.
\end{array} \right.
\]

Let \( L^*(x) = -x + X_M + X_m \) with slope \( k = -1 \). Since \( -1 < k^* < 0 \), it is easy to know that between \( L^* \) and the parallel lines \( L_M, L_m \) there exist two intersections \( (y_m, y_M) \) and
where \( y_M \geq y_m \), which implies that \( L_M(y_m) = y_M \) and \( L_n(y_m) = y_m \). It is easy to know that \( y_m < X_m \leq X_M < y_n \). Then, it can be concluded that
\[
\begin{align*}
&f_{ij}(x) \leq L_M(x) \leq y_M, \quad y_m \leq x \leq X_M \\
&f_{ij}(x) \geq L_n(x) \geq y_m, \quad X_m \leq x \leq y_M.
\end{align*}
\]

Combining (16) and (17), we can get that \( y_m \leq f_{ij}(x) \leq y_M \), \( y_m \leq x \leq y_M \). By Theorem 3, it shows that the system (1) has at least one asymptotically stable equilibrium.

Assume one of equilibria is \( e^* = \{e^*_1, \ldots, e^*_n\}^T \), and denote the error between the state \( x(t) \) and equilibrium \( e^* \) by \( e_i(t) = x_i(t) - e^*_i \). By (15), it turns out that for all \( j \in \mathbb{N}, i \in \mathcal{N}_j \)
\[
\begin{align*}
&\epsilon^*_i \leq f_{ij}(e_i + \epsilon^*_i) - f_{ij}(e_i) \leq k^* \epsilon^*_i, \quad \epsilon^*_i < 0 \\
&k^* \epsilon^*_i \leq f_{ij}(e_i + \epsilon^*_i) - f_{ij}(e_i) \leq \epsilon^*_i, \quad \epsilon^*_i > 0.
\end{align*}
\]
Since \( k^* \in (-1, 0) \), it shows that \( k^*k^{*} \leq 1 \). Hence, the Condition 3) of Theorem 4 holds. Because \( \bigcap_{(i,j) \in e} \Theta_{ij} = \emptyset \) and for any \( j \in \mathbb{N}, i \in \mathcal{N}_j \) and \( \omega \neq 0 \)
\[
-1 < \frac{f_{ij}(x + \omega) - f_{ij}(x)}{\omega} < 1, \quad x \notin \Theta_{ij}
\]
we can conclude that for any \( \epsilon^* \neq 0 \), there exist \( j \in \mathbb{N}, i \in \mathcal{N}_j \) and \( \delta > 0 \) such that \( f_{ij}(e_i + \epsilon^*) - f_{ij}(e_i) \neq 0 \forall \epsilon^* \in (\epsilon^* - \delta, \epsilon^* + \delta) \). Hence, the Condition 4) of Theorem 4 holds.

Apply Theorem 4, it shows that the equilibrium \( e \) is a unique, asymptotically stable equilibrium.

\[\text{B. Proof of Corollary 2}\]

\textbf{Proof}: Because \( f_{ij}(x_i) \) is a continuous and piecewise linear function with its slopes \( k_{ij} \in (-1, 1) \) and \( m_{ij} = 0 \) when \( k_{ij} = 1 \), it turns out that for any \( \omega \neq 0, -1 < \frac{f_{ij}(x + \omega) - f_{ij}(x)}{\omega} < 1, \quad x \notin \Theta_{ij} \), Apply Theorem 5, this corollary is proved.

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