ANOTHER PROOF OF THE CORONA THEOREM

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Dedicated to the memory of Junzo and Sadako Wada

Abstract. Let $H^\infty(\Delta)$ be the uniform algebra of bounded analytic functions on the open unit disc $\Delta$, and let $\mathfrak{M}(H^\infty)$ be the maximal ideal space of $H^\infty(\Delta)$. By regarding $\Delta$ as an open subset of $\mathfrak{M}(H^\infty)$, the corona problem asks whether $\Delta$ is dense in $\mathfrak{M}(H^\infty)$, which was solved affirmatively by L. Carleson. Extending the cluster value theorem to the case of finitely many functions, we provide a direct proof of the corona theorem: Let $\phi$ be a homomorphism in $\mathfrak{M}(H^\infty)$, and let $f_1, f_2, \ldots, f_N$ be functions in $H^\infty(\Delta)$. Then there is a sequence $\{\zeta_j\}$ in $\Delta$ satisfying $f_k(\zeta_j) \to \phi(f_k)$ for $k = 1, 2, \ldots, N$. On the other hand, the corona problem remains unsolved in many general settings, for instance, certain plane domains, polydiscs and balls, our approach is so natural that it may be possible to deal with such cases from another point of view.

1. Introduction

The corona problem was posed by S. Kakutani in 1941 and finally settled in 1962 by L. Carleson [2], where he introduced important techniques to solve the problem. Many new methods have been exploited since then, especially, T. Wolff [7] presented a new proof of the corona theorem in 1979. However, the author learned from O. Hatori that Kakutani had often said there would be a simple proof of the corona problem. Indeed, E. L. Stout also wrote in [18, p 32]: Carleson’s proof uses only classical analysis. It would be of great interest to have a solution to the corona problem that draws less on classical methods and more on algebraic analysis, but to the best of my knowledge, no such proof has been discovered yet. Our approach may make headway to some degree in this direction. Roughly speaking, the corona problem could be solvable only with the knowledge of Hoffman’s book [9, Chapter 10].

The usual Lebesgue and Hardy spaces in the unit circle $\mathbb{T}$ are denoted by $L^p(\mathbb{T})$ and $H^p(\mathbb{T}), 1 \leq p \leq \infty$, respectively. We usually identify $\mathbb{T}$ with $[0, 2\pi)$ and, for a function $f$ on $\mathbb{T}$, we write $f(\theta)$ for $f(e^{i\theta})$. By boundary value identification, $H^\infty(\Delta)$ may be considered as the closed subalgebra $H^\infty(\mathbb{T})$ of $L^\infty(\mathbb{T})$. Regarding $H^\infty(\Delta)$ as a uniform algebra on $\mathfrak{M}(H^\infty)$, we observe that its Shilov boundary $X$ is the maximal ideal space.
\( \mathcal{M}(L^\infty) \) of \( L^\infty(\mathbb{T}) \), which is totally disconnected. Denote by \( m \) the normalized Lebesgue measure \( dm(\theta) = d\theta/2\pi \) on \( \mathbb{T} \). Since \( L^\infty(\mathbb{T}) \) is identified with \( C(X) \), \( m \) is lifted to a measure \( \widehat{m} \) on \( X \), where each measurable set \( E \) in \( \mathbb{T} \) corresponds to an open-closed subset \( U(E) \) of \( X \). Then the family \( \{U(E)\} \) of all such open-closed subsets forms a basis for the topology of \( X \), and satisfies that \( m(E) = \widehat{m}(U(E)) \) and \( \widehat{m}(U(E)) > 0 \) unless \( U(E) = \emptyset \) (see [4, Chapter I]). Recall that the Poisson kernel for \( z \) in \( \Delta \) is given by \( P_z(\theta) = \text{Re}[(e^{i\theta} + z)/(e^{i\theta} - z)] \), so the Poisson integral by \( P_z(\theta)dm(\theta) \) is also regarded as a measure on \( X \). From now on, we identify each function in \( H^\infty(\Delta) \) with its Gelfand transform, and regard \( H^\infty(\Delta) \) as a uniformly closed subalgebra of \( C(\mathcal{M}(H^\infty)) \).

When \( \alpha \) is in \( \mathbb{T} \), the fiber \( \mathcal{M}_\alpha \) of \( \mathcal{M}(H^\infty) \) over \( \alpha \) is defined to be
\[
\mathcal{M}_\alpha = \{ \xi \in \mathcal{M}(H^\infty) \mid \xi(z) = \alpha \},
\]
where \( z \) is the coordinate function. For each function \( f \) in \( H^\infty(\Delta) \), the cluster set of \( f \) at \( \alpha \) is
\[
\text{Cl}(f, \alpha) = \bigcap_{r>0} f(\Delta \cap \{ |z-\alpha| < r \}).
\]

Then the cluster value theorem asserts that
\[
(1.1) \quad \text{Cl}(f, \alpha) = f(\mathcal{M}_\alpha), \quad f \in H^\infty(\Delta),
\]
consequently, if \( \phi \) is in \( \mathcal{M}_\alpha \), then there is a sequence \( \{\zeta_j\} \) in \( \Delta \) satisfying \( \zeta_j \to \alpha \) and \( f(\zeta_j) \to f(\phi) \). Notice that this sequence \( \{\zeta_j\} \) depends on \( f \) and \( \phi \), however, the same property holds on the uniformly closed subalgebra of \( H^\infty(\Delta) \) generated by \( f \). Recall that the open unit disc \( \Delta \) is homeomorphically embedded in \( \mathcal{M}(H^\infty) \) by identifying each \( z \) in \( \Delta \) with the point evaluation \( \phi_z(f) = f(z) \) (see [9, Chapter 10]). We then have the decomposition
\[
\mathcal{M}(H^\infty) \setminus \Delta = \bigcup_{|\alpha|=1} \mathcal{M}_\alpha.
\]
Since \( \mathcal{M}_\alpha \) is a peak set with peaking function \((1 + \bar{\alpha}z)/2\), the restriction of \( H^\infty(\Delta) \) to \( \mathcal{M}_\alpha \) is a uniform algebra on \( \mathcal{M}_\alpha \), which is denoted by \( A_\alpha \). Then the Shilov boundary of \( A_\alpha \) is \( X \cap \mathcal{M}_\alpha \) (see [9, 187p - 193p] for the algebras \( A_\alpha \)). Each \( \phi \) in \( \mathcal{M}_\alpha \) has a unique representing measure \( \mu \) on \( X \cap \mathcal{M}_\alpha \) with minimal support \( S_\phi \) (see [4, Chapter II, Theorem 2.3] and [9, Chapter 10, Exercise 4] for minimal support sets). Since various fibers are homeomorphic to one another, we restrict our attention to the fiber \( \mathcal{M}_1 \) over \( z = 1 \) to look into the structure of fringe \( \mathcal{M}(H^\infty) \setminus \Delta \).

Our objective in this note is to provide a strong version of the cluster value theorem \((1.1)\), from which the corona theorem follows directly:

**Theorem.** Let \( B(\mathfrak{F}) \) be the uniformly closed subalgebra of \( H^\infty(\Delta) \) generated by a countable family \( \mathfrak{F} \) of \( H^\infty(\Delta) \). If \( \phi \) is a homomorphism in the fiber \( \mathcal{M}_1 \) over \( z = 1 \), then there is a sequence \( \{\zeta_j\} \) in \( \Delta \), depending on \( \mathfrak{F} \) and \( \phi \), such that
\[
(1.2) \quad \zeta_j \to 1 \quad \text{and} \quad f(\zeta_j) \to f(\phi)
\]
for each \( f \) in \( B(\mathfrak{F}) \).

Let us make a comment on Theorem. It is not necessary that the homomorphism \( \phi \) lies in the closure of \( \{ \zeta_j \} \) in \( \mathfrak{M}(H^\infty) \), in other words, there may exist \( h \) in \( H^\infty(\Delta) \) with the property that \( h(\phi) = 1 \) while \( |h(\zeta_j)| < 1/2 \) for \( j = 1, 2, \ldots \). Of course, each \( f \) in \( B(\mathfrak{F}) \) values constant \( f(\phi) \) on the set of adherent points of \( \{ \zeta_j \} \) in \( \mathfrak{M}(H^\infty) \). Since \( \{ \zeta_j \} \) may be chosen to be sufficiently sparse, there appears a relation between interpolating sequences and analytic discs (see Section 2 for details).

Recall that a basic neighborhood of \( \phi \) in \( \mathfrak{M}(H^\infty) \) is given by

\[
W(\phi, f_1, \ldots, f_N, \varepsilon) = \{ \xi \in \mathfrak{M}(H^\infty) \mid |f_k(\xi) - f_k(\phi)| < \varepsilon, \ k = 1, 2, \ldots, N \},
\]

for \( \varepsilon > 0 \) and for \( f_1, f_2, \ldots, f_N \) in \( H^\infty(\Delta) \). By definition, the family of all such neighborhoods forms a basis for the (weak-star) topology of \( \mathfrak{M}(H^\infty) \). Since \( W(\phi, f_1, \ldots, f_N, \varepsilon) \cap \Delta \neq \emptyset \) by Theorem, it follows immediately that the open set \( \Delta \) is dense in \( \mathfrak{M}(H^\infty) \). This fact is interpreted as a well-known formulation in function theory:

**Corollary.** If \( f_1, f_2, \ldots, f_N \) in \( H^\infty(\Delta) \) satisfy

\[
|f_1(z)| + |f_2(z)| + \cdots + |f_N(z)| \geq \delta > 0, \quad z \in \Delta,
\]

then there exist \( g_1, g_2, \ldots, g_N \) in \( H^\infty(\Delta) \) such that

\[
f_1(z)g_1(z) + f_2(z)g_2(z) + \cdots + f_N(z)g_N(z) \equiv 1, \quad z \in \Delta.
\]

It would be helpful to understand the basic idea behind our proof of the corona theorem. Let \( \phi \) be a homomorphism in the fiber \( \mathfrak{M}_1 \), and let \( \mu \) be the representing measure for \( \phi \). Then the minimal support \( S_\phi \) for \( \mu \) is contained in \( X \cap \mathfrak{M}_1 \). Since \( f_1, f_2, \ldots, f_N \) in \( H^\infty(\Delta) \) are continuous on \( X \), we may choose disjoint open-closed subsets \( U_i = U(E_i), i = 1, 2, \ldots, \ell \), of \( X \) such that \( S_\phi = \bigcup_{i=1}^\ell (U_i \cap S_\phi) \) and all \( f_k \) vary little on each \( U_i \). Here \( E_i \) denotes the measurable set in \( T \) corresponding to \( U_i \). Denoting by \( \chi_{E_i} \) the characteristic function of \( E_i \), we then choose a nonnegative simple function of the form \( s(\theta) = \sum_{i=1}^\ell a_i \chi_{E_i}(\theta) \) satisfies that \( \int_T s(\theta)dm(\theta) = 1 \) and the value of

\[
\left| \int_{S_\phi} f_k(x)d\mu(x) - \int_T f_k(\theta)s(\theta)dm(\theta) \right|
\]

is as small as desired. Therefore there is a sequence \( \{ s_j(\theta) \} \) of such simple functions satisfying

\[
\lim_{j \to \infty} \left| \phi(f_k) - \int_T f_k(\theta)s_j(\theta)dm(\theta) \right| = 0.
\]

With the aid of certain Blaschke products, we then see that \( s_j(\theta)dm(\theta) \) is close to the Poisson integral of \( \zeta_j \) asymptotically on the algebra generated by \( f_1, f_2, \ldots, f_N \). This shows that the sequence \( \{ \zeta_j \} \) in \( \Delta \) satisfies

\[
\lim_{j \to \infty} \left| f_k(\zeta_j) - \int_T f_k(\theta)s_j(\theta)dm(\theta) \right| = 0,
\]
from which the conclusion of Theorem follows (see Lemma 2.3 and Section 5 for more details).

In the next section, we establish some notation and elementary facts on the structure of $\mathcal{M}(H^\infty)$. In Section 3, among other things, Hoffman maps are discussed by the relation to interpolating sequences in $\Delta$. Section 4 is devoted to constructing auxiliary Blaschke products of which zeros determine desired sequences. In Section 5, the proof of Theorem is provided. We close with two remarks in Section 6.

We refer the reader to [1], [2], [3], [8, Chapter VIII] and [15, Appendix 3] for further details and recent developments on the corona problem. Related results concerning the Hardy space theory can be found in [4], [8], [9] and [15].

2. Analytic discs and Hoffman maps

We begin with showing that the Shilov boundary $X$ of $H^\infty(\Delta)$ is contained in the closure of $\Delta$ in $\mathcal{M}(H^\infty)$, which is well-known. This fact enables us to restrict our attention to the homomorphisms lying in $\mathcal{M}(H^\infty) \setminus X$.

Lemma 2.1. Let $\phi$ be a homomorphism in $\mathcal{M}(H^\infty)$, and let $f_1, f_2, \ldots, f_N$ be functions in $H^\infty(\Delta)$. Denote by $S_\phi$ the minimal support of representing measure $\mu$ for $\phi$. If $f_1, f_2, \ldots, f_N$ are constant on $S_\phi$, then we have

$$W(\phi, f_1, \ldots, f_N, \varepsilon) \cap \Delta \neq \emptyset,$$

for any $\varepsilon > 0$. Consequently, the Shilov boundary $X$ lies in the closure of $\Delta$ in $\mathcal{M}(H^\infty)$.

Proof. Since each $f_k$ is continuous on $X$, $f_k(\phi) = f_k(x)$ for all $x \in S_\phi$. Fix an $x \in S_\phi$, and choose an open-closed neighborhood $U = U(E)$ of $S_\phi$ such that

$$|f_k(\psi) - f_k(x)| < \varepsilon/2, \quad \psi \in U,$$

for $k = 1, 2, \ldots, N$. Since the subset $E$ of $\mathbb{T}$ satisfies that $m(E) = \hat{m}(U(E)) > 0$, we observe that

$$|f_k(\theta) - f_k(x)| < \varepsilon/2, \quad m - a.e. \quad \theta \in E.$$

Thus it follows from Fatou’s theorem that, for some $\theta$ in $E$, there is a $z = re^{i\theta}$ in $\Delta$ satisfying that

$$|f_k(z) - f_k(\phi)| \leq |f_k(z) - f_k(\theta)| + |f_k(\theta) - f_k(x)| < \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

for $k = 1, 2, \ldots, N$, so the proof is complete. \hfill \Box

Let us make a remark on this lemma. Since $H^\infty(\Delta)$ is a logmodular algebra on $X$, $\mu$ is a Jensen measure, meaning that $\mu$ satisfies the inequality

$$\log |\phi(f)| \leq \int_{S_\phi} \log |f(x)| \, d\mu(x), \quad f \in H^\infty(\Delta),$$

where $S_\phi$ is the minimal support of representing measure $\mu$ for $\phi$. This fact enables us to restrict our attention to the homomorphisms lying in $\mathcal{M}(H^\infty) \setminus X$.
so if $f$ vanishes on a Borel subset $K$ with $\mu(K) > 0$, then $f(\phi) = 0$. This provides that if each $f_k$ is constant $c_k$ on such a $K$, then the conclusion of Lemma \[\text{2.1}\] holds.

We notice that, except for analytic discs, there may exist a function $f$ in $H^\infty(\Delta)$ such that $f$ is not constant on $S_\phi$ and the right side of the above inequality diverges.

For $\eta$ and $\xi$ in $\mathcal{M}(H^\infty)$, the pseudo-hyperbolic distance $\rho(\eta, \xi)$ between $\eta$ and $\xi$ is defined to be

$$
\rho(\eta, \xi) = \sup \{ |f(\eta)| ; f \in H^\infty(\Delta), f(\xi) = 0 \text{ and } \|f\| \leq 1 \}.
$$

Then the relation $\rho(\eta, \xi) < 1$ is an equivalence relation in $\mathcal{M}(H^\infty)$ and the equivalence class $P(\xi) = \{ \eta \in \mathcal{M}(H^\infty); \rho(\eta, \xi) < 1 \}$ is called the Gleason part of $\xi$. A Gleason part $P$ is an analytic disc if there exists a continuous bijective map $L$ of $\Delta$ onto $P$ such that $f \circ L$ is analytic on $\Delta$ for all $f$ in $H^\infty(\Delta)$, and such a map $L$ is called an analytic map. Since $H^\infty(\Delta)$ is a logmodular algebra on $X$, it follows from Wermer’s embedding theorem that each part is either a single point or an analytic disc.

Furthermore, K. Hoffman [10] characterized analytic discs in $\mathcal{M}(H^\infty)$ by using interpolating sequences in $\Delta$. Recall that a sequence $\{z_j\}$ in $\Delta$ is an interpolating sequence if, for any bounded sequence $\{w_j\}$, there exists a function $f$ in $H^\infty(\Delta)$ such that $f(z_j) = w_j$, for $j = 1, 2, \ldots$. Such a sequence is characterized by the condition

$$
\inf_k \prod_{j: j \neq k} \left| \frac{z_k - z_j}{1 - \overline{z}_j z_k} \right| > 0.
$$

Especially, an interpolating sequence $\{z_j\}$ is said to be thin (sparse), if it satisfies

$$
\lim_{k \to \infty} \prod_{j: j \neq k} \left| \frac{z_k - z_j}{1 - \overline{z}_j z_k} \right| = 1.
$$

For an an interpolating sequence $\{z_j\}$, the associated Blaschke product

$$
B(z) = \prod_{j=1}^\infty \frac{\overline{z}_j}{|z_j|} \frac{z - z_j}{1 - z_j z}, \quad (2.1)
$$

is called the interpolating Blaschke product (where $\overline{z}_j/|z_j| = -1$, if $z_j = 0$). When $B(z)$ is a Blaschke product, let us agree to also call $e^{i\gamma}B(z)$ a Blaschke product, for a real constant $\gamma$.

The set $\mathcal{M}(H^\infty)^\Delta$ of all maps of $\Delta$ into $\mathcal{M}(H^\infty)$ is a compact Hausdorff space in the product topology. Observe that, in this topology, a net $(F_\beta)$ in $\mathcal{M}(H^\infty)^\Delta$ has limit $F$ if and only if $f \circ F_\beta(\zeta) \to f \circ F(\zeta)$ for all $f$ in $H^\infty(\Delta)$ and all $\zeta$ in $\Delta$. For a sequence $\{c_n\}$ in $\Delta$, we put

$$
L_n(\zeta) = \frac{\zeta + c_n}{1 + \overline{c}_n \zeta}, \quad \zeta \in \Delta. \quad (2.2)
$$

Then $L_n$ is an analytic map of $\Delta$ onto the part $\Delta$ in $\mathcal{M}(H^\infty)$. From the sequence $\{L_n\}$ in $\mathcal{M}(H^\infty)^\Delta$, we take a convergent subnet $(L_\beta)$ with limit $L$ in $\mathcal{M}(H^\infty)^\Delta$, which is called the Hoffman map determined by $(L_\beta)$. 
Let $P(\psi)$ be a Gleason part of $\psi$ in $\mathcal{M}(H^\infty)$. Then Hoffman showed that $P(\psi)$ is an analytic disc if and only if the analytic map for $P(\psi)$ is the Hoffman map $L_\psi = \lim_\beta L_\beta$, where $(L_\beta)$ is a subnet of $\{L_n\}$ for an interpolating sequence $\{c_n\}$. We notice that the proof of the “only if” part requires the corona theorem. Whenever $P$ is an analytic disc in $\mathcal{M}(H^\infty) \setminus \Delta$, the closure of $P$ in $\mathcal{M}(H^\infty)$ never meets the Shilov boundary $X$, because of the existence of a Blaschke product vanishing identically on $P$ (see [10, 102p]).

The following result is well-known, and used repeatedly in what follows (see [8 Chapter X, Exercise 8]). However, let us provide here an easy proof. It is noteworthy that the argument does not depend on the corona theorem.

**Lemma 2.2.** Let $\{c_n\}$ be a thin interpolating sequence in $\Delta$ with $c_n \to 1$, and let $L$ be the Hoffman map by a convergent subnet $(L_\beta)$ of the sequence $\{L_n\}$ by (2.2). We put $\xi = L(0)$, a homomorphism in $\mathcal{M}_1$. Then the Gleason part $P(\xi)$ of $\xi$ is an analytic disc in $\mathcal{M}_1$, where $L$ is an analytic homeomorphism of $\Delta$ onto $P(\xi)$.

**Proof.** Since $\mathcal{M}_1$ is a peak set, $P(\xi)$ is contained in $\mathcal{M}_1$. It follows from [8 Chapter X, Lemma 1.1] that $L$ is an analytic map of $\Delta$ to $P(\xi)$. Now we show that $L$ maps $\Delta$ onto $P(\xi)$. Let $B$ be the Blaschke product with zeros $\{c_n\}$. Observe that $B \circ L_\beta(\xi)$ converges uniformly to $B \circ L(\xi)$ on compact subsets of $\Delta$. Since $B \circ L(0) = 0$, and since

$$|(B \circ L)'(0)| = \lim_\beta |(B \circ L_\beta)'(0)| = \lim_\beta (1 - |c_\beta|^2)|B'(c_\beta)| = 1,$$

Schwarz’s lemma shows that $(B \circ L)(z) = z$. On the other hand, Wermer’s embedding theorem assures the existence of an analytic map $\tau$ of $\Delta$ onto $P(\xi)$ with $\tau(0) = \xi$. Let $f = \tau^{-1} \circ L$ and $g = B \circ \tau$. Then these functions map $\Delta$ into itself, and vanish at 0 in $\Delta$. Since $H^\infty(\Delta)$ is a logmodular algebra on $X$, $\tau^{-1}$ is approximated by a bounded sequence in $H^\infty(\Delta)$ (compare to (a) of Section 6). Observe that $g(f(z)) = (B \circ L)(z) = z$. Since $|g'(f(z))f'(z)| = 1$, $|f'(0)| = 1$ by Schwarz’s lemma. This shows that $L$ maps $\Delta$ onto $P(\xi)$. Since $B = L^{-1}$ is continuous on $P(\xi)$, $L$ is a desired homeomorphism. \qed

In our argument it would be useful to understand the following observation: Let $f_1, f_2, \cdots, f_N$ be in $H^\infty(\Delta)$, and let $\{c_n\}$ be a sequence in $\Delta$. By taking a suitable subsequence $\{c_{n_k}\}$ of $\{c_n\}$, it follows from normal families argument that $f_k \circ L_{n_k}$ converges uniformly to $F_k$ on compact subsets of $\Delta$, for $k = 1, 2, \cdots, N$. We also assume that $\{c_{n_k}\}$ is a thin interpolating sequence. If $L$ is the Hoffman map by a convergent subnet of $\{L_{n_k}\}$, then $L$ is a homeomorphism with $F_k = f_k \circ L$. Moreover, $f_k$ extends to the closure of Gleason part of $L^{-1}(0)$ in $\mathcal{M}(H^\infty)$.

**Lemma 2.3.** Let $\phi$ be a homomorphism in $\mathcal{M}_1$. Then $\phi$ lies in the closure of $\Delta$ in $\mathcal{M}(H^\infty)$ if and only if, for every at most countable family $\mathcal{F}$ in $H^\infty(\Delta)$, there is a thin interpolating sequence $\{c_\xi\}$ such that (1.2) holds for each $f$ in $B(\mathcal{F})$, the uniformly closed subalgebra generated by $\mathcal{F}$. Moreover, there is a $\xi$ in $\mathcal{M}_1$ whose Gleason part $P(\xi)$ is homeomorphic to $\Delta$ such that $f(\phi) = f(\xi)$ for each $f$ in $B(\mathcal{F})$. 

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Proof. Suppose that $\phi$ lies in the closure of $\Delta$ in $\mathfrak{M}(H^\infty)$. Put $\mathfrak{F} = \{f_1, f_2, \ldots\}$, and let $\mathfrak{F}_0$ be the family of finite sums of functions of the form $r \cdot f_1^{n_1} f_2^{n_2} \cdots f_k^{n_k}$ with a rational number $r$ and with nonnegative integers $n_1, n_2, \ldots, n_k$. Observe that $\mathfrak{F}_0$ is also a countable family. Replacing $\mathfrak{F}$ with $\mathfrak{F}_0$, we may assume $B(\mathfrak{F})$ is the uniform closure of $\mathfrak{F}$ in $H^\infty(\Delta)$. Let $\{\varepsilon_j\}$ be a decreasing sequence of positive numbers with $\varepsilon_j \to 0$. Since $\mathfrak{M}_1$ is a peak set in $\mathfrak{M}(H^\infty)$, it then follows from our assumption that

$$W(\phi, f_1, \ldots, f_j, \varepsilon_j) \cap \{z; |z - 1| < \varepsilon_j\} \neq \emptyset,$$

where $W(\phi, f_1, \ldots, f_j, \varepsilon_j)$ is defined as in (1.3). We then fix a $\zeta_j$ in this set. Taking a subsequence if necessary, we choose a sequence $\{\zeta_j\}$ for which (1.2) holds on $\mathfrak{F}$. Since $\mathfrak{F}$ is uniformly dense in $B(\mathfrak{F})$, the same conclusion holds on $B(\mathfrak{F})$. The converse is obvious, so the proof is finished. \hfill \square

The point of our argument on the corona problem is to find such an analytic disc $P(\xi)$ for a given $\phi$ in $\mathfrak{M}_1$ and for $f_1, f_2, \ldots, f_N$ in $H^\infty(\Delta)$. Together with the corona theorem, Lemma 2.3 shows that the union of all homeomorphic analytic discs is dense in $\mathfrak{M}(H^\infty)$.

3. APPROXIMATION TO REPRESENTING MEASURES

In this section we prepare three lemmas, which play an important role in our argument. Recall that the restriction $A_1$ of $H^\infty(\Delta)$ to $\mathfrak{M}_1$ is a uniform algebra, whose Shilov boundary is $X \cap \mathfrak{M}_1$, and also that an open-closed set $U = U(E)$ in $X$ satisfies that $m(E) = \hat{m}(U) > 0$ whenever $U \neq \emptyset$. Here $\hat{m}$ is the lifting of Lebesgue measure $m$ to $X$. Let $\phi$ be a homomorphism in $\mathfrak{M}_1$, and let $S_\phi$ be the minimal support of the representing measure $\mu$ for $\phi$. Then $S_\phi$ is a compact subset of $X \cap \mathfrak{M}_1$. For a simple function $s(\theta)$, the subset $E = \{\theta : s(\theta) \neq 0\}$ is called the support (carrier) of $s(\theta)$.

**Lemma 3.1.** Let $f_1, f_2, \ldots, f_N$ be in $H^\infty(\Delta)$, and let $\varepsilon > 0$. Then there is a simple function $s(\theta) = \sum_{i=1}^\ell a_i \chi_{E_i}(\theta)$ on $T$ with $a_i > 0$ such that the support $E = \cup_{i=1}^\ell E_i$ is contained in $(-\varepsilon, \varepsilon)$, $\int_E s(\theta)dm(\theta) = 1$ and

$$\left| \phi(f_k) - \int_E f_k(\theta)s(\theta)dm(\theta) \right| < \varepsilon, \quad k = 1, 2, \ldots, N. \tag{3.1}$$

Consequently, if $\{\varepsilon_j\}$ is a decreasing sequence of positive numbers with $\varepsilon_j \to 0$, then there is a sequence $\{s_j(\theta)\}$ of nonnegative simple functions with supports $E^{(j)}$ such that each $s_j(\theta)dm(\theta)$ is a probability measure on $(-\varepsilon_j, \varepsilon_j)$ and

$$\phi(f_k) = \lim_{j \to \infty} \int_{E^{(j)}} f_k(\theta)s_j(\theta)dm(\theta).$$
Proof. Since $f_1, f_2, \cdots, f_N$ are continuous on $X$, we may choose disjoint open-closed subsets $U_i = U(E_i)$ of $X$, $i = 1, 2, \cdots, \ell$, such that $\bigcup_{i=1}^{\ell} U_i$ contains $S_{\phi}$, $U_i \cap S_{\phi} \neq \emptyset$, and
\[
|f_k(x) - f_k(y)| < \frac{\varepsilon}{2}, \quad x, y \in U_i,
\]
for $k = 1, 2, \cdots, N$. Since $S_{\phi}$ is the minimal support, we observe that $\mu(U_i \cap S_{\phi}) > 0$. We may assume the corresponding sets $E_i$ of $T$ are disjoint subsets of $(-\varepsilon, \varepsilon)$. If we fix an $x_i$ in $U_i \cap S_{\phi}$, then
\[
\left| \int_{S_{\phi}} f_k(x) \, d\mu(x) - \sum_{i=1}^{\ell} f_k(x_i) \mu(U_i \cap S_{\phi}) \right| = \left| \sum_{i=1}^{\ell} \left( \int_{U_i \cap S_{\phi}} f_k(x) \, d\mu(x) - \int_{U_i \cap S_{\phi}} f_k(x_i) \, d\mu(x) \right) \right|
\leq \sum_{i=1}^{\ell} \int_{U_i \cap S_{\phi}} |f_k(x) - f_k(x_i)| \, d\mu(x)
\leq \frac{\varepsilon}{2} \sum_{i=1}^{\ell} \mu(U_i \cap S_{\phi}) = \frac{\varepsilon}{2}.
\]
Let
\[a_i = \frac{\mu(U_i \cap S_{\phi})}{m(E_i)} \quad \text{and} \quad s(\theta) = \sum_{i=1}^{\ell} a_i \chi_{E_i}(\theta).\]
Since $|f_k(\theta) - f_k(x_i)| < \varepsilon/2$ for $m$-a.e. $\theta$ in $E_i$, we obtain
\[
\left| \int_{E} f_k(\theta) s(\theta) \, dm(\theta) - \sum_{i=1}^{\ell} f_k(x_i) a_i m(E_i) \right| < \frac{\varepsilon}{2}.
\]
Thus the simple function $s(\theta)$ on $T$ satisfies the desired inequality (3.1). \hfill \Box

We note that the above simple function $s(\theta)$ may have the form $c \chi_{E}(\theta)$ with $c > 0$. Indeed, if we choose each $U_i = U(E_i)$ satisfying that
\[
\frac{\mu(U_i \cap S_{\phi})}{\mu(U_m \cap S_{\phi})} = \frac{m(E_i)}{m(E_m)} \quad \text{for} \quad i, m = 1, 2, \cdots, \ell,
\]
by cutting down suitably a part of $U_i \setminus S_{\phi}$, then $s(\theta)$ has the form $s(\theta) = c \chi_{E}(\theta)$ with $c = \mu(U_i \cap S_{\phi})/m(E_i) > 0$.

Let us now turn to a minor extension of Lemma 3.1.

**Lemma 3.2.** Under the notation of Lemma 3.1, let $\mathcal{F}^z = \{f_1, f_2, \cdots, f_N\} \cup \{1, z, B\}$ with a fixed Blaschke product $B$ and the coordinate function $z$. Then we have:

(a) There is a nonnegative simple function $s^z(\theta)$ supported on a subset $F$ of $E$ such that $\int_{F} s^z(\theta) \, dm(\theta) = 1$, and
\[
(3.2) \quad \left| \phi(g) - \int_{F} g(\theta) s^z(\theta) \, dm(\theta) \right| < \varepsilon, \quad g \in \mathcal{F}^z.
\]
(b) Let $B(\mathfrak{F}^2)$ be the uniformly closed subalgebra of $H^\infty(\Delta)$ generated by $\mathfrak{F}^2$. Then there is a sequence $\{s_j^2(\theta)\}$ of nonnegative simple functions with supports $F^{(j)}$ such that
\[
\phi(g) = \lim_{j \to \infty} \int_{F^{(j)}} g(\theta)s_j^2(\theta)dm(\theta), \quad g \in B(\mathfrak{F}^2).
\]

Proof. (a) Let $s(\theta) = \sum_{i=1}^\ell a_i \chi_{E_i}(\theta)$ be the simple function in Lemma 3.1. Since each $g$ in $\{1, z, B\}$ is continuous on each $U(E_i)$, there are disjoint subsets $F^{(i)}_j$ of $E_i$, $j = 1, 2, \ldots, m_i$, such that $S_\phi \cap U(F^{(i)}_j) \neq \emptyset$,
\[
|g(x) - g(y)| < \frac{\varepsilon}{2}, \quad x, y \in U(F^{(i)}_j),
\]
and the family $\{U(F^{(i)}_j); i = 1, 2, \ldots, \ell, j = 1, 2, \ldots, m_i\}$ forms a finite covering of $S_\phi$. We then write $\{U(F_j); k = 1, 2, \ldots, m\}$ for this family $\{U(F^{(i)}_j)\}$, and put $b_j = \mu(U(F_j) \cap S_\phi)/m(F_j)$. By the same way as in the proof of Lemma 3.1 we see that the simple function $s^2(\theta) = \sum_{j=1}^m b_j \chi_{F_j}(\theta)$ satisfies (3.2) and the support $F = \cup_{j=1}^m F_j$ of $s^2(\theta)$ is a subset of $E = \cup_{i=1}^\ell E_i$.

(b) Since, for $f, g$ in $\mathfrak{F}^2$,
\[
|(fg)(x) - (fg)(y)| \leq |(fg)(x) - f(x)g(y)| + |f(x)g(y) - (fg)(y)|
\]
\[
< (\|f\|_\infty + \|g\|_\infty) \frac{\varepsilon}{2}, \quad x, y \in U(F_i),
\]
we observe that
\[
\left| \phi(fg) - \int_F (fg)(\theta)s^2(\theta)dm(\theta) \right| < (\|f\|_\infty + \|g\|_\infty) \varepsilon.
\]
Let $h$ be a finite product of functions in $\mathfrak{F}^2$, that is, $h = f_{m_1}^m f_{m_2}^m \cdots f_{m_N}^m z^m B^n$, where $m_1, m_2, \ldots, m_N, m$ and $n$ are nonnegative integers. Extending the above argument, we may choose a constant $C$, depending on $f_k, B, m_k, m$ and $n$, such that $|h(x) - h(y)| < C\varepsilon/2$ for $x, y$ in $U(F_i)$ and
\[
\left| \phi(h) - \int_F h(\theta)s^2(\theta)dm(\theta) \right| < C\varepsilon.
\]
Notice that $C$ never depends on $\varepsilon, F_i$, and $s^2(\theta)$. Let $\{\varepsilon_j\}$ be a decreasing sequence of positive numbers with $\varepsilon_j \to 0$. It follows from (a) and the argument above that there is a sequence $\{s_j^2(\theta)\}$ of simple functions with decreasing supports $F^{(j)}$ such that
\[
\left| \phi(h) - \int_{F^{(j)}} h(\theta)s_j^2(\theta)dm(\theta) \right| < C\varepsilon_j,
\]
for the above finite product $h$ of functions in $\mathfrak{F}^2$, so the equality (3.3) holds for the function $h$. Let $\mathfrak{F}_0^2$ be the space of all finite linear combinations of such product functions $h = f_{m_1}^m f_{m_2}^m \cdots f_{m_N}^m z^m B^n$. Since $\phi$ and the integrals by $s_j^2(\theta)dm(\theta)$ are linear, (3.3) holds for all functions in $\mathfrak{F}_0^2$. Observe that $\mathfrak{F}_0^2$ is an algebra, in other words,
it is closed under the formation of multiplications. Since \( B(\mathbb{H}) \) is the uniform closure of \( \mathbb{H}_0 \), the equation (3.3) extends to \( B(\mathbb{H}) \), which completes the proof.

Since \( \phi(fg) = \phi(f)\phi(g) \) for \( f, g \) in \( B(\mathbb{H}) \), (3.3) shows that
\[
\left| \int_{F_i} f(\theta)s_j^*(\theta)dm(\theta) \cdot \int_{F_i} g(\theta)s_j^*(\theta)dm(\theta) - \int_{F_i} (fg)(\theta)s_j^*(\theta)dm(\theta) \right|
\]
tends to 0, as \( j \to \infty \). Since the restriction \( \phi|_B \) of \( \phi \) to \( B(\mathbb{H}) \) is a homomorphism on the uniform algebra, the property (b) of Lemma 3.2 shows that, in a sense, \( s_j^*(\theta)dm(\theta) \) is close to a representing measure for \( \phi \).

Let \( s(\theta) = \sum_{i=1}^e a_iX_{E_i}(\theta) \) be the simple function obtained in Lemma 3.1. Since each term \( a_iX_{E_i}(\theta) \) may be replaced by \( (a_i/m(E_i))X_{F_i}(\theta) \) for a subset \( F_i \) of \( E_i \) with \( m(F_i) > 0 \), there are many ways to represent such simple functions. Let us always assume that \( \mu(S_\phi \cap U(E_i)) > 0 \) and each \( E_i \) is contained in either \([0, \varepsilon]\) or \((-\varepsilon, 0]\). Hence, if \( E_i \) is a subset of \([0, \varepsilon]\), then \( m(E_i \cap [0, \delta]) > 0 \) for all \( \delta > 0 \). In order to discuss the relation between sequences of such simple functions and Poisson kernels, we need to choose certain analytic discs in \( \mathfrak{M}_1 \). Since \( s(\theta)dm(\theta) \) is a continuous probability measure on \( T \), there are \( \alpha \) and \( \beta \) with \(-\varepsilon < \alpha < \beta < \varepsilon \) such that
\[
\int_{-\pi}^\alpha s(\theta)dm(\theta) = \frac{1}{4} = \int_{\beta}^\pi s(\theta)dm(\theta).
\]
Observe that \( \alpha \) and \( \beta \) satisfy that
\[
-\varepsilon < \alpha < \beta \leq 0, \quad \text{if} \quad 0 \leq \mu(S_\phi \cap U([0, \varepsilon])) \leq 1/4,
\]
\[
-\varepsilon < \alpha \leq 0 < \beta, \quad \text{if} \quad 1/4 < \mu(S_\phi \cap U([0, \varepsilon])) \leq 3/4,
\]
\[
0 < \alpha < \beta < \varepsilon, \quad \text{if} \quad 3/4 < \mu(S_\phi \cap U([0, \varepsilon])) \leq 1.
\]
By our assumption on \( E_i \), we also observe that if \( \mu(S_\phi \cap U([0, \varepsilon])) = 3/4 \) or \( 1/4 \), then \( \alpha = 0 \) or \( \beta = 0 \), respectively. It is sometimes useful to modify \( \alpha \) and \( \beta \) suitably.

Let \( \alpha \) and \( \beta \) be as above, and let \( C \) be the circular arc from \( e^{i\alpha} \) to \( e^{i\beta} \) orthogonal to the unit circle \( T \) lying in \( \Delta \), and put \( c \) to the point in \( C \) meeting the line \( \ell(t) = t e^{i(\alpha + \beta)/2} \) with \( 0 \leq t \leq 1 \). We call \( c \) the mid-point of the arc \( C \). If \( L_c(\zeta) = (\zeta + c)/(1 + \bar{c}\zeta) \) as in (2.2), then \( L_c^{-1}(z) = (z - c)/(1 - \bar{c}z) \).

Let us turn to certain properties of M"obius transformations to investigate the desired analytic discs. The next lemma is so fundamental that we omit the proof:

**Lemma 3.3.** Under the above hypotheses, \( L_c^{-1} \) maps the closed unit disc \( \overline{\Delta} = \Delta \cup T \) onto itself such that \( L_c^{-1}(c) = 0 \) and \( L_c^{-1}(e^{i\alpha}) = -L_c^{-1}(e^{i\beta}) \), that is, symmetric with respect to 0. Moreover, if \( g \) is a function in \( L^1(T) \), then we have
\[
\int_{L_c^{-1}(A)} |g \circ L_c(\theta)| (L_c)'(\theta) |dm(\theta) = \int_A g(\theta) dm(\theta)
\]
for all Borel sets $A$ in $T$. Particularly, if we set $u(\theta) = (s \circ L_c)(\theta) \cdot |(L_c)'(\theta)|$, then
\[
\int_{L_c^{-1}(E)} (f \circ L_c)(\theta)u(\theta) \, dm(\theta) = \int_{E} f(\theta)s(\theta) \, dm(\theta), \quad f \in H^\infty(\Delta),
\]
where $E$ is the support of $s(\theta)$.

4. Construction of auxiliary Blaschke products

In this section we derive certain Blaschke products from given ones, which play an important role in the proof of Theorem. For a Blaschke product $B$, we denote by $Z(B)$ the set of all zeros of $B$ repeated multiplicity for each zero. Let us show some elementary properties related to $Z(B)$.

**Lemma 4.1.** Let $0 < \eta < 1$ and let $\varepsilon > 0$. Then there is a $\delta = \delta(\varepsilon, \eta) > 0$ such that, for any Blaschke product $B$ with $Z(B) = \{z_k\}$, the condition
\[
\sum_{k=1}^{\infty} (1 - |z_k|) < \delta,
\]
on $Z(B)$ implies that
\[
|B(z)| > 1 - \varepsilon, \quad \text{for } |z| \leq \eta.
\]

**Proof.** When $|z| \leq \eta$, we observe that
\[
1 - \left| \frac{z_k - z}{1 - \bar{z}_k z} \right| \leq 1 - \frac{z_k - z}{1 - \bar{z}_k z} \cdot \frac{|z_k|}{1 - \eta} \leq 1 - \frac{\eta}{1 - \eta} (1 - |z_k|).
\]
Since
\[
- \log t \leq -\frac{2 \log a}{1 - a^2} (1 - t) \leq (1 + 2 \log \frac{1}{a})(1 - t)
\]
is valid for $a^2 < t < 1$ (see [8] Chapter VII, Lemma 1.2),
\[
-\log |B(z)| \geq -\sum_{k=1}^{\infty} \log \left| \frac{z_k - z}{1 - \bar{z}_k z} \right| \leq C_1 \sum_{k=1}^{\infty} \left(1 - \left| \frac{z_k - z}{1 - \bar{z}_k z} \right| \right) \leq C_2 \sum_{k=1}^{\infty} (1 - |z_k|)
\]
for some constants $C_1$ and $C_2$. Then we have
\[
|B(z)| \geq e^{-C_2\delta} > 1 - \varepsilon \quad \text{for } |z| \leq \eta,
\]
with sufficiently small $\delta = \delta(\varepsilon, \eta) > 0$. \(\square\)

Recall that if $L_c(\zeta) = (\zeta + c)/(1 + \bar{c}\zeta)$ with $|c| < 1$, then $L_c^{-1}(z) = (z - c)/(1 - \bar{c}z)$. 

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Lemma 4.2. Let $L_c$ and $L_c^{-1}$ be as above, and let $B$ be a Blaschke product with $Z(B) = \{z_k\}$. Then $B \circ L_c$ itself is the Blaschke product with $Z(B \circ L_c) = \{L_c^{-1}(z_k)\}$. In particular, for a given $\delta > 0$, there is an $N$ such that
\[
\sum_{k=N}^{\infty} (1 - |L_c^{-1}(z_k)|) < \delta.
\]
Proof. By the similar way as in the proof of (4.1), we observe that
\[
1 - \left| \frac{z_k - c}{1 - \overline{c}z_k} \right| = 1 - \frac{z_k - c}{1 - c\overline{z}_k} \cdot \frac{|z_k|}{1 - |c|} \leq 1 + \frac{|c|}{1 - |c|} (1 - |z_k|).
\]
So $\zeta_k = (z_k - c)/(1 - \overline{c}z_k)$ is a Blaschke sequence, meaning that $\sum_{k=1}^{\infty} (1 - |\zeta_k|) < \infty$. On the other hand, if $S(\zeta)$ is a nonconstant singular function, then so is $S \circ L_c^{-1}(z)$, because it has no zeros on $\Delta$. This shows that the inner function $B \circ L_c$ cannot have a singular factor, so $B \circ L_c$ is the Blaschke product with $Z(B \circ L_c) = \{\zeta_k\}$, as desired. \qed

We notice that $B \circ L_c(z)$ has the form of original Blaschke product multiplied a constant of modulus one, while $L_c \circ B(z)$ may happen to be a singular function by the Frostman theorem [8, Chapter II, Theorem 6.4].

Let $\{c_n\}$ be a sequence in $\Delta$ with $c_n \to 1$. For an $\eta > 0$, we denote by $K(c_n, \eta)$ the noneuclidean disc
\[
K(c_n, \eta) = \left\{ z \in \Delta; \rho(z, c_n) = \left| \frac{z - c_n}{1 - \overline{c_n}z} \right| < \eta \right\} = L_n(\{|\zeta| < \eta\}),
\]
where $L_n$ is the map on $\Delta$ in (2.2). Then $K(c_n, \eta)$ is the euclidean disc with center $a_n = (1 - \eta^2)c_n/(1 - \eta^2|c_n|^2)$ and radius $r_n = \eta(1 - |c_n|^2)/(1 - \eta^2|c_n|^2)$ (see [8, Chapter I, §1]). Observe that $a_n \to 1$ and $r_n \to 0$, as $c_n \to 1$.

Suppose $\phi$ is a homomorphism in the fiber $\mathcal{M}_1$ outside the Shilov boundary $X$ of $H^\infty(\Delta)$. Then Newman’s theorem assures the existence of a Blaschke product $B_0$ vanishing at $\phi$ (see [9, Chapter 10] or [8, Chapter V, Theorem 2.2]). By modifying $B_0$ suitably, we construct a certain Blaschke product $B$ with $B(\phi) = 0$ such that, for a subsequence $\{c_n\}$ of $\{c_n\}$, the function, $\lim_{j \to \infty} B \circ L_{n_j}(\zeta)$, generates the disc algebra $A(\Delta)$, the uniform algebra on $\Delta$ generated by $G(\zeta) = \zeta$.

Let $0 < \ell < 1$, and let $[s, t)$ be the interval with $\ell \leq s < t \leq 1$. Then $S[s, t)$ denotes the sector
\[
S[s, t) = \left\{ re^{i\theta}; r \in [s, t), |\theta| < \frac{1 - \ell}{2} \right\}.
\]
Since each Blaschke product with zeros outside $S[s, 1)$ is continuous on $\{re^{i\theta}; |\theta| < (1 - \ell)/2\}$, we assume $Z(B_0)$ is contained in $S[\ell, 1)$, for the above $B_0$. Notice that $S[\ell, t) \cap Z(B_0)$ is always finite whenever $\ell < t < 1$, and that each Blaschke product with zeros $S[t, 1) \cap Z(B_0)$ has always the value 0 at $\phi$.

Lemma 4.3. Let $\phi$, $\{c_n\}$ and $L_n$ be as above. Then we may choose a Blaschke product $B$ with $B(\phi) = 0$ such that, for some subsequence $\{c_{n_j}\}$ of $\{c_n\}$, $B(c_{n_j}) = 0$ and $B \circ L_{n_j}(\zeta)$ converges uniformly to $G(\zeta) = \zeta$ on compact subsets of $\Delta$. 

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Proof. Let \( \{\varepsilon_n\} \) be a decreasing sequence of positive numbers with \( \varepsilon_n \to 0 \), and let \( \{\eta_n\} \) be an increasing sequence of positive numbers with \( \eta_n \to 1 \). If we put \( Z(B_0) = \{z_k\} \) for the above \( B_0 \), then \( B_0 \circ L_n \) is a Blaschke product with \( Z(B_0 \circ L_n) = \{L_n^{-1}(z_k)\} \) by Lemma 4.1. Observe that \( |L_n^{-1}(z_k)| \to 1 \), as \( c_n \to 1 \).

Let \( s_1 = \ell \) and \( r_1 = 1 - 2(1 - s_1)/3 = (2s_1 + 1)/3 \). It follows from Lemma 4.1 that there is a \( \delta_1 > 0 \) for which (4.1) holds with \( \varepsilon_1 \) and \( \eta_1 \). Since \( S[\ell, r_1] \cap Z(B_0) \) is finite, there is a \( c_{n_1} \) in \( \{c_n\} \) such that

\[
\sum_{S[\ell, r_1] \cap Z(B_0) \ni z_k} (1 - |L_n^{-1}(z_k)|) < \frac{\delta_1}{2}.
\]

We fix such a \( c_{n_1} \) in \( \{c_n\} \). Observe that, for any \( \rho_1 > 0 \), there is a \( \rho_2 > 0 \) such that \( |L_n^{-1}(z_k)| > \rho_1 \) whenever \( |z_k| > \rho_2 \). Hence there is an \( s_2 \) with \( r_1 < s_2 < 1 \) such that

\[
\sum_{S[s_2, 1] \cap Z(B_0) \ni z_k} (1 - |L_n^{-1}(z_k)|) < \frac{\delta_1}{2},
\]

which is an infinite sum. Let \( B^{(1)} \) be the Blaschke product with zeros \( z_k \) in \( S[\ell, r_1] \cup S[s_2, 1] \), that is, \( Z(B^{(1)}) = Z(B_0) \cap (S[\ell, r_1] \cup S[s_2, 1]) \). It follows from Lemmas 4.1 and 4.2 that

\[
|B^{(1)} \circ L_n(\zeta)| > 1 - \varepsilon_1 \quad \text{for} \quad |\zeta| \leq \eta_1.
\]

We then put \( r_2 = (2s_2 + 1)/3 \). By repetitions of the process on ad infinitum, we choose the sequences \( \{s_j\} \), \( \{c_{n_j}\} \) and \( \{B^{(j)}\} \) satisfying that, for

\[
\ell = s_1 < r_1 < s_2 < r_2 < \cdots < s_j < r_j < \cdots < 1
\]

with \( r_j = (2s_j + 1)/3 \), the zero-set of \( B^{(j)} \) is \( Z(B_0) \cap (S[\ell, r_j] \cup S[s_j+1, 1]) \), and the Blaschke product \( B^{(j)} \circ L_{n_j} \) satisfies

\[
|B^{(j)} \circ L_{n_j}(\zeta)| > 1 - \varepsilon_j \quad \text{for} \quad |\zeta| \leq \eta_j.
\]

Notice that if a Blaschke product has the zero-set contained in \( Z(B^{(j)}) \), then it satisfies the same inequality (4.3), and also notice that \( \cup_{j=1}^{\infty} S[s_j, r_j] \), \( \cup_{j=1}^{\infty} S[r_{2j-1}, s_{2j}] \) and \( \cup_{j=1}^{\infty} S[r_{2j}, s_{2j+1}] \) are disjoint one another. We then consider the three Blaschke products \( B_1, B_2 \) and \( B_3 \) whose zero-sets are given by

\[
Z(B_1) = Z(B_0) \cap (S[s_1, r_1] \cup S[s_2, r_2] \cup \cdots \cup S[s_j, r_j] \cup \cdots),
\]

\[
Z(B_2) = Z(B_0) \cap (S[r_1, s_2] \cup S[r_3, s_4] \cup \cdots \cup S[r_{2j-1}, s_{2j}] \cup \cdots),
\]

\[
Z(B_3) = Z(B_0) \cap (S[r_2, s_3] \cup S[r_4, s_5] \cup \cdots \cup S[r_{2j}, s_{2j+1}] \cup \cdots),
\]

respectively. Since \( B_0(\phi) = 0 \) and \( B_0 = B_1B_2B_3 \), we observe that either \( B_1B_2 \) or \( B_1B_3 \) vanishes at \( \phi \). Now let us assume that \( (B_1B_2)(\phi) = 0 \), because the other case is dealt with similarly. Since we see easily

\[
Z(B_1B_2) = \bigcap_{j=1}^{\infty} [Z(B_0) \cap (S[\ell, r_{2j}] \cup S[s_{2j+1}, 1])] = \bigcap_{j=1}^{\infty} Z(B^{(2j)}),
\]

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it follows that
\[ |(B_1B_2) \circ L_{n_{2j}}(\zeta)| > 1 - \varepsilon_{2j} \text{ for } |\zeta| \leq \eta_{2j}, \]
for \( j = 1, 2, \ldots \). Replacing \( \{c_{2j}\} \) with its suitable subsequence and multiplying some unimodular constant, we assume that \( (B_1B_2) \circ L_{n_{2j}}(\zeta) \) converges uniformly to the constant 1 on compact subsets of \( \Delta \). We may also assume \( B = B_1B_2B_4 \) with \( B_1 = B \) a thin Blaschke product \( B \). It follows that
\[ \{\zeta\} \text{ for } \lim_{k \to \infty} \phi_k \text{ converges to } G(\zeta) = \zeta \text{ on compact subsets of } \Delta. \]
Let us write \( \{c_{n_j}\} \) for \( \{c_{n_{2j}}\} \). Then the Blaschke product \( B = B_1B_2B_4 \) satisfies the desired properties. \( \square \)

Let us make some remarks on Lemma 4.3. When \( B_1(\phi) = 0 \), we may replace \( B_1B_2B_4 \) with \( B_1B_4 \) in the argument above. It is rather easy to find such a \( B \) whenever there exists an interpolating Blaschke product \( B_0 \) with \( B_0(\phi) = 0 \) (compare with [8, Chapter IX, Lemma 3.3]). We next consider the relation between the boundary value functions of \( B \circ L_{n_j}(\zeta) \) and \( G(\zeta) \). Usually, \( B \circ L_{n_j}(\zeta) \) has a lot of zeros near to the boundary \( T \) of \( \Delta \), in contrast to \( G(\zeta) \).

**Lemma 4.4.** Under the notation of Lemma 4.3, we may choose a subsequence \( \{c_{n_j}\} \) of \( \{c_{n_j}\} \) such that \( B \circ L_{n_j}(\theta) \) converges to \( G(\theta) = e^{i\theta} \), m-a.e. \( \theta \) in \( T \), consequently, \( B \circ L_{n_j}(x) \) converges to \( G(x) \), \( \hat{m} \)-a.e. \( x \) in \( X \), where \( \hat{m} \) denotes the lifting of \( m \) to \( X \).

**Proof.** Since each Blaschke product \( B \circ L_{n_j} \) lies in \( L^2(T) \), its Taylor series shows that
\[ B \circ L_{n_j}(\theta) = \sum_{k=0}^{\infty} a_k^{(j)} e^{ik\theta} \quad \text{and} \quad \|B \circ L_{n_j}\|_2^2 = \sum_{k=0}^{\infty} |a_k^{(j)}|^2 = 1. \]
It follows from Lemma 4.3 and the Weierstrass theorem of double series that \( a_1^{(j)} = (B \circ L_{n_j})'(0) \) tends to \( G'(0) = 1 \). Since \( a_0^{(j)} = B \circ L_{n_j}(0) = B(c_{n_j}) = 0 \), we have
\[ \|B \circ L_{n_j} - G\|^2_2 = |a_1^{(j)} - 1|^2 + \sum_{k=2}^{\infty} |a_k^{(j)}|^2 \to 0, \quad j \to \infty. \]
Thus, we may choose a subsequence \( \{c_{n_k}\} \) of \( \{c_{n_j}\} \) such that \( B \circ L_{n_k}(\theta) \) converges to \( G(\theta) = e^{i\theta} \), m-a.e. \( \theta \) in \( T \). \( \square \)

5. Existence of desired sequences

Let \( \phi \) be a homomorphism in \( \mathfrak{M}(H^\infty) \setminus \Delta \), and let \( f_1, f_2, \ldots, f_N \) be functions in \( H^\infty(\Delta) \). What should be shown is the existence of a sequence \( \{\zeta_j\} \) in \( \Delta \) for which \( \lim_{j \to \infty} f_k(\zeta_j) = f_k(\phi) \) for \( k = 1, 2, \ldots, N \). By Lemma 2.1, it suffices to consider the case where \( \phi \) lies in \( \mathfrak{M}_1 \setminus X \). So the representing measure \( \mu \) for \( \phi \) is a continuous measure, and its minimal support \( S_\phi \) is a compact subset of \( \mathfrak{M}_1 \cap X \). Then we know that there exists a Blaschke product vanishing at \( \phi \) in \( \mathfrak{M}(H^\infty) \).
Let \( \{ \varepsilon_n \} \) be a decreasing sequence of positive numbers with \( \varepsilon_n \to 0 \), and let \( s_n(\theta) = \sum_{i=1}^{c(\varepsilon_n)} a_i^{(n)} \chi_{E_i^{(n)}}(\theta) \) denote the simple function determined by Lemma \([3.1]\) of which support \( E^{(n)} = \bigcup_{i=1}^{c(\varepsilon_n)} E_i^{(n)} \) is contained in \( (-\varepsilon_n, \varepsilon_n) \). Then there are \( \alpha_n \) and \( \beta_n \) with \( -\varepsilon_n < \alpha_n < \beta_n < \varepsilon_n \) satisfying the properties of \([3.4]\) and \([3.5]\). Let \( c_n = t_n e^{i(\alpha_n + \beta_n)/2} \) in \( \Delta \) be the mid-point of the circular arc \( C_n \) from \( e^{i\alpha_n} \) to \( e^{i\beta_n} \) orthogonal to \( T \). Observe that \( c_n \to 1 \), as \( n \to \infty \). We now choose a thin interpolating subsequence \( \{ c_{n_j} \} \) of \( \{ c_n \} \) for which a Blaschke product \( B \) with \( B(\phi) = 0 \) has the property of Lemma \([4.3]\). For simplicity of notation, we write \( \{ \varepsilon_j \}, \{ s_j(\theta) \} \) and \( \{ c_j \} \) for \( \{ \varepsilon_{n_j} \}, \{ s_{n_j}(\theta) \} \) and \( \{ c_{n_j} \} \), respectively.

Let \( \mathfrak{F} = \{ f_1, f_2, \ldots, f_N \} \cup \{ 1, z, B \} \) with the Blaschke product \( B \) above, and let \( B(\mathfrak{F}) \) be the uniformly closed subalgebra of \( H^\infty(\Delta) \) generated by \( \mathfrak{F} \). Let \( L_j \) be the map \( L_j(\zeta) = (\zeta + c_j)/(1 + \overline{c_j}\zeta) \) as usual. Replacing \( \{ c_j \} \) with a suitable subsequence if necessary, we may assume, by normal families argument, that \( f_k \circ L_j(\zeta) \) converges uniformly to \( F_k(\zeta) \) on compact subsets of \( \Delta \), as \( j \to \infty \). Since each function in \( \{ 1, z, B \} \) also has the same property, it holds on all \( B(\mathfrak{F}) \), that is, if \( h \) is in \( B(\mathfrak{F}) \), then there is an \( H \) in \( H^\infty(\Delta) \) such that

\[
\lim_{j \to \infty} h \circ L_j(\zeta) = H(\zeta)
\]

uniformly on compact subsets of \( \Delta \). Modifying the equation, we obtain the following:

Lemma 5.1. Let \( B \) be the Blaschke product with the property of Lemma \([4.3]\). Under the notation of \([5.1]\), here are some properties:

(a) We have

\[
\lim_{j \to \infty} h \circ L_j(B \circ L_j(\zeta)) = H(\zeta) = \lim_{j \to \infty} (H \circ B) \circ L_j(\zeta)
\]

uniformly on compact subsets of \( \Delta \). Moreover, we also observe that

\[
\lim_{j \to \infty} | h \circ L_j(\zeta) - (H \circ B) \circ L_j(\zeta) | = 0
\]

uniformly on compact subsets of \( \Delta \).

(b) Regarding \( H \) as a function in \( L^\infty(T) \), we may choose a sequence \( \{ p_n \} \) of polynomials with \( \| p_n \|_\infty \leq \| H \|_\infty \) such that \( \lim_{n \to \infty} \| p_n - H \|_1 = 0 \), consequently,

\[
\lim_{n \to \infty} p_n(\zeta) = H(\zeta)
\]

uniformly on compact subsets of \( \Delta \).

Proof. The first equation of (a) is a direct consequence of \([5.1]\) and the property of \( B \). The second one follows from the inequality

\[
| h \circ L_j(\zeta) - (H \circ B) \circ L_j(\zeta) | \leq | h \circ L_j(\zeta) - H(\zeta) | + |H(\zeta) - (H \circ B) \circ L_j(\zeta) |.
\]

On the other hand, it is known that the Cesaro means \( p_n \) of the Fourier (Taylor) series of \( H \) converges to \( H \) in \( L^1(T) \) (see \([9, 16p]\)). Since \( H \) is bounded on \( \Delta \), we observe that \( \| p_n \|_\infty \leq \| H \|_\infty \). Hence, the conclusion of (b) follows from Poisson integral formulas for analytic functions on \( \Delta \). \( \square \)
As we shall see later, \( H(\zeta) \) is identified with the restriction of \( h \) or \( H \circ B \) in \( H^\infty(\Delta) \) to a suitable homeomorphic part in \( \mathfrak{M}_1 \) (see the remarks below).

Moreover, it follows from Lemma 3.2 that there is a sequence \( \{s_j^\circ(\theta)\} \) of simple functions with supports \( F^{(j)} \) satisfying (3.3) on \( B(\mathfrak{H}) \). We especially have

\[
(5.2) \quad \phi(B) = \phi(B^n) = \lim_{j \to \infty} \int_{F^{(j)}} B(\theta)^n s_j^\circ(\theta) \, dm(\theta), \quad n = 0, 1, 2, \cdots,
\]

which vanishes for each \( n \geq 1 \).

Let \( M(X) \) be the space of all regular complex Borel measures on \( X = \mathfrak{M}(L^\infty) \). Then the Riesz representation theorem shows that \( M(X) \) is identified with the dual space of \( C(X) = L^\infty(T) \). Since each \( s_j^\circ(\theta) \, dm(\theta) \) extends to a probability measure \( s_j^\circ(x) \, d\hat{m}(x) \) on \( X \), we may choose an adherent point \( \hat{\nu} \) of \( \{s_j^\circ(x) \, d\hat{m}(x)\} \) in \( M(X) \) in the weak*-topology. Then \( \hat{\nu} \) is a representing measure for the homomorphism which is the restriction of \( \phi \) to \( B(\mathfrak{H}) \). Indeed, we have

\[
(5.3) \quad h(\phi) = \lim_{j \to \infty} \int_{-\pi}^\pi h(\theta) s_j^\circ(\theta) \, dm(\theta) = \lim_{j \to \infty} \int_X h(x) s_j^\circ(x) \, d\hat{m}(x) = \int_X h(x) \, d\hat{\nu}(x)
\]

for all \( h \in B(\mathfrak{H}) \), because Lemma 3.2 assures the existence of above limits on \( B(\mathfrak{H}) \). It would be a crucial point that the sequence of \( \{s_j^\circ(\theta) dm(\theta)\} \) may be replaced by the one of Poisson integrals for functions in \( B(\mathfrak{H}) \).

Let \( \mathbb{N}^\infty = \mathbb{N} \cup \{\infty\} \) be the one-point compactification of all positive integers \( \mathbb{N} \), and let \( H^\infty(\Delta \times \mathbb{N}^\infty) \) be the space of all bounded sequences \( g(\zeta, j) \), for \( j \in \mathbb{N}^\infty \), of analytic functions on \( \Delta \) such that \( g(\zeta, j) \) converges to \( g(\zeta, \infty) \) uniformly on compact subsets of \( \Delta \). It then follows from Fatou’s theorem that each function \( \zeta \to g(\zeta, j) \) extends to the closed unit disc \( \overline{\Delta} = \Delta \cup T \), of which the boundary function is denoted by \( g(\theta, j) \).

As a sequence \( \{g(\theta, j)\} \) in \( L^\infty(T) = L^1(T)^* \), the dual space of \( L^1(T) \), it is easy to see that \( g(\theta, j) \) converges to \( g(\theta, \infty) \) in the weak*-topology (compare with Lemma 4.4). Let \( u_j(\theta) = s_j^\circ \circ L_j(\theta) |(L_j)'(\theta)| \). Then Lemma 3.3 and (5.3) assure that the following equation

\[
(5.4) \quad h(\phi) = \lim_{j \to \infty} \int_{-\pi}^\pi h(\theta) s_j^\circ(\theta) \, dm(\theta) = \lim_{j \to \infty} \int_{-\pi}^\pi h \circ L_j(\theta) u_j(\theta) \, dm(\theta)
\]

holds on \( B(\mathfrak{H}) \). We have to represent the measure \( \hat{\nu} \) in (5.3) as the one on \( T \times \{\infty\} \).

We are now ready for the proof of Theorem.
Proof of Theorem. Let us define a closed subalgebra \((SB)(\mathfrak{F}^2)\) of \(H^\infty(\Delta \times \mathbb{N}^\infty)\). With the above notation, for each function \(h\) in \((\mathfrak{F}^2)\), we put

\[
(5.5) \quad g(\zeta, j) = \begin{cases} 
  h \circ L_j(\zeta), & j \in \mathbb{N}, \\
  H(\zeta), & j = \infty,
\end{cases}
\]

in the equation \((5.1)\). Then \(g(\zeta, j)\) is an element of \(H^\infty(\Delta \times \mathbb{N}^\infty)\). We set \((SB)(\mathfrak{F}^2)\) to be the space of all such functions \(g(\zeta, j)\) by \((5.1)\). Observe that \((SB)(\mathfrak{F}^2)\) is a uniformly closed subalgebra of \(H^\infty(\Delta \times \mathbb{N}^\infty)\). We may regard \(u_j(\theta)dm(\theta) = s_j^2 \circ L_j(\theta) |(L_j)'(\theta)|dm(\theta)\) as a probability measure on \(T \times \mathbb{N}^\infty\) concentrated on \(T \times \{j\}\).

Let \(H(\Delta)\) denote the restriction of \((SB)(\mathfrak{F}^2)\) to \(\Delta \times \{\infty\}\). Then if \(H\) lies in \(H(\Delta)\), then there is an \(h\) in \(B(\mathfrak{F}^2)\) such that \(H(\zeta) = \lim_{j \to \infty} h \circ L_j(\zeta)\). We claim that the value \(h(\phi)\) may be represented as

\[
(5.6) \quad h(\phi) = H(0) = \int_{-\pi}^{\pi} H(\theta) \, dm(\theta),
\]

where \(H(\theta)\) is the boundary value function of \(H(\zeta)\). Indeed, recall that the Blaschke product \(B\) in \(B(\mathfrak{F}^2)\) satisfies that \(B \circ L_j(\zeta)\) converges to \(G(\zeta) = \zeta\) uniformly on compact subsets of \(\Delta\), so does \(p(B \circ L_j)(\zeta)\) to \(p(G)(\zeta) = p(\zeta)\), for each polynomial \(p\) on \(\Delta\). Since \(p(B)\) lies in \(B(\mathfrak{F}^2)\), the function,

\[
g(\zeta, j) = \begin{cases} 
  p(B \circ L_j)(\zeta), & j \in \mathbb{N}, \\
  p(\zeta), & j = \infty,
\end{cases}
\]

is an element of \((SB)(\mathfrak{F}^2)\), the space \(H(\Delta)\) contains all polynomials. Since the disc algebra \(A(\Delta)\) is the uniform closure of polynomials, the subalgebra \(H(\Delta)\) of \(H^\infty(\Delta)\) also contains \(A(\Delta)\). Recall that we have \(B(\phi) = 0 = G(0)\) by the property of Lemma \(4.3\). Then \((5.2)\) and \((5.4)\) show that \(P \circ B(\phi) = P(G(0)) = P(0)\).

On the other hand, for each \(H\) in \(H(\Delta)\), there is a sequence \(\{p_n\}\) of polynomials with the property (b) of Lemma \(5.1\) so \(p_n(0) \to H(0)\). Since

\[
\lim_{j \to \infty} \int_{-\pi}^{\pi} (h \circ L_j(\theta) - p_n(B \circ L_j)(\theta)) \, u_j(\theta) \, dm(\theta) = h(\phi) - p_n \circ B(\phi)
\]

\[
= h(\phi) - p_n(0),
\]

we have

\[
|h(\phi) - H(0)| \leq |h(\phi) - p_n \circ B(\phi)| + |p_n(0) - H(0)| \to 0,
\]

as \(n \to \infty\). Thus \((5.6)\) holds for all \(h\) in \(B(\mathfrak{F}^2)\).

Let \(\zeta_j = L_j(0) = c_j\). We show that the sequence \(\{\zeta_j\}\) in \(\Delta\) satisfies the property \((1.2)\) for \(f_1, f_2, \ldots, f_N\). Indeed, for all \(h\) in \(B(\mathfrak{F}^2)\), Lemma \(5.1\) implies that

\[
H(\zeta) = \lim_{j \to \infty} h \circ L_j(B \circ L_j(\zeta)),
\]
uniformly on compact subsets of $\Delta$. Hence, since $B \circ L_j(0) = 0$, it follows from (5.6) that
\[ h(\phi) = H(0) = \lim_{j \to \infty} h \circ L_j(0) = \lim_{j \to \infty} h(\zeta_j). \]

Since each $f_k$ lies in $B(\hat{\mathcal{F}})$, we obtain $\lim_{j \to \infty} f_k(\zeta_j) = f_k(\phi)$. Therefore, by virtue of Lemma 2.3, the property (1.2) holds on the uniform algebra $B(\mathcal{F})$ generated by a countable family $\mathcal{F}$ of $H^\infty(\Delta)$, as desired. \hfill \Box

Some remarks are in order on the above proof. Let us explain the relation between $\Delta \times \{\infty\}$ and a homeomorphic part in $\mathcal{M}_1 \setminus \Delta$, which reveals the structure of $H(\Delta)$. Since $\{c_j\}$ is a thin interpolating sequence in $\Delta$, any adherent point $\xi$ of $\{c_j\}$ lies in the homeomorphic part $P(\xi)$ in $\mathcal{M}_1 \setminus \Delta$. Let $L = \lim_\beta L_\beta$ be the Hoffman map with $L(0) = \xi$ by Lemma 2.2, where $(L_\beta)$ is a convergent subnet of $\{L_j\}$. Then $H(\Delta)$ is the algebra of all $h \circ L(\zeta)$ for $h$ in $B(\hat{\mathcal{F}})$. So (5.6) holds for $H(\zeta) = h \circ L(\zeta)$, by regarding $H(\zeta)$ as a function on $\Delta$. Although $\xi$ and $\phi$ are usually different, the values $h(\phi)$ and $h(\xi)$ coincide for all $h$ in $B(\hat{\mathcal{F}})$. Moreover we see that the measure $\widehat{\nu}$ in (5.5) may be regarded as a representing measure $\widehat{\mu}$ on homeomorphic part $P(\xi)$, and the sequence of $u_j^*(\theta)dm(\theta)$ may be replaced by the one of Poisson kernels $P_{c_j}$ on $T$. Since $B(\phi) = 0 = G(0)$, $H \to h(\phi)$ is the evaluation homomorphism at $0$ for $A(\Delta)$, which extends uniquely to the one on $H^\infty(\Delta)$, so does for $H(\Delta)$. Thus it seems to be natural that (5.6) holds. However we do not know whether $H(\Delta)$ is uniformly closed, although it is strictly smaller than $H^\infty(\Delta)$. This fact follows easily from Lusin’s theorem, since $H(\Delta)$ is the algebra generated by $A(\Delta)$ and finite set $\{F_k(\zeta)\}$ with $F_k(\zeta) = \lim_{j \to \infty} f_k \circ L_j(\zeta)$.

6. Remarks

(a) As far as we restrict our attention to analytic discs in $\mathcal{M}(H^\infty) \setminus \Delta$, it is rather easy to show that each of them belongs to the closure of $\Delta$ in $\mathcal{M}(H^\infty)$. Indeed, let $P(\phi)$ be a nontrivial Gleason part in $\mathcal{M}_1$, and let $\mu$ be the representing measure for $\phi$ on the Shilov boundary $X$. Denote by $H^p(\mu), 1 \leq p < \infty$, the closure of $H^\infty(\Delta)$ in $L^p(\mu)$. Then Wermer’s embedding theorem assures the existence of an inner function $Z$ in $H^2(\mu)$ such that $Z$ has a bijective extension $\tilde{Z}$ to $P(\phi)$ with $\tilde{Z}(\phi) = 0$, with which $\tau(z) = \tilde{Z}^{-1}(z)$ is an analytic map on $\Delta$, meaning that $f \circ \tau(z)$ is analytic on $\Delta$ for all $f$ in $H^\infty(\Delta)$, and
\[ f(\xi) = \sum_{n=0}^\infty a_n \tilde{Z}^n(\xi), \quad \xi \in P(\phi), \]
(see, for example, [12, Chapter 6, §6.4]). Since $Z$ is an inner function in $H^2(\mu)$, there is a sequence $\{q_i\}$ in $H^\infty(\Delta)$ such that $|q_i(x)| \leq 1$ on $X$ and $\|q_i - Z\|_{L^2(\mu)} \to 0$, as $i \to \infty$ (see the proof of [3, Chapter II, Theorem 7.2]). Let $f_1, f_2, \cdots, f_N$ be in $H^\infty(\Delta)$, and put $\mathcal{F} = \{f_1, f_2, \cdots, f_N\} \cup \{q_i : i = 1, 2, \cdots\}$. Let $\varepsilon > 0$, and denote by
For analyticity in ergodic theory, we refer to [13], [14] and [16]. Let \( f_k \circ \tau(z) = \sum_{n=0}^{\infty} a_n^{(k)} z^n \) the Taylor expansion of \( f_k \circ \tau \) on \( \Delta \). By (6.1) we choose a \( q_i \) in \( \{ q_i \} \) and an integer \( \ell_k \geq 0 \) such that

\[
\phi(f_k) - \sum_{n=0}^{\infty} a_n^{(k)} q_i^n(\phi) < \varepsilon, \quad k = 1, 2, \ldots, N.
\]

It follows from Lemmas 3.2 and 3.3 that we find a thin interpolating sequence \( \{ c_j \} \) for which the maps \( L_j \) by (2.2) satisfy that there is a sequence \( \{ u_j(\theta) \} \) of nonnegative functions such that \( \int u_j(\theta) \, dm(\theta) = 1 \) and

\[
| \phi(f) - \int_{-\pi}^{\pi} f \circ L_j(\theta) u_j(\theta) \, dm(\theta) | \to 0, \quad j \to \infty,
\]

for all \( f \) in \( \mathcal{F} \). Taking a subsequence of \( \{ c_j \} \) suitably, we may assume that each \( f \) in \( \mathcal{F} \) satisfies \( f \circ L_j(\zeta) \) converges uniformly on compact subsets in \( \Delta \). Let \( L \) be the Hoffman map by a convergent subnet of \( \{ L_j \} \) in \( \mathcal{M}(H^\infty) \). Then \( P_1 = L(\Delta) \) is an analytic disc homeomorphic to \( \Delta \). We then see that \( e^{\gamma_j} q_j \circ L(\zeta) \), with some real \( \gamma_j \), converges to \( G(\zeta) = \zeta \), so the measure \( u_j(\theta) \, dm(\theta) \) converges to the representing measure \( dm(\theta) \) at \( \zeta = 0 \). Then the sequence \( \{ c_j \} \) in \( \Delta \) has the property that \( f(c_j) \to f(\phi) \) for \( f \) in \( \mathcal{F} \), consequently, we see that

\[
W(\phi, f_1, \ldots, f_N, \varepsilon) \cap \Delta \neq \emptyset,
\]

for any \( \varepsilon > 0 \).

By a similar argument as above, it enables us to show Hoffman’s characterization of analytic discs by interpolating sequences. Under the notation above, let \( \mathcal{F}_m = \{ q_1, q_2, \ldots, q_m \} \). Then there is a thin interpolating sequence \( \{ \zeta_j^{(m)} \} \) in \( \Delta \) and a homeomorphic part \( P(\phi^{(m)}) \) such that \( \lim_{j \to \infty} g(\zeta_j^{(m)}) = g(\phi^{(m)}) = g(\phi) \) for any \( g \) in the algebra generated by \( \mathcal{F}_m \). It suffices to consider the case where \( \{ \zeta_j^{(m)} \} \cap \{ \zeta_j^{(k)} \} = \emptyset \) if \( m \neq k \). With the aide of diagonal argument, we choose an interpolating sequence \( \{ \zeta_j \} \) so that

\[
\{ \zeta_j \} = \{ \zeta_{jm}, \zeta_{jm+1}, \ldots, \zeta_{jm+1-1} \; ; \; m = 1, 2, \ldots \},
\]

for which such \( \phi^{(m)} \) may be regarded as an adherent point for \( \{ \zeta_j \} \) in \( \mathcal{M}(H^\infty) \). Modifying the argument in Lemma 3.3, we may see that \( P(\phi) \) lies in the closure of \( \cup_{m=1}^{\infty} P(\phi^{(m)}) \) in \( \mathcal{M}(H^\infty) \). Thus \( \{ \zeta_j \} \) is a desired interpolating sequence. Conversely, if \( \phi \) lies in the closure of an interpolating sequence \( \{ \zeta_j \} \), then we see easily that \( \phi \) lies in a nontrivial Gleason part \( P(\phi) \). Then by a similar argument as above, we find a Hoffman map \( L \) by converging subnet of \( \{ L_j \} \) such that \( P(\phi) = L(\Delta) \). For our characterization of analytic discs, full details and further developments will appear elsewhere.

(b) We may represent concretely a large portion of the fiber \( \mathcal{M}_1 \) by a continuous flow. It is useful to study representing measures in connection with invariant measures. For analyticity in ergodic theory, we refer to [13], [14] and [16]. Let \( H^\infty(\mathbb{R}_+^2) \) be the space of all bounded analytic functions on the upper half-plane \( \mathbb{R}_+^2 \). Then \( H^\infty(\mathbb{R}_+^2) \) is identified with \( H^\infty(\Delta) \) via the conformal map \( z(w) = (w - i)/(w + i) \). Setting
where \( t \) denotes the largest integer not exceeding \( t \). We write \( x \) for \( (y, s) \) in \( X \), and the translate \( S_t x \) is denoted by \( x + t \). Let \( A(X) \) be the uniform algebra of all functions \( f \) in \( C(X) \) satisfying that each \( t \to f(x + t) \) lies in \( H^\infty(\mathbb{R}) \), the space of all boundary value functions in \( H^\infty(\mathbb{R}^2_+) \). Then \( A(X) \) is a logmodular algebra on \( X \) whose maximal ideal space is identified with a certain quotient space of \( X \times [0, \infty] \). Recall that the Poisson kernel \( P_{ir} \) for \( \mathbb{R}^2_+ \) is defined by \( P_{ir}(t) = r / \pi (t^2 + r^2) \). For a bounded Borel function \( g \) on \( X \), we put

\[
g(x, r) = g \ast P_{ir}(x) = \int_{-\infty}^{\infty} g(x + t)P_{ir}(t)dt, \quad (x, r) \in X \times (0, \infty).
\]

This decides the representing measures for \( A(X) \) on \( X \times (0, \infty) \), while representing measures lying in \( X \times \{ \infty \} \) are invariant measures being multiplicative on \( A(X) \) (see [14], [19], [20] for representing measures for \( A(X) \)). Denote by \( H^\infty(X) \) the algebra of all bounded Borel functions \( g \) for which \( x \to g(x, r) \) lies in \( A(X) \) for each \( r > 0 \). Since \( H^\infty(X) \) is isometrically isomorphic to \( H^\infty(\Delta) \), the subset \( X \times (0, \infty) \setminus \mathbb{R}^2_+ \) represents a portion of the fiber \( \mathfrak{M}_1 \), from which we observe immediately that either nontangential point or orocycular point is in the closure of an interpolating sequence (compare with [8], Chapter X, Exercises 1 and 2). Let \( M \) be a minimal set in \( (X, \{ S_t \}_{t \in \mathbb{R}} \) (see [16] for minimal sets). Observe that each \( O(x) \times (0, \infty) \) corresponds to an analytic disc, where \( O(x) \) denotes the orbit \( \{ x + t; t \in \mathbb{R} \} \). If \( x \) is in \( M \), then the analytic disc by \( O(x) \times (0, \infty) \) is never homeomorphic to \( \Delta \). We also see that every representing measure on \( M \) not being point mass has the same support set \( M \), on which there are many representing measures. Since \( M \) is an intersection of peak sets, the restriction \( A_M \) of \( A(X) \) to \( M \) is a uniform algebra equipped with many interesting properties (see [13], [14] and [21] for more details).

References

[1] H. Arai, Two problems originated in Japan: Kakeya’s problem and the corona problem, (Japanese), Sūri-Kagaku 12 (2000), 56-65.
[2] L. Carleson, Interpolations by bounded analytic functions and the corona problem, Ann. of Math. 76 (1962), 547–559.
[3] R. Douglas, S. Krantz, E. Sawyer, S. Treil, and B. Wick, A history of the corona problem, in The corona problem, Springer-Verlag, Berlin and New York 2014, 1–29.
[4] T. Gamelin, Uniform algebras, Prentice-Hall, Englewood Cliffs, New Jersey, 1969.
[5] ______, Localization of the corona problem, Pacific J. Math. 34 (1970), 73–81.
[6] ______, The algebra of bounded analytic functions, Bull. Amer. Math. Soc. 79 (1973), 1095–1107.
[7] ______, Wolff’s proof of the corona theorem, Isael J. Math. 37 (1980), 113–119.
[8] J. Garnett, Bounded analytic functions, Springer-Verlag, Berlin and New York, 2007.
[9] H. Hoffman, Banach space of analytic functions, Prentice-Hall, Englewood Cliffs, New Jersey, 1962.
[10] ______, Bounded analytic functions and Gleason parts, Ann. of Math. 86 (1967), 74–111.
[11] K. Izuchi, Structure of the maximal ideal space of $H^\infty$, (Japanese), Sugaku 54 (2001), 24–36.
[12] G. Leibowitz, Lectures on complex function algebras, Scott, Foresman and Company, Glenview, Illinois, 1970.
[13] P. Muhly, Function algebras and flows, Acta Sci. Math. (Szeged) 35 (1973), 111–121.
[14] ______, Function algebras and flows II, Ark. Mat. 11 (1973), 203–213.
[15] N. K. Nikol’skii, Treatise on the shift operator, Springer-Verlag, Berlin Heidelberg, 1986.
[16] K. Petersen, Ergodic Theory, Cambridge University Press, Cambridge, 1983.
[17] I. J. Schark The maximal ideals of bounded analytic functions, J. Math. Mech. 10 (1961), 735–746.
[18] E. L. Stout, The theory of uniform algebras, Bogden and Quigley, Tarrytown-on-Hudson, 1971.
[19] J. Tanaka, On a theorem of P.S. Muhly, Proc. Amer. Math. Soc. 142 (1977), 119–123.
[20] ______, Corona problem and flows, J. Funct. Anal. 102 (1991), 360–378.
[21] ______, Flows in fibers, Trans. Amer. Math. Soc. 343 (1994), 779–804.

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