A comprehensive survey on 3-equitable and divisor 3-equitable labeling of graphs

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Abstract. This article presents a short survey on 3-equitable and divisor 3-equitable labeling of graphs. For any graph $G(V,E)$ and $k > 0$, assign vertex labels from $\{0, 1, \ldots, k - 1\}$ such that when the edge labels induced by the absolute value of the difference of the vertex labels, the number of vertices labeled with $i$ and the number of vertices labeled with $j$ differ by at most one and the number of edges labeled with $i$ and the number of edges labeled with $j$ differ by at most one. We call a graph $G$ with such an assignment of labels $k$-equitable. When $k = 3$, it becomes a 3-equitable labeling. In 2019, Sweta Srivastav et al. introduced the notion of divisor 3-equitable labeling of graphs. A bijection $f : V(G) \rightarrow \{1, 2, \ldots, n\}$ induces a function $f^* : E(G) \rightarrow \{0, 1, 2\}$ defined by for each edge $e = xy$, (i) $f^*(e) = 1$ if $f(x)f(y)$ or $f(y)f(x)$, (ii) $f^*(e) = 2$ if $f(x)f(y) = 2$ or $f(y)f(x) = 2$, and (iii) $f^*(e) = 0$ otherwise such that $|e_f(i) - e_f(j)| \leq 1$ for all $0 \leq i, j \leq 2$. A graph which admits a divisor 3-equitable labeling is called a divisor 3-equitable graph. This article stands divided into five sections. The first and fifth sections are reserved respectively for introduction and some important references. The second section deals with the 3-equitable labeling of graphs wherein some important known results have been recalled. The third section deals with the divisor 3-equitable labeling of graphs wherein a few known results have been outlined. In the fourth section we highlight certain conjectures and open problems in respect of the above mentioned labeling that still remain unsolved.

Keywords: 3-Equitable Labeling, 3-Equitable Graphs, Divisor 3-Equitable Labeling, Divisor 3-Equitable Graphs

1. Introduction

We consider only “non-trivial, simple, finite, connected, and undirected graph” in this paper. A graph labeling is an “assignment of labels” (mostly, numbers or colors) to the edges and/or vertices of a graph. For the standard terminology and notations we follow [1]. For $G = (V,E)$, a function $f$ from $V$ to $\{0, 1, 2\}$ with an “induced function” $f^*$ from $E$ to $\{0, 1, 2\}$ given by $f^*(e = uv) = |f(u) - f(v)|$ is 3-equitable labeling if the “number of vertices with label $i$ and $j$ differ by at most 1” and in the same way the “number of edges with label $i$ and $j$ differ by at most 1”, $0 < i, j < 2, i \neq j$. Also $|v_f(i) - v_f(j)| < 1$ and $|e_f(i) - e_f(j)| < 1, 0 < i, j < 2$. A graph which permits 3-equitable labeling is called a “3-equitable graph”. In this paper, we have summarized the results concerning the 3-equitable and divisor 3-equitable labeling of graphs. We also provide a brief introduction to the concerned labeling. Further, we recall a few relevant definitions & other necessary results which are important for the present study.

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Definition 1.
If the vertices (edges) of $G$ are assigned values (mostly integers) subject to some restriction(s) is known as a vertex (edge) labeling of $G$.

Definition 2.
A “ternary vertex labeling (TVL)” of $G$ is called a “3-equitable labeling” if $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for all $0 < i, j > 2$. A graph which permits a 3-equitable labeling is called a “3-equitable graph”.

Definition 3.
A chord of a cycle $C_n$ is an “edge joining two non-adjacent vertices of $C_n$”.

Definition 4.
If "$a$ divides $b$" (denoted by $a \mid b$), then there is a “positive integer $k$ such that $b = ka$”. If "$a$ does not divide $b$", then it is denoted by $a \nmid b$.

Definition 5.
The divisor function $d(n)$ of $n$, an integer, is the number of divisors of $n$.

Definition 6.
Let $n, x$ denote an integer and a real number, respectively. The DSF (“divisor summability function”) $D(x)$ is the “sum of the number of divisors of $n$” for $n \leq x$.

Definition 7.
Let $f : V \rightarrow \{1, 2, \ldots, |V|\}$ be a bijection. For every edge $xy$, assign the label “1” if either $f(x) \mid f(y)$ or $f(y) \mid f(x)$ and the label “0” otherwise. Then $f$ is called a “divisor cordial labeling” (DCL) if $|e_f(0) - e_f(1)| \leq 1$. A graph which permits a DCL is called a “divisor cordial graph”.

Definition 8.
A “binary vertex labeling” (BVL) of $G$ is said to be a cordial labeling if $|v_f(0) - v_f(1)| \leq 1$ & $|e_f(0) - e_f(1)| \leq 1$. A graph which permits a cordial labeling is known as “cordial”.

Definition 9.
A TVL of $G$ is said to be a “3-equitable labeling” if $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for all $0 \leq i, j \leq 2$. A graph $G$ is 3-equitable if it permits a 3-equitable labeling.

Definition 10.
A function $f : V \rightarrow \{0, 1, 2\}$ such that each edge $uv$ receives $|f(v_i) - f(v_j)|$ where $v_i, v_j \in V$ is said to be a “3-equitable labeling” if $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for all $0 \leq i, j \leq 2$ where $v_f(i)$ is the number of vertices labelled with $i$ and $e_f(i)$ is the number of edges labelled with $i$.

Definition 11.
A function $f$ from $V$ of $G$ to $\{0, 1, 2\}$ is called a 3-equitable labeling if the labels on the edges labels produced by “absolute difference of the labels of end vertices” of the respective edges in such a way that “the number of edges with label $i$ and $j$ differ by at most 1”.

Definition 12.
A mapping $f : V(G) \rightarrow \{0, 1, 2\}$ is called a TVL of $G$ and $f(v)$ is called “the label of the vertex $v$ of $G$” under $f$.

2. Certain well-known results on 3-equitable labeling

In this section, we highlight a few important results proved by different authors concerning the 3-equitable labeling of graphs.

Cahit [2] proved the following results.

Theorem 2.1.
(i) A cycle $C_n$ is 3-equitable iff $C_n \not\equiv 3 \pmod{6}$.
(ii) A graph $G$ which is Eulerian with $q \equiv 3 \pmod{6}$ is not 3-equitable.
(iii) Every caterpillar admits a 3-equitable labeling.
(iv) Every tree (with “fewer than five end vertices”) is 3-equitable.
Seoud et al. [13] proved the following results.

**Theorem 2.2.**
A graph $G(p, q)$ in which every vertex is of odd degree is not 3-equitable if $p \equiv 0 \text{ (mod 3)}$ and $q \equiv 3 \text{ (mod 6)}$.

**Theorem 2.3.**
(i) All fans except $P_2 + K_1$ are 3-equitable.
(ii) $P_n^2$ is 3-equitable for all $n$ except 3.
(iii) $K_{m,n}$ ($3 \leq m \leq n$) is 3-equitable iff $(m, n) = (4, 4)$.

Bapat et al. [1] proved the following.

**Theorem 2.4.**
(i) Helms $H_n$ (where $n \geq 4$) admit a 3-equitable labeling.
(ii) Flower graph is 3-equitable.
(iii) The “one-point union of any number of helms” is 3-equitable graph.
(iv) The “one-point union of any number of copies of $K_4$” is a 3-equitable graph.

Youssef gave the following result in [23].

**Theorem 2.5.**
The wheel $W_n = C_n + K_1$ is 3-equitable for all $n \geq 4$.

Vaidya et al. [17] have proved the following results.

**Theorem 2.6.**
(i) $D_2(C_n)$ is 3-equitable except for $n = 3 \& 5$.
(ii) $D_2(P_n)$ is 3-equitable except for $n = 3$.
(iii) $M(P_n)$ is 3-equitable.
(iv) $M(C_n)$ is 3-equitable for $n$ even & not 3-equitable for $n$ odd.

Vaidya et al. have also discussed the 3-equitable labeling of wheel related graphs in [15], some shell related graphs in [16], and some star related graphs in [21].

**Theorem 2.7.**
(i) All caterpillars are 3-equitable.
(ii) The graph $S'(K_{1,n})$ is 3-equitable.

Illustration: 3-equitable labeling of $S'(K_{1,7})$. 
Theorem 2.8.

The graph $S'(B_{n,n})$ is 3-equitable.

Illustration: 3-equitable labeling of $S'(B_{6,6})$.

Theorem 2.9.

The graph $D_2(B_{n,n})$ is 3-equitable.

Illustration: 3-equitable labeling of $D_2(B_{5,5})$.

Theorem 2.10.

(i) For $n \geq 6$, the graph $C_n + K_n$ is 3-equitable if and only if $n$ is even.
(ii) $C_n^2$ is 3-equitable if and only if $n \geq 8$.

I. Cahit [3] proved that $C_n$ is 3-equitable, $n \not\equiv 3 \pmod{6}$. M. V. Bapat et al. in [1] proved that Helms $H_n$, ($n \geq 4$) are 3-equitable. S.K. Vaidya et al. in [14] have shown that $B_{n,n}$ is 3-equitable.

Theorem 2.11.

(i) Switching of any rim vertex of $W_n$ (except for $n \equiv 1, 3, 5 \pmod{6}$) is 3-equitable.
(ii) The graph $Gn \oplus K_{1,n}$ is 3-equitable for all $n$.
(iii) The graph $G \oplus K_{1,n}$ is 3-equitable for all $n$, where $G$ is “cycle having twin chords $C_{n,3}$”.

3. Some known results on divisor 3-equitable labeling

In this section, we present the divisor 3-equitable graphs.

Theorem 3.1.

Split graph $spl K(1,n)$ is 3-equitable prime cordial graph.

Theorem 3.2.

The graph obtained by “joining two copies of $S_n$ by path $P_k$” permits a divisor cordial labeling for $n \geq 4$. 

**Theorem 3.3.**
The jewel graph $J_n$ is a divisor cordial.

**Theorem 3.4.**
The double cone $C_n + 2K_1$ is a divisor cordial graph.

**Theorem 3.5.**
Every complete graph is a divisor graph.

**Theorem 3.6.**
Every tree is a divisor graph.

**Theorem 3.7.**
The jewel graph $J_n$ is divisor cordial.

**Theorem 3.8.**
Every full binary tree is divisor cordial.

**Theorem 3.9.**
Given $n$ (a positive integer), there is a divisor cordial graph $G$ with $n$ vertices.

**Theorem 3.10.**
The cycle $C_n$ is a divisor cordial.

**Theorem 3.11.**
The graph $K_{3,n}$ is divisor cordial.

**Theorem 3.12.**
Path $P_n$ is divisor 3-equitable graph.

**Illustration 3.1**
n=3

**Illustration 3.2**
n=4

**Illustration 3.3**
n=9

**Illustration 3.4**
n=17

**Theorem 3.13.**
Cycle $C_n$ is a divisor 3-equitable cordial.

**Illustration 3.5**
$C_{12}$
Illustration 3.6
$C_{24}$

4. Open Problems
In this section we highlight a few important open problems which are remaining unsolved.
1. Deriving new classes of 3-equitable graphs and divisor 3-equitable graphs in the context of other graph operations.
2. Investigating the “necessary and sufficient conditions” for a graph to admit a 3-equitable labeling and a divisor 3-equitable labeling.
3. Analysis of these two labeling on subgraphs/supergraphs of graphs that admit these labeling.
4. Complete characterization of 3-equitable and divisor 3-equitable graphs.
5. Exploring the exclusive applications of 3-equitable and divisor 3-equitable labeling.

Conclusion
In this article two important graph labeling called 3-equitable and divisor 3-equitable labeling are discussed in details. Certain important results concerning these two labeling are also given. The study of different graph labeling techniques plays a vital role in many areas of science and technology especially in the field of communication networks and network security. Establishing 3-equitable labeling and divisor 3-equitable labeling of other classes of graphs are still open and this is for future work.

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