METRIC SPACES ADMITTING ONLY TRIVIAL WEAK CONTRACTIONS

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Abstract. If \((X, d)\) is a metric space then the map \(f: X \to X\) is defined to be a weak contraction if \(d(f(x), f(y)) < d(x, y)\) for all \(x, y \in X, x \neq y\). We determine the simplest non-closed sets \(X \subseteq \mathbb{R}^n\) in the sense of descriptive set theoretic complexity such that every weak contraction \(f: X \to X\) is constant. In order to do so, we prove that there exists a non-closed \(F_\sigma\) set \(F \subseteq \mathbb{R}\) such that every weak contraction \(f: F \to F\) is constant. Similarly, there exists a non-closed \(G_\delta\) set \(G \subseteq \mathbb{R}\) such that every weak contraction \(f: G \to G\) is constant. These answer questions of M. Elekes.

We use measure theoretic methods, first of all the concept of generalized Hausdorff measure.

1. Introduction

We use the following descriptive set theoretical notation in this section.

Notation 1.1. The class of open, closed, \(F_\sigma\), and \(G_\delta\) sets are denoted by \(\Sigma^0_1\), \(\Pi^0_1\), \(\Sigma^0_2\), and \(\Pi^0_2\), respectively. The simultaneously \(F_\sigma\) and \(G_\delta\) sets are denoted by \(\Delta^0_2\).

M. Elekes \[6\] introduced the next definition.

Definition 1.2. We say that the metric space \(X\) possesses the Banach Fixed Point Property (BFPP) if every contraction \(f: X \to X\) has a fixed point.

The Banach Fixed Point Theorem implies that every complete metric space has the BFPP. E. Behrends \[2\] pointed out that the converse implication does not hold. He presented the following example, which he referred to as ‘folklore’.

Theorem 1.3. Let \(X = \text{graph}(\sin(1/x)|_{(0,1)})\). Then \(X \subseteq \mathbb{R}^2\) is a non-closed simultaneously \(F_\sigma\) and \(G_\delta\) set possessing the Banach Fixed Point Property.

M. Elekes \[6\] described the simplest non-closed sets having the BFPP in the sense of descriptive set theoretic complexity. He proved the following theorems.

Theorem 1.4 (M. Elekes). Every open subset of \(\mathbb{R}^n\) with the Banach Fixed Point Property is closed. Every simultaneously \(F_\sigma\) and \(G_\delta\) subset of \(\mathbb{R}\) with the Banach Fixed Point Property is closed.

Theorem 1.5 (M. Elekes). There exist non-closed \(F_\sigma\) and non-closed \(G_\delta\) subsets of \(\mathbb{R}\) with the Banach Fixed Point Property.

2010 Mathematics Subject Classification. Primary: 54H25, 47H10, 28A78, 54H05; Secondary: 03E15.

Key words and phrases. Banach, Borel class, contraction, delta, dimension function, fixed point, gauge, Hausdorff measure, sigma.

Supported by the Hungarian Scientific Research Fund grant no. 72655.
The above three theorems answer the question about the lowest possible Borel classes of $\mathbb{R}^n$ having a non-closed element with the BFPP. In the language of descriptive set theory, if $n \geq 2$ then $\Delta_0^2$ is the best possible class, since there are no $\Sigma_1^0$ and $\Pi_1^0$ examples. If $n = 1$ then $\Sigma_2^0$ and $\Pi_2^0$ are possible, but $\Delta_0^2$ is not.

Note that if every weak contraction $f : X \to X$ is constant then $X$ has the BFPP. There are infinite complete metric spaces that admit only trivial weak contractions, for example the metric spaces $X = \mathbb{Z} \times \{0\}^{n-1} \subseteq \mathbb{R}^n$ clearly have this property (there is a non-degenerate connected compact example in $\mathbb{R}^n$ for every $n \geq 2$, see later). Therefore it is natural to ask the following question.

**Question 1.6** (M. Elekes). *What are the lowest possible Borel classes of $\mathbb{R}^n$ having a non-closed element $X$ such that every weak contraction $f : X \to X$ is constant?*

The main goal of our paper is to answer Question 1.6.

On the one hand, Theorem 1.4 yields that there are no $\Sigma_1^0$ and $\Pi_1^0$ examples in the cases $n \geq 2$.

On the other hand, T. Dobrowolski [5] pointed out the connection between our question and the so called *Cook continuum*; that is, a non-degenerate connected compact topological space $C$ such that every continuous map $f : C \to C$ is either constant or the identity. It was named after H. Cook [4], who first constructed such an object. Cook’s example cannot be embedded in $\mathbb{R}^2$, only in $\mathbb{R}^3$. Later T. Maćkowiak [8, Cor. 32.] has shown that there exists an arc-like (snake-like) Cook continuum, and arc-like continua are embeddable in the plane by [3, Thm. 4.]. The next theorem is straightforward, it follows that the answer is $\Delta_0^2$ if $n \geq 2$.

**Theorem 1.7** (Maćkowiak, Dobrowolski). *Let $X = C \setminus \{c_0\}$, where $C \subseteq \mathbb{R}^2$ is a Cook continuum and $c_0 \in C$ is arbitrary. Then $X \subseteq \mathbb{R}^2$ is non-closed, simultaneously $F_\sigma$ and $G_\delta$, and every weak contraction $f : X \to X$ is constant.*

If $n = 1$ then Theorem 1.4 implies that there is no $\Delta_0^2$ example for Question 1.6.

In the positive direction M. Elekes obtained the following partial result.

**Theorem 1.8** (M. Elekes). *There exists a non-closed $G_\delta$ set $G \subseteq \mathbb{R}$ such that every contraction $f : G \to G$ is constant.*

The proof of Theorem 1.8 is based on the following theorem, that is interesting in its own right.

**Theorem 1.9** (M. Elekes). *For the generic compact set $K \subseteq \mathbb{R}$ (in the sense of Baire category) for any contraction $f : K \to \mathbb{R}$ the set $f(K)$ does not contain a non-empty relatively open subset of $K$.*

In order to answer Question 1.6 it is enough to show that there are non-closed $\Sigma_2^0$ and $\Pi_2^0$ subsets of $\mathbb{R}$ that admit only trivial weak contractions. Therefore we prove the following theorems.

**Theorem 6.1** (Main Theorem, $F_\sigma$ case). *There exists a non-closed $F_\sigma$ set $F \subseteq \mathbb{R}$ such that every weak contraction $f : F \to F$ is constant.*

**Theorem 6.2** (Main Theorem, $G_\delta$ case). *There exists a non-closed $G_\delta$ set $G \subseteq \mathbb{R}$ such that every weak contraction $f : G \to G$ is constant.*
The heart of the proof is the following theorem, that is a partial, measure theoretic analogue of Theorem 1.9. For a gauge function \( h \) let us denote by \( \mathcal{H}^h \) the \( h \)-Hausdorff measure.

**Theorem 5.1** (simplified version). There exists a compact set \( K \subseteq \mathbb{R} \) and a continuous gauge function \( h \) such that \( 0 < \mathcal{H}^h(K) < \infty \), and for every weak contraction \( f : K \to \mathbb{R} \) we have \( \mathcal{H}^h(K \cap f(K)) = 0 \).

Based on this paper, A. Máthé and the author show in [II] the following more general theorem. If \( X \) is a Polish space, then the generic compact set \( K \subseteq X \) is either finite or there is a continuous gauge function \( h \) such that \( 0 < \mathcal{H}^h(K) < \infty \), and for every weak contraction \( f : K \to X \) we have \( \mathcal{H}^h(K \cap f(K)) = 0 \). If \( X \) is perfect, then the generic compact set \( K \subseteq X \) is infinite, so the first case does not occur. This is the measure theoretic analogue of Theorem 1.9 which also answers a question of C. Cabrelli, U. B. Darji, and U. M. Molter. This is the reason why we will work in Polish spaces instead of \( \mathbb{R} \).

The structure of the paper will be as follows. In the Preliminaries section we introduce some notation and definitions. In Section 3 we define balanced compact sets in a Polish space \( X \), and we prove its existence if \( X \) is uncountable. In Section 4 we show that every balanced compact set \( K \subseteq X \) has a continuous gauge function \( h \) such that \( 0 < \mathcal{H}^h(K) < \infty \). In Section 5 we show that \( \mathcal{H}^h(K \cap f(K)) = 0 \) for every weak contraction \( f : K \to X \), which completes the proof of Theorem 5.1. In Section 6 we prove our Main Theorems based on Theorem 5.1 and ideas from [I].

2. Preliminaries

Let \((X, d)\) be a metric space, and let \( A, B \subseteq X \) be arbitrary sets. We denote by \( \text{int} A \) and \( \text{diam} A \) the interior and the diameter of \( A \), respectively. We use the convention \( \text{diam} \emptyset = 0 \). The **distance** of the sets \( A \) and \( B \) is defined by \( \text{dist}(A, B) = \inf \{ d(x, y) : x \in A, y \in B \} \). The function \( h : [0, \infty) \to [0, \infty) \) is defined to be a **gauge function** if it is non-decreasing, right-continuous, and \( h(x) = 0 \) iff \( x = 0 \). For all \( A \subseteq X \) and \( \delta > 0 \) consider

\[
\mathcal{H}_\delta^h(A) = \inf \left\{ \sum_{i=1}^{\infty} h(\text{diam} A_i) : A \subseteq \bigcup_{i=1}^{\infty} A_i, \forall i \ \text{diam} A_i \leq \delta \right\},
\]

\[
\mathcal{H}^h(A) = \lim_{\delta \to 0^+} \mathcal{H}_\delta^h(A).
\]

We call \( \mathcal{H}^h \) the \( h \)-Hausdorff measure. For more information on these concepts see [I].

A metric space \( X \) is **perfect** if it has no isolated points. A metric space \( X \) is **Polish** if it is complete and separable.

Given two metric spaces \((X, d_X)\) and \((Y, d_Y)\), a function \( f : X \to Y \) is called **Lipschitz** if there exists a constant \( C \in \mathbb{R} \) such that \( d_Y(f(x_1), f(x_2)) \leq C \cdot d_X(x_1, x_2) \) for all \( x_1, x_2 \in X \). The smallest such constant \( C \) is the **Lipschitz constant** of \( f \) and denoted by \( \text{Lip}(f) \). If \( \text{Lip}(f) \leq 1 \) then \( f \) is a **1-Lipschitz map**, if \( \text{Lip}(f) < 1 \) then \( f \) is a **contraction**. We say that \( f \) is a **weak contraction** if \( d_Y(f(x_1), f(x_2)) < d_X(x_1, x_2) \) for all \( x_1, x_2 \in X, x_1 \neq x_2 \).

Stand \( \lambda \) for the Lebesgue measure of \( \mathbb{R} \). Let us denote the positive odd numbers by \( 2N + 1 \).
3. The definition and existence of balanced compact sets

**Definition 3.1.** If \( a_n \ (n \in \mathbb{N}^+) \) are positive integers then let us consider for all \( n \in \mathbb{N}^+ \),
\[
I_n = \prod_{k=1}^{n} \{1, \ldots, a_k\} \quad \text{and} \quad \mathcal{I} = \bigcup_{n=1}^{\infty} I_n.
\]
We say that a map \( \Phi : 2\mathbb{N} \to \mathcal{I} \) is an index function according to the sequence \( \langle a_n \rangle \) if it is surjective and \( \Phi(n) \in \bigcup_{k=1}^{n} I_k \) for every odd \( n \).

**Definition 3.2.** Let \( X \) be a Polish space. A compact set \( K \subseteq X \) is balanced if it is of the form
\[
K = \bigcap_{n=1}^{\infty} \left( \bigcup_{i_1=1}^{a_1} \cdots \bigcup_{i_n=1}^{a_n} C_{i_1 \ldots i_n} \right),
\]
where \( a_n \) are positive integers and \( C_{i_1 \ldots i_n} \subseteq X \) are non-empty closed sets with the following properties. There are positive reals \( b_n \) and there is an index function \( \Phi : 2\mathbb{N} + 1 \to \mathcal{I} \) according to the sequence \( \langle a_n \rangle \) such that for all \( n \in \mathbb{N}^+ \) and \( (i_1, \ldots, i_n), (j_1, \ldots, j_n) \in \mathcal{I}_n \)

(i) \( a_1 \geq 2 \) and \( a_{n+1} \geq na_1 \cdot \ldots \cdot a_n \),
(ii) \( C_{i_1, i_{n+1}} \subseteq C_{i_1 \ldots i_n} \),
(iii) \( \text{diam} \ C_{i_1 \ldots i_n} \leq b_n \),
(iv) \( \text{dist} \ (C_{i_1 \ldots i_n}, C_{j_1 \ldots j_n}) > 2b_n \) if \( (i_1, \ldots, i_n) \neq (j_1, \ldots, j_n) \).
(v) If \( n \) is odd, \( C_{i_1 \ldots i_n} \subseteq C_{\Phi(n)} \) and \( C_{j_1 \ldots j_n} \not\subseteq C_{\Phi(n)} \), then for all \( s, t \in \{1, \ldots, a_{n+1}\} \), \( s \neq t \) we have
\[
\text{dist} \ (C_{i_1 \ldots i_ns}, C_{i_1 \ldots i_nt}) > \text{diam} \left( \bigcup_{j_{n+1}=1}^{a_{n+1}} C_{j_1 \ldots j_n j_{n+1}} \right).
\]

**Remark 3.3.** The only reason why the domain of \( \Phi \) is \( 2\mathbb{N} + 1 \) instead of \( \mathbb{N}^+ \) is that we refer to this construction in [1], where this slight difference is important.

**Remark 3.4.** In a countable Polish space \( X \) there is no balanced compact set \( K \subseteq X \), since every balanced compact set has cardinality \( 2^{\aleph_0} \).

**Theorem 3.5.** If \( X \) is an uncountable Polish space, then there exists a balanced compact set \( K \subseteq X \).

**Proof.** Every uncountable Polish space contains a non-empty perfect subset, see [2] (6.4 Thm.), so we may assume by shrinking that \( X \) is also perfect. Let us fix positive integers \( a_n \) according to [1] and an index function \( \Phi \) according to the sequence \( \langle a_n \rangle \). We need to construct non-empty closed sets \( C_{i_1 \ldots i_n} \) and positive reals \( b_n \) that satisfy properties (iii)-(v), then the set \( K = \bigcap_{n=1}^{\infty} \left( \bigcup_{i_1=1}^{a_1} \cdots \bigcup_{i_n=1}^{a_n} C_{i_1 \ldots i_n} \right) \) will be a balanced compact set. Let \( n \in \mathbb{N} \) and assume that \( b_k \) and \( C_{i_1 \ldots i_k} \) with \( \text{int} \ C_{i_1 \ldots i_k} \neq \emptyset \) are already defined for all \( k \leq n \) and \( (i_1, \ldots, i_k) \in \mathcal{I}_k \), where we use the convention \( \mathcal{I}_0 = \{\emptyset\} \), \( C_\emptyset = X \), and \( b_0 = \infty \). It is enough to construct \( b_{n+1} \) and \( C_{i_1 \ldots i_{n+1}} \) such that \( \text{int} \ C_{i_1 \ldots i_{n+1}} \neq \emptyset \) for all \( (i_1, \ldots, i_{n+1}) \in \mathcal{I}_{n+1} \).

We define distinct points \( x_{i_1 \ldots i_{n+1}} \in \text{int} \ C_{i_1 \ldots i_n} \) for all \( (i_1, \ldots, i_{n+1}) \in \mathcal{I}_{n+1} \). First assume that \( n \) is even. As \( X \) is perfect and \( \text{int} \ C_{i_1 \ldots i_n} \neq \emptyset \), we can fix distinct points \( x_{i_1 \ldots i_{n+1}} \in \text{int} \ C_{i_1 \ldots i_n} \) for all \( (i_1, \ldots, i_{n+1}) \in \mathcal{I}_{n+1} \). Now assume that \( n \) is odd. First consider those \( (i_1, \ldots, i_n) \) for which \( C_{i_1 \ldots i_n} \subseteq C_{\Phi(n)} \), then let us
fix distinct points \( x_{i_1} \ldots i_{n+1} \in \text{int} \, C_{i_1} \ldots i_n \) for all \( i_{n+1} \in \{1, \ldots, a_{n+1}\} \). Let \( \delta \) be the minimum distance between the points \( x_{i_1} \ldots i_{n+1} \) we have defined so far. Now consider those \((i_1, \ldots, i_n)\) for which \( C_{i_1} \ldots i_n \not \subseteq C_{\Phi(n)} \). For each of them, fix distinct points \( x_{i_1} \ldots i_{n+1} \in \text{int} \, C_{i_1} \ldots i_n \) for all \( i_{n+1} \in \{1, \ldots, a_{n+1}\} \) such that

\[
\text{diam} \left( \bigcup_{i_{n+1}=1}^{a_{n+1}} \{x_{i_1} \ldots i_{n+1}\} \right) \leq \frac{\delta}{2}.
\]

For \((i_1, \ldots, i_{n+1}) \in \mathcal{I}_{n+1}\) consider the non-empty closed sets

\[
C_{i_1} \ldots i_{n+1} = B\left(x_{i_1} \ldots i_{n+1}, b_{n+1}/2\right),
\]

where \( b_{n+1} > 0 \) is sufficiently small. Then the sets \( C_{i_1} \ldots i_{n+1} \) satisfy properties (i)-(v), and clearly \( \text{int} \, C_{i_1} \ldots i_{n+1} \neq \emptyset \) for all \((i_1, \ldots, i_{n+1}) \in \mathcal{I}_{n+1}\). \( \square \)

**Fact 3.6.** If \( K \subseteq \mathbb{R} \) is a balanced compact set, then \( K \) has zero Lebesgue measure.

**Proof.** For all \( n \in \mathbb{N}^+ \) and \((i_1, \ldots, i_n) \in \mathcal{I}_n\) let \( I_{i_1} \ldots i_n \subseteq \mathbb{R} \) be compact intervals such that \( C_{i_1} \ldots i_n \subseteq I_{i_1} \ldots i_n \) and \( \text{diam} \, I_{i_1} \ldots i_n = \text{diam} \, C_{i_1} \ldots i_n \). Set \( I_n^* = \bigcup_{i_1=1}^{a_1} \cdots \bigcup_{i_n=1}^{a_n} I_{i_1} \ldots i_n \). Properties (iii) and (iv) imply that \( \lambda(I_n^*) / 2 \leq \lambda(I_{n+1}^*) / 2 \) for all \( n \in \mathbb{N}^+ \), thus \( K \subseteq \bigcap_{n=1}^{\infty} I_n^* \) has zero Lebesgue measure. \( \square \)

4. Balanced compact sets admit exact continuous gauge functions

The main goal of this section is to prove Theorem 4.2.

Assume that \( X \) is a Polish space and \( K \subseteq X \) is a fixed balanced compact set. Let \( a_n, b_n, C_{i_1} \ldots i_n, \Phi \) be the objects witnessing that \( K \) is balanced according to Definition 3.2

**Definition 4.1.** Let \( K_{i_1} \ldots i_n = K \cap C_{i_1} \ldots i_n \) for all \((i_1, \ldots, i_n) \in \mathcal{I}_n\) and \( n \in \mathbb{N}^+ \). These sets are called the \( n \)th level elementary pieces of \( K \). For a set \( A \subseteq K \) we call the \( n \)th level elementary pieces of \( A \) the \( n \)th level elementary pieces of \( K \) that intersect \( A \).

**Theorem 4.2.** There exists a continuous gauge function \( h \) such that \( \mathcal{H}^h(K) = 1 \). Moreover,

\[
\mathcal{H}^h(K_{i_1} \ldots i_n) = \frac{1}{a_1 \cdots a_n}
\]

for all \( n \in \mathbb{N}^+ \) and \((i_1, \ldots, i_n) \in \mathcal{I}_n\).

**Proof.** Consider \( h : [0, \infty) \rightarrow [0, \infty) \),

\[
h(x) = \begin{cases} 
1 & \text{if } x \geq 2b_1, \\
\frac{1}{a_1 \cdots a_n} & \text{if } 2b_{n+1} \leq x \leq b_n \text{ for all } n \in \mathbb{N}^+, \\
\text{linear} & \text{if } b_n \leq x \leq 2b_n \text{ for all } n \in \mathbb{N}^+, \\
0 & \text{if } x = 0.
\end{cases}
\]

As \( a_n \geq 2 \) for all \( n \in \mathbb{N}^+ \), properties (i)-(iv) yield that \( 2b_{n+1} < b_n \) for all \( n \in \mathbb{N}^+ \). Thus \( b_n < b_1/2^{n-1} \rightarrow 0 \) as \( n \rightarrow \infty \). These imply that \( h \) is well-defined. Clearly, \( h \) is non-decreasing, continuous, and \( h(x) = 0 \) iff \( x = 0 \). Therefore \( h \) is a continuous gauge function. It is enough to prove that \( \mathcal{H}^h(K) = 1 \), because
Proof. Let \( \lambda \) be the natural condition the one side, the definition of \( h \). Therefore (4.3) holds. Finally, (4.3) and (4.2) yield
\[
\sum_{j=1}^{k} h(\text{diam } U_j) \geq \sum_{j=1}^{k} \frac{s_j}{a_1 \cdots a_m} \geq 1,
\]
and the proof is complete.

Remark 4.3. Note that property (v) and the notion of an index function \( \Phi \) are not needed for the proof of Theorem 4.2. We used only the natural condition \( a_n \geq 2 \) (\( n \in \mathbb{N}^+ \)) instead of property (ii).

Fact 4.4. Let \( K \subseteq \mathbb{R} \) be a balanced compact set, and let \( h \) be the gauge function for \( K \) according to (4.1). Then \( \lambda \) is absolutely continuous for \( \mathcal{H}^h \).

Proof. Let \( I \) be a compact interval such that \( \bigcup_{i=1}^{a_n} C_i \subseteq I \), and assume \( \text{diam } I = c \). Set \( g(x) = x/c \). First we prove that \( h(x) \geq g(x) \) for all \( x \in [0, b_i] \). Let \( n \in \mathbb{N}^+ \). On the one side, the definition of \( h \) implies \( h(b_n) = \frac{1}{a_1 \cdots a_m} \). On the other side, (iv) yields \( 2b_n (\#\mathcal{I}_n - 1) \leq \text{diam } I \), so \( b_n \leq \frac{\text{diam } I}{2(\#\mathcal{I}_n - 1)} \leq \frac{c}{a_1 \cdots a_m} \). Thus \( h(b_n) \geq b_n/c = g(b_n) \).

As \( h \) is concave and \( g \) is linear on \([b_{n+1}, b_n]\) for all \( n \in \mathbb{N}^+ \), we have \( h(x) \geq g(x) \) for all \( x \in [0, b_1] \).

Finally, \( h|_{[0,b_1]} \geq g|_{[0,b_1]} \) implies that for all \( A \subseteq \mathbb{R} \) we have \( \mathcal{H}^h(A) \geq \mathcal{H}^g(A) = \lambda(A)/c \), so \( \lambda \) is absolutely continuous for \( \mathcal{H}^h \).
5. The proof of Theorem 5.1

The goal of this section is to prove the following theorem.

**Theorem 5.1.** Let $X$ be a Polish space, and let $K \subseteq X$ be a balanced compact set. Then there exists a continuous gauge function $h$ such that $0 < \mathcal{H}^h(K) < \infty$, and for every weak contraction $f : K \to X$ we have $\mathcal{H}^h(K \cap f(K)) = 0$.

**Proof.** Let $a_n, b_n, C_{i_1, \ldots, i_n}, \Phi$ be the objects witnessing that $K$ is balanced according to Definition 3.2. Let $h$ be the continuous gauge function for $K$ according to (3.3). Theorem 4.2 implies $\mathcal{H}^h(K) = 1$. Let $f : K \to X$ be a weak contraction, it is enough to prove that $\mathcal{H}^h(K \cap f(K)) = 0$. For all $n \in \mathbb{N}^+$ let

$$A_n = \bigcup_{i_1=1}^{a_1} \cdots \bigcup_{i_n=1}^{a_n} (K_{i_1, \ldots, i_n} \cap f(K \setminus K_{i_1, \ldots, i_n})).$$

First we prove

$$K \cap f(K) \subseteq \text{Fix}(f) \cup \bigcup_{n=1}^\infty A_n,$$

where $\text{Fix}(f) = \{x \in K : f(x) = x\}$. Assume that $y \in K \cap f(K)$ and $y \notin \text{Fix}(f)$, we need to prove that $y \notin \bigcup_{n=1}^\infty A_n$. There is an $x \in K$ such that $f(x) = y$ and $x \neq y$. Then $\text{diam}K_{i_1, \ldots, i_n} \leq b_n$ and $b_n \to 0$ imply that there is an $n \in \mathbb{N}^+$ and $(i_1, \ldots, i_n) \in \mathcal{I}_n$ such that $y \in K_{i_1, \ldots, i_n}$ and $x \in K \setminus K_{i_1, \ldots, i_n}$, so $y \in A_n$. Thus $y \notin \bigcup_{n=1}^\infty A_n$, hence (5.1) holds.

As $f$ is a weak contraction, $\text{Fix}(f)$ has at most 1 element. Therefore (5.1) implies that it is enough to prove that $\mathcal{H}^h(\bigcup_{n=1}^\infty A_n) = 0$. Property (ii) easily yields that $A_n \subseteq A_{n+1}$ for all $n \in \mathbb{N}^+$, so we need to prove that

$$\lim_{n \to \infty} \mathcal{H}^h(A_n) = 0.$$

Let us fix $n \in \mathbb{N}^+$ and $(i_1, \ldots, i_n) \in \mathcal{I}_n$. The definition of $\Phi$ yields that there is an odd number $m \geq n$ such that $\Phi(m) = (i_1, \ldots, i_n)$. Let us denote by $\Delta_m$ the set of $m$th level elementary pieces of $K \setminus K_{i_1, \ldots, i_n}$. Set $E \in \Delta_m$. As $f$ is a weak contraction, $\text{diam}f(E) \leq \text{diam}E$. Therefore (vi) together with (iii) and (iv) imply that $f(E)$ can intersect at most one $m+1$st level elementary piece of $K_{i_1, \ldots, i_n}$. Thus $f(\bigcup \Delta_m) = f(K \setminus K_{i_1, \ldots, i_n})$ can intersect at most $\#\Delta_m \leq a_1 \cdots a_m$ many $m+1$st level elementary pieces of $K_{i_1, \ldots, i_n}$. Theorem 4.2 yields that every $m+1$st level elementary piece of $K$ has $\mathcal{H}^h$ measure $1/(a_1 \cdots a_{m+1})$, and $m \geq n$ implies $a_{m+1} \geq a_{n+1}$. Therefore

$$\mathcal{H}^h(K_{i_1, \ldots, i_n} \cap f(K \setminus K_{i_1, \ldots, i_n})) \leq \frac{a_1 \cdots a_m}{a_1 \cdots a_{m+1}} = \frac{1}{a_{m+1}} \leq \frac{1}{a_{n+1}}.$$

Finally, equation (5.3), the definition of $A_n$, the subadditivity of $\mathcal{H}^h$, and property (ii) yield

$$\mathcal{H}^h(A_n) \leq \frac{a_1 \cdots a_n}{a_{n+1}} \leq \frac{1}{n}.$$

Thus (5.2) follows, and the proof is complete.
6. The proof of our Main Theorems

Let us recall that the main goal of our paper is to answer the following question.

**Question 1.6.** What are the lowest possible Borel classes of \(\mathbb{R}^n\) having a non-closed element \(X\) such that every weak contraction \(f: X \to X\) is constant?

If \(n \geq 2\) then the answer is \(\Delta^n_2\), and there is no non-closed \(\Delta^n_2\) example in \(\mathbb{R}\), see the Introduction. If \(n = 1\) then the following theorems show that \(\Sigma^0_2\) and \(\Pi^0_2\) are the lowest possible Borel classes satisfying Question 1.6.

**Theorem 6.1** (Main Theorem, \(F_\sigma\) case). There exists a non-closed \(F_\sigma\) set \(F \subseteq \mathbb{R}\) such that every weak contraction \(f: \mathbb{R} \to \mathbb{R}\) is constant.

**Proof.** By Theorem 3.5 there exists a balanced compact set \(K \subseteq \mathbb{R}\). Let \(a_n\) be the positive integers and let \(h\) be the continuous gauge function for \(K\) according to Definition 3.2 and equation (4.1), respectively. Set \(Q = \{q_n : n \in \mathbb{N}^+\}\). For all \(n \in \mathbb{N}^+\) let \(K_n^*\) be an \(n\)th level elementary piece of \(K\), see Definition 4.1. Consider

\[
F_0 = \bigcup_{n=1}^{\infty} (K_n^* + q_n).
\]

Clearly, \(F_0\) is an \(F_\sigma\) set, thus \(\mathcal{H}^h\) measurable. The countable subadditivity and the translation invariance of \(\mathcal{H}^h\), and Theorem 4.2 imply

\[
\mathcal{H}^h(F_0) \leq \sum_{n=1}^{\infty} \mathcal{H}^h(K_n^* + q_n) = \sum_{n=1}^{\infty} \mathcal{H}^h(K_n^*) = \sum_{n=1}^{\infty} \frac{1}{a_1 \cdots a_n} \leq \sum_{n=1}^{\infty} \frac{1}{2^n} = 1.
\]

As \(F_0\) is a \(\mathcal{H}^h\)-measurable set with finite measure, there is a \(G_\delta\) set \(G_0 \subseteq \mathbb{R}\) such that

\[
F_0 \subseteq G_0 \quad \text{and} \quad \mathcal{H}^h(G_0 \setminus F_0) = 0,
\]

see [9] Thm. 27.] for the proof. Set \(F = \mathbb{R} \setminus G_0\). Clearly, \(F\) is an \(F_\sigma\) set. First we prove that \(F\) is non-closed. Fact 5.0 yields \(\lambda(K) = 0\), so the translation invariance and the countable subadditivity of the Lebesgue measure imply \(\lambda(F_0) = 0\). Fact 4.4 and (6.2) imply \(\lambda(G_0 \setminus F_0) = 0\). Hence \(\lambda(G_0) = 0\). Therefore \(G_0 \neq \emptyset\) yields that \(G_0\) is not open, so \(F = \mathbb{R} \setminus G_0\) is non-closed. These imply also that \(F\) is of full Lebesgue measure, therefore it is dense in \(\mathbb{R}\).

Assume to the contrary that there exists a non-constant weak contraction \(f: \mathbb{R} \to \mathbb{R}\). As \(F\) is dense in \(\mathbb{R}\), \(f\) has a unique 1-Lipschitz extension \(\tilde{f}: \mathbb{R} \to \mathbb{R}\). First we prove that \(\tilde{f}\) is a weak contraction. Assume to the contrary that there are \(a, b \in \mathbb{R}\), \(a < b\) such that \(|\tilde{f}(b) - \tilde{f}(a)| = |b - a|\). Since \(\tilde{f}\) is 1-Lipschitz, for all \(x, y \in [a, b]\) we have

\[
|\tilde{f}(x) - \tilde{f}(y)| = |x - y|.
\]

Since \(F\) is dense in \(\mathbb{R}\), there are \(x_0, y_0 \in F \cap [a, b]\), \(x_0 \neq y_0\). Applying (6.3) for \(x_0, y_0\) contradicts that \(f\) is a weak contraction. Thus \(\tilde{f}\) is a weak contraction.

As \(f\) is non-constant, \(I = \tilde{f}(\mathbb{R})\) is a non-degenerate interval. Then \(\tilde{f}(F) = f(F) \subseteq F\) and the definition of \(F\) implies \(F_0 \cap I \subseteq I \setminus F \subseteq \tilde{f}(\mathbb{R} \setminus F) = \tilde{f}(G_0)\), so

\[
F_0 \cap I \subseteq F_0 \cap \tilde{f}(G_0).
\]
Equation (iii) and $b_n \to 0$ yield $\text{diam } K_n^* \to 0$ as $n \to \infty$. Thus there exists an $n \in \mathbb{N}^+$ such that $K_n^* + q_n \subseteq I$, and Theorem 4.2 implies $\mathcal{H}^b (K_n^*) > 0$. Therefore the translation invariance of $\mathcal{H}^b$ yields
\begin{equation}
\mathcal{H}^b (F_0 \cap I) \geq \mathcal{H}^b (K_n^* + q_n) = \mathcal{H}^b (K_n^*) > 0.
\end{equation}

Theorem 5.1 implies that for all $p, q \in \mathbb{Q}$ we have $
\mathcal{H}^b (K^* + p) \cap \hat{f}(K^* + q) = 0,$

as $\hat{f}(K + q)$ is a weak contractive image of $K + p$. Therefore $F_0 \subseteq K + \mathbb{Q}$ and the countable subadditivity of $\mathcal{H}^b$ yield
\begin{equation}
\mathcal{H}^b \left( F_0 \cap \hat{f}(F_0) \right) \leq \mathcal{H}^b \left( (K + \mathbb{Q}) \cap \hat{f}(K + \mathbb{Q}) \right) \\
\leq \sum_{p, q \in \mathbb{Q}} \mathcal{H}^b \left( (K + p) \cap \hat{f}(K + q) \right) \\
= 0.
\end{equation}

As $\hat{f}$ is a weak contraction and (6.2) holds, we obtain
\begin{equation}
\mathcal{H}^b \left( \hat{f}(G_0 \setminus F_0) \right) \leq \mathcal{H}^b (G_0 \setminus F_0) = 0.
\end{equation}

Finally, equations (6.5), (6.4), the subadditivity of $\mathcal{H}^b$, (6.6), and (6.7) imply
\begin{align*}
0 < \mathcal{H}^b (F_0 \cap I) &\leq \mathcal{H}^b \left( F_0 \cap \hat{f}(G_0) \right) \\
&\leq \mathcal{H}^b \left( F_0 \cap \hat{f}(F_0) \right) + \mathcal{H}^b \left( \hat{f}(G_0 \setminus F_0) \right) \\
&= 0.
\end{align*}

This is a contradiction, so the proof is complete. \qed

**Theorem 6.2** (Main Theorem, $G_\delta$ case). There exists a non-closed $G_\delta$ set $G \subseteq \mathbb{R}$ such that every weak contraction $f : G \to G$ is constant.

**Proof.** Let $G = \mathbb{R} \setminus F_0$, for the definition of $F_0$ see (6.1). Clearly, $G$ is a $G_\delta$ set. Since $\lambda(F_0) = 0$, $F_0$ is not open, so $G$ is non-closed. Assume to the contrary that $f : G \to G$ is a non-constant weak contraction. Repeating the same arguments as in the proof of Theorem 6.1 leads us to a contradiction, only write $G$ and $F_0$ instead of $\bar{F}$ and $G_0$, respectively. \qed

**Acknowledgement** The author is indebted to A. Máté for some helpful suggestions. He also pointed out that the original proof of Theorem 5.1 can be significantly shortened.

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