Coupled Cavity Model: Correctness and Limitations

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Abstract

Results of analysis of correctness and limitations of the classical Coupled Cavity Model are presented in the paper. It is shown that in the case of an infinite chain of resonators, there are spurious solutions of the characteristic equation. These spurious solutions do not violate the correctness of direct numerical calculations, but their existence makes it difficult (or even impossible) to use approximate WKB methods for analysing chains with slowly varying parameters.

1 Introduction

The classical Coupled Cavity Model (CCM) is widely used for design of RF/microwave devices, in particular, the accelerating structures (see, for example, [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20], the travelling wave tubes (see, for example, [21]), the narrow-band bandpass filters (see, for example, [22, 23, 24]) and so on. It is also found wide application in optic and metamaterial structures (see, for example, [25, 26]).

These devises are based on chains of coupled resonators. Such chains belong to the class of closed structured waveguides - waveguides that consist of similar, but not always identical, cells that couples through openings in dividing walls. Each cell couples only with two neighbouring cells. In opening structured waveguides (optic and metamaterial structures [25, 26]) each cell couples with all cells of the chain.

Despite the differences, the general approach was the same - it is necessary to find eigen modes of noncoupling cells and construct the coupling coefficients of intercoupled resonators and the external quality factors of the input and output resonators. The general coupling matrix is of importance for representing a wide range of coupled resonators topologies.

On the base of simple model of infinite chain of cylindrical resonators that couple through circular openings in thing dividing walls we show that the classical CCM has problems in describing field distributions in closed structured waveguides.

2 Finite Chain of Resonators. Main equations

Consider a chain of $N_R$ cylindrical resonators with annular discs of zero thickness. The first and last resonators are connected through cylindrical openings to semi-infinite cylindrical waveguides.
We will consider only axially symmetric TM fields with $E_z$, $E_r$, $H_\phi$ components. Time dependence is $\exp(-i\omega t)$.

In each resonator we expand the electromagnetic field with the short-circuit resonant cavity modes

$$
\vec{E}^{(k)} = \sum_q e_q^{(k)} \vec{E}_q^{(k)}(\vec{r}), \\
\vec{H}^{(k)} = i \sum_q h_q^{(k)} \vec{H}_q^{(k)}(\vec{r}),
$$

where $q = \{0, m, n\}$.

$\vec{E}_q^{(k)}$, $\vec{H}_q^{(k)}$ are the solutions of homogenous Maxwell equations

$$
\text{rot } \vec{E}_q^{(k)} = i\omega_q^{(k)} \mu_0 \vec{H}_q^{(k)}, \\
\text{rot } \vec{H}_q^{(k)} = -i\omega_q^{(k)} \varepsilon_0 \vec{E}_q^{(k)}
$$

with boundary condition $\vec{E}_r = 0$ on the metal surface

$$
E_{m,n,z}^{(k)} = J_0 \left( \frac{\lambda_m}{b_k} r \right) \cos \left( \frac{\pi}{d_k} n \left( z - z_k \right) \right),
$$

$$
H_{m,n,\phi}^{(k)} = -i\omega_{m,n}^{(k)} \frac{\varepsilon \varepsilon_0 b_k}{\lambda_m} J_1 \left( \frac{\lambda_m}{b_k} r \right) \cos \left( \frac{\pi}{d_k} n \left( z - z_k \right) \right),
$$

$$
E_{m,n,r}^{(k)} = \frac{b_k}{\lambda_m} \frac{\pi}{d_k} n J_1 \left( \frac{\lambda_m}{b_k} r \right) \sin \left( \frac{\pi}{d_k} n \left( z - z_k \right) \right),
$$

$$
\omega_{0,m,n}^{(k)} = \frac{\varepsilon_0}{\varepsilon} \left( \frac{\lambda_m}{b_k} \right)^2 + \left( \frac{\pi}{d_k} n \right)^2,
$$

where $J_0(\lambda_m) = 0$, $b_k$, $d_k$ - k-th resonator radius and length, $a_k$ - opening radius between k-th and (k-1)-th resonators.

Amplitudes $e_q^{(k)}$ can be found if the tangential electric fields on the openings are known

$$
(\omega_q^{(k)} - \omega_0^{(k,2)} e_q^{(k)}) = \frac{i \omega_q^{(k)} N_{0,m,n}^{(k)}}{N_{0,m,n}^{(k)}} \left( \oint_{S_k} \left[ \vec{E}_q^{(k)} \vec{H}_q^{(k)*} \right] d\vec{S} + \oint_{S_{k+1}} \left[ \vec{E}_q^{(k+1)} \vec{H}_q^{(k+1)*} \right] d\vec{S} \right),
$$

$$
\omega_{0,m,n}^{(k,2)} = \frac{2c^2 \lambda_m^2}{\varepsilon_0 |\varepsilon| \varepsilon \sigma_n \lambda_k \pi d_k J_1^2(\lambda_m)}.
$$

In the semi-infinite waveguides the electromagnetic field can be expanded in terms of the TM eigenmodes $\vec{E}_{s}^{(w,p)}$, $\vec{H}_{s}^{(w,p)}$ of a circular waveguide ($p = 1, 2$)

$$
\vec{H}^{(w,p)} = \sum_s \left( G_s^{(p)} \vec{H}_s^{(w,p)} + G_{-s}^{(p)} \vec{H}_{-s}^{(w,p)} \right),
$$

$$
\vec{E}^{(w,p)} = \sum_s \left( G_s^{(p)} \vec{E}_s^{(w,p)} + G_{-s}^{(p)} \vec{E}_{-s}^{(w,p)} \right)
$$

(9)
where \( z_{w,1} = z_1 \), \( z_{w,2} = z_{N_R + 1} \),

\[
\mathcal{E}_{s,z}^{(w,p)} = J_0 \left( \frac{\lambda_s}{b_{w,p}} r \right) \exp \left\{ \gamma_s^{(w,p)} (z - z_{w,p}) \right\},
\]

(10)

\[
\mathcal{E}_{s,r}^{(w,p)} = -\frac{b_{w,p}}{\lambda_s} \gamma_s^{(w,p)} J_1 \left( \frac{\lambda_s}{b_{w,p}} r \right) \exp \left\{ \gamma_s^{(w,p)} (z - z_{w,p}) \right\},
\]

(11)

\[
\mathcal{H}_{s,\varphi}^{(w,p)} = -i \omega \varepsilon_0 b_{w,p} J_1 \left( \frac{\lambda_s}{b_{w,p}} r \right) \exp \left\{ \gamma_s^{(w,p)} (z - z_{w,p}) \right\},
\]

(12)

\[
\gamma_s^{(w,k)} = \frac{1}{b_{w,k}^2} \left( \frac{2}{\lambda_s^2} - \frac{b_{w,k}^2 \omega^2}{c^2} \right).
\]

(13)

Further consideration will be based on the Moment Method. We shall use the Bessel functions as the testing functions \((\psi_s(x) = J_1(\lambda_s x), \ x \in [0, 1])\) and the complete set of functions that fulfil the edge condition on the diaphragm rims as the basis functions (the Meixner basis). We use such Meixner basis [27]

\[
\varphi_s(r) = 2\sqrt{\pi} \frac{\Gamma(s + 1)}{\Gamma(s - 0.5)} \frac{1}{\sqrt{1 - r^2}} P_{2s-1}^{0} \left( \sqrt{1 - r^2} \right),
\]

(14)

where \( P_n^m(x) \) are Legendre functions (or spherical functions) of the first kind,

\[
\varphi_s \left( \frac{r}{a_k} \right) \xrightarrow{r \rightarrow a_k} \frac{C_s}{\sqrt{1 - (\frac{r}{a})^2}}.
\]

Tangential electric fields on the openings we expand into such series

\[
E_r^{(k)} = \sum_s \mathcal{E}_s^{(k)} \varphi_s \left( \frac{r}{\lambda} \right).
\]

(15)

Using a procedure proposed in [28, 29, 30, 31, 32], we get such matrix equations

\[
\begin{align*}
(T^{(1,1)} - \varepsilon^{-1} W^{(1)}) C^{(1)} - T^{(1,3)} C^{(2)} &= -R^{(1,1)} \rho_{010}^{(1)} + \varepsilon^{-1} V, \\
(T^{(k,1)} + T^{(k,2)}) C^{(k)} - T^{(k,3)} C^{(k+1)} - T^{(k,4)} C^{(k-1)} &= -R^{(k,1)} \rho_{010}^{(k)} + R^{(k,2)} \rho_{010}^{(k-1)}, \\
&\quad k = 2, 3, ..., N_R
\end{align*}
\]

(16)

\[
- T^{(N_R+1,4)} C^{(N_R)} + (T^{(N_R+1,2)} - \varepsilon^{-1} W^{(2)}) C^{(N_R+1)} = R^{(1N_R+1,2)} \rho_{010}^{(N_R)},
\]

(17)

\[
\begin{align*}
\left( \omega_{010}^{(k)} - \omega^2 \right) \rho_{010}^{(k)} &= \\
-2 \frac{b_k}{d_k} \frac{\omega_{010}^{(k,2)}}{J_1^2(\lambda_1)} \left( \frac{a_k^2}{b_k^2} \sum_s C_s^{(k)} j_{2s-1} \left( \frac{\lambda_1 a_k}{b_k} \right) + \frac{a_{k+1}^2}{b_k^2} \sum_s C_s^{(k+1)} j_{2s-1} \left( \frac{\lambda_1 a_{k+1}}{b_{k+1}} \right) \right),
\end{align*}
\]

(18)

\[
k = 1, 2, ..., N_R.
\]
Matrices $T$, $W$ and vector $R$ are defined in Appendix 1. We introduce the fundamental solution of system (16) - (18) as the solution of such difference matrix equations

\[
\begin{align*}
(T^{(1,1)} - \varepsilon^{-1}W^{(1)}) Y^{(1,k_0)} - T^{(1,3)} Y^{(2,k_0)} &= -R^{(1,1)} \delta_{1,k_0}, \\
(T^{(k,1)} + T^{(k,2)}) Y^{(k,k_0)} - T^{(k,3)} Y^{(k+1,k_0)} - T^{(k,4)} Y^{(k-1,k_0)} &= -R^{(k,1)} \delta_{k,k_0} + R^{(k,2)} \delta_{k-1,k_0},
\end{align*}
\]

\[\delta_{k,k_0}, k = 2, 3, ..., N_R,\]

\[-T^{(N_R+1,4)} Y^{(N_R,k_0)} + (T^{(N_R+1,2)} - \varepsilon^{-1}W^{(2)}) Y^{(N_R+1,k_0)} = R^{(N_R+1,2)} \delta_{N_R,k_0},\]

Then the solution of (16) - (18) is written as

\[C_s^{(k)} = \sum_{k_0=1}^{N_R} Y_s^{(k,k_0)} e_{010}^{(k_0)} + Y_s^{(k)}\]

(22)

Its substitution in (19) gives the basic CCM equation

\[
\left(\omega_{010}^{(k)} - \omega^2\right) e_{010}^{(k)} = \omega_{010}^{(k)} \sum_{k_0=1}^{N_R} \alpha^{(k,k_0)} e_{010}^{(k_0)} + \omega_{010}^{(k)} F^{(k)},
\]

(23)

where

\[
\alpha^{(k,k_0)} = -2 \frac{b_k}{d_k J_1^2(\lambda_1) \lambda_1} \left(-\frac{a_k^2}{b_k} \sum_s Y_s^{(k,k_0)} j_{2s-1} \left(\frac{\lambda_1 a_k}{b_k}\right) + \frac{a_{k+1}^2}{b_k^2} \sum_s Y_s^{(k+1,k_0)} j_{2s-1} \left(\frac{\lambda_1 a_{k+1}}{b_k}\right)\right)
\]

(24)

\[
F^{(k)} = -2 \frac{b_k}{d_k J_1^2(\lambda_1) \lambda_1} \left(-\frac{a_k^2}{b_k^2} \sum_s Y_s^{(k)} j_{2s-1} \left(\frac{\lambda_1 a_k}{b_k}\right) + \frac{a_{k+1}^2}{b_k^2} \sum_s Y_s^{(k+1)} j_{2s-1} \left(\frac{\lambda_1 a_{k+1}}{b_k}\right)\right)
\]

(25)

Mode amplitudes in cylindrical waveguides can be found using the following formulas

\[
G_{s}^{(1)} = 1 + 2 \frac{a_2^2 \lambda_1}{J_1^2(\lambda_1) b_{w,1}^2 \gamma_s^{(1)} b_{w,1}} \sum_{s'} C_{s'}^{(1)} j_{2s'-1} \left(\frac{\lambda_2 a_1}{b_{w,1}}\right)
\]

(26)

\[
G_{s}^{(1)} = 2 \frac{a_2^2 \lambda_1}{J_1^2(\lambda_2) b_{w,2}^2 \gamma_s^{(2)} b_{w,2}} \sum_{s'} C_{s'}^{(1)} j_{2s'-1} \left(\frac{\lambda_2 a_1}{b_{w,2}}\right), \quad s = 2, 3, ...
\]

\[
G_{s}^{(2)} = -2 \frac{a_{N_R+1}^2 \lambda_s}{J_1^2(\lambda_1) b_{w,1}^2 \gamma_s^{(2)} b_{w,2}} \sum_{s'} C_{s'}^{(N_R+1)} j_{2s'-1} \left(\frac{\lambda_s a_{N_R+1}}{b_{w,2}}\right), \quad s = 1, 2, ...
\]

(27)

The model described above has no limitations. We get the exact coupling matrix (24) and the vector of distributed external sources (25).
3 Infinitive Chain of Resonators

Consider an infinitive ($N_R \to \infty$) homogeneous ($a_k = a$, $b_k = b$, $d_k = d$) chain. Equations (17), (19) we write as

$$e^{(k)}_{010} = -q \sum_{s=1}^{\infty} R_s \left( \mathcal{E}_s^{(k)} - \mathcal{E}_s^{(k+1)} \right), \quad (28)$$

$$2T^{(1)} \mathcal{E}^{(k)} - T^{(2)} \mathcal{E}^{(k+1)} - T^{(2)} \mathcal{E}^{(k-1)} = R \left( e^{(k)}_{010} - e^{(k-1)}_{010} \right), \quad (29)$$

where

$$q = \frac{\omega^2_0}{\omega^2_0 - \omega^2} \frac{a^3}{b^2d J^2_1(\lambda_1)}, \quad (30)$$

$$R_s = \frac{1}{2} \frac{J_0 \left( \frac{a_1}{b} \right)}{(\frac{a_1}{b})^2 - (\lambda_s)^2}, \quad (31)$$

3.1 Systems of matrix equations

Substitution (28) into (29) gives

$$2 \bar{T}^{(1)} \mathcal{E}^{(k)} - \bar{T}^{(3)} \mathcal{E}^{(k+1)} - \bar{T}^{(3)} \mathcal{E}^{(k-1)} = 0, \quad (32)$$

where

$$T^{(1,2)}_{s,s'} = \frac{a^2}{b^2} \sum_{m} \bar{\lambda}^{(1,2)} \frac{\lambda_m}{J^2_m(\lambda_m)} \frac{J_0 \left( \frac{a}{b} \lambda_m \right)}{\left( \frac{a}{b} \lambda_m \right)^2 - (\lambda_s)^2} \bar{U}_s, \quad (33)$$

$$\bar{\Lambda}^{(1)}_m = \frac{d}{bdh_m} \frac{\cosh(dh_m)}{\sinh(dh_m)}, \quad (34)$$

$$\bar{\Lambda}^{(2)}_m = \frac{d}{bdh_m} \frac{\sinh(dh_m)}{\sinh(dh_m)}. \quad (35)$$

It can be shown that the matrix $T^{(2)}$ is invertible and the equation (32) can be rewritten as

$$\bar{T} \mathcal{E}^{(k)} = \mathcal{E}^{(k+1)} + \mathcal{E}^{(k-1)}, \quad (36)$$

where

$$\bar{T} = 2\bar{T}^{(2)-1} \bar{T}^{(1)}. \quad (37)$$

The general solution of the difference equation (32) is (31)

$$\mathcal{E}^{(k)} = \sum_{s=1}^{\infty} \bar{C}_s^{(1)} \bar{\lambda}^{(1)}_s \bar{U}_s + \sum_{s=1}^{\infty} \bar{C}_s^{(2)} \bar{\lambda}^{(2)}_s \bar{U}_s \quad (38)$$

where

$$\bar{T} \bar{U}_s = \bar{\mu}_s \bar{U}_s, \quad (39)$$

$$\bar{\lambda}_s^2 - \bar{\mu}_s \bar{\lambda}_s + 1 = 0, \quad (40)$$

$$\bar{\lambda}_s^{(1,2)} = \frac{\bar{\mu}_s}{2} \pm \frac{1}{2} \sqrt{\bar{\mu}_s^2 - 4}, \quad (41)$$

$$\bar{\mu}_s = \bar{\lambda}_s^{(1)} + \bar{\lambda}_s^{(2)}, \quad (42)$$

$$\bar{\lambda}_s^{(1)} \bar{\lambda}_s^{(2)} = 1. \quad (43)$$
We used EVCRG program from IMSL Fortran Numerical Library to calculate $\bar{\lambda}_s$. Results are presented in Tab.1 (first and second columns, angles are given in degrees). It can be seen that chosen frequency $f=2.856$ GHz lay in the first passband ($E_{01}$) and others modes ($E_{02}, E_{03}...$) are evanescent with zero angles. These results are fitted well the existing theories of periodic waveguides. For the frequency that lay in the first propagation zone we have two propagating waves and the infinitive number of evanescent waves with characteristic multipliers with zero phases. Between the first and the second propagation zones there are only evanescent oscillations with the phase shift per cell equals $\pi$. The other evanescent oscillations have zero phase shift. At the end of the second stop band evanescent $E_{01}$ wave with $\pi$-shift transforms into propagating $E_{02}$ wave with $\pi$-shift too [33]. Therefore, in the second passband $E_{02}$ wave has negative dispersion.

| Table 1 Calculated values of $\lambda_s$ |
|----------------------------------------|
| $f=2.856$ GHz, $a=0.99$ cm, $b=4.08896$ cm, $d=3.4989$ cm |
| Matrix $4 \times 4$ | Matrix $3 \times 3$ | CCM (9) | CCM (7) |
| 3.85E+08 $\angle 0^\circ$ | 194.06 $\angle 0^\circ$ | 6.18E-07 $\angle 0^\circ$ |
| 0.00E+00 $\angle 0^\circ$ | 5.15E-03 $\angle 0^\circ$ | 1.64E-04 $\angle 0^\circ$ |
| 1.62E+06 $\angle 0^\circ$ | 1.64E+06 $\angle 0^\circ$ | 1.0 $\angle -119.994^\circ$ |
| 6.18E-07 $\angle 0^\circ$ | 5.16E-03 $\angle 0^\circ$ | 1.0 $\angle 119.994^\circ$ |
| 6.09E+03 $\angle 0^\circ$ | 5.16E-03 $\angle 89.96^\circ$ | 1.0 $\angle 199.997^\circ$ |
| 1.64E-04 $\angle 0^\circ$ | 5.16E-03 $\angle 59.96^\circ$ | 1.0 $\angle -119.997^\circ$ |

### 3.2 Coupled Cavity Model

The equation (29) we rewrite as

$$2T^{(1)}E^{(k)} - T^{(3)}E^{(k+1)} - T^{(3)}E^{(k-1)} = R \sum_{j=-\infty}^{\infty} \left( e_{010}^{(j)} - e_{010}^{(j-1)} \right) \delta_{k,j} = R \sum_{j=-\infty}^{\infty} e_{010}^{(j)} \left( \delta_{k,j} - \delta_{k,j+1} \right).$$  \hspace{1cm} (41)

Let’s introduce the fundamental solution as the solution of such difference equation (i - the number of diaphragm, j - the number of cell with a source)

$$2T^{(1)}Y^{(i,j)} - T^{(3)}Y^{(i+1,j)} - T^{(3)}Y^{(i-1,j)} = R \delta_{k,j}$$  \hspace{1cm} (42)

We will suppose that eigenvalues $\mu_s$ of the matrix $T = 2T^{(2)} - 1 T^{(1)}$ ($TU_l = \mu_l U_l$) take such values that the quantities

$$\rho_s^{(1,2)} = \frac{\mu_s}{2} \pm \frac{1}{2} \sqrt{\mu_s^2 - 4}$$  \hspace{1cm} (43)

are the real numbers. In this case, we can impose the condition at infinity

$$|Y^{(i,j)}| \underset{i \to \pm \infty}{\longrightarrow} 0$$  \hspace{1cm} (44)

and the solution of (26) we can write as

$$Y^{(g,0)} = \sum_{l=1}^{\infty} A_l U_l \begin{cases} \lambda_l^{(1)g}, & g < 0, \\ \lambda_l^{(2)g}, & g \geq 0, \end{cases}$$  \hspace{1cm} (45)
where
\[ A_l = \frac{r_l}{\lambda_{l}^{(2)} - \lambda_{l}^{(1)}}, \]
\[ R = \sum_{l=1}^{\infty} r_l U_l. \] (46)

The solution of equation (41) can be written as
\[ \mathcal{E}_s^{(k)} = \sum_{j=-\infty}^{j=-\infty} \left( Y_s^{(k,j)} - Y_s^{(k-1,j)} \right) e_{010}^{(j)} \] (47)
and the equation (28) takes the form
\[ e_{010}^{(k)} = -q \sum_{j=-\infty}^{j=-\infty} e_{010}^{(j)} \sum_{s=1}^{\infty} R_s \left( 2Y_s^{(k,j)} - Y_s^{(k-1,j)} - Y_s^{(k+1,j)} \right) = \]
\[ = -q \sum_{j=-\infty}^{j=-\infty} (k+j) e_{010}^{(k+j)} \sum_{s=1}^{\infty} R_s \left( 2Y_s^{(k,j+k)} - Y_s^{(k-1,j+k)} - Y_s^{(k+1,j+k)} \right). \] (48)

From (45) it follows that \( Y^{(i,j+g)} = Y^{(i,g,j)} \), \( Y^{(i,0)} = Y^{(-i,0)} \) and we get the final equation of CCM
\[ \left( \omega_{010}^2 - \omega^2 \right) e_{010}^{(k)} = -\omega_{010}^2 \sum_{k_0=-\infty}^{\infty} e_{010}^{(k+k_0)} \alpha^{(k_0)} \] (49)
where
\[ \alpha^{(k_0)} (\omega) = \frac{a^3}{b^2 d J_1^2 (\lambda_1)} \sum_{s=1}^{\infty} R_s \left( 2Y_s^{(k_0,0)} - Y_s^{(k_0-1,0)} - Y_s^{(k_0+1,0)} \right). \] (50)
It is often assumed that we can neglect "long couplings"
\[ \left( \omega_{010}^2 - \omega^2 \right) e_{010}^{(k)} = -\omega_{010}^2 \sum_{k_0=-N}^{N} e_{010}^{(k+k_0)} \alpha^{(k_0)}, \] (51)
It was also assumed [18, 19] that the sum in (50) can be also truncated
\[ \alpha^{(k_0)} = \frac{a^3}{b^2 d J_1^2 (\lambda_1)} \sum_{s=1}^{S} R_s \left( 2Y_s^{(k_0,0)} - Y_s^{(k_0-1,0)} - Y_s^{(k_0+1,0)} \right). \] (52)

Equation (51) is widely used for description of different objects. We made assumption that \( \rho_s^{(1,2)} \) are the real numbers. Tab.2 shows that it is true.
Table 2 Calculated values of $\mu_s$ and $\rho_s$

| $f=2.856$ GHz, $a=0.99$ cm, $b=4.08896$ cm, $d=3.4989$ cm | $I=6$, $N=5$, $S=5$ |
|---|---|
| $\mu_s$ | $\rho_s^{(2)}$ | $\rho_s^{(1)}$ |
| (-1.93E+02,0.0E+0) | 5.17E-03 | 1.93E+02 |
| (2.16E+04,0.0E+0) | 4.61E-05 | 2.16E+04 |
| (2.02E+07,0.0E+0) | 4.93E-08 | 2.02E+07 |
| (2.02E+07,0.0E+0) | 4.93E-08 | 2.02E+07 |
| (5.32E+12,0.0E+0) | 7.91E-11 | 1.26E+10 |
| (5.32E+12,0.0E+0) | 1.88E-13 | 5.32E+12 |

If we seek the solution of $e^{(k)}_{010} \sim \lambda^k$, we get the characteristic equation

$$\sum_{k_0=-N}^{N} \lambda^{k_0}\alpha^{(k_0)}(\omega) + \left(\frac{\omega^2_{010} - \omega^2}{\omega^2_{010}}\right) = 0.$$  \hspace{1cm} (54)

Solutions of this equation are given in Table 1 (third and fourth columns) for different $N$. We see that propagating $E_{01}^1$ wave has the same phase shift for different approaches, but other solutions are very different. We obtain solutions that represent "non-physical" evanescent waves with phase shifts that differ from 0 and $\pi$. Amplitudes of these solutions are nearly the same, but phases strongly depend on the number of interacting oscillators.

To understand the reasons for such solutions, we consider the simplest case when the tangential electric field is described by one coefficient

$$E_r^{(k)} = E_1^{(k)} \varphi_1 (r/a).$$  \hspace{1cm} (55)

Then (28),(29) transform into

$$e_{010}^{(k)} = -qR_1 \left( E_1^{(k)} - E_1^{(k+1)} \right),$$  \hspace{1cm} (56)

$$T^{(E)} E_1^{(k)} = R_1 \left( e_{010}^{(k)} - e_{010}^{(k-1)} \right),$$  \hspace{1cm} (57)

where operator $T^{(E)} = 2T_{11}^{(1)} - T_{11}^{(3)} \sigma^{(+)} - T_{11}^{(3)} \sigma^{(-)}, \sigma^{(+)}, \sigma^{(-)}$ - shift operators ($\sigma^{(+)} x^{(k)} = x^{(k+1)}, \sigma^{(-)} x^{(k)} = x^{(k-1)}$).

Substitution (56) into (57) gives following equation

$$2 \left( T_{11}^{(1)} + qR_1^2 \right) E_1^{(k)} - \left( T_{11}^{(3)} + qR_1^2 \right) E_1^{(k+1)} - \left( T_{11}^{(3)} + qR_1^2 \right) E_1^{(k-1)} = 0.$$  \hspace{1cm} (58)

The solution of this equation is

$$E_1^{(k)} = C_1 \lambda_1^k + C_2 \lambda_2^k,$$  \hspace{1cm} (59)

where $\lambda_1$ and $\lambda_2$ are the solutions of the characteristic equation

$$\lambda^2 - 2 \left( T_{11}^{(1)} + qR_1^2 \right) \lambda + 1 = 0.$$  \hspace{1cm} (60)

Solutions to this equation are ($f=2.856$ GHz, $a=0.99$ cm, $b=4.08896$ m, $d=3.4989$ cm): $\lambda_1 = 1\angle 119.41^\circ$, $\lambda_2 = 1\angle -119.41^\circ$ (compare with results in Table 1).
We can also apply the operator $T^{(E)}$ to the right and left sides of equation (56) and get the equation like (58)

$$\frac{1}{2} \left( T^{(1)}_{11} + qR_1^2 \right) e_{010}^{(k)} - \left( T^{(3)}_{11} + qR_1^2 \right) e_{010}^{(k+1)} - \left( T^{(1)}_{11} + qR_1^2 \right) e_{010}^{(k-1)} = 0. \quad (61)$$

The solution of equation (67) can be also written using the solution $Y_1^{(k,j)}$ of the fundamental equation

$$2T^{(1)}_{11} Y_1^{(k,j)} - T^{(3)}_{11} Y_1^{(k+1,j)} - T^{(3)}_{11} Y_1^{(k-1,j)} = R_1 \delta_{k,j}. \quad (62)$$

If $|\chi_2| < 1$, where

$$\chi_{1,2} = \frac{T^{(1)}_{11}}{T^{(3)}_{11}} \pm \sqrt{\left( \frac{T^{(1)}_{11}}{T^{(3)}_{11}} \right)^2 - 1}, \quad (63)$$

then the solution of (62) is

$$Y_1^{(k,j)} = \begin{cases} A\chi_1^{k-j}, & k - j < 0, \\ A\chi_2^{k-j}, & k - j \geq 0, \end{cases} \quad (64)$$

$$A = \frac{R_1}{2 \left( T^{(1)}_{11} - \chi_2 T^{(3)}_{11} \right)}. \quad (65)$$

The solution of (57) is

$$\epsilon_1^{(k)} = \sum_j Y_1^{(k,j)} \left( e_{010}^{(j)} - e_{010}^{(j-1)} \right). \quad (66)$$

Substitution (66) into (56) gives the final equation

$$e_{010}^{(k)} = -qR_1 \sum_{j=-\infty}^{\infty} e_{010}^{(k+j)} \left( 2Y_1^{(k-j,k)} - Y_1^{(k-j-1,k)} - Y_1^{(k-j+1,k)} \right) =$$

$$= -qR_1 A e_{010}^{(k)} \left( 1 - \chi_2 \right) - qR_1 A \sum_{j=1}^{\infty} \left( e_{010}^{(k+j)} + e_{010}^{(k-j)} \right) \left( 2\chi_2^j - \chi_2^{j+1} - \chi_2^{j-1} \right) =$$

$$= -qR_1 2 A e_{010}^{(k)} \left( 1 - \chi_2 \right) - qR_1 A \left( 2 - \chi_2 - \chi_1 \right) \sum_{j=1}^{\infty} \chi_2^j \left( e_{010}^{(k+j)} + e_{010}^{(k-j)} \right). \quad (67)$$

The characteristic equation of the difference equation (67) coincide with the equation

$$1 + qR_1 2 A \left( 1 - \chi_2 \right) - qR_1 A \frac{(1 - \chi_2)^2}{\chi_2} \sum_{j=1}^{N} \left( \lambda^j + \lambda^{-j} \right) \chi_2^j = 0, \quad (68)$$

when $N \to \infty$.

The equation (68) we can rewrite as

$$\lambda^2 - 2 \frac{\left[ T^{(1)}_{11} + qR_1^2 \right]}{\left( T^{(3)}_{11} + qR_1^2 \right)} \lambda + 1 = -qR_1^2 \frac{(1 - \chi_2)}{(1 + \chi_2)} \times \left[ (\chi_2 - \lambda) \lambda (\lambda \chi_2)^N + (\lambda \chi_2 - 1) (\chi_2 / \lambda)^N \right]. \quad (69)$$
If we suppose that
$$|\lambda \chi_2| < 1, \ |\chi_2/\lambda| < 1$$
and the number $N$ tends to infinity, the right hand side of the equation (69) tends to zero and this $2(N+1)$ order characteristic equation transforms into the two order equation that coincide with the equation (60).

The equation (69) under the conditions (70) has two solutions very close to the solutions of the equation (60), and 2N additional solutions.

For considered above parameter values $f=2.856$ GHz, $a=0.99$ cm, $b=4.08896$ cm, $d=3.4989$ cm $\chi_2 = 5.03E-003$. Therefore, conditions (70) are fulfilled in passband $(|\lambda| = 1)$ and even in some part of stopband.

A distinctive feature of the difference equation (67) is that despite the presence of an infinite sum (infinite order), its characteristic equation is of the second order (at least in some frequency domain).

The above consideration clarifies the reason for the appearance of the spurious solutions in the main equation of the Coupled Cavity Model (51) - this is a consequence of the truncation of the infinite sum.

It is well known that without the evanescent eigenfields we cannot get the correct characteristics of inhomogeneities in waveguides. Since we cannot operate numerically with infinite sums, using this approach for study non-homogeneous infinite chains may become problematic because the CCM does not correctly describe the evanescent eigenfields.

To clarify the correctness of using CCM to study infinite inhomogeneous resonator chains it is necessary to compare the values of the coupling coefficients calculated on the basis of the approximate equations (52)-(53) with the values obtained using the exact equations (16) - (24). The number of resonators in the finite chain must be sufficient to exclude the influence of boundaries on the coupling coefficients of cells in the middle of the chain. We have made a series of calculations for different inhomogeneities inside the finite chain and the same non-uniform insertions in the infinitive chain. Their results show that there is such number $N$ in (53) that the coupling coefficients calculated on the basis of approaches (52) and (24) are almost the same.

Therefore, the use of the CCM for the numerical study of the characteristics of infinitive resonator chains gives correct results. The model of an infinite chain of resonators plays an important role in calculating the electrodynamic characteristics of inhomogeneities in structured waveguides. Indeed, based on the CCM of an infinitive chain, one can study the processes of wave propagation in inhomogeneous resonator chains. This can be done if we assume that there are homogeneous fragments before and after inhomogeneities. In this case, in homogeneous fragments at a sufficient distance from the interfaces (when all evanescent waves are damped), one can look for amplitudes in the form
$$e^{(k)}_{010} = \begin{cases} \exp \{i\varphi_1 (k - k_1)\} + R \exp \{-i\varphi_1 (k - k_1)\}, & k < k_1, \\ T \exp \{i\varphi_2 (k - k_2)\}, & k > k_2, \end{cases}$$
where $R$ and $T$ are the reflection and transition coefficients. This approach avoids the use of an ideal load or tuned couplers. This is especially useful when we are studying frequency dependencies.

But existence of spurious solutions make it difficult (or even impossible) to use the approximate methods for analysing the chains with slow varying parameters (WKB approach). As shown by [34], in order to obtain the WKB equations from the original difference equation, it is necessary to know the local characteristic multipliers. It is unknown how the spurious solutions will influence on WKB equations, since their number can be great. But from the physical point of view, including the spurious solutions in consideration will be wrong.
It is necessary to use other approaches with correct eigenmode basis for analysing the chains with slow varying parameters \([32, 35, 36]\).

4 Conclusions

Summarizing the above, we can draw the following conclusions:

1. We can use the CCM to numerical study the characteristics of infinitive inhomogeneous resonator chains.

2. We cannot use the CCM to obtain approximate equations (the WKB approach) for the analysis of chains with slowly varying parameters.

5 Appendix 1

\[
T^{(1,1)}_{s,s'} = \frac{a_k^2}{b_k^2} \sum_m \Lambda_m^{(1,k)} \frac{\lambda_m}{J_1^2(\lambda_m)} \frac{j_{2s'-1} (\frac{\lambda_m a_k}{b_k})}{(\frac{a_k \lambda_m}{b_k})^2} - (\lambda_s)^2
\]

\[
T^{(1,2)}_{s,s'} = \frac{a_k^2}{b_k^2} \sum_m \Lambda_m^{(1,k-2)} \frac{\lambda_m}{J_1^2(\lambda_m)} \frac{j_{2s'-1} (\frac{\lambda_m a_k}{b_k})}{(\frac{a_k \lambda_m}{b_k})^2} - (\lambda_s)^2
\]

\[
T^{(1,3)}_{s,s'} = \frac{a_k^2}{b_k^2} \sum_m \Lambda_m^{(1,k-2)} \frac{\lambda_m}{J_1^2(\lambda_m)} \frac{j_{2s'-1} (\frac{\lambda_m a_k}{b_k})}{(\frac{a_k \lambda_m}{b_k})^2} - (\lambda_s)^2
\]

\[
T^{(1,4)}_{s,s'} = \frac{a_k^2}{b_k^2} \sum_m \Lambda_m^{(1,k-2)} \frac{\lambda_m}{J_1^2(\lambda_m)} \frac{j_{2s'-1} (\frac{\lambda_m a_k}{b_k})}{(\frac{a_k \lambda_m}{b_k})^2} - (\lambda_s)^2
\]
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