Nucleon QCD sum rules in instanton medium

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Abstract

We calculate the polarization operator of the nucleon current in the instanton medium. We solve the QCD sum rules equations and find the value of the nucleon mass. Successive inclusion of the instanton effects provides the values around the physical one. Uncertainties of the approach are discussed.

1 Introduction

1.1 QCD sum rules

The QCD sum rules (SR) approach suggested in [1] succeeded in describing the characteristics of mesons in terms of the vacuum expectation values of the products of quark or (and) gluon operators (QCD condensates). This approach succeeded in describing the characteristics of nucleons as well [2, 3, 4, 5]. It is based on the dispersion relation for the function describing the propagation of the system which carries the quantum numbers of the hadron. In the case of nucleon (we consider the proton) this function, usually referred to as the "polarization operator" is

\[ \Pi(q) = \hat{q}\Pi^q(q^2) + I\Pi^I(q^2), \]

(1)

with \( q \) the four-momentum of the system, \( \hat{q} = q_\mu\gamma^\mu \), \( I \) is the unit matrix. The dispersion relations

\[ \Pi^i(q^2) = \frac{1}{\pi} \int dk^2 \frac{\text{Im}\Pi^i(k^2)}{k^2 - q^2}; \quad i = q, I \]

(2)

are considered at \( q^2 \to -\infty \). This enables to expand the left hand sides (LHS) of Eq.(2) in powers of \( 1/q^2 \). The coefficients of the expansion are the QCD condensates. This is known as the Operator Product Expansion (OPE) [6]. This provides the perturbative expansion of the short distance effects, while the nonperturbative physics is contained in the condensates. The higher terms of the OPE contain the condensates of the higher dimension.

The right-hand side (RHS) of Eq. (2) is usually approximated by the "pole+continuum" model [1, 2] in which the lowest lying pole is written down exactly, while the higher states are
described by continuum. Thus Eqs.\([2]\) take the form

\[
\Pi^i_{\text{OPE}}(q^2) = \frac{\lambda^2_{\text{NS}}}{m^2 - q^2} + \frac{1}{2\pi i} \int_{W^2}^{\infty} dk^2 \frac{\Delta \Pi^i_{\text{OPE}}(k^2)}{k^2 - q^2}; \quad i = q, I.
\]  

(3)

Here \(\xi^q = 1, \xi^I = m\). The upper index OPE means that several lowest OPE terms are included. Note that the “pole+continuum” presentation of the RHS makes sense only if its first term, treated exactly is larger than the second term, which approximates the higher states. The position of the lowest pole \(m\), its residue \(\lambda^2_{\text{NS}}\) and the continuum threshold \(W^2\) are the unknowns of Eqs. \((3)\).

In the next step one usually applies the Borel transform which converts the functions of \(q^2\) to the functions of the Borel mass \(M^2\). An important assumption is that there is an interval of the values of \(M^2\) where the two sides of the SR have a good overlap, approximating also the true functions. This interval is in the region of 1 MeV\(^2\). Thus actually one tries to expand the OPE expansion from the high momentum region to the region of 1 MeV\(^2\).

The polarization operator depends on the form of the local operator \(j(x)\) with the proton quantum numbers, often referred to as “current”

\[
\Pi(q^2) = i \int d^4xe^{i(q\cdot x)} \langle 0|T[j(x)\bar{j}(0)]|0\rangle,
\]  

(4)

The form of \(j(x)\) is not unique. One can write

\[
j(t, x) = j_1(x) + tj_2(x),
\]  

(5)

with

\[
j_1(x) = (u^T_\alpha(x)C\bar{d}_b(x))\gamma_5u_c(x)\varepsilon^{abc}, \quad j_2(x) = (u^T_\alpha(x)C\gamma_5\bar{d}_b(x))u_c(x)\varepsilon^{abc},
\]

while \(t\) is an arbitrary coefficient. Following \([2, 7]\) we use the current determined by Eq. \((5)\) with \(t = -1\), which can be written (up to a factor \(1/2\)) as

\[
J(x) = (u^T_\alpha(x)C\gamma_\mu\bar{u}_b(x))\gamma_5\gamma^\mu\gamma_c(x)\varepsilon^{abc}.
\]  

(6)

This current is often used in the QCD SR calculations. One of the strong points of the choice \([7]\) is that it makes the domination of the lowest pole over the higher states on the LHS of Eq.\((3)\) more pronounced. We shall use only this current in the present paper.

For calculation of the OPE expansion for the polarization operator one needs to expand the integrand in powers of \(x^2\). In the lowest orders it is sufficient to expand the expectation value of the product of the quark operators \(S^{ab}_{ij}(x) = \langle 0|q^a_i(x)\bar{q}^b_j(0)|0\rangle\), which describes the time-space propagation of the quark

\[
S^{ab}_{ij}(x) = S^{0ab}_{ij}(x) - \frac{x^2}{12} I_{ij} \delta_{ab} \langle 0|\bar{q}(0)q(0)|0\rangle + O(x^4)
\]  

(7)

Here \(S^{0ab}_{ij}(x) = iG^{0ab}_{ij}(x)\), while \(G^{0ab}(x) = \hat{x}\delta_{ab}/(2\pi^2x^4)\) is the Green function of the free massless quark. In momentum space it is \(G^{0ab}(p) = \hat{p}\delta_{ab}/p^2\). The second term on the RHS of Eq.\((7)\) contains the scalar quark condensate \(\langle 0|\bar{q}(0)q(0)|0\rangle\). The terms \(O(x^4)\) can be treated as coming from the nonlocality of the expectation value \(\langle 0|\bar{q}(0)q(x)|0\rangle\). In the higher terms one should include the four-quark and six-quark expectation values \(\langle 0|qq\bar{q}\bar{q}|0\rangle\), etc. One should also take into account that the quarks move in the field of the gluon condensate.
1.2 Instanton effects

We include the instantons into the nucleon QCD SR, basing on the treatment of light quarks (i.e., $u$ and $d$ quarks) in the instanton medium, developed in [8]. It is assumed that the QCD vacuum is described by the system of instantons and antiinstantons which have the fixed size $\rho \sim 0.3 \text{ fm} \approx (600 \text{ MeV})^{-1}$, being separated by the distances $R \approx 3\rho$. At large distances one can assume that the instantons just create the "mean field" condensate [9]. However at the distances of the order of the instanton size $\rho$ one deals with some relatively small size spots of quarks created by each instanton. The size of the spots is of the order of the inverse Borel mass $M \sim 1 \text{ GeV}$. Thus there is no small parameter and we must calculate the effect of inhomogeneous structure of vacuum using explicit form of zero mode light quark wave function. The connection with OPE can be established by introducing effective condensates of higher dimension $d \geq 5$.

The attempts to include the instantons into the SR were made in [10], [11], [12]. Actually the authors were interested in description of the quark correlations in the polarization operator due to interaction with the same instanton. In other words the authors considered the instanton correction to the Green function of two quarks but neglect the instanton effect in the propagation of one quark where just the "mean field" scalar condensate $<0|\bar{q}(0)q(0)|0>$ was included. Note also that, as it was demonstrated in [13], that the SIA is not good for the single-quark propagators.

On the other hand one not exclude a strong impact of the instantons on the nucleon pole in the SR approach since their average size is $\rho \sim 0.3 \text{ fm} \approx (600 \text{ MeV})^{-1}$, and thus $1/\rho$ is of the order of the nucleon mass $m_N$, i.e., $m_N \rho \sim 1$. On the LHS of the SR the parameter $M^2 \rho^2$ can be of the order of unity. This can manifest itself as nonlocalities of the scalar quark condensate. We include the instantons into the nucleon QCD SR, basing on the treatment of light quarks (i.e., $u$ and $d$ quarks) in the instanton medium, developed in [8]. Construction of the quark Green function in this approach is based on averaging of the propagator in the single-instanton field over the system of instantons and antiinstantons separated by the distances $R \approx 3\rho$. A closed form for the propagator in instanton medium was obtained in [8] in the limit $N_c \to \infty$ with $N_c$ standing for the number of colors. In the lowest order of expansion in powers of $1/N_c$ the light quark propagates in instanton medium as a free one, obtaining, however, the dynamical mass $m$, which depends on the number of colors $N_c$, on the average instanton size $\rho$ and on the average distance between the instantons $R$. The latter is connected with the value of the gluon condensates which is assumed to be totaly produced by the instantons

$$\frac{1}{32\pi^2} \langle 0|G^{a\mu\nu}G^{a}_{\mu\nu}|0 \rangle = \frac{N}{V} \equiv \frac{1}{R^4}. \quad (9)$$

The mass $m(p)$ can be presented as the Fourier transform of a combination of products of the modified Bessel functions. It reaches the largest value $m(0) \approx 350 \text{ MeV}$, dropping as $p^{-6}$ at
large $p$. The quark condensate is created by the instanton field and can be expressed in terms of the quark propagator $S(p)$ by the general relation

$$\langle 0|\bar{q}(0)q(0)|0\rangle = \int \frac{d^4p}{(2\pi)^4} \text{Tr}S(p) = -4N_c \int \frac{d^4p}{(2\pi)^4} \frac{m(p)}{p^2 + m^2(p)}.$$  \hspace{1cm} (10)

Here we kept the Minkowsky metric for the quark operators in the vacuum expectation values. Thus we must calculate the polarization operator $\Pi(q)$ employing the quark propagator determined by Eq. (8).

Including the instanton field in the propagator of one quark we obtain the nucleon mass $m = 1.47$ GeV. Describing all quarks by the product of the single-quark propagators \[8\] we find $m = 1.12$ GeV. This can be viewed as the "independent particle approximation" (IPA). Inclusion of the gluon condensate can be treated as the first step beyond the IPA, since the gluons interact with different quarks of the polarization operator \[14\]. This provides $m = 800$ MeV. Thus the value of the nucleon mass variate around its physical value while we improve the accuracy of our approach.

Note, however, that the propagator \[8\] is obtained in the lowest nonvanishing order of expansion in powers of $1/N_c$. Thus one can not exclude that there are the errors of the order $1/N_c = 1/3$. Another weak point is that our calculations involve the region of the quark momenta $p \approx 2$ GeV, where the accuracy of Eq. (8) is obscure. However, the reasonable results stimulate us to continue the studies by taking into account interactions between the quarks which compose the polarization operator.

We demonstrate that the SR still provide a reasonable value for the nucleon mass. However, the LHS of the SR obtains a more complicated structure than it was without inclusion of the instantons.

In Sec. 2 we recall the main features of the nucleon SR. In Sec. 3 we calculate the polarization operator in instanton medium. In Sec. 4 we solve the SR equations. We discuss the uncertainties of the approach in Sec. 5 and summarize in Sec. 6.

2 QCD sum rules without instantons

The LHS of Eq. (3) can be written as

$$\Pi^OPE(q^2) = \sum_{n=0} A_n(q^2); \quad \Pi^I OPE(q^2) = \sum_{n=3} B_n(q^2)$$ \hspace{1cm} (11)

where the lower index $n$ is the dimension of the corresponding QCD condensate ($A_0$ stands for the three-quark loop). The most important terms for $n \leq 9$ were obtained earlier \[2],\[5\]

$$A_0 = -\frac{Q^4 \ln Q^2}{64 \pi^4}; \quad A_4 = -\frac{b \ln Q^2}{128 \pi^4}; \quad A_6 = \frac{1}{24 \pi^4} \frac{a^2}{Q^2}; \quad B_3 = \frac{a Q^2 \ln Q^2}{16 \pi^4},$$ \hspace{1cm} (12)

with $Q^2 = -q^2 > 0$, while $a$ and $b$ are the scalar and gluon condensates multiplied by certain numerical factors

$$a = -(2\pi)^2 \langle 0|\bar{q}q|0\rangle; \quad b = (2\pi)^2 \langle 0|\frac{\alpha_s}{\pi} G_{\mu\nu} G^{\mu\nu}_{\mu\nu}|0\rangle,$$ \hspace{1cm} (13)
The leading contribution to the chirality-even structure $A_0$ is the loop containing three free quarks. The leading contribution to the chirality-odd structure $B_3$ is proportional to the scalar quark condensate. Here the free $u$ quarks form a loop, while the $d$ quarks are exchanged with the vacuum condensate—Fig. 1.

Actually one usually considers the SR for the operators $\Pi_1^{OPE}(q^2) = 32\pi^4 \Pi^{OPE}(q^2)$. The factor $32\pi^4$ is introduced in order to deal with the values of the order of unity (in GeV units). After the Borel transform $B$ we write (11) as

$$B\Pi_1(q^2) = \sum_{n=0} A'_n(M^2); \quad B\Pi_1(q^2) = \sum_{n=3} B'_n(M^2); \quad A'_n(M^2) = 32\pi^4 \mathcal{B}A_n(q^2). \quad (14)$$

Here we present the most important terms

$$A'_0(M^2) = M^6; \quad A'_4(M^2) = \frac{bM^2}{4}; \quad A'_6 = \frac{4}{3} a^2; \quad B'_3(M^2) = 2aM^4. \quad (15)$$

The Borel transformed SR (14) can be written now as

$$B\Pi_1(M^2) = \mathcal{F}_p(M^2) + \mathcal{F}_c(M^2), \quad (16)$$

where the two terms on the RHS

$$\mathcal{F}_p(M^2) = \xi_i\lambda^2 e^{-m^2/M^2}; \quad \mathcal{F}_c(M^2) = \int_{M^2}^{\infty} dk^2 e^{-k^2/M^2} \Delta[B\Pi_1(k^2)], \quad (17)$$

are the contribution of the pole and that of continuum.

The conventional form of the SR is

$$L^q(M^2, W^2) = R^q(M^2), \quad (18)$$

and

$$L^i(M^2, W^2) = R^i(M^2). \quad (19)$$

Here $L^q$ and $R^q$ are the Borel transforms of the LHS and of the RHS of Eqs. (3) correspondingly

$$R^q(M^2) = \lambda^2 e^{-m^2/M^2}; \quad R^i(M^2) = m\lambda^2 e^{-m^2/M^2}, \quad (20)$$

with $\lambda^2 = 32\pi^4 \lambda_N^2$. The contribution of continuum is moved to the LHS of Eqs. (18, 19) which can be written as

$$L^q = \sum_{n=0} \bar{A}_n(M^2, W^2); \quad L^i = \sum_{n=3} \bar{B}_n(M^2, W^2), \quad (21)$$

-see Eq.(14). Here

$$\bar{A}_0 = \frac{M^6 E_2(\gamma)}{L(M^2)}; \quad \bar{A}_4 = \frac{bM^2 E_0(\gamma)}{4L(M^2)}; \quad \bar{A}_6 = \frac{4}{3} a^2 L(M^2); \quad \bar{B}_3 = 2aM^4 E_1(\gamma); \quad \gamma = \frac{W^2}{M^2};$$

with $E_0(\gamma)$ and $E_1(\gamma)$ being the contributions of the poles and of the continuum.
with

\[ E_0(\gamma) = 1 - e^{-\gamma}, \quad E_1(\gamma) = 1 - (1 + \gamma)e^{-\gamma}, \quad E_2(\gamma) = 1 - (1 + \gamma + \frac{\gamma^2}{2})e^{-\gamma}. \]  

(23)

The factor

\[ L(M^2) = \left( \frac{\ln M^2/\Lambda^2}{\ln \mu^2/\Lambda^2} \right)^{4/9} \]  

(24)

includes the most important radiative corrections of the order \( \alpha_s \ln Q^2 \) (LLA). These contributions were summed to all orders of \( (\alpha_s \ln Q^2)^n \). In Eq.(24) \( \Lambda = \Lambda_{QCD} \) is the QCD scale, while \( \mu \) is the normalization point, the standard choice is \( \mu = 0.5 \) GeV.

After inclusion of several condensates of the higher dimension and of the lowest order radiative corrections beyond the leading logarithmic approximation [15] the SR provide solution (for \( \Lambda_{QCD} = 230 \) MeV) \( m = 928 \) MeV, \( \lambda^2 = 2.36 \) GeV\(^6\), \( W^2 = 2.13 \) GeV\(^2\).

3 Polarization operator in instanton medium

The pseudoeuclidean metric is known to be more convenient for instanton physics. It will be used through the paper. Recall that in this metric any Minkowsky vector \( a = (a_0, a) \) is represented as \( a = (a, a_4) \), with \( a_4 = ia_0 \). In this metric Eq.(1) is

\[ \Pi(Q) = \hat{Q} \Pi^Q(Q^2) + i \Pi^I(Q^2), \]  

(25)

with

\[ \Pi(Q) = \int d^4x e^{-i(Q-x) \cdot 0} T[J(x)\bar{J}(0)]|0\rangle, \]  

(26)

In the first step we can assume that the quarks are described by the single-particle propagators in instanton medium given by Eq.(8) where we omit the diagonal color factor \( \delta_{ab} \)

\[ S(p) = S_I(p) + S_Q(p), \]  

(27)

with \( S_I \) and \( S_Q \) corresponding to the structures proportional to the unit matrix \( I \) and to \( \hat{Q} \)

\[ S_I(p) = i \frac{m(p)}{p^2 + m^2(p)}; \quad S_Q(p) = S_0(p) + S_1(p), \]  

(28)

with

\[ S_0(p) = \frac{\hat{p}}{p^2} \]  

(29)

describing the propagation of free massless quark, while

\[ S_1(p) = -S_0(p)g(p); \quad g(p) = \frac{m^2(p)}{p^2 + m^2(p)}, \]  

(30)

Now Eq.(10) for the scalar condensate can be written as

\[ a = 6 \int_0^\infty dpp^3g_I(p); \quad a = -(2\pi)^2 \langle 0|\bar{q}q|0\rangle \]  

(31)
while
\[ g_I(p) = \frac{m(p)}{p^2 + m^2(p)}. \] (32)

Note that the value of scalar condensate in Eq. (32) and in further calculations corresponds to the normalization point \( \mu = 1/\rho = 600 \) MeV. The value at a conventional point \( \mu_c = 500 \) MeV is thus \( a_c = aL(\mu_c^2) = 0.90a \).

Now we calculate the polarization operator (26) with all quarks described by the propagators determined by Eq.(8)-see Fig. 2. Since all propagators are diagonal in color, we shall present the contributions summed over the color indices.

We shall include the influence of instantons step by step. This means that in the first step we describe only one of the quarks by the propagators \( S_1 \) or \( S_I \) while two other ones are described by the propagator \( S_0 \).

### 3.1 One quark in the instanton field

Start with the contribution to \( \hat{Q} \) structure of Eq.(25). Two quarks are described by the functions \( S_0 \), while the third one is described by the function \( S_Q \). If both \( u \) quarks are free, while the \( d \) one moves in the instanton medium, the contribution to the polarization operator is

\[ X_d = 12 \int \frac{d^4 p}{(2\pi)^4} \gamma_\mu S_Q(p) \gamma_\nu T_{\mu\nu}(Q - p), \] (33)

The factor 12 on the RHS of Eq.(33) comes from the sum over the colors and includes permutations of the \( u \) quarks, while \( T_{\mu\nu} \) describes the \( u \) quark loop. We present it as

\[ T_{\mu\nu}(Q - p) = \int d^4 x e^{-i(Q-p\cdot x)} Tr[t_{\mu\nu}(x)], \] (34)

with

\[ t_{\mu\nu}(x) = \gamma_\mu G_0(x) \gamma_\nu G_0(x). \] (35)

Here

\[ G_0(x) = -\frac{1}{2\pi^2 x^4} \] (36)

is the Fourier transform of the propagator \( S_0 \) determined by Eq.(29). Thus

\[ X_d = \frac{3}{\pi^4} \int \frac{d^4 p}{(2\pi)^4} d^4 x e^{-i(Q-p\cdot x)} (1 - g(p)) \frac{\gamma_\mu \hat{p}\gamma_\nu}{p^2} Tr[\gamma_\mu \hat{x}/x^4 \gamma_\nu \hat{x}/x^4]. \] (37)

The terms 1 and \(-g(p)\) in the brackets correspond to the terms \( S_0 \) and \( S_1 \) of the quark propagator in instanton medium-Eq.(30).

The contribution \( X_u \) corresponding to one of the \( u \) quarks in the instanton medium can be obtained by interchange of the factor \( \hat{p}/p^2 \) and one of the factors \( \hat{x}/x^4 \) in the integrand on the RHS of Eq.(37). For the total contribution we find

\[ X = X_d + 2X_u = X^0 + X^{ins}, \quad X^0 = \frac{3}{\pi^8} \int d^4 p d^4 x e^{-i(Q-p\cdot x)} \frac{2(px)\hat{x} + x^2 \hat{p}}{p^2 x^8}; \] (38)
The term $X^0$ corresponds to contribution of three massless quarks. It can be obtained by calculation of the integral over $p$

$$\int d^4p e^{i(p\cdot x)} \frac{2(px)\hat{x} + x^2\hat{p}}{p^2 x^8} = 24\pi^2 \frac{\hat{x}}{x^2}.$$  

Further integration over $x$ provides (omitting polynomials in $Q^2$ which will be eliminated by the Borel transform)

$$X^0 = -\frac{\hat{Q}}{64\pi^4} Q^4 \ln Q^2,$$

in agreement with \[2\]

We start calculation of $X^{\text{ins}}$ by integration over $x$

$$\int d^4x e^{i(k\cdot x)} \frac{1}{x^6} = \frac{\pi^2}{8}\kappa^2 \ln \kappa^2; \quad \int d^4x e^{i(k\cdot x)} \frac{x\mu x\nu}{x^8} = \frac{\pi^2}{12} (g_{\mu\nu}\kappa^2 + 2\kappa_{\mu}\kappa_{\nu}) \ln \kappa^2,$$

and thus find

$$X^{\text{ins}} = \frac{1}{8\pi^6} \int \frac{d^3p}{p^2} g(p)(\hat{p}\kappa^2 + 2(p\kappa)\hat{\kappa}) \ln \kappa^2.$$  

Integrating over the angular variables we obtain

$$X^{\text{ins}} = \frac{\hat{Q}}{32\pi^4} \left( 36I_3 \ln Q^2 + 24\frac{I_5}{Q^2} - 3\frac{I_7}{Q^4} \right).$$

Here we defined

$$I_n = \int_0^\infty dp \gamma I(p)p^n,$$

with the function $g(p)$ defined by Eq.(30). The integrals $I_n$ are saturated by $p \sim 1/\rho$.

Combining Eqs.(40) and (43) we obtain, following the definition (15)

$$A'_0(M^2) = M^6 F_0(M^2); \quad F_0(M^2) = 1 - 36I_3/M^4 + 24I_5/M^6 - 3I_7/M^8.$$

The function $F_0$ is shown in Fig. 3.

Now we calculate the contribution to the structure $\Pi'$. Similar to Eq.(33) we can write

$$X = X_d = i12 \int \frac{d^4p}{(2\pi)^4} \gamma_\mu S_I(p)\gamma_\nu T_{\mu\nu}(Q - p),$$

with $S_I(p)$ defined by Eq.(28), while $T_{\mu\nu}(Q - p)$ is given by Eq.(34). Integration over $x$ provides

$$X = 3 \pi^2 \int \frac{d^4p}{(2\pi)^4} g_I(p)\kappa^2 \ln \kappa^2,$$

with $g_I$ defined by Eq.(32) Note that putting $\kappa = Q$ in the integrand on the RHS of Eq.(47) we would obtain

$$X = i\frac{3Q^2 \ln Q^2}{8\pi^4} \int_0^\infty dp \gamma I(p) = iB_3(Q^2).$$
with $B_3(Q^2)$ defined by Eq. (12). Thus we can view calculation of the contribution given by Eq. (46) as inclusion of inhomogeneity of the scalar quark condensate.

Caring out integration over the angular variables we find

$$X = \frac{3i}{8\pi^4} \int_0^\infty dpp^3 g_1(p)f(p), \quad (49)$$

with

$$f(p) = f_1(p, Q)\theta(Q - p) + f_2(p, Q)\theta(p - Q); \quad f_1(p, Q) = (Q^2 + p^2) \ln Q^2 + \frac{3p^2}{2} + \frac{p^4}{6Q^2}; \quad (50)$$

$$B_3' = 2aM^4F_3(M^2); \quad F_3(M^2) = e^{-\beta}(1 - \beta) + \beta^2\mathcal{E}(\beta); \quad \beta = \eta^2/M^2. \quad (53)$$

In literature our function $\mathcal{E}$ is usually denoted as $E_1$. We avoid this notation since in QCD SR publications it has another meaning—see Eq. (23). The function $F_3$ is shown in Fig. 3.

### 3.2 Two quarks in the instanton field

Start with the contribution to the structure $\hat{Q}$. Now two quarks are described by the propagator $S_1$ while one is described by the free propagator $S_0$. Since the contribution contains the high power $m^4$ we expect it to be numerically small.

If the $d$ quark is described by $S_0$, we can write for the contribution to the polarization operator

$$X_d = -12 \int d^4xe^{-i(Q\cdot x)}\gamma_\mu G_0(x)\gamma_\nu T_{\mu\nu}(x), \quad (55)$$

With $G_0(x)$ is the Fourier transform of the propagator $S_0$ given by Eq. (36), while

$$T_{\mu\nu}(x) = Tr[\gamma_\mu G_1(x)\gamma_\nu G_1(x)], \quad (56)$$
where

\[ G_1(x) = \frac{d^4 p}{(2\pi)^4} \frac{g(p)\hat{p}}{p^2} e^{i(p \cdot x)}, \]  

(57)

is the Fourier transform of the propagator \( S_1 \). Note that \( G_0(x) \) and \( G_1(x) \) are \( 4 \times 4 \) matrices. The contribution \( X_u \) of configuration in which one of \( u \) quarks is described by \( S_0 \) can be obtained by permutation of propagators \( G_0 \) and \( G_1 \) on the RHS of Eq. (55). If the \( d \) quark is described by \( S_0 \), we can write for the contribution to the polarization operator

\[ X = X_d + 2X_u = \frac{96}{\pi^2} \int \frac{d^4 p_1}{(2\pi)^4} \frac{d^4 p_2}{(2\pi)^4} \frac{g(p_1) g(p_2)}{p_1^2 p_2^2} d^4 x \hat{p}_1 (p_2 x) + \hat{p}_2 (p_1 x) + \hat{x} (p_1 p_2) e^{i((p_1 + p_2 - Q) \cdot x)}. \]

(58)

Carrying out integration over \( x \) we obtain

\[ X = 64 \cdot 3 \int \frac{d^4 p_1}{(2\pi)^4} \frac{d^4 p_2}{(2\pi)^4} \frac{g(p_1) g(p_2)}{p_1^2 p_2^2} F(p_1, p_2) \frac{F(p_1, p_2)}{(Q - p_1 - p_2)^2}, \]

(59)

with \( F(p_1, p_2) = \hat{p}_1 (p_2 Q) + \hat{p}_2 (p_1 Q) + \hat{Q} (p_1 p_2) - \hat{p}_1 p_2 + \hat{p}_2 p_1^2 - 2(\hat{p}_1 + \hat{p}_2)(p_1 p_2). \)

The integrals are saturated by \( p_{1,2} \sim 1/\rho \ll q \). Thus the result can be obtained by expansion of denominator in powers of \( p_i/Q \). We obtain that the contribution adds to polarization operator

\[ \delta A_0 = \frac{3}{(2\pi)^4} \frac{I_3^2}{Q^4}, \]

(60)

adding

\[ \delta A_0' = \frac{48I_3^2}{M^2}, \]

(61)

to the term \( A_0' \).

Now we consider the case, when two quarks are described by the propagators \( S_I \). They should be \( u \) quarks. The contribution can be written as

\[ X = 12 \cdot 8 \int \frac{d^4 p_1}{(2\pi)^4} \frac{d^4 p_2}{(2\pi)^4} g_I(p_1) g_I(p_2) \frac{\hat{Q} - \hat{p}_1 - \hat{p}_2}{(Q - p_1 - p_2)^2}. \]

(62)

The integrals are saturated by \( p_1 \sim 1/\rho \ll q \). Neglecting momenta \( p_i \) in the last factor on the RHS we would find

\[ X = \frac{a^2}{24\pi^4} \frac{\hat{Q}}{Q^2}, \]

corresponding to the term \( A_6' = \text{Eq. (12)} \). Employing parametrization (51) and carrying out integrations we obtain the contribution to the structure \( Q \) with

\[ A_6 = \frac{a^2}{24\pi^4} \int \frac{dtt^4}{(\eta^2 + t(1 - t)Q^2)^3}, \]

(63)

and

\[ A_6' = \frac{4}{3} a^2 F_6(M^2); \quad F_6(M^2) = \frac{\eta^4}{M^4} \int_0^1 \frac{dtt \exp(-\eta^2/t(1 - t)M^2)}{(1 - t)^3}. \]

(64)
Without instantons we would obtain \( F_6(M^2) = 1 \). Note that in the limit \( M^2 \gg \eta^2 \) the small interval of integration \( 1-t \sim \eta^2/M^2 \) provides \( F_6(M^2) = 1 \). The function \( F_6 \) is shown in Fig. 3.

There is a contribution to the term \( B_3 \) from the configuration, where \( d \) quark is described by the propagator \( S_I \), while one of \( u \) quarks is described by \( S_1 \). Performing calculations similar to those presented above we find \( \delta B_3' = -4aI_3 \).

### 3.3 Three quarks in the instanton field

In the dominant contribution all three quarks are described by propagators \( S_I \). One of the quarks should carry large momentum of the order \( Q \). The contribution can be found immediately by replacing free propagator \( S_0(Q) \) in Eq.(22) for \( A_6 \) without instantons by propagator \( S_0(Q) \). Thus we obtain

\[
B_6 = \frac{1}{32\pi^4} \frac{4}{3} a^2 \frac{3A}{Q^8}; \quad B_6' = \frac{2}{3} a^2 \frac{A}{M^6};
\]

### 3.4 Polarization operator

Combining the results, we can write

\[
A'_0 = M^6 F_0(M^2); \quad F_0(M^2) = 1 - 36I_3/M^4 + 24I_5/M^6 - 3(I_7 - 16I_3^2)/M^8; \quad (66)
\]

\[
A'_6 = \frac{4}{3} a^2 F_6(M^2)
\]

\[
F_6(M^2) = \frac{\eta^4}{M^4} \int_0^1 \frac{dt \exp(-\eta^2/t(1-t)M^2)}{(1-t)^3}.
\]

\[
B_3' = 2aM^4 F_3(M^2); \quad F_3(M^2) = e^{-\beta}(1-\beta) + \beta^2 E(\beta);
\]

\[
\beta = \eta^2/M^2; \quad B_6' = \frac{2}{3} a^2 \frac{A}{M^6}.
\]

We present the main results for

\[
R = R_0 = 1 \text{ fm}; \quad \rho = \rho_0 = 0.33 \text{ fm}.
\]

The parameters of parametrization Eq.(51) are chosen as \( A = 0.59 \text{ GeV}; \quad \eta^2 = 0.65 \text{ GeV} \); corresponding to \( \langle 0|\bar{q}q|0 \rangle = (-260 \text{MeV})^3 \), i.e., \( a = 0.70 \text{ GeV}^3 \). Also, direct computations provides

\[
I_3 = 1.06 \cdot 10^{-2} \text{ GeV}^4; \quad I_5 = 5.65 \cdot 10^{-3} \text{ GeV}^6; \quad I_7 = 7.06 \cdot 10^{-3} \text{ GeV}^8.
\]

The functions \( F_i(M^2) \) \((i = 0, 3, 6)\) are shown in Fig 3.

These values are obtained in "independent particle approximation", i.e we assume the three quarks which contribute to polarization operator are not correlated. In next step we must include their interaction, and also include the possibility that two quarks may interact with the same instanton.

In the approach of \([8]\) the gluon condensate is considered to be due to the instantons. The quark system interacts with the gluon condensate by exchange the even number of gluons. Such
exchanges with the same quark are already included in the quark propagator. Exchanges with the different quarks is the simplest quark correlation. In the chiral limit only these contribution survive in the gauge with $x_\mu A_\mu = 0$ \cite{14}, which is just the case in approach developed in \cite{8}. Thus in the simplest version we can take it into account by including the term.

$$\bar{A}'_4 = \frac{b M^2 E_0(\gamma)}{4 L (M^2)} - \frac{3 b I_3}{2 M^2}; \quad \gamma = \frac{W^2}{M^2}.$$  \hspace{1cm} \text{(69)}$$

The parameter $b$ is defined by Eq. (13). Employing the definition \text{(9)} we obtain $b = 0.75$ GeV$^4$ at $\alpha_s(1 \text{ GeV}^2) = 0.475$.

Note that inclusion of the "mean field" gluon condensate $\langle G^{\mu \nu} G_{\mu \nu} \rangle |0\rangle$ is not selfconsistent in the present model. The gluon field is also inhomegeneous and should be described by an explicit instanton configuration. Therefore this estimate should be considered as a "hint" but not as a real numerical result.

\section{Solution of the sum rules equations}

Now we return to the Minkowsky metric and solve Eqs.(18) and (19) with

$$\mathcal{L}^q = \bar{A}'_6(M^2, W^2) + \bar{A}'_4(M^2, W^2) + \bar{A}'_6(M^2); \quad \mathcal{L}^I = \bar{B}'_3(M^2, W^2) + \bar{B}'_6(M^2).$$  \hspace{1cm} \text{(70)}$$

Here

$$\bar{A}'_6(M^2, W^2) = M^4 F'_6(M^2, W^2)/L(M^2),$$  \hspace{1cm} \text{(71)}$$

$$F'_6(M^2) = E_2(\gamma) - \frac{36 I_3 E_0(\gamma)}{M^4} + \frac{24 I_5}{M^6} - \frac{3 (I_7 - 16 I_3^2)}{M^8}; \quad \gamma = \frac{W^2}{M^2},$$

and

$$\bar{A}'_4 = \frac{b M^2 E_0(\gamma)}{4 L (M^2)} - \frac{3 b I_3}{2 M^2},$$  \hspace{1cm} \text{(72)}$$

while

$$\bar{B}'_3 = 2 a M^4 F'_3(M^2); \quad F'_3(M^2) = e^{-\beta (E_1(\gamma) - E_0(\gamma) + \beta^2 (\mathcal{E}(\beta) - \mathcal{E}(\gamma)) - \frac{2 I_3}{M^4}; \quad \beta = \frac{\eta^2}{M^2}.}$$  \hspace{1cm} \text{(73)}$$

Here the functions $E_i(i = 0, 1, 2)$ are determined by Eq. \text{(23)}. The factor $L$ includes the main radiative corrections, the normalization point $\mu = 0.6$ GeV is the inverse instanton size. The terms $\bar{A}'_6(M^2) = A'_6(M^2) L(M^2)$ and $\bar{B}'_6(M^2) = A'_6(M^2) L(M^2)$ do not contribute to continuum.

We shall present the results for the values $R = R_0$ and $\rho = \rho_0$-Eq. \text{(67)}. Start with the case when only one quark is in the instanton field. This means that we must put $\bar{A}'_6 = \bar{B}'_6 = 0$, and also $\bar{A}'_4 = 0$. We must omit also the last term in expression for $F'_3$.

We can see that the SR equations can be solved with good accuracy in the broad interval of the values of the Borel mass $0.8 \text{ GeV}^2 \leq M^2 \leq 2.2 \text{ GeV}^2$. The solution $m, \lambda^2, W^2$ do not modify much if we choose only the parts of this interval-see Table.1. However we must examine the pole-to-continuum ratio

$$r_i(M^2) = \mathcal{F}_i^p(M^2)/\mathcal{F}_i^c(M^2); \quad i = q, I$$  \hspace{1cm} \text{(74)}$$
of the two contributions to the RHS of Eq. (16). One can see that \( r_q \) becomes unacceptably small while the value of \( M^2 \) increases. Thus we focus on the interval \( 0.8 \text{ GeV}^2 \leq M^2 \leq 1.4 \text{ GeV}^2 \), and find

\[
m = 1.47 \text{ GeV} \quad \lambda^2 = 2.12 \text{ GeV}^6; \quad W^2 = 2.84 \text{ GeV}^2. \tag{75}
\]

The LHS and RHS of the SR are shown in Fig. 4.

In our next step the instanton field is included in the propagators of all three quarks. This means that we keep all terms in Eq. (70) except the contribution \( \tilde{A}'_4 \). The accuracy of the fit of the LHS and RHS of Eqs. (18) and (19) is not as good as in the previous case-see Table 3. However the value of \( \chi^2 \) per point does not vary much with the variation of the interval of the values of \( M^2 \). Again the ration \( r_q \) becomes too small at large value of \( M^2 \)-see Table 4, and we work in the interval \( 0.8 \text{ GeV}^2 \leq M^2 \leq 1.4 \text{ GeV}^2 \). Here we find

\[
m = 1.13 \text{ GeV} \quad \lambda^2 = 1.25 \text{ GeV}^6; \quad W^2 = 2.46 \text{ GeV}^2. \tag{76}
\]

Note that fixing the value \( m = 0.94 \text{ GeV} \) we find \( \lambda^2 = 1.00 \text{ GeV}^6 \) and \( W^2 = 2.54 \text{ GeV}^2 \) from Eqs. (18) and (19). The value \( \chi^2/N = 0.90 \) becomes twice larger.

Now we include the correlations between the quarks moving in the instanton medium. This means that we include the term \( \tilde{A}'_4 \) in Eq. (70). As well as in previous cases we can find several intervals with \( \chi^2/N \approx 0.5 \). However, the satisfactory pole-to continuum ratio can be obtained only in the interval \( 0.8 \text{ GeV}^2 \leq M^2 \leq 1.4 \text{ GeV}^2 \). We find

\[
m = 0.82 \text{ GeV} \quad \lambda^2 = 0.80 \text{ GeV}^6; \quad W^2 = 2.05 \text{ GeV}^2, \tag{77}
\]

with \( \chi^2/N = 0.54 \). The LHS and RHS of the SR are shown in Fig. 5. The pole-to-continuum ratio is presented in Table 5. Note that fixing the value \( m = 0.94 \text{ GeV} \) we find \( \lambda^2 = 0.99 \text{ GeV}^6 \) and \( W^2 = 2.20 \text{ GeV}^2 \) from Eqs. (18) and (19) with a larger value \( \chi^2/N = 0.72 \).

5 Discussion and Outlook

Now we describe the uncertainties of our approach. A part of contribution to the chirality-violating structure comes from the integration over large moments \( p \sim 2.5 \text{ GeV} \), where the validity of Eq. (8) is obscure. Thus, the role of inhomogeneity of the scalar quark expectation value can be overestimated.

The propagator (8) is obtained in the lowest nonvanishing order of expansion in powers of \( N_c \). Thus there may be additional contributions of the order \( 1/N_c = 1/3 \).

There is an interaction (repulsion) between the instantons at small distances, which are dominate in the nucleon polarization operator. Thus we deal mainly with the only one instanton. Therefore the factorization hypothesis is not well justified. It may be even more realistic to consider the QCD sum rules in the presence of the one instanton only. Next, actually the size of the instanton \( \rho \) is not fixed. There is some distribution over \( \rho \). Part of the quark condensate caused by a large size instantons may have a rather weak \( x \) dependence and looks as a the homogeneous ”mean field” indeed, while another part will corresponds to a smaller size spots. Finally there should be some contribution from the higher (not just the ‘zero’) modes of the quark wave function in the instanton field. This will slightly modify the behavior of the quark propagator, in particular, the asymptotics of the constituent quark mass \( m(p) \) at a large \( p^2 \).

The role of all these effects should be considered in a future calculations.
6 Summary

We calculated the polarization operator for the nucleon current in instanton medium. The propagation of quarks was described by the single-particle Green functions determined by Eq. (8). We included also the simplest correlation between the quarks, taking into account the configuration in which the two different quarks exchange gluons with the gluon condensate. We employed the usual anzac of the QCD SR, in which the nucleon pole is treated exactly, while the higher lying states are approximated by continuum.

In the limit of very large Borel masses $M^2 \to \infty$ the contributions to the LHS of the SR became equal to those of the OPE sum rules. However, at $M^2 \sim 1 \text{ GeV}^2$ the LHS became a more complicated function of $M^2$ than it was in the case of the OPE sum rules. All contributions are smaller than their OPE analogs. Also, the relative role of the four-quark counterpart is now smaller than that of the four-quark contribution in the OPE case. By successive inclusion of the instanton field into the quark propagators we demonstrated that the value of the nucleon mass varyate around its physical value. The value of the continuum threshold is close to that obtained in the OPE case, while the value of the residue is more than twice smaller.

Anyway, we expect that further calculations with more rigorous treatment of interactions between the quarks which compose the polarization operator will clarify the situation.

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Table 1: Solutions of the sum rules equations with one quark in the instanton field.

| $M^2$, GeV$^2$ | $m$, GeV | $\lambda^2$, GeV$^6$ | $W^2$, GeV$^2$ | $\chi^2_N$ |
|----------------|----------|-----------------|-----------------|-----------|
| 0.8-1.4        | 1.47     | 2.11            | 2.84            | 0.28·10$^{-2}$ |
| 1.4-2.2        | 1.51     | 2.43            | 2.97            | 0.59·10$^{-2}$ |
| 1.0-1.6        | 1.48     | 2.19            | 2.87            | 0.62·10$^{-2}$ |

Table 2: Pole-to-continuum ratio $r(M^2)$ for solutions of the sum rules equations with one quark in the instanton field.

| $M^2$, GeV$^2$ | $r_q(M^2)$ | $r_I(M^2)$ |
|----------------|-------------|-------------|
| 0.8            | 0.94        | 4.48        |
| 1.0            | 0.55        | 2.46        |
| 1.2            | 0.36        | 1.57        |
| 1.4            | 0.25        | 1.10        |

Table 3: Solutions of the sum rules equations with all quarks in the instanton field.

| $M^2$, GeV$^2$ | $m$, GeV | $\lambda^2$, GeV$^6$ | $W^2$, GeV$^2$ | $\chi^2_N$ |
|----------------|----------|-----------------|----------------|-----------|
| 0.8-1.4        | 1.13     | 1.26            | 2.46            | 0.45      |
| 1.4-2.0        | 0.95     | 0.96            | 2.09            | 0.18      |
| 1.0-1.6        | 1.03     | 0.88            | 2.05            | 0.26      |

Table 4: Pole-to-continuum ratio $r(M^2)$ for solutions of the sum rules equations with three quarks in the instanton field.

| $M^2$, GeV$^2$ | $r_q(M^2)$ | $r_I(M^2)$ |
|----------------|-------------|-------------|
| 1.4            | 0.26        | 0.79        |
| 1.6            | 0.18        | 0.56        |
| 1.8            | 0.13        | 0.41        |
| 2.0            | 0.10        | 0.32        |
Table 5: Pole-to-continuum ratio $r(M^2)$ for solutions of the sum rules equations with three quarks in the instanton field and the correlations included.

| $M^2$, GeV$^2$ | $r_q(M^2)$ | $r_I(M^2)$ |
|----------------|-------------|-------------|
| 0.8            | 1.34        | 2.90        |
| 1.0            | 0.64        | 1.33        |
| 1.2            | 0.36        | 0.74        |
| 1.4            | 0.23        | 0.47        |
Figure captions

Fig. 1. The set of the diagrams for the lowest OPE terms of the nucleon sum rules. Wavy lines are for the nucleon current, solid lines stand for the quarks, dashed lines denote the gluons. The circles stand for the quark and gluon condensates.

Fig. 2. The set of the diagrams for the quarks in the instanton medium. Large blobs on the quark lines mean that the quarks move in the instanton medium. Other notations are the same as in Fig. 1.

Fig. 3. The functions $F_0(M^2)$, $F_3(M^2)$ and $F_6(M^2)$ -see Eqs. (45), (53), (64).

Fig. 4. The consistency of the LHS and RHS of the sum rules (given in powers of GeV) for the case when instanton field is included in the propagator of a single quark. Solid and dashed curves are for Eqs. (18) and (19) correspondingly.

Fig. 5. The consistency of the LHS and RHS of the sum rules (given in powers of GeV) for the case when instanton field is included in the propagators of all quarks and the correlations caused by the gluon condensate are taken into account. Solid and dashed curves are for Eqs. (18) and (19) correspondingly.
Figure 1:

Figure 2:
Figure 3:

Figure 4:
