Probing hybrid modified gravity by stellar motion around Galactic Centre

D. Borka, S. Capozziello, P. Jovanović, and V. Borka Jovanović

Abstract. We consider possible signatures for the so called hybrid gravity within the Galactic Central Parsec. This modified theory of gravity consists of a superposition of the metric Einstein-Hilbert Lagrangian with an $f(R)$ term constructed à la Palatini and can be easily reduced to an equivalent scalar-tensor theory. The present analysis is based on the S2 star orbital precession around the massive compact dark object at the Galactic Centre where the simulated orbits in hybrid modified gravity are compared with astronomical observations. These simulations result with strong constraints on the range of hybrid gravity interaction parameter $\phi_0$ and show that its most probable value, in the case of S2 star, is around $-0.0009$ to $-0.0002$. At the same time, we are also able to obtain reliable constrains on the effective mass parameter $m_\phi$ of hybrid modified gravity. Its most probable value, in the case of S2 star, is around $-0.0034$ to $-0.0025$. Furthermore, the hybrid gravity potential induces precession of S2 star orbit in the same direction as General Relativity. In previous papers, we considered other types of extended gravities, like metric power law $f(R) \propto R^n$ gravity, inducing Yukawa and Sanders-like gravitational potentials, but it seems that hybrid gravity is the best among these models to explain different gravitational phenomena at different astronomical scales.
Contents

1 Introduction 1
2 Basic theory and fitting procedure 2
3 Results: simulations vs observations 5
4 Conclusions 6

1 Introduction

The existence of different anomalous astrophysical and cosmological phenomena like the cosmic acceleration, the dynamics of galaxies and gas in clusters of galaxies, the galactic rotation curves, etc. recently boosted the growth of several long-range modifications of the usual laws of gravitation. These mentioned phenomena did not find satisfactory explanations in terms of the standard Newton-Einstein gravitational physics, unless exotic and still undetected forms of matter-energy are postulated: dark matter and dark energy. A recent approach is to try to explain these phenomena without using new material ingredients like dark matter and dark energy, but using well-motivated generalization and extensions of General Relativity. Several alternative gravity theories have been proposed (see e.g. [1–7] for reviews), such as: MOND [8], scalar-tensor [9–12], conformal [13, 14], Yukawa-like corrected gravity theories [15–18], theories of "massive gravity" [19–25]. Alternative approaches to Newtonian gravity in the framework of the weak field limit [26] of fourth-order gravity theory have been proposed and constraints on these theories have been discussed [27–38].

The philosophy is to search for alternative form of gravity, i.e. of the Einstein-Hilbert theory, so that such modifications could naturally explain some astrophysical and cosmological phenomena without invoking the presence of new material ingredients that, at the present state of the art, seem hard to be detected. Besides, this extended approach can be connected to effective theories that emerge both from the quantization on curved spacetimes and from several unification schemes [2–4].

The simplest extension of the Einstein-Hilbert action is based on straightforward generalizations of the Einstein theory where the gravitational action (the Einstein-Hilbert action) is assumed to be linear in the Ricci curvature scalar $R$. If this action consists in modifying the geometric part considering a generic function $f(R)$, we get so called $f(R)$ gravity which was firstly proposed in 1970 by Buchdahl [39]. Generally, the most serious problem of $f(R)$ theories is that these theories cannot easily pass the standard Solar System tests [40, 41]. However, there exists some classes of them that can solve this problem [42]. It can be shown that $f(R)$ theories, in principle, could explain the evolution of the Universe, from a matter dominated early epoch up to the present, late-time self accelerating phase. Several debates are open in this perspective [43–46] but the crucial point is that suitable self-consistent model can be achieved. $f(R)$ theories have also been studied in the Palatini approach, where the metric and the connection are regarded as independent fields [47]. Metric and Palatini approaches are certainly equivalent in the context of General Relativity, i.e., in the case of the linear Einstein-Hilbert action. This is not so for extended gravities. The Palatini variational approach leads to second order differential field equations, while the resulting field equations
in the metric approach are fourth order coupled differential equations. These differences also extend to the observational aspects.

A novel approach, that consists of adding to the metric Einstein-Hilbert Lagrangian an \( f(R) \) term constructed within the framework of the Palatini formalism, was recently proposed [48, 49]. For a brief review of the hybrid metric-Palatini theory, we refer the reader to [50]. The hybrid metric-Palatini theory opens up new possibilities to approach, in the same theoretical framework, the problems of both dark energy and dark matter. To probe this issue, star dynamics around the Galactic Centre could be a useful test bed.

In particular, S-stars are the young bright stars which move around the centre of our Galaxy [51–56] where the compact radio source Sgr A* is located. For more details about S2 star see references [56, 57]. There are some observational indications that the orbits of some of them, like S2, could deviate from the Keplerian case [53, 58], but the current astrometric limit is not sufficient to unambiguously confirm such a claim [36, 59].

Here we study a possible application of hybrid modified gravity within Galactic Central Parsec, in order to explain the observed precession of orbits of S-stars. This paper is a continuation of previous studies where we considered different extended gravities, such as power law \( f(R) \) gravity [29, 38], \( f(R, \phi) \) gravity implying Yukawa and Sanders-like gravitational potentials in the weak field limit [36, 37]. Results obtained using hybrid gravity point out that, very likely, such a theory is the best candidate among those considered to explain (within the same picture) different gravitational phenomena at different astronomical scales.

The present paper is organized as follows: in Sec. 2, we sketch the theory and describe our simulations of stellar orbits in the gravitational potential derived in the weak field limit of hybrid gravity; the fitting procedure to simulate orbits with respect to astrometric observations of S2 star is described in Sec. 3; results are presented in Sec. 4. Conclusions are drawn in Sec. 5.

## 2 Basic theory and fitting procedure

In this Section, we present the basic formalism for the hybrid metric-Palatini gravitational theory within the equivalent scalar-tensor representation (we refer the reader to [49, 50, 60, 61] for more details). The \( f(R) \) theories are the special limits of the one-parameter class of theories where the scalar field depends solely on the stress energy trace \( T \) (Palatini version) or solely on the Ricci curvature \( R \) (metric version). Here, we consider a one-parameter class of scalar-tensor theories where the scalar field is given as an algebraic function of the trace of the matter fields and the scalar curvature [60]:

\[
S = \int d^Dx \sqrt{-g} \left[ \frac{1}{2} \phi R - \frac{D - 1}{2(D - 2)} \frac{\Omega_A}{\phi} (\partial \phi)^2 - V(\phi) \right]. \tag{2.1}
\]

The theories can be parameterized by the constant \( \Omega_A \). The limiting values \( \Omega_A = 0 \) and \( \Omega_A \to \infty \) correspond to scalar-tensor theories with the Brans-Dicke parameter \( \omega = -(D - 1)/(D - 2) \) and \( \omega = 0 \). These limits reduce to \( f(R) \) gravity in the Palatini and the metric formalism, respectively. For any finite value of \( \Omega_A \), its value depends both on the matter and curvature. In the limit \( \Omega_A \to \infty \) the propagating mode is given solely by the curvature, \( \phi(R, T) \to \phi(R) \), and in the limit \( \Omega_A \to 0 \) solely the matter fields \( \phi(R, T) \to \phi(T) \). In the general case the field equations are fourth order both in the matter and in the metric derivatives.
More specifically, the intermediate theory with $\Omega_A = 1$, corresponds to the hybrid metric-Palatini gravity theory proposed in [48, 49], where the action is given by

$$ S = \int d^Dx \sqrt{-g} \left[ R + f(R) + 2\kappa^2 L_m \right]. \quad (2.2) $$

For $D = 4$, it can be easily recast into a scalar-tensor representation [48, 61] given by the action

$$ S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left[ (1 + \phi)R + \frac{3}{2\phi} \nabla_{\mu}\phi\nabla^{\mu}\phi - V(\phi) \right] + S_m, \quad (2.3) $$

where $S_m$ is the matter action, $\kappa^2 = 8\pi G/c^2$, and $V(\phi)$ is the scalar field potential given by the specific form of $f(R)$.

In the weak field limit and far from the sources, the scalar field behaves as $\phi(r) \approx \phi_0 + (2G\phi_0 M/3r)e^{-m_\phi r}$; the effective mass is defined as $m_\phi^2 \equiv (2V - \phi_0 - \phi(1 + \phi)V_{\phi\phi})/3|_{\phi=\phi_0}$, where $\phi_0$ is the amplitude of the background value. The metric perturbations yield

$$ h^{(2)}_{00}(r) = \frac{2G_{\text{eff}} M}{r} + \frac{V_0}{1 + \phi_0} r^2, \quad h^{(2)}_{ij}(r) = \left( \frac{2\gamma G_{\text{eff}} M}{r} - \frac{V_0}{1 + \phi_0} r^2 \right) \delta_{ij}, \quad (2.4) $$

where the effective Newton constant $G_{\text{eff}}$ and the post-Newtonian parameter $\gamma$ are defined as

$$ G_{\text{eff}} \equiv \frac{G}{1 + \phi_0} \left[ 1 - (\phi_0/3) e^{-m_\phi r} \right], \quad \gamma \equiv \frac{1 + (\phi_0/3) e^{-m_\phi r}}{1 - (\phi_0/3) e^{-m_\phi r}}. \quad (2.5) $$

The coupling of the scalar field to the local system depends on $\phi_0$. If $\phi_0 \ll 1$, then $G_{\text{eff}} \approx G$ and $\gamma \approx 1$ regardless of the value of $m_\phi^2$. This is contrast with the result obtained in the metric version of $f(R)$ theories. For sufficiently small $\phi_0$, this modified theory allows to pass the Solar System tests, even if the scalar field is very light [50]. Modified gravitational potential has the form:

$$ \Phi(r) = -\frac{G}{1 + \phi_0} \left[ 1 - (\phi_0/3) e^{-m_\phi r} \right] M/r. \quad (2.6) $$

We use eq. (2.6) to simulate orbits of S2 star in the hybrid modified gravity potential and then we compare the obtained results with the set of S2 star observations obtained by the New Technology Telescope/Very Large Telescope (NTT/VLT). The simulated orbits of S2 star are obtained by numerical integration of equations of motion where the hybrid gravitational potential is adopted, i.e.

$$ \dot{r} = v, \quad \mu \ddot{r} = -\nabla \Phi(r), \quad (2.7) $$

where $\mu$ is the reduced mass in the two-body problem. In our calculations, we assume that the mass of central object is $M = 4.3 \times 10^6 M_\odot$ and that the distance to the S2 star is $d_\ast = 8.3$ kpc [53]. We fit the orbits of S2 star to the NTT/VLT astrometric observations for different combinations of a priori given values of $\phi_0$ and $m_\phi$ parameters of the hybrid modified gravity. Each simulated orbit is defined by the following four initial conditions: two components of the initial position and two components of the initial velocity in orbital plane at the epoch of the first observation. We perform fitting procedure using LMDIF1 routine from MINPACK-1 Fortran 77 library which solves the nonlinear least squares problems by a modification of Marquardt-Levenberg algorithm [62]. For each combination of $\phi_0$ and $m_\phi$, we obtain the best fit initial conditions corresponding to a simulated orbit with the lowest discrepancy with respect to the observed one. A detailed description of simulation and fitting procedure is given in the paper [36].
Figure 1. (Color online) The precession per orbital period for $\phi_0$ in the range $[-0.0009,-0.0002]$ and $m_{\phi}$ in $[-0.0034,-0.0025]$ (left panel), and $\phi_0$ in the range $[-0.0004,-0.0002]$ and $m_{\phi}$ in $[-0.0029,-0.0027]$ (right panel) in the case of hybrid modified gravity potential. With a decreasing value of angle of precession colors are darker.
Figure 2. (Color online) The maps of the reduced $\chi^2$ over the $\phi_0 - m_\phi$ parameter space for all simulated orbits of S2 star which give at least the same or better fits than the Keplerian orbits. With a decreasing value of $\chi^2$ (better fit) colors in grey scale are darker. A few contours are presented for specific values of reduced $\chi^2$ given in the figure’s legend.

Figure 3. (Color online) The same as in Fig. 2, but for the zoomed range of parameters.

3 Results: simulations vs observations

We simulated orbits of S2 star around the central object considering both the hybrid gravitational potential and and the Newtonian potential. Our analysis shows that the hybrid modified gravity potential induces the precession of S2 star orbit in the same direction of General Relativity. We used these simulated orbits to fit the observed orbits of S2 star. The best fit (according to NTT/VLT data) is obtained for the $\phi_0$ from between -0.0009 and -0.0002, and for the $m_\phi$ between -0.0034 and -0.0025. We calculate orbital precession in hybrid modified gravity potential and results are reported in Fig. 1 as a function of $\phi_0$ and $m_\phi$. Assuming that the hybrid potential does not differ significantly from Newtonian potential, we derive the perturbed potential as

$$V(r) = \Phi(r) - \Phi_N(r) ; \Phi_N(r) = -\frac{GM}{r}.$$  (3.1)

The obtained perturbing potential is of the form:
\[ V(r) = \frac{G}{1 + \phi_0} \left[ 1 + \frac{1}{3} e^{-m_\phi r} \right] M\phi_0/r. \]  

(3.2)

and it can be used for calculating the precession angle according to Eq. (30) in Ref. [63]:

\[ \Delta\theta = -\frac{2L}{GMc^2} \int_{-1}^{1} \frac{z \cdot dz \cdot dV(z)}{\sqrt{1 - z^2}}. \]  

(3.3)

where \( r \) is related to \( z \) via: \( r = \frac{L}{1 + ez} \). By differentiating the perturbing potential \( V(z) \) and substituting its derivative and \((L = a (1 - e^2))\) in the above Eq. (3.3), and taking the same values for orbital elements of S2 star like in Ref. [29] we obtain numerically, for \( \phi_0 = -0.00033 \) and \( m_\phi = -0.0028 \), that the precession per orbital period is 3°.26.

Graphical representation of precession per orbital period for \( \phi_0 \) in the range \([-0.0009, -0.0002]\) and \( m_\phi \) in \([-0.0034, -0.0025]\) is given in the left panel of Fig. 1. As one can see, the pericenter advance (like in General Relativity) is obtained. The precession per orbital period for \( \phi_0 \) in the range \([-0.0004, -0.0002]\) and \( m_\phi \) in \([-0.0029, -0.0027]\) is given in the right panel of Fig. 1.

General Relativity predicts that the pericenter of S2 star should advance by 0°.18 per orbital revolution [54] which is much less than the value of precession per orbital period in hybrid modified gravity potential, and the direction of the precession is the same.

Figs. 2 and 3 present the maps of the reduced \( \chi^2 \) over the \( \phi_0 - m_\phi \) parameter space for all simulated orbits of S2 star which give at least the same or better fits than the Keplerian orbits. These maps are obtained by the same fitting procedure as before. As it can be seen from Figs. 2 and 3, the most probable value for the parameter \( \phi_0 \) in the case of NTT/VLT observations of S2 star is between -0.0009 and -0.0002 and for the parameter \( m_\phi \) is between -0.0034 and -0.0025 (see the darkest regions in Figs. 3). In other words, we obtain reliable constrains on the parameters \( \phi_0 \) and \( m_\phi \) of hybrid modified gravity. The absolute minimum of the reduced \( \chi^2 \) (\( \chi^2 = 1.503 \)) is obtained for \( \phi_0 = -0.00033 \) and \( m_\phi = -0.0028 \), respectively.

4 Conclusions

In this paper, the orbit of S2 star around the galactic Centre has been investigated in the framework of the hybrid modified gravity. Using the observed positions of S2 star, we constrained the parameters of hybrid modified gravity. Our simulation resulted with:

1. strong constraints on the range of hybrid modified gravity interaction \( \phi_0 \), showing that its most probable value, in the case of S2 star, is between -0.0009 and -0.0002;

2. reliable constrains on the parameter \( m_\phi \) of hybrid modified gravity. Its most probable value, in the case of S2 star, is between -0.0034 and -0.0025;

3. precession of S2 star orbit in the hybrid modified gravity potential with the same direction as in General Relativity, but with a value which is much bigger compared to GR.
In conclusion, the comparison of the observed orbits of S2 star and theoretical calculations performed by the hybrid modified gravity model can provide a powerful method for the observational test of the theory, and for observationally discriminating among the different modified gravity theoretical models. It seems that hybrid modified gravity potential is sufficient in addressing completely the problem of dark matter at galactic scales [50], and it gives indications that alternative theories of gravity could be viable in describing galactic dynamics.

In other words, orbital solutions derived from such a potential are in good agreement with the reduced $\chi^2$ deduced for Keplerian orbits. This fact allows to fix the range of variation for $\phi_0$ and $m_\phi$, the two parameters characterizing the hybrid modified gravity potential. The percession of S2 star orbit obtained for the best fit parameter values ($\phi_0$ from -0.0009 to -0.0002 and $m_\phi$ from -0.0034 to -0.0025) has the positive direction, as in General Relativity, but we obtained much larger orbital percession of S2 star in hybrid modified gravity compared to prediction of General Relativity.

Using hybrid modified type of gravity we get constrains on both parameter $\phi_0$ and $m_\phi$. If we take into account that hybrid gravity also gets good results at other astronomical and cosmological scales [50], we can conclude that hybrid type of gravity is probably the best candidate among the other considered types of gravities such as e.g. $R^n$ [29, 38], Yukawa-like [36] and Sanders-like [37] to explain gravitational phenomena at different astronomical scales.

Acknowledgments

D.B., P.J. and V.B.J. wish to acknowledge the support by the Ministry of Education, Science and Technological Development of the Republic of Serbia through the project 176003. S.C. acknowledges the support of INFN (iniziative specifiche QGSKY and TEONGRAV). D.B. would like to thank to dr A.F. Zakharov for many usefull discussions.

References

[1] E. Fischbach and C. L. Talmadge, The Search for Non-Newtonian Gravity, 305p., Heidelberg–New York, Springer (1999).
[2] T. Clifton, P. G. Ferreira, A. Padilla, and C. Skordis, Modified gravity and cosmology, Physics Reports 513 (2012) 1.
[3] S. Capozziello, and M. de Laurentis, Extended Theories of Gravity, Physics Reports 509 (2011) 167.
[4] S. Capozziello, V. Faraoni, Beyond Einstein Gravity: A Survey of Gravitational Theories for Cosmology and Astrophysics, Fundamental Theories of Physics 170, Springer (2011).
[5] S. Kopeikin, I. Vlasov, Parametrized post-Newtonian theory of reference frames, multipolar expansions and equations of motion in the N-body problem, Physics Reports 400 (2004) 209.
[6] T. P. Sotiriou and V. Faraoni, f(R) theories of gravity, Rev. Mod. Phys. 82 (2010) 451.
[7] S. Nojiri and S. D. Odintsov, Unified cosmic history in modified gravity: from f(R) theory to Lorentz non-invariant models, Physics Reports 505 (2011) 59.
[8] M. Milgrom, A modification of the Newtonian dynamics as a possible alternative to the hidden mass hypothesis, Astrophys. J. 270 (1983) 365.
[9] C. Brans and H. Dicke, Mach’s principle and a relativistic theory of gravitation, Phys. Rev. 124 (1961) 925.
[10] J.W. Moffat, *Gravitational theory, galaxy rotation curves and cosmology without dark matter*, JCAP 05 (2005) 22.

[11] J.W. Moffat, *Scalar-tensor-vector gravity theory*, JCAP 03 (2006) 004.

[12] T. Biswas, E. Gerwick, T. Koivisto, and A. Mazumdar, *Towards Singularity and Ghost-Free Theories of Gravity*, Phys. Rev. Lett. 108 (2012) 031101.

[13] D. Behnke, D. B. Blaschke, V. N. Pervushin, and D. Proskurin, *Description of supernova data in conformal cosmology without cosmological constant*, Phys. Lett. B 530 (2002) 20.

[14] B. M. Barbashov, V. N. Pervushin, A. F. Zakharov, and V. A. Zinchuk, *Hamiltonian cosmological perturbation theory*, Phys. Lett. B 633 (2006) 458.

[15] E. Fischbach and C. Talmadge, *Six years of the fifth force*, Nature 356 (1992) 207.

[16] V. F. Cardone and S. Capozziello, *Systematic biases on galaxy haloes parameters from Yukawa-like gravitational potentials*, Mon. Not. R. Astron. Soc. 414 (2011) 1301.

[17] A. Stabile and S. Capozziello, *Galaxy rotation curves in f(R, φ) gravity*, Phys.Rev. D 87 (2013) 6, 064002.

[18] A. Stabile, *The most general fourth order theory of Gravity at low energy*, Phys. Rev. D 82, (2010) 124026.

[19] V. A. Rubakov and P. G. Tinyakov, *Infrared-modified gravities and massive gravitons*, Phys. Usp. 51 (2008) 759.

[20] E. Babichev, C. Deffayet, and R. Ziou, *Recovery of general relativity in massive gravity via the Vainshtein mechanism*, Phys. Rev. D 82 (2010) 104008.

[21] J. B. Pitts and W. C. Schieve, *Universally coupled massive gravity*, Theor. Math. Phys. 151(2) (2007) 700.

[22] E. Babichev, C. Deffayet, and R. Ziou, *Recovering General Relativity from Massive Gravity*, Phys. Rev. Lett. 103 (2009) 201102.

[23] C. de Rham, G. Gabadadze, and A. J. Tolley, *Resummation of massive gravity*, Phys. Rev. Lett. 106 (2011) 231101.

[24] G. Yun-Gui, *Cosmology in Massive Gravity*, Commun. Theor. Phys. 59 (2013) 319.

[25] E. Babichev and A. Fabbri, *Instability of black holes in massive gravity*, Class. Quantum Grav. 30 (2013) 152001.

[26] T. Clifton and J. D. Barrow, *The power of general relativity*, Phys. Rev. D 72 (2005) 103005.

[27] S. Capozziello, V. F. Cardone, and A. Troisi, *Gravitational lensing in fourth order gravity*, Phys. Rev. D 73 (2006) 104019.

[28] S. Capozziello, V. F. Cardone, and A. Troisi, *Low surface brightness galaxy rotation curves in the low energy limit of R^n gravity: no need for dark matter?*, Mon. Not. R. Astron. Soc. 375 (2007) 1423.

[29] D. Borka, P. Jovanović, V. Borka Jovanović, and A. F. Zakharov, *Constraints on R^n gravity from precession of orbits of S2-like stars*, Phys. Rev. D 85 (2012) 124004.

[30] C. Frigerio Martins and P. Salucci, *Analysis of rotation curves in the framework of R^n gravity*, Mon. Not. R. Astron. Soc. 381 (2007) 1103.

[31] A. F. Zakharov, A. A. Nucita, F. De Paolis, and G. Ingrosso, *Solar system constraints on R^n gravity*, Phys. Rev. D 74 (2006) 107101.

[32] A. F. Zakharov, A. A. Nucita, F. De Paolis, and G. Ingrosso, *Apoastron shift constraints on dark matter distribution at the Galactic Center*, Phys. Rev. D 76 (2007) 062001.
[33] A. A. Nucita, F. De Paolis, G. Ingrosso, A. Qadir, and A. F. Zakharov, Sgr A*: A laboratory to measure the central black hole and stellar cluster parameters, Publ. Astron. Soc. Pac. 119 (2007) 349.

[34] S. Capozziello, A. Stabile, and A. Troisi, A general solution in the Newtonian limit of f(R) - gravity, Mod. Phys. Lett. A 24 (2009) 659.

[35] L. Iorio, Galactic orbital motions in the dark matter, modified Newtonian dynamics and modified gravity scenarios, Mon. Not. R. Astron. Soc. 401 (2010) 2012.

[36] D. Borka, P. Jovanović, V. Borka Jovanović, and A. F. Zakharov, Constraining the range of Yukawa gravity interaction from S2 star orbits, JCAP 11 (2013) 050.

[37] S. Capozziello, D. Borka, P. Jovanović and V. Borka Jovanović, Constraining Extended Gravity Models by S2 star orbits around the Galactic Centre, Phys. Rev. D 90 (2014) 044052.

[38] A. F. Zakharov, D. Borka, V. Borka Jovanović and P. Jovanović, Constraints on $R^n$ gravity from precession of orbits of S2-like stars: case of bulk distribution of mass Adv. Sp. Res. 54 (2014) 1108.

[39] H. A. Buchdahl, Non-linear Lagrangians and cosmological theory, Mon. Not. R. Astron. Soc. 150 (1970) 1.

[40] T. Chiba, 1/R gravity and scalar-tensor gravity, Phys. Lett. B 575 (2003) 1.

[41] G. J. Olmo, The Gravity Lagrangian According to Solar System Experiments, Phys. Rev. Lett. 95 (2005) 261102.

[42] S. Nojiri and S. D. Odintsov, Modified gravity with negative and positive powers of curvature: Unification of inflation and cosmic acceleration, Phys. Rev. D 68 (2003) 123512.

[43] L. Amendola, R. Gannouji, D. Polarski and S. Tsujikawa, Conditions for the cosmological viability of f(R) dark energy models, Phys. Rev. D 75 (2007) 083504.

[44] N. Goheer, J. Larena and P. Dunsby, Power-law cosmic expansion in f(R) gravity models, Phys. Rev. D 80 (2009) 061301.

[45] L. Amendola, D. Polarski and S. Tsujikawa, Are f(R) Dark Energy Models Cosmologically Viable?, Phys. Rev. Lett. 98 (2007) 131302.

[46] S. Capozziello, S. Nojiri, S.D. Odintsov, A. Troisi, Cosmological viability of f(R)-gravity as an ideal fluid and its compatibility with a matter dominated phase Phys.Lett. B 639 (2006) 135.

[47] G. J. Olmo, Palatini Approach to Modified Gravity: f(R) Theories and Beyond , Int. J. Mod. Phys. D 20 (2011) 413

[48] T. Harko, T. S. Koivisto, F. S. N. Lobo and G.J. Olmo, Metric-Palatini gravity unifying local constraints and late-time cosmic acceleration, Phys. Rev. D 85 (2012) 084016.

[49] S. Capozziello, T. Harko, T. S. Koivisto, F. S. N. Lobo and G. J. Olmo, Cosmology of hybrid metric-Palatini f(X)-gravity, JCAP 1304 (2013) 011.

[50] S. Capozziello, T. Harko, F. S. N. Lobo, and G. J. Olmo, Hybrid modified gravity unifying local tests, galactic dynamics and late-time cosmic acceleration, Int. J. Mod. Phys. D 22 (2013) 1342006.

[51] A. M. Ghez, M. Morris, E. E. Becklin, A. Tanner, and T. Kremenek, The accelerations of stars orbiting the Milky Way’s central black hole, Nature 407 (2000) 349.

[52] A. M. Ghez, S. Salim, N. N. Weinberg, J. R. Lu, T. Do, J. K. Dunn, K. Matthews, M. R. Morris, S. Yelda, E. E. Becklin, T. Kremenek, M. Milosavljević, and J. Naiman, Measuring distance and properties of the Milky Way’s central supermassive black hole with stellar orbits, Astrophys. J. 689 (2008) 1044.
[53] S. Gillessen, F. Eisenhauer, T. K. Fritz, H. Bartko, K. Dodds-Eden, O. Pfuhl, T. Ott, and R. Genzel, The orbit of the star S2 around SGR A* from very large telescope and Keck data, Astrophys. J. 707 (2009) L114.

[54] S. Gillessen, F. Eisenhauer, S. Trippe, T. Alexander, R. Genzel, F. Martins, and T. Ott, Monitoring stellar orbits around the massive black hole in the Galactic Center, Astrophys. J. 692 (2009) 1075.

[55] R. Schödel, T. Ott, R. Genzel, et al., Closest star seen orbiting the supermassive black hole at the Centre of the Milky Way, Nature 419 (2002) 694.

[56] R. Genzel, F. Eisenhauer, and S. Gillessen, The Galactic Center massive black hole and nuclear star cluster, Rev. Mod. Phys. 82 (2010) 3121.

[57] S. Gillessen, R. Genzel, T. K. Fritz et al., A gas cloud on its way towards the supermassive black hole at the Galactic Centre, Nature 481 (2012) 51.

[58] L. Meyer, A. M. Ghez, R. Schödel, et al., The Shortest-Known-Period Star Orbiting Our Galaxy’s Supermassive Black Hole, Science 338 (2012) 84.

[59] T. Fritz, S. Gillessen, S. Trippe, T. Ott, H. Bartko, O. Pfuhl, K. Dodds-Eden, R. Davies, F. Eisenhauer, R. Genzel, What is limiting near-infrared astrometry in the Galactic Centre?, Mon. Not. R. Astron. Soc. 401 (2010) 1177.

[60] T. S. Koivisto, Cosmology of modified (but second order) gravity, AIP Conf. Proc. 1206 (2010) 79.

[61] S. Capozziello, T. Harko, T. S. Koivisto, F. S. N. Lobo and G. J. Olmo, Wormholes supported by hybrid metric-Palatini gravity, Phys. Rev. D 86 (2012) 127504.

[62] J. J. Moré, B. S. Garbow, and K. E. Hillstrom, User Guide for MINPACK-1, Argonne National Laboratory Report ANL-80-74, Argonne, Ill. (1980).

[63] G. S. Adkins and J. McDonnell, Orbital precession due to central-force perturbations, Phys. Rev. D 75 (2007) 082001.