CANONICAL AND FUNCTIONAL SCHRÖDINGER QUANTIZATION OF TWO–DIMENSIONAL DILATON GRAVITY

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Abstract

We discuss the relation between canonical and Schrödinger quantization of the CGHS model. We also discuss the situation when background charges are added to cancel the Virasoro anomaly. New physical states are found when the square of the background charges vanishes.
The quantization of reparametrization invariant theories is an open problem with many unanswered questions. Even in two space–time dimensions many delicate questions remain without a clear answer. In the last few years much effort has been spent in the study of several two–dimensional dilaton gravity models as prototypes of reparametrization invariant theories. In particular the string inspired CGHS (Callan, Giddings, Harvey and Strominger) model \[1\] was intensively investigated.

The CHGS model consists of a particular coupling of two–dimensional gravity and a dilaton field. It is described by the action

\[
S = \int d^2x \sqrt{-g} e^{-2\phi} (R + 4g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \Lambda),
\]

where \(R\) is the scalar curvature, \(\phi\) is the dilaton and \(\Lambda\) the cosmological constant. Matter fields can also be added but we will not consider them. The interest in the CGHS model stem from the fact that the model allows black hole formation and Hawking radiation at the semi–classical level \[2\]. From a more formal point of view the CGHS model has also interesting properties. It can be reformulated as a topological field theory of the BF type with the gauge group being the extended Poincaré group \[3\]. Its supersymmetric version is also known \[4\]. It is possible to go beyond the semi–classical level and to quantize the model without any approximation. Most surprisingly, the reparametrization constraints of the CGHS model can be set in a quadratic form after a conformal transformation followed by a canonical transformation \[3, 4\]. In this form there are two scalar fields \(r^a(x), a = 1, 2\) corresponding to combinations of the gravity and dilaton fields with a vanishing Hamiltonian which is characteristic of reparametrization invariant theories. The first order form of the Lagrangian in the absence of matter and with the cosmological constant \(\Lambda = 1\) is

\[
L = \pi_a \dot{r}^a + \lambda_0 H_0 + \lambda_1 H_1,
\]

where \(\pi_a(x)\) is the canonical momentum of \(r^a(x)\) and \(\lambda_a(x)\) are the Lagrange multipliers which implement the reparametrization constraints

\[
H_0 = \frac{1}{2} (\pi^a \pi_a + r^a r^a'),
\]
In our notation a dot (dash) indicates differentiation with respect to time (space). The “internal” indices $a, b, \ldots$ are raised and lowered with a Minkowskian metric $\eta_{ab} = \text{diag}(1, -1)$ (not the space–time metric) so that the canonical variables $\pi_a, r^a$ appear in an indefinite quadratic form in Eqs.(2, 3). We will then say that the field $r^0(x)$ has positive signature while $r^1(x)$ has negative signature. Note that the constraints Eqs.(3) are just the components of the energy–momentum tensor of two massless scalar fields with opposite signature.

The theory described by Eqs.(2, 3) looks very simple since there are no interaction terms. In the gauge $\lambda_0 = 0, \lambda_1 = 1$ it describes two massless scalar fields with opposite signature and with a vanishing energy–momentum tensor. We would expect that the physical states should be the direct product of states for each degree of freedom separately. However, there are subtle correlations due to the constraints Eqs.(3) and the Hilbert space has not a direct product structure.

The canonical quantization of the theory is upset with anomalies. Due to the normal ordering in the constraints Eqs.(3) there appears the well known Virasoro anomaly in the algebra of the energy–momentum tensor. It is possible to cancel the anomaly in three different ways and the resulting quantum theories are not equivalent. The first possibility is to make a non conventional choice of the vacuum for one of the fields $r^a(x)$ [7]. It is non conventional in the sense that the usual creation and annihilation operators have their role reversed. For this field the resulting central charge changes sign. Then the overall central charge vanishes and no anomaly is present. In the second possibility we add background charges to the scalar fields [8]. By choosing appropriately the value of the background charges the anomaly can be made to cancel. Ghosts can also be added in this case. The third procedure consists in modifying the constraints in order to cancel the anomaly [8] but we will not take this route here.

In this paper we will concentrate on the first and second procedures. We will find new physical states in the presence of background charges.
The usual way to incorporate background charges is to consider an improved energy–momentum tensor. To derive this improved energy–momentum tensor consider the Lagrangian of a free massless scalar field $\phi$ with a surface term linear in the field $Q \Box \phi$, where $Q$ is the background charge. From this Lagrangian we can find the appropriate energy–momentum tensor. When this is done for the fields $r^a$, taking into account their signature, we find

$$T_{\mu\nu} = \frac{1}{2} \partial_\mu r^a \partial_\nu r_a - \frac{1}{4} \eta_{\mu\nu} \partial^\rho r^a \partial_\rho r_a + \frac{1}{2} Q_a \partial_\mu \partial_\nu r^a - \frac{1}{4} \eta_{\mu\nu} Q_a \Box r^a. \quad (4)$$

The constraints $H_0 = (T_{++} + T_{--})/2$ and $H_1 = (T_{++} - T_{--})/2$ are now

$$H_0 = \frac{1}{2} \left( \pi^a r_a + r^a r'_a \right) + Q_a r^{''a}, \quad (5)$$

$$H_1 = \pi_a r^a + Q_a \pi^a. \quad (6)$$

As usual the Poisson bracket algebra of the constraints acquires a classical central charge due to the surface term

$$\{H_0(x), H_0(y)\} = (H_1(x) + H_1(y)) \delta'(x - y),$$

$$\{H_0(x), H_1(y)\} = (H_0(x) + H_0(y)) \delta'(x - y) - Q^a Q_a \delta'''(x - y),$$

$$\{H_1(x), H_1(y)\} = (H_1(x) + H_1(y)) \delta'(x - y). \quad (7)$$

Therefore the new constraints have a first class algebra only if $Q_a Q^a = 0$. A careful analysis of the canonical transformation which brings the original CGHS model Eq.(1) (written in terms of the dilaton and gravity fields) to the quadratic form Eq.(2) (written in terms of $r^a$) shows that the two Lagrangians differ by a surface term $[5]$. This surface term can be written in the form $Q_a \Box r^a$ with $Q_a Q^a = 0$. This is necessary to retain the reparametrization symmetry of the original model. However, since the quantum theory is afflicted with anomalies we are allowed to modify it by adding a Wess–Zumino field to cancel the anomaly. Since in two–dimensions a scalar field can serve as its own Wess–Zumino field we can add an improvement term to the constraints with an appropriate value of $Q_a Q^a$ to cancel the
anomaly. So, in general, $Q_a Q^a$ will no longer vanish and the classical theory will lose reparametrization invariance. However, it will be recovered at the quantum level.

Before performing the canonical quantization with the new constraints let us first consider a single massless scalar field. In the canonical approach it has an expansion in terms of Fock space operators $a^\dagger(k)$ and $a(k)$ associated with particles of positive and negative energy respectively. The conventional vacuum is defined as $a(k)|0>=0$ so that the Hilbert space is positive definite and the energy of the states is also positive. This gives rise to a central charge $c = 1$ in the energy–momentum tensor algebra when normal ordering is taken into account. An alternative choice for the vacuum is to take $a^\dagger(k)|0>=0$. In this case the Hilbert space is no longer positive definite, the energy of the states is negative and the central charge is $c = -1$. For conventional theories this choice of the vacuum is not allowed.

Let us now consider a single scalar field with negative signature. In the canonical approach there is a crucial change of sign in the canonical momentum which leads to a change of sign in the algebra of creation and annihilation operators. Now if the vacuum is defined as $a(k)|0>=0$ then the Hilbert space is not positive definite but the energy of the states is positive and the central charge is $c = 1$. For the other choice of the vacuum $a^\dagger(k)|0>=0$ the Hilbert space is positive definite, the energy is negative and $c = -1$. Then the quantum theory of a scalar field with negative signature has troubles for any choice of the vacuum.

When a background charge $Q$ is added its effect is just to shift the value of the central charge. A short calculation shows that for the conventional scalar field we have for the usual choice of the vacuum $a(k)|0>=0$ the value $c = 1 + 12\pi Q^2$ while for the vacuum $a^\dagger(k)|0>=0$, $c = -1 + 12\pi Q^2$. For the scalar field with negative signature and vacuum $a(k)|0>=0$ we have $c = 1 - 12\pi Q^2$, while for the vacuum $a^\dagger(k)|0>=0$ we find $c = -1 - 12\pi Q^2$. This is summarized in Table I.

As remarked before the CGHS model written in the form Eq.(2) involves two free massless scalar fields with opposite signature as can be seen when the gauge $\lambda_0 = 1, \lambda_1 = 0$ is choosen. Then canonical quantization allows several possibilities for the vanishing of the total central charge. If no background charges are present we can achieve $c = 0$ by choosing the vacuum...
\[ a_0(k)|0 >= a_1^\dagger(k)|0 >= 0. \] Note that since our Hamiltonian is zero we have no troubles with the positivity of the energy. If background charges with \( Q_a Q^a = 0 \) are present we must do the same vacuum choice. If the background charges have \( Q_a Q^a \neq 0 \) then the vanishing of the central charge requires \( Q_a Q^a = \pm 1/(6\pi) \) depending on which vacuum is choosen. There are two possibilities: \( a_0(k)|0 >= a_1(k)|0 >= 0 \) or \( a_0^\dagger(k)|0 >= a_1^\dagger(k)|0 >= 0. \) Either possibility is troublesome since positivity of the Hilbert space is compromised. We will also meet difficulties for the case \( Q_a Q^a \neq 0 \) in the Schrödinger representation. These results are presented in Table II.

Physical states have been explicitely constructed for the case \( Q_a = 0 \) \[7\]. For the case \( Q_a \neq 0 \) they have been found when the topology of space–time is non–trivial. We will comment on this at the end of the paper. Ghosts can also be added. They simply change the value of the background charges and the same analysis carries through.

We now consider the Schrödinger representation. The Scrödinger functional \( \Psi \) is a functional of \( r^a, \Psi(r^a) \), and \( \pi_a \) is realized as a functional derivative \( \pi_a(x) = -i\delta/\delta r^a(x) \). In the Schrödinger representation there is no normal products to be taken into account. The only source of ambiguity is in the operator ordering. So the questions about the value of the central charge are difficult to be posed in this formalism. The relevant point here is whether there is a first class algebra of quantum constraints so that physical states can be properly defined.

When the Poisson bracket algebra of the constraints Eqs.\( \{7\} \) is replaced by the respective commutator algebra we obtain the same central charge proportional to \( Q_a Q^a \). The algebra of the constraints is not first class and physical states can not be defined unless \( Q_a Q^a = 0 \). Alternatively we could try to modify the constraints to take normal ordering into account in order to recover a first class algebra. So let us consider the effect of normal ordering in each term of the constraints. Let us assume again that we have a single scalar field \( \phi \). Assuming that \( \phi(x) \) and \( \pi(x) \) have canonical commutation relations we find that

\[
: \phi'(x)\pi(y) : = \phi'(x)\pi(y) - \frac{i}{2}\delta'(x-y),
\] (8)
for any choice of the vaccum and for any signature of the field. This means that

\[ H_1(x) := r^a \pi_a + Q_a \pi^a - i \lim_{y \to x} \delta'(x - y), \]

which does not depend on which vaccum is choosen. This is the same ambiguity that we find if we consider the operator ordering in \( H_1 \). Since \( \pi_a \) and \( r^a \) have canonical commutation relations there is an ambiguity in the term \( \pi_a r^a \) in Eq.(8) with the same form as in Eq.(9).

Then the coefficient of the \( \delta'(x - y) \) term is not fixed. For each choice of this coefficient we have an operator ordering prescription. This is also consistent with the commutator algebra of the constraints. It is independent of the value for this coefficient as it is easily verified.

Let us now consider the effect of normal ordering in \( H_0 \). If the field \( \phi \) has positive signature

\[ :\pi(x)\pi(y) := \pi(x)\pi(y) + \frac{1}{2} \omega(x - y), \]

where

\[ \omega(x - y) = \frac{1}{2\pi} \int dk |k| e^{i k(x-y)}. \]

The upper sign in Eq.(10) is for the usual vaccum \( a|0 >= 0 \) while the lower sign is for the unusual vaccum \( a^\dagger|0 >= 0 \). If the field \( \phi \) has negative signature then

\[ :\pi(x)\pi(y) := \pi(x)\pi(y) \pm \frac{1}{2} \omega(x - y), \]

with the upper (lower) sign for the usual (unusual) vaccum. The same structure holds for \( \phi'(x)\phi'(y) \). Therefore we find that

\[ :H_0 := \frac{1}{2}(\pi^a \pi_a + r^a r_a') + Q_a r^a + \frac{c}{2} \lim_{y \to x} \omega(x - y), \]

where \( c = 0, \pm 2 \) is the sum of the central charges of \( r^0 \) and \( r^1 \). This takes into account all possible choices of the vaccum.

Therefore the constraints in the form Eqs.(8,13) are now first class and must be used for seeking solutions in the Schrödinger representation. The equations for the physical states are then
\[ r^a(x) \frac{\delta \Psi}{\delta r^a(x)} + Q^a \left( \frac{\delta \Psi}{\delta r^a(x)} \right)' - \alpha \lim_{y \to x} \delta'(x - y) \Psi = 0, \tag{14} \]

\[ \frac{1}{2} \left( - \frac{\delta^2 \Psi}{\delta r^a(x) \delta r^a(x)} + r^a(x) r^a(x) \Psi \right) + Q_a r^a(x) \Psi + \frac{c}{2} \lim_{y \to x} \omega(x - y) \Psi = 0, \tag{15} \]

where \( \alpha \) is a constant which will select the operator ordering prescription. The simplest choice is to take \( \alpha = 0 \) and, as we will see, we can find solutions with this prescription. So we will adopt it from now on.

We will now look for the vacuum state in the Schrödinger representation. This is most easily done going to the canonical formalism and expressing the creation and annihilation operators in terms of \( r^a \) and \( \pi_a \). We find

\[ a_a(k) = \frac{1}{\sqrt{4\pi |k|}} \int dx \left( |k| r^a(x) \pm i\pi_a(x) \right), \tag{16} \]

where the upper sign holds for \( a = 0 \) and the lower sign for \( a = 1 \). As we have seen before the vacuum is the same for \( Q_a = 0 \) and \( Q_a Q^a = 0 \) cases. It is defined by

\[ a_0(k) \Psi_{vaccum} = a_1^\dagger(k) \Psi_{vaccum} = 0, \tag{17} \]

and the solution is known \[7\]

\[ \Psi_{vaccum} = \det \frac{1}{2} \left( \frac{\omega}{\pi} \right) \exp \left[ - \frac{1}{2} \int dx dy \left( r^0(x) \omega(x - y) r^0(y) + r^1(x) \omega(x - y) r^1(y) \right) \right]. \tag{18} \]

Since \( \omega(x - y) \) Eq.(11) has a positive kernel this vacuum is normalizable. For the case \( Q_a Q^a \neq 0 \) the vacuum would be defined by

\[ a_0(k) \Psi_{vaccum} = a_1^\dagger(k) \Psi_{vaccum} = 0 \tag{19} \]

or

\[ a_0^\dagger(k) \Psi_{vaccum} = a_1^\dagger(k) \Psi_{vaccum} = 0. \tag{20} \]

These equations do not have normalizable solutions. The solution of Eq.(19), for example, is given by \[4\]
\begin{equation}
\Psi = \exp \left[ -\frac{1}{2} \int dx \, dy \left( r^0(x)\omega(x - y)r^0(y) - r^1(x)\omega(x - y)r^1(y) \right) \right]. \tag{21}
\end{equation}

The minus sign in front of the second term makes the wave functional non normalizable since the kernel is positive. This shows that there is no vacuum state for \( Q_a Q^a \neq 0 \) in the Schrödinger representation. In the canonical analysis we found that in this case the Hilbert space is not positive definite.

We now look for physical states in Eqs.(14,15). For the case \( Q_a = 0 \) they are already known \[7\]. Taking \( Q_a = c = 0 \) in Eq.(14,15) we obtain

\begin{equation}
\Psi_{Q=0} = \exp \left( \pm \frac{i}{2} \int dx \, \epsilon_{ab} r^a(x) r^b(x) \right). \tag{22}
\end{equation}

For the case \( Q_a Q^a = 0 \) we still have \( c = 0 \) in Eq.(15) and we find

\begin{equation}
\Psi_{Q_aQ^a=0} = \exp \left[ \pm \frac{i}{2} \int dx \, \epsilon_{ab} r^a(x) r^b(x) \pm i \int dx \, \epsilon_{ab} Q^a r^b(x) \ln \left( \epsilon_{cd} Q^c r^d(x) \right) \right]. \tag{23}
\end{equation}

This solution reduces to the former solution when \( Q_a = 0 \). It is possible to rewrite it in many other forms thanks to the identity

\begin{equation}
\epsilon_{ab} Q^a r^b Q^c r^c = \epsilon_{ab} r^b Q^c r^c, \tag{24}
\end{equation}

which holds for \( Q_a Q^a = 0 \). Another form for the argument of the second term in the exponential in Eq.(23) could be \( \epsilon_{ab} Q^a r^b \ln (Q_c r^c) \). It is easier to work with the form given in Eq.(23). We suspect that the Fock space state corresponding to the wave functional Eq.(23) will have a very cumbersome form due to the presence of the logarithmic term.

These new physical states Eq.(23) have not been previously identified in the BRST formulation \[9\]. The reason is that the general BRST techniques \[10\] that have been applied to the problem hold only when the background charges have \( Q_a Q^a \neq 0 \) which is not our case.

As we have seen there are no physical states for \( Q_a Q^a \neq 0 \). However, they have been found when the space–time topology is \( R \times S^1 \) \[1\]. These states depend on the zero–mode momenta and seems not to have a well behaved limit when the space–time topology is taken to be trivial.
Having found new physical states still leaves open the main difficulty of this approach: how to extract the space–time geometric properties from the Hilbert space. It is also necessary to compare the non-perturbative results obtained in this approach with the semi-classical results. We must solve these issues in the two-dimensional models, where the problems are tractable, before embarking in realistic four or higher dimensional gravity theories with propagating gravitons.

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TABLE I. Hilbert space norm, energy sign and central charge for all possible vacuum choices of a scalar field

| Signature | Vacumm | Norm               | Energy  | Central Charge       |
|-----------|--------|--------------------|---------|----------------------|
| positive  | $a|0>=0$ | positive definite  | positive | $1 + 12\pi Q^2$     |
| positive  | $a^†|0>=0$ | not positive definite | negative | $-1 + 12\pi Q^2$ |
| negative  | $a|0>=0$ | not positive definite | positive | $1 - 12\pi Q^2$     |
| negative  | $a^†|0>=0$ | positive definite  | negative | $-1 - 12\pi Q^2$ |

TABLE II. Choices of the background charge and vacuum for vanishing central charge in the CGHS model

| Background Charges | Vacumm | Norm               |
|--------------------|--------|--------------------|
| $Q_a = 0$          | $a_0|0>=a_1^†|0>=0$ | positive definite  |
| $Q_a = 0$          | $a_0^†|0>=a_1|0>=0$ | not positive definite |
| $Q_a Q^a = 0$      | $a_0|0>=a_1^†|0>=0$ | positive definite  |
| $Q_a Q^a = 0$      | $a_0^†|0>=a_1|0>=0$ | not positive definite |
| $Q_a Q^a = -\frac{1}{6\pi}$ | $a_0|0>=a_1|0>=0$ | not positive definite |
| $Q_a Q^a = \frac{1}{6\pi}$ | $a_0^†|0>=a_1^†|0>=0$ | not positive definite |
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