Actions of discrete amenable groups into the normalizers of full groups of ergodic transformations

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Dedicated to Professor Yasuyuki Kawahigashi on the occasion of his 60th birthday

Abstract
We apply Evans-Kishimoto’s intertwining argument to the classification of actions of discrete amenable groups into the normalizer of a full group of an ergodic transformation. Our proof does not depend on the types of ergodic transformations.

1 Introduction
The purpose of this article is the study actions of discrete amenable groups into the normalizer of a full group of an ergodic transformation on the Lebesgue space. The study of such objects has been motivated by the theory of operator algebras. In fact, many examples of von Neumann algebras arise from ergodic transformation through Krieger’s construction.

The study of automorphism groups of operator algebras is one of the central subjects for the theory of operator algebras, and the classification of automorphisms and group actions has been developed since Connes’ seminal works \[5\], \[4\]. In particular classification of actions of discrete amenable groups on injective factors has been completed by many hands \[12\], \[17\], \[13\], \[18\], \[19\], \[15\], \[14\]. These works heavily depend on the types of factors. However, we present the unified approach in \[16\] based on Evans-Kishimoto’s method \[8\], and gave proof that is independent of the types of factors.

There are corresponding results in ergodic theory. The first result is due to Connes and Krieger \[6\]. They developed the technique of use of ultraproduct to measure spaces and their transformation, and classified transformations (i.e., actions of \(\mathbb{Z}\)) in the normalizer of a full group of type II. Connes-Krieger’s result has been generalized by \[2\] in the case of type II transformation and general discrete amenable groups, by \[1\] in the case of type \(\text{III}_1\) transformation (\(\lambda \neq 0\)) and general discrete amenable groups and finally by \[3\] in the case of type \(\text{III}_0\) transformation and general discrete amenable groups. (See Theorem 2.4 below for the classification theorem.) These results mentioned above depend on the types of transformations, and it is natural to expect that our unified approach \[16\] is valid for classification of actions of discrete amenable groups into the normalizers of full groups on Lebesgue spaces. In fact, the answer is affirmative, and this is the main result of this
article. This classification result is very similar to that of the classification of actions of discrete amenable groups on injective factors. Indeed, classification result mentioned above can be regarded as the classification of actions that fix Cartan subalgebras of Krieger factors.

To apply the Evans-Kishimoto type intertwining argument, we need the characterization of full groups and their closures given by in [6] and [9]. In the study of group actions on operator algebras, two classes of automorphisms play important roles, i.e., centrally trivial automorphisms and approximately inner automorphisms. In our case, full groups and their closures correspond to centrally trivial automorphism groups, and approximately inner automorphism groups, respectively. Another main tool is the Rohlin type theorem. Combining these results, we first show the cohomology vanishing theorem. Then we obtain classification theorem by applying the Evans-Kishimoto type intertwining argument.

This paper is organized as follows. In §2 we collect basic facts which will be used in this paper, and state the main results. In §3 we recall the ultraproduct construction of Connes-Krieger, and Ocneanu’s Rohlin type theorem. In §4 we show the second cohomology vanishing theorem. In §5 we apply the Evans-Kishimoto type intertwining argument and classify actions of discrete amenable groups into the normalizer of a full group.

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2 Preliminaries

2.1 Full groups of ergodic transformations and their normalizers

In this subsection, we collect known facts on full groups of ergodic transformations and their normalizers, which will be used in this article.

Let $(X, \mathcal{B}, \mu)$ be a nonatomic Lebesgue space with $\mu(X) = 1$. (Throughout this article, we treat only nonatomic Lebesgue spaces.) We denote by $\text{Aut}(X, \mu)$ the set of all nonsingular transformations. Fix an ergodic transformation $T \in \text{Aut}(X, \mu)$. Let $[T]_s$ be a set of all nonsingular bijection $R : A \to B$ for some $A, B \in \mathcal{B}$ such that $Rx \in \{T^n x\}_{n \in \mathbb{Z}}, \ x \in A$. Define a full group of $T$ by $[T] := [T]_s \cap \text{Aut}(X, \mu)$, i.e.,

$$[T] = \{ R \in [T]_s \mid \text{the domain and the range of } R \text{ are both } X \}.$$  

We say $E, F \in \mathcal{B}$ are $T$-equivalent if there exists $R \in [T]_s$ whose domain is $E$ and range is $F$. A set $E \in \mathcal{B}$ is said to be $T$-infinite if there exists $F \subset E$ such that $\mu(E \setminus F) > 0$ and $F$ is $T$-equivalent to $E$. A set $E \in \mathcal{B}$ is said to be $T$-finite if it is not $T$-infinite.

When $T$ is of type II, there exists the unique $T$-invariant measure $m$ on $X$ ($m(X) < \infty$ when $T$ is of type II$_1$, and $m(X) = \infty$ when $T$ is of type II$_\infty$). In this case, the following two statements hold: (1) $E \in \mathcal{B}$ is $T$-finite if and only if $m(E) < \infty$, (2) $E, F \in \mathcal{B}$ are $T$-equivalent if and only if $m(E) = m(F)$. When $T$ is of type II$_1$, we always assume $\mu$ is the unique $T$-invariant probability measure.

When $T$ is of type III, then any $E \in \mathcal{B}$ with $\mu(E) > 0$ is $T$-infinite. Hence if $E, F \in \mathcal{B}$ satisfy $\mu(E), \mu(F) > 0$, then $E$ and $F$ are $T$-equivalent.
Let \( N[T] \subset \text{Aut}(X, \mu) \) be the normalizer of \([T]\). In the following, we use the notation \( \hat{\alpha}(t) = \alpha t \alpha^{-1} \) for \( t \in [T] \) and \( \alpha \in N[T] \).

For \( \alpha \in \text{Aut}(X, \mu) \) and \( \xi \in L^1(X, \mu) \), define \( \alpha_\mu(\xi) \in L^1(X, \mu) \) by

\[
\alpha_\mu(\xi)(x) := \xi(\alpha^{-1}x) \frac{d(\mu \circ \alpha^{-1})}{d\mu}(x), \; \xi \in L^1(X, \mu).
\]

Then \( \alpha_\mu \) is an isometry of \( L^1(X, \mu) \), and \( (\alpha \beta)_\mu = \alpha_\mu \beta_\mu \) holds for \( \alpha, \beta \in \text{Aut}(X, \mu) \).

Let \( M(X, \mu) \) (resp. \( M_1(X, \mu) \)) be the set of complex-valued measures (resp. probability measures) which are absolutely continuous with respect to \( \mu \). For \( \nu \in M(X, \mu) \), let \( \|\nu\| = |\nu|(X) \), where \( |\nu| \) is the total variation of \( \nu \). Then \( M(X, \mu) \) is a Banach space with respect to the norm \( \|\nu\| \). For \( \xi \in L^1(X, \mu) \), let \( \nu_\xi(f) = \int_X \xi(x)f(x)d\mu(x) \). Note that \( L^1(X, \mu) \) and \( M(X, \mu) \) are isomorphic as Banach spaces by \( \xi \mapsto \nu_\xi \). Via this identification, \( \alpha_\mu(\xi) \) corresponds to \( \alpha(\nu_\xi) = \nu_\xi \circ \alpha^{-1} \). In what follows, we freely use this identification, and we simply denote \( \alpha_\mu(\xi) \) by \( \alpha(\xi) \) for \( \xi \in L^1(X, \mu) \). Thus \( \xi(A), A \in \mathcal{B}, \) means \( \nu_\xi(A) \).

Recall the topology of \( N[T] \) introduced in [11]. For \( \alpha, \beta \in \text{Aut}(X, \mu) \), \( \{\alpha \neq \beta\} \) denotes the set \( \{x \in X \mid \alpha x \neq \beta x\} \). We say a sequence \( \{\alpha_n\}_n \subset N[T] \) converges to \( \beta \in N[T] \) weakly if \( \lim_{n \to \infty} \|\alpha_n(\xi) - \beta(\xi)\| = 0 \) for all \( \xi \in M(X, \mu) \). Define a metric \( d_\mu \) by \( d_\mu(\alpha, \beta) := \|\mu(\alpha) - \beta(\xi)\| = 0 \). This definition does not depend on the choice of equivalence classes of \( \mu \in M_1(X, \mu) \). It is shown in [11] that \( [T] \) is a Polish group by \( d_\mu \).

Now we gift a topology of \( N[T] \) as follows. We say a sequence \( \{\alpha_n\}_n \subset N[T] \) converges to \( \beta \in N[T] \) if \( \{\alpha_n\}_n \) converges to \( \beta \) weakly, and \( \alpha_n(t) \) converges to \( \beta(t) \) uniformly for all \( t \in [T] \). (In fact, we only have to require convergence for \( t \in \{T^n\}_{n \in \mathbb{Z}} \).) This is the right topology for \( N[T] \). In fact, this topology coincides with the \( u \)-topology for a Krieger factor \( \mathcal{R}_T \) constructed from \( (X, \mu, T) \). So we also call this topology the \( u \)-topology. It is shown that \( N[T] \) is a Polish group in the \( u \)-topology [11]. Indeed, let \( \{\xi_k\}_{k=1}^\infty \subset L^1(X, \mu) \) be a countable dense subset, and define a metric \( d \) on \( N[T] \) by

\[
d(\alpha, \beta) := \sum_{k=1}^\infty \frac{1}{2^k} \left[ \frac{1}{1 + \|\alpha(\xi_k) - \beta(\xi_k)\|} + \sum_{k \in \mathbb{Z}} \frac{1}{2^k} \left( \frac{1}{1 + d_\mu(\hat{\alpha}(T^k), \hat{\beta}(T^k))} \right) \right].
\]

Then this \( d \) makes \( N[T] \) a Polish group, and the topology defined by \( d \) is nothing but the \( u \)-topology on \( N[T] \).

We collect elementary results which will be frequently used in what follows. Since proof is easy, we leave it to the readers.

**Lemma 2.1** The following statements hold.

1. \( d_\mu(\theta \alpha, \theta \beta) = d_\mu(\alpha, \beta), \; d_\mu(\alpha \theta, \beta \theta) = d_\mu(\alpha, \beta), \; \alpha, \beta, \theta \in N[T]. \) In particular we have \( d_\mu(\alpha, \text{id}) = d_\mu(\text{id}, \alpha^{-1}) = d_\mu(\alpha^{-1}, \text{id}), \) and \( d_\mu(\hat{\alpha}(t), \hat{\beta}(t')) = d_{\alpha^{-1}}(\mu)(t, t'), \; \alpha \in N[T], \; t, t' \in [T]. \)

2. \( d_{\nu_1}(\alpha, \beta) \leq \|\nu_1 - \nu_2\| + d_{\nu_2}(\alpha, \beta), \; \nu_1, \nu_2 \in M_1(X, \mu), \; \alpha, \beta \in N[T]. \)

3. Let \( \nu \in M_1(X, \mu), \; A, B, C, D \in \mathcal{B}. \) Then we have

\[
\nu((A \cup B)\triangle(C \cup D)) \leq \nu(A \triangle C) + \nu(B \triangle D),
\]

\[
\nu((A \cap B)\triangle(C \cap D)) \leq \nu(A \triangle C) + \nu(B \triangle D).
\]
Recall the definition of the fundamental homomorphism \[^{[10]}\]. Let \( \tilde{X} := X \times \mathbb{R} \), and \( \mu_L \) be the Lebesgue measure on \( \mathbb{R} \). For \( R \in \text{Aut}(X, \mu) \) and \( t \in \mathbb{R} \), define \( \tilde{R}, F_t \in \text{Aut}(\tilde{X}, \mu \times \mu_L) \) by
\[
\tilde{R}(x, u) = \left(Rx, u - \log \frac{d(\mu \circ R)}{d\mu}(x)\right), \quad F_t(x, u) = (x, u + t).
\]

Let \( (Y, \nu_Y) \) be the quotient space by \( \tilde{T} \). Since \( \tilde{T} \) and \( F_t \) commute, we get the ergodic flow \( (Y, \nu_Y, F_t) \), which is called the associated flow of \( (X, T) \). Let
\[
\text{Aut}_F(Y, \nu_Y) := \{ P \in \text{Aut}(Y, \nu_Y) \mid PF_t = F_tP, t \in \mathbb{R} \}.
\]

When \( R \) is in \( N[T] \), \( \tilde{R} \) induces \( \text{mod}(R) \in \text{Aut}_F(Y, \nu) \), which is called the fundamental homomorphism. If we lift \( R \) to an automorphism of a Krieger factor \( \mathcal{R}_T \), \( \text{mod}(R) \) is nothing but a Connes-Takesaki module for \( R \) \[^{[7]}\].

In this article, we do not use the above definition of \( \text{mod}(R) \) explicitly, and what we need is the fact \( \text{Ker}(\text{mod}) = [\mathcal{T}] \) (closure is taken in the \( u \)-topology) and the surjectivity of \( \text{mod} \) \[^{[10]}, [9]\].

### 2.2 Main results

**Definition 2.2** Let \( G \) be a countable discrete group.

1. A map (or 1-cochain) \( v : G \to [T] \) is said to be normalized if \( v(e) = \text{id} \). We denote the set of all normalized maps from \( G \) into \( [T] \) by \( C^1(G, [T]) \).
2. A cocycle crossed action of \( G \) into \( N[T] \) is a pair of maps \( \alpha : G \to N[T] \), and \( c : G \times G \to [T] \) such that \( \alpha_g \alpha_h = c(g, h) \alpha_{gh} \), \( \alpha_e = \text{id} \), \( c(e, h) = c(g, e) = \text{id} \). When \( c(g, h) = \text{id} \) for all \( g, h \in G \), we say \( \alpha \) is an action of \( G \) into \( N[T] \).
3. Let \( (\alpha, c) \) be a cocycle crossed action of \( G \) into \( N[T] \), and \( v \in C^1(G, [T]) \). A perturbed crossed action \( (\nu\alpha, \nu c) \) of \( (\alpha, c) \) by \( v \) is defined by
\[
\nu\alpha_g := v(g)\alpha_g, \quad \nu c(g, h) = v(g)\alpha_g(v(h))c(g, h)v(gh)^{-1}.
\]
4. Let \( \alpha \) be an action of \( G \) into \( N[T] \). We say a map \( v \in C^1(G, [T]) \) is a 1-cocycle for \( \alpha \) if \( v \) satisfies the 1-cocycle identity \( v(g)\alpha_g(v(h)) = v(gh) \). It is equivalent to that \( \nu\alpha \) is an action.
5. Let \( \alpha \) and \( \beta \) be actions of \( G \) into \( N[T] \). We say they are cocycle conjugate if there exist \( \theta \in N[T] \) and 1-cocycle \( \nu(\cdot) \) such that \( \nu\alpha_g = \theta \beta_g \theta^{-1} \) for all \( g \in G \). If \( \theta \) is chosen in \( [T] \), then we say they are strongly cocycle conjugate.

**Remark** (1) Let \( (\alpha, c) \) be a cocycle crossed action of \( G \). (Notion of a \( p \)-action is used in \[^{[3]}\].) By \( (\alpha_g \alpha_h)\alpha_k = \alpha_g(\alpha_h \alpha_k) \), we can deduce the 2-cocycle identity \( c(g, h)c(gh, k) = \bar{\alpha}_g(c(h, k))c(g, hk) \).

(2) In many works, cocycle conjugacy is said to be outer conjugacy. In fact, we must distinguish these two notions for group actions on operator algebras, However, in ergodic theory, we do not have to distinguish them. (We have the canonical homomorphism \( u \in [T] \) into the normalizer of a Krieger factor arising from \( (X, \mu, T) \).)

At first, we show the following theorem.
Theorem 2.3 Let \((\alpha, c)\) be a cocycle crossed action of a discrete amenable group into \(N[T]\) with \(\alpha_\gamma \not\in [T], g \neq e\). Then \(c(g, h)\) is a coboundary, that is, there exists \(v \in C^1(G, [T])\) such that \(v(c(g, h)) = id\), equivalently \(\alpha\) is a genuine action of \(G\). If \(c(g, h)\) is close to id, then we can choose \(v\) so that it is also close to id.

See below for a more precise statement.

Let \(N_\alpha := \{g \in G \mid \alpha_\gamma \in [T]\}\), which is a normal subgroup of \(G\). Our main result in this article is the following.

Theorem 2.4 Let \((X, \mu)\) be a Lebesgue space with \(\mu(X) = 1\), \(T\) an ergodic transformation on \((X, \mu)\). Let \(G\) be a countable discrete amenable group, and \(\alpha, \beta\) actions of \(G\) into \(N[T]\). Then \(\alpha\) and \(\beta\) are strongly cocycle conjugate if and only if \(N_\alpha = N_\beta\) and \(mod(\alpha) = \mod(\beta)\).

If \(\alpha\) and \(\beta\) are strongly cocycle conjugate, then it is obvious that \(N_\alpha = N_\beta\) and \(\mod(\alpha_\gamma) = \mod(\beta_\gamma)\). (Amenability of \(G\) is unnecessary for this implication.) Thus the problem is to prove the converse implication, and a proof will be presented in subsequent sections. Here we only state the following corollary, which can be easily verified by Theorem 2.4.

Corollary 2.5 Let \(\alpha\) and \(\beta\) be actions of \(G\) into \(N[T]\). Then \(\alpha\) and \(\beta\) are cocycle conjugate if and only if \(N_\alpha = N_\beta\) and \(\mod(\alpha_\gamma) = \theta \mod(\beta_\gamma)\theta^{-1}\) for some \(\theta \in Aut_F(Y, \nu_Y)\).

Proof. Since “only if part” is clear, we only have to prove “if part”. Suppose \(N_\alpha = N_\beta\) and \(\mod(\alpha_\gamma) = \theta \mod(\beta_\gamma)\theta^{-1}\) for some \(\theta \in Aut_F(Y, \nu_Y)\). By the surjectivity of \(\mod[5]\), we can take \(\sigma \in N[T]\) with \(\mod(\sigma) = \theta\). Then \(\mod(\alpha_\gamma) = \mod(\sigma_\beta(\sigma)^{-1})\) holds, and hence \(\alpha_\gamma\) and \(\sigma_\beta(\sigma)^{-1}\) are strongly cocycle conjugate by Theorem 2.4. 

3 Ultraproduct of a Lebesgue space and Rohlin type theorem

We recall ultraproduct construction in [6].

Let \(\omega \in \beta \mathbb{N}\) be a free ultrafilter on \(\mathbb{N}\). For sequences \((A_n)_n, (B_n)_n \subset \mathcal{B}\), define an equivalence relation \((A_n)_n \sim (B_n)_n\) by \(\lim_{n \to \omega} \mu(A_n \Delta B_n) = 0\). Let \(\mathcal{B}^\omega := \{(A_n)_n \in \mathcal{B}\}/\sim\) This definition depends only on the equivalence class of \(\mu\), and \(\mathcal{B}^\omega\) is a boolean algebra.

Any \(\alpha \in N[T]\) induces a transformation \(\alpha^\omega\) on \(\mathcal{B}^\omega\) by \(\alpha^\omega((A_n)_n) := (\alpha(A_n))_n\). Let \(\mathcal{B}_\omega := \{\hat{A} \in \mathcal{B}^\omega : t^\omega \hat{A} = \hat{A}, t \in [T]\}\). We denote by \(\alpha^\omega\) the restriction of \(\alpha^\omega\) on \(\mathcal{B}_\omega\).

Let \(\hat{A} = (A_n) \in \mathcal{B}_\omega\). Then \(\lim_{n \to \omega} \chi_{A_n}\) exists in weak-\(\ast\) topology on \(L^\infty(X, \nu)\). By the ergodicity of \(T\), this limit is in \(\mathbb{C}\), and does not depend on the choice of representative \(A = (A_n)\). Thus we can define \(\tau : \mathcal{B}_\omega \to \mathbb{C}\) by \(\tau(A) := \lim_{n \to \omega} \chi_{A_n}\). We can see \(\tau \circ \alpha^\omega = \tau\) for \(\alpha \in N[T]\). By [6] Lemma 2.4, for \(\alpha \in N[T]\), \(\alpha^\omega = id\) if and only if \(\alpha \in [T]\). In fact, we have a stronger result. For \(R \in N[T]\), if there exists \(\hat{A} \in \mathcal{B}_\omega\) such that \(R_{\hat{B}} = \hat{B}\) for any \(\hat{B} \subset \hat{A}\), \(\hat{B} \in \mathcal{B}_\omega\), then \(R_{\hat{B}} = id\), and hence \(R \in [T]\), [6] Lemma 2.3. This means that \(R_{\hat{B}}\) is a free transformation if \(R_{\hat{B}} = id\).

The main tool of this article is the following Rohlin type Theorem, essentially due to Ocneanu [17]. (The following formulation is presented in [16].)
Theorem 3.1 Let \((\alpha, c)\) be a cocycle crossed action of a discrete amenable group \(G\) into \(N[T]\) such that \(\alpha \neq c\) for all \(g \neq e\). Let \(K \subseteq G\), \(\varepsilon > 0\), and \(S\) be a \((K, \varepsilon)\)-invariant set. (The notation \(K \subseteq G\) means that \(F\) is a finite subset of \(G\).) Then there exists a partition of unity \(\{\hat{E}_s\}_{s \in S} \subset \mathcal{B}_\omega\) such that

\[
\begin{align*}
(1) & \sum_{s \in S_g} \tau(\alpha \hat{E}_g \triangle \hat{E}_{g_s}) < 5\varepsilon^\frac{1}{2}, \ g \in K, \\
(2) & \sum_{s \in S \setminus S_{g-1}} \tau(\hat{E}_s) < 3\varepsilon^\frac{1}{2},
\end{align*}
\]

where \(S_g := S \cap g^{-1}S\).

Note that we have \(g \in S_{g-1} = S \cap gS\) for \(s \in S_g\).

The proof of [17] is based on the following two facts, i.e., the freeness of actions on central sequence algebras, and ultraproduct technique. In our case, freeness holds as we remarked before Theorem 3.1. Hence the proof of [17] can be applied in our case by the suitable modification.

In what follows, we say \(\alpha\) is an ultrafree action of \(G\) if \(\alpha \neq c\) for any \(g \in G\), \(g \neq e\), to distinguish from the usual freeness of actions on Lebesgue spaces.

Lemma 3.2 Let \(A, B\) be finite sets, \(\{E_a\}_{a \in A} \subset \mathcal{B}_\omega\) a partition of \(X\), and \(\{P_{a,b}\}_{a \in A, b \in B} \subset [T]\). Choose representative \(E_a = (E_a^n)_{n \in \omega}\) such that \(E_a^n \cap E_a^{n'} = \emptyset\) for \(a \neq a'\), \(\bigcup_{a \in A} E_a^n = X\).

Then for any \(\varepsilon > 0\), \(\Phi \in M_1(X, \mu)\), there exists \(N \in \omega\), \(\{Z^n_a\}_{a \in A} \subset \mathcal{B}\), \(R^n_b \in [T]\), \(n \in N\), \(b \in B\), such that

1. \(\nu(P_{a,b}^{-1}E_a^n \triangle E_a^n) < \varepsilon\), \(n \in N\), \(\nu \in \Phi\),
2. \(Z^n_a \subseteq E_a^n\), \(P_{a,b}Z^n_a \subseteq E_a^n\), \(n \in N\),
3. \(\nu(E_a^n \setminus Z^n_a) < \varepsilon\), \(\nu(E_a^n \setminus P_{a,b}Z^n_a) < \varepsilon\), \(n \in N\), \(\nu \in \Phi\),
4. \(R^n_b \cdot x = P_{a,b} \cdot x\), \(n \in N\), \(x \in Z^n_a\).

Proof. Since \(P_{a,b}E_a = E_a\) by [6] Lemma 2.4, there exists \(N \in \omega\) such that

\[
P_{a,b}(\nu)
\left(
\left(E_a^n \cup \bigcup_{b \in B} P_{a,b}^{-1}E_a^n\right) \setminus \left(E_a^n \cap \bigcap_{b \in B} P_{a,b}^{-1}E_a^n\right)
\right) < \varepsilon \frac{1}{2}
\]

for \(n \in N\), \(a \in A\), \(b \in B\), \(\nu \in \Phi\).

Let \(Y^n_a := E_a \cap \bigcap_{b \in B} P_{a,b}^{-1}E_a^n\). Clearly we have \(Y^n_a \cdot P_{a,b}Y^n_a \subseteq E_a^n\). Moreover

\[
\nu(P_{a,b}^{-1}E_a^n \triangle E_a^n) < \varepsilon \frac{1}{2}, \ \nu(E_a^n \setminus Y^n_a) < \varepsilon \frac{1}{2}, \ \nu(E_a^n \setminus P_{a,b}Y^n_a) = P_{a,b}(\nu)(P_{a,b}^{-1}E_a^n \setminus Y^n_a) < \varepsilon \frac{1}{2}
\]

hold for \(n \in N\), \(\nu \in \Phi\). Let \(Y^n := \bigsqcup_{a \in A} Y^n_a\). Thus we can define \(R^n_{a,b} \in [T]\) with \(\text{Dom}(R^n_{a,b}) = Y^n\)

by \(R^n_{a,b} \cdot x = P_{a,b} \cdot x\), \(x \in Y^n_a\). If \(X \setminus Y^n\) and \(X \setminus R^n_{a,b}Y^n\) are \(T\)-equivalent, then we can extend \(R^n_{a,b}\) to an element \(R^n_{b} \in [T]\).

At first, let us assume that \(Y^n\) is \(T\)-finite. (Thus so is \(R^n_{a,b}Y^n\).) Such a case can happen if \(T\) is of type II. Then \(X \setminus Y^n\) and \(X \setminus R^n_{a,b}Y^n\) are \(T\)-equivalent. Hence we can extend \(R^n_{a,b}\) to \(R^n_{b} \in [T]\). Set \(Z^n_a := Y^n_a\). Then all the statements in the lemma are satisfied.
Next, let us assume that $Y^n$ is $T$-infinite. (Hence so is $R^n_{0,1}Y^n$.) Take $W_k \subset Y^n$, $k \in \mathbb{N}$, such that $W_k \subset W_{k+1}$, $\bigcup_k W_k = Y^n$ and $Y^n \setminus W_k$ are $T$-infinite for all $k$. Set $Z_{a,k}^n := Y^n \cap W_k$. Of course we have $Z_{a,k}^n \subset Z_{a,k+1}^n$, $\bigcup_k Z_{a,k}^n = Y^n$, $\bigcup_{a \in A} Z_{a,k}^n = Y^n \cap W_k = W_k$, and $Z_{a,k}^n P_a Z_{a,k}^n \subset E_a^n$. Thus $\{Z_{a,k}^n\}_{a \in A}$ satisfies the condition (2).

Take sufficiently large $k$ such that
\[
\nu(Y_a^n \setminus Z_{a,k}^n) < \frac{\varepsilon}{2}, \quad \nu(P_a Z_{a,k}^n) < \frac{\varepsilon}{2}
\]
for $a \in A$, $b \in B$, $\nu \in \Phi$. Then it is clear that $\{Z_{a,k}^n\}$ satisfies the condition (3). By the choice of $\{W_k\}$, $X \setminus \bigcup_{a \in A} Z_{a,k}^n \supset Y^n \setminus W_k$ and $X \setminus R_{0,b}^n \bigcup_{a \in A} Z_{a,k}^n \supset R_{0,b}^n (Y^n \setminus W_k)$. It follows that $X \setminus \bigcup_{a \in A} Z_{a,k}^n$ and $X \setminus R_{0,b}^n \bigcup_{a \in A} Z_{a,k}^n$ are both $T$-infinite and hence are equivalent. Thus $Z_a \colonequals Z_{a,k}$ satisfies all statements in the lemma.

Now we can combine Theorem 3.1 and Lemma 3.2 as follows.

**Proposition 3.3** Let $G$ be a discrete amenable group, $(\alpha, c)$ an ultrafree cocycle crossed action of $G$ into $\mathcal{N}[T]$. Let $K \subset G$ and $\varepsilon > 0$ be given, and $S$ a $(K, \varepsilon)$-invariant set. Let $B, C$ be finite set, $\{P_b\}_{b \in B} \subset \{T\}$, $\{\nu^c_b\}_{b \in C} \subset M_1(X, \mu)$. Then for any $\delta > 0$, there exists a partition $\{E_s\}_{s \in S} \subset B$ of $X$, $E_s \supset Z_s$ and $R_b \in \{T\}$, $b \in B$, such that

\begin{align*}
(1) & \quad \sum_{s \in S_g} \nu^c_s(\alpha g E_s \triangle E_g) < 5\varepsilon^\frac{1}{2}, \quad g \in K, \quad c \in C, \\
(2) & \quad \sum_{s \in S \setminus S_g} \nu^c_s(E_s) < 3\varepsilon^\frac{1}{2}, \quad g \in K, \quad c \in C, \\
(3) & \quad \nu^c_s(P_{s,b}^{-1} E_{s,b}^{-1} \triangle E_s) < \delta, \quad s \in S, \quad b \in B, \quad c \in C, \\
(4) & \quad P_{s,b} Z_s \subset E_s, \quad s \in S, \quad b \in B, \\
(5) & \quad \nu^c_s(E_s \setminus Z_s) < \delta, \quad s \in S, \quad b \in B, \quad c \in C, \\
(6) & \quad R_b x = P_{s,b} x, \quad s \in S, \quad b \in B, \quad x \in Z_s.
\end{align*}

**Proof.** Let $\{E_s\}_{s \in S} \subset B_\omega$ be a Rohlin partition as in Theorem 3.1. Since $\tau(\hat{A}) = \lim_{n \to \omega} \chi_{A_n}$ for $\hat{A} = (A_n)_{n \in \mathbb{N}} \subset B_\omega$, $\tau(\hat{A}) = \lim_{n \to \omega} \nu(A^n)$ for any $\nu \in M_1(X, \mu)$. Choose representative $\hat{E}_s = (E^n_s)_n$ such that $E^n_s \cap E^n_{s'} = \emptyset$, $\bigcup_{s \in S} E^n_s = X$. By Theorem 3.1,

\begin{align*}
(1) & \quad \lim_{n \to \omega} \sum_{s \in S_g} \nu^c_s(\alpha g E^n_s \triangle E^n_g) < 5\varepsilon^\frac{1}{2}, \quad g \in K, \\
(2) & \quad \lim_{n \to \omega} \sum_{s \in S \setminus S_g} \nu^c_s(E^n_s) < 3\varepsilon^\frac{1}{2}, \quad g \in K
\end{align*}

holds for any $\{\nu^c_s\}_{s \in S, c \in C} \subset M_1(X, \mu)$. Thus there exists $N_1 \in \omega$ such that

\[
\sum_{s \in S_g} \nu^c_s(\alpha g E^n_s \triangle E^n_g) < 5\varepsilon^\frac{1}{2}, \quad g \in K, \quad c \in C, \\
\sum_{s \in S \setminus S_g} \nu^c_s(E^n_s) < 3\varepsilon^\frac{1}{2}, \quad g \in K, \quad c \in C
\]
for all \( n \in N_1 \). By Lemma 4.2 there exists \( N_2 = \omega, Z^n_s \subset E^n_s \) and \( R^n_b \in [T], (n \in N_2) \), such that

\[
\nu^c_s(P^{-1}_s E^n_s \Delta E^n_s) < \delta, \ s \in S, \ b \in B, \\
P_{s,b} Z^n_s \subset E^n_s, \ s \in S, b \in B, \\
\nu^c_s(E^n_s \setminus Z^n_s) < \delta, \nu^c_s(E^n_s \setminus P_{s,b} Z^n_s) < \delta, \ s \in S, b \in B, \ c \in C, \\
R^n_b x = P_{s,b} x, \ s \in S, b \in B, \ x \in Z^n_s
\]

for any \( n \in N_2 \). Fix \( n \in N_1 \cap N_2 \), and set \( E_s := E^n_s, Z_s := Z^n_s, R_b := R^n_b \). Then these \( E_s, Z_s, R_b \) are desired objects. \( \square \)

4 Cohomology vanishing

At first, we show the following second cohomology vanishing result, which is shown in [3, Theorem 1.3]. We present the proof for readers’ convenience.

**Theorem 4.1** Let \( T \) be a transformation of type II\(_\infty\) or type III, and \( (\gamma, c) \) a cocycle crossed action of a discrete group \( G \) into \( N[T] \). Then \( c(g, h) \) is a coboundary, i.e., there exists \( u \in C^1(G, [T]) \) such that \( u c(g, h) = \text{id} \).

**Proof.** Since \( T \) is of type II\(_\infty\), or type III, there exists a partition \( \{E_h\}_{h \in G} \) of \( X \) such that each \( E_h \) is \( T \)-infinite. Let \( \{f_{g,h}\}_{g,h \in G} \subset [T] \) be an array for \( \{E_g\}_{g \in G} \). Take \( v^0_g \in [T] \), with \( \text{Dom}(v^0_g) = \gamma_g E_e \) and \( \text{Ran}(v^0_g) = E_e \). Define \( v(g) \in [T] \) by \( f_{g,h} v^0_g \gamma_g(f_{e,h}) \) on \( \gamma_g E_h \).

Then we have \( v \gamma_g : E_h \to E_h \) and \( \gamma_g(f_{k,h}) = f_{k,h} \) for any \( g, h, k \in G \). Replacing \( (\gamma, c) \) with \( (\nu \gamma, \nu c) \), we may assume \( \gamma_g(E_k) = E_k \) and \( \gamma_g(f_{h,k}) = f_{h,k} \).

Since \( \gamma_g \gamma_h = c(g, h) \gamma_{gh} \), we also have \( c(g, h) E_k = E_k \) and \( c(g, h)(f_{k,l}) = f_{k,l} \).

Next define \( u(g) \in [T] \) by \( u(g) = c(g, l)^{-1} f_{g, l} \) on \( E_l \). Note \( u(g) \) sends \( E_l \) to \( E_{g l} \), hence so does \( u \gamma_g \).

Hence for \( x \in E_l \),

\[
u^c_g u \gamma_g x = u(g) \gamma_g c(h, l)^{-1} f_{h l, l} x = u(g) \gamma_g c(h, l)^{-1} \gamma_{h l} f_{h l, l} x
= u(g) \gamma_g c(h, l)^{-1} \gamma_{h l} f_{h l, l} x = u(g) \gamma_g c(h, l)^{-1} \gamma_{h l} f_{h l, l} x
= c(g, h l)^{-1} f_{g h l, h l} \gamma_g c(h, l)^{-1} \gamma_{h l} f_{h l, l} x
= c(g, h l)^{-1} \gamma_g c(h, l)^{-1} \gamma_{h l} f_{h l, l} x = u(g h) \gamma_{g l} x.
\]

This implies that \( u \gamma_g \) is an action, and \( u c(g, h) = u(g) \gamma_g u(h) c(g, h) u(g h) \gamma_{g l} = \text{id} \) holds. \( \square \)

In the Theorem 4.1 we have no estimation on the choice of \( u(g) \), even if \( c(g, h) \) is close to \( \text{id} \). The rest of this section is devoted to solving this problem. From now on, we always assume that \( G \) is a discrete amenable group.

For all \( g \in G \) and \( S \in G \), fix a bijection \( l(g) : S \to S \) such that \( l(g) s = gs \) if \( gs \in S \).

**Lemma 4.2** Let \( (\gamma, c) \) be an ultrafree cocycle crossed action of \( G \). For any \( \varepsilon > 0, K \subset G, \mu \in \Phi \in M_1(X, \mu) \), there exists \( u \in C^1(G, [T]) \) such that

\[
d_u(u c(g, h), \text{id}) < \varepsilon, \ g, h \in K, \nu \in \Phi.
\]
Moreover for given \( \varepsilon > 0 \), \( e \in K \subseteq G \), there exist \( \delta > 0 \) and \( S \subseteq G \), which depends only on \( K \) and \( \varepsilon > 0 \), such that if

\[
\|c(g, h)(\xi) - \xi\| < \delta, \quad d_{\nu}(c(g, h)(t), t) < \delta, \quad g, h \in S, \quad t \in \Lambda, \quad \xi, \nu \in \Phi
\]

for some cocycle crossed action \((\gamma', c), \Lambda \in [T]\) and \( \Phi \in M_1(X, \mu) \), then we can choose \( w \in C^1(G, [T]) \) so that it further satisfies

\[
\|w(g)(\xi) - \xi\| < \varepsilon, \quad d_{\nu}(w(g)(t), t) < \varepsilon, \quad g \in K, \quad \xi, \nu \in \Phi, \quad t \in \Lambda.
\]

**Proof.** Choose \( \varepsilon' > 0 \) with \( 11 \sqrt{\varepsilon'} < \varepsilon \), and let \( S' \subseteq G \) be a \((K \cup K^2, \varepsilon')\)-invariant set, and \( S = S' \cup K \). Choose \( \delta \) such that \( 5\delta|S| + 11\sqrt{\varepsilon'} < \varepsilon \).

By applying Proposition 3.3, we can take Rohlin partition \( \{E_s\}_{s \in S'} \subseteq \mathcal{B}, \quad Z_s \subseteq E_s, \quad w(g) \in [T], \quad g \in K \), such that

1. \( E_{l(s)} \supset (c(g, s)^{-1}Z_l(s), \quad g \in K \cup K^2, \quad s \in S' \),
2. \( \nu(E_s \setminus Z_s) < \delta, \quad \nu(E_{l(s)} \setminus (c(g, s)^{-1}Z_l(s)), \quad g \in K \cup K^2, \quad s \in S', \quad \nu \in \Phi),
3. \( \nu((c(gh, k)^{-1}c(g, h)^{-1}\gamma_g(c(h, k))(E_{ghk} \setminus Z_{ghk})) < \delta, \quad g, h \in K, \quad k \in S'_{gh} \cap S_h', \quad \nu \in \Phi),
4. \( \nu((c(gh, k)^{-1}c(g, h)^{-1}\gamma_g(E_{ghk} \setminus Z_{hkk})) < \delta, \quad g, h \in K, \quad k \in S'_{gh} \cap S_h', \quad \nu \in \Phi),
5. \( \nu(E_{ghk} \setminus c(g, h)^{-1}c(g, h)^{-1}(E_{ghh} \setminus \gamma_gE_{ghh})) < \delta, \quad g, h \in K, \quad k \in S'_{gh} \cap S_h', \quad \nu \in \Phi),
6. \( \nu((c(gh, k)^{-1}c(g, h)^{-1}(E_{ghk} \setminus Z_{ghk})) < 5\sqrt{\varepsilon'}, \quad g, h \in K, \quad \nu \in \Phi),
7. \( \sum_{k \in S'_{gh} \cap S_h'} \nu((c(gh, k)^{-1}c(g, h)^{-1}(E_{ghh} \setminus \gamma_gE_{ghh})) < 3\sqrt{\varepsilon'}, \quad g \in K \cup K^2, \quad \nu \in \Phi),
8. \( \sum_{k \in S' \setminus (S'_{gh})^{-1}} \nu(E_s) < 3\sqrt{\varepsilon'}, \quad g \in K \cup K^2, \quad \nu \in \Phi),
9. \( w(g)x = c(g, s)^{-1}x, \quad x \in Z_l(s), \quad g \in K, \quad s \in S' \).

Here we applied Proposition 3.3 for

\[
B = \{c(g, s)^{-1} | g \in K, \quad s, \in S'\} \cup \{c(gh, k)^{-1}c(g, h)^{-1} | g, h \in K, \quad k \in S'\}
\]

\[
\cup \{c(gh, k)^{-1}c(g, h)^{-1}\gamma_g(c(h, k)) | g, h \in K, \quad k \in S'\},
\]

and

\[
C = \Phi \cup \{\nu((c(gh, k)^{-1}c(g, h)^{-1}\gamma_g(c(h, k)) \cdot ) | \nu \in \Phi, \quad g, h \in K, \quad k \in S'\}
\]

\[
\cup \{\nu((c(gh, k)^{-1}c(g, h)^{-1}\gamma_g \cdot ) | \nu \in \Phi, \quad g, h \in K, \quad k \in S'\}
\]

\[
\cup \{\nu((c(gh, k)^{-1}c(g, h)^{-1} \cdot ) | \nu \in \Phi, \quad g, h \in K, \quad k \in S'\}.
\]

We define \( w(g) = \text{id} \) if \( g \notin G \).

Let

\[
W_{g,h}^0 = c(g, h)^{-1}Z_{ghk} \cap c(g, h)^{-1}c(g, h)^{-1}\gamma_gZ_{hk} \cap c(g, h)^{-1}c(g, h)^{-1}\gamma_g(c(h, k))Z_{ghk}
\]

for \( k \in S'_{gh} \cap S_h' \) and

\[
W_{g,h} = \bigcup_{k \in S'_{gh} \cap S_h'} W_{g,h}^0.
\]
We can verify $w(g)\hat{\gamma}_g(w(h))c(g, h)w(gh)^{-1} = \text{id}$ on $W_{g,h}$ as follows. Take $x \in W_{g,h}^0$. Since $x \in c(g, h)^{-1}Z_{ghk}$, we have $w(gh)^{-1}x = c(g, h)x$. Thus we have $\gamma_g^{-1}c(g, h)w(gh)^{-1}x = \gamma_g^{-1}c(g, h)c(g, k)x$. Since $x \in c(g, h)^{-1}c(g, h)^{-1}\gamma_g Z_{hk}$, $\gamma_g^{-1}c(g, h)c(g, k)x \in Z_{hk}$ holds. Hence we have

$$w(h)\gamma_g^{-1}c(g, h)c(g, k)x = c(g, h)^{-1}\gamma_g^{-1}c(g, h)c(g, k)x.$$

Since $x \in c(g, h)^{-1}c(g, h)^{-1}c(g, h)Z_{ghk}$,

$$\hat{\gamma}_g(w(h))c(g, h)w(gh)^{-1}x = \hat{\gamma}_g(c(h, k))^{-1}c(g, h)c(g, k)x \in Z_{ghk}$$

holds, and hence we have

$$w(g)\gamma_g c(g, h)^{-1}c(g, h)c(g, k)x = c(g, h)^{-1}\gamma_g c(g, h)^{-1}c(g, h)c(g, k)x = x$$

by the 2-cocycle identity. This shows $w,c(g, h) = w(g)\hat{\gamma}_g(w(h))c(g, h)w(gh)^{-1} = \text{id}$ on $W_{g,h}$. Thus we have $\{w,c(g, h) \neq \text{id}\} \subset X \setminus W_{g,h}$.

We will show $\nu(X \setminus W_{g,h}) < \varepsilon$ for $\nu \in \Phi$. By (2), we have

$$\nu(E_{ghk}\setminus c(g, h, k)^{-1}Z_{ghk}) < \delta, \ \nu \in \Phi, g, h \in K k \in S'_{gh}.$$

For $g, h \in K, k \in S'_{gh} \cap S'_{h}$, $\nu \in \Phi$, we have

$$\nu(E_{ghk}\Delta c(g, h, k)^{-1}c(g, h)^{-1}\gamma_g Z_{hk})$$

$$\leq \nu(E_{ghk}\Delta c(g, h, k)^{-1}c(g, h)^{-1}E_{ghk}) + \nu\left(c(g, h, k)^{-1}c(g, h)^{-1}(E_{ghk}\Delta \gamma_g Z_{hk})\right)$$

$$\leq \delta + \nu\left(c(g, h, k)^{-1}c(g, h)^{-1}(E_{ghk}\Delta \gamma_g Z_{hk})\right) \quad \text{(by (6))}$$

$$\leq \delta + \nu\left(c(g, h, k)^{-1}c(g, h)^{-1}(E_{ghk}\Delta \gamma_g E_{hk})\right) + \nu\left(c(g, h, k)^{-1}c(g, h)^{-1}\gamma_g (E_{hk}\Delta Z_{hk})\right)$$

$$\leq 2\delta + \nu\left(c(g, h, k)^{-1}c(g, h)^{-1}(E_{ghk}\Delta \gamma_g E_{hk})\right) \quad \text{(by (4))}$$

and

$$\nu\left(E_{ghk}\Delta c(g, h, k)^{-1}c(g, h)^{-1}\hat{\gamma}_g(c(h, k))\right)Z_{ghk}$$

$$\leq \nu\left(E_{ghk}\Delta c(g, h, k)^{-1}c(g, h)^{-1}\hat{\gamma}_g(c(h, k))\right)E_{ghk}$$

$$+ \nu\left(c(g, h, k)^{-1}c(g, h)^{-1}\hat{\gamma}_g(c(h, k))\right)(E_{ghk}\Delta Z_{ghk})$$

$$< 2\delta \quad \text{(by (5) and (3))}.$$

Thus

$$\nu(E_{ghk}\Delta W_{g,h,k}^0) \leq \nu(E_{ghk}\setminus c(g, h, k)^{-1}Z_{ghk}) + \nu(E_{ghk}\Delta c(g, h, k)^{-1}c(g, h)^{-1}\gamma_g Z_{hk})$$

$$+ \nu\left(E_{ghk}\Delta c(g, h, k)^{-1}c(g, h)^{-1}\gamma_g c(h, k)\gamma_g^{-1}Z_{ghk}\right)$$

$$< 5\delta + \nu\left(c(g, h, k)^{-1}c(g, h)^{-1}(E_{ghk}\Delta \gamma_g E_{hk})\right)$$

follows. Then

$$\nu\left(\left(\bigcup_{k \in S'_{gh} \cap S'_{h}} E_{ghk}\right)\Delta W_{g,h}\right) \leq \sum_{k \in S'_{gh} \cap S'_{h}} \nu\left(E_{ghk}\Delta W_{g,h,k}^0\right)$$

$$< 5\delta |S| + \sum_{k \in S'_{gh} \cap S'_{h}} \nu\left(c(g, h, k)^{-1}c(g, h)^{-1}(E_{ghk}\Delta \gamma_g E_{hk})\right)$$

$$< 5\delta |S| + 5\sqrt{\varepsilon} \quad \text{(by (7))}.$$
holds.

Finally, we have

\[
\nu(X \setminus W_{g,h}) \leq \nu \left( X \setminus \bigcup_{k \in S_{gh}'} E_{ghk} \right) + \nu \left( \bigcup_{k \in S_{gh}'} E_{ghk} \right) \Delta W_{g,h} \]

\[
= \nu \left( \bigcup_{k \in S \left( S_{gh} \setminus S_{(gh)}' \right)} E_k \right) + \nu \left( \bigcup_{k \in S \left( S_{gh} \setminus S_{(gh)}' \right)} E_{gh} \right) \Delta W_{g,h} \]

\[
< \sum_{k \in S \setminus S_{gh}'} \nu(E_k) + \sum_{k \in S \setminus S_{(gh)}'} \nu(E_{gh}) + 5\delta |S| + 5\sqrt{\varepsilon'}
\]

\[
< \varepsilon.
\]

(Note \( ghk \in S_{gh} \cap S_{gh} \) for \( k \in S_{gh} \).) This implies \( d_\nu(w(c(g,h), id)) < \varepsilon \) for \( g, h \in K, \nu \in \Phi \).

Assume

\[
\|c(g,h)(\xi) - \xi\| < \delta, \quad d_\nu(c(g,h)(t), (t)) < \delta, \quad g, h \in S, \quad t \in \Lambda, \quad \xi, \nu \in \Phi.
\]

We show

\[
\|w(g)(\xi) - \xi\| < \varepsilon, \quad d_\nu(w(g)(t), (t)) < \varepsilon, \quad g \in K, \quad \xi, \nu \in \Phi, \quad t \in \Lambda.
\]

Let \( Z := \bigcup_{s \in S'} Z_s \). By the definition of \( w(g), w(g)Z = \bigcup_{s \in S'} c(g,s)^{-1}Z_s \) holds. Then we have

\[
\|w(g)(\xi) - \xi\|
\]

\[
= \int_X |w(g)(\xi)(x) - \xi(x)|d\mu(x)
\]

\[
= \int_{w(g)Z} |w(g)(\xi)(x) - \xi(x)|d\mu(x) + \int_{X \setminus w(g)Z} |w(g)(\xi)(x) - \xi(x)|d\mu(x)
\]

\[
= \sum_{s \in S'} \int_{c(g,s)^{-1}Z_{(g)s}} |w(g)^{-1}(\xi)(x) - \xi(x)|d\mu(x) + \int_{X \setminus w(g)Z} |w(g)(\xi)(x) - \xi(x)|d\mu(x).
\]

Note

\[
w(g)(\xi)(x) = \xi(w(g)^{-1}x)\frac{d(\mu \circ w(g)^{-1})}{d\mu}(x) = \xi(c(g,s)x)\frac{d(\mu \circ c(g,s))}{d\mu}(x) = c(g,s)^{-1}(\xi)(x)
\]

for \( x \in c(g,s)^{-1}Z_{(g)s}, \) when we regard \( \xi \) as an element of \( L^1(X, \mu) \). Thus the first term is estimated as follows;

\[
\sum_{s \in S'} \int_{c(g,s)^{-1}Z_{(g)s}} |w(g)^{-1}(\xi)(x) - \xi(x)|d\mu(x)
\]

\[
= \sum_{s \in S'} \int_{c(g,s)^{-1}Z_{(g)s}} |c(g,s)^{-1}(\xi)(x) - \xi(x)|d\mu(x)
\]

\[
\leq \sum_{s \in S'} \|c(g,s)^{-1}(\xi) - \xi\| < \delta |S|.
\]
To estimate the second term, one should note
\[ \xi(X \setminus Z) = \sum_{s \in S^{'}} \xi(E_s \setminus Z_s) < \delta|S'|, \quad \xi(X \setminus w(g)Z) = \sum_{s \in S^{'}} \xi(E_s \setminus c(g, s)^{-1}Z_s) < \delta|S'| \]
by (2). Hence
\[
\int_{X \setminus w(g)Z} |w(g)(\xi)(x) - \xi(x)|d\mu(x)
\leq \int_{X \setminus w(g)Z} w(g)(\xi)(x)d\mu(x) + \int_{X \setminus w(g)Z} \xi(x)d\mu(x)
= \int_{X \setminus w(g)Z} \xi(w(g)^{-1}x)d(\mu \circ w(\pi_g)^{-1})d\mu(x) + \xi(X \setminus w(g)Z)
= \int_{X \setminus Z} \xi(x)d\mu(x) + \xi(X \setminus w(g)Z) = \xi(X \setminus Z) + \xi(X \setminus w(g)Z) < 2|S'|\delta,
\]
and we obtain \( \|w(g)(\xi) - \xi\| < 3\delta|S'| < \varepsilon \).

We next show
\[ d_\nu(w(g)(t), t) < \varepsilon, \ g \in G, \ \nu \in \Phi, \ t \in \Lambda. \]
By the assumption
\[ \|c(g, s)(\nu) - \nu\| < \delta, \ d_\nu(c(g, s)(t), t) < \delta, \ t \in \Lambda, \ g, s \in S, \ \nu \in \Phi, \]
\[ d_\nu(c(g, s)^{-1}(t), t) = d_\nu(c(g, s)(\nu), t, c(g, s)(t)) \]
\[
\leq \|c(g, s)(\nu) - \nu\| + d_\nu(c(g, s)(t), t) < 2\delta
\]
holds. We can further assume
\[ \nu \left( E_{l(g)s}\Delta c(g, s)^{-1}t^{-1}E_{l(g)s} \right) < \delta, \ \nu \left( c(g, s)^{-1}t^{-1} \left( E_{l(g)s}\Delta Z_{l(g)s} \right) \right) < \delta \]
for \( t \in \Lambda, \ s \in S^{'}, \ g \in K \) in the choice of \( Z_s \) and \( E_s \).

Let \( B_{g,s,t} := \{ c(g, s)^{-1}(t) = t \} \). Then we have \( \nu(X \setminus B_{g,s,t}) < 2\delta, \ \nu \in \Phi \). We can see \( w(g)(t) = c(g, s)^{-1}(t) \) on \( c(g, s)^{-1}Z_{l(g)s} \cap c(g, s)t^{-1}Z_{l(g)s} \) as above. Thus \( w(g)(t) = t \) holds on
\[ \bigcup_{s \in S^{'}} c(g, s)^{-1}Z_{l(g)s} \cap c(g, s)t^{-1}Z_{l(g)s} \cap B_{g,s,t}. \]
We will show
\[ \nu \left( X \setminus \bigcup_{s \in S^{'}} c(g, s)^{-1}Z_{l(g)s} \cap c(g, s)t^{-1}Z_{l(g)s} \cap B_{g,s,t} \right) < \varepsilon. \]
At first, we have
\[
\nu \left( E_{l(g)s}\Delta \left( c(g, s)^{-1}Z_{l(g)s} \cap c(g, s)^{-1}t^{-1}Z_{l(g)s} \right) \right)
\leq \nu(E_{l(g)s}\Delta c(g, s)^{-1}Z_{l(g)s}) + \nu(E_{l(g)s}\Delta c(g, s)^{-1}t^{-1}Z_{l(g)s})
< \delta + \nu(E_{l(g)s}\Delta \cap c(g, s)^{-1}t^{-1}E_{l(g)s}) + \nu \left( c(g, s)^{-1}t^{-1} \left( E_{l(g)s}\Delta Z_{l(g)s} \right) \right) \quad \text{(by (2))}
< 3\delta.
\]
Thus

\[
\nu \left( X \setminus \bigcup_{s \in S'} c(g, s)^{-1} Z_{l(g)s} \cap c(g, s)^{-1} t^{-1} Z_{l(g)s} \cap B_{g,s,t} \right)
\]

\[
\leq \nu \left( X \setminus \bigcup_{s \in S'} c(g, s)^{-1} Z_{l(g)s} \cap c(g, s)^{-1} t^{-1} Z_{l(g)s} \right) + \nu \left( X \setminus \bigcup_{s \in S'} B_{g,s,t} \right)
\]

\[
\leq \sum_{s \in S'} \nu \left( E_{l(g)s} \triangle (c(g, s)^{-1} Z_{l(g)s} \cap c(g, s)^{-1} t^{-1} Z_{l(g)s}) \right) + \sum_{s \in S'} \nu \left( X \setminus B_{g,s,t} \right)
\]

\[
< 5|S'| \delta < \varepsilon
\]

holds, and we obtain \( d_\nu(\widehat{wx}(g)(t), t) < \varepsilon \) for \( g \in K, \nu \in \Phi, t \in \Lambda \). \( \square \)

**Lemma 4.3** For any \( e \in K \subset G \) and \( \varepsilon > 0 \), there exist \( S \subset G \) and \( \delta > 0 \) satisfying the following property: for any \( \mu \in \Phi \subset M_1(X, \mu) \), an ultrafree cocycle crossed action \((\gamma, c)\) of \( G \), and \( u \in C^1(G, [T]) \) with

\[
d_\nu(\widehat{\gamma}_g(u(s))^{-1}u(g)^{-1}u(gs), \text{id}) < \delta, \ g, s \in S, \nu \in \Phi,
\]

there exists \( w \in [T] \) such that

\[
d_\nu(w^{-1}u(g)\widehat{\gamma}_g(w), \text{id}) < \varepsilon, \ \nu \in \Phi, \ g \in K.
\]

**Proof.** Let \( K \subset G, \varepsilon > 0 \) be given. Take \( \varepsilon' > 0 \) such that \( 8\sqrt{\varepsilon'} < \varepsilon \). Let \( S' \) be a \((K, \varepsilon')\)-invariant set, and set \( S = S' \cup K \). Choose \( \delta > 0 \) such that \( 4|S'| \delta + 8\sqrt{\varepsilon'} < \varepsilon \). Let a cocycle crossed action \((\gamma, c), \Phi \subset M_1(X, \mu) \), and \( u \in C^1(G, [T]) \) satisfying the condition

\[
d_\nu(\widehat{\gamma}_g(u(s))^{-1}u(g)^{-1}u(gs), \text{id}) < \delta, \ g, s \in S, \nu \in \Phi
\]

be given. By Proposition 3.3, choose a partition \( \{E_s\}_{s \in S} \) of \( X, \ E_s \supset Z_s \) and \( w \in [T] \) such that

1. \( u(s)Z_s \subset E_s \),
2. \( \nu(\gamma_g(E_s \setminus Z_s)) < \delta, \nu(E_s \setminus u(s)Z_s) < \delta, \ g \in K, \nu \in \Phi \),
3. \( \nu(\widehat{\gamma}_g(u(s))^{-1}u(g)^{-1}(E_{gs} \setminus Z_{gs})) < \delta, \ g \in K, s \in S_g', \nu \in \Phi \),
4. \( \nu(E_{gs} \triangle \widehat{\gamma}_g(u(s))^{-1}g^{-1}E_{gs}) < \delta, \ g \in K, s \in S_g', \nu \in \Phi \),
5. \( \sum_{s \in S_g'} \nu(E_{gs} \triangle \gamma_g E_s) < 5\sqrt{\varepsilon'}, \ g \in K, \nu \in \Phi \),
6. \( \sum_{s \in S \setminus S_{g-1}} \nu(E_s) < 3\sqrt{\varepsilon'}, \ g \in K, \nu \in \Phi \),
7. \( wx = u(s)x, \ x \in Z_s \).

Let

\[
W_g := \bigcup_{s \in S_g'} \{\widehat{\gamma}_g(u(s))^{-1}u(g)^{-1}u(gs) = \text{id}\} \cap \gamma_g Z_s \cap \widehat{\gamma}_g(u(s))^{-1}u(g)^{-1}Z_{gs}.
\]

We can verify that \( w^{-1}u(g)\widehat{\gamma}_g(w) = \text{id} \) on \( W_g, \ g \in K \), as in the proof of Lemma 4.2
Next we show $\nu(X \setminus W_g) < \varepsilon$. We have

$$
\nu(E_{gs} \triangle \gamma_s Z_s) \leq \nu(E_{gs} \triangle \gamma g E_s) + \nu(\gamma g E_s \setminus \gamma_s Z_s) < \nu(E_{gs} \triangle \gamma g E_s) + \delta
$$

by (2), and

$$
\nu(E_{gs} \triangle \gamma_g(u(s))^{-1}u(g)^{-1}Z_{gs})
\leq \nu(E_{gs} \triangle \gamma_g(u(s))^{-1}u(g)^{-1}E_{gs}) + \nu(\gamma_g(u(s))^{-1}u(g)^{-1}(E_{gs} \setminus Z_{gs})) < 2\delta
$$

by (3) and (4). Hence we have

$$
\nu(E_{gs} \triangle (\gamma_s Z_s \cap \gamma_g(u(s))^{-1}u(g)^{-1}E_{gs})) < 3\delta + \nu(E_{gs} \triangle \gamma g E_s).
$$

Then we have

$$
\nu \left( \bigcup_{s \in S'_g} E_{gs} \triangle \bigcup_{s \in S'_g} (\gamma_s Z_s \cap \gamma_g(u(s))^{-1}u(g)^{-1}E_{gs}) \right)
\leq \sum_{s \in S'_g} \nu(E_{gs} \triangle (\gamma_s Z_s \cap \gamma_g(u(s))^{-1}u(g)^{-1}E_{gs}))
< \sum_{s \in S'_g} (3\delta + \nu(E_{gs} \triangle \gamma g E_s)) < 3|S'|\delta + 5\sqrt{\varepsilon'}
$$

by (5). Hence we get

$$
\nu \left( X \setminus \bigcup_{s \in S'_g} (\gamma_s Z_s \cap \gamma_g(u(s))^{-1}u(g)^{-1}E_{gs}) \right)
\leq \sum_{s \in S \setminus S'_{g-1}} \nu(E_s) + \nu \left( \bigcup_{s \in S'_g} E_{gs} \triangle \bigcup_{s \in S'_g} (\gamma_s Z_s \cap \gamma_g(u(s))^{-1}u(g)^{-1}E_{gs}) \right)
< 3|S'|\delta + 8\sqrt{\varepsilon'}
$$

by (6). By the assumption

$$
d_\nu(\gamma_g(u(s))^{-1}u(g)^{-1}u(gs), id) < \delta, \ g, s \in S, \ \nu \in \Phi,
$$

we have $\nu(X \setminus \bigcup_{s \in S'_g} \{\gamma_g(u(s))^{-1}u(g)^{-1}u(gs) = id\}) < |S'|\delta$. Hence

$$
\nu(X \setminus W_g) < 4|S'|\delta + 8\sqrt{\varepsilon'} < \varepsilon
$$

holds.

\[\square\]

**Theorem 4.4** Let $(\gamma, c)$ be an ultrafree cocycle crossed action of $G$. Then there exists $u \in C^1(G, [T])$ such that $u c(g, h) = id$, and hence $u \gamma$ is an action.

Moreover, for any $e \in K \subseteq G, \ \varepsilon > 0$, there exists $S \subseteq G, \ \delta > 0$, which depends only on $K$ and $\varepsilon$, nor on cocycle crossed action $(\gamma, c)$, such that if

$$
d_\nu(c(g, h), id) < \delta, \ g, h \in S, \ \nu \in \Phi
$$

for some $\Phi \subseteq M_1(X, \mu)$ with $\mu \in \Phi$, then we can choose $u \in C^1(G, [T])$ so that

$$
d_\nu(u(g), id) < \varepsilon, \ g \in K, \ \nu \in \Phi.
$$
Proof. At first, we treat type \( \Pi_\infty \) or type III case.

Let \( e \in K \subset G \) and \( \varepsilon > 0 \) be given, and take \( S \subset G \) and \( \delta > 0 \) as in Lemma 4.3. Assume \( d_\nu(c(g,h), id) < \delta \) for \( g, h \in S, \nu \in \Phi \in M_1(X, \mu) \). There exists \( v \in C^1(G, [T]) \) such that \( v c(g,h) = id \) by Theorem 4.11. Hence \( c(g,h) = \gamma_g(v(h))^{-1} v(g)^{-1} v(gh) \) holds, and

\[
d_\nu(\gamma_g(v(h))^{-1} v(g)^{-1} v(gh), id) < \delta, \quad g, h \in S, \nu \in \Phi.
\]

By Lemma 4.3 there exists \( w \in [T] \) such that

\[
d_\nu(w^{-1} v(g) \gamma_g(w), id) < \varepsilon, \quad \nu \in \Phi, \quad g \in K.
\]

Define \( u(g) := w^{-1} v(g) \gamma_g(w) \). Then we obtain \( d_\nu(u(g), id) < \varepsilon \) for \( g \in K, \nu \in \Phi, \) and

\[
u c(g,h) = u(g) \gamma_g(u(h)) c(g,h) u(gh)^{-1} = w^{-1} v(g) \gamma_g(v(h)) c(g,h) v(gh)^{-1} w = id.
\]

Hence we have proved the theorem for type \( \Pi_\infty \) and type III case.

Next, we assume \( T \) is of type \( \Pi_1 \). In this case, we can assume that \( \mu \) is the unique \( T \)-invariant probability measure, and choose \( \Phi \) as \( \Phi = \{ \mu \} \). Let us take an increasing sequence \( \{K_n\}_n \subset G \), and decreasing sequence \( \{\varepsilon_n\}_n \) such that \( e \in K_n, \bigcup_{n=1}^\infty K_n = G; \) and \( \sum_n \varepsilon_n < \infty \). Take \( S_n \) and \( \delta_n \) for \( K_n \) and \( \varepsilon_n > 0 \) as in Lemma 4.3. We can choose \( S_{n+1}, \delta_n \) so that \( S_n \subset S_{n+1}, \delta_n > \delta_{n+1} \).

For given \( K \subset G, \) and \( \varepsilon > 0 \), choose \( N \in \mathbb{N} \) such that \( K \subset K_N, \varepsilon > \sum_{k=N}^\infty \varepsilon_k \). By Lemma 4.2 take \( S_N \subset G \) and \( \delta_N > 0 \) for \( K_N \) and \( \varepsilon_N > 0 \). Again by Lemma 4.2 we can perturb \( (\gamma, c) \) by some \( w \in C^1(G, [T]) \) so that

\[
d_\mu(w c(g,h), id) < \varepsilon_N, \quad g, h \in K_N; \quad d_\mu(w c(g,h), id) < \delta_N, \quad g, h \in S_N.
\]

Set

\[
(\gamma^{(N)}, c_N) := (w \gamma, w c), \quad u_N(g) = 1.
\]

We will inductively construct a family of cocycle crossed actions \( (\gamma^{(n)}, c_n) \) and normalized maps \( \{u_n\} \subset C^1(G, [T]) \) \( n \geq N \), such that

\[
(1.n) \quad (\gamma^{(n)}, c_n) = (u_n \gamma^{(n-1)}, u_n c_{n-1}),
\]

\[
(2.n) \quad d_\mu(c_n(g,h), id) < \varepsilon_n, \quad g, h \in K_n,
\]

\[
(3.n) \quad d_\mu(c_n(g,h), id) < \delta_n, \quad g, h \in S_n,
\]

\[
(4.n) \quad d_\mu(u_n(g), id) < \varepsilon_{n-1}, \quad g \in K_{n-1}.
\]

Here we regard \( \gamma^{(N-1)} = \gamma^{(N)}, c_{N-1}(c,h) = c_N(g,h) \). Clearly we have (1.N), (2.N), (3.N) and (4.N).

Assume we have done up to the \( n \)-th step.

By Lemma 4.2 we choose \( u_{n+1} \in C^1(G, [T]) \) such that

\[
(a.n+1) \quad d_\mu \left( \gamma^{(n)}(u_{n+1}(g)) c_{n}(g,h) u_{n+1}(gh)^{-1}, id \right) < \varepsilon_{n+1}, \quad g, h \in K_{n+1},
\]

\[
(b.n+1) \quad d_\mu \left( \gamma^{(n)}(u_{n+1}(g)) c_{n}(g,h) u_{n+1}(gh)^{-1}, id \right) < \frac{\delta_{n+1}}{2}, \quad g, h \in S_{n+1}.
\]
By (b.n + 1), we have
\[ d_\mu \left( \gamma_g^{(n)}(\bar{u}_{n+1}(h))^{-1}\bar{u}_{n+1}(g)^{-1}\bar{u}_{n+1}(gh), c_n(g, h) \right) < \frac{\delta_{n+1}}{2}, \ g, h \in S_{n+1}. \]
Combining with (3.n), we get
\[ d_\mu \left( \gamma_g^{(n)}(\bar{u}_{n+1}(h))^{-1}\bar{u}_{n+1}(g)^{-1}\bar{u}_{n+1}(gh), \text{id} \right) < \delta_n, \ g, h \in S_n. \]
By Lemma 4.3, there exists \( w \in [T] \) such that \( d_\mu(w^{-1}\bar{u}_{n+1}(g)\gamma_g^{(n)}(w), \text{id}) < \varepsilon_n \) for \( g \in K_n \).
Here set \( u_{n+1}(g) := w^{-1}\bar{u}_{n+1}(g)\gamma_g^{(n)}(w) \). Then we get (4.n + 1). Define a cocycle crossed action \((\gamma^{(n+1)}, c_{n+1})\) as \((1.n + 1)\). Then we get (2.n + 1) and (3.n + 1) from (a.n + 1) and (b.n + 1), respectively, and complete induction.

Let \( v_n(g) := u_n(g)u_{n-1}(g)\cdots u_1(g) \). We have \((\gamma^n, c_n) = (v_n\gamma^{(n)}, v_n \gamma^{(n)}c_n)\) by the construction. Fix \( L \in \mathbb{N} \), and take any \( g \in K_L \). By (4.n),
\[ d_\mu(v_n(g), v_{n-1}(g)) = d_\mu(u_n(g), \text{id}) < \varepsilon_{n-1}, \ n \geq L + 1 \]
holds. So \( \{v_n(g)\}_n \) is a Cauchy sequence, and hence \( v_n(g) \) converges to some \( v(g) \in [T] \) uniformly. Note that \( v_n(g)^{-1} \) converges to \( v(g)^{-1} \) automatically, since \( \mu \) is the invariant measure for \([T]\). Combining with (2.n), we obtain \( v(c, h) = \text{id} \) for all \( g, h \in G \).

If \( g \in K_N \), then
\[ d_\mu(v_n(g), \text{id}) = d_\mu(v_n(g), v_N(g)) \leq \sum_{k=N}^{n-1} d_\mu(v_{k+1}(g), v_k(g)) < \sum_{k=N}^{n-1} \varepsilon_k. \]
Hence we have \( d_\mu(v(g), \text{id}) \leq \sum_{k=N}^{\infty} \varepsilon_k < \varepsilon \). Set \( S := S_N \cup K_N \), \( \delta := \min\{\delta_N/2, \varepsilon_N\} \). If \( d_\mu(g, h) < \delta \) for \( g, h \in S \), then we have \( d_\mu(v(g), \text{id}) < \varepsilon \) for \( g \in K_N \). Note that \( S \) and \( \delta \) are determined only on \( K \) and \( \varepsilon \).

5 Classification

**Lemma 5.1** Let \( \alpha \) and \( \beta \) be actions of \( G \) into \( N[T] \) with \( \text{mod}(\alpha_g) = \text{mod}(\beta_g) \). Then for any \( \varepsilon > 0 \), \( K \in G \), \( \mu \in \Phi \in M_1(X, \mu) \), \( \Lambda \in [T] \), there exists \( w \in C^1(G, [T]) \) such that

1. \( \|w\alpha_g(\xi) - \beta_g(\xi)\| < \varepsilon, \ g \in K, \ \xi \in \Phi, \)
2. \( d_\nu(w\alpha_g(t), \beta_g(t)) < \varepsilon, \ g \in K, \ t \in \Lambda, \nu \in \Phi, \)
3. \( \nu(c(g, h))(\xi - \xi) < \varepsilon, \ d_\nu(c(g, h)(t), (t)) < \varepsilon, \ g, h \in K, \xi, \nu \in \Phi, \ t \in \Lambda. \)

**Proof.** By enlarging \( K \), we may assume \( e \in K = K^{-1} \subseteq G \). Let
\[ \tilde{\Phi} := \{\beta_{gh}(\xi) \mid g, h \in K, \xi \in \Phi\}, \ \tilde{\Lambda} := \{\beta_{gh}(t) \mid g, h \in K, t \in \Lambda\}. \]
By the assumption, \( \beta_g \alpha_g^{-1} \in \text{Ker}(\text{mod}) = [T] \). Hence we can take \( w \in C^1(G, [T]) \) so that
\[ \|w\alpha_{gh}(\xi) - \beta_{gh}(\xi)\| < \frac{\varepsilon}{7}, \ d_\nu(w\alpha_{gh}(t), \beta_{gh}(t)) < \frac{\varepsilon}{7}. \]
for \( g, h \in K, \nu, \xi \in \bigcup_{g \in K} \beta_g(\Phi) \), \( t \in \bigcup_{g \in K} \beta_g(\Lambda) \). Obviously, we have conditions (1), (2).

Then for \( g, h \in K, \eta \in \Phi \), we have
\[
\| v_{\alpha_g} \alpha_h(\eta) - \beta_{gh}(\eta) \| \leq \| v_{\alpha_g} \alpha_h(\eta) - v_{\alpha_h} \beta_h(\eta) \| + \| v_{\alpha_h} \beta_h(\eta) - \beta_{gh}(\eta) \| \\
\leq \| v_{\alpha_h} \beta_h(\eta) \| + \| v_{\alpha_h} \beta_h(\eta) - \beta_{gh}(\eta) \| < \frac{2\varepsilon}{t}.
\]
Thus
\[
\| c(g, h)\beta_{gh}(\eta) - \beta_{gh}(\eta) \| \leq \| c(g, h)\beta_{gh}(\eta) - c(g, h)\alpha_{gh}(\eta) \| + \| c(g, h)\alpha_{gh}(\eta) - \beta_{gh}(\eta) \| \\
< \frac{3\varepsilon}{7}
\]
holds for \( g, h \in K, \eta \in \Phi \). Hence we get \( \| c(g, h)(\xi) - \xi \| < \frac{3\varepsilon}{7} \) for \( g, h \in K, \xi \in \Phi \).

For \( g \in K, t \in \Lambda, \nu \in \Phi \), we have
\[
d_{\nu}(\hat{c}(g, h)\nu) = d_{\nu}(\hat{v}_{\alpha_g} \alpha_h(t), \hat{\beta}_{gh}(t)) \\
\leq d_{\nu}(\hat{v}_{\alpha_g} \alpha_h(t), \hat{v}_{\alpha_h} \beta_h(t)) + d_{\nu}(\hat{v}_{\alpha_h} \beta_h(t), \hat{\beta}_{gh}(t)) \\
\leq d_{\nu}(\hat{v}_{\alpha_g} \nu(t), \hat{\beta}_{gh}(t)) + \frac{\varepsilon}{t} \\
\leq d_{\nu}(\hat{\beta}_{gh}(t), \hat{\beta}_{gh}(t)) + \frac{2\varepsilon}{t} < \frac{3\varepsilon}{7}.
\]
By noting \( \| c(g, h)(\nu) - \nu \| \leq \frac{3\varepsilon}{7} \) for \( g, h \in K, \nu \in \Phi \), we have
\[
d_{\nu}(\hat{c}(g, h)\nu) < \frac{\varepsilon}{t} \quad \text{holds for} \quad g, h \in K, \nu \in \Phi.
\]

**Lemma 5.2** Let \( \alpha \) and \( \beta \) be actions of \( G \) into \( N[T] \) with \( \text{mod}(\alpha_g) = \text{mod}(\beta_g) \). For any \( \varepsilon > 0, K \subseteq G, \Lambda \subseteq [T], \Phi \subseteq M_1(X, \mu) \), there exists \( v \in C^1(G, [T]) \) such that
\[
\| v_{\alpha_g}(\xi) - \beta_{gh}(\xi) \| < \varepsilon, \ g \in K, \nu \in \Phi,
\]
\[
d_{\nu}(\hat{v}_{\alpha_g}(t), \hat{\beta}_g(t)) < \varepsilon, \ g \in K, t \in \Lambda, \nu \in \Phi,
\]
\[
d_{\nu}(v(g)\alpha_g(v(h))v(gh)^{-1}, \text{id}) < \varepsilon, \ g, h \in K, \nu \in \Phi.
\]

**Proof.** Let \( \tilde{\Phi} := \{ \beta_g(\xi) \mid g \in K, \xi \in \Phi \}, \tilde{\Lambda} := \{ \hat{\beta}_g(t) \mid g \in K, t \in \Lambda \} \). Choose \( \delta > 0 \) and \( S \) for \( \varepsilon/3 > 0 \) and \( K \) as in Lemma 4.2. By Lemma 5.1 there exists \( u \in C^1(G, [T]) \) such that
\[
\| v_{\alpha_g}(\xi) - \beta_g(\xi) \| < \frac{\varepsilon}{3}, \ g \in K, \xi \in \Phi,
\]
\[
\| c(g, h)(\xi) - \xi \| < \delta, \ g, h \in S, \xi \in \Phi,
\]
\[
d_{\nu}(\hat{c}(g, h)\nu(t), \nu) < \delta, \ g, h \in S, t \in \Lambda, \nu \in \Phi.
\]
where \( c(g, h) = u(g)\alpha_g(u(h))u(gh)^{-1} \). By Lemma 4.2 there exists \( w \in C^1(G, [T]) \) such that
\[
d_\nu(w(g)u\hat{\alpha}_g(w(h)))c(g, h)w(gh)^{-1}, \text{id}) < \frac{\varepsilon}{3}, \quad g, h \in K, \nu \in \Phi
\]
and
\[
\|w(g)(\xi) - \xi\| < \frac{\varepsilon}{3}, \quad d_\nu(w(g)(t), t) < \frac{\varepsilon}{3}, \quad g \in K, \xi, \nu \in \tilde{\Phi}, \ t \in \tilde{\Lambda}.
\]

Let \( v(g) := w(g)u(g) \). Then we have
\[
d_\nu(v(g)\hat{\alpha}_g(v(h))v(gh)^{-1}, \text{id}) < \varepsilon, \quad g, h \in K, \nu \in \Phi.
\]
We can verify the first inequality as follows. For \( g \in K, \xi \in \Phi \),
\[
\|v\alpha_g(\xi) - \beta_g(\xi)\| \leq \|w(g)u(\alpha_g(\xi)) - w(g)\beta_g(\xi)\| + \|w(g)\beta_g(\xi) - \beta_g(\xi)\|
\]
\[
< \frac{2\varepsilon}{3} < \varepsilon
\]
since \( \beta_g(\xi) \in \tilde{\Phi} \). Similarly, we have
\[
d_\nu(v\hat{\alpha}_g(t), \hat{\beta}_g(t)) \leq d_\nu(w(g)u\hat{\alpha}_g(t), w(g)\hat{\beta}_g(t)) + d_\nu(w(g)\hat{\beta}_g(t), \hat{\beta}_g(t))
\]
\[
\leq d_\nu(w(g)(v\hat{\alpha}_g(t)), \hat{\beta}_g(t)) + \frac{\varepsilon}{3}
\]
\[
\leq \|w(g)(v) - v\| + d_\nu(u\hat{\alpha}_g(t), \hat{\beta}_g(t)) + \frac{\varepsilon}{3} < \varepsilon
\]
for \( g \in K, \ t \in \Lambda, \nu \in \Phi \).

\[\square\]

**Theorem 5.3** Let \( \alpha \) and \( \beta \) be ultrafree actions of \( G \) into \( N[T] \) with \( \text{mod}(\alpha_g) = \text{mod}(\beta_g) \). Then there exists a sequence \( \{u_n(\cdot)\} \) of 1-cocycles for \( \alpha_g \) such that \( \lim_{n \to \infty} u_n \alpha_g = \beta_g \) in the \( u \)-topology.

**Proof.** By Lemma 5.2 there exists a sequence \( \{v_n\} \subset C^1(G, [T]) \) of normalized maps such that \( \lim_{n \to \infty} v_n \alpha_g = \beta_g \) in the \( u \)-topology, and \( \lim_{n \to \infty} d_\nu(v_n(g)\hat{\alpha}_g(v_n(h))v_n(gh)^{-1}, \text{id}) = 0 \).

Let \( \alpha^{(n)} = v_n \alpha \) and \( c_n(e, h) = v_n(g)\hat{\alpha}_g(v_n(h))v_n(gh)^{-1} \). By Theorem 4.4 there exists a sequence \( \{w_n\} \subset C^1(G, [T]) \) such that
\[
w_n(g)\hat{\alpha}_g^{(n)}(w_n(h))c_n(g, h)w_n(gh)^{-1} = 1, \quad \lim_{n \to \infty} d_\mu(w_n(g), \text{id}) = 0.
\]
Then it turns out that \( u_n(g) := w_n(g)v_n(g) \) is a 1-cocycle for \( \alpha_g \), and \( \lim_{n \to \infty} u_n \alpha_g = \beta_g \) holds in the \( u \)-topology.

\[\square\]

**Lemma 5.4** Let \( K \subseteq G \) and \( \varepsilon > 0 \) be given. Then there exist \( S \subseteq G \) and \( \delta > 0 \) satisfying the following; for any action \( \gamma \) of \( G \), a 1-cocycle \( u(\cdot) \) for \( \gamma \), \( \Phi \subseteq M_1(X, \mu) \) with \( \mu \in \Phi \) and \( \Lambda \subseteq [T] \) satisfying
\[
\|u(s)(\xi) - \xi\| < \delta, \quad d_\nu(u(s)(t), t) < \delta, \quad s \in S, \xi, \nu \in \Phi, \ t \in \Lambda,
\]
there exists \( w \in [T] \) such that
\[
d_\nu(u(g)\hat{\gamma}_g(w)w^{-1}, 1) < \varepsilon, \quad \|w(\xi) - \xi\| < \varepsilon, \quad d_\nu(w(t), t) < \varepsilon, \quad g \in K, \xi, \nu \in \Phi, \ t \in \Lambda.
\]
Proof. Take $\varepsilon_1 > 0$ with $8\varepsilon_1^{\frac{1}{2}} < \varepsilon$, and let $S$ be a $(K, \varepsilon_1)$-invariant set. Choose $\delta > 0$ with $8\varepsilon_1^{\frac{1}{2}} + 3|S|\delta < \varepsilon$, $4|S|\delta < \varepsilon$.

By Proposition 3.3 take a partition $\{E_s\}_{s \in S}$ of $X$, $Z_s \subseteq E_s$ and $w \in [T]$ such that

1. $u(s)Z_s \subseteq E_s$, $s \in S$,
2. $\nu(E_s \setminus Z_s) < \delta$, $\nu(E_s \setminus u(s)Z_s) < \delta$, $s \in S$, $\nu \in \Phi$,
3. $\nu(u(gs)\gamma_g(E_s \setminus Z_s)) < \delta$, $g \in K$, $s \in S_g$, $\nu \in \Phi$,
4. $\nu(u(s)E_s \setminus t^{-1}(E_s \setminus Z_s)) < \delta$, $s \in S$, $t \in \Lambda$, $\nu \in \Phi$,
5. $\nu(u(s)E_s \setminus E_s) < \delta$, $s \in S$, $\nu \in \Phi$,
6. $\nu(E_{gs} \setminus \gamma_g(u(s))E_{gs}) < \delta$, $s \in S_g$, $\nu \in \Phi$,
7. $\nu(E_{gs} \setminus u(s)t^{-1}E_{gs}) < \delta$, $s \in S$, $t \in \Lambda$, $\nu \in \Phi$,
8. $\sum_{s \in S_g} u(gs)^{-1}(\nu)(\gamma_g E_s \setminus E_{gs}) < 5\varepsilon_1^{\frac{1}{2}}, g \in K, \nu \in \Phi$,
9. $\sum_{s \in S \setminus S_{g-1}} \nu(E_s) < 3\varepsilon_1^{\frac{1}{2}}, g \in K$,
10. $wx = u(s)x$, $x \in Z_s$.

In the following proof, the letter $g$, $s$, and $\nu$ denote an element in $K$, $S$, and $\Phi$, respectively. As in the proof of Lemma 4.2 we can see that

$$u(g)\gamma_g(w)w^{-1}x = u(g)\gamma_g(u(s)\gamma_g^{-1}u(gs)^{-1}x = x$$

for $x \in u(gs)Z_{gs} \cap u(gs)\gamma_gZ_s$.

We have

$$\nu(E_{gs} \setminus (u(gs)Z_{gs} \cap u(gs)\gamma_gZ_s))$$
$$\leq \nu(E_{gs} \setminus u(gs)Z_{gs}) + \nu(E_{gs} \setminus u(gs)\gamma_gZ_s)$$
$$< \delta + \nu(E_{gs} \setminus u(gs)\gamma_gE_s) + \nu(u(gs)\gamma_g(E_s) \setminus u(gs)\gamma_gZ_s) \quad \text{(by (2))}$$
$$< 2\delta + \nu(E_{gs} \setminus u(gs)E_{gs}) + \nu(u(gs)(E_{gs} \setminus \gamma_gE_s)) \quad \text{(by (3))}$$
$$< 3\delta + u(gs)^{-1}(\nu)(E_{gs} \setminus \gamma_gE_s) \quad \text{(by (5))}.$$
We next show \(|w(\xi) - \xi| < \varepsilon\) and \(d_\nu(\hat{w}(t), t) < \varepsilon\). Let \(Z = \bigcup_{s \in S} Z_s\). As in the proof of Lemma 4.2 we can see \(w(\xi)(x) = u(s)(\xi)(x)\) on \(u(s)Z_s\), and

\[
\int_{X \setminus wZ} |w(\xi)(x) - \xi(x)|d\mu(x) < 2|S|\delta
\]

by using (2) and (10). If \(u(s)\) satisfies \(|u(s)(\xi) - \xi| < \delta\) for \(s \in S\), then

\[
||w(\xi) - \xi|| = \sum_{s \in S} \int_{u(s)Z_s} |w(\xi)(x) - \xi(x)|d\mu(x) + \int_{X \setminus wZ} |w(\xi)(x) - \xi(x)|d\mu(x)
\]

\[
< \sum_{s \in S} \int_{u(s)Z_s} |u(s)(\xi)(x) - \xi(x)|d\mu(x) + 2|S|\delta < 3|S|\delta < \varepsilon
\]

holds for \(\xi \in \Phi\).

For \(t \in \Lambda \subset [T]\), and \(x \in u(s)Z_s \cap u(s)t^{-1}Z_s\), \(w^{-1}x = u(s)^{-1}x \in Z_s \cap t^{-1}Z_s\). Hence \(tw^{-1}x = u(s)^{-1}x \in tZ_s \cap Z_s\), and \(wtw^{-1}x = u(s)tu(s)^{-1}x\) holds.

Then

\[
\nu\left(E_s \Delta (u(s)Z_s \cap u(s)t^{-1}Z_s)\right) \leq \nu(E_s \setminus u(s)Z_s) + \nu(E_s \setminus u(s)t^{-1}Z_s)
\]

\[
< \delta + \nu(E_s \setminus u(s)z_t^{-1}E_s) + \nu(u(s)t^{-1}(E_s \setminus Z_s)) \quad \text{(by (2))}
\]

\[
< 3\delta \quad \text{(by (4) and (7)).}
\]

Let us assume \(d_\nu(u(s)(t), t) < \delta\). Hence \(A_{s,t} := \{u(s)(t) = t\}\) satisfies \(\nu(X \setminus A_{s,t}) < \delta\). Thus

\[
\nu\left(X \setminus \bigcup_{s \in S} (u(s)Z_s \cap u(s)t^{-1}Z_s \cap A_{s,t})\right)
\]

\[
\leq \sum_{s \in S} \nu(E_s \Delta (u(s)Z_s \cap u(s)t^{-1}Z_s)) + \sum_{s \in S} \nu(X \setminus A_{s,t})
\]

\[
\leq 4\delta|S| < \varepsilon
\]

and we have \(\nu(\{\hat{w}(t) \neq t\}) < \varepsilon\), equivalently \(d_\nu(\hat{w}(t), t) < \varepsilon\).

**Remark.** In Lemma 5.4 we can choose \(\delta\) and \(S\) so that \(\delta < \delta'\) and \(S' \subset S\) for any given \(\delta' > 0\) and \(S' \in G\).

Now we can classify ultrafree actions.

**Theorem 5.5** Let \(\alpha\) and \(\beta\) be ultrafree actions of \(G\) into \(N[T]\) with \(\text{mod}(\alpha_g) = \text{mod}(\beta_g)\). Then they are strongly cocycle conjugate.

**Proof.** Let \(\{\xi_i\}_{i=0}^{\infty}\) be a countable dense subset of \(M(X, \mu)\) with \(\xi_0 = \mu\). Take \(\varepsilon_n > 0\) and \(K_n \in G\) such that \(\sum_{n=0}^{\infty} \varepsilon_n < \infty\), \(\varepsilon_n > \varepsilon_{n+1}\), \(\varepsilon \in K_n\), \(K_n \subset K_{n+1}\), \(\bigcup_{n=0}^{\infty} K_n = G\).

Then choose \(S_n \in G\), \(\delta_n > 0\) for \(K_n\), \(\varepsilon_n\) as in Lemma 5.4. We can assume \(S_n \subset S_{n+1}\) and \(\delta_{n+1} < \delta_n\). (See a remark after Lemma 5.4.)
Set $\gamma_g^{(0)} := \alpha_g$, $\gamma_g^{(-1)} := \beta_g$, and construct actions $\gamma_g^{(n)}$ of $G$, $v_n(g), \bar{v}_n(g), w_n, \theta_n \in [T]$, $\Phi_n \in M_1(X, \mu)$ and $\Lambda_n \in [T]$ as follows;

1. $\gamma_g^{(n)} = \bar{v}_n(g)w_n\gamma_g^{(n-2)}w_n^{-1}$.
2. $\theta_n = w_n\theta_{n-2}$.
3. $v_n(g) = \bar{v}_n(g)\bar{w}_n(v_{n-2}(g))$.
4. $\|\gamma_g^{(n)}(\xi) - \gamma_g^{(n-1)}(\xi)\| < \varepsilon_n$, $g \in K_n$, $\xi \in \Phi_{n-1}$.
5. $d_\mu\left(\gamma_g^{(n)}(t), \gamma_g^{(n-1)}(t)\right) < \varepsilon_n$, $g \in K_n$, $t \in \Lambda_{n-1}$.
6. $\|\gamma_g^{(n)}(\xi) - \gamma_g^{(n-1)}(\xi)\| < \frac{\delta_n-1}{2}$, $g \in S_{n-1}$, $\xi \in \bigcup_{g \in S_{n-1}} \gamma_g^{(n-1)}(\Phi_{n-1})$.
7. $d_\nu\left(\gamma_g^{(n)}(t), \gamma_g^{(n-1)}(t)\right) < \frac{\delta_n-1}{2}$, $g \in S_{n-1}$, $t \in \bigcup_{s \in S_{n-1}} \gamma_s^{(n-1)}(\Lambda_{n-1})$, $\nu \in \Phi_{n-1}$.
8. $d_\nu(\bar{v}_n(g), id) < \varepsilon_n$, $g \in K_{n-2}$, $\nu \in \Phi_{n-2}$, $(n \geq 2)$.
9. $\left|\lambda_n(\xi) - \xi\right| < \varepsilon_n$, $\xi \in \Phi_{n-2}$, $(n \geq 2)$.
10. $d_\nu(\bar{w}_n(t), t) < \varepsilon_n$, $\nu \in \Phi_{n-2}$, $t \in \Lambda_{n-2}$, $(n \geq 2)$.
11. $\Phi_n = \{\xi\}_{i=0}^n \cup \{\theta_n(\xi)\}_{i=0}^n \cup \{v_n(g)(\mu)\}_{g \in K_n}$.
12. $\Lambda_n = \{T_i\}_{i=-n}^n \cup \{\theta_n(T_i)\}_{i=-n}^n \cup \{v_n(g), v_n(g)^{-1}\}_{g \in K_n}$.

1st step. Let $\theta_1 = \theta_0 = id$, $v_1(g) = v_0(g) = id$. By Theorem 5.3 take a 1-cocycle $u_1(\cdot)$ for $\gamma^{(-1)}$ such that

(a.1) $\|u_1\gamma_g^{(-1)}(\xi) - \gamma_g^{(0)}(\xi)\| < \varepsilon_1$, $g \in K_1$, $\xi \in \Phi_0$.
(b.1) $d_\mu\left(u_1\gamma_g^{(-1)}(t), \gamma_g^{(0)}(t)\right) < \varepsilon_1$, $g \in K_1$, $t \in \Lambda_0$.
(c.1) $\|u_1\gamma_g^{(-1)}(\xi) - \gamma_g^{(0)}(\xi)\| < \frac{\delta_0}{2}$, $g \in S_0$, $\xi \in \bigcup_{g \in S_0} \gamma_g^{(0)}(\Phi_0)$.
(d.1) $d_\nu\left(u_1\gamma_g^{(-1)}(t), \gamma_g^{(0)}(t)\right) < \frac{\delta_0}{2}$, $g \in S_0$, $t \in \bigcup_{g \in S_0} \gamma_g^{(0)}(\Lambda_0)$, $\nu \in \Phi_0$.

Set $w_1 = id$, $v_1(g) = u_1(g)$, and define

$$\gamma_g^{(1)} := \bar{v}_1(g)w_1\gamma_g^{(-1)}w_1^{-1} = u_1\gamma_g^{(-1)},$$
$$\theta_1 := w_1\theta_1 = id,$$
$$v_1(g) := \bar{v}_1(g)\bar{w}_1(v_1(g)) = u_1(g)$$

as in (1.1), (1.2), (1.3), respectively. By (a.1), (b.1), (c.1), (d.1), we get (4.1), (5.1), (6.1) and (7.1), respectively. Define $\Phi_1$ and $\Lambda_1$ as in (11.1), (12.1), respectively. Then we finished the 1st step of induction.

Assume that we have done up to the $n$-th step. By Theorem 5.3 let us take a $\gamma^{(n-1)}$-
cocycle $u_{n+1}(\cdot)$ such that

\[(a.n + 1) \| u_{n+1} (\gamma_g^{(n-1)}(\xi) - \gamma_g^{(n)}(\xi)) \| < \varepsilon_{n+1}, \ g \in K_{n+1}, \xi \in \Phi_n,\]

\[(b.n + 1) d_\mu \left( u_{n+1} \gamma_g^{(n-1)}(t), \gamma_g^{(n)}(t) \right) < \varepsilon_{n+1}, \ g \in K_{n+1}, t \in \Lambda_n,\]

\[(c.n + 1) \| u_{n+1} \gamma_g^{(n-1)}(\xi) - \gamma_g^{(n)}(\xi) \| < \frac{\delta_n}{2}, \ g \in S_n, \xi \in \bigcup_{g \in S_n} \gamma_g^{(n)}(\Phi_n),\]

\[(d.n + 1) d_\nu \left( u_{n+1} \gamma_g^{(n-1)}(t), \gamma_g^{(n)}(t) \right) < \frac{\delta_n}{2}, \ g \in S_n, t \in \bigcup_{g \in S_n} \gamma_g^{(n)}(\Lambda_n), \nu \in \Phi_n,\]

\[(e.n + 1) \| u_{n+1} \gamma_g^{(n-1)}(\xi) - \gamma_g^{(n)}(\xi) \| < \frac{\delta_n}{2}, \ g \in S_{n-1}, \xi \in \bigcup_{g \in S_{n-1}} \gamma_g^{(n-1)}(\Phi_{n-1}),\]

\[(f.n + 1) d_\nu \left( u_{n+1} \gamma_g^{(n-1)}(t), \gamma_g^{(n)}(t) \right) < \frac{\delta_n}{2}, \ g \in S_{n-1}, t \in \bigcup_{g \in S_{n-1}} \gamma_g^{(n-1)}(\Lambda_{n-1}), \nu \in \Phi_{n-1}.\]

By (6.n) and (e.n + 1), we have

$$\| u_{n+1}(g) \gamma_g^{(n-1)}(\xi) - \gamma_g^{(n-1)}(\xi) \| < \delta_{n-1}, \ g \in S_{n-1}, \xi \in \bigcup_{g \in S_{n-1}} \gamma_g^{(n-1)}(\Phi_{n-1})$$

and hence

$$\| u_{n+1}(g)(\xi) - \xi \| < \delta_{n-1}, \ g \in S_{n-1}, \xi \in \Phi_{n-1}.$$

By (7.n) and (f.n + 1),

$$d_\nu \left( u_{n+1}(g) \gamma_g^{(n-1)}(t), \gamma_g^{(n-1)}(t) \right) < \delta_{n-1}, \ g \in S_{n-1}, t \in \bigcup_{g \in S_{n-1}} \gamma_g^{(n-1)}(\Lambda_{n-1}), \nu \in \Phi_{n-1},$$

and hence

$$d_\nu \left( u_{n+1}(g)(t), t \right) < \delta_{n-1}, \ g \in S_{n-1}, t \in \Lambda_{n-1}, \nu \in \Phi_{n-1}.$$

By Lemma 5.4 there exists $w_{n+1} \in [T]$ such that

$$d_\nu \left( u_{n+1}(g) \gamma_g^{(n-1)}(w_{n+1}) w_{n+1}^{-1}, \text{id} \right) < \varepsilon_{n-1}, \ g \in K_{n-1}, \nu \in \Phi_{n-1},$$

$$\| w_{n+1}(\xi) - \xi \| < \varepsilon_{n-1}, \ d_\nu (w_{n+1}(t), t) < \varepsilon_{n-1}, \xi, \nu \in \Phi_{n-1}, t \in K_{n-1}.$$

Set

$$\bar{v}_{n+1}(g) := u_{n+1}(g) \gamma_g^{(n-1)}(w_{n+1}) w_{n+1}^{-1},$$

$$\gamma_g^{(n+1)} := u_{n+1} \gamma_g^{(n-1)} = \bar{v}_{n+1}(g) w_{n+1} \gamma_g^{(n-1)} w_{n+1}^{-1},$$

$$\theta_{n+1} := w_{n+1} \theta_{n-1}.$$

We clearly have (1.n + 1), (2.n + 1), (3.n + 1), (8.n + 1), (9.n + 1), and (10.n + 1). From (a.n + 1), (b.n + 1), (c.n + 1) and (d.n + 1), we obtain (4.n + 1), (5.n + 1), (6.n + 1)
and \((7.n + 1)\), respectively. We define \(\Phi_{n+1}\) and \(\Lambda_{n+1}\) as in (11\(n + 1)\) and (12\(n + 1)\), respectively. Then we finished the \((n + 1)\)-st step, and completed induction.

By the construction, we have
\[
\gamma_g^{(2n)} = v_{2n}(g)\theta_{2n}\alpha_g\theta_{2n}^{-1}, \quad \gamma_g^{(2n+1)} = v_{2n+1}(g)\theta_{2n+1}\beta_g\theta_{2n+1}^{-1}.
\]
We will show that sequences \(\{\theta_{2n}\}_n\), \(\{\theta_{2n+1}\}_n\), \(\{v_{2n}(g)\}_n\) and \(\{v_{2n+1}(g)\}_n\) will converge.
Fix \(k \in \mathbb{N}\), and take \(\xi \in \{\xi\}_1^k\), \(t \in \{T\}_{|l| \leq k}\).

For \(n > k + 2\), we have \(\xi, \theta_{n-2}(\xi) \in \Phi_{n-2}\), \(\theta_{n-2}(t) \in \Lambda_{n-2}\). Then
\[
\|\theta_n(\xi) - \theta_{n-2}(\xi)\| = \|w_n(\theta_{n-2}(\xi)) - \theta_{n-2}(\xi)\| < \varepsilon_{n-2},
\]
and
\[
d_\mu(\theta_n(t), \theta_{n-2}(t)) = d_\mu\left(\hat{w}_n(\theta_{n-2}(t)), \theta_{n-2}(t)\right) < \varepsilon_{n-2}
\]
hold by (9\(n)\) and (10\(n)\). It follows that \(\{\theta_{2n}\}_n\) and \(\{\theta_{2n+1}\}_n\) are both Cauchy sequences with respect to the metric \(d\) on \(N[T]\). (See \(\S 2.4\) on the definition of \(d\).) Hence both \(\{\theta_{2n}\}_n\) and \(\{\theta_{2n+1}\}_n\) converge to some \(\sigma_0, \sigma_1 \in [T]\), respectively in the \(u\)-topology.

Fix \(l \in \mathbb{N}\) and take any \(g \in K_l\). Then for \(n > l + 2\), we have \(v_{n-2}(g), v_{n-2}(g)^{-1} \in \Lambda_{n-2}\), \(v_{n-2}(g)(\mu) \in \Phi_{n-2}\). Thus
\[
d_\mu(v_n(g), v_{n-2}(g))
\]
\[
\leq d_\mu(\bar{v}_n(g), \bar{v}_n(v_{n-2}(g))), \bar{v}_n(g)v_{n-2}(g) + d_\mu(\bar{v}_n(g)v_{n-2}(g), v_{n-2}(g))
\]
\[
= d_\mu(\bar{w}_n(v_{n-2}(g)), v_{n-2}(g)) + d_{v_{n-2}(g)}(\bar{v}_n(g), \bar{v}_n(g), \text{id})
\]
\[
< 2\varepsilon_{n-2}
\]
and
\[
d_\mu(v_n(g)^{-1}, v_{n-2}(g)^{-1})
\]
\[
\leq d_\mu(\bar{w}_n(v_{n-2}(g)^{-1}), \bar{w}_n(v_{n-2}(g)^{-1})) + d_\mu(\bar{w}_n(v_{n-2}(g)^{-1}), v_{n-2}(g)^{-1})
\]
\[
= d_\mu(\bar{v}_n(g)^{-1}, \text{id}) + d_\mu(\bar{w}_n(v_{n-2}(g)^{-1}), v_{n-2}(g)^{-1})
\]
\[
< 2\varepsilon_{n-2}
\]
by (8\(n)\) and (10\(n)\). Thus both \(\{v_{2n}(g)\}_n\) and \(\{v_{2n+1}(g)\}_n\) are Cauchy sequences with respect to \(d_\mu\), and hence converge to some \(z_0(g), z_1(g) \in [T]\) uniformly, respectively.

Summarizing these results, we have
\[
\lim_{n \to \infty} \gamma_{g}^{(2n)} = \lim_{n \to \infty} v_{2n}(g)\theta_{2n}\alpha_g\theta_{2n}^{-1} = z_0(g)\sigma_0\alpha_g\sigma_0^{-1},
\]
\[
\lim_{n \to \infty} \gamma_{g}^{(2n+1)} = \lim_{n \to \infty} v_{2n+1}(g)\theta_{2n+1}\beta_g\theta_{2n+1}^{-1} = z_1(g)\sigma_1\beta_g\sigma_1^{-1}.
\]

By (4\(n)\) and (5\(n)\), we have \(z_0(g)\sigma_0\alpha_g\sigma_0^{-1} = z_1(g)\sigma_1\beta_g\sigma_1^{-1}\). Hence \(\alpha\) and \(\beta\) are cocycle conjugate. □

**Proof of Theorem [2.4]** Let \(N := N_\alpha = N_\beta, Q := G/N\), and \(\pi : G \to Q\) be the quotient map. Fix a section \(s : Q \to G\) such that \(s(\varepsilon) = \varepsilon\). Then \(s_\varepsilon\) is an ultrafree cocycle crossed action of \(Q\). By Theorem [4.4], there exists \(v \in C^1(Q, [T])\) such that
\[ \bar{\alpha}_p := v(p)\alpha_{s(p)} \] is a genuine action of \( Q \). Here define \( v(g) := v(p)\alpha_n^{-1} \in [T] \), where \( g = ns(p) \) with \( p = \pi(g) \) and \( n \in N \). Then \( v(g)\alpha_g = v(p)\alpha_{s(p)} = \bar{\alpha}_{\pi(g)} \) and \( \bar{\alpha}_{\pi(g)} \) is an action of \( G \). Thus \( \alpha_g \) is strongly cocycle conjugate to \( \bar{\alpha}_{\pi(g)} \) for some ultrafree action \( \bar{\alpha} \) of \( Q \). In the same way, \( \beta_g \) is strongly cocycle conjugate to \( \bar{\beta}_{\pi(g)} \) for some ultrafree action \( \bar{\beta} \) of \( Q \). Since \( \text{mod}(\bar{\alpha}_p) = \text{mod}(\bar{\beta}_p) \), \( \bar{\alpha} \) and \( \bar{\beta} \) are strongly cocycle conjugate as actions of \( Q \) by Theorem 6.3 and hence so are as actions of \( G \). Therefore two actions \( \alpha \) and \( \beta \) of \( G \) are strongly cocycle conjugate.

\[ \Box \]

References

[1] Bezuglyi, S. I., *Outer conjugation of the actions of countable amenable groups*, Mathematical physics, functional analysis (Russian), vol. 145, (1986), 59–63.

[2] Bezuglyi, S. I. and Golodets, V. Ya., *Outer conjugacy of actions of countable amenable groups on a space with measure*, Izv. Akad. Nauk SSSR Ser. Mat. 50 (1986), 643–660.

[3] Bezuglyi, S. I. and Golodets, V. Ya., *Type III\( \theta \) transformations of measure space and outer conjugacy of countable amenable groups of automorphisms*, J. Operator Theory 21 (1989), 3–40.

[4] Connes, A., *Outer conjugacy classes of automorphisms of factors*, Ann. Sci. Eco. Norm. Sup. 8 (1975), 383–419.

[5] Connes, A., *Periodic automorphisms of the hyperfinite factor of type II\( _1 \)*, Acta Sci. Math 39 (1977), 39–66.

[6] Connes, A. and Krieger, W., *Measure space automorphisms, the normalizers of their full groups, and approximate finiteness*, J. Funct. Anal. 24 (1977), 336–352.

[7] Connes, A. and Takesaki, M., *The flow of weights on factors of type III*, Tohoku J. Math. 29 (1977), 473–555.

[8] Evans, D. E. and Kishimoto, A., *Trace scaling automorphisms of certain stable AF algebras*, Hokkaido Math J. 26 (1997), 211–224.

[9] Hamachi, T., *The normalizer group of an ergodic automorphism of type III and the commutant of an ergodic flow*, J. Funct. Anal. 40 (1981), 387–403.

[10] Hamachi, T. and Osikawa, M., *Fundamental homomorphism of normalizer group of ergodic transformation*, Ergodic theory (Proc. Conf., Math. Forschungsinst., Oberwolfach, 1978), Lecture Notes in Math., vol. 729, (1979), 43–57.

[11] Hamachi, T. and Osikawa, M., *Ergodic groups of automorphisms and krieger’s theorems*, Seminar on Mathematical Sciences, vol. 3, Keio University, (1981).

[12] Jones, V. F. R., *Actions of finite groups on the hyperfinite type II\( _1 \) factor*, Memoirs of Amer. Math. Soc 237 (1980).
[13] Jones, V. F. R. and Takesaki, M., *Actions of compact abelian groups on semifinite injective factors*, Acta Math. **153** (1984), 213–258.

[14] Katayama, Y., Sutherland, C. E., and Takesaki, M., *The characteristic square of a factor and the cocycle conjugacy of discrete group actions on factors*, Invent. Math. **132** (1998), 331–380.

[15] Kawahigashi, Y., Sutherland, C. E., and Takesaki, M., *The structure of the automorphism group of an injective factor and the cocycle conjugacy of discrete abelian group actions*, Acta Math. **169** (1992), 105–130.

[16] Masuda, T., *Unified approach to classification of actions of discrete amenable groups on injective factors*, J. Reine Angew. Math. **683** (2013), 1–47.

[17] Ocneanu, A., *Action of discrete amenable groups on von Neumann algebras*, Lecture Notes in Math., vol. 1138, Springer, Berlin, (1985).

[18] Sutherland, C. E. and Takesaki, M., *Actions of discrete amenable groups and groupoids on von Neumann algebras*, Publ. Res. Inst. Math. Sci. **21** (1985), 1087–1120.

[19] Sutherland, C. E. and Takesaki, M., *Actions of discrete amenable groups on injective factors of type III_λ, λ \neq 1*, Pacific. J. Math. **137** (1989), 405–444.