Triple-Block Generalized Inverses for Control Applications with Mixed Consistency Requirements

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Abstract

Previous work on block-partitioned mixed generalized inverses is extended from two subsets of system variables with distinct consistency requirements to three subsets. Does not include any significant theoretical contributions.

INTRODUCTION

Previous work has identified the need to select appropriate generalized inverses to enforce critical application-specific assumptions, such as whether system behavior should be invariant with respect to rigid rotations or with respect to changes of units on state variables [3]. For example, the familiar Moore-Penrose (MP) pseudoinverse is only applicable when invariance is needed with respect to orthogonal/unitary transformations of the input and/or output of the system. The unit-consistent (UC) generalized inverse, by contrast, is only applicable when invariance is needed with respect to diagonal transformations of the input and/or output of the system, e.g., as determined by choices of units on state variables.

It is not widely recognized that the choice of generalized inverse is critical for robust performance of a control system. Consequently, it is common to see control-system implementations that simply replace all matrix inverses with the MP inverse (e.g., replace inv with pinv when using Matlab) to defend against singular states, such as arise during gimball lock or other loss-of-rank configurations. The fact is that there are an infinite number of generalized inverses, each of which sacrifices a specific subset of properties of the true matrix inverse [2, 3].

The term consistency is used to describe the behavior of a system with respect to a set of transformations [4, 5]. For example, if the behavior of a robotic system is expected to be invariant up to arbitrary orthogonal transformations of an assumed Euclidean coordinate frame, then the MP inverse is appropriate. In other words, the behavior of the system should be identical up to a rotation of the coordinate frame. If, however, the MP inverse is applied with variables involving arbitrary choices of units, e.g., centimeters versus meters for lengths, then the system will exhibit different behavior depending on the choices of units, which is clearly undesirable [7, 8, 6, 1].

A challenge that arises in most nontrivial systems is that there exist distinct subsets of variables with different consistency requirements. For example, one subset may be defined in arbitrary units of length, whereas another may be defined in an arbitrarily-oriented Euclidean coordinate frame. This means that reliable control (i.e., insensitive to the choice of units or rotation of coordinate frame) must involve a generalized inverse that treats the different subsets of variables in different ways. This can be expressed in terms of a block partition of the system matrix $M$ with $m$ variables requiring unit consistency and the remaining $n$ variables requiring rotation consistency:

$$M = \begin{bmatrix} W & X \\ Y & Z \end{bmatrix}_{m \times n}$$

It has been shown [4, 5] that the mixed inverse can be obtained from this block-partitioned form as

$$M^{-1} = \begin{bmatrix} (W - XZ^{-1}Y)^{-U} & -W^{-U}X(Z - YW^{-U}X)^{-P} \\ -Z^{-1}Y(W - XZ^{-1}Y)^{-U} & (Z - YW^{-U}X)^{-P} \end{bmatrix}$$

where superscript $-U$ denotes the UC inverse, the superscript $-P$ denotes the MP pseudoinverse, and $-J$ denotes the specialized joint inverse of the system matrix. This dual-block mixed inverse form
The dual-block solution of Eq. (1) can be concisely expressed. Unfortunately, the complexity of the solution for the $3 \times 3$ block matrix is considerably greater. These expressions were therefore derived laboriously by hand, but have not been empirically verified. To avoid introducing transcription mistakes, no factoring of expressions has been performed.

**THE TRIPLE-BLOCK MIXED INVERSE**

The triple-block mixed inverse demands a formulation of the generalized inverse of a matrix of the following form:

$$M = \begin{bmatrix} R & S & T \\ U & V & W \\ X & Y & Z \end{bmatrix}$$

The dual-block solution of Eq. (1) can be concisely expressed. Unfortunately, the complexity of the solution for the $3 \times 3$ block matrix is considerably greater. The following are expressions for each block of $M^{-1}$ in unsimplified form with superscripts $\{-a,-b,-c\}$ denoting three distinct generalized inverses:

$$M_{11}^{-1} = \frac{(R^{-a}S((-UR^{-a}S + V)^{-b} + (-UR^{-a}S + V)^{-b}(-UR^{-a}T + W))}{(-X^{-a}S + Y)((-UR^{-a}S + V)^{-b}(-UR^{-a}T + W) - XR^{-a}T + Z)^{-c}}$$

$$M_{12}^{-1} = \frac{-R^{-a}S((-UR^{-a}S + V)^{-b} + (-UR^{-a}S + V)^{-b}(-UR^{-a}T + W))}{(-X^{-a}S + Y)((-UR^{-a}S + V)^{-b}(-UR^{-a}T + W) - XR^{-a}T + Z)^{-c}}$$

$$M_{13}^{-1} = \frac{R^{-a}S((-UR^{-a}S + V)^{-b}(-UR^{-a}T + W))}{(-X^{-a}S + Y)((-UR^{-a}S + V)^{-b}(-UR^{-a}T + W) - XR^{-a}T + Z)^{-c}}$$

$$M_{21}^{-1} = \frac{-(((-UR^{-a}S + V)^{-b} + (-UR^{-a}S + V)^{-b}(-UR^{-a}T + W))}{(-X^{-a}S + Y)((-UR^{-a}S + V)^{-b}(-UR^{-a}T + W) - XR^{-a}T + Z)^{-c}}$$

$$M_{22}^{-1} = \frac{(-UR^{-a}S + V)^{-b} + (-UR^{-a}S + V)^{-b}(-UR^{-a}T + W))}{(-X^{-a}S + Y)((-UR^{-a}S + V)^{-b}(-UR^{-a}T + W) - XR^{-a}T + Z)^{-c}}$$

$$M_{23}^{-1} = \frac{(-UR^{-a}S + V)^{-b}(-UR^{-a}T + W))}{(-X^{-a}S + Y)((-UR^{-a}S + V)^{-b}(-UR^{-a}T + W) - XR^{-a}T + Z)^{-c}}$$

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3It does not appear that any of the current computer algebra systems, e.g., Mathematica and Maple, are capable of performing the kinds of symbolic block-matrix manipulations needed to automate the solution of problems of this kind. These expressions were therefore derived laboriously by hand, but have not been empirically verified. To avoid introducing transcription mistakes, no factoring of expressions has been performed.
(-UR^{-a}S + V)^{-b}(-UR^{-a}T + W) - XR^{-a}T + Z)^{-c}

\[ M_{11}^1 = -(-(-XR^{-a}S + Y)(UR^{-a}S + V)^{-b}(-UR^{-a}T + W) - XR^{-a}T + Z)^{-c}(-XR^{-a}S + Y)(UR^{-a}S + V)^{-b}U + (-(-XR^{-a}S + Y)(UR^{-a}S + V)^{-b}(-UR^{-a}T + W) - XR^{-a}T + Z)^{-c}X)R^{-a} \]

\[ M_{12}^1 = -(-(-XR^{-a}S + Y)(UR^{-a}S + V)^{-b}(-UR^{-a}T + W) - XR^{-a}T + Z)^{-c}(-(-XR^{-a}S + Y)(UR^{-a}S + V)^{-b}(-UR^{-a}T + W) - XR^{-a}T + Z)^{-c} \]

For purposes of redundancy, and to facilitate practical implementation, the following is a conversion from the same derivation as above into functional form. The three user-selected generalized inverses are denoted as InvA, InvB, and InvC. Indentation and square brackets for functions are purely for ease of parsing.

\[ M_{11} = \]
\[
\begin{align*}
(\text{InvA}[R]*S* & \\
& \{ \\
& \text{InvB}[\text{-U}*\text{InvA}[R]*S+V]*\text{Inverse}[\text{-U}*\text{InvA}[R]*S+V]* \\
& \text{-U}*\text{InvA}[R]*T+W)* \\
& \text{Inverse}[\text{-} \\
& \text{-X}*\text{InvA}[R]*S+Y)*\text{InvB}[\text{-U}*\text{InvA}[R]*S+V]* \\
& \text{-U}*\text{InvA}[R]*T+W)-\text{X}*\text{InvA}[R]*T+Z \\
& \})*(\text{-X}*\text{InvA}[R]*S+Y)*\text{InvB}[\text{-U}*\text{InvA}[R]*S+V] \\
& \}\text{U} \\
& (- \\
& \text{InvA}[R]*S*\text{InvB}[\text{-U}*\text{InvA}[R]*S+V]* \\
& \text{-U}*\text{InvA}[R]*T+W)* \\
& \text{Inverse}[\text{-} \\
& \text{-X}*\text{InvA}[R]*S+Y)*\text{InvB}[\text{-U}*\text{InvA}[R]*S+V]* \\
& \text{-U}*\text{InvA}[R]*T+W)\text{-X}*\text{InvA}[R]*T+Z \\
& \}+\text{InvA}[R]*T* \\
& \text{InvC}[\text{-} \\
& \text{-X}*\text{InvA}[R]*S+Y)*\text{InvB}[\text{-U}*\text{InvA}[R]*S+V]* \\
& \text{-U}*\text{InvA}[R]*T+W)\text{-X}*\text{InvA}[R]*T+Z \\
& \} \})*X \\
& \})*\text{InvA}[R]+\text{InvA}[R] ;
\end{align*}
\]

\[ M_{12} = - \]
\[
\text{InvA}[R]*S* \\
\{ \\
\text{InvB}[\text{-U}*\text{InvA}[R]*S+V]*\text{InvB}[\text{-U}*\text{InvA}[R]*S+V]* \\
\text{-U}*\text{InvA}[R]*T+W)* \\
\}
\]
\( \text{InvC}[-
\begin{align*}
&(-X*\text{InvA}[R]*S+Y)*\text{InvB}[-U*\text{InvA}[R]*S+V]* \\
&(-U*\text{InvA}[R]*T+W)-X*\text{InvA}[R]*T+Z \\
&\}*(-X*\text{InvA}[R]*S+Y)* \\
&\text{InvB}[-U*\text{InvA}[R]*S+V])+\text{InvA}[R]*T* \\
&\text{InvC}[-
\begin{align*}
&(-X*\text{InvA}[R]*S+Y)*\text{InvB}[-U*\text{InvA}[R]*S+V]* \\
&(-U*\text{InvA}[R]*T+W)-X*\text{InvA}[R]*T+Z \\
&\}*(-X*\text{InvA}[R]*S+Y)*\text{InvB}[-U*\text{InvA}[R]*S+V] \\
\end{align*}
\] \\
Mi13 = \\
\text{InvA}[R]*S*\text{InvB}[-U*\text{InvA}[R]*S+V]* \\
(-U*\text{InvA}[R]*T+W)* \\
\text{Inverse}[-
\begin{align*}
&(-X*\text{InvA}[R]*S+Y)*\text{InvB}[-U*\text{InvA}[R]*S+V]* \\
&(-U*\text{InvA}[R]*T+W)-X*\text{InvA}[R]*T+Z \\
&\}*-\text{InvA}[R]*T* \\
&\text{Inverse}[-
\begin{align*}
&(-X*\text{InvA}[R]*S+Y)*\text{InvB}[-U*\text{InvA}[R]*S+V]* \\
&(-U*\text{InvA}[R]*T+W)-X*\text{InvA}[R]*T+Z \\
\} \\
\end{align*}
\] \\
Mi21 = - \\
( \\
( \\
\text{Inverse}[-U*\text{InvA}[R]*S+V]+\text{InvB}[-U*\text{InvA}[R]*S+V]* \\
(-U*\text{InvA}[R]*T+W)* \\
\text{InvC}[-
\begin{align*}
&(-X*\text{InvA}[R]*S+Y)*\text{InvB}[-U*\text{InvA}[R]*S+V]* \\
&(-U*\text{InvA}[R]*T+W)-X*\text{InvA}[R]*T+Z \\
&\}*(-X*\text{InvA}[R]*S+Y)*\text{InvB}[-U*\text{InvA}[R]*S+V] \\
&(-U*\text{InvA}[R]*T+W)-X*\text{InvA}[R]*T+Z \\
\}X \\
)\}*-\text{InvA}[R] \\
) \\
Mi22 = \\
\text{InvB}[-U*\text{InvA}[R]*S+V]+\text{InvB}[-U*\text{InvA}[R]*S+V]* \\
(-U*\text{InvA}[R]*T+W)* \\
\text{InvC}[-
\begin{align*}
&(-X*\text{InvA}[R]*S+Y)*\text{InvB}[-U*\text{InvA}[R]*S+V]* \\
&(-U*\text{InvA}[R]*T+W)-X*\text{InvA}[R]*T+Z \\
&\}*(-X*\text{InvA}[R]*S+Y)*\text{InvB}[-U*\text{InvA}[R]*S+V] \\
\} \\
\end{align*}
\] \\
Mi23 = - \\
\text{InvB}[-U*\text{InvA}[R]*S+V]*(-U*\text{InvA}[R]*T+W)* \\
\text{InvC}[-
\begin{align*}
&(-X*\text{InvA}[R]*S+Y)*\text{InvB}[-U*\text{InvA}[R]*S+V]* \\
&(-U*\text{InvA}[R]*T+W)-X*\text{InvA}[R]*T+Z \\
\} \\
\end{align*}
\] \\
Mi31 = - \\
( \\
( \\
\text{InvC}[-
\((\text{-}X \cdot \text{InvA}[R] \cdot S + Y) \cdot \text{InvB}[\text{-}U \cdot \text{InvA}[R] \cdot S + V] \cdot \text{U+} \cdot \text{Inverse}[-] \cdot \text{(-}X \cdot \text{InvA}[R] \cdot S + Y) \cdot \text{InvB}[\text{-}U \cdot \text{InvA}[R] \cdot S + V] \cdot \text{U} \cdot \text{X} \cdot \text{InvA}[R] \); 

\text{Mi32} = \text{Inverse}[-] \cdot \text{(-}X \cdot \text{InvA}[R] \cdot S + Y) \cdot \text{InvB}[\text{-}U \cdot \text{InvA}[R] \cdot S + V] \cdot \text{U} \cdot \text{(-}X \cdot \text{InvA}[R] \cdot S + Y) \cdot \text{InvB}[\text{-}U \cdot \text{InvA}[R] \cdot S + V]; 

\text{Mi33} = \text{Inverse}[-] \cdot \text{(-}X \cdot \text{InvA}[R] \cdot S + Y) \cdot \text{InvB}[\text{-}U \cdot \text{InvA}[R] \cdot S + V] \cdot \text{U} \cdot \text{(-}X \cdot \text{InvA}[R] \cdot S + Y) \cdot \text{InvB}[\text{-}U \cdot \text{InvA}[R] \cdot S + V]; 

The formatting of the above reveals several small and large subexpressions that can be factored out for efficiency. However, this explicit form can serve as a baseline for verifying optimized variants.

**ILLUSTRATIVE EXAMPLE**

The MP inverse provides consistency with respect to left or right unitary transforms of the system matrix. The UC inverse provides consistency with respect to left or right nonsingular diagonal transforms of the system matrix. These two are by far the most common forms of required consistency that arise in practical applications.

Another class of transforms are more structured with respect to the kind of consistency demanded of the left and right transformations. One is similarity, i.e., requiring consistency with respect to transformations of the system matrix \( M \) as \( SMS^{-1} \) for arbitrary nonsingular matrix \( S \).

A similarity-consistent (SC) generalized inverse can be obtained by decomposing \( M \) as

\[ M = FCF^{-1}, \]

where \( C \) is the Frobenius Canonical Form (FCF) of \( M \), which then permits the SC inverse to be evaluated using the MP inverse of \( C \) as

\[ M^{-S} = FCF^{-P}F^{-1}. \]

This form is theoretically tractable to compute, unlike the Jordan Normal Form (JNF), to provide similarity consistency.

Now, given a system with \( m \) variables requiring Euclidean rotational consistency; \( n \) variables requiring diagonal unit consistency; and \( p \) variables requiring similarity consistency:

\[ M = \begin{bmatrix} R & S & T \\ U & V & W \\ X & Y & Z \end{bmatrix}, \]

\( \text{m} \quad \text{n} \quad \text{p} \)

\(^{2}\text{The Drazin inverse can also be used in cases in which it produces a result with the same rank as } M^{[3]. \text{ There are also various speculative approaches } [5]. \text{ One class involves the scaling of } M \text{ by a positive value large enough relative to the precision of the entries that they can be treated as integers to permit the Frobenius Rational Canonical Form or Smith Normal Form to be used, both of which can be efficiently computed. However, the numerical implications of integerising } M \text{ are not well understood.} \)
the appropriate generalized matrix inverse of $M$ can be constructed using the formulation of the previous section with $\text{InvA}$ provided in the form of the MP inverse; $\text{InvB}$ provided in the form of the UC inverse; and $\text{InvC}$ provided in the form of the SC inverse.

**DISCUSSION**

We have provided an explicit formulation of the triple-block mixed generalized matrix inverse. This complements the dual-block formulation for cases in which three subsets of system variables have distinct consistency requirements.

The triple-block case is in fact redundant because the general solution can be recursively constructed from repeated applications of the dual-block solution. Specifically, given $k$ sets of variables $S_1...S_k$, with corresponding sizes $n_1...n_k$, a mixed inverse for sets $S_1$ and $S_2$ can be constructed using the dual-block solution. With this, their respective state variables can be treated as a single set of size $n_1+n_2$ with consistency conditions satisfied by the newly-constructed mixed inverse. In other words, the number of sets of variables can be reduced by 1. This can then be repeated to the base case of two sets.

The motivation for deriving an explicit triple-block solution is purely for convenience. Specifically, while the dual-block inverse is much more likely to find use in practical control applications, we anticipate a likelihood that increasing awareness of consistency issues will lead to recognition of cases in which the triple-block formulation is needed. Having an explicit plug-and-play solution will hopefully reduce the perceived effort to apply that recognition.

Future work will examine consistency considerations arising in physics models, e.g., dynamical systems for which it is necessary to rigorously establish that key consistency/invariance properties are preserved across reduced-rank manifolds (or where a covariant derivative becomes singular).

**References**

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