Curvature Quintessence

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Abstract

The issues of quintessence and cosmic acceleration can be discussed in the framework of higher order theories of gravity. We can define effective pressure and energy density directly connected to the Ricci scalar of curvature of a generic fourth order theory and then ask for the conditions to get an accelerated expansion. Exact accelerated expanding solutions can be achieved for several fourth order theories so that we get an alternative scheme to the standard quintessence scalar field, minimally coupled to gravity, usually adopted. We discuss also conformal transformations in order to see the links of quintessence between the Jordan and Einstein frames.

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I. INTRODUCTION

One of the astonishing recent result in cosmology is the fact that the universe is accelerating instead of decelerating along the scheme of standard Friedmann model as everyone has learned in textbooks. Type Ia supernovae (SNe Ia) allow to determine cosmological parameters probing the today values of the Hubble constant $H_0$ and the deceleration parameter $q_0$ [1]. Besides, data coming from clusters of galaxies at low red shift (including the mass - to - light methods, baryon fraction and abundance evolution) [2], and data coming from the CMBR investigation (e.g. BOOMERANG) [3] give observational constraints from which we deduce the picture of a spatially flat, low density universe dominated by some kind of non-clustered dark energy. Such an energy, which is supposed to have dynamics, should be the origin of the cosmic acceleration.

In terms of density parameter, we have

$$
\Omega_{(\text{matter})} \simeq 0.3, \quad \Omega_{\Lambda} \simeq 0.7, \quad \Omega_k \simeq 0.0
$$

where the matter is the non-relativistic baryonic and non-baryonic (dark) matter, $\Lambda$ is the dark energy (cosmological constant, quintessence,...), $k$ is the curvature parameter of Friedmann-Robertson-Walker (FRW) metric of the form

$$
ds^2 = dt^2 - a(t)^2 \left[ \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right],
$$

where $a(t)$ is the scale factor of the universe.

The luminosity distance can be deduced from SNe Ia used as standard candles. For $z \leq 1$, by the luminosity distance $d_L \simeq H_0^{-1}[z + (1 - q_0)z^2/2]$, the observational results indicate

$$
-1 \leq q_0 < 0
$$

which is a clear indication for the acceleration.

The deceleration parameter can be given in terms of density parameter and then we have for FRW models

$$
q_0 = -\frac{\ddot{a}}{a^2} = \frac{1}{2}(3\gamma + 1)\Omega_{(\text{matter})} - \Omega_{\Lambda},
$$

where $\gamma$ is the constant of state equation $p = \gamma \rho$. Immediately, we realize that acceleration or deceleration depends on the value of $\gamma$. For standard fluid matter, it is defined the Zeldovich range $0 \leq \gamma_{(\text{matter})} \leq 1$ where $\gamma_{(\text{matter})} = 0$ indicates dust (i.e. non-relativistic matter), $\gamma_{(\text{matter})} = 1/3$ radiation (i.e. relativistic matter). Another interesting case has been widely considered in literature, it is $\gamma = -1$ which points out a scalar field fluid where dynamics is dominated by self-interaction potential or cosmological constant. Inserting a standard matter fluid into Friedmann–Einstein cosmological equations gives
rise to decelerated dynamics. Due to this fact, non-standard forms of matter–energy have
to be taken into account if one wants to insert observations into a theoretical frame.

Several approaches can be pursued in order to realize this goal. All of them can be
summarized into three great families: the cosmological constant, the variable cosmologi-
cal constant and quintessence. Essentially they are linked but, in order to match the
observations, several issue have to be satisfied. Below we give a short summary of this
three pictures.

The cosmological constant has become one of the main issue of modern physics since
by fixing its value should provide the gravity vacuum state, should make to understand
the mechanism which led the early universe to the today observed large scale structures
and to predict what will be the fate of the whole universe (no–hair conjecture).

From the cosmological point of view, the main feature of inflationary models is the
presence of a finite period during which the expansion is de Sitter (or quasi–de Sitter or
power law): this fact implies that the expansion of the scale factor \( a(t) \) is superluminal
(at least \( a(t) \sim t \), in general \( a(t) \sim \exp H_0 t \) where \( H_0 \) is the Hubble parameter nearly
constant for a finite period) with respect to the comoving proper time \( t \). Such a situation
arises in presence of an effective energy–momentum tensor which is approximately pro-
portional (for a certain time) to the metric tensor and takes place in various gravitational
theories: i.e. the Einstein gravity minimally coupled with a scalar field, fourth or
higher–order gravity, scalar–tensor gravity .

Several inflationary models are affected by the shortcoming of "fine tuning", that
is inflationary phase proceeds from very special initial conditions, while a natural issue
would be to get inflationary solutions as attractors for a large set of initial conditions.
Furthermore, the same situation should be achieved also in the future: if a remnant of
the cosmological constant is today observed, the universe should evolve toward a final de
Sitter stage. A more precise formulation of such a conjecture is possible for a restricted
class of cosmological models, as discussed in . We have to note that the conjecture
holds when any ordinary matter field, satisfies the dominant and strong energy conditions
However it is possible to find models which explicitly violate such conditions but
satisfies no–hair theorem requests. Precisely, this fact happens if extended gravity theo-
ries are involved and matter is in the form of scalar fields, besides the ordinary perfect
fluid matter .

In any case, we need a time variation of cosmological constant to get successful in-
flationary models, to be in agreement with observations, and to obtain a de Sitter stage
toward the future. In other words, this means that cosmological constant has to acquire
a great value in early epochs (de Sitter stage), has to undergo a phase transition with a
graceful exit and has to result in a small remnant toward the future . The today ob-
erved accelerated cosmological behaviour should be the result of this dynamical process
where the value of cosmological constant is not fixed exactly at zero.

In this context, a fundamental issue is to select the classes of gravitational theo-
ries and the conditions which ”naturally” allow to recover an effective time–dependent
cosmological constant without considering special initial data.

The third approach is quintessence \cite{19}. Quintessence is a time-varying, spatially inhomogeneous component of cosmic density with negative pressure $-1 \leq \gamma_Q \leq 0$. Formally, vacuum energy density is quintessence in the limit $\gamma_Q \to -1$ so that the three approaches present in literature (cosmological constant, variable cosmological constant and quintessence) are strictly linked.

However all of them claim for an ingredient which, a part a pure cosmological constant, comes from a matter-energy counter part. In this paper, we want to investigate if the quintessential scheme can be achieved in a geometrical way by taking into account higher order theories of gravity.

There is no a priori reason to restrict the gravitational Lagrangian to a linear function of the Ricci scalar $R$ minimally coupled with matter \cite{20}. Additionally, we have to note that, recently, some authors have taken into serious consideration the idea that there are no ”exact” laws of physics but that the Lagrangians of physical interactions are ”stochastic” functions with the property that local gauge invariances (i.e. conservation laws) are well approximated in the low energy limit and physical constants can vary \cite{9}. This scheme was adopted in order to treat the quantization on curved spacetimes and the result was that the interactions among quantum scalar fields and background geometry or the gravitational self–interactions yield corrective terms in the Einstein–Hilbert Lagrangian \cite{21}. Futhermore, it has been realized that such corrective terms are inescapable if we want to obtain the effective action of quantum gravity on scales closed to the Planck length \cite{22}. They are higher–order terms in curvature invariants as $R^2$, $R_{\mu\nu}R^{\mu\nu}$, $R^\alpha_{\mu\nu\beta}R_{\mu\nu\alpha\beta}$, $\Box R$, or $\Box^k R$, or nonminimally coupled terms between scalar fields and geometry as $\phi^2 R$. Terms of these kinds arise also in the effective Lagrangian of strings and Kaluza–Klein theories when the mechanism of dimensional reduction is working \cite{23}.

Besides fundamental physics motivations, all these theories have acquired a huge interest in cosmology due to the fact that they ”naturally” exhibit inflationary behaviours and that the related cosmological models seem very realistic \cite{8,12}. Furthermore, it is possible to show that, via conformal transformations, the higher–order and nonminimally coupled terms (Jordan frame) always corresponds to the Einstein gravity plus one or more than one minimally coupled scalar fields (Einstein frame) \cite{24,28} so that these geometric contributions can always have a ”matter” interpretation.

Quintessence can be achieved also in the framework of higher-order theories of gravity, that is in a geometrical way.

In Sec.II, we derive the Friedmann-Einstein equations for generic fourth-order models. Sec.III is devoted to the discussion of conditions to obtain quintessence while exact solutions satisfying these prescriptions are shown in Sec.IV. Conclusions are drawn in Sec.V.
II. FOURTH-ORDER GRAVITATIONAL THEORIES AND COSMOLOGICAL EQUATIONS

A generic fourth–order theory in four dimensions can be described by the action

\[ A = \int d^4x \sqrt{-g} \left[ f(R) + L_{(\text{matter})} \right] , \]  

where \( f(R) \) is a function of Ricci scalar \( R \) and \( L_{(\text{matter})} \) is the standard matter Lagrangian density. We are using physical units \( 8\pi G_N = c = \hbar = 1 \). The field equations are

\[ f'(R) R_{\alpha\beta} - \frac{1}{2} f(R) g_{\alpha\beta} = f'(R)^{\alpha\beta} \left( g_{\alpha\mu} g_{\beta\nu} - g_{\alpha\beta} g_{\mu\nu} \right) + \tilde{T}_{\alpha\beta}^{(\text{matter})} , \]  

which can be recast in the more expressive form

\[ G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = T_{\alpha\beta}^{(\text{curv})} + T_{\alpha\beta}^{(\text{matter})} , \]  

where

\[ T_{\alpha\beta}^{(\text{curv})} = \frac{1}{f'(R)} \left\{ \frac{1}{2} g_{\alpha\beta} \left[ f(R) - R f'(R) \right] + f'(R)^{\alpha\beta} \left( g_{\alpha\mu} g_{\beta\nu} - g_{\alpha\beta} g_{\mu\nu} \right) \right\} , \]  

and

\[ T_{\alpha\beta}^{(\text{matter})} = \frac{1}{f'(R)} \tilde{T}_{\alpha\beta}^{(\text{matter})} , \]  

is the stress-energy tensor of matter where we have taken into account the nontrivial coupling to geometry. The prime means the derivative with respect to \( R \).

However, if \( f(R) = R + 2\Lambda \), the standard second–order gravity is recovered. Reducing the action to a point-like, FRW one, we have to write

\[ A_{(\text{curv})} = \int dt \mathcal{L}(a, \dot{a}; R, \dot{R}) , \]  

where dot means derivative with respect to the cosmic time. The scale factor \( a \) and the Ricci scalar \( R \) are the canonical variables. This position could seem arbitrary since \( R \) depends on \( a, \dot{a}, \ddot{a} \), but it is generally used in canonical quantization \[10,29,30\]. The definition of \( R \) in terms of \( a, \dot{a}, \ddot{a} \) introduces a constraint which eliminates second and higher order derivatives in action (II.3), and gives a system of second order differential equations in \( \{a, R\} \). Action (II.3) can be written as

\[ A_{(\text{curv})} = 2\pi^2 \int dt \left\{ a^3 f(R) - \lambda \left[ R + 6 \left( \frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} \right) + \frac{k}{a^2} \right] \right\} , \]  

where the Lagrange multiplier \( \lambda \) is derived by varying with respect to \( R \). It is
\[ \lambda = a^3 f'(R). \]  

(II.8)

The point-like Lagrangian is then

\[ \mathcal{L} = \mathcal{L}_{\text{curv}} + \mathcal{L}_{\text{matter}} = a^3 \left[ f(R) - R f'(R) \right] + 6a \dot{a}^2 f'(R) + 
+ 6a^2 \ddot{a} \dot{R} f''(R) - 6ka f'(R) + a^3 p_{\text{matter}}, \]  

(II.9)

where we have taken into account also the fluid matter contribution which is, essentially, a pressure term \([\text{p}]\).

The Euler-Lagrange equations are

\[ 2 \left( \frac{\ddot{a}}{a} \right) + \left( \frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2} = -p_{\text{tot}}, \]  

(II.10)

and

\[ f''(R) \left[ R + 6 \left( \frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} \right) \right] = 0. \]  

(II.11)

The dynamical system is completed by the energy condition

\[ \left( \frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2} = \frac{1}{3} \rho_{\text{tot}}. \]  

(II.12)

### III. CURVATURE QUINTESSENCE

Combining Eq. (II.10) and Eq. (II.12), we obtain

\[ \left( \frac{\ddot{a}}{a} \right) = -\frac{1}{6} \left[ \rho_{\text{tot}} + 3p_{\text{tot}} \right], \]  

(III.1)

where it is clear that the accelerated or decelerated behaviour depends on the rhs. However

\[ p_{\text{tot}} = p_{\text{curv}} + p_{\text{matter}} \quad \rho_{\text{tot}} = \rho_{\text{curv}} + \rho_{\text{matter}}, \]  

(III.2)

where we have distinguished the curvature and matter contributions.

From the curvature-stress-energy tensor, we can define a curvature pressure

\[ p_{\text{curv}} = \frac{1}{f'(R)} \left\{ 2 \left( \frac{\dot{a}}{a} \right) \dot{R} f''(R) + \dot{R}^2 f''(R) - \frac{1}{2} \left[ f(R) - R f'(R) \right] \right\}, \]  

(III.3)

and a curvature density

\[ \rho_{\text{curv}} = \frac{1}{f'(R)} \left\{ \frac{1}{2} \left[ f(R) - R f'(R) \right] - 3 \left( \frac{\dot{a}}{a} \right) \dot{R} f''(R) \right\}. \]  

(III.4)
From Eq. (III.1), the accelerated behaviour is achieved if
\[ \rho_{(\text{tot})} + 3p_{(\text{tot})} < 0, \] (III.5)
which means
\[ \rho_{(\text{curv})} > \frac{1}{3}\rho_{(\text{tot})}, \] (III.6)
assuming that all matter components have non-negative pressure.

In other words, conditions to obtain acceleration depends on the relation
\[ \rho_{(\text{curv})} + 3p_{(\text{curv})} = \frac{3}{f'(R)} \left\{ R^2 f'''(R) + \left( \frac{\dot{a}}{a} \right) \dot{R} f''(R) + \ddot{R} f''(R) - \frac{1}{3} [f(R) - Rf'(R)] \right\}, \] (III.7)
which has to be compared with matter contribution. However, it has to be
\[ \frac{p_{(\text{curv})}}{\rho_{(\text{curv})}} = \gamma_{(\text{curv})}, \quad -1 \leq \gamma_{(\text{curv})} < 0. \] (III.8)
The form of \( f(R) \) is the main ingredient to obtain this curvature quintessence.

**IV. EXACT SOLUTIONS**

As simple choice in order to fit the above prescriptions, we ask for solutions of the form
\[ f(R) = f_0 R^n, \quad a(t) = a_0 \left( \frac{t}{t_0} \right)^\beta. \] (IV.1)
However, the interesting cases are for \( n \neq 1 \) (Einstein gravity) and \( \beta \geq 1 \) (accelerated behaviour). Inserting Eqs. (IV.1) into the above dynamical system, we obtain the exact solutions
\[ \beta = 2; \quad n = -1, 3/2; \quad k = 0. \] (IV.2)
In both cases, the deceleration parameter is
\[ q_0 = -\frac{1}{2}, \] (IV.3)
in perfect agreement with the observational results.

The case \( n = 3/2 \) deserves further discussion. It is interesting in conformal transformations from Jordan frame to Einstein frame \cite{31,32} since it is possible to give explicit form of scalar field potential. In fact, if
\[ \tilde{g}_{\alpha\beta} \equiv f'(R)g_{\alpha\beta}, \quad \varphi = \sqrt{\frac{3}{2}} \ln f'(R), \] (IV.4)

we have the conformal equivalence of the Lagrangians

\[ \mathcal{L} = \sqrt{-g} f_0 R^{3/2} \longleftrightarrow \tilde{\mathcal{L}} = \sqrt{-\tilde{g}} \left[ -\tilde{R} + \frac{1}{2} \nabla_\mu \varphi \nabla^\mu \varphi - V_0 \exp \left( \sqrt{\frac{2}{3}} \varphi \right) \right] , \] (IV.5)

in our physical units. This is the so-called Liouville field theory and it is one of the few cases where a fourth-order Lagrangian can be expressed, in the Einstein frame, in terms of elementary functions under a conformal transformation. It is possible to obtain the general cosmological solution \[33\] which is

\[ a(t) = a_0 \left[ c_4 t^4 + c_3 t^3 + c_2 t^2 + c_1 t + c_0 \right]^{1/2} . \] (IV.6)

The constants \( c_i \) are combinations of the initial conditions. Their values determine the type of cosmological evolution. For example, \( c_4 \neq 0 \) gives a power law inflation while, if the regime is dominated by the linear term in \( c_1 \), we get a radiation-dominated stage.

V. CONCLUSIONS

In this paper, we have shown that the quintessence "paradigm" can be recovered in the framework of higher-order theories of gravity. In other words, as it is possible for inflationary models, we can ask for a sort of curvature quintessence which can be recovered by taking into account curvature geometric invariants. The interest of this approach is that quintessence could be related to some effective theory of quantum gravity where curvature invariants are widely derived \[34\].

In any case, the models have to be improved by the comparison with observations in order to see if it is possible to constrain the form of \( f(R) \) without making \textit{ad hoc} choices. In this sense, some results are present in literature where the form of \( f(R) \) is selected by the CMBR constraint \[35\].

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