Off-shell higher-spin gauge supermultiplets and conserved supercurrents

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Abstract

This thesis presents the general structure of non-conformal higher-spin supercurrent multiplets in three and four spacetime dimensions. Such supercurrents are in one-to-one correspondence with off-shell massless higher-spin gauge supermultiplets, some of which are constructed in this thesis for the first time. Explicit realisations of these conserved current multiplets in various supersymmetric theories are worked out in detail.

In the first part of the thesis, we begin by reviewing the key properties of known massless higher-spin $\mathcal{N}=1$ supermultiplets in four-dimensional (4D) Minkowski and anti-de Sitter (AdS) backgrounds. We then propose a new off-shell gauge formulation for the massless integer superspin multiplet. Its novel feature is that the gauge-invariant action involves an unconstrained complex superconformal prepotential, in conjunction with two types of compensators. Its dual version is obtained by applying a superfield Legendre transformation. Next, we deduce the structure of consistent non-conformal higher-spin $\mathcal{N}=1$ supercurrents associated with these massless supersymmetric gauge theories. Explicit closed-form expressions for such supercurrents are derived for various supersymmetric theories in 4D $\mathcal{N}=1$ Minkowski and AdS superspaces. These include a model of $N$ massive chiral superfields with an arbitrary mass matrix, along with free theories of tensor and complex linear multiplets.

The second part of the thesis is devoted to a detailed study of $\mathcal{N}=1$ and $\mathcal{N}=2$ supersymmetric higher-spin theories in three-dimensional AdS space. By analogy with our 4D $\mathcal{N}=1$ constructions, we derive two dually equivalent off-shell Lagrangian formulations for the massless multiplets of arbitrary superspin in (1,1) AdS superspace. These formulations allow us to determine the most general higher-spin supercurrent multiplets and provide their examples for models of chiral superfields. With regards to (2,0) AdS supersymmetry, our approach is to first identify a multiplet of conserved higher-spin currents in simple models for a chiral superfield. This is then used to construct two series of a massless half-integer superspin multiplet in (2,0) AdS superspace. Finally, our (2,0) AdS higher-spin supermultiplets are reduced to (1,0) AdS superspace, which yield four series of $\mathcal{N}=1$ supersymmetric massless higher-spin models. We illustrate the duality transformations relating some of these dynamical systems. We also perform the component reduction of two new $\mathcal{N}=1$ higher-spin actions in flat superspace. Further applications of these off-shell $\mathcal{N}=1$ models are discussed, one of which is related to the construction of two new off-shell formulations for the massive $\mathcal{N}=1$ gravitino supermultiplet in AdS.
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Permission has been granted to use this work.

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Chapter 1

Introduction

Relativistic quantum field theory gives a powerful framework to describe all known elementary particles to a good accuracy (see [7,8] for reviews). At present, our theoretical understanding of the fundamental interactions of Nature is based on the Standard Model of particle physics and Einstein’s theory of gravity. The Standard Model accounts for three of the four fundamental interactions, i.e. the electromagnetic, weak and strong forces. It is a renormalisable quantum field theory, and is described in terms of the Yang-Mills gauge fields coupled to some matter sector, with the gauge group SU(3) × SU(2) × U(1). All the fundamental particles encoded in such a formulation have spin $s \leq 1$. Gravity is described by general relativity at the classical level, which is a non-Abelian gauge theory of a spin-2 field possessing a diffeomorphism invariance. While the gravitational interaction is irrelevant at the energy scale of the Standard Model, quantum gravitational effects are not negligible at the Planck scale, which is of the order of $10^{19}$ GeV. Since general relativity is non-renormalisable, one of the main challenges of modern theoretical physics is to reconcile quantum field theory and general relativity in order to arrive at a consistent quantum theory of gravity. Given that the regions associated to the Planck energy are not directly accessible using existing experiments, the development of this subject has been mainly driven by symmetry principles and other theoretical ideas.

Supersymmetry in four dimensions was discovered by Golfand and Likhtman [9], Volkov and Akulov [10], and Wess and Zumino [11]. It is a symmetry relating two types of particles in Nature, bosons and fermions, which were previously unrelated in field theories. The generators of supersymmetry transformations are fermionic, thus they obey anticommutation relations. The supersymmetric extension of the Poincaré group is known as $\mathcal{N}$-extended super-Poincaré group. This involves the addition of $\mathcal{N}$ Majorana spinor generators to the generators of the Poincaré algebra, where $\mathcal{N}$ is a positive integer.

Despite the lack of experimental evidence, supersymmetry remains attractive for its phenomenological and theoretical applications. It offers some solutions to long standing
problems, such as the hierarchy problem, gauge unification, etc, see e.g. [12] for a review. On the theoretical side, it is interesting to study a supersymmetric extension of general relativity\footnote{As discovered by Volkov and Soroka [13], gauging the $\mathcal{N} = 1$ super-Poincaré group leads to the supergravity action with nonlinearly realised local supersymmetry.} which is supergravity \cite{14,15}. The latter is the gauge theory of supersymmetry. It arises if one makes supersymmetry transformations local, with the gravitino (spin 3/2) being the corresponding gauge field. Its superpartner, the graviton (spin 2), is associated to the diffeomorphism invariance. Supergravity multiplet consists of the graviton and $\mathcal{N}$ gravitino fields (for $\mathcal{N} > 1$, the multiplet also contains some vector and scalar fields).

Supergravity has some remarkable properties, see \cite{16,17} for reviews. Most notably, it is a low-energy limit of superstring theory (see \cite{18,19} for reviews), which is currently the leading candidate to give a unified description of the four fundamental forces. Superstring theory includes in its spectrum an infinite tower of massive higher-spin excitations, which lead to improved ultraviolet behaviour of the theory. In recent years, there has also been an incredibly productive area of research to understand the relationship between supergravity and the dynamics of such higher-spin states. In the past, most of the studies were restricted to $\mathcal{N} \leq 8$ case. Theories with $\mathcal{N} > 8$ would include massless fields with higher-spin (helicity) $s > 2$, and at that time it was not known how to couple such fields consistently to gravity \cite{20}.

The study of higher-spin fields has been carried out quite independently of string theory, initiated in the works of Dirac \cite{21}, Fierz and Pauli \cite{22}, Rarita and Schwinger \cite{23} and many others. As follows from Wigner’s classification \cite{24}, free elementary particles are associated with the unitary irreducible representations of the Poincaré group, the latter are classified by mass and spin. In any physical system, the spin can take arbitrary integer or half-integer values. The term “higher-spin” refers to fields with spin $s > 2$, which are higher-rank tensor representations of the Poincaré group. Of particular interest are massless higher-spin fields and their gauge symmetries underlying their dynamics, which pose challenges in the construction of interaction vertices. We now provide a brief historical overview of higher-spin theory, see e.g. \cite{20,25,28} for complete reviews.

Despite the pioneering works of Dirac \cite{21}, Fierz and Pauli \cite{22} who formulated relativistic wave equations for free massive fields of arbitrary spin, it took over thirty years until the corresponding Lagrangian description were constructed by Singh and Hagen \cite{29,30}. In the massless limit, consistent Lagrangians were constructed in 4D flat and (anti)-de Sitter ((A)dS) spaces. They were proposed by Fronsdal \cite{31,32} in the bosonic case, and by Fang and Fronsdal for fermionic fields \cite{33,34}. Section 6.9 of \cite{35} contains a pedagogical review of the (Fang-)Fronsdal actions in Minkowski space \cite{31,33} in the two-component spinor formalism.\footnote{Such a formalism will be used in this thesis. Not only is this formalism useful for higher-spin calculations, but it is also well adopted to the framework of supersymmetry.}
Constructing consistent deformations of Fronsdal’s Lagrangian, which should lead to fully interacting higher-spin gauge theories, proved to be very challenging. This was due to a large number of highly restrictive no-go theorems (e.g. [36–38]) which, under certain assumptions, rule out any gravitational interactions of massless higher-spin fields in flat space. In the 1980s, several cubic vertices for higher-spin fields interacting with each other were constructed in flat space [39–41]. It was also observed that any consistent interacting theory must involve fields of all spins and higher derivative terms.

The first successful result on higher-spin gravitational interactions was achieved by Fradkin and Vasiliev in 1987 [42]. They constructed a Lagrangian describing consistent cubic vertices in the presence of a cosmological term, i.e. in an (A)dS background where the no-go results can be evaded. This line of research culminated in Vasiliev’s unfolded formulation [43], in which the fully interacting theories were described in terms of consistent nonlinear equations of motion for an infinite spectrum of higher-spin fields. This was later extended to arbitrary spacetime dimensions [44]. However, one of the issues is that a Lagrangian formulation for these nonlinear theories is still unknown, thus constraining our understanding of their quantum properties.

Since the early 1990s, higher-spin gauge theory has been an intense field of research in modern theoretical and mathematical physics. That AdS space is the natural setup for interacting higher-spin fields has motivated further studies in the context of the AdS/CFT correspondence, leading to some conjectures relating higher-spin theories to weakly coupled conformal field theories [45,46]. There are many other important research directions in higher-spin theory, such as the unfolded formulation, (topologically) massive higher-spin, BRST approach, etc (see e.g. [20, 25, 26, 47–52] and the references therein). As mentioned previously, a major motivation for current research stems from its close connection with (super)string theory. It is conjectured that the latter is a spontaneously broken phase of a massless higher-spin theory. In this respect, it is of interest to study supersymmetric extensions of the massless higher-spin fields of [32,34].

A powerful approach to construct supersymmetric field theories makes use of the concept of superspace. Volkov and Akulov [53] introduced an $\mathcal{N}$-extended superspace in the framework of nonlinear realisations. Salam and Strathdee [54] proposed to use superspace and superfields, as tools to construct and study supersymmetric theories. Superspace is an extension of spacetime by anticommuting (Grassmann) coordinates, while superfields are functions of the superspace coordinates. One has to impose certain constraints on superfields to describe irreducible representations of the $\mathcal{N}$-extended super-Poincaré algebra, known as supermultiplets. A series expansion of a superfield in the Grassmann coordinates will be finite due to their anticommuting nature. The coefficients in such an expansion correspond to component fields of the superfield, which are ordinary bosonic and fermionic functions of spacetime coordinates. A supersymmetric theory is written
compactly in terms of superfields, thus supersymmetry is kept manifest. Throughout this thesis, we will employ the superspace approach. For a thorough introduction to this formalism, the reader is referred to [35, 55, 56].

In general, the superspace approach is not widely used by higher-spin practitioners. This thesis is primarily devoted to higher-spin multiplets of conserved currents in supersymmetric field theories. As first shown by Ferrara and Zumino [57], the conserved energy-momentum tensor and spin-vector supersymmetry current(s) associated with the supertranslations are embedded in a supermultiplet, called the supercurrent. It should be pointed out that this multiplet of currents is always \textit{off-shell} by construction [58]. Given an on-shell superfield, one may construct another multiplet by taking bilinear combinations of the component fields of the superfield. One then considers their variations under supersymmetry transformations to find the other components of the composite multiplet. The on-shell condition is not preserved when taking such products. The resulting multiplet of bilinears thus forms an off-shell multiplet. An example for this is the supercurrent, which is a composite of the underlying on-shell matter superfield. Off-shell supermultiplets require auxiliary fields in their description. In order to efficiently formulate off-shell supersymmetric theories, superspace techniques are absolutely essential, since all the appropriate auxiliary fields are included automatically.

\textbf{Off-shell massless higher-spin supermultiplets}

In 4D $\mathcal{N} = 1$ supersymmetric field theory, a massless superspin-$\hat{s}$ ($\hat{s} = \frac{1}{2}, 1, \ldots$) multiplet describes two ordinary massless fields of spin $\hat{s}$ and $\hat{s} + \frac{1}{2}$. Such a supermultiplet is often denoted by $(\hat{s}, \hat{s} + \frac{1}{2})$. The three lowest superspin values, $\hat{s} = \frac{1}{2}, 1$ and $\frac{3}{2}$, correspond to the vector, gravitino and supergravity multiplets, respectively.

It follows from first principles that the sum of two actions for free massless spin-$\hat{s}$ and spin-($\hat{s} + \frac{1}{2}$) fields should possess an on-shell supersymmetry. Thus, there is no problem of constructing on-shell massless higher-spin supermultiplets ($\hat{s} > \frac{3}{2}$), for one only needs to work out the supersymmetry transformations leaving invariant the pair of (Fang-)Fronsdal actions [31, 33]. In four dimensions, this task was completed first by Curtright [59] who made use of the (Fang-)Fronsdal actions, and soon after by Vasiliev [60] with his frame-like reformulation of the (Fang-)Fronsdal models. The nontrivial problem, however, is to construct off-shell massless higher-spin supermultiplets. Early attempts to construct such off-shell realisations [61, 62] were unsuccessful, as explained in detail in [63].

The problem of constructing gauge \textit{off-shell} superfield realisations for free massless higher-spin supermultiplets was solved in the early 1990s by Kuzenko, Postnikov and

\textsuperscript{3}A supermultiplet is called \textit{off-shell} if the algebra of supersymmetry transformations closes off the mass shell, \textit{i.e.} without imposing the equations of motion. Otherwise, the supermultiplet under consideration is called on-shell.
Sibiryakov [64, 65]. For each superspin value \( \hat{s} > \frac{3}{2} \), half-integer [64] and integer [65], these publications provided two dually equivalent off-shell actions formulated in 4D \( \mathcal{N} = 1 \) Minkowski superspace. At the component level, each of the two superspin-\( \hat{s} \) actions [64, 65] reduces, upon imposing a Wess-Zumino-type gauge and eliminating the auxiliary fields, to a sum of the spin-\( \hat{s} \) and spin-(\( \hat{s} + \frac{1}{2} \)) actions [31, 33]. The models [64, 65] thus provided the first manifestly supersymmetric extensions of the (Fang-)Fronsdal actions for massless higher-spin fields. A pedagogical review of the results obtained in [64, 65] can be found in section 6.9 of [35]. In [63], the massless higher-superspin theories of [64, 65] were generalised to 4D \( \mathcal{N} = 1 \) AdS superspace, AdS\( ^4 \), and their quantisation was carried out in [66]. Building on the \( \mathcal{N} = 1 \) analysis, off-shell massless \( \mathcal{N} = 2 \) supermultiplets were proposed in [67]. Models describing off-shell \( \mathcal{N} = 1 \) superconformal higher-spin multiplets were constructed in [68], which made use of the gauge prepotentials introduced in [69].

The structure of the (Fang-)Fronsdal gauge-invariant actions and their \( \mathcal{N} = 1 \) supersymmetric counterparts of half-integer superspin share one common feature. For each of them, the action is written in terms of two multiplets: a (super)conformal gauge (super)field coupled to certain compensators. The (Fang-)Fronsdal actions [31, 33] can be interpreted as gauge-invariant models described by the Fradkin-Tseytlin conformal gauge fields [70, 71] and an appropriate set of compensators.\(^4\) The massless half-integer superspin actions of [64] involve not only the real superconformal gauge prepotential [68], but also some complex compensating superfields. Such a description was previously unknown for the massless multiplet of integer superspin [65]. This was the motivation for the recent work [2], where we proposed a new off-shell formulation for the massless multiplet of integer superspin. Its properties are: (i) the gauge freedom matches that of the complex superconformal integer superspin multiplet introduced in [68]; and (ii) the action involves two compensating multiplets, in addition to the superconformal integer superspin multiplet. Upon imposing a partial gauge fixing, this action reduces to the so-called longitudinal formulation for the integer superspin [65]. This construction was later lifted to \( \mathcal{N} = 1 \) AdS supersymmetry [3].

**Higher-spin supercurrents**

As previously pointed out, the concept of supercurrent was introduced by Ferrara and Zumino [57] in the context of \( \mathcal{N} = 1 \) supersymmetry. This was later extended to 4D \( \mathcal{N} = 2 \) supersymmetry by Sohnius [72].

The multiplet of currents in superconformal field theories has a simpler structure. The \( \mathcal{N} = 1 \) conformal supercurrent multiplet contains the symmetric traceless energy-momentum tensor \( T_{ab} \), the spin-vector supersymmetry current \( S_a \) and the \( R \)-symmetry (i.e. \( U(1)_R \)) current \( j_a \).

\(^4\)See the beginning of chapter 3 for the details.
In the non-superconformal case (e.g. $\mathcal{N} = 1$ Poincaré supersymmetric theories), the supercurrent multiplet also includes the trace multiplet containing the trace of the energy-momentum tensor and the $\gamma$-trace of the supersymmetry current. In some cases, the trace multiplet also contains the divergence of the $U(1)_R$ current, $\partial_\alpha j^\alpha$. Different supersymmetric theories may possess different trace multiplets. This means that the problem of classifying inequivalent non-conformal supercurrent multiplets needs to be addressed. Ten years ago, there appeared numerous papers devoted to studying consistent $\mathcal{N} = 1$ supercurrents in four dimensions [73–79].

Supercurrent can be viewed as the source of supergravity [80,81], in complete analogy with the energy-momentum tensor as the source of gravity. This idea proves to be powerful in deriving various consistent supercurrents. Given a linearised off-shell supergravity action, the supercurrent conservation equation is obtained by coupling the supergravity prepotentials to external sources and demanding invariance of the resulting action under the linearised supergravity gauge transformations. Using this procedure, the general structure of consistent supercurrents are presented in [77,78,82] for $\mathcal{N} = 1$ and $\mathcal{N} = 2$ super-Poincaré cases in four dimensions, and in [83] for $\mathcal{N} = 2$ supersymmetric theories in three dimensions.

On the other hand, supercurrent can be used to deduce the off-shell structure of a massless supermultiplet which is associated to it. The procedure to follow is concisely described by Bergshoeff et al. [58]: “One first assigns a field to each component of the current multiplet, and forms a generalised inner product of field and current components.” Indeed, this approach has been used in the past to construct off-shell supergravity multiplets in diverse dimensions [58,72,84–88]. The point is that the currents always form an off-shell multiplet, thus the fields to which they couple must also be off-shell. For example, in [72] the $\mathcal{N} = 2$ supercurrent for the massive hypermultiplet model was used to derive the minimal multiplet of $\mathcal{N} = 2$ Poincaré supergravity.

All off-shell formulations for $\mathcal{N} = 1$ Poincaré supergravity are described in terms of the real gravitational superfield $H^{\alpha\dot{\alpha}}$ and a compensator. The gravitational superfield couples to the supercurrent $J_{a\dot{a}}$, while the source associated with the compensator is the trace multiplet. Thus, different choices of compensator lead to variant non-conformal supercurrents. Since the linearised off-shell $\mathcal{N} = 1$ supergravity actions have been classified [89], all minimal consistent supercurrents are readily derivable [77]. Reducible supercurrents, such as the $S$-multiplet introduced by Komargodski and Seiberg [76], can be obtained by combining some of the minimal ones.

Various aspects of field theories with $\mathcal{N} = 1$ AdS supersymmetry have been studied in detail over the last forty years, see e.g. [90–98] and references therein. The works of Ivanov and Sorin [92,93] are fundamental in the development of superfield techniques. They classified off-shell superfield representations of the OSp(1|4) group (i.e. the isometry
group of $\text{AdS}^{4|4}$). Furthermore, they constructed $\text{OSp}(1|4)$-invariant actions generalising the Wess-Zumino model and $N = 1$ super Yang-Mills theory.

The structure of consistent supercurrent multiplets in $\text{AdS}^{4|4}$ considerably differs from that in the $N = 1$ super-Poincaré case. There exist three minimal supercurrents with $(12 + 12)$ degrees of freedom in Minkowski superspace. As discussed in [99], there are only two irreducible AdS supercurrents: minimal $(12 + 12)$ and non-minimal $(20 + 20)$, which are related via a well-defined improvement transformation. The minimal supercurrent is the AdS extension of the Ferrara-Zumino supercurrent [57].

We remark that these consistent AdS supercurrents are closely related to two classes of supersymmetric gauge theories: (i) the known off-shell formulations for pure $N = 1$ AdS supergravity, minimal (see e.g. [35, 56] for reviews) and non-minimal [99]; and (ii) the two dually equivalent series of massless higher-spin supermultiplets in AdS [63]. More precisely, the minimal supercurrent is associated with the longitudinal action $S^\parallel_{(3/2)}$ for a massless superspin-3/2 multiplet in AdS. The non-minimal supercurrent is associated with the dual formulation $S^\perp_{(3/2)}$. The functional $S^\parallel_{(3/2)}$ proves to be the linearised action for minimal $N = 1$ AdS supergravity. The dual action $S^\perp_{(3/2)}$ results from the linearisation around the AdS background of non-minimal $N = 1$ AdS supergravity. Both actions represent the lowest superspin limits of the off-shell massless supermultiplets of half-integer superspin in AdS [63].

Higher-spin supercurrent multiplet is a higher-spin extension of the ordinary supercurrent. In its component expansion, it contains conserved bosonic and fermionic currents. Conserved higher-spin currents for scalar and spinor fields in 4D Minkowski space have been studied in numerous publications. To the best of our knowledge, the first construction of currents for both scalar and spinor fields was given by Kibble [101]. Migdal [102] and Makeenko [103] later also described the spinor case. Further examples of conserved higher-spin currents were given in [103–108]. Higher-spin extension (in the half-integer superspin case) of the conformal supercurrent [57] was proposed more than thirty years ago by Howe, Stelle and Townsend [69]. Recently in Ref. [68], the structure of such a supercurrent was described in more detail, and examples for supercurrents in some superconformal models were also given. In 3D Minkowski space, explicit constructions of conserved higher-spin supercurrents in free superconformal theories were obtained in [109].

As regards non-conformal higher-spin supercurrents, their properties had not been analysed. The primary goal of this thesis is to study the general structure of non-conformal higher-spin supercurrent multiplets in three and four dimensions from the viewpoint of off-shell higher-spin gauge supermultiplets. As demonstrated in [77,83,99], the general structure of the supercurrents in AdS differ significantly from their counterparts in Minkowski space. This motivated us to look for realisations of higher-spin supercurrents in field theories with Poincaré and AdS supersymmetry.
For this purpose, we developed a higher-spin extension of the general superfield approach advocated in [77, 83]. Here the task was simpler in four dimensions since such off-shell actions already existed [63–65]. In three dimensions, off-shell massless $\mathcal{N} = 1$ [50] and $\mathcal{N} = 2$ [49] supermultiplets were constructed in Minkowski space, but only for the half-integer superspin case. In [51], there appeared two off-shell actions corresponding to half-integer and integer $\mathcal{N} = 1$ supermultiplets in AdS. More general off-shell massless higher-spin supermultiplets in $\mathcal{N} = 1$ and $\mathcal{N} = 2$ AdS superspaces were presented in [4–6].

In four dimensions, we only concentrated on off-shell massless higher-spin multiplets and their associated conserved currents with $\mathcal{N} = 1$ Poincaré and AdS supersymmetry [1–3]. In three dimensions, however, we considered both $\mathcal{N} = 1$ and $\mathcal{N} = 2$ AdS cases [4–6]. In contrast to the four-dimensional case where pure $\mathcal{N} = 1$ AdS supergravity [110] is unique on-shell, the specific feature of three dimensions is the existence of two distinct $\mathcal{N} = 2$ AdS supergravity theories [111, 112]. They are known as the (1,1) and (2,0) AdS supergravity theories, originally constructed as Chern-Simons theories [111]. In Ref. [83], various aspects of (1,1) and (2,0) AdS supergravity theories (including the general structure of supercurrents) were studied in detail, using the superspace formalism developed by Kuzenko, Lindström and Tartaglino-Mazzucchelli [113]. Thus, it is natural to extend the analysis of [83] to the higher-spin case [4–5].

Let us discuss some applications of our results presented in this thesis. In accordance with the standard Noether method (see e.g. [114] for a review), construction of conserved higher-spin supercurrents for various supersymmetric theories is equivalent to generating consistent cubic vertices of the type $\int H J$. Here $H$ denotes some off-shell higher-spin gauge multiplet, and $J = D^p\Phi D^q\Psi$ is the higher-spin conserved current multiplet, constructed in terms of some matter multiplets $\Phi$ and $\Psi$, and superspace covariant derivatives $D$. In 4D Minkowski superspace, several cubic vertices involving the off-shell higher-spin multiplets of [64, 65] were constructed recently [115, 120] using the superfield Noether procedure [73]. For instance, conserved supercurrents and cubic interactions between massless higher-spin supermultiplets and a single chiral superfield were constructed by Buchbinder, Gates and Koutrolikos [115]. This analysis was soon extended by Koutrolikos, Koči and von Unge to study cubic vertices in the case of a free complex linear superfield [116]. The corresponding component higher-spin currents were also computed [116].

Making use of the gauge off-shell formulations for massless higher-spin supermultiplets of [63], higher-spin extensions of the AdS supercurrents [82] were formulated in 4D $\mathcal{N} = 1$ AdS superspace [3] for the first time. Their realisations for various supersymmetric theories in AdS were also presented, including a model of $N$ massive chiral scalar superfields with an arbitrary mass matrix. Such a program was a natural extension of the earlier flat-space results [1, 2], in which we built on the structure of higher-spin supercurrent multiplets in models for superconformal chiral superfields [68].
In the non-supersymmetric case, conserved higher-spin currents for scalar fields in AdS were studied, e.g. in [121–125]. The nonvanishing curvature of AdS space makes explicit calculations of conserved higher-spin currents much harder than in Minkowski space. Refs. [121,122] studied only the conformal scalar, and only the first order correction to the flat-space expression was given explicitly. The construction presented in [125] is more complete since all conserved higher-spin currents were computed exactly for a free massive scalar field using the so-called ambient space formulation. All these works dealt with bosonic currents. The important feature of supersymmetric theories is that they also possess fermionic currents. The conserved higher-spin supercurrents computed in [3] can readily be reduced to components. This leads to closed-form expressions for conserved higher-spin bosonic and fermionic currents in models with massive scalar and spinor fields.

**Thesis outline**

The purpose of chapter 2 is to introduce various technical aspects and essential background materials. First, a brief account of the $\mathcal{N} = 1$ superspace formalism of [35] is given. We then recall the structure of the non-conformal supercurrent multiplets in 4D $\mathcal{N} = 1$ Minkowski and AdS superspaces following [77, 82]. Finally, we briefly review the two dually equivalent off-shell Lagrangian formulations for massless multiplets of arbitrary superspin in 4D $\mathcal{N} = 1$ Minkowski superspace [64,65].

Chapter 3 presents a new off-shell formulation for the massless superspin-$s$ multiplet in 4D $\mathcal{N} = 1$ Minkowski superspace, where $s = 2, 3, \ldots$ and for the massless gravitino multiplet ($s = 1$). The non-conformal higher-spin supercurrent multiplets associated with the massless (half-)integer superspin gauge theories are derived. In addition, we compute higher-spin supercurrents that originate in the models for a single massless and massive chiral superfield, as well as the massive $\mathcal{N} = 2$ hypermultiplet. This chapter is based on the original works [1],[2].

Chapter 4 is concerned with the extension of the flat-space results in chapter 3 to the case of 4D $\mathcal{N} = 1$ AdS supersymmetry. The dual formulations for massless (half-)integer superspin multiplets in AdS [63] are reviewed and a novel formulation for the massless integer superspin is proposed. Making use of these gauge off-shell models, higher-spin supercurrent multiplets are formulated. Their explicit constructions are presented for various supersymmetric in AdS, including the case of $N$ chiral scalar superfields with an arbitrary mass matrix $M$. We further elaborate on several nontrivial applications of the construction of higher-spin supercurrents. This chapter is based on the original work [3].

In chapter 5, we turn our attention to $\mathcal{N} = 2$ supersymmetric higher-spin theories in 3D AdS space. First, some of the important facts concerning (1,1) and (2,0) AdS superspaces, including superfield representations of the corresponding isometry groups, are reviewed. By analogy with our 4D $\mathcal{N} = 1$ analysis, we construct two dually equivalent
off-shell Lagrangian formulations for every massless higher-spin supermultiplet in (1,1) AdS superspace, and subsequently generate consistent higher-spin supercurrents. In the context of (2,0) AdS supersymmetry, we begin with some simple supersymmetric models in (2,0) AdS superspace to deduce a multiplet of conserved higher-spin currents, from which the corresponding supermultiplet of higher-spin fields can be determined. This results in two off-shell gauge formulations for a massless multiplet of half-integer superspin \((s + \frac{1}{2})\), for arbitrary integer \(s > 0\). This chapter is based on the original works [4,5].

In chapter 6, a manifestly supersymmetric setting to reduce every field theory in (2,0) AdS superspace to \(\mathcal{N} = 1\) AdS superspace is developed. As nontrivial examples, we consider supersymmetric nonlinear sigma models described in terms of \(\mathcal{N} = 2\) chiral and linear supermultiplets. This \((2,0) \rightarrow (1,0)\) AdS reduction technique is then applied to our off-shell massless higher-spin supermultiplets described in chapter 5. This results in four series of \(\mathcal{N} = 1\) supersymmetric higher-spin models in AdS, two of which are new gauge theories. This chapter is based on the original work [6].

Finally, in chapter 7, we conclude this thesis by summarising its key outcomes.

There are three appendices. Our notation and conventions are summarised in appendix A. Appendix B reviews the conserved higher-spin currents for free \(N\) scalars and Majorana spinors with arbitrary mass matrices. In appendix C, we analyse the component structure of the two new \(\mathcal{N} = 1\) supersymmetric higher-spin models constructed in chapter 6.
Chapter 2

Field theories in $\mathcal{N} = 1$ Minkowski and AdS superspaces

In this chapter we collect some technical background materials required to understand subsequent chapters. We begin with a brief introduction to some aspects of field theories in 4D $\mathcal{N} = 1$ Minkowski superspace following [35] \footnote{Although we only give a review of 4D $\mathcal{N} = 1$ supersymmetry, most of its structure can be readily generalised to 3D $\mathcal{N} = 2$ super-Poincaré case.}. The next two sections are intended to illustrate the differences between supercurrent multiplets with $\mathcal{N} = 1$ Poincaré and AdS supersymmetry [77, 99, 100]. Finally, we review the off-shell formulations for massless higher-spin $\mathcal{N} = 1$ supermultiplets in Minkowski space, which were developed in [64, 65].

2.1 Field theories in $\mathcal{N} = 1$ Minkowski superspace

A more detailed and pedagogical introduction to various aspects covered in this section can be found in [35, 55, 56]. Our 4D notation and conventions are essentially those of [35] and are summarised in appendix A.

2.1.1 The Poincaré superalgebra

The simplest supersymmetric extension of the Poincaré group in four dimensions is the $\mathcal{N} = 1$ super-Poincaré group. Associated to this super-Lie group is the $\mathcal{N} = 1$ Poincaré superalgebra [9] with the following (anti-)commutation relations

$$[P_a, P_b] = 0 \ , \quad [M_{ab}, P_c] = i(\eta_{ac}P_b - \eta_{bc}P_a) \ ,$$
$$[M_{ab}, M_{cd}] = i(\eta_{ac}M_{bd} - \eta_{ad}M_{bc} + \eta_{bd}M_{ac} - \eta_{bc}M_{ad}) \ ,$$
$$[M_{ab}, Q_\alpha] = i(\sigma_{ab})^{\alpha\beta}Q_\beta \ , \quad [P_a, Q_\alpha] = 0 \ ,$$
\[ [M_{ab}, \bar{Q}_a] = i(\bar{\sigma}_{ab})_{\dot{\alpha} \dot{\beta}} \bar{Q}_{\dot{\beta}} , \quad [P_a, \bar{Q}_a] = 0 , \tag{2.1.1} \]

\[ \{Q_\alpha, Q_\beta\} = 0 , \quad \{\bar{Q}_\dot{\alpha}, \bar{Q}_{\dot{\beta}}\} = 0 , \]

\[ \{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 2(\sigma^a)_{a\dot{\alpha}} P_a . \]

Here \( P_a \) and \( M_{ab} \) denote the generators of the translation and Lorentz group, respectively.

The supersymmetry generators \( Q_\alpha, \bar{Q}_{\dot{\alpha}} \) (\( \alpha, \dot{\alpha} = 1, 2 \)) are Weyl spinors which transform respectively as \( (1/2, 0) \) and \( (0, 1/2) \) of the Lorentz group. The automorphism group (\( R \)-symmetry group) of the \( \mathcal{N} = 1 \) Poincaré superalgebra \((2.1.1)\) is \( \text{U}(1) \), which act on the supercharges in the following way

\[ Q'_\alpha = e^{i\tau} Q_\alpha , \quad \bar{Q}'_{\dot{\alpha}} = e^{-i\tau} \bar{Q}_{\dot{\alpha}} , \quad \tau \in \mathbb{R} . \tag{2.1.2} \]

In the extended \( (\mathcal{N} > 1) \) supersymmetry case, the \( R \)-symmetry group is \( \text{U}(N) \).

### 2.1.2 \( \mathcal{N} = 1 \) Minkowski superspace

We denote the 4D \( \mathcal{N} = 1 \) Minkowski superspace \([53, 54]\) by \( \mathbb{M}^{4|4} \). It can be identified with the coset space

\[ \mathbb{M}^{4|4} = S\Pi/\text{SL}(2, \mathbb{C}) . \tag{2.1.3} \]

Here \( S\Pi \) is the \( \mathcal{N} = 1 \) super-Poincaré group, and \( \text{SL}(2, \mathbb{C}) \) is the double cover of the restricted Lorentz subgroup \( \text{SO}_0(3, 1) \). Any element of the supergroup \( S\Pi \) can be represented in an exponential form

\[ \exp\left( -ix^a P_a + i(\theta^a Q_\alpha + \bar{\theta}_{\dot{\alpha}} \bar{Q}^{\dot{\alpha}}) \right) \exp\left( \frac{i}{2} \omega^{ab} M_{ab} \right) . \tag{2.1.4} \]

The points of the coset space \( \mathbb{M}^{4|4} \) are

\[ \exp\left( -ix^a P_a + i(\theta^a Q_\alpha + \bar{\theta}_{\dot{\alpha}} \bar{Q}^{\dot{\alpha}}) \right) . \tag{2.1.5} \]

Thus, \( \mathbb{M}^{4|4} \) can be parametrised by the local coordinates \( z^A = (x^a, \theta^\alpha, \bar{\theta}_{\dot{\alpha}}) \), where \( x^a \) are real commuting numbers and \( (\theta^\alpha)^* = \bar{\theta}^{\dot{\alpha}} \) are complex anticommuting numbers.

The action of supersymmetry transformations on the superspace coordinates can be determined using the algebra \((2.1.1)\). It is given by

\[ \exp\left( i(\epsilon^a Q_\alpha + \bar{\epsilon}_{\dot{\alpha}} \bar{Q}^{\dot{\alpha}}) \right) \exp\left( -ix^a P_a + i(\theta^a Q_\alpha + \bar{\theta}_{\dot{\alpha}} \bar{Q}^{\dot{\alpha}}) \right) \]

\[ = \exp\left( -i(x^a + i\theta^a \bar{\epsilon} - i\epsilon^a \bar{\theta}) P_a + i[(\theta^a + \epsilon^a) Q_\alpha + (\bar{\theta}_{\dot{\alpha}} + \bar{\epsilon}_{\dot{\alpha}}) \bar{Q}^{\dot{\alpha}}] \right) , \tag{2.1.6} \]

from which we can read off

\[ x'^a = x^a + i(\theta^a \epsilon - \epsilon^a \bar{\theta}) , \quad \theta'^\alpha = \theta^\alpha + \epsilon^\alpha . \tag{2.1.7} \]
One may also compute the action of translations, $\exp\left(-ib^aP_a\right)$, and Lorentz transformations, $\Lambda = \exp\left(\frac{i}{2}\omega^{ab}M_{ab}\right)$, in a similar manner. The result is

$$x'^a = (\Lambda(N))_c^a x^c + b^a, \quad \theta'^\alpha = \theta^\beta (N^{-1})_\beta^\alpha,$$

(2.1.8)

where $\Lambda : \text{SL}(2, \mathbb{C}) \to \text{SO}_0(3, 1)$ is the well-known homomorphism given by

$$(\Lambda(N))_c^a = -\frac{1}{2}\text{tr}(\tilde{\sigma}^a N \sigma_c N^\dagger), \quad N \in \text{SL}(2, \mathbb{C}).$$

(2.1.9)

### 2.1.3 Superfields

Supersymmetric field theories on superspace are naturally formulated in terms of tensor superfields. A tensor superfield $V$ of Lorentz type $(\frac{n}{2}, \frac{m}{2})$ is a superfield carrying $n$ undotted and $m$ dotted spinor indices, which are separately symmetrised. Furthermore, it transforms in the following way under the action of an infinitesimal $\mathcal{N} = 1$ super-Poincaré group (note that we have suppressed its tensor indices):

$$\delta V = i \left( -b^a P_a + \frac{1}{2} \omega^{ab} J_{ab} + \epsilon^\alpha Q_\alpha + \bar{\epsilon}^\dot{\alpha} \bar{Q}_{\dot{\alpha}} \right) V. $$

(2.1.10)

The generators take the form

$$P_a = -i \partial_a, \quad J_{ab} = i(x_b \partial_a - x_a \partial_b + (\sigma_{ab})^{\alpha\beta} \theta_\alpha \partial_\beta - (\tilde{\sigma}_{ab})^{\dot{\alpha}\dot{\beta}} \bar{\theta}_{\dot{\alpha}} \bar{\partial}_{\dot{\beta}} - M_{ab}), \quad Q_\alpha = i \partial_\alpha + \tilde{\theta}^{\dot{\alpha}} (\sigma^a)_{\alpha\dot{\alpha}} \partial_a = i \partial_\alpha + \bar{\theta}^{\dot{\alpha}} \partial_{\alpha \dot{\alpha}}, \quad \bar{Q}_{\dot{\alpha}} = -i \bar{\partial}_{\dot{\alpha}} - \theta^\alpha \partial_{a \alpha}. $$

(2.1.11a-d)

We have also introduced the following notation:

$$\partial_{a \dot{\alpha}} := (\sigma^a)_{a \dot{\alpha}} \partial_a, \quad \partial_\alpha := \frac{\partial}{\partial \theta^\alpha}, \quad \bar{\partial}_{\dot{\alpha}} := \frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}}. $$

(2.1.12)

We denote the set of covariant derivatives of $\mathcal{N} = 1$ Minkowski superspace by $D_A = (\partial_a, D_\alpha, \bar{D}^{\dot{\alpha}})$, which have the form

$$D_a = \partial_a + i \tilde{\theta}^{\dot{\alpha}} \partial_{a \dot{\alpha}}, \quad \bar{D}^{\dot{\alpha}} = \bar{\partial}^{\dot{\alpha}} + i \theta_\alpha \partial^{a \dot{\alpha}}, \quad \bar{\partial}^{\dot{\alpha}} = -\varepsilon^{\dot{\alpha} \dot{\beta}} \bar{\partial}_{\dot{\beta}}. $$

(2.1.13)

They obey the (anti)commutation relations:

$$\{D_\alpha, D_\beta\} = \{\bar{D}^{\dot{\alpha}}, \bar{D}^{\dot{\beta}}\} = [D_a, D_\alpha] = [\bar{D}^{\dot{\alpha}}, \partial_a] = 0, \quad \{D_a, \bar{D}^{\dot{\alpha}}\} = -2i \partial_{a \dot{\alpha}}. $$

(2.1.14)

The latter indicates that flat superspace has non-vanishing torsion.

---

2See appendix A for some important properties of the covariant derivatives.
Expanding a tensor superfield \( V(x, \theta, \bar{\theta}) \) with respect to its fermionic coordinates \((\theta, \bar{\theta})\), one obtains its corresponding component fields as the coefficients of the series. Due to the property \( \theta_\alpha \theta_\beta \theta_\gamma = \bar{\theta}_\dot{\alpha} \bar{\theta}_\dot{\beta} \bar{\theta}_\dot{\gamma} = 0 \), such a series will be finite. As an example, consider a Taylor expansion of a real scalar superfield, \( \bar{V}(z) = V(z) \), but otherwise unconstrained:

\[
V(x, \theta, \bar{\theta}) = A(x) + \theta^\alpha \psi_\alpha(x) + \bar{\theta}^\dot{\alpha} \bar{\psi}^\dot{\alpha}(x) + \theta^2 F(x) + \bar{\theta}^2 \bar{F}(x) + \theta^\alpha \bar{\theta}^\dot{\alpha} C_{\alpha\dot{\alpha}}(x) + \theta^2 \bar{\theta}^2 \lambda_\alpha(x) + \theta^2 \bar{\theta} \bar{\lambda}^\dot{\alpha}(x) + \theta^2 \bar{\theta}^2 D(x) . \tag{2.1.15}
\]

To get insight into the physical content of the superfield \( V(z) \), a more systematic and convenient way is to use space projection (also often called bar-projection in some literature) and covariant differentiation. By space projection we mean the zeroth order term in the power series expansion in \( \theta \) and \( \bar{\theta} \):

\[
V := V(x, \theta = 0, \bar{\theta} = 0) . \tag{2.1.16}
\]

In the case of a real scalar superfield above, we may define the components using the bar-projection:

\[
A(x) = V|, \quad \psi_\alpha(x) = D_\alpha V|, \quad \bar{\psi}^\dot{\alpha}(x) = \bar{D}_{\dot{\alpha}} V|, \\
F(x) = -\frac{1}{4} D^2 V(z)|, \quad \bar{F}(x) = -\frac{1}{4} \bar{D}^2 V(z)|, \quad C_{\alpha\dot{\alpha}}(x) = \frac{1}{2} [D_\alpha, \bar{D}_{\dot{\alpha}}] V(z)|, \\
\lambda_\alpha(x) = -\frac{1}{4} D_\alpha D^2 V(z)|, \quad \bar{\lambda}^\dot{\alpha}(x) = -\frac{1}{4} \bar{D}_{\dot{\alpha}} D^2 V(z)|, \\
H(x) = \frac{1}{32} \{ D^2, \bar{D}^2 \} V(z)| . \tag{2.1.17}
\]

In addition, one may work out how the component fields transform under the infinitesimal supersymmetry transformations

\[
\delta V(z) = i(\epsilon^\alpha Q_\alpha + \bar{\epsilon}_{\dot{\alpha}} \bar{Q}^{\dot{\alpha}}) V(z) , \tag{2.1.18}
\]

by taking various numbers of covariant derivatives of \( \text{(2.1.18)} \) and then bar-project them. One should also note that the supersymmetry generator anticommutes with the spinor covariant derivatives, which implies \([D_\alpha, \epsilon Q + \bar{\epsilon} \bar{Q}] = [\bar{D}_{\dot{\alpha}}, \epsilon Q + \bar{\epsilon} \bar{Q}] = 0\).

In appendix [C] we will describe in more detail the 3D \( \mathcal{N} = 1 \) analogue of the above component reduction. Such a procedure is useful to study the field contents of some supersymmetric higher-spin models.

An unconstrained superfield is a reducible representation of supersymmetry. In order to obtain an irreducible representation, we need to impose certain constraints on the superfield which are also consistent with the supersymmetry transformations. This can be done with the help of the covariant derivatives \( D_\alpha, \bar{D}_{\dot{\alpha}} \). The simplest example is the chirality constraint,

\[
\bar{D}_{\dot{\alpha}} \Phi(x, \theta, \bar{\theta}) = 0 , \tag{2.1.19}
\]
with $\Phi(x, \theta, \bar{\theta})$ being a complex scalar superfield. It can be shown that the above constraint implies that

$$\bar{\partial}_\alpha (e^{-i\theta \sigma^a \bar{\partial}_a} \Phi(x, \theta, \bar{\theta})) = 0.$$  \hspace{1cm} (2.1.20)

In order to see this, one may act on both sides of (2.1.19) with the operator $e^{-i\theta \sigma^a \bar{\partial}_a}$, and make use of the identity

$$e^{-i\theta \sigma^a \bar{\partial}_a} D_\alpha e^{i\theta \sigma^a \bar{\partial}_a} = -\bar{\partial}_\alpha .$$  \hspace{1cm} (2.1.21)

It follows from (2.1.20) that $e^{-i\theta \sigma^a \bar{\partial}_a} \Phi(x, \theta, \bar{\theta})$ is independent of $\bar{\theta}$, so it can be written as

$$e^{-i\theta \sigma^a \bar{\partial}_a} \Phi(x, \theta, \bar{\theta}) = \Phi(x, \theta),$$  \hspace{1cm} (2.1.22)

for an arbitrary superfield $\Phi(x, \theta)$. Thus, the solution to the chirality constraint (2.1.19) is given by

$$\Phi(x, \theta, \bar{\theta}) = e^{i\theta \sigma^a \bar{\partial}_a} \Phi(x, \theta) = \Phi(x^a + i\theta \sigma^a \bar{\theta}, \theta).$$  \hspace{1cm} (2.1.23)

A superfield of the form (2.1.23), which depends only on $(x, \theta)$, is called chiral superfield.

Analogously, one can also impose the anti-chirality constraint

$$D_\alpha \bar{\Phi}(x, \theta, \bar{\theta}) = 0 ,$$  \hspace{1cm} (2.1.24)

which is solved by

$$\bar{\Phi}(x, \theta, \bar{\theta}) = e^{-i\theta \sigma^a \partial_a} \bar{\Phi}(x, \theta) = \bar{\Phi}(x^a - i\theta \sigma^a \bar{\theta}, \bar{\theta}).$$  \hspace{1cm} (2.1.25)

In contrast to the chiral superfield (2.1.23), we see that the anti-chiral superfield essentially depends on $(x, \bar{\theta})$ only. Any function of a chiral superfield only is also chiral, that is

$$D_\alpha F(\Phi) = F'(\Phi) \bar{D}_\alpha \Phi = 0 .$$  \hspace{1cm} (2.1.26)

The same goes with anti-chiral superfield.

The component structure of a chiral superfield can be studied by first decomposing $\Phi(x, \theta)$ in terms of $\theta$:

$$\Phi(x, \theta) = A(x) + \theta^a \psi_\alpha(x) + \theta^2 F(x) ,$$  \hspace{1cm} (2.1.27)

or, equivalently

$$A(x) = \Phi| , \quad \psi_\alpha(x) = D_\alpha \Phi| , \quad F(x) = -\frac{1}{4} D^2 \Phi| .$$  \hspace{1cm} (2.1.28)

As a result, we have that

$$\Phi(x, \theta, \bar{\theta}) = e^{i\theta \sigma^a \bar{\partial}_a} \Phi(x, \theta)$$
\[ = A(x) + \theta^\alpha \psi_\alpha + \theta^2 F(x) + i \theta \sigma^a \partial_a A(x) + \frac{i}{2} \theta^2 \bar{\theta} \sigma^a \partial_a \psi(x) + \frac{1}{4} \theta^2 \bar{\theta}^2 \Box A(x) . \] (2.1.29)

The component fields of an anti-chiral superfield can be worked out from (2.1.29) by conjugation.

There are other types of constrained superfields, such as complex linear and real linear superfields. They will be described in section 2.2.

### 2.1.4 Supersymmetric action principle

In superspace formalism, any supersymmetric field theory is described by a set of superfields, with the corresponding action functional written as an integral over the superspace of a Lagrangian superfield \( \mathcal{L} \). The (classical) superfield equations of motion can be obtained using a supersymmetric action principle.

Let us first understand some basics of integration over the anticommuting coordinates \( (\theta, \bar{\theta}) \), which was first given by Berezin [126]. The Berezin integral is equivalent to differentiation. We further note some properties:

\[
\int d\theta^\alpha \theta^\beta = \partial^\beta \theta^\alpha = \delta^\beta_\alpha ,
\]

\[
d^2 \theta = \frac{1}{4} \varepsilon^{\alpha\beta} d\theta^\alpha d\theta^\beta \implies \int d^2 \theta = \frac{1}{4} \partial^\alpha \partial^\alpha , \int d^2 \theta \theta^2 = 1 . \] (2.1.30)

Similarly,

\[
\int d\bar{\theta}^\alpha \bar{\theta}_\beta = \bar{\partial}^\alpha \bar{\theta}_\beta = \delta^\alpha_\beta ,
\]

\[
d^2 \bar{\theta} = \frac{1}{4} \varepsilon_{\dot{\alpha}\dot{\beta}} d\bar{\theta}^\dot{\alpha} d\bar{\theta}^\dot{\beta} \implies \int d^2 \bar{\theta} = \frac{1}{4} \bar{\partial}_{\dot{\alpha}} \bar{\partial}^\dot{\alpha} , \int d^2 \bar{\theta} \bar{\theta}^2 = 1 . \] (2.1.31)

The measure of full \( \mathcal{N} = 1 \) Minkowski superspace is denoted \( d^8 z = d^4 x d^2 \theta d^2 \bar{\theta} \), while the measures on chiral and antichiral subspaces are given by \( d^6 z = d^4 x d^2 \theta \) and \( d^6 \bar{z} = d^4 x d^2 \bar{\theta} \) respectively.

There are many useful properties of integration in full superspace or chiral subspace. First, for an arbitrary superfield \( V(z) \), we have that

\[
\int d^8 z \ D_A(V(z)) = 0 . \] (2.1.32)

Integration in an (anti-)chiral subspace can be reduced to Minkowski space,

\[
\int d^6 z \ V(z) = -\frac{1}{4} \int d^4 x \ D^2 V(z) \bigg| , \] (2.1.33a)

16
\[
\int d^6 \bar{z} \ V(z) = -\frac{1}{4} \int d^4 x \ \bar{D}^2 V(z) . \tag{2.1.33b}
\]

Given an integration in full superspace, it can be written either in (anti-)chiral subspace or Minkowski space:
\[
\int d^8 z V(z) = -\frac{1}{4} \int d^6 \bar{z} \ D^2 \bar{V}(z) = \frac{1}{16} \int d^4 x \ D^2 \bar{D}^2 V(z) . \tag{2.1.34}
\]

The most general supersymmetric action functional takes the form
\[
S = \int d^4 x d^2 \theta d^2 \bar{\theta} L + \int d^4 x d^2 \theta L_c + \int d^4 x d^2 \bar{\theta} \bar{L}_c . \tag{2.1.35}
\]

Here \( L \) is a real scalar superfield, while \( L_c \) and \( \bar{L}_c \) are chiral and anti-chiral scalar superfields, respectively. Performing integration over all the Grassmann variables in (2.1.35) results in component form of the action, which is expressed as an integral over the 4D Minkowski space. This procedure yields
\[
S = \int d^4 x \left( \frac{1}{16} D^2 \bar{D}^2 L - \frac{1}{4} D^2 L_c - \frac{1}{4} \bar{D}^2 \bar{L}_c \right) . \tag{2.1.36}
\]

For completeness, let us prove the invariance of (2.1.35) under the \( \mathcal{N} = 1 \) super-Poincare transformations. We will explicitly show this for the first term. The invariance of the (anti-)chiral action can also be proved in a similar way. For this we vary the Lagrangian \( L \), which is a scalar superfield according to the rule (2.1.18):
\[
\delta_s S = \int d^4 x d^2 \theta d^2 \bar{\theta} i (\epsilon^\alpha Q_\alpha + \bar{\epsilon}_a \bar{Q}^a) L = \frac{1}{16} \int d^4 x D^2 \bar{D}^2 i (\epsilon^\alpha Q_\alpha + \bar{\epsilon}_a \bar{Q}^a) \mathcal{L} \]
\[
= \frac{1}{16} \int d^4 x i (\epsilon^\alpha Q_\alpha + \bar{\epsilon}_a \bar{Q}^a) D^2 \bar{D}^2 L = -\frac{1}{16} \int d^4 x (\epsilon^\alpha D_\alpha + \bar{\epsilon}_a \bar{D}^a) D^2 \bar{D}^2 L \]
\[
= -\frac{1}{16} \int d^4 x \bar{\epsilon}_a [\bar{D}^a, D^2] \bar{D}^2 L = \frac{1}{4} \int d^4 x \partial_{\alpha\alpha} (\epsilon^\alpha D_\alpha \bar{D}^2 L) , \tag{2.1.37}
\]

i.e. the Lagrangian changes by a total spacetime derivative.

Consider a simple superfield model for a free massless chiral scalar superfield \( \Phi \). Its action is given by
\[
S = \int d^4 x d^2 \theta d^2 \bar{\theta} \bar{\Phi} \Phi , \quad \bar{D}_a \Phi = 0 . \tag{2.1.38}
\]

The superfield equations of motion can be derived by varying the action with respect to \( \Phi \), which is defined as an integral over the chiral subspace. This leads to \( D^2 \bar{\Phi} = 0 \). Similarly, one gets \( D^2 \Phi = 0 \) by varying the anti-chiral superfield \( \bar{\Phi} \). Recalling the definition of the
components of $\Phi$ given in (2.1.28), the corresponding component action is easily found to be

$$S = \int d^4x \left( -\partial^a \bar{A} \partial_a A - \frac{i}{2} \psi^\alpha \partial_a \bar{\psi}^{\dot{\alpha}} + \bar{F} F \right) .$$

(2.1.39)

The component fields $F$ and $\bar{F}$ are auxiliary fields. They enter the action without derivatives (or kinetic terms), thus they have no non-trivial dynamics. One further finds that the component fields $A, \psi, F$ (and their conjugates) transform linearly under the infinitesimal supersymmetry transformations:

$$\delta A = -\epsilon^\alpha \psi_\alpha ,$$
$$\delta \psi_\alpha = -2\epsilon_\alpha F - 2i\bar{\epsilon}^{\dot{\alpha}} \partial_a \bar{\psi}^{\dot{\alpha}} A ,$$
$$\delta F = -i\bar{\epsilon}^{\dot{\alpha}} \partial_a \psi_\alpha .$$

(2.1.40)

Suppose the auxiliary fields $F, \bar{F}$ are eliminated through their equations of motion (in this case it is $F = \bar{F} = 0$). Computing the commutators of two infinitesimal supersymmetry transformations, one finds that the supersymmetry algebra is broken when the auxiliary fields are eliminated. More precisely, the result is of the form

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}] A = c^m \partial_m A ,$$
$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}] \psi_\alpha = c^m \partial_m \psi_\alpha + i c_{\alpha \dot{\alpha}} \frac{\delta S}{\delta \bar{\psi}^{\dot{\alpha}}} ,$$

(2.1.41)

with $c^m = 2i(\epsilon_1^\alpha \sigma^m_{\alpha \dot{\alpha}} \bar{\epsilon}_2^{\dot{\alpha}} - \epsilon_2^\alpha \sigma^m_{\alpha \dot{\alpha}} \bar{\epsilon}_1^{\dot{\alpha}})$. This algebra is closed only on the equations of motion for spinor fields $\delta S/\delta \bar{\psi}^{\dot{\alpha}} = 0$. Therefore, the supersymmetry algebra in the theory without auxiliary fields is closed only on-shell (on equations of motion). This explains the role of auxiliary fields, which are to ensure off-shell closure of the supersymmetry algebra on the component fields.

### 2.2 Linearised $\mathcal{N} = 1$ Poincaré supergravity and variant supercurrents

Given an $\mathcal{N} = 1$ supersymmetric theory, one can derive the conserved spin-vector current associated to rigid supersymmetry. By computing Noether currents in the massive Wess-Zumino model, Ferrara and Zumino demonstrated that the conserved energy-momentum tensor $T_{ab}$ and the spin-vector current $S_a$ belong to a supermultiplet, called the supercurrent $[57]$. Additionally, the supercurrent contains the axial U(1)$_R$ current, $j_a$, which is only conserved for a theory with U(1)$_R$ symmetry. The trace of the energy-momentum tensor, the gamma-trace of the spinor current $\gamma^a S_a$, and the axial current divergence $\partial^a j_a$ form a smaller supermultiplet, i.e. the trace supermultiplet (also called supertrace).
For 4D \( \mathcal{N} = 1 \) supersymmetric theories in Minkowski space, the most general supercurrent multiplet is subject to the following conservation law [77,78]

\[
\bar{D}_\dot{a} \chi_\alpha + i \eta_\alpha + D_\alpha T = 0, \quad D^\alpha \chi_\alpha - D_\dot{a} \bar{\chi}_\dot{a} = 0.
\]

Here the real vector superfield \( J_{\alpha \dot{a}} = \bar{J}_{\dot{a} \alpha} \) is the supercurrent. The chiral superfields \( T, \chi_\alpha \) and \( \eta_\alpha \) are the trace supermultiplets.

Depending on supersymmetric theories, some of the trace supermultiplets might vanish. In the case of superconformal theories, we can set all of them to zero. The three terms on the right-hand side of (2.2.1) correspond to the fact that there exist exactly three linearised off-shell formulations for \textit{minimal} \( (12+12) \) \( \mathcal{N} = 1 \) Poincaré supergravity, which have been studied in [89]. These off-shell models are related by duality transformations, i.e. they are equivalent on-shell. More precisely, the authors of [89] classified the following off-shell \( \mathcal{N} = 1 \) superfield models for linearised supergravity: (i) three minimal formulations with \( (12+12) \) off-shell degrees of freedom; (ii) three reducible realisations with \( (16+16) \) components; and (iii) one non-minimal formulation with \( (20+20) \) components. Each formulation corresponds to a different way of gauge-fixing \( \mathcal{N} = 1 \) conformal supergravity to describe Poincaré supergravity (see [35, 56] for reviews). These seven supergravity models give rise to variant supercurrent multiplets.

We recall that \( \mathcal{N} = 1 \) Poincaré supergravity [14,15] describes interacting spin-2 \( h_{mn} \) (the graviton) and spin-3/2 fields \( \psi^\alpha_m, \bar{\psi}^\dot{a}_m \) (the gravitino), with local translational and supersymmetry invariance. All off-shell formulations for linearised \( \mathcal{N} = 1 \) Poincaré supergravity are described by two types of dynamical superfields: the real gravitational gauge superfield \( H_{\alpha \dot{a}} = \bar{H}_{\dot{a} \alpha} \) and a compensating superfield. Each off-shell description contains the graviton and gravitino as dynamical fields, but differs in the set of auxiliary fields. The graviton and gravitino fields can be identified with the components of \( H_{\alpha \dot{a}} \). Switching to the two-component spinor notation (see appendix A), they read

\[
h_{\alpha \dot{a} \beta \dot{\beta}} \sim \left| D(\beta \dot{a}, D(\dot{\beta} \alpha) H_{\alpha \dot{a}} \right|, \quad \psi_{\beta \alpha \dot{a}} \sim \bar{D}^2 D(\dot{\beta} \alpha H_{\alpha \dot{a}}).
\]

The real gravitational superfield \( H_{\alpha \dot{a}} \) has the following linearised gauge transformation

\[
\delta H_{\alpha \dot{a}} = \bar{D}_{\dot{a}} L_\alpha - D_\alpha \bar{L}_{\dot{a}},
\]

where \( L_\alpha \) is an unconstrained spinor superfield. Upon imposing the Wess-Zumino gauge, the remaining transformations correspond to superconformal transformations [127], see also [35,56] for reviews. A compensating superfield is required in order to remove this extra symmetry, and thus describing Poincaré supergravity. The difference between the off-shell models is thus encoded in the choice of the compensators.
Let us now focus on the structure of irreducible superscurrents and the linearised off-shell minimal supergravity formulations that they correspond to. There are three irreducible supercurrent multiplets with (12+12) off-shell degrees of freedom.

- Setting $\chi_\alpha = \eta_\alpha = 0$ leads to the well-known Ferrara-Zumino multiplet \[57\], which corresponds to the old minimal formulation for $\mathcal{N} = 1$ supergravity \[128-130\], with the following supergravity gauge transformation law:

$$
\delta H_{a\dot{a}} = \bar{D}_\dot{a} L_\alpha - D_\alpha \bar{L}_{\dot{a}} ,
\delta \sigma = -\frac{1}{12} \bar{D}^2 D^\alpha L_\alpha .
$$

(2.2.4)

Here $\sigma$ is the chiral compensator, $\bar{D}_\dot{a} \sigma = 0$.

- The case $T = \eta_\alpha = 0$ is known as the $R$-multiplet, which exists if the model has an $R$-symmetry. This multiplet corresponds to the new minimal supergravity \[88\]. New minimal supergravity uses a real linear superfield $G$ as a compensator, $G - \bar{G} = \bar{D}^2 G = 0$. The constrained superfield $G$ is the gauge-invariant field strength of a chiral spinor potential $\Psi_\alpha$

$$
G = D^\alpha \Psi_\alpha + \bar{D}_\dot{a} \bar{\Psi}_{\dot{a}} ,
\bar{D}_\dot{a} \Psi_\beta = 0 ,
$$

(2.2.5)

which is defined modulo gauge freedom

$$
\delta \Psi_\alpha = i \bar{D}^2 D_\alpha K ,
K = \bar{K} .
$$

(2.2.6)

The gauge transformation law is

$$
\delta H_{a\dot{a}} = \bar{D}_\dot{a} L_\alpha - D_\alpha \bar{L}_{\dot{a}} ,
\delta G = \frac{1}{4} (D^\alpha \bar{D}^2 L_\alpha + \bar{D}_\dot{a} D^2 \bar{L}_{\dot{a}}) \implies \delta \Psi_\alpha = \frac{1}{4} \bar{D}^2 L_\alpha .
$$

(2.2.7)

- The third option with $T = \chi_\alpha = 0$, corresponds to the Virial multiplet, which was studied quite recently \[131\]. It corresponds to another minimal supergravity theory introduced in \[132\]. It might have fewer applications because it is known only at linearised level, unlike the old and new minimal theories. This theory also makes use of a real linear compensator superfield $F$, which is the gauge-invariant field strength of a chiral spinor potential $\rho_\alpha$

$$
F = D^\alpha \rho_\alpha + \bar{D}_\dot{a} \bar{\rho}_{\dot{a}} ,
\bar{D}_\dot{a} \rho_\beta = 0 .
$$

(2.2.8)

The supergravity transformation is given by

$$
\delta H_{a\dot{a}} = \bar{D}_\dot{a} L_\alpha - D_\alpha \bar{L}_{\dot{a}} ,
\delta F = \frac{i}{12} (D^\alpha \bar{D}^2 L_\alpha - \bar{D}_\dot{a} D^2 \bar{L}_{\dot{a}}) \implies \delta \rho_\alpha = \frac{i}{12} \bar{D}^2 L_\alpha .
$$

(2.2.9)

\(^3\)A supercurrent multiplet is called irreducible if it is associated with an off-shell formulation for pure supergravity.
If only one of the trace multiplets is zero, the supercurrent multiplet contains bigger (16+16) components and is said to be reducible. The most famous one is the so-called $S$-multiplet, introduced by Komargodski and Seiberg \[65\]. The $S$-multiplet is subject to the conservation equation

$$\bar{D}^\dot{\alpha} J_{a\dot{\alpha}} = D_\alpha T + \chi_\alpha ,$$
$$\bar{D}_\alpha T = \bar{D}_\alpha \chi_\alpha = 0 , \quad D^\alpha \chi_\alpha - \bar{D}_\alpha \chi^{\dot{\alpha}} = 0 . \quad (2.2.10)$$

It has been shown in \[89\] that such models with 16+16 off-shell degrees of freedom can be written as a sum of two of the three minimal models discussed above.

Let us show how to derive a supercurrent multiplet and its conservation equation, starting from a linearised off-shell formulation for $\mathcal{N} = 1$ supergravity, for instance the old minimal supergravity. This approach is based on \[77\], in which variant $\mathcal{N} = 1$ supercurrent multiplets are derived. The analysis for the other formulations should be analogous. First, the following linearised action \[56\]

$$S^{(I)}[H, \sigma] = \int d^4x d^2\theta d^2\bar{\theta} \left\{ - \frac{1}{16} H^{\dot{\alpha}\dot{\beta}} D^\dot{\beta} D^\dot{\beta} D_{a\dot{\alpha}} - \frac{1}{4} \left( \partial_{a\dot{\alpha}} H^{\dot{\alpha}\dot{\beta}} \right)^2 + \frac{1}{48} \left( [D_a, \bar{D}_{\dot{a}}] H^{a\dot{\alpha}} \right)^2 - i (\sigma - \bar{\sigma}) \partial^{\alpha\dot{\alpha}} H_{a\dot{\alpha}} - 3 \bar{\sigma} \sigma \right\} , \quad (2.2.11)$$

is invariant under the linearised gauge transformations \[2.2.4\] of the supergravity prepotentials. Next, we consider the following coupling of the dynamical variables to external sources

$$S^{(I)} \to S^{(I)}[H, \sigma] - \frac{1}{2} \int d^4x d^2\theta d^2\bar{\theta} H^{\dot{\alpha}\dot{\beta}} J_{a\dot{\alpha}} - \frac{3}{2} \left\{ \int d^4x d^2\theta \sigma T + c.c. \right\} . \quad (2.2.12)$$

Demanding invariance of the above action under \[2.2.4\], it is straightforward to show that the sources $J_{a\dot{\alpha}}$ and $T$ must satisfy the conservation equation \[2.2.1\], that is

$$\bar{D}^\dot{\alpha} J_{a\dot{\alpha}} = D_\alpha T , \quad \bar{D}_\alpha T = 0 . \quad (2.2.13)$$

In the case of conformal supergravity, the coupling becomes very simple

$$S_{\text{source}} = \int d^4x d^2\theta d^2\bar{\theta} H^{a\dot{\alpha}} J_{a\dot{\alpha}} , \quad (2.2.14)$$

which leads to the conservation condition

$$\bar{D}^\dot{\alpha} J_{a\dot{\alpha}} = 0 \iff D^a J_{a\dot{\alpha}} = 0 , \quad (2.2.15)$$

as a consequence of imposing invariance under \[2.2.3\]. The independent components of the conformal supercurrent $J_{a\dot{\alpha}}$ are

$$j_{a\dot{\alpha}} := J_{a\dot{\alpha}} , \quad S_{\alpha\dot{\beta}\dot{\alpha}} := D_\beta J_{a\dot{\alpha}} = S_{(\alpha\beta)\dot{\alpha}} , \quad T_{\alpha\dot{\beta}\dot{\gamma}} := [D_(\beta, \bar{D}_\gamma J_{a\dot{\alpha}}] . \quad (2.2.16)$$
Here \( j_{a\dot{a}} \) is the \( R \)-symmetry current, which is not always conserved. The supersymmetry current \( S_{\alpha\dot{\beta}} \), \( \bar{S}_{\dot{\alpha}\beta} \) and the energy-momentum tensor \( T_{\alpha\dot{\beta}\dot{\gamma}} \) are conserved,

\[
\partial^{\alpha\dot{\alpha}} S_{\alpha\dot{\beta}} = 0, \quad \partial^{\alpha\dot{\alpha}} T_{\alpha\dot{\beta}\dot{\gamma}} = 0. \tag{2.2.16}
\]

Let us look at some simple supersymmetric theories in which the above (non-)conformal supercurrents are realised. We first consider a superconformal model for a massless chiral scalar \( \Phi \), \( \bar{D}_{\dot{a}} \Phi = 0 \), with action

\[
S = \int d^4x d^2\theta d^2\bar{\theta} \Phi \Phi \tag{2.2.17}
\]

This theory is characterised by the conformal supercurrent \( \text{[57]} \)

\[
J_{a\dot{a}} = D_a \Phi \bar{D}_{\dot{a}} \Phi + 2i(\Phi \partial_{a\dot{a}} \Phi - \bar{\Phi} \partial_{a\dot{a}} \Phi), \tag{2.2.18}
\]

which obeys the conservation equation (2.2.15), provided the matter superfield is put on-shell: \( D^2 \Phi = 0 \), \( \bar{D}^2 \Phi = 0 \).

A single massive chiral superfield can be coupled to the old minimal supergravity, which is reflected in the existence of the Ferrara-Zumino supercurrent \( \text{[57]} \). A massive chiral superfield is described by the action

\[
S = \int d^4x d^2\theta d^2\bar{\theta} \Phi \Phi + \left\{ \frac{m}{2} \int d^4x d^2\theta \Phi^2 + \text{c.c.} \right\}. \tag{2.2.19}
\]

For this model, \( J_{a\dot{a}} \) can be chosen to have the same functional form as in the massless case, eq. (2.2.18). The trace multiplet is given by

\[
T = m\Phi^2. \tag{2.2.20}
\]

It may be shown that the conservation equation

\[
\bar{D}^\dot{a} J_{a\dot{a}} = D_a T, \quad \bar{D}_\dot{a} T = 0 \tag{2.2.21}
\]

holds on the use of the massive equations of motion

\[
-\frac{1}{4} \bar{D}^2 \Phi + m\Phi = 0, \quad -\frac{1}{4} D^2 \Phi + m\bar{\Phi} = 0. \tag{2.2.22}
\]

As explained in \( \text{[79]} \), the Ferrara-Zumino supercurrent multiplet is not well defined in some supersymmetric theories. On the other hand, the \( S \)-multiplet always exists in all known rigid supersymmetric theories in Minkowski space.

Given real scalar superfields \( U \) and \( V \), the non-conformal supercurrent multiplets (2.2.1) can be transformed by the rule \( \text{[78]} \)

\[
J_{a\dot{a}} \rightarrow J_{a\dot{a}} + [D_a, \bar{D}_{\dot{a}}]V - 2\partial_{a\dot{a}} U, \tag{2.2.23a}
\]

\footnote{This follows from the fact that the gravitational superfield does not couple to the superpotential \( \text{[133]} \).}
\[ T \rightarrow T + \frac{1}{2} \bar{D}^2 (V - iU) , \]  
(2.2.23b)

\[ \chi_\alpha \rightarrow \chi_\alpha + \frac{3}{2} \bar{D}^2 D_\alpha V , \]  
(2.2.23c)

\[ \eta_\alpha \rightarrow \eta_\alpha + \frac{1}{2} \bar{D}^2 D_\alpha U , \]  
(2.2.23d)

while keeping the conservation equation \((2.2.1)\) unchanged. Such a transformation is called an improvement.

### 2.3 Field theories in \( \mathcal{N} = 1 \) AdS superspace

In four dimensions, \( \mathcal{N} = 1 \) supersymmetry in anti-de Sitter space was first studied by Keck [90], and analysis of its nonlinear realisations was given by Zumino [91]. Ivanov and Sorin [93] extensively developed the concept of 4D \( \mathcal{N} = 1 \) AdS superspace and superfield techniques. The \( \mathcal{N} = 1 \) AdS superspace, \( \text{AdS}^{4|4} \), is the simplest member of the family of \( \mathcal{N} \)-extended AdS superspaces

\[ \text{AdS}^{4|4} = \frac{\text{OSp}(\mathcal{N}|4)}{\text{SO}(3,1) \times \text{SO}(\mathcal{N})} . \]  
(2.3.1)

In the following we give a summary of the results which are absolutely essential for constructing \( \mathcal{N} = 1 \) supersymmetric field theory in AdS in a manifestly OSp\((1|4)\)-invariant way. We mostly follow the presentation in [63]. Our notation and two-component spinor conventions agree with [35], except for the superspace integration measures.

Let \( z^M = (x^m, \theta^\mu, \bar{\theta}^{\dot{\mu}}) \) be local coordinates for \( \text{AdS}^{4|4} \). The geometry of \( \text{AdS}^{4|4} \) may be described in terms of covariant derivatives of the form

\[ D_A = (D_a, D_\alpha, \bar{D}^{\dot{\alpha}}) = E_A + \Omega_A , \quad E_A = E_A^M \partial_M , \]  
(2.3.2)

where \( E_A^M \) is the inverse superspace vielbein, and

\[ \Omega_A = \frac{1}{2} \Omega_A^{bc} M_{bc} = \Omega_A^{\beta\gamma} M_{\beta\gamma} + \bar{\Omega}_A^{\dot{\beta}\dot{\gamma}} \bar{M}_{\dot{\beta}\dot{\gamma}} \]  
(2.3.3)

is the Lorentz connection. The Lorentz generators \( M_{bc} \leftrightarrow (M_{\beta\gamma}, \bar{M}_{\dot{\beta}\dot{\gamma}}) \) act on two-component spinors, see appendix [A]. In particular, they act on the spinor covariant derivatives by the rule

\[ [M_{\alpha\beta}, D_\gamma] = \varepsilon_{\gamma(\alpha} D_{\beta)} , \quad [\bar{M}_{\dot{\alpha}\dot{\beta}}, \bar{D}_{\dot{\gamma}}] = \varepsilon_{\dot{\gamma}(\dot{\alpha}} \bar{D}_{\dot{\beta})} , \]  
(2.3.4)

while \([M_{\alpha\beta}, \bar{D}_{\dot{\alpha}}] = [\bar{M}_{\dot{\alpha}\dot{\beta}}, D_\gamma] = 0\). The covariant derivatives of \( \text{AdS}^{4|4} \) satisfy the following algebra

\[ \{ D_\alpha, \bar{D}_{\dot{\alpha}} \} = -2i D_{\alpha\dot{\alpha}} , \]  
(2.3.5a)
\{D_\alpha, D_\beta\} = -4\mu_\alpha\beta , \quad \{\bar{D}_\bar{\alpha}, \bar{D}_\bar{\beta}\} = 4\mu_\bar{\alpha}\bar{\beta} , \quad (2.3.5b)

\[D_\alpha, D_\beta\] = i\mu \varepsilon_\alpha\beta \bar{D}_\beta , \quad \[\bar{D}_\bar{\alpha}, D_\beta\] = -i\mu \varepsilon_\bar{\alpha}\beta D_\beta , \quad (2.3.5c)

\[D_{\alpha\dot{\alpha}}, D_{\beta\dot{\beta}}\] = -2\mu_\alpha\dot{\alpha}\beta \bar{M}_{\dot{\alpha}\dot{\beta}} + \varepsilon_{\alpha\dot{\alpha}} M_{\beta\dot{\beta}} , \quad (2.3.5d)

with \(\mu \neq 0\) being a complex parameter, which is related to the scalar curvature \(R\) of AdS space by the rule \(R = -12|\mu|^2\).

The isometry group of \(N = 1\) AdS superspace is OSp(1|4). The isometries transformations of AdS\(^4\) are generated by the Killing vector fields \(\Lambda^A E_A\) which are defined to solve the Killing equation

\[[\Lambda + \frac{1}{2} \omega^{bc} M_{bc}, D_A] = 0 , \quad \Lambda := \lambda^B D_B = \lambda^b D_b + \lambda^{\dot{\beta}} \bar{D}_{\dot{\beta}} + \bar{\lambda}_{\dot{\beta}} D_{\dot{\beta}} , \quad (2.3.6)\]

for some Lorentz superfield parameter \(\omega^{bc} = -\omega^{cb}\). As shown in [35], the equations in (2.3.6) are equivalent to

\[D_{(\alpha} \lambda_{\beta)\dot{\beta}} = 0 , \quad \bar{D}^{\dot{\beta}} \lambda_{\alpha\dot{\beta}} + 8i\lambda_\alpha = 0 , \quad (2.3.7a)\]

\[D_\alpha \lambda^\alpha = 0 , \quad \bar{D}_\alpha \lambda_\alpha + \frac{i}{2} \mu_\alpha\dot{\alpha}\beta \lambda_{\dot{\beta}} = 0 , \quad (2.3.7b)\]

\[\omega_{\alpha\beta} = D_\alpha \lambda_\beta . \quad (2.3.7c)\]

The solution to these equations is given in [35]. If \(T\) is a tensor superfield (with suppressed indices), its infinitesimal OSp(1|4) transformation is

\[\delta T = \left(\Lambda + \frac{1}{2} \omega^{bc} M_{bc}\right) T . \quad (2.3.8)\]

In Minkowski space, we have seen that there are two ways to generate supersymmetric invariants, one of which corresponds to the integration over the full superspace and the other over its chiral subspace. In AdS superspace, every chiral integral can be always recast as a full superspace integral. Associated with a scalar superfield \(L\) is the following OSp(1|4) invariant

\[\int \text{d}^4 x \text{d}^2 \theta \text{d}^2 \bar{\theta} \ E \mathcal{L} = -\frac{1}{4} \int \text{d}^4 x \text{d}^2 \theta \mathcal{E} (\bar{D}^2 - 4\mu) \mathcal{L} , \quad E^{-1} = \text{Ber} (E_\lambda^M) , \quad (2.3.9)\]

where \(\mathcal{E}\) denotes the chiral integration measure\(^5\). Let \(\mathcal{L}_c\) be a chiral scalar, \(\bar{D}_\alpha \mathcal{L}_c = 0\). It generates the supersymmetric invariant \(\int \text{d}^4 x \text{d}^2 \theta \mathcal{E} \mathcal{L}_c\). The specific feature of AdS superspace is that the chiral action can equivalently be written as an integral over the full superspace \(^{134}\)

\[\int \text{d}^4 x \text{d}^2 \theta \mathcal{E} \mathcal{L}_c = \frac{1}{\mu} \int \text{d}^4 x \text{d}^2 \theta d^2 \bar{\theta} E \mathcal{L}_c . \quad (2.3.10)\]

Unlike the flat superspace case, the integral on the right does not vanish in AdS.

\(^5\)In the chiral representation \(^{35,56}\), the chiral measure is \(\mathcal{E} = \varphi^3\), where \(\varphi\) is the chiral compensator of old minimal supergravity \(^{134}\).
2.4 Variant supercurrents in AdS space

We now turn to describing the structure of $\mathcal{N} = 1$ supercurrent multiplets in AdS. In contrast to the variant supercurrents in Minkowski space, there exist only two irreducible AdS supercurrents, with $(12 + 12)$ and $(20 + 20)$ degrees of freedom [100]. The former is associated with the old minimal AdS supergravity (see e.g. [35, 56] for reviews). This supercurrent is the AdS extension of the Ferrara-Zumino multiplet satisfying the conservation equation

$$\bar{D}^\dot{\alpha} J_{\alpha a} = D_\alpha T, \quad \bar{D}_\dot{\alpha} T = 0.$$  \tag{2.4.1}

The latter corresponds to non-minimal AdS supergravity [99], with the following conservation law

$$\bar{D}^\dot{\alpha} J_{\alpha a} = -\frac{1}{4} \bar{D}^2 \zeta_\alpha, \quad \mathcal{D}(\beta \zeta_\alpha) = 0.$$  \tag{2.4.2}

The vector superfields $J_\alpha$ and $J_a$ are real.

The non-minimal supercurrent (2.4.2) is equivalent to the Ferrara-Zumino multiplet (2.4.1), since there exists a well-defined improvement transformation that turns (2.4.2) into (2.4.1), as demonstrated in [99]. In AdS superspace, the constraint on the trace multiplet $\zeta_\alpha$, $\mathcal{D}(\beta \zeta_\alpha)$, can always be solved as

$$\zeta_\alpha = \mathcal{D}_\alpha (V + iU),$$  \tag{2.4.3}

for well-defined real operators $V$ and $U$.\footnote{This follows from the properties of linear superfields which will be discussed further in section 4.1} If we now introduce

$$J_{a\dot{a}} := \bar{J}_{a\dot{a}} + \frac{1}{6}[D_\alpha, \bar{D}_\dot{\alpha}]V - \mathcal{D}_{a\dot{a}} U, \quad T := \frac{1}{12}(\bar{D}^2 - 4\mu)(V - 3iU),$$  \tag{2.4.4}

then the operators $J_{a\dot{a}}$ and $T$ prove to satisfy the conservation equation (2.4.1). For the Ferrara-Zumino supercurrent (2.4.1), there exists an improvement transformation that is generated by a chiral scalar operator $\Omega$. Specifically, using the operator $\Omega$ allows one to introduce new supercurrent $\bar{J}_{a\dot{a}}$ and chiral trace multiplet $\bar{T}$ defined by

$$\bar{J}_{a\dot{a}} = J_{a\dot{a}} + i\mathcal{D}_{a\dot{a}}(\Omega - \bar{\Omega}), \quad \bar{D}_\dot{\alpha} \Omega = 0,$$  \tag{2.4.5a}

$$\bar{T} = T + 2\mu \Omega + \frac{1}{4}(\bar{D}^2 - 4\mu)\bar{\Omega}.$$  \tag{2.4.5b}

The operators $\bar{J}_{a\dot{a}}$ and $\bar{T}$ obey the conservation equation (2.4.1).
2.5 Off-shell higher-spin multiplets: a brief review

In later chapters, we are going to construct higher-spin extensions of supercurrents in three and four dimensions. For this we require off-shell formulations for massless higher-spin supermultiplets. In the framework of 4D $\mathcal{N} = 1$ Poincaré and AdS supersymmetry, such gauge theories have already been developed in a series of papers [63–65]. Here we briefly review the constructions in Minkowski space [64,65]. Their AdS counterparts [63] will be reviewed in the beginning of chapter 4.

2.5.1 Massless half-integer superspin multiplets

There exist two dually equivalent off-shell formulations for a free massless superspin-$(s + \frac{1}{2})$ multiplet, with $s = 1, 2, \ldots$. They are referred to as transverse and longitudinal formulations [64]. In these two off-shell gauge models, the main feature is the use of the so-called transverse and longitudinal linear superfields as one of the dynamical variables. Both are complex superfields and subject to different constraints. More generally, a complex tensor superfield $\Gamma_{\alpha(m)\dot{\alpha}(n)}$ is called transverse linear, if it obeys the constraint

$$\bar{D}^2 \Gamma_{\alpha(m)\dot{\alpha}(n-1)} = 0, \quad n > 0.$$  \hspace{1cm} (2.5.1)

A longitudinal linear superfield $G_{\alpha(m)\dot{\alpha}(n)}$ is defined to satisfy the constraint

$$\bar{D}_{(\dot{\alpha}_1} G_{\alpha(m)\dot{\alpha}_2...\dot{\alpha}_{n+1})} = 0.$$  \hspace{1cm} (2.5.2)

The above constraints imply that $\Gamma_{\alpha(m)\dot{\alpha}(n)}$ and $G_{\alpha(m)\dot{\alpha}(n)}$ are linear superfields,

$$\bar{D}^2 \Gamma_{\alpha(m)\dot{\alpha}(n)} = \bar{D}^2 G_{\alpha(m)\dot{\alpha}(n)} = 0.$$  \hspace{1cm} (2.5.3)

In the case $n = 0$, the constraint (2.5.1) has to be replaced with the standard linear constraint $\bar{D}^2 \Gamma_{\alpha(m)} = 0$. The constraint (2.5.2) for $n = 0$ is the chirality condition $\bar{D}\dot{\beta} G_{\alpha(m)} = 0$.

In the case of 4D $\mathcal{N} = 1$ AdS supersymmetry, longitudinal linear and transverse linear superfields were first described in [93] to realise the irreducible representations of the AdS isometry group OSp(1|4) (see [135] for a nice review of the results of [93]). In the framework of 4D $\mathcal{N} = 1$ conformal supergravity, primary longitudinal linear and transverse linear supermultiplets were introduced for the first time by Kugo and Uehara [136]. Such superfields were used in [3,4,49,63–65] for the description of off-shell massless gauge theories in three and four dimensions.

---

7 All Lorentz tensor (super)fields considered in this thesis are completely symmetric in their undotted spinor indices and separately in their dotted ones. We use the shorthand notation $V_{\alpha(m)\dot{\alpha}(n)} := V_{\alpha_1...\alpha_m\dot{\alpha}_1...\dot{\alpha}_n}(z) = V_{(\alpha_1...\alpha_m)(\dot{\alpha}_1...\dot{\alpha}_n)}(z)$ and $V_{\alpha(m)\dot{\alpha}(n)} = V_{\alpha_1...\alpha_m\dot{\alpha}_1...\dot{\alpha}_n} U_{\alpha(m)\dot{\alpha}(n)}; \quad V_{\alpha(m)\dot{\alpha}(n)} = V_{\alpha_1...\alpha_m\dot{\alpha}_1...\dot{\alpha}_n}.$ Parentheses denote symmetrisation of indices; the undotted and dotted spinor indices are symmetrised independently. Indices sandwiched between vertical bars (for instance, $|\gamma|$) are not subject to symmetrisation.
2.5.1.1 Transverse formulation

The transverse formulation is realised in terms of the following dynamical variables:

\[ V_{s+1/2}^\perp = \left\{ H_{\alpha(s)\dot{\alpha}(s)} , \, \Gamma_{\alpha(s-1)\dot{\alpha}(s-1)} , \, \bar{\Gamma}_{\alpha(s-1)\dot{\alpha}(s-1)} \right\} . \] (2.5.4)

Here \( H_{\alpha(s)\dot{\alpha}(s)} \) is a real unconstrained superfield. The complex superfield \( \Gamma_{\alpha(s-1)\dot{\alpha}(s-1)} \) is transverse linear,

\[ \bar{D}^\beta \Gamma_{\alpha(s-1)\dot{\beta}\dot{\alpha}(s-2)} = 0 \quad \Rightarrow \quad \bar{D}^2 \Gamma_{\alpha(s-1)\dot{\alpha}(s-1)} = 0 . \] (2.5.5)

The constraint (2.5.5) can be solved in terms of an unconstrained prepotential:

\[ \Gamma_{\alpha(s-1)\dot{\alpha}(s-1)} = \bar{D}^\beta \Phi_{\alpha(s-1)}(\dot{\beta}\dot{\alpha}_1\ldots\dot{\alpha}_{s-1}) . \] (2.5.6)

The prepotential is defined modulo gauge transformation of the form

\[ \delta_\xi \Phi_{\alpha(s-1)}(\dot{\alpha}(s)) = \bar{D}^\beta \xi_{\alpha(s-1)}(\dot{\beta}\dot{\alpha}_1\ldots\dot{\alpha}_s) , \] (2.5.7)

with the gauge parameters \( \xi_{\alpha(s-1)}(\dot{\alpha}+1) \) being unconstrained.

It was postulated in \[64\] that the linearised gauge transformations for the superfields \( H_{\alpha(s)\dot{\alpha}(s)} \) and \( \Gamma_{\alpha(s-1)\dot{\alpha}(s-1)} \) are

\[ \delta_\Lambda H_{\alpha_1\ldots\alpha_s\dot{\alpha}_1\ldots\dot{\alpha}_s} = \bar{D}(\alpha_1\Lambda_{\alpha_1\ldots\alpha_s\dot{\alpha}_2\ldots\dot{\alpha}_s}) - D(\alpha_1\bar{\Lambda}_{\alpha_2\ldots\alpha_s\dot{\alpha}_1}) , \] (2.5.8a)

\[ \delta_\Lambda \Gamma_{\alpha_1\ldots\alpha_{s-1}\dot{\alpha}_1\ldots\dot{\alpha}_{s-1}} = -\frac{1}{4} \bar{D}^\beta D^2 \bar{\Lambda}_{\alpha_1\ldots\alpha_{s-1}\dot{\beta}\dot{\alpha}_1\ldots\dot{\alpha}_{s-1}} , \] (2.5.8b)

where the complex gauge parameter \( \Lambda_{\alpha_1\ldots\alpha_s\dot{\alpha}_1\ldots\dot{\alpha}_{s-1}} = \Lambda_{(\alpha_1\ldots\alpha_s)}(\dot{\alpha}_1\ldots\dot{\alpha}_{s-1}) \) is unconstrained. It follows from (2.5.8b) that the transformation law of the prepotential \( \Phi_{\alpha(s-1)}(\dot{\alpha}(s)) \) is

\[ \delta_\Lambda \Phi_{\alpha_1\ldots\alpha_{s-1}\dot{\alpha}_1\ldots\dot{\alpha}_s} = -\frac{1}{4} D^2 \bar{\Lambda}_{\alpha_1\ldots\alpha_{s-1}\dot{\alpha}_1\ldots\dot{\alpha}_s} . \] (2.5.9)

The action invariant under the gauge transformations (2.5.8a) and (2.5.8b) is

\[ S_{s+1/2}^\perp[H,\Gamma,\bar{\Gamma}] = \left( -\frac{1}{2} \right)^s \int d^4x d^2\theta d^2\bar{\theta} \left\{ \frac{1}{8} H^{\alpha(s)\dot{\alpha}(s)} D^\beta D^2 D_\beta H_{\alpha(s)\dot{\alpha}(s)} 
+ H^{\alpha(s)\dot{\alpha}(s)} \left( D_{\alpha_1} \bar{D}_{\dot{\alpha}_1} \Gamma_{\alpha(s-1)\dot{\alpha}(s-1)} - \bar{D}_{\dot{\alpha}_1} D_{\alpha_1} \bar{\Gamma}_{\alpha(s-1)\dot{\alpha}(s-1)} \right) 
+ \left( \bar{\Gamma}_{\alpha(s-1)\dot{\alpha}(s-1)} \right) \Gamma_{\alpha(s-1)\dot{\alpha}(s-1)} 
+ \frac{s+1}{s} \Gamma^{\alpha(s-1)\dot{\alpha}(s-1)} \Gamma_{\alpha(s-1)\dot{\alpha}(s-1)} + \text{c.c.} \right\} . \] (2.5.10)

We now briefly comment on the limiting \( s = 1 \) case which should correspond to linearised supergravity. The transverse linear constraint (2.5.5) cannot be used for \( s = 1 \), however its corollary \( \bar{D}^2 \Gamma_{\alpha(s-1)\dot{\alpha}(s-1)} = 0 \) can be used,

\[ \bar{D}^2 \Gamma = 0 . \] (2.5.11)

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This constraint defines a complex linear superfield. In accordance with (2.5.8b), the gauge transformation of \( \Gamma \) is

\[
\delta_\Lambda \Gamma = \frac{1}{4} \bar{D}_\beta D^2 \bar{\Lambda}^\beta .
\] (2.5.12)

The action (2.5.10) for \( s = 1 \) coincides with the linearised action for the \( n = -1 \) non-minimal supergravity, see [35,89] for reviews.

## 2.5.1.2 Longitudinal formulation

The longitudinal formulation is described in terms of the following dynamical variables:

\[
\mathcal{V}_{s+1/2}^\parallel = \left\{ H_{\alpha(s)} \alpha(s) , \ G_{\alpha(s-1)} \alpha(s-1) , \ \bar{G}_{\alpha(s-1)} \alpha(s-1) \right\},
\] (2.5.13)

where the real superfield \( H_{\alpha(s)} \alpha(s) \) is unconstrained, and the compensating superfield \( G_{\alpha(s-1)} \alpha(s-1) \) is longitudinal linear,

\[
\bar{D}_{\dot{\alpha}_1} G_{\alpha(s-1) \dot{\alpha}_2 \ldots \dot{\alpha}_x} = 0 \implies \bar{D}^2 G_{\alpha(s-1) \alpha(s-1)} = 0 .
\] (2.5.14)

The constraint (2.5.14) can be solved in terms of an \textit{unconstrained} prepotential

\[
G_{\alpha(s-1) \alpha(s-1)} = \bar{D}_{(\dot{\alpha}_1} \Psi_{\alpha(s-1) \dot{\alpha}_2 \ldots \dot{\alpha}_{s-1})} .
\] (2.5.15)

The prepotential is defined modulo gauge transformations of the form

\[
\delta_\zeta \Psi_{\alpha(s-1) \alpha(s-2)} = \bar{D}_{(\dot{\alpha}_1} \zeta_{\alpha(s-1) \dot{\alpha}_2 \ldots \dot{\alpha}_{s-2})} ,
\] (2.5.16)

with the gauge parameter \( \zeta_{\alpha(s-1) \alpha(s-3)} \) being unconstrained.\(^8\)

The gauge transformations for the dynamical superfields are given by

\[
\delta_\Lambda H_{\alpha_1 \ldots \alpha_s \dot{\alpha}_1 \ldots \dot{\alpha}_s} = \bar{D}_{(\dot{\alpha}_1} \Lambda_{\alpha_1 \ldots \alpha_s \dot{\alpha}_2 \ldots \dot{\alpha}_s)} = D_{(\alpha_1} \bar{\Lambda}_{\alpha_2 \ldots \alpha_s) \dot{\alpha}_1 \ldots \dot{\alpha}_s} ,
\] (2.5.17a)

\[
\delta_\Lambda G_{\alpha_1 \ldots \alpha_{s-1} \dot{\alpha}_1 \ldots \dot{\alpha}_{s-1}} = -\frac{1}{2} \bar{D}_{(\dot{\alpha}_1} \bar{D}^{\bar{\beta}} \bar{D}^\beta \Lambda_{\beta \alpha_1 \ldots \alpha_{s-1} \dot{\alpha}_2 \ldots \dot{\alpha}_{s-1})} \dot{\beta} + i(s-1) \bar{D}_{(\dot{\alpha}_1} \partial_{\beta \bar{\beta}} \Lambda_{\beta \alpha_1 \ldots \alpha_{s-1} \dot{\alpha}_2 \ldots \dot{\alpha}_{s-1})} \dot{\beta} ,
\] (2.5.17b)

The symmetrisation in (2.5.17b) is extended only to the indices \( \dot{\alpha}_1, \dot{\alpha}_2, \ldots, \dot{\alpha}_{s-1} \). It follows from (2.5.17b) that the transformation law of the prepotential \( \Psi_{\alpha(s-1) \alpha(s-2)} \) is

\[
\delta_\Lambda \Psi_{\alpha_1 \ldots \alpha_{s-1} \dot{\alpha}_1 \ldots \dot{\alpha}_{s-2}} = -\frac{1}{2} \left( \bar{D}^{\beta} \bar{D}^\beta - 2i(s-1) \partial_{\beta \bar{\beta}} \right) \Lambda_{\beta \alpha_1 \ldots \alpha_{s-1} \dot{\alpha}_1 \ldots \dot{\alpha}_{s-2}} .
\] (2.5.18)

The action invariant under the gauge transformations (2.5.17a) and (2.5.17b) is

\[
S_{s+1/2}^\parallel [H, G, \bar{G}] = \left( -\frac{1}{2} \right)^8 \int d^4 x d^2 \theta d^2 \bar{\theta} \left\{ -\frac{1}{8} H_{\alpha(s)} \alpha(s) \bar{D}^\beta \bar{D}^\beta D_\beta H_{\alpha(s)} \alpha(s) \right\} .
\]

\(^8\)For \( s = 2 \) the gauge transformation law (2.5.16) has to be replaced with \( \delta \Psi_\alpha = \zeta_\alpha \), with the gauge parameter \( \zeta_\alpha \) being chiral, \( D_\beta \zeta_\alpha = 0 \).
We assume that (2.5.21) can be solved to express $V$ rewritten in the form $V$ can integrate out $S$ first-order action follows. Let us start with the transverse theory (2.5.10) and associate with it the following a superfield Legendre transformation described in [64]. In general, the procedure works as follows. Let us start with the transverse theory (2.5.10) and associate with it the following first-order action

$$S[H, V, \bar{V}, G, \bar{G}] = S_{(s+\frac{1}{2})}^+ [H, V, \bar{V}] + \int d^4 x d^2 \theta d^2 \bar{\theta} \left( V^{\alpha(s-1)\dot{\alpha}(s-1)} G_{\alpha(s-1)\dot{\alpha}(s-1)} + \text{c.c.} \right), \quad (2.5.20)$$

with $V_{\alpha(s-1)\dot{\alpha}(s-1)}$ being unconstrained complex. Here $S_{(s+\frac{1}{2})}^+ [H, V, \bar{V}]$ is obtained from $S_{(s+\frac{1}{2})}^+ [H, \Gamma, \bar{\Gamma}]$ by the replacement $\Gamma_{\alpha(s-1)\dot{\alpha}(s-1)} \rightarrow V_{\alpha(s-1)\dot{\alpha}(s-1)}$. The Lagrange multiplier $G_{\alpha(s-1)\dot{\alpha}(s-1)}$ is longitudinal linear. Varying the first-order action $S[H, V, \bar{V}, G, \bar{G}]$ with respect to the Lagrange multiplier $G_{\alpha(s-1)\dot{\alpha}(s-1)}$ gives $V_{\alpha(s-1)\dot{\alpha}(s-1)} = \Gamma_{\alpha(s-1)\dot{\alpha}(s-1)}$, thus $S[H, V, \bar{V}, G, \bar{G}]$ reduces to the original action $S_{(s+\frac{1}{2})}^+ [H, \Gamma, \bar{\Gamma}]$. On the other hand, one can integrate out $V_{\alpha(s-1)\dot{\alpha}(s-1)}$ using its equation of motion

$$\frac{\delta}{\delta V_{\alpha(s-1)\dot{\alpha}(s-1)}} S_{(s+\frac{1}{2})}^+ [H, V, \bar{V}] + G_{\alpha(s-1)\dot{\alpha}(s-1)} = 0 \quad (2.5.21)$$

We assume that (2.5.21) can be solved to express $V_{\alpha(s-1)\dot{\alpha}(s-1)}$ in terms of $G_{\alpha(s-1)\dot{\alpha}(s-1)}$ and its conjugate. Plugging this solution back into (2.5.20) leads to the dual action $S_D[H, G, \bar{G}]$, given by the expression (2.5.19).

The constraint (2.5.14) is the chirality condition for $s = 1$, $\bar{D}_\alpha G = 0$. The gauge transformation law (2.5.17b) cannot directly be used for $s = 1$. Nevertheless, it can be rewritten in the form

$$\delta_{\Lambda} G_{\alpha_1 \ldots \alpha_{s-1} \dot{\alpha}_1 \ldots \dot{\alpha}_{s-1}} = -\frac{1}{4} \bar{D}^2 D^\beta \Lambda_{\beta \alpha_1 \ldots \alpha_{s-1} \dot{\alpha}_1 \ldots \dot{\alpha}_{s-1}}$$

$$+ i(s - 1) \bar{\theta}^{\beta \dot{\beta}} \bar{D}_{(\dot{\alpha}_1} \Lambda_{\beta \alpha_1 \ldots \alpha_{s-1} \dot{\alpha}_2 \ldots \dot{\alpha}_{s-1}) \dot{\beta}}, \quad (2.5.22)$$

which is well defined for $s = 1$:

$$\delta_{\Lambda} G = -\frac{1}{4} \bar{D}^2 D^\beta \Lambda_{\beta \dot{\beta}}. \quad (2.5.23)$$

The action (2.5.19) for $s = 1$ coincides with the linearised action for the old minimal supergravity, see [35, 89] for reviews.
2.5.2 Massless integer superspin multiplets

We now recall the two off-shell gauge models for a massless multiplet of integer superspin \( s \geq 2 \), which were originally constructed in \([65]\). In each of the formulations, the dynamical variables consist of a real unconstrained prepotential \( H_{\alpha(s-1)} \hat{a}(s-1) \) in conjunction with some compensating supermultiplets.

2.5.2.1 Longitudinal formulation

The longitudinal theory is described by the following set of superfields

\[
\mathcal{V}_\parallel = \left\{ H_{\alpha(s-1)} \hat{a}(s-1) , \ G_{\alpha(s)} \hat{a}(s) , \ \bar{G}_{\alpha(s)} \hat{a}(s) \right\} .
\] (2.5.24)

The superfield \( H_{\alpha(s-1)} \hat{a}(s-1) \) is unconstrained real, while the compensator \( G_{\alpha(s)} \hat{a}(s) \) is longitudinal linear. The latter is a field strength associated with a complex unconstrained prepotential \( \Psi_{\alpha(s)} \hat{a}(s-1) \),

\[
G_{\alpha_1...\alpha_s \hat{a}_1...\hat{a}_s} := \bar{D}(\hat{a}_1 \Psi_{\alpha_1...\alpha_s \hat{a}_2...\hat{a}_s}) \implies \bar{D}(\hat{a}_1 G_{\alpha_1...\alpha_s \hat{a}_2...\hat{a}_{s+1}}) = 0 .
\] (2.5.25)

The dynamical superfields are defined modulo gauge transformations of the form

\[
\delta_L H_{\alpha(s-1)} \hat{a}(s-1) = D^\beta L_{\beta \alpha(s-1)} \hat{a}(s-1)) , \quad \delta_L \Psi_{\alpha(s)} \hat{a}(s-1) = \frac{1}{2} D_{(\alpha_1} D^{\beta_1} L_{\alpha_2...\alpha_s \beta_1...\beta_{s-1})} \Psi_{\alpha(s)} \hat{a}(s-1) ,
\] (2.5.26a)

with the gauge parameter \( L_{\alpha(s)} \hat{a}(s-1) \) being complex unconstrained. The action functional which is quadratic in the superfields \( H, G, \bar{G} \) and invariant under the gauge transformations \((2.5.26)\) is given by

\[
S_{(s)}^\parallel = \left( -\frac{1}{2} \right)^s \int d^4x d^2\theta d^2\bar{\theta} \left\{ \frac{1}{8} H^\alpha(s-1) \hat{a}(s-1) D^\beta \bar{D}^\gamma D_\beta H_{\alpha(s-1)} \hat{a}(s-1) \right. \\
+ \frac{s}{s+1} H^\alpha(s-1) \hat{a}(s-1) \left( D^\beta \bar{D}^\gamma G_{\beta \alpha(s-1)} \hat{a}(s-1) - \bar{D}^\beta D_\beta \bar{G}_{\beta \alpha(s-1)} \hat{a}(s-1) \right) \\
+ 2 \bar{G}^\alpha(s) \hat{a}(s) G_{\alpha(s)} \hat{a}(s) + \frac{s}{s+1} \left( G^\alpha(s) \hat{a}(s) G_{\alpha(s)} \hat{a}(s) + \bar{G}^\alpha(s) \hat{a}(s) \bar{G}_{\alpha(s)} \hat{a}(s) \right) \right\}.
\] (2.5.27)

2.5.2.2 Transverse formulation

The transverse formulation is realised by the following set of superfields

\[
\mathcal{V}_\perp^\parallel = \left\{ H_{\alpha(s-1)} \hat{a}(s-1) , \ \Gamma_{\alpha(s)} \hat{a}(s) , \ \bar{\Gamma}_{\alpha(s)} \hat{a}(s) \right\} .
\] (2.5.28)

Here the compensating multiplet is described by a transverse linear superfield \( \Gamma_{\alpha(s)} \hat{a}(s) \) (and its conjugate \( \bar{\Gamma}_{\alpha(s)} \hat{a}(s) \)) constrained by

\[
\bar{D}^\beta \Gamma_{\alpha(s)} \hat{a}(s-1) = 0 \implies \bar{D}^2 \Gamma_{\alpha(s)} \hat{a}(s) = 0 .
\] (2.5.29)
The dynamical superfields are defined modulo gauge transformations of the form

\[ \delta_L H_{\alpha(s-1)\dot{\alpha}(s-1)} = D^\beta L_{\beta \alpha(s-1)\dot{\alpha}(s-1)} - \bar{D}^\beta \bar{L}_{\alpha(s-1)\dot{\beta}(s-1)} \quad (2.5.30a) \]
\[ \delta_L \Gamma_{\alpha(s)\dot{\alpha}(s)} = \frac{s + 1}{2(s + 2)} \bar{D}^\beta \left\{ \bar{D}_{(\beta} \partial_{(\alpha_1 \dot{\beta}(s-1)\dot{\beta})} \bar{L}_{\alpha_2 ... \alpha_s \dot{\alpha}_1 ... \dot{\alpha}_s} \right\} \quad (2.5.30b) \]

The gauge-invariant action is given by

\[ S_\perp^{(s+\frac{1}{2})} = -\left( -\frac{1}{2} \right)^s \int d^4x d^2\theta d^2\bar{\theta} \left\{ -\frac{1}{8} H^{\alpha(s-1)\dot{\alpha}(s-1)} D^\beta \bar{D}^\delta D_\beta H_{\alpha(s-1)\dot{\alpha}(s-1)} + \frac{1}{8 (s + 1)(2s + 1)} [D^\beta, \bar{D}^\delta] H^{\alpha_1 ... \alpha_{s-1} \dot{\alpha}_1 ... \dot{\alpha}_{s-1}} [D_{(\beta}, \bar{D}_{(\delta}] H_{\alpha_1 ... \alpha_{s-1} \dot{\alpha}_1 ... \dot{\alpha}_{s-1})}
\]
\[ + \frac{1}{2 s + 1} \partial^\beta H^{\alpha_1 ... \alpha_{s-1} \dot{\alpha}_1 ... \dot{\alpha}_{s-1}} \partial_{(\beta} H_{\alpha_{s-1} \dot{\alpha}_1 ... \dot{\alpha}_{s-1})} + \frac{2i s}{2s + 1} H^{\alpha(s-1)\dot{\alpha}(s-1)} \partial^\beta \left( \Gamma_{\beta \alpha(s-1)\dot{\beta}(s-1)} - \bar{\Gamma}_{\beta \alpha(s-1)\dot{\beta}(s-1)} \right)
\]
\[ + \frac{1}{2s + 1} \left( \Gamma^{\alpha(s)\dot{\alpha}(s)} \Gamma_{\alpha(s)\dot{\alpha}(s)} - \frac{s}{s + 1} \Gamma^{\alpha(s)\dot{\alpha}(s)} \Gamma_{\alpha(s)\dot{\alpha}(s)} + c.c. \right) \right\} \quad (2.5.31) \]

As demonstrated in \[65\], the two actions (2.5.27) and (2.5.31) are classically equivalent, for they are related by a superfield Legendre transformation.
Chapter 3

Non-conformal higher-spin supercurrents in Minkowski space

In the previous chapter, the off-shell structure of the massless superspin-\(s\) \(\mathcal{N}=1\) multiplets was described. In the case of half-integer superspin \(\hat{s}=s+\frac{1}{2}\), with \(s=2,3,\ldots\) both models involve one and the same gauge superfield \(H_{\alpha(s)\hat{\alpha}(s)}\), but differ in the compensating multiplets used, i.e. transverse or longitudinal superfield. The real unconstrained pre-potential \(H_{\alpha(s)\hat{\alpha}(s)}\) is the higher-spin superconformal gauge multiplet introduced in [69].

In the \(s=1\) case, the gauge transformation (2.5.8a) corresponds to linearised conformal supergravity [81]. It is important to note that the non-supersymmetric higher-spin theories proposed by (Fang-)Fronsdal [31, 33] and their supersymmetric counterparts of half-integer superspin share one common feature. Specifically, for each of them, the gauge-invariant action is formulated in terms of a (super)conformal gauge (super)field coupled to certain compensators. Such a description was not known for the massless multiplet of integer superspin until recently [2], in which a reformulation of the integer superspin action (2.5.27) was given. We now make these points more precise.

3.1 Conformal gauge (super)fields and compensators

Given an integer \(s \geq 2\), the conformal spin-\(s\) field [70, 71] is described by a real potential \(h_{\alpha_1...\alpha_s\hat{\alpha}_1...\hat{\alpha}_s} = h_{(\alpha_1...\alpha_s)(\hat{\alpha}_1...\hat{\alpha}_s)} \equiv h_{\alpha(s)\hat{\alpha}(s)}\) with the gauge freedom

\[
\delta h_{\alpha_1...\alpha_s\hat{\alpha}_1...\hat{\alpha}_s} = \partial_{(\alpha_1(\hat{\alpha}_1\lambda_{\alpha_2...\alpha_s}\hat{\alpha}_2...\hat{\alpha}_s)} ,
\]

for an arbitrary real gauge parameter \(\lambda_{\alpha_1...\alpha_{s-1}\hat{\alpha}_1...\hat{\alpha}_{s-1}} = \lambda_{(\alpha_1...\alpha_{s-1})(\hat{\alpha}_1...\hat{\alpha}_{s-1})} \equiv \lambda_{\alpha(s-1)\hat{\alpha}(s-1)}\). In addition to the gauge field \(h_{\alpha(s)\hat{\alpha}(s)}\), the massless spin-\(s\) action [31] also involves a real compensator \(h_{\alpha(s-2)\hat{\alpha}(s-2)}\) with the gauge transformation

\[
\delta h_{\alpha_1...\alpha_{s-2}\hat{\alpha}_1...\hat{\alpha}_{s-2}} = \partial^{\beta\hat{\beta}}\lambda^{\beta}_{\alpha_1...\alpha_{s-2}\beta\hat{\alpha}_1...\hat{\alpha}_{s-2}} .
\]
In the fermionic case, the conformal spin-\(s + \frac{1}{2}\) field \([70, 71]\) is described by a potential \(\psi_{\alpha(s+1)\dot{a}(s)}\) and its conjugate \(\bar{\psi}_{\alpha(s)\dot{a}(s+1)}\) with the gauge freedom
\[
\delta \psi_{\alpha_1...\alpha_{s+1}\dot{a}_1...\dot{a}_s} = \partial_{(\alpha_1}(\dot{a}_1\xi_{\alpha_2...\alpha_{s+1})\dot{a}_2...\dot{a}_s)} \; ,
\]
for an arbitrary gauge parameter \(\xi_{\alpha(s)\dot{a}(s-1)}\). In addition to the gauge fields \(\psi_{\alpha(s+1)\dot{a}(s)}\) and \(\bar{\psi}_{\alpha(s)\dot{a}(s+1)}\), the massless spin-\(s + \frac{1}{2}\) action \([33]\) also involves two compensators \(\psi_{\alpha(s-1)\dot{a}(s)}\) and \(\bar{\psi}_{\alpha(s)\dot{a}(s-2)}\) and their conjugates, with the following gauge transformations
\[
\begin{align*}
\delta \psi_{\alpha_1...\alpha_{s-1}\dot{a}_1...\dot{a}_{s-1}} &= \partial^\beta (\dot{a}_1\xi_{\beta\alpha_1...\alpha_{s-1}}\dot{a}_2...\dot{a}_{s-1}) \; , \\
\delta \bar{\psi}_{\dot{\alpha}_1...\dot{\alpha}_{s-1}\dot{a}_1...\dot{a}_{s-2}} &= \partial^{\dot{\beta}} \xi_{\dot{\beta}\dot{\alpha}_1...\dot{\alpha}_{s-1}\dot{\beta}\dot{a}_1...\dot{a}_{s-2}} \; .
\end{align*}
\]

In the case of an integer superspin \(\hat{s} = s\), with \(s = 2, 3, \ldots\) the superconformal multiplet introduced in \([68]\) is described in terms of an unconstrained prepotential \(\Psi_{\alpha(s)\dot{a}(s-1)}\) and its complex conjugate with the gauge freedom
\[
\delta \Psi_{\alpha_1...\alpha_s\dot{a}_1...\dot{a}_{s-1}} = \frac{1}{2} D_{(\alpha_1} \mathcal{Y}_{\alpha_2...\alpha_s)\dot{a}_1...\dot{a}_{s-1}} + \bar{D}_{(\dot{a}_1} \zeta_{\alpha_1...\alpha_s\dot{a}_2...\dot{a}_{s-1})} \; .
\]
Here the gauge parameters \(\mathcal{Y}_{\alpha(s-1)\dot{a}(s-1)}\) and \(\zeta_{\alpha(s)\dot{a}(s-2)}\) are both unconstrained. As shown in subsection \([2.5.2]\) (see also \([65]\)), the prepotential \(\Psi_{\alpha(s)\dot{a}(s-1)}\) naturally appears as one of the dynamical variables in the longitudinal formulation for the massless superspin-\(s\) multiplet, in addition to the real unconstrained prepotential \(H_{\alpha(s-1)\dot{a}(s-1)}\). However, the gauge transformation of \(\Psi_{\alpha(s)\dot{a}(s-1)}\) given in eq. \([2.5.26b]\) differs from eq. \([3.1.3]\). The difference is that the parameter \(\mathcal{Y}_{\alpha(s-1)\dot{a}(s-1)}\) in \([65]\) is not unconstrained, but instead takes the form \(\mathcal{Y}_{\alpha(s-1)\dot{a}(s-1)} = D^\beta L(\beta\alpha_1...\alpha_{s-1})\dot{a}(s-1)\). Furthermore, the prepotential \(\Psi_{\alpha(s)\dot{a}(s-1)}\) enters the action functional \([2.5.27]\) only via the constrained field strength \(G_{\alpha(s)\dot{a}(s)} := \bar{D}_{(\dot{a}_1} \Psi_{\alpha(s)\dot{a}_2...\dot{a}_s)}\), which is longitudinal linear. It is then natural to look for a new formulation by properly generalising the off-shell supersymmetric action \([2.5.27]\).

In this chapter we propose a new off-shell realisation for the massless superspin-\(s\) multiplet with the following properties: (i) the gauge freedom of \(\Psi_{\alpha(s)\dot{a}(s-1)}\) is given by \([3.1.3]\); and (ii) the original longitudinal formulation \([2.5.27]\) emerges upon imposing a gauge condition. The new model is shown to possess a dual formulation obtained by applying a superfield Legendre transformation. We then introduce non-conformal higher-spin supercurrents associated to the off-shell actions for the massless \(\mathcal{N} = 1\) supermultiplets in 4D Minkowski space. Explicit realisations for these conserved higher-spin supercurrents are given for models for a single massless and massive chiral superfield, as well as the massive \(\mathcal{N} = 2\) hypermultiplet.
3.2 The massless integer superspin multiplets revisited

Here we present a new off-shell gauge formulation for the massless superspin-$s$ multiplet, as well as for the massless gravitino multiplet ($s = 1$) which requires special consideration.

3.2.1 Reformulation of the longitudinal theory

Given a positive integer $s \geq 2$, we propose to describe the massless superspin-$s$ multiplet in terms of the following superfield variables: (i) an unconstrained prepotential $\Psi_{\alpha(s-1)\dot{a}(s-1)}$ and its complex conjugate $\bar{\Psi}_{\alpha(s-1)\dot{a}(s-1)}$; (ii) a real superfield $H_{\alpha(s-1)\dot{a}(s-1)} = \bar{H}_{\alpha(s-1)\dot{a}(s-1)}$; and (iii) a complex superfield $\Sigma_{\alpha(s-1)\dot{a}(s-2)}$ and its conjugate $\bar{\Sigma}_{\alpha(s-2)\dot{a}(s-1)}$, where $\Sigma_{\alpha(s-1)\dot{a}(s-2)}$ is constrained to be transverse linear,

$$\bar{D}^\delta \Sigma_{\alpha(s-1)\dot{a}(s-2)} = 0 \ .$$  \hspace{1cm} (3.2.1)

In the $s = 2$ case, for which (3.2.1) is not defined, $\Sigma_\alpha$ is constrained to be complex linear,

$$\bar{D}^2 \Sigma_\alpha = 0 \ .$$  \hspace{1cm} (3.2.2)

The constraint (3.2.1), or its counterpart (3.2.2) for $s = 2$, can be solved in terms of a complex unconstrained prepotential $Z_{\alpha(s-1)\dot{a}(s-1)}$ by the rule

$$\Sigma_{\alpha(s-1)\dot{a}(s-2)} = \bar{D}^\delta (3.2.5a)$$

This prepotential is defined modulo gauge transformations

$$\delta_\xi Z_{\alpha(s-1)\dot{a}(s-1)} = \bar{D}^\delta \xi_{\alpha(s-1)\dot{a}(s-1)} \ ,$$  \hspace{1cm} (3.2.5c)

with the gauge parameter $\xi_{\alpha(s-1)\dot{a}(s)}$ being unconstrained complex.

The gauge freedom of $\Psi_{\alpha_1...\alpha_s\dot{a}_1...\dot{a}_{s-1}}$ is chosen to coincide with that of the superconformal superspin-$s$ multiplet [68], which is

$$\delta_\Psi \Psi_{\alpha_1...\alpha_s\dot{a}_1...\dot{a}_{s-1}} = \frac{1}{2} D_{(\alpha_1} \Psi_{\alpha_2...\alpha_s)\dot{a}_1...\dot{a}_{s-1}} + \bar{D}_{\dot{a}_1(\alpha_1...\alpha_s\dot{a}_2...\dot{a}_{s-1})} \ ,$$  \hspace{1cm} (3.2.5a)

with *unconstrained* complex gauge parameters $\Psi_{\alpha(s-1)\dot{a}(s-1)}$ and $\bar{\Psi}_{\alpha(s-1)\dot{a}(s-2)}$. The $\Psi$-transformation is defined to act on the superfields $H_{\alpha(s-1)\dot{a}(s-1)}$ and $\Sigma_{\alpha(s-1)\dot{a}(s-2)}$ as follows

$$\delta_\Psi H_{\alpha(s-1)\dot{a}(s-1)} = \Psi_{\alpha(s-1)\dot{a}(s-1)} + \bar{\Psi}_{\alpha(s-1)\dot{a}(s-1)} \ ,$$  \hspace{1cm} (3.2.5b)

$$\delta_\Psi \Sigma_{\alpha(s-1)\dot{a}(s-2)} = \bar{D}^\delta \Psi_{\alpha(s-1)\dot{a}(s-2)} \implies \delta_\Psi Z_{\alpha(s-1)\dot{a}(s-1)} = \bar{\Psi}_{\alpha(s-1)\dot{a}(s-1)} \ .$$  \hspace{1cm} (3.2.5c)
The longitudinal linear superfield

\[ G_{\alpha_1 \ldots \alpha_s \dot{\alpha}_1 \ldots \dot{\alpha}_s} := \bar{D}_{(\dot{\alpha}_1} \Psi_{\alpha_1 \ldots \alpha_s \dot{\alpha}_2 \ldots \dot{\alpha}_s)} \ ; \quad \bar{D}_{(\dot{\alpha}_1} G_{\alpha_1 \ldots \alpha_s \dot{\alpha}_2 \ldots \dot{\alpha}_{s+1})} = 0 \]  

(3.2.6)

is invariant under the \( \zeta \)-transformation (3.2.5) and varies under the \( \mathcal{W} \)-transformation as

\[ \delta_{\mathcal{W}} G_{\alpha_1 \ldots \alpha_s \dot{\alpha}_1 \ldots \dot{\alpha}_s} = \frac{1}{2} \bar{D}_{(\dot{\alpha}_1} D_{(\alpha_1} \mathcal{W}_{\alpha_2 \ldots \alpha_s) \dot{\alpha}_2 \ldots \dot{\alpha}_s)} . \]  

(3.2.7)

It may be checked that the following action

\[ S_{(s)}^\parallel = \left( -\frac{1}{2} \right)^s \int d^4x d^2 \theta d^2 \bar{\theta} \left\{ \frac{1}{8} H^\alpha(s-1) \bar{\alpha}(s-1) D^\beta \bar{D}^2 D_\beta H_{\alpha(s-1) \bar{\alpha}(s-1)} + \frac{s}{s+1} H^\alpha(s-1) \bar{\alpha}(s-1) \left( D^\beta \bar{D}^\delta G_{\beta \alpha(s-1) \bar{\alpha}(s-1)} - \bar{D}^\delta \bar{D}^\beta \bar{G}_{\beta \alpha(s-1) \bar{\alpha}(s-1)} \right) + 2 \bar{G}^\alpha(s) \bar{\alpha}(s) G_{\alpha(s) \bar{\alpha}(s)} + \frac{s}{s+1} \left( G^\alpha(s) \bar{\alpha}(s) G_{\alpha(s) \bar{\alpha}(s)} + \bar{G}^\alpha(s) \bar{\alpha}(s) \bar{G}_{\alpha(s) \bar{\alpha}(s)} \right) \right\} \]

(3.2.8)

is invariant under the gauge transformations (3.2.5). By construction, the action is also invariant under (3.2.4). It should be pointed out that in constructing the new action (3.2.8), one has to check explicitly its invariance under the \( \zeta \)-gauge transformation (3.2.5). In contrast, the original action (2.5.27) is formulated in terms of \( H_{\alpha(s-1) \bar{\alpha}(s-1)} \) and the field strength \( G_{\alpha(s) \bar{\alpha}(s)} \). The latter is manifestly invariant under the \( \zeta \)-transformation.

It is important to keep in mind the following identity, which is used quite often in proving the gauge-invariance (and also for other higher-spin calculations in this thesis)

\[ D_\beta U_{\alpha_1 \ldots \alpha_m \dot{\alpha}_1 \ldots \dot{\alpha}_n} = D_\beta U_{\alpha_1 \ldots \alpha_m \dot{\alpha}_1 \ldots \dot{\alpha}_n} + \frac{1}{m+1} \sum_{k=1}^{m} \varepsilon_{\beta \alpha_k} D^\alpha U_{\gamma |\alpha_1 \ldots \alpha_k \ldots \alpha_m \dot{\alpha}_1 \ldots \dot{\alpha}_n} \]

\[ = D_\beta U_{\alpha_1 \ldots \alpha_m \dot{\alpha}_1 \ldots \dot{\alpha}_n} + \frac{m}{m+1} \varepsilon_{\beta \alpha_1} D^\alpha U_{\gamma |\alpha_2 \ldots \alpha_m \dot{\alpha}_1 \ldots \dot{\alpha}_n} \]  

(3.2.9)

The reader is referred to appendix A for the symmetrisation convention used in this work.

The \( \mathcal{W} \)-gauge freedom (3.2.5) may be used to impose the condition

\[ \Sigma_{\alpha(s-1) \bar{\alpha}(s-2)} = 0 . \]  

(3.2.10)

In this gauge, the action (3.2.8) reduces to that describing the longitudinal formulation for the massless superspin-\( s \) multiplet (2.5.27). The gauge condition (3.2.10) does not
fix completely the $\mathfrak{g}$-gauge freedom. There remains a residual gauge transformation generated by

$$\mathfrak{v}_{\alpha(s-1)\dot{a}(s-1)} = D^\beta L_{(\bar{\beta}\alpha_1...\alpha_{s-1})\dot{a}(s-1)}; \quad (3.2.11)$$

with the parameter $L_{\alpha(s)\dot{a}(s-1)}$ being an unconstrained superfield. With this expression for $\mathfrak{v}_{\alpha(s-1)\dot{a}(s-1)}$, the gauge transformations (3.2.5a) and (3.2.5b) coincide with those given in (2.5.26). Our consideration implies that the action (3.2.8) indeed provides an off-shell formulation for the massless superspin-$s$ multiplet.

Instead of choosing the condition (3.2.11), one can impose an alternative gauge fixing

$$H_{\alpha(s-1)\dot{a}(s-1)} = 0. \quad (3.2.12)$$

In accordance with (3.2.5b), in this gauge the residual gauge freedom is described by

$$\mathfrak{v}_{\alpha(s-1)\dot{a}(s-1)} = i\mathfrak{r}_{\alpha(s-1)\dot{a}(s-1)}; \quad \mathfrak{r}_{\alpha(s-1)\dot{a}(s-1)} = \mathfrak{r}_{\alpha(s-1)\dot{a}(s-1)}. \quad (3.2.13)$$

The action (3.2.8) includes a single term which involves the ‘naked’ gauge field $\Psi_{\alpha(s)\dot{a}(s-1)}$ and not the field strength $G_{\alpha(s)\dot{a}(s)}$. This is actually a BF term, for it can be written in two different forms

$$\frac{1}{s} \int d^4x d^2\theta d^2\bar{\theta} \Psi^{(s)\dot{a}(s-1)} \left( D_{\alpha_1} \bar{D}_{\dot{a}_1} - 2i(s-1)\partial_{\alpha_1\dot{a}_1} \right) \Sigma_{a_2...a_s\dot{a}_2...\dot{a}_{s-1}}$$

$$= -\frac{1}{s+1} \int d^4x d^2\theta d^2\bar{\theta} G^{(s)\dot{a}(s)} \left( D_{\alpha_1} D_{\alpha_2} + 2i(s+1)\partial_{\alpha_1\dot{a}_1} \right) Z_{a_2...a_s\dot{a}_2...\dot{a}_s}. \quad (3.2.14)$$

The former makes the $\xi$-gauge symmetry (3.2.4) manifestly realised, while the latter turns the $\zeta$-transformation (3.2.5a) into a manifest symmetry. Making use of (3.2.14) allows us to write the action (3.2.8) in the following form:

$$S_{(s)}^\parallel = \left( -\frac{1}{2} \right)^s \int d^4x d^2\theta d^2\bar{\theta} \left\{ \frac{1}{8} H^{(s-1)\dot{a}(s-1)} D^\beta \bar{D}_\beta H_{\alpha(s-1)\dot{a}(s-1)} \right. + \frac{s}{s+1} H^{(s)\dot{a}(s)} \left( D^\beta \bar{D}_\beta G_{\beta\alpha(s-1)\dot{a}(s-1)} - \bar{D}^\beta \bar{D}_\beta \bar{G}_{\beta\alpha(s-1)\dot{a}(s-1)} \right)$$

$$+ 2G^{(s)\dot{a}(s)} G_{\alpha(s)\dot{a}(s)} + \frac{s}{s+1} \left( G^{(s)\dot{a}(s)} G_{\alpha(s)\dot{a}(s)} + \bar{G}^{(s)\dot{a}(s)} \bar{G}_{\alpha(s)\dot{a}(s)} \right)$$

$$+ \frac{s-1}{4s} H^{(s-1)\dot{a}(s-1)} \left( D_{\alpha_1} \bar{D}_{\dot{a}_1} \right) Z_{a_2...a_s\dot{a}_2...\dot{a}_s}$$

$$+ \frac{1}{s+1} \left( G^{(s)\dot{a}(s)} \right) \left( D_{\alpha_1} \bar{D}_{\dot{a}_1} + 2i(s+1)\partial_{\alpha_1\dot{a}_1} \right) Z_{a_2...a_s\dot{a}_2...\dot{a}_s}$$

$$+ \frac{s-1}{8s} \left( \Sigma^{(s-1)\dot{a}(s-2)} D^2 \Sigma_{\alpha(s-1)\dot{a}(s-2)} - \Sigma^{(s-2)\dot{a}(s-1)} \bar{D}^2 \Sigma_{\alpha(s-2)\dot{a}(s-1)} \right)$$

$$- \frac{1}{s^2} \left( \Sigma^{(s-2)\dot{a}(s-2)} \right) \left( \frac{1}{2} (s^2 + 1) D^\beta \bar{D}_\beta + i(s-1)^2 \partial^\beta \bar{\partial}_\beta \right) \Sigma_{\beta\alpha(s-2)\dot{a}(s-2)} \right\}. \quad (3.2.15)$$
3.2.2 Dual formulation

The theory with action (3.2.15) possesses a dual formulation that can be obtained by applying the duality transformation described in subsection 2.5.1.2. We now associate with our theory (3.2.15) the following first-order action

\[ S_{\text{first-order}} = S_{(s)}^{\parallel} \left[ U, \bar{U}, H, Z, \bar{Z} \right] \]

\[ + \left( -\frac{1}{2} \right)^s \int d^4 x d^2 \theta d^2 \bar{\theta} \left( \frac{2}{s+1} \Gamma_{\alpha(s)\dot{\alpha}(s)} U_{\alpha(s)\dot{\alpha}(s)} + \text{c.c.} \right) , \quad (3.2.16) \]

where \( S_{(s)}^{\parallel} \left[ U, \bar{U}, H, Z, \bar{Z} \right] \) is obtained from the action (3.2.15) by replacing \( G_{\alpha(s)\dot{\alpha}(s)} \) with an unconstrained complex superfield \( U_{\alpha(s)\dot{\alpha}(s)} \). The Lagrange multiplier \( \Gamma_{\alpha(s)\dot{\alpha}(s)} \) is a transverse linear superfield,

\[ \bar{D}^\beta \Gamma_{\alpha(s)\dot{\beta}1\ldots\dot{\alpha}_{s-1}} = 0 . \quad (3.2.17) \]

The specific normalisation of the Lagrange multiplier in (3.2.16) is chosen to match that of [63,65].

The first-order model introduced is equivalent to the original theory (3.2.15), which can be seen by varying \( S_{\text{first-order}} \) with respect to the Lagrange multiplier. The action (3.2.16) is invariant under the gauge \( \xi \)-transformation (3.2.4) which acts on \( U_{\alpha(s)\dot{\alpha}(s)} \) and \( \Gamma_{\alpha(s)\dot{\alpha}(s)} \) by the rule

\[ \delta_\xi U_{\alpha(s)\dot{\alpha}(s)} = 0 , \quad (3.2.18a) \]

\[ \delta_\xi \Gamma_{\alpha(s)\dot{\alpha}(s)} = \bar{D}^\beta \left\{ \frac{s+1}{2(s+2)} D_\beta D_{(\alpha_1 \xi_{a_2\ldots a_s})\dot{\alpha}_1\ldots\dot{\alpha}_s} + i(s+1) \partial_{(\alpha_1(\beta \xi_{a_2\ldots a_s})\dot{\alpha}_1\ldots\dot{\alpha}_s)} \right\} . \quad (3.2.18b) \]

The first-order action (3.2.16) is also invariant under the gauge \( \mathfrak{H} \)-transformation (3.2.5b) and (3.2.5c), which acts on \( U_{\alpha(s)\dot{\alpha}(s)} \) and \( \Gamma_{\alpha(s)\dot{\alpha}(s)} \) as

\[ \delta_\mathfrak{H} U_{\alpha(s)\dot{\alpha}(s)} = \frac{1}{2} \bar{D}_{(\dot{\alpha}_1} D_{(\alpha_1(\mathfrak{H}_{a_2\ldots a_s})\dot{\alpha}_2\ldots\dot{\alpha}_s)} , \quad (3.2.19a) \]

\[ \delta_\mathfrak{H} \Gamma_{\alpha(s)\dot{\alpha}(s)} = 0 . \quad (3.2.19b) \]

On the other hand, eliminating the auxiliary superfields \( U_{\alpha(s)\dot{\alpha}(s)} \) and \( \bar{U}_{\alpha(s)\dot{\alpha}(s)} \) from (3.2.16) using their equations of motion leads to

\[ S_{(s)}^{\parallel} = - \left( -\frac{1}{2} \right)^s \int d^4 x d^2 \theta d^2 \bar{\theta} \left\{ - \frac{1}{8} H^{\alpha(s-1)\dot{\alpha}(s-1)} D^\beta \bar{D}^2 D_\beta H_{\alpha(s-1)\dot{\alpha}(s-1)} \right. \]

\[ + \frac{1}{8} \left( \frac{s^2}{s+1} \right) \left[ D^\beta , \bar{D}^\bar{\beta} \right] H^{\alpha(s-1)\dot{\alpha}(s-1)} \left[ D_\beta , \bar{D}_{\bar{\beta}} \right] H_{\alpha(s-1)\dot{\alpha}(s-1)} \]

\[ + \frac{1}{2} \left( \frac{s^2}{s+1} \right) \theta^{\alpha(s-1)\dot{\alpha}(s-1)} \partial_{(\beta \bar{\beta}} H_{\alpha(s-1)\dot{\alpha}(s-1))} \right\} . \]

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where we have defined

$$
\Gamma_{\alpha(s-1)\dot{\alpha}(s-1)} = \Gamma_{\alpha(s)\dot{\alpha}(s)} - \frac{1}{2} \tilde{D}_{\dot{\alpha}_{1}} D_{\alpha_{1}} Z_{\alpha_{2}...\alpha_{s}} \dot{\alpha}_{2}...\dot{\alpha}_{s} - i(s+1) \partial_{\alpha_{1}} \dot{\alpha}_{1} Z_{\alpha_{2}...\alpha_{s}} \dot{\alpha}_{2}...\dot{\alpha}_{s} \ . (3.2.21)
$$

In accordance with (3.2.5c), the $\mathcal{W}$-gauge freedom may be used to impose the condition

$$
Z_{\alpha(s-1)\dot{\alpha}(s-1)} = 0 \ . \ (3.2.22)
$$

In this gauge the action (3.2.20) reduces to the one defining the transverse formulation for the massless superspin-s multiplet, eq. (2.5.31). The gauge condition (3.2.22) is preserved by residual local $\mathcal{W}$- and $\xi$-transformations of the form

$$
\delta \tilde{D}^\beta \xi_{\alpha(s-1)\dot{\alpha}(s-1)} + \tilde{\xi}_{\alpha(s-1)\dot{\alpha}(s-1)} = 0 \ . \ (3.2.23)
$$

Making use of the parametrisation (3.2.11), the residual gauge freedom is

$$
\delta H_{\alpha(s-1)\dot{\alpha}(s-1)} = D^\beta L_{\beta\alpha(s-1)\dot{\alpha}(s-1)} - \tilde{D}^\beta \tilde{L}_{\alpha(s-1)\dot{\alpha}(s-1)} \ , \ (3.2.24a)
$$

$$
\delta \Gamma_{\alpha(s)\dot{\alpha}(s)} = \frac{s+1}{2(s+2)} \tilde{D}^\beta \left\{ \tilde{D}_{\beta} D_{(\alpha_{1} + 2i(s+2)\partial_{(\alpha_{1})\beta}} \tilde{L}_{\alpha_{2}...\alpha_{s} \dot{\alpha}_{1}...\dot{\alpha}_{s}} \right\} \ , \ (3.2.24b)
$$

which is exactly the gauge symmetry of the transverse formulation for the massless superspin-s multiplet, eq. (2.5.30).

We note that the action (3.2.8) involves the transverse linear compensator $\Sigma_{\alpha(s-1)\dot{\alpha}(s-2)}$ and its conjugate $\bar{\Sigma}_{\alpha(s-2)\dot{\alpha}(s-1)}$. These superfields cannot be dualised into a longitudinal linear supermultiplet without destroying the locality of the theory, for the action (3.2.8) contains terms with derivatives of $\Sigma_{\alpha(s-1)\dot{\alpha}(s-2)}$ and $\bar{\Sigma}_{\alpha(s-2)\dot{\alpha}(s-1)}$.  

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3.2.3 Massless gravitino multiplet

The original longitudinal and transverse actions for the massless superspin-s multiplet, given by (2.5.27) and (2.5.31) respectively, are well defined for $s = 1$, in which case they describe two off-shell formulations for the massless gravitino multiplet. However, the new action functional (3.2.8) is not defined in the $s = 1$ case. The point is that the gauge transformation law (3.2.5a) is not defined for $s = 1$. Instead, one should replace the gauge transformation (3.2.5a) with

$$\delta \Psi_\alpha = \frac{1}{2} D_\alpha \Psi + \zeta_\alpha , \quad \bar{D}_\dot{\beta} \zeta_\alpha = 0 ,$$  \hspace{1cm} (3.2.25a)

in accordance with the superconformal gravitino model [68]. This transformation law of $\Psi_\alpha$ coincides with the one occurring in the off-shell model for the massless gravitino multiplet proposed in [137]. In addition to the gauge superfield $\Psi_\alpha$, this model also involves two compensators: a real scalar $H$ and a chiral scalar $\Phi$, $\bar{D}_\dot{\alpha} \Phi = 0$, with the gauge transformations

$$\delta H = \Psi + \bar{\Psi} ,$$  \hspace{1cm} (3.2.25b)

$$\delta \Phi = -\frac{1}{2} \bar{D}^2 \Psi .$$  \hspace{1cm} (3.2.25c)

The gauge-invariant action of [137] takes the form (see also [35] for a review):

$$S_{GM}^{(I)} = S_{(1, \frac{3}{2})}^\parallel [\Psi, \bar{\Psi}, H] - \frac{1}{4} \int d^4x d^2\theta d^2\bar{\theta} \left( \Phi \bar{\Phi} + \Phi D^\alpha \Psi_\alpha + \bar{\Phi} \bar{D}_\dot{\alpha} \bar{\Psi}^{\dot{\alpha}} \right) ,$$  \hspace{1cm} (3.2.26)

where $S_{(1, \frac{3}{2})}^\parallel [\Psi, \bar{\Psi}, H]$ denotes the longitudinal action for the gravitino multiplet, which is obtained from (3.2.8) by choosing the gauge (3.2.10) and setting $s = 1$. At the component level, this model corresponds to the Fradkin-Vasiliev-de Wit-van Holten formulation for the gravitino multiplet [138,139].

There exists a dual formulation for (3.2.26). This is obtained by performing a superfield Legendre transformation [140], which gives the dual action [140]

$$S_{GM}^{(II)} = S_{(1, \frac{3}{2})}^\parallel [\Psi, \bar{\Psi}, H] + \frac{1}{4} \int d^4x d^2\theta d^2\bar{\theta} \left( G + D^\alpha \Psi_\alpha + \bar{D}_\dot{\alpha} \bar{\Psi}^{\dot{\alpha}} \right)^2 ,$$  \hspace{1cm} (3.2.27)

where $G = \bar{G}$ is a real linear superfield, $\bar{D}^2 G = D^2 G = 0$. The gauge freedom in this theory is given by eqs. (3.2.25a), (3.2.25b) and

$$\delta G = -D^\alpha \zeta_\alpha - \bar{D}_\dot{\alpha} \bar{\zeta}^{\dot{\alpha}} .$$  \hspace{1cm} (3.2.28)

It may be used to impose two conditions $H = 0$ and $G = 0$. We then end up with the Ogievetsky-Sokatchev formulation for the gravitino multiplet [141] (see section 6.9.5 of the book [35] for the technical details).
There exists one more dual formulation for (3.2.26) that is obtained by performing the complex linear-chiral duality transformation. It leads to

\[ S_{GM}^{(III)} = S \parallel (\frac{s}{2}, \frac{s}{2})[\Psi, \bar{\Psi}, H] + \frac{1}{2} \int d^4x d^2\theta d^2\bar{\theta} (\Sigma + D^\alpha \Psi_\alpha)(\bar{\Sigma} + \bar{D}_\dot{\alpha} \bar{\Psi}_{\dot{\alpha}}) , \]  

(3.2.29)

where \( \Sigma \) is a complex linear superfield constrained by \( \bar{D}^2 \Sigma = 0 \). The gauge freedom in this theory is given by eqs. (3.2.25a), (3.2.25b) and

\[ \delta \Sigma = -D^\alpha \zeta_\alpha . \]  

(3.2.30)

This gauge freedom does not allow one to gauge away \( \Sigma \) off the mass shell. To the best of our knowledge, the supersymmetric gauge theory (3.2.29) is a new off-shell realisation for the massless gravitino multiplet.

Let us also remark that all the constructions considered in this section can naturally be lifted to the case of anti-de Sitter supersymmetry to extend the results of [63]. This will be studied in chapter 4.

3.3 Higher-spin multiplets of conserved currents

This section is devoted to the study of non-conformal higher-spin supercurrent multiplets in Minkowski space, as an extension of the superconformal case which was first described in [69] and further elaborated in [68]. Our approach will be a higher-spin extension of that used in subsection 2.2 to derive consistent supercurrents associated with a linearised off-shell formulation for \( \mathcal{N} = 1 \) Poincaré supergravity. Here we will demonstrate that the off-shell actions for the massless half-integer superspin multiplet described in section 2.5 along with the new integer superspin action (3.2.15), allow us to formulate \( \mathcal{N} = 1 \) non-conformal higher-spin supercurrents in 4D Minkowski space [1,2].

3.3.1 Non-conformal supercurrents: Half-integer superspin

Let us proceed with the massless half-integer superspin case and derive the current multiplet corresponding to the longitudinal formulation (2.5.19). The first step is to add a source (or coupling) term to the action \( S^\parallel (s+\frac{1}{2})[H, G, \bar{G}] \), eq. (2.5.19)

\[
S^\parallel (s+\frac{1}{2})[H, G, \bar{G}] - \int d^4x d^2\theta d^2\bar{\theta} \left\{ H^{(s)}\dot{\alpha}(s) J_{\alpha(s)} + \left( \Psi^{(s-1)}\dot{\alpha}(s-2) T_{\alpha(s-1)}\dot{\alpha}(s-2) + \text{c.c.} \right) \right\} .
\]  

(3.3.1)

Next, requiring the above to be invariant under the \( \zeta \)-transformation (2.5.16)

\[ \delta_\zeta \Psi_{\alpha(s-1)} = \bar{D}_{\dot{\alpha}(s-1)} \zeta_{\alpha(s-1)} \dot{\alpha}(s-2) . \]  

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implies that $T_{\alpha(s-1)\bar{\alpha}(s-2)}$ is a transverse linear superfield,

$$D^\beta T_{\alpha(s-1)\bar{\alpha}(s-2)} = 0 \quad \text{(3.3.2)}$$

The action (3.3.1) should also respect the $\Lambda$-gauge freedom given in (2.5.17a) and (2.5.17b):

$$\delta_\Lambda H_{a_1\ldots a_s\bar{\alpha}_1\ldots \bar{\alpha}_s} = \bar{D}_{\bar{\alpha}_1\ldots \bar{\alpha}_s} \Lambda_{a_1\ldots a_s\bar{\alpha}_1\ldots \bar{\alpha}_s}$$

$$\delta_\Lambda \Psi_{a_1\ldots a_s\bar{\alpha}_1\ldots \bar{\alpha}_s} = -\frac{1}{2} \left( \bar{D}^\beta D^\beta - 2i(s-1)\partial^\beta \right) \Lambda_{\beta} a_1\ldots a_s\bar{\alpha}_1\ldots \bar{\alpha}_s \quad \text{(3.3.3a)}$$

This demands the sources to satisfy the following conservation equation

$$\bar{D}^\beta J_{a_1\ldots a_s\bar{\alpha}_1\ldots \bar{\alpha}_{s-1}} + \frac{1}{2} (D_{a_1} \bar{D}_{\bar{\alpha}_1} - 2i(s-1)\partial_{a_1}(\bar{\alpha}_1)) T_{\bar{\alpha}_1 a_2\ldots a_s} = 0 \quad \text{(3.3.3a)}$$

For completeness, we also give the conjugate equation

$$D^\beta J_{\bar{\alpha}_1\ldots \bar{\alpha}_s a_1\ldots a_{s-1}} = -\frac{1}{2} \left( \bar{D}_{\alpha_1} D_{a_1} - 2i(s-1)\partial_{\alpha_1}(a_1) \right) T_{a_1\ldots a_s \bar{\alpha}_1\ldots \bar{\alpha}_{s-1}} = 0 \quad \text{(3.3.3b)}$$

Associated with the transverse model (2.5.10) is the following non-conformal supercurrent multiplet

$$\bar{D}^\beta J_{a_1\ldots a_s\bar{\alpha}_1\ldots \bar{\alpha}_{s-1}} - \frac{1}{4} \bar{D}^2 F_{a_1\ldots a_s\bar{\alpha}_1\ldots \bar{\alpha}_{s-1}} = 0 \quad \text{(3.3.4a)}$$

$$D_{a_1} F_{a_2\ldots a_{s+1}\bar{\alpha}_1\ldots \bar{\alpha}_{s-1}} = 0 \quad \text{(3.3.4b)}$$

Thus the trace multiplet $\bar{F}_{\alpha(s-1)\bar{\alpha}(s)}$ is longitudinal linear.

When working with higher-spin supercurrents, it proves to be convenient to make use of a condensed notation. Let us introduce auxiliary commuting complex variables $\zeta^\alpha \in \mathbb{C}^2$ and their conjugates $\bar{\zeta}^{\bar{\alpha}}$. Given a tensor superfield $U_{\alpha(p)\bar{\alpha}(q)}$, we associate with it the following index-free field on $\mathbb{C}^2$

$$U_{(p,q)}(\zeta,\bar{\zeta}) := \zeta^\alpha_1 \ldots \zeta^\alpha_p \bar{\zeta}^{\bar{\alpha}_1} \ldots \bar{\zeta}^{\bar{\alpha}_q} U_{\alpha_1\ldots \alpha_p\bar{\alpha}_1\ldots \bar{\alpha}_q} \quad \text{(3.3.5)}$$

which is a homogeneous polynomial of degree $(p,q)$ in $\zeta^\alpha$ and $\bar{\zeta}^{\bar{\alpha}}$. Furthermore, we make use of the bosonic variables $(\zeta^\alpha, \bar{\zeta}^{\bar{\alpha}})$ and their corresponding partial derivatives $(\partial/\partial \zeta^\alpha, \partial/\partial \bar{\zeta}^{\bar{\alpha}})$ to convert the spinor and vector covariant derivatives into index-free operators. We introduce operators that increase the degree of homogeneity in $\zeta^\alpha$ and $\bar{\zeta}^{\bar{\alpha}}$:

$$D_{(1,0)} := \zeta^\alpha D_\alpha \quad \text{and} \quad \bar{D}_{(0,1)} := \bar{\zeta}^{\bar{\alpha}} \bar{D}_{\bar{\alpha}} \quad \text{(3.3.6)}$$

and their descendants

$$A_{(1,1)} := -D_{(1,0)} \bar{D}_{(0,1)} + (s-1)\partial_{(1,1)} \quad \text{and} \quad \bar{A}_{(1,1)} := \bar{D}_{(0,1)} D_{(1,0)} - (s-1)\partial_{(1,1)} \quad \text{(3.3.7)}$$

The fermionic operators $D_{(1,0)}$ and $\bar{D}_{(0,1)}$ are nilpotent, $D_{(1,0)}^2 = 0$ and $\bar{D}_{(0,1)}^2 = 0$. Additionally, we also have the following nilpotent operators which decrease the degree of homogeneity in $\zeta^\alpha$ and $\bar{\zeta}^{\bar{\alpha}}$:

$$D_{(-1,0)} := D^\alpha \frac{\partial}{\partial \zeta^\alpha} \quad \text{and} \quad D_{(-1,0)}^2 = 0 \quad \text{(3.3.8a)}$$
\[ D_{(0,-1)} := \bar{D}^\dot{\alpha} \frac{\partial}{\partial \zeta^{\dot{\alpha}}} \quad D_{(0,-1)}^2 = 0. \]  

(3.3.8b)

Making use of the above notation, the transverse linearity condition (3.3.2) and its conjugate become

\[ \bar{D}_{(0,-1)} T_{(s-1,s-2)} = 0, \]  

(3.3.9a)

\[ D_{(-1,0)} \bar{T}_{(s-2,s-1)} = 0. \]  

(3.3.9b)

The conservation equations (3.3.3a) and (3.3.3b) turn into

\[ \bar{D}_{(0,-1)} J_{(s,s)} - \frac{1}{2} \bar{A}_{(1,1)} T_{(s-1,s-2)} = 0, \]  

(3.3.10a)

\[ D_{(-1,0)} J_{(s,s)} - \frac{1}{2} \bar{A}_{(1,1)} \bar{T}_{(s-2,s-1)} = 0. \]  

(3.3.10b)

Since the operator \( \bar{D}_{(0,-1)} J_{(s,s)} \) is nilpotent, the conservation equation (3.3.10a) is consistent provided

\[ \bar{D}_{(0,-1)} A_{(1,1)} T_{(s-1,s-2)} = 0. \]  

(3.3.11)

This is indeed true, as a consequence of (3.3.9a).

### 3.3.1.1 Examples of higher-spin supercurrents

Consider a free massless chiral scalar superfield \( \Phi \) with the action

\[ S = \int d^4x d^2\theta d^2\bar{\theta} \Phi \bar{\Phi}, \quad \bar{D}_\dot{\alpha} \Phi = 0. \]  

(3.3.12)

The conserved higher-spin supercurrent multiplet associated to the model (3.3.12) was first constructed in [68]. It is

\[
J_{\alpha(s)\dot{\alpha}(s)} = (2i)^{s-1} \sum_{k=0}^{s} (-1)^k \binom{s}{k} \left\{ \partial_{(\alpha_1(\dot{\alpha}_1) \ldots \partial_{\alpha_k(\dot{\alpha}_k)} D_{(k+1)} \Phi \bar{D}_{(k+1)} \partial_{\alpha_{k+1}\dot{\alpha}_{k+1}}\ldots \partial_{\alpha_s(\dot{\alpha}_s)} \Phi + 2i \binom{s}{k} \partial_{(\alpha_1(\dot{\alpha}_1) \ldots \partial_{\alpha_k(\dot{\alpha}_k)} \Phi \partial_{(k+1)}(\alpha_{k+1} \dot{\alpha}_{k+1})\ldots \partial_{\alpha_s)(\dot{\alpha}_s)} \Phi} \right\}. \tag{3.3.13}
\]

Using our notation, it reads

\[
J_{(s,s)} = \sum_{k=0}^{s} (-1)^k \binom{s}{k} \left\{ \binom{s}{k+1} \partial_{(1,1)} T_{(1,0)} \Phi \partial_{(1,1)}^{s-k-1} D_{(0,1)} \Phi + \binom{s}{k} \partial_{(1,1)}^{k} \Phi \partial_{(1,1)}^{s-k} \bar{\Phi} \right\}, \tag{3.3.14}
\]
which obeys the conservation equations on-shell

\[ D_{(-1,0)} J_{(s,s)} = 0 \iff \bar{D}_{(0,-1)} J_{(s,s)} = 0 \quad (3.3.15) \]

It is useful to understand the construction of the conformal higher-spin supercurrent \((3.3.14)\). For this we need to discuss a few important notions of \( \mathcal{N} = 1 \) superconformal multiplet following the presentation of [68].

A tensor superfield \( T \) (with suppressed indices) is called superconformal primary of weight \((p,q)\) if it transforms as

\[ \delta_{\xi} T = \left( \xi + \frac{1}{2} \omega^{bc}[\xi] M_{bc} \right) T + \left( p\sigma[\xi] + q\bar{\sigma}[\xi] \right) T, \quad (3.3.16) \]

for some parameters \( p \) and \( q \). Here \( \xi = \xi^A D_A \) is the \( \mathcal{N} = 1 \) conformal Killing real supervector field generating superconformal transformations in Minkowski space. The superfields \( \omega^{bc}[\xi] \) and \( \sigma[\xi] \) denote some local Lorentz and super-Weyl parameters, respectively. The dimension of \( T \) is \((p+q)\) and its \( R \)-symmetry charge is proportional to \((p-q)\). If \( T \) is chiral, \( q = 0 \), and we say that \( T \) is superconformal primary of dimension \( p \).

For example, by requiring that both \( H_{\alpha(s)} \dot{\alpha}(s) \) and the gauge parameter \( \Lambda_{\alpha(s)} \dot{\alpha}(s-1) \) in \((2.5.17a)\) to be superconformal primary, the superconformal transformation law for \( H_{\alpha(s)} \dot{\alpha}(s) \) can be derived [68]. It is

\[ \delta_{\xi} H_{\alpha(s)} \dot{\alpha}(s) = \left( \xi + \frac{1}{2} \omega^{bc}[\xi] M_{bc} \right) H_{\alpha(s)} \dot{\alpha}(s) - \frac{s}{2} \left( \sigma[\xi] + \bar{\sigma}[\xi] \right) H_{\alpha(s)} \dot{\alpha}(s). \quad (3.3.17) \]

Given a real scalar \( L \), the action functional over the full superspace,

\[ S = \int d^4x d^2\theta d^2\bar{\theta} L \quad (3.3.18) \]

is invariant under the superconformal transformations if \( L \) is superconformal primary of weight \((1,1)\). On the other hand, the chiral action

\[ S = \int d^4x d^2\theta L_c, \quad \bar{D}_\dot{\alpha} L_c = 0 \quad (3.3.19) \]

is superconformally invariant provided \( L_c \) is superconformal primary of dimension +3. For instance, the massless model \((3.3.12)\) is superconformal provided the chiral scalar superfield is superconformal primary of dimension +1.

Now, in order to describe the structure of \( J_{\alpha(s)} \dot{\alpha}(s) \), the authors of [68] first consider coupling of the form

\[ S_{\text{source}}^{(s+\frac{1}{2})} = \int d^4x d^2\theta d^2\bar{\theta} H_{\alpha(s)} \dot{\alpha}(s) J_{\alpha(s)} \dot{\alpha}(s) \quad (3.3.20) \]

and require invariance under the superconformal transformations \((3.3.17)\). From \((3.3.17)\), one sees that \( H_{\alpha(s)} \dot{\alpha}(s) \) is superconformal primary of weight \((-\frac{s}{2}, -\frac{s}{2})\), thus the real superfield \( J_{\alpha(s)} \dot{\alpha}(s) \) must be of weight \((1 + \frac{s}{2}, 1 + \frac{s}{2})\). Next, the requirement of gauge-invariance
under (2.5.17a) leads to the conservation equations (3.3.15). Since Φ is superconformal primary of dimension +1, the following ansatz for \( J_{\alpha(s)}^{\dot{\alpha}(s)} \) as composites of \( \Phi \) and \( \bar{\Phi} \) was considered [68]:

\[
J_{(s,s)} = \sum_{k=0}^{s} \left\{ a_k \partial^{k}_{(1,1)} D_{(1,0)} \Phi \partial^{s-k-1}_{(1,1)} \bar{D}_{(0,1)} \bar{\Phi} + b_k \partial^{k}_{(1,1)} \bar{\Phi} \partial^{s-k}_{(1,1)} \Phi \right\} .
\] (3.3.21)

The coefficients \( a_k \) and \( b_k \) can be fixed uniquely by imposing two conditions: (i) \( J_{\alpha(s)}^{\dot{\alpha}(s)} \) must be real; and (ii) it must obey the conservation equation (3.3.15). Indeed, setting \( s = 1 \) leads to the Ferrara-Zumino supercurrent [57] which we reviewed in subsection 2.2:

\[
J_{\alpha \dot{\alpha}} = D_{\alpha} \Phi \bar{D}_{\dot{\alpha}} \bar{\Phi} + 2i(\Phi \partial_{\alpha \dot{\alpha}} \bar{\Phi} - \bar{\Phi} \partial_{\alpha \dot{\alpha}} \Phi) .
\] (3.3.22)

Our aim is to construct non-conformal higher-spin supercurrent arising in the model for a massive chiral superfield

\[
S = \int d^4x d^2\theta d^2\bar{\theta} \bar{\Phi} \Phi + \frac{m^2}{2} \int d^4x d^2\theta \Phi^2 + c.c. .
\] (3.3.23)

As will be demonstrated below, it is the longitudinal higher-spin supercurrent multiplet described by (3.3.9) and (3.3.10), which naturally arises in (3.3.23). Guided by the structure of the Ferrara-Zumino supercurrent for the model (3.3.23), we assume that \( J_{(s,s)} \) has the same functional form as in the massless case, eq. (3.3.14). We first compute the left-hand side of (3.3.10a) and use the massive equation of motion,

\[
- \frac{1}{4} \bar{D}^2 \bar{\Phi} + m \Phi = 0 .
\]

This gives

\[
\bar{D}_{(0,-1)} J_{(s,s)} = F_{(s,s-1)} ,
\] (3.3.24a)

where we have denoted

\[
F_{(s,s-1)} = 2m(s+1) \sum_{k=0}^{s} (-1)^{s-1+k} \binom{s}{k} \binom{s}{k+1} \times \left\{ 1 + (-1)^s \frac{k+1}{s-k+1} \right\} \partial_{(1,1)}^k \Phi \partial_{(1,1)}^{s-k-1} D_{(1,0)} \Phi .
\] (3.3.24b)

Let us now determine the trace multiplet \( T_{(s-1,s-2)} \). For this we consider a general ansatz in the form

\[
T_{(s-1,s-2)} = (-1)^s m \sum_{k=0}^{s-2} c_k \partial_{(1,1)}^k \Phi \partial_{(1,1)}^{s-k-2} D_{(1,0)} \Phi .
\] (3.3.25)

This ansatz is chosen based on the following requirements: (i) \( T_{(s-1,s-2)} \) must be transverse linear (3.3.9a); and (ii) it solves the equation (3.3.10a),

\[
F_{(s,s-1)} = \frac{1}{2} A_{(1,1)} T_{(s-1,s-2)} .
\] (3.3.26)
For $k = 1, 2, \ldots s - 2$, the first condition implies that the coefficients $c_k$ must satisfy

$$kc_k = (s-k-1)c_{s-k-1}.$$  \hspace{1cm} (3.3.27a)

Imposing condition (ii) leads to

$$c_{s-k-1} + sc_k + (s-1)c_{k-1} = -4(s+1)(-1)^k \binom{s}{k} \binom{s}{k+1} \left( 1 + (-1)^s \frac{k+1}{s-k+1} \right).$$  \hspace{1cm} (3.3.27b)

In addition, it also follows from (ii) that

$$(s-1)c_{s-2} + c_0 = 4(-1)^s(s+1) \left\{ 1 + (-1)^s \frac{s}{2} \right\},$$  \hspace{1cm} (3.3.27c)

and

$$c_0 = -4(s+1)(-1)^s.$$  \hspace{1cm} (3.3.27d)

We find that the set of equations (3.3.27) leads to a unique expression for $c_k$,

$$c_k = -\frac{4(s+1)(s-k-1)}{s-1} \sum_{l=0}^{k} \frac{(-1)^k \binom{s}{l} \binom{s}{l+1} \left\{ 1 + (-1)^s \frac{l+1}{s-l+1} \right\}}{s-l},$$  \hspace{1cm} (3.3.28)

$k = 0, 1, \ldots s-2$.

If the parameter $s$ is odd, $s = 2n + 1$, with $n = 1, 2, \ldots$, one can check that the equations (3.3.27a)–(3.3.27c) are identically satisfied. However, if the parameter $s$ is even, $s = 2n$, with $n = 1, 2, \ldots$, there appears an inconsistency: the right-hand side of (3.3.27c) is positive, while the left-hand side is negative, $(s-1)c_{s-2} + c_0 < 0$. As a result, our solution (3.3.28) is only consistent for $s = 2n + 1, n = 1, 2, \ldots$.

Relations (3.3.14), (3.3.25) and (3.3.28) determine the non-conformal higher-spin supercurrent in the massive chiral model (3.3.23), with the trace multiplet $T_{(s-1,s-2)}$ being the higher-spin extension of (2.2.20). Unlike the conformal higher-spin supercurrent (3.3.14), the non-conformal one exists only for the odd values of $s$, $s = 2n + 1$, with $n = 1, 2, \ldots$. The same conclusion was also reached by the authors of [115] who employed the superfield Noether procedure.

### 3.3.2 Non-conformal supercurrents: Integer superspin

Having derived a new off-shell gauge formulation for the massless superspin-$s$ multiplet, we turn to describing the structure of the non-conformal higher-spin supercurrents associated to the model (3.2.15).

As in the half-integer superspin case, let us couple the prepotentials $H_{\alpha(s-1)\dot{\alpha}(s-1)}$, $Z_{\alpha(s-1)\dot{\alpha}(s-1)}$ and $\Psi_{\alpha(s-1)\dot{\alpha}(s-1)}$ to external sources

$$S_{\text{source}}^{(s)} = \int d^4x d^2\theta d^2\bar{\theta} \left\{ \Psi^{\alpha(s)\dot{\alpha}(s-1)} J_{\alpha(s)\dot{\alpha}(s-1)} - \bar{\Psi}^{\alpha(s-1)\dot{\alpha}(s)} \bar{J}_{\alpha(s-1)\dot{\alpha}(s)} \right\}$$
We see that the superfields \( J \) transformation (3.2.5) gives the following conservation equation

\[
+ H^{\alpha(s-1)\dot{\alpha}(s-1)} S^{\alpha(s-1)\dot{\alpha}(s-1)} + Z^{\alpha(s-1)\dot{\alpha}(s-1)} T^{\alpha(s-1)\dot{\alpha}(s-1)} + \bar{Z}^{\alpha(s-1)\dot{\alpha}(s-1)} \bar{T}^{\alpha(s-1)\dot{\alpha}(s-1)} \right) . \tag{3.3.29}
\]

In order for the source term \( s^{(s)}_{\text{source}} \) to be invariant under the \( \zeta \)-transformation in \((3.2.5a)\), the source \( J^{(s)}_{\alpha(s)\dot{\alpha}(s-1)} \) must obey

\[
-\frac{1}{2} D^\beta J_{\alpha(s)\dot{\alpha}(s-2)} = 0 \iff D^\beta \bar{J}_{\alpha(s)\dot{\alpha}(s-2)} = 0 . \tag{3.3.30}
\]

Next, in order for \( s^{(s)}_{\text{source}} \) to be invariant under the transformation \((3.2.4)\), we require the superfield \( T^{\alpha(s-1)\dot{\alpha}(s-1)} \) to satisfy

\[
\bar{D}^{(\alpha_1} T^{\alpha_2...\alpha_s)}_{\alpha(s-1)\dot{\alpha}(s-1)} = 0 \iff D^{(\alpha_1} \bar{T}^{\alpha_2...\alpha_s)}_{\alpha(s-1)\dot{\alpha}(s-1)} = 0 . \tag{3.3.31}
\]

We see that the superfields \( J^{(s)}_{\alpha(s)\dot{\alpha}(s-1)} \) and \( T^{\alpha(s-1)\dot{\alpha}(s-1)} \) are transverse linear and longitudinal linear, respectively. Finally, requiring \( s^{(s)}_{\text{source}} \) to be invariant under the \( \Psi \)-transformation \((3.2.5)\) gives the following conservation equation

\[
-\frac{1}{2} D^\beta J^{\beta\alpha(s-1)}_{\dot{\alpha}(s-1)} + S^{\alpha(s-1)\dot{\alpha}(s-1)} + \bar{T}^{\alpha(s-1)\dot{\alpha}(s-1)} = 0 . \tag{3.3.32a}
\]

and its conjugate

\[
\frac{1}{2} \bar{D}^\beta \bar{J}_{\alpha(s-1)\dot{\alpha}(s-1)} + S^{\alpha(s-1)\dot{\alpha}(s-1)} + T^{\alpha(s-1)\dot{\alpha}(s-1)} = 0 . \tag{3.3.32b}
\]

As a consequence of \((3.3.31)\), from \((3.3.32a)\) we deduce

\[
\frac{1}{4} D^2 J^{\alpha(s)\dot{\alpha}(s-1)} + D^{(\alpha_1 S^{\alpha_2...\alpha_s)}_{\alpha(s-1)\dot{\alpha}(s-1)} = 0 . \tag{3.3.33}
\]

The equations \((3.3.30)\) and \((3.3.33)\) describe the conserved current supermultiplet which corresponds to our theory in the gauge \((3.2.10)\).

Taking the sum of \((3.3.32a)\) and \((3.3.32b)\) leads to

\[
\frac{1}{2} D^\beta J^{\beta\alpha(s-1)}_{\dot{\alpha}(s-1)} + \frac{1}{2} \bar{D}^\beta \bar{J}_{\alpha(s-1)\dot{\alpha}(s-1)} + T^{\alpha(s-1)\dot{\alpha}(s-1)} - \bar{T}^{\alpha(s-1)\dot{\alpha}(s-1)} = 0 . \tag{3.3.34}
\]

The equations \((3.3.30)\), \((3.3.31)\) and \((3.3.34)\) describe the conserved current supermultiplet which corresponds to our theory in the gauge \((3.2.12)\). As a consequence of \((3.3.31)\), the conservation equation \((3.3.34)\) implies

\[
\frac{1}{2} D^{(\alpha_1 \left\{ D^\beta J^{\beta\alpha_2...\alpha_s}_{\dot{\alpha}(s-1)} + \bar{D}^\beta \bar{J}_{\alpha_2...\alpha_s)\dot{\alpha}(s-1)} \right\} + D^{(\alpha_1 T^{\alpha_2...\alpha_s)}_{\dot{\alpha}(s-1)} = 0 . \tag{3.3.35}
\]

Using our condensed notation, the transverse linear condition \((3.3.30)\) turns into

\[
\bar{D}^{(0,-1)} J^{(s,s-1)}_{s(s-1)} = 0 , \tag{3.3.36}
\]

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while the longitudinal linear condition (3.3.31) takes the form

\[ \bar{D}_{(0,1)} T_{(s-1,s-1)} = 0 . \] (3.3.37)

The conservation equation (3.3.32a) becomes

\[ -\frac{1}{2s} D_{(-1,0)} J_{(s,s-1)} + S_{(s-1,s-1)} + T_{(s-1,s-1)} = 0 . \] (3.3.38)

and (3.3.35) takes the form

\[ \frac{1}{2s} D_{(1,0)} \left\{ D_{(-1,0)} J_{(s,s-1)} + \bar{D}_{(0,-1)} J_{(s-1,s)} \right\} + D_{(1,0)} T_{(s-1,s-1)} = 0 . \] (3.3.39)

### 3.3.2.1 Examples of higher-spin supercurrents

Let us consider the Fayet-Sohnius model \[142,143\] for a free massive hypermultiplet

\[ S = \int d^4 x d^2 \theta d^2 \bar{\theta} \left( \bar{\Phi} \Phi + \bar{\Phi} \Phi - + \bar{\Phi} \Phi - \right) + \left\{ m \int d^4 x d^2 \theta \bar{\Phi} \Phi + c.c. \right\} , \] (3.3.40)

where the superfields \( \Phi_\pm \) are chiral, \( \bar{D}_\alpha \Phi_\pm = 0 \), and the mass parameter \( m \) is chosen to be positive.

In the massless case, \( m = 0 \), the fermionic higher-spin supercurrent \( J_{\alpha(s)\bar{\alpha}(s-1)} \) was first constructed in \[68\]. In our notation it reads

\[ J_{(s,s-1)} = \sum_{k=0}^{s-1} (-1)^k \binom{s-1}{k} \left\{ \binom{s}{k+1} \partial_{(1,1)}^{k} D_{(1,0)} \Phi_+ \partial_{(1,1)}^{s-k-1} \Phi_- 
- \binom{s}{k} \partial_{(1,1)}^{k} \Phi_+ \partial_{(1,1)}^{s-k-1} D_{(1,0)} \Phi_- \right\} . \] (3.3.41)

One may check that \( J_{(s,s-1)} \) obeys, for \( s > 1 \), the conservation equations

\[ D_{(-1,0)} J_{(s,s-1)} = 0, \quad \bar{D}_{(0,-1)} J_{(s,s-1)} = 0 , \] (3.3.42)

are satisfied as a consequence of the massless equations of motion, \( D^2 \Phi_\pm = 0 \).

Let us construct conserved fermionic supercurrent corresponding to the massive model (3.3.40). Assuming that \( J_{(s,s-1)} \) has the same functional form as in the massless case, eq. (3.3.41), and making use of the equations of motion

\[ -\frac{1}{4} D^2 \Phi_+ + m \bar{\Phi}_- = 0, \quad -\frac{1}{4} D^2 \Phi_- + m \bar{\Phi}_+ = 0, \] (3.3.43)

we obtain

\[ D_{(-1,0)} J_{(s,s-1)} = 2m(s + 1) \sum_{k=0}^{s-1} (-1)^{k+1} \binom{s-1}{k+1} \binom{s}{k} \]
On the other hand, for \( k \)

Imposing the first condition, we find that the coefficients must be related by

\[
\sum_{k=0}^{s-1} (-1)^{k+1} \binom{s-1}{k} \frac{k}{k+1} \partial_{(1,1)}^{s-k-1}\Phi_- + \partial_{(1,1)}^{s-k}\Phi_+ \partial_{(1,1)}^{s-k-1}\Phi_+ + 2m(s+1) \sum_{k=1}^{s-1} (-1)^{k+1} \binom{s-1}{k} \frac{k}{k+1} \partial_{(1,1)}^{k-1} \bar{D}_{(0,1)} \Phi_- \partial_{(1,1)}^{s-k-1} D_{(1,0)} \Phi_-
\]

\[
+ 2m(s+1) \sum_{k=0}^{s-2} (-1)^{k+1} \binom{s-1}{k} \frac{k}{k+1} \partial_{(1,1)}^{k} \bar{D}_{(0,1)} \Phi_+ \partial_{(1,1)}^{s-k-2} \bar{D}_{(0,1)} \Phi_+ .
\]

(3.3.44)

It can be shown that the massive supercurrent \( J_{(s,s-1)} \) also obeys \((3.3.36)\).

As the next step, we need to construct a superfield \( T_{(s-1,s-1)} \), which has the following properties: (i) it is longitudinal linear \((3.3.37)\); and (ii) it satisfies \((3.3.39)\), which is a consequence of the conservation equation \((3.3.38)\). Within these conditions, our ansatz takes the form

\[
T_{(s-1,s-1)} = \sum_{k=0}^{s-1} c_k \partial_{(1,1)}^{k} \bar{\Phi}_- \partial_{(1,1)}^{s-k-1} \Phi_-
\]

\[
+ \sum_{k=0}^{s-1} d_k \partial_{(1,1)}^{k} \Phi_+ \partial_{(1,1)}^{s-k-1} \bar{\Phi}_+
\]

\[
+ \sum_{k=1}^{s-1} \bar{f}_k \partial_{(1,1)}^{k-1} D_{(1,0)} \Phi_- \partial_{(1,1)}^{s-k-1} \bar{D}_{(0,1)} \Phi_-
\]

\[
+ \sum_{k=1}^{s-1} g_k \partial_{(1,1)}^{k-1} D_{(1,0)} \Phi_+ \partial_{(1,1)}^{s-k-1} \bar{D}_{(0,1)} \Phi_+ .
\]

(3.3.45)

Imposing the first condition, we find that the coefficients must be related by

\[
c_0 = d_0 = 0 , \quad f_k = c_k , \quad g_k = d_k .
\]

(3.3.46a)

On the other hand, for \( k = 1, 2, \ldots s - 2 \), condition (ii) yields the following recurrence relations:

\[
c_k + c_{k+1} = \frac{m(s+1)}{s} (-1)^{s+k} \binom{s-1}{k} \binom{s}{k}
\]

\[
\times \frac{1}{(k+2)(k+1)} \left\{ (2k+2-s)(s+1) - k - 2 \right\} .
\]

(3.3.46b)

\[
d_k + d_{k+1} = \frac{m(s+1)}{s} (-1)^{k} \binom{s-1}{k} \binom{s}{k}
\]

\[
\times \frac{1}{(k+2)(k+1)} \left\{ (2k+2-s)(s+1) - k - 2 \right\} .
\]

(3.3.46c)

Condition (ii) also implies that

\[
c_1 = -(-1)^{s-1} \frac{m(s^2-1)}{2} , \quad c_{s-1} = -\frac{m(s^2-1)}{s} ;
\]
\[ d_1 = -\frac{m(s^2-1)}{2}, \quad d_{s-1} = -(-1)^s \frac{m(s^2-1)}{s}. \]  

(3.3.46e)

The above relations lead to simple expressions for \( c_k \) and \( d_k \):

\[ d_k = \frac{m(s+1)}{s} \frac{k}{k+1} (-1)^k \binom{s-1}{k} \binom{s}{k}, \]  
\[ c_k = -(-1)^s d_k, \]  

(3.3.47a)

(3.3.47b)

where \( k = 1, 2, \ldots s-1 \). Now that we have already derived an expression for the trace multiplet \( T_{(s-1,s-1)} \), the superfield \( S_{(s-1,s-1)} \) can be computed using the conservation equation (3.3.38). This leads to

\[ S_{(s-1,s-1)} = -m(s+1) \sum_{k=0}^{s-1} (-1)^{k+1} \binom{s-1}{k} \binom{s}{k} \frac{1}{k+1} \times \left\{ \partial^k_{(1,1)} \bar{\Phi}_- \partial^{s-k-1}_{(1,1)} \Phi_- + (-1)^s \partial^k_{(1,1)} \bar{\Phi}_+ \partial^{s-k-1}_{(1,1)} \Phi_+ \right\}. \]  

(3.3.48)

One may verify that \( S_{(s-1,s-1)} \) is a real superfield.

### 3.4 Discussion

A novel off-shell formulation for the massless superspin-\( s \) multiplet has been proposed in this chapter. In addition, we derived consistent higher-spin supercurrents associated with the off-shell gauge theories of massless \( \mathcal{N} = 1 \) supermultiplets in Minkowski space. Several supercurrents were constructed explicitly, paying particular attention to models of free chiral scalar superfields.

Actually, the theory of a free massive chiral superfield (3.3.23) proves to possess conserved fermionic higher-spin supercurrents only for \textit{even} integer superspin \( s = 2, 4, \ldots \). Indeed, one can extract from eq. (3.3.41) (by setting \( \Phi_+ = \Phi_- = \Phi \)) the following supercurrent \( J_{(s,s-1)} \), which is a complex fermionic superfield:

\[ J_{(s,s-1)} = \sum_{k=0}^{s-1} (-1)^k \binom{s-1}{k} \left\{ \binom{s}{k+1} \partial^k_{(1,1)} D_{(1,0)} \Phi \partial^{s-k-1}_{(1,1)} \Phi \\
- \binom{s}{k} \partial^k_{(1,1)} \Phi \partial^{s-k-1}_{(1,1)} D_{(1,0)} \Phi \right\}. \]  

(3.4.1)

The above expression can be further simplified by changing the index of summation of the second term (i.e. let \( k' = s - k - 1 \)). We obtain

\[ J_{(s,s-1)} = \sum_{k=0}^{s-1} (-1)^k \binom{s-1}{k} \binom{s}{k+1} \{1 + (-1)^s\} \partial^k_{(1,1)} D_{(1,0)} \Phi \partial^{s-k-1}_{(1,1)} \Phi. \]  

(3.4.2)
This implies that \( J_{(s, s-1)} = 0 \) if \( s \) is odd. Thus, for even values of \( s \), we have

\[
J_{(s, s-1)} = 2 \sum_{k=0}^{s-1} (-1)^k \binom{s-1}{k} \binom{s}{k+1} \partial^k_{(1,1)} D_{(1,0)} \Phi \partial^{s-k-1} \Phi .
\]

(3.4.3)

The corresponding trace multiplet \( T_{(s-1, s-1)} \) is given by

\[
T_{(s-1, s-1)} = \sum_{k=0}^{s-1} c_k \partial^k_{(1,1)} \Phi \partial^{s-k-1} \bar{\Phi}
+ \sum_{k=1}^{s-1} d_k \partial^{s-k-1}_{(1,1)} D_{(1,0)} \Phi \partial^{s-k-1} D_{(0,1)} \bar{\Phi} ,
\]

(3.4.4)

with the coefficients \( c_k \) and \( d_k \) given by [3.3.47]. It may be checked that the conservation equations

\[
\bar{D}_{(0, -1)} J_{(s, s-1)} = 0 , \quad \bar{D}_{(1,0)} T_{(s-1, s-1)} = 0 ,
\]

\[
\frac{1}{2s} D_{(1,0)} \left( D_{(-1,0)} J_{(s, s-1)} + \bar{D}_{(0, -1)} \bar{J}_{(s-1, s)} \right) = - D_{(1,0)} T_{(s-1, s-1)} ,
\]

(3.4.5)

are satisfied for the even values of \( s, s = 2n \), with \( n = 1, 2, \ldots \). An alternative approach based on the superfield Noether procedure [73] was recently developed in [115–120] to study supercurrents and cubic vertices between various matter and massless higher-spin multiplets in 4D Minkowski superspace.

An interesting open question is to classify all non-conformal deformations of the higher-spin supercurrents (3.3.42), along the lines of the recent analysis of non-conformal \( \mathcal{N} = (1, 0) \) supercurrents in six dimensions [144]. Our results provide the setup required for developing a program to derive higher-spin supersymmetric models from quantum correlation functions, as an extension of the non-supersymmetric approaches pursued, e.g., in [145–147]. Another interesting project would be to study \( \mathcal{N} = 2 \) supercurrents corresponding to the off-shell massless higher-spin \( \mathcal{N} = 2 \) supermultiplets in 4D Minkowski space constructed in [67].
Chapter 4

Higher-spin supercurrents in AdS space

An interesting feature of our results in the previous chapter is the existence of a selection rule for higher-spin supercurrents in $\mathcal{N} = 1$ supersymmetric theories. We recall that in the case of a massless half-integer superspin multiplet, the bosonic supercurrent $J_{\alpha(s)\dot{\alpha}(s)}$ for a massless chiral superfield is defined for all values of $s$, while those corresponding to the massive chiral model exists only for odd $s$. The situation turns out to be different for the integer superspin case. For a single (massless or massive) chiral superfield, the fermionic supercurrent $J_{\alpha(s)\dot{\alpha}(s-1)}$ exists only for even values of $s$, yet it is defined for arbitrary $s$ in the massive hypermultiplet model. It is thus natural to look for a generalisation of these flat space results to various supersymmetric theories in 4D $\mathcal{N} = 1$ AdS superspace $\text{AdS}^{4|4}$, for instance a model of $N$ massive chiral scalar superfields with an arbitrary mass matrix. A large part of this chapter will be devoted to this analysis.

This chapter is organised as follows. In section 4.1 we review the general properties of transverse and longitudinal linear superfields. Novel off-shell gauge formulations for the massless integer superspin multiplet in AdS are presented in section 4.2. They are shown to reduce to those proposed in [63] upon partially fixing the gauge freedom. We also describe off-shell formulations (including a novel one) for the massless gravitino multiplet in AdS. In section 4.3 we introduce higher-spin supercurrent multiplets in AdS and describe improvement transformations for them. Sections 4.4 and 4.5 are devoted to the explicit constructions of higher-spin supercurrents for $N$ chiral superfields. Several nontrivial applications of the results obtained are given in section 4.6.
4.1 Linear superfields

Before we describe superfield formulations for off-shell massless higher-spin gauge multiplets in AdS\textsuperscript{4} \cite{35}, it is important to first recall the notion of transverse and longitudinal superfields \cite{93}. Complex tensor superfields $\Gamma_{\alpha(m)\dot{\alpha}(n)} := \Gamma_{\alpha_1...\alpha_n\dot{\alpha}_1...\dot{\alpha}_n} = \Gamma_{(\alpha_1...\alpha_m)(\dot{\alpha}_1...\dot{\alpha}_n)}$ and $G_{\alpha(m)\dot{\alpha}(n)}$ are called transverse linear and longitudinal linear respectively, if the constraints\footnote{Our 4D AdS notation and two-component spinor conventions correspond to \cite{35}. For concise results concerning field theories in AdS\textsuperscript{4}, see subsection 2.3.}

\[
\begin{align*}
\bar{D}^\dot{\beta} \Gamma_{\alpha(m)\dot{\alpha}(n-1)} &= 0 \, , \quad n \neq 0 \, , \quad (4.1.1a) \\
\bar{D}_{(\dot{\alpha}_1} G_{\alpha(m)\dot{\alpha}_2...\dot{\alpha}_{n+1})} &= 0 \tag{4.1.1b} \\
\end{align*}
\]

are satisfied. For $n = 0$ the latter constraint coincides with the condition of covariant chirality, $\bar{D}_{\dot{\beta}} G_{\alpha(m)} = 0$. The relations $[4.1.1]$ lead to the linearity conditions

\[
\begin{align*}
(\bar{D}^2 - 2(n + 2)\mu) \Gamma_{\alpha(m)\dot{\alpha}(n)} &= 0 \, , \quad (4.1.2a) \\
(\bar{D}^2 + 2n\mu) G_{\alpha(m)\dot{\alpha}(n)} &= 0 \, . \quad (4.1.2b) \\
\end{align*}
\]

The transverse condition (4.1.1a) is not defined for $n = 0$. However, its corollary (4.1.2a) remains consistent for the choice $n = 0$ and corresponds to complex linear superfields $\Gamma_{\alpha(m)}$ constrained by

\[
(\bar{D}^2 - 4\mu) \Gamma_{\alpha(m)} = 0 \, . \quad (4.1.3)
\]

In the family of constrained superfields $\Gamma_{\alpha(m)}$ introduced, the scalar multiplet, $m = 0$, is used most often in applications. One can define projectors $P_n^\perp$ and $P_n^\parallel$ on the spaces of transverse linear and longitudinal linear superfields respectively:

\[
\begin{align*}
P_n^\perp &= \frac{1}{4(n+1)\mu} (\bar{D}^2 + 2n\mu) \, , \quad (4.1.4a) \\
P_n^\parallel &= -\frac{1}{4(n+1)\mu} (\bar{D}^2 - 2(n + 2)\mu) \, , \quad (4.1.4b) \\
\end{align*}
\]

with the properties

\[
(\begin{array}{c}
P_n^\perp \\
P_n^\parallel \\
\end{array})^2 = \begin{array}{c} P_n^\perp \\
P_n^\parallel \\
\end{array} \, , \quad (\begin{array}{c}
P_n^\perp \\
P_n^\parallel \\
\end{array})^2 = \begin{array}{c} P_n^\parallel \\
P_n^\parallel \\
\end{array} \, , \quad P_n^\perp P_n^\parallel = P_n^\parallel P_n^\perp = 0 \, , \quad P_n^\perp + P_n^\parallel = 1 \, . \quad (4.1.5)
\]

Given a complex tensor superfield $V_{\alpha(m)\dot{\alpha}(n)}$ with $n \neq 0$, it can be represented as a sum of transverse linear and longitudinal linear multiplets,

\[
V_{\alpha(m)\dot{\alpha}(n)} = -\frac{1}{2\mu(n + 2)} \bar{D}^{\dot{\gamma}} \bar{D}_{(\dot{\alpha}_1} V_{\alpha(m)\dot{\alpha}_1...\dot{\alpha}_n)} - \frac{1}{2\mu(n + 1)} \bar{D}_{(\dot{\alpha}_1} \bar{D}^{\dot{\gamma}]} V_{\alpha(m)\dot{\alpha}_2...\dot{\alpha}_n)} \, . \tag{4.1.6}
\]
Choosing $V_{\alpha(m)\dot{\alpha}(n)}$ to be transverse linear ($\Gamma_{\alpha(m)\dot{\alpha}(n)}$) or longitudinal linear ($G_{\alpha(m)\dot{\alpha}(n)}$), the above relation gives

$$\Gamma_{\alpha(m)\dot{\alpha}(n)} = \mathcal{D}^\beta \Phi_{\alpha(m)(\dot{\beta}\dot{\alpha}_1\ldots\dot{\alpha}_n)} ;$$  \hfill (4.1.7a)

$$G_{\alpha(m)\dot{\alpha}(n)} = \mathcal{D}_{(\dot{\alpha}_1} \Psi_{\alpha(m)\dot{\alpha}_2\ldots\dot{\alpha}_n)} ;$$  \hfill (4.1.7b)

for some prepotentials $\Phi_{\alpha(m)\dot{\alpha}(n+1)}$ and $\Psi_{\alpha(m)\dot{\alpha}(n-1)}$. The constraints \[4.1.1\] hold for unconstrained $\Phi_{\alpha(m)\dot{\alpha}(n+1)}$ and $\Psi_{\alpha(m)\dot{\alpha}(n-1)}$. These prepotentials are defined modulo gauge transformations of the form:

$$\delta \xi \Phi_{\alpha(m)\dot{\alpha}(n+1)} = \mathcal{D}^\dot{\gamma} \xi_{\alpha(m)(\dot{\beta}\dot{\gamma}\dot{\alpha}_1\ldots\dot{\alpha}_{n+1})} ;$$  \hfill (4.1.8a)

$$\delta \xi \Psi_{\alpha(m)\dot{\alpha}(n-1)} = \mathcal{D}_{(\dot{\alpha}_1} \xi_{\alpha(m)\dot{\alpha}_2\ldots\dot{\alpha}_{n-1})} ;$$  \hfill (4.1.8b)

with the gauge parameters $\xi_{\alpha(m)\dot{\alpha}(n+2)}$ and $\xi_{\alpha(m)\dot{\alpha}(n-2)}$ being unconstrained.

### 4.2 Massless integer superspin multiplets

Let $s$ be a positive integer. The longitudinal formulation for the massless superspin-$s$ multiplet in AdS was realised in \[63\] in terms of the following dynamical variables

$$\mathcal{Y}^{(s)} = \{ H_{\alpha(s-1)\dot{\alpha}(s-1)} ; G_{\alpha(s)\dot{\alpha}(s)} ; \bar{G}_{\alpha(s)\dot{\alpha}(s)} \} .$$  \hfill (4.2.1)

Here $H_{\alpha(s-1)\dot{\alpha}(s-1)}$ is an unconstrained real superfield, while $G_{\alpha(s)\dot{\alpha}(s)}$ is a longitudinal linear superfield. The latter is the field strength associated with a complex unconstrained prepotential $\Psi_{\alpha(s)\dot{\alpha}(s)}$.

$$G_{\alpha_1\ldots\alpha_s\dot{\alpha}_1\ldots\dot{\alpha}_s} := \mathcal{D}_{(\dot{\alpha}_1} \Psi_{\alpha_1\ldots\alpha_s\dot{\alpha}_2\ldots\dot{\alpha}_s} \implies \mathcal{D}_{(\dot{\alpha}_1} G_{\alpha_1\ldots\alpha_s\dot{\alpha}_2\ldots\dot{\alpha}_{s+1})} = 0 .$$  \hfill (4.2.2)

The gauge freedom postulated in \[63\] is given by

$$\delta H_{\alpha(s-1)\dot{\alpha}(s-1)} = \mathcal{D}^\beta L_{\beta\alpha(s-1)\dot{\alpha}(s-1)} - \mathcal{D}^\dot{\beta} \bar{L}_{\alpha(s-1)\dot{\beta}\dot{\alpha}(s-1)} ;$$  \hfill (4.2.3a)

$$\delta G_{\beta\alpha(s-1)\dot{\beta}\dot{\alpha}(s-1)} = \frac{1}{2} \mathcal{D}_{(\beta} \mathcal{D}^{[\dot{\gamma}]} L_{\alpha(s-1)\dot{\gamma}\dot{\alpha}(s-1))} ;$$  \hfill (4.2.3b)

where the gauge parameter is $L_{\alpha(s)\dot{\alpha}(s-1)}$ is unconstrained.

The goal of this section is to reformulate the longitudinal theory by enlarging the gauge freedom \[4.2.3\] at the cost of introducing a new compensating superfield, in addition to $H_{\alpha(s-1)\dot{\alpha}(s-1)}$, $\Psi_{\alpha(s)\dot{\alpha}(s-1)}$ and $\bar{\Psi}_{\alpha(s-1)\dot{\alpha}(s)}$. In such a setting, the gauge freedom of $\Psi_{\alpha(s)\dot{\alpha}(s-1)}$ coincides with that of a superconformal multiplet of superspin-$s$ \[68\]. This new formulation will be an extension of the one given in \[2\] (and described in section 3.2) in flat superspace case.
4.2.1 New formulation

We fix an integer $s \geq 2$. Our task is to derive an AdS extension of the gauge-invariant action (3.2.8) in Minkowski superspace. The geometry of AdS$^{4,4}$ is completely determined by the covariant derivatives algebra (2.3.5). To start with, we consider the following action functional, which is a minimal lift of (3.2.8) to AdS$^{4,4}$

$$S_{(s)}^\parallel = \left( -\frac{1}{2} \right)^s \int d^4x d^2\theta d^2\bar{\theta} \ E \left\{ \frac{1}{8} H^{\alpha(s-1)\dot{\alpha}(s-1)} D^\beta \bar{D}^\dot{\beta} \mathcal{D}_\beta H_{\alpha(s-1)\dot{\alpha}(s-1)} \right\}$$

$$+ \frac{s}{s+1} H^{\alpha(s-1)\dot{\alpha}(s-1)} \left( D^\beta \bar{D}^\dot{\beta} G_{\beta\alpha(s-1)\dot{\beta}\dot{\alpha}(s-1)} - \bar{D}^\dot{\beta} D^\beta G_{\bar{\beta}\bar{\alpha}(s-1)\bar{\dot{\beta}}\dot{\alpha}(s-1)} \right)$$

$$+ 2 G^{\dot{\alpha}(s)\dot{\beta}(s)} G_{\dot{\alpha}(s)\dot{\beta}(s)} + \frac{s}{s+1} \left( G^{\dot{\alpha}(s)\dot{\beta}(s)} G_{\dot{\alpha}(s)\dot{\beta}(s)} + G^{\dot{\alpha}(s)\dot{\beta}(s)} G_{\dot{\alpha}(s)\dot{\beta}(s)} \right)$$

$$+ \frac{s-1}{4s} H^{\alpha(s-1)\dot{\alpha}(s-1)} \left( D_{\alpha_1} D_{\dot{\alpha}_1} - 2i(s-1) D_{\alpha_1} \right) \Sigma_{\alpha_2...\alpha_s \dot{\alpha}_2...\dot{\alpha}_s - 1}$$

$$+ \frac{s-2}{8s} \left( \Sigma_{\alpha(s-2)\dot{\alpha}(s-2)} D^2 \Sigma_{\alpha(s-2)\dot{\alpha}(s-2)} - \Sigma_{\alpha(s-2)\dot{\alpha}(s-2)} D^2 \Sigma_{\alpha(s-2)\dot{\alpha}(s-2)} \right)$$

$$+ \frac{1}{s^2} \Sigma_{\alpha(s-2)\dot{\alpha}(s-2)} \left( \frac{1}{2} (s^2 + 1) D^\beta \bar{D}^\dot{\beta} + i(s-1)^2 D^\beta \bar{D}^\dot{\beta} \right) \Sigma_{\beta\dot{\alpha}(s-2)\dot{\alpha}(s-2)} \right\} + \ldots \quad (4.2.4)$$

In accordance with section 3.2, our dynamical superfields consist of a complex unconstrained prepotential $\Psi_{\alpha(s-1)\dot{\alpha}(s-1)}$, a real superfield $H_{\alpha(s-1)\dot{\alpha}(s-1)}$ and a complex superfield $\Sigma_{\alpha(s-1)\dot{\alpha}(s-2)}$ constrained to be transverse linear,

$$\bar{D}^\dot{\beta} \Sigma_{\alpha(s-1)\dot{\alpha}(s-3)} = 0 \quad (4.2.5)$$

In the $s = 2$ case, for which (4.2.5) is not defined, $\Sigma_\alpha$ is instead constrained by

$$(\bar{D}^2 - 4\mu) \Sigma_\alpha = 0 \quad (4.2.6)$$

The constraint (4.2.5), or its counterpart (4.2.6) for $s = 2$, can be solved in terms of a complex unconstrained prepotential $Z_{\alpha(s-1)\dot{\alpha}(s-1)}$,

$$\Sigma_{\alpha(s-1)\dot{\alpha}(s-2)} = \bar{D}^\dot{\beta} Z_{\alpha(s-1)\dot{\beta} \dot{\alpha}(s-1)} \quad (4.2.7)$$

which is defined modulo gauge shifts

$$\delta_\xi Z_{\alpha(s-1)\dot{\alpha}(s-1)} = \bar{D}^\dot{\beta} \xi_{\alpha(s-1)\dot{\beta} \dot{\alpha}(s-1)} \quad (4.2.8)$$

Here the gauge parameter $\xi_{\alpha(s-1)\dot{\alpha}(s)}$ is unconstrained.

The gauge-invariant action in AdS is expected to differ from (4.2.4) by some $\mu$-dependent terms. These are required to ensure invariance under the linearised gauge transformations which we postulate to be of the form

$$\delta \theta_\xi \Psi_{\alpha_1...\alpha_s \dot{\alpha}_1...\dot{\alpha}_s - 1} = \frac{1}{2} D_{\alpha_1} \Psi_{\alpha_2...\alpha_s \dot{\alpha}_1...\dot{\alpha}_s - 1} + \bar{D}_{\dot{\alpha}_1} \xi_{\alpha_1...\alpha_s \dot{\alpha}_2...\dot{\alpha}_s - 1} \quad (4.2.9a)$$
\[ \delta_y H_{\alpha(s-1)\dot{\alpha}(s-1)} = \mathfrak{D}_{\alpha(s-1)\dot{\alpha}(s-1)} + \mathfrak{D}_{\alpha(s-1)\dot{\alpha}(s-1)}, \quad (4.2.9b) \]
\[ \delta_y \Sigma_{\alpha(s-1)\dot{\alpha}(s-2)} = D^\beta \mathfrak{D}_{\alpha(s-1)\dot{\beta}(s-2)} \implies \delta_y Z_{\alpha(s-1)\dot{\alpha}(s-1)} = \mathfrak{D}_{\alpha(s-1)\dot{\alpha}(s-1)}, \quad (4.2.9c) \]

with unconstrained gauge parameters \( \mathfrak{D}_{\alpha(s-1)\dot{\alpha}(s-1)} \) and \( \zeta_{\alpha(s)}(s-2) \). We note that the gauge freedom of \( \Psi_{\alpha_1...\alpha_s \dot{\alpha}_1...\dot{\alpha}_s} \) is chosen to coincide with that of the superconformal superspin-s multiplet \( \bar{\mathfrak{D}} \). The longitudinal linear superfield \( G_{\alpha(s)\dot{\alpha}(s)} \) defined by \( (4.2.2) \) is invariant under the \( \zeta \)-transformation \( (4.2.9a) \) and varies under the \( \mathfrak{D} \)-transformation as

\[ \delta_y G_{\alpha_1...\alpha_s \dot{\alpha}_1...\dot{\alpha}_s} = \frac{1}{2} \bar{D}_{(\dot{\alpha}_1} D_{(\alpha_1} \mathfrak{D}_{\alpha_2...\alpha_s)\dot{\alpha}_2...\dot{\alpha}_s). \quad (4.2.10) \]

Let us compute the variation of \( (4.2.4) \) under \( (4.2.9) \) and iteratively add certain \( \mu \)-dependent terms to achieve a gauge-invariant action. The following identities are derived from the covariant derivatives algebra \( (2.3.5) \) and prove to be useful in carrying out such calculations:

\[ D_{\alpha} D_{\beta} = \frac{1}{2} \varepsilon_{\alpha\beta} D^2 - 2\bar{\mu} M_{\alpha\beta}, \quad \bar{D}_{\alpha} \bar{D}_{\beta} = -\frac{1}{2} \varepsilon_{\alpha\beta} \bar{D}^2 + 2\mu \bar{M}_{\alpha\beta}, \quad (4.2.11a) \]
\[ D_{\alpha} D^2 = 4\bar{\mu} D^2 M_{\alpha\beta} + 4\bar{\mu} D_{\alpha}, \quad \bar{D}^2 D_{\alpha} = -4\mu \bar{D}^2 M_{\alpha\beta} - 2\bar{\mu} D_{\alpha}, \quad (4.2.11b) \]
\[ \bar{D}_{\alpha} \bar{D}^2 = 4\mu \bar{D}^2 M_{\alpha\beta} + 4\mu \bar{D}_{\alpha}, \quad \bar{D}^2 \bar{D}_{\alpha} = -4\mu \bar{D}^2 M_{\alpha\beta} - 2\mu \bar{D}_{\alpha}, \quad (4.2.11c) \]
\[ [\bar{D}^2, D_{\alpha}] = 4iD_{\alpha\beta} \bar{D}^\beta + 4\mu D_{\alpha} = 4i\bar{D}^\beta D_{\alpha\beta} - 4\mu D_{\alpha}, \quad (4.2.11d) \]
\[ [D^2, \bar{D}_{\alpha}] = -4iD_{\beta\dot{\alpha}} \bar{D}^\beta + 4\bar{\mu} \bar{D}_{\dot{\alpha}} = -4i\bar{D}^\beta \bar{D}_{\beta\dot{\alpha}} - 4\bar{\mu} \bar{D}_{\dot{\alpha}}, \quad (4.2.11e) \]

where \( D^2 = D^\alpha D_{\alpha} \) and \( \bar{D}^2 = \bar{D}_{\dot{\alpha}} \bar{D}^{\dot{\alpha}} \).

This procedure leads to the following action in AdS, which is invariant under \( (4.2.9) \) and, by construction, \( (4.2.8) \):

\[ S^\parallel_{(s)} = \left( -\frac{1}{2} \right)^s \int d^4x \alpha^2 d^2\theta d^2\bar{\theta} E \left\{ \frac{1}{8} H^{\alpha(s-1)\dot{\alpha}(s-1)} D^2 (\bar{D}^2 - 4\mu) D_{\beta} H_{\alpha(s-1)\dot{\alpha}(s-1)} \right. \]
\[ + \frac{s}{s+1} \frac{H^{\alpha(s-1)\dot{\alpha}(s-1)} \left[ D^2 \bar{D}^2 G_{\beta\alpha(s-1)} \dot{\beta}(s-1) - \bar{D}^2 \bar{D}^2 \bar{G}_{\beta\alpha(s-1)} \dot{\beta}(s-1) \right]}{2} \]
\[ + \frac{(s+1)^2}{2} \frac{\bar{\mu} M^{\alpha(s-1)\dot{\alpha}(s-1)} H_{\alpha(s-1)\dot{\alpha}(s-1)}}{4s+1} \]
\[ + \frac{s-1}{4} \frac{H^{\alpha(s-1)\dot{\alpha}(s-1)}}{4s} \left( D_{\alpha_1} \bar{D}_{\dot{\alpha}_1} - 2i(s-1) D_{\alpha_1\dot{\alpha}_1} \right) \Sigma_{\alpha_2...\alpha_{s-1}} \dot{\alpha}_2...\dot{\alpha}_{s-1} \]
\[ + \frac{1}{s} \frac{\Psi^{\alpha(s-1)\dot{\alpha}(s-1)}}{4s} \left( D_{\alpha_1} \bar{D}_{\dot{\alpha}_1} - 2i(s-1) D_{\alpha_1\dot{\alpha}_1} \right) \Sigma_{\alpha_2...\alpha_{s-1}} \dot{\alpha}_2...\dot{\alpha}_{s-1} \]
\[ - \frac{s^2+4s-1}{2s} \frac{H^{\alpha(s-1)\dot{\alpha}(s-1)}}{4s} \left( D_{\alpha_1} \bar{D}_{\dot{\alpha}_1} - 2i(s-1) D_{\alpha_1\dot{\alpha}_1} \right) \Sigma_{\alpha_2...\alpha_{s-1}} \dot{\alpha}_2...\dot{\alpha}_{s-1} \]
\[ + \frac{s^2+4s-1}{2s} \frac{H^{\alpha(s-1)\dot{\alpha}(s-1)}}{4s} \left( D_{\alpha_1} \bar{D}_{\dot{\alpha}_1} - 2i(s-1) D_{\alpha_1\dot{\alpha}_1} \right) \Sigma_{\alpha_2...\alpha_{s-1}} \dot{\alpha}_2...\dot{\alpha}_{s-1} \]
In accordance with (4.2.9b), in this gauge the residual gauge freedom is

\[ + s - \frac{1}{8s} \left( \sum_{\alpha(s-1)\bar{\alpha}(s-2)} D^2 \sum_{\alpha(s-1)\bar{\alpha}(s-2)} - \sum_{\alpha(s-2)\bar{\alpha}(s-1)} D^2 \sum_{\alpha(s-2)\bar{\alpha}(s-1)} \right) \]

\[ - \frac{1}{s^2} \sum_{\alpha(s-2)\bar{\alpha}(s-2)} \left( \frac{1}{2} (s^2 + 1) D^\beta \bar{D}_\beta + i (s - 1)^2 D^\beta \bar{D}_\beta \right) \sum_{\beta(s-2)\bar{\alpha}(s-2)} \]

\[ + \frac{\mu}{4} \left( s^2 + 4s - 1 \sum_{\alpha(s-2)\bar{\alpha}(s-1)} \bar{D}_\alpha \sum_{\alpha(s-1)\bar{\alpha}(s-2)} \sum_{\alpha(s-1)\bar{\alpha}(s-2)} \right) \]

(4.2.12)

The above action is real due to the identity

\[ D^\alpha (\bar{D}^2 - 4\mu) D_\alpha = \bar{D}_\alpha (D^2 - 4\mu) \bar{D}^\alpha . \]

(4.2.13)

In the limit of vanishing curvature of the AdS superspace (\( \mu \to 0 \)), we see that (4.2.12) reduces to (3.2.8).

The \( \mathfrak{W} \)-gauge freedom (4.2.9) allows us to gauge away \( \Sigma_{\alpha(s-1)\bar{\alpha}(s-2)} \),

\[ \Sigma_{\alpha(s-1)\bar{\alpha}(s-2)} = 0 . \]

(4.2.14)

In this gauge, the action (4.2.12) reduces to that describing the longitudinal formulation for the massless superspin-\( s \) multiplet [63]. The gauge condition (4.2.14) does not fix completely the \( \mathfrak{W} \)-gauge freedom. The residual gauge transformations are generated by

\[ \mathfrak{W}_{\alpha(s-1)\bar{\alpha}(s-1)} = D^\beta L_{(\bar{\beta}\alpha_1...\alpha_{s-1})\bar{\alpha}(s-1)} \]

(4.2.15)

with \( L_{\alpha(s)\bar{\alpha}(s-1)} \) being an unconstrained superfield. With this expression for \( \mathfrak{W}_{\alpha(s-1)\bar{\alpha}(s-1)} \), the gauge transformations (4.2.9a) and (4.2.9b) coincide with (4.2.3). Thus, the action (4.2.12) indeed provides an off-shell formulation for the massless superspin-\( s \) multiplet in AdS\(^{4|4} \).

Alternatively, one can impose a gauge fixing

\[ H_{\alpha(s-1)\bar{\alpha}(s-1)} = 0 . \]

(4.2.16)

In accordance with (4.2.9b), in this gauge the residual gauge freedom is

\[ \mathfrak{W}_{\alpha(s-1)\bar{\alpha}(s-1)} = i \mathfrak{M}_{\alpha(s-1)\bar{\alpha}(s-1)} ; \quad \mathfrak{R}_{\alpha(s-1)\bar{\alpha}(s-1)} = \mathfrak{M}_{\alpha(s-1)\bar{\alpha}(s-1)} . \]

(4.2.17)

The gauge-invariant action (4.2.12) includes a single term which involves the prepotential \( \Psi_{\alpha(s)\bar{\alpha}(s-1)} \) and not the field strength \( G_{\alpha(s)\bar{\alpha}(s)} \), the latter being defined by (4.2.2) and invariant under the \( \zeta \)-transformation (4.2.9a). This is actually a BF term, for it can be written in two different forms

\[ \frac{1}{s} \int d^4x d^2\theta d^2\bar{\theta} E \Psi^{\alpha(s)\bar{\alpha}(s-1)} \left( \bar{D}_{\alpha_1} D_{\bar{\alpha}_1} - 2i(s - 1)D_{\alpha_1\bar{\alpha}_1} \right) \Sigma_{\alpha_2...\alpha_s\bar{\alpha}_2...\bar{\alpha}_{s-1}} \]

\[ = - \frac{1}{s + 1} \int d^4x d^2\theta d^2\bar{\theta} E G^{\alpha(s)\bar{\alpha}(s)} \left( \bar{D}_{\alpha_1} D_{\alpha_1} + 2i(s + 1)D_{\alpha_1\bar{\alpha}_1} \right) Z_{\alpha_2...\alpha_s\bar{\alpha}_2...\bar{\alpha}_{s}} . \]
The former makes the gauge symmetry (4.2.8) manifestly realised, while the latter turns the ζ-transformation (4.2.9a) into a manifest symmetry. Making use of (4.2.18) leads to a different representation for the action (4.2.12). It is

\[
S^\parallel_{(s)} = \left( -\frac{1}{2} \right)^s \int d^4x d^2\theta d^2\bar{\theta} \left\{ \frac{1}{8} H^\alpha(s-1)\dot{\alpha}(s-1) D^\beta(D^2 - 4\mu) D_\beta H_{\alpha(s-1)\dot{\alpha}(s-1)} \right. \\
+ \frac{s}{s + 1} H^\alpha(s-1)\dot{\alpha}(s-1) \left( D^\beta\bar{D}^\beta G_{\beta\alpha(s-1)\dot{\beta}\dot{\alpha}(s-1)} - \bar{D}^\beta D^\beta \bar{G}_{\beta\alpha(s-1)\dot{\beta}\dot{\alpha}(s-1)} \right) \\
+ \frac{(s + 1)^2}{2} \bar{\mu} H^\alpha(s-1)\dot{\alpha}(s-1) \\
+ 2\bar{G}^\alpha(s)\dot{\alpha}(s) G_{\alpha(s)\dot{\alpha}(s)} + \frac{s}{s + 1} \left( G^\alpha(s)\dot{\alpha}(s) G_{\alpha(s)\dot{\alpha}(s)} + \bar{G}^\alpha(s)\dot{\alpha}(s) \bar{G}_{\alpha(s)\dot{\alpha}(s)} \right) \\
+ \frac{s - 1}{4s} H^\alpha(s-1)\dot{\alpha}(s-1) \left( D_{\alpha 1} \bar{D}\bar{\alpha}_1 + 2i(s + 1) D_{\alpha 1} \dot{\alpha} \right) \bar{Z}_{a_2...a_{s-1}} \\
- \frac{1}{s + 1} \left( G^\alpha(s-1) \left( D_{\alpha 1} \bar{D}\bar{\alpha}_1 + 2i(s + 1) D_{\alpha 1} \dot{\alpha} \right) \bar{Z}_{a_2...a_{s-1}} \right) \\
- \frac{s^2 + 4s - 1}{2s} \left( H^\alpha(s-1)\dot{\alpha}(s-1) D_{\alpha 1} \Sigma_{a_2...a_{s-1}} \right) \\
+ \bar{\mu} \left( \frac{s^2 + 4s - 1}{2s} \left( H^\alpha(s-1)\dot{\alpha}(s-1) D_{\alpha 1} \Sigma_{a_2...a_{s-1}} \right) \right) \\
+ \frac{1}{8s} \left( \Sigma^\alpha(s-1)\dot{\alpha}(s-2) D^2 \Sigma_{a(s-1)}\dot{\alpha}(s-2) - \Sigma^\alpha(s-2)\dot{\alpha}(s-1) D^2 \Sigma_{a(s-2)}\dot{\alpha}(s-1) \right) \\
- \frac{1}{s^2} \Sigma^\alpha(s-2)\dot{\alpha}(s-2) \left( \frac{1}{2} (s^2 + 1) D^\beta \bar{D}^\beta + i(s - 1)^2 D^\beta \bar{D}^\beta \right) \Sigma_{\beta\alpha(s-2)}\dot{\alpha}(s-2) \\
+ \bar{\mu} \left( \frac{s^2 + 4s - 1}{4s} \Sigma^\alpha(s-2)\dot{\alpha}(s-1) \right) \\
+ \bar{\mu} \left( \frac{s^2 + 4s - 1}{4s} \Sigma^\alpha(s-1)\dot{\alpha}(s-2) \right) \right\}. \tag{4.2.19}
\]

4.2.2 Dual formulation

By analogy with the flat superspace case, the action (4.2.19) can be reformulated in terms of a transverse linear superfield by applying the duality transformation [63]. Let us associate with our theory (4.2.19) the following first-order action

\[
S_{\text{first-order}} = S^\parallel_{(s)}[U, \bar{U}, H, Z, \bar{Z}] \\
+ \left\{ \left( -\frac{1}{2} \right)^s \int d^4x d^2\theta d^2\bar{\theta} \left( \frac{2}{s + 1} \Gamma^\alpha(s)\dot{\alpha}(s) U_{\alpha(s)\dot{\alpha}(s)} + \text{c.c.} \right) \right\}, \tag{4.2.20}
\]

where \(S^\parallel_{(s)}[U, \bar{U}, H, Z, \bar{Z}]\) is obtained from the action (4.2.19) by replacing \(G^\alpha(s)\dot{\alpha}(s)\) with an unconstrained complex superfield \(U_{\alpha(s)\dot{\alpha}(s)}\). The Lagrange multiplier \(\Gamma^\alpha(s)\dot{\alpha}(s)\) is transverse linear,

\[
\bar{D}^\beta \Gamma_{\alpha(s)\dot{\alpha}(s)\dot{\alpha}_1...\dot{\alpha}_{s-1}} = 0. \tag{4.2.21}
\]
We note that the specific normalisation of the Lagrange multiplier in (4.2.20) is chosen to match that of [63]. Varying [4.2.20] with respect to the Lagrange multiplier and taking into account the constraint [4.2.21] yields \( U_{\alpha(s)\dot{\alpha}(s)} = G_{\alpha(s)\dot{\alpha}(s)} \). As a result, \( S_{\text{first-order}} \) turns into the original action (4.2.19). On the other hand, we can eliminate the auxiliary superfields \( U_{\alpha(s)\dot{\alpha}(s)} \) and \( \bar{U}_{\alpha(s)\dot{\alpha}(s)} \) from (4.2.20) using their equations of motion. This leads to the dual action

\[
S_{\text{s}(s)}^\perp = \left( -\frac{1}{2} \right)^s \int \! d^4 x d^2 \theta d^2 \bar{\theta} \, E \left\{ -\frac{1}{8} \, H^{\alpha(s-1)\bar{\alpha}(s-1)} \bar{D}^\beta (\bar{D}^2 - 4\mu) \bar{D}_\beta H^{\alpha(s-1)\bar{\alpha}(s-1)}
\right.
\]

\[ + \frac{1}{8} \frac{s^2}{(s+1)(2s+1)} \left[ \bar{D}^\beta, \bar{D}^\bar{\beta} \right] H^{\alpha(s-1)\bar{\alpha}(s-1)} \left[ \bar{D}_\beta, \bar{D}_\bar{\beta} \right] H^{\alpha(s-1)\bar{\alpha}(s-1)} \]

\[ + \frac{1}{2} \frac{s^2}{s+1} \bar{D}^\beta \bar{H}^{\alpha(s-1)\bar{\alpha}(s-1)} \bar{D}_\beta H^{\alpha(s-1)\bar{\alpha}(s-1)} \]

\[ - \frac{(s+1)^2}{2} \bar{\mu} H^{\alpha(s-1)\bar{\alpha}(s-1)} H^{\alpha(s-1)\bar{\alpha}(s-1)} \]

\[ + \frac{2s}{2s+1} \bar{H}^{\alpha(s-1)\bar{\alpha}(s-1)} \bar{D}^\beta \left( \bar{\Gamma}_{\beta \alpha(s-1)\bar{\beta}} H^{\alpha(s-1)\bar{\alpha}(s-1)} - \bar{\Gamma}_{\beta \alpha(s-1)\bar{\beta}} \bar{H}^{\alpha(s-1)\bar{\alpha}(s-1)} \right) \]

\[ + \frac{2}{2s+1} \bar{\Gamma}^{\alpha(s)} \Gamma_{\alpha(s)\dot{\alpha}(s)} - \frac{s}{(s+1)(2s+1)} \left( \bar{\Gamma}^{\alpha(s)\dot{\alpha}(s)} \Gamma_{\alpha(s)\dot{\alpha}(s)} + \bar{\Gamma}^{\alpha(s)\dot{\alpha}(s)} \Gamma_{\alpha(s)\dot{\alpha}(s)} \right) \]

\[ - \frac{s-1}{2(2s+1)} \bar{H}^{\alpha(s-1)\bar{\alpha}(s-1)} \left( \bar{D}_{\alpha_1} \bar{D}^\beta \bar{\Sigma}_{\alpha_2...\alpha_{s-1} \bar{\alpha}} H^{\alpha(s-1)\bar{\alpha}(s-1)} - \bar{D}_{\alpha_1} \bar{D}^2 \bar{\Sigma}_{\alpha(s-1)\bar{\alpha}_2...\bar{\alpha}_{s-1}} \right) \]

\[ + \frac{1}{2(2s+1)} \bar{H}^{\alpha(s-1)\bar{\alpha}(s-1)} \left( \bar{D}^2 \bar{D}_{\alpha_1} \bar{\Sigma}_{\alpha(s-1)\bar{\alpha}_2...\bar{\alpha}_{s-1}} - \bar{D}^2 \bar{D}_{\alpha_1} \bar{\Sigma}_{\alpha_2...\alpha_{s-1} \bar{\alpha}} \right) \]

\[ - \frac{i}{s} \frac{(s-1)^2}{2s+1} \bar{H}^{\alpha(s-1)\bar{\alpha}(s-1)} \bar{D}_{\alpha_1} \bar{\Sigma} \left( \bar{D}^3 \bar{\Sigma}_{\alpha_2...\alpha_{s-1} \bar{\alpha}_2...\bar{\alpha}_{s-1}} + \bar{D}^3 \bar{\Sigma}_{\alpha_2...\alpha_{s-1} \bar{\alpha}_2...\bar{\alpha}_{s-1}} \right) \]

\[ + \mu \frac{(s+2)(s+1)}{2s+1} \bar{H}^{\alpha(s-1)\bar{\alpha}(s-1)} \bar{D}_{\alpha_1} \bar{\Sigma}_{\alpha_2...\alpha_{s-1} \bar{\alpha}} \]

\[ - \bar{\mu} \frac{(s+2)(s+1)}{2s+1} \bar{H}^{\alpha(s-1)\bar{\alpha}(s-1)} \bar{D}_{\alpha_1} \bar{\Sigma}_{\alpha(s-1)\bar{\alpha}_2...\bar{\alpha}_{s-1}} \]

\[ - \frac{s-1}{8s} \left( \bar{\Sigma}^{\alpha(s-1)\bar{\alpha}(s-2)} \bar{D}^2 \bar{\Sigma}_{\alpha(s-1)\bar{\alpha}(s-2)} - \bar{\Sigma}^{\alpha(s-2)\bar{\alpha}(s-1)} \bar{D}^2 \bar{\Sigma}_{\alpha(s-2)\bar{\alpha}(s-1)} \right) \]

\[ + \frac{1}{s^2} \bar{\Sigma}^{\alpha(s-2)\bar{\alpha}(s-2)} \left( \bar{D}^3 \bar{\Sigma}_{\bar{\alpha}(s-2)\bar{\alpha}(s-2)} + i(s-1)^2 \bar{D}^3 \bar{\Sigma}_{\bar{\alpha}(s-2)\bar{\alpha}(s-2)} \right) \]

\[ - \mu \frac{s^2 + 4s - 1}{4s} \bar{\Sigma}_{\alpha(s-1)\bar{\alpha}(s-2)} \bar{\Sigma}_{\alpha(s-1)\bar{\alpha}(s-2)} \]

\[ - \bar{\mu} \frac{s^2 + 4s - 1}{4s} \bar{\Sigma}_{\alpha(s-1)\bar{\alpha}(s-2)} \bar{\Sigma}_{\alpha(s-1)\bar{\alpha}(s-2)} \right\}, \tag{4.2.22} \]

where we have defined

\[
\Gamma_{\alpha(s)\dot{\alpha}(s)} = \Gamma_{\alpha(s)\dot{\alpha}(s)} - \frac{1}{2} \bar{D}_{\alpha_1} \bar{D}_{\alpha_1} Z_{\alpha_2...\alpha_s \bar{\alpha}_2...\bar{\alpha}_s} - i(s+1) \bar{D}_{\alpha_1} \bar{\Sigma}_{\alpha_2...\alpha_s \bar{\alpha}_2...\bar{\alpha}_s} \tag{4.2.23} \]

The first-order model introduced is equivalent to the original theory (4.2.19). The action (4.2.20) is invariant under the gauge \( \xi \)-transformation (4.2.8) which acts on \( U_{\alpha(s)\dot{\alpha}(s)} \).
and \( \Gamma_{\alpha(s)} \dot{\alpha}(s) \) by the rule
\[
\delta \xi U_{\alpha(s)} \dot{\alpha}(s) = 0 , \tag{4.2.24a}
\]
\[
\delta \xi \Gamma_{\alpha(s)} \dot{\alpha}(s) = \bar{D}^\beta \left\{ \frac{s + 1}{2(s + 2)} \bar{D}_{(\alpha_1 \xi_{\alpha_2...\alpha_s}) \dot{\alpha}_1...\dot{\alpha}_s} + i(s + 1) \bar{D}_{(\alpha_1 \beta \xi_{\alpha_2...\alpha_s}) \dot{\alpha}_1...\dot{\alpha}_s} \right\} . \tag{4.2.24b}
\]

Here we point out that \( \Gamma_{\alpha(s)} \dot{\alpha}(s) \) is invariant under the gauge transformations (4.2.8) and (4.2.24b). The first-order action (4.2.20) is also invariant under the gauge \( \mathfrak{V} \)-transformation (4.2.9b) and (4.2.9c), which acts on \( U_{\alpha(s)} \dot{\alpha}(s) \) and \( \Gamma_{\alpha(s)} \dot{\alpha}(s) \) as
\[
\delta \mathfrak{V} U_{\alpha(s)} \dot{\alpha}(s) = \frac{1}{2} \bar{D}_{(\dot{\alpha}_1 \mathfrak{V}_{\alpha_2...\alpha_s}) \dot{\alpha}_1...\dot{\alpha}_s} , \tag{4.2.25a}
\]
\[
\delta \mathfrak{V} \Gamma_{\alpha(s)} \dot{\alpha}(s) = 0 . \tag{4.2.25b}
\]

The \( \mathfrak{V} \)-gauge freedom in (4.2.9c) may be used to impose the condition
\[
Z_{\alpha(s-1)} \dot{\alpha}(s-1) = 0 . \tag{4.2.26}
\]

As a result, the action (4.2.22) reduces to that describing the transverse formulation for the massless superspin-\( s \) multiplet [63]. The gauge condition (4.2.26) is preserved by residual local \( \mathfrak{V} - \) and \( \xi \)-transformations of the form
\[
\bar{D}^\beta \xi_{\alpha(s-1)\beta\dot{\alpha}(s-1)} + \bar{\mathfrak{V}}_{\alpha(s-1)} \dot{\alpha}(s-1) = 0 . \tag{4.2.27}
\]

Making use of the parametrisation (4.2.15), the residual gauge freedom is
\[
\delta H_{\alpha(s-1)\dot{\alpha}(s-1)} = \bar{D}^\beta L_{\beta\alpha(s-1)\dot{\alpha}(s-1)} - \bar{D}^\beta \bar{L}_{\alpha(s-1)\beta\dot{\alpha}(s-1)} , \tag{4.2.28a}
\]
\[
\delta \Gamma_{\alpha(s)} \dot{\alpha}(s) = \frac{s + 1}{2(s + 2)} \bar{D}^\beta \left\{ \bar{D}_{(\alpha_1 \beta \xi_{\alpha_2...\alpha_s}) \dot{\alpha}_1...\dot{\alpha}_s} + 2i(s + 2) \bar{D}_{(\alpha_1 \beta \xi_{\alpha_2...\alpha_s}) \dot{\alpha}_1...\dot{\alpha}_s} \right\} . \tag{4.2.28b}
\]

This is exactly the gauge symmetry of the transverse formulation for the massless superspin-\( s \) multiplet [63].

### 4.2.3 Models for the massless gravitino multiplet in AdS

The massless gravitino multiplet (\textit{i.e.} the massless superspin-1 multiplet) was excluded from the above consideration. Here we will fill the gap.

The (generalised) longitudinal formulation for the gravitino multiplet is described by the action
\[
S^\parallel_{\text{GM}} = - \int \, d^4 x d^2 \theta d^2 \bar{\theta} E \left\{ \frac{1}{16} HD^\alpha (\bar{D}^2 - 4 \mu) D_\alpha H + \frac{1}{4} H (D^\alpha D^\beta G_{\alpha \dot{\alpha}} - \bar{D}^\alpha D^\beta \bar{G}_{\alpha \dot{\alpha}}) \right. \\
+ \bar{G}^{\alpha \dot{\alpha}} G_{\alpha \dot{\alpha}} + \frac{1}{4} (G^{\alpha \dot{\alpha}} G_{\alpha \dot{\alpha}} + \bar{G}^{\alpha \dot{\alpha}} \bar{G}_{\alpha \dot{\alpha}})
\]

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\[ +|\mu|^2 \left( H - \frac{\Phi}{\mu} - \frac{\bar{\Phi}}{\bar{\mu}} \right)^2 + \left( \frac{\Phi}{\mu} + \frac{\bar{\Phi}}{\bar{\mu}} \right) \left( \mu D^a \Psi_\alpha + \bar{\mu} \bar{D}_\alpha \bar{\Psi}^\alpha \right) \] , \hspace{1cm} (4.2.29a)

where \( \Phi \) is a chiral scalar superfield, \( \bar{D}_\alpha \Phi = 0 \), and

\[ G_{\alpha \dot{\alpha}} = \bar{D}_\alpha \Psi_\alpha , \quad \bar{G}_{\alpha \dot{\alpha}} = - \bar{D}_\alpha \bar{\Psi}^\alpha . \] \hspace{1cm} (4.2.29b)

This action is invariant under gauge transformations of the form

\[ \delta H = \Psi + \bar{\Psi} , \] \hspace{1cm} (4.2.30a)

\[ \delta \Psi_\alpha = - \frac{1}{2} \bar{D}_\alpha \Psi + \eta_\alpha , \quad \bar{D}_\alpha \eta_\alpha = 0 , \] \hspace{1cm} (4.2.30b)

\[ \delta \Phi = - \frac{1}{4} (\bar{D}^2 - 4 \mu) \bar{\Psi} . \] \hspace{1cm} (4.2.30c)

This is one of the two models for the massless gravitino multiplet in AdS introduced in \[100\]. In a flat superspace limit, the action reduces to that given in \[137\]. Imposing the gauge condition \( \Phi = 0 \) reduces the action (4.2.29) to the original longitudinal formulation for the massless gravitino multiplet in AdS \[63\].

The action (4.2.29) involves the chiral scalar \( \Phi \) and its conjugate only in the combination \( (\varphi + \bar{\varphi}) \), where \( \varphi = \Phi/\mu \). This means that the model (4.2.29) possesses a dual formulation realised in terms of a real linear superfield \( L \),

\[ (\bar{D}^2 - 4 \mu) L = 0 , \quad \bar{L} = L . \] \hspace{1cm} (4.2.31)

The dual model is described by the action \[100\]

\[ S_{\text{GM}} = - \int d^4 x d^2 \theta d^2 \bar{\theta} E \left\{ \frac{1}{16} H D^a (\bar{D}^2 - 4 \mu) D_a H + \frac{1}{4} H (D^a \bar{D}^\dot{a} G_{a \dot{\alpha}} - \bar{D}^\dot{a} D^a \bar{G}_{a \dot{\alpha}}) + \bar{G}^{a \dot{\alpha}} G_{a \dot{\alpha}} + \frac{1}{4} (G^{a \dot{\alpha}} G_{a \dot{\alpha}} + \bar{G}^{a \dot{\alpha}} \bar{G}_{a \dot{\alpha}}) + |\mu|^2 H^2 \right. \]
\[ \left. - \frac{1}{4} \left( 2 |\mu| H + L - \frac{\mu}{|\mu|} D^a \Psi_\alpha - \frac{\bar{\mu}}{|\mu|} \bar{D}_\alpha \bar{\Psi}^\alpha \right)^2 \right\} . \] \hspace{1cm} (4.2.32)

This action is invariant under the gauge transformations (4.2.30a), (4.2.30b) and

\[ \delta L = \frac{1}{|\mu|} \left( \mu D^a \eta_\alpha + \bar{\mu} \bar{D}_\alpha \bar{\eta}^\dot{\alpha} \right) . \] \hspace{1cm} (4.2.33)

In a flat superspace limit, the action (4.2.32) reduces to that given in \[140\].

In Minkowski superspace, there exists one more dual realisation for the massless gravitino multiplet model \[2\], which is obtained by performing a Legendre transformation converting \( \Phi \) into a complex linear superfield. This formulation cannot be lifted to the AdS case, the reason being the fact that the action (4.2.29) involves the chiral scalar \( \Phi \) and its conjugate only in the combination \( (\varphi + \bar{\varphi}) \), where \( \varphi = \Phi/\mu \).
The dependence on $\Psi_\alpha$ and $\bar{\Psi}_{\dot{\alpha}}$ in the last term of (4.2.29) can be expressed in terms of $G_{\alpha\dot{\alpha}}$ and $\bar{G}_{\alpha\dot{\alpha}}$ if we introduce a complex unconstrained prepotential $U$ for $\Phi$ in the standard way

$$\Phi = -\frac{1}{4}(\bar{D}^2 - 4\mu)U \ . \quad (4.2.34)$$

Then making use of (4.2.11d) gives

$$\int d^4x d^2\theta d^2\bar{\theta} E \Phi \bar{D}^{\alpha} \Psi_\alpha = -\int d^4x d^2\theta d^2\bar{\theta} E G^{\alpha\dot{\alpha}} \left(\frac{1}{4} \bar{D}_\alpha D_\alpha + i D_{\alpha\dot{\alpha}}\right) U \ . \quad (4.2.35)$$

Since the resulting action depends on $G_{\alpha\dot{\alpha}}$ and $\bar{G}_{\alpha\dot{\alpha}}$, we can introduce a dual formulation for the theory that is obtained turning $G_{\alpha\dot{\alpha}}$ and $\bar{G}_{\alpha\dot{\alpha}}$ into a transverse linear superfield

$$\Gamma_{\alpha\dot{\alpha}} = \bar{D}^{\dot{\beta}} \Phi_{\alpha\dot{\alpha} \dot{\beta}} \ , \quad \Phi_{\alpha\dot{\beta}} = \Phi_{\alpha \dot{\alpha} \dot{\beta}} \quad (4.2.36)$$

and its conjugate using the scheme described in [63]. The resulting action is

$$S_{\text{GM}}^\perp = \int d^4x d^2\theta d^2\bar{\theta} E \left\{ -\frac{1}{16} H D^{\alpha}(\bar{D}^2 - 4\mu)D_\alpha H 
+ \frac{1}{96} [D^\alpha, \bar{D}^{\dot{\alpha}}] H [D_\alpha, \bar{D}_{\alpha}] H + \frac{1}{8} D^{\alpha\dot{\alpha}} H D_{\alpha\dot{\alpha}} H 
+ \frac{1}{3} \bar{\Gamma}^{\alpha\dot{\alpha}} \Gamma_{\alpha\dot{\alpha}} - \frac{1}{12} \left( \Gamma^{\alpha\dot{\alpha}} \Gamma_{\alpha\dot{\alpha}} + \bar{\Gamma}^{\alpha\dot{\alpha}} \bar{\Gamma}_{\alpha\dot{\alpha}} \right) + \frac{i}{3} \left( \bar{\Gamma}^{\alpha\dot{\alpha}} - \Gamma^{\alpha\dot{\alpha}} \right) D_{\alpha\dot{\alpha}} H 
- \frac{1}{6} \Phi D^2 H - \frac{1}{6} \bar{\Phi} \bar{D}^2 H - |\mu|^2 \left( H - \frac{\Phi}{\mu} - \bar{\Phi} \right)^2 \right\} \ , \quad (4.2.37)$$

where we have defined

$$\Gamma_{\alpha\dot{\alpha}} := \Gamma_{\alpha\dot{\alpha}} - \frac{1}{2} D_\alpha D_\alpha U - 2i D_{\alpha\dot{\alpha}} U \ . \quad (4.2.38)$$

The action (4.2.37) is invariant under the following gauge transformations

$$\delta_\xi U = D_\alpha \xi^\alpha \ , \quad (4.2.39a)$$
$$\delta_\xi \Gamma_{\alpha\dot{\alpha}} = -\frac{1}{3} \bar{D}^{\dot{\beta}} \left( D_{(\beta} D_{\alpha \dot{\beta} \dot{\alpha})} + 6i D_{\alpha(\beta \dot{\alpha}\dot{\beta})} \right) \ . \quad (4.2.39b)$$

Both $\Phi$ and $\Gamma_{\alpha\dot{\alpha}}$ are invariant under $\xi$-gauge transformations. The action (4.2.37) is also invariant under the gauge transformations (4.2.30a), (4.2.30c) and

$$\delta_\mathcal{G} U = \bar{\mathcal{G}} \ , \quad (4.2.40a)$$
$$\delta_\mathcal{G} \Gamma_{\alpha\dot{\alpha}} = 0 \ . \quad (4.2.40b)$$

Imposing the gauge condition $U = 0$ reduces the action (4.2.37) to the original transverse formulation for the massless gravitino multiplet in AdS [63].

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4.3 Higher-spin supercurrents

In this section we introduce higher-spin supercurrent multiplets in AdS. First, we recall the structure of the gauge superfields in terms of which the massless superspin-$(s+1/2)$ multiplet $(s = 2, 3, \ldots)$ are described \[63\].

4.3.1 Massless half-integer superspin multiplets

For a massless superspin-$(s+1/2)$ multiplet in AdS, there exist two dually equivalent off-shell formulations (i.e. transverse and longitudinal), which were first constructed in \[63\]. The corresponding dynamical variables are \[63\]

\[
\mathcal{V}^{\perp}_{s+1/2} = \left\{ H_{\alpha(s)} \hat{a}(s) , \; \Gamma_{\alpha(s-1)\dot{\alpha}(s-1)} , \; \bar{\Gamma}_{\alpha(s-1)\dot{\alpha}(s-1)} \right\} , \tag{4.3.1a}
\]

\[
\mathcal{V}^{\parallel}_{s+1/2} = \left\{ H_{\alpha(s)} \hat{a}(s) , \; G_{\alpha(s-1)\dot{\alpha}(s-1)} , \; \bar{G}_{\alpha(s-1)\dot{\alpha}(s-1)} \right\} . \tag{4.3.1b}
\]

Here $H_{\alpha(s)} \hat{a}(s)$ is a real unconstrained superfield. The complex superfields $\Gamma_{\alpha(s-1)\dot{\alpha}(s-1)}$ and $G_{\alpha(s-1)\dot{\alpha}(s-1)}$ are transverse and longitudinal superfields, respectively, \[4.3.2a\]

\[
\bar{\mathcal{D}}^{\dot{\beta}} \Gamma_{\alpha(s-1)\dot{\beta}(s-2)} = 0 , \tag{4.3.2a}
\]

\[
\bar{\mathcal{D}}_{(\dot{\alpha}_1} \; G_{\alpha(s-1)\dot{\alpha}_2\ldots\dot{\alpha}_s)} = 0 . \tag{4.3.2b}
\]

These constraints are solved in terms of unconstrained prepotentials as follows:

\[
\Gamma_{\alpha(s-1)\dot{\alpha}(s-1)} = \bar{\mathcal{D}}^{\dot{\beta}} \Phi_{\alpha(s-1)} (\dot{\beta}\dot{\alpha}_1\ldots\dot{\alpha}_{s-1}) , \tag{4.3.3a}
\]

\[
G_{\alpha(s-1)\dot{\alpha}(s-1)} = \bar{\mathcal{D}}_{(\dot{\alpha}_1} \; \Psi_{\alpha(s-1)} \dot{\alpha}_2\ldots\dot{\alpha}_{s-1}) . \tag{4.3.3b}
\]

The prepotentials are defined modulo gauge transformations of the form:

\[
\delta^{\xi} \Phi_{\alpha(s-1)} (\dot{\beta}\dot{\alpha}_1\ldots\dot{\alpha}_s) = \mathcal{D}^{\dot{\beta}} \xi_{\alpha(s-1)} (\dot{\beta}\dot{\alpha}_1\ldots\dot{\alpha}_s) , \tag{4.3.4a}
\]

\[
\delta^{\xi} \Psi_{\alpha(s-1)} (\dot{\alpha}_2\ldots\dot{\alpha}_{s-2}) = \mathcal{D}_{(\dot{\alpha}_1} \; \xi_{\alpha(s-1)} \dot{\alpha}_2\ldots\dot{\alpha}_{s-2}) , \tag{4.3.4b}
\]

with the gauge parameters $\xi_{\alpha(s-1)} (\dot{\alpha}(s+1))$ and $\xi_{\alpha(s-1)} (\dot{\alpha}(s-3))$ being unconstrained.

The gauge transformations of the superfields $H$, $\Gamma$ and $G$ are

\[
\delta_{\Lambda} H_{\alpha_1\ldots\alpha_s \dot{\alpha}_1\ldots\dot{\alpha}_s} = \mathcal{D}_{(\dot{\alpha}_1} \; \Lambda_{\alpha_1\ldots\alpha_s \dot{\alpha}_2\ldots\dot{\alpha}_s)} - \mathcal{D}_{(\dot{\alpha}_1} \; \bar{\Lambda}_{\alpha_1\alpha_2\ldots\alpha_s \dot{\alpha}_1\ldots\dot{\alpha}_s)} , \tag{4.3.5a}
\]

\[
\delta_{\Lambda} \Gamma_{\alpha_1\ldots\alpha_{s-1} \dot{\alpha}_1\ldots\dot{\alpha}_{s-1}} = - \frac{s}{2(s+1)} \mathcal{D}^{\dot{\beta}} \mathcal{D}^{\dot{\beta}} \mathcal{D} (\dot{\alpha}_{s-1}) \dot{\beta} \dot{\alpha}(s-1) \]

\[
- \frac{1}{4} \mathcal{D}^{\dot{\beta}} \mathcal{D}^{\dot{\beta}} \bar{\Lambda}_{\alpha_1\ldots\alpha_{s-1} \dot{\beta} \dot{\alpha}_1\ldots\dot{\alpha}_{s-1}} \]

\[
- \frac{1}{2} \mathcal{D} (\dot{\alpha}_{s-1}) \mathcal{D}^{\dot{\beta}} \bar{\Lambda}_{\alpha_1\ldots\alpha_{s-1} \dot{\beta} \dot{\alpha}_1\ldots\dot{\alpha}_{s-1}} , \tag{4.3.5b}
\]

\[
\delta_{\Lambda} G_{\alpha_1\ldots\alpha_{s-1} \dot{\alpha}_1\ldots\dot{\alpha}_{s-1}} = - \frac{1}{2} \mathcal{D}_{(\dot{\alpha}_1} \mathcal{D}^{\dot{\beta}} \mathcal{D}^{\dot{\beta}} \bar{\Lambda}_{\alpha_1\ldots\alpha_{s-1} \dot{\beta}} \dot{\alpha}_1\ldots\dot{\alpha}_{s-1} \dot{\beta} \dot{\alpha} .
\]
Here the gauge parameter $\Lambda_{\alpha_1...\alpha_s} = \Lambda_{(\alpha_1...\alpha_s)(\hat{\alpha}_1...\hat{\alpha}_{s-1})}$ is unconstrained. The symmetrisation in (4.3.5c) is extended only to the indices $\hat{\alpha}_1,\hat{\alpha}_2,\ldots,\hat{\alpha}_{s-1}$. It follows from (4.3.5b) and (4.3.5c) that the transformation laws of the prepotentials $\Phi_{\alpha(s-1)\hat{\alpha}(s)}$ and $\Psi_{\alpha(s-1)\hat{\alpha}(s-2)}$ are

$$
\delta_{\Lambda} \Phi_{\alpha_1...\alpha_{s-1}\alpha_1...\alpha_s} = -\frac{1}{4} D^2 \Lambda_{\alpha_1...\alpha_{s-1}\alpha_1...\alpha_s} - \frac{1}{2} \bar{\mu} (s - 1) \bar{\Lambda}_{\alpha_1...\alpha_{s-1}\alpha_1...\alpha_s} ,
\tag{4.3.6a}
$$

$$
\delta_{\Lambda} \Psi_{\alpha_1...\alpha_{s-1}\hat{\alpha}_1...\hat{\alpha}_{s-2}} = -\frac{1}{2} \left( \bar{D}^\beta D^\beta - 2i(s - 1)D^\beta \right) \bar{\Lambda}_{\beta\alpha_1...\alpha_{s-1}\hat{\alpha}_1...\hat{\alpha}_{s-2}} .
\tag{4.3.6b}
$$

### 4.3.2 Non-conformal supercurrents: Half-integer superspin

In the framework of the longitudinal formulation (4.3.1b), let us couple the prepotentials $H_{\alpha(s)\hat{\alpha}(s)}, \Psi_{\alpha(s-1)\hat{\alpha}(s-2)}$ and $\bar{\Psi}_{\alpha(s-2)\hat{\alpha}(s-1)}$, to external sources

$$
S^{(s+\frac{1}{2})}_\text{source} = \int d^4x d^2\theta d\bar{\theta} E \left\{ H_{\alpha(s)\hat{\alpha}(s)} J_{\alpha(s)\hat{\alpha}(s)} + \Psi_{\alpha(s-1)\hat{\alpha}(s-2)} T_{\alpha(s-1)\hat{\alpha}(s-2)} + \bar{\Psi}_{\alpha(s-2)\hat{\alpha}(s-1)} \bar{T}_{\alpha(s-2)\hat{\alpha}(s-1)} \right\} .
\tag{4.3.7}
$$

Requiring $S^{(s+\frac{1}{2})}_\text{source}$ to be invariant under (4.3.4b) gives

$$
\bar{D}^\beta T_{\alpha(s-1)\beta\hat{\alpha}_1...\hat{\alpha}_{s-3}} = 0 ,
\tag{4.3.8a}
$$

and therefore $T_{\alpha(s-1)\hat{\alpha}(s-2)}$ is a transverse linear superfield. Requiring $S^{(s+\frac{1}{2})}_\text{source}$ to be invariant under the gauge transformations (4.3.5a) and (4.3.6b) gives the following conservation equation:

$$
\bar{D}^\beta J_{\alpha_1...\alpha_s\beta\hat{\alpha}_1...\hat{\alpha}_{s-1}} + \frac{1}{2} \left( \bar{D}_{(\alpha_1} \bar{D}_{(\hat{\alpha}_1} - 2i(s - 1) \bar{D}_{(\alpha_1} \bar{D}_{\hat{\alpha}_1)} \right) T_{\alpha_2...\alpha_s\hat{\alpha}_2...\hat{\alpha}_{s-1})} = 0 .
\tag{4.3.8b}
$$

For completeness, we also give the conjugate equation

$$
\bar{D}^\beta J_{\beta\alpha_1...\alpha_{s-1}\hat{\alpha}_1...\hat{\alpha}_s} - \frac{1}{2} \left( \bar{D}_{(\hat{\alpha}_1} \bar{D}_{\alpha_1} - 2i(s - 1) \bar{D}_{(\hat{\alpha}_1} \bar{D}_{\alpha_1)} \bar{T}_{\hat{\alpha}_2...\hat{\alpha}_{s-1}\hat{\alpha}_2...\hat{\alpha}_s)} = 0 .
\tag{4.3.8c}
$$

Similar considerations for the transverse formulation (4.3.1a) lead to the following non-conformal supercurrent multiplet

$$
\bar{D}^\beta \bar{J}_{\alpha_1...\alpha_s\beta\hat{\alpha}_1...\hat{\alpha}_{s-1}} - \frac{1}{4} \left( \bar{D}^2 + 2\bar{\mu}(s - 1) \right) F_{\alpha_1...\alpha_s\hat{\alpha}_1...\hat{\alpha}_{s-1}} = 0 ,
\tag{4.3.9a}
$$

$$
\bar{D}_{(\alpha_1} \bar{F}_{\alpha_2...\alpha_{s+1})\hat{\alpha}_1...\hat{\alpha}_{s-1})} = 0 .
\tag{4.3.9b}
$$

It follows from (4.3.9b) that the trace multiplet $\bar{F}_{\alpha(s-1)\hat{\alpha}(s)}$ is longitudinal linear. In the flat-superspace limit, the higher-spin supercurrent multiplets (4.3.8) and (4.3.9) reduce to those described in subsection 4.3.1.
Let us recall our condensed notation used in subsection 3.3.1. Associated with any tensor superfield \( U_{\alpha(m)\dot{\alpha}(\nu)} \) is the following index-free field on \( \mathbb{C}^2 \)

\[
U_{(m,n)}(\zeta, \bar{\zeta}) := \zeta^{\alpha_1} \cdots \zeta^{\alpha_m} \bar{\zeta}^{\dot{\alpha}_1} \cdots \bar{\zeta}^{\dot{\alpha}_n} U_{\alpha_1 \cdots \alpha_m \dot{\alpha}_1 \cdots \dot{\alpha}_n}, \tag{4.3.10}
\]

We also introduce the AdS analogues of the operators (3.3.6):

\[
\mathcal{D}_{(1,0)} := \zeta^\alpha \partial_{\zeta^\alpha}, \tag{4.3.11a}
\]

\[
\bar{\mathcal{D}}_{(0,1)} := \bar{\zeta}^{\dot{\alpha}} \partial_{\bar{\zeta}^{\dot{\alpha}}}, \tag{4.3.11b}
\]

\[
\mathcal{D}_{(1,1)} := 2i \zeta^\alpha \bar{\zeta}^{\dot{\alpha}} \partial_{\zeta^\alpha \bar{\zeta}^{\dot{\alpha}}} = -\{\mathcal{D}_{(1,0)}, \bar{\mathcal{D}}_{(0,1)}\}. \tag{4.3.11c}
\]

The following operators decrease the degree of homogeneity in the variables \( \zeta^\alpha \) and \( \bar{\zeta}^{\dot{\alpha}} \), specifically

\[
\mathcal{D}_{(-1,0)} := \mathcal{D}_{(-1,0)}, \tag{4.3.12a}
\]

\[
\bar{\mathcal{D}}_{(0,-1)} := \bar{\mathcal{D}}_{(0,-1)}. \tag{4.3.12b}
\]

Making use of the above notation, the transverse linear condition (4.3.13a) and its conjugate become

\[
\bar{\mathcal{D}}_{(0,-1)} T_{(s-1,s-2)} = 0, \tag{4.3.13a}
\]

\[
\mathcal{D}_{(-1,0)} \bar{T}_{(s-2,s-1)} = 0. \tag{4.3.13b}
\]

The conservation equations (4.3.8) and (4.3.9) turn into

\[
\frac{1}{s} \mathcal{D}_{(0,-1)} J_{(s,s)} - \frac{1}{2} A_{(1,1)} T_{(s-1,s-2)} = 0, \tag{4.3.14a}
\]

\[
\frac{1}{s} \mathcal{D}_{(-1,0)} J_{(s,s)} - \frac{1}{2} \bar{A}_{(1,1)} \bar{T}_{(s-2,s-1)} = 0. \tag{4.3.14b}
\]

where

\[
A_{(1,1)} := -\mathcal{D}_{(1,0)} \mathcal{D}_{(0,1)} + (s-1) \mathcal{D}_{(1,1)}; \quad \bar{A}_{(1,1)} := \bar{\mathcal{D}}_{(0,1)} \mathcal{D}_{(1,0)} - (s-1) \bar{\mathcal{D}}_{(1,1)}. \tag{4.3.15}
\]

Since \( \mathcal{D}_{(0,-1)}^2 J_{(s,s)} = 0 \), the conservation equation (4.3.14a) is consistent provided

\[
\mathcal{D}_{(0,-1)} A_{(1,1)} T_{(s-1,s-2)} = 0. \tag{4.3.16}
\]

This is indeed true, as a consequence of the transverse linear condition (4.3.13a).

### 4.3.3 Improvement transformations

The conservation equations (4.3.8) and (4.3.9) define two consistent higher-spin super-currents in AdS. Similar to the two irreducible AdS supercurrents \[99\], with \((12 + 12)\) and...
(20 + 20) degrees of freedom (see also the review in section 2.4), the higher-spin supercurrents (4.3.8) and (4.3.9) are equivalent in the sense that there always exists a well-defined improvement transformation that converts (4.3.8) into (4.3.9). Such an improvement transformation is constructed below.

Since the trace multiplet \( T_{\alpha(s-1)\dot{a}(s-2)} \) is transverse, eq. (4.3.8), there exists a well-defined complex tensor operator \( X_{\alpha(s-1)\dot{a}(s-1)} \) such that

\[
T_{\alpha(s-1)\dot{a}(s-2)} = \bar{D}^\beta X_{\alpha(s-1)\dot{b}\dot{a}_1...\dot{a}_{s-2}} . \tag{4.3.17}
\]

Let us introduce the real \( U_{\alpha(s-1)\dot{a}(s-1)} \) and imaginary \( V_{\alpha(s-1)\dot{a}(s-1)} \) parts of \( X_{\alpha(s-1)\dot{a}(s-1)} \),

\[
X_{\alpha(s-1)\dot{a}(s-1)} = U_{\alpha(s-1)\dot{a}(s-1)} + i V_{\alpha(s-1)\dot{a}(s-1)} . \tag{4.3.18}
\]

Then it may be checked that the operators

\[
\mathbb{J}_{\alpha(s)\dot{a}(s)} := J_{\alpha(s)\dot{a}(s)} + s \left[ D_{(\alpha_1,\dot{a}_1)} U_{\alpha_2...\alpha_s\dot{a}_2...\dot{a}_s} + s D_{(\alpha_1,\dot{a}_1) V_{\alpha_2...\alpha_s\dot{a}_2...\dot{a}_s}} \right] , \tag{4.3.19a}
\]

\[
F_{\alpha(s)\dot{a}(s-1)} := D_{(\alpha_1} \left\{ (2s + 1) U_{\alpha_2...\alpha_s\dot{a}(s-1)} - i V_{\alpha_2...\alpha_s}\dot{a}(s-1) \right\} \tag{4.3.19b}
\]

enjoy the conservation equation (4.3.9) and the constraint (4.3.9b). It is also not difficult to construct an inverse improvement transformation converting the higher-spin supercurrent (4.3.9) to (4.3.8).

In accordance with the result obtained, for all applications it suffices to work with the longitudinal supercurrent (4.3.8). This is why in the integer superspin case, which will be studied in the next subsection, we will introduce only a higher-spin supercurrent corresponding to the new gauge formulation (4.2.19).

There exists an improvement transformation\(^2\) for the supercurrent multiplet (4.3.8) Given a chiral scalar superfield \( \Omega \), we introduce

\[
\tilde{J}_{(s,s)} := J_{(s,s)} + D^{s}_{(1,1)} \left( \Omega + (-1)^s \bar{\Omega} \right) , \quad \bar{D}_s \Omega = 0 , \tag{4.3.20a}
\]

\[
\tilde{T}_{(s-1,s-2)} := T_{(s-1,s-2)} + \frac{2(-1)^s}{s(s-1)} \bar{D}_{(0,-1)} D^{s-1}_{(1,1)} \Omega
+ \frac{4(s+1)}{s} \mu D^{s-3}_{(1,1)} D_{(1,0)} \Omega . \tag{4.3.20b}
\]

The operators \( \tilde{J}_{(s,s)} \) and \( \tilde{T}_{(s-1,s-2)} \) prove to obey the conservation equation (4.3.8).

### 4.3.4 Non-conformal supercurrents: Integer superspin

We now make use of the new gauge formulation (4.2.12), or equivalently (4.2.19), for the integer superspin-s multiplet to derive the AdS analogue of the non-conformal higher-spin supercurrents in subsection 3.3.2.

\(^2\)One may compare this to (2.4.5) in the lower-spin case.
Let us couple the prepotentials $H_{\alpha(s-1)\dot{\alpha}(s-1)}$, $Z_{\alpha(s-1)\dot{\alpha}(s-1)}$ and $\Psi_{\alpha(s)\dot{\alpha}(s-1)}$ to external sources

$$S_{\text{source}}^{(s)} = \int d^4x d^2\theta d^2\bar{\theta} E \left\{ \Psi^{\alpha(s)\dot{\alpha}(s-1)} J_{\alpha(s)\dot{\alpha}(s-1)} - \Psi^{\alpha(s-1)\dot{\alpha}(s)} \bar{J}_{\alpha(s-1)\dot{\alpha}(s)} ight. + H^{\alpha(s-1)\dot{\alpha}(s-1)} S_{\alpha(s-1)\dot{\alpha}(s-1)} + Z^{\alpha(s-1)\dot{\alpha}(s-1)} T_{\alpha(s-1)\dot{\alpha}(s-1)} + \bar{Z}^{\alpha(s-1)\dot{\alpha}(s-1)} \bar{T}_{\alpha(s-1)\dot{\alpha}(s-1)} \right\} . \quad (4.3.21)$$

In order for $S_{\text{source}}^{(s)}$ to be invariant under the $\zeta$-transformation in (4.2.9), the source $J_{\alpha(s)\dot{\alpha}(s-1)}$ must satisfy

$$\mathcal{D}^\beta J_{\alpha(s)\dot{\beta}(s-2)} = 0 \iff \mathcal{D}^\beta \bar{J}_{\beta(s-2)\dot{\alpha}(s)} = 0 . \quad (4.3.22)$$

Next, requiring $S_{\text{source}}^{(s)}$ to be invariant under the transformation (4.2.8) leads to

$$\mathcal{D}_{(\alpha(s)\dot{\alpha}(s-1)\dot{\alpha}_2...\dot{\alpha}_s)} T_{\alpha(s-1)\dot{\alpha}_2...\dot{\alpha}_s} = 0 \iff \mathcal{D}_{(\alpha(s)\dot{\alpha}_2...\dot{\alpha}_s)} T_{\alpha(s-1)\dot{\alpha}_2...\dot{\alpha}_s} = 0 . \quad (4.3.23)$$

We see that the superfields $J_{\alpha(s)\dot{\alpha}(s-1)}$ and $T_{\alpha(s-1)\dot{\alpha}(s-1)}$ are transverse linear and longitudinal linear, respectively. Finally, requiring $S_{\text{source}}^{(s)}$ to be invariant under the $\mathfrak{V}$-transformation (4.2.9) gives the following conservation equation

$$-\frac{1}{2} \mathcal{D}^\beta J_{\beta\alpha(s-1)\dot{\alpha}(s-1)} + S_{\alpha(s-1)\dot{\alpha}(s-1)} + \bar{T}_{\alpha(s-1)\dot{\alpha}(s-1)} = 0 \quad (4.3.24a)$$

as well as its conjugate

$$\frac{1}{2} \mathcal{D}^\beta \bar{J}_{\alpha(s-1)\dot{\beta}(s-1)} + S_{\alpha(s-1)\dot{\alpha}(s-1)} + T_{\alpha(s-1)\dot{\alpha}(s-1)} = 0 . \quad (4.3.24b)$$

As a consequence of (4.3.23), from (4.3.24a) we deduce

$$\frac{1}{4} \mathcal{D}^2 J_{\alpha(s)\dot{\alpha}(s-1)} - \frac{1}{2} \hat{\mu}(s+2) J_{\alpha(s)\dot{\alpha}(s-1)} + \mathcal{D}_{(\alpha(s)\dot{\alpha}_2...\dot{\alpha}_s)} S_{\alpha(s-1)\dot{\alpha}_2...\dot{\alpha}_s} = 0 . \quad (4.3.25)$$

The equations (4.3.22) and (4.3.25) describe the conserved current supermultiplet which corresponds to our theory in the gauge (4.2.16).

Taking the sum of (4.3.24a) and (4.3.24b) leads to

$$\frac{1}{2} \mathcal{D}^\beta J_{\beta\alpha(s-1)\dot{\alpha}(s-1)} + \frac{1}{2} \mathcal{D}^\beta \bar{J}_{\alpha(s-1)\dot{\beta}(s-1)} + T_{\alpha(s-1)\dot{\alpha}(s-1)} - \bar{T}_{\alpha(s-1)\dot{\alpha}(s-1)} = 0 . \quad (4.3.26)$$

The equations (4.3.22), (4.3.23) and (4.3.26) describe the conserved current supermultiplet which corresponds to our theory in the gauge (4.2.14). As a consequence of (4.3.23), the conservation equation (4.3.26) implies

$$\frac{1}{2} \mathcal{D}_{(\alpha(s)\dot{\alpha}_2...\dot{\alpha}_s)} \left\{ \mathcal{D}^\beta J_{\alpha(s)\dot{\alpha}_2...\dot{\alpha}_s} + \mathcal{D}^\beta \bar{J}_{\alpha(s)\dot{\alpha}_2...\dot{\alpha}_s} \right\} + \mathcal{D}_{(\alpha(s)\dot{\alpha}_2...\dot{\alpha}_s)} T_{\alpha(s-1)\dot{\alpha}_2...\dot{\alpha}_s} = 0 . \quad (4.3.27)$$

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Written in the condensed notation, the transverse linear condition (4.3.22) turns into
\[ \bar{D}_{(0,-1)} J_{(s,s-1)} = 0 , \] (4.3.28)
while the longitudinal linear condition (4.3.23) takes the form
\[ \bar{D}_{(0,1)} T_{(s-1,s-1)} = 0 . \] (4.3.29)
The conservation equation (4.3.24a) becomes
\[ -\frac{1}{2s} \mathcal{D}_{(-1,0)} J_{(s,s-1)} + S_{(s-1,s-1)} + T_{(s-1,s-1)} = 0 \] (4.3.30)
and (4.3.27) takes the form
\[ \frac{1}{2s} \mathcal{D}_{(1,0)} \{ \mathcal{D}_{(-1,0)} J_{(s,s-1)} + \mathcal{D}_{(0,-1)} \bar{J}_{(s-1,s)} \} + \mathcal{D}_{(1,0)} T_{(s-1,s-1)} = 0 . \] (4.3.31)

### 4.3.5 Improvement transformation

There exist an improvement transformation for the supercurrent multiplet (4.3.24). Given a chiral scalar superfield \( \Omega \), we introduce
\[ \tilde{J}_{(s-1)} := J_{(s,s-1)} + D^s_{(1,1)} D^s_{(1,0)} \Omega , \quad \bar{D}_a \Omega = 0 , \] (4.3.32a)
\[ \tilde{T}_{(s-1,s-1)} := \bar{T}_{(s-1,s-1)} + \frac{s}{4s} D^s_{(1,1)} (D^2 - 4\bar{\mu}) \Omega \]
\[ + (-1)^s (s - 1) \left( \bar{\mu} + \frac{\mu}{s} \right) D^s_{(1,1)} \tilde{\Omega} , \] (4.3.32b)
\[ \tilde{S}_{(s-1,s-1)} := S_{(s-1,s-1)} + \mu (s - 1) D^s_{(1,1)} \Omega + (-1)^{s-1} \bar{\mu} (s - 1) D^s_{(1,1)} \tilde{\Omega} \]
\[ + \bar{\mu} \frac{s-1}{s} D^s_{(1,1)} \Omega + (-1)^{s-1} \frac{\mu}{s} D^s_{(1,1)} \tilde{\Omega} . \] (4.3.32c)

It may be checked that the operators \( \tilde{J}_{(s,s-1)} \), \( \tilde{T}_{(s-1,s-1)} \) and \( \tilde{S}_{(s-1,s-1)} \) obey the conservation equation (4.3.30), as well as (4.3.23) and (4.3.28).

### 4.4 Higher-spin supercurrents for chiral superfields: Half-integer superspin

In the remainder of this chapter we will study explicit realisations of the higher spin supercurrents introduced above in various supersymmetric field theories in AdS.
4.4.1 Superconformal model for a chiral superfield

Let us consider the superconformal theory of a single chiral scalar superfield

$$S = \int d^4x d^2\theta d^2\bar{\theta} \, E \, \Phi \Phi ,$$  \hspace{1cm} (4.4.1)

where $\Phi$ is covariantly chiral, $\bar{\mathcal{D}}_\alpha \Phi = 0$. We can define the conformal supercurrent $J_{(s,s)}$ in direct analogy with the flat superspace case \[1,68\]

$$J_{(s,s)} = \sum_{k=0}^{s} (-1)^k \binom{s}{k} \left\{ \binom{s}{k+1} \mathcal{D}_{(1,1)}^k \mathcal{D}_{(1,0)} \Phi \, \mathcal{D}_{(1,1)}^{s-k-1} \bar{\mathcal{D}}_{(0,1)} \bar{\Phi} + \binom{s}{k} \mathcal{D}_{(1,1)}^k \mathcal{D}_{(1,1)}^{s-k} \right\} .$$  \hspace{1cm} (4.4.2)

Making use of the massless equations of motion, $(\mathcal{D}^2 - 4\bar{\mu}) \Phi = 0$, one may check that $J_{(s,s)}$ satisfies the conservation equation

$$\mathcal{D}_{(-1,0)} J_{(s,s)} = 0 \iff \bar{\mathcal{D}}_{(0,-1)} J_{(s,s)} = 0 .$$  \hspace{1cm} (4.4.3)

The calculation of (4.4.3) in AdS is much more complicated than in flat superspace due to the fact that the algebra of covariant derivatives (2.3.5) is nontrivial. Let us sketch the main steps in evaluating the left-hand side of eq. (4.4.3) with $J_{(s,s)}$ given by (4.4.2). We start with the obvious relations

$$\frac{\partial}{\partial \zeta^\alpha} \mathcal{D}_{(1,1)}^k = 2i \bar{\zeta}^\alpha \mathcal{D}_{a\dot{a}} \mathcal{D}_{a\dot{a}} ,$$  \hspace{1cm} (4.4.4a)

$$\frac{\partial}{\partial \zeta^\alpha} \mathcal{D}_{(1,1)}^k = \sum_{n=1}^{k} \mathcal{D}_{(1,1)}^{n-1} 2i \bar{\zeta}^\alpha \mathcal{D}_{a\dot{a}} \, \mathcal{D}_{(1,1)}^{k-n} , \hspace{1cm} k > 1 .$$  \hspace{1cm} (4.4.4b)

To simplify eq. (4.4.4b), we may push $\bar{\zeta}^\alpha \mathcal{D}_{a\dot{a}}$, say, to the left provided that we take into account its commutator with $\mathcal{D}_{(1,1)}$: \[68\]

$$[\bar{\zeta}^\alpha \mathcal{D}_{a\dot{a}} \, , \mathcal{D}_{(1,1)}] = -4i \bar{\mu} \mu \zeta_\alpha \bar{\zeta}^\alpha \bar{\zeta}^\beta \bar{M}_{\dot{\alpha}\dot{\beta}} .$$  \hspace{1cm} (4.4.5)

Associated with the Lorentz generators are the operators

$$\bar{M}_{(0,2)} := \bar{\zeta}^\alpha \bar{\zeta}^\beta \bar{M}_{\dot{\alpha}\dot{\beta}} ,$$  \hspace{1cm} (4.4.6a)

$$M_{(2,0)} := \zeta^\alpha \zeta^\beta M_{\alpha\beta} ,$$  \hspace{1cm} (4.4.6b)

where $\bar{M}_{(0,2)}$ appears in the right-hand side of (4.4.5). These operators annihilate every superfield $U_{(m,n)}(\zeta, \bar{\zeta})$ of the form (4.3.10) \[3\]

$$\bar{M}_{(0,2)} U_{(m,n)} = 0 , \hspace{1cm} M_{(2,0)} U_{(m,n)} = 0 .$$  \hspace{1cm} (4.4.6c)

\[3\]These properties are analogous to those that play a fundamental role for the consistent definition of covariant projective supermultiplets in 5D $\mathcal{N} = 1$ \[148\] and 4D $\mathcal{N} = 2$ \[149\] supergravity theories.
From the above consideration, it follows that

\[
\left[\tilde{\alpha}\mathcal{D}_{\alpha\bar{\alpha}}, \mathcal{D}^k_{(1,1)}\right] U_{(m,n)} = 0 , \tag{4.4.7a}
\]

\[
\left(\frac{\partial}{\partial \zeta^\alpha} \mathcal{D}^k_{(1,1)}\right) U_{(m,n)} = 2ik \tilde{\zeta}^\alpha \mathcal{D}_{\alpha\bar{\alpha}} \mathcal{D}^{k-1}_{(1,1)} U_{(m,n)} . \tag{4.4.7b}
\]

We also state some other properties which we often use throughout our calculations

\[
\mathcal{D}^2_{(0,1)} = -2\bar{\mu} M_{(2,0)} , \tag{4.4.8a}
\]

\[
\left[\mathcal{D}_{(1,0)}, \mathcal{D}_{(1,1)}\right] = \left[\tilde{\mathcal{D}}_{(0,1)}, \mathcal{D}_{(1,1)}\right] = 0 , \tag{4.4.8b}
\]

\[
\left[\mathcal{D}^\alpha, \mathcal{D}^k_{(1,1)}\right] = -2\bar{\mu}\zeta^\alpha \mathcal{D}^{k-1}_{(1,1)} \tilde{\mathcal{D}}_{(0,1)} , \tag{4.4.8c}
\]

\[
\left[\mathcal{D}^\alpha, \tilde{\alpha}\mathcal{D}^k_{(1,1)}\right] = -2\bar{\mu}\zeta^\alpha \mathcal{D}^{k-1}_{(1,1)} \mathcal{D}_{(1,1)} \Phi . \tag{4.4.8d}
\]

\[
\left[\mathcal{D}^\alpha, \bar{\zeta}^\beta \mathcal{D}^k_{(1,1)}\right] = i\bar{\mu} \delta^\alpha_\beta \mathcal{D}_{(0,1)} . \tag{4.4.8e}
\]

The above identities suffice to prove that the supercurrent \(4.4.2\) does obey the conservation equation \(4.4.3\).

### 4.4.2 Non-superconformal model for a chiral superfield

Let us now add the mass term to \(4.4.1\) and consider the following action

\[
S = \int d^4x d^2\theta d^2\bar{\theta} E \Phi \Phi + \left\{ \frac{m}{2} \int d^4x d^2\theta E \Phi^2 + c.c. \right\} , \tag{4.4.9}
\]

with \(m\) a complex mass parameter. The real supercurrent \(J_{(s,s)}\) takes the same form as in the massless case, \(4.4.2\). However, in the massive case \(J_{(s,s)}\) satisfies a more general conservation equation \(4.3.14a\) for some superfield \(T_{(s-1,s-2)}\), which we need to determine. Indeed, making use of the equations of motion

\[
-\frac{1}{4}(\mathcal{D}^2 - 4\bar{\mu})\Phi + \bar{m}\Phi = 0 , \quad -\frac{1}{4}(\tilde{\mathcal{D}}^2 - 4\mu)\bar{\Phi} + m\Phi = 0 , \tag{4.4.10}
\]

we obtain

\[
\mathcal{D}_{(0,-1)} J_{(s,s)} = F_{(s,s-1)} , \tag{4.4.11a}
\]

where we have denoted

\[
F_{(s,s-1)} = 2m(s+1) \sum_{k=0}^{s} (-1)^{s+1+k} \binom{s}{k} \binom{s}{k+1} \times \left\{ 1 + (-1)^s \frac{k+1}{s-k+1} \right\} \mathcal{D}^k_{(1,1)} \Phi \mathcal{D}^{s-k-1}_{(1,1)} \mathcal{D}_{(1,0)} \Phi . \tag{4.4.11b}
\]

We now look for a superfield \(T_{(s-1,s-2)}\) such that (i) it obeys the transverse linear constraint \(4.3.13a\); and (ii) it satisfies the equation

\[
F_{(s,s-1)} = \frac{s}{2} A_{(1,1)} T_{(s-1,s-2)} . \tag{4.4.12}
\]
Our analysis will be similar to the one performed in 3.3.1.1 in flat superspace. We consider a general ansatz

\[ T_{(s-1,s-2)} = (-1)^s m \sum_{k=0}^{s-2} c_k D_{(1,1)}^k \Phi D_{(1,1)}^{s-k-2} D_{(1,0)} \Phi \]  

(4.4.13)

with some coefficients \( c_k \) which have to be determined. For \( k = 1, 2, ... s - 2 \), condition (i) implies that the coefficients \( c_k \) must satisfy

\[ kc_k = (s - k - 1)c_{s-k-1} \]  

(4.4.14a)

while (ii) gives the following equation

\[ c_{s-k-1} + sc_k + (s - 1)c_{k-1} = -4(-1)^k \frac{s+1}{s} \left( \begin{array}{c} s \\ k \end{array} \right) \left( \begin{array}{c} s \\ k+1 \end{array} \right) \times \left\{ 1 + (-1)^s \frac{k+1}{s-k+1} \right\} . \]  

(4.4.14b)

Condition (ii) also implies that

\[ (s - 1)c_{s-2} + c_0 = 4(-1)^s(s+1) \left\{ 1 + (-1)^s \frac{s}{2} \right\} , \]  

(4.4.14c)

\[ c_0 = -\frac{4}{s}(s + 1 + (-1)^s) . \]  

(4.4.14d)

It turns out that the equations (4.4.14) lead to a unique expression for \( c_k \) given by

\[ c_k = -\frac{4(s+1)(s-k-1)}{s(s-1)} \sum_{l=0}^{k} \frac{(-1)^k}{s-l} \left( \begin{array}{c} s \\ l \end{array} \right) \left( \begin{array}{c} s \\ l+1 \end{array} \right) \left\{ 1 + (-1)^s \frac{l+1}{s-l+1} \right\} , \]  

(4.4.15)

\[ k = 0, 1, \ldots s - 2 \ . \]

If the parameter \( s \) is odd, \( s = 2n + 1 \), with \( n = 1, 2, \ldots \), one can check that the equations (4.4.14a)–(4.4.14c) are identically satisfied. However, if the parameter \( s \) is even, \( s = 2n \), with \( n = 1, 2, \ldots \), there appears an inconsistency: the right-hand side of (4.4.14c) is positive, while the left-hand side is negative, \( (s - 1)c_{s-2} + c_0 < 0 \). Therefore, our solution (4.4.15) is only consistent for \( s = 2n + 1, n = 1, 2, \ldots \). Relations (4.4.2), (4.4.13) and (4.4.15) determine the non-conformal higher-spin supercurrents in the massive chiral model (4.4.9). Unlike the conformal higher-spin supercurrents (4.4.2), the non-conformal ones exist only for the odd values of \( s, s = 2n + 1 \), with \( n = 1, 2, \ldots \). In the flat superspace limit, the above results reduce to those derived in 3.3.1.1 and in Ref. [115].

**4.4.3 Superconformal model with \( N \) chiral superfields**

We now generalise the superconformal model (4.4.1) to the case of \( N \) covariantly chiral scalar superfields \( \Phi^i, i = 1, \ldots N \),

\[ S = \int d^4x d^2\theta d^2\bar{\theta} E \Phi^i \Phi^i , \quad \mathcal{D}_a \Phi^i = 0 . \]  

(4.4.16)
The novel feature of the $N > 1$ case is that there exist two different types of conformal supercurrents, which are:

$$J^+ = S^{ij} \sum_{k=0}^{s} (-1)^k \left( \begin{array}{c} s \\ k \\ \end{array} \right) \left\{ \left( \begin{array}{c} s \\ k+1 \\ \end{array} \right) D^{k}_{(1,1)} D^{s-k-1}_{(1,1)} \phi^i D^{s-k}_{(0,1)} \phi^j \right. \right.$$ 

$$+ \left( \begin{array}{c} s \\ k \\ \end{array} \right) D^{k}_{(1,1)} D^{s-k}_{(1,1)} \phi^i \phi^j \right\}, \quad S^{ij} = S^{ji} \quad (4.4.17)$$

and

$$J^- = i A^{ij} \sum_{k=0}^{s} (-1)^k \left( \begin{array}{c} s \\ k \\ \end{array} \right) \left\{ \left( \begin{array}{c} s \\ k+1 \\ \end{array} \right) D^{k}_{(1,1)} D^{s-k-1}_{(1,1)} \phi^i D^{s-k}_{(0,1)} \phi^j \right. \right.$$ 

$$+ \left( \begin{array}{c} s \\ k \\ \end{array} \right) D^{k}_{(1,1)} D^{s-k}_{(1,1)} \phi^i \phi^j \right\}, \quad A^{ij} = -A^{ji} \quad (4.4.18)$$

Here $S$ and $A$ are arbitrary real symmetric and antisymmetric constant matrices, respectively. We have put an overall factor $\sqrt{-1}$ in eq. (4.4.18) in order to make $J^-$ real. One can show that the currents (4.4.17) and (4.4.18) are conserved on-shell:

$$D^{(-1,0)} J^\pm = 0 \iff D^{(0,-1)} J^\pm = 0 \quad (4.4.19)$$

The above results can be recast in terms of the matrix conformal supercurrent $J_{(s,s)} = (J_{(s,s)}^{ij})$ with components

$$J_{(s,s)}^{ij} := \sum_{k=0}^{s} (-1)^k \left( \begin{array}{c} s \\ k \\ \end{array} \right) \left\{ \left( \begin{array}{c} s \\ k+1 \\ \end{array} \right) D^{k}_{(1,1)} D^{s-k-1}_{(1,1)} \phi^i D^{s-k}_{(0,1)} \phi^j \right. \right.$$ 

$$+ \left( \begin{array}{c} s \\ k \\ \end{array} \right) D^{k}_{(1,1)} D^{s-k}_{(1,1)} \phi^i \phi^j \right\}, \quad (4.4.20)$$

which is Hermitian, $J_{(s,s)}^\dagger = J_{(s,s)}$. The chiral action (4.4.16) possesses rigid U(N) symmetry acting on the chiral column-vector $\Phi = (\Phi^i)$ by $\Phi \rightarrow g\Phi$, with $g \in U(N)$, which implies that the supercurrent (4.4.20) transforms as $J_{(s,s)} \rightarrow g J_{(s,s)} g^{-1}$.

### 4.4.4 Massive model with $N$ chiral superfields

Consider a theory of $N$ massive chiral multiplets with action

$$S = \int d^4 x d^2 \theta d^2 \bar{\theta} \ E \, \bar{\Phi}^i \Phi^i + \left\{ \frac{1}{2} \int d^4 x d^2 \theta E \, M^{ij} \Phi^i \Phi^j + \text{c.c.} \right\} \quad (4.4.21)$$

where $M^{ij}$ is a constant symmetric $N \times N$ mass matrix. The corresponding equations of motion are

$$-\frac{1}{4} (D^2 - 4\mu) \Phi^i + M^{ij} \Phi^j = 0 \quad , \quad -\frac{1}{4} (\bar{D}^2 - 4\bar{\mu}) \bar{\Phi}^i + M^{ij} \bar{\Phi}^j = 0 \quad (4.4.22)$$
First we will consider the case where $S$ is a real and symmetric matrix. Making use of the equations of motion, we obtain

$$ \mathcal{D}_{(-1,0)} J_{(s,s)} = 2(s + 1)(\tilde{M})^{ji} \sum_{k=0}^{s-1} (-1)^k \binom{s}{k} \binom{s}{k+1} $$

\[ \times \frac{k}{k+1} \mathcal{D}_{(1,1)}^{k-1} \bar{\Phi}^i \mathcal{D}_{(1,1)}^{s-k} \bar{\Phi}^j + 2(s + 1)(\tilde{M})^{ji} \sum_{k=0}^{s-1} (-1)^k \binom{s}{k} \binom{s}{k+1} \]

\[ \times \mathcal{D}_{(1,1)}^k \bar{\Phi}^i \mathcal{D}_{(0,1)}^{s-k-1} \bar{\Phi}^j. \] (4.4.23)

Now, suppose the product $S\tilde{M}$ is symmetric, which implies $[S, \tilde{M}] = 0$. Then, (4.4.23) becomes

$$ \mathcal{D}_{(-1,0)} J_{(s,s)} = 2(s + 1)(\tilde{M})^{ji} \sum_{k=0}^{s-1} (-1)^k \binom{s}{k} \binom{s}{k+1} $$

\[ \times \left\{ 1 + (-1)^s \frac{k+1}{s-k+1} \right\} \mathcal{D}_{(1,1)}^k \bar{\Phi}^i \mathcal{D}_{(0,1)}^{s-k-1} \bar{\Phi}^j. \] (4.4.24)

We now look for a superfield $\tilde{T}_{(s-2,s-1)}$ such that (i) it obeys the transverse antilinear constraint (4.3.13b); and (ii) it satisfies the conservation equation (4.3.14b):

$$ \mathcal{D}_{(-1,0)} J_{(s,s)} = \frac{s}{2} \tilde{A}_{(1,1)} \tilde{T}_{(s-2,s-1)}. \] (4.4.25)

As in the single field case we consider a general ansatz

$$ \tilde{T}_{(s-2,s-1)} = (\tilde{M})^{ij} \sum_{k=0}^{s-2} c_k \mathcal{D}_{(1,1)}^k \bar{\Phi}^i \mathcal{D}_{(1,1)}^{s-k-2} \bar{\Phi}^j. \] (4.4.26)

Then for $k = 1, 2, ... s - 2$, condition (i) implies that the coefficients $c_k$ must satisfy

$$ kc_k = (s - k - 1)c_{s-k-1}, \] (4.4.27a)

while (ii) gives the following equation

$$ c_{s-k-1} + sc_k + (s-1)c_{k-1} = -4(-1)^k \frac{s+1}{s} \binom{s}{k} \binom{s}{k+1} $$

\[ \times \left\{ 1 + (-1)^s \frac{k+1}{s-k+1} \right\}. \] (4.4.27b)

Condition (ii) also implies that

$$ (s-1)c_{s-2} + c_0 = 4(-1)^s(s+1) \left\{ 1 + (-1)^s \frac{s}{2} \right\}, \] (4.4.27c)

$$ c_0 = -\frac{4}{s}(s + 1 + (-1)^s). \] (4.4.27d)

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The above conditions coincide with eqs. (4.4.14a)–(4.4.14d) in the case of a single, massive chiral superfield, which are satisfied only for \( s = 2n + 1, n = 1, 2, \ldots \). Hence, the solution for the coefficients \( c_k \) is given by (4.4.15) for odd values of \( s \) and there is no solution for even \( s \).

On the other hand, if \( S \bar{M} \) is antisymmetric (which is equivalent to \( \{S, \bar{M}\} = 0 \)), eq. (4.4.24) is slightly modified

\[
D_{(-1,0)} J_{(s,s)} = 2(s + 1)(S \bar{M})^{ij} \sum_{k=0}^{s-1} (-1)^k \binom{s}{k} \left( \binom{s}{k+1} \left\{-1 + (-1)^s \frac{k + 1}{s - k + 1} \right\} D_{(1,1)}^{k,i} \bar{\Phi}^i D_{(-1,1)}^{s-k-1,1} \bar{\Phi}^j .
\]

Starting with a general ansatz

\[
\bar{T}_{(s-2,s-1)} = (S \bar{M})^{ij} \sum_{k=0}^{s-2} d_k D_{(1,1)}^{k,i} \bar{\Phi}^i D_{(-1,1)}^{s-k-2,1} \bar{\Phi}^j \]

and imposing conditions (i) and (ii) yield the following equations for the coefficients \( d_k \)

\[
kd_k = -(s - k - 1)d_{s-k-1} .
\]

\[
-d_{s-k-1} + sd_k + (s - 1)d_{k-1} = -4(-1)^k s + 1 s \left( \binom{s}{k} \left\{-1 + (-1)^s \frac{k + 1}{s - k + 1} \right\} \right)
\]

\[
(s - 1)d_{s-2} - d_0 = 4(-1)^s(s + 1) \left\{-1 + (-1)^s \frac{s}{2} \right\} .
\]

The equations (4.4.30) lead to a unique expression for \( d_k \) given by

\[
d_k = -4(s + 1)(s - k - 1) s(s - 1) \sum_{l=0}^{k} (-1)^k \binom{s}{l} \left\{-1 + (-1)^s \frac{l + 1}{s - l + 1} \right\} ,
\]

\[
k = 0, 1, \ldots s - 2 .
\]

If the parameter \( s \) is even, \( s = 2n \), with \( n = 1, 2, \ldots \), one can check that the equations (4.4.30a)–(4.4.30d) are identically satisfied. However, if the parameter \( s \) is odd, \( s = 2n + 1 \), with \( n = 1, 2, \ldots \), there appears an inconsistency: the right-hand side of (4.4.30c) is positive, while the left-hand side is negative, \( (s - 1)d_{s-2} - d_0 < 0 \). Therefore, our solution (4.4.31) is only consistent for \( s = 2n, n = 1, 2, \ldots \).

Finally, we consider \( A^{ij} = -A^{ji} \) with the corresponding \( J_{(s,s)} \) given by (4.4.18). The analysis in this case is similar to the one presented above and we will simply state the
results. If \( s \) is odd the non-conformal higher-spin supercurrents exist if \( \{ A, \tilde{M} \} = 0 \). The trace supercurrent \( \tilde{T}_{(s-2,s-1)} \) is given by (4.4.26) with the coefficients \( c_k \) given by

\[
c_k = i \frac{4(s+1)(s-k-1)}{s(s-1)} \sum_{l=0}^{k} \frac{(-1)^k}{s-l} \binom{s}{l} \binom{s}{l+1} \left\{ 1 + (-1)^s \frac{l+1}{s-l+1} \right\}, \quad (4.4.32)
\]

\( k = 0, 1, \ldots s - 2 \).

If \( s \) is even the non-conformal higher-spin supercurrents exist if \( [A, \tilde{M}] = 0 \). The trace supercurrent \( \tilde{T}_{(s-2,s-1)} \) is given by (4.4.29) with the coefficients \( d_k \) given by

\[
d_k = i \frac{4(s+1)(s-k-1)}{s(s-1)} \sum_{l=0}^{k} \frac{(-1)^k}{s-l} \binom{s}{l} \binom{s}{l+1} \left\{ -1 + (-1)^s \frac{l+1}{s-l+1} \right\}, \quad (4.4.33)
\]

\( k = 0, 1, \ldots s - 2 \).

Note that the coefficients \( c_k \) in (5.5.6) differ from similar coefficients in (4.4.15) by a factor of \(-i\). This means that for odd \( s \) we can define a more general supercurrent

\[
J_{(s,s)} = H^{ij} \sum_{k=0}^{s} (-1)^k \binom{s}{k} \left\{ \binom{s}{k+1} D_{(1,1)}^{i} \Phi^{j} D_{(1,1)}^{s-k-1} \bar{D}_{(0,1)} \bar{\Phi}^{j} + \binom{s}{k} D_{(1,1)}^{i} \Phi^{j} D_{(1,1)}^{s-k} \bar{D}_{(0,1)} \bar{\Phi}^{j} \right\}, \quad (4.4.34)
\]

where \( H^{ij} \) is a generic matrix which can be split into the symmetric and antisymmetric parts \( H^{ij} = S^{ij} + iA^{ij} \). Here both \( S \) and \( A \) are real and we put an \( i \) in front of \( A \) because \( J_{(s,s)} \) must be real. From the above consideration it then follows that the corresponding more general solution for \( \tilde{T}_{(s-2,s-1)} \) reads

\[
\tilde{T}_{(s-2,s-1)} = (\tilde{H} \tilde{M})^{ij} \sum_{k=0}^{s-2} c_k D_{(1,1)}^{i} \bar{\Phi}^{j} D_{(1,1)}^{s-k-2} \bar{D}_{(0,1)} \bar{\Phi}^{j}, \quad (4.4.35)
\]

where \( [S, \tilde{M}] = 0, \{ A, \tilde{M} \} = 0 \) and \( c_k \) are, as before, given by eq. (4.4.15). Similarly, the coefficients \( d_k \) in (4.4.33) differ from similar coefficients in (4.4.31) by a factor of \(-i\). This means that for even \( s \) we can define a more general supercurrent (4.4.34), where \( H^{ij} \) is a generic matrix which we can split as before into the symmetric and antisymmetric parts, \( H^{ij} = S^{ij} + iA^{ij} \). From the above consideration it then follows that the corresponding more general solution for \( \tilde{T}_{(s-2,s-1)} \) reads

\[
\tilde{T}_{(s-2,s-1)} = (\tilde{H} \tilde{M})^{ij} \sum_{k=0}^{s-2} d_k D_{(1,1)}^{i} \bar{\Phi}^{j} D_{(1,1)}^{s-k-2} \bar{D}_{(0,1)} \bar{\Phi}^{j}, \quad (4.4.36)
\]

where \( \{ S, \tilde{M} \} = 0, \{ A, \tilde{M} \} = 0 \) and \( d_k \) are given by eq. (4.4.31).
4.5 Higher-spin supercurrents for chiral superfields: Integer superspin

In this section we provide explicit realisations for the fermionic higher-spin supercurrents (integer superspin) in models described by chiral scalar superfields.

4.5.1 Massive hypermultiplet model

Consider a free massive hypermultiplet in AdS

\[ S = \int d^4x d^2\theta d^2\bar{\theta} E \left( \bar{\Psi}_+ \Psi_+ + \bar{\Psi}_- \Psi_- + c.c. \right) + \left\{ m \int d^4x d^2\theta E \bar{\Psi}_+ \Psi_+ + c.c. \right\} , \]

where the superfields \( \Psi_\pm \) are covariantly chiral, \( \bar{D}_\alpha \Psi_\pm = 0 \) and \( m \) is a complex mass parameter.\(^4\) By a change of variables it is possible to make \( m \) real. Let us introduce another set of fields \( \Phi_\pm \), \( \bar{D}_\alpha \Phi_\pm = 0 \), related to \( \Psi_\pm \) by the following transformations

\[ \Phi_\pm = e^{i\alpha/2} \Psi_\pm, \quad m = M e^{i\alpha} . \]

Under the transformations (4.5.2), the action (4.5.1) turns into

\[ S = \int d^4x d^2\theta d^2\bar{\theta} E \left( \bar{\Phi}_+ \Phi_+ + \bar{\Phi}_- \Phi_- + c.c. \right) + \left\{ M \int d^4x d^2\theta E \bar{\Phi}_+ \Phi_+ + c.c. \right\} , \]

where the mass parameter \( M \) is now real. In the massless case, \( M = 0 \), the conserved fermionic supercurrent \( J_{(s,s-1)} \) was constructed in [68] and is given by

\[ J_{(s,s-1)} = \sum_{k=0}^{s-1} (-1)^k \binom{s-1}{k} \left\{ \binom{s}{k+1} D_{(1,1)}^k D_{(1,0)} \Phi_+ D_{s-k-1}^s D_{(1,0)} \Phi_- + \binom{s}{k} D_{(1,1)}^k D_{(1,0)} \Phi_+ D_{s-k-1}^s D_{(1,0)} \Phi_- \right\} . \] \hspace{1cm} (4.5.4)

Making use of the massless equations of motion, \(-\frac{1}{3} (D^2 - 4\bar{\mu}) \Phi_\pm = 0\), one may check that \( J_{(s,s-1)} \) obeys, for \( s > 1 \), the conservation equations

\[ D_{(1,0)} J_{(s,s-1)} = 0, \quad D_{(0,-1)} J_{(s,s-1)} = 0 . \] \hspace{1cm} (4.5.5)

We will construct fermionic higher-spin supercurrents corresponding to the massive model (4.5.3). Making use of the massive equations of motion

\[ -\frac{1}{4} (D^2 - 4\bar{\mu}) \Phi_+ + M \Phi_- = 0, \quad -\frac{1}{4} (D^2 - 4\bar{\mu}) \Phi_- + M \Phi_+ = 0, \] \hspace{1cm} (4.5.6)

\(^4\)This model possesses off-shell \( \mathcal{N} = 2 \) AdS supersymmetry [96][150].
we obtain

\[
D_{(-1,0)} J_{(s,s-1)} = 2M(s + 1) \sum_{k=0}^{s-1} (-1)^{k+1} \binom{s-1}{k} \binom{s}{k} \times \left\{ -\frac{s-k}{k+1} D^k_{(1,1)} \Phi_- D^{s-k-1}_{(1,1)} \Phi_- + D^k_{(1,1)} \Phi_+ D^{s-k-1}_{(1,1)} \bar{\Phi}_+ \right\}
\]

\[
+ 2M(s + 1) \sum_{k=1}^{s-1} (-1)^{k+1} \binom{s-1}{k} \binom{s}{k} \frac{k}{k+1}
\times D^{k-1}_{(1,1)} D_{(0,1)} \Phi_- D^{s-k-1}_{(1,1)} D_{(0,1)} \Phi_-
\]

\[
+ 2M(s + 1) \sum_{k=0}^{s-2} (-1)^{k+1} \binom{s-1}{k} \binom{s-1-k}{k} \frac{s-1-k}{k+1}
\times D^k_{(1,1)} D_{(1,0)} \Phi_+ D^{s-k-2}_{(1,1)} D_{(0,1)} \bar{\Phi}_+.
\]

(4.5.7)

It can be shown that the massive supercurrent \( J_{(s,s-1)} \) also obeys (4.3.28).

We now look for a superfield \( T_{(s-1,s-1)} \) such that (i) it obeys the longitudinal linear constraint (4.3.29); and (ii) it satisfies (4.3.31), which is a consequence of the conservation equation (4.3.30). For this we consider a general ansatz

\[
T_{(s-1,s-1)} = \sum_{k=0}^{s-1} c_k D^k_{(1,1)} \Phi_- D^{s-k-1}_{(1,1)} \bar{\Phi}_-
\]

\[
+ \sum_{k=0}^{s-1} d_k D^k_{(1,1)} \Phi_+ D^{s-k-1}_{(1,1)} \bar{\Phi}_+
\]

\[
+ \sum_{k=1}^{s-1} f_k D^{k-1}_{(1,1)} D_{(1,0)} \Phi_- D^{s-k-1}_{(1,1)} D_{(0,1)} \bar{\Phi}_-
\]

\[
+ \sum_{k=1}^{s-1} g_k D^{k-1}_{(1,1)} D_{(1,0)} \Phi_+ D^{s-k-1}_{(1,1)} D_{(0,1)} \bar{\Phi}_+.
\]

(4.5.8)

Condition (i) implies that the coefficients must be related by

\[
c_0 = d_0 = 0 , \quad f_k = c_k , \quad g_k = d_k ,
\]

(4.5.9a)

while for \( k = 1, 2, \ldots s - 2 \), condition (ii) gives the following recurrence relations:

\[
c_k + c_{k+1} = \frac{M(s + 1)}{s} (-1)^{s+k} \binom{s-1}{k} \binom{s}{k}
\times \frac{1}{(k+2)(k+1)} \left\{ (2k+2-s)(s+1) - k - 2 \right\},
\]

(4.5.9b)

\[
d_k + d_{k+1} = \frac{M(s + 1)}{s} (-1)^{k} \binom{s-1}{k} \binom{s}{k}
\times \frac{1}{(k+2)(k+1)} \left\{ (2k+2-s)(s+1) - k - 2 \right\}.
\]

(4.5.9c)
Condition (ii) also implies that

\[ c_1 = -(-1)^s \frac{M(s^2 - 1)}{2}, \quad c_{s-1} = \frac{M(s^2 - 1)}{s}; \quad (4.5.9d) \]

\[ d_1 = -\frac{M(s^2 - 1)}{2}, \quad d_{s-1} = -(-1)^s \frac{M(s^2 - 1)}{s}. \quad (4.5.9e) \]

The above conditions lead to simple expressions for \( c_k \) and \( d_k \):

\[ d_k = \frac{M(s + 1)}{s} \frac{k}{k + 1} (-1)^k \binom{s - 1}{k} \binom{s}{k}, \quad (4.5.10a) \]

\[ c_k = (-1)^s d_k, \quad (4.5.10b) \]

where \( k = 1, 2, \ldots, s - 1 \).

### 4.5.2 Superconformal model with \( N \) chiral superfields

In this subsection we will generalise the above results for \( N \) chiral superfields \( \Phi^i, i = 1, \ldots N \). We first consider the superconformal model (4.4.16). Let us construct the following fermionic supercurrent

\[ J_{(s,s-1)} = C^{ij} \sum_{k=0}^{s-1} (-1)^k \binom{s-1}{k} \left\{ \left( \frac{s}{k+1} \right) D_{(1,1)}^k D_{(1,0)}^i \Phi^i D_{(1,0)}^{s-k-1} \Phi^j ight. \]

\[ \left. - \left( \frac{s}{k} \right) D_{(1,1)}^k \Phi^i D_{(1,0)}^{s-k-1} D_{(1,0)}^{i} \Phi^j \right\}, \quad (4.5.11) \]

where \( C^{ij} \) is a constant complex matrix. By changing the summation index it is not hard to show that \( J_{(s,s-1)} = 0 \) if (i) \( s \) is odd and \( C^{ij} \) is symmetric; and (ii) \( s \) is even and \( C^{ij} \) is antisymmetric, that is

\[ C^{ij} = C^{ji}, \quad s = 1, 3, \ldots \implies J_{(s,s-1)} = 0; \quad (4.5.12a) \]

\[ C^{ij} = -C^{ji}, \quad s = 2, 4, \ldots \implies J_{(s,s-1)} = 0. \quad (4.5.12b) \]

This means that we have to consider the two separate cases: the case of even \( s \) with symmetric \( C \), and the case of odd \( s \) with antisymmetric \( C \). Using the massless equation of motion, \( -\frac{1}{4} (D^2 - 4\bar{\mu}) \Phi^i = 0 \), one may check that \( J_{(s,s-1)} \) satisfies the conservation equations (4.5.5)

\[ D_{(-1,0)} J_{(s,s-1)} = 0, \quad D_{(0,-1)} J_{(s,s-1)} = 0. \quad (4.5.13) \]

In the case of a single chiral superfield, the supercurrent (4.5.11) exists for even \( s \),

\[ J_{(s,s-1)} = 2 \sum_{k=0}^{s-1} (-1)^k \binom{s-1}{k} \binom{s}{k+1} D_{(1,1)}^k D_{(1,0)}^i \Phi D_{(1,0)}^{s-k-1} \Phi \quad (4.5.14) \]

The flat-superspace version of (4.5.14) is given by eq. (3.4.3) and Ref. [119].

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4.5.3 Massive model with $N$ chiral superfields

Let us turn to the massive model (4.4.21). As was discussed in previous subsection, to construct the conserved currents we first have to calculate $D_{(−1,0)}J_{(s,s−1)}$ using the equations of motion in the massive theory. The calculation depends on whether $C^{ij}$ is symmetric or antisymmetric.

4.5.3.1 Symmetric $C$

If $C^{ij}$ is a symmetric matrix, using the massive equation of motion, we obtain

\[
D_{(−1,0)}J_{(s,s−1)} = -2(s + 1)(CM)^{ij} \sum_{k=0}^{s-1} (-1)^{k+1} \binom{s - 1}{k} \frac{s - k}{k + 1} \\
\times D_{(1,1)}^k \bar{\Phi}^i D_{(1,1)}^{s-k-1} \Phi^j \\
+ 2(s + 1)(CM)^{ij} \sum_{k=0}^{s-1} (-1)^{k+1} \binom{s - 1}{k} \frac{k}{k + 1} \\
\times D_{(1,1)}^k \bar{\Phi}^i D_{(1,1)}^{s-k-1} \Phi^j \\
+ 2(s + 1)(CM)^{ij} \sum_{k=1}^{s-2} (-1)^{k+1} \binom{s - 1}{k} \frac{s - 1 - k}{k + 1} \\
\times D_{(1,1)}^{k-1} D_{(0,1)} \bar{\Phi}^i D_{(1,1)}^{s-k-2} D_{(0,1)} \bar{\Phi}^j.
\] (4.5.15)

Here we have two cases to consider:

1. $CM$ is symmetric $\iff [C, \bar{M}] = 0$, $s$ even.
2. $CM$ is antisymmetric $\iff \{C, \bar{M}\} = 0$, $s$ even.

Case 1: Eq. (4.5.15) can be simplified to yield

\[
D_{(−1,0)}J_{(s,s−1)} = 4(s + 1)(CM)^{ij} \sum_{k=0}^{s-1} (-1)^{k+1} \binom{s - 1}{k} \frac{s}{k} \\
\times D_{(1,1)}^k \bar{\Phi}^i D_{(1,1)}^{s-k-1} \Phi^j \\
+ 4(s + 1)(CM)^{ij} \sum_{k=1}^{s-2} (-1)^{k+1} \binom{s - 1}{k} \frac{k}{k + 1} \\
\times D_{(1,1)}^{k-1} D_{(0,1)} \bar{\Phi}^i D_{(1,1)}^{s-k-2} D_{(0,1)} \Phi^j.
\] (4.5.16)
We now look for a superfield $T_{(s-1,s-1)}$ such that (i) it obeys the longitudinal linear constraint \[4.3.29\]; and (ii) it satisfies \[4.3.31\], which is a consequence of the conservation equation \[4.3.30\]. The precise form of eq. \[4.3.31\] in the present case is

$$
\frac{1}{2s} D_{(1,0)} \left\{ D_{(-1,0)} J_{(s,s-1)} + D_{(0,-1)} J_{(s-1,s)} \right\}
= \frac{2}{s+1} D_{(1,0)} \sum_{k=0}^{s-1} (-1)^{k+1} \binom{s-1}{k} \binom{s}{k} 
\times \left\{ \frac{s}{k+1} (CM)^{ij} - \frac{(s+1)(s-k)}{(k+1)(k+2)} (CM)^{ij} \right\}
\times D_{(1,1)}^{k} \Phi^{i} D_{(1,1)}^{s-k-1} \Phi^{j}
= -D_{(1,0)} T_{(s-1,s-1)}. \tag{4.5.17}
$$

To find $T_{(s-1,s-1)}$ we consider a general ansatz

$$
T_{(s-1,s-1)} = \sum_{k=0}^{s-1} (c_{k})^{ij} D_{(1,1)}^{k} \Phi^{i} D_{(1,1)}^{s-k-1} \Phi^{j}
+ \sum_{k=1}^{s-1} (d_{k})^{ij} D_{(1,1)}^{k} D_{(1,0)} \Phi^{j} D_{(1,1)}^{s-k-1} \Phi^{j}. \tag{4.5.18}
$$

It is possible to show that no solution for $T_{(s-1,s-1)}$ can be found unless we impose\[5\]

$$
CM = \bar{C} M. \tag{4.5.19}
$$

Furthermore, condition (i) implies that the coefficients must be related by

$$
(c_{0})^{ij} = 0, \quad (c_{k})^{ij} = (d_{k})^{ij}, \tag{4.5.20a}
$$

while for $k = 1, 2, \ldots s - 2$, while condition (ii) and eq. \[4.5.19\] gives the following recurrence relations

$$
(d_{k})^{ij} + (d_{k+1})^{ij} = -2 \frac{(s+1)}{s} (CM)^{ij} (-1)^{k+1} \binom{s-1}{k} \binom{s}{k}
\times \frac{1}{k+1} \left\{ s - \frac{(s+1)(s-k)}{k+2} \right\}. \tag{4.5.20b}
$$

Condition (ii) also implies that

$$
(d_{1})^{ij} = (1-s^{2})(CM)^{ij}, \quad (d_{s-1})^{ij} = \frac{2(1-s^{2})}{s} (CM)^{ij}. \tag{4.5.20c}
$$

The above conditions lead to simple expressions for $d_{k}$:

$$
(d_{k})^{ij} = \frac{2(s+1)}{s} (CM)^{ij} \frac{k}{k+1} (-1)^{k} \binom{s-1}{k} \binom{s}{k}, \tag{4.5.21}
$$

where $k = 1, 2, \ldots s - 1$ and $s$ is even.

**Case 2:** If we take $CM$ to be antisymmetric, a similar analysis shows that no solution for $T_{(s-1,s-1)}$ exists for even values of $s$.

---

\[5\] Since $C$ and $M$ commute we can take them both to be diagonal, $C = \text{diag}(c_{1}, \ldots, c_{N})$, $M = \text{diag}(m_{1}, \ldots, m_{N})$. Then the condition \[4.5.19\] means that $\arg(c_{i}) = \arg(m_{i}) = n_{i} \pi$ for some integers $n_{i}$.
4.5.3.2 Antisymmetric \( C \)

If \( C^{ij} \) is antisymmetric we get:

\[
D_{(-1,0)} J_{(s,s-1)} = 2(s + 1)(CM)^{ij} \sum_{k=0}^{s-1} (-1)^{k+1} \binom{s - 1}{k} \binom{s}{k} \frac{s - k}{k + 1} \\
\times D^k (1,1) \Phi^i D^{s-k-1} \Phi^j \\
+ 2(s + 1)(CM)^{ij} \sum_{k=0}^{s-1} (-1)^{k+1} \binom{s - 1}{k} \binom{s}{k} \\
\times D^k (1,1) \Phi^i D^{s-k-1} \Phi^j \\
- 2(s + 1)(CM)^{ij} \sum_{k=1}^{s-1} (-1)^{k+1} \binom{s - 1}{k} \binom{s}{k} \frac{k}{k + 1} \\
\times D^{k-1} (0,1) \Phi^i D^{s-k-1} D_{(1,0)} \Phi^j \\
+ 2(s + 1)(CM)^{ij} \sum_{k=0}^{s-2} (-1)^{k+1} \binom{s - 1}{k} \binom{s}{k} \frac{s - 1 - k}{k + 1} \\
\times D^k (1,1) D_{(1,0)} \Phi^i D^{s-k-2} D_{(0,1)} \Phi^j .
\] (4.5.22)

As in the symmetric \( C \) case, there are also two cases to consider:

1. \( CM \) is symmetric \( \iff \{ C, M \} = 0, s \) odd.
2. \( CM \) is antisymmetric \( \iff [ C, M ] = 0, s \) odd.

**Case 1:** Using eq. (4.5.22) and keeping in mind that \( s \) is odd, we obtain

\[
D_{(-1,0)} J_{(s,s-1)} = 4(s + 1)(CM)^{ij} \sum_{k=0}^{s-1} (-1)^{k+1} \binom{s - 1}{k} \binom{s}{k} \\
\times D^k (1,1) \Phi^i D^{s-k-1} \Phi^j \\
- 4(s + 1)(CM)^{ij} \sum_{k=1}^{s-1} (-1)^{k+1} \binom{s - 1}{k} \binom{s}{k} \frac{k}{k + 1} \\
\times D^{k-1} (0,1) \Phi^i D^{s-k-1} D_{(1,0)} \Phi^j .
\] (4.5.23)

Then it follows that eq. (4.3.31) becomes

\[
\frac{1}{2s} D_{(1,0)} \left\{ D_{(-1,0)} J_{(s,s-1)} + D_{(0,-1)} J_{(s-1,s)} \right\} \\
= \frac{2}{s + 1} D_{(1,0)} \sum_{k=0}^{s-1} (-1)^{k+1} \binom{s - 1}{k} \binom{s}{k} \\
\times \left\{ \frac{s}{k + 1} (CM)^{ij} - \frac{(s + 1)(s - k)}{(k + 1)(k + 2)} (CM)^{ij} \right\}
\]
\[ \sum_{D} k^i \Phi^i D_{s-k-1} \Phi_j = -D_{(1,0)} T_{(s-1,s-1)}. \]  

Note that it is the equation same as eq. (4.5.17) which means that the solution for \( T_{(s-1,s-1)} \) is the same as in Case 1. That is, the matrices \( C \) and \( M \) must satisfy \( CM = \bar{C}M \), \( T_{(s-1,s-1)} \) is given by eq. (4.5.18) and the coefficients \((c_k)_{ij}, (d_k)_{ij}\) are given by eqs. (4.5.20).

**Case 2:** If we take \( C \bar{M} \) to be antisymmetric, a similar analysis shows that no solution for \( T_{(s-1,s-1)} \) exists for odd values of \( s \).

### 4.5.3.3 Massive hypermultiplet model revisited

As a consistency check of our general method, let us reconsider the case of a hypermultiplet studied previously. For this we will take \( N = 2 \), the mass matrix in the form

\[ M = \begin{pmatrix} 0 & m \\ m & 0 \end{pmatrix}, \]  

and denote \( \Phi^i = (\Phi_+, \Phi_-) \). If \( s \) is even we will take \( C \) in the form

\[ C = \begin{pmatrix} 0 & c \\ c & 0 \end{pmatrix}. \]  

Note that \( C \) commutes with \( M \). The condition \( CM = \bar{C}M \) is equivalent to \( \arg(c) = \arg(m) + n\pi \). For simplicity, let us choose both \( c \) and \( m \) to be real. Under these conditions eq. (4.5.11) for \( J_{(s,s-1)} \) becomes

\[ J_{(s,s-1)} = c \sum_{k=0}^{s-1} (-1)^k \begin{pmatrix} s-1 \\ k \end{pmatrix} \begin{pmatrix} s \\ k+1 \end{pmatrix} \left\{ D_{(1,0)}^k D_{(1,0)} \Phi_+ + D_{(1,1)}^{s-k-1} \Phi_- \right\} \\
+ D_{(1,1)}^k D_{(1,0)} \Phi_- D_{(1,1)}^{s-k-1} \Phi_+ \right\} \\
+ c \sum_{k=0}^{s-1} (-1)^{k+1} \begin{pmatrix} s-1 \\ k \end{pmatrix} \begin{pmatrix} s \\ k \end{pmatrix} \left\{ D_{(1,1)}^k \Phi_+ + D_{(1,1)}^{s-k-1} D_{(1,0)} \Phi_- \right\} \\
+ D_{(1,1)}^k \Phi_- D_{(1,1)}^{s-k-1} D_{(1,0)} \Phi_+ \right\}. \]  

Introducing a new summation variable \( k' = s - 1 - k \) for the second and fourth terms, we obtain

\[ J_{(s,s-1)} = c \sum_{k=0}^{s-1} (-1)^k \begin{pmatrix} s-1 \\ k \end{pmatrix} \begin{pmatrix} s \\ k+1 \end{pmatrix} \left[(1 + (-1)^s) D_{(1,1)}^k D_{(1,0)} \Phi_+ + D_{(1,1)}^{s-k-1} \Phi_- \right] \\
- c \sum_{k=0}^{s-1} (-1)^k \begin{pmatrix} s-1 \\ k \end{pmatrix} \begin{pmatrix} s \\ k \end{pmatrix} \left[(1 + (-1)^s) D_{(1,1)}^k \Phi_+ + D_{(1,1)}^{s-k-1} D_{(1,0)} \Phi_- \right]. \]
We see that for even $s$ it coincides with the hypermultiplet supercurrent given by \(4.5.4\) up to an overall coefficient $2c$. If $s$ is odd we have to choose $C$ to be antisymmetric
\[
C = \begin{pmatrix} 0 & c \\ -c & 0 \end{pmatrix}.
\] (4.5.29)

Note that $C$ now anticommutes with $M$. For simplicity, we again choose $c$ and $m$ to be real. Now the expression \(4.5.11\) for $J(s,s-1)$ becomes
\[
J(s,s-1) = c \sum_{k=0}^{s-1} (-1)^k \binom{s-1}{k} \binom{s}{k+1} \left[ (1 - (-1)^s) D_{(1,1)}^k D_{(1,0)}^\Phi + D_{(1,1)}^{s-k-1} D_{(1,0)}^\Phi \right] \\
-c \sum_{k=0}^{s-1} (-1)^k \binom{s-1}{k} \binom{s}{k} \left[ (1 - (-1)^s) D_{(1,1)}^k D_{(1,1)}^\Phi + D_{(1,0)}^{s-k-1} D_{(0,1)}^\Phi \right].
\] (4.5.30)

We see that for odd $s$ it coincides with the hypermultiplet supercurrent given by \(4.5.4\) up to an overall coefficient $2c$. To summarise, we reproduced the hypermultiplet supercurrent \(4.5.4\) for both even and odd values of $s$. However, for even $s$ it came from a symmetric matrix \(4.5.26\) and for odd $s$ it came from an antisymmetric matrix \(4.5.29\).

Let us now consider $T_{(s-1,s-1)}$. First, we will note that the product $C\bar{M}$ is given by
\[
C\bar{M} = cm \begin{pmatrix} 1 & 0 \\ 0 & (-1)^s \end{pmatrix}.
\] (4.5.31)

This means that $T_{(s-1,s-1)}$ is given by the following expression valid for all values of $s$
\[
T_{(s-1,s-1)} = \sum_{k=0}^{s-1} (d_k)^{ij} \left[ D_{(1,1)}^k D_{(1,1)}^\Phi D_{(1,0)}^{s-k-1} \bar{\Phi}^j + D_{(1,1)}^{s-k-1} D_{(1,0)}^\Phi D_{(1,1)}^\Phi D_{(0,1)}^{s-k} \bar{\Phi}^j \right],
\] (4.5.32)

where the matrix $(d_k)^{ij}$ is given by
\[
(d_k)^{ij} = 2cm \frac{s+1}{s} \frac{k}{k+1} (-1)^k \binom{s-1}{k} \binom{s}{k} \begin{pmatrix} 1 & 0 \\ 0 & (-1)^s \end{pmatrix}.
\] (4.5.33)

It is easy to see that this expression for $T_{(s-1,s-1)}$ coincides with the one obtained for the hypermultiplet in the previous subsections in e.g. \(4.5.8\), \(4.5.9\), \(4.5.10\) up to an overall factor $2c$.

4.6 Summary and applications

In this chapter, we have described higher-spin conserved supercurrents for $\mathcal{N} = 1$ supersymmetric theories in four-dimensional anti-de Sitter space. We have explicitly constructed such supercurrents in the case of $N$ chiral scalar superfields with an arbitrary
mass matrix $M$. The structure of the supercurrents depends on whether the superspin is integer or half-integer, as well as on the value of the superspin, and the mass matrix. Let us summarise our results.

In the case of half-integer superspin-$\left(s + \frac{1}{2}\right)$, the supercurrent has the structure $J_{(s,s)} = H_{ij} J^{ij}_{(s,s)}$, where $i, j = 1, \ldots, N$ and $H_{ij}$ is a Hermitian matrix. The precise form of $J^{ij}_{(s,s)}$ was discussed in section 4.4. In the massless theory it is conserved for all values of $s$. In the massive theory, the conservation equation involves an additional complex multiplet $T_{(s-1,s-2)}$ whose existence depends on the value of $s$ and the mass matrix. For odd values of $s$, it exists provided $[S, \bar{M}] = 0$, $\{A, \bar{M}\} = 0$, where $S$ and $A$ are the symmetric and antisymmetric parts of $H$, respectively. When $s$ is even, it exists provided $\{S, \bar{M}\} = 0$, $[A, \bar{M}] = 0$.

In the case of integer superspin-$s$, the fermionic supercurrent was discussed in section 4.5. It has the form $J_{(s,s-1)} = C_{ij} J^{ij}_{(s,s-1)}$. In the massless theory it exists for even values of $s$ if $C$ is symmetric and for odd values of $s$ if $C$ is antisymmetric. In the massive theory the conservation equation involves an additional complex multiplet $T_{(s-1,s-1)}$ and a real multiplet $S_{(s-1,s-1)}$. Their existence also depends on the value of $s$. For $s$ even they exist provided $CM = \bar{C}M$, $[C, \bar{M}] = 0$ and for $s$ odd provided $CM = \bar{C}M$, $\{C, \bar{M}\} = 0$.

It should be mentioned that in the non-supersymmetric case, conserved higher-spin currents for scalar and spinor fields in Minkowski space have been studied extensively in the past. Appendices B.1 and B.2 review the construction of conserved higher-spin currents for $N$ scalars and spinors, respectively, with arbitrary mass matrices. These results are scattered in the literature, including [102–105].

In the rest of this section, we will discuss several applications of the results obtained.

### 4.6.1 Higher-spin supercurrents for a tensor multiplet

Let us consider a special case of the non-supersymmetric model (4.4.9) with the mass parameter $m = \mu$,

$$S[\Phi, \bar{\Phi}] = \frac{1}{2} \int d^4x d^2\theta d^2\bar{\theta} E (\Phi + \bar{\Phi})^2, \quad D\alpha \Phi = 0. \quad (4.6.1)$$

This theory is known to be dual to a tensor multiplet model [151]

$$S[L] = -\frac{1}{2} \int d^4x d^2\theta d^2\bar{\theta} E L^2, \quad (4.6.2)$$

which is realised in terms of a real linear superfield $L = \bar{L}$, constrained by $(\bar{D}^2 - 4\mu)L = 0$, which is the gauge-invariant field strength of a chiral spinor superfield

$$L = D^\alpha \eta_\alpha + \bar{D}_\beta \eta^\beta, \quad \bar{D}_\beta \eta_\alpha = 0. \quad (4.6.3)$$
We recall that the duality between (4.6.1) and (4.6.2) follows, e.g., from the fact the off-shell constraint

\[(\bar{D}^2 - 4\mu)(\Phi + \bar{\Phi}) = 0\]  \hspace{1cm} (4.6.4a)

and the equation of motion for \(\Phi\)

\[(\bar{D}^2 - 4\mu)(\Phi + \bar{\Phi}) = 0\]  \hspace{1cm} (4.6.4b)

are equivalent to the equation of motion for \(\eta_\alpha\)

\[(\bar{D}^2 - 4\mu)L = 0\]  \hspace{1cm} (4.6.5a)

and the off-shell constraint

\[(\bar{D}^2 - 4\mu)L = 0\], \hspace{1cm} (4.6.5b)

respectively.

Higher-spin supercurrents for the tensor model (4.6.2) can be obtained from the results derived in subsection 4.4.2 in conjunction with an improvement transformation of the type (4.3.20) with \(\Omega = -\frac{1}{2}\Phi^2\). Given an odd \(s = 3, 5, \ldots\), for the supercurrent we get

\[J_{(s,s)} = -L \: [D^s_{(1,1)}, \bar{D}^s_{(0,1)}]L\]

\[+ \sum_{k=0}^{s-1} (-1)^k \binom{s}{k} \binom{s}{k+1} D^k_{(1,1)} [D^s_{(1,0)}, \bar{D}^k_{(0,1)}]L \]

\[+ \frac{1}{2} \sum_{k=1}^{s-1} \left\{ -1 + (-1)^k \binom{s}{k} \right\} \left( \binom{s}{k} D^{s-k-1}_{(1,1)} [D^s_{(1,0)}, \bar{D}^{s-k}_{(0,1)}]L D^{s-k}_{(1,1)} \right). \]  \hspace{1cm} (4.6.6)

The corresponding trace multiplet proves to be

\[T_{(s-1,s-2)} = -\frac{4\mu}{s} L \: [D^{s-2}_{(1,1)}, D^s_{(1,0)}]L + 4\mu \frac{s+1}{s} D^s_{(1,0)} L D^{s-2}_{(1,1)} [D^s_{(1,0)}, \bar{D}^1_{(0,1)}]L \]

\[+ \frac{2}{s} D^{s-2}_{(1,1)} \left\{ D^s_{(1,0)} \bar{D}^1_{(0,1)} L \right\} \]

\[+ \mu \sum_{k=1}^{s-2} c_k D^{s-k}_{(1,1)} [D^s_{(1,0)}, \bar{D}^k_{(0,1)}]L D^{s-k}_{(1,1)} \]

\[+ \frac{4\mu}{s} \sum_{k=1}^{s-2} \binom{s-2}{k} D^{s-k-1}_{(1,1)} [D^s_{(1,0)}, \bar{D}^{s-k}_{(0,1)}]L D^{s-k-2}_{(1,1)} \]

\[+ 2\mu \frac{s+1}{s} \sum_{k=1}^{s-3} \binom{s-2}{k} \left\{ D^k_{(1,1)} [D^s_{(1,0)}] L D^{s-k-3}_{(1,1)} [D^s_{(1,0)}, \bar{D}^{s-k}_{(0,1)}]L \right\} + \bar{D}^{k-1}_{(1,1)} [D^s_{(1,0)}] L D^{s-k-2}_{(1,1)} [D^s_{(1,0)}]L \]

\hspace{1cm} (4.6.7)
The coefficient $c_k$ is given by eq. (4.4.15), $s$ is odd. The Ferrara-Zumino supercurrent $(s = 1)$ for the model (4.6.2) in an arbitrary supergravity background was derived in section 6.3 of [35]. Modulo normalisation, the AdS supercurrent is

$$J_{\alpha \dot{\alpha}} = \bar{D}_\alpha LD_{\alpha}L + L[D_{\alpha}, \bar{D}_\alpha]L,$$

and the corresponding trace multiplet is

$$T = \frac{1}{4}(\bar{D}^2 - 4\mu)L^2.$$  

The supercurrent obeys the conservation equation (2.4.1).

### 4.6.2 Higher-spin supercurrents for a complex linear multiplet

Conserved higher-spin supercurrents for a complex linear multiplet in Minkowski superspace were first studied by Koutrolikos, Koči and von Unge [116], as an extension of the lower-spin case [152]. In AdS, the superconformal non-minimal scalar multiplet is described by the action

$$S[\Gamma, \bar{\Gamma}] = -\int d^4x d^2\theta d^2\bar{\theta} E \bar{\Gamma} \Gamma,$$

where $\Gamma$ is a complex linear scalar, $(\bar{D}^2 - 4\mu)\Gamma = 0$. This is a dual formulation for the superconformal chiral model (4.4.1). As is well known, the duality between (4.4.1) and (4.6.9) follows from the fact that the off-shell constraint

$$(D^2 - 4\bar{\mu})\bar{\Gamma} = 0,$$

and the equation of motion for $\Gamma$

$$\bar{D}_\alpha \bar{\Gamma} = 0$$

are equivalent to the equation of motion for $\Phi$, $(D^2 - 4\mu)\Phi = 0$, and the off-shell constraint $\bar{D}_\alpha \Phi = 0$, respectively. In other words, on the mass shell we can identify $\bar{\Gamma}$ with $\Phi$.

The higher-spin supercurrents, $J_{(s,s)}$ and $J_{(s,s-1)}$, for the model (4.6.9) are obtained from (4.4.2) and (4.5.14), respectively, by replacing $\Phi$ with $\bar{\Gamma}$. The fermionic supercurrent $J_{(s,s-1)}$ exists for even values of $s$. Indeed, in Minkowski superspace, the expression for $J_{(s,s)}$ obtained coincides with the main result of Ref. [116], which applied the Noether procedure to generate cubic vertices between massless higher-spin supermultiplets and the free complex linear superfield model

$$S[\Gamma, \bar{\Gamma}] = -\int d^4x d^2\theta d^2\bar{\theta} E \bar{\Gamma} \Gamma, \quad D^2 \Gamma = 0.$$ 

See also [153] for the discussion of the fermionic supercurrent $J_{(s,s-1)}$.  

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4.6.3 Gauge higher-spin multiplets and conserved supercurrents

For each of the two off-shell formulations for the massless multiplet of half-integer superspin-\((s + \frac{1}{2})\), with \(s = 2, 3, \ldots\), which we reviewed in section 4.3.1 it was shown in [63] that there exists a gauge-invariant field strength \(W_{\alpha(2s+1)}\) which is covariantly chiral, \(\mathcal{D}_\beta W_{\alpha(2s+1)} = 0\), and is given by the expression

\[
W_{\alpha(2s+1)} = -\frac{1}{4}(\mathcal{D}^2 - 4\mu)\mathcal{D}_{(\alpha_1} \beta_1 \ldots \mathcal{D}_{(\alpha_s} \beta_s \mathcal{D}_{\alpha_{s+1}} H_{\alpha_{s+2} \ldots \alpha_{2s+1})\beta_1 \ldots \beta_s} .
\]  
(4.6.12)

It was also shown in [63] that on the mass shell it holds that (i) \(W_{\alpha(2s+1)}\) and its conjugate \(\bar{W}_{\dot{\alpha}(2s+1)}\) are the only independent gauge-invariant field strengths; and (ii) \(W_{\alpha(2s+1)}\) obeys the irreducibility condition

\[
\mathcal{D}^\beta W_{\beta\alpha(2s)} = 0 .
\]  
(4.6.13)

The relations (4.6.12) and (4.6.13) also hold for the cases \(s = 0\) and \(s = 1\), which correspond to the vector multiplet and linearised supergravity, respectively. In terms of \(W_{\alpha(2s+1)}\) and \(\bar{W}_{\dot{\alpha}(2s+1)}\), we can define the following higher-spin supercurrent

\[
J_{\alpha(2s+1)\dot{\alpha}(2s+1)} = W_{\alpha(2s+1)\bar{W}_{\dot{\alpha}(2s+1)}} , \quad s = 0, 1, \ldots ,
\]  
(4.6.14)

which obeys the conservation equation

\[
\mathcal{D}_{(0,1)} J_{(2s+1,2s+1)} = 0 \iff \mathcal{D}_{(-1,0)} J_{(2s+1,2s+1)} = 0 .
\]  
(4.6.15)

In the case of the longitudinal formulation for the massless multiplet of integer superspin-\(s\), with \(s = 2, 3, \ldots\), which we described in section 4.2 it was shown in [63] that there exists a gauge-invariant field strength \(W_{\alpha(2s)}\) which is covariantly chiral, \(\mathcal{D}_\beta W_{\alpha(2s)} = 0\), and is given by the expression

\[
W_{\alpha(2s)} = -\frac{1}{4}(\mathcal{D}^2 - 4\mu)\mathcal{D}_{(\alpha_1} \dot{\beta}_1 \ldots \mathcal{D}_{(\alpha_s} \dot{\beta}_s \mathcal{D}_{\alpha_{s+1}} \Psi_{\alpha_{s+2} \ldots \alpha_{2s+1})\beta_1 \ldots \beta_{s-1}} .
\]  
(4.6.16)

As demonstrated in [63], on the mass shell it holds that (i) \(W_{\alpha(2s)}\) and its conjugate \(\bar{W}_{\dot{\alpha}(2s)}\) are the only independent gauge-invariant field strengths; and (ii) \(W_{\alpha(2s)}\) obeys the irreducibility condition

\[
\mathcal{D}^\beta W_{\beta\alpha(2s-1)} = 0 .
\]  
(4.6.17)

The relations (4.6.16) and (4.6.17) also hold for the case \(s = 1\), which corresponds to the gravitino multiplet. In terms of \(W_{\alpha(2s)}\) and \(\bar{W}_{\dot{\alpha}(2s)}\), we can define the higher-spin supercurrent

\[
J_{\alpha(2s)\dot{\alpha}(2s)} = W_{\alpha(2s)\bar{W}_{\dot{\alpha}(2s)}} , \quad s = 1, 2, \ldots ,
\]  
(4.6.18)

\[\text{The flat-superspace version of (4.6.16) is given in section 6.9 of [35].}\]
which obeys the conservation equation

\[ D_{(0,-1)}J_{(2s,2s)} = 0 \iff D_{(-1,0)}J_{(2s,2s)} = 0. \]  

(4.6.19)

The conserved supercurrents \( J_{\alpha(n)\bar{\alpha}(n)} = W_{\alpha(n)}\bar{W}_{\bar{\alpha}(n)} \), with \( n = 1, 2, \ldots \), are the AdS extensions of those introduced many years ago by Howe, Stelle and Townsend [69].

Now, for any positive integer \( n > 0 \), we can try to generalise the higher-spin supercurrent (4.4.2) as follows:

\[ J_{(s+n,s+n)} = \sum_{k=0}^{s} (-1)^k \binom{s}{n+1} \binom{s+n}{n+k} \left\{ (-1)^n \frac{s-k}{n+k+1} D_{(1,1)}^k D_{(1,0)} W_{(n,0)} D_{(1,1)}^{s-k} D_{(0,1)} \bar{W}_{(0,n)} + D_{(1,1)}^k W_{(n,0)} D_{(1,1)}^{s-k} \bar{W}_{(0,n)} \right\}. \]  

(4.6.20)

Making use of the on-shell condition

\[ D_{(-1,0)} W_{(n,0)} = 0 \iff (D^2 - 2(n + 2)\mu) W_{(n,0)} = 0, \]  

(4.6.21)

one may check that

\[ D_{(-1,0)} J_{(s+n,s+n)} = 2n\mu \sum_{k=0}^{s-1} (-1)^{n+k} \frac{s-k}{n+k+1} \binom{s}{n+1} \binom{s+n}{n+k} \]

\[ \times D_{(1,1)}^k W_{(n,0)} D_{(1,1)}^{s-k} D_{(0,1)} \bar{W}_{(0,n)} \cdot \]  

(4.6.22)

This demonstrates that \( J_{(s+n,s+n)} \) is not conserved in AdS\(^4\).

In the flat-superspace limit, \( \mu \to 0 \), the right-hand side of (4.6.22) vanishes and \( J_{(s+n,s+n)} \) becomes conserved. In Minkowski superspace, the conserved supercurrent \( J_{(s+n,s+n)} \) was recently constructed in [119] as an extension of the non-supersymmetric approach [154].

As a generalisation of the conserved supercurrents \( J_{\alpha(n)\bar{\alpha}(n)} = W_{\alpha(n)}\bar{W}_{\bar{\alpha}(n)} \), one can introduce

\[ J_{\alpha(n)\bar{\alpha}(m)} = W_{\alpha(n)}\bar{W}_{\bar{\alpha}(m)} \]  

(4.6.23)

with \( n \neq m \). They obey the conservation equations

\[ \mathcal{D}_{(0,-1)} J_{(n,m)} = 0, \quad D_{(-1,0)} J_{(n,m)} = 0 \]  

(4.6.24)

and can be viewed as Noether currents for the generalised superconformal higher-spin multiplets introduced in [68]. Starting from the conserved supercurrents (4.6.23), one can construct a generalisation of (4.6.20). We will not elaborate on a construction here.
Chapter 5

$\mathcal{N} = 2$ supersymmetric higher-spin gauge theories and current multiplets in three dimensions

In four dimensions, there exists a correspondence between $\mathcal{N} = 1$ anti-de Sitter (AdS) supergravity \cite{110} and the two dually equivalent series of massless multiplets of half-integer superspin-$(s + \frac{1}{2})$, with $s = 1, 2, \ldots$ \cite{63}. Specifically, there are two off-shell formulations for pure $\mathcal{N} = 1$ AdS supergravity: minimal (see e.g. \cite{35,56} for reviews) and non-minimal \cite{99}. These theories possess a single maximally supersymmetric solution, which is the $\mathcal{N} = 1$ AdS superspace AdS$^{4|4}$. For the lowest superspin value corresponding to $s = 1$, the longitudinal series yields the linearised action for minimal AdS supergravity, while the transverse one leads to linearised non-minimal AdS supergravity.

In three dimensions, the AdS group is a product of two simple groups, $\text{SO}(2, 2) \cong (\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R}))/\mathbb{Z}_2$, and so are its simplest supersymmetric extensions, $\text{OSp}(p|2; \mathbb{R}) \times \text{OSp}(q|2; \mathbb{R})$. This implies that $\mathcal{N}$-extended AdS supergravity exists in several incarnations \cite{111}. These are known as the $(p, q)$ AdS supergravity theories, where the non-negative integers $p \geq q \geq 0$ are such that $\mathcal{N} = p + q$. Superspace approach to 3D $\mathcal{N}$-extended conformal supergravity was developed by Kuzenko, Lindström and Tartaglino-Mazzucchelli \cite{113}, and used to construct off-shell $\mathcal{N} \leq 4$ supergravity-matter couplings. The formalism of \cite{113} was then applied to study the geometry of $(p, q)$ AdS superspaces \cite{155}. The so-called $(p, q)$ AdS superspace \cite{155}

$$\text{AdS}^{(3|p,q)} = \frac{\text{OSp}(p|2; \mathbb{R}) \times \text{OSp}(q|2; \mathbb{R})}{\text{SL}(2, \mathbb{R}) \times \text{SO}(p) \times \text{SO}(q)}$$

can be realised as a maximally symmetric solution of $(p, q)$ AdS supergravity (see \cite{155} for the technical details).

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In the case of 3D $\mathcal{N} = 2$ supersymmetry, there exist two distinct AdS superspaces, $\text{AdS}^{(3|1,1)}$ and $\text{AdS}^{(3|2,0)}$. The former is the 3D counterpart of the 4D $\mathcal{N} = 1$ AdS superspace, while the latter has no 4D analogue. The existence of these superspaces and their superconformal flatness were studied for the first time in [112]. Ref. [83] presented superfield formulations for 3D $\mathcal{N} = 2$ AdS supergravity theories and their corresponding supercurrent multiplets. Two off-shell formulations for (1,1) AdS supergravity have been developed: minimal [83,113,156–160] and non-minimal [83,160] theories; and one for (2,0) AdS supergravity [83,113,160,161]. $\text{AdS}^{(3|1,1)}$ is the unique maximally symmetric solution of the two dually equivalent (1,1) AdS supergravity theories, minimal and non-minimal ones. $\text{AdS}^{(3|2,0)}$ is the unique maximally symmetric solution of the (2,0) AdS supergravity. This supergravity theory was originally formulated in [161] in the component setting. The early superspace descriptions of the minimal (1,1) supergravity were given in [157,158].

It has recently been pointed out [4] that the correspondence between AdS supergravity theories and massless higher-spin supermultiplets in 3D anti-de Sitter space, $\text{AdS}_3$, might occur in the $\mathcal{N} = 2$ case. Since there are three off-shell $\mathcal{N} = 2$ AdS supergravity theories, one might expect the existence of three series of massless higher-spin gauge supermultiplets. Two series of massless higher-spin actions associated with the minimal and the non-minimal (1,1) AdS supergravity theories were presented in [4]. These generalise similar constructions in the super-Poincaré case [49]. As will be explained in sections 5.6 and 5.7, the off-shell higher-spin supermultiplets in (2,0) AdS superspace [5] were constructed using a different approach.

Pure $\mathcal{N} = 2$ supergravity (massless superspin-3/2 multiplet) and its higher-spin extensions have no propagating degrees of freedom in three dimensions. Nevertheless, there are at least two nontrivial applications of the massless higher-spin gauge supermultiplets. Firstly, one can follow the pattern of topologically massive (super)gravity [162–165] and construct massive higher-spin supermultiplets by combining a massless action with a higher-spin extension of the action for linearised conformal supergravity. This has been achieved in [49] in the $\mathcal{N} = 2$ super-Poincaré case, and similar ideas have been implemented in the frameworks of $\mathcal{N} = 1$ Poincaré and AdS supersymmetry [50,51]. Topologically massive higher-spin supermultiplets in (1,1) and (2,0) AdS superspaces have been formulated in [4] and [5], respectively. The second application is to develop a 3D extension of the higher-spin supercurrents presented in chapters [3] and [4]. Specifically, making use of the off-shell formulations for massless higher-spin supermultiplets in $\text{AdS}_3$, one can define consistent higher-spin supercurrent multiplets that contain ordinary bosonic and fermionic conserved currents in $\text{AdS}_3$. One can then look for explicit realisations of such higher-spin supercurrents in concrete supersymmetric theories in $\text{AdS}_3$ [4].

This chapter can be divided into two parts: sections 5.1 to 5.5 focus on rigid supersymmetric higher-spin gauge theories in (1,1) AdS superspace which were studied in [4], while
sections 5.6 and 5.7 are concerned with the construction of off-shell massless higher-spin
gauge multiplets with (2,0) AdS supersymmetry as described in \[5\]. In section 5.1 we
review the superspace geometry of 3D \( \mathcal{N} = 2 \) conformal supergravity. We then introduce
primary linear supermultiplets and conformal higher-spin gauge superfields coupled to
\( \mathcal{N} = 2 \) conformal supergravity, the latter being one of the key ingredients in constructing
massless higher-superspin actions. Section 5.2 reviews the two inequivalent \( \mathcal{N} = 2 \) AdS
superspaces. Two dual off-shell Lagrangian formulations for every massless higher-spin
supermultiplet in (1,1) AdS superspace will be presented in sections 5.3 and 5.4. As in
the 4D AdS constructions, the two cases of half-integer and integer superspin, as well
as massless gravitino multiplet have to be treated separately. Section 5.5 is devoted to
constructing non-conformal higher-spin supercurrent multiplets in models for chiral scalar
superfields. The materials presented in sections 5.1, 5.3 and subsections 5.4.1–5.4.2 are
based on the work by Kuzenko and Ogburn \[4\]. Here I only include a summary of those
results which are essential for constructing a new off-shell model for the massless integer
superspin, as well as describing (1,1) AdS higher-spin supercurrents.

Starting with simple models for a chiral scalar supermultiplet in (2,0) AdS superspace,
in section 5.6 we obtain the conservation equation obeyed by the multiplet of higher-spin
currents. This will allow us to determine the off-shell gauge superfields which couple
to the current multiplet. Two off-shell formulations for a massless multiplet of half-
integer superspin in (2,0) AdS superspace are developed in section 5.7. Our results, their
implications and possible extensions are discussed in section 5.8.

5.1 Superconformal higher-spin multiplets

Before presenting superconformal higher-spin multiplets, let us first give a succinct
review of the formulation for \( \mathcal{N} = 2 \) conformal supergravity following \[113\]. There exists a
more general formulation for conformal supergravity \[166\], the so-called \( \mathcal{N} = 2 \) conformal
superspace. However, for our purposes it suffices to use the formulation of \[113\], which
is obtained from the \( \mathcal{N} = 2 \) conformal superspace by partially fixing the gauge freedom.
The reader is referred to appendix A.2 for more details on our 3D conventions.

5.1.1 Conformal supergravity

All known off-shell formulations for 3D \( \mathcal{N} = 2 \) supergravity \[83,113\] can be realised in
a curved superspace \( \mathcal{M}^{3|4} \) with the structure group \( \text{SL}(2,\mathbb{R}) \times \text{U}(1)_R \). Here \( \text{SL}(2,\mathbb{R}) \) and
\( \text{U}(1)_R \) stand for the spin group and the \( R \)-symmetry group, respectively. We parametrise
the superspace by local bosonic \((x^m)\) and fermionic \((\theta^\mu, \bar{\theta}_\mu)\) coordinates \( z^M = (x^m, \theta^\mu, \bar{\theta}_\mu) \),
where \( m = 0, 1, 2, \mu = 1, 2 \). The Grassmann variables \( \theta^\mu \) and \( \bar{\theta}_\mu \) are related to each other by complex conjugation: \( \bar{\theta}^\mu = \bar{\theta}^\mu \).

The superspace covariant derivatives have the form

\[
\mathcal{D}_A = (\mathcal{D}_a, \mathcal{D}_\alpha, \mathcal{D}^\alpha) = E_A + \Omega_A + i\Phi_A J.
\]

Here \( E_A \) is the inverse supervielbein, while \( \Omega_A \) and \( \Phi_A \) denote the Lorentz and \( \mathrm{U}(1)_R \) connections, respectively,

\[
E_A = E_A^M \frac{\partial}{\partial \nu^M}, \quad \Omega_A = \frac{1}{2} \Omega_A^{bc} M_{bc} = -\Omega_A^b M_b = \frac{1}{2} \Omega_{\alpha}^{\beta\gamma} M_{\beta\gamma}.
\]

The explicit relations between Lorentz generators with two vector indices \( (M_{ab} = -M_{ba}) \), one vector index \( (M_a) \) and two spinor indices \( (M_{\alpha\beta} = M_{\beta\alpha}) \) are defined in appendix \[A.2\].

The actions of the generators \( \mathrm{SL}(2, \mathbb{R}) \times \mathrm{U}(1)_R \) on the covariant derivatives are defined as

\[
[J, D_a] = D_a, \quad [J, \bar{D}^a] = -\bar{D}^a, \quad [J, D_\alpha] = 0, \quad [M_{\alpha\beta}, D_\gamma] = \varepsilon_{\gamma(a} \bar{D}_{\beta)}, \quad [M_{\alpha\beta}, \bar{D}_\gamma] = \varepsilon_{\gamma(a} D_{\beta)}, \quad [M_{ab}, D_c] = 2\eta_{[a} D_{b]}.
\]

The covariant derivatives obey (anti-)commutation relations

\[
\{D_A, D_B\} = T_{AB} \epsilon D_C + \frac{1}{2} R_{AB} M_{cd} + i R_{AB} J.
\]

In the above, \( T_{AB} \epsilon \) is the torsion, while \( R_{AB} M_{cd} \) and \( R_{AB} J \) describe the curvature. In order to describe \( \mathcal{N} = 2 \) conformal supergravity, the torsion has to obey the covariant constraints proposed in [161]. Solving the constraints gives the following algebra of covariant derivatives [83, 113]

\[
\{D_a, D_\beta\} = -4\mathcal{R} M_{a\beta}, \quad (5.1.5a)
\]

\[
\{D_a, \bar{D}_\beta\} = -2i(\gamma^\beta)_{a\beta} D_a - 2\epsilon_a^b S J + 4i\epsilon_a^b S J + 4i S M_{a\beta} - 2\epsilon_{a\beta}^{\gamma\delta} M_{\gamma\delta}, \quad (5.1.5b)
\]

\[
[D_a, D_\beta] = i\epsilon_a^{abc}(\gamma^\beta)_{a\beta} C_{\gamma} + (\gamma^\beta)_{a\beta}^{\gamma} S \bar{D}_\gamma - i(\gamma_a)_{\beta\gamma} \bar{D}_\gamma + (\gamma^\beta)_{a\beta} C_{\gamma} \epsilon_{\gamma\delta \rho} M_{\rho} - \frac{1}{3} (2\mathcal{R}_a S + i \bar{D}_\beta \mathcal{R}) M_a + \frac{2}{3} \varepsilon_a^{\gamma b} (\gamma^\beta)_{a\beta} S (2D_a \mathcal{S} + i \bar{D}_\beta \mathcal{R}) M_c
\]

\[
+ \frac{i}{2} \left( (\gamma_a)_{a\beta} \bar{D}_a C_{\beta \gamma} + \frac{1}{3} (\gamma_{a\beta})_{a\beta} (8i D_a S - \bar{D}_a \mathcal{R}) \right) J, \quad (5.1.5c)
\]

We thus see that the algebra is parametrised by three torsion superfields: a real scalar \( S \), a complex scalar \( \mathcal{R} \) and its conjugate \( \bar{\mathcal{R}} \), and a real vector \( C_{a\beta} := (\gamma^a)_{a\beta} C_a \). The \( \mathrm{U}(1)_R \) charges of the torsion superfields \( \mathcal{R}, \bar{\mathcal{R}} \) and \( C_{a\beta} \) are \(-2, +2\) and \(0\), respectively. They satisfy the Bianchi identities

\[
D_a \bar{\mathcal{R}} = 0, \quad (\mathcal{D}^2 - 4\mathcal{R}) S = 0 \quad D^a C_{a\beta} = -\frac{1}{2} (\bar{D}_a \bar{\mathcal{R}} + 4i D_a S),
\]

Throughout this chapter, we define \( \mathcal{D}^2 := D^a D_a \) and \( \bar{\mathcal{D}}^2 := \bar{D}_a \bar{D}^a \).
The algebra of covariant derivatives given by (5.1.5) is invariant under the super-Weyl transformation \[83,113\]

\[
D'_{\alpha} = e^{\frac{1}{2}\sigma} \left( D_\alpha + D^\gamma \sigma M_{\gamma \alpha} - D_{\alpha \sigma} J \right),
\]

(5.1.7a)

\[
\bar{D}'_{\alpha} = e^{\frac{1}{2}\sigma} \left( \bar{D}_\alpha + \bar{D}^\gamma \sigma M_{\gamma \alpha} + D_{\alpha \sigma} J \right),
\]

(5.1.7b)

\[
D'_{a} = e^{\sigma} \left( D_a - \frac{i}{2}(\gamma_a)^{\gamma \delta} D_\gamma \sigma \bar{D}_\delta - \frac{i}{2}(\gamma_a)^{\gamma \delta} \bar{D}_\gamma \sigma D_\delta + \varepsilon_{abc} \bar{D}^b \sigma M^c 
- \frac{i}{2}(\gamma_a)^{\gamma \delta} \bar{D}_\gamma \sigma M_a - \frac{i}{24}(\gamma_a)^{\gamma \delta} e^{-3\sigma} [\bar{D}_\gamma, \bar{D}_\delta] e^{3\sigma} J \right),
\]

(5.1.7c)

which induces the following transformation of the torsion tensors:

\[
S' = e^{\sigma} \left( S + \frac{i}{4} D^\gamma \sigma \bar{D}_\gamma \sigma \right),
\]

(5.1.7d)

\[
C'_{a} = \left( C_a + \frac{1}{8}(\gamma_a)^{\gamma \delta} [\bar{D}_\gamma, \bar{D}_\delta] \right) e^{\sigma},
\]

(5.1.7e)

\[
R' = -\frac{1}{4} e^{2\sigma} (D^2 - 4\mathcal{R}) e^{-\sigma}.
\]

(5.1.7f)

The parameter \(\sigma\) is an arbitrary real scalar superfield. The super-Weyl invariance \(5.1.7\) is intrinsic to conformal supergravity. For every supergravity-matter system, its action is required to be a super-Weyl invariant functional of the supergravity multiplet coupled to certain conformal compensators, see \(83,113\) for more details.

The \(\mathcal{N} = 2\) supersymmetric extension of the Cotton tensor \(167\) is

\[
\mathcal{W}_{\alpha \beta} = -\frac{i}{4} [D^\gamma, \bar{D}_\gamma] C_{\alpha \beta} + \frac{1}{2} [D_{(\alpha}, \bar{D}_{\beta)}] S + 2SC_{\alpha \beta}.
\]

(5.1.8)

It transforms homogeneously under \(5.1.7\),

\[
\mathcal{W}'_{\alpha \beta} = e^{2\sigma} \mathcal{W}_{\alpha \beta},
\]

(5.1.9)

and obeys the Bianchi identities \(166\)

\[
\bar{D}^\beta \mathcal{W}_{\alpha \beta} = D^\beta \mathcal{W}_{\alpha \beta} = 0.
\]

(5.1.10)

The curved superspace is conformally flat if and only if \(\mathcal{W}_{\alpha \beta} = 0\) \(166\).

### 5.1.2 Primary superfields

Let \(T_{(\alpha n)} := T_{(\alpha_1...\alpha_n)} = T_{(\alpha_1...\alpha_n)}\) be a symmetric rank-\(n\) spinor superfield of \(U(1)_R\) charge \(q\),

\[
JT_{(\alpha n)} = qT_{(\alpha n)}.
\]

(5.1.11)
The superfield $T_{\alpha(n)}$ is called super-Weyl primary of dimension $d$ if it transforms under the infinitesimal super-Weyl transformation law as

$$
\delta_{\sigma} T_{\alpha(n)} = d\sigma T_{\alpha(n)}.
$$

As an example, the super-Cotton tensor is super-Weyl primary of dimension $+2$. Let us introduce several types of primary superfields which will be important for our subsequent analysis.

A symmetric rank-$n$ spinor superfield $G_{\alpha(n)}$ is called longitudinal linear if it obeys the following first-order constraint

$$
\bar{D}^{(\alpha_1} G_{\alpha_2...\alpha_{n+1})} = 0,
$$

which implies

$$
(\bar{D}^2 + 2n R) G_{\alpha(n)} = 0.
$$

If $G_{\alpha(n)}$ is super-Weyl primary, the constraint (5.1.13) is consistent provided the dimension $d_{G_{\alpha(n)}}$ and U$(1)_{R}$ charge $q_{G_{\alpha(n)}}$ of $G_{\alpha(n)}$ are related as

$$
d_{G_{\alpha(n)}} = -\frac{n}{2} - q_{G_{\alpha(n)}}.
$$

In the scalar case, $n = 0$, the constraint (5.1.13) becomes the condition of covariant chirality, $\bar{D}_\alpha G = 0$. The dimension $d_G$ and U$(1)_{R}$ charge $q_G$ of any primary chiral scalar superfield $G$ are related as $d_G + q_G = 0$, in accordance with [113].

Given a positive integer $n$, a symmetric rank-$n$ spinor superfield $\Gamma_{\alpha(n)}$ is called transverse linear if it obeys the first-order constraint

$$
\bar{D}^{(\beta} \Gamma_{\beta\alpha_1...\alpha_{n-1})} = 0, \quad n \neq 0,
$$

which implies

$$
(\bar{D}^2 - 2(n+2)R) \Gamma_{\alpha(n)} = 0.
$$

If $\Gamma_{\alpha(n)}$ is super-Weyl primary, then the constraint (5.1.16) is consistent provided the dimension $d_{\Gamma_{\alpha(n)}}$ and U$(1)_{R}$ charge $q_{\Gamma_{\alpha(n)}}$ of $\Gamma_{\alpha(n)}$ are related to each other as follows:

$$
d_{\Gamma_{\alpha(n)}} = 1 + \frac{n}{2} - q_{\Gamma_{\alpha(n)}}.
$$

In the $n = 0$ case, the constraint (5.1.16) is not defined. However, its corollary (5.1.17) is perfectly consistent,

$$
(\bar{D}^2 - 4R) \Gamma = 0,
$$

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and defines a covariantly linear scalar superfield $\Gamma$. The dimension $d_\Gamma$ and $U(1)_R$ charge $q_\Gamma$ of any primary linear scalar $\Gamma$ are related as $d_\Gamma + q_\Gamma = 1$, in accordance with [113].

The constraints (5.1.13) and (5.1.16) are solved in terms of prepotentials $\Psi_{\alpha(n-1)}$ and $\Phi_{\alpha(n+1)}$ as follows:

$$G_{\alpha(n)} = \bar{D}_{(\alpha_1 \Psi_{\alpha_2 ... \alpha_n})} ;$$  \hspace{1cm} (5.1.20a)

$$\Gamma_{\alpha(n)} = \bar{D}^\gamma \Phi_{(\beta_1 ... \alpha_n)} ;$$  \hspace{1cm} (5.1.20b)

Provided the constraints (5.1.13) and (5.1.16) are the only conditions imposed on $G_{\alpha(n)}$ and $\Gamma_{\alpha(n)}$ respectively, the prepotentials $\Psi_{\alpha(n-1)}$ and $\Phi_{\alpha(n+1)}$ can be chosen to be unconstrained complex, and are defined modulo gauge transformations of the form:

$$\delta_\zeta \Psi_{\alpha(n-1)} = \bar{D}_{(\alpha_1 \zeta \alpha_2 ... \alpha_{n-1})} ;$$  \hspace{1cm} (5.1.21a)

$$\delta_\zeta \Phi_{\alpha(n+1)} = \bar{D}^\gamma \zeta_{(\gamma \alpha_1 ... \alpha_{n+1})} ;$$  \hspace{1cm} (5.1.21b)

with the gauge parameters $\zeta_{\alpha(n-2)}$ and $\zeta_{\alpha(n+2)}$ being unconstrained. If the linear superfields $G_{\alpha(n)}$ and $\Gamma_{\alpha(n)}$ are super-Weyl primary, then their prepotentials $\Psi_{\alpha(n-1)}$ and $\Phi_{\alpha(n+1)}$ can also be chosen to be super-Weyl primary.

In the $n = 0$ case, the prepotential solution (5.1.20b) is still valid. The prepotential $\Phi_{\alpha}$ can be chosen to be unconstrained complex provided the constraint (5.1.19) is the only condition imposed on $\Gamma$. However, if we are dealing with a real linear superfield,

$$(\bar{D}^2 - 4\mathcal{R})L = 0 , \hspace{1cm} \bar{L} = L ,$$

then the constraints are solved [155] in terms of an unconstrained real prepotential $V$,

$$L = i\bar{D}^\alpha D_\alpha V , \hspace{1cm} \bar{V} = V ,$$

which is defined modulo gauge transformations of the form:

$$\delta V = \lambda + \bar{\lambda} , \hspace{1cm} J\lambda = 0 , \hspace{1cm} D_\alpha \lambda = 0 .$$

If $L$ is super-Weyl primary, then eq. (5.1.18) tells us that the dimension of $L$ is $+1$. In this case it is consistent to consider the gauge prepotential $V$ to be inert under the super-Weyl transformations [113], $\delta_\sigma V = 0$.

### 5.1.3 Conformal gauge superfields

Let $n$ be a positive integer. A real symmetric rank-$n$ spinor superfield $\mathfrak{H}_{\alpha(n)}$ is said to be a conformal gauge supermultiplet if (i) it is super-Weyl primary of dimension $(-n/2)$,

$$\delta_\sigma \mathfrak{H}_{\alpha(n)} = -\frac{n}{2} \sigma \mathfrak{H}_{\alpha(n)} ;$$

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and (ii) it is defined modulo gauge transformations of the form

\[ \delta_{\lambda} \mathcal{H}_{\alpha(n)} = \mathcal{D}_{(\alpha_1 \lambda \alpha_2 ... \alpha_n)} - (-1)^n \mathcal{D}_{(\alpha_1 \lambda \alpha_2 ... \alpha_n)} , \]

with the gauge parameter \( \lambda_{\alpha(n-1)} \) being unconstrained complex. The dimension of \( \mathcal{H}_{\alpha(n)} \) in (5.1.25) is uniquely fixed by requiring the longitudinal linear superfield \( g_{\alpha(n)} = \bar{\mathcal{D}}_{(\alpha_1 \bar{\lambda} \alpha_2 ... \alpha_n)} \) in the right-hand side of (5.1.26) to be super-Weyl primary. Indeed, the gauge parameter \( g_{\alpha(n)} \) must be neutral with respect to the \( R \)-symmetry group \( U(1)_R \) since \( \mathcal{H}_{\alpha(n)} \) is real. Hence, the dimension of \( g_{\alpha(n)} \) is equal to \((-n/2)\), in accordance with (5.1.15).

5.2 Geometry of \( \mathcal{N} = 2 \) AdS superspaces

Let us briefly discuss maximally supersymmetric backgrounds in the off-shell \( \mathcal{N} = 2 \) supergravity theories, since the superspaces \( \text{AdS}^{(3|1,1)} \) and \( \text{AdS}^{(3|2,0)} \) are special examples of such supermanifolds. The most general maximally supersymmetric backgrounds are characterised by several conditions [160] on the torsion superfields \( R, S \) and \( C_a \), which parametrise the superspace geometry of \( \mathcal{N} = 2 \) conformal supergravity, see 5.1.1. These requirements are as follows:

\[
\begin{align*}
\mathcal{R} S &= 0 , & \mathcal{R} C_a &= 0 , \\
\mathcal{D}_A \mathcal{R} &= 0 , & \mathcal{D}_A S &= 0 , & \mathcal{D}_a C_b &= 0 \implies \mathcal{D}_a C_b = 2 \varepsilon_{abc} C_c S .
\end{align*}
\]

The \((1,1)\) AdS superspace is singled out by the conditions \( S = 0 \) and \( C_a = 0 \), with \( \mathcal{R} \) and its conjugate \( \bar{\mathcal{R}} \) having non-zero constant values [83]. On the other hand, the solution with \( \mathcal{R} = 0, C_a = 0 \) and \( S \neq 0 \) corresponds to the \((2,0)\) AdS superspace [83]. It may be shown that the \( U(1)_R \) connection is flat if and only if \( S = 0 \) [155]. The non-vanishing \( U(1)_R \) curvature is the main reason why the structure of massless higher-spin gauge supermultiplets in \((2,0)\) AdS superspace [5] considerably differs from their counterparts with \((1,1)\) AdS supersymmetry. This will be the subject of section 5.7.

5.2.1 \((1,1)\) AdS superspace

In this subsection we collect salient facts about the geometry of \((1,1)\) AdS superspace [83], \( \text{AdS}^{(3|1,1)} \), as well as elaborate on superfield representations of the isometry group.

The geometry of \( \text{AdS}^{(3|1,1)} \) is characterised by covariant derivatives

\[ \mathcal{D}_A = (\mathcal{D}_a, \mathcal{D}_\alpha, \bar{\mathcal{D}}^\alpha) = E_A^M \frac{\partial}{\partial z^M} + \frac{1}{2} \Omega_A^{cd} M_{cd} \]

obeying the following graded commutation relations [83]:

\[ \{ \mathcal{D}_\alpha, \bar{\mathcal{D}}_\beta \} = -2i \mathcal{D}_{\alpha\beta} , \]

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\{ \mathcal{D}_\alpha, \mathcal{D}_\beta \} = -4 \mu M_{\alpha\beta} , \quad \{ \tilde{\mathcal{D}}_\alpha, \tilde{\mathcal{D}}_\beta \} = 4 \mu M_{\alpha\beta} , \quad (5.2.3b)

[\mathcal{D}_{\alpha\beta}, \mathcal{D}_\gamma] = -2 \mu \varepsilon_{\gamma(\alpha} \tilde{\mathcal{D}}_{\beta)} , \quad [\tilde{\mathcal{D}}_{\alpha\beta}, \tilde{\mathcal{D}}_\gamma] = 2 \mu \varepsilon_{(\alpha} M_{\beta)\gamma} , \quad (5.2.3c)

[\mathcal{D}_{\alpha\beta}, \mathcal{D}_{\gamma\delta}] = 4 \mu \mu \left( \varepsilon_{(\alpha} M_{\beta)\delta} + \varepsilon_{\delta(\alpha} M_{\beta)\gamma} \right) , \quad (5.2.3d)

with \( \mu \neq 0 \) being a complex parameter. As compared with (5.1.5), we have denoted \( \mathcal{R} = \mu \). In particular, of some use during calculations are the following identities, which can derived from the algebra (5.2.3):

\[ \mathcal{D}_\alpha \tilde{\mathcal{D}}_\beta = \frac{1}{2} \varepsilon_{\alpha\beta} \bar{\mathcal{D}}^2 - 2 \bar{\mu} M_{\alpha\beta} , \quad \tilde{\mathcal{D}}_\alpha \mathcal{D}_\beta = -\frac{1}{2} \varepsilon_{\alpha\beta} \bar{\mathcal{D}}^2 + 2 \mu M_{\alpha\beta} , \quad (5.2.4a) \]

\[ \mathcal{D}_\alpha \mathcal{D}^2 = 4 \bar{\mu} \bar{\mathcal{D}}^2 M_{\alpha\beta} + 4 \mu \mathcal{D}_\alpha , \quad \bar{\mathcal{D}}^2 \mathcal{D}_\alpha = -4 \bar{\mu} \bar{\mathcal{D}}^2 M_{\alpha\beta} - 2 \bar{\mu} \mathcal{D}_\alpha , \quad (5.2.4b) \]

\[ \tilde{\mathcal{D}}_\alpha \bar{\mathcal{D}}^2 = 4 \mu \bar{\mathcal{D}}^2 M_{\alpha\beta} + 4 \bar{\mu} \tilde{\mathcal{D}}_\alpha , \quad \bar{\mathcal{D}}^2 \tilde{\mathcal{D}}_\alpha = -4 \mu \bar{\mathcal{D}}^2 M_{\alpha\beta} - 2 \mu \tilde{\mathcal{D}}_\alpha , \quad (5.2.4c) \]

\[ [\bar{\mathcal{D}}^2, \mathcal{D}_\alpha] = 4i \mathcal{D}_\alpha \bar{\mathcal{D}}^2 + 6 \mu \mathcal{D}_\alpha = 4i \bar{\mathcal{D}}^2 \mathcal{D}_\alpha - 6 \mu \mathcal{D}_\alpha , \quad (5.2.4d) \]

\[ [\bar{\mathcal{D}}^2, \tilde{\mathcal{D}}_\alpha] = -4i \tilde{\mathcal{D}}_\alpha \bar{\mathcal{D}}^2 + 6 \bar{\mu} \tilde{\mathcal{D}}_\alpha = -4i \bar{\mathcal{D}}^2 \tilde{\mathcal{D}}_\alpha - 6 \bar{\mu} \tilde{\mathcal{D}}_\alpha , \quad (5.2.4e) \]

These relations imply the identity

\[ \mathcal{D}^\alpha (\bar{\mathcal{D}}^2 - 6 \mu ) \mathcal{D}_\alpha = \mathcal{D}_\alpha (\bar{\mathcal{D}}^2 - 6 \bar{\mu} ) \bar{\mathcal{D}}^\alpha , \quad (5.2.5) \]

which guarantees the reality of the actions considered in later sections.

The covariantly transverse linear and longitudinal linear superfields on an arbitrary supergravity background were described in the previous section. In the case of (1,1) AdS superspace, such superfields play an important role. One can define projectors \( P^\perp_n \) and \( P_n^\parallel \) on the spaces of transverse linear and longitudinal linear superfields, respectively. The projectors are

\[ P_n^\perp = \frac{1}{4(n+1)\mu} ( \bar{\mathcal{D}}^2 + 2n\mu ) , \quad (5.2.6a) \]

\[ P_n^\parallel = -\frac{1}{4(n+1)\mu} ( \bar{\mathcal{D}}^2 - 2(n+2)\mu ) , \quad (5.2.6b) \]

with the properties

\[ (P_n^\perp)^2 = P_n^\perp , \quad (P_n^\parallel)^2 = P_n^\parallel , \quad P_n^\perp P_n^\parallel = P_n^\parallel P_n^\perp = 0 , \quad P_n^\perp + P_n^\parallel = \mathbb{1} . \quad (5.2.7) \]

Given a complex tensor superfield \( V_\alpha(n) \) with \( n \neq 0 \), it can be represented as a sum of transverse linear and longitudinal linear multiplets,

\[ V_\alpha(n) = -\frac{1}{2\mu(n+2)} \bar{\mathcal{D}}^\gamma \bar{\mathcal{D}}_\gamma V_{\alpha_1...\alpha_n} - \frac{1}{2\mu(n+1)} \tilde{\mathcal{D}}_\alpha \tilde{\mathcal{D}}^\gamma |V_{\alpha_2...\alpha_n}\rangle_\gamma . \quad (5.2.8) \]

Choosing \( V_\alpha(n) \) to be longitudinal linear (\( G_\alpha(n) \)) or transverse linear (\( \Gamma_\alpha(n) \)), the above identity gives the relations (5.1.20a) and (5.1.20b) for some prepotentials \( \Psi_\alpha(n-1) \) and \( \Phi_{\alpha(n+1)} \), respectively.
In order to study rigid supersymmetric field theories in (1,1) AdS superspace, a superfield description of the corresponding isometry transformations is required. There exists a universal formalism to determine isometries of curved superspace backgrounds in diverse dimensions [35]. Real supervector fields \( \lambda^A E_A \) on AdS\((3|1),1)\) are called Killing supervector fields if

\[
\left[ \Lambda + \frac{1}{2} t^{ab} M_{ab}, \mathcal{D}_C \right] = 0 , \quad \Lambda := \lambda^a \mathcal{D}_a + \lambda^\alpha \mathcal{D}_\alpha + \bar{\lambda}^\alpha \bar{\mathcal{D}}^\alpha , \quad \Lambda^C = \lambda^C ,
\]

and \( t^{ab} \) corresponds to some local Lorentz parameter. As demonstrated in [83], the master equation (5.2.9) implies that the parameters \( \lambda^a \) and \( l^{ab} \) are uniquely expressed in terms of the vector \( \lambda^a \),

\[
\lambda^a = \frac{i}{6} \bar{\mathcal{D}}^\beta \lambda_{\alpha \beta} , \quad l_{\alpha \beta} = 2 \mathcal{D}_{(\alpha} \lambda_{\beta)} , \quad (5.2.10)
\]

and the vector parameter obeys the equation

\[
\mathcal{D}_{(\alpha \lambda_{\beta \gamma})} = 0 \iff \mathcal{D}_{(\alpha \lambda_{\beta \gamma})} = 0 . \quad (5.2.11)
\]

In comparison with the 3D \( \mathcal{N} = 2 \) Minkowski superspace, the specific feature of AdS\((3|1),1)\) is that any two of the three parameters \( \{ \lambda_{\alpha \beta}, \lambda^{\alpha}, l^{\alpha \beta} \} \) are expressed in terms of the third parameter, in particular

\[
\lambda_{\alpha \beta} = \frac{i}{\mu} \bar{\mathcal{D}}^\beta \lambda_{\alpha \beta} , \quad \lambda^{\alpha} = \frac{1}{12 \mu} \bar{\mathcal{D}}^\beta l_{\alpha \beta} . \quad (5.2.12)
\]

From (5.2.10) and (5.2.12) we deduce

\[
\bar{\mathcal{D}}^\alpha \lambda^\alpha = \mathcal{D}_\alpha \lambda^\alpha = 0 . \quad (5.2.13)
\]

These Killing supervector fields can be shown to generate the isometry group of AdS\((3|1),1)\), which is \( \text{OSp}(1|2; \mathbb{R}) \times \text{OSp}(1|2; \mathbb{R}) \).

In Minkowski superspace \( \mathbb{M}^{3|4} \), there are two ways to generate supersymmetric invariants, one of which corresponds to the integration over the full superspace and the other over its chiral subspace. In (1,1) AdS superspace, every chiral integral can always be recast as a full superspace integral. Associated with a scalar superfield \( \mathcal{L} \) is the following supersymmetric invariant

\[
\int d^3x d^2 \theta d^2 \bar{\theta} \mathcal{E} \mathcal{L} = -\frac{1}{4} \int d^3 x d^2 \theta \mathcal{E} (\bar{\mathcal{D}}^2 - 4 \mu) \mathcal{L} , \quad E^{-1} = \text{Ber} (E_A^M) , \quad (5.2.14)
\]

where \( \mathcal{E} \) denotes the chiral integration measure. Let \( \mathcal{L}_c \) be a covariantly chiral scalar Lagrangian, \( \mathcal{D}_a \mathcal{L}_c = 0 \). It generates a supersymmetric invariant of the form \( \int d^3 x d^2 \theta \mathcal{E} \mathcal{L}_c \). The specific feature of (1,1) AdS superspace is that the chiral action can equivalently be written as an integral over the full superspace [83]

\[
\int d^3 x d^2 \theta \mathcal{E} \mathcal{L}_c = \frac{1}{\mu} \int d^3 x d^2 \theta d^2 \bar{\theta} \mathcal{E} \mathcal{L}_c . \quad (5.2.15)
\]
Unlike the flat superspace case, the integral on the right does not vanish in AdS.

Supersymmetric invariant (5.2.14) can be reduced to component fields by the rule
\[
\int d^3x d^2\theta d^2\bar{\theta} \mathcal{E} \mathcal{L} = \frac{1}{16} \int d^3x e^{-1}(\mathcal{D}^2 - 16\bar{\mu})(\mathcal{D}^2 - 4\mu)\mathcal{L} ,
\]
with $e^{-1} := \det(e_a^m)$. Here $e_a^m$ is the inverse vielbein, which determines the torsion-free covariant derivative of AdS space
\[
\nabla_a = e_a + \frac{1}{2}\omega_a^{bc}(e)M_{bc} , \quad e_a := e_a^m \partial_m .
\]
In general, the $\theta, \bar{\theta}$-independent component, $T|_{\theta=\bar{\theta}=0}$, of a superfield $T(x, \theta, \bar{\theta})$ is denoted by $T|_{\theta=\bar{\theta}=0}$.

To complete the formalism of component reduction, we only need the following relation
\[
(\mathcal{D}_a T)| = \nabla_a T| .
\]

In what follows, we will work with full superspace integrals only and make use of the notation $d^3z := d^3x d^2\theta d^2\bar{\theta}$.

### 5.2.2 (2,0) AdS superspace

Let us briefly review the key results concerning (2,0) AdS superspace, $\text{AdS}^{(3|2,0)}$; see [83, 168] for the details. There are two ways to describe the geometry of (2,0) AdS superspace, which correspond to making use of either a real or complex basis for the spinor covariant derivatives. Here we first consider the formulation in the complex basis.

The geometry of $\text{AdS}^{(3|2,0)}$ is described by covariant derivatives
\[
\mathcal{D}_A = (\mathcal{D}_a, \mathcal{D}_\alpha, \mathcal{D}^\alpha) = E_A^M \frac{\partial}{\partial z^M} + \frac{1}{2} \Omega_A^{cd} M_{cd} + i\Phi_A J
\]

obeying the following algebra
\[
\{\mathcal{D}_\alpha, \mathcal{D}_\beta\} = 0 , \quad \{\mathcal{D}_\alpha, \mathcal{D}_\beta\} = 0 ,
\]
\[
\{\mathcal{D}_a, \mathcal{D}_b\} = -2i(\mathcal{D}_{a\beta} - 2SM_{a\beta}) - 4i\epsilon_{a\beta}\mathcal{S}J ,
\]
\[
[\mathcal{D}_a, \mathcal{D}_\beta] = (\gamma_a)_{\beta}^\gamma \mathcal{S}\mathcal{D}_\gamma , \quad [\mathcal{D}_a, \mathcal{D}_{\bar{\beta}}] = (\gamma_a)_{\bar{\beta}}^\gamma \mathcal{S}\mathcal{D}_\gamma ,
\]
\[
[\mathcal{D}_a, \mathcal{D}_\bar{b}] = -4\mathcal{S}^2M_{ab} .
\]

Here the parameter $\mathcal{S}$ is related to the AdS scalar curvature as $R = -24\mathcal{S}^2$.

The covariant derivatives of (2,0) AdS superspace hold various identities, which can be easily derived from the algebra (5.2.20). Some of the useful ones include
\[
[\mathcal{D}^\alpha, \mathcal{D}^\beta] = 4i\mathcal{D}^\alpha\mathcal{D}^\beta + 4i\mathcal{S}^2\mathcal{D}^\alpha - 8i\mathcal{S}\mathcal{D}^\alpha J - 8i\mathcal{S}\mathcal{D}_\beta M^{\alpha\beta} ,
\]
\[
[D^\alpha, D^2] = -4iD^{\alpha\beta}D_\beta - 4iSD^\alpha - 8iS'D'^\alpha J + 8iSD_\beta M^\alpha_\beta , \quad (5.2.21b)
\]
\[
[D^a, D^2] = 0 \, , \quad [\bar{D}^a, D^2] = 0 . \quad (5.2.21c)
\]

These relations imply
\[
D^\alpha \bar{D}^2 D_\alpha = \bar{D}_a D^2 \bar{D}^a , \quad (5.2.22)
\]
which guarantees the reality of the actions considered in the later sections.

In accordance with the general formalism of \cite{35}, the isometries of (2,0) AdS superspace are generated by the Killing supervector fields $\zeta^A E_A$, which are defined to solve the master equation
\[
[\zeta + \frac{1}{2} l^{bc} M_{bc} + i\tau J , D_A] = 0 , \quad (5.2.23a)
\]
where
\[
\zeta = \zeta^B D_B = \zeta^b D_b + \zeta^\beta D_\beta + \bar{\zeta}_\beta \bar{D}^\beta , \quad \bar{\zeta}^b = \zeta^b , \quad (5.2.23b)
\]
and $\tau$ and $l^{bc}$ are some real $U(1)_R$ and Lorentz superfield parameters, respectively. It follows from eq. \ref{5.2.23} that the parameters $\zeta_\alpha$, $\tau$ and $l_{\alpha\beta}$ are uniquely expressed in terms of the vector parameter $\zeta_{\alpha\beta}$ as follows:
\[
\zeta_\alpha = \frac{i}{6} \bar{D}^\beta \zeta_{\beta\alpha} , \quad \tau = \frac{i}{2} D^\alpha \zeta_\alpha , \quad l_{\alpha\beta} = 2(D_{(\alpha} \zeta_{\beta)} - S \zeta_{\alpha\beta}) . \quad (5.2.24)
\]
The vector parameter $\zeta_{\alpha\beta}$ satisfies the equation
\[
D_{(\alpha} \zeta_{\beta)} = 0 . \quad (5.2.25)
\]
This implies the standard Killing equation,
\[
D_a \zeta_b + D_b \zeta_a = 0 . \quad (5.2.26)
\]
One may also prove the following relations
\[
\bar{D}_a \tau = \frac{1}{3} \bar{D}^\beta l_{\alpha\beta} = 4S \zeta_\alpha , \quad \bar{D}_a \zeta_\beta = 0 , \quad D_{(\alpha} l_{\beta)} = 0 , \quad (5.2.27)
\]
see \cite{83} for derivations. The Killing supervector fields prove to generate the supergroup $\text{OSp}(2|2; R) \times \text{Sp}(2, R)$, which is the isometry group of (2,0) AdS superspace. Rigid supersymmetric field theories in (2,0) AdS superspace are required to be invariant under the isometry transformations. An infinitesimal isometry transformation acts on a tensor superfield $U$ (with suppressed indices) by the rule
\[
\delta_{\zeta} U = (\zeta + \frac{1}{2} l^{bc} M_{bc} + i\tau J) U . \quad (5.2.28)
\]
Associated with a real scalar superfield $L$ is the following supersymmetric invariant
\[
\int d^3 x d^2 \theta d^2 \bar{\theta} E L = -\frac{1}{4} \int d^3 x d^2 \theta E \bar{D}^2 L . \quad (5.2.29)
\]
5.3 Massless half-integer superspin gauge theories in (1,1) AdS superspace

The results presented in this section were obtained by Daniel Ogburn [4].

The conformal higher-spin gauge superfields $\mathcal{H}_{\alpha(n)}$ (see 5.1.3) at least for $n = 2s$, with $s = 1, 2, \ldots$, can be used to construct massless actions in two of the three $\mathcal{N} = 2$ maximally symmetric backgrounds, which are Minkowski superspace [49] and (1,1) AdS superspace [4]. Such actions, however, involve not only $\mathcal{H}_{\alpha(n)}$ but also some compensators.

It is worth pointing out that all massless higher-spin supermultiplets in 3D (1,1) AdS superspace may be obtained from their counterparts in 4D $\mathcal{N} = 1$ AdS superspace [63] by dimensional reduction. In practice, however, carrying out such a reduction proves to be a non-trivial technical task. To explain this, let us consider the longitudinal formulation for massless superspin-$(s + \frac{1}{2})$ multiplets, with $s = 1, 2, \ldots$, in four and three dimensions. In the 4D $\mathcal{N} = 1$ AdS case [63], the massless superspin-$(s + \frac{1}{2})$ multiplet is described by a real unconstrained gauge superfield $H_{a_1\ldots a_s\dot{a}_1\ldots\dot{a}_s} = H_{(a_1\ldots a_s)(\dot{a}_1\ldots\dot{a}_s)}$, a complex longitudinal linear compensator $G_{a_1\ldots a_{s-1}a_{s-1}\dot{a}_1\ldots\dot{a}_{s-1}} = G_{(a_1\ldots a_{s-1})(\dot{a}_1\ldots\dot{a}_{s-1})}$ and its conjugate. The dimensional reduction of $H_{a_1\ldots a_{s-1}a_{s-1}\dot{a}_1\ldots\dot{a}_{s-1}}$ leads to a family of real unconstrained symmetric superfields $H_{a_1\ldots a_{2s}}$, $H_{a_1\ldots a_{2s-2}}$, $\ldots$, $H$. Next, the dimensional reduction of $G_{a_1\ldots a_{s-1}a_{s-1}\dot{a}_1\ldots\dot{a}_{s-1}}$ leads to a family of constrained 3D superfields, which include a complex longitudinal linear compensator $G_{a_1\ldots a_{2s-2}}$ and some lower-spin supermultiplets.

As will be shown later, the massless superspin-$(s + \frac{1}{2})$ multiplet in 3D (1,1) AdS superspace is described by the gauge superfield $H_{a_1\ldots a_{2s}}$, the compensator $G_{a_1\ldots a_{2s-2}}$ and its conjugate. The above consideration makes it clear that the naive $4D \rightarrow 3D$ dimensional reduction leads to the massless superspin-$(s + \frac{1}{2})$ multiplet intertwined with lower-superspin multiplets. The non-trivial technical task is to disentangle the pure superspin-$(s + \frac{1}{2})$ multiplet from the rest. This was explicitly done in [83] for the $s = 1$ case, for which dimensional reduction leads to two supermultiplets in (1,1) AdS superspace: a massless superspin-$\frac{3}{2}$ multiplet and a massless vector supermultiplet. Instead of carrying out dimensional reduction, it proves to be more efficient to recast the 4D gauge principle of [63] in a 3D form and use it to construct gauge-invariant actions. This is the approach advocated in [4,49].

We recall the constructions presented in [49]. There exist two off-shell formulations for the massless $\mathcal{N} = 2$ multiplet of superspin-$(s + \frac{1}{2})$, $s = 2, 3, \ldots$, which describe two propagating massless fields with spin-$(s + 1)$ and spin-$(s + \frac{1}{2})$ on Minkowski space [49]. These dually equivalent formulations, known as transverse and longitudinal, differ in the compensators used.

\[\text{1}\] The $s = 1$ case corresponds to linearised supergravity.
Let us extend these gauge theories to (1,1) AdS superspace. There exist two formulations which are described in terms of the following dynamical variables

\[ \mathcal{V}^\perp_{(s+\frac{1}{2})} = \left\{ \mathcal{H}_{(2s)}, \Gamma_{(2s-2)}, \bar{\Gamma}_{(2s-2)} \right\}, \quad (5.3.1a) \]
\[ \mathcal{V}^\parallel_{(s+\frac{1}{2})} = \left\{ \mathcal{H}_{(2s)}, G_{(2s-2)}, \bar{G}_{(2s-2)} \right\}. \quad (5.3.1b) \]

Here \( \mathcal{H}_{(2s)} \) is an unconstrained real superfield. The complex superfields \( \Gamma_{(2s-2)} \) and \( G_{(2s-2)} \) are transverse linear and longitudinal linear in the sense that they obey the constraints \( (5.1.16) \) and \( (5.1.13) \), respectively. In accordance with \( (5.1.20) \), these constraints can be solved in terms of unconstrained complex prepotentials as follows:

\[ \Gamma_{(2s-2)} = \mathcal{D}^\beta \Phi_{(\beta \alpha_1 \ldots \alpha_{2s-2})}, \quad (5.3.2a) \]
\[ G_{(2s-2)} = \mathcal{D}(\alpha_1 \Psi_{\alpha_2 \ldots \alpha_{2s-2}}). \quad (5.3.2b) \]

These prepotentials are defined modulo gauge transformations of the form

\[ \delta_\xi \Phi_{(2s-1)} = \mathcal{D}^\beta \xi_{(\beta \alpha_1 \ldots \alpha_{2s-1})}, \quad (5.3.3a) \]
\[ \delta_\xi \Psi_{(2s-3)} = \mathcal{D}(\alpha_1 \zeta_{\alpha_2 \ldots \alpha_{2s-3}}), \quad (5.3.3b) \]

with the gauge parameters \( \xi_{(2s)} \) and \( \zeta_{(2s-4)} \) being unconstrained complex.

The dynamical superfields \( \mathcal{H}_{(2s)} \) and \( \Gamma_{(2s-2)} \) are postulated to be defined modulo gauge transformations of the form

\[ \delta_\lambda \mathcal{H}_{(2s)} = \mathcal{D}(\alpha_1 \lambda_{\alpha_2 \ldots \alpha_{2s}}) - \mathcal{D}(\alpha_1 \bar{\lambda}_{\alpha_2 \ldots \alpha_{2s}}) \equiv g_{(2s)} + \bar{g}_{(2s)}, \quad (5.3.4a) \]
\[ \delta_\lambda \Gamma_{(2s-2)} = -\frac{1}{4} \mathcal{D}^\beta (\mathcal{D}^2 + 2(2s-1)\bar{\mu}) \lambda_{\beta \alpha_{2s-2}} = \frac{s}{2s+1} \mathcal{D}^\beta \mathcal{D}^7 \bar{g}_{(\beta \gamma \alpha_1 \ldots \alpha_{2s-2})}, \quad (5.3.4b) \]
\[ \delta_\lambda G_{(2s-2)} = -\frac{1}{4}(\mathcal{D}^2 - 4s\mu) \mathcal{D}^\beta \lambda_{(2s-2)\beta} + i(s-1)\mathcal{D}(\alpha_1) \mathcal{D}^{1|\beta|\gamma} \lambda_{\alpha_2 \ldots \alpha_{2s-2})\beta \gamma}, \quad (5.3.4c) \]

where the complex gauge parameter \( \lambda_{(2s-1)} \) is unconstrained. The gauge transformation of \( \mathcal{H}_{(2s)} \) coincides with \( (5.1.26) \) for \( n = 2s \). From \( \delta_\lambda \Gamma_{(2s-2)} \), we can read off the gauge transformation of the prepotential \( \Phi_{(2s-1)} \), which is

\[ \delta_\lambda \Phi_{(2s-1)} = -\frac{1}{4}(\mathcal{D}^2 + 2(2s-1)\bar{\mu}) \bar{\lambda}_{(2s-1)}. \quad (5.3.5) \]

In the transverse formulation, the quadratic action invariant under the gauge transformations \( (5.3.4a) \) and \( (5.3.4b) \) is

\[ S^\perp_{(s+\frac{1}{2})} = \left( -\frac{1}{2} \right)^s \int d^{3|4}z \left\{ \frac{1}{8} \mathcal{H}^{\alpha(2s)} \mathcal{D}^\beta (\mathcal{D}^2 - 6\mu) \mathcal{D}_\beta \mathcal{H}_{(2s)} \right. \]
\[ + 2s(s-1)\bar{\mu} \mathcal{H}^{\alpha(2s)} \mathcal{H}_{(2s)} + \mathcal{H}^{\alpha(2s)} \left( \mathcal{D}_{\alpha_1} \mathcal{D}_{\alpha_2} \Gamma_{\alpha_3 \ldots \alpha_{2s}} - \mathcal{D}_{\alpha_1} \mathcal{D}_{\alpha_2} \bar{\Gamma}_{\alpha_3 \ldots \alpha_{2s}} \right) \]
\[ + \left. \frac{2s-1}{s} \bar{\Gamma}_{(2s-2)} \Gamma_{(2s-2)} + \frac{2s+1}{2s} (\Gamma^{\alpha(2s-2)} \Gamma_{(2s-2)} + \bar{\Gamma}^{\alpha(2s-2)} \bar{\Gamma}_{(2s-2)} ) \right\}. \quad (5.3.6) \]
In the flat superspace limit, this action reduces to the one derived in \[49\].

The \( s = 1 \) choice was excluded from the above consideration, since the constraint \((5.1.16)\) is not defined for \( n = 0 \). However, the corollary \((5.1.17)\) of \((5.1.16)\) is perfectly consistent for \( n = 0 \) and defines a covariantly transverse linear scalar superfield \((5.1.19)\),

\[
(\bar{\mathcal{D}}^2 - \mu)\Gamma = 0 .
\] (5.3.7)

We therefore postulate \( \Gamma \) and its conjugate \( \bar{\Gamma} \) to be the compensators in the \( s = 1 \) case. The gauge transformations \((5.3.4a)\) and \((5.3.4b)\) then become

\[
\delta_{\lambda} H_{\alpha\beta} = \bar{\mathcal{D}}(\alpha, \lambda) - \mathcal{D}(\beta, \bar{\lambda}), \tag{5.3.8a}
\]

\[
\delta_{\lambda} \Gamma = -\frac{1}{4} \bar{\mathcal{D}}(\beta, 2\bar{\mathcal{D}} + 2\mu) \bar{\lambda} . \tag{5.3.8b}
\]

The variation \( \delta_{\lambda} \Gamma \) is compatible with the constraint \((5.3.7)\), that is \((\bar{\mathcal{D}}^2 - \mu)\delta_{\lambda} \Gamma = 0\).

Finally, choosing \( s = 1 \) in \((5.3.6)\) gives the linearised action for non-minimal \((1,1)\) AdS supergravity, which was originally derived in section 9.2 of \[83\].

In the longitudinal formulation, the action invariant under the gauge transformations \((5.3.4a)\) and \((5.3.4c)\) is

\[
S^\parallel_{(s+\frac{1}{2})} = \left( -\frac{1}{2} \right)^s \int d^{3+1}z \mathcal{E}\left\{ \frac{1}{8} \bar{\mathcal{D}}^\alpha(2s) \mathcal{D}^\beta(\bar{\mathcal{D}}^2 - 6\mu) \mathcal{D}_\beta \bar{\mathcal{D}}_\alpha(2s) \right.
\]

\[
+ 2s(s - 1)\mu \bar{\mathcal{D}}^\alpha(2s) \bar{\mathcal{D}}_\alpha(2s) - \frac{1}{16} ([\mathcal{D}_\beta, \mathcal{D}_\gamma] \mathcal{D}^\beta(\bar{\mathcal{D}}^2 - 2\mu) [\mathcal{D}^\delta, \mathcal{D}^\rho] \mathcal{D}_\delta \mathcal{D}_\rho(2s-2)) 
\]

\[
+ \frac{s}{2} (\mathcal{D}_\beta, \mathcal{D}_\gamma) \mathcal{D}^\beta(\bar{\mathcal{D}}^2 - 2\mu) \mathcal{D}^\gamma \mathcal{D}_\delta \mathcal{D}_\rho(2s-2) 
\]

\[
+ \frac{2s - 1}{2s} \left[ i(\mathcal{D}_\beta, \mathcal{D}_\gamma) \mathcal{D}^\beta(\bar{\mathcal{D}}^2 - 2\mu) \left( \mathcal{D}_\delta \mathcal{D}_\rho(2s-2) - \bar{\mathcal{D}}_\delta \bar{\mathcal{D}}_\rho(2s-2) \right) + \frac{1}{s} \mathcal{D}^\alpha(2s-2) \mathcal{D}_\alpha(2s-2) \right] 
\]

\[
- \frac{2s + 1}{4s^2} \left( \mathcal{D}^\alpha(2s-2) \mathcal{D}_\alpha(2s-2) + \bar{\mathcal{D}}^\alpha(2s-2) \bar{\mathcal{D}}_\alpha(2s-2) \right) \right\} . \tag{5.3.9}
\]

As shown in \[4\], the longitudinal action may be derived from the transverse one by performing a superfield duality transformation.

In the \( s = 1 \) case, the compensator \( G \) becomes covariantly chiral, \( \mathcal{D}_a G = 0 \). Choosing \( s = 1 \) in \((5.3.9)\) gives the linearised action for minimal \((1,1)\) AdS supergravity, which was originally derived in section 9.1 of \[83\], provided we identify \( G = 3\sigma \). The corresponding gauge transformations are

\[
\delta_{\lambda} H_{\alpha\beta} = \bar{\mathcal{D}}(\alpha, \lambda) - \mathcal{D}(\beta, \bar{\lambda}) , \tag{5.3.10a}
\]

\[
\delta_{\lambda} G = -\frac{1}{4} (\bar{\mathcal{D}}^2 - 4\mu) \mathcal{D}^\beta \lambda_\beta . \tag{5.3.10b}
\]

It is clear that the variation \( \delta_{\lambda} G \) is covariantly chiral.
5.4 Massless integer superspin gauge theories in (1,1) AdS superspace

The results in subsections 5.4.1 and 5.4.2 were obtained by Daniel Ogburn [4].

When attempting to develop a Lagrangian formulation for a massless multiplet of superspin \( s \), where \( s = 1, 2, \ldots \), a naive expectation is that the dynamical variables of such a theory should consist of a conformal gauge superfield \( \tilde{\mathcal{H}}_{\alpha(2s-1)} = \tilde{H}_{\alpha(2s-1)} \), introduced in subsection 5.1.3, in conjunction with some compensator(s). Instead, our approach in this section will be based on developing 3D \( N = 2 \) analogues of the two dually equivalent off-shell formulations, the so-called longitudinal and transverse ones, for the massless \( N = 1 \) multiplets of integer superspin in AdS\(_4\) [63]. As the next step, we will construct a generalised longitudinal model, in a way similar to the one proposed in the 4D \( N = 1 \) AdS case in subsection 4.2.1. Such a reformulation naturally leads to the appearance of the conformal gauge superfield \( \tilde{\mathcal{H}}_{\alpha(2s-1)} \).

5.4.1 Longitudinal formulation

Given an integer \( s \geq 1 \), the longitudinal formulation for the massless superspin-\( s \) multiplet is realised in terms of the following dynamical variables:

\[
\mathcal{V}^\parallel_{(s)} = \left\{ U_{\alpha(2s-2)}, G_{\alpha(2s)}, \bar{G}_{\alpha(2s)} \right\} .
\]  
(5.4.1)

Here \( U_{\alpha(2s-2)} \) is an unconstrained real superfield, and the complex superfield \( G_{\alpha(2s)} \) is longitudinal linear, eq. (5.1.13). In accordance with (5.1.20a), the constraint (5.1.13) can be solved in terms of an unconstrained complex prepotential \( \Psi_{\alpha(2s-1)} \),

\[
G_{\alpha_1 \ldots \alpha_{2s}} := \mathcal{D}_{(\alpha_1} \Psi_{\alpha_2 \ldots \alpha_{2s})} ,
\]  
(5.4.2)

which is defined modulo gauge transformations of the form

\[
\delta_\zeta \Psi_{\alpha(2s-1)} = \mathcal{D}_{(\alpha_1} \zeta_{\alpha_2 \ldots \alpha_{2s-1})} ,
\]  
(5.4.3)

with the gauge parameter \( \zeta_{\alpha(2s-2)} \) being unconstrained complex.

We postulate the dynamical superfields \( U_{\alpha(2s-2)} \) and \( G_{\alpha(2s)} \) to be defined modulo gauge transformations of the form

\[
\delta_L U_{\alpha(2s-2)} = \mathcal{D}^\beta L_{\beta \alpha_1 \ldots \alpha_{2s-2}} - \mathcal{D}^\beta \bar{L}_{\beta \alpha_1 \ldots \alpha_{2s-2}} \equiv \tilde{\gamma}_{\alpha(2s-2)} + \gamma_{\alpha(2s-2)} ,
\]  
(5.4.4a)

\[
\delta_L G_{\alpha(2s)} = -\frac{1}{2} \mathcal{D}_{(\alpha_1} \left( \mathcal{D}^2 - 2(2s + 1)\bar{\mu} \right) L_{\alpha_2 \ldots \alpha_{2s}} = \mathcal{D}_{(\alpha_1} \mathcal{D}_{\alpha_2} \tilde{\gamma}_{\alpha_3 \ldots \alpha_{2s}} .
\]  
(5.4.4b)
Here the gauge parameter $L_{\alpha(2s-1)}$ is an unconstrained complex superfield, and $\gamma^{\alpha(2s-2)} := \overline{\mathbf{D}}_{\beta} \tilde{L}^{\beta\alpha(2s-2)}$ is transverse linear. From (5.4.4b) we read off the gauge transformation law of the prepotential,

$$
\delta_L \Psi_{\alpha(2s-1)} = -\frac{1}{2} \left( \mathbf{D}^2 - 2(2s + 1) \mu^2 \right) L_{\alpha(2s-1)} = \mathbf{D}_{(\alpha_1} \mathbf{D}^{\beta)} L_{\alpha_2...\alpha_{2s-1})\beta} .
$$

(5.4.5)

Modulo an overall normalisation factor, there is a unique quadratic action which is invariant under the gauge transformations (5.4.4). The action is

$$
S_{(s)}^{\parallel} = \left( -\frac{1}{2} \right)^s \int d^4z \frac{1}{8} U^{\alpha(2s-2)} \mathbf{D}^\gamma \overline{\mathbf{D}} (\mathbf{D}^2 - 6\mu) \mathbf{D}_\gamma U_{\alpha(2s-2)} + \frac{s}{2s+1} U^{\alpha(2s-2)} \left( \mathbf{D}_\beta \mathbf{D}^\gamma G_{\beta\gamma\alpha(2s-2)} - \overline{\mathbf{D}}_\beta \mathbf{D}^\gamma \tilde{G}_{\beta\gamma\alpha(2s-2)} \right) + \frac{2s}{2s-1} U_{\alpha(2s-2)} \left( G^{\alpha(2s)} G_{\alpha(2s)} + \tilde{G}^{\alpha(2s)} \tilde{G}_{\alpha(2s)} \right) + 2s(s + 1) \mu \overline{\mu} U^{\alpha(2s-2)} U_{\alpha(2s-2)} \right) .
$$

(5.4.6)

The special $s = 1$ case, which corresponds to the massless gravitino multiplet, will be studied in more detail in subsection 5.4.4.

### 5.4.2 Transverse formulation

The transverse formulation for the massless superspin-$s$ multiplet is realised in terms of the following dynamical variables:

$$
\Psi_{(s)}^{\perp} = \left\{ U_{\alpha(2s-2)}, \Gamma_{\alpha(2s)} , \tilde{\Gamma}_{\alpha(2s)} \right\} .
$$

(5.4.7)

Here $U_{\alpha(2s-2)}$ is the same as in (5.4.1), and the complex superfield $\Gamma_{\alpha(2s)}$ is transverse linear, eq. (5.1.16). In accordance with (5.1.20b), the constraint on $\Gamma_{\alpha(2s)}$ is solved in terms of an unconstrained prepotential $\Phi_{\alpha(2s+1)}$,

$$
\Gamma_{\alpha(2s)} = \overline{\mathbf{D}} \Phi_{(\beta\alpha_1...a_{2s})} ,
$$

(5.4.8)

which is defined modulo gauge transformations of the form

$$
\delta_\xi \Phi_{\alpha(2s+1)} = \overline{\mathbf{D}} \xi_{(\beta\alpha_1...a_{2s+1})} ,
$$

(5.4.9)

with the gauge parameter $\xi_{\alpha(2s+2)}$ being unconstrained.

The transverse formulation for the massless superspin-$s$ multiplet is described by the following action

$$
S_{(s)}^{\perp} = \left( -\frac{1}{2} \right)^s \int d^4z \frac{1}{8} U^{\alpha(2s-2)} \mathbf{D}^\gamma (\mathbf{D}^2 - 6\mu) \mathbf{D}_\gamma U_{\alpha(2s-2)}
$$

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\[-\frac{2s-1}{16(2s+1)}(8s\mathcal{D}^{a_1a_2}U^{a_3...a_{2s}}\mathcal{D}_{(a_1a_2}U_{a_3...a_{2s})}
+\mathcal{D}^{a_1,\bar{D}^{a_2}}U^{a_3...a_{2s}}[\mathcal{D}_{(a_1,\bar{D}^{a_2})}U_{a_3...a_{2s}}])
+2s(s+1)\mu\bar{U}^{a_1(2s-2)}U_{a_1(2s-2)}-iU^{a_1...a_{2s-2}}\mathcal{D}^{a_2...a_{2s}}(\Gamma_{a(2s)}-\bar{\Gamma}_{a(2s)})
-\frac{2}{2s-1}\Gamma_{a(2s)}+\frac{1}{2s+1}(\Gamma_{a(2s)}+\bar{\Gamma}_{a(2s)})\right), \quad (5.4.10)\]

which is invariant under the gauge transformation [5.4.4] accompanied with

\[\delta_L \Gamma_{a(2s)} = -\frac{1}{4}(\tilde{\mathcal{D}}^2+4s\mu)\mathcal{D}_{(a_1}L_{a_2...a_{2s})} + \frac{1}{2}(2s+1)\tilde{\mathcal{D}}\gamma\mathcal{D}^{a_1}L_{a_2...a_{2s})}\]
\[= \frac{1}{2}\mathcal{D}_{(a_1}\tilde{\mathcal{D}}_{a_2}\gamma_{a_3...a_{2s})} - \frac{1}{2}(2s-1)\mathcal{D}_{(a_1a_2}\gamma_{a_3...a_{2s})}, \quad (5.4.11)\]

where \(\gamma_{a(2s-2)} = -\tilde{\mathcal{D}}^\beta L_{\beta a_1...a_{2s-2}}\).

### 5.4.3 Reformulation of the longitudinal theory

Let us take a step further and consider a generalisation of the longitudinal formulation [5.4.6]. This can be achieved by enlarging the gauge freedom [5.4.4], where we choose to work with an unconstrained complex gauge superfield \(\gamma_{a(2s-2)}\). As a result, we are required to introduce a new purely gauge superfield, in addition to \(U_{a(2s-2)}, \Psi_{a(2s-1)}\) and \(\bar{\Psi}_{a(2s-1)}\). In such a setting, the gauge freedom of \(\Psi_{a(2s-1)}\) coincides with that of a complex conformal gauge superfield.

Given a positive integer \(s \geq 2\), a massless superspin-\(s\) multiplet in AdS\(^{(3+1)}\) can be described using a complex unconstrained prepotential \(\Psi_{a(2s-1)}\), a real superfield \(U_{a(2s-2)}\) and a complex superfield \(\Sigma_{a(2s-3)}\) constrained to be transverse linear,

\[\tilde{\mathcal{D}}^\beta\Sigma_{a(2s-4)} = 0. \quad (5.4.12)\]

The constraint [5.4.12] is solved in terms of a complex unconstrained prepotential \(Z_{a(2s-2)}\),

\[\Sigma_{a(2s-3)} = \tilde{\mathcal{D}}^\beta Z_{(\beta a_1...a_{2s-3})}, \quad (5.4.13)\]

which is defined modulo gauge shift

\[\delta_\xi Z_{a(2s-2)} = \tilde{\mathcal{D}}^\beta \xi_{(\beta a_1...a_{2s-2})}. \quad (5.4.14)\]

Here the gauge parameter \(\xi_{a(2s-1)}\) is unconstrained.

The gauge freedom of \(\Psi_{a(2s-1)}\) is given by

\[\delta_{\partial_\xi} \Psi_{a_1...a_{2s-1}} = \mathcal{D}_{(a_1} \Psi_{a_2...a_{2s-1})} + \tilde{\mathcal{D}}_{(a_1} \xi_{a_2...a_{2s-1})}, \quad (5.4.15a)\]
with unconstrained gauge parameters $\mathfrak{W}_{\alpha(2s-2)}$ and $\zeta_{\alpha(2s-2)}$. We further postulate the linearised gauge transformations for the superfields $U_{\alpha(2s-2)}$ and $\Sigma_{\alpha(2s-3)}$ as follows

$$\delta_{\mathfrak{W}} U_{\alpha(2s-2)} = \mathfrak{W}_{\alpha(2s-2)} + \mathfrak{W}_{\alpha(2s-2)} \ , \quad \delta_{\mathfrak{W}} \Sigma_{\alpha(2s-3)} = D^{\beta} \mathfrak{W}_{\beta(2s-3)} \implies \delta_{\mathfrak{W}} Z_{\alpha(2s-2)} = \mathfrak{W}_{\alpha(2s-2)} \ . \quad (5.4.15c)$$

The longitudinal linear superfield $G_{\alpha(2s)}$ defined by (5.4.2) is invariant under the $\zeta$-transformation (5.4.15a). It varies under the $\mathfrak{W}$-transformation as

$$\delta_{\mathfrak{W}} G_{\alpha_1 ... \alpha_{2s}} = \mathfrak{W}_{(a_1} \mathfrak{W}_{a_2} \mathfrak{W}_{a_3} ... a_{2s)} \ . \quad (5.4.16)$$

The action

$$S^\mathfrak{W}_{(s)} = \left( -\frac{1}{2} \right)^{s} \int d^{3|4} x \ E \left\{ \frac{1}{8} U^{\alpha(2s-2)} D^\beta \left( D^2 - 6 \mu \right) D_\beta U_{\alpha(2s-2)} \right. \right.$$ \nonumber 

$$\left. + \frac{s}{2s+1} U^{\alpha(2s-2)} \left( D^\beta \bar{D}^\gamma G_{\beta \gamma(2s-2)} - \bar{D}^\beta D^\gamma \bar{G}_{\beta \gamma(2s-2)} \right) \right.$$ \nonumber 

$$\left. + 2s(s+1) \mu \mu U^{\alpha(2s-2)} U_{\alpha(2s-2)} \right.$$ \nonumber 

$$\left. + \frac{s}{2s-1} G^{\alpha(2s)} G_{\alpha(2s)} + \frac{s}{2(2s+1)} \left( G^{\alpha(2s)} G_{\alpha(2s)} + \bar{G}^{\alpha(2s)} \bar{G}_{\alpha(2s)} \right) \right.$$ \nonumber 

$$\left. + \frac{1}{2s} U^{\alpha(2s-2)} \left( \bar{D}_{a_1} \bar{D}^2 \bar{\Sigma}_{a_2 ... a_{2s-2}} - \bar{D}_{a_1} \bar{D}^2 \Sigma_{a_2 ... a_{2s-2}} \right) \right.$$ \nonumber 

$$\left. + \frac{1}{2s} \Psi^{\alpha(2s-1)} \left( \bar{D}_{a_1} \bar{D}_{a_2} - 2i(s-1) \bar{D}_{a_1 a_2} \right) \Sigma_{a_3 ... a_{2s-1}} \right.$$ \nonumber 

$$\left. + \frac{1}{2s} \bar{\Psi}^{\alpha(2s-1)} \left( \bar{D}_{a_1} \bar{D}_{a_2} - 2i(s-1) \bar{D}_{a_1 a_2} \right) \Sigma_{a_3 ... a_{2s-1}} \right.$$ \nonumber 

$$\left. - \mu(s+3) U^{\alpha(2s-2)} \bar{D}_{a_1} \bar{\Sigma}_{a_2 ... a_{2s-2}} + \bar{\mu}(s+3) U^{\alpha(2s-2)} \bar{D}_{a_1} \Sigma_{a_2 ... a_{2s-2}} \right.$$ \nonumber 

$$\left. + \frac{s-1}{4(2s-1)} \left( \Sigma^{\alpha(2s-3)} \bar{D}^2 \Sigma_{a_2 ... a_{2s-2}} \right) \bar{\Sigma}_{a_3 ... a_{2s-2}} \right.$$ \nonumber 

$$\left. - \frac{1}{2s} \Sigma^{\alpha(2s-3)} \left( 2s^2 - s + 1 \right) D^\beta \bar{D}_{a_1} + 2i \left( s - 1 \right) \left( 2s - 3 \right) \bar{D}^\beta \bar{D}_{a_1} \right.$$ \nonumber 

$$\left. \Sigma_{a_2 ... a_{2s-3}} + \mu(s+3) \Sigma_{\alpha(2s-3)} \bar{\Sigma}_{a_2 ... a_{2s-3}} + \bar{\mu}(s+3) \Sigma_{\alpha(2s-3)} \Sigma_{a_2 ... a_{2s-3}} \right\} , \quad (5.4.17)$$

possesses the gauge invariance (5.4.15) and, by construction, (5.4.14). The above action is real due to the identity (5.2.22).

Due to the $\mathfrak{W}$-gauge freedom (5.4.15), we are free to make the gauge choice

$$\Sigma_{\alpha(2s-3)} = 0 \ , \quad (5.4.18)$$

by which we regain the original longitudinal action for the massless superspin-$s$ multiplet (5.4.6). The gauge condition (5.4.18) does not fix completely the $\mathfrak{W}$-gauge freedom. The residual gauge transformations are generated by

$$\mathfrak{W}_{\alpha(2s-2)} = D^\beta L(\beta \alpha_1 ... \alpha_{2s-2}) \ , \quad (5.4.19)$$

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with \( L_{\alpha(2s-1)} \) being an unconstrained superfield. With this expression for \( \mathfrak{Q}_{\alpha(2s-2)} \), the gauge transformations (5.4.15a) and (5.4.15b) coincide with (5.4.4b). Thus, the action (5.4.17) indeed provides an off-shell formulation for the massless superspin-s multiplet in (1,1) AdS superspace.

The action (5.4.17) contains a single term which involves the gauge prepotential \( \bar{\Psi}_{\alpha(2s-1)} \) and not the field strength \( \bar{\Sigma} \). This term can be written as

\[
\int d^{3/4}z \ E \bar{\Psi}^\alpha(2s-1) \left( \bar{D}_{\alpha_1} \bar{D}_{\alpha_2} - 2i(s-1)\bar{D}_{\alpha_1\alpha_2} \right) \bar{\Sigma}_{\alpha_3...\alpha_{2s-1}}
\]

\[
= -\frac{2s}{2s+1} \int d^{3/4}z E \bar{G}^\alpha(2s) \left( \bar{D}_{\alpha_1} \bar{D}_{\alpha_2} + i(2s+1)\bar{D}_{\alpha_1\alpha_2} \right) \bar{Z}_{\alpha_3...\alpha_{2s}}. \tag{5.4.20}
\]

The former makes the gauge symmetry (5.4.14) manifestly realised, while the latter turns the \( \zeta \)-transformation (5.4.15a) into a manifest symmetry. If we instead wish to make use of (5.4.20), we obtain a different representation for the action (5.4.17). It is

\[
S^\|_{(s)} = \left( -\frac{1}{2} \right)^s \int d^{3/4}z E \left\{ \frac{1}{8} U^\alpha(2s-2) \bar{D}^\beta \left( \bar{D}^2 - 6\mu \right) \bar{D}_\beta U_{\alpha(2s-2)} 
\right.
\]

\[
+ \frac{s}{2s+1} U^\alpha(2s-2) \left( \bar{D}^\beta \bar{D}^\gamma G_{\beta\gamma\alpha(2s-2)} - \bar{D}^\beta \bar{D}^\gamma G_{\beta\gamma\alpha(2s-2)} \right) 
\]

\[
+ 2s(s+1) \bar{\mu} U^{\alpha(2s-2)} U_{\alpha(2s-2)} 
\]

\[
+ \frac{s}{2s-1} U^\alpha(2s-2) \left( \bar{D}_{\alpha_1} \bar{D}^2 \bar{\Sigma}_{\alpha_2...\alpha_{2s-2}} - \bar{D}_{\alpha_1} \bar{D}^2 \Sigma_{\alpha_2...\alpha_{2s-2}} \right) 
\]

\[
+ \frac{2s}{(2s-1)(2s+1)} G^{\alpha(2s)} \left( \bar{D}_{\alpha_1} \bar{D}_{\alpha_2} + i(2s+1)\bar{D}_{\alpha_1\alpha_2} \right) \bar{Z}_{\alpha_3...\alpha_{2s}} 
\]

\[
- \frac{2s}{(2s-1)(2s+1)} \bar{G}^{\alpha(2s)} \left( \bar{D}_{\alpha_1} \bar{D}_{\alpha_2} + i(2s+1)\bar{D}_{\alpha_1\alpha_2} \right) \bar{Z}_{\alpha_3...\alpha_{2s}} 
\]

\[
- \mu(s+3) U^{\alpha(2s-2)} \bar{D}_{\alpha_1} \bar{\Sigma}_{\alpha_2...\alpha_{2s-2}} + \bar{\mu}(s+3) U^{\alpha(2s-2)} \bar{D}_{\alpha_1} \Sigma_{\alpha_2...\alpha_{2s-2}} 
\]

\[
+ \frac{s}{4(2s-1)} \left( \bar{\Sigma}^{\alpha(2s-3)} \bar{\Sigma}^{\alpha(2s-3)} - \bar{\Sigma}^{\alpha(2s-3)} \bar{\Sigma}^{\alpha(2s-3)} \right) 
\]

\[
- \frac{1}{2s-1} \bar{\Sigma}^{\alpha(2s-3)} \left( (2s^2 - s + 1)\bar{D}^\beta \bar{D}_{\alpha_1} + 2i \frac{(s-1)(2s-3)}{2s-1} \bar{D}^\beta \bar{D}_{\alpha_1} \right) \Sigma_{\beta\alpha_2...\alpha_{2s-3}} 
\]

\[
+ \mu(s+3) \bar{\Sigma}^{\alpha(2s-3)} \bar{\Sigma}^{\alpha(2s-3)} + \bar{\mu}(s+3) \Sigma^{\alpha(2s-3)} \Sigma^{\alpha(2s-3)} \right\}. \tag{5.4.21}
\]

It is worth discussing the structure of the dynamical variable \( \Psi_{\alpha(2s-1)} \). This superfield is unconstrained complex, and its gauge transformation law is given by eq. (5.4.15a). Comparing (5.4.15a) with the gauge transformation law (5.1.26) with \( n = 2s-1 \), which corresponds to the conformal gauge superfield \( \mathfrak{H}_{\alpha(2s-1)} \), we see that \( \Psi_{\alpha(2s-1)} \) may be interpreted as a complex conformal gauge superfield.
5.4.4 Massless gravitino multiplet

The massless gravitino multiplet, which corresponds to the \( s = 1 \) case, was excluded from our consideration of the previous subsection. Here we will fill the gap.

The (generalised) longitudinal formulation for the gravitino multiplet is described by the action

\[
S^\parallel_{GM} = -\frac{1}{2} \int d^3 z E \left\{ \frac{1}{8} U \mathcal{D}^\beta (\mathcal{D}^2 - 6 \mu) \mathcal{D}_\beta U + \frac{1}{3} U (\mathcal{D}^\alpha \mathcal{D}^\beta G_{\alpha\beta} - \mathcal{D}^\alpha \mathcal{D}^\beta \bar{G}_{\alpha\beta}) 
+ G^{\alpha\beta} G_{\alpha\beta} + \frac{1}{6} (G^{\alpha\beta} G_{\alpha\beta} + \bar{G}^{\alpha\beta} \bar{G}_{\alpha\beta}) 
+ |\mu|^2 \left( 2U - \frac{\Phi}{\mu} - \frac{\bar{\Phi}}{\bar{\mu}} \right)^2 + 2 \left( \frac{\Phi}{\mu} + \frac{\bar{\Phi}}{\bar{\mu}} \right) \left( \mu \mathcal{D}^\alpha \Psi_\alpha + \bar{\mu} \bar{\mathcal{D}}^\alpha \bar{\Psi}_\alpha \right) \right\},
\]

(5.4.22)

where \( \Phi \) is a covariantly chiral scalar superfield, \( \bar{\mathcal{D}}^\alpha \Phi = 0 \), and

\[
G_{\alpha\beta} = \bar{\mathcal{D}}_{(\alpha} \Psi_{\beta)} , \quad \bar{G}_{\alpha\beta} = - \mathcal{D}_{(\alpha} \bar{\Psi}_{\beta)} .
\]

(5.4.23)

This action is invariant under gauge transformations of the form

\[
\delta U = \mathcal{V} + \bar{\mathcal{V}} ,
\]

(5.4.24a)

\[
\delta \Psi_\alpha = = \mathcal{D}_\alpha \mathcal{V} + \bar{\mathcal{D}}_\alpha \zeta ,
\]

(5.4.24b)

\[
\delta \Phi = - \frac{1}{4} (\bar{\Phi}^2 - 4 \mu) \bar{\mathcal{V}} ,
\]

(5.4.24c)

where the gauge parameters \( \mathcal{V} \) and \( \zeta \) are unconstrained complex superfields.

The gauge \( \mathcal{V} \)-freedom (5.4.24) allows us to impose the condition \( \Phi = 0 \). In this gauge the action (5.4.22) turns into (5.4.6) with \( s = 1 \), and the residual gauge \( \mathcal{V} \)-freedom is described by \( \mathcal{V} = \mathcal{D}^\beta L_\beta \), where the spinor gauge parameter \( L_\alpha \) is unconstrained complex.

The action (5.4.22) involves the chiral scalar \( \Phi \) and its conjugate only in the combination \( (\varphi + \bar{\varphi}) \), where \( \varphi = \Phi/\mu \). This means that the model (5.4.22) possesses a dual formulation realised in terms of a real linear superfield subject to the constraint (5.1.22).

5.4.5 \( \mathcal{N} = 2 \rightarrow \mathcal{N} = 1 \) superspace reduction

Every supersymmetric field theory in (1,1) AdS superspace \( \text{AdS}^{3(1,1)} \) may be reformulated in terms of superfields on \( \mathcal{N} = 1 \) AdS superspace.\(^2\) Let us briefly discuss how to perform such a reduction.

First, it proves to be advantageous to switch to the real basis for the (1,1) AdS spinor covariant derivatives. Following \[155\], we can introduce a real basis for the spinor covariant\(^2\)

\(^2\)In the case of \( \mathcal{N} = 1 \) AdS supersymmetry, both notations \( (1,0) \) and \( \mathcal{N} = 1 \) are used in the literature. We will also use the notation \( \text{AdS}^{3(1,1)} \) for \( \mathcal{N} = 1 \) AdS superspace.
derivatives which is obtained by replacing the complex operators $\mathcal{D}_\alpha$ and $\bar{\mathcal{D}}_\alpha$ with $\nabla^I_\alpha$, where $I = 1, 2$, defined by

$$
\mathcal{D}_\alpha = \frac{e^{i\varphi}}{\sqrt{2}}(\nabla^1_\alpha - i\nabla^2_\alpha) , \quad \bar{\mathcal{D}}_\alpha = -\frac{e^{-i\varphi}}{\sqrt{2}}(\nabla^1_\alpha + i\nabla^2_\alpha),
$$

where we have represented $\mu = -ie^{2i\varphi}|\mu|$. The new covariant derivatives can be shown to obey the algebra:

$$
\{\nabla^1_\alpha, \nabla^1_\beta\} = 2i\nabla_{\alpha\beta} - 4i|\mu|M_{\alpha\beta}, \quad \{\nabla^2_\alpha, \nabla^2_\beta\} = 2i\nabla_{\alpha\beta} + 4i|\mu|M_{\alpha\beta},
$$

(5.4.26a)

$$
\{\nabla^1_\alpha, \nabla^2_\beta\} = 0,
$$

(5.4.26b)

$$
\{\nabla_\alpha, \nabla^1_\beta\} = |\mu|(\gamma_\alpha)_\beta^\gamma \nabla^1_\gamma, \quad \{\nabla_\alpha, \nabla^2_\beta\} = -|\mu|(\gamma_\alpha)_\beta^\gamma \nabla^2_\gamma,
$$

(5.4.26c)

$$
\{\nabla_\alpha, \nabla_\beta\} = -4|\mu|^2 M_{\alpha\beta}.
$$

(5.4.26d)

The graded commutation relations for the operators $\nabla_\alpha$ and $\nabla^I_\alpha$ form a closed algebra. Indeed, they are isomorphic to those defining the $\mathcal{N} = 1$ AdS superspace, see [155] for the details. These properties mean that $(1,0)$ AdS superspace is naturally embedded in $(1,1)$ AdS superspace as a subspace. The Grassmann variables $\theta_I^\mu = (\theta_1^\mu, \theta_2^\mu)$ may be chosen in such a way that $(1,0)$ AdS corresponds to the surface defined by $\theta_2^\mu = 0$. It is thus possible to carry out a consistent $(1,1) \to (1,0)$ AdS superspace reduction for all the higher-spin supersymmetric gauge theories constructed in sections 5.3 and 5.4. Implementation of this program will be described elsewhere.

For concreteness, let us consider the $\mathcal{N} = 2 \to \mathcal{N} = 1$ superspace reduction of the longitudinal model for massless superspin-$s$ multiplet (5.4.6). Here our analysis is restricted to the flat superspace case for simplicity.

In order to be consistent with the previous work [50], in which the $\mathcal{N} = 2 \to \mathcal{N} = 1$ superspace reduction of the massless superspin-$(s + \frac{1}{2})$ models of [49] was studied, we denote by $\mathcal{D}_\alpha$ and $\bar{\mathcal{D}}_\alpha$ the spinor covariant derivatives of $\mathcal{N} = 2$ Minkowski superspace $\mathbb{M}^{3|4}$. They obey the anti-commutation relations

$$
\{\mathcal{D}_\alpha, \bar{\mathcal{D}}_\beta\} = -2i \partial_{\alpha\beta} , \quad \{\mathcal{D}_\alpha, \mathcal{D}_\beta\} = \{\bar{\mathcal{D}}_\alpha, \bar{\mathcal{D}}_\beta\} = 0.
$$

(5.4.27)

In order to carry out the $\mathcal{N} = 2 \to \mathcal{N} = 1$ superspace reduction, it is useful to introduce real Grassmann coordinates $\theta^I_\alpha$ for $\mathbb{M}^{3|4}$, where $I = 1, 2$. We define these coordinates by choosing the corresponding spinor covariant derivative derivatives $D^I_\alpha$ as in [169]:

$$
\mathcal{D}_\alpha = \frac{1}{\sqrt{2}}(D^1_\alpha - iD^2_\alpha), \quad \bar{\mathcal{D}}_\alpha = -\frac{1}{\sqrt{2}}(D^1_\alpha + iD^2_\alpha).
$$

(5.4.28)

From (5.4.27) we deduce

$$
\{D^I_\alpha, D^J_\beta\} = 2i \delta^{IJ}(\gamma^m)_{\alpha\beta} \partial_m , \quad I, J = 1, 2.
$$

(5.4.29)
Given an $\mathcal{N} = 2$ superfield $U(x, \theta_I)$, we define its $\mathcal{N} = 1$ bar-projection

\[
U| := U(x, \theta_I)|_{\theta_2=0},
\]  

which is a superfield on $\mathcal{N} = 1$ Minkowski superspace $\mathbb{M}^{3|2}$ parametrised by real Cartesian coordinates $z^A = (x^\alpha, \theta^\alpha)$, where $\theta^\alpha := \theta_1^\alpha$. The spinor covariant derivative of $\mathcal{N} = 1$ Minkowski superspace $D_\alpha := D_\alpha^1$ obeys the anti-commutation relation

\[
\{ D_\alpha, D_\beta \} = 2i (\gamma^m)_{\alpha\beta} \partial_m .
\]  

Finally, the $\mathcal{N} = 2 \rightarrow \mathcal{N} = 1$ superspace reduction of the $\mathcal{N} = 2$ supersymmetric action is carried out using the rule \[50\]

\[
S = \int d^{3|4}z \ L(\mathcal{N}=2) = \int d^{3|2}z \ L(\mathcal{N}=1) , \quad L(\mathcal{N}=1) := -\frac{i}{4}(D^2)^2 L(\mathcal{N}=2) | .
\]  

Given an integer $s \geq 1$, the longitudinal formulation for the massless superspin-$s$ multiplet is realised in terms of the following dynamical variables:

\[
\mathcal{V}^\parallel_{(s)} = \left\{ U_\alpha(2s-2), G_\alpha(2s), \bar{G}_\alpha(2s) \right\} .
\]  

Here $U_\alpha(2s-2)$ is an unconstrained real superfield, and the complex superfield $G_\alpha(2s)$ is longitudinal linear,

\[
\bar{D}_{(a_1} G_{a_2...a_{2s-1})} = 0 .
\]  

The dynamical superfields are defined modulo gauge transformations of the form

\[
\delta U_\alpha(2s-2) = \bar{\gamma}_\alpha(2s-2) + \gamma_\alpha(2s-2) ,
\]

\[
\delta G_\alpha(2s) = \bar{D}_{(a_1} D_{a_2} \bar{\gamma}_{a_3...a_{2s}}) ,
\]  

where the gauge parameter $\gamma_\alpha(2s-2)$ is an arbitrary transverse linear superfield,

\[
\bar{D}^\beta \gamma_{\beta a_1...a_{2s-3}} = 0 .
\]  

The gauge-invariant action is

\[
S^\parallel_{(s)} = \left(-\frac{1}{2}\right)^s \int d^{3|4}z \left\{ \frac{1}{8} U^{(2s-2)} \bar{D}^\gamma \bar{D}^2 D_\gamma U_\alpha(2s-2) \\
+ \frac{s}{2s+1} U^{(2s-2)} \left( \bar{D}^\beta \bar{D}^\gamma G_{\beta \gamma a(2s-2)} - \bar{D}^\beta \bar{D}^\gamma \bar{G}_{\beta \gamma a(2s-2)} \right) \\
+ \frac{s}{2s-1} \bar{G}^{(2s)} G_\alpha(2s) + \frac{s}{2(2s+1)} \left( G^{(2s)} G_\alpha(2s) + \bar{G}^{(2s)} \bar{G}_\alpha(2s) \right) \right\} .
\]  

Making use of the representation (5.4.28), the transverse linear constraint (5.4.36) takes the form

\[
D^2 \gamma_{\beta a_1...a_{2s-3}} = iD^\parallel \gamma_{\beta a_1...a_{2s-3}} .
\]  

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It follows that $\gamma_{\alpha(2s-2)}$ has two independent $\theta_2$-components, which are:

$$\gamma_{\alpha(2s-2)}, \quad D^2_{(\alpha_1 \gamma_{\alpha_2...\alpha_{2s-1})}} \ .$$  \hspace{1cm} (5.4.39)

The gauge transformation of $U_{\alpha(2s-2)}$, eq. (5.4.35), allows us to impose two conditions

$$U_{\alpha(2s-2)} = 0 \ , \quad D^2_{(\alpha_1 U_{\alpha_2...\alpha_{2s-1})}} = 0 \ .$$  \hspace{1cm} (5.4.40)

In this gauge we define the following unconstrained real $N = 1$ superfields:

$$U_{\alpha(2s-3)} := \frac{i}{s} D^{2\beta} U_{\beta\alpha(2s-3)} \ ,$$  \hspace{1cm} (5.4.41a)

$$U_{\alpha(2s-2)} := -\frac{i}{4s} (D^2)^2 U_{\alpha(2s-2)} \ .$$  \hspace{1cm} (5.4.41b)

The residual gauge freedom, which preserves the gauge conditions (5.4.40), is described by unconstrained real $N = 1$ superfield parameters $\zeta_{\alpha(2s-2)}$ and $\lambda_{\alpha(2s-1)}$ defined by

$$\gamma_{\alpha(2s-2)} = \frac{i}{2} \zeta_{\alpha(2s-2)} \ , \quad \tilde{\zeta}_{\alpha(2s-2)} = \zeta_{\alpha(2s-2)} \ ,$$  \hspace{1cm} (5.4.42a)

$$D^2_{(\alpha_1 \gamma_{\alpha_2...\alpha_{2s-1})}} = \frac{i}{2} \lambda_{\alpha(2s-1)} \ , \quad \tilde{\lambda}_{\alpha(2s-1)} = \lambda_{\alpha(2s-1)} \ .$$  \hspace{1cm} (5.4.42b)

The gauge transformation laws of the superfields (5.4.41) are

$$\delta U_{\alpha(2s-3)} = -\frac{i}{s} D^2 \zeta_{\alpha(2s-3)} \ ,$$  \hspace{1cm} (5.4.43a)

$$\delta U_{\alpha(2s-2)} = \frac{1}{2s} D^2 \lambda_{\alpha(2s-2)} \ .$$  \hspace{1cm} (5.4.43b)

We now turn to reducing $G_{\alpha(2s)}$ to $\mathcal{N} = 1$ superspace. From the point of view of $\mathcal{N} = 1$ supersymmetry, $G_{\alpha(2s)}$ is equivalent to two unconstrained complex superfields, which we define as follows:

$$G_{\alpha(2s)} = -\frac{1}{2} (G_{\alpha(2s)} + iH_{\alpha(2s)}) \ ,$$  \hspace{1cm} (5.4.44a)

$$iD^2_{\beta} G_{\beta\alpha(2s-1)} = \Phi_{\alpha(2s-1)} + i\Psi_{\alpha(2s-1)} \ .$$  \hspace{1cm} (5.4.44b)

Making use of the gauge transformation (5.4.35) gives

$$\delta G_{\alpha(2s)} = -i \nabla_{(\alpha_1 \alpha_2 \gamma_{\alpha_3...\alpha_{2s})}} + iD^2_{(\alpha_1 \alpha_2 \gamma_{\alpha_3...\alpha_{2s})}} \ ,$$  \hspace{1cm} (5.4.45a)

$$iD^2_{\beta} \delta G_{\beta\alpha(2s-1)} = i \left\{ -\frac{2s-1}{2s} \nabla^\beta \nabla_{(\alpha_1 \alpha_2 \gamma_{\alpha_3...\alpha_{2s-1})} \beta} \right.\left. + \frac{s-1}{s} \sqrt{(\alpha_1 \alpha_2) \sqrt{\beta \gamma_{\alpha_3...\alpha_{2s-1})} \beta}} - 2D^\beta \nabla_{(\alpha_1 \alpha_2 \gamma_{\alpha_3...\alpha_{2s-1})} \beta} \right\} \ .$$  \hspace{1cm} (5.4.45b)

At this stage one should recall that upon imposing the $\mathcal{N} = 1$ supersymmetric gauge conditions (6.2.3) the residual gauge freedom is described by the gauge parameters (5.4.42a).
and (5.4.42b). From (5.4.45) we read off the gauge transformations of the $\mathcal{N} = 1$ complex superfields (5.4.44)

$$
\delta G_\alpha(2s) = -\frac{1}{2} \left\{ \partial_\alpha \zeta_{\alpha_3...\alpha_{2s}} + i D_\alpha \lambda_{\alpha_2...\alpha_{2s}} \right\},
$$

(5.4.46a)

$$
i D^2 \delta G_\beta \alpha(2s-1) = -\frac{2s-1}{4s} \partial^\beta (\alpha_1 \lambda_{\alpha_2...\alpha_{2s-1}})\beta - \frac{2s+1}{8s} D^2 \lambda_\alpha(2s-1)
+ \frac{s-1}{2s} \partial_\alpha \zeta_{\alpha_3...\alpha_{2s-1}}\beta - D^\beta \partial_\beta (\alpha_1 \zeta_{\alpha_2...\alpha_{2s-1}}),
$$

(5.4.46b)

In the $\mathcal{N} = 1$ supersymmetric gauge (5.4.40), $U_\alpha(2s-2)$ is described by two unconstrained real superfields $U_\alpha(2s-3)$ and $U_\alpha(2s-2)$ defined according to (5.4.41), and their gauge transformation laws are given by eqs. (5.4.43a) and (5.4.43b), respectively. It follows from the gauge transformations (5.4.43a), (5.4.43b) and (5.4.46) that in fact we are dealing with two different gauge theories. One of them is formulated in terms of the unconstrained real gauge superfields

$$
\{G_\alpha(2s), U_\alpha(2s-3), \Psi_\alpha(2s-1)\}
$$

(5.4.47)

which are defined modulo gauge transformations of the form

$$
\delta G_\alpha(2s) = \partial_\alpha \zeta_{\alpha_3...\alpha_{2s}},
$$

(5.4.48a)

$$
\delta U_\alpha(2s-3) = -\frac{1}{s} D^\beta \zeta_\beta \alpha(2s-3),
$$

(5.4.48b)

$$
\delta \Psi_\alpha(2s-1) = -\frac{1}{2s} \partial_\alpha D^\beta \zeta_{\alpha_3...\alpha_{2s-1}} \beta + i D^\beta \partial_\beta (\alpha_1 \zeta_{\alpha_2...\alpha_{2s-1}}),
$$

(5.4.48c)

where the gauge parameter $\zeta_\alpha(2s-2)$ is unconstrained real. The other theory is described by the gauge superfields

$$
\{H_\alpha(2s), U_\alpha(2s-2), \Phi_\alpha(2s-1)\}
$$

(5.4.49)

with the following gauge freedom

$$
\delta H_\alpha(2s) = D_\alpha \lambda_{\alpha_2...\alpha_{2s}},
$$

(5.4.50a)

$$
\delta U_\alpha(2s-2) = \frac{1}{2s} D^\beta \lambda_\beta \alpha(2s-2),
$$

(5.4.50b)

$$
\delta \Phi_\alpha(2s-1) = -\frac{1}{8s} \left\{ (4s-2) \partial^\beta (\alpha_1 \lambda_{\alpha_2...\alpha_{2s-1}})\beta + i (2s+1) D^2 \lambda_\alpha(2s-1) \right\}.
$$

(5.4.50c)

Applying the reduction rule (5.4.32) to the action (5.4.37) gives two decoupled $\mathcal{N} = 1$ supersymmetric actions, which are described in terms of the dynamical variables (5.4.47) and (5.4.49), respectively. In the former case, the superfield $\Psi_\alpha(2s-1)$ is auxiliary. Integrating it out, we arrive at the following action:

$$
S = -\left( -\frac{1}{2} \right)^s \frac{s^2 (s-1)}{2s-1} \frac{i}{2} \int d^{3/2} z \left\{ \frac{1}{2s} G^\alpha(2s) D^2 G_\alpha(2s) \right\}
$$

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\[
\begin{align*}
&\frac{i}{s-1} G^{\alpha(2s-1)\beta} \partial_\beta \gamma G_{\alpha(2s-1)\gamma} - 2i U^{\alpha(2s-3)} \partial^{\beta\gamma} D^\delta G_{\beta\gamma\delta\alpha(2s-3)}
+ 2 U^{\alpha(2s-3)} D U_{\alpha(2s-3)} + \frac{(2s - 3)(s - 2)}{2s - 1} \partial_\delta \lambda U^{\delta\lambda\alpha(2s-5)} \partial^{\beta\gamma} U_{\beta\gamma\alpha(2s-5)}
- \frac{1}{2} \frac{2s - 3}{2s - 1} D_\beta U^{\alpha(2s-4)\beta} D^2 D^\gamma U_{\gamma\alpha(2s-4)} \right).
\end{align*}
\]

(5.4.51)

This action is invariant under the gauge transformations (5.4.48a) and (5.4.48b).

In the latter case, the superfield \(\Phi_{\alpha(2s-1)}\) is auxiliary. Integrating it out, we obtain the following gauge-invariant action:

\[
S = \left( -\frac{1}{2} \right)^s \frac{s}{2s - 1} \int d^{3/2} z \left\{ \frac{1}{2} H^{\alpha(2s)} D^2 H_{\alpha(2s)} + i H^{\alpha(2s-1)\beta} \partial_\beta \gamma H_{\alpha(2s-1)\gamma}
+ 2(2s - 1) U^{\alpha(2s-2)} \partial^{\beta\gamma} H_{\beta\gamma\alpha(2s-2)} + (2s - 1) U^{\alpha(2s-2)} D^2 U_{\alpha(2s-2)}
+ 2(2s - 1)(s - 1) D_\beta U^{\alpha(2s-3)\beta} D^2 D^\gamma U_{\gamma\alpha(2s-3)} \right\}.
\]

(5.4.52)

This action is invariant under the gauge transformations (5.4.50a) and (5.4.50b). Modulo an overall normalisation factor, (5.4.52) coincides with the off-shell \(\mathcal{N} = 1\) supersymmetric action for massless superspin-\(s\) multiplet [50] in the form given in [51].

The action (5.4.51) defines a new \(\mathcal{N} = 1\) supersymmetric higher-spin theory which did not appear in the analysis of [50]. It may be shown that at the component level it reduces, upon imposing a Wess-Zumino gauge and eliminating the auxiliary fields, to a sum of two massless actions. One of them is the bosonic Fronsdal-type spin-\(s\) model and the other is the fermionic Fang-Fronsdal-type spin-(\(s + 1/2\)) model.

### 5.5 Higher-spin (1,1) AdS supercurrents

Inspired by the analysis of Dumitrescu and Seiberg [79], the most general supercurrent multiplets for theories with (1,1) AdS or (2,0) AdS supersymmetry were introduced in [83], with the (1,1) AdS case being a natural extension of the 4D \(\mathcal{N} = 1\) AdS supercurrents classified in [99,100]. Here we will formulate higher-spin supercurrent multiplets in (1,1) AdS superspace by making use of the off-shell massless supersymmetric higher-spin theories constructed in the previous two sections. Our analysis will be mostly analogous to that in the 4D case.

#### 5.5.1 Non-conformal supercurrents: Half-integer superspin

The two formulations for the massless half-integer superspin which were described in section 5.3 lead to different higher-spin supercurrent multiplet. Following similar derivations as in subsections 3.3.1 and 4.3.2 one may show that the most general half-integer
superspin current multiplet is described by the conservation equation

\[
\mathcal{D}^{\beta} J_{\beta \alpha(2s-1)} = -\frac{1}{2} \left( \mathcal{D}_{(\alpha_1} \mathcal{D}_{\alpha_2} - 2i(s-1) \mathcal{D}_{(\alpha_1\alpha_2)} \right) T_{\alpha_3 \ldots \alpha_{2s-1}} + \frac{1}{4} \left( \mathcal{D}^2 + 2\mu(2s-1) \right) F_{\alpha(2s-1)} \quad (5.5.1a)
\]

Here the higher-spin supercurrent \( J_{\alpha(2s)} \) is a real superfield. The trace multiplets \( T_{\alpha(2s-3)} \) and \( F_{\alpha(2s-1)} \) are complex superfields constrained by

\[
\mathcal{D}^{\beta} T_{\beta \alpha(2s-4)} = 0 \quad (5.5.1b)
\]
\[
\mathcal{D}_{(\alpha_1} F_{\alpha_2 \ldots \alpha_{2s})} = 0 \quad (5.5.1c)
\]

and therefore \( T_{\alpha(2s-3)} \) is a transverse linear superfield, while \( \bar{F}_{\alpha(2s-1)} \) is longitudinal linear. The multiplet with \( F_{\alpha(2s-1)} = 0 \) corresponds to the longitudinal formulation for massless superspin-(\(s + \frac{1}{2}\)) multiplet \((5.3.9)\). The case \( T_{\alpha(2s-3)} = 0 \) is associated with the transverse formulation \((5.3.6)\). In this way, we have 3D counterparts of the 4D half-integer superspin current multiplets given by \((4.3.8)\) and \((4.3.9)\).

We can also construct a well-defined improvement transformation which converts the longitudinal higher-spin supercurrent to the transverse one, thus showing that they are indeed equivalent. The most general higher-spin supercurrent \((5.5.1)\) can be modified by an improvement transformation

\[
J_{\alpha(2s)} \rightarrow J_{\alpha(2s)} + \frac{s}{2} \left[ \mathcal{D}_{(\alpha_1} \mathcal{D}_{\alpha_2]} U_{\alpha_3 \ldots \alpha_{2s}} + s \mathcal{D}_{(\alpha_1\alpha_2} V_{\alpha_3 \ldots \alpha_{2s})} \right] \quad (5.5.2a)
\]
\[
T_{\alpha(2s-3)} \rightarrow T_{\alpha(2s-3)} - \mathcal{D}^{\beta} \left( U_{\beta \alpha(2s-3)} + iV_{\beta \alpha(2s-3)} \right) \quad (5.5.2b)
\]
\[
F_{\alpha(2s-1)} \rightarrow F_{\alpha(2s-1)} + \mathcal{D}_{(\alpha_1} \left( 2s U_{\alpha_2 \ldots \alpha_{2s} (2s)} - iV_{\alpha_2 \ldots \alpha_{2s} (2s)} \right) \quad (5.5.2c)
\]

with \( U_{\alpha(2s-2)} \) and \( V_{\alpha(2s-2)} \) well-defined operators.

The transverse linearity constraint \((5.5.1b)\) can always be solved in the \((1,1)\) AdS geometry as

\[
T_{\alpha(2s-3)} = \mathcal{D}^{\beta} \left( U_{\beta \alpha(2s-3)} + iV_{\beta \alpha(2s-3)} \right) \quad (5.5.3)
\]

for well-defined real tensor operators \( U_{\alpha(2s-2)} \) and \( V_{\alpha(2s-2)} \). This property means that we can always set \( T_{\alpha(2s-3)} \) to zero by applying a certain improvement transformation \((5.5.2)\). The above analysis shows that the longitudinal and transverse supercurrents are equivalent. The situation proves to be analogous in the integer superspin case, for which we will formulate in the next subsection a higher-spin supercurrent associated with the new gauge formulation \((5.4.21)\). Therefore, it suffices to work with one of them, say, the longitudinal supercurrent multiplet \((J_{\alpha(2s)}, T_{\alpha(2s-3)})\), which obeys the conservation equation

\[
\mathcal{D}^{\beta} J_{\beta \alpha(2s-1)} = -\frac{1}{2} \left( \mathcal{D}_{(\alpha_1} \mathcal{D}_{\alpha_2} - 2i(s-1) \mathcal{D}_{(\alpha_1\alpha_2)} \right) T_{\alpha_3 \ldots \alpha_{2s-1}} \quad (5.5.4)
\]
For completeness, we also give the conjugate equation

\[ \mathcal{D}^{\beta} J_{\alpha}(2s-1) = \frac{1}{2} \left( \bar{\mathcal{D}}_{(\alpha_1} \mathcal{D}_{\alpha_2} - 2i(s-1)\mathcal{D}_{(\alpha_1\alpha_2)} \right) \bar{T}_{\alpha_3...\alpha_{2s-1}} \, . \]  

(5.5.5)

Before we proceed to the construction of higher-spin supercurrents for (1,1) AdS supersymmetric field theories, let us first recall our condensed notation in complete analogy with the four-dimensional analysis. We introduce auxiliary real variables \( \zeta^\alpha \in \mathbb{R}^2 \) and associate with any tensor superfield \( U_{\alpha(m)} \) the following index-free field

\[ U_{(m)}(\zeta) := \zeta^{\alpha_1} \cdots \zeta^{\alpha_m} U_{\alpha_1...\alpha_m} \, , \]  

(5.5.6)

which is a homogeneous polynomial of degree \( m \) in \( \zeta^\alpha \). Furthermore, we make use of the bosonic variables \( \zeta^\alpha \) and the corresponding partial derivatives \( \partial/\partial \zeta^\alpha \) to convert the spinor and vector covariant derivatives into index-free operators. In the case of (1,1) AdS superspace, we introduce operators that increase the degree of homogeneity in \( \zeta^\alpha \),

\[ \mathcal{D}_{(1)} := \zeta^\alpha \mathcal{D}_\alpha \, , \quad \bar{\mathcal{D}}_{(1)} := \zeta^\alpha \bar{\mathcal{D}}_\alpha \, , \]  

(5.5.7a)

\[ \mathcal{D}_{(2)} := i\zeta^\alpha \zeta^\beta \mathcal{D}_{\alpha\beta} = -\frac{1}{2} \{ \mathcal{D}_{(1)}, \bar{\mathcal{D}}_{(1)} \} \, . \]  

(5.5.7b)

We also introduce two operators that decrease the degree of homogeneity in the variable \( \zeta^\alpha \), specifically

\[ \mathcal{D}_{(-1)} := \mathcal{D}^\alpha \frac{\partial}{\partial \zeta^\alpha} \, , \quad \bar{\mathcal{D}}_{(-1)} := \bar{\mathcal{D}}^\alpha \frac{\partial}{\partial \zeta^\alpha} \]  

(5.5.8)

The transverse linear condition (5.5.1b) and its conjugate can be written as

\[ \bar{\mathcal{D}}_{(-1)} T_{(2s-3)} = 0 \, , \]  

(5.5.9a)

\[ \mathcal{D}_{(-1)} \bar{T}_{(2s-3)} = 0 \, . \]  

(5.5.9b)

The conservation equations (5.5.4) and (5.5.5) turn into

\[ \frac{1}{2s} \bar{\mathcal{D}}_{(-1)} J_{(2s)} = \frac{1}{2} A_{(2)} T_{(2s-3)} \, , \]  

(5.5.10a)

\[ \frac{1}{2s} \mathcal{D}_{(-1)} J_{(2s)} = \frac{1}{2} \bar{A}_{(2)} \bar{T}_{(2s-3)} \, . \]  

(5.5.10b)

where

\[ A_{(2)} := -\mathcal{D}_{(1)} \bar{\mathcal{D}}_{(1)} + 2(s-1)\mathcal{D}_{(2)} \, , \quad \bar{A}_{(2)} := \bar{\mathcal{D}}_{(1)} \mathcal{D}_{(1)} - 2(s-1)\bar{\mathcal{D}}_{(2)} \, . \]  

(5.5.11)

Since \( (\mathcal{D}_{(-1)})^2 J_{(2s)} = 0 \), the conservation equation (5.5.10a) is consistent provided

\[ \mathcal{D}_{(-1)} A_{(2)} T_{(2s-3)} = 0 \, . \]  

(5.5.12)

This is indeed true, as a consequence of the transverse linear condition (5.5.9a).
5.5.1.1 Models for a chiral superfield

We now give several examples of higher-spin supercurrents introduced above by studying rigid supersymmetric field theories in (1,1) AdS superspace.

Our first example is the superconformal theory of a single chiral scalar superfield

\[ S = \int d^{3|4}z \, \mathcal{E} \Phi \Phi, \quad (5.5.13) \]

where \( \Phi \) is covariantly chiral, \( \bar{\mathcal{D}}_\alpha \Phi = 0 \). The corresponding conformal higher-spin supercurrent is given by

\[ J_{(2s)} = \sum_{k=0}^{s} (-1)^k \left\{ \frac{1}{2} \left( \frac{2s}{2k+1} \right) \mathcal{D}_{(2)}^k \bar{\Phi} \mathcal{D}_{(2)}^{s-k-1} \mathcal{D}_{(1)} \Phi + \left( \frac{2s}{2k} \right) \mathcal{D}_{(2)}^k \bar{\Phi} \mathcal{D}_{(2)}^{s-k} \Phi \right\}, \quad (5.5.14) \]

which is a minimal extension of the conserved supercurrent constructed in flat \( \mathcal{N} = 2 \) Minkowski superspace \([109]\). It may be checked that for \( s > 0 \), the real higher-spin supercurrent \( J_{(2s)} \) satisfies the conservation equation

\[ \mathcal{D}_{(-1)} J_{(2s)} = 0 \iff \bar{\mathcal{D}}_{(-1)} J_{(2s)} = 0, \quad (5.5.15) \]

by virtue of the massless equations of motion, \( (\mathcal{D}^2 - 4\mu) \Phi = 0 \).

Let us now add the mass term to \((5.5.13)\) and consider the following action

\[ S = \int d^{3|4}z \, \mathcal{E} \Phi \Phi + \left\{ \frac{1}{2} \int d^{3|4}z \, \mathcal{E} \frac{m}{\mu} \Phi^2 + \text{c.c.} \right\}, \quad (5.5.16) \]

with \( m \) a complex mass parameter. The equations of motion are

\[ -\frac{1}{4} (\mathcal{D}^2 - 4\bar{\mu}) \Phi + \bar{m} \bar{\Phi} = 0, \quad -\frac{1}{4} (\bar{\mathcal{D}}^2 - 4\mu) \bar{\Phi} + m \Phi = 0. \quad (5.5.17) \]

After some lengthy calculations (see \([4]\) for the derivation), the equations of motion imply that on-shell the higher-spin supercurrent multiplet takes the form

\[ J_{(2s)} = \sum_{k=0}^{s} (-1)^k \left\{ \frac{1}{2} \left( \frac{2s}{2k+1} \right) \mathcal{D}_{(2)}^k \bar{\Phi} \mathcal{D}_{(2)}^{s-k-1} \mathcal{D}_{(1)} \Phi + \left( \frac{2s}{2k} \right) \mathcal{D}_{(2)}^k \bar{\Phi} \mathcal{D}_{(2)}^{s-k} \Phi \right\} \quad (5.5.18a) \]

\[ \bar{T}_{(2s-3)} = \bar{m} \sum_{k=0}^{s-2} c_k \mathcal{D}_{(2)}^k \bar{\Phi} \mathcal{D}_{(2)}^{s-k-2} \mathcal{D}_{(1)} \Phi, \quad (5.5.18b) \]

with the coefficients \( c_k \) given by

\[ c_k = (-1)^{s+k-1} \left( \frac{2s+1}{2s(s-1)} \right) \frac{1}{2s(s-1)} \sum_{l=0}^{k} \frac{1}{s-l} \left( \frac{2s}{2l+1} \right) \left\{ 1 + (-1)^s \frac{2l+1}{2s-2l+1} \right\}, \quad (5.5.18c) \]

\( k = 0, 1, \ldots s-2 \).
This is the (1,1) AdS analogue of the non-conformal supercurrents presented in 4.4.2. Indeed, the same selection rules also emerge since one can verify that the conservation equation (5.5.10b) and the transverse linearity constraint (5.5.9b) are identically satisfied only for the odd values of \(s, s = 2n + 1\), with \(n = 1, 2, \ldots\). In this sense our (1,1) AdS higher-spin supercurrents are very similar to the 4D \(\mathcal{N} = 1\) Minkowski and AdS cases studied in subsections 3.3.1.1 and 4.4.2 respectively.

5.5.1.2 Superconformal model with \(N\) chiral superfields

Another interesting example is a generalisation of the superconformal model (5.5.13) to the case of \(N\) covariantly chiral scalar superfields \(\Phi^i, i = 1, \ldots N\),

\[
S = \int d^{3}z \, E \bar{\Phi}^i \Phi^i, \quad \bar{D}_a \Phi^i = 0. \tag{5.5.19}
\]

This model is characterised by two different types of conformal supercurrents, which we denote by

\[
J^+_{\langle 2s \rangle} = S^{ij} \sum_{k=0}^{s} (-1)^k \left\{ \frac{1}{2} \left( \frac{2s}{2k+1} \right) \bar{D}_{(2)}^k \bar{D}_{(1)} \bar{\Phi}^i \bar{D}_{(2)}^{s-k-1} D_{(1)} \Phi^j \\
+ \left( \frac{2s}{2k} \right) D_{(2)}^k \bar{\Phi}^i \bar{D}_{(2)}^{s-k} \Phi^j \right\}, \quad S^{ij} = S^{ji} \tag{5.5.20}
\]

and

\[
J^-_{\langle 2s \rangle} = i A^{ij} \sum_{k=0}^{s} (-1)^k \left\{ \frac{1}{2} \left( \frac{2s}{2k+1} \right) \bar{D}_{(2)}^k \bar{D}_{(1)} \bar{\Phi}^i \bar{D}_{(2)}^{s-k-1} D_{(1)} \Phi^j \\
+ \left( \frac{2s}{2k} \right) D_{(2)}^k \bar{\Phi}^i \bar{D}_{(2)}^{s-k} \Phi^j \right\}, \quad A^{ij} = -A^{ji} \tag{5.5.21}
\]

Here \(S\) and \(A\) are arbitrary real symmetric and antisymmetric constant matrices, respectively. We have put an overall factor \(\sqrt{-1}\) in eq. (5.5.21) in order to make \(J^-_{\langle 2s \rangle}\) real. The currents (5.5.20) and (5.5.21) obey the conservation equation

\[
\bar{D}_{\langle -1 \rangle} J^+_{\langle 2s \rangle} = 0 \iff \bar{D}_{\langle -1 \rangle} J^-_{\langle 2s \rangle} = 0. \tag{5.5.22}
\]

The above results can be recast in terms of the matrix conformal supercurrent \(J_{\langle 2s \rangle} = (J^i_{\langle 2s \rangle})\) with components

\[
J^{ij}_{\langle 2s \rangle} := \sum_{k=0}^{s} (-1)^k \left\{ \frac{1}{2} \left( \frac{2s}{2k+1} \right) \bar{D}_{(2)}^k \bar{D}_{(1)} \bar{\Phi}^i \bar{D}_{(2)}^{s-k-1} D_{(1)} \Phi^j \\
+ \left( \frac{2s}{2k} \right) D_{(2)}^k \bar{\Phi}^i \bar{D}_{(2)}^{s-k} \Phi^j \right\}, \tag{5.5.23}
\]

which is Hermitian, \(J^i_{\langle 2s \rangle} = J^i_{\langle 2s \rangle}\). The chiral action (5.5.19) possesses rigid \(U(N)\) symmetry acting on the chiral column-vector \(\Phi = (\Phi^i)\) by \(\Phi \rightarrow g\Phi\), with \(g \in U(N)\). This implies that the supercurrent (5.5.23) transforms as \(J_{\langle 2s \rangle} \rightarrow gJ_{\langle 2s \rangle}g^{-1}\).
5.5.2 Non-conformal supercurrents: Integer superspin

Let us now consider the new gauge formulation (5.4.17), or equivalently (5.4.21), for the integer superspin-s multiplet to derive the 3D analogue of the non-conformal higher-spin supercurrents formulated in 4.3.4.

As usual, we first couple the prepotentials $U_{\alpha(2s-2)}$, $Z_{\alpha(2s-2)}$ and $\Psi_{\alpha(2s-1)}$ to some external sources through the action

$$S^{(s)}_{\text{source}} = \int d^{3|4}z \left\{ \Psi^{\alpha(2s-1)}J_{\alpha(2s-1)} - \bar{\Psi}^{\alpha(2s-1)}\bar{J}_{\alpha(2s-1)} + U^{\alpha(2s-2)}S_{\alpha(2s-2)} + Z^{\alpha(2s-2)}T_{\alpha(2s-2)} + \bar{Z}^{\alpha(2s-2)}\bar{T}_{\alpha(2s-2)} \right\} .$$

The action $S^{(s)}_{\text{source}}$ should be invariant under the $\zeta$-transformation (5.4.15a), which demands the source $J_{\alpha(2s-1)}$ to be transverse linear,

$$\bar{D}^{\beta}J_{\beta\alpha(2s-2)} = 0 \iff D^{\beta}\bar{J}_{\beta\alpha(2s-2)} = 0 .$$

Next, the action $S^{(s)}_{\text{source}}$ should also preserve the $\xi$-gauge freedom (5.4.14). This requires $T_{\alpha(2s-2)}$ to be longitudinal linear

$$\bar{D}_{(\alpha_{1}}T_{\alpha_{2}...\alpha_{2s-1})} = 0 \iff D_{(\alpha_{1}}\bar{T}_{\alpha_{2}...\alpha_{2s-1})} = 0 .$$

Finally, imposing the invariance of $S^{(s)}_{\text{source}}$ under the $\mathcal{V}$-transformation (5.4.15) leads to the following conservation equation

$$-D^{\beta}J_{\beta\alpha(2s-2)} + S_{\alpha(2s-2)} + T_{\alpha(2s-2)} = 0$$

as well as its conjugate

$$\bar{D}^{\beta}\bar{J}_{\beta\alpha(2s-2)} + S_{\alpha(2s-2)} + \bar{T}_{\alpha(2s-2)} = 0 .$$

Taking the sum of (5.5.27a) and (5.5.27b) leads to

$$D^{\beta}J_{\beta\alpha(2s-2)} + \bar{D}^{\beta}\bar{J}_{\beta\alpha(2s-2)} + T_{\alpha(2s-2)} - \bar{T}_{\alpha(2s-2)} = 0 .$$

As a consequence of (5.5.26), the conservation equation (5.5.28) implies

$$\mathcal{D}_{(\alpha_{1}} \left\{ D^{\beta}J_{\alpha_{2}...\alpha_{2s-1})\beta} + \bar{D}^{\beta}\bar{J}_{\alpha_{2}...\alpha_{2s-1})\beta} \right\} + \mathcal{D}_{(\alpha_{1}}T_{\alpha_{2}...\alpha_{2s-1})} = 0 .$$

Employing the condensed notation, the transverse linear condition (5.5.25) turns into

$$\bar{D}_{(-1)}J_{(2s-1)} = 0 ,$$

while the longitudinal linear condition (5.5.26) takes the form

$$\mathcal{D}_{(1)}T_{(2s-2)} = 0 .$$
The conservation equation (5.5.27a) becomes
\[- \frac{1}{(2s - 1)} \mathcal{D}_{(1)}(\mathcal{D}_{(-1)}J_{(2s-1)} + S_{(2s-2)} + \bar{T}_{(2s-2)}) = 0 \tag{5.5.32}\]

and (5.5.29) takes the form
\[\frac{1}{(2s - 1)} \mathcal{D}_{(1)} \left\{ \mathcal{D}_{(-1)}J_{(2s-1)} + \bar{\mathcal{D}}_{(-1)}\bar{J}_{(2s-1)} \right\} + \mathcal{D}_{(1)}T_{(2s-2)} = 0 . \tag{5.5.33}\]

As an example, let us go back to the massive chiral multiplet model (5.5.16)
\[S = \int d^4z E \bar{\Phi} \Phi + \left\{ \frac{1}{2} \int d^4z E M \Phi^2 + c.c. \right\} , \tag{5.5.34}\]
where the mass parameter $M$ is now real.

In the massless case, $M = 0$, this model is characterised by a fermionic supercurrent $J_{\alpha(2s-1)}$, which only exists for even values of $s$. In condensed notation, it has the form
\[J_{(2s-1)} = 2 \sum_{k=0}^{s-1}(-1)^k \left( \frac{2s - 1}{2k + 1} \right) \mathcal{D}_k^{(2)} \Phi \mathcal{D}_s^{s-k-1} \bar{\Phi} \tag{5.5.35}\]

The above is the (1,1) AdS counterpart of the integer supercurrent (4.5.14). One may check that for $s > 1$, the conservation equations
\[\mathcal{D}_{(-1)}J_{(2s-1)} = 0, \quad \bar{\mathcal{D}}_{(-1)}\bar{J}_{(2s-1)} = 0 \tag{5.5.36}\]
hold on-shell.

In the massive case, we need to solve a more general conservation equation given by (5.5.33). After some calculations, one may show that the on-shell conditions (5.5.17) imply
\[\bar{\mathcal{D}}_{(-1)}J_{(2s-1)} = 0 , \tag{5.5.37a}\]
\[\mathcal{D}_{(-1)}J_{(2s-1)} = 8Ms \sum_{k=0}^{s-1}(-1)^{k+1} \left( \frac{2s - 1}{2k + 1} \right) \]
\[\times \left\{ \mathcal{D}_k^{(2)} \Phi \mathcal{D}_s^{s-k-1} \bar{\Phi} + \frac{k}{2k + 1} \mathcal{D}_{k-1}^{(2)} \bar{\Phi} \mathcal{D}_s^{s-k-1} \mathcal{D}_{(1)} \Phi \right\} . \tag{5.5.37b}\]

The latter allows us to deduce the explicit form of the trace multiplet $T_{(2s-2)}$, which is a longitudinal linear superfield (5.5.31) and obeys (5.5.33), as a consequence of the conservation equation (5.5.32). This guides us to choose an ansatz of the form
\[T_{(2s-2)} = \sum_{k=0}^{s-1} c_k \mathcal{D}_k^{(2)} \Phi \mathcal{D}_s^{s-k-1} \bar{\Phi} \]

\[\text{This is analogous to the massive hypermultiplet model considered in (4.5.1), where it is always possible to make the mass parameter real by changing of variables.}\]
\[ + \sum_{k=1}^{s-1} d_k \mathcal{D}_{(2)}^{k-1} \mathcal{D}_{(1)} \Phi \mathcal{D}_{(2)}^{s-k-1} \bar{\mathcal{D}}_{(1)} \bar{\Phi} . \] (5.5.38)

Condition (5.5.33) implies that the coefficients must be related by
\[ c_0 = 0 , \quad c_k = 2d_k . \] (5.5.39a)

For \( k = 1, 2, \ldots s - 2 \), the following recurrence relations are obtained by the requirement (5.5.32):
\[ d_k + d_{k+1} = - \frac{8Ms}{2s-1} (-1)^{k+1} \left( \frac{2s-1}{2k} \right) \frac{4ks + 3s - 1 - 2s^2}{(2k+1)(2k+3)} . \] (5.5.39b)

It also follows from (5.5.33) that
\[ d_1 = - \frac{8}{3} Ms(s-1) , \quad d_{s-1} = - \frac{8}{2s-1} Ms(s-1) . \] (5.5.39c)

The above conditions lead to a simple expression for \( d_k \):
\[ d_k = \frac{8Ms}{2s-1} \frac{k}{2k+1} (-1)^k \left( \frac{2s-1}{2k} \right) , \] (5.5.40)
where \( k = 1, 2, \ldots s - 1 \) and the parameter \( s \) is even for \( J_{(2s-1)} \) to be non-zero.

5.6 Higher-spin supercurrents for chiral matter in (2,0) AdS superspace

We now turn to describing the off-shell constructions of higher-spin gauge supermultiplets in (2,0) AdS superspace [5], which prove to be less trivial. As pointed out in the introduction, the massless 3D constructions of [4, 49], were largely modelled on the 4D results of [63, 64]. With respect to 3D (2,0) AdS supersymmetry, unfortunately there is no 4D intuition to guide us, and new ideas are required in order to construct higher-spin gauge supermultiplets. The approach employed in [5] was based on an observation that has often been used in the past to formulate off-shell supergravity multiplets [58, 84–88]. The idea is to make use of a higher-spin extension of the supercurrent. Specifically, for a simple supersymmetric model in (2,0) AdS superspace we identify a multiplet of conserved higher-spin currents. In general, the multiplet of currents is always off-shell. Using the constructed higher-spin supercurrent, we may identify a corresponding off-shell supermultiplet of higher-spin fields.

We begin with some simple models for a chiral scalar supermultiplet in (2,0) AdS superspace and try to derive the corresponding higher-spin supercurrent multiplet.
5.6.1 Massless models

Let us first consider a massless model. Its action

\[ S = \int d^3 x d^2 \theta d^2 \bar{\theta} E \bar{\Phi} \Phi, \quad \bar{D}_\alpha \Phi = 0 \]  

is invariant under the isometry transformations of (2,0) AdS superspace for any U(1)$_R$ charge $r$ of the chiral superfield,

\[ J\Phi = -r\Phi, \quad r = \text{const}. \]  

The action is superconformal provided $r = \frac{1}{2}$.

Let us first consider the superconformal case, $r = \frac{1}{2}$. The analysis given in subsection 5.5.1.1 implies that the theory possesses a real, bosonic supercurrent $\mathbb{J}^{(2s)} = \bar{\mathbb{J}}^{(2s)}$, for any positive integer $s$, which obeys the conservation equation

\[ D(-1)\mathbb{J}^{(2s)} = 0. \]  

This supercurrent proves to have the same form as in the (1,1) AdS case, given by (5.5.14). Specifically, the higher-spin supercurrent is given by

\[ \mathbb{J}^{(2s)} = \sum_{k=0}^{s} (-1)^k \left\{ \frac{1}{2} \left( \frac{2s}{2k + 1} \right) D^{(2)}_s \bar{\Phi} \ D^{s-k-1}_s D^{(2)_s}_s \Phi + \left( \frac{2s}{2k} \right) D^{(2)}_s \bar{\Phi} \ D^{s-k}_s \Phi \right\}. \]  

Making use of the massless equations of motion, $D^2 \Phi = 0$, one may check that (5.6.4) does obey the conservation equation (5.6.3). In the flat superspace limit, the supercurrent (5.6.4) reduces to the one constructed in [109].

Now we turn to the non-superconformal case, $r \neq \frac{1}{2}$. Direct calculations give

\[ D(-1)\bar{\mathbb{J}}^{(2s)} = D^{(1)} \mathbb{T}^{(2s-2)} \]  

where we have denoted

\[ \mathbb{T}^{(2s-2)} = 2i(1 - 2r)S(2s + 1)(s + 1) \sum_{k=0}^{s-1} \frac{1}{2s - 2k + 1} (-1)^k \left( \frac{2s}{2k + 1} \right) \]

\times D^{k}_s \bar{\Phi} \ D^{s-k-1}_s \Phi. \]  

The trace multiplet $\mathbb{T}^{(2s-2)}$ is covariantly linear,

\[ \bar{D}^2 \mathbb{T}^{(2s-2)} = 0, \quad D^2 \mathbb{T}^{(2s-2)} = 0, \]  

as a consequence of the equations of motion and the identity (5.2.21c). It is seen that $\mathbb{T}^{(2s-2)}$ has nonzero real and imaginary parts,

\[ \mathbb{T}^{(2s-2)} = Y^{(2s-2)} + iZ^{(2s-2)}, \quad \bar{Y}^{(2s-2)} = Y^{(2s-2)}, \quad \bar{Z}^{(2s-2)} = Z^{(2s-2)}, \]
except for the $s = 1$ case which is characterised by $\Psi = 0$. For $s = 1$ the above results agree with [83].

The above results can be used to derive higher-spin supercurrents in a non-minimal scalar supermultiplet model described by the action

$$S = - \int d^3x d^2\theta d^2\bar{\theta} E \Gamma \Gamma , \quad D^2 \Gamma = 0 ,$$  \hspace{1cm} (5.6.6)

with $\Gamma$ being a complex linear superfield. The non-minimal theory (5.6.6) proves to be dual to (5.6.1) provided the $U(1)_R$ weight of $\Gamma$ is opposite to that of $\Phi$, $J \Gamma = r \Gamma$. \hspace{1cm} (5.6.7)

Replacing $\Phi \rightarrow \bar{\Gamma}$ and $\bar{\Phi} \rightarrow \Gamma$ in (5.6.5) gives the higher-spin supercurrents in the non-minimal theory (5.6.6), which is similar to the 4D case [3,116].

Let us also mention that in deriving eq. (5.6.5), one may find the following identities useful. We start with the obvious relations

$$\frac{\partial}{\partial \zeta^\alpha} D_{(2)} = 2i \zeta^\beta D_{\alpha\beta} ,$$  \hspace{1cm} (5.6.8a)

$$\frac{\partial}{\partial \zeta^\alpha} D_{k(2)} = \sum_{n=1}^{k} D_{(2)}^{n-1} 2i \zeta^\beta D_{\alpha\beta} D_{k-n(2)} , \quad k > 1 .$$  \hspace{1cm} (5.6.8b)

To simplify eq. (5.6.8b), we may push $\zeta^\beta D_{\alpha\beta}$, say, to the left provided that we take into account its commutator with $D_{(2)}$:

$$[\zeta^\beta D_{\alpha\beta} , D_{(2)}] = -4i S^2 \zeta^\alpha \zeta^\beta \zeta^\gamma M_{\beta\gamma} .$$  \hspace{1cm} (5.6.9)

Associated with the Lorentz generators are the operators

$$M_{(2)} := \zeta^\alpha \zeta^\beta M_{\alpha\beta} ,$$  \hspace{1cm} (5.6.10)

where $M_{(2)}$ appears in the right-hand side of (5.6.9). This operator annihilates every superfield $U_{(m)}(\zeta)$ of the form (5.5.6),

$$M_{(2)} U_{(m)} = 0 .$$  \hspace{1cm} (5.6.11)

From the above consideration, it follows that

$$[\zeta^\beta D_{\alpha\beta} , D_{k(2)}] U_{(m)} = 0 ,$$  \hspace{1cm} (5.6.12a)

$$\left( \frac{\partial}{\partial \zeta^\alpha} D_{k(2)} \right) U_{(m)} = 2i k \zeta^\beta D_{\alpha\beta} D_{k-1(2)} U_{(m)} .$$  \hspace{1cm} (5.6.12b)

We also state some other properties which we often use throughout our calculations

$$D_{(1)}^2 = 0 ,$$  \hspace{1cm} (5.6.13a)

$$[D_{(1)} , D_{(2)}] = [\bar{D}_{(1)} , D_{(2)}] = 0 ,$$  \hspace{1cm} (5.6.13b)

$$[D^\alpha , D_{(2)}] = 2i S \zeta^\alpha D_{(1)} ,$$  \hspace{1cm} (5.6.13c)

$$[D^\alpha , D_{k(2)}] = 2i S k \zeta^\alpha D_{k-1(2)} D_{(1)} ,$$  \hspace{1cm} (5.6.13d)

$$[D^\alpha , \zeta^\beta D_{\alpha\beta}] = 3 S D_{(1)} .$$  \hspace{1cm} (5.6.13e)
5.6.2 Massive model

We consider the addition of a mass term to the functional \[ (5.6.1) \]

\[ S = \int d^3x d^2\theta d^2\bar{\theta} E \bar{\Phi}\Phi + \left\{ \frac{m}{2} \int d^3x d^2\theta \ E \Phi^2 + \text{c.c.} \right\}, \tag{5.6.14} \]

with \( m \) a complex mass parameter. In the \( m \neq 0 \) case, the U(1)\(_R\) weight of \( \Phi \) is uniquely fixed to be \( r = 1 \), in order for the action to be \( R \)-invariant.

Making use of the massive equations of motion

\[
\begin{align*}
-\frac{1}{4} D^2 \Phi + \bar{m} \Phi &= 0, \\
-\frac{1}{4} D^2 \Phi + m \Phi &= 0, \tag{5.6.15}
\end{align*}
\]

we obtain

\[
D_{(-1)^s} \mathbb{J}_{(2s)} = -2i S (2s + 1)(s + 1) D_{(1)} \sum_{k=0}^{s-1} \frac{1}{2s - 2k + 1} (-1)^k \left( \frac{2s}{2k + 1} \right) \]

\[
\times D_{(2)}^k \Phi \ D_{(2)}^{s-k-1} \Phi
\]

\[
+ \bar{m} (1 - 1)^s (2s + 1) \sum_{k=0}^{s-1} \left\{ 1 + (-1)^s \left( \frac{2k + 1}{2s - 2k + 1} \right) \right\} (-1)^k \left( \frac{2s}{2k + 1} \right)
\]

\[
\times D_{(2)}^k \Phi \ D_{(2)}^{s-k-1} \bar{D}_{(1)} \Phi, \tag{5.6.16}
\]

where \( \mathbb{J}_{(2s)} \) is defined by \(5.6.4\). We observe that \(5.6.16\) can also be written in the form

\[
D_{(-1)^s} \mathbb{J}_{(2s)} = \frac{1}{2} (-1)^s D_{(-1)} \sum_{k=0}^{s-1} (-1)^k \left( \frac{2s}{2k + 1} \right) D_{(2)}^k D_{(1)} \Phi \ D_{(2)}^{s-k-1} \bar{D}_{(1)} \Phi
\]

\[
- \frac{1}{2} D_{(1)} \sum_{k=0}^{s-1} (2k + 1)(-1)^k \left( \frac{2s}{2k + 1} \right) D_{(2)}^k D_{(2)} \Phi \ D_{(2)}^{s-k-1} \bar{D}_a \bar{\Phi}
\]

\[
+ 2i S D_{(1)} \sum_{k=0}^{s-1} \left[ (2k + 1) + (-1)^s (2s - 2k - 1) \right]
\]

\[
\times (-1)^k \left( \frac{2s}{2k + 1} \right) D_{(2)}^k \Phi \ D_{(2)}^{s-k-1} \Phi
\]

\[
+ i [1 + (-1)^s] \sum_{k=0}^{s-1} (2k + 1)(-1)^k \left( \frac{2s}{2k + 1} \right)
\]

\[
\times D_{(2)}^k \Phi \ D_{(2)}^{s-k-1} \zeta^\beta \bar{D}_{\alpha\beta} \Phi. \tag{5.6.17}
\]

Thus, for all odd values of \( s \),

\[ s = 2n + 1, \quad n = 0, 1, \ldots, \tag{5.6.18a} \]

we end up with the conservation equation

\[ D_{(-1)^s} \mathbb{J}_{(2s)} = D_{(1)} \bar{T}_{(2s-2)} \tag{5.6.18b} \]
where we have denoted
\[
\hat{J}_{(2s)} = J_{(2s)} - \frac{1}{2} \sum_{k=0}^{s} (-1)^k \binom{2s}{2k+1} D^{k}_{(2)} \bar{D}^{s-k-1}_{(1)} \bar{D}_{(1)} \Phi ,
\]
(5.6.18c)
\[
\hat{T}_{(2s-2)} = -\frac{1}{2} \sum_{k=0}^{s-1} (2k+1)(-1)^k \binom{2s}{2k+1} D^{k}_{(2)} \bar{D}^{s-k-1}_{(1)} \bar{D}_{(1)} \Phi 
+ 2iS \sum_{k=0}^{s-1} \left[(1-s)(2k+1) + 2s^2\right] (-1)^k \binom{2s}{2k+1} D^{k}_{(2)} \bar{D}^{s-k-1}_{(1)} \bar{D}_{(1)} \Phi .
\]
(5.6.18d)

The trace multiplet \( \hat{T}_{(2s-2)} \) is covariantly linear,
\[
\bar{D}^2 \hat{T}_{(2s-2)} = 0 , \quad D^2 \hat{T}_{(2s-2)} = 0 .
\]
(5.6.18e)

The conservation equation defined by eqs. (5.6.18b) and (5.6.18c) coincides with that defined by eqs. (5.6.5a) and (5.6.5c).

The above analysis demonstrates that in the massive case, the higher-spin supercurrent \( \hat{J}_{(2s)} \) exists only for the odd values of \( s \). This conclusion is again analogous to our previous results in 4D and (1,1) AdS superspace. As demonstrated in the construction of AdS higher-spin supercurrents (see 4.4.4), the even values of \( s \) are also allowed provided there are several massive chiral superfields in the theory. This analysis may be extended to the (2,0) AdS case.

### 5.7 Massless higher-spin gauge theories in (2,0) AdS superspace

The explicit structure of the higher-spin supercurrent multiplet defined by eqs. (5.6.5a) and (5.6.5c) allows us to develop two off-shell formulations for a massless multiplet of half-integer superspin-\((s + \frac{1}{2})\), with \( s = 2, 3, \ldots \). We will call them type II and type III series\(^4\) to comply with the terminology introduced in [83] for the minimal off-shell formulations for \( \mathcal{N} = 2 \) supergravity (\( s = 1 \)).

#### 5.7.1 Type II series

Given a positive integer \( s \geq 2 \), we propose to describe a massless multiplet of superspin-\((s + \frac{1}{2})\) in terms of two unconstrained real superfields
\[
Y^{(\text{II})}_{(s + \frac{1}{2})} = \left\{ \mathcal{H}_{\alpha(2s)}, \mathcal{L}_{\alpha(2s-2)} \right\} .
\]
(5.7.1)

\(^4\)Type I series will be referred to as the longitudinal formulation for the gauge massless half-integer superspin multiplets in (1,1) AdS superspace [5.3.9] and Minkowski superspace [49]. The type I series and its dual are naturally related to the off-shell formulations for massless higher-spin \( \mathcal{N} = 1 \) supermultiplets in four dimensions [63] [65]. The type II and type III series have no four-dimensional counterpart.
Here $\mathcal{H}_\alpha(2s) = \mathcal{H}_{(\alpha_1...\alpha_{2s})}$ and $\mathcal{L}_\alpha(2s-2) = \mathcal{L}_{(\alpha_1...\alpha_{2s-2})}$ are symmetric in their spinor indices.

We postulate gauge transformations for the dynamical superfields:

$$
\delta_\lambda \mathcal{H}_\alpha(2s) = \hat{D}_{(\alpha_1...\alpha_{2s})} \bar{\lambda}_{(\alpha_1...\alpha_{2s})} - D_{(\alpha_1...\alpha_{2s})} \bar{\lambda}_{(\alpha_1...\alpha_{2s})} \equiv g_\alpha(2s) + \bar{g}_\alpha(2s) , \tag{5.7.2a}
$$

$$
\delta_\lambda \mathcal{L}_\alpha(2s-2) = -\frac{i}{2} \left( \hat{D}^\beta \lambda_{\beta\alpha(2s-2)} + D^\beta \bar{\lambda}_{\beta\alpha(2s-2)} \right) , \tag{5.7.2b}
$$

where the gauge parameter $\lambda_{(2s-1)}$ is unconstrained complex. Eq. (5.7.2a) implies that the complex gauge parameter $g_\alpha(2s)$ is a covariantly longitudinal linear superfield,

$$
g_\alpha(2s) := \hat{D}_{(\alpha_1...\alpha_{2s})} \bar{\lambda}_{(\alpha_1...\alpha_{2s})} , \quad \hat{D}_{(\alpha_1...\alpha_{2s-1})} g_\alpha(2s) = 0 . \tag{5.7.3}
$$

The gauge transformation of $\mathcal{H}_\alpha(2s)$, eq. (5.7.2a), corresponds to the conformal superspin-$(s + \frac{1}{2})$ gauge prepotential reviewed in subsection 5.1.3. It is natural to interpret $\mathcal{L}_\alpha(2s-2)$ as a compensating multiplet. In order for $\delta_\lambda \mathcal{H}_\alpha(2s)$ and $\delta_\lambda \mathcal{L}_\alpha(2s-2)$ to be real, $\lambda_{(2s-1)}$ must be charged under the $R$-symmetry group $U(1)_R$:

$$
J \lambda_{\alpha(2s-1)} = \lambda_{(2s-1)} , \quad J \bar{\lambda}_{\alpha(2s-1)} = -\bar{\lambda}_{(2s-1)} . \tag{5.7.4}
$$

In addition to (5.7.2b), the compensator $\mathcal{L}_\alpha(2s-2)$ also possesses its own gauge freedom

$$
\delta_\xi \mathcal{L}_\alpha(2s-2) = \xi_{\alpha(2s-2)} + \bar{\xi}_{\alpha(2s-2)} , \quad \hat{D}_\beta \xi_{\alpha(2s-2)} = 0 , \tag{5.7.5}
$$

with the gauge parameter $\xi_{\alpha(2s-2)}$ being covariantly chiral, but otherwise arbitrary. It should be pointed out that in (1,1) AdS superspace covariantly chiral superfields exist only in the scalar case, since the constraint $\hat{D}_\beta \Psi_{\alpha(n)} = 0$ is inconsistent for $n > 0$. Therefore, the gauge transformation law (5.7.5) is specific for the (2,0) AdS supersymmetry.

Associated with $\mathcal{L}_\alpha(2s-2)$ is the real field strength

$$
\mathcal{L}_\alpha(2s-2) = i \hat{D}^\beta \bar{D}_\beta \mathcal{L}_\alpha(2s-2) , \quad \mathcal{L}_\alpha(2s-2) = \mathcal{L}_\alpha(2s-2) , \tag{5.7.6}
$$

which is a covariantly linear superfield,

$$
\mathcal{D}^2 \mathcal{L}_\alpha(2s-2) = 0 \iff \hat{D}^2 \mathcal{L}_\alpha(2s-2) = 0 . \tag{5.7.7}
$$

It is inert under the gauge transformation (5.7.5), $\delta_\xi \mathcal{L}_\alpha(2s-2) = 0$. From (5.7.2b) we can read off the $\lambda$-gauge transformation of the field strength

$$
\delta_\lambda \mathcal{L}_\alpha(2s-2) = \frac{1}{4} \left( \mathcal{D}^\beta \mathcal{D}^2 \lambda_{\beta\alpha(2s-2)} - \hat{\mathcal{D}}^\beta \mathcal{D}^2 \bar{\lambda}_{\beta\alpha(2s-2)} \right) = -\frac{s}{2s+1} \mathcal{D}^\beta \mathcal{D}^\gamma \left( g_{\beta\gamma\alpha(2s-2)} + \bar{g}_{\beta\gamma\alpha(2s-2)} \right) - \frac{2is}{2s+1} \mathcal{D}^\beta \mathcal{D}^{\gamma\alpha(2s-2)} \bar{g}_{\beta\gamma\alpha(2s-2)} . \tag{5.7.8}
$$

The reason why we express the gauge transformations of $\mathcal{H}_\alpha(2s)$ and $\mathcal{L}_\alpha(2s-2)$ in terms of the constrained superfield $g_\alpha(2s)$ is that such representation will be useful to carry out the $(2,0) \rightarrow (1,0)$ AdS reduction in chapter 6.
Modulo an overall normalisation factor, there is a unique quadratic action which is invariant under the gauge transformations (5.7.2). It is given by

\[ S^{(II)}_{\frac{s}{2}}[\mathcal{H}_\alpha(2s), \mathcal{V}_\alpha(2s-2)] = \left( -\frac{1}{2} \right)^s \int d^3x d^2\theta d^2\bar{\theta} \mathcal{E} \left\{ \frac{1}{8} \delta y^{(2s)}D^\beta \bar{D}^\gamma D_\beta \bar{D}_\gamma \delta \mathcal{H}_\alpha(2s) \\
- \frac{s}{8} ([D_\beta, \bar{D}_\gamma] \delta y^{(2s-2)} + [D^\beta, \bar{D}^\gamma] \delta \mathcal{H}_\alpha(2s-2) \\
+ \frac{s}{2} (D_\beta y^{(2s-2)} - [D^\beta, \bar{D}^\gamma] \delta \mathcal{H}_\alpha(2s-2) + 2i S \delta y^{(2s)} D^\beta \bar{D}_\beta \delta \mathcal{H}_\alpha(2s) \\
- \frac{2s-1}{2} \left( [L^{(2s-2)}_\alpha, \bar{D}_\gamma] \delta \mathcal{H}_\alpha(2s-2) + 2L^{(2s-2)}_\alpha L_\alpha(2s-2) \right) \\
- \frac{(s-1)(2s-1)}{4s} \left( D_\beta \mathcal{V}^{(2s-3)}_{\beta \alpha} \bar{D}^2 \bar{D}_\gamma \mathcal{V}_{\gamma \alpha(2s-3)} + c.c. \right) \\
- 4(2s-1) \delta \mathcal{V}^{(2s-2)}_{\alpha} L_\alpha(2s-2) \right\} \right) . \]

(5.7.9)

By construction, the action is also invariant under (5.7.5). This action differs from the massless half-integer superspin actions in (1,1) AdS superspace, (5.3.6) and (5.3.9), due to the presence of a Chern-Simons-type term.

Setting \( s = 1 \) in (5.7.9) gives the linearised action for (2,0) AdS supergravity, which was originally derived in section 10.1 of [83]. Ref. [83] made use of the curvature parameter \( \rho \), which is related to our \( S \) as \( \rho = 4S \). It should be remarked that the structure \( D_\beta \mathcal{V}^{(2s-3)}_{\beta \alpha} \bar{D}^2 \bar{D}_\gamma \mathcal{V}_{\gamma \alpha(2s-3)} \) in (5.7.9) is not defined for \( s = 1 \). However, this term contains an overall numerical factor \((s-1)\) and therefore it does not contribute for \( s = 1 \).

### 5.7.2 Type III series

Our second model for the massless multiplet of superspin-(\( s + \frac{1}{2} \)) is realised in terms of dynamical variables that are completely similar to (5.7.1),

\[ \mathcal{V}^{(III)}_{\frac{s}{2} + \frac{1}{2}} \left\{ \mathcal{H}_\alpha(2s), \mathcal{V}_\alpha(2s-2) \right\} . \]

(5.7.10)

Here \( \mathcal{H}_\alpha(2s) \) and \( \mathcal{V}_\alpha(2s-2) \) are unconstrained real tensor superfields.

The dynamical superfields are defined modulo gauge transformations of the form

\[ \delta \lambda \mathcal{H}_\alpha(2s) = \mathcal{D}_{\alpha \lambda} \mathcal{H}_{\alpha_{a_2...a_2s}} - \mathcal{D}_{\alpha_1 \lambda_1} \mathcal{H}_{\alpha_{a_2...a_2s}} = g_{\alpha} + \bar{g}_{\alpha} , \]

(5.7.11a)

\[ \delta \lambda \mathcal{V}_\alpha(2s-2) = \frac{1}{2s} \left( \bar{D}^\beta \lambda_{\beta \alpha(2s-2)} - D^\beta \bar{\lambda}_{\beta \alpha(2s-2)} \right) , \]

(5.7.11b)

where the gauge parameter \( \lambda_{\alpha(2s-1)} \) is unconstrained complex, and the longitudinal linear parameter \( g_{\alpha(2s)} \) is defined as in (5.7.3). As in the type II case, \( \mathcal{H}_\alpha(2s) \) is the superconformal gauge multiplet, while \( \mathcal{V}_\alpha(2s-2) \) is a compensating multiplet. The only difference from the type II case occurs in the gauge transformation law for the compensator \( \mathcal{V}_\alpha(2s-2) \).
The compensator $\mathcal{W}_{\alpha(2s-2)}$ is required to have its own gauge freedom of the form

$$\delta \xi \mathcal{W}_{\alpha(2s-2)} = \xi_{\alpha(2s-2)} + i \bar{\xi}_{\alpha(2s-2)}$$  \hspace{1cm} (5.7.12)

with the gauge parameter $\xi_{\alpha(2s-2)}$ being covariantly chiral, but otherwise arbitrary.

Associated with $\mathcal{W}_{\alpha(2s-2)}$ is the real field strength

$$\nabla_{\alpha(2s-2)} = i D^\beta \bar{D}_\beta \mathcal{W}_{\alpha(2s-2)}$$  \hspace{1cm} (5.7.13)

which is inert under (5.7.12), $\delta \xi \nabla_{\alpha(2s-2)} = 0$. It is not difficult to see that $\nabla_{\alpha(2s-2)}$ is covariantly linear,

$$\mathcal{D}^2 \nabla_{\alpha(2s-2)} = 0 \iff \bar{\mathcal{D}}^2 \nabla_{\alpha(2s-2)} = 0.$$  \hspace{1cm} (5.7.14)

Modulo normalisation, there exists a unique action being invariant under the gauge transformations (5.7.11) as

$$\delta_{\lambda} \nabla_{\alpha(2s-2)} = \frac{i}{4s} \left[ D^\beta \bar{D}^2 \lambda_{\beta\alpha(2s-2)} + \bar{D}^\beta D^2 \bar{\lambda}_{\beta\alpha(2s-2)} \right].$$  \hspace{1cm} (5.7.15)

Although the structure $D_\beta \mathcal{W}^{\alpha(2s-3)} \bar{D}^2 D^\gamma \mathcal{W}_{\alpha(2s-3)}$ in (5.7.16) is not defined for $s = 1$, it comes with the factor $(s - 1)$ and drops out from (5.7.16) for the $s = 1$ case. In this case the action coincides with the type III supergravity action\footnote{Type III supergravity is known only at the linearised level. In the super-Poincaré case, it is a 3D analogue of the massless superspin-3/2 multiplet proposed in [132].} in (2,0) AdS superspace, which was originally derived in section 10.2 of [83].

5 Type III supergravity is known only at the linearised level. In the super-Poincaré case, it is a 3D analogue of the massless superspin-3/2 multiplet proposed in [132].
5.8 Summary and discussion

Let us summarise the main results obtained thus far. Sections 5.3 and 5.4 are devoted to the superfield descriptions of off-shell massless higher-spin gauge theories in (1,1) AdS superspace, which are essentially analogous to their 4D $\mathcal{N} = 1$ AdS counterparts. A useful application includes the possibility to derive off-shell massless higher-spin $\mathcal{N} = 1$ supermultiplets by performing $\mathcal{N} = 2 \rightarrow \mathcal{N} = 1$ superspace reduction. As an example, we carried out reduction of the longitudinal theory for the massless superspin-$s$ multiplet (5.4.6) in the super-Poincaré limit and ended up with a new model for massless $\mathcal{N} = 1$ higher-spin supermultiplet that was not described in [50, 51]. In section 5.5, the off-shell gauge formulations enabled us to derive consistent higher-spin supercurrent multiplets with (1,1) AdS supersymmetry. By studying models for chiral scalar superfields, we presented explicit expressions of such supercurrents.

With regards to (2,0) AdS supersymmetry, we employed a “bottom-up” approach. The starting point was some simple dynamical systems in (2,0) AdS superspace, i.e models for a free chiral scalar superfield. In such models, we deduced that the corresponding multiplet of higher-spin currents is described by the conservation equations

$$ D^\beta \mathcal{J}^{\alpha_1 \ldots \alpha_{s-1}} = D_{(\alpha_1} T_{\alpha_2 \ldots \alpha_{s-1})} , \quad \bar{D}^\beta \mathcal{J}^{\alpha_1 \ldots \alpha_{s-1}} = \bar{D}_{(\alpha_1} \bar{T}_{\alpha_2 \ldots \alpha_{s-1})} , \quad \tag{5.8.1a} $$

with the real superfield $\mathcal{J}_\alpha^{(2s)}$ denoting the higher-spin supercurrent, and $T_{\alpha(2s-2)}$ the corresponding trace supermultiplet constrained to be covariantly linear

$$ \bar{D}^2 T_{\alpha(2s-2)} = 0 , \quad D^2 T_{\alpha(2s-2)} = 0 . \quad \tag{5.8.1b} $$

In general, the trace supermultiplet is complex,

$$ T_{\alpha(2s-2)} = Y_{\alpha(2s-2)} - iZ_{\alpha(2s-2)} , \quad \text{Im} Y_{\alpha(2s-2)} = 0 , \quad \text{Im} Z_{\alpha(2s-2)} = 0 . \quad \tag{5.8.1c} $$

In the $s = 1$ case, the above conservation equation coincides with that for the (2,0) AdS supercurrent [83].

We did not carry out a systematic analysis (similar to that given by Dumitrescu and Seiberg [79] for ordinary supercurrents in Minkowski space) of the higher-spin supercurrent (5.8.1). However, the formal consistency of (5.8.1) follows from the structure of the massless superspin-$(s + \frac{1}{2})$ gauge theories constructed in section 5.7. For instance, within the framework of the type II formulation, let us couple the prepotentials $\mathcal{H}_\alpha^{(2s)}$ and $\mathcal{L}_\alpha^{(2s-2)}$ to external sources

$$ S_{\text{source}}^{(s+\frac{1}{2})} = \int d^3x d^2\theta d^2\bar{\theta} \mathcal{E} \left\{ \mathcal{H}_\alpha^{(2s)} \mathcal{J}_\alpha^{(2s)} - 2\mathcal{L}_\alpha^{(2s-2)} Z_{\alpha(2s-2)} \right\} , \quad \tag{5.8.2} $$

Requiring $S_{\text{source}}^{(s+\frac{1}{2})}$ to be invariant under the gauge transformations (5.7.5) tells us that the real supermultiplet $Z_{\alpha(2s-2)}$ is covariantly linear,

$$ \bar{D}^2 Z_{\alpha(2s-2)} = 0 . \quad \tag{5.8.3} $$

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If we also require \( S^{(s+\frac{1}{2})}_{\text{source}} \) to be invariant under the gauge transformations (5.7.2), we obtain the conservation equation

\[
\mathcal{D}^\beta J_{\beta \alpha_1 \ldots \alpha_{2s-1}} = i \mathcal{D}_{(\alpha_1} Z_{\alpha_2 \ldots \alpha_{2s-1})} \, .
\] (5.8.4)

Additionally, taking the type III formulation into account leads to the general conservation equation

\[
\bar{\mathcal{D}}^\beta \bar{J}_{\beta \alpha_1 \ldots \alpha_{2s-1}} = \mathcal{D}_{(\alpha_1} \left( \bar{Y}_{\alpha_2 \ldots \alpha_{2s-1})} + i \bar{Z}_{\alpha_2 \ldots \alpha_{2s-1})} \right) \, ,
\] (5.8.5)

where the real trace supermultiplets \( Y_{\alpha(2s-2)} \) and \( Z_{\alpha(2s-2)} \) are covariantly linear. The off-shell construction of a massless multiplet of integer superspin with (2,0) AdS supersymmetry would definitely deserve further study.

An improvement transformation exists for the higher-spin supercurrent multiplet (5.8.1). Let us introduce

\[
\tilde{J}_{\alpha(2s)} := J_{\alpha(2s)} + [\mathcal{D}_{(\alpha_1}, \bar{\mathcal{D}}_{\alpha_2]} S_{\alpha_3 \ldots \alpha_{2s}}] + 2 \mathcal{D}_{(\alpha_1} \alpha_2] R_{\alpha_3 \ldots \alpha_{2s})} \, ,
\] (5.8.6a)

\[
\tilde{Y}_{\alpha(2s-2)} := Y_{\alpha(2s-2)} - i \mathcal{D}^\gamma \mathcal{D}_\gamma R_{\alpha(2s-2)} + 4(s+1) S R_{\alpha(2s-2)} + \frac{2}{s}(s - 1) \mathcal{D}^\beta (a_1 R_{\alpha_2 \ldots \alpha_{2s-2}}) \beta \, ,
\] (5.8.6b)

\[
\tilde{Z}_{\alpha(2s-2)} := Z_{\alpha(2s-2)} - i \frac{s+1}{s} \mathcal{D}^\gamma \mathcal{D}_\gamma S_{\alpha(2s-2)} - 4(s+1) S S_{\alpha(2s-2)} - \frac{2}{s}(s - 1) \mathcal{D}^\beta (a_1 S_{\alpha_2 \ldots \alpha_{2s-2}}) \beta \, ,
\] (5.8.6c)

with \( S_{\alpha(2s-2)} \) and \( R_{\alpha(2s-2)} \) real linear superfields. One may check that \( \tilde{J}_{\alpha(2s)}, \tilde{Y}_{\alpha(2s-2)} \) and \( \tilde{Z}_{\alpha(2s-2)} \) obey the conservation equation and constraints described by (5.8.1). In the \( s = 1 \) case, we reproduce the result given in section 10.4 of [83].

As a final remark, there is one special feature of the supergravity case, \( s = 1 \), for which the supercurrent conservation equation takes the form [83]

\[
\mathcal{D}^\beta \bar{J}_{\beta \alpha} = \bar{D}_{\alpha} (Y + iZ) \, ,
\] (5.8.7)

with the real trace supermultiplets \( Y \) and \( Z \) being covariantly linear. Building on the thorough analysis of [79], it was pointed out in [83] that there exists a well-defined improvement transformation that results with \( Y = 0 \). For all the supersymmetric field theories in (2,0) AdS superspace considered in [83], the supercurrent is characterised by the condition \( Y = 0 \). Actually, this condition is easy to explain. The point is that every 3D \( \mathcal{N} = 2 \) supersymmetric field theory with U(1) \( R \)-symmetry may be coupled to the (2,0) AdS supergravity, which implies \( Y = 0 \) upon freezing the supergravity multiplet to its maximally supersymmetric (2,0) AdS background. There is another way to explain why \( Y \) may always be improved to zero. For simplicity, let us consider the case
of $\mathcal{N} = 2$ Poincaré supersymmetry, with $D_\alpha$ and $\bar{D}_\alpha$ being the flat-superspace covariant derivatives. In Minkowski superspace eq. (5.8.7) implies $\partial^{\alpha\beta}J_{\alpha\beta} = iD^\alpha \bar{D}_\alpha Y$, and therefore $Y = iD^\alpha \bar{D}_\alpha R$, for some real linear superfield $R$. If we now apply the flat-superspace version of (5.8.6) with $S = 0$, we will end up with $Y = 0$. However, in the higher-spin case it no longer seems possible to improve the trace supermultiplet $\mathcal{Y}_{\alpha(2s-2)}$ to vanish, as our analysis in section 5.6 indicates.
Chapter 6

Field theories with (2,0) AdS supersymmetry in \( \mathcal{N} = 1 \) AdS superspace

In the preceding chapter, it was pointed out that 3D \( \mathcal{N} \)-extended AdS supergravity exists in several incarnations and they are known as \((p,q)\) AdS supergravity theories. Various aspects of \( \mathcal{N} = 2 \) supersymmetric higher-spin gauge theories in 3D anti-de Sitter space, AdS\(_3\), have also been elaborated in some detail.

This chapter has two main objectives. The first is to present a formalism which was developed in [6] to reduce every field theory with (2,0) AdS supersymmetry to \( \mathcal{N} = 1 \) AdS superspace. This formalism is then applied to carry out the (2,0) \( \rightarrow \) (1,0) AdS reduction of the two off-shell massless higher-spin supermultiplets constructed in section 5.7. Our motivation came from certain theoretical arguments which suggest the existence of more general off-shell massless higher-spin \( \mathcal{N} = 1 \) supermultiplets in AdS\(_3\) than those described in [51]. The second objective is to study \( \mathcal{N} = 1 \) supermultiplets of conserved higher-spin currents in AdS\(_3\), which were derived for the first time in [6].

6.1 (2,0) \( \rightarrow \) (1,0) AdS superspace reduction

The aim of this section is to elaborate on the details of procedure for reducing field theories in (2,0) AdS superspace to \( \mathcal{N} = 1 \) AdS superspace. Explicit examples of such a reduction are given by considering supersymmetric nonlinear \( \sigma \)-models.

\(^1\)It should be pointed out that the superconformal multiplets of conserved currents in Minkowski superspace [109] can readily be lifted to AdS\(_3\).
6.1.1 Geometry of (2,0) AdS superspace: Real basis

In section 5.2.2 the geometry of (2,0) AdS superspace was described in terms of the complex basis for the spinor covariant derivatives, eq. (5.2.20). It proves to be more convenient to switch to a real basis in order to carry out reduction to $\mathcal{N} = 1$ AdS superspace $\text{AdS}^{3|2}$. Following [155], such a basis is introduced by replacing the complex operators $D_\alpha$ and $\bar{D}_\alpha$ with $\nabla^I_\alpha = (\nabla^1_\alpha, \nabla^2_\alpha)$ defined as follows:

$$D_\alpha = \frac{1}{\sqrt{2}}(\nabla^1_\alpha - i \nabla^2_\alpha), \quad \bar{D}_\alpha = -\frac{1}{\sqrt{2}}(\nabla^1_\alpha + i \nabla^2_\alpha).$$  (6.1.1)

In a similar way, we introduce real coordinates, $z^M = (x^m, \theta^I)$, to parametrise (2,0) AdS superspace. Defining $\nabla_a = D_a$, the algebra of (2,0) AdS covariant derivatives (5.2.20) turns into

$$\{\nabla^I_\alpha, \nabla^J_\beta\} = 2i\delta^{IJ}\nabla_{\alpha\beta} - 4i\delta^{IJ}SM_{\alpha\beta} + 4\varepsilon_{\alpha\beta}\varepsilon^{IJ}SJ,$$  (6.1.2a)

$$[\nabla_a, \nabla^I_\beta] = S(\gamma_a)_\beta^\gamma\nabla^J_\gamma,$$  (6.1.2b)

The action of the $U(1)_R$ generator on the spinor covariant derivatives is given by

$$[J, \nabla^I_\alpha] = -i\varepsilon_{IJ}\nabla^J_\alpha.$$  (6.1.3)

As may be seen from (6.1.2), the graded commutation relations for the operators $\nabla_a$ and $\nabla^1_\alpha$ have the following properties:

1. These (anti-)commutation relations do not involve $\nabla^2_\alpha$,

$$\{\nabla^1_\alpha, \nabla^1_\beta\} = 2i\nabla_{\alpha\beta} - 4iS M_{\alpha\beta}, \quad [\nabla_a, \nabla^1_\beta] = S(\gamma_a)_\beta^\gamma\nabla^1_\gamma,$$  (6.1.4a)

$$[\nabla_a, \nabla^1_\alpha] = -4S^2 M_{ab}.$$  (6.1.4b)

2. Relations (6.1.4) are isomorphic to the algebra of the covariant derivatives of $\text{AdS}^{3|2}$, see [155] for the details.

We thus see that $\text{AdS}^{3|2}$ is naturally embedded in (2,0) AdS superspace as a subspace. The real Grassmann variables of (2,0) AdS superspace, $\theta^I = (\theta^1, \theta^2)$, may be chosen in such a way that $\text{AdS}^{3|2}$ corresponds to the surface defined by $\theta^2 = 0$. We also note that no $U(1)_R$ curvature is present in the algebra of $\mathcal{N} = 1$ AdS covariant derivatives. These properties make possible a consistent (2,0) $\rightarrow$ (1,0) AdS superspace reduction.

Now we will recast the fundamental properties of the (2,0) AdS Killing supervector fields in the real representation (6.1.1). The isometries of (2,0) AdS superspace are described in terms of those first-order operators

$$\zeta := \zeta^B \nabla_B = \zeta^b \nabla_b + \zeta^I \nabla^I, \quad J = 1, 2.$$  (6.1.5a)

The antisymmetric tensors $\varepsilon^{IJ}$ and $\varepsilon_{IJ}$ are normalised as $\varepsilon^{12} = \varepsilon_{12} = 1$. 

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which solve the equation
\[
[\zeta + \frac{1}{2} l^{bc} M_{bc} + i \tau J, \nabla_A] = 0 ,
\]
(6.1.5b)
for some real parameters \(\tau\) and \(l^{ab} = -l^{ba}\). Equation (6.1.5b) is equivalent to
\[
\nabla^I_a \zeta^J_\beta = -\varepsilon_{\alpha \beta} \varepsilon^I_J \tau + S \delta^I_J \zeta_{\alpha \beta} + \frac{1}{2} \delta^I_J l_{\alpha \beta} ,
\]
(6.1.6a)
\[
\nabla^I_a \zeta^\beta = 2i \zeta^I_\beta (\gamma_\beta) ,
\]
(6.1.6b)
\[
\nabla^I_a \tau = -4i S \varepsilon^I_J \zeta_\alpha J ,
\]
(6.1.6c)
\[
\nabla^I_a l_{\beta \gamma} = 8i S \varepsilon_\alpha (\beta \zeta_\gamma) ,
\]
(6.1.6d)
and
\[
\nabla_a \zeta_b = l_{ab} = -l_{ba} ,
\]
(6.1.7a)
\[
\nabla_a \zeta^\beta = -S \zeta^\alpha_\beta (\gamma_\alpha) ,
\]
(6.1.7b)
\[
\nabla_a \tau = 0 ,
\]
(6.1.7c)
\[
\nabla_a l^{bc} = 4 S^2 (\delta^b_a \zeta^c - \delta^c_a \zeta^b) .
\]
(6.1.7d)

Some nontrivial implications of the above equations which will be important for our subsequent consideration are:
\[
\nabla^I_{(a} \zeta_{b)} = 0 , \quad \nabla^I_{(a} \zeta_{b \gamma)} = 0 ,
\]
(6.1.8a)
\[
\nabla^I_{(a} \zeta^J_{b)} = 2S \delta^I_J \zeta_{\alpha \beta} , \quad \nabla^\gamma (I \zeta^J_\gamma) = 0 ,
\]
(6.1.8b)
\[
\zeta^I_\alpha = \frac{i}{6} \nabla^I_{\beta \alpha} = \frac{i}{12S} \nabla^I_{\beta \alpha} = -\frac{i}{4S} \varepsilon^I_J \nabla_\gamma^J \tau ,
\]
(6.1.8c)
\[
\tau = -\frac{1}{4} \varepsilon_{IJ} \nabla^I \zeta^J_\gamma .
\]
(6.1.8d)

Equation (6.1.7a) implies that \(\zeta_a\) is a Killing vector field,
\[
\nabla_a \zeta_b + \nabla_b \zeta_a = 0 ,
\]
(6.1.9)
while (6.1.7b) is a Killing spinor equation. The real parameter \(\tau\) is constrained by
\[
(\nabla^2)^2 \tau = (\nabla^1)^2 \tau = 8S \tau , \quad \nabla_a \tau = 0 .
\]
(6.1.10)

### 6.1.2 Reduction from (2,0) to (1,0) AdS superspace

Given a tensor superfield \(U(x, \theta_I)\) on (2,0) AdS superspace, its \(N = 1\) projection (or bar-projection) is defined by
\[
U| := U(x, \theta_I)|_{\theta_2 = 0}
\]
(6.1.11)
in a special coordinate system to be specified below. By definition, $U|$ depends on the real coordinates $z^M = (x^m, \theta^\mu)$, with $\theta^\mu := \theta^\mu_1$, which will be used to parametrise $\mathcal{N} = 1$ AdS superspace AdS$_{3|2}$. For the (2,0) AdS covariant derivative
\[
\nabla_A = (\nabla_a, \nabla_a^I) = E_A^M \frac{\partial}{\partial z^M} + \frac{1}{2} \Omega_A^{bc} M_{bc} + i \Phi_A J , \tag{6.1.12}
\]
its bar-projection is defined as
\[
\nabla_A| = E_A^M \frac{\partial}{\partial z^M} + \frac{1}{2} \Omega_A^{bc}| M_{bc} + i \Phi_A| J . \tag{6.1.13}
\]

We use the freedom to perform general coordinate, local Lorentz and U(1)$_R$ transformations to choose the following gauge condition
\[
\nabla_a| = \nabla_a , \quad \nabla_a^1| = \nabla_a , \tag{6.1.14}
\]
where
\[
\nabla_A = (\nabla_a, \nabla_a) = E_A^M \frac{\partial}{\partial z^M} + \frac{1}{2} \omega_A^{bc} M_{bc} \tag{6.1.15}
\]
represents the set of covariant derivatives for AdS$_{3|2}$, which obey the following graded commutation relations:
\[
\begin{align*}
\{ \nabla_\alpha, \nabla_\beta \} &= 2i \nabla_{\alpha \beta} - 4i S_{\alpha \beta} , \quad \tag{6.1.16a} \\
[\nabla_a, \nabla_\beta] &= S(\gamma_a)_\beta^\gamma \nabla_\gamma , \quad [\nabla_a, \nabla_b] = -4 S^2 M_{ab} . \quad \tag{6.1.16b}
\end{align*}
\]
In such a coordinate system, the operator $\nabla_a^1|$ contains no partial derivative with respect to $\theta_2$. As a consequence, $(\nabla_a^1 \cdots \nabla_a^k U)| = \nabla_a \cdots \nabla_a^k U|$, for any positive integer $k$, where $U$ is a tensor superfield on (2,0) AdS superspace. Let us study how the $\mathcal{N} = 1$ descendants of $U$ defined by $U_{a_1 \cdots a_k} := (\nabla_a^1 \cdots \nabla_a^k U)|$ transform under the (2,0) AdS isometries, with $k$ a non-negative integer.

We introduce the $\mathcal{N} = 1$ projection of the (2,0) AdS Killing supervector field (6.1.5)
\[
\zeta| = \xi^b \nabla_b + \xi^\beta \nabla_\beta + \epsilon^\beta \nabla_2^\beta , \quad \xi^b := \zeta^b| , \quad \xi^\beta := \zeta_1^\beta , \quad \epsilon^\beta := \zeta_2^\beta . \tag{6.1.17}
\]
We also introduce the $\mathcal{N} = 1$ projections of the Lorentz and U(1)$_R$ parameters in (6.1.5):
\[
\lambda^{bc} := \tau^{bc} , \quad \epsilon := \tau| . \tag{6.1.18}
\]
It follows from (6.1.5) that the $\mathcal{N} = 1$ parameters $\xi^B = (\xi^b, \xi^\beta)$ and $\lambda^{bc}$ obey the equation
\[
\left[ \xi + \frac{1}{2} \lambda^{bc} M_{bc}, \nabla_A \right] = 0 , \quad \xi = \xi^B \nabla_B = \xi^b \nabla_b + \xi^\beta \nabla_\beta , \tag{6.1.19}
\]
which tells us that $\xi^B$ is a Killing supervector field of $\mathcal{N} = 1$ AdS superspace [155]. This equation is equivalent to
\[
\nabla_{(\alpha} \xi_{\beta \gamma)} = 0 , \quad \nabla_\beta \xi^{\beta \alpha} = -6i \xi^\alpha , \quad \tag{6.1.20a}
\]

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\[ \nabla_{\alpha \beta} = \frac{1}{2} \lambda_{\alpha \beta} + S \xi_{\alpha \beta} , \quad (6.1.20b) \]
\[ \nabla_{(\alpha \beta)} = 0 , \quad \nabla_\beta \lambda^{\alpha \beta} = -12i S \xi^{\alpha} . \quad (6.1.20c) \]

These relations automatically follow from the (2,0) AdS Killing equations, eqs. (6.1.6a) – (6.1.6d), upon \( \mathcal{N} = 1 \) projection. Thus \( (\xi^a, \xi^\alpha, \lambda^{ab}) \) parametrise the infinitesimal isometries of AdS\(^{3\mid2} \) \cite{155} (see also \cite{51}).

The remaining parameters \( \epsilon^a \) and \( \epsilon \) generate the second supersymmetry and U(1)\(_R\) transformations, respectively. Using the Killing equations \( (6.1.8) \), it can be shown that they satisfy the following properties
\[ \epsilon_\alpha = \frac{i}{4S} \nabla_\alpha \epsilon , \quad \epsilon = -\frac{1}{2} \nabla_\alpha \epsilon_\alpha , \quad (6.1.21a) \]
\[ (i \nabla^2 + 8S) \epsilon = 0 , \quad \nabla_a \epsilon = 0 . \quad (6.1.21b) \]
These imply that the only independent components of \( \epsilon \) are \( \epsilon_{|\theta=0} \) and \( \nabla_\alpha \epsilon_{|\theta=0} \). They correspond to the U(1)\(_R\) and second supersymmetry transformations, respectively.

Given a matter tensor superfield \( U \), its (2,0) AdS transformation law
\[ \delta_\xi U = (\xi + \frac{1}{2} \lambda^{bc} M_{bc} + i \tau J) U \quad (6.1.22) \]
turns into
\[ \delta_\xi U| = \delta_\xi U + \delta_\epsilon U| , \quad (6.1.23a) \]
\[ \delta_\xi U| = \left( \xi^b \nabla_b + \xi^\beta \nabla_\beta + \frac{1}{2} \lambda^{bc} M_{bc} \right) U| , \quad (6.1.23b) \]
\[ \delta_\epsilon U| = \epsilon^\beta (\nabla^2_\beta U) + i \epsilon J U| . \quad (6.1.23c) \]
It follows from \( (6.1.5) \) and \( (6.1.23) \) that every \( \mathcal{N} = 1 \) descendant \( U_{a_1 \ldots a_k} := (\nabla^2_{a_1} \cdots \nabla^2_{a_k} U)| \) is a tensor superfield on AdS\(^{3\mid2} \),
\[ \delta_\xi U_{a_1 \ldots a_k} = \left( \xi^b \nabla_b + \xi^\beta \nabla_\beta + \frac{1}{2} \lambda^{bc} M_{bc} \right) U_{a_1 \ldots a_k} . \quad (6.1.24) \]
For the \( \epsilon \)-transformation we get
\[ \delta_\epsilon U_{a_1 \ldots a_k} = \epsilon^\beta (\nabla^2_\beta \nabla^2_{a_1} \cdots \nabla^2_{a_k} U)| + i \epsilon (J \nabla^2_{a_1} \cdots \nabla^2_{a_k} U)| \quad (6.1.25) \]
\[ = \epsilon^\beta U_{\beta a_1 \ldots a_k} - \epsilon \sum_{l=1}^k \nabla^2_{a_{l+1}} \cdots \nabla^2_{a_{l-1}} \nabla^2_{a_l} \nabla^2_{a_{l+1}} \cdots \nabla^2_{a_k} U| + i q \epsilon U_{a_1 \ldots a_k} , \]
where \( q \) is the U(1)\(_R\) charge of \( U \) defined by \( J U = q U \). In the second term on the right, we have to push \( \nabla^2_{a_l} \) to the far left through the \( (l - 1) \) factors of \( \nabla^2 \)'s by making use of the relation \( \{ \nabla^2_{a_1}, \nabla^2_{a_2} \} = 4 \epsilon_{a_\beta} SJ \) and taking into account the relation
\[ (\nabla^2_{a_1} \cdots \nabla^2_{a_{l-1}} \nabla^2_{a_{l+1}} \cdots \nabla^2_{a_k} U)| = \nabla^2_{a_1} U_{a_1 \ldots a_{l-1} a_{l+1} \ldots a_k} . \quad (6.1.26) \]
As the next step, the U(1) generator $J$ should be pushed to the right until it hits $U$ producing on the way insertions of $\nabla^1$. Then the procedure should be repeated. As a result, the variation $\delta \epsilon U_{\alpha_1...\alpha_k}$ is expressed in terms of the superfields $U_{\alpha_1...\alpha_{k+1}}, U_{\alpha_1...\alpha_k}, \ldots U_{\alpha_1}, U$.

So far we have been completely general and discussed infinitely many descendants $U_{\alpha_1...\alpha_k}$ of $U$. However only a few of them are functionally independent. Indeed, eq. (6.1.27) tells us that

$$\{\nabla^2_{\alpha}, \nabla^2_{\beta}\} = 2i \nabla_{\alpha\beta} - 4i S M_{\alpha\beta} ,$$

and thus every $U_{\alpha_1...\alpha_k}$ for $k > 2$ can be expressed in terms of $U$, $U_{\alpha}$ and $U_{\alpha_1\alpha_2}$. Therefore, it suffices to consider $k \leq 2$.

Let us give two examples of matter superfields on (2,0) AdS superspace. We first consider a covariantly chiral scalar superfield $\phi$, $\bar{D}_\alpha \phi = 0$, with an arbitrary U(1)$_R$ charge $q$ defined by $J \phi = q \phi$. It transforms under the (2,0) AdS isometries as

$$\delta \zeta \phi = (\zeta + i q \tau) \phi .$$

When expressed in the real basis (6.1.1), the chirality constraint on $\phi$ means

$$\nabla^2_{\alpha} \phi = i \nabla^1_{\alpha} \phi ,$$

As a result, there is only one independent $\mathcal{N} = 1$ superfield upon reduction,

$$\varphi := \phi | .$$

We then get the following relations

$$\nabla^2_{\alpha} \varphi = i \nabla^1_{\alpha} \varphi ,$$

$$\nabla^1_\beta \nabla^2_\beta \varphi = 0 .$$

The $\epsilon$-transformation (6.1.25) is given by

$$\delta \epsilon \varphi = i \epsilon^\beta \nabla_\beta \varphi + i q \epsilon \varphi .$$

Our second example is a real linear superfield $L = \bar{L}$, $\bar{D}^2 L = 0$. The real linearity constraint relates the $\mathcal{N} = 1$ descendants of $L$ as follows:

$$\nabla^2 L = (\nabla^1)^2 L ,$$

$$\nabla^1_\alpha \nabla^2_{\beta} L = 0 .$$

Thus, $L$ is equivalent to two independent, real $\mathcal{N} = 1$ superfields:

$$X := L | , \quad W_\alpha := i \nabla^2_{\alpha} L | .$$
Here $X$ is unconstrained, while $W_\alpha$ obeys the constraint (6.1.33b)
\[ \nabla^\alpha W_\alpha = 0 , \]  
(6.1.35)
which means that $W_\alpha$ is the field strength of an $\mathcal{N} = 1$ vector multiplet. Since $\mathbb{L}$ is neutral under the $R$-symmetry group $U(1)_R$, $J \mathbb{L} = 0$, the second supersymmetry and $U(1)_R$ transformation laws of the $\mathcal{N} = 1$ descendants of $\mathbb{L}$ are as follows:
\[ \delta_\epsilon X = \delta_\epsilon \mathbb{L}| = \epsilon^\beta (\nabla_\beta \mathbb{L})| = -i \epsilon W_\beta , \]  
(6.1.36a)
\[ \delta_\epsilon W_\alpha = i (\nabla_\beta \delta_\epsilon \mathbb{L}|) = i \epsilon (\nabla_\beta \nabla_\alpha \mathbb{L})| - \epsilon [J, \nabla_\alpha \mathbb{L}]| \\
= -\epsilon^\beta \nabla_\alpha \beta X - \frac{i}{2} \epsilon_\alpha \nabla X - i \epsilon \alpha X . \]  
(6.1.36b)

### 6.1.3 The $(2,0)$ AdS supersymmetric actions in $\text{AdS}^3$

Every rigid supersymmetric field theory in $(2,0)$ AdS superspace may be reduced to $\mathcal{N} = 1$ AdS superspace. Here we provide the key technical details of the reduction.

In accordance with [83, 113, 160, 168], there are two ways of constructing supersymmetric actions in $(2,0)$ AdS superspace: (i) either by integrating a real scalar $\mathcal{L}$ over the full $(2,0)$ AdS superspace
\[ S = \int d^3x d^2\theta d^2\bar{\theta} \mathcal{E} \mathcal{L} = \frac{1}{16} \int d^3x e D^2 \bar{D}^2 \mathcal{L} \bigg|_{\theta=0} = \frac{1}{16} \int d^3x e D^2 \bar{D}^2 \mathcal{L} \bigg|_{\theta=0} \]  
(6.1.37)
\[ = \int d^3x e \left( \frac{1}{16} D^\alpha D^2 \mathcal{D}_\alpha + i S D^\alpha D_\alpha \right) \mathcal{L} \bigg|_{\theta=0} = \int d^3x e \left( \frac{1}{16} \bar{D}_\alpha D^2 \bar{D}^\alpha + i S \bar{D}^\alpha \bar{D}_\alpha \right) \mathcal{L} \bigg|_{\theta=0} , \]
with $E^{-1} = \text{Ber}(E_d^M)$; or (ii) by integrating a covariantly chiral scalar $\mathcal{L}_c$ over the chiral subspace of the $(2,0)$ AdS superspace,
\[ S_c = \int d^3x d^2\theta \mathcal{E} \mathcal{L}_c = -\frac{1}{4} \int d^3x e D^2 \mathcal{L}_c \bigg|_{\theta=0} , \]  
(6.1.38)
\[ D^\alpha \mathcal{L}_c = 0 , \]
with $\mathcal{E}$ being the chiral density. The superfield Lagrangians $\mathcal{L}$ and $\mathcal{L}_c$ are neutral and charged, respectively with respect to the group $U(1)_R$:
\[ J \mathcal{L} = 0 , \]  
\[ J \mathcal{L}_c = -2 \mathcal{L}_c . \]  
(6.1.39)

The two types of supersymmetric actions are related to each other by the rule
\[ \int d^3x d^2\theta d^2\bar{\theta} \mathcal{E} \mathcal{L} = \int d^3x d^2\theta \mathcal{E} \mathcal{L}_c , \]  
(6.1.40)
\[ \mathcal{L}_c := -\frac{1}{4} \bar{D}^2 \mathcal{L} . \]

Instead of reducing the above actions to components, in this paper we need their reduction to $\mathcal{N} = 1$ AdS superspace. We remind the reader that the supersymmetric

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3 The component inverse vierbein is defined as usual, $e_a^m(x) = E_a^m|_{\theta=0}$, with $e^{-1} = \text{det}(e_a^m)$.
action in AdS$^{3|2}$ makes use of a real scalar Lagrangian $L$. The superspace and component forms of the action are:

$$S = \int d^3z \, E \, L = \frac{1}{4} \int d^3x \, e\left(i\nabla^2 + 8S\right) L \bigg|_{\theta=0}.$$  (6.1.41)

For the action (6.1.37) we get

$$S = \int d^3x d^2\theta d^2\bar{\theta} \, E \, L = -\frac{i}{4} \int d^3z \, E \, (\nabla^2)^2 L \bigg|_{\theta=0},$$  (6.1.42)

with $E^{-1} = \text{Ber}(E_A \, M)$. The chiral action (6.1.38) reduces to (1,0) AdS as follows:

$$S_c = \int d^3x d^2\theta \, E \, L_c = 2i \int d^3z \, E \, L_c \bigg|_{\theta=0}.$$  (6.1.43)

Making use of the (2,0) AdS transformation law $\delta L = \zeta L, \delta L_c = (\zeta - 2i\tau) L_c$, and the Killing equation (6.1.5b), it can be checked explicitly that the $N=1$ action defined by the right-hand side of (6.1.42), or (6.1.43) are invariant under the (2,0) AdS isometry transformations.

### 6.1.4 Supersymmetric nonlinear sigma models

To illustrate the $(2,0) \to (1,0)$ AdS superspace reduction described above, here we discuss two interesting examples.

Our first example is a general nonlinear $\sigma$-model with (2,0) AdS supersymmetry [83, 168]. It is described by the action

$$S = \int d^3x \, d^2\theta \, d^2\bar{\theta} \, E \, K(\phi^i, \bar{\phi}^\bar{j}) + \left\{ \int d^3x d^2\theta \, E \, W(\phi^i) + c.c. \right\}, \quad \bar{D}_\alpha \phi^i = 0,$$  (6.1.44)

where $K(\phi^i, \bar{\phi}^\bar{j})$ is the Kähler potential of a Kähler manifold and $W(\phi^i)$ is a superpotential. The $U(1)_R$ generator is realised on the dynamical superfields $\phi^i$ and $\bar{\phi}^\bar{j}$ as

$$iJ = \mathcal{J}^i(\phi) \partial_i + \bar{\mathcal{J}}^\bar{j}(\bar{\phi}) \bar{\partial}_{\bar{j}},$$  (6.1.45)

where $\mathcal{J}^i(\phi)$ is a holomorphic Killing vector field such that

$$\mathcal{J}^i(\phi) \partial_i K = -\frac{i}{2} \mathcal{D}(\phi, \bar{\phi}), \quad \bar{\mathcal{D}} = \mathcal{D},$$  (6.1.46)

for some Killing potential $\mathcal{D}(\phi, \bar{\phi})$. The superpotential has to obey the condition

$$\mathcal{J}^i(\phi) \partial_i W = -2iW$$  (6.1.47)

in order for the action (6.1.44) to be invariant under the (2,0) AdS isometry transformations

$$\delta \phi^i = (\zeta + i\tau J) \phi^i.$$  (6.1.48)
In the real representation (6.1.1), the chirality condition on $\phi^i$ turns into
\[ \nabla^2_\alpha \phi^i = i \nabla_\alpha \phi^i . \] (6.1.49)

It follows that upon $\mathcal{N} = 1$ reduction, $\phi^i$ leads to just one superfield,
\[ \varphi^i := \phi^i | . \] (6.1.50)

In particular, we have the following relations
\[ \nabla^2_\alpha \phi^i | = i \nabla_\alpha \varphi^i , \] (6.1.51a)
\[ (\nabla^2_\alpha)^2 \phi^i | = -\nabla^2 \varphi^i - 8 \mathcal{S} \mathcal{J}^i (\varphi) . \] (6.1.51b)

Using the reduction rules (6.1.42) and (6.1.43), we obtain
\[ S = \int d^{3|2} z \left\{ -i K_{ij} (\varphi, \bar{\varphi}) \nabla^a \varphi^i \nabla_a \bar{\varphi}^j + S \mathcal{D} (\varphi, \bar{\varphi}) + (2i W (\varphi) + c.c.) \right\} , \] (6.1.52)

where we have made use of the standard notation
\[ K_{i_1 \ldots i_p j_1 \ldots j_q} := \frac{\partial^{p+q} K (\varphi, \bar{\varphi})}{\partial \varphi^{i_1} \ldots \partial \varphi^{i_p} \partial \bar{\varphi}^{j_1} \ldots \partial \bar{\varphi}^{j_q}} . \] (6.1.53)

The action (6.1.52) is manifestly $\mathcal{N} = 1$ supersymmetric. One may explicitly check that it is also invariant under the second supersymmetry and $R$-symmetry transformations generated by a real scalar parameter $\epsilon$ subject to the constraints (6.1.21), which are:
\[ \delta_{\epsilon} \varphi^i = i \epsilon^a \nabla_a \varphi^i + \epsilon \mathcal{J}^i (\varphi) . \] (6.1.54)

The family of supersymmetric $\sigma$-models (6.1.44) includes a special subclass which is specified by the two conditions: (ii) all $\phi$’s are neutral, $J_{\phi^i} = 0$; and (ii) no superpotential is present, $W (\phi) = 0$. In this case no restriction on the Kähler potential is imposed by eq. (6.1.46), and the action (6.1.44) is invariant under arbitrary Kähler transformations
\[ K \to K + \Lambda + \bar{\Lambda} , \] (6.1.55)

with $\Lambda (\phi^i)$ a holomorphic function. The corresponding action in $\mathcal{N} = 1$ AdS superspace is obtained from (6.1.52) by setting $\mathcal{D} (\varphi, \bar{\varphi}) = 0$ and $W (\varphi) = 0$, and thus the action is manifestly Kähler invariant.

Let us also consider a supersymmetric nonlinear $\sigma$-model formulated in terms of several Abelian vector multiplets with action [83]
\[ S = -2 \int d^3 x \, d^2 \theta \, d^2 \bar{\theta} \, E \, F (L^i) , \quad \bar{D}^2 L^i = 0 , \quad L^i = L^i , \] (6.1.56)

where $F (x^i)$ is a real analytic function of several variables, which is defined modulo linear inhomogeneous shifts
\[ F (x) \to F (x) + b_i x^i + c , \] (6.1.57)
with real parameters $b_i$ and $c$. The real linear scalar $L^i$ is the field strength of a vector multiplet. Upon reduction to $\mathcal{N} = 1$ AdS superspace, $L^i$ generates two different $\mathcal{N} = 1$ superfields:

$$X^i := L^i |, \quad W^i_\alpha := i \nabla_\alpha^2 L^i |. \quad (6.1.58)$$

Here the real scalar $X^i$ is unconstrained, while the real spinor $W^i_\alpha$ obeys the constraint

$$\nabla^\alpha W^i_\alpha = 0 , \quad (6.1.59)$$

which means that $W^i_\alpha$ is the field strength of an $\mathcal{N} = 1$ vector multiplet. Reducing the action (6.1.56) to $\mathcal{N} = 1$ AdS superspace gives

$$S = -\frac{i}{2} \int d^{3|2} z E g^i(X) \left\{ \nabla^\alpha X^i \nabla_\alpha X^j + W^\alpha W^j_\alpha \right\} , \quad (6.1.60)$$

where we have introduced the target-space metric

$$g^i(X) = \frac{\partial^2 F(X)}{\partial X^i \partial X^j} . \quad (6.1.61)$$

The vector multiplets in (6.1.60) can be dualised into scalar ones, which gives

$$S_{\text{dual}} = -\frac{i}{2} \int d^{3|2} z E \left\{ g^i(X) \nabla^\alpha X^i \nabla_\alpha Y^j + g^{ij}(X) \nabla^\alpha Y_i \nabla_\alpha Y_j \right\} , \quad (6.1.62)$$

with $g^{ij}(X)$ being the inverse metric.

### 6.2 Massless higher-spin models: Type II series

In accordance with section 5.7, there exist two off-shell formulations for a massless multiplet of half-integer superspin-$(s+\tfrac{1}{2})$ in $(2,0)$ AdS superspace, with $s = 2, 3, \ldots$, which are called the type II and type III series. In this section we describe the $(2,0) \rightarrow (1,0)$ AdS superspace reduction of the type II theory. The reduction of the type III theory will be given in section 6.3.

#### 6.2.1 Reduction of the gauge prepotentials to AdS$^{3|2}$

Let us turn to reducing the gauge prepotentials (5.7.1) to $\mathcal{N} = 1$ AdS superspace.\footnote{In the super-Poincaré case, the $\mathcal{N} = 2 \rightarrow \mathcal{N} = 1$ reduction of $\mathcal{F}_{s(2s)}$ has been carried out in [50].} Our first task is to work out such a reduction for the superconformal gauge multiplet $\mathcal{F}_{s(2s)}$. In the real representation (6.1.1), the longitudinal linear constraint (5.7.3) takes the form

$$\nabla^2_{(a_1 a_2 \ldots a_{2s+1})} = i \nabla^\dagger_{(a_1 a_2 \ldots a_{2s+1})} . \quad (6.2.1)$$
It follows that $g_{\alpha(2s)}$ has two independent $\theta_2$-components, which are
\[
g_{\alpha(2s)} |, \quad \nabla^2 g_{\alpha(2s-1)\beta} |. \tag{6.2.2}
\]

The gauge transformation of $\mathfrak{f}_{\alpha(2s)}$, eq. \[5.7.2a\], allows us to choose two gauge conditions
\[
\mathfrak{f}_{\alpha(2s)} | = 0 , \quad \nabla^2 g_{\alpha(2s-1)\beta} | = 0 . \tag{6.2.3}
\]

In this gauge we stay with the following unconstrained real $\mathcal{N} = 1$ superfields:
\[
H_{\alpha(2s+1)} := i \nabla^2 (\alpha_1 \mathfrak{f}_{\alpha_2...\alpha_{2s+1}}) | , \tag{6.2.4a}
\]
\[
H_{\alpha(2s)} := \frac{i}{4} (\nabla^2)^2 \mathfrak{f}_{\alpha(2s)} | . \tag{6.2.4b}
\]

There exists a residual gauge freedom which preserves the gauge conditions \[6.2.3\]. It is described by unconstrained real $\mathcal{N} = 1$ superfields $\zeta_{\alpha(2s)}$ and $\bar{\zeta}_{\alpha(2s)}$ defined by
\[
g_{\alpha(2s)} | = -\frac{i}{2} \bar{\zeta}_{\alpha(2s)} , \quad \bar{\zeta}_{\alpha(2s)} = \zeta_{\alpha(2s)} ; \tag{6.2.5a}
\]
\[
\nabla^2 g_{\alpha(2s-1)\beta} | = \frac{2s+1}{2s} \zeta_{\alpha(2s-1)} , \quad \bar{\zeta}_{\alpha(2s-1)} = \zeta_{\alpha(2s-1)} . \tag{6.2.5b}
\]

The gauge transformation laws of the superfields \[6.2.4\] are given by
\[
\delta H_{\alpha(2s+1)} = i \nabla (\alpha_1 \zeta_{\alpha_2...\alpha_{2s+1}}) , \tag{6.2.6a}
\]
\[
\delta H_{\alpha(2s)} = \nabla (\alpha_1 \zeta_{\alpha_2...\alpha_{2s}}) . \tag{6.2.6b}
\]

Our next step is to reduce the compensator $\mathcal{L}_{\alpha(2s-2)}$ to $\mathcal{N} = 1$ AdS superspace. Making use of the representation \[6.1.1\], we observe that the chirality condition \[5.7.5\] reads
\[
\nabla^2 \zeta_{\alpha(2s-2)} = i \nabla^2 \zeta_{\alpha(2s-2)} . \tag{6.2.7}
\]

The gauge transformation \[5.7.5\] allows us to impose a gauge condition
\[
\mathcal{L}_{\alpha(2s-2)} | = 0 . \tag{6.2.8}
\]

Thus, upon reduction to $\mathcal{N} = 1$ superspace, we have the following real superfields
\[
\Psi_{\beta;\alpha(2s-2)} := i \nabla^2 \zeta_{\alpha(2s-2)} | , \tag{6.2.9a}
\]
\[
L_{\alpha(2s-2)} := \frac{i}{4} (\nabla^2)^2 \mathcal{L}_{\alpha(2s-2)} | . \tag{6.2.9b}
\]

Here $\Psi_{\beta;\alpha(2s-2)}$ is a reducible superfield which belongs to the representation $2 \otimes (2s - 1)$ of $\text{SL}(2, \mathbb{R})$, $\Psi_{\beta;\alpha_1...\alpha_{2s-2}} = \Psi_{\beta;\alpha_1...\alpha_{2s-2}}$. The condition \[6.2.8\] is preserved by the residual gauge freedom generated by a real unconstrained $\mathcal{N} = 1$ superfield $\eta_{\alpha(2s-2)}$ defined by
\[
\zeta_{\alpha(2s-2)} | = -\frac{i}{2} \eta_{\alpha(2s-2)} , \quad \bar{\eta}_{\alpha(2s-2)} = \eta_{\alpha(2s-2)} . \tag{6.2.10}
\]
We may now determine how the \( \eta \)-transformation acts on the superfields (6.2.9a) and (6.2.9b). We obtain
\[
\delta_\eta \Psi_{\beta; \alpha(2s-2)} = i \nabla_\beta \eta_{\alpha(2s-2)} ,
\]
(6.2.11a)
\[
\delta_\eta L_{\alpha(2s-2)} = 0 ,
\]
(6.2.11b)
where we have used the chirality constraint (6.2.7) and the expression (6.2.10) for the residual gauge transformation.

Next, we analyse the \( \lambda \)-gauge transformation and reduce the \( \mathcal{N} = 2 \) field strength \( L_{\alpha(2s-2)} \) to AdS\(^3\). In the real basis for the covariant derivatives, the real linearity constraint (5.7.7) is equivalent to two constraints:
\[
(\nabla^2)^2 L_{\alpha(2s-2)} = (\nabla^1)^2 L_{\alpha(2s-2)} ,
\]
(6.2.12a)
\[
\nabla^1 \beta \nabla^2 \beta L_{\alpha(2s-2)} = 0 .
\]
(6.2.12b)
These constraints imply that the resulting \( \mathcal{N} = 1 \) components of \( L_{\alpha(2s-2)} \) are given by
\[
L_{\alpha(2s-2)}| , \quad i \nabla_\beta^2 L_{\alpha(2s-2)}| ,
\]
(6.2.13)
of which the former is unconstrained and the latter is a constrained \( \mathcal{N} = 1 \) superfield that proves to be a gauge-invariant field strength, as we shall see below. The relation between \( L_{\alpha(2s-2)} \) and the prepotential \( \mathcal{L}_{\alpha(2s-2)} \) is given by (5.7.6), which can be expressed as
\[
L_{\alpha(2s-2)} = -\frac{i}{2} \left\{ (\nabla^1)^2 + (\nabla^2)^2 \right\} \mathcal{L}_{\alpha(2s-2)} .
\]
(6.2.14)
We now compute the bar-projection of (6.2.14) in the gauge (6.2.8) and make use of the definition (6.2.9b) to obtain
\[
L_{\alpha(2s-2)}| = -2 L_{\alpha(2s-2)} .
\]
(6.2.15)
Making use of (6.2.14) and (6.2.9a), the bar-projection of \( i \nabla_\beta^2 L_{\alpha(2s-2)} \) leads to the \( \mathcal{N} = 1 \) field strength
\[
\mathcal{W}_{\beta; \alpha(2s-2)} := i \nabla_\beta^2 L_{\alpha(2s-2)}| = -i \left( \nabla^\gamma \nabla_\beta - 4i \mathcal{S}_{\delta} \right) \Psi_{\gamma; \alpha(2s-2)} .
\]
(6.2.16)
Here \( \mathcal{W}_{\beta; \alpha(2s-2)} \) is a real superfield, \( \mathcal{W}_{\beta; \alpha(2s-2)} = \mathcal{W}_{\beta; \alpha(2s-2)}^* \), and is a descendant of the real unconstrained prepotential \( \Psi_{\beta; \alpha(2s-2)} \) defined modulo gauge transformation (6.2.11a).
The field strength proves to be gauge invariant under (6.2.11a). It also obeys
\[
\nabla^\beta \mathcal{W}_{\beta; \alpha(2s-2)} = 0 ,
\]
(6.2.17)
as a consequence of (6.2.12b) and the identity (A.2.12b). Let us express the gauge transformation of \( L_{\alpha(2s-2)} \), eq. (5.7.8) in terms of the real basis for the covariant derivatives,
\[
\delta L_{\alpha(2s-2)} = \frac{is}{2s+1} \left\{ \nabla^1 \beta \nabla^2 \gamma \left( g_{\beta \gamma \alpha(2s-2)} + \bar{g}_{\beta \gamma \alpha(2s-2)} \right) \right\}
\]
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In a similar way, one should also rewrite \( \nabla^2 \delta L_{\alpha(2s-2)} \) in the real basis. This allows us to derive the gauge transformations for \( L_{\alpha(2s-2)} \) and \( W_{\beta;\alpha(2s-2)} \)

\[
\delta L_{\alpha(2s-2)} = -\frac{s}{2(2s + 1)} \nabla^\beta \zeta_{\beta\gamma\alpha(2s-2)} ,
\]

\[
\delta W_{\beta;\alpha(2s-2)} = i \left( \nabla^\gamma \nabla_\beta - 4iS_\delta^{\gamma} \right) \zeta_{\gamma\alpha(2s-2)} .
\]

We can then read off the transformation law for the prepotential \( \Psi_{\beta;\alpha(2s-2)} \)

\[
\delta \Psi_{\beta;\alpha(2s-2)} = -\zeta_{\beta\alpha(2s-2)} + i\nabla_\beta \eta_{\alpha(2s-2)} ,
\]

where we have also taken into account the \( \eta \)-gauge freedom (6.2.11a).

Applying the \( \mathcal{N} = 1 \) reduction rule (6.1.42) to the type II action (5.7.9) and using the commutation relation

\[
\left[ (\nabla^1)^2 (\nabla^2)^2 - 4iS(\nabla^1)^2, \nabla^2_\alpha \right] = 16S \nabla_\alpha \nabla^2 \beta - 16S^2 \nabla^2_\alpha - 32S^2 \nabla^2_\beta M_{\alpha\beta} - 32iS^2 \nabla^1 \eta J ,
\]

we find that (5.7.9) becomes a sum of two actions,

\[
S^{(\Pi)}_{(s+\frac{1}{2})}[^{\|}\partial_{\alpha(2s)}, \mathcal{L}_{\alpha(2s-2)}] = S^{\|}_{(s+\frac{1}{2})}[H_{\alpha(2s+1)}, L_{\alpha(2s-2)}] + S^{\|}_{(s)}[H_{\alpha(2s)}, \Psi_{\beta;\alpha(2s-2)}] .
\]

Explicit expressions for these \( \mathcal{N} = 1 \) actions will be given in the next subsection.

### 6.2.2 Massless higher-spin \( \mathcal{N} = 1 \) supermultiplets in AdS

The gauge transformations (6.2.6a), (6.2.6b), (6.2.19a) and (6.2.19c) tell us that in fact we are dealing with two different \( \mathcal{N} = 1 \) supersymmetric higher-spin gauge theories.

Given a positive integer \( n > 0 \), we say that a supersymmetric gauge theory describes a multiplet of superspin \( n/2 \) if it is formulated in terms of a superconformal gauge prepotential \( H_{\alpha(n)} \) and possibly a compensating multiplet. The gauge freedom of the real tensor superfield \( H_{\alpha(n)} \) is

\[
\delta_\zeta H_{\alpha(n)} = i^n (-1)^{[n/2]} \nabla_{(\zeta_{\alpha_1 \cdots \alpha_n})} ,
\]

with the gauge parameter \( \zeta_{\alpha(n-1)} \) being real but otherwise unconstrained.

#### 6.2.2.1 Longitudinal formulation for massless superspin-(\( s + \frac{1}{2} \)) multiplet

One of the two \( \mathcal{N} = 1 \) theories provides an off-shell formulation for the massless superspin-(\( s + \frac{1}{2} \)) multiplet. It is formulated in terms of the real unconstrained gauge superfields

\[
\mathcal{V}^{\|}_{(s+\frac{1}{2})} = \left\{ H_{\alpha(2s+1)}, L_{\alpha(2s-2)} \right\} ,
\]

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which are defined modulo gauge transformations

\[
\begin{align*}
\delta H_{\alpha(2s+1)} &= i \nabla(\alpha_1 \zeta_{\alpha_2 \ldots \alpha_{2s+1}}), \\
\delta L_{\alpha(2s-2)} &= - \frac{s}{2(2s+1)} \nabla^\beta \zeta_{\beta \gamma \alpha(2s-2)},
\end{align*}
\]

where the parameter \( \zeta_{\alpha(2s)} \) is unconstrained real. The gauge-invariant action is

\[
S_{(s+\frac{1}{2})}^\parallel [H_{\alpha(2s+1)}, L_{\alpha(2s-2)}] = \left( -\frac{1}{2} \right)^s \int d^{3|2}z \, E \left\{ -\frac{i}{2} H_{\alpha(2s+1)}^\parallel Q H_{\alpha(2s+1)} + \frac{i}{8} \nabla_\beta H^{\beta \alpha(2s)} \nabla^2 \nabla^\gamma H_{\gamma \alpha(2s)} + \frac{18}{4} \nabla_\gamma H^{\beta \gamma \alpha(2s-1)} \nabla^\rho \nabla^\delta H_{\rho \delta \alpha(2s-1)} - (2s-1) L^\alpha(2s-2) \nabla^\gamma \nabla^\delta H_{\beta \gamma \delta \alpha(2s-2)} + 2(2s-1) \left( L^\alpha(2s-2) (i \nabla^2 - 4S) L_{\alpha(2s-2)} - \frac{i}{s} (s-1) \nabla_\beta L^{\beta \alpha(2s-3)} \nabla^\gamma L_{\gamma \alpha(2s-3)} \right) + S \left( s \nabla_\beta H^{\beta \alpha(2s)} \nabla^\gamma H_{\gamma \alpha(2s)} + \frac{1}{2} (2s+1) H_{\alpha(2s+1)}^\parallel (\nabla^2 - 4iS) H_{\alpha(2s+1)} \right) \right\},
\]

where \( Q \) is the quadratic Casimir operator of the 3D \( \mathcal{N} = 1 \) AdS supergroup, see eq. (A.2.14). The action (6.2.25) coincides with the off-shell \( \mathcal{N} = 1 \) supersymmetric action for massless half-integer superspin in AdS\(^{3|2} \) in the form given in [51]. Its flat superspace limit was presented earlier in [50]. In what follows, we will refer to the above theory as the longitudinal formulation for the massless superspin-(\( s + \frac{1}{2} \)) multiplet.

The structure \( \nabla_\beta L^{\beta \alpha(2s-3)} \nabla^\gamma L_{\gamma \alpha(2s-3)} \) in (6.2.25) is not defined for \( s = 1 \). However it comes with the factor \( (s-1) \) and drops out from (6.2.25) for \( s = 1 \). The resulting action

\[
S_{(\frac{1}{2})}^\parallel [H_{\alpha(3)}, L] = -\frac{1}{2} \int d^{3|2}z \, E \left\{ -\frac{i}{2} H_{\alpha(3)}^\parallel Q H_{\alpha(3)} + \frac{1}{8} \nabla_\beta H^{\beta \alpha(2)} \nabla^2 \nabla^\gamma H_{\gamma \alpha(2)} + \frac{i}{4} \nabla_\gamma H^{\beta \gamma \alpha} \nabla^\rho \nabla^\delta H_{\rho \delta \alpha} + L \nabla^\beta \nabla^\delta H_{\beta \gamma \delta \alpha} + 2L (i \nabla^2 - 4S) L + S \left( \nabla_\beta H^{\beta \alpha(2)} \nabla^\gamma H_{\gamma \alpha(2)} + \frac{3}{2} H_{\alpha(3)}^\parallel (\nabla^2 - 4iS) H_{\alpha(3)} \right) \right\}
\]

is the linearised action for \( \mathcal{N} = 1 \) AdS supergravity. In the flat superspace limit, the action is equivalent to the one given in [56].

### 6.2.2.2 Transverse formulation for massless superspin-s multiplet

The other \( \mathcal{N} = 1 \) theory provides a formulation for the massless superspin-s multiplet. It is described by the unconstrained real superfields

\[
\Psi_{(s)}^\perp = \left\{ H_{\alpha(2s)}, \Psi_{\beta; \alpha(2s-2)} \right\},
\]

which are defined modulo gauge transformations of the form

\[
\delta H_{\alpha(2s)} = \nabla (\alpha_1 \zeta_{\alpha_2 \ldots \alpha_{2s}}),
\]

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\[ \delta \Psi;_\beta;_α(2s-2) = -\zeta_βα(2s-2) + i\nabla_βη_α(2s-2) \; , \]  

(6.2.28b)

where the gauge parameters \( \zeta_α(2s-1) \) and \( η_α(2s-2) \) are unconstrained real. The gauge-invariant action is given by

\[
S_{(a)}^{-}[H_α(2s), \Psi_β;_α(2s-2)] = \left( -\frac{1}{2} \right)^s \int d^3|z| E \left\{ \frac{1}{2} H_α(2s)(i\nabla^2 + 8sS)H_α(2s) \right.
\]

\[
- i s \nabla_β H^βα(2s-1)\nabla^γ H_γα(2s-1) - (2s - 1)W^β;_α(2s-2)\nabla^γ H_γβα(2s-2) \\
- i \left( \frac{1}{2} (2s - 1) \right) \left( W^β;_α(2s-2)W^γ;_β,_α(2s-2) + \frac{s - 1}{s} W^β;_α(2s-3)W^γ;_β,γα(2s-3) \right) \\
- 2i(2s - 1)S_β;_α(2s-2)W^γ;_β,α(2s-2) \right\} ,
\]

(6.2.29a)

where \( W^β;_α(2s-2) \) denotes the field strength

\[
W^β;_α(2s-2) = -i \left( \nabla^γ \nabla_β - 4iS^γ;_β \right) \Psi_γ;_α(2s-2) , \quad \nabla^γ W^β;_α(2s-2) = 0 . \]

(6.2.29b)

The action (6.2.29) defines a new \( \mathcal{N} = 1 \) supersymmetric higher-spin theory which was not present in [4,50,51] even in the super-Poincaré case.

The structure \( W^β;_α(2s-3)W^γ;_β,α(2s-3) \) in (6.2.29a) is not defined for \( s = 1 \). However it comes with the factor \( (s - 1) \) and drops out from (6.2.29a) for \( s = 1 \). The resulting gauge-invariant action

\[
S_{(1)}^{-}[H_α(2), \Psi_β] = -\frac{1}{2} \int d^3|z| E \left\{ \frac{1}{2} H_α(2)(i\nabla^2 + 8S)H_α(2) - i\nabla_β H^βα\nabla^γ H_γα \\
- W^β\nabla^γ H_γβ - \frac{1}{2} W^β W^γ - 2iS^β W^γ \right\}
\]

(6.2.30)

provides an off-shell realisation for a massless gravitino multiplet in AdS₃. In the flat-superspace limit, this model reduces to the one described in [50].

In the \( s > 1 \) case, the gauge freedom of the prepotential \( \Psi_β;_α(2s-2) \) (6.2.28) allows us to impose a gauge condition

\[
\Psi_{(α_1; α_2...α_{2s-1})} = 0 \quad \iff \quad \Psi_β;_α(2s-2) = \sum_{k=1}^{2s-2} \varepsilon_βα_k ϕ_{α_1...α_k...α_{2s-2}} ,
\]

(6.2.31)

for some field \( ϕ_{α(2s-3)} \). Since we gauge away the symmetric part of \( \Psi_β;_α(2s-2) \), the two gauge parameters \( \zeta_α(2s-1) \) and \( η_α(2s-2) \) are related. The theory is now realised in terms of the following dynamical variables

\[
\left\{ H_α(2s), \; ϕ_{α(2s-3)} \right\} ,
\]

(6.2.32)

with the gauge freedom

\[
\delta H_α(2s) = -\nabla_{(α_1α_2...α_{2s-3})} η_α(2s) ,
\]

(6.2.33a)

\[
\delta ϕ_{α(2s-3)} = i\nabla^β η_βα(2s-3) .
\]

(6.2.33b)

It follows that in the flat superspace limit, \( S = 0 \), and in the gauge (6.2.31), the action (6.2.29) reduces to (5.4.51). The component structure of this model will be discussed in appendix C.1.
6.3 Massless higher-spin models: Type III series

In this section we carry out the $\mathcal{N} = 1$ AdS superspace reduction of the type III theory following the procedure employed in section 6.2.

6.3.1 Reduction of the gauge prepotentials to AdS$^3|2$

The reduction of the superconformal gauge multiplet $\mathcal{H}_{\alpha(2s)}$ to AdS$^3|2$ has been carried out in the previous section. We saw that in the gauge (6.2.3), $\mathcal{H}_{\alpha(2s)}$ is described by the two unconstrained real superfields $H_{\alpha(2s+1)}$ and $H_{\alpha(2s)}$ defined according to (6.2.4), with their gauge transformation laws given by eqs. (6.2.6a) and (6.2.6b), respectively. Now it remains to reduce the prepotential $\mathcal{V}_{\alpha(2s-2)}$ to $\mathcal{N} = 1$ AdS superspace, following the same approach as outlined in the type II series. The gauge transformation (5.7.12) allows us to choose a gauge condition

$$\mathcal{V}_{\alpha(2s-2)} = 0.$$  \hspace{1cm} (6.3.1)

The compensator $\mathcal{V}_{\alpha(2s-2)}$ is then equivalent to the following real $\mathcal{N} = 1$ superfields, which we define as follows:

$$\Upsilon_{\beta; \alpha(2s-2)} := i\nabla^2_{\beta} \mathcal{V}_{\alpha(2s-2)} |,$$  \hspace{1cm} (6.3.2a)

$$V_{\alpha(2s-2)} := \frac{i}{4} (\nabla^2) \mathcal{V}_{\alpha(2s-2)} |.$$  \hspace{1cm} (6.3.2b)

The residual gauge freedom, which preserves the gauge condition (6.3.1) is described by a real unconstrained $\mathcal{N} = 1$ superfield $\eta_{\alpha(2s-2)}$ defined by

$$\xi_{\alpha(2s-2)}| = -\frac{i}{2} \eta_{\alpha(2s-2)} ; \quad \bar{\eta}_{\alpha(2s-2)} = \eta_{\alpha(2s-2)}.$$  \hspace{1cm} (6.3.3)

As a result, we may determine how (6.3.2a) and (6.3.2b) vary under $\eta$-transformation

$$\delta_{\eta} \Upsilon_{\beta; \alpha(2s-2)} = i\nabla_{\beta} \eta_{\alpha(2s-2)} ;$$  \hspace{1cm} (6.3.4a)

$$\delta_{\eta} V_{\alpha(2s-2)} = 0 .$$  \hspace{1cm} (6.3.4b)

Next, we analyse the $\lambda$-gauge transformation and reduce the field strength $\mathcal{V}_{\alpha(2s-2)}$ to AdS$^3|2$. In the real basis for the covariant derivatives, the real linearity constraint (5.7.14) turns into:

$$(\nabla^2)^2 V_{\alpha(2s-2)} = (\nabla^1)^2 V_{\alpha(2s-2)} ,$$  \hspace{1cm} (6.3.5a)

$$(\nabla^1_{\beta} \nabla^2_{\beta} V_{\alpha(2s-2)} = 0 .$$  \hspace{1cm} (6.3.5b)

This tells us that $\mathcal{V}_{\alpha(2s-2)}$ is equivalent to two real $\mathcal{N} = 1$ superfields

$$\mathcal{V}_{\alpha(2s-2)} |, \quad i\nabla^2_{\beta} \mathcal{V}_{\alpha(2s-2)} | .$$  \hspace{1cm} (6.3.6)
The relation between the field strength $\nabla_{\alpha(2s-2)}$ and the prepotential $\mathfrak{U}_{\alpha(2s-2)}$ is given by (6.7.13), which can be expressed as

$$\nabla_{\alpha(2s-2)} = -\frac{i}{2} \left\{ (\nabla_{\perp})^2 + (\nabla_{\parallel})^2 \right\} \mathfrak{U}_{\alpha(2s-2)} \ .$$  (6.3.7)

We now compute the bar-projection of (6.3.7) in the gauge (6.3.1) and make use of the definition (6.3.2b) to obtain

$$\nabla_{\alpha(2s-2)} = -2\nabla_{\alpha(2s-2)} \ .$$  (6.3.8)

The bar-projection of $i\nabla_{\beta}^2 \nabla_{\alpha(2s-2)}$ leads to the $\mathcal{N} = 1$ field-strength

$$\Omega_{\beta;\alpha(2s-2)} := i\nabla_{\beta}^2 \nabla_{\alpha(2s-2)} = -i\left( \nabla^\gamma \nabla_{\gamma} - 4iS^\gamma_{\delta} \right) \Upsilon_{\gamma;\alpha(2s-2)} \ ,$$  (6.3.9)

which is a real superfield, $\Omega_{\beta;\alpha(2s-2)} = \bar{\Omega}_{\beta;\alpha(2s-2)}$, and is a descendant of the real unconstrained prepotential $\Upsilon_{\beta;\alpha(2s-2)}$ defined modulo gauge transformation (6.3.4a). One may check that the field strength is invariant under (6.3.4a) and obeys the condition

$$\nabla^\beta \Omega_{\beta;\alpha(2s-2)} = 0 \ .$$  (6.3.10)

Let us express the gauge transformation of $\nabla_{\alpha(2s-2)}$, eq. (5.7.15), in terms of the real basis for the covariant derivatives. This leads to

$$\delta \nabla_{\alpha(2s-2)} = -\frac{1}{2s+1} \left\{ \nabla^\beta \nabla_{\beta} \left( g_{\beta\gamma\alpha(2s-2)} - \bar{g}_{\beta\gamma\alpha(2s-2)} \right) ight. \left. + \nabla^\gamma \left( g_{\beta\gamma\alpha(2s-2)} + \bar{g}_{\beta\gamma\alpha(2s-2)} \right) \right\} \ .$$  (6.3.11)

One should also express its corollary $\nabla_{\beta}^2 \delta \nabla_{\alpha(2s-2)}$ in the real basis for the covariant derivatives. We determine the gauge transformations law for $V_{\alpha(2s-2)}$ and $\Omega_{\beta;\alpha(2s-2)}$ to be

$$\delta V_{\alpha(2s-2)} = \frac{1}{2s} \nabla^\beta \zeta_{\beta\alpha(2s-2)} \ ,$$  (6.3.12a)

$$\delta \Omega_{\beta;\alpha(2s-2)} = \frac{1}{2s+1} \left( \nabla^\gamma \nabla_{\beta} \nabla^\delta - 4iS^\delta_{\gamma} \delta_{\beta} \gamma \right) \zeta_{\delta\alpha(2s-2)} \ .$$  (6.3.12b)

From (6.3.12b) we read off the transformation law for the prepotential $\Upsilon_{\beta;\alpha(2s-2)}$:

$$\delta \Upsilon_{\beta;\alpha(2s-2)} = \frac{i}{2s+1} \left( \nabla^\gamma \zeta_{\gamma\beta\alpha(2s-2)} + (2s+1)\nabla_{\beta} \eta_{\alpha(2s-2)} \right) \ ,$$  (6.3.13)

where we have also taken into account the $\eta$-gauge freedom (6.3.4a).

Performing $\mathcal{N} = 1$ reduction to the original type III action (5.7.16), we arrive at two decoupled $\mathcal{N} = 1$ actions

$$S^{(III)}_{(s+\frac{1}{2})}[\mathfrak{B}_{\alpha(2s)}, \mathfrak{V}_{\alpha(2s-2)}] = S^{\perp}_{(s+\frac{1}{2})}[H_{\alpha(2s+1)}, \Upsilon_{\beta;\alpha(2s-2)}] + S^{\parallel}_{(s)}[H_{\alpha(2s)}, \nabla_{\alpha(2s-2)}] \ .$$  (6.3.14)

We will present the exact form of these actions in the next subsection.
6.3.2 Massless higher-spin $\mathcal{N} = 1$ supermultiplets in AdS$_3$

Upon reduction to $\mathcal{N} = 1$ superspace, the type III theory leads to two $\mathcal{N} = 1$ supersymmetric gauge theories.

6.3.2.1 Longitudinal formulation for massless superspin-$s$ multiplet

One of the two $\mathcal{N} = 1$ theories provides an off-shell realisation for massless superspin-$s$ multiplet described in terms of the real unconstrained superfields\n\[ \mathcal{V}^{(s)} = \left\{ H_{\alpha(2s)}, V_{\alpha(2s-2)} \right\}, \] (6.3.15)

which are defined modulo gauge transformations of the form\n\[ \delta H_{\alpha(2s)} = \nabla (\alpha_1 \zeta_{\alpha_2 \ldots \alpha_{2s}}), \] (6.3.16a)\n\[ \delta V_{\alpha(2s-2)} = \frac{1}{2s} \nabla^\beta \zeta_{\beta\alpha(2s-2)}, \] (6.3.16b)

where the gauge parameter $\zeta_{\alpha(2s-1)}$ is unconstrained real. The gauge-invariant action is given by\n\[ S^{(1)}_{(s)}[H_{\alpha(2s)}, V_{\alpha(2s-2)}] = \left( -\frac{1}{2} \right)^s \int d^3z E \left\{ \frac{1}{2} H^{\alpha(2s)} (i \nabla^2 + 4S) H_{\alpha(2s)} - \frac{i}{2} \nabla_\beta H^{\beta\alpha(2s-1)} \nabla^\gamma H_{\gamma\alpha(2s-1)} - (2s - 1) V^{\alpha(2s-2)} \nabla^\beta H_{\beta\gamma\alpha(2s-2)} \right. \]
\[ \left. + (2s - 1) \left( \frac{1}{2} V^{\alpha(2s-2)} (i \nabla^2 + 8S) V_{\alpha(2s-2)} + i(s - 1) \nabla_\beta V^{\beta\alpha(2s-3)} \nabla^\gamma V_{\gamma\alpha(2s-3)} \right) \right\}. \] (6.3.17)

Modulo an overall normalisation factor, (6.3.17) coincides with the off-shell $\mathcal{N} = 1$ supersymmetric action for massless superspin-$s$ multiplet in the form given in $[51]$. In the flat superspace limit it reduces to the action derived in $[50]$. Although the structure $\nabla_\beta V^{\beta\alpha(2s-3)} \nabla^\gamma V_{\gamma\alpha(2s-3)}$ in (6.3.17) is not defined for $s = 1$, it comes with the factor $(s - 1)$ and thus drops out from (6.3.17) for $s = 1$. The resulting gauge-invariant action\n\[ S^{(1)}_{(1)}[H_{\alpha(2)}, V] = -\frac{1}{2} \int d^3z E \left\{ \frac{1}{2} H^{\alpha(2)} (i \nabla^2 + 4S) H_{\alpha(2)} - \frac{i}{2} \nabla_\beta H^{\beta\alpha} \nabla^\gamma H_{\gamma\alpha} \right. \]
\[ \left. - V \nabla^\beta H_{\beta\gamma} + \frac{1}{2} V (i \nabla^2 + 8S) V \right\}, \] (6.3.18)

describes an off-shell massless gravitino multiplet in AdS$_3$. In the flat superspace limit, it reduces to the gravitino multiplet model described in $[133]$ (see also $[50]$).
6.3.2.2 Transverse formulation for massless superspin-\((s + \frac{1}{2})\) multiplet

The other theory provides an off-shell formulation for massless superspin-\((s + \frac{1}{2})\) multiplet. It is described by the unconstrained superfields

\[ \mathcal{V}_{(s + \frac{1}{2})} = \left\{ H_\alpha(2s+1), \, \Upsilon_\beta ; \alpha(2s-2) \right\}, \]

which are defined modulo gauge transformations of the form

\[ \delta H_\alpha(2s+1) = i\nabla_{(\alpha_1} \zeta_{\alpha_2...\alpha_{2s+1})} , \]
\[ \delta \Upsilon_\beta ; \alpha(2s-2) = \frac{i}{2s+1} \left( \nabla^\gamma \zeta_{\gamma\beta\alpha(2s-2)} + (2s+1)\nabla_\beta \eta_\alpha(2s-2) \right). \]

The gauge-invariant action is

\[ S_{(s + \frac{1}{2})}^\perp [H_\alpha(2s+1), \, \Upsilon_\beta ; \alpha(2s-2)] = \left( -\frac{1}{2} \right)^s \int d^{3|2}z \; E \left\{ -\frac{i}{2} H_\alpha^{(2s+1)}(Q) H_\alpha(2s+1) \right. \]
\[ \left. -\frac{i}{8} \nabla_\beta H^{\beta\alpha(2s)} \nabla^\gamma H_{\gamma\alpha(2s)} + \frac{i}{8} \nabla_\beta H^{\beta\gamma\alpha(2s-1)} \nabla^\rho \Omega_\rho \alpha(2s-1) \right. \]
\[ \left. -\frac{i}{4} (2s-1) \Omega^{\beta;\alpha(2s-2)} \nabla^\gamma H_{\gamma\delta\alpha(2s-2)} \right. \]
\[ \left. -\frac{i}{8} (2s-1) \left( \Omega^{\beta;\alpha(2s-2)} \Omega_\gamma ; \alpha(2s-2) - 2(s-1) \Omega^{\beta;\alpha(2s-3)} \Omega_\gamma ; \alpha(2s-3) \right) \right. \]
\[ \left. + S \left( H^{(2s+1)} \left( \nabla^2 - 4iS \right) H_\alpha(2s+1) + \frac{1}{2} \nabla_\beta H^{\beta\alpha(2s)} \nabla^\gamma H_{\gamma\alpha(2s)} \right) \right. \]
\[ \left. + is(2s-1)S \Upsilon^{\beta;\alpha(2s-2)} \Omega_\beta ; \alpha(2s-2) \right\}, \]

where \( \Omega^{\beta;\alpha(2s-2)} \) denotes the real field strength

\[ \Omega^{\beta;\alpha(2s-2)} = -i \left( \nabla^\gamma \nabla_\beta - 4iS \delta_\beta^\gamma \right) \Upsilon_\gamma ; \alpha(2s-2) , \quad \nabla^\beta \Omega_\beta ; \alpha(2s-2) = 0 . \]

This action defines a new \( \mathcal{N} = 1 \) supersymmetric higher-spin theory which did not appear in [4][50][51].

The structure \( \Omega^{\beta;\alpha(2s-3)} \Omega_\gamma ; \alpha(2s-3) \) in \( (6.3.21a) \) is not defined for \( s = 1 \). However it comes with the factor \((s-1)\) and hence drops out from \( (6.3.21a) \) for \( s = 1 \). The resulting gauge-invariant action

\[ S_{(\frac{3}{2})}^\perp [H_\alpha(3), \, \Upsilon_\beta] = -\frac{1}{2} \int d^{3|2}z \; E \left\{ -\frac{i}{2} H^{(3)}(Q) H_\alpha(3) - \frac{i}{8} \nabla_\beta H^{\beta\alpha(2)} \nabla^\gamma H_{\gamma\alpha(2)} \right. \]
\[ \left. + \frac{i}{8} \nabla_\beta H^{\beta\gamma\alpha} \nabla^\rho \Omega_\rho \alpha - \frac{i}{4} \Omega^{\beta;\gamma\alpha} \Omega_\beta \right. \]
\[ \left. + S \left( H^{(3)} \left( \nabla^2 - 4iS \right) H_\alpha(3) + \frac{1}{2} \nabla_\beta H^{\beta\alpha(2)} \nabla^\gamma H_{\gamma\alpha(2)} \right) \right. \]
\[ \left. - \frac{i}{8} \Omega^{\beta;\gamma} \Omega_\beta + iS \Upsilon^{\beta;\gamma} \Omega_\beta \right\} \]

provides an off-shell formulation for a linearised supergravity multiplet in AdS\(_3\). In the flat superspace limit, it reduces to the linearised supergravity model proposed in [50].
6.4 Analysis of the results

Let $s > 0$ be a positive integer. For each superspin value, integer $(s)$ or half-integer $(s + \frac{1}{2})$, we have constructed two off-shell formulations which have been called longitudinal and transverse. Now we have to explain this terminology.

Consider a field theory in $\mathcal{N} = 1$ AdS superspace that is described in terms of a real tensor superfield $V_{\alpha(n)}$. We assume the action to have the form

$$S^{\parallel}[V_{\alpha(n)}] = \int d^{3|2}z \, E \, \mathcal{L}(i^{n+1}\nabla_{\beta}V_{\alpha(n)}) \ . \quad (6.4.1)$$

It is natural to call $\nabla_{\beta}V_{\alpha(n)}$ a longitudinal superfield, by analogy with a longitudinal vector field. This theory possesses a dual formulation that is obtained by introducing a first-order action

$$S_{\text{first-order}} = \int d^{3|2}z \, E \left\{ \mathcal{L}(\Sigma_{\beta;\alpha(n)}) + i^{n+1}\mathcal{W}_{\beta;\alpha(n)}\Sigma_{\beta;\alpha(n)} \right\} \ , \quad (6.4.2)$$

where $\Sigma_{\beta;\alpha(n)}$ is unconstrained and the Lagrange multiplier is

$$\mathcal{W}_{\beta;\alpha(n)} = i^{n+1}\left(\nabla^{\gamma}\nabla_{\beta} - 4i\delta^{\gamma}_{\beta}\right)\Psi_{\gamma;\alpha(n)} \ , \quad \nabla^{\beta}\mathcal{W}_{\beta;\alpha(n)} = 0 \ , \quad (6.4.3)$$

for some unconstrained prepotential $\Psi_{\gamma;\alpha(n)}$. Varying (6.4.2) with respect to $\Psi_{\gamma;\alpha(n)}$ gives

$$\nabla^{\beta}\nabla_{\gamma}\Sigma_{\beta;\alpha(n)} - 4i\Sigma_{\gamma;\alpha(n)} = 0 \quad \Rightarrow \quad \Sigma_{\beta;\alpha(n)} = i^{n+1}\nabla^{\beta}V_{\alpha(n)} \ , \quad (6.4.4)$$

and then $S_{\text{first-order}}$ reduces to the original action (6.4.1). On the other hand, we may start from $S_{\text{first-order}}$ and integrate $\Sigma_{\beta;\alpha(n)}$ out. This will lead to a dual action of the form

$$S^{\perp}[\Psi_{\gamma;\alpha(n)}] = \int d^{3|2}z \, E \, \mathcal{L}_{\text{dual}}(\mathcal{W}_{\beta;\alpha(n)}) \ . \quad (6.4.5)$$

This is a gauge theory since the action is invariant under gauge transformations

$$\delta\Psi_{\gamma;\alpha(n)} = i^{n+1}\nabla_{\gamma}\eta_{\alpha(n)} \ . \quad (6.4.6)$$

The gauge-invariant field strength $\mathcal{W}_{\beta;\alpha(n)}$ can be called a transverse superfield, due to the constraint (6.4.3) it obeys. It is natural to call the dual formulations (6.4.1) and (6.4.5) as longitudinal and transverse, respectively.

Now, let us consider the transverse and longitudinal formulations for the massless superspin-$s$ models, which are given by eqs. (6.2.29) and (6.3.17), respectively. These actions depend parametrically on $S$, the curvature of AdS superspace. We denote by $S_{(s)}^{\perp}[H_{\alpha(2s)}, \Psi_{\beta;\alpha(2s-2)}]_{\text{FS}}$ and $S_{(s)}^{\parallel}[H_{\alpha(2s)}, V_{\alpha(2s-2)}]_{\text{FS}}$ these actions in the limit $S = 0$, which corresponds to a flat-superspace. The dynamical systems $S_{(s)}^{\perp}[H_{\alpha(2s)}, \Psi_{\beta;\alpha(2s-2)}]_{\text{FS}}$ and $S_{(s)}^{\parallel}[H_{\alpha(2s)}, V_{\alpha(2s-2)}]_{\text{FS}}$ prove to be related to each other by the Legendre transformation.
described above. Thus $S_{(s)}^\perp[H_\alpha(2s), \Psi_{\beta; \alpha(2s-2)]FS$ and $S_{(s)}^\parallel[H_\alpha(2s), V_{\alpha(2s-2)]FS$ are dual formulations of the same theory. This duality does not survive if $S$ is non-vanishing.

The same feature characterises the longitudinal and transverse formulations for the massless superspin-$\frac{s}{2}$ multiplet, which are described by the actions (6.2.25) and (6.3.21), respectively. The flat-superspace counterparts of these higher-spin models, which we denote by $S_{(s+\frac{1}{2})}^\parallel[H_\alpha(2s+1), L_\alpha(2s-2)]FS$ and $S_{(s+\frac{1}{2})}^\perp[H_\alpha(2s+1), \Upsilon_{\beta; \alpha(2s-2)]FS$, are dual to each other. However, this duality does not survive if we turn on a non-vanishing AdS curvature.

### 6.5 Non-conformal higher-spin supercurrents

In the previous sections, we have shown that there exist two different off-shell formulations for the massless higher-spin $N=1$ supermultiplets. Massless half-integer superspin theory can be realised in terms of the dynamical variables (6.2.23) and (6.3.19), while the models (6.2.27) and (6.3.15) define massless multiplet of integer superspin $s$, with $s > 1$. These models lead to different $N=1$ higher-spin supercurrent multiplets. Our aim in this section is to describe the general structure of $N=1$ supercurrent multiplets in AdS.

#### 6.5.1 $N=1$ supercurrents: Half-integer superspin case

Our half-integer supermultiplet in the longitudinal formulation (6.2.23) can be coupled to external sources

$$S_{source}^{(s+\frac{1}{2})} = \int d^3z E \left\{ i H^{\alpha(2s+1)} J_\alpha(2s+1) + 4 L^{\alpha(2s-2)} S_\alpha(2s-2) \right\}. \quad (6.5.1)$$

The condition that the above action is invariant under the gauge transformations (6.2.24) gives the conservation equation

$$\nabla^\beta J_\beta\alpha(2s) = -\frac{2s}{2s+1} \nabla_{(\alpha_1\alpha_2 S_{\alpha_3\ldots\alpha_{2s})}}. \quad (6.5.2)$$

For the transverse theory (6.3.19) described by the prepotentials $\{H_\alpha(2s+1), \Upsilon_{\beta; \alpha(2s-2)}\}$, we construct an action functional of the form

$$S_{source}^{(s+\frac{1}{2})} = \int d^3z E \left\{ i H^{\alpha(2s+1)} J_\alpha(2s+1) + 2is \Upsilon^{\beta; \alpha(2s-2)} U_{\beta; \alpha(2s-2)} \right\}. \quad (6.5.3)$$

Requiring that the action is invariant under the gauge transformations (6.3.20) leads to

$$\nabla^\beta J_\beta\alpha(2s) = \frac{2s}{2s+1} \nabla_{(\alpha_1 U_{\alpha_2\ldots\alpha_{2s}})}, \quad \nabla^\beta U_{\beta; \alpha(2s-2)} = 0. \quad (6.5.4)$$
From the above consideration, it follows that the most general conservation equation in the half-integer superspin case takes the form

\[ \nabla^\beta J_{\beta \alpha}(2s) = \frac{2s}{2s + 1} \left( \nabla_{(\alpha_1} U_{\alpha_2...\alpha_{2s})} - \nabla_{(\alpha_1\alpha_2} S_{\alpha_3...\alpha_{2s})} \right), \]  
(6.5.5a)

\[ \nabla^\beta U_{\beta; \alpha}(2s-2) = 0 \]  
(6.5.5b)

6.5.2 \( \mathcal{N} = 1 \) supercurrents: Integer superspin case

In complete analogy with the half-integer superspin case, we couple the prepotentials \(6.3.15\) in terms of which the integer superspin-\(s\) is described, to external sources

\[ S_{\text{source}}^{(s)} = \int d^3z \mathcal{E} \left\{ H_{(\alpha}(2s) J_{\alpha)(2s)} + 2s \nabla^{(2s-2)} R_{\alpha(2s-2)} \right\}. \]  
(6.5.6)

For such an action to be invariant under the gauge freedom \(6.3.16\), the sources must be conserved

\[ \nabla^\beta J_{\beta \alpha}(2s-1) = \nabla_{(\alpha_1} R_{\alpha_2...\alpha_{2s-1})} . \]  
(6.5.7)

Next, we turn to the transverse formulation \(6.2.27\) characterised by the prepotentials \(\{H_{\alpha(2s)}, \Psi_{\beta; \alpha(2s-2)}\}\) and construct an action functional

\[ S_{\text{source}}^{(s)} = \int d^3z \mathcal{E} \left\{ H_{\alpha(2s)} J_{\alpha(2s)} + i\Psi_{\beta; \alpha(2s-2)} T_{\beta; \alpha(2s-2)} \right\}. \]  
(6.5.8)

Demanding that the action be invariant under the gauge transformations \(6.2.28\), we derive the following conditions

\[ \nabla^\beta J_{\beta \alpha}(2s-1) = iT_{\alpha(2s-1)} , \quad \nabla^\beta T_{\beta; \alpha(2s-2)} = 0 . \]  
(6.5.9)

From the above consideration, the most general conservation equation for the multiplet of currents in the integer superspin case is given by

\[ \nabla^\beta J_{\beta \alpha}(2s-1) = \nabla_{(\alpha_1} R_{\alpha_2...\alpha_{2s-1})} + iT_{\alpha(2s-1)} , \]  
(6.5.10a)

\[ \nabla^\beta T_{\beta; \alpha(2s-2)} = 0 . \]  
(6.5.10b)

6.5.3 From \( \mathcal{N} = 2 \) supercurrents to \( \mathcal{N} = 1 \) supercurrents

In the previous chapter (see section 5.6), we formulated the general conservation equation for the \( \mathcal{N} = 2 \) higher-spin supercurrent multiplets in \((2,0)\) AdS superspace, which takes the form

\[ \tilde{D}^\beta J_{\beta \alpha(2s-1)} = \tilde{D}_{(\alpha_1} \left( \Psi_{\alpha_2...\alpha_{2s-1})} + iZ_{\alpha_2...\alpha_{2s-1})} \right) . \]  
(6.5.11)
Here $\mathcal{J}_{\alpha(2s)}$ denotes the higher-spin supercurrent, while the trace supermultiplets $\mathcal{Y}_{\alpha(2s-2)}$ and $\mathcal{Z}_{\alpha(2s-2)}$ are both real and covariantly linear superfields,

$$
\mathcal{Y}_{\alpha(2s-2)} - \mathcal{Y}_{\alpha(2s-2)} = \mathcal{Z}_{\alpha(2s-2)} - \mathcal{Z}_{\alpha(2s-2)} = 0, \quad \mathcal{D}^2 \mathcal{Y}_{\alpha(2s-2)} = \mathcal{D}^2 \mathcal{Z}_{\alpha(2s-2)} = 0. \quad (6.5.12)
$$

The explicit form of this multiplet of currents was presented by considering simple $\mathcal{N} = 2$ supersymmetric models for a chiral scalar superfield. Unlike in 4D $\mathcal{N} = 1$ supergravity where every supersymmetric matter theory can be coupled to only one of the off-shell supergravity formulations (either old-minimal or new-minimal), here in the (2,0) AdS case our trace multiplets require both type II and type III compensators to couple to.

The general conservation equation (6.5.11) naturally gives rise to the $\mathcal{N} = 1$ higher-spin supercurrent multiplets discussed in the previous subsection. One may show that in the real basis, (6.5.11) turns into:

$$
\nabla^\beta \mathcal{J}_{\alpha(2s-1)} = \nabla^\beta \left( \mathcal{Y}_{\alpha(2s-2)} \alpha_{\alpha_2\ldots\alpha_{2s-1}} - \nabla^2 \mathcal{Z}_{\alpha(2s-2)} \right), \quad (6.5.13a)
$$

$$
\nabla^2 \mathcal{Z}_{\alpha(2s-1)} = \nabla^2 \left( \mathcal{Y}_{\alpha(2s-2)} \alpha_{\alpha_2\ldots\alpha_{2s-1}} + \nabla^2 \mathcal{Z}_{\alpha(2s-2)} \right), \quad (6.5.13b)
$$

The real linearity constraints on the trace supermultiplets, eq. (6.5.12), are equivalent to

$$
(\nabla^2)^2 \mathcal{Y}_{\alpha(2s-2)} = (\nabla^2)^2 \mathcal{Y}_{\alpha(2s-2)} = 0, \quad (6.5.14a)
$$

$$
(\nabla^2)^2 \mathcal{Z}_{\alpha(2s-2)} = (\nabla^2)^2 \mathcal{Z}_{\alpha(2s-2)} = 0. \quad (6.5.14b)
$$

It follows from (6.5.13) and (6.5.14) that $\mathcal{J}_{\alpha(2s)}$ contains two independent real $\mathcal{N} = 1$ supermultiplets:

$$
J_{\alpha(2s)} := |\mathcal{J}_{\alpha(2s)}|, \quad (6.5.15a)
$$

$$
J_{\alpha(2s+1)} := i \nabla^2 \left( \mathcal{J}_{\alpha(2s)} \alpha_{\alpha_2\ldots\alpha_{2s+1}} \right), \quad (6.5.15b)
$$

while the independent real $\mathcal{N} = 1$ components of $\mathcal{Y}_{\alpha(2s-2)}$ and $\mathcal{Z}_{\alpha(2s-2)}$ are defined by

$$
R_{\alpha(2s-2)} := \mathcal{Y}_{\alpha(2s-2)}, \quad U_{\beta;\alpha(2s-2)} := i \nabla^2 \mathcal{Y}_{\alpha(2s-2)}, \quad (6.5.16a)
$$

$$
S_{\alpha(2s-2)} := \mathcal{Z}_{\alpha(2s-2)}, \quad T_{\beta;\alpha(2s-2)} := i \nabla^2 \mathcal{Z}_{\alpha(2s-2)}. \quad (6.5.16b)
$$

Making use of (6.5.14), one may readily show that

$$
\nabla^\beta U_{\beta;\alpha(2s-2)} = 0, \quad (6.5.17a)
$$

$$
\nabla^\beta T_{\beta;\alpha(2s-2)} = 0. \quad (6.5.17b)
$$

On the other hand, eq. (6.5.13) implies that the $\mathcal{N} = 1$ superfields obey the following conditions

$$
\nabla^\beta J_{\beta;\alpha(2s)} = \frac{2s}{2s+1} \left( \nabla_{\alpha_1} U_{\alpha_2\ldots\alpha_{2s}} - \nabla_{\alpha_1} S_{\alpha_3\ldots\alpha_{2s}} \right), \quad (6.5.18a)
$$

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\[ \nabla^\beta J_{\beta\alpha(2s-1)} = \nabla_{(\alpha_1 R_{\alpha_2...\alpha_{2s-1})} + iT_{\alpha(2s-1)}. \]  

(6.5.18b)

Indeed, the right-hand side of eq. (6.5.18a) coincides with (6.5.5a). Therefore, eqs. (6.5.17a) and (6.5.18a) define the \( N = 1 \) higher-spin current multiplets associated with the massless half-integer superspin formulations (6.2.23) and (6.3.19). In a similar way, it can be observed that eqs. (6.5.17b) and (6.5.18b) correspond to the \( N = 1 \) higher-spin supercurrents for the two integer superspin models (6.2.27) and (6.3.15).

6.6 Examples of \( N = 1 \) higher-spin supercurrents

In this section we give an explicit realisation of the \( N = 1 \) multiplet of higher-spin supercurrent introduced earlier.

We recall the action (6.6.1) for a massless chiral scalar in (2,0) AdS superspace

\[ S = \int d^3x d^2\theta d^2\bar{\theta} E \Phi \Phi, \quad \bar{D}_\alpha \Phi = 0. \]  

(6.6.1)

The chiral superfield is charged under the \( R \)-symmetry group \( U(1)_R \),

\[ J\Phi = -r\Phi, \quad r = \text{const}. \]  

(6.6.2)

This action is a special case of the supersymmetric nonlinear sigma model studied in subsection (6.1.3) with a vanishing superpotential, \( W(\Phi) = 0 \). Making use of (6.1.52), the reduction of the action (6.6.1) to \( N = 1 \) AdS superspace is given by

\[ S = \int d^3z E \left\{ -i\nabla^\alpha \bar{\varphi} \nabla_\alpha \varphi + 4rS \bar{\varphi} \varphi \right\}, \]  

(6.6.3)

where we have denoted \( \varphi := \Phi \). This action is manifestly \( N = 1 \) supersymmetric. It also possesses hidden second supersymmetry and \( U(1)_R \) invariance. These transformations are

\[ \delta_\epsilon \varphi = i\epsilon^\alpha \nabla_\alpha \varphi - i\epsilon r \varphi, \quad \delta_\epsilon \bar{\varphi} = i\epsilon^\alpha \nabla_\alpha \bar{\varphi} + i\epsilon r \bar{\varphi}, \]  

(6.6.4)

where \( \epsilon^\alpha \) is given in terms of \( \epsilon \) according to (6.1.21a), and the real parameter \( \epsilon \) is constrained by (6.1.21b). It can be seen that \( \varphi \) and \( \bar{\varphi} \) obey the equations of motion

\[ (i\nabla^2 + 4rS)\varphi = 0, \quad (i\nabla^2 + 4rS)\bar{\varphi} = 0. \]  

(6.6.5)

Making use of our condensed notation, we may define the operators associated with the real spinor (2,0) AdS covariant derivatives \( \nabla_\alpha^I \)

\[ \nabla_1^I := \zeta^\alpha \nabla_\alpha^I, \quad \nabla_2^I := i\zeta^\alpha \zeta^\beta \nabla_{\alpha\beta}, \]  

\[ \nabla_{(-1)}^I := \nabla^{Ia} \frac{\partial}{\partial \zeta^a}. \]  

(6.6.6)

(6.6.7)
Analogous operators are introduced in the case of $\mathcal{N} = 1$ AdS superspace. They are

\[ \nabla^{(1)} := \zeta^\alpha \nabla_\alpha, \quad \nabla^{(2)} := i \zeta^\alpha \zeta^\beta \nabla_{\alpha \beta}, \] (6.6.8)

\[ \nabla^{(-1)} := \nabla^\alpha \frac{\partial}{\partial \zeta^\alpha}. \] (6.6.9)

It was shown in section [5.6] that by using the massless equation of motion, $D^2 \Phi = 0$, the $\mathcal{N} = 2$ higher-spin supercurrent multiplet associated with the theory (6.6.1) is described by the conservation equation

\[ D^{(-1)} J^{(2s)} = D^{(1)} T^{(2s-2)}. \] (6.6.10a)

Here the real supercurrent $J^{(2s)} = \bar{J}^{(2s)}$ is given by

\[ J^{(2s)} = \sum_{k=0}^s (-1)^k \left\{ \frac{1}{2} \left( \frac{2s}{2k+1} \right) D^{k}_{(2)} \bar{D}^{(1)} \Phi D^{s-k-1}_{(2)} D^{(1)} \Phi + \left( \frac{2s}{2k} \right) D^{k}_{(2)} \bar{D}^{(1)} \Phi D^{s-k}_{(2)} \Phi \right\}, \] (6.6.10b)

while the trace multiplet $T^{(2s-2)}$ has the form

\[ T^{(2s-2)} = 2i S(1 - 2r)(2s + 1)(s + 1) \sum_{k=0}^{s-1} \frac{1}{2s - 2k + 1} (-1)^k \left( \frac{2s}{2k + 1} \right) \times D^{k}_{(2)} \bar{D}^{(1)} \Phi D^{s-k-1}_{(2)} \Phi. \] (6.6.10c)

One may check that $T^{(2s-2)}$ is covariantly linear,

\[ \bar{D}^2 T^{(2s-2)} = 0, \quad D^2 T^{(2s-2)} = 0. \] (6.6.10d)

As is seen from (6.6.10c), $T^{(2s-2)}$ vanishes for $r = \frac{1}{2}$, in which case $\Phi$ is an $\mathcal{N} = 2$ superconformal multiplet.

The complex trace multiplet $T^{(2s-2)}$ may be split into its real and imaginary parts:

\[ T^{(2s-2)} = Y^{(2s-2)} - iZ^{(2s-2)}, \] (6.6.11a)

with

\[ Y^{(2s-2)} = 2i S(1 - 2r)(2s + 1)(s + 1) \sum_{k=0}^{s-1} \frac{2k - s + 1}{(2k + 3)(2s - 2k + 1)} \times (-1)^k \left( \frac{2s}{2k + 1} \right) D^{k}_{(2)} \bar{D}^{(1)} \Phi D^{s-k-1}_{(2)} \Phi, \] (6.6.11b)

\[ Z^{(2s-2)} = -2S(1 - 2r)(2s + 1)(s + 1)(s + 2) \sum_{k=0}^{s-1} \frac{1}{(2k + 3)(2s - 2k + 1)} \times (-1)^k \left( \frac{2s}{2k + 1} \right) D^{k}_{(2)} \bar{D}^{(1)} \Phi D^{s-k-1}_{(2)} \Phi. \] (6.6.11c)
In accordance with (6.5.15), the supercurrent $J^{(2s)}$ reduces to two different multiplets upon projection to $\mathcal{N} = 1$ superspace:

$$J^{(2s)} := J^{(2s)} \bigg|_{\mathcal{N} = 1} = \sum_{k=0}^{s} (-1)^{k+1} \left\{ \begin{array}{c} 2s \\ 2k + 1 \end{array} \right\} \nabla^{k}_{(2)} \nabla_{(1)} \bar{\varphi} \nabla^{s-k-1}_{(2)} \nabla_{(1)} \varphi - \left( \begin{array}{c} 2s \\ 2k \end{array} \right) \nabla^{k}_{(2)} \bar{\varphi} \nabla^{s-k}_{(2)} \varphi \right\} ,$$

(6.6.12a)

$$J^{(2s+1)} := \imath \nabla^{2}_{(1)} J^{(2s)} \bigg|_{\mathcal{N} = 1} = -\frac{1}{\sqrt{2}} \left( D_{(1)} + \bar{D}_{(1)} \right) J^{(2s)} \bigg|_{\mathcal{N} = 1} ,$$

(6.6.12b)

of which the former corresponds to the integer superspin current and the latter half-integer superspin current.

In the case of half-integer superspin, the conservation equation (6.5.5) is satisfied provided we impose (6.6.5):

$$\nabla^{(−1)} J^{(2s+1)} = \frac{2s}{2s+1} \left( \nabla_{(1)} U^{(2s−1)} + \imath \nabla_{(2)} S^{(2s−2)} \right) , \quad \nabla^{\beta} U_{\beta ; (2s−2)} = 0 ,$$

(6.6.13a)

with

$$S^{(2s−2)} := Z^{(2s−2)} \bigg|_{\mathcal{N} = 1} = -2S(1−2r)(2s+1)(s+1)(s+2) \sum_{k=0}^{s−1} \frac{1}{(2k+3)(2s−2k+1)} \times (-1)^{k} \left( \begin{array}{c} 2s \\ 2k + 1 \end{array} \right) \nabla^{k}_{(2)} \bar{\varphi} \nabla^{s-k-1}_{(2)} \varphi ,$$

(6.6.13b)

$$U_{\beta ; (2s−2)} := -\frac{1}{\sqrt{2}} \left( D_{\beta} + \bar{D}_{\beta} \right) \mathcal{Y}^{(2s−2)} \bigg|_{\mathcal{N} = 1} ,$$

(6.6.13c)

It may also be verified that the $\mathcal{N} = 1$ supercurrent multiplet for integer superspin obeys the conditions (6.5.10) on-shell:

$$\nabla^{(−1)} J^{(2s)} = \nabla_{(1)} R^{(2s−2)} + \imath T^{(2s−1)} , \quad \nabla^{\beta} T_{\beta ; (2s−2)} = 0 ,$$

(6.6.14a)
with

\[ R_{(2s-2)} := Y_{(2s-2)} | \\
= 2iS(1 - 2r)(2s + 1)(s + 1) \sum_{k=0}^{s-1} \frac{2k - s + 1}{(2k + 3)(2s - 2k + 1)} \]
\[ \times (-1)^k \left( \frac{2s}{2k + 1} \right) \nabla_{(2)}^k \nabla_{(2)}^{s-k-1} \varphi , \] (6.6.14b)

\[ T_{\beta; (2s-2)} := -\frac{1}{\sqrt{2}} (D_\beta + \bar{D}_\beta) Y_{(2s-2)} | , \]
\[ = 2S(1 - 2r)(2s + 1)(s + 1)(s + 2) \sum_{k=0}^{s-1} \frac{1}{(2k + 3)(2s - 2k + 1)} (-1)^k \left( \frac{2s}{2k + 1} \right) \]
\[ \times \left\{ \nabla_{(2)}^k \varphi \nabla_{(2)}^{s-k-1} \nabla_\beta \varphi + (-1)^s \nabla_{(2)}^k \varphi \nabla_{(2)}^{s-k-1} \nabla_\beta \bar{\varphi} \right. \]
\[ + 2iS(s - k - 1) \zeta_\beta \left( \nabla_{(2)}^k \varphi \nabla_{(2)}^{s-k-2} \nabla_{(1)} \varphi \right. \]
\[ \left. \left. + (-1)^s \nabla_{(2)}^k \varphi \nabla_{(2)}^{s-k-2} \nabla_{(1)} \bar{\varphi} \right) \right\} . \] (6.6.14c)

The above technique can also be used to construct \( \mathcal{N} = 1 \) higher-spin supercurrents for the Abelian vector multiplets model described by the action (6.1.56). We will not elaborate on such a construction here.

### 6.7 Applications and open problems

Let us briefly summarise the results obtained in this chapter. In section 6.1, a formalism to reduce every field theory with (2,0) AdS supersymmetry to \( \mathcal{N} = 1 \) AdS superspace was developed. As nontrivial examples, we considered supersymmetric nonlinear sigma models formulated in terms of \( \mathcal{N} = 2 \) chiral and linear supermultiplets. In sections 6.2 and 6.3, we applied the reduction technique and presented the \( \mathcal{N} = 1 \) superfield descriptions of the off-shell massless higher-spin supermultiplets with (2,0) AdS supersymmetry, which were constructed in chapter 5. For each superspin value \( \hat{s} \), integer \( (\hat{s} = s) \) or half-integer \( (\hat{s} = s + \frac{1}{2}) \), with \( s = 1, 2, \ldots \), the reduction produced two off-shell gauge formulations (called longitudinal and transverse) for a massless \( \mathcal{N} = 1 \) superspin-\( \hat{s} \) multiplet in AdS3. The transverse formulations are new gauge theories. In section 6.4, we proved that for each superspin value the longitudinal and transverse theories are dually equivalent only in the flat superspace limit. In section 6.5 we formulated the non-conformal higher-spin \( \mathcal{N} = 1 \) supercurrent in AdS3. In section 6.6 we provided the explicit examples of these supercurrents in simple models of a chiral scalar superfield.

There are several interesting applications of the results presented in this chapter. In particular, the massless higher-spin \( \mathcal{N} = 1 \) supermultiplets in AdS3, which were derived
in sections 6.2 and 6.3, can be used to construct new topologically massive higher-spin off-shell supermultiplets in AdS$_3$ by extending the approaches advocated in [49,51]. Such a massive supermultiplet is described by a gauge-invariant action being the sum of massless and superconformal higher-spin actions.$^5$ This procedure follows the philosophy of topologically massive theories [133,162,163,170].

We now present two off-shell formulations for the massive $\mathcal{N} = 1$ gravitino supermultiplet in AdS$_3$ and analyse the corresponding equations of motion. The massive extension of the longitudinal theory (6.3.18) is described by the action

$$S_{(1),\mu}^{||} = -\frac{1}{2} \int d^3z \frac{E}{2} \left\{ \frac{i}{2} H^{\alpha\beta} \nabla^2 H_{\alpha\beta} - \frac{i}{2} \nabla_\beta H^{\alpha\beta} \nabla^\gamma H_{\gamma\alpha} - V \nabla^{\alpha\beta} H_{\alpha\beta} \right. \\
\left. + \frac{i}{2} V \nabla^2 V + (\mu + 2S) H^{\alpha\beta} H_{\alpha\beta} - 2(\mu - 2S) V^2 \right\},$$

(6.7.1)

with $\mu$ a real mass parameter. The massive gravitino action is thus constructed from the massless one by adding mass-like terms. In the limit $\mu \to 0$, the action reduces to (6.3.18). The equations of motion for the dynamical superfields $H^{\alpha\beta}$ and $V$ are

$$2 \nabla^\gamma (\alpha H_{\beta\gamma}) - i \nabla^2 H_{\alpha\beta} - 2 \nabla_\alpha V - 4\mu H_{\alpha\beta} = 0,$$  

(6.7.2a)

$$\nabla^{\alpha\beta} H_{\alpha\beta} = (i\nabla^2 + 8S - 4\mu) V.$$  

(6.7.2b)

Multiplying (6.7.2a) by $\nabla^{\alpha\beta}$ and noting that $[\nabla_{\alpha\beta}, \nabla^2] = 0$ yields

$$-i \nabla^2 \nabla^{\alpha\beta} H_{\alpha\beta} + 4\Box V - 4\mu \nabla^{\alpha\beta} H_{\alpha\beta} = 0.$$  

(6.7.3)

Substituting (6.7.2b) into (6.7.3) leads to

$$V = 0.$$  

(6.7.4)

Now that $V = 0$ on-shell, eq. (6.7.2b) turns into

$$\nabla^{\alpha\beta} H_{\alpha\beta} = 0,$$  

(6.7.5)

while (6.7.2a) can equivalently be written as

$$-i \nabla^\gamma \nabla_\alpha H_{\beta\gamma} - (2\mu + 4S) H_{\alpha\beta} = 0.$$  

(6.7.6)

Making use of the identity (A.2.12b), it immediately follows from (6.7.6) that

$$\nabla^{\alpha} H_{\alpha\beta} = 0.$$  

(6.7.7)

$^5$ We will not review such a construction in this thesis, see [6] for details.

$^6$ The construction of the models (6.7.1) and (6.7.10) is similar to those used to derive the off-shell formulations for massive superspin-1 and superspin-3/2 multiplets in four dimensions [171,181].
and then (6.7.6) is equivalent to

$$\frac{-i}{2} \nabla^2 H_{\alpha \beta} = (\mu + 2S)H_{\alpha \beta}.$$  \hspace{1cm} (6.7.8)

Therefore, we have demonstrated that the model (6.7.1) leads to the following conditions on the mass shell:

$$V = 0, \quad (6.7.9a)$$

$$\nabla^\alpha H_{\alpha \beta} = 0 \implies \nabla^{\alpha \beta} H_{\alpha \beta} = 0, \quad (6.7.9b)$$

$$-\frac{i}{2} \nabla^2 H_{\alpha \beta} = (\mu + 2S)H_{\alpha \beta}. \quad (6.7.9c)$$

Such conditions are required to describe an irreducible on-shell massive gravitino multiplet in 3D $\mathcal{N} = 1$ AdS superspace \cite{182}.

In the transverse formulation (6.2.30), the action for a massive gravitino multiplet is given by

$$S_{(1),\mu}^\perp = -\frac{1}{2} \int d^3 z E \left\{ \frac{i}{2} H^{\alpha \beta} \nabla^2 H_{\alpha \beta} - i \nabla_\beta H^{\alpha \beta} \nabla^\gamma H_{\gamma \alpha} - H^{\alpha \beta} \nabla_\alpha \mathcal{W}_\beta - \frac{i}{2} \mathcal{W}^\alpha \mathcal{W}_\alpha \\
+ (\mu + 4S) H^{\alpha \beta} H_{\alpha \beta} - i(\mu + 2S) \left( \Psi^\alpha W_\alpha + 2\mu \Psi^\alpha \Psi_\alpha \right) \right\}. \quad (6.7.10)$$

In the limit $\mu \to 0$, the action reduces to (6.2.30). One may check that the equations of motion for this model imply that

$$\Psi_\alpha = 0, \quad (6.7.11a)$$

$$\nabla^\alpha H_{\alpha \beta} = 0 \implies \nabla^{\alpha \beta} H_{\alpha \beta} = 0, \quad (6.7.11b)$$

$$-\frac{i}{2} \nabla^2 H_{\alpha \beta} = (\mu + 4S)H_{\alpha \beta}. \quad (6.7.11c)$$

The actions (6.7.1) and (6.7.10) can be made into gauge-invariant ones using the Stueckelberg construction.

In the Minkowski superspace limit, the massive models (6.7.1) and (6.7.10) lead to the identical equations of motion described in terms of $H_{\alpha \beta}$:

$$D^\alpha H_{\alpha \beta} = 0, \quad -\frac{i}{2} D^2 H_{\alpha \beta} = \mu H_{\alpha \beta}. \quad (6.7.12)$$

In the AdS case, the equations (6.7.9) and (6.7.11) lead to equivalent dynamics modulo a redefinition of $\mu$. It is an interesting open problem to understand whether there exists a duality transformation relating these models.

It should be pointed out that there also exists an on-shell construction of gauge-invariant Lagrangian formulations for massive higher-spin supermultiplets in 3D Minkowski
and AdS spaces, which were developed in \cite{183,184}. It is obtained by combining the massive bosonic and fermionic higher-spin actions \cite{185,186}, and therefore this construction is intrinsically on-shell. The formulations given in \cite{183,184,185,186} are based on the gauge-invariant approaches to the dynamics of massive higher-spin fields, which were advocated by Zinoviev \cite{47} and Metsaev \cite{187}. It is an interesting open problem to understand whether there exists an off-shell uplift of these models.
Chapter 7

Conclusion

Over the course of this thesis, we have presented various non-conformal higher-spin supercurrents and their associated off-shell massless higher-spin supermultiplets in three and four spacetime dimensions. All of our analyses were performed using the superspace approach, which is an efficient means of formulating supercurrent multiplets and off-shell supersymmetric theories. In four dimensions, a major part of this work was devoted to the explicit construction of conserved higher-spin currents with $\mathcal{N} = 1$ Poincaré and AdS supersymmetry. In three dimensions, we studied both the $\mathcal{N} = 1$ and $\mathcal{N} = 2$ AdS cases. A number of avenues which could be explored for future studies have been highlighted at the end of each chapter. Here we summarise the key results of this thesis and discuss their implications.

Higher-spin supercurrents in 4D $\mathcal{N} = 1$ Minkowski and AdS superspaces [1–3] were studied in chapter 3 and 4, respectively. The main ingredients in deriving the supercurrents are the known off-shell massless higher-spin supermultiplets [63–65] and their gauge symmetries. We also formulated the higher-spin supercurrent multiplet associated with the new off-shell model for the massless integer superspin. Having understood the structure of the current multiplets and their improvement transformations, we obtained closed-form expressions of conserved supercurrents for various supersymmetric theories in AdS. For instance, a model with $N$ massive chiral scalar superfields with an arbitrary mass matrix, and the free theories of tensor and complex linear superfields. For the latter cases, we employed the complex linear-chiral and the minimal scalar-tensor dualities. The structure of the conserved supercurrents is determined by the type (integer or half-integer) and value (even or odd) of the superspin, as well as the mass matrix. This has been summarised in section 4.6.

A natural extension of the analysis presented in chapters 3 and 4 would be to construct 4D $\mathcal{N} = 2$ higher-spin supercurrents, by making use of the known off-shell gauge supermultiplets [67].
In regards to the off-shell massless higher-spin $\mathcal{N} = 1$ supermultiplets in 4D Minkowski and AdS backgrounds, we also developed a new off-shell formulation for the massless integer superspin multiplet. It was shown that the gauge-invariant action generalises that of the longitudinal theory. It is described in terms of the complex superconformal higher-spin prepotential $\Psi_{\alpha(s)\delta(s-1)}$, in conjunction with two compensating superfields. Making use of the superfield Legendre transformation, we constructed its dual action and demonstrated that it reduces to the transverse formulation.

Chapter 5 discussed $\mathcal{N} = 2$ supersymmetric higher-spin gauge theories in AdS$_3$ based on the results of [4,5]. Along the same lines, we generalised the 4D gauge principles used in chapter 4 to construct off-shell linearised actions for massless higher-spin supermultiplets around the (1,1) AdS background. In addition, we derived the corresponding consistent supercurrents and gave their explicit expressions for models of chiral superfields. Within the framework of (2,0) AdS supersymmetry, the problem of constructing off-shell massless higher-spin gauge supermultiplets has not been fully resolved. In section 5.6 we identified a multiplet of conserved higher-spin currents, which allowed us to construct two off-shell actions for the massless half-integer superspin multiplet. In each of the formulations, the corresponding gauge-invariant action contains a higher-spin extension of a Chern-Simons term. In the limit of $s = 1$, these actions reduce to the linearised actions for (2,0) AdS supergravity [83]. It remains an open problem to construct an off-shell formulation for a massless integer superspin multiplet. For completeness, it would be useful to study the component actions of (5.7.9) and (5.7.16) in order to understand their actual differences with the (1,1) AdS actions.

In chapter 6 we derived four series of off-shell massless higher-spin $\mathcal{N} = 1$ supermultiplets in AdS$_3$, two of which were new supersymmetric gauge theories. This was accomplished via the (2,0) → (1,0) AdS superspace reduction procedure [9]. Further analysis showed that these off-shell models are related by a superfield Legendre transformation in the flat superspace limit, but the duality is not lifted to the AdS case. The massless $\mathcal{N} = 1$ supersymmetric higher-spin actions in AdS$_3$ were used to formulate (i) conserved $\mathcal{N} = 1$ higher-spin supercurrents; and (ii) two new off-shell massive $\mathcal{N} = 1$ gravitino supermultiplets in AdS$_3$. Additionally, we elaborated on the component structure of the two new $\mathcal{N} = 1$ supersymmetric higher-spin models (6.2.29) and (6.3.21) in flat superspace (see appendix C). Whilst it was shown that the action (6.2.29) reduces to the 3D (Fang-)Fronsdal actions upon elimination of the auxiliary fields, an interesting feature appeared in the analysis of (6.3.21). At the component level, the corresponding multiplet is a 3D counterpart of the so-called (reducible) higher-spin triplet systems. In AdS$_D$ an action for higher-spin triplets was constructed in [188] and [189,190], for the bosonic and fermionic cases, respectively. This demonstrates that our superfield construction provides a manifestly supersymmetric generalisation of these systems.
All supersymmetric higher-spin models presented in this thesis are linearised actions for higher-spin multiplets. We believe that they can be used to construct interacting theories, which will allow us to make contact with the 3D gauge theories developed by Vasiliev and collaborators [191–193]. As a next step towards complete superfield formulation for higher-spin supergravity, an important problem is to go beyond the linearised approximation, i.e. finding the relevant deformations of superfield higher-spin actions, gauge transformations and their corresponding supercurrents.

It would be of particular interest to examine the off-shell structure of supersymmetric higher-spin multiplets and their associated conserved supercurrents in 3D with $\mathcal{N} > 2$ supersymmetry. It is also expected that new techniques are required. For example, in order to study the $\mathcal{N} = 3$ case in AdS$_3$, one may apply the projective-superspace formalism developed in [155].

All off-shell higher-spin $\mathcal{N} = 2$ supermultiplets in AdS$_3$ presented in chapter 5 are reducible gauge theories (in the terminology of the Batalin-Vilkovisky quantisation [194]), similar to the massless higher-spin supermultiplets in AdS$_4$ [63]. The Lagrangian quantisation of such theories is nontrivial, as demonstrated in [66] in the 4D case. All off-shell higher-spin $\mathcal{N} = 1$ supermultiplets in AdS$_3$, which were constructed in chapter 6, are irreducible gauge theories. They can be quantised using the Faddeev-Popov procedure [195] as in the non-supersymmetric case, see e.g. [196,197].

As a final comment, all off-shell supersymmetric massless higher-spin models presented in this thesis (both in three and four dimensions) share a common feature. After we reformulated the massless integer superspin theories, one obtains a universal picture in which every gauge-invariant action is now realised in terms of two dynamical variables: a superconformal gauge prepotential and an appropriate set of compensating superfield(s).
Appendix A

Notation and conventions

In this appendix we collect important definitions and identities that have been used throughout the thesis. For a more rigorous presentation, the reader is referred to [35], which our 4D notation and conventions mainly follow. Below are the types of indices that we use:

- Lower case letters from the beginning (middle) of the Latin alphabet, i.e. $a, b, \ldots (m, n, \ldots)$ correspond to flat (curved) spacetime indices.
- Lower case letters from the beginning of the Greek alphabet, i.e. $\alpha, \beta, \ldots$ denote indices for two-component Weyl spinors.
- Likewise, upper case letters from the beginning and middle of the Latin alphabet denote flat and curved superspace coordinates respectively.

A.1 4D spinor and tensor identities

We use the mostly positive convention for the Minkowski metric:
\[ \eta_{ab} := \text{diag}(-1, 1, 1, 1), \tag{A.1.1} \]
in order to raise and lower spacetime indices of tangent space tensors, $V_a = \eta_{ab}V^b$, $V^a = \eta^{ab}V_b$, with $a, b = 0, 1, 2, 3$. On the other hand, the indices of curved spacetime tensors can be raised or lowered using the curved metric $g_{mn}$,
\[ g_{mn} = e^m_a e^n_b \eta_{ab}, \tag{A.1.2} \]
where $e^m_a$ is the vierbein. The inverse vierbein $e^m_a$ is introduced by $e^m_a e_m^b = \delta^b_a$ and $e_m^a e^m_a = \delta^m_n$. 

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The (brackets) parentheses denote (anti-)symmetrisation of tensor or spinor indices, which include a normalisation factor, for instance

\[ V_{[a_1a_2...a_n]} := \frac{1}{n!} \sum_{\pi \in S_n} \text{sgn}(\pi) V_{a_{\pi(1)}...a_{\pi(n)}} \] , \hspace{1cm} V_{(a_1...a_n)} := \frac{1}{n!} \sum_{\pi \in S_n} \text{sgn}(\pi) V_{a_{\pi(1)}...a_{\pi(n)}} , \quad (A.1.3) \]

with \( S_n \) being the symmetric group of \( n \) elements.

The totally antisymmetric Levi-Civita tensor, \( \varepsilon_{abcd} \equiv \varepsilon_{[abcd]} \), is normalised such that

\[ \varepsilon^{0123} = -\varepsilon_{0123} = 1 \] . \hspace{1cm} (A.1.4) 

A product of Levi-Civita tensors can be written as

\[ \varepsilon^{abcd} \varepsilon_{a'b'c'd'} = -4! \delta^{a}_{[a'} \delta^{b}_{b'} \delta^{c}_{c'} \delta^{d}_{d']} . \hspace{1cm} (A.1.5) \]

Central to the description of 4D \( \mathcal{N} = 1 \) supersymmetry is the formalism of two-component Weyl spinors. These are representations of \( \text{SL}(2, \mathbb{C}) \), which is the covering group of the restricted Lorentz group \( \text{SO}_0(3,1) \). Specifically, an object \( \psi_{\alpha} (\alpha = 1, 2) \) which transforms under the fundamental representation of \( \text{SL}(2, \mathbb{C}) \),

\[ \psi_{\alpha} = N_{\alpha}^{\beta} \psi_{\beta} , \quad N_{\alpha}^{\beta} \in \text{SL}(2, \mathbb{C}) \] \hspace{1cm} (A.1.6)

is called a left-handed Weyl spinor. This is denoted by \( (\frac{1}{2},0) \) and is known as the left-handed spinor representation of the Lorentz group. On the other hand, a right-handed Weyl spinor \( \bar{\psi}_{\dot{\alpha}} (\dot{\alpha} = \dot{1}, \dot{2}) \) transforms in the conjugate representation

\[ \bar{\psi}_{\dot{\alpha}} = \bar{N}_{\dot{\alpha}}^{\dot{\beta}} \bar{\psi}_{\dot{\beta}} , \quad (N_{\alpha}^{\beta})^* = \bar{N}_{\dot{\alpha}}^{\dot{\beta}} . \hspace{1cm} (A.1.7) \]

This is denoted by \( (0,\frac{1}{2}) \) and is called the right-handed spinor representation of the Lorentz group.

The undotted and dotted indices of two-component spinors may be raised and lowered with the help of \( \varepsilon \) tensors:

\[ \psi_{\alpha} = \varepsilon_{\alpha\beta} \psi^{\beta} , \quad \psi^{\alpha} = \varepsilon^{\alpha\beta} \psi_{\beta} , \quad \bar{X}_{\dot{\alpha}} = \varepsilon_{\dot{\alpha}\dot{\beta}} \chi^{\dot{\beta}} , \quad \bar{\chi}^{\dot{\alpha}} = \varepsilon^{\dot{\alpha}\dot{\beta}} \bar{X}_{\dot{\beta}} . \hspace{1cm} (A.1.8) \]

The antisymmetric tensors, \( \varepsilon_{\alpha\beta} = -\varepsilon_{\beta\alpha} \) and \( \varepsilon_{\dot{\alpha}\dot{\beta}} = -\varepsilon_{\dot{\beta}\dot{\alpha}} \) are invariant under \( \text{SL}(2, \mathbb{C}) \). They are defined by

\[ \varepsilon^{\alpha\beta} \varepsilon_{\beta\gamma} = \delta^{\alpha}_{\gamma} , \quad \varepsilon^{\dot{\alpha}\dot{\beta}} \varepsilon_{\dot{\beta}\dot{\gamma}} = \delta^{\dot{\alpha}}_{\dot{\gamma}} , \quad \varepsilon^{12} = \varepsilon_{21} = 1 , \quad \varepsilon^{1\dot{2}} = \varepsilon_{1\dot{2}} = 1 . \hspace{1cm} (A.1.9) \]

We will adopt the following rules for contraction of spinor indices:

\[ \psi \chi := \psi_{\alpha} X_{\alpha} = \chi \psi , \quad \bar{\psi} \bar{\chi} := \bar{\psi}_{\dot{\alpha}} \bar{X}^{\dot{\alpha}} = \bar{\chi} \bar{\psi} . \hspace{1cm} (A.1.10) \]
and \( \psi^2 = \psi \bar{\psi} \), \( \bar{\psi}^2 = \bar{\psi} \bar{\psi} \). Here spinor conjugation is understood as Hermitian conjugation,

\[
(\psi \chi)^* = (\psi^\alpha \chi_\alpha)^* = (\chi_\alpha)^*(\psi^\alpha)^* = \bar{\chi}_\dot{\alpha} \bar{\psi}. \tag{A.1.11}
\]

We define the sigma matrices \( \sigma_a := (\sigma_a)_{a\dot{a}} \) as

\[
(\sigma_a) := (1, \vec{\sigma}), \quad (\tilde{\sigma}_a) := (1, -\vec{\sigma}) \tag{A.1.12}
\]

i.e.

\[
\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.1.13}
\]

The tilded sigma matrices with raised indices are denoted by

\[
(\tilde{\sigma}_a)_{\dot{\alpha}\dot{\beta}} = \varepsilon_{\dot{\alpha}\dot{\beta}} \varepsilon^{\alpha\beta} (\sigma_a)_{\beta\dot{\beta}}. \tag{A.1.14}
\]

The sigma matrices satisfy some useful properties, for instance

\[
(\sigma_a \tilde{\sigma}_b + \tilde{\sigma}_b \sigma_a)_{\alpha \dot{\beta}} = -2 \eta_{ab} \delta_{\alpha \dot{\beta}}, \quad \text{Tr}(\sigma_a \tilde{\sigma}_b) = -2 \eta_{ab}, \tag{A.1.15}
\]

\[
(\tilde{\sigma}_a \sigma_b + \sigma_b \tilde{\sigma}_a)_{\dot{\alpha} \beta} = -2 \eta_{ab} \delta_{\dot{\alpha} \beta}, \quad (\sigma_a)_{a\dot{a}}(\tilde{\sigma}_a)_{\beta\dot{\beta}} = -2 \delta_{\dot{\alpha}}^\beta \delta_{\dot{\beta}}^\alpha. \tag{A.1.16}
\]

We can introduce the antisymmetric traceless matrices

\[
(\sigma_{ab})_{\alpha \beta} = -\frac{1}{4} (\sigma_a \tilde{\sigma}_b - \tilde{\sigma}_b \sigma_a)_{\alpha \beta}, \tag{A.1.16a}
\]

\[
(\tilde{\sigma}_{ab})_{\dot{\alpha} \dot{\beta}} = -\frac{1}{4} (\tilde{\sigma}_a \sigma_b - \sigma_b \tilde{\sigma}_a)_{\dot{\alpha} \dot{\beta}}, \tag{A.1.16b}
\]

which are (anti) self-dual,

\[
\frac{1}{2} \varepsilon_{abcd} \sigma^{cd} = -i \sigma_{ab}, \quad \frac{1}{2} \varepsilon_{abcd} \tilde{\sigma}^{cd} = i \tilde{\sigma}_{ab}. \tag{A.1.17}
\]

They also satisfy the Lorentz algebra

\[
[\sigma_{ab}, \sigma_{cd}] = \eta_{ad} \sigma_{bc} - \eta_{ac} \sigma_{bd} + \eta_{bc} \sigma_{ad} - \eta_{bd} \sigma_{ac}. \tag{A.1.18}
\]

Given a vector \( V_a \), one can convert the vector index to a pair of spinor indices using the \( \sigma \)-matrices. The rules are as follows

\[
V_{\alpha \dot{\beta}} = (\sigma^a)_{a\dot{\beta}} V_a, \quad V_a = -\frac{1}{2} (\tilde{\sigma}_a)^{\dot{\beta} \alpha} V_{\alpha \dot{\beta}}. \tag{A.1.19}
\]

As an example, let us consider a real and antisymmetric rank-2 tensor, \( F_{ab} = -F_{ba} \). The decomposition is

\[
F_{a\dot{\alpha};\beta\dot{\beta}} = (\sigma_a)_{a\dot{\alpha}} (\sigma^b)_{\beta\dot{\beta}} F_{ab} = 2 \varepsilon_{a\dot{\alpha}} \tilde{F}_{\dot{\alpha} \beta} + 2 \varepsilon_{\dot{\alpha} \beta} F_{a\dot{\alpha}}. \tag{A.1.20a}
\]
Here we have defined
\[ F_{\alpha\beta} = \frac{1}{2} (\sigma^{ab})_{\alpha\beta} F_{ab}, \quad \tilde{F}_{\dot{\alpha}\dot{\beta}} = -\frac{1}{2} (\tilde{\sigma}^{ab})_{\dot{\alpha}\dot{\beta}} F_{ab}. \] (A.1.20b)

In particular, this applies to the Lorentz generators, \( M_{ab} = -M_{ba} \leftrightarrow (M_{\alpha\beta}, \tilde{M}_{\dot{\alpha}\dot{\beta}}) \), which satisfy the same algebra \((A.1.18)\) as the \( \sigma^{ab} \) matrices. They act on arbitrary spinors as follows:

\[ M_{\alpha\beta}(\psi_{\gamma}) = \frac{1}{2} (\epsilon^{\gamma\alpha} \psi_{\beta} + \epsilon_{\gamma\beta} \psi_{\alpha}), \quad M_{\alpha\beta}(\bar{\psi}_{\dot{\gamma}}) = 0, \] (A.1.21)

\[ \bar{M}_{\dot{\alpha}\dot{\beta}}(\bar{\psi}_{\dot{\gamma}}) = \frac{1}{2} (\epsilon_{\dot{\gamma}\dot{\alpha}} \bar{\psi}_{\dot{\beta}} + \epsilon_{\dot{\gamma}\dot{\beta}} \bar{\psi}_{\dot{\alpha}}), \quad \bar{M}_{\dot{\alpha}\dot{\beta}}(\bar{\psi}_{\dot{\gamma}}) = 0. \]

Let \( D_A = (\partial_a, D_\alpha, \bar{D}_{\dot{\alpha}}) \) be the set of covariant derivatives of \( \mathcal{N} = 1 \) Minkowski superspace. The spinor covariant derivatives, \( D_\alpha \) and \( \bar{D}_{\dot{\alpha}} \), are related by complex conjugation, which works as follows

\[ D_\alpha V = (-1)^{\epsilon(V)} \bar{D}_{\dot{\alpha}} \bar{V}, \quad D^2 V = \bar{D}^2 \bar{V}. \] (A.1.22)

Here \( \bar{V} \) is the complex conjugate of \( V \). The Grassmann parity of \( V \) is denoted by \( \epsilon(V) \) i.e. \( \epsilon(V) = 0 \) for a bosonic superfield and \( \epsilon(V) = 1 \), if \( V \) is fermionic. We also note that \( D^2 = D^a D_a \) and \( \bar{D}^2 = \bar{D}_{\dot{a}} \bar{D}^{\dot{a}} \). It is important to keep in mind the following rules when doing calculations with the covariant derivatives:

\[ D_A(UV) = D_A(U)V + (-1)^{\epsilon(U)\epsilon(D_A)} UD_A(V), \]
\[ \epsilon(D_A V) = \epsilon(D_A) + \epsilon(V) \quad (\text{mod} \ 2) \] (A.1.23)

for arbitrary superfields \( U \) and \( V \).

## A.2 3D notation and AdS identities

We summarise our 3D notation and conventions following [113][169]. The 3D Minkowski metric is \( \eta_{ab} = \text{diag}(-1, 1, 1) \). The spinor indices are raised and lowered by the rule

\[ \psi^\alpha = \epsilon^{\alpha\beta} \psi_\beta, \quad \psi_\alpha = \epsilon_{\alpha\beta} \psi^\beta. \] (A.2.1)

Here the antisymmetric SL(2, \( \mathbb{R} \)) invariant tensors \( \epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha} \) and \( \epsilon^{\alpha\beta} = -\epsilon^{\beta\alpha} \) are normalised as \( \epsilon_{12} = -1, \epsilon^{12} = 1 \).

We make use of real Dirac \( \gamma \)-matrices, \( \gamma_a := (\gamma_a)^{\alpha} \beta \) defined by

\[ (\gamma_a)_{\alpha}^\beta := \epsilon^{\beta\gamma}(\gamma_a)_{\alpha\gamma} = (-i\sigma_2, \sigma_3, \sigma_1). \] (A.2.2)

They obey the algebra

\[ \gamma_a \gamma_b = \eta_{ab} \mathbb{1} + \epsilon_{abc} \gamma^c, \] (A.2.3)

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where the Levi-Civita tensor is normalised as $\varepsilon^{012} = -\varepsilon_{012} = 1$. Some useful relations involving $\gamma$-matrices are

$$
(\gamma^a)_{\alpha\beta}(\gamma_a)^{\rho\sigma} = - (\delta_\rho^\alpha \delta_\sigma^\beta + \delta_\rho^\beta \delta_\sigma^\alpha), \tag{A.2.4a}
$$
$$
\varepsilon_{abc}(\gamma^b)_{\alpha\beta}(\gamma^c)^{\gamma\delta} = \varepsilon_{\alpha(\gamma_a)\beta\gamma} + \varepsilon_{\delta(\gamma_a)\beta\gamma}, \tag{A.2.4b}
$$
$$
\text{tr}[\gamma^a \gamma^b \gamma^c \gamma^d] = 2\eta_{ab}\eta_{cd} - 2\eta_{ac}\eta_{db} + 2\eta_{ad}\eta_{bc}. \tag{A.2.4c}
$$

Given a three-vector $x_a$, it can be equivalently described by a symmetric second-rank spinor $x_{\alpha\beta}$ defined as

$$
x_{\alpha\beta} := (\gamma^a)_{\alpha\beta} x_a = x_{\beta\alpha}, \quad x_a = -\frac{1}{2}(\gamma_a)^{\alpha\beta} x_{\alpha\beta}. \tag{A.2.5}
$$

In the 3D case, an antisymmetric tensor $F_{ab} = -F_{ba}$ is Hodge-dual to a three-vector $F_a$, specifically

$$
F_a = \frac{1}{2}\varepsilon_{abc} F^{bc}, \quad F_{ab} = -\varepsilon_{abc} F^c. \tag{A.2.6}
$$

Then, the symmetric spinor $F_{\alpha\beta} = F_{\beta\alpha}$, which is associated with $F_a$, can equivalently be defined in terms of $F_{ab}$:

$$
F_{\alpha\beta} := (\gamma^a)_{\alpha\beta} F_a = \frac{1}{2}(\gamma_a)^{\alpha\beta} \varepsilon_{abc} F^{bc}. \tag{A.2.7}
$$

These three algebraic objects, $F_a$, $F_{ab}$ and $F_{\alpha\beta}$, are in one-to-one correspondence to each other, $F_a \leftrightarrow F_{ab} \leftrightarrow F_{\alpha\beta}$. The corresponding inner products are related to each other as follows:

$$
-F^a G_a = \frac{1}{2} F^{ab} G_{ab} = \frac{1}{2} F^{\alpha\beta} G_{\alpha\beta}. \tag{A.2.8}
$$

The Lorentz generators with two vector indices ($M_{ab} = -M_{ba}$), one vector index ($M_a$) and two spinor indices ($M_{\alpha\beta} = M_{\beta\alpha}$) are related to each other by the rules:

$$
M_{ab} = -\varepsilon_{abc} M^c, \quad M_a = \frac{1}{2} \varepsilon_{abc} M^c, \quad M_{\alpha\beta} = (\gamma^a)_{\alpha\beta} M_a, \quad M_a = -\frac{1}{2} (\gamma_a)^{\alpha\beta} M_{\alpha\beta}. \tag{A.2.9}
$$

These generators act on a vector $V_c$ and a spinor $\Psi_\gamma$ as follows:

$$
M_{ab} V_c = 2\eta_{[a} V_{b]} , \quad M_{\alpha\beta} \Psi_\gamma = \varepsilon_{\gamma(\alpha} \Psi_{\beta)}. \tag{A.2.10}
$$

We collect some useful properties for $\mathcal{N} = 1$ AdS covariant derivatives, which we denote by $\nabla_A = (\nabla_a, \nabla_{\alpha})$. We first note the unusual complex conjugation property of the spinor covariant derivative, which can be compared with the 4D case, see (A.1.22). Given an arbitrary superfield $V$ and its complex conjugate $\bar{V}$, it holds that

$$
\nabla_A \bar{V} = -(-1)^{i(V)} \nabla_A \bar{V}, \tag{A.2.11}
$$

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where $\epsilon(V)$ denotes the Grassmann parity of $V$.

Making use of the (anti)-commutation relation (6.1.2a) and (6.1.2b), we obtain the following identities

\[
\nabla_\alpha \nabla_\beta = \frac{1}{2} \epsilon_{\alpha\beta} \nabla^2 + i \nabla_{\alpha\beta} - 2i S M_{\alpha\beta}, \quad (A.2.12a)
\]

\[
\nabla^\beta \nabla_\alpha \nabla_\beta = 4i S \nabla_\alpha, \quad (A.2.12b)
\]

\[
\nabla^2 \nabla_\alpha = -\nabla_\alpha \nabla^2 + 4i S \nabla_\alpha = 2i \nabla_{\alpha\beta} \nabla^\beta + 2i S \nabla_\alpha - 4i S \nabla^\beta M_{\alpha\beta}, \quad (A.2.12c)
\]

\[
-\frac{1}{4} \nabla^2 \nabla^2 = -\nabla_\alpha \nabla^\alpha = -\frac{1}{2} \nabla^2 \nabla_{\alpha\beta}. \quad (A.2.12d)
\]

where $\nabla^2 = \nabla^\alpha \nabla_\alpha$ and $\Box = \nabla^a \nabla_a = -\frac{1}{2} \nabla_\alpha \nabla^\alpha \nabla_{\alpha\beta}$. An important corollary of (A.2.12a) and (A.2.12c) is

\[
[\nabla^\alpha \nabla_{\beta}, \nabla^2] = 0 \quad \implies \quad [\nabla_{\alpha\beta}, \nabla^2] = 0. \quad (A.2.13)
\]

The left-hand side of (A.2.12d) can be expressed in terms of the quadratic Casimir operator of the 3D $\mathcal{N} = 1$ AdS supergroup [51]:

\[
\mathcal{Q} = -\frac{1}{4} \nabla^2 \nabla^2 + i S \nabla^2, \quad [\mathcal{Q}, \nabla_A] = 0. \quad (A.2.14)
\]
Appendix B

Conserved higher-spin currents in four dimensions

In appendix B.1 we consider the construction of conserved higher-spin currents in free scalar field theory in flat space. Similar analysis for free fermions will be done in the next section B.2. This material has been drawn from [3].

B.1 Free real scalars

Given an integer \( s \geq 2 \), the massless spin-\( s \) field \([31]\) is described by real potentials

\[
\begin{align*}
  h^{\alpha(s)} & \equiv h^{\alpha_1 \ldots \alpha_s} , \\
  h^{(s-2)\alpha_{s-2}} & \equiv h^{\alpha_1 \ldots \alpha_{s-2}}
\end{align*}
\]

with the gauge freedom \( \frac{1}{2} \delta h^{\alpha_1 \ldots \alpha_s} = \partial (\alpha_1 \ldots \alpha_s) , \) \( \delta h^{\alpha_1 \ldots \alpha_{s-2}} = \frac{s-1}{s^2} \partial^{\beta_1 \beta_2 \ldots \beta_{s-2}} \lambda^{\alpha_1 \ldots \alpha_{s-2}} \beta_1 \ldots \beta_{s-2} \),

for an arbitrary real gauge parameter \( \lambda^{\alpha(s-1)\beta(s-1)} \). The field \( h^{\alpha(s)} \) may be interpreted as a conformal spin-\( s \) field \([70,71]\).

To construct non-conformal higher-spin currents, we couple \( h^{\alpha(s)} \) and \( h^{(s-2)\alpha_{s-2}} \) to external sources

\[
S^{(s)}_{\text{source}} = \int d^4 x \left\{ h^{\alpha(s)} \partial \alpha(s) + h^{(s-2)\alpha_{s-2}} t^{\alpha(s-2)} \right\} .
\]

Requiring that \( S^{(s)}_{\text{source}} \) be invariant under the \( \lambda \)-transformation in (B.1.1) gives the conservation equation

\[
\partial^{\beta_1 \beta_2 \ldots \beta_s} j_{\beta_1 \ldots \beta_{s-1} \alpha_{s-1} \ldots \alpha_{s-1}} + \frac{s-1}{s^2} \partial (\alpha_1 t_{\alpha_2 \ldots \alpha_{s-1}}) = 0 .
\]

1We follow the description of Fronsdal’s theory \([31]\) given in section 6.9 of \([35]\).
Our derivation of (B.1.3) is analogous to that given in [106].

Let us introduce the following operators

\[
\partial_{(1,1)} := 2i \zeta^\alpha \bar{\zeta}^\dot{\alpha} \partial_{\alpha \dot{\alpha}} , \\
\partial_{(-1,-1)} := 2i \partial_{\alpha \dot{\alpha}} \frac{\partial}{\partial \zeta^\alpha} \frac{\partial}{\partial \bar{\zeta}^\dot{\alpha}} .
\]

The conservation equation (B.1.3) then becomes

\[
\partial_{(-1,-1)} j_{(s,s)} + (s - 1) \partial_{(1,1)} t_{(s-2,s-2)} = 0
\]

(B.1.5)

Note that both \( j_{(s,s)} \) and \( t_{(s-2,s-2)} \) are real.

Let us now consider the model for \( N \) massless real scalar fields \( \phi^i \), with \( i = 1, \ldots, N \), in Minkowski space

\[
S = -\frac{1}{2} \int d^4x \partial_\mu \phi^i \partial^\mu \phi^i ,
\]

(B.1.6)

which admits conserved higher spin currents of the form

\[
j_{(s,s)} = i^s C^{ij} \sum_{k=0}^{s} (-1)^k \binom{s}{k} \binom{s}{k} \partial_{(1,1)}^{s-k} \phi^i \partial_{(1,1)}^k \phi^j ,
\]

(B.1.7)

where \( C^{ij} \) is a constant matrix. It can be shown that \( j_{(s,s)} = 0 \) if \( s \) is odd and \( C^{ij} \) is symmetric. Similarly, \( j_{(s,s)} = 0 \) if \( s \) is even and \( C^{ij} \) is antisymmetric. Thus, we have to consider two separate cases: the case of even \( s \) with symmetric \( C \) and, the case of odd \( s \) with antisymmetric \( C \). Using the massless equation of motion \( \Box \phi^i = 0 \), one may show that \( j_{(s,s)} \) satisfies the conservation equation

\[
\partial_{(-1,-1)} j_{(s,s)} = 0 .
\]

(B.1.8)

We now turn to the massive model

\[
S = -\frac{1}{2} \int d^4x \left\{ \partial_\mu \phi^i \partial^\mu \phi^i + (M^2)^{ij} \phi^i \phi^j \right\} ,
\]

(B.1.9)

where \( M = (M^{ij}) \) is a real, symmetric \( N \times N \) mass matrix. In the massive theory, the conservation equation is described by (B.1.5) and so we first need to compute \( \partial_{(-1,-1)} j_{(s,s)} \) using the massive equations of motion

\[
\Box \phi^i - (M^2)^{ij} \phi^j = 0 .
\]

(B.1.10)

For symmetric \( C \), we obtain

\[
\partial_{(-1,-1)} j_{(s,s)} = -8(s + 1)^2 (CM^2)^{ij} \sum_{k=0}^{s-1} (-1)^k \binom{s}{k} \binom{s}{k}
\]

\]

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\[
\times \frac{(s-k)^2}{(k+1)(k+2)} \partial^k_{(1,1)} \phi^j \partial^{s-k-1}_{(1,1)} \phi^i . \quad \text{(B.1.11)}
\]

If \( C^{ij} \) is antisymmetric, we get

\[
\partial_{(-1,-1)} j_{(s,s)} = 8i(s+1)^2 (CM^2)^{ij} \sum_{k=0}^{s-1} (-1)^k \begin{pmatrix} s \\ k \end{pmatrix} \begin{pmatrix} s \\ s-k \end{pmatrix} 
\times \frac{(s-k)^2}{(k+1)(k+2)} \partial^k_{(1,1)} \phi^j \partial^{s-k-1}_{(1,1)} \phi^i . \quad \text{(B.1.12)}
\]

Thus, in the massive real scalars there are four cases to consider:

1. Both \( C \) and \( CM^2 \) are symmetric \( \iff [C,M^2] = 0, \ s \text{ even.} \)
2. \( C \) is symmetric; \( CM^2 \) is antisymmetric \( \iff \{C,M^2\} = 0, \ s \text{ even.} \)
3. \( C \) is antisymmetric; \( CM^2 \) is symmetric \( \iff \{C,M^2\} = 0, \ s \text{ odd.} \)
4. Both \( C \) and \( CM^2 \) are antisymmetric \( \iff [C,M^2] = 0, \ s \text{ odd.} \)

**Case 1:** Eq. \( \text{[B.1.11]} \) is equivalent to

\[
\partial_{(-1,-1)} j_{(s,s)} = -4(s+1)^2 (CM^2)^{ij} \sum_{k=0}^{s-1} (-1)^k \begin{pmatrix} s \\ k \end{pmatrix} \begin{pmatrix} s \\ s-k \end{pmatrix} (s-k) 
\times \left\{ \frac{s-k}{(k+1)(k+2)} + (-1)^{s-1} \frac{1}{s-k+1} \right\} \partial^k_{(1,1)} \phi^j \partial^{s-k-1}_{(1,1)} \phi^i . \quad \text{(B.1.13)}
\]

We look for \( t_{(s-2,s-2)} \) such that (i) it is real; and (ii) it satisfies the conservation equation \( \text{[B.1.5]} \). We consider a general ansatz

\[
t_{(s-2,s-2)} = -(CM^2)^{ij} \sum_{k=0}^{s-2} d_k \partial^k_{(1,1)} \phi^j \partial^{s-k-2}_{(1,1)} \phi^i . \quad \text{(B.1.14)}
\]

For \( k = 1, 2, \ldots s-2 \), condition (ii) gives

\[
d_{k-1} + d_k = -4 \frac{(s+1)^2}{s-1} (-1)^k \begin{pmatrix} s \\ k \end{pmatrix} \begin{pmatrix} s \\ s-k \end{pmatrix} (s-k) 
\times \left\{ \frac{s-k}{(k+1)(k+2)} + (-1)^{s-1} \frac{1}{s-k+1} \right\} . \quad \text{(B.1.15a)}
\]

Condition (ii) also implies that

\[
d_{s-2} + d_0 = -4s(s+1)(s+2) , \quad \text{(B.1.15b)}
\]

Equations \( \text{[B.1.15]} \) lead to the following expression for \( d_k, k = 1, 2, \ldots s-2 \)

\[
d_k = (-1)^k d_0 - 4 \frac{(s+1)^2}{s-1} \sum_{l=1}^{k} (-1)^l \begin{pmatrix} s \\ l \end{pmatrix} \begin{pmatrix} s \\ s-l \end{pmatrix} \left\{ \frac{s-l}{(l+1)(l+2)} - \frac{1}{s-l+1} \right\} , \quad \text{(B.1.16a)}
\]
\[ d_0 = d_{s-2} = -2s(s+1)(s+2) . \]  \hfill (B.1.16b)

One can check that the equations [B.1.15a]–[B.1.15b] are identically satisfied if \( s \) is even.

**Case 2:** If we take \( CM^2 \) to be antisymmetric, a similar analysis shows that no solution for \( t_{(s-2,s-2)} \) exists for even \( s \).

**Case 3:** Now we consider the case where \( C \) is antisymmetric and \( CM^2 \) symmetric. Again, similar consideration shows that no solution for \( t_{(s-2,s-2)} \) exists for odd \( s \).

**Case 4:** Eq. [B.1.12] is equivalent to
\[
\partial_{(-1,-1)} \tilde{f}_{(s,s)} = 4i(s+1)^2 (CM^2)^{ij} \sum_{k=0}^{s-1} (-1)^k \binom{s}{k} \binom{s}{k} (s-k) \\
\times \left\{ \frac{s-k}{(k+1)(k+2)} - \frac{1}{s-k+1} \right\} \partial_{(1,1)^j} \partial_{(1,1)^k} \phi^i . \tag{B.1.17}\]

We consider a general ansatz
\[
t_{(s-2,s-2)} = -i (CM^2)^{ij} \sum_{k=0}^{s-2} d_k \partial_{(1,1)^j} \partial_{(1,1)^k} \phi^i . \tag{B.1.18}\]

Imposing (i) and (ii) and keeping in mind that \( s \) is odd, we obtain the following conditions for \( d_k \):
\[
d_{k-1} + d_k = 4 \frac{(s+1)^2}{s-1} (-1)^k \binom{s}{k} \binom{s}{k} (s-k) \\
\times \left\{ \frac{s-k}{(k+1)(k+2)} - \frac{1}{s-k+1} \right\} . \tag{B.1.19a}\]

Condition (ii) also implies that
\[
d_{s-2} - d_0 = -4s(s+1)(s+2) , \tag{B.1.19b}\]

Equations [B.1.19] lead to the following expression for \( d_k \), \( k = 1, 2, \ldots, s-2 \)
\[
d_k = (-1)^k d_0 + \frac{4(s+1)^2}{s-1} \sum_{l=1}^{k} (-1)^l \binom{s}{l} \binom{s}{l} \left\{ \frac{(s-l)^2}{(l+1)(l+2)} - \frac{s-l}{s-l+1} \right\} , \tag{B.1.20a}\]
\[
d_0 = -d_{s-2} = 2s(s+1)(s+2) . \tag{B.1.20b}\]

One can check that the equations [B.1.19a]–[B.1.19b] are identically satisfied if \( s \) is odd.
B.2 Free Majorana fermions

Let us now consider \( N \) free massless Majorana fermions
\[
S = -i \int d^4x \, \psi^{\dot{a}i} \partial_{a\dot{a}} \bar{\psi}^{\dot{a}i},
\]
with the equation of motion
\[
\partial_{a\dot{a}} \bar{\psi}^{\dot{a}i} \Longrightarrow \Box \bar{\psi}^i_\alpha = 0, \quad i = 1, \ldots N.
\]

We can construct the following higher spin currents
\[
j_{(s,s)} = C^{ij} \sum_{k=0}^{s-1} (-1)^k \left( \frac{s}{k} \right) \partial_{(1,1)}^k \bar{\psi}_\alpha^{\dot{a}j} \partial_{(1,1)}^{k-1} \bar{\zeta}_\alpha^{\dot{a}i}, \quad C^{ij} = C^{ji}, \quad (B.2.3)
\]
\[
j_{(s)} = i C^{ij} \sum_{k=0}^{s-1} (-1)^k \left( \frac{s}{k} \right) \partial_{(1,1)}^k \bar{\psi}_\alpha^{\dot{a}i} \partial_{(1,1)}^{k-1} \bar{\zeta}_\alpha^{\dot{a}j}, \quad C^{ij} = -C^{ji}, \quad (B.2.4)
\]
where we put an extra \( i \) in eq. (B.2.4) since \( j_{(s,s)} \) has to be real. Using the equation of motion (6.6.5), it can be shown that the currents (B.2.3), (B.2.4) are conserved
\[
\partial_{(-1,-1)} j_{(s,s)} = 0. \quad (B.2.5)
\]

We now look at the massive model
\[
S = - \int d^4x \left\{ i \psi^{\dot{a}i} \partial_{a\dot{a}} \bar{\psi}^{\dot{a}i} + \left( \frac{1}{2} M^{ij} \psi^{\dot{a}i} \bar{\psi}^{\dot{a}j} + \frac{1}{2} \bar{M}^{ij} \bar{\psi}^{\dot{a}i} \psi^{\dot{a}j} \right) \right\}, \quad (B.2.6)
\]
where \( M^{ij} \) is a constant symmetric \( N \times N \) mass matrix. To construct the conserved currents, we compute \( \partial_{(-1,-1)} j_{(s,s)} \) using the massive equations of motion \( i = 1, \ldots, N \)
\[
i \partial_{a\dot{a}} \bar{\psi}^{\dot{a}i} + M^{ij} \psi^{\dot{a}j} = 0 \Longrightarrow \Box \bar{\psi}^i_\alpha = (\bar{M} M)^{ij} \bar{\psi}^j_\alpha, \quad (B.2.7a)
\]
\[-i \partial_{a\dot{a}} \psi^{\dot{a}i} + \bar{M}^{ij} \bar{\psi}^j_\alpha = 0 \Longrightarrow \Box \psi^i_\alpha = (\bar{M} M)^{ij} \psi^j_\alpha. \quad (B.2.7b)
\]
If \( C^{ij} \) is a real symmetric matrix, we find
\[
\partial_{(-1,-1)} j_{(s,s)} = -2(s + 1) \sum_{k=0}^{s-1} \frac{k + 1}{s - k + 1} (-1)^k \left( \frac{s}{k} \right) \left( \frac{s}{k + 1} \right) \times \left\{ (CM)^{ij} \partial_{(1,1)}^k \bar{\psi}^{\dot{a}i} \partial_{(1,1)}^{s-k-1} \bar{\psi}^{\dot{a}j} + (-1)^s (CM)^{ij} \partial_{(1,1)}^k \bar{\psi}^{\dot{a}i} \partial_{(1,1)}^{s-k-1} \bar{\zeta}^{\dot{a}j} \right\}
\]
\[+ 4(s + 1)(s + 2) \sum_{k=1}^{s-1} k(-1)^k \left( \frac{s}{k} \right) \left( \frac{s}{k + 1} \right) \times \left\{ \frac{1}{k + 2} (CM \bar{C})^{ij} - \frac{k + 1}{(s - k + 2)(s - k + 1)} (CM \bar{M})^{ij} \right\}
\]
\[\times \partial_{(1,1)}^{k-1} \bar{\psi}^{\dot{a}i}_\alpha \partial_{(1,1)}^{s-k-1} \bar{\zeta}^{\dot{a}j}_\alpha. \quad (B.2.8)
\]

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If $C^{ij}$ is antisymmetric, we have

$$\partial_{(-1,-1)} j_{(s,s)} = -2i(s + 1) \sum_{k=0}^{s-1} \frac{k + 1}{s-k+1} (-1)^k \binom{s}{k} \binom{s}{k+1}$$

$$\times \left\{ (CM)^{ij} \partial^{k}_{(1,1)} \psi_{\alpha}^i \partial^{s-k-1}_{(1,1)} \psi_{\alpha}^j + (-1)^{s-1} (CM)^{ij} \partial^{k}_{(1,1)} \bar{\psi}_{\dot{\alpha}}^i \partial^{s-k-1}_{(1,1)} \bar{\psi}_{\dot{\alpha}}^j \right\}$$

$$+ 4i(s + 1)(s + 2) \sum_{k=1}^{s-1} k(-1)^k \binom{s}{k} \binom{s}{k+1}$$

$$\times \left\{ \frac{1}{k + 2} (M\bar{M}C)^{ij} - \frac{k + 1}{(s - k + 2)(s - k + 1)} (CM\bar{M})^{ij} \right\}$$

$$\times \partial^{k-1}_{(1,1)} \zeta_{\alpha}^i \partial^{s-k-1}_{(1,1)} \zeta_{\dot{\alpha}}^j .$$

(B.2.9)

There are four cases to consider:

1. $C, CM, CM\bar{M}$ are symmetric $\iff [C, M] = [C, \bar{M}] = 0, [M, \bar{M}] = 0$.
2. $C, CM\bar{M}$ symmetric; $CM$ antisymmetric $\iff \{C, M\} = \{C, \bar{M}\} = 0, [M, \bar{M}] = 0$.
3. $C, CM\bar{M}$ antisymmetric; $CM$ symmetric $\iff \{C, M\} = \{C, \bar{M}\} = 0, [M, \bar{M}] = 0$.
4. $C, CM, CM\bar{M}$ are antisymmetric $\iff [C, M] = [C, \bar{M}] = 0, [M, \bar{M}] = 0$.

Case 1: Eq. (B.2.8) becomes

$$\partial_{(-1,-1)} j_{(s,s)} = -(s + 1) \sum_{k=0}^{s-1} (-1)^k \binom{s}{k} \binom{s}{k+1}$$

$$\times \left\{ \frac{k + 1}{s - k + 1} + (-1)^{s-1} \frac{s - k}{k + 2} \right\} (CM)^{ij} \partial^{k}_{(1,1)} \psi_{\alpha}^i \partial^{s-k-1}_{(1,1)} \psi_{\alpha}^j$$

$$+ (-1)^{s-1} (s + 1) \sum_{k=0}^{s-1} (-1)^k \binom{s}{k} \binom{s}{k+1}$$

$$\times \left\{ \frac{k + 1}{s - k + 1} + (-1)^{s-1} \frac{s - k}{k + 2} \right\} (CM)^{ij} \partial^{k}_{(1,1)} \bar{\psi}_{\dot{\alpha}}^i \partial^{s-k-1}_{(1,1)} \bar{\psi}_{\dot{\alpha}}^j$$

$$+ 4(s + 1)(s + 2) \sum_{k=1}^{s-1} k(-1)^k \binom{s}{k} \binom{s}{k+1}$$

$$\times \left\{ \frac{1}{k + 2} - \frac{k + 1}{(s - k + 2)(s - k + 1)} \right\}$$

$$\times (CM\bar{M})^{ij} \partial^{k-1}_{(1,1)} \zeta_{\alpha}^i \partial^{s-k-1}_{(1,1)} \zeta_{\dot{\alpha}}^j .$$

(B.2.10)

We look for $t_{(s-2,s-2)}$ such that (i) it is real; and (ii) it satisfies the conservation equation (B.1.5):

$$\partial_{(-1,-1)} j_{(s,s)} = -(s - 1) \partial_{(1,1)} t_{(s-2,s-2)} .$$

(B.2.11)
Consider a general ansatz

\[ t_{(s-2,s-2)} = (CM)^{ij} \sum_{k=0}^{s-2} c_k \partial_{(1,1)}^k \psi_{\alpha i} \partial_{(1,1)}^{s-k-2} \psi_{\alpha j}^{\dagger} + (-1)^s (CM)^{ij} \sum_{k=0}^{s-2} c_k \partial_{(1,1)}^k \bar{\psi}_{\alpha}^i \partial_{(1,1)}^{s-k-2} \bar{\psi}_{\alpha}^{\dagger j} + (CM\bar{M})^{ij} \sum_{k=1}^{s-2} g_k \partial_{(1,1)}^{k-1} \psi_{\alpha}^i \partial_{(1,1)}^{s-k-1} \bar{\psi}_{\alpha}^{\dagger j} . \] (B.2.12)

For \( k = 1, 2, \ldots, s-2 \), condition (i) gives

\[ g_k = (-1)^{s-1} g_{s-1-k} , \] (B.2.13a)

while condition (ii) gives

\[ c_{k+1} + c_k = \frac{s+1}{s-1} (-1)^k \binom{s}{k} \left( \frac{s}{k+1} \right) \left\{ \frac{k+1}{s-k+1} + (-1)^{s-1} \frac{s-k}{k+2} \right\} , \] (B.2.13b)

\[ g_{k+1} + g_k = -4 \frac{(s+1)(s+2)}{s-1} (-1)^k \binom{s}{k} \left( \frac{s}{k+1} \right) k \left\{ \frac{1}{k+2} - \frac{k+1}{(s-k+2)(s-k+1)} \right\} . \] (B.2.13c)

Condition (ii) also implies that

\[ c_{s-2} + c_0 = \frac{1}{s-1} \left\{ 2s + (-1)^{s-1} s^2(s+1) \right\} , \] (B.2.13d)

\[ g_1 = \frac{2s(s-2)}{3} (s^2 + 5s + 6) , \] (B.2.13e)

\[ g_{s-2} = (-1)^{s-1} \frac{2s(s-2)}{3} (s^2 + 5s + 6) . \] (B.2.13f)

The above conditions lead to the following expressions for \( c_k \) and \( g_k \) \((k = 1, 2, \ldots, s-2)\)

\[ c_k = (-1)^k c_0 + \frac{s+1}{s-1} \sum_{l=1}^{k} (-1)^k \binom{s}{l} \left( \frac{s}{l+1} \right) \left\{ \frac{l+1}{s-l+1} + (-1)^{s-1} \frac{s-l}{l+2} \right\} , \] (B.2.14a)

\[ g_k = 4 (-1)^k \frac{(s+1)(s+2)}{s-1} \sum_{l=1}^{k} \binom{s}{l} \left( \frac{s}{l+1} \right) \left\{ \frac{l(l+1)}{(s-l+1)(s-l+2)} - \frac{l}{l+2} \right\} . \] (B.2.14b)

If the parameter \( s \) is even, (B.2.14a) gives

\[ c_{s-2} = c_0 = -\frac{1}{2} s(s+2) \] (B.2.14c)
and \( \text{(B.2.13a)-(B.2.13f)} \) are identically satisfied. However, when \( s \) is odd, there appears an inconsistency: the right-hand side of \( \text{(B.2.13d)} \) is positive, while the left-hand side is negative, \( c_{s-2} + c_0 < 0 \). Therefore, our solution \( \text{(B.2.14)} \) is only consistent for \( s = 2n, n = 1, 2, \ldots \).

**Case 2:** If \( CM \) is antisymmetric while \( CM\bar{M} \) symmetric, eq. \( \text{(B.2.8)} \) is slightly modified

\[
\partial_{(1,-1)} \dot{j}_{(s,s)} = -(s + 1) \sum_{k=0}^{s-1} (-1)^k \left( \begin{array}{c} s \\ k \end{array} \right) \left( \begin{array}{c} s \\ k + 1 \end{array} \right) \\
\times \left\{ \frac{k + 1}{s - k + 1} + (-1)^s \frac{s - k}{k + 2} \right\} (CM)^{ij} \partial_{(1,1)}^{k \alpha i} \partial_{(1,1)}^{s-1 \bar{j} \alpha j} \\
+ (-1)^{s-1} (s + 1) \sum_{k=0}^{s-1} (-1)^k \left( \begin{array}{c} s \\ k \end{array} \right) \left( \begin{array}{c} s \\ k + 1 \end{array} \right) \\
\times \left\{ \frac{k + 1}{s - k + 1} + (-1)^s \frac{s - k}{k + 2} \right\} (CM)^{ij} \partial_{(1,1)}^{k \alpha i} \partial_{(1,1)}^{s-1 \bar{j} \alpha j} \\
+ 4(s + 1)(s + 2) \sum_{k=1}^{s-1} k(-1)^k \left( \begin{array}{c} s \\ k \end{array} \right) \left( \begin{array}{c} s \\ k + 1 \end{array} \right) \\
\times \left\{ \frac{1}{k + 2} - \frac{k + 1}{(s - k + 2)(s - k + 1)} \right\} \\
\times (CM\bar{M})^{ij} \partial_{(1,1)}^{k \alpha i} \partial_{(1,1)}^{s-1 \bar{j} \alpha j}.
\]  

(B.2.15)

Starting with a general ansatz

\[
t_{(s-2,s-2)} = (CM)^{ij} \sum_{k=0}^{s-2} d_k \partial_{(1,1)}^{k \alpha i} \partial_{(1,1)}^{s-2 \bar{j} \alpha j} \\
+ (-1)^s (CM\bar{M})^{ij} \sum_{k=0}^{s-2} d_k \partial_{(1,1)}^{k \alpha i} \partial_{(1,1)}^{s-2 \bar{j} \alpha j} \\
+ (CM\bar{M})^{ij} \sum_{k=1}^{s-2} g_k \partial_{(1,1)}^{k \alpha i} \partial_{(1,1)}^{s-2 \bar{j} \alpha j}
\]

(B.2.16)

and imposing conditions (i) and (ii) yield

\[
g_k = (-1)^{s-1} g_{s-1-k},
\]

(B.2.17a)

\[
d_{k-1} + d_k = \frac{s + 1}{s - 1} (-1)^k \left( \begin{array}{c} s \\ k \end{array} \right) \left( \begin{array}{c} s \\ k + 1 \end{array} \right) \left\{ \frac{k + 1}{s - k + 1} - (-1)^{s-1} \frac{s - k}{k + 2} \right\},
\]

(B.2.17b)

\[
g_{k-1} + g_k = -4 \frac{(s + 1)(s + 2)}{s - 1} (-1)^k \left( \begin{array}{c} s \\ k \end{array} \right) \left( \begin{array}{c} s \\ k + 1 \end{array} \right) k \left\{ \frac{1}{k + 2} - \frac{k + 1}{(s - k + 2)(s - k + 1)} \right\},
\]

(B.2.17c)

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\[ d_0 - d_{s-2} = \frac{1}{s-1} \left\{ 2s + (-1)^s s^2 (s+1) \right\} , \]  
(B.2.17d)

\[ g_1 = \frac{2s(s-2)}{3} (s^2 + 5s + 6) , \]  
(B.2.17e)

\[ g_{s-2} = (-1)^s - \frac{2s(s-2)}{3} (s^2 + 5s + 6) . \]  
(B.2.17f)

As a result, the coefficients \( d_k \) and \( g_k \) are given by \((k = 1, \ldots, s-2)\)

\[ d_k = (-1)^k d_0 + \frac{s+1}{s-1} \sum_{l=1}^{k} (-1)^k \binom{s}{l} \left( \frac{s}{l+1} \right) \left\{ \frac{l+1}{s-l+1} + (-1)^{s-1} \frac{s-l}{l+2} \right\} , \]  
(B.2.18a)

\[ g_k = 4(-1)^k \frac{(s+1)(s+2)}{s-1} \sum_{l=1}^{k} \binom{s}{l} \left( \frac{s}{l+1} \right) \left\{ \frac{l(l+1)}{(s-l+1)(s-l+2)} - \frac{l}{l+2} \right\} . \]  
(B.2.18b)

When the parameter \( s \) is odd, \( \text{(B.2.18a)} \) gives

\[ d_{s-2} = -d_0 = \frac{1}{2} s(s+2) \]  
(B.2.18c)

and \( \text{(B.2.17a)-(B.2.17f)} \) are identically satisfied. However, when \( s \) is even, there appears an inconsistency: the right-hand side of \( \text{(B.2.17d)} \) is positive, while the left-hand side is negative, \( d_0 - d_{s-2} < 0 \). Therefore, our solution \( \text{(B.2.18)} \) is only consistent for \( s = 2n+1, n = 1, 2, \ldots \).

Finally, we consider \( C^{ij} = -C^{ji} \) with the corresponding \( j(s,s) \) given by \( \text{(B.2.4)} \). Similar considerations show that in \text{Case 3}, the non-conformal currents exist only if \( s \) is even. The trace \( t_{(s-2,s-2)} \) is given by \( \text{(B.2.12)} \) with the coefficients \( c_k \) and \( g_k \) given by

\[ c_k = i(-1)^k c_0 + i \frac{s+1}{s-1} \sum_{l=1}^{k} (-1)^k \binom{s}{l} \left( \frac{s}{l+1} \right) \left\{ \frac{l+1}{s-l+1} + (-1)^{s-1} \frac{s-l}{l+2} \right\} , \]  
(B.2.19a)

\[ g_k = 4i (-1)^k \frac{(s+1)(s+2)}{s-1} \sum_{l=1}^{k} \binom{s}{l} \left( \frac{s}{l+1} \right) \left\{ \frac{l(l+1)}{(s-l+1)(s-l+2)} - \frac{l}{l+2} \right\} . \]  
(B.2.19b)

In \text{Case 4}, the non-conformal currents exist only for odd values of \( s \). The trace \( t_{(s-2,s-2)} \) is given by \( \text{(B.2.16)} \) with the coefficients \( d_k \) and \( g_k \) given by

\[ d_k = i(-1)^k d_0 + i \frac{s+1}{s-1} \sum_{l=1}^{k} (-1)^k \binom{s}{l} \left( \frac{s}{l+1} \right) \left\{ \frac{l+1}{s-l+1} + (-1)^{s-1} \frac{s-l}{l+2} \right\} , \]  
(B.2.20a)

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\[ g_k = 4i \left( -1 \right)^k \frac{(s+1)(s+2)}{s-1} \sum_{l=1}^{k} \binom{s}{l} \binom{s}{l+1} \left\{ \frac{l(l+1)}{(s-l+1)(s-l+2)} - \frac{l}{l+2} \right\} . \]  

(B.2.20b)

We observe that the coefficients \( c_k \) and \( g_k \) in eq. (B.2.19a) and (B.2.19b), respectively, differ from similar coefficients in (B.2.14a) and (B.2.14b) by a factor of \( i \). Hence, for even \( s \) we may define a more general supercurrent

\[ j_{(s,s)} = C^{ij} \sum_{k=0}^{s-1} (-1)^k \binom{s}{k} \binom{s}{k+1} \partial_{(1,1)}^{k} \xi_{\alpha}^{\bar{i}} \partial_{(1,1)}^{s-k-2} \bar{\psi}_{\bar{\alpha}}^{j} , \]  

(B.2.21)

where \( C^{ij} \) is a generic matrix which can be split into the symmetric and antisymmetric parts: \( C^{ij} = S^{ij} + iA^{ij} \). Here both \( S \) and \( A \) are real and we put an \( i \) in front of \( A \) because \( j_{(s,s)} \) must be real. From the above consideration it then follows that the corresponding more general solution for \( t_{(s-2,s-2)} \) reads

\[ t_{(s-2,s-2)} = (CM)^{ij} \sum_{k=0}^{s-2} c_k \partial_{(1,1)}^{k} \psi_{\alpha}^{i} \partial_{(1,1)}^{s-k-2} \bar{\psi}_{\bar{\alpha}}^{j} \]

\[ +(-1)^s(\bar{CM})^{ij} \sum_{k=0}^{s-2} c_k \partial_{(1,1)}^{k} \bar{\psi}_{\bar{\alpha}}^{i} \partial_{(1,1)}^{s-k-2} \psi_{\alpha}^{j} \]

\[ +(CM\bar{M})^{ij} \sum_{k=1}^{s-2} g_k \partial_{(1,1)}^{k-1} S_{\alpha}^{\bar{i}} \partial_{(1,1)}^{s-k-1} \bar{\psi}_{\bar{\alpha}}^{j} , \]  

(B.2.22)

where \([S, M] = [S, \bar{M}] = 0\), \([A, M] = [A, \bar{M}] = 0\) and \([M, \bar{M}] = 0\). The coefficients \( c_k \) and \( g_k \) are given by eqs. (B.2.14a) and (B.2.14b), respectively. Similarly, the coefficients \( d_k \) and \( g_k \) in (B.2.20a) and (B.2.20b) differ from similar coefficients in (B.2.18a) and (B.2.18b) by a factor of \( i \). This means that for odd \( s \) we can define a more general supercurrent (B.2.21), where \( C^{ij} \) is a generic matrix which we can split as before into the symmetric and antisymmetric parts, \( C^{ij} = S^{ij} + iA^{ij} \). From the above consideration it then follows that the corresponding more general solution for \( t_{(s-2,s-1)} \) reads

\[ t_{(s-2,s-1)} = (CM)^{ij} \sum_{k=0}^{s-2} d_k \partial_{(1,1)}^{k} \psi_{\alpha}^{i} \partial_{(1,1)}^{s-k-2} \bar{\psi}_{\bar{\alpha}}^{j} \]

\[ +(-1)^s(\bar{CM})^{ij} \sum_{k=0}^{s-2} d_k \partial_{(1,1)}^{k} \bar{\psi}_{\bar{\alpha}}^{i} \partial_{(1,1)}^{s-k-2} \psi_{\alpha}^{j} \]

\[ +(CM\bar{M})^{ij} \sum_{k=1}^{s-2} g_k \partial_{(1,1)}^{k-1} S_{\alpha}^{\bar{i}} \partial_{(1,1)}^{s-k-1} \bar{\psi}_{\bar{\alpha}}^{j} , \]  

(B.2.23)

where \([S, M] = [S, \bar{M}] = 0\), \([A, M] = [A, \bar{M}] = 0\) and \([M, \bar{M}] = 0\). The coefficients \( d_k \) and \( g_k \) are given by eqs. (B.2.18a) and (B.2.18b), respectively.
Appendix C

Component analysis of $\mathcal{N} = 1$ higher-spin actions in three dimensions

In this appendix we discuss the component structure of the two new off-shell $\mathcal{N} = 1$ supersymmetric higher-spin theories in three dimensions: the transverse massless superspin-$s$ multiplet (6.2.29), and the transverse massless superspin-$(s + \frac{1}{2})$ multiplet (6.3.21). The longitudinal actions (6.2.25) and (6.3.17) can be reduced to components in a similar fashion. For simplicity we will carry out our analysis in flat Minkowski superspace. This material has been drawn from [6].

C.1 Massless superspin-$s$ action

In accordance with (6.1.41), the component form of an $\mathcal{N} = 1$ supersymmetric action is computed by the rule

$$ S = \int \text{d}^3z \, L = \left. \frac{i}{4} \int \text{d}^3x \, D^2 L \right|_{\theta = 0}, \quad L = \bar{L}. \quad \text{(C.1.1)} $$

Let us first work out the component structure of the massless integer superspin model (6.2.29). In the flat superspace limit, the transverse action (6.2.29) takes the form

$$ S_{\perp}^{(s)}[H_{\alpha(2s)}, \Psi_{\beta; \alpha(2s-2)}] = \left( -\frac{1}{2} \right)^s \int \text{d}^3z \left\{ \frac{i}{2} H^{\alpha(2s)} D^2 H_{\alpha(2s)} - i s D_{\beta} H^{\alpha(2s-1)} \bar{D} H_{\gamma\alpha(2s-1)} - (2s - 1) W^{\beta\alpha(2s-2)} D^\gamma H_{\gamma\beta\alpha(2s-2)} - \frac{i}{2} (2s - 1) \left( W^{\beta; \alpha(2s-2)} W_{\beta; \alpha(2s-2)} + s - 1 \right) W_{\beta; \alpha(2s-3)} W_{\gamma; \gamma\alpha(2s-3)} \right\}. \quad \text{(C.1.2)} $$

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As described in \([6.2.31]\), it is possible to choose a gauge condition \(\Psi_{(a_1; a_2 \ldots a_{2s-1})} = 0\), such that the above action turns into

\[
S_{(s)}^{\perp}[H_{\alpha(2s)}, \Psi; \alpha(2s-2)] = \left( -\frac{1}{2} \right)^s \int d^{3/2}z \left\{ \frac{i}{2} H^{\alpha(2s)} D^2 H_{\alpha(2s)} - is D_{\beta} H^{\beta\alpha(2s-1)} D^\gamma H_{\gamma\alpha(2s-1)} - 2(s - 1) \varphi^{\alpha(2s-3)} \partial^\beta D^\delta H_{\beta\gamma\delta\alpha(2s-3)} - \frac{2i}{s} (s - 1) \varphi^{\alpha(2s-3)} \partial_{\alpha} \varphi_{(2s-3)} - \frac{i(s - 1)(2s - 3)}{s(2s - 1)} \partial_{\alpha} \varphi_{(2s-3)} \right\}.
\]  

(C.1.3)

It is invariant under the following gauge transformations

\[
\delta H_{(2s)} = -\partial_{(a_1 a_2 \ldots a_{2s})}, \\
\delta \varphi_{(2s-3)} = i D^\beta \eta_{(a_2 \ldots a_{2s-2})}.
\]

(C.1.4a, 1.4b)

where the gauge parameter \(\eta_{(2s-2)}\) is a real unconstrained superfield.

The gauge freedom (C.1.4) can be used to impose a Wess-Zumino gauge

\[
\varphi_{(2s-3)}|_{\theta = 0} = 0, \quad D_{(a_1 \varphi_{a_2 \ldots a_{2s-2})}}|_{\theta = 0} = 0.
\]  

(C.1.5)

In order to preserve these gauge conditions, the residual gauge freedom has to be constrained by

\[
D^\beta \eta_{(a_2 \ldots a_{2s-3})}|_{\theta = 0} = 0, \quad D^2 \eta_{(2s-2)}|_{\theta = 0} = 2i \partial^\beta (\alpha_1 \eta_{a_2 \ldots a_{2s-2}})|_{\theta = 0}.
\]  

(C.1.6)

These imply that there remain two independent, real components of \(\eta_{(2s-2)}\):

\[
\xi_{(2s-2)} := \eta_{(2s-2)}|_{\theta = 0}, \quad \lambda_{(2s-1)} := i D_{(a_1 \eta_{a_2 \ldots a_{2s-1})}}|_{\theta = 0}.
\]  

(C.1.7)

In the gauge \([4.5.15]\), the independent component fields of \(\varphi_{(2s-3)}\) can be chosen as

\[
y_{(2s-4)} := -\frac{2s - 2}{2s - 1} D^\beta \varphi_{a_1 \ldots a_{2s-4}}|_{\theta = 0}, \quad y_{(2s-3)} := \frac{i}{2} D^2 \varphi_{(2s-3)}|_{\theta = 0}.
\]  

(C.1.8)

We define the component fields of \(H_{(2s)}\) as

\[
h_{(2s)} := -H_{(2s)}|_{\theta = 0}, \\
h_{(2s+1)} := \frac{i}{s} \frac{2s + 1}{2s} D_{(a_1 H_{a_2 \ldots a_{2s+1}})}|_{\theta = 0}, \quad y_{(2s-1)} := i D^\beta H_{\beta a_1 \ldots a_{2s-1}}|_{\theta = 0}, \\
F_{(2s)} := \frac{i}{4} D^2 H_{(2s)}|_{\theta = 0}.
\]  

(C.1.9, 1.10, 1.11)

Applying the reduction rule (C.1.1) to the \(\mathcal{N} = 1\) action (C.1.3), we find that it splits into bosonic and fermionic parts:

\[
S_{(s)}^{\perp}[H_{(2s)}, \Psi; \alpha(2s-2)] = S_{\text{bos}} + S_{\text{ferm}}.
\]  

(C.1.12)
The bosonic action is given by

\[
S_{\text{bos}} = \left( -\frac{1}{2} \right)^s \int d^3x \left\{ 2(1 - s) F^{\alpha(2s)} F_{\alpha(2s)} + 2s F^{\alpha(2s-1)\beta} \partial_{[\gamma} h_{\alpha(2s-1)\gamma]} + \frac{1}{2} (s - 1) h^{\alpha(2s)} \Box h_{\alpha(2s)} - \frac{(2s - 1)(2s - 3)}{2s(s - 1)} g^{\alpha(2s-4)} \Box g_{\alpha(2s-4)} \\
- \frac{(2s - 1)(2s - 3)}{4(s - 1)} g^{\alpha(2s-4)} \partial^{[\beta_1} \partial^{\beta_2] \partial_\lambda \partial_\gamma h_{\beta_1 \gamma \lambda \alpha(2s-4)} \\
- \frac{(s - 2)(2s - 1)(2s - 3)(2s - 5)}{16s(s - 1)^2} \partial_\beta \partial_\gamma \partial_\delta \partial_\lambda \partial_\alpha(2s-6) \partial_{[\beta_1} \partial^{\beta_2]} y_{\beta_1 \gamma \alpha(2s-6)} \right\} .
\] (C.1.13)

Integrating out the auxiliary field \( F_{\alpha(2s)} \) leads to

\[
S_{\text{bos}} = \left( -\frac{1}{2} \right)^s \int d^3x \left\{ \frac{2s - 1}{2s - 2} h^{\alpha(2s)} \Box h_{\alpha(2s)} - \frac{s}{2} \partial_\beta \partial_\gamma \partial_\delta \partial_\lambda h_{\beta_1 \gamma \lambda \alpha(2s-2)} \\
- \frac{2s - 3}{2s} \left[ s g^{\alpha(2s-4)} \partial^{[\beta_1} \partial^{\beta_2] \partial_\delta \partial_\lambda h_{\beta_1 \delta \lambda \alpha(2s-4)} + 2 g^{\alpha(2s-4)} \Box y_{\alpha(2s-4)} \\
+ \frac{(s - 2)(2s - 5)}{4(s - 1)} \partial_\beta \partial_\gamma \partial_\delta \partial_\lambda \partial_\alpha(2s-6) \partial_{[\beta_1} \partial^{\beta_2]} y_{\beta_1 \gamma \alpha(2s-6)} \right] \right\} .
\] (C.1.14)

This action is invariant under the gauge transformations

\[
\delta_\xi h_{\alpha(2s)} = \partial_{(\alpha_1 \alpha_2} \xi_{\alpha_3 \ldots \alpha_{2s})} ;
\] (C.1.15)

\[
\delta_\xi y_{\alpha(2s-4)} = \frac{2s - 2}{2s - 1} \partial^{[\beta_1} \partial^{\beta_2]} \xi_{\beta_1 \gamma \alpha_1 \ldots \alpha_{2s-4}] .
\] (C.1.16)

The gauge transformations for the fields \( h_{\alpha(2s)} \) and \( y_{\alpha(2s-4)} \) can be easily read off from the gauge transformations of the superfields \( H_{\alpha(2s)} \) and \( \varphi_{\alpha(2s-3)} \), respectively. Modulo an overall normalisation factor, \( \text{(C.1.14)} \) corresponds to the massless Fronsdal spin-s action \( S_F^{(2s)} \) described in \[51\].

The fermionic sector of the component action is described by the real dynamical fields \( h_{\alpha(2s+1)}, y_{\alpha(2s-1)}, y_{\alpha(2s-3)} \), defined modulo gauge transformations of the form

\[
\delta_\lambda h_{\alpha(2s+1)} = \partial_{(\alpha_1 \alpha_2} \lambda_{\alpha_3 \ldots \alpha_{2s+1})} ;
\] (C.1.17)

\[
\delta_\lambda y_{\alpha(2s-1)} = \frac{1}{2s + 1} \partial_{[\beta_1} \partial^{\beta_2]} (\alpha_1 \lambda_{\alpha_2 \ldots \alpha_{2s-1})\beta} ;
\] (C.1.18)

\[
\delta_\lambda y_{\alpha(2s-3)} = \partial^{\beta_1} \lambda_{\beta_1 \alpha_1 \ldots \alpha_{2s-3}} .
\] (C.1.19)

The gauge-invariant action is

\[
S_{\text{ferm}} = \left( -\frac{1}{2} \right)^s \int d^3x \left\{ h^{\alpha(2s)\beta} \partial_{[\gamma} \gamma h_{\alpha(2s)\gamma]} + 2(2s - 1) g^{\alpha(2s-1)\beta} \partial_{[\gamma} h_{\alpha(2s-1)\gamma]} \\
+ 4(2s - 1) g^{\alpha(2s-2)\beta} \partial_{[\gamma} y_{\alpha(2s-2)\gamma]} + 2 \frac{(s - 1)(2s - 3)}{2s(s + 1)} g^{\alpha(2s-3)\beta} \partial_{[\gamma} y_{\beta_1 \gamma \alpha(2s-3)}} .
\] (C.1.20)
It may be shown that $S_{\text{term}}$ coincides with the Fang-Fronsdal spin-$(s + \frac{1}{2})$ action, $S_{FF}^{(2s+1)}$ [51].

We have thus proved that at the component level and upon elimination of the auxiliary field, the transverse theory (4.5.11) is equivalent to a sum of two massless models: the bosonic Fronsdal spin-$s$ model and the fermionic Fang-Fronsdal spin-$(s + \frac{1}{2})$ model.

### C.2 Massless superspin-$(s + \frac{1}{2})$ action

We now elaborate on the component structure of the massless half-integer superspin model in the transverse formulation (6.3.21a). The theory is described in terms of the real unconstrained prepotentials $H_{\alpha(2s+1)}$ and $\Upsilon_{\beta; \alpha(2s-2)}$. In Minkowski superspace, the action (6.3.21a) simplifies into

$$S^\perp_{(s+\frac{1}{2})}[H_{\alpha(2s+1)}, \Upsilon_{\beta; \alpha(2s-2)}] = \left( -\frac{1}{2} \right)^s \int d^3|z|^2 \left\{ -\frac{i}{2} H^{\alpha(2s+1)} \Box H_{\alpha(2s+1)} 
- \frac{i}{8} D_\beta H^{\beta \alpha(2s)} D^\gamma D^\gamma H_{\gamma \alpha(2s)} + \frac{i}{8} \partial_{\beta \gamma} H^{\beta \gamma \alpha(2s-1)} \partial^{\rho \delta} H_{\rho \delta \alpha(2s-1)} 
- \frac{i}{4} (2s-1) \Omega^{\beta \alpha(2s-2) \gamma \alpha(2s-2)} - 2(s-1) \Omega^{\beta \alpha(2s-3) \gamma \alpha(2s-3)} \right\}, \tag{C.2.1}$$

with the following gauge symmetry

$$\delta H_{\alpha(2s+1)} = i D_{(\alpha_1 \zeta_{a_2 \ldots a_{2s+1}})} , \tag{C.2.2a}$$

$$\delta \Upsilon_{\beta; \alpha(2s-2)} = \frac{i}{2s+1} \left( D^\gamma \zeta_{\gamma \beta \alpha(2s-2)} + (2s+1) D^\gamma \eta_{\gamma \beta \alpha(2s-2)} \right) . \tag{C.2.2b}$$

The action (C.2.1) involves the real field strength $\Omega_{\beta; \alpha(2s-2)}$

$$\Omega_{\beta; \alpha(2s-2)} = -i D^\gamma D_\beta \Upsilon_{\gamma \alpha(2s-2)} , \quad D^\beta \Omega_{\beta; \alpha(2s-2)} = 0 . \tag{C.2.3}$$

The gauge transformations (C.2.2) allow us to impose a Wess-Zumino gauge on the prepotentials:

$$H_{\alpha(2s+1)}|_{\theta=0} = 0 , \quad D_\beta H_{\beta \alpha_1 \ldots \alpha_{2s}}|_{\theta=0} = 0 , \quad \Upsilon_{\beta; \alpha(2s-2)}|_{\theta=0} = 0 , \quad D_\beta \Upsilon_{\beta; \alpha(2s-2)}|_{\theta=0} = 0 . \tag{C.2.4}$$

The residual gauge symmetry preserving the conditions (C.2.4) is characterised by

$$D_{(\alpha_1 \zeta_{a_2 \ldots a_{2s+1}})}|_{\theta=0} = 0 , \quad D^2 \zeta_{\alpha(2s)}|_{\theta=0} = -\frac{2is}{s+1} \partial^{\beta \gamma} (\alpha_1 \zeta_{a_2 \ldots a_{2s}})_{\beta}|_{\theta=0} , \tag{C.2.5a}$$

$$D_\beta \eta_{\alpha(2s-2)}|_{\theta=0} = D_{(\beta \eta_{\alpha(2s-2)}}|_{\theta=0} = -\frac{1}{2s+1} D^\gamma \zeta_{\gamma \beta \alpha(2s-2)}|_{\theta=0} , \tag{C.2.5b}$$

$$D^2 \eta_{\alpha(2s-2)}|_{\theta=0} = -\frac{1}{2s+1} \partial^{\beta \gamma} \zeta_{\beta \gamma \alpha(2s-2)}|_{\theta=0} . \tag{C.2.5c}$$
As a result, there are three independent, real gauge parameters at the component level, which we define as

\[ \xi_{\alpha(2s)} := \zeta_{\alpha(2s)}|_{\theta=0}, \quad \lambda_{\alpha(2s-1)} := -i \frac{s}{2s+1} D^{3}_{\bar{\alpha}} \zeta_{\alpha(2s-1)}|_{\theta=0}, \quad \rho_{\alpha(2s-2)} := \eta_{\alpha(2s-2)}|_{\theta=0} \] (C.2.6)

Let us now represent the prepotential \( \Upsilon_{\beta;\alpha(2s-2)} \) in terms of its irreducible components,

\[ \Upsilon_{\beta;\alpha(2s-2)} = Y_{\beta\alpha_1...\alpha_{2s-2}} + \sum_{k=1}^{2s-2} \varepsilon_{\beta\alpha_k} Z_{\alpha_1...\alpha_k...\alpha_{2s-2}}, \] (C.2.7)

where we have introduced the two irreducible components of \( \Upsilon_{\beta;\alpha(2s-2)} \) by the rule

\[ Y_{\beta\alpha_1...\alpha_{2s-2}} := \Upsilon_{(\beta;\alpha_1...\alpha_{2s-2})}, \quad Z_{\alpha_1...\alpha_{2s-3}} := \frac{1}{2s-1} \Upsilon^{\beta;\alpha_1...\alpha_{2s-3}}. \] (C.2.8)

The next step is to determine the remaining independent component fields of \( H_{\alpha(2s+1)} \) and \( \Upsilon_{\beta;\alpha(2s-2)} \) in the Wess-Zumino gauge (C.2.4).

In the bosonic sector, we have the following set of fields:

\[
\begin{align*}
\theta_{\alpha(2s+2)} & := -D_{(\alpha_1} H_{\alpha_2...\alpha_{2s+2})}|_{\theta=0}, \quad \text{(C.2.9a)} \\
y_{\alpha(2s)} & := D_{(\alpha_1} Y_{\alpha_2...\alpha_{2s})}|_{\theta=0}, \quad \text{(C.2.9b)} \\
z_{\alpha(2s-2)} & := -\frac{1}{s} (2s-1) D_{(\alpha_1} Z_{\alpha_2...\alpha_{2s-2})}|_{\theta=0}, \quad \text{(C.2.9c)} \\
z_{\alpha(2s-4)} & := -(2s-1) D^3 \theta Z_{\beta\alpha(2s-4)}|_{\theta=0}. \quad \text{(C.2.9d)}
\end{align*}
\]

Reduction of the action (C.2.1) to components leads to the following bosonic action:

\[
S_{\text{bos}} = \left( -\frac{1}{2} s \right) \int d^3x \left\{ -\frac{1}{4} h^{\alpha(2s+2)} \Box h_{\alpha(2s+2)} + \frac{3}{16} \partial_{\bar{\delta}\lambda} h_{\delta\lambda(2s)} \partial^{\beta\gamma} h_{\beta\gamma(2s)} \\
+ \frac{1}{4} (2s-1) \partial_{\delta\lambda} h_{\delta\lambda(2s)} \partial^{(\alpha_1} y_{\alpha_2...\alpha_{2s})}\beta - \frac{1}{4} (2s-1) (s-1) z_{\delta\lambda(2s-2)} \partial^{\beta\gamma} h_{\beta\gamma(2s)} \\
- \frac{1}{4} (2s-1) y^{\alpha(2s)} \Box y_{\alpha(2s)} - \frac{1}{8} (s-2) (2s-1) \partial_{\delta\lambda} y_{\delta\lambda(2s-2)} \partial^{\beta\gamma} y_{\beta\gamma(2s-2)} \\
- (s-1) (2s-1) z^{\alpha(2s)} \Box z_{\alpha(2s)} \\
- \frac{1}{4} (s-1) (s+2) (2s-1) (2s-3) \partial_{\delta\lambda} z_{\delta\lambda(2s-4)} \partial^{\beta\gamma} z_{\beta\gamma(2s-4)} \\
+ (s-1) (2s-1) \partial_{\beta\gamma} y_{\beta\gamma(2s-2)} \partial^{\delta} (\alpha_1} z_{\alpha_2...\alpha_{2s-2})\delta \\
- \frac{s}{4} \frac{2s-3}{(s-1)(2s-1)} (4s^2 - 12s + 11) z^{\alpha(2s-4)} \Box z_{\alpha(2s-4)} \\
+ \frac{3s}{8(s-1)(2s-1)} (s-2) (2s-3) (2s-5) \partial_{\delta\lambda} z_{\delta\lambda(2s-6)} \partial^{\beta\gamma} z_{\beta\gamma(2s-6)} \\
+ \frac{1}{4} (s+1) (2s-3) z^{\alpha(2s-4)} \partial^{\beta\gamma} y_{\beta\gamma(2s-4)} \\
+ \frac{1}{2} (s-2) (2s+1) (2s-3) \partial_{\beta\gamma} z^{\beta\gamma(2s-4)} \partial^{\delta} (\alpha_1} z_{\alpha_2...\alpha_{2s-4})\delta \right\}, \quad \text{(C.2.10)}
\]
which proves to be invariant under gauge transformations of the form

\[
\begin{align*}
\delta \xi h_{\alpha(2s+2)} &= \partial_{(\alpha_1 \alpha_2 \xi_{\alpha_3 \cdots \alpha_{2s+2}})} , \\
\delta \xi y_{\alpha(2s)} &= -\frac{1}{s+1} \partial^\beta (\alpha_1 \xi_{\alpha_2 \cdots \alpha_{2s}})\beta - \partial_{(\alpha_1 \alpha_2 \rho_{\alpha_3 \cdots \alpha_{2s}})} , \\
\delta \xi,\rho y_{\alpha(2s-2)} &= \frac{1}{2s(2s+1)} \partial^\beta \xi_{\beta \gamma \alpha(2s-2)} + \frac{1}{s} \partial^\beta (\alpha_1 \rho_{\alpha_2 \cdots \alpha_{2s-2}})\beta , \\
\delta \rho z_{\alpha(2s-4)} &= \partial^\gamma \rho_{\beta \gamma \alpha(2s-4)} .
\end{align*}
\]

\[\text{(C.2.11a)}\]

\[\text{(C.2.11b)}\]

\[\text{(C.2.11c)}\]

\[\text{(C.2.11d)}\]

Let us consider the fermionic sector. We find that the independent fermionic fields are:

\[
\begin{align*}
h_{\alpha(2s+1)} := & \frac{i}{4} D^2 H_{\alpha(2s+1)}|_{\theta=0} , \\
y_{\alpha(2s-1)} := & \frac{i}{8} D^2 Y_{\alpha(2s-1)}|_{\theta=0} , \\
y_{\alpha(2s-3)} := & \frac{i}{2} s(2s - 1) D^2 Z_{\alpha(2s-3)}|_{\theta=0} ,
\end{align*}
\]

\[\text{(C.2.12a)}\]

\[\text{(C.2.12b)}\]

\[\text{(C.2.12c)}\]

and their gauge transformation laws are given by

\[
\begin{align*}
\delta \lambda h_{\alpha(2s+1)} &= \partial_{(\alpha_1 \alpha_2 \lambda_{\alpha_3 \cdots \alpha_{2s+2}})} , \\
\delta \lambda y_{\alpha(2s-1)} &= \frac{1}{2s+1} \partial^\beta (\alpha_1 \lambda_{\alpha_2 \cdots \alpha_{2s-1}})\beta , \\
\delta \lambda y_{\alpha(2s-3)} &= \partial^\gamma \lambda_{\beta \gamma \alpha(2s-3)} .
\end{align*}
\]

\[\text{(C.2.13a)}\]

\[\text{(C.2.13b)}\]

\[\text{(C.2.13c)}\]

The above fermionic fields correspond to the dynamical variables of the Fang-Fronsdal spin-\((s + \frac{1}{2})\) model. As follows from \[\text{(C.2.13a)}, \text{(C.2.13b)}\] and \(\text{(C.2.13c)}\), their gauge freedom is equivalent to that of the massless spin-\((s + \frac{1}{2})\) gauge field. Indeed, direct calculations of the component action give the standard massless gauge-invariant spin-\((s + \frac{1}{2})\) action \(S_{FF}^{(2s+1)}\).

The component structure of the obtained supermultiplets is a three-dimensional counterpart of so-called (reducible) higher-spin triplet systems. In AdS\(_D\) an action for bosonic higher-spin triplets was constructed in \[188\] and for fermionic triplets in \[189, 190\]. Our superfield construction provides a manifestly off-shell supersymmetric generalisation of these systems. It might be of interest to extend it to AdS\(_4\).
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